THE MULTIDIMENSIONAL TRUNCATED MOMENT PROBLEM: 
THE MOMENT CONE

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Abstract. Let $A = \{a_1, \ldots, a_m\}$, $m \in \mathbb{N}$, be measurable functions on a measurable space $(\mathcal{X}, \mathcal{A})$. If $\mu$ is a positive measure on $(\mathcal{X}, \mathcal{A})$ such that $\int a_i d\mu < \infty$ for all $i$, then the sequence $(\int a_1 d\mu, \ldots, \int a_m d\mu)$ is called a moment sequence.

By Richter’s Theorem each moment sequence has a $k$-atomic representing measure with $k \leq m$. The set $S_A$ of all moment sequences is the moment cone. The aim of this paper is to analyze the various structures of the moment cone. The main results concern the facial structure (exposed faces, facial dimensions) and lower and upper bounds of the Carathéodory number (that is, the smallest number of atoms which suffices for all moment sequences) of the convex cone $S_A$. In the case when $\mathcal{X} \subseteq \mathbb{R}^n$ and $a_i \in C^1(\mathcal{X}, \mathbb{R})$, the differential structure of the moment map and regularity/singularity properties of moment sequences are analyzed. The maximal mass problem is considered and some applications to other problems are sketched.

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1. Introduction

The moment problem was first formulated and investigated by T. Stieltjes in his famous memoir [Sti94]. However, the use of moments for the study of integrals of functions can be traced back to P. L. Chebyshev and A. A. Markov [KN77], and
even to C. F. Gauß [Gau15]. The moment problem is now a well-studied classical problem which has deep connections with many mathematical fields (see e.g. [Lan80], [Las09], [FN10], [BBCM13], and [Sm17]) and a broad scope of applications (see e.g. [Las15]). In its most general form the moment problem is the following:

**Generalized Moment Problem:** Let \((X, \mathcal{A})\) be a measurable space and \(A = \{a_i\}_{i \in I}\) be a (finite or infinite) set of measurable (real-valued) functions on \((X, \mathcal{A})\). Given a (real) sequence \(s = (s_i)_{i \in I}\), does there exist a (positive) measure \(\mu\) on \((X, \mathcal{A})\) such that

\[
s_i = \int_X a_i(x) \, d\mu(x) \quad \forall i \in I?
\]

This general problem has a number of natural and important subproblems:

1) **Existence:** When does there exist a measure \(\mu\) such that (1) holds?
2) **Determinacy:** Is the measure \(\mu\) satisfying (1) unique?
3) **Characterization of solutions:** In the case when the measure \(\mu\) is not unique, how can one describe all solutions?
4) **Simplification:** Are there natural classes of “simplest” solutions?

The name moment problem stems from the classical case of monomials on \(\mathbb{R}\):

The number \(s_n = \int_{\mathbb{R}} x^n \, d\mu(x)\) is called the \(n\)-th moment of the measure \(\mu\) on \(\mathbb{R}\). If \(A\) is the set of all monomials \(x^n\) for \(n \in \mathbb{N}_0\), then the problem is called **multi-dimensional full moment problem**. If \(A\) consists of all monomials \(x^n\) up to a fixed degree \(|\alpha| \leq m\), or more generally, if the set \(A = \{a_1, \ldots, a_m\}\) is finite, we obtain the **truncated moment problem**, which is the subject of this paper. Even if many results are formulated for general functions, the case of monomials \(x^n, |\alpha| \leq m\), is our main guiding example. By an important theorem of J. Stochel [Sto01], solving all truncated power moment problems implies solving the full power moment problem.

There is a huge literature of thousands research papers about versions and applications of moment problems. Even for the truncated moment problem there are an extensive literature and interactions with various fields in pure and applied mathematics, see [AK62], [Lau80], [Lau09], [Las15], and [Sm17]. Let us mention some of the main lines of such interactions. As in the case of the full moment problem, there is a close interplay with real algebraic geometry, especially with nonnegative polynomials, see e.g. [Rob69], [CLR80], [Rez92], [Mat92], [Har99], and [Mar08]. Semi-algebraic sets and convex hulls of curves occur in a natural manner: The moment cone is semi-algebraic (Example 31), the set of atoms is a real algebraic set (the core variety), and the moment cone is the convex hull of the moment curve. There are connections with sums of squares and Waring decompositions, see e.g. [Rez92], [Lau09], [Ble12], [Ble15], [BT15]. An important method of solving truncated moment problems is based on flat extensions of Hankel matrices, see e.g. [CF95], [CF96], [LM09], [Vas12], [CF13], [MS16], and [Sau]. Another related topic is polynomial optimization [Las15]. Determining the maximal mass at a given atom leads to a convex optimization problem [Sm15], [Sm17, Section 18.4]. By Richter’s Theorem, each truncated moment problem has a finitely atomic solution. This builds the bridge to the theory of numerical integration and cubature formulas for the approximation of integrals, see e.g. [Str81], [Mol76], [DR84], [Put97], [Put09], [FN10], [Vas14]. A related problem deals with the smallest number of atoms for a moment sequence, the so-called Carathéodory number [RS18]. Convex hulls of curves [St18] enter the study of truncated moment problems in a natural manner. This was noted already early in [Kem68], [Kem87] and developed in [dDS18a] and [dDS18b] Chapter 18. Most questions concerning solutions of the truncated moment problem depends
This paper is about the *moment cone of the truncated moment problem* for a finite set $A = \{a_1, \ldots, a_m\}$ of measurable functions on a measurable space $(\mathcal{X}, \mathcal{A})$. Since each moment sequence has a finitely atomic representing measure, the corresponding moment functional is a positive linear combination of point evaluations. This in turn suggests other interesting subproblems such as:

5) Number of atoms: How many atoms are needed to represent a given moment sequence? What is the smallest number $n$ such that all moment sequences can be represented by $k \leq n$ atoms?

6) Position of atoms: Which points $x \in \mathcal{X}$ can appear as atoms of a representing measure of a given moment sequence?

Question 5 leads to the Carathéodory number of a moment sequence and of the whole moment cone, while the answer to question 6 is given by the core variety.

This paper continues our study of truncated moment problems in [Sm15], [dDS18a] and [dDS18b]. As noted above, the central topic of the present paper is the cone of all moment sequences, the so-called *moment cone* $S_A$. This is a convex but not necessarily closed cone in $\mathbb{R}^m$. It spans the vector space $\mathbb{R}^m$ if the set $A$ is linearly independent. Our main aim is to analyze the various structures of the moment cone. These are

- the structure of the convex set (boundary, interior, exposed faces),
- the differential structure (given by the derivative of the moment map if the functions $a_i$ are differentiable),
- the internal structure (regular and singular points).

We try to develop these topics and properties of the moment cone as complete as possible (to the best of our knowledge) and hope that the paper is also readable by non-experts. In some sense, this paper is a mixture of a survey of known facts and of a research paper with new results. To make the presentation self-contained, we occasionally repeat results and examples from our previous papers, sometimes with new proofs based on arguments from convex analysis. At the end of the paper a few open problems are listed. We hope that our treatment of the moment cone encourages further research on this topic.

Let us briefly describe the structure of this paper. In Section 3 we introduce the moment cone $S_A$. A crucial result is Richter’s Theorem which states that each moment sequence has a $k$-atomic representing measure with $k \leq m$. In Section 5 we reprove this theorem and present a detailed historical discussion around this result. In Section 4 we define the moment map and analyze its differential structure in the case when $\mathcal{X} \subseteq \mathbb{R}^n$ and the functions $a_i$ are differentiable.

Sections 6 and 7 are the main sections of this paper. In Section 6 we investigate the face structure and the geometry of the cone $S_A$, in particular the face $F_s$, the exposed face $E_s$ of a moment sequence $s$, and face dimensions. Section 7 is about Carathéodory numbers. We present new lower and upper bounds based on face dimensions and show that $F_s$ and $E_s$ are closely related to the set of atoms $W(s)$ and the set $V(s)$. Using Harris’ polynomial we develop in the polynomial case without gaps the first example of a moment sequence $s$ for which $V(s) \neq W(s)$.

In Section 8 we study the interior of the moment cone and the regularity and singularity of moment sequences. Section 9 contains various applications (SOS and tensor decomposition, Pythagoras numbers for polynomials in one variable with gaps). Section 10 deals with the maximal mass and its relation to conic optimization. The maximal mass of a moment sequence $s$ at $x$ is equal to the infimum of $L_s(p)$ over $p \geq 0$ with $p(x) = 1$. Using again Harris’ polynomial we
give an example where the infimum is not a minimum. The final Section 11 collects some open problems which might be starting points for future investigations.

2. Preliminaries on Integration

We briefly summarize some facts and notations from integration theory. For a deeper treatment we refer to standard texts such as [Bog07].

Throughout, \((\mathcal{X}, \mathcal{A})\) denotes a measurable space. This means that \(\mathcal{A}\) is a \(\sigma\)-algebra on a non-empty set \(\mathcal{X}\), \(\mu\) is a positive measure on \(\mathcal{A}\), i.e., \(\mu : \mathcal{A} \rightarrow [0, \infty]\), if it is a countably additive set function on \(\mathcal{A}\). For any \(x \in \mathcal{X}\), the Dirac (delta) measure 
\[\delta_x(A) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \not\in A \end{cases}\]
is a measure on \((\mathcal{X}, \mathcal{A})\); this holds even when the one point set \(\{x\}\) is not in \(\mathcal{A}\).

A measure \(\mu\) is called \(k\)-atomic if there are \(k\) pairwise different points \(x_1, \ldots, x_k \in \mathcal{X}\) and positive numbers \(c_1, \ldots, c_k\) such that
\[\mu = \sum_{j=1}^{k} c_j \cdot \delta_{x_j}.\]
The points \(x_i\) are called atoms and the numbers \(c_i\) masses of \(\mu\). If the numbers \(c_i\) in (2) are real rather than positive, then \(\mu\) is called a \(k\)-atomic signed measure.

A function \(a : \mathcal{X} \rightarrow \mathbb{R}\) is \(\mathcal{A}\)-measurable if \(a^{-1}(B) := \{x \in \mathcal{X} : a(x) \in B\} \in \mathcal{A}\) for all Borel subsets \(B\) of \(\mathbb{R}\) and \(a\) is called \(\mu\)-integrable if
\[\int_{\mathcal{X}} a(x) \, d\mu(x) \in (-\infty, \infty).\]
By the definition of the Lebesgue integral, (3) is equivalent to
\[
\begin{align*}
(4a) & \iff \int |a(x)| \, d\mu(x) < \infty \\
(4b) & \iff \int a_+(x) \, d\mu(x), \int a_-(x) \, d\mu(x) < \infty,
\end{align*}
\]
where \(a_+(x) := \max\{a(x), 0\}\) and \(a_-(x) := \max\{-a(x), 0\}\), i.e., \(a = a_+ - a_-\) and \(|a| = a_+ + a_-\).

Throughout, \(\mathcal{A} = \{a_1, \ldots, a_m\}\) denotes a finite set of real-valued measurable functions on \((\mathcal{X}, \mathcal{A})\).

**Remark 1.** In measure theory, measurable functions can have values in \([-\infty, \infty]\). Since all moments have to be finite, the functions \(a_i\)'s are integrable, so it is sufficient to work on the measurable space 
\[\tilde{\mathcal{X}} := \{x \in \mathcal{X} \mid a_i(x) \in \mathbb{R} \; \forall i = 1, \ldots, m\}.\]

On the measurable space \((\tilde{\mathcal{X}}, \mathcal{A}|_{\tilde{\mathcal{X}}})\) all functions \(a_i\) have finite real values. We will assume this in what follows.

**Definition 2.**
\[\mathcal{M}_\mathcal{A} := \{\mu \text{ positive measure on } (\mathcal{X}, \mathcal{A}) \mid a_1, \ldots, a_m \text{ are } \mu\text{-integrable}\}.\]

Since the functions \(a_i\) are finite, \(\delta_x \in \mathcal{M}_\mathcal{A}\) for any \(x \in \mathcal{X}\), so \(\mathcal{M}_\mathcal{A}\) is not empty.

**Lemma 3.** \(\delta_x \in \mathcal{M}_\mathcal{A}\) for all \(x \in \mathcal{X}\).

It is eminent that the following holds:
\[a(x) \geq 0 \text{ on } \mathcal{X} \implies \int a(x) \, d\mu(x) \geq 0 \quad \forall \mu \in \mathcal{M}_\mathcal{A}.\]
Example 4. Let \( \mathcal{X} = [0, 2] \) and \( \mathfrak{A} = \{\emptyset, [0, 1), [1, 2], \mathcal{X}\} \). Then all measurable functions on \((\mathcal{X}, \mathfrak{A})\) are of the form \( a(x) = c_1 \chi_{[0,1]}(x) + c_2 \chi_{[1,2]}(x) \), where \( \chi_A \) denotes the characteristic function on \( A \in \mathfrak{A} \). Then \( \delta_x \in \mathcal{M}_A \) for \( x \in [0, 2] \) despite the fact that \( \{x\} \not\in \mathfrak{A} \) and we have \( \delta_{x_1} = \delta_{x_2} \) for all \( x_1, x_2 \in [0, 1) \) or \( x_1, x_2 \in [1, 2] \).

As mentioned in the introduction, our guiding example will be the polynomial Throughout the rest of this paper we assume the following:

\begin{align*}
A_n,d := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid |\alpha| = \alpha_1 + \cdots + \alpha_n \leq d\} & \quad \text{on } \mathcal{X} = \mathbb{R}^n, \\
B_n,d := \{x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n} \mid |\alpha| = \alpha_0 + \cdots + \alpha_n = d\} & \quad \text{on } \mathcal{X} = \mathbb{P}^n, \\
A_{n,d} := \text{lin } A_{n,d}, \text{ and } B_{n,d} := \text{lin } B_{n,d}. \quad \text{We have of course } |A_{n,d}| = |B_{n,d}| = \binom{n+d}{n}.
\end{align*}

3. The Moment Cone

The following definition collects a number of basic notions on truncated moment problems. Lemma 3 motivates the following definition.

**Definition 5.** The moment curve \( s_A \) is defined by

\[ s_A : \mathcal{X} \to \mathbb{R}^m, \quad x \mapsto s_A(x) := \begin{pmatrix} a_1(x) \\ \vdots \\ a_m(x) \end{pmatrix} \]

and the moment cone \( \mathcal{S}_A \) is

\[ \mathcal{S}_A := \left\{ \int s_A(x) \, d\mu(x) \mid \mu \in \mathcal{M}_A \right\} \subseteq \mathbb{R}^m. \]

A sequence \( s \in \mathcal{S}_A \) is called a moment sequence and a measure \( \mu \in \mathcal{M}_A \) with \( s = \int s_A(x) \, d\mu(x) \) is called a representing measure of \( s \). We denote the set of representing measures of \( s \) by:

\[ \mathcal{M}_A(s) := \left\{ \mu \in \mathcal{M}_A \mid s = \int s_A(x) \, d\mu(x) \right\}. \]

From the preceding definition the following is obvious.

**Lemma 6.** The moment cone \( \mathcal{S}_A \) is a convex cone in \( \mathbb{R}^m \).

Since \( s_A(x) = \int s_A(y) \, d\delta_x(y) \), each vector \( s_A(x) \) belongs to the moment cone \( \mathcal{S}_A \) and the Delta measure \( \delta_x \) is a representing measure of \( s_A(x) \).

**Definition 7.** \( \text{Pos}(A) := \{p \in \text{lin } A \mid p(x) \geq 0 \ \forall x \in \mathcal{X}\} \).

**Lemma 8.** The following are equivalent:

i) The convex cone spanned by \( s_A(\mathcal{X}) \) is \( k \)-dimensional.

ii) \( \mathcal{S}_A \) is \( k \)-dimensional.

iii) \( \text{lin } A \) is a \( k \)-dimensional vector space.

iv) \( \exists b_1, \ldots, b_k \in \text{lin } A \) linearly independent with \( \text{lin } A = \text{lin } \{b_1, \ldots, b_k\} \).

**Proof.** iv) \( \iff \) iii) \( \Rightarrow \) i) \( \Rightarrow \) ii) are clear.

ii) \( \Rightarrow \) iii): Assume to the contrary that \( \text{lin } A \) is \( l \)-dimensional with \( l < k \), then \( \text{dim } \mathcal{S}_A \leq l < k \), which is a contradiction.

Throughout the rest of this paper we assume the following:

\( \ast \) **The set of functions** \( A = \{a_1, \ldots, a_m\} \) **is linearly independent.**

By Lemma 8 this is no loss of generality. It should be emphasized that basic properties of the moment cone (facial structure, Carathéodory number etc.) do not depend on the particular choice of the linearly independent set \( A \).
By (\(\ast\)), \(\dim \text{lin } A = m\). For each real sequence \(s = (s_1, \ldots, s_m)\) there is a unique linear functional \(L_s\) on \(\text{lin } A\), called the Riesz functional of \(s\), defined by
\[
L_s(a_i) = s_i, \quad i = 1, \ldots, m.
\]
Obviously, if \(s \in \mathcal{S}_A\) and \(\mu \in \mathfrak{M}_A(s)\), then \(L_s(p) = \int p(x) \, d\mu\) for \(p \in \text{lin } A\).

