Emerging cooperation on the road by myopic local interactions

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Abstract

We study a combinatorial problem inspired by the following scenario: fully autonomous vehicles drive on a multi-lane \( m \geq 2 \) road. Each vehicle heads to its own destination and is allowed to exit the road only through a single designated off-ramp lane. However, an individual vehicle has a severely limited memory and sensing capabilities, and, moreover, does not communicate with its peers. In this work we present a distributed algorithm that, nonetheless, allows vehicles to get to the desired lane without collisions and in timely manner.

1 Introduction

In recent years the research in the field of autonomous vehicles has gained considerable momentum, and the idea of relieving the burden of driving from humans starts to lose its futuristic science fiction aura. Some people believe that autonomous traffic is “our last hope” of relief from the frequent road-jams, we now witness in even mid-size urban areas. We envision roads of the future with fully autonomous vehicles, that not only track the lane, keep safe distance and assist the driver, but essentially liberate humans from driving related activities altogether.

Contemporary vehicles already posses a relatively high-degree autonomy. Today, low-level procedures for collision-free driving are routinely executed by the consumer vehicles. Furthermore, the advancement on the strategic level is also impressive. We can plan optimal routes from point A to point B. Moreover, we can accomplish such planning, accounting for numerous rules, e.g. considering current road conditions, intermediate point visiting requirements and meeting stringent time constraints. These are high-level tasks that technology provided solutions for us even before the DARPA proposed the Urban Challenge back in 2007 (3).

However, before revolutionary changes will greatly improve our quality of transportation, a different line of challenges should also be addressed. Cur-
intermediate level responsibilities are still handed back to the driver. The list of such tasks includes merging into the traffic on the entry ramps and exiting it via the off-ramps. Some initial efforts to address the associated complexity were already made by the automotive community. Various approaches attempted include modeling the problem, developing communication protocols and numerous centralized controller-based solutions. On the other hand, we still lack truly scalable distributed solutions addressing these challenges, solutions to mitigate single points of failure of centralized controllers.

In this work we envision autonomous vehicle traffic on a multi-lane freeway. A road is assumed to connect a number of locations \( D_1, D_2, \ldots, D_K \) along the road. We can consider a specific location, say \( D_K = S \), as the source and model, w.l.o.g., a circular road to the other locations as destinations (Fig. 1a). A similar model of integrated system of corridors and the urban network was previously mentioned in [1]. However, our interest will mainly be focused on a single sliding section of the road after an on-ramp entrance and before the next off-ramp road exit.

We consider the following scenario: autonomous cars enter the freeway through an on-ramp entry at some location. The specific car entrance point is of no importance. Each car heads on its own, forward along the road, and it is expected to depart through a desired off-ramp exit. We shall assume further that a vehicle is required to move on the right-most lane in order to exit at the destination off-ramp (see Fig. 1b for a depiction of a departure-merger section). Hence, we assume Right-hand Traffic (RHT), however, clearly, the very same principles could be applied to the Left-hand Traffic roads with appropriate changes (if in England, for example!)

Throughout the work, we assume vehicles possess full autonomy. They execute all the necessary low level maneuvers, which in our model boil down to

Figure 1: (a) A circular road around few urban destinations \( (D_1, D_2, \ldots, D_K) \). (b) An exit/entry section of the road.
“follow the lane at constant speed”. Additionally, we require a vehicle to be able
to execute some high-level complex commands. The list of supported commands
is as follows: do nothing (“continue to move in the same lane at constant speed
\( v \)”), change lane to the left or right, move forward/backward, where forward
and backward is relative to vehicles that continue to move at constant speed \( v \)
on neighboring lanes.

In this paper we develop a series of completely distributed, polynomial time,
communication-less local algorithms to re-arrange vehicles on the road. The
work culminates in a geometric sorting algorithm, that allows autonomous ve-
hicles to exit the freeway using myopic local interactions only, without explicitly
communicating with others.

2 Related work

2.1 Traffic management

Highway congestion poses a serious burden on the road infrastructure and its
users all over the globe. Until recent technological advances we were helpless
while car fleets were surging at unprecedented rates.

In a prototype sketch [24] and later in a seminal paper [23], Varaiya proposed
a concept of an Intelligent Vehicle Highway System (IHVS). The proposed 4-
layer IHVS makes a heavy use of platooning, where groups of vehicles move in
tightly-spaced coordinated fashion towards possibly different destinations, while
sharing the same road. The platooning concept came to solve a problem of road
infrastructure waste incurred by road safety gap requirements.

However, an actual protocol for this centralized multi-layer controller was
estimated to be a huge 500 thousand states big state machine ([7]). Furthermore,
Kurzhanskiy et al. have shown, in [13], that bad management of the existing
highways is the main cause of congestion, while excessive demand is only the
next culprit. Thus, designing a simple and, at the same time, efficient local
traffic management system becomes a crucial challenge.

2.2 Vehicular control

Automated longitudinal control of vehicles gained track since at least 60s. Cur-
rently, with the design of (semi-) autonomous vehicles, the topic of lane-changing
algorithms and protocols raised to prominence. Lane-changing could be divided
into three main scenarios. First, merging - the entrance of the vehicle to the
freeway from an on-ramp. Second, - diverging - the exit of the vehicle from the
freeway and the subsequent integration into the urban mesh traffic. Integration
into, and departure from a freeway traffic are even more complicated at “weav-
ing” sections, where both scenarios are executed simultaneously by a number
of vehicles, and lane-changing has to be carried out in-between. Currently, the
lane-changing process, while mostly a discretionary activity causes a significant
portion of road accidents and is a leading cause of the freeway congestion. In
fully autonomous settings the requirement to change lanes in a safe way will clearly remain. Although, in the future, we should expect a higher traffic efficiency and stability due to cooperation, better sensing, faster reaction, and mostly due to an automated deterministic decision-making.

A centralized optimal freeway merging strategy of fully automated vehicles is presented in [14]. A dedicated roadside short-range communication controller assigns merging priority through adjusting acceleration profiles (optimizing vehicle trajectories). The proposed algorithms are shown to maximize average travel speed under safety gap requirements. A gap metering approach is suggested by Jin et al., in [11], and later improved by Jiang et al., in [10]. The gap metering system, installed near the merging section on the road, advises drivers to provide a gap in favor of incoming merging traffic. A significant on-ramp delay decrease was shown in simulations even under low driver compliance rate.

In the diverging scenario a significant effort was invested to develop lane-changing strategies that minimize congestion and increase safety. Huang et al. conducted a series of simulations [8] to investigate the cause of heavy congestions in two Beijing off-ramp areas. It was found that lane-spacings and the lane-changing spacings are the main factors that define traffic throughput, safety and efficacy. Moreover, as average traffic speeds rise both should increase accordingly in order to allow the same off-ramp throughput flow. Zheng et al., [27], proposed a cooperative strategy between the exiting vehicles and the through traffic vehicles near the off-ramp. The strategy calls for cooperative vehicles deceleration to create safe lane gaps that allow smoother lane changing for off-ramp vehicles.

A research spotlight in the third lane-changing scenario focuses on the safety of the maneuver and the opportunities it creates to minimize travel time. Ji et al., in [9], found that individuals tend to maximize their “payoffs” in terms of safety and travel times, given others are doing the same. Hence, lane-changing is seen as an opportunity to increase “gains”. While different drivers weigh factors differently and adopt diverse behavior strategies, the whole traffic system eventually reaches an equilibrium. Those findings were reinforced in [25], as it was found that mixing self and overall reward boosts an overall traffic efficiency.

Mainstream works in the field suggest lane changing communication protocols as a solution to cooperative lane changing. A vehicle broadcasts to the surroundings, or some limited neighborhood, its position, speed and acceleration, and sends its intentions to change the lane to the desired one. Lombard et al. suggest, in [15], one such “automatic courtesy” protocol. A lane changing vehicle initiates the protocol and becomes a virtual local “intersection” server that broadcasts intersection coordinates to neighbors. Surrounding vehicles could opt to become followers and allow safety gap creations by voluntary deceleration, thus providing the space for the initiator’s desired maneuver. A promising decentralized cooperative lane changing decision model was proposed by Nie et al., in [16]. Decision making in the model is based on three module framework installed on each Connected Autonomous Vehicle (CAV). Here the authors show that their approach improves traffic stability, homogeneity and efficiency.
2.3 Tile-sliding puzzles

The research into tile-sliding puzzles traces back to the nineteenth century. The 15-puzzle is a classical tile-sliding puzzle game where a player slides arbitrarily placed tiles with numbers from 1 to 15 in a $4 \times 4$ matrix. It was shown using permutation parity arguments ([12]), that only half of the initial configurations are solvable in the sense that tiles could be arranged in a specific predefined order. However, finding a solution that requires a minimum number of tile slides in a generalized $n$-puzzle version was shown to be an NP-hard problem ([20]). Despite the hardness, a number of efficient sub-optimal solutions exist, and, in fact, are pretty-straight-forward, see e.g., [19], [26].

An interesting application of tile-sliding puzzles was proposed in a two-lane congested road settings by Toy et al., in [22]. An emergency vehicle (EV) moves through the traffic on the IHVS (see reference [23] above). Two scenarios are addressed: a) a rapid transit of EV through traffic, and b) an EV transition through stopped traffic. A vertex maneuver, i.e. traffic that circulates around the EV, is discussed and is shown in simulations to provide a solution for the first scenario. For the second scenario two solutions are suggested: the part-and-go maneuver, that allows platoon creation in both lanes, thereby ensuring enough space for the EV; and the zig-zag maneuver in case the part-and-go maneuver is inapplicable due to lack of initial free space. Moreover, the latter maneuver could be performed even in the absence of the free space. Some necessary stationary, platoon-forward-join maneuvers can be initiated under all circumstances.

A beautiful problem was suggested by Petig et al., in [17]. Suppose, we observe a two-lane road in the autonomous cars scenario. Under mild restrictions, cars are allowed to change lanes before the upcoming intersection, according to their intentions to leave the road. Define a makespan, determined by the total number of vehicle slots occupied in the final configuration and a total cost - the number of steering maneuvers executed to reach that configuration. The paper provides a polynomial-time algorithm that finds a sequence of maneuvers that minimizes the makespan and at the same time is a 1.5-approximation of the minimal total cost. Additionally, it is shown that a natural multi-lane extension of the problem is NP-hard. However, the algorithm based on the viewpoint of an omniscient being, i.e. it is a centralized solution.

A surprisingly simple solution to a tile-sliding sorting problem was recently proposed in [18]. A set of stones is laid on the lower row of a two-row grid. The goal is to split stones according to their color (black or white) by moving them from one grid cell to another. Black stones should be moved to the left side, while white ones to the right. Stones are moved at discrete time ticks, where every moving stone can change its current cell to one of its unoccupied neighbors, in a way that no two stones share the same cell at any point in time. The paper provides a lower time bound for a solution and presents an amazingly simple instance-optimal approach for the stated problem, i.e. no other algorithm can sort stones faster for any initial stone configuration on the grid. Furthermore, authors propose a distributed variant of the algorithm, that could require a
single additional time tick to complete, compared to the centralized version.

3 The exit the road (ER) problem

3.1 The freeway road model

We here assume that in the future no human-controlled vehicles will be present on the road. In other words, all vehicles on the road will be fully autonomous and running the same standardized distributed traffic control algorithm. At present, due to the driver reaction time safety concerns, road infrastructure is deeply under-utilized, or it is completely jammed, hence has an extremely low throughput rate. However, autonomous vehicle reaction time could be made orders of magnitude lower than that of humans. Therefore, safety distances could be substantially decreased. Moreover, it can be shown, that the safety distance \( d \) of a vehicle moving at the speed \( v \), behind another vehicle, is given by

\[
d = v \cdot \Delta_t,
\]

where \( \Delta_t \) - is the reaction time of the driver (or the system controlling the vehicle).

Human drivers react somewhat slowly to the abrupt changes on the road. For example, an emergency breaking of the vehicle in front initiates a reactive breaking delayed by 1 – 2 seconds. Therefore, a driver is taught to hold a safety distance of about 2 seconds, as a rule of thumb. (According to the California’s Performance Measurement System (PeMS) data [4] the average vehicle headway is 1.63 seconds). At the speed of 35 m·sec\(^{-1}\), the safety distance becomes 35 – 70 meters. Assuming, that an average vehicle length is around 5 meters and drivers adhere to the “2 seconds” rule of thumb, we conclude that road occupancy is bounded from above by a mere 12.5\% (see [21] for similar back-of-the-envelope calculations).

In the autonomous settings the occupancy could be boosted significantly. A reaction time of 100ms translates to a 3.5m safety distance and to an occupancy upper bound of nearly 60\%. Define a safety buffer to be this 3.5m long empty space of one lane. Then, a vehicle itself, a safety buffer in front of it, and an empty space around the vehicle in the same lane may be defined as a vehicle slot or as a slot for brevity. We shall allow slots to be empty or “vehicle-less”, and require all slots to be of the same physical dimensions.

Thus, we model, superimposed on the road, a possibly infinite, collection of vehicle slots. Slots are considered to move at some predefined constant speed \( v \) along the road. We look to maximize the utilization of the road, therefore we may set \( v \) to equal the speed limit of the road.

A vehicle occupies a slot, and moves within the slot at speed \( v \). A slot will always move along a lane, however a vehicle is allowed to change lanes. Therefore, from time to time a vehicle is allowed to need two slots at the same time, while executing a lane-change maneuver, i.e., moving to another neighboring slot in the same lane or to a slot in a neighboring lane.
Complex maneuvers resulting in a change of a currently occupied slot to a slot behind and left will be executed in two steps, hence only one move will be allowed in any given time interval. First, a vehicle could proceed to the left, temporarily moving from the current slot and to its direct left neighbor. Then, by slowing down (and then accelerating to speed $v$) a vehicle will move one slot back (see [6] for an optimal trapezoidal acceleration profile). Note, that we shall allow a vehicle to temporarily exceed the speed limit $v$ to accomplish side and forward-move maneuvers.

**Definition 3.1.** A frame is a sliding rectangular grid of slots of size $n \times m$ on an $m$ lane wide road.

A frame, proposed also by Chouchan et al., in [5], is depicted in Fig. 2b. In our assumed road model frames move at constant speed $v$, hence slots are static with respect to the moving frame. Vehicles move from one slot to another, however, we assume that they never cross frame boundaries. Thus, vehicle-frame association remains constant over time. We index the slots in the frame by row-column index pairs, where rows are counted from the front side of the frame to the back, and columns from the left side of the road to the right side. For example, in Fig. 2a light colored (green in the pdf version) cars are located at positions: $(2,6), (3,6), (4,3)$ in the frame.

### 3.2 Vehicles as Agents on the Grid

We model a vehicle as an agent moving on a grid of slots (cells). As discussed above, we assume that agents posses a number of capabilities that allow them to solve various challenges, by moving from one grid cell to another. An important agent ability is sensing in the neighborhood. At discrete time ticks, agents sense their surroundings, then decide on actions, and, finally, execute them (the so called LOOK-COMPUTE-MOVE paradigm).

We define *sensing* as a process of mapping the visible surroundings to a neighborhood representation. The visibility range can be defined in several possible ways: from no sensing at all to everything that is not obscured can
Figure 3: Different visibility ranges’ visualization. Depiction of a visibility range around a circled vehicle, (a) an $L_1$ visibility range of 1 cell, (b) an infinite visibility range.

be sensed (see Fig. 3b). We use an $L_1$-metric with a given visibility range and accounting for occlusions, hence obstacles (i.e., other agents) obscure everything beyond them. In discrete setting, where only the cell state matters, it becomes necessary to precisely define what is visible. We here exclusively work under a visibility range of 1 step (see Fig. 3a). Under this assumption, an agent observes only its immediate 4 neighborhood, and no grid cells inside the visibility range are obscured by “closer” objects.

Note, that, it is impossible to execute any cooperative strategy under a 0-visibility regime, since even immediate neighbors are beyond the horizon. We of course want to model physical reality, and collisions must be avoided. This imposes a minimum visibility range requirement of 1 cell, which is also employed by algorithms in this work.

In our model, the actions of an agent depend not only on the neighborhood, but also on the “internal agent state”. The facility that enables holding the agent state is the agent memory, and measure it in bits. Memory allows different agent actions given the same visible surroundings. An agent at state $X$ in a given neighborhood could remain motionless, while at state $Y$ it could make a simple move from its current row to the row with one lower index, for example!

In physical reality all the vehicles move in the same direction on the road. Hence, we assume, that their motion/orientation naturally provides them a common North, i.e., a preferred direction. Due to the limited visibility agents are unable to assess the dimensions of the grid frame, or their current location within the frame, i.e., their row and column indexes. However, the preferred direction allows to state facts along the following lines: “the cell to the north of the current cell I am in, i.e., the cell in front of me has a lower grid row index, but the same grid column index”.

In our multi-agent paradigm, agents do not communicate with each other. In order, to prevent possible collisions, we assume time synchronization, i.e. that all agents sense and reset their neighborhood at the same moment, once in an arbitrary defined unit time interval. Then all agents act according to their current location, neighborhood occupancy pattern and state.

