Chain rules for quantum Rényi entropies

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Abstract

We present chain rules for the quantum Rényi entropies first defined in [MLDSFT13; WWY14] and sometimes called the “sandwiched” Rényi entropies. More precisely, we prove analogues of the equation $H_{p}(AB|C) = H(A|BC) + H(B|C)$, which holds as an identity for the von Neumann entropy. In the case of the Rényi entropy, this relation no longer holds as an equality, but survives as an inequality of the form $H_{\alpha}(AB|C) \geq H_{\beta}(A|BC) + H_{\gamma}(B|C)$, where the parameters $\alpha, \beta, \gamma$ obey the relation $\frac{1}{\alpha-1} = \frac{1}{\beta-1} + \frac{1}{\gamma-1}$, and the direction of the inequality depends on the parameters.

1 Introduction

The Shannon entropy is one of the central concepts in information theory: it quantifies the amount of uncertainty contained in a random variable, and is used to characterize a wide range of information theoretical tasks. However, it is primarily useful for making asymptotic statements about problems involving repeated experiments, such as i.i.d. sources or channels. If one is interested not only in asymptotic rates, but in how quickly we can approach these rates with increasing block sizes, the natural quantities that arise are Rényi entropies. Likewise, in the case of “one-shot” problems, such as those arising in cryptographic settings, the (smooth) min- and max-entropies are usually the relevant quantities.

For all of these quantities, it is possible to define quantum counterparts. However, these counterparts are not unique: the fact that quantum states can fail to commute leads to multiple non-equivalent definitions that all reduce to the classical version in the commutative case, and one has to pick the “right” choice based on which version possesses the properties we want. Recently, the question of choosing the “right” quantum version of the Rényi entropy has seen renewed interest, with the development of a new definition in [MLDSFT13; WWY14] which seems to possess better properties than the traditional definition in certain parameter regimes. In particular, this new definition characterizes the strong converse exponent in hypothesis testing [MO13] (at least in some parameter regimes) and in classical-quantum channel coding [MO14], and can be used to prove strong converses for a variety of information theoretical problems [WWY14; CMW14; TWW14]. Furthermore, since this new quantity has a wide range of applications, its fundamental properties are being investigated: a number of properties have

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been proven in [MLDSFT13], including the data processing inequality for \(1 < \alpha \leq 2\) and a duality property (see Fact 4 below, also independently proven in [Bei13]). The data processing inequality was also proven in [Bei13] for \(\alpha > 1\) and in [FL13] for the full range \(\alpha \geq \frac{1}{2}\). Moreover, in [AD13], the quantity was generalized to so-called \(\alpha\)-\(\varepsilon\)-entropies, and in [TBH14] the authors presented a duality property that involves both the new “sandwiched” definition and the traditional definition.

The present work is in this vein. One of the most fundamental properties of the von Neumann entropy is the so-called chain rule: given a tripartite state \(\rho_{ABC}\), we can break down the conditional entropy \(H(AB|C)\rho\) into two parts: \(H(AB|C)\rho = H(A|BC)\rho + H(B|C)\rho\). While this rule no longer holds as an equality for the Rényi entropy, one can nonetheless hope to salvage it in the form of inequalities, as was done in [VDTR13] for the smooth min-/max-entropies. The result is the main theorem of this paper:

**Theorem 1.** Let \(\rho_{ABC} \in D(A \otimes B \otimes C)\) be a normalized density operator, and let \(\alpha, \beta, \gamma \in (\frac{1}{2}, 1) \cup (1, \infty)\) be such that \(\frac{1}{\alpha - 1} = \frac{\beta}{\beta - 1} + \frac{\gamma}{\gamma - 1}\). Then, we have that

\[
H_\alpha(AB|C)\rho \geq H_\beta(A|BC)\rho + H_\gamma(B|C)\rho
\] (1)

if \(\alpha, \beta, \gamma > 1\), and

\[
H_\alpha(AB|C)\rho \leq H_\beta(A|BC)\rho + H_\gamma(B|C)\rho
\] (2)

if \(\alpha > 1\) but one of \(\beta\) or \(\gamma\) is less than 1.

