The Möbius Function on Affine Grassmannian elements

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Abstract

To any saturated chain in the affine Weyl group whose translation parts are sufficiently regular, we associate a near path and a far path in the quantum Bruhat graph. Using this, working in the Bruhat order on the minimal-length representatives of the cosets in the affine Weyl group with respect to the finite Weyl group, we characterize the pairs of elements for which the Möbius function is nonzero. This is applied to obtain explicit expansions in the $K$-theory of affine Grassmannians, of the basis of ideal sheaves into the basis of structure sheaves of Schubert varieties.

1 Introduction

1.1 Möbius Function on Affine Grassmannian elements

Let $W$ be the Weyl group of a root system of untwisted affine type or the dual of untwisted affine type. For ease of exposition we shall work with untwisted affine type, leaving the statements of the dual case to §4. The group $W$ is generated by simple reflections $s_i$ for $i$ in the affine Dynkin node set $I$, with distinguished affine node $0 \in I$. Say that $w \in W$ is affine Grassmannian (denoted $w \in W_0$) if $ws_i \succ w$ for all $i \in I_0 = I \setminus \{0\}$. Let $\tilde{\mu}$ be the Möbius function of the Bruhat order on $W_0$. The following is the affine Grassmannian case of Deodhar’s general formula [3] for the Möbius function of a parabolic quotient of the Bruhat order. For $u, v \in W$ with $u \preceq v$, denote by $[u, v] = \{z \in W \mid u \preceq z \preceq v\}$ the interval between $u$ and $v$ in the Bruhat order on $W$.

Theorem 1 ([3, Theorem 1.2]). For $u, v \in W_0$ with $u \preceq v$,

$$\tilde{\mu}(u, v) = \begin{cases} (-1)^{\ell(v) - \ell(u)} & \text{if } [u, v] \subset W_0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Moreover the first case holds if and only if there is no $i \in I_0$ such that $us_i \preceq v$.

For $y \in W_0$ let $B_y = \{x \in W_0 \mid [x, y] \subset W_0\}$. Our main theorem (Theorem 3) is an explicit description of the subposet $B_y$ of $W_0$ in the case that $y \in W_0$ is
superregular, that is, its translation part is sufficiently regular. For superregular 
y, we show that $B_y$ bijects with the classical Weyl group $W_0$. Moreover, all of 
the possible poset structures on $W_0$ which arise this way, come from a single 
directed graph structure $\Gamma$ on $W_0$. This directed graph, first studied by Brenti, 
Fomin, and Postnikov [2], has become known as the quantum Bruhat graph 
due to its connection with quantum cohomology of flag varieties; it indicates 
which Schubert classes appear in the quantum product of a Schubert class with 
a Schubert divisor.

### 1.2 Quantum Bruhat Graph

For a positive classical root $\alpha \in \Phi_0^+$ let $r_\alpha \in W_0$ be the associated reflection and 
$\alpha^\vee$ the associated coroot. Let $\rho = (1/2) \sum_{\alpha \in \Phi_0^+} \alpha$.

The quantum Bruhat graph (QBG) [2] is the directed graph $\Gamma$ with vertex 
set $W_0$ and a directed edge $w \to wr_\alpha$ for each pair $(w, \alpha) \in W_0 \times \Phi_0^+$ such that 
one of the following holds.

1. $\ell(wr_\alpha) = \ell(w) + 1$. Such edges are called Bruhat edges.

2. $\ell(wr_\alpha) = \ell(w) - \langle \alpha^\vee, 2\rho \rangle + 1$. Such edges are called quantum edges.

The label of the edge $w \to wr_\alpha$ is $\alpha \in \Phi_0^+$. The weight of the edge $w \to wr_\alpha$ in $\Gamma$ is $\alpha^\vee$ if the edge is quantum and 0 if it is Bruhat. The weight of a path $P$, denoted $\text{wt}(P)$, is the sum of the weights of its edges. For any $u, v \in W_0$, let $M(u, v)$ be the weight of any shortest path in $\Gamma$ from $u$ to $v$; this is well defined by [8, Lemma 1].

