Quantum codes derived from cyclic codes

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Abstract

In this note, we present a construction of new nonbinary quantum
codes with good parameters. These codes are obtained by applying the
Calderbank-Shor-Steane (CSS) construction. In order to do this, we show
the existence of (classical) cyclic codes whose defining set consists of only
one cyclotomic coset containing at least two consecutive integers.

1 Introduction

The class of cyclic codes is well-known in the literature \cite{9,4}. Recently, it has
been extensively employed in the construction of quantum codes \cite{2,10,5,6,7,8,11}. Let \( q \) be a prime power. Recall that a \( q \)-ary quantum code \( Q \) of length \( n \)
is a \( K \)-dimensional subspace of the \( q^n \)-dimensional Hilbert space \((\mathbb{C}^q)^\otimes n\), where
\( \otimes n \) denotes the tensor product of vector spaces. If \( K = q^k \) we write \([n,k,d]_q\)
to denote a \( q \)-ary quantum code of length \( n \) and minimum distance \( d \). Let
\([n,k,d]_q\) be a quantum code. The Quantum Singleton Bound (QSB) asserts
that \( k + 2d \leq n + 2 \). If the equality holds then the code is called a maximum
distance separable (MDS) code. For more details on quantum codes, the reader
can consult \cite{10,5}.

In this note, we present constructions of new quantum codes by applying
the well-known CSS construction. In order to do this, we show the existence of
(classical) cyclic codes whose defining set consists of only one cyclotomic coset
containing at least two consecutive integers. This fact induces the construction
of quantum codes with good parameters. More precisely, the code parameters
are good in the sense of the Singleton bound.

The paper is arranged as follows. In Section 2 some preliminaries results
are provided. In Section 3 we present the contributions of the paper, i.e.,
constructions of new quantum codes derived from (classical) cyclic codes. In
Section 4 we give some examples of the new codes and we compare the new
code parameters with the ones available in the literature. Finally, in Section 5
the final remarks are drawn.

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2 Basic Concepts

As usual, $\mathbb{F}_q$ represents a finite field with $q$ elements, where $q$ is a prime power. The parameters of a linear code over $\mathbb{F}_q$ are denoted by $[n, k, d]_q$, where $n$ is the code length, $k$ is the dimension and $d$ is the minimum distance of the code. If $C$ is a linear code then $C^\perp$ denotes its (Euclidean) dual code. In this paper, we assume that $\gcd(n, q) = 1$ (simple root cyclic codes). The multiplicative order of $q$ modulo $n$ is denoted by $m = \text{ord}_n(q)$. The $q$-cyclotomic coset ($q$-coset for short) of $s$, modulo $n$, is defined as $C_s = \{s, sq, \ldots, sq^{m_s-1}\}$, where $m_s$ is the smallest positive integer such that $sq^{m_s} \equiv s \mod n$. A primitive $n$th root of unity is denoted by $\alpha$.

Let $R_n = \mathbb{F}_q[x]/(x^n - 1)$ be the quotient ring of polynomials modulo $(x^n - 1)$. A cyclic code $C$ of length $n$ is a non zero ideal in $R_n$. There exists only one polynomial $g(x)$ with minimal degree in $C$ such that $g(x)$ is a generator polynomial of $C$, where $g(x)$ is a factor of $x^n - 1$. The dimension of $C$ equals $n - \deg(g(x))$. The dual of a cyclic code is also cyclic. Recall the well-known BCH bound:

**Theorem 2.1** [4] (The BCH bound) Let $C$ be a cyclic code with generator polynomial $g(x)$ such that, for some integers $b \geq 0$, $\delta \geq 1$, and for $\alpha$ belongs to some extension field of $\mathbb{F}_q$, we have $g(\alpha^b) = g(\alpha^{b+1}) = \ldots = g(\alpha^{b+\delta-1}) = 0$, i.e., the code has a sequence of $\delta - 1$ consecutive powers of $\alpha$ as zeros. Then the minimum distance of $C$ is, at least, $\delta$.

