Quantization of Fermionic Fields with Two Mass States in the First Order Formalism

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Abstract

The relativistic 20-component wave equation, describing particles with spin 1/2 and two mass states, is analyzed. The projection operators extracting states with definite energy and spin projections, and density matrix are obtained. The canonical quantization of the field with two mass states in the formalism of the first order is performed and the chronological pairing of the 20-component operators was found.

1 Introduction

It is known that the problem of quark and lepton generations, and their mass spectrum has not been solved yet. The standard model (SM) of electroweak interactions does not explain the number of quark and lepton generations, mass spectrum and contains many free parameters. Barut suggested [1], (see also [2], [3], [4]) to describe $e$, $\mu$-leptons by the generalized Dirac equation of the second order. He interpreted this equation, based on the non-perturbative approach to quantum electrodynamics (QED), as an effective equation for partly “dressed” fermions. Some investigations of the generalized Dirac equation of the second order were made in [5] We have formulated this equation in the form of the first order generalized Dirac equation (FOGDE) [6]. This form is convenient for analyzing symmetry properties of the theory and different calculations. Here we continue to explore this matrix form of the equation.

The paper is organized as follows. In Sec. 2, we formulate FOGDE and corresponding Lagrangian. The projection operators extracting solutions with definite energy and spin projections for free particles are given in Sec. 3. In Sec. 4, we perform the canonical quantization of fields. The commutation relations for 20-component functions and the chronological pairing
of operators were obtained. We make a conclusion in Sec. 6. Appendix contains some useful formulas. We use notations as in \[7\] and the system of units $\hbar = c = 1$ is chosen.

2 First Order Field Equation

The Barut second order generalized Dirac equations \[1\], describing spin-1/2 particles, which possess two mass states, can be represented in the form \[6\]:

$$\left(\gamma_\nu \partial_\nu - \frac{a}{m} \partial_\mu^2 + m\right) \psi(x) = 0. \quad (1)$$

where $\partial_\nu = \partial/\partial x_\nu = (\partial/\partial x_m, \partial/\partial (it))$, $\psi(x)$ is a Dirac spinor and the Dirac matrices $\gamma_\mu$ obey the commutation relations $\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_\mu\nu$. The $m$ is a parameter with the dimension of the mass, and $a$ is a massless parameter in Eq. (1).

Masses of spin-1/2 particles are given by

$$m_1 = \pm m \left(1 - \sqrt{4a + 1}\right), \quad m_2 = \pm m \left(1 + \sqrt{4a + 1}\right). \quad (2)$$

For a given parameter $a$, one can chose the positive values (for $a \geq -1/4$) of masses $m_1, m_2$ in Eq. (2).

According to Barut’s idea \[1\], the states of fermions with two masses are interpreted as an electron and a muon. One can apply Eq. (1) for describing quarks or massive neutrinos.

It was shown in \[6\] that if we introduce the 20-component function

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} \psi(x) \\ \psi_\mu(x) \end{pmatrix}, \quad (3)$$

where

$$\psi_\mu(x) = -\frac{1}{m} \partial_\mu \psi(x)$$

and index $A$ runs values $A = 0, \mu$, Eq. (1) becomes

$$(\alpha_\nu \partial_\nu + m) \Psi(x) = 0. \quad (4)$$

The 20-dimensional matrices of the first order wave equation (4) are

$$\alpha_\nu = \left(\varepsilon^{\nu,0} + a\varepsilon^{0,\nu}\right) \otimes I_4 + \varepsilon^{0,0} \otimes \gamma_\nu, \quad (5)$$

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where $I_4$ is the unit $4 \times 4$ matrix, and $\otimes$ means the direct product of matrices. The elements of the entire matrix algebra $\varepsilon^{A,B}$ obey equations as follows:

$$
\left( \varepsilon^{M,N} \right)_{AB} = \delta_{MA}\delta_{NB}, \quad \varepsilon^{M,A} \varepsilon^{B,N} = \delta_{AB} \varepsilon^{M,N},
$$

where $A, B, M, N = 0, 1, 2, 3, 4$. The first equation in (6) shows the matrix elements of the matrix $\varepsilon^{M,N}$, and the second equation presents the product of two matrices. The relativistic wave equation of the first order (4) is convenient for different applications because the matrices of the equation, $\alpha_{\nu}$, expressed through the elements of the entire matrix algebra with simple properties (6).

