GEOMETRY OF COMPLEX BOUNDED DOMAINS WITH
FINITE-VOLUME QUOTIENTS

KEFENG LIU AND YUNHUI WU

Abstract. We first show that for a bounded pseudoconvex domain with a manifold quotient of finite-volume in the sense of Kähler-Einstein measure, the identity component of the automorphism group of this domain is semi-simple without compact factors. This partially answers an open question in [Fra95]. Then we apply this result in different settings to solve several open problems, for examples,

1. We prove that the automorphism group of the Griffiths domain [Gri71] in $\mathbb{C}^2$ is discrete. This gives a complete answer to an open question raised four decades ago.

2. We show that for a contractible HHR/USq complex manifold $D$ with a finite-volume manifold quotient $M$, if $D$ contains a one-parameter group of holomorphic automorphisms and the fundamental group of $M$ is irreducible, then $D$ is biholomorphic to a bounded symmetric domain. This theorem can be viewed as a finite-volume version of Nadel-Frankel’s solution for the Kazhdan conjecture, which has been open for years.

3. We show that for an irreducible bounded convex domain $D \subset \mathbb{C}^n$ of $C^1$-smooth boundary, if $D$ has a finite-volume manifold quotient with an irreducible fundamental group, then $D$ is biholomorphic to the unit ball in $\mathbb{C}^n$, which partially solves an old conjecture of Yau.

For (2) and (3) above, if the complex dimension is equal to 2, more refined results will be provided.

1. INTRODUCTION

D. Kazhdan conjectured that any irreducible bounded domain with a one-parameter group of holomorphic automorphisms and a compact quotient is biholomorphic to a bounded symmetric domain. Frankel [Fra89] first proved this conjecture for the case that the bounded domain is convex. Subsequent works by Nadel [Nad90] and Frankel [Fra95] completely confirmed Kazhdan’s conjecture. How to extend it to the finite-volume quotient case, containing the Teichmüller space of Riemann surfaces, is a well-known open problem in geometry and complex analysis. The main purpose of this article is to study this open problem and related topics.

Recall that the proof of Kazhdan’s conjecture consists of two parts:

- 2010 Mathematics Subject Classification. Primary 32Q30, 53C24 Secondary 32M15, 32G15, 53C55.
(1). It was shown that the identity component of the automorphism group of the bounded domain in Kazhdan’s conjecture is semi-simple (see [Fra89, Theorem 10.1] or [Nad90, Theorem 0.1]) without compact factors (see [Nad90, Theorem 0.1]).

(2). Frankel in [Fra95] applied part (1) above and strong harmonic map techniques (see Theorems 1.3, 2.3, 3.1 and Prop. 4.2 in [Fra95]) to complete the solution of Kazhdan’s conjecture.

In this paper, we will prove certain finite-volume versions of the Nadel-Frankel theorem [Fra95, Theorem 0.1] on the solution of Kazhdan’s conjecture. As in part (1), in the finite-volume case we will firstly show that the identity component \( \text{Aut}_0(D) \) of the automorphism group of the bounded domain \( D \) is semi-simple without compact factors. For this part, we will follow some ideas of Frankel [Fra89]. One may see the following subsection 1.1 and section 3 for more details. For the second part, it is not easy to extend the work of Frankel in [Fra95] to the finite-volume case by using harmonic map techniques. In this paper we will develop a complete different method (without using harmonic map techniques) as in [Fra95]. Except the complex two dimensional case, we will use certain Lie group theory and \( \ell^2 \) cohomology theory to show that if \( \text{Aut}(D) \) is not discrete,

\[
\dim(D) = \dim(\text{Aut}_0(D)/K)
\]

where \( K \) is the maximal compact subgroup of \( \text{Aut}_0(D) \). This in particular implies that \( D \) is biholomorphic to a bounded symmetric domain. One may see the proof of Theorem 1.7 in section 5 on details.

1.1. Semisimple without compact factors. Let \( M \) be a connected compact complex manifold with ample canonical bundle, \( \tilde{M} \) be the universal covering space of \( M \) and \( \text{Aut}_0(\tilde{M}) \) be the identity component of the automorphism group \( \text{Aut}(\tilde{M}) \) of \( \tilde{M} \). Nadel proved

**Theorem** (Nadel). [Nad90, Theorem 0.1] The group \( \text{Aut}_0(\tilde{M}) \) is a real semisimple Lie group without compact factors.

In the important special case that \( \tilde{M} \) is a bounded domain in \( \mathbb{C}^n \), this theorem was obtained by Frankel [Fra89, Theorem 10.1]. And the theorem above is crucial in [Fra95] to complete the confirmation of Kazhdan’s conjecture. And Frankel asked

**Question 1.1.** [Fra95, Page 296] *How to extend the theorem of Nadel to the finite volume case?*

It is known that a bounded domain with a compact manifold quotient is pseudoconvex. And the works of Cheng-Yau [CY80] and Mok-Yau [MY83] tell that there always exists a complete Kähler-Einstein metric on a bounded
pseudoconvex domain. Clearly the Kähler-Einstein metric induces a natural measure which is called the Kähler-Einstein measure. We denote it by $\text{Vol}_{KE}$. Our first result is to give a positive answer to Question 1.1 for the case that $\tilde{M}$ is a bounded pseudoconvex domain. More precisely,

**Theorem 1.2.** Let $D$ be a bounded pseudoconvex domain with a manifold quotient $M$ satisfying $\text{Vol}_{KE}(M) < \infty$. Then $\text{Aut}_0(D)$ is a real semisimple Lie group without compact factors.

The group $\text{Aut}_0(D)$ above could be trivial. In the following subsections applications of Theorem 1.2 in different settings will be discussed.

1.2. **Bounded domains in $\mathbb{C}^2$.** It is known that any bounded symmetric domain in $\mathbb{C}^2$ is biholomorphic to either the bi-disk $\mathbb{D} \times \mathbb{D}$ or the complex two dimensional unit ball $B$. The first application of Theorem 1.2 is the following rigidity result in the complex two dimensional case, which may be viewed as a finite-volume version of [Nad90, Theorem 0.2] for the case that $\tilde{M}$ is a bounded pseudoconvex domain. More precisely,

**Theorem 1.3.** Let $D \subset \mathbb{C}^2$ be a contractible bounded pseudoconvex domain with a manifold quotient $M$ satisfying $\text{Vol}_{KE}(M) < \infty$ and the Euler characteristic number $\chi(M) > 0$. Then exactly one of the following is valid:

(i) $D$ is biholomorphic to the complex two dimensional unit ball $B$.

(ii) $D$ is biholomorphic to the bi-disk $\mathbb{D} \times \mathbb{D}$.

(iii) The group $\text{Aut}(D)$ is discrete.

Where $\text{Aut}(D)$ is the automorphism group of $D$.

Griffiths [Gri71] constructed a complex two dimensional contractible bounded domain $D$ as the universal covering space of a Zariski open set. He proved that this domain is biholomorphic to a bounded pseudoconvex domain by using the theory of simultaneous uniformization of Riemann surfaces due to Bers. This domain $D$ is a disc fibration over the unit open disc, which holomorphically covers a manifold $M$ which is a surface fibration over a surface $S$. One may refer to [GD08, GR15, Ima83, Sha77] for related topics. The following question was listed by Fornaess and Kim, which has been open for four decades.

**Question 1.4.** [FK15, Problem 18] Is $\text{Aut}(D)$ discrete?

Shabat [Sha77, Theorem 3] showed that $\text{Aut}(D)$ is discrete provided that either the base or each fiber of $M$ is compact. The difficult part of Question 1.4 is the case that both the base and the fibers of $M$ are open surfaces. As a direct application of Theorem 1.3, in this paper we give an affirmative answer to Question 1.4.
Theorem 1.5. Let $D$ be the complex two dimensional bounded domain constructed by Griffiths \([\text{Gri71}]\) which is not biholomorphic to the bi-disk $\mathbb{D} \times \mathbb{D}$. Then the automorphism group $\text{Aut}(D)$ is discrete.

1.3. HHR/USq complex manifolds. As in \([\text{LSY04, LSY05}]\), a complex manifold $D$ of dimension $n$ is said to be holomorphic homogeneous regular (HHR) if there exists a constant $a \in (0, 1]$ such that for any $p \in D$ there is a holomorphic map $f_p : D \to \mathbb{C}^n$ satisfying

(i) $f_p(p) = 0 \in \mathbb{C}^n$;
(ii) $f_p : D \to f_p(D) \subset \mathbb{C}^n$ is biholomorphic;
(iii) $B(0; a) \subset f_p(D) \subset B(0; 1)$ where $B(0; a)$ is the Euclidean geodesic ball of radius $r$ centered at $0$ in $\mathbb{C}^n$.

In \([\text{Yeu09}]\) a HHR complex manifold is also called a manifold with the uniform squeezing property (USq). The motivation of HHR/USq complex manifolds can go back to Morrey’s work \([\text{Mor08, Chapter 10}]\) on higher dimensional plateau problems. Examples of HHR/USq complex manifolds contain

(i) bounded homogeneous domains;
(ii) Bound domains which covers compact manifolds—the ones in Kazhdan’s conjecture;
(iii) \([\text{Ber60}]\) Teichmüller space of Riemann surfaces;
(iv) \([\text{Fra91, KZ16}]\) Bounded convex domains;
(v) Products of domains as above.

It was shown in \([\text{LSY04, LSY05, Yeu09}]\) that on a HHR/USq complex manifold $D$, the Carathéodory metric, Kobayashi metric, Bergman metric and Kähler-Einstein metric are equivalent. The automorphism group of $D$ acts as isometries on $D$ endowed with any one of these four metrics. For the case that $D$ is the Teichmüller space $T_{g,m}$ of Riemann surfaces of genus $g$ with $m$ punctures, one may also refer to \([\text{Che04, McM00}]\) for more equivalent Kähler metrics. When we say a HHR/USq complex manifold covers a manifold $M$ of finite-volume, the measure on $M$ is the one induced by any one of the four classical metrics above.

Let $D$ be a HHR/USq complex manifold. In particular, by definition one may view $D$ as a bounded domain. It is known \([\text{Yeu09, Lemma 2}]\) that $D$ is a bounded pseudoconvex domain. When $D$ is of complex dimension two, we have the following result which is a consequence of Theorem 1.3 by checking $\chi(M) > 0$.

Theorem 1.6. Let $D$ be a contractible, complex two dimensional, HHR/USq complex manifold with a finite-volume manifold quotient $M$. Then exactly one of the following is valid:
(i) $D$ is biholomorphic to the complex two dimensional unit ball $B$.
(ii) $D$ is biholomorphic to the bi-disk $\mathbb{D} \times \mathbb{D}$.
(iii) The group $\text{Aut}(D)$ is discrete.

When $D$ has complex dimension greater than or equal to 3, if we let $D = B \times \mathcal{T}_{g,m}$ ($3g + m \geq 5$) where $B$ is a bounded symmetric domain, then it is easy to see that $D$ is HHR/USq, and admits a finite-volume quotient because both $B$ and $\mathcal{T}_{g,m}$ are HHR/USq and do admit finite-volume manifold quotients. Moreover, $\text{Aut}(D)$ is not discrete because $\text{Aut}(B) \subset \text{Aut}(D)$ is not discrete. However, $D$ is not symmetric because $\mathcal{T}_{g,m}$ is not symmetric. So it requires more assumption for any possible generalization of Theorem 1.6 to higher dimensions.

We say that a group $\Gamma$ is irreducible if any finite index subgroup of $\Gamma$ cannot split, that is, any finite index subgroup $\Gamma'$ of $\Gamma$ is not of form $\Gamma_1 \times \Gamma_2$ where $\Gamma_i$ ($i = 1, 2$) cannot be trivial. Another application of Theorem 1.2 is the following one, which may be viewed as a finite-volume version of the Nadel-Frankel theorem [Fra95, Theorem 0.1] for the case that $\tilde{M}$ is HHR/USq and $\dim_{\mathbb{C}}(\tilde{M}) \geq 3$.

**Theorem 1.7.** Let $D$ be a contractible, complex $n$ ($n \geq 3$)-dimensional, HHR/USq complex manifold with a finite-volume manifold quotient $M$ whose fundamental group $\pi_1(M)$ is irreducible. Then either

(i) $D$ is biholomorphic to a bounded symmetric domain, or
(ii) the group $\text{Aut}(D)$ is discrete. Moreover, $[\text{Aut}(D) : \pi_1(M)] < \infty$.