Below we summarize agent capabilities and knowledge about the world:
• Sensing ($L_1$-visibility $r$).
• Memory ($b$ bits, hence $2^b$ states).
• Common North (well defined forward, backward, right and left).
• Synchronized operation (all agents operate in synchronized unit intervals).

Further we assume that agents

• **Do not** know their location on the grid.
• **Do not** know the frame dimensions.
• **Do not** have explicit communication capabilities (i.e., can not communicate to surrounding agents or with any central controller).

Moreover, we assume all agents to be identical and anonymous, i.e. all the agents execute the same algorithm, and neighbors can not be distinguished one from the other.

### 3.3 The agent position on a grid

As mentioned previously, at the beginning of a unit time interval, each agent senses its surroundings. The sensors are capable of detecting other agents, the edges of the grid (i.e., the frame boundaries) and empty grid cells within the agent’s visibility range. Note, that detecting boundaries in reality is trivial near road shoulders, but is not at all obvious on the front and back boundaries of the frame. We assume, that these boundaries detection is based on some infrastructure provided support, for example a laser pulses from one side of the road to the other at predefined locations. A vehicle then register a laser pulse or the absence thereof and decides if it moves near the frame front or the frame back, or not near the frame boundary at all.

We have 9 different types of positions on the grid, according to sensor readings of a simple 4-neighborhood of an agent (see Fig. 18a). An agent can be inside a frame, in one of the 4 corners of it or in one of the 4 boundary rows. For example, on Fig. 18b the agent senses position 6, since only the West (left) sensor reading is a wall/non-grid cell). Occasionally, we shall refer the reader to Fig. 18a, otherwise we shall mention the position type in text.

### 3.4 A local algorithm

We denote exiting agents, i.e., agents modeling vehicles with destination set at the next exit, by 1; continue agents, i.e., agents corresponding to vehicles, that continue down the road, by $-1$; and, finally, empty cells will be denoted by 0. Hence, we can model a frame as a rectangular grid filled with three possible numbers from the set $\{+1, -1, 0\}$, where road lanes become columns of the grid.

**Definition 3.2.** A grid configuration $C$, is a particular state of an $n \times m$ grid, over the number set $\{1, -1, 0\}$. 9
Definition 3.3. An $n \times m$ grid configuration $C$ is a target grid configuration if all the 1s are located in the right-most, $m$th, column of a grid.

Definition 3.4. A legal move of an agent $A$ is an exchange of positions of an agent $A$ (represented by 1 or $-1$) into one of the 4 neighboring empty spaces (0). A legal move results in an empty space “taking” the place of an agent $A$, and an agent $A$ moving into that empty space.

In particular, a legal move leaves an agent on the grid, i.e. agents do not fall off the grid. But, not every set of legal moves, executed at the same time tick, constitutes a legal configuration change of the grid. Two standalone legal moves executed at the same time could lead to a vehicle collision. Therefore,

Definition 3.5. A legal move sequence is a set of legal vehicle moves, such that no two legal moves in the set, that overlap in time, overlap also in the spatial dimension.

Definition 3.6. A legal move sequence $S$ is said to transform an $n \times m$ grid from configuration $C$ to configuration $T$, if an execution of $S$ moves changes grid configuration from $C$ to $T$.

We make the following assumptions to ensure that there exists a legal move sequence, which transforms a grid from an initial grid configuration to a target grid configuration. Naturally, legal moves assume that there is at least one empty grid cell (cell with value 0). Additionally, the number of exiting agents (1s) should not exceed the number of available grid cells in the right-most grid column. We summarize:

\[ N_0 > 0 \]
\[ N_1 \leq n \]
\[ N_0 + N_1 + N_{-1} = n \cdot m, \tag{C_0} \]

where $N_0$ - is the number of 0s (empty frame slots), $N_1$ - is the number of 1s (exiting vehicles) and $N_{-1}$ - is the number of $-1$s (continue vehicles).

Consider an $n \times m$ grid, an initial configuration $C$ and a non-empty set of target configurations $T = \{T_1, T_2, \ldots, T_k\}$. We shall ask if there exists a legal move sequence that transforms a grid from an initial configuration $C$ to some target configuration $T_1, T_2, \ldots, T_k$? Further, we shall refer to this question as the Exit the Road (ER) Problem.

In the further discussion we shall make use of the following list of notations:

- $a_{i,j}$ - a grid cell at position $(i,j)$.
- $a_{i,j}(t)$ - an agent occupying grid cell $a_{i,j}$ at time $t$.
- $\mathcal{N}(a_{i,j}(t))$ - a visible neighborhood of an agent $a_{i,j}(t)$ at time $t$.
- $M_{i,j}(t)$ - an agent $a_{i,j}(t)$ b-bit memory state.
And more generally, for agent $A$, $N(A)$ and $M(A)$ will denote the visible neighborhood and the memory state of agent $A$, where agent implicit location $a_{i,j}$, and time $t$ are obvious from the context.

**Definition 3.7.** Let $C$ be an $n \times m$ grid configuration at time $t$. A *local rule of exchanges* is a function that defines the location and the memory state of every agent $A \in C$ at the next time tick $t + 1$ given $N(A)$ and $M(A)$.

In this paper we develop a particular local rule of exchanges $A$. We show that an iterative execution of $A$ by the agents on the grid $G$ at times 1, 2, 3, ... in a distributed manner eventually solves the Exit the Road Problem, i.e., transforms a grid to a target configuration.

## 4 Sorting a multi-column grid

We here propose a novel distributed algorithm that, if applied, transforms an $n \times m$ grid from any possible initial configuration $IC$ to a target configuration.

We consider a specific restricted implementation of an agent. An agent is missing any communication capabilities, and additionally does not posses the ability to distinguish between 1 and $-1$ neighbor agents. Therefore, from an agent point of view, grid neighbors in the $N(a_{i,j}(t))$ appear in three different types. Those are either other agent (indistinguishable) cells, empty cells or border cells.

We assume that agents act under the LOOK - COMPUTE - MOVE paradigm at discrete times 1, 2, 3, ... No agent occupies more than one cell on the grid at a time, i.e. every move from a cell to a neighbor cell could be accomplished in a single time tick.

We adopt the following convention: North-to-South is the direction of increasing row indices, while West-to-East is the direction of increasing column indices. Therefore, the exit lane corresponds to the eastern-most column of the grid, and the front frame row to the northern row of the grid. The exact local rules are listed in Appendix B.1 and in Appendix B.2; below we provide the main properties and characteristics of Algorithm $A$.

- An agent possesses a 3 bit memory.
- An agent utilizes its 3 bits in the following way: 2 bits are used as the modulo 4 timer, i.e. an agent counts times 0, 1, 2, 3, 0, 1, ... We follow the Biham et al. approach (see [2]) to ensure that agents do not move in colliding directions at the same time. An additional bit $d$ is utilized to store an agent “heading” direction.
- An agent memory could be initialized in an arbitrary way, except, that timer bits should be initialized to the same value by all the agents, i.e., agent timers are synchronized. Therefore, we canonically initialize all the memory bits to 0.
• An agent senses an $L_1$ neighborhood of radius 1, i.e. the standard 4-neighborhood.

• An agent strives to continue the movement in the direction encoded by $d$.

• An agent movement direction could change, if an agent is unable to proceed in the direction encoded by $d$. However, a movement direction change occurs after a cool-down period.

• An exiting agent does not leave the last column $m$ (see the absence of such “movement” patterns on Fig. 4a).

• Therefore, a continue agent must leave column $m$ westward (see Fig. 4b).

• Agents of both types move in a clockwise manner on the boundary of the grid (Fig. 4c, i.e. empty spaces (0) “move” in a counter-clockwise direction).

• Agents of both types “enter” non-boundary columns, and traverse them in North, then South directions (Fig. 4c).

• An exiting agent exploits the opportunity to move closer to the first column and goes westward, in the interior columns of the grid. This behavior ensures, that an exiting agent is not stuck indefinitely in the middle of a column (distinctive right arrow “movements” of empty spaces on Fig. 4a).

The proposed algorithm $A$ is a local rule of exchanges. Agents apply $A$ simultaneously and independently at discrete times $t = 1, 2, 3, \ldots$. Moreover, the algorithm does not require agents to communicate their intentions tactically (signaling a lane change) or strategically (designating the type of an agent). Therefore, it is enough to sense only the presence of other agents, or the absence thereof. However, an infrastructure support is required to appropriately implement the algorithm on the road. Specifically, a moving frame boundaries should be marked in some way.

We show that Algorithm $A$ brings a general $n \times m$ grid to a target configuration. On the other hand, Algorithm $A$ pose additional requirements and possesses certain shortcomings and limitations. First of all, the algorithm is inapplicable to two-column grids, and some other algorithm should be applied in that specific case.

Additionally, continue agents demonstrate a somewhat “unreasonable” behavior, in particular, from time to time a continue agent moves to the first row on the grid, then, immediately, in the next time tick, moves back to its previous position (see the top row down/up movement arrows on Fig. 4b). Although, the behavior may seem irrational, those agents have no way to avoid it. Agents with a limited visibility range can not identify the current row, unless it is a row on the edges of the grid. Therefore, agents, located on the interior grid cells can detect an edge row (first or last) only after making a step into an edge row. It is when an agent understands it should have not leave its previous position and decides to return back.
Figure 4: Empty space “movement” visualization. (c) A common pattern and “moves inspired by” (a) exiting and (b) continue agent only. Overall, a common pattern fused with a specific “movement” pattern produces a full empty spaces “movement” pattern caused by (d) exiting and (e) continue agents.
On the other hand, exiting agents exploit the above “eccentric” behavior, and do not return back into the interior cell (see one-way down arrows on Fig. 4a). Instead, they continue the movement in the first row, which is a significant shortcut for them on the way to the right-most exit lane.

Finally, we assume a slightly more severe constraint

\[ N_0 > 0 \]
\[ N_1 < n \]
\[ N_0 + N_1 + N_{-1} = n \cdot m, \]

i.e., for the algorithm to work, the number of exiting agents should be strictly less than the number of cells in the right-most grid column.

We claim that the proposed Algorithm solves the general ER Problem.

**Theorem 4.1 (Algorithm A Soundness Theorem).** Let \( IC \) be an initial grid configuration of an \( n \times m \) grid \( G \), where \( m \geq 3, n \geq 2 \). Suppose \( IC \) meets \( C_1 \) criterion. Then Algorithm \( A \) solves the ER Problem and transforms \( G \) from \( IC \) to a target configuration.

### 4.1 Time complexity

In this section we provide an analysis of the proposed Algorithm time complexity as a function of various parameters.

Recall that we are given an \( n \times m \) grid \( G \) with \( N_0 \) - empty cells and \( N_1 \) - cells occupied by exiting agents. An Algorithm \( A \) takes an initial grid configuration \( C \) and iteratively transforms it to a target configuration \( T \). Then, the number of time ticks it takes \( A \) to execute the transformation - is the number we are looking for. However, a rigorous time complexity analysis is complicated due to the intricacy of agent interactions with empty slots.

**Definition 4.2.** A \((n, m, N_0, N_1)\) problem is an ER Problem, where an initial configuration \( C \) of an \( n \times m \) grid is parameterized by \( N_0 \) empty spaces and \( N_1 \) exiting agents.

We informally define an empty space cycle - a period of time it takes a designated empty space to “visit” a specific cell on the grid, e.g. \( a_{1,1} \). In the context of a single empty space initial configuration the designated empty space tracking is an unambiguous task.

Below we provide a worst case time complexity analysis for this uncomplicated \( N_0 = 1 \) case.

**Lemma 4.3.** Consider a \((n, m, 1, 1)\) problem. The worst case time complexity of Algorithm \( A \) is \( \Theta(n^2 m + m^2 n) \).

**Proof.** We have noted in Observation C.7 that an exiting vehicle makes at most \( 2m \) West/East moves until it reaches the right-most lane. Additionally, an exiting vehicle executes at most \( n \) North moves in the left-most column of the grid.
By our informal definition an empty space cycle is a period of time until \( a_{1,1} \) becomes empty once again. Let us ignore an exiting agent shortcut route, i.e. an opportunity of an exiting agent to immediately move into the first row from the current column (second row). Then, during every empty space cycle an exiting vehicle, either makes one East/West move, or it makes one North move in the first column. Therefore, at most \( 2m + n \) cycles are required to bring an exiting vehicle to the right-most column of the grid.

Each empty space cycle takes time proportional to the parameters \( m, n \). The time is mostly spent on internal columns double “traversals”. Moreover, each empty space “move” takes a full four-time-ticks round to be executed (with a rare exception in the last column, or the u-turns at \( a_{2,i} \) by continue agents, which are all require less time). Hence, we can bound the time needed for a full empty space cycle by \( 8mn \) ticks from above.

Therefore, in total, it takes at most \( 16m^2n + 8mn^2 = \Theta(n^2m + m^2n) \) time ticks, as claimed.

Lemma 4.4. Let \( T_e \) be the first time tick, when internal columns have become empty of the exiting agents in the \((n, m, 1, N_1)\) problem, i.e. all exiting agents are located in the first/last column or the first row after \( T_e \) ticks. Then \( T_e = \Theta(n + N_1) \) empty space cycles or \( \Theta((n + N_1) \cdot mn) \) time ticks.

In order to prove Lemma 4.4 we consider the following problem. Suppose, we put \( M \) boxes indexed by positive integers \( 1, 2, \ldots, M \) and a sink box indexed 0. Then we arbitrary put \( N \) identical balls into the boxes, including sink box. At discrete time ticks \( t = 1, 2, 3, \ldots \) we pick a single ball from each non-empty box \( k \) and move it to box \( k - 1 \) (see a one time tick simulation of two ball configurations in [Fig. 5]). Balls in the sink box remain in the sink box forever. Denote this ball movement protocol as \( BMP_1 \).

Question: When will all the balls disappear in the sink?

The answer depends on the initial distribution of the balls. However, we are more interested in the upper bound on the time, and we claim it to be
Figure 6: Two simulations run according to the slowest possible mode.

\[ O(M + N) \].

\( \text{Proof.} \) Put all the balls in the box indexed \( M \). We remove one ball at a time, hence the last ball will be removed at \( t = N \). A standalone ball can “travel” at most one box at a tick. Therefore, \( M \) ticks are required to move the last ball from box \( M \) to box 0. Note, that throughout the execution, all the balls that have already left the \( M \) box are alone in their current location (except for balls in the sink). Thus, all the balls are constantly moving towards the sink at box 0. Finally, the last ball will disappear in sink at \( t = M + N \).

We claim that an arbitrary placement of balls will lead to a faster ball disappearance in sink. Label the balls from 1 to \( N \), in such a manner, that a ball at the box \( k \) will be labeled with a higher ordinal, than all the balls in the lower indexed boxes.

For example, let \( M = 7 \), \( N = 9 \), such that two balls are at 1 - we shall label them 1 and 2; four balls are at 3 - we shall label them 3, 4, 5 and 6, and the other balls are at 7 - and we shall label them 7, 8, 9. See Fig. 6 at \( t = 0 \).

We compare ball disappearance time versus the “All in \( M \) initially” scenario duration. Denote the former original scenario as \( O \) and the latter “All in \( M \) initially” scenario as \( P \). In \( P \) we always remove the lowest indexed ball from the box. However, in the \( O \) scenario we proceed in the following way: we remove the ball with label \( L \) from the box \( k \), at the same time when the ball with label \( L \) is removed from the box \( k \) in the \( P \). We denote this changed protocol as \( BMP_2 \).

Since all the balls in \( P \) do eventually end up in sink, we ensure that all the balls will end up in sink in \( O \) also under the protocol \( BMP_2 \). However, an actual \( BMP_1 \) run on \( O \) will be at least as fast as \( BMP_2 \). Hence, the time required to move all the \( O \) balls into the sink at 0 under \( BMP_1 \) is at most \( N + M \). \( \square \)

Let get back to the ER Problem and the Algorithm \( A \) time complexity analysis.
An empty space “executes” a counter-clockwise walk on the grid. It “traverses” internal columns North then South, then “moves” East. East movement from column $i$ to column $i+1$ originates from three different agent movement rules:

- Exiting vehicles in the internal column $i+1$ “Rush West” if neighbored by an empty space from the West.
- Continue vehicles in the last column $m$ execute the rule “Leave the Exit Lane” under the same conditions.
- Both types of agents leave $a_{n,i+1}$ West.

The second option is irrelevant in the context of exiting vehicles. Hence, we shall concentrate on the two remaining cases.

The last type of agent Westward movement is not initiated upon empty space “arrival” to $a_{n,i}$ for the first time. Only the second encounter triggers an appropriate agent movement rule. This allows execution of the following scenario: an empty space “arrives” from $a_{n,i-1}$ to $a_{n,i}$, then “enters” column $i$ and “traverses” it in the “regular” North-South fashion. Finally, an empty space “moves” to $a_{n,i+1}$ on the second arrival to $a_{n,i}$.

