On the trajectory of the nonlinear pendulum: 
Exact analytic solutions via power series.

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Abstract

We provide an exact infinite power series solution that describes the trajectory of a nonlinear simple pendulum undergoing librating and rotating motion for all time. Although the series coefficients were previously given in [V. Fairén, V. López, and L. Conde, \textit{Am. J. Phys} \textbf{56} (1), (1988), pp. 57?61], the series itself – as well as the optimal location about which an expansion should be chosen to assure series convergence and maximize the domain of convergence – was not examined, and is provided here. By virtue of its representation as an elliptic function, the pendulum function has singularities that lie off of the real axis in the complex time plane. This, in turn, imposes a radius of convergence on the physical problem in real time. By choosing the expansion point at the top of the trajectory, the power series converges all the way to the bottom of the trajectory without being affected by these singularities. In constructing the series solution, we re-derive the coefficients using an alternative approach that generalizes to other nonlinear problems of mathematical physics. Additionally, we provide an exact resummation of the pendulum series – Motivated by the asymptotic approximant method given in [Barlow et al., \textit{Q. J. Mech. Appl. Math.}, \textbf{70} (1) (2017), pp. 21-48] – that accelerates the series’ convergence uniformly from the top to the bottom of the trajectory. We also provide an accelerated exact resummation of the infinite series representation for the elliptic integral used in calculating the period of a pendulum’s trajectory. This allows one to preserve analyticity in the use of the period to extend the pendulum series for all time via symmetry.
I. INTRODUCTION

The pendulum equation is often presented in undergraduate science and math courses as a physically-relevant nonlinear ordinary differential equation (ODE) whose general solution cannot be written in terms of elementary functions. The ODE is used to motivate phase-plane and stability analyses [1, 2] and, when linearized for small deflections, admits a classical harmonic oscillator solution. It has been known for over a century that the analytic solution to the pendulum equation, expressed as angle as a function of time, may be written in terms of Jacobi elliptic functions [3–5]. The inverse solution (time as a function of angle) is expressed in terms of elliptic integrals and has been well documented for even longer; its application towards the evaluation of the period of a pendulum’s trajectory was established alongside those of elliptic integrals in the 18th century [6, 7]. The singularities of the Jacobi elliptic functions (and thus the pendulum equation itself) were studied during the early development of the field of complex analysis [7] and are now well-documented [8]. The efficient representation of Jacobi elliptic functions via power series is limited by the radius of convergence imparted by such singularities. While the use of power series solutions for linear ODEs is commonplace, the analogous treatment for their nonlinear counterparts is only found in scattered pockets of the literature. In general, power series solutions to nonlinear ODEs have been thought by many (in the words of [9] p. 263) to “play minor or nonexistent roles in the lives of most people”. It is perhaps not surprising that a power series solution – for the pendulum angle as a function of time – is virtually absent from the literature, especially since numerical algorithms for elliptic functions are readily available in many scientific computing packages. An early attempt at finding the power series solution to the pendulum equation is given in [9] (p. 264), where substitution of the initial conditions directly into the ODE and then successive (and objectively arduous) applications the chain rule leads one after four terms to (again in the author’s words) “feed the wastebasket and go to work on the next chapter”. A few decades later, a methodology was developed [10] to obtain the recursion for all coefficients of the pendulum series through a clever decomposition of a system of recursions. That said, the series of [10] was implied as being approximate (by the paper’s title), and a study of the radius of convergence of the series was not provided. In fact, the solution itself is incomplete for that reason, as the initial conditions are left arbitrary, and convergence is not guaranteed. The aim of this work is to show that the
power series representation of the pendulum function given by [10] is, in fact, exact and analytic over the entire physical domain if one judiciously chooses the initial conditions (or equivalently, expansion point) along a given trajectory to assure convergence. Once a convergent series is constructed through this choice of initialization, the series may be re-summed exactly to accelerate convergence, and thus can be used as an alternative to elliptic functions. It is relevant to note here that, although expansions for Jacobi elliptic functions are well-documented [8], the number of arithmetic steps needed needed to convert such expansions to a single power series for the pendulum solution is larger than those to directly find the power series solution of the original nonlinear ODE; we demonstrate this through the analysis in the current work. More generally, insights provided in this paper regarding power series solutions to nonlinear ODEs are relevant to problems that cannot be expressed by special functions such as Jacobi elliptic functions.

In general, power series solutions to ODEs are limited by their radius of convergence as mentioned above. In linear ODEs, the radius of convergence may be anticipated based on the distance between the expansion point and singularities in ODE coefficients. In nonlinear ODE’s, singularities may be spontaneous and cannot be anticipated by inspection. As such, the radius of convergence imparted to power series solutions of nonlinear ODEs can restrict an analytic representation over the whole domain of interest - or, can lead to slow convergence properties. Thus, power series solutions are often deficient, and with the prevalence of numerical techniques nowadays, often ignored as a viable solution technique. However, techniques have been developed to accelerate convergence and even sum divergent series beyond the convergence radius (i.e., analytic continuation). Approximant techniques, such as the well-known Padé, have been utilized to this end [11]. A relatively new approach, asymptotic approximants [12], has been successfully applied to power series expansions to create highly accurate approximate solutions of nonlinear ODEs arising in many areas of mathematical physics [13–17]. In particular, if one knows the asymptotic behavior in the vicinity of the boundaries of the physical domain, the method of asymptotic approximants may be used to constrain analytic continuation by enforcing these behaviors at both ends, thus providing efficient and accurate analytic infinite series solutions which are in closed form when truncated. In cases where a power series solution converges over the whole domain of interest, but convergence is slow, the same approach may be used to re-sum the series exactly and enhance convergence [16, 17]. In this paper, we establish that, with a judicious
choice of initial conditions, a convergent power series solution to the nonlinear pendulum ODE may be obtained. The convergence of the series solution is then accelerated by exactly re-summing it in a form motivated by the asymptotic approximant technique. We should note that the word ‘approximant’ is not to be confused with ‘approximate’ and is only used here to tie to previous literature. The infinite series solution we present here, both in its original and re-summed form, is an exact solution.

While one goal of this work is to assemble the above-described solution elements from across the centuries in order to give a full treatment of the pendulum series for direct use, another aim here is to add this example to a growing list of problems \[12–17\] in which power series solutions are demonstrated to be a viable solution technique for nonlinear ODEs, even in the more typical cases where singularity locations are unknown. The paper is organized as follows. In section II we detail the key structural elements of the pendulum equation, as well as its solution in terms of Jacobi elliptic functions. In section III we demonstrate an equivalent but alternative approach to \[10\] for deriving the power series solution to the differential equation. Using the known placement of the singularities in the complex time plane, we illustrate that the point of expansion – or equivalently, initial conditions – can be chosen such that the series solution encompasses the entire physical domain, using reflections about a judiciously chosen effective period. In section IV the method of asymptotic approximants is used to motivate an exact resummation of the pendulum series, and details are provided on its construction and use. Appendix A provides an efficient way of implementing truncations of this resummation. In section V we develop an exact series resummation for the complete elliptic integral of the first kind that is used in period calculations. This period resummation, in turn, is utilized to periodically extend the solution forms of sections III and IV. Concluding remarks are given in section VI on the pendulum series solution provided here. This solution adds to the growing body of literature in which nonlinear ODEs are solved using series solutions, and so some closing remarks on this connection are provided as well.

