RE-PARAMETERIZING AND REDUCING FAMILIES OF NORMAL OPERATORS

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ABSTRACT. We present a new proof of (a slight generalization of) rather recent results of Kurdyka & Paunescu, and of Rainer, about multi-parameters generalizations of classical results by Rellich and Kato about the reduction in families of univarinate deformations of normal operators over real or complex vector spaces of finite dimensions.

Given a real analytic normal operator $L : F \to F$, where $F$ is a connected real analytic (real or complex) vector bundle of finite rank equipped with a fibered-Euclidean/Hermitian structure, there exists a locally finite composition of blowings-up of the basis manifold $N$, with smooth centers, $\sigma : \tilde{N} \to N$, such that at each point $\tilde{y}$ of the source manifold $\tilde{N}$ it is possible to find a neighbourhood of $\tilde{y}$ over which exists a real analytic orthonormal frame in which the pull back of the operator $\sigma^*L : \sigma^*F \to \sigma^*F$ is in reduced form.

A free by-product of our proof is the local real analyticity of the eigen-values, which in all prior works was a prerequisite step to get local regular reducing bases. We deal only with the bouquet of the eigen-spaces as a whole, and understand their accumulation over the points of the discriminant locus of the characteristic polynomial of the operator.

1. INTRODUCTION

Given a regular (continuous, $C^k$, $C^\infty$, real analytic) family $t \to A(t) \in \text{Hom}(\mathbb{K}^n, \mathbb{K}^n)$ of normal operators over the real or complex field, for each parameter $t$ the operator $A(t)$ is in reduced form in some orthonormal basis $f(t)$ of $\mathbb{K}^n$. It is natural to ask what is the regularity of the mapping $t \to f(t) \in \text{Frame}(\mathbb{K}^n)$?

Given a real analytic family $\mathbb{R} \ni t \to A(t)$ of symmetric operators over $\mathbb{R}^n$, Rellich shows in \cite{Rel}, that the eigen-values can be chosen locally real analytic at any a priori given parameter $t_0$, and that there exists a real analytic section $(\mathbb{R}, t_0) \ni t \to f(t) \in \text{Frame}(\mathbb{R}^n)$ which is orthonormal and consisting of eigen-vectors of $A(t)$. In \cite[Chapter 2]{Kat}, Kato investigates similar problems for perturbations (analytic or not) in a single complex variable of a given operator over $\mathbb{C}^n$, and gets some regularity results for the eigen-values as well as for the family of diagonalizing bases.

More recently, the multi-parameter problem was in some sense solved in the best possible ways: given a real analytic family of symmetric matrices $\mathbb{R}^{k(\geq 2)} \ni t \to A(t) \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, it is in general not possible to find at every point of the parameter space a neighbourhood over which exists a real analytic (or even continuous) orthonormal frame of eigen-vectors. Nevertheless (and principally) as a consequence of their study of roots of real analytic monic hyperbolic polynomials, Kurdyka and Paunescu show that \cite{KuPa}, up to a real analytic re-parameterization (i.e. surjective) of the family, $\sigma : N \to \mathbb{R}^k$, where $N$ is a real analytic connected manifold, choosing locally the eigen-values in a really analytic fashion as well as providing a real analytic orthonormal frame of $\mathbb{R}^n$ consisting of eigen-vectors of the family $\tilde{t} \to A(\sigma(\tilde{t}))$ become possible. They deduce the analog result for real analytic families of anti-symmetric matrices. Around the same time, Rainer have started a series of papers about (mostly) roots of regular monic complex polynomials \cite{Rai}, and proves an analogue of Kurdyka & Paunescu result when considering multi-variate quasi-analytic families of monic complex polynomials, which when applied to a quasi-analytic family of complex normal matrices, provides, up to a quasi-analytic re-parameterization of the family, the possibility to choose locally quasi-analytically the eigen-values as well as a local orthonormal frame of eigen-vectors.

Date: November 15, 2018.

Key words and phrases. normal operator, reduction of endomorphism, desingularization, blowings-up, real analytic coherent sheaves, eigen-bouquet, quadratic ideals.

The author is most grateful to IMPAN for their working conditions while visiting Warsaw and working on this paper. This work was partially supported by CNPq-Brazil grant 305614/2015-0.
The aforementioned results obtained in the multi-parameter case, using re-parameterization, proceed in the following way: First, re-parameterize the domain so that the eigen-values can be locally chosen real/quasi-analytic; Second, solve the corresponding eigen-value linear system and arrange it by further re-parameterizations if needs be. The results of Kurdyka & Paunescu and Rainer are that good because quasi-analytic functions admit a resolution of singularities [Hir, BM1, BM2] or at the least a local uniformization. Indeed their re-parameterization mappings are (locally) finite composition of geometrically admissible blowings-up [KuPa] and of ramifications [Rai2]. The most consistent part of the work in [KuPa, Rai2] to follow this strategy lies in the local regularization of the eigen-values.

Before starting to describe the content of our paper let us make the following:

**Observations:**
1) As already noticed by Rellich, the problem of the regularity of the eigen-vectors basis is harder to handle than that of the regularity of the eigen-values. The local regularity of the eigen-values and that of the eigen-frame are not really linked. Indeed it should be so, since the latter is of projective nature, that is insensitive to rescaling (see for instance [Kur] p. 111).

2) Let $t \rightarrow A(t) \in Hom(\mathbb{K}^n, \mathbb{K}^n)$ be a regular (real/quasi-analytic) family of operators. Let $t_0$ be a point of the parameter space (a regular manifold) for which there exists a neighbourhood $\mathcal{V}$ over which exists a regular mapping $t \rightarrow f(t) \in Frame(\mathbb{K}^n)$ of (orthonormal) frames consisting of eigen-vectors of the family $(A(t))_{t \in \mathcal{V}}$. Then the eigen-values functions can be chosen regular by just evaluating the regular operator $A$ along the regular frame.

Beside our initial motivation to be interested in this problem (postponed to the last section), the above observation is the starting point of the material presented here.

Let $F$ be a real analytic $\mathbb{K}$-vector bundle of finite rank $n$ over a real analytic connected manifold $N$. We further assume that $F$ is equipped with a real analytic fibered Euclidean ($\mathbb{K} = \mathbb{R}$) or Hermitian ($\mathbb{K} = \mathbb{C}$) structure. The discriminant locus $D_L$ of an operator $L : F \rightarrow F$ is the locus of points of $N$ where the number of distinct (complex) eigen-values of $L$ is not locally constant, in other words it is the discriminant locus of the characteristic polynomial of $L$. A non-null vector of $\mathbb{K}^n$ is a characteristic vector of the normal operator $A : \mathbb{K}^n \rightarrow \mathbb{K}^n$, if it is either an eigen-vector or, when $\mathbb{K} = \mathbb{R}$ and if it is not an eigen-vector, it is lying in a real plane invariant by $A$ along which the restriction of $A$ is a similitude (composition of a rotation with a dilation).

The problem we are interested in solving is:

**Problem to be solved:** Find a re-parameterization $\sigma : \tilde{N} \rightarrow N$ of an à-priori given real analytic normal operator $L : F \rightarrow F$ such that: (1) at every point $\tilde{y} \in \tilde{N}$ there exists a neighbourhood $\tilde{\mathcal{V}}$ over which exists a local real analytic frame of $\sigma^* F)|\tilde{\mathcal{V}}$ made of orthonormal eigen-vector of $\sigma^* L$ which reduces $L \circ \sigma(x)$ at each $\tilde{x} \in \tilde{\mathcal{V}}$; and (2) Do so in working only with the eigen-bouquet.

In order to have an impression about what we are doing in this note, let us present the following

**Example.** We re-visit [KuPa, Example 6.1.] with our point of view: Let $F := N \times \mathbb{R}^2$ with $N = \mathbb{R}^2$, and let $L : F \rightarrow F$ be given by the symmetric matrix

$$M(x, y) = \begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix}.$$ 

The eigen-values are 0 and $x^2 + y^2$, and the eigen-spaces are $(1, \frac{y}{x})$ and $(1, -\frac{x}{y})$. There is a $\mathbb{P}^1$ of distinct pairs of eigen-spaces accumulating at $(0, 0)$. The eigen-bouquet bundle of $L$ is the subset:

$$\{(x, y) ; (X, Y) \in F : M(x, y) : (X, Y) \wedge (X, Y) = 0\} = \{(yY + xX)(yX - xY) = 0\}.$$

Let $x = (x, y)$ so that the ideal $J_x$ of the polynomials over $F_x$ vanishing along the eigen-bouquet of $L(x)$ is reduced, principal and given by

$$J_x = (yY + xX)(yX - xY).$$
As expected we see that $J_{(0,0)} = (0)$. The quadratic part of the family of ideals $J_x$ is a real analytic section of the vector bundle

$$S^2F^\vee = N \times H_2[X, Y] = \mathbb{R}^2 \times (\mathbb{R}X^2 \oplus \mathbb{R}XY \oplus \mathbb{R}Y^2) \subset SF^\vee = N \times \mathbb{R}[X, Y].$$

Blowing-up the origin $(0, 0)$ in $N$, the base manifold of the vector bundle $F$, and considering the chart $\sigma : (u, v) \rightarrow (u, uv)$ (the other chart will lead to a similar computation), gives

$$\sigma^*(J_x) = u^2(v^2 - X)(vX + Y) = (u^2 \cdot J'_{(u,v)}) \text{ with } J'_{(u,v)} = (vY + X)(vX - Y)$$

the vanishing locus of $J'_{(u,v)}$ is a union of two orthogonal lines of the fiber $\sigma^*F(u, v)$, each of which invariant by $(\sigma^*)^i(u, v)$. Those pairs of lines “move” analytically in the parameter $(u, v) \in \tilde{N}$.

**Remark about the example.** The problem of controlling point-wise all the eigen-spaces, the eigen-bouquet, was solved by means of a single object: the family of ideals $(J_x)_{x \in N}$. This method did not involve resolving simultaneously the equations of the eigen-spaces, neither it needed to do anything about the eigen-values.

The main result of this paper is the following:

**Theorem 9.1.** Let $L : F \rightarrow F$ be a real analytic normal operator over a real analytic $\mathbb{K}$-vector bundle $F$ over a connected real analytic manifold $N$, and equipped with a real analytic fiber Euclidean/Hermitian structure.

(i) There exists a locally finite composition of geometrically admissible blowings-up $\sigma : (\tilde{N}, \tilde{E}) \rightarrow (N, D_L)$ such that for any $\tilde{y} \in \tilde{N}$, there exists $\tilde{V}$ a neighbourhood of $\tilde{y}$ and real analytic vector sub-bundles $R_1, \ldots, R_s$ of $\sigma^*F|\tilde{V}$ such that

(a) they are pair-wise orthogonal and everywhere in direct sum;

(b) The restriction of $\sigma^*L$ to each $R_i$ is reduced: either it is a multiple of identity or $R_i$ is real of real rank 2, and the restriction of $\sigma^*L$ is a fiber-wise a similitude.

