Bound states of spin–orbit coupled cold atoms in a Dirac delta-function potential

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Abstract
Dirac delta-function potential is widely studied in quantum mechanics because it usually can be exactly solved and at the same time is useful in modeling various physical systems. Here we study a system of delta-potential trapped spin–orbit coupled cold atoms. The spin–orbit coupled atomic matter wave has two kinds of evanescent modes, one of which has a pure imaginary wavevector and is an ordinary evanescent wave; while the other with a complex number wave vector is recognized as an oscillating evanescent wave. We identified the eigenenergy spectra and the existence of bound states in this system. The bound states can be constructed analytically using the two kinds of evanescent modes and we found that they exhibit typical features of stripe phase, separated phase or zero-momentum phase. In addition to that, the properties of semi-bound states are also discussed, which is a localized wave packet on a plane wave background.

Keywords: spin–orbit coupling, cold atoms, delta-function potential

(Some /f_igures may appear in colour only in the online journal)

1. Introduction
Spin–orbit (SO) coupling has been widely studied in diverse branches of physics including nanotechnology [1, 2], nuclear physics [3–5], optics [6–8], condensed matter physics [9, 10] and cold atom physics [11–13]. For a charged particle with non-zero spin, its spin magnetic momentum will interact with the magnetic field induced by its movement, thus generating a coupling between its orbital motion and spin degree of freedom. For neutral cold atom systems, SO-coupling can be artificially generated via a Raman coupling scheme [14]. The occurrence of SO-coupling will greatly enrich the physics of atomic matter wave [12, 15, 16]. Many emergent phenomena such as spin Hall effect [17], topological insulator [18], Zitterbewegung [19–22], supersolid [23–26], solitons [27–30], Beliaev damping [31] and spin-dependent atom optics [32–43] have been reported.

It was found that in the presence of SO-coupling the atomic system can display a rich phase diagram in which the ground state wavefunction can favor stripe, spin separated or zero momentum phase [14, 44–47]. Specifically, the stripe phase can be regarded as a signature of supersolid [23–26] which receives much recent attention. Physical insight into the properties of eigenfunction of the SO-coupled ultracold atoms can be gained via bound state solutions. Previous work has solved bound state in a one-dimensional short-range potential, and predicts a type of SO induced extra states [48]. More interestingly, bound state in the continuum can exist under appropriate trapping potentials [49].

In this article we analytically study the bound states of SO-coupled atomic matterwave in a δ-function potential. Physical models with δ-function potential play significant roles in quantum mechanics. Analytical or partially analytical solutions can be deduced for these models and in the meanwhile they provide physical insight into real systems. Typical examples include...
the Kronig–Penney model with δ-function potential [50], the hydrogen-like atoms and diatomic molecules [51], the interatom interaction in ultra cold atomic cloud [52, 53], very narrow potential [54–57], obstacle [58, 59] and impurity [60, 61]. It is well known that a δ-function potential well supports only one bound state which is constructed with free particle evanescent waves. For SO-coupled matter wave, there are three different types of free particle modes: plane wave, ordinary evanescent wave and oscillating evanescent waves [36, 62]. We found that there exist two types of bound states which can be constructed using the oscillating evanescent and ordinary evanescent waves, respectively. The bound state constructed with oscillating evanescent waves has stripe structure on its density profile, while the bound state with ordinary evanescent waves is a zero momentum wave packet. Due to the spin-1/2 nature of the system, a δ-function potential well can (but not always) support bound states both in ground state and excited state. A separated phase bound state can then be constructed by superposing the ground and excited bound state. Besides these bound states, we found that there also exists a kind of semi-bound state, which is a localized state on a plane wave background. The interference between the localized state and the plane wave background will produce a dip (bump) on the density profile for a δ-potential well (barrier).

