Analytical solution of $D$ dimensional Schrödinger equation for Eckart potential with a new improved approximation in centrifugal term

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Abstract

Analytical solutions are presented for eigenvalues, eigenfunctions of $D$-dimensional Schrodinger equation having Eckart potential within Nikiforov-Uvarov method. This uses a new, improved approximation for centrifugal term, from a combination of Greene-Aldrich and Pekeris approximations. Solutions are obtained in terms of hypergeometric functions. It facilitates an accurate representation in entire domain. Its validity is illustrated for energies in an arbitrary $\ell \neq 0$ quantum state. Results are compared for a chosen set of potential parameters in different dimensions. In short, a simple accurate approximation is offered for Eckart and other potentials in quantum mechanics, in higher dimension.

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I. INTRODUCTION

The exponential diatomic molecular Eckart potential [1],

\[ v_e(r) = -\frac{\alpha e^{-r/a}}{1 - e^{-r/a}} + \frac{\beta e^{-r/a}}{(1 - e^{-r/a})^2}, \quad \alpha, \beta > 0. \]  

(1)
is used as an important model in chemical physics [2]. The depth is controlled by two positive real parameters \( \alpha \) and \( \beta \), while the range is governed by a positive parameter, \( \alpha \), having the dimension of inverse length. It is related to the Schöenberg potential by a change of parameter. It has a minimum value of \( -(\alpha - \beta)^2 / 4\beta \) at \( r = a \ln \left( \frac{\alpha + \beta}{\alpha - \beta} \right) \), for \( \alpha > \beta \). Like the other few familiar exponential-type model potentials such as, Morse, Woods-Saxon, Hulthén, Manning-Rosen, the relevant Schrödinger equation (SE) in this case too offers exact analytical solution for \( \ell = 0 \), but not for \( \ell \neq 0 \) states. However, a number of elegant approximations have been put forth for non-zero \( \ell \) states with varied success. It may be noted that the Eckart potential and its PT-symmetric version can be considered as special cases of the multi-parameter exponential type potential [3].

Analytical bound state energy expressions and radial wave functions for arbitrary \( \ell \) states are derived by using various approximations to the centrifugal term in amalgamation with a host of techniques, such as, supersymmetric shape invariance approach combined with a functional analysis, asymptotic iteration, tridiagonal representation, Feynman path integral approach, etc. An approximate solution of the SE with Eckart potential and its parity-time symmetric version was investigated using a Nikiforov-Uvarov (NU) method. The same for scattering states were expressed in terms of the generalized hypergeometric functions \( _2F_1(a, b; c; z) \), and phase shifts were analyzed. The relativistic bound- and scattering-state solutions of Klein-Gordon equation have been reported. Apart from energies, information theoretical studies were pursued, viz., Shannon entropy [18], Rényi entropy [9], etc. Variants of this potential, like deformed hyperbolic Eckart, Eckart-like, Hua plus modified Eckart, generalized Deng-Fan plus deformed Eckart potential have been considered as well. The thermodynamic properties of anharmonic Eckart potential have also been the subject of investigation. The above discussion suggests that, a reasonable number of good-quality methods are available for low-lying (especially, \( \ell = 0 \)) states of this potential. But such attempts are rather limited for high-lying states belonging to the non-trivial \( \ell \neq 0 \) case. Also it is worth mentioning that a majority of the articles are devoted to eigenvalues and eigenfunctions, mainly in 3D. A review of the available methods further indicates that, a decent number of works resort to the approximation of centrifugal term from physical/mathematical consideration. Herein lies the objectives of this work, which is
two-fold. At first, we would like to make an analysis and test performance of a recently developed, simple approximation for centrifugal term in the context of Eckart potential. So far, this has been applied to shifted Deng-Fan, Manning-Rosen and Pöschl-Teller molecular potentials \[23, 24\]. This will extend the scope of applicability to a wider range of physical systems. Another objective is to report the results in D dimension, for which the references are very rare \[25\].

