Abstract

The studies of influence of spin on a photon’s motion in a Schwarzschild spacetime is continued. In the previous paper [13] the first order correction to the geodesic motion is found for the first half of the photon world line. The system of equations for the first order correction to the geodesic motion is reduced to a non-uniform linear ordinary differential equation. The equation obtained is solved by the standard method of integration of the Green function.

1 Introduction

Existence of spin-gravitational interaction was proved by A. Papapetrou in his analysis of motion of deformable body [1, 2]. More recently we have derived Papapetrou equations as a reduction of equations of motion of (tangent) rigid body [3]. However, analysis based on classical mechanics does not admit passage to the limit of zero mass. The problem of motion of a massless spinning particle is especially interesting due to importance of electromagnetic waves in astronomical observations. Importance of the problem of photon world line with account of the spin-gravitational interaction was under discussion for decades [4]-[7], however it could not be formulated properly until equations of motion for massless spinning particle were derived.

Passage to classical mechanical limit from a non-scalar field equation is an alternative approach tried by numerous authors [8]-[10]. Straightforward derivation of photon’s equations of motion from Lagrangian of electromagnetic field was made in our works [11, 12]. As
the result we have obtained Papapetrou equation in the form
\[ \omega \frac{D\hat{x}^a}{ds} = R_{\alpha\beta\sigma}^a \hat{x}^\sigma \sigma^{\alpha\beta}, \] (1)
where \( \omega \) is frequency and \( \sigma^{\alpha\beta} \) – spin of photon which lies on tangent vector subspace of polarization vectors \( \hat{n}_{\alpha}, \alpha, \beta, \ldots = 2, 3 \). The spin remains parallel to itself in the polarization subspace along the world line. It is seen from this equation that the effect of spin-gravitational interaction grows linearly with the wavelength, so, under large enough wavelengths it may well be observable.

The next step is to find out photon world lines in a given model of space-time. The simplest way is to construct world lines via computing the first order correction to isotropic geodesic. These computation in case of Schwartzschild space-time was completed in our recent work [13] where, however, the desired result was not reached because the method used is valid only on a half of the world line which starts from infinitely distant source and ends up at the minimal value of the radial coordinate \( r \) (periastr). To construct the whole world line one needs to match two such halves built in different coordinate systems that is quite difficult. The goal of the present work is to work out a method of building the whole of the world line of a massless spin 1 particle in Schwartzschild space-time.

In the framework of the accepted approximation scheme we chose an isotropic geodesic (hereafter: reference geodesic) which is a world line of a spinless particle, then include spin into the right-hand side of the equation (1) putting it parallel to itself along the geodesic. It is convenient to introduce a geodetic flow in the equatorial plane (\( \theta = \pi/2 \) in spherical coordinates) as a vector field of velocities and its small variation. By construction the vector field satisfies geodesic equation, so it remains to find out its small variation such that the total vector satisfies the equation (1). This yields a linear equation of second order for the variation which can be solved on a geodesic of the flow. Solution of this equation is the first order correction to be found. The only independent variable in the equation is parameter \( s \) on the reference geodesic.

The metric of the Schwarzschild space-time can be specified by a field of orthonormal frame. A sample of pair of dual to each other covector and vector standard (with no isotropic elements) orthonormal frames is
\[ \begin{align*}
\theta^0 &= (1 - r_g/r)^{1/2} dt, & \tilde{e}_0 &= (1 - r_g/r)^{-1/2} \partial_t, \\
\theta^1 &= (1 - r_g/r)^{-1/2} dr, & \tilde{e}_1 &= (1 - r_g/r)^{1/2} \partial_r, \\
\theta^2 &= rd\theta, & \tilde{e}_2 &= r^{-1} \partial_\theta, \\
\theta^3 &= r \sin \theta d\varphi; & \tilde{e}_3 &= (r \sin \theta)^{-1} \partial_\varphi
\end{align*} \] (2)
where orthonormality of the frames means
\[ <\theta^a, \theta^b> = \eta^{ab}, \quad <\tilde{e}_a, \tilde{e}_b> = \eta_{ab}, \quad \theta^a(\tilde{e}_b) = \delta^a_b, \]
\( \eta_{ab} \) the standard Minkowski metric. The corresponding connection 1-form is
\[ \begin{align*}
\omega_1^0 &= \frac{r_g}{2r^2 \sqrt{1 - r_g/r}} \theta^0, & \omega_2^1 &= -\frac{\sqrt{1 - r_g/r}}{r} \theta^2, \\
\omega_2^0 &\equiv 0, & \omega_3^2 &= -\frac{\tan^{-1} \theta}{r} \theta^3, \\
\omega_3^0 &\equiv 0; & \omega_1^3 &= \frac{\sqrt{1 - r_g/r}}{r} \theta^3.
\end{align*} \] (3)
The 2-form of curvature referred to the frame is:

