Some estimates for solutions of the Dirichlet problem for second-order elliptic equations are obtained in this paper. Here the leading coefficients are locally VMO functions, while the hypotheses on the other coefficients and the boundary conditions involve a suitable weight function.

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1. Introduction

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, and let

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} + a(x)$$

be a uniformly elliptic operator with measurable coefficients in $\Omega$. Several bounds for the solutions of the problem

$$Lu \geq f, \quad f \in L^p(\Omega),$$
$$u \in W^{2,p}(\Omega) \cap C^\alpha(\bar{\Omega}),$$
$$u|_{\partial \Omega} \leq 0,$$

($p \in ]n/2, +\infty[)$ have been given, and the application of such estimates allows to obtain certain uniqueness results for (D).

For instance, if $p \geq n$, $a_i, a \in L^p(\Omega)$ (with $a \leq 0$), a classical result of Pucci [4] shows that any solution $u$ of the problem (D) verifies the bound

$$\sup_{\Omega} u \leq K \|f\|_{L^p(\Omega)},$$

where $K \in \mathbb{R}_+$ depends on $\Omega, n, p, \|a_i\|_{L^p(\Omega)}$ and on the ellipticity constant.
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The case $p < n$, where additional hypotheses on the leading coefficients are necessary, has been studied by several authors. Recently, a uniqueness result has been obtained in [3] under the assumption that the $a_{ij}$'s are of class VMO, $a_i = a = 0$ and $p \in ]1, +\infty[$. This theorem has been extended to the case $a_i \neq 0, a \neq 0$ in [7].

If $\Omega$ is an arbitrary open subset of $\mathbb{R}^n$ and $p \in ]n/2, +\infty[$, a bound of type (1.2) and a consequent uniqueness result can be found in [1]. In fact, it has been proved there that if the coefficients $a_{ij}$ are bounded and locally VMO, the coefficients $a_i, a$ satisfy suitable summability conditions and $\text{esssup}_\Omega a < 0$, then for any solution $u$ of the problem

$$Lu \geq f, \quad f \in L^p_{\text{loc}}(\Omega),$$
$$u \in W^{2,p}_{\text{loc}}(\Omega) \cap C^0(\Omega),$$
$$u_{|\partial\Omega} \leq 0,$$
$$\limsup_{|x| \to +\infty} u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded},$$

there exist a ball $B \subset \subset \Omega$ and a constant $c \in \mathbb{R}_+$ such that

$$\sup_{\Omega} u \leq c\left(\int_B |f^-|^p \, dx\right)^{1/p}, \quad (1.3)$$

where $f^-$ is the negative part of $f$,

$$\int_B |f^-|^p \, dx = \frac{1}{|B|} \int_B |f^-|^p \, dx,$$  \quad (1.4)

and $c$ depends on $n, p$, on the ellipticity constant, and on the regularity of the coefficients of $L$.

The aim of this paper is to study a problem similar to that considered in [1], but with boundary conditions depending on an appropriate weight function. More precisely, fix a weight function $\sigma \in A^{100}(\Omega) \cap C^\infty(\Omega)$ (see Section 2 for the definition of $A^{100}(\Omega)$) and $s \in \mathbb{R}$, we consider a solution $u$ of the problem

$$Lu \geq f, \quad f \in L^p_{\text{loc}}(\Omega),$$
$$u \in W^{2,p}_{\text{loc}}(\Omega),$$
$$\limsup_{x \to x_o} \sigma^s(x) u(x) \leq 0 \quad \forall x_o \in \partial\Omega,$$
$$\limsup_{|x| \to +\infty} \sigma^s(x) u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded},$$  \quad (1.5)

If the coefficients $a_{ij}$ are bounded and locally VMO, the functions $\sigma a_i$ and $\sigma^2 a$ are bounded and $\text{esssup}_{\Omega} \sigma^2 a < 0$, we will prove that there exist a ball $B \subset \subset \Omega$ and a constant $c_o \in \mathbb{R}_+$ such that

$$\sup_{\Omega} \sigma^s u \leq c_o \left(\int_B |\sigma^{s+2} f^-|^p \, dx\right)^{1/p},$$  \quad (1.6)

where $c_o$ depends on $n, p, s, \sigma$, on the ellipticity constant, and on the regularity of the coefficients of $L$. As a consequence, some uniqueness results are also obtained.
2. Notation and function spaces

Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and let \( \Sigma(\Omega) \) be the collection of all Lebesgue measurable subsets of \( \Omega \). For each \( E \in \Sigma(\Omega) \), we denote by \( |E| \) the Lebesgue measure of \( E \) and put

\[
E(x,r) = E \cap B(x,r) \quad \forall x \in \mathbb{R}^n, \; \forall r \in \mathbb{R}_+,
\]

where \( B(x,r) \) is the open ball in \( \mathbb{R}^n \) of radius \( r \) centered at \( x \).