However, consider the former case. During a North-South “traversal” an empty space is “rushed” East by an exiting vehicle from column $i+1$. Then, an agent in the $a_{n,i+1}$ cell is left waiting for its second encounter with an empty space in the following empty space cycles. Moreover, the next East “move” of an empty space from $a_{n,i-1}$ to $a_{n,i}$ could lead to a different “movement” pattern. Specifically, an empty space could “continue” to $a_{n,i+1}$. Though, an empty space will repeat a regular column $i$ North-South pattern in the very next cycle. It follows, that at least once per two consecutive empty space cycles, a group of column $i+1$ exiting vehicles decreases by 1.

Then, a ball-box analogy can be applied. We shall set a “ball-box time tick” to count a consecutive empty space cycle pairs. Since more than one exiting agent can leave a column (a box) in the same time tick, the ball-box model application is not immediate. However, removing a single ball restriction could only decrease the “All in the Sink” completion time. Thus, a complexity of time required for all the exiting vehicles to leave the internal columns (except, maybe, second) is still $\Theta(m + N_1)$ empty space cycles.

An exiting agent visible behavior in the second column differs slightly from the behavior in the other internal columns. We have previously noted, that an exiting agent in the internal column moves West once neighbored by an empty space. And that a vertical “movement” of an empty space in the internal column followed by a West movement of an exiting agent is possible at most once per clock cycle. However, in the first column an empty space can “move” South more than once during the same 4 successive ticks. In fact, in the first column an exiting agent can move North 3 out of 4 ticks in one clock cycle. Therefore, an empty space could “pass” in the first column near an exiting vehicle in the second column without that said exiting agent moving West in the same clock cycle.
This second column complication is, however, caused by another exiting agent moving North in the first column. Let $A$ be an exiting vehicle, that "misses" a neighbor empty space. Denote the $A$ row index by $k$. In the simplest, single empty space, model, the maximum number of "new" exiting vehicles that could occupy $a_{k,1}$ during the same empty space cycle is 1. Moreover, continue vehicles can not prevent $A$’s move into $a_{k,1}$ during the clock cycle when an empty space “moved” into $a_{k,1}$.

Since there are at most $N_1$ exiting agents we conclude that in at most $\Theta(m + N_1)$ empty space cycles all the exiting agents leave the internal columns. Thus we have established the correctness of Lemma 4.4.

We summarize the statements above and we come up with a specific bound of $2m + 3N_1$ cycles. Recall that 2 comes from the movement rule of agents at position 5 (i.e. empty space double encounter), and one additional $N_1$ was added as a second column complication payment. However, this last summand will be later counted towards the cycles required for exiting vehicles to leave the first column.

Suppose, at $T$ all exiting agents have already left the internal columns. Then an exiting agent in the first row or the first column moves exactly one step clockwise in the first row/first column during a single empty space cycle. Hence, in at most $(m + n)$ empty space cycles all the exiting vehicles will move into the last column. The Algorithm $\mathcal{A}$ will stop then upon reaching a target configuration. We conclude, that the time complexity of the original Algorithm $\mathcal{A}$ is $\Theta(n + m + N_1)$ empty space cycles. Specifically, the upper bound is $n + 3m + 2N_1$ cycles.

Finally, we shall analyze the most general $(n, m, N_0, N_1)$ problem, where $N_0$ is the number of empty spaces on the grid. Initially, we shall assume, that empty spaces do not “interfere” with each other, i.e. empty spaces do not “wait”, because neighbor agents interact with other empty spaces. Then we get a reciprocity: $N_0$ empty spaces pass through designated cell $a_{i,j}$ during one empty space cycle. Hence each exiting agent observes $N_0$ empty spaces during the same period of time which is equivalent to observing a single empty space, but $N_0$ times.

In reality an “empty space cycle” is not length-constant over time. But depends on the actual “interference” of empty spaces during the execution of the Algorithm. Moreover, due to the certain positions agent movement rules the cycles are even shorter comparatively to the expected $1/N_0$ rate. Indeed, if one empty space “moves” into an internal column $i$ from the position 5, then the next empty space “skips” to the next $i + 1$ column instead of entering the column $i$, as the “previous” empty space.

Thus, an overall time upper bound is as follows:

$$T \leq \frac{3m + n + 2N_1}{N_0} \cdot 8mn,$$

where $8mn$ is an upper bound on the “empty space cycle” length.
4.1.1 Average case time complexity

In some scenarios, not only can we bound the worst-case time complexity, but we can also estimate an average case time complexity.

Consider a special case \( N_0 = N_1 = 1 \), an “empty space cycle” is bounded by \( 8mn \). We can also estimate a number of cycles required to move an exiting vehicle from any cell of the grid to the right-most column of the grid.

We shall analyze a number of exiting agent initial positions:

1. The right-most column \( m \). Agents, starting in this column, need 0 cycles to finish the algorithm. There are \( n \) distinct starting positions in this column.

2. The first row. Agents, starting in the first row, are expected to make a number of steps to the right, and the number of steps is exactly the distance between the agent initial column and the \( n \)th column. However, an agent is only able to make a single step during one full empty space cycle. Hence, e.g. it will take a whole \( (m - k) \) empty space cycles for an agent starting at cell \( a_{1,k} \) to reach the right-most column.

There are \( (m - 1) \) distinct starting positions in the first row (not already mentioned in the previous bullet).

3. The second row. An agent starting in the second row has a shortcut opportunity to move into the first row. And then to behave like a first row agent from the previous bullet. In other words, an additional empty space cycle will be required to complete move to the last column for an agent starting at cell \( a_{2,k} \) versus an agent starting at \( a_{1,k} \).

However, in the worst case, the first encounter of an empty space will end up in a West movement, i.e. an Algorithm will require additional two empty space cycles to accomplish the task in comparison to the optimistic scenario of an “immediate shortcut”, or three cycles in comparison to an agent at \( a_{1,k} \).

4. The first column. An agent, starting in the first column, will need to make all the North steps to reach the first row. Then an additional \( (m - 1) \) East steps in the first row. The number of required steps in the first column is a function of a starting row: agents starting at \( a_{k,1} \) will be required to make \( (k - 1) \) steps North.

5. The last and the most numerous group of agents, are agents starting in the general internal column position. An agent starting at \( a_{i,j} \) will need to move West \( (j - 1) \) steps, then \( (i - 1) \) steps North, before, finally, proceed \( (m - 1) \) East steps in the first row.

To estimate the bound on the average completion time we sum up the completion times for all the possible initial placements of an exiting vehicle. The below times are expressed in the empty space cycles.
First group can be skipped. Agents from the second group will need a total number of cycles equal to:

\[ \sum_{j=1}^{m-1} (m - j) = \frac{m \cdot (m - 1)}{2}, \]

While, agents from the third group in the optimistic scenario

\[ \sum_{j=1}^{m-1} (m - j + 1) = \frac{(m - 1) \cdot (m + 2)}{2}, \]

and in the pessimistic case:

\[ m + \sum_{j=2}^{m-1} (m - j + 3) = \frac{(m - 1) \cdot (m + 4)}{2} + (m - 3), \]

where \( m \) cycles are paid for a North-then-East movement of an agent initially located at \( a_{2,1} \).

In the forth group, we pay cycles according to:

\[ \sum_{i=3}^{n} ((i - 1) + (m - 1)) = (m - 1)(n - 2) + \frac{n(n - 1)}{2} - 1. \]

In the last group, the calculate payment in two steps. An agent should move West from \( a_{i,j} \) to \( a_{i,1} \), total in

\[ (n - 2) \cdot \sum_{j=2}^{m-1} (j - 1) = (n - 2) \cdot \frac{(m - 1) \cdot (m - 2)}{2}. \]

From \( a_{i,1} \) the cost was already calculated, hence this is \( m - 2 \) copies of the forth group:

\[ (m - 2) \left[ (m - 1)(n - 2) + \frac{n(n - 1)}{2} - 1 \right]. \]

The total number of initial positions is equal to the number of cells on the grid. Though in the worst case we shall additionally pay a whole “empty space cycle” due to the random initial placement of an empty space. After summing previous terms and multiplying by the cycle length we get an upper bound on the average time:

\[ \bar{T} \leq 4(3m + n)mn = O(m^2 n + mn^2) \]

On the other hand, the second and the third groups in the optimistic scenario require \( m^2 - 1 \) cycles. While, two last groups total time could be bounded from below by \( \frac{1}{2}(m-1)(n-2)(m+1/2n-1) \) cycles. Therefore, \( \mathbb{E}(T) = \Omega(m^2 n + mn^2) \). Hence,

\[ \mathbb{E}(T) = \Theta(m^2 n + mn^2). \]
We conclude that the average and the worst cases of the \((m, n, 1, 1)\) problem have the same time complexity of \(\Theta(m^2n + mn^2)\) ticks.

In the general case, \(N_1 > 1\), the Algorithm completion time is intuitively determined by an agent in the internal column with the highest index and/or highest row index. For an arbitrary initial configuration the probability that an agent is located in the bottom half could be bounded from below by:

\[
P(\text{agent is in the grid bottom half}) = \frac{n/2 \cdot (m-1)}{n \cdot m} = \frac{m-1}{2m} \geq \frac{1}{4}.
\]

Therefore, the probability that no agent is located in the bottom half area is:

\[
P(\text{grid bottom half is agent free}) \leq (1 - 1/4)^{N_1} \to 0,
\]
and the complementary event probability is high, in particular at least \(1/4\).

Hence, in at least \(1/4\) cases the time it takes the Algorithm to solve the problem is \(\Theta(m^2n + n^2m)\), as in the worst case. It follows, that in the average case

\[
\mathbb{E}(T) = \Theta(m^2n + mn^2).
\]

Moreover, our previous observations hold, therefore

\[
\mathbb{E}(T) = \Theta \left( \frac{1}{N_0} \cdot (m + n + N_1) \cdot mn \right).
\]

### 4.2 Simulation results

Alongside, the theoretical complexity, we are extremely keen to investigate “an average case” behavior of the Algorithm in the settings of practical interest. The number of lanes on the real roads is not unbounded or extremely high. In fact, \(3 \leq m \leq 6\) cover the vast majority of the modern infrastructure.

We have conducted a series of simulations to estimate an average Algorithm completion time. Note, that a single time tick in a simulation represents roughly one second in the real-world environment required to execute a lane change or a “move”-forward/backward maneuver by a real vehicle on the road.

We have run the simulations on the proprietary software implemented in C++. The agents were implemented in the OOP paradigm with a full memory separation between the grid state and the agent state. Every time tick an agent sensed the immediate neighborhood and decided to act based on the visible neighborhood and the agent internal state, i.e. according to the Algorithm \(A\). Exactly by the Algorithm, an internal state change kept secret, while a “move” decision was communicated to the grid. After all agent move decisions were collected, the grid moved all the agents at once.

Each test run started with a random initial configuration parametrized by a tuple \((n, m, N_0, N_1)\). Then the test was executed until a target configuration was reached. Completion times were saved and processed later, with the graphical results presented below. If not otherwise stated, we have set \(N_1 = n - 1\) throughout the tests.
Figure 7: The average completion time on an $n \times m$ grid vs. the number of lanes ($m$). The number of empty spaces $N_0 = 5$. The best fit second degree polynomial is depicted by a dash line for every data series.

Figure 8: The completion time on an $n \times m$ grid vs. $m$ (lane number) ($n = 18$, $N_0 = 3$ and $N_0 = 21$). Minimum and maximum times are depicted along the averages as on Fig. 7. The best fit second degree polynomials (a) $T_{est} = \frac{11.637}{N_0} \cdot n \cdot m^2 + O(m)$ and (b) $T_{est} = \frac{5.796}{N_0} \cdot n \cdot m^2 + O(m)$ are depicted by a dash line. Loosely dotted line is the time upper bound $T_{ub} = \frac{24}{N_0} \cdot n \cdot m^2 + O(m)$ (see [1]).
In Fig. 7 and Fig. 9 we have measured the time against the number of grid columns and rows respectfully. In the previous section we have claimed that the time to solution is upper-bounded by:

\[ T \leq \frac{3m + n + 2N_1}{N_0} \cdot 8mn, \]

i.e.

\[ T \leq \frac{1}{N_0} \cdot \left( 24n \cdot m^2 + 8m \cdot n^2 + 16mn \cdot N_1 \right). \]  

Therefore, we have fitted a second degree polynomial line (dashed lines on Fig. 7 and Fig. 9). Both parameter graphs nicely support our findings from the above section.

On the other hand, our time upper bound estimates seem way too cautious. We provide a number of plausible explanations below:

- Exiting vehicles in the neighboring internal columns rush the same empty space without that empty space “finishing” double traversal of an internal column, i.e. in less time than needed for double traversal completion.

- A pair of empty space “moving” in close proximity could “save” each other half of double traversals. Once the first empty space “enters” the column to double “pass” it, the second empty space “skips” that same column.
However, the second empty space in the pair “traverses” the next right column twice. Then roles switch once again. Thus, in practice, a single empty space in a pair “traverses” every second column twice, but also “skips” every second column. Therefore, we should expect $m - n$-long traversals instead of $2m$. Increasing the empty space numbers increases the chances for such “closely running” empty space pair/triplet etc. The effect become prominent after all the first algorithm stage, i.e. after the exiting vehicles exodus from the internal columns.

Indeed, in relative terms, Fig. 8b featuring $N_0 = 21$ (large number of empty spaces) is almost twice faster than Fig. 8a showing results corresponding to $N_0 = 3$ (small number of empty spaces).

- Due to the practical limitations, the number of simulations that we have executed was not high enough to cover enough “bad” initial configurations. Thus the set of problem parameters we have chosen to test, led to better times than we should have expect.

We have assumed that

$$T(N_0) = c \cdot \frac{1}{N_0}.$$

Therefore, a dependency relation between $N_0$ and $T$ in the Log-Log plane should
n = 6; $\beta = -1.170$
n = 9; $\beta = -1.137$
n = 12; $\beta = -1.124$
n = 15; $\beta = -1.106$
n = 18; $\beta = -1.101$

Figure 11: An average completion time vs. an empty space number ($N_0$) on $n \times 4$ grids. In a log X-log Y scale. The best fit line slope $\beta$ is shown.

be a line with a slope equal to $-1$, i.e. taking logarithms on both sides produces

$$\log T(N_0) = -1 \cdot \log N_0 + \log c.$$ 

And after standard algebraic transformations

$$\frac{\log T - \log c}{\log N_0} = -1$$

This reciprocity can be observed on Fig. 11 (the slope $\beta$ is depicted in the legend).

The last parameter that we have tested is $N_1$ - the exiting agent number. We have predicted, that a completion time is linear in $N_1$. Our findings support nicely the theoretical analysis (see in Fig. 12). However, the Algorithm has been proven to complete the task in a limited range of $N_1$ values (namely for $N_1 \in \{1, \ldots, n-1\}$).

An interesting question raises from our daily lives: “what is the algorithm behavior not as a function of $m, n$ and $N_0$”, but rather “what is the algorithm behavior on the road with a constant fraction of empty slots”? I.e. what happens when the road is half-empty, or the congestion is high? How well the time scales as a function of grid dimensions? We provide an answer in Fig. 13 and Fig. 14. Denote the ratio of empty spaces on grid as $\rho$, we have that $N_0 = \rho \cdot mn$, hence
the upper bound in this case is:

$$T \leq \frac{3m + n + 2N_1}{\rho \cdot mn} \cdot 8mn.$$ 

After simple algebraic reduction we get, that time is expected to be linearly dependent in $m$ and $n$ for roads with a constant free spaces ratio.

$$T \leq \frac{8}{\rho} (3m + n + 2N_1)$$
Figure 13: An average completion time vs. \( m \) (columns) on \( n \times m \) grid with a constant ratio \( \rho = 0.6 \) of empty spaces.

Figure 14: An average completion time vs. \( n \) (rows) on \( n \times m \) grid with a constant ratio \( \rho = 0.6 \) of empty spaces.
5 Sorting a two-column grid

In this section we consider a special case: a freeway road with two lanes \((n = 2)\). It could be shown that an Algorithm \(A\), presented in Section 4, does not sort all the possible bi-lane grid configurations. In particular, in some initial configurations exiting agents in the first column are “starved” in the following sense: after some time \(t\) an agent never observes an empty space, and, hence, is unable to move into the exiting lane right next to it.

Below we present a different 3-bit visibility 1 algorithm \(A_2\), that solves ER(Exit the Road) problem on the road with two lanes. Like previously, an agent is not required to communicate its intentions in any way, nor does it require centralized orders to execute in-frame movement. The actual cellular automata state machines that makes the solution possible could be found in Appendix D.1 and Appendix D.2.

