Anderson localization in a partially random Bragg grating and a conserved area theorem

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ABSTRACT

We investigate the gradual emergence of the disorder-related phenomena in intermediate regimes between a deterministic periodic Bragg grating and a fully random grating and highlight two critical properties of partially disordered Bragg gratings. First, the integral of the logarithm of the transmittance over the reciprocal wavevector space is a conserved quantity. Therefore, adding disorder merely redistributes the transmittance over the reciprocal space. Second, for any amount of disorder, the average transmittance decays exponentially with the number of grating layers in the simple form of $\exp(-\eta N)$ for sufficiently large $N$, where $\eta$ is a constant and $N$ is the number of layers. Conversely, the simple exponential decay form does not hold for small $N$ except for a highly disordered system. Implications of these findings are demonstrated.

I. INTRODUCTION

An optical Bragg grating is created by periodic variations in the permittivity of dielectric material. Depending on the size of the periodicity, the grating can act as a highly reflective (bandgap) or a highly transmittive (bandpass) mirror [1]. A bandgap or bandpass is not limited to an optical Bragg grating: in solid-state physics, a periodic lattice of atoms or molecules in a crystal modulates the background potential for electron wave function and similar badgap and bandpass characteristics are observed. Understanding the electronic band structure is at the foundation of solid-state physics.

In practice, no potential is perfectly periodic and no refractive index modulation is disorder-free. Therefore, it is important to investigate the impact of disorder on the band structure in electronic systems, optical gratings, and other coherent wave systems [2]. There have been numerous studies of disordered coherent wave systems, much of which follows the seminal work of Philip Anderson [3]. He showed that for a sufficiently high level of disorder the electronic wavefunction is exponentially attenuated, resulting in diminished transport and conductivity [4], as also confirmed in subsequent work by others [5]–[16].

A one-dimensional (1D) periodic Bragg grating is simple and elegant—its band structure is easy to understand and analytical calculations can often be performed to study its characteristics [17]. Moreover, many of its properties are generic and can be extended to 2D and 3D gratings. A highly disordered 1D grating made from a random stack of dielectric slabs has also been treated analytically by Berry and Klein [18]. They showed that the average transmitted intensity drops exponentially with the number of slabs.

The problem of a perfectly periodic Bragg grating is well understood, and so is that of a highly disordered grating. The intention of this article is to shed further light on intermediate situations with moderate disorder [18]–[20]. We explore the transition from a deterministic periodic Bragg grating to a fully random grating, and highlight the gradual emergence of the disorder-related phenomena in intermediate regimes. In particular, we present two critical properties of partially disordered 1D Bragg gratings. Because our analysis is carried out for a stack of dielectric slabs similar to Ref. [18], we present these properties in the language of dielectric slabs for simplicity:

First we find a globally conserved quantity in the reciprocal wavevector space that is quite instrumental in visualizing the wave localization behavior in a partially disordered stack of dielectric slabs. We show that the total area under the curve of the logarithm of the transmittance plotted in the reciprocal wavevector space is the same for all Bragg gratings, regardless of the amount of disorder. As a consequence of this conservation law, if the transmittance is higher in some region of the reciprocal space, it has to be lower in another region to conserve the total area. For example, the presence of a strong bandgap for a periodic Bragg grating necessitates a strong bandpass in a different region of the reciprocal space.

Second we observe that for any amount of disorder, the average transmitted intensity decays exponentially with the number of slabs as $\exp(-\eta N)$ for sufficiently large $N$. $\eta$ is a constant whose value depends on the amount of disorder, and $N$ is the number of slabs. Even in the absence of disorder, the simple exponential decay law holds in the bandgap region for large $N$. Conversely, the simple exponential decay form does not hold for small $N$ except for a highly disordered system.

In the following, these ideas will be presented in the framework of optical transmission through a 1D stack of dielectric slabs. The slabs are assumed to be identical with refractive index $n_2$ and are embedded in a background dielectric of refractive index $n_1$. The disorder is introduced by randomizing the location of the slabs.