(ii) For any $\tilde{y} \in \tilde{N}$, there exists $\tilde{V}$ a neighbourhood of $\tilde{y}$ such that the eigen-values of $\tilde{x} \rightarrow L(\sigma(\tilde{x}))$ are real analytic.

As we announced in the example above, we **neither computed eigen-values nor solved any eigen-value linear system.** Our proof does not fix/better (re-parameterize) the eigen-values of the operator $L \circ \sigma$. Point (ii) is just the consequence of point (i) after evaluating $\sigma^*L$ along the sub-bundles.

Finding locally a real analytic frame of eigen-vectors is problematic only at points of the discriminant locus $D_L$ of $L$. At such a discriminant point, different eigen-value functions take the same value and the accumulation at the point of their corresponding eigen-spaces can be quite complicated. Ideally, we would like to guarantee that at each point $y$ of the discriminant locus, a given eigen-space accumulates only on a single subspace of the fiber of $Fy$, but in general it is not so (see the example aforementioned). Nevertheless, such a situation - separation of too many regular limits - can often be handled through resolution of singularities techniques. The problem here is not only to separate different limits of a sub-bundle (eigen-space) but do that simultaneously for several. Outside the discriminant locus we can define a real analytic multi-valued mapping

$$N \setminus D_L \xrightarrow{\Gamma} \bigcup_{k=1}^{s_L} \mathcal{G}(k, F) \quad \mathbf{x} \quad \mapsto \quad \{[E_1(\mathbf{x})], \ldots, [E_{s_L}(\mathbf{x})]\}$$

where $s_L$ is the number of distinct complex eigen-values outside $D_L$ and $E_1(\mathbf{x}), \ldots, E_{s_L}(\mathbf{x})$ are the eigen-spaces (when diagonalizable), characteristic spaces (when not diagonalizable) of $L(\mathbf{x})$. Although there is no systematic resolution of singularities of multivalued objects, “resolving the multivalued mapping $\Gamma$ over the points $D_L$” (should be possible once produced a real analytic “uni-valued” object defined over $N$ containing the very same information as $\Gamma$).

Our proof of Theorem 9.1 just does that: we encode the same information at each point $y$ as that of $\Gamma(y)$, namely the bouquet of eigen-spaces or eigen-bouquet, defined as the union of the eigen-spaces of $L(y)$, by the polynomial ideal $J_y$ vanishing along the bouquet of eigen-spaces of $L(y)$. Any such ideal $J_y$ is reduced and generated by quadratic polynomials. Let $Q_y \subset (S^2F^\vee)_y$ (the second symmetric power of the dual bundle $F^\vee$.
of \( F \) be the subspace of the quadratic polynomials over \( F \) generated by the quadratic polynomials belonging to \( J \). Outside \( D_L \) its dimension is constant equal to \( d_L \). To this family of subspaces \((Q_\ell)_\ell\) we associate an \( O_N \)-coherent module, say \( A_L \), with co-support exactly the discriminant \( D_L \). This module corresponds in some sense to the vector sub-bundle over \( N \) with fibers the sub-spaces \((Q_\ell)_\ell\). The \( O_N \)-ideal of the \( d_L \times d_L \) minors of the module \( A_L \) is denoted \( F_L \), and can be principalized and monomialized by the classical Theorems of embedded resolution of singularities by means of \( \sigma : (\tilde{N}, \tilde{E}) \to (N, D_L) \), a locally finite composition of geometrically admissible blowings-up. The divided \( O_N \)-module (weak transform) \( A_L' \) obtained from \( \sigma^* A_L \) is locally free of rank \( d_L \) with empty co-support. The module \( A_L' \) corresponds to a vector sub-bundle of \( \bigwedge^d F \) of rank \( d_L \) which generates the reduced ideals \( J'_\ell \subset (\sigma^*F)_\ell \), which at every point \( x \) vanishes along a bouquet of eigen-subspaces of \( L \circ \sigma(x) \) (in the diagonalizable case) which are in orthogonal direct sum. It becomes then a simple matter to construct locally finitely many real analytic vector-sub-bundles of \( \sigma^* F \) which are in orthogonal direct sum, and over each of which the restriction of \( \sigma^* L \) is a multiple of the identity operator (in the diagonalizable case), thus providing the final result. The real normal but not symmetric case is dealt straightforwardly with from the real symmetric case.

The paper is organized as follows: Section 2 introduces very quickly some of the fundamental object to be used in the paper. Section 3 recalls classical fact about normal operators and insists about the real ones when not diagonalizable. Section 4 and Section 5 present very elementary algebraic facts about bouquets of linear subspaces which are in direct sums and how these properties relates to diagonalization of an endomorphism in relation to the ideal of the polynomials vanishing along the bouquet. Section 6 presents the Essential Lemma and its corollary. But, although mostly technical, this Lemma catches the very nature of the structure we are looking for. The most important work is done in Section 7, which after studying thoroughly some local properties of the (pointed)-eigen-bouquet bundle, provides a proof of the main result in the local case, that is when \( F \) is trivial under some special hypotheses that we know achievable by composition of blowings-up if need be. Section 8 finishes to treat the whole local case, Section 9 deals with the general case presented above and Section 10 ends the real case. We would like to single out Remark 9.3 to emphasize what our method can do that the regularization of eigen-values strategy would struggled to achieve. Finally in Section 11 we say a few words about our motivations to come to such a problem and also point out at the quasi-analytic setting.

2. Miscellaneous preliminaries

- Let \( \mathbb{K} \) be either \( \mathbb{R} \) or \( \mathbb{C} \).
- A \( \mathbb{K} \)-vector bundle \( F \) over a topological space \( S \) is just that: a ”bundle of \( \mathbb{K} \)-vector spaces” over each point of \( S \), namely
  \[
  F = \bigcup_{y \in S} \{y\} \times F_y
  \]
  where each \( F_y \) is a \( \mathbb{K} \)-vector space.

  If \( R \) is any subset of \( S \), let us denote \( F|R := \bigcup_{y \in R} \) the restriction of \( F \) to \( R \).

  If all fibers \( F_y \) have dimension (over \( \mathbb{K} \)) \( 0 \leq r < +\infty \), then we speak of a vector bundle or rank \( r \).

  Given a \( \mathbb{K} \)-vector bundle over \( F \) we associate the projective bundle \( \mathbb{P} F := \bigcup_{y \in S} \{y\} \times \mathbb{P} F_y \). A subset \( Z := \bigcup_{y \in S} \{y\} \times Z_y \) with \( \emptyset \neq Z_y \subset \mathbb{P} F_y \) is called a sub-bundle of \( \mathbb{P} F \). To such sub-bundle of \( \mathbb{P} \), we can define \( \text{Cone}(Z) := \bigcup_{y \in S} \{y\} \times \text{Cone}(Z_y) \) a subset of \( F \) called the cone bundle over \( Z \), where \( \text{Cone}(Z_y) \) is the \( \mathbb{K} \)-cone of \( F_y \) over \( Z_y \).

  Let \( S \) be a connected manifold of regularity \( C \) (continuous, \( C^k \) for \( k = 1, \ldots, \infty \), real analytic). A real \( C \)-\( \mathbb{K} \)-vector bundle over \( S \) of rank \( r \) is a \( \mathbb{K} \)-vector bundle \( F \) over \( S \) which is a \( C \)-manifold and which is locally \( C \)-trivial: that is for each \( y \in S \), there exists an open neighbourhood \( U \) of \( y \) in \( S \) such that:

  (i) there exists a \( C \)-diffeomorphism \( \Phi : F|U \to U \times \mathbb{K}^r \);

  (ii) the mapping \( \Phi \) is such that \( \Phi(x; v) = (\varphi(x), G(x)v) \), where \( \varphi \) is a \( C \)-diffeomorphism of \( U \) onto \( \varphi(U) \), and \( G(x) : F_x \to \mathbb{K}^r \) is a \( \mathbb{K} \)-linear isomorphism.
Given $F$ a $\mathbb{C}$-$\mathbb{K}$-regular vector bundle of rank $r$ over the $\mathbb{C}$-manifold $S$, let $\tilde{F}$ (also denoted $F^*$ or $(F)^*$) be the pointed $\mathbb{C}$-vector bundle of $F$, namely the fiber $\mathbb{C}$-sub-bundle of $F$ obtained by removing its zero section $S \times 0$.

Let $F$ a $\mathbb{C}$-$\mathbb{K}$-regular vector bundle of rank $r$ over the $\mathbb{C}$-manifold $\mathcal{U}$. For every $0 \leq s \leq r$, let $\mathbf{G}(s, F)$ be the Grassmann fiber $\mathbb{C}$-bundle of $F$ of the $\mathbb{K}$-vector subspaces of $F$ of dimension $s$, in other words $\mathbf{G}(s, F)_y := \mathbf{G}(s, F_y)$ at any $y \in S$. We will also use $\mathbf{PF} = \mathbf{G}(1, F)$ the projective bundle obtained from $F$.

Let $SF^\vee$ be the symmetric algebra vector bundle built from $F^\vee$ the dual of $F$, that is

$$SF^\vee = (\mathcal{U} \times \mathbb{K}) \oplus F^\vee \oplus S^2F^\vee \oplus \ldots \oplus S^kF^\vee \oplus \ldots$$

where $S^kF^\vee$, for $k \geq 2$ is the symmetric bundle of order $k$.

- In the real analytic category we will sometimes use regular for a subset or a mapping to mean it is real analytically regular.

Let $S$ be a connected real analytic manifold of dimension $n$. A simple normal crossing divisor $D$ is a real analytic subset of $S$ which is a union of real analytic and regular hypersurfaces of $S$, such that at any point of $S$, the germ of $D$ is (using a local chart at the point) the union of some coordinates hyperplanes of $\mathbb{R}^n$, (none is allowed) and such that $D$ intersects any compact subset into finitely many connected components. The empty set is also a simple normal crossing divisor by convention.

A coherent $\mathcal{O}_S$-ideal sheaf $\mathcal{I}$ is principal and monomial in the simple normal crossing divisor $D$, if at each $y$ of $S$, there exists real analytic coordinates $(x, z) = (x_1, \ldots, x_p, z_1, \ldots, z_{n-p})$ centered at $y$ such that

(i) $(D, y) = \{x_1 \cdots x_p = 0\}$;

(ii) $\mathcal{I}$ is generated nearby $y$ by a monomial $x_1^{a_1} \cdots x_p^{a_p}$ where $a_1, \ldots, a_p$ are non-negative integers.

The point $y$ is called a $p$-point of $D$, where the integer number $p$ corresponds to the number of components of $D$ containing $y$.

The blowing-up of a given regular submanifold $C$ is geometrically admissible with respect to $(S, D)$ (shortened later as geometrically admissible), for $D$ a simple normal crossing divisor, if for each $y \in S$ there are local regular coordinates $(x, w, z) = (x_1, \ldots, x_p, w_1, \ldots, w_q, z_1, \ldots, z_{n-p-q})$ at $y$ such that

(i) $(D, y) = \{x_1 \cdots x_p = 0\}$;

(ii) there exists a non negative integers $r \leq p$ such that $(C, y) = \{x_1 = \ldots = x_r = w_1 = \ldots = w_q = 0\}$.

