Abstract—We consider the problem of inferring the structure and dynamics of an unknown network driven by unknown noise inputs. For linear, time-invariant systems of minimal order, with hidden states and one noise source per measured state, we characterise under what conditions this problem is well-posed and present two main results. First, for networks with stable interactions between measured states and where the noise sources act through minimum phase dynamics, we prove that the solution for the network is unique and that the solution for the input dynamics is unique up to sign. If the assumptions of stability and phase-minimality are relaxed, we characterise the set of all solutions and prove that if the noise acts directly on the measured states, then the number of solutions is finite and bounded by a function of the number of hidden states. Lastly we present a method to obtain this set of solutions from data.

I. INTRODUCTION

Many phenomena are naturally described as networks of interconnected dynamical systems and the identification of the dynamics and structure of a network has recently become an important problem. A variety of model classes exists to model the dynamics and structure of a network. We focus here on Linear, Time-Invariant (LTI) systems, for which there are still many interesting theoretical questions outstanding, and leave the network structure unrestricted.

Previous work characterised identifiability in the deterministic case where targeted, known inputs may be applied to the network [1]. This also extends to biologically-inspired knock-out type manipulations [2], [3]. In practice however, and often in biological applications, these types of experiments are not possible or are expensive to conduct. One may simply be faced with the outputs of an existing network driven by its own intrinsic variation. Noise is endemic in biological networks and its sources are numerous [4]; making use of this natural variation as a non-invasive means of identification is an appealing prospect, for example in gene regulatory networks [5].

This problem has been considered in various forms in the literature. In [6] for example, networks of known, identical subsystems are considered, which can be identified using an exhaustive grounding procedure similar to that in [3]. A solution is presented in [7] for identifying the undirected structure for a restricted class of polytree networks; and in [8] for “self-kin” networks. In [9] the problem is posed as a closed-loop system identification problem for a more general, but known, topology; in [10], the authors claim that their method can also be applied to networks with unknown topology.

In all of the above-cited work, the intrinsic variation is modelled as known noise sources applied only to the states that are measured. Whilst being an unrealistic assumption in some applications, this input requirement has been shown to be necessary for solution uniqueness even in the deterministic case [1]. Here we make the same assumption and consider the same model class as the above. The more general case of noise applied to every state is considered in a separate paper currently in preparation. Using the Dynamical Structure Function (DSF) framework, introduced in the following section, we derive results with no restrictions on the network topology and characterise under what circumstances the problem may be solved.

Section II provides an overview of DSFs and then presents some new results that will be used in later sections. Section III formulates the network reconstruction problem as a variant of the spectral factorization problem. The main results are then presented in Section IV concluding with a numerical example. Conclusions are given in Section V and some additional proofs are included in the Appendix.

Notation

For a matrix $A$, $A(i,j)$ denotes element $(i,j)$, $A(:,i)$ denotes its $i^{th}$ row and $A(:,j)$ its $j^{th}$ column. For a vector $x$, the $j^{th}$ element is denoted $x(i)$. The transpose and conjugate transpose of $A$ are denoted respectively by $A^T$ and $A^*$. The matrix identity of appropriate dimension is denoted by $I$; similarly the zero matrix is denoted by $0$ with implicit dimension. The diagonal matrix comprising the diagonal elements of $A$ is denoted diag$(A)$. Free choice of the signs of the elements of $A$ is denoted by $\pm A$; hence if the solution to $A$ is unique up to sign, the solution to $\pm A$ is unique.

II. DYNAMICAL STRUCTURE FUNCTIONS

We consider systems defined by the following linear, time-invariant (LTI) representation:

$$\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{align*}$$

(1)

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$ and $C = [I \ 0]$. The matrices $A$ and $B$ define a digraph whose nodes comprise the system states and inputs and whose edges correspond to nonzero elements in those matrices. For example, there is an edge from node $j$ to node $i$ if $A(i,j) \neq 0$. These matrices are in general not obtainable from measured data, hence we consider a different representation of the network.
direct causal dependencies between the p measured states, y, where typically p < n. Partition (1) as follows:

\[
\begin{bmatrix}
\dot{y}_z
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
y \\
z
\end{bmatrix} + \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} u
\]

where \( x = [y^T \ z^T]^T \) is the full state vector and \( z \in \mathbb{R}^h \) are the \( h = n - p \) “hidden” states. Taking the Laplace transform of (2) and eliminating \( Z \) yields \( sY = WY + VU \), where

\[
W := A_{11} + A_{12} (sI - A_{22})^{-1} A_{21}
\]

\[
V := B_1 + A_{12} (sI - A_{22})^{-1} B_2
\]

Now define \( D := \text{diag}(W) \), subtract \( DY \) from both sides and rearrange to give:

\[
Y = QY + PU
\]

where

\[
Q := (sI - D)^{-1} (W - D)
\]

\[
P := (sI - D)^{-1} V
\]

are strictly proper transfer matrices. Note that Q always has diagonal elements equal to zero (it is hollow).

