HIGGS PHENOMENON FOR THE GRAVITON IN
ADS SPACE

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Abstract

In this review, we summarize recent findings that show how standard 4-d Ein-
stein gravity coupled to a conformal field theory can become massive in Anti de
Sitter Space. Key ingredients in this phenomenon are non-standard “transparent”
boundary condition given to the CFT fields and the fact that AdS space is not
globally hyperbolic, due to the presence of a time-like boundary.

1 Ward Identities in AdS

A widely held misconception about gauge theories in general, and generally covariant ones
in particular, is that gauge invariance forbids an explicit mass term for the gauge field. In
the case of gravity, the gauge field is the graviton, and invariance under general coordinate
transformations translates into a set of Ward identities. Actually, these identities do
not depend on the fact that gravity is dynamical; they also hold for a field theory on
an arbitrary fixed space-time background. In case the field theory possesses additional
symmetries, additional Ward identities will hold.

Let us be specific and consider a 4-d conformal field theory on an arbitrary background
with metric $g_{mn}$. Let us denote by $W[g_{mn}]$ the generating functional of the connected
Green functions of the stress-energy tensor. To wit:

$$\frac{\delta W}{\delta g_{mn}} = \langle 0 | T^{mn} | 0 \rangle,$$

$$\frac{\delta^2 W}{\delta g_{mn} \delta g_{pq}} = \langle 0 | T^{mn} T^{pq} | 0 \rangle - \langle 0 | T^{mn} | 0 \rangle \langle 0 | T^{pq} | 0 \rangle \equiv \Sigma^{mn,pq},$$

(1)
and so on.

Let us expand $W$ around a background $\bar{g}_{mn}$ ($h_{mn} \equiv g_{mn} - \bar{g}_{mn}$):

$$W[\bar{g}_{mn} + h_{mn}] = W[\bar{g}_{mn}] + \int d^4x \sqrt{-g(0)} T_{mn}(0) h_{mn} + \frac{1}{2} \int d^4x \sqrt{-g} \int d^4y \sqrt{-g} h_{mn}(x) \Sigma_{mn,pq}(x,y) h_{pq}(y) + ...$$  \hspace{1cm} (2)

It is important to recall that $W[g]$ is only defined up to local terms in $g_{mn}$; equivalently, the Green functions are determined only up to local contact terms.

General covariance and conformal invariance translate into two Ward Identities:

$$g_{mn} \frac{\delta W}{\delta g_{mn}} = A[g], \quad D_m \frac{\delta W}{\delta g_{mn}} = 0.$$  \hspace{1cm} (3)

Here $A[g]$ is the conformal anomaly, whose specific form will be given later to the necessary extent. $D_m$ is the standard Riemann covariant derivative. By expanding around $\bar{g}_{mn}$ up to quadratic order in the fluctuation $h_{mn}$, and by denoting with $\bar{D}_m$ the background covariant derivative, we find

$$\bar{g}_{mn} \Sigma_{mn,pq}(x,y) = 0, \quad \bar{D}_m \Sigma_{mn,pq}(x,y) = 0, \quad x \neq y.$$  \hspace{1cm} (4)

We can do better than that. Indeed, by a judicious choice of finite, local counter-terms, we can completely cancel the contact terms when the background is any of the three Lorentzian-signature, maximally symmetric spaces in 4-d: de Sitter (dS), Minkowsky (M) or Anti de Sitter (AdS) \[1\].

First of all, we notice that maximal symmetry of the background implies that the VEV of the stress energy tensor is proportional to the background metric, $\langle 0 | T_{mn} | 0 \rangle = C \bar{g}_{mn}$. Furthermore, we can use the ambiguity in the definition of $W[g]$ and define a new generating functional

$$W'[g] = W[g] + 2C \int d^4x \sqrt{-g}.$$  \hspace{1cm} (5)

This new functional does not contain any term linear in the fluctuations around the background, i.e. $W'[\bar{g}] = W'[\bar{g}] + O(h^2)$. The Ward identity due to general covariance, Eq. (3), now implies that $\Sigma_{mn,pq}$ is exactly transverse, without any contact term.

