Massive Feynman integrals and electroweak corrections

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Abstract

There are steady advances in the calculation of electroweak corrections to massive scattering problems at colliders, from the very beginning in the nineteen seventies until contemporary developments. Recent years brought a remarkable progress due to new calculational technologies. This was motivated by demands from phenomenological applications at particle accelerators: higher multiplicities of the final states, extreme kinematics, need of higher precision and thus of higher orders in perturbation theory. We describe selected contributions from the project “Massive particle production” of Sonderforschungsbereich/Transregio 9 of Deutsche Forschungsgemeinschaft.

Keywords: Quantum field theory, perturbation theory, Feynman integrals, collider physics

1. Introduction

The calculation of observable quantities for high energy colliders became more and more involved in recent years, although the basic understanding of perturbative quantum field theory has been settled decades ago. The term “calculation” has two sides here to be taken into account, of quite different origin. First, one has to derive formulae for the quantity of interest, with a sufficient accuracy in order to match experimental needs. This is part of theoretical work in the classical understanding. But, by time the answers get more involved, both in quantity and in complexity. Also, the singularity behaviour becomes worse. As a consequence, the result of theoretical research to be disseminated is often not only an analytical formula written in an article, but also a piece of more or less sophisticated software. This is fine, but it raises new questions of cooperation. Software has to be supported in a rapidly developing world of computing. How to distribute software in an appropriate manner, thereby respecting the authors’ rights in a satisfactory way, but at the same time not too much hindering its use? Let us remind that software use in nearly all realistic cases means also adaptation and so changing the original creation.

Since we are working in the field of particle phenomenology since decades, we collected some experience with all these aspects, to some extent we even contributed to the culture of practicing. We come back to the point in section 7.

In sections 2 to 6 we survey part of research performed in the research group B1 “Massive particle production” of Sonderforschungsbereich/Transregio 9 of Deutsche Forschungsgemeinschaft. Due to the calculational difficulties it took few years after the concise formulation of the perturbative renormalization of the electroweak theory by t’ Hooft and Veltman [1, 2] and after the invention of SCHOONSCHIP [3]. A famous piece of work was Veltman’s study of the $\rho$ parameter with the observation that high particle masses may show up at low energy [4]. First detailed studies of the calculational techniques and of the consequences for phenomenology came out soon, notably [5, 6]. Since then, much effort has been concentrated to the refinement of predictions of perturbative effects in the Standard Model and beyond.

Calculations have been done for many quantities, notably the weak corrections to the $Z$ boson parameters $\rho_Z$ and $\sin^2\theta_W$; at one loop e.g. in [7, 8, 9, 10], and later also with higher order corrections predicted by the elec-
troweak theory and by QCD [11, 12, 13, 14, 15]. These higher-order calculations have to be performed, but they have also to be inserted into phenomenological tools.

Although a lot of the material presented here is applied also to LHC physics, we will concentrate on higher-order contributions to $e^+e^-$ annihilation, mainly arising from loop corrections:

$$e^+e^- \rightarrow f^+f^- , \ f^+f^- (\gamma) , \ f^+f^- \gamma , \ f^+f^- \gamma(\gamma).$$  \hfill (1)

A large part of the present study is devoted to the treatment of single Feynman integrals. They are the building blocks of Feynman diagrams related to some observable. We will consider an arbitrary $L$–loop integral $G(X)$ with loop momenta $k_l$ , with $E$ external legs with momenta $p_e$ and with $N$ internal lines with masses $m_i$ and propagators $1/D_i$:

$$G(X) = \frac{1}{(i\pi)^{L/2}} \int \frac{d^d k_1 \ldots d^d k_L}{D_1^{\nu_1} \ldots D_L^{\nu_L}} , \hfill (2)$$

with

$$d = 4 - 2e, \hfill (3)$$

$$D_i = q_i^2 - m_i^2 = \left[ \sum_{l=1}^L e_l^i k_l + \sum_{e=1}^M d_e^i p_e \right] - m_i^2 , \hfill (4)$$

where $X(k_1, \ldots , k_L)$ stands for tensors in the loop momenta.

2. ZFITTER

ZFITTER [16, 17] is a long-term project, dating back to the nineteen seventies. The aim is a state-of-the-art description of

$$e^+e^- \rightarrow (\gamma , Z) \rightarrow f^+f^- (\nu \nu)$$  \hfill (5)

in the Standard Model. A description of the project has been published quite recently [18]. Since 1989 ZFITTER is among the standard software packages for the description of the $Z$ boson resonance at LEP. Further, it was used for predictions of the top-quark and Higgs-boson masses from radiative corrections in the Standard Model prior to their discoveries. Until about 1992, ZFITTER rested mainly on theoretical work done by its authors on complete one-loop electroweak corrections in the Standard Model. In the nineteen nineties it become more and more important to integrate higher-order corrections derived by other authors and to support the users from experimental groups, notably from DELPHI, L3, OPAL, and also from the LEPEWWG. This is documented in the “LEP electroweak working group report” of 1995 [11] and references therein. The seminal review studies of (5) by the LEP community for LEP 1 in 2005 [19] and LEP 2 in 2013 [20] rest to a large extent on ZFITTER v.6.42 [16, 17]. ZFITTER became the “etalon” software for the $Z$-boson resonance studied for many years at LEP 1 und at LEP 2. Among the main results of LEP are the following, quoted from the “Review of Particle Physics” (2012) [21]:

$$M_Z = 91.1876 \pm 0.0021 \text{ GeV},$$

$$\Gamma_Z = 2.4952 \pm 0.0023 \text{ GeV},$$

$$\sin^2 \theta_{\text{weak}} = 0.22296 \pm 0.00028,$$

$$\sin^2 \theta^\text{eff}_{\text{lep}} = 0.23146 \pm 0.00012,$$

$$\sin^2 \theta^\text{MS}_Z = 0.23116 \pm 0.00012,$$

$$N_e = 2.989 \pm 0.007. \hfill (6)$$

A similar analysis, also based on ZFITTER, has been published by ALEPH, DELPHI, L3, OPAL, LEPEWWG in 2013 [20]. The constraint

$$m_t = 178^{+11}_{-8} \text{ GeV} \hfill (7)$$

is obtained from the virtual corrections, in good agreement with the much more precise direct measurement of about $m_t = 173.2 \pm 1 \text{ GeV}$ [22]. For the Higgs boson mass, they predict:

$$M_H = 118^{+203}_{-164} \text{ GeV, only LEP},$$

$$M_H = 122^{+59}_{-41} \text{ GeV, plus } m_t,$$

$$M_H = 148^{+237}_{-81} \text{ GeV, plus } M_W , \Gamma_W ,$$

$$M_H = 94^{+29}_{-26} \text{ GeV, plus } m_t , M_W , \Gamma_W . \hfill (8)$$

An update is [23], where it is quoted $M_H = 89^{+8}_{-12} \text{ GeV, or } M_H < 127 \text{ GeV} (90\% \text{ c.l.}).$ In 2012, the LHC collaborations discovered a scalar particle with a mass of about 125 GeV [24, 25]. The present best value is $M_H = 125.6\pm0.3 \text{ GeV} [26].$ This might be illustrated by the famous blue band plot of the LEPEWWG [27, 28], which we reproduce in figure 2, together with the presumably first proposal of an electroweak precision plot in figure 1. The development of precision predictions is nicely illustrated in figures 3 to 5.

