A generalized forward-backward splitting operator: nonexpansiveness, convergence rates and applications

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Abstract

In this paper, we consider a generalized forward-backward splitting (G-FBS) operator for solving the monotone inclusions, and analyze its nonexpansive properties in a context of arbitrary variable metric. Then, for the associated fixed-point iterations (i.e. the G-FBS algorithms), the global ergodic and pointwise convergence rates of metric distance are obtained from the nonexpansiveness. The convergence in terms of objective function value is also investigated, when the G-FBS operator is applied to a minimization problem. A main contribution of this paper is to show that the G-FBS operator provides a simplifying and unifying framework to model and analyze a great variety of operator splitting algorithms, where the convergence behaviours can be easily described by the fixed-point construction of this simple operator.

Keywords Generalized forward-backward splitting (G-FBS), nonexpansiveness, convergence rates, operator splitting algorithms

AMS subject classifications 68Q25, 47H05, 90C25, 47H09

1 Introduction

1.1 Forward-backward splitting algorithms

The forward-backward splitting (FBS) algorithm has become a standard solver for finding a zero point of the sum of a maximally monotone operator $A : \mathcal{H} \mapsto 2^\mathcal{H}$ and a $\beta^{-1}$-cocoercive operator $B : \mathcal{H} \mapsto \mathcal{H}^1$:

$$0 \in (A + B)b^*$$

It reads as [33, 54, 62]:

$$b^{k+1} := (I + \tau A)^{-1}(I - \tau B)b^k := J_{\tau A} (I - \tau B)b^k$$

(2)

where $\tau > 0$ is a step size, $J_{\tau A}$ denotes a resolvent of $\tau A$. Here, $I - \tau B$ performs the forward (explicit) computation w.r.t. the operator $B$, followed by a backward (implicit) step of $(I + \tau A)^{-1}$ w.r.t. the

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Note that (1) essentially encompasses $v \in (A' + B)b^*$ for any constant vector $v$. Indeed, $0 \in (A + B)b^*$ is equivalent to $v \in (A' + B)b^*$, if we define $A' : \mathcal{H} \mapsto 2^\mathcal{H} : b \mapsto Ab + v$. It is easy to see that $A'$ is monotone, if $A$ is monotone.
operator $\mathcal{A}$ [3]. The scheme (2) can be seen as a fixed point iteration of the classical FBS operator $T := J_{\tau \mathcal{A}} \circ (I - \tau \mathcal{B})$. In this paper, we extend the operator (2) from scalar $\tau$ to arbitrary metric $Q$:

$$T := (Q + \mathcal{A})^{-1}(Q - \mathcal{B}) := J_{Q^{-1} \mathcal{A}} \circ (I - Q^{-1} \mathcal{B})$$  \hspace{1cm} (3)

and study the associated generalized FBS (G-FBS) algorithm [28]:

$$b^{k+1} := (Q + \mathcal{A})^{-1}(Q - \mathcal{B})b^k := J_{Q^{-1} \mathcal{A}} \circ (I - Q^{-1} \mathcal{B})b^k \hspace{1cm} (4)$$

If $\mathcal{A} = \partial g$, $\mathcal{B} = \nabla f$ for some functions $f$ and $g$, the problem (1) becomes finding a minimizer of $f + g$: $b^* \in \text{Argmin}(f + g)$. The algorithm (2) becomes the well-known proximal FBS (P-FBS) algorithm [30, 25]:

$$b^{k+1} := (I + \tau \partial g)^{-1}(I - \tau \nabla f)b^k = \text{prox}_{\tau g} (b^k - \tau \nabla f(b^k)) \hspace{1cm} (5)$$

where $\text{prox}_{\tau g}$ is defined as: $\text{prox}_{\tau g} : \mathcal{H} \mapsto \mathcal{H} : x \mapsto \arg \min_u g(u) + \frac{1}{2\tau} \|u - x\|^2$ [5, Definition 12.23],[30, Eq. (2.13)]. The generalized FBS operator (3) yields the so-called variable metric P-FBS algorithm:

$$b^{k+1} := (\partial g + Q)^{-1}(Q - \nabla f)b^k = \text{prox}_Q^g (b^k - Q^{-1} \nabla f(b^k)) \hspace{1cm} (6)$$

where the generalized proximity operator $\text{prox}_Q^g$ is defined as: $\text{prox}_Q^g : \mathcal{H} \mapsto \mathcal{H} : x \mapsto \arg \min_u g(u) + \frac{1}{2} \|u - x\|_Q^2$ [51]. The basic P-FBS (5) has been extensively studied in the literature, e.g. [30, 25] in a convex setting, and further discussed in [1, 2, 10] for the nonconvex case. The generalized scheme (6) was recently studied in [23, 22] in a nonconvex setting. Though the above FBS schemes have been well understood, the nonexpansive properties of the G-FBS operator (3) are rarely discussed, which is the first focus of the paper.

Nowadays, there has been a revived interest in the design and analysis of the first-order operator splitting algorithms [68], typically including the aforementioned P-FBS, Douglas-Rachford splitting (DRS) [34], alternating direction method of multipliers (ADMM) [40, 39], primal-dual splitting (PDS) [75, 19] and Bregman methods [60, 73, 41, 74]. The convergence analysis of these algorithms is often performed case-by-case. Though some unified frameworks and tools have recently been proposed, e.g. [53, 29, 68, 49, 50, 48, 8], these works do not connect the operator splitting algorithms with a simple and unified nonexpansive mapping. The second purpose of this paper is to show that most classes of splitting algorithms can be simply expressed by the G-FBS operator (3) or its relaxed version.

1.2 Contributions

We first study the nonexpansiveness of the G-FBS operator (3). Then, for the associated fixed-point Banach-Picard and Krasnosel’skii-Mann iterations, we establish the global pointwise/nonergodic and ergodic convergence rates in terms of the solution distance and asymptotic regularity. The convergence rates in terms of objective function value are further presented, when the operators $\mathcal{A}$ and $\mathcal{B}$ are associated with the functions $f$ and $g$.

The main results can be applied to many existing operator splitting methods. In particular, we show that a great variety of popular algorithms can be uniformly represented by the G-FBS operator (3), by specifying the operators $\mathcal{A}$ and $\mathcal{B}$, variable metric $Q$ (and relaxation matrix $M$, if necessary). This unification and simplification helps to understand these algorithms with substantially simplified analysis, compared to the original proofs in the literature.
1.3 Related work

Nonexpansive mappings  The nonexpansive properties in a context of arbitrary variable metric have recently been revisited in [70], which lays the foundation for analyzing the G-FBS operator (3). This work can also be seen an extension of the previous note [71], which focused on the metric resolvent $J_{Q^{-1}A}$—a special case of G-FBS operator with $B : b \mapsto 0, \forall b \in \mathcal{H}$. It should be stressed that the metric resolvent cannot interpret the P-FBS algorithm (5) or (6), which, on the contrary, perfectly fits into the G-FBS operator (3).

Unified frameworks  Several frameworks have recently been proposed to unify the existing optimization algorithms. Fejér sequence [29] is rather abstract, and not directly connected to the specific algorithms at hand. In a most recent paper [27], various popular operator splitting methods were revisited by a fixed-point construction, including P-FBS, DRS, ADMM and PDS algorithms. The fixed-point iteration of nonexpansive mapping has also been investigated in [53], with more emphasis on the applications to the operator splitting algorithms. Even more complicated algorithms, e.g. three-operator splitting algorithm [33], GFBS algorithm [63], can be viewed as a nonexpansive mapping. However, in these works, the nonexpansive mapping depends on specific algorithm, for which the nonexpansive properties have to be analyzed case-by-case. Our work differs from them in that: we represent the existing algorithms using a unified and explicit form of the G-FBS operator (3), which provides a unified treatment of many classes of algorithms.

The AFBA framework was proposed in [49, 50], to generalize the classical splitting schemes, e.g. DRS and FBS. However, it fails to cover the multiple block algorithms, e.g. extended ADMM algorithms [4, 47, 20], and does not simplify the algorithms into a simple nonexpansive mapping. The works of [68, 16] discussed the Bregman proximal mapping and the associated Bregman proximal gradient algorithm. It remains unclear that how to use the framework to analyze the splitting algorithms.

Operator splitting algorithms  We show that most splitting algorithms essentially take the (relaxed) G-FBS structure (often equipped with a designed product space), despite of their seemingly different splitting strategies and algorithm structures. Consequently, the convergence behaviours can be analyzed by a unified treatment.

Variable metric proximal point algorithms (PPA)  The fixed-point iteration of the G-FBS operator can be rewritten in an inclusion form:

$$0 \in Ab^{k+1} + Bb^k + Q(b^{k+1} - b^k)$$

which encompasses the variable metric PPA: $0 \in Ab^{k+1} + Q(b^{k+1} - b^k)$ [61, 17, 11] as a special case of (7) with the operator $B : b \mapsto 0, \forall b \in \mathcal{H}$. The variable metric PPA were also discussed in [45, 44, 57] using the tool of variational inequality. With the success of reinterpretations of DRS, ADMM and PDHG by the variable metric PPA, these works, however, fail to cover the P-FBS and many PDS algorithms, which can be addressed by incorporating the cocoercive operator $B$.

1.4 Notations and definitions

We use standard notations and concepts from convex analysis and variational analysis, which, unless otherwise specified, can all be found in the classical and recent monographs [64, 65, 5, 6].
A few more words about our notations are in order. The classes of positive semi-definite (PSD) and positive definite (PD) matrices are denoted by \( \mathcal{M}^+ \) and \( \mathcal{M}^{++} \), respectively. The classes of symmetric, symmetric and PSD, symmetric and PD matrices are denoted by \( \mathcal{M}_S, \mathcal{M}^+_S \), and \( \mathcal{M}^{++}_S \), respectively. For our specific use, the \( Q \)-based inner product (where \( Q \) is an arbitrary square matrix) is defined as: 
\[
\langle a, b \rangle_Q := \langle Qa, b \rangle = \langle a, Q^\top b \rangle, \quad \forall (a, b) \in H \times H; \quad \text{the Q-norm is defined as:} \quad \|a\|_Q^2 := \langle Qa, a \rangle, \quad \forall a \in H.
\]
Note that unlike the conventional treatment in the literature, symmetric and PSD, symmetric and PD matrices are denoted by \( M \), positive definite (PD) matrices are denoted by \( M^+ \), respectively. For our specific use, the \( Q \)-based inner product (where \( Q \) is an arbitrary square matrix) is defined as: 
\[
\langle a, b \rangle_Q := \langle Qa, b \rangle = \langle a, Q^\top b \rangle, \quad \forall (a, b) \in H \times H; \quad \text{the Q-norm is defined as:} \quad \|a\|_Q^2 := \langle Qa, a \rangle, \quad \forall a \in H.
\]
Note that unlike the conventional treatment in the literature, \( Q \) is not assumed to be symmetric and PSD here (e.g. see Section 4), and hence, \( \| \cdot \|_Q \) is not always well–defined.