**Example 2.** The quantum Bruhat graph of type $A_2$ is pictured in Figure 1. Let $w_0 = s_1 s_2 s_1$ and $\alpha^\vee_2 = \alpha^\vee_1 + \alpha^\vee_2$. The quantum edges are given in red and their weights are indicated. We have $M(w_0, s_1) = \alpha^\vee_1 + \alpha^\vee_2$, which is realized by either of the shortest paths $w_0 \to \text{id} \to s_1$ or $w_0 \to s_1 s_2 \to s_1$.

### 1.3 Main Theorem

An element $\lambda$ of the classical coroot lattice $Q^\vee$ is superregular if $|\langle \lambda, \alpha_i \rangle| \gg 0^2$ 
for all $i \in I_0$. Any element $x \in W$ can be written uniquely in the form $x = wt_\lambda$ 
with $w \in W_0$ and $\lambda \in Q^\vee$ where $t_\lambda \in W$ is the translation element. For $w \in W_0$ 
we define the affine Weyl group element $wt_\lambda$ to be superregular if $\lambda$ is.

**Theorem 3.** Let $x, y \in W^0$ with $x = w't_{\lambda'}$, $y = wt_\lambda$ with $w, w' \in W_0$, $\lambda, \lambda' \in Q^\vee$ and $\lambda$ superregular. Then

$$
\tilde{\mu}(x, y) = \begin{cases} 
-1^{\ell(y) - \ell(x)} & \text{if } \lambda' = \lambda + M(w, w') \\
0 & \text{else}
\end{cases}
$$

where $M$ is the weight of a shortest path in $\Gamma$ from $w$ to $w'$.

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1. All our paths are directed.
2. Lower bounds for superregularity are discussed in §3.
Corollary 4. Let $y = wt_\lambda \in W^0$ be superregular. Then the set

$$B_y = \{ x \in W^0 \mid x \leq y \text{ and } \bar{\mu}(x, y) \neq 0 \}$$

is given by

$$\{ u_{t_\lambda + M(u, u)} \mid u \in W_0 \}.$$  \hfill (2)

Remark 5. The poset structure on $B_y$ can be constructed from the shortest paths from $w$ to all the other vertices in $\Gamma$. This poset structure is given by the $w$-tilted Bruhat order $\preceq_w$ of [2], which is defined by $u \preceq_w v$ if $u$ is on some shortest path in $\Gamma$ from $w$ to $v$.

1.4 Application to the $K$-theory of the affine Grassmannian

Let $\mathcal{O} = \mathbb{C}[t]$ be the ring of formal power series with coefficients in $\mathbb{C}$, $\mathcal{K} = \mathbb{C}[[t]]$ the field of formal Laurent series, $G$ a simple Lie group over $\mathbb{C}$, $Gr_G = G(\mathcal{K})/G(\mathcal{O})$ the affine Grassmannian, $I$ the Iwahori subgroup of the affine Kac-Moody group. For $x \in W^0$ let $C_x = L x G(\mathcal{O})/G(\mathcal{O}) \subset Gr_G$ be the affine Grassmannian Schubert cell and $X_x = \overline{C_x} \subset Gr_G$ the Schubert variety. For $x \in W^0$ let $O_x$ and $I_x$ be the elements of $K(Gr_G)$ given by the $K$-classes of the structure sheaf of $X_x$ and the ideal sheaf of the boundary $\partial X_x$.

The following is due to Kumar [5], who produced the proof (for Kac-Moody partial flag varieties) upon being asked about it by the second author.

Proposition 6 ([5]). For any $y \in W^0$

$$O_y = \sum_{x \in W^0 \atop x \leq y} I_x.$$ \hfill (3)

By the definition of the Möbius function and Theorem 3 we deduce:
Corollary 7. Let \( y = wt_\lambda \in W^0 \) be superregular. Then

\[
I_y = \sum_{u \in W_0} (-1)^{\ell(w,u)} O_{wt_\lambda + M(w,u)}
\]

where \( \ell(w,u) \) is the length of a shortest path from \( w \) to \( u \) in \( \Gamma \).