The Lagrangian is given by

$$
\mathcal{L} = -\frac{1}{2} \left[ \overline{\Psi}(x)\alpha_{\mu} \partial_{\mu} \Psi(x) - \left( \partial_{\mu} \overline{\Psi}(x) \right) \alpha_{\mu} \right] - m \overline{\Psi}(x) \Psi(x),
$$

where $\overline{\Psi}(x) = \Psi^+(x)\eta$, $\eta$ is the Hermitianizing matrix, and $\Psi^+(x)$ is the Hermitian-conjugate wave function. The Euler-Lagrange equation (4) follows from the Lagrangian (7). The Hermitianizing matrix $\eta$ obeys the relations

$$
\eta \alpha_m = -\alpha_m^+ \eta^+, \quad \eta \alpha_4 = \alpha_4^+ \eta^+, \quad \eta^+ = \eta \quad (m = 1, 2, 3).
$$

and is given by

$$
\eta = \left( a \varepsilon^{m,m} - a \varepsilon^{4,4} - \varepsilon^{0,0} \right) \otimes \gamma_4.
$$

Eq. (8) guarantees that the Lagrangian (7) is the real function. By varying the action $S = \int \mathcal{L} d^3x$ with respect to $\Psi(x)$, one may obtain the second Euler-Lagrange equation:

$$
\left( \partial_{\nu} \overline{\Psi}(x) \right) \alpha_{\nu} - m \overline{\Psi}(x) = 0.
$$

This equation may also be obtained by Hermitian conjugating Eq. (4), and multiplying it on $\eta$, and taking into account Eqs. (8).

3 Mass and Spin Projection Operators

Now we consider states of particles with definite energy, $p_0$, and momentum, $\mathbf{p}$. It should be noted that for definite mass state, one has

$$
p_0 = \sqrt{\mathbf{p}^2 + m_1^2}, \quad \text{or} \quad p_0 = \sqrt{\mathbf{p}^2 + m_2^2}.
$$
This means that there is an additional quantum number $\tau$, so that $\tau = 1, 2$ for states with masses $m_1$ and $m_2$. We omit this index $\tau$ in four momentum $p = (p, ip_0)$ keeping this in mind. Solutions of Eq. (4) with definite energy and momentum in the form of plane waves are given by

$$\Psi_{\pm p}^{(\pm)}(x) = \frac{1}{\sqrt{2p_0 V}}U(\pm p)\exp(\pm ipx),$$

where $V$ is the normalization volume. There are two solutions (11) corresponding to Eqs. (10). Replacing Eq. (11) into Eg. (4), one obtains

$$(\pm i\hat{p} + m) U(\pm p) = 0,$$

where $\hat{p} = \alpha_{\mu}p_{\mu}$. The solution $\Psi_{+ p}^{(\pm)}(x)$ with positive frequency $\omega = p_0$ describes particles and the solution $\Psi_{- p}^{(\pm)}(x)$ with negative frequency $\omega = -p_0$ corresponds to antiparticles. We use here the normalization condition

$$\int_V \Psi_{\pm p}^{(\pm)}(x)\alpha_{\mu}\Psi_{\pm p}^{(\pm)}(x)d^3x = 1,$$

where $\Psi_{\pm p}^{(\pm)}(x) = \left(\Psi_{\pm p}^{(\pm)}(x)\right)^+ \eta$. It is implied the integration over the volume $V$ in Eq. (13). The difference from QED is that all functions in Eqs. (11)-(13) are 20-component functions and $\eta \neq \alpha_4$. In addition, Eqs. (11)-(13) are valid for two mass values ($\tau = 1, 2$), Eq. (10). Normalization conditions for 20-component functions $U(\pm p)$ follow from Eqs. (11)-(13):

