On Classical and Quantum Cryptography

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Abstract

Lectures on classical and quantum cryptography. Contents: Private key cryptosystems. Elements of number theory. Public key cryptography and RSA cryptosystem. Shannon's entropy and mutual information. Entropic uncertainty relations. The no cloning theorem. The BB84 quantum cryptographic protocol. Security proofs. Bell's theorem. The EPRBE quantum cryptographic protocol.

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1 Introduction

Cryptography is the art of code-making, code-breaking and secure communication. It has a long history of military, diplomatic and commercial applications dating back to ancient societies. In these lectures an introduction to basic notions of classical and quantum cryptography is given.

A well known example of cryptosystem is the Caesar cipher. Julius Caesar allegedly used a simple letter substitution method. Each letter of Caesar's message was replaced
by the letter that followed it alphabetically by 3 places. This method is called the
Caesar cipher. The size of the shift (3 in this example) should be kept secret. It is
called the key of the cryptosystem. It is an example of the traditional cryptosystem.
It is also called the private key cryptography. Anyone who knew the enciphering key
can decipher the message. Mathematical theory of classical cryptography has been
developed by C. Shannon.

There is a problem in the private key cryptography which is called the problem
of key distribution. To establish the key, two users must use a very secure channel.
In classical world an eavesdropper in principle can monitor the channel without the
legitimate users being aware that an eavesdropping has taken place.

In 1976 W. Diffie and M. Hellman\footnote{1} discovered a new type of cryptosystem and
invented public key cryptography. In this method the problem of key distribution was
solved. A public key cryptosystem has the property that someone who knows only
how to encipher cannot use the enciphering key to find the deciphering key without
a prohibitively lengthy computation. The best-known public key cryptosystem, RSA
\footnote{2}, is widely used in Internet and other business. The system relies on the difficulty of
factoring large integers.

In the 1970's S. Wiesner\footnote{3} and C.H. Bennett and G. Brassard\footnote{4} (their method
is the called BB84 protocol) have proposed the idea of quantum cryptography. They
used the sending of single quantum particles. The method of quantum cryptography
also can solve the key distribution problem. Moreover it can detect the presence of
an eavesdropper. In 1991 A. Ekert\footnote{5} proposed to use in quantum cryptography the
phenomena of entanglement and Bell's inequalities.

Experimental quantum key distribution was demonstrated for the first time in 1989
and since then tremendous progress has been made. Several groups have shown that
quantum key distribution is possible, even outside the laboratory. In particular it was
reported the creation of a key over the distance of several dozens kilometers\footnote{6}.

First we discuss Caesar's cryptosystem and then in Sect.3 elements of number
theory needed for cryptography are discussed. In Section 4 the public key distribution
and the RSA cryptosystem is considered. The BB84 quantum cryptographic protocol
is discussed in Sect.8. Some useful notions of the mutual information and Shannon's
entropy are included and proofs of security of the protocol is discussed. In Sect 9.
the Einstein-Podolsky-Rosen-Bell-Ekert (EPRBE) quantum cryptographic protocol is
considered. The security of the protocol is based on Bell's theorem describing nonlocal
properties of entangled states. The importance of consideration of entangled states
in space and time is stressed. A modification of Bell's equation which includes the
spacetime variables is given and the problem of security of the EPRBE protocol in real
spacetime is discussed.
2 Private Key Cryptosystems

Cryptography is the art of sending messages in disguised form. We shall use the following notions.

**Alphabet** - a set of letters.
**Plaintext** - the message we want to send.
**Ciphertext** - the disguised message.

The plaintext and ciphertext are broken up into *message units*. A message unit might be a single letter, a pair of letters or a block of *k* letters.

An *enciphering* transformation is a function \( f \) from the set \( \mathcal{P} \) of all possible plaintext message units to the set \( \mathcal{C} \) of all possible ciphertext units. We assume that \( f \) is a 1- to 1 correspondence. \( f : \mathcal{P} \to \mathcal{C} \). The *deciphering* transformation is the map \( f^{-1} \) which goes back and recovers the plaintext from the ciphertext. Schematically one has the diagram

\[
\mathcal{P} \xrightarrow{f} \mathcal{C} \xrightarrow{f^{-1}} \mathcal{P}
\]

Any such set-up is called a *cryptosystem*.

2.1 Julius Caesar’s cryptosystem

Let us discuss the Caesar cryptosystem in more detail. Suppose we use the 26-letter Latin alphabet \( A, B, \ldots, Z \) with numerical equivalents \( 0, 1, \ldots, 25 \). Let the letter \( x \in \{0, 1, \ldots, 25\} \) stands for a plaintext message unit. Define a function

\[
f : \{0, \ldots, 25\} \to \{0, \ldots, 25\}
\]

by the rule

\[
f(x) = \begin{cases} 
  x + 3, & \text{if } x < 23 \\
  x + 3 - 26 = x - 23, & \text{if } x \geq 23 
\end{cases}
\]

In other words \( f(x) \equiv x + 3 \pmod{26} \).

To decipher a message one subtracts 3 modulo 26.

**Exercise.** According to the Caesar’s cryptosystem the word ”COLD” reads ”FROG”.

More generally consider the congruence (see Sect. 3 about the properties of congruences)

\[
f(x) = x + b \pmod{N}
\]

i.e.

\[
\begin{cases} 
  x + b, & \text{if } x < N - b \\
  x - (N - b) = x + b - N, & \text{if } x \geq N - b 
\end{cases}
\]
In the case of Caesar's cryptosystem $N = 26$, $b = 3$. To decipher the message one subtracts $b$ modulo $N$.

We could use a more general affine map, i.e. $f(x) = ax + b \pmod{N}$. To decipher a message $y = ax + b \pmod{N}$ one solves for $x$ in terms of $y$ obtaining

$$x = a'y + b' \pmod{N}$$

where $a'$ is the inverse of $a$ modulo $N$ and $b' = -a^{-1}b \pmod{N}$. Assume $a$ is relatively prime to $N$, then there exists $a^{-1}$ (see Sect.3).

In this example the enciphering function $f$ depends upon the choice of parameters $a$ and $b$. The values of parameters are called the enciphering key $K_E = (a, b)$. In order to compute $f^{-1}$ (decipher) we need a deciphering key $K_D$. In our example $K_D = (a', b')$ where $a' = a^{-1} \pmod{N}$ and $b' = -a^{-1}b \pmod{N}$.