**Definition 9.** \(\text{Pos}(A) := \{p \in \text{lin } A \mid p(x) \geq 0 \ \forall x \in X\}\).

Let \(\mathcal{L}_A\) denote the set of all Riesz functionals \(L_s\) for \(s \in \mathcal{S}_A\). Clearly, \(\mathcal{L}_A\) and
\[
\text{Pos}(A) \setminus (\text{lin } A)^* \text{ are cones in the dual space } (\text{lin } A)^* \text{ and } \mathcal{S}_A \cong \mathcal{L}_A \subseteq \text{Pos}(A)^*.
\]

**Example 10.** Let \(A = \{1, x, x^2\} \text{ on } X = \mathbb{R} \) and \(s = (0, 0, 1). \) Then \(L_s \in \text{Pos}(A)^*, \) but \(L_s \notin \mathcal{L}_A\) (i.e., \(s \notin \mathcal{S}_A\)). (Indeed, otherwise \(s_0 = 0\) and \(\mu \in \mathfrak{M}_A(s)\) would imply \(\mu = 0\), which contradicts \(s_2 = 1\).)

The dual cone \(\text{Pos}(A)^* \) coincides with the closure of \(\mathcal{L}_A\) in \((\text{lin } A)^*\). Sufficient conditions for \(\mathcal{L}_A\) being closed, equivalently \(\mathcal{L}_A = \text{Pos}(A)^*\), are given in [Sm17], see Theorems 1.26, 1.30, and Proposition 1.27 therein.

Since \(\mathcal{S}_A\) is a convex cone, we repeat basic definitions and facts from the theory of convex sets and fix some notation, see e.g. [Roc72], [Sol15], or [Sil14] for more details.

For a \(v \in \mathbb{R}\) we set
\[
H_v := \{x \in \mathbb{R}^m \mid \langle v, x \rangle = 0\} \quad \text{and} \quad H_v^+ := \{x \in \mathbb{R}^m \mid \langle v, x \rangle \geq 0\}.
\]
If \(v \neq 0\), then \(H_v\) is called the hyperplane with normal vector \(v\) and \(H_v^+\) the corresponding halfspace. We shall say that \(H_v^+\) is a containing halfspace for \(\mathcal{S}_A\) iff \(\mathcal{S}_A \subseteq H_v^+\), that \(H_v\) is a supporting hyperplane iff \(\mathcal{S}_A \subseteq H_v^+\) and \(\mathcal{S}_A \cap H_v \neq \emptyset\) and finally that \(H_v\) supports \(\mathcal{S}_A\) at \(s \in \mathcal{S}_A\) iff \(H_v\) is a supporting hyperplane and \(s \in H_v\).

For a convex cone \(K \subseteq \mathbb{R}^m\) we denote by \(K^* := \{v \in \mathbb{R}^m \mid K \subseteq H_v^+\}\) its dual cone. An extreme face (briefly, a face) of a convex set \(K\) is a subset \(F\) such that \(\lambda x + (1 - \lambda)y \in F\) for some \(x, y \in K\) and \(\lambda \in [0, 1]\) implies \(x, y \in F\). An exposed face \(F\) is a face of \(K\) such that there exists a hyperplane \(H\) with \(F = K \cap H\). A cone \(K\) is called line-free (or pointed) iff \(K\) does not contain a line \(x + y \cdot \mathbb{R}\) with \(x \in \mathbb{R}^m\) and \(y \in \mathbb{R}^m \setminus \{0\}\).

For \(s \in \mathcal{S}_A\) we define the normal cone \(\text{Nor}_A(s)\) by
\[
\text{Nor}_A(s) := \{v \in \mathbb{R}^m \mid H_v\text{ is a supporting hyperplane of }\mathcal{S}_A\text{ at }s\}.
\]
It is a closed convex cone, see e.g. [Sol15]. Finally, since our moment cone \(\mathcal{S}_A\) is not necessarily closed, it is necessary to distinguish between the boundary \(\partial \mathcal{S}_A\) of \(\mathcal{S}_A\) and the boundary points \(\partial^* \mathcal{S}_A\) belonging to \(\mathcal{S}_A\), that is, \(\partial^* \mathcal{S}_A := \partial \mathcal{S}_A \cap \mathcal{S}_A\).

The following lemma states that any proper exposed face (\(\neq \mathcal{S}_A\)) of the moment cone \(\mathcal{S}_A\) is again a moment cone \(\mathcal{S}_B\) on a subset \(Z \subseteq X\) and a basis \(B = \{b_1, \ldots, b_k\} \subseteq \text{lin } A\) with \(k < m\).

**Lemma 11.** Let \(H_v\) be a supporting hyperplane of \(\mathcal{S}_A\). Then \(\mathcal{S}_A \cap H_v\) is a moment cone on \((Z, \mathfrak{A}|_Z)\) with \(Z := Z(\langle v, s_A(\cdot) \rangle)\) and \(\dim \mathcal{S}_A \cap H_v < \dim \mathcal{S}_A\).

**Proof.** It is clear that \(\dim \mathcal{S}_A \cap H_v < \dim \mathcal{S}_A\). By a basis change in the vector space \(\text{lin } A\) we can assume without loss of generality that \(v = (0, \ldots, 0, 1)\), so that \(a_m \geq 0\).

Therefore, \(s = (s_1, \ldots, s_m) \in \mathcal{S}_A \cap H_v\) implies \(s_m = 0\) and
\[
0 = s_m = \int a_m(y) \, d\mu(y) \quad \forall \mu \in \mathfrak{M}_A(s) \quad \forall s \in \mathcal{S}_A \cap H_v.
\]
Then \(Z = Z(a_m)\) and \(\mathfrak{A}|_Z = \{M \cap Z \mid M \in \mathfrak{A}\}\). For any \(x \notin Z\) we have \(a_m(x) \neq 0\), so \(s_m \neq 0\), and \(\mathcal{S}_A(x) = s \notin \mathcal{S}_A \cap H_v\). Hence \(\mathcal{S}_A \cap H_v\) is the moment cone on \((Z, \mathfrak{A}|_Z)\). \(\square\)
One might also ask which sequences are obtained by allowing signed atomic measures. The next proposition (see [131b, p. 1608]) says that each vector $s \in \mathbb{R}^m$ is a signed moment sequence and that a representing signed measure can easily be calculated by using linear algebra.

**Proposition 12.** There exist points $x_1, \ldots, x_m \in X$ such that every vector $s \in \mathbb{R}^m$ has a $k$-atomic signed representing measure with $k \leq m$ and all atoms are from the set $\{x_1, \ldots, x_m\}$, that is, $s \in \mathbb{R}^m$ is a signed moment sequence.

**Proof.** Since the set $A$ of functions is linearly independent by the above assumption, there are points $x_1, \ldots, x_m \in X$ such that the matrix $(s_A(x_1), \ldots, s_A(x_m)) \in \mathbb{R}^{m \times m}$ has full rank. Therefore, for any $s \in \mathbb{R}^m$, we have

$$s = c_1 \cdot s_A(x_1) + \cdots + c_m \cdot s_A(x_m) = \int s_A(y) \, d\left(\sum_{i=1}^{m} c_i \cdot \delta_{x_i}\right)$$

with $(c_1, \ldots, c_m)^T = (s_A(x_1), \ldots, s_A(x_m))^{-1}s$. □

Thus, the crucial and hard part of the truncated moment problem is to find positive representing measures.

### 4. The Moment Map and its Differential Structure

**Definition 13.** For $k \in \mathbb{N}$ the moment map $S_{k,A}$ is defined by

$$S_{k,A} : \mathbb{R}_{\geq 0}^k \times X^k \to \mathbb{R}^m, \quad (C, X) \mapsto S_{k,A}(C, X) := \sum_{i=1}^{k} c_i \cdot s_A(x_i)$$

with $C = (c_1, \ldots, c_k)$ and $X = (x_1, \ldots, x_k)$.

More explicitly, the moment map $S_{k,A}$ has the form

$$S_{k,A}(C, X) = \begin{pmatrix} c_1a_1(x_1) + c_2a_1(x_2) + \cdots + c_ka_1(x_k) \\ \vdots \\ c_1a_m(x_1) + c_2a_m(x_2) + \cdots + c_ka_m(x_k) \end{pmatrix} \in \mathbb{R}^m.$$  \hspace{1cm} (6)

The moment map $S_{k,A}$ is differentiable in all coefficients $c_i$. From (6) we obtain

$$\partial_{c_i} S_{k,A}(C, X) = \begin{pmatrix} a_1(x_i) \\ \vdots \\ a_m(x_i) \end{pmatrix} = s_A(x_i).$$

If in addition $X \subseteq \mathbb{R}^n$ is open and all functions $a_i$ are differentiable, that is, $A \subseteq C^1(X, \mathbb{R})$, then $S_{k,A}$ is also differentiable in all coordinates $x_{i,j}$ of any atom $x_i = (x_{i,1}, \ldots, x_{i,n})$ and from (6) we get

$$\partial_{x_{i,j}} S_{k,A}(C, X) = \begin{pmatrix} c_i \partial_j a_1(x_i) \\ \vdots \\ c_i \partial_j a_m(x_i) \end{pmatrix} = c_i \partial_j s_A(x_i).$$
where \( \partial_j \) denotes the partial derivative with respect to the \( j \)-th coordinate. Then \( S_{k,A} \) is differentiable with respect to all \( c_i \) and \( x_{i,j} \) and the total derivative is

\[
(7) \quad DS_{k,A}(C,X) = (\partial_{c_1}S_{k,A}, \partial_{x_{1,1}}, \ldots, \partial_{x_{1,n}}S_{k,A}, \partial_{x_{2,1}}, \ldots, \partial_{x_{k,n}}S_{k,A})
\]

\[
= (s_A(x_1), c_1 \partial_1 s_A|_{x=x_1}, \ldots, c_1 \partial_n s_A|_{x=x_1}, s_A(x_2), \ldots, c_k \partial_n s_A|_{x=x_1})
\]

\[
= \begin{pmatrix}
    a_1(x_1) & c_1 \partial_1 a_1(x_1) & \cdots & c_1 \partial_n a_1(x_1) & a_1(x_2) & \cdots & c_k \partial_n a_1(x_k) \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    a_m(x_1) & c_1 \partial_1 a_m(x_1) & \cdots & c_1 \partial_n a_m(x_1) & a_m(x_2) & \cdots & c_k \partial_n a_m(x_k)
\end{pmatrix}.
\]

Here we ordered the variables as \( c_1, x_{1,1}, \ldots, x_{1,n}, c_2, x_{2,1}, \ldots, x_{k,n} \), i.e., the first column is the derivative with respect to \( c_1 \), followed by the derivatives with respect to the coordinates \( x_{1,1}, \ldots, x_{1,n} \) of the 1st atom \( x_1 \), and so on. Since \( DS_{k,A} \) is a matrix, we can calculate the rank of \( DS_{k,A}(C,X) \) at some atomic measure \( (C,X) \). As \( k \) increases, there exists a \( k \in \mathbb{N} \) such that \( DS_{k,A}(C,X) \) has rank \( m \) at some \( (C,X) \). This leads to the definition of the following important number.

**Definition 14.** Suppose \( \mathcal{X} \subseteq \mathbb{R}^n \) is open and \( a_i \in C^1(\mathcal{X}, \mathbb{R}) \) for all \( i \). We define

\[
N_A := \min\{k \in \mathbb{N} | \exists (C,X) \in \mathbb{R}^k \times \mathcal{X}^k : \text{rank } DS_{k,A}(C,X) = m\}.
\]

**Remark 15.** Instead of an open subset \( \mathcal{X} \) of \( \mathbb{R}^n \), this definition and the following considerations can be extended almost verbatim to differentiable manifolds \( \mathcal{X} \). By choosing a chart \( \phi : U \subseteq \mathbb{R}^n \rightarrow \mathcal{X} \) of the manifold, \( U \) open, the preceding definition makes sense for \( A \circ \phi = \{a_i \circ \phi | i = 1, \ldots, m\} \).

The number \( N_A \) is well-defined, because the total derivative \( DS_{k,A}(C,X) \) contains the column vectors \( s_A(x_1), \ldots, s_A(x_k) \). Since the functions \( a_i \) are assumed to be linearly independent, we can find \( x_1, \ldots, x_m \in \mathcal{X} \) s.t. \( s_A(x_1), \ldots, s_A(x_m) \) are linearly independent, i.e., \( N_A \leq m \). Of course, the rank of \( DS_{k,A}(C,X) \) does not depend on the numbers \( c_i \neq 0 \). Thus, we can set without loss of generality \( c_i = 1 \) for all \( i \), i.e., \( C = (1, \ldots, 1) =: 1_k = 1 \). We have the following lower bound.

**Proposition 16** ([DLS18b] Prop. 23). \([\frac{m}{n+1}] \leq N_A\).

We illustrate the preceding by an example (taken from [DLS18b] Exm. 50).

**Example 17.** Let \( A = A_{2,2} \). Then

\[
DS_{2,A}(C,X) = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & c_1 & 0 & x_{1,1} & c_2 & 0 \\
0 & c_1 & x_{1,2} & x_{2,1} & 0 & c_2 \\
0 & c_1 x_{1,1} & 2 c_1 x_{2,1} & (x_{2,1})^2 & 2 c_2 x_{2,1} & 0 \\
0 & c_1 x_{1,2} & c_1 x_{1,1} & x_{1,2} x_{2,1} & c_2 x_{2,1} & c_2 x_{2,1} \\
0 & 2 c_1 x_{1,1} & (x_{2,1})^2 & 0 & 2 c_2 x_{2,2} & 0
\end{pmatrix},
\]

where \( C = (c_1, c_2) \) and \( X = (x_1, x_2) \), \( x_i = (x_{i,1}, x_{i,2}) \). From this we find that

\[
\ker DS_{2,A}(C,X) = R \cdot v(C,X) \quad \text{with} \quad v(C,X) := \begin{pmatrix}
-2 \\
-c_1^{-1}(x_{1,1} - x_{2,1}) \\
-c_1^{-1}(x_{1,2} - x_{2,2}) \\
2 \\
c_2^{-1}(x_{1,1} - x_{2,1}) \\
c_2^{-1}(x_{1,2} - x_{2,2})
\end{pmatrix}.
\]

Hence \( \text{rank } DS_{2,A_{2,2}} = 5 \) for each point \((x_1, x_2), x_1 \neq x_2\). In order to get full rank \(6\) we need a third atom. Therefore, it follows that \( N_A = 3 \).
Determining the number $N_A$ for general functions requires explicit calculations. But in the polynomial case (that is, for $A = A_{n,d}$ or $B_{n,d}$ on $\mathbb{R}^n$, $\mathbb{P}^n$ or on an open subset) the Alexander–Hirschowitz Theorem [AH95] gives the following important result.

**Theorem 18** ([DJS18b] Thm. 53). We have

\[(8) \quad N_{A_{n,d}} = \left\lceil \frac{1}{n+1} \left( n + d \right) \right\rceil,
\]

except for the following five cases:

1. $d = 2$: $N_{A_{2,2}} = n + 1$.
2. $d = 4$: $N_{A_{4,4}} = 8$.
3. $d = 3$: $N_{A_{3,3}} = 6$.
4. $n = 3$: $d = 4$: $N_{A_{3,4}} = 10$.
5. $n = 4$: $d = 4$: $N_{A_{4,4}} = 15$.

Example 17 is just one of the exceptional cases of the Alexander–Hirschowitz Theorem which shows that $N_A$ is not given by (8) in general.

5. Richter’s Theorem

The following theorem due to Hans Richter (1957) is probably the most important general result on truncated moment problems and it is the starting point for many problems as well. For this reason we include the proof of Richter [Ric57] rewritten in terms of convex analysis, see also [Sm17, Thm. 1.24]. By Remark 1 we can assume that all measurable functions have finite values.

**Theorem 19** (Richter [Ric57] Satz 4). Let $A = \{a_1, \ldots, a_m\}$ be (finite) measurable functions on a measurable space $(\mathcal{X}, \mathfrak{A})$. Then every moment sequence $s \in S_A$ has a $k$-atomic representing measure with at most $k \leq m$ atoms. Thus,

\[(9) \quad S_A = \text{range } S_{m,A} = \text{conv cone } s_A(\mathcal{X}).
\]

If $s$ is a boundary point of $S_A$, then $m - 1$ atoms are sufficient.