II. BACKGROUND

A schematic of the problem to be examined is provided in figure II which shows a pendulum bob of point mass \(m\) anchored by a mass-less string of length \(\ell\). During motion, the bob is displaced from the vertical by an angle \(\theta\), acted upon by a downward gravitational
FIG. 1. Setup of the simple pendulum problem. The mass, \( m \), at the end of the string (of length \( \ell \)) swings freely from some initial angle and angular velocity. In this reference frame, the acceleration of gravity, \( g \), acts vertically downward.

force \( mg \) (\( g \) being the acceleration of gravity), and moves with an angular acceleration of \( \frac{d^2\theta}{dt^2} \) where \( t \) is time. Subject to the above assumptions, a moment balance leads to the “simple pendulum equation” [9]:

\[
\frac{d^2\theta}{dt^2} + \frac{g}{\ell} \sin(\theta) = 0.
\]

Defining the dimensionless time \( \bar{t} = t \sqrt{\frac{\ell}{g}} \) as done in [2], the initial value problem for the simple pendulum then becomes

\[
\frac{d^2\theta}{d\bar{t}^2} + \sin(\theta) = 0 \quad (1a)
\]

\[
\theta(0) = \theta_0, \quad \frac{d\theta}{d\bar{t}}(0) = \bar{\omega}_0, \quad (1b)
\]

where \( \theta_0 \) and \( \bar{\omega}_0 \) are the initial angular displacement and dimensionless angular velocity\(^1\) respectively. An energy constant for the simple pendulum is obtained by multiplying both sides of (1a) by \( \frac{d\theta}{d\bar{t}} \), integrating both sides with respect to \( \bar{t} \), and applying (1b) to obtain

\[
\frac{1}{2} \left( \frac{d\theta}{d\bar{t}} \right)^2 + 1 - \cos(\theta) = \frac{1}{2} \bar{\omega}_0^2 + 1 - \cos(\theta_0) \equiv \bar{E} \geq 0. \quad (2)
\]

In (2), \( \bar{E} \) is the dimensionless total energy\(^2\) which is zero when the pendulum is resting at zero angle.

Equation (1) – and the types of solutions it admits – have been studied extensively in prior literature [5, 18–20]. All possible moving (ignoring fixed points) solutions to (1) are shown in figure 2. Initial conditions used for figure 2 are purposely chosen to be \( \theta_0=0 \) while letting

\(^1\) The dimensionless angular velocity is defined in terms of its dimensional equivalent, \( \omega_0 \), as \( \bar{\omega}_0 = \omega_0 \sqrt{\frac{\ell}{g}} \).

\(^2\) The dimensionless total energy is defined in terms of its dimensional equivalent, \( E \), as \( \bar{E} = \frac{E}{mg\ell} \).
ω₀ vary to determine E according to (2) (i.e., pushing the pendulum from the bottom) since these constraints are consistent with all possible pendulum motion. For other combinations of initial conditions, one only needs to compute Ē, locate the initial conditions in figure 2 (shifting the t̄ origin to that point), and then follow the fixed Ē trajectory from that point forward - this is consistent with the properties of autonomous ODEs. The sgn(ω₀) in figure 2 is used as a pre-factor on θ to indicate that ω₀ > 0 leads to the surface of solutions as shown while ω₀ < 0 leads to a reflection of the surface about the t − E plane.

The types of pendulum solutions admitted by (1) and shown in figure 2 are classified by their value of Ē ∈ [0, ∞), approaching the standard closed-form linear approximations as Ē → 0 and Ē → ∞. As Ē → 0, simple harmonic motion arises in the form θ ≈ θ₀ cos(t̄) + ω₀ sin(t̄), this limit being accessed for small θ in (1) such that sin(θ) ≈ θ. As Ē → ∞, uniform circular motion arises in the form θ ≈ ω₀t̄, this limit being accessed by making dθ/dt̄ large in (2) such that the potential energy term 1 − cos(θ₀) (always of O(1)) can be neglected, allowing direct integration of (2).

In reality, the system (1) admits 3 types of pendulum motion, as shown in figure 2: the librating (i.e., “swinging” back and forth) trajectory for Ē < 2, the separatrix orbit for Ē = 2 (dashed line in figure) in which the pendulum rises towards the top, never reaching it yet never falling back down, and the rotating or “overturning” case for Ē > 2.

For definiteness, we define the pendulum’s period T as follows. For libration (Ē < 2), we use the usual definition of period, such that θ satisfies θ(t̄) = θ(t̄ ± T) while for rotation (Ē > 2) we use the unwound definition θ(t̄) = θ(t̄ ± T) ± sgn(ω₀)2π where sgn(ω₀) is 1 for counter-clockwise rotation and −1 for clockwise rotation. For the separatrix orbit (Ē = 2), T is infinite and the solution is not periodic. As mentioned in section 1, the evaluation of T itself dates back to the 18th century [6, 7]; it is obtained by applying separation of variables to (2), solving for t̄ in terms of an integral in θ, and then either: (for Ē < 2) letting ω₀=0, integrating between the quarter-period limits (θ = 0 to θ = θ₀), and multiplying by 4; or (for Ē > 2) letting θ₀ = 0, integrating between the half-period limits (θ = 0 to θ = π) and multiplying by 2. Although these integrals cannot be evaluated in closed-form, they may be manipulated into elliptic integral form by applying the half-angle identity, employing a
FIG. 2. The solution to (1) describing the trajectory of a pendulum over time for several energy levels \( \bar{E} \), encompassing the 3 possibilities: libration (\( \bar{E} < 2 \)), the separatrix orbit (\( \bar{E} = 2 \)), and rotation (\( \bar{E} > 2 \)). The location of \( \bar{t} = 0 \) is chosen to be at the bottom of the pendulum’s trajectory, such that all energy states can be shown from the same starting point. The \( \text{sgn}(\omega_0) \) is used as a pre-factor on \( \theta \) to indicate that \( \omega_0 > 0 \) leads to the surface of solutions as shown while \( \omega_0 < 0 \) leads to a reflection of the surface about the \( t - \bar{E} \) plane. The solutions shown may be considered exact to within machine precision and are generated using analytic formulae (23), (24), and (28) developed in sections III C, IV, and V.

variable substitution\(^3\) and a rewriting the resulting expression in terms of \( \bar{E} \) to obtain

\[
T = \begin{cases} 
4 \left( \sqrt{\frac{E}{2}} \right) K(\sqrt{\frac{E}{2}}) & \text{if } \bar{E} < 2 \\
2 \sqrt{\frac{2}{\bar{E}}} K(\sqrt{\frac{2}{\bar{E}}}) & \text{if } \bar{E} > 2.
\end{cases}
\]  

\(^3\) For \( \bar{E} < 2 \), let \( \sin \frac{\omega_0}{2} \sin u = \sin \frac{\theta}{2} \). For \( \bar{E} > 2 \), let \( u = \frac{\theta}{2} \).
where
\[ K(k) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2(\phi)}}. \tag{4} \]

In (4), \( K(k) \) is the complete elliptic integral of the first kind \([8]\).

While figure 2 shows trajectory solutions of (1) over several periods, all information above and below the separatrix for all \( \bar{t} \) is, in fact, obtainable from reflections and translations of the solution in a finite interval \( t \in [0, T^*] \), afforded by the symmetry of the problem. We define the effective period \( T^* \) to be the minimum portion of the period necessary to map out the entire solution. In the case of libration, the pendulum sweeps out the same (albeit negative) angles in time as it moves from \( \theta = 0 \) to \( -\theta_0 \) as it does going from \( \theta = 0 \) to \( \theta = \theta_0 \). Thus \( T^* \) for a librating pendulum is the quarter-period. In the case of rotation, instead of the back-and-forth motion described above, the pendulum swings “over the top” and ends up back where it started from. Here, the pendulum sweeps out the same angles (albeit with a \( \pi \) offset) in time as it moves from \( \theta = \theta_0 \) to \( \theta_0 + \text{sgn}(\bar{\omega}_0)\pi \) as it does going from \( \theta = \theta_0 + \text{sgn}(\bar{\omega}_0)\pi \) to \( \theta_0 + 2 \text{sgn}(\bar{\omega}_0)\pi \). Thus \( T^* \) for a rotating pendulum is the half-period.