\[
\begin{align*}
\Omega^0_1 &= \frac{r_g}{r^3} \theta^0 \wedge \theta^1, \\
\Omega^1_2 &= -\frac{r_g}{2r^3} \theta^1 \wedge \theta^2, \\
\Omega^0_2 &= -\frac{r_g}{2r^3} \theta^0 \wedge \theta^2, \\
\Omega^2_3 &= \frac{r_g}{r^3} \theta^2 \wedge \theta^3, \\
\Omega^3_0 &= -\frac{r_g}{2r^3} \theta^3 \wedge \theta^1.
\end{align*}
\] (4)

2 Isotropic geodesic flow, canonical parameter and the co-moving frame

We construct a congruence of isotropic geodesic lines lying wholly on \( \theta = \pi/2 \) hypersurface (an equatorial plane) by solving the Hamilton-Jacobi equation for isotropic geodesics on it

\[
\langle d\Psi, d\Psi \rangle = 0
\] (5)

where the function to be found is of the form

\[
\Psi = Et - L\phi + R(r)
\] (6)

and contains two constant parameters \( E \) and \( L \). Accordingly to Hamilton-Jacobi theorem the integrated equations of geodesics are

\[
\frac{\partial \Psi}{\partial E} = \text{const}, \quad \frac{\partial \Psi}{\partial L} = \text{const}.
\]

These equations specify the variables \( t \) and \( \phi \) as functions of the variable \( r \):

\[
t = -\frac{\partial R}{\partial E}, \quad \phi = \phi_0 + \frac{\partial R}{\partial L}, \quad \theta = \pi/2
\] (7)

where the function \( R \) is found by integrating the Hamilton-Jacobi equation (5):

\[
R(r) = -\varepsilon(t) \int_{r_0}^{r} \sqrt{E^2 r^2 - L^2 (1 - r_g/r)} \frac{dr}{r(1 - r_g/r)}
\] (8)

where \( \varepsilon(t) \) is the well-known step function equal +1 for positive \( t \) and -1 for negative \( t \).

Each geodesic of the congruence starts at \( r = \infty \) under \( t = -\infty \) and reaches the periastr \( r = r_0 \) at the moment \( t = 0 \). After this moment the coordinate \( r \) grows and reaches infinity under \( t \rightarrow +\infty \). We choose initial conditions such a way that \( \phi|_{t \rightarrow -\infty} = 0 \), so that \( \phi_0 = \left. \frac{\partial R}{\partial L} \right|_{r \rightarrow \infty} \). It must be noted that the isotropic geodesics of the congruence lie wholly on surfaces of level of the function \( \Psi \) or, in other words, \( \Psi \) is constant on each of them. Therefore this function cannot be used as a parameter on the geodesics. Instead, a canonic parameter on the geodesics will be introduced below.