**Proof.** We proceed by induction on $m$. By Lemma 8 we can assume that the set $A$ is linearly independent.

For $m = 1$, let $s \in S_A \subseteq \mathbb{R}$ and $s \neq 0$. Since $s = \int a_1(x) \, d\mu(x) \neq 0$ there is because of (5) an $x \in \mathcal{X}$ such that $a_1(x) \neq 0$ and $\text{sgn } s = \text{sgn } a_1(x)$. Then $e := s \cdot a_1(x)^{-1}$ is a 1-atomic representing measure of $s$.

Let $m \geq 1$. Suppose the assertion of the theorem holds for all $k = 1, \ldots, m - 1$. By Lemma 8 $S_A$ and range $S_{m,A}$ have non-empty interior. First we show that int $S_A = \text{int range } S_{m,A}$. Assume to the contrary that int $S_A \neq \text{int range } S_{m,A}$. Then, since range $S_{m,A} \subseteq S_A$ and both sets are convex cones, int $S_A \setminus \text{int range } S_{m,A}$ has inner points. Let $s$ be such an inner point with representing measure $\mu$. Then there exists a separating linear functional $l(\cdot)$ such that $l(s) < 0$ and $l(t) > 0$ for all $t \in \text{int range } S_{m,A}$, i.e., $0 \leq l(s_A(x)) = \langle \tilde{l}, s_A(x) \rangle = \langle \tilde{l}, x \rangle a(x)$ for all $x \in \mathcal{X}$. In other words,

\[(10) \quad a(x) \geq 0 \text{ on } \mathcal{X} \text{ but } \int a(x) \, d\mu(x) = \int \langle \tilde{l}, s_A(x) \rangle \, d\mu(x) = \langle \tilde{l}, s \rangle = l(s) < 0.
\]

This is a contradiction to (5) and therefore we have int $S_A = \text{int range } S_{m,A}$. This proves the assertion for inner points $s$ of $S_A$. 

If $s$ is a boundary point of the moment cone $S_A$, $s$ is contained in an exposed face of the cone $S_A$. By Lemma 11 this is again a moment cone of dimension $k \leq m - 1$ and the assertion follows from the induction hypothesis.

Thus, by Theorem 19 any moment sequence $s \in S_A$ can be written as

\begin{equation}
 s = \sum_{i=1}^{k} c_i \cdot s_A(x_i) = \int_X s_A(x) \, d\left(\sum_{i=1}^{k} c_i \cdot \delta_{x_i}\right)(x)
\end{equation}

for some points $x_1, \ldots, x_k \in \mathcal{X}$, positive real numbers $c_1, \ldots, c_k$, and $k \leq m$.

Requiring that $(\mathcal{X}, \mathfrak{A})$ is a measurable space and $A$ are measurable functions is necessary to define integration. \{x\} \in \mathfrak{A}$ is not required. On the other side, a number of generalizations can be made for $s_A(x) \in \mathbb{R}^m$. We can replace $\mathbb{R}^m$ by any finite dimensional vector space $V$ (for instance, matrices or tensors) and $A = \{a_1, \ldots, a_m\}$ by the coordinate functions of any basis $v_1, \ldots, v_m$: $v = a_1 v_1 + \cdots + a_m v_m \in V$.

It should be emphasized that Theorem 19 holds for arbitrary measurable spaces $(\mathcal{X}, \mathfrak{A})$ and measurable functions $A$, see Example 4 for \{x\} \not\in \mathfrak{A}$ for all $x \in \mathcal{X}$. We illustrate its use by another example.

**Example 20.** Let $s$ be a moment sequence on a measure space $(\mathcal{X}, \mathfrak{A})$ with representing measure $\mu$. If $M \in \mathfrak{A}$ such that $\mu(M) = 0$, then it follows from Richter’s Theorem (Theorem 19), applied to $\mathcal{X} \setminus M$, that the points $x_1$ in (11) can be chosen from the set $\mathcal{X} \setminus M$. For instance, suppose $\mu$ is the Lebesgue measure of a closed set $\mathcal{X}$ of $\mathbb{R}^m$. If the Lebesgue measure of the boundary of $\mathcal{X}$ is zero, we can take the atoms $x_1$ from the interior of $\mathcal{X}$. Also, if $M$ is a finite subset of $\mathcal{X}$, we can choose the atoms outside $M$.

The history of Richter’s Theorem 19 is confusing and intricate and often the corresponding references in the literature are misleading. For this reason we take this opportunity to discuss this history in detail. First we collect several versions of Theorem 19 occurring in the literature in chronological order.

A) A. Wald 1939 [Wal39 Prop. 13]: Let $\mathcal{X} = \mathbb{R}$ and $a_0(x) := |x - x_0|^d_i$ with $d_i \in \mathbb{N}_0$ and $0 \leq d_1 < d_2 < \cdots < d_m < \infty$ for an $x_0 \in \mathcal{X}$. Then (9) holds.

B) P. C. Rosenbloom 1952 [Ros52 Cor. 38c]: Let $(\mathcal{X}, \mathfrak{A})$ be a measurable space and $a_i$ bounded measurable functions. Then (9) holds.

C) H. Richter 1957 [Ric57 Satz 4]: Let $(\mathcal{X}, \mathfrak{A})$ be a measurable space and let $a_i$ be measurable functions. Then (9) holds.

D) M. V. Tchakaloff 1957 [Tch57]: Let $\mathcal{X} \subset \mathbb{R}^n$ be compact and $a_\alpha(x) = x^\alpha$, $|\alpha| \leq d$. Then (9) holds.

E) W. W. Rogosinski 1958 [Rog58 Thm. 1]: Let $(\mathcal{X}, \mathfrak{A})$ be a measurable space and let $a_i$ be measurable functions. Then (9) holds.

From this list we see that Tchakaloff’s result [1] from 1957 is a special case of Rosenbloom’s result [3] from 1952 and that the general case was proved by Richter and Rogosinski almost at the same time, see the exact dates in the footnotes. If one reads Richter’s paper, one might think at first glance that he treats only the one-dimensional case, but a closer look reveals that his Proposition (Satz) 4 covers actually the general case of measurable functions. Rogosinski treats the one-dimensional case, but he also states that his proof works for general measurable spaces. The above proof of Theorem 19 and likewise the one in Sm17 Thm. 1.24, are nothing but modern formulations of the proofs of Richter and Rogosinski.

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1Received: December 27, 1956. Published: April, 1957.
2Published: July-September, 1957.
3Received: August 22, 1957. Published: May 6, 1958.
without additional arguments. Note that Rogosinski’s paper [Rog58] was submitted about a half year after the appearance of Richter’s [Ric57].

It might be of interest to note that the general results of Richter and Rogosinski can be easily derived from Rosenbloom’s Theorem by the following simple trick. Let \( A = \{a_1, \ldots, a_m\} \) be (finite) measurable functions on \((\mathcal{X}, \mathcal{M})\) and set \( B = \{b_1, \ldots, b_m\} \), where \( b_i := \frac{a_i}{f} \) with \( f := 1 + \sum_{i=1}^m a_i^2 \). Then

\[
s \in S_B \iff \exists \nu \in \mathcal{M}_B : s = \int s_B(x) \, d\nu(x) \iff s = \int \frac{s_A(x)}{f(x)} \, d\nu(x) = \int s_A(x) \, d\mu(x)
\]

\[
\iff \exists \mu \in \mathcal{M}_A : s = \int s_A(x) \, d\mu(x) \iff s \in S_A \text{ with } d\mu = f^{-1} \, d\nu.
\]

Since all functions \( b_i \) are bounded, Rosenbloom’s Theorem applies to \( B \), so each sequence \( s \in S_B = S_A \) has a \( k \)-atomic representing measure \( \nu \in \mathcal{M}_B(s) \) with \( k \leq m \) and scaling by \( f^{-1} \) yields a \( k \)-atomic representing measure \( \mu \in \mathcal{M}_A(s) \):

\[
s = \sum_{i=1}^k c_i \cdot s_B(x_i) = \sum_{i=1}^k \frac{c_i}{f(x_i)} \cdot s_A(x_i).
\]

Theorem 19 was overlooked in the modern literature on truncated polynomial moment problems. It was reproved in several papers in weaker forms and finally in the polynomial case in [BT06]. But Theorem 19 for general measurable functions was known and cited by J. H. B. Kemperman in [Kem68, Thm. 1] and attributed therein to Richter and Rogosinski. In the moment problem community succeeding Kemperman the general form of Theorem 19 was often used, see e.g. [Kem71, eq. (2.3)], [Kem87, p. 29], [IPP01, Thm. 1, p. 198], [Ana06, Thm. 1], and [Las15, Thm. 2.50]. In [Sm17] Theorem 19 was called Richter–Tchakaloff Theorem. After the preceding discussion and comparing the proofs and the precise publication data we are convinced that it is fully justified to use the name Richter Theorem. If one wants to take the broader history of this result into account, it might be also fair and appropriate to call it Richter–Rogosinski–Rosenbloom Theorem.

6. FACIAL STRUCTURE OF THE MOMENT CONE \( S_A \) AND SET OF ATOMS

Let \( s \) denote a fixed sequence of the moment cone \( S_A \).

**Definition 21.** The set of atoms of \( s \) is the set

\[ W(s) := \{ x \in \mathcal{X} \mid \exists \mu \in \mathcal{M}_A(s) : \mu(A_x) > 0 \} \]

with \( A_x := s_A^{-1}(s_A(x)) \equiv \{ y \in \mathcal{X} \mid s_A(y) = s_A(x) \} \).

The name set of atoms for \( W(s) \) is justified by the following lemma.

**Lemma 22.**

\[ x \in W(s) \iff s = c \cdot s_A(x) + \sum_{i=1}^k c_i \cdot s_A(x_i) \text{ with } x_i \in \mathcal{X}, \ c > 0, c_i > 0. \]

**Proof.** \( \Leftarrow \): Clear.

\( \Rightarrow \): Let \( \mu \in \mathcal{M}_A(s) \) such that \( \mu(A_x) > 0 \). Then

\[ s = \int_{\mathcal{X}} s_A(y) \, d\mu(y) = \int_{A_x} s_A(y) \, d\mu(y) + \int_{\mathcal{X} \setminus A_x} s_A(y) \, d\mu(y) = \mu(A_x)s_A(x) + s'
\]

with \( s' = \int_{\mathcal{X} \setminus A_x} s_A(y) \, d\mu(y) \). Hence \( s' \in S_A \) with representing measure \( \mu|_{\mathcal{X} \setminus A_x} \). By Richter’s Theorem (Theorem 19), \( s' \) has an atomic representing measure which gives the assertion. \( \square \)
The next definition introduces two other fundamental notions, the cone $\mathcal{N}(s)$ and the set $\mathcal{V}(s)$.

**Definition 23.**
\[
\mathcal{N}(s) := \{ p \in \text{Pos}(A) \mid L_s(p) = \langle \vec{p}, s \rangle = 0 \},
\]
\[
\mathcal{V}(s) := \bigcap_{p \in \mathcal{N}(s)} Z(p).
\]

Basic properties of these concepts are collected in the next propositions. Note that in the important case when $A$ consists of polynomials and $\mathcal{X} = \mathbb{R}^d$, $\mathcal{V}(s)$ is a real zero set of polynomials and hence a real algebraic set, see Proposition 25.

**Proposition 24.**

i) $p \in \text{Pos}(A) \iff S_A \subseteq H_p^+$.

ii) $p \in \mathcal{N}(s) \iff S_A \subseteq H_p^+ \land s \in H_p \iff \vec{p} \in \text{Nor}_A(s)$.

iii) Let $p \in \text{Pos}(A)$. Then $L_s(p) = \langle \vec{p}, s \rangle = 0 \Rightarrow W(s) \subseteq Z(p)$.

iv) $\mathcal{W}(s) \subseteq \mathcal{V}(s)$.

v) There exists a $p \in \text{Pos}(A)$ with $\mathcal{W}(s) = Z(p)$.

vi) Suppose that there exist a function $e \in \text{lin}A$ s.t. $e(x) > 0$ on $\mathcal{X}$. Then: $0 \in \mathfrak{M}_A(s) \iff s = 0 \iff W(s) = \emptyset$.

Let $\mathcal{X}$ be a locally compact Hausdorff space. Suppose that $A$ is contained in $C(\mathcal{X}, \mathbb{R})$ and there exists a function $e \in \text{lin}A$ such that $e(x) > 0$ on $\mathcal{X}$. Then:

vii) Let $p \in \text{Pos}(A)$. If $L_s(p) = \langle \vec{p}, s \rangle = 0$, then supp $\mu \subseteq Z(p)$ $\forall \mu \in \mathfrak{M}_A(s)$.

viii) $s = 0 \iff \mathfrak{M}_A(s) = \{0\}$.

**Proof.**

i): $p \in \text{Pos}(A) \iff p(x) = \langle \vec{p}, s_A(x) \rangle \geq 0 \forall x \in \mathcal{X} \iff \langle \vec{p}, c_1s_A(x_1) + \cdots + c_ms_A(x_m) \rangle \forall x_1, \ldots, x_m \in \mathcal{X}$ and $c_1, \ldots, c_m \in (0, \infty)$ \(\subseteq\) $(\vec{p}, s) \geq 0 \forall s \in S_A \iff S_A \subseteq H_p^+$, where $(s)$ holds by Richter’s Theorem 19.

ii): It is easily verified that all three assertions mean that $\langle \vec{p}, s \rangle = 0$ and $S_A \subseteq H_p^+$.

Then they are equivalent.

iii): Let $x \in W(s)$. Then by Lemma 22 there are $c_i, c_i > 0$ and $x_i \in \mathcal{X}$ such that $\mu = c \cdot \delta_x + \sum_{i=1}^{k} c_i \cdot s_A(x_i) \in \mathfrak{M}_A(s)$. Therefore, since $p \geq 0$ on $\mathcal{X}$, we obtain

\[
0 = L_s(p) = \int_{\mathcal{X}} p(y) \, d\mu(y) = c \cdot p(x) + \sum_{i=1}^{k} c_i \cdot p(x_i) \Rightarrow x, x_i \in Z(p).
\]

iv): Follows at once from iii).

v): [D18, Prop. 29].

vi): The first equivalence is clear and in the second the direction $\Leftarrow$ follows from Richter’s Theorem 19. For the remaining implication assume there is a $x \in W(s)$. Then, by Lemma 22 we have $s = c \cdot s_A(x) + \sum_{i=1}^{k} c_i \cdot s_A(x_i)$ and

\[
0 = L_{s=0}(e) = c \cdot e(x) + \sum_{i=1}^{k} c_i \cdot e(x_i) \geq c \cdot e(x) > 0,
\]

which is a contradiction. Thus, $W(s) = \emptyset$.

vii): We repeat this well-known argument (see e.g. [Sm17, Proposition 1.23]). Suppose $x \notin Z(p)$. Since $p$ is continuous on $\mathcal{X}$, there are an open neighborhood $U$ of $x$ and a constant $\delta > 0$ such that $p(x) \geq \delta > 0$ on $U$. Then

\[
0 = L(p) = \int_{\mathcal{X}} p(x) \, d\mu \geq \int_{U} p(x) \, d\mu \geq \delta \mu(U) \geq 0.
\]

Hence $\mu(U) = 0$, so that $x \notin \text{supp} \mu$.

viii): Set $p = e$ in vii).

**Proposition 25** ([D18 Thm. 4.6]). If $\vec{p} \in \text{relint} \text{Nor}_A(s)$, then $\mathcal{V}(s) = Z(p)$. 

\[\square\]
Proof. Let \( v \in \text{Nor}_A(s) \). Since \( \bar{p} \in \text{relint} \text{Nor}_A(s) \), there is a number \( \varepsilon_v > 0 \) such that \( \bar{q}_v := \bar{p} - \varepsilon_v \cdot v \in \text{Nor}_A(s) \), i.e., \( q_v := \langle \bar{q}_v, s_A(\cdot) \rangle \geq 0 \). Then
\[
Z(p) \subseteq Z(q_v + \varepsilon_v(v, s_A(\cdot))) \subseteq Z((v, s_A(\cdot))) \quad \forall v \in \text{Nor}_A(s),
\]
i.e.,
\[
Z(p) \subseteq \bigcap_{v \in \text{Nor}_A(s)} Z((v, s_A(\cdot))) =: \mathcal{V}(s) \subseteq Z(p). \quad \square
\]

The next theorem is [IDS18a Thm. 30]. It is valid in the measurable case as well with verbatim the same proof using Lemma 22.

**Theorem 26.** \( \mathcal{V}(s) = \mathcal{W}(s) \iff s \in \text{relint} F \) for an exposed face \( F \) of \( S_A \).

By definition, the moment cone \( S_A \) itself is also an exposed face and we have \( \mathcal{W}(s) = \mathcal{X} \) for all \( s \in \text{int} S_A \). Theorem 26 shows that exposed faces and faces \( F \) of \( S_A \) such that \( s \in \text{relint} F \) play a central role for the study of the set of atoms and the structure of \( S_A \).

