Emergence of space from non-equilibrium thermodynamics in \( f(R) \) gravity

Hassan Basari V. T.,* P. B. Krishna,† and Titus K. Mathew.‡

Department of Physics, Cochin University of Science
and Technology, Kochi, Kerala 682022, India

Abstract

It has shown that the accelerated expansion of the FRW Universe can be explained as the quest towards the holographic equipartition \( (N_{\text{sur}} = N_{\text{bulk}}) \), satisfies the expansion law \( \frac{dV}{dt} = l_P^2 (N_{\text{sur}} - \epsilon N_{\text{bulk}}) \), from which one can derive the Friedmann equation of the FRW Universe in Einstein gravity [1]. We introduce a generic derivation of the expansion law from the generalized first law of thermodynamics \( -dE = TdS \) and \( dE = TdS + WdV \). The generic derivation provides an expression for \( N_{\text{sur}} \) in terms of entropy \( S \) and the expansion law consistent with gravity theories having different entropy \( S \), like Gauss-Bonnet and more general Lovelock gravity. We extended the same idea to the non-equilibrium situation and obtained the expansion law in \( f(R) \) gravity as a specific case. For this, we used the first law of thermodynamics in non-equilibrium description having the extra entropy production term \( Td_i S \).

* basari@cusat.ac.in
† krishnapb@cusat.ac.in
‡ titus@cusat.ac.in
I. INTRODUCTION

Historically, the connection between gravity and thermodynamics emerges from studies on black hole physics. In 1973, Bekenstein introduced the concept of black hole entropy as a measure of the horizon area of the black hole [2]. Using this, the same author generalised the second law of thermodynamics for a region containing a massive black hole [3]. Parallelly Bardeen et al. formulated the four laws of black hole mechanics, which are analogous to the laws of thermodynamics of an ordinary macroscopic system [4]. During the same time, Hawking [5] showed that the black holes, formed by the collapse of matter, can radiate particles with a thermal spectrum having temperature \( T = \frac{\kappa}{2\pi} \), where \( \kappa \) is the surface gravity at the horizon. The temperature is not a peculiar feature of the black hole alone but a general feature of all spacetime horizons [6–8]. Remarkably, an accelerating observer in a flat spacetime, who possesses a horizon, can attribute a temperature proportional to the acceleration \( a \) as \( T = \frac{a}{2\pi} \) (Unruh temperature) [9]. All these imply that Einstein’s field equations of gravity on the spacetime horizon possess a thermodynamic structure.

Jacobson [10] showed that Einstein’s field equations can be derived by projecting the Clausius relation, \( \delta Q = T dS \) to a local Rindler horizon, where the horizon entropy is proportional to horizon area as proposed by Bekenstein [2]. Here \( \delta Q \) is the energy flux through the local Rindler horizon and \( T \) is the Unruh temperature [9] seen by the accelerated observer near the horizon. Later, Cai and Kim [11] derived the Friedmann equations by applying the first law of the form \( -dE = TdS \) to the apparent horizon of an \((n+1)\)-dimensional FRW Universe. Here \( S = A/4G \) is the horizon entropy following the Bekenstein formula, \( T = \frac{1}{2\pi r_A} \) is the temperature of the horizon and \( -dE = A(\rho + p)Hr_A dt \) is the energy flux through the apparent horizon of radius \( r_A \). These authors extended this result in Gauss-Bonnet gravity and the more general Lovelock gravity by adopting the corresponding entropy.

Padmanabhan has shown that the structure of the Einstein equation near any spherically symmetric horizon assumes the form \( dE = TdS - PdV \), where \( E \) is the energy associated with the horizon, which is proportional to the horizon’s radius, \( P \) here is the pressure term due to the gravitational source (which is different form \( p \) used before) and \( T \) is horizon temperature proportional to surface gravity [12, 13]. Later, Akbar and Cai derived the Friedmann equations by applying the first law of thermodynamics of the form, \( dE = TdS + WdV \) [14], often called as the unified first law, to the apparent horizon of FRW Universe.
For this unified first law, $dE$ here is the distortion in the Misner-Sharp energy of the matter inside the apparent horizon, temperature $T = \kappa/2\pi$, a measure of the surface gravity as said before, and $W = -\frac{1}{2}T^{ab}h_{ab} = \frac{1}{2}(\rho - p)$ is the work density, the orthogonal projection of energy-momentum tensor $T^{\mu\nu}$ to the horizon. Interestingly this form of the first laws found to have the same structure as the first law for the trapping horizons proposed by Hayward [15, 16]. It should be noted that for spherically symmetric horizon the work density $W$, is the effective pressure as defined by Padmanabhan, i.e. $W = -(1/2)T_{ab}h^{ab} = -P$, in the relation $dE = TdS - PdV$ [17].

The results shown above have clearly indicated that gravity is intimately connected to the thermodynamics of spacetime. The thermodynamics of any ordinary system, say like a gas, is described using the macroscopic variables like pressure, temperature, etc., having no significance in the microscopic domain. Hence these macroscopic variables of a system are considered to be ‘emerged’ from the collective behaviour of the constituent microscopic degrees of freedom associated with the system. Similarly, the connection between gravity and thermodynamics implies that macroscopic properties like metric, curvature, etc. are irrelevant in the microscopic domain, but could be emerged in the macroscopic domain due to collective behaviour of some underlying fundamental microscopic degrees of freedom. These motivate the reformulation of gravity as an emergent phenomenon. In this context, Padmanabhan [18] derived Newton’s law of gravity by combining the equipartition law of energy for the degrees of freedom at the horizon and the thermodynamic relation for entropy, $S = E/2T$, where $E$ is the effective gravitational mass, and $T$ is the horizon temperature. On the other hand, following string theory considerations, Verlinde reformulated gravity as an entropic force arising from the natural tendency of material distribution to maximize the entropy [19]. These works enlightened the concept that gravity could be an emergent property in a pre-existing spacetime.