The proposed algorithm is simple and is based on few principles:

- Agents move **only** into empty slots.
- All agents, that move at time \(t\), move in the non-conflicting directions at time \(t\). This rule and the one above ensure no collision could happen at time \(t\).
- Exiting agents move horizontally inside a frame **only** from the non-exiting lane to the exiting one.
- Continue agents, on the contrary, move horizontally **only** from the exiting lane to the non-exiting one. The rule and the previous sibling rule ensure, that agents sort themselves to the target lane (TC) according to own type.
- Agent that moved in the certain direction does not return back in less than one cycle of timer (we further denote by CC(clock cycle); spans 4 time ticks). The rule ensures that an agent is not starved while waiting for an empty slot neighbor, and, eventually, the rule will serve us to show that an empty space “visits” every cell that should be visited.

An agent that executes \(A_2\) is exactly the same agent as in the multi-lane ER Problem. With exactly the same capabilities: one unit \(L_1\) visibility and 3 bits of memory employed to track the timer and the last movement direction.

Memory bits are split into a \(1 – 2\) tuple. A couple of bits is used to track CC(“modulo 4 timer”). These bits are initialized to 0, so all agents are initially synchronized. An implementation should merely add 1 to the current timer value, so an agent “proceeds” to the next time tick. Since all the agents implement the same logic, they remain time-synchronized during the algorithm execution. We arbitrary treat timer as the lower bits of the state id encoding (see Fig. 31 and following table representation of \(A_2\)). The remaining bit, aka a Movement Direction bit, is utilized to store the last memorized movement direction. This bit allows agents to continue unobstructed movement in the same direction. However, an agent will eventually change the direction to the
Figure 15: Two lane road. A sorted configuration (a) right lane is filled by exiting vehicles, (b) vehicles are split according to type, (c) left lane is fully occupied by continue vehicles.

opposite if the movement is blocked (by another agent or by the edge of the frame). To prevent sporadic changes in direction, we require a whole cycle to pass between the last movement tick and a Direction Bit flip tick.

**Theorem 5.1.** [The Algorithm $A_2$ Soundness Theorem] Consider an initial configuration $C$ of an $n \times 2$ grid $G$ that contains at least a single empty slot. Let the number of exiting vehicles be $N_1$. Then Algorithm $A_2$ solves the ER Problem and transitions $G$ to a target configuration. Where target configuration is defined in the following sense: in case $N_1 \leq n$, then no exiting vehicles remain in the first column, otherwise no continue vehicles remain in the second column.

### 5.1 Time complexity

In this section we analyze the time complexity of the two-lane Algorithm and provide time-to-solution evaluation based on computer simulations.

We are interested to estimate the time span of the Algorithm for some initial configuration of an $n \times 2$ grid with $N_0$ empty cells and $N_1$ exiting agents. Note, that if both lanes are exiting, then, after the vehicle rearrangement, all exiting vehicles will be able to exit the road. Hence, there is no limitation on the $N_1$, except, that $N_0 + N_1 \leq 2n$ and that $N_0 > 0$, i.e. it is possible to rearrange vehicles at all. However, in case there a single exiting lane is present, the obvious limitation is $N_1 \leq n$.

We first consider a simple case: an initial configuration has a single empty slot. $N_0 = 1$. An empty slot “travels” at the speed of one slot per timer cycle (4 time ticks). Exiting vehicles are expected to move into the right (exiting) lane, unless $N_1 > n$. Then we expect exactly $n$ exiting vehicles to move on the right, and the rest to stay in the left lane. In the opposite case, where $N_1 < n$ we expect all the exiting vehicles to move on the right. See Fig. 15 for different possible sorted configurations.

All sorted configurations have a common property: no further progress of time will witness a lane change by an agent. Vehicles will continue to jiggle in their respective column, however horizontal moves have ceased.
In the worst case scenario an empty slot will need to traverse a whole grid column at most twice before it could “find” a next candidate for a lane change, i.e. a single vehicle could be moved in at most 2n cycles. We need to move no more than all the agents on the grid (2n), therefore the Algorithm running time is upper bounded by

\[ T \leq 16n^2 \]

time ticks.

We assume that empty slots are “placed” and “moving” independently. And, as such, each additional empty slot should “increase” the overall agent movement count in a linear manner. Until the point, when empty slots “start” to interfere with each other. For example, a line of empty slots can not “process” an agent faster than a one cell at a cycle, no matter how many empty slots are queued in line. Therefore, we should observe some phase transition effect at the critical empty slot saturation point.

Moreover, there is no need to move all the agents. It is enough to move at most \( \min\{N_1, n\} \) exiting agents to the exiting lane. However, slots should be freed in the exiting lane, i.e. the same number of continue agents should be moved to the left. We expect roughly half of exiting vehicles to reside initially in the right lane. Moreover, agent spread in the column should be expected to be almost uniform on the average. Which, in turn, requires empty slots to “run” for only half a distance on the average. Hence,

\[ \mathbb{E}(T) \leq 4 \cdot \frac{1}{N_0} \cdot n \cdot N_1. \]

5.1.1 Simulation Results

We further ran simulations to check our hypothesis. The simulation framework used was the same C++ implementation that we have utilized in the preceding sections to assess Algorithm \( A \) runtime complexity.

We have studied a functional dependency between \( \mathbb{E}(T) \) - an average time until \( A_2 \) sorts a \( n \times 2 \) grid of a random initial vehicle configuration. Fig. 16 presents a time as a function of \( N_0 \) - the number of empty slots on the grid. A regular linear representation (on the left) depicts an inverse dependency rate, i.e. the time required to sort a random initial vehicle configuration significantly drops once enough empty slots are “floating” around. The log-log representation (on the right) allows us to observe the predicted phase transition effect, when \( N_0 \rightarrow n \). Recall,

\[ \mathbb{E}(T) \leq 4 \cdot \frac{1}{N_0} \cdot n \cdot N_1. \]

Therefore,

\[ \log \mathbb{E}(T) \propto - \log N_0, \]

where the proportionality coefficient \( \beta \rightarrow -1 \), and could be expected to be a limiting behavior for a higher values of \( n \).

Two other parameters: the number of rows in the frame \( n \) and the number of exiting vehicles \( N_1 \) do produce a linear best fit line dependency as predicted
Figure 16: $A_2$ makespan as a function of $N_0$ (a number of empty slots on the $n \times 2$ grid), in a (a) linear and a (b) log-log scale representation. A phase-transition is evident on the (b), where $N_0 \rightarrow n$. The $\beta$ is calculated according to $\beta = \frac{\log T - c}{\log N_0}$, and is the best-fit line slope, i.e. the power of $N_0$ in the $E(T)$. ($N_1 = n$).

Figure 17: $A_2$ makespan as a function of (a) row number $n$ (exiting vehicle number $N_1$ is equal to $n$) and (b) exiting vehicle number $N_1$ (for a constant $N_0 = 10$).
(see Fig. 17). Note, that on Fig. 17a the number of exiting vehicles up-scales with the number of rows, hence the quadratic outlook of the image. Overall, the findings are in line with our theoretical reasoning.

6 Discussion

We studied the problem of sorting vehicles on a multi-lane road, sending vehicles with destination at the nearest exit to the departing lane and leaving vehicles, that should continue down the road on the other lanes. We have shown that the problem is solvable in the distributed settings without communication. Moreover, the hardware implementation requirements are minimal, while the time to solution produced by the proposed algorithms is convenient.

We proved the correctness of the algorithms, thus ensuring that any feasible initial configuration gets solved under physical constraints of the real world. We note, that sorting, constrained by physical reality, become more and more relevant with the advent of automation in such fields as warehouse management, traffic management, massive deployment of aerial shipping solutions etc. Our work unequivocally shows that distributed approach is viable in the traffic context. Moreover, the technological requirements are few orders of magnitude less demanding than previously thought.

However, we note that our approach to the problem is not without a limitation. Our model relies on road infrastructure to support the so-called moving frames, where algorithms meant to be applied. Therefore, it would be interesting to search for solutions that do not rely on infrastructure and inter-vehicle communication altogether.
Appendices

A Local sensing algorithm

The algorithm that we here present is a set of rules. A rule in this set is an action of the following kind: “What an agent of type $T$ at state $S$ should do if it observes a specific neighborhood $N$”. Where state $S$ is the current value of an agent internal memory and type $T$ can either describe an exiting or a continue agent. The rule additionally specifies how an agent should adjust its internal memory state after an action got executed.

From the mathematical perspective Algorithm $A$ is a function from triplets $\text{type-state-neighborhood}$ to pairs $\text{state-action}$. Where type is set arbitrary to 1 for exiting vehicles, and to $-1$ for continuing vehicles, state is one of the $2^b$ possible memory values between 0 and $2^b - 1$, and action is either do nothing, i.e., remain at place, or move in one of the four primary directions North, East, South or West.

Further we shall explain what is a neighborhood, and how does an agent use it as an input to the algorithm. Per definition, an agent neighborhood at time $t$ is a collection of all the cells that an agent senses at time $t$. We here develop an algorithm assuming an $L_1$ visibility of radius 1. An example of one such $L_1$ neighborhood of radius 1 is presented on Fig. 18b.

In our model, an agent, with the $L_1$ visibility of radius 1, can distinguish up to nine different types of grid positions (see Fig. 18a). Position detection is done according to sensor readings of a simple 4-neighborhood of an agent, where the presence of border type neighbor cells is that what determines the agent’s grid position. E.g. on Fig. 18c an agent detects only one border cell neighbor on its West, hence an agent self-identified grid position is 6.
Some visible neighboring cells could be irrelevant to the agent decision making process. The relevance of a neighboring cell depends on its position relative to the agent location. For brevity, we replace irrelevant neighbors in the agent’s sensor diagram with the X mark (see (do not care)), where X is set in place of East and South neighbors on Fig. 18c. However, in the actual algorithm implementation, a rule corresponding to a neighborhood with an X mark should be replaced by a pair of rules. One rule, corresponding to a neighborhood, where the X mark is replaced by an empty cell, and another rule, with an identical outcome, where X is replaced by an agent occupied cell. Note, that a rule, corresponding to $k$ X marks at different neighborhood locations, should be replaced by $2^k$ rules, following the same substitution principles.

For example, we present an agent sensor reading in Fig. 18c, which includes: an occupied cell on the North, a border cell on the West and two do not care readings on the East and South. Any rule corresponding to this neighborhood template is expanded into four possible rules in the full description of the algorithm. Where every possible combination of an empty cell and/or of an occupied cell instead of a do not care is listed. The expanded rules share the same (original) output, i.e. a common state-direction pair.

The algorithm description minimization approach resembles a similar procedure in the Karnaugh Map manipulation of incompletely specified switching functions, where the same output for two possible states of the same variable eliminates the variable from the input.

Some rules in the short description of the algorithm are left undefined (see Fig. 19). We later show that input triplets, corresponding to the missing rules, are not attainable by the serial execution of the algorithm, therefore they could be arbitrary filled.
Figure 19: Rules for an exiting agent at grid position 3. All memory states are listed in binary, i.e. $011 \equiv 011_2 = 3$. Following agent actions are possible: $N$ - *move* North, $E$ - *move* East, $S$ - *move* South, $W$ - *move* West, $\emptyset$ - *stay* in place. Undefined rules are left empty in the table.
B Multi-column sorting algorithm state machine

B.1 An exit agent

| Position 1 | Position 5 |
|------------|------------|
| 000        | 000        |
| 001        | 001,∅      |
| 010        | 011,∅      |
| 011        | 000,∅      |
| 100        | 101,∅      |
| 101        | 110,∅      |
| 110        | 111,∅      |
| 111        | 100,∅      |

Figure 20: State machine(s) (a) at position 1, (b) at position 5.
|   | 000, N | 001, Φ |
|---|--------|--------|
| 001 | 010, N | 010, Φ |
| 010 | 011, N | 011, Φ |
| 011 | 000, Φ | 000, Φ |
| 100 | 001, N | 001, Φ |
| 101 | 010, N | 010, Φ |
| 110 | 011, N | 011, Φ |
| 111 | 000, Φ | 000, Φ |

(a) Position 2

|   | 000, E | 001, Φ |
|---|--------|--------|
| 001 | 010, E | 010, Φ |
| 010 | 011, E | 011, Φ |
| 011 | 000, Φ | 000, Φ |
| 100 | 001, E | 001, Φ |
| 101 | 010, E | 010, Φ |
| 110 | 011, E | 011, Φ |
| 111 | 000, Φ | 000, Φ |

(b) Position 3

|   | 000, S | 001, Φ |
|---|--------|--------|
| 001 | 010, S | 010, Φ |
| 010 | 011, S | 011, Φ |
| 011 | 000, Φ | 000, Φ |
| 100 | 001, S | 001, Φ |
| 101 | 010, S | 010, Φ |
| 110 | 011, S | 011, Φ |
| 111 | 000, Φ | 000, Φ |

(c) Position 4

Figure 21: State machine(s) (a) at position 2, (b) at position 3, (c) at position 4.
Figure 22: State machine(s) (a) at position 6, (b) at position 7, (c) at position 8.
Figure 23: State machine(s) (a) at position 9.
B.2 A continue agent

Figure 24: State machine(s) (a) at position 1, (b) at position 5.
Figure 25: State machine(s) (a) at position 2, (b) at position 3, (c) at position 4.
Figure 26: State machine(s) (a) at position 6, (b) at position 7.
(a) Position 8

(b) Position 9

Figure 27: State machine(s) (a) at position 8, (b) at position 9.
C Theorem 4.1 proof

Theorem 4.1 (Algorithm \( A \) Soundness Theorem). Consider an initial configuration \( C \) of an \( n \times m \) grid \( G \), where \( m \geq 3, n \geq 2 \). Suppose \( C \) meets constraint criteria. Then Algorithm \( A \) solves the ER Problem and transitions \( G \) to a target configuration.

C.1 Impossible inputs and collision avoidance

Movement rules defined in short form in Appendix B.1 and Appendix B.2 result in a movement into an adjacent empty cells only. A fast enumeration ensures: no rule directs an agent to move into an occupied cell or into a neighboring border cell. However, a different type of collision could occur if two or more agents decide to move into the same empty cell on the grid at the same time (see Fig. 28 for an illustration of a possible collision situation). Therefore, we shall thoroughly check all the potential moving agent pairs. Unsurprisingly, Algorithm \( A \) was designed with collision prevention in mind. Since agent does not possess abilities to detect another agent “over the gap”, agent movement should be synchronized in one way or another. We have implemented synchronization using a modulo 4 timer, i.e. a timer counting 0, 1, 2, 3, then count (cycle) repeats. Under the 4 cycles clock regime, we allocate specific times for the movement in the specific directions as follows:

- Cycles 0 and 2 will be assigned to southbound traffic.
- Cycle 1 will be allocated to northbound traffic.

![Figure 28: Possible collision under the \( L_1 \) visibility of radius 1. Two agents decide to move into the same empty space. Observed neighborhoods and movement decisions.](image-url)
Table 1: Possible movement direction for exiting vehicle. Times on the left are given in the binary format. Columns enumeration is done according to agents’ position on the grid (from 1 to 9). *N,E,S,W* are four possible directions (north, east, south, and west).

| Position | 00 (0) | 01 (1) | 10 (2) | 11 (3) |
|----------|--------|--------|--------|--------|
|          | - N E S | - N S N W N S N | - N E S | - E S W |

Table 2: Possible movement direction for continuing vehicle. Times on the left are given in the binary format. Columns enumeration is done according to agents’ position on the grid (from 1 to 9). *N,E,S,W* are four possible directions (north, east, south, and west).

| Position | 00 (0) | 01 (1) | 10 (2) | 11 (3) |
|----------|--------|--------|--------|--------|
|          | - N - S | W - - S N W - S N | - E S - | - E S W - |

- The remaining cycle 3 will be utilized for westward movement.

Nonetheless, we shall allow traffic in additional directions, if such traffic does not interfere with potential traffic in the designated directions at the same cycle. For example, northbound traffic that enters the last *n*th row is impossible. Therefore a westward movement in the last row at tick 1 does not pose a collision threat. Moreover, a southbound traffic could be allowed in the last column at the same tick. Since, the last column agents are unaffected by the two previous rules at tick 1.

Under the same principle, we can allow eastern movement in the first row at all times, except at cycle 1. The already mentioned southern movement at the last column is possible around the clock. Finally, we could speed up the movement in the first column. However, to make sure that the Algorithm resolves the ER Problem, we prefer to implement speedup only for the exiting agents.

We list all possible movement directions in Table 1 and Table 2. Thus agents stay at place if the movement in the desired direction could lead to a possible collision with agents moving in the designated direction. We summarize the above ideas in the following Lemma.

**Lemma C.1 (No Collisions).** Let *C* be a legal initial configuration of an *n* × *m* grid *G*. Then, simultaneous Algorithm *A* execution by the agents on the grid does not lead to collision at any future time *t* > 0.
Lemma C.1 does not assume \((C_0)\) or \((C_1)\) requirements. Thus it is a claim about Algorithm \(\mathcal{A}\) and not about the geometry of the grid \(G\).