**Remark 1.8.** It is known that the Teichmüller space $\mathcal{T}_{g,m}$ of Riemann surfaces of genus $g$ with $m$ punctures is contractible and has a finite-volume manifold quotient; the mapping class group is irreducible; and $\mathcal{T}_{g,m}$ ($3g + m \geq 5$) is not symmetric. Thus, a direct consequence of the theorem above is that $\text{Aut}(\mathcal{T}_{g,m})$ ($3g + m \geq 5$) is discrete, which is due to Royden [Roy71].

**Remark 1.9.** If the Kähler-Einstein metric (or any $\text{Aut}(D)$-invariant Riemannian metric which is equivalent to the Kähler-Einstein metric) on $D$ has nonpositive sectional curvature, the works in [Bal85, BS87, Ebe82, EH90] imply that $D$ is isometric to a symmetric space provided that $\text{Aut}(D)$ is not discrete. Here we do not have any assumption on the sectional curvature of the Kähler-Einstein metric, although it is known that the sectional curvatures are bounded (one may see Theorem 2.2 for more details).

**Remark 1.10.** To our best knowledge, Theorem 1.7 is new even for the case that $D$ is a strictly convex bounded domain.

If $M$ is compact, as stated above, Theorem 1.7 is due to Frankel-Nadel [Nad90, Fra95]. One may refer to [CFKW02, IK99, Siu91, Won77,
Won81, Yau11, Zim17b] for related topics. Throughout this article we always assume that the quotient manifold (also including the subsequent ones) is open.

We enclose this subsection by the following characterization for bounded symmetric domains, which is also an application of Theorem 1.2.

**Theorem 1.11.** Let $D$ be a contractible HHR/USq complex manifold with a finite volume quotient manifold $M$ such that the fundamental group $\pi_1(M) < Aut_0(D)$. Then $D$ is biholomorphic to a bounded symmetric domain.

Comparing to Theorem 1.7, the fundamental group of the quotient manifold in Theorem 1.11 is not required to be irreducible.

### 1.4. Bounded convex domains.

A remarkable theorem of Frankel [Fra89] says that a bounded convex complex domain with a compact quotient is biholomorphic to a bounded symmetric domain, which confirmed a conjecture of S.-T. Yau [Yau87]. It is an open problem that whether the condition on a compact quotient in Frankel’s theorem can be replaced by a finite-volume quotient. One may see [Siu91, Page 124] in Siu’s survey for more details. We state the following conjecture which is well-known to experts.

**Conjecture 1.12.** A bounded convex domain with a finite-volume manifold quotient is biholomorphic to a bounded symmetric domain.

Recall that it is known by [Fra91, KZ16] that a bounded convex complex domain is always HHR/USq. As stated before, the measure on the finite-volume quotient is induced by a metric which is equivalent to the classical Kobayashi metric, such as the Kähler-Einstein metric.

A special case of Conjecture 1.12 (e.g. [Siu91, Conjecture 3.7]) is that the Teichmüller space $T_{g,m}$ $(3g + m \geq 5)$ is not biholomorphic to a bounded convex domain. Kim [Kim04] showed that the image of the Bers embedding is not convex in $\mathbb{C}^{3g+m}$. Recently Markovic completely [Mar] solved this conjecture by showing that the Kobayashi metric and the Carathéodory metric do not coincide on $T_{g,m}$. Then by work of Lempert [Lem87] the Teichmüller space $T_{g,m}$ can not be convex.

The following two corollaries give positive evidences to Conjecture 1.12. The first one is a direct consequence of Theorem 1.7 and 1.11.

**Corollary 1.13.** Let $D \subset \mathbb{C}^n (n \geq 3)$ be a bounded convex domain with a finite-volume manifold quotient $M$. If either

(i) the domain $D$ contains a one-parameter group of holomorphic automorphisms and the fundamental group $\pi_1(M)$ is irreducible,

(ii) or the fundamental group $\pi_1(M) < Aut_0(D)$,

then $D$ is biholomorphic to a bounded symmetric domain.
The second one is a direct consequence of Theorem 1.6.

**Corollary 1.14.** Let $D \subset \mathbb{C}^2$ be a bounded convex domain with a finite-volume manifold quotient. Then $D$ is biholomorphic to either $B$ or $\mathbb{D} \times \mathbb{D}$ if and only if the domain $D$ contains a one-parameter group of holomorphic automorphisms.

A remarkable theorem of Wong-Rosay [Won77, Ros79] says that a bounded domain $D$ in $\mathbb{C}^n$ with a compact quotient is biholomorphic to the unit ball provided that the boundary of $D$ is $C^2$-smooth. It is stated in [Won77, Page 257] that S.-T. Yau suggested that the condition on a compact quotient in Wong-Rosay’s theorem can be replaced by a finite-volume quotient. More precisely,$^1$

**Conjecture 1.15** (Yau). Let $D \subset \mathbb{C}^n (n \geq 2)$ be a bounded pseudoconvex domain whose boundary is $C^2$-smooth. Assume that $D$ has an open quotient of finite-volume (in the sense of Kähler-Einstein measure). Then $D$ is biholomorphic to the unit ball in $\mathbb{C}^n$.

If the bounded domain is convex, we have the following two rigidity results, which are partial answers to Conjecture 1.12 and 1.15. And the hypothesis only assumes that the convex domain has $C^1$-smooth boundary.

**Theorem 1.16.** Let $D \subset \mathbb{C}^n (n \geq 3)$ be an irreducible bounded convex domain of $C^1$-smooth boundary. If $D$ has a finite-volume manifold quotient whose fundamental group is irreducible, then $D$ is biholomorphic to the complex $n$-dimensional unit ball in $\mathbb{C}^n$.

For complex two dimensional case, the condition on irreducible in Theorem 1.16 can be removed. More precisely, we have

**Theorem 1.17.** Let $D \subset \mathbb{C}^2$ be an irreducible bounded convex domain of $C^1$-smooth boundary. If $D$ has a finite-volume manifold quotient, then $D$ is biholomorphic to the complex two dimensional unit ball $B$.

We remark here that there is no regularity assumption on the boundaries of the complex domains in this article, except the ones in Theorem 1.16 and 1.17. And we also remark that the manifold quotients in the theorems in this introduction are always assumed to be open.

Recently, A. Zimmer [Zim18] claims a solution of Conjecture 1.15.

**Plan of the paper.** In Section 2 we give some necessary backgrounds for bounded pseudoconvex domains and HHR/USq complex manifolds. And we

---

$^1$We are grateful to Prof. B. Wong for bringing this question to our attention.
also provide some necessary propositions for $\text{Aut}(D)$ and the fundamental group of the quotient manifold. In Section 3 we will complete the proof of Theorem 1.2, that is to show that for a bounded pseudoconvex domain $D$ with a finite-volume manifold quotient, the identity component of $\text{Aut}(D)$ is a real semisimple Lie group without compact factors. Then we will apply Theorem 1.2 to different settings in the subsequent sections. In Section 4 we will finish the proofs of Theorem 1.3 and 1.5. In Section 5 we will complete the proofs of Theorem 1.6, 1.7 and 1.11. In the last section we will prove Theorem 1.16 and 1.17 by using Theorem 1.6 and 1.7.

Acknowledgement. The authors would like to thank Prof. W. Ballmann, S. Krantz, B. Wong, S. T. Yau and K. Zuo for their interests. We especially would like to thank to Prof. B. Wong and S. T. Yau for their invaluable comments and suggestions which greatly improve this article. The first author is partially supported by the NSFC, Grant No. 11531012 and a NSF grant. And the second author is partially supported by a grant from Tsinghua university.

2. Notations and Preliminaries

This section contains general facts and necessary propositions for the proofs in subsequent sections. The general notation we use is as follows:

(i) $D$ is a bounded pseudoconvex domain or a HHR/USq complex manifold;
(ii) $M$ is a finite-volume manifold quotient of $D$;
(iii) $\Gamma := \pi_1(M)$;
(iv) $\text{Aut}(D)$ is the automorphism group of $D$ (clearly containing $\Gamma$);
(v) $\text{Aut}_0(D)$ is the identity component of $\text{Aut}(D)$;
(vi) $\Gamma_0 := \Gamma \cap \text{Aut}_0(D)$.

2.1. Kähler-Einstein metric. Our work highly relies on the Kähler-Einstein metric. We summarize the results needed.

Let $D$ be a bounded pseudoconvex domain. One may refer to the book [Dem] for general theory for bounded pseudoconvex domains. Cheng-Yau [CY80] showed that there always exists a complete Kähler-Einstein metric on a bounded pseudoconvex domain of $C^2$-smooth boundary. Later Mok-Yau [MY83] removed the assumption on $C^2$-smoothness for the boundary. More precisely,

Theorem 2.1 (Cheng-Mok-Yau). Let $D$ be a bounded pseudoconvex domain. Then there exists a complete Kähler metric $\omega$ on $D$ such that

(i) The Ricci curvature $\text{Ric}_\omega = -1$;
(ii) The automorphism group $\text{Aut}(D)$ acts on $(D, \omega)$ by isometries.

Throughout the article we always assume that the complex manifold $D$ is endowed with this Kähler-Einstein metric. We write $D$ for $(D, \omega)$ for simplicity.

2.2. HHR/USq complex manifolds. The definition for a HHR/USq complex manifold is given in the introduction. Let $D$ be a contractible HHR/USq complex manifold. In particular, by definition one may view $D$ as a bounded domain. It is known [Yeu09, Lemma 2] that $D$ is a bounded pseudoconvex domain. And by Theorem, 2.1 $D$ admits a complete Kähler-Einstein metric $\omega$ which is $\text{Aut}(D)$-invariant.

Assume that $D$ has a finite-volume manifold quotient $M$, that is, $D$ holomorphically covers $M$ and $M$ has finite volume in the sense of a measure induced from certain metric $ds^2$ which is equivalent to the Kähler-Einstein metric on $D$. In particular, the works in [LSY04, Yeu09] tell us that the metric $ds^2$ can be chosen to be any one of the Carathéodory metric, Kobayashi metric, Bergman metric and Kähler-Einstein metric. We consider the complete Kähler-Einstein metric on $M$, which is induced from the Kähler-Einstein metric $\omega$ on $D$.

We say that $M$ has bounded geometry if

(i) $M$ is complete and has finite volume;
(ii) The sectional curvature of $M$ is bounded from below and above;
(iii) The injectivity radius of $D$ is bounded from below by a positive constant.

We say that $M$ is Kähler-hyperbolic if

(i) $M$ has bounded geometry;
(ii) on $D$, the Kähler form $\omega = d\beta$ for some bounded 1-form $\beta$.

The following result is part of [Yeu09, Theorem 2]. One may also refer to [LSY05, Section 4] for the case that $D$ is the Teichmüller space of Riemann surfaces.

**Theorem 2.2.** Let $D$ be a contractible HHR/USq complex manifold with a finite-volume manifold quotient $M$. Then $M$ is Kähler-hyperbolic.

From [Yeu09, Corollary 2] we know that $M$ is a quasi-projective variety. It is well-known that a quasi-projective variety is a finite CW-complex (one may see [Dim92] for more details).

We enclose several properties for the above groups, which will be used in subsequent sections.

The following lemma is well-known.

**Lemma 2.3.** If $D$ is contractible, then the group $\Gamma$ is torsion-free, so is $\Gamma_0$. 
Proof. It directly follows from the classical Smith Theorem. Or let $A$ be a finite subgroup of $\Gamma$. Since $D$ is contractible, the cohomology dimension of $D/A$ is the same as the cohomology dimension of $A$. Since $M = D/\Gamma$ is a manifold, $D/A$ is a manifold. In particular $D/A$ has finite cohomology dimension. On the other hand, since $A$ is finite, the group $A$ has infinite cohomology dimension, which is a contradiction.

**Proposition 2.4.** If $D$ is contractible, then the Euler characteristic number satisfies that the signature

$$\text{sign}(\chi(M)) = (-1)^n.$$ 

In particular,

$$\chi(\Gamma) \neq 0.$$

Proof. We follow a similar argument as in [McM00]. By Theorem 2.2 we know that $M$ is Kähler-hyperbolic. Gromov shows that the $L^2$-cohomology group of a Kähler-hyperbolic is concentrated in the middle dimension. Since $M$ is Kähler-hyperbolic of complex dimension $n$, from the generalized Atiyah’s Covering Index Theorem [CG85] one may get that the signature satisfies

$$\text{sign}(\chi(M)) = (-1)^n.$$ 

Since $D$ is contractible, $\chi(\Gamma) = \chi(M)$. So the conclusion follows.

The following proposition will be applied to prove Theorem 1.5 and 1.11.

