REALIZATION OF AN EQUIVARIANT HOLOMORPHIC
HERMITIAN LINE BUNDLE
AS A QUILLEN DETERMINANT BUNDLE

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Abstract. Let $M$ be an irreducible smooth complex projective variety
equipped with an action of a compact Lie group $G$, and let $(\mathcal{L}, h)$ be a $G$–
equivariant holomorphic Hermitian line bundle on $M$. Given a compact connected Riemann surface $X$, we construct a $G$–equivariant holomorphic Hermitian line bundle $(L, H)$ on $X \times M$ (the action of $G$ on $X$ is trivial) such that the corresponding Quillen determinant line bundle $(Q, h_Q)$, which is a $G$–equivariant holomorphic Hermitian line bundle on $M$, is isomorphic to the given $G$–equivariant holomorphic Hermitian line bundle $(\mathcal{L}, h)$. This proves a conjecture by Dey and Mathai (2013).

1. Introduction

This work was inspired by [DM], where the following result is proved. Let $M$ be an irreducible smooth complex projective variety and $\mathcal{L}$ an ample line bundle on $M$ equipped with a Hermitian structure $h$ of positive curvature. There is a natural family of Cauchy–Riemann operators on $\mathbb{C}P^1$, parametrized by $M$, such that the corresponding Quillen determinant line bundle, which is a holomorphic Hermitian line bundle on $M$, is holomorphically isomorphic to a positive tensor power of $(\mathcal{L}, h)$ [DM, p. 785, Theorem 1.1]. It is conjectured in [DM] that an equivariant version also holds (see [DM, p. 793, §5]).

Let $M$ be as before. Assume that it is equipped with a $C^\infty$ action of a compact Lie group $G$ via holomorphic automorphisms of $M$. Let $(\mathcal{L}, h)$ be any $G$–equivariant holomorphic Hermitian line bundle on $M$.

Let $X$ be a compact connected Riemann surface equipped with a Kähler form $\omega_X$. Let $L_0$ be a holomorphic line bundle on $X$ of degree genus($X) - 2$ such that $H^0(X, L_0) = 0$. Fix a Hermitian structure $h_0$ on $L_0$.

The action of $G$ on $M$ and the trivial action of $G$ on $X$ together produce an action of $G$ on $X \times M$. Let $p_1$ and $p_2$ be the projections of $X \times M$ on $X$ and $M$ respectively. Consider the Hermitian structure $H := (p_1^*h_0) \otimes (p_2^*h)$ on the holomorphic line bundle

$$L := (p_1^*L_0) \otimes (p_2^*\mathcal{L})$$

over $X \times M$. The action of $G$ on $\mathcal{L}$ and the trivial action of $G$ on $L_0$ together produce an action of $G$ on $L$, thus making $L$ a $G$–equivariant holomorphic Hermitian line bundle on $X \times M$. We will consider $(L, H)$ as a family of holomorphic Hermitian
line bundles on $X$. Let $(Q, h_Q)$ be the Quillen determinant line bundle associated to the triple $(L, H, \omega_X)$. It is a $G$–equivariant holomorphic Hermitian line bundle on $M$.

We prove the following (see Theorem 2.4):

The two $G$–equivariant holomorphic Hermitian line bundle on $M$, namely $(L, h)$ and $(Q, h_Q)$, are isomorphic.

We note that this proves the earlier mentioned conjecture in [DM].

2. A holomorphic family and its determinant bundle

Let $X$ be a compact connected Riemann surface. Let $g$ be the genus of $X$. Fix a holomorphic line bundle $L_0$ on $X$ of degree $g - 2$ such that

\begin{equation}
H^0(X, L_0) = 0.
\end{equation}

We note that such a line bundle exists. Indeed, if $g \leq 1$, then any holomorphic line bundle of degree $g - 2$ works; if $g = 2$, then any nontrivial holomorphic line bundle of degree zero works; if $g \geq 2$, then any point outside the image of the Abel-Jacobi map $\text{Sym}^{g-2}(X) \to \text{Pic}^{g-2}(X)$ works. From Riemann–Roch it follows that

\begin{equation}
\dim H^1(X, L_0) = 1.
\end{equation}

Fix a $C^\infty$ Hermitian structure $h_0$ on $L_0$. Also, fix a Kähler form $\omega_X$ on $X$.