The article is organized as follows: Sec. \(\text{II}\) gives an overview of various approximations to the centrifugal term, including the newly proposed one \[23, 24\], for solving the SE for Eckart potential, within the NU method. The resultant analytic expressions are derived for radial wave functions, eigenvalues, normalization constants. Wherever possible, the computed energies in Sec. \(\text{III}\) are discussed and compared with available literature results. Finally, a few remarks are made in Sec. \(\text{IV}\) along with the future prospect of our scheme.

II. THEORETICAL ANALYSIS

Let us consider the following SE,

\[
- \frac{\hbar^2}{2\mu} \nabla_D^2 \psi + V(r)\psi(r) = E\psi(r),
\]

where \(\nabla\) is \(D\)-dimensional gradient operator, \(r = (x_1, x_2, ..., x_D) = (r, \theta_1, \theta_2, ..., \theta_{D-1}) = (r, \Omega_{D-1})\), \(r = |r| = \sqrt{\sum_{i=1}^{D} x_i^2}\), \(x_i = r \cos \theta_i \prod_{k=1}^{i-1} \sin \theta_k\), \(0 \leq \theta_i < \pi\), \(i = 1, 2, ..., D - 2\), \(0 \leq \theta_{D-1} < 2\pi\), while,

\[
\nabla_D = \left( \frac{\partial}{\partial r}, \frac{1}{r} \frac{\partial}{\partial \theta_1}, \frac{1}{r \sin \theta_1} \frac{\partial}{\partial \theta_2}, ..., \frac{1}{r \prod_{j=1}^{k-1} \sin \theta_j} \frac{\partial}{\partial \theta_k}, ..., \frac{1}{r \prod_{j=1}^{D-2} \sin \theta_j} \frac{\partial}{\partial \theta_{D-1}} \right),
\]

is the \(D\)-dimensional gradient operator. Then the Laplacian operator \(\nabla_D^2\) is defined by,

\[
\nabla_D^2 = \frac{1}{r^{D-1}} \frac{\partial}{\partial r} \left( r^{D-1} \frac{\partial}{\partial r} \right) - \frac{\Lambda_D^2}{r^2},
\]

where \(\Lambda_D\) is the \(D\)-dimensional generalized angular momentum operator, given by,

\[
\Lambda_D^2 = - \sum_{i=1}^{D-1} \frac{(\sin \theta_i)^{i+1-D}}{(\prod_{j=1}^{i-1} \sin \theta_j)^2} \frac{\partial}{\partial \theta_i} \left( \frac{(\sin \theta_i)^{D-1-i}}{\partial \theta_i} \right),
\]

\(\mu\) is reduced mass of diatomic molecule, and \(r\) is the internuclear distance. Let

\[
\psi(r) = r^{-\frac{D-1}{2}} R(r) Y_{\ell,\{\mu\}}(\Omega),
\]

be the solution of Eq. (2). Then it is obtained that,

\[
\frac{d^2 R}{dr^2} + \left[ \frac{2\mu}{\hbar^2} E - \frac{2\mu}{\hbar^2} V(r) - \frac{L(L+1)}{r^2} \right] R = 0,
\]

(7)
and

$$\Lambda_D^2 Y_{\ell,\{\mu\}} = \ell(\ell + D - 2)Y_{\ell,\{\mu\}},$$  \hspace{1cm} (8)

where $\ell(\ell + D - 2)$ is the separation constant, and $L = \ell + \frac{D-3}{2}$. Then following \[26, 27\],

$$Y_{\ell,\{\mu\}} = N_{\ell,\{\mu\}} e^{i m \theta} D_{\ell}^{\frac{D}{2} - 1} \prod_{j=1}^{D-2} C_{\mu_j - \mu_{j+1}}^{\alpha_j + \mu_j + 1} \cos \theta_j \sin \theta_j^{\mu_j + 1},$$  \hspace{1cm} (9)

where $C_i(t)$ is the Gegenbauer polynomial in $t$ of degree $i$ with parameter $j$, whereas

$$N_{\ell,\{\mu\}} = \frac{1}{2\pi} \prod_{j=1}^{D-2} \frac{\Gamma(\alpha_j + \mu_j + 1)}{\Gamma(2\alpha_j + \mu_j + \mu_{j+1})}^{2},$$  \hspace{1cm} (10)