The proof is given in Section 3, where the two cases are split into Propositions 7 and 8, and where the statements proven are slightly more general. The proof is based on the Riesz-Thorin-type norm interpolation techniques developed in [Bei13], which are then applied to convenient expressions for the Rényi entropy.

## 2 Preliminaries

### 2.1 Notation

In the table below, we summarize the notation used throughout the paper:

| Symbol | Definition |
|--------|------------|
| \(A, B, C, \ldots\) | Quantum systems |
| \(A, B, \ldots\) | Hilbert spaces corresponding to systems \(A, B, \ldots\) |
| \(L(A, B)\) | Set of linear operators from \(A\) to \(B\) |
| \(L(A)\) | \(L(A, A)\) |
| \(X_{AB}\) | Operator in \(L(A \otimes B)\) |
| \(X_{A\rightarrow B}\) | Operator in \(L(A, B)\) |
| \(\text{Pos}(A)\) | Set of positive semidefinite operators on \(A\) |
| \(\text{D}(A)\) | Set of positive semidefinite operators on \(A\) with unit trace |
| \(\text{id}_A\) | Identity operator on \(A\) |
| \(\alpha', \beta', \gamma'\) | \(\alpha' = \frac{\alpha - 1}{\alpha - 2}, \beta' = \frac{\beta - 1}{\beta - 2}, \gamma' = \frac{\gamma - 1}{\gamma - 2}\) |
| \(\hat{\alpha}, \hat{\beta}, \hat{\gamma}\) | Dual parameter of \(\alpha, \beta, \gamma: \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2\), etc. |
| \(\text{OP}_{A\rightarrow B}(\{i\}_A \otimes \{i\}_B)\) | for computational basis vectors \(\{i\}_A, \{j\}_B\) |
| \(X^\dagger\) | Adjoint of \(X\). |
| \(X^T\) | Transpose of \(X\) with respect to the computational basis. |
| \(X_A \geq Y_A\) | \(X - Y \in \text{Pos}(A)\). |
| \(\|X\|_p\) | \(\text{Tr}[(X^\dagger X)^{\frac{p}{2}}]^{\frac{1}{p}}\). Note that if \(p < 1\), this is not a norm. |
Note in particular the use of the shorthand $\alpha'$ to denote $\frac{\alpha-1}{\alpha}$; it will be used extensively. Using this shorthand, $\alpha \in (\frac{1}{2}, 1)$ corresponds to $\alpha' \in (-1, 0)$ and $\alpha \to \infty$ corresponds to $\alpha' \to 0$.

Another convention used extensively throughout the paper is the implicit tensorization by the identity: if we have two operators $X_{AB}$ and $Y_B$, by $X_{AB} Y_B$ we mean $X_{AB}(\text{id}_A \otimes Y_B)$. We will also often omit subscripts when doing so should not create confusion.

We will also make use of the operator-vector correspondence. We will endow every Hilbert space with its own computational basis, denoted by $|i\rangle_A$ for the space $A$ and so on, and we define the linear map $\text{Op}_{A \to B}: A \otimes B \to \mathcal{L}(A, B)$ by its action on the computational basis as follows: $\text{Op}_{A \to B}(|i\rangle_A \otimes |j\rangle_B) = |j\rangle_B \langle i|_A$.

### 2.2 The quantum Rényi entropy

We now present the definition of the quantum Rényi divergence, first defined in [MLDSFT13; WWY14] and sometimes called the ”sandwiched” Rényi divergence:

**Definition 2** (Quantum Rényi divergence). Let $\rho, \sigma \in \text{Pos}(A)$, and let $\alpha \in [\frac{1}{2}, 1) \cup (1, \infty)$. Then, we define their Rényi $\alpha$-divergence as

$$D_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \left( \frac{1}{\text{Tr}[\rho]} \text{Tr} \left( \left( \frac{\sigma^{\alpha} \rho^{1-\alpha}}{\| \sigma^{\alpha} \rho^{1-\alpha} \|_1} \right)^\alpha \right) \right),$$

where, in the above, $\log 0$ is understood to be $-\infty$.