**Proposition 2.5.** The cardinality of $\Gamma$ satisfies

$$|\Gamma| = \infty.$$ 

Proof. Since $M$ has finite volume, it suffices to show that

$$\text{Vol}(D) = \infty$$

where we use the Kähler-Einstein measure.

By Theorem 2.2 we know that $D$ has bounded geometry. In particular, the sectional curvature of $D$ (in the sense of the Kähler-Einstein metric) is bounded and we may assume that $\epsilon_0 > 0$ is a lower bound for the injectivity radius of $D$. Then the standard comparison theorem in Riemannian geometry gives that for any $p \in D$ there exists a constant $c(\epsilon_0) > 0$ such that the volume

$$\text{Vol}(B(p, \epsilon_0)) \geq c(\epsilon_0) > 0$$

where $B(p, \epsilon_0) \subset D$ is the geodesic ball of radius $\epsilon_0$ centered at $p$.

By Theorem 2.2 we know that $D$ is complete. Since $D$ is non-compact, we may choose a geodesic ray $\gamma : [0, \infty) \to D$ with an increasing sequence $\{t_i\}_{i \geq 1}$ such that for all $t_i \neq t_j$,

$$\text{dist}(\gamma(t_i), \gamma(t_j)) \geq 4\epsilon_0.$$
It is clear that the triangle inequality gives that
\[ B(\gamma(t_i), \epsilon_0) \cap B(\gamma(t_j), \epsilon_0) = \emptyset, \quad \forall t_i \neq t_j. \]
Recall that \( \text{Vol}(B(p, \epsilon_0)) \geq c(\epsilon_0) \) for all \( p \in D \). Thus, we have
\[
\begin{align*}
\text{Vol}(D) & \geq \text{Vol}(\bigcup B(\gamma(t_i), \epsilon_0)) \\
& = \sum \text{Vol}(B(\gamma(t_i), \epsilon_0)) \\
& = \infty.
\end{align*}
\]
The proof is complete. \( \square \)

3. Semisimple and No Compact Factor

Let \( D \) be a bounded pseudoconvex domain with a manifold quotient \( M \) of finite volume in the sense of the Kähler-Einstein measure. In this section we complete the proof of Theorem 1.2, which is divided into the following two propositions.

**Proposition 3.1** (Semisimple). The group \( \text{Aut}_0(D) \) is semisimple.

**Proposition 3.2** (No Compact Factor). The group \( \text{Aut}_0(D) \) has no nontrivial compact factor.

Recall that \( \Gamma = \pi_1(M) \) and \( \Gamma_0 = \text{Aut}_0(D) \cap \Gamma \). Before proving the two propositions above, we firstly provide the following result, which is crucial in the proofs of Proposition 3.1 and 3.2. It roughly says that the information on finite-volume of \( M \) can be transferred to \( \Gamma_0 \) in some sense. More precisely,

**Lemma 3.3.** The group \( \Gamma_0 \) is a lattice of \( \text{Aut}_0(D) \). In particular, \( \Gamma_0 \) is an infinite group if \( \text{Aut}_0(D) \) is nontrivial.

**Proof.** Let \( D = (D, \omega) \) where \( \omega \) is the unique complete Kähler-Einstein metric on \( D \). From Theorem 2.1 we know that \( \text{Aut}(D) \) acts on \( D \) by isometries. Then the conclusion follows by entirely the same argument for the proof of [FW10, Step-1 on page 94], where no special properties of \( \mathcal{T}_{g,n} \) and the mapping class group are applied, except that the moduli space of Riemann surfaces endowed with the candidate metric has finite volume. For completeness, we give an outline for the proof here.

Let \( \dim_{\mathbb{C}}(D) = n > 0 \). Since \( D \) is a complete Kähler manifold, there is a natural unit sphere-bundle over \( D \), whose fiber over each \( x \in D \) is the unit sphere \( S_x \) of the tangent bundle of \( D \). We also have the associated bundle \( E \to D \) whose fiber is the \( 2n \)-fold product of \( S_x^{2n} \). Let \( \mathcal{F}(D) \) denote the subbundle of this bundle, with fiber the set of \( 2n \)-tuples of distinct points of \( S_x \) that span the tangent space \( T_x D \) of \( D \) at \( x \). Recall that the exponential map on a complete Riemannian manifold is a local diffeomorphism. Since
an isometry of $D$ take geodesic rays to geodesic rays, one may see that the set of points of $D$ for which an element in $\text{Aut}(D)$ is the identity and has derivative the identity, is both open and closed. Thus, the action of $\text{Aut}(D)$ on $\mathcal{F}(D)$ is free.

There is a natural $\text{Aut}(D)$-invariant measure on $\mathcal{F}(D)$, which is induced from the natural measure on $E$. More precisely, the bundle $E \rightarrow D$ discussed above is locally a product of form $U \times S^{2n}$, where $U$ is a neighbourhood in $D$ and $S \subset \mathbb{R}^{2n}$ is the unit sphere. The Kähler-Einstein metric on $D$ determines the Kähler-Einstein measure on $D$, which induces the Kähler-Einstein measure $\nu$ on $U$. On $S$, we have an induced measure $\mu$ which is given infinitesimally by the rule that, for a subset $A \subset S \times x$, the measure is given by the measure of the Euclidean cone of $A$, normalized so that the measure of $S \times x$ is equal to 1. The local product measure $\nu \times \mu$ gives an $\text{Aut}(D)$-invariant measure on $E$, which induces an $\text{Aut}(D)$-invariant measure of $\mathcal{F}(D)$. By construction, the pushforward of this measure under the natural projection $\mathcal{F}(D) \rightarrow D$ is the Kähler-Einstein measure on $D$ induced by the Kähler-Einstein metric on $D$.

By Myers-Steenrod [MS39] we know that $\text{Aut}(D)$ is a Lie group which acts properly discontinuously on $D$. Let $x \in \mathcal{F}(D)$ and $\text{Aut}(D) \cdot x$ be the $\text{Aut}(D)$-orbit. The Slice Theorem for proper group action (e.g. [DK00, Theorem 2.4.1]) implies that there is an $\text{Aut}(D)$-invariant tubular neighbourhood $V \subset \mathcal{F}(D)$ of $\text{Aut}(D) \cdot x$ that is a homogeneous vector bundle

$$\pi : V \rightarrow \text{Aut}(D) \cdot x \subset \mathcal{F}(D).$$

The measure on $\mathcal{F}(D)$ constructed above reduces to a measure on $V$, and the pushforward of the measure on $V$ under the projection above is a left-invariant measure on $\text{Aut}(D) \cdot x$, which can be identified with a left-invariant measure on $\text{Aut}(D) \cdot x$. Thus, this measure on $\text{Aut}(D) \cdot x$ is proportional to the unique Haar measure on $\text{Aut}(D)$. In particular, if a subset $A \subset \text{Aut}(D)$ has infinite measure then $\pi^{-1}(A)$ has infinite measure.

Choose a fiber $F$ of the bundle $V \rightarrow \text{Aut}(D) \cdot x$ such that $V = \text{Aut}(D) \cdot F$. Since $\text{Aut}_0(D) < \text{Aut}(D)$ is a connected closed subgroup, one may write $V$ as a disjoint union of $\text{Aut}_0(D)$-orbits of $F$, one for each element of $\pi_0(\text{Aut}(D))$. Thus, $V/\Gamma$ is given by the image of the $\text{Aut}_0(D)$-orbit $W$ of $D$ under the projection

$$\mathcal{F}(D) \rightarrow \mathcal{F}(D)/\Gamma = \mathcal{F}(M).$$

Since $\text{Aut}(D)$ acts freely on $\mathcal{F}(D)$, when restricted to $W$ this projection is a measure-preserving homeomorphism.

Now we argue by contradiction. Assume that $\text{Aut}_0(D)/\Gamma_0$ has infinite measure, by the discussion above $W$ would also have infinite measure, so would $\mathcal{F}(M)$. However, the pushforward of the measure under the natural
projection $\mathcal{F}(M) \rightarrow M$ is the Kähler-Einstein measure on $M$, which in particular tells that $M$ has infinite Kähler-Einstein measure, contradicting to our assumption that $M$ has finite volume with respect to the Kähler-Einstein measure. Therefore, we conclude that $\text{Aut}_0(D)/\Gamma_0$ has finite measure. That is, $\Gamma_0$ is a lattice of $\text{Aut}_0(D)$.

3.1. $\text{Aut}_0(D)$ is semisimple. We follow the idea in [Fra89, Section 10], although the cocompactness assumption is essential in the proof of [Fra89, Theorem 10.1]. Actually the cocompactness assumption was used to apply the maximum principle twice for subharmonic functions to show the semisimplicity of $\text{Aut}_0(D)$. In our setting since $M$ is open, the maximum principle can not be applied. Therefore we need to develop a new method to overcome this difficulty.

First by the work of Myers-Steenrod [MS39] we know that the automorphism group $\text{Aut}(D)$ is a Lie group. If $\text{Aut}(D)$ is discrete, $\text{Aut}_0(D)$ is trivial. For this case, we are done with the proof of Proposition 3.1. So from now on we assume that $\text{Aut}(D)$ is a Lie group of positive dimension. Thus, $\text{Aut}_0(D)$ is a closed connected Lie group which also has positive dimension.

We refer to [Fra89, Hel01, Rag72] for the basic facts of Lie groups. First let us recall the following well-known definition.

**Definition 3.4.**

(i) Let $\mathfrak{g}$ be the Lie algebra of $\text{Aut}_0(D)$. The nilpotent radical $\mathfrak{n}$ of $\mathfrak{g}$ is its maximal nilpotent ideal. We call the center of $\mathfrak{n}$ the abelian radical of $\mathfrak{g}$ which is denoted by $\mathfrak{c}$.

(ii) Let $\mathfrak{C} = \exp \mathfrak{c}$ and $\mathfrak{N} = \exp \mathfrak{n}$ be the corresponding subgroups in $\text{Aut}_0(D)$. We call $\mathfrak{C}$ is the abelian radical of $\text{Aut}_0(D)$ and $\mathfrak{N}$ is the nilpotent radical of $\text{Aut}_0(D)$.

For any subgroup $H < \text{Aut}(D)$ we let $\mathcal{N}(H)$ denote the normalizer of $H$ and $\mathfrak{h}$ be the Lie algebra of $H$ which is a subalgebra of $\mathfrak{g}$. Recall that given any $\gamma \in \mathcal{N}(H)$, $\text{Ad}_H(\gamma) : \mathfrak{h} \rightarrow \mathfrak{h}$ is defined by

$$\text{Ad}_H(\gamma)(h) = \gamma^{-1} \cdot h \cdot \gamma.$$ 

The derivative of $\text{Ad}_H(\gamma)$, denoted by $\text{ad}_H(\gamma)$, is given by

$$\text{ad}_H(\gamma) = d(\text{Ad}_H(\gamma))(e) : \mathfrak{h} \rightarrow \mathfrak{h}.$$ 

Recall that $\text{Aut}_0(D)$ is semisimple if and only if $\mathfrak{C}$ is trivial, which is equivalent to $\mathfrak{c} = 0$. One may refer to [Fra89, Lemma 10.3] for more details. We assume that

$$\dim(\mathfrak{c}) = l$$

where $l$ is a nonnegative integer.

Our aim is to show that $l = 0$. 

From now on we assume that $l > 0$, and our strategy is to arrive at a contradiction.

We outline the proof of Proposition 3.1 into two steps.

(i) We follow a similar idea in step-4 in the proof of [Fra89, Theorem 10.1] to apply a machinery of discrete subgroups of Lie groups in [Rag72] to show that $C/C \cap \Gamma$ is compact. Essentially we will check the condition $\circ$ in Theorem 3.5. The idea is: if condition $\circ$ in Theorem 3.5 is not true, then one follows Frankel’s method to construct a non-constant subharmonic function $g_K$ on $M$. However, from the structure of $M$ one can also show that such a function does not exist, which will arrive at a contradiction.

(ii) Applying the result in step-1, saying that $C/C \cap \Gamma$ is compact, to construct a function $g_C$ on $D$ such that $g_C$ is $\Gamma$-invariant. So this function can be also viewed as a function $g_C$ on $M$. The classical Bochner-Weitzenböck type formula could tell that $g_C$ is a subharmonic function on $M$. Then similar to step-1, we use the structure of $M$ to show that $g_C \equiv 0$ on $M$. However, it is known from step-2 in the proof of [Fra89, Theorem 10.1] that $g_C \neq 0$ on $M$, which will arrive at a contradiction.