On further extending this idea of emergent paradigm, Padmanabhan has shown that, for an evolving spacetime, the rate of change of gravitational momentum is proportional to the difference in the number of degrees of freedom between the horizon and the bulk within the horizon, i.e. $N_{\text{sur}} - N_{\text{bulk}}$. The evolution will come to a stop when $N_{\text{sur}} = N_{\text{bulk}}$ and is known as the holographic equipartition. So ultimately, it is the departure from the holographic equipartition, which drives the time evolution of spacetime. In line with this, Padmanabhan took one step further to propose that space itself possess an emergent nature.
in the cosmological context [1]. He states that the expansion of the Universe (expansion of the Hubble volume) can be explained as the emergence of space with the progress of cosmic time $t$. It is conceptually difficult in general to consider the time has emerged from any pre-geometric variables. However, this difficulty will not be surfaced in cosmology due to the existence of proper time for the geodesic observers for whom cosmic background radiation appears to be homogeneous and isotropic [1]. It was proposed that the time evolution of the Universe in Einstein’s gravity can be described using the equation, $\frac{dV}{dt} = l_p^2 (N_{\text{sur}} - \epsilon N_{\text{bulk}})$, known as the holographic equipartition principle and later many called it as expansion law, where $V$ is the volume of the apparent horizon of the Universe and $l_p$ is the Planck length. The emergence of space happens to equalize the degrees of freedom ($\text{DoF}$) on the horizon with the $\text{DoF}$ in bulk enclosed by the horizon. Based on this paradigm, Padmanabhan derived the Friedmann equation from the expansion law for a flat FRW Universe in (3+1) Einstein gravity [1]. The expansion law was extended to higher dimensional gravity theories like (n+1) dimensional Einstein gravity, Gauss-Bonnet gravity and more general Lovelock gravity [20] by appropriately modifying the surface degrees of freedom on the boundary surface. An extension of this procedure to non-flat FRW Universe was done by Sheykhi[21].

More investigations in this line employing Padmanabhan’s idea of emergent paradigm can be found in references [22–31]. In recent studies, one of us has shown that the expansion law effectively implies the entropy maximization in Einstein’s gravity [32] and more general forms of gravity like Gauss-Bonnet and Lovelock gravities [33]. Consequently, one can interpret the emergence of space in the expansion law as equivalent to the rise in horizon entropy with the progress of cosmic time and the Holographic equipartition, $N_{\text{sur}} = N_{\text{bulk}}$ as the state in which the horizon entropy is in its upper bound given in the references [34, 35].

In reference [36], the authors conjectured that the first law of thermodynamics could be considered the origin of the expansion law. They derived the expansion law using an appropriate form of the unified first law in (3+1) Einstein’s gravity. In different approaches, the authors also used the same idea for Horava-Lifshitz gravity and $f(R)$ gravity. However, they failed to extend the idea to gravity theories like Gauss-Bonnet and Lovelock gravities. Recently it has been shown that the expansion law can be derived from the first law of thermodynamics in Einstein’s gravity, Gauss-Bonnet and Lovelock gravities [37]. During the derivation process, the above authors have used the Friedmann equation on the midway. But the more important point is that the law of thermodynamics used is basically the
equilibrium version of it. Hence there arise two questions, (i) whether the expansion law can be derived from the thermodynamic law without explicitly using the Friedmann equation and (ii) how to obtain the expansion law from the non-equilibrium version of the thermodynamic law. The first point is important in its own way, since it has already known that expansion law otherwise implies the Friedmann equation. So deriving the expansion law using the Friedmann equation is seems to be a circular process. The second point is relevant with reference to gravity theories in which higher-order curvature corrections have been taken into account. In the presence of higher-order curvature corrections, the entropy becomes a polynomial function of the Ricci scalar. This in turn requires a non-equilibrium treatment, equivalent to non-equilibrium thermodynamics [38]. In such a situation, one has to use a modified entropy balance relation $\delta Q/T = dS + d_i S$, (or equivalently $TdS + Td_i S = -dE$) often known as the non-equilibrium Clausius relation (or the first law of thermodynamics), where $d_i S$ corresponds to an additional entropy production due to non-equilibrium evolution happening in the system. The corresponding form of the unified first law in non-equilibrium will be $dE = TdS + W dV + T d_i S$. The $f(R)$ gravity is a typical example of considering the higher curvature corrections [39–41], in which the action is an arbitrary function of the curvature scalar $R$. In the present work we first derive the expansion law from the equilibrium thermodynamic law without explicitly using the Friedmann equation. We then extend this process to obtain the expansion law from the non-equilibrium thermodynamic law.

The paper is organized as follows. In section II, we derive the general expansion law from the first law of thermodynamics for equilibrium description in a generic approach in section III, we extend this approach to the non-equilibrium description. In section IV, we find the expansion law in $f(R)$ gravity. Finally, we conclude our work in section V.