**Definition 1** (Dynamical Structure Function). Given any system (2), the Dynamical Structure Function (DSF) is uniquely defined as \((Q, P)\) and characterizes direct, causal relations among measured states \( Y \) and inputs \( U \).

The DSF defines a digraph with only the measured states and inputs as nodes, where \( Q(i, j) \neq 0 \) implies an edge from node \( j \) to node \( i \) with open-loop dynamics given by the transfer function \( Q(i, j) \). The problem of network reconstruction is then that of identifying \((Q, P)\) and is well-posed if the solution for \((Q, P)\) is unique. The transfer matrix from \( U \) to \( Y \), defined as \( G(s) := C(sI - A)^{-1} B \), is related to \( Q \) and \( P \) as follows:

\[
G = (I - Q)^{-1} P
\]

which provides a vehicle for assessing the identifiability of \((Q, P)\) from data. The following definitions will be useful.

**Definition 2** (Consistency). A DSF \((Q, P)\) is consistent with transfer matrix \( G \) if (6) is satisfied.

**Definition 3** (Directness). A causal interaction between two variables is direct if the appropriate element of \( A \) or \( B \) is nonzero, and indirect otherwise.

For example, \( Q(i, j) \neq 0 \) and \( A(i, j) = 0 \) implies a causal, but indirect, interaction from state \( j \) to state \( i \).

A. Network Reconstruction from \( G \)

The problem of solving for \( Q \) and \( P \) from \( G \) was considered in [1] and the main results are stated here. Under the assumptions that there is no \textit{a priori} knowledge about the nonzero elements of \( Q \) and \( P \) and that \( G \) is full rank, the following theorem gives necessary and sufficient conditions for network reconstruction:

**Theorem 1** ([11]). The dynamical structure function, \((Q, P)\), can be obtained uniquely from the transfer matrix, \( G \), if and only if \( p \) elements in each row of \((Q, P)\) are known.

By applying targeted input experiments, the structure of \( P \) can be chosen such that this condition is satisfied. The following corollary outlines a protocol for reconstruction in this manner:

**Corollary 1** ([11]). The DSF \((Q, P)\) can be obtained uniquely from \( G \) if \( P \) is square, diagonal and full rank.

The knowledge that each input uniquely affects a single measured state provides sufficient knowledge of the zero elements of \( P \) to satisfy the condition of Theorem [11]. As mentioned, \( P \) diagonal is a common assumption in the literature [6], [8], [9].

**Example 1.** Consider the following stable, minimal system with two measured states and one hidden state:

\[
A = \begin{bmatrix}
-1 & 0 & 4 \\
0 & -2 & 5 \\
-6 & 0 & -3
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
0 & 0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad D = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}
\]

where the parameter values were chosen arbitrarily. The system transfer matrix is:

\[
G(s) = \begin{bmatrix}
s + 3 \\
(s + 2)(s + 4s + 27)
\end{bmatrix}
\]

and may be realized by an infinite variety of \( A \) and \( B \) matrices. The DSF is given by:

\[
Q(s) = \begin{bmatrix}
0 & 0 \\
\frac{-30}{(s + 2)(s + 3)} & 0
\end{bmatrix}, \quad P(s) = \begin{bmatrix}
\frac{s + 3}{s + 4s + 27} & 0 \\
0 & \frac{s + 2}{s + 2}
\end{bmatrix}
\]

and is the only valid \( Q \) and diagonal \( P \) that is consistent with \( G \). This system is represented graphically for state-space (a) and DSF (b) like so:

![Diagram](image)

The DSF describes causal interactions among measured states that may be indirect in the underlying system.

B. Representations of Structure

A given system with state-space realization \((A, B, C, D)\) with \( C = [I \ 0] \) and \( D = 0 \) can also be represented by its DSF \((Q, P)\) and transfer function \( G \), with a decreasing amount of structural information. The relationship between these three representations is explained in Figure [1] which shows that a state space realization uniquely defines both a DSF and a transfer function. However, multiple DSFs are
consistent with a given transfer function, and a given DSF can be realized by multiple state space realizations. Theorem 1 proves that there is only one DSF with $P$ diagonal that is consistent with $G$.

All realizations of a particular $G$ are parameterized by the set of invertible matrices $T \in \mathbb{R}^{n \times n}$ as follows:

$$A' = TAT^{-1}, \quad B' = TB, \quad C' = CT^{-1}, \quad D' = D$$

which in general does not admit a DSF. Partition the system as in (2); in order to admit a DSF, the transformed system must have $C' = [I \ 0]$, which requires $T$ to be of the form:

$$T = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix}$$

(7)

This characterises all transformed systems that admit a DSF – the set of realizations in Figure 1(a). What is the subset of transformations that result in realizations that have the same DSF, corresponding to the blue region in Figure 1(a)? The following is a subset of this.

**Lemma 1.** Invertible state transformations of the form $T = \begin{bmatrix} I & 0 \\ 0 & T_2 \end{bmatrix}$ are $(Q, P)$-invariant.

*Proof.* The proof follows directly from the definitions of $W$ and $V$ in (3), which are unchanged under the transformation and hence $Q$ and $P$ are also unchanged.  