In order to reduce the number of non-zero components of spin of the particle it is convenient to introduce a co-moving frame on the geodesics like that done in work [13]. Since the geodesics are isotropic the co-moving frame contains isotropic elements \( \vec{n}_\pm \). One of
them, \( \vec{n}_- \) is orthogonal to surfaces \( \Psi = \text{const} \), (thus, due to properties of isotropic vectors, tangent to them everywhere):
\[
\vec{n}_- = \left\langle dx^i, d\Psi \right\rangle \partial_i
\]
and another isotropic vector \( \vec{n}_+ \) is chosen only due to the normalization condition \( \langle \vec{n}_-, \vec{n}_+ \rangle = 1 \). Two other vectors which constitute polarization subspace \( \vec{n}_\alpha, \alpha = 2, 3 \) are normalized to unit, space-like, orthogonal to each other and to the isotropic vectors:
\[
\begin{align*}
\vec{n}_- &= (1 - r_g/r)^{-1} \partial_t + Dr^{-2} \partial_\varphi - R'E^{-1}(1 - r_g/r) \partial_r, \\
\vec{n}_+ &= (1 - r_g/r)/2 \left[(1 - r_g/r)^{-1} \partial_t - Dr^{-2} \partial_\varphi + R'E^{-1}(1 - r_g/r) \partial_r\right], \\
\vec{n}_2 &= r^{-1} \partial_\theta, \\
\vec{n}_3 &= (r^2 - D^2(1 - r_g/r)) \varphi h \left(r^{-2} \partial_\varphi - \frac{\partial R'}{\partial L}(1 - r_g/r) \partial_r\right),
\end{align*}
\]
where
\[
R' = \epsilon(t) \frac{\sqrt{E^2 r^2 - L^2(1 - r_g/r)}}{r(1 - r_g/r)}
\]
and \( D = L/E \) is the impact parameter. The vector \( \vec{n}_- \) is obtained from the 1-form of momentum \( d\Psi \), by index lifting therefore, it satisfies the geodesic equation:
\[
\vec{n}_- \circ \vec{n}_- \equiv (\vec{n}_- \cdot \nabla)\vec{n}_- \equiv 0
\]
and hence \( \vec{n}_- \) is the canonical velocity on the geodesic \([15]\). Thus, the canonical parameter can be found by equating \( \vec{n}_- \) to \( \dot{x} = dx^i/ds \partial_i \) that gives:
\[
\frac{dr}{ds} = -R'/E(1 - r_g/r), \quad \frac{d}{ds} = \frac{dr}{ds} \frac{d}{dr}.
\]
It is seen that sign of \( dr/ds \) changes when passing through the hypersurface \( t = 0 \) which separates the isotropic geodesic into two halves. Under \( t < 0 \) coordinate \( r \) decreases along the geodesic until at \( t = 0 \) it reaches its minimal value \( r_0 \) and afterwards grows up to infinity under \( t > 0 \). Therefore \( r \) can be used as a parameter only on a half of the geodesic where the coordinate \( r \) runs the range \( r \in (r_0, \infty) \).

In order to compute the first order correction to the whole of the geodesic we need a monotonic parameter \( s \) on it. It is convenient to define it such that at the moment of time \( t = 0 \) when the coordinate \( r \) takes its minimal value \( (r = r_0) \) the value of \( s \) is zero, for example:
\[
s = \epsilon(t) \int_{r_0}^{r} \frac{r \, dr}{\sqrt{r^2 - D^2(1 - r_g/r)}}. \tag{10}
\]
Now we can find the relation between two parameterizations on the geodesic for each of its halves:
\[
ds = \frac{\epsilon(s)r \, dr}{\sqrt{r^2 - D^2(1 - r_g/r)}}. \tag{11}
\]
Since hereafter we shall use the values of the function \( R \) on the geodesic it is convenient to introduce an alternating function \( \bar{R}(s) \) with its derivative w.r.t \( L \) :
\[
\frac{\partial \bar{R}(s)}{\partial L} = \epsilon(s) \left| \frac{\partial R(r)}{\partial L} \right|, \quad \left| \frac{\partial R}{\partial L} \right| = \int_{r_0}^{r} \frac{D \, dr}{r \sqrt{r^2 - D^2(1 - r_g/r)}}. \tag{12}
\]
Besides, we need expressions of vectors \( \vec{n}_p = n^a_p \vec{e}_a \) via vectors \( \vec{e}_a \) of standard frame:

\[
\begin{align*}
\vec{n}_- &= (1 - r_g/r)^{-1/2} \vec{e}_0 + Dr^{-1} \vec{e}_3 - R'E^{-1}(1 - r_g/r)^{1/2} \vec{e}_1, \\
\vec{n}_+ &= (1 - r_g/r)/2 \left[(1 - r_g/r)^{-1/2} \vec{e}_0 Dr^{-1} \vec{e}_3 + R'E^{-1}(1 - r_g/r)^{1/2} \vec{e}_1\right], \\
\vec{n}_2 &= \vec{e}_2, \\
\vec{n}_3 &= (r^2 - D^2(1 - r_g/r))^{1/2} \left[r^{-1} \vec{e}_3 - \frac{\partial R'}{\partial L}(1 - r_g/r)^{1/2} \vec{e}_1\right].
\end{align*}
\]

(13)