$$\bar{U}(\pm p)\alpha_{\mu}U(\pm p) = -2i\hat{p}_{\mu},$$

$$\bar{U}(\pm p)U(\pm p) = \mp 2\frac{p^2}{m}.$$  

For $\tau = 1$, $p^2 = -m_1^2$, and for $\tau = 2$, $p^2 = -m_2^2$. Eqs. (14), (15) are analogs of normalization conditions for bispinors in QED. But, using the definition of the 20-component function $\Psi(x)$, Eq. (3), we find, from Eq. (15), the normalization condition for bispinor $u(\pm p)$ (see also [4] for comparison):

$$\bar{u}(\pm p)u(\pm p) = \pm \frac{2mp^2}{m^2 - ap^2},$$

where $\bar{u}(\pm p) = u^+(\pm p)\gamma_4$. The normalization condition (16) for bispinors is different from QED. For the case of one mass state, $a = 0$, $p^2 = -m^2$ (see Eq.
(1)), we come to QED, and Eq. (16) becomes the standard normalization condition \[1\].

Now we construct the projection matrix extracting solutions of Eq. (12) corresponding to definite energy and momentum. With the help of Eqs (5), (6) (see Appendix), we obtain the minimal equation for the matrix \(\tilde{p} = p_\mu \alpha_\mu\):

\[
\tilde{p}^5 - (1 + 2a) p^2 \tilde{p}^3 + a^2 p^4 \tilde{p} = 0.
\] (17)

Using Eq. (17) it is not difficult to verify that the matrices

\[
\Pi_\pm = \frac{\pm i\tilde{p} (m \mp i\tilde{p}) [\tilde{p}^2 - (1 + 2a) p^2 - m^2]}{2m^2 [(1 + 2a) p^2 + 2m^2]}
\] (18)

obey equations

\[
(\pm i\tilde{p} + m) \Pi_\pm = 0,
\] (19)

\[
\Pi_\pm^2 = \Pi_\pm, \quad \Pi_+ \Pi_- = 0.
\] (20)

So, matrices \(\Pi_\pm\) are projection matrices \[1\] and extract solutions of Eq. (12). For different mass states the energy of particles, \(p^0\) takes two values, Eq. (10).

In \([10]\) (see also \([6]\)), we have obtained the spin projection operators

\[
P_{\pm 1/2} = \mp \frac{1}{2} \left( \sigma_p \pm \frac{1}{2} \right) \left( \sigma_p^2 - \frac{9}{4} \right),
\] (21)

where the operator of the spin projections on the direction of the momentum, \(p\), is given by

\[
\sigma_p = -\frac{i}{2|p|} \epsilon_{abc} p_a J_{bc}.
\] (22)

We use the notation \(|p| = \sqrt{p_1^2 + p_2^2 + p_3^2}\), and the generators of the Lorentz group in the 20-dimensional reducible representation are \[8\]

\[
J_{\mu\nu} = J_{\mu\nu}^{(1)} \otimes I_4 + I_5 \otimes J_{\mu\nu}^{(1/2)},
\] (23)

\[
J_{\mu\nu}^{(1)} = \epsilon^{\mu\nu\rho} \epsilon_{\nu\rho\mu},
\] (24)

\[
J_{\mu\nu}^{(1/2)} = \frac{1}{4} (\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).
\] (25)

\[1\]To have the same normalization condition as in QED for \(a = 0\), \(p^2 = -m^2\), one should make the replacement \(\eta \rightarrow -\eta\), which does not change Eqs. (8)
The projection operator \( P_{\pm 1/2} \) obeys the equation

\[
\sigma_p P_{\pm 1/2} = \pm \frac{1}{2} P_{\pm 1/2}.
\]  

(26)

Eq. (26) guarantees that the operator \( P_{\pm 1/2} \) extracts spin projections \( s = \pm 1/2 \). The operators \( \hat{p} \) and \( \sigma_p \) commute: \([\hat{p}, \sigma_p] = 0\) and, therefore, these operators have the common eigenfunction in the momentum space.