S. Karlin and L. S. Shapley [KS53] investigated the face structure of the moment cone \( S_A \) in the special case \( A = \{1, x, \ldots, x^d\} \) on \( \mathcal{X} = [0, 1] \). We will generalize this study to the multivariate truncated moment problem.

**Definition 27.** For \( s \in S_A \) we define the face \( F_s \) of \( s \) as the face of \( S_A \) such that \( s \in \text{relint} F_s \) and the dimension \( D_s \) of \( s \) is \( D_s := \dim F_s \). Additionally, we define the exposed face \( E_s \) of \( s \) as the smallest exposed face of \( S_A \) containing \( s \).

It is clear from the definition that \( F_s \) and \( E_s \) are unique and \( F_s \subseteq E_s \). In [KS53] \( E_s \) is called contact set and \( F_s \) is the reduced contact set, since it is obtained by an iterated cutting out of \( S_A \), see e.g. Theorem 30 and the discussion afterwards.

**Lemma 28.** Any face \( F \) of \( S_A \) is of the form \( F_s \) for some \( s \in S_A \).

**Proof.** Take an element \( s \in \text{relint} F \).

The next proposition connects the set \( \mathcal{W}(s) \) with \( F_s \) and \( \mathcal{V}(s) \) with \( E_s \).

**Proposition 29.**

i) \( s_A(x) \in F_s \iff x \in \mathcal{W}(s) \), i.e.,
\[
\mathcal{W}(s) = s_A^{-1}(F_s) \quad \text{and} \quad F_s = \text{conv cone} s_A(\mathcal{W}(s)).
\]

ii) \( s_A(x) \in E_s \iff x \in \mathcal{V}(s) \), i.e.,
\[
\mathcal{V}(s) = s_A^{-1}(E_s) \quad \text{and} \quad E_s = \text{conv cone} s_A(\mathcal{V}(s)).
\]

iii) The sets \( \mathcal{W}(s) \) and \( \mathcal{V}(s) \) are measurable.

**Proof.** i): “\( \Rightarrow \)”: Since \( s \in \text{relint} F_s \) and \( s_A(x) \in F_A \) there is an \( \varepsilon > 0 \) such that \( s' := s - \varepsilon \cdot s_A(x) \in F_s \subseteq S_A \). Let \( \mu' \in \mathcal{M}_A(s') \), then \( \mu := \varepsilon \cdot \delta_x + \mu' \in \mathcal{M}_A(s) \) and therefore \( x \in \mathcal{W}(s) \) by Lemma 22.

“\( \Leftarrow \)”: Let \( x \in \mathcal{W}(s) \). Then, by Lemma 22 \( s = c \cdot s_A(x) + \sum_{i=1}^k c_i \cdot s_A(x_i) \) for some \( c, c_i > 0 \) and \( x_i \in \mathcal{X} \). But since \( s_A(x), s_A(x_i) \in S_A \) and \( F_s \) is a face of \( S_A \) containing \( s \), also \( s_A(x), s_A(x_i) \in F_s \).

ii): Let \( s' \in \text{relint} E_s \) and apply i) with \( \mathcal{W}(s') = \mathcal{V}(s) \).

iii): Since the functions of \( A \) are measurable and \( A \) is a finite set, \( s_A \) is a measurable function and therefore \( \mathcal{W}(s) = s_A^{-1}(F_s) \in \mathcal{X} \) is measurable by i). \( \mathcal{V}(s) \) is measurable by ii) or by \( \mathcal{V}(s) = Z(p) \) from Proposition 25.

So \( \mathcal{W}(s) \subseteq \mathcal{V}(s) \) means geometrically that \( F_s \subseteq E_s \) and Theorem 26 means geometrically that \( F_s = E_s \). From the preceding considerations we find the following useful description of the set of atoms \( \mathcal{W}(s) \).

**Theorem 26 (continued).**
Theorem 30. For any \( s \in S_A \) there exist functions \( p \in \text{Pos}(A) \) (i.e., \( p \) with \( V(s) = Z(p) \) from Proposition 22) and \( p_1, \ldots, p_k \in \text{lin} A \) with \( k \leq m - D_A - 1 \) such that
\[
W(s) = Z(p) \cap Z(p_1) \cap \cdots \cap Z(p_k).
\]

Proof. First let \( s \in \text{int} S_A \). Then, as noted above, \( W(s) = X \), so \( W(s) = Z(p) \) with \( p = 0 \). Now let \( s \in \partial^* S_A \). Then there is a supporting hyperplane \( H_v \) of \( S_A \) such that \( s \in H_v \) (w.l.o.g. p.s.t. \( V(s) = Z(p) \) from Proposition 25). Additionally, there are \( v_1, \ldots, v_k \in \mathbb{R}^m \) with \( k \leq m - D_A - 1 \) such that \( \text{lin} F_s = \text{lin} s_A(W(s)) = H_v \cap H_{v_1} \cap \cdots \cap H_{v_k} \). Then
\[
W(s) = s_A^{-1}(F_s) = s_A^{-1}(S_A \cap \text{lin} F_s) = s_A^{-1}(S_A \cap H_v \cap H_{v_1} \cap \cdots \cap H_{v_k})
\]
\[
= Z(p) \cap Z(p_1) \cap \cdots \cap Z(p_k)
\]
with \( p(x) := \langle v, s_A(x) \rangle \) and \( p_i(x) := \langle v_i, s_A(x) \rangle \).

W.l.o.g. we can assume \( p_i \in \text{Pos}(A, Z(p) \cap Z(p_1) \cap \cdots \cap Z(p_{i-1})) \) in the proof. Theorem 30 says that we can cut \( F_s \) out of \( S_A \) by finitely many hyperplanes. Note that for \( s \in \text{int} S_A \) we have \( W(s) = X \) by [DS18, Thm. 16] and therefore \( p = 0 \) and \( k \leq -1 \), i.e., no function \( p_i \) is required. The inequality \( k \leq m - D_A - 1 \) gives an upper bound for \( k \), but Example 35 or Theorem 37 show that it may happen that \( k < m - D_A - 1 \). For a fixed \( S_A \) let \( k_{\text{max}} \leq m - 2 \) be the maximal \( k \) needed such that \( (12) \) holds for any \( s \in S_A \). By taking an \( s \in \text{relint} S_A \cap H_{v_1} \cap \cdots \cap H_{v_k} \) we see for any \( k' \), \( 0 \leq k' \leq k_{\text{max}} \) there is an \( s \in S_A \) such that \( k' \) is the minimal \( k \) with \( (12) \).

First we note that Richter’s Theorem 19 implies that the moment cone \( S_A \) with \( A = \{a_1, \ldots, a_n\} \subset \mathbb{R}[x_1, \ldots, x_n] \) on a semi-algebraic \( X \subset \mathbb{R}^n \) and its dual cone \( \text{Pos}(A) \) are semi-algebraic, i.e., there exist finitely many polynomial inequalities for the components \( s_i \) of the sequence \( s = (s_i)_{i=1}^m \) or the coefficients \( c_i \) of the polynomial \( p = \sum_{i=1}^m c_i a_i \) for deciding \( s \in S_A \) or \( p \in \text{Pos}(A) \).

Example 31. Let \( A = \{1, x, x^2\} \). Set \( q(x) = 1 + x^2 \) and \( B = \{1 + x^2, x, x^2\} \), so \( S_B = \text{conv cone range } s_B(\mathbb{R}) \), i.e.,
\[
\text{conv range } \left\{ \begin{array}{c}
x \frac{x}{1 + x^2} \\
\frac{1}{1 + x^2}
\end{array} \right\}
\]
is a base of the cone \( S_B \). Note that \( (13) \) is a circle in \( \mathbb{R}^2 \) with center \( (0, 1/2) \) and radius \( 1/2 \) without the point \( (0, 1) \). The point \( (0, 1) \) corresponds to an “atom at infinity”. The semi-algebraic description of \( (13) \) is
\[
\{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1/2)^2 \leq 1/4 \land y < 1\},
\]
where \( x = \frac{s_1}{s_0 + s_2} \) and \( y = \frac{s_2}{s_0 + s_2} \). Then \( s = (s_0, s_1, s_2) \in S_A \setminus \{0\} \) if and only if
\[
s_0 > 0, \quad s_2 \geq 0 \quad \text{and} \quad \left( \frac{s_1}{s_0 + s_2} \right)^2 + \left( \frac{s_2}{s_0 + s_2} - \frac{1}{2} \right)^2 \leq \frac{1}{4}.
\]

Of course, if \( S_A \) is semi-algebraic, so are \( E_s \) and \( F_s \) for any \( s \in S_A \). For polynomials on semi-algebraic sets Theorem 30 has the following corollary.

Corollary 32. Let \( X \) be a (semi-)algebraic set in \( \mathbb{R}^n \) and \( A \) a set of polynomials. Then the set of atoms \( W(s) \) is (semi-)algebraic for any moment sequence \( s \in S_A \). The same is true if \( X \) is (semi-)algebraic in \( \mathbb{P}^n \) and \( A \) consists of homogeneous polynomials.

We return to the face structure of \( S_A \).

Definition 33. A convex set \( K \subseteq \mathbb{R}^m \) is called perfect iff every face of \( K \) is also an exposed face.
An immediate consequence of Theorem 26 yields the following.

**Corollary 34.** The following are equivalent:

i) The moment cone $S_A$ is perfect.

ii) $\mathcal{V}(s) = \mathcal{W}(s)$ for all $s \in S_A$.

iii) $\mathcal{F}_s = \mathcal{E}_s$ for all $s \in S_A$.

In the one-dimensional monomial case $A = \{1, x, x^2, \ldots, x^d\}$ on a closed interval of $\mathbb{R}$ we always have $\mathcal{V}(s) = \mathcal{W}(s)$, see e.g. [IDS18a Exm. 14]. The first examples for which $\mathcal{W}(s) \neq \mathcal{V}(s)$ were [IDS18a Examples 38 and 39], see also [Sm17 Example 18.25]; all known examples of this kind were one-dimensional monomials with gaps. The following is the first example for polynomials without gaps.

**Example 35 (Harris polynomial $\mathcal{V}(s) \neq \mathcal{W}(s)$).** W. R. Harris [Har99] proved that the polynomial

\[
h(x_0, x_1, x_2) = 16(x_0^{10} + x_1^{10} + x_2^{10}) - 36(x_0^6x_2^2 + x_0^4x_1^2 + x_1^8x_2 + x_0^2x_2 + x_0^8x_2 + x_1^2x_2^2) + 20(x_0^6x_2^4 + x_0^4x_1^4 + x_0^4x_2^4 + x_1^4x_2^4) + 57(x_0^6x_1^2x_2^2 + x_0^2x_1^2x_2^2 + x_0^6x_1^2x_2^2) - 38(x_0^6x_1^2x_2 + x_0^4x_1^4x_2 + x_0^6x_1^4x_2^3)
\]

in $\mathcal{B}_{2,10}$ is nonnegative on $\mathbb{P}^2$ and has the projective zero set

\[
\mathcal{Z}(h) = \{(1, 1, 0)^*, (1, 1, \sqrt{2})^*, (1, 1, 1/2)^*\} = \{z_1 = (1, 1, 0), z_2 = (1, -1, 0), z_3 = (1, 0, 1), z_4 = (1, 0, -1), z_5 = (0, 1, 1), z_6 = (0, 1, -1), z_7 = (1, 1, 1/2), z_8 = (1, 1, -1/2), z_9 = (1, -1, 1/2), z_{10} = (1, -1, -1/2), z_{11} = (1, 1/2, 1), z_{12} = (1, 1/2, -1), z_{13} = (1, -1/2, 1), z_{14} = (1, -1/2, -1), z_{15} = (1/2, 1, 1), z_{16} = (1/2, 1, -1), z_{17} = (1/2, -1, 1), z_{18} = (1/2, -1, -1), z_{19} = (1, 1/2, \sqrt{2}), z_{20} = (1, 1, -\sqrt{2}), z_{21} = (1, -1/2, \sqrt{2}), z_{22} = (1, -1, -\sqrt{2}), z_{23} = (1, \sqrt{2}, 1), z_{24} = (1, \sqrt{2}, -1), z_{25} = (1, -\sqrt{2}, 1), z_{26} = (1, -\sqrt{2}, -1), z_{27} = (\sqrt{2}, 1, 1), z_{28} = (\sqrt{2}, 1, -1), z_{29} = (\sqrt{2}, -1, 1), z_{30} = (\sqrt{2}, -1, -1)\}.
\]

Here the symbol $(a, b, c)^*$ denotes all permutations of $(a, b, c)$ including sign changes. Hence, $h$ has exactly 30 projective zeros. Set $Z_k := \{1, \ldots, z_k\}$. Note that the full rank of $DS_{k, \mathcal{B}_{2,10}}$ is $[\mathcal{B}_{2,10}] = 66$. Table 1 shows the rank of $DS_{k,A}(1, Z_k)$ for $Z_k$.

| Table 1. Rank of $DS_{k,A}(1, Z_k)$ of subsets $Z_k = \{z_1, \ldots, z_k\}$ of the zero set of the Harris polynomial. $[\mathcal{B}_{2,10}] = 66$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| rank $DS_{k,A}(1, Z_k)$ | $Z_1$ | $Z_2$ | $Z_3$ | $Z_4$ | $Z_5$ | $Z_6$ | $Z_7$ | $Z_8$ | $Z_9$ | $Z_{10}$ |
| increase | +3 | +3 | +3 | +3 | +3 | +3 | +3 | +3 | +2 | +1 |
| rank $DS_{k,A}(1, Z_k)$ | $Z_11$ | $Z_{12}$ | $Z_{13}$ | $Z_{14}$ | $Z_{15}$ | $Z_{16}$ | $Z_{17}$ | $Z_{18}$ | $Z_{19}$ | $Z_{20}$ |
| increase | +3 | +3 | +3 | +3 | +3 | +3 | +3 | +3 | +3 | +3 |
| rank $DS_{k,A}(1, Z_k)$ | $Z_{21}$ | $Z_{22}$ | $Z_{23}$ | $Z_{24}$ | $Z_{25}$ | $Z_{26}$ | $Z_{27}$ | $Z_{28}$ | $Z_{29}$ | $Z_{30}$ |
| increase | +2 | +1 | +2 | +0 | +0 | +0 | +0 | +0 | +0 | +0 |
In [1DS18b, Exm. 63] we showed by using a computer algebra program that the set \( \{ s_{2,10}(z_i) \}_{i=1}^{30} \) is linearly independent, i.e., the exposed face of \( h \) has dimension 30: \( \dim B_{2,10} \cap H = 30 \). Table 4 shows that already the zeros \( z_1, \ldots, z_{23} \) give rank \( DS_{23,B_{2,10}}(1, Z_{23}) = 65 \). Hence each sequence \( s = \sum_{i=1}^{23} c_i \cdot s_{2,10}(z_i) \) with \( c_i > 0 \) gives \( W(s) = Z(h) = \{ z_1, \ldots, z_{30} \} \). But \( W(s) = \{ z_1, \ldots, z_{23} \} \), so that \( V(s) = W(s) \).

Since \( \{ s_{2,10}(z_i) \}_{i=1}^{30} \) is linearly independent, for any \( I \subset Z(h) \) there is a polynomial \( p_I \in \text{lin } B_{2,10} \) such that \( p_I(z) = 0 \) for all \( z \in I \) and \( p_I(z) \neq 0 \) for \( z \in J := Z(h) \setminus I \). Therefore, \( W(s) = Z(h) \cap Z(p_{W(s)}) \) in Theorem 36, i.e., \( k = 1 < m - D_s - 1 = 34 \).

An immediate consequence of Example 35 is the following.

**Corollary 36.**

a) \( S_{A,d} \) and \( S_{B_{n,24}} \) are perfect for all \( d \geq 1 \).

b) \( S_{A_{10}} \) and \( S_{B_{n,10}} \) for \( n \geq 2 \) are not perfect.

That \( S_{A_{10}} \) and \( S_{B_{n,10}} \) are not perfect indicates that also \( S_{A_{d}} \) and \( S_{B_{n,24}} \) for \( n \geq 2 \) and \( d \geq 1 \) are not perfect (see Problem 4).

**Proposition 37.** If \( s \in S_A \) such that the set \( V(s) \) is finite, then

\[
W(s) = Z(p) \quad \text{or} \quad W(s) = Z(p) \cap Z(q)
\]

for some \( p \in \text{Pos}(A) \) and \( q \in \text{lin } A \) indefinite.

**Proof.** Set \( S := \text{conv cone}_A(V(s)) \). Then \( S_A \) is a simplicial cone, i.e., every extreme face is an exposed face and the assertion follows from Theorem 26.

The conclusion of the previous theorem does not hold if \( V(s) \) is infinite. As shown by Example 35, the set \( W(s) \) of atoms can be smaller than \( V(s) \). To repair this the procedure of defining \( V(s) \) will be iterated.