2. Model and eigensolution of free SO-coupled cold atomic matter wave

As schematically shown in figure 1, we consider a system of quasi-one-dimensional SO-coupled cold atoms subjected to a spin-independent δ-function potential. The SO coupling is realized by a typical Raman scattering scheme [14]. And the spin-independent δ-potential is generated by a far off-resonant laser beam [63] which is tightly focused at the center of the atomic cloud [54, 55]. Such a system can be described by the Hamiltonian

\[
H = H_0 - V_0 \delta(x),
\]

with \(V_0\) being the depth of δ-function potential, and \(H_0\) being the free particle Hamiltonian of SO-coupled matter wave

\[
H_0 = \left( \frac{\hbar^2}{2m}(k_x^2 - k_z^2) + \frac{\hbar\Omega}{2} \right) \frac{\hbar\Omega/2}{\hbar^2 k_z^2} \left( \frac{k_x^2}{2m} + k_z^2 \right),
\]

which can be implemented with a Raman coupling scheme in cold atom system [14]. Here \(\hbar\) is the reduced Planck constant, \(m\) is the mass of an atom, \(p_x = \hbar k_x = -i\hbar \delta/\delta x\) is the x-direction momentum operator, \(k_z\) signals SO-coupling strength and \(\Omega\) is the Rabi coupling strength. For simplicity, we have assumed that the interatomic collision interaction is eliminated using the technique of Feshbach resonance [64, 65].

Since the total Hamiltonian \(H\) equals \(H_0\) except at the point \(x = 0\), we first give a brief discussion on the properties of free particle Hamiltonian \(H_0\). The eigenenergy \(E\) of \(H_0\) is given by the equation

\[
\left( E - \frac{\hbar^2 (k_x^2 + k_z^2)}{2m} \right)^2 - \left( \frac{\hbar^2 k_z k_x}{m} \right)^2 - \left( \frac{\hbar\Omega}{2} \right)^2 = 0.
\]

For a given energy \(E\), this quartic algebraic equation has four solutions

\[
k_x = \pm \sqrt{\frac{k_z^2 + 2mE}{\hbar^2} \pm \sqrt{8\sin^2 mE + m^2\Omega^2} \frac{\hbar}{m}},
\]

These four solutions can generally be written as \(k_x = \beta + i\alpha\), with \(\alpha\) and \(\beta\) being two real numbers. In the wavefunction \(e^{ik_x x} = e^{-i\alpha}e^{i\beta}\), the real part \(\beta\) of wavevector \(k_x\) contributes a plane wave factor, while the imaginary part \(\alpha\) contributes an exponential decay factor. Thus, depending on the values of \(\alpha\) and \(\beta\), the corresponding free particle waves can be divided into three types: (1) plane wave with \(\alpha = 0, \beta \neq 0\); (2) ordinary evanescent wave with \(\alpha \neq 0, \beta = 0\); and (3) oscillating evanescent wave when \(\alpha, \beta \neq 0\).

From equation (3), eigenenergy can be further split into two spectrum branches (referred as upper and lower branch)

\[
E_{\pm}(k_x) = \frac{\hbar^2 (k_x^2 + k_z^2)}{2m} \pm \frac{\sqrt{8\sin^2 mE + m^2\Omega^2}}{m\hbar} \frac{\hbar}{m}.
\]

with the corresponding eigenstates

\[
\Psi_{\pm} = \chi_{\pm}(k_x)e^{ik_x x} = C_{\pm} \left( \frac{\zeta_{\pm}}{1} \right) e^{ik_x x},
\]

in which \(\zeta_{\pm} = -\left(2\hbar k_z k_x + \sqrt{4\hbar^2 k_z^2 + m^2\Omega^2}\right) / m\Omega\) and \(C_{\pm} = 1/\sqrt{1 + |\zeta_{\pm}|^2}\) is the normalization constant.

Equation (4) indicates that there are four states for any energy \(E\), the eigenenergy dispersion display two typically different structures depending on the SO-coupling strength:

(a) The strong SO-coupling case with \(k_x^2 > m^2\Omega^2/4\hbar\). In this case, when \(E > E_{+}(0) = \hbar^2 k_z^2/2m + \hbar\Omega/2\), both the upper and lower branches support two plane wave states with \(k_x\) real. In the region \(E \in [E_{-}(0) = \hbar^2 k_z^2/2m - \hbar\Omega/2, E_{-}(0)]\), there are two plane wave states in the lower spectrum branch while the other two are ordinary evanescent states coming from either the upper branch when...
The energy spectra as a function of $k^2$ can be constructed using the free particle modes by matching the boundary conditions at $x = 0$. Because the plane wave modes extend to infinity, it cannot be used to construct a bound state. While ordinary and oscillating evanescent wave modes decay to zero when $x$ approaches $+\infty$ or $-\infty$, they are the candidates for bound state constructing. So based on the discussion in section 2, it is concluded that bound states can exist in energy range $[-\infty, E_-(k_0)]$ for both strong and weak SO-coupling, and in energy range $[E_-(k_0), E_+(0)]$ for only weak SO-coupling, since in these energy ranges there exist the candidates modes. And in energy range $[E_-(0), E_+(0)]$, the plane wave and ordinary evanescent wave modes exist simultaneously, this gives a chance to construct a kind of semi-bound state, which shows as a localized wave packet on plane wave background. These states will be discussed in the rest content of this section one by one.

