Finite temperature corrections
to weak rates prior to nucleosynthesis

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Abstract

We have reexamined the electromagnetic corrections to the weak interaction rates for the transformation of neutrons to protons, and protons to neutrons, in the early universe, before freeze-out. We derive compact expressions for these rates in terms of thermal expectation values of products of fields, and we give explicit constructions of the terms to order $e^2$. We disagree in several respects with results in the literature.
I. INTRODUCTION

In the early universe, in the era just before nucleosynthesis, the weak interaction rates combined with the expansion rate determine the freeze-out neutron-proton ratio at a temperature on the order of 0.3 MeV. This freeze-out ratio in turn essentially determines the primordial He abundance. In the region of temperature 0.3 — 3.0 MeV, in which the weak interaction rates are not fast enough to preserve chemical equilibrium, it is necessary to calculate these rates reasonably accurately. The Born approximation rates, as used by Peebles in the first calculations of the freeze-out ratio and in the later refinements by Wagoner, nearly suffice for the purposes of this calculation. But it is possible that the primordial He abundance will be measured to such a precision that the electromagnetic corrections to the rates will play a role in the interpretation.

There have been a number of calculations of the electromagnetic corrections to the freeze-out ratio. These corrections come from both Coulomb and radiative effects as modified by the hot plasma in the early universe and depend on its temperature. Roughly speaking, the method that has been used in the calculations has been the calculation of all of the T matrix elements in which $n \rightarrow p$ or $p \rightarrow n$, as calculated to order $e^2 G^2 W$. The squares of these T matrix elements are then weighted with the appropriate statistical factors for the incoming and outgoing electrons, positrons, neutrinos, anti-neutrinos and photons, and integrated over the phase space for the particles. We call this the “exclusive channel” approach.

In the present paper we re-address the electromagnetic correction problem. Our motivations are several:

1) There is unresolved disagreement in the literature over the correct treatment of one artifact of the exclusive channel approach, the so-called “temperature-dependent wave-function renormalization”. And even if the differences with respect to the the electron’s wave function renormalization were resolved, there is, in exclusive calculations done in Feynman gauge, the necessity of considering the proton’s temperature-dependent wave-function renormalization” as well. This has not been dealt with in the calculations in the literature, which were all done in Feynman gauge.

2) In the exclusive approach followed by all previous authors, two reactions have been left out that give contributions of the same order as the corrections that have been included, namely the reactions $\nu + e^+ + n \leftrightarrow p + \gamma$. We believe that this is mere oversight, originating from the fact that these reactions are not corrections to processes that take place in the absence of the photon coupling, with a photon added.

3) It would appear that the reported results of the two most complete reworking of the electromagnetic effects, do not obey the appropriate detailed balance relation between proton and neutron rates.

4) The rate correction problem is completely well defined within the standard model of the weak and electromagnetic interactions, and it is addressable through perturbation theory. Indeed, for the temperature-dependent corrections considered in the present paper, the weak interaction can be taken to be of the local four-Fermi form. We shall endeavor to

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1For a simple pedagogical treatment of the abundance calculation see Ref. [1].
present a definitive and final result. In view of all of the past confusion, we shall take pains to give complete arguments, to give the analytical results in detail, and to explain all of the steps.

We shall develop the perturbation theory for the neutron and proton disappearance rates directly, without going through the intermediary of the calculation of T matrix elements for exclusive reactions. Instead, we begin from equations for the inclusive neutron and proton transformation rates, expressed as the appropriate integrals over thermal expectation values of electron field operators. We then apply the standard methods of thermal field theory to obtain the electromagnetic corrections of order \( e^2 \) and to reduce the answers to a form allowing easy computation. We prove that the simple detailed balance relation between the proton and neutron rates remains valid to order \( e^2 \). This relation serves as a useful check on results. There is no role for “temperature-dependent wave-function renormalization” in our approach. Our calculation ends with numerical results that are quite different from those of previous treatments.

We shall assume complete thermal equilibrium for each species. The effects of a non-thermal neutrino distribution have been dealt with separately elsewhere \cite{6}, \cite{7}. Their effect on the terms of order \( e^2 \), a correction to a correction, is not consequential, particularly so since the deviations from a thermal spectrum become appreciable only at temperatures very near the freeze-out point.

However the system can be arbitrarily far from chemical equilibrium. It will suffice to consider a single neutron in the medium in order to calculate the neutron transformation rate, and a single proton in the medium to calculate the proton transformation rate. The electron and neutrino chemical potentials are taken to vanish.

II. GENERAL FORMULATION

The rates of neutron and proton appearance and disappearance can be expressed in terms of the inclusive rates of \( \nu_e \) and \( \bar{\nu}_e \) appearance and disappearance. Since we are concerned only with the modifications of the rates which arise from plasma effects that involve low energies — which are ‘soft processes’ — we may take the nucleon mass to be infinite\(^2\). We take the proton or neutron to be situated at the origin. We begin by taking the initial nucleon to be a neutron. As described in detail in Appendix A, the rate for a neutron to change into a proton may be expressed as

\[
\Gamma_n = \frac{G_W^2}{(2\pi)^2} \left( g_W^2 + 3 g_A^2 \right) \int_{-\infty}^{+\infty} dE_{\nu_e} E_{\nu_e}^2 n(E_{\nu_e}) \int_{-\infty}^{+\infty} dt \ e^{iE_{\nu_e}t} W(t) ,
\]

where

\[
W(t) = \left\langle \psi_{\nu_e}(0,t)T_{\nu_e}(t)\bar{\psi}_{\nu_e}(0)T_{\nu_e}(0) \right\rangle_T.
\]

\(^2\)The zero temperature rates, which we shall delete from our results, involve a domain of very hard virtual photons that requires the inclusion of nucleon recoil. The zero temperature radiative corrections are summarized in Ref’s. \cite{3} and \cite{4}. There are also small temperature-dependent recoil effects that do not involve the electromagnetic couplings. These are evaluated in ref. \cite{3}.
Here the angular brackets with the $T$ subscript denote the thermal average including all plasma and electromagnetic radiative interactions as well as the single particle state of the (infinitely heavy) neutron placed at $r = 0$. The electron field operator is denoted by $\psi_e(x)\text{ and } T_\pm(t)$ are the isospin raising and lowering operators, with all the operators in the Heisenberg picture — their time dependence is controlled by the full Hamiltonian of the system which also defines the thermal average. We shall assume that the chemical potentials of all the leptons vanish so that they involve the Fermi occupation density\footnote{These apply to the free neutrinos and to the electrons once the electromagnetic corrections appear as a perturbation. The electron distribution is, of course, modified by the electromagnetic interactions, but this is accounted for automatically in our formalism.}

$$n(E) = \frac{1}{e^{\beta E} + 1}. \quad (2.3)$$

Note that

$$n(-E) = e^{\beta E} n(E) = 1 - n(E) \quad (2.4)$$

is the Pauli blocking factor for a particle produced with (positive) energy $E$.

The integral over positive neutrino energies $E_\nu$ in the rate $\Gamma_n$ describes neutrinos absorbed from the thermal bath with a population governed by $n(E_\nu)$. Including corrections of order $e^2$, it is the sum of the rates for the processes:

1. $\nu + n \rightarrow p + e^-$
2. $\nu + n + \gamma \rightarrow p + e^-$
3. $\nu + n \rightarrow p + e^- + \gamma$
4. $\nu + n + e^+ \rightarrow p + \gamma$

The last process in this list, $\nu + n + e^+ \rightarrow p + \gamma$, and the reversed process, have been omitted in all previous calculations in the early universe application. Their contribution will turn out to be at about the same level as some of the processes considered in Ref’s. \[4\] – \[7\].

The integral over negative neutrino energies $E_\nu$ in the rate $\Gamma_n$ describes antineutrinos emitted into the thermal bath with the Pauli blocking factor $n(-|E_\nu|)$. Again to order $e^2$, it is the sum of the rates for the processes:

1. $n \rightarrow p + e^- + \bar{\nu}$
2. $n + \gamma \rightarrow p + e^- + \bar{\nu}$
3. $n \rightarrow p + e^- + \gamma + \bar{\nu}$
4. $n + e^+ \rightarrow p + \bar{\nu}$
5. $n + e^+ + \gamma \rightarrow p + \bar{\nu}$
6.  \( n + e^+ \rightarrow p + \gamma + \bar{\nu} \)

Again as shown in Appendix A, the rate for the disappearance of an initial proton is given by

\[
\Gamma_p = \frac{G_W^2}{(2\pi)^2} \left( g_\nu^2 + 3g_\lambda^2 \right) \int_{-\infty}^{\infty} dE_\nu E_\nu^2 n(E_\nu) \int_{-\infty}^{\infty} dt e^{-iE_\nu t} V(t),
\]

(2.5)

where

\[
V(t) = \left\langle \psi_e^\dagger(0) T_+(0) \psi_e(t) T_-(0) \right\rangle_T .
\]

(2.6)

The integral over positive \( E_\nu \) describes the rates involving the absorption of antineutrino, the processes in the second list above with the direction of the reaction arrow reversed. The integral over negative \( E_\nu \) describes the emission of neutrinos, the reverse of the first list of processes above. For future reference, we note that the simple change \( t \rightarrow -t \) gives

\[
\Gamma_p = \frac{G_W^2}{(2\pi)^2} \left( g_\nu^2 + 3g_\lambda^2 \right) \int_{-\infty}^{\infty} dE_\nu E_\nu^2 n(E_\nu) \int_{-\infty}^{\infty} dt e^{+iE_\nu t} \bar{V}(t),
\]

(2.7)

where \( \bar{V}(t) = V(-t) \). We make use of the time-translation invariance of the expectation value to add the time \( t \) to all operators to obtain the form

\[
\bar{V}(t) = \left\langle \psi_e^\dagger(t) T_+(t) \psi_e(0) T_-(0) \right\rangle_T .
\]

(2.8)

This expresses the proton rate \( \Gamma_p \) in exactly the same form as the neutron rate \( \Gamma_n \) except for the interchanges \( \psi_e^\dagger \leftrightarrow \psi_e \) and \( T_+ \leftrightarrow T_- \).

It should be emphasized that our formalism takes account of all these individual processes and unifies them in terms of a simple expression. This expression forms the basis in which the plasma effects on the radiative corrections are easily computed in a complete and unambiguous fashion.

To the order \( e^2 \) to which we work, the neutron and proton proton rates, \( \Gamma_n \) and \( \Gamma_p \), obey the detailed balance relation

\[
\Gamma_p = e^{-\beta \Delta} \Gamma_n ,
\]

(2.9)

where

\[
\Delta = M_n - M_p
\]

(2.10)

is the neutron–proton mass difference. Thus only one of the two rates \( \Gamma_n \) or \( \Gamma_p \) need be calculated.

