Constructing all entanglement witnesses from density matrices

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We demonstrate a general procedure to construct entanglement witnesses for any entangled state. This procedure is based on the trace inequality and a general form of entanglement witnesses, which is in the form \(W = \rho - c_\rho I\), where \(\rho\) is a density matrix, \(c_\rho\) is a non-negative number related to \(\rho\), and \(I\) is the identity matrix. The general form of entanglement witnesses is deduced from Choi-Jamiołkowski isomorphism, that can be reinterpreted as that all quantum states can be obtained by a maximally quantum entangled state pass through certain completely positive maps. Furthermore, we provide the necessary and sufficient condition of the entanglement witness \(W = \rho - c_\rho I\) in operation, as well as in theory.

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Quantum entanglement, which is applied to various types of quantum information processing such as quantum computation\textsuperscript{1}, quantum dense coding\textsuperscript{2}, quantum teleportation\textsuperscript{3}, quantum cryptography\textsuperscript{4}, etc., has been incorporated as a central notion in quantum information theory\textsuperscript{5}. It is well known that entanglement can be identified by applying all positive but not completely positive (PNCP) maps to a given state\textsuperscript{6,7}. However, it is not easy to find and physically realize PNCP maps\textsuperscript{8}. An equivalent approach of identifying entanglement is based on entanglement witnesses (EWs)\textsuperscript{5,7}. EWs are observables that completely characterize separable (not entangled) states and allow us to detect entanglement physically\textsuperscript{8}. This make EWs one of the main methods of physically detecting entanglement.

Constructing the EW for an entangled state is a difficult task, and the determination of EWs for all entangled states is a nondeterministic polynomial-time (NP) hard problem\textsuperscript{8,11}. Concerning this topic, much work has been done for constructing special EWs (for example, Refs.\textsuperscript{9,10,13}). In this Brief Report, we show a general form of EWs from density matrices. This general form of entanglement witnesses is deduced from a known relation; that is, any quantum state can actually be generated from a maximally entangled quantum state with a completely positive map. The trace inequality indicates an EW can be built up as commuting with a given entangled state. Therefore, we provide a general procedure for detecting entangled quantum states. Furthermore, we provide the necessary and sufficient condition of the entanglement witness \(W = \rho - c_\rho I\) both in operation and in theory.

For our purpose, we first consider quantum states on the finite dimensional Hilbert space \(\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B\). Let \(\dim(\mathcal{H}_A) = d_A\), \(\dim(\mathcal{H}_B) = d_B\), and \(\dim(\mathcal{H}_{AB}) = d_{AB}\). Denote \(P_+\) as the density matrix of the maximally quantum entangled state \(|\beta\rangle = d_A^{-1/2} \sum_i |i\rangle \otimes |i\rangle\) on \(\mathcal{H}_A \otimes \mathcal{H}_A\), where \(|i\rangle\}_{i=0}^{d_A-1}\) are computational bases in \(\mathcal{H}_A\), i.e. \(P_+ = |\beta\rangle \langle \beta|\).

The relation between any quantum state and a maximally entangled state. Quantum entanglement, which is a fascinating feature of quantum theory, underlines the intrinsic order of statistical relations between subsystems of a compound quantum system\textsuperscript{14}. In the following, we show a relation between all quantum states and a maximally entangled state. This relation comes from a well-known feature called “channel-state duality” or “Jamiołkowski isomorphism” or “Choi-Jamiołkowski isomorphism”. This result first appeared in Ref. \textsuperscript{15} with a proof. We now show it in a different manner.

**Lemma 1.** Any matrix \(H\) on \(\mathcal{H}_{AB}\) is a Hermitian matrix if and only if it can always be written as

\[
H = (I \otimes \Phi)(P_+)
\]

where \(\Phi\) is a hermiticity-preserving linear map.

**Lemma 2.**\textsuperscript{16} A linear map \(\Lambda: \mathcal{H}_A \rightarrow \mathcal{H}_B\) is completely positive if and only if the matrix \(D \in \mathcal{H}_{AB}\) given by

\[
D = (I \otimes \Lambda)(P_+)
\]

is positive semidefinite.