In the energy range $E < E_-(k_0)$, as having been discussed above there exist four oscillating evanescent modes and bound states can be constructed using them. One can find that the wave vector (4) of these four modes have symmetrical form $k_i = \beta \pm i \alpha$ and we label them as $k_1 = \beta + \alpha$, $k_2 = -\beta + \alpha$, $k_3 = \beta - \alpha$, $k_4 = -\beta - \alpha$, so the bound state can be written as

$$
\Psi_b = \begin{cases} 
A_1 e^{i\chi_1} + A_2 e^{i\chi_2} e^{i\alpha} - A_3 e^{-i\chi_3} e^{-i\alpha}, & x > 0, \\
A_1 e^{i\chi_1} + A_4 e^{-i\chi_4} e^{i\alpha}, & x < 0,
\end{cases}
$$

(7)

with $A_1, A_2, A_3, A_4$ and eigenenergy $E$ (note that $k_{1,2,3,4}$ are determined by $E$ according to formula (4)) to be determined by normalization constraint

$$
\int_{-\infty}^{\infty} |\Psi_b(x)|^2 dx = 1,
$$

(8)

and boundary conditions at $x = 0$: continuity of the wave function

$$
\Psi_b|_{x=0} = \Psi_b|_{x=0^-},
$$

(9)

with

$$
\Psi_b|_{x=0^+} = A_1 e^{i\chi_1} + A_2 e^{i\chi_2},
$$

(10)

$$
\Psi_b|_{x=0^-} = A_3 e^{-i\chi_3} + A_4 e^{-i\chi_4},
$$

(11)

Table 1. Plane (P), ordinary evanescent (Ev), and oscillating evanescent (OE) mode numbers of SO-coupled atomic matter wave in different energy ranges. Strong coupling means $k^2 > m^2\Omega^2/(4\hbar)$, while weak coupling means $k^2 < m^2\Omega^2/(4\hbar)$. Letters ‘U’ and ‘L’ are used to label the upper and lower branch of the spectrum. In the table $\alpha_0$ and $k_0$ are $\alpha_0 = m/2\hbar k_0$, $k_0 = \sqrt{k^2 - m^2\Omega^2/(4\hbar^2)}$.

| Energy range   | Strong coupling | Weak coupling |
|----------------|-----------------|--------------|
|                | P Ev OE         | P Ev OE      |
| $[E_+(0), +\infty]$ | 2 2             | 2 2          |
| $[E_-(i\alpha_0), E_+(0)]$ | 2 2             | 2 2          |
| $[E_-(0), E_-(i\alpha_0)]$ | 2 2             | 2 2          |
| $[E_-(k_0), E_-(0)]$ | 4 4             | 4 4          |
| $[-\infty, E_-(k_0)]$ | 4 4             | 4 4          |

Figure 2. Free particle energy spectra of SO-coupled atomic matter waves. Left panel: strong SO-coupling case with parameters $k^2 > m^2\Omega^2/(4\hbar)$ ($k = 1, \Omega = 1$). Right panel: weak SO-coupling case with parameters $k^2 < m^2\Omega^2/(4\hbar)$ ($k = 1, \Omega = 3$). The upper branch spectrum $E_+$ is plotted in red color, while the lower branch $E_-$ is plotted in blue color. The plane, ordinary evanescent and oscillating evanescent wave modes are plotted with solid, dash-dot and dash lines respectively. For plane wave mode, $k_i$ is a real number, x-axis is simply set to $f(k_i) = k_i$; for ordinary evanescent wave mode, $k_i$ has no real part, x-axis is set to its imaginary part $f(k_i) = \text{Im}(k_i)$; and for oscillating evanescent wave mode, $k_i = \pm \beta \pm i \alpha$ is a complex number, x-axis is set to $f(k_i) = \text{sgn}(|\text{Im}(k_i)|) |k_i|$ (note that in such a way the points for $+\beta$ and $-\beta$ will overlap with each other, thus the seemingly two curves in the figure are in fact four curves). Other parameters used are $m = \hbar = 1$.