Now we begin the first step which is to show that the abelian radical $C$ has a cocompact action. Similar as step-4 in [Fra89, Section 10] we will apply the following result.

**Theorem 3.5.** [Fra89, Theorem 10.14] or [Rag72, Corollary 8.28] Let $G$ be a connected Lie group and $A \subset G$ be a lattice. Let $R$ be the radical of $G$, $N$ be the nilpotent radical, and let $S \subset G$ be a semisimple subgroup such that $G = SR$ is a Levi-Malcev decomposition. Let $\sigma$ be the action of $S$ on $R$, that is for all $s \in S$ and $r \in R$,

$$\sigma_s(r) = s^{-1} \cdot r \cdot s.$$

$\circ$ Assume that the kernel of $\sigma$ has no compact factors in its identity component.

If $C$ is the center $N$, then $C/C \cap A$ is compact.

In our settings we let $G = \text{Aut}_0(D)$ and $A = \Gamma_0$. From Lemma 3.3 we know that $\Gamma_0$ is a lattice of $\text{Aut}_0(D)$. Since $C \subset \text{Aut}_0(D)$, $C \cap \Gamma = C \cap \Gamma_0$. Thus, by Theorem 3.5 we know that the following result directly follows by verifying $\circ$.

**Proposition 3.6.** The quotient $C/C \cap \Gamma$ is compact.
Verifying \(\odot\). We argue by contradiction. Assume that \(\odot\) is not correct. First we use a similar argument in step-4 in the proof of [Fra89, Theorem 10.1] to construct a \(\Gamma\)-invariant subharmonic function on \(D\), and then use a similar argument in [Wu17] to conclude that this function is the constant zero function, which in particular implies that any compact factor in the kernel of \(\sigma\) is trivial.

More precisely, similar to Theorem 3.5, let \(S \subset \text{Aut}_0(D)\) be a semisimple subgroup such that \(\text{Aut}_0(D) = SR\) is a Levi-Malcev decomposition where \(R\) is the radical of \(\text{Aut}_0(D)\). Since \(S\) is semisimple, there exists a unique maximal compact factor \(K\) of \(\ker \sigma\). In particular \(K\) is also semisimple. On the level of Lie algebras, the Lie algebra \(\mathfrak{k}\) of \(K\) is a factor of \(\mathfrak{g}\). It is clear that \(\Gamma \subset \mathcal{N}(K)\) because \(K\) is characteristic in \(S\). Thus, for any \(\gamma \in \Gamma\), the map

\[
\text{ad}_K(\gamma) : \mathfrak{k} \to \mathfrak{k}
\]

is well-defined. Actually it is an isometry since the Killing form is a canonical bi-invariant metric on \(\mathfrak{k}\).

Let \(\{X_i\}_{1 \leq i \leq k}\) be an orthonormal basis for \(\mathfrak{k}\) where \(k = \dim(\mathfrak{k})\). Define a function

\[
f_K : D \to \mathbb{R}
\]

by

\[
f_K(x) = \sum_{i=1}^{k} |X_i(x)|^2
\]

where \(X_i(x) = \frac{d}{dt}(\exp(tX_i) \cdot x)|_{t=0}\).

For any \(\gamma \in \Gamma\),

\[
X_i(\gamma \cdot x) = \frac{d}{dt}(\gamma^{-1} \cdot \exp(tX_i) \cdot \gamma \cdot x)|_{t=0} = d\gamma(x) \cdot \text{ad}_K(\gamma) \cdot X_i(x).
\]

As above we know that \(\text{ad}_K(\gamma)\) acts on \(\mathfrak{k}\) as an isometry. By Theorem 2.1 we also know that \(\gamma\) acts on \(D\) as an isometry. Then we have,

\[
|X_i(\gamma \cdot x)|^2 = <d\gamma(x) \cdot \text{ad}_K(\gamma) \cdot X_i(x), d\gamma(x) \cdot \text{ad}_K(\gamma) \cdot X_i(x)> = |X_i(x)|^2.
\]

Thus, we have for any \(\gamma \in \Gamma\) and \(x \in D\),

\[
f_K(\gamma \cdot x) = f_K(x).
\]

So \(f_K\) descends to a function, still denoted by \(f_K\), on \(M = D/\Gamma\).
Recall that the Kähler-Einstein metric on $M$ has constant Ricci curvature $-1$. It follows from [Fra89, Lemma 10.15] that for all $p \in M$,

$$\Delta f_K(p) = \sum_{i=1}^{k}(|\nabla X_i(p)|^2 + |X_i(p)|^2)$$

where $\Delta$ is the Beltrami-Laplace operator in the sense of the Kähler-Einstein metric on $M$. In particular we have for all $p \in M$,

(3.1) \hspace{1cm} \Delta f_K(p) \geq f_K(p) \geq 0.

Next we will show that $f_K$ is a constant function.

Let $g_t$ denote the flow generated by the vector field $\text{grad} f_K$. From Theorem 2.1 we know that $D$ (endowed with the Kähler-Einstein metric) is complete. In particular $M$ is complete. Thus, $g_t$ is well defined for all $t \geq 0$.

Suppose that $f_K$ is not a constant. We let $p_0 \in M$ such that $\text{grad} f_K(p_0) \neq 0$. Along the flow line of $g_t$ starting at $p_0$, $f_K$ is increasing since for all $s_2 > s_1 \geq 0$,

(3.2) \hspace{1cm} f_K(g_{s_2}(p_0)) - f_K(g_{s_1}(p_0)) = \int_{s_1}^{s_2} ||\text{grad} f_K(g_t(p_0))|| \, dt \\
\hspace{1cm} \geq 0.

That is,

$$f_K(g_{s_2}(p_0)) \geq f_K(g_{s_1}(p_0)) \quad \forall s_2 > s_1 \geq 0.$$ 

Since we assume that $\text{grad} f_K(p_0) \neq 0$, let $s_2 = 1$ and $s_1 = 0$ we have

$$f_K(g_1(p_0)) > f_K(p_0) \geq 0.$$ 

Therefore there exists a small enough constant $r_0 > 0$ such that

$$\inf_{q \in B(p_0, r_0)} f_K(g_1(q)) > \sup_{q \in B(p_0, r_0)} f_K(q)$$

where $B(p_0, r_0) \subset M$ is the geodesic ball centered at $p_0$ of radius $r_0$.

In particular we have

(3.3) \hspace{1cm} B(p_0, r_0) \cap g_1(B(p_0, r_0)) = \emptyset.

Inequality (3.2) and equation (3.3) give that

(3.4) \hspace{1cm} B(p_0, r_0) \cap g_n(B(p_0, r_0)) = \emptyset \quad \forall n \in \mathbb{Z}^+.

Which also implies

(3.5) \hspace{1cm} g_n(B(p_0, r_0)) \cap g_m(B(p_0, r_0)) = \emptyset \quad \forall n \neq m \in \mathbb{Z}^+.

Otherwise there exist two positive integers $n_0 > m_0 \geq 1$ and $q_1, q_2 \in B(p_0, r_0)$ such that $g_{n_0}(q_1) = g_{m_0}(q_2)$. Since $g_t$ is a flow, $g_{n_0-m_0}(q_1) = q_2$ which contradicts equation (3.4).
On the other hand, for any $t_0 > 0$ (we use Proposition 18.18 in [Lee13]), we have

\[ \frac{d}{dt} \left|_{t=t_0} \right. \text{Vol}(g_t(B(p_0, r_0))) = \int_{B(p_0, r_0)} \frac{d}{dt} \left|_{t=t_0} \right. g_t^*(d\text{Vol}) \]

\[ = \int_{B(p_0, r_0)} g_0^* (L_{\text{grad} f_K} (d\text{Vol})) \]

\[ = \int_{B(p_0, r_0)} g_0^* (\text{Div}(\text{grad}(f_K)) d\text{Vol}) \]

\[ = \int_{g_0(B(p_0, r_0))} \Delta f_K d\text{Vol}. \]

From equation (3.1) we have

\[ \frac{d}{dt} \left|_{t=t_0} \right. \text{Vol}(g_t(B(p_0, r_0))) |_{t=t_0} \geq 0 \quad \forall t_0 > 0. \]

That is the flow $g_t$ is volume non-decreasing.

Thus, equation (3.5) and inequality (3.7) give that

\[ \text{Vol}(M) \geq \text{Vol}(\bigcup_{k=1}^{\infty} g_k(B(p_0, r_0))) = \sum_{k=1}^{\infty} \text{Vol}(g_k(B(p_0, r_0))) \geq \sum_{k=1}^{\infty} \text{Vol}(B(p_0, r_0)) = \text{Vol}(B(p_0, r_0)) \]

which contradicts our assumption that $M$ has finite volume.

Thus, $f_K$ is a constant function. By equation (3.1) we know that

\[ f_K \equiv 0 \text{ on } M. \]

Therefore, $K$ is trivial.

The verification of condition $\circ$ is complete.

We now begin the second step in the outline of the proof of Proposition 3.1 to complete the proof.

**Proof of Proposition 3.1.** We first recall a subharmonic function $g_C$ constructed in step-1 in the proof of [Fra89, Theorem 10.1] and then use the result in our first step to show that this function $g_C$ is the zero constant function. On the other hand, by work in [Fra89, Section 10] one knows that this function is not always zero, which will arrive at a contradiction.
Assume that $\text{Aut}_0(D)$ is not semisimple. Recall the $c$ is the abelian radical and we may assume that
\[ \dim(c) = l > 0 \]
for some integer $l$.

It follows from [Fra89, Lemma 10.3] that
\[ \Gamma \subset \mathcal{N}(C). \]

Thus, for any $\gamma \in \Gamma$ the map
\[ \text{ad}_C(\gamma) : c \to c \]
is well-defined.

Same as [Fra89, Definition 10.10], we define the modular function
\[ \phi_C : \Gamma \to \mathbb{R} \]
by
\[ \phi_C(\gamma) = \det(\text{ad}_C(\gamma)). \]

It is clear that $\phi_C$ is a homomorphism.

From Proposition 3.6 we know that the quotient $C/C \cap \Gamma$ is compact. By [Fra89, Lemma 10.3] we know that
\[ \Gamma \subset \mathcal{N}(C \cap \Gamma). \]

Then it follows from [Fra89, Lemma 10.12] that for all $\gamma \in \Gamma$,
\[ |\phi_C(\gamma)| = 1 \]
(3.9)
(\text{where the group $N$ in [Fra89, Lemma 10.12] is the abelian radical $C$ of $\text{Aut}_0(D)$ in our case.})

Let $\{X_i(x) \in T^{1,0}D\}_{1 \leq i \leq l}$ be complete holomorphic vector fields on $D$ giving a basis for the Lie algebra $c$ (tensor over $\mathbb{C}$). For $x \in D$, we define
\[ w_C(x) := \wedge_i X_i(x) \in \wedge^k T^{1,0}D \]
and
\[ g_C(x) = \langle w_C(x), w_C(x) \rangle. \]

Similar as in the proof of step-1 above, we have for any $\gamma \in \Gamma$ and $x \in D$,
\[ X_i(\gamma \cdot x) = \frac{d}{dt}(\gamma \cdot (\gamma^{-1} \cdot \exp(t X_i) \cdot x)|_{t=0}) = d\gamma(x) \cdot \text{ad}_C(\gamma) \cdot X_i(x). \]

Thus, for any $\gamma \in \Gamma$ and $x \in D$
\[ g_C(\gamma \cdot x) = \langle w_C(\gamma \cdot x), w_C(\gamma \cdot x) \rangle \]
\[ = \langle \wedge_i X_i(\gamma \cdot x), \wedge_i X_i(\gamma \cdot x) \rangle \]
\[ = (\det(\text{ad}_C(\gamma)))^2 \langle \wedge_i X_i(x), \wedge_i X_i(x) \rangle \] (since $\gamma$ is an isometry)
= g_C(x)

where we apply equation (3.9) for the last equality.

Thus, $g_C$ descends to a function, still denoted by $g_C$, on $M$. Let $\Delta$ is the Beltrami-Laplace operator in the sense of the Kähler-Einstein metric on $M$. Recall that the Kähler-Einstein metric on $M$ has constant Ricci curvature $-1$. It follows from the classical Bochner-Weitzenböck type formula [Fra89, Lemma 10.5] that for all $p \in M$,

$$\frac{1}{2} \Delta g_C(p) = |\nabla w|^2(p) + l \cdot g_C(p).$$

In particular we have for all $p \in M$,

$$\Delta g_C(p) \geq g_C(p) \geq 0. \tag{3.10}$$

Thus we get a subharmonic function $g_C$ on $M$. Recall that $M$ is complete and has finite volume. Then we apply the totally same argument in step-1 to conclude that

$$g_C \equiv 0 \text{ on } M. \tag{3.11}$$

On the other hand, it follow from [Fra89, Corollary 10.9] that

$$g_C \neq 0 \text{ on } M \tag{3.12}$$

which contradicts equation (3.11).

