METRIC STABILITY FOR RANDOM WALKS
WITH APPLICATIONS IN RENORMALIZATION THEORY

CARLOS GUSTAVO MOREIRA AND DANIEL SMANIA

Abstract. Consider deterministic random walks \( F: I \times \mathbb{Z} \to I \times \mathbb{Z} \), defined by \( F(x, n) = (f(x), \psi(x) + n) \), where \( f \) is an expanding Markov map on the interval \( I \) and \( \psi: I \to \mathbb{Z} \). We study the universality (stability) of ergodic (for instance, recurrence and transience), geometric and multifractal properties in the class of perturbations of the type \( F(x, n) = (f_n(x), \psi(x, n) + n) \) which are topologically conjugate with \( F \) and \( f_n \) are expanding Markov maps exponentially close to \( f \) when \( |n| \to \infty \). We give applications of these results in the study of the regularity of conjugacies between (generalized) infinitely renormalizable maps of the interval and the existence of wild attractors for one-dimensional maps.

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1. Introduction

1.1. Metric stability for random walks. In the study of a dynamical system, some of the most important questions concerns the stability of their dynamical properties under (most of the) perturbations: how robust are they?

Here we are mainly interested in the stability of metric (measure-theoretical) properties of dynamical systems. A well-known example is given by \((C^2)\) expanding
maps on the circle: this is a class stable under perturbations and all of them have an absolutely continuous and ergodic invariant probability satisfying certain decay of correlations estimatives. In particular, in the measure theoretical sense, most of the orbits are dense in the phase space.

Now let us study a slightly more complicated situation: consider a $C^2$ Markov almost onto expanding map of the interval $f: I \to I$ with bounded distortion and large images (see Section 2 for details) and let $\psi: I \to \mathbb{Z}$ be a function which is constant in each interval of the Markov partition of $f$. We can define $F: I \times \mathbb{Z} \to I \times \mathbb{Z}$ as

$$ F(x, n) := (f(x), \psi(x) + n). $$

The second entry of $(x, n)$ will be called its state. We also assume that

$$ \inf \psi > -\infty $$

and that $F$ is topologically mixing.

The map $F$ is refereed to in literature in many ways: as a "skew-product between $f$ and the translation on the group $\mathbb{Z}$", a "group extension of $f$", or even a "deterministic random walk generated by $f$", and its metric behavior is very well studied: for instance, are most the orbits recurrent? Everything depends on the mean drift

$$ M = \int \psi d\mu, $$

where $\mu$ is the absolutely continuous invariant probability of $f$ (the function $\psi$ will be called drift function). Indeed, note that

$$ F^n(x, i) = (f^n(x), i + \sum_{k=0}^{n-1} \psi(f^k(x))). $$

By the Birkhoff Ergodic Theorem

$$ \lim_{n \to \infty} \frac{\pi_2(F^n(x, i)) - \pi_2(x, i)}{n} = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \psi(f^k(x)) = M. $$

for almost every $x \in I$ (here $\pi_2(x, n) := n$). In particular if $M \neq 0$ then almost every point $(x, i) \in I \times \mathbb{Z}$ is transient: in other words we have

$$ \lim_{n \to \infty} |\pi_2(F^n(x, i))| = \infty. $$

So most of the points are not recurrent.
On the other hand, if \( M = 0 \), most of points are recurrent (see Guivarc’h [G]): by the Central Limit Theorem for expanding maps (here we need to assume that \( \psi \) is not constant and \( f \in On \); see Section 2) of the interval

\[
\sup_{x \in \mathbb{R}} |\mu(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\epsilon} e^{-u^2/2} du| \leq C \sqrt{n}.
\]

Given \( \delta > 0 \) we can easily obtain, taking \( \epsilon = n^{-1/4} \) and applying Borel-Cantelli Lemma, that

\[
\mu(A_+) := \limsup_{n \to \infty} \frac{\sum_{k=0}^{n-1} \psi(f^k(x))}{n^{1/2}} = \infty \geq \frac{1}{2},
\]

\[
\mu(A_-) := \liminf_{n \to \infty} \frac{\sum_{k=0}^{n-1} \psi(f^k(x))}{n^{1/2}} = -\infty \geq \frac{1}{2}.
\]

Clearly \( A_+ \) and \( A_- \) are invariant sets: the ergodicity of \( f \) implies that

\[
\mu(A_+ \cap A_-) = 1.
\]

By the conditions on \( \psi \) in Eq. (1), that \( f \) is expanding with distortion control and that \( F \) is transitive, we can easily conclude that almost every point in \( I \times \mathbb{Z} \) is a \( F \)-recurrent point.

Note that the random walk \( F \) is a dynamical system quite similar to expanding circle maps: \( F \) is an expanding map, with good bounded distortion properties; but the lack of compactness of the phase space allows the non-existence of an invariant probability absolutely continuous with respect to the Lebesgue measure on \( I \times \mathbb{Z} \). Moreover, in general the random walk is not even recurrent and the recurrence property lost its stability: given a recurrent random walk \((f, \psi)\), it is possible to obtain a transient random walk by changing a little bit \( f \) and \( \psi \).

Since the non compactness of the phase space seems to be the origin of the lack of stability of recurrence and transience properties, a natural question is to ask if such properties are stable by compact perturbations. The answer is yes. Indeed, as we are going to see in Theorems 1-4, the transience and recurrence are preserved even by non-compact perturbations which decreases fast away from state 0. For instance, we can choose perturbations like

\[
\tilde{F}(x,n) = (f_n(x), \psi(x) + n),
\]

where, for some \( \lambda \in [0, 1) \),

\[
|f_n - f|_{C^3} \leq \lambda |n|.
\]

The notations and conventions are more or less obvious: we postponed the rigorous definitions to the next section.

With respect to the stability of transience and recurrence, there is a previous quite elegant result by R. L. Tweedie [I]: if \( p_{ij} \) are the transition probabilities of a Markov chain on \( \mathbb{Z} \), then any perturbation \( \tilde{p}_{ij} \) so that

\[
(1 + \epsilon_i)^{-1} p_{ij} \leq \tilde{p}_{ij} \leq p_{ij}(1 + \epsilon_i), \ j \neq i,
\]

and

\[
\prod_{i=0}^{\infty} (1 + \epsilon_i) < \infty
\]
preserves the recurrence or transience of the original Markov chain. But Tweedie argument does not seem to work in our setting. Our result coincides with Tweedie result in the very special case where \( f \) and \( f_n \) are linear Markov maps and \( \epsilon_i \sim C \lambda^{|i|} \).

In the transient case we can tell a little more: there will be a conjugacy between the original random walk \( f \) and its perturbation which is a martingale strongly quasisymmetric map (for short, mSQS-map) with respect to certain dynamically defined set of partitions. Unlike the usual class of one-dimensional quasisymmetric functions, which does not share many of most interesting properties of higher dimensional quasisymmetric maps, the one-dimensional mSQS-maps are much closer to their high-dimensional cousins, as quasiconformal maps in dimension 2. For instance, they are absolutely continuous.

We also study the behavior of the Hausdorff dimension of dynamically defined sets: Denote by \( \Omega_+ (F) \) the set of points which have non-negative states along the positive orbit by \( F \). We prove that \( \Omega_+ (F) \) has Hausdorff dimension strictly smaller than one if and only if \( \Omega_+ (\tilde{F}) \) has dimension less than one for all perturbation satisfying Eq. (5). Furthermore we give a variational characterization for the Hausdorff dimension \( \text{HD}(\Omega_+ (F)) \) as the minimum of \( \text{HD}(\Omega_+ (\tilde{F})) \), where \( \tilde{F} \) runs on the set of such perturbations. For these results we study the stability of the multifractal spectrum of the random walk \( F \) under those perturbations.

### 1.2. Applications to (generalized) renormalization theory.

An unimodal map is a map with an unique critical point. Under reasonable conditions (real-analytic maps with negative Schwarzian derivative and non-flat critical point) two non renormalizable unimodal maps with the same topological entropy are indeed topologically conjugated. A key question in one-dimensional dynamics is about the regularity of the conjugacy: is it Hölder? Is it absolutely continuous? Since Dennis Sullivan work in the 80’s the quasisymmetry of the conjugacy became a very useful tool to obtain deep results in one-dimensional dynamics. Lyubich proved that under the reasonable condition above the conjugacy between two non renormalizable unimodal maps is quasisymmetric. Later on, the density of the hyperbolic maps in the real quadratic family was proved verifying the quasisymmetry of the conjugacies for all combinatorics, including infinitely renormalizable ones.

Note that quasisymmetric maps are not, in general, absolutely continuous: they do not even preserve (in general) sets of Hausdorff dimension one. Are the conjugacy between unimodal maps absolutely continuous? The answer is no: M. Martens and W. de Melo [MdM] proved that under the reasonable conditions above an absolutely continuous conjugacy is actually \( C^\infty \), provided the unimodal maps

1. do not have a periodic attractor,
2. are not infinitely renormalizable,
3. do not have a wild attractor (the topological and measure-theoretical attractor must coincide).

Since we can change the eigenvalues of the periodic points of maps preserving its topological class, and the eigenvalues are preserved by \( C^1 \) conjugacies, we conclude that in general a conjugacy between unimodal maps is not absolutely continuous.
Condition i. is clearly necessary. This work (Theorem 8) shows that the Condition ii. is necessary proving that the conjugacy between two arbitrary Feigenbaum unimodal maps with same critical order is always absolutely continuous. Actually the conjugacy is martingale strongly quasisymmetric with respect to a set of dynamically defined partitions.

Condition iii. is never violated when the critical point is quadratic. But for certain topological classes of unimodal maps wild attractor appears when the order of the critical point increases: Fibonacci maps are the simplest kind of such maps [BKNvS][Br]. We are going to prove (Theorem 11) that a Fibonacci map with even order has a wild attractor if and only if all Fibonacci maps with the same even order are conjugated to each other by an absolutely continuous mapping (in particular all these Fibonacci maps have a wild attractor). So Condition iii. is necessary.

To show that conditions i. and ii. are necessary, the (generalized) renormalization theory for unimodal maps and the study of perturbations of transient and recurrent random walks are going to be crucial. Feigenbaum and Fibonacci unimodal maps admit induced maps which are essentially perturbations of deterministic random walks (Section 8). In the Fibonacci case the transience of this random walk is equivalent to the existence of a wild attractor. Random walks associated to a Feigenbaum map will always be transient.

For both Feigenbaum and Fibonacci maps there are infinitely many periodic points (indeed in the Fibonacci case the periodic points are also dense in the maximal invariant set). It is well known that the conjugacy between critical circle maps with same irrational rotation number and satisfying certain Diophantine condition is absolutely continuous, but we think that these are the first interesting examples of a similar phenomena for maps with many periodic points.

2. Expanding Markov maps, random walks and its perturbations

In this article we will deal with maps

$$F: I \times \mathbb{Z} \to I \times \mathbb{Z}$$

which are piecewise $C^2$ diffeomorphisms, which means that there is a partition $\mathcal{P}^0$ of $I \times \mathbb{Z}$ so that each element $J \in \mathcal{P}^0$ is an open interval where $F|_J$ is a $C^2$ diffeomorphism. Denote $I_n = I \times \{n\}$. Denote by $m$ the Lebesgue measure in the in $I \times \mathbb{Z}$, that is, if $A \subset I \times \mathbb{Z}$ is a Borelian set then

$$m(A) = \sum_n m_1(\pi(A \cap I_n)),$$

where $m_1$ is the Lebesgue measure in the interval $I$ and $\pi(n, x) = x$.

If $A_J$ denotes the unique affine transformation which maps the interval $J$ to $[0, 1]$ and preserves orientation, then define, for each $J \in \mathcal{P}^0$,

$$\tau^F_J := A_J \circ F^{-1} \circ A^{-1}_J.$$

Throughout this article we will assume that $F$ satisfies some of the following properties:

- **Markovian (Mk)**: For each $J \in \mathcal{P}^0$, $F(J)$ is a connected union of elements in $\mathcal{P}^0$. In particular we can write $F(x, n) = (f_n(x), n + \psi(x, n))$, where $f_n: I \to I$ is a piecewise $C^2$ diffeomorphism relative to the partition
\[ \mathcal{P}^0_0 := \{ J \in \mathcal{P}^0 : J \subset I_n \} \] and \( \psi : I \times \mathbb{Z} \to \mathbb{Z} \), called the **drift function**, is constant on each element of \( \mathcal{P}^0 \).

- **Lower Bounded Drift (LBD)**: \( F \) is Markovian and \( \min \psi > -\infty \).
- **Large Image (LI)**: \( F \) is Markovian and there exists \( \delta > 0 \) so that for each \( J \in \mathcal{P}^0 \) we have \( |F(J)| \geq \delta \).
- **Onto (On)**: \( F \) is Markovian and for each \( J \in \mathcal{P}^0 \) we have \( F(J) = I^n \), for some \( n \in \mathbb{Z} \).
- **Bounded Distortion (BD)**: There exists \( C > 0 \) so that every \( J \in \mathcal{P}^0_n \) and map \( \tau_J \) is a \( C^2 \) function satisfying
  \[ \sup_J \left| \frac{D^2 \tau_J}{(D\tau_J)^2} \right| \leq C. \]
- **Strong Bounded Distortion (sBD)**: There exists \( C > 0 \) so that every \( J \in \mathcal{P}^0_n \) and map \( \tau_J \) is a \( C^2 \) function satisfying
  \[ \sup_J \left| \frac{D^2 \tau_J}{(D\tau_J)^2} \right| \leq C|J|. \]
- **Expansivity (Ex)**: If \( J \in \mathcal{P}^0_n := \{ J \in \mathcal{P}^0 : J \subset I_n \} \), denote \( \phi_J := f_n^{-1}|f_n(J) \). Then either \( \phi_J \) can be extended to a function in a \( \delta \)-neighborhood of \( J \) so that
  \[ S\phi_J > 0, \]
  where \( S\phi_J \) denotes the Schwarzian derivative of \( \phi_J \), or there exists \( \theta \in (0,1) \) so that for all \( n \) and \( J \in \mathcal{P}^0_n \) we have
  \[ |\phi'_J| < \theta \]
on \( f_n(J) \).
- **Regularity a (Ra)**: There exists \( N \in \mathbb{N} \), \( \delta > 0 \) and \( C > 0 \) with the following properties: the intervals in \( \mathcal{P}^0_n \) are positioned in \( I_n = [a,b] \) in such way that the complement of
  \[ \bigcup_{J \in \mathcal{P}^0_n} \text{int } J \]
contains at most \( N \) accumulation points
  \[ c^1_i < c^2_i < \cdots < c^n_i, \]
with \( i_n \leq N \), which are in the interior of \( I_n \). Furthermore \( |c^i_{i+1} - c^i_i| \geq \delta \) and \( |a - c^i_1|, |b - c^i_n| \geq \delta \). Moreover, given \( P \) and \( Q \in \mathcal{P}^0_n \) so that \( \overline{P} \cap \overline{Q} \neq \emptyset \) then
  \[ \frac{1}{C} \leq \frac{|P|}{|Q|} \leq C. \]
• **Regularity b (Rb):** Assume Ra. There exists $C > 0$, $\lambda \in (0, 1)$, $\delta > 0$ so that for each $1 < i < i_n$ we can find a point

$$d^n_i \in (c^n_i, \pi_{i+1}^n),$$

which does not belong to any $P \in \mathcal{P}_n$, and

$$\min\{|c^n_{i+1} - d^n_i|, |d^n_i - c^n_i|\} \geq \delta$$

with the following property: If $J$ is a connected component of $I_n \setminus \{d^n_i, c^n_i\}\_{i,j}$

then we can enumerate the set

$$\{P\}_{P \in \mathcal{P}_n, P \subseteq J} = \{J_i\}_{i \in \mathbb{N}}$$

in such way that $\partial J_i \cap \partial J_{i+1} \neq \emptyset$ for each $i$ and

$$\frac{|J_{i+j}|}{|J_i|} \leq C\lambda^j$$

for $i \geq 0$, $j > 0$.

• **Good Drift (GD):** if $\psi$ is the drift function of the random walk then for each $n \in \mathbb{Z}$ there exists $x$ such that $\psi(x,n) > 0$. Moreover there exists $\gamma \in (0, 1)$ and $C > 0$ so that for every $k \geq 0$

$$m(\{(x,n) \text{ s.t. } \psi(x,n) \geq k\}) \leq C\gamma^k.$$  

• **Transitive (T):** $F$ has a dense orbit.

