Disorder-induced critical phenomena
—new universality classes in Anderson localization—

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The Anderson metal-insulator transition is a continuous phase transition driven by disorder. It remains a challenging problem to theoretically determine universal critical properties at the transition. The Anderson transition in a model with a discrete sublattice or particle-hole symmetry belongs to one of seven universality classes which are different from the three well-known standard ones. Here we review our recent theoretical work on these new universality classes in (quasi) one and two dimensions.

I. ANDERSON LOCALIZATION AND THE DYSON SINGULARITY

According to Landau's Fermi-liquid theory, low-energy electronic excitations in a solid are weakly interacting quasiparticles that are adiabatically connected to excitations in a free electron gas. By combining Bloch's theorem with the Fermi statistics obeyed by electrons, one knows if the crystalline state is a perfect metal (having an infinite conductivity) or a band-gap insulator at temperatures low enough that the electron gas is degenerate. It is an empirical fact that there are no perfect metals. This fact can easily be reconciled with the (nearly) free electron model by the following arguments due to Drude and Sommerfeld. One assumes that the electrons in a metal are accelerated by an electric field for times smaller than the average time \( \tau \) needed to scatter off an impurity. As a result, the distribution of electron velocity reaches a steady state that is characterized by the finite Drude-Sommerfeld conductivity

\[
\sigma_D = \frac{ne^2\tau}{m},
\]

where \( e, m, \) and \( n \) are the electron charge, mass, and density, respectively. This argument is essentially classical. The assumption underlying Eq. (1) is that electrons are particles, not waves. It took two pioneering papers, written by Dyson in 1953 and Anderson in 1958, to call into question this assumption on theoretical grounds.

Anderson's 1958 paper dealt with a quantum particle hopping between neighboring sites on a three-dimensional lattice. Anderson added on-site potentials to each lattice site, which were taken to be random with the standard deviation of the random on-site potentials far exceeding the hopping amplitude \( t \). For this case, Anderson showed that any time-dependent wave packet that is initially localized both in space and in energy almost certainly remains so after an infinite time-evolution. Anderson further showed that the wavefunction envelopes decay exponentially fast away from their maxima, a phenomenon now called Anderson localization. Disorder was thus shown to play a role that far exceeds the one that it plays in the Drude-Sommerfeld picture. Disorder alone can turn the nearly free electron gas (the metallic state) into an insulator where wavefunctions are localized. This (Anderson) insulating state is completely different in nature from the band-gap insulator.

In the late 1960s, Mott argued that, if the randomness of the on-site potential is not too strong, a mobility edge should separate delocalized states with energies close to the band center from localized states near the band edges. Upon increasing the disorder strength, the two mobility edges approach each other until they meet at a critical value of the disorder strength. The possibility raised by Mott that tuning the Fermi energy through the mobility edge triggers a transition from the metallic to an insulating state solely due to weak disorder in three dimensions raised the issue of the nature of such a transition.

The decisive argument in support of interpreting the Anderson metal-insulator transition as a continuous transition amenable to scaling ideas developed for critical phenomena in equilibrium phase transitions came from the scaling theory of Abrahams et al. Whereas localization occurs in three dimensions only for strong enough disorder, in one and two-dimensional versions of the Anderson model, eigenfunctions are always localized, irrespective of the degree of the randomness of the on-site potentials. In one dimension, the localization length \( \xi \) is as small as the mean free path \( \ell \). In two dimensions, \( \xi \) is typically much larger than \( \ell \).

Dyson's earlier 1953 paper dealt with a quantum particle hopping along a one-dimensional lattice (if we use the language of the Anderson model). Dyson, however, considered the case that there are no on-site potentials. Instead, he took the hopping amplitudes \( t_{j,j+1} = t_{j+1,j} \) between neighboring sites \( j \) and \( j + 1 \) to be random variables. We refer to Dyson's model as the "random hopping chain". The random variations of \( t_{j,j+1} \) cause scattering between Bloch waves, which is described by means of a mean free time \( \tau \). Dyson calculated the density of states (DOS) per unit length \( \rho \) in the thermodynamic limit and
found that it diverges near the band center $\varepsilon = 0$,
\begin{equation}
\rho(\varepsilon) = \frac{\rho_0(\varepsilon)}{|\varepsilon \ln^3 |\varepsilon||} \quad (|\varepsilon| \ll 1).
\end{equation}
Here, $\rho_0$ is the DOS in the absence of randomness, which is finite and nearly constant near $\varepsilon = 0$. In contrast, there is no divergence in the DOS, which is roughly equal to $\rho_0(\varepsilon)$, in Anderson’s model with random on-site potentials. Further, it was found that the conductance $g$ of the random hopping chain of length $L$ is not exponentially small (as is the case in Anderson’s model), but that it has an anomalously wide distribution precisely at $\varepsilon = 0$, with average $\langle g \rangle \sim (\ell/L)^{1/2}$. Again, the disorder has far more profoundly changed the properties of the chain than one would anticipate based on the Drude-Sommerfeld model.