The Rényi entropy is then defined as follows:

**Definition 3** (Quantum Rényi entropy). Let $\rho_{AB} \in \text{Pos}(A \otimes B)$ and $\sigma_B \in \text{D}(B)$, and let $\alpha \in [\frac{1}{2}, 1) \cup (1, \infty)$. Then,

$$H_\alpha(A|B)_\rho := -D_\alpha(\rho_{AB}||\text{id}_A \otimes \sigma_B),$$

and

$$H_\alpha(A|B)_\rho := -\inf_{\omega_B \in \text{D}(B)} D_\alpha(\rho_{AB}||\text{id}_A \otimes \omega_B).$$

By taking the limit as $\alpha \to 1$, we recover the von Neumann entropy; by taking the limit $\alpha \to \infty$, we get the min-entropy [Ren05]; by choosing $\alpha = \frac{1}{2}$, we get the max-entropy [KRS09]; and by choosing $\alpha = 2$ we get the collision entropy from [Ren05].

We also recall the duality property of quantum Rényi entropies proven in [MLDSFT13, Theorem 9] and also independently in [Bei13, Theorem 9], which is the Rényi analogue of the duality between min- and max-entropy:

**Fact 4** (Duality of Rényi entropies). Let $|\psi\rangle_{ABC} \in A \otimes B \otimes C$ be a normalized pure state, and let $\rho_{ABC} := |\psi\rangle\langle\psi|_{ABC}$. Then, we have that for any $\alpha \in (\frac{1}{2}, 1) \cup (1, \infty)$,

$$H_\alpha(A|B)_\rho = -H_\alpha(A|C)_\rho,$$

where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 2$.

Note here that it is particularly convenient to rephrase the condition $\frac{1}{\alpha} + \frac{1}{\alpha'} = 2$ using the shorthand given in the table in Section 2.1: it corresponds to $\alpha' = -\alpha'$. 

3
3 Proof of the main result

The first ingredient of the proof is the following generalization of Hadamard’s three-line theorem proven by Beigi in [Bei13] as part of his proof of the data processing inequality of the sandwiched Rényi divergence for \( \alpha > 1 \):

**Theorem 5** (Theorem 2 from [Bei13]). Let \( F : S \to L(A) \), where \( S := \{ z \in \mathbb{C} : 0 \leq \text{Re}(z) \leq 1 \} \), be a bounded map that is holomorphic on the interior of \( S \) and continuous on the boundary. Let \( 0 < \theta < 1 \) and define \( p_\theta \) by

\[
\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}.
\]

For \( k = 0, 1 \) define

\[
M_k = \sup_{t \in \mathbb{R}} \| F(k + it) \|_{p_k}.
\]

Then, we have

\[
\| F(\theta) \|_{p_\theta} \leq M_0^{1 - \theta} M_1^\theta.
\]

Note that in [Bei13], the theorem is stated for more general norms involving a positive operator \( \sigma \) (which can be chosen to be \( \sigma = \text{id} \) to obtain this version) and adds a condition that \( p_0 \leq p_1 \) which is not necessary, as can be seen by applying the theorem with \( \hat{F}(z) := F(1 - z) \) and \( \hat{\theta} = 1 - \theta \).