$$
E > E_-(i\alpha_0) = \frac{(4\hbar^2k_0^4 - m^2\Omega^2)}{8mk_0^2},
$$

(6)

or lower branch when $E < E_-(i\alpha_0)$. For $E \in [E_-(k_0), E_-(0)]$ with $k_0 = \sqrt{k^2 - m^2\Omega^2/(4\hbar^2)}$, there are four plane wave states in the lower branch. Four oscillating evanescent states possess minimum energies with $E < E_-(k_0)$ and they are linked to the energy minimum of the upper energy spectra at the points $|k_i| = k_0$.

(b) The weak SO-coupling case with $k^2 < m^2\Omega^2/(4\hbar)$. It differs from the strong coupling case only in the energy region $[E_-(k_0), E_-(0)]$, in which all four eigenstates come from the lower branch are ordinary evanescent waves with $k_i$ imaginary.

3. **Bound states and semi-bound states with delta-function potential**

The bound state of a $\delta$-function potential well $V(x) = -V_0\delta(x)$ can be constructed using the free particle modes by matching the boundary conditions at $x = 0$. Because the plane wave
functions. Bottom panels are the corresponding densities $|\Psi|^2$. The energies of the ground and excited states are $-0.1391$ and $-0.1267$ respectively. Natural unit $m = \hbar = 1$ is applied.

![Figure 3](image)

**Figure 3.** Oscillating evanescent wave bound states. Ground (‘G’) and excited (‘E’) states of SO-coupled matter wave in $\delta$-function potential well are plotted for parameters $V_0 = 0.1, k_c = 1, \Omega = 1$. Top panels are real (solid) and imaginary (dashed) parts of the wave function $\Psi$ with different spins (labeled with ‘$\uparrow$’ and ‘$\downarrow$’). Bottom panels are the corresponding densities $|\Psi|^2$. The energies of the ground and excited states are $-0.1391$ and $-0.1267$ respectively. Natural unit $m = \hbar = 1$ is applied.

and jump of the first-order derivation caused by divergence of $\delta$-function potential

$$
\frac{\partial \Psi_0}{\partial x} \bigg|_{0+} - \frac{\partial \Psi_0}{\partial x} \bigg|_{0-} = -\frac{2mV_0}{\hbar^2} \Psi_0 \quad (x = 0),
$$

with

$$
\frac{\partial \Psi_{b,G}}{\partial x} \bigg|_{0+} = ik_1 A_1 \chi_{\downarrow} (k_1) + ik_2 A_2 \chi_{\downarrow} (k_2),
$$

$$
\frac{\partial \Psi_{b,E}}{\partial x} \bigg|_{0-} = ik_3 A_3 \chi_{\downarrow} (k_3) + ik_4 A_4 \chi_{\downarrow} (k_4).
$$

Solving equations (9) and (12), one can have two solutions (the lower energy one is the ground state and the other one is an excited state) fulfill properties $|A_1| = |A_2| = |A_3| = |A_4|$, indicating that the bound state is a spin symmetric state with $|\Psi_{b,G}|^2 = |\Psi_{b,E}|^2$. From equation (7) one can understand that the interference between $e^{i\chi x}$ and $e^{-i\chi x}$ terms will produce an interference stripe on the density profile of bound states. This type of bound state has very similar properties as the stripe phase state in free space or harmonically trapped SO-coupled matter wave [46]. In figure 3, examples of such bound states are plotted for parameters $V_0 = 0.1, k_c = 1, \Omega = 1$, clearly demonstrating spin mixed stripe phase structure. We also note that no-node theorem does not hold here because of SO-coupling [66, 67].

The bound states can also be constructed via the linear superposition of the ground and excited states. In figure 4, the superposition state $\Psi_{b,G+} = (\Psi_{b,G} + \Psi_{b,E})/\sqrt{2}$ is plotted which is a spin-$\uparrow$ component dominated state. Similarly a spin-$\downarrow$ component dominated state can also be constructed by superposing $\Psi_{b,G}$ and $\Psi_{b,E}$ with opposite phase $\Psi_{b,G-E} = (\Psi_{b,G} - \Psi_{b,E})/\sqrt{2}$. This resembles the separated phase discussed in [46] with nonzero spin polarization. But, it should be noted that because the ground and excited states have different eigenenergies, these superposition states are not stationary states of the system.