2 Near and far paths in \( \Gamma \) associated to superregular saturated chains in \( W \)

2.1 Sufficiently Regular Bruhat covers

Say that \( \lambda \in Q^\vee \) is sufficiently regular if \( | \langle \lambda, \alpha_i \rangle | \gg 0 \) for all \( i \in I_0 \) and that \( y = wt_\lambda \in W \) is sufficiently regular if \( \lambda \) is\(^3\). The question of sufficient regularity to qualify will be addressed in §3. Say that \( \lambda \in Q^\vee \) is antidominant if \( \langle \lambda, \alpha_i \rangle \leq 0 \) for all \( i \in I_0 \). Let \( \tilde{Q} \) be the set of antidominant elements in \( Q^\vee \).

Proposition 8 ([6, Proposition 4.1]). Let \( \lambda \in \tilde{Q} \) be sufficiently regular and \( y = wt_\lambda \) where \( w, v \in W_0 \). Then \( x = yr_{\alpha+\rho} \leq y \) for some \( \alpha \in \Phi_0^\vee \) and \( n \in \mathbb{Z} \) if and only if one of the following conditions hold:

1. \( \ell(wv) = \ell(wvr_\alpha) - 1 \) and \( n = \langle \lambda, \alpha \rangle \), giving \( x = wr_\alpha t_v \).
2. \( \ell(wv) = \ell(wvr_\alpha) + \langle \alpha^\vee, 2\rho \rangle - 1 \) and \( n = \langle \lambda, \alpha \rangle + 1 \), giving \( x = wr_\alpha t_v(\lambda + \alpha^\vee) \).
3. \( \ell(v) = \ell(vr_\alpha) + 1 \) and \( n = 0 \), giving \( x = wr_\alpha t_{vr_\alpha}(\lambda) \).
4. \( \ell(v) = \ell(vr_\alpha) - \langle \alpha^\vee, 2\rho \rangle + 1 \) and \( n = -1 \), giving \( x = wr_\alpha t_{vr_\alpha}(\lambda + \alpha^\vee) \).

That is, every sufficiently regular cover in the Bruhat order on \( W \) has a corresponding edge in \( \Gamma \). In the terminology introduced in [6], the first two cases are called near, since the chamber of the translation component remains fixed, and the last two cases far.

Corollary 9. The four possibilities in the previous Proposition may be replaced by the following two:

1. \( wv \rightarrow wvr_\alpha \) is an edge \( E \) in \( \Gamma \) and \( x = wr_\alpha t_v(\lambda + \text{wt}(E)) \). We call this a near covering.
2. \( vr_\alpha \rightarrow v \) is an edge \( E \) in \( \Gamma \) and \( x = wr_\alpha t_{vr_\alpha}(\lambda + \text{wt}(E)) \). We call this a far covering.

\(^3\)This definition follows the definition of superregular given in [6]. In this paper, superregularity has a different requirement.
2.2 Superregular saturated Bruhat chains

We extend this idea to saturated chains in the Bruhat order. We say that $y = \text{wt}_\lambda \in W$ is superregular with respect to $x$ if for every saturated chain $x < z_{k-1} < \cdots < z_1 < y$ it is true that $y$ and $z_i$ for $i = 1, \ldots, k - 1$ are all sufficiently regular. We say a saturated chain $C$ is superregular if its top element is superregular with respect to its bottom element. Suppose that

$$x \overset{\beta_k}{\rightarrow} z_{k-1} \overset{\beta_{k-1}}{\rightarrow} \cdots \overset{\beta_2}{\rightarrow} z_1 \overset{\beta_1}{\rightarrow} y$$

is a superregular saturated chain in $W$, where $\beta_i \in \Phi_0^+$ is the root corresponding to $\alpha$ in Proposition 8 which labels the corresponding edge in $\Gamma$. Let $1 \leq f_1 < f_2 < \cdots < f_i \leq k$ be the ordered subindices relating to far coverings and $1 \leq n_1 < n_2 < \cdots < n_j \leq k$ be the ordered subindices relating to near coverings. Define the far product $r_f = r_{\beta_{f_1}}r_{\beta_{f_2}}\cdots r_{\beta_i}$ and near product $r_n = r_{\beta_{n_1}}r_{\beta_{n_2}}\cdots r_{\beta_{n_j}}$.