Note that this is only a sufficient condition and that intuitively such transformations should not change $(Q, P)$ as only the hidden states are being transformed.

### C. Realizations with $P$ Square, Diagonal and Full Rank

From Corollary 1 the solution for $(Q, P)$ from $G$ is unique if $P$ is square, diagonal and full rank. We seek to define a canonical form of realization for such systems, making use of the following two lemmas. First we convert the requirement that $P$ be diagonal into an algebraic condition on $A$ and $B$.

**Lemma 2.** The matrix $P$ is diagonal if and only if the matrices:

$$B_1 \quad \text{and} \quad A_{12} A_{22}^k B_2$$

for $k = 0, 1, \ldots, h - 1$ are diagonal, where $h = \text{dim}(A_{22})$.

The proof is omitted but follows from expressing $V$ in (3) as a Neumann series and making use of the Cayley-Hamilton Theorem. From this, $B_1$ must be diagonal; we can further simplify $B$ by removing dependent columns of $B_2$ as follows.

**Lemma 3.** Dependent columns of $B_2$ have no effect on $P$ if it is diagonal.

The proof is omitted but follows from the contradiction of a column of $B_2$ being dependent on other columns of $B_2$ and every column of $B_2$ affecting only the corresponding column of $V$ in (3), which is diagonal. A consequence of Lemma 3 is that dependent columns of $B_2$ can be set to zero without changing $Q$ or $P$.

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**Lemma 4.** Every DSF with $P$ square, diagonal and full rank has a realization with $B$ in the following form:

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & B_{22} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \\ T & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{31} \\ 0 & 0 & 0 \end{bmatrix}$$

(8)

for some partitioning $B_{22}$ square, diagonal and full rank and $B_{42} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ for some other partitioning.

The proof is achieved via Lemmas 1 - 3 and is given in Appendix VI-A. Given $B$ in canonical form, we now define a canonical form for $A_{12}$ and $A_{22}$.

**Lemma 5.** Every DSF with $P$ square, diagonal and full rank and with $B$ in the canonical form of (8), has a realization with $A_{12}$ and $A_{22}$ given as follows:

$$A_{12} = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & X \end{bmatrix}, \quad A_{22} = \begin{bmatrix} \beta_{11} & 0 & X & X \\ X & X & X & X \\ 0 & \beta_{32} & X & X \end{bmatrix}$$

(9)

where the dashed partitions are drawn in accordance with 8 but the block rows and columns may be of different dimension. The matrix $\alpha_{13}$ is square, full rank and diagonal, $X$ denotes an unspecified element, $\beta_{11}$ and $\beta_{32}$ are both square and diagonal, but not necessarily full rank and

$$\begin{bmatrix} \beta_{32} \\ \beta_{42} \end{bmatrix} = \begin{bmatrix} \gamma & 0 \\ 0 & 0 \\ X & I \\ X & 0 \end{bmatrix}$$

(10)

for some partitioning, where $\gamma$ is square, diagonal and full rank.

The proof is given in Appendix VI-B. In the following section we will compare realizations of systems with different DSFs and diagonal $P$. Since any such system can be realized in this canonical form, we can without loss of generality restrict ourselves to such realizations.

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Fig. 1: Graphical representation of relationship between state space, DSF space and transfer function space for a particular system $(A, B, C, D)$. In (a) is contained the set of realizations $\{TAT^{-1}, TB, [I \ 0], 0\}$ for all $T$ as in (7); in red is the particular realization with $T = I$ and in blue the set of realizations with the same DSF $(Q, P)$. In (b) is the set of all DSFs that have realizations in (a); in blue is the particular DSF $(Q, P)$. In (c) is the single transfer function $G$, with which are consistent all DSFs in (b) and which can be realized by all realizations in (a).

(a) State space

(b) DSF space

(c) TF space
III. PROBLEM FORMULATION

Consider an LTI system \((A, B, C, D)\) and associated transfer function:

\[
G(s) = C(sI - A)^{-1}B + D
\]

(11)
with \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}\) and \(D \in \mathbb{R}^{p \times m}\) under the following assumptions:

Assumption 1. The matrix \(A\) is Hurwitz.

Assumption 2. The system is driven by unknown white noise \(e(t)\) with covariance \(E[e(t)e^T(t)] = I\delta(t - \tau)\).

Assumption 3. The output \(y(t)\) is observed in steady state.

Assumption 4. The system \((A, B, C, D)\) is globally minimal.

From \(y(t)\), the most information that can be obtained is the output spectral density:

\[
\Phi(s) = E[Y(s)Y^*(s)] = G(s)G^*(s).
\]

The problem of obtaining a solution \(G(s)\) from \(\Phi(s)\) is the spectral factorization problem, considered by Youla [11] in 1961. Anderson [12] provides an algebraic characterisation of all globally minimal state-space realizations of a given \(\Phi(s)\), where the globally minimal degree is the smallest dimension of all minimal solutions. First, for a given \(\Phi(s)\), define the positive real matrix \(Z(s)\) as follows:

\[
Z(s) + Z^*(s) = \Phi(s)
\]

(12)
Minimal realizations of \(Z(s)\) are related to solutions \(G(s)\) to \(\Phi(s) = G(s)G^*(s)\) by the following lemma.