To prove the detailed balance statement, we first define the projection operators \( P_n \) and \( P_p \) for the point-like heavy neutron and proton states located at \( \mathbf{r} = 0 \). In terms of these operators, we have the explicit expressions

\[
W(t) = Z^{-1}_n \text{Tr} e^{-\beta \mathbf{H}} \psi_e(0) T_-(t) \psi_e^\dagger(0) T_+(0) P_n ,
\]

(2.11)

and
\[ V(t) = Z_p^{-1} \text{Tr} e^{-\beta H} \psi_e^\dagger(0) T_+(0) \psi_e(0, t) T_-(t) P_p, \] (2.12)

where
\[ Z_n = \text{Tr} e^{-\beta H} P_n, \quad Z_p = \text{Tr} e^{-\beta H} P_p. \] (2.13)

The projection operators \( P_n \) and \( P_p \) are time independent (commute with \( H \)), and they commute with the electron fields, while
\[ T_-(t) P_p = P_n T_-(t). \] (2.14)

Using this result, the cyclic symmetry of the trace, and
\[ \psi_e(0, t) T_-(t) e^{-\beta H} = e^{-\beta H} \psi_e(0, t - i\beta) T_-(t - i\beta), \] (2.15)
we obtain
\[
V(t) = Z_p^{-1} \text{Tr} e^{-\beta H} \psi_e(0, t - i\beta) T_-(t - i\beta) \psi_e^\dagger(0) T_+(0) P_n
= \left( \frac{Z_n}{Z_p} \right) W(t - i\beta).
\] (2.16)

Placing this result in the rate formula (2.5) for the proton and shifting the time integration variable \( t \rightarrow t + i\beta \), we find that
\[
\Gamma_p = \frac{Z_n}{Z_p} G_{W^2} \left( g_V^2 + 3g_A^2 \right) \int_{-\infty}^{+\infty} dE_{\nu} E_{\nu}^2 n(E_{\nu}) e^{\beta E_{\nu}} \int_{-\infty}^{+\infty} dt e^{-iE_{\nu}t} W(t).
\] (2.17)

Finally we recall Eq. (2.4) which states that \( n(E_{\nu}) \exp \{ \beta E_{\nu} \} = n(-E_{\nu}) \). Accordingly, we reflect the neutrino energy integration variable, \( E_{\nu} \rightarrow -E_{\nu} \), to place this result in precisely the form (2.1) of the neutron rate except for the initial overall factor, and hence
\[
\Gamma_p = \left( \frac{Z_n}{Z_p} \right) \Gamma_n.
\] (2.18)

In lowest order,
\[
\frac{Z_n}{Z_p} = e^{-\beta \Delta},
\] (2.19)
and the detailed balance statement (2.9) is obtained to order \( e^2 \) if there are no corrections to this ratio to order \( e^2 \). It is a familiar result from statistical mechanics that there is no order \( e^2 \) correction to the partition function for a dilute gas. The same considerations show that there is no \( e^2 \) correction when a single charged particle is introduced into a plasma. This is made explicit in Ref. [22] where it is also shown that the first correction is given by
\[
\frac{\delta Z_p}{Z_p} = \frac{1}{2} \beta \alpha \kappa_D,
\] (2.20)
where \( \kappa_D \) is the Debye wave number. The nature of this result is clear: The introduction of a charged particle into a plasma alters its electrostatic field from a pure Coulomb potential.
\( e/4\pi r \) to the Debye screened potential \((e/4\pi r) \exp\{-\kappa_D r\}\), and thus changes the particle’s self-energy by

\[
\delta E = \lim_{r \to 0} \frac{1}{2} \frac{\alpha}{r} [e^{-\kappa_D r} - 1] = -\frac{1}{2} \alpha \kappa_D , \quad (2.21)
\]
giving \( \delta Z_p/Z_p = -\beta \delta E \). For the electron-positron plasma that is of our concern, the squared Debye wave number is given by

\[
\kappa_D^2 = \beta e^2 \Delta \ell^2 = e^2 \frac{\partial \ell}{\partial \mu} \bigg|_{\mu = 0} , \quad (2.22)
\]
where \( \ell \) is the leptonic charge density,

\[
\ell = 2 \int \frac{d^3 p}{(2\pi)^3} \left[ \frac{1}{\exp\{\beta[E(p) - \mu]\} + 1} - \frac{1}{\exp\{\beta[E(p) + \mu]\} + 1} \right] . \quad (2.23)
\]

In the high temperature limit in which the temperature is much greater than the electron mass, \( T \gg m \), this gives \( \kappa_D^2 = 4\pi \alpha T^2/3 \) and so the plasma correction \( (2.20) \) is a negligible change

\[
\frac{\delta Z_p}{Z_p} = \frac{\alpha^{3/2}}{\sqrt{3}} \approx 0.00064 . \quad (2.24)
\]

At lower temperatures, this correction vanishes exponentially \((\sim \exp\{-\beta m\})\) as the temperature becomes small in comparison with the electron mass, and is thus even smaller.

To demonstrate the workings of our formalism, we exhibit the rates in the absence of plasma interactions and radiative corrections. In this limit,

\[
T_-(t) = e^{+i\Delta t} T_-(0) , \quad (2.25)
\]
and using the free field correlators \((A17)\) for the thermal values of the electron fields in Eq. \((2.1)\) we get

\[
\Gamma_n^{(0)} = G_W^2 \left(g_V^2 + 3g_A^2\right) \frac{1}{\pi} \int \frac{d^3 p}{(2\pi)^3} \left[ n(-E) (E - \Delta)^2 n(E - \Delta) + n(E) (E + \Delta)^2 n(-E - \Delta) \right] , \quad (2.26)
\]
where we use the short-hand notation \( E = E(p) \). The first term in the square brackets in the region \( m_e < E < \Delta \) describes free neutron decay \( n \to p + e^- + \bar{\nu} \) with the final leptons having the Pauli blocking factors \( n(-E), n(-[\Delta - E]) \). For \( E > m_e \), the term corresponds to \( \nu + n \to p + e^- \) with the initial thermal neutrino population governed by \( n(E - \Delta) \) and the final electron having the Pauli blocking factor \( n(-E) \). The second term in the square brackets represents the process \( e^+ + n \to p + \bar{\nu} \) with the initial positron drawn from the thermal bath with population factor \( n(E) \) and the final antineutrino having the Pauli blocking factor \( n(-E - \Delta) \). In this limit of no corrections, the proton disappearance rate \((2.5)\) becomes
\[ \Gamma_p^{(0)} = G_W^2 \left( g_V^2 + 3g_A^2 \right) \frac{1}{\pi} \int \frac{d^3 p}{(2\pi)^3} \left[ n(E) (E - \Delta)^2 n(\Delta - E) + n(-E) (E + \Delta)^2 n(E + \Delta) \right], \]  

(2.27)

with the terms in the square bracket describing the previous neutron processes in the reverse direction. Since

\[ n(\pm E) = e^{\mp E} n(\mp E), \quad n(\Delta \mp E) = e^{\pm E} e^{-\beta \Delta} n(\pm E - \Delta), \]  

(2.28)

these rates do indeed obey the detailed balance relation \( \Gamma_p^{(0)} = \exp\{\beta \Delta\} \Gamma_n^{(0)} \). Note that the neutron and proton rates \( \Gamma_n \) and \( \Gamma_p \) are related by the interchange of \( \Delta \leftrightarrow -\Delta \).

In view of the detailed balance relation (2.3), we shall concentrate on the neutron disappearance rate \( \Gamma_n \). To write the lowest-order result in a compact form, we define

\[ \chi(E) = n(E - \Delta)n(-E)(E - \Delta)^2, \]  

(2.29)

and integrate over the electron solid angle to obtain

\[ \Gamma_n^{(0)} = 4 \frac{G_W^2}{(2\pi)^3} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty p^2 dp \left[ \chi(E) + \chi(-E) \right]. \]  

(2.30)

The general form for the neutron disappearance rate is given by Eq’s. (2.1) and (2.2). Since the rate involves the operators \( \psi_e(0,t)T_-(t) \) and \( \psi_e^\dagger(0)T_+(0) \) which are electrically neutral (they commute with the electromagnetic current), the rate is gauge invariant. Since we treat the nucleons as being infinitely heavy, it is convenient to perform calculations in the radiation or Coulomb gauge, and this we shall do. In this gauge and with a very heavy proton, the second-order electromagnetic correction involving the proton alone is simply a mass shift that is removed by the usual renormalization. To the order \( e^2 \) to which we work, and with the radiation gauge, the only non-trivial corrections come from the Coulomb interaction between the electron and proton and the radiative and Coulomb corrections to the electron (and positron) system.

We define the \( T = 0 \) rates as the rates that one would get summing the contributions from all of the processes listed in the introduction, where thermal distributions are taken for all initial and final electrons, positrons, neutrinos and antineutrinos, but the temperature is taken to vanish everywhere else. This completely excludes participation by the thermal bath of photons. Thus the only real photon effect in our definition of the \( T = 0 \) rates is a process with a bremsstrahlung photon in a final state. The \( T = 0 \) terms include all the radiative and Coulomb corrections that produce ultraviolet divergences which are removed by the usual renormalization of the vacuum amplitudes.

\[ \text{This equivalence for the unperturbed rates follows directly from the alternative form (2.7) for } \Gamma_p. \]  

Because of charge-conjugation invariance, the free-field expectation values of \( \psi^\dagger(t)\psi(0) \) and \( \psi(t)\psi^\dagger(0) \) are equal (in the Majorana representation of the Dirac matrices). Replacing \( T_-(t) \) by \( T_+(t) \) changes the factor \( e^{+i\Delta t} \) to \( e^{-i\Delta t} \). The expectation value of \( T_-T_+ \) in the neutron state equals that of \( T_+T_- \) in the proton state.
We shall subtract all such “$T = 0$” corrections to define and work only with temperature-dependent corrections to the rates that are free of the renormalizations needed to deal with the ultraviolet infinities.

This definition of $T = 0$ corrections agrees with the definition as used in Ref’s. \cite{4}–\cite{7} except in one regard: If we consider that part of the reaction rate for the reaction $n + e^+ + \nu \rightarrow p + \gamma$ in which the Bose factor for emission $[1 - \exp(-\beta \omega)]^{-1}$ is replaced by unity, this term falls under the above definition of a $T=0$ term, but it is not included in the $T = 0$ terms of Ref’s. \cite{4}–\cite{7}, for the reason that the process was not considered at all in these works.

### III. ELECTRON-PROTON INTERACTION

We shall first look at the plasma correction to the Coulomb interaction between the electron and proton,

$$H_C(t) = -\int (d^3 r')(d^3 r) \, \Psi_p^\dagger(r', t) \Psi_p(r', t) \frac{e^2}{4\pi |r' - r|} \psi_e^\dagger(r, t) \psi_e(r, t).$$

(3.1)

We start with this piece because it serves as an easy introduction to the method that will later be applied in the more complex electron self-energy correction. To account for the Coulomb perturbation in first order, we temporarily continue to imaginary time, $t \rightarrow -i\tau$, with $\tau > 0$. Then the perturbative rules of thermal field theory in imaginary time apply, and the electron-proton Coulomb interaction correction to Eq. (2.2) becomes

$$W_{ep}(-i\tau) = -\int_0^\beta d\tau' \left\langle \left(\psi_e(0, -i\tau) T_-( -i\tau) H_C(-i\tau') \psi_e(0) T_+ (0) \right)_+ \right\rangle_T,$$

(3.2)

where the $(\cdots)_+$ denotes time ordering in the imaginary time. The operator $H_C(-i\tau')$ makes no contribution when $\tau' > \tau$ because the ordering places it adjacent to the neutron state on which it vanishes. Taking this into account, we may now continue back to real time and write

$$W_{ep}(t) = -i \int_0^t dt' \left\langle \left(\psi_e(0, t) T_- (t) H_C(t') \psi_e(0) T_+ (0) \right)_+ \right\rangle_T,$$

(3.3)

where now no time ordering is involved. Since the perturbation is explicitly taken into account, we may now use the non-interacting results. Since the expectation value of $\Psi_p^\dagger(r', t) \Psi_p(r', t)$ is sharply localized at $r' = 0$, the coordinate $r'$ may be taken to vanish in the Coulomb potential factor. Hence the total charge operator of the proton appears which acts on a proton state of unit charge, and within the expectation value we have

$$T_-(t) \int (d^3 r') \Psi_p^\dagger(r', t) \Psi_p(r', t) \, T_+ (0) = T_-(t) \, T_+ (0) \rightarrow e^{+i\Delta t}.$$

(3.4)

Moreover, the electron field thermal expectation value may be replaced by the product of free fields defined in Eq. (A13) of Appendix A. Thus

$$W_{ep}(t) = +ie^{+i\Delta t} \int_0^t dt' \int (d^3 r) \frac{e^2}{4\pi |r|} \, \text{tr} \, S^+(\,-r, t - t') \gamma^0 \, S^+(r, t') \gamma^0,$$

(3.5)
and using the explicit form (A17) for the functions that appear here,

\[ W_{ep}(t) = +4e^{i\Delta t} \int \frac{(d^3p)}{(2\pi)^3} \frac{1}{2E} \int \frac{(d^3p')}{(2\pi)^3} \frac{1}{2E'} \frac{e^2}{(p - p')^2} \]

\[ \left\{ \frac{EE' + m^2 + p \cdot p'}{E - E'} \left[ n(-E)n(-E') \left[ e^{-iE't} - e^{-iEt} \right] - n(E)n(E') \left[ e^{iEt} - e^{-iEt} \right] \right] \right. \]

\[ +2\frac{EE' - m^2 - p \cdot p'}{E + E'} n(E)n(-E') \left[ e^{iEt} - e^{-iE't} \right] \} , \quad (3.6) \]

where \( E = E(p) \) and \( E' = E(p') \).