The origin of the difference between the random hopping chain at $\varepsilon = 0$ and Anderson’s model with random on-site potentials is a symmetry that is present in the former, but absent in the latter. In the random hopping chain, for each eigenfunction $\psi_j$ with energy $\varepsilon$, $(-1)^j \psi_j$ is also an eigenfunction, but with energy $-\varepsilon$, and hence the energy eigenvalues always appear in pairs, $\pm \varepsilon$. This symmetry property follows from the fact that the Hamiltonian of the random hopping chain only has matrix elements that connect sites belonging to two distinct sublattices. Under this “sublattice symmetry” (SLS), the band center $\varepsilon = 0$ is special, and wavefunctions at that energy have anomalous localization properties.

The band center of the random hopping chain can be viewed as an example of two mobility edges merged together. Thus, it can serve as a relatively simple model to study the metal-insulator transition in the Anderson model in higher dimensions. The random hopping chain has a diverging localization length $\xi(\varepsilon) \sim \ell \ln |\varepsilon|$ upon approaching the band center $\varepsilon = 0$. In the parlance of critical phenomena, it is convention to call the exponent $z$ entering the scaling relation $L \sim E^{-1/z}$ between a length scale $L$ and an energy scale $E$ the dynamical scaling exponent, in which case the scaling relation $\xi(\varepsilon) \sim \ell \ln |\varepsilon|$ corresponds to $z = \infty$ and signals an “infinite disorder fixed point.”

Whenever the (typical) localization length $\xi$ is much larger than the mean free path $\ell$, universality is expected. By universality is meant that transport and thermodynamic characteristics of a disordered sample only depend on dimensionality and some intrinsic properties such as the presence or absence of certain symmetries (an example is the SLS, but also time-reversal symmetry (TRS) and spin-rotation symmetry (SRS) play a role). Microscopic details such as the band parameters and the detailed form of the disorder should be irrelevant. For example, the Dyson singularity is universal in that it is independent of the form of the probability distribution of the hopping randomness (as long as the standard deviation of the logarithm of the hopping amplitude is finite).

The outstanding open question in the problem of Anderson localization is to compute the evolution of the full probability distribution of physical observables as a function of the disorder strength, and in particular to compute scaling exponents at the metal-insulator transition in two and three dimensions. New insights on disorder induced critical behavior in the problem of Anderson localization has occurred on two fronts that we shall review below. The first front stems from the renewed interest in the random hopping problem in samples of dimensions larger than one, and in its three incarnations associated to TRS and SRS. The second front stems from the introduction by Altland and Zirnbauer of four symmetry classes associated to dirty superconductors. In both cases, the existence of disorder induced critical points was established in a quasi-one dimensional geometry (sample with a length $L$ much larger than its width) and shown to be identical to the Dyson critical point of the random hopping chain. Progress also took place in two dimensions with the exact computation of some critical exponents among these additional symmetry classes of Anderson localization. We shall review these developments in the following sections.

II. TRANSFER MATRIX ANALYSIS IN QUASI-ONE DIMENSION

Most detailed theoretical results for the localization problem have been obtained in one-dimensional and quasi one-dimensional disordered systems. The width of a quasi one-dimensional conductor is defined by the number of propagating channels $N$ at the Fermi level. A one-dimensional conductor has $N = 1$.

The localization problem in (quasi) one dimension has been tackled using a variety of theoretical methods, most notably field theory, network models, and transfer matrix approaches. In the field-theoretical approach the localization problem is mapped onto a field theory known as the (one-dimensional) “non-linear sigma model” (NLSM). The NLSM has been the canonical framework to establish universality, as the same field theory is obtained for a wide class of microscopic disorder. An essential step in the derivation of the universal field theory is a separation of length scales (in this case $\ell \ll \xi = O(N\ell)$) which restricts the applicability of this method to quasi one-dimensional systems with many channels, $N \gg 1$. The other two methods do not require the limit $N \gg 1$. However, equivalence with the NLSM and, thus, universality, is to be expected for $N \gg 1$ only.