If $\sigma_C : (S', D') \rightarrow (S, D \cup C)$ denotes the geometrically admissible blowing-up of $C$, then $D' = \sigma_C^{-1}(D \cup C)$ is a simple normal crossing divisor, with a new component namely $\sigma_C^{-1}(C)$.

3. ABOUT NORMAL OPERATORS

The material presented here is classic, but may be for some aspect of the terminology. The real part will be used in Section [10].

Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and let $F$ be a $\mathbb{K}$-vector space of finite dimension $n$, equipped with a scalar product (meaning Hermitian if $\mathbb{K} = \mathbb{C}$), denoted $(\cdot, \cdot)$ and simply $| \cdot |$ for the associated norm.

A $\mathbb{K}$-endomorphism $L \in Hom(F, F)$ is normal, if the adjoint $L^*$ of $L$ commutes with $L$, where $L^*$ is uniquely defined by the following condition

$$\forall u, v \in F \; \text{we have} \; \langle Lu, v \rangle = \langle u, L^*v \rangle.$$  

Let $\text{Sp}(L)$ be the (complex) spectrum of the normal operator $L : F \rightarrow F$. Thus: (i) $\lambda \in \text{Sp}(L)$ if and only if $\bar{\lambda} \in \text{Sp}(L^*)$; and (ii) Let $\lambda \in \text{Sp}(L)$. Then $\ker(L - \lambda I) = \ker(L^* - \bar{\lambda} I)$. Normal endomorphisms and their adjoint which are diagonalizable, are simultaneously so in an orthonormal basis of eigen-vectors: Any complex normal endomorphism is diagonalizable in the unitary group.

Let $V$ be the real Euclidean space of dimension $n$. Given $L : V \rightarrow V$ a normal endomorphism, we recall

$$V = R \oplus C$$
where $R$ and $C$ are orthogonal subspaces of $E$ such that: (i) The subspaces $R$ and $C$ are invariant by $L$; (ii) the restriction $L|R : R \to R \subset V$ is diagonalizable, once $R$ is not the null space; (iii) the restriction $L|C : C \to C \subset V$ is an isomorphism of $C$ with no real eigen-values, once $C$ is not reduced to the null space.

Let us call $C$ the anti-real characteristic space of $L$. Whenever $C$ is not the null space, it decomposes as an orthogonal direct sum

$$C = C_1 \oplus \ldots \oplus C_t$$

of vector subspaces $C_1, \ldots, C_t$ of dimension 2 invariant by $L$, such that the restriction $L|C_i$ is a similitude: a composition of a rotation and a dilation. Any such a plane $C_i$ is called an anti-real characteristic plane (ARCP for short) of $L$. Equivalently there exist $\alpha_i \in \mathbb{R}$ and $\beta_i \in \mathbb{R}^*$, and an orthonormal basis of $C_i$ such that the matrix of $L|C_i$ is of the form

$$\begin{bmatrix}
\alpha_i & \beta_i \\
-\beta_i & \alpha_i
\end{bmatrix}$$

In particular we can write

$$L = A + B, \text{ where } A := \frac{L + L^*}{2} \text{ and } B = \frac{L - L^*}{2}$$

so that

$$A = A^* \text{ and } B^* = -B.$$ 

The eigen-values of $B$ are pure imaginary complex numbers and square roots of those of $B^2 = -BB^*$. Any eigen-space $W$ of $BB^*$ is contained in an eigen-space of $A$ and there exist real numbers $a, b$ such that

$$A|W = aId_W \text{ and } BB^*|W = b^2Id_W$$

and thus $a + \sqrt{-1}b$ and $a - \sqrt{-1}b$ are complex eigen-values of (the complexification of) $L$.

Let us define the following operators $J$ and the symmetric operator $B_2 := J(B \oplus B)$:

$$J : V \oplus V \to V \oplus V \quad B_2 : V \oplus V \to V \oplus V$$

$$u \oplus v \to (-v) \oplus u \quad u \oplus v \to (-Bv) \oplus Bu$$

We recall

**Lemma 3.1.**

1) $b \in \text{Sp}(B_2)$ if and only if $-b \in \text{Sp}(B_2)$ if and only if $-b^2 \in \text{Sp}(B^2)$.

2) Let $b \neq 0$ be an eigen value of $B_2$ and let $E_b$ be the corresponding eigen-space, and let $u \oplus v \in E_b \setminus 0 \oplus 0$.

The following hold true:

(i) $u, v$ are orthogonal and the plane $\mathbb{R}u + \mathbb{R}v$ of $V$ is invariant by $B$

(ii) $v \oplus u \in E_{-b}$;

(iii) $J(u \oplus v) = (-v) \oplus u \in E_{b}$;

(iv) $J E_b = E_b$.

A normal endomorphism $L : F \to F$ is reduced in the basis $B = (b_1, \ldots, b_n)$ of $F$, if it consists of orthonormal characteristic vectors of $L$, that is diagonalizes $L$ if $K = \mathbb{C}$, and if $K = \mathbb{R}$ it is a basis corresponding to the orthogonal direct sum $F = R \oplus C_1 \oplus \ldots \oplus C_s$ such that $L|R$ is diagonal in the basis $(b_1, \ldots, b_r)$ of $R$, and each $C_i$ is an ARCP.

4. Some very elementary polynomial algebra

Let $F$ be a $\mathbb{K}$-vector space of dimension $n$. Let $E_1, \ldots, E_s$ be pairwise distinct non-trivial vector subspaces of $F$, that is each $E_i$ is neither the null-space nor is $F$.

The union $E := E_1 \cup \ldots \cup E_s$ is called a bouquet of subspaces of $F$.

Let $S F^\vee$ be the $\mathbb{K}$-algebra of polynomials over $F$ (which is isomorphic to $\mathbb{K}[X] := \mathbb{K}[X_1, \ldots, X_n]$) and let $S^2 F^\vee$ be the vector space of quadratic polynomials of $F$ (which we identify with $H_2[X] \subset \mathbb{K}[X]$, the $\mathbb{K}$-vector subspace of $\mathbb{K}[X]$ consisting only of quadratic polynomials).
Let us consider the subset
\[ S_n^k := \{(e_1, \ldots, e_k) \in \{1, \ldots, n\}^k : e_1 \leq \ldots \leq e_k \text{ and } \sum_{i=1}^k e_i = n \}. \]
Given \( \mathbf{e} \in S_n^k \) is associated a unique partition of \( \{1, \ldots, n\} \), namely \( n_{\mathbf{e}} = \{n_{\mathbf{e}}^1, \ldots, n_{\mathbf{e}}^k\} \) for
\[ n_{\mathbf{e}}^i := \{j : 1 + e_1 + \ldots + e_{i-1} \leq j \leq 1 + e_1 + \ldots + e_i\} \]
with the convention \( e_0 := 0 \).

Let \( \mathbf{x} = (x_1, \ldots, x_n) \) be a linear system of coordinates in \( F \), and let \( E_1, \ldots, E_s \) be \( s \geq 2 \) coordinates subspaces which are in direct sum
\[ E_1 \oplus \ldots \oplus E_s = F. \]
Let \( e_i \) be the dimension of \( E_i \).
\[ E_i := \{(0, \ldots, 0, x_{1+e_1+\ldots+e_{i-1}}, \ldots, x_{1+e_1+\ldots+e_i}, 0, \ldots, 0) \in F : x_k \in \mathbb{K} \} \]
Let \( J_i \) be the ideal of \( SF^v \) of polynomial vanishing onto \( E_i \), that is \( J_i = (X_j)_{j \notin n_{\mathbf{e}}^i} \).

Let \( E := \bigcup_{i=1}^s E_i \) and let \( J \) be the ideal of \( SF^v \), vanishing along the bouquet \( E \). Let \( \mathbf{e} \in S_n^k \) corresponding to the dimensions of \( E_1, \ldots, E_k \) (after permutation is necessary). We do have the elementary

**Lemma 4.1.** (i) Let \( J = \bigcap_{i=1}^s J_i \).
(ii) \( J \) is minimally generated by all the products \( x_i x_j \), with \( i < j \) and \( (i, j) \in \bigcup_{1 \leq i < m \leq s} n_{\mathbf{e}}^i \times n_{\mathbf{e}}^m \).
(iii) We have
\[ \dim J \cap S^2 F^v = \sum_{j=1}^{s-1} e_j (e_{j+1} + \ldots + e_s) = \sum_{1 \leq i < j \leq s} \dim E_i \cdot \dim E_j \leq \frac{n(n-1)}{2}. \]

**Proof.** With our notation we have \( J_i \) is generated by \( (X_k)_{k \notin n_{\mathbf{e}}^i} \). An induction on \( s \) (or on \( n - s \)) will do. \( \square \)

**Corollary 4.2.** Let us consider two direct sums decompositions of \( \mathbb{K}^n \) of the following form:
\[ \mathbb{K}^n = E_1 \oplus \ldots \oplus E_r \oplus F_1 \oplus F_s \quad \text{and} \quad \mathbb{K}^n = E'_1 \oplus \ldots \oplus E'_r \oplus G_1 \oplus G_t \]
such that: (i) \( r, s, t \geq 1 \) with \( t < s \).
(ii) \( \dim E_i = \dim E'_i \);
(iii) there exists a \( t \)-tuple \( (d_1, \ldots, d_t) \in \mathbb{N}^t_{\geq 1} \) such that \( d_1 + \ldots + d_t = s \) and a partition \( \bigcup_{k=1}^t A_k \) of \( \{1, \ldots, s\} \) with \( |A_k| = d_k \) such that for each \( k = 1, \ldots, t \), we have
\[ \dim G_k = \sum_{j \in A_k} \dim F_j. \]
Let \( J \) be the ideal of \( SF^v \) of polynomials vanishing over \( (\bigcup_i E_i) \cup (\bigcup_j F_j) \) and \( J' \) be the ideal of polynomials vanishing over \( (\bigcup_i E'_i) \cup (\bigcup_k G_k) \). Then
\[ \dim J' \cap S^2 F^v < \dim J \cap S^2 F^v. \]

**Proof.** It is an elementary computation from Equation (4.1). \( \square \)

If \( K \) is any ideal of \( SF^v \), let \( K := K \cap S^2 F^v \) be the vector subspace of the quadratic polynomials contained in \( K \). Let \( d_2(K) \) be the dimension of \( K_2 \). We can now state the following

**Lemma 4.3.** Let \( J \) as in Lemma 4.1. Let \( K \) be an ideal of \( SF^v \) generated by quadratic polynomials and such that \( d_2(K) = d_2(J) \). If \( K \) vanishes along \( E \) then \( K = J \).