Denote by \( \mathcal{A}(\Omega) \) the class of measurable functions \( \rho : \Omega \to \mathbb{R}_+ \) such that

\[
\beta^{-1} \rho(y) \leq \rho(x) \leq \beta \rho(y) \quad \forall y \in \Omega, \; \forall x \in \Omega(y, \rho(y)),
\]

where \( \beta \in \mathbb{R}_+ \) is independent of \( x \) and \( y \). For \( \rho \in \mathcal{A}(\Omega) \), we put

\[
S_\rho = \{ z \in \partial \Omega : \lim_{x \to z} \rho(x) = 0 \}.
\]

It is known that

\[
\rho \in L_{\text{loc}}^\infty(\bar{\Omega}), \quad \rho^{-1} \in L_{\text{loc}}^\infty(\bar{\Omega} \setminus S_\rho),
\]

and, if \( S_\rho \neq \emptyset \),

\[
\rho(x) \leq \text{dist}(x, S_\rho) \quad \forall x \in \Omega,
\]

(see [2, 6]). Having fixed \( \rho \in \mathcal{A}(\Omega) \) such that \( S_\rho = \partial \Omega \), it is possible to find a function \( \sigma \in \mathcal{A}(\Omega) \cap C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega}) \) which is equivalent to \( \rho \) and such that

\[
\sigma \in L_{\text{loc}}^\infty(\bar{\Omega}), \quad \sigma^{-1} \in L_{\text{loc}}^\infty(\Omega),
\]

\[
\sigma(x) \leq \text{dist}(x, \partial \Omega) \quad \forall x \in \Omega,
\]

\[
|\partial^\alpha \sigma(x)| \leq c_\alpha \sigma^{1-|\alpha|}(x) \quad \forall x \in \Omega, \; \forall \alpha \in \mathbb{N}_+^n,
\]

\[
y^{-1} \sigma(y) \leq \sigma(x) \leq y \sigma(y) \quad \forall y \in \Omega, \; \forall x \in \Omega(y, \sigma(y)),
\]

where \( c_\alpha, y \in \mathbb{R}_+ \) are independent of \( x \) and \( y \) (see [6]). For more properties of functions of \( \mathcal{A}(\Omega) \) we refer to [2, 6]. If \( \Omega \) has the property

\[
|\Omega(x,r)| \geq Ar^n \quad \forall x \in \Omega, \; \forall r \in [0,1],
\]

where \( A \) is a positive constant independent of \( x \) and \( r \), it is possible to consider the space \( \text{BMO}(\Omega, t) \), \( t \in \mathbb{R}_+ \), of functions \( g \in L_{\text{loc}}^1(\bar{\Omega}) \) such that

\[
[g]_{\text{BMO}(\Omega,t)} = \sup_{x \in \Omega} \sup_{r \in [0,t]} \left( \int_{\Omega(x,r)} \left| g - \int_{\Omega(x,r)} g \right| dy \right) < +\infty,
\]

where \( \int_{\Omega(x,r)} g dy = 1/|\Omega(x,r)| \int_{\Omega(x,r)} g dy \). If \( g \in \text{BMO}(\Omega) = \text{BMO}(\Omega, t_A) \), where

\[
t_A = \sup \left\{ t \in \mathbb{R}_+ : \sup_{x \in \Omega} \sup_{r \in [0,t]} \frac{r^n}{|\Omega(x,r)|} \leq \frac{1}{A} \right\},
\]
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we will say that \( g \in \text{VMO}(\Omega) \) if \([g]_{\text{BMO}(\Omega,t)} \to 0 \) for \( t \to 0^+ \). A function \( \eta[g] : \mathbb{R}_+ \to \mathbb{R}_+ \) is called a modulus of continuity of \( g \) in \( \text{VMO}(\Omega) \) if

\[
\eta[g](t) \leq \lim_{t \to 0^+} [g]_{\text{BMO}(\Omega)} \quad \forall t \in \mathbb{R}_+,
\]

\[
\lim_{t \to 0^+} \eta[g](t) = 0.
\]

We say that \( g \in \text{VMO}_{\text{loc}}(\Omega) \) if \( (\zeta g)_o \in \text{VMO}(\mathbb{R}^n) \) for any \( \zeta \in C_\infty(\Omega) \), where \( (\zeta g)_o \) denotes the zero extension of \( \zeta g \) outside of \( \Omega \). A more detailed account of properties of the above defined spaces \( \text{BMO}(\Omega) \) and \( \text{VMO}(\Omega) \) can be found in [5].