II. EXPANSION LAW FROM FIRST LAW OF THERMODYNAMICS - EQUILIBRIUM CASE.

In cosmology, the first law of thermodynamics is usually used either in the form $dE = T dS + W dV$, the unified first law, or a form without pressure term, as $-dE = T dS$. The first form is applied to the whole volume within the horizon, in which the energy is the Misner-Sharp energy, $E = \rho V$ contained within the given volume $V$ [14], while the second form is applied at the horizon, where energy $E$ refers to the flux crossing the apparent horizon of
the Universe [11]. The expansion law was derived from both these forms of the first law, in reference[37], by explicitly using the Friedmann equations.

Let us consider an (n+1) dimensional FRW Universe with metric

\[ ds^2 = h_{ab}dx^a dx^b + a^2 r^2 d\Omega_{n-1}^2, \]

where \( h_{ab} = \text{diag} [-1, a(t)^2/1-kr^2] \) is the two dimensional metric of the \( t-r \) surface, \( a \) is the scale factor of expansion, \( r \) is the co-moving radial distance, and \( d\Omega_{n-1} \) is the metric of \( (n-1) \)-dimensional sphere with unit radius. The spatial curvature constant have values \( k = 1, 0 \) and \( -1 \), corresponding to a closed, flat and open Universe and, \( a(t) \) is the scale of expansion. The apparent horizon of the Universe satisfies the condition, \( h_{ab}\partial_a \tilde{r} \partial_b \tilde{r} = 0 \), (where \( \tilde{r} = a(t)r \)), which gives the apparent horizon radius as,

\[ \tilde{r}_A = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}}, \]

where \( H \) is the Hubble parameter. From the standard relation for the surface gravity \( \kappa \), the horizon temperature can have the form [15, 16],

\[ T = \frac{1}{2\pi} \left[ -\frac{1}{\tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right) \right], \]

where the over-dot represents a derivative with respect to cosmic time.

Let us formulate the unified first law at the apparent horizon. The cosmic component is assumed to be a perfect fluid, such that the time and spacial components of the energy-momentum tensor are, \( T^0_0 = -\rho; \ T^i_i = p \) with density \( \rho \) and pressure \( p \) of the cosmic components. Thus the energy within the volume \( V \) enclosed by the apparent horizon is \( E = \rho V \). Then the unified first law can be expressed as [14],

\[ \frac{1}{2\pi} \left[ -\frac{1}{\tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A} \right) \right] dS = (\rho dV + V d\rho) - \frac{(\rho - p)}{2} dV. \]

Using the continuity equation in \( (n+1) \) FRW Universe, \( \dot{\rho} + nH (\rho + p) = 0 \), the above equation becomes

\[ -\frac{1}{2\pi H\tilde{r}_A^{n+1}} \frac{dS}{dt} = -n\Omega_n (\rho + p). \]

In obtaining the above relation, we took \( V = \Omega_n \tilde{r}_A^n \), the volume of \( (n+1) \) FRW Universe enclosed by the apparent horizon, where \( \Omega_n \) is the areal volume of an \( n \)-dimensional sphere with unit radius.
Now let us turn to the second form of the law, \(-dE = TdS\) and see how it reduces at the horizon of an \((n+1)\)-dimensional Friedmann Universe. It should be noted that, unlike in the previous form of the law, the energy \(dE\) here is the energy flux across the apparent horizon. The observer measuring this flux is located on the apparent horizon, for whom the apparent horizon is virtually stationary. Relative to this local observer, the temperature of the apparent horizon becomes \(T = 1/2\pi \tilde{r}_A\). The energy flux through the apparent horizon during a small interval of time \(dt\) is given by

\[
- dE = A (\rho + p) H \tilde{r}_A dt,
\]

where \(A = n\Omega_n \tilde{r}_A^{n-1}\) is the area of the horizon of an \((n+1)\) dimensional FRW Universe. Then the first law at the apparent horizon of the FRW Universe will take the form,

\[
A (\rho + p) H \tilde{r}_A dt = \frac{1}{2\pi \tilde{r}_A} dS.
\]

On substituting the area of the apparent horizon with some suitable rearrangements, the above equation becomes exactly similar to Eq. (5), obtained for the unified form of the law of thermodynamics. The fact that Eq. (5) represents both forms of the thermodynamic laws at the horizon, is not surprising. However we use this result in our later considerations.

The integration of Eq. (5) using the continuity equation, \(\dot{\rho} + nH (\rho + p) = 0\) leads to

\[
- \frac{1}{\pi} \int \frac{1}{\tilde{r}_A^{n+1}} dS = 2\Omega_n (\rho + \rho_\Lambda).
\]

Here the dark energy density \(\rho_\Lambda\) arises naturally as the integration constant and is equivalent to the cosmological constant. It should be noted that, in reference [43], Padmanabhan have obtained the cosmological constant density as an integration constant, following the principles of the emergent gravity paradigm. Combining the Eqs. (5) and (8), multiplay both the sides by \(\frac{4\pi l_P^{n-1}}{n-2} H \tilde{r}_A^{n+2}\) and rearranging suitably, will lead to

\[
\alpha \frac{4\pi l_P^{n-1}}{(n-1)} \frac{\tilde{r}_A dS}{\tilde{r}_A} = l_P^{n-1} \frac{\tilde{r}_A}{H} \left[-\alpha \frac{2}{(n-1)} 4\tilde{r}_A^{n+1} \int \frac{1}{\tilde{r}_A^{n+1}} dS + \left(\frac{(n-2)\rho + np - 2\rho_\Lambda}{n-2}\right) \frac{V}{\tilde{r}_A^{n+1}}\right] \left(\frac{1}{4\pi \tilde{r}_A}\right)^{-1}.
\]