$$\bar{D}_m \Sigma_{mn,pq}(x,y) = 0 \text{ everywhere}, \quad \Sigma_{mn,pq} \equiv \frac{\delta^2 W'}{\delta g_{mn} \delta g_{pq}}.$$  \hspace{1cm} (6)

The conformal Ward identity for $W'[g]$ is

$$g_{mn} \frac{\delta W'}{\delta g_{mn}} = A[g] - A[\bar{g}] = \alpha C_{abcd} C^{abcd} + \beta (R_{ab} - \frac{1}{4} g_{ab} R)^2 + \gamma (R^2 - \bar{R}^2),$$  \hspace{1cm} (7)
with \( \alpha, \beta, \gamma \) known constants. Expanding \( W' \) to linear order in \( h_{mn} \), we find
\[
\bar{g}_{mn} \Sigma^{mn,pq} \ast h_{pq} = 2\gamma \tilde{R} R^L(h) + O(h^2).
\] (8)
\( R^L(h) \) is the linearized scalar curvature and \( \ast \) is the convolution in the 4-d coordinates \( x^m \). We can finally define a generating functional \( W''[g] \) by
\[
W''[g] = W'[g] + 2\gamma \bar{R} \int d^4x \sqrt{-g} \left( R - \frac{1}{2} \tilde{R} \right).
\] (9)
The conformal Ward identity on \( W''[g] \), expanded to linear order in \( h_{mn} \) implies
\[
\bar{g}_{mn} \Sigma^{mn,pq} = 0 \quad \text{everywhere.} \quad \text{(10)}
\]
We can summarize what we did so far as follows: in a CFT on a maximally symmetric space, we can make the 2-point function of the stress-energy tensor transverse-traceless everywhere, including at contact points, by adding finite contact terms. These terms explicitly break conformal invariance. Notice that these terms are the standard Einstein action and the cosmological constant term. So, when we will let the metric \( g_{mn} \) fluctuate dynamically, they will generate finite renormalizations of the cosmological and Newton constants.

## 2 The Graviton Mass: General Properties

As anticipated at the end of the previous section, we want to couple our (cutoff) CFT to gravity. We will treat gravity perturbatively, by discarding graviton loops, but we will include all matter loops. This approximation is consistent whenever the cutoff \( \Lambda \) is well below the Planck scale (\( \Lambda \ll M_{Pl} \)).

The result of integrating out the CFT fields is the effective action
\[
\Gamma_{\text{eff}}[g] = 16\pi G_B \int d^4x \sqrt{-g} (R - 2\Lambda_B) + W[g].
\] (11)
\( W[g] \) is the result of integrating out the CFT degrees of freedom, so it is precisely the generating functional of the connected Green functions of the stress-energy tensor, as in the previous section. Since \( W[g] \) is defined only up to contact terms, the splitting of \( \Gamma_{\text{eff}}[g] \) in between the “bare” Einstein action and \( W[g] \) is ambiguous. We can fix this ambiguity by defining
\[
\Gamma_{\text{eff}}[g] = 16\pi G \int d^4x \sqrt{-g} (R - 2\Lambda) + W''[g].
\] (12)
With this choice, \( G \) is the physical Newton constant, and \( \Lambda \) is the physical cosmological constant. By this we mean in particular that \( \bar{R}_{ab} = \Lambda \bar{g}_{ab} \) solves the equations of motion...
derived from the action $\Gamma_{\text{eff}}[g]$. As before we denote by $\bar{g}_{mn}$ the metric of a maximally symmetric space.