For convenience of the notation if for instance $F$ is Markovian and it has Bounded Distortion, we will write $F \in Mk + BD$.

A **deterministic random walk** (or simply random walk) is a map

$$F \in Mk + LBD + LI + Ex + BD + GD.$$  

It is generated by the pair $(\{f_n\}, \psi)$ if

$$F(x,n) := (f_n(x), \psi(x,n) + n).$$

When $f_n = f \in Mk$ and $\psi(x,n) = \psi(x)$, we say that $F$ is the **homogeneous deterministic random walk** generated by the pair $(f, \psi)$. There is a large literature about such random walks. We will sometimes assume the following property:

• **Almost Onto (aO):** For every $i, j \in \Lambda$ there exists a finite sequence $i = i_0, i_1, i_2, \ldots, i_{n-1}, i_n = j \in \Lambda$ so that

$$f(I_{i_k}) \cap f(I_{i_{k+1}}) \neq \emptyset$$

for each $k < j$.  

Denote $\pi(x, n) := n$. A random walk is called transient if for almost every $(x, n) \in I \times \mathbb{Z}$

$$\lim_{k \to \infty} |\pi_2(F^k(x, n))| = \infty,$$

and it is recurrent if for almost every $(x, n) \in I \times \mathbb{Z}$

$$\#\{k: \pi_2(F^k(x, n)) = n\} = \infty.$$

Making use of usual bounded distortion tricks it is easy to show that every $F \in Mk + LI + Ex + BD + T$ is either recurrent or transient.

A (topological) perturbation of a random walk is a random walk $\tilde{F}$, generated by a pair $(\{\tilde{f}_i\}, \tilde{\psi})$, so that $H \circ F = \tilde{F} \circ H$ for some homeomorphism $H: I \times \mathbb{Z} \to I \times \mathbb{Z}$ which preserves states: $\pi_2(H(x, i)) = i$.

Define $P_n(F) := \vee_{i=0}^{n-1} F^i$. If $F$ and $\tilde{F}$ are random walks and $H$ is a topological conjugacy that preserves states between $F$ and $\tilde{F}$, then for each interval $L$ such that $L \subset J \in P_n^{-1}(F)$, define

$$dist_n(L) := \sup_{y \in L} \left| \ln \frac{D\tilde{F}^n(H(y))}{DF^n(y)} \right|,$$

Similarly, define

$$dist_n(x) := \left| \ln \frac{D\tilde{F}^n(H(x))}{DF^n(x)} \right|$$

and

$$dist_\infty(x) := \sup_n dist_n(x).$$

Another kind of random walk which will have a central role in our results are those which are asymptotically small perturbations: these are perturbations $(\{\tilde{f}_i\}, \tilde{\psi})$ of a deterministic random walk $(\{f_i\}, \psi)$ such that there exists $\lambda \in (0, 1)$ and $C > 0$ satisfying either

(7) $$|\log \frac{D\tilde{F}(H(p))}{DF(p)}| \leq C\lambda^{\pi_2(p)};$$

if $\psi$ is bounded, or

(8) $$|\log \frac{D\tilde{F}(H(p))}{DF(p)}| \leq C\lambda^{\pi_2(p)},$$

for $\pi_2(p) \geq 0$ and $D\tilde{F}(H(p)) = DF(p)$ otherwise, if $\psi$ has only a lower bound.

It is easy to see that properties $Ra$, $Rb$ and $GD$ are invariant by asymptotically small perturbations (if we allow to change the constants described in these properties).

Let $F = (\{f_i\}, \psi)$ be a random walk, where $\psi$ is Lebesgue integrable on compact subsets of $I \times \mathbb{Z}$. We say that $F$ is strongly transient if $K > 0$ and

$$E(\psi \circ F^n | P_n^{-1}(F)) > K$$

for every $n \geq 1$. We will also say that $F$ is $K$-strongly transient. Here we are considering conditional expectations relative to the Lebesgue measure. As the notation suggest, every strongly transient random walk is transient. Moreover we have the following large deviations result:
Proposition 2.1. Let $F = (f, \psi) \in On + sBD + Ra + Rb$ be a homogeneous random walk with positive mean drift. Let $K := \int \psi \, dm > 0$. Then $F$ is transient and for every small $\epsilon > 0$ there exist $\lambda \in [0, 1)$ and $C > 0$ so that for each $P \in P^0$ we have

$$m(p \in P: \pi_2(F^n(p)) - \pi_2(p) < (K - \epsilon)n) \leq C\lambda^n |P|.$$

Proposition 2.2. Every $K$-strongly transient random walk $F \in Ra + Rb$ is transient. Furthermore for every small $\epsilon > 0$ there exist $\lambda \in [0, 1)$ and $C > 0$ so that for each $P \in P^0$ we have

$$m(p \in P: \pi_2(F^n(p)) - \pi_2(p) < (K - \epsilon)n) \leq C\lambda^n |P|.$$

We will postpone the proof of Propositions 2.2 and 2.1 to Section 5.

Remark 2.3. By the Birkhoff Ergodic Theorem it is easy to see that a sufficiently high iteration of a homogeneous random walk with positive mean drift is strongly transient (see the proof of Proposition 5.1 for details).

3. Statements of results

3.1. Stability of transience.

Theorem 1 (Stability of Transience I). Assume that the random walk $F$ defined by the pair $(\{f_i\}, \psi)$ is strongly transient. Then every asymptotically small perturbation $G$ of $F$ is also transient. Indeed there is a topological conjugacy between $F$ and $G$ which is an absolutely continuous map and preserves the states.

We have a similar theorem for all transient homogeneous random walks:

Theorem 2 (Stability of Transience II). Suppose that the homogeneous random walk $F$ defined by the pair $(f, \psi)$ has positive mean drift. Then every asymptotically small perturbation of $F$ is topologically conjugated to $F$ by an absolutely continuous map which preserves the states.

We can be more precise regarding the regularity of the conjugacy if the drift is non-negative:

Let $A_0, A_1, \ldots, A_n, A_{n+1}, \ldots$ be a succession of partitions by intervals of $I \times \mathbb{Z}$, such that $A_{n+1}$ refines $A_n$ and whose union generates the Borelian algebra of $\cup_n I_n$. We say that $h: \cup_n I_n \rightarrow \cup_n I_n$ is a martingale strongly quasisymmetric (mSQS) map with respect to the stochastic basis $\cup_n A_n$ if there exist $C > 0$ and $\alpha \in (0, 1]$ so that

$$\frac{m(h(B))}{|h(J)|} \leq C \left( \frac{m(B)}{|J|} \right)^\alpha$$

for all Borelian $B \subset J \in \cup_n A_n$, and the same inequality holds replacing $h$ by $h^{-1}$ and $\cup_n A_n$ by $\cup_n h(A_n)$.

Theorem 3 (Strongly quasisymmetric rigidity). Let $F$ be either a strongly transient random walk or a transient homogeneous random walk with positive mean drift. Moreover assume in both cases that $\psi \geq 0$. Then every asymptotically small perturbation $G$ of $F$ is topologically conjugated to $F$ by an absolutely continuous map $h$ which preserves the states. Furthermore $h$ on $\cup_{i\geq 0} I_i$ is a martingale strongly quasisymmetric mapping with respect to the stochastic basis $\cup_i P^i$. 

3.2. Stability of recurrence. In the recurrent case, we are going to restrict ourselves to the stability of the metric properties of homogeneous random walks under asymptotically small perturbations: it is easy to see that the recurrence is not stable by perturbations which are not asymptotically small. Nevertheless

**Theorem 4** (Stability of Recurrence). Suppose that \( F \in \text{On} + T \) is a recurrent homogeneous random walk generated by the pair \((f, \psi)\). Then every asymptotically small perturbation of \( F \) is also recurrent.

If \( p \) is a periodic point with prime period \( n \) then \( DF^n(p) \) is called the spectrum of the periodic point \( p \). Note that we can not expect, as in the transient case, an absolutely continuous conjugacy which preserves states between \( F \) and \( G \), once asymptotic small perturbations do not preserve (in general) the spectrum of the periodic points and:

**Proposition 3.1** (Rigidity). Suppose that the random walk \( F \in \text{On} \) generated by a pair \((\{f_i\}, \psi)\) is recurrent. If there is an absolutely continuous conjugacy which preserves states \( H \) between \( F \) and a random walk \( G \), then \( H \) is \( C^1 \) in each state. In particular the spectrum of the corresponding periodic points of \( F \) and \( G \) are the same.

The reader should compare this result with similar results by Shub and Sullivan [ShSu] for expanding maps on the circle and de Melo and Martens [MdM] for unimodal maps.

3.3. Stability of the multifractal spectrum. Let \( F \) be a random walk and denote

\[
\Omega_+(F) := \{ p : \pi_2(F^j p) \geq 0, \text{ for } j \geq 0 \},
\]

\[
\Omega^k_+(F) := \{ (x,k) : \pi_2(F^j (x,k)) \geq 0, \text{ for } j \geq 0 \}
\]

and

\[
\Omega^k_{+\beta}(F) := \{ (x,k) \in \Omega^k_+ \text{ s.t } \lim_{n \to \infty} \frac{\pi_2(F^n(x,k))}{n} \geq \beta \}
\]

**Theorem 5.** Let \( F \in Ra + Rb + On \) be a random walk. Then, for all \( k \in \mathbb{Z} \) and \( \beta > 0 \) the Hausdorff dimension \( HD(\Omega^k_{+\beta}(F)) \) is invariant by asymptotically small perturbations.

We will need

**Proposition 3.2.** Let \( F \in Ra + Rb + On \) be a homogeneous random walk. Then

\[
HD(\Omega^k_+(F)) = \lim_{\beta \to 0^+} HD(\Omega^k_{+\beta}(F)).
\]

and as a consequence of Theorem 5 and Proposition 3.2:

**Theorem 6.** Let \( F \in Ra + Rb + On \) be a homogeneous random walk. If \( G \) is an asymptotically small perturbation of \( F \) then

\[
HD(\Omega^k_+(G)) \geq HD(\Omega^k_+(F)). \tag{9}
\]

We can not replace the inequality in Eq. (9) by an equality. Indeed, even if \( HD(\Omega^k_+(F)) < 1 \), we have that \( \sup HD(\Omega^k_+(G)) = 1 \), where the supremum is taken on all asymptotically small perturbations \( G \) of \( F \). Nevertheless:
**Theorem 7.** Let \(F \in R_a + R_b + R_0 + T\) be the homogeneous random walk generated by the pair \((f, \psi)\). Consider \(M = \int \psi d\mu\), where \(\mu\) is the unique absolutely continuous invariant measure of \(f\).

- If \(M > 0\) then for all asymptotically small perturbations \(G\) of \(F\) we have \(m(\Omega_+ (G)) > 0\).

- If \(M = 0\) then for all asymptotically small perturbations \(G\) of \(F\) we have \(HD(\Omega_+ (G)) = 1\) but \(m(\Omega_+ (G)) = 0\).

- If \(M < 0\) then for all asymptotically small perturbations \(G\) of \(F\) we have \(HD(\Omega_+ (G)) < 1\).

**Remark 3.3.** Since the authors are more familiar with deterministic rather than stochastic terminology, we stated and proved the results in this work for deterministic random walks. However we believe that the above results could be easily translated to the theory of chains with complete connections (g-measures, chains of infinite order) and one-sided shifts on an infinite alphabet.

**3.4. Applications to renormalization theory of one-dimensional maps.**

**Theorem 8.** Let \(f\) and \(g\) be unimodal maps which are infinitely renormalizable with the same bounded combinatorial type and even critical order. Then the continuous conjugacy \(h\) between \(f\) and \(g\) is a strongly quasisymmetric mapping with respect to a certain stochastic basis of intervals \(\mathcal{P}\).

The set of intervals \(\mathcal{P}\) is defined using a map induced by \(f\). See the details in Section 8.1.

**Remark 3.4.** D. Sullivan [Su][dMvS] show that on the assumptions of Theorem 8 the conjugacy \(h\) is a quasisymmetric map. However it is known that quasisymmetric maps on the real line are not in general absolutely continuous maps.

Let \(\mathcal{F}_d\) be the class of analytic maps with negative Schwarzian derivative which are infinitely renormalizable in the Fibonacci sense with even critical order \(d\) (see Section 8.2 for definitions). If \(f\) is a Fibonacci map, denote by \(J_R(f)\) the maximal invariant set of \(f\). Let \(\mathcal{F}^{uni}_d\) be the class of Fibonacci unimodal maps with negative Schwarzian derivative.

**Theorem 9 (Metric Universality).** For each even critical order \(d, d \geq 4\), one of the following statements holds:

- \(HD(J_R(f)) < 1\), for all \(f \in \mathcal{F}_d\).
- \(HD(J_R(f)) = 1\) and \(m(J_R(f)) = 0\) for all \(f \in \mathcal{F}_d\).
- \(HD(J_R(f)) = 1\) and \(f\) has a wild attractor (in particular, \(m(J_R(f)) > 0\)) for all \(f \in \mathcal{F}_d\).

**Theorem 10 (Measurable Deep Point).** Let \(f \in \mathcal{F}_d\), where \(d \geq 4\) is an even integer, and assume that \(0\) is its critical point. If \(J_R(f)\) has positive Lebesgue measure then there exists \(\alpha > 0\) and \(C > 0\) so that

\[m(x \in (-\delta, \delta): x \notin J_R(f)) \leq C\delta^{1+\alpha} \text{.}\]

**Remark 3.5.** Indeed \(\alpha\) can be taken depending only on \(d\).

**Theorem 11.** For each even critical order \(d, d \geq 4\), the following statements are equivalent:
There exists $f \in F_d$ such that $m(J_R(F)) > 0$.

There exists $f \in F_d$ with a wild attractor.

There exist maps $f, g \in F_d^{uni}$ which are conjugated by a continuous absolutely continuous maps $h$, but $f$ has a periodic point $p$ whose eigenvalue is different from the eigenvalue of the periodic point $h(p)$ of $g$.

All maps in $F_d^d$ have wild attractors.

All maps in $F_d^{uni}$ can be conjugated with each other by an absolutely continuous conjugacy.

4. Preliminaries

4.1. Probabilistic tools. We are going to collect here a handful of probabilistic tools which are going to be useful along the article. A good reference for these results is [B].

Most of the probabilistic results in dynamical systems (large deviation, central limit theorem) assumes the observables is quite regular: usual regularity assumptions are either Holder continuity or bounded variation. Fix $((f, \psi) \in On + sBD + Ra + Rb + GD$. Then $f$ has a unique absolutely continuous invariant probability $\mu$. Moreover this invariant measure is ergodic (see [B, page 29]). We are interested in $P_0$-measurable observables with integer values which do not have such regularity. Fortunately this is almost true: Denote by $O(f)$ the class of $P_0$-measurable functions $\phi : I \to \mathbb{Z}$ so that $-\phi \in L^2(\mu)$.

If $P$ denotes the Perron-Frobenius-Ruelle operator of $f$, then $P\phi$ has bounded variation.

Then $\psi \in O(f)$. Up to simple modifications in the proofs in [B], we have

**Proposition 4.1** (Large Deviations Theorem [B]). Suppose $((f, \psi) \in On + sBD + Ra + Rb + GD$. For every $\psi \in O(f)$ and $\epsilon > 0$ there exists $\gamma \in (0,1)$ and $C \geq 0$ so that

$$\mu(\{x \in I : |\frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(x)) - \int \psi d\mu| \geq \epsilon\}) \leq C\gamma^n$$

**Proposition 4.2** (Proposition 6.1 of [B]). Suppose $((f, \psi) \in On + sBD + Ra + Rb + GD$. For every $\psi \in O(f)$ the limit

$$\sigma^2 := \lim_{n \to \infty} \int \left(\frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \psi(f^k(x)) \right)^2 d\mu$$

exists. Furthermore $\sigma^2 = 0$ if and only if there exists a function $\alpha \in L^2(\mu)$ so that $\psi = \alpha \circ f - \alpha$.

and

**Proposition 4.3** (Central Limit Theorem: Theorem 8.1 in [B]). Suppose $((f, \psi) \in On + Mk + sBD + Ra + Rb + GD$. For every $\psi \in O(f)$ so that $\sigma^2 \neq 0$ we have

$$\sup_{\epsilon \in \mathbb{R}} \left| \mu(x \in I: \frac{\sum_{k=0}^{n-1} \psi(f^k(x))}{\sigma \sqrt{n}} \leq \epsilon) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\epsilon} e^{-\frac{u^2}{2}} du \right| \leq \frac{C}{\sqrt{n}}.$$
Indeed we are going to see that the assumption $\sigma^2 \neq 0$ is very weak: to this end we need the following result:

**Proposition 4.4** (Theorem 3.1 in [AD]). Let $f: \cup_i I_i \to I$ be a map in $M_k + BD + Ex + Ra + Rb$. Let $\psi: \cup_i I_i \to \mathbb{S}^1$ be a $\mathcal{P}_0$-measurable function. If

$$\psi = \frac{\alpha \circ f}{\alpha},$$

where $\alpha$ is measurable, then $\alpha$ is $\mathcal{P}^*$-measurable, where $\mathcal{P}^*$ is the finest partition of $I$ so that $f(I_i)$ is included in an atom of $\mathcal{P}^*$ for each $i \in \Lambda$.