In placing this result in formula (2.1) for \( \Gamma_n \), the time integration produce energy-conserving \( \delta \) functions. When the resulting neutrino occupancy factors appear in the form of \( n(E - \Delta) \) multiplying \( n(-E) \) or \( n(-E - \Delta) \) multiplying \( n(E) \) [or the same forms with \( E \to E' \)], we have a "\( T = 0 \)" structure that, as described above, is to be removed. These are just the neutrino and electron occupancy factors that appear in the free rate (2.30). The terms involve a further factor of \( n(-E) = 1 - n(E) \) [or \( n(-E') = 1 - n(E') \)] with the "1" parts giving the (Coulomb corrected) "\( T = 0 \)" part that is to be removed. Our definition thus produces the temperature-dependent rate correction

\[ \Gamma^{(T)}_{n, ep} = G_W^2 \left( g_V^2 + 3g_A^2 \right) \frac{2}{\pi} \int \frac{(d^3p)}{(2\pi)^3} \frac{1}{2E} \int \frac{(d^3p')}{(2\pi)^3} \frac{1}{2E'} \frac{e^2}{(p - p')^2} \]

\[ \left\{ -\frac{EE' + m^2 + p \cdot p'}{E - E'} \left[ n(E)n(-E') (E' - \Delta)^2 n(E' - \Delta) \right. \right. \]

\[ -n(-E)n(E') (E - \Delta)^2 n(E - \Delta) \]

\[ +n(E)n(E') \left[ (E' + \Delta)^2 n(-E' - \Delta) - (E + \Delta)^2 n(-E - \Delta) \right] \]

\[ -2\frac{EE' - m^2 - p \cdot p'}{E + E'} \left[ n(E)n(E') (E + \Delta)^2 n(-E - \Delta) \right. \]

\[ +n(-E)n(E') (E - \Delta)^2 n(E - \Delta) \} . \quad (3.7) \]

We make use of the definition (2.29) of \( \chi(E) \), use \( pdp = EdE \), and perform the integrals over angles to obtain

\[ \Gamma^{(T)}_{n, ep} = -\frac{e^2 G_W^2}{(2\pi)^5} \left( g_V^2 + 3g_A^2 \right) \int_{-\infty}^{\infty} dE \int_{-\infty}^{\infty} dE' \]

\[ \left\{ \frac{1}{2} \left[ (E + E')^2 \ln \left( \frac{p + p'}{p - p'} \right)^2 - 4pp' \right] \frac{1}{E' - E} \]

\[ + \left[ n(E')\chi(E) - n(E)\chi(E') + n(E')\chi(-E) - n(E)\chi(-E') \right] \]

\[ + \left[ 4pp' - (E' - E)^2 \ln \left( \frac{p + p'}{p - p'} \right)^2 \right] \frac{1}{E' + E} \]
Each term in the integrand has a Fermi distribution function which falls exponentially for large $E$ or $E'$, so that there is no ultraviolet divergence. The terms in the second square bracket are odd under the interchange $E \leftrightarrow E'$ and thus vanish when $E = E'$ so as to remove the singularity in the overall $(E - E')^{-1}$ factor, leaving an integrand that is integrable at $E = E'$. Thus we are left with a well-defined double integral that may be done numerically. However, to save writing, and to place this result in a form that will prove useful later, we exploit the symmetry of the double integral to express the result in an asymmetrical form with the understanding that the potentially singular terms are defined by a principal part prescription:

$$\Gamma(T)_{n,\text{ep}} = \frac{e^2 G_W^2}{(2\pi)^3} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty dE \int_0^\infty dE' n(E') \left[ \chi(E) + \chi(-E) \right]$$

$$\frac{E}{E'^2 - E^2} \left[ 4pp' - (3E'^2 + E^2) \ln \left( \frac{p + p'}{p - p'} \right)^2 \right].$$

(3.9)

It is not difficult to confirm that the principal part prescription is equivalent to the original symmeterization of the double integral.

Making use of $n(E) = e^{-\beta E}$, we can write the first form (3.7) of the result as

$$\Gamma(T)_{n,\text{ep}} = e^{-\beta \Delta} \Gamma(T)_{n,\text{ep}}(\Delta) (3.12)$$

In this guise, it follows immediately that the formal change $\Delta \rightarrow -\Delta$ produces

$$\Gamma(T)_{n,\text{ep}}(-\Delta) = e^{-\beta \Delta} \Gamma(T)_{n,\text{ep}}(\Delta).$$

(3.11)

This is just the detailed balance relation (2.9) that relates the neutron and proton rates. Since the subtracted "$T = 0$" terms are proportional to the uncorrected rates that obey this relation, we conclude that the corresponding "$T \neq 0$" part of the Coulomb electron-proton correction to the proton rate may be obtained simply by the interchange $\Delta \rightarrow -\Delta$. That is:

$$\Gamma(T)_{p,\text{ep}} = \Gamma(T)_{n,\text{ep}}(\Delta \rightarrow -\Delta)$$

(3.12)
IV. ELECTRON SELF-ENERGY AND RADIATIVE CORRECTIONS

We turn now to examine the remaining leading-order electromagnetic correction that involves only the electron (positron) in the neutron rate defined in Eq.’s (2.1), (2.2). Since this effect involves no electromagnetic interaction with the proton, we have, effectively,

\[ T_-(t) T_+(0) \to e^{i\Delta t}. \]  

It is convenient to translate the energy integration variable by

\[ E \nu \to E = E \nu + \Delta \]  

and re-express the neutron rate formulas (2.1), (2.2) as

\[ \Gamma_{n e}^{e e} = \frac{G_W^2}{(2\pi)^2} \int_{-\infty}^{+\infty} dE (E - \Delta)^2 n(E - \Delta) \tilde{W}(E), \]  

in which

\[ \tilde{W}(E) = \int_{-\infty}^{+\infty} dt e^{iEt} \langle \psi(0, t) \psi^\dagger(0) \rangle_T. \]  

The angular brackets now denote the thermal average only for electrons and photons and, to reduce notational clutter, we have removed the \( e \) subscript on the electron field operators. Note that \( E > 0 \) describes outgoing electrons.

As we have just seen, the effects of interactions are most conveniently dealt with in the imaginary time formulation. However, the continuation to real time is now no longer trivial. Hence we pause to review very briefly the relationship between the unordered real-time function (4.3) and its counterpart, the time-ordered in imaginary time electron thermal Green’s function. Since the thermal average \( \langle \cdots \rangle_T \) is the normalized trace \( Z^{-1} \text{Tr} \exp\{-\beta H\} \cdots \), the cyclic symmetry of the trace may be invoked to interchange the order of operators by passing then through the evolution operator in imaginary time \( \text{exp}\{-\beta H\} \). Therefore

\[ \langle \psi(0, t) \psi^\dagger(0) \rangle_T = \langle \psi^\dagger(0) \psi(0, t + i\beta) \rangle_T. \]  

In term of Fourier transforms using the sign convention \( e^{-iEt} \), this relates the transforms of the two operator ordering by a factor of \( e^{\beta E} \). Therefore,

\[ \langle \psi(0, t) \psi^\dagger(0) \rangle_T = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{-iEt} n(-E) A(E), \]  

and

\[ \langle \psi^\dagger(0) \psi(0, t) \rangle_T = \int_{-\infty}^{+\infty} \frac{dE}{2\pi} e^{-iEt} n(+E) A(E), \]  

where \( A(E) \) is the Fourier transform of the thermal average of the anticommutator and \( n(E) \) is the Fermi distribution function with vanishing chemical potential that we have already made much use of. The thermal Green’s function, time-ordered in imaginary time, are computed by the usual quantum field theory rules, but in Euclidean space, \( t \to -i\tau \), and with Bosonic functions periodic and Fermionic functions anti-periodic in the imaginary time interval \( 0, \beta \). Thus the electron’s thermal Green’s function has the Euclidean 4-th momentum component \( p_4 = (2m + 1)\pi T \), and using the representation (4.5), we have
\[ \int \frac{(d^3 p)}{(2\pi)^3} G(p, ip_4) \gamma^0 = \int_0^\beta d\tau e^{ip_4\tau} \left\langle \psi(0, t = -i\tau)\psi^+(0) \right\rangle_T \]
\[ = \int_{-\infty}^{+\infty} dE' \frac{A(E')}{2\pi} E' - ip_4. \] 
\[ (4.7) \]

Continuing back to Minkowski space-time, with \( p_4 \to -iE + \epsilon = -i(E + i\epsilon), \epsilon \to 0^+ \), and taking the imaginary part produces

\[ A(E) = 2 \int \frac{(d^3 p)}{(2\pi)^3} \text{Im} G(p, E + i\epsilon) \gamma^0, \]
\[ (4.8) \]

and so, in view of Eq’s. (4.3) and (4.5),

\[ \tilde{W}(E) = 2n(-E) \int \frac{(d^3 p)}{(2\pi)^3} \text{Im tr} G(p, E + i\epsilon) \gamma^0. \]
\[ (4.9) \]

The thermal electron Green’s function has the structure

\[ G(p, ip_4) = \gamma_{\mu}p_{\mu} + m + \Sigma(p), \]
\[ (4.10) \]

where \( \gamma_4 = -i\gamma^0 \), and we shall compute the self-energy function \( \Sigma(p) \) to first order in \( \epsilon^2 \). We separate out the divergent terms that are removed by the renormalization process by writing

\[ \Sigma(p) = \Sigma^{(0)}(p) + \Sigma^{(T)}(p), \]
\[ (4.11) \]

in which the first term on the right-hand side is the vacuum function and the second term vanishes when the temperature vanishes. The vacuum part gives the renormalization terms. We avoid dealing with these renormalization effects by computing only the “intrinsic” temperature-dependent corrections, the temperature-dependent correction for the internal lines of order \( \epsilon^2 \) graphs that have no divergences. We continue back to Minkowski space-time and write

\[ \frac{1}{\gamma p + m} = \frac{m - \gamma p}{E(p)^2 - (E + i\epsilon)^2}, \]
\[ (4.12) \]

where now \( \gamma p = \gamma_k p_k - \gamma^0 E, \) and \( E(p) = \sqrt{p^2 + m^2}. \) To our order,

\[ \text{Im } G(p, E + i\epsilon) = -(m - \gamma p) \left\{ \frac{1}{[E(p)^2 - E^2]^2} \text{Im } \Sigma(p, E + i\epsilon) \right. \]
\[ + \Sigma^{(T)}(p, E) \text{Im } \frac{1}{[E(p)^2 - (E + i\epsilon)^2]^2} \left\} (m - \gamma p). \]
\[ (4.13) \]

In the second term on the right here, only the temperature-dependent part of the self-energy function enters — the vacuum contribution is removed by renormalization. To deal with this second term, we note that

\[ \text{Im } \frac{1}{[E(p)^2 - (E + i\epsilon)^2]^2} = \frac{1}{2E} \frac{\partial}{\partial E} \text{Im } \frac{1}{E(p)^2 - (E + i\epsilon)^2} \]
\[ = \frac{1}{2E} \frac{\partial}{\partial E} \frac{\pi}{2E(p)} [\delta(E - E(p)) - \delta(E + E(p))]. \]
\[ (4.14) \]
To summarize, we recall the definition (2.29) of $\chi(E)$, and use Eq’s. (4.14), (4.13), and (4.9) in the rate formula (4.2) to obtain

$$\Gamma_{n-e^-} = \frac{G_W^2}{(2\pi)^2} \left( g_Y^2 + 3g_A^2 \right) \int_{-\infty}^{+\infty} dE \int \frac{(d^3p)}{(2\pi)^3} \text{tr} \gamma^0 \left\{ \begin{array}{c}
- (m - \gamma p) \frac{\chi(E)}{[E(p)^2 - E^2]^2} 2 \text{Im} \Sigma(p, E + i\epsilon)(m - \gamma p) \\
+ [\delta(E - E(p)) - \delta(E + E(p))] \\
\times \frac{\pi}{2E(p)} \frac{\partial}{\partial E} \left[ E^{-1} \chi(E)(m - \gamma p)\Sigma^{(T)}(p, E)(m - \gamma p) \right] \end{array} \right\}.$$  

(4.15)

In the last line, the derivative of the delta function has been integrated by parts. When this derivative acts upon $\chi(E)$, the remaining factor is gauge invariant, and the result corresponds to a temperature-dependent shift of the electron’s energy in the plasma. The terms that result when the derivative acts on the remaining factors could be associated with some sort of “temperature-dependent wave-function renormalization”, but these terms are not gauge invariant, and so they cannot be given any independent physical interpretation. The first line, involving the imaginary part of the complete self-energy function describes bremsstrahlung processes in the plasma and, since an imaginary part is taken, it has no ultraviolet divergences. The first line does, however, contain “$T = 0$” contributions that must be removed. This is discussed in the next section and in more detail in Appendix B.

