Fibre bundle formulation of nonrelativistic quantum mechanics

IV. Mixed states and evolution transport’s curvature

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Abstract

We propose a new systematic fibre bundle formulation of nonrelativistic quantum mechanics. The new form of the theory is equivalent to the usual one but it is in harmony with the modern trends in theoretical physics and potentially admits new generalizations in different directions. In it a pure state of some quantum system is described by a state section (along paths) of a (Hilbert) fibre bundle. It’s evolution is determined through the bundle (analogue of the) Schrödinger equation. Now the dynamical variables and the density operator are described via bundle morphisms (along paths). The mentioned quantities are connected by a number of relations derived in this work.

The present fourth part of this series is devoted mainly to the fibre bundle description of mixed quantum states. We show that to the conventional density operator there corresponds a unique density morphism (along paths) for which the corresponding equations of motion are derived. It is also investigated the bundle description of mixed quantum states in the different pictures of motion. We calculate the curvature of the evolution transport and prove that it is curvature free iff the values of the Hamiltonian operator at different moments commute.
1. Introduction

This is the fourth part of our series of works considering the application of the theory of fibre bundles to nonrelativistic quantum mechanics.

The paper, which is a straightforward continuation of [1, 2, 3], is organized as follows.

The bundle approach to the quantum mechanics of mixed states is investigated in Sect. 2. Subsection 2.1 is a brief review of the conventional concepts of mixed state(s) and density operator (matrix). Their bundle description is presented in Subsection 2.2. It turns out that to the density operator there corresponds a suitable density morphism along paths. The equations for its time evolution are derived. In Subsection 2.3 are studied problems connected with the representations and description of mixed quantum states in the different pictures of motion. The equations of motion for density morphisms and operators are derived.

Sect. 3 is devoted to the curvature of the (bundle) evolution transport. The work ends with some remarks in Sect. 4.

The notation of the present part of the series is identical with the one of the preceding parts for which the reader is referred to [1, 2, 3].

The references to sections, equations, footnotes etc. from the previous three parts of the series, namely [1], [2] and [3], are denoted by the corresponding sequential reference numbers in these parts preceded by the Roman number of the part in which it appears and a dot as a separator.

For instance, Sect. I.5 and (III.3.16) mean respectively section 5 of part I, i.e. of [1], and equation (3.16) (equation 16 in Sect. 3) of part III, i.e. of [3].

Below, for reference purposes, we present a list of some essential equations of [1, 2, 3] used in this paper. Following the above convention, we retain their original reference numbers.

\[
\psi(t) = l_{\gamma(t)}(\Psi_\gamma(t)) \in \mathcal{F}, \quad (I.4.1)
\]

\[
l_\gamma^t = l_\gamma^{-1}, \quad (I.4.9)
\]

\[
U_\gamma(t,s) = l_{\gamma(s)}^{-1} \circ U(t,s) \circ l_{\gamma(s)}, \quad s, t \in J, \quad (I.5.10)
\]

\[
\Gamma_\gamma(t) := \left[ \Gamma_{ab}^\gamma(t; \gamma) \right] = -\frac{1}{i\hbar} H_\gamma^m(t), \quad (II.2.21)
\]

\[
A_\gamma(t) = l_{\gamma(t)}^{-1} \circ A(t) \circ l_{\gamma(t)}: F_{\gamma(t)} \to F_{\gamma(t)}, \quad (II.3.1)
\]

\[
\left[ \tilde{D}_\gamma^\gamma(C) \right] = \frac{d}{dt} C_t + [\Gamma_\gamma(t), C_t]_\gamma, \quad (II.3.27)
\]

\[
A_{H,0}^\gamma(t_0) := U_\gamma^{-1}(t_0) \circ A_\gamma(t_0) \circ U_\gamma(t_0): F_{\gamma(t_0)} \to F_{\gamma(t_0)}, \quad (III.2.4)
\]

\[
A_{H}^\gamma(t_0) := U(t_0, t) \circ A(t) \circ U(t, t_0): \mathcal{F} \to \mathcal{F}, \quad (III.2.7)
\]

\[
i\hbar \frac{\partial A_{H,t}^\gamma(t_0)}{\partial t} = \left[ A_{H,t}^\gamma(t_0), H_{\gamma,t}^\gamma(t_0) \right] + \hbar \left( \frac{\partial A}{\partial t} \right)_H^\gamma(t_0), \quad (III.2.10)
\]