Note that our expositions in Sections 2 and 3 are largely based on the nonexpansive properties in the context of arbitrary variable metric \( Q \), which have been thoroughly discussed in [70]. For sake of completeness and convenience, a key notion of \( Q \)-based \( \xi \)-Lipschitz \( \alpha \)-averaged is restated here.

**Definition 1.1.** [70, Definition 2.2] An operator \( T : H \mapsto H \) is said to be \( Q \)-based \( \xi \)-Lipschitz \( \alpha \)-averaged with \( \xi \in ]0, +\infty[ \) and \( \alpha \in ]0, 1[ \), denoted by \( T \in \mathcal{F}_{Q, \xi, \alpha}^\xi \), if there exists a \( Q \)-based \( \xi \)-Lipschitz continuous operator \( K : H \mapsto H \), such that \( T = (1 - \alpha)I + \alpha K \). In particular, if \( \xi \in ]1, +\infty[ \), \( T \) is \( Q \)-weakly averaged; if \( \xi \in ]0, 1[ \), \( T \) is \( Q \)-strongly averaged.

**Lemma 1.2.** Let an operator \( B : H \mapsto H \) be \( \beta^{-1} \)-cocoercive. If \( Q \in \mathcal{M}^{++}_S \) and \( Q \succeq \nu I \) with \( \nu > 0 \), then, \( Q^{-1}B \) is \( Q \)-based \( \frac{\nu}{\beta} \)-cocoercive.

**Proof.** We deduce that:
\[
\langle b_1 - b_2 | Q^{-1}Bb_1 - Q^{-1}Bb_2 \rangle_Q = \langle b_1 - b_2 | BBb_1 - BBb_2 \rangle \geq \frac{1}{\beta} \| BBb_1 - BBb_2 \|^2 = \frac{1}{\beta} \| Q^{-1}Bb_1 - Q^{-1}Bb_2 \|^2 = \frac{\nu}{\beta} \| Q^{-1}Bb_1 - Q^{-1}Bb_2 \|^2_Q 
\]

\[\blacksquare\]

## 2 The nonexpansiveness of G-FBS operator

In this part, we will focus on the variable metric \( Q \) satisfying the following assumption:

**Assumption 2.1.** \( Q \in \mathcal{M}^{++}_S, \quad \nu_1 I \preceq Q \preceq \nu_2 I, \quad (\nu_1, \nu_2) \in ]0, +\infty[^2 \)

### 2.1 Nonexpansive properties

**Lemma 2.2.** Given the operator \( T : H \mapsto H \) defined in (3), then, under Assumption 2.1, the following hold:

(i) \( \| TB_1 - TB_2 \|^2_Q \leq \langle b_1 - b_2 | TB_1 - TB_2 \rangle_Q - \langle BBb_1 - BBb_2 | TB_1 - TB_2 \rangle \)

(ii) \( \langle b_1 - b_2 | TB_1 - TB_2 \rangle_Q \geq \| TB_1 - TB_2 \|^2_Q - \frac{\beta}{4} \| (I - T)b_1 - (I - T)b_2 \|^2 \)

(iii) \( \| TB_1 - TB_2 \|^2_Q + \left( 1 - \frac{\beta}{2\nu_1} \right) \| (I - T)b_1 - (I - T)b_2 \|^2_Q \leq \| b_1 - b_2 \|^2_Q \)
Proof. (i) We develop:
\[
\|Tb_1 - Tb_2\|_Q^2 = \langle QTb_1 - QTb_2 | Tb_1 - Tb_2 \rangle \\
\leq \langle QTb_1 - QTb_2 | Tb_1 - Tb_2 \rangle + \langle ATb_1 - ATb_2 | Tb_1 - Tb_2 \rangle \text{ by monotonicity of } A \\
= \langle (Q - B)b_1 - (Q - B)b_2 | Tb_1 - Tb_2 \rangle_Q \text{ since } Q - B \in QT + AT \text{ by (3)} \\
= \langle b_1 - b_2 | Tb_1 - Tb_2 \rangle_Q - \langle b_1 - b_2 | Tb_1 - Tb_2 \rangle \\
\]
Substituting it into (i) obtains (ii).

(ii) Observe in (i) that the averagedness of \(\gamma\)–firmly nonexpansive, if \(T \in F\).

(iii) We develop:
\[
\|Tb_1 - Tb_2\|_Q^2 = \langle QTb_1 - QTb_2 | Tb_1 - Tb_2 \rangle \\
\leq \langle QTb_1 - QTb_2 | Tb_1 - Tb_2 \rangle + \langle ATb_1 - ATb_2 | Tb_1 - Tb_2 \rangle \text{ by monotonicity of } A \\
= \langle (Q - B)b_1 - (Q - B)b_2 | Tb_1 - Tb_2 \rangle_Q \text{ since } Q - B \in QT + AT \text{ by (3)} \\
= \langle b_1 - b_2 | Tb_1 - Tb_2 \rangle_Q - \langle b_1 - b_2 | Tb_1 - Tb_2 \rangle \\
\]
which yields (iii).

Proposition 2.3. Let \(\mathcal{T}\) be defined as (3). Under Assumption 2.1, if \(\beta < 2\nu_1\), then, the following hold.

(i) \(\mathcal{T} \in \mathcal{F}_{\frac{2\nu_1}{4\nu_1 - \beta}}^Q\).

(ii) \(\mathcal{T}\) is \(Q\)-strongly averaged and \(Q\)-nonexpansive, but not \(Q\)-firmly nonexpansive.

(iii) \(I - \gamma(I - \mathcal{T}) \in \mathcal{F}_{\frac{4\nu_1}{2\nu_1 - \beta}}^Q\), if \(\gamma \in [0, 2 - \frac{\beta}{2\nu_1}].\)

(iv) \(I - \gamma(I - \mathcal{T})\) is \(Q\)-strongly averaged and \(Q\)-nonexpansive. In particular, \(I - \gamma(I - \mathcal{T})\) is \(Q\)-firmly nonexpansive, if \(\gamma \in [0, 1 - \frac{\beta}{4\nu_1}].\)

Proof. (i) Lemma 2.2–(iii) and Definition 1.1.

(ii) Observe in (i) that the averagedness of \(\mathcal{T}\) is \(\alpha = \frac{2\nu_1}{4\nu_1 - \beta} > \frac{1}{2}\).

(iii)–(iv): [70, Lemma 2.7–(iii) and (iv)].

Remark 1. The condition \(\beta < 2\nu_1\) in Proposition 2.3 is equivalent to \(Q \succ \frac{\beta}{2} I\) or \(Q - \frac{\beta}{2} I > 0\).
2.2 Another view on the averagedness of the operator

Lemma 2.2 can also be verified by the composition of $T$ given by (3).

**Lemma 2.4.** Given $T$ defined as (3), then, under Assumption 2.1, the following hold.

(i) $J_{Q^{-1},A} \in F_{1,1/2}^{Q}$

(ii) $I - Q^{-1}B \in F_{1,1/2}^{Q}$, if $\beta < 2\nu_1$. In particular, if $\beta \leq \nu_1$, $I - Q^{-1}B$ is $Q$–firmly nonexpansive.

(iii) $T \in F_{1,2\nu_1/\beta}^{Q}$, if $\beta < 2\nu_1$.

**Proof.** (i) Noting that $Q \in (A + Q)J_{Q^{-1},A}$, we have:

\[
\left\| J_{Q^{-1},A}(b_1) - J_{Q^{-1},A}(b_2) \right\|^2_Q = \left\langle QJ_{Q^{-1},A}(b_1) - QJ_{Q^{-1},A}(b_2), J_{Q^{-1},A}(b_1) - J_{Q^{-1},A}(b_2) \right\rangle \\
\leq \left\langle QJ_{Q^{-1},A}(b_1) - QJ_{Q^{-1},A}(b_2), J_{Q^{-1},A}(b_1) - J_{Q^{-1},A}(b_2) \right\rangle + \left\langle AJ_{Q^{-1},A}(b_1) - AJ_{Q^{-1},A}(b_2), J_{Q^{-1},A}(b_1) - J_{Q^{-1},A}(b_2) \right\rangle \\
= \left\langle b_1 - b_2, J_{Q^{-1},A}(b_1) - J_{Q^{-1},A}(b_2) \right\rangle_Q
\]

which implies that $J_{Q^{-1},A}$ is $Q$–firmly nonexpansive, i.e. $Q$–based $1/2$–averaged.

(ii) First, $Q^{-1}B$ is $Q$–based $\frac{4\beta}{\nu_1}$–cocoercive, by Lemma 1.2. Then, if $\frac{\beta}{\nu_1} \in ]0, 2[$, $Q^{-1}B \in F_{2\nu_1 - \beta, 2\nu_1}^{Q}$, and $I - Q^{-1}B \in F_{1,2\nu_1/\beta}^{Q}$, by [70, Lemma 2.11].

(iii) Since $T = J_{Q^{-1},A} \circ (I - Q^{-1}B)$, with $\alpha_1 = \frac{1}{2}$ and $\alpha_2 = \frac{\beta}{2\nu_1}$, the averagedness of $T$, by [58, Theorem 3] or [31, Proposition 2.4], is given as: $\alpha = \frac{\alpha_1 + \alpha_2 - 2\alpha_1 \alpha_2}{1 - \alpha_1 \alpha_2} = \frac{2\nu_1}{4\nu_1 - \beta}$. \hfill \□

**Remark 2.** Lemma 2.4–(iii) is an extended version of [63, Proposition 4.14], from a scalar parameter $\gamma$ to any variable metric $Q$. As observed in [53, Remark 1], Lemma 2.4–(iii) is sharper than [5, Proposition 4.32], which gives the averagedness of $T$ as:

\[
\alpha = \frac{2}{1 + \max\{\frac{1}{2}, \frac{2\nu_1}{3\beta}\}} = \frac{2\beta}{\beta + 2 \times \min\{\beta, \nu_1\}} = \max\left\{\frac{2}{3}, \frac{2\beta}{\beta + 2\nu_1}\right\}
\]

2.3 The nonexpansive properties of $I - T$

**Proposition 2.5.** Given $T$ defined as (3), denote $R := I - T$. Under Assumption 2.1, if $\beta < 2\nu_1$, then, the following hold.

(i) $R$ is $Q$–based $(1 - \frac{\beta}{4\nu_1})$–cocoercive;

(ii) $R \in F_{\frac{2\nu_1}{2\nu_1 - \beta}, \frac{4\nu_1}{2\nu_1 - \beta}}^{Q}$;

(iii) $R$ is $Q$–weakly averaged, with the averagedness $\alpha \in ]0, \frac{1}{2}[$;

(iv) $I - \gamma R \in F_{1,\frac{2\nu_1}{4\nu_1 - \beta}}^{Q}$, if $\gamma \in ]0, 2 - \frac{\beta}{2\nu_1}[$.