With the aid of Eqs. (19), (26), we obtain the projection operators for pure spin states in the form of matrix-dyads (density matrices)

\[
U_s(\pm p) \cdot \overline{U}_s(\pm p) = NP_{\pm 1/2} \Pi_\pm,
\]  

(27)

where \( N \) is the normalization constant, and we imply the matrix elements of matrix-dyads to be: \( (U_s(\pm p) \cdot \overline{U}_s(\pm p))_{AB} = (U_s(\pm p))_A (\overline{U}_s(\pm p))_B \). There is no summation over spin indexes \( s \) in Eq. (27). The operator \( U_s(\pm p) \cdot \overline{U}_s(\pm p) \) extracts the solution of Eq. (12) as well as the equation:

\[
\sigma_p U_s(\pm p) = sU_s(\pm p),
\]  

(28)

where \( s = \pm (1/2) \). The density matrices (27) are 20×20 matrices describing the polarization of particles for pure spin states.

For pure spin states we use the normalization:

\[
\overline{U}_s(\pm p)U_r(\pm p) = \mp \delta_{sr} \frac{p^2}{m}, \quad \overline{U}_s(p)U_r(-p) = 0.
\]  

(29)

Eqs. (29) generalize the normalization condition (15) on the case of states with definite spin projections. From Eq. (27), we obtain the expression for matrix density summed over spin projections

\[
\sum_s U_s(\pm p) \cdot \overline{U}_s(\pm p) = N_\pm \Pi_\pm.
\]  

(30)

Here we took into consideration the relationship (see Appendix)

\[
\hat{p} \left( P_{1/2} + P_{-1/2} \right) = \hat{p}.
\]  

(31)

Taking the trace in both sides of Eq. (30), and using the equality (see Appendix) \( tr \Pi_\pm = 2 \), one finds

\[
\sum_s \overline{U}_s(\pm p)U_s(\pm p) = 2N_\pm.
\]  

(32)
If we compare Eq. (32) with the expression (29) summed over two states $s = \pm 1/2$, we get

$$N_{\pm} = \mp \frac{p^2}{m}, \quad (33)$$

for two values of $p^2$: $p^2 = -m_1^2$ or $p^2 = -m_2^2$.

4 Quantization of Fermionic Fields with Two Mass States

Following the general prescription [11], we obtain from Eq. (7) the momenta:

$$\pi(x) = \frac{\partial L}{\partial (\partial_0 \Psi(x))} = \frac{i}{2} \Psi(x) \alpha_4, \quad (34)$$

$$\bar{\pi}(x) = \frac{\partial L}{\partial (\partial_0 \bar{\Psi}(x))} = -\frac{i}{2} \alpha_4 \Psi(x). \quad (35)$$

We consider here the fields $\Psi(x)$, $\bar{\Psi}(x)$ as independent “coordinates”. The Poisson bracket $\{.,.\}^P$ of the “coordinate” $\Psi(x)$ and the momentum $\pi(x)$ is given by the standard equation

$$\{\Psi_M(x, t), \pi_N(y, t)\}^P = \delta_{MN} \delta(x - y), \quad (36)$$

and indexes $M$, $N$ run 20 components. In the quantum field theory, one should make the substitution for fermionic fields

$$\{\Psi_M(x, t), \pi_N(y, t)\}^P \rightarrow -i \{\Psi_M(x, t), \pi_N(y, t)\}, \quad (37)$$

where the $\{\Psi_M, \pi_N\} = \Psi_M \pi_N + \pi_N \Psi_M$ is quantum anticommutator. Replacing Eqs. (34), (35) into Eqs. (36), and taking into consideration Eq. (37), we arrive at

$$\{\Psi_M(x, t), (\bar{\Psi}(y, t) \alpha_4)_N\} = 2\delta_{MN} \delta(x - y). \quad (38)$$

From Eqs. (3), (9), (38), we obtain anticommutators of Dirac spinors:

$$\{\psi_{\mu}(x, t), \partial_0 \bar{\psi}_\nu(y, t)\} = -2i \frac{m}{a} \delta_{\mu\nu} \delta(x - y), \quad (39)$$

$$\{\psi_{\mu}(x, t), \psi^*_\nu(y, t)\} = 0, \quad (40)$$
where \( \partial_0 = \partial/\partial t \), \( \overline{\psi}(x) = \psi^+(x)\gamma_4 \). Commutation relations (39), (40) are different from those of QED because the equations considered possess higher derivatives, Eq. (1). In standard QED, anticommutator (40) does not equal zero, and there is no the equation (39). The equation like (39), including anticommutator of the field and its time derivative, is typical for bosonic fields with the replacement of anticommutators on commutators because of different statistics. This is due to the fact that bosonic fields obey second order equations as well as Eq. (1).