It should be emphasized that the electron-electron contribution (4.15) given here, together with the electron-proton contribution (3.9) given at the end of the previous section, give a complete and unified description of all the thermal, plasma effects on the leading electromagnetic corrections to the neutron disappearance rate. Some distinct and separate physical processes may be identified in the total rate contribution such as the energy shift mentioned above which will soon be described in more detail. However, other terms, such as those that may be associated with a ‘wave function renormalization’ contribution, are not gauge invariant and have no physical meaning whatsoever. Indeed, the electron-proton Coulomb correction computed in the previous section is not gauge invariant and has no physical significance by itself. We turn in the next section to assemble our results in terms of physically significant parts. We do this so as to obtain formulae that are easily read and not so long as to tire the eye, but still have well-defined physical significance. The details of the calculation of these results appears in Appendix B.

Before going into the details of our results in the next section, we note that the alternative form (2.7) shows that the proton rate may be obtained from the neutron rate by the interchanges $\psi \leftrightarrow \psi^\dagger$ and $T_+ \leftrightarrow T_-$. Since $\Gamma_{e-e^-}$ involves no interaction with the proton, charge conjugation invariance holds for the electron field thermal expectation values, and so they are not altered by the interchange $\psi \leftrightarrow \psi^\dagger$. The interchange $T_+ \leftrightarrow T_-\text{ just changes the associated time dependence from } e^{i\Delta t} \text{ to } e^{-i\Delta t}$. Hence the proton rate is related to the neutron rate by

$$\Gamma_{p-e^-} = \Gamma_{n-e^-}(\Delta \to -\Delta).$$  

(4.16)
V. RESULTS

As we have just noted, the first line of Eq. (4.15) describes real photon processes. One piece involves a factor of the thermal photon distribution function

\[ f(k) = \frac{1}{e^{\beta k} - 1}, \]  

(5.1)

and the remaining part does not involve this function. This remaining part has pieces corresponding to the real photon emission processes \( \nu + n \rightarrow p + e^- + \gamma, \nu + e^+ + n \rightarrow p + \gamma, \) \( n \rightarrow p + \bar{\nu} + e^- + \gamma, \) and \( e^+ + n \rightarrow p + \bar{\nu} + \gamma. \) As described in Appendix B, these pieces correspond to parts of the “\( T = 0 \)” rate that are easily identified and removed to produce the desired \( T \neq 0 \) rate contribution. The remaining part of the real photon processes described by the first line of Eq. (4.15) is the photon thermal bath contribution (B30) computed in Appendix B. This result reads

\[
\Gamma^{(\gamma,\gamma)}_{\nu e \rightarrow e} = 2 \frac{e^2 G_\nu^2}{(2\pi)^5} \left( g_\nu^2 + 3 g_A^2 \right) \int_0^\infty \frac{dk}{k} f(k) \int_{m}^{\infty} d\mathcal{E} \left[ [n(-\mathcal{E}) \tilde{\chi}(\mathcal{E} - k) + n(\mathcal{E}) \tilde{\chi}(-\mathcal{E} + k)] F_- + [n(-\mathcal{E}) \tilde{\chi}(\mathcal{E} + k) + n(\mathcal{E}) \tilde{\chi}(-\mathcal{E} - k)] F_+ \right].
\]  

(5.2)

Here

\[ \tilde{\chi}(E) = n(E - \Delta) (E - \Delta)^2, \]  

(5.3)

according to Eq. (B29). The functions

\[ F_\pm = A \pm k B \]  

(5.4)

appear in Eq. (B20), with

\[ A = \left[ 2 E^2 + k^2 \right] \ln \left( \frac{E + q}{E - q} \right) - 4 q E, \]  

(5.5)

and

\[ B = 2 E \ln \left( \frac{E + q}{E - q} \right) - 4 q, \]  

(5.6)

where

\[ q = \sqrt{E^2 - m^2}. \]  

(5.7)

Although this set of terms in Eq. (5.2) has no ultraviolet divergences, since \( f(k) k^{-1} \rightarrow k^{-2} \) as \( k \rightarrow 0, \) it is infrared divergent. This contribution is rendered finite in the infrared by a part of the contribution contained in the last two lines of Eq. (4.15). It is one of the

---5 The fact that the infrared divergences must cancel, order by order, for these rate calculations was proved in [18].
parts that arises from all the terms that remain when the energy derivative $\partial/\partial E$ does not act on $\chi(E)$. This piece, which involves an overall factor of the photon distribution function $f(k)$, is calculated in Eq. (B53) of Appendix B to be

$$\Gamma_{\gamma e \rightarrow e} = \frac{2\alpha}{(2\pi)^5} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty \frac{dk}{k} f(k) \int_m^\infty dE A \left\{ -2n(-E) \tilde{\chi}(-E) - 2n(E) \tilde{\chi}(E) \right\}. \quad (5.8)$$

Adding Eq. (5.8) to the real photon rate (5.2), and writing our rationalized charge in terms of the fine structure constant, $\alpha = e^2/4\pi$, gives

$$\Gamma_{\gamma e \rightarrow e}^{(2)} = \frac{2\alpha}{(2\pi)^3} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty \frac{dk}{k} f(k) \int_m^\infty dE \left\{ A \left[ n(-E) \{ \tilde{\chi}(E - k) + \tilde{\chi}(E + k) - 2\tilde{\chi}(E) \} ight] + n(E) \{ \tilde{\chi}(-E + k) + \tilde{\chi}(-E - k) - 2\tilde{\chi}(-E) \} \right) - kB \left\{ n(-E) \{ \tilde{\chi}(E - k) - \tilde{\chi}(E + k) \} + n(E) \{ \tilde{\chi}(-E + k) - \tilde{\chi}(-E - k) \} \right\} \right\}. \quad (5.9)$$

The weight $k^{-1}f(k)$ diverges as $k^{-2}$ when $k$ vanishes, $k \rightarrow 0$. The set of terms in the first square brackets is of order $k^2$ for small $k$. The terms in the final square brackets are of order $k$ for small $k$, but they are multiplied by an explicit factor of $k$. Hence, this contribution to the neutron rate has no infrared divergence. The $k$ and $E$ integrals are damped for large $k$ and $E$ by the Bose and Fermi phase-space density functions, and so it is also well behaved in the ultraviolet as well as the infrared. Since this is the only part of the rate that is proportional to the photon density $f(k)$, it is a well-defined, gauge invariant part of the neutron disappearance rate.

The final pieces that arise when the energy derivative $\partial/\partial E$ does not act on $\chi(E)$ in the last two lines in Eq. (3.13) is presented in Eq. (B54) in Appendix B. Adding this to the modification (3.3) of the neutron rate brought about by the Coulomb interaction between the electron and proton and again using $\alpha = e^2/4\pi$ gives

$$\Gamma_{n,ee \rightarrow e} = \frac{2\alpha}{(2\pi)^3} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty dE \left[ \chi(E) + \chi(-E) \right] \int_m^\infty dE' n(E') \frac{E}{E'^2 - E^2} \left\{ 8pp' - [E'^2 + E^2] \ln \left( \frac{p + p'}{p - p'} \right)^2 - \frac{E'}{E} [E'^2 + E^2] \ln \left( \frac{[EE' + pp']^2 - 4m^2}{[EE' - pp']^2 - 4m^2} \right) \right\}. \quad (5.10)$$

Since the only remaining term involves the gauge-invariant energy shift shown in Eq. (5.11) below, and all the other previous contributions are gauge invariant, we conclude that Eq. (5.10) is a well-defined, gauge-invariant correction to the neutron disappearance rate. Just as we have done in the writing of Eq. (3.3), it is left implicit that the potentially singular terms are defined by a principal part prescription.
Finally, there is the term when the energy derivative in the last line of Eq. \((11.13)\) acts on \(\chi(E)\). In this case, the electron self-energy function is evaluated on mass shell and thus describes a gauge-invariant, temperature-dependent energy shift correction which, after performing the angular integrations, has the form
\[
\Gamma_{n\bar{e}e}^{(\Delta E)} = 4 \frac{G_W^2}{(2\pi)^3} \left( g_V^2 + 3g_A^2 \right) \int_{-\infty}^{\infty} dE \int_0^{\infty} p^2 dp \left[ \delta(E - E(p)) + \delta(E + E(p)) \right] \Delta E^{(T)}(p) \frac{\partial}{\partial E} \chi(E). \tag{5.11}
\]

As explained in more detail in Appendix B, the energy shift \(\Delta E^{(T)}(p)\) is a shift in the position of the pole in the interacting thermal electron propagator — it gives the energy shift to create an electron (positron) of momentum \(p\) in the plasma relative to the energy needed to create the particle in the vacuum. The form of Eq. \((5.11)\) is precisely the change in the free rate \((2.30)\) when the energy–momentum relation of the electron (positron) is altered. The energy shifts is computed in Appendix B. The results displayed in Eq’s. \((B45)\) and \((B47)\) give
\[
\Delta E^{(T)}(p) = \frac{2\alpha}{\pi E} \left\{ \int_0^{\infty} k^2 dk \frac{1}{k} f(k) + \int_0^{\infty} q^2 dq \frac{1}{E(q)} n(E(q)) \left[ 1 - \frac{m^2}{4qp} \ln \left( \frac{q + p}{q - p} \right)^2 \right] \right\}. \tag{5.12}
\]

The energy derivative in the energy-shift term \((5.11)\) may be removed by an integration by parts using \(\partial/\partial E = (E/p) \partial/\partial p\). Using the explicit form \((5.12)\) of the energy shift, we find that
\[
\Gamma_{n\bar{e}e}^{(\Delta E)} = -\frac{8\alpha}{\pi} \frac{G_W^2}{(2\pi)^3} \left( g_V^2 + 3g_A^2 \right) \int_{m}^{\infty} \frac{E}{p} dE \left[ \chi(E) + \chi(-E) \right] \left\{ \int_0^{\infty} dk k f(k) + \int_m^{\infty} dE' p' n(E') \left[ 1 - \frac{m^2}{E'^2 - E^2} \right] \right\}. \tag{5.13}
\]

Again it has been left implicit that integrand in the second double integral in \(E\) and \(E'\) is defined by the principal part prescription.

The non-trivial plasma effects on the one-loop electromagnetic correction to the total neutron disappearance rate is the sum of the results that we have enumerated:
\[
\Delta \Gamma_n^{(T)} = \Gamma_{n\bar{e}e}^{(\gamma,f)} + \Gamma_{n,ep+ee} + \Gamma_{n\bar{e}e}^{(\Delta E)}, \tag{5.14}
\]
in which the successive contributions are given in Eq’s. \((5.9)\), \((5.10)\), and \((5.13)\). We should emphasize one last time that this result follows from a straightforward evaluation of the corrections to the basic rate formulas \((2.1)\) and \((2.2)\). We have broken up the complete result into smaller pieces in large part so as to have reasonably short expressions that do have gauge-invariant, physical meanings. But we have not patched together a set of various physical processes. Rather, we have started from a well-defined general formulation and just worked out the consequences.

The second set of terms on the right-hand side of Eq. \((5.14)\) involve singular integrands that require principal part definitions. To write the result in a less singular form, we perform
some algebraic rearrangement and partial integrations to express the sum of these two terms in the form

\[
\Gamma_{n, ep+ee} + \Gamma_{n\rightarrow e-e} = \frac{2 \alpha}{\pi} \frac{G_W^2}{(2\pi)^3} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty dE \left[ \chi(E) + \chi(-E') \right] \\
\left\{ -\frac{2\pi^2}{3\beta^2} \frac{E}{p} + \int_m^\infty dE' F(E, E') \right\},
\]

(5.15)

with

\[
F(E, E') = -\frac{1}{4} \ln^2 \left( \frac{p + p'}{p - p'} \right)^2 \left\{ n'(E') \left( \frac{E^2}{p} \right)^2 \frac{E}{E'} \left[ E + E' \right] + n(E') \frac{E^2}{pp'} \left[ E' + \frac{m^2E}{E'^2} \right] \right\} \\
+ \ln \left( \frac{p + p'}{p - p'} \right)^2 \left\{ n'(E') \left[ \frac{p^2}{p^2} \left( \frac{m^2}{p^2} + 2 \right) - E^2 \frac{p'}{p} \right] L(E, E') \right\} \\
+ n(E') \left\{ \frac{E^2}{p^2E'^2} \left( E'^2 + 2p^2 + m^2 \right) - \frac{E^2 + E'^2}{E + E'} - \frac{E^2E'}{pp'} L(E, E') \right\} \\
- n(E') \left\{ 4E \frac{p'}{p} + 2E' L(E, E') \right\}.
\]

(5.16)

Here in the first equality we have explicitly evaluated the simple integration over the photon distribution \( f(k) \) in Eq. (5.13), and in the second equality we have used the definitions

\[
n'(E') = \frac{dn(E')}{dE'},
\]

(5.17)

and

\[
L(E, E') = \ln \left( \frac{EE' + pp' + m^2}{EE' - pp' + m^2} \right).
\]

(5.18)

This non-singular expression of these rates is a convenient form to use in numerical evaluations as we shall do below.