\[
A_{V,t}^\gamma(t_1) := V_\gamma(t_1,t) \circ A_\gamma(t) \circ V_\gamma^{-1}(t_1,t): F_{\gamma(t_1)} \to F_{\gamma(t_1)}, \quad (III.2.27)
\]
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\[ i\hbar \frac{\partial A^V(t)}{\partial t} = [A^V(t), V H^V(t(t))] - i\hbar \left( \frac{\partial A(t)}{\partial t} \right)^V(t), \quad \text{(III.2.39)} \]

\[ i\hbar \frac{\partial A^V_{\gamma,t}(t)}{\partial t} = [A^V_{\gamma,t}(t), V H^V_{\gamma,t}(t)] - i\hbar \left( \frac{\partial A(t)}{\partial t} \right)^V_{\gamma,t}(t). \quad \text{(III.2.41)} \]

2. Mixed states

In the framework of quantum mechanics the most general description of a state of a quantum system is provided via the so-called density (or statistical) matrix (or operator) by means of which is achieved a uniform description of pure and mixed states (see [4, chapter VIII, sect. IV] and [5, §33]; for mathematically rigorous exposition of the problem see [6, chapter IV, sect. 8]). This formalism has also a bundle analogue which is described in this section.

2.1. Hilbert space description (review)

Here we briefly recall the notions of a mixed state and density operator in the conventional Hilbert space description of quantum mechanics [4] (see also [7, 8]).

Consider a quantum system which at a moment \( t \) with a probability \( p_i \) can be found in a state with a state vector \( \psi_i(t) \in \mathcal{F} \). Here \( i \) belongs to some set of indexes \( I \) and the statistical weights \( p_i \) are assumed time-independent:

\[ 0 \leq p_i \leq 1, \quad \sum_{i \in I} p_i = 1, \quad \frac{\partial p_i}{\partial t} = 0. \quad (2.1) \]

The state of such a system is described by the density operator

\[ \rho(t) := \sum_{i \in I} \psi_i(t) \frac{p_i}{\langle \psi_i(t) | \psi_i(t) \rangle} \psi_i(t)^\dagger : \mathcal{F} \to \mathcal{F}. \quad (2.2) \]

Here with a dagger as a superscript we denote the dual Hermitian conjugate vectors and spaces with respect to the inner product \( \langle \cdot | \cdot \rangle \), i.e. if \( \psi \in \mathcal{F} \), then \( \psi^\dagger \in \mathcal{F}^\dagger \) is a map \( \psi^\dagger : \mathcal{F} \to \mathbb{C} \) such that \( \psi^\dagger : \chi \mapsto \langle \psi | \chi \rangle \) for \( \chi \in \mathcal{F} \), and a product like \( \psi \chi^\dagger \), \( \psi, \chi \in \mathcal{F} \) is defined as an operator \( \psi \chi^\dagger : \mathcal{F} \to \mathcal{F} \) via \( (\psi \chi^\dagger)(\varphi) := \langle \chi \varphi | \psi \rangle \) for \( \varphi \in \mathcal{F} \). The density operator (2.2) is Hermitian, positive definite, trace-class, and of unit trace [4, 6]. So we have

\[ \rho^\dagger(t) = \rho(t), \quad \langle \psi(t) | \rho(t) \psi(t) \rangle \geq 0, \quad \text{Tr} \rho(t) = 1, \quad (2.3) \]

where \( \text{Tr} \) denotes the trace of an operator. Conversely, any such operator is a density operator (in the absence of superselection rules) [6, chapter IV, subsect. 8.6].

1In Dirac’s notation \( \psi, \psi^\dagger \), and \( \psi \chi^\dagger \) will look like \( |\psi\rangle \), \( \langle \psi | \) and \( |\psi\rangle \langle \chi | \) respectively [5, 4]. Notice that \( (\psi \chi^\dagger)^\dagger = \chi \psi^\dagger \) corresponds to \( (|\psi\rangle \langle \chi |)^\dagger = \langle \chi | \psi \rangle \) in Dirac’s notation.
By definition the mean (expectation) value of an observable $A$ is
\[ \langle A(t) \rangle_\rho^{t} := \text{Tr}(\rho(t) \circ A(t)) \] (2.4)
for a system whose state is described by a density operator $\rho(t)$.