**Proof.** The ideal \( J \) is generated by \( d_2(J) \) quadratic monomials. Since it is reduced, it contains \( K \), so that any quadratic generators of \( K \) is a linear combination of the monomials generating \( J \), since \( K_2 \subset J_2 \). But they have the same dimension so \( J_2 = K_2 \) and then \( K = J \). \( \square \)
The following Lemma will be quite useful later in the paper.

**Lemma 4.4.** Let $J$ as in Lemma 4.1. Let $K$ be an ideal of $SF^\vee$ generated by quadratic polynomials. If the vanishing locus $V(K)$ of $K$ is exactly the vanishing locus $V(J)$ of $J$, then $K = J$.

**Proof.** Since $J$ is reduced, we must have $K \subset J$. It is enough to show that $K \cap S^2F^\vee = J \cap S^2F^\vee$. In particular $K_2$ is a vector subspace of $J_2$. Assume that $J_2 = K_2 \oplus L_2$ for $L_2$ a subspace of $S^2F^\vee$ of positive dimension. If $L$ denotes the ideal of $SF^\vee$ generated by $L_2$ we find

$$V(J) = V(K) \cap V(L)$$

which implies that $V(L) \cap V(J) = V(K)$ by hypothesis. But if this latter fact were to be true it would contradict the minimality of the generators of $J$, which are explicitly listed above.

The reason why we are interested in these elementary results is the following elementary

**Proposition 4.5.** Let $A \in M_n(\mathbb{K})$ be a $n \times n$ matrix with $\mathbb{K}$-entries.

1) Assume that $A$ is not a multiple of the identity. Let $E_1, \ldots, E_s$ be the eigen-spaces of $A$, and let $J$ be the ideal of polynomials of $\mathbb{K}[X]$ vanishing over $\bigcup_{i=1}^s E_i$.

The following conditions are equivalent:

(i) $A$ is diagonalizable;

(ii) $J$ is generated by quadratic polynomials

(iii) The ideal generated by $J_2 := J \cap H_2[X]$ is equal to $J$.

(iv) The ideal generated by $J_2$ is reduced.

2) If $A$ is a multiple of the identity, then the ideal $J$ is the null ideal.

**Proof.** Obvious using the previous two Lemmas.

5. SOME LINEAR AND MULTI-LINEAR ALGEBRA

Let $F$ be a $\mathbb{K}$-vector space of dimension $n$, and let $F^\vee$ be its dual. We assume $F$ is equipped with a Euclidean ($\mathbb{K} = \mathbb{R}$) or a Hermitian product ($\mathbb{K} = \mathbb{C}$).

Let $L : F \rightarrow F$ be a normal operator which we assume to be diagonalizable. Further assuming that $L$ is not a multiple of the identity map, let $E_1, \ldots, E_s$ be all the eigen-vector subspaces of $L$, with $s \geq 2$.

We consider the mapping

$$Q_L : F \rightarrow \wedge^2 F$$

$$\xi \rightarrow L\xi \wedge \xi$$

Since $S^2F^\vee$ is the space of quadratic polynomials over $F$, we can consider also that $Q_L \in (S^2F^\vee)^{n(n-1)/2}$.

Let us define

$$EV_L := \{\xi \in \{0 : Q_L(\xi) = 0\} = (E_1) \cup \ldots \cup (E_s)\}.$$

the union of the pointed eigen-vector subspaces of $L$, which we rename as the (pointed) eigen-bouquet of $L$. It is a smooth submanifold of the pointed vector space $F^\vee$. Since it is invariant by the action of $\mathbb{K}^*$, it gives rise to a projective submanifold of $PF$ with at least two connected/irreducible components.

Let $DQ_L : TF^\vee = F^\vee \times F \rightarrow T(\wedge^2 F)$ be the differential mapping of $Q_L$. Looking at $DQ_L$ as a mapping $F^\vee \rightarrow Hom(F, \wedge^2 F)$ with linear coefficients, it is a regular section of some regular vector bundle over $F^\vee$.

Assume we are given an orthonormal basis $B = (b_1, \ldots, b_n)$ of $F$. Let $M$ be the matrix of $L$ in this basis. Let us write

$$Lv \wedge v = Mv \wedge v =: Q(v) = \sum_{1 \leq i < j \leq n} Q_{i,j}(v) b_i \wedge b_j.$$
with \( Q_{i,j}(v) := (Mv)_{i}v_j - (Mv)_{j}v_i \). We also do see that
\[
D_w Q = 2 \sum_{1 \leq i < j \leq n} \omega_{(i,j)}(Q, w) b_i \wedge b_j
\]
where \( \omega_{(i,j)}(Q, w) \) is a \( K \)-linear form over \( F \) with linear (in \( w \)) coefficients in \( F \).

A non-zero vector \( v \) lies in \( EV_L \) if and only if \( Q_{i,j}(v) = 0 \) for all \( i < j \). Thus we deduce that
\[
D_v Q \cdot v = 0
\]
Assume that \( B \) is such that the matrix \( M \) is diagonal with diagonal coefficients \( a_1, \ldots, a_n \). Thus there exist \( i < j \) with \( a_i \neq a_j \). We obtain
\[
Q(v) := Mv \wedge v = \sum_{i < j} (a_i - a_j)v_i v_j b_{(i,j)}
\]
for \( b_{(i,j)} = b_i \wedge b_j \). Let \( w \in EV_L \), so that \( Mw = aw \) for some diagonal coefficient \( a \) of \( M \). Thus we deduce that
\[
D_w Q = 2 \sum_{a_i = a, a_j \neq a \& i \neq j} (a - a_j)w_i \, dv_j \, b_{(i,j)}
\]
Let \( e \) be the dimension of the eigen-space \( E_a \) of \( M \) containing \( w \). We can assume that \( a = a_1 \) and \( w_1 \neq 0 \) and \( w_j = 0 \) once \( j \geq 2 \), so that
\[
D_w Q = 2 \sum_{j \geq e + 1} (a_1 - a_j)w_1 \, dv_j \, b_{(1,j)}
\]
Form this presentation we deduce the obvious following

**Lemma 5.1.** Let \( w \) be an eigen-vector of \( M \) whose eigen-space has dimension \( e \). Thus \( \wedge^{n-e+1} D_w Q_L \) is identically null while \( \wedge^{n-e} D_w Q_L \) is not identically null. Thus there \( i < j \) such that \( \omega_{(i,j)}(Q, w) \) is not a null 1-form over \( \mathbb{R}^n \).

**Proof.** It is indeed a property on \( Q_L \), and computing with \( Q \) provides the answer. \( \square \)

To follow on Lemma 5.3 and justify the material presented in Section 4 we deduce

**Lemma 5.2.** The coefficients \( Q_{i,j} \in S^2 F^\vee \) of the mapping \( Q_L \) generates the ideal of the polynomials over \( F \) (that would be \( SF^\vee \)) vanishing along \( EV_L \).

### 6. The Essential Lemma

We present here a result of technical nature which is the substance of the structure about real analytic families of normal operators we wish to exhibit. Having in mind the material presented in Section 4 and Section 5 will provide hint about how to use Lemma 6.1 and Corollary 6.2 below.

Let \( \mathcal{U} \) be a connected open subset of \( \mathbb{R}^n \) and let \( J : \mathcal{U} \to H_2[X]^d \) be real analytic mapping with values in the quadratic polynomials over \( \mathbb{K}^n \), where \( d \) is some positive integer number.

Let \( J_\mathbf{x} \) be the ideal of \( \mathbb{K}[X] \) generated by the quadratic polynomials \( J_1(\mathbf{x})(X), \ldots, J_d(\mathbf{x})(X) \). Let \( J_{\mathbf{x},2} := J_\mathbf{x} \cap H_2[X] \) the \( \mathbb{K} \)-vector subspace of \( H_2[X] \) generated by \( J_\mathbf{x} \).

We further assume that the mapping \( J \) satisfies the following hypotheses:

For each \( \mathbf{x} \in \mathcal{U} \), the ideal \( J_\mathbf{x} \) is reduced and its vanishing locus is a bouquet of vector subspaces of \( \mathbb{K}^n \),
\[
E_\mathbf{x} := V(J_\mathbf{x}) = \bigcup_{i=1}^{s_\mathbf{x}} E_{i,\mathbf{x}}
\]
such that

(i) there exists \( s \geq 1 \) such that \( s = s_\mathbf{x} \) for all \( \mathbf{x} \in \mathcal{U} \).

(ii) For each \( \mathbf{x} \in \mathcal{U} \), the subspaces \( (E_{i,\mathbf{x}})^s \) are in direct sum.
The generators of \( J \) are the \( s \)-tuple of ordered dimension of the spaces \( E_{i,x} \) (after a possible permutation on the indices, which a-priori depends on \( x \)). Then the mapping \( x \to e_x \) is constant equal to \( e = (e_1, \ldots, e_s) \) with \( e_i \geq 1 \) and \( e_1 + \ldots + e_s = n \).

The subset of \( F := \mathcal{U} \times \mathbb{K}^n \) defined as
\[
E := \bigcup_{x \in \mathcal{U}} \{ x \} \times V(J_x) = \bigcup_{x \in \mathcal{U}} \{ x \} \times E_x
\]
is a real analytic subset of \( F \) and is called a bundle of bouquets of type \( e \).

Each \( E_x \) is a \( \mathbb{K} \)-cone, thus induces a projective sub-variety \( Z_x \) of \( \mathbb{P} \mathbb{K}^{n-1} \) consisting of the disjoint union of \( s \) linear subspaces \( \mathbb{P} \mathbb{K}^{e_i-1} = \mathbb{P} E_{i,x} \). Let be the following sub-bundle of \( \mathcal{P} F := \mathcal{U} \times \mathbb{P} \mathbb{K}^{n-1} \)
\[
Z := \bigcup_{x \in \mathcal{U}} \{ x \} \times Z_x \subset \mathcal{U} \times \mathbb{P} \mathbb{K}^{n-1}
\]
It is a real analytic subset of the trivial real analytic bundle \( \mathcal{P} F \).

The cone bundle over \( Z \) is
\[
E := \bigcup_{x \in \mathcal{U}} \{ x \} \times (E_x) = \bigcup_{x \in \mathcal{U}} \{ x \} \times \text{Cone}(Z_x) \subset F.
\]
It is a real analytic subset of \( F \). Let \( (E_x)^+ \) be the pointed bouquet \( E_x \setminus 0 \). The subset \( E^- := \bigcup_{x \in \mathcal{U}} \{ x \} \times (E_x)^+ \) of \( F^- \) is a the pointed cone bundle over \( Z \) and is a real analytic subset of \( F^- \).

We will refer later in these notes to the next result as the Essential Lemma, since everything we need to do is to produce a situation where it can be used.

**Lemma 6.1.** The subset \( E^- \) is a real analytic submanifold of \( F^- \).

**Proof.** By the results of Section 4 let \( d_j \) be the dimension of any of the vector-subspaces \( J_{x,2} \subset H_2[\mathcal{X}] \), for all \( x \in \mathcal{U} \). We will use some of the notations of Section 4 below.