II. BACKGROUND

The transmission (transfer) matrix for a lossless reciprocal mirror (dielectric slab) can be expressed as

$$M_t = \begin{bmatrix} 1/r^* & r/t \\ r^*/t & 1/t \end{bmatrix} = \frac{1}{\tau} \begin{bmatrix} 1 & i\rho \\ -i\rho & 1 \end{bmatrix}.$$ (1)
t and r are the amplitude transmittance and reflectance, respectively, and are functions of the dielectric constants and geometric properties of the slab and frequency and incidence angle of light \[ |t|^2 + |r|^2 = 1. \] They also satisfy the losslessness condition \[ |t|^2 + |r|^2 = 1. \] The latter simplified form is obtained by formally incorporating a thin layer of the background \( n_1 \) dielectric in the \( n_2 \) slab, where \( r = |t| \) and \( \rho = |r| \). The latter form is used in the following discussions as it does not affect the generalities of the presented arguments.

The transfer matrix of an array of \( N \) identical dielectric slabs with varying separations (gaps) of the background dielectric material can be expressed as

\[
M^{(N)} = \prod_{n=1}^{N} (M_{t}\cdot M_{r}^{(n)}), \quad M_{\Phi}^{(n)} = \begin{bmatrix} e^{i\varphi_n} & 0 \\ 0 & e^{-i\varphi_n} \end{bmatrix}, \tag{2}
\]

where \( M_{r}^{(n)} \) is the translation matrix accounting for the phase accumulation \( \varphi_n \) in the \( n \)th gap. Here, \( M_{\Phi}^{(N)} \) is assumed to be the identity matrix. The total wave transfer matrix of the system is given by

\[
M^{(N)} = \frac{1}{\tau^N} \prod_{n=1}^{N} \begin{bmatrix} e^{i\varphi_n} & i\rho e^{-i\varphi_n} \\ -i\rho e^{i\varphi_n} & e^{-i\varphi_n} \end{bmatrix}, \tag{3}
\]

where the total intensity transmittance is given by the \((2,2)\) element of the \( M^{(N)} \): \( T(N, \tau; \{\varphi_n\}) = |M_{22}^{(N)}|^{-2} \).

### III. TRANSMISSION THROUGH A PERIODIC VS. RANDOM STACK

The most common application of the preceding analysis is to study periodic Bragg gratings. For a periodic grating where the gap thickness between consecutive slabs is identical, the accumulated phase in each gap takes a common value \( \bar{\varphi} \). In Fig. 1, the total transmittance \( T \) is plotted for a periodic stack of \( N = 20 \) dielectric mirrors with \( \tau = 0.8 \) as a function \( \bar{\varphi} \), where one can clearly see the familiar bandpass and bandgap regions.

In practice, it is impossible to maintain a uniform gap (fixed \( \bar{\varphi} \)) between all slabs, and some randomness is inevitable. A convenient way to parametrize a partially disordered grating is to assume that the accumulated phase in each gap is a random number with an average value of \( \bar{\varphi} \) with some probability for variation around the average. We adopt the definition in Eq. [2] which states that \( \varphi_n \) is chosen from a uniform random distribution between \( \bar{\varphi} - \alpha \pi \) and \( \bar{\varphi} + \alpha \pi \). The disorder level is parametrized by \( \alpha \): the deterministic periodic grating is identified by \( \alpha = 0 \) and a fully random grating by \( \alpha = 1 \).

\[
\varphi_n = \bar{\varphi} + \theta_n, \quad \theta_n \in \alpha \times \text{unif}[-\pi, \pi], \quad 0 \leq \alpha \leq 1. \tag{4}
\]

The transmittance for different levels of randomness characterized by \( \alpha \) is plotted in Fig. 2 as a function of \( \bar{\varphi} \). Here, \( N = 50 \) and \( \tau = 0.5 \) is assumed and the vertical scale is logarithmic. For each value of \( \alpha \), the transmittance curve is properly averaged (proper averaging will be explained shortly), for an ensemble of 10,000 different gratings, each for a different set of \( \{\varphi_n\} \) values.