Where \(\alpha = \frac{(n-1)}{2(n-2)}\). The first part of the second term on the right-hand side of the above equation is proportional to the Komar energy, while the second part is inversely proportional to the temperature as, \((1/4\pi \tilde{r}_A)^{-1} = ((1/2)k_BT)^{-1}\) [20, 21, 44]. Hence the whole second
term on the right-hand side can be conveniently identified as the DoF of the bulk within the apparent horizon, \( N_{\text{bulk}} \). Then the first term on the right-hand side can be the surface DoF, \( N_{\text{sur}} \). And the above equation can then be re-expressed as,

\[
\frac{4l_p^{n-1}\hat{r}_A}{(n-1)} \frac{dS}{dt} = \frac{l_p^{n-1}}{\hat{r}_A} (N_{\text{sur}} - \epsilon N_{\text{bulk}}) .
\] (10)

Where \( N_{\text{sur}} \) and \( N_{\text{bulk}} \) are in the form,

\[
N_{\text{sur}} = -\alpha \frac{2}{(n-1)} \frac{4\hat{r}_A^{n+1}}{n+1} \int \frac{1}{\hat{r}_A^{n+1}} dS \quad \text{and} \quad N_{\text{bulk}} = -\epsilon \left( \frac{[(n-2)\rho + np - 2\rho_\Lambda]V}{n-2} \right) \left( \frac{1}{4\pi\hat{r}_A} \right)^{-1} .
\] (11) (12)

Here the equation for \( N_{\text{sur}} \), is expressed as an integral over entropy. For instance, in the \((n+1)\) Einstein’s gravity, \( S = A/4l_p^{n-1} \). On substituting this into the equation for \( N_{\text{sur}} \) and using \( A = n\Omega_n\hat{r}_A^{n-1} \), it can easily be shown that \( N_{\text{sur}} = \alpha A/l_p^{n-1} \) is the relation for the horizon surface DoF used previously[21]. The L.H.S. of Eq. (10) will reduce to \( \alpha(dV/dt) \) for the same gravity. As a result, the equation of emergence will be reduced to the standard form [1, 20, 21].

\[
\alpha \frac{dV}{dt} = \frac{l_p^{n-1}}{\hat{r}_A} (N_{\text{sur}} - \epsilon N_{\text{bulk}}) .
\] (13)

In the case of Gauss-Bonnet gravity, the entropy is of the form [21, 45, 46],

\[
S = \frac{A}{4l_p^{n-1}} \left( 1 + \frac{n-1}{n-3} \frac{2\alpha}{\hat{r}_A^2} \right) .
\] (14)

which having an additional correction term in comparison with that of Einstein’s gravity. The corresponding surface DoF can then be obtained using Eq. (11) as,

\[
N_{\text{sur}} = -\alpha \frac{2}{(n-1)} \frac{4\hat{r}_A^{n+1}}{n+1} \int \frac{1}{\hat{r}_A^{n+1}} dS
= \frac{\alpha n \Omega_n}{l_p^{n-1}} \hat{r}_A^{-1} \left[ 1 + \alpha \hat{r}_A^{-2} \right] .
\] (15)

Now by using the \( N_{\text{bulk}} \) as given in the second equality of Eq. (12), the generalized expansion law in Gauss-Bonnet gravity can be obtained as,

\[
(1 + \alpha \hat{r}_A^{-2}) \hat{r}_A^{-2} - (1 + 2\alpha \hat{r}_A^{-2}) \hat{r}_A H^{-1} \hat{r}_A^{-3} = \\
- \frac{8\pi l_p^{n-1}}{n(n-1)} [(n-2)\rho + np - 2\rho_\Lambda] .
\] (16)
Which is same as the expansion law in Gauss-Bonnet gravity, obtained in [21]. Similarly, for more general Lovelock gravity the entropy has the form [21, 47],

$$S = \frac{A}{4l_P^{n-1}} \sum_{i=1}^{m} \frac{i(n-1)}{(n-2i+1)} \dot{\bar{r}}_A^{2-2i}$$

(17)
corresponding to which the surface DoF (11) takes the form,

$$N_{\text{sur}} = \frac{\alpha A}{l_P^{n-1}} \sum_{i=1}^{m} \dot{\bar{r}}_A^{2-2i}.$$  

(18)

With the above forms of the surface DoF, the generalized expansion law (10) in Lovelock gravity is

$$\sum_{i=1}^{m} \left( \dot{\bar{r}}_A^{-2i} - i \dot{\bar{r}}_A^{-2i-1} \dot{\bar{r}} A H^{-1} \right) =$$

$$- \frac{8\pi l_P^{n-1}}{n(n-1)} \left[ (n-2)\rho + np - 2\rho_\Lambda \right].$$

(19)

From the above two equations of expansion law (16) and (19), the respective Friedmann equations can be obtained, first by multiplying both sides of the equations with the factor $2a\dot{a}$ and then integrate the result using the continuity equation [21].