3. An a priori bound

Fix \( p \in ]n/2, +\infty[ \). Let \( B \) be an open ball of \( \mathbb{R}^n \), \( n \geq 3 \), of radius \( \delta \). We consider in \( B \) the differential operator

\[
L_B = \sum_{i,j=1}^n \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i} + \alpha(x),
\]

(3.1)

with the following condition on the coefficients:

\[
\alpha_{ij} = \alpha_{ji} \in L^\infty(B) \cap \text{VMO}(B), \quad i, j = 1, \ldots, n,
\]

\[
\exists \mu \in \mathbb{R}_+ : \sum_{i,j=1}^n \alpha_{ij} \zeta_i \zeta_j \geq \mu |\zeta|^2 \quad \text{a.e. in } B, \ \forall \zeta \in \mathbb{R}^n,
\]

(3.2)

\[
(\text{h}_B)
\]

\[
\alpha_i \in L^\infty(B), \quad i = 1, \ldots, n, \ \alpha \in L^\infty(B), \ \alpha \leq 0 \text{ a.e. in } B.
\]

Note that under the assumption \( (\text{h}_B) \), the operator \( L_B \) from \( W^{2,p}(B) \) into \( L^p(B) \) is bounded and the estimate

\[
\|L_B u\|_{L^p(B)} \leq c_1 \|u\|_{W^{2,p}(B)} \quad \forall u \in W^{2,p}(B)
\]

(3.3)

holds, where \( c_1 \in \mathbb{R}_+ \) depends on \( n, p, \mu_0, \mu_1, \mu_2 \).

**Lemma 3.1.** Suppose that condition \( (\text{h}_B) \) is verified, and let \( u \) be a solution of the problem

\[
u \in W^{2,p}(B), \quad L_B u \geq \phi, \quad \phi \in L^p(B), \quad u_{iabs} \leq 0.
\]

(3.4)

Then there exists \( c \in \mathbb{R}_+ \) such that

\[
\sup_B u \leq c \delta^{2-n/p} \|\phi\|_{L^p(B)},
\]

(3.5)
where \( c \) depends on \( n, p, \mu, \mu_0, \mu_1, \mu_2 \), \( \{p(\alpha_{ij})\}_{\text{BMO}(\mathbb{R}^n)} \), and where \( p(\alpha_{ij}) \) is an extension of \( \alpha_{ij} \) to \( \mathbb{R}^n \) in \( L^\infty(\mathbb{R}^n) \cap \text{VMO}(\mathbb{R}^n) \).

**Proof.** Put \( B = B(y, \delta) \), where \( y \) is the centre of \( B \), and \( B^* = B(y, 1) \).

Consider the function \( T : B \to B^* \) defined by the position

\[
T(x) = y + \frac{x - y}{\delta} = z, \tag{3.6}
\]

and for each function \( g \) defined on \( B \), put \( g^* = g \circ T^{-1} \).

We observe that

\[
L_B^* u^* = \delta^2 (L_B u^*)^*, \tag{3.7}
\]

where

\[
L_B^* = \sum_{i,j=1}^{n} \alpha_{ij}^*(z) \frac{\partial^2}{\partial z_i \partial z_j} + \delta \sum_{i=1}^{n} \alpha_i^*(z) \frac{\partial}{\partial z_i} + \delta^2 \alpha^*(z). \tag{3.8}
\]

Denote by \( p(\alpha_{ij}) \) an extension of \( \alpha_{ij} \) to \( \mathbb{R}^n \) such that

\[
p(\alpha_{ij}) \in L^\infty(\mathbb{R}^n) \cap \text{VMO}(\mathbb{R}^n) \tag{3.9}
\]

(for the existence of such function see [5, Theorem 5.1]). Since

\[
p(\alpha_{ij})^* \in L^\infty(\mathbb{R}^n) \cap \text{VMO}(\mathbb{R}^n), \quad p(\alpha_{ij})^* \vert_{B^*} = \alpha_{ij}^*, \tag{3.10}
\]

it follows that

\[
\alpha_{ij}^* \in L^\infty(B^*) \cap \text{VMO}(B^*). \tag{3.11}
\]

Moreover, the condition \((h_B)\) yields that

\[
\alpha_{ij}^* = \alpha_{ji}^*, \quad i, j = 1, \ldots, n,
\]

\[
\sum_{i,j=1}^{n} \alpha_{ij}^* |\zeta_i \zeta_j|^2 \geq \mu |\zeta|^2 \quad \text{a.e. in } B^*, \quad \forall \zeta \in \mathbb{R}^n, \tag{3.12}
\]

\[
\alpha_i^* \in L^\infty(B^*), \quad i = 1, \ldots, n, \quad \alpha^* \in L^\infty(B^*), \quad \alpha^* \leq 0 \quad \text{a.e. in } B^*.
\]

We observe that the condition (3.12) implies that for \( r, s \in ]1, +\infty[ \) the modulus of continuity of \( \delta \alpha_i^* \) in \( L^r(B^*) \) and that of \( \delta^2 \alpha^* \) in \( L^s(B^*) \) depend only on \( \|\delta \alpha_i^*\|_{L^\infty(B^*)} \) and \( \|\delta^2 \alpha^*\|_{L^s(B^*)} \), respectively.