The second ingredient is the following lemma, which gives a particularly useful expression for the Rényi entropy:

**Lemma 6.** Let \( \langle \psi \rangle_{ABCD} \in A \otimes B \otimes C \otimes D \) be a normalized pure state, let \( X_{AD \to BC} = \text{Op}_{AD \to BC}(\langle \psi \rangle) \), and let \( \alpha \in (\frac{1}{2}, 1) \cup (1, \infty) \), with \( \alpha \) such that \( \frac{1}{\alpha} + \frac{1}{\alpha'} = 2 \), and let \( \rho_{ABCD} = |\psi\rangle\langle \psi| \) and \( \sigma_C \in D(D) \). Then,

\[
\begin{align*}
H_\alpha(AB|C)_{\rho|\sigma} &= -\log \sup_{\tau_D \in D(D)} \left\| \sigma_C^{-\alpha} X^{\frac{2}{\alpha}} \tau_D^{\frac{1}{2\alpha'}} \right\|_2, \quad (3) \\
H_\alpha(B|C)_{\rho|\sigma} &= -\log \left\| \sigma_C^{-\alpha} X^{\frac{1}{2\alpha}} \right\|_2, \quad (4) \\
H_\alpha(A|BC)_{\rho} &= -\log \sup_{\tau_D \in D(D)} \left\| X^{\frac{1}{2\alpha'}} \tau_D^{\frac{1}{\alpha'}} \right\|_2. \quad (5)
\end{align*}
\]

**Proof.** We start by proving (3) from the representation in equation (19) from [MLDSFT13]:

\[
H_\alpha(AB|C)_{\rho|\sigma} = \begin{cases} 
\left( -\frac{1}{\alpha} \right) \log \inf_{\tau_D \in D(D)} \langle \psi | \text{id}_{AB} \otimes \sigma_C^{-\alpha} \otimes \tau_D | \psi \rangle & \text{if } \alpha < 1 \\
\left( -\frac{1}{\alpha} \right) \log \sup_{\tau_D \in D(D)} \langle \psi | \text{id}_{AB} \otimes \sigma_C^{-\alpha} \otimes \tau_D | \psi \rangle & \text{if } \alpha > 1,
\end{cases}
\]

Putting the prefactor in the exponent, we get

\[
H_\alpha(AB|C)_{\rho|\sigma} = -\log \sup_{\tau_D \in D(D)} \langle \psi | \text{id}_{AB} \otimes \sigma_C^{-\alpha} \otimes \tau_D | \psi \rangle \frac{1}{\alpha'}
\]

for all \( \alpha \in [\frac{1}{2}, 1) \cup (1, \infty) \). Now, consider the vector \( \sigma_C^{-\alpha'} \otimes \tau_D | \psi \rangle \): by Lemma 11, we have that

\[
\text{Op}_{AD \to BC} \left( \sigma_C^{-\alpha'} \otimes \tau_D | \psi \rangle \right) = \sigma_C^{-\alpha} X^{\frac{1}{\alpha'}} \tau_D^{\frac{1}{\alpha'}}
\]
and therefore
\[
\langle \psi | \text{id}_{AB} \otimes \sigma_C^{-\alpha'} \otimes \tau_D^{\alpha'} | \psi \rangle^{\frac{1}{2}} = \left\| \mathcal{O}_{AB \rightarrow BC} \left( \sigma_C^{-\alpha'} \otimes \tau_D^{\alpha'} | \psi \rangle \right) \right\|_2^{\frac{1}{2}} = \left\| \sigma_C^{-\alpha'} X \tau_D^{\alpha'} \right\|_2^{\frac{1}{2}}
\]

by Lemma 12, from which (3) follows. We now derive (4) from (3):
\[
2^{-H_\alpha(B|C)_{\rho|\sigma}} = \sup_{\tau_{AD} \in (A \otimes D)} \left\| \sigma_C^{-\alpha'} X \sigma_C^{\beta'} \right\|_2^{\frac{1}{2}}
= \sup_{\tau_{AD} \in (A \otimes D)} \text{Tr} \left[ X^{\dagger} \sigma_C^{-\alpha'} \sigma_C^{\beta'} X \tau_{AD} \right]^{\frac{1}{2}}
= \left\| X^{\dagger} \sigma_C^{-\alpha'} X \right\|^2_{\alpha}
= \left\| \sigma_C^{-\alpha'} X \right\|^2_{2\alpha},
\]
where the third line follows from Lemma 10, and (4) follows. Finally, we prove (5) via duality:
\[
H_\alpha(A|BC)_{\rho} = -H_\beta(A|D)_{\rho}
= - \sup_{\omega_D \in D(D)} H_\beta(A|D)_{\rho|\omega}
= \log \inf_{\omega_D \in D(D)} \left\| X \omega_D \right\|_2^{\frac{1}{2}}
= \log \inf_{\omega_D \in D(D)} \left\| X \omega_D \right\|_2^{\frac{1}{2}}
= - \log \sup_{\omega_D \in D(D)} \left\| X \omega_D \right\|_2^{\frac{1}{2}},
\]
where we used the fact that \( \hat{\alpha'} = -\alpha' \), and invoked Lemma 9 to get the third line. This concludes the proof.