**Proposition 4.5.** Let $\psi: \cup_i I_i \to \mathbb{Z}$ be a $\mathcal{P}_0$-measurable function. If $\psi = \alpha \circ f - \alpha$, where $\alpha$ is measurable, then $\alpha$ is constant on $f(I_i)$, for each $i \in \Lambda$.

**Proof.** Note that we can assume that $\alpha(x) \in \mathbb{Z}$, for every $x$. Indeed, the relation $\psi = \alpha \circ f - \alpha$ implies that the function $\beta(x) = \alpha(x) \mod 1$ is $f$-invariant, so we can replace $\alpha$ by $\alpha - \beta$, if necessary. Fix an irrational number $\gamma$. Then

$$e^{2\pi \gamma \psi(x)i} = e^{2\pi \gamma \alpha(f(x))i},$$

so by Proposition 4.4 we have that $e^{2\pi \gamma \alpha(x)i}$ is a $\mathcal{P}^*$-measurable function. Since $j \in \mathbb{Z} \to e^{2\pi j i} \in \mathbb{S}^1$ is one-to-one, we get that $\alpha$ is $\mathcal{P}^*$-measurable. □

A Markov map $f$ is almost onto if and only if $\mathcal{P}^*_0 = \{I\}$, so

**Corollary 4.6.** On the conditions of Proposition 4.5, if $f$ is almost onto then $\alpha$ is constant.

**Corollary 4.7.** For every nonconstant $\psi \in \mathcal{O}(f)$ we have that $\sigma^2 \neq 0$. In particular the Central Limit Theorem as given in Eq. (10) holds for every non-constant $\psi$.

Let $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \ldots$ be an increasing sequence of $\sigma$-subalgebras of a probability space $(\Omega, \mathcal{A}, \mu)$. A martingale difference sequence is a sequence of functions $\psi_n: \Omega \to \mathbb{R}$, where $\psi_n$ is $\mathcal{A}_n$-measurable for $n \geq 1$, so that

$$E(\psi_n | \mathcal{A}_{n-1}) = 0$$

for every $n$. Here $E(\psi | \mathcal{B})$ denotes de conditional expectation of $\psi$ relative to the sub-algebra $\mathcal{B}$. When $\mathcal{B}$ is generated by atoms $\{J_i\}_i$ then $E(\psi | \mathcal{B})$ is the function defined as

$$E(\psi | \mathcal{B})(x) = \frac{1}{\mu(J_i)} \int_{J_i} \psi \, d\mu$$

for every $x \in J_i$.

The following Proposition is the classic Azuma-Hoeffding inequality: see, for instance Exercise E14.2 in [W]:

**Proposition 4.8** (Azuma-Hoeffding inequality). Let $\psi_n$ be a martingale difference sequence and furthermore assume that

$$\|\psi_i\|_{\infty} = c_i < \infty.$$

Define

$$\psi := \sum_{i=1}^n \psi_i.$$

Then
\[ \mu(x \in \Omega: |\psi - \mathbb{E}(\psi)| > t) \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n} \mu^2}\right). \]

4.2. How to construct asymptotically small perturbations. As we will see in the next Proposition, it is easy to construct asymptotically small perturbations of a random walk:

**Proposition 4.9.** Let \( F \) and \( G \) be random walks satisfying the properties LI, Ex, sBD, Ra and Rb, where \( G \) is a topological perturbation of \( F \). Assume that there exist \( C > 0 \) and \( \lambda \in (0, 1) \) with the following properties: if \( I_j^n \) is as in properties Ra and Rb, then

i. For every \( I_j^n \in \mathcal{P}_n^0 \) we have
\[
|\log \frac{|I_{j+1}^n|}{|I_j^n|} \frac{|H(I_{j+1}^n)|}{|H(I_j^n)|}| \leq C\lambda^n|+|j|. 
\]

ii. For every \( J \in \mathcal{P}_n^0 \) we have
\[
|\tau^G_J - \tau^G_H| \leq C\lambda^n. 
\]

iii. If \( I_i^n = [a_i^n, b_i^n] \) then
\[
\max_{i} \max \{|a_i^n - H(a_i^n)|, |b_i^n - H(b_i^n)|\} \leq C\lambda^n. 
\]

iv. Either \( \psi \) is a bounded function or \( \psi \) has a lower bound and \( F = G \) on \( \bigcup_{n<0} I_n \).

Then \( G \) is an asymptotically small perturbation of \( F \). Furthermore there exist \( \beta \in [0, 1) \) and \( C > 0 \) so that
\[ |H(p) - p| \leq C\beta^{|\pi_2(p)|}. \]

**Proof.** We will assume that \( \psi \) is bounded: the other case is analogous. Consider \( (x, n) \in I \times \mathbb{Z} \) and \( (y, n) = H(x, n) \). Denote \((x_i, n_i) := F^i(x, n), (y_i, n_i) := G^i(y, n)\).

Denote \( \delta_i = |y_i - x_i| \) and \( \hat{\delta}_i = |A_G(H(J_{i-1}))(y_i) - A_F(J_{i-1})(x_i)| \). Here \((x_i, n_i) \in J_i \in \mathcal{P}_n^0 \).

It is easy to conclude, using iii. and property LI, that
\[
\hat{\delta}_i \leq \frac{\delta_i}{|F(J_{i-1})|} + C\lambda^n. 
\]

and making use of ii. to get
\[
|\tau^G_{H(J_{i-1})}(A_G(H(J_{i-1}))(y_i)) - \tau^F_{J_{i-1}}(A_F(J_{i-1})(x_i))| \leq D\tau^F_{J_{i-1}}(z_i) \left| \frac{\delta_i}{|F(J_{i-1})|} + C\lambda^n \right|. 
\]

Here \( z_i \in [0, 1] \). Since \( D\tau^F_{J_{i-1}}(z_i)|F(J_{i-1})||J_{i-1}| \leq \lambda \) (property Ex), we get, using again iii.
\[
\delta_{i-1} \leq \lambda \delta_i + C\lambda^n. 
\]
Because $\psi$ is bounded, $|n_{i+1} - n_i| \leq B = \max |\psi|$. So if $i < n/2B$ then $|n_i| > |n_0|/2$. Since $\delta_{|n_i|} \leq 1$, Eq. (12) implies

\[ |H(x, n) - (x, n)| = |y_0 - x_0| \leq C\lambda^{|n_i|}. \]

In particular, by Eq. (11) and property ii., we have

\[ |D_{\tau^G_{H(J_0)}(G_U(H(J_0)))}(y_1)) - D_{\tau^F_{H(J_0)}(H(U)(x_1))}| \leq C\lambda^{|n_i|}. \]

By $Ra + Rb$ there exists $\theta \in (0, 1)$ so that

\[ \theta^{|i|} \leq |I^n_i|. \]

Let $i$ be so that $J = I^n_i$.

Case A. $|i| > |n|/2(\log \lambda/ \log \theta)$: Due to i. and iii. and property $Ra$, there exists $C > 0$ so that

\[ |\log \frac{|H(I^n_i)|}{|I^n_i|}| \leq C\lambda^n. \]

Together with $sBD + LI$ and $iii.$, this implies that for every $p \in I^n_i$, with $|i| > |n|/2(\log \lambda/ \log \theta)$, we have

\[ |\log \frac{DG(H(p))}{DF(p)}| \leq C\lambda^{\left|\frac{|i|}{\log \lambda/ \log \theta}\right|}. \]

Case B. $|i| < |n|/2(\log \lambda/ \log \theta)$: In this case, by iii. and Eq. (15) we have

\[ \log \frac{|H(I^n_i)|}{|I^n_i|} \leq C\left|\frac{|H(b^n_i) - b^n_i| + |H(a^n_i) - a^n_i|}{b^n_i - a^n_i}\right| \leq C\lambda^{\left|\frac{|i|}{\log \lambda/ \log \theta}\right|}. \]

Now using Eq. (13) and Eq. (14) we can easily obtain

\[ |\log \frac{DG(H(p))}{DF(p)}| \leq C\lambda^{\left|\frac{|i|}{\log \lambda/ \log \theta}\right|}. \]

\[ \square \]

5. Stability of transience

We will begin this section with the large deviations result to transient homogeneous random walks and strongly transient random walks:

**Proof of Proposition 2.1.** Let $P \in P^0(F)$ be such that $F(P) = I_\ell$. By Proposition 4.1 we have that for every $\epsilon > 0$ there exist $\gamma < 1$ such that

\[ m(p \in I_\ell): \pi_2(F^{n-1}p) - \pi_2(p) < (K - \frac{\epsilon}{2}) (n - 1) \leq C\gamma^n - 1, \]

for every $\ell$. By the property BD we have

\[ m(p \in P): \pi_2(F^{n}p) - \pi_2(p) < (K - \frac{\epsilon}{2}) (n - 1) \leq C\gamma^n - 1 |P|. \]

Denote by $\Lambda^{n}_p$ the set in the l.h.s. of Eq. (16). Let $n_0$ be such that $\min \psi > -\epsilon(n_0 - 1)/2 - \epsilon + K$. Then for $n \geq n_0$ we have

\[ \tilde{\Lambda}^{n}_p := \{ p \in P: \pi_2(F^{n}p) - \pi_2(p) < (K - \epsilon) n \} \subset \Lambda^{n}_p. \]

Indeed

\[ \pi_2(F^{n}p) - \pi_2(p) < (K - \epsilon)n \]

implies

\[ \pi_2(F^{n}p) - \pi_2(F(p)) < (K - \epsilon/2) (n - 1). \]
So
\[ m(p \in P : \pi_2(F^n p) - \pi_2(p) < (K - \epsilon) n) \leq C_2 \gamma^n |P| \]
for every \( n \). This completes the proof. \( \square \)

**Proof of Proposition 2.2.** Fix \( \epsilon > 0 \) small. We intend to apply the Azuma-Hoeffding inequality, but since \( \psi \) is not necessarily bounded, we need to make some adjustments first: Fix \( P \in \mathcal{P}(F) \) and define \( F_0 := \{ P \} \) and \( F_n := \{ Q \in P : Q \in \mathcal{P}(F) \} \). Since \( F \in GD \), by the usual distortion control tricks for \( F \), we can find \( M > \min \psi \) such that \( \alpha(x) := \min \{ \psi(x), M \} \) satisfies
\[
(18) \quad \mathbb{E}(\alpha \circ F^n | F_{n-1}) \geq K - \epsilon/4
\]
for every \( n \geq 1 \). Here we are considering conditional expectations relative to the probability
\[
\mu_P(A) := \frac{m(A)}{|P|},
\]
where \( m \) is the Lebesgue measure.

Define the martingale difference sequence
\[
\Psi_n := \alpha \circ F^n - \mathbb{E}(\alpha \circ F^n | F_{n-1}).
\]
Of course \( ||\Psi_n||_\infty \leq M \), if \( M \) is large enough. By the Azuma-Hoeffding inequality we have
\[
m(p \in P : | \sum_{i=1}^{n} \Psi_i(p) | > t) \leq 2 \exp(-\frac{t^2}{2nM^2}) |P|.
\]
Taking \( t = \epsilon n/4 \) we obtain
\[
(19) \quad m(p \in P : | \sum_{i=1}^{n} \Psi_i(p) | > \frac{\epsilon}{4} n) \leq 2 \exp(-\frac{\epsilon^2 n}{32M^2}) |P|.
\]

Since
\[
\pi_2(F^{n+1} p) - \pi_2(F(p)) = \sum_{i=1}^{n} \psi(F^i(p)) \geq \sum_{i=1}^{n} \alpha(F^i(p)) = \sum_{i=1}^{n} \Psi_i(p) + \sum_{i=1}^{n} \mathbb{E}(\alpha \circ F^i | F_{i-1})(x)
\]
\[
\geq \sum_{i=1}^{n} \Psi_i(p) + (K - \epsilon/4)n.
\]
Due Eq. (19), this implies that
\[
m(p \in P : \pi_2(F^n p) - \pi_2(F(p)) = \sum_{i=1}^{n-1} \psi(F^i(p)) < (K - \epsilon/2) (n-1)) \leq C_1 \exp(-\frac{\epsilon^2 n}{32M^2}) |P|.
\]
Let \( n_0 \) be such that \( \min \psi > -\epsilon(n_0 - 1)/2 - \epsilon + K \). Then for \( n \geq n_0 \) we have that
\[
\pi_2(F^n p) - \pi_2(p) < (K - \epsilon)n
\]
implies
\[
\pi_2(F^n p) - \pi_2(F(p)) < (K - \epsilon/2) (n - 1).
\]
So
\[
m(p \in P : \pi_2(F^n p) - \pi_2(p) < (K - \epsilon) n) \leq C_2 \exp(-\frac{\epsilon^2 n}{32M^2}) |P|
\]
for every \( n \). This completes the proof. \( \square \)
Proposition 5.1. Let $F$ be either a homogeneous random walk with positive mean drift or a strongly transient random walk. Then any asymptotically small perturbation $G$ of $F$ has the following property: there exists $\lambda \in [0, 1)$, $C > 0$ and $K > 0$ so that for every $P \in \mathcal{P}^0(G)$

$$m(p \in P: \sum_{i=0}^{n-1} \psi(G^i(p)) < Kn) \leq C\lambda^n |P|.$$ 

In particular $G$ is also transient.

Proof. We will carry out the proof assuming the strong transience: the homogeneous case with positive mean drift is analogous: Fix $\epsilon > 0$. Let $\delta_1 > 0$ be small enough such that

$$(1 - \delta_1)(K - \epsilon) + \delta_1 \min \psi > K - 2\epsilon.$$ 

Due to the bounded distortion of $G$, there exists $\delta_1 > 0$ such that for every $n \geq 1$ and every $P \in \mathcal{P}^{n-1}(G)$, interval $Q \subset G^n(P)$, and set $A \subset Q$ satisfying

$$\frac{m(A)}{m(Q)} \geq 1 - \delta_1$$

we have

$$\frac{m(P \cap G^{-n}A)}{m(P \cap G^{-n}Q)} \geq 1 - \delta_1.$$ 

By Proposition 2.2 we have

$$(21) \quad m(p \in P: \sum_{i=0}^{n-1} \psi(F^i(p)) < (K - \epsilon)n \text{ for some } n \geq n_0) \leq C_1 \exp(-C_2n_0)|P|,$$

for every $P \in \mathcal{P}^0_j(F)$. Since $G$ is an asymptotically small perturbation, Eq. (7) implies that

$$(22) \quad m(p \in H(P): \sum_{i=0}^{n-1} \psi(G^i(p)) < (K - \epsilon)n \text{ for some } n \geq n_0) \leq C_3 \exp(-C_4n_0)|H(P)|$$

provided that $P \in \mathcal{P}^0_j(F)$, $j \geq 2 |\min \psi| n_0$. Indeed, the set in the l.h.s. of Eq. (21) can the written as the pairwise disjoint union of the sets $\Delta_j$, $j \geq n_0$, where $\Delta_j$ is defined as

$$\{p \in P: \sum_{i=0}^{n-1} \psi(F^i(p)) \geq (K - \epsilon)n \text{ for every } n_0 \leq n < k \text{ and } \sum_{i=0}^{k-1} \psi(F^i(p)) < (K - \epsilon)k\}$$

So by Eq. (7) and Eq. (8) we have that

$$\text{dist}_k(p) \leq C n_0 \lambda |\min \psi| n_0 + \sum_{i=n_0+1}^{\infty} C \lambda^{(K - \epsilon)i} \leq \tilde{C} < \infty$$

for every $p \in \Delta_k$, $k \geq n_0$, and $j \geq 2 |\min \psi| n_0$. In particular

$$(23) \quad m(H(\Delta_k)) \leq \tilde{C} m(\Delta_k).$$

Note that the set in the l.h.s. of Eq. (22) is the pairwise disjoint union of $H(D_j)$. Since $P \in \mathcal{P}^0_j$ we have $m(P) \leq C m(H(P))$, so from Eq. (23) we obtain Eq. (22).
In particular there exists \( n_0 = n_0(\delta_1) \) such that for every \( P \in \mathcal{P}^0_j(G) \), \( j \geq 2 \), \( \min \psi \), \( n_0 \), we have

\[
m(\hat{\Omega}_P) \geq (1 - \delta_1)|P|,
\]

where \( \hat{\Omega}_P \) is the set of points \( p \in P \) such that \( \pi(G^n(p)) \geq \min \psi \), \( n_0 \) for all \( n \geq 0 \) and \( \pi(G^n(p)) - \pi(p) \geq (K - \epsilon)n \) for all \( n \geq n_0 \).