To summarize:
\[ T^* = \begin{cases} \frac{T}{4} & \text{if } \bar{E} < 2 \\ \frac{T}{2} & \text{if } \bar{E} > 2, \end{cases} \tag{5} \]

where the separatrix orbit \( (\bar{E} = 2) \) is omitted from the above definition since it is not periodic.

Interestingly, initial conditions corresponding to \( \bar{E} = 2 \) lead to simplifications such that the elliptic integral for \( \bar{t}(\theta) \) (obtained by separating (2)) can be evaluated in closed form and \( \theta(t) \) can be solved for explicitly \([18]\) as
\[ \theta = -\pi + 4 \text{arctan} \left[ e^{\bar{t}} \tan \left( \frac{\theta_0 + \pi}{4} \right) \right], \text{ for } \bar{E} = 2. \tag{6a} \]

For the libration and rotation cases, no known solution exists for (1) in terms of elementary functions; however, as mentioned in section 1, the solution can be written in terms of Jacobi elliptic functions. These solutions are obtained by making the substitution \( y = \sin(\theta/2) \) into the energy equation (2), enforcing initial conditions for either \( \bar{E} < 2 \) or \( \bar{E} > 2 \), and

\[ 4 \text{ Note that, in arriving at (6a) through the route of elliptic integrals described here, } \theta_0 \text{ is taken to be } 0 \text{ in [18], whereas we leave } \theta_0 \text{ to be arbitrary. An alternative derivation (starting from elliptic functions) is } \]

\[ 8 \text{ given in [5] where } \theta_0 \text{ is also taken to be } 0. \]
recognizing the resulting ODE as that governing an elliptic function [20]. For libration and rotation, the solutions are respectively,

\[ \theta = 2 \arcsin \left[ \sqrt{\frac{E}{2}} \cd \left( \tilde{t}, \sqrt{\frac{E}{2}} \right) \right], \text{ for } \tilde{E} < 2, \quad \tilde{\omega}_0 = 0, \tag{6b} \]

and

\[ \theta = 2 \arcsin \left[ \sn \left( \sqrt{\frac{E}{2}} \tilde{t}, \sqrt{\frac{2}{E}} \right) \right], \text{ for } \tilde{E} > 2, \quad \theta_0 = 0 \tag{6c} \]

for a given \( \tilde{E} \). In (6a), (6b), and (6c), note that, once the energy \( \tilde{E} \) is set, only one initial condition (either \( \theta_0 \) or \( \tilde{\omega}_0 \)) is needed to specify the solution. The remaining initial condition may then be extracted from (2). As stated previously, all initial conditions can be accessed by shifting solutions of the same energy \( \tilde{E} \). As there can be notational differences in the literature, for definiteness we utilize the convention [8, 20] in (6b) and (6c) that the Jacobi elliptic functions \( \cd(u, k) \) and \( \sn(u, k) \) satisfy the nonlinear initial value problems

\[ \left( \frac{d \cd}{du} \right)^2 = \left( 1 - \cd^2 \right) \left( 1 - k^2 \cd^2 \right), \quad \cd(0, k) = 1 \]

and

\[ \left( \frac{d \sn}{du} \right)^2 = \left( 1 - \sn^2 \right) \left( 1 - k^2 \sn^2 \right), \quad \sn(0, k) = 0. \]

The solution of these equations is not known analytically in closed form; however, efficient numerical algorithms exist to evaluate the Jacobi elliptic functions in many scientific computing packages. As an alternative to the numerical solution of (1) or the solutions forms (6b) and (6c), in section III of this paper we generate an exact solution to (1) via a convergent power series, whose convergence is subsequently accelerated in section IV. Although the solution of the pendulum equation in this paper provide an alternative to the numerical evaluation of the elliptic functions, the expressions (6b) and (6c) are important when making exact statements about singularity locations in section III. Note that there are several other equivalent representations of the solution to (1) involving other Jacobi elliptic functions. In fact, one may also express the solution as a single elliptic function that governs all pendulum movement [19].

III. THE PENDULUM SERIES AND COMPLEX-TIME SINGULARITIES

The power series solution of (1) follows a standard approach [9], and is developed in section IIIA; this leads to a recurrence formula for its coefficients, equivalent to that obtained
via a different technique in [10]. However, the previous solution [10] leaves implementation
to the reader with arbitrary initial conditions and does not examine convergence properties
of the series, leaving the solution itself as incomplete. The radius of convergence of the
series based on closest singularities is determined in section III B through the placement of
judicious expansion points (or, equivalently, initial conditions) this enables us to create a
convergent series in the time range $t \in [0, T^*]$ (see (5) and surrounding discussion). This
solution is subsequently utilized to construct the solution to (1) for all time in section III C.

A. Power Series Solution to (1) for $\theta(t)$

The power series solution may be generated by assuming

$$\theta = \sum_{n=0}^{\infty} a_n \bar{t}^n,$$

substituting this into (1), and equating like-powers of $\bar{t}$ on each side. The result is a recur-
rence expressed as

$$a_{n+2} = \frac{-s_n}{(n+1)(n+2)}, \quad a_0 = \theta_0, \quad a_1 = \bar{\omega}_0$$

(7)

where $s_n$ corresponds to coefficients one obtains by expanding the sine of an infinite series,

i.e.,

$$\sin \left( \sum_{n=0}^{\infty} a_n \bar{t}^n \right) = \sum_{n=0}^{\infty} s_n \bar{t}^n.$$  

(8)

The expansion (8) is obtained using the same method as that for raising a a series to a
power [21], taking the log of a series [14], and taking the exponential of a series [15]. In fact,
we make direct use of the latter formula in what follows, and repeat its construction here.

The exponential of a series

$$\exp \left( \sum_{n=0}^{\infty} a_n \bar{t}^n \right) = \sum_{n=0}^{\infty} b_n \bar{t}^n$$

(9)

is obtained by first noting that $b_0 = e^{a_0}$ and then differentiating both sides of (9) such that

$$\left[ \sum_{n=0}^{\infty} (n+1) a_{n+1} \bar{t}^n \right] \exp \left( \sum_{n=0}^{\infty} a_n \bar{t}^n \right) = \sum_{n=0}^{\infty} (n+1) b_{n+1} \bar{t}^n,$$

(10)

which can be rewritten, using (9), as

$$\left[ \sum_{n=0}^{\infty} (n+1) a_{n+1} \bar{t}^n \right] \sum_{n=0}^{\infty} b_n \bar{t}^n = \sum_{n=0}^{\infty} (n+1) b_{n+1} \bar{t}^n.$$  