The Calderbank-Shor-Steane (CSS) quantum code construction is well-known in the literature (see [10, 2]). In the case of the classical code to be dual-containing, the CSS construction reads as follows:

**Lemma 2.2** [1] Lemma 17] If there exists a classical linear $[n, k, d]_q$ code $C$ such that $C^\perp \subseteq C$, then there exists an $[[n, 2k - n, \geq d]]_q$ stabilizer code that is pure to $d$.

3 The New Codes

In this section we show how to guarantee the existence of cyclic codes whose defining set contains only one $q$-coset containing at least two consecutive integers. This fact produces conditions to construct quantum codes with good parameters (in the sense of the QSB). Theorem 3.1 given in the following, is the main result of this note.

**Theorem 3.1** Let $q \geq 3$ be a prime power and $n > m$ be a positive integer such that $\gcd(q, n) = 1$ and $\gcd(q^{ai} - 1, n) = 1$ for every $i = 1, 2, \ldots, r$, where $m = \text{ord}_n(q) \geq r + 2$ and $1 \leq r, a_1, a_2, \ldots, a_r < m$ are integers. If $n|\gcd(t_2, \ldots, t_r)$, where $t_j = [(j - (j - 1)q^{a_j})(q^{a_j} - 1)^{-1} - (q^{a_1} - 1)^{-1}]$ for every $j = 2, \ldots, r$ (the operations are performed modulo $n$), then there exists an $[n, n - m^*, d \geq r + 2]_q$ cyclic code, where $m^*$ is the cardinality of the $q$-coset containing $r + 1$ consecutive integers.
Proof: We want to investigate the following system of congruences

\[ \begin{align*}
  xq^{a_1} &\equiv (x + 1) \mod n \\
  (x + 1)q^{a_2} &\equiv (x + 2) \mod n \\
  (x + 2)q^{a_3} &\equiv (x + 3) \mod n \\
  \vdots \\
  (x + r - 1)q^{a_r} &\equiv (x + r) \mod n,
\end{align*} \]

where \(1 \leq r, a_1, a_2, \ldots, a_r < m\). Since \(\gcd(q^{a_i} - 1, n) = 1\) for every \(i = 1, 2, \ldots, r\), it follows that the above system is equivalent to

\[ \begin{align*}
  x &\equiv (q^{a_1} - 1)^{-1} \mod n \\
  x &\equiv (2 - q^{a_2})(q^{a_2} - 1)^{-1} \mod n \\
  x &\equiv (3 - 2q^{a_3})(q^{a_3} - 1)^{-1} \mod n \\
  \vdots \\
  x &\equiv [r - (r - 1)q^{a_r}](q^{a_r} - 1)^{-1} \mod n,
\end{align*} \]

where \((q^{a_i} - 1)^{-1}\) denotes the multiplicative inverse of \((q^{a_i} - 1)\) modulo \(n\).

The system has a solution if and only if

\[ [j - (j - 1)q^{a_j}](q^{a_j} - 1)^{-1} \equiv [i - (i - 1)q^{a_i}](q^{a_i} - 1)^{-1}( \mod n) \]

for all \(i, j = 2, \ldots, r\) and

\[ (q^{a_i} - 1)^{-1} \equiv [i - (i - 1)q^{a_i}](q^{a_i} - 1)^{-1}( \mod n) \]

for all \(i = 2, \ldots, r\). This means that

\[ n\left[^{j - (j - 1)q^{a_j}}_{(q^{a_j} - 1)^{-1}}(q^{a_j} - 1)^{-1} - (q^{a_i} - 1)^{-1}\right] \]

for every \(j = 2, \ldots, r\), i.e., \(n \mid \gcd(t_2, \ldots, t_r)\), where \(t_j = [j - (j - 1)q^{a_j}](q^{a_j} - 1)^{-1} - (q^{a_i} - 1)^{-1}\) for all \(j = 2, \ldots, r\).