**Proof.** (i) By Lemma 2.2–(i), and $R = I - T$, we have:

\[
0 \leq \left\langle Rb_1 - Rb_2, J_{b_1} - J_{b_2} \right\rangle_Q - \left\langle B(b_1 - b_2), J_{b_1} - J_{b_2} \right\rangle
\]

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Adding $\|Rb_1 - Rb_2\|_Q^2$ on both sides, we obtain:

$$\|Rb_1 - Rb_2\|_Q^2 \leq \langle b_1 - b_2 | Rb_1 - Rb_2 \rangle_Q - \langle Bb_1 - Bb_2 | Tb_1 - Tb_2 \rangle_Q - \langle Bb_1 - Bb_2 | Tb_1 - Tb_2 \rangle_Q$$

$$= \langle b_1 - b_2 | Rb_1 - Rb_2 \rangle_Q + \frac{1}{\beta} \|Bb_1 - Bb_2\|^2 + \frac{\beta}{4} \|Rb_1 - Rb_2\|^2 - \frac{1}{\beta} \|Bb_1 - Bb_2\|^2$$

which yields (i).

(ii): [70, Lemma 2.11].

(iii): note that $\xi = \frac{2\mu}{2\nu_1 - \beta} > 1$, $\alpha = \frac{2\mu - \beta}{4\nu_1 - \beta} < \frac{1}{2}$.

(iv) Lemma 2.3–(iii) or combine Proposition 2.5–(i) with [70, Lemma 2.7–(vi)].

\[\square\]

2.4 The case of strongly monotone $\mathcal{A}$

Under the condition of $\mu$–strongly monotone $\mathcal{A}$, Lemma 2.2 and Proposition 2.5 can be strengthened as follows.

**Lemma 2.6.** Define the operator $T$ as (3), denote $R = I - T$. Under Assumption 2.1, if $\mathcal{A}$ in (3) is $\mu$–strongly monotone, the following hold:

(i) $(1 + \frac{\mu}{\nu_2}) \|Tb_1 - Tb_2\|_Q^2 \leq \langle b_1 - b_2 | Tb_1 - Tb_2 \rangle_Q - \langle Bb_1 - Bb_2 | Tb_1 - Tb_2 \rangle_Q - \langle Bb_1 - Bb_2 | Tb_1 - Tb_2 \rangle_Q$

(ii) $\langle b_1 - b_2 | Tb_1 - Tb_2 \rangle_Q \geq (1 + \frac{\mu}{\nu_2}) \|Tb_1 - Tb_2\|_Q^2 - \frac{\beta}{4} \|I - T\|b_1 - (I - T)b_2\|_Q^2$

(iii) $\langle b_1 - b_2 | Rb_1 - Rb_2 \rangle_Q \geq \left(1 - \frac{\beta}{2\nu_1}\right) \|Rb_1 - Rb_2\|_Q^2 + \frac{\mu}{\nu_2} \|Tb_1 - Tb_2\|_Q^2$

(iv) $(1 + \frac{2\mu}{\nu_2}) \|Tb_1 - Tb_2\|_Q^2 + \left(1 - \frac{\beta}{2\nu_1}\right) \|I - T\|b_1 - (I - T)b_2\|_Q^2 \leq \|b_1 - b_2\|_Q^2$

**Proof.** Combining Lemma 2.2 and Proposition 2.5 with the $\mu$–strongly monotone condition of $\mathcal{A}$:

$$\langle Ab_1 - Ab_2 | b_1 - b_2 \rangle \geq \mu \|b_1 - b_2\|_Q^2 \geq \frac{\mu}{\nu_2} \|b_1 - b_2\|_Q^2$$

completes the proof.

\[\square\]

**Proposition 2.7.** Under the condition of Lemma 2.6, if $\beta < 2\nu_1$, the following hold:

(i) $T \in \mathcal{F}_Q^{\xi_1, \alpha_1}$ with $\xi_1 = \sqrt{1 + \frac{\mu}{\nu_1 \nu_2}}$ and $\alpha_1 = \frac{1 + \beta}{2 - \frac{\beta}{2\nu_1} + \frac{\mu}{\nu_2}}$; $R \in \mathcal{F}_Q^{\xi_2, \alpha_2}$ with $\xi_2 = \sqrt{1 + \frac{\mu}{\nu_1 \nu_2}}$ and $\alpha_2 = \frac{1 - \beta}{2 - \frac{\beta}{2\nu_1} + \frac{\mu}{\nu_2}}$.

(ii) $T$ is $Q$–strongly averaged with $\alpha_1 \in \left[\frac{1}{2}, 1\right]$, $R$ is $Q$–weakly averaged with $\alpha_2 \in \left[0, \frac{1}{2}\right]$.

(iii) Neither $T$ nor $R$ is $Q$–firmly nonexpansive.

(iv) $T$ is not $Q$–cocoercive; $R$ is $Q$–based ($1 - \frac{\beta}{4\alpha_1}$)–cocoercive.

**Proof.** (i) The averagedness of $T$ follows by Lemma 2.6–(iv) and [70, Theorem 2.8–(i)].
(ii) It is easy to see that \( \xi_1 \in ]0, 1[ \) and \( \xi_2 \in ]1, +\infty[ \), by recognizing that

\[
1 + \frac{\mu \beta}{\nu_1 \nu_2} = \left(1 + \frac{2\mu}{\nu_2}\right)^2 - \frac{2\mu}{\nu_2} \left(2 - \frac{\beta}{2\nu_1} + \frac{2\mu}{\nu_2}\right) = \left(1 - \frac{\beta}{2\nu_1}\right)^2 + \frac{\beta}{2\nu_1} \left(2 - \frac{\beta}{2\nu_1} + \frac{2\mu}{\nu_2}\right)
\]

(iii) follows from \([70, \text{Theorem 2.8–(ii)}]\), by noting that \( \xi_1 > \frac{1-\alpha_1}{\alpha_1} \) and \( \xi_2 > 1 \).

(iv) follows from \([70, \text{Theorem 2.8–(iii)}]\), by noting that \( \xi_1 \in ]\frac{1-\alpha_1}{\alpha_1}, 1[ \) and \( \xi_2 \in ]1, \frac{1-\alpha_2}{\alpha_2}]. \) The cocoerciveness of \( \mathcal{R} \) coincides with Proposition 2.5–(i). \( \square \)

3 The fixed-point iterations

3.1 The Banach-Picard iteration (4)

3.1.1 Convergence of metric distance

The convergence properties of (4) are given below.

**Theorem 3.1** (Convergence in terms of metric distance). Let \( b^0 \in \mathcal{H}, \{b^k\}_{k \in \mathbb{N}} \) be a sequence generated by (4). Under Assumption 2.1, if \( \nu_1 > \frac{\beta}{2} \), then, the following hold.

(i) \( T \) is \( \mathcal{Q} \)–asymptotically regular.

(ii) [Basic convergence] There exists \( b^* \in \text{zer}(A + B) \), such that \( b^k \to b^* \), as \( k \to \infty \).

(iii) [Sequential error] \( \|b^{k+1} - b^k\|_Q \) has the pointwise sublinear convergence rate of \( O(1/\sqrt{k}) \):

\[
\|b^{k+1} - b^k\|_Q \leq \frac{1}{\sqrt{k+1}} \sqrt{\frac{2\nu_1}{2\nu_1 - \beta}} \|b^0 - b^*\|_Q, \forall k \in \mathbb{N}
\]

**Proof.** Note that \( T \in \mathcal{F}^Q_{\frac{2\nu_1}{1+\frac{2\nu_1}{\nu_2}}} \), by Proposition 2.3–(i). Then, (i) and (iii) follows from \([70, \text{Theorem 3.3}]. \)

(ii) follows by noting that \( \text{Fix}T = \text{zer}(A + B) \). Indeed, \( b^* \in \text{Fix}T \iff b^* = (A + Q)^{-1}(Q - B)b^* \iff (Q - B)b^* \in (A + Q)b^* \iff 0 \in (A + B)b^* \iff b^* \in \text{zer}(A + B). \) \( \square \)

If \( A \) in (4) is \( \mu \)–strongly monotone, then we have the following results.

**Proposition 3.2** (Convergence in terms of metric distance). Let \( b^0 \in \mathcal{H}, \{b^k\}_{k \in \mathbb{N}} \) be a sequence generated by (4). Under Assumption 2.1, if \( A \) is \( \mu \)–strongly monotone, \( \nu_1 > \frac{\beta}{2} \), then, the following hold.

(i) [\( q \)–linear convergence] Both \( \|b^k - b^*\|_Q \) and \( \|b^k - b^{k+1}\|_Q \) are \( q \)–linearly convergent with the rate of \( \frac{1}{\sqrt{1+\frac{2\nu_1}{\nu_2}}} \).

(ii) [\( r \)–linear convergence] If \( \mu \geq \frac{\nu_2}{\sqrt{5+1}}, \|b^k - b^{k+1}\|_Q \) is globally \( r \)–linearly convergent w.r.t.
\[ \|b^0 - b^*\|_Q : \]
\[ \|b^k - b^{k+1}\|_Q \leq \sqrt{\frac{2\mu}{\nu_2}} \sqrt{\frac{1 + \frac{2\mu}{\nu_2}}{1 - \frac{\beta}{\nu_1}} \cdot \left(1 + \frac{2\mu}{\nu_2}\right)^{-\frac{k+1}{4}} \|b^0 - b^*\|_Q} \]

The above inequality is also locally satisfied for \( k \geq \ln((1+\sqrt{5})/2) / \ln(\sqrt{1+\frac{2\nu_2}{\nu_1}} - 1) \), if \( \mu < \nu_1 \).

**Proof.** Combine Proposition 2.7–(i) with [70, Theorem 3.3–(iv) and (v)]. \( \square \)

**Remark 3.** If \( Q = \frac{1}{r}I \), the result of [5, Example 27.12] is recovered from Proposition 3.2.

### 3.1.2 Convergence of objective value

If \( A = \partial g, B = \nabla f \), (4) becomes the scheme (6) for minimizing the objective function \( f + g \). We make the following assumptions on the functions \( f \) and \( g \).

**Assumption 3.3.** (i) \( f : \mathcal{H} \rightarrow \mathbb{R} \) and \( g : \mathcal{H} \rightarrow \mathbb{R} \cup \{+\infty\} \) are proper, lower semi-continuous;
(ii) \( f \) is differentiable with \( \beta \)-Lipschitz continuous gradient;
(iii) \( g \) is convex.

The following lemma is important for proving the convergence, which extends the sufficient decrease property [10, Lemma 2] to the case of arbitrary variable metric \( Q \).

**Lemma 3.4** (Sufficient decrease property). Let \( b^0 \in \mathcal{H}, \{b^k\}_{k \in \mathbb{N}} \) be a sequence generated by (6). Then, the following hold.