[Fra89, Corollary 10.9] follows directly by [Fra89, Lemma 10.8]. We remark here that in the proof of [Fra89, Lemma 10.8] the manifold $\Omega$ is only required to satisfy that $\Omega$ does not contain any holomorphic embedding of a complex line. Since $D$ is a bounded pseudoconvex domain, by the classical Liouville’s theorem one knows that $D$ can not contain any holomorphic embedding of a complex line. The proof is complete.

Remark 3.7. In the proof of Proposition 3.1, besides the existence of a complete Kähler-Einstein metric of negative Ricci curvature on $D$, the assumption that $D$ is a bounded domain is only applied in (3.12) to arrive at a contradiction to (3.11). Actually the proof of Proposition 3.1 yields the following result.

Theorem 3.8. Let $\overline{M}$ be a complex manifold which admits a complete Kähler-Einstein metric of negative Ricci curvature and an open manifold quotient of finite volume with respect to the Kähler-Einstein measure. If $\overline{M}$ does not contain any holomorphic embedding of a complex line, then $\text{Aut}_0(\overline{M})$ is semisimple.
3.2. $\text{Aut}_0(D)$ has no compact factor. In this subsection we will finish the proof of Proposition 3.2.

**Proof of Proposition 3.2.** We argue by contradiction.

Assume that $\text{Aut}_0(D)$ contains a nontrivial compact factor $I$.

By Proposition 3.1 we know that $\text{Aut}_0(D)$ is semisimple. Then we may assume that $K \subset \text{Aut}_0(D)$ is the maximal compact factor containing $I$. On the level of Lie algebras, the Lie algebra $\mathfrak{k}$ of $K$ is a factor of $\mathfrak{g}$. Since $K$ is characteristic in $\text{Aut}_0(D)$, $\Gamma \subset N(K)$. So the map $\text{ad}_K(\gamma) : \mathfrak{k} \to \mathfrak{k}$ is well-defined for any $\gamma \in \Gamma$. Since the Killing form is a canonical bi-invariant metric on $\mathfrak{k}$, for any $\gamma \in \Gamma$

$$\text{ad}_K(\gamma) : \mathfrak{k} \to \mathfrak{k}$$

is an isometry.

Similar to the argument as in step-1 in the proof of Proposition 3.1 we let \{X_i\}_{1 \leq i \leq k} be an orthonormal basis for $\mathfrak{k}$ where $k = \dim(\mathfrak{k})$. And define a function

$$f_K : D \to \mathbb{R}$$

by

$$f_K(x) = \sum_{i=1}^{k} |X_i(x)|^2$$

where $X_i(x) = \frac{d}{dt}(\exp(tX_i) \cdot x)|_{t=0}$.

For any $\gamma \in \Gamma$,

$$X_i(\gamma \cdot x) = \frac{d}{dt}(\gamma \cdot (\gamma^{-1} \cdot \exp(tX_i) \cdot \gamma \cdot x))|_{t=0}
= d\gamma(x) \cdot \text{ad}_K(\gamma) \cdot X_i(x).$$

As above we know that $\text{ad}_K(\gamma)$ acts on $\mathfrak{k}$ as an isometry. By Theorem 2.1 we also know that $\gamma$ acts on $D$ as an isometry. Thus, we have for any $\gamma \in \Gamma$ and $x \in D$,

$$f_K(\gamma \cdot x) = f_K(x).$$

So $f_K$ descends to a function, still denoted by $f_K$, on $M = D/\Gamma$.

Then we apply [Fra89, Lemma 10.15] to get that for all $p \in D$,

$$\Delta f_K(p) \geq f_K(p) \geq 0.$$ 

Again we get a nonnegative subharmonic function on $M$. Recall that Theorem 2.1 tells that $D$ is complete. In particular $M$ is complete and has finite volume. Then we can apply the same argument as in step-1 in the proof of Proposition 3.1 to get

$$f_K \equiv 0 \text{ on } M$$

which is a contradiction since $k = \dim(K) \geq 1$. \qed
Similar to Remark 3.7, the proof of Proposition 3.2 yields the following result.

**Theorem 3.9.** Let \( \overline{M} \) be a complex manifold which admits a complete Kähler-Einstein metric of negative Ricci curvature and an open manifold quotient of finite volume with respect to the Kähler-Einstein measure. If \( \overline{M} \) does not contain any holomorphic embedding of a complex line, then \( \text{Aut}_0(\overline{M}) \) has no nontrivial compact factor.

**Remark 3.10.** Following Question 1.1, it is very interesting to know whether the assumption, that \( \overline{M} \) does not contain any holomorphic embedding of a complex line, in Theorem 3.8 and 3.9 can be removed. And we also hope that Theorem 3.8 and 3.9 can be applied to study other related problems.

4. **Bounded domains with finite-volume quotients in \( \mathbb{C}^2 \)**

In this section we finish the proofs of Theorem 1.3 and 1.5. We begin by recalling the following theorem of Nadel, which is crucial in the proofs of Theorem 1.3 and 1.5.

**Theorem 4.1.** [Nad90, Theorem 5.1] Let \((M, g)\) be a connected, simply connected, complete Kähler-Einstein surface and let \(G\) be a connected Lie group acting biholomorphically and isometrically on \((M, g)\). Assume that \(G\) acts effectively and that \(\dim \mathbb{R}G \geq 6\). Then \((M, g)\) is a Hermitian symmetric space.

Let \(D \subset \mathbb{C}^2\) be a contractible bounded pseudoconvex domain. By Theorem 2.1 one knows that there always exists a complete Kähler-Einstein metric \(\omega\) on \(D\) and \(\text{Aut}(D)\) acts biholomorphically and isometrically on \((D, \omega)\). So we can view \(D\) as a complete Kähler-Einstein surface. By the classification of Hermitian symmetric space of complex dimension two it follows that \(D\) is biholomorphic to either \(B\) or \(D \times D\). Therefore, by Theorem 4.1 it remains to prove Theorem 1.3 for the case that

\[
\dim \mathbb{R}(\text{Aut}(D)) \leq 5.
\]

Since \(D\) has a finite-volume manifold quotient \(M\), it follows by Theorem 3.8 and 3.9 that \(\text{Aut}_0(D)\) is semisimple without compact factor. As in [Nad90, Section 6], by the classification of complex semisimple Lie algebras it is not possible for \(\text{Aut}_0(D)\) to have real dimension 1, 2, 4 or 5. For the remaining of this section we always assume that

\[
(4.1) \quad \dim \mathbb{R}(\text{Aut}_0(D)) = 3.
\]

We will arrive at a contradiction.

The following result of Shabat is crucial in this section.
Theorem 4.2. [Sha77, Theorem 2] Let $D \subset \mathbb{C}^2$ be a contractible bounded pseudoconvex domain. Then one of the following three assertions is valid:

(i) The quotient $D/\text{Aut}_0(D)$ is a separable manifold and $D \to D/\text{Aut}_0(D)$ is a locally trivial fibration.
(ii) There exists a point in $D$ fixed under the action of $\text{Aut}(D)$.
(iii) There exists a complex analytically imbedded one-dimensional disk $D$ in $D$ which is $\text{Aut}(D)$-invariant.

In the proof of Theorem 4.2 in [Sha77] assertion (i) happens only if the dimensions of the orbits of all the points are the same, which is [Sha77, Proposition 1]. Assertion (ii) and (iii) happen only if the dimensions of the orbits are not the same, which is [Sha77, Proposition 2]. Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. First recall that

\begin{equation}
\dim_{\mathbb{R}}(\text{Aut}_0(D)) = 3.
\end{equation}

Let $\Gamma = \pi_1(M)$. It follows from (4.2) and Lemma 3.3 that $\Gamma$ is an infinite group. That is,

\begin{equation}
|\Gamma| = \infty.
\end{equation}

Since $\Gamma$ acts properly discontinuously on $D$, assertion (ii) of Theorem 4.2 can not hold; otherwise it contradicts to (4.3).

If assertion (iii) of Theorem 4.2 holds, then $D/\Gamma$ is a complex one-dimensional surface. Since $D$ is contractible, by our assumption we know that $\chi(\Gamma) = \chi(M) > 0$. The complex one-dimensional surface of positive Euler characteristic number is homeomorphism to either a sphere or a disk, which in particular implies that $\Gamma$ is trivial, which contradicts (4.3).

Thus, only assertion (i) may happen. From the discussion above (or [Sha77, Proposition 1]) we may assume that the dimensions of the orbits of all the points are the same. We will arrive at a contradiction.

By Proposition 3.8 we know that $\text{Aut}_0(D)$ is semisimple. Thus, we apply [Nad90, Lemma 6.1] to get the maximal compact subgroup $K$ of $\text{Aut}_0(D)$ is real one dimensional. Since the isotropy groups of $\text{Aut}_0(D)$ are compact, in particular they have dimension $\leq 1$. Thus, from (4.2) there are only two cases to consider:

Case 1. $\forall x \in D$, $\dim_{\mathbb{R}}(\text{Aut}_0(D) \cdot x) = 2$.
Case 2. $\forall x \in D$, $\dim_{\mathbb{R}}(\text{Aut}_0(D) \cdot x) = 3$.

First we consider Case 1.
In this case for every point \( x \in D \), the isotropy group
\[
K_x := \{ \phi \in \text{Aut}_0(D); \; \phi(x) = x \}
\]
is a one-dimensional compact subgroup of \( \text{Aut}_0(D) \). Define
\[
F := \{ y \in D; \; \phi(y) = y, \; \forall \phi \in K_x \}.
\]
Then \( F \) is a closed complex submanifold of \( D \). Consider the map
\[
H : \text{Aut}_0(D) \cdot x \times F \to D
\]
\[
(\gamma(x), y) \mapsto \gamma(y).
\]
It follows from \([\text{Nad90}, \text{Lemma 6.1}]\) that for all \( x \in D \), the isotropy group \( K_x \) is a maximal subgroup of \( \text{Aut}_0(D) \). Thus, this map \( H \) is bijective. Next we will show that the map \( H \) is biholomorphic. It suffices to show that \( H \) is holomorphic.

The following argument is due to Frankel \([\text{Fra89}, \text{Lemma 11.9}]\). For completeness, we give an outline of the proof for the holomop hicity of \( H \) here. One may refer to \([\text{Fra89}, \text{Lemma 11.9}]\) or \([\text{Nad90}, \text{Page 2018}]\) for more details.

To prove that \( H \) is holomorphic, by the classical Hartogs’ or Osgood’s theorem it suffices to show that \( H \) is holomorphic separately in each factor. Firstly it is clear that the map \( H \) is holomorphic in the second variable because \( H(\gamma(x), \cdot) = \gamma(\cdot) \).

**Proof that \( H \) is holomorphic in the first variable.** It suffices to show that for a fixed point \( y \in F \) the induced map, still denoted by \( H \), \( H : \text{Aut}_0(D)/K_x \to D \) defined by \( H(*) = H(*)(x), y \) is holomorphic (for one of the two choices of homogeneous complex structures on \( \text{Aut}_0(D)/K_x \)). It is reduced to show that the orbit \( \text{Aut}_0(D) \cdot y \) is a complex submanifold of \( D \). For this by homogeneity it suffices to show that the real tangent space to the orbit \( \text{Aut}_0(y) \) at \( y \) is \( J \)-invariant where \( J \) is the complex structure tensor for \( D \). At the point \( x \in D \) we have the following direct sum decomposition of real tangent vector spaces as
\[
T_x(D) = T_x(\text{Aut}_0(D) \cdot x) \oplus T_x(F).
\]
Since \( K_x \) acts trivially on the second summand the nontrivially on the first, we see that the summands are \( J \)-invariant since the action of \( K_x \) on \( T_x(D) \) commutes with the action of \( J \).