The expansion law we have obtained in Eq. (10) (with Eqs. (11, 12)) have the following advantages. First, it can be taken as the general form of the expansion law, from which the expansion law for different gravity theories can be obtained by using the respective form of entropy. Second, it naturally selects what is known as areal volume instead of proper invariant volume, hence eliminating the discrepancy in the use of proper invariant volume. There exists a discrepancy in choosing the volume of the horizon in expressing the expansion law. The expansion law in a non-flat Universe can not be properly formulated using the proper invariant volume, but it can only be done using the areal volume. This is discussed in reference [48, 49]. In our derivation, since it is written directly in terms of entropy, Eq. (10) naturally selects the areal volume rather than the proper invariant volume and avoid such disparity. Thirdly, the more prominent advantage is that this form of the expansion law can be easily generalized to the non-equilibrium thermodynamic situations, which we consider in the next section. In the next section, we use the same idea to derive the expansion law in the general gravity models like $f(R)$ gravity, including non-equilibrium description.

It should also be noted that, in formulating a general law of expansion in flat FRW universe, Cai has postulated the relation for the rate of change of the Hubble volume in both
Gauss-Bonnet and Lovelock gravities as \( \alpha \frac{1}{(n-1)H} \frac{dA_{\text{eff}}}{dt} \), in order to arrive at the Friedmann equation in both these gravity theories \([20]\). Here the effective area, \( A_{\text{eff}} = 4l_p^{n-1}S \).

Latter Sheykhi extended the law of expansion to non-flat FRW Universe by postulating \( \alpha \frac{r_A}{(n-1)} \frac{dA_{\text{eff}}}{dt} \) \([21]\). In our approach, the L.H.S of equation (10), which is a consequence from the first law of thermodynamics, will naturally reduce to a form similar to the one postulated by Cai and Sheykhi, hence giving it a basic physical motivation.

### III. EXPANSION LAW FROM FIRST LAW OF THERMODYNAMICS - NON EQUILIBRIUM CASE

In this section, we extend the above procedure to obtain the expansion law from the generalized first law in a non-equilibrium situation. It has been shown that thermodynamics of gravity theories with higher-order curvature corrections requires a non-equilibrium treatment. The field equations in such theories can be derived using a modified entropy balance relation \( dS = \delta Q/T + d_i S \) \([38]\), where \( d_i S \) is the entropy generated due to the system being out of equilibrium. The \( f(R) \) gravity is the typical example for theory with higher-order curvature correction, in which action is an arbitrary function of curvature scalar, \( R \) \([39–41]\). Akbar and Cai \([50]\) have found that the Friedmann equations of \( f(R) \) gravity on the horizon assumes the non-equilibrium first law of thermodynamics,

\[
dE = TdS + WdV + Td_i S. \tag{20}
\]

This indicates that it is needed to use the non-equilibrium version of the generalized first law to derive the expansion law in gravity theories like \( f(R) \).

For a general derivation of the expansion law in non-equilibrium, we choose the unified formulation for the modified gravity \([51]\), which accounts for the non-equilibrium characteristics of the modified gravities by adopting the effective coupling strength \( G_{\text{eff}} \) instead of \( G \) in the Einstein gravity. On this approach, the field equation in the gravity theories having higher curvature corrections can generally be expressed as

\[
G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G_{\text{eff}}T^{(\text{eff})}_{\mu\nu}. \tag{21}
\]

where \( G_{\text{eff}} \) depends on time in general and \( T^{(\text{eff})}_{\mu\nu} \) is an effective stress-energy tensor, which can be bifurcated as, \( T^{(\text{eff})}_{\mu\nu} = T^{(m)}_{\mu\nu} + T^{(MG)}_{\mu\nu} \). Here \( T^{(m)}_{\mu\nu} \) is due to matter and \( T^{(MG)}_{\mu\nu} \) is that
arises due to the modified gravity part, which reflects the dependence on the higher-order curvature. The density and pressure can also be bifurcated into the matter and modified gravity part as, 
\[ \rho_t = \rho_m + \rho_{MG} \] and pressure \[ p_t = p_m + p_{MG}. \]

The generalized continuity equation in these kinds of gravity theories can be suitably be expressed as \[ 51],
\[ \dot{\rho}_t + nH (\rho_t + p_t) = -\frac{\dot{G}_{eff}}{G_{eff}} \rho_t. \] (22)

The term in the R.H.S of Eq. (22) is the energy dissipation arises due to the non-equilibrium, which balances the energy flow. The above continuity equation can effectively be rewritten as,
\[ \dot{\rho}_t + nH (\rho_t + p_{eff}) = 0, \] (23)
where \[ p_{eff} \equiv p_t + \frac{\dot{G}_{eff}}{nHG_{eff}} \rho_t \] can be considered as the effective pressure in the non-equilibrium description having the energy dissipation. In literature, a similar idea was used in the bulk viscous model of the Universe to define the so-called effective pressure \[ 52].

Now lets assume the (n+1) FRW Universe having the metric as in Eq. (1), the non-equilibrium first law (20) takes the form,
\[ \rho_t dV + V d\rho_t = \frac{-1}{2\pi \tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H \tilde{r}_A} \right) d\tilde{S} + \frac{(\rho_t - p_{eff})}{2} dV, \] (24)

where we took the total change in energy inside the horizon as \[ E = \rho_t V \], the temperature of the horizon as, \[ T = \frac{-1}{2\pi \tilde{r}_A} \left( 1 - \frac{\dot{\tilde{r}}_A}{2H \tilde{r}_A} \right) \], effective work density \[ W = \frac{(\rho_t - p_{eff})}{2} \] and \[ d\tilde{S} = dS + d_iS \]. Using the continuity equation (23), the above equation can be reduced to the simple form,
\[ -\frac{1}{2\pi H \tilde{r}^{n+1}_A} \frac{d\tilde{S}}{dt} = -n\Omega_n (\rho_t + p_{eff}), \] (25)
where used \[ V = \Omega_n \tilde{r}_A^n \] as in the last section.