We are now ready to prove Theorem 1. We break it down into its two cases, with Proposition 7 below corresponding to Equation (1) and Proposition 8 further down corresponding to Equation (2).

**Proposition 7** (First case of Theorem 1). Let \( \rho_{ABC} \in D(A \otimes B \otimes C) \), and let \( \alpha, \beta, \gamma > 1 \) be such that \( \frac{1}{\alpha} = \frac{1}{\beta} + \frac{1}{\gamma} \). Then, we have that
\[
H_\alpha(AB|C)_{\rho|\sigma} \geq H_\beta(A|BC)_{\rho} + H_\gamma(B|C)_{\rho|\sigma}
\]
for any \( \sigma_C \in D(C) \). In particular,
\[
H_\alpha(AB|C)_{\rho} \geq H_\beta(A|BC)_{\rho} + H_\gamma(B|C)_{\rho}.
\]
Proof. Let $|\psi\rangle_{ABCD} \in A \otimes B \otimes C \otimes D$ be a purification of $\rho$, let $X_{AD\rightarrow BC} = \text{Op}_{AD\rightarrow BC}(|\psi\rangle)$, and let $\tau_D \in D(D)$. We use Theorem 5 with the following choices:

$$F(z) = \sigma_C^{-\frac{\alpha'}{\beta'}} X_{\tau_D}^{\frac{(1-\gamma)\alpha'}{\beta'}}$$

$$\frac{1}{\rho_0} = 1 - \frac{1}{2\beta} = \frac{1}{2\beta}$$

$$p_1 = 2\gamma$$

$$\theta = \frac{\alpha'}{\gamma}.$$ 

Now, note that $p_\theta = 2$, that $F(\theta) = \sigma_C^{-\frac{\alpha'}{\beta'}} X_{\tau_D}^{\frac{\alpha'}{\beta'}}$, that $1 - \theta = \frac{\alpha'}{\beta'}$, and that

$$M_0 = \sup_{t \in \mathbb{R}} \left\| \sigma_C^{-\frac{\alpha'}{\beta'}} X_{\tau_D}^{\frac{(1-t\gamma)\alpha'}{\beta'}} \right\|_{p_0}$$

$$= \left\| X_{\tau_D}^{\frac{\alpha'}{\beta'}} \right\|_{p_0}$$

due to the fact that $\sigma_C^{-\frac{\alpha'}{\beta'}}$ and $\tau_D^{-\frac{\alpha'}{\beta'}}$ are unitary for all $t \in \mathbb{R}$. Likewise,

$$M_1 = \left\| \sigma_C^{-\frac{\alpha'}{\beta'}} X \right\|_{p_1}.$$ 

Hence, for any choice of normalized $\sigma$ and $\tau$, we have that

$$\left\| \sigma_C^{-\frac{\alpha'}{\beta'}} X_{\tau_D}^{\frac{\alpha'}{\beta'}} \right\|_2^2 \leq \left\| X_{\tau_D}^{\frac{\alpha'}{\beta'}} \right\|_{2\beta} \left\| \sigma_C^{-\frac{\alpha'}{\beta'}} X \right\|_{2\gamma}.$$ 