By the GD condition, there exists \( n_1 \) such that for \( n \geq n_1 \) we have

\[
m(p \in P: \text{there exists } i \leq n \text{ s.t. } \psi(F^i(p)) \geq n) \leq \frac{\delta_1}{4}
\]

By Eq. (21) there exists \( n_2 > n_1 \) such that

\[
m(p \in P: \sum_{i=0}^{n_2-1} \psi(F^i(p)) > (K - \epsilon)n_2) \geq (1 - \frac{\delta_1}{4})|P|.
\]

So

\[
m(p \in P: \sum_{i=0}^{n_2-1} \psi(F^i(p)) > (K - \epsilon)n_2 \text{ and } \psi(F^i(p)) < n_2 \text{ for every } i \leq n_2)
\]

\[
\geq (1 - \frac{\delta_1}{2})|P|.
\]

Note that for \( p \) in the set in Eq. (26) we have \( \pi(G^i(p)) - \pi_2(p) \leq (n_2)^2 \) for every \( i \leq n_2 \). Since \( G \) is an asymptotically small perturbation of \( F \), this observation and Eq. (26) implies that there exists \( n_3 >> (n_2)^2 \) such that for \( P \in \mathcal{P}^0_j(G) \), with \( j \leq -n_3 \), we have

\[
m(p \in P: \sum_{i=0}^{n_2-1} \psi(G^i(p)) > (K - \epsilon)n_2 \text{ and } \psi(G^i(p)) < n_2 \text{ for every } i \leq n_2)
\]

\[
\geq (1 - \delta_1)|P|.
\]

So for \( P \in \mathcal{P}^0_j(G) \), with \( j \leq -n_3 \), we have

\[
m(p \in P: \sum_{i=0}^{n_2-1} \psi(G^i(p)) > (K - \epsilon)n_2) \geq (1 - \delta_1)|P|.
\]

Claim A: Almost every point \( x \in I \times \{j\}, j \leq -n_3 \), visits at least once (and consequently infinitely many times) the set

\[
\bigcup_{j \geq -n_3} I \times \{j\}
\]

Indeed, define a new random walk \( \tilde{G}: I \times \mathbb{Z} \rightarrow I \times \mathbb{Z} \)

\[
\tilde{G}(x, n) := (\tilde{g}_n(x), n + \tilde{\psi}(x, n))
\]

in the following way. Let \( T \) be an integer larger than \( n_2(K - \epsilon) \). If \( n \geq -n_3 \) then define \( \tilde{g}_n: I \rightarrow I \) as an affine expanding map, onto on each element of \( \mathcal{P}^n_i \), and \( \tilde{\psi}(x, n) = T \).
For \((x, n)\), with \(n < -n_3\), define \(\tilde{G}(x, n) = G^{n_2}(x, n)\). In this case
\[
\tilde{\psi}(x, n) = \sum_{i=0}^{n_2-1} \psi(G^i(x, n)).
\]

It is not difficult to see that the \(\tilde{G}\)-orbit of a point \((x, n)\), with \(n < -n_3\), visits the set in Eq. (29) at least once then the \(G\)-orbit of \((x, n)\) visits the same set at least once.

To prove the claim, it is enough to show that \(\tilde{G}\) is strongly transient. Indeed, let \(P\) be an element of the Markov partition \(\mathcal{P}_{j}^{k-1}(G)\). If \(\pi_2(G^i(P)) \geq -n_3\), for some \(i \leq k\) then \(\pi_2(\tilde{G}^k(P)) \geq -n_3\), so by the distortion control in Eq. (30) we have
\[
\frac{1}{|P|} \int_P \tilde{\psi} \circ \tilde{G}^k dm = \frac{1}{|P|} \int_P T dm \geq (K - \epsilon)n_2.
\]

Otherwise \(\pi_2(\tilde{G}^i(P)) < -n_3\) for every \(i \leq k\). In particular \(\tilde{G}^i = G^{in_2}\) on \(P\), for every \(i \leq k\). Note that
\[
\tilde{G}^k P = \bigcup_i Q_i,
\]
where \(\{Q_i\}_i\) is the family of all interval \(Q\) such that \(Q \in \mathcal{P}_j^0(G)\) for some \(j > -n_3\) and \(Q \cap \tilde{G}^k P \neq \emptyset\) (this is a consequence of the Markovian property of \(G\)). By Eq. (28) we have
\[
m(q \in Q_i; \tilde{\psi}(q) \geq (K - \epsilon)n_2) \geq (1 - \delta_1)|Q_i|,
\]
so by the distortion control in Eq. (20) we obtain
\[
m(p \in P \cap \tilde{G}^{-k}Q_i; \tilde{\psi}(\tilde{G}^k p) \geq (K - \epsilon)n_2) \geq (1 - \tilde{\delta}_1)|P \cap \tilde{G}^{-k}Q_i|,
\]
consequently
\[
\int_P \tilde{\psi} \circ \tilde{G}^k dm = \sum_i \int_{P \cap \tilde{G}^{-k}Q_i} \tilde{\psi} \circ \tilde{G}^k dm \\
\geq \sum_i ((1 - \tilde{\delta}_1)(K - \epsilon)n_2 + \tilde{\delta}_1 n_2 \min \tilde{\psi})|P \cap \tilde{G}^{-k}Q_i| \\
\geq \sum_i (K - 2\epsilon)n_2|P \cap \tilde{G}^{-k}Q_i| = (K - 2\epsilon)n_2|P|
\]

Eq. (30) and (31) imply that \(\tilde{G}\) is strongly transient, so by Proposition 2.2, \(\tilde{G}\) is transient. This concludes the proof of the claim.

Claim B: The \(G\)-orbit of almost every point of \(I \times \mathbb{Z}\) eventually arrives at \(\hat{\Omega}_P\), for some \(P \in \mathcal{P}_j^0\), with \(j > 2\min \psi|n_0\).

Since \(F\) is transient and \(G\) is topologically conjugate to \(F\) the set
\[
\Omega := \{p: -n_3 \leq \pi_2(p) \leq 2\min \psi|n_0 and \lim_n \pi_2(G^n(p)) = +\infty\}
\]
is dense on
\[
\bigcup_{j=-n_3}^{2\min \psi|n_0} I \times \{j\}.
\]
This implies that for every non-empty open set $O \subset I_j$, with $-n_3 \leq j \leq 2\min \psi|n_0$ we have

$$m((x,j) \in O : \exists k \geq 0 \ s.t. \ G^k(x,j) \in \tilde{\Omega}_P, \ with \ P \in \mathcal{P}_{\tilde{\Omega}}(G), \ q > 2\min \psi|n_0) > 0,$$

where $\tilde{\Omega}_P$ is as in Eq. (24). Indeed, pick a point $p \in O \cap \Omega$. By property $E_x$ and the definition of $\Omega$, there exists $k$ and $Q \in \mathcal{P}_k(G)$ such that $Q \subset O$, $P = G^k(Q) \in \mathcal{P}_{\tilde{\Omega}}$, with $q > 2\min \psi|n_0$. By Eq. (24) we have $m(\Omega_P) > 0$, so

$$m(O \cap G^{-k}\Omega_P) \geq m(Q \cap G^{-k}\Omega_P) > 0.$$

In particular there exists $\tilde{\delta} > 0$ such that for every interval $J \subset I_j$, with $-n_3 \leq j \leq 2\min \psi|n_0$ and $|J| \geq \tilde{\delta}$, where $\tilde{\delta}$ is as in the LI property, we have

$$m((x,j) \in J : \exists k \geq 0 \ s.t. \ G^k(x,j) \in \tilde{\Omega}_P, \ with \ P \in \mathcal{P}_{\tilde{\Omega}}(G), \ q > 2\min \psi|n_0) > \tilde{\delta}|J|,$$

It follows that there exists $\delta_3 > 0$ such that for every $i$ and every $Q \in \mathcal{P}^{i-1}(G)$ such that $\pi_2(G^iQ) \geq -n_3$ we have that

$$m(p \in Q : \exists k \geq 0 \ s.t. \ G^kP \in \tilde{\Omega}_P, \ with \ P \in \mathcal{P}_{\tilde{\Omega}}(G), \ q > 2\min \psi|n_0) \geq \delta_3|Q|.$$

Indeed, if $\pi_2(G^iQ) \leq 2\min \psi|n_0$ we can apply Eq. (33), BD and LI property. Otherwise apply Eq. (24) and BD property.

We will show Claim $B$ by contradiction. Suppose that it does not hold. Then there is a set $W$ of positive measure whose $G$-orbit of its elements never hits $\tilde{\Omega}_P$ for any $P \in \mathcal{P}_j$, with $j > 2\min \psi|n_0$. Pick a Lebesgue density point $p$ of $W$ whose $G$-orbit visits

$$\bigcup_{j \geq -n_3} I \times \{j\}$$

infinitely many times, which is possible due Claim A. In particular there exists a sequence $Q_k \in \mathcal{P}_{n_k-1}(G)$ such that $|Q_k| \to_n 0$, $p \in Q_k$, $\pi_2(G^{n_k}Q_k) \geq -n_3$ and

$$\lim_k \frac{m(Q_k \cap W)}{|Q_k|} = 1.$$

That contradicts Eq. (34). This concludes the proof of Claim $B$.

Note that Claim $B$ implies the following: almost every point in $I \times \{j\}$ belongs to the set

$$\Lambda_j := \bigcup_{k \geq 0} \Lambda_j^k,$$

where

$$\Lambda_j^k := \{p \in I \times \{j\} : \pi_2(G^n(p)) - \pi_2(G^k(p)) \geq (K-\epsilon)(n-k), \ for \ every \ n \geq k+n_0\}.$$

Let $k_0$ be large enough such that for every $-n_3 \leq j \leq 2\min \psi|n_0$ we have

$$m(A \cap \bigcup_{k \leq k_0} \Lambda_j^k) \geq (1 - \delta_1)|A|$$

for every interval $A \subset I \times \{j\}$ satisfying $|A| \geq \delta$, where $\delta > 0$ is as in the property $LI$. Pick $n_4$ satisfying $n_4 \geq k_0 + n_0$ and

$$n_4 > \frac{-k_0 \min \psi}{\epsilon} - k_0.$$
It is easy to see that if \( p \in \bigcup_{k \leq k_0} \Lambda_k \) then

\[
\pi_2(G^n p) - \pi_2(p) = \sum_{i=0}^{n_4-1} \psi(G^i p) \geq (K - 2\epsilon)n_4.
\]

In a argument similar to the proof of Claim A, consider the random walk \( \hat{G} \) defined in the following way: if \( \pi_2(p) \leq -n_3 \) define \( \hat{G}(p) = G^{n_2} \). If \( \pi_2(p) \geq 2|\min \psi|n_0 \) define \( \hat{G}(p) = G^{n_0} \). Finally if \( -n_3 < \pi_2(p) < 2|\min \psi|n_0 \) define \( \hat{G}(p) = G^{n_4} \). The random walk \( \hat{G} \) is \( 3\hat{K} \)-strongly transient, for some \( \hat{K} > 0 \). The proof is quite similar to the proof of the strong transience of \( G \), so we let it to the reader. So \( \hat{G} \) is transient. It is easy to see that this implies that \( G \) is transient. Finally Proposition 2.2 implies that

\[
m(p \in P : \pi_2(\hat{G}^m(p)) - \pi_2(p) < 2\hat{K}m) \leq C\hat{\lambda}^n |P|,
\]

for some \( \hat{\lambda} \in (0, 1) \), which implies

\[
m(Y^n_p) \leq C\hat{\lambda}^n |P|,
\]

where

\[
Y^n_p := \left\{ p \in P : \exists m \geq n \text{ s.t. } \pi_2(\hat{G}^m(p)) - \pi_2(p) < 2\hat{K}m \right\}.
\]

Let \( n_5 = \max\{n_0, n_4, n_2\} \). Let \( p \in P \) be such that

\[
\pi_2(G^i(p)) - \pi_2(p) < \frac{\hat{K}}{n_5} m.
\]

There exists \( m \) and \( j \) such that \( \hat{G}^m(p) = G^j(p) \), with \( i \geq j, |i - j| \leq n_5 \). Note that

\[
m \leq i \leq j + n_5 \leq (m + 1)n_5,
\]

so we can find \( i_0 \) such that for every \( i \geq i_0 \) we have

\[
\frac{-n_5 \min \psi}{m} + \frac{\hat{K}m + 1}{m} < 2\hat{K}.
\]

So

\[
\pi_2(\hat{G}^m(p)) - \pi_2(p) = \pi_2(G^i(p)) - \pi_2(G^j(p)) + \pi_2(G^j(p)) - \pi_2(p)
\]

\[
\leq -n_5 \min \psi + \frac{\hat{K}}{n_5} i \leq -n_5 \min \psi + \hat{K}(m + 1) < 2\hat{K}m,
\]

where

\[
m \geq \frac{i}{n_5} - 1.
\]

This implies

\[
\left\{ p \in P : \pi_2(G^i(p)) - \pi_2(p) < \frac{\hat{K}}{n_5} i \right\} \subset Y^{n_5 - 1}_p,
\]

so

\[
m(p \in P : \pi_2(G^i(p)) - \pi_2(p) < \frac{\hat{K}}{n_5} i) \leq C\hat{\lambda}^{i/n_5} |P|
\]

This completes the proof. \( \square \)

Let \( n > 0 \) and \( j \) be integers and \( F \) be a deterministic random walk. Then any connected component \( C \) of \( F^{-n} \text{ int } I_j \) is called a cylinder. It follows from the Markovian property of \( F \) that a cylinder is a disjoint union of intervals in \( \mathcal{P}^{n-1} \). The length \( \ell(C) \) of the cylinder \( C \) is \( n \). If \( C \) is a cylinder of length \( n \) so that \( F^n(C) \subset I_{j_n} \), for \( i < n \), we will denote \( C = C(j_0, j_1, \ldots, j_n) \).
Proposition 5.2. Let $F = (\{f_i\}, \psi) \in M_k + \text{LBD} + \text{LI} + \text{Ex} + \text{BD}$. Assume that there exists $\epsilon > 0$ so that for $K > 0$, we have

$$m(\{p \in I_n : \psi(p) < -K\}) \leq \frac{1}{K^2 + \epsilon},$$

provided $n \geq n_0$. Then

$$\lim_k m(\{p \in I_{n_k} : \text{there exists } i \leq k^2 \text{ so that } \psi(F_i(p)) < -k\}) = 0,$$

uniformly for all sequences satisfying $n_k > k^3 + n_0$.