Lemma 6 ([12]). Let \((A, B_z, C, D_z)\) be a minimal realization of the positive real matrix \(Z(s)\) of \((12)\), then the minimal system \((A, B, C, D)\) has spectral density \(\Phi(s)\) if and only if there exists a real, positive definite and symmetric matrix \(R \in \mathbb{R}^{n \times n}\) such that the following equations hold:

\[
RA^T + AR = -BB^T
\]

\[
RC^T = B_z - BD^T
\]

\[
2D_z = DD^T
\]

(13)
This result was used by Glover and Willems [13] to provide conditions of equivalence between any two globally minimal realizations, which are stated below.

Lemma 7 ([13]). Two globally minimal systems \((A, B, C, D)\) and \((A', B', C', D')\) have equal output spectral density:

\[
\Phi(s) = G(s)G^*(s) = G'(s)G'^*(s)
\]

if and only if there exists invertible \(T \in \mathbb{R}^{n \times n}\) and symmetric \(S \in \mathbb{R}^{n \times n}\) such that:

\[
A' = T^{-1}AT
\]

(14a)
\[
C' = CT
\]

(14b)
\[
SA^T + AS = -BB^T + TB'B'^T T^T
\]

(14c)
\[
SC^T = -BD^T + TB'D'^T
\]

(14d)
\[
DD^T = D'D'^T
\]

(14e)

For any two systems that satisfy Lemma 7 for a particular \(S\), all additional solutions for this \(S\) may be parameterized as follows.

Corollary 2. Given two systems \((A, B, C, D)\) and \((A', B', C', D')\) that satisfy Lemma 7 for a particular \(S\), the globally minimal system \((A'', B'', C'', D'')\) is also a solution for this \(S\) if and only if

\[
A'' = T'A'T'^{-1}
\]

(15a)
\[
B'' = T'B'U
\]

(15b)
\[
C'' = C'T'^{-1}
\]

(15c)
\[
D'' = D'U
\]

(15d)
for some invertible \(T' \in \mathbb{R}^{n \times n}\) and orthogonal \(U \in \mathbb{R}^{p \times p}\). If \(G(s)\) is square and “minimum phase” (full rank for all \(s\) with \(\text{Re}(s) > 0\)), then for \(S = 0\), \((15)\) characterises all minimum phase solutions \(G''(s)\).

Clearly in general the solution to the spectral factorization problem is not unique, even in the minimum phase case. We make the following additional assumptions on the underlying system and consider the problem of identifying its DSF.

Assumption 5. The matrices \(C = [I \quad 0]\) and \(D = 0\).

Assumption 6. The matrix \(P\) is square, diagonal and full rank.

Assumption 5 is necessary for the system to admit a DSF. Assumption 6 then states that there are \(p\) noise sources and each of the \(p\) measured states is uniquely driven by exactly one of these. This excitation may occur indirectly via hidden states and hence have dynamics; the system is therefore driven by filtered white noise. This is a standard assumption in the literature for the following reason.

Remark 1. If nothing is known about \(Q\) or the nonzero entries of \(P\), then if \(P\) is square but not diagonal for some ordering of the inputs, the solution \((Q, P)\) is not unique. This follows from Theorem 7 – the solution is not unique given \(G\), hence it cannot be unique given only \(GG^*\).

The network reconstruction problem may now be stated as a variant of the spectral factorization problem.

Problem 1. Given a system \((A, B, C, D)\) under Assumptions 7–6 with DSF \((Q, P)\) and output spectral density \(\Phi(s)\), construct all systems \((A', B', C', D')\) with equal spectral density that also satisfy these assumptions. Hence obtain all DSFs \((Q', P')\) with globally minimal realization and equal spectral density.

We will first use the spectral factorization framework to assess the uniqueness of the solution \((Q, P)\) to this problem. Then for a class of system with a finite number of solutions, we will provide a method to construct all solutions using Lemma 6.
IV. NETWORK RECONSTRUCTION BY SPECTRAL FACTORIZATION

A. Minimum Phase G

We first consider this simpler case, the results of which will be useful later for the general case. Make the following additional assumption:

**Assumption 7.** The matrix G is minimum phase.

First we will show that a sufficient condition for G to be minimum phase is that Q is stable and P is minimum phase. Then we will see that under Assumption 7, the solution to Problem 1 is unique and the solution P unique up to sign.

**Lemma 8.** If Q is stable and P is square, diagonal and minimum phase then G is minimum phase.

**Proof.** If Q is stable, then $I - Q$ is also stable as it has the same poles. For any invertible transfer function, $z_0$ is a transmission zero if and only if it is a pole of the inverse transfer function [14]. Therefore $(I - Q)^{-1}$ is minimum phase if and only if Q is stable. If in addition P is minimum phase, then $G = (I - Q)^{-1}P$ is also minimum phase since P is diagonal.

Hence for a significant class of systems – those with stable interactions among manifest variables – that are driven by filtered noise, where the filters are minimum phase, the matrix G is minimum phase.

