Effect of chiral anomaly on the circular dichroism and Hall angle in doped and tilted Weyl semimetals

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From the Kubo formula for transport in a tilted Weyl semimetal we calculate the absorptive part of the dynamic conductivity for both right and left handed circular polarized light. These depend on the real part of the longitudinal conductivity and the imaginary part of the transverse (Hall) conductivity. We include the effect of the chiral anomaly which pumps charge from negative to positive chirality node when the usual $E \cdot B$ term is included in the electrodynamics and obtain analytic expressions. To calculate the Hall angle we further provide expressions for the imaginary part of longitudinal and real part of the transverse conductivity and compare results with and without the pumping term. We also consider the case of a non centrosymmetric Weyl semimetal in which the chiral nodes are displaced in energy by an amount $\pm \Omega_0$. This leads to modification in dichroism and Hall angle which parallel the pumping case.

I. INTRODUCTION

In the last decade an area of research referred to as topological materials has seen phenomenal growth and has introduced many new ideas and concepts as well as materials with new functionalities\textsuperscript{1–6}. Of particular interest here are the Weyl semimetals\textsuperscript{7–15}. Most were first predicted in density functional band structure calculations and then found experimentally. The Dirac cones in these materials can be tilted with respect to the energy axis and there are two types. For type I (undertilted), the tilt $\mathcal{C}$ is less than the Fermi velocity, $v_F$. For $\mathcal{C} = v_F$ the cones have fully tipped and for $\mathcal{C} > v_F$ they are over-tilted (type II)\textsuperscript{12}. At charge neutrality the Fermi surface remains a single point at the node of the Dirac cone for type I while electron-hole pockets form at zero energy for type II.

Tilting can have important effects on the electromagnetic properties of Weyl semimetals. With finite doping, the interband optical absorption background, $\text{Re}\left[\sigma_{xx}(\Omega)\right]$\textsuperscript{16} (due to the longitudinal conductivity), which is linear in photon energy will be cut off at twice the value of the chemical potential $\mu_0$. There is no absorption for $\Omega < 2\mu_0$ at which point there is an abrupt increase to its undoped value. In type I Weyl, tilting will allow absorption for $\Omega > 2\mu_0/(1 + \mathcal{C})$ and provide modifications in the background till $\Omega \leq 2\mu_0/(1 - \mathcal{C})$ at which point the undoped background is restored and continues to high energies. The jump at $\Omega = 2\mu_0$ of the untilted case has now been replaced by a quasi linear smooth rise in the interval $2\mu_0/(1 + \mathcal{C}) < \Omega < 2\mu_0/(1 - \mathcal{C})$. In this same photon energy interval with tilt and finite doping the imaginary part of the dynamic transverse Hall conductivity $\text{Im}\left[\sigma_{xy}(\Omega)\right]$\textsuperscript{17} becomes finite. This leads to dichroism in the absorption of circular polarized\textsuperscript{18} light, since right and left hand polarizations will have different conductivities ($\text{Re}\left[\sigma_{\pm}(\Omega)\right]$, see Fig. 1(b) and Fig. 1(c)). One needs to remember that while the contributions to $\text{Im}\left[\sigma_{xy}(\Omega)\right]$ of both negative chirality node (NCN) and positive chirality node (PCN) have the same magnitude for a given $\mu_0$ and $\mathcal{C}$, they carry opposite sign (Fig.1(b)).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{Fig1.pdf}
\caption{(a) Schematic of two tilted Weyl cones with nonzero chiral pumping. The positive chirality node (PCN) and the negative chirality node (NCN) are tilted anticlockwise and clockwise respectively and we include chiral pumping, both affect the Hall conductivity. The charge imbalance between the two nodes, which changes their chemical potential is described by the parameter $\mu_0$, which is directly proportional to the $E \cdot B$ term and the inter-nodal scattering time ($\tau_v$). The effective chemical potential of node $s$ is given in Eq. (4). (b) The total longitudinal ($\sigma_{xx}$) and Hall ($\sigma_{xy}$) conductivity is a sum of the contribution from each chiral node. The upward and downward arrows refer to the positive and the negative contribution respectively. The length of the arrows signifies the different doping levels in each of the two Weyl nodes. (c) The charge imbalance makes a non-vanishing contribution to longitudinal and Hall conductivity which leads to changes in the circular dichroism, i.e. the right (red) and left (green) circularly polarized light gets absorbed by a different amount. Furthermore, bending the cones results in strong dependence on the tilt parameter.
\end{figure}
Further changing the relative sign of the tilt on one cone again changes its sign so that, if both are tilted clockwise or both anti-clockwise, the sum of Im $[\sigma_{xy}(\Omega)]$ coming from the two cones will cancel if they have the same magnitude of $\mu_0$ and $C$ and there will be no dichroism. To get a finite effect we need the cones to be oppositely tilted, one clockwise the other anticlockwise and this is the case we treat here.

An effect of particular importance which is associated with Weyl semimetal is the chiral anomaly. The application of a parallel electric ($E$) and magnetic ($B$) field drives charge from NNC to PCN. This leads to the two Weyl cones to effectively have different values of chemical potential and this alters their relative contribution to the dichroism when tilting is considered. Tilting is known to importantly modify the anomalous Hall effect, collective effects, anomalous Nernst, and modify disorder effects. Here we turn to the effect of tilt in Type I Weyl semimetals on both the dichroism and to the Hall angle.