### 2.2 Symmetric Cryptosystems - DES and GOST

Suppose that the algorithm of the cryptosystem is publicly known but the keys are kept in secret. It is a private key cryptography. Examples of such cryptosystems are Data Encryption Standard (DES), with 56-bit private key (USA, 1980) and a more secure GOST-28147-89 which uses 256-bit key (Russia, 1989). In such cryptosystems anyone who knows an enciphering key can determine the deciphering key. Such cryptosystems are called symmetric cryptosystems.

### 3 Elements of Number Theory

In this section we collect some relevant material from number theory [7].

**Euclid's Algorithm.** Given two integers $a$ and $b$, not both zero, the greatest common divisor of $a$ and $b$, denoted $\text{g.c.d}(a, b)$ is the biggest integer $d$ dividing both $a$ and $b$. For example, $\text{g.c.d}(9, 12) = 3$.

There is the well known Euclid's algorithm of finding the greatest common divisor. It proceeds as follows.

- Find $\text{g.c.d}(a, b)$ where $a > b > 0$.

1) Divide $b$ into $a$ and write down the quotient $q_1$ and the remainder $r_1$:

$$a = q_1 b + r_1, \quad 0 < r_1 < b,$$

2) Next, perform a second division with $b$ playing the role of $a$ and $r_1$ playing the role of $b$:

$$b = q_2 r_1 + r_2, \quad 0 < r_2 < r_1,$$
3) Next:

\[ r_1 = q_3 r_2 + r_3, \quad 0 < r_3 < r_2. \]

Continue in this way. When we finally obtain a remainder that divides the previous
remainder, we are done: that final nonzero remainder is the \( g.c.d. \) of \( a \) and \( b \):

\[
\begin{align*}
 r_t &= q_{t+2} r_{t+1} + r_{t+2}, \\
r_{t+1} &= q_{t+3} r_{t+2}.
\end{align*}
\]

We obtain: \( r_{t+2} = d = g.c.d.(a,b) \).

**Example.** Find \( g.c.d.(128,24) \):

\[
\begin{align*}
128 &= 5 \cdot 24 + 8, \\
24 &= 3 \cdot 8
\end{align*}
\]

We obtain that \( g.c.d.(128,24) = 8 \).

Let us prove that Euclid’s algorithm indeed gives the greatest common divisor. 
Note first that \( b > r_1 > r_2 > \ldots \) is a sequence of decreasing positive integers which can 
not be continued indefinitely. Consequently Euclid’s algorithm must end.

Let us go up throughout Euclid’s algorithm. \( r_{t+2} = d \) divides \( r_{t+1}, r_t, \ldots, r_1, b, a \). 
Thus \( d \) is a common divisor of \( a \) and \( b \).

Now let \( c \) be any common divisor of \( a \) and \( b \). Go downward throughout Euclid’s 
algorithm. \( c \) divides \( r_1, r_2, \ldots, r_{t+2} = d \). Thus \( d \), being a common divisor of \( a \) and \( b \), 
is divisible by any common divisor of these numbers. Consequently \( d \) is the greatest 
common divisor of \( a \) and \( b \).

Another (but similar) proof is based on the formula 

\[
g.c.d.(q b + r, b) = g.c.d.(b, r).
\]

**Corollary.** Note that from Euclid’s algorithm it follows (go up) that if \( d = g.c.d.(a,b) \) 
then there are integers \( u \) and \( v \) such that

\[
d = u a + v b. \tag{1}
\]

In particular one has

\[
u a \equiv d \pmod{b} \tag{2}
\]

One can estimate the efficiency of Euclid’s algorithm. By *Lame’s theorem* the 
number of divisions required to find the greatest common divisor of two integers is 
never greater that five-times the number of digits in the smaller integer.
**Congruences.** An integer \( a \) is *congruent to* \( b \) modulo \( m \),

\[
a \equiv b \pmod{m}
\]
iff \( m \) divides \((a - b)\). In this case \( a = b + km \) where \( k = 0, \pm 1, \pm 2, \ldots \).

**Proposition.** Let us be given two integers \( a \) and \( m \). The following are equivalent

(i) There exists \( u \) such that \( au \equiv 1 \pmod{m} \).

(ii) \( \gcd(a, m) = 1 \).

**Proof.** From (i) it follows

\[
ab - mk = 1.
\]

Therefore the \( \gcd(a, m) = 1 \), i.e. we get (ii).

Now if (ii) is valid then one has the relation (2) for \( d = 1, b = m \):

\[
au \equiv 1 \pmod{m}
\]

which gives (i). \( \square \)

Let us solve in integers the equation

\[
ax \equiv c \pmod{m}
\]  \hspace{1cm} (3)

We suppose that \( \gcd(a, m) = 1 \). Then by the previous proposition there exists such \( b \) that

\[
ab \equiv 1 \pmod{m}.
\]

Multiplying Eq (3) to \( b \) we obtain the solution

\[
x \equiv bc \pmod{m}
\]  \hspace{1cm} (4)

or more explicitly

\[
x = bc + km, \quad k = 0, \pm 1, \pm 2, \ldots
\]

**Exercise.** Find all of the solutions of the congruence

\[
3x \equiv 4 \pmod{7}.
\]

**Chinese Remainder Theorem.** Suppose there is a system of congruences to different moduli:

\[
x \equiv a_1 \pmod{m_1},
\]

\[
x \equiv a_2 \pmod{m_2},
\]

\[
\ldots
\]

\[
x \equiv a_t \pmod{m_t}
\]

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Suppose $g.c.d.(m_i, m_j) = 1$ for $i \neq j$. Then there exists a solution $x$ to all of the congruences, and any two solutions are congruent to one another modulo $M = m_1 m_2 \ldots m_t$.

**Proof.** Let us denote $M_i = M/m_i$. There exist $N_i$ such that

$$M_i N_i \equiv 1 \pmod{m_i}$$

Let us set

$$x = \sum_i a_i M_i N_i$$

This is the solution. Indeed we have

$$\sum_i a_i M_i N_i \equiv a_1 + a_2 + \ldots \equiv a_1 \pmod{m_1}$$

and similarly for other congruences. $\square$

We will need also

**Fermat's Little Theorem.** Let $p$ be a prime number. Any integer $a$ satisfies

$$a^p \equiv a \pmod{p}$$

and any integer $a$ not divisible by $p$ satisfies

$$a^{p-1} \equiv 1 \pmod{p}.$$

**Proof.** Suppose $a$ is not divisible by $p$. Then $\{0a, 1a, 2a, \ldots, (p-1)a\}$ form a complete set of residues modulo $p$, i.e. $\{a, 2a, \ldots, (p-1)a\}$ are a rearrangement of $\{1, 2, \ldots, p-1\}$ when considered modulo $p$. Hence the product of the numbers in the first sequence is congruent modulo $p$ to the product of the members in the second sequence, i.e.

$$a^{p-1}(p - 1) \equiv (p - 1)! \pmod{p}$$

Thus $p$ divides $(p - 1)(a^{p-1} - 1)$. Since $(p - 1)!$ is not divisible by $p$, it should be that $p$ divides $(a^{p-1} - 1)$. $\square$

**The Euler function.**

The Euler function $\varphi(n)$ is the number of nonnegative integers $a$ less than $n$ which are prime to $n$:

$$\varphi(n) = \#\{0 \leq a < n : g.c.d.(a, n) = 1\}$$

In particular $\varphi(1) = 1$, $\varphi(2) = 1$, $\ldots$, $\varphi(6) = 2$, $\ldots$. One has $\varphi(p) = p - 1$ for any prime $p$.