(11)
Cauchy’s product rule is then applied to the left-hand side of (11) to arrive at
\[ \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (k+1) a_{k+1} b_{n-k} \right) \bar{t}^{n} = \sum_{n=0}^{\infty} (n+1) b_{n+1} \bar{t}^{n}, \]  
(12)
after which, equating like powers of \( \bar{t} \) in (12) leads to a recursive expression for the coefficients of the expansion of an exponential of an infinite series, collected with (9) as
\[ \exp \left( \sum_{n=0}^{\infty} a_{n} \bar{t}^{n} \right) = \sum_{n=0}^{\infty} b_{n} \bar{t}^{n}, \]

\[ b_{n+1} = \frac{1}{(n+1)} \sum_{k=0}^{n} (k+1) a_{k+1} b_{n-k}, \quad b_{0} = e^{a_{0}}. \]  
(13)
To find the coefficients of (8), we now define the cosine of a series as
\[ \cos \left( \sum_{n=0}^{\infty} a_{n} \bar{t}^{n} \right) = \sum_{n=0}^{\infty} c_{n} \bar{t}^{n}, \]  
(14)
and apply Euler’s identity
\[ \exp \left( i \sum_{n=0}^{\infty} a_{n} \bar{t}^{n} \right) = \cos \left( \sum_{n=0}^{\infty} a_{n} \bar{t}^{n} \right) + i \sin \left( \sum_{n=0}^{\infty} a_{n} \bar{t}^{n} \right), \]
directly to (13) (using the notation of (8) and (14)) to yield:
\[ c_{n+1} + i s_{n+1} = \frac{1}{(n+1)} \sum_{k=0}^{n} (k+1) i a_{k+1} \left( c_{n-k} + i s_{n-k} \right). \]  
(15)
In (15), it is clear that \( s_{0} = \sin(a_{0}) \) and \( c_{0} = \cos(a_{0}) \) from (8) and (14), respectively. Before continuing, it is useful to note that \( a_{n}, c_{n}, \) and \( s_{n} \) are real for all \( n \). Now, equating the real part of each side of (15) we obtain a recursion for the coefficients of (14). Similarly, equating the imaginary part of each side of (15), we obtain a recursion for the coefficients of (8). Collecting these (coupled) recursions with (7), we may now write the series solution for the pendulum equation (1) as
\[ \theta = \sum_{n=0}^{\infty} a_{n} \bar{t}^{n}, \quad |\bar{t}| < \bar{t}_{\text{ROC}} \]  
(16a)
\[ a_{n+2} = \frac{-s_{n}}{(n+1)(n+2)}, \quad a_{0} = \theta_{0}, \quad a_{1} = \bar{\omega}_{0}, \]  
(16b)
\[ s_{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} (k+1) a_{k+1} c_{n-k}, \quad s_{0} = \sin(\theta_{0}) \]  
(16c)
\[ c_{n+1} = \frac{-1}{n+1} \sum_{k=0}^{n} (k+1) a_{k+1} s_{n-k}, \quad c_{0} = \cos(\theta_{0}). \]  
(16d)
where $\tilde{t}_{\text{ROC}}$ is the radius of convergence of the series, to be discussed in section \textbf{III.B}.

An alternative derivation of the recursion in series (16) is given in \cite{10} by writing (11) as a coupled system of 4 ODEs in dependent variables $\theta$, $d\theta/dt$, $\sin(\theta)$, and $\cos(\theta)$ and relating the coefficients of their series expansions. As another alternative to the approach used here to derive the series solution to (1), one could substitute the well-known series expansions \cite{8, 22} of the Jacobi elliptic functions into (6b) and (6c). However, an arcsine of the resulting series is then required to extract the solution for $\theta(t)$, and for this reason, it is easier to proceed as presented above.

A comparison of the numerical solution\textsuperscript{5} to (1) and the $(N + 1)$-term partial sums of the series solution (16) are shown in figure 3 for both libration and rotation. Note that the series diverges at a finite $\tilde{t}_{\text{ROC}}$. In generating the expansion (16) for figure 3 we chose initial conditions at the bottom of the pendulum trajectory (i.e., $\theta_0 = 0$ in figure 3a). Figure 3 indicates that for these choices of initial conditions, the pendulum series (16) cannot fully describe the solution over the effective period $T^*$ due to divergence. In the following section, we examine how $\tilde{t}_{\text{ROC}}$ is affected by initial conditions, and determine an optimal choice to ensure convergence over the full range 0 to $T^*$. Note that such an exploration is not restrictive, since the pendulum trajectory is invariant for a given choice of $\tilde{E}$ as discussed previously - the locus of all initial conditions that achieve a given $\tilde{E}$ yield trajectories that are simply shifted in time but sweep through the same angles.

\textsuperscript{5} All numerical solutions in this work are generated using the 4\textsuperscript{th}-order Runge-Kutta method with $\Delta \tilde{t} = 10^{-5}$. Points on figures have been down-sampled for readability.
FIG. 3. Series solution (16) (dashed lines) taken up to (and including) the $\bar{t}^N$ term, shown for $N=10$, $10^2$, and $10^3$ and compared with the numerical solution (○) to (1) over 1 period. The solution is initialized at the bottom of the pendulum’s trajectory for cases of (a) libration ($\bar{E} < 2$) with $\bar{E} = 1.71$; and (b) rotation ($\bar{E} > 2$) with $\bar{E} = 2.02$. Note that, for either case, the series does not converge to the effective period $\bar{t} = T^*$ (indicated by a vertical line in the plots).

B. Singularities of $\theta(t)$ and series convergence

Although any initial conditions ($\theta_0, \omega_0$) may be chosen in (16), the value of $\bar{t}_{ROC}$ is affected by this choice. Here, we show that there is an optimal choice of ($\theta_0, \omega_0$) such that (16) converges beyond the effective period ($\bar{t} = T^*$) for all energy levels $\bar{E}$ – and thus all possible motions – of the pendulum. This enables the series solution (16) to be used as an exact solution that describes the pendulum trajectory for all $\bar{t}$.

In order to determine the radius of convergence, $\bar{t}_{ROC}$, of the pendulum series (16), the singularities of the pendulum function in the complex-time plane are examined. The value of $\bar{t}_{ROC}$ is equal to the distance between the expansion point, chosen to be $\bar{t}=0$ in (16), and its closest singularity in the complex plane. For the case of the separatrix orbit ($\bar{E}=2$), the arctan in (6a) admits logarithmic branch point singularities, the smallest of which are at $\bar{t} = ±i\pi/2$ (for $\theta_0 = 0$ in (6a)), such that $\bar{t}_{ROC} = \pi/2$. However, since an exact solution is known for the separatrix case in terms of elementary functions, this restriction is illustrative but not relevant. Unlike the separatrix case for which a closed-form solution exists, or for the inverse solution $t(\theta)$ where the locations of singularities are known from the integrands in the elliptic integrals involved (such as (4)), the time singularities of $\theta(t)$ for $\bar{E} \neq 2$ are not obvious. As mentioned in section I, it is typical that nonlinear ODEs such as (1) contain
“spontaneous” singularities that are not discernible by inspection of the ODE itself \[12\]-\[15\] \[23\]. In such problems, an estimate for $\bar{t}_{ROC}$ may be obtained from numerical versions of the root or ratio tests, the latter of which is often represented graphically as a Domb-Sykes plot \[24\].

Since the Jacobi elliptic functions have been well studied, their pole locations are known exactly, and thus the singularities in the pendulum function are also known as logarithmic branch points \[6\] at the same locations through \((6b)\) and \((6c)\). This enables an exact determination of $\bar{t}_{ROC}$ for the pendulum series \((16)\). For libration, the Jacobi elliptic function $\text{cd}(\bar{t}, \sqrt{\bar{E}/2})$ appearing in \((6b)\) has an infinite number of poles \[8\] at the complex time values

$$
\bar{t}_{\text{poles}} = n_o K \left( \sqrt{\frac{\bar{E}}{2}} \right) + n'_o i K' \left( \sqrt{\frac{\bar{E}}{2}} \right), \text{ for } \bar{E} < 2, \bar{\omega}_0 = 0
$$

where $n_o$ and $n'_o$ are odd integers, $K(k)$ is given by \(4\), and $K'(k)$ is (in general) defined as

$$
K'(k) \equiv K(\sqrt{1-k^2}).
$$

In \[18\], $K'(k)$ is the magnitude of the “imaginary quarter-period” \[4\] \[20\]. The singularities prescribed by \(17\) are shown in the complex $\bar{t}$-plane in figure 4 for a constant value of $\bar{E} < 2$; note that the singularity structure relative to $\bar{t} = 0$ is qualitatively the same for all values of $\bar{E} < 2$ so one figure suffices to draw general conclusions. The series expansion about $\bar{t} = 0$ given by \((16)\) encounters the closest singularity when $n_o = n'_o = 1$ in \(17\) and the distance to it (i.e., the radius of convergence), $\bar{t}_{ROC}$, is given as follows:

$$
\bar{t}_{ROC} = \sqrt{T^*^2 + \left[ K' \left( \sqrt{\frac{\bar{E}}{2}} \right) \right]^2} > T^*; \text{ valid for } \bar{E} < 2, \bar{\omega}_0 = 0.
$$