Let \( L \) be a linear functional on \( \text{lin } A \). We define inductively linear subspaces \( N_k(L), k \in \mathbb{N}, \) of \( \text{lin } A \) and subsets \( V_j(L), j \in \mathbb{N}_0, \) of \( X \) by \( V_0(L) = X, \)

\[
N_k(L) := \{ p \in \text{lin } A : L(p) = 0, \ p(x) \geq 0 \ \text{ for } \ x \in V_{k-1}(L) \},
\]

\[
V_j(L) := \{ x \in X : p(x) = 0 \ \text{ for } \ p \in N_j(L) \}.
\]

If \( V_k(L) \) is empty for some \( k \), we set \( V_j(L) = \emptyset \) for \( j \geq k \). Clearly, if \( s \in S_A \), then \( V_1(L_s) = V(s) \). From this definition it is obvious that \( V_j(L) \subseteq V_{j-1}(L) \) for \( j \in \mathbb{N} \).

**Definition 38.** The core variety \( V_C(L) \) of the linear functional \( L \) on \( \text{lin } A \) is

\[
V_C(L) := \bigcap_{j=0}^{\infty} V_j(L).
\]

The core variety was introduced by L. Fialkow [Fia77] and studied in [1DS18a, Sm17, BF]. If \( A \) consists of real polynomials and \( X = \mathbb{R}^d \), then \( V_C(L) \) is the zero set of real polynomials and hence a real algebraic set in \( \mathbb{R}^d \).

Let \( L_s \) be the Riesz functional of a moment sequence \( s \in S_A \). Then, as proved in [1DS18a, Theorem 33], see also [Sm17, Theorem 1.49], we have \( V_C(L_s) = W(s) \), that is, the core variety coincides with the set of atoms, see the most general form Theorem 30. Further, there exists a \( k \in \mathbb{N}_0 \) such that \( V_C(L_s) = V_k(L_s) \). The core variety is treated in [Sm17, Sections 1.2.5, 18.3].

Let us resume the investigation of the face structure of the moment cone.

**Definition 39.** For \( s \in S_A \) we define

\[
\Gamma_s := \{ f \in \text{lin } A \mid W(s) \subseteq Z(f) \} \quad \text{and} \quad \gamma_s := \dim \Gamma_s.
\]

**Proposition 40.** \( D_s = |A| - \gamma_s \).
Proof. Clearly, \( f \in \Gamma_s \) if and only if \( \mathcal{F}_s \subseteq \text{aff} \mathcal{F}_s \subseteq H_{\gamma} \). This gives the assertion. \( \square \)

**Proposition 41.** Let \( A = A_{n,d} \) on \( \mathcal{X} = \mathbb{R}^n \) or \( A = B_{n,2d} \) on \( \mathcal{X} = \mathbb{P}^n \). If \( s \in \partial^* S_A \), then
\[
\mathcal{D}_s \leq |A| - n - 1.
\]

**Proof.** Since \( s \in \partial^* S_A \), there is a \( p \in \text{Pos}(A) \setminus \{0\} \) such that \( \mathcal{W}(s) \subseteq \mathcal{Z}(p) \). Since \( p \neq 0 \), there is an \( i \) such that \( \partial_i p \neq 0 \) and the functions \( \partial_i p \) (resp. \( x_0 \partial_i p \) in the projective case), \( x_1 \partial_i p, \ldots, x_n \partial_i p \) are non-zero, linearly independent, and in \( \Gamma_s \). Hence, \( \gamma_s \geq n + 1 \) and \( \mathcal{D}_s \leq |A| - \gamma_s \leq |A| - n - 1 \). \( \square \)

The preceding shows that the face \( \mathcal{F}_s \) of \( s \) and its dimension \( \mathcal{D}_s \) play an important role in the study of the moment cone \( S_A \) and its boundary. Proposition 41 contains an upper bound for \( \mathcal{D}_s \). To give some lower bounds we consider two special cases:

a) \( \mathcal{X} = \mathbb{R}^n \) (or \( \mathbb{P}^n \)) with \( A = A_{n,2d} \) (or \( B_{n,2d} \), respectively). Set
\[
(16) \quad p := p_1 + \cdots + p_n \quad \text{with} \quad p_i(x) := \prod_{j=0}^{d-1} (x_i - j)^2,
\]

i.e., \( p \geq 0 \) on \( \mathbb{R}^n \) and \( \mathcal{Z}(p) = \{0, 1, \ldots, d - 1\}^n \) with \( \#\mathcal{Z}(p) = d^n \).

b) \( \mathcal{X} = [0, d]^n \) with \( A = A_{n,2d} \). Set
\[
(17) \quad q := q_1 + \cdots + q_n \quad \text{with} \quad q_i(x) := x \prod_{j=1}^{d-1} (x_i - j)^2 \cdot (d - x),
\]

i.e., \( q \geq 0 \) on \( [0, d]^n \) and \( \mathcal{Z}(q) = \{0, 1, \ldots, d\}^n \) with \( \#\mathcal{Z}(q) = (d + 1)^n \).

Both cases are the simplest cases of non-negative polynomials with large numbers, but finitely many zeros. The first case works on the whole space \( \mathbb{R}^n \) (or \( \mathbb{P}^n \)). The second case on \( [0, d]^n \) is important in numerical analysis. Since \( \mathcal{Z}(p) \) and \( \mathcal{Z}(q) \) are finite, we can set \( s := \sum_{p \in \mathcal{Z}(p)} s_A(x) \) and \( s' := \sum_{q \in \mathcal{Z}(q)} s_A(x) \). Then
\[
\mathcal{D}_s = \mathcal{R}_{n,2d} := \text{rank} (s_{A,2d}(x))_{x \in \mathcal{Z}(p)}
\]
and
\[
\mathcal{D}_{s'} = \mathcal{R}'_{n,2d} := \text{rank} (s_{A,2d}(x))_{x \in \mathcal{Z}(q)}.
\]

In addition, we set \( \mathcal{W}_{n,2d} := \frac{\mathcal{R}_{n,2d}}{\dim S_{\mathcal{A}}} \) and \( \mathcal{Z}_{n,2d} := \frac{\mathcal{R}_{n,2d}}{|\mathcal{Z}(p)|} \). The numbers \( \mathcal{W}_{n,2d} \) and \( \mathcal{Z}_{n,2d} \) are defined in the same way for the second case b). By these definitions, \( \mathcal{W}_{n,2d} \) and \( \mathcal{Z}_{n,2d} \) are the ratios of the dimension of the exposed face \( \mathcal{F}_s \) by the dimension of \( \dim S_{\mathcal{A}} = |A_{n,2d}| = n + 2d \) and the cardinality of the zero set \( \mathcal{Z}(p) \), respectively.

For \( n = 1 \) we can use the formula for the Vandermonde determinant and obtain the following.

**Lemma 42.**

i) \( \mathcal{R}_{1,2d} = d \), \( \mathcal{W}_{1,2d} = \frac{d}{2d+1} \), and \( \mathcal{Z}_{1,2d} = 1 \).

ii) \( \mathcal{R}'_{1,2d} = d + 1 \), \( \mathcal{W}'_{1,2d} = \frac{d+1}{2d+1} \), and \( \mathcal{Z}'_{1,2d} = 1 \).

For \( n = 2 \), C. Riener and M. Schweighofer in \( \text{RSTS}^8 \) proved the following result.

**Lemma 43 (RSTS^8 Lem. 8.6).** \( \mathcal{R}_{2,2d} = d^2 \), \( \mathcal{W}_{2,2d} = \frac{d^2}{(d+1)(2d+1)} \), and \( \mathcal{Z}_{2,2d} = 1 \).

For \( n \geq 3 \) very little is known about these numbers. In table 2 we collect several numerical examples which have been calculated by a computer algebra program, see also fig. 1.

Some simple facts are collected in the following lemma.

**Lemma 44.**

i) \( \mathcal{R}_{n,2} = 1 \) and \( \mathcal{Z}_{n,2} = 1 \).

ii) \( \mathcal{W}_{n,2} > \mathcal{W}_{n',2} \) for \( 1 \leq n \leq n' \).

iii) \( \mathcal{R}_{n,2d} \leq \mathcal{R}_{n,2d'} \) and \( \mathcal{R}'_{n,2d} \leq \mathcal{R}'_{n,2d'} \) for \( 1 \leq d \leq d' \).
Table 2. Values of $D_s = R_{n,2d}$, $w_{n,2d}$, and $z_{n,2d}$ for $A_{n,2d}$ on $\mathbb{R}^n$; as well as $D_s' = R'_{n,2d}$, $w'_{n,2d}$, and $z'_{n,2d}$ for $A_{n,2d}$ on $[0,d]^n$ calculated by a computer algebra program.

| n | $d$ | $|A_{n,2d}|$ | $|\mathcal{Z}(p)|$ | $R_{n,2d}$ | $w_{n,2d}$ | $z_{n,2d}$ | $|\mathcal{Z}(q)|$ | $R'_{n,2d}$ | $w'_{n,2d}$ | $z'_{n,2d}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 1 | 10 | 1 | 10.0% | 100.0% | 8 | 7 | 70.0% | 87.5% | |
| 2 | 35 | 8 | 8 | 22.9% | 100.0% | 27 | 23 | 65.7% | 85.2% | |
| 3 | 84 | 27 | 27 | 32.1% | 100.0% | 64 | 54 | 63.4% | 84.4% | |
| 4 | 165 | 64 | 63 | 38.2% | 98.4% | 125 | 105 | 63.6% | 84.0% | |
| 5 | 286 | 125 | 121 | 42.3% | 96.8% | 216 | 181 | 63.3% | 83.8% | |
| 6 | 455 | 216 | 206 | 45.3% | 95.4% | 343 | 287 | 63.1% | 83.7% | |
| 7 | 680 | 343 | 323 | 47.5% | 94.2% | 512 | 428 | 62.9% | 83.6% | |
| 8 | 969 | 512 | 477 | 49.2% | 93.2% | 729 | 609 | 62.8% | 83.5% | |
| 9 | 1330 | 729 | 673 | 50.6% | 92.3% | 1000 | 835 | 62.8% | 83.5% | |
| 10 | 1771 | 1000 | 916 | 51.7% | 91.6% | 1331 | 1111 | 62.7% | 83.5% | |
| 11 | 2300 | 1331 | 1211 | 52.7% | 91.0% | 1728 | 1442 | 62.7% | 83.4% | |
| 12 | 2925 | 1728 | 1563 | 53.4% | 90.5% | 2197 | 1833 | 62.7% | 83.4% | |
| 13 | 3654 | 2197 | 1977 | 54.1% | 90.0% | 2744 | 2289 | 62.7% | 83.4% | |
| 14 | 4495 | 2744 | 2458 | 54.7% | 89.6% | 3375 | 2815 | 62.6% | 83.4% | |
| 15 | 5456 | 3375 | 3011 | 55.2% | 89.2% | 4096 | 3416 | 62.6% | 83.4% | |
| 4 | 1 | 15 | 1 | 1 | 6.6% | 100.0% | 16 | 11 | 73.3% | 68.8% | |
| 5 | 70 | 16 | 16 | 22.9% | 100.0% | 81 | 50 | 71.4% | 61.7% | |
| 3 | 210 | 81 | 76 | 36.2% | 93.8% | 256 | 150 | 71.4% | 58.6% | |
| 4 | 495 | 256 | 221 | 44.6% | 86.3% | 625 | 355 | 71.7% | 56.8% | |
| 5 | 1001 | 625 | 503 | 50.2% | 80.5% | 1296 | 721 | 72.0% | 55.6% | |
| 6 | 1820 | 1296 | 986 | 54.2% | 76.1% | 2401 | 1316 | 72.3% | 54.8% | |
| 7 | 3060 | 2401 | 1746 | 57.1% | 72.7% | 4096 | 2220 | 72.5% | 54.2% | |
| 8 | 4845 | 4096 | 2871 | 59.3% | 70.1% | 6561 | 3525 | 72.8% | 53.7% | |
| 5 | 1 | 21 | 1 | 1 | 4.7% | 100.0% | 32 | 16 | 76.2% | 50.0% | |
| 2 | 126 | 32 | 31 | 24.6% | 96.9% | 243 | 96 | 76.2% | 39.5% | |
| 3 | 462 | 243 | 192 | 41.6% | 79.0% | 1024 | 357 | 77.3% | 34.9% | |
| 4 | 949 | 489 | 221 | 49.3% | 83.8% | 1536 | 355 | 77.3% | 34.9% | |
| 5 | 1001 | 625 | 503 | 50.2% | 80.5% | 1296 | 721 | 72.0% | 55.6% | |
| 6 | 1820 | 1296 | 986 | 54.2% | 76.1% | 2401 | 1316 | 72.3% | 54.8% | |
| 7 | 3060 | 2401 | 1746 | 57.1% | 72.7% | 4096 | 2220 | 72.5% | 54.2% | |
| 8 | 4845 | 4096 | 2871 | 59.3% | 70.1% | 6561 | 3525 | 72.8% | 53.7% | |
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Figure 1. Graphic representation of \( w_{n,2d} \) from table 2.

iv) \( \mathcal{R}_{n,2d} \leq \mathcal{R}'_{n,2d} \) and \( \mathcal{R}'_{n,2d} \leq \mathcal{R}'_{n',2d} \) for \( 1 \leq n \leq n' \).

In table \( \mathcal{R}' \) we observe that in all calculated cases \( (n = 3, 4, \ldots, 10) \) we have \( \mathcal{R}'_{n,2} = \left( \frac{n+2}{2} \right) - n \). In the next lemma we prove that this holds for all \( n \in \mathbb{N} \).

**Lemma 45.** \( \mathcal{R}'_{n,2} = \left( \frac{n+2}{2} \right) - n = \frac{n}{2}(n^2 + n + 2) \) for \( n \in \mathbb{N} \).

**Proof.** Since \( q = q_1 + \cdots + q_n \), we have \( \gamma_{\nu'} \geq n \) and therefore \( D_{\nu'} = \mathcal{R}'_{n,2} \leq \left( \frac{n+2}{2} \right) - n \).

For the converse direction, let \( e_i \) be the \( i \)-th unit vector in \( \mathbb{R}^n \). Take the \( \left( \frac{n+2}{2} \right) - n \) points \( x_i \) as

\[
P = \{ 0, e_1, \ldots, e_n, e_1 + e_2, \ldots, e_1 + e_2, \ldots, e_{n-1} + e_n \} \subseteq \{ 0, 1 \}^n.
\]

Then \( \mathcal{R}'_{n,2} \geq \text{rank} \left( s_A(x) \right)_{x \in P} = \text{rank} \ M \) with

\[
M = \begin{pmatrix}
s_A(0)^T \\
s_A(e_1)^T - s_A(0)^T \\
\vdots \\
s_A(e_n)^T - s_A(0)^T \\
s_A(e_1 + e_2)^T - s_A(e_1)^T - s_A(e_2)^T + s_A(0)^T \\
\vdots \\
s_A(e_{n-1} + e_n)^T - s_A(e_{n-1})^T - s_A(e_n)^T + s_A(0)^T
\end{pmatrix}.
\]

But

\[
s_A(0) = \begin{cases} 
1 & \text{at position 1} \\
0 & \text{elsewhere}
\end{cases}, \quad s_A(e_i) - s_A(0) = \begin{cases} 
1 & \text{at position } x_i \text{ and } x_i^2 \\
0 & \text{elsewhere}
\end{cases},
\]

and

\[
s_A(e_1 + e_j)^T - s_A(e_1)^T - s_A(e_j)^T + s_A(0)^T = \begin{cases} 
1 & \text{at position } x_i x_j \\
0 & \text{else}
\end{cases}.
\]

So the rows in \( M \) are linearly independent and \( \text{rank} \ M = \left( \frac{n+2}{2} \right) - n \). \( \square \)

For all pairs \( (n,d) \) of numbers occurring in table \( \mathcal{R}' \) and fig. 1 we gather the inequalities:

\[
(18a) \quad w_{n,2d} < w_{n,2d'} \quad \text{for all } 1 \leq d < d',
\]
We don’t know whether or not these inequalities hold in general (see Problem 2). Do the limits
\begin{align}
\lim_{n \to \infty} w_{n, 2d}, \quad \lim_{d \to \infty} w_{n, 2d}, \quad \lim_{n \to \infty} w'_{n, 2d}, \quad \lim_{d \to \infty} w'_{n, 2d},
\end{align}
exist and if so, what are these limits (see Problem 3)? From Lemma 45 we already obtain
\[
\lim_{n \to \infty} w_{n, 2} = 1.
\]

In the next section we will see how the facial structure and especially table 2 affect the Carathéodory number \( C_A \).