Let \( y \) be a given point in \( \mathcal{U} \). The ideal \( J_y \) is generated by exactly \( d_j \) quadratic polynomials, which are \( \mathbb{K} \)-linear combinations of \( J_1(y), \ldots, J_{d_j}(y) \). Up to a linear change in \( \mathbb{K}^n \), we can assume that for each \( i = 1, \ldots, s \) we have
\[
E_{i,y} = \bigcap_{j \notin n^i} \{ u = (u_1, \ldots, u_n) : u_j = 0 \}.
\]
The generators of \( J_y \) can be chosen to be the polynomials \( c_{k,l}(y)(X) = X_kX_l \) for \( k, l \) as in 4.1 in such a way that each mapping \( x \to c_{k,l}(x)(X) \in H_2[\mathcal{X}] \) is real analytic over \( \mathcal{U} \) and for each \( x \) gives a quadratic polynomial of \( J_x \). Since these polynomials generate \( J_y \), the family \( (c_{k,l}(x))_{k,l} \) generates \( J_x \) for \( x \) nearby \( y \). Since the result we are looking for is local at \( y \) we can assume that these polynomials generates \( J_x \) for all \( x \in \mathcal{U} \). Thus, for \( u \in \mathbb{K}^n \), we can write
\[
c_{k,l}(x)(u) = u_ku_l + \sum_{1 \leq i < j \leq n} c_{i,j}^{k,l}(x)u_iu_j
\]
where each function \( x \to c_{i,j}^{k,l}(x) \) is real analytic and vanishes at \( y \). For any \( u \in E_y \), we have \( c_{k,l}(y)(u) = 0 \). Let \( 1 \leq i \leq s \) be given. For \( u \in E_{i,y} \), we do find
\[
\begin{align*}
\partial_u c_{k,l}(u; y) &= 0 & \text{if } k, l \notin n^i \\
\partial_u c_{k,l}(u; y) &= 0 & \text{if } k, l \in n^i \\
\partial_u c_{k,l}(u; y) &= 0 & \text{if } k \in n^i_e \text{ and } l \notin n^i_e \\
\partial_u c_{k,l}(u; y) &= 0 & \text{if } k \in n^i_e \text{ and } l \notin n^i_e
\end{align*}
\]
In particular we see that for each \( k \in n^i_e \)
\[
\omega_k(u; y) := \sum_{l \notin n^i_e} D c_{k,l}(u; y) = u_k u_l \sum_{l \notin n^i_e} \omega_k(u; y)
\]
Taking \( u \in E_y \setminus 0 \), there is there is \( 1 \leq i \leq s \) such that \( u \in E_{i,y} \) and there exists \( k \in n^i_e \) such that \( \omega_k(u; y) \neq 0 \).

Let us consider the real analytic mapping
\[
C^e : F \to \mathbb{K}^{d_j}
\]
\[
(x; u) \to (c_{k,l}(x)(u))_{k,l}
\]
so that \((C^n)^{-1}(\{0\}^{d_j}) = E\). Given \(1 \leq r \leq n\), let \(M_r\) be the ideal of \(\mathcal{O}_\mathcal{U}[X]\) generated by the \((n-r) \times (n-r)\) minors of the \(n \times d_j\) matrix \([\partial_{um} C^n] = [\partial_{um} c_k,l]_{m,(k,l)}\). Let \(V(M_r) \subset F\) be the vanishing locus of this ideal and let \(V(M_r)\) be \(V(M_r) \cap F\).

Let \(e = (e_1, \ldots, e_s)\) be the \(s\)-tuple of ordered dimensions of the bouquet-bundle \(E\). Let \(e \in \{e_1, \ldots, e_s\}\). Let us consider the following subsets

\[
E^{(e)}_x := \bigcup_{j: e_j = e} E_{x,j} \quad \text{and} \quad E^{(e)} := \bigcup_{x \in \mathcal{U}} \{x\} \times E^{(e)}_x.
\]

In particular we see that

\[
\bigcup_{k \leq e}(E^{(k)}) = E' \setminus V(M_e).
\]

We deduce that each \(E^{(r)}\) is a closed semi-analytic set of \(F\) and is, if not empty also a bouquet bundle of some type \((r, \ldots, r)\) where there are as many \(r\) as the number of subspace in a(ny) fiber \(E^{(r)}_x\). Thus it is of dimension \(u+r\) and of local dimension \(u+r\) at each of its point. The local computation above producing the \((n-r)\)-forms \(\omega_k\) shows that the projection \((E^{(r)})\) onto \(\mathcal{U}\) is a submersion, so that \(E^{(r)}\) is a real analytic submanifold of dimension \(u+r\) of \(F\).

Since \(E'\) is a real analytic submanifold of \(F\), so is \(Z = PE\). Let \(e \in \{e_1, \ldots, e_s\}\), and let \(Z^{(e)} := PE^{(e)},\) which is a real analytic submanifold of \(PF\) whose fibers consists of a (disjoint) union of \(d_e := \#\{1 \leq j \leq s : e_j = e\}\) linearly embedded projective spaces \(\mathbb{P}^{e-1}\). We deduce that, \(Z\) has exactly \(s\) connected components, namely

\[
P_e := P_1 \cup \ldots \cup P_s
\]

such that each \(P_i\) has dimension \(u + e_i - 1\). Each \(P_i\) is a real analytic submanifold of \(PF\) such that for each \(x \in \mathcal{U}\), the fiber \(P_{i,x}\) is the projective space of one of the subspaces of the bouquet of \(V(J_x) \subset F_x\) of dimension \(e_i\). Thus we obtain the following

**Corollary 6.2.** For each \(i\), the projection \(PF \rightarrow \mathcal{U}\) induces a submersion \(P_i \rightarrow \mathcal{U}\). More precisely each \(P_i\) is a connected real analytic sub-bundle of \(PF\), whose fiber is \(\mathbb{P}^{e_i-1} \subset \mathbb{P}^{n-1}\). In other words \(\text{Cone}(P_i) \subset F\), the cone-bundle over \(P_i\), is a real analytic vector sub-bundle of \(F\) over \(\mathcal{U}\).

**Proof.** This is just a consequence of the fact that the projection of \(Z\) onto \(\mathcal{U}\) is a submersion as a result of the local computations done in the proof of Lemma [6.1] since \(E\) is the cone bundle over \(Z\). \(\square\)

### 7. Reduced model

Let \(F := \mathcal{U} \times \mathbb{K}^n\) where \(\mathcal{U}\) is a connected open subset of \(\mathbb{R}^n\). We assume that \(F\) is equipped with the Euclidean/Hermitian structure in each of its fiber. Let \(\text{Hom}(F,F) = \mathcal{U} \times \text{Hom}(\mathbb{K}^n, \mathbb{K}^n)\) be the \(\mathbb{K}\)-vector bundle over \(\mathcal{U}\) of \(\mathbb{K}\)-endomorphisms of \(F\).

The tangent bundle \(TF\) decomposes as the direct sum

\[
T^h F \oplus T^v F
\]

of \(T^h F\), the horizontal vector sub-bundle, with \(T^v F\), the vertical vector sub-bundle, both being real analytic \(\mathbb{K}\)-vector bundles over \(F\), where, for \(p = (y,w)\), the respective fibers are defined as \(T^h_p F := T_y \mathcal{U}\), and \(T^v_p F := T_w F_y\). Of course we also find that

\[
T\hat{F} = T^h F \oplus T^v F.
\]

A normal operator over \(F\) is a real analytic section of \(L : \mathcal{U} \rightarrow \text{Hom}(F,F)\) which is normal in every fiber of \(F\). We will consider, without change of notation, the restriction of \(L\) to the pointed vector bundle \(\hat{F}\).

Let \(V_L\) be the zero locus of \(L\), that is the set of points \(x \in \mathcal{U}\) where \(L(x)\) it the null operator. Given a basis of \(\mathbb{K}^n\), let \(I_L\) be the \(\mathcal{O}_\mathcal{U}\)-ideal of coefficients of the matrix \(L\) in the given basis. Since it is defined by global sections it is a coherent \(\mathcal{O}_\mathcal{U}\)-ideal. If \(V_L\) were a simple normal crossing divisor and \(I_L\) were principal we could
(locally) divide $L$ by the generator of $I_L$ in order to get a new normal operator which would vanish nowhere. For the time being we will work under the following

**Hypothesis 0.** Assume that $L$ is normal if $K = \mathbb{C}$ and symmetric if $K = \mathbb{R}$.

In order to postpone the use of blowings-up we demand

**Hypothesis I.** Assume that $L$ is nowhere the null operator and is not a multiple of the identity.

As previously, we define the real analytic mapping of $K$-vector bundles over $U$

$$Q_L^i : F \to \wedge^2 F$$

$$(x; v) \to (x; Q_L(x; v) := L(x)v \wedge v)$$

We will still write $Q_L^i$ and $Q_L$ for their restriction to $\hat{F}$.

To such a normal operator we associates the pointed eigen-bouquet bundle $EV_L$ defined as

$$EV_L := \{(x; v) \in \hat{F} : Q_L(x; v) = 0\}.$$ Each fiber $EV_{L,y}$ over $y \in U$ is the (pointed) eigen-bouquet of $L(y)$. Let $s(y)$ be the number of distinct eigen-spaces of $L(y)$. Let $s_L := \max_U s(y) \leq n$. Hypothesis I implies that $s_L \geq 2$. Let

$$D_L := \{y \in U : s(y) < s_L\}$$

which is the discriminant locus of the characteristic polynomial of $L$, thus a closed real analytic subset of $U$ of positive codimension if not empty. The function $s(y)$ is then Zariski-analytic semi-continuous, that is constant and equal to $s_L$ over $U \setminus D_L$.

Let $(U^i)_{i \in I_L}$ be the connected components of $U \setminus D_L$, which are locally finite. Let $i \in I_L$ be given. For any $x \in U^i$ the operator $L(x)$ has exactly $s_L$ eigen-spaces $E_{j,x}^i$ for $j = 1, \ldots, s_L$.

We recall the following well known fact:

**Proposition 7.1.** For each $i \in I_L$ and each $j = 1, \ldots, s_L$ the subset $E_j^i := \cup_{x \in U^i} E_{j,x}^i$ of $F^i := F|U^i$ is a real analytic $K$-vector bundle over $U^i$.

**Proof.** Assume that $i$ is fixed. Let $y$ be a point of $U^i$.

Let $Q_{i,j}$ be the components of the mapping $Q_L$. By Lemma 5.2 we know that the quadratic polynomials $Q_{i,j}(y)$ generate the ideal $J_y$ of the polynomials vanishing over $EV_{L,y} \cup 0 \subset F_y$. Thus $J_y$ is reduced and we can apply The Essential Lemma 6.1 and Corollary 6.2 to conclude. □

**Remark 7.2.** The proof of Proposition 7.1 did not use the eigen-values.

Since $F$ is the trivial vector bundle $U \times \mathbb{K}^n$, we also find that

$$S^2 F^\vee = U \times H_2[X] \text{ and } SF^\vee = U \times \mathbb{K}[X].$$

The ordered $s_L$-tuple of dimensions of the eigen-spaces of $L$ is constant outside $D_L$. Let us denotes $e_L \in S^n_{s_L}$ this ordered $s_L$-tuple of dimensions.