It is pointed out in [20] that there are, besides ZFITTER, two alternative approaches for precision Standard Model tests available. One approach is practiced in the “Review of particle physics” of the Particle Data Group [29], which traces to a large extent back to ZFITTER. The second approach is the Gfitter project. In fact, at the webpage http://gfitter.desy.de/ one finds a lot of data similar to the blue band plot of the LEPEWWG, figure 2. We reproduce here as an example the figure 6. The Gfitter plots are
not due to independent calculations. They have been made with the software Gfitter/gsm [30] which originates from the Standard Model library of ZFITTER v.6.42 [16, 17], see http://zfitter-gfitter.desy.de.1 Like the diploma thesis [30], all the Gfitter publications, talks and proceedings contributions from December 2007 until July 2011 make use of Gfitter/gsm, see webpage http://fh.desy.de/projekte/gfitter01/Gfitter01.htm.

So we are faced with the unfortunate situation that for a large extent of precision predictions in the Standard Model there is basically only ZFITTER as a supported tool with reproducible features.2

ZFITTER is among the software tools with longest history of open access in the particle physics community. The program traces back to times when there was no public internet, and the files were exchanged by floppy disks. We estimate the human investment into ZFITTER as follows:

- 2.2 million Euro derived from 30 years FTE (staff researcher full time equivalent with 74,000 Euro per FTE)
- 1.1 million Euro 1/2 of the total amount for project management, publications, numerical tests, user support etc.
- 1.1 million Euro 1/2 of the total amount for software = QED corrections + Standard Model library, resulting in:
  - 550,000 Euro → QED corrections
  - 550,000 Euro → Standard Model library

The ZFITTER webpage with lots of information on older versions of ZFITTER is http://zfitter.education. ZFITTER v.6.42 is exclusively available from http://cpc.cs.qub.ac.uk/summaries/ADMJ. A download is possible only after active agreement on the licence conditions shown in the pop-up window “CPC LICENCE ALERT”.

In recent years, new regions of application have been explored. There are quite interesting analyses of
the Drell-Yan cross section by the CDF collaboration [32, 33]; see also the CMS study [34]. The high luminosity data expected at BELLE 2 will also deserve the application of ZFITTER; see also section 3. In many cases, a support by the authors, including theoretical adaptations, has been welcomed by the experimentalists.

The theoretical description of the $Z$ resonance measurements at a future Giga-Z factory will deserve electroweak two-loop precision. Here, certain multi-scale two-loop vertex-type Feynman integrals are hard to calculate, but seem feasible with present technology. Once this is done, the complete electroweak two-loop corrections to the $Z$ resonance will be known. For further remarks on this subject see section 6.

3. TOPFIT

ZFITTER assumes the external fermions in (1) and (5) to be light. In $\mu$ and $\tau$ pair production at meson factories potentially, and naturally in top pair production in the continuum at the ILC, the final state mass is essential. For the description of complete electroweak corrections with exact account of the final state mass, one may use TOPFIT [35]. The virtual corrections and a semi-analytical treatment of hard photonic corrections with certain experimental cuts has been described in [36]. As an example of the sophisticated situation we reproduce a Dalitz plot with an acollinearity cut in figure 7. The variables are: $\xi = \pi - \langle f^- f^+ \rangle$, $x = 2p(\gamma)p(f^+)/s$, and $r = 1 - 2E(\gamma)/\sqrt{s}$. 

Figure 4: Top quark mass measurements. Reprinted from [18], with permission from Springer Verlag under licence number 3494820307523.

Figure 5: Higgs boson mass measurements. The upper limits and the fit values for $M_H$ derive from a combination of virtual corrections to LEP and similar data, top and $W$ mass measurements, performed by the LEPEWWG. The lower mass limit is due to LEP direct searches. The lower limits from data combinations are not shown. Reprinted from [18], with permission from Springer Verlag under licence number 3494820307523.

Figure 6: Gfitter plot on the Higgs boson mass, reproduced from figure 6.8 of [30].

Figure 7: Kinematic region of $r$ and $x$ for different values of the acollinearity angle $\xi$ for $r_m = 4m_f^2/s = 0.2$. Figure from [37].
Phenomenological applications were worked out in [38, 37, 39, 40, 41, 42]. The Born cross section depends on the running electromagnetic coupling \( \alpha_{em} \) and, for massless fermion pair production, on four electroweak form factors \( \rho_2, \sin^2 \theta_W, \sin^2 \beta, \sin^2 \gamma \). Final state mass adds two degrees of freedom:

\[
\frac{d\sigma_{\text{Born}}}{d\cos \theta} = \frac{\pi \alpha^2}{2s} c_f \beta 2 \mathcal{R} e \left[ (u^2 + t^2 + 2m_f^2 s) \left( \bar{F}_1^{11} F_1^{11, \ast} + \bar{F}_1^{51} F_1^{51, \ast} \right) \\
+ (u^2 + t^2 - 2m_f^2 s) \left( \bar{F}_1^{15} F_1^{15, \ast} + \bar{F}_1^{55} F_1^{55, \ast} \right) \\
+ (u^2 - t^2) \left( \bar{F}_1^{55} F_1^{11, \ast} + \bar{F}_1^{15} F_1^{51, \ast} + \bar{F}_1^{51} F_1^{15, \ast} + \bar{F}_1^{11} F_1^{55, \ast} \right) \\
+ 2m_f(tu - m_f^2) \left( \bar{F}_3^{11} F_1^{11, \ast} + \bar{F}_3^{51} F_1^{51, \ast} \right) \right].
\]

The \( s, t, u \) are the Mandelstam variables, \( c_f \) the color factor, and the \( F, \bar{F} \) are form factors [36]. The project was originally devoted to physics at the ILC, and several phenomenological studies were performed for the corresponding energy range [42, 39]. In tables 1 for \( \tau \) leptons and 2 for top quarks we reproduce comparisons of differential cross sections and in table 3 integrated observables with hard photons included for top quark production. The tables demonstrate the impressive accuracy level achieved by comparing the numerics of different groups.

Recently, a quite interesting application of ZFITTER and TOPFIT became relevant at the Belle experiment at KEK in Japan. In Section 5.14 “Electroweak physics” of “Physics at Super B Factory” [43], it is worked out that Belle II will measure about \( 10^0 \, \mu^+\mu^- \) pairs at \( \sqrt{s} = 10.58 \, GeV \), with a need of theoretical precision of about \( 10^{-3} \) or even better. Evidently, to some extent the account of the final state muon mass is needed, \( m_{\mu}^2 \lesssim 10^{-4} \). This may be exactly controlled by TOPFIT, after some necessary adaptations of the package. Although the application of ZFITTER was originally excluded at meson factory energies, here it is nevertheless possible due to the suppression of contributions from meson resonances in the experimental set-up. So, the experiment gives access to electroweak physics at \( \sqrt{s} \approx 10 \, GeV \), from a measurement of the forward-backward asymmetry \( A_{FB}(\mu^+\mu^-) \). In the usual calculationale frames, the weak mixing angle is not accessible, but the measurement is sensitive to the \( \rho_2 \) parameter of the Z boson. The \( A_{FB} \) will be a single parameter measurement of \( \rho_2 \) [44, 45, 46] with high accuracy, see figure 8.