Let \(C\) be the cyclic code whose defining is the \(q\)-coset \(C_x\). From construction, the defining set of \(C\), i.e., the coset \(C_x\), contains the sequence \(x, x + 1, \ldots, x + r\) of \(r + 1\) consecutive integers. From the BCH bound, the minimum distance \(d\) of \(C\) satisfies \(d \geq r + 2\). Since \(|C_x| = m^*\), the dimension of \(C\) equals \(n - m^*\). Then, one can get an \([n, n - m^*, d \geq r + 2]_q\) code, as required. \(\square\)

Corollary 3.2 Let \(q \geq 3\) be a prime power and \(n > m\) be a prime number such that \(\gcd(q, n) = 1\), where \(m = \ord_n(q) \geq r + 2\) and \(1 \leq r, a_1, a_2, \ldots, a_r < m\) are integers. If \(n \mid \gcd(t_2, \ldots, t_r)\), where \(t_j = [j - (j - 1)q^{a_j}](q^{a_j} - 1)^{-1} - (q^{a_i} - 1)^{-1}\) for every \(j = 2, \ldots, r\) and \(a_1, a_2, \ldots, a_r\) are integers such that \(1 \leq a_1 + a_2 + \ldots + a_r < m\) (the operations are performed modulo \(n\)), then there exists an \([n, n - m^*, d \geq r + 2]_q\) cyclic code.
Proof: Notice that since \( n \) is prime, it follows that \( \gcd(q^{a_i} - 1, n) = 1 \) for every \( i = 1, 2, \ldots, r \), because \( a_1, a_2, \ldots, a_r < m \). We next apply Theorem 3.1 and the result follows.

Let \( C_x \) be the \( q \)-coset of \( x \). We denote by \( C_{-x} \) the coset of \(-x\), where \(-x\) is taken modulo \( n \). With this notation we have:

**Theorem 3.3** Assume all the hypotheses of Theorem 3.1 hold. Let \( C \) be the cyclic code with defining set \( C_x \), where \( C_x \) is a coset containing \( r + 1 \) consecutive integers. If \( C_x \neq C_{-x} \) then there exists an \([[n, n - 2m^*, d \geq r + 2]]_q \) quantum code.

Proof: From Lemma 1, \( C \) contains its (Euclidean) dual code \( C^\perp \). The dimension and the minimum distance of the corresponding quantum code follow directly from Theorem 3.1 and from Lemma 2.2.

4 Examples and Code Comparison

**Example 4.1** Consider that \( q = 5 \) and \( n = 11; m = \text{ord}_{11}(5) = 5 \). The 5-cosets are \( C_0 = \{0\}, C_1 = \{1, 5, 3, 4, 9\} \) and \( C_2 = \{2, 10, 6, 8, 7\} \). If \( C \) is the cyclic code with defining set \( C_1 \), then it is a dual-containing code with parameters \([11, 6, d \geq 4]_5 \). From Lemma 2.3 one can get an \([[11, 1, d \geq 4]]_5 \) code. Similarly, take \( q = 17 \) and \( n = 19; m = \text{ord}_{19}(17) = 9 \). If \( C \) is the code with defining set \( C_1 = \{1, 17, 4, 11, 16, 6, 7, 5, 9\} \) one has an \([[19, 1, d \geq 5]]_17 \) quantum code. We have an \([61, 56, d \geq 3]_9 \) code with defining set \( C_8 = \{8, 11, 38, 37, 28\} \). It is a dual-containing code, so an \([[61, 51, d \geq 3]]_9 \) quantum code exists. There exists an \([67, 64, d \geq 3]_{29} \) dual-containing code \( C \) with defining set \( C_{12} = \{12, 13, 42\} \). Hence, there exists an \([67, 61, d \geq 3]_{29} \) quantum code. The existence of an \([35, 31, d \geq 3]_{13} \) dual-containing code \( C \) generates an \([[35, 27, d \geq 3]]_{13} \) quantum code. An \([35, 31, d \geq 3]_{27} \) dual-containing code \( C \) with defining set \( C_3 = \{3, 11, 17, 4\} \) guarantees the existence of an \([[35, 27, d \geq 3]]_{27} \) quantum code. An \([73, 70, d \geq 3]_{64} \) dual-containing code with defining set \( C_{21} = \{22, 21, 30\} \) exists, so there exists an \([73, 67, d \geq 3]_{64} \) quantum code.