**Lemma 10.** Suppose $x < y = \text{wt}_\lambda$ with $\lambda \in \tilde{Q}$ and superregular chain as in Equation (4). Let $r_f$ and $r_n$ be defined as above. Then there exist well defined paths $P_f : vr_f \rightarrow v$ and $P_n : wv \rightarrow wvr_n$ such that

$$x = wvr_n(vr_f)^{-1}vr_f(\lambda + \text{wt}(P_n) + \text{wt}(P_f))$$

**Example 11.** Let 1 denote the identity element of $W_0$. Consider the following superregular saturated chain in type $A_2^{(1)}$.

$$x = s_2s_1s_2s_1(-2\alpha_1' - 3\alpha_2')^{\alpha_1 + \alpha_2}$$

$$z_2 = s_1s_2t_{-3\alpha_1'} - 4\alpha_2'

\begin{align*}
z_1 &= s_1s_2s_1t_{-4\alpha_1'} - 4\alpha_2' \\
y &= s_1s_2t_{-4\alpha_1'} - 4\alpha_2'
\end{align*}$$

We observe that $x < z_2$ is a far covering that bijects with the quantum edge $s_1s_2s_1 \rightarrow 1$ with path weight $\alpha_1' + \alpha_2'$. $z_2 < z_1$ is a near covering that bijects with the quantum edge $s_1s_2s_1 \rightarrow s_1s_2$ with path weight $\alpha_1'$. $z_1 < y$ is a near covering that bijects with the Bruhat edge $s_1s_2 \rightarrow s_1s_2s_1$ with path weight 0.

In terms of Lemma 10, this implies that

- $w = s_1s_2$, $v = 1$, and $\lambda = -4\alpha_1' - 4\alpha_2'$
- $r_f = s_1s_2s_1$ and $r_n = (s_1)(s_1) = 1$
- $P_f : s_1s_2s_1 \rightarrow 1$ with $\text{wt}(P_f) = \alpha_1' + \alpha_2'$
- $P_n : s_1s_2 \rightarrow s_1s_2$ with $\text{wt}(P_n) = \alpha_1'$
We find $x$ can be expressed in these terms.

$$x = wvr_n(vr_f)^{-1}r_{vr_f}^{-1}(vr_f)^{-1}(vr_f) r_{vr_f}^{-1}(vr_f)^{-1} = wvr_n r_{vr_f}(vr_f)^{-1}$$

If $x < z_k$ by a near covering, then $r' = r_f$, $r'_n r_{vr_f} = r_n$, and $P'_n = P_f$. The covering yields an edge $E$ from $wvr_n(vr_f)^{-1}(vr_f)^{-1} = wvr_n r_{vr_f} = wvr_n$ and shows that $P'_n + E = P_n$ and that $wt(P'_n) + wt(E) = wt(P_n)$. Therefore

$$x = wvr_n r_{vr_f}(vr_f)^{-1}t_{vr_f}(\lambda + wt(P_n) + wt(E))$$

If $x < z_k$ by a far edge, then $r' = r_f$, $r'_n r_{vr_f} = r_f$, and $P'_n = P_n$. The covering yields an edge $E$ from $vr_f r_{vr_f} = vr_f$ and shows that $E + P'_n = P_f$ and $wt(P'_n) + wt(E) = wt(P_f)$. Therefore

$$x = wvr_n r_{vr_f}(vr_f)^{-1}t_{vr_f}(\lambda + wt(P_n) + wt(E))$$

So by induction, the result is true for all superregular saturated chains.