All three approaches have contributed to the present understanding of the localization problem. The field-theoretical approach suggested the existence of critical points induced by disorder belonging to different universality classes. It has also played an important role in the characterization of the mesoscopic fluctuations. The mapping onto a network model allowed the numerical computation of the correlation (localization) length scaling exponent in the plateau transition corresponding to the lowest Landau level in the integer quantum Hall effect (IQHE) and has also been applied to the new uni-
versatility classes. The transfer matrix approach was implemented numerically to compute the scaling exponents of the Anderson localization in three dimensions. It also yielded important analytical results for the quasi one-dimensional case, which we now review.

Let us first illustrate how the transfer matrix approach is applied to the random hopping problem in one dimension ($N = 1$). We ignore the electron spin and assume that disorder is weak compared with the band width. Since we are interested in properties close to the band center $\varepsilon = 0$, the energy spectrum ($\varepsilon(k) \propto -\cos k$) can be linearized about it (i.e., about $k = \pm \pi/2$) and the wavefunction $\psi_j$ can be written as

$$\psi_j \approx e^{i\frac{x}{\ell}} \psi_1(y) + e^{-i\frac{x}{\ell}} \psi_0(y)$$

with $y = j \times$ (lattice spacing), and $\psi_r$ and $\psi_l$ corresponding to waves moving to the right and to the left, respectively, at velocity $v_F$. The continuum limit of the Hamiltonian for the random hopping chain can then be represented by the stationary Schrödinger equation (\(\hbar = 1\)) for a spinor $\Psi = (\psi_r, \psi_l)^T$,

$$\left(\mathcal{H} - \varepsilon\right) \Psi = 0, \quad \mathcal{H} = -iv_F \tau_3 \partial_y - V(y),$$

where $\tau_{1,2,3}$ are the Pauli matrices in the left-mover/right-mover grading. From Eq. (3) we infer that multiplying $\psi_j$ by the factor $(-1)^j = e^{i\pi j}$ amounts to exchanging $\psi_0$ and $\psi_1$, or equivalently, to the transformation $\Psi \rightarrow \tau_1 \Psi$. Hence the SLS of the Hamiltonian, that an eigenfunction $\psi_j$ with energy $\varepsilon$ has its partner $(-1)^j \psi_j$ with energy $-\varepsilon$, reads, in the continuum language,

$$\tau_1 \mathcal{H} \tau_1 = -\mathcal{H},$$

which implies that the scattering potential $V$ can have terms proportional to $\tau_2$ and $\tau_3$ only. Without SLS, $V$ also has terms proportional to $\tau_1$ and the $2 \times 2$ unit matrix, which do not satisfy the symmetry.

The transfer matrix $\mathcal{M}_\varepsilon(L, 0)$ relates the wavefunctions $\Psi(0)$ and $\Psi(L)$,

$$\Psi(L) = \mathcal{M}_\varepsilon(L, 0) \Psi(0).$$

Knowledge of $\mathcal{M}$ is sufficient to calculate the Landauer conductance $g$: the eigenvalues of the product $\mathcal{M}_\varepsilon \mathcal{M}_\varepsilon^\dagger$ come in the pair $\exp(\pm 2i\varepsilon x)$, which is related to $g$ by

$$g = \frac{1}{\cosh^2 x}.$$  

Transfer matrices satisfy a multiplicative rule: the transfer matrix of a disordered segment of length $L_1 + L_2$ is simply the product of the transfer matrices of the disordered segments of lengths $L_1$ and $L_2$,

$$\mathcal{M}_\varepsilon(L_1 + L_2, 0) = \mathcal{M}_\varepsilon(L_1 + L_2, L_1) \mathcal{M}_\varepsilon(L_1, 0).$$

Using the Born approximation to calculate the transfer matrix of a segment of length much smaller than the mean free path $\ell$, the transfer matrix of the full system is then found by repeated matrix multiplications. What makes an analytical solution possible is that each multiplication changes the transfer matrix only slightly. As we increase the system size from $L$ to $L + \delta L$, the random potential $V(y)$ in the interval $y \in [L, L + \delta L]$ changes the transfer matrix from $\mathcal{M}_\varepsilon(L, 0)$ into $\mathcal{M}_\varepsilon(L + \delta L, L) \mathcal{M}_\varepsilon(L, 0)$. Drawing an analogy with the Brownian motion, we may regard $L$ as a time and the random potential $V(y)$ as a random force; $\mathcal{M}_\varepsilon(L, 0)$ then performs, as a function of length $L$, a “random walk” in the manifold of allowed transfer matrices.