| \( \cos \theta \) | \( \frac{dr}{d\cos \theta} \)_{Born} | \( \frac{dr}{d\cos \theta} \)_{B+\omega+QED+soft} |
|---|---|---|
| -0.9 | 0.094591 02171 86329 | 0.092419 02671 14061 |
| -0.9 | 0.094591 02171 86327 | 0.092419 02671 14065 |
| -0.5 | 0.089298 53117 79585 | 0.086699 48248 65248 |
| -0.5 | 0.089298 53117 79586 | 0.086699 48248 69477 |
| 0.0 | 0.15032 16282 75192 | 0.14335 79492 08648 |
| 0.0 | 0.15032 16282 75192 | 0.14335 79492 08648 |
| 0.5 | 0.28649 90174 53525 | 0.28258 86777 50811 |
| 0.5 | 0.28649 90174 53525 | 0.28258 86777 59161 |
| 0.9 | 0.44955 18970 14604 | 0.47648 29191 20338 |
| 0.9 | 0.44955 18970 14604 | 0.47648 29191 19623 |

Table 1: Differential cross-sections in picobarn for selected scattering angles for \( \tau \)-production at \( \sqrt{s} = 500 \, GeV \). The three columns contain the Born cross-section, Born including only the weak \( O(\alpha) \) corrections, and Born including the weak and soft photonic \( O(\alpha) \) corrections. For each angle, the first row represents the TOPFIT result of the Zeuthen group while the second contains the Feynarts/FeynCalc calculation of the Munich group. Table shortened from SFB/CPP-03-13 [42].

| \( \sqrt{s} \) | \( \sigma^0 \) | \( A^0_{FB} \) | \( \sigma_{\text{tot}} \) | \( A_{FB} \) |
|---|---|---|---|---|
| 500 | T : 0.5122744 | 0.4160039 | 0.526337 | 0.362929 |
| G : 0.5122751 | 0.416042 | 0.526371 | 0.363140 |
| 1000 | T : 0.1559185 | 0.5641706 | 0.171916 | 0.488869 |
| G : 0.1559175 | 0.5641710 | 0.171931 | 0.488872 |

Table 2: Various differential cross sections, last column includes hard photon emission. The upper and lower rows correspond to the TOPFIT and GRACE approach, respectively, \( \sqrt{s} = 500 \, GeV \). Table shortened from [39].
Figure 8: Measurements of $A_{FB}(e^+e^- \rightarrow \mu^+\mu^-)$ at different energies corrected for QED effects by the respective authors (see [47] and references therein), theoretical Standard Model prediction at lowest order and the expected Belle and Belle II statistical uncertainties (scaled up by a factor of 1000) at $\sqrt{s} = 10.58$ GeV. Figure and caption from T. Ferber, in [45].

4. Upgrading Monte-Carlo programs for Bhabha scattering and $\mu^+\mu^-$ production at the ILC and at meson factories

4.1. Heavy fermion two-loop corrections to Bhabha scattering, Babayaga.

For a variety of scattering processes we need cross section predictions with a two-loop accuracy. Among them is Bhabha scattering, which is interesting by itself, as a very clean and simple reaction. Any deviation from predictions would be a strong indication for New Physics. Further, small angle Bhabha scattering is important for luminosity determinations at LEP and meson factories, and both for small and wide angle scattering also at the ILC. Here, for many applications one may concentrate at QED corrections. An important step was the prediction of the two-loop photonic corrections [48, 49] by relating them to the massless case. The alternative approach of calculating the virtual two-loop contributions to massive Bhabha scattering proved to be much more involved and its proponents have not been succeeded so far. We refer to [50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64], and to the recent report [65] on planar integrals. The difficulty is mainly related to the non-planar double box diagrams. We come to some aspects and techniques of the evaluation of more complicated Feynman diagrams in section 6. Much easier, and finally calculated by several groups, are the heavy fermion (and hadronic) two-loop contributions to Bhabha scattering shown in figure 9 [61, 66, 67, 68]. The numerical influence is, both at meson factories and at the ILC, of the order of a per mil, see figure 10. The software package related to [61, 66] has been attached to the BabaYaga package [69, 70, 71, 72] in order to stabilize its numerical precision at the NNLO level. At the time, there was dedicated scientific activity related to the subject by several groups. We should mention, in addition to the extensive list of references quoted in [66], the references [73] and [74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84].

4.2. 5-point functions in muon pair production. PHOKHARA.

Mainly motivated by the need of efficient and numerically stable calculations of higher-point Feynman integrals for the description of final states with higher multiplicities at LEP 2 and at the LHC, there was much
research activity in recent years on tensor integral reduction. For a review see e.g. [86]. Section 5 is devoted to an advantageous technique, based on dimensional shifts. An application of the corresponding software library PJFry is the calculation of the complete QED NLO contributions to

\[ e^+ e^- \rightarrow \mu^+ \mu^- \gamma(y), \]  

(10)

which is an important reaction with precision measurements at meson factories. It contains the contributions from five-point functions which were not studied before. The correct interpretation of the recent (g-2) measurements depends on the reliable knowledge of pion production, and (10) is used as a normalization for that. The state-of-the-art Monte Carlo program is PHOKHARA, where the resulting code finally was integrated. Crucial for stability and efficiency were the inclusion of both the muon and electron masses, and a clever treatment of numerical problems related to the appearance of inverse Gram determinants from tensor reduction [87, 72, 85, 69, 88]. Figure 11 demonstrates this. Figure 12 shows the improvement from the inclusion of pentagon diagrams on the muon charge asymmetry for KLOE energies for two different momentum intervals. Finally, the numerical effects are below the present experimental concern, what was not evident before.

5. Tensor reduction for Feynman integrals. PJFry and OLEC.

An efficient approach to the systematic reduction of arbitrary one-loop tensor Feynman integrals to scalar integrals relies on dimensional shifts. Basic ideas have been formulated in [89, 90, 91], and the approach has been worked out in all necessary details and numerical and algebraic tools were created in recent years [92, 93, 94, 95, 88, 96, 97, 98, 99, 100, 101, 99, 102,
Tensor integrals in terms of scalar integrals

where denominators $c_j = (k-q_j)^2 - m_j^2 + i\epsilon$ have indices $\nu_j$ and chords $q_j$. An example is shown in figure 13.

The aim is to get for $n > 4$ tensor reductions with:

- arbitrary masses;
- all inverse pentagon Gram determinants being eliminated;
- treatment of full kinematics, also with vanishing sub-diagram Gram determinants;
- higher $n$ point functions, $n \geq 6$ [99];
- as an option: multiple sums over tensor coefficients made efficient by contracting with external momenta [97].

Tensor integrals in terms of scalar integrals $I_n^{d+\nu}$ in higher dimensions, $D = d + 2\nu = 4 - 2\epsilon, 6 - 2\epsilon, \cdots$ were derived in [89], see also [92]. With $n_{ij} = \nu_{ij} = 1 + \delta_{ij}, n_{ijk} = \nu_{ijk}, \nu_{ijk} = 1 + \delta_{ik} + \delta_{jk},$ one gets for a tensor of rank four:

$$I_n^{d+\nu} = \int \frac{d^d{k}}{(2\pi)^d} \frac{1}{\prod_{i=1}^n c_{i}^{-1}}$$

$$= \sum_{i,j,k,l=1}^n d_i' q_j' d_k' q_l' n_{ijkl} I_{nijkl}^{d+\nu}$$

Figure 13: The one-loop $n$ point function.