**Proof.** We proceed by induction. The base case for $k = 1$ is given explicitly by Corollary 9: the near case yields $P_f$ as trivial while $P_n$ is the edge described, and the far case yields $P_n$ as trivial while $P_f$ is the edge described. Suppose the result is true for any fixed length difference $k \geq 1$ and that $\ell(y) - \ell(x) = k+1$. By induction, there are elements $r'_{n}, r'_{f}$ and associated paths $P'_n : wvr \rightarrow wvr_{n}', P'_f : vr_f \rightarrow v$ such that $z_k = wvr_{n}'(vr_f)^{-1}t_{vr_f}(\lambda + wt(P'_n) + wt(P'_f))$. Let $\beta = \beta_{k+1}$. We will perform cases based on whether the covering $x \preccurlyeq z_k$ is a near or far covering. Note that in either case, we observe the following is the resulting classical Weyl induction, there are elements $r'_{n}, r'_{f}$ and associated paths $P'_n : wvr \rightarrow wvr_{n}', P'_f : vr_f \rightarrow v$ such that $z_k = wvr_{n}'(vr_f)^{-1}t_{vr_f}(\lambda + wt(P'_n) + wt(P'_f))$. Let $\beta = \beta_{k+1}$. We will perform cases based on whether the covering $x \preccurlyeq z_k$ is a near or far covering. Note that in either case, we observe the following is the resulting classical Weyl component of $x$:

$$wvr_{n}'(vr_f)^{-1}r_{vr_f}(\beta) = wvr_{n}'(vr_f)^{-1}(vr_f)^{-1}r_{vr_f}(vr_f)^{-1} = wvr_{n}'r_{vr_f}(vr_f)^{-1}$$

For a superregular saturated chain (4), we call $P_f$ the associated far path and $P_n$ the associated near path.

The following Corollary is the specialization of Lemma 10 to $W^0$.

**Corollary.** Suppose $x, y \in W^0$, $x < y = wt_{\lambda}$ and let (4) be a superregular saturated chain where $\beta_{i}, r_{f}, r_{n}, P_f, P_n$ are defined as in Lemma 10. Then $P_f$ goes from 1 to 1, $P_n$ goes from $w$ to $wr_n$, and

$$x = wvr_{n}t_{\lambda + wt(P_n) + wt(P_f)}$$
Proof. Since \( y \in W^0 \), \( v = 1 \) in Lemma 10. Similarly, since \( x \in W^0 \), \( vr_f = r_f = 1 \). Therefore \( P_f \) is a closed loop at the identity. The proof is complete. \( \square \)

2.3 Loops of length 2 in \( \Gamma \)

We note two facts about the Bruhat order.

**Theorem 13 ([1, Chain Property]).** If \( u < w \) and \( u, w \in W^0 \), then there exists a saturated chain \( u = w_0 < w_1 < w_2 < \cdots < w_k = w \) with \( w_i \in W^0 \).

**Lemma 14 ([1, Lemma 2.7.3]).** If \( x < y \) in \( \Gamma \) with \( \ell(y) = \ell(x) + 2 \), then \( [x, y] = \{x, u, z, y\} \) consists of four distinct elements with \( x < u < y \) and \( x < z < y \).

We observe that any loop of length 2 in \( \Gamma \) has a simple root as the label for both of its edges; apply both quantum and Bruhat edge conditions to \( \alpha \).

**Lemma 15.** Let \( x, y \in W^0 \) with \( y \) superregular, \( \ell(y) = \ell(x) + 2 \) and \( x < y \). Let \( z \in W^0 \) be such that \( x < z < y \) is the chain guaranteed to exist by Theorem 13. Then \( [x, y] \not\in W^0 \) if and only if the near path associated to \( x < z < y \) is a 2-loop.