Let $M$ be a connected complex projective manifold (meaning a connected smooth complex projective variety). Assume that a compact Lie group $G$ acts smoothly on $M$ via holomorphic automorphisms. Let $(L, h)$ be a $G$–equivariant holomorphic Hermitian line bundle on $M$. This means that the holomorphic line bundle $L$ is equipped with an action of $G$ such that

1. for each element $z \in G$, the action of $z$ on $L$ is a holomorphic automorphism of the line bundle $L$ over the automorphism of $M$ given by the action of $z$ on $M$,

2. the action of $G$ on $L$ is $C^\infty$ and it preserves $h$.

Let $p_1$ and $p_2$ be the projections on $X \times M$ to $X$ and $M$ respectively. Consider the holomorphic line bundle

$$L := (p_1^*L_0) \otimes (p_2^*L) \to X \times M.$$ 

It is equipped with the Hermitian structure

\begin{equation}
H := (p_1^*h_0) \otimes (p_2^*h).
\end{equation}

The action of $G$ on $M$ and the trivial action of $G$ on $X$ together define an action of $G$ on $X \times M$. Similarly, the action of $G$ on $L$ and the trivial action of $G$ on $L_0$ together define an action of $G$ on $L$. This action of $G$ on $L$ clearly preserves $H$.

Consider $(L, H)$ as a family of holomorphic Hermitian line bundles on $X$ parametrized by $M$. Let

$$(Q, h_Q) \to M$$

be the Quillen determinant line bundle associated to $(L, H, \omega_X)$ [Qu]. For any point $y \in M$, let $(L^y, H^y)$ be the holomorphic Hermitian line bundle on $X$ obtained by restricting $(L, H)$ to $X \times \{y\}$. Note that $(L^y, H^y)$ is isomorphic to the holomorphic Hermitian line bundle $(L_0, h_0)$. We recall that the fiber $Q_y$ is identified with the complex line $\bigwedge^{\text{top}} H^0(X, L^y)^* \otimes \bigwedge^{\text{top}} H^1(X, L^y)$ [Qu]. In view of (2.1) and (2.2), the fiber $Q_y$ is identified with $H^1(X, L^y)$. \[3184\]
The action of $G$ on $L$ produces an action of $G$ on $Q$. The action of any $z \in G$ on $Q$ is a holomorphic automorphism of the line bundle $Q$ over the automorphism of $M$ given by $z$. The action of $G$ on $Q$ preserves the Hermitian structure $h_Q$ on $Q$ because the action of $G$ on $L$ preserves $H$ and the trivial action on $X$ preserves $\omega_X$.

Let
\begin{equation}
\xi := M \times H^1(X, L_0) \rightarrow M
\end{equation}
be the holomorphically trivial line bundle with fiber $H^1(X, L_0)$ (see (2.2)). The trivial action of $G$ on $H^1(X, L_0)$ and the action of $G$ on $M$ together define an action of $G$ on $\xi$. The actions of $G$ on $L$ and $\xi$ together produce an action of $G$ on $L \otimes \xi$ that is a lift of the action of $G$ on $M$.

**Lemma 2.1.** The holomorphic line bundle $Q$ over $M$ is identified with $L \otimes \xi$. This identification is $G$-equivariant.

**Proof.** From (2.1) it follows that
\[ R^0 p_{2*} L = 0 \]
(recall that $L^y$ is isomorphic to $L_0$). By the projection formula [Ha, p. 253, Ex. 8.3], we have
\[ R^1 p_{2*} L = L \otimes R^1 p_{2*} (p_1^* L_0) . \]
But $R^1 p_{2*} (p_1^* L_0) = \xi$. Therefore, we get an isomorphism
\begin{equation}
\tau : Q := \text{Det}(L) = R^1 p_{2*} L \sim L \otimes \xi .
\end{equation}
From the construction of $\tau$ it follows immediately that the isomorphism intertwines the actions of $G$ on $Q$ and $\mathcal{L} \otimes \xi$. \hfill \square

Let $\nabla^Q$ be the Chern connection on $Q$ for the Hermitian structure $h_Q$. The curvature of $\nabla^Q$ will be denoted by $\mathcal{K}(\nabla^Q)$. The curvature $\mathcal{K}(\nabla^Q)$ can be computed using [Qu], [BGS].

Let $\nabla^L$ denote the Chern connection for $(\mathcal{L}, h)$. Its curvature will be denoted by $\mathcal{K}(\nabla^L)$.

