SPATIALLY HOMOGENEOUS MODELS STÄCKEL SPACES OF TYPE (2.1)

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All classes of spatially homogeneous space-time models have been found that allow the integration of the equations of motion of test particles and of the eikonal equation by the method of complete separation of variables according to type (2.1). Four classes of models are obtained that can be applied in any modified metric theory of gravity. Two of them allow solutions of the Einstein equations with a cosmological constant and radiation. For the obtained solutions, the Hamilton–Jacobi equation of motion of test particles and the eikonal equation for radiation are integrated by the method of variable separation.

Keywords: theory of gravity, exact solutions, group of motions, homogeneous spaces, gravitational waves, Petrov classification, Bianchi classification.

As is well known, spatially homogeneous models of space-time play an important role in constructing realistic models of evolution of the Universe in any metric theory of gravity. One of the important tools for studying these models is the study of geodetic lines in these spaces, including isotropic (light) ones. From this point of view, in the study of spatially homogeneous models of interest are the possibilities of analytical integration in these models of the eikonal equation and of the equations of test particle motion in the Hamilton–Jacobi formalism by the method of variable separation. The spaces that allow the existence of the coordinate systems in which the Hamilton–Jacobi equations of motion of test particles are integrated by the method of complete separation of variables are called the Stäckel spaces (named after Paul Stäckel, see [1]). The spaces admitting a complete separation of variables in the eikonal equation are called the conformal Stäckel spaces. According to the general theory of the Stäckel spaces (SS) developed by V. N. Shapovalov [2], these spaces are determined by the so-called complete set of the Killing fields consisting of the Killing vectors and the Killing tensors of the second rank corresponding to sets of integrals of motion of test particles and meeting some algebraic conditions. A brief description of the main results of the SS theory can be found in [3]. The spatially homogeneous models that allow integration of the eikonal equation (radiation) and of the equation of motion of test particles (dust matter) are of interest both for Einstein’s classical theory of gravity and for other modified metric theories of gravity [4], including a comparative analysis of the behavior of these models in various modified theories of gravity.

In this work, we consider the Stäckel spaces of type (2.1) admitting two Killing commuting vectors in the complete set; therefore, in a privileged coordinate system (CS) where the separation of variables is allowed, the metric of the SS of type (2.1) can be written so that it depends only on two variables – $x^0$ and $x^1$: 

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\[
g^{ij} = \frac{1}{\Delta} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & f_1(x^1) & 1 \\ 0 & f_1(x^1) & A(x^0, x^1) & b_0(x^0) \\ 0 & 1 & b_0(x^0) & c_0(x^0) \end{pmatrix}, \tag{1}
\]

where \(A(x^0, x^1) = a_0(x^0) + a_1(x^1)\), the function \(\Delta\) is an arbitrary function of all four variables in the case of the conformal Stäckel spaces, and in the case of the Stäckel spaces of type (2.1) it has the form \(\Delta = t_0(x^0) + t_1(x^1)\). The variables in a privileged CS on which the metrics do not depend are called ignored (cyclic).

The function of action of test particles \(S\) for the SS can be written in a privileged CS in a split form. In the same system of coordinates, it allows complete separation of variables and of the eikonal equation; in this case, the metric \(\tilde{g}^{ij}\) will differ from \(g^{ij}\) by the presence of an arbitrary conformal factor \(\Delta\). For the Stäckel spaces of type (2.1) with metric (1) in a privileged CS, the function of test particle action (with ignored variables that enter linearly) has the form

\[
S = \phi_0(x^0) + \phi_1(x^1) + px^2 + qx^3, \quad p, q, r - \text{const}. \tag{2}
\]