Thus, applying (3.10), (3.12), and [7, Theorem 2.1], it follows that the problem

\[
L_B^* \nu = \psi \in L^p(B^*),
\]

\[
\nu \in W^{2,p}(B^*) \cap W^{1,p}(B^*) \tag{3.13}
\]
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has a unique solution \( v \) satisfying the estimate

\[
\|v\|_{W^{2,p}(B^+)} \leq K \|\psi\|_{L^p(B^+)},
\]

(3.14)

where \( K \) depends on \( n, p, \mu, \mu_0, \mu_1, \mu_2, [p(\alpha_{ij})^*]_{\text{BMO}(R^n)} \).

The estimate (3.5) follows now from (3.14) using the same arguments of the proof of Lemma 3.2 [1] in order to obtain there \((e_B)\) from [1, (3.23)]. \( \square \)

4. Hypotheses and preliminary results

Let \( \Omega \) be an open subset of \( \mathbb{R}^n, n \geq 3 \). Fix \( \rho \in \mathcal{S}(\Omega) \cap L^\infty(\Omega) \) such that \( S_\rho = \partial \Omega \).

Consider a function \( g \in C_0^\infty(\bar{\mathbb{R}}^+) \) satisfying the condition

\[
0 \leq g \leq 1, \quad g(t) = 1 \quad \text{if } t \geq 1, \quad g(t) = 0 \quad \text{if } t \leq \frac{1}{2}.
\]

(4.1)

For any \( k \in \mathbb{N} \), we put

\[
\eta_k(x) = \frac{1}{k} \zeta_k(x) + (1 - \zeta_k(x)) \sigma(x), \quad x \in \Omega,
\]

(4.2)

where \( \zeta_k(x) = g(k\sigma(x)), x \in \Omega \). Clearly, \( \eta_k \in C^\infty(\Omega) \) for any \( k \in \mathbb{N} \) and

\[
\eta_k(x) = \begin{cases} \frac{1}{k} & \text{if } x \in \bar{\Omega}_k, \\ \sigma(x) & \text{if } x \in \Omega \setminus \bar{\Omega}_2k, \end{cases}
\]

(4.3)

where

\[
\Omega_k = \left\{ x \in \Omega : \sigma(x) > \frac{1}{k} \right\}, \quad k \in \mathbb{N}.
\]

(4.4)

In the following we will use the notation

\[
f_x = \left( \sum_{i=1}^n f_{x_i}^2 \right)^{1/2}, \quad f_{xx} = \left( \sum_{i,j=1}^n f_{x_i x_j}^2 \right)^{1/2}.
\]

(4.5)

It is easy to show that for each \( k \in \mathbb{N} \),

\[
\sigma(x) \leq \eta_k(x) \leq 2\sigma(x), \quad x \in \Omega \setminus \bar{\Omega}_k, \quad (4.6)
\]

\[
c'_k \sigma(x) \leq \eta_k(x) \leq \sigma(x), \quad x \in \Omega_k, \quad (4.7)
\]

\[
(\eta_k(x))_x \leq c_1 (\sigma(x))_x, \quad x \in \Omega, \quad (4.8)
\]

\[
(\eta_k(x))_{xx} \leq c_2 \frac{(\sigma(x))^2 + \sigma(x)(\sigma(x))_{xx}}{\sigma(x)}, \quad x \in \Omega, \quad (4.9)
\]
where \( c'_k \in \mathbb{R}_+ \) depends on \( k \) and \( \sigma \), and \( c_1, c_2 \in \mathbb{R}_+ \) depend only on \( n \). Moreover, for any \( s \in \mathbb{R} \), we have

\[
\frac{(\eta^k_s(x))_x}{\eta^k_s(x)} \leq c_3 \frac{(\eta_k(x))_x}{\sigma(x)}, \quad x \in \Omega, \tag{4.10}
\]

\[
\frac{(\eta^k_s(x))_{xx}}{\eta^k_s(x)} \leq c_3 \frac{(\eta_k(x))^2 + \eta_k(x)(\eta_k(x))_{xx}}{\sigma^2(x)}, \quad x \in \Omega, \tag{4.11}
\]

where \( c_3 \in \mathbb{R}_+ \) depends on \( s \) and \( n \).

We consider in \( \Omega \) the differential operator

\[
L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x_i} + a(x), \quad \tag{4.12}
\]

and put

\[
L_0 = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}. \quad \tag{4.13}
\]

We will make the following assumption on the coefficients of \( L \):

\[
\begin{align*}
\forall \nu, \nu_0 \in \mathbb{R}_+: \quad & \sum_{i,j=1}^{n} \| a_{ij} \|_{L^\infty(\Omega)} \leq \nu_0, \quad \sum_{i,j=1}^{n} a_{ij} \xi_i \xi_j \geq \nu |\xi|^2 \quad \text{a.e. in } \Omega, \quad \forall \xi \in \mathbb{R}^n, \\
\text{(h1)} & \exists \nu_1, \nu_2 \in \mathbb{R}_+: \quad \text{ess sup}_{\Omega} \left( \sigma(x) \sum_{i=1}^{n} |a_i(x)| \right) \leq \nu_1, \quad \text{ess sup}_{\Omega} (\sigma^2(x)|a(x)|) \leq \nu_2, \\
& \exists a_o \in \mathbb{R}_+: \quad \text{ess sup}_{\Omega} (\sigma^2(x)a(x)) = -a_o.
\end{align*}
\]