It was shown in Eq. (3.12) that the changes in the neutron and proton rates brought about by the electron-proton Coulomb interaction are related by the formal substitution \( \Delta \leftrightarrow -\Delta \). The same substitution relates the total electron self-energy and radiative corrections to rates as shown in Eq. (4.16). A glance at the details of the “\( T = 0 \)” subtractions made to the electron self-energy and radiative corrections shows that those for the proton and neutron are also related by the interchange \( \Delta \leftrightarrow -\Delta \). We therefore conclude that the complete “\( T \neq 0 \)” corrections to the proton rate are given by

\[
\Delta \Gamma_p^{(T)} = \Delta \Gamma_n^{(T)}(\Delta \rightarrow -\Delta).
\]

(5.19)

\[\text{An ingredient needed for this is the identity}\]

\[
\frac{[EE' + pp']^2 - m^4}{[EE' - pp']^2 - m^4} = \left( \frac{EE' + pp' + m^2}{EE' - pp' + m^2} \right)^2 \left( \frac{p + p'}{p - p'} \right)^2.
\]
As was discussed in the Introduction, the process $\nu + e^+ + n \rightarrow p + \gamma$ has been hitherto inadvertently omitted in the literature. This contribution to the neutron rate is identified in Eq. (B31), which we repeat here:

$$\Gamma_{n F} = \frac{2\alpha}{\pi} \frac{G_W^2}{(2\pi)^3} \left(g_{\nu e}^2 + 3g_A^2\right) \int_m^\infty d\mathcal{E} \frac{n(\mathcal{E})}{\mathcal{E} + \Delta} \int_{\mathcal{E} + \Delta}^\infty \frac{dk}{k} \left[-f(-k)\right] \tilde{\chi}(-\mathcal{E} + k) F_- . \quad (5.20)$$

As discussed further in Appendix B, the $T = 0$ part of this rate is obtained by taking $-f(-k) \rightarrow 1$,

$$\Gamma_{n F}^{(T=0)} = \frac{2\alpha}{\pi} \frac{G_W^2}{(2\pi)^3} \left(g_{\nu e}^2 + 3g_A^2\right) \int_m^\infty d\mathcal{E} \frac{n(\mathcal{E})}{\mathcal{E} + \Delta} \int_{\mathcal{E} + \Delta}^\infty \frac{dk}{k} \tilde{\chi}(-\mathcal{E} + k) F_- . \quad (5.21)$$

VI. DISCUSSION

We have derived expressions for the rate shifts, to order $e^2$, in terms of thermal expectation values involving only electron field operators. The perturbative expansion of these expectation values has been carried out using the standard methods of thermal field theory. The "$T = 0$" terms have been carefully defined and removed. The remainder has been expressed in the form of convergent two-dimensional integrals.

We gave formulae for only the neutron rates $\Gamma_n$; as we stressed earlier, if we had calculated the whole rate, rather than the finite temperature piece, we could use detailed balance directly to get a proton rate $\Gamma_p$ from $\Gamma_n$. But since the present work is limited to the finite temperature part, we use the transformation $\Delta \rightarrow -\Delta$ in the integrands to get the corrections to the proton rate $\Delta \Gamma_p^{(T)}$ from those for the neutron rate. We have performed the integrals in Eq's. (5.9), (5.10), and (5.13) to compute the total temperature-dependent part of the leading electromagnetic correction to the neutron rate as shown in Eq. (5.14). Their counterparts for the proton process have been computed in the same way after making the substitution $\Delta \rightarrow -\Delta$. We exhibit the results of the integration in the form of the fractional changes of the neutron rate $\Delta \Gamma_n^{(T)}/\Gamma_n^{(0)}$ and proton rate $\Delta \Gamma_p^{(T)}/\Gamma_p^{(0)}$, where $\Gamma_n^{(0)}$ is the free neutron rate given in Eq. (2.30), while the free proton rate $\Gamma_p^{(0)}$ is obtained by the same calculation after the making the replacement $\Delta \rightarrow -\Delta$. The results are shown as a function of temperature in Fig. 1.

We cannot compare these results directly with those in the literature, since previous authors have omitted the processes $n + e^+ + \nu \rightarrow p + \gamma$ and $p + \gamma \rightarrow n + e^+ + \nu$. The first of these processes, since it has an outgoing photon, generates a $T = 0$ part that has been subtracted as a part of the general $T = 0$ subtractions in (5.14). We define a "revised" $T$ part of the rate as that which must be added to the previous literature’s $T = 0$ part to get the complete answer. The “revised” $T$ neutron rate is found by adding the contribution (5.21) to the the $T$ rate plotted in Fig. 1. There is no $T = 0$ piece of the reaction, $p + \gamma \rightarrow n + e^+ + \nu$, since here the photon is in the initial state, and therefore the proton “revised” $T$ rate is the same as that plotted in Fig. 1.

We plot the fractional “revised” rate corrections in Fig. 2. For comparison we also show data reconstructed from the figures of [6], shown as dashed curves; the latter are derived by combining the numbers shown in Fig. 11 of [6], called the “finite T corrections”, with those
of Fig. 16 of that reference, called the “finite temperature electron mass corrections.” We see that at a temperature of 0.3 MeV our results nearly coincide with those of [6], but that the two families of results diverge rapidly at higher temperatures. We find positive rate corrections that are larger than those of Ref. [6] for both processes, and we find much less difference between the neutron and proton corrections. Both of these changes lead to faster equilibration.

Since the electromagnetic corrections are quite small, the detailed balance relation (2.9), expressed in terms of fractional corrections, is simply

\[
\frac{\Delta \Gamma_p^{(T)} + \Delta \Gamma_p^{(T=0)}}{\Gamma_p^{(0)}} = \frac{\Delta \Gamma_n^{(T)} + \Delta \Gamma_n^{(T=0)}}{\Gamma_n^{(0)}}.
\]  

(6.1)

We can, in principle, try to check our answers with this statement by adding to our results the \( T = 0 \) results of [6], shown in Fig. 8 of this reference, to our finite \( T \) corrections. Unfortunately this gets us into a spot of guesswork, since we cannot read the \( \Delta \Gamma_p^{(T=0)} - \Delta \Gamma_n^{(T=0)} \) differences off of this plot quite precisely enough to get a clear answer. It would appear that these differences, as shown in the plot, are a little too small to compensate for the \( \Delta \Gamma_p^{(T)} - \Delta \Gamma_n^{(T)} \) differences that we calculate and that are plotted in our Fig. 1. The same detailed balance test using the finite \( T \) results of [6], which shows a larger difference in the \( n \) and \( p \) corrections, appears to fail by more. We conclude that the approximations that were made in the calculation of the \( T = 0 \) terms in Ref’s. [4] and [6] may deserve some further scrutiny.

It is harder to make a detailed comparison of our results with those of Ref. [7] because of the way the latter are presented. However in Fig. 12 of Ref. [7] we see large violations of the detailed balance requirement (6.1).

We believe that with our formalism we have avoided the questions that have led to confusion and ambiguity in the literature. We should comment a little further on the wave-function renormalization questions that have plagued previous calculations. In the T matrix approach, with exclusive channels, the need for this renormalization arises from processes on an external line. Conceptually at least, one needs to sum the bubbles on the external line and calculate therefrom the residue of the energy shifted pole. The renormalization constant so obtained can then be expanded to second order in \( e \). The complication in the case of finite temperature, with its preferred coordinate system, is that the multiplicative factor contains a transformation on the Dirac spinor indices, as worked out in [17]. It is instructive to consider how these considerations would be embodied if we separated our result (4.13) into exclusive channels. The “external line” problem would present itself if we tried to evaluate the imaginary part of the square of the electron propagator, encountered in the third line of Eq. (4.13) (with the omission of the spinor numerators) as \( \text{Im}[E(p)^2 - (E + ie)^2]^{-2} \), by taking the imaginary part of one factor (a delta function) times the real part of the other (a denominator that is singular at the point of vanishing of the delta function). This is the structure that comes from taking a Feynman graph with a correction on an external line. We can see some of the benefits of the inclusive approach from the operations that follow our Eq. (4.13), in which the representation of the imaginary part given by Eq. (4.14) avoids entirely the \( ie \) delicacies required in treating exclusive channels.

But there appears to be a deeper problem with the methods that have been used in the literature. In our Eq. (4.13) the propagators in question are the ordinary real time vacuum functions; the thermal effects have all been incorporated through the preceding formalism.
However, in [4] and [5] as well as in the subsequent papers on this subject, the individual graphs were calculated with real time thermal propagators of the form (for electrons)

$$S(p) = (\gamma p + m)^{-1} + 2\pi n(|E|)(m - \gamma p)\delta(p^2 + m^2).$$

When this propagator gets squared, it is truly meaningless; the interpretation is no longer just a question of following the $i\epsilon$’s carefully. It appears to us that the square of the delta function has been implicitly treated in an arbitrary way in the treatments based on real time thermal propagators. This defect in the the real-time approach has been noted before in other contexts, for example in [19] and [20].

Finally, as we remarked earlier, the wave function renormalization is gauge dependent. In Feynman gauge, in which photons are emitted and absorbed by the proton, there is a temperature dependent part of the proton wave-function renormalization that must be calculated in order to get the correct answer in the exclusive approach. This appears to be missing in all of the published calculations. We conclude, from the above discussion, that it would be futile to try to make term by term comparisons of our results with the previous ones. However, we can say that most of the discrepancy comes from sources other than the new channel that we noted, $\gamma + p \leftrightarrow \nu + e^+ + n$. This fact is made clear by Fig. 3, which shows the complete $T = 0 + T \neq 0$ contributions from these reactions. Since these terms obey detailed balance by themselves, there is no difference between the neutron and the proton fractional corrections shown in Fig. 3.

A further use for the formalism developed in this paper would be in the calculation of the electromagnetic corrections to the charged current neutrino opacity in the environment of the supernova core. The neutrino reactions to be considered are the same ones that we have dealt with in the present paper. But the electrons are very degenerate, with the electron chemical potential in the 10’s of MeV. Hardy [21] has found, using some of the technology from the early universe calculations that we have discussed above, some evidence of significant electromagnetic effects in the rates. To get the complete answer to order $e^2$ one can adapt our equations to get the neutrino absorption rate in the presence of a non-vanishing electron chemical potential.

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FIG. 1. The lower curve is \((10^3 \text{ times})\) the ratio of the finite-temperature part of the leading electromagnetic corrections to the neutron disappearance rate \(\Delta \Gamma^{(T)}_n\), given in Eq. (5.14), to the free neutron rate \(\Gamma^{(0)}_n\), given in Eq. (2.30). The upper curve is the same ratio for the proton.
FIG. 2. Our results shown as solid lines compared to the results of Lopez and Turner [6] shown as dashed lines. As in Fig. 1, the curves describe the temperature-dependent part of the leading electromagnetic corrections to the rates divided by the corresponding uncorrected rates, $\Delta\Gamma(T)/\Gamma(0)$. The upper curves refer to the proton rates, the lower curves to the neutron rates. As described in the text, we plot our “revised” result for the neutron rate obtained by adding the contribution (5.21) so as to account for the previously omitted process. To obtain the Lopez and Turner comparison figures, we combined their results for the “finite temperature radiative corrections”, as shown in their Fig. 11, with their results for the “finite temperature electron-mass correction”, as shown in their Fig. 16.
FIG. 3. Ratio of the previously omitted processes $\nu + e^+ + n \leftrightarrow \gamma + p$ contribution to the neutron disappearance rate $\Delta \Gamma_{n_F}$, as given in Eq. (5.20), to the uncorrected rate $\Gamma_n^{(0)}$ given in Eq. (2.30), multiplied by $10^4$. Since the complete contributions of a new set of processes obey detailed balance, the corresponding ratio for a proton is the same. Note that the contribution of these processes is about an order-of-magnitude smaller than the $\Delta \Gamma^{(T)}/\Delta \Gamma^{(0)}$ ratios displayed in Fig. 2, and thus they cannot account for the discrepancies appearing in Fig. 2.
APPENDIX A: RATE FORMULA DERIVATION

As discussed in the text, plasma modifications of the rates involve low-energy, soft processes. Thus they may be calculated using the limit in which the nucleons are treated as being very heavy and described by operators $\Psi_r^\dagger(r, t)$ and $\Psi(r, t)$ that create and destroy a particle at the spatial point $r$ at time $t$. The weak interaction Hamiltonian may be expressed as

$$H_W(t) = \mathcal{K}(t) + \mathcal{K}^\dagger(t), \quad \text{(A1)}$$

where

$$\mathcal{K}(t) = \frac{G_W}{\sqrt{2}} \int (d^3r) \left\{ g_V \psi_\nu^\dagger(x) (1 - \gamma_5) \psi_\nu(x) \Psi_n^\dagger(x)\Psi_p(x) + g_A \psi_\nu^\dagger(x) \alpha_l (1 - \gamma_5) \psi_\nu(x) \Psi_n^\dagger(x)\sigma_l\Psi_p(x) \right\}. \quad \text{(A2)}$$

Here $\psi_\nu$ and $\psi_\nu^\dagger$ are the Dirac fields for the electron and (electron) neutrino, and $\alpha_l = \gamma^0\gamma_l$ is the usual Dirac matrix.