**Theorem 2.** Two systems $(A, B, C, D)$ and $(A', B', C', D')$ under Assumptions 7-10 with DSFs $(Q, P)$ and $(Q', P')$ have equal output spectral density:

$$\Phi(s) = G(s)G^*(s) = G'(s)G'^*(s)$$

if and only if Q = Q' and P = ±P'. Given $\Phi(s)$, there is therefore only one solution $(Q, P)$ with minimum phase G.

**Proof.** From Corollary 2 two globally minimal systems with minimum phase transfer functions have equal output spectral density if and only if they satisfy (15) for some invertible $T \in \mathbb{R}^{n \times n}$ and orthogonal $U \in \mathbb{R}^{n \times p}$. We shall derive necessary conditions for (15) to hold and show that this constrains U to the desired extent.

First, (15a) is satisfied if and only if $T = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix}$. Take $A, A', B$ and $B'$ in the canonical form of Lemmas 4 and 5 and partition them as in (9). Then a necessary condition to satisfy (15d): $B' = TBU$ is that $U$ and $T_2$ are in the following forms:

$$U = \begin{bmatrix} U_1 & 0 \\ 0 & \pm I \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & 0 & 0 \\ U_{21} & U_{22} & 0 & 0 \\ 0 & 0 & \pm I & 0 \\ 0 & 0 & 0 & \pm I \end{bmatrix}$$

(16)

where $U_1$ is orthogonal and X denotes a free element:

$$T_2 = \begin{bmatrix} U_1^* & X \\ 0 & X \end{bmatrix} = \begin{bmatrix} U_{11}^* & U_{21}^* & X & X \\ U_{12}^* & U_{22}^* & X & X \\ 0 & 0 & X & X \\ 0 & 0 & X & X \end{bmatrix}$$

(17)

From (15a) the matrix $A'_{12}$ is given as follows:

$$A'_{12} = A_{12}T_2^{-1} = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & X & X \\ 0 & 0 & X & X \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & X & X \\ U_{21} & U_{22} & X & X \\ 0 & 0 & X & X \\ 0 & 0 & 0 & X \end{bmatrix}$$

which is in canonical form (9) only if:

$$T_2 = \begin{bmatrix} \pm I & 0 & 0 & 0 \\ 0 & U_{22} & X & X \\ 0 & 0 & I & 0 \\ 0 & 0 & X & X \end{bmatrix}$$

(18)

which implies:

$$U_1 = \begin{bmatrix} \pm I & 0 \\ 0 & U_{22} \end{bmatrix}$$

(19)

for some orthogonal $U_{22}$. The matrix $A'_{22}$ is given by:

$$A'_{22} = T_2A_{22}T_2^{-1} + T_1A'_{12} = \begin{bmatrix} \ddots & 0 & X & X \\ X & X & X & X \\ 0 & \beta_{32}U_{22} & X & X \\ X & \beta_{42}U_{22} & X & X \end{bmatrix}$$

where \(\ddots\) denotes a square, diagonal element. With $\beta_{32}$ and $\beta_{42}$ in the form of (10). $A'_{22}$ is in canonical form only if $U_{22} = \pm I$.

The matrix U must therefore be equal to a signed identity in order for (15) to be satisfied. From (15), equality of spectral densities implies:

$$GU = (I - Q)^{-1}PU = (I - Q')^{-1}P' = G'$$

Inverting the above and equating diagonal elements yields $P' = \pm P$ and hence $Q' = Q$.

The network reconstruction problem for stable systems with minimum phase P therefore has a unique solution irrespective of topology. The ambiguity in the sign of $\Phi(s)$ is finite. The following lemma characterises all solutions $(Q', P')$ to Problem 1.

**Lemma 9.** Two systems with DSFs $(Q, P)$ and $(Q', P')$ under Assumptions 7-10 have equal output spectral density:

$$\Phi(s) = G(s)G^*(s) = G'(s)G'^*(s)$$

if and only if for any realizations $(A, B, \begin{bmatrix} I & 0 \end{bmatrix}, 0)$ and $(A', B', \begin{bmatrix} I & 0 \end{bmatrix}, 0)$ with B and $B'$ in the canonical form of Lemma 4 there exists invertible $T = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \in \mathbb{R}^{n \times n}$ and symmetric $S_2 \in \mathbb{R}^{h \times h}$ such that:

$$A' = T^{-1}AT$$

$$B'_1B'_1^T = B_1B_1^T$$

$$A_{12}S_2 = -B_1B'_2^T + B'_1B_2^TT_2^T + B_1B_1^TT_1^T$$

$$S_2\ddot{A} + \dddot{A}^T S_2 - S_2 \dddot{BB}^TS_2 + \dddot{Q} = 0$$
where $\hat{A}$, $\hat{B}$ and $\hat{Q}$ are defined as:

$$A := (A_{22} - B_1 B_1^T A_{12})^T$$

$$\hat{B} := (B_1^T A_{12})^T$$

$$\hat{Q} := B_3 B_3^T - T_2 B_3 B_3^T T_2^T$$

and $B_1^+$ and $B_3$ are defined as:

$$B_1^+ := \begin{bmatrix} I & 0 \\ 0 & B_{22}^{-1} \end{bmatrix}$$

$$B_3 := \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

partitioned as $B_1$ and $B_2$ in (\ref{eq:partition}) respectively.