Here \(S = S(x^1)\) is the function of test particle action, \(g^{ij}\) is the space-time metric, \(m\) is the mass of a test particle, and the functions \(\phi_0(x^0)\) and \(\phi_1(x^1)\) are the solutions of the ordinary differential equations:

\[
\dot{\phi}_0^2 = m^2 t_0(x^0) - p^2 a_0(x^0) - 2pq b_0(x^0) - q^2 c_0(x^0) - r, \tag{3}
\]

\[
2\dot{\phi}_1(pf_1(x^1) + q) = m^2 t_1(x^1) - p^2 a_1(x^1) + r, \tag{4}
\]

where \(m, p, q,\) and \(r\) are constant parameters of test particles, and the dot atop means the ordinary derivative of the function of one variable. Thus, in the considered space-time models, we can integrate in quadratures the equations of test particle motion and radiation in the gravitational field. In the case of the eikonal equation, the vector \(l_k = \partial \Psi / \partial x^k\) (where \(\Psi = \Psi(x^1)\) is the eikonal function) sets the radiation wave vector, and the equations \(\Psi(x^1) = \text{const}\) set the radiation wave surfaces (wave front).

**CLASSES OF THE SPATIALLY HOMOGENEOUS SS MODELS OF TYPE (2.1)**

To determine the spatially homogeneous models from the family of the Stäckel spaces of type (2.1), we assume that the number of the pairwise commuting Killing vectors of considered models remains equal to two, so that the metric in a privileged CS depends on two non-ignoreable variables, and the commuting Killing vectors \(X^0\) and \(X^1\) from the complete set of SS (2.1) can be represented in the form

\[
X^0_i = (0, 0, 0, 1), \quad X^1_i = (0, 0, 1, 0). \tag{5}
\]

Note that the vector \(X^0\) is isotropic, and the vector \(X^1\) is spatially similar.

The additional two Killing vectors providing spatial homogeneity of the models have the form
\[ X_2 = \xi^i \partial_i, \quad X_3 = \eta^i \partial_i. \] (6)

The models under study that allow the dependence of the metric on one of the non-ignorable variables in a privileged CS only through the conformal factor are referred to the subtype B; in contrast, the models that depend on both non-ignorable variables are referred to the subtype A. The models of type B refer to the intersection of the Stäckel space of type (2.1) and the set of the conformal Stäckel spaces of type (3.1) we considered earlier in [5]. The spaces that allow integration of equations of motion of test particles by the method of complete separation of variables in the Hamilton–Jacobi formalism make it possible to construct precisely integrable gravitational models both in Einstein’s theory and in modified theories of gravity with different types of matter [6–11].

In the present work, Einstein’s equations with the cosmological constant \( \Lambda \) and the energy-momentum tensor of pure radiation with energy density \( \varepsilon \) and wave vector \( l^k \):

\[ R_{ij} - \frac{1}{2} R g_{ij} = \Lambda g_{ij} + \varepsilon l^i l_j, \quad l^k l_k = 0, \] (7)

are integrated below for each A model type. For the models that satisfy Einstein’s equation (7), the eikonal equation and the Hamilton–Jacobi equation of motion of test particles are also integrated below.

**MODEL A.1 OF THE SPATIALLY HOMOGENEOUS SS OF TYPE (2.1)**

For this class of spatially homogeneous models of the SS (2.1), we have

\[ f_1 = 0, \quad a_0 = 0, \quad A = a_1 = (x^1)^2, \quad b_0 = \alpha x^0, \quad c_0 = 0, \quad t_1 = 0, \quad \Delta = t_0 = 1/ (x^0)^2, \quad \alpha \neq 0. \] (8)

The metric and the additional Killing vectors of the model can be written in the form

\[
g^{ij} = x^0^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & (x^1)^2 & \alpha x^0 \\ 0 & 1 & \alpha x^0 & 0 \end{pmatrix}, \quad \xi^i = \begin{pmatrix} 0, \\ x^1, \\ x^2, \\ -x^3, \end{pmatrix}, \quad \eta^i = \begin{pmatrix} x^0, \\ 2x^1, \\ 3x^2, \\ 0, \end{pmatrix} \] (9)

where \( \alpha \) is a constant parameter of the model. For the space-time interval, we have

\[
d s^2 = \frac{d x^0^2}{x^0^2} + \left( \frac{d x^1}{x^1} - \frac{d x^2}{x^0 x^1} \right)^2 + 2 \frac{d x^1 d x^3}{x^0^2}. \] (10)