The operator

$$N_\nu(t) = \int (d^3r) \psi_\nu^\dagger(x)\psi_\nu(x) \quad \text{(A3)}$$

measures the leptonic charge of the neutrinos, the number of $\nu$ minus the number of $\bar{\nu}$. The generic equal-time anticommutator for Fermi fields reads

$$\{ \psi(r, t) , \psi^\dagger(r', t) \} = \delta(r - r') \quad \text{(A4)}$$

where the $\delta$ function on the right implicitly includes the Dirac field indices. Using this anticommutator for the neutrino field gives the time derivative

$$\dot{N}_\nu(t) = i [H_W(t) , N_\nu(t)] = -i \left[ \mathcal{K}(t) - \mathcal{K}^\dagger(t) \right]. \quad \text{(A5)}$$

The operators which appear here create and destroy electrons and neutrinos, and hence their expectation vanishes in the unperturbed plasma ensemble, which is diagonal in these particle numbers. The reaction rate $\Gamma_\nu$ appears in the additional linear response to the effect of the perturbation $H_W(t)$,

$$\Gamma_\nu = -i \int_{-\infty}^0 dt \left\langle \left[ \dot{N}_\nu(0), H_W(t) \right] \right\rangle_T = - \int_{-\infty}^0 dt \left\langle \left[ \mathcal{K}(0) - \mathcal{K}^\dagger(0), \mathcal{K}(t) + \mathcal{K}^\dagger(t) \right] \right\rangle_T. \quad \text{(A6)}$$

Since $\mathcal{K}$ does not conserve particle number,

\footnote{We use here the general method advocated by Brown and Sawyer [22].}
\[ \langle [\mathcal{K}(0), \mathcal{K}(t)] \rangle_T = 0 = \langle [\mathcal{K}^\dagger(0), \mathcal{K}^\dagger(t)] \rangle_T. \]  

(A7)

In virtue of the time transitional invariance of the ensemble,
\[ \langle [\mathcal{K}(0), \mathcal{K}^\dagger(t)] \rangle_T = \langle [\mathcal{K}(-t), \mathcal{K}^\dagger(0)] \rangle_T. \]  

(A8)

Thus, we change the integration variable, \( t \rightarrow -t \), use the antisymmetry of the commutator, and combine terms combine to obtain
\[ \Gamma_\nu = -\int_{-\infty}^{\infty} dt \langle [\mathcal{K}(t), \mathcal{K}^\dagger(0)] \rangle_T. \]  

(A9)

Although the expectation value that appears here describes a thermal ensemble of electrons and neutrinos, both of which are taken to have zero chemical potentials, we take the expectation value to have only a single nucleon, which we treat as a heavy particle located at the coordinate origin, \( \mathbf{r} = 0 \). Thus the nucleon destruction operators in the weak Hamiltonian give a vanishing contribution unless their coordinate \( \mathbf{r} = 0 \), and so the spatial coordinates of the electron and neutrino fields can be taken to vanish in the local interaction (A2). Rotational invariance of the thermal ensemble implies that the (time-component) vector – (space-component) axial-vector interference terms vanish and that there can be no dependence upon the nucleon spin orientation. Hence the result is unchanged by averaging over the initial nucleon spin as well as performing the required sum over the final nucleon spin. Thus, we may replace the (direct product) \( \sigma_l \sigma_m \) in the axial-vector – axial-vector contribution by \( \delta_{lm} \). Moreover, as one can verify in detail from the structure of the neutrino field thermal expectation values that will soon be presented, one can also replace the resulting (direct product) \( \alpha_l \alpha_l \) by 3. With these replacements made, both the vector and the axial-vector parts of the weak Hamiltonian involve the isospin lowering operator
\[ T_-(t) = \int (d^3 \mathbf{r}) \Psi^\dagger_p(x) \Psi_n(x), \]  

(A10)

and its Hermitian adjoint, the isospin raising operator
\[ T_+(t) = T_+^\dagger(t) = \int (d^3 \mathbf{r}) \Psi^\dagger_n(x) \Psi_p(x), \]  

(A11)

with
\[ \Gamma_\nu = -\frac{1}{2} G_W^2 \left( g_\nu^2 + 3 g_A^2 \right) \int_{-\infty}^{+\infty} dt \left\langle \left[ \psi^\dagger_\nu(\mathbf{0}, t) (1 - \gamma_5) \psi_e(\mathbf{0}, t) T_-(t), \psi^\dagger_e(0) (1 - \gamma_5) \psi_\nu(0) T_+(0) \right] \right\rangle_T. \]  

(A12)

We define the real-time, generic free Fermi field thermal expectation values with vanishing chemical potential as
\[ \langle \psi_\alpha(x) \psi^\dagger_\beta(x') \rangle_T^{(0)} = \left[ S^{(+)}(x - x') \gamma_0 \right]_{\alpha\beta}, \]  

(A13)

and
\[ \langle \psi^\dagger_\beta(x') \psi_\alpha(x) \rangle_T^{(0)} = \left[ S^{(-)}(x - x') \gamma_0 \right]_{\alpha\beta}. \]  

(A14)
These functions satisfy the free Dirac equation. Since these thermal expectation values involve $\text{Tr} e^{-\beta H \cdots}$, with

$$e^{-\beta H} \psi(r, t) = \psi(r, t + i\beta) e^{-\beta H}, \quad (A15)$$

the cyclic symmetry of the trace provides the boundary condition

$$[S^{(+)}(\mathbf{r} - \mathbf{r}', t - t')]_{\alpha \beta} = [S^{(-)}(\mathbf{r} - \mathbf{r}', t + i\beta - t')]_{\alpha \beta}. \quad (A16)$$

This boundary condition, plus that provided by the equal-time anticommutation relation (A4), determines the solution of the free Dirac equation to be

$$S^{(\pm)}(x - x')\gamma^0 = \int \frac{(d^3 \mathbf{p})}{(2\pi)^3} \frac{1}{2E(p)} e^{i\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')} \left\{ [E(p) + \gamma^0 m + \alpha \cdot \mathbf{p}] n(\mp E(p)) e^{-iE(p)(t-t')} \right. \right.$$

$$+ [E(p) - \gamma^0 m - \alpha \cdot \mathbf{p}] n(\pm E(p)) e^{iE(p)(t-t')} \right\}. \quad (A17)$$

Here

$$E(p) = \sqrt{p^2 + m^2}, \quad (A18)$$

and

$$n(E) = \frac{1}{e^{\beta E} + 1} \quad (A19)$$

is the Fermi thermal distribution appropriate for an initial state, while

$$n(-E) = \frac{e^{\beta E}}{e^{\beta E} + 1} = 1 - n(E) \quad (A20)$$

gives the Pauli blocking factor appropriate for final states.

Since the rate (A12) already has the weak interaction taken into account, we may use the free field correlations (A17) with $m = 0$ for the neutrino fields. Since the electron thermal ensemble is parity invariant, a single factor of $\gamma_5$ does not contribute while $\gamma_5^2 = 1$. Integrating over the neutrino solid angles and writing $p = E_\nu$ for the massless neutrino, we get

$$\Gamma_\nu = -G_W^2 \left( g_\nu^2 + 3g_\lambda^2 \right) \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dE_\nu E_\nu^2 n(E_\nu) \int_{-\infty}^{+\infty} dt \langle e^{+iE_\nu t} \psi_\nu(\mathbf{0}, t) T_- (t) \psi_\nu^\dagger (0) T_+ (0) e^{-iE_\nu t} \psi_\nu^\dagger (0) T_- (0) \psi_\nu (\mathbf{0}, t) T_+ (t) \rangle_T. \quad (A21)$$

The first term in the thermal expectation value only contributes when the initial nucleon state is a neutron which is changed into a proton by the action of $T_+ (0)$; this term corresponds to the process $\nu + n \rightarrow p + e^{-}$ plus the other processes related by crossing the leptons and/or including photon emission. These processes decrease the number $\nu$ minus $\bar{\nu}$ in accord with
the overall minus sign out in front. Thus the positive rate $\Gamma_n$ for the disappearance of an initial neutron is given by

$$\Gamma_n = G^2 W \left( g^2 V + 3 g^2 A \right) \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{+\infty} dE_\nu E^2_\nu n(E_\nu) \int_{-\infty}^{+\infty} dt \left\langle e^{+iE_\nu t} \psi_e(0,t)T_-(t)\psi_e^\dagger(0)T_+(0) \right\rangle_T. \quad (A22)$$

Similarly, the rate for the disappearance of an initial proton is given by

$$\Gamma_p = G^2 W \left( g^2 V + 3 g^2 A \right) \left( \frac{1}{2\pi} \right)^2 \int_{-\infty}^{+\infty} dE_\nu E^2_\nu n(E_\nu) \int_{-\infty}^{+\infty} dt \left\langle e^{-iE_\nu t} \psi_e^\dagger(0)T_+(0)\psi_e(0,t)T_-(t) \right\rangle_T. \quad (A23)$$

**APPENDIX B: ELECTRON SELF-ENERGY CONTRIBUTIONS**

The electron self-energy function for the thermal electron Green’s function, is given, to one-loop order, by

$$\Sigma(p) = -T \sum_{n=-\infty}^{n=+\infty} \int \frac{d^3k}{(2\pi)^3} \gamma^\mu \frac{m - \gamma(p+k)\cdot \gamma}{(p+k)^2 + m^2} \gamma^\nu \epsilon^2 D_{\mu\nu}(k). \quad (B1)$$

The four vectors here are in Euclidean space with the scalar product $\gamma(p+k) = \gamma \cdot (p + k) + \gamma_4 (p+k)_4$, $\gamma_4 = -i\gamma^0$, and $k_4 = \omega_n = 2\pi n T$ while the external energy takes on the values $p_4 = (2m + 1)\pi T$. The radiation gauge is chosen where

$$D_{lm}(k) = \begin{bmatrix} \delta_{lm} - \frac{k_l k_m}{k^2} \end{bmatrix} \frac{k}{k^2}, \quad (B2)$$

and

$$D_{44}(k) = \frac{1}{k^2}. \quad (B3)$$

The frequency sum is conveniently performed by expressing it as a contour integral,

$$T \sum_{n=-\infty}^{n=+\infty} \cdots = \int_C \frac{d\omega}{2\pi i} \frac{1}{\cot \left( \frac{\beta \omega}{2} \right)} \cdots, \quad (B4)$$

where the contour $C$ runs from $-\infty$ to $+\infty$ just below the real axis and then returns to $-\infty$ just above the real axis. For the part of the contour just below the real axis, we write

$$\frac{1}{2i} \cot \left( \frac{\beta \omega}{2} \right) = \frac{1}{2} + \frac{1}{e^{\beta \omega} - 1}, \quad (B5)$$

with the second term exponentially damped in the lower-half plane where we shall close the contour for it. For the upper-half portion of the contour, we write
\[
\frac{1}{2i} \cot \left( \frac{\beta \omega}{2} \right) = -\frac{1}{2} - \frac{1}{e^{-i\beta \omega} - 1}, \tag{B6}
\]

and close the contour for the second term in the upper-half plane where it is exponentially damped. In this way, we achieve the separation

\[
\Sigma(p) = \Sigma^{(0)}(p) + \Sigma^{(T)}(p), \tag{B7}
\]

where

\[
\Sigma^{(0)}(p) = -e^2 \int \frac{d\omega}{2\pi} \int \frac{(d^3k)}{(2\pi)^3} \gamma_\mu \frac{m - \gamma \cdot (p + k)}{(p + k)^2 + m^2} \gamma_\nu D_{\mu\nu}(k) \tag{B8}
\]

is the vacuum self-energy function (in Euclidean space). This vacuum part must be regulated and renormalized. The remaining, temperature-dependent piece \(\Sigma^{(T)}(p)\) is a finite, well-defined function which vanishes when the temperature vanishes.