Therefore, we have that the map \( H \) is biholomorphic. From \([\text{Nad90}, \text{Lemma 6.1}]\) we know that the orbit \( \text{Aut}_0(D) \cdot x = (\text{Aut}_0(D)/K_x) \cdot x \) is biholomorphic to the unit disk \( \mathbb{D} \). Since \( D \) is contractible, \( F \) is simply-connected. The uniformization theorem of Riemann surfaces implies that \( F \)...
must be biholomorphic to \( \mathbb{P}^1, \mathbb{C} \) or \( \mathbb{D} \). Thus, \( \text{Aut}_0(F) \geq 2 \). Therefore, we have

\[
\dim_{\mathbb{R}} \text{Aut}_0(D) \geq \dim_{\mathbb{R}} \text{Aut}_0(\mathbb{D}) + \dim_{\mathbb{R}} \text{Aut}_0(F) > 3
\]

which contradicts to (4.2).

Now we consider Case 2.

As above we know that only assertion (i) of Theorem 4.2 may happen. Thus, \( D \to D/\text{Aut}_0(D) \) is a locally trivial fibration. Since \( D \) is contractible, it follows from the standard exact homotopy sequence of the fibration \( D \to D/\text{Aut}_0(D) \) that \( D/\text{Aut}_0(D) \) is contractible (One may also apply the work in [Oli76] in a more general setting to conclude that \( D/\text{Aut}_0(D) \) is contractible). Thus, by our assumptions that \( \dim_{\mathbb{R}} \{ \text{Aut}_0(D) \cdot x \} = 3 \) for all \( x \in D \) and \( \dim_{\mathbb{R}}(D) = 4 \) we have \( D/\text{Aut}_0(D) \) is homeomorphic to the real line. That is,

\[
D/\text{Aut}_0(D) \cong \mathbb{R}.
\]

Recall that \( \Gamma = \pi_1(M) \) and \( \Gamma_0 = \Gamma \cap \text{Aut}_0(D) \). The following effective action of the group \( \Gamma/\Gamma_0 \) on \( D/\text{Aut}_0(D) \) is well defined:

\[
\gamma \Gamma_0 \times D/\text{Aut}_0(D) \to D/\text{Aut}_0(D) \]

\[
(\gamma \Gamma_0, [x]) \mapsto [\gamma(x)].
\]

Set

\[
T = (D/\text{Aut}_0(D))/(\Gamma/\Gamma_0).
\]

It is not hard to see that \( T \) is a manifold and the action above induces a natural map

\[
\theta : M \to T
\]

defined by \( \theta(p) = [\hat{p}] \) where \( \hat{p} \in D \) is a lift point of \( p \), which is a locally trivial fibration (one may see [Sha77, Page 140] for more details). Since \( D/\text{Aut}_0(D) \cong \mathbb{R} \), the manifold \( T \) is homeomorphic to either \( \mathbb{R} \) or the unit circle \( S^1 \).

Case 2-1. \( T \cong \mathbb{R} \). Then we have \( \Gamma/\Gamma_0 \) is trivial. That is, \( \Gamma = \Gamma_0 \). It follows from Lemma 3.3 that \( \Gamma = \Gamma_0 \) is a lattice of \( \text{Aut}_0(D) \). Let \( K \) be a maximal compact subgroup of \( \text{Aut}_0(D) \). Thus, \( \Gamma \) is also a lattice of \( \text{Aut}_0(D)/K \). From [Nad90, Lemma 6.1] we know that \( \Gamma/\text{Aut}_0(D)/K \) is a hyperbolic surface of finite-volume. In particular, we have that the Euler characteristic number

\[
\chi(\Gamma) < 0
\]

which contradicts to our assumption that \( \chi(M) = \chi(\Gamma) > 0 \) because \( D \) is contractible.
Case 2-2. $T \cong S^1$. It is clear that $\chi(T) = 0$. Since $\theta : M \to T$ is a fibration, we have $\chi(M) = \chi(T) \times \chi(B) = 0$ where $B$ is a fiber. Then, we get a contradiction since $\chi(M) = \chi(\Gamma) > 0$ because $D$ is contractible.

Before proving Theorem 1.5, let us recall some basic facts of the bounded pseudoconvex domain constructed by Griffiths [Gri71]. Let $V$ be an irreducible, smooth, quasi-projective algebraic variety over the complex numbers. The main results in [Gri71] are

**Theorem 4.3** (Griffiths). *Given a point $p \in V$, there is a Zariski neighborhood $U$ of $p$ in $V$ such that*

(i) the universal covering $D$ of $U$ is topologically a cell, in particular it is contractible.

(ii) $D$ is biholomorphic to a bounded pseudoconvex domain.

(iii) There exists a complete Kähler metric $ds^2$ on $U$ such that $(U, ds^2)$ has finite-volume and uniformly negative holomorphic sectional curvatures.

We just consider the case that $\dim_C U = 2$.

It follows from [Gri71, Lemma 2.2] that the Zariski neighborhood $U$ in Theorem 4.3 satisfies that there exists a Riemann surface $S_{g_1, n_1}$ of genus $g_1$ with $n_1$ punctures and a rational holomorphic map

$$\pi : U \to S_{g_1, n_1},$$

which is a locally trivial smooth fibration such that each fiber $\pi^{-1}(s)$ is a Riemann surface $S_{g_2, n_2}$ of genus $g_2$ with $n_2$ punctures. It is clear that both $S_{g_1, n_1}$ and $S_{g_2, n_2}$ have negative Euler characteristic numbers. Thus, the universal covering $D$ of $U$ is a disc fibration over the unit open disc.

Now we are ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** First by Theorem 4.3 one knows that $D$ is biholomorphic to a bounded pseudoconvex domain. By Theorem 2.1 there always exists a complete Kähler-Einstein metric $\omega$ on $D$, which descends to a complete Kähler-Einstein metric, still denoted by $\omega$ on $U$ because $\pi_1(U) \subset \text{Aut}(D)$. In particular, the Ricci form $\text{Ric}_{(U, \omega)} = -\omega$. Then one may apply [Gri71, Proposition 7.3] to get that

$$\text{Vol}((U, \omega)) < \infty.$$

Since both the base and fibers of locally trivial smooth fibration $\pi : U \to S_{g_1, n_1}$ are Riemann surfaces of negative Euler characteristic numbers, the Euler characteristic number

$$\chi(U) = \chi(S_{g_1, n_1}) \times \chi(S_{g_2, n_2}) > 0.$$
By Theorem 4.3 of Griffiths we know that the universal covering $D$ of $(U, \omega)$ is biholomorphic to a contractible bounded pseudoconvex domain. Thus, one may apply Theorem 1.3 to get that either the automorphism group $\text{Aut}(D)$ is discrete, or $D$ is biholomorphic to $B$, or biholomorphic to $\mathbb{D} \times \mathbb{D}$. From [IN05, Theorem 1] we know that $D$ cannot be biholomorphic to the complex two-dimensional unit ball $B$. From our assumption we know that $D$ is not biholomorphic to the bi-disk $\mathbb{D} \times \mathbb{D}$. Therefore, the conclusion follows. That is, $\text{Aut}(D)$ is discrete.

5. HHR/USq Complex Manifolds with Finite-Volume Quotients

In this section we firstly finish the proof of Theorem 1.6 by applying Theorem 1.3, and then prove Theorem 1.7 and 1.11.

Proof of Theorem 1.6. Since $D \subset \mathbb{C}^2$ is HHR/USq, it follows from [Yeu09, Lemma 2] that $D$ is a bounded pseudoconvex domain. Since $\dim_\mathbb{C}(D) = 2$, it follows from Proposition 2.4 that the signature

$$\text{sign}(\chi(M)) = (-1)^2 = 1 > 0.$$ 

Then the conclusion directly follows by Theorem 1.3.

The proof of Theorem 1.6 highly depends on the assumption $\dim_\mathbb{C}(D) = 2$. For higher dimensional case, before we prove Theorem 1.7 and 1.11, we prepare two propositions which have their own interests: one is to show that the group $\text{Aut}_0(D)$ has finite center; and the other one is to show that up to a finite-index subgroup, $\Gamma$ must split such that one factor is just from $\Gamma_0$.

We first show that

**Proposition 5.1** (Finite Center). The group $\text{Aut}_0(D)$ has finite center.

**Proof.** Let $\Gamma^{\text{sol}}$ denote the unique maximal normal solvable subgroup of $\Gamma_0$. Since $\Gamma^{\text{sol}}$ is unique, it is a characteristic subgroup in $\Gamma_0$. So it is normal in $\Gamma$. Since $\Gamma^{\text{sol}}$ is solvable, it is amenable. From Proposition 2.4 and [LÖ2, Theorem 7.2(1),(2)] we have $\Gamma^{\text{sol}}$ is finite. By Lemma 2.3 we know that $\Gamma^{\text{sol}}$ is torsion-free. Thus,

$$\Gamma^{\text{sol}} = \{e\}.$$ 

Let $\text{Aut}_0^{\text{sol}}(D)$ be the solvable radical of $\text{Aut}_0(D)$ and $\text{Aut}_0^{ss}(D)$ be the connected semisimple Lie group $\text{Aut}_0(D)/\text{Aut}_0^{\text{sol}}(D)$. Then we apply a formula of Prasad [Pra76, Part (2) of Lemma 6] to get

$$\text{rank}(\Gamma^{\text{sol}}) = \chi(\text{Aut}_0^{\text{sol}}(D)) + \text{rank}(Z(\text{Aut}_0^{ss}(D)))$$

where $\chi(\text{Aut}_0^{\text{sol}}(D))$ is the dimension of $\text{Aut}_0^{\text{sol}}(D)$ minus that of its maximal compact subgroup, and $\text{rank}(Z(\text{Aut}_0^{ss}(D)))$ is the rank of the center of $\text{Aut}_0^{ss}(D)$. 

Since $\Gamma^{sol} = \{e\}$, we have

(i) $\chi(\text{Aut}^{sol}_0(D)) = 0$. In particular, $\text{Aut}^{sol}_0(D)$ is compact.

(ii) $\text{rank}(Z(\text{Aut}^{ss}_0(D))) = 0$. Thus, $Z(\text{Aut}^{ss}_0(D))$ is finite.

Since $\text{Aut}^{sol}_0(D)$ is both compact and solvable, it is a torus $T$. Thus, the automorphism group of $\text{Aut}^{sol}_0$ is discrete. Meanwhile, it follows from the exact sequence

\[
\{e\} \to \text{Aut}^{sol}_0(D) \to \text{Aut}_0(D) \to \text{Aut}^{ss}_0(D) \to \{e\}
\]

that the natural conjugation action of the group $\text{Aut}^{ss}_0(D)$ on $\text{Aut}^{sol}_0(D)$ is trivial. In particular, $\text{Aut}^{sol}_0(D)$ is a compact factor of $\text{Aut}_0(D)$. Thus, from Proposition 3.2 we know that

$\text{Aut}^{sol}_0(D) = \{e\}$.

From the exact sequence above we get

$\text{Aut}_0(D) = \text{Aut}^{ss}_0(D)$.

By (ii) we know that $Z(\text{Aut}_0(D)) = Z(\text{Aut}^{ss}_0(D))$ is finite. \hfill \Box

The following result is crucial in the proof of Theorem 1.7.

**Theorem 5.2** (Split). Let $D$ be a contractible HHR/USq complex manifold with a finite-volume manifold quotient whose fundamental group is $\Gamma$. Then there exists a finite index subgroup $\Gamma'$ of $\Gamma$ such that

$\Gamma' \cong \Gamma_0' \times \Gamma'/\Gamma_0'$

where $\Gamma_0' = \Gamma' \cap \text{Aut}_0(D)$ which is a finite index subgroup of $\Gamma_0$.

**Proof.** Consider the exact sequence

(5.1) \[
\{e\} \to \Gamma_0 \to \Gamma \to \Gamma/\Gamma_0 \to \{e\}.
\]

Our aim is to show that after replacing $\Gamma$ by a finite index subgroup $\Gamma'$ if necessary, the exact sequence above splits as a direct product.

It is well-known [ML95, Chapter IV, Theorem 8.8] that such an extension like equation (5.1) is determined by

(i) a representation $\rho : \Gamma/\Gamma_0 \to \text{Out}(\Gamma_0)$, and

(ii) a cohomology class in $H^2(\Gamma/\Gamma_0; Z(\Gamma_0))_\rho$ where $Z(\Gamma_0)|_{\rho}$ is a $\Gamma/\Gamma_0$-module via $\rho$.

In particular, if the representation $\rho$ and the center $Z(\Gamma_0)$ are both trivial, we get the trivial extension. That is, $\Gamma = \Gamma_0 \times \Gamma/\Gamma_0$.