The structure of the symmetric space is determined by the symmetries of the transfer matrix $\mathcal{M}_\varepsilon$. These follow from the fundamental symmetries of the Hamiltonian $\mathcal{H}$: Hermiticity, TRS, and the SLS. For the transfer matrix, these imply

$$\mathcal{M}_\varepsilon^\dagger(L) \tau_3 \mathcal{M}_\varepsilon(L) = \tau_3,$$

$$\tau_1 \mathcal{M}_\varepsilon^\dagger(L) \tau_1 = \mathcal{M}_\varepsilon(L),$$

$$\tau_1 \mathcal{M}_\varepsilon^\dagger(L) \tau_1 = \mathcal{M}_\varepsilon(L),$$

respectively. Here we have used the shorthand notation $\mathcal{M}_\varepsilon(L) := \mathcal{M}_\varepsilon(L, 0)$. These conditions define a manifold (Lie group) on which the transfer matrices $\mathcal{M}_\varepsilon$ live.

In view of Eq. (4), we are interested in the radial diffusion only, related to the eigenvalues $\exp(\pm 2i\varepsilon x)$ of $\mathcal{M}_\varepsilon \mathcal{M}_\varepsilon^\dagger$. Mathematically, this random walk is reformulated into a random walk on an associated “symmetric space” which is obtained after identifying transfer matrices $\mathcal{M}_\varepsilon$ that have the same product $\mathcal{M}_\varepsilon \mathcal{M}_\varepsilon^\dagger$ (Fig. 1). Symmetric spaces have a well-defined natural metric, and if the random potential has a Gaussian distribution without long-range correlations, one finds that the random walk can be described by the diffusion equation for the symmetric space.

The SLS poses an extra condition (commutativity with $\tau_1$) on the transfer matrix at $\varepsilon = 0$, that is absent for
transfer matrices at finite energy or for transfer matrices of a wire without SLS. In other words, \( \epsilon \) breaks the extra symmetry of \( \mathcal{M}_{\pm 0} \), which we also call SLS. This fundamentally changes the structure of the symmetric space and, hence, the solution of the localization problem: at finite energy \( \epsilon \), the radial coordinate \( x \) (i.e., the logarithm of the eigenvalue of \( \mathcal{M}_{\pm} \)) performs a biased random walk, resulting in an exponentially small conductance for large \( L \): generically, all wavefunctions are localized in one dimension. On the other hand, at \( \epsilon = 0 \) one finds that as \( x \) performs an unbiased random walk, the root-mean-square of the radial coordinate \( x \) grows proportionally to \( \sqrt{L} \) whereas the typical values of the conductance decays as \( \exp(-a \sqrt{L/\ell}) \) where \( a \) is a positive constant. 

Since there is a finite probability density \( \propto L^{-1/2} \) to find \( x \) at the origin, the ensemble averaged conductance decays only algebraically, \( g \propto L^{-1/2} \).

A quasi one-dimensional system with spinful electrons is obtained by replacing the wavefunctions \( \psi_z \) and \( \psi_\uparrow \) by \( 2N \)-component vectors. The disorder potential \( V \) and the transfer matrix \( \mathcal{M} \) then become \( 4N \times 4N \) matrices. The eigenvalues of \( \mathcal{M} \) come in \( D \)-degenerate pairs \( \exp(\pm 2\pi i n) \), \( n = 1, 2, \ldots, N^* \), where the degeneracy \( D \) depends on the symmetries and \( N^* = 2N/D \). The Landauer conductance reads

\[
g = D \sum_{n=1}^{N^*} \frac{1}{\cosh^2 x_n}.
\]

Upon increasing \( L \), \( \mathcal{M} \) performs a random walk in the manifold of allowed transfer matrices. If the matrix elements of the disorder potential have identical and independent Gaussian distribution without long-range correlations, this random walk again is described as diffusion on an associated symmetric space. The corresponding diffusion equation (or Fokker-Planck equation) for the probability density of the radial coordinates \( x_n \), \( n = 1, \ldots, N^* \) reads

\[
\frac{\partial P}{\partial L} = \frac{1}{2\gamma \ell} \sum_{j=1}^{N^*} \frac{\partial}{\partial x_j} J \frac{\partial}{\partial x_j} J^{-1} P,
\]

where \( J \) is a Jacobian describing the transformation to radial coordinates. It has the functional form

\[
J = \prod_j \sinh^{m_j}(2x_j) \prod_{k<j} \sinh^{m_{kj}}(x_j \pm x_k),
\]

where \( m_j \) and \( m_{kj} \) are numerical constants describing the geometric structure of the symmetric space (see Table I). Finally, \( \gamma = (m_{++} + m_{--})(N^* - 1)/2 + 1 + m_l \) is another numerical constant.