In closing the contours for the thermal part of the self energy, poles are encountered at \(\omega = \pm ik\), where \(k = |k|\), producing a factor of the Bose distribution

\[
f(k) = \frac{1}{e^{\beta k} - 1}. \tag{B9}
\]

Poles are also encountered at \(\omega = \pm iE(p + k) - p_4\), where \(E(p + k) = \sqrt{(p + k)^2 + m^2}\). Since \(\beta p_4 = (2m + 1)\pi\), with \(\exp\{i\beta p_4\} = -1\), these poles are accompanied by a factor of the Fermi distribution

\[
n(E(p + k)) = \frac{1}{e^{\beta (E(p + k))} + 1}. \tag{B10}
\]

Separating out the Coulomb and radiative pieces, and continuing to real energy by setting \(p_4 = -iE\), we obtain

\[
\Sigma^{(T)}(p) = \Sigma_R^{(T)}(p) + \Sigma_C^{(T)}(p), \tag{B11}
\]

where

\[
\Sigma_R^{(T)}(p, E) = -e^2 \int \frac{(d^3k)}{(2\pi)^3} \left[ \delta_{lm} - \hat{k}_l \hat{k}_m \right] \gamma_l \left\{ \frac{1}{2k} f(k) \left[ \frac{m - \gamma \cdot (p + k) + \gamma^0 (E + k)}{E(p + k)^2 - (E + k)^2} + \frac{m - \gamma \cdot (p + k) + \gamma^0 (E - k)}{E(p + k)^2 - (E - k)^2} \right] 
- \frac{1}{2E(p + k)} n(E(p + k)) \left[ \frac{m - \gamma \cdot (p + k) + \gamma^0 E(p + k)}{k^2 - (E - E(p + k))^2} + \frac{m - \gamma \cdot (p + k) - \gamma^0 E(p + k)}{k^2 - (E + E(p + k))^2} \right] \gamma_m \right\}, \tag{B12}
\]

and

\[
\Sigma_C^{(T)}(p, E) = -2e^2 \int \frac{(d^3k)}{(2\pi)^3} \frac{1}{2E(p + k)} n(E(p + k)) \left[ \frac{m + \gamma \cdot (p + k)}{k^2} \right]. \tag{B13}
\]
A partial fraction decomposition in the radiative piece produces a convenient alternative form:

\[
\Sigma_R^{(T)}(p, E) = -e^2 \int \frac{(d^3k)}{(2\pi)^3} \left( \delta_{lm} - \hat{k}_l \hat{k}_m \right) \frac{1}{4k E(p + k)} \gamma_l \left[ m - \gamma \cdot (p + k) + \gamma^0 E(p + k) \right] \frac{f(k) + n(E(p + k))}{E(p + k) - E - k} + \frac{f(k) - n(E(p + k))}{E(p + k) + E - k} + \frac{m - \gamma \cdot (p + k) - \gamma^0 E(p + k)}{E(p + k) + E - k} \right] \gamma_m .
\]

(B14)

Thus we find that the gamma-matrix trace in the basic rate formula (4.15) involves, for the radiative part,

\[
r_{\pm} = \frac{1}{4} \left( \delta_{lm} - \hat{k}_l \hat{k}_m \right) \text{tr} (m - \gamma p) \gamma^0 (m - \gamma p)
\]

\[
\gamma_l \left[ m - \gamma \cdot (p + k) \pm \gamma^0 E(p + k) \right] \gamma_m = -4E m^2 + E k^2 - (E/k^2) \left[ E^2(p + k) - E^2(p) \right]^2 + 2E(p + k) \left[ E^2 + E^2(p) \right] .
\]

(B15)

The trace for the Coulomb part is given by

\[
c = \frac{1}{4} \text{tr} (m - \gamma p) \gamma^0 (m - \gamma p) \left[ m + \gamma \cdot (p + k) \right] = 2E \left[ E^2(p) + p \cdot k \right] .
\]

(B16)

We begin by calculating the \( \text{Im} \Sigma \) contribution in the rate formula (1.13), which we shall call \( \Gamma_{n_{e-e}}^{(\gamma)} \). The complete self-energy function (with no \( T = 0 \) subtraction) enters here. Returning for a moment to recall the discussion of our evaluation of the frequency sum (B15), it is easy to see that the complete \( \Sigma \) function can be obtained from the result (B14) for \( \Sigma^{(T)} \) by making the replacements, \( f(k) = [f(k) + 1/2] \) and \( n(E(p + k)) \rightarrow [n(E(p + k)) - 1/2] \). We incorporate this replacement and insert the imaginary part of the result (B14) evaluated with the trace formula (B15) into the basic rate formula (4.13). We interchange the \( p \) and \( k \) integrals and then replace the \( p \) integration variable by \( q = p + k \). In this way, we obtain

\[
\Gamma_{n_{e-e}}^{(\gamma)} = \frac{e^2 G_W^2}{(2\pi)^2} (g_V^2 + 3g_A^2) \int_{-\infty}^{+\infty} dE \chi(E) \int \frac{(d^3k)}{(2\pi)^3} \int \frac{(d^3q)}{(2\pi)^3} \frac{1}{E(q)}\left[ E^2(q - k) - E^2 \right]^2 \left\{ [f(k) + n(E(q))] \left[ r_{\pm} \delta(E + k - E(q)) - r_{\pm} \delta(E - k + E(q)) \right] + [f(k) - n(E(q)) + 1] \left[ r_{\pm} \delta(E - k - E(q)) - r_{\pm} \delta(E + k + E(q)) \right] \right\} .
\]

(B17)
The only angular dependence that appears here is in

\[ E(q-k)^2 = E(q)^2 + k^2 + q \cdot k, \]  

which enters in the denominator and in the definition (B15) of \( r_\pm \). Taking \( k \) as the \( z \) axis and performing the solid angle integral for \( q \) and then the remaining solid angle integral for \( k \), and using the delta functions to remove the \( E \) integral gives, after one final variable change to \( E = \sqrt{q^2 + m^2} \), and some algebraic effort,

\[ \Gamma^{(\gamma)}_{n e-e} = 2 \frac{e^2 G_W^2}{(2\pi)^5} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty \frac{dk}{k} \int_m^\infty d\epsilon \left\{ [f(k) + n(\epsilon)] [\chi(\epsilon - k) + \chi(-\epsilon + k)] F_- \\
+ [f(k) - n(\epsilon) + 1] [\chi(\epsilon + k) + \chi(-\epsilon - k)] F_+ \right\}, \]

where

\[ F_\pm = A \pm k B, \]

with

\[ A = 2 \epsilon^2 + k^2 \ln \left( \frac{\epsilon + q}{\epsilon - q} \right) - 4q \epsilon, \]

and

\[ B = 2 \epsilon \ln \left( \frac{\epsilon + q}{\epsilon - q} \right) - 4q. \]

We have also made use the function defined in Eq. (2.29), which we repeat here for convenience,

\[ \chi(\epsilon) = n(\epsilon - \Delta)n(-\epsilon)(\epsilon - \Delta)^2. \]

To facilitate the removal of the “\( T = 0 \)” piece and to also put the result in a form that clarifies its structure, we note that

\[ f(k) + n(\epsilon) = f(k) n(-\epsilon) n(-\epsilon + k)^{-1} = -f(-k) n(\epsilon) n(\epsilon - k)^{-1}, \]

and

\[ f(k) - n(\epsilon) + 1 = -f(-k) n(\epsilon) n(-\epsilon - k)^{-1} = f(k) n(\epsilon) n(\epsilon + k)^{-1}. \]

Here

\[ -f(-k) = 1 + f(k) \]
and
\[ n(-\mathcal{E}) = 1 - n(\mathcal{E}) \]  

(B27)
describe the occupation factors appropriate for final states: \(1 + f(k)\) gives the Bose enhancement factor and \(1 - n(\mathcal{E})\) the Pauli blocking factor. Using these identities, we obtain
\[
\Gamma_{n\rightarrow e}^{(\gamma)} = 2 \frac{e^2 G_W^2}{(2\pi)^5} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty \frac{dk}{k} \int_m^\infty \frac{d\mathcal{E}}{\mathcal{E}} \left\{ [f(k)n(-\mathcal{E})\tilde{\chi}(\mathcal{E} - k) - f(-k)n(\mathcal{E})\tilde{\chi}(-\mathcal{E} + k)] F_- 
+ [-f(-k)n(-\mathcal{E})\tilde{\chi}(\mathcal{E} + k) + f(k)n(\mathcal{E})\tilde{\chi}(-\mathcal{E} - k)] F_+ \right\},
\]  

(B28)
where
\[
\tilde{\chi}(\mathcal{E}) = n(\mathcal{E} - \Delta)(\mathcal{E} - \Delta)^2.
\]  

(B29)
The energy integration variable \(\mathcal{E}\) in Eq. (B17) differs from the neutrino energy by a positive constant, \(\mathcal{E} = E_\nu + \Delta\). Neutrinos in the initial state are described by positive values of the neutrino energy, \(E_\nu > 0\). Thus the terms in Eq. (B17) involving \(\delta(\mathcal{E} - k + \cdots)\), where \(k = |k|\) is an on mass shell, real photon energy, correspond to the production of photons in the final state. These real photon contributions appear in Eq. (B28) in the terms involving \(\tilde{\chi}(-\mathcal{E} + k)\) and \(\tilde{\chi}(\mathcal{E} + k)\). These are just the terms that have the occupancy factors \(f(-k)\) that are appropriate for produced photons. It is a simple matter to check that the Fermi factors \(n(\pm \mathcal{E})\) are just those needed in the various photon absorption and emission processes described by Eq. (B28), but we shall not bother to enumerate then here. We just note that the \("T = 0"\) part of the rate is obtained by setting \(f(k) = 0\). Subtracting this part is equivalent to the substitution \(-f(-k) \rightarrow f(k)\), and so the \(T \neq 0\), photon thermal bath contribution is given by
\[
\Gamma_{n\rightarrow e}^{(\gamma,\gamma)} = 2 \frac{e^2 G_W^2}{(2\pi)^5} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty \frac{dk}{k} f(k) \int_m^\infty \frac{d\mathcal{E}}{\mathcal{E}} \left\{ [n(-\mathcal{E})\tilde{\chi}(\mathcal{E} - k) + n(\mathcal{E})\tilde{\chi}(-\mathcal{E} + k)] F_- + [n(-\mathcal{E})\tilde{\chi}(\mathcal{E} + k) + n(\mathcal{E})\tilde{\chi}(-\mathcal{E} - k)] F_+ \right\}.
\]  

(B30)
Although this set of terms has no ultraviolet divergences, since \(f(k) k^{-1} \rightarrow k^{-2}\) as \(k \rightarrow 0\), it is infrared divergent. This contribution is rendered finite in the infrared by other contributions that we are about to compute.

We pause for a moment from our development to identify the contribution from the rate \(e^+ + \nu + n \rightarrow \gamma + p\) which has been omitted from all previous papers. Since it involves real photon production, it must involve the terms containing \(-f(-k)\) in Eq. (B28). The second term involving the \(\tilde{\chi}(\mathcal{E} + k)\) function describes an incident neutrino of energy \(E_\nu = \mathcal{E} + k - \Delta\) corresponding to a produced electron of energy \(\mathcal{E}\) rather than an initial positron of this energy. Hence it is the first term containing \(-f(-k)\) that contributes to the process in question, with the photon energy starting at \(k = \mathcal{E} + \Delta\) corresponding to an incident
neutrino with zero energy. Thus the previously forgotten contribution to the rate is given by

$$\Gamma_{nF} = \frac{e^2 G_W^2}{(2\pi)^5} (g_\nu^2 + 3g_\lambda^2) \int_{m}^{\infty} d\mathcal{E} n(\mathcal{E}) \int_{\epsilon+\Delta}^{\infty} \frac{dk}{k} \left[ -f(-k) \right] \tilde{\chi}(\mathcal{E} + k) F_. \quad (B31)$$

The $T = 0$ part of this rate is obtained by taking $-f(-k) \to 1$,

$$\Gamma_{nF}^{(T=0)} = \frac{e^2 G_W^2}{(2\pi)^5} (g_\nu^2 + 3g_\lambda^2) \int_{m}^{\infty} d\mathcal{E} n(\mathcal{E}) \int_{\epsilon+\Delta}^{\infty} \frac{dk}{k} \tilde{\chi}(\mathcal{E} + k) F_. \quad (B32)$$

We turn now to evaluate the on mass shell $E = \pm E(p)$ contributions that appear in the last two lines of the rate formula (4.15). This piece of the rate formula involves the temperature-dependent self-energy function in the form

$$\tilde{\Sigma}^{(T)}(p, E) = \text{tr} \left[ m - \gamma p \right] \gamma^0 \left( m - \gamma p \right) \Sigma^{(T)}(p, E). \quad (B33)$$