**Exercise.** Prove: $\varphi(p^n) = p^n - p^{n-1}$ for any $n$ and prime $p$. 

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The Euler function is multiplicative, meaning that

$$\varphi(mn) = \varphi(m)\varphi(n)$$

whenever $g.c.d.(m,n) = 1$.

If

$$n = p_1^{a_1}p_2^{a_2}...p_k^{a_k}$$

then

$$\varphi(n) = n(1 - \frac{1}{p_1})...(1 - \frac{1}{p_k})$$

In particular, if $n$ is the product of two primes, $n = pq$, then

$$\varphi(n) = \varphi(p)\varphi(q) = (p - 1)(q - 1)$$

There is the following generalization of Fermat’s Little Theorem.

**Euler’s theorem.** If $g.c.d.(a,m) = 1$ then

$$a^{\varphi(m)} \equiv 1 \pmod{m}.$$ 

**Proof.** Let $r_1, r_2, ..., r_{\varphi(m)}$ be classes of integers relatively prime to $m$. Such a system is called a reduced system of residues mod $m$. Then $ar_1, ar_2, ..., ar_{\varphi(m)}$ is another reduced system since $g.c.d.(a,m) = 1$. Therefore

$$ar_1 \equiv r_{\pi(1)}, ar_2 \equiv r_{\pi(2)}, ..., ar_{\varphi(m)} \equiv r_{\pi(m)} \pmod{m}$$

On multiplying these congruences, we get

$$a^{\varphi(m)}r_1r_2...r_{\varphi(m)} \equiv r_1r_2...r_{\varphi(m)} \pmod{m}$$

Now since $r_1r_2...r_{\varphi(m)}$ is relatively prime to $m$ the theorem is proved.$\square$

4 Public Key Cryptography and RSA Cryptosystem

First let us define some extra notions that we will use along with ones defined in the previous sections.

**Information channel** - a way to transmit information from one endpoint to another.

**Trusted channel** - an information channel where it is believed that is impossible to eavesdrop the transmitted information. For example military optical communication channels.
Public channel - an information channel where the transmitted information could be quite easily overheard. An example is the Internet.

Let us introduce our main characters: Alice, Bob and Eve. Alice wants to send ciphertext to Bob. Eve, the eavesdropper, wants to catch the ciphertext and break it, i.e. decipher without knowing the deciphering key. In our scheme in order to produce a ciphertext from the plaintext Alice has to have an enciphering key. In turn, Bob to read (decipher) the Alice’s ciphertext needs a deciphering key. If Alice and Bob use a private key cryptosystem, i.e. a cryptosystem where enciphering and deciphering keys could be easily produced one from another they come to the key distribution problem. Indeed Alice and Bob should use a trusted channel to share the keys.

From the first glance it seems to be impossible to get rid of the need of the secret channel. However in 1976 W. Diffie and M. Hellman \cite{1} discovered a new type of cryptosystem called public key cryptosystem where there is no key distribution problem at all. A public key cryptosystem has the property that having the enciphering key one cannot find the deciphering key without a prohibitively lengthy computation. In other words the enciphering function $f : \mathcal{P} \rightarrow \mathcal{C}$ is easy to compute if the enciphering key $K_E$ is known, but it is very hard to compute the inverse function $f^{-1} : \mathcal{C} \rightarrow \mathcal{P}$ without knowing the deciphering key $K_D$ even having the enciphering key $K_E$.

One of the most widely used public key cryptosystem is RSA - a cryptosystem named after the three inventors, Ron Rivest, Adi Shamir, and Leonard Adleman \cite{2}. The RSA cryptosystem is based on the fact that in order to factorise a big natural number with $N$ digits any classical computer needs at least a number of steps that grows faster than any polynomial in $N$. Faithfully speaking there is no rigorous proof of this fact but all known factoring algorithms obey this fact.

Let us describe RSA cryptosystem in more detail. First we describe the protocol, i.e. the steps our characters Alice and Bob should perform in order to allow Alice send enciphered messages to Bob. The mathematical basis of the RSA cryptosystem will be described in the next section.

### 4.1 The RSA Protocol

The RSA protocol solves the following problem. Bob wants to announce publicly a public key such that Alice using this key will send to him an enciphering message and nobody but Bob will be able to decipher it.

1. Bob generates public and private keys - each of them is a pair of two natural numbers - $(e, n)$ and $(d, n)$. Here $K_e = (e, n)$ is the enciphering key (public) and $K_d = (d, n)$ is the deciphering key (private).

   In order to generate public and private keys Bob does the following:

   a) Takes any two big prime numbers $p$ and $q$ and compute $n = pq$ and the value of the Euler function $\varphi(n) = (p - 1)(q - 1)$. In modern cryptosystems one uses $\log p \approx \log q \approx 1000$. 

   1.
b) Takes any \( e < n \), such that \( \gcd(e, \varphi(n)) = 1 \).

c) Computes \( d = e^{-1} \pmod{\varphi(n)} \), i.e. finds natural \( d \) such that

\[
ed \equiv 1 \pmod{\varphi(n)}, \ 1 \leq d < \varphi(n) \tag{5}\]

2. Bob sends a public key \((n, e)\) to Alice via a public channel.

3. Alice having Bob’s public key \((n, e)\) and a plaintext \(m\) (assume \(m\) is a natural number and \(m < n\)) that she wants to send to Bob computes

\[
c = m^e \pmod{n}
\]

and sends \(c\) (ciphertext) to Bob.

4. When Bob receives \(c\) from Alice he computes

\[
c^d \pmod{n}
\]

and gets the Alice’s plaintext \(m\), because \(m = c^d \pmod{n}\)

Nobody but Bob will be able to decipher Alice’s message.

4.2 Mathematical Basis of the RSA Protocol

In this section we will show why the RSA cryptosystem works. Then we will discuss the security of the protocol, i.e. how hard for Eve, the eavesdropper, to decipher the Alice’s message without knowing the private key.