The density of the Hamiltonian (the energy density) may be found from the equation

\[
\mathcal{H} = \pi(x)\partial_0\Psi(x) + \left(\partial_0\overline{\Psi}(x)\right)\pi(x) - \mathcal{L}.
\]

Replacing Eqs. (34), (35) into Eq. (41), and taking into consideration that \( \mathcal{L} = 0 \) for fields satisfying Eq. (4), one obtains

\[
\mathcal{H} = \frac{i}{2}\overline{\Psi}(x)\alpha_4\partial_0\Psi(x) - \frac{i}{2}\left(\partial_0\overline{\Psi}(x)\right)\alpha_4\Psi(x).
\]

The density of the Hamiltonian (42) coincides with fourth components, \( \mathcal{H} = T_{44} \), of the energy-momentum tensor [6]:

\[
T_{\mu\nu} = \frac{1}{2} \left( \partial_\nu \overline{\Psi}(x) \right) \alpha_\mu \Psi(x) - \frac{1}{2} \overline{\Psi}(x) \alpha_\mu \partial_\nu \Psi(x)
\]

\[
= \frac{1}{2} \overline{\psi}(x) \gamma_\mu \partial_\nu \psi(x) - \frac{1}{2} \left( \partial_\nu \overline{\psi}(x) \right) \gamma_\mu \psi(x) + \frac{a}{2m} \left( \partial_\nu \overline{\psi}(x) \right) \partial_\mu \psi(x)
\]

\[
- \frac{a}{2m} \left( \partial_\mu \partial_\nu \overline{\psi}(x) \right) \psi(x) + \frac{a}{2m} \left( \partial_\nu \overline{\psi}(x) \right) \partial_\mu \psi(x) - \frac{a}{2m} \overline{\psi}(x) \partial_\nu \partial_\mu \psi(x).
\]

The electric current density are given by [6]

\[
j_\mu(x) = i\overline{\Psi}(x)\alpha_\mu \Psi(x)
\]

\[
= -i\overline{\psi}(x) \gamma_\mu \psi(x) + \frac{ia}{m} \overline{\psi}(x) \partial_\mu \psi(x) - \frac{ia}{m} \left( \partial_\mu \overline{\psi}(x) \right) \psi(x).
\]

This expression includes the usual Dirac current and Barut’s convective terms. The charge density follows from Eq. (44): \( j_0 = -ij_4 = \overline{\Psi}(x)\alpha_4 \Psi(x) \). Therefore, the normalization condition (13) is the normalization on the charge.

In the second quantized theory the general solution of Eq. (4) may be written as

\[
\Psi_s(x) = \sum_s \left[a_{\tau,s} \Psi_{\tau,s}^+(x) + b_{\tau,s} \Psi_{\tau,s}^-(x)\right]
\]
\[
\sum_{p,s} \frac{1}{\sqrt{2p_0 V}} \left[ a_{\tau,p,s} U_{\tau,s}(p) \exp(ipx) + b_{\tau,p,s}^+ U_{\tau,s}(-p) \exp(-ipx) \right].
\]  \hspace{1cm} (45)