This leads to

$$\left\| \sigma_C^{-\frac{\alpha'}{\beta'}} X_{\tau_D}^{\frac{\alpha'}{\beta'}} \right\|_2^2 \leq \left\| X_{\tau_D}^{\frac{\alpha'}{\beta'}} \right\|_{2\beta} \left\| \sigma_C^{-\frac{\alpha'}{\beta'}} X \right\|_{2\gamma}.$$ 

Maximizing over $\tau_D$ on both sides yields:

$$\sup_{\tau_D \in D(D)} \left\| \sigma_C^{-\frac{\alpha'}{\beta'}} X_{\tau_D}^{\frac{\alpha'}{\beta'}} \right\|_2^2 \leq \sup_{\tau_D \in D(D)} \left\| X_{\tau_D}^{\frac{\alpha'}{\beta'}} \right\|_{2\beta} \left\| \sigma_C^{-\frac{\alpha'}{\beta'}} X \right\|_{2\gamma}.$$ 

Finally, using Lemma 6, we get that

$$H_\alpha(AB|C)_{\rho\sigma} \geq H_\beta(A|BC)_{\rho} + H_\gamma(B|C)_{\rho\sigma},$$

as advertised. \qed

We now turn to the second case of Theorem 1, corresponding to Equation (2), using essentially the same proof:

**Proposition 8** (Second case of Theorem 1). Let $\rho_{ABC} \in D(A \otimes B \otimes C)$, and let $\alpha' > 0$, and $\beta'$ and $\gamma'$ have opposite signs, and be such that $\frac{1}{\alpha'} = \frac{1}{\beta'} + \frac{1}{\gamma'}$. Then, we have that

$$H_\alpha(AB|C)_{\rho\sigma} \leq H_\beta(A|BC)_{\rho} + H_\gamma(B|C)_{\rho\sigma}$$

for every $\sigma_C \in D(C)$. In particular,

$$H_\alpha(AB|C)_{\rho} \leq H_\beta(A|BC)_{\rho} + H_\gamma(B|C)_{\rho}.$$
Proof. Let $|\psi\rangle_{ABCD} \in A \otimes B \otimes C \otimes D$ be a purification of $\rho$, let $X_{AD\rightarrow BC} = \text{Op}_{AD\rightarrow BC}(|\psi\rangle)$, and let $\tau_D \in \mathcal{D}(D)$. We split the proof into two cases: first, we assume that $\frac{\theta'}{\gamma'} < 1$. We use Theorem 5 and the following choices:

\[
F(z) = \frac{\alpha' z + \beta' z^2}{2\gamma'} X_{\tau_D} \alpha' z + \beta' z^2
\]

\[
p_0 = 2
\]

\[
p_1 = 2\gamma
\]

\[
\theta = -\frac{\beta'}{\gamma'}
\]

Now, note that $F(\theta) = X_{\tau_D} \theta'$, that $1 - \theta = \frac{\theta'}{\gamma'}$, that $p_0 = 2\beta$, and that

\[
M_0 = \sup_{t \in \mathbb{R}} \left| \frac{\alpha' t - \beta' t^2}{2\gamma'} X_{\tau_D} \frac{\alpha' t - \beta' t^2}{2\gamma'} \right|_2
\]

\[
= \left| \frac{\alpha' t}{2\gamma'} X_{\tau_D} \right|_2
\]

due to the fact that $\frac{\alpha' t - \beta' t^2}{2\gamma'}$ and $\tau_D \frac{\alpha' t - \beta' t^2}{2\gamma'}$ are unitary for all $t \in \mathbb{R}$. Likewise,

\[
M_1 = \left| \frac{\alpha' t}{2\gamma'} X \right|_{2\gamma}
\]

Hence, for any choice of normalized $\sigma$ and $\tau$, we have that

\[
\left| X_{\tau_D} \frac{\alpha'}{2\beta} \right|_{2\beta} \leq \left| \frac{\alpha' t - \beta' t^2}{2\gamma'} X_{\tau_D} \frac{\alpha' t - \beta' t^2}{2\gamma'} \right|_2 \left| \frac{\alpha' t}{2\gamma'} X \right|_{2\gamma}
\]