is the normalization constant. Here $(\ell, \{\mu\}) = (\mu_1, \mu_2, \ldots, \mu_{D-1})$, $\ell = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_{D-2} \geq |\mu_{D-1}| = |m|$, $\ell = 0, 1, 2, \ldots$, $m = 0, \pm 1, \pm 2, \ldots$, $\alpha_j = (D - j - 1)/2$, $D = 3, 4, 5, 6, \ldots$. To find the general solution of resulting radial SE of Eq. (7), we will use a transformation, $s = e^{-r/a}$, and a set of functions, \{1, $\frac{1}{1-s}, \frac{s^2}{1-s^2}$\}, for which the centrifugal term is approximated in different forms, such as Greene-Aldrich \[28\], Pekeris-type \[29\]. Recently, the authors have proposed an intuitive combination of these two approximations \[23, 24\] with considerable success.

A. Approximation to the centrifugal term

Let us consider the following relation:

$$\frac{1}{r^2} \approx f_i(r) = \frac{1}{a^2} \left( x_{1i} + \frac{x_{2i}s}{1-s} + \frac{x_{3i}s^2}{(1-s)^2} \right), \ \text{s} = e^{-r/a}, \ i = 1, 2, 3.$$  \hspace{1cm} (11)

The approximation $f_1(r)$, with $x_{11} = 0, x_{21} = x_{31} = 1$, is commonly used in references \[4, 11, 15, 28\], while for $x_{12} = 0, x_{22} = \xi_1, x_{32} = \xi_2$ and $\xi_1, \xi_2$ as two adjustable dimensionless parameters, the approximation $f_2(r)$ is considered by the authors of \[6, 9\]. Apart from that, the approximation $f_3(r)$ is employed in reference \[30–34\], having $x_{13} = \frac{1}{12}, x_{23} = x_{33} = 1$. The approximations $f_1, f_2, f_3$ are good near the origin \[4, 6, 9, 11, 15, 28\]. Moreover, the centrifugal term $\frac{1}{r^2}$ is approximated near the point $r = r_0$, as below, following the references \[5, 29, 35, 36\],

$$\frac{1}{r^2} \approx f_4(r) = \frac{1}{a^2} \left( x_{14} + \frac{x_{24}s}{1-s} + \frac{x_{34}s^2}{(1-s)^2} \right),$$  \hspace{1cm} (12)
where

\[
\begin{align*}
x_{14} &= \frac{1}{u^4} \left[ (3 + u)s_0^2 + (2u - 6)s_0 + (3 - 3u + u^2) \right], \\
x_{24} &= \frac{2}{u^4} (1 - s_0)^2 \left[ 3 + u + \frac{2u - 3}{s_0} \right], \\
x_{34} &= -\frac{1}{u^4} (1 - s_0)^3 \left[ \frac{3 + u}{s_0} + \frac{u - 3}{s_0^2} \right], \\
s_0 &= e^{-u}, \quad u = r_0/a.
\end{align*}
\]

(13)

Here \( r_0 \) is a positive real number, having the dimension of length. In particular, if \( r_0 = a \ln \left[ \frac{\alpha + \beta}{\alpha - \beta} \right] \), with \( \alpha > \beta \), then the potential has a minimum value of \(-\frac{(\alpha - \beta)^2}{4}\). By construction, \( f_4 \) approximation is more effective around \( r = r_0 \) region [5, 29, 35, 36]. In this article, we propose a new approximation to the centrifugal term, following our previous works [23, 24],

\[
\frac{1}{r^2} \approx f_5 (r) = \sum_{j=1}^{4} \lambda_j f_j (r) = \frac{1}{a^2} \left( y_1 + \frac{y_2 s}{1 - s} + \frac{y_3 s^2}{(1 - s)^2} \right),
\]

(14)

where

\[
y_i = \sum_{j=1}^{4} \lambda_j x_{ij}, \quad i = 1, 2, 3; \quad \sum_{j=1}^{4} \lambda_j = 1.
\]

(15)