If order to prove that RSA cryptosystem works we have to prove that the computations that Bob does on the step d). of the protocol is inverse to the computations that Alice does on the step c). That is

\[
c^d \equiv m \pmod{n}
\]

From (5) we have

\[
ed = 1 + k\varphi(n), \ k \in \mathbb{Z}
\]

We have

\[
c^d = m^{ed} = m \cdot m^{k\varphi(n)} \tag{6}
\]

Finally using the Euler’s theorem for the rhs of (5) we obtain

\[
c^d \equiv m \pmod{n}. \Box
\]
Now let us investigate the security of the RSA cryptosystem. It seems to be rather straightforward for Eve to obtain the Bob’s private key having his public key. The only thing she has to do is having \( n \) and \( e \) solve the congruence

\[
de e \equiv 1 \pmod{\varphi(n)}, \quad 1 \leq d < \varphi(n)
\]

The problem that Eve would face here is to compute \( \varphi(n) \). To this end she has to know \( p \) and \( q \), i.e. she has to solve the factoring problem. The practical solution of this problem is not possible with modern technology. For a discussion of this problem see for example [8].

5 Shannon’s Entropy and Mutual Information

Here we summarize some notions from information theory [10, 12, 13] used in quantum cryptography for the consideration of security of quantum cryptographic protocols.

Privacy is often expressed in terms of Shannon’s entropy or mutual information. Let \( (\Omega, \mathcal{F}, P) \) be a probability space and \( X, Y \) and \( Z \) three random variables taking values in a discrete set on the real line. Let \( p(x, y, z) = P(X = x \land Y = y \land Z = z) \) is the joint distribution, \( p(x, y) = P(X = x \land Y = y) \) is the marginal distribution, \( p(x|y) = P(X = x \mid Y = y) \) is the conditional distribution, and \( p(x) = P(X = x), p(y) = P(Y = y) \).

The Shannon entropy of \( X \) is given by

\[
H(X) = -\sum_x p(x) \log p(x).
\]

The mutual information between \( X \) and \( Y \) is given by

\[
I(X; Y) = \sum_{x, y} p(x, y) \log \left( \frac{p(x, y)}{p(x)p(y)} \right).
\]

The conditional Shannon entropy of \( X \) given \( Y \) is given by

\[
H(X \mid Y) = -\sum_{x, y} p(x, y) \log p(x \mid y).
\]

One has

\[
I(X; Y) = H(X) - H(X \mid Y) = H(Y) - H(Y \mid X).
\]

The conditional mutual information between \( X \) and \( Y \) given \( Z \) is

\[
I(X; Y \mid Z) = \sum_{x, y, z} p(x, y, z) \log \left( \frac{p(x, y \mid z)}{p(x \mid z)p(y \mid z)} \right).
\]
Quantum entropy of an observable $A$ in the state $\rho$ is defined by

$$H(A, \rho) = -\sum_i p(i, \rho) \log p(i, \rho)$$

where $p(\cdot, \rho)$ is the probability distribution of an observable $A$ in the state $\rho$. If the state $\rho$ is pure, i.e. $\rho = |\varphi\rangle\langle\varphi|$, where $\varphi$ is a unit vector in a Hilbert space, one can rewrite (7) as

$$H(A, \varphi) = -\sum_i |\langle\xi_i|\varphi\rangle|^2 \log |\langle\xi_i|\varphi\rangle|^2$$

where $\{|\xi_i\rangle\}$ is an orthonormal basis consisting from eigenvectors of the observable $A$.

6 Entropic Uncertainty Relations

The fundamental Heisenberg uncertainty relation is a particular case of the Robertson inequality

$$\Delta(A, \psi)\Delta(B, \psi) \geq \frac{1}{2} |\langle \psi | [A, B] | \psi \rangle|$$

where $A$ and $B$ are two observables and

$$\Delta(A, \psi) = \sqrt{\langle \psi | (A - \langle \psi | A | \psi \rangle)^2 | \psi \rangle}$$

Here we discuss a generalization of the uncertainty relation which uses the notions of entropy and mutual information.

**Theorem 1.** For any nondegenerate observables $A$ and $B$ in the finite dimensional Hilbert space the entropic uncertainty relation holds \[9, 10\]

$$H(A, \rho) + H(B, \rho) \geq -2 \log c$$

where $c$ is defined as the maximum possible overlap of the eigenstates of $A$ and $B$

$$c \equiv \max_{a,b} |\langle a | b \rangle|$$

Here $\{|a\rangle\}$ and $\{|b\rangle\}$ are orthonormal bases consisting from eigenvectors of $A$ and $B$ respectively.

One can check that for any nondegenerate observable $A$ in $N$-dimensional Hilbert space there exists an upper bound on the entropy

$$H(A, \rho) \leq \log N$$
Let us illustrate the entropic uncertainty relation on a simple spin-$\frac{1}{2}$ particle. Taking Pauli matrices
\begin{equation}
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{equation}
as observables with eigenstates
\begin{equation}
h_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad h_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\end{equation}
we compute $c = 1/\sqrt{2}$. Now taking 2 as a base of the logarithm, the relation (9) states that for any unit vector $\varphi \in \mathbb{C}^2$ it holds
\begin{equation}
\sum_{i=1,2} (|\langle e_i|\varphi\rangle|^2 \log |\langle e_i|\varphi\rangle|^2 + |\langle h_i|\varphi\rangle|^2 \log |\langle h_i|\varphi\rangle|^2) \leq -1
\end{equation}

Now we will formulate the uncertainty relation using the mutual information. Consider a quantum system which is described by density operator $\rho_i$ with probability $p_i$. Then the density operator of the whole ensemble $E = \{\rho_i\}$ of all possible states of the system is given by
\begin{equation}
\rho = \sum_i p_i \rho_i
\end{equation}
The mutual information corresponding to a measurement of an observable $A$ is given by
\begin{equation}
I(A, E) = H(A, \rho) - \sum_i p_i H(A, \rho_i)
\end{equation}
From (9) using (11) one can obtain the following theorem (information exclusion relation [11])

**Theorem 2.** Let $A$ and $B$ be arbitrary observables in $N$-dimensional Hilbert space, then
\begin{equation}
I(A, E) + I(B, E) \leq 2 \log Nc
\end{equation}
where $c$ is defined by (10).

### 7 The No Cloning Theorem

The eavesdropper, Eve, wants to have a perfect copy of Alice’s message. However Wootters and Zurek [14] proved that perfect copying is impossible in the quantum world.