For $y \in U$, let

$$\Sigma_L^y := \{(x, u, P) \in F \oplus S^2 F^\vee : L(x)u \wedge u = 0 \text{ and } P(x, u) = 0\},$$

the bundle of quadratic polynomials over $F$ vanishing along the (pointed) eigen-bouquet bundle $EV_L$. Let $\pi_{S^2 F^\vee}$ be the projection of $F \oplus S^2 F^\vee$ over $S^2 F^\vee$. Let $J_y$ be the vector subspace of $S^2 F^\vee$ of the quadratic polynomials lying in $J_y$, as was done in Section 4. Let $d(y) = d_0(J_y) \geq 2$ be the dimension of $J_y$ as in Section 4. Following Lemma 4.1 $d(y)$ is the number of minimal generators of $J_y$. 

"
At each point \( y \) outside of \( D_L \), the vector subspace \( J_{y,2} \subset (S^2 F^\vee)_y = H_2[X] \simeq \mathbb{K}^{n(n+1)/2} \) is of constant dimension \( d_L \leq \frac{n(n-1)}{2} \) since its zero locus is the union of the eigen-spaces of \( L(y) \) and the ordered \( s_L \)-tuples of the dimensions of these eigen-spaces is constant over \( U \setminus D_L \) and equal to \( e_L \). At each point \( y \) of \( D_L \) we do find that \( d(y) < d_L \) since there are strictly fewer than \( s_L \) eigen-spaces at \( y \) and they satisfy the hypotheses of Corollary 4.2.

Once an orthonormal basis of \( \mathbb{K}^n \) is given, we obtain an explicit expression \( Q \) of the mapping \( Q_L \). Since the mappings \( y \to Q_{i,j}(y) \) are real analytic sections of \( S^2 F^\vee \), for any pair \( 1 \leq i < j \leq n \), let \( F_L \) be the \( \mathcal{O}_U \)-ideal sheaf generated by the \( d_L \times d_L \) minors of the \( \frac{n(n-1)}{2} \times \frac{n(n+1)}{2} \)-matrix formed by the vectors \( Q_{i,j}(y) \in S^2 F^\vee \). In other words, it is the ideal of the coefficients of the \( \mathcal{O}_U \)-module \( A_L \) generated by \( \wedge_{k=1}^{d_L} Q_{i_k,j_k} \) taken over all possible \( d_L \)-tuples of vectors \( Q_{i,j}(y) \).

At any point \( y \) of \( D_L \), the ideal \( F_L \) vanishes. Another way to look at this ideal is by using the Plücker embedding of

\[
\text{pl} : G(d_L, S^2 F^\vee) \to \mathbb{K}P^{N_L}
\]

for some \( N_L \), depending only on \( d_L \) and \( n \). The composition of the mapping \( A : x \to (Q_{i,j}(x))_{i < j} \in S^2 F^\vee 
\)

with \( \text{pl} \) is not well defined at points of \( D_L \) since all \( d_L \times d_L \)-minors made of the vectors \( Q_{i,j} \) vanish along \( D_L \). Nevertheless we have a well defined real analytic mapping

\[
U \setminus D_L \to G(d_L, S^2 F^\vee) \setminus U \setminus D_L.
\]

Thus up to resolving the ideal \( F_L \) we assume for the time being the following.

**Hypothesis II.** The discriminant locus \( D_L \) is a simple normal crossing divisor and the ideal sheaf \( F_L \) is principal and monomial in \( D_L \).

Let \( A'_L \) be the \( \mathcal{O}_U \)-module \( F_L^{-1} A_L \). It is a locally \( \mathcal{O}_U \)-free module of rank \( d_L \). In particular the mapping \( \text{pl} \circ A \) extends analytically to the whole \( U \) since we can divide by the local generator of \( F_L \) at every point. Using the (trivial) real analytic universal bundle of \( G(d_L, S^2 F^\vee) \), to every point \( y \in U \) corresponds a \( d_L \)-dimensional subspace \( J_{y,2}' \) of \( S^2 F^\vee = H_2[X] \) given by the extension to \( U \) of the mapping \( \text{pl} \circ A \). For a point \( y \notin D_L \) we have \( J_{y,2} = J_{y,2}' \). Let \( A_L \) be the (non-trivial) real analytic vector sub-bundle of \( S^2 F^\vee \) obtained thanks to the mapping \( \text{pl} \circ A \).

Let \( x \to (A'_L(x))_{r=1,\ldots,d_L} \) be a local real analytic frame at \( y \in U \) of the vector sub-bundle \( A_L \subset S^2 F^\vee 
\)

lifting the mapping \( x \to \text{pl} \circ A(x) \). Suppose it is defined over a neighbourhood \( V \) of \( y \). Let \( Q'(x; v) \) be the quadratic polynomial over \( F_X \) defined by the vector \( A'_L(x) \). Thus

\[
\cap \{ (x; v) \in F_{V} | Q'(x; v) = 0 \} \subset EV_L \cap \pi^{-1}(V)
\]

and if \( V \cap D_L \) is empty then

\[
\cap \{ (x; v) \in F_{V} | Q'(x; v) = 0 \} = EV_L \cap \pi^{-1}(V)
\]

Let \( J'_Y \) be the quadratic ideal of \( S^2 F^\vee = \mathbb{K}[X] \) generated by the quadratic polynomials \( Q'_L(y) \), in other words generated by \( J'_{Y,2} \). Let \( (x_k)_{k} \) be a sequence of \( U \setminus D_L \) converging to \( y \in D_L \). Let \( E_{1,k}, \ldots, E_{s_L,k} \) be the eigen-spaces of \( L(x_k) \). Up to taking a sub-sequence, we can assume that each \( E_{i,k} \) converges (in the right Grassmannian) to \( E_i \subset F_Y \). In particular the subspaces \( E_{1}, \ldots, E_{s_L} \) are in orthogonal direct sum and the restriction of \( L(y) \) to any \( E_i \) is a multiple of the identity mapping restricted to \( E_i \). By a continuity argument all the functions \( Q'_L \) must vanish along the union \( E_Y := \bigcup_{i} E_i \), in other words the vanishing locus of \( J'_Y \) contains \( E_Y \). The ideal \( K_Y \) of quadratic polynomials vanishing along \( E_Y \) generates (see Section [B]) a vector subspace \( K'_{Y,2} \) of \( S^2 F^\vee \) which following Lemma 4.1 is of dimension \( d_L \), since the \( s_L \)-tuple of the ordered dimensions of \( E_1, \ldots, E_{s_L} \) is obviously \( e_L \). The reduced ideal \( K_Y \) contains \( J'_Y \) and by Lemma 4.3 we deduce that \( J'_Y = K_Y \).

The following result although not necessary in order to have a complete proof, is of interest since it shows that at every point of the discriminant locus, in the current post-resolution of singularities setting we are working with, there exists a unique limit eigen-bouquet. So the wish expressed in the introduction is satisfied.
Lemma 7.3. Let \( y \) be a point of \( D_L \).

(i) The zero locus of \( J'_y \) consists of the union of pairwise orthogonal subspaces \( G_1, \ldots, G_{s_L} \) which also form a direct sum.

(ii) Let \( U_k \) be a connected component of \( U \setminus D_L \) whose closure contains \( y \). Let \( E^i_{j,x} \) be the eigen-space of \( L(x) \mid U_k \). Let \( (x_k)_k \) be any sequence of points of \( U_k \) converging to \( y \). Assume that each \( E^i_{j,x_k} \) converges to \( P_j \).

There exists \( \varepsilon \) a permutation of \( \{1, \ldots, s_L\} \), independent of the sequence \( (x_k)_k \) taken, such that for the set of corresponding limit \( (P_j)_j \) at \( y \) we have \( P_j = G_{\varepsilon(j)} \).

In other words the real analytic vector bundles \( E^i_j \) for \( j = 1, \ldots, s_L \) over \( U_k \) extends as vector bundles of constant rank over \( \text{clos}(U_k) \).

Proof. Point (i) comes from the fact \( J'_y = K_y \).

For any sequence like in (ii), the zero locus of \( J'_y \) must vanish along \( \bigcup_{i=1}^{s_L} P_i \). The uniqueness of the permutation \( \varepsilon \) comes from the pairwise orthogonality of the vector sub-bundles \( (E^i_j)_{j=1}^{s_L} \).

At this point we have produced a real analytic vector sub-bundle of \( S^2 F^\vee \) of dimension \( d_L \) lifting the mapping \( p \circ A \). We have thus produced a real analytic family of quadratic ideals \( (J'_y)_{y \in U} \) of \( S^2 F^\vee \), that is a real analytic section \( U \to (S^2 F^\vee)^{d_L} \) with image in \( (S^2 F^\vee)^{d_L} \). For each \( y \), the zero locus \( EV'_j \cup \{0\} \subset \tilde{F}_y \cup \{0\} \) of \( J'_y \) in \( F_y = \mathbb{K}^n \) consists of a bouquet of vector subspaces of \( F_y \), forming an orthogonal direct sum, each of which being contained in an eigen-space of \( L(y) \). We define

\[
EV'_L := \bigcup_{y \in U} \bigcup_{i=1}^{s_L} E^i_{j,y} \subset EV_L \subset \tilde{F}
\]

the "bundle-d" union taken over \( U \) of the (pointed) vanishing loci of the ideals \( J'_y \). We call \( EV'_L \) the reduced (pointed) eigen-bouquet bundle of \( L \). The subset \( EV'_L \) is a closed real analytic subset of \( EV_L \) thus of \( \tilde{F} \) as can be checked at every point of \( EV_L \), using locally a real analytic normal frame for lifting in \( S^2 F^\vee \) the mapping \( p \circ A \). More precisely we have the following

Lemma 7.4. The subset \( EV'_L \) is real analytic submanifold of \( \tilde{F} \).

Proof. The goal of obtaining the ideals \( (J'_x)_{x \in U} \) was exactly to be in a situation to apply the Essential Lemma 6.1. Since we are satisfying such hypotheses we get the announced result.

Let us recall, in the current context, the preliminaries to the statement of Corollary 6.2. Since \( EV'_L \) is fiber-wise a (pointed) bouquet of subspaces of \( F \), it defines a subset

\[
P'_L := PEV'_L \subset PF.
\]

It is a real analytic subset and by Lemma 7.4 it is also a real analytic submanifold of \( P \) consisting exactly of \( s_L \) connected components, namely

\[
P'_L := \mathcal{P}_1 \sqcup \ldots \sqcup \mathcal{P}_{s_L}
\]

such that each \( \mathcal{P}_i \) has dimension \( u + e_i - 1 \), where \( (e_1, \ldots, e_{s_L}) = e_L \). Each \( \mathcal{P}_i \) is a real analytic submanifold of \( PF \) such that for each \( x \in U \), the fiber \( \mathcal{P}_{i,x} \) is the projective space of one of the bouquet of subspaces of \( V(J_x) \subset F_x \) of dimension \( e_i \), and is contained in an eigen-space of \( L(x) \). Concluding the proof of our main result under the conditions of this section is a consequence of Corollary 6.2.