Figure 1 compares these approximations with exact centrifugal term \( \frac{\ell (\ell + 1)}{r^2} \), for \( \ell = 2 \) having following parameters: \((h, \mu) = (1, 1), \quad (a, \alpha, \beta) = (1/0.025, 1/a, 0.0001), \quad (\xi_1, \xi_2) = (1.1, 0.98), \quad D = 3. \) The upper and lower panels correspond to \( r \to 0 \) and \( r \to r_0 \) regions. In each row, five approximations are illustrated in five panels—the first four refer to \( f_1, f_2, f_3, f_4 \), while two rightmost panels imply \( f_5^{(a)} \) and \( f_5^{(b)} \) having \( (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, 0, 0.98, 0.02) \) and \( (0, 0, 0.02, 0.98) \) respectively. Clearly, panels (C), (E) show that \( f_3 \) and \( f_5^{(a)} \) are good approximations near origin. Likewise, from panels (I), (J), it is inferred that \( f_4, f_5^{(b)} \) are superior in the neighborhood of \( r = r_0 \). Thus \( f_5 \) remains quite accurate over the entire effective domain of \( r \) [23, 24], which has been defined in our previous communications [37, 38]. Throughout the article, for sake of convenience, \( f_5 \) approximations will be used for various sets of \( \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\} \). These will be indicated by alphabetic superscripts.

**B. Ro-vibrational energy by NU method**

This is a well-established mathematical tool for various eigenvalue problems [39]. It has been utilized by a number of researchers for a broad range central and non-central potentials in non-relativistic and relativistic quantum mechanics [40–43]. Under the transformation, \( s = e^{-r/a} \),
FIG. 1: Plot of the difference, \( \ell(\ell+1) \left( \frac{1}{r^2} - f_i(r) \right) \), for \( i = 1, 2, 3, 4 \). In (A), (F) \( i = 1 \); (B), (G) \( i = 2 \); (C), (H) \( i = 3 \); (D), (I) \( i = 4 \). (E) and (J) refer to \( f_5^{(a)} \) and \( f_5^{(b)} \) respectively. The parameters are: \( \ell = 2, \) \((h, \bar{\mu}) = (1, 1), (a, \alpha, \beta) = (1/0.025, 1/a, 0.0001), (\xi_1, \xi_2) = (1.1, 0.98), D = 3 \). See text for details.

Eq. (7) becomes,

\[
\frac{d^2 R}{ds^2} + \frac{\bar{\tau}(s)}{\sigma(s)} \frac{dR}{ds} + \frac{\bar{\sigma}(s)}{[\sigma(s)]^2} R = 0,
\]  

(16)
where
\[
\bar{\tau}(s) = 1 - s, \quad \sigma(s) = s - s^2, \quad \bar{\sigma}(s) = -As^2 + Bs - C,
\]
\[
A = \frac{2\mu a^2 \alpha}{h^2} - L(L + 1)(y_2 - y_3) + C,
\]
\[
B = \frac{2\mu a^2(\alpha - \beta)}{h^2} - L(L + 1)y_2 + 2C,
\]
\[
C = \epsilon^2_0 + L(L + 1)y_1,
\]
\[
\epsilon^2_0 = -\frac{2\mu a^2 E}{\hbar^2}.
\]

(17)

Let us suppose,
\[
R(s) = \phi_1(s)\phi_2(s),
\]
be the solution of Eq. (16). Following (23, 24, 39), it is then obtained that,
\[
\sigma(s)\phi_2''(s) + \tau(s)\phi_2'(s) + \nu \phi_2(s) = 0,
\]
and
\[
\phi_1(s) = e^{\int \frac{\pi(s)}{\sigma(s)}ds},
\]
where
\[
\pi(s) = \frac{\sigma'(s) - \bar{\tau}(s)}{2} \pm \sqrt{\left(\frac{\sigma'(s) - \bar{\tau}(s)}{2}\right)^2 - \bar{\sigma}(s) + k \sigma(s)},
\]
\[
\tau(s) = \bar{\tau}(s) + 2\pi(s), \quad \bar{\nu} = k + \pi'(s),
\]
with \(\bar{\nu}\) and \(k\) being real constants. Since \(\pi(s)\) is a polynomial in \(s\), one has to find \(k\) in such a way that, \(\left(\frac{\sigma'(s) - \bar{\tau}(s)}{2}\right)^2 - \bar{\sigma}(s) + k \sigma(s)\) is the square of a polynomial in \(s\). Then the solutions of Eq. (19) can be written as follows,
\[
\phi_{2, n_r}(s) = \frac{1}{\rho(s)} \frac{d^{n_r}}{ds^{n_r}} [\sigma^{n_r}(s)\rho(s)],
\]
and the corresponding eigenvalues are obtained as,
\[
\nu_{n_r} = -n_r \tau'(s) - \frac{\sigma''(s)}{2} \sigma'(s), \quad n_r = 0, 1, 2, \ldots,
\]
where
\[
\rho(s) = [\sigma(s)]^{-1} e^{\int \frac{\pi(s)}{\sigma(s)}ds}.
\]