The time evolution of the density operator is described by postulating the Schrödinger equation for it, called also von Neumann’s or Liouville equation:
\[ i\hbar \frac{d\rho(t)}{dt} = [\mathcal{H}(t), \rho(t)] := \mathcal{H}(t) \circ \rho(t) - \rho(t) \circ \mathcal{H}(t) \] (2.5)
where $\mathcal{H}(t)$ is system’s Hamiltonian. If $\rho(t_0)$ is known for some instant of time $t_0$, then
\[ \rho(t) = \mathcal{U}(t, t_0) \circ \rho(t_0) \circ \mathcal{U}^{-1}(t, t_0), \] (2.6)
where $\mathcal{U}(t, t_0)$ is the system’s evolution operator (see Sect. I.2). In fact, (2.6) is the general solution of (2.5) with respect to $\rho(t)$.

If the sum in (2.2) contains only one term or terms which are proportional up to a phase factor to one of them, the system is said to be in a pure state (described by (one of) the corresponding state vector(s) in (2.2)), otherwise the system’s state is called mixed. A criterion for a state to be pure is $\rho^2(t) = \rho(t)$ [4, 6].

2.2. Hilbert bundle description

Now we are ready to apply the bundle approach to systems described via some density operator $\rho(t)$. The easiest way to introduce the bundle analogue of $\rho(t)$ is via (2.4). Expressing $A$ from (II.3.1) and substituting the result into (2.4), we find
\[ \langle A(t) \rangle_\rho^{t} = \langle A_\gamma(t) \rangle_{P_\gamma}^{t} \] (2.7)
where
\[ \langle A_\gamma(t) \rangle_{P_\gamma}^{t} = \text{Tr}(P_\gamma(t) \circ A_\gamma(t)) \] (2.8)
is the (bundle) mean value of the morphism $A$, corresponding to $A$, in the state characterized by the density morphism along paths $P : \gamma \mapsto P_\gamma : \gamma(t) \mapsto \{P_\gamma(s) : s \in J, \gamma(s) = \gamma(t)\}$ defined via (cf. (II.3.1))
\[ P_\gamma(t) := l_{-1}^{-1}_\gamma(t) \circ \rho(t) \circ \rho(t) : F_\gamma(t) \rightarrow F_\gamma(t). \] (2.9)

\textsuperscript{4}Equation (2.5) is equivalent to the assumption that every vector $\psi_i(t)$ in (2.2) evolves according to the Schrödinger equation (I.2.6). Respectively, equation (2.6) is equivalent to (I.2.1) for the vectors $\psi_i(t)$. 
Meanwhile, equation (2.7) expresses the natural requirement that the expectation value of a dynamical variable \( A \) must be independent of the (mathematical) way we calculate it.

The bundle density operator has also a representation like (2.2). In fact, substituting (I.4.1) and its Hermitian conjugate, i.e.
\[
\psi^\dagger(t) = \Psi^\dagger x(t) = \gamma(t) \circ P_{\gamma}(t) \circ \gamma^{-1}(t),
\]
where \( \Psi^\dagger x : F_x \to C \) is defined by \( \Psi^\dagger x : \Phi_x \mapsto \langle \Psi_x | \Phi_x \rangle x \) for \( \Psi_x, \Phi_x \in F_x, x \in M \), which is a consequence of the unitarity of \( l_x \) (see (I.4.9)), for the vectors \( \psi_i(t) \) (appearing in (2.2)) into (2.2), we get
\[
\rho(t) = l_{\gamma}(t) \circ P_{\gamma}(t) \circ l_{\gamma}^{-1}(t)
\]
which is equivalent to (2.9) with
\[
P_{\gamma}(t) = \sum_{i \in I} \Psi_{i,\gamma}(t) \frac{p_i}{\langle \Psi_{i,\gamma}(t) | \Psi_{i,\gamma}(t) \rangle} \Psi_{i,\gamma}^\dagger(t).
\]
where a product like \( \Phi_x \Psi^\dagger x \) is considered as an operator \( \Phi_x \Psi^\dagger x : F_x \to F_x \) such that \( (\Phi_x \Psi^\dagger x)X_x := \langle \Psi_x | X_x \rangle x \Phi_x, X_x \in F_x \).