**Example 4.2** In this example, we construction cyclic codes whose defining set consists of two \( q \)-cosets (the idea is the same as that presented in Theorem 3.1). An \([35, 27, d \geq 4]_{27} \) dual-containing code \( C \) with defining set consisting of \( C_2 \) and \( C_3 \) ensures the existence of an \([[35, 19, d \geq 4]]_{27} \) quantum code. Taking the cosets \( C_{14} = \{14, 20, 30\} \) and \( C_{21} \) one has an \([[73, 61, d \geq 4]]_{64} \) code. Similarly, an \([[63, 51, d \geq 3]]_{11} \) code (coset \( C_{43} \)) and an \([[63, 39, d \geq 4]]_{11} \) code (cosets \( C_{43} \) and \( C_{20} \)) can be constructed. Analogously, an \([[63, 51, d \geq 3]]_{23} \) and an \([[63, 45, d \geq 4]]_{23} \) code (cosets \( C_4 \) and \( C_{27} \)) can be constructed.

It is usual the comparison of the new code parameters with the ones presented in the literature. However, it seems that there is no source available in
Table 1: New quantum codes

| Parameters of the new codes |
|-----------------------------|
| $[11,1,d \geq 4]_5$         |
| $[19,1,d \geq 5]_{17}$      |
| $[35,27,d \geq 3]_{13}$     |
| $[35,27,d \geq 3]_{27}$     |
| $[35,19,d \geq 4]_{27}$     |
| $[51,35,d \geq 3]_{32}$     |
| $[61,51,d \geq 3]_{11}$     |
| $63,51,d \geq 3]_{11}$      |
| $63,39,d \geq 4]_{11}$      |
| $63,51,d \geq 3]_{23}$      |
| $63,45,d \geq 4]_{23}$      |
| $67,61,d \geq 3]_{29}$      |
| $73,67,d \geq 3]_{64}$      |
| $[73,61,d \geq 4]_{64}$     |

literature for codes over large alphabets. More precisely, the procedure adopted in [1] does not generate codes with relatively small length with respect to large alphabets. In [8], it is possible to derive good quantum codes of minimum distance three only if the length is a prime number, whereas here we can construct codes whose lengths are not necessarily prime and with minimum distances greater than three. Further, in [6], only primitive codes were constructed.

All quantum codes shown in Table 1 seem to be new. Recall that an $[[n,k,d]]_q$ quantum code satisfies $k + 2d \leq n + 2$ (QSB). Note that the new $[[67,61,d \geq 3]]_{29}$ and $[[73,67,d \geq 3]]_{64}$ codes have parameters satisfying $n + 2 - k - 2d \leq 2$; the parameters of the new $[[11,1,d \geq 4]]_5$, $[[35,27,d \geq 3]]_{13}$ and $[[35,27,d \geq 3]]_{27}$ codes satisfy $n + 2 - k - 2d \leq 4$. The new $[[11,1,d \geq 4]]_5$ code is comparable to the $[[17,9,4]]_5$ code shown in [3], and the new $[[61,51,d \geq 3]]_9$ code is comparable to the $[[65,51,4]]_9$ code shown in [3].

5 Final Remarks

We have constructed new quantum codes with good parameters by means of the CSS construction. The existence of such codes are due to the existence of suitable (classical) cyclic codes whose defining set consists of only one $q$-coset which contains at least two consecutive integers. This new method brings new ideas in order to construct more new quantum (classical) cyclic codes.

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