(24)

According to NU method, and following the prescription (23, 24, 39), one gets a pair of \(k\) as,
\[
k_{\pm} = \frac{2\mu a^2(\alpha - \beta)}{h^2} - L(L + 1)y_2 \pm \sqrt{C(2L_1 - 1)},
\]
(25)
We have chosen

\[ k = k_+ \text{ with } k_+ - B < 0, \text{ and selected,} \]

\[ \pi(s) = -\frac{\sqrt{C}}{2} \pm \left\{ \begin{array}{l}
(\sqrt{C} - L_1 + \frac{1}{2}) s + \sqrt{C}, \ k = k_+, \ k_+ - B > 0 \\
(\sqrt{C} + L_1 - \frac{1}{2}) s + \sqrt{C}, \ k = k_-, \ k_- - B > 0 \\
(\sqrt{C} - L_1 + \frac{1}{2}) s - \sqrt{C}, \ k = k_+, \ k_+ - B < 0 \\
(\sqrt{C} + L_1 - \frac{1}{2}) s - \sqrt{C}, \ k = k_-, \ k_- - B < 0
\end{array} \right. \].

(27)

We have chosen \( k = k_- \) with \( k_- - B < 0 \), and selected,

\[ \pi(s) = -\left( \sqrt{C} + L_1 \right) s + \sqrt{C}. \]

(28)

This leads to [44],

\[ \rho(s) = s^2 \sqrt{C} (1 - s)^{2L_1 - 1}, \]

(29)

\[ \phi_1(s) = s^{\sqrt{C}} (1 - s)^{L_1}, \]

and

\[ \phi_{2,n_r} = (n_r)! P_{n_r}^{(2\sqrt{C},2L_1 - 1)}(1 - 2s), \]

(30)

\[ = \frac{\Gamma(n_r + 2\sqrt{C} + 1)}{\Gamma(2\sqrt{C} + 1)} \binom{2L_1 - 2}{-n_r, n_r + 2\sqrt{C} + 2L_1; 2\sqrt{C} + 1; s}, \]

where \( P_n^{(a,b)}(x) \) is the Jacobi polynomial in \( x \) of degree \( n \) with parameters \( a, b \), while \( 2F_1(a, b; c; x) \) is the Hypergeometric function, defined by \( 2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k x^k}{(c)_k k!} \) and \( (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} \) is the Pochhammer symbol. The eigenvalues \( E_{n_r,\ell} \) are then given as,

\[ E_{n_r,\ell} = \frac{\hbar^2 L(L+1) y_1}{2\mu a^2} - \frac{\hbar^2}{2\mu a^2} \left( \frac{\omega a^2 n - L(L+1)(y_2 - y_1)/2}{n_r + L_1} - \frac{n_r + L_1}{2} \right)^2. \]

(31)

Therefore, the radial wave function can be expressed as [44],

\[ R_{n_r,\ell}(r) = N_{n_r,\ell} s^{\sqrt{C_{n_r,\ell}}(1 - s)^{L_1}} 2F_1 \left( -n_r, n_r + 2\sqrt{C_{n_r,\ell}} + 2L_1; 2\sqrt{C_{n_r,\ell}} + 1; s \right), \]

(32)

where

\[ N_{n_r,\ell} = \left[ \frac{2\sqrt{C_{n_r,\ell}}(n_r + \sqrt{C_{n_r,\ell}} + L_1) \Gamma(n_r + 2\sqrt{C_{n_r,\ell}} + 1) \Gamma(n_r + 2\sqrt{C_{n_r,\ell}} + 2L_1)}{a(n_r)!(n_r + \sqrt{C_{n_r,\ell}}) \Gamma(n_r + 2L_1) \left[ \Gamma(2\sqrt{C_{n_r,\ell}} + 1) \right]^2} \right]^{\frac{1}{2}}. \]