Corollary 7.5. (i) For each \( i \), the projection \( PF \to U \) induces a submersion \( \mathcal{P}_i \to U \). More precisely each \( \mathcal{P}_i \) is a connected real analytic sub-bundle of \( PF \), whose fiber is \( P\mathbb{K}^{e_i-1} \subset P\mathbb{K}^{n-1} \). In other words \( \mathcal{E}_i := \text{Cone}(\mathcal{P}_i ) \subset F \), the cone-bundle over \( \mathcal{P}_i \), is a real analytic vector sub-bundle of \( F \) over \( U \).

(ii) Under the hypotheses 0 and I and II, any point \( y \) admits a neighbourhood \( \mathcal{W} \) over which exist \( n \) real analytic sections of \( F \) which form an orthonormal frame of \( F \) at every \( x \) of \( \mathcal{W} \) consisting of eigen-vectors of \( L \).

(iii) The eigen-values can be locally chosen real analytic.
Proof. Point (i) is the statement of Corollary 8.2 which can be applied here.

Points (ii) and (iii) becomes obvious since the real analytic vector sub-bundles $\mathcal{E}_i$ are pairwise orthogonal and in direct sum and contained point-wise in eigen-spaces of $L$.

\square

8. General Model and Hypothesis 0

We are pursuing what was done in Section 7 but dropping hypotheses (I) and (II). Our work here will consist mostly, by means of blowings-up, to recreate a situation where we can work under these dropped hypotheses.

We fix once for all an orthonormal basis $B = (b_1, \ldots, b_n)$ of $\mathbb{K}^n$. We recall that for $i < j$ the notations $b_{(i,j)}$ stands for the 2-vector $b_i \wedge b_j \in \wedge^2 \mathbb{K}^n$.

The normal operators $L(x)$ are given by complex normal (or real symmetric) matrices $M(x)$, in the basis $B$, whose coefficients are real analytic functions over $\mathcal{U}$, thus global sections of $\mathcal{O}_\mathcal{U}$. In order to have some work to do here we can assume that

**Hypothesis.** The operator $L$ is not a multiple of the identity.

The ideal $\mathcal{I}_L$ of the coefficients of $L(x)$ is generated by finitely many global sections, thus is $\mathcal{O}_\mathcal{U}$-coherent. If $\chi_L(x; T)$ denotes the characteristic polynomial of the normal matrix $M(x)$, then its discriminant locus (as a subset of $\mathcal{U}$) is just $D_L$. Under Hypothesis 0, we have $\mathcal{V}_L \subset D_L$ is of codimension one or more in $\mathcal{U}$.

Let as before be $Q : F \to \wedge^2 F$ defined as $Q(x; v) = M(x)v \wedge v$. Since $Q = \sum_{i < j} Q_{i,j} e_{i,j}$, the coefficients $Q_{i,j}$ are real analytic quadratic forms over $\mathcal{U}$.

Let again $\mathcal{A}_L$ be the $\mathcal{O}_\mathcal{U}$-sub-module of $\wedge^d L \wedge^2 F^\vee = \mathcal{U} \times \bigwedge^d \mathcal{H}^2 [X]$ generated by the $\wedge^d L \wedge^2 H_2 [X]$. It is coherent since the $Q_{i,j}$ are global sections of $S^2 F^\vee$ (when the latter is identified canonically with $\mathcal{U} \times \mathcal{K}^2 \mathbb{K}^2$). In particular its ideal of coefficients $\mathcal{F}_L$, generated by the $d_L \times d_L$-minors of the $\frac{n(n-1)}{2}$ matrices of the vector $Q_{i,j}$, also $\mathcal{O}_\mathcal{U}$-coherent with co-support exactly $D_L$.

**Lemma 8.1.** There exists $\sigma : (\widetilde{\mathcal{U}}, \widetilde{\mathcal{E}}) \to (\mathcal{U}, D_L)$ a locally finite composition of geometrically admissible blowings-up such that

(i) the $\mathcal{O}_{\widetilde{\mathcal{U}}}$-ideal $\sigma^* \mathcal{F}_L$ is principal and monomial in the simple normal crossing divisor $\widetilde{\mathcal{E}}$.

(ii) The $\mathcal{O}_{\widetilde{\mathcal{U}}}$-module $\mathcal{A}'_L := (\sigma^* \mathcal{F}_L)^{-1} \sigma^* \mathcal{A}_L$ is locally free of rank $d_L$ with empty co-support.

**Proof.** The first point is a direct application of the embedded resolution of ideals [Hir, BM].

The module $\mathcal{A}'_L$ has empty co-support by definition of $\mathcal{A}_L$ and $\mathcal{F}_L$. It is locally free as was already seen in Section 7.

We can now state the main result given the context of this section.

**Theorem 8.2.** Let $L : \mathcal{U} \to F = \mathcal{U} \times \mathbb{K}^n$ be a real analytic normal operators (real symmetric if $\mathbb{K} = \mathbb{R}$), where $\mathcal{U}$ is a connected subset of $\mathbb{R}^n$. Assume that $L$ is not a multiple of identity.

There exists a locally finite composition of geometrically admissible blowings-up $\sigma : (\widetilde{\mathcal{U}}, \widetilde{\mathcal{E}}) \to (\mathcal{U}, D_L)$ such that for any $\tilde{y} \in \widetilde{\mathcal{U}}$, there exists $\mathcal{V}$ a neighbourhood of $\tilde{y}$ such that there exists real analytic sections of $(\sigma^* F)|_\mathcal{V}$ which form an orthonormal basis of $L(\sigma(x))$ for each $x \in \mathcal{V}$.

**Proof.** Let $\sigma$ be that of Lemma 8.1.

From here on we proceed as in Section 7 since all we need to do from now is literally of local nature, thus was done there, and will work as well here, since $\mathcal{A}'_L$ has the exact same properties. In other words, by Corollary 7.5, for each $\tilde{y} \in \widetilde{\mathcal{U}}$ there exists a neighbourhood $\mathcal{V}$ of $\tilde{y}$ such that there exists $R_1, \ldots, R_{s_L}$ real analytic vector sub-bundles of $\sigma^* F|_\mathcal{V}$ which are contained in $\sigma^* E \mathcal{V}_L$ and such that they form an orthogonal direct sum of $\sigma^* F|_\mathcal{V}$. Shrinking $\mathcal{V}$ if necessary so that it is a chart domain of each $R_i$, those vector bundles are all real analytically trivial over $\mathcal{V}$, thus the statement.
Now we deduce the following

**Corollary 8.3.** For any $\tilde{y} \in \tilde{U}$, there exists $\tilde{V}$ a neighbourhood of $\tilde{y}$ such that the eigen-values of $\tilde{x} \to L(\sigma(\tilde{x}))$ are real analytic.

**Proof.** Just evaluate the mapping $(\tilde{x}, v) \to \frac{L(\sigma(\tilde{x})).v}{|v|}$ over each $R_i \setminus 0$. □

9. **General Case**

Let $N$ be a connected real analytic manifold of dimension $u \geq 1$.

Let $F$ be a real analytic $\mathbb{K}$-vector bundle over $N$ equipped with a real analytic fiber metric structure, that is endowed with a real analytic section $N \to S^2 F^\vee$, the 2nd symmetric power of $F$, which is positive definite ($\mathbb{K} = \mathbb{R}$) or Hermitian ($\mathbb{K} = \mathbb{C}$) over every fiber of $F$. This means the local model of $F$ is $\mathbb{R}^u \times \mathbb{K}^n$ but with transitions taken into the orthogonal group $O(\mathbb{R}^n)$ or the unitary group $U(\mathbb{C}^n)$.

Let $L$ be a real analytic normal operator $F \to F$, which we suppose symmetric if $\mathbb{K} = \mathbb{R}$. Since $N$ is connected, the set of points where the number of distinct eigen-values of $L$ is not locally maximal is a real analytic subset $D_L$ of $N$ of positive codimension. Let $s_L$ be this maximal number of distinct eigen-values.

Let $U$ be an open subset of $N$ over which $F$ is trivial. We can then choose an orthonormal basis $B^U$, and then do as in Section 8 for $F^U$. Let $L_{1U}$ be the restriction of $L$ to $F^U$. Let $Q_U$ and $Q_{i,j}^U$ be $Q$ and $Q_{i,j}$ of Section 7 for $I_{1U}$. In particular we can construct a coherent $O_U$-module $A_U := A_{I_{1U}}$ which is locally free of rank $d_U$ at any point outside $D_U := D_L \cap U$. Note that all these $d_U$ are equal to $d_L \geq 1$ since $D_L$ is nowhere dense in $N$. Let $F_U$ be the coherent $O_U$-ideal of coefficients of $A_U$, which we know to be defined globally, that is generated by the $d_L \times d_L$-minors of the matrix of the $Q_{i,j}^U$.

Let $W_1$ and $W_2$ be two open subsets of $N$ intersecting each other and over each of which exist respectively a basis $B_1$ and a basis $B_2$. Taking trivializations of $F$ over $W_1$ and over $W_2$ we see that $A_{I_{W_1}}|W_1 \cap W_2$ and $A_{I_{W_2}}|W_1 \cap W_2$ are isomorphic. Let $P$ be the orthogonal/unitary matrix passing from the basis over $W_1$ to the basis over $W_2$. Thus at any point $y \in W_1 \cap W_2$, we check that $P$ induces a linear automorphism $P_2[n]$ over $H_2[X]$ such that the vector subspace of $S^2 F_\tilde{x}$ spanned by $(Q_{i,j}^{W_1}(y))_{i<j}$ is mapped onto the vector subspace spanned by $(Q_{i,j}^{W_2}(y))_{i<j}$. In particular this implies that we can extend each $A_{I_{W_i}}$ as a $O_{W_i} \cup W_2$-module, which is still coherent since locally free of rank $d_i$ outside $D_{W_i} \cup D_{W_2}$. Thus we produce a coherent $O_N$-module $A_L$ locally free of rank $d_L$ outside $D_L$ and vanishing only over $D_L$. Thus its coefficient ideal $F_L$ is also $O_N$-coherent locally generated by the minors of the $d_L \times d_L$-matrix of the vectors $Q_{i,j}^U$.

And we arrive at the main result of the paper:

**Theorem 9.1.** Let $L : F \to F$ be a real analytic normal operator over a real analytic $\mathbb{K}$-vector bundle $F$ of finite rank over a real analytic and connected manifold $N$, and equipped with a real analytic fiber Hermitian (Euclidean if $\mathbb{K} = \mathbb{R}$) metric.