**Proof.** Let \( y = wt_\alpha \). Suppose \( u \not\in W^0 \). Then by Corollary 12, \( x < u < y \) associates to a trivial near path, a far path that is a loop of length 2 at the identity with weight \( \alpha^\vee \) for some \( \alpha \in \Delta_0 \), \( u = ws_\alpha t_{s_\alpha(\lambda)} \), and \( x = wt_{\lambda + \alpha^\vee} \). Thus by Corollary 12, the associated near path for \( x < z < y \) is a loop of length 2 at \( w \). Now suppose the near path associated to \( x < z < y \) is a loop \( P_\alpha : w \rightarrow ws_\alpha \rightarrow w (\alpha \in \Delta_0) \). Then \( x = wt_{\lambda + \alpha^\vee} \) by Corollary 12. If \( u \in W^0 \), this and Lemma 12 would imply a different \( \beta^\vee \) such that

\[
wt_{\lambda + \beta^\vee} = x = wt_{\lambda + \alpha^\vee} \Rightarrow \alpha = \beta
\]

But this contradicts \( u \neq z \). \( \square \)

**Example 16.** In Example 11, we find that \( z_1 < z_2 < y \) corresponds with a near path that is a 2-loop at \( s_1s_2 \) and \( z_1, z_2, y \in W^0 \). In this case, \( u = s_1s_2s_1 t_{s_1(-3\sigma)^{-1}} \) satisfies \( u \not\in W^0 \) and \( u \in [z_1, y] \).

**Remark 17.** It is always the case that if the near path 2-loop from Lemma 15 utilizes the simple root \( \alpha \), then the associated far path which induces \( u \) is the far path is the loop \( 1 \rightarrow r_\alpha \rightarrow 1 \).

2.4 Interval equivalent paths

We now discuss and extend properties of paths in \( \Gamma \) uncovered by Postnikov [8]. Following Dyer [4], a reflection ordering is equivalent to a total order on \( \Phi_0^+ \) such that for all \( \alpha, \beta \in \Delta_0 \) with \( \alpha < \beta \) and \( \alpha + \beta \in \Phi_0^+ \), then \( \alpha < \alpha + \beta < \beta \). Given a reflection ordering on \( \Phi_0^+ \), then the following is true.

**Lemma 18 (Lemma 6.7 [2]).** If \( a \xrightarrow{\alpha_1} x \xrightarrow{\beta_1} c \) is a path in \( \Gamma \) with \( \alpha_1 > \beta_1 \), then there exists a unique \( y \in W_0 \) such that \( a \xrightarrow{\alpha_2} y \xrightarrow{\beta_2} c \) where \( \beta_1 < \beta_2 > \alpha_2 < \alpha_1 \).
In essence, any descent in the label sequence of a path can be uniquely exchanged for an ascent without changing the endpoints of the path. Note that the reversed ordering of a reflection ordering (α < β in the reversed ordering if and only if β < α in the original reflection ordering) is itself a reflection ordering. By applying Lemma 18 to the reversed ordering, we find that we may also uniquely exchange any descent in the label sequence of a path with an ascent.

Given a path P, we can apply these swaps to sort the label sequence of a path. Pairing this information with [2, Theorem 6.4], we obtain two key facts.

Corollary 19 ([8, proof of Lemma 1], [2]). Suppose P : w → w′ is a non-minimal path in Γ.

1. P is interval equivalent to a path containing a loop of length 2.
2. wt(P) − M(w, w′) is a positive sum of coroots.

We say that two paths in Γ are interval equivalent if the paths can be obtained from each other by a series of exchanges using Lemma 18. In Postnikov’s proof of [8, Lemma 1], he notes Lemma 18 also preserves the weight of the path.

Lemma 20. Let C be a superregular saturated chain from x to y which utilizes only near (resp. far) coverings, P be the associated near (resp. far) path, and P′ be interval equivalent to P. Then there is a unique superregular saturated chain C′ from x to y which utilizes only near (resp. far) coverings with corresponding near (resp. far) path P′.

Proof. By Proposition 8, each edge of P bijects to a covering in C. By Lemma 10, the exact element x can be found using the weight and endpoint of P. If P′ is interval equivalent, then P′ shares the same endpoint, length, and weight of P. Thus, if we biject the edges of P′ with near (resp. far) coverings then Lemma 10 states that the lower element from the resulting saturated chain C′ must be x as well.

This previous lemma is sufficient to prove Theorem 3. However, it can be strengthened by the following proposition.