Algorithm \(\mathcal{A}\) is only partially defined in Appendix B.1 and Appendix B.2. However, the missing input tuples are unattainable from any initial configuration \(C\) of the grid \(G\). I.e., we further show that an agent type-state-neighborhood tuple \(I = (\tau, S, N)\) with an undefined value \(\mathcal{A}(I)\) is not an attainable input tuple. Specifically, an agent of agent type \(\tau\), sensing a neighborhood \(N\) can not possibly be in the state \(S\). Moreover, some other rules are in fact also unreachable, but are left in the tables for convenience.

Lemma C.2 (Algorithm is Well-Defined Lemma). Distributed parallel execution of Algorithm \(\mathcal{A}\) does not lead to an undefined behavior, due to the partial definition of the local rule of exchanges \(\mathcal{A}\).

Proof. Recall, that we conceptually treat an agent memory state, as the binary value resulting from concatenation of the Direction Bit \((d)\) and the current value of a 4-tick timer held by an agent, where \(d\) is treated as an MSB bit. Denote an agent memory state according to the MSB value of its binary representation as 0-state or 1-state respectively.

We notice, that rules for both agent types are fully defined for all 0-states (i.e. for all states where Direction Bit \((d)\) is set to 0). Moreover, for a given pair agent type - grid position the following dichotomy holds:

- agent movement rules are defined for all possible neighborhoods and for all possible agent memory states, incl. 1-states.
- only zero rules are defined for 1-states in total.

Therefore, if an agent of type \(\tau\) moves in direction \(D\) on the grid and its Direction Bit \(d\) is set to 1, we need to check that the first option holds.

The full enumeration of rules is given in Table 3 and Table 4. In summary, every possible input tuple has an appropriate movement rule defined. Hence, no agent is stuck in the undefined state throughout the Algorithm execution. □

C.2 Empty spaces “turn-around” on a boundary and a hypothetical unsolvable ER problem Instance

The Algorithm is designed in a way to enforce a clock-wise agent movement on the edges of the grid. We elaborate this fact in the following straightforward observations.

Observation C.3. Agents move East in the first (upper) row only.

Observation C.4. Exiting agents do not leave the last (right-most) column.

Observation C.5. Agents in the first (left-most) column move North only.

However, sometimes agents move into the first row “unintentionally”, i.e. not as a part of the routine clock-wise motion around the frame from the left
| Position | Direction | new Position | handles \( d = 1 \) | new Position | handles \( d = 1 \) |
|----------|-----------|--------------|-----------------|--------------|-----------------|
| 5        | ∅         | 5            | ✓               |              |                 |
| 5        | N         | 7            | ✓               | 9            | ✓               |
| 5        | W         | 2            | ✓               | 5            | ✓               |
| 9        | ∅         | 9            |                 |              |                 |
| 9        | N         | 7            | ✓               | 9            | ✓               |
| 9        | W         | 6            | ✓               | 9            | ✓               |

Table 3: Exit vehicle rules that set Direction Bit(\( d \)) to 1. An agent follows the prescribed motion and moves to, a possibly, different position on the grid (e.g. an agent moving North from position 5 could move into exactly two grid positions: 7 and 9 respectively). In case that the agent position’s after the move rules support memory state(s) of the form 1xx, ✓ is set, otherwise ✗ is set. Possible agent actions[move to] are N - north, E - east, S - south, W - west and ∅ - stay.

| Position | Direction | new Position | handles \( d = 1 \) | new Position | handles \( d = 1 \) |
|----------|-----------|--------------|-----------------|--------------|-----------------|
| 1        | ∅         | 1            | ✓               |              |                 |
| 1        | W         | 5            |                 |              |                 |
| 5        | ∅         | 5            | ✓               |              |                 |
| 5        | N         | 7            | ✓               | 9            | ✓               |
| 5        | W         | 2            | ✓               | 5            | ✓               |
| 8        | W         | 9            |                 |              |                 |
| 9        | ∅         | 9            |                 |              |                 |
| 9        | N         | 7            | ✓               | 9            | ✓               |

Table 4: Continue agent rules that set Direction Bit(\( d \)) to 1. An agent follows the prescribed motion and moves to, a possibly, different position on the grid (e.g. an agent moving North from position 5 could move into exactly two grid positions: 7 and 9 respectively). In case that the agent position’s after the move rules support memory state(s) of the form 1xx, ✓ is set, otherwise ✗ is set. Possible agent actions[move to] are N - north, E - east, S - south, W - west and ∅ - stay.
most column. In this case, exiting agents exploit the opportunity to minimize the traveling time to the target column and remain in the first row. On the other hand, continue agents have no immediate benefit to reside in the first row and move back. We summarize this insight below:

**Lemma C.6 (No Accidental Shortcuts to the Front Row Lemma).** Suppose continue agent $A$ occupies grid cell $a_{2,k}$ on an $n \times m$ grid, where $1 < k < m$. Assume $A$ moves North at time $t \equiv 01_2 \mod 4$. Then, $A$ moves back from $a_{1,k}$ to $a_{2,k}$ at time $t + 1$.

**Proof.** $a_{2,k}$ is a position 9 grid cell, while $a_{1,k}$ - is a position 7 cell. A continue agent moving North from position 9 sets the Direction bit to 1. Therefore, an agent $A$ is in the state $110_2$ at $t + 1$. According to Fig. 26b, a continue agent will move S(outh) at $(t + 1) \equiv 10_2 \mod 4$, since two possible neighborhood readings at time $t + 1$ are: \[ \text{or} \], as claimed. 

Let us consider $P(n,m,N_1,N_0,C(0))$ an instance of the ER problem, such that $C(0)$ is an initial configuration of an $n \times m$ grid with $N_1$ exiting agents and $N_0$ empty cells. We assume problem parameters $n, m, N_1$ and $N_0$ are natural numbers satisfying \([C_1]\).

Let $C(t)$ be a grid configuration at time $t \in \mathbb{N}$.

We shall follow the proof by contradiction approach to show the correctness of Theorem 4.1. Assume the opposite, i.e. that there exists an instance of the problem $P(n,m,N_1,N_0,C(0))$, such that no grid configuration $C(0), C(1), C(2), \ldots$ is a grid target configurations. Where configuration $C(t+1)$ succeeds the grid configuration $C(t)$ at time $t+1$, and, is in fact, a result of decentralized simultaneous execution of Algorithm $A$ by grid agents at time $t \in \mathbb{N}$.

We now proceed to characterize a macroscopic emergent agent behavior. Exiting agents do not leave the first row, except to the last column (South movement from position 4 to position 8). And never leave the last column. Moreover, the first row is the only row in which agents execute eastward motion. Therefore, once an exiting agent reaches to the first row, it either gets “stuck” in the first row forever, or, proceeds in the first row eastward and gets “stuck” in the last column.

Hence, we can make the following observation:

**Observation C.7.** An exiting agent $A$ executes at most $2m$ East/West moves during Algorithm $A$ execution.

The observation holds true according to the behavior characteristic provided above, i.e. an exiting agent ceases to move West after a single East step. Furthermore, the number of moves in each direction is bounded by the number of columns.

Let $T_0$ - be the time tick just after the last column change by any exiting agent. $T_0$ is well defined according to the above observation.
Since \( C(T_0) \) is not a target configuration by our assumption, all the exiting agents on the grid are “stuck” in their current column, i.e. they move in the North-South direction only, if at all. And at least one exiting agent is not located in the right-most column.

We claim, that :

**Lemma C.8** (No “Parking” at \( a_{1,m} \), Lemma). Grid cell \( a_{1,m} \) becomes empty infinitely often, i.e. \(|\{t|t > T_0 \text{ and } a_{1,m}(t) = 0\}| = \infty\).

Suppose this is not the case, let \( T_1 > T_0 \) be the time such that the upper right corner cell of the grid \((a_{1,n})\) is occupied forever after \( T_1 \), i.e.

\[
\forall t \geq T_1 \quad a_{1,m}(t) \neq 0
\]

Then none of the last column cells is empty after \( T_1 \) due to the fact that all the agents in the last column of the grid (position 4 and 8) move South once neighbored by an empty cell. However, continue agents could also leave westward. Nonetheless, the observed behavior is a “North propagation” of an empty slot at the pace of one cell at a single time tick. Moreover, no rule allows eastward moves into the last column. Except in the first row, where an assumed “stuck” agent is located \((a_{1,m})\). Therefore, an iterative application of Algorithm \( \mathcal{A} \) by agents in the last column will inevitably “bring” an empty slot to neighbor an agent occupying \( a_{1,m} \) from the South.

Since we assumed that an agent at \( a_{1,m} \) is “stuck”, we are forced to conclude the whole \( m \)th column is “stuck”, i.e. no movement to/from/in it is possible after \( T_1 \).

We observe, that the “flow” of empty spaces counters the movement of agents and could be described as a motion in the counter clock-wise direction. Moreover, the only area where empty spaces could “move” West is the first row. Therefore, it is unavoidable that East-West “movement” will cease, i.e. there is no source or sink for empty spaces on the grid. Denote \( T_2 > T_1 \) - the time tick of the last East/West move by any agent.

Let \( k_g \) - be the highest column index, such that for \( t > T_2 \) columns \( k_g + 1, k_g + 2, \ldots, m \) do not contain, but column \( k_g \) does. \( k_g \) is well-defined, since there is at least one empty cell on the grid. And, according to the previous proposition \( 1 \leq k_g < m \).

**Observation C.9.** For \( 1 < i < m \) an agent \( A \) moving North from \( a_{n,i} \) at time \( t > T_2 \), does not return back before \( t + 5 \).

**Proof.** Any agent in a column, except the first one, leaving North from \( a_{n,i} \) sets its state to \( 110_2 \). After the East-West movement cessation, agent state could according to the following sequence only:

\[
110_2 \rightarrow 111_2 \rightarrow 100_2 \rightarrow 101_2
\]

(see [Fig. 23a] and [Fig. 27b]). And only then a agent at position 9 could change its state to \( 010_2 \). In case \( a_{n,i} \) remains empty, it is possible for agent \( A \) to move back.
However only an exiting agent could move back South at $t + 5$. A continuing agent could not move back before $t + 7$.

The only exception, is the $n = 2$ case. An agent leaving $a_{n,i}$ moves from position 5 on the grid to the position 7 and then immediately back. Although, a first row agent in the highest index column $j < i$ will immediately move East in the next tick, breaking no East-West movement assumption. In the absence of such $j$, we have that an agent in the lowest-indexed non-empty column either:

(a) moves West to the empty column on the left,

(b) moves North, if $j = 1$, then eventually East.

In both cases, after $T_2$ the above behavior will contradict our no East-West movement assumption.

**Observation C.10.** The previous claim holds true also for agents moving from cell $a_{j,i}$ at Position 9 to cell $a_{j-1,i}$ at Position 9. In other words, it takes at least five time ticks for a North-bound leaving agent to return to its previous position.

**Observation C.11.** A similar claim holds for agents moving South in those internal columns. From position 9 to either position 5 or 9. With roles exchanged, i.e. it takes at least 7 time ticks for an exiting agent to return to its previous position, and at least 5 time ticks for a continue agent.

Let us denote a grid cell that remains occupied (forever) after some time $T$ as a **SOFT cell**. We then continue to investigate the dynamics in the column $k_g$, after all the SOFT grid cells were occupied for the last time. Curiously, we shall show, that all the $k_g$ column cells are non-SOFT cells. Except (maybe) for the $a_{1,k_g}$ cell.

**Lemma C.12 (No Partially Stuck Columns Lemma).** Let $k_g < n$ be the index of a column, that contains empty cells after $T_2$, East-West movement termination point, but all columns right to the column $k_g$ are SOFT cell columns. Then all cells in the $k_g$ column are non-SOFT cells. Except (maybe) for the $a_{1,k_g}$ cell.

**Proof.** Suppose $a_{i,k_g}$ is a SOFT cell, that is occupied forever after w.l.o.g. $T_2$, by an agent $A$, and $a_{i+1,k_g}$ is a non-SOFT cell. Suppose at some time $t = 00_2$, $a_{i+1,k_g}$ become empty [it either become empty, since an agent occupying it moved South, or it was empty for all $t \geq T_2$]. By the observations above, an agent that left $a_{i+1,k_g}$ could not return to $a_{i+1,k_g}$ before $t + 5$. Moreover, since East-Wast movement ceased at $T_1$ it is not possible that the cell $a_{i+1,k_g}$ becomes occupied by an agent moving west from the column $k_g + 1$.

If $A$ is a continue agent, it will either move North at $t + 1$, or will set the state to 010__ and move South at $t + 4$. Recall, that $A$ occupies a SOFT cell, hence leaving $a_{i,k_g}$ is a contradiction to assumptions.

If $A$ is an exiting agent, the analysis is a bit more involved. Readings

and at time tick $t + 1$ could be sensed if one of the following is true i) an
agent departing from cell $a_{i,k_g-1}$ South at time $t$, ii) cell $a_{i,k_g-1}$ was empty at $t-1$. However, in the former case $A$ will leave West, in contradiction to the definition of $T_1$. And in the latter case, $A$ should have moved West at $t-1$ (or $t-3$, depends on the $t \equiv 1, 3 \mod 4$). Therefore, leading to a contradiction in one of our assumptions.

Additionally, an exiting agent would behave exactly like a continue agent in two remaining cases $\begin{array}{c}X \end{array}$ and $\begin{array}{c}X' \end{array}$, resulting in $A$ leaving $a_{i,k_g}$ no later than $t+4$. In contradiction to our assumption that $a_{i,k_g}$ is a SOFT cell.

We still need to take care of a SOFT cell occupant with a non-SOFT Northern neighbor cell. W.l.o.g. we assume that the South neighbor is also a SOFT cell agent.

Suppose $A$ at $a_{i,k_g}$ is a continue agent, and an agent $B$ that occupied $a_{i-1,k_g}$ and just left it at $t \equiv 01_2 \mod 4$. Then, depending on the agent type, $B$ can move back to $a_{i-1,k_g}$ at $t+7$ if it is a continue agent, or at $t+5$ if $B$ is an exiting agent. Although, $A$ will move North at $t+4$, contradicting our assumption, that $A$ resides at $a_{i,k_g}$ forever.

However, a special case $i = n$ should be treated separately. We denote by $t \equiv 01_2 \mod 4$ a time tick when $B$, the North neighbor, moves further North. Then, the following three cases are possible. i) the cell $a_{n,k_g-1}$ is empty at $t$, and therefore was empty at $t-1$. Although, in this case $A$ observes $\begin{array}{c}X \end{array}$ at $t-1$ and should have moved West unconditionally at time $t$. ii) the cell $a_{n,k_g-1}$ becomes empty at $t$, and an agent $C$, leaving it, could not return back until at least $t+5$. Therefore, $A$ will unconditionally move either West or North at $t+4$ according to its internal state. iii) the cell $a_{n,k_g-1}$ remains occupied, and $A$ observes $\begin{array}{c}X \end{array}$ between $t+1$ and $t+4$. Consequently, $A$ moves North at $t+4$. In all the above cases $A$ moves in contradiction to our assumption, that $A$ stays in $a_{n,k_g}$ forever.

If $A$ is an exiting agent, we could w.l.o.g. suppose that $a_{n,k_g-1}$ is always occupied (i.e. a SOFT cell), since otherwise $A$ will move West in at most two time ticks, contradicting the definition of $T_0$. Moreover, an exiting agent in the observed neighborhood $\begin{array}{c}X \end{array}$ will leave North at $t+4$, in contradiction to our assumption.

We have just shown that all cells in the column $k_g$ become empty eventually, and therefore will become empty infinitely often (i.e. non-SOFT cells). □

We have just established, that every cell in the $k_g$ column is empty infinitely often (i.e. is a non-SOFT cell, and except maybe to the cell $a_{1,k_g}$). In particular,
a grid cell $a_{n,k}$, hence an agent in $a_{n,k+1}$ at SOFT cell column $k+1$ observes a neighborhood from time to time. Yet, after an occupying agent leaves the cell $a_{n,k}$ and moves North, an agent at cell $a_{n,k+1}$ (position 5 on the grid) should move West in at most 4 time ticks. The above conclusion contradicts the definition of $T_2$, except, whenever $k = m - 1$. In the latter case, an agent occupying $a_{n,m}$ could be an exiting agent, and will not leave the target column. Note, that in the edge case whenever $n = 2$, we have that the current configuration is a target configuration and we are done.

However, by assumption, not all exiting agent moved into the last column at $T_0$. Therefore, there is at least one continue agent occupying one of the cells $a_{2,m}, \ldots, a_{n,m}$. Suppose, this is a continue agent $B$ at some cell $a_{i,m}$.