For a periodic Bragg grating with \( \alpha = 0 \), an analytical formula exists for transmittance:

\[
T(N, \tau, \bar{\varphi}) = \frac{\tau^2}{\tau^2 + \rho^2 \Psi_N^2}, \tag{5}
\]

\[
\Psi_N = \frac{\sin(N\Phi)}{\sinh(\Phi)} \quad \Phi = \cos^{-1}\left(\frac{\cos(\bar{\varphi})}{\tau}\right).
\]

The transmittance curve in Fig. 1 and the black solid curve in Fig. 2 follow Eq. [5] The oscillatory bandpass regions correspond to \( |\cos(\bar{\varphi})| < \tau \) where \( \Phi \) is real, while the bandgap region correspond to \( |\cos(\bar{\varphi})| > \tau \), where \( \Phi \) is imaginary. In the bandgap region, it is more convenient to redefine \( \Psi_N \) and \( \Phi \) as

\[
\Psi_N = \frac{\sinh(N\Phi)}{\sinh(\Phi)}, \quad \Phi = \cosh^{-1}\left(\frac{\cos(\bar{\varphi})}{\tau}\right). \tag{6}
\]

For a partially random grating with \( \alpha > 0 \), the transmittance curve would look rather noisy. The smooth transmittance curves in Fig. 2 are only obtained after properly averaging over many gratings. Because the transmittance is a multiplicative quantity, proper averaging of the random transmitted intensity is done by averaging the logarithm of the transmittance \([18]\). Using Eq. [3] we can formally express the averaging as

\[
\log\left(\bar{T}_{\text{avg}}\right) = \left(\prod_{n=1}^{N} \frac{d\varphi_n}{2\pi\alpha}\right) \log\left(T(\{\varphi_n\})\right)
\]

\[
= 2N \log(\tau) + \Omega(N, \tau, \bar{\varphi}, \alpha), \tag{7}
\]

where \( 2N \log(\tau) \) comes from the overall multiplicative factor in Eq. [5] and \( \Omega(\tau, N, \{\varphi_n\}) \) comes from averaging the matrix multiplication part.
For the case of a *totally random* dielectric stack with \( \alpha = 1 \), Berry and Klein [18] have rigorously shown that \( \Omega(N, \tau, \varphi, 1) = 0 \); therefore, \( T_{\text{avg}} = \exp(2N \log(\tau)) \) and is independent of the value of \( \varphi \). The numerical simulation of Fig. 2 in red dotted line agrees with the analytical results, where \( \log(T_{\text{avg}}) = 2N \log(\tau) = -69.31 \) for \( N = 50 \) and \( \tau = 0.5 \).

For intermediate values of \( \alpha \) between 0 and 1, a closed-form solution is not available for \( T_{\text{avg}} \) and one must resort to numerical plots similar to those presented in Fig. 2. However, all these curves follow a remarkable conservation law, which is a consequence of \( \int_0^\pi d\varphi \Omega = 0 \). It can be verified that by changing the amount of randomness via \( \alpha \), the averaged transmittance profile redistributes itself over the mean accumulated gap phase \( \bar{\varphi} \) in a such way that it preserves the area under the logarithm-transmittance curves in Fig. 2:

\[
\mathcal{A} = \int_0^\pi d\varphi \log \left( T_{\text{avg}}(N, \tau, \varphi, \alpha) \right) = 2\pi N \log(\tau). \tag{8}
\]

The area is independent of the randomness level \( \alpha \) and is the same for all curves. This *conserved transmittance area theorem* gives special status to the case of a totally random dielectric stack marked with \( \alpha = 1 \) and the conventional periodic Bragg grating marked with \( \alpha = 0 \). The totally disordered grating incorporates all possible values of phase and \( \varphi \) loses its special standing; therefore, the transmittance becomes independent of \( \bar{\varphi} \) by uniformly spreading the available conserved area \( \mathcal{A} \) over all values of \( \bar{\varphi} \). Conversely, the periodic Bragg grating with \( \alpha = 0 \) provides the most nonuniform distribution of the available conserved area \( \mathcal{A} \) over the space of \( \bar{\varphi} \) with highly depressed values of transmittance in the bandgap, accompanied by a large transmittance in the bandpass to compensate.