To determine the appropriate symmetric space, one begins from the three symmetries listed above (TRS, SRS, and SLS). A statistical ensemble of Hamiltonian is classified into symmetry classes according to its invariance properties under the transformations induced by these symmetries. This gives 6 symmetry classes with the corresponding symmetric spaces given in the first 6 rows from Table I. In the context of quasi one-dimensional localization for the standard symmetry classes (O, U, S in Table I, Eq. [11]) was first derived by Dorokhov. It was re-derived independently by Mello, Pereyra, and Kumar and is now called the DMPK equation. The three symmetry classes with SLS are called chiral classes. Of the three chiral random matrix theories that describe the energy level statistics on the scale of the mean-level spacing of the random hopping problem, the chU class has also been applied to the chiral phase transition in quantum chromodynamics. The diffusion equation associated to \( \mathcal{M}, \mathcal{M}^\dagger \) for the localization problem with SLS was derived by Altland, Simons, and two of the authors.

Four more symmetry classes are obtained when considering the effect of disorder on Bogoliubov quasiparticles in a superconductor described at the mean-field level. In a superconductor, the quasiparticle wavefunctions at excitation energy \( \epsilon \) are eigenfunctions of the Bogoliubov-de Gennes (BdG) equation,

\[
(\mathcal{H} - \epsilon) \Psi = 0, \quad \mathcal{H} = \begin{pmatrix} \hat{h} & \hat{\Delta} \\ -\hat{\Delta}^\dagger & -\hat{h}^\dagger \end{pmatrix},
\]

which has an additional \( 2 \times 2 \) grading, the particle-hole grading, when compared to the Schrödinger equation. Here \( \hat{h} (\hat{\Delta}) \) is a \( 4N \times 4N \) Hermitian (antisymmetric) matrix representing a single-particle Hamiltonian (superconducting order parameter). The Hamiltonian \( \mathcal{H} \) in the BdG equation satisfies the symmetry relation \( \gamma_1 \mathcal{H}^\dagger \gamma_1 = -\mathcal{H} \) (particle-hole symmetry), where \( \gamma_1 \) is the Pauli matrix in the particle-hole grading. For the transfer matrix particle-hole symmetry implies

\[
\gamma_1 \mathcal{M}_{\pm \epsilon}(L) \gamma_1 = \mathcal{M}_{-\epsilon}^\dagger(L).
\]

At \( \epsilon = 0 \), the symmetry requirement modifies the structure of the manifold of allowed transfer matrices and thus qualitatively modifies the localization behavior. The presence of particle-hole symmetry gives four more symmetry classes, referred to as C, CI, D, and DIII, depending on the presence or absence of TRS and SRS, see Table I. At the level of mean-field theory, charge is not a good quantum number. Only energy is conserved as is spin in the presence of SRS. Anderson localization applied to BdG quasiparticles thus aims to describe the statistics of the global and local DOS of BdG quasiparticles and the thermal or spin conductance. The diffusion equation for the four BdG symmetry classes was reported at the end of the 20th century.

In total, there are 10 different symmetry classes, yielding 10 symmetric spaces for the transfer matrix ensemble. For four of those symmetry classes (U, chU, CI, DIII), an exact solution of the diffusion equation is possible. For the remaining classes, asymptotic solutions for \( L \ll N \ell \) and \( L \gg N \ell \) are known. We first review the asymptotic for small and large lengths, and then discuss the exact solutions.

In the regime \( \ell \ll L \ll N \ell \) where transport is diffusive, moments of the conductance \( g \) can be obtained
as a power series in $L/N\ell$. The leading term in the expansion for the average $\langle g \rangle$ is the Drude conductance ($\propto N\ell/L$), cf. Eq. 11. The leading correction to the Drude conductance is known as the weak localization correction which, unlike the Drude conductance, depends on the absence or presence of SRS, a situation similar to the one for the orthogonal and symplectic cases in the standard symmetry classes, but the behavior at $L \gg N\ell$ is very different from the standard classes.

The probability distribution of the conductance becomes very broad if $L \gg N\ell$ and is poorly characterized by its mean. For the standard symmetry classes, the chiral classes with an even number of channels, and for the BdG symmetric classes CI and C, $ln g$ becomes self averaging and thus represents the distribution well. In these classes, $ln g$ increases linearly with $L$, corresponding to exponential localization. We refer to Table II for the relevant results. For the remaining cases, $ln g$ scales proportionally to $(L/N\ell)^{1/2}$ (implying the diverging localization length), and its fluctuations are of the same order as its mean. This is a signature of anomalous localization that reflects itself with the anomalously slow decay $\sim N\ell/L$ of the mean conductance.