This paper is organized as follows: In Sec. II, we present the theoretical formulation including the discussion on node dependent chemical potential, introduction to the system’s Hamiltonian, and the formalism for calculating interband optical conductivity. In Sec. III we calculate the optical conductivity for type I Weyl semimetals and provide analytical expressions for Re $[\sigma_{xx}(\Omega)]$ and Im $[\sigma_{xy}(\Omega)]$ and discuss the circular dichroism. The remaining parts, i.e. Im $[\sigma_{xx}(\Omega)]$ and Re $[\sigma_{xy}(\Omega)]$ are calculated using Kramers-Kronig relation and their analytical expressions along with the discussion on Hall angle are provided in Sec. IV. In Sec. V, we consider the case of broken inversion symmetry. In this case the NNC is pushed up in energy by an amount $Q_0$ while the PCN is pushed down by the same amount. This leads to an effective doping in the two cones describe by the chemical potential $\mu_\pm = \mu_0 \pm Q_0$. This is analogous to the pumping case with a different relationship between the two chemical potentials. Finally in Sec. VI we summarize and draw conclusion.

II. THEORETICAL MODEL

A. Charge non-conservation: nodal chemical potential

The chiral anomaly term, which is proportional to $E \cdot B$, pumps charge from negative chirality to the PCN. In doing so, the density of charge in the region of Weyl nodes is changed by

\[
\Delta n_s = s \frac{e^2}{4\pi^2\hbar^2} (E \cdot B) \tau_\nu, \tag{1}
\]

where $s$ is the chirality, $\tau_\nu$ is the inter-nodal relaxation time and is assumed large compared with the intra-node scattering rate so that thermal equilibrium is reached inside each nodal cone. To start with, we consider the case of tilted Weyl semimetal when the chiral term is not present, such that the total charge carrier is given by

\[
n_0 = \frac{1}{6\pi^2} \frac{\mu_0^3}{\hbar^3 v_F^3} \frac{\mu_0^3}{(C_s^2/v_F^2 - 1)^2}, \tag{2}
\]

where $\mu_0$ is the chemical potential without the chiral term which is taken to be positive and $C_s$ is the tilt velocity for chiral node $s$. This is modified when the chiral pumping is included and the total number of charge carriers in node $s$ is given by

\[
n_s = n_0 + s \frac{e^2}{4\pi^2\hbar^2} (E \cdot B) \tau_\nu. \tag{3}
\]

Now using Eq. (2) and Eq. (3), we find the chiral chemical potential to be

\[
\mu_s = (\mu_0^3 + s\mu_p)^{1/3}, \tag{4}
\]

where, $\mu_p^3 = 3e^2h v_F^3 (E \cdot B) \tau_\nu (C_s^2/v_F^2 - 1)^2/2$ and is assumed to be positive. For negligible chiral pumping, $\mu_p \to 0$, which implies $\mu_\pm \approx \mu_0$. In the NNC, the effective chemical potential becomes negative for $\mu_p > \mu_0$ and this node is hole doped. In order to proceed further in this case, we simply replace $\mu_- \to -|\mu_-|$.

B. Hamiltonian and optical conductivity

The Hamiltonian for low-energy quasi-particles in a tilted Weyl semimetal with nodal index $s$ is given as,

\[
\hat{H}^s = C_s k_z \mathbb{I}_{2 \times 2} + s \sigma \cdot k, \tag{5}
\]

where $C_s$ denotes the tilt parameter, $\mathbb{I}_{2 \times 2}$ is the $2 \times 2$ unit matrix, the Fermi velocity, $v_F$ and $\hbar$ are set to be equal to unity for simplicity, $\sigma$ consists of Pauli matrix triplets and $k = (k_x, k_y, k_z)$, is the quasi-particle momentum. The energy eigenvalue is given as, $\varepsilon_k^s = C_s k_z + \lambda |k|$, with $\lambda = 1 (-1)$ for conduction (valence) band. The chirality $s$ is chosen to be positive (negative) for anti-clockwise (clockwise) tilted node. The $|C_s| < 1$ case is characterized as a type-I Weyl semimetal whereas $|C_s| > 1$ refers to type-II Weyl semimetal. In this work, we restrict ourselves to the former class of materials only, as the extension of these results to type-II semimetals is straightforward.

The chiral optical conductivity can be given as, $\sigma_{\alpha\beta}^s = i\Pi_{\alpha\beta}^s (i\Omega_m)/\Omega$, where the finite temperature ($T$) current-current correlation function is defined as,

\[
\Pi_{\alpha\beta}^s = \sum_{\omega_n} \frac{dk}{\beta} \text{Tr} \left[ \hat{J}_{k\alpha} \hat{G}_k (i\omega_n + i\Omega_m) \hat{J}_{k\beta} \hat{G}_k (i\omega_n) \right], \tag{6}
\]