7. Carathéodory Numbers

By the Richter Theorem (Theorem 19) every \( s \in S_A \) has a \( k \)-atomic representing measure with \( k \leq m \). This justifies the following definitions.

Definition 46. The Carathéodory number \( C_A(s) \) of a moment sequence \( s \in S_A \) is the smallest \( k \in \mathbb{N} \) such that \( s \) has a \( k \)-atomic representing measure:
\[
C_A(s) := \min \{ k \in \mathbb{N} \mid s \in \text{range } S_{k, A} \} \leq m.
\]
The Carathéodory number \( C_A \) of the moment cone \( S_A \) is the maximum of numbers \( C_A(s) \) for \( s \in S_A \), or equivalently the smallest number such that every \( s \in S_A \) has an at most \( k \)-atomic representing measure:
\[
C_A := \max_{s \in S_A} C_A(s) = \min \{ k \in \mathbb{N} \mid S_A \subseteq \text{range } S_{k, A} \} \leq m.
\]

For the univariate polynomial moment problem the following classical result is already contained in [Ric57].

Theorem 47 ([Ric57, Satz 11]). For \( X = \mathbb{R} \), \([a, b] \), \([a, b) \), \((a, b] \), or \((a, b) \) and \( A = \{1, x, \ldots, x^d\} \) with \( d \in \mathbb{N} \) and \( -\infty \leq a < b \leq -\infty \) we have
\[
C_A = |N_A| = \left\lfloor \frac{d + 1}{2} \right\rfloor.
\]

Here, as usual, \( \lfloor r \rfloor \) denotes the smallest integer which is larger or equal to \( r \).

For monomials on \( \mathbb{R} \) with gaps, that is, \( A = \{1, x^{d_2}, \ldots, x^{d_m}\} \) with \( 0 < d_2 < \cdots < d_m \), formula \( C_A = |N_A| \) is no longer valid in general. Sufficient conditions for \( C_A = |N_A| \) to hold are given in [dDS18, Theorem 45]. We restate this result without proof. Let
\[
A = \{1, x^{d_1}, \ldots, x^{d_m}\} \text{ with } 1 \leq d_1 < \cdots < d_m = 2d \text{ on } X = \mathbb{R}.
\]
In [DST18b] Lem. 40] it was shown that
\[
\det DS_{k,A}(c_1, \ldots, c_k, x_1, \ldots, x_k) =
\]
\[
c_1 \cdots c_k \cdot (x_1 \cdots x_k)^{2d_1} \prod_{1 \leq i < j \leq k} (x_j - x_i)^4 \cdot q_A(x_1, \ldots, x_k),
\]
and
\[
\det(DS_{k-1,A}, s_A(x_k)) =
\]
\[
c_1 \cdots c_{k-1} \cdot (x_1 \cdots x_{k-1})^{2d_1} x_k^{d_1} \cdot \prod_{1 \leq i < j \leq k-1} (x_j - x_i)^4 \cdot \prod_{i=1}^{k-1} (x_k - x_i)^2 \cdot q_{A,k}(x_1, \ldots, x_k),
\]
where \(q_A\) and \(q_{A,k}\) are polynomials with non-negative coefficients.

**Theorem 48 ([DST18b] Thm. 45).** Let \(A\) be as in (29) and \(Z := Z(q_A) \subseteq \mathbb{R}^k\) if \(m = 2k\) is even and \(Z := Z(q_{A,1}) \cap \cdots \cap Z(q_{A,k}) \subseteq \mathbb{R}^k\) if \(m = 2k - 1\) is odd. Suppose
\[
(x_1, \ldots, x_k) \in Z \Rightarrow \exists i \neq j : x_i = x_j.
\]
Then
\[
C_A = \left\lceil \frac{m}{2} \right\rceil.
\]

An example where (24) fails is given in [DST18b] Example 48]. For one-dimensional monomial systems with gaps, A. Wald [Wal39] proved already in 1939 the following.

**Theorem 49 ([Wal39] Prop. 13].** Let \(0 \leq d_1 < \cdots < d_m\) be integers, \(X = [0, \infty)\), and \(A = \{x^{d_1}, \ldots, x^{d_m}\}\). Then any \(s \in S_A\) has a \(k\)-atomic representing measure with \(k \leq \frac{m+1}{2}\).

The next result deals with the half-line \(X = (0, \infty)\).

**Theorem 50 ([DST18a] Thm. 40].** Suppose \(d_1, \ldots, d_m \in \mathbb{N}_0, m \in \mathbb{N}, d_1 < \cdots < d_m\), and \(A = \{x^{d_1}, \ldots, x^{d_m}\}\) on \(X = (0, \infty)\). Then \(C_A = \left\lceil \frac{m}{2} \right\rceil\).

For other cases than one-dimensional truncated power moment problems only a few Carathéodory numbers are known, but upper and lower bounds can be given.

First we note that in the general case the bound \(C_A \leq m\) in Theorem 19 is sharp as we see from the next result. Recall that \(m = \dim S_A\).

**Theorem 51 ([DST18b] Thm. 3.31].** If \(s_A(X)\) is countable, then
\[
C_A = m.
\]

**Proof.** Since \(A\) is linearly independent, \(S_A\) is full-dimensional and \(\text{int } S_A \neq \emptyset\). Set \(h_{x_1, \ldots, x_{m-1}} := \text{lin } \{s_A(x_1), \ldots, s_A(x_{m-1})\}\) with \(x_i \in X\). Then \(h_{x_1, \ldots, x_{m-1}}\) is a closed subspace of \(\mathbb{R}^m\) of dimension at most \(m - 1\). Since \(s_A(X)\) is countable, so is \(s_A(X)^{m-1}\) and \(H := \bigcup_{x_1, \ldots, x_{m-1} \in X} h_{x_1, \ldots, x_{m-1}}\) is a countable union of closed subspaces with dimension at most \(m - 1\). Hence \(H\) does not contain inner points. Therefore, \(\text{int } S_A \setminus H \neq \emptyset\). Any sequences \(s \in \text{int } S_A \setminus H\) need at least \(m\) atoms, since otherwise it would be contained in some hyperplane \(h_{x_1, \ldots, x_{m-1}}\). \(\square\)

The truncated moment problem on \(X = \mathbb{N}_0\) was studied in [KLST17] and the previous theorem completely solves the Carathéodory number problem in this case: Since \(\mathbb{N}_0\) is countable, so is \(s_A(\mathbb{N}_0)\) and therefore \(C_A = m\) by Theorem 51.

A slightly better upper bound than \(m\) is given in the following result, see [DST18a Thm. 12]. It is a version of [DST18b Thm. 13] with weaker conditions, but its proof is verbatim the same.
Theorem 52. If the set $\{x \in X \mid s_A(y) \neq 0\}$ has at most $m-1$ path-connected components, then

$$C_A \leq m - 1.$$  

For (homogeneous) polynomials on $\mathbb{R}^2$ or $\mathbb{P}^2$ upper bounds for the Carathéodory number $C_A$ have been obtained in [dDS18b] and [RS18]. They are based on zeros of nonnegative polynomials and use deep results of Petrovski [Pet38] on Hilbert’s 6th problem. We summarize the main results in the following theorem.

Theorem 53. i) For $X = \mathbb{R}^2$ we have $C_{A_{\mathbb{R}^2}} \leq \frac{3}{2}d(d-1) + 1$ for $d \in \mathbb{N}$. ii) For $X = \mathbb{P}^2$ we have $C_{A_{\mathbb{P}^2}} \leq \frac{3}{2}d(d-1) + 2$ for $d \geq 5$.

Proof. i) is [RS18, Cor. 8.4] and ii) is [dDS18b, Thm. 62]. □

A general lower bound for sufficiently differentiable functions $a_i$ was given in [dDS18b, Thm. 27]. It is based on Sard’s Theorem [Sar42].

Theorem 54 ([dDS18b, Prop. 23 and Thm. 27]). Suppose that $A \subset C^r(\mathbb{R}^n, \mathbb{R})$ with $r > N_A \cdot (n+1) - m$. Then

$$\left\lceil \frac{m}{n+1} \right\rceil \leq N_A \leq C_A.$$  

Further, the set of moment sequences $s$ which can be represented by less than $N_A$ atoms has $|A|$-dimensional Lebesgue measure zero in $\mathbb{R}^m$.

Though the previous theorem was stated for $X = \mathbb{R}^n$, it remains for differential manifolds by Remark 15.

In the preceding we mainly reviewed the recent developments on Carathéodory numbers from [dDS18b] and [RS18]. Now we apply the considerations on the facial structure of the moment cone $S_A$ from the previous section to derive some new results.

Lemma 55. $C_A(s) \leq D_s$ for all $s \in S_A$.

Proof. The assertion follows from Richter’s Theorem 19 applied to $X = W(s)$. □

Corollary 56. Suppose $A = A_{\mathbb{R}^n,d}$ on $X = \mathbb{R}^n$ or $A = B_{\mathbb{R}^n,d}$ (d even) on $X = \mathbb{P}^n$. Then, for $s \in \partial^* S_A := \partial S_A \cap S_A$, we have

$$C_A(s) \leq \left(\frac{n+d}{n}\right) - n - 1.$$  

Proof. Combine Lemma 55 and Proposition 41. □

Theorem 57. For $A = B_{\mathbb{R}^n,2d}$ on $X = \mathbb{P}^n$ we have

$$C_A \leq \left(1 + \frac{2d}{n}\right) - n.$$  

Proof. Let $s \in S_A$. By [dDS18b, Prop. 8] we can write $s = c \cdot s_A(x) + s'$ for some $x \in \mathbb{P}^n$ and $c > 0$ such that $s' \in \partial S_A = \partial^* S_A$. Then, by Corollary 56

$$C_A(s') \leq \left(\frac{n+2d}{n}\right) - n - 1.$$  

Therefore, $C_A(s) \leq \left(\frac{n+2d}{n}\right) - n$ which implies the assertion. □

This slightly improves the upper bounds in the projective case. Lower bound improvements are also possible by using table 2 and the results obtained at the end of Section 6.

Theorem 58. Let $A = A_{\mathbb{R}^n,2d}$. For an open subset $X$ of $\mathbb{R}^n$ or $X = [0,1]^n$, we have

$$C_A \geq \mathfrak{R}_{\mathbb{R}^n,2d} \text{ or } \mathfrak{R}_{\mathbb{R}^n,2d},$$  

respectively.
Proof. For $X \subseteq \mathbb{R}^n$ we take $p$ from (16) and consider the moment problem on $X' = \mathbb{Z}(p)$. For $X = [0,1]^n$ we take $q$. Then Theorem 51 shows that there is a moment sequence $s \in S_A$ such that $C_A(s) = \dim \text{conv} \, s_A(X')$ is $\mathbb{R}_{n,2d}$ or $\mathbb{R}_{n,2d'}$, respectively.

Let us briefly discuss these results. From Theorem 54 we recall the lower bound (26)
\[ \frac{C_A}{|A|} \geq \frac{1}{n+1} \]
which decreases with increasing $n$. Theorem 58 yields (27)
\[ \frac{C_A}{|A|} \geq w_{n,2d} \text{ and } w'_{n,2d}, \]
As seen from table 2, the numbers $w_{n,2d}$ and $w'_{n,2d}$ give much better estimates than (26), but they have to be calculated for each case. For instance, in the case $A = A_{5,14}$ on $\mathbb{R}^5$ we have
\[ \frac{C_A}{|A|} \geq \frac{1}{6} \approx 0.17 \text{ from (26), } \quad \text{but} \quad \frac{C_A}{|A|} \geq w_{5,14} \approx 0.66 \text{ from (27),} \]
and in the case $A = A_{10,4}$ on $[0,1]^{10}$ we even have
\[ \frac{C_A}{|A|} \geq \frac{1}{11} \approx 0.09 \text{ from (26), } \quad \text{but} \quad \frac{C_A}{|A|} \geq w'_{10,4} \approx 0.89 \text{ from (27).} \]

For $d = 1$ (i.e., $A = A_{n,2}$) on $[0,1]^n$ there is the following explicit result.

**Theorem 59.** For $A = A_{n,2}$ on $X = [0,1]^n$ we have
\[ \left( \frac{n+2}{2} \right) - n \leq C_A \leq \left( \frac{n+2}{2} \right) - 1 \]
and therefore
\[ \lim_{n \to \infty} \frac{C_{A_{n,2}}}{|A_{n,2}|} = 1. \]

Proof. The upper bound is Theorem 52 while the lower bound is Theorem 58 combined with Lemma 45. The limit follows by a straightforward calculation. □

From Theorem 59 we see that $(n+2d) - n$ is a lower bound on the Carathéodory number for $A = A_{n,2}$ on $X = [0,1]^n$, but it is also an upper bound for $A = B_{n,2d}$ on $X = \mathbb{P}^n$ by Theorem 57. In fact, $(n+2d) - n$ is also a lower bound on the face dimension of the moment cone $S_A$ for $A = A_{n,2}$ ($d = 1$) on $X = [0,1]^n$, but for $A = A_{n,2d}$ or $B_{n,2d}$ on $X = \mathbb{R}^n$ or $\mathbb{P}^n$ we have an upper bound of the face dimension of $(n+2d) - n - 1$ by Corollary 56. Thus, changing the set $X$ from $\mathbb{R}^n$ (or $\mathbb{P}^n$) to $[0,1]^n$ has drastic effects on the moment cone and its Carathéodory number.

To demonstrate the drastic effect of higher dimensions $n$ we give also the following flat extension example.

**Example 60.** Let $(n,d) = (5,7)$ and $s$ be from table 2, i.e., $s = \sum_{x \in \mathbb{Z}(p)} s_A(x)$ is from the end of Section 6. Then all 11628 moments of $s$ are collected in a 792 × 792 Hankel matrix since $(7+5) = 792$. But from table 3 we find that $C_A(s) = 7678$, i.e., $s$ needs at least 7678 atoms in a representing measure. So applying flat extension we have to extend the original 792 × 792 Hankel matrix to an at least 7679 × 7679 Hankel matrix. We have all moments up to degree 2d = 14 and must extend them to at least degree 24 since $(12+5) = 6188$ but at most degree 26 since $(13+5) = 8568$ if all additional $k$ moments are optimally chosen. Then $k$ is bounded by
\[ \left( \frac{24+5}{5} \right) - \left( \frac{14+5}{5} \right) = 107127 \leq k \leq \left( \frac{26+5}{5} \right) - \left( \frac{14+5}{5} \right) = 158283. \]
The previous example shows that the application of flat extension to larger systems might not be possible.

There are several reasons which damp the hope of finding upper bounds that are significantly lower than those given in Theorem 53. First, the proofs in [dDS18b] and [RS18] are tight and based on Petrovski’s deep result [Pet38]; it seems hardly possible to improve the corresponding bounds in this manner since the number of isolated zeros exceed the number \( m \) of monomials as seen from table 2. Secondly, Theorems 58 and 59 combined with the lower bounds in table 3 indicate that strong improvement cannot be expected, see also the growth of the lower bounds \( w_{n,2d} \) in fig. 1. This indicates that further investigations of the inequalities in (18) and the possible limits in (19) are important.

In table 2 only the simple polynomials \( p \) and \( q \) with large but finite numbers of zeros have been used. It is natural to ask whether or not the lower bounds of the Carathéodory number \( C_\mathcal{A} \) in table 2 can be (significantly) improved by using other non-negative polynomials with finitely many zeros (see Problem 1)?

Another variant of the Carathéodory number problem is to allow signed measures and to study signed Carathéodory numbers \( C_{\mathcal{A},\pm} \). By Proposition 12, every vector \( s \in \mathbb{R}^m \) has a representing \( k \)-atomic measure with \( k \leq m \). This leads to the following definition.

**Definition 61.** The signed Carathéodory number \( C_{\mathcal{A},\pm}(s) \) of \( s \in \mathbb{R}^m \) is
\[
C_{\mathcal{A},\pm}(s) := \min\{ k \in \mathbb{N} \mid s \in S_{k, \mathcal{A}}(\mathbb{R}^k \times \mathcal{X}^k) \}
\]
and the signed Carathéodory number \( C_{\mathcal{A},\pm} \) is
\[
C_{\mathcal{A},\pm} := \min\{ k \in \mathbb{N} \mid S_{k, \mathcal{A}}(\mathbb{R}^k \times \mathcal{X}^k) = \mathbb{R}^m \} = \max_{s \in \mathbb{R}^m} C_{\mathcal{A},\pm}(s).
\]

For the signed Carathéodory number \( C_{\mathcal{A},\pm} \) we have the following result.

**Theorem 62** (dDS18b, Thm. 25). Suppose \( \mathcal{X} \) is an open subset \( \mathbb{R}^n \) and \( a_i \in C^1(\mathcal{X}, \mathbb{R}) \) for all \( i \). Then
\[
C_{\mathcal{A},\pm} \leq 2N_{\mathcal{A}}.
\]

Comparing (26) and (27) we see that \( w_{n,2d} \) gives a much better lower bound than \( N_{\mathcal{A}} \).