In Appendix, we offer a proof of the conserved area theorem. It is shown that the conserved area theorem applies to the transmittance for a general partially random grating and averaging over \( \{\theta_n\} \) is not required (\( \{\theta_n\} \) was defined in Eq. 4). In the absence of ensemble averaging, the transmittance curve would look rather noisy but still satisfies the conserved area theorem.

The conserved area theorem provides a very intuitive and useful approach to visualize the impact of partial disorder. It is important to note that the accumulated phase values in the gaps are proportional to the vector \( k \), where \( \varphi_n = n_k d_n \) and \( d_n \) is the *random* thickness of the \( n \)th gap. Each curve in Fig. 2 should be regarded as corresponding to a grating with an average gap thickness \( \bar{d} \), where \( \bar{\varphi} = n_k k \bar{d} \). Therefore, the horizontal axis \( \varphi \) in Fig. 2 actually represents the wavevector, and the conserved area theorem is a statement about the integral of the average of the logarithm of the transmittance over the reciprocal space.

**IV. TRANSMITTANCE SCALING WITH THE NUMBER OF MIRRORS**

The exponential decay of transmittance (relative optical intensity) is regarded as the main signature of Anderson localization. For the case of a totally random stack marked with \( \alpha = 1 \), it was previously shown that the average transmittance scales like \( T_{\text{avg}} = \exp(-\eta N) \), where the exponent is proportional to the number of mirrors. The scaled exponent \( \eta \) is independent of \( N \) and is given by \( \eta = -2\log(\tau) \geq 0 \).

In this section, it is argued that neither the exponential decay nor its universal scaling with \( N \) is specific to the case of a totally random stack marked with \( \alpha = 1 \). Rather, for a sufficiently large number of mirrors \( N \), \( T_{\text{avg}} \) always scales like \( \exp(-\eta N) \) for all gratings. The only exception is for the periodic Bragg grating marked with \( \alpha = 0 \) in the bandpass region where \( |\cos(\bar{\varphi})| < \tau \).

We start by studying the periodic Bragg grating with \( \alpha = 0 \). According to Eq. 5, the transmittance in the bandpass is a non-decaying oscillatory function of \( N \). However, in the bandgap, it is an exponentially decaying function of \( N \) for large \( N \). This can be seen by using Eq. 6 and noting that \( \sinh(N\Phi) \sim \exp(N\Phi)/2 \) for large \( N \). After a few simple algebraic steps, it can be shown that \( \log(T) \sim 2N\Phi \); therefore, \( \eta = -2\Phi \) for large \( N \). For \( N = 1 \), Eq. 5 shows that \( \log(T) = 2\log(\tau) \).

In Fig. 3 \( \eta \) is plotted as a function of \( N \) for different values of \( \alpha \), all for \( \varphi = 0 \) and \( \tau = 0.5 \). The black solid line corresponds to \( \alpha = 0 \), where \( \eta \) starts at \( 2\log(\tau) \approx -1.37 \) for \( N = 1 \) and saturates at \( \eta = 2 \cosh^{-1}(1/\tau) \approx 2.63 \) for large \( N \).

The plots of \( \eta \) as a function of \( N \) for different values of \( \alpha \) follow a similar behavior to that of the periodic grating explained above. They all start at \( \eta = 2\log(\tau) \) for \( N = 1 \) (see Eq. 3) and asymptotically approach a limit for large \( N \). Therefore, the average transmittance decays exponentially for all cases at large \( N \). In Fig. 3 which is specific to \( \varphi = 0 \), the largest asymptotic value for \( \eta \) is obtained for \( \alpha = 0 \). This is not surprising, because \( \bar{\varphi} = 0 \) corresponds to the bottom of the bandgap of the periodic Bragg grating in Fig. 2.