Figure 14: A six-point topology leading to a four-point function with realistically vanishing Gram determinant. Example from [111].
Grams) with their subsequent dimensional recurrence, improved finally by a Pade approximation [95].

The C++ package PJJrty for tensor reduction is due to V. Yundin and is available as open-source software [113]. It has been used for a variety of cross section calculations for LHC and meson factories. The C++ package OLEC [114] for the replacement of tensor reduction by tensor contractions is due to J. Gluza, M. Gluza, I. Dubovyk. Its basics have been worked out also by A. Almasy, J. Fleischer, T. Riemann [115, 103]. It is not yet published.

6. Calculation of master integrals with Mellin-Barnes representations. AMBRE.

There are several powerful techniques for the evaluation of higher order Feynman integrals. We mentioned already the reduction to (an algebraic system of) simpler integrals by recurrence relations. A similar idea is followed by solving (systems of) difference or differential equations. Single Feynman integrals may be solved by a multiple Feynman parameter integral representation for the propagators,

\[
\frac{1}{D_1^{n_1}D_2^{n_2}...D_N^{n_N}} = \frac{\Gamma(n_1 + ... + n_N)}{\Gamma(n_1)...\Gamma(n_N)} \int_0^1 dx_1 \\
... \int_0^1 dx_N \frac{x_1^{n_1-1}...x_N^{n_N-1}\delta(1-x_1-...-x_N)}{(x_1D_1+...+x_ND_N)^{n_1}}, \tag{16}
\]

where \(N_e = n_1 + ... + n_N\). After performing the momentum integrations in (2), the \(x\)-parameters are left:

\[
G(X) = \frac{(-1)^N\Gamma(N_e - \frac{dL}{2})}{\Gamma(n_1)...\Gamma(n_N)} \int_0^1 \prod_{i=1}^{N} dx_i \frac{1}{x_1^{n_1-1}...x_N^{n_N-1}}.
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\]

The functions \(U\) and \(F\) are called graph or Symanzik polynomials. Sector decomposition techniques allow to isolate endpoint singularities related to infrared poles. Another technique is the replacement of sums by products of \(x\)-monomials in the Symanzik polynomials \(U\) and \(F\) by a sequence of Mellin-Barnes (MB-) integrals [116] with well-defined integration paths in the complex plane, see figure 16.

\[
\times \delta(1 - \sum_{i=1}^{N} x_j) \frac{U(x)^{N_e - dL + 1/2}}{F(x)^{N_e - dL/2}}, \tag{17}
\]

The Mellin-Barnes approach has certain limitations, partly resting in its basic features. We like to mention the dimensionality. The dimensionality of the sector decomposition [123, 124, 125, 121, 126, 127, 128] is essentially the number of Feynman parameters, while in the MB-approach it depends on the complexity of the problem and may be quite high. Further, only recently there was substantial progress for the treatment of non-planar topologies with AMBRE [129, 130], while numerics in the Minkowskian region is not yet implemented. An MB-treatment of mixed virtual and real IR-
singers have been proposed in [131]. An advantageous feature of MB-integrals is the potential to find analytical solutions. This may be tackled by deriving infinite series over residues with the Cauchy theorem [132]. Some software is under development [130]. The bottleneck is the need to sum up the multiple series. While the algorithms of e.g. MATHEMATICA are limited, there are dedicated approaches like SUMMER [133] and XSUMMER [134], for massless problems of a not too high dimensionality. Both packages are written in FORM [135]. Presently we are exploring the potential of the package family of the RISK (Linz) group around FORM [135]. Presently we are exploring the potential of the package family of the RISK (Linz) group around FORM [135]. Presently, we are exploring the potential of the package family of the RISK (Linz) group around FORM [135].

Automation we have to implement several steps:

1. Determine if the topology is planar or non-planar; PlanarityTest [129].
2. Construct MB representations; AMBRE, MB, and related packages [117, 118, 119, 120, 121].
3. Change the MB-integrals into nested sums; MB-sums package [130].
4. Try to perform the multiple sums analytically [130, 138].
5. Accept Minkowskian kinematics.

As mentioned, there are intrinsic limitations to the MB-approach, due to the number of loops, the number of scales, the number of legs, and last but not least the complexity of a given integral.

In a practical study, one will combine several methods in order to calculate all the necessary Feynman integrals. One or the other application [58, 56] was mentioned already in the foregoing sections. As an interesting state-of-the-art problem we mention here the calculation of the complete two-loop electroweak vertex form factors of the Z boson where one has to determine several hundreds of Minkowskian Feynman integrals of different dimensionality with up to four different dimensionless scales (arising from $s$, $M_Z$, $M_W$, $M_H$, $m_t$) and with an accuracy of several numerical digits, typically about five. Alternatively, one might try to determine a minimal set of master integrals, but this raises other non-trivial problems. Concerning multi-scale problems, notably arising in electroweak multi-loop calculations, the numerical approach has been proved to be powerful. We can quote here only a selection of relevant articles [139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 153, 154].

7. The role of software in the dissemination of scientific results

The dissemination of the results of theoretical research in elementary particle physics is an essential element of any single project. We mentioned in the introduction that the publication of an article in a peer-reviewed scientific journal is not sufficient in view of the complexity of contemporary calculations. Additionally, one has to deliver some piece of software code ready for use by third parties, notably experimentalists. It has to be sufficiently robust to be safely used by non-specialists and during longer periods, maybe even decades. Further, it is often needed to deliver some direct individual support by debugging, new adaptations, etc. Some aspects of long-term support have been described for the example of the ZFITTER project [18], see also section 2.

There are the rapidly rising opportunities of interconnections by the internet. More and more often they are used anonymously, essentially because the number of researchers is also rapidly rising. Just compare the number of scientists per experiment at LEP (few hundreds) and at LHC (three thousand). This raises questions related to the foundations of academic basic research like proper quotation, and related to copyright law, like licence problems.

There is an ongoing discussion in a broader scientific community in Germany on software use in scientific cooperation. We quote from a statement by the Ombudsman for Science in Germany, formulated with respect to software use in high energy physics (3 July 2012) [155]:

“The proper legal treatment of such software, in the field of tension of the rules of good scientific practice, has not been the subject of rule making until now (if I see right) . . .”

A similar understanding was expressed by the Editor-in-Chief of “European Physical Journal C” (26 January 2012) [156]:

“We note that a subtlety may remain in the question as of what “scientific usage of the code” includes in the broader sense, namely if it is restricted to using the code as-is, or if copying and altering the original code is also permitted. Here we refer to the common practice of e.g. using Monte Carlo generator code by a large number of scientists who, as we observe, not only run that original code, but alter and copy parts of it according to their specific (scientific) needs. Such Monte Carlo codes exist, in a wide variety, under similar or identical license terms, as Open Source software . . . ”

These quotes might be contrasted by the scientific practice in our international community. Let us refer to
the write-up of the results of the ACAT (May 2013, Beijing) round table discussion on “Open-source, knowledge sharing and scientific collaboration” [157, 158] where many interesting reflections and ideas, together with facts of life, where communicated. The abstract summarizes:

“Although the discussion was, in part, controversial, the participants agreed unanimously on several basic issues in software sharing:

• The importance of having various licensing models in academic research;
• The basic value of proper recognition and attribution of intellectual property, including scientific software;
• The user respect for the conditions of use, including licence statements, as formulated by the author.”