The commutators of the Killing vectors of model A.1 have the form

\[
[X_0, X_1] = 0, \quad [X_0, X_2] = -X_0, \quad [X_0, X_3] = 0, \quad [X_1, X_2] = X_1, \quad [X_1, X_3] = 3X_1, \quad [X_2, X_3] = 0,
\]

where \( X_0 = \delta_x^i \), \( X_1 = \delta_y^i \), \( X_2 = \xi^i \), and \( X_3 = \eta^i \); \( (X_1, X_2, X_3) \) is the vector of the subgroup of spatial homogeneity. The sign definiteness of the metric on the orbits of the spatial homogeneity subgroup imposes restrictions
on the range of permissible values of the employed coordinates: $x^1x^3 < 0, x^0 > 2|x^1x^3|$. The scalar curvature is constant and negative ($R = -12$), and the Weyl tensor is $C_{ijkl} \neq 0$. The model belongs to type III according to the Bianchi classification and to type N according to the Petrov classification.

If we introduce new variables $z = \ln x^0$ and $y = \ln x^1$, we obtain the interval

$$ds^2 = dz^2 + \left(\alpha dy - e^{-(z+y)} dx^2\right)^2 + 2e^{(y-2z)} dy dx^3.$$  \hspace{1cm} (11)

For metric (9), Einstein’s equations (7) leads to the condition $\alpha = 0$. Thus, the field equations have no solution for model A.1.

**MODEL A.2 OF THE SPATIALLY HOMOGENEOUS SS OF TYPE (2.1)**

For this class of spatially homogeneous models of the SS (2.1), we obtain

$$f_i = 0, \quad a_0 = 0, \quad A = a_i = e^{2x^i}, \quad b_0 = 0, \quad c_0 = -\alpha^2 (x^0)^2, \quad t_1 = 0, \quad \Delta = t_0 = 1/(x^0)^2,$$

$$g^{ij} = x^0^2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & e^{2x^1} & 0 \\ 0 & 1 & 0 & -\alpha^2 x^0^2 \end{pmatrix}, \quad \xi^i = \begin{pmatrix} x^0 \\ -1 \\ 0 \\ 2x^3 \end{pmatrix}, \quad \eta^i = \begin{pmatrix} 0 \\ 1 \\ x^2 \\ 0 \end{pmatrix}. \hspace{1cm} (13)$$

where $\alpha$ is a constant parameter of the model. With $\alpha = \pm 1$, the space admits three pairwise commuting Killing vectors and degenerates into the Stäckel space of type (3.1); we do not consider here this case. The sign-definiteness of the metric on the orbits of the spatial homogeneity group leads to the restrictions on the range of the allowed values of the employed coordinates: $(1 + \alpha^2)x^0^2 - 4x^3 > 0, (\alpha^2 x^0^4 - 4x^2) > 0$. From this it follows that $\alpha \neq 0$. The space-time interval has the form

$$ds^2 = \frac{(dx^0)^2}{(x^0)^2} + \alpha^2 (dx^1)^2 + 2\frac{dx^1 dx^3}{(x^0)^2} + \frac{e^{-2x^1}}{x^0} (dx^2)^2, \quad \alpha \neq 0, \pm 1.$$ \hspace{1cm} (14)

The commutators of the Killing vectors of model A.2 have the form

$$[X_0, X_1] = 0, \quad [X_0, X_2] = 2X_0, \quad [X_0, X_3] = 0, \quad [X_1, X_2] = 0, \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = 0.$$  

The scalar curvature is constant and negative ($R = -12$), and the components of the Weil vector are $C_{ijkl} \sim (\alpha^2 - 1) \neq 0$. The model belongs to type III according to the Bianchi classification and to type N according to the Petrov classification. The solution of Einstein’s equations (7) for metric (13) gives the following result:

$$\Lambda = 3, \quad l_0 = l_2 = l_3 = 0, \quad \varepsilon l_1^2 = \alpha^2 - 1, \quad |\alpha| > 1.$$ \hspace{1cm} (15)
Thus, we obtain a spatially homogeneous Universe with the cosmological constant $\Lambda$ filled with radiation with energy density $\varepsilon$ and wave vector $l^k = (0,0,0,l^3)$, $l^3 = x^0 l_1$.