The above results show that the transition from Hilbert space to Hilbert bundle description of mixed states is achieved simply via a replacement of the vectors (resp. operators) of (resp. acting on) \( F \) with sections (resp. morphisms) along paths of \( (F, \pi, M) \) according to the general rules of sections I.4 and II.3. As we shall see below, this observation has a general validity.

From (2.3), (2.9) and (I.4.9) follows that the density operator \( P_{\gamma}(t) \) in \( F_{\gamma}(t) \) is Hermitian, positive definite, trace-class, and of trace one, i.e.
\[
P_{\gamma}^\dagger(t) = P_{\gamma}(t), \quad \langle \Psi_{\gamma}(t) | P_{\gamma}(t) | \Psi_{\gamma}(t) \rangle_{\gamma(t)} \geq 0, \quad \text{Tr} (P_{\gamma}(t)) = 1.
\]

The time evolution of the density morphism along paths, i.e. the relation between \( P_{\gamma}(t) \) and \( P_{\gamma}(t_0) \) for any \( t, t_0 \in J \), can be found as follows. Substituting (2.11) for \( t = t_0 \) into (2.6), then substituting the result into (2.9), and, at last, applying (I.5.10), we get
\[
P_{\gamma}(t) = U_{\gamma}(t, t_0) \circ P_{\gamma}(t_0) \circ U_{\gamma}^{-1}(t, t_0), \quad t, t_0 \in J,
\]
where \( U_{\gamma}(t, t_0) \) is the evolution transport from \( F_{\gamma(t_0)} \) to \( F_{\gamma(t)} \).

The differential equation, corresponding to the evolution law (2.14), can be derived in the following way. Differentiating the matrix form of (2.14) with respect to \( t \) and applying equation (II.2.14), we obtain the matrix-bundle Schrödinger equation for the density morphism as
\[
i \hbar \frac{dP_{\gamma}(t)}{dt} = [H_{\gamma}^m(t), P_{\gamma}(t)]_-
\]
with $H^\gamma_m(t)$ being the matrix-bundle Hamiltonian (given by (II.2.12)). This equation is the matrix-bundle analogue of (2.5), to which it is equivalent as it can be proved via the substitution of the matrix form of (2.11) into the one of (2.5) (see also (II.3.10)). Consequently, the results of Sect. II.2 show that (2.14) gives the general solution (2.15) with respect to $P^\gamma(t)$.

The matrix equation (2.15) can be written into an invariant form too. To this end we shall use the following result which is a simple corollary of (2.15), (II.2.19), and (II.2.21): If $\Psi$ is a section along paths and one of the equations $D^\gamma_t(\Psi^\gamma(t)) = 0$, $D^\gamma_t[P^\gamma(t)(\Psi^\gamma(t))] = 0$, or (2.15) is valid, then the remaining two of them are equivalent. From here follows that (2.15) is equivalent to the system

\begin{align}
(D^\gamma_t \circ P^\gamma(t))\Psi^\gamma(t) &= 0, \\
D^\gamma_t(\Psi^\gamma(t)) &= 0.
\end{align}

(Note: $\gamma$ is not a summation index here and below!)

If we denote by $\tilde{P}_\gamma(t)$ the restriction of $P^\gamma(t): F^\gamma(t) \to F^\gamma(t)$ on the set of (state) sections along $\gamma$ which are (linearly) transported along $\gamma$ by means of the evolution transport, i.e the ones satisfying (2.16b), then (2.16) is equivalent to

\[ \tilde{D}^\gamma_t(\tilde{P}_\gamma(t)) = 0 \]  

(2.17)

where $\tilde{D}$ is the defined by (II.3.25) differentiation along paths of bundle morphisms (along paths in the present case). The above discussion shows the equivalence of (2.17) and (2.15), a fact which is also an evident corollary of (II.3.27) and (II.2.21). The equation (2.17) can be called a \textit{(bundle) Schrödinger equation for the density morphism.}

A simple verification proves that the linear map $P^\gamma(t_0) \mapsto P^\gamma(t)$, defined by (2.14), satisfies (I.3.2) and (I.3.3). So, freely speaking, we may say that this is a ‘transport-like’ map by means of which $P^\gamma$ is ‘transported’ along $\gamma$. This is something like ‘representation’ of the evolution transport in the space of morphisms along $\gamma$. The rigorous study of this problem (see the end of subsection III.2.1 and [9, sect. 3]) reviles that the pointed map is a linear transport along $\gamma$ in the fibre bundle of bundle morphisms (along paths) of $(F, \pi, M)$. With respect to this transport along paths the density morphism (along $\gamma$) is a linearly transported section (along $\gamma$) of the bundle of morphisms (along $\gamma$) of $(F, \pi, M)$.