(33)
is the normalization constant, and \( C_{n_r,\ell} = -\frac{2\bar{\mu}^2 E_{n_r,\ell}}{\hbar^2} + (\ell + \frac{D-3}{2})(\ell + \frac{D-1}{2}) x_1 \). Finally, the explicit form of eigenfunctions can be written as,

\[
\psi_{n_r,\ell,\{\mu\}}(r) = N_{n_r,\ell} r^{\frac{1-D}{2}} s \sqrt{C_{n_r,\ell}} (1 - s) L_{1/2}(-n_r + 2 \sqrt{C_{n_r,\ell}} + 2 L_{1/2} 2 \sqrt{C_{n_r,\ell}} + 1; s) Y_{\ell,\{\mu\}}(\Omega),
\]

\[
s = e^{-r/a}.
\]

(34)

It is to be noted that the approximation (14) is well defined for convex combination \((0 \leq \lambda_j \leq 1, j = 1(1)4, \sum_{j=1}^4 \lambda_j = 1)\) for Eckart potential, and one may consider a linear combination (may assume negative values) for different values of \(\lambda_j, j = 1(1)4\), if they satisfy the following relations,

\[
\sum_{j=1}^4 \lambda_j = 1, \\
\sum_{j=1}^4 \lambda_j x_{1j} \geq 0, \\
\frac{1}{4} + \frac{2\mu a^2 \beta}{\hbar^2} + (\ell + \frac{D-3}{2})(\ell + \frac{D-1}{2}) \sum_{j=1}^4 \lambda_j x_{3j} \geq 0.
\]

(35)

III. RESULTS AND DISCUSSION

At first, Table I reports 9 low-lying energies corresponding to first three non-zero-\(\ell\) (1-3) values of first three \(n_r\) (1-3). These are given for \(D = 3\) having a fixed \(a = 1/0.025\); the upper and lower portions correspond to \(\beta = 0.0001\) and 0.0005 respectively. The four columns 3–6 employ four different approximations \(viz., f_1, f_2, f_3, f_4\) from Eqs. (11) and (12), while columns 7 and 8 provide those for approximations \(f_5^{(c)}\) and \(f_5^{(d)}\) in Eq. (14), with fixed \((\xi_1, \xi_2) = (1.1, 0.98)\) set, but having different \(\lambda_i\)'s, namely, \((\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0.5, 0.2, 0.2, 0.1)\) and \((0.1, 0, 0, 0.9)\). These are compared with accurate numerical results of [4, 6, 45] and from the generalized pseudospectral (GPS) method [46–48]. The latter has been found to be very successful for a variety of potentials of physical and chemical relevance. In case of \(\beta = 0.0005\), no reference energies could be found except the GPS [46–48]. The approximated energies quite favorably compare with literature results.

In our analysis, for \(D = 3\), under \(f_1\) approximation, the zero-energy states characterized by quantum numbers \((n_r, \ell)\), for a particular \(a = a_0\), can be obtained from the following equation,

\[
\left[\frac{8\mu a_0^2 (\alpha_0 - \beta)}{\hbar^2} - (2n_r + 1)^2 - (2\ell + 1)^2\right]^2 = (2n_r + 1)^2 \left[(2\ell + 1)^2 + \frac{8\mu a_0^2 \beta}{\hbar^2}\right],
\]

(36)

where \(\alpha_0\) is a function of \(a_0\). Our analysis also reveals that, for a given \(a = a_{12}\), two states identified by quantum numbers \((n_{r1}, \ell_1)\) and \((n_{r2}, \ell_2)\) are degenerate possessing same energy. This can be
TABLE I: Energies, $E_{n_r,\ell}$, within different approximations, for $\hbar = \mu = 1, \alpha = 1/a, a = 1/0.025, D = 3$. The upper and lower portions refer to $\beta = 0.0001$ and 0.0005. For details, see text.