(i) Under Assumptions 2.1 and 3.3–(i) and (ii), we have:
\[ (f + g)(b^k) - (f + g)(b^{k+1}) \geq \frac{1}{2} \left(1 - \frac{\beta}{\nu_1}\right) \|b^{k+1} - b^k\|_Q^2 \]

(ii) Under Assumptions 2.1 and 3.3, we have:
\[ (f + g)(b^k) - (f + g)(b^{k+1}) \geq \left(1 - \frac{\beta}{2\nu_1}\right) \|b^{k+1} - b^k\|_Q^2 \]

**Proof.** Denote \( h := f + g \). Rewrite (6) in an inclusion form:
\[ 0 \in \nabla f(b^k) + \partial g(b^{k+1}) + Q(b^{k+1} - b^k) \]

By \( \beta \)-Lipschitz continuity of \( \nabla f \) and Descent Lemma [5, Theorem 18.15], we have:
\[ h(b^{k+1}) = f(b^{k+1}) + g(b^{k+1}) \leq f(b^k) + \langle \nabla f(b^k)|b^{k+1} - b^k\rangle + \frac{\beta}{2} \|b^{k+1} - b^k\|_Q^2 + g(b^{k+1}) \]

(i) By the definition of generalized proximity operator, we have:
\[ b^{k+1} = \arg\min_b g(b) + \frac{1}{2} \|b - b^k\|_Q^2 + \langle b - b^k|\nabla f(b^k)\rangle \]
Taking \( b = b^k \), we obtain:

\[
g(b^{k+1}) + \frac{1}{2}||b^{k+1} - b^k||_Q^2 + \langle b^{k+1} - b^k | \nabla f(b^k) \rangle \leq g(b^k) \quad (10)
\]

Combining (9) with (10) yields:

\[
h(b^{k+1}) \leq h(b^k) - \frac{1}{2}||b^k - b^{k+1}||_Q^2 \leq h(b^k) - \frac{1}{2}\left(1 - \frac{\beta}{\nu_1}\right)||b^k - b^{k+1}||_Q^2
\]

(ii) By convexity \( g \), we have:

\[
h(b^k) = f(b^k) + g(b^k) \geq f(b^k) + g(b^{k+1}) + \langle \partial g(b^{k+1}) | b^k - b^{k+1} \rangle \quad (11)
\]

Combining (9) with (11) yields:

\[
h(b^k) - h(b^{k+1}) \geq -\langle \nabla f(b^k) + \partial g(b^{k+1}) | b^{k+1} - b^k \rangle - \frac{\beta}{2}||b^k - b^{k+1}||_Q^2
\]

\[
= \langle b^{k+1} - b^k | b^{k+1} - b^k \rangle_Q - \frac{\beta}{2}||b^k - b^{k+1}||_Q^2 \quad \text{by (8)}
\]

\[
= ||b^{k+1} - b^k||_Q^2 \geq \left(1 - \frac{\beta}{2\nu_1}\right)||b^{k+1} - b^k||_Q^2
\]

\(\square\)

**Remark 4.** Without convexity of \( g \) (i.e. Lemma 3.4–(i)), the sufficient decreasing requires \( \nu_1 \geq \beta \) (i.e. \( Q \succeq I \)). If \( g \) is convex (i.e. Lemma 3.4–(ii)), this condition is relaxed to \( \nu_1 \geq \frac{\beta}{2} \) (i.e. \( Q \succeq \frac{\beta}{2}I \)). This coincides with the observation in [10, Remark 4–(iii)]. In addition, combining (8) with (11), we obtain:

\[
g(b^{k+1}) + \langle b^{k+1} - b^k | \nabla f(b^k) \rangle + ||b^{k+1} - b^k||_Q^2 \leq g(b^k)
\]

which is in agreement with the **sufficient decrease condition** [22, Eq.(7a)].

**Theorem 3.5 (Convergence in terms of objective value).** Let \( b^0 \in \mathcal{H}, \{b^k\}_{k \in \mathbb{N}} \) be a sequence generated by (6). Under Assumptions 2.1 and 3.3, if \( \nu_1 \geq \beta \), the following hold.

(i) **Basic convergence** The sequence \( \{(f + g)(b^k)\}_{k \in \mathbb{N}} \) is non-increasing, and converges to its minimum, which is attained at some point \( b^* \in \text{Arg min}(f + g) \).

(ii) **Non-ergodic rate** The objective value \( h(b^k) \) converges to \( h(b^*) \) with the **non-ergodic rate** of \( O(1/k) \), i.e.

\[
h(b^k) - h(b^*) \leq \frac{1}{2k}||b^0 - b^*||_Q^2
\]

**Proof.** (i) Theorem 3.1, Lemma 3.4–(ii).

(ii) Denote \( h := f + g \). By convexity of \( f \) and \( g \), we have:

\[
h(b^*) = f(b^*) + g(b^*) \geq f(b^k) + \langle \nabla f(b^k) | b^* - b^k \rangle + g(b^{k+1}) + \langle \partial g(b^{k+1}) | b^* - b^{k+1} \rangle \quad (12)
\]

\[
\]
Combining (12) with (9) yields:

$$
\begin{align*}
    h(b^*) - h(b^{k+1}) & \geq -\langle \nabla f(b^k) + \partial g(b^{k+1})|b^{k+1} - b^*\rangle - \frac{\beta}{2} \|b^k - b^{k+1}\|^2 \\
    &= \langle b^{k+1} - b^k|b^{k+1} - b^*\rangle_{\mathcal{Q}} - \frac{\beta}{2} \|b^k - b^{k+1}\|^2 \quad \text{by (8)} \\
    &= \frac{1}{2} \|b^{k+1} - b^k\|^2_{\mathcal{Q} - \beta \mathcal{I}} + \frac{1}{2} \|b^{k+1} - b^*\|^2_{\mathcal{Q}} - \frac{1}{2} \|b^k - b^*\|^2_{\mathcal{Q}} \\
    &\geq \frac{1}{2} \|b^{k+1} - b^*\|^2_{\mathcal{Q}} - \frac{1}{2} \|b^k - b^*\|^2_{\mathcal{Q}} \quad \text{by } \nu_1 \geq \beta
\end{align*}
$$

(13)

The rest of the proof adopts similar techniques with [7, Theorem 3.1]. Multiplying Lemma 3.4–(ii) by $k$, adding (13) from $k = 0$ to $k = K - 1$, and combining them together, we obtain:

$$
K(h(b^K) - h(b^*)) \leq \frac{1}{2} \|b^0 - b^*\|^2_{\mathcal{Q}} - \frac{1}{2} \|b^K - b^*\|^2_{\mathcal{Q}} - \sum_{k=0}^{K-1} k \|b^{k+1} - b^k\|^2_{\mathcal{Q} - \beta \mathcal{I}} \leq \frac{1}{2} \|b^0 - b^*\|^2_{\mathcal{Q}}
$$

which completes the proof.

We further develop the linear convergence in terms of objective value.

**Proposition 3.6** (Convergence in terms of objective value). Under the conditions of Theorem 3.5, Assumptions 2.1 and 3.3, if $g$ is $\mu$–strongly convex, then, the following hold.

(i) [Basic convergence] $h(b^k) - h(b^*) \leq \frac{\mu}{2\nu_2} \cdot \frac{1}{(1 + \frac{\beta}{\mu})^{k-1}} \cdot \|b^0 - b^*\|^2_{\mathcal{Q}}$.

(ii) [r-linear convergence] If $\mu \geq \frac{\sqrt{\nu_1 + 1}}{2} \nu_2$, $h(b^k)$ is globally r-linearly convergent to $h(b^*)$:

$$
    h(b^k) - h(b^*) \leq \frac{\mu}{2\nu_2} \cdot \left(1 + \frac{\nu_1}{\nu_2}\right)^{-\frac{k}{2}} \|b^0 - b^*\|^2_{\mathcal{Q}}
$$

The above r-linear convergence is also locally satisfied, for $k \geq \frac{\ln((1 + \sqrt{\nu_1})/2)}{\ln(1 + \frac{\sqrt{\nu_1}}{\nu_2})}$, if $\mu \in \overline{0, \frac{\sqrt{\nu_1 + 1}}{2} \nu_2}$.

**Proof.** Considering the case of $\mu$-strongly monotone $\mathcal{A}$, the above results are modified as:

$$
\begin{cases}
    h(b^*) - h(b^{k+1}) \geq \frac{1}{2} \|b^{k+1} - b^k\|^2_{\mathcal{Q} - \beta \mathcal{I}} + \frac{1}{2} \left(1 + \frac{\mu}{\nu_2}\right) \|b^{k+1} - b^*\|^2_{\mathcal{Q}} - \frac{1}{2} \|b^k - b^*\|^2_{\mathcal{Q}} \\
    h(b^k) - h(b^{k+1}) \geq \|b^{k+1} - b^k\|^2_{\mathcal{Q} + \frac{\mu}{\nu_2} \mathcal{I}}
\end{cases}
$$

(14)

where the first inequality of (14) uses $\| \cdot \|^2 \geq \frac{1}{\nu_2} \| \cdot \|^2_{\mathcal{Q}}$.

Denoting $\Delta_k = h(b^k) - h(b^*)$, multiplying the second of (14) by $\theta_k \geq 0$, and adding the first of (14), one obtains:

$$
\begin{align*}
    \theta_k \Delta_k - (\theta_k + 1) \Delta_{k+1} & \geq \|b^{k+1} - b^k\|^2_{\theta_k + \frac{1}{2} + \frac{\mu}{\nu_2} \mathcal{I}} + \frac{1}{2} \left(1 + \frac{\mu}{\nu_2}\right) \|b^{k+1} - b^*\|^2_{\mathcal{Q}} - \frac{1}{2} \|b^k - b^*\|^2_{\mathcal{Q}} \\
    &\geq \frac{1}{2} \left(1 + \frac{\mu}{\nu_2}\right) \|b^{k+1} - b^*\|^2_{\mathcal{Q}} - \frac{1}{2} \|b^k - b^*\|^2_{\mathcal{Q}}
\end{align*}
$$
Multiplying by \( \zeta_k \geq 0 \) obtains:

\[
2 \left( \frac{\zeta_k \theta_k}{t_k} \Delta_k - \frac{2 \zeta_k (\theta_k + 1)}{t_{k+1}} \right) \Delta_{k+1} \geq \zeta_k \left( 1 + \frac{\mu}{\nu_2} \right) \| b_k^{k+1} - b^* \| Q - \zeta_k \| b_k - b^* \| Q^2
\]

which yields:

\[
2 t_{k+1} \Delta_{k+1} + \zeta_{k+1} \| b_k^{k+1} - b^* \| Q^2 \leq 2 t_k \Delta_k + \zeta_k \| b_k - b^* \| Q^2 \leq \cdots \leq 2 t_1 \Delta_1 + \zeta_1 \| b_1 - b^* \| Q^2
\]

By the first inequality of (14), we have:

\[
2 t_1 \Delta_1 = 2 t_1 (h(b^1) - h(b^*)) \leq t_1 \| b_0 - b^* \| Q - \left( 1 + \frac{\mu}{\nu_2} \right) t_1 \| b_1 - b^* \| Q - t_1 \| b_1 - b_0 \| Q^2
\]

Substituting (17) into (16) yields:

\[
2 t_k \Delta_k \leq 2 t_1 \Delta_1 + \zeta_1 \| b_1 - b^* \| Q^2
\]

\[
\leq t_1 \| b_0 - b^* \| Q - \left( 1 + \frac{\mu}{\nu_2} \right) t_1 \| b_1 - b^* \| Q + \zeta_1 \| b_1 - b^* \| Q^2
\]

\[
\leq t_1 \| b_0 - b^* \| Q - (t_1 + \frac{\mu}{\nu_2} t_1 - \zeta_1) \| b_1 - b^* \| Q^2
\]

Now, we evaluate \( t_k \). From (15), we have:

\[
\zeta_{k+1} = \left( 1 + \frac{\mu}{\nu_2} \right) \zeta_k; \quad \theta_{k+1} = \frac{1}{1 + \frac{\mu}{\nu_2}} (\theta_k + 1)
\]

which leads to:

\[
\theta_k = \left( \theta_0 - \frac{\eta}{1 - \eta} \right) \eta^k + \frac{\eta}{1 - \eta}; \quad \zeta_k = \zeta_0 \eta^{-k}
\]

where \( \eta := (1 + \frac{\mu}{\nu_2})^{-1} \).