The exact solution of the diffusion equation is possible if the multiplicity of the ordinary root $m_o$ is two. The exact solution has been found to the first and second moments of the conductance for all lengths. For the chU, CI, and DIII symmetry classes, the first moment is given by

$$\langle g \rangle_{chU} = \frac{1}{2s} + \frac{1}{s} \sum_{n=1}^{\infty} (-1)^{n(N+1)} e^{-\pi^2 n^2/8s},$$

$$\langle g \rangle_{CI} = \frac{1}{4} - \frac{1}{3} - 4 \sum_{n=1}^{\infty} e^{-\pi^2 n^2/4s} \left( \frac{1}{s} + \frac{2}{\pi^2 n^2} \right),$$

$$\langle g \rangle_{DIII} = \frac{1}{2} + \frac{2}{3} - 4 \sum_{n=1}^{\infty} e^{-\pi^2 n^2/2s} \left( \frac{1}{\pi^2 n^2} \right),$$

respectively, where $s = L/4\pi N\ell$ in the universal limit $N \to \infty$. The exact result shows explicitly that localization is non-perturbative in $L$, explaining why only very little progress can be made with standard perturbative techniques.

In Fig. 2 we show results of numerical simulations of the mean and variance of the conductance for the so-called quasi-1D random flux model where $t_{i,j} = e^{i\theta_{i,j}}$ with random $\theta_{i,j}$ and which has the SLS but no TRS, together with the exact results with $\ell$ as a fitting parameter. What is specific to the chiral symmetry classes realized at $\varepsilon = 0$ is an even-odd effect in $N$: as in the random hopping chain, $\langle g \rangle$ decreases algebraically as $L^{-1/2}$ with $L$ for odd $N$ whereas it decays exponentially fast with $L$ for even $N$. This remarkable even-odd effect is reminiscent of the one for the spin gap in the number of legs of $S = 1/2$ spin ladder systems. For the diffusive regime $(\ell \ll L \ll N\ell)$, the second cumulant of $g$ is called the universal conductance fluctuations, and its value in the chiral classes is twice as large as the one in the corresponding standard symmetry classes. The numerical results at $\varepsilon \neq 0$ show the behavior of the unitary class as
b) variance $\langle \epsilon \rangle$ of the conductance in the random flux model. The numerical results at the band center $\varepsilon = 0$ are denoted by $\circ$ ($N = 15$) and $\square$ ($N = 16$), and the exact solution (at $N \gg 1$) from the diffusion equation are shown by full ($N$: odd) and dotted ($N$: even) lines. For the variance, the result off the band center $\varepsilon = 0.1$ with $N = 32$ is also displayed by $\triangle$ for comparison. It is well-fitted by the analytical result (broken line) of the unitary symmetric class which is obtained by solving the NLSM exactly in quasi-one dimensions.

a result of the breaking of the SLS $\omega$. $\omega$

What is the relation between the divergence of the localization length and the DOS singularity originally discovered by Dyson? In order to see this, one has to adapt the transfer matrix approach to the calculation of scattering phase shifts. Phase shifts determine the DOS through the Friedel sum rule. Interestingly, information about the scattering phase shifts can be obtained from the standard transfer matrix approach if we reconsider the transport problem in the presence of an imaginary energy $\varepsilon = i\omega$. Unlike a real energy, an imaginary energy does not change the symmetry of the transport problem. The energy $i\omega$ simply yields a drift term in the diffusion equation (11), but otherwise allows the transport problem to be solved by essentially the same methods as outlined previously. Hence, just like the conductance, the DOS is thus completely determined by the geometry of the corresponding symmetric spaces!

For energies of the order of the inverse of the time $\tau_c \sim N^2\ell/v_F$ needed to diffuse through a wire of length $\sim N\ell$, the DOS is constant and given by $\rho_0 = ND/(\pi v_F)$ for all symmetry classes. This is also the behavior of the DOS for the standard symmetry classes when $|\varepsilon \tau_c| < 1$. For small energies, $|\varepsilon \tau_c| < 1$, the remaining symmetry classes, for which the energy $\varepsilon = 0$ plays a special role, are much more interesting. The expectation that level repulsion would enforce a vanishing of the DOS captured by random matrix theory is wrong whenever the long root $m_l$ vanishes. As shown in table III, $\rho(\varepsilon)$ shows a power law behavior with logarithmic corrections whose exponent for the chiral classes with $N$ even is not captured by the chiral random matrix theory. For the chiral classes with $N$ odd and the BdG classes D and DIII, the DOS exhibits the Dyson singularity.