where $i\Omega_m \to \Omega + i0^+$, $\beta = (k_BT)^{-1}$, $k_B$ being the Boltzmann constant, $dk = d^3k/(2\pi)^3$, the current operator, $\hat{J}_k = \nabla_k \hat{H}^s$ and the Green’s function is expressed as,
\[ \hat{G}(k, z) = \left( z \hat{H}_{2 \times 2} - \hat{H} \right)^{-1}, \]
which for the Hamiltonian described in Eq. (5), can be expressed explicitly as,
\[ \hat{G}_k(z) = \frac{1}{\Delta} \begin{pmatrix} z + (s - C_z) k_z & s(k_x - ik_y) \\ s(k_x + ik_y) & z - (s + C_z) k_z \end{pmatrix}, \]
where we have \( \Delta = (z - C_z k_z)^2 - |k|^2 \). The spectral function can be extracted using the following relation,
\[ \hat{A}_k(z) = \int_{-\infty}^{\infty} d\omega \hat{A}_k(\omega). \]
Using Eq. (7) and Eq. (8), we have,
\[ \Pi_{xx}(i\Omega_n) = \frac{e^2}{\beta} \int dk \frac{d\omega}{2\pi} \int d\omega' \frac{f(\omega') - f(\omega)}{2\pi i\Omega_n - \omega + \omega'} \\
\times \text{Tr} \left[ \hat{\sigma}_x \hat{A}_k(\omega) \hat{\sigma}_x \hat{A}_k(\omega') \right], \]
where \( f(x) = (1 + e^{\beta(x-\mu)})^{-1}, \mu \) being the chemical potential. In this work, we are interested only in the interband optical conductivity. Therefore, focusing only on the interband contribution, the real part of the conductivity can be simplified and is given as
\[ \frac{\text{Re}[\sigma_{xx}(\Omega)]}{e^2\mu_0/(8\pi)} = \frac{\Omega}{64\pi\mu_0} \sum_{s=\pm} \int_{-1}^{1} dx \left( 1 + x^2 \right) \\
\times \left( f \left[ -\frac{\Omega}{2} (1 - C_s x) \right] - f \left[ \frac{\Omega}{2} (1 + C_s x) \right] \right), \]
In the similar manner we can evaluate the Hall conductivity.

### III. OPTICAL CONDUCTIVITY: TILTED WEYL SEMIMETAL

In this section we apply the formalism discussed in the previous section to calculate the longitudinal and Hall conductivities in a tilted Weyl semimetal. Using Eq. 11, the real part of dynamic longitudinal conductivity at \( T = 0 \) for chiral node \( s \) is given as,
\[ \frac{\text{Re}[\sigma_{xx}(\Omega)]}{e^2\mu_0/(8\pi)} = \frac{\Omega}{8\mu_0} \int_{-1}^{1} dx \left( 1 + x^2 \right) \left( \Theta \left[ \mu_s + \frac{\Omega}{2} (1 - C_s x) \right] - \Theta \left[ \mu_s - \frac{\Omega}{2} (1 + C_s x) \right] \right), \]
and the total longitudinal optical conductivity is given by, \( \text{Re}[\sigma_{xx}(\Omega)] = \text{Re}[\sigma_{xx}^+(\Omega)] + \text{Re}[\sigma_{xx}^-(\Omega)] \). For definitiveness, we have considered \( C_- = -C_+ = C \). The integral in Eq. (12) can be solved analytically, with the following defining cases: 1) \( (1 + C)\mu_- > (1 - C)\mu_+ \), and 2) \( (1 + C)\mu_- < (1 - C)\mu_+ \). For the first case, we have,
\[ \frac{\text{Re}[\sigma_{xx}(\Omega)]}{e^2\mu_0/(8\pi)} = \begin{cases} 0, & \Omega < \Omega_L^- \\
\sum_{s=\pm} \frac{G_{\mu_s} + 2F_1}{1 + 3\Omega^2/(4\mu_0^2)}, & \Omega > \Omega_L^+ \end{cases} \]
where we have defined,
\[ G_{\mu_s} = \mu_s^2 \frac{(3\Omega - 2\mu_s)}{6\mu_0 C^2 \Omega^2} - \frac{(1 + C^2)\mu_s}{4\mu_0 C^3}, \]
For the second case, we have,
\[ \frac{\text{Re}[\sigma_{xx}(\Omega)]}{e^2\mu_0/(8\pi)} = \begin{cases} 0, & \Omega < \Omega_L^- \\
\sum_{s=\pm} \frac{G_{\mu_s} + 2F_2}{1 + 3\Omega^2/(4\mu_0^2)}, & \Omega > \Omega_L^+ \end{cases} \]
The imaginary part of Hall conductivity is given as,
\[ \frac{\text{Im}[\sigma_{xy}(\Omega)]}{e^2/(8\pi)} = \frac{-\Omega}{4} \sum_{s=\pm} \int_{-1}^{1} dx \left( \Theta \left[ \mu_s + \frac{\Omega}{2} (1 - C_s x) \right] - \Theta \left[ \mu_s - \frac{\Omega}{2} (1 + C_s x) \right] \right), \]
which can easily be integrated out and we have,
\[ \frac{\text{Im}[\sigma_{xy}(\Omega)]}{e^2\mu_0/(8\pi)} = \begin{cases} 0, & \Omega < \Omega_L^- \\
\sum_{s=\pm} H_{\mu_s}, & \Omega > \Omega_L^+ \end{cases} \]
where,
\[ H_{\mu_s} = \frac{\mu_s (\Omega - \mu_s)}{2\mu_0 C^2} + \frac{(C^2 - 1)\Omega}{8\mu_0 C^2}, \]
and for the other case, \( \text{Im}[\sigma_{xy}(\Omega)] \) is given by \( H_{\mu_-} \) in the frequency regime, \( \Omega_L^- < \Omega < \Omega_L^+ \), and by \( H_{\mu_+} \) in the frequency regime, \( \Omega_L^+ < \Omega < \Omega_U^+ \). These are the only two interval where the \( \text{Im}[\sigma_{xy}(\Omega)] \) is non-zero.
For the Hall conductivity, we need to replace $\rho$ redundant as in this particular case we have, $\Omega$.

(c) $\mu_+ > \mu_-$.