According to Lemma C.12, cell $a_{i,m-1}$ is a non-SOFT cell, hence it is empty infinitely often. Therefore, we have the following dichotomy: either no agent moves into cell $a_{i,m-1}$ from North, then $B$ moves West at the next $11_2 \mod 4$ time tick, upon $a_{i,m-1}$ is vacated. In contradiction to the definition of $T_2$. In the other case, agents periodically move into $a_{i,m-1}$ from North. However, it should be that at some time tick $t \equiv 01_2 \mod 4$, an agent occupying $a_{i,m-1}$ leaves North. According to our previous observations the same agent could not return to $a_{i,m-1}$ before $t + 5$, nor could any other agent move into $a_{i,m-1}$ from $a_{i+1,m-1}$ before $t + 4$. Hence, an agent $B$, that finds itself at position 8 on the grid, moves West at $t + 2$. This contradicts the assumption, that an East-West movement ceased. Thus finally proving Lemma C.8

### C.3 Inner grid column

We have shown, that cell $a_{1,m}$ is empty infinitely often. Consequently, every agent in the first row or in the first column will eventually reach $a_{1,m}$, as all the agents in this “Great Outer Arc” move East and North respectively. However, we are not yet done. We have established, that the first row or the first column are free of exiting agents, but they (exiting agents) could be located in the “inner” columns, i.e. columns with indices $1 < k < m$. We, therefore, shall show, that these agents eventually move West.

Recall, that cell $a_{1,m}$ is empty infinitely often, and that, by assumption, empty spaces “move” East in the last row only. Hence, we have that $a_{n,1}$, $a_{n,2}, \ldots, a_{n,m-1}$ are also empty infinitely often. Therefore, as a corollary of Lemma C.12 all cells in columns $2, 3, \ldots, m - 2$ are empty infinitely often. Note, that the Lemma C.12's assumptions do not hold for column $m - 1$, since continue agents do move West from column $m$ to column $m - 1$ in any row besides the first.

In the below analysis we shall show, that an exiting agent $A$, located in the column $1 < k < m$, is eventually “neighbored” by an empty cell from the West. Moreover, we claim that an exiting agent at positions 5 and 9 that faces an empty cell from the West, necessarily moves into this empty cell. However,
establishing the former is a much more involved task than the latter.

We introduce a column spatio-temporal plane \( D_k \) and show an existence of a “continuous” path along the column. That is we show that there exists a monotone function \( h_e \), defined on some time interval \( TI \triangleq \{t_0, t_0 + 1, \ldots, t_1\} \), that describes the row location of a “designated” empty space in the column. Later, we combine \( h_e \) with a function \( h_A \) - the row tracking function of an exiting agent \( A \). It is a straightforward exercise in algebra to establish that function \( g \triangleq h_e - h_A \) has discrete changes bounded by 1. Finally, The Discrete Intermediate Value Theorem is utilized to demonstrate that \( g \) takes a value of 0 on \( TI \). In other words, that \( A \) neighbors an empty cell on its West.

**Lemma C.13** (No Missed Opportunities Lemma). Let \( A^* \) be a west-most exiting agent on the grid at time \( T_0 \). Suppose \( A^* \) occupies cell \( a_{i,k+1} \), where \( 2 < i \leq n \) and \( 1 \leq k \leq m - 1 \). Suppose \( t \geq T_0 \) is the earliest time when \( A^* \) faces an empty cell on its West. Then \( A^* \) moves West before \( t + 4 \).

**Proof.** Note, that all agents to the West of \( A^* \) are continue agents. We analyze possible local changes around \( A^* \) in the near past that “led” an empty space to neighbor \( A^* \) from the West.

(a) A continue agent \( B \) left cell \( a_{i,k} \) at time \( t - 1 \). Then \( a_{i,k} \) could be a cell at position 2, 5, 6 or 9 on the grid. However, a continue agent could leave \( a_{i,k} \) at only two time ticks 00\(_2\) mod 4 or 01\(_2\) mod 4. Since agents to the West of \( A^* \) are all continue agents, none could re-occupy \( a_{i,k} \) before \( t + 4 \). On the other hand, \( A^* \) is at position 5 or 9 and will move West no later than at the next 11\(_2\) mod 4 time tick, which is at most \( t + 3 \).

(b) Agent \( A^* \) moved to \( a_{i,k+1} \) at time \( t - 1 \) and at time \( t \) faces the empty cell \( a_{i,k} \). An exiting agent moves North at times 01\(_2\) and South at times 10\(_2\). Though, continue agents do not move at positions 2, 5, 6 and 9 at times 10\(_2\) and 11\(_2\). Therefore, \( a_{i,k} \) could not become occupied before \( A^* \)’s move West at the next time tick 11\(_2\) (either \( t + 1 \) or \( t + 2 \)).

In both cases, \( A^* \) moves into an empty cell it faces on the West before any other agent have an opportunity to do so. \( \Box \)

The above Lemma C.13 holds true for an exiting agent at \( a_{2,k+1} \), in case an adversary agent left \( a_{2,k} \) to South. For \( k = 1 \) this is always the case. Later we shall show, that an agent occupying \( a_{2,k-1} \) eventually moves South also in the interior columns 2, 3, \ldots, \( m - 2 \).

From now on we focus on the column \( k \) immediately West of \( A^* \).

We define a spatio-temporal block diagram \( D_k \) of column \( k \) on the \( \{T, T + 1, \ldots\} \times \{1, 2, \ldots, n\} \) in the following way: \( D_k(t, i) = \text{empty} \iff a_{i,k}(t) = 0 \), the block \( D_k(t, i) \) is defined to be occupied otherwise (see Fig. 29).

**Definition C.14.** Suppose \( B = (t, i) \) is a singleton empty block in a spatio-temporal block diagram \( D_k \). A \( B \)’s following block is another empty block \( C = (t', i') \) in \( D_k \), where \( |i - i'| \leq 1 \) and \( t \leq t' \leq t + 1 \).
In a layman terms, we would say: a following block is a block sharing at least one common point with a given block, and not the block itself or blocks in a prior times.

**Definition C.15.** A sequence of empty blocks in the spatio-temporal plane $D_k$ will be called a path, if for each block $S$ in the sequence, except maybe for the last one, there is another block $S'$ in the sequence, such that $S'$ is a following block of $S$.

In other words, there are no isolated blocks on a path, nor jumps. See two piece-wise lines in Fig. 29 for path examples.

We associate the time dimension of spatio-temporal plane $D_k$ with $x$-axis, extending from left to right. The row dimension of column $k$ is expressed on $y$-axis of $D_k$, stretching from top to bottom. We define a right step of the path to be a pair of two empty blocks $B_1$ and $B_2$ in $D_k$, such that $B_2$ is a following block of $B_1$, and both have the same $y$-coordinate, i.e. are on the same row. Thus, a right step on the path is equivalent to a specific cell of the grid remaining empty for two consecutive time ticks.

We define an up step and a down step in the similar manner. Except, that we shall assume $B_1$ and $B_2$ share the same $x$-coordinate. In grid terms, this is equivalent to a pair of neighboring empty cells on the grid in the same column at the same time tick.

Finally, we define a diagonal up and diagonal down steps of the path to be a pair of empty blocks $B_1$ and $B_2$ that share a single common point in $D_k$. Both steps are equivalent to an agent moving either South or North respectively in the $k$th column.

**Definition C.16.** A path consisting entirely of up, right and diagonal up steps is called an ascending path.

**Definition C.17.** A path consisting entirely of down, right and diagonal down steps is called a descending path.

Figure 29: Spatio-temporal diagram of a grid column.
We state and prove a following series of claims under the previous general assumption, that the East-West movement of exiting agents ceased, but at least one exiting agent is “stuck” in some internal column, i.e. all the claims are true for \( t > T_0 \).

**Claim C.18** (First Column Path Lemma). *Suppose, that \( a_{1,1} \) is empty at time \( t_0 \). Then there exists a descending path in \( D_1 \) from cell \((t_0,1)\) to cell \((t_1,n)\) for some \( t_1 \geq t_0 \).*

*Proof.* We observe the first column of the grid. By the definition of \( T_0 \) the first column could only contain continue agents. Let us designate an empty space that is “located” at \( a_{1,1} \) at time tick \( t_0 \). Note that the corresponding block in \( D_1 \) is \((t_0,1)\).

We interpret an agent move into \( a_{1,1} \), as though the designated empty space “moved” to \( a_{2,1} \), and so on. Such interpretation is equivalent to a sequence of right steps followed by a diagonal down step in the \( D_1 \) plane. In case the designated empty space reaches the \( n \)th row, we are done.

However, it could be possible that after \( T_0 \) only finitely many agents will pass through \( a_{1,1} \). In this case agents will stop entering the first column (through \( a_{n,1} \), since by assumption only continue agents are moving in the first column). Then all cells beneath the designated empty space are already empty (denote that time: \( t_1 \)). Therefore, the path from that point \((t_1,i)\) in \( D_1 \) extends in a series of down steps to \((t_1,n)\).

Unfortunately, the movement of agents in the column \( k > 1 \) is more complicated. The agents could move North and South, moreover from time to time an agent at \( a_{2,k} \) could “borrow” an empty space from the first row. The unexpectedly strange behavior is a direct consequence of the agent characteristics in our model. The agents are quite myopic. A visibility range of 1 does not allow an agent to discriminate between second, third, or, for the matter, *any* non-edge rows.

Moreover, the paths from \((t_0,2)\) could end before reaching \((t_1,n)\), where \( t_1 > t_0 \), due to unfortunate conjunction of a grid configuration and an agent internal state (see Fig. 30 for a specific example).

Further we show, that for an internal column \( k \in \{2, \ldots, m - 1\} \) there exists an ascending \( D_k \) path that starts at \((t_0,n-1)\) and ends at \((t_1,2)\) for times \( t_1 \geq t_0 \geq T_0 \). We later demonstrate, that at least some paths from above could be pre-extended by an empty block at \((t_0-1,n)\) or \((t_0,n)\), while remaining an ascending path, i.e. that there exists a path going up from \((t_0,n)\) to \((t_1,2)\). However, the latter will be useful only in the context of the columns containing continue agents only.

**Claim C.19** (Penetrable Column Lemma). *Suppose, that an instance of the ER Problem is unsolvable by Algorithm A, and the East-West movement of exiting agents ceased. Then cell \( a_{n-1,k} \) is empty infinitely often, i.e. \( a_{n-1,k} \) is not a SOFT-cell.*
Figure 30: Tempo-spatial diagram of \( k^{th} \) column. The path started from the second row, but not ending in the last row.

| Time tick | 01_2 | 10_2 | 11_2 | 00_2 | 01_2 | 10_2 |
|-----------|------|------|------|------|------|------|
| Neighborhood | ![Diagram](image)
| initial \( d = 0 \) | 001_2 | 010_2 | 011_2 | 000_2 | 101_2 | moved N |
| initial \( d = 1 \) | 101_2 | 110_2 | 111_2 | 100_2 | 101_2 | moved N |

Table 5: Agent at \( a_{n-1,k} \). North neighbor leaves at \( t \equiv 01_2 \mod 4 \). Agent’s state and neighborhood readings, in case South neighbor does not leave.

Proof. Claim C.19 is closely related to Lemma C.12 however the assumptions are slightly different. We have already proved Lemma C.8, i.e. that empty spaces “circulate” on the grid. Therefore, we have established that \( a_{n,k} \) is empty infinitely often.

Suppose, that \( a_{n-1,k} \) is a SOFT cell and is constantly occupied after time \( T \) by agent \( A \), w.l.o.g. assume \( T = T_0 \). We consider the following scenario: there is an empty cell in the column \( k \) at some time \( t \equiv 00_2 \mod 4 \), i.e. an empty space that is not a “borrowed” empty space. Then it could be shown, based on the ideas presented in the proof of Lemma C.12, that all cells \( a_{2,k}, \ldots, a_{n-2,k} \) are empty infinitely often. Therefore, we list two possible sequences of \( A \)'s states and decisions based on them (see Table 5 and Table 6). In all the possible scenarios \( A \) moves in at most 4 time ticks, in contradiction to our assumption.

In the remaining case, there are no non-“borrowed” empty spaces in column \( k \). Therefore, \( A \) only ever faces an empty space from South. However, after at most the second “encounter” \( A \) will unconditionally move South before an empty space could be occupied by an agent from cell \( a_{n,k+1} \).

However, two special cases still require our attention. The case of \( n = 4 \) and the case of \( n = 3 \).

In the former case, \( n = 4 \), the North neighbor of \( A \) at cell \( a_{2,k} \) could “borrow” an empty cell “passing” in the first row and leave North. However, a continue agent that “erroneously” enters the first row immediately returns back. Hence, \( A \) is regularly neighbored by an empty space from North at 10_2 ticks. On the
| Time tick | 01₂ | 10₂ | 11₂ | 00₂ | 01₂ | 10₂ |
|-----------|-----|-----|-----|-----|-----|-----|
| Neighborhood | | | | | | |
| initial $d = 0$ | 001₂ | 010₂ | 011₂ | 000₂ | moved S |
| initial $d = 1$ | 101₂ | 110₂ | 111₂ | 100₂ | 101₂ | moved N |

Table 6: Agent at $a_{n-1,k}$. Both North and South neighbors leave at $t \equiv 01₂ \mod 4$. Agent’s state and neighborhood readings.

The other hand, A’s Direction Bit could only change at 00₂ ticks. Therefore, A leaves South on at most the second encounter with an empty cell from South.

In the latter case, $n = 3$, we claim that there exists a time tick $T$, such that for any $t > T$ such that $t \equiv 00₂$ A’s Direction Bit is set to 0. Indeed, if A ever moved North, then A immediately returned and set its Direction Bit to 0. Moreover, A resets the Direction Bit to 1 only at times $t \equiv 00₂$. Hence, in case $a_{n,k}$ is empty at time $t = 00₂$, we have that A will immediately move South (see Fig. 27b).

Hence, $a_{n-1,k}$ eventually becomes empty in all the cases.

We conclude from Lemma C.12 and Claim C.19 that all cells in columns 1 through $m-2$ are empty infinitely often.

**Claim C.20 (Start of the Path).** Consider grid column $k$. There exists $t \geq T₀$ such that at least one of the following is true: 1) cell $a_{n-1,k}$ and cell $a_{n-2,k}$ are empty at time $t$ 2) cell $a_{n-1,k}$ is empty at time $t$, and cell $a_{n-2,k}$ is empty at the next time tick $t + 1$.

**Proof.** Assume that former never happens, and that no agent moves from $a_{n-2,k}$ into $a_{n-1,k}$ after $T₀$. Recall, that according to Claim C.19 cell $a_{n-2,k}$ is empty infinitely often.

Consider a time tick $t \equiv 10₂ \mod 4$, a moment when $a_{n-2,k}$ became empty due to the northbound move of the agent $B$ occupying $a_{n-2,k}$ at $t - 1$. In line with our assumption cell $a_{n-1,k}$ is occupied at time $t$. Let $A$ be an agent occupying cell $a_{n-1,k}$ at time $t$.

In case $a_{n,k}$ is empty at $t$, we have that $A$ necessarily executes one of the following moves. $A$ will either move South at $t + 2$, and since $B$ could not return before time tick $t + 4$, we have a contradiction - both $a_{n-1,k}$ and $a_{n-2,k}$ are empty at $t + 3$.

Otherwise, $A$ could move North at $t + 3$. Since continue agents at position 9 could not leave a column, the number of such northbound moves (unidirectional moves) is finite. Denote the last claim the “Finite Column Capacity Argument”. Ultimately, some agent ought to move in the opposite direction. However, such a move from cell $a_{n-2,k}$ to cell $a_{n-1,k}$ proves the claim.

In the complementary case, $a_{n,k}$ is occupied at time $t$, therefore $A$ moves North at $t + 3$. See the “Finite Column Capacity Argument” above, that leads
to the desired conclusion.

Claim C.20 clears up the path to the existence of an ascending $D_k$ path, starting at $(t_0, n - 1)$ and all the way up to $(t_1, 2)$.

**Definition C.21.** Suppose $a_{i,j}$ is empty at time $t$. We define the leading agent of $a_{i,j}$ to be the closest northern agent located at the same column $j$ at time $t$.

**Definition C.22.** A leading block of an empty block $(t, i)$ in the spatio-temporal plane $D_j$ is a block $(t, i')$ corresponding to the leading agent of an empty cell $a_{i,j}$.

Note, that the leading agent should not necessarily exist. Though in that case, cells $a_{i,j}, a_{i-1,j}, \ldots, a_{1,j}$ are simultaneously empty at time $t$.

**Claim C.23** (Path Extension Lemma). Let $P$ be an ascending path in the tempo-spatial plane $D_k$, starting at $(t_0, n - 1)$ and ending at $(t_1, n - 2)$. Then the path extends to an ascending path ending at $(t_2, 2)$. Where $t_2 \geq t_1 \geq t_0 \geq T_0$.

**Proof.** Define a leading block (and a corresponding leading agent) of a path at time $t$ to be the leading block of one of the empty blocks on the path at time $t$.

There are two possible ways a leading agent could move and “extend” the path. First, it could go south, then the path could be extended by a diagonal step, followed by possibly one or more up steps, due to empty spaces configuration. This move necessarily forces the leading agent change. Second, a leading agent could move North, then the path extends by a diagonal step, though the leading agent remains the same.

Otherwise the leading agent does not move, and the path could be extended by the right steps. However, we should take caution and show that another agent does not move North at the same just under the leading agent, thus terminating the ascending path.