Fixed \( s \in \mathbb{R} \), let \( u \) be a solution of the problem

\[
Lu \geq f, \quad f \in L^p_{\text{loc}}(\Omega), \quad u \in W^{2,p}_{\text{loc}}(\Omega),
\]

\[
\limsup_{x \to x_o} \sigma'(x)u(x) \leq 0 \quad \forall x_o \in \partial \Omega, \quad \tag{P}
\]

\[
\limsup_{|x| \to +\infty} \sigma^2(x)u(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded.}
\]

For any \( k \in \mathbb{N} \), we put

\[
w_k(x) = \eta_k^s(x)u(x), \quad x \in \Omega. \quad \tag{4.14}
\]
Lemma 4.1. Suppose that condition (h₁) holds. Then, for any \( k \in \mathbb{N} \) there exist functions \( b^k_i (i = 1, \ldots, n) \), \( b^k \), \( g^k \) and positive constants \( \beta_1 \) and \( \beta_2 \) such that

\[
\text{esssup}_\Omega \left( \sigma(x) \sum_{i=1}^n |b^k_i(x)| \right) \leq \beta_1, \tag{4.15}
\]
\[
\text{esssup}_\Omega \left( \sigma^2(x) |b^k(x)| \right) \leq \beta_2, \tag{4.16}
\]
\[
g^k \in L^p_{\text{loc}}(\Omega), \tag{4.17}
\]

where \( \beta_1 \) depends on \( s, n, \nu_0, \nu_1 \) and \( \beta_2 \) depends on \( s, n, \nu_0, \nu_2 \). Moreover, the function \( w_k, k \in \mathbb{N} \), satisfies the following conditions:

\[
w_k \in W^{2,p}_{\text{loc}}(\Omega), \quad \limsup_{x \to x_0} w_k(x) \leq 0 \quad \forall x_0 \in \partial \Omega, \tag{4.18}
\]
\[
\limsup_{|x| \to +\infty} w_k(x) \leq 0 \quad \text{if } \Omega \text{ is unbounded}, \tag{4.19}
\]

Proof. Fix \( k \in \mathbb{N} \). From (4.6)–(4.11) and from (2.6), (2.8), it easily follows that the function \( w_k \), defined by (4.14), verifies (4.18).

Moreover, observe that

\[
L_0 w_k \sum_{i=1}^n b^k_i (w_k)_{x_i} + b^k w_k \geq g^k \quad \text{in } \Omega. \tag{4.20}
\]

Since

\[
(\eta^i_k)_{x_i} u_{x_i} = (\eta^i_k u)_{x_i} - (\eta^i_k \eta^i_k)_{x_i} (\eta^i_k)^2 u, \tag{4.21}
\]

from (4.20), (4.19) follows, where we have put

\[
b^k_i = a_i - 2 \sum_{j=1}^n a_{ij} \frac{(\eta^i_k)_{x_j}}{\eta^i_k}, \quad i = 1, \ldots, n,
\]
\[
b^k = a + 2 \sum_{i,j=1}^n a_{ij} \frac{(\eta^i_k)_{x_i}}{(\eta^i_k)^2} \frac{(\eta^i_k)_{x_j}}{\eta^i_k} - \sum_{i,j=1}^n a_{ij} \frac{(\eta^i_k)_{x_i x_j}}{\eta^i_k}, \tag{4.22}
\]
\[
g^k = \eta^i_k f + \sum_{i=1}^n a_i \frac{(\eta^i_k)_{x_i}}{\eta^i_k} w_k.
\]

On the other hand, using the hypothesis (h₁), (4.6)–(4.11), and (2.8) it is easy to show that there exist \( \beta_1 \in \mathbb{R}_+ \) depending on \( s, n, \nu_0, \nu_1 \) and \( \beta_2 \in \mathbb{R}_+ \) depending on \( s, n, \nu_0, \nu_2 \), such that (4.15), (4.16), (4.17) hold. \( \square \)
Now we suppose that the following hypothesis on $\rho$ holds:

$$\lim_{k \to +\infty} \left( \sup_{\Omega, \Omega_k} \left( (\sigma(x))_x + \sigma(x)(\sigma(x))_{xx} \right) \right) = 0. \tag{h_2}$$

An example of function $\rho$ such that $\sigma$ satisfies $(h_2)$ is provided in [2].