**Proposition 2.2.** The two $1,1$-forms $\mathcal{K}(\nabla^Q)$ and $\mathcal{K}(\nabla^L)$ on $M$ coincide.

**Proof.** The Chern connection on the holomorphic Hermitian line bundle $(L_0, h_0)$ (respectively, $(L, H)$) will be denoted by $\nabla^{L_0}$ (respectively, $\nabla^L$). Let $\mathcal{K}(\nabla^{L_0})$ (respectively, $\mathcal{K}(\nabla^L)$) be the curvature of $\nabla^{L_0}$ (respectively, $\nabla^L$). From the definition of $H$ (see (2.3)) it follows immediately that
\begin{equation}
\mathcal{K}(\nabla^L) = p_1^* \mathcal{K}(\nabla^{L_0}) + p_2^* \mathcal{K}(\nabla^L) .
\end{equation}
Let $\mathcal{K}(\omega_X) \subset C^\infty(X; \Omega^{1,1}_X)$ be the curvature of $TX$ for the Kähler form $\omega_X$.

A theorem due to Quillen and Bismut–Gillet–Soulé says that $\mathcal{K}(\nabla^Q)$ is given by the following fiber integral along $X$:
\begin{equation}
\mathcal{K}(\nabla^Q) = - \frac{1}{2\pi \sqrt{-1}} \left( \int_{(X \times M)/M} (\mathcal{K}(\nabla^L) + \frac{1}{2} \mathcal{K}(\nabla^L)^2) \wedge (1 + \frac{1}{2} p_1^* \mathcal{K}(\omega_X)) \right)_2 .
\end{equation}
[BGS] p. 51, Theorem 0.1, [Qu], where $(\beta)_2$ denotes the component of the differential form $\beta$ of degree two; note that $\frac{1}{4\pi \sqrt{-1}} \mathcal{K}(\omega_X)$ is the Todd form on $X$ for
the Kähler form $\omega_X$ that represents the Todd class $\frac{1}{2}c_1(TX)$. Using (2.6), the expression in (2.7) reduces to

$$2\pi\sqrt{-1} \cdot K(\nabla^Q) = -K(\nabla^L) \cdot \int_X (K(\nabla^{L_0}) + \frac{1}{2}K(\omega_X)) .$$

Now note that

$$\frac{1}{2\pi\sqrt{-1}} \int_X (K(\nabla^{L_0}) + \frac{1}{2}K(\omega_X)) = \deg(L_0) + \frac{1}{2}\deg(TX) = g - 2 + 1 - g = -1 .$$

Using this, from (2.8) we conclude that $K(\nabla^Q) = K(\nabla^L)$.

The Hermitian structure $L_0$ and the Kähler form $\omega_X$ together produce an inner product on $H^1(X, L_0)$. This inner product defines a Hermitian structure $h_\xi$ on the holomorphic line bundle $\xi$ in (2.4). Note that the Chern connection on $\xi$ for $h_\xi$ is flat.

The Hermitian structure $h$ on $\mathcal{L}$ and the Hermitian structure $h_\xi$ on $\xi$ together produce a Hermitian structure $\tilde{h}$ on $\mathcal{L} \otimes \xi$.

**Proposition 2.3.** For the isomorphism $\tau$ in (2.5), there is a positive real number $t$ such that $\tau^*\tilde{h} = t \cdot h_Q$.

**Proof.** There is a real valued $C^\infty$ function $f$ on $M$ such that

$$\tau^*\tilde{h} = \exp(f) \cdot h_Q .$$

From Proposition 2.2 it follows the two holomorphic Hermitian line bundles $(\mathcal{Q}, h_Q)$ and $(\mathcal{Q}, \exp(f) \cdot h_Q)$ have the same curvature. This implies that $f$ is a harmonic function. Since $M$ is compact and connected, any harmonic function on it is a constant one.

**Theorem 2.4.** The two $G$–equivariant holomorphic Hermitian line bundles $(\mathcal{L}, h)$ and $(\mathcal{Q}, h_Q)$ are isomorphic.

**Proof.** Consider the isomorphism

$$\frac{1}{\sqrt{t}} \cdot \tau : \mathcal{Q} \rightarrow \mathcal{L} \otimes \xi ,$$

where $\tau$ is the isomorphism in (2.5), and $t$ is the constant in Proposition 2.3. From Lemma 2.1 and Proposition 2.3 it follows immediately that this is a $G$–equivariant holomorphic isomorphism that takes the Hermitian structure $h_Q$ on $\mathcal{Q}$ to the Hermitian structure $\tilde{h}$ on $\mathcal{L} \otimes \xi$.

The two $G$–equivariant holomorphic Hermitian line bundles $(\mathcal{L}, h)$ and $(\mathcal{L} \otimes \xi, \tilde{h})$ are clearly isomorphic. Therefore, the two $G$–equivariant holomorphic Hermitian line bundles $(\mathcal{L}, h)$ and $(\mathcal{Q}, h_Q)$ are isomorphic.

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