For special case when we have $(1 + \mu_+) \mu_- = (1 - \mu) \mu_+$, the expression for the longitudinal and the Hall conductivity is still given by Eq. (13) and Eq. (17) respectively, except that the condition $\Omega_L^\mu_+ < \Omega < \Omega_U^\mu_-$ now becomes redundant as in this particular case we have, $\Omega_L^\mu_+ = \Omega_U^\mu_-$. Another case that could be realized in practice is when we have $\mu_+ > \mu_-$. In this case, for $\text{Re}[\sigma_{xx}(\Omega)]$, the expressions provided in Eq. (13) and Eq. (15) can be used with only a replacement, $\mu_- \rightarrow \mu_-$, where $\mu_- = (\mu_+^3 - \mu_0^3)^{1/3}$. For the Hall conductivity, we need to replace $\mathcal{H}_{\mu_-}$ with $-\mathcal{H}_{\mu_-}$ in Eq. (17) and elsewhere.

In Fig. 2 we show explicit results for two values of $\mu_+$ namely $\mu_+ = 0.8\mu_0$ in frames (b) and $\mu_+ = 0.99\mu_0$ in frames (c). The left frames in (b) and (c) are for the real part of the dynamic longitudinal conductivity while the right are for imaginary part of the dynamic Hall conductivity. The top frame is a schematic for the optical transition possible in the presence of charge imbalance. Returning to frames (b) and (c) the four dashed vertical lines identify the critical frequencies $\Omega_L^\mu_+ = 2\mu_+/(1 + C)$ and $\Omega_U^\mu_- = 2\mu_+/(1 - C)$. With $C = 1/2$ for definiteness, $\Omega_L^\mu_-$ is red, $\Omega_U^\mu_+$ is indigo and $\Omega_U^\mu_-$ is cyan. In frame (b) $(1 + \mu) \mu_- > (1 - \mu) \mu_+$, with $\mu_+ = 0.8\mu_0$ neither chirality node contributes to the longitudinal conductivity (left frame) below $\Omega_L^\mu_-$, in the next interval $\Omega_L^\mu_+ - \Omega_U^\mu_-$, only the NCN is involved. In the third and fourth both nodes enter and in the fifth above $\Omega_U^\mu_+$ we get back the no tilt no doping interband background of the longitudinal conductivity.

For transverse Hall conductivity shown in the right frame the situation is similar but now the contribution from the NCN ends at $\Omega_U^\mu_+$ (blue vertical line). Above this energy in the fourth interval the PCN continues till $\Omega_U^\mu_+$ (cyan vertical line) above which the Hall conductivity vanishes. For no pumping $\mu_+ = \mu_- = \mu_0$ and red and indigo vertical dashed lines merge as do blue and cyan and both chirality nodes contribute equally in the energy interval where they do not vanish. By contrast pumping has created two new energy intervals in which only one of the two nodes contribute to the Hall conductivity. This will have a direct effect on the dichroism for absorption of circular polarized light as we will see later.

In frame (c) of Fig. 2 we present as additional set of results for the case $(1 + \mu) \mu_- < (1 - \mu) \mu_+$ ($\mu_+ = 0.99\mu_0$), for which Eq. (15) applies rather than Eq. (13). Continuing with our discussion of the Hall conductivity first (right frame) the non zero contribution to $\text{Im}[\sigma_{xy}(\Omega)]$ in the second energy interval between red and blue vertical lines is entirely due to the NCN which is followed by a third region where $\text{Im}[\sigma_{xy}(\Omega)]$ vanishes. Beyond this in the fourth energy interval only the PCN contributes and finally above this in a fifth interval there is no Hall conductivity. Thus, in the case for $\mu_+ = 0.99\mu_0$ we have managed to separate out the contributions of PCN and NCN to distinct regions of frequencies. In reference to the longitudinal conductivity (left frame) we note that in the third energy region only the NCN is involved and we are above $\Omega_L^\mu_-$ so that the no doping no tilting interband background applies and this straight line is $\Omega/3\mu_0$ exactly half the value which apply above the last cyan vertical line where the two nodes are involved.

Till now we have discussed only the interband conductivity. In the low frequency regime, the intraband contribution dominates which, in the clean limit is expressed as:

$$\mathcal{D} = \frac{e^2 \mu_0^2}{8\pi C^3} \left( \frac{2C}{1 - C^2} - \log \left[ \frac{1 + C}{1 - C} \right] \right),$$

such that the total Drude weight is given as,
$\mathcal{D} = \mathcal{D}^+ + \mathcal{D}^-$. There is no such intraband contribution to the Hall conductivity.

**Dichroism.**- The right and left circularly polarized light gets absorbed by different amounts due to the presence of transverse optical conductivity. If the absorptive part of the optical conductivities are denoted as, $\sigma_+$ and $\sigma_-$ respectively, we define,

$$\sigma_\pm(\Omega) = \text{Re}[\sigma_{xx}(\Omega)] \pm \text{Im}[\sigma_{xy}(\Omega)], \quad (20)$$

hence one can get the analytical expression by simply adding/subtracting the $\text{Re}[\sigma_{xx}(\Omega)]$ and $\text{Im}[\sigma_{xy}(\Omega)]$ parts.