This leads to

\[
\left| X_{\tau_D} \frac{\alpha'}{2\beta} \right|_{2\beta} \leq \left| \frac{\alpha' t}{2\gamma'} X_{\tau_D} \frac{\alpha' t}{2\gamma'} \right|_2 \left| \frac{\alpha' t}{2\gamma'} X \right|_{2\gamma}
\]

and therefore

\[
\sup_{\tau_D \in \mathcal{D}(D)} \left| X_{\tau_D} \frac{\alpha'}{2\beta} \right|_{2\beta} \leq \sup_{\tau_D \in \mathcal{D}(D)} \left| \frac{\alpha' t}{2\gamma'} X_{\tau_D} \frac{\alpha' t}{2\gamma'} \right|_2 \left| \frac{\alpha' t}{2\gamma'} X \right|_{2\gamma}
\]

Using Lemma 6, we get that

\[
H_\beta(A|BC)_\rho \geq H_\alpha(AB|C)_{\rho|\sigma} - H_\gamma(B|C)_{\rho|\sigma}
\]

or,

\[
H_\alpha(AB|C)_{\rho|\sigma} \leq H_\beta(A|BC)_\rho + H_\gamma(B|C)_{\rho|\sigma}
\]

We now turn to the case where $\frac{\beta'}{\gamma'} > 1$. We again use Theorem 5, but with these choices:

\[
F(z) = \frac{\alpha' z + \beta' z^2}{2\gamma'} X_{\tau_C} \alpha' z + \beta' z^2
\]

\[
p_0 = 2
\]

\[
p_1 = 2\beta
\]

\[
\theta = \frac{-\gamma'}{\beta'}
\]
Now, note that $F(\theta) = \sigma_C^{-\frac{\theta}{\alpha}} X$, that $1 - \theta = \frac{\beta}{\alpha}$, and that $p_\theta = 2\gamma$, and that
\[
M_0 = \sup_{t \in \mathbb{R}} \left\| \sigma_C^{-\frac{\theta}{\alpha}} X \tau_D^{\frac{-\theta t + \frac{\beta}{\alpha}}}{2t} \right\|_2 = \left\| \sigma_C^{-\frac{\theta}{\alpha}} X \tau_D^{\frac{-\theta t}{2t}} \right\|_2.
\]
due to the fact that $\sigma_C^{-\frac{\theta}{\alpha}}$ and $\tau_D^{\frac{-\theta t + \frac{\beta}{\alpha}}}{2t}$ are unitary for all $t \in \mathbb{R}$. Likewise,
\[
M_1 = \left\| X \tau_D^{\frac{-\theta t}{2t}} \right\|_{2\beta}.
\]
Hence, for any choice of normalized \(\sigma\) and \(\tau\), we have that
\[
\left\| \sigma_C^{-\frac{\theta}{\alpha}} X \right\|_{2\gamma} \leq \left\| \sigma_C^{-\frac{\theta}{\alpha}} X \tau_D^{\frac{-\theta t + \frac{\beta}{\alpha}}}{2t} \right\|_2 \left\| X \tau_D^{\frac{-\theta t}{2t}} \right\|_{2\beta}.
\]
This leads to
\[
\left\| \sigma_C^{-\frac{\theta}{\alpha}} X \right\|_{2\gamma} \leq \left\| \sigma_C^{-\frac{\theta}{\alpha}} X \tau_D^{\frac{-\theta t}{2t}} \right\|_2 \left\| X \tau_D^{\frac{-\theta t}{2t}} \right\|_{2\beta}.
\]
Moving the rightmost term to the left-hand side and maximizing over $\tau_D$ on both sides, we get:
\[
\left\| \sigma_C^{-\frac{\theta}{\alpha}} X \right\|_{2\gamma}^2 \left( \sup_{\tau_D \in \mathcal{D}(D)} \left\| X \tau_D^{\frac{-\theta t}{2t}} \right\|_2^{\frac{\beta}{\alpha}} \right) \leq \sup_{\tau_D \in \mathcal{D}(D)} \left\| \sigma_C^{-\frac{\theta}{\alpha}} X \tau_D^{\frac{-\theta t}{2t}} \right\|_2^{\frac{\beta}{\alpha}}.
\]
Using Lemma 6, we get that
\[
H_\gamma(B|C)_\rho|\sigma + H_\beta(A|BC)_\rho \geq H_\alpha(AB|C)_\rho|\sigma.
\]
This concludes the proof. \(\square\)