By Lemmas 1 and 2, we can get the following result.

**Theorem 1.** Any matrix \(\rho\) on \(\mathcal{H}_{AB}\) is a bipartite density matrix if and only if it can always be written as

\[
\rho = (I \otimes \Lambda)(P_+),
\]

where \(\Lambda: \mathcal{H}_A \rightarrow \mathcal{H}_B\) is a completely positive map.

Note that \(\Lambda\) may not be trace preserving and \(\rho\) may not be normalized. A similar result on \(\mathcal{H}_A \otimes \mathcal{H}_A\) was shown by DiVincenzo et al. \textsuperscript{17}. If \(\Lambda\) is trace preserving, \(\Lambda\) is an entanglement-breaking channel (EBC)\textsuperscript{18}, but not vice versa\textsuperscript{19}.

**The general form of entanglement witnesses.** Starting from the positive maps\textsuperscript{5}, the concept of EW was applied to detecting the presence of entanglement\textsuperscript{20}. EWs
are observables whose expectation value can reveal something about the entanglement in a given state. A Hermitian matrix \( W = W^\dagger \) on \( \mathcal{H}_{AB} \) is an EW if it has (i) at least one negative eigenvalue and (ii) nonnegative mean values in all separable quantum states, or equivalently satisfy

\[
\langle \mu_A \nu_B | W | \mu_A \nu_B \rangle \geq 0
\]

for all product states \( | \mu_A \nu_B \rangle \). If we have negative mean value in a quantum state for an EW, the quantum state is entangled. In that case, we say that the EW “witnesses” (detects) the quantum state. To balance out the “not trace-preserving” property of the completely positive map in this Brief Report, we need another property of EWs; (iii) if \( W \) is an EW, \( \gamma W \) keeps all properties of \( W \) as an EW for a non-negative number \( \gamma \). Note that the third property is different from the definition by Lewenstein et al. for comparing the action of different EWs.

**Theorem 2.** Any bipartite density matrix \( \pi \) is entangled if and only if there exists a density matrix \( \rho \) and a non-negative number \( c_\rho \) such that the matrix

\[
W = \rho - c_\rho I
\]

satisfies \( \text{tr}(W \pi) < 0 \) and \( \text{tr}(W \sigma) \geq 0 \) for all separable states \( \sigma \).

This result shows that every entangled state in a composite system has an EW in the simple form \( W = \rho - c_\rho I \), where \( c_\rho \) is a non-negative number and \( \rho \) is a density matrix. This result also shows that the research on density matrices can reduce the research on EWs since \( c_\rho I \) is simple.

**Proof.** By Lemma 1, any EW \( W' \) can be written as

\[
W' = (I \otimes \Theta)(P_+),
\]

where \( \Theta \) is a positive map. By property (iii) of EWs,

\[
W = (1 - p)W' = (1 - p)(I \otimes \Theta)(P_+)
\]

is the same EW as \( W' \) for \( 0 < p < 1 \).

We could mix \( (1 - p) \otimes \Theta \) with a simple completely positive map \( \rho \Lambda_\Theta \): \( (1 - p)\Theta + p\Lambda_\Theta \). By structurally completely positive approximation (SCPA) and structural physical approximation (SPA) \([23, 24]\) for proper \( p \),

\[
[I \otimes ((1 - p)\Theta + p\Lambda_\Theta)](P_+) = I \otimes \Lambda'(P_+) = \rho,
\]

where \( \Lambda' = (1 - p)\Theta + p\Lambda_\Theta \) is a completely positive map and \( \rho \) is a density matrix by Theorem 1. Note that \( \Lambda' \) could not be non trace preserving. Rewriting Eq. \([8]\),

\[
\rho - p(I \otimes \Lambda_\Theta)(P_+) = (1 - p)(I \otimes \Theta)(P_+) = (1 - p)W'.
\]

Without loss of generality, let \( \Lambda_\Theta(x) = \sum_{ij} E_{ij} \rho_{ij} E_{ij}^\dagger \), where \( E_{ij} = |i\rangle \langle j| \) and \( \{|i\rangle\}_{i=0}^{d_A-1} \) are computational bases in \( \mathcal{H}_A \), \( \{|j\rangle\}_{j=0}^{d_B-1} \) in \( \mathcal{H}_B \). We have \( I \otimes \rho \Lambda_\Theta(P_+) = \frac{\rho}{\text{d}_A \text{d}_B} \).