![Optical conductivity](image)

**FIG. 3.** Optical conductivity, $(\sigma_\pm)$ as seen by left and right circularly polarized light as a function of frequency for (a) $\mu_p = 0$, (b) $\mu_p = 0.8\mu_0$, (c) $\mu_p = 0.99\mu_0$, and (d) $\mu_p = \mu_0$. For $\mu_+ \to \mu_0$, $\mu_- \to 0$, implying that for the negative node optical transitions start at zero frequency. Increasing frequency beyond $\Omega^{\mu_+}$, we see that both the nodes contribute, as expected. Color coded numbers identify the values of the critical frequencies defining the various photon ranges. Other parameters are the same as those of Fig. 2.

In Fig. 3 we present results for the absorptive part of the dynamic conductivity associated with right and left handed polarized light. For vanishing chiral pumping the chemical potential in each Weyl node is the same and each nodes contribute equally to both longitudinal and transverse conductivity. There is no absorption below photon energy of $4\mu_0/3$ (frame (a)). This is followed with a region of dichroism. Above photon energy of $4\mu_0$ there remains no dichroism and the conductivity is due entirely to the longitudinal part, as the Hall contribution is zero. In the units used, this contribution is linear with slope $2/3$. When $\mu_p$ is increased to $0.8\mu_0$, shown in frame (b), the region of no absorption below the first vertical line at energy $4\mu_-/3$ is reduced as compared with the no pumping case. It is followed by a region where only the NCN contributes to the absorption and the dichroism is due only to this node. At the second vertical line the PCN begins to contribute and both nodes contribute to the dichroism. At the third vertical line the NCN no longer has a finite off diagonal (transverse) part and the dichroism is due entirely to the PCN. Above the fourth vertical dashed line the system returns to its no doping with no tilting value as in frame (a).

In frame (c) the pumping has been increased to $0.99\mu_0$. The small photon energy region of no absorption has again been reduced. It is followed by a region of dichroism extending between the first two vertical dashed lines. Here only the NCN contributes. This is followed by a third region where there is no dichroism and the conductivity is linear in photon energy with slope $1/3$ as again only the NCN is involved. In the region above the third vertical line both nodes are now involved but the dichroism comes only from the PCN. For the final region above the fourth vertical line there is no dichroism at all since the transverse Hall conductivity is zero for both nodes. Finally we point out that the region between the first two vertical lines behaves exactly as in frame (a) but now only the NCN is involved. For the last frame the doping in the NCN is zero and the absorptive Hall conductivity for this node is zero at all photon energies. This contribution now starts at zero energy and the conductivity shows no dichroism till the second vertical dashed line is reached and the PCN also contributes. There is dichroism in this region due only to the PCN until the third vertical dashed line is reached at which point the conductivity returns to its no doping, no tilting value.

Pumping has made important changes to both longitudinal and transverse (Hall) dynamic conductivity. While for $\mu_p = 0$ both Weyl nodes contribute equally to both these quantities, this is no longer the case when $\mu_p$ is non zero. Finite $\mu_p$ can separates out regions of photon energies in which a single chirality node contributes and others where both contribute but not equally. In particular the dichroism can be entirely due to one node or the other or a combination of both but not in equal proportion. In all cases there are kinks in the conductivities as we move from one region of photon energy to another.

**IV. BULK HALL ANGLE**

Having discussed the absorption of circularly polarized light, we now proceed to study the bulk Hall angle. The dynamic bulk Hall angle can be defined as\textsuperscript{33–35},

$$\Theta_H(\Omega) = \tan^{-1} \left( \frac{\text{Re}[\sigma_{xy}(\Omega)]}{\text{Re}[\sigma_{xx}(\Omega)]} \right), \quad (21)$$

where $\sigma_{\alpha\beta}(\Omega) = \text{Re}[\sigma_{\alpha\beta}(\Omega)] + i\text{Im}[\sigma_{\alpha\beta}(\Omega)]$, such that $(\alpha, \beta) \in \{x, y\}$. Apart from the real part of longitudinal and imaginary part of Hall conductivity discussed in the previous sections, we also need the imaginary part of longitudinal and real part of Hall conductivity for the calculation of the Hall angle. In order to calculate these
quantities we rely on the Kramers-Kronig relation for conductivities which are given as,

\[
\text{Im } [\sigma_{xx} (\Omega)] = -\frac{2\Omega}{\pi} \mathcal{P} \int_0^\Lambda d\Omega' \frac{\text{Re } [\sigma_{xx} (\Omega')]}{\Omega'^2 - \Omega^2},
\]

\[
\text{Re } [\sigma_{xy} (\Omega)] = \frac{2}{\pi} \mathcal{P} \int_0^\Lambda d\Omega' \frac{\Omega' \text{Im } [\sigma_{xy} (\Omega')]}{\Omega'^2 - \Omega^2},
\]

where \(\mathcal{P}\) refers to the principle value of the integral and \(\Lambda (\gg \Omega)\) is the ultra-violet cutoff. For the case \((1 + \mathcal{C}) \mu^- > (1 - \mathcal{C}) \mu^+\), substituting Eq. (13) and Eq. (17) in Eq. (22), we find the corresponding Kramers-Kronig counterparts as given below,

\[
\text{Im } [\sigma_{xx} (\Omega)] = -\frac{\Omega}{24\mu_0 \pi C^3} \sum_{s=\pm} \left( \frac{8\mu_c^3}{\Omega^3} + 6 \frac{(1 + C^2) \mu_s}{\Omega} \right) \log \left( \frac{\Omega_{L,s}^{\mu_s} - \Omega}{\Omega_{L,s}^{\mu_s} + \Omega} \right) \left( \frac{\Omega_{L,s}^{\mu_s} + \Omega}{\Omega_{L,s}^{\mu_s} - \Omega} \right)
\]