It is instructive to start with the following
Proposition. If \( \mathcal{H} \) is a Hilbert space and \( \phi_0 \) is a vector from \( \mathcal{H} \) then there is no a linear map \( M : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H} \) with the property \( M(\psi \otimes \phi_0) = \psi \otimes \psi \) for any \( \psi \).

Proof. Indeed we would have

\[
M(2\psi \otimes \phi_0) = 2\psi \otimes 2\psi = 4\psi \otimes \psi
\]

But because of linearity we should have

\[
M(2\psi \otimes \phi_0) = 2M(\psi \otimes \phi_0) = 2\psi \otimes \psi
\]

This contradiction proves the claim. Now let us prove the no cloning theorem.

Theorem. Let \( \mathcal{H} \) and \( \mathcal{K} \) be two Hilbert spaces, \( \dim \mathcal{H} \geq 2 \). Let \( M \) be a a linear map (copy machine)

\[
M : \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{K} \to \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{K}
\]

with the property

\[
M(\psi \otimes \phi_0 \otimes \xi_0) = \psi \otimes \psi \otimes \eta_\psi
\]

for any \( \psi \in \mathcal{H} \) and some nonzero vectors \( \psi_0 \in \mathcal{H} \) and \( \xi_0 \in \mathcal{K} \) where \( \eta_\psi \in \mathcal{K} \) can depend on \( \psi \). Then \( M \) is a trivial map, \( M = 0 \) (i.e. \( \eta_\psi = 0 \) for any \( \psi \)).

Proof. Let \( \{e_i\} \) be an orthonormal basis in \( \mathcal{H} \). We have

\[
M(e_i \otimes \phi_0 \otimes \xi_0) = e_i \otimes e_i \otimes \eta_i
\]

where \( \eta_i \) are some vectors in \( \mathcal{K} \). To prove the theorem we prove that \( \eta_i = 0 \). If \( i \neq j \) then \( (e_i + e_j)/\sqrt{2} \) is a unit vector (here we use that \( \dim \mathcal{H} \geq 2 \)). We have the equality

\[
\frac{1}{\sqrt{2}}(e_i + e_j) \otimes \phi_0 \otimes \xi_0 = \frac{1}{\sqrt{2}}e_i \otimes \phi_0 \otimes \xi_0 + \frac{1}{\sqrt{2}}e_j \otimes \phi_0 \otimes \xi_0
\]

Let us apply the map \( M \) to both sides of this equality. Then we get

\[
\frac{1}{\sqrt{2}}(e_i + e_j) \otimes \phi_0 \otimes \xi_0 \otimes \eta_{ij} = \frac{1}{\sqrt{2}}e_i \otimes \phi_0 \otimes \xi_0 \otimes \eta_i + \frac{1}{\sqrt{2}}e_j \otimes \phi_0 \otimes \xi_0 \otimes \eta_j
\]

(15)

where \( \eta_{ij} \) is a vector in \( \mathcal{K} \). We can rewrite (15) as

\[
e_i \otimes e_i \otimes (\eta_{ij} - \eta_i) + e_i \otimes e_j \otimes \eta_{ij} + e_j \otimes e_i \otimes \eta_{ij} + e_j \otimes e_j \otimes (\eta_{ij} - \eta_j) = 0
\]

Now taking into account that \( e_i \) and \( e_j \) belong to a basis in \( \mathcal{H} \) we get

\[
\eta_{ij} - \eta_i = 0, \quad \eta_{ij} = 0, \quad \eta_i - \eta_j = 0
\]

Hence \( \eta_i = 0 \) for any \( i \) and Theorem is proved.

Remark. If \( \dim \mathcal{H} = 1 \), i.e. \( \mathcal{H} = \mathbb{C} \), then Theorem is not valid. For \( \phi_0 = 1 \) and \( \psi \in \mathbb{C} \) one can set \( M(\psi \xi_0) = \psi \xi_0 = \psi^2 \eta_\psi \) where \( \eta_\psi = \xi_0/\psi \) for \( \psi \neq 0 \).
We proved that Eve can not get a perfect quantum copy because perfect quantum copy machines can not exist. The possibility to copy classical information is one of the most crucial features of information needed for eavesdropping. The quantum no cloning theorem prevents Eve from perfect eavesdropping, and hence makes quantum cryptography potentially secure.

Note however that though there is no a perfect quantum cloning machine but there are cloning machines that achieve the optimal approximate cloning transformation compatible with the no cloning theorem, see [13, 14].

8 The BB84 Quantum Cryptographic Protocol

Quantum cryptographic protocols differ from the classical ones in that their security is based on the laws of quantum mechanics, rather than the conjectured computational difficulty of certain functions. In this section we will describe the Bennett and Brassard (BB84) quantum cryptographic protocol [4].

8.1 The BB84 Protocol

First let us describe the physical devices used by Alice and Bob.

Alice has a photon emitter - a device which is capable to emit single photons that are linear polarized in one of four directions. The polarizations are described by the four unit vectors in $\mathbb{C}^2$ here they are $e_1, e_2, h_1, h_2$ given in (13). We will call the polarizations vertical, horizontal, diagonal, anti-diagonal ones and denote them respectively ($|, —, \backslash, /$). We have two bases in $\mathbb{C}^2$. One basis, $G_z = \{e_1, e_2\}$, describes the vertical and horizontal polarizations. Another basis, $G_x = \{h_1, h_2\}$, describes the diagonal and anti-diagonal polarizations. Note that one has

$$|(e_i, h_j)| = 1/\sqrt{2}, \quad i, j = 1, 2$$

(16)

Bases with such a property are called conjugate. Note also that the vectors $e_1, e_2$ from the basis $G_z$ and $h_1, h_2$ from the basis $G_x$ are the eigenvectors of the Pauli matrices $\sigma_z$ and $\sigma_x$ respectively, see (12).

Bob has a photon detector - a device that detects single photons in one of the two bases.

Alice can send photons emitted by the photon emitter to Bob and Bob detects the photons with the photon detector.

The Protocol.

1. Alice chooses a random polarization basis and prepares photons with a random polarization that belongs to the chosen basis. She sends the photons to Bob.
2. For each photon Bob chooses at random which polarization basis he will use, and measures the polarization of the photon. (If Bob chooses the same basis as Alice he can for sure identify the polarization of the photon).

3. Alice and Bob use the public channel to compare the polarization bases they used. They keep only the polarization data for which the polarization bases are the same. In the absence of errors and eavesdropping these data should be the same on both sides, it is called a raw key.

4. At the last step Alice and Bob use methods of classical information theory to check whether their raw keys are the same. For example, they choose a random subset of the raw key and compare it using the public channel. They compute the error rate (that is, the fraction of data for which their values disagree). If the error rate is unreasonably high - above, say, 10% - they abort the protocol and may be try again later. If the error rate is not that high they could use error correction codes.