Proof. For each $k$ and $i \leq k^2$, denote

$$\Lambda_{n_k}^i = \{p \in I_{n_k} : \psi(F_j(p)) \geq -k \text{ for every } j < i \text{ and } \psi(F_i(p)) < -k\}.$$

The set in the l.h.s. of Eq. (36) is the union of the sets $\Lambda_{n_k}^i$. The interval $I_{n_k}$ is the union of the cylinders in $P_{n_k}^{-1}$. Let $Q \in P_{n_k}^{-1}$ and suppose that $Q \cap \Lambda_{n_k}^i \neq \emptyset$. Then $\pi_2(F(Q)) \geq n_0$. By the property LI and Eq. (35) we get

$$m(p \in F_i(Q) : \psi(p) < -k) \leq C \frac{1}{k^2 + \epsilon}.$$

By the BD property

$$m(Q \cap \Lambda_{n_k}^i) \leq \frac{C}{k^2 + \epsilon} m(Q).$$

As a consequence

$$m(I_{n_k} \cap \Lambda_{n_k}^i) \leq \frac{C}{k^2 + \epsilon}.$$  

So

$$m(I_{n_k} \cap \bigcup_{i \leq k^2} \Lambda_{n_k}^i) \leq \frac{C}{k^2 + \epsilon}.  \square$$

Remark 5.3. For a homogeneous random walk, the condition on $\psi$ is equivalent to $1_{I_0} \cdot \psi \in L^{2+\epsilon}(m)$.

Let $F$ and $G$ be random walks which are topologically conjugated by a homeomorphism $h$ that preserves states. For any $p \in I \times \mathbb{Z}$ define

$$C_p := \sup_{i \geq 0} \text{dist}_i(p).$$

For each $n_0 \in \mathbb{Z} \cup \{-\infty\}$ define

$$\Omega_{n_0+}(F) := \{p : \pi_2(F^n(p)) \geq n_0, \text{ for all } n \geq n_0\}.$$  

In particular $\Omega_{-\infty+}(F) = I \times \mathbb{Z}$.

Proposition 5.4. Let $F$ and $G$ be random walks which are conjugated by a homeomorphism $h$ which preserves states. Suppose that there exists a $F$-forward invariant set $\Lambda$ so that

$$-H1: C_p := \sup_{i \geq 0} \text{dist}_i(p) < \infty, \text{ for each } p \in \Lambda.$$  

Then $h$ is absolutely continuous on $\cup_i F^{-i} \Lambda$ and $h^{-1}$ is absolutely continuous on $\cup_i G^{-i}h(\Lambda)$. Furthermore if also
-H2: There exists $C > 0$, $M > 0$ and $n_0 \in \mathbb{Z} \cup \{-\infty\}$ so that for every $n \geq n_0$
with $n \in \mathbb{Z}$ and $P \in \mathcal{P}_n^0$,

$$m(p \in P \cap \Lambda: \ C_p \leq C) \geq M|P|.$$ 

Then $h$ is absolutely continuous on $\cup_i F^{-i}(\Omega_{n_0+}(F))$ and $h^{-1}$ is absolutely continuous on $\cup_i G^{-i}(\Omega_{n_0+}(G))$. In particular when $n_0 = -\infty$ we have that $h$ and $h^{-1}$ are absolutely continuous on $I \times \mathbb{Z}$.

Proof. For each $j \in \mathbb{N}$ denote

$$\Lambda_j := \{p \in \Lambda: \sup_i \text{dist}_i(p) \leq j\}.$$ 

Note that $\Lambda_i$ is forward invariant.

We claim that $h$ is absolutely continuous on $\Lambda_j$ and $h^{-1}$ is absolutely continuous on $h(\Lambda_j)$. Indeed, for each $p \in \Lambda_j$ and $k \in \mathbb{N}$, denote $F^k p = (x_k, n_k)$. Denote by $J_k(x) \in \mathcal{P}^k$ the unique interval which contains $x$ so that $F^k$ maps $J_k(x)$ diffeomorphically onto $Q_k \subset I_{n_k}$. There is some ambiguity here if $x$ is in the boundary of $J_k(x)$, but these points are countable, so they are irrelevant for us.

If we use the analogous notation to $h(x)$ and $G$, we have $h(J_k(x)) = J_k(h(x))$ and, due the BD+LI property of the random walks $F$ and $G$, there exist $C_1, C_2 > 0$ such that

$$C_1 e^{-\text{dist}_k(p)} \leq \frac{|h(J_k(x))|}{|J_k(x)|} \leq C_2 e^{\text{dist}_k(p)}.$$ 

So, if $p \in \Lambda_j$ then

$$C_1 e^{-j} \leq \frac{|h(J_k(x))|}{|J_k(x)|} \leq C_2 e^j, \quad \text{for all } k \in \mathbb{N}. \quad (37)$$

Let $A \subset \Lambda_j$ be a set with positive Lebesgue measure. We claim that $h(A)$ also has positive Lebesgue measure. Indeed, choose a compact set $K \subset A$ with positive Lebesgue measure. Denote $U_k := \cup_{x \in K} J_k(x)$. Since $|J_k(x)| \leq \lambda^k$, we have that $\lim_{k \to \infty} m(U_k) = m(K)$ and $\lim_{k \to \infty} m(h(U_k)) = m(h(K))$. Since $U_k$ is a countable disjoint union of intervals of the type $J_k(x)$, by Eq. (38)

$$C_1 e^{-j} \leq \frac{m(h(U_k))}{m(U_k)} \leq C_2 e^j, \quad \text{so } C_1 e^{-j} \leq \frac{m(h(K))}{m(K)} \leq C_2 e^j, \quad (39)$$

and we conclude that $h(K)$ also has positive Lebesgue measure. An identical argument shows that, if $A \subset \Lambda_j$ has positive Lebesgue measure, then $h^{-1}A$ also has positive Lebesgue measure. The proof of the claim is finished and so $h$ and $h^{-1}$ are absolutely continuous on $\Lambda = \cup_j \Lambda_j$ and $h(\Lambda) = \cup_j h(\Lambda_j)$.

Now it is easy to conclude that $h$ and $h^{-1}$ are absolutely continuous on $\cup_i F^{-i} \Lambda$ and $\cup_i G^{-i} h(\Lambda)$.

Now assume H2. We claim that $\cup_i F^{-i} \Lambda$ has full Lebesgue measure on $\Omega_{n_0+}(F)$. Indeed, Assume that $m(\Omega_{n_0+}(F) \setminus \cup_i F^{-i} \Lambda) > 0$ and choose a Lebesgue density point $p$ of this set. Then

$$\lim_{k \to \infty} \frac{m(J_k(p) \cap \Omega_{n_0+}(F) \setminus \cup_i F^{-i} \Lambda)}{|J_k(x)|} = 1.$$ 

Due the bounded distortion of $F$, if $F^k(p) = (x_k, n_k)$ and $F^k(J_k(x)) = Q_k \subset I_{n_k}$, with $n_k \geq n_0$, where $Q_k$ is a union of intervals in $\mathcal{P}_{n_k}^0$, then
\[ \limsup_k \frac{m(Q_k \cap \Lambda)}{|Q_k|} \leq C(1 - \liminf_k \frac{m(J_k(x) \cap \Omega_n + (F) \setminus \cup_i F^{-i \Lambda})}{|J_k(x)|}) = 0, \]

which contradicts H2.

Since dist\(_k\)(p) is uniformly bounded with respect to \( k \) and \( p \) on the set \( \{ p \in P \cap \Lambda : C_p \leq C \} \), we can use an argument identical to the proof of Eq. (39) to conclude that

\[ \frac{m(p \in P \cap \Lambda : C_p \leq C)}{m(h(p) \in h(P) \cap h(\Lambda) : C_p \leq C)} \leq C_1, \]

so \( m(h(P \cap \Lambda : C_p \leq C)) \geq \tilde{C}M|h(P)| \), for all \( P \in \mathcal{P}_n^0 \), \( n \geq n_0 \) and using an argument as above, we conclude that \( \cup_i G^{-i}h(\Lambda) \) has full Lebesgue measure on \( \Omega_{n_0} + (G) \). Since \( h(h^{-1}) \) is absolutely continuous on \( \cup_i F^{-i \Lambda} (\cup_i G^{-i}h(\Lambda)) \) and \( m(\Omega_{n_0} + (F) \setminus \cup_i F^{-i \Lambda}) = m(h(\Omega_{n_0} + (F) \setminus \cup_i F^{-i \Lambda})) = m(\Omega_{n_0} + (G) \setminus \cup_i G^{-i}h(\Lambda)) = 0 \), we have that \( h \) and \( h^{-1} \) are absolutely continuous on \( \Omega_{n_0} + (F) \) and \( \Omega_{n_0} + (G) \). Now it is easy to prove that \( h \) is absolutely continuous on \( \cup_i F^{-i} \Omega_{n_0} + (F) \) and \( h^{-1} \) is absolutely continuous on \( \cup_i G^{-i} \Omega_{n_0} + (G) \).

**Proof of Theorem 1.** By Proposition 5.1, \( G \) is transient. In particular for all \( n_0 \in \mathbb{Z} \) the sets

\[ \cup_i F^{-i} \Omega_{n_0} + (F) \text{ and } \cup_i G^{-i} \Omega_{n_0} + (G) \]

have full Lebesgue measure. So by Proposition 5.4, to prove that \( h \) and \( h^{-1} \) are absolutely continuous, it is enough to find a forward invariant set satisfying the assumptions H1 and H2 for some \( n_0 \in \mathbb{Z} \). Indeed, fix \( \delta > 0 \) (we will choose \( \delta \) later).

Consider the \( F \)-forward invariant set

\[ \Lambda = \Lambda_\delta := \{ p : \liminf_k \frac{\pi_2(F^k(p)) - \pi_2(p)}{k} \geq \frac{\delta}{3} \}. \]

We claim that \( \Lambda \) satisfies H1. Indeed take \( p \in \Lambda \). Then, for \( k \geq k_0(p) \) we have

\[ n_k := \pi_2(F^k(p)) \geq k\delta/4. \]

So

\[ \text{dist}_k(p) \leq \sum_{i=0}^{k-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| \]

\[ \leq \sum_{i=0}^{k_0-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| + \sum_{i=k_0}^{k-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| \]

\[ \leq \sum_{i=0}^{k_0-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| + \sum_{i=k_0}^{k-1} \lambda^{n_i} \]

\[ \leq \sum_{i=0}^{k_0-1} |\log \frac{DF(F^{i+1}(p))}{DG(h(F^{i+1}(p)))}| + \sum_{i=k_0}^{\infty} \lambda^{\delta/4} \]

\[ \leq K_p + C(\delta). \]

Here \( \lambda \) is as in Eq. (7) and Eq. (8). To prove that \( \Lambda \) satisfies H2, By Proposition 2.2 for each \( P \in \mathcal{P}_n^0 \) we have

\[ m(p \in P : \pi_2(F^k(p)) - \pi_2(p) < \delta k) \leq C\lambda^k |P|, \]

Here \( \lambda \) is as in Eq. (7) and Eq. (8). To prove that \( \Lambda \) satisfies H2, By Proposition 2.2 for each \( P \in \mathcal{P}_n^0 \) we have

\[ m(p \in P : \pi_2(F^k(p)) - \pi_2(p) < \delta k) \leq C\lambda^k |P|, \]

Here \( \lambda \) is as in Eq. (7) and Eq. (8). To prove that \( \Lambda \) satisfies H2, By Proposition 2.2 for each \( P \in \mathcal{P}_n^0 \) we have

\[ m(p \in P : \pi_2(F^k(p)) - \pi_2(p) < \delta k) \leq C\lambda^k |P|, \]
provided $\delta$ is small enough. From Eq. (41) we obtain
\begin{equation}
\mu(p \in P: \pi_2(F^n(p)) - \pi_2(p) \geq \delta n \text{ for all } n \geq n_0 \geq (1 - C\lambda^m)|P|.
\end{equation}
In particular, we have that, for every $n$,
\begin{equation}
\pi_2(F^n(p)) \geq \delta(n - n_0) + \pi_2(p) + n_0 \min \psi.
\end{equation}
in the set in Eq. (42). Using the same argument as in Eq. (40) we can easily obtain $H2$ from Eq. (43) and Eq. (42), choosing $n_0$ large enough. \hfill \square

**Proof of Theorem 2.** Since the mean drift is positive, by the Birkhoff Ergodic Theorem $F$ is transient. By Proposition 5.1, $G$ is also transient. Now the proof goes exactly as the Theorem 1, except that to obtain Eq. (41) we use Proposition 2.1 instead of Proposition 2.2. \hfill \square

**Proof of Theorem 3.** Let $F$ be either a $K$-strongly recurrent random walk or a homogeneous random walk with mean drift $K = \int \psi \, dm$. Let $\epsilon < K$. By Proposition 5.1 there exists $\theta < 1$ such that for every $i$ we have
\begin{equation}
m(p \in I_i : \frac{\pi_2(F^k(p)) - \pi_2(p)}{k} \leq \epsilon) \leq C\theta^k.
\end{equation}
Using an argument as in the proof of Theorem 1 we can conclude that
\begin{equation}
m(p \in I_i : \frac{\pi_2(F^k(p)) - \pi_2(p)}{k} \geq \epsilon \text{ for } k \geq k_0 \geq 1 - C\theta^{k_0}
\end{equation}
for every $i \geq 0$. By Theorem 1 and Theorem 2 the conjugacy $h$ is absolutely continuous. Let $\delta = \sup_p dist_1(p)$. Then Eq. (45) implies that there exist $C > 0$ such that
\begin{equation}
m(p \in I_i : dist_k(p) \geq \delta n + C \text{ for some } k) \leq C\theta^n,
\end{equation}
for $i \geq 0$. Denote $\Lambda_1 := \{p \in I_i : h'(p) \leq 1\}$ and, for $n \geq 1$
\begin{equation}
\Lambda_n := \{p \in I_i : e^{\delta(n-1)} < h'(p) \leq e^{\delta n}\}.
\end{equation}
By Eq. (46) we have $m(\Lambda_n) \leq C\theta^n$. Indeed, Let $J_k(p)$ be as in the proof of Proposition 5.4. By the Lebesgue differentiation Theorem for almost every $p$ we have
\begin{equation}
\lim_{k} \frac{|h(J_k(p))|}{|J_k(p)|} = h'(p).
\end{equation}
On the other hand, by Eq. (37) we have that for almost every $p \in I_i$ outside the set in Eq. (46)
\begin{equation}
C_1 e^{-(n\delta + C)} \leq \frac{|h(J_k(p))|}{|J_k(p)|} \leq C_2 e^{n\delta + C},
\end{equation}
so
\begin{equation}
C_1 e^{-(n\delta + C)} \leq h'(p) \leq C_2 e^{n\delta + C},
\end{equation}
in a subset of $I_i$ with measure larger than $1 - C\theta^n$. Of course this implies $m(\Lambda_n) \leq C\theta^n$. Let $B \subset I_i$ be an arbitrary Lebesgue measurable set. Let $k_1$ be so that
\begin{equation}
\theta^{k_1+1} < |B| \leq \theta^{k_1}.
\end{equation}
First we prove Theorem 3 assuming that $e^{\delta} \theta < 1$. Since $h$ is absolutely continuous we have
\begin{equation}
|h(B)| = \int_B h' \, dm
\end{equation}
\[= \sum_{n=0}^{k_1} \int_{B \cap \Lambda_n} h' \ dm + \sum_{n=k_1+1}^{\infty} \int_{B \cap \Lambda_n} h' \ dm \]

\[\leq \sum_{n=0}^{k_1} \theta^{k_1} e^{\delta n} + \sum_{n=k_1+1}^{\infty} C(e^{\delta \theta})^n\]

\[\leq C(e^{\delta \theta})^{k_1} \leq C|B|^{1 + \frac{1}{\delta \theta}}.\]

Now if \(B \subset J \in \mathcal{P}^n\) and \(F^n(J) = Q \subset I_i\), with \(|Q|, |h(Q)| \geq C\) (due Property LI for \(F\) and \(G\)), then due the bounded distortion of \(F\) and \(G\)

\[\frac{|h(B)|}{|h(J)|} \leq C\frac{|h(F^n(B))|}{|h(Q)|} \leq C\left(\frac{|F^n(B)|}{|Q|}\right)^{1 + \frac{1}{\delta \theta}} \leq C\left(\frac{|B|}{|J|}\right)^{1 + \frac{1}{\delta \theta}}.\]

To prove a similar inequality to \(h^{-1}\), define

\[\tilde{\Lambda}_n := \{p \in I_i : e^{\delta(n-1)} < (h^{-1})'(p) \leq e^{\delta n}\}.\]

of course

\[h^{-1}\tilde{\Lambda}_n = \{p \in I_i : e^{-\delta n} < h'(p) \leq e^{-\delta(n-1)}\},\]

so by Eq. (46) and Eq. (47) we obtain

\[m(h^{-1}\tilde{\Lambda}_n) \leq C\theta^n.\]

In particular

\[m(\tilde{\Lambda}_n) = \int_{h^{-1}\tilde{\Lambda}_n} h'(x) \ dm \leq C(e^{-\delta \theta})^n\]

Note that this argument gives us an exponential upper bound even if \(\delta\) is large.