By Eqs. \([7]\) and \([9]\), we have

\[
W = \rho - p(I \otimes \Lambda_\Theta)(P_+) = \rho - c_\rho I.
\]

where \( c_\rho \) is a non-negative number.

Let \( F = \rho + (1 - c_\rho)I \). By Eq. \([10]\), \( W = I - F \). This is the form of EWs in Refs. \([7]\) and \([25]\). Clearly, all EWs in \([25]\) can be constructed in the form of \( W = \rho - c_\rho I \). A general discussion can be found in Refs. \([7]\) and \([25]\). Moreover, we can also easily obtain Theorem 2 mathematically from the EW in the form \( \rho - c_\rho I \) (Hermitian matrix), where \( \rho \) is a positive matrix. However, they all have no physical interpretation.

We can prove that \( \lambda_{\text{min}}(\rho) < c_\rho \leq d_{\text{min}}(\rho) \) if \( W = \rho - c_\rho I \) is an EW, where \( \lambda_{\text{min}}(\rho) \) and \( d_{\text{min}}(\rho) \) are the minimum eigenvalue of \( \rho \) and the minimum diagonal element in \( \rho \), respectively. However, for any density matrix \( \rho \), such as the diagonal state (its density matrix is a diagonal matrix), the EW in the form \( W = \rho - c_\rho I \) does not always exist. Generally, it is not easy for any density matrix \( \rho \) to find \( c_\rho \) to make \( W = \rho - c_\rho I \) an EW, but we have the following operational result.

**Theorem 3.** A Hermitian matrix \( W = \rho - c_\rho I \) is an EW for any density matrix \( \rho \), if \( \lambda_{\text{min}}(\rho) < c_\rho \leq \rho_{\text{max}} = \sum_{ij} |d_i|^2 |f_j|^2 \rho_{ij} - 2\sum_{i<j} 2(|d_i|^2 i^2 |f_i|^2 |f_j|^2) \overline{\text{Re}(\rho_{ij})} - \sum_{i<k,j<l} 2(|d_i|^2 |d_k|^2 + |f_j|^2 |f_l|^2) \overline{\text{Re}(\rho_{ijkl})} \), where \( \sum_i |d_i|^2 = 1, \sum_j |f_j|^2 = 1 \) and \( \overline{\text{Re}(\rho_{ijkl})} \) is the real part of the element \( \rho_{ijkl} \) of \( \rho \).

Since any density matrix \( \rho \) can be written as

\[
\rho = \sum_{ijkl} a_{ijkl} + \sum_{ijkl,i\neq k,j\neq l} b_{ijkl},
\]

where \( a_{ijkl} \) and \( b_{ijkl} \) are real, \( b_{ijkl} = -b_{klij} \), and \( \overline{\text{Re}(\rho_{ijkl})} \sum_{ijkl,i\neq k,j\neq l} b_{ijkl} = 0 \), we can only consider real parts of a density matrix.

**Proof.** Any density matrix \( \rho \) on \( \mathcal{H}_{AB} \) can be defined as

\[
\rho = \sum_{ijkl} |i\rangle \langle j| \langle k| \otimes |l\rangle
\]

by computational (real orthonormal) bases \( \{|i\rangle\}_{i=0}^{d_A-1} \) and \( \{|k\rangle\}_{k=0}^{d_B-1} \) in \( \mathcal{H}_A \), and \( \{|j\rangle\}_{j=0}^{d_B-1} \) and \( \{|l\rangle\}_{l=0}^{d_B-1} \) in \( \mathcal{H}_B \).