As a result of the protocol Alice and Bob share the same random data. This data could now be used as a private key in the symmetric cryptosystems.

Instead of polarized photons one can use any two level quantum system. One can consider also a generalized quantum key distribution protocol using a $d$-dimensional Hilbert space with $k$ bases, each basis has $d$ states, $[31, 32, 33, 34]$.

8.2 BB84 Security

In transmitting information, there are always some errors and Alice and Bob must apply some classical information processing protocols to improve their data. They can use error correction to obtain identical keys and privacy amplification to obtain a secret key. To solve the problem of eavesdropping one has to find a protocol which, assuming that Alice and Bob can only measure the error rate of the received data, either provides Alice and Bob with a secure key, or aborts the protocol and tells the parties that the key distribution has failed. There are various eavesdropping problems, depending in particular on the technological power which Eve could have and on the assumed fidelity of Alice and Bob’s devices, $[8, 16, 17]$.

There is a simple eavesdropping strategy, called intercept-resend. Eve measures each qubit in one of the two basis and resends to Bob a qubit in the state corresponding to the result of her measurement. This attack belongs to the class of the so called individual attacks. In this way Eve will get 50% information. However Alice and Bob can detect the actions of Eve because they will have 25% of errors in their sifted key. But it would be not so easy to detect eavesdropping if Eve applies the intercept-resend strategy to only a fraction of the Alice’s sending.

In this case one can use methods of classical cryptography. We suppose that once Alice, Bob and Eve have made their measurements, they will get classical random variables $\alpha, \beta$ and $\epsilon$ respectively, with a joint probability distribution $p(x, y, z)$. Let $I(\alpha, \beta)$ be the mutual information of Alice and Bob and $I(\alpha, \epsilon)$ and $I(\beta, \epsilon)$ the mutual
information of Alice and Eve and Bob and Eve respectively. Intuitively, it is clear that only if Bob has more information on Alice’s bits then Eve then it could be possible to establish a secret key between Alice and Bob. In fact one can prove (see [35, 3]) the following

**Theorem 1.** Alice and Bob can establish a secret key (using error correction and privacy amplification) if, and only if

\[ I(\alpha, \beta) \geq I(\alpha, \epsilon) \text{ or } I(\alpha, \beta) \geq I(\beta, \epsilon). \]

Let \( D \) be the error rate. Then one can prove that the BB84 protocol is secure against individual attacks if one has the following bound

\[ D < D_0 \equiv \frac{1 - 1/\sqrt{2}}{2} \approx 15\% \]

There have been discussed also more general coherent or joint attacks when Eve measures several qubits simultaneously. An important problem of the eavesdropping analysis is to find quantum cryptosystems for which one can prove its ultimate security. Ultimate security means that the security is guaranteed against the whole class of eavesdropping attacks, even if Eve uses any conceivable technology of future.

We assume that Eve has perfect technology which is only limited by the laws of quantum mechanics. This means she can use any unitary transformation between any number of qubits and an arbitrary auxiliary system. But Eve is not allowed to come to Alice’s lab and read all her data.

### 8.3 Ultimate Security Proofs

Main ideas on how to prove security of BB84 protocol were presented by D. Mayers [13] in 1996. The security issues are considered in recent papers [13, 18, 20, 21, 31]-[45]. We describe here a simple and general method proposed in [6, 32, 33]. The method is based on Theorem 1 from Sect. 8.2 on classical cryptography and on Theorem 2 from Sect. 6 on information uncertainty relations.

The argument runs as follows. Suppose Alice sends out a large number of qubits and Bob receives \( n \) of them in the correct basis. The relevant Hilbert space dimension is then \( 2^n \). Let us re-label the bases used for each of the \( n \) qubits in such a way that Alice used \( n \) times the \( x \)-basis. Hence, Bob’s observable is the \( n \)-time tensor product \( \sigma_x \otimes \ldots \otimes \sigma_x \). Since Eve had no way to know the correct bases, her optimal information on the correct ones is precisely the same as her optimal information on the incorrect ones. Hence one can bound her information assuming she measures \( \sigma_z \otimes \ldots \otimes \sigma_z \). Therefore \( c = 2^{-n/2} \) and Theorem 2 from Sect. 6 implies:

\[ I(\alpha, \epsilon) + I(\alpha, \beta) \leq 2 \log_2(2^n 2^{-n/2}) = n \] (17)
Next, combining the bound (17) with Theorem 1 from Sect. 8.2, one deduces that a secret key is achievable if \( I(\alpha, \beta) \geq n/2 \). Using
\[
I(\alpha, \beta) = n (1 - D \log_2 D - (1 - D) \log_2(1 - D))
\]
one obtains the sufficient condition on the error rate \( D \):
\[
-D \log_2 D - (1 - D) \log_2(1 - D) \leq \frac{1}{2}
\]
i.e. \( D \leq 11\% \). This bound was obtained in Mayers proof (after improvement by P. Shor and J. Preskill[21]). It is compatible with the 15\% bound found for individual attacks.

One can argue, however, that previous arguments lead in fact to another result: \( c = 2^{-n/4} \). Indeed, Bob’s observable is the \( n \)-time tensor product \( \sigma_x \otimes \ldots \otimes \sigma_x \). Now, since Eve had no way to know the correct basis it was assumed that she measures \( \sigma_z \otimes \ldots \otimes \sigma_z \). However it seems if Eve does not know the correct basis then her observables \( \sigma_i \) will be complementary observables to \( \sigma_x \) only in the half of cases. In the other half of cases her observables \( \sigma_i \) will be the same as Bob’s, i.e. \( \sigma_x \). Therefore one gets: \( c = (1/\sqrt{2})^{n/2} = 2^{-n/4} \). This leads to a lower error rate, instead of 11\% one gets 4\%.

9 The EPRBE Quantum Cryptographic Protocol

9.1 Quantum Nonlocality and Cryptography

Bell’s theorem [22] states that there are quantum correlation functions that cannot be represented as classical correlation functions of separated random variables. It has been interpreted as incompatibility of the requirement of locality with the statistical predictions of quantum mechanics [22]. For a recent discussion of Bell’s theorem see, for example [23] - [30] and references therein. It is now widely accepted, as a result of Bell’s theorem and related experiments, that ”local realism” must be rejected.

Bell’s theorem constitutes an important part in quantum cryptography [5]. It is now generally accepted that techniques of quantum cryptography can allow secure communications between distant parties. The promise of some secure cryptographic quantum key distribution schemes is based on the use of quantum entanglement in the spin space and on quantum no-cloning theorem. An important contribution of quantum cryptography is a mechanism for detecting eavesdropping.