By expanding the action to quadratic order around $\bar{g}_{mn}$, we get the linearized equations of motion for the fluctuation $h_{mn}$. In particular, the transverse-traceless part of the fluctuation, $h_{mn}^{\text{TT}}$, obeys

$$(\Delta - 2\Lambda)h_{mn}^{\text{TT}} + 32\pi G\Sigma_{mn, pq}^{*}h_{pq}^{\text{TT}} = 0.$$  \hfill (13)

Here, $\Delta$ denotes the Lichnerowicz operator $[2]$, $\Delta = -\Box - 2R_{mnpq}h_{pq}^{r} + 2R_{(m}h_{n)r}$. \hfill (14)

In AdS space, a free spin-2 field $\phi_{mn}^{M}$ of mass $M$ obeys the equation of motion

$$(\Delta - 2\Lambda)\phi_{mn}^{M, TT} = -M^{2}\phi_{mn}^{M, TT}.$$  \hfill (15)

Also, $\Sigma_{mn, pq}$ is the two-point function of the stress-energy tensor on the background metric, so it commutes with $\Delta$ and can be diagonalized simultaneously with it. \hfill (1)

If we combine Eqs. (14, 15), and we define $\Sigma_{mn, pq}^{*} \equiv \Sigma(M^{2})\phi_{pq}^{M, TT}$, we find that $h_{mn}$ describes spin-2 excitations with mass given by the solution of the equation

$$-M^{2} + 32\pi G\Sigma(M^{2}) = 0.$$  \hfill (16)

This equation says that a massless graviton exists when $\Sigma(M^{2}) = 0$. On the other hand, when $\Sigma(M^{2})$ is continuous around zero and $\Sigma(0) > 0$, Einstein gravity coupled to a CFT does not possess any massless graviton. The mass of the lightest spin-2 excitation is instead

$$M_{\text{min}} \approx \sqrt{32\pi G\Sigma(0)}.$$  \hfill (17)

This is at best a very small mass! To appreciate this fact, consider that $\Sigma(0)$ is independent of $G$: it is a correlator in a CFT on a fixed background with cosmological constant $\Lambda$. Purely by dimensional analysis, we have $\Sigma(0) = O(\Lambda^{2})$, since $\Lambda$ is the only scale of the CFT. Equation (17) then gives the following estimate for the mass of the lightest spin-2 excitation

$$M_{\text{min}} = O\left(\sqrt{G\Lambda^{2}}\right) = O\left(\frac{|\Lambda|}{M_{\text{Pl}}}\right) \ll \sqrt{|\Lambda|}.$$  \hfill (18)

The last inequality holds whenever the radius of curvature of the space, $L \equiv \sqrt{3/|\Lambda|}$ is much larger than the Planck length $L_{\text{Pl}} \equiv M_{\text{Pl}}^{-1}$.\footnote{To prove this, use a spectral representation obtained by inserting a complete set of intermediate states: $\Sigma_{mn, pq}(x, y) = \sum_{E}(0|T^{mn}(x)|E)(E|T_{pq}(y)||0)$. In the case of interest here, AdS space, this basis will be described in more details in the next section.}
Up to now, we have not proven that $\Sigma(0) \neq 0$; indeed, nothing we said so far is specific to AdS space. To prove that the graviton does indeed get a mass we need to study more carefully the properties of (free) fields in AdS space. This is the subject of next section.

3 Free Particles in AdS Space

To proceed further we need to review some facts about positive-energy representations of the $AdS_4$ isometry group, $SO(2, 3)$.

These representations were classified in [3] (see also [4] for a clear review). In the decomposition $SO(2, 3) \rightarrow SO(2) \times SO(3)$, the generator of $SO(2)$ is the $AdS_4$ energy; while angular momentum is given by the generators of $SO(3)$. A unitary, irreducible, positive-weight representation of $SO(2, 3)$ (UIR), $D(E, s)$, is labeled by the energy $E$ and spin $s$ of its (unique, up to spin degeneracy) lowest-energy state $^2$. Free fields form irreducible representations of $SO(2, 3)$. A conformal scalar can belong to either the $D(1, 0)$ or the $D(2, 0)$. A conformal (massless) spin-$1/2$ fermion belongs to a $D(3/2, 1/2)^\pm$, while a massless vector (also conformal) belongs to a $D(2, 1)^\pm$ [5, 6]. The label $\pm$ denotes the parity of the UIR.