Conjugated function \( \bar{\Psi}(x) \) reads
\[
\bar{\Psi}_\tau(x) = \sum_s \left[ a_{\tau,s}^+ \bar{\Psi}_{\tau,s}^\dagger(x) + b_{\tau,s}^+ \bar{\Psi}_{\tau,s}(x) \right] = \sum_{p,s} \frac{1}{\sqrt{2p_0 V}} \left[ a_{\tau,p,s}^+ U_{\tau,s}(p) \exp(-ipx) + b_{\tau,p,s} U_{\tau,s}(-p) \exp(ipx) \right],
\]  \hspace{1cm} (46)

where \( a_{\tau,p,s}^+ \), \( a_{\tau,p,s} \) are the creation and annihilation operators of particles, and \( b_{\tau,p,s}^+ \), \( b_{\tau,p,s} \) are the creation and annihilation operators of antiparticles. The quantum number \( \tau = 1, 2 \) corresponds to two mass states (10). Commutation relations are given by
\[
\{a_{\tau,p,s}, a_{\tau',p',s'}^+\} = \delta_{ss'} \delta_{\tau\tau'} \delta_{pp'}, \quad \{a_{\tau,p,s}, a_{\tau',p',s'}\} = 0, \quad \{a_{\tau,p,s}^+, a_{\tau',p',s'}^+\} = 0,
\]
\[
\{b_{\tau,p,s}, b_{\tau',p',s'}^+\} = \delta_{ss'} \delta_{\tau\tau'} \delta_{pp'}, \quad \{b_{\tau,p,s}, b_{\tau',p',s'}\} = 0, \quad \{b_{\tau,p,s}^+, b_{\tau',p',s'}^+\} = 0, \quad (47)
\]
\[
\{a_{\tau,p,s}, b_{\tau',p',s'}\} = \{a_{\tau,p,s}^+, b_{\tau',p',s'}^+\} = \{a_{\tau,p,s}^+, b_{\tau',p',s'}\} = \{a_{\tau,p,s}, b_{\tau',p',s'}\} = 0.
\]

With the help of Eqs. (45)-(47), and normalization condition (13), one obtains from Eq. (42) the energy of particles-antiparticles fields
\[
H = \int \mathcal{H} d^3x = \sum_{\tau,p,s} p_0 \left( a_{\tau,p,s}^+ a_{\tau,p,s} - b_{\tau,p,s} b_{\tau,p,s}^+ \right). \quad (48)
\]

Like QED, we find from Eqs. (45)-(47) commutation relations for fields \( \Psi_\tau(x), \bar{\Psi}_\tau(x) \):
\[
\{\Psi_{\tau M}(x), \Psi_{\tau N}(x')\} = \{\bar{\Psi}_{\tau M}(x), \bar{\Psi}_{\tau N}(x')\} = 0, \quad (49)
\]
\[
\{\Psi_{\tau M}(x), \bar{\Psi}_{\tau N}(x')\} = K_{\tau MN}(x, x'), \quad (50)
\]
\[
K_{\tau MN}(x, x') = K_{\tau MN}^+(x, x') + K_{\tau MN}^-(x, x'), \quad (51)
\]
\[
K_{\tau MN}^+(x, x') = \sum_s \left( \Psi_{\tau,s}(x) \right)_M \left( \Psi_{\tau,s}^\dagger(x') \right)_N, \quad K_{\tau MN}^-(x, x') = \sum_s \left( \Psi_{\tau,s}^\dagger(x) \right)_M \left( \Psi_{\tau,s}(x') \right)_N.
\]
From Eqs. (45), (46), one finds

\[
K_{\tau MN}^+(x, x') = \sum_{p,s} \frac{1}{2p_0 V} \left( U_{\tau,s}(p) \right)_M \left( U_{\tau,s}(-p) \right)_N \exp[ip(x - x')],
\]

(52)

\[
K_{\tau MN}^-(x, x') = \sum_{p,s} \frac{1}{2p_0 V} \left( U_{\tau,s}(-p) \right)_M \left( U_{\tau,s}(p) \right)_N \exp[-ip(x - x')].
\]

Taking into account Eqs. (30), (33), and the relation \( p^2 = -m^2_\tau \), we obtain from Eqs. (52) functions as follows:

\[
K_{\tau MN}^+(x) = \sum_p \left( \frac{i\hat{p} (m - \hat{p} \hat{p}) \left[ \hat{p}^2 + (1 + 2a) m^2_\tau - m^2 \right] m^2_\tau}{4p_0 V m^3 [2m^2 - (1 + 2a) m^2_\tau]} \right)_{MN} \exp(ipx),
\]