**INTEGRATION OF THE HAMILTON–JACOBI EQUATION AND OF THE EIKONAL EQUATION FOR THE MODEL A.2**

Now we integrate Hamilton–Jacobi equations (3) and (4) for metric (13) and obtain the explicit form of the function $S$ for the test particle action given by Eq. (2) (at $q \neq 0$):

$$S(x^k) = m \ln (x^0) + \frac{1}{2} \left[ m^2 + \alpha^2 q^2 x^0 x^4 - r x^0 - m \ln \left( 2m \sqrt{m^2 + \alpha^2 q^2 x^4} - r x^2 \right) \right]$$

$$- \frac{r}{4 \alpha q} \ln \left[ 2q \left( \sqrt{m^2 + \alpha^2 q^2 x^4} - r x^2 \right) \right] + \frac{1}{4q} \left( 2rx^4 - p^2 e^{2x^4} \right) + px^2 + qx^3 + F(p,q,r), \quad (16)$$

where $p, q$, and $r$ are constant parameters of test particle motion and $F(p,q,r)$ is an arbitrary function of constants.

When the test particle moves along the coordinate $x^0$, turning points may arise. When $r < 2 | \alpha q m |$, there are no turning points, with $r = 2 | \alpha q m |$ there are two turning points with different signs and a forbidden zone at $x^0$ between them (infinite motion in $\pm \infty$ along $x^0$ from the turning points of opposite signs), and with $r > 2 | \alpha q m |$, there are four turning points (two permitted zones of motion along $x^0$ with different signs; in this case, we obtain two regions of finite motions of the test particle along $x^0$).

The eikonal function for metric (13) (at $q \neq 0$) has the following form:

$$\Psi = \frac{1}{2} \sqrt{\alpha^2 q^2 x^4 - r x^2} - \frac{r}{4 \alpha q} \ln \left[ 2 \alpha q \sqrt{\alpha^2 q^2 x^4 - r x^2 + \alpha q x^2} \right] - r$$

$$+ \frac{1}{4q} \left( 2rx^4 - p^2 e^{2x^4} \right) + px^2 + qx^3 + F(p,q,r). \quad (17)$$

When $r \leq 0$, there are no turning points along $x^0$. When $r > 0$, there are two turning points and a forbidden zone between them (infinite motion along $x^0$ in $\pm \infty$ from the turning points of opposite signs). In the particular case when the motion constant $q$ vanishes, Eq. (4) leads to conditions on constants

$$q = 0 \rightarrow p = r = 0. \quad (18)$$

The action for $q = p = r = 0$ takes the form

$$S = m \ln x^0. \quad (19)$$

If we introduce the new variable $z = \ln x^0$, then for the interval we obtain ($x^3$ is an isotropic variable)

$$ds^2 = dz^2 + \alpha^2 dx^2 + 2 e^{-z} dx^4 dx^3 + e^{-2(z+i^3)} dx^2. \quad (20)$$
For the action function of the test particle in the space-time with metric (20), we obtain (when \( q \neq 0 \))

\[
S = mz + \frac{1}{2}m^2 - re^{2z} + q^2 \alpha^2 e^{4z} - m \ln \left[ 4m\left( m^2 - re^{2z} + q^2 \alpha^2 e^{4z} + m \right) - 2re^{2z} \right] - \frac{r}{4q} \ln \left[ 2q\alpha \left( m^2 - re^{2z} + q^2 \alpha^2 e^{4z} + q\alpha e^{2z} \right) - r \right] + \frac{r}{2q} x^1 - \frac{P^2}{4q} e^{2x^1} + px^2 + qx^3 + F(p, q, r). \tag{21}
\]

In the particular case when the constant \( q \) of test particle motion vanishes, we obtain for the particle action function \( S = mz \) – the motion of the particle with a constant momentum along the coordinate \( z \).