The bound states can also exist in the energy region $E_{\uparrow}$ for a weak SO-coupling ($k_c^2 < m^2\Omega^2/4\hbar^2$). In such a case, there exist four ordinary evanescent modes for a given energy $E$, the corresponding wavevectors (4) are pure imaginary and have form $k_{1,3} = \pm ik_1, k_{2,4} = \pm ik_2$. The bound state can then be written in the following form

$$
\Psi_{b} = \begin{cases} 
A_1 \chi_{\downarrow} (k_1) e^{-i\chi x} + A_2 \chi_{\downarrow} (k_2) e^{-i\chi x}, & x > 0, \\
A_3 \chi_{\downarrow} (k_3) e^{i\chi x} + A_4 \chi_{\downarrow} (k_4) e^{i\chi x}, & x < 0,
\end{cases}
$$

which decays exponentially and is very similar to the bound states in the SO-uncoupled case. Applying the boundary conditions and normalization constraints (similar to equations (8), (9) and (12)) one can solve the bound states. An example is given in figure 5, the bound state is spin symmetric and can be viewed as the ‘zero momentum’ states discussed in [46].
and imaginary (dashed) parts of the spin-$\uparrow$ and spin-$\downarrow$ wave functions. Bottom panels are the corresponding densities $|\Psi|^2$. The energy of this state is $-1.0625$. Natural unit $m = \hbar = 1$ is applied.

We also examined the spectrum with $\delta$-function potential well. In the top two and bottom left panels of figure 6, the binding energies of the ground and excited bound states are plotted as a function of $\delta$-function potential well depth $V_0$ for Rabi coupling strength $\Omega = 0, 1$ and 3. When $\Omega = 0$, the two spin components are not coupled with each other, each component can be separately treated as a usual $\delta$-function potential problem (except that the momentum is shifted by $\pm \hbar k_0$), thus there exist two bound states with degenerate energy $E_{b,G} = E_{b,E} = -mV_0^2/(2\hbar^2)$. When $\Omega \neq 0$, this degeneracy is eliminated. For a small value of $\Omega$ (1 or in other words, a strong $\delta$-coupling since $k_0^2 > m^{2}\Omega^2/4\hbar$ is fulfilled), both the ground and excited states can exist regardless of the value of $V_0$. Even when $V_0 \rightarrow 0$ (approaching the free particle limit), the system has two solutions corresponding to the two minimums of the lower dispersion branch. However, for a large value of $\Omega = 3$ (weak $\delta$-coupling since $k_0^2 < m^{2}\Omega^2/4\hbar$), the excited state can be lifted so high that a shallow potential well can no longer trap it. Thus, we see that for $V_0$ smaller than a critical value, the excited state disappears. This also coincides with the fact that for weak $\delta$-coupling the lower dispersion branch of the free particle spectrum only has one minimum. And in the bottom right panel, we show the critical value of Rabi coupling strength $\Omega_c$ for the disappearing of the excited state as a function of $V_0$, see the solid black line. Below this critical line, both the ground and excited states can exist. While above this line, the potential well can no longer support an excited bound state, only the ground state can exist. In this panel, we also noticed that when $V_0 \rightarrow 0$, the critical Rabi coupling $\Omega_c \rightarrow 2$ which is just the demarcation point between strong and weak $\delta$-coupling strength ($4\hbar^2k_0^2/m^2 = 2$). This also agrees with the above free particle limit discussion.

In the energy region $[E_-(0), E_+(0)]$, for a given energy $E$ there are two ordinary evanescent modes and two plane wave modes, the corresponding wavevectors (4) have form $k_{1,3} = \pm ik$ and $k_{2,4} = \pm k$. One can then construct a semi-bound state as follows

$$\Psi_{sb} = \Psi_P + \Psi_E,$$

where $\Psi_P$ is a plane wave background consisting of incident, transmission and reflection waves (their amplitudes are $t, r, r$ respectively)

$$\Psi_P = \begin{cases} t\chi_-(k_0)e^{i\kappa x}, & x > 0, \\ \chi_-(k_0)e^{i\kappa x} + r\chi_-(k_0)e^{-i\kappa x}, & x < 0, \end{cases}$$

and $\Psi_E$ is a localized wave packet constructed using the two ordinary evanescent modes

$$\Psi_E = \begin{cases} A_1\chi_\pm(k_1)e^{-\kappa x}, & x > 0, \\ A_3\chi_\pm(k_3)e^{i\kappa x}, & x < 0. \end{cases}$$