A leading agent observes exactly one of the two possible neighborhoods: 

\[
\begin{array}{c}
X \quad \quad \quad X \\
\end{array}
\quad \quad \quad
\begin{array}{c}
X \quad \quad \quad X \\
\end{array}
\].

This is due to our path extension analysis we have done above.

Suppose a leading agent $L_1$ decides to move South at time $t \equiv 00_2$. Thus $L_2$ becomes a new leading agent, and additionally $L_1$ will not be able to move back until $t + 5$. $L_2$ then either moves North at $t + 1$ or otherwise South at $t + 4$. In the former case, $L_2$ opens up a two-block distance at least from the next southern agent in the column. And the distance to the nearest southern agent will be kept at value at least 2, as long as $L_2$ continues to move North. In case $L_2$ stops North movement, it will change the movement direction and move South before $L_1$ (or the agent that could have replaced $L_1$ from the $k + 1$ column) could occupy the southern neighbor empty cell. However, if $L_2$ moves South we apply the same reasoning to the new leading agent.

Note, that in the absence of the leading agent, we have that all cells above the last leading agent are currently empty. And the path extends vertically in $D_k$ to the first row.
Notoriously, all the agents in the $k$th column become leading agents of the same ascending path. Until either no leading agent is defined, or the leading agent is the agent located in the first row. Thus, we have just showed that an ascending path extends to the second row.

The claim above establishes that any non-horizontal ascending path that starts at the row $n - 1$ extends to at least the second row. However, this path ends when a leading agent at time $t_2$ “borrows” an empty space from the first row of the grid, and then immediately returns back.

Denote this path $P_0$. We claim that there exists another path $P_1$ in $D_k$, coinciding with $P_0$ from time $t_0$ to $t_2 - 1$. $P_1$ then could be extended to the second row at time $t_2 + 3$. Moreover, cell $a_{2,k}$ will remain empty throughout time interval $\{t_2 + 3, \ldots, t_2 + 8\}$.

We assume that at time $t_2$ a leading agent $L_1$ moved North from position 9 to position 7. According to Algorithm $A$, $L_1$ returns back immediately at $t_2 + 1$. In addition $L_1$ sets its Direction bit to 0 and will move South at $t_2 + 3$, since the gap between the agent $L_1$ and the former leading agent $L_2$ stayed of size at least 1 (the gap was either of size at least two before the agent $L_1$ decided to push North into the first row. The gap could have narrowed down at $t_2 + 1$ in case $L_2$ also moved North at $t_2$. Or the gap was of size 1 at $t_2$. However, in the latter case, it is the result of the lead change at time $t_2 - 1$. Consequently, $L_2$ is unable to return to the third row until at least $t_2 + 4$).

Once the agent $L_1$ at the second row moves South at $t_2 + 3$ it could not return back until at least $t_2 + 8$, nor any agent could move South into $a_{2,k}$ from $a_{1,k}$.

Therefore, we define a new path $P_1$ in the following manner. The path $P_1$ coincides exactly with $P_0$ in $D_k$ from $(t_0, n - 1)$ through $(t_1, n - 2)$ until $(t_2, 3)$. Then $P_1$ deviates horizontally right until $(t_2, 3)$, followed by a diagonal step to $(t_2 + 3, 2)$, and finally to $(t_2 + 8, 2)$ where it ends. Note that by definition $P_1$ is also an ascending path.

We claim, that at least one such ascending path $P_1$ in $D_k$ could be pre-extended to a longer ascending path by a diagonal-up from $(t_0 - 1, n)$ or by an up step from $(t_0, n)$. Thus, we claim that there exists an ascending path in $D_k$ that spans from the row $n$ to the row 2.

We assume below, that column $k$ agents do not leave cell $a_{n-1,k}$ in the southern direction, i.e. that the column $k$ is kind of a plugged from below column. Additionally, we shall assume that $a_{n,k}$ and $a_{n-1,k}$ are not simultaneously empty. Denote these assumptions above $C_2$.

**Lemma C.24 (Limited Agent Throughput).** Let $A$ be an agent that occupies $a_{n-1,k}$ during time interval $(t_0, t_1]$, i.e. $A$ moves to $a_{n-1,k}$ at time $t_0$ and leaves North at $t_1$. Then, under the $C_2$ assumptions, at most two different agents occupy $a_{n,k}$ during $(t_0, t_1]$.

**Proof.** Suppose the opposite is true, and at least three different agents occupy cell $a_{n,k}$ during time interval $(t_0, t_1]$. 

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If \( A \) neighbors an empty cell on the North for the 4 continuous ticks it leaves the \( a_{n-1,k} \) (North or South, depends on the actual neighborhood). Therefore, we conclude that \( A \) senses only the following two neighborhoods during time interval \((t_0 + 4, t_1 - 4]\) and \( a_{n-1,k} \).

However, in that time interval at least two agents left \( a_{n,k} \) to the West, leaving \( a_{n,k} \) empty for at least 4 consecutive time ticks each time. In the first 4 consecutive time ticks term \( A \) should have changed its Direction bit to 0. Subsequently, it should have left South during the second term, in contradiction to \( C_2 \) assumption.

**Lemma C.25** (No Closed Columns Lemma). *Suppose that the western-most exiting agent is located at column \( k + 1 \). Then there exists \( t \geq T_0 \), such that cell \( a_{n,k} \) is empty at time \( t \), and cell \( a_{n-1,k} \) is either empty at the same time \( t \), or is empty at the next time tick \( t + 1 \).*

Note, that the latter condition implies that an agent leaves South from \( a_{n-1,k} \) to \( a_{n,k} \).

**Proof.** We note, that cells \( a_{n,k} \) and \( a_{n-1,k} \) could become empty simultaneously, if agents occupying them leave West and North respectively at the same time. On the other hand, the cells could be empty in two consecutive time ticks, if an agent occupying cell \( a_{n-1,k} \) leaves South to cell \( a_{n,k} \).

Assume \( C_2 \) on the contrary, that such time tick does not exist, i.e. no agent leaves cell \( a_{n-1,k} \) to the South after the time \( T_0 \), nor both cells are empty at the same time.

Note, that an agent \( M \) moving from cell \( a_{n,k+1} \) to \( a_{n,k} \) sets its own Direction bit to 1. If cell \( a_{n-1,k} \) is empty at \( t + 1 \), then we have the following dichotomy: either \( a_{n-1,k} \) was empty at time \( t \) or it became empty following an agent departing North at time \( t \). In the former case, at time \( t \) \( a_{n,k} \) and \( a_{n-1,k} \) were simultaneously empty, in contradiction to \( C_2 \).

In the latter case, \( M \) will move North at \( t + 4 \). However, in the periodic state of the grid, agents could not enter the column \( k \) from \( a_{n,k} \) in the North direction, without some agents eventually leaving in the opposite direction (according to Algorithm \( A \)). Hence, at some point in time an agent moves South to the \( a_{n,k} \), which is also in contradiction to the assumption.

Therefore, we conclude that cell \( a_{n-1,k} \) is occupied at time \( t + 1 \) (and was occupied at time \( t \)). Let \( A \) be an agent occupying cell \( a_{n-1,k} \) at time \( t \).

Once the agent \( M \) moved to \( a_{n,k} \), it could only change its Direction Bit value to 0 if faced by an empty cell on the West, while facing an occupied cell on the North. Note, that the second option of going up the column and then back, is assumed to be non-existent (see the latter case above).

We want to emphasis, that the opposite case when \( M \) faces an empty cell on the North, and an occupied cell on the West, leads to an imminent move of the agent \( M \) to the North, and a contradiction to assumptions.
Moreover, suppose that $M$ faces an empty cell on its North and/or West at time $T \equiv 00_2 \mod 4$. Then, the only viable way $M$ does not leave cell $a_{n,k}$ at $T + 1$ is in the case that all the following conditions are satisfied:

- the $M$’s Direction bit is set to value 0 at time $T$.
- at time $T$ a North-West (invisible-to-$M$) neighbor moves South and becomes a West neighbor of $M$ (visible only from time $T + 1$). We further denote the invisible-at-$T$ agent as agent $B$.

Then, we conclude based on Lemma C.24 that the number of agents leaving $a_{n-1,k}$ to the North is at least equal to the number of agents leaving $a_{n,k}$ to the West. Moreover, each such northbound excursion of the agent $A$ happens when an agent at $a_{n,k}$ already set its Direction bit value to 0.

However, $M$ sets the Direction bit to 0 at time $t \equiv 00_2$ and moves to the West at the next time tick, unless $a_{n-1,k-1}$ is already occupied by an agent $B$ that moved at $t$ from $a_{n-1,k-1}$. We note, that the southbound motion of an agent in the $k-1$ column is only possible following the westward departure of an agent that previously occupied $a_{n,k-1}$ at $t - 3$, otherwise agent $B$ was the last agent that left $a_{n,k-1}$ and it should have happen at time $t - 7$. Though in this case $M$ would have already moved West to occupy $a_{n,k-1}$.

We summarize, $M$ observes at least two agents leaving westward (the fact unknown to agent $M$) from column $k-1$ before finally leaving itself to the $k-1$ column. Therefore, for every agent leaving column $k$, at least two agents leave column $k-1$, which is unsustainable in the long run, when the only source of those column $k-1$ “leaving” agents are the agents moving West from cell $a_{n,k}$.

Since this is an obvious fallacy, we conclude, that $a_{n,k}$ become empty at one of those times $T + 1$. Consequently, agent $A$ would eventually move South, contradicting our assumptions.

Therefore, our initial assumption turned out to not hold. Except if $n = 3$ or $n = 4$. However, In the former case, an agent at $a_{2,k}$ sets the value of the Direction bit to 0 each time it “borrows” an empty space, and each time it neighbors an empty space from the South. Hence, inevitably, an agent occupying cell $a_{2,k}$ will move South to $a_{3,k}$ on the next tick after cell $a_{3,k}$ becomes empty.

In the latter case, an agent $A$ at $a_{3,k}$ either never observes an empty space at time tick 012, hence will move eventually South, as in case $n = 3$. Otherwise, an agent at $a_{3,k}$ is the agent moving back and forth from $a_{3,k}$ to $a_{2,k}$ (and, possibly, “borrowing” an empty space from $a_{1,k}$ from time to time). However, in this specific case, denote by $T$ - time when an occupying agent leaves $a_{4,k}$ to the West. If $A$ at time $T$ is located in cell $a_{2,k}$ - we are done. Hence suppose $A$ occupied $a_{3,k}$ at time $T$. It could either move North at time $T$, or to move South at $T + 3$ - in both cases the claim is established.

Recall our assumption that $A$ does not lead this specific ER Problem instance $P$ to target configuration. Therefore, at least one exiting agent remains in the same non-edge column forever. However, if at least one such “stuck” exiting agent moves to a new column, we deduce that our assumption was wrong, and
\[ \Delta f = \Delta H - \Delta G. \]

Table 7: Changes in functions \( f, G, H \) split by time ticks. \( G \) - tracks the \( V \)’s row; \( H \) - tracks the “designated” empty space’s row; \( \Delta f = \Delta H - \Delta G \).

| Time tick | Virtual tick | \( \Delta G \) | \( \Delta H \) | \( \Delta f \) |
|-----------|--------------|----------------|----------------|----------------|
| 0 mod 4   | no           | 0, -1          | 0, -1          |                |
| 1 mod 4   | no           | 0, -1          | 0, -1          | 0, 1, -1       |
| 2 mod 4   | no           | 0, 1           | 0              | 0, 1, -1       |
| 3 mod 4   | no           | 0              | 0              | 0              |
| \( \hat{t} \) | yes         | 0, -1          | -1             | -1             |

\( \mathcal{A} \) does lead \( P \) to a target configuration. We then plan to show that a left-most exiting agent \( V \) faces an empty cell as its West neighbor and subsequently leaves to the West [Lemma C.13], thus breaking our assumption and proving, that \( \mathcal{A} \) brings grid of an arbitrary size \( (m \geq 3, n \geq 2) \) to a target configuration.

Re-define the time (in \( D_k \) plane), so that up steps on the path become diagonal steps, i.e. we add virtual time ticks to make this happen. Define a function \( H(\hat{t}) \) - that designates the row index of an empty space on the path at new adjusted time \( \hat{t} \). Finally, define function \( G \) - that tracks the row index of an exiting agent \( V \) (in column \( k + 1 \)) at \( \hat{t} \). Note that both functions are discrete functions, such that \( H \) is monotone, and both have increments of 0 and 1. Though \( G \) could also decrease by 1.

Let \( T_s \) - be the time tick when cell \( a_{n,k} \) is empty, and \( a_{n-1,k} \) is empty either at \( T_s \) or at \( T_s + 1 \) (see Lemma C.25). Note, that we define a virtual version of time \( \hat{t} \) after the ascending path \( \mathcal{P} \) has been started, i.e. at either \( T_s \) or \( T_s + 1 \). Define, \( f := H - G \). Let \( T_e \) - be the time when the ascending path \( \mathcal{P} \) reaches the second row (see Claim C.23 and the conclusion thereafter).

Note, that in case one of the following equalities \( G(T_s) = n \) or \( G(T_e) = 2 \) hold, we are done [Lemma C.13], since an exiting agent \( V \) neighbors an empty space on its West. Therefore, we can assume w.l.o.g. that

\[ G(T_s) < n \quad G(T_e) > 2. \]

However, according to the definition of times \( T_s \) and \( T_e \) we have that

\[ H(T_s) = n \quad H(T_e) = 2. \]

We also claim that \( |f(\hat{t}+1) - f(\hat{t})| \leq 1 \). See Table 7 for \( f \) changes summary. The changes are as follows: if \( \hat{t} \) is derived from the real time (e.g. on diagonal up step of an ascending path), than the change is a consequence of a leading agent move (time ticks 0, 1 mod 4), \( V \)’s move (time ticks 1, 2 mod 4), or both. In case the time tick is a virtual time tick then only the function \( H \) value changes, as it tracks a “designated” empty space.
Therefore, we can apply The Discrete Intermediate Value Theorem to the function $f$.

**Theorem C.26** (The Discrete Intermediate Value Theorem). For integers $a < b$, let $f$ be a function from integers in $[a, b]$ to $\mathbb{Z}$, that satisfies the property

$$|f(i + 1) - f(i)| \leq 1$$

for all $i$. If $f(a) < 0 < f(b)$, then there exists $c \in (a, b)$, such that $f(c) = 0$.

**Proof.** Let $S = \{x \in \mathbb{Z} \cap [a, b] : f(x) < 0\}$ and let $c = \max S + 1$. We claim that $f(c) = 0$.

Suppose otherwise $f(c) < 0$, then $c \in S$, a contradiction to the fact that $c - 1$ is an upper bound on $S$. If $f(c) > 0$, we have that $f(c - 1) \geq 0$ (see the “continuity” property of $f$). But by definition of $c$, $c - 1 \in S$. Contradiction.

The application of the The Discrete Intermediate Value Theorem is as follows: we conclude that there exist a time $\hat{T}$ when $H(\hat{T}) = G(\hat{T})$. In other words, $V$ an exiting agent is located East to an empty cell. If $\hat{T}$ happen to originate in real time, we conclude that an agent at column $k$ just moved North or South, and it will take at least one full cycle to another column $k$ agent to occupy the “designated” empty space. However, it will take $V$ at most 3 time ticks to move West.

Otherwise, the origin of time tick $\hat{T}$ is in the “virtualization” of the time, i.e. a group of empty cells was linked to the path under construction. According to the way we extended the ascending path, this event could happen after a continue agent moved South. The next possible move by continue agent in column $k$ could be North movement, but the just-linked empty space is not the destination of such move. Hence, an empty space will “remain” near $V$, until the latter moves West in two time ticks.

We have just proved, that $V$ moves West, in contradiction to the definition of $T_0$.

Therefore, we conclude, that execution of Algorithm $A$ leads to the solution of the ER Problem for any given grid ($m \geq 3$) initialized to any initial configuration under the following requirements: the number of exiting agents is at most $n - 1$, and the number of empty cells is at least 1.
D Two-column sorting algorithm state machine

D.1 An exit agent

Figure 31: State machine(s) (a) at position 1, (b) at position 2, (c) at position 4.
Figure 32: State machine(s) (a) at position 3, (b) at position 8.
Figure 33: State machine(s) (a) at position 6.
D.2 A continue agent

Figure 34: State machine(s) (a) at position 1, (b) at position 2, (c) at position 3.
Figure 35: State machine(s) (a) at position 4, (b) at position 6.
Figure 36: State machine(s) at position 8.
E Theorem 5.1 proof

In this section we provide a proof for the following theorem:

**Theorem 5.1** (Algorithm $A_2$ Soundness Theorem). Consider an initial configuration $IC$ of an $n \times 2$ grid $G$ that contains at least a single empty slot. Let the number of exiting vehicles be $N_1$. Then Algorithm $A_2$ solves the ER Problem and transitions $G$ to a target configuration. Where target configuration is defined in the following sense: in case $N_1 \leq n$, then no exiting vehicles remain in the first column, otherwise no continue vehicles remain in the second column.