In the US, there is governmental interest in the sensible topic. On June 3, 2011, the “Report of the HEPAP Sub-Committee on the Dissemination of Research Results” [159] was published which answers a request by the Director of the US DOE Office of Science to summarize the current practices of researchers funded by the Office of High Energy Physics (OHEP) for disseminating their results. We quote from there:

“Although not technically “digital data”, it’s important to note that some theoretical research produces results besides the published articles. Examples include simulation programs (e.g. lattice gauge theory simulations like USQCD or MILC and Monte Carlo simulation programs like PYTHIA, HERWIG, SHERPA, or ALPGEN), computation programs (e.g. MCFM or MadGraph), and global fits to a large corpus of data (e.g. CTEQ, ZFITTER, or CKMFITTER). Typically the computer code itself is disseminated in an open access manner via the internet. The release of the computer code is usually accompanied by a publication in a peer reviewed journal describing the functionality of the code and, if relevant, specific results obtained using the code. . . . The Version of Record is taken to be the latest version besides the published articles. Examples include

the ACAT round table summary [157, 158] and in the HEPAP Sub-Committee report [159], although there was absolutely no interaction between their authors.

These quotes might stimulate further thinking on the subject.

Acknowledgements

We remember thankful the cooperation with Prof. Dr. Jochem Burkhard Fleischer (17 December 1937 - 1 April 2013).3

We profited essentially from several long-term cooperations. Instead of listing here a list of names, we would like to refer to the exhaustive list of references where we give full account to our co-authors.

Most of the work presented would not have been possible without the support by DFG Sonderforschungsbereich Transregio 9, Computergestützte Theoretische Teilchenphysik. We also acknowledge support by the Research Executive Agency (REA) of the European Union under the Grant Agreement number PITN-GA-2010-264564 (LHCPhenoNet) and by the Polish National Center of Science (NCN) under the Grant Agreement number DEC-2013/11/B/ST2/04023.

References

[1] G. 't Hooft, M. Veltman, Regularization and Renormalization of Gauge Fields, Nucl. Phys. B44 (1972) 189–213. doi:10.1016/0550-3213(72)90279-9.
[2] G. 't Hooft, M. Veltman, Combinatorics of gauge fields, Nucl. Phys. B50 (1972) 318–353. doi:10.1016/0550-3213(72)80021-X.
[3] H. Strubbe, Manual for Schoonschip: A CDC 6000/7000 program for symbolic evaluation of algebraic expressions, Comput. Phys. Commun. 8 (1974) 1–30. doi:10.1016/0010-4655(74)90081-2.
[4] M. Veltman, Limit on Mass Differences in the Weinberg Model, Nucl. Phys. B123 (1977) 89. doi:10.1016/0550-3213(77)90342-X.
[5] G. 't Hooft, M. Veltman, Scalar One Loop Integrals, Nucl. Phys. B153 (1979) 365–401. doi:10.1016/0550-3213(79)90605-9.
[6] G. Passarino, M. Veltman, One loop corrections for e+e− annihilation into μ+μ− in the Weinberg model, Nucl. Phys. B160 (1979) 151. doi:10.1016/0550-3213(79)90234-7.
[7] D. Bardin, P. Khristova, O. Fedorenko, On the lowest order electroweak corrections to spin 1/2 fermion scattering. 1. The one loop diagrammar, Nucl. Phys. B175 (1980) 435. doi:10.1016/0550-3213(80)90021-8.
[8] D. Y. Bardin, P. K. Khristova, O. Fedorenko, On the Lowest Order Electroweak Corrections to Spin 1/2 Fermion Scattering. 2. The One Loop Amplitudes, Nucl. Phys. B197 (1982) 1. doi:10.1016/0550-3213(82)90152-3.

3http://www-zeuthen.desy.de/~riemann/FleischerJ
[9] A. Akhundov, D. Bardin, T. Riemann, Hunting the hidden standard Higgs, Phys. Lett. B166 (1986) 111. doi:10.1016/0370-2693(86)91166-4.

[10] A. Akhundov, D. Bardin, T. Riemann, Electroweak One Loop Corrections to the Decay of the Neutral Vector Boson, Nucl. Phys. B276 (1986) 1. doi:10.1016/0550-3213(86)90014-3.

[11] D. Bardin, W. Beenakker, M. Bilenky, W. Hollik, M. Martinez, G. Montagna, O. Nicrosini, V. Novikov, L. Okun, A. Olchevsky, G. Passarino, F. Peccei, S. Riemann, T. Riemann, A. Rozanov, F. Teubert, M. Vysotsky, Electroweak working group report (1997) 7–162, CERN 95–03A. In: D. Bardin, W. Hollik, G. Passarino (eds.), Reports of the working group on precision calculations for the Z resonance, CERN 95–03, p. 7–162 (31 March 1995). arXiv:hep-ph/9709229.

[12] M. Awramik, M. Czakon, A. Freitas, G. Weiglein, Complete two-loop electroweak fermionic corrections to $\sin^2\theta_W$ and indirect determination of the Higgs boson mass, Phys. Rev. Lett. 93 (2004) 201805. arXiv:hep-ph/0407317, doi:10.1103/PhysRevLett.93.201805.

[13] A. Akhundov, A. Arbuzov, S. Riemann, T. Riemann, The $Z$-boson width and production rate, Phys. Lett. B730 (2014) 50–52. doi:10.1016/j.physletb.2014.01.017.

[14] A. Freitas, Y.-C. Huang, Electroweak two-loop corrections to $\sin^2\theta_W$ and $R_D$, using numeric Mallin-Barnes integrals, JHEP 1208 (2012) 050. arXiv:1205.0299, doi:10.1007/JHEP08(2012)050.

[15] A. Freitas, Two-loop fermionic electroweak corrections to the Z-boson width and production rate, Phys. Lett. B730 (2014) 50–52. arXiv:1310.2256, doi:10.1016/j.physletb.2014.01.017.

[16] M. Awramik, M. Czakon, A. Freitas, Electroweak two-loop corrections to the effective weak mixing angle, JHEP 0611 (2006) 048. arXiv:hep-ph/0608099, doi:10.1088/1126-6708/2006/11/048.

[17] A. Freitas, Y.-C. Huang, Electroweak two-loop corrections to $\sin^2\theta_W$ and $R_D$ using numeric Mellin-Barnes integrals, JHEP 1208 (2012) 050. arXiv:1205.0299, doi:10.1007/JHEP08(2012)050.

[18] A. Freitas, Two-loop fermionic electroweak corrections to the Z-boson width and production rate, Phys. Lett. B730 (2014) 50–52. arXiv:1310.2256, doi:10.1016/j.physletb.2014.01.017.

[19] M. Awramik, M. Czakon, A. Freitas, G. Weiglein, Complete two-loop electroweak fermionic corrections to $\sin^2\theta_W$ and indirect determination of the Higgs boson mass, Phys. Rev. Lett. 93 (2004) 201805. arXiv:hep-ph/0407317, doi:10.1103/PhysRevLett.93.201805.

[20] A. Akhundov, D. Bardin, T. Riemann, Hunting the hidden standard Higgs, Phys. Lett. B166 (1986) 111. doi:10.1016/0370-2693(86)91166-4.