\[
+ \frac{16\mu_c^3}{\Omega^2} \left( \frac{1}{\Omega_{L,s}^{\mu_s}} - \frac{1}{\Omega_{L,s}^{\mu_s}} \right) - \frac{1}{\pi \mu_0} \left( \frac{2 \log \left( \frac{\Omega_{L,s}^{\mu_s} - \Omega}{\Omega_{L,s}^{\mu_s} + \Omega} \right) + \log \left( \frac{\Omega_{L,s}^{\mu_s} - \Omega}{\Omega_{L,s}^{\mu_s} + \Omega} \right) + \frac{12\mu_c^3}{\Omega^2} \log \left( \frac{\Omega_{L,s}^{\mu_s} - \Omega}{\Omega_{L,s}^{\mu_s} + \Omega} \right) \right) \]

\[
+ \sum_{s=\pm} \frac{\mu_s}{2\mu_0 \pi C^3} \log \left( \frac{\Omega_{L,s}^{\mu_s} - \Omega}{\Omega_{L,s}^{\mu_s} + \Omega} \right) - \frac{\mu_s^2}{2\mu_0 \pi C^3} \Omega \]

\[
\text{Re } [\sigma_{xy} (\Omega)] = \frac{(C^2 - 1)}{4\mu_0 \pi C^2} \left( \frac{\Omega}{2} \log \left( \frac{\Omega_{L,s}^{\mu_s} + \Omega}{\Omega_{L,s}^{\mu_s} - \Omega} \right) \left( \frac{\Omega_{L,s}^{\mu_s} + \Omega}{\Omega_{L,s}^{\mu_s} - \Omega} \right) + \frac{\mu_s^2}{\Omega_{L,s}^{\mu_s} - \Omega} \right) \]

\[
+ \sum_{s=\pm} \frac{\mu_s}{2\mu_0 \pi C^3} \log \left( \frac{\Omega_{L,s}^{\mu_s} - \Omega}{\Omega_{L,s}^{\mu_s} + \Omega} \right) - \frac{\mu_s^2}{2\mu_0 \pi C^3} \Omega \right).
\]

Similarly, we can obtain the Kramers-Kronig pairs for other cases. In the DC limit (photon energy \(\Omega = 0\)) the expression for \(\text{Re } [\sigma_{xy} (\Omega)]\) greatly simplifies and reduces to,

\[
\text{Re } [\sigma_{xy} (0)] = \frac{(\mu_+ + \mu_-)}{\mu_0 \pi C} \left( -2 + \frac{1}{\Omega} \log \left( \frac{1 + C^2}{1 - C} \right) \right),
\]

which agrees with the previous result reported in reference [26]. Note that \(Q_0\) drops out of the D.C. limit which depends only on \(\mu_0\). Eq. (25) shows that the tilt makes an important contribution to \(\text{Re } [\sigma_{xy} (0)]\). In the limit of small tilt the leading order in \(C\) is linear and consequently this contribution is only non-zero when there is a tilt. From Eq. (21), we can see that, \(\tan \Theta_H (\Omega)\)

\[
= \frac{\text{Re } [\sigma_{xx} (\Omega)] \text{Re } [\sigma_{xy} (\Omega)] + \text{Im } [\sigma_{xx} (\Omega)] \text{Im } [\sigma_{xy} (\Omega)]}{(\text{Re } [\sigma_{xx} (\Omega)])^2 + (\text{Im } [\sigma_{xx} (\Omega)])^2},
\]

Note from the structure of the numerator in Eq. (26) that, because both \(\text{Re } [\sigma_{xx} (\Omega)]\) and \(\text{Im } [\sigma_{xy} (\Omega)]\) are zero for photon energies less that \(2\mu_-/(1 + C)\), the Hall angle will be zero as well provided of course \(\mu_-\) is non zero. We will return to this point later. Note further that \(\text{Im } [\sigma_{xy} (\Omega)]\) has a natural cutoff at energy \(2\mu_+/(1 - C)\) so that the second contribution to the Hall angle in Eq. (26) vanishes beyond this point. This is not the case for the first contribution. The real part of \(\sigma_{xx} (\Omega)\) is positive and grows linearly at large \(\Omega\) while the real part of Hall conductivity (Fig. 4(b)) becomes negative and small but is not zero. Consequently we expect the Hall angle to get small but remain negative.

![Fig. 4](image_url)

FIG. 4. (a) The imaginary part of longitudinal and (b) real part of the Hall conductivity is shown as a function of frequency. The bubbled line in (a) refers to the total sum of the contribution from the inter (red) and the intra-band (blue) conductivity, black dashed line is for \(\mu_p = 0\) and green circles for \(\mu_p = 0.5\mu_0\). In (b), the D.C. value of Hall conductivity is consistent with Eq. (25). Other parameters are the same as those of Fig. 2.
pumping). Also shown separately are the interband (red curve) and the intraband (blue curve) contribution for $\mu_p = 0.85\mu_0$. In Fig. 4(b), corresponding results are shown for $\text{Re} \sigma_{xy}(\Omega)$. In this case, the differences between green circles and the black dashed curve are more significant, particularly in the region of the maximum and minimum of the black dashed curve. Note that neither of these quantities have a sharp cutoff at small or large $\Omega$ although $\text{Re} \sigma_{xy}(\Omega)$ does become small and negative as $\Omega \to \infty$, as we have already commented on.