A similar behavior is observed in Fig. 4 which is the same as Fig. 3 except \( \varphi = \pi/2 \). The main difference is for the solid black line, which relates to the periodic Bragg grating with \( \alpha = 0 \); \( \eta \) oscillates and asymptotically approaches zero as \( 1/N \); therefore, transmittance does not decay exponentially even for large \( N \). This behavior is expected because \( \bar{\varphi} = \pi/2 \) is at the center of the bandpass for the Bragg grating. When a slight amount of disorder marked by \( \alpha = 0.1 \) is introduced, the \( \eta \)-curve (long-dashed blue) goes through a brief oscillation and then approaches asymptotically to a nonzero value, indicating
that for large $N$ the $\mathcal{T}_{\text{avg}} = \exp(-\eta N)$ behavior is observed. Increasing randomness to $\alpha = 0.5$ increases the scaled decay exponent monotonically. Increasing $\alpha$ beyond 0.5 initially increases $\eta$, but eventually rolls it back to make the $\eta$-curve for $\alpha = 1.0$ the same as that of $\alpha = 0.5$.

The universal exponential decay behavior at large $N$ is expected because all 1D disordered systems are Anderson localized [4]. However, the converse observation that for small $N$, the transmittance curve can significantly deviate from the simple exponential decay form despite being Anderson-localized [4]. However, the converse observation that for small $N$, the transmittance curve can significantly deviate from the simple exponential decay form despite being Anderson-localized is worthy of special attention. The exponential decay in the first few layers do not exclude the possibility of Anderson localization.

V. CONCLUSIONS

The transition from a deterministic periodic Bragg grating to a fully random grating and the gradual emergence of the disorder-related phenomena in the intermediate regimes is studied in detail. It is shown that for any amount of disorder the average transmittance decays exponentially with the number of grating layers in the simple form of $\exp(-\eta N)$ for sufficiently large $N$, where $\eta$ is a constant. For a highly disordered grating, this simple form is true regardless of the number of mirrors $N$. Conversely, for small $N$, the transmittance curve can deviate from the simple exponential decay form and can even oscillate as it decays. Whether increasing the disorder increases or decreases the decay exponent depends on the location of the grating in the reciprocal wavevector space, the amount of disorder, and other parameters defining the grating.

It is also shown that the integral of the logarithm of the transmittance over the reciprocal wavevector space is a conserved quantity. The randomness in the examples used in this article is introduced through a uniform distribution for convenience. However, the conserved area theorem works for other methods of randomization such as Gaussian and Poisson distributions and is valid even in the absence of ensemble averaging. It is plausible that the conserved area theorem can be extended to 2D and 3D photonic crystals [21], which will be the subject of future studies.

VI. APPENDIX

Here, we present a proof of the conserved area theorem, i.e. $\int_\pi^0 d\varphi \log \left(|M_{22}^{(N)}|^{-2}\right) = 2N\log(\tau)$, where $M_{22}^{(N)}$ is defined in Eq. 3. Equivalently, we can show $\int_0^\pi d\varphi \log \left(P_{22}^{(N)}|^{-2}\right) = 0$, where $P_{22}^{(N)}$ is defined as:

$$P_{22}^{(N)} = \tau^N \times M^{(N)} = \left(\prod_{n=1}^N e^{i \varphi_n}\right) \times \prod_{n=1}^N \left[1 + i\rho e^{-2i\varphi_n}\right].$$

Using Eq. 9 and defining $z = \exp(-2i\varphi)$, it can be shown that

$$\int_0^\pi d\varphi \log \left|P_{22}^{(N)}\right|^{-2} = -\int_0^\pi d\varphi \log \left(P_{22}^{(N)}\right) + c.c. = 0,$$

which completes the proof (c.c. stands for the complex conjugate). In the last step, we have used the fact that $\gamma$ is a purely imaginary number. Note that averaging over $\{\theta_n\}$ is not needed to prove the theorem; therefore, it is valid for each random transmission curve.

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