(53)

\[
= \left( \frac{\alpha_\mu \partial_\mu (m - \alpha_\nu \partial_\nu) \left[ (1 + 2a) m^2_\tau - m^2 - (\alpha_\mu \partial_\mu)^2 \right] m^2_\tau}{2m^3 [2m^2 - (1 + 2a) m^2_\tau]} \right)_{MN}
\]

\[
\times \sum_p \frac{1}{2p_0 V} \exp(ipx),
\]

\[
K_{\tau MN}^-(x) = \sum_p \left( \frac{i\hat{p} (m + \hat{p} \hat{p}) \left[ \hat{p}^2 + (1 + 2a) m^2_\tau - m^2 \right] m^2_\tau}{4p_0 V m^3 [2m^2 - (1 + 2a) m^2_\tau]} \right)_{MN} \exp(-ipx),
\]

(54)

\[
= - \left( \frac{\alpha_\mu \partial_\mu (m - \alpha_\nu \partial_\nu) \left[ (1 + 2a) m^2_\tau - m^2 - (\alpha_\mu \partial_\mu)^2 \right] m^2_\tau}{2m^3 [2m^2 - (1 + 2a) m^2_\tau]} \right)_{MN}
\]

\[
\times \sum_p \frac{1}{2p_0 V} \exp(-ipx).
\]

Using the singular functions \[5\]

\[
\Delta_+(x) = \sum_p \frac{1}{2p_0 V} \exp(ipx), \quad \Delta_-(x) = \sum_p \frac{1}{2p_0 V} \exp(-ipx),
\]

\[
\Delta_0(x) = i \left( \Delta_+(x) - \Delta_-(x) \right),
\]

we obtain from Eqs. (51), (53), (54)

\[
K_{\tau MN}(x) = -i \left( \frac{\alpha_\mu \partial_\mu (m - \alpha_\nu \partial_\nu) \left[ (1 + 2a) m^2_\tau - m^2 - (\alpha_\mu \partial_\mu)^2 \right] m^2_\tau}{2m^3 [2m^2 - (1 + 2a) m^2_\tau]} \right)_{MN} \times \Delta_0(x),
\]

(55)

\[
= \left( \frac{\alpha_\mu \partial_\mu (m - \alpha_\nu \partial_\nu) \left[ (1 + 2a) m^2_\tau - m^2 - (\alpha_\mu \partial_\mu)^2 \right] m^2_\tau}{2m^3 [2m^2 - (1 + 2a) m^2_\tau]} \right)_{MN} \times \Delta_0(x).
\]
There is no summation in index $\tau$ in Eqs. (49)-(55). As in QED, due to the properties of the function $\Delta_0(x)$, anticommutator $\{\Psi_M(x), \Psi_N(x')\}$ equals zero if the points $x$ and $x'$ are separated by the space-like interval $(x-x') > 0$. For equal times, $t = t'$, one has $\{\Psi_M(x,0), \Psi_N(x')\} = K_{\tau MN}(x-x',0)$, where the function $K_{\tau MN}(x-x',0)$ may be obtained from Eq. (55) with the help of equalities

$$
\partial_0^n \Delta_0(x)|_{t=0} = 0, \quad \partial_0^n \Delta_0(x)|_{t=0} = 0, \quad \partial_0 \Delta_0(x)|_{t=0} = \delta(x), \quad (56)
$$

where $n = 1, 2, 3, \ldots$. It is easy to verify, using Eq. (17), that the equations

$$
(\alpha_\mu \partial_\mu + m) K^-_\tau (x) = 0, \quad (\alpha_\mu \partial_\mu + m) K^+_\tau (x) = 0 \quad (57)
$$

are valid. The chronological pairing of the operators in our formalism are given by [7]

$$
\Psi^a_{\tau M}(x) \overline{\Psi}^a_{\tau N}(y) = K^c_{\tau MN}(x-y) \quad (58)
$$

$$
= \theta(x_0 - y_0) K^+_{\tau MN}(x-y) - \theta(y_0 - x_0) K^-_{\tau MN}(x-y),
$$