| $n_r$ | $\ell$ | $-E_{n_r,\ell}[f_1]$ | $-E_{n_r,\ell}[f_2]$ | $-E_{n_r,\ell}[f_3]$ | $-E_{n_r,\ell}[f_4]$ | $-E_{n_r,\ell}[f_5^{(c)}]$ | $-E_{n_r,\ell}[f_5^{(d)}]$ | Ref. [45] | GPS$^c$ |
|-------|-------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|-----------------------|--------|--------|
| 0     | 1     | 0.1008879             | 0.1015944             | 0.1008358             | 0.1010119             | 0.1008410             | 0.1008358             |        |        |
| 0     | 2     | 0.0415198             | 0.0414791             | 0.0413635             | 0.0414643             | 0.0413792             | 0.0413642             |        |        |
| 0     | 3     | 0.0193308             | 0.0189461             | 0.0190183             | 0.0190183             | 0.0190087             | 0.0189346             |        |        |
| 1     | 1     | 0.0410768             | 0.0401247             | 0.0401247             | 0.0401247             | 0.0401299             | 0.0401250             |        |        |
| 1     | 2     | 0.0190752             | 0.0189774             | 0.0189190             | 0.0189190             | 0.0190087             | 0.0189216             |        |        |
| 1     | 3     | 0.0091303             | 0.0088504             | 0.0088504             | 0.0088504             | 0.0088544             | 0.0088576             |        |        |
| 2     | 1     | 0.0185142             | 0.0184622             | 0.0184622             | 0.0184622             | 0.0185050             | 0.0184632             |        |        |
| 2     | 2     | 0.0090066             | 0.0089303             | 0.0089303             | 0.0089303             | 0.0089444             | 0.0088660             |        |        |
| 2     | 3     | 0.0040362             | 0.0038908             | 0.0038908             | 0.0038908             | 0.0038908             | 0.0038908             |        |        |

**a**These compare with those from [4].  
**b**These compare with those from [6], where $\xi = 1.10$ and $\lambda = 0.98$.  
**c**These are calculated here for this work, using GPS [46–48].

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found from following relation,

$$
\frac{2\mu a^2_{12} \alpha_{12}}{\hbar^2} \left( \frac{1}{n_r + \frac{1}{2} + \sqrt{1 + \left( \ell_1 + \frac{1}{4} + \frac{2\mu a^2_{12} \beta}{\hbar^2} \right)^2}} + \frac{1}{n_r + \frac{1}{2} + \sqrt{1 + \left( \ell_2 + \frac{1}{4} + \frac{2\mu a^2_{12} \beta}{\hbar^2} \right)^2}} \right)
\equiv 1 \mp \frac{1}{2} + n_r \mp n_r + \sqrt{\ell_1 (\ell_1 + 1) + \frac{1}{4} + \frac{2\mu a^2_{12} \beta}{\hbar^2}} \mp \sqrt{\ell_2 (\ell_2 + 1) + \frac{1}{4} + \frac{2\mu a^2_{12} \beta}{\hbar^2}}.
$$

(37)

Considering the positive sign, $a_{12}$ seems to satisfy the equation as below,

$$
\frac{2\mu a^2_{12} \alpha_{12}}{\hbar^2} = \left( n_r + \frac{1}{2} + \sqrt{\left( \ell_1 + \frac{1}{2} \right)^2 + \frac{2\mu a^2_{12} \beta}{\hbar^2}} \right) \left( n_r + \frac{1}{2} + \sqrt{\left( \ell_2 + \frac{1}{2} \right)^2 + \frac{2\mu a^2_{12} \beta}{\hbar^2}} \right).
$$

(38)
The degenerate energy levels can be obtained numerically under $f_5$ approximation. Using Eq. (38), one can find these levels, $E_{n\tau,\ell} = E_{n\tau,\ell}$ which are equal at the points, $a = a_{12}$.