E.1 Synchronization and Collision Prevention

Below we show that a distributed execution of the Algorithm $A_2$ does not lead to an agent collision. Recall that the Algorithm $A_2$ is based on the LOOK-COMPUTE-MOVE paradigm, and is executed by the agents at discrete time ticks. I.e., the Algorithm $A_2$ is a synchronized distributed algorithm. The movement direction summary is presented in Table 8 and Table 9. A succinct agent flow overview is as follows: movement in North-South and East-West directions is equally split on the CC (see Fig. 37). Column movement is allowed according to the East-West clock split, i.e. an agent is allowed to leave column $C$ at tick $t$ if the movement is allowed in column $C$ at tick $t$.

E.2 Non-starvation

The 2-column case is much simpler than the general $m > 2$ one. We are essentially required to show that an agent, which should be in a target column $TC$, eventually leaves its initial non-target column $IC$. The motion on the East-West axes is a one-way only: East for exiting agents and West for continue agents. Hence, we conclude that every agent that has changed its initial column - is already located in the column it is expected to be in a target configuration.

Denote an agent located in its $TC$ as a settled agent, otherwise we denote an agent as an unsettled agent.

| Time  | Position | 1 | 2 | 3 | 4 | 6 | 8 |
|-------|----------|---|---|---|---|---|---|
| 002 = 0_{10} |               | - | N, E | E | - | N, E | - |
| 012 = 2_{10} |               | - | - | S, E | - | S, E | - |
| 102 = 1_{10} |               | - | - | - | S | - | S |
| 112 = 3_{10} |               | N | - | - | - | - | N |

Table 8: Possible exiting agent movement direction. Columns enumeration is done according to agents’ position on the two-column grid (from 1 to 8). $N, E, S, W$ are four possible movement directions (north, east, south, and west).
Figure 37: (left) East-West clock split: East movement is allowed at times 0 and 1; West movement is possible at times 2 and 3. (right) North-South clock split. Column movement is aligned with the East-West split.

| Time | Position | 1  | 2  | 3  | 4  | 6  | 8  |
|------|----------|----|----|----|----|----|----|
| 00₂ = 0₁₀ | - | N | - | - | N | - |
| 0₁₂ = 2₁₀ | - | - | S | - | S | - |
| 1₀₂ = 1₁₀ | W | - | S, W | - | S, W |
| 1₁₂ = 3₁₀ | N, W | - | - | - | - | N, W |

Table 9: Possible continue vehicle movement direction. Columns enumeration is done according to agents’ position on the two-column grid (from 1 to 8). N, E, S, W are four possible movement directions (north, east, south, and west).
Table 10: Cell $a_{i,j}$ vacancy/refill state. Cells in the first column are in position 2, 3, 6, and in the second column, 1, 4 and 8. An agent moves from one position to another; movement direction is the → column. Direction bit at departure and arrival are shown in the next two columns. Then briefly followed by the earliest possible return time, or the earliest arrival time of a following agent that moves in the North/South direction.

| Category         | Direction | $D$ bit | Earliest move | Earliest return |
|------------------|-----------|---------|---------------|-----------------|
| first column     | 2 3, 6    | N 0     | 1             | t + 5           | -               |
|                  | 3 2, 6    | S 1     | 0             | t + 7           | -               |
|                  | 6 2, 6    | S 1     | 0             | t + 7           | t + 4           |
|                  | 6 3, 6    | N 0     | 1             | t + 5           | t + 4           |
| second column    | 1 4, 8    | N 0     | 1             | t + 7           | -               |
|                  | 4 1, 8    | S 1     | 0             | t + 5           | -               |
|                  | 8 1, 8    | S 1     | 0             | t + 5           | t + 4           |
|                  | 8 4, 8    | N 0     | 1             | t + 7           | t + 4           |

Lemma E.1. Let $j \in \{1, 2\}$ and $1 \leq i \leq n$. Suppose an agent $A$ left cell $a_{i,j}$ at time $t$ to either North or South. Then $a_{i,j}$ can not become occupied by an agent moving North or South until at least $t + 4$.

Proof. The proof is done by enumeration. We refer the reader to Table 10.

Let us name grid cell $a_{i,3-j}$ the side neighbor cell of cell $a_{i,j}$.

Lemma E.2. Consider an unsettled agent $A$ at cell $a_{i,j}$. Suppose that the side neighbor cell $a_{i,3-j}$ became empty at $t$, i.e. $A$ detects an empty cell on the East/West at the beginning of time tick $t+1$. Then $A$ moves to its target column no later than at $t + 2$.

Proof. Note, that the side neighbor cell became empty due to the North-South movement of an agent, that occupied the $a_{i,3-j}$ cell at time $t$ and subsequently left. According to Lemma E.1, $a_{i,3-j}$ will not be the target of the North-South agent movement at least until $t+4$. However, according to Table 8 and Table 9, an unsettled agent moves on the East-West axes in at most two time ticks after the side neighbor cell became empty. We observe, that agents in columns move in “bursts”: first column agents move at ticks 00 and 01, followed by two ticks dedicated to the movement of the second column agents. Moreover, in the first tick out of two consecutive column-related ticks, an agent is allowed to move to its target column. Hence, $A$ will move to its target column no later than at $t + 2$.

Notice, that agents are active for two consecutive ticks during one CC (4 time ticks). Agents in the first column at ticks 00 and 01, while agents in the
second column during the remaining two ticks. Let denote those tick pairs as column activity interval(s).

An unsettled agent that neighbors an empty space at the start of its current column activity interval, will unconditionally move to its target column immediately. Furthermore, an unsettled agent that moved in the North-South direction at the start of its column activity interval, and consequently moved to neighbor an empty space, will also unconditionally move to its target column in the closing tick of this activity interval.

We note, that an unsettled exiting agents move North in the first tick of the column activity interval, while an unsettled continuing agents move South. Therefore, we have eliminated East-West movement options for positions 2 and 4 in the closing activity interval tick. This is due to the fact, that an unsettled agent can not physically move to these two specific positions at the start of an activity interval.

Our next goal is to show that at least one unsettled agent in the unsorted grid configuration eventually moves to its target column.

Assume that an initial configuration BC can not be sorted by Algorithm A_2 execution. Therefore, there exists a time T such that for any t > T no agent moves East or West. Moreover, grid configurations at all times t are not sorted.

Below we show, that empty cells are quite in demand despite the East-West movement termination. We prove that an agent near an empty cell does not ignore it for any substantial amount of time.

**Lemma E.3.** Suppose that an initial grid configuration is BC. Consider an agent A at cell a_{i,j}. Suppose, that a neighbor cell a_{i-1,j} (or a_{i+1,j}) becomes empty at time t > T, then A leaves a_{i,j} no later than at t + 4.

The proof is by enumeration of all the possible states (see Table 11 [Table 14].

An immediate corollary of Lemma E.3 is the following:

**Lemma E.4.** Suppose that an initial grid configuration is BC. Let i \in \{1, \ldots, n\}, j \in \{1, 2\}. Suppose, there exists i_0, such that a_{i_0,j} is empty at time t > T, then grid cell a_{i,j} is empty infinitely often for every i.

**Proof.** Assume that i is the lowest index such that a_{i,j} is not empty i.o., but a_{i-1,j} is. And in particular w.l.o.g. we can assume a_{i,j} is occupied for all t > T (denote an agent at a_{i,j} by A). Suppose agent B leaves a_{i-1,j} (North) at t, then by Lemma E.1 B does not return back until at least t + 5. However, according to Lemma E.3 A will leave a_{i,j} no later than at t + 4. This is a contradiction to the assumptions.

The case, when there are no cells that are constantly occupied with an i.o. empty North neighbor can be handled in a similar manner.

We shall need to define a number of terms for simplicity:

**Definition E.5.** A finite/infinite sequence of real numbers (x_0, x_1, x_2, \ldots) is called a bounded sequence, if there exists a real number M, such that |x_k| < M for every k.
| Position | Neighborhood | Memory state |
|----------|--------------|--------------|
|          | $t$          | $t+1$        | $t+2$ | $t$     | $t+1$ | $t+2$ | $t+3$ | $t+4$ |
| 1        | ![Image](1)   | ![Image](1)  | ![Image](1) | 011_2  | 010_2  | 000_2  | 001_2  | 010_2  | N      |
| 4        | ![Image](1)   | ![Image](1)  | ![Image](1) | 010_2  | 110_2  | 111_2  | 100_2  | 101_2  | S      |
| 8        | ![Image](1)   | ![Image](1)  | ![Image](1) | 010_2  | 110_2  | 111_2  | 100_2  | 101_2  | S      |

Table 11: Enumeration of possible neighborhoods and memory state sequences of an exiting agent. After East-West movement cessation. Grid position; visible $L_1$ neighborhood and memory state in the binary format are provided for short periods of time starting at $t$. Where applicable agent movement direction is given instead of a memory state. Relevant agent positions: 1, 4 and 8. Impossible states are excluded from enumeration.
| Position | Neighborhood | Memory state |
|----------|--------------|--------------|
|          | $t$          | $t + 1$      | $t + 2$      |
|          |              |              |              |
| 1        | ![Image](image1) | 011_2        | 000_2        |
|          |              | 111_2        | 001_2        |
|          |              |              | 010_2        |
|          |              |              | N            |

| 4        | ![Image](image2) | 010_2        | 111_2        |
|          |              | 110_2        | 100_2        |
|          |              |              | 101_2        |
|          |              |              | S            |

| 8        | ![Image](image3) | 010_2        | 011_2        |
|          |              | 110_2        | 100_2        |
|          |              |              | 101_2        |
|          |              |              | S            |

|          | ![Image](image4) | 010_2        | 111_2        |
|          |              | 110_2        | 000_2        |
|          |              |              | 001_2        |
|          |              |              | 010_2        |
|          |              |              | N            |

Table 12: Enumeration of possible visible neighborhoods and memory state sequences of a continue agent. After East-West movement cessation. Grid position; visible $L_1$ neighborhood and memory state in the binary format are provided for short periods of time starting at $t$. Where applicable agent movement direction is given instead of a memory state. Relevant agent positions: 1, 4 and 8. Impossible states are excluded from enumeration.
| Position | Neighborhood | Memory state |
|----------|--------------|--------------|
|          | $t$ | $t + 1$ | $t + 2$ | $t$ | $t + 1$ | $t + 2$ | $t + 3$ | $t + 4$ |
| 2        | ![Position 2](image1) | ![Position 2](image2) | ![Position 2](image3) | 000$_2$ | 001$_2$ | 010$_2$ | 011$_2$ | N |
| 3        | ![Position 3](image4) | ![Position 3](image5) | ![Position 3](image6) | 001$_2$ | 010$_2$ | 111$_2$ | 100$_2$ | S |
| 6        | ![Position 6](image7) | ![Position 6](image8) | ![Position 6](image9) | 000$_2$ | 001$_2$ | 010$_2$ | 011$_2$ | N |
|          | ![Position 6](image10) | ![Position 6](image11) | ![Position 6](image12) | 000$_2$ | S | |
|          | ![Position 6](image13) | ![Position 6](image14) | ![Position 6](image15) | 001$_2$ | 110$_2$ | 111$_2$ | 100$_2$ | S |

Table 13: Enumeration of possible visible neighborhoods and memory state sequences of an exiting agent. After East-West movement cessation. Grid position; visible $L_1$ neighborhood and memory state in the binary format are provided for short periods of time starting at $t$. Where applicable agent movement direction is given instead of a memory state. Relevant agent positions: 2, 3 and 6. Impossible states are excluded from enumeration.
Table 14: Enumeration of possible visible neighborhoods and memory state sequences of a continue agent. After East-West movement cessation. Grid position; visible $L_1$ neighborhood and memory state in the binary format are provided for short periods of time starting at $t$. Where applicable agent movement direction is given instead of a memory state. Relevant agent positions: 2, 3 and 6. Impossible states are excluded from enumeration.
Definition E.6. A finite/infinite sequence of real numbers \((x_0, x_1, x_2, \ldots)\) is called a *bounded difference sequence*, if the sequence of differences \(y_k := x_{k+1} - x_k\) is a bounded sequence.

Definition E.7. A bounded sequence \(\{x_k\}\) is called a *special bounded sequence* if \(x_k \leq 1\) for all \(k\).

Definition E.8. A bounded difference sequence \(\{x_k\}\) is called a *special bounded difference sequence*, if a sequence of differences is a special bounded sequence.

Definition E.9. A cell sequence \(a_{i,k,j}(t_k)_{k=1}^N\) will be called a *path*, if the following holds:

1. \(\{t_k\}_k\) is monotone non-decreasing special bounded difference time sequence.
2. \(\{i_k\}_k\) is a special bounded difference index sequence.

Definition E.10. A path is called an *empty path* if every cell in a sequence is empty.

We shall prove [Theorem 5.1] in the following manner. First, we shall show the existence of an empty path that extends from one end of the grid column to the other (in columns with at least one empty space). Then, we shall reveal, that an unsettled agent in the neighbor column necessarily observes an empty space near it in the target column. Finally, an unsettled agent will be demonstrated to move into its target column.

*Proof.* We assume that an initial grid configuration is BC, and the column \(j\) contains at least one empty space at \(t > T\) (i.e., the East-West agent movement ceased).

Denote the leading agent of the cell \(a_{i,j}\) at time \(t\) to be the agent at cell \(a_{k,j}\) at time \(t\), such that \(k > i\) and there are no other agent at cell \(a_{k',j}\) at time \(t\) where \(i < k' < k\). It could be possible no cell leading agent at time \(t\) exists, i.e., all the cells with a higher row index are empty at time \(t\).

Our goal is to show the existence of an empty path that spans from \(a_{1,j}\) at time \(t_0\) to \(a_{n,j}\) at time \(t_1\). By [Lemma E.3], every cell \(a_{i,j}\) is empty infinitely often. In particular, \(a_{1,j}\). Fix time \(t_0\), when \(a_{1,j}\) is empty. We add \(a_{1,j}(t_0)\) as the first cell in the sequence. In case \(a_{2,j}, a_{3,j}, \ldots\) are empty, add them to the sequence in that order. Inductively, we add cells to the sequence in the following manner. Denote \(a_{i,j}(t - 1)\) - the last added cell to the sequence. If the leading agent of \(a_{i,j}(t - 1)\) moved South at time \(t - 1\), we add \(a_{i+1,j}(t)\) to the sequence and repeat the inductive step. If the leading agent of \(a_{i,j}(t - 1)\) moved North, we add \(a_{i+1,j}(t)\) and all the empty cells between \(a_{i+1,j}\) and its (new) leading agent (or all the cells to the South, if no leading agent exists). Otherwise, i.e., the leading agent did not move at all, we add \(a_{i,j}(t)\) to the sequence.

By the process description, we have just built an empty path. Note, that we still need to verify that a leading agent moves necessarily before the previous leading agent could move back and ruin the construction process. However, see
Lemma E.3 together with Lemma E.1 that ensures this is not the case. For the first cell in the sequence the fact is vacuously true.

The path in the other direction could be build in the exact same way with appropriate changes.

Now, suppose w.l.o.g. that an unsettled agent $A$ is an exiting agent in the first column, and that an empty space is located at time $T$ in the second column. By the above claim, there exist an empty path in the second column (extending through space and time). Moreover, by definition, such path is a tempo-spatial sequence of empty cells, that either extends to spatially the south by at least one cell (in case a leading agent moves), or it extends in time otherwise. If $A$ was in cell $a_{11}$ when the path started, it should have already left East. Otherwise, $A$ is at row indexed at least 2, i.e. $A$ is spatially below the path. At time tick when the path reaches $a_{n2}$, $A$ is either at $a_{n1}$ and will leave East at the next time tick, or $A$ is spatially above the path.

Is it possible for $A$ to somehow miss an empty side neighbor? We claim that this is impossible. Agents in the first column (such as $A$ above) move North at the start of the column activity time interval, and South in the end. Hence, if $A$ is located below the path at the start of the column activity interval, it($A$) can not find itself spatially above the path at the end of the column activity interval. In case $A$ moves North at the start of such interval, it could only neighbor an empty cell belonging to the path, but it will move East then, in contradiction to our assumption.

Therefore, we can safely assume that at the very start of the second column activity interval $A$ is neighbored by some agent $B$. The $B$ could be the leading agent of the path and move South at the start of some appropriate second column activity interval. In that case, according to Lemma E.1 the side neighbor of $A$ will remain empty for at least 4 ticks, hence $A$ will not miss an empty neighbor. Otherwise, $B$ could move North in the end of its column activity interval, but then $A$ will also observe an empty neighbor during the following first column activity interval.

In all cases, we have shown $A$ is neighbored by an empty cell, and, hence, will inevitably move East. This contradicts our assumption, that BC is one such unlucky initial configuration, that could not be sorted by Algorithm $A$. \hfill $\square$

Moreover, the similar reasoning, but using the upward empty paths in the first column, shows, that Algorithm $A$ pushes continue vehicles out of the second column. Therefore, Algorithm $A$ moves the maximum number of unsettled agents to their target columns.
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