### 2.3. Representations in the different pictures of motion

Let us turn now our attention to the description of mixed states in the different pictures of motion (see Sect. III.2).

In the Schrödinger picture of the Hilbert bundle (resp. space) description of quantum mechanics, which, in fact, was investigated until now in this section, the motion of a quantum system is described by pairs like $(P^\gamma(t), A^\gamma(t))$
and operator $\rho$ and operators in $F$ pairs like $\rho(\gamma,t)$. The Hilbert bundle and Hilbert space descriptions are by means of (2.14) and (2.6) were used. Consequently in the Heisenberg picture (III.2.4) and (III.2.7) for bundle morphisms along paths in $(F, \pi, M)$ and operators in $F$, respectively. In particular, for the density morphism $P$ and operator $\rho$ they, respectively, give:

$$P^H_{\gamma,t}(t_0) := U^{-1}_\gamma(t, t_0) \circ P_\gamma(t) \circ U_\gamma(t, t_0) = P_\gamma(t_0) : F_\gamma(t_0) \to F_\gamma(t_0),$$

$$\rho^H_t(t_0) := U^{-1}(t, t_0) \circ \rho(t) \circ U(t, t_0) = \rho_\gamma(t_0) : \mathcal{F} \to \mathcal{F}$$

where (2.14) and (2.6) were used. Consequently in the Heisenberg picture the density morphisms and operators are constant(s of motion), do not evolve in time, while the observer’s evolution is governed by the Heisenberg equation of motion for them (see (III.2.10), or (III.2.8), or (III.2.24)).

Of course, in the Heisenberg picture the mean values remain unchanged:

$$\langle A^H_{\gamma,t}(t_0) \rangle_{P^H_{\gamma,t}}^{t_0} = \langle A^H_\gamma(t_0) \rangle_{P^H_\gamma}^{t_0} = \langle A_\gamma(t) \rangle_{P^H_\gamma}^t = \langle A(t) \rangle_{\rho^H_t}^t,$$

where

$$\langle A^H_{\gamma,t}(t_0) \rangle_{P^H_{\gamma,t}}^{t_0} := \text{Tr} (P_\gamma(t_0) \circ A^H_{\gamma,t}(t_0)), \quad \langle A^H_\gamma(t_0) \rangle_{P^H_\gamma}^{t_0} := \text{Tr} (\rho(t_0) \circ A^H_\gamma(t_0)).$$

The chain equation (2.20) is a corollary of the invariance of the trace of a product (composition) of operators with respect to a cyclic permutation of the multipliers.

The shift from the Schrödinger to ‘general’ picture is done by the general equations (III.2.27) and (III.2.30). Hence, in the $V$-picture of motion the density morphism $P$ and operator $\rho$, respectively, are

$$P^V_{\gamma,t}(t_1) = V_\gamma(t_1, t) \circ P_\gamma(t) \circ V^{-1}_\gamma(t_1, t) : F_\gamma(t_1) \to F_\gamma(t_1),$$

$$\rho^V_t(t_1) = V_\gamma(t_1, t) \circ \rho_\gamma(t) \circ V^{-1}(t_1, t) : \mathcal{F} \to \mathcal{F}.$$