**MODEL A.3 OF THE SPATIALLY HOMOGENEOUS SS OF TYPE (2.1)**

For this class of spatially homogeneous models of the SS of type (2.1), we have

\[
f_1 = 0, \quad a_0 = 0, \quad A = a_1 = \alpha \cos^{-1-\lambda}(x^1 - \beta) \sin^{-1+\lambda}(x^1 + \beta), \quad \epsilon = \pm 1,
\]

\[
b_0 = 0, \quad c_0 = \epsilon x^0 \gamma, \quad t_1 = 0, \quad \Delta = t_0 = 1/(x^0)^2, \quad \Lambda = t_0 = 1/(x^0)^2, \tag{22}
\]

\[
g_{ij} = x^0 \gamma = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & \alpha \cos^{-1-\lambda}(x^1 - \beta) \sin^{-1+\lambda}(x^1 + \beta) & 0 \\
0 & 1 & 0 & \epsilon x^0 \gamma
\end{pmatrix}, \tag{23}
\]

\[
\xi^i = \begin{pmatrix}
x^0, \\
0, \\
x^2, \\
2x^3,
\end{pmatrix}
\]

\[
\eta^i = \begin{pmatrix}
x^0 \cos \epsilon x^1, \\
\sin \epsilon x^1 + \sin \epsilon 2\beta, \\
\gamma x^2, \\
\epsilon x^0 \sin \epsilon x^1,
\end{pmatrix}
\]

Here \( \alpha, \beta, \) and \( \lambda \) are constant model parameters. When \( \epsilon = -1 \), the harmonic functions are replaced by the hyperbolic ones: \( \sin_{(-1)}(x) = \sinh(x), \cos_{(-1)}(x) = \cosh(x) \).

If we introduce the new variable \( z = \ln x^0 \), then for the interval we obtain

\[
dx^2 = dz^2 - \epsilon dx^1 \gamma + e^{-2z} \left[ 2dx^1 dx^3 + (1/\alpha) \sin^{1-\lambda}(x^1 + \beta) \cos^{1+\lambda}(x^1 - \beta) dx^2 \right]. \tag{24}
\]

The commutators of the Killing vectors of the model A.3 have the form

\[
[X_0, X_1] = 0, \quad [X_0, X_2] = 2X_0, \quad [X_0, X_3] = 0, \quad [X_1, X_2] = X_1, \quad [X_1, X_3] = \gamma X_1, \quad [X_2, X_3] = 0,
\]

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the scalar curvature is constant and negative ($R = -12$), and all components of the Weyl tensor are proportional to the factor $(\lambda^2 - 1) \cos(2\beta)$. The model belongs to type III according to the Bianchi classification and to type N according to the Petrov classification.

Solving Einstein’s equations (7) for metric (23), we obtain

\[
\Lambda = 3, \quad l_0 = l_2 = l_3 = 0, \quad \epsilon l_1^2 = \frac{(1 - \lambda^2) \cos^2(2\beta)}{\left(\sin(\beta) + \sin(2\beta)\right)^2}, \quad |\lambda| \leq 1. \tag{25}
\]

Thus, we obtain a spatially homogeneous Universe with interval (24) and the cosmological constant $\Lambda$ filled with radiation with energy density $\epsilon$ and wave vector $l^k = (0, 0, 0, l^3)$, $l^3 = x^0 \frac{\lambda}{l_1}$. When $\lambda = \pm 1$ and/or $\cos(2\beta) = 0$, the Weyl tensor vanishes, and the space-time model under consideration degenerates and becomes vacuum and conformally flat.

INTEGRATION OF THE HAMILTON–JACOBI EQUATION AND OF THE EIKONAL EQUATION FOR THE MODEL A.3 (HARMONIC FUNCTIONS IN THE METRIC)

Integrating the Hamilton–Jacobi equation for $\epsilon = 1$ (harmonic functions $x^1$ in the metric), we obtain for the function of action of the test particles given by Eq. (2) when $q \neq 0$ the expression

\[
S = \phi_0(x^0) + \phi_1(x^1) + px^2 + qx^3 + F(p, q, r), \quad p, q, r - \text{const}, \tag{26}
\]

\[
\phi_0(x^0) = m \ln x^0 + \frac{1}{2} \sqrt{m^2 - q^2 x^0^4 - r x^0^2} - \frac{r}{4q} \arcsin \left( \frac{2q^2 x^0^2 + r}{\sqrt{4m^2 q^2 + r^2}} \right) - \frac{m}{2} \ln \left( 2m^2 - r x^0^2 + 2m \sqrt{m^2 - q^2 x^0^4 - r x^0^2} \right), \tag{27}
\]

\[
0 < \left( x^0 \right)^2 \leq \left( x^0_{\text{max}} \right)^2, \quad \left( x^0_{\text{max}} \right)^2 = \frac{r + \sqrt{r^2 + 4q^2 m^2}}{2q^2}. \tag{28}
\]