Massless representations of spin $s > 0$ have $E = s + 1$ [5, 6, 4]. Massive unitary representations of spin larger than zero have $E > s + 1$. In the limit $E \rightarrow s + 1$, the UIR $D(E, s), s \geq 1$ becomes reducible [3, 5]:

$$D(E, s) \rightarrow D(s + 1, s) \oplus D(s + 2, s - 1), \quad E \rightarrow s + 1.$$  \hspace{1cm} (19)

Eq. (19) encodes the group theoretical aspect of the Higgs phenomenon in $AdS_4$: when a spin-$s$ field, $s \geq 1$, becomes massive, it “eats” a spin-$(s - 1)$ boson. Notice that for $s = 0$ this boson is in a $D(3, 0)$, i.e. it is a minimally-coupled scalar [6]. For spin 2, it is a massive vector in the $D(4, 1)$ $^3$.

Now, this $E = 4$, spin-$1$ field is the Goldstone field for the graviton, so it must have a nonzero matrix element with the state $T_{mn}|0\rangle$. Equivalently, the connected two-point function of the graviton must have a pole corresponding to a vector belonging to $D(4, 1)$. The wave equation obeyed by such a vector is $(\Delta - 2\Lambda)A_m = 0$ [1]. So, a Goldstone vector exists when

$$\Sigma_{mn}^{\mu \nu} h_{\mu \nu} = \frac{2C}{\Delta - 2\Lambda} D_{(m}D^{n)l} h_{l} + \ldots, \hspace{1cm} (20)$$

$^2$E is the dimensionless energy, measured in units of the $AdS$ curvature radius $L$.

$^3$Recall that in Minkowsky space, instead, the Higgs phenomenon for a spin 2 requires a massless vector and a massless scalar. They together provide 3 degrees of freedom, as it does our massive vector in $AdS_4$. 

and $C$ is a nonzero constant, that determines the strength of the Goldstone boson coupling. Since $\Sigma_{mn}^{\prime\prime} h_{pq}$ is transverse and traceless, it is proportional to the projector over $TT$ states, $\Pi_{mn}^{pq}$. This projector was computed, for instance, in [1]. Its relevant part reads

$$\Pi_{mn}^{pq} h_{pq} = h_{mn} + \frac{2}{\Delta - 2\Lambda} D_{(m} D_{n)} h_{l} + \ldots \quad (21)$$

In the previous section, we saw that $\Sigma_{mn}^{\prime\prime} = \Sigma(\Delta - 2\Lambda) \Pi_{mn}^{pq}$, so Eqs. (20,21) and the continuity of $\Sigma(M^2)$ near $M^2 = 0$ imply $C \approx \Sigma(0)$.

Not surprisingly, we get the same condition that we found at the end of Section 2: a Goldstone vector exists – i.e. the graviton acquires a mass – iff $\Sigma(0) \neq 0$.

The projector $\Pi_{mn}^{pq}$ has also an additional, spin-zero pole at $\Delta = 4\Lambda/3$ [7]. To see whether this scalar is a physical degree of freedom or a ghost, it is useful to introduce an external source with (covariantly conserved) stress-energy tensor $T_{mn}$, and use the fact that action $\Gamma_{\text{eff}}$ in Eq. (11) is invariant under general coordinate transformations to decompose the metric fluctuations as $h_{mn} = h_{mn}^{TT} + \tilde{g}_{mn} \psi$. In this gauge, the equations of motion derived from $\Gamma_{\text{eff}}$ in the presence of the external source are

$$(\Delta - 2\Lambda) h_{mn}^{TT} + 32\pi G \Sigma(\Delta - 2\Lambda) * h_{pq}^{TT} = 2\Pi_{mn}^{pq} * T_{pq},$$

$$3(\Delta - 4\Lambda) \psi = T. \quad (22)$$

On the conserved source $T_{mn}$ we also have

$$\Pi_{mn}^{pq} * T_{pq} = T_{mn} + \frac{1}{3} \tilde{g}_{mn} T + (D_{m} D_{n} + \tilde{g}_{mn} \Lambda/3)(3\Delta - 4\Lambda)^{-1} T. \quad (23)$$