For any unit product vector \( |\mu_A \nu_B\rangle = \sum_{ij} d_i f_j |i\rangle |j\rangle \) on \( \mathcal{H}_{AB} \) with \( \sum_i |d_i|^2 = 1 \) and \( \sum_j |f_j|^2 = 1 \), by Eq.
\[ \text{tr} [\rho (|\mu_A \nu_B \rangle \langle \mu_A \nu_B |)] = \langle \mu_A \nu_B | \rho (|\mu_A \nu_B \rangle \langle \mu_A \nu_B |) |\mu_A \nu_B \rangle \langle \mu_A \nu_B |] \]

\[ = \langle \mu_A \nu_B | \sum_{ijkl} (ij)(kl) \langle i| \otimes \langle j| \langle k| \otimes \langle l| \rangle |\mu_A \nu_B \rangle \langle \mu_A \nu_B |] \]

\[ = \sum_{ijkl} d_{ijkl} f_{ijkl} \rho_{ijkl} \]

\[ = \sum_{ijkl} 2 \text{Re}(d_{ijkl}^* f_{ijkl}) \rho_{ijkl} \]

\[ + \sum_{ijkl} 2 \text{Re}(d_{ijkl}^* f_{ijkl}) \rho_{ijkl} \]

\[ + \sum_{ijkl} 2 \text{Re}(d_{ijkl}^* f_{ijkl}) \rho_{ijkl} \]

\[ = \sum_{ijkl} 2 \text{Re}(d_{ijkl}^* f_{ijkl}) + |d_{ijkl}|^2 + |f_{ijkl}|^2 - |d_{ijkl}|^2 |f_{ijkl}|^2 \]

\[ - |d_{ijkl}|^2 |f_{ijkl}|^2 \rho_{ijkl} \]

\[ \geq \sum_{ijkl} |d_{ijkl}|^2 |f_{ijkl}|^2 \rho_{ijkl} - \sum_{ijkl} 2(|d_{ijkl}|^2 + |f_{ijkl}|^2) \rho_{ijkl} \]

\[ - \sum_{ijkl} 2(|d_{ijkl}|^2 |f_{ijkl}|^2) \rho_{ijkl} \]

where \( \rho_{ijkl} \) are the diagonal elements of \( \rho \). Thus, if \( W = \rho - c_p I \), \( \text{tr}(W \sigma) \geq 0 \) for all separable states \( \sigma \).

A general procedure for detecting entangled states. Generally, for any entangled state \( \pi \), it is not easy to find \( p \) to make \( W = \rho - c_p I \) detecting it, but we have the following operational result. Let us recall a well-known trace inequality for Hermitian matrices.

**Lemma 3.** \((\text{20}, \text{27})\) For any two Hermitian matrices \( H, K \) in \( \mathcal{H}_{AB} \),

\[ \sum_{i=0}^{d_{AB}-1} \lambda_i(H) \lambda_{d_{AB}-i}(K) \leq \text{tr}(HK) \leq \sum_{i=0}^{d_{AB}-1} \lambda_i(H) \lambda_i(K), \]

where \( \lambda_1(H) \geq \cdots \geq \lambda_{d_{AB}-1}(H), \lambda_1(K) \geq \cdots \geq \lambda_{d_{AB}-1}(K) \), and \( \lambda_i(H) \) and \( \lambda_i(K) \) are the eigenvalues of \( H \) and \( K \), respectively.

If \( W = \rho - c_p I \) is the EW for any entangled density matrix \( \pi \), \( \text{tr}(W \pi) = \text{tr}(\rho \pi) - c_p < 0 \). It requires \( \text{tr}(\rho \pi) \) as small as possible and \( c_p \) as big as possible to make \( \text{tr}(\rho \pi) - c_p < 0 \). By Lemma 3, it is not difficult to conclude that the minimum of \( \text{tr}(\rho \pi) \) is equal to \( \sum_{i=0}^{d_{AB}-1} \lambda_i(\rho) \lambda_{d_{AB}-i}(\pi) \) if and only if \( \rho \) and \( \pi \) are simultaneously diagonalizable. Therefore, we have the following result.

**Theorem 4.** An EW can be built up as commuting with a given entangled state.

By Theorem 4, we have a general procedure of constructing EW for any density matrix.
to be an EW in theory? Let us start from the following results.