**Lemma 4.2.** Suppose that conditions $(h_1)$ and $(h_2)$ hold. Then there exists $k_0 \in \mathbb{N}$ such that

$$\text{esssup}_{\Omega} \left( \sigma(x) \sum_{i=1}^{n} \left| b_{k_i}^k(x) \right| \right) \leq \nu_1 + \frac{a_0}{2},$$

$$\text{esssup}_{\Omega} \left( \sigma^2(x) b_{k_i}^k(x) \right) \leq -\frac{a_0}{2},$$

$$g_{k_i}^k(x) \geq \frac{a_0}{2} \sigma^{-2}(x) \left| w_{k_i}(x) \right|, \quad x \in \Omega.$$  \hspace{1cm} (4.23)

**Proof.** From (4.10), (4.11), and hypothesis $(h_1)$, we deduce that

$$\sigma \left| \sum_{i,j=1}^{n} a_{ij} \frac{(\eta_k^i)_{xj}}{\eta_k^i} \right| \leq c_4(\eta_k)_x,$$

$$\sigma^2 \left| \sum_{i,j=1}^{n} a_{ij} \frac{(\eta_k^i)_x (\eta_k^j)_{xj}}{(\eta_k^i)^2} \right| + \sigma^2 \left| \sum_{i,j=1}^{n} a_{ij} \frac{(\eta_k^i)_{xj}}{(\eta_k^i)^2} \right| \leq c_5((\eta_k)_x^2 + \eta_k(\eta_k)_{xx}),$$

$$\sigma^2 \left| \sum_{i=1}^{n} a_i \frac{(\eta_k^i)_x}{\eta_k^i} \right| \leq c_6(\eta_k)_x,$$  \hspace{1cm} (4.24)

where $c_4, c_5 \in \mathbb{R}_+$ depend on $s, n, \nu_0$ and $c_6 \in \mathbb{R}_+$ depends on $s, n, \nu_1$. Observing that $(\eta_k)_x = (\eta_k)_{xx} = 0$ in $\tilde{\Omega}_k$, the statement follows now from (4.8), (4.9), $(h_1)$, $(h_2)$, and (4.24). \hfill \Box

5. Main results

It is well known that there exists a function $\tilde{\alpha} \in C^\infty(\Omega) \cap C^{0,1}(\bar{\Omega})$ which is equivalent to $\text{dist}(\cdot, \partial \Omega)$ (see, e.g., [8]). For every positive integer $m$, we define the function

$$\psi_m : x \in \Omega \rightarrow g(m\tilde{\alpha}(x)) \left( 1 - g\left( \frac{|x|}{2m} \right) \right), \tag{5.1}$$

where $g \in C^\infty(\mathbb{R}_+)$ verifies (4.1). It is easy to show that $\psi_m$ belongs to $C^\infty_0(\Omega)$ for every $m \in \mathbb{N}$ and

$$0 \leq \psi_m \leq 1, \quad \text{supp} \psi_m \subseteq E_{2m}, \quad \psi_m|_{E_m} = 1, \tag{5.2}$$

where

$$E_m = \left\{ x \in \Omega : |x| < m, \tilde{\alpha}(x) > \frac{1}{m} \right\}.$$ \hspace{1cm} (5.3)
Remark 5.1. It follows from hypothesis \((h_1)\) and from [5, Lemma 4.2] that for any \(m \in \mathbb{N}\) the functions \((\psi ma_{ij})_o\) (obtained as extensions of \(\psi ma_{ij}\) to \(\mathbb{R}^n\) with zero values out of \(\Omega\)) belong to \(\text{VMO}(\mathbb{R}^n)\) and
\[
\left[\left(\psi ma_{ij}\right)_o\right]_{\text{BMO}(\mathbb{R}^n,t)} \leq \left[\psi ma_{ij}\right]_{\text{BMO}(\Omega,t)},
\]
for \(t\) small enough.

In the following we denote by \(w, b_i, b,\) and \(g\) the functions defined by (4.14), (4.22), respectively, corresponding to \(k = k_o\), where \(k_o\) is the positive integer of Lemma 4.2.

We can now prove the main result of the paper.

**Theorem 5.2.** Suppose that conditions \((h_1)\) and \((h_2)\) hold, and let \(u\) be a solution of the problem \((P)\). Then there exist an open ball \(B \subset \subset \Omega\) and a constant \(c_o \in \mathbb{R}_+\) such that
\[
\sup_{\Omega} \sigma^s(x)u(x) \leq c_o \left(\int_B \left| \sigma^{s+2} f - \left| p \right| dx\right)^{1/p},
\]
where \(c_o\) depends only on \(n, p, s, \gamma, \nu, \nu_0, \nu_1, \nu_2, a_o, \eta[\psi ma_{ij}] (m \in \mathbb{N})\).

**Proof.** It can be assumed that \(\sup_{\Omega} \sigma^i(x) u(x) > 0\). Thus it follows from (4.14) and (4.18) that there exists \(y \in \Omega\) such that \(\sup_{\Omega} w(x) = w(y)\); moreover, there exists \(R_o \in ]0, \text{dist}(y, \partial \Omega)\] such that \(w(x) > 0\) for all \(x \in B(y, R_o)\).