Now we can switch the roles of \(F\) and \(G\) to obtain the inequality to \(h^{-1}\), which shows that \(h\) is a mSQS-homeomorphism relative to the stochastic basis \(\cup_n \mathcal{P}^n\).

To complete the proof when \(e^{\delta \theta} \geq 1\) we do the following: find a continuous path of random walks \(F_t\) with \(F_0 = F\) and \(F_1 = G\) and so that for every \(t \in [0, 1]\) we have that \(F_t\) is an asymptotically small perturbation of \(F\) and moreover there exist \(\epsilon > 0\) and \(\theta < 1\) such that Eq. (44) holds for every random walk in this family. Using the compactness of \([0, 1]\) we can find a finite sequence of random walks \(F_{t_0} = F, F_{t_1}, F_{t_2}, \ldots, F_{t_n} = G\) so that \(F_{t_i}\) and \(F_{t_{i+1}}\) are conjugated by a map \(H_i\) such that

\[\delta_i := \sup_p \left|\frac{DF_{t_{i+1}}(H_i(p))}{DF_{t_i}(p)}\right|,\]

satisfy \(e^{\delta_i \theta} < 1\). So the conjugacy \(H_i\) is mSQS with respect some dynamically defined stochastic basis. Composing these conjugacies we find a mSQS-conjugacy between \(F\) and \(G\).

6. Stability of recurrence

Let \(F = (f, \psi)\) be a homogeneous random walk and let \(G\) be an asymptotically small perturbation of \(F\). To avoid a cumbersome notation, in this section we make the convention that all inequalities holds only for large \(n\). Moreover in this section we assume that \(\psi\) is unbounded. Recall that in this case we assume that asymptotically small perturbations \(G\) coincides with \(F\) on negative states. The case where \(\psi\) is bounded is similar.  

\[\square\]
The following is an easy consequence of the Central Limit Theorem for Birkhoff sums (Proposition 4.3)

**Corollary 6.1.** Let $a_n$ be a positive increasing sequence. Then

$$
\mu\left(\left|S_n\right| > a_n \right) \leq Ce^{-\frac{a_n^2}{2n}} + C\frac{1}{\sqrt{n}}.
$$

Here

$$S_n(x) = \sum_{k=0}^{n-1} \psi(f^k(x)).$$

**Proof.** Use Proposition 4.3 and note that the estimative

$$\int_{-\infty}^v e^{-\frac{u^2}{2}} du \leq Ce^{-\frac{v^2}{2}}$$

holds for $v << 0$.

□

Given $n \in \mathbb{N}$, split $[0, 2n] \cap \mathbb{N}$ in $\sqrt{\log n}$ blocks (called main blocks), denoted $B_j$, with length

$$\frac{n}{\log^{j+1} n}, \quad j = 1, \ldots, \sqrt{\log n},$$

and between the main blocks we put little blocks $H_j$, called holes, of length $\log^4 n$. These holes will warranty the independence between the events in distinct main blocks. Put these blocks in the following order:

$$\cdots < B_{j+1} < H_{j+1} < B_j < H_j < \cdots,$$

with $\min B_j \sqrt{\log n} = 0$. Note that we let most of the second half of the interval $[0, 2n] \cap \mathbb{N}$ uncovered.

Define

$$S(j) = \sum_{i \in B_j} \psi \circ f^i,$$

$$H(j) = \sum_{i \in H_j} \psi \circ f^i$$

Denote $|B_j| := \max B_j - \min B_j$.

**Lemma 6.2.** We have

$$\mu\left(\sum_{i=0}^{\left|B_j\right|} \psi \circ f^i \geq \frac{\sqrt{n}}{\log^{j+1} n} \log^3 n \right) \leq C\frac{\log^{4j} n}{\sqrt{n}}.$$

**Proof.** This follows from Corollary 6.1. □

**Proposition 6.3.** For every $\epsilon > 0$ we have

$$\mu(S(j) > \frac{\sqrt{n}}{\log^{j+1} n} \log^3 n, \text{ for some } j \leq \sqrt{\log n}) \leq C\frac{1}{\sqrt{n}},$$

provided $n$ is large enough.
Proof. For \( j \leq \sqrt{\log n} \) define
\[
\Lambda_j := \{ x \in I : S(j)(x) > \frac{\sqrt{n}}{\log^j n} \log^3 n \}
\]
\[
= \{ x \in I : \sum_{i < |B_j|} \psi \circ f^{i+\min B_j}(x) > \frac{\sqrt{n}}{\log^j n} \log^3 n \}
\]
and for each \( P \in \mathcal{P} \) denote \( \Lambda_j(P) := \Lambda_j \cap P \).

Due Lemma 6.2 and the bounded distortion of \( f^{\min B_j} \) on \( P \) we have
\[
m(\Lambda_j(P)) \leq C \frac{\log^j n}{\sqrt{n}} |P|.
\]

Summing on \( j \) and \( P \)
\[
m(\bigcup_j \bigcup_P \Lambda_j(P)) \leq \sqrt{\log n} \frac{\log^j n}{\sqrt{n}} \ll C \frac{1}{n^d}.
\]

\( \square \)

Proposition 6.4. For every \( \epsilon > 0 \) and \( d > 0 \) we have
\[
(48) \quad \mu(\sum_{i \in H_j} \psi(f^i(x))) > \log^8 n, \text{ for some } j \leq \sqrt{\log n} \leq C \frac{1}{n^d},
\]
provided \( n \) is large enough.

Proof. For \( i \in H_j - 1 \), with \( j \leq \sqrt{\log n} \), define
\[
\Lambda^i_j := \{ x \in I : |\psi(f^i(x))| > \log^4 n \}.
\]

By expanding and bounded distortion properties of \( f \) and condition GD we have that
\[
\mu(\Lambda^i_j) \leq C \lambda \log^4 n.
\]

Since \( |H_j| = \log^4 n \), if \( x \) belongs to the set in Eq. (48) then \( x \in \Lambda^i_j \), for some \( i \in H_j - 1 \), with \( j \leq \sqrt{\log n} \). So
\[
\mu(\sum_{i \in H_j} \psi(f^i(x))) > \log^8 n, \text{ for some } j \leq \sqrt{\log n}
\]
\[
\leq \mu(\bigcup_{j \leq \sqrt{\log n}} \bigcup_{i \in H_j - 1} \Lambda^i_j)
\]
\[
\leq \sqrt{\log n} \log^4 n \log^{\lambda \log^3 n}
\]
\[
<< \frac{1}{n^d},
\]
where the last inequality holds for \( n \) large enough. \( \square \)

Proposition 6.5 (Independence between distant events). There exists \( \lambda < 1 \) so that the following holds: Let \( C_1 \) be a disjoint union of elements of \( \mathcal{P}^{n-1} \) and let \( C_2 \) be a disjoint union of elements of \( \mathcal{P}^{k-1} \). We have
\[
\mu(C_1 \cap f^{-(n+d)} C_2) = \mu(C_1) \mu(C_2)(1 + O(\lambda^d)).
\]
Here \( n = \ell(C_1) \).
Proof. Let $J \in \mathcal{P}^{n-1}$. Since $F \in \mathcal{O}_n$ we have $f^n(J) = I$. Define the measure $ho(A) := \mu(f^{-n}A \cap J)/\mu(J)$. Note that by the bounded distortion property of $f$, we have that $\log d\rho/dm$ is uniformly $\alpha$-Holder, that is,

\begin{equation}
|\log \frac{d\rho}{dm}(x) - \log \frac{d\rho}{dm}(y)| \leq C|x - y|^{\alpha},
\end{equation}

where $C$ and $\alpha$ do not depend on $n$ and $C_1$. Furthermore it is bounded by above by a constant which does not depend on $n$. By the well-know theory of Ruelle-Perron-Frobenius operators for Markov expanding maps (see for instance [V]), if $P$ is the Perron-Frobenius-Ruelle operator of $f$, then there exists $\lambda < 1$ so that

\[ Pd\rho = (1 + O(\lambda^d))d\mu. \]

So

\[ \frac{\mu(J \cap f^{-(n+d)}C_2)}{\mu(J)} = \rho(f^{-d}C_2) = \int 1_{C_2} \circ f^d \frac{d\rho}{dm} dm \]

\[ = \int 1_{C_2} P^d \frac{d\rho}{dm} dm \]

\[ = (1 + O(\lambda^d)) \int 1_{C_2} \frac{d\mu}{dm} dm \]

\[ = (1 + O(\lambda^d)) \mu(C_2). \]

The constant $\lambda$ is the contraction of the Ruelle-Perron-Frobenious operator in certain cone of positive functions and whose logarithm is $\alpha$-Holder continuous (see [V]). Since all functions $\log \frac{d\rho}{dm}$ belongs to the very same cone (Due Eq. (49)), $\lambda$ does not depend on $C_1$. Since $C_1$ is a disjoint union of intervals $J \in \mathcal{P}^{n-1}$, we finished the proof. \hfill $\square$

**Corollary 6.6.** There exists $M > 0$ so that

\[ \mu(S_j < \frac{\sqrt{n}}{\log^{3/2} n} M \text{ for all } j \leq \sqrt{\log n}) \leq C \left( \frac{2}{3} \right)^{\sqrt{\log n}} \]

Proof. Choose $M > 0$ so that

\[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} du < \frac{2}{3} \]

Consider

\[ C_j := \{ x \text{ s.t. } \sum_{i=0}^{\lfloor |B_j| \rfloor} \psi \circ f^i(x) < \frac{\sqrt{n}}{\log^{3/2} n} M \}. \]

Note that $C_j$ is a the disjoint union of elements of $\mathcal{P}^{|B|^{-1}}$. The Central Limit Theorem tells us that if $n$ is large enough then

\[ \mu(C_j) < \frac{2}{3} \]

for every $j \leq \sqrt{\log n}$.

Recall that between $B_j$ and $B_{j+1}$ there is a hole $H_{j+1}$ with length $\log^{4/3} n$. Denote

\[ A_j := \bigcap_{i=1}^{j} f^{-\sum_{k=i+1}^{j} (|B_k| + |H_k|)} C_i \]
Note that $\Lambda_j$ is a disjoint union of elements of $P|B_1|+\sum_{k=2}^{|B_k|+|H_k|}-1$ and
\[ \Lambda_j = C_j \cap f^{-|B_j|-|H_j|}\Lambda_{j-1}. \]
Moreover
\[ \Lambda_{\sqrt{\log n}} = \{ x \text{ s.t. } S_j < \frac{\sqrt{n}}{\log^j n} M \text{ for all } j \leq \sqrt{\log n} \}. \]
By Proposition 6.5, we obtain
\[ \mu(\Lambda_j) = (1 + O(\lambda|H_j|))\mu(C_j)\mu(\Lambda_{j-1}) \]
So by Eq. (50)
\[ \mu(S_j < \frac{\sqrt{n}}{\log^j n} M \text{ for all } j \leq \sqrt{\log n}) \]
\[ \leq \left(\frac{2}{3}\right)^{\sqrt{\log n}}(1 + O(\lambda^n))\sqrt{\log n} \leq C\left(\frac{2}{3}\right)^{\sqrt{\log n}} \]

Proposition 6.7. There exists $C > 0$ so that
\[ \mu(x \in I: \text{there exists } i < \ell^3 \text{ so that } \sum_{k=0}^i \psi \circ f^k(x) > \frac{\ell}{2}) \geq 1 - C\left(\frac{2}{3}\right)^{\sqrt{\log \ell}}. \]

Proof. Let $M$ be as in Corollary 6.6. Denote $n = \ell^3$ and define
\[ A_\ell := \{ x: \text{there exists } i < \ell^3 \text{ so that } \sum_{k=0}^i \psi \circ f^k(x) > \frac{\ell}{2} \}, \]
\[ B_\ell := \{ x: |S_j| < \frac{\sqrt{n}}{\log^j n} \log^3 n, \text{ for all } j \leq \sqrt{\log n} \}, \]
\[ C_\ell := \{ x: S_j \geq \frac{\sqrt{n}}{\log^j n} M, \text{ for some } j \leq \sqrt{\log n} \}, \]
\[ D_\ell := \{ x: |H_j(x)| \leq \log^8 n, \text{ for all } j \leq \sqrt{\log n} \}. \]
We claim that if $\ell$ is large then $B_\ell \cap C_\ell \cap D_\ell \subset A_\ell$. Indeed, let $x \in B_\ell \cap C_\ell \cap D_\ell$. Then for some $j_0 \leq \sqrt{\log n}$,
\[ S_{j_0}(x) \geq \frac{\sqrt{n}}{\log^{j_0} n} M. \]
We claim that, if $m = \max B_{j_0}$, then
\[ \sum_{0}^m \psi \circ f^i(x) > \frac{\ell}{2}. \]
Indeed, since $x \in D_\ell$, \[ |\sum_{i \in H_j, j > j_0} \psi \circ f^i(x)| \leq \sqrt{\log n} \log^8 n = o(\ell). \]
Moreover, since $x \in B_\ell$, \[ |\sum_{i \in B_j, j > j_0} \psi \circ f^i(x)| \leq \sum_{j > j_0} \frac{\sqrt{n}}{\log^j n} \log^3 n \leq C\frac{\sqrt{n}}{\log^{j_0+1} n}. \]
So
\[ \sum_{i=0}^{m} \psi \circ f^i(x) = \sum_{i \in B_{j_0}} \psi \circ f^i(x) + \sum_{i \in B_j, j > j_0} \psi \circ f^i(x) + \sum_{i \in B_{j_0}, j > j_0} \psi \circ f^i(x) \geq (M - \frac{C}{\log n}) \frac{\sqrt{n}}{\log^{4\alpha} n} + o(\ell) > C\ell^\frac{1}{2} - o(\ell) > \ell \frac{C}{2} \]
and we finished the proof of the claim. To finish the proof, note that by Proposition 6.3, Corollary 6.6 and Proposition 6.4
\[ \mu(A) \geq \mu(B_{j_0} \cap C_{j_0} \cap D_{j_0}) \geq 1 - C\left(\frac{2}{3}\right)^{\frac{\sqrt{\log n}}{\log n}} - C \frac{1}{n^c} \geq 1 - C\left(\frac{2}{3}\right)^{\sqrt{\log n}}. \]

Let \( C > 0 \) and \( \lambda \in (0, 1) \) be as in Eq. (7) and Eq. (8). Define
\[ Dist_n(p) := n - 1 \sum_{i=0}^{k} C \lambda^{\frac{3}{2}i} \pi_2(F_j^i(p)). \]
Of course \( dist_n(p) \leq Dist_n(p). \)

**Proposition 6.8.** There exist \( \epsilon \) and \( D \) so that for every \( \ell \geq 0 \),
\[ \mu\{p \in I_{\ell} : \text{there exists } i \text{ so that } F^i(p) \in \bigcup_{t \in [\min \psi, -\min \psi]} I_t \text{ and } dist_i(p) \leq D\} \geq \epsilon \]

**Proof.** For \( \ell \geq 0 \) and \( k \) define \( B_{k} \) as the set of all \( p \in I_{\ell} \) such that there exists
\[ j \leq \sum_{i=0}^{k} \frac{\ell^3}{2^{3i}} \]
satisfying
\[ \pi_2(F_j^i(p)) \leq \frac{\ell}{2^k} \text{ and } Dist_j(p) \leq \sum_{i=0}^{k} C \frac{\ell^3}{2^{3i}} \lambda^{\frac{3}{2}i+\min \psi}. \]

We are going to prove by ascending induction on \( k \geq 0 \) that there is \( C > 0 \) so that for every \( \ell \geq 0 \) we have
\[ \mu(B_{k}^i) \geq \prod_{i=0}^{k-1} \left(1 - C\left(\frac{2}{3}\right)^{\frac{\sqrt{\log n}}{\log n}}\right), \]
for all \( k \geq 1 \) and \( \mu(B_{k}^0) = 1. \)

Note that \( B_{0}^i = I_{\ell} \) so \( \mu(B_{0}^i) = 1. \) Now assume the induction hypothesis for some \( k \geq 0. \) Take \( p \in B_{\ell}^k. \) Let \( p \in L = C(i_0, i_1, \dots, i_{j-1}), \) where \( j \) is the smallest integer as in the definition of \( B_{k}^i. \) In particular
\[ \frac{\ell}{2^k} + \min \psi \leq \pi_2(F_j^i(p)) \leq \frac{\ell}{2^k}. \]

Note that \( L \subset B_{\ell}^k \) and \( F_j^i(L) = I_r, \) with \( r := \pi_2(F_j^i(p)). \) Applying Proposition 6.7 to \(-\psi\) we get
\[ \mu(x \in I_r : \text{there exists } i < \frac{\ell^3}{2^k} \text{ so that } \sum_{n=0}^{i} \psi \circ f^n(x) < -\frac{\ell}{2^{k+r}}) \]
\[ \geq 1 - C \left( \frac{2}{3} \right)^{\sqrt{\log \frac{\ell}{2k}}} . \]

Denote
\[ D_L := \{ x \in L : \text{there exists } i < \frac{\ell^3}{2k} \text{ so that } \sum_{n=0}^{i} \psi \circ f^n(f^j(x)) < -\frac{\ell}{2k+1} \} . \]

Due to the bounded distortion property for \( F \), the estamative in Eq. (53) implies
\[ \mu(D_L) \geq 1 - C \left( \frac{2}{3} \right)^{\sqrt{\log \frac{\ell}{2k}}} . \]

We claim that \( D_L \subset B_{k+1}^\ell \). Indeed, let \( x \in D_L \). Take the smallest \( i \) so that
\[ \sum_{n=0}^{i} \psi \circ f^n(f^j(x)) < -\frac{\ell}{2k+1} . \]

Then by Eq. (52) we have
\[ \pi_2(F^j(p)) \geq \frac{\ell^3}{2k} \lambda^{+ \psi} + \min \psi , \]

for every \( 0 \leq h < i \), so
\[ \text{Dist}_i(F^j(p)) \leq \sum_{h=0}^{i} C \lambda^{\pi_2(F^{j+h}(p))} \leq C \frac{\ell^3}{2k} \lambda^{\pi_2(F^j(p)) + \min \psi} . \]

So \( D_L \subset B_{k+1}^\ell \). Since \( B_k^\ell \) is a disjoint union of cylinders \( L \), the estamative in Eq. (54) implies that Eq. (51) holds replacing \( k \) by \( k + 1 \). This concludes the induction step.