We separate out the parts that involve the photon and electron phase-space densities by writing

$$\tilde{\Sigma}^{(T)}(p, E) = \tilde{\Sigma}_{\gamma}^{(T)}(p, E) + \tilde{\Sigma}_{e}^{(T)}(p, E). \quad (B34)$$

We use of the previous results (B13) – (B16) and a little algebra to write the two parts as

$$\tilde{\Sigma}_{\gamma}^{(T)}(p, E) = 4e^2 \int \frac{(d^3 k)}{(2\pi)^3} \frac{1}{k} f(k) \left\{ 2E \left[ m^2 + p \cdot k + \frac{(p \cdot k)^2}{k^2} \right] - \frac{1}{E^2(q) - (E + k)^2} + \frac{1}{E^2(q) - (E - k)^2} \right\}, \quad (B35)$$

and

$$\tilde{\Sigma}_{e}^{(T)}(p, E) = -2e^2 \int \frac{(d^3 q)}{(2\pi)^3} \frac{1}{E(q)} n(E(q)) \left\{ \frac{4E}{k^2} \left[ E^2(p) + p \cdot k \right] + E \left[ k^2 - 4m^2 - \frac{1}{k^2} \left[ E^2(q) - E^2(p) \right] \right] \right\} \frac{1}{(E(q) - E)^2 - k^2} + \frac{1}{(E(q) + E)^2 - k^2}$$

$$+ 2E(q) \left[ E^2 + E^2(p) \right] \left\{ \frac{1}{(E(q) - E)^2 - k^2} - \frac{1}{(E(q) + E)^2 - k^2} \right\}. \quad (B36)$$

Here we have again introduced the variable $q = p + k$ and used it as the integration variable for the electron contribution to the self energy (B36). We have used the different
integration variables \( k \) and \( q \) for the photon and electron contributions because then the angular integrations appear in a simple form. The terms on the first line on the right-hand side of Eq. (B30) come from the Coulomb contribution (B13) to the self energy.

To proceed with our computation, we note that the self-energy function evaluated on the mass shell is gauge invariant\(^8\) and gives an energy shift.\(^9\) For positive energies (particles) with spinors that obey
\[
\sum_\lambda u_\lambda(p) \bar{u}_\lambda(p) = m - \gamma p, \tag{B37}
\]
we have
\[
2E \Delta E = \bar{u}_\lambda(p) \Sigma u_\lambda(p), \tag{B38}
\]
while for negative energies (antiparticles) with spinors
\[
\sum_\lambda v_\lambda(p) \bar{v}_\lambda(p) = m + \gamma p, \tag{B39}
\]
the energy shift is
\[
2E \Delta E = \bar{v}_\lambda(p) \Sigma v_\lambda(p), \tag{B40}
\]
Thus, in either case, one of the pieces of the rate formula (4.15) may be expressed as
\[
8 E^2 \Delta E^{(T)}(p) = \text{tr} \gamma^0 \left( m - \gamma p \right) \Sigma^{(T)}(p, E) \left( m - \gamma p \right) \bigg|_{E = \pm E(p)}
\]
\[
= \bar{\Sigma}^{(T)}(p, E) \bigg|_{E = \pm E(p)} \tag{B41}
\]
Accordingly, we write the last, on-mass-shell contribution to the rate formula (4.15) as \( \Gamma_{n^- e^-}^{(\Delta E)} + \Gamma_{n^- e^-}^{(Z)} \), where
\[
\Gamma_{n^- e^-}^{(\Delta E)} = G_W^2 \left( g_V^2 + 3g_A^2 \right) \frac{1}{\pi} \int_{-\infty}^{+\infty} dE \int \frac{(d^3p)}{(2\pi)^3} \left[ \delta(E - E(p)) + \delta(E + E(p)) \right] \Delta E^{(T)}(p) \frac{\partial}{\partial E} \chi(E), \tag{B42}
\]
and
\[
\Gamma_{n^- e^-}^{(Z)} = \frac{G_W^2}{(2\pi)^2} \left( g_V^2 + 3g_A^2 \right) \int_{-\infty}^{+\infty} dE \int \frac{(d^3p)}{(2\pi)^3} \frac{\pi}{2E(p)} \left[ \delta(E - E(p)) - \delta(E + E(p)) \right] \chi(E) \frac{\partial}{\partial E} \left[ E^{-1} \Sigma^{(T)}(p, E) \right] . \tag{B43}
\]
\(^8\)That is, the result would not be altered if we changed our use of the radiation (or Coulomb) gauge to a relativistic gauge such as Landau of Feynman gauge.

\(^9\)See, for example, the discussion of Eq. (8.7.13) in Section 8 of [23].
Referring back to the form of the uncorrected neutron rate (2.30), we see that Eq. (B42) describes precisely the correction to that rate which results from the (gauge-invariant) energy shift $\Delta E^{(T)}(p)$. As we shall see, this temperature-dependent correction is well defined with no infrared divergences. The remaining contribution (B43), which has the form of a temperature-dependent wave function renormalization, is not gauge invariant, and it also suffers from an infrared divergence. This contribution by itself has no physical meaning. It combines with previous contributions to form a gauge-invariant result with no infrared divergence.

In parallel with the partition (B34), we write

$$\Delta E^{(T)}(p) = \Delta E^{(T)}_\gamma(p) + \Delta E^{(T)}_e(p),$$  \hspace{1cm} (B44)

with the total energy shift given by Eq. (B41). Using Eq. (B35) for the photon part gives, after some calculation,

$$\Delta E^{(T)}_\gamma(p) = \frac{e^2}{E} \int \frac{(d^3k)}{(2\pi)^3} \frac{1}{k} f(k), \hspace{1cm} (B45)$$

where the positive/negative energy has the mass shell values $E = \pm E(p)$. Similarly, some work with the electron part (B36) yields

$$\Delta E^{(T)}_e(p) = \frac{e^2}{E} \int \frac{(d^3q)}{(2\pi)^3} \frac{1}{E(q)} n(E(q)) \left\{ 1 \right. \left. + \frac{m^2}{[E(q) + E(p)]^2 - k^2} + \frac{m^2}{[E(q) - E(p)]^2 - k^2} \right\}. \hspace{1cm} (B46)$$

Averaging over the solid angle of $q$ produces

$$\Delta E^{(T)}_e(p) = \frac{e^2}{E} \int \frac{(d^3q)}{(2\pi)^3} \frac{1}{E(q)} n(E(q)) \left\{ 1 - \frac{m^2}{4qp} \ln \left( \frac{q + p}{q - p} \right)^2 \right\}. \hspace{1cm} (B47)$$

These energy shift corrections are perfectly finite: They have neither ultraviolet nor infrared divergences. They also vanish in the zero temperature limit. Moreover, as remarked before, the energy shifts are gauge invariant. This gauge invariance may be confirmed by explicitly computing the self-energy function in the Feynman gauge, $\Sigma^{(T)}_F(p, E)$. This function is obtained by making the replacement $\delta_{lm} - \hat{k}_l \hat{k}_m \to g_{\mu\nu}$ in Eq. (B12), where $g_{\mu\nu}$ is the Minkowski four-dimensional space-time metric with the spatial gamma matrices $\gamma_l, \gamma_m$ replaced by the space-time matrices $\gamma_\mu, \gamma_\nu$. The Coulomb contribution (B13) is now, of course, omitted. It is a simple matter to evaluate this replacement. Passing to the mass shell, $-p^\mu p_\mu = m^2$, with effectively $\gamma^\mu \to p^\mu/m$, it is easy to show that

$$\Sigma^{(T)}_F(p, E) \to \frac{E}{m} \Delta E^{(T)}(p), \hspace{1cm} (B48)$$

where $\Delta E^{(T)}(p)$ is given by Eq’s. (B44) – (B46). (Our previous work shows that $E/m$ is the correct factor to relate the energy shift to the mass-shell self energy.) Gauge invariance is the reason behind the cancelation of the $1/k^2$ terms when the electron contribution (B36) to the self energy is evaluated on the mass shell. The resulting energy shifts are the shifts in
the positions of the poles of the thermal electron Green’s functions — they give the change in the energy needed to create an electron or positron of momentum \( \mathbf{p} \) in the plasma relative to the energy needed to create them in the vacuum.

The energy shifts are physical quantities. Thus we let them stand as separate corrections and turn to the temperature-dependent “wave function renormalization” contribution \((B43)\) which is gauge-dependent and which is not infrared finite, a contribution that must be combined with the previous result to obtain a physically relevant contribution to the neutron rate. Some calculation with the photon part \((B35)\) of the self energy yields

\[ \frac{\partial}{\partial E} \left[ E^{-1} \Sigma_\gamma(T)(\mathbf{p}, E) \right] \bigg|_{E=\pm E(p)} = \pm \frac{2e^2}{\pi^2} \frac{1}{p} \int_0^\infty \frac{dk}{k} f(k) A, \quad (B49)\]

where \( A \) is the function defined before in Eq. \((B24)\) but with different labels,

\[ A = \left[ 2 E(p)^2 + k^2 \right] \ln \left( \frac{E(p) + p}{E(p) - p} \right) - 4 p E(p), \quad (B50)\]

The electron part \((B36)\) of the self energy yields

\[ \frac{\partial}{\partial E} \left[ E^{-1} \Sigma_\gamma(T)(\mathbf{p}, E) \right] \bigg|_{E=\pm E(p)} = \pm \frac{2e^2}{\pi^2} \int_0^\infty dq \frac{q^2}{E(q)} n(E(q)) \frac{E(p)}{E^2(q) - E^2(p)} \left\{ 2 + \frac{E(q)^2}{pq} \ln \left( \frac{p + q}{p - q} \right)^2 \right. \\
\left. - \frac{E(q)}{E(p)} \frac{E^2(q) + E^2(p)}{2pq} \ln \left( \frac{[E(p)E(q) + pq]^2 - m^4}{[E(p)E(q) - pq]^2 - m^4} \right) \right\}. \quad (B51)\]

For the photon contribution \((B49)\) to the wave function term \((B43)\), we integrate over \( E \) and make the simple notational change \( \mathbf{p} \rightarrow \mathbf{q} \) with, as before, \( \mathcal{E} = E(q) \). Then, since \( q dq = \mathcal{E}d\mathcal{E} \) and, integrating over solid angle,

\[ \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \frac{\pi}{2E(p)} \cdots = \frac{1}{4\pi} \int_m^\infty q d\mathcal{E} \cdots. \quad (B52)\]

Thus, the photon contribution to the wave function correction \((B43)\) is

\[ \Gamma_{\gamma,e-e}^{\gamma,Z} = 2 \frac{e^2 G_W^2}{(2\pi)^5} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty \frac{dk}{k} f(k) \int_m^\infty d\mathcal{E} A \left\{ -2\chi(\mathcal{E}) - 2\chi(-\mathcal{E}) \right\} = 2 \frac{e^2 G_W^2}{(2\pi)^5} \left( g_V^2 + 3g_A^2 \right) \int_0^\infty \frac{dk}{k} f(k) \int_m^\infty d\mathcal{E} A \left\{ -2n(-\mathcal{E}) \tilde{\chi}(\mathcal{E}) - 2n(\mathcal{E}) \tilde{\chi}(-\mathcal{E}) \right\}. \quad (B53)\]

As discussed in the text, this adds simply with the real photon contribution Eq. \((B30)\) to produce an infrared finite result. The second wave function contribution comes from placing Eq. \((B51)\) in Eq. \((B43)\). Removing the first energy integral by the delta functions, performing the angular integral, and changing integration variables from momentum to energy with the notational change \( q \rightarrow p' \), \( E(q) \rightarrow E' \), gives the result.
\[
\Gamma_{n_e-e}^n = 2 \frac{e^2 G_W^2}{(2\pi)^5} \left( g_V^2 + 3g_A^2 \right) \int_m^\infty dE \int_m^\infty dE' n(E') \left[ \chi(E) + \chi(-E) \right] \\
\frac{E}{E'^2 - E^2} \left[ 4pp' + 2E^2 \ln \left( \frac{p + p'}{p - p'} \right)^2 \right. \\
- \frac{E'}{E} \left[ E'^2 + E^2 \right] \ln \left( \frac{[EE' + pp']^2 - m^4}{[EE' - pp']^2 - m^4} \right) \right].
\]

(B54)

This piece by itself is not gauge invariant. As discussed in the text, adding it to Coulomb interaction between the electron and proton, the rate contribution (3.9), produces a well-defined, gauge-invariant contribution.
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