3 First order approximation of the Papapetrou equation

Assuming that interaction of spin with gravitation is weak and deviation of the ray from the reference geodesic: \( \delta x^a/r \ll 1 \) is small we shall derive the equation for the deviation from the Papapetrou equation and solve it. It must be noted that analysis made in the work [13] contains one more inaccuracy: when considering deviation from a geodesic one should take into account the difference between values of connection on the geodesic and on the deviation from it. If it is done properly instead of the l.h.s. of the geodesic equation \( \vec{n}_- \circ \vec{n}_- \) one has that of the Jacobi equation

\[
\frac{D^2}{ds^2} \delta x^a - R^{a}_{bcd} n^b_{-} n^c_{-} \delta x^d
\]

[11]. Therefore the desired equation for the small deviation \( \delta x^a \) extracted form the Papapetrou equation (1) is

\[
\left[\frac{D^2}{ds^2} \delta x^a - R^{a}_{bcd} n^b_{-} n^c_{-} \delta x^d\right] = -\lambda R^{a}_{bcd} n^b_{-} n^c_{-} n^d_{-} \sigma^{23},
\]

(14)

where \( \lambda = \omega^{-1} = \text{const} \) is the (asymptotical) wavelength.

It was shown in the work [13] that the world line of a photon lie initially in the equatorial plane \( \theta = \pi/2 \) the vector of deviation has in the frame (13) the only non-zero component \( \delta x^2 \). Therefore it is convenient to have explicit form of the following contractions of the Riemann tensor:

\[
R^2_{bcd} n^b_{-} n^c_{-} = -\frac{3r_gD^2}{2r^5};
\]

(15)

\[
R^2_{bcd} n^b_{-} n^c_{-} n^d_{-} = \frac{3r_gD}{2r^5} \sqrt{r^2 - D^2(1 - r_g/r)}. 
\]

(16)

The latter is needed because in the co-moving frame (13) the spin of photon has only one non-zero component \( \sigma^{23} \). Taking into account that in the co-moving frame all components of the connection 1-form are zero we substitute partial derivative of the deviation vector for its covariant derivative and, referring to (4) and (13) we reduce the Papapetrou equation for the deviation vector in Schwarzschild background to the following ordinary differential equation

\[
\frac{d^2 y}{ds^2} + \frac{3r_gD^2}{2r^5} y = -\frac{3\lambda r_gD}{2r^5} \sqrt{r^2 - D^2(1 - r_g/r)} \sigma^{23},
\]

(17)
where the function \( y(s) \) to be found stands for \( \delta x^2 \). Accordingly to fact that under \( s \to -\infty \) the ray coincides with reference geodesic sought solution of the eq. must obey asymptotical conditions as follows:

\[
y(-\infty) = 0, \quad \frac{dy}{ds}(-\infty) = 0.
\]

(17)

It is convenient to rewrite the approximated Papapetrou equation in the standard denotations

\[
\frac{d^2 y}{ds^2} + Q(s)y(s) = F(s),
\]

(18)

with

\[
Q(s) = \frac{3rgD^2}{2r^5},
\]

\[
F(s) = -C \frac{D\sqrt{r^2 - D^2(1 - r_g/r)}}{r^5},
\]

\[
C = 3\lambda r_g \sigma_{23}/2.
\]

(19)

This is a well-known linear non-uniform ordinary differential equation of second order which can be solved by standard methods. To do it we need first the solution of the uniform equation which corresponds to commonplace deviation of geodesics, thus, to the Jacobi equation in which spin-gravitational is ‘switched off’ by putting \( C = 0 \). Then as the uniform equation is solved we can use the solution for constructing solution of the non-uniform equation and obtain deviation caused by the spin-gravitational interaction.

### 4 Integration of the Jacobi equation

In this section we make straightforward calculation of the deviation vector which in the field of local frames \([13]\) has only one non-zero component \( \delta x^2 \). For this end we consider a geodesic flow with small 2-component found from the corresponding solution of the Hamilton-Jacobi equation. Unlike the solutions considered above this solution depends on all four coosdinates \( \tilde{\Psi} = Et - M\varphi + R(r) + \Theta(\theta) \)

(20)

and contains three arbitrary constants \( E, L \) and \( M \). The function \( R(r) \) is the same as before and the new function \( \Theta(\theta) \) has the form

\[
\Theta(\theta) = \pm \int_\theta^\theta \sqrt{L^2 - M^2 / \sin^2 \theta}.
\]