Proposition 21. Let C be a superregular saturated chain from x to y, Pn be the associated near path, and Pf be the associated far path. If Pn is interval equivalent to P′n, and Pf is interval equivalent to P′f, then

1. there exists a saturated chain C′ from x to y with associated near and far paths P′n and P′f.
2. any saturated chain C′ with upper element y, associated near path P′n, and associated far path P′f has lower element x.

Proof. Note that the proof by induction of Lemma 10 shows that each near covering only modifies r_n and P_n while leaving the far components fixed. Similarly, each far covering only modifies r_f and P_f. Thus, having a near covering then
from implies there exists a saturated chain \( C_1 \) given by Lemma 20 from \( z \) to \( y \) generated by \( P'_n \), then append the saturated chain \( C_2 \) from \( x \) to \( z \) given by \( P'_f \). □

**Example 22.** Consider the sub-chain \( x \xrightarrow{\alpha_1+\alpha_2} z_2 \xrightarrow{\alpha_1} z_1 \) from Example 11. The left covering was a far covering corresponding to the quantum edge \( s_1s_2s_1 \rightarrow 1 \) while the right covering was a near covering corresponding to the quantum edge \( s_1s_2s_1 \rightarrow s_1s_2 \). If instead we used the quantum edge \( s_1s_2s_1 \rightarrow 1 \) of a far covering with upper element \( z_1 \), we generate the element \( z_3 = t_{s_1s_2s_1(-3\alpha_1'−3\alpha_2')} \). We also observe that \( x \xrightarrow{\alpha_1} z_3 \) by a near covering. This demonstrates that changing the order of near and far covers with respect to each other results in a chain still contained in the interval: \( z_3 \in [x, z_1] \).

### 2.5 Proof of Theorem 3

Note that proving Theorem 3 is equivalent to showing that for \( x = w't_X, y = wt_\lambda \in W^0 \) and \( y \) superregular with respect to \( x \), then \( [x, y] \subset W^0 \) if and only if \( \lambda' = \lambda + M(w, w') \).

Suppose \( [x, y] \subset W^0 \). Theorem 13 implies there exists a saturated chain in \( W^0 \) from \( x \) to \( y \), which by Corollary 12 implies \( x = w't_{\lambda + wt(P_n)} \) for the associated near path. If \( P_n \) has non-minimal length, then by Corollary 19 part 1, there exists a \( P'_n \) which is interval equivalent to \( P_n \) but contains a loop of length 2. However, Lemma 15 implies the induced saturated chain in \( [x, y] \) is not contained in \( W^0 \), which is a contradiction. Thus \( P_n \) has minimal length, i.e. \( wt(P_n) = M(w, w') \).

Suppose \( \lambda' = \lambda + M(w, w') \). If \( [x, y] \not\subset W^0 \), then there exists a saturated chain from \( x \) to \( y \) which utilizes far coverings. Corollary 12 supplies that \( \lambda' = \lambda + wt(P_n) + wt(P_f) \) for the associated near and far paths. \( P_f \) must be a non-empty loop at the identity and \( P_n : w \rightarrow w' \). But this implies \( wt(P_n) - M(w, w') = -wt(P_f) \) with \( wt(P_f) \neq 0 \) since \( P_f \) is a non-empty loop, which is a contradiction to Corollary 19 part 2. □

### 3 Bounds on Regularity

Initially, the definition of sufficiently regular in [6] left a large bound of \( |\langle \lambda, \alpha \rangle| > 2|W_0| + 2 \) for all \( \alpha \in \Delta_0 \). Later, in [7, Proposition 4.2], it was found that one only requires \( |\langle \lambda, \alpha \rangle| \geq 2\ell(w_0) \) if \( W_0 \neq G_2 \) (change to \( 3\ell(w_0) \) in the case of \( G_2 \)) for all such \( \alpha \in \Delta_0 \). Recently in [9], it was shown that Proposition 8 holds when \( W_0 \) is simply laced and \( |\langle \lambda, \alpha \rangle| \geq 3 \) for all \( \alpha \in \Delta_0 \). Furthermore, this hints at a similar, smaller bound for the non-simply laced cases using a similar proof to
For the following discussion, suppose the regularity bound for Proposition 8 is $k$.