Back to (18). It is easy to check that \( t_1 + \frac{\mu}{\nu_2} t_1 \geq \xi_1 \), as long as \( \theta_0 \geq 0 \). Thus, (18) becomes:

\[
\Delta_k \leq \frac{t_1}{2t_k} \| b_0 - b^* \| Q^2 = \frac{\theta_1 \zeta_1}{2 \theta_k \zeta_k} \| b_0 - b^* \| Q^2 = \frac{\theta_0 + 1}{2 (\theta_0 + \nu_2 (\eta^{-k} - 1))} \| b_0 - b^* \| Q^2
\]

Since \( \kappa \) is increasing with \( \theta_0 \), the best estimate of \( \Delta_k \) follows by letting \( \theta_0 = 0 \).

(ii) It is easy to check that \( (1 + \frac{\mu}{\nu_2})^k - 1 \geq (1 + \frac{\mu}{\nu_2})^{k/2} \), if \( k \geq \frac{\ln(1 + \frac{\nu_2}{\sqrt{2}})}{\ln 1 + \frac{\nu_2}{\sqrt{2}}} \). Then, the r-linear convergence immediately follows from (i).

\[\Box\]

### 3.2 The Krasnosel’skii-Mann iteration

The Krasnosel’skii-Mann iteration of \( \mathcal{T} \) in (3) is given as:

\[
b^{k+1} := b^k + \gamma \left( \mathcal{J}_{Q^{-1}A} (b^k - Q^{-1} \mathcal{J}_b b^k) - b^k \right)
\]

where \( \gamma \) is a relaxation parameter. The scheme (19) is also a Banach-Picard iteration of \( \mathcal{T}_\gamma := \mathcal{I} + \gamma (\mathcal{T} - \mathcal{I}) \). The convergence properties of (19) are given below.
Theorem 3.7 (Convergence in terms of metric distance). Let \( b^0 \in \mathcal{H} \), \( \{b^k\}_{k \in \mathbb{N}} \) be a sequence generated by (19). Under Assumptions 2.1 and 3.3, if \( \nu_1 > \frac{\beta}{2} \) and \( \gamma \in ]0, 2 - \frac{\beta}{2\nu_1}[ \), then, the following hold.

(i) [Basic convergence] There exists \( b^* \in \text{zer}(\mathcal{A} + \mathcal{B}) \), such that \( b^k \to b^* \), as \( k \to \infty \).

(ii) [Sequential error] \( \|b^{k+1} - b^k\|_Q \) has the pointwise sublinear convergence rate of \( O(1/\sqrt{k}) \):

\[
\|b^{k+1} - b^k\|_Q \leq \frac{1}{\sqrt{k+1}} \sqrt{\frac{2\gamma \nu_1}{(4-2\gamma)\nu_1 - \beta}} \|b^0 - b^*\|_Q, \forall k \in \mathbb{N}
\]

Proof. First, \( T_\gamma := I - \gamma(I - T) \in \mathcal{F}^Q_{1, \frac{2\nu_1}{\nu_2} + \frac{\gamma}{2\nu_1}} \), if \( \gamma \in ]0, 2 - \frac{\beta}{2\nu_1}[ \), by Proposition 2.3–(iii). Then, the proof is completed by [70, Theorem 3.3]. Note that (ii) follows by noting that \( \text{Fix} T = \text{Fix} T = \text{zer}(\mathcal{A} + \mathcal{B}) \). \( \square \)

Remark 5. While extending the existing result of the relaxed PFBS [5, Theorem 25.8] to arbitrary variable metric \( Q \), the convergence condition is less restrictive than the above one. In [5, Theorem 25.8], the condition is \( \gamma < \min\{1, \nu_2\} + \frac{1}{2} \), which is obtained by the rough estimate of averagedness of \( T_\gamma : \alpha = \gamma \cdot \max\{\frac{\nu_2}{2(1 + \frac{\nu_2}{2\mu})}, \frac{\mu}{2}\} \) (see Remark 2). By contrast, our result corresponds to the sharper estimate of \( \alpha = \frac{2\nu_1}{4\nu_1 - \beta} \) (i.e. Proposition 2.3–(iii)).

If \( \mathcal{A} \) in (19) is \( \mu \)-strongly monotone, the linear rate is achieved.

Proposition 3.8 (Convergence in terms of \( Q \)-based distance). Let \( b^0 \in \mathcal{H} \), \( \{b^k\}_{k \in \mathbb{N}} \) be a sequence generated by (19). Under Assumptions 2.1 and 3.3, if \( \mathcal{A} \) is \( \mu \)-strongly monotone, \( \beta < \nu_1 \), \( \gamma \leq 1 + \frac{1 - \frac{\beta}{2\nu_1}}{1 + \frac{\nu_2}{\nu_1}} \), then, the following hold.

(i) [\( q \)-linear convergence] Both \( \|b^k - b^*\|_Q \) and \( \|b^k - b^{k+1}\|_Q \) are \( q \)-linearly convergent with the rate of \( \sqrt{1 - \frac{2\gamma \nu_1}{2\mu + \nu_2}} \).

(ii) [\( r \)-linear convergence] If \( \mu \in ]\sqrt{\nu_2} + \gamma, +\infty[ \), \( \gamma \in ]3 - \nu_2(1 + \frac{\nu_2}{2\mu}), 1[ \), \( \|b^k - b^{k+1}\|_Q \) is globally \( r \)-linearly convergent w.r.t. \( \|b^0 - b^*\|_Q \):

\[
\|b^k - b^{k+1}\|_Q \leq \gamma \sqrt{\frac{2\mu}{\nu_2}} \sqrt{\frac{1 + \frac{2\mu}{\nu_2}}{1 - \frac{\beta}{2\nu_1}}} \cdot \left(1 - \frac{2\mu \gamma}{2\mu + \nu_2}\right)^{\frac{k+1}{4}} \|b^0 - b^*\|_Q
\]

Proof. First, \( T_\gamma \in \mathcal{F}^Q_{\xi, \alpha} \) with \( \xi = \sqrt{\frac{1 + \frac{2\mu}{\nu_2}}{1 + \frac{\mu}{\nu_1}}} \) and \( \alpha = \frac{\gamma(1 + \frac{2\mu}{\nu_2})}{2 - \frac{\beta}{2\nu_1} + \frac{2\mu}{\nu_2}} \) by Proposition 2.7–(i) and [70, Theorem 2.8–(iv)]. Then, the results follow from [70, Theorem 3.3]. \( \square \)

4 Further extension: A relaxed version of G-FBS operator

In this sequel, we further consider a relaxed version of G-FBS operator:

\[
\tilde{T} := I + \mathcal{M}\left((\mathcal{A} + \mathcal{Q})^{-1}(\mathcal{Q} - \mathcal{B}) - I\right) = I + \mathcal{M}(\mathcal{T} - I)
\]
where $\mathcal{M}$ is a relaxation matrix. Then, the fixed-point iteration $b^{k+1} := \mathcal{T}b^k$ is equivalent to the following generalized variable metric FBS algorithm:

\[
\begin{bmatrix}
0 \\
\mathcal{A}b^k + \mathcal{B}b^k + Q(\bar{b}^k - b^k)
\end{bmatrix} \quad \text{variable metric FBS step}
\]
\[
b^{k+1} := b^k + \mathcal{M}(\bar{b}^k - b^k) \quad \text{relaxation step}
\]

(21)

To the best of our knowledge, (21) has never been discussed before in the literature. The applications of (21) will be illustrated in Section 5.

Lemma 4.1 presents several key ingredients, which are the ‘recipe’ for proving the convergence of (21).

**Lemma 4.1.** Let $\mathcal{T}$ defined as (3). Let $b^* \in \text{zer} \mathcal{A}$ and $\{b^k\}_{k \in \mathbb{N}}$ be a sequence generated by (21). Denote $S := \mathcal{Q}\mathcal{M}^{-1}$, $\mathcal{G} := \mathcal{Q} + \mathcal{Q}^T - \mathcal{M}^T \mathcal{Q}$, and the operator $\mathcal{R} := \mathcal{I} - \mathcal{T}$. If $\mathcal{A}$ is maximally monotone and $S \in \mathcal{M}_S$. Then, the following hold.

(i) $\|b^{k+1} - b^*\|_S^2 \leq \|b^k - b^*\|_S^2 - \|b^k - b^{k+1}\|_{\mathcal{M}^{-1}(\mathcal{G} - \frac{\beta}{\mathcal{W}_1})\mathcal{M}^{-1}}$

(ii) $\|\mathcal{R}b^k|\mathcal{R}b^k - \mathcal{R}b^{k+1}\|_{\mathcal{M}^T S \mathcal{M}} \geq \frac{1}{2}(1 - \frac{\beta}{\mathcal{W}_1}) \|\mathcal{R}b^k - \mathcal{R}b^{k+1}\|_{\mathcal{Q} + \mathcal{Q}^T}^2$

(iii) $\|b^k - b^{k+1}\|_S^2 - \|b^{k+1} - b^{k+2}\|_S^2 \geq \|\mathcal{R}b^k - \mathcal{R}b^{k+1}\|_{(1 - \frac{\beta}{\mathcal{W}_1})(\mathcal{Q} + \mathcal{Q}^T) - \mathcal{M}^T S \mathcal{M}}^2$