As noted in \(19\), the radius of convergence of \((16)\) is always larger than the effective period $T^*$ (defined in \(5\)) for the given initial conditions, since the integral $K'(k)$ is never 0. This circle of convergence is shown as a solid curve in figure 4 centered around the expansion point $\bar{t}=0$ with initial conditions at the top of the pendulum’s trajectory ($\theta_0 \neq 0, \bar{\omega}_0 = 0$); this is an optimal expansion point with respect to convergence for a given $\bar{E} < 2$, as indicated in the figure. If one were to expand about $\bar{t} = \pm T^*$ or, equivalently, redefine $\bar{t} = 0$ on the

\[6\] The change of singularity type from a pole to a logarithmic branch point is due to the arcsin in \((6b)\) and \((6c)\).
real-axis and choose initial conditions at the bottom of the trajectory \((\theta_0 = 0, \omega_0 \neq 0)\) for the same \(\bar{E} < 2\), the radius of convergence of \((16)\) is less than the effective period \(T^*\), as shown by the smaller dashed circles in figure 4; this is observed in figure 3a, where the series diverges prior to reaching \(t = T^*\). Note that the radius of convergence decreases monotonically for a given \(E < 2\) as the initial conditions are moved from the top of the trajectory \((\theta_0 \neq 0, \omega_0 = 0)\) to the bottom of the trajectory \((\theta_0 = 0, \omega_0 \neq 0)\) for the case of a librating pendulum. This statement is made in the context of Equation (2), where \(\theta_0\) and \(\bar{\omega}_0\) are not independent quantities for a given \(\bar{E}\)– hence the omission of \(\theta_0\) in (19).

For the case of rotation \((E > 2)\), a similar expression for pole locations is available for the Jacobi elliptic function \(\operatorname{sn}(\sqrt{\bar{E}/2} \bar{\tau}, \sqrt{2/\bar{E}})\) that appears in (6c); here, the infinite number of singularities in the pendulum function are placed in the complex \(\bar{\tau}\)-plane at

\[\bar{\tau}_{\text{poles}} = n_e \bar{K}\left(\sqrt{\frac{2}{\bar{E}}}\right) + n'_o i\bar{K'}\left(\sqrt{\frac{2}{\bar{E}}}\right), \quad \text{for} \quad \bar{E} > 2, \theta_0 = 0, \quad (20)\]

where \(n_e\) and \(n'_o\) respectively denote even and odd integers and we define \(\bar{K} (k) \equiv (\sqrt{2/\bar{E}}) K(k)\) and \(\bar{K}'(k) \equiv (\sqrt{2/\bar{E}}) K'(k)\) with \(K'(k)\) given by (18). The singularities prescribed by (20) are shown in the complex \(\bar{\tau}\)-plane in figure 5 for a constant value of \(\bar{E} > 2\). The singularity structure for rotation is qualitatively similar for all values of \(\bar{E} > 2\) so one figure suffices to draw general conclusions. The circle of convergence for the series expansion (16) taken about \(t = 0\), with initial conditions specified by (20) at the bottom of the pendulum trajectory, is shown as the leftmost dashed circle in figure 5. A circle of convergence of the same size is shown centered around \(\bar{t} = T^*\), where the pendulum is also at the bottom of its trajectory. In both cases, the radius of convergence\(^7\) is shown to be less than \(T^*\), such that the series diverges prior to reaching the effective period. This is observed in figure 3b, where the series (16) is initialized with \(\theta_0 = 2\pi\) and diverges prior to reaching \(t = T^*\). All other points initialized at the bottom of a rotating pendulum’s trajectory are equally detrimental to convergence for a given \(\bar{E} > 2\). As shown in figure 5 an optimal expansion point (or initial condition) for a given energy \(\bar{E} > 2\) with respect to convergence is one that is again taken at the top of the pendulum’s trajectory (e.g., \(\theta_0 = \pi, \bar{\omega}_0 \neq 0\)). Using the solid circle centered

\(^7\) The precise value of the radius of the smaller circles in figure 5 may be obtained by letting \(n_e = 0\) and \(n'_o = 1\) in (20) and taking its modulus; this radius is only larger than \(T^*\) if \(\bar{E} > 4\). Thus, the pendulum series for rotation cannot converge beyond \(\bar{t} = T^*\) for \(\bar{E} \in [2, 4]\) if initialized at the bottom of the trajectory.
FIG. 4. Schematic showing periodic structure of the singularities (shown as •’s) of (6b) for the librating pendulum at a constant value of $\bar{E} < 2$, as specified by (17). The integrals $K(k)$ and $K'(k)$ are defined by (4) and (18), respectively. The radius (19) and placement of the solid circle corresponds to the original initial conditions given by (6b), initializing the pendulum at the top of its trajectory with zero speed; this is the optimal radius of convergence $\bar{t}_{ROC}$ for the series solution (16) of a librating pendulum. By contrast, dashed circles show the radius of convergence when a librating pendulum is initialized at the bottom of the trajectory with nonzero angular speed.

at $t = T^*$ and the surrounding lattice in figure 3, the optimal radius of convergence and initial conditions for rotation ($\bar{E} > 2$) is given by

$$\bar{t}_{ROC} = \sqrt{T^*^2 + (\bar{K}' \left(\sqrt{\frac{2}{\bar{E}}}\right))^2} > T^*; \text{ valid for } \bar{E} > 2, \theta_0 = \pi,$$

where again, the radius of convergence is always larger than the effective period $T^*$, since $\bar{K}'(k)$ is nonzero for finite $\bar{E}$. As a reminder when interpreting these equations, note that, by inspection of (2), $\theta_0$ and $\bar{\omega}_0$ are not independent quantities for a given $\bar{E}$, allowing for
the omission of $\bar{\omega}_0$ in (21).

\[ \tilde{\theta}(\bar{\theta}) = \sum_{n=0}^{\infty} a_n \bar{\theta}^n, \quad (\bar{\theta}_0, \bar{\omega}_0) = \begin{cases} \arccos(1 - \bar{E}), & \bar{E} < 2 \\ \pi, & \bar{E} > 2 \end{cases}, \quad |\bar{\theta}| \leq T^* \quad (22) \]

FIG. 5. Schematic showing the periodic structure of the singularities (shown as $\bullet$'s) of (6b) of (6c) for the rotating pendulum at a constant value of $\bar{E} > 2$, as specified by (20). $\tilde{K}(k) \equiv \sqrt{2/\bar{E}} \, K(k)$ and $\tilde{K}'(k) \equiv \sqrt{2/\bar{E}} \, K'(k)$, where the integrals $K(k)$ and $K'(k)$ are defined by (4) and (18), respectively. The leftmost dashed circle corresponds to the original initial conditions given by (6c), initializing the pendulum at the bottom of its trajectory. The radius (21) and placement of the solid circle correspond to a shifted initialization at the top of the pendulum’s trajectory, which is the optimal radius of convergence $\bar{t}_{ROC}$ for the series solution (16) of a rotating pendulum.