We can say that $y$ is superregular with respect to $x$ if we are able to use Proposition 8 for every covering between $x$ and $y$ in a saturated chain. If $\ell(y) - \ell(x) = m$, then we must use Proposition 8 $(m - 1)$-times. We note that each use of Proposition 8 can modify the translation by adding $\alpha^\vee$. Thus we can decrease the regularity by at most the maximum of $\langle \beta, \alpha^\vee \rangle$ over $\beta \in \Delta_0$. If $j$ is the mentioned maximum, then one functioning bound for $\lambda$ is $k + (m - 1)j$.

Theorem 3 requires the use of Proposition 8 up to the length of the longest shortest path starting at $w$. Note that $|W_0|$ is an upper bound on the longest shortest path. One functioning lower bound on the regularity of $\lambda$ in Theorem 3 is $k + |W_0| j e$

## 4 Dual untwisted affine root systems

Consider the dual of an untwisted affine root system. There is an associated quantum Bruhat graph defined in the same way as for the untwisted affine root systems, except that the roles of roots and coroots are exchanged. For dual untwisted type $s_0 = t_\phi s_\phi$ where $\phi$ is the short dominant root.

Let $\rho^\vee = (1/2) \sum_{\alpha^\vee \in \Phi^+_0} \alpha^\vee$.

The quantum Bruhat graph (QBG) is the directed graph $\Gamma$ with vertex set $W_0$ and a directed edge $w \to wr_\alpha$ for each pair $(w, \alpha) \in W_0 \times \Phi^+_0$ such that one of the following holds.

1. (Bruhat edge) $\ell(wr_\alpha) = \ell(w) + 1$.
2. (Quantum edge) $\ell(wr_\alpha) = \ell(w) - \langle 2\rho^\vee, \alpha \rangle + 1$.

The weight of the edge $w \to wr_\alpha$ in $\Gamma$ is $\alpha$ if the edge is quantum and 0 if it is Bruhat. For any $u, v \in W_0$, let $M(u, v)$ be the weight of any shortest path in $\Gamma$ from $u$ to $v$.

With these definitions in place, analogues of the statements for the untwisted affine root systems hold for the dual root systems with essentially the same proofs.

**Example 23.**

1. Consider the root system $A^{(2)}_{2,3-1}$, which is dual to $B^{(1)}_3$.

The classical subrootsystem has type $C_3$ realized in $\mathbb{R}^3$ with simple classical roots $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$, $\alpha_3 = (0, 0, 2)$, simple coroots $\alpha_i^\vee = \alpha_i$ for $i \in \{1, 2\}$ and $\alpha_3^\vee = (0, 0, 1)$, and $\varphi = (1, 1, 0)$. We have $2\rho^\vee = (5, 3, 1)$, $\langle 2\rho^\vee, \varphi \rangle = 8$, $s_\varphi = s_2s_1s_3s_2s_1s_3s_2$ and $s_0 = t_\varphi s_\varphi$. $\Gamma$ has a quantum edge from $s_\varphi$ to 1 since $\ell(s_\varphi) = 7 = \langle 2\rho^\vee, \varphi \rangle - 1$.

2. Consider the root system $D^{(2)}_{4+1}$, which is dual to $C^{(1)}_3$. The classical subrootsystem is type $B_4$ realized in $\mathbb{R}^3$ by $\alpha_1 = (1, -1, 0)$, $\alpha_2 = (0, 1, -1)$, and $\alpha_3 = (0, 0, 1)$, $\alpha_i^\vee = \alpha_i$ for $i \in \{1, 2\}$, $\alpha_3^\vee = (0, 0, 2)$, $2\rho^\vee = (6, 4, 2)$, $\varphi = (1, 0, 0)$, $\langle 2\rho^\vee, \varphi \rangle = 6$, and $s_\varphi = s_1s_2s_3s_2s_1$. $\Gamma$ has a quantum edge from $s_\varphi$ to 1 since $\ell(s_\varphi) = 5 = \langle 2\rho^\vee, \varphi \rangle - 1$.
Figure 2: Some dual untwisted affine Dynkin diagrams

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