**Proof.** The lemma is shown to be a special case of Lemma \ref{lem:algebraic} for the systems in question. Under Assumption 6, (\ref{eq:partition}) simplifies to:

$$A' = T^{-1}AT$$

$$C' = CT$$

$$SA' + AS = -BB'T + TB'B'TT'$$

$$STC' = 0$$

where $T = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix}$ is necessary and sufficient to satisfy (\ref{eq:3b}) and (\ref{eq:3a}). Equation (\ref{eq:2e}) gives $S = \begin{bmatrix} 0 & 0 \\ 0 & S_2 \end{bmatrix}$ partitioned in accordance with $C$ for some symmetric $S_2$. Equation (\ref{eq:2f}) then defines three equations:

$$B'_1 B'_T = B_1 B_1^T$$

$$A_{12} S_2 = -B_1 B'_2 + B'_1 (T_1 B'_1 + T_2 B_2)^T$$

$$S_2 A_{22} + A_{22} S_2 = (T_1 B'_1 + T_2 B_2)^T (T_1 B'_1 + T_2 B_2)^T$$

where (\ref{eq:2f}) is equal to (\ref{eq:2c}) and substitution of $B_1 B_1^T$ into (\ref{eq:2b}) shows equality with (\ref{eq:2c}). Pre-multiply (\ref{eq:2f}) by $B_1^+ := \begin{bmatrix} I & 0 \\ 0 & B_{22}^{-1} \end{bmatrix}$ and substitute into (\ref{eq:2f}) to obtain the Algebraic Riccati Equation (ARE) (\ref{eq:algebraic}) with coefficients given by (\ref{eq:coefficients}).

**Remark 2.** The number of solutions $(Q', \pm P')$ is equal to the number of solutions for $S_2$. This can be seen from Corollary \ref{cor:algebraic} and Theorem \ref{thm:algebraic} – for any solution $S_2$, all solutions $(Q', \pm P')$ are characterised as in the minimum phase case, in which case the solution is unique from Theorem \ref{thm:algebraic}. Note that the dimension of $S_2$ is $h$ – the number of hidden states; in the absence of hidden states the solution is therefore unique.

**Remark 3.** Given a system $(Q, P)$, in order to construct an equivalent system $(Q', P')$ it is necessary to find matrices $T_1$, $T_2$, $B_1'$ and $B_2'$ that satisfy both (\ref{eq:algebraic}) and Lemma \ref{lem:algebraic} which can also be formulated in terms of these parameters.

In general (\ref{eq:algebraic}) defines an infinite set of Algebraic Riccati Equations (AREs) parameterized by $T_2$, hence the number of solutions is expected to be unbounded. For a slightly less general case, in which each input affects a measured state directly, the number of solutions to Problem \ref{prob:algebraic} is finite.

**Theorem 3.** If $B_1$ is invertible, the number of solutions $(Q', \pm P')$ to Problem \ref{prob:algebraic} is finite and bounded by $2^h C$ where $h$ is the number of hidden states.

**Proof.** If $B_1$ is invertible, then the following matrices simplify: $B_1^T = B_1^{-1}$ and $B_3 = 0$. The ARE coefficients (\ref{eq:coefficients}) then only depend on the original system $(A, B)$ and (\ref{eq:algebraic}) is therefore a single ARE of order $h$. This has at most $2^h C$ solutions parameterized by combinations of the eigenvectors of its Hamiltonian [14], hence there are at most this many solutions for $S_2$. The number of solutions $(Q', \pm P')$ is equal to the number of distinct, symmetric solutions for $S_2$ from Remark 2.

Therefore, if the noise drives only the measured states directly, then the number of solutions for the network structure and dynamics is finite. Furthermore, given any one solution, Lemma \ref{lem:algebraic} allows all possible solutions to be constructed.

**C. An Algorithm**

If $B_1$ is invertible, then the number of solutions $(Q, \pm P)$ to Problem \ref{prob:algebraic} is finite. Lemma \ref{lem:algebraic} may then be used to construct all solutions $Q$ and $\pm P$ to Problem \ref{prob:algebraic} from data as follows.

1. Estimate $\Phi(s)$
2. Construct positive real $Z(s)$ such that: $Z + Z^* = \Phi$
3. Make a minimal realization of $Z$ of the form: $(A, B, [I \ 0], 0)$
4. Find all solutions to the following equations for symmetric $R > 0$ and $B'$:

$$RA'T + AR = -B'B'TT'$$

$$RC'T = B$$

5. For each solution, $B'$, the system $(A, B', [I \ 0], 0)$ with transfer function $G'$ is minimal and has spectral density $\Phi$

6. Apply the state transformation $T = \begin{bmatrix} I & 0 \\ -B_1 B_1^{-1} & I \end{bmatrix}$ to obtain system $(A'', B'', [I \ 0], 0)$, which has the same transfer function $G''$, spectral density $\Phi$ and has $P''$ diagonal. The system $(Q'', P'')$ is then a solution to Problem \ref{prob:algebraic}.