(i) There exists a locally finite composition of geometrically admissible blowings-up $\sigma : (\tilde{N}, \tilde{E}) \to (N, D_L)$ such that for any $\tilde{y} \in \tilde{N}$, there exists $\tilde{V}$ a neighbourhood of $\tilde{y}$ and real analytic vector sub-bundles $R_1, \ldots, R_s$ of $\sigma^* F|\tilde{V}$ such that

(a) they are pair-wise orthogonal and everywhere in direct sum;

(b) The restriction of $\sigma^* L$ to each $R_i$ is reduced: either it is a multiple of identity, or $R_i$ is real of real rank 2, and the restriction of $\sigma^* L$ is a fiber-wise a similitude.

(ii) For any $\tilde{y} \in \tilde{N}$, there exists $\tilde{V}$ a neighbourhood of $\tilde{y}$ such that the eigen-values of $\tilde{x} \to L(\sigma(\tilde{x}))$ are real analytic.

**Proof.** The proof of the real and non symmetric case will be dealt with in Section 10.

First if $s_L = 1$, that is $L$ is a multiple of the identity operator then there is nothing to do.

Assume that $s_L \geq 2$, so that $D_L$ is not empty.
Assume also that $L$ is complex normal or real symmetric.

The ideal $\mathcal{F}_L$ is $\mathcal{O}_N$-coherent, there exists a locally finite composition of geometrically admissible blowings-up $\sigma : (\tilde{U}, \tilde{E}) \rightarrow (N, D_L)$ such that the ideal for any $\tilde{y} \in \tilde{U}$, there exists $\tilde{V}$ a neighbourhood of $\tilde{y}$ such that there exists real analytic sections of $\langle \sigma^*(F) \rangle$ which form an orthonormal basis of $L(\sigma(\tilde{x}))$ for each $\tilde{x} \in \tilde{V}$.

And the rest is as we did in Section 7 and the proof of Theorem 8.2.

As for the second point, it also works as in Corollary 8.3

Remark 9.2. The reader would have noticed that we never uses the condition of monomialization of the ideals we desingularize. We just need to have there weak transforms principal.

The next remark is inspired by our original motivation, sketched in the last Section.

Remark 9.3. What we did here for a normal operator of Remark 9.3. we desingularize. We just need to have there weak transforms principal.

The reader would have noticed that we never uses the condition of monomialization of the ideals we desingularize.

Proof of Theorem 10.1. Let $L : F \rightarrow F$ be a real analytic normal operator over a real analytic and connected manifold $N$, and equipped with a real analytic fiber Euclidean structure.

(i) There exists a locally finite composition of geometrically admissible blowings-up $\sigma : (\tilde{N}, \tilde{E}) \rightarrow (U, D_L)$ such that for any $\tilde{y} \in \tilde{N}$, there exist $\tilde{V}$ a neighbourhood of $\tilde{y}$ and real analytic sections of $\sigma^*F$ which form an orthonormal basis of characteristic vectors of $L(\sigma(x))$ for each $x \in \tilde{V}$.

(ii) For any $\tilde{y} \in \tilde{N}$, there exists $\tilde{V}$ a neighbourhood of $\tilde{y}$ such that the eigen-values of $\tilde{x} \rightarrow L(\sigma(\tilde{x}))$ are real analytic.

Proof of Theorem 10.1. As introduced at the end of Section 3, let us write

$$L = A + B,$$

where $A := \frac{L + L^*}{2}$ and $B = \frac{L - L^*}{2}$.

It is sufficient to have the result for anti-symmetric operators $F \rightarrow F$, since it is already true for symmetric ones.

Assume the normal operator $L : F \rightarrow F$ is anti-symmetric, that is $L = B$. The real analytic vector bundle $F \oplus F$ is equipped with the product of the scalar product of each embedding $O \oplus F$ and $F \oplus O$, where $O$ is the zero-section of $F$. We construct, as in Section 3, a symmetric operator over $F \oplus F$ from $L$, namely

$$B_2 : F \oplus F \rightarrow F \oplus F$$

$$u \oplus v \rightarrow -Bv \oplus Bu$$

Its kernel is the direct sum $\text{ker} L \oplus \text{ker} L$. In order to avoid introducing new notations we can already assume that $B_2$ satisfies Theorem 10.1 for symmetric operator.

Let $y$ be any point of $N$. There exists a neighbourhood $V$ of $y$ such that there exists a real analytic section

$$f \in V \rightarrow \text{Frame}(F \oplus F)$$

such that for each $x$ in $V$ the frame $f(x)$ of $F_x \oplus F_x$ is orthonormal and consists of eigen-vectors of $B_2(x)$. We can always assume that $F$ is trivial over $V$.

There exists a (maximal) real analytic vector sub-bundle $R_0$ of $(F \oplus F)|_V$ consisting only of eigen-vector of $B_2$ associated to the eigen-value $b \neq 0$ (outside $D_L$). Thus $R_0$ is generated by $f_1, \ldots, f_{2d}$ sections of $F \oplus F$ over $V$ and vectors of the frame $f$. Let $f_j = u_j \oplus v_j$ for pairs of orthogonal sections $u_j, v_j$ of $F$ and let $f'_j := u_j \oplus (-v_j)$. Following the classical material presented in Section 3, The sections $f'_1, \ldots, f'_{2d}$ form an orthonormal basis of $R_{-b}$, the eigen-space bundle of $B_2$ corresponding to $-b$. We observe that $u_j$ and $v_j$ never
vanish over \( V \). We can assume that \( f_1', \ldots, f_d' \) are also in the frame \( f \). By Lemma 3.1, the section \((-v_1) \oplus u_1\) lies in \( R_b \) and is orthogonal to \( f_1 \). So we can assume that \( f_b = (-v_1) \oplus u_1 \) is a vector of the frame \( f \).

Let \( f_b \) be the free family of vectors of the frame \( f \) generating \( R_b \), and \( f_{-b} \) be the free family of \( f \) generating \( R_{-b} \) (orthogonal to \( f_b \)). Thus \( f_b = \{f_1', f_2', f_3'\} \) where \( f_b' \) is orthonormal and orthogonal to \( f_1, f_2, \) and \( f_{-b} = \{f_1', f_2', f_3'\} \) where \( f_{-b}' \) is orthonormal and orthogonal to \( f_1', f_2' \). Let \( R_1^b \) (resp. \( R_1^{−b} \)) be the real analytic subbundle of \( R_b \) (resp. \( R_{−b} \)) generated by \( f_1^b \) (resp. \( f_{−1}^b \)). Note that the plane \( \mathbb{R}f_1 + \mathbb{R}f_2 = \mathbb{R}f_1' + \mathbb{R}f_2' \subset F \oplus F \) and \( R_1^b, R_1^{−b} \) are both invariant by \( B_2 \).

A descending induction process produces a real orthogonal and maximal free family \( u_1, v_1, \ldots, u_d, v_d \) of \( F \), such that each plane \( C_{b,j} := \mathbb{R}u_j + \mathbb{R}v_j \subset V \) is an ARCP of \( B \) and the eigen-vector space of \( B^2 \) associated to \(-b^2\) is the direct sum of the \( C_{b,j} \) \( j = 1, \ldots, 2 \).

And this concludes point (i) in the anti-symmetric case.

The eigen-values of \( B \) are exactly the \( \pm \sqrt{-1}b_i \), which are already analytic. \( \square \)

11. COMMENTS

- A key point in our result is the category of functions used. Real analytic objects over a connected domain vanish in positive codimension or everywhere. Thus, outside the discriminant locus of a given normal operator, the number of real eigen-values and the number of non-real eigen-values, and their respective multiplicities are constant. Our proof should work - within the known variations of such a modification - in the local case (trivial vector bundle) if we substitute real analyticity for any quasi-analytic regularity admitting a resolution of singularities or just a local uniformization.

- The origin of our interest in the problem treated here is explained below.

To keep it simple, let \( X \) be an analytic subset of \( \mathbb{K}^n \) with a non-empty singular part \( Y \). Suppose \( \mathbb{K}^n \) is equipped with \( g \) the canonical Euclidean/Hermitian structure.

Let \( T X \) be the closure of \( T(X \setminus Y) \) taken into \( T\mathbb{K}^n|X \): At points of \( Y \) are added to \( T(X \setminus Y) \), all the limits of tangent spaces of \( X \) coming from the regular part \( X \setminus Y \). The restriction \( g_X \) of \( g \) to \( T X \) is well defined, but is a real analytic Euclidean/Hermitian only over \( X \setminus Y \). It is possible to tailor a resolution of singularities of \( X \), given by a surjective \( \mathbb{K} \)-analytic mapping \( \sigma : (\tilde{X}, \tilde{E}) \rightarrow (X, Y) \subset \mathbb{K}^n \), realizing an isomorphism \( \tilde{X} \setminus \tilde{E} \) onto \( X \setminus Y \), so that \( \tilde{X} \) is \( \mathbb{K} \)-analytic manifold and \( \tilde{E} \) is a simple normal crossing divisor, such that \( \tilde{g} := \sigma^*(g_X) \) extends real analytically as as section of \( S^2 T^* \tilde{X} \). (In the previous sections we would have worked over \( \sigma^*(T(X \setminus Y)) \).

Since \( \sigma \) is not a submersion at any point of \( \tilde{E} \), the 2-symmetric tensor \( \tilde{g} \) is just semi-definite along \( T \tilde{X} | \tilde{E} \). An important issue for application to Analysis and Topology over complex projective varieties is to describe \( \tilde{g} \) (see \[HsPa\] in the case of complex normal projective surfaces). Nevertheless such a description is possible in (real and complex) dimensions 2 and 3 performing further purposely-labeled blowings-up (\[BBGM\]).

Within the same context, let \( \kappa \) be a any metric over \( \mathbb{K}^n \), that is a \( \mathbb{K} \)-analytic section of \( S^2 T^* \mathbb{K}^n \). The restriction of \( \kappa \) to \( X \), namely \( \kappa_X := \kappa|TX \), is a well defined object. After resolving \( X \) it is possible to have that \( \sigma^*(\kappa_X) \) and \( \sigma^*(g_X) \) both extend as sections of \( S^2 T^* \tilde{X} \), respectively as \( \tilde{\kappa} \) and \( \tilde{g} \). We can ask whether it is possible to reduce locally \( \tilde{\kappa} \) with respect to \( \tilde{g} \). Our preprint \[Gra\], when \( X \) is a real analytic surface, shows it is possible to give locally a meaning to the notion of reduction of the section \( \tilde{\kappa} \) with respect to the semi-definite structure generated over \( \tilde{X} \) by \( \tilde{g} \). A key step in this process is first to present well the section \( \kappa_X \circ \sigma \) of \( S^2 \sigma^*(T^*(X \setminus Y)) \), that is we can find a locally free module of rank two of sections of \( T \tilde{X} \) such that (the real analytic extension of) \( \kappa_X \circ \sigma \) is everywhere locally diagonalizable in a basis of local generators, orthogonal for (the real analytic extension of) \( g_X \circ \sigma \).

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