Let us stress that the very formulation of the problem of locality in quantum mechanics is based on ascribing a special role to the position in ordinary three-dimensional space. However the space dependence of the wave function is neglected in many discussions of the problem of locality in relation to Bell’s inequalities. Actually it is the
space part of the wave function which is relevant to the consideration of the problem of locality.

It was pointed out in [25] that the space part of the wave function leads to an extra factor in quantum correlation which changes the Bell equation. It was suggested a criterion of locality (or nonlocality) of quantum theory in a realist model of hidden variables. In particular predictions of quantum mechanics can be consistent with Bell’s inequalities for some Gaussian wave functions.

If one neglects the space part of the wave function in a cryptographic scheme then such a scheme could be insecure in the real three-dimensional space.

We will discuss how one can try to improve the security of quantum cryptography schemes in space by using a special preparation of the space part of the wave function, see [29].

9.2 Bell’s Inequalities

In the presentation of Bell’s theorem we will follow [25] where one can find also more references, see [30] for more details. The mathematical formulation of Bell’s theorem reads:

\[
\cos(\alpha - \beta) \neq E\xi_{\alpha}\eta_{\beta}
\]  

where \(\xi_{\alpha}\) and \(\eta_{\beta}\) are two random processes such that \(|\xi_{\alpha}| \leq 1\), \(|\eta_{\beta}| \leq 1\) and \(E\) is the expectation. Let us discuss in more details the physical interpretation of this result.

Consider a pair of spin one-half particles formed in the singlet spin state and moving freely towards two detectors (Alice and Bob). If one neglects the space part of the wave function then the quantum mechanical correlation of two spins in the singlet state \(\psi_{\text{spin}}\) is

\[
D_{\text{spin}}(a, b) = \langle \psi_{\text{spin}} | \sigma \cdot a \otimes \sigma \cdot b | \psi_{\text{spin}} \rangle = -a \cdot b
\]  

Here \(a\) and \(b\) are two unit vectors in three-dimensional space, \(\sigma = (\sigma_1, \sigma_2, \sigma_3)\) are the Pauli matrices and

\[
\psi_{\text{spin}} = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)
\]

Bell’s theorem states that the function \(D_{\text{spin}}(a, b)\) Eq. (19) can not be represented in the form

\[
P(a, b) = \int \xi(a, \lambda)\eta(b, \lambda) d\rho(\lambda)
\]  

i.e.

\[
D_{\text{spin}}(a, b) \neq P(a, b)
\]  

20
Here $\xi(a, \lambda)$ and $\eta(b, \lambda)$ are random fields on the sphere, $|\xi(a, \lambda)| \leq 1$, $|\eta(b, \lambda)| \leq 1$ and $d\rho(\lambda)$ is a positive probability measure, $\int d\rho(\lambda) = 1$. The parameters $\lambda$ are interpreted as hidden variables in a realist theory. It is clear that Eq. (21) can be reduced to Eq. (18).

One has the following Bell-Clauser-Horn-Shimony-Holt (CHSH) inequality

$$|P(a, b) - P(a, b') + P(a', b) + P(a', b')| \leq 2 \quad (22)$$

From the other hand there are such vectors $(ab = a'b = a'b' = -ab' = \sqrt{2}/2)$ for which one has

$$|D_{\text{spin}}(a, b) - D_{\text{spin}}(a, b') + D_{\text{spin}}(a', b) + D_{\text{spin}}(a', b')| = 2\sqrt{2} \quad (23)$$

Therefore if one supposes that $D_{\text{spin}}(a, b) = P(a, b)$ then one gets the contradiction.

It will be shown below that if one takes into account the space part of the wave function then the quantum correlation in the simplest case will take the form $g \cos(\alpha - \beta)$ instead of just $\cos(\alpha - \beta)$ where the parameter $g$ describes the location of the system in space and time. In this case one can get the representation $|25|

$$g \cos(\alpha - \beta) = E\xi_\alpha\eta_\beta \quad (24)$$

if $g$ is small enough (see below). The factor $g$ gives a contribution to visibility or efficiency of detectors that are used in the phenomenological description of detectors.

### 9.3 Localized Detectors

In the previous section the space part of the wave function of the particles was neglected. However exactly the space part is relevant to the discussion of locality. The complete wave function is $\psi = (\psi_{\alpha,\beta}(\mathbf{r}_1, \mathbf{r}_2))$ where $\alpha$ and $\beta$ are spinor indices and $\mathbf{r}_1$ and $\mathbf{r}_2$ are vectors in three-dimensional space.

We suppose that Alice and Bob have detectors which are located within the two localized regions $\mathcal{O}_A$ and $\mathcal{O}_B$ respectively, well separated from one another.

Quantum correlation describing the measurements of spins by Alice and Bob at their localized detectors is

$$G(a, \mathcal{O}_A, b, \mathcal{O}_B) = \langle \psi | \sigma \cdot a P_{\mathcal{O}_A} \otimes \sigma \cdot b P_{\mathcal{O}_B} | \psi \rangle \quad (25)$$

Here $P_{\mathcal{O}}$ is the projection operator onto the region $\mathcal{O}$.

Let us consider the case when the wave function has the form of the product of the spin function and the space function $\psi = \psi_{\text{spin}} \phi(\mathbf{r}_1, \mathbf{r}_2)$. Then one has

$$G(a, \mathcal{O}_A, b, \mathcal{O}_B) = g(\mathcal{O}_A, \mathcal{O}_B) D_{\text{spin}}(a, b) \quad (26)$$
where the function
\[ g(O_A, O_B) = \int_{O_A \times O_B} |\phi(r_1, r_2)|^2 dr_1 dr_2 \] (27)
describes correlation of particles in space. It is the probability to find one particle in the region \( O_A \) and another particle in the region \( O_B \).

One has
\[ 0 \leq g(O_A, O_B) \leq 1 \] (28)

**Remark.** In relativistic quantum field theory there is no nonzero strictly localized projection operator that annihilates the vacuum. It is a consequence of the Reeh-Schlieder theorem. Therefore, apparently, the function \( g(O_A, O_B) \) should be always strictly smaller than 1.

Now one inquires whether one can write the representation
\[ g(O_A, O_B) D_{\text{spin}}(a, b) = \int \xi(a, O_A, \lambda) \eta(b, O_B, \lambda) d\rho(\lambda) \] (29)

Note that if we are interested in the conditional probability of finding the projection of spin along vector \( a \) for the particle 1 in the region \( O_A \) and the projection of spin along the vector \( b \) for the particle 2 in the region \( O_B \) then we have to divide both sides of Eq. (29) by \( g(O_A, O_B) \).