**Theorem 5.** For any density matrix $\rho$ with spectral decomposition $\rho = \sum_r \lambda_r \langle \psi_r | \psi_r \rangle$, if $c_\rho \leq c_{\rho}^{\text{max}}$ for any unit product vector $|\mu_{AB}\rangle$, $\text{tr}(W\sigma) \geq 0$ for all separable states $\sigma$, where $c_{\rho}^{\text{max}} = \inf \|\mu_{AB}\| = 1, \|\sigma\| = 1 \sum_r \lambda_r \| \langle \psi_r | \mu_{AB} \rangle \|^2$ and $W = \rho - c_\rho I$.

**Proof.** For any unit product vector $|\mu_{AB}\rangle$, by Eq. (19) and $\rho = \sum_r \lambda_r \langle \psi_r | \psi_r \rangle$, we have

$$\text{tr}[W(|\mu_{AB}\rangle \langle \mu_{AB}|)] = \langle \mu_{AB}| \rho |\mu_{AB}\rangle - c_\rho,$$

$$= \langle \mu_{AB}| \sum_r \lambda_r \langle \psi_r | \psi_r \rangle |\mu_{AB}\rangle - c_\rho$$

$$= \sum_r \lambda_r \| \psi_r \| |\mu_{AB}\| \| \langle \psi_r | \mu_{AB} \rangle \|^2 - c_\rho$$

$$\geq c_{\rho}^{\text{max}} \sum_r \lambda_r \delta - c_\rho \geq 0,$$

where $c_{\rho}^{\text{max}} = \inf \|\mu_{AB}\| = 1, \|\sigma\| = 1 \sum_r \lambda_r \| \langle \psi_r | \mu_{AB} \rangle \|^2$.

**Corollary 1.** For any density matrix $\rho$, $W = \rho - c_\rho I$ is an EW if and only if $\lambda_{\text{min}}(\rho) < c_\rho \leq c_{\rho}^{\text{max}}$, where $\lambda_{\text{min}}(\rho)$ is the the minimum eigenvalue and $c_{\rho}^{\text{max}} = \inf \|\mu_{AB}\| = 1, \|\sigma\| = 1 \sum_r \lambda_r \| \langle \psi_r | \mu_{AB} \rangle \|^2$.

Before giving our result in theory, we need the following lemmas.

**Lemma 4.** A linear map $\Theta: \mathcal{H}_A \to \mathcal{H}_B$ is positive if and only if there exists $C_0, \ldots, C_{k-1}, D_0, \ldots, D_{l-1}$ on $\mathcal{H}_{AB}$ such that $\{D_j\}_{j=0}^{l-1}$ is a contractive locally linear combination of $\{C_i\}_{i=0}^{k-1}$ and

$$\Theta(X) = \sum_{i=0}^{k-1} C_i X C_i^\dagger J - \sum_{j=0}^{l-1} D_j X D_j^\dagger J$$

for all $X$ on $\mathcal{H}_{AB}$.

**Lemma 5.** Furthermore, $\Theta$ in Eq. (22) is completely positive if and only if $\{D_j\}_{j=0}^{l-1}$ is a contractive locally linear combination of $\{C_i\}_{i=0}^{k-1}$ with a contractive coefficient matrix.

**Theorem 6.** A Hermitian matrix $W = \rho - c_\rho I$ is an EW for any density matrix $\rho = (I \otimes \Lambda)(P_\rho)$ if and only if $\{\sqrt{d_{AB}c_\rho E_{ij}}\}_{i=0}^{k-1}$ is a contractive locally linear combination of $\{U_r\}_{r=0}^{k-1}$ but not a linear combination of $\{U_r\}_{r=0}^{k-1}$ with a contractive coefficient matrix, where $\Lambda(\cdot) = \sum_{r=0}^{k-1} U_r(\cdot) U_r^\dagger$, $E_{ij} = |i\rangle \langle j|$ and $\{|i\rangle\}_{i=0}^{d_{AB}-1}$ are computational bases in $\mathcal{H}_A, \{|j\rangle\}_{j=0}^{d_{AB}-1}$ in $\mathcal{H}_B$.