Our results for the Hall angle, $\Theta_H$ (in radians) are presented in Fig. 5(a) for several values of pumping namely, $\mu_p = 0$ (red), $\mu_p = 0.85\mu_0$ (green), $\mu_p = 0.99\mu_0$ (blue) and $\mu_p = 1.05\mu_0$ (indigo). For no pumping, the chemical potential of the two nodes are equal and we observe that the Hall angle is always negative except for a very small positive peak at the onset. This peak can be understood by looking at the numerator of Eq. (26) which has two positive peak at the onset. This peak can be understood by looking at the numerator of Eq. (26) which has two positive terms and their sum sets the sign on the Hall angle. Just above the lower energy cut off, the first term is positive because both $\text{Re} \sigma_{xx}(\Omega)$ and $\text{Re} \sigma_{xy}(\Omega)$ are positive. But $\text{Re} \sigma_{xy}(\Omega)$ has a peak in this region (Fig. 4(b), black dashed curve) and then changes sign. The second term in Eq. (26) is always negative in the region of interest. Consequently the sum of these two terms can only be positive in a small interval above the lower cutoff, and then become negative. At large $\Omega$ the Hall angle remains negative as it approaches zero (solid red curve in Fig. 5(a)). The continuous green curve for $\mu_p = 0.85\mu_0$ behaves in much the same way as the red but the peak above the onset is now much more pronounced, but in the negative region for the Hall angle, $\Theta_H$, the two curves track each other well. The solid blue curve is for $\mu_p = 0.99\mu_0$, shows two small kinks identified in the figure as $\Omega_{L}^{\mu_0}$ and $\Omega_{L}^{\mu_+}$. These arise due to the unequal chemical potential in the two nodes. The region between these kinks is one where there is no dichroism (see Fig. 3 panel (c)). In the Hall angle this manifests as a plateau type behaviour around the end of this frequency interval, as the second kink is approached. The indigo curve is for $\mu_p = 1.05\mu_0$ and shows yet another possible variation for the Hall angle. Here the first peak is negative and then it has a second peak which is positive. This variation is understood as follows. For this case the effective chemical potential of the NCN become negative (hole doping). This changes the sign of $\text{Im}(\sigma_{xy}(\Omega))$ associated with this node and this changes the sign of the numerator in Eq. (26) so the first peak is a valley (negative) followed at larger values of $\Omega$ by a positive peak.

V. BROKEN INVERSION SYMMETRY

In this section we consider the case of broken inversion symmetry. This leads to the energy of the NCN to be pushed up in energy by an amount $Q_0$, while the positive energy node is pushed down by the same amount. This is illustrated in Fig. 6(a). Just as with pumping, there is a different effective chemical potential in the two Weyl cones and the formulas developed so far can still be used. Differences will arise only from the different relationship between the two chemical potentials involved.

In Fig. 6(b), we show results for the absorptive part of circularity polarized light ($\sigma_{\pm}$). Four values of $Q_0$ have been considered, namely, 0.5$\mu_0$, 1.5$\mu_0$, 2$\mu_0$ and 2.5$\mu_0$. Note that $Q_0 = 0$ and $Q_0 = \mu_0$ are identical to case of $\mu_p = 0$ and $\mu_p = \mu_0$ respectively. In the first frame on the left, the red curve for $\sigma_{+}$ is always below the green curve for $\sigma_{-}$ because the imaginary part of the Hall conductivity is positive for both nodes. There is no absorption below photon energy of 2$\mu_0$/3 (first vertical red dashed line). Above this energy till we reach a photon energy of 2$\mu_0$, the NCN has a non zero imaginary part of the dynamic Hall conductivity and it is only this node that contributes to the dichroism and indeed to the absorption. The PCN contributes only above $\Omega = 2\mu_0$. While in this energy range both node contribute to the absorption, only the positive chirality contributes to the dichroism. Beyond $\Omega = 6\mu_0$, there is no dichroism and both nodes contribute equally to the absorption and this dependence on $\Omega$ is that of the no doping no tilting case. In the second frame for $Q_0 = 1.5\mu_0$, there is a region at small energy where there is no absorption followed by a second region where the red curve is now above the green. This feature is due to the fact that the effective chemical potential in the NCN is negative (hole doping)

FIG. 5. Bulk Hall angle as a function of frequency for (a) the non-vanishing chiral pumping case, where the red curve is the no pumping case ($\mu_p = 0$) and is for comparison, and (b) the broken inversion symmetry case. Other parameters are the same as those of Fig. 2.
FIG. 6. (a) Schematic for various transitions involved when inversion symmetry is broken. (b) Absorptive part of the conductivity for right and left circularly polarized light as a function of frequency for $Q_0 = 0.5 \mu_0$, $Q_0 = 1.5 \mu_0$, $Q_0 = 2 \mu_0$ and $Q_0 = 2.5 \mu_0$. This case is analogous to the chiral pumping case shown in Fig. 3, except that the red and green curves are less distorted here. Other parameters are the same as those of Fig. 2.

and this changes the sign of $\text{Im}[\sigma_{xy}(\Omega)]$. At yet higher photon energies there is a region of no dichroism and the absorption is only from the NCN. Above this energy the PCN kicks in and red and green curves switch. The dichroism is now due to this node. The other two frames serve to show the richness of behaviours that can arise.