If we put \( L = M \) thes function vanishes and the corresponding geodetic flow reduces to the flow considered above. So, if we put \((L^2 - M^2)/L^2 = \varepsilon^2\) an infinitesimal parameter we obtain a family of geodetic flows which are infinitesimally close to the geodetic flow in the equatorial plane. Applying the Hamilton-Jacobi theorem to the solution \((20)\) gives:

\[
\frac{\partial \tilde{\Psi}}{\partial M} = -\varphi \mp \int_\theta^\theta \frac{M / \sin^2 \theta}{\sqrt{L^2 - M^2 / \sin^2 \theta}} d\theta = \text{const},
\]

(21)

\[
\frac{\partial \tilde{\Psi}}{\partial L} = \pm \int_\theta^\theta \frac{L d\theta}{\sqrt{L^2 - M^2 / \sin^2 \theta}} + \frac{\partial R}{\partial L} = \text{const}
\]
and all the rest equations remain unchanged.

The second line specifies the component $\delta \theta$ of the deviation

$$\delta \theta = \frac{\sqrt{L^2 - M^2}}{L} \sin \left( \frac{\partial R}{\partial L} + \text{const} \right) = \varepsilon \sin \left( \frac{\partial R}{\partial L} + \zeta \right),$$

and the first line contains the same integral, therefore, neglecting the second power of the parameter $\varepsilon$ we obtain the same equation as under $\varepsilon = 0$:

$$-\varphi + \frac{M}{L} \frac{\partial R}{\partial L} = \text{const},$$

consequently, the component $\delta \varphi$ of the deviation is zero. Thus, the deviation vector has only one non-zero component which is $\delta \theta$:

$$\delta r = \delta t = \delta \varphi \equiv 0, \quad \delta \theta = \varepsilon \sin \left( \frac{\partial R}{\partial L} + \zeta \right),$$

where $\zeta = \text{const}$ and $\varepsilon$ is an arbitrary infinitesimal parameter. Parameter $\zeta$ specifies deviation from the geodesic at $t = -\infty$ which can also be put non-zero. In our denotations $y = \delta x^2$ solution of the Jacobi equation just obtained is:

$$y = \varepsilon r \sin \left( R_L + \zeta \right)$$

where $|R_L(r_1)|$ stands for the elliptical integral (12).

## 5 Integration of the non-uniform equation

In this section we solve the non-uniform equation (18) by standard method of integrating the Green function. By definition the Green function of this ordinary differential equation (18) is a piecewise-smooth function composed of two solutions of the uniform equation. Since the differential equation is of second order the relevant Green function $G(s|s_1)$ should be continuous in the point $s = s_1$ while its first derivative makes a unit jump [16]. This way we construct the desired Green function by matching two solutions of the uniform equation as follows:

$$G(s|s_1) = \begin{cases} 0 & \text{if } s < s_1, \\ \frac{r_1}{D} r \sin \{\varepsilon(s)|R_L(r)| - \varepsilon(s_1)|R_L(r_1)|\} & \text{if } s_1 < s; \end{cases}$$

(23)

Here the factor $r_1/D$ normalizes the jump of the derivative, $r_1 = r(s_1)$, $r = r(s)$ are specified by the function inverse to the function $s(r)$ defined in the equation (10) and the variable $s$ enters only the step function $\varepsilon(s)$ which specifies only the sign depending on the half of the world line. Besides, solution under $s < s_1$ is trivial that provides validity of our asymptotical conditions (17) at $s \to -\infty$.

Now, following the standard procedure we can find out solution of the nonuniform Jacobi equation (18) as follows:

$$y(s) = \int_{-\infty}^{\infty} G(s|s_1) F(s_1) ds_1 =$$

$$= -C r \int_{-\infty}^{s} \sin \{\varepsilon(s)|R_L(r)| - \varepsilon(s_1)|R_L(r_1)|\} \frac{r^2}{r_1^2} D^2(1 - r_g/r_1) ds_1,$$