The diffusion equation (11) is a beautiful example of one-parameter scaling. Here we’d like to exert one word of caution about the role played by symmetry: our formal discussion was conditional on the assumption that all matrix elements of the weak potential $V$ have identical, independent, and Gaussian distributions. If these conditions are not met, the conductance and DOS distributions will depend on microscopic details. Obviously, in such a case, symmetry need not be the only factor in determining the conductance and DOS distributions. In particular, if symmetries are only weakly broken (for example, when TRS is broken by a weak magnetic field or when a small $\varepsilon > 0$ breaks the SLS of $\mathcal{M}(\omega)$), one may end up in a cross-over between two or more of the symmetry classes listed in Table I. However, in the limit of large channel number $N$, universality is believed to be restored. Symmetry is the only player in the limit $N \gg 1$ (keeping $L/N\ell$ fixed, to ensure a fixed position in the crossover to localization), so that the notion of a symmetry class can be elevated to that of a universality class.

III. PROGRESS IN TWO DIMENSIONS

One of the best studied disorder-induced critical points in two dimensions is the plateau transition in the IQHE. Upon changing the filling fraction of the (lowest) Landau levels by applying a magnetic field, say, the localization length for states at the Fermi level diverges at some critical value $B_c$,

$$\xi \sim |B - B_c|^{-\nu},$$  

(18)

at which the Hall conductance jumps by one in the unit of $e^2/h$. Here, the exponent for the localization length $\nu$ is known numerically to be $2.3 < \nu < 2.4$ within the framework of Anderson localization. The single-particle wavefunction $\psi(r)$ at the plateau transition is multifractal, i.e., the so-called inverse participation ratio $P_q$ scales with the linear size $L$ of the system as a power law with the non-linear function of scaling exponents $\tau(q)$,

$$P_q = \int_{L^2} d^2r |\psi(r)|^{2q} \sim L^{-\tau(q)}.$$  

(19)
The spectrum of exponents $\tau(q)$ is believed to be universal. The successful theory of the plateau transition in the IQHE must predict $\nu$ and $\tau(q)$. Decrypting the critical (conformal field?) theory describing the plateau transition remains a tantalizing open problem. On the other hand, there has been some progress on two-dimensional criticality in the chiral and BdG universality classes. In this section, we will review these results with main emphasis on exact results.

Within the two-dimensional chiral symmetry class, the exact computation of the multifractal scaling exponents $\tau(q)$ was carried out for the Hamiltonian describing a particle with relativistic dispersion in a random white-noise correlated vector potential of vanishing mean $A_{\mu}(r)$,

$$H = \sum_{\mu=1,2} \sigma_{\mu} \left( i \partial_{\mu} + A_{\mu}(r) \right)$$

($\sigma_{\mu}$ is a Pauli matrix). This problem of Anderson localization is a fine-tuned model with symmetry of the chU type that realizes a line of critical points as a function of the variance $g_{A}$ of $A_{\mu}$. The band center $\varepsilon = 0$ is a mobility edge, where a critical state is located, as was the case in the random hopping chain; see Fig. 3. The exact computation of the $\tau(q)$-spectrum for this critical wavefunction is possible by exploiting similarities to the problem of directed polymers in random media. For $|q| \leq \sqrt{2\pi/g_{A}}$, $\tau(q)$ is a quadratic function of $q$ whereas it is linear in $q$ outside this region (Fig. 3). A similar behavior was also observed numerically for critical wavefunctions in the IQHE. The DOS too can be computed exactly close to the band center. Its dependence on energy $\varepsilon$ is algebraic, $\rho(\varepsilon) \sim |\varepsilon|^z$. The exponent $\beta$ also displays a non-analytic dependence on $g_{A}$. Both singularities can be ascribed to the non-analyticity at $g_{A} = 2\pi$ of the dynamical exponent $\kappa$.

$$z(g_{A}) = \begin{cases} 
1 + \frac{g_{A}}{2\pi}, & \text{for } g_{A} < 2\pi, \\
4\sqrt{\frac{g_{A}}{2\pi}} - 1, & \text{for } g_{A} \geq 2\pi.
\end{cases}$$

(21)

This non-analyticity as a function of $g_{A}$ can be interpreted as a freezing transition by analogy to the non-analytic dependence on temperature of the free energy in the random-energy model for spin glasses.