Let \(\lambda, \alpha, a_o \in \mathbb{R}_+\), with \(\alpha_o > 1\) (that will be chosen late), such that
\[
\lambda \alpha \leq \min\{R_o, \sigma(y)\}, \quad \alpha = \alpha_o \sigma(y).
\]

In the following we denote by \(B\) the open ball \(B(y, a\lambda)\).

We put
\[
\varphi(x) = 1 + \lambda^2 - \frac{|x - y|^2}{\alpha^2}, \quad x \in \bar{B},
\]
and observe that
\[
1 \leq \varphi(x) \leq 1 + \lambda^2 \leq 2, \quad x \in \bar{B},
\]
\[
\varphi_i \leq \frac{2\lambda}{\alpha}, \quad \varphi_i \varphi_j \leq \frac{4\lambda^2}{\alpha^2}, \quad i, j = 1, \ldots, n,
\]
\[
\varphi_{x_i x_j} = 0 \quad \text{if} \quad i \neq j, \quad \varphi_{x_i x_j} = -\frac{2}{\alpha^2} \quad \text{if} \quad i = j.
\]

Consider now the function \(v\) defined by
\[
v(x) = \varphi(x)w(x) - w(y), \quad x \in \bar{B}.
\]

Obviously,
\[
v_{|_{\partial B}} = w_{|_{\partial B}} - w(y) \leq 0, \quad v(y) = \lambda^2 w(y).
\]
It is easy to show that
\[
L_0(\varphi w) - wL_0\varphi - 2 \sum_{i,j=1}^{n} a_{ij} \varphi x_j w x_i + \sum_{i=1}^{n} b_i(\varphi) x_i
\]
\[
- \sum_{i=1}^{n} b_i \varphi x_i w + b \varphi w = \varphi \left( L_0 w + \sum_{i=1}^{n} b_i w x_i + bw \right) \geq \varphi g \quad \text{in } B.
\]

Thus
\[
L_0(\varphi w) + \sum_{i=1}^{n} d_i (\varphi w) x_i + d \varphi w \geq \varphi g + \sum_{i=1}^{n} b_i \varphi x_i w \quad \text{in } B,
\]
where
\[
d_i = b_i - 2 \sum_{j=1}^{n} a_{ij} \frac{\varphi x_j}{\varphi}, \quad i = 1, \ldots, n,
\]
\[
d = b + 2 \sum_{i,j=1}^{n} a_{ij} \frac{\varphi x_i \varphi x_j}{\varphi^2} - \sum_{i,j=1}^{n} a_{ij} \frac{\varphi x_i x_j}{\varphi}.
\]

Therefore we obtain from (5.14) that
\[
L_0 v + \sum_{i=1}^{n} d_i v x_i + dv \geq h,
\]
where
\[
h = \varphi g + w \sum_{i=1}^{n} b_i \varphi x_i - dw(y).
\]

Clearly, (2.9), (5.6), and (5.9) yield that
\[
|\varphi x_i| \leq 2 \gamma \frac{\sigma}{\alpha_o \sigma^2(y)} \quad \text{in } B,
\]
and hence it follows from Lemma 4.2 that
\[
h \geq \varphi \eta_{k,f} \frac{d_o}{8} \sigma^{-2} \varphi w(y) - 2 y w(y) \left( v_1 + \frac{a_o}{2} \right) \frac{1}{\alpha_o^2} \sigma^{-2}(y) - dw(y)
\]
\[
\geq \varphi \eta_{k,f} \left[ - d - \left( \frac{a_o}{4} \frac{\gamma v_1}{\alpha_o^2} + 2 \frac{\gamma a_o}{\alpha_o^2} + \frac{\gamma a_o}{\sigma^2} \right) \sigma^{-2}(y) \right] w(y).
\]

The constant \( \alpha_o \) can be chosen in such a way that \( d < -d_o \sigma^{-2}(y) \) in \( B \), where
\[
d_o = \frac{a_o}{4} \gamma \sigma^2 + 2 \frac{\gamma v_1}{\alpha_o^2} + \frac{\gamma a_o}{\sigma^2}.
\]
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In fact, by Lemma 4.2, (5.9) and (5.10), we have

\[\begin{align*}
    d + d_o \sigma^{-2}(y) &= b + 2 \sum_{i,j=1}^{n} a_{ij} \frac{\Phi_{x_i} \Phi_{x_j}}{\Phi} - \sum_{i,j=1}^{n} a_{ij} \frac{\Phi_{x_i} \Phi_{x_j}}{\Phi} + d_o \sigma^{-2}(y) \\
    &\leq -\frac{d_o}{2} \sigma^{-2} + 8\nu_o \frac{\lambda^2}{\alpha^2} + 2\nu_o \frac{1}{\alpha^2} + d_o \sigma^{-2}(y) \\
    &\leq \left[ -\gamma^2 \frac{d_o}{4} + (10\nu_o + 2\gamma\nu_1 + \gamma d_o) \frac{1}{\alpha_0^2} \right] \sigma^{-2}(y),
\end{align*}\]