(24)
Our goal is to describe behavior of the solution obtained under \( s > 0 \) corresponding to deviation the ray off the reference geodesic after the periastr. It is convenient to represent it by the angle of deviation:

\[
\delta \theta(s) = \frac{y(s)}{r} = -C \int_{-\infty}^{s} \sin \left\{ \varepsilon(s) |R_L(r)| - \varepsilon(s_1) |R_L(r_1)| \right\} \frac{1}{r_1^4} \sqrt{\frac{r_1^2 - D^2(1-r_g/r_1)}{r_1^2 - D^2(1-r_g/r_1)}} ds_1 = \\
\int_{-\infty}^{0}(...) ds_1 + \int_{0}^{s}(...) ds_1
\]

where we divide the domain of integration into parts of definite signs of the step function of \( s_1 \):

\[
\delta \theta(s) = \int_{\infty}^{r_0}(...) \frac{-r_1 dr_1}{\sqrt{r_1^2 - D^2(1-r_g/r_1)}} + \int_{r_0}^{r}(...) \frac{r_1 dr_1}{\sqrt{r_1^2 - D^2(1-r_g/r_1)}}.
\]

Here the square roots in the denominators appear when passing from the variable \( s_1 \) to \( r_1 \). At the same time we substitute the limit = \( \infty \) of \( r_1 \) for the limit = \( -\infty \) of \( s_1 \). After cancelling the square roots the expression simplifies and takes the form

\[
\delta \theta(s) = -C \int_{r_0}^{\infty} \frac{\sin \{\|R_L(r)\| + |R_L(r_1)|\}}{r_1^3} dr_1 - C \int_{r_0}^{r} \frac{\sin \{\|R_L(r)\| - |R_L(r_1)|\}}{r_1^3} dr_1.
\]

Introducing the notion of asymptotical angle \( \delta \theta_{\infty} = \delta \theta(+\infty) \) which describes the final angular divergence between the ray and its reference geodesic we obtain expression for the deviation on \( s > 0 \) branche of the ray as follows:

\[
y(s) = r \delta \theta_{\infty} + r \frac{\sin |R_L(r)| - \sin |R_L(\infty)|}{\sin |R_L(\infty)|} \delta \theta_{\infty} + C r \int_{r}^{\infty} \frac{\sin \{|R_L(r)| - |R_L(r_1)|\}}{r_1^3} dr_1.
\]

Evidently, the last term vanishes under \( s \to \infty \). Now let’s find expression for \( \delta \theta_{\infty} \). After some algebraic operations we obtain:

\[
\delta \theta |_{s \to \infty} = -2C \sin |R_L(\infty)| \int_{r_0}^{\infty} \cos |R_L(r_1)| \ r_1^{-3} dr_1. \tag{25}
\]

It is seen that the integral \( |R_L(r_1)| \) cannot be expressed in terms of convenient functions. However, since we are seeking for the first order on \( r_g/r_0 \ll 1 \) and the constant factor \( C = 3r_g \sigma^{23}/2E \) already contains the factor \( r_g \), it suffices to take the integral in zeroth order approximation on \( r_g/r_0 \). In other words, to obtain the desired result we can put \( r_g = 0 \) in \( |R_L(r_1)| \). This is possible because though the integrand has square root singularity at \( r_1 = r_0 \) all the integrals in \( \tag{25} \) are convergent. Moreover, the integrals as functions of \( r_g \) are continuous and differentiable at \( r_g = 0 \).

Now, simplifying the integral \( \tag{12} \) we obtain

\[
|R_L(r)|_{r_g=0} = \arccos(r_0/r).
\]

In this approximation we have \( |R_L(\infty)| = \pi/2 \). Substituting these expressions into \( \tag{25} \) simplifies the integration and finally we have:

\[
\delta \theta |_{s \to \infty} = -2C \int_{r_0}^{\infty} \cos \arccos(r_0/r_1) \frac{dr_1}{r_1^3} = -2C \int_{r_0}^{\infty} \frac{r_0 dr_1}{r_1^4} = -\frac{r_g \sigma^{23}}{E r_0^4} = -\frac{r_g \sigma^{23}}{E D^2}.
\]
where we take into account that $r_0 |_{r_s=0} = D$. Substituting this estimation to formula for the deviation we obtain under $s > 0$ following asymptotical presentation:

$$y(s) = r \delta \theta_\infty + \frac{D^2}{2r} \delta \theta_\infty + O(Cr^{-2}).$$

This expression is valid for the second half of the world line and represents photon world line under large $r$. Since the deviation on the second half grows linearly even if there is no interaction, just because the initial plane of the ray is lost during the first half, this linear growth yields about one order grater deviation at infinity than that on the periastr obtained in the work [13].

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