where $\theta(x)$ is the well known theta-function. With the aid of Eqs. (53), (54), one finds

$$
\Psi^a_{\tau M}(x) \overline{\Psi}^a_{\tau N}(y)
$$

$$
= \left( \frac{\alpha_\mu \partial_\mu (m - \alpha_\nu \partial_\nu) [(1 + 2a) m^2 - m^2 - (\alpha_\mu \partial_\mu)^2] m^2}{2m^3 [2m^2 - (1 + 2a) m^2]} \right)_{MN} \Delta_c(x-y)
$$

$$
(59)
$$

and the function $\Delta_c(x-y)$ is given by

$$
\Delta_c(x-y) = \theta(x_0 - y_0) \Delta_+(x-y) + \theta(y_0 - x_0) \Delta_-(x-y). \quad (60)
$$

5 Conclusion

In the formalism of the first order, we have obtained the projection operators extracting states with definite energy and spin projections of the generalized Dirac equation, describing particles with spin 1/2 and two mass states. The density matrix was found for pure spin states. The canonical quantization was performed and anticommutators of 20-component fields were obtained in this formalism. So, FOGDE is convenient for a consideration of the conserving currents as well as for quantization of fields. The density matrix and the chronological pairing of the operators found allow us to calculate different quantum possesses in the formalism of the first order.
6 Appendix

For convenience, we write down some matrices entering the matrix density and singular functions. From Eq. (5), one finds

\[ \hat{p} \equiv p_\nu \alpha_\nu = I^{(0)} \otimes \hat{p} + I^{(1)} \otimes I_4, \tag{61} \]

where \( \hat{p} \equiv p_\nu \gamma_\nu \), and

\[ I^{(0)} \equiv \varepsilon^{0,0}, \quad I^{(1)} \equiv p_\nu \left( \varepsilon^{\nu,0} + a \varepsilon^{0,\nu} \right), \quad I^{(2)} \equiv p_\mu p_\nu \varepsilon^{\mu,\nu}. \tag{62} \]

With the aid of Eq. (6), we obtain matrices as follows:

\[ \hat{p}^2 = (1 + a) p^2 I^{(0)} \otimes I_4 + I^{(1)} \otimes \hat{p} + a I^{(2)} \otimes I_4, \tag{63} \]

\[ \hat{p}^3 = (1 + 2a) p^2 I^{(0)} \otimes \hat{p} + (1 + a) p^2 I^{(1)} \otimes I_4 + a I^{(2)} \otimes \hat{p}, \tag{64} \]

\[ \hat{p}^4 = \left( 1 + 3a + a^2 \right) p^4 I^{(0)} \otimes I_4 + (1 + 2a) p^2 I^{(1)} \otimes \hat{p} + a (1 + a) p^2 I^{(2)} \otimes I_4, \tag{65} \]

\[ \hat{p}^5 = \left( 1 + 4a + 3a^2 \right) p^4 I^{(0)} \otimes \hat{p} + \left( 1 + 3a + a^2 \right) p^4 I^{(1)} \otimes I_4 \]

\[ + a (1 + 2a) p^2 I^{(2)} \otimes \hat{p}, \tag{66} \]

\[ \hat{p}^6 = \left( 1 + 5a + 6a^2 + a^3 \right) p^6 I^{(0)} \otimes I_4 + \left( 1 + 4a + 3a^2 \right) p^4 I^{(1)} \otimes \hat{p} \]

\[ + a \left( 1 + 3a + a^2 \right) p^4 I^{(2)} \otimes I_4. \tag{67} \]

The matrices (62) obey the equations:

\[ I^{(0)2} = I^{(0)}, \quad I^{(0)} I^{(1)} + I^{(1)} I^{(0)} = I^{(1)}, \quad I^{(2)2} = p^2 I^{(2)}, \tag{68} \]

\[ I^{(0)} I^{(2)} = I^{(2)} I^{(0)} = 0, \quad I^{(2)} I^{(1)} + I^{(1)} I^{(2)} = p^2 I^{(1)}. \]
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