The random vector potential problem is the simplest two-dimensional continuum model with randomness that encodes the chiral symmetry. For more generic models with SLS, the critical line is unstable to marginal perturbations (in the renormalization group sense) compatible with the chiral symmetry, which renormalize $g_{A}$ to strong coupling. Gade introduced a NLSM that describes generic two-dimensional random hopping problems with SLS, for which she found the diverging DOS

$$\rho(\varepsilon) \sim \frac{1}{|\varepsilon|} \exp(-c|\ln|\varepsilon|)^{\beta}$$

(22)

sufficiently close to the band center. Here, $c$ is a non-universal constant and $\kappa = 1/2$ according to Gade. This result can be derived by dimensional analysis if one assumes that energy and length scales are related by $\ln|\varepsilon| \sim -z(L)\ln L$ with $z(L) \sim \ln L$. However, Motrunich et al. conjectured that $\kappa = 1/2$ is only a transient as the true asymptotic dependence of the effective dynamical exponent on $L$ is $z(L) \sim \sqrt{\ln L}$, as would follow if the $L$ dependence of $z(L)$ was given by Eq. (21) with an effective running coupling constant $g_{A} \sim \ln L$. If so, the scaling exponent $\kappa = 2/3$ follows instead of $\kappa = 1/2$. The latter scaling analysis was confirmed by a functional renormalization group calculation for a two-dimensional random hopping problem in Ref. 43.

According to the theorem of Anderson on the effect of disorder in an $s$-wave BCS superconductor, non-magnetic impurities are largely an irrelevant effect. In an unconventional BCS superconductor, here defined by an order parameter with a nonzero angular momentum, non-magnetic impurities have a much stronger effect. With the realizations of unconventional superconductivity such as high-$T_{c}$ superconductors, spin triplet superconductivity in ruthenates, two-dimensional organic materials, paired states in the fractional quantum Hall effects etc., searching for non-perturbative effects in dirty, unconventional superconductors gained in urgency in the mid 90s. With SRS and no strong breaking of TRS, BdG quasi-particles are localized in two dimensions. In this case the DOS is depressed close to the band center as expected from level repulsion, $\rho(\varepsilon) \sim \varepsilon$ with TRS (CI class) and $\rho(\varepsilon) \sim \varepsilon^2$ without TRS (C class). On the other hand, as is the case with the IQHE, strong breaking of TRS opens up the possibility of a new critical point induced by disorder, the thermal (spin) Hall transition. Using a network model representation of the spin Hall transition in Class C Gruzberg et al. reduced the problem of computing the exponent $\nu$ for the diverging localization length and the exponent $\beta$ for the power law decay of the DOS to the classical problem of bond percolation. They found $\nu = 4/3$ and $\beta = 1/7$ at the plateau transition. No results of that magnitude are presently available when SRS is broken in a dirty BdG but the existence of a plateau transition for the Hall thermal current

![FIG. 3: (a) Typical wavefunction amplitude $|\psi(r)|^2$ and (b) its exact $\tau(q)$-spectrum (solid line) for the random Hamiltonian at $g_{A} = 1.2$.](image)
has been established numerically together with a very rich phase diagram.

**IV. SUMMARY AND OPEN PROBLEMS**

Recent theoretical progress has established that critical phenomena at mobility edges in quasi-one dimension are of the Dyson type. Their properties are much better understood than with Anderson localization in two dimensions in all but few cases. In this article, we have reviewed two-dimensional models for which the multifractal spectrum or some critical indices can be obtained exactly. Within a classification based on symmetries of disorder induced critical points in the problem of Anderson localization, the chiral symmetry class appears to be the simplest both in quasi one and two dimensions.

The physics of the chiral symmetry classes is closely related to the physics of the pseudogap in Peierls systems and to the physics of the chiral phase transition in quantum chromodynamics, which we could not discuss in this limited space. Chiral and BdG classes are also related to classical and quantum random spin systems. For example, the Dyson singularity in the random hopping chain is connected to the random singlet phase in one-dimensional quantum spin systems, the two-dimensional chiral symmetry classes are related to the two-dimensional gauge glass model and the classical random bond Ising model in two dimensions can be mapped onto a two-dimensional network model belonging to the symmetry class D. In the last example, this connection was used to demonstrate the multifractal behavior of the correlation function of a dual order parameter and a freezing phenomenon.

Another important open issue is that of the competition between disorder and interactions in the chiral and BdG universality classes. The situation seems to be under better control for the BdG symmetry classes than in the standard symmetry classes.

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