The one-graviton interaction between two conserved sources, $T_{mn}$, $T'_{mn}$ is given by the amplitude $A = (1/2) T_{mn}^{nm} h_{mn}(T)$, which has indeed a scalar pole at $\Delta = 4\Lambda/3$:

$$A = T_{mn}^{nm} h_{mn} \approx 8\pi G \frac{\Sigma(-2\Lambda/3)}{\Lambda} T'(\Delta - 4\Lambda/3)^{-1} T. \quad (24)$$

The residue is always small, $O(L_{Pl}^2/L^2)$. It is physical when $\Sigma(-2\Lambda/3) \leq 0$. Since positivity of the graviton residue requires instead $\Sigma(0) \geq 0$, we conclude that $\Sigma(M^2)$ cannot be constant in the range $0 \leq M^2 \leq -2\Lambda/3$ in a unitary theory. This is possible in our case, since the CFT coupled to gravity provides us with an infinite number of degrees of freedom with mass $O(\sqrt{\Lambda})$.

Let us come back to the search for the Goldstone vector, in the special case where the CFT is free (before coupling it to gravity, i.e. in the limit $M_{Pl} \to \infty$). In a free conformal field theory, $T_{mn}$ is quadratic in the fields, so the Goldstone vector is a bound state in the product of two free fields $^4$. A necessary condition for the existence of a Goldstone

$^4$In Minkowsky space this is obviously absurd since non-interacting two-particle states form a continuum. In Anti de Sitter space, instead, free particles do form bound states, since the AdS energy spectrum is discrete.
Let us examine separately free conformally-coupled fields of spin 1, 1/2, and 0.

spin 1 Massless spin-1 fields are conformal; they belong to the $D(2,1)$ \cite{5,6}. The tensor product of $SO(2,3)$ UIRs was found by Heidenreich in \cite{8}. For $D(2,1)$, he found

$$D(2,1) \otimes D'(2,1) = \sum_{n=0}^{\infty} D(4+n,0) \oplus \sum_{n=0}^{\infty} D(4+n,1) \oplus \sum_{S=0}^{\infty} \left[ D(4+S,2+S) \oplus \sum_{n=0}^{\infty} 2D(5+S+n,2+S) \right].$$

(26)

In our case, since we are tensoring two identical bosons, some of the representations that appear in the tensor product above are absent. For instance, the ground state of the $D(4,1)$ that appears in the tensor product above is antisymmetric in its arguments ($\sim \epsilon_{ijk} A_i A_j$), so it is forbidden by Bose statistics. This means that the entire $D(4,1)$ is absent.

spin 1/2 The massless (conformal) spin-1/2 field belongs to the $D(3/2,1/2)$. Tensoring two different $D(3/2,1/2)$, Heidenreich finds \cite{8}

$$D(3/2,1/2) \otimes D'(3/2,1/2) = \sum_{n=0}^{\infty} D(3+n,0) \oplus \sum_{S=0}^{\infty} D(3+S,1+S) \oplus \sum_{n=0}^{\infty} 2D(4+S+n,1+S).$$

(27)

In this tensor product, the $D(4,1)$ appears twice. By taking into account Fermi statistics when tensoring two identical representations, we get rid of one of them. The other one cannot appear in the stress-energy tensor since it has the wrong parity \cite{11}. To arrive at this result we first notice that the stress-energy tensor of a free CFT made of several fields of spin $s \leq 1$ is given by the sum $T_{mn} = \sum_i T^i_{mn}$. The $i$-component of this sum is the stress-energy tensor of either a real vector, a real scalar, or a Majorana fermion. To preserve the Majorana condition ($\psi = C\psi^*$, $C =$ charge conjugation), the field $\psi(x)$ must transforms as follows under parity: $\psi(t,x) \rightarrow \eta \gamma^0 \psi(t,-x)$, $\eta = \pm 1$. The fermion field $\psi$ can be expanded in spherical waves. Its positive-frequency part (with respect to the global AdS time $t$) is \cite{6}

$$\psi_{pf} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} e^{-i\omega t/L} a_{\omega jm} \chi^\pm_{\omega jm}, \quad j = 1/2 + k, \quad \omega = 1 + j + n.$$ 