Step 1 may be achieved by standard methods, the details of which are not considered here. Every spectral density matrix has a decomposition of the form of step 2, as described in [12] and Step 3 is always possible. Step 4 necessarily has a finite number of solutions obtainable as follows.

First note that (\ref{eq:algebraic}) is obtained from (\ref{eq:algebraic}) under Assumption 5. Then (\ref{eq:algebraic}) gives:

$$R = \begin{bmatrix} B_1 & B_2^T \\ B_2 & R_2 \end{bmatrix}$$

for some symmetric $R_2$. Equation (\ref{eq:algebraic}) then defines three equations:

$$B_1 B_1^T = -B_1 A_{11}^T - B_2^T A_{12}^T - A_{11} B_1 - A_{12} B_2$$

$$R_2 A_{22}^T + A_{23} R_2 + B_2 A_{21}^T + A_{21} B_2^T + B_2^T B_2^T = 0$$

$$R_2 A_{12} = -B_2 A_{11} - A_{21} B_1 - A_{22} B_2 - B_2^T B_2^T$$
Both sides of (27a) must be diagonal, since a solution with \( B_1' \) diagonal is known to exist; hence \( B_1' \) can be found up to a sign. For any such symmetric \( B_1' \) that is full rank, \( B_2' \) can be eliminated from (27b) using (27c), yielding the following ARE:

\[
R_2 \tilde{A} + \tilde{A}^T R_2 + R_2 \tilde{B} \tilde{B}^T R_2 + \tilde{Q} = 0
\]

with:

\[
\tilde{A} = (A_{22} + F B_1'^{-2} A_{12})^T
\]

\[
\tilde{B} = (B_1'^{-1} A_{12})^T
\]

\[
\tilde{Q} = B_2 A_{21}' + A_{21} B_2' + F F^T
\]

and \( F = B_2 A_{21}' + A_{21} B_2 + A_{22} B_2' \). Since the parameters of (28) are known, we can compute all solutions \( R_2 \). For every such symmetric \( R_2 \) with \( R > 0 \), the matrix \( B_2' \) is given uniquely by (27c) as:

\[
B_2' = - (R_2 A_{21}' + A_{21} B_2 + A_{22} B_2') B_1'^{-1}
\]

The system \( (A, B', I, 0, 0) \) therefore has spectral density \( \Phi \) but will in general not have \( P' \) diagonal. Since \( B_1' \) is full rank, we can always transform by \( T = \begin{bmatrix} I & 0 \\ -B_2' B_1'^{-1} & I \end{bmatrix} \), resulting in \( B_2'' = 0 \) and hence \( P'' \) diagonal from Lemma 2. Since state transformations do not affect \( \Phi \), the system \( (Q'', P'') \) is a solution to Problem 1. This method is demonstrated by the following example, in which all solutions are constructed for a particular \( \Phi \).

**Example 2.** Start with the output spectral density \( \Phi(s) \) for the system of Example 1 and from it construct the positive real matrix \( Z(s) \), such that \( Z(s) + Z^*(s) = \Phi(s) \):

\[
Z(s) = \begin{bmatrix} 0.17(s+1) & 0.032(s+19) \\ 0.032(s+8.2)(s-17.2) & 0.578^2 + 2.9s + 15.1 \\ (s+2)(s^2+4s+27) & (s+2)(s^2+4s+27) \end{bmatrix}
\]

where all numbers are shown to two significant figures. Construct any minimal realization of \( Z \) by standard methods, such as:

\[
A = \begin{bmatrix} -3.9 & -0.97 & 1.9 \\ -3.6 & -3.2 & 2.4 \\ -15.5 & -1.5 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.17 & 0.032 \\ 0.032 & 0.578 \\ 0.092 & 0.60 \end{bmatrix},
\]

with \( C = [I \ 0] \) and \( D = 0 \). First solve for \( B_1' \) as in (27a):

\[
B_1' = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}
\]

and choose the signs to be positive for simplicity. Next construct and solve the ARE (28), which has the following two solutions:

\[
R_2 = 1.02 \quad \text{and} \quad 1.65
\]