First from Proposition 3.1, 3.2 and Lemma 3.3 we know that $\Gamma_0$ (or any finite index subgroup of $\Gamma_0$) is a lattice in a semisimple Lie group $\text{Aut}_0(D)$ without compact factors. From Proposition 5.1 we know that $\text{Aut}_0(D)$ has finite center. So the center $Z(\Gamma_0)$ is finite. By Lemma 2.3 we know that $\Gamma_0$
is torsion-free. So the center $Z(\Gamma_0)$ is trivial. Thus, it suffices to show that after replacing $\Gamma$ by a finite index subgroup $\Gamma'$ if necessary, the representation 

$$\rho : \Gamma/\Gamma_0 \to \text{Out}(\Gamma_0)$$

is trivial.

Consider the exact sequence

$$\{e\} \to \text{Aut}_0(D) \to <\text{Aut}_0(D),\Gamma> \to \Gamma/\Gamma_0 \to \{e\}$$

where $<\text{Aut}_0(D),\Gamma>$ is the smallest subgroup of Aut($D$) containing Aut$_0(D)$ and $\Gamma$. The conjugation action of $\Gamma$ on Aut$_0(D)$ induces a representation

$$\rho_1 : \Gamma/\Gamma_0 \to \text{Out(Aut}_0(D)).$$

From Proposition 3.1 we know that Aut$_0(D)$ is semisimple. By [Hel01, Chapter IX, Theorem 5.4] we know that Out(Aut$_0(D))$ is finite. Up to a finite index subgroup of $\Gamma$ if necessary, we may assume that the representation $\rho_1$ is trivial. This gives a representation

$$\rho_2 : \Gamma/\Gamma_0 \to \text{Aut}_0(D)/Z(\text{Aut}_0(D)).$$

Since the conjugation action of $\Gamma$ on Aut$_0(D)$ preserves $\Gamma_0$, the image

$$\rho_2(\Gamma/\Gamma_0) \subset N_H(\Gamma_0)/\Gamma_0$$

where $H = \text{Aut}_0(D)/Z(\text{Aut}_0(D)).$

By Proposition 3.1, 3.2 and Lemma 3.3 we know that $\Gamma_0$ is a lattice in a semisimple Lie group Aut$_0(D)$ without compact factors. Let $K < \text{Aut}_0(D)$ be a maximal compact subgroup. From Lemma 3.3 we know that the manifold $\Gamma_0/\text{Aut}_0(D)/K$ is a local symmetric space of nonpositive sectional curvature with finite-volume. It is clear that

$$N_H(\Gamma_0)/\Gamma_0 \subset \text{Isom}(\Gamma_0/\text{Aut}_0(D)/K)$$

It is well-known that Isom$(\Gamma_0/\text{Aut}_0(D)/K)$ is a finite group (one may refer to [Yam85, Theorem 2] for a more general statement). Thus, the image $\rho_2(\Gamma/\Gamma_0)$ is finite. Up to a finite index subgroup of $\Gamma$ if necessary, we may assume that the representation $\rho_2$ is trivial. Thus, the conjugation action of $\Gamma$ on $\Gamma_0$ is only by inner automorphisms of $\Gamma_0$. As above we know that the center $Z(\Gamma_0)$ is trivial. Therefore, the representation

$$\rho : \Gamma/\Gamma_0 \to \text{Out}(\Gamma_0)$$

is trivial. The proof is complete.

Now we are ready to prove Theorem 1.7 and 1.11.
Proof of Theorem 1.7. Case 1: $\text{Aut}(D)$ is not discrete.

First from Theorem 5.2 we get a finite index subgroup $\Gamma'$ of $\Gamma$ such that
\[ \Gamma' \cong \Gamma'_0 \times \Gamma'/\Gamma'_0. \]

Recall that we assume that $\Gamma$ is irreducible. Thus, either $\Gamma'_0$ is trivial or $\Gamma'/\Gamma'_0$ is trivial. Since $\text{Aut}(D)$ is not discrete, from Lemma 3.3 $\Gamma_0$ has infinite elements. So we have $\Gamma'/\Gamma'_0$ is trivial. Thus,
\[ \Gamma' \cong \Gamma'_0. \]

In particular,
\[ [\Gamma : \Gamma_0] < \infty. \]

Let $K < \text{Aut}_0(D)$ be a maximal compact subgroup. By Proposition 3.1, 3.2 and 5.1 we know that the quotient $\text{Aut}_0(D)/K$ is a noncompact type symmetric space without compact or Euclidean factors. Thus, from Lemma 3.3 we know that $\Gamma_0 \setminus \text{Aut}_0(D)/K$ is aspherical and has bounded geometry. Actually the injectivity radius of the universal cover $\text{Aut}_0(D)/K$ is infinite because it is nonpositively curved.

On the other hand, by our assumption that $D$ is contractible and Theorem 2.2 we know that the quotient $D/\Gamma_0$ is also aspherical and has bounded geometry (in the sense of Kähler-Einstein metric).

By Proposition 2.4 we know that the Euler characteristic number
\[ \chi(\Gamma) \neq 0. \]

Since $\Gamma_0$ is a subgroup of $\Gamma$ of finite index,
\[ \chi(\Gamma_0) \neq 0. \]

By applying [CG86, Corollary 5.2] we know that
\[ \dim(D) = \dim(\text{Aut}_0(D)/K). \]

For any $x \in D$ we let $K_x < \text{Aut}_0(D)$ be the isotropy group fixing $x$. It is clear that
\[ \dim(\text{Aut}_0(D)/K) \leq \dim(\text{Aut}_0(D)/K_x) \]
and
\[ \dim(\text{Aut}_0(D)/K_x) \leq \dim(D). \]

Therefore, we get
\[ \dim(\text{Aut}_0(D)/K) = \dim(\text{Aut}_0(D)/K_x) \]
which gives that
\[ K_x = K, \quad \forall x \in D. \]

That is, $D$ is homogenous. Since it has a quotient of finite-volume, $D$ is symmetric (one may see works of Borel-Hano-Koszul [Han57] for details).
Case 2: Aut(D) is discrete. It suffices to show that
\[ [\text{Aut}(D) : \pi_1(M)] < \infty. \]

The following argument is standard. Let \( F_D \) be a fundamental domain for the action of Aut(D) on D. We choose the Kähler-Einstein measure induced by the Kähler-Einstein metric on D. By Theorem 2.1 of Mok-Yau we know that Aut(D) acts on D as isometries. Since Aut(D) is discrete,
\[
0 < \text{Vol}(F_D) < \infty.
\]

Similarly we let \( F_M \) be a fundamental domain for the action of \( \pi_1(M) \) on D. Since M has finite volume,
\[
0 < \text{Vol}(F_M) < \infty.
\]

Hence,
\[ [\text{Aut}(D) : \pi_1(M)] < \infty. \]

Otherwise; let \( \{ \gamma_i \}_{i \geq 1} \) be a sequence of coset representatives for \( \pi_1(M) \) in \( \text{Aut}(D) \), then
\[
F_M = \bigcup_{i \geq 1} \gamma_i \cdot F_D.
\]

Since \( F_D \) is a fundamental domain, \( \text{Vol}(\gamma \cdot F_D \cap F_D) = 0 \) and \( \text{Vol}(\gamma \cdot F_D) = \text{Vol}(F_D) \) for all \( \gamma \in \text{Aut}(D) \). Thus,
\[
\text{Vol}(F_M) = \sum_{i \geq 1} \text{Vol}(\gamma_i \cdot F_D) = \infty,
\]
which is a contradiction. \( \Box \)

Proof of Theorem 1.11. Since \( \Gamma < \text{Aut}_0(D) \),
\[
\Gamma = \Gamma_0.
\]

By Proposition 2.5 we know that \( \text{Aut}_0(D) \) is a Lie group of positive dimension. Similar to the proof of Theorem 1.7, let \( K < \text{Aut}_0(D) \) be a maximal compact subgroup. By Proposition 3.1, 3.2 and 5.1 we know that \( \Gamma_0 \backslash \text{Aut}_0(D)/K \) is aspherical and has bounded geometry. Meanwhile, by Theorem 2.2 and our assumption on finite-volume we have that \( D/\Gamma_0 \) is also aspherical and has bounded geometry (in the sense of Kähler-Einstein metric). By Proposition 2.4 and [CG86, Corollary 5.2] we know that
\[
\dim(D) = \dim(\text{Aut}_0(D)/K).
\]

Then we use the same argument in the end of the proof of Theorem 1.7 to finish the proof. For any \( x \in D \) we let \( K_x < \text{Aut}_0(D) \) be the isotropy group fixing \( x \). It is clear that
\[
\dim(\text{Aut}_0(D)/K_x) \leq \dim(D)
\]
and
\[ \dim(\text{Aut}_0(D)/K) \leq \dim(\text{Aut}_0(D)/K_x). \]

Therefore, we get
\[ \dim(\text{Aut}_0(D)/K) = \dim(\text{Aut}_0(D)/K_x) \]

implying that
\[ K_x = K, \quad \forall x \in D. \]

That is, \( D \) is homogenous. Since it has a quotient of finite-volume, \( D \) is symmetric by Borel-Hano-Koszul [Han57].

In the proofs of Theorem 1.7 and 1.11 the key step is to show that \( \Gamma_0 \) is a lattice of a semisimple Lie group without compact factors of finite center. It is unclear for the relation between the HHR/USq manifold \( D \) and the semisimple Lie group \( \text{Aut}_0(D) \). The following question is interesting.

**Question 5.3.** Let \( D \) be a contractible HHR/USq complex manifold which holomorphically covers a manifold of finite-volume with the fundamental group \( \Gamma \). If \( \Gamma \) is isomorphic to a lattice in an irreducible Hermitian symmetric space \( N \) of noncompact type other than the hyperbolic plane, is \( D \) (anti)biholomorphic to \( N \)?

**Remark 5.4.** If the Kähler-Einstein metric on \( D \) has nonpositive sectional curvature, [BE87, Theorem D] of Ballmann-Eberlein tells that \( D \) and \( N \) are isometric with respect to the Kähler-Einstein metrics.

We end this section by the following result whose proof is a combination of several known results. It gives a positive answer to Question 5.3.

**Proposition 5.5** (Holomorphicity Rigidity). Let \( D \) be a contractible HHR/USq complex manifold which holomorphically covers a manifold of finite-volume whose fundamental group is \( \Gamma \). If \( \Gamma \) is isomorphic to a lattice in an irreducible Hermitian symmetric space \( N \) of noncompact type without Euclidean de Rham factor other than the hyperbolic plane, then \( D \) is (anti)biholomorphic to \( N \).

**Proof.** From Theorem 2.2 we know that \( D/\Gamma \) has bounded geometry (in the sense of Kähler-Einstein metric). It is clear that \( N/\Gamma \) also has bounded geometry. Meanwhile, by Proposition 2.4 we know that the Euler characteristic number
\[ \chi(\Gamma) \neq 0. \]

Since both \( D/\Gamma \) and \( N/\Gamma \) are aspherical of bounded geometry, we apply [CG86, Corollary 5.2] to get
\[ \dim(D) = \dim(N) \]
because $\chi(\Gamma) \neq 0$.

Since $D$ is a HHR/USq complex manifold and $D/\Gamma$ has finite volume, it follows from [Yeu09, Corollary 2] that $D/\Gamma$ is quasi-projective variety of log-general type. Finally, thanks to Jost-Zuo [JZ97, Theorem 2.1] we get that $D$ is (anti)biholomorphic to $N$. \hfill \Box

6. ONE CONJECTURE

In this last section, we begin with a folklore conjecture which is stated in the introduction. And then we apply Theorem 1.7 to provide two partial answers, which are Theorem 1.16 and 1.17.

**Conjecture 6.1** (=Conjecture 1.12). A bounded convex domain with a finite-volume quotient is biholomorphic to a bounded symmetric domain.

In light of Theorem 1.7, whether a one-parameter of automorphism groups of $D$ exists is essential to study Conjecture 6.1. If the boundary of $D$ has certain regularity, it is known that the works in [Fra89, Kim04] can produce a continuous parameter of automorphisms. Now we are ready to prove Theorem 1.16 and 1.17.

**Proof of Theorem 1.16.** Since the fundamental group $\pi_1(M) < \text{Aut}(D)$, firstly by Proposition 2.5 we know that the automorphism group Aut($D$) is non-compact. Thus, from our assumption that the boundary of $D$ is $C^1$-smooth, it follows from the so-called rescaling method in [Fra89, Kim04] that Aut($D$) contains a continuous one parameter subgroup. One may also see [Zim17a, Proposition 5.1] for this point. In particular, Aut($D$) is not discrete. Recall that a bounded convex domain is HHR/USq. Then, by Theorem 1.7 we know that $D$ is biholomorphic to a bounded symmetric domain.