(28)
The operators $a_{\omega jm}^\dagger, a_{\omega jm}$ respectively create and annihilate states of definite energy $\omega/L$ and angular momentum $j$, belonging to the $D(3/2, 1/2)^\pm$. The superscript $\pm$ labels the parity of the representation. More precisely, when $\psi$ belongs to the $D(3/2, 1/2)^+$, the spherical waves $\chi_{\omega jm}^+$ in Eq. (28) transforms under parity as $\chi_{\omega jm}^+ \rightarrow i(-)^{\omega - 3/2} \chi_{\omega jm}^+$. Analogously, for $D(3/2, 1/2)^-$, $\chi_{\omega jm}^- \rightarrow -i(-)^{\omega - 3/2} \chi_{\omega jm}^-$. The parity of the ground state of the $D(4, 1)$ is fixed by the parity of its ground state. The assignments given above show immediately that the parity of the ground state of the $D(4, 1)$ in the tensor product of either $D(3/2, 1/2)^+ \otimes D(3/2, 1/2)^+$ or $D(3/2, 1/2)^- \otimes D(3/2, 1/2)^-$ is $+1$. This is the parity of a pseudo-vector, while the $D(4, 1)$ contained in $T_{mn}$ must be a true vector, with parity $-1$. This can be seen most easily by noticing that $T_{mn}$ is a true tensor and that the $D(4, 1)$ we are after must appear in it as follows

$$T_{mn} = D(mA_n) + \ldots, \quad (\Delta - 2\Lambda) A_m = 0. \quad (29)$$

Equivalently, we may notice that with a single Majorana fermion we cannot form a vector, as $\bar{\psi}\gamma^m\gamma^5\psi = 0$, but we can form the pseudo-vector $\bar{\psi}\gamma^m\gamma^5\psi$.

Spin 0 Scalars belong to $D(E, 0), E \geq 1/2$. The tensor product of two spin zero representations of $SO(2, 3)$ is

$$D(E_1, 0) \otimes D(E_2, 0) = \sum_{S=0}^{\infty} \sum_{n=0}^{\infty} D(E_1 + E_2 + S + 2n, S). \quad (30)$$

Here $E_1, E_2 > 1/2$. When $E = 1/2$, the representation degenerates, becoming a singleton $[9, 10]$, namely a representation that propagates only boundary degrees of freedom and cannot be represented as a standard local field living in the bulk of $AdS_4$. We will not consider it further. When $E_1 = E_2 = E > 1/2$, a $D(4, 1)$ exists in the tensor product $D(E, 0) \otimes D'(E, 0)$ only for $E = 3/2$. If the two representations are identical, $D(4, 1)$ is eliminated by Bose statistics [its would be ground state is in reality a descendant belonging to the $D(3, 0)$].

This is not the end of the story though, since the $D(4, 1)$ appears in the tensor product of two representations of different energy. In particular, it appears in the product $D(1, 0) \otimes D(2, 0)$. A conformally-coupled scalar field belongs entirely either to the $D(1, 0)$ or the $D(2, 0)$ when it is given reflecting boundary conditions at the boundary of $AdS_4$. This is not the most general condition one can give. In particular, one can give so-called “transparent” boundary conditions. They are obtained by noticing that $AdS_4$ is conformal to half the Einstein static universe. So, one can obtain an (over-complete) set of solutions for the conformal scalar wave equation by solving it in the static Einstein universe, and then conformally transforming the solution to AdS. These boundary conditions
were studied first in [11]. Physically, these conditions correspond to the possibility of letting the boundary exchange energy and momentum with the interior of the AdS space. This could be achieved by introducing a 3-d defect CFT localized at the boundary. This possibility may seem bizarre, but it is actually required [16] in order to give a holographic interpretation to certain warped compactifications of 5-d gravity [12, 13, 14, 15, 16, 17].

4 The Graviton Mass: Explicit Computation

Transparent boundary conditions imply that a conformal scalar can belong to the reducible representation $D(1,0) \oplus D(2,0)$. This changes the form of its two-point function.