For the signed Carathéodory number \( C_{\mathcal{A},\pm} \) we have the upper bound 2\( N_{\mathcal{A}} \) by Theorem 62. Combining Theorem 62 with Theorem 52 we get
\[
C_{\mathcal{A},\pm} \leq \min\{ 2N_{\mathcal{A}}, m - 1 \}.
\]

In dDS18b Sec. 7] we used the apolar scalar product to relate the signed Carathéodory number \( C_{\mathcal{A},\pm} \) to the real Waring rank. Note that (28) gives a sharp bound, since G. Blekherman [Ble15] showed that there is a \( f \in \mathbb{R}[x, y]_d \) which can be written as
\[
f(x, y) = \sum_{i=1}^k c_i (a_i x + b_i y)^d
\]
with \( k = d \), but not with \( k < d \).

### 8. Internal Structure of \( S_{\mathcal{A}} \)

In Definition 13 we already defined the moment map \( S_{k, \mathcal{A}} \). For \( \mathcal{X} \subseteq \mathbb{R}^n \) open and \( a_i \in C^1(\mathcal{X}, \mathbb{R}) \) the moment map is a \( C^1 \)-mapping and we can investigate the atomic measures \( \mu = (C, X) = \sum_{i=1}^k c_i \delta_{x_i} \). The following important definition is taken from dDS18b Def. 26).

**Definition 63.** An atomic measure \( \mu(C, X) = \sum_{i=1}^k c_i \delta_{x_i} \) is called
- **regular** if the matrix \( D S_{k, \mathcal{A}}(C, X) \) is regular (i.e., it has full rank),
- **singular** if the matrix \( D S_{k, \mathcal{A}}(C, X) \) is singular (i.e., does not have full rank).
A moment sequence \( s \in S_A \) is called regular iff \( S_{A}^{-1}(s) \) is empty or consists only of regular measures. Otherwise, \( s \) is called singular.

Being singular or regular might depend on the number \( k \) according to the preceding definition. Obviously, if \( s \) is singular for \( k \), so is for all \( k' \geq k \). We don't know whether or not it is possible that \( s \) is regular for \( k \) and singular for some \( k' > k \) (see Problem 5 below).

**Example 64.** In the one-dimensional case \( A = \{1, x, \ldots, x^d\} \) on \( X = \mathbb{R} \) it follows from [dDS18b, Lem. 35] that

\[
(29) \quad s \in \text{int } S_A \iff s \text{ is regular.}
\]

In this section we want to investigate this relation (29).

For the one-dimensional polynomial case \( A = \{1, x^2, \ldots, x^{dm}\} \) (\( dm = 2m \)) with gaps we observed in [dDS18b, Thm. 45] that when \( DS_{k,A}(\mu) \) is singular then \( k < \left\lceil \frac{m}{2} \right\rceil \) if the condition in [dDS18b, eq. (32)] is fulfilled. But a singular moment sequence might be an inner point of \( S_A \), as the following example shows.

**Example 65.** Let \( A = \{1, x^2, x^3, x^5, x^6\} \) on \( X = \mathbb{R} \). In [dDS18b, Exm. 46] we showed with Theorem 48 that \( C_A = N_A = 3 \). We will prove here that (29) does not hold, so there exists an inner point of \( S_A \) which is singular.

Let \( x_1 = 1, x_2 = 2 \) and \( s = c_1 s_A(x_1) + c_2 s_A(x_2) \) for some \( c_1, c_2 > 0 \). Then \( s \) is singular, since \( \text{rank } DS_{2, A}((c_1, c_2), (x_1, x_2)) = 4 < 5 = |A| \). But \( s \) is not a boundary moment sequence. If \( s \) would be a boundary sequence, then there exists a \( p \in \text{Pos}(A) \) with \( x_1, x_2 \in Z(p) \). From

\[
\ker DS_{2, A}(\mathbb{1}, (x_1, x_2))^T = \ker \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 5 & 6 \\ 1 & 4 & 8 & 32 & 64 \\ 0 & 4 & 12 & 80 & 192 \end{pmatrix} = v \cdot \mathbb{R}
\]

with \( v = (52, -231, 225, -63, 17)^T \), we find that

\[
p(x) = \langle v, s_A(x) \rangle = (x - 1)^2(x - 2)^2(17x^2 + 39x + 13)
\]

is the only possible polynomial in \( A \) with \( p(x_i) = p'(x_i) = 0 \). But it is indefinite since \( p(-1) = -324 \) and \( p(0) = 52 \), i.e., \( s \) is no boundary point but an inner point.

Thus, Example 65 is an example where \( C_A = N_A \) holds but (29) does not. Next we show that it is also possible that (29) holds but \( C_A \neq N_A \).

**Example 66.** Let \( A = \{1, x, x^2, x^6\} \) on \( X = \mathbb{R} \). Then we have seen in [dDS18b, Exm. 48] that \( C_A = 3 > N_A = 2 \). By a straightforward calculation we verify that

\[
\det DS_{2, A}(\mathbb{1}, (x, y)) = 2(x - y)^4 q_A(x, y) \quad \text{with} \quad q_A(x, y) = (x + y)(2x^2 + xy + 2y^2)
\]

and \( Z(q_A) = \{(a, -a) \mid a \in \mathbb{R} \} \). Hence \( c_1 s_A(x) + c_2 s_A(y) \) is regular iff \( x \neq y \) or \( -y \).

Let \( s \) be singular. Then \( s = c_1 s_A(a) + c_2 s_A(-a) \) and \( s \in \partial^* S_A \), since

\[
p_a(x) := x^6 - 3a^2 x^2 + 2a^6 = (x - a)^2(x + a)^2(2a^2 + x^2) \in \text{Pos}(A),
\]

that is, (29) holds.

At last, we give an example where neither \( C_A = N_A \) nor (29) hold.

**Example 67.** Let

\[
A = B_{2, 6} = \{x^6, x^5 y, x^4 y^2, x^3 y^3, x^2 y^4, x y^5, y^6, x^5 z, x^4 y z, x^3 y^2 z, x^2 y z^2, x y z^3, y z^4, x y^2 z^2, x^2 y^2 z, y^2 z^3, y^3 z^4, x^2 z^3, y z^4, x^2 z^4, x^3, y^3, z^6 \}
\]
on \( X = \mathbb{P}^2 \). \( \mathcal{N}_A = 10 \) by Theorem 13, but it is known that \( \mathcal{C}_A = 11 \) [Kun14]. Let
\[
X = \{ (1, 1, 1), (0, 1, -1), (1, 0, -1), (1, 1, 0), (1, 0, 1), (0, 1, 0), (0, 0, 1), (0, 1, 0), (1, 0, 1), (1, 1, 0), (1, 1, -1), (1, -1, 1) \}
\]
be 10 projective points. Set \( s := \sum_{i=1}^{10} c_i s_A(\xi_i) \) for some numbers \( c_1, \ldots, c_{10} > 0 \).

Since \( \mathrm{rank} \ DS_{10,A}(1, X) = 27 < 28 = |A| \), the moment sequence \( s \) is singular.

But \( s \) is not a boundary moment sequence. Assume the contrary. Then there exists a \( p \in \mathrm{Pos}(A) \) with \( X \subseteq \mathbb{Z}(p) \). From \( \ker DS_{10,A}(1, X)^T = v : \mathbb{R} \) with
\[
v = (0, 0, 0, 0, 0, 0, 0, 1, 0, -1, 0, 0, -1, 0, 2, 1, 0, 0, -2, -2, 0, 1, 0, 0, 0, 0)^T
\]
we find that
\[
p(x) = \langle v, s_A(x, y, z) \rangle = y(x + y)(x - z)(y - z)z(x - y + z)
\]
or a multiple of it. Since \( p(1, 2, 3) = 72 \) and \( p(6, 2, 3) = -1008 \), \( p \) is indefinite, a contradiction. This proves that \( s \) is not a boundary point.

Therefore, \( s \) is a singular inner point of the moment cone. Thus, \( \mathcal{B}_2,6 \) on \( X = \mathbb{P}^2 \) is an example such that \( \mathcal{C}_A \neq \mathcal{N}_A \) and relation 29 does not hold.

Table 3 shows that \( \mathcal{C}_A = \mathcal{N}_A \) (true T or false F) and 29 (T or F) are independent properties. Example 67 indicates that (F, F) is probably the largest and most common class. Our only examples for \( (T, T) \) are \( A = \{1, x, \ldots, x^d\} \) on \( \mathbb{R} \) (Example 64) and \( A = \{x^1, \ldots, x^m\} \) on \( (0, \infty) \) (Theorem 50). Several natural questions arise:

Are Example 64 and Theorem 50 the only examples in \( (T, T) \), that is, for which \( \mathcal{C}_A = \mathcal{N}_A \) and 29 holds (Problem 1)? If there are others, can they be completely described? Besides the two independent properties \( \mathcal{C}_A = \mathcal{N}_A \) and 29, are there other basic properties to distinguish moment problems?

9. APPLICATIONS TO SOS AND TENSOR DECOMPOSITIONS

In this section and the next we discuss some applications of the previous results. Let us begin with sums of squares. The following definition is an adaption of Definition 13 to the study of sums of squares. For a set \( A = \{a_1, \ldots, a_m\} \) of \( m \) elements (e.g. measurable function) we set \( A^2 := \{a_i a_j \mid i, j = 1, \ldots, m\} \).

**Definition 68.** Let \( A = \{a_1, \ldots, a_m\} \). The square curve \( p_A \) is defined by
\[
p_A : \mathbb{R}^m \to \mathrm{lin} A^2, \ y = (y_1, \ldots, y_m) \mapsto p_A(y) := (y_1 a_1 + \cdots + y_m a_m)^2
\]
and the square map \( P_{k,A} \) is
\[
P_{k,A} : \mathbb{R}^k \to \mathrm{lin} A^2, \ Y = (Y_1, \ldots, Y_m) \mapsto P_{k,A}(Y) := \sum_{i=1}^{k} p_A(Y_i)
\]
with \( Y_i = (y_{i1}, \ldots, y_{im}) \). \( \Sigma A^2 \) denotes the sum of squares in \( \mathrm{lin} A^2 \).
The following lemma collects straightforward results adapted from results in the previous sections.

Lemma 69. Let $A = \{a_1, \ldots, a_m\}$.

i) $m \leq \dim \text{lin} A^2 \leq m^2$. If $a_i a_j = a_j a_i$, then $\dim \text{lin} A^2 \leq \frac{m(m+1)}{2}$.

ii) For all $k \geq \dim \text{lin} A^2$ we have

$$\Sigma A^2 = \text{range } P_{k,A^2} = P_{k,A^2}(\mathbb{R}^{m-k})$$

iii) $\Sigma A^2$ is a closed full-dimensional cone in $\text{lin} A^2$.

iv) $\text{Pos}(A^2)$ is a closed full-dimensional cone in $\text{lin} A^2$.

Definition 70. The Pythagoras number $\mathcal{P}_{A^2}$ of $A^2$ is

$$\mathcal{P}_{A^2} := \min\{k \in \mathbb{N} \mid (30) \text{ holds}\}.$$ 

With the next proposition we illustrate the use of the square map by giving a simple proof of the well-known fact that the Pythagoras number of $\mathbb{R}[x_1, \ldots, x_n]$ is $\infty$ [PDD01, Sec. 8.1]. This proof follows some arguments from [DST05 Prop. 23 and Thm. 27].

Proposition 71. i) $\left\lceil \frac{\dim \text{lin} A^2}{|A|} \right\rceil \leq \mathcal{P}_{A^2}$.

ii) For $n \geq 2$, the Pythagoras number of $\mathbb{R}[x_1, \ldots, x_n]$ is $\infty$.

Proof. i): The map $P_k A : \mathbb{R}^{k \cdot m} \to \text{lin} A^2$ is a $C^\infty$-map. Since $\Sigma A^2$ is full-dimensional by Lemma [69ii], it has a non-empty interior. By Sard’s Theorem [Sar42] the set of regular values in $\Sigma A^2$ is dense. Hence $k \geq \left\lceil \frac{\dim \text{lin} A^2}{|A|} \right\rceil$.

ii): Set $A = A_{n,d}$. Then $\text{lin} A^2 = \text{lin} A_{n,2d}$. So for $\Sigma A_{n,2d}$ with $n \geq 2$ there is a sum of squares with at least $\left\lceil \frac{|A_{n,2d}|}{|A_{n,d}|} \right\rceil$ many squares and the required number of squares does not decrease with increasing $d$ since higher terms have non-negative coefficients. Therefore, in $\mathbb{R}[x_1, \ldots, x_n]$ we have

$$\left\lceil \frac{|A_{n,2d}|}{|A_{n,d}|} \right\rceil = \left\lceil \frac{n+2d}{d} \right\rceil \cdot \left\lceil \frac{n+d}{n} \right\rceil^{-1} \xrightarrow{n \to \infty} \infty.$$

Using algebraic versions of Sard’s Theorem (see e.g. [BCR98 Sec. 9.6]) the preceding proof carries over to $R[x_1, \ldots, x_n]$ for $R$ a real closed field. For $n = 1$ (i.e., $A = A_{1,d} = \{1, x, \ldots, x^d\}$ for all $d \in \mathbb{N}$) the minimal number of squares in $\mathbb{R}[x]$ is

$$\left\lceil \frac{|A_{1,2d}|}{|A_{1,d}|} \right\rceil = \left\lceil \frac{2d+1}{d+1} \right\rceil = 2 \quad \forall d \in \mathbb{N}.$$ 

This is also the maximal number which is needed [Mar08 Prop. 1.2.1].

The following example shows that for univariate polynomials with gaps the Pythagoras number can be arbitrary large.

Example 72. Let $A = \{1, x, x^4, x^7\}$, then $A^2 = \{1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^{10}, x^{14}\}$, i.e., $|A| = 4$ and $|A^2| = 10$. The upper bound in Lemma [69i] is attained since $10 = \frac{4(4+1)}{2}$. Hence, by Proposition [71],

$$\mathcal{P}_{A^2} \geq \left\lceil \frac{10}{4} \right\rceil = 3.$$ 

There is a sum of squares $(y_1 + y_2 \cdot x + y_2 \cdot x^3 + y_4 \cdot x^7)^2$ which cannot be written as a sum of less than 3 squares.

In general, by choosing appropriate numbers $d_1 < d_2 < \cdots < d_n$, $d_i \in \mathbb{N}_0$, we have $|A^2| = \frac{m(m+1)}{2}$, so that $m+1 \leq \mathcal{P}_{A^2}$ and hence $\mathcal{P}_{A^2} \to \infty$ as $m \to \infty$. 
Of course, the preceding example is equivalent to \( \tilde{A} = \{1, x, y, z\} \) when we set \( y = x^3 \) and \( z = x^7 \) and no cancellations in \((\tilde{A})^2\) appear. Thus, studying univariate cases with gaps might give new insight into multivariate cases. Which \( P_h = k \in \mathbb{N} \) can be realized is an open problem [Problem 8].

For univariate polynomials with gaps nonnegative polynomials are not necessarily sum of squares, as shown by the next example.

**Example 73.** Let \( A = \{1, x^2, x^3\} \). Then \( A^2 = \{1, x^2, x^3, x^4, x^5, x^6\} \) and 
\[
p(x) := x^6 - x^4 + 10 \in \operatorname{Pos}(A^2) \setminus \Sigma A^2.
\]

That \( p \) is not a sum of squares follows immediately from 
\[
(a + bx^2 + cx^3)^2 = a^2 + 2abx^2 + 2acx^3 + b^2x^4 + 2bcx^5 + c^2x^6
\]
since the coefficient \( b^2 \) of \( x^4 \) is non-negative.

So, again, univariate polynomial systems with gaps bear properties of the multivariate polynomials.

Our second application concerns the tensor decomposition, see e.g. [BBCM13]. A tensor \( T \) is a multilinear map 
\[
T : \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d} \rightarrow \mathbb{R} \quad \text{with} \quad T = (t_{i_1, \ldots, i_d})_{i_1=1, \ldots, n_1}^{i_d=1, \ldots, n_d}.
\]
The simplest tensor is the rank 1 tensor 
\[
x_1 \otimes \cdots \otimes x_d := (x_{1,i_1} \cdots x_{d,i_d})_{i_1=1, \ldots, n_1}^{i_d=1, \ldots, n_d}
\]
with \( x_i = (x_{i,1}, \ldots, x_{i,n_i}) \in \mathbb{R}^{n_i} \). In order to bring tensors into the framework of moment problems we set \( A = \{x_{1,i_1} \cdots x_{d,i_d} \mid i_j = 1, \ldots, n_j, j = 1, \ldots, d\} \). Then the tensor decomposition
\[
(31) \quad T = \sum_{j=1}^d v_{j,1} \otimes \cdots \otimes v_{j,d} \quad \text{with} \quad v_{j,i} \in \mathbb{R}^{n_i}
\]
is nothing but determining a signed representing \( k' \)-atomic measure \( k' \leq k \). From Theorem [54] we get the lower bound on the number \( R \) of rank 1 tensors
\[
\left[ \frac{n_1 \cdots n_d}{n_1 + \cdots + n_d - d + 1} \right] \leq R.
\]
Thus, finding a signed atomic representing measure (or an approximation) for \( T \) is equivalent to finding a tensor decomposition (31) (or an approximation).