[21] J. Beringer, et al., Review of Particle Physics (RPP), Phys. Rev. D88 (2013) 072002, 10.1103/PhysRevD.88.072002, 10.1103/PhysRevD.88.079905.

[22] T. Aaltonen, et al., Indirect measurement of $\sin^2\theta_W(M_W)$ using $e^+e^-$ pairs in the Z-boson region with $pp$ collisions at a center-of-momentum energy of 1.96 TeV, Phys. Rev. D87 (8) (2013) 072002. arXiv:1307.0770, doi:10.1103/PhysRevD.87.072002.

[23] T. Aaltonen, et al., Indirect measurement of $\sin^2\theta_W$ (or $M_W$) using $\mu^+\mu^-$ pairs from $\gamma^*Z$ bosons produced in $pp$ collisions at a center-of-momentum energy of 1.96 TeV, Phys. Rev. D89 (2014) 072005. arXiv:1402.2239, doi:10.1103/PhysRevD.89.072005.

[24] A. Freitas, Two-loop fermionic electroweak corrections to the Z-boson width and production rate, Phys. Lett. B730 (2014) 50–52. arXiv:1310.2256, doi:10.1016/j.physletb.2014.01.017.

[25] A. Freitas, Y.-C. Huang, Electroweak two-loop corrections to $\sin^2\theta_W$ and $R_D$ using numeric Mellin-Barnes integrals, JHEP 1208 (2012) 050. arXiv:1205.0299, doi:10.1007/JHEP08(2012)050.

[26] A. Freitas, Two-loop fermionic electroweak corrections to the Z-boson width and production rate, Phys. Lett. B730 (2014) 50–52. arXiv:1310.2256, doi:10.1016/j.physletb.2014.01.017.

[27] M. Ciuchini, E. Franco, S. Mishima, L. Silvestrini, Electroweak Precision Observables, New Physics and the Nature of a 126 GeV Higgs Boson, JHEP 1308 (2013) 106. arXiv:1306.4644, doi:10.1007/JHEP08(2013)106.

[28] S. Schael, et al., Precise weak measurements of the Z resonance, Phys. Rept. 427 (2006) 257–454. arXiv:hep-ex/0509008, doi:10.1016/j.physrep.2005.12.006.

[29] J. Fleischer, A. Leike, T. Riemann, A. Werthenbach, fortran program toppfit.F v.0.91 (06 March 2002), http://www-zeuthen.desy.de/~riemann.

[30] J. Fleischer, A. Leike, T. Riemann, A. Werthenbach, Electroweak one loop corrections for $e^-e^+$ annihilation to $\gamma$ including hard bremsstrahlung, Eur. Phys. J. C31 (2003) 37. arXiv:hep-ph/0302259, doi:10.1140/epjc/s2003-01263-8.

[31] J. Fleischer, A. Leike, T. Riemann, A. Werthenbach, Status of electroweak corrections to top pair production (2002) 275–279. arXiv:hep-ph/0211428.

[32] J. Fleischer, T. Hahn, W. Hollik, T. Riemann, C. Schappacher, A. Werthenbach, Complete electroweak one loop radiative corrections to top pair production at TESLA: A Comparison. arXiv:hep-ph/0202109.

[33] J. Fleischer, J. Fujimoto, T. Ishikawa, A. Leike, T. Riemann, Y. Shimizu, A. Werthenbach, One loop corrections to the process $e^-e^+ \rightarrow t\bar{t}$ including hard bremsstrahlung, in: Y. Kurihara (ed.), Second Symposium on Computational Particle Physics (CPP, Tokyo, 28–30 Nov 2001), KEK Proceedings 2002-11 (2002), p. 153. arXiv:hep-ph/0203220.

[34] J. Fleischer, T. Hahn, W. Hollik, T. Riemann, C. Schappacher, A. Werthenbach, Complete electroweak one loop radiative corrections to top pair production at TESLA: A Comparison. arXiv:hep-ph/0202109.

[35] J. Fleischer, J. Fujimoto, T. Ishikawa, A. Leike, T. Riemann, Y. Shimizu, A. Werthenbach, One loop corrections to the process $e^-e^+ \rightarrow t\bar{t}$ including hard bremsstrahlung, in: Y. Kurihara (ed.), Second Symposium on Computational Particle Physics (CPP, Tokyo, 28–30 Nov 2001), KEK Proceedings 2002-11 (2002), p. 153. arXiv:hep-ph/0203220.

[36] E. Accomando, T. Riemann, A. Werthenbach, et al., Physics at the CLIC multi-TeV linear collider. Report of the CLIC Physics Working Group, Section 7.2: $e^+e^- \rightarrow t\bar{t}$ Cross Sec-
tions, CERN-2004-005, p. 71. arXiv:hep-ph/0412251.

[41] J. Fleischer, A. Leike, A. Lorca, T. Riemann, A. Werthenbach, fortran program topfit.p. F.042.2.01 July 2003, http:/www-
zeuthen.desy.de/theory/num.html.

[42] T. Hahn, W. Hollik, A. Lorca, T. Riemann, A. Werthenbach, $O(a)$ electroweak corrections to the processes $e^+e^- \rightarrow \tau^+\tau^-$, $c\bar{c}, b\bar{b}, t\bar{t}$: A comparison.
http://www-fc.desy.de/lnotes/notes/LC-TH-2003-083.pdf.

[43] T. Asaliev, W. Bartel, A. Bondar, J. Brodzicka, T. Browder, et al., Physics at Super B Factory.
arxiv:1406.4539.

[44] O. Fedorenko, T. Riemann, Analytic Bremsstrahlung Integration for the Process $e^+e^- \rightarrow \mu^+\mu^-$ in QED, Acta Phys.

[45] J. M. Henn, V. A. Smirnov, Analytic results for two-loop massive two-loop $B$habha scattering, Nucl. Phys. Proc. Suppl. 135 (2004) 83–87.
arxiv:hep-ph/0412164.

[46] J. Fleischer, A. Leike, A. Lorca, T. Riemann, Planar four-point master integrals for massive two-loop $B$habha scattering: $N_f = 1$ and $N_f = 2$, Nucl. Phys. Proc. Suppl. 160 (2006) 91–100. arXiv:hep-ph/0609051.

[47] S. Actis, M. Czakon, J. Gluza, T. Riemann, Fermionic NNLO corrections to Bhabha Scattering, Nucl. Phys. Proc. Suppl. 135 (2004) 83–87.
arxiv:hep-ph/0412251.

[48] J. Fleischer, A. Leike, A. Lorca, T. Riemann, First order radiative corrections to Bhabha scattering in d dimensions, Eur.
J. Phys. 48 (2006) 35–52. arxiv:hep-ph/0606210, doi:10.1140/epjc/s10052-006-0008-6.

[49] S. Actis, M. Czakon, J. Gluza, T. Riemann, Planar two-loop master integrals for massive $B$habha scattering: $N_f = 1$ and $N_f = 2$, Nucl. Phys. Proc. Suppl. 160 (2006) 91–100. arXiv:hep-ph/0609051.

[50] S. Actis, M. Czakon, J. Gluza, T. Riemann, Two-Loop Fermionic Corrections to Massive Bhabha Scattering, Nucl.