Since in the $V$-picture the Hilbert bundle (resp. space) description is via pairs like $(P^V_{\gamma,t}(t_1), A^V_{\gamma,t}(t_1))$ (resp. $(\rho^V(t_1), A^V(t_1))$), the mean values of the observables remain unchanged, as in the Heisenberg picture:

$$\langle A^V_{\gamma,t}(t_1) \rangle_{P^V_{\gamma,t}}^{t_1} = \langle A^V_\gamma(t_1) \rangle_{\rho^V_t}^{t_1} = \langle A_\gamma(t) \rangle_{P^V_\gamma}^t = \langle A(t) \rangle_{\rho^V_t}^t,$$
where

\[ \langle A^V_{\gamma,t}(t_1) \rangle_{P^V_{\gamma,t}}^{t_1} := \text{Tr} \left( P^V_{\gamma,t}(t_1) \circ A^V_{\gamma,t}(t_1) \right), \]

\[ \langle A^V_{\gamma,\rho}(t_1) \rangle_{P^V_{\gamma,t}}^{t_1} := \text{Tr} \left( \rho^V_{\gamma,t}(t_1) \circ A^V_{\gamma,t}(t_1) \right). \] (2.25)

In the V-picture the density morphisms, operators, and observables generally change in time. For all of them this change is governed by the equations (III.2.41) and (III.2.39), but for the density morphisms and operators they can be written in a more concrete form. For this purpose we have to calculate the last terms in the r.h.s. of (III.2.41) and (III.2.39).

Using (III.2.27) and (2.5), we obtain

\[ \left( \frac{\partial \rho(t)}{\partial t} \right)^V_{\gamma,t} (t_1) = V(t_1,t) \circ \rho(t) \circ V^{-1}(t_1,t) = \frac{1}{i\hbar} \left[ H^V_{\gamma,t}(t_1), \rho^V_{\gamma,t}(t_1) \right]. \]

Analogously, applying (III.2.33) and the just get equation, we find

\[ \left( \frac{\partial \rho(t)}{\partial t} \right)^V_{\gamma,t} (t_1) = I_{\gamma,t}^{-1}(t_1) \circ \left( \frac{\partial \rho(t)}{\partial t} \right)^V_{\gamma,t} (t_1) = \frac{1}{i\hbar} \left[ H^V_{\gamma,t}(t_1), P^V_{\gamma,t}(t_1) \right]. \]

At last, substituting the above two equations into (III.2.41) and (III.2.39), we, respectively, get

\[ i\hbar \frac{\partial P^V_{\gamma,t}(t_1)}{\partial t} = \left[ H^V_{\gamma,t}(t_1), P^V_{\gamma,t}(t_1) \right], \] (2.26)

\[ i\hbar \frac{\partial \rho^V_{\gamma,t}(t_1)}{\partial t} = \left[ H^V_{\gamma,t}(t_1), \rho^V_{\gamma,t}(t_1) \right]. \] (2.27)

where (III.2.37) and (III.2.42) were taken into account. These are the equations of motion for the density morphism and operator in the V-picture.

If the evolution transport and operator are known (in the V-picture), then combining, from one hand, (2.22), (2.14) and (III.2.48) and, from the other hand, (2.23), (2.6), and (III.2.47), we get the general solution of (2.26) and (2.27), respectively, in the form

\[ P^V_{\gamma,t}(t_1) = U^V_{\gamma}(t,t_1,t_0) \circ P_{\gamma,t_0}(t_1) \circ U^V_{\gamma}(t,t_1,t_0)^{-1}, \] (2.28)

\[ \rho^V_{\gamma,t}(t_1) = U^V(t,t_1,t_0) \circ \rho_{t_0}(t_1) \circ U^V(t,t_1,t_0)^{-1}. \] (2.29)

As one can expect, in the case of Heisenberg picture, due to (III.2.49), these formulae reproduce (2.18) and (2.19) respectively.

### 3. Curvature of the evolution transport

Let \( \eta: J \times J' \rightarrow M \) with \( J \) and \( J' \) being \( \mathbb{R} \)-intervals. According to [10, Sect. 3] the curvature of the (bundle) evolution transport \( U \) is \( R: \eta \mapsto R^\eta \),
with \( R^0: (s, t) \mapsto R^0(s, t), (s, t) \in J \times J', \) where

\[
R^0(s, t) := D^\eta(s, \cdot) \circ D^\eta(s, \cdot) - D^\eta(s, \cdot) \circ D^\eta(s, \cdot): \text{Sec}^2(F, \pi, M) \to \pi^{-1}(\eta(s, t))
\]

(3.1)

Here \( D \) is the differentiation along paths assigned to \( U \) by (II.2.18).