Next we consider first law of the form
\[ -dE = T (dS + d_iS) = T d\tilde{S} \] (26)
in FRW Universe with \[ dE \] as the energy flux \[ A (\rho_t + p_{eff}) H \tilde{r}_A dt \] for the horizon with temperature \[ T = 1/2\pi \tilde{r}_A \]. With little algebra it can show that, this form of the first will leads to the same as in Eq. (25) at the horizon.
Using the continuity equation (23), the integral form of the above equation can be expressed as

\[ \frac{-1}{\pi} \int \frac{1}{\tilde{r}_A^{n+1}} d\tilde{S} = 2\Omega_n (\rho_t + \rho_\Lambda). \]  

(27)

Combining the Eqs. (25) and (27) and multiply both sides by \( \frac{4\pi l_p^{n-1}}{n-2} H \tilde{r}_A^{n+2} \) will leads to

\[ \alpha \frac{4l_p^{n-1}\tilde{r}_A d\tilde{S}}{(n-1)} dt = l_p^{n-1} \tilde{r}_A / H^{-1} (N_{\text{sur}} - \epsilon N_{\text{bulk}}), \]

(28)

with the DoF on surface, \( N_{\text{sur}} \) and that of bulk, \( N_{\text{bulk}} \) as

\[ N_{\text{sur}} = -\alpha \frac{2}{(n-1)} 4\tilde{r}_A^{n+1} \int \frac{1}{\tilde{r}_A^{n+1}} d\tilde{S} \text{ and} \]

(29)

\[ N_{\text{bulk}} = -\epsilon \left( \frac{[(n-2)\rho_t + np_{\text{eff}} - 2\rho_\Lambda] V}{(n-2)} \right) \left( \frac{1}{4\pi \tilde{r}_A} \right)^{-1}. \]

(30)

The Eq. (28) represents the expansion law in the non-equilibrium situation with DoF defined in Eqs. (29, 30). Interestingly, the above equation of the expansion law is similar in form to the corresponding Eq. (10) obtained for the equilibrium description. Even though the formal appearance looks the same, the entropy, density and pressure in the present equation are different in such a way that they contain the contribution due to the non-equilibrium nature.

**IV. EXPANSION LAW FOR F(R) GRAVITY IN (N+1) FRW UNIVERSE**

In this section, we derive the expansion law in the f(R) gravity using the generalized expansion law in the non-equilibrium description, obtained in the previous section. The f(R) gravity is one of the main candidates from the gravity models with higher-order curvature corrections [39–41], which can explain the late acceleration of the Universe. The Einstein-Hilbert action of f(R) gravity has the form,

\[ A = \int d^{(n+1)}x \sqrt{-g} (f(R) + 2\kappa L_m), \]

(31)

where \( \kappa = 8\pi G \). From the variational principle, \( \delta A = 0 \) the field equations can be obtained as given in Eq. (21). The matter and gravity components of the effective energy-momentum tensor, \( T^{(\text{eff})}_{\mu\nu} \) are,

\[ T^{(m)}_{\mu\nu} = (\rho_m + p_m) U_\mu U_\nu + p_m g_{\mu\nu} \]

(32)
\begin{align}
T^{(MG)}_{\mu \nu} &= \frac{1}{8\pi G} \times \\
&\left( \frac{f(R) - Rf'(R)}{2} g_{\mu \nu} + \nabla_\mu \nabla_\nu f'(R) - g_{\mu \nu} \nabla^2 f'(R) \right) 
\end{align}

(33)

Where \( \rho_m \) is the density of matter and \( p_m \) is its pressure, Then the total effective energy-momentum tensor implies a total density \( \rho_t \) and total pressure \( p_t \) as

\begin{align}
\rho_t &= \rho_m + \frac{1}{8\pi G} \left[ \frac{Rf'(R) - f(R)}{2} - nH f'(R) \right] \\
p_t &= p_m + \frac{1}{8\pi G} \left[ \frac{f(R) - Rf'(R)}{2} + f''(R) \right. \\
&\quad \left. + (n - 1)H f'(R) \right]. 
\end{align}

(34) (35)

The total effective density \( \rho_t \) (34) coupled to \( G_{\mu \nu} \) through the coupling strength \( G_{eff} = G/f'(R) \). The time evolution of the total density can be obtained as,

\begin{align}
\dot{\rho}_t &= \dot{\rho}_m + \frac{1}{8\pi G} \left[ \frac{R \dot{f}'(R)}{2} - nH \dot{f}'(R) - nH \ddot{f}'(R) \right]. 
\end{align}

(36)

Using this, one can obtain a continuity equation analogous to the general expression (22),

\begin{align}
\dot{\rho}_t + nH (\rho_t + p_t) = \frac{n(n - 1) \dot{f}'(R)}{16\pi G r_H^2}. 
\end{align}

(37)

The continuity equation shows dissipation characteristics of the effective total density \( \rho_t \).

The above continuity equation can be rewritten as

\begin{align}
\dot{\rho}_t + nH (\rho_t + p_{eff}) = 0, 
\end{align}

(38)

where \( p_{eff} \equiv p_t - \frac{(n - 1) \dot{f}'(R)}{16\pi G r_H^2} \) is the effective pressure due to the energy dissipation.