[51] C. Carloni Calame, H. Czyz, J. Gluza, M. Gunia, G. Montagna, O. Nicrosini, F. Piccinini, T. Riemann, M. Worek, NNLO leptonic and hadronic corrections to Bhabha scattering and their implementation into BabaYaga Monte Carlo generator, Acta Phys. Polon. B42 (2011) 2469–2476. doi:10.5506/APhysPolB.42.2469.

[52] C. Carloni Calame, H. Czyz, J. Gluza, M. Gunia, G. Montagna, O. Nicrosini, F. Piccinini, T. Riemann, M. Worek, NNLO leptonic and hadronic corrections to Bhabha scattering and their implementation into BabaYaga Monte Carlo generator, Acta Phys. Polon. B42 (2011) 2469–2476. doi:10.5506/APhysPolB.42.2469.

[53] J. Gluza, M. Gunia, T. Riemann, M. Worek, Theoretical Improvements for Luminosity Monitoring at Low Energies, PoS RADCOR2011 (2011) 034.

[54] J. J. H. Kuhn, S. Uccirati, Two-loop heavy fermion corrections to Bhabha scattering, eConf C0705302 (2007) TEV02, http://www-library.desy.de/...desy07-192.pdf.

[55] J. M. Henn, V. A. Smirnov, Analytic results for two-loop master integrals for massive $B$habha scattering I, JHEP 1311 (2013) 041.

[56] S. Actis, M. Czakon, J. Gluza, T. Riemann, Virtual Hadronic and Heavy-Fermion $O(a^2)$ Corrections to Bhabha Scattering, Phys. Rev. D78 (2008) 085019. arxiv:0807.4691, doi:10.1103/PhysRevD.78.085019.

[57] J. Bonciani, A. Ferroglia, A. Penin, Heavy-flavor contribution to Bhabha Scattering, Phys. Rev. Lett. 100 (2008) 131601. arxiv:0707.1475, doi:10.1103/PhysRevLett.100.131601.

[58] J. Bonciani, A. Ferroglia, A. Penin, Calculation of the Two-Loop Heavy-Flavor Contribution to Bhabha Scattering, JHEP 0802 (2008) 080. arxiv:0802.2215, doi:10.1088/1126-6708/2008/02/080.

[59] J. H. Kuhn, S. Uccirati, Two-loop QED hadronic corrections to Bhabha Scattering, Nucl. Phys. B806 (2009) 300–326. arxiv:0807.1284, doi:10.1016/j.nuclphysb.2008.08.002.

[60] J. M. Henn, V. A. Smirnov, Analytic results for two-loop master integrals for massive Bhabha scattering I, JHEP 1311 (2013) 041.

[61] S. Actis, M. Czakon, J. Gluza, T. Riemann, Virtual Hadronic and Heavy-Fermion $O(a^2)$ Corrections to Bhabha Scattering, Phys. Rev. D78 (2008) 085019. arxiv:0807.4691, doi:10.1103/PhysRevD.78.085019.

[62] S. Actis, M. Czakon, J. Gluza, T. Riemann, Two-loop heavy fermion corrections to Bhabha scattering, eConf C0705302 (2007) TEV02, http://www-library.desy.de/...desy07-192.pdf.

[63] J. M. Henn, V. A. Smirnov, Analytic results for two-loop master integrals for massive $B$habha scattering I, JHEP 1311 (2013) 041.

[64] S. Actis, M. Czakon, J. Gluza, T. Riemann, Virtual Hadronic and Heavy-Fermion $O(a^2)$ Corrections to Bhabha Scattering, Phys. Rev. D78 (2008) 085019. arxiv:0807.4691, doi:10.1103/PhysRevD.78.085019.

[65] J. Bonciani, A. Ferroglia, A. Penin, Heavy-flavor contribution to Bhabha Scattering, Phys. Rev. Lett. 100 (2008) 131601. arxiv:0707.1475, doi:10.1103/PhysRevLett.100.131601.

[66] J. Bonciani, A. Ferroglia, A. Penin, Calculation of the Two-Loop Heavy-Flavor Contribution to Bhabha Scattering, JHEP 0802 (2008) 080. arxiv:0802.2215, doi:10.1088/1126-6708/2008/02/080.

[67] J. M. Henn, V. A. Smirnov, Analytic results for two-loop master integrals for massive $B$habha scattering I, JHEP 1311 (2013) 041.

[68] S. Actis, M. Czakon, J. Gluza, T. Riemann, Virtual Hadronic and Heavy-Fermion $O(a^2)$ Corrections to Bhabha Scattering, Phys. Rev. D78 (2008) 085019. arxiv:0807.4691, doi:10.1103/PhysRevD.78.085019.

[69] J. H. Kuhn, S. Uccirati, Two-loop QED hadronic corrections to Bhabha Scattering, Nucl. Phys. B806 (2009) 300–326. arxiv:0807.1284, doi:10.1016/j.nuclphysb.2008.08.002.

[70] J. M. Henn, V. A. Smirnov, Analytic results for two-loop master integrals for massive $B$habha scattering I, JHEP 1311 (2013) 041.

[71] S. Actis, M. Czakon, J. Gluza, T. Riemann, Virtual Hadronic and Heavy-Fermion $O(a^2)$ Corrections to Bhabha Scattering, Phys. Rev. D78 (2008) 085019. arxiv:0807.4691, doi:10.1103/PhysRevD.78.085019.

[72] J. H. Kuhn, S. Uccirati, Two-loop QED hadronic corrections to Bhabha Scattering, Nucl. Phys. B806 (2009) 300–326. arxiv:0807.1284, doi:10.1016/j.nuclphysb.2008.08.002.

[73] J. M. Henn, V. A. Smirnov, Analytic results for two-loop master integrals for massive $B$habha scattering I, JHEP 1311 (2013) 041.
A. Denner, Techniques and concepts for higher order calculations, Introductory Lecture at DESY Theory Workshop on Collider Phenomenology, Hamburg, 29 Sep - 2 Oct 2009.

V. Yundin, Talk at Kick-off meeting of the LHCPHENONet Initial Training Network, Jan/Feb 2011, Valencia, Spain.

V. Yundin, C++ package PLFry. Available at https://github.com/Vayu/PLFry/.

J. Fleischer, J. Gluza, M. Gluza, and T. Riemann, Automated analytic continuation of Mellin-Barnes representations, multiple sums, PoS LL2014 (2014) 052. arXiv:1407.7832.

J. Gluza, T. Riemann, A New treatment of mixed virtual and real IR-singularities, PoS RADCOR2007 (2007) 007. arXiv:0801.4228.

J. Gluza and T. Riemann, Simple Feynman diagrams and simple sums, Talk held at RISC-DESY Workshop on Advanced Summation Techniques and their Applications in Quantum Field Theory, on the occasion of the 5th year jubilee of the RISC-DESY cooperation, May 7-8, 2012. RISC Institute, Castle of Hagenberg, Linz, Austria. Conference link: http://www.risc.jku.at/conferences/RISCDESYS12/ talk link: http://www-wezuchen.desy.de/~riemann/Talks/riemann-Linz-2012.pdf.

J. Vermaseren, Harmonic sums, Mellin transforms and integrals, Int. J. Mod. Phys. A14 (1999) 2037–2076. arXiv:hep-ph/9806280.

J. Ablinger, J. Blümlein, C. G. Raab, C. Schneider, Nested inverse binomial sums and new iterated integrals for massive Feynman diagrams, PoS LL2014 (2014) 052. arXiv:1312.5603.