The 2-point function of a conformally coupled scalar field is best written by representing $AdS_4$ as the cover of a hyperboloid in a 5-d space with signature $(2,3)$:

$$X^M X^N \eta_{MN} = -L^2, \quad \eta = \text{DIAG}(-1, -1, 1, 1, 1).$$

Here, $X^M$ is a homogeneous coordinate. The distance between two points on the hyperboloid, with coordinates $X^M$ and $Y^M$, is a function of $Z \equiv X^M Y^M / L^2$ only. The scalar propagator, for arbitrary boundary conditions, reads

$$\Delta(Z) = \frac{1}{4\pi^2 L^2} \left( \alpha \frac{1}{Z^2 - 1} + \beta \frac{Z}{Z^2 - 1} \right).$$

Standard, reflecting boundary conditions are $\alpha = 0, \beta = 1$ [D(1,0)] or $\alpha = 1, \beta = 0$ [D(2,0)]. “Transparent” boundary conditions are instead $\alpha = \beta = 1/2$ [1].

Once the two-point function of the scalar is known, it is relatively straightforward to find the two-point function of its (improved) stress-energy tensor, and from this last quantity the mass of the graviton. The most subtle point in the computation is that, in order to determine the mass unambiguously, one must look for the non-local term [cfr. Eq. (20)]

$$\langle 0 | T_{mn}(0)T_{pq}(x)|0 \rangle = \frac{1}{2} \langle 0 | \frac{C}{\Delta - 2\Lambda} \bar{g}_{np} D_n D_p | x \rangle + (m \leftrightarrow n, p \leftrightarrow q) + \ldots$$

In fact, it is only by looking at a non-local term that one can distinguish unambiguously between a true mass term and a spurious term that can canceled by redefining the contact terms in the two-point function of the stress-energy tensor.

In homogeneous coordinates, at large distances ($Z \to \infty$), the right-hand side of Eq. (33) goes as $Z^{-4} + O(Z^{-6})$. The coefficient of the $Z^{-4}$ term is $\Sigma(0)$, from which we can get the graviton mass thanks to Eq. (16). The computation of ref. [1] gives

$$\Sigma(0) = \frac{K}{L^4 \alpha \beta}.$$
The nonzero numerical constant $K$ is independent of $L, \alpha, \beta$. Eq. (34) manifestly shows that a nonzero graviton mass arises only when the matter CFT is given non-standard, non-reflecting boundary conditions 5.

Ref. [7] extended the computation of $K$ to the case of a free $N = 4$ super Yang-Mills theory. The numerical value for $K$ found there is the same that one obtains when the $N = 4$ theory is strongly coupled, in which case the computation can be done using the holographic duality [14] with the Karch-Randall compactification [12, 13, 18]. The reason behind this equality is quite mysterious, since it is not due to any known non-renormalization theorem. At present, indeed, it is not clear if this equality persists when the gauge coupling of the $N = 4$ super Yang-Mills theory is finite and nonzero.

5 Conclusions

In this review, we re-examined the possibility of giving a mass to the graviton in Anti de Sitter space. We pointed out that Ward identities do not forbid a graviton mass. We then proceeded to examine the conditions that allow a gravitational Higgs mechanism in AdS, and we gave a model-independent estimate of the graviton mass induced by a CFT.

Next, we recalled that unlike Minkowsky space, $AdS_4$ allows even a free theory to form bound states, owing to the discreteness of the AdS energy spectrum. We investigated free CFTs with spin not greater than 1. We found that, in order to have a Goldstone particle in the stress-energy tensor, we had to impose non-standard (i.e. non-reflecting) boundary conditions on the fields of our free CFT. Similar boundary conditions are not only allowed, but indeed necessary, to interpret holographically the KR model [12, 14, 13, 16, 17].

We considered next a free conformal scalar in $AdS_4$, and we found that, even in that very simple example, nonstandard boundary conditions do produce a Goldstone boson that gives a mass to the graviton, when the CFT is coupled to standard Einstein’s gravity.

In [12, 14], it was shown that the same phenomenon happens when gravity is coupled to a strongly interacting CFT. In both cases the key ingredient is the boundary conditions imposed on the CFT. They must allow energy and momentum to flow in and out of $AdS_4$ through its boundary.

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5 The graviton has instead standard boundary conditions!
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