(5.22)

and hence, fixed \(\alpha_0\) such that

\[\frac{1}{\alpha_0^2} \leq \gamma^2 \frac{d_o}{4(10\nu_o + 2\gamma\nu_1 + \gamma d_o)},\]

(5.23)

it follows that

\[d < -d_o \sigma^{-2}(y) \quad \text{in } B.\]

(5.24)

By (5.11), (5.12), and (5.15)–(5.17), we deduce that the problem

\[\begin{align*}
    v \in W^{2,p}(B), \\
    L_o v + \sum_{i=1}^{n} d_i v_{x_i} + dv \geq \varphi \eta_{k_o}^s f, \\
    f \in L^p(B),
\end{align*}\]

(5.25)

satisfies the hypotheses of Lemma 3.1. Therefore, it follows from (5.6), (4.15), and (4.16) that there exists a constant \(c_1 \in \mathbb{R}_+\), depending on \(n, p, s, \gamma, \nu, \nu_0, \nu_1, \nu_2, [p(a_{ij}^s)]_{BMO(\mathbb{R}^n)},\) such that

\[v(x) \leq c_1 (\lambda \alpha)^{-n/p} \left\| (\varphi \eta_{k_o}^s f)^{-} \right\|_{L^p(B)} \quad \forall x \in B.\]

(5.26)

So it follows from (5.8) and from (5.26) with \(x = y\) that

\[\lambda^2 w(y) \leq c_1 (\lambda \alpha)^{-n/p} \left\| (\varphi \eta_{k_o}^s f)^{-} \right\|_{L^p(B)} \leq 2c_1 (\lambda \alpha)^{-n/p} \left\| \eta_{k_o}^s f^- \right\|_{L^p(B)};\]

(5.27)

Thus by (5.6) and (5.27) we have

\[w(y) \leq c_2 (\lambda \alpha)^{-n/p} \alpha_0^2 \sigma^2(y) \left\| \eta_{k_o}^s f^- \right\|_{L^p(B)} \leq c_3 (\lambda \alpha)^{-n/p} \alpha_0^2 2 \left\| \sigma^2 \eta_{k_o}^s f^- \right\|_{L^p(B)},\]

(5.28)

where \(c_2, c_3 \in \mathbb{R}_+\) depend on the same parameters as \(c_1\). Finally from (4.6), (4.7), (4.14), and (5.28) we obtain

\[\sup_{\Omega \sigma^s u \leq c_4 (\lambda \alpha)^{-n/p} \left( \int_B \sigma^{2+s} f^- p \, dx \right)^{1/p} \leq c_5 \left( \int_B \sigma^{s+2} f^- \, dx \right)^{1/p},\]

(5.29)
where $c_4, c_5 \in \mathbb{R}_+$ depend on the same parameters as $c_1$ and on $a_o$. Then, if we choose

$$p\left(a_{ij}|_{B}\right) = (\psi_{m_1} a_{ij})_o,$$

(5.30)

where $m_1$ is a positive integer such that $\psi_{m_1}|_{B} = 1$, (5.5) follows from (5.29), (5.30), and from Remark 5.1.

**Corollary 5.3.** Suppose that conditions $(h_1)$ and $(h_2)$ hold, and let $u$ be a solution of the problem

$$Lu = f, \quad \sigma^{s+2} f \in L^\infty(\Omega), \quad u \in W^{2,p}_\text{loc}(\Omega),$$

$$\limsup_{x \to x_o} \sigma^i(x)u(x) = 0 \quad \forall x_o \in \partial \Omega, \quad (p')$$

$$\limsup_{|x| \to +\infty} \sigma^i(x)u(x) = 0 \quad \text{if } \Omega \text{ is unbounded.}$$

Then

$$\sup_{\Omega} \sigma^i |u| \leq c_o \|\sigma^{s+2} f\|_{L^\infty(\Omega)},$$

(5.31)

where $c_o \in \mathbb{R}_+$ is the constant of the statement of Theorem 5.2.

**Proof.** The result can be obtained applying Theorem 5.2 to the functions $u$ and $-u$. □

The following uniqueness result is an obvious consequence of Corollary 5.3.

**Corollary 5.4.** If the hypotheses $(h_1)$ and $(h_2)$ hold, then the problem

$$Lu = 0, \quad u \in W^{2,p}_\text{loc}(\Omega),$$

$$\limsup_{x \to x_o} \sigma^i(x)u(x) = 0 \quad \forall x_o \in \partial \Omega, \quad (p'')$$

$$\limsup_{|x| \to +\infty} \sigma^i(x)u(x) = 0 \quad \text{if } \Omega \text{ is unbounded}$$

has only the zero solution.

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