The factor \( g \) is important. In particular one can write the following representation for \( 0 \leq g \leq 1/2 \):
\[ g \cos(\alpha - \beta) = \int_0^{2\pi} \sqrt{2g \cos(\alpha - \lambda)} \sqrt{2g \cos(\beta - \lambda)} d\lambda \frac{d\lambda}{2\pi} \] (30)

Let us now apply these considerations to quantum cryptography.

### 9.4 The EPRBE Quantum Key Distribution

Ekert [5] showed that one can use the Einstein-Podolsky-Rosen correlations to establish a secret random key between two parties ("Alice" and "Bob"). Bell’s inequalities are used to check the presence of an intermediate eavesdropper ("Eve"). We will call this method the Einstein-Podolsky-Rosen-Bell-Ekert (EPRBE) quantum cryptographic protocol. There are two stages to the EPRBE protocol, the first stage over a quantum channel, the second over a public channel.

The quantum channel consists of a source that emits pairs of spin one-half particles, in a singlet state. The particles fly apart towards Alice and Bob, who, after the particles have separated, perform measurements on spin components along one of three
directions, given by unit vectors $a$ and $b$. In the second stage Alice and Bob communicate over a public channel. They announce in public the orientation of the detectors they have chosen for particular measurements. Then they divide the measurement results into two separate groups: a first group for which they used different orientation of the detectors, and a second group for which they used the same orientation of the detectors. Now Alice and Bob can reveal publicly the results they obtained but within the first group of measurements only. This allows them, by using Bell’s inequality, to establish the presence of an eavesdropper (Eve). The results of the second group of measurements can be converted into a secret key. One supposes that Eve has a detector which is located within the region $O_E$ and she is described by hidden variables $\lambda$.

We will interpret Eve as a hidden variable in a realist theory and will study whether the quantum correlation Eq. (26) can be represented in the form Eq. (20). From (22), (23) and (24) one can see that if the following inequality

$$g(O_A, O_B) \leq 1/\sqrt{2} \quad (31)$$

is valid for regions $O_A$ and $O_B$ which are well separated from one another then there is no violation of the CHSH inequalities (22) and therefore Alice and Bob can not detect the presence of an eavesdropper. On the other side, if for a pair of well separated regions $O_A$ and $O_B$ one has

$$g(O_A, O_B) > 1/\sqrt{2} \quad (32)$$

then it could be a violation of the realist locality in these regions for a given state. Then, in principle, one can hope to detect an eavesdropper in these circumstances.

Note that if we set $g(O_A, O_B) = 1$ in (29) as it was done in the original proof of Bell’s theorem, then it means we did a special preparation of the states of particles to be completely localized inside of detectors. There exist such well localized states (see however the previous Remark) but there exist also another states, with the wave functions which are not very well localized inside the detectors, and still particles in such states are also observed in detectors. The fact that a particle is observed inside the detector does not mean, of course, that its wave function is strictly localized inside the detector before the measurement. Actually one has to perform a thorough investigation of the preparation and the evolution of our entangled states in space and time if one needs to estimate the function $g(O_A, O_B)$.

### 9.5 Gaussian Wave Functions

Now let us consider the criterion of locality for Gaussian wave functions. We will show that with a reasonable accuracy there is no violation of locality in this case. Let us take the wave function $\phi$ of the form $\phi = \psi_1(r_1)\psi_2(r_2)$ where the individual wave functions
have the moduli

\[ |\psi_1(r)|^2 = \left(\frac{m^2}{2\pi}\right)^{3/2} e^{-m^2 r^2}, \quad |\psi_2(r)|^2 = \left(\frac{m^2}{2\pi}\right)^{3/2} e^{-m^2 (r-\ell)^2/2} \]

We suppose that the length of the vector \(\ell\) is much larger than \(1/m\). We can make measurements of \(P_{O_A}\) and \(P_{O_B}\) for any well separated regions \(O_A\) and \(O_B\). Let us suppose a rather nonfavorite case for the criterion of locality when the wave functions of the particles are almost localized inside the regions \(O_A\) and \(O_B\) respectively. In such a case the function \(g(O_A, O_B)\) can take values near its maximum. We suppose that the region \(O_A\) is given by \(|r_i| < 1/m\), \(r = (r_1, r_2, r_3)\) and the region \(O_B\) is obtained from \(O_A\) by translation on \(\ell\). Hence \(\psi_1(r_1)\) is a Gaussian function with modules appreciably different from zero only in \(O_A\) and similarly \(\psi_2(r_2)\) is localized in the region \(O_B\). Then we have

\[ g(O_A, O_B) = \left(\frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-x^2/2} dx\right)^6 \]

One can estimate (34) as

\[ g(O_A, O_B) < \left(\frac{2}{\pi}\right)^3 \]

which is smaller than \(1/2\). Therefore the locality criterion (31) is satisfied in this case.

Let us remind that there is a well known effect of expansion of wave packets due to the free time evolution. If \(\epsilon\) is the characteristic length of the Gaussian wave packet describing a particle of mass \(M\) at time \(t = 0\) then at time \(t\) the characteristic length \(\epsilon_t\) will be

\[ \epsilon_t = \epsilon \sqrt{1 + \frac{\hbar^2 t^2}{M^2 \epsilon^4}}. \]

It tends to \((\hbar/M\epsilon)t\) as \(t \to \infty\). Therefore the locality criterion is always satisfied for nonrelativistic particles if regions \(O_A\) and \(O_B\) are far enough from each other.

### 10 Conclusions

In quantum cryptography there are many interesting open problems which require further investigations. In quantum cryptographic protocols with two entangled photons (such as the EPRBE protocol) to detect the eavesdropper’s presence by using Bell’s inequality we have to estimate the function \(g(O_A, O_B)\). In order to increase the detectability of the eavesdropper one has to do a thorough investigation of the process
of preparation of the entangled state and then its evolution in space and time towards Alice and Bob. One has to develop a proof of the security of such a protocol.

In the previous section Eve was interpreted as an abstract hidden variable. However one can assume that more information about Eve is available. In particular one can assume that she is located somewhere in space in a region $O_E$. It seems that one has to study a generalization of the function $g(O_A, O_B)$, which depends not only on the Alice and Bob locations $O_A$ and $O_B$ but also on Eve’s location $O_E$. Then one can try to find a strategy which leads to an optimal value of this function.

In quantum cryptographic protocols with single photons (such as the BB84 protocol) further investigation of the security under various types of attacks, including attacks from real space, would be desirable.

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