**Proof.** (i) First, we have:

\[
0 \leq \langle \mathcal{A}b^k - \mathcal{A}b^*|\bar{b}^k - b^*\rangle
\]

\[
= \langle \mathcal{A}\mathcal{M}^{-1}(b^k - b^*)|\bar{b}^k - b^*\rangle - \langle \mathcal{B}b^k - \mathcal{B}b^*|\bar{b}^k - b^*\rangle \quad \text{by (21)}
\]

\[
= \langle \mathcal{Q}|b^k - \bar{b}^k\rangle|\bar{b}^k - b^*\rangle - \langle \mathcal{B}b^k - \mathcal{B}b^*|\bar{b}^k - b^*\rangle \quad \text{by (21)}
\]

\[
= \langle \mathcal{S}(b^k - b^+)|b^k - b^* + \mathcal{M}^{-1}(b^{k+1} - b^k)) - \langle \mathcal{S}b^k - \mathcal{S}b^*|\bar{b}^k - b^*\rangle \quad \text{by (21)}
\]

\[
= \langle \mathcal{S}(b^k - b^+)|\mathcal{B}b^k - \mathcal{B}b^*|\bar{b}^k - b^*\rangle
\]

\[
\geq \frac{1}{2}\|b^k - b^+\|_S^2 + \frac{1}{2}\|b^k - b^*\|_S^2 - \frac{1}{2}\|b^{k+1} - b^*\|_S^2 - \frac{1}{2}\|b^k - b^{k+1}\|_{\mathcal{M}^{-1} S \mathcal{S} \mathcal{M}^{-1}}^2
\]

\[
- \langle \mathcal{B}b^k - \mathcal{B}b^*|\bar{b}^k - b^*\rangle
\]

\[
= \frac{1}{2}\|b^k - b^*\|_S^2 - \frac{1}{2}\|b^{k+1} - b^*\|_S^2 - \frac{1}{2}\|b^k - b^{k+1}\|_{\mathcal{M}^{-1} S \mathcal{S} \mathcal{M}^{-1} - S}^2
\]

\[
- \langle \mathcal{B}b^k - \mathcal{B}b^*|\bar{b}^k - b^*\rangle
\]

\[
= \frac{1}{2}\|b^k - b^*\|_S^2 - \frac{1}{2}\|b^{k+1} - b^*\|_S^2 - \frac{1}{2}\|b^k - b^{k+1}\|_{\mathcal{M}^{-1} \mathcal{G}^{-1} \mathcal{M}^{-1} - \mathcal{G}^{-1} \mathcal{M}^{-1} - \mathcal{G}^{-1} \mathcal{M}^{-1} \mathcal{S} \mathcal{S} \mathcal{M}^{-1} - S}^2
\]

\[
- \langle \mathcal{B}b^k - \mathcal{B}b^*|\bar{b}^k - b^*\rangle
\]

\[
= \frac{1}{2}\|b^k - b^*\|_S^2 - \frac{1}{2}\|b^{k+1} - b^*\|_S^2 - \frac{1}{2}\|b^k - b^{k+1}\|_{\mathcal{M}^{-1} \mathcal{G}^{-1} \mathcal{M}^{-1} - \mathcal{G}^{-1} \mathcal{M}^{-1} \mathcal{S} \mathcal{S} \mathcal{M}^{-1} - \mathcal{G}^{-1} \mathcal{M}^{-1} \mathcal{S} \mathcal{S} \mathcal{M}^{-1} - S}^2
\]

\[
- \langle \mathcal{B}b^k - \mathcal{B}b^*|\bar{b}^k - b^*\rangle
\]
1), the last term becomes:

\[-\langle Bb^k - Bb^* | \hat{b}^k - b^* \rangle = -\langle Bb^k - Bb^* | \hat{b}^k - b^k + b^k - b^* \rangle \]

\[= -\langle Bb^k - Bb^* | \hat{b}^k - b^k \rangle - \langle Bb^k - Bb^* | b^k - b^* \rangle \]

\[\leq -\langle Bb^k - Bb^* | \hat{b}^k - b^k \rangle - \frac{1}{\beta} \| Bb^k - Bb^* \|^2 \]

\[\leq \frac{1}{\beta} \| Bb^k - Bb^* \|^2 + \frac{\beta}{4} \| \hat{b}^k - b^k \|^2 - \frac{1}{\beta} \| Bb^k - Bb^* \|^2 \]

\[= \frac{\beta}{4} \| \hat{b}^k - b^k \|^2 = \frac{\beta}{4} \| b^k - b^{k+1} \|^2 \| \mathcal{M}^{-\top} \mathcal{M}^{-1} \] by (21)

which is substituted into (22) yields:

\[\frac{1}{2} \| b^k - b^* \|^2_S - \frac{1}{2} \| b^{k+1} - b^* \|^2_S - \frac{1}{2} \| b^k - b^{k+1} \|^2_{\mathcal{M}^{-\top} \mathcal{M}^{-1}} + \frac{\beta}{4} \| b^k - b^{k+1} \|^2_{\mathcal{M}^{-\top} \mathcal{M}^{-1}} \geq 0 \]

which is equivalent to (i).

(ii) Note that Lemma 2.2–(i) is valid for any \( Q \), not limited to \( Q \in \mathcal{M}^T_S \). Proposition 2.5–(i) can be modified to:

\[\| Rb_1 - Rb_2 \|^2_{Q^T} \leq \langle b_1 - b_2 | Rb_1 - Rb_2 \rangle_{Q^T} + \frac{\beta}{4} \| Rb_1 - Rb_2 \|^2 \]

which leads to:

\[\langle b^k - b^{k+1} | Rb^k - Rb^{k+1} \rangle_{Q^T} \geq \left( 1 - \frac{\beta}{4\nu_1} \right) \| Rb^k - Rb^{k+1} \|^2_{Q^T} = \frac{1}{2} \left( 1 - \frac{\beta}{4\nu_1} \right) \| Rb^k - Rb^{k+1} \|^2_{Q^T} \]

Then, (ii) follows from \( b^k - b^{k+1} = \mathcal{M} Rb^k \) by (21) and \( Q^T = \mathcal{M}^T S \).

(iii) From Lemma 4.1–(ii), we have:

\[\| b^k - b^{k+1} \|^2_S - \| b^{k+1} - b^{k+2} \|^2_S \]

\[= \| \mathcal{M} Rb^k \|^2_S - \| \mathcal{M} Rb^{k+1} \|^2_S \] by (21)

\[= 2 \langle Rb^k | Rb^k - Rb^{k+1} \rangle_{\mathcal{M}^T S M} - \| Rb^k - Rb^{k+1} \|^2_{\mathcal{M}^T S M} \]

\[\geq \| Rb^k - Rb^{k+1} \|^2_{(1 - \frac{\beta}{4\nu_1}) (Q + Q^T) - M^T S M} \] by Lemma 4.1–(ii)

This completes the proof.

The following theorem gives the convergence result.

**Theorem 4.2** (Convergence in terms of metric distance). Let \( b^0 \in \mathcal{H}, \{ b^k \}_{k \in \mathbb{N}} \) be a sequence generated by (21), where \( \mathcal{A} \) is maximally monotone, \( B \) is \( \beta^{-1} \)-cocoercive. If \( Q \mathcal{M}^{-1} \in \mathcal{M}^T_S \) and \( Q + Q^T = \mathcal{M}^T Q = \frac{\beta}{2} I \in \mathcal{M}^T_S \), then the following hold.

(i) [Basic convergence] There exists \( b^* \in \text{zer} \mathcal{A} \), such that \( b^k \to b^* \) as \( k \to \infty \).

(ii) [Sequential convergence] If \( (1 - \frac{\beta}{4\nu_1}) \mathcal{Q} + \mathcal{Q}^T \geq \mathcal{M}^T Q \), then \( \| b^k - b^{k+1} \|_S \) has the non-
ergodic convergence rate of $O(1/\sqrt{k})$, i.e.
\[
\|b^{k+1} - b^k\|_S \leq \frac{1}{\sqrt{k+1}} \sqrt{\frac{\lambda_{\max}(S)}{\lambda_{\min}(M^{-T}(G - \frac{\beta}{2}I)M^{-1})}} \|b^0 - b^*\|_S, \quad k \in \mathbb{N}
\]

**Proof.** In view of Lemma 4.1–(i), by the similar arguments of [71, Theorem 5.3], invoking Opial’s lemma [59], the convergence of (21) is guaranteed, if $S, G - \frac{\beta}{2}I \in \mathcal{M}_S^{++}$.

(ii) In view of Lemma 4.1–(i), we have:
\[
\|b^{k+1} - b^*\|_S^2 \leq \|b^k - b^*\|_S^2 - \frac{\lambda_{\min}(M^{-T}(G - \frac{\beta}{2}I)M^{-1})}{\lambda_{\max}(S)} \|b^k - b^{k+1}\|_S^2
\]
where $\lambda_{\max}$ and $\lambda_{\min}$ denote the largest and smallest eigenvalues of a matrix.

On the other hand, the sequence $\{\|b^k - b^{k+1}\|_S\}_{k \in \mathbb{N}}$ is non-increasing, if $(1 - \frac{\beta}{4\nu_1})(Q + Q^T) \succeq \mathcal{M}^T Q$, by Lemma 4.1-(iii). Finally, (ii) is obtained, following the similar proof of [70, Theorem 3.3–(iii)].

**Remark 6.** In particular, if $M = \gamma I$, Lemma 4.1 is simplified as:
\[
\|b^{k+1} - b^*\|_Q^2 \leq \|b^k - b^*\|_Q^2 - \frac{1}{\gamma}(2 - \gamma - \frac{\beta}{2\nu_1}) \|b^k - b^{k+1}\|_Q^2
\]
and
\[
\|b^k - b^{k+1}\|_Q^2 - \|b^{k+1} - b^{k+2}\|_Q^2 \geq \gamma(2 - \gamma - \frac{\beta}{2\nu_1}) \|\mathcal{R}b^k - \mathcal{R}b^{k+1}\|_Q^2
\]
Then, Theorem 3.7 is exactly recovered.

## 5 Applications to the first-order operator splitting algorithms

The focus of this part is to show that many popular operator splitting algorithms fall into the G-FBS category. In each example, we only show $A, B, Q$ and $M$ associated with the specific algorithm, without further presenting the convergence properties, which can be readily obtained by the results in Sections 3 and 4.

### 5.1 The ADMM algorithms

ADMM is one of the most commonly used algorithms for solving the structured constrained optimization [13]:
\[
\min_{x,u} f(x) + g(u), \quad \text{s.t.} \quad Ax + Bu = c
\]
(23)
where $x \in \mathbb{R}^N, u \in \mathbb{R}^L, A : \mathbb{R}^N \mapsto \mathbb{R}^M, B : \mathbb{R}^L \mapsto \mathbb{R}^M, f : \mathbb{R}^N \mapsto \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^L \mapsto \mathbb{R} \cup \{+\infty\}$ are proper, lower semi-continuous and convex. Two typical ADMM algorithms are listed here to show the corresponding PPA interpretations.
Example 1 (Relaxed-ADMM). The relaxed ADMM, or equivalent relaxed DRS applied to the dual problem [38], is given as [37, Eq.(3)]:

\[
\begin{align*}
x^{k+1} & := \arg \min_x f(x) + \frac{\tau}{2} \|Ax + Bu^k - c - \frac{1}{\tau} s^k\|^2 \\
u^{k+1} & := \arg \min_u g(u) + \frac{\tau}{2} \|B(u + u^k) + \gamma(Ax^{k+1} + Bu^k - c) - \frac{1}{\tau} s^k\|^2 \\
s^{k+1} & := s^k - \tau B(u^{k+1} - u^k) - \tau \gamma(Ax^{k+1} + Bu^k - c)
\end{align*}
\]

which fits into the relaxed G-FBS operator (20) as:

\[
Q = \begin{bmatrix} P_1 & 0 & 0 \\
0 & P_2 + \tau B^T B & 0 \\
0 & -B & \frac{1}{\tau} I_M \end{bmatrix}, \quad M = \begin{bmatrix} I_N & 0 & 0 \\
0 & I_L & 0 \\
0 & -\tau B & \gamma I_M \end{bmatrix}
\]

Example 2 (Proximal-ADMM). The proximal-ADMM is given as [52, modified SPADMM], [66, Eq.(10)]:

\[
\begin{align*}
x^{k+1} & := \arg \min_x f(x) + \frac{\tau}{2} \|Ax + Bu^k - c - \frac{1}{\tau} s^k\|^2 + \frac{1}{2} \|x - x^k\|_P^2 \\
u^{k+1} & := \arg \min_u g(u) + \frac{\tau}{2} \|A x^{k+1} + Bu - c - \frac{1}{\tau} s^k\|^2 + \frac{1}{2} \|u - u^k\|_P^2 \\
s^{k+1} & := s^k - \tau \gamma(Ax^{k+1} + Bu^k - c)
\end{align*}
\]

which fits into the relaxed G-FBS operator (20) as:

\[
Q = \begin{bmatrix} P_1 & 0 & 0 \\
0 & P_2 + \tau B^T B & 0 \\
0 & -B & \frac{1}{\tau} I_M \end{bmatrix}, \quad M = \begin{bmatrix} I_N & 0 & 0 \\
0 & I_L & 0 \\
0 & -\tau B & \gamma I_M \end{bmatrix}
\]

with the same \(b^k, A\) and \(B\) as Example 1.