Constants $p, q, r$ are determined by the initial conditions of motion of the test particle. Note that when the particle moving along the coordinate $x^0$ reaches the maximum value $x^0_{\text{max}}$, the rate of change of the action function of the test particles $x^0$ (zero component of the four-momentum of the test particles) changes its sign. Thus, $x^0_{\text{max}}$ is the tuning point of test particle motion along the coordinate $x^0$. For $\phi_1(x^1)$, we obtain

\[
\phi_1(x^1) = \frac{r}{2q} x^1 - \frac{\alpha p^2}{2q \lambda \cos(2\beta)} \left( \frac{\sin(x^1 + \beta)}{\cos(x^1 - \beta)} \right)^{\lambda}. \tag{29}
\]

In the degenerate case when $q = 0$, the condition $p = r = 0$ follows from the equations of test particle motion (since the function $\alpha p(x^1)$ is not constant). Then the function $\phi_0(x^0)$ takes the trivial form (motion of the particle with a constant momentum along $z$)
\[ \phi_0 = m \ln x^0 = mz. \]  

For the eikonal function, we obtain the expression \( r < 0 \)

\[ \Psi = \frac{1}{2} \sqrt{-rx^0 - q^2 x^4} - \frac{r}{4q} \arcsin \left( \frac{2q^2 x^0 + r}{\sqrt{4m^2 q^2 + r^2}} \right) + \frac{r}{2q} x^1 - \frac{\alpha p^2}{2q \lambda \cos(2\beta)} \left( \frac{\sin(x^1 + \beta)}{\cos(x^1 - \beta)} \right)^\lambda + F(p,q,r). \]  

INTEGRATION OF THE HAMILTON–JACOBI EQUATION AND OF THE EIKONAL EQUATION FOR THE MODEL A.3 (HYPERBOLIC FUNCTIONS IN THE METRIC)

Integrating the Hamilton–Jacobi equation for \( \epsilon = -1 \) (hyperbolic functions \( x^1 \) in the metric), we obtain

\[ \phi_0(x^0) = m \ln x^0 + \frac{1}{2} \sqrt{m^2 + q^2 x^4 - rx^0} - \frac{r}{4q} \ln \left( m^2 + 2q^2 x^0 + 2q \sqrt{m^2 + q^2 x^4 - rx^0} \right) \]

\[ - \frac{m}{2} \ln \left( 2m^2 - rx^0 + 2m \sqrt{m^2 + q^2 x^4 - rx^0} \right). \]

When \( r \geq 2 |qm| > 0 \), the motion of the test particles along the variable \( x^0 \) has turning points \( x^0_{\min} \leq x^0 \leq x^0_{\max} \):

\[ \phi_1(x^1) = \frac{r}{2q} x^1 - \frac{\alpha p^2}{2q \lambda \cosh(2\beta)} \left( \frac{\sinh(x^1 + \beta)}{\cosh(x^1 - \beta)} \right)^\lambda. \]

Then for the action function of the test particles, we have (where \( F(p,q,r) \) is an arbitrary function of the parameters)

\[ S = \phi_0(x^0) + \phi_1(x^1) + px^2 + qx^3 + F(p,q,r), \quad p,q,r - \text{const}. \]

For the eikonal function, we obtain the expression

\[ \Psi = \frac{1}{2} \sqrt{q^2 x^0 - rx^0} - \frac{r}{4q} \ln \left( qx^0 + \sqrt{q^2 x^0 - rx^0} \right) \]

\[ + \frac{rx^1}{2q} - \frac{\alpha p^2}{2q \lambda \cosh(2\beta)} \left( \frac{\sinh(x^1 + \beta)}{\cosh(x^1 - \beta)} \right)^\lambda + F(p,q,r). \]

MODEL A.4 OF THE SPATIALLY HOMOGENEOUS SS OF TYPE (2.1)