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**A Auxillary lemmas**

**Lemma 9.** Let $\rho_{AB} \in \mathcal{D}(A \otimes B)$. Then, we have that
\[
H_\alpha(A|B) = -\log \inf_{\sigma_B \in \mathcal{D}(B)} \left\| X \sigma_B^{-\frac{\theta}{\alpha}} \right\|_{2\alpha}^{\frac{\beta}{\alpha}},
\]
where $X := \text{Op}_{AB \rightarrow D}(|\psi\rangle)$ for a purification $|\psi\rangle_{ABD}$ of $\rho$. 

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Proof. First, we write

\[
H_\alpha(A|B)_{\rho,\sigma} = D_\alpha(\rho_{AB}||id_A \otimes \sigma_B) \\
= \frac{1}{1-\alpha} \log \text{Tr} \left[ \left( \frac{\rho_{B}^{\alpha'}}{\sigma_{B}^{\alpha'}} \right)^{\alpha'} \right] \\
= \frac{-1}{\alpha'} \log \left\| \frac{\rho_{B}^{\alpha'}}{\sigma_{B}^{\alpha'}} \right\|_\alpha \\
= -\log \left\| \frac{\rho_{B}^{\alpha'}}{\sigma_{B}^{\alpha'}} X^\dagger X \left( \frac{\rho_{B}^{\alpha'}}{\sigma_{B}^{\alpha'}} \right)^{\alpha'} \right\|_\alpha \\
= -\log \left\| X \sigma_{B}^{\alpha'} \right\|_{2/\alpha}^{\alpha'}
\]

where we have used the fact that \( \rho_{AB} = X^\dagger X \). \( \square \)

Lemma 10. Let \( \alpha \in [\frac{1}{2}, 1) \cup (1, \infty) \). Then, for any \( X \in \text{Pos}(A) \), we have that

\[
\| X \|_{\alpha} = \sup_{Y \in D(A)} \text{Tr}[Y^{\alpha'} X] 
\]

if \( \alpha > 1 \), and

\[
\| X \|_{\alpha} = \inf_{Y \in D(A)} \text{Tr}[Y^{\alpha'} X] 
\]

if \( \alpha < 1 \). (As usual, \( \alpha' = \frac{\alpha-1}{\alpha} \).)

Proof. This is simply a reformulation of Lemma 11 in [MLDSFT13]. \( \square \)

Lemma 11. Let \( |\psi\rangle \in A \otimes B \), and let \( X_A \in \text{L}(A) \) and \( Y_B \in \text{L}(B) \). Then, we have that

\[
\text{Op}_{A \rightarrow B}(X_A \otimes Y_B|\psi\rangle) = Y_B \text{Op}(|\psi\rangle) X_A^\dagger.
\]

Proof. This can be shown by a simple manipulation of indices; see for example [Wat11, Section 2.4]. \( \square \)

Lemma 12. Let \( |\psi\rangle \in A \otimes B \). Then,

\[
\langle \psi|\psi\rangle = \text{Tr}[\text{Op}_{A \rightarrow B}(|\psi\rangle)^\dagger \text{Op}_{A \rightarrow B}(|\psi\rangle)]
\]

and therefore,

\[
\| |\psi\rangle \| = \| \text{Op}_{A \rightarrow B}(|\psi\rangle) \|_2.
\]

Proof. Again, see [Wat11, Section 2.4]. \( \square \)

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