In both the case of libration and rotation, initializing the series solution to the pendulum at the highest point along the trajectory leads to the largest radius of convergence. This allows us to exploit the symmetry of the problem such that the series solution (16), when prescribed initial conditions as follows,
is an exact representation of the pendulum trajectory for all \( t \) between 0 and \( T^* \) and for any \( \bar{E} \) describing librating and rotating motions of the pendulum. In (22), the \( a_n \) coefficients are given in (16), the \( \pm \) symbol specifies either counterclockwise or clockwise rotation, and (2) has been used to represent the initial conditions (taken at the top of the trajectory) in terms of \( \bar{E} \).

C. Periodic extension of the series solution

The convergent series solution (22), justified in section III B for \( \bar{t} \leq T^* \), allows for a convergent solution to (1) for all \( \bar{t} \) as follows. For any given initial condition \((\theta_0, \bar{\omega}_0)\), one first computes the energy \( \bar{E} \) given by (2). Using this value of \( \bar{E} \), the convergent series for the solution up to the effective period \( T^* \) is constructed from (22) (or the resummation (24) provided in section IV) as

\[
\tilde{\theta}(\bar{t}) \equiv \theta(\bar{t} \leq T^*).
\]

The solution may then be extended for all \( \bar{t} \) using the symmetry of the problem described in section II, such that the periodic solution is given by

\[
\theta(\bar{t}) = \begin{cases} 
\tilde{\theta}(\bar{t}), & 0 \leq \bar{t} \leq T^* \\
-\tilde{\theta}(2T^* - \bar{t}), & T^* \leq \bar{t} \leq 2T^* \\
-\tilde{\theta}(\bar{t} - 2T^*), & 2T^* \leq \bar{t} \leq 3T^* \\
\tilde{\theta}(4T^* - \bar{t}), & 3T^* \leq \bar{t} \leq 4T^*
\end{cases}, \quad \bar{E} < 2
\]

\[
\pm \frac{2\pi}{T}(\hat{t} - \hat{t}) + \tilde{\theta}(\hat{t}), \quad 0 \leq \hat{t} \leq T^* \\
\pm \frac{2\pi}{T}(\hat{t} - \hat{t}) - \tilde{\theta}(2T^* - \hat{t}) + 2\pi \pm 2\pi, \quad T^* \leq \hat{t} \leq 2T^*
\]

\[
\bar{E} > 2
\]

where \( \hat{t} = t \) modulo \( T \), \( \bar{t} \geq 0 \).

As in (22), the \( \pm \) symbol in (23b) specifies either counterclockwise or clockwise rotation. The modulo operation\(^8\) in (23b) is used such that \( \hat{t} \) may be interpreted as the additional time beyond the smallest integer multiple of the period \( T \). In section V we provide an exact and rapidly converging series for the computation of \( T \) (and by extension, \( T^* \)), thus

---

\(^8\) The modulo operation is available in most programming languages. For example, the MATLAB syntax is \( \text{mod}(\bar{t}, T) \).
preserving the analyticity of the solution when used in (23). The re-initializing and period extension given by (23) is shown in figures 6 through 8. Note that the formerly-divergent series solutions of figures 3a and 3b have been re-initialized using (22) such that they are now convergent, and whose prior domains are now contained within the periodically extended domains of figures 6 and 7. A final step in this solution process is to shift the origin such that the originally intended initial conditions are at \( \bar{\theta} = 0 \).

The norm of the error for \( \bar{\theta} \in [0, T^*] \) incurred by series (22) is plotted versus \( \bar{E} \) (shown as dashed lines) in figure 9 for various values of series truncation. Although the error increases as \( \bar{E} \to 2 \), for any given \( \bar{E} \), the error can be reduced to machine precision by increasing the number of terms in the series. This is expected, since there are no singularities of the pendulum function in the circle \( |\bar{\theta}| < T^* \) for the initial conditions used in (22).

![Graph showing series solution](image)

**FIG. 6.** Series solution (23) (using (22) for \( \bar{\theta}(t) \)) taken up to (and including) the \( \bar{\theta}^N \) term, shown for \( N=2, 4, \) and 6 and compared with the numerical solution (●) to (1) for the case of libration (\( \bar{E} \approx 2 \)). The energy \( \bar{E} = 1.71 \) is same as that in figure 3a but here the series is initialized at the top of the trajectory such that it converges. The solution is shown over 2 periods above, encompassing the solution in figure 3a.
FIG. 7. Series solution \(23\) (using \(22\) for \(\tilde{\theta}(t)\)) taken up to (and including) the \(\tilde{t}^N\) term, shown for \(N=3, 6,\) and 36 and compared with the numerical solution (\(\bullet\)) to \((1)\) for the case of rotation \((\tilde{E} > 2)\). The energy \(\tilde{E} = 2.02\) is same as that in figure 3b but here the series is initialized at the top of the trajectory such that it converges. The solution is shown over 2 periods above, encompassing the solution in figure 3b.
IV. EXACT RESUMMATION OF THE SERIES SOLUTION

Although (22) is an exact analytic solution to the pendulum problem, its convergence may be accelerated by a similarly exact resummation technique. The resummation is motivated by the method of asymptotic approximants [12] that assures that it asymptotically approaches the exact solution at each end of the domain, taken here to be $\bar{t} \in [0, T^*]$. We match the low-order behavior at $\bar{t} = T^*$ (up to the 2nd derivative) and up any any desired order at $t = 0$. Using the choice of initial conditions prescribed by (22) to preserve analyticity, the resummation of (22) is

$$
\theta(\bar{t}) = \bar{\omega}^* (\bar{t} - T^*) + (\bar{t} - T^*)^2 \sum_{n=0}^{\infty} \hat{a}_n \bar{t}^n, \quad |\bar{t}| \leq T^*,
$$

$$
(\theta_0, \bar{\omega}_0) = \begin{cases} 
\arccos(1 - \bar{E}), & \bar{E} < 2 \\
\pi, & \bar{E} > 2 
\end{cases},
$$

(24a)

where

$$
\bar{\omega}^* \equiv \begin{cases} 
\sqrt{2\bar{E}}, & \bar{E} < 2 \\
\pm \sqrt{2\bar{E}}, & \bar{E} > 2
\end{cases}
$$

(24b)

enforces the correct angular velocity at $\bar{t} = T^*$ and the $\pm$ symbol specifies either counterclockwise or clockwise rotation. In (24a), the coefficients $\hat{a}_n$ are related to the known
coefficients $a_n$ (given by (16b)) as follows. We set (24a) equal to (16a) to obtain
\[ \sum_{n=0}^{\infty} a_n \bar{t}^n = \bar{\omega}^* (t - T^*) + (t - T^*)^2 \sum_{n=0}^{\infty} \bar{a}_n t^n, \]
and then isolate the $\hat{a}_n$ series onto one side to obtain
\[ \left( (t-T^*)^{-2} \right. \left[ \sum_{n=0}^{\infty} \left( (n + 1) \left( \frac{1}{T^*} \right)^{n+2} \right) \bar{t}^n \right] \sum_{n=0}^{\infty} b_n \bar{t}^n = \sum_{n=0}^{\infty} \hat{a}_n \bar{t}^n. \]
As indicated in the above, $(\bar{t} - T^*)^{-2}$ is expanded explicitly as an arithmetico-geometric series, and the coefficients
\[ b_0 = a_0 + T^* \bar{\omega}^*, \quad b_1 = a_1 - \bar{\omega}^*, \quad b_{n>1} = a_n \tag{24c} \]
are introduced to enable the left-hand side of (24) to be combined into a single Cauchy product series. The result is an expression for the $\hat{a}_n$ coefficients as
\[ \hat{a}_n = \sum_{k=0}^{n} b_{n-k} \left( (k + 1) \left( \frac{1}{T^*} \right)^{k+2} \right). \tag{24d} \]
Together, (24a), (24c), and (24d) form an exact resummation of (22) that explicitly enforces the asymptotic behaviors as $\bar{t} \to 0$ (from the power series (22)) and $\bar{t} \to T^*$, whereas (22) does not have this latter feature. We remind the reader that the restriction on initial conditions in (24) is not a restriction on energy; one can use (24) to provide convergent solutions for all librating and rotating energy states of the pendulum.