**Proof.** Any density matrix $\rho$ can be written as

$$\rho = \sum_r \lambda_r \langle \psi_r | \psi_r \rangle$$

by means of its spectral decomposition with nonnegative eigenvalues $\lambda_r$. Taking $V_r$ such that $I \otimes V_r |\beta\rangle = |\psi_r\rangle$, any density matrix $\rho$ can be written as

$$\rho = (I \otimes \Lambda)(P_\rho),$$

where $\Lambda(\cdot) = \sum_r U_r(\cdot) U_r^\dagger$ is a completely positive map and $U_r = \sqrt{\text{tr}(V_r I \otimes V_r |\beta\rangle \langle \beta |)}$, $\langle n | V_r | m \rangle = |\langle n | U_r | m \rangle|$, and $|\psi_r\rangle = \sum_m a_{mn}(\rho) |m\rangle |n\rangle$.

By Eq. (9), $W = I \otimes \Lambda(P_\rho) - I \otimes \Lambda'(P_\rho)$, where $\Lambda'(\cdot) = \sum_{t=0}^{k-1} \sqrt{d_{AB}c_\rho E_{ij}^{(t)}}|\sqrt{d_{AB}c_\rho E_{ij}^{(t)}}\rangle - \langle \sqrt{d_{AB}c_\rho E_{ij}^{(t)}}|$. By Lemmas 4 and 5, $\{\sqrt{d_{AB}c_\rho E_{ij}^{(t)}}\}_{i=0}^{k-1}$ is a contractive locally linear combination of $\{U_r\}_{r=0}^{k-1}$ but not a linear combination of $\{U_r\}_{r=0}^{k-1}$ with a contractive coefficient matrix if and only if $\Phi = \Lambda - \Lambda'$ is a PNCP map, and $W = \rho - c_\rho I$ is an EW.

In conclusion, we have demonstrated that any EW can be constructed from a certain density matrix. This result shows that the research on density matrices can replace the research on entanglement witnesses. The trace inequality reveals the general procedure of constructing EW for any density matrix. Both in operation and in theory, the necessary and sufficient condition of an EW in the form $W = \rho - c_\rho I$ and some examples are given. Here we only consider the bipartite case on the finite dimensional Hilbert space, but we can also generalize our results to multipartite system and infinite dimensional Hilbert space.

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### Appendix: The Procedure of Computing $\epsilon_{\rho_q}^{\text{max}}$

Clearly,

$$\rho_q = \begin{pmatrix} \frac{1+q}{4} & 0 & 0 & 0 \\ 0 & \frac{1+q}{4} & 0 & 0 \\ 0 & 0 & \frac{1+q}{2} & \frac{1+q}{4} \\ 0 & 0 & \frac{1+q}{2} & \frac{1+q}{4} \end{pmatrix},$$

where $-\frac{1}{4} \leq q < 0$.

By Theorem 3, $|\mu_A\rangle\langle\mu_B|)$ for two qubits can be written as $|\mu_A\rangle\langle\mu_B|) = d_0 f_0 (00) + d_0 f_1 (01) + d_1 f_0 (10) + d_1 f_1 (11)$ with $|d_0|^2 + |d_1|^2 = 1$, $|f_0|^2 + |f_1|^2 = 1$.

$$\text{tr}[\rho_q(|\mu_A\rangle\langle\mu_B|)] = \langle \mu_A | \rho_q | \mu_B \rangle$$

$$= (d_0 f_0 (00) + d_0 f_1 (01) + d_1 f_0 (10) + d_1 f_1 (11)) \times \rho_q \times (d_0 f_0 (00) + d_0 f_1 (01) + d_1 f_0 (10) + d_1 f_1 (11))$$

$$= |d_0|^2 f_0^2 \frac{1+q}{4} + |d_0|^2 f_1^2 \frac{1+q}{4} + |d_1|^2 f_0^2 \frac{1+q}{4}$$

$$+ |d_1|^2 f_1^2 \frac{1+q}{4} + d_0^2 f_0 d_1 f_0 \frac{q}{2} + d_1^2 f_1 d_0 f_0 \frac{q}{2}$$