J. Blümlein, I. Dubovyk, J. Gluza, M. Ochman, C. G. Raab, et al., Non-planar Feynman integrals, Mellin-Barnes representations, multiple sums, PoS LL2014 (2014) 052. arXiv:1407.7832.

J. Vermaseren, New features of FORM. arXiv:math-ph/0102025.

C. Schneider, Symbolic summation assists combinatorics, Sém. Lothar. Combin. 56 (2007) 1–36, Article B56b, http://www.emis.de/journals/SLC/wpapers/s56.html.

J. Blümlein, J. Blümlein, C. G. Raab, C. Schneider, Nested (inverse) binomial sums and new iterated integrals for massive Feynman diagrams, PoS LL2014 (2014) 020. arXiv:1407.4721.

G. Passarino, An Approach toward the numerical evaluation of multiloop Feynman diagrams, Nucl. Phys. B619 (2001) 257–312. arXiv:hep-ph/0108252.

G. Passarino, S. Uccirati, Algebraic numerical evaluation of Feynman diagrams: Two loop selfenergies, Nucl. Phys. B629 (2002) 97–187. arXiv:hep-ph/0112004.

A. Ferroglio, M. Passera, G. Passarino, S. Uccirati, All purpose numerical evaluation of one loop multilieg Feynman diagrams, Nucl.Phys. B650 (2003) 162–228. arXiv:hep-ph/0209219.

S. Becker, S. Weinzierl, Direct numerical integration for multiloop integrals, Eur. Phys. J. C73 (2) (2013) 2321. arXiv:1211.0509.

S. Becker, S. Weinzierl, Two loop vertices in quantum field theory: Infrared convergent scalar configurations, Nucl. Phys. B860 (2004) 199–270. arXiv:hep-ph/0311186.

C. Schneider, Symbolic summation assists combinatorics, Sém. Lothar. Combin. 56 (2007) 1–36, Article B56b, http://www.emis.de/journals/SLC/wpapers/s56.html.

J. Blümlein, J. Blümlein, C. G. Raab, C. Schneider, Nested (inverse) binomial sums and new iterated integrals for massive Feynman diagrams, PoS LL2014 (2014) 020. arXiv:1407.4721.

G. Passarino, An Approach toward the numerical evaluation of multiloop Feynman diagrams, Nucl. Phys. B619 (2001) 257–312. arXiv:hep-ph/0108252.

G. Passarino, S. Uccirati, Algebraic numerical evaluation of Feynman diagrams: Two loop selfenergies, Nucl. Phys. B629 (2002) 97–187. arXiv:hep-ph/0112004.

A. Ferroglio, M. Passera, G. Passarino, S. Uccirati, All purpose numerical evaluation of one loop multilieg Feynman diagrams, Nucl.Phys. B650 (2003) 162–228. arXiv:hep-ph/0209219.

S. Becker, S. Weinzierl, Direct numerical integration for multiloop integrals, Eur. Phys. J. C73 (2) (2013) 2321. arXiv:1211.0509.

S. Becker, S. Weinzierl, Two loop vertices in quantum field theory: Infrared convergent scalar configurations, Nucl. Phys. B860 (2004) 199–270. arXiv:hep-ph/0311186.

A. V. Smirnov, FIESTA 3: cluster-parallelizable multiloop numerical calculations in physical regions, Comput. Phys. Commun. 185 (2014) 2090–2104. arXiv:1312.3186.
[144] S. Actis, A. Ferroglia, G. Passarino, M. Passera, S. Uccirati, Two-loop tensor integrals in quantum field theory, Nucl. Phys. B703 (2004) 3–126. arXiv:hep-ph/0402132, doi:10.1016/j.nuclphysb.2004.10.018.

[145] C. Anastasiou, A. Daleo, Numerical evaluation of loop integrals, JHEP 0610 (2006) 031. arXiv:hep-ph/0511176, doi:10.1088/1126-6708/2006/10/031.

[146] S. Pozzorini, E. Remiddi, Precise numerical evaluation of the two loop sunrise graph master integrals in the equal mass case, Comput. Phys. Commun. 175 (2006) 381–387. arXiv:hep-ph/0505041, doi:10.1016/j.cpc.2006.05.005.

[147] G. Passarino, S. Uccirati, Two-loop vertices in quantum field theory: Infrared and collinear divergent configurations, Nucl. Phys. B747 (2006) 113–189. arXiv:hep-ph/0603121, doi:10.1016/j.nuclphysb.2006.04.014.

[148] S. Actis, G. Passarino, S. Uccirati, Two Loop QFT in the Making, Nucl. Phys. Proc. Suppl. 160 (2006) 145–149. arXiv:hep-ph/0608294, doi:10.1016/j.nuclphysbps.2006.09.106.

[149] C. Anastasiou, S. Beerli, A. Daleo, Evaluating multi-loop Feynman diagrams with infrared and threshold singularities numerically, JHEP 0705 (2007) 071. arXiv:hep-ph/0703282, doi:10.1088/1126-6708/2007/05/071.

[150] A. Freitas, Y.-C. Huang, On the Numerical Evaluation of Loop Integrals With Mellin-Barnes Representations, JHEP 1004 (2010) 074. arXiv:1001.3243, doi:10.1007/JHEP04(2010)074.

[151] R. K. Ellis, Z. Kunst, K. Melnikov, G. Zanderighi, One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts, Phys. Rept. 518 (2012) 141–250. arXiv:1105.4319, doi:10.1016/j.physrep.2012.01.008.

[152] J. P. Carter, Higher Order Corrections in Perturbative Quantum Field Theory via Sector Decomposition, http://etheses.dur.ac.uk/3370/.

[153] S. Becker, S. Weinzierl, Direct contour deformation with arbitrary masses in the loop, Phys. Rev. D86 (2012) 074009. arXiv:1208.4088, doi:10.1103/PhysRevD.86.074009.

[154] A. Freitas, Numerical evaluation of multi-loop integrals using subtraction terms, JHEP 1207 (2012) 132. arXiv:1205.3515, doi:10.1007/JHEP07(2012)132, 10.1007/JHEP09(2012)129.

[155] W. Löwer, English translation of Schiedsspruch (3 July 2012), Schiedsspruch_ProfLöwer-en-03Jul2012.pdf.

[156] Letter by Editor-in-Chief of EPJC (26 January 2012), EPJC-Letter.pdf.

[157] F. Carminati, D. Perret-Gallix, T. Riemann, Summary of the ACAT 2013 round table discussion: Open-source, knowledge sharing and scientific collaboration. Red report DESY 14–079. arXiv:1407.0560.

[158] F. Carminati, D. Perret-Gallix, T. Riemann, Summary of the ACAT 2013 round table discussion: Open-source, knowledge sharing and scientific collaboration, J. Phys. Conf. Ser. 523 (2014) 012066. doi:10.1088/1742-6596/523/1/012066.

[159] M. Artuso, A. Cohen, L. Dixon, D. Glenzinski, P. McBride, I. Shipsey, M. Shochet, Report of the HEPAP Subcommittee on the Dissemination of Research Results, http://inspirehep.net/record/1399830.

[160] K. Olive, et al., Review of Particle Physics, Chin. Phys. C38 (2014) 090001. doi:10.1088/1674-1137/38/9/090001.