Now we will derive the expansion law for \( f(R) \) gravity using the non-equilibrium first law (26). The total entropy change has two components; the first corresponds to change in the Wald entropy of the horizon, \( S = Af'(R)/4l_p^{n-1} \) [53] and the second is \( d \delta S \), the additional entropy change generated due to non-equilibrium evolution. We need to include additional entropy production term \( T d \delta S \) in the generalized first law in the volume inside the apparent horizon for \( f(R) \) gravity [38, 50],

\begin{align}
-dE = T dS + T d \delta S. 
\end{align}

(39)
Where \(-dE = A(\rho_t + p_{eff})H\tilde{r}_A dt\) is flux across the horizon and \(T = \frac{1}{2\pi\tilde{r}_A}\) is the horizon temperature. The variation of the Wald entropy \(S\) is,
\[dS = \frac{A}{4l_p^{n-1}} \left[ df'(R) + \frac{(n-1)f'(R) d\tilde{r}_A}{\tilde{r}_A} \right].\]
Then the additional entropy generated, \(d_i S\) can be obtained by substituting \(dE\) and \(dS\) in to the above form of the first law. A little algebra with the help of respective Einstein equations, it can be shown that,
\[d_i S = -\frac{(n+1)A}{8l_p^{n-1}} df'(R).\] (40)
Hence the total entropy change should be
\[d\tilde{S} = \frac{(n-1)A}{4l_p^{n-1}} \left[ \frac{f'(R)d\tilde{r}_A}{\tilde{r}_A} - \frac{df'(R)}{2} \right].\] (41)
Now we can express the expansion law in \(f(R)\) gravity from the general expression (28) using Eq. (41) as
\[\alpha A\tilde{r}_A \left[ \frac{f'(R)\dot{\tilde{r}}_A}{\tilde{r}_A} - \frac{\ddot{f}'(R)}{2} \right] = l_p^{n-1} \tilde{r}_A (N_{sur} - \epsilon N_{bulk}).\] (42)
Here we can estimate the DoF associated with the horizon surface of the FRW Universe in \(f(R)\) gravity using the general expression (29),
\[N_{sur} = \alpha A f'(R) \frac{n\Omega_{n-1}}{l_p} \tilde{r}_A^{n+1} \int \left[ -\frac{2f'(R)d\tilde{r}_A}{\tilde{r}_A^3} + \frac{df'(R)}{\tilde{r}_A^2} \right].\] (43)
The integrand in the above equation will reduces to the form \(d \left( \frac{f'(R)}{\tilde{r}_A^2} \right)\), and hence the surface DoF in \(f(R)\) gravity reduces to
\[N_{sur} = \alpha A \frac{f'(R)}{l_p^{n-1}}.\] (44)
For Einstein gravity, \(f'(R) = 1\) and hence have one DoF in the unit Plank area \(l_p^{n-1}\) seems a special case. But in general there have \(f'(R)\) number of DoF associated with the unit Plank area for \(f(R)\) gravity in the \((n+1)\) FRW Universe. Similarly, the number of bulk DoF in \(f(R)\) gravity can be obtained as
\[N_{bulk} = -\epsilon \left( \frac{(n-2)\rho_t + np_{eff} - 2\rho_\Lambda}{(n-2)} \right) \left( \frac{1}{4\pi\tilde{r}_A} \right)^{-1}.\] (45)
Then We can write the expansion law in f(R) gravity (42) using the surface degrees of freedom, \(N_{sur}\) in Eq. (44) and the bulk degrees of freedom, \(N_{bulk}\) in f(R) gravity (45) as,
\[\alpha A\ddot{r}_A \left[ \frac{f'(R)\dot{r}_A}{\ddot{r}_A} - \frac{f'(R)}{2} \right] = l_p^{n-1}\ddot{r}_AH \left[ \alpha Af'(R) + \left( \frac{(n-2)\rho_t + np_{eff} - 2\rho_\Lambda}{(n-2)} \right) Vl_p^{n-1} \left( \frac{1}{4\pi\ddot{r}_A} \right)^{-1} \right].\]

Which can be reduced to the form,

\[\frac{\dot{r}_A}{H\dot{r}_A^3} - \frac{1}{\dot{r}_A^2} = \frac{\dot{f}'(R)}{2f'(R)H\dot{r}_A^2} + \frac{8\pi G_{eff}}{(n-1)}(\rho_t + p_{eff}) - \frac{16\pi G_{eff}}{n(n-1)}[\rho_t + \rho_\Lambda].\] (47)

On substituting back the effective pressure \(p_{eff} = p_t - \frac{(n-1)f'(R)}{16\pi GH\dot{r}_A^2}\) to the above expansion law, the first term in the R.H.S. of the above equation will cancelled out with the additional pressure term from the \(p_{eff}\) and hence we finally get the expansion law in f(R) gravity as

\[-\frac{\dot{r}_A}{H\dot{r}_A^3} + \frac{1}{\dot{r}_A^2} = \frac{16\pi G_{eff}}{n(n-1)}(\rho_t + \rho_\Lambda) - \frac{8\pi G_{eff}}{(n-1)}(\rho_t + p_t).\] (48)