Taking the resummation (24) as $\tilde{\theta}(t)$ in the periodically extended solution (23), the $(N+1)$-term partial sums are shown in figure 10. Inspection of figure 8 and figure 10 generated for the same physical parameters, indicates that the resummation (24) accelerates convergence, in that less terms are needed to achieve the same accuracy. This convergence acceleration extends to other energy values, as shown in figure 9, where the norm of the error in (24) is shown (by a solid line) for various $(N+1)$-term partial sums of the resummation series (24). Note that further manipulations, shown in Appendix A, may be used to increase computational speed for any finite truncation of the resummation series (24).
FIG. 9. The error here is taken to be the difference between the numerical solution to [1] and either the series solution [22] or exact resummation [24] (as indicated in the figure) taken up to (and including) the $\bar{t}^N$ term and shown for various $N$. The infinity norm of this error is taken over the interval $\bar{t} \in [0, T^*]$ for each value of $\bar{E}$ shown, except at $\bar{E} = 2$ where an exact solution exists. The difference between the exact solution [6a] and the numerical solution for $\bar{E} = 2$ is shown as a horizontal line, indicating the effective machine precision, as a baseline of comparison with all the other error calculations shown (for which the same numerical scheme and time-step is used).

FIG. 10. Exact resummation [24] taken up to (and including) the $\bar{t}^N$ term, shown for $N=5$, 10, and 20, periodically extended using [23], and compared with the numerical solution (●) to [1] for same conditions of figure 8. A direct comparison with figure 8 shows the convergence advantage imparted by the resummation series [24] over the original series [22].
V. EXACT RESUMMATION OF THE PENDULUM PERIOD

In both the series solution (22) and its resummation (24), the series converge for $t \in [0, T^*]$, and $T^*$ explicitly appears in both (22) and the periodic extension formulae given in (23); thus, the value of $T^*$ (and by extension, the value of $T$) must be determined accurately. As can be seen in (3) and (5), in every case of period motion, $T^*$ is written in terms of the complete elliptic integral of the first kind, $K(k)$, where $k$ depends on the energy of the pendulum. Thus we are faced with computing the period either by numerical integration, using an approximation (of which there are many [25]), or using a truncation of the long-known infinite series representation of $K(k)$ [8, 26], namely:

$$K(k) = \frac{\pi}{2} \sum_{n=0}^\infty \frac{(2n)!}{n!^2} 16^{-n} k^{2n}, \quad k^2 < 1. \quad (25)$$

In (25), note that the $k^2 < 1$ constraint is always satisfied for the pendulum problem (see [2], for example). The error in calculating $T^*$ using (25) in (5) is shown by the dashed lines in figure 11, plotted versus $\bar{E}$ for partial sums up to and including the $k^{2N}$ term (i.e, $N+1$ terms in (25)). Note that the error increases as $\bar{E} \to 2$ where the period becomes infinite, as it corresponds to the separatrix in figure 2.

Figure 12 shows results close to $E = 2$ (above and below) that show the effect on the periodic extension of solution (24) via (23) when (25) is used to evaluate $T^*$ (see dashed curves). In the figure, a truncation of $N=100$ is used in (25) in order to illustrate that one would need to take more than 100 terms in the series (25) before a solution using this value of $T^*$ can be considered accurate.

In order to make practical use of (25), we accelerate its convergence using a resummation approach. To do so, we begin with an equivalent definition to (4) for the complete elliptic integral of the first kind [8]:

$$K(k) = \int_0^1 \left[(1 - v^2)(1 - k^2 v^2)\right]^{-1/2} dv. \quad (26)$$

We follow a similar approach to that used in [27] to strip off the singular behavior of (26) by first rewriting the integral as

$$K(k) = \int_0^1 \frac{\sqrt{1 - k^2 v^2} - \sqrt{1 - v^2}}{(1 - k^2 v^2)\sqrt{1 - v^2}} \frac{dv}{\int_{t_1}^{t_2}} + \int_0^1 \frac{1}{1 - k^2 v^2} dv.$$
FIG. 11. The error is shown for evaluating a pendulum’s effective period $T^*$ (and by extension, actual period) given by (5). In evaluating $T^*_{\text{series}}$, the elliptic integral $K(k)$ given by (4), is taken to be either the original series (25) (dashed line) or its resummation (28) (solid line) taken up to (and including) the $k^{2N}$ term. The exact solution, denoted here as $T^*_{\text{exact}}$, uses a numerical evaluation of (4) that has converged to within machine precision.

so that $I_2$ naturally handles the apparent integrand singularity at $k = 1$. Using partial fractions, this integral may be evaluated exactly as

$$ I_2 = \frac{1}{2k} \ln \left( \frac{1 + k}{1 - k} \right), $$

and Taylor expanded as

$$ I_2 = \frac{1}{2k} \ln \left( \frac{1 + k}{1 - k} \right) = \frac{\text{arctanh}(k)}{k} = \sum_{n=0}^{\infty} \frac{1}{2n + 1} k^{2n}. $$

Thus, we may use (27) to add 0 to (25) in such a way that the singular behavior (responsible for slow convergence) is exactly summed, leading to an improved series for $K(k)$, given as

$$ K(k) = \sum_{n=0}^{\infty} \left[ \frac{\pi}{2} \left( \frac{(2n)!}{n!^2} \right)^2 16^{-n} - \frac{1}{2n + 1} \right] k^{2n} + \frac{1}{2k} \ln \left( \frac{1 + k}{1 - k} \right), \quad k^2 < 1. $$

When used to compute $T^*$ (or $T$) in (5) (or (3)) the convergence of the resummation (28) is significantly improved, compared with that of (25), as seen in figure 11 for a range of $\bar{E}$ and in figure 12 for $\bar{E}$ close to 2.
FIG. 12. Here, we demonstrate the effect of period accuracy on the periodically extending via (23) (taken up to 20th order), shown for librating and rotating cases near \( \bar{E} = 2 \) as indicated in the figure. The numerical solution is given by ●. The long-dashed curves show a solution that uses a period computed using (3) with \( N = 100 \). The short-dashed and solid curves show a solution that uses a period computed using (3) with (28) and letting \( N = 1 \) and \( N = 10 \), respectively.

VI. CONCLUSIONS

The exact series solution to the pendulum problem, whether librating or rotating, attains optimal convergence when the initial conditions are chosen to be at the top of the pendulum’s trajectory, allowing the series to converge beyond the bottom of the trajectory. Through symmetry, this convergent solution is then used to construct the solution for all time. Taking inspiration from the method of asymptotic approximants, the exact power series solution is re-summed to accelerate convergence, based on the asymptotic behavior at the bottom of the trajectory. The resulting exact power series representation is shown to be more accurate than the original series for any given series truncation and pendulum energy.

We close this paper with some general comments about the methodology used here as well as the solution itself. When finding the series solution to a nonlinear ODE, there are (at least) 3 scenarios: (i) the radius of convergence is larger than the physical domain, (ii) the radius of convergence is precisely the physical domain length (e.g., series for Rayleigh’s
equation [16]), and (iii) the radius of convergence lies within the physical domain (e.g., Blasius
series [28]). The exact pendulum solution provided in this paper falls into case (i), once the
initial conditions are chosen with care. Here our analysis benefited from the properties of the
Jacobi elliptic functions whose singularity locations are known. In case (ii), a singularity is
known at the boundary of the domain and this is the closest singularity in the complex plane
to the expansion point. Case (iii) arises from spontaneous singularities and is commonplace
in the solution of nonlinear ODEs. Note that even when case (iii) occurs, resummation
methods such as asymptotic approximants may be used to analytically continue a series
over the whole domain to obtain a useful convergent solution (see [12–15]).