Define
\[ D := \sum_{i=0}^{\infty} C \frac{\ell^3}{2k} \lambda^{\pi_2(F^j(p)) + \min \psi} < \infty . \]

Let \( k \) be so that \( 2^k \leq \ell \leq 2^{k+1} \). Now it is easy to check that
\[ \mu(\{ x \in I_\ell : \text{there exists } i \text{ so that } F^i(p) \in I_0 \text{ and } \text{dist}(p) \leq D \}) \]
\[ \geq C \mu(B_k^\ell) \geq \prod_{i=0}^{k-1} \left( 1 - C \left( \frac{2}{3} \right)^{\sqrt{\log \frac{\ell}{2k}}} \right) \geq \prod_{i=0}^{k-1} \left( 1 - C \left( \frac{2}{3} \right)^{\sqrt{\log \frac{\ell}{2k}}} \right) \]
\[ \geq \exp(-C \sum_{i=1}^{\infty} (\frac{2}{3})^{\sqrt{\log \frac{\ell}{2k}}}) > \tilde{C} > 0 , \]

which finishes the proof. \( \square \)

**Proof of the Stability of Recurrence (Theorem 4).** Since \( F \) is recurrent, its mean drift is zero. By Corollary 4.7 we can apply the Central Limit Theorem as in the introduction to conclude Eq. (2) and Eq. (4). Because \( G \) coincides with \( F \) on negative states, the orbit by \( G \) of almost every point \( p \) satisfying \( \pi_2(p) < 0 \) will entry
\[ \cup_{i \geq 0} I_i^j . \]

As a consequence the orbit by \( G \) of almost every point \( p \) visits this set infinitely many times. Let \( \ell \geq 0 \).
By Proposition 6.8, there exist $D > 0$ and $\epsilon > 0$ so that

$$A_\ell := \{ p \in I_\ell : \text{there exists } i \text{ so that } F^i(p) \in \bigcup_{t=\min \psi}^{\min \psi} I_t \text{ and } Dist_i(p) < D \}$$

satisfies $\mu(A_\ell) > \epsilon$, for all $\ell \geq 0$.

Consider a cylinder $C_F = C_F(\ell, k_1, \ldots, k_{i-1}, k_i) \subset A_\ell$, with $C_F \neq \emptyset$, satisfying $|k_j| \geq \min \psi$ for $0 < j < i$, $\min \psi \leq k_i \leq \min \psi$ and $Dist_i(x) < D$, for every $x \in C_F$. We claim that that corresponding cylinder $C_G = C_G(\ell, k_1, \ldots, k_{i-1}, k_i)$ for the perturbed random walk $G$ satisfies

$$\frac{1}{C} \leq \frac{|C_G|}{|C_F|} \leq C,$$

where $C$ depends only on $D$. Because we used $Dist_i(p)$ instead of $dist_i(p)$ in the definition of $A_\ell$, the set $A_\ell$ is a disjoint union of cylinders of this type, so we obtain that $B_\ell = H(A_\ell)$ satisfies $m(B_\ell) > \epsilon > 0$, for all $\ell \geq 0$.

To prove that the set of points whose orbits returns infinitely many times to

$$\bigcup_{t=\min \psi}^{\min \psi} I_\ell$$

has full Lebesgue measure, it is enough to prove that $\Lambda := \bigcup_{j \geq 0, \ell} G^{-j} B_\ell$ has full Lebesgue measure.

Indeed, assume by contradiction that $\Lambda$ is not full. Choose a Lebesgue density point $p$ of the complement of $\Lambda$ and also satisfying $\limsup \psi(x) G^k(p) \geq 0$. Then there exist a sequence of cylinders $C_k \subset P^{k-1}$ so that $p \in C_k$

$$m(C_k \setminus \Lambda) \to_k 1.$$

But $G^k(C_k) = I_{\ell_k}$, with $\ell_k = \pi_2(G^k(C_k))$, and $m(I_{\ell_k} \cap B_{\ell_k}) \geq \epsilon_l |I_{\ell_k}|$. By the bounded distortion property

$$\frac{m(\Lambda \cap C_k)}{|C_k|} > \frac{m(G^{-k} B_{\ell_k} \cap C_k)}{|C_k|} > \tilde{C} \epsilon,$$

which contradicts Eq. (55). Now we can use that $G$ is transitive and has bounded distortion to prove that $G$ is recurrent. \hfill \square

**Proof of Proposition 3.1.** Since $F$ is recurrent, almost every point of $I^0$ returns to $I^0$ at least once. So the first return map $R_F : I^0 \to I^0$ is defined almost everywhere is $I^0$ and the same can be said about $R_G$. Of course, the absolutely continuous conjugacy $H$ also conjugates the expanding Markovian maps $R_F$ and $R_G$. Using the same argument used in Shub and Sullivan [ShSu] and Martens and de Melo [MdM], we can prove that $H$ is actually $C^1$ on $I^0$. Using the dynamics, it is easy to prove that $H$ is $C^1$ everywhere. \hfill \square

7. Stability of the multifractal spectrum

7.1. Dynamical defined intervals and root cylinders. When we are dealing with Markov expanding maps with finite Markov partitions, for each arbitrary interval $J$ we can find an element of $\cup_j P^j$ which covers $J$ and has more or less the same size that $J$. Note that this is no longer true when the Markov partitions is
infinite. Since coverings by intervals are crucial in the study of the Hausdorff dimension of an one-dimensional set, this trick is very useful to estimate the dimension of dynamically defined sets, once we can replace an arbitrary covering by intervals by another one with essentially the same metric properties but whose elements are themselves dynamically defined sets (cylinders).

Consider \( j \geq 0 \) and let \( \{C_i\}_{i \in \Theta} \subset P^j \) be a finite or countable family of cylinders such that \( W := \bigcup_i C_i \) is connected, \( W \subset J \in P^{j-1} \) and \( F^j(\text{int } W) \) does not contain any point \( d_i^n \) (as defined in property \( R_b \)). Then \( W \) is called a dynamically defined interval (dd-interval, for short) of level \( j \). Define the root cylinder of \( W \) as the unique cylinder \( C_{i_0} \) with the following property: if \( \#\Theta = \infty \) then \( W \) is a semi-open interval and \( C_{i_0} \) will be the cylinder so that \( \partial C_{i_0} \cap \partial W \neq \emptyset \). Otherwise \( W \) is closed and let \( C_{i_0} \) be the unique cylinder such that \( F^j = \partial C_{i_0} \cap \partial W \) is the boundary of a semi-open dd-interval which contains \( W \). The following Lemmas are an easy consequence of the regularity properties \( Ra + R_b \) and it will be useful to recover the trick described above for (certain) infinite Markov partitions. The proof is very simple.

**Lemma 7.1.** For every \( d \in (0,1) \) there exists \( K > 1 \) so that for every dd-interval \( W := \bigcup_i C_i \) with root cylinder \( C_{i_0} \) we have

\[
\frac{1}{K} \leq \frac{|W|^\alpha}{\sum_i |C_i|^\alpha} \leq K
\]

(56)

\[
\frac{1}{K} \leq \frac{|C_{i_0}|^\alpha}{\sum_i |C_i|^\alpha} \leq K
\]

(57)

for every \( 1 \geq \alpha \geq d \). Indeed the constant \( K \) depends only on \( d \) and constants in the properties \( Ra + R_b + Ex + BD \).

**Proof.** Due Property \( Ra \), we can enumerate \( C_i \) in such way that \( C_0 \) is the root cylinder of \( W \) and \( \partial C_{i+1} \cap \partial C_i \neq \emptyset \). Moreover if \( j \) is the level of \( W \) then \( W \subset J \in P^{j-1} \). Let \( F^j(J) = I_n \). In particular \( F^j(C_i) \in P_0^n \) and \( F^j(W) = \bigcup_i F^j(C_i) \) is a dd-interval of level 0, with root cylinder \( F^j(C_0) \) and \( \partial F^j(C_{i+1}) \cap \partial F^j(C_i) \neq \emptyset \). By property \( R_b \) we have

\[
\frac{|F^j(C_i)|}{|F^j(C_0)|} \leq C\lambda^i.
\]

By the property \( BD \) we have that

\[
\frac{|C_i|}{|C_0|} \leq C\lambda^i,
\]

so we obtain Eq. (57) since

\[
|C_0|^\alpha \leq \sum_i |C_i|^\alpha \leq |C_0|^\alpha \sum_{i=0}^{\infty} C\lambda^{di}.
\]

In particular for \( \alpha = 1 \) we have

\[
1 \leq \frac{|W|}{|C_0|} \leq C,
\]

(58)

From Eq. (58) and Eq. (57) we can easily get Eq. (56) for every \( d \leq \alpha \leq 1 \). \( \square \)
Lemma 7.2. Let $N$ be as in Properties $Ra + Rb$. For every $d \in (0, 1)$ there exists $K > 1$ so that the following holds: For every interval $J \subset I \times \mathbb{Z}$ there exists $m$ $dd$-intervals $W_j$, all of same level, with $m \leq 2N$, satisfying the following properties:

- The interior of these $dd$-intervals are pairwise disjoint.
- The closure of the union of $W_j$ covers $J$:

$$J \subset \bigcup_j W_j.$$

- We have

$$\frac{1}{K} \leq \frac{\sum_{i=1}^m |W_i|^\alpha}{|J|^\alpha} \leq K$$

for every $1 \geq \alpha > d$.

Indeed the constant $K$ depends only on $d$ and constants in the properties $Ra + Rb + Ex + BD$.

Proof. Let $\mathcal{P}^{-1} = \{I_n\}_n$. Define the sequence of partitions $Q^j$, $j \geq 0$, of $I \times \mathbb{Z}$ in the following way: $Q^0$ is the family of the connected components of

$$I \times \mathbb{Z} \setminus \{c^n_i, d^n_i\}_{i,n},$$

and an interval $Q$ belongs to $Q^j$, $j \geq 1$ if there exists $P \in \mathcal{P}^{j-1}$ such that $Q$ is one of the connected components of

$$P \cap F^{-j}\{c^n_i, d^n_i\}_i,$$

where $n = \pi_2(F^j P)$. Here $c^n_i, d^n_i$ are as in Property $Ra$ and $Rb$. Note that each $P \in \mathcal{P}^{j-1}$ contains at most $2N$ intervals in $Q^j$ and each $Q \in Q^j$ is a $dd$-interval of level $j$. First we consider the case

$$\{j \geq 0 \text{ such that } \#\{Q \in Q^j : J \cap Q \neq \emptyset\} \leq 2\} = \emptyset.$$

Let $n = \pi_2(J)$. Then $J$ intersects at least three connected components of

$$I_n \setminus \{c^n_i, d^n_i\}_i,$$

so it contains one of the connected components of this set. In particular if

$$\{W_j\}_j := \{Q \in Q^0 \text{ such that } Q \cap J \neq \emptyset\},$$

then by Property $Ra + Rb$ we have $\max_j |W_j| \geq \delta$, so

$$1 \leq \frac{\sum_{i=1}^m |W_i|^\alpha}{|J|^\alpha} \leq \frac{2N}{\delta}$$

If Eq. (59) does not hold, let

$$j_0 = \max\{j \geq 0 \text{ such that } \#\{Q \in Q^j : J \cap Q \neq \emptyset\} \leq 2\}.$$

Let $Q, R \in Q^{j_0}$ be such that $J \subset \overline{Q \cup R}$. Then

$$Q = \bigcup_i D_i, \quad R = \bigcup_i E_i,$$

with $D_i, E_i \in \mathcal{P}^{j_0}$, $\partial D_i \cap \partial D_{i+1} \neq \emptyset$, $\partial E_i \cap \partial E_{i+1} \neq \emptyset$ and $D_0$ and $E_0$ are the root intervals of $Q$ and $R$. Let

$$i_Q := \min\{i \geq 0 \text{ such that } D_i \cap J \neq \emptyset\},$$

$$i_R := \min\{i \geq 0 \text{ such that } E_i \cap J \neq \emptyset\},$$

$$Q_{i_Q} = \bigcup_{i \geq i_Q} D_i, \quad R_{i_R} = \bigcup_{i \geq i_R} E_i,$$
Note that $Q_i, Q_j$ are dd-intervals of level $j_0$. Without lost of generality, suppose that $|D_{iQ}| \geq |E_{iR}|$. Then $Q_i \cup R_{iR} \supset J$ is an interval. Let $K$ be the constant given by Proposition 7.1 for $\alpha > d$. Then

$$(60) \quad (1 + C_1)|D_{iQ}| \leq |D_{iQ}| + |D_{iQ+1}| \leq |Q_{iQ}| + |R_{jR}| \leq 2K|D_{iQ}|,$$

where the first inequality follows from Eq. (6). We have three cases.