In Table II we offer representative state energies, for different dimensions (3–5) for particular values of $a = 1/0.025$, $\beta = 0.0001$, within the $f_5^{(d)}$ approximation, having $\lambda_1 - \lambda_4$ as (0.1, 0, 0, 0.9). The $D = 3$ results are slightly different from those in previous table, as the $\lambda_i$’s employed for $f_5$ are different. No reference energies can be found for comparison. Note that, the state $E_{2,4}$ does not exist for $D = 5$, for the $a$ considered in this table. Furthermore, under $f_1$ approximation, for $D \geq 3$, there occurs an inter-dimensional degeneracy, corresponding to energy levels $E_{n\tau,\ell_1}(D_i) = E_{n\tau,\ell_2}(D_j)$ at the point $a = a_{ij}$, given by the equation,

$$\frac{2\mu}{\hbar^2} \left( \frac{D_{ij}}{a_{12}} \right)^2 \frac{D_{ij}}{a_{12}} \left( n_{r1} + \frac{1}{2} + \sqrt{\left( \frac{\ell_1}{2} + \frac{D_i - 2}{2} \right)^2 + \frac{2\mu}{\hbar^2} \left( \frac{D_{ij}}{a_{12}} \right)^2 \beta} \right)$$

$$= \frac{1 + 1}{2} + n_{r1} + \sqrt{\left( \frac{\ell_1}{2} + \frac{D_i - 2}{2} \right)^2 + \frac{2\mu}{\hbar^2} \left( \frac{D_{ij}}{a_{12}} \right)^2 \beta}$$

$$\left( n_{r2} + \frac{1}{2} + \sqrt{\left( \frac{\ell_2}{2} + \frac{D_j - 2}{2} \right)^2 + \frac{2\mu}{\hbar^2} \left( \frac{D_{ij}}{a_{12}} \right)^2 \beta} \right)$$

where $a_{12}^{D_{ij}}$ is again a function of $a_{12}^{D_{ij}}$. Similarly, considering the positive sign, one obtains,

$$\frac{2\mu}{\hbar^2} \left( \frac{D_{ij}}{a_{12}} \right)^2 \frac{D_{ij}}{a_{12}} = \left( n_{r1} + \frac{1}{2} + \sqrt{\left( \frac{\ell_1}{2} + \frac{D_i - 2}{2} \right)^2 + \frac{2\mu}{\hbar^2} \left( \frac{D_{ij}}{a_{12}} \right)^2 \beta} \right)$$

$$\left( n_{r2} + \frac{1}{2} + \sqrt{\left( \frac{\ell_2}{2} + \frac{D_j - 2}{2} \right)^2 + \frac{2\mu}{\hbar^2} \left( \frac{D_{ij}}{a_{12}} \right)^2 \beta} \right).$$

The value of $a = a_{0}^D$, corresponding to a zero-energy state occurs for $(n_{r}, \ell)$ quantum numbers,
which for \( D > 3 \), under \( f_1 \) approximation, can be estimated from,

\[
\left( \frac{8\tilde{\mu}(a_0^D)(a_0^D - \beta)}{\hbar^2} - (2n_r + 1)^2 - (2\ell + D - 2)^2 \right)^2 = (2n_r + 1)^2 \left( (2\ell + D - 2)^2 + \frac{8\tilde{\mu}(a_0^D)^2\beta}{\hbar^2} \right).
\]

From Eqs. (40) and (41) we see that, \( a_{D_{ij}} = a_{12} \), if \( D_i = D_j = 3 \) and \( a_0^D = a_0 \) if \( D = 3 \).

IV. CONCLUSION

Analytical eigenvalues and eigenfunctions are obtained for Eckart potential from a solution of \( D \)-dimensional SE within the rubric of Nikiforov-Uvarov method. A new, improved approximation to the centrifugal term is proposed for this potential, which is based on physical grounds. This general form is intuitively arrived from an amalgamation of two well-known existing approximations for centrifugal potential, namely, Greene-Aldrich and Pekeris-type. This required a linear combination of four approximating functions. The condition on the involved parameters is also discussed. It offers an accurate representation of the potential throughout the entire effective domain of \( r \).

Energy states corresponding to arbitrary quantum numbers (\( \ell \neq 0 \)) are presented and compared critically amongst each other, as well as, with available theoretical reference results. The approach can be easily extended to other exponential-type potentials of interest, such as Deng-Fan, Manning-Rosen, Pöschl-Teller, Hulthén etc. Some of these may be taken up in future communications.

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