If $D$ is of rank one, that is, the domain $D$ is biholomorphic to the unit ball. Then, we are done.

Assume that $D$ is of rank $\geq 2$, we will arrive at a contradiction. Since $D$ is convex, by the work of Mok and Tsai [MT92, Main Theorem] one may assume that $D$ is the image of the classical Harish-Chandra embedding up to an affine linear transformation of $\mathbb{C}^n$. That is, $D = T \circ \tau \circ \phi(X_0)$ where $T$ is an affine linear transformation of $\mathbb{C}^n$, $\tau$ is the classical Harish-Chandra embedding, $\phi$ is an automorphism of $X_0$ and $X_0$ is a standard Hermitian symmetric manifold of non-compact type and of rank $\geq 2$. It is known that the boundary of the Harish-Chandra embedding $\tau \circ \phi(X_0)$ can not be $C^1$-smooth since it has corners. In particular, $D$ can not have $C^1$-smooth boundary, which contradicts our assumption. \hfill \Box
Proof of Theorem 1.17. It follows from the same argument as in the proof of Theorem 1.16 above, except the step that we apply Theorem 1.6 instead of applying Theorem 1.7 because we do not assume that the fundamental group of the quotient is irreducible.

Remark 6.2. If the bounded domain $D$ has $C^2$ smooth boundary, it is known that there exists a strongly pseudoconvex point $p$ on the boundary of $D$ near which the geometry behaves similarly as the one in the complex hyperbolic unit ball. Under the same conditions in Theorem 1.16 or 1.17, it is interesting to know that without using Theorem 1.6 and 1.7 in this article, whether one can find an orbit in $D$ converging to $p$, which would also imply that $D$ is biholomorphic to the unit ball by works in [Won77].

References

[Bal85] Werner Ballmann, Nonpositively curved manifolds of higher rank, Ann. of Math. (2) 122 (1985), no. 3, 597–609.

[BE87] Werner Ballmann and Patrick Eberlein, Fundamental groups of manifolds of nonpositive curvature, J. Differential Geom. 25 (1987), no. 1, 1–22.

[Ber60] Lipman Bers, Spaces of Riemann surfaces as bounded domains, Bull. Amer. Math. Soc. 66 (1960), 98–103.

[BS87] Keith Burns and Ralf Spatzier, Manifolds of nonpositive curvature and their buildings, Inst. Hautes Études Sci. Publ. Math. (1987), no. 65, 35–59.

[CFKW02] Wing Sum Cheung, Siqi Fu, Steven G. Krantz, and Bun Wong, A smoothly bounded domain in a complex surface with a compact quotient, Math. Scand. 91 (2002), no. 1, 82–90.

[CG85] Jeff Cheeger and Mikhail Gromov, On the characteristic numbers of complete manifolds of bounded curvature and finite volume, Differential geometry and complex analysis, Springer, Berlin, 1985, pp. 115–154.

[CG86] L2-cohomology and group cohomology, Topology 25 (1986), no. 2, 189–215.

[Che04] Bo-Yong Chen, The Bergman metric on Teichmüller space, Internat. J. Math. 15 (2004), no. 10, 1085–1091.

[CY80] Shiu Yuen Cheng and Shing Tung Yau, On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman’s equation, Comm. Pure Appl. Math. 33 (1980), no. 4, 507–544.

[Dem] Jean-Pierre Demailly, Complex analytic and differential geometry, https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.

[Dim92] Alexandru Dimca, Singularities and topology of hypersurfaces, Universitext, Springer-Verlag, New York, 1992.

[DK00] J. J. Duistermaat and J. A. C. Kolk, Lie groups, Universitext, Springer-Verlag, Berlin, 2000.

[Ebe82] Patrick Eberlein, Isometry groups of simply connected manifolds of nonpositive curvature. II, Acta Math. 149 (1982), no. 1-2, 41–69.

[EH90] Patrick Eberlein and Jens Heber, A differential geometric characterization of symmetric spaces of higher rank, Inst. Hautes Études Sci. Publ. Math. (1990), no. 71, 33–44.
[FK15] John-Erik Fornaess and Kang-Tae Kim, *Some problems*, Complex analysis and geometry, Springer Proc. Math. Stat., vol. 144, Springer, Tokyo, 2015, pp. 369–377.

[Fra89] Sidney Frankel, *Complex geometry of convex domains that cover varieties*, Acta Math. **163** (1989), no. 1-2, 109–149.

[Fra91] ———, *Applications of affine geometry to geometric function theory in several complex variables. I. Convergent rescalings and intrinsic quasi-isometric structure*, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), Proc. Sympos. Pure Math., vol. 52, Amer. Math. Soc., Providence, RI, 1991, pp. 183–208.

[Fra95] ———, *Locally symmetric and rigid factors for complex manifolds via harmonic maps*, Ann. of Math. (2) **141** (1995), no. 2, 285–300.

[FW10] Benson Farb and Shmuel Weinberger, *The intrinsic asymmetry and inhomogeneity of Teichmüller space*, Duke Math. J. **155** (2010), no. 1, 91–103.

[GD08] Gabino González-Diez, *Belyi’s theorem for complex surfaces*, Amer. J. Math. **130** (2008), no. 1, 59–74.

[GR15] G. González-Diez and S. Reyes-Carocca, *Families of Riemann Surfaces, Uniformization and Arithmeticity*, ArXiv e-prints (2015).

[Gri71] Phillip A. Griffiths, *Complex-analytic properties of certain Zariski open sets on algebraic varieties*, Ann. of Math. (2) **94** (1971), 21–51.

[Han57] Jun-ichi Hano, *On Kaehlerian homogeneous spaces of unimodular Lie groups*, Amer. J. Math. **79** (1957), 885–900.

[Hel01] Sigurdur Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, vol. 34, American Mathematical Society, Providence, RI, 2001, Corrected reprint of the 1978 original.

[IK99] A. V. Isaev and S. G. Krantz, *Domains with non-compact automorphism group: a survey*, Adv. Math. **146** (1999), no. 1, 1–38.

[Ima83] Yoichi Imayoshi, *Universal covering spaces of certain quasiprojective algebraic surfaces*, Osaka J. Math. **20** (1983), no. 3, 581–598.

[IN05] Yoichi Imayoshi and Minoru Nishimura, *A remark on universal coverings of holomorphic families of Riemann surfaces*, Kodai Math. J. **28** (2005), no. 2, 230–247.

[JZ97] Jürgen Jost and Kang Zuo, *Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasiprojective varieties*, J. Differential Geom. **47** (1997), no. 3, 469–503.

[Kim04] Kang-Tae Kim, *On the automorphism groups of convex domains in \( \mathbb{C}^n \)*, Adv. Geom. **4** (2004), no. 1, 33–40.

[KZ16] Kang-Tae Kim and Liyou Zhang, *On the uniform squeezing property of bounded convex domains in \( \mathbb{C}^n \)*, Pacific J. Math. **282** (2016), no. 2, 341–358.

[Lö2] Wolfgang Lück, *\( L^2 \)-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 44, Springer-Verlag, Berlin, 2002.

[Lee13] John M. Lee, *Introduction to smooth manifolds*, second ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013.

[Lem87] László Lempert, *Complex geometry in convex domains*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 759–765.
RIGIDITY

[LSY04] Kefeng Liu, Xiaofeng Sun, and Shing-Tung Yau, Canonical metrics on the moduli space of Riemann surfaces. I, J. Differential Geom. 68 (2004), no. 3, 571–637.

[LSY05] Kefeng Liu, Xiaofeng Sun, and Shing-Tung Yau, Canonical metrics on the moduli space of Riemann surfaces. II, J. Differential Geom. 69 (2005), no. 1, 163–216.

[Mar] Vladimir Markovic, Caratheodory’s Metrics on Teichmüller Spaces and L-shaped pillowcases, to appear on Duke Math. J.

[McM00] Curtis T. McMullen, The moduli space of Riemann surfaces is Kähler hyperbolic, Ann. of Math. (2) 151 (2000), no. 1, 327–357.

[ML95] Saunders Mac Lane, Homology, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1975 edition.

[Mor08] Charles B. Morrey, Jr., Multiple integrals in the calculus of variations, Classics in Mathematics, Springer-Verlag, Berlin, 2008, Reprint of the 1966 edition.

[MS39] S. B. Myers and N. E. Steenrod, The group of isometries of a Riemannian manifold, Ann. of Math. (2) 40 (1939), no. 2, 400–416.

[MT92] Ngaiming Mok and I Hsun Tsai, Rigidity of convex realizations of irreducible bounded symmetric domains of rank \( \geq 2 \), J. Reine Angew. Math. 431 (1992), 91–122.

[MY83] Ngaiming Mok and Shing-Tung Yau, Completeness of the Kähler-Einstein metric on bounded domains and the characterization of domains of holomorphy by curvature conditions, The mathematical heritage of Henri Poincaré, Part 1 (Bloomington, Ind., 1980), Proc. Sympos. Pure Math., vol. 39, Amer. Math. Soc., Providence, RI, 1983, pp. 41–59.

[Nad90] Alan Michael Nadel, Semisimplicity of the group of biholomorphisms of the universal covering of a compact complex manifold with ample canonical bundle, Ann. of Math. (2) 132 (1990), no. 1, 193–211.

[Oli76] Robert Oliver, A proof of the Conner conjecture, Ann. of Math. (2) 103 (1976), no. 3, 637–644.

[Pra76] Gopal Prasad, Discrete subgroups isomorphic to lattices in Lie groups, Amer. J. Math. 98 (1976), no. 4, 853–863.

[Rag72] M. S. Raghunathan, Discrete subgroups of Lie groups, Springer-Verlag, New York-Heidelberg, 1972, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68.

[Ros79] Jean-Pierre Rosay, Sur une caractérisation de la boule parmi les domaines de \( \mathbb{C}^n \) par son groupe d'automorphismes, Ann. Inst. Fourier (Grenoble) 29 (1979), no. 4, ix, 91–97.

[Roy71] H. L. Royden, Automorphisms and isometries of Teichmüller space, Advances in the Theory of Riemann Surfaces (Proc. Conf., Stony Brook, N.Y., 1969), Ann. of Math. Studies, No. 66. Princeton Univ. Press, Princeton, N.J., 1971, pp. 369–383.

[Sha77] G. B. Shabat, The complex structure of domains that cover algebraic surfaces, Functional Anal. Appl. 11 (1977), no. 2, 135–142.

[Siu91] Yum Tong Siu, Uniformization in several complex variables, Contemporary geometry, Univ. Ser. Math., Plenum, New York, 1991, pp. 95–130.

[Won77] B. Wong, Characterization of the unit ball in \( \mathbb{C}^n \) by its automorphism group, Invent. Math. 41 (1977), no. 3, 253–257.

[Won81] B. Wong, The uniformization of compact Kähler surfaces of negative curvature, J. Differential Geom. 16 (1981), no. 3, 407–420 (1982).
[Wu17] Yunhui Wu, *Scalar curvatures of Hermitian metrics on the moduli space of Riemann surfaces*, Comm. Anal. Geom. 25 (2017), no. 2, 465–484.

[Yam85] Takao Yamaguchi, *The isometry groups of manifolds of nonpositive curvature with finite volume*, Math. Z. 189 (1985), no. 2, 185–192.

[Yau87] Shing-Tung Yau, *Nonlinear analysis in geometry*, Enseign. Math. (2) 33 (1987), no. 1-2, 109–158.

[Yau11] ———, *A survey of geometric structure in geometric analysis*, Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, Surv. Differ. Geom., vol. 16, Int. Press, Somerville, MA, 2011, pp. 325–347.

[Yeu09] Sai-Kee Yeung, *Geometry of domains with the uniform squeezing property*, Adv. Math. 221 (2009), no. 2, 547–569.

[Zim17a] A. Zimmer, *The automorphism group and limit set of a bounded domain II: the convex case*, ArXiv e-prints (2017).

[Zim17b] Andrew M. Zimmer, *Characterizing domains by the limit set of their automorphism group*, Adv. Math. 308 (2017), 438–482.

[Zim18] A. Zimmer, *Smoothly bounded domains covering finite volume manifolds*, ArXiv e-prints (2018).

School of Mathematics, Capital Normal University, Beijing, 100048, China

Department of Mathematics, University of California, Los Angeles, Los Angeles, CA 90095-1555, USA

E-mail address: liu@math.ucla.edu

(Y. W.) Yau Mathematical Sciences Center, Tsinghua University, Haidian District, Beijing 100084, China

E-mail address: yunhui_wu@mail.tsinghua.edu.cn