Remark 7. Other ADMM algorithms proposed in [42, 46, 47, 57, 37, 20, 43, 4] can also be interpreted by the (relaxed) G-FBS operator, with the same \(b, A\) and \(B\) as Example 1, but with different \(Q\) and \(M\). Interested readers can verify it case-by-case, which is omitted here.

Also notice that the monotone operator \(A\) bears the typical (diagonal) monotone + (off-diagonal) skew structure:

\[
A = \begin{bmatrix} \partial f & 0 & -A^T \\
0 & \partial g & -B^T \\
A & B & 0 \end{bmatrix} = \begin{bmatrix} \partial f & 0 & 0 \\
0 & \partial g & 0 \\
0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & -A^T \\
0 & 0 & -B^T \\
A & B & 0 \end{bmatrix}
\]

which coincides with the observations in [15, 14, 27].

---

\(^2\)The standard ADMM can be recovered by letting \(\gamma = 1\) [13].
5.2 Steepest descent, P-FBS and PDS algorithms

Consider the primal problem [69, Problem 4.1]:

$$
\min_x f(x) + \sum_{i=1}^m (g_i(x) + h(x) + \langle x | z_i \rangle)
$$

where $x \in \mathbb{R}^N$, $A_i : \mathbb{R}^N \rightarrow \mathbb{R}^{M_i}$, $r_i \in \mathbb{R}^{M_i}$, $z \in \mathbb{R}^N$. The functions $f : \mathbb{R}^N \rightarrow \mathbb{R}$, $g_i : \mathbb{R}^{M_i} \rightarrow \mathbb{R} \cup \{+\infty\}$, $l_i : \mathbb{R}^{M_i} \rightarrow \mathbb{R} \cup \{+\infty\}$, $h : \mathbb{R}^N \rightarrow \mathbb{R} \cup \{+\infty\}$ are proper, lower semi-continuous and convex, $f$ is differentiable with $\beta$-Lipschitz continuous gradient, $l_i$ is $\mu_i$-strongly convex, and thus, by [5, Theorem 18.15], $l_i^*$ is differentiable with $\mu_i^{-1}$-Lipschitz continuous gradient. There are various classes of algorithms for solving (24) or the special cases, listed below.

**Example 3** (Steepest descent). For solving $\min_x f(x)$, which is a special case of (24) with $m = 1$, $l = \iota_{\{0\}} : a \mapsto \begin{cases} 0, & \text{if } a = 0 \\ +\infty, & \text{otherwise} \end{cases}$ (i.e. indicator function of the set $C = \{0\}$), $g : a \mapsto 0$, $h : a \mapsto 0$, $z = 0$, the steepest descent is given by [27, Proposition 63]:

$$
x^{k+1} := x^k - \tau \nabla f(x^k)
$$

which fits into the G-FBS operator (3) with:

$$
b = x, \quad A = \nabla f, \quad B = 0, \quad Q = \frac{1}{\tau} I_N
$$

**Example 4** (Classical PPA). For solving $\min_x h(x)$, which is a special case of (24) with $m = 1$, $l = \iota_{\{0\}}$, $f : a \mapsto 0$, $g : a \mapsto 0$, $z = 0$, the classical PPA is given by [27, Proposition 64]:

$$
x^{k+1} := x^k + \gamma (\text{prox}_h(x^k) - x^k)
$$

which fits the relaxed G-FBS operator (20) with:

$$
b = x, \quad A = \partial h, \quad B = 0, \quad Q = \frac{1}{\tau} I_N, \quad M = \gamma I_N
$$

**Example 5** (Classical P-FBS [30, 25]). For solving $\min_x f(x) + h(x)$, which is a special case of (24) with $m = 1$, $l = \iota_{\{0\}}$, $g : a \mapsto 0$, $z = 0$, the error-free version of the classical P-FBS is given by [30, Eq.(3.6)]:

$$
x^{k+1} := x^k + \gamma (\text{prox}_h(x^k - \tau \nabla f(x^k)) - x^k)
$$

which fits the relaxed G-FBS operator (20) with:

$$
b = x, \quad A = \partial h, \quad B = \nabla f, \quad Q = \frac{1}{\tau} I_N, \quad M = \gamma I_N
$$

**Example 6** (PDHG [36, 19]). For solving $\min_x h(x) + g(Ax)$, which is a special case of (24) with $m = 1$, $l = \iota_{\{0\}}$, $f : a \mapsto 0$, $r = 0$, $z = 0$, the PDHG is given by [36, PDHGMp]:

$$
\begin{cases}
    s^{k+1} := \text{prox}_{\sigma g^*} (s^k + \sigma A x^k) & \text{dual step} \\
    x^{k+1} := \text{prox}_{\tau h} (x^k - \tau A^T (2s^{k+1} - s^k)) & \text{primal step}
\end{cases}
$$
the associated G-FBS operator (3) is:

\[
b^k = \begin{bmatrix} s^k \\ x^k \end{bmatrix}, \quad A = \begin{bmatrix} \partial g^* + A^\top \partial h \\ -A \end{bmatrix}, \quad B : b \mapsto 0, \quad Q = \begin{bmatrix} \frac{1}{\sigma} I_M & -A \\ -A & \frac{1}{\tau} I_N \end{bmatrix}
\]

Another form of PDHG is [36, PDHGMu]:

\[
\begin{aligned}
 s^{k+1} &:= \text{prox}_{\sigma g^*}(s^k + \sigma A(2x^k - x^{k-1})) \quad \text{dual step} \\
x^{k+1} &:= \text{prox}_{\tau h}(x^k - \tau A^\top s^{k+1}) \quad \text{primal step}
\end{aligned}
\]

The corresponding G-FBS operator is\(^3\):

\[
b^k = \begin{bmatrix} s^k \\ x^{k-1} \end{bmatrix}, \quad A = \begin{bmatrix} \partial g^* + A^\top \partial h \\ -A \end{bmatrix}, \quad B : b \mapsto 0, \quad Q = \begin{bmatrix} \frac{1}{\sigma} I_M & -A \\ -A & \frac{1}{\tau} I_N \end{bmatrix}
\]

**Example 7 (PDS algorithm [15]).** For solving the primal problem \( \min_x h(x) + g(Ax - r) + \langle x|z \rangle \), which is a special case of (24) with \( m = 1, l = \iota(0) \), \( f : a \mapsto 0 \), the error-free version of [15, Proposition 4.2] is given as:

\[
\begin{aligned}
 \tilde{x}^k &:= \text{prox}_{\tau h}(x^k - \tau A^\top s^k - \tau z) \\
\tilde{s}^k &:= \text{prox}_{\sigma g^*}(s^k + \sigma A x^k - \tau r) \\
x^{k+1} &:= \tilde{x}^k - \tau A^\top (\tilde{s}^k - s^k) \\
 s^{k+1} &:= \tilde{s}^k + \tau A(\tilde{x}^k - x^k)
\end{aligned}
\]

which corresponds to the following relaxed G-FBS operator:

\[
b^k = \begin{bmatrix} x^k \\ s^k \end{bmatrix}, \quad A : b \mapsto \begin{bmatrix} \partial f + A^\top s^k \\ -A \end{bmatrix} b + \begin{bmatrix} z \\ r \end{bmatrix}, \quad B : b \mapsto 0, \quad Q = \begin{bmatrix} \frac{1}{\tau} I_N & -A^\top \\ -A & \frac{1}{\tau} I_M \end{bmatrix}, \quad M = \begin{bmatrix} I_N & -\tau A^\top \\ \tau A & I_{M} \end{bmatrix}
\]

**Example 8 (Generalized Dykstra-like algorithm [24]).** For solving [24, Problem 1.2]:

\[
\min_x h(x) + g(Ax - r) + \frac{1}{2}\|x - u\|^2
\]

which is a special case of (24) with \( m = 1, l = \iota(0) \), \( f : a \mapsto \frac{1}{2}\cdot\|u\|^2 \), \( z = 0 \), [24, Algorithm 3.5] is given as:

\[
\begin{aligned}
x^k &:= \text{prox}_f(z - L^\top s^k) \\
s^{k+1} &:= s^k + \gamma\left( \text{prox}_{\sigma g^*}(s^k + \gamma(Lx^k - r)) - s^k \right)
\end{aligned}
\]

Using Fenchel duality and Moreau’s decomposition identity, (25) is equivalent to a simple P-FBS for solving the dual problem:

\[
s^{k+1} := s^k + \gamma\left( \text{prox}_{\sigma g^*}(s^k - \tau r - \tau \nabla q(s^k)) - s^k \right)
\]

where the function \( q \) is defined as: \( q : s \mapsto \frac{1}{2}\|u - A^\top s\|^2 - \left( \min_x h(x) + \frac{1}{2}\|x - u + A^\top s\|^2 \right) \). This algorithm fits the relaxed G-FBS operator with:

\[
b = s, \quad A : s \mapsto \partial g^*(s) + r, \quad B = \nabla q, \quad Q = \frac{1}{\tau} I_M, \quad M = \gamma I_M
\]

\(^3\text{Note that we use a mismatch of iteration indices between }x \text{ and } s : b^k := (s^k, x^{k-1}) \text{. This technique can also be found in [9].}\)
Example 9 (PAPC [21, 35, 56]). For solving $\min_x f(x) + g(Ax)$, which is a special case of (24) with $m = 1$, $h : b \mapsto 0$, $l = I(0)$, $r = 0$, $z = 0$, the PAPC scheme is given as:

$$
\begin{aligned}
 s^k &:= \text{prox}_{\sigma g^*} \left( (I - \sigma \tau AA^\top)s^k + \sigma A(x^k - \tau \nabla f(x^k)) \right) \\
x^{k+1} &:= x^k - \tau \nabla f(x^k) - \tau A^\top s^{k+1}
\end{aligned}
$$

which can be interpreted by the G-FBS operator (3):

$$
b^k = \begin{bmatrix} s^k \\ x^k \end{bmatrix}, \quad A = \begin{bmatrix} \partial g^* & -A \\ A^\top & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \nabla f \end{bmatrix}, \quad Q = \begin{bmatrix} 1/\sigma & 0 \\ 0 & 1/\tau \end{bmatrix}, \quad M = \begin{bmatrix} I_M & 0 \\ -\tau A^\top & I_N \end{bmatrix}
$$