Here $t$, $r$ and $A_1, A_3$ are parameters to be determined by boundary conditions. We note that such a semi-bound state is actually a scattering state, thus it can not only exist for a $\delta$-potential well but also a $\delta$-potential barrier. In figure 7, examples of such semi-bound states are shown for parameters $k_1 = 1$, $\Omega = 3$ and $V_0 = \pm 0.25 (+0.25$ for a potential well, while $-0.25$ for a potential barrier). The density profiles of both spin components are plotted in this figure. In the left-half space ($x < 0$), the interference between incident and reflected waves produces an interference fringe. In the right-half space ($x > 0$), the transmission wave produces a flat density background. And around the location of $\delta$-potential ($x = 0$), a dip of density on plane wave ground can be observed for a potential well, while a bump is observed for a potential barrier. This semi-bound state represents the coupling between the bound state and the plane wave propagating state, thus bound states in the continuum do not exist in the present single-particle system [68].

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**Figure 5.** Ordinary evanescent wave bound state. Ground state of SO-coupled matter wave in $\delta$-function potential well is plotted for parameters $V_0 = 0.25, k_0 = 1, \Omega = 3$. Top panels are the real (solid) and imaginary (dashed) parts of the spin-$\uparrow$ and spin-$\downarrow$ wave functions. Bottom panels are the corresponding densities $|\Psi|^2$. The energy of this state is $-1.0625$. Natural unit $m = \hbar = 1$ is applied.

**Figure 6.** $\delta$-function potential well bounded spectrum. The binding energies of the ground (‘G’) and excited (‘E’) states are plotted as a function of potential well depth $V_0$ for parameters $\Omega = 0, 1, 3$ in the top two and bottom left panels respectively. In the bottom right panel, the critical value of Rabi coupling $\Omega_c$ for the disappearing of the excited state is plotted as a function of potential well depth $V_0$, the solid black line. Below this line, in the light blue color filled area, there exist both the ground and excited states. While, above this line, in the light red color filled area, only the ground state can exist. In all the panels, SO-coupling strength is set to $k_c = 1$ and natural unit $m = \hbar = 1$ is applied.
− potential well, while \( \delta \) the SO-uncoupled case. For the SO-coupled matter wave, a evanescent wave is an ordinary one having a similar feature to wave packet on a plane wave background. For a \( \delta \)-function potential well can (but not always) support both a ground and excited state. By superposing these two states, a separated phase state can also be constructed. Besides the bound states, there also exists a kind of semi-bound state (a localized wave packet on a plane wave background). For a \( \delta \)-function potential well, a dip emerges on the plane wave background. While for a \( \delta \)-function potential barrier, a bump is formed on the plane wave background.

4. Summary

In summary, we studied the bound and semi-bound states of SO-coupled matter wave in a \( \delta \)-function potential. We found that there are two kinds of bound state in the system, one of which is a stripe constructed using an oscillating evanescent wave, while the other one constructed using an ordinary evanescent wave is an ordinary one having a similar feature to the SO-uncoupled case. For the SO-coupled matter wave, a \( \delta \)-potential well can (but not always) support both a ground and an excited bound state. By superposing these two states, a separated phase state can also be constructed. Besides the bound states, there also exists a kind of semi-bound state (a localized wave packet on a plane wave background). For a \( \delta \)-function potential well, a dip emerges on the plane wave background. While for a \( \delta \)-function potential barrier, a bump is formed on the plane wave background.

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Figure 7. Semi-bound states of SO-coupled matter wave in \( \delta \)-function potential well ('W', top panels) and barrier ('B', bottom panels). Density profiles of spin-\( \uparrow \) and spin-\( \downarrow \) components are plotted. Parameters used are \( E_0 = -0.9, V_0 = \pm 0.25\) (\( +0.25 \) for a potential well, while \( -0.25 \) for a potential barrier), \( k_c = 1, \Omega = 3 \) and \( m = \hbar = 1 \).
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