For $Q_0 = 2 \mu_0$, in the region between first and second horizontal lines only the NCN is involved and its effective chemical potential is negative so the green and red curves switch. In the third interval the PCN has kicked in while the negative chirality node has no Hall contribution. Consequently in this region the dichroism is entirely due to the PCN. The final value chosen for $Q_0 = 2.5 \mu_0$ shows yet another different behaviour. In the interval between the first two vertical lines we have the same behaviour as in the case $Q_0 = 2 \mu_0$ but beyond that there is an important difference because the contribution of the NCN now extends to the third vertical line and the second region involves a finite imaginary Hall contribution from both nodes with opposite signs (negative for NCN and positive for PCN). This is why the crossing point between $\sigma_+$ and $\sigma_-$ does not fall on one of the vertical lines but is determined by the crossing of NCN and PCN Hall conductivity which have the opposite sign. In the fourth interval only the PCN makes a contribution to the dichroism.

FIG. 7. (a) The imaginary part of longitudinal and (b) real part of the Hall conductivity is shown as a function of frequency for broken inversion symmetry case. The bubbled line in (a) refers to the total sum of the contribution from the inter and the intra-band conductivity for $Q_0 = 0.5 \mu_0$. In panel (b), $\text{Re}[\sigma_{xy}(\Omega)]$ is shown for the same value of $Q_0$. Please note that its D.C. value is consistent with Eq. (25). Other parameters are the same as those of Fig. 2.

In Fig. 7 we show results for the imaginary part of longitudinal conductivity in frame (a) and real part of Hall conductivity in frame (b), as in Fig. 4 but now we have $Q_0 = 0.5 \mu_0$ (green circles) compared with $Q_0 = 0$ case (see Fig. 4(b)). As Eq. (25) tells us, $Q_0$ drops out of the D.C. limit for $\text{Re}[\sigma_{xy}(\Omega)]$ but introduces new structure as a function of $\Omega$ when compared with frame (b) of Fig. 4 and this leads to important differences in the behaviour of the Hall angle.

The results for Hall angle are shown in Fig. 5(b) for four values of $Q_0$, namely, $Q_0 = 0.5 \mu_0$ (red), $Q_0 = 1.5 \mu_0$ (green), $Q_0 = 2 \mu_0$ (blue) and $Q_0 = 2.5 \mu_0$ (indigo). Only the first (red) curve corresponds to electron doping for both nodes. In the other cases the doping in the NCN involve holes and as expected all these curves show, first a valley followed by a peak at higher energies as found for the chiral pumping case with $\mu_p = 1.05 \mu_0$. The characteristics of the curves are similar to those already described in detail for the pump case and will not be emphasized further here. We point out one feature namely that, for the green curve for which there is a region of no dichroism between second and third vertical line in Fig. 6(b) (right top frame), we again see a plateau type structure ending in a kink as we emphasized in the blue curve in Fig. 5(a).
VI. SUMMARY

In this work we have considered the optical properties of a doped and tilted Weyl semi metal. With tilting and a finite chemical potential, such systems acquire a finite transverse dynamic Hall conductivity in a photon energy interval between $2\mu_0/(1+C)$ and $2\mu_0/(1-C)$. This leads to dichroism for the absorption of circular polarized light. When a parallel electric and magnetic field is applied to the sample, there is a charge transfer from negative to PCN due to the chiral anomaly. This leads to a new steady state in which the effective chemical potential of the NCN is smaller than $\mu_0$ and that in the PCN is larger. Another way of producing a difference in chemical potential between the nodes is to brake inversion symmetry and consider the noncentro symmetric case. In both these instances the dichroism is modified as is the dynamic Hall angle. Here we have considered both cases and have provided analytic results for both the real and imaginary part of the dynamic longitudinal and transverse Hall conductivity from which both the dichroism and Hall angle follow. We have found a rich pattern of possible behaviours depending on the size of the differences in effective chemical potential of the two nodes.

When considering absorption, there can be regions of photon energies in which only one of the nodes contributes to the longitudinal conductivity, and others where both contribute. This is also the case for the imaginary part of the Hall conductivity. A case considered in detail shows a first region at low energy for which there is no absorption of both right and left handed circularly polarized light, followed by a region to which only the NCN contributes, but there is nevertheless dichroism. This is followed by another region which still is a result of only the NCN but now displays no dichroism and the absorptive part of the conductivity is linear in photon energy with slope $1/3$ in our reduced units. At still higher energies both nodes contribute to the absorption but only the PCN is involved in dichroism. Finally there is a region at higher energies where the well known, no doping no tilting behaviour of a Weyl semimetal is recovered where the slope of the linear behaviour is equal to $2/3$. Other interesting patterns also emerge.

The dynamic Hall angle is strongly modified by pumping due to the chiral anomaly or due to broken inversion symmetry. As an example for no pumping and inversion symmetry, the Hall angle is negative for photon energies between $2\mu_0/(1+C)$ and $2\mu_0/(1-C)$ and zero everywhere else except for a small peak at its onset. With pumping this all changes, although for small $\mu_p$ values, the changes from its zero pumping value are small except that the initial peak can be much larger. When $\mu_p = 0.99\mu_0$, there is a positive peak above the onset energy and then a slower decay towards zero Hall angle as $\Omega \to \infty$. There is also a region where the Hall angle plateaux and shows relatively little change with changing $\Omega$. This region corresponds to a region of frequency where there is no dichroism. For $\mu_p = 1.05\mu_0$, the behaviour is similar but with an initial negative dip before a large positive peak is seen. The reason for a dip rather than a peak is traced to the contribution from the NCN which is hole doped. A second rich pattern of possible behaviours for the Hall angle is predicted for the case of no pumping but broken inversion symmetry.

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