This model is the unique spatially homogeneous model with \( f_1 \neq 0 \). The metric and the additional Killing vectors can be written as follows:

\[ f_1 = x^1, \quad a_1 = 0, \quad A = a_0 = x^0, \quad b_0 = 0, \quad c_0 = 0, \quad t_1 = 0, \quad \Delta = t_0 = 1/(x^0)^2, \]
If we introduce the variable $z = \ln x^0$, the interval takes the form

$$ds^2 = dz^2 + 2e^{-2z}dx^1dx^3 + e^{-4z}(dx^2 - x^1dx^3)^2,$$

where $x^1$ is the isotropic wave variable. The commutators of the Killing vector of the model A.4 have the form

$$[X_0, X_1] = 0, [X_0, X_2] = X_0, [X_0, X_3] = 0, [X_1, X_2] = 0, [X_1, X_3] = 2X_1, [X_2, X_3] = 0.$$

The sign definiteness of the metric on the orbits of the spatial homogeneity subgroup imposes restrictions on the range of the permissible values of the employed coordinates:

$$x^1x^3 < 0, x^0 > 2|x^1x^3|.$$

The scalar curvature is constant and negative ($R = -43/2$), and the Weyl tensor $\tilde{C}_{ijkl} \neq 0$. The model has type D according to the Petrov classification and type III according to the Bianchi classification. There are no solutions of the Einstein equations with pure radiation (7) for the model A.4 with metric (37).

**CONCLUSIONS**

In this work, we have classified the spatially homogeneous space-time models that allow the existence of privileged coordinate systems for which the Hamilton–Jacobi equation of test particle motion and of the eikonal equation allow exact integration by the method of complete separation of variables according to type (2.1). The classification does not include subsets of spaces related to the conformal Stäckel spaces of type (3.1) we considered earlier in [5]. In total, 4 models of the considered type were obtained, which exhaust the classification.

All models are of type III according to the Bianchi classification. The model A.4 refers to type D according to the Petrov classification, the remaining models are of type N according to the Petrov classification. The model A.4 refers to wave-like models, i.e., to the space-time models the metrics of which in a privileged system of coordinates (where the separation of variables is possible) depend on the isotropic wave variable. The models obtained can be applied not only in the Einstein theory of gravity, but also in other modified metric theories of gravity.

The obtained space-time models allow two exact solutions of the Einstein equations with the cosmological constant and the energy-momentum tensor of pure radiation in the explicit form (for two models – A.1 and A.4 – there are no solutions of the Einstein equations with pure radiation). For the obtained spatially homogeneous models of the Universe with radiation and the cosmological constant, the explicit form of the eikonal function and the form of the complete integrals for action function of test particles were found.

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REFERENCES

1. P. Stäckel, Über die Integration der Hamilton-Jacobischen-Differentialgleichung mittels der Separation der Variabeln.
2. V. N. Shapovalov, Sov. J. Math., 20, 1117 (1979).
3. V. V. Obukhov and K. E. Osetrin, Proc. Sci. (WC2004) 027.
4. K. E. Osetrin, A. E. Filippov, and E. K. Osetrin, Russ. Phys. J., 61, No. 8, 1383–1391 (2018).
5. E. K. Osetrin, K. E. Osetrin, and A. E. Filippov, Russ. Phys. J., 63, No. 3 (2020).
6. K. Osetrin, A. Filippov, and E. Osetrin, Mod. Phys. Lett. A, 31, No. 06, 1650027 (2016).
7. E. Osetrin and K. Osetrin, J. Math. Phys., 58, No. 11, 112504 (2017).
8. V. G. Bagrov, V. V. Obukhov, and K. E. Osetrin, Gen. Relativ. Gravit., 20, No. 11, 1141–1154 (1988).
9. V. V. Obukhov, K. E. Osetrin, and A. E. Filippov, Russ. Phys. J., 45, No. 1, 42–48 (2002).
10. K. E. Osetrin, V. V. Obukhov, and A. E. Philippov, J. Phys. A, 39, No. 21, 6641–6647 (2006).
11. E. K. Osetrin, K. E. Osetrin, and A. E. Filippov, Russ. Phys. J., 62, No. 2, 292–301 (2019).