Using the continuity equation (22) the above equation can be expressed as

\[-\frac{\dot{r}_A}{H\dot{r}_A^3} + \frac{1}{\dot{r}_A^2} = \frac{16\pi G_{eff}}{n(n-1)}(\rho_t + \rho_\Lambda) + \frac{8\pi G_{eff}}{n(n-1)H}\dot{\rho}_t + \frac{8\pi \dot{G}_{eff}}{n(n-1)H}(\rho_t + \rho_\Lambda).\] (49)

Once we have the expansion law as in the above Eq. (49), it is easy to derive the Friedmann equations. Multiply both sides of the above equation with \(2\dot{a}\dot{a}\), we get (using \(H = \dot{a}/a\))

\[-\frac{2\ddot{r}_A}{\dot{r}_A^3}a^2 + \frac{1}{\dot{r}_A^2}2\dot{a}\dot{a} = \frac{16\pi}{n(n-1)} \left[ 2\dot{a}aG_{eff}(\rho_t + \rho_\Lambda) + a^2G_{eff}\dot{\rho}_t + a^2\dot{G}_{eff}(\rho_t + \rho_\Lambda) \right].\] (50)

The above equation can be identified as the exact differential of the form

\[\frac{d}{dt} \left( \frac{a^2}{\dot{r}_A^2} \right) = \frac{16\pi}{n(n-1)} \frac{d}{dt} \left[ a^2G_{eff}(\rho_t + \rho_\Lambda) \right].\] (51)

On integrating the above equation we get the first Friedmann equation in f(R) gravity

\[H^2 + \frac{k}{a^2} = \frac{16\pi G_{eff}}{n(n-1)}(\rho_t + \rho_\Lambda).\] (52)

From which, one can get the second Friedmann equation by taking differential of Eq. (52), using the continuity equation (37),

\[\dot{H} - \frac{k}{a^2} = -\frac{8\pi G_{eff}}{(n-1)}(\rho_t + p_t).\] (53)
Hence the expansion law is consistent with Friedmann equation in f(R) gravity.

In reference [29], the authors attempt to derive the expansion law in f(R) gravity by taking $N_{\text{sur}} = 4S = Af'(R)/l_p^{n-1}$. However, the resulting dynamical equation seems erroneous and hence is not consistent with the standard Friedmann equation. In extending this work, authors in [28] generalized $\frac{dV}{dt}$ in the expansion law with $\frac{\alpha}{(n-1)H} \frac{dN_{\text{sur}}}{dt}$ and proposed a dynamic equation (Eq.16 in the reference), which is valid only in equilibrium conditions, $\dot{f}_R = 0$. Also, in the process of derivation, the authors seem to have omitted a term, $Hf'(R)/2f(R)$, with which it is impossible to arrive at the Friedmann equation in f(R) gravity[28]. In contrast to these, our approach generalizes the expansion in non-equilibrium conditions and is consistent with the Friedmann equation.

V. CONCLUSIONS

Padmanabhan proposed the law of expansion based on the concept of the emergence of cosmic space as cosmic time evolves. However, later this law was derived by applying the first law of thermodynamics to the horizon of expanding universe [36, 37]. In this paper, our main aim was to obtain the expansion law using the non-equilibrium first law of thermodynamics. For gravity theories having higher-order curvature corrections, the first law indeed has the non-equilibrium form due to an additional entropy generation because of the non-equilibrium evolution. Earlier, the expansion law was derived by explicitly using the corresponding Friedman equation [37]. To avoid this circular process, we first described a method to derive the expansion law by projecting the first law of thermodynamics at the horizon of an FRW Universe without explicitly using the Friedmann equations. The final form we arrived at is structurally different from one usually seen in the literature. We then extended this method to the non-equilibrium first law of thermodynamics.

We have first formulated the expansion law as direct time evolution of the entropy, which is caused by the discrepancy between the DoF on the horizon and that within the bulk enclosed by the horizon. As a matter of fact, we have derived the expansion law from both forms of the thermodynamics law, $-dE = TdS$ applicable to the locality of the horizon and $dE = TdS + WdV$, which is true for the entire volume of the horizon. The respective surface DoF, $N_{\text{sur}}$ is formulated as the integral over the respective entropy as given in Eq. (11), which automatically guarantees the use of the areal volume for the correct formulation.
of the expansion law. Since many have shown that the use of proper invariant volume does not lead to the expansion law in any gravity theory [48, 49].

Further, we derived the expansion law from the modified first law in non-equilibrium, 
\[-dE = TdS + Td_iS\] and 
\[dE = TdS + Td_iS + WdV,\] by extending the procedure developed for the equilibrium situation. Then we particularly formulated the expansion law in f(R) gravity using the generalized expansion law in non-equilibrium (28). We found that the expansion law obtained for f(R) gravity (42) is consistent with the Friedmann equations in f(R) gravity. This shows that the adequate entropy evolution (or the emergence of areal volume) in f(R) gravity is proportional to \(N_{sur} - \epsilon N_{bulk}\), the discrepancy in the DoF.

It is to be noted that the generic derivation of the expansion law in non-equilibrium, we used the unified formulation for modified gravities with an effective dynamic coupling strength \(G_{eff}\) [51] to keep the generality of the derivation. Hence the resulting expansion law (28) can be widely applicable to several minimally coupled modified gravity theories like f(R), generalized Brans-Dicke, scalar-tensor-chameleon and \(f(R, G)\) generalized Gauss-Bonnet gravity. Thus, in general, we can conclude that there exists a strong correlation between generalized first law and the expansion law, both in equilibrium and non-equilibrium thermodynamic conditions.

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