Example 10 (AFBA [49, 50]). For solving $\min_x f(x) + h(x) + g(Ax)$, which is a special case of (24) with $m = 1$, $l = I(0)$, $r = 0$, $z = 0$, the AFBA scheme is given by:

$$
\begin{aligned}
 s^{k+1} &:= \text{prox}_{\sigma g^*} \left( s^k + \sigma A w^k \right) \\
x^{k+1} &:= w^k - \tau A^\top(s^{k+1} - s^k) \\
w^{k+1} &:= \text{prox}_{\tau h} \left( x^{k+1} - \tau \nabla f(x^{k+1}) - \tau A^\top s^{k+1} \right)
\end{aligned}
$$

To interpret it by the G-FBS operator, we remove $w$ and obtain the equivalent form:

$$
\begin{aligned}
 s^{k+1} &:= \text{prox}_{\sigma h^*} \left( s^k + \sigma A(x^{k+1} + \tau A^\top(s^{k+1} - s^k)) \right) \\
x^{k+1} &:= \text{prox}_{\tau g} \left( x^k - \tau \nabla f(x^k) - \tau A^\top s^k \right) - \tau A^\top(s^{k+1} - s^k)
\end{aligned}
$$

which is exactly the relaxed G-FBS operator (20):

$$
b^k = \begin{bmatrix} s^k \\ x^k \end{bmatrix}, \quad A = \begin{bmatrix} \partial g^* & -A \\ A^\top & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \nabla f \end{bmatrix}, \quad Q = \begin{bmatrix} 1/\sigma & 0 \\ 0 & 1/\tau \end{bmatrix}, \quad M = \begin{bmatrix} I_M & 0 \\ -\tau A^\top & I_N \end{bmatrix}
$$

Example 11 (PDS algorithm [32]). For solving the same problem as in Example 10, [32, Algorithm 3.1] is given as:

$$
\begin{aligned}
 \hat{x}^k &:= \text{prox}_{\tau h} \left( x^k - \tau \nabla f(x^k) - \tau A^\top s^k \right) \\
\hat{s}^k &:= \text{prox}_{\sigma g^*} \left( s^k + \sigma A(2\hat{x}^k - x^k) \right) \\
x^{k+1} &:= x^k + \gamma_k(\hat{x}^k - x^k) \\
s^{k+1} &:= s^k + \gamma_k(\hat{s}^k - s^k)
\end{aligned}
$$

which falls into the relaxed G-FBS operator (20):

$$
b^k = \begin{bmatrix} x^k \\ s^k \end{bmatrix}, \quad A = \begin{bmatrix} \partial h & A^\top \\ -A & \partial g^* \end{bmatrix}, \quad B = \begin{bmatrix} \nabla f \\ 0 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1/\tau & 0 \\ -A & 1/\sigma \end{bmatrix}, \quad M = \begin{bmatrix} \gamma I_N & 0 \\ 0 & \gamma I_M \end{bmatrix}
$$

Another algorithm is [32, Algorithm 3.2]:

$$
\begin{aligned}
 \hat{s}^k &:= \text{prox}_{\sigma g^*} \left( s^k + \sigma A x^k \right) \\
\hat{x}^k &:= \text{prox}_{\tau h} \left( x^k - \tau \nabla f(x^k) - \tau A^\top(2\hat{s}^k - s^k) \right) \\
x^{k+1} &:= x^k + \gamma(\hat{x}^k - x^k) \\
s^{k+1} &:= s^k + \gamma(\hat{s}^k - s^k)
\end{aligned}
$$

which falls into the relaxed G-FBS operator (20):

$$
b^k = \begin{bmatrix} s^k \\ x^k \end{bmatrix}, \quad A = \begin{bmatrix} \partial g^* & -A \\ A^\top & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0 & \nabla f \end{bmatrix}, \quad Q = \begin{bmatrix} 1/\sigma & A \\ A^\top & 1/\tau \end{bmatrix}, \quad M = \begin{bmatrix} \gamma I_M & 0 \\ 0 & \gamma I_N \end{bmatrix}
$$
which can be compactly expressed in a relaxed G-FBS form (20):

\[
\begin{align*}
\tilde{x}^k &= \text{prox}_{\tau h}(x^k - \tau \nabla f(x^k) - \tau \sum_{i=1}^{m} A_i^\top s_i^k) \\
\tilde{s}_i^k &= \text{prox}_{\sigma g_i^*}(s_i^k + \sigma A_i (2\tilde{x}^k - x^k)), \quad i = 1, 2, \ldots, m \\
x^{k+1} &= x^k + \gamma(\tilde{x}^k - x^k) \\
s_i^{k+1} &= s_i^k + \gamma(\tilde{s}_i^k - s_i^k), \quad i = 1, 2, \ldots, m
\end{align*}
\]

which can be compactly expressed in a relaxed G-FBS form (20):

\[
Q = \begin{bmatrix}
\tau^{-1}I_N & -A_1^\top & \cdots & -A_m^\top \\
-A_1 & \sigma^{-1}I_{M_1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-A_m & 0 & \cdots & \sigma^{-1}I_{M_m}
\end{bmatrix}
\]

Another algorithm is given by [32, Algorithm 5.2]:

\[
\begin{align*}
\tilde{s}_i^k &= \text{prox}_{\sigma g_i^*}(s_i^k + \sigma A_i x^k), \quad i = 1, 2, \ldots, m \\
\tilde{x}^k &= \text{prox}_{\tau h}(x^k - \tau \nabla f(x^k) - \tau \sum_{i=1}^{m} A_i^\top (2\tilde{s}_i^k - s_i^k)) \\
x^{k+1} &= x^k + \gamma(\tilde{x}^k - x^k) \\
s_i^{k+1} &= s_i^k + \gamma(\tilde{s}_i^k - s_i^k), \quad i = 1, 2, \ldots, m
\end{align*}
\]

whose corresponding relaxed G-FBS form is with the same \(b, A, B\) and \(M\) as the above example, and \(Q\) is given as:

\[
Q = \begin{bmatrix}
\tau^{-1}I_N & A_1^\top & \cdots & A_m^\top \\
A_1 & \sigma^{-1}I_{M_1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_m & 0 & \cdots & \sigma^{-1}I_{M_m}
\end{bmatrix}
\]

### 5.3 Other examples

Other classes of algorithms can also be expressed by the G-FBS operator. Let us now consider a typical optimization problem with a linear equality constraint:

\[
\min_{x} h(x), \quad \text{s.t. } Ax = c
\]

where \(A : \mathbb{R}^N \mapsto \mathbb{R}^M\), \(h : \mathbb{R}^N \mapsto \mathbb{R} \cup \{+\infty\}\) is proper, lower semi-continuous and convex.

**Example 13 (Basic ALM).** The augmented Lagrangian method (ALM) is (see [57, Eq.(1.2)] and [67, Eq.(7.2)] for example):

\[
\begin{align*}
x^{k+1} &:= \arg\min_{x} h(x) + \frac{\tau}{2} \|Ax - c - \frac{1}{2}s^k\|^2 \\
s^{k+1} &:= s^k - \tau(Ax^{k+1} - c)
\end{align*}
\]
Its G-FBS interpretation is given as:

\[
\begin{align*}
\mathbf{b}^k &= \begin{bmatrix} x^k \\ s^k \end{bmatrix},
\mathcal{A}: \mathbf{b} \mapsto \begin{bmatrix} \partial h \\ A \\ 0 \\ \tau I \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix},
\mathcal{B}: \mathbf{b} \mapsto 0,
Q = \begin{bmatrix} 0 & 0 \\ 0 & 1/\tau I \end{bmatrix}
\end{align*}
\]

**Example 14 (Linearized ALM).** The linearized ALM is given as [72]:

\[
\begin{align*}
x^{k+1} &= \text{arg min}_x h(x) + \frac{\rho}{2} \| x - x^k + \frac{1}{\rho} A^T (\tau (Ax^k - c) - s^k) \|^2 \\
s^{k+1} &= s^k - \tau (Ax^{k+1} - c)
\end{align*}
\]

Its G-FBS interpretation is given as:

\[
\begin{align*}
\mathbf{b}^k &= \begin{bmatrix} x^k \\ s^k \end{bmatrix},
\mathcal{A}: \mathbf{b} \mapsto \begin{bmatrix} \partial h \\ A \\ 0 \\ \tau \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix},
\mathcal{B}: \mathbf{b} \mapsto 0,
Q = \begin{bmatrix} \rho I - \tau A^T A & 0 \\ 0 & \frac{1}{\tau} I \end{bmatrix}
\end{align*}
\]

**Example 15 (Linearized Bregman algorithm [18]).** The scheme reads as (see also [74, Eq.(1.11)]):

\[
\begin{align*}
x^{k+1} &= \text{arg min}_x \rho \tau h(x) + \frac{1}{2} \| x - \rho A^T s^k \|^2 \\
s^{k+1} &= s^k - (Ax^{k+1} - c)
\end{align*}
\]

Its G-FBS interpretation is given as:

\[
\begin{align*}
\mathbf{b}^k &= \begin{bmatrix} x^k \\ s^k \end{bmatrix},
\mathcal{A}: \mathbf{b} \mapsto \begin{bmatrix} \tau \partial h + \frac{1}{\rho} I_N - A^T A \\ A \\ 0 \\ \tau \end{bmatrix} \begin{bmatrix} \mathbf{b} \\ \mathbf{c} \end{bmatrix},
\mathcal{B}: \mathbf{b} \mapsto 0,
Q = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}
\end{align*}
\]

**Remark 8.** (1) It is the first time to show that most first-order operator splitting algorithms boil down to the simple G-FBS operator.

(2) The convergence conditions of these algorithms can be easily obtained by reformulating them as the G-FBS operator. It is much simpler than their original analysis in the literature.

(3) Many results given in Sections 3 and 4 can be applied to the listed algorithms, and thus, more properties can be explored than the previous works. Interested readers can make further investigations along this direction.

(4) We did not enumerate all the examples here. One can verify more existing algorithms.

### 6 Conclusions

In this paper, we considered the G-FBS operator and analyzed the nonexpansive properties. The fixed-point iterations were studied, and their convergence rates in terms of metric distance and objective value were established, which extended the existing results in some aspects. A great variety of operator splitting algorithms were illustrated as the concrete examples of the G-FBS operator.

Last, it seems interesting to further extend the proposed framework to the case when the variable metric \( Q \) and relaxation matrix \( \mathcal{M} \) are allowed to vary over the iterations. Another limitation of this G-FBS operator is that it fails to cover Bregman proximal algorithms [68, 16] and a few PDS algorithms [26, 12], since \( Q \) and \( \mathcal{M} \) presented here denote the linear transforms only. It is worthwhile to extend the G-FBS to nonlinear operators \( Q \) and \( \mathcal{M} \).
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