In contrast to the problem examined here, there is often no pre-knowledge of singularity
locations in the solution of nonlinear ODEs. For such cases, one can find the radius of
convergence for a series around a given expansion point numerically using a Domb-Sykes
plot [24] or root test. For autonomous equations such as this one, then, an optimal expansion
point would need to be deduced numerically; whether or not this expansion point leads an
exact solution depends on the final radius of convergence achieved. Note, however, that
even in cases where the radius of convergence is not larger than the domain of interest,
approximant techniques may be utilized to analytically continue such series and make them
useful. Coupled with approximants (when needed), the power series solution method is a
prevailing technique that can lead to efficient, useful, and many times exact solutions to
nonlinear ODEs.

Appendix A: Efficient computation of (24)

Recognizing that the exact resummation of the pendulum series, given by (24), will be
implemented in any code as a partial sum, we provide an efficient means of calculating the
deflection angle. One may observe that the truncation of (24a), denoted here as $\theta_N(t)$, has
the following structure

$$\theta_N(t) = \hat{\omega}^* \left( t - T^* \right) + \left( t - T^* \right)^2 \sum_{n=0}^{N} \hat{a}_n t^n. \quad (A1a)$$
Since the approximant given by (A1a) is just a degree $N + 2$ polynomial, we may rewrite it as
\[ \theta_N(t) = \sum_{n=0}^{N} a_n t^n + \alpha t^{N+1} + \beta t^{N+2}. \] (A1b)

Upon equating the above to the original series (16a), one finds that the $a_n$ coefficients are precisely the original series coefficients given in (16). Hence, in order to obtain the approximant in the form of (A1b), we first determine the $a_n$ from the formulae given in (16), and then use those to compute $\alpha$ and $\beta$ so that the first and second derivatives of (A1a) and (A1b) match at $\bar{t} = T^*$. This leads to a system of two linear equations in $\alpha$ and $\beta$, whose solution is
\[ \alpha = -(N + 2)\sigma_N(T^*)T^{*N-1} - (\bar{\omega} - \sigma_N'(T^*)) T^{*N} \] (A1c)
\[ \beta = (N + 1)\sigma_N(T^*)T^{*N-2} + (\bar{\omega} - \sigma_N'(T^*)) T^{*N-1} \] (A1d)

where
\[ \sigma_N(t) \equiv \sum_{n=0}^{N} a_n t^n. \] (A1e)

For comparison, the entire process to compute the resummation coefficients (for a given truncation $N$) via the formulae given in (24) (including the computation of series coefficients $a_n$ from (16)) requires two quadratic time steps. The first step is the recursion in (16) used to get the series solution, and the second is Cauchy’s product rule, used in (24). The method given in this appendix - recasting (24) as (A1) - removes the latter of those quadratic time steps, which leads to a substantial reduction in computational time - such as when the series is re-used for different values of $\bar{E}$ to construct the solution surface in figure 2.

[1] G. F. Simmons. *Differential Equations with Applications and Historical Notes*. McGraw-Hill, 2 edition, 1994.
[2] S. H. Strogatz. *Nonlinear Dynamics and Chaos*. Westview, 1994.
[3] H. von Helmholtz. *Vorlesungen über theoretische Physik*. Leipzig, 1898.
[4] P. Appell. Sur une interprétation des valeurs imaginaires du temps en mécanique. *Comptes Rendus.*, 87(1):1074–1077, 1878.

[5] E.T. Whittaker. *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies; With an Introduction to the Problem of Three Bodies.* Cambridge University Press, 2 edition, 1917.

[6] J. Stillwell. *Mathematics and its History.* Springer, 2 edition, 2002.

[7] J. Gray. *The Real and the Complex: A History of Analysis in the 19th Century.* Springer, 2015.

[8] P. F. Byrd and M. D. Friedman. *Handbook of Elliptic Integrals for Engineers and Scientists.* Springer-Verlag, 2 edition, 1971.

[9] R. P. Agnew. *Differential Equations.* McGraw-Hill, 1960.

[10] V. Fairén, V. López, and L. Conde. Power series approximation to solutions of nonlinear systems of differential equation. *Am. J. Phys.*, 56(1):57–61, 1988.

[11] J. P. Boyd. Padé approximant algorithm for solving nonlinear ordinary differential equation boundary value problems on an unbounded domain. *Comp. Phys.*, 11:299–303, 1997.

[12] N. S. Barlow, C. R. Stanton, N Hill, S. J. Weinstein, and A. G. Cio. On the summation of divergent, truncated, and underspecified power series via asymptotic approximants. *Q. J. Mech. Appl. Math.*, 70(1):21–48, 2017.

[13] E. R. Belden, Z. A. Dickman, S. J. Weinstein, A. D. Archibee, E. Burroughs, and N. S. Barlow. Asymptotic approximant for the falkner-skan boundary-layer equation. *Q. J. Mech. Appl. Math.*, 73(1):36–50, 2020.

[14] N. S. Barlow and S. J. Weinstein. Accurate closed-form solution of the SIR epidemic model. *Physica D*, 408:132540:1–4, 2020.

[15] S. J. Weinstein, M. S. Holland, K. E. Rogers, and N. S. Barlow. Analytic solution of the SEIR epidemic model via asymptotic approximant. *Physica D*, 411:132633:1–6, 2020.

[16] A. Harkin, A. Giammarese, N. S. Barlow, and S. J. Weinstein. The rayleigh collapse of two spherical bubbles. arXiv:2102.05222.

[17] E. Ramé, S. J. Weinstein, and N. S. Barlow. Free surface shapes in rigid body rotation: Exact solutions, asymptotics and approximants. *IMA J. Appl. Math.*, doi:10.1093/imamat/hxab038, 2021.

[18] F. M. S. Lima. Analytical study of the critical behavior of the nonlinear pendulum. *Am. J. Phys.*, 78(11):1146–1151, 2010.
[19] K. Ochs. A comprehensive analytical solution of the nonlinear pendulum. *Eur. J. Phys.*, 32:479–490, 2011.

[20] W. A. Schwalm. *Lectures on Selected Topics in Mathematical Physics: Elliptic Functions and Elliptic Integrals*. Morgan & Claypool, 2015.

[21] P. Henrici. Automatic computations with power series. *JACM*, 3:10–15, 1956.

[22] S. Wrigge. Calculation of the Taylor series expansion coefficients of the Jacobian elliptic function sn(x, k). *Math. Comp.*, 36(154):555–564, 1981.

[23] C. M. Bender and S. A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers I: Asymptotic Methods and Perturbation Theory*. McGraw-Hill, 1978.

[24] C. Domb and M. F. Sykes. On the susceptibility of a ferromagnetic above the Curie point. *Proc. Roy. Soc. London A*, 240(1221):214–228, 1957.

[25] P. F. Hinrichsen. Review of approximate equations for the pendulum period. *Eur. J. Phys.*, 42:015005:1–19, 2021.

[26] M. Abramowitz and I Stegun. *Handbook of Mathematical Functions*, page 298. Dover, 1972.

[27] De-Sheng Xue, Zhao Zhou, and Mei-Zhen Gao. Accurate period approximation for any simple pendulum amplitude. *Chinese Physics Letters*, 29(4):044601, April 2012.

[28] H. Blasius. Grenzschichten in flussigkeiten mit kleiner reibung. *Zeitschrift fur Mathematik und Physik*, 56:1–37, 1908.