10. **Maximal masses and conic optimization**

**Definition 74.** For \( s \in S_A \) the maximal mass function \( \rho_s \) is 
\[
\rho_s(x) := \sup\{\mu(A_x) \mid \mu \in \mathcal{M}_A(s)\}, \quad x \in \mathcal{X},
\]
with \( A_x := s_A^{-1}(s_A(x)) = \{y \in \mathcal{X} \mid s_A(x) = s_A(y)\} \).

We have \( \mathcal{W}(s) = \{x \in \mathcal{X} \mid \rho_s(x) > 0\} \) from Lemma [22]. Another important quantity is defined by
\[
(32) \quad \kappa_s(x) = \inf\{L_s(p) \mid p \in \operatorname{Pos}(A), p(x) = 1\}
\]
where \( \frac{\kappa}{c} := +\infty \) for \( c \geq 0 \).

The nonnegative number \( \kappa_s(x) \) in (32) is defined by a conic optimization problem: It is the infimum of the Riesz functional \( L_s \) over the cone \( \operatorname{Pos}(A, \mathcal{K}) \) under the constraint \( p(x) = 1 \). The following results is contained in [DD18], Prop. 8].
Proposition 75. Suppose the moment cone $\mathcal{S}_A$ is pointed (that is, line-free) and $x \in \mathcal{X}$. Then

$$(33) \quad \rho_s(x) = \sup \{ c \in \mathbb{R}_+ \mid s - c \cdot s_A(x) \in \mathcal{S}_A \} \leq \kappa_s(x) < \infty.$$ 

If $\mathcal{S}_A$ is also closed, then $\kappa_s(x) = \rho_s(x)$, the supremum in $(33)$ is a maximum, and $s - \rho_s(x)s_A(x) \in \partial \mathcal{S}_A$.

Proof. The first equality in $(33)$ is clear from Lemma 22. Let $p \in \text{Pos}(A)$ and $p(x) = 1$. Then

$$L_s(p) = \int_{\mathcal{X}} p(y) \, d\mu(y) \geq p(x)\mu(A_x) = \mu(A_x).$$

Taking the infimum over $s$ that

$$\text{on a compact space, then the maximal mass function}$$

are maximal.

Suppose there exists a function $p \in \text{Pos}(A)$ such that $p(x_j) = 1$ and $p(x_i) = 0$ for all $j \neq i$. Then, for any $\nu \in \mathcal{M}_A(s)$, we have

$$(34) \quad \nu(A_{x_i}) \leq \int p(x) \, d\nu = L_s(p) = \int p(x) \, d\mu = c_i.$$ 

Therefore, $c_i = \rho_s(x_i)$ and $\mu$ has maximal mass at $x_i$.

Definition 76. Let $s \in \mathcal{S}_A$ and let $\mu = \sum_{j=1}^k c_j \delta_{x_j}$, $c_j > 0$, be a $k$-atomic representing measure of $s$. We say that $\mu$ has maximal mass at $x_i$ if $c_i = \rho_s(x_i)$ and that $\mu$ is a maximal mass measure for $s$ if $c_j = \rho_s(x_j)$ for all $j = 1, \ldots, k$.

Let $\mu = \sum_{j=1}^k c_j \delta_{x_j}$, $c_j > 0$, be a representing measure of $s$. Fix $i \in \{1, \ldots, k\}$. Suppose there exists a function $f \in \text{Pos}(A)$ such that $p(x_i) = 1$ and $p(x_j) = 0$ for all $j \neq i$. Then, for any $\nu \in \mathcal{M}_A(s)$, we have

$$(34) \quad \nu(A_{x_i}) \leq \int p(x) \, d\nu = L_s(p) = \int p(x) \, d\mu = c_i.$$ 

Therefore, $c_i = \rho_s(x_i)$ and $\mu$ has maximal mass at $x_i$.

Definition 77. We say that points $x_1, \ldots, x_k \in \mathcal{X}$ satisfy the positive separation property (PSP)$_A$ if there exist functions $p_1, \ldots, p_k \in \text{Pos}(A)$ such that

$$(35) \quad p_i(x_j) = \delta_{i,j} \quad \text{for} \quad i, j = 1, \ldots, k.$$ 

By the reasoning preceding Definition 77 it follows that if $x_1, \ldots, x_k \in \mathcal{X}$ obey (PSP)$_A$, then $\mu = \sum_{j=1}^k c_j \delta_{x_j} \in \mathcal{M}_A(s)$ is a maximal mass measure.

If $\mathcal{S}_A$ is closed and pointed and $s$ is an inner point of $\mathcal{S}_A$, then the converse of the preceding statement is true, that is, if $\mu = \sum_{j=1}^k c_j \delta_{x_j} \in \mathcal{M}_A(s)$ is a maximal mass measure, then the points $x_1, \ldots, x_k$ satisfy (PSP)$_A$. Indeed, the assumption imply that $\rho_s(x_j) = \kappa_s(x_j)$ and the infimum $(32)$ for $x = x_j$ is attained at some function $p_j$. One easily verifies that $(34)$ holds for the functions $p_1, \ldots, p_k$. Thus, $x_1, \ldots, x_k$ satisfy (PSP)$_A$. More details and examples on this matter can be found in [Sm17] Sec. 18.4.

The remaining part of this section is devoted to the question when the infimum in $(32)$ is a minimum. As noted by [Sm17] Prop. 18.28, this holds if $s$ is an interior point of the moment cone. The following example shows that for boundary points the infimum in $(32)$ is not necessarily attained.

Example 78. Let $\mathcal{X} = \mathbb{R}$, $\alpha \in [1, \infty)$, and $A = \{1, x, f_\alpha(x)\}$, where

$$f_\alpha(x) := \begin{cases} 
0 & \text{for } x \leq 0, \\
x^\alpha & \text{for } x > 0.
\end{cases}$$
consider the moment sequence \( s := (2, -2, 0) = s_A(0) + s_A(-2) \). Then we have \( \rho_s(-2) = \kappa_s(-2) = 1 \).

First suppose \( \alpha = 1 \). Set \( p(x) = -x/2 + f_s(x) \). Then \( p \in Pos(A) \), \( p(-2) = 1 \) and \( L_s(p) = 1 \), that is, the infimum in (32) at \( x = -2 \) is attained for \( p \).

Now suppose \( \alpha > 1 \). We show that the infimum (32) for \( x = -2 \) is not attained.

Assume the contrary. Then there exists \( p(x) = a + bx + cf_s(x) \in Pos(A) \) such that \( L_s(p) = 1 \), so \( 2a - 2b = 1 \), and \( p(-2) = 1 \), so \( a - 2b = 1 \). Hence \( a = 0 \) and \( b = -\frac{1}{2} \). We consider the function \( p(x) = -x/2 + cf_s(x) \in Pos(A) \) on \((0, \varepsilon)\) for small \( \varepsilon > 0 \) and conclude that \( c = 0 \). Thus, \( p(x) = -x/2 \in Pos(A) \), a contradiction.

Since \( L_s(f_s) = 0 \) and \( f_s \in Pos(A) \), \( s \) is a boundary point of the moment cone.

We illustrate the fact that the infimum (32) in the case \( \alpha > 1 \) is not attained from a slightly different viewpoint. There exists a sequence \((p_n)_{n \in \mathbb{N}}\) of functions \( p_n(x) = a_n + b_n x + c_n f_s(x) \in Pos(A) \) such that \( p_n(-2) = 1 \) and \( \lim_{n \to \infty} L_s(p_n) = \kappa_s(-2) = 1 \). From \( p_n(-2) = a_n - 2b_n \) and \( L_s(p_n) = 2a_n - 2b_n \) we conclude that \( \lim_{n \to \infty} a_n = 0 \) and \( \lim_{n \to \infty} b_n = \frac{1}{2}(a_n - 1) = -\frac{1}{2} \). We claim that \( \lim_{n \to \infty} c_n = \infty \). Indeed, otherwise there is a subsequence \((c_{n_k})\) of \((c_n)\) which has a finite limit \( c \). Then

\[
\lim_{k \to \infty} p_{n_k}(x) = \lim_{k \to \infty} a_{n_k} + b_{n_k} x + c_{n_k} f_s(x) = -x/2 + cx^\alpha \geq 0 \quad \text{for small} \ x > 0,
\]

which is a contradiction. Hence \( \lim_{n} c_n = \infty \), so the sequence \((p_n)\) does not converge.

In Example 78 we have seen two boundary points of the moment cone, one for which the infimum (32) is a minimum, for the other it is not. The difference lies in the geometry of moment cone at the boundary point. While \( s_A(0) \) is an “edge” of the moment cone, the moment cone is “round” in a neighborhood of \( s \) for \( \alpha > 1 \). This shows that whether or not the optimization problem (32) has a solution depends on the geometry of the boundary of the moment cone.

**Example 79.** Let \( \mathcal{X} = \mathbb{R}, \ A = \{1, x, x^2(x+2)^2\} \), and \( \mathcal{B} = \{1, x, f(x)\} \), where

\[
f(x) := \begin{cases} x^2(x+2)^2 & \text{for } x > 0 \text{ or } x < -2 \\ 0 & \text{for } x \in (-2,0) \end{cases}.
\]

Then \( \mathcal{S}_B = \mathcal{S}_A \) and both cones have the same local behavior at \( s_A(0) = s_B(0) = (1,0,0) \). Setting

\[
s := (2, -2, 0) = s_A(0) + s_A(-2) = s_B(0) + s_B(-2),
\]

it follows by a similar reasoning as in Example 78 that for \( A \) and \( B \) the infimum (32) at \( x = -2 \) is not attained.

The next result characterizes the case when the infimum (32) is a minimum. In particular, it implies [Sm14, Prop. 18.28].

**Theorem 80.** Suppose \( \mathcal{S}_A \) is closed and pointed (i.e., line-free). Let \( s \in \mathcal{S}_A \) and \( x \in \mathcal{X} \). Set \( s' := s - \rho_s(x)s_A(x) \in \partial \mathcal{S}_A \). The following are equivalent:

i) The infimum (32) for \( \kappa_s(x) \) is attained at some function \( p \in Pos(A) \).

ii) \( x \notin \mathcal{V}(s') \).

**Proof.** By Proposition 25 there is a \( p \in Pos(A) \) such that \( \mathcal{V}(s') = \mathcal{Z}(p) \). By scaling we can assume \( p(x) = 1 \). By Proposition 75 we have \( \kappa_s(x) = \rho_s(x) < \infty \). Then

i) \( \leftrightarrow p(x) = 1 \land L_s(p) = \kappa_s(x) \leftrightarrow p(x) = 1 \land L_{s'}(p) = 0 \leftrightarrow ii) \). \( \square \)

Let us retain the assumptions and the notation of Theorem 80. Then, by this theorem, \( x \in \mathcal{V}(s') \) if and only if the infimum (32) for \( \kappa_s(x) \) is *not* a minimum. Further, if \( \mathcal{V}(s') = \mathcal{W}(s') \), then the infimum (32) is attained. Hence the case that
the infimum (32) is not attained appears only when \( V(s') \neq W(s') \). That is, \( \kappa_s(x) \) is a minimum for all \( x \in \mathcal{X} \) and \( s \in \mathcal{S}_A \) if and only if the moment cone \( \mathcal{S}_A \) is perfect (according to Definition 52).

From the definition of \( s' \) it is clear that \( x \not\in W(s') \). Hence, each example \( s \in \mathcal{S}_A \) such that the infimum (32) is not attained is of the following form: \( s' \in \mathcal{S}_A \) with \( V(s') \neq W(s') \) and \( s = s' + c \cdot s_A(x) \) for \( x \in V(s') \setminus W(s') \) and \( c > 0 \).

Note that the proof of Theorem 24 in \[Sm15\] contains a gap. The following example shows that this result does not hold without additional assumptions.

**Example 81** (Example 35 revisited). Let \( A = B_{2,10} \) on \( \mathcal{X} = \mathbb{P}^2 \) and retain the notation of Example 23. Recall that \( z_i, i = 1, \ldots, 30 \), are the projective zeros of the Harris polynomial. Let \( x_i = z_i \) and \( c_i > 0 \) for \( i = 1, \ldots, 23 \) and \( x = x_j \) for some \( j \in \{24, \ldots, 30\} \) and \( c > 0 \).

Set

\[
s := \sum_{i=1}^{23} c_i \cdot s_A(x_i) + c \cdot s_A(x) = s' + c \cdot s_A(x).
\]

As shown in Example 35, \( V(s') = \{z_1, \ldots, z_{30}\} \) and \( W(s') = \{z_1, \ldots, z_{23}\} \).

Then the infimum (32) for \( \kappa_s(x) = c = \rho_s(x) \) is not attained. Indeed, if the infimum (32) would be attained at some \( p \in \text{Pos}(A) \) with \( p(x) = 1 \), then

\[
c = L_s(p) = L_{s'}(p) + L_{c \cdot s_A(x)}(p) = L_{s'}(p) + c \cdot p(x) = L_{s'}(p) + c,
\]

so that \( L_{s'}(p) = 0 \). Hence \( p \) is a multiple of the Harris polynomial and therefore, \( p(x) = 0 \), a contradiction.

### 11. Open Problems

The final section is devoted to a list of open problems which are related to the topics treated in this paper.

As shown in Corollary 30, the cones \( \mathcal{S}_{A_{n,10}} \) and \( \mathcal{S}_{B_{n,10}} \) for \( n \geq 2 \) are not perfect. It is likely to expect that this holds for polynomials of higher degrees as well.

**Problem 1.** Are the cones \( \mathcal{S}_{A_{n,10}} \) and \( \mathcal{S}_{B_{n,10}} \) perfect for \( k \in \mathbb{N} \)?

In Section 6, the zero sets of the polynomials \( p \) in (16) and \( q \) in (17) played a crucial role and a number of inequalities (18) were stated for the pairs in Table 2. This leads to the following problems.

**Problem 2.** Do the inequalities in (18) hold for arbitrary pairs \((n,d)\)?

**Problem 3.** Do the limits in (19) and (20) exist? If yes, what are these limits?

In Section 7, the polynomials \( p \) and \( q \) were used to derive bounds for Carathéodory numbers.

**Problem 4.** Can the lower bounds of the Carathéodory numbers \( C_A \) in Table 2 be (significantly) improved by using other nonnegative polynomials with finitely many zeros than \( p \) and \( q \)? What happens then with the limits in (19) and (20)?

In Section 8, we investigated the inner structure of the moment cone \( \mathcal{S}_A \). The first question comes from the definition of regular/singular moment sequences.

**Problem 5.** Do the regularity/singularity notions in Definition 63 depend on \( k \)? Is it possible that a moment sequence is regular for \( k \in \mathbb{N} \), but singular for some \( k' > k \)?

There are two (independent) distinguished properties of “nice” behavior of sets \( A \). This first is that “\( \mathcal{C}_A = \mathcal{N}_A \)”, while the second is stated as (29): “\( s \in \text{int} \mathcal{S}_A \Leftrightarrow s \) is regular”. As shown in Section 8, both properties are valid for \( A = \{x^1, \ldots, x^d\} \) on \( \mathbb{R} \) and \( A = \{x^{d_1}, \ldots, x^{d_m}\} \) on \( (0, \infty) \).
Problem 6. Are there other finite sets $A$ of polynomials for which $C_A = N_A$ and \[ (29) \] hold? Are there other useful properties to distinguish "nice" moment problems?

Using the Harris polynomial we constructed in Example 35 a moment sequence $s$ for $A = B_2,10$ on $X = P^2$. Since $W(s) = V_C(s)$, this means that $V_1(s)$ is not the core variety. This suggests the following problem, see also the discussion after the proof of Theorem 30.

Problem 7. Let $A = B_n,2k$ on $X = P^n$ for $k,n \in \mathbb{N}$. Given $m \in \mathbb{N}$, does there exist a moment sequence $s$ such that $V_{m-1}(s) \neq V_C(s)$ and $V_m(s) = V_C(s)$?

While $\mathbb{R}[x]$ has the Pythagoras number 2, univariate polynomials with gaps can have arbitrary large Pythagoras numbers (see Example 72).

Problem 8. What is the Pythagoras number of $A = \{x_{d_1}, \ldots, x_{d_m}, x_{d_m+1} \cdots \}$ for $d_1 < d_2 < \cdots < d_m$, $d_i \in \mathbb{N}_0$.

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