In each case, solve for \( B_2' \) using (30):

\[
B_2' = \begin{bmatrix} 1.49 & 0.5 \\ 0.28 & -1.01 \end{bmatrix}
\]

and transform both systems by \( T = \begin{bmatrix} I & 0 \\ -B_2' B_1'^{-1} & I \end{bmatrix} \) to yield two systems with DSFs with \( P \) diagonal. The first corresponds to the system of Example 1 with DSF:

\[
Q(s) = \begin{bmatrix} 0 & 0 \\ \frac{s+3}{s^2+4s+27} & 0 \end{bmatrix}, \quad P(s) = \begin{bmatrix} \frac{s^2+0.34s+23.3}{s^2+2.659s+1.3} & 0 \\ 0 & s-3 \end{bmatrix}
\]

and the second to the following, stable, minimal system:

\[
A' = \begin{bmatrix} -3.3 & -2.9 & 4 \\ -2.9 & -5.7 & 5 \\ -8.3 & -3.7 & 3 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},
\]

with \( C' = [I \ 0] \), \( D' = 0 \) and DSF:

\[
Q'(s) = \begin{bmatrix} 0 \\ \frac{-2.9(s+2.0)}{s^2+2.659s+1.3} \end{bmatrix},
\]

\[
P'(s) = \begin{bmatrix} \frac{s-3}{s^2+0.34s+23.3} & 0 \\ 0 & \frac{s^2+0.34s+23.3}{s^2+2.659s+1.3} \end{bmatrix}
\]

Note that this system has a different network structure for both state-space (a) and DSF (b):

![Network diagrams](image)

The reader may verify that these systems do indeed have the same output spectral density \( \Phi(s) \). It may also be verified that all four equations (29) are satisfied for the following matrices \( S \) and \( T \):

\[
S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -0.15 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -0.59 & -0.73 \end{bmatrix}
\]

The transfer matrix for the second system is given by:

\[
G'(s) = \begin{bmatrix} \frac{s+0.66}{s^2+4s+27} & \frac{-2.9}{(s+11.1)(s+13)} \\ \frac{-2.9}{(s+11.1)(s+13)} & \frac{s^2+4s+27}{(s+11.1)(s+13)} \end{bmatrix}
\]

which (from Theorem 2) is necessarily non-minimum phase – it has a transmission zero at \( s = 3 \). Hence if we restrict ourselves to minimum phase transfer functions, the solution for \( (Q, \pm P) \) is unique. In this example these are the only two solutions for \( Q \) to Problem 1; additional solutions for \( P \) may be obtained by changing the signs of \( B_1' \) in (31).

**V. Conclusion**

The problem of identifying the structure and dynamics of an unknown network driven by unknown intrinsic noise has been considered. For LTI systems with standard assumptions on the noise, this problem was shown to be a variant of the spectral factorization problem with solutions parameterized by the solutions of a set of Algebraic Riccati Equations. Solvability of this problem, irrespective of method, was then assessed for networks with arbitrary topology. Important results for two broad classes of system were presented: in the first the solution is unique, and in the second the number of solutions is finite and an algorithm presented to construct them, given the spectral density. We are currently investigating this problem for more general classes of system and for a more general noise model.
A. Proof of Lemma 4

Proof. A. Proof of Lemma 4

\[ B'_2 = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \] (32)

which is always invertible, transforms \( B_2 \) to:

\[ B'_2 = \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} \]

B. Proof of Lemma 5

Proof. The proof is given for \( B_{12} = 0 \) in (38) for the sake of clarity, in which case \( B_1 \) and \( B_2 \) may be written as:

\[ B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_{33} \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (33)

partitioned as \( (9) \), where, abusing notation, \( b_{33} \) and \( b_{44} \) are square and full rank and comprise \( b_{33} \) in (38). The proof holds in the general case but requires further partitioning.

We first define state transformations that preserve the structure of \( B \), then use these and Lemma 2 to prove the result. Now \((Q,P)\)-invariant state transformations with:

\[ T = \begin{bmatrix} I \\ 0 \\ T_2 \end{bmatrix} \]

\[ T_2 = \begin{bmatrix} I & X \\ X & 0 \\ 0 & X \\ 0 & 0 & X \\ 0 & 0 & 0 \end{bmatrix} \] (34)

partitioned as \( B_2 \) above, are also \( B \)-invariant. From Lemma 2 \( A_{12}B_2 \) must be diagonal, which proves the first two columns of \( A_{12} \) for some ordering of measured states. Note that the first two rows of \( A_{12} \) must be full (row) rank in order that \( V \) is. Hence we can apply a transformation of the form (33) to transform \( A_{12} \) to:

\[ A_{12} = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & X & X \\ 0 & 0 & 0 & X \end{bmatrix} \] (35)

Any further \((Q,P)\)-invariant state transformation with \( T_2 \) as:

\[ T_2 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & X \\ 0 & 0 & I \\ 0 & 0 & X \end{bmatrix} \] (36)

will preserve the form of \( A_{12} \) in (35).

From Lemma 2, \( A_{12}A_{22}B_2 \) must also be diagonal and hence \( A_{22} \) must have the following form:

\[ A_{22} = \begin{bmatrix} \beta_{11} & 0 & X & X' \\ X & X & X \\ 0 & \beta_{32} & X & X \\ X & \beta_{42} & X & X \end{bmatrix} \] (37)

where \( \beta_{11} \) and \( \beta_{32} \) are both square and diagonal, but not necessarily full rank. If \( \beta_{32} = 0 \), it is straightforward to show (for example by constructing the transfer function from elements of \( A_{22} \) that \( \beta_{32} \) must be full column rank in order for block \((2,2)\) of \( V \) to be full rank. For a general \( \beta_{32} \), by application of a particular matrix transformation \( T_2 \) of the form (36), we obtain (10).