$$= |d_0|^2 (|f_0|^2 + |f_1|^2) \frac{1+q}{4} + |d_1|^2 (|f_0|^2 + |f_1|^2) \frac{1+q}{4}$$

$$+ |d_0|^2 (|f_0|^2 + |f_1|^2) - \frac{2q}{4} + 2 \text{Re}(d_0^* d_1 f_1) \frac{q}{2}$$

$$\geq (|d_0|^2 + |d_1|^2) (|f_0|^2 + |f_1|^2) \frac{1+q}{4} + |\text{Re}(d_0^* f_1) - \text{Re}(d_1 f_0)|^2 \frac{q}{2}$$

$$= \frac{1+q}{4} + |\text{Re}(d_0^* f_1) - \text{Re}(d_1 f_0)|^2 \frac{q}{2}$$

$$\geq \frac{1+q}{4}$$

Therefore, we have $\epsilon_{\rho_q}^{\text{max}} = \frac{1+q}{4}$. In addition, we have another method for computing $\epsilon_{\rho_q}^{\text{max}}$.

Any qubit pure state $|\psi\rangle$ can be written as $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$, where $\alpha$ and $\beta$ are complex number and $|\alpha|^2 + |\beta|^2 = 1$. Because $|\alpha|^2 + |\beta|^2 = 1$, $|\psi\rangle$ can be rewritten as

$$|\psi\rangle = e^{i\theta} (\cos \frac{\theta}{2} |0\rangle + e^{i\phi} \sin \frac{\theta}{2} |1\rangle),$$

where $\theta$, $r$ and $s$ are real numbers. The factor of $e^{i\theta}$ out the front can be ignored since it has no observable effects.

Therefore, $|\mu_A\rangle\langle\mu_B|$ for two qubits can be written as

$$|\mu_A\rangle\langle\mu_B| = (\cos \frac{\theta_1}{2} |0\rangle + e^{i\phi_1} \sin \frac{\theta_1}{2} |1\rangle) (\cos \frac{\theta_2}{2} |0\rangle + e^{i\phi_2} \sin \frac{\theta_2}{2} |1\rangle)$$

$$= \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |00\rangle + e^{i\phi_1} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |01\rangle$$

$$+ e^{i\phi_2} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} |10\rangle + e^{i(\phi_1 + \phi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} |11\rangle.$$

$$\text{tr}[\rho_q(|\mu_A\rangle\langle\mu_B|)] = \frac{1+q}{4} \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \frac{q}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} e^{i(\phi_1 + \phi_2)}$$

$$+ \frac{1-q}{4} \cos^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} + \frac{1}{4} \sin^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2}$$

$$+ \frac{q}{2} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} e^{-i(\phi_1 + \phi_2)} + \frac{1+q}{4} \sin^2 \frac{\theta_1}{2} \sin \frac{\theta_2}{2}$$

$$= \frac{1+q}{4} \cos^2 \frac{\theta_1}{2} (\cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_2}{2}) - \frac{2q}{4} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2}$$

$$+ \frac{1+q}{4} \sin^2 \frac{\theta_1}{2} (\cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_2}{2}) - \frac{2q}{4} \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2}$$

$$+ \frac{q}{2} \sin \frac{\theta_1}{2} \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \cos \frac{\theta_2}{2} e^{-i(\phi_1 + \phi_2)} + e^{i(\phi_1 + \phi_2)}$$

$$= \frac{1+q}{4} - \frac{q}{2} \cos \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}$$

$$+ \sin^2 \frac{\theta_1}{2} \cos \frac{\theta_2}{2} + 2 \cos (\phi_1 + \phi_2) \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin \frac{\theta_2}{2} \cos \frac{\theta_2}{2}$$

$$\geq \frac{1+q}{4} - \frac{q}{2} \cos \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \sin \frac{\theta_2}{2} \cos \frac{\theta_2}{2} \sin \frac{\theta_2}{2} \cos \frac{\theta_2}{2}$$

$$\geq \frac{1+q}{4}.$$