In a local field of bases the local components of the curvature are [10, equation 3.3]

\[
(R^0(s, t))^a_b = \frac{\partial}{\partial s} [\Gamma^a_b(t; \eta(s, \cdot))] - \frac{\partial}{\partial t} [\Gamma^a_b(s; \eta(\cdot, t))] + \Gamma^a_c(s; \eta(\cdot, t)) [\Gamma^c_b(t; \eta(s, \cdot)) - \Gamma^c_b(t; \eta(s, \cdot)) \Gamma^c_b(s; \eta(\cdot, t))].
\]

(3.2)

Physically we interpret \( \eta(s, \cdot): t \mapsto \eta(s, t) \) and \( \eta(\cdot, t): s \mapsto \eta(s, t) \) as world lines (trajectories) of observers with proper times \( t \) and \( s \) respectively. So, using the fundamental relation (II.2.21) we can explicitly calculate the curvature. The easiest way to do this is to choose the bases \( \{e_a(x)\} \) and \( \{f_a(t)\} \) such that \( L_x(t) = [\delta^a_b] = 1 \) (see remark II.2.1). Then \( E(t) = 0 \) and \( \mathcal{H}^0(t) = \mathcal{H}(t) = \mathcal{H}(t) \). Hence, now (3.2) reduces to

\[
R^0(s, t) = \frac{1}{(-\hbar)^2} [\mathcal{H}(s), \mathcal{H}(t)]_\mathbb{L}.
\]

(3.3)

where we have assumed, as usual, that \( \mathcal{H}(s) \) is independent of the observers trajectory \( \gamma \). (This equality is valid only in the special basis in which it is derived!)

From here it follows that the evolution transport is curvature free if and only if the values of the Hamiltonian operator at different moments commute, viz.

\[
R^0 = 0 \iff [\mathcal{H}(s), \mathcal{H}(t)]_{\mathbb{L}} = 0.
\]

(3.4)

In particular this is true for time-independent Hamiltonians, i.e. for \( \partial \mathcal{H}(t)/\partial t = 0 \). According to (II.3.23) the bundle formulation of (3.4) is

\[
R^0 = 0 \iff [\dot{\mathcal{H}}_{\gamma,s}(r), \dot{\mathcal{H}}_{\gamma,t}(r)]_{\mathbb{L}} = 0.
\]

(3.5)

for some (and hence any) path \( \gamma: J \to M \). Here \( r, s, t \in J \) and \( \dot{\mathcal{H}}_{\gamma,t}(r) \) is (the transported by means of \( \Gamma^\gamma_{[s,t]} \), Hamiltonian which is) calculated via (II.3.20).

Consider now a curvature free evolution transport on \( W = \eta(J, J') \), i.e. \( R^0(s, t) \equiv 0 \) for every \( (s, t) \in J \times J' \). From [10, proposition 3.3] we know that in this case there exists a field of base \( \{e_i\} \) over \( W \) in which the transport’s coefficients vanish along any path \( \gamma \) in \( W \). In it, due to (II.2.14) and (II.2.21), we have

\[
\widetilde{U}_\gamma(t, t_0) = 1, \quad \tilde{\Gamma}_\gamma(t) = 0, \quad \tilde{\mathcal{H}}^\gamma(t) = 0 \quad \text{for every } \gamma.
\]

(3.6)
Notice, equation (III.2.3) is valid for arbitrary evolution transports along any fixed path in appropriate bases along it, while (3.6) holds only for curvature free evolution transports in a suitably chosen field of bases in a whole set $W$. The connection of the special bases in which (3.6) is true with the Heisenberg picture is the same as discussed in Subsect III.2.2 for the bases in which (III.2.3) are satisfied.

We want to emphasize on the fact that according to (II.2.12) the local vanishment of the matrix-bundle Hamiltonian does not imply the same property for the Hamiltonian as an operator or morphism.

4. Conclusion

In the present work we have seen that the bundle formulation of quantum mechanics admits natural description of mixed states. We have derived different versions of the equation of motion for the density morphism along paths, which now replaces the conventional density operator. The mixed states were considered from the view-point of different bundle pictures of motion. Here we have calculated the curvature of the evolution transport. It turns to be curvature free iff the values of the Hamiltonian at different moments commute.

This paper ends the introduction of general formalism of Hilbert bundle description of nonrelativistic quantum mechanics. The interpretation of this description and its possible further developments will be given elsewhere.

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