Case 1. Suppose that $Q \neq R$ and the intervals are in the order

$$D_{iQ} < D_{iQ+1} < \cdots < E_{iR+1} < E_{iR},$$

then $|J| \geq C_1|D_{iQ}|$, otherwise

$$J \subset D_{iQ} \cup D_{iQ+1},$$

which contradicts $J \cap E_{iR} \neq \emptyset$. So

$$1 \leq \frac{|Q_{iQ}|^\alpha + |R_{iR}|^\alpha}{|J|^\alpha} \leq \frac{(4K|D_{iQ}|)^\alpha}{C_1^\alpha|D_{iQ}|^\alpha} \leq \frac{4K}{C_1}.$$

Case 2. Suppose that $Q \neq R$ and the intervals are in the order

$$\cdots < D_{iQ+1} < D_{iQ} < E_{iR} < E_{iR+1} < \cdots$$

Then $i_Q = i_R = 0$ and there exists $y \in \partial D_0 \cap \partial E_0$. By Properties $Ra + Rb + BD$ there exist $[d, y], [y, e] \in \mathcal{P}^j$, with $[d, y] \subset D_0, [y, e] \subset E_0$ such that

$$C_5|D_0| \leq |d - y|, |e - y| \leq C_2|D_0|.$$

since $J$ intersects $D_0$ and $Q_0$ and at least three intervals in $Q_{iQ+1}$ intersect $J$, we have that either $[d, y]$ or $[y, e]$ is contained on $J$. So $|J| \geq C_2|D_0|$. We conclude

$$1 \leq \frac{|Q_{iQ}|^\alpha + |R_{iR}|^\alpha}{|J|^\alpha} \leq \frac{(4K|D_{iQ}|)^\alpha}{C_2^\alpha|D_{iQ}|^\alpha} \leq \frac{4K}{C_2}.$$

Case 3. Suppose that $R = Q$, that is $J \subset Q_{iQ}$. By Properties $Ra + Rb + BD$

$$C_5|D_{iQ}| \leq |D_{iQ+1}| \leq C_4|D_{iQ}|,$$

Using $Ra + Rb + BD$ again, for every interval $S \subset D_{iQ} \cup D_{iQ+1}$ such that $S \in Q_{iQ+1}$ we have

$$|S| \geq C_6|D_{iQ}|.$$

Since at least three intervals in $Q_{iQ+1}$ intersect $J$ there is $S \subset D_{iQ} \cup D_{iQ+1}$ with $S \in Q_{iQ+1}$ such that $S \subset J$. So $|J| \geq C_6|D_{iQ}|$. We conclude

$$1 \leq \frac{|Q_{iQ}|^\alpha}{|J|^\alpha} \leq \frac{(4K|D_{iQ}|)^\alpha}{C_6^\alpha|D_{iQ}|^\alpha} \leq \frac{4K}{C_6}.$$

$\square$

### 7.2. Dimension of dynamically defined sets

Let $f \in Mk + BD + Ex$ and denote by $\mathcal{P}^0$ its Markov partition. Let

$$\mathcal{I} := \{C_i\}_{i} \subset \cup_{n} \mathcal{P}^n$$

be a finite or countable family of disjoint cylinders. Define the induced Markov map $f_{\mathcal{I}}: \cup_{i} C_i \to I$ by

$$f_{\mathcal{I}}(x) = f^{\ell(C_i)-1}(x), \text{ if } x \in C_i.$$
We can also define an induced drift function \( \Psi : \cup_i C_i \to \mathbb{Z} \) in the following way: Define, for \( x \in C \in \mathcal{P}_0^n \),

\[
\Psi_I(x) := \sum_{i=0}^{n-1} \psi(f^i(x)).
\]

Under the same conditions on \( x \), define \( N_I(x) = n \). The maximal invariant set of \( f_I \) is

\[
\Lambda(I) := \{ x \in I : f^j(x) \in \bigcup_i C_i, \text{ for all } j \geq 0 \}.
\]

Denote by \( HD(I) \) the Hausdorff dimension of the maximal invariant set of \( f_I \).

We are going to use the following result

**Proposition 7.3** (Theorem 1.1 in [MU2]). We have

\[
HD(J) = \sup \{ HD(I) : I \subset J, I \text{ finite} \}.
\]

Before to give the proof of Proposition 3.2 we need to introduce some tools which are useful to estimate the Hausdorff dimension.

Let \( I \) be a finite collection of disjoint cylinders. Then there exists \( \beta \) such that

\[
\sum_{C \in I} |C|^\beta = 1,
\]

we will call \( \beta \) the **virtual Hausdorff dimension** of \( f_I \), denoted \( VHD(I) \). The virtual Hausdorff dimension is a nice way to estimate \( HD(I) \): indeed if \( f_I \) is linear on each interval of the Markov partition then these values coincide. When the distortion is positive, these values remain related, as expressed in the following result (which is included, for instance, in the proof of Theorem 3, Section 4.2 of [PT]).

**Proposition 7.4.** Let \( I \) be a finite family of disjoint cylinders. Then

\[
|HD(I) - VHD(I)| \leq \frac{d}{\log \lambda - d},
\]

where

\[
d := \sup_{C \in I} \sup_{x, y \in C} \log \frac{Df_C(y)}{Df_C(x)} \text{ and } \lambda := \inf_{C \in I} \inf_{x \in C} |Df_C(x)|.
\]

Recall that if \( I \) is finite then \( f_I \) has an invariant probability measure \( \mu_I \) supported on its maximal invariant set \( \Lambda(I) \) such that for any subset \( S \subset \Lambda(I) \) satisfying \( \mu_I(S) = 1 \) we have \( HD(S) = HD(I) \) (see for instance [PU]).

Note that for a homogeneous random walk \( F \)

\[
\Omega^k_>(F) = \{ k \} \times \{ x \in I \text{ s.t. } \sum_{i=0}^j \psi(f^i(x)) + k \geq 0, \text{ for } j \geq 0 \}
\]

and

\[
\Omega^k_{+\beta}(F) = \{ k \} \times \{ x \in I \text{ s.t. } \sum_{j=0}^{n-1} \psi(f^j(x)) + k \geq 0 \text{ for all } n \geq 0 \text{ and } \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \psi(f^j(x)) \geq \beta \}.
\]

Define \( \pi_1(x, n) := x \). The following is an easy consequence of this observation:
Lemma 7.5. If $F$ is a homogeneous random walk then $\pi_1(\Omega^0_+(F)) \subset \pi_1(\Omega^k_+(F))$ and $\pi_1(\Omega^0_{+\beta}(F)) \subset \pi_1(\Omega^k_{+\beta}(F))$, for all $k \geq 0$. Furthermore

$$HD(\Omega^0_+(F)) = HD(\Omega^k_+(F))$$

and

$$HD(\Omega^0_{+\beta}(F)) = HD(\Omega^k_{+\beta}(F)).$$

Proposition 7.6. Let $F$ be a homogeneous random walk. Then there exists a sequence of finite families of cylinders

$$F_s \subset \bigcup_i P^i$$

so that

- $\Lambda(F_s) \subset \Omega^0_+(F)$,
- Denote $\beta_n := \int \Psi_{F_s} \, d\mu_{F_s}$. Then $\beta_n > 0$.
- $\lim_{s \to \infty} HD(F_s) = HD(\Omega^0_+(F))$.

Proof. Denote $d = HD(\Omega^0_+(F)) \leq 1$. Given any $s \in \mathbb{N}^*$, $m_d(\Omega_+(F)) = \infty$, where $d_s := d(1 - 1/s) < 1$. Here $m_D$ denotes the $D$-dimensional Hausdorff measure.

By Theorem 5.4 in [F], for each positive number $M$ we can find a compact subset $\Lambda_s \subset \Omega^0_+(F)$ satisfying $m_d(\Lambda_s) = M$. We may assume that $\Lambda_s$ does not have isolated points. We will specify $M$ later.

In particular, for each $\epsilon$ small enough the following holds:

i. For every family of intervals $\{J_i\}_i$ which covers $\Lambda_s$, with $|J_i| < \epsilon$ we have

$$\frac{M}{2} \leq \sum_i |J_i|^{d_s}.$$

ii. There exists a family of intervals $\{J_i\}_i$, with $|J_i| \leq \epsilon$, which covers $\Lambda_s$ and

$$\sum_i |J_i|^{d_s} \leq 2M.$$

Furthermore we can assume that $\partial J_i \subset \Lambda_s$.

Assume that $d_s \geq d/2$. By Lemma 7.1 and Lemma 7.2, there exists some $K$ such that we can replace the special covering $\{J_i\}$ in ii. by a new covering by $d_d$-intervals $\{W^\ell_i\}_i$, $\epsilon$, with root cylinders $R^\ell_i$, where

\begin{align}
J_i \cap \Lambda_s & \subset \bigcup_{\ell} W^\ell_i, \\
W^\ell_i & := \bigcup_k C^\ell_k, \text{ for each } \ell \leq m_d \leq 2N,
\end{align}

\begin{align}
\frac{1}{K} \leq \sum_{i} |R^\ell_i|^{d_s} |J_i|^{d_s} \leq K,
\end{align}
\[ \frac{1}{K} \leq \sum_k \frac{|C_k^t|^d_s}{|R_k^t|^d_s} \leq K, \]

Indeed we can replace \( W_\ell^t \) by a \( dd \)-subinterval of it, if necessary, in such way that \( R_\ell^t \cap \Lambda_s \neq \emptyset \) and Eq. (61), Eq. (62), Eq. (63) and Eq. (64) hold, except perhaps the lower bound in Eq. (63), since the new root cylinder could be smaller than the original one. The above estimates, together with the fact that \( \{W_\ell^t\} \) covers \( \Lambda_s \) (up to a countable set) gives

\[ \frac{M}{2K^2} \leq \sum_{i,\ell,k} |C_{i\ell}^k|^d_s \leq 2K^2M. \]

The lower bound in Eq. (65) follows from \( i \). Since these intervals are cylinders, if necessary we can replace this family of cylinders by a subfamily of disjoint cylinders which covers \( \Lambda_s \) up to a countable number of points and such that each cylinder intersects \( \Lambda_s \). Indeed we can choose a finite subfamily \( F_s := \{C_r\}_r \) satisfying

\[ \frac{M}{3K^2} \leq \sum_r |C_r|^d_s \leq 2K^2M. \]

Let’s call this finite subfamily \( F_s \). Note that, since \( C_r \cap \Lambda_s \neq \emptyset \) we have that

\[ \sum_{t=0}^{\ell} \psi(f^t(x)) \geq 0 \]

for every \( x \in C_r \) and \( \ell \leq \ell(C_r) \). If

\[ \sum_{t=0}^{\ell(C_r)} \psi(f^t(x)) = 0 \]

for every \( C_r \), choose a very small cylinder \( \tilde{C} \) satisfying

\[ \tilde{C} \cap \bigcup_r C_r = \emptyset \]

and such that

\[ \sum_{t=0}^{\ell} \psi(f^t(x)) \geq 0 \]

for every \( x \in \tilde{C} \) and \( \ell < \ell(\tilde{C}) \), and

\[ \sum_{t=0}^{\ell(\tilde{C})} \psi(f^t(x)) > 0 \]

on \( \tilde{C} \), and moreover

\[ \frac{M}{3K^2} \leq |\tilde{C}|^d_s + \sum_r |C_r|^d_s \leq 3K^2M. \]

Add \( \tilde{C} \) to the family \( F_s \). Then, if \( \mu_s \) is the geometric invariant measure of \( f_{\mathcal{F}_s} \), we have

\[ \int \Psi_{\mathcal{F}_s} d\mu_s > 0. \]
We can find such \( \tilde{C} \) because \( F \in \Omega_n + GD \) implies that there is at least a point \( x_0 \) such that
\[
\min_{k \geq 0} \sum_{i=0}^k \psi(f^i(x_0)) > 0.
\]
and \( x_0 \not\in \Lambda_s \). By Proposition 7.4 and Eq. (67)
\[
|HD(\Lambda(f_{\mathcal{F}})) - d_s| \leq -\frac{C}{\log \epsilon}.
\]
Since \( \epsilon \) can be taken arbitrary, we can choose \( F_s \) such that
\[
HD(\Lambda(f_{\mathcal{F}})) \to s d.
\]
\[\square\]

Corollary 7.7. If \( F \) is a homogeneous random walk we have that
\[
HD(\Omega_+(F)) = \lim_{\beta \to 0^+} HD(\Omega_{+\beta}(F)) = \sup_{\beta > 0} HD(\Omega_{+\beta}(F)).
\]

Proof. Due Lemma 7.5, it is enough to prove the Corollary for \( k = 0 \). Of course
\[
\Omega^0_{+\beta}(F) \subset \Omega^0_{+}(F) \quad \text{and} \quad \beta_0 \leq \beta_1 \implies \Omega^0_{+\beta_1}(F) \subset \Omega^0_{+\beta_0}(F),
\]
so
\[
\lim_{\beta \to 0^+} HD(\Omega^0_{+\beta}(F)) = \sup_{\beta > 0} HD(\Omega^0_{+\beta}(F)) \leq HD(\Omega^0_{+}(F)).
\]
To obtain the opposite inequality, let \( \mathcal{F}_s \) be as in Proposition 7.6. Denote
\[
\gamma_s := \int \Psi_{\mathcal{F}_s} d\mu_{\mathcal{F}_s}, \quad \text{and} \quad W_n := \int N_{\mathcal{F}_s} d\mu_{\mathcal{I}}
\]
and \( \beta_s := \gamma_s/W_s \). Then by the Birkhoff Ergodic Theorem there is subset \( T_s \subset \Lambda(I_n) \) such that \( \mu_{\mathcal{F}_s}(T_s) = 1 \) and
\[
\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \psi(f^i(x)) = \lim_{k \to \infty} \frac{\sum_{j=0}^{k-1} \Psi_{\mathcal{I}_n}(f_{\mathcal{F}_s}^j(x))}{\sum_{j=0}^{k-1} N_{\mathcal{I}_n}(f_{\mathcal{F}_s}^j(x))} = \frac{\gamma_s}{W_s} = \beta_s > 0.
\]
for every \( x \in T_s \). Since the Hausdorff dimension of \( \mu_{\mathcal{F}_s} \) is equal to \( HD(\mathcal{F}_s) \), we have that \( HD(T_s) = HD(\mathcal{F}_s) \). Note also that
\[
T_s \subset \Omega^0_{+\beta_s},
\]
which implies \( HD(\mathcal{F}_s) \leq HD(\Omega^0_{+\beta_s}) \), so by the choice of \( \mathcal{F}_s \), we conclude that
\[
HD(\Omega^0_{+}) = \lim_{s} HD(\mathcal{F}_s) \leq HD(\Omega^0_{+\beta_s}) \leq \sup_{\beta > 0} HD(\Omega^0_{+\beta}).
\]
\[\square\]

Proof of Theorem 5. Define
\[
\Gamma_n(F) := \{ x \in \Omega^k_{+\beta}(F) \text{ s.t. } \pi_2(f^i(x,k)) \geq \frac{\beta}{2} i, \text{ for all } i \geq n \}.
\]
Of course
\[
\Omega^k_{+\beta}(F) = \bigcup_n \Gamma_n(F).
\]
To prove the Theorem, it is enough to verify that \( HD(\Gamma_n(F)) = HD(\Gamma_n(G)) \). Indeed, for every \( \epsilon > 0 \) and \( \alpha \in (HD(\Gamma_n(F)), 1) \) there exists a covering of \( \Gamma_n(F) \) by intervals \( A_i \) so that

\[
\sum_j |A_j|^{\alpha} \leq \epsilon.
\]

Note that we can assume that \( \partial A_j \subset \Gamma_n(F) \). Since \( G \) is an asymptotically small perturbation of \( F \), it is easy to see that \( G \) also satisfies the properties \( Ra + Rb \), replacing the points \( c_i^n \) and \( d_i^n \) by \( h(c_i^n) \) and \( h(d_i^n) \), and modifying the constant \( K \). Indeed, we can choose constants in the definitions of the properties \( Ex + BD + Ra + Rb \) which works for both random walks, so we can take \( K > 0 \) in the statements of Lemma 7.2 and Lemma 7.1 in such way that it works for both random walks.

In particular (as in the proof of Proposition 7.6) for each \( A_j \) we can find at most \( 2N \) dd-intervals

\[
W_j^\ell := \bigcup_k C_j^{\ell k}, \text{ with } \ell \leq m_j \leq 2N
\]

which satisfy

\[
A_i \cap \Gamma_n(F) \subset \bigcup_{\ell} W_j^\ell,
\]

and

\[
\sum_{k,\ell} |C_j^{\ell k}|^{\alpha} \leq K|A_j|^{\alpha}.
\]

Furthermore, we can assume that the root \( R_j^\ell \) of \( W_j^\ell \) satisfies

\[
\frac{1}{K} \leq \frac{|R_j^\ell|^{\alpha}}{\sum_k |C_j^{\ell k}|^{\alpha}} \leq K
\]

and \( R_j^\ell \cap \Gamma_n(F) \neq \emptyset \).

The constant \( K \) does not depend on \( \alpha, j \) or \( \ell \). In particular the union of all cylinders \( C_j^{\ell k} \) covers \( \Gamma_n(F) \) up to a countable set and

\[
\sum_{j,k,\ell} |C_j^{\ell k}|^{\alpha} \leq K\epsilon.
\]

Note that if \( x \in \Gamma_n(F) \) then

\[
dist_i(x) \leq r_n := Cn + C\lambda^n
\]

for every \( i \in \mathbb{N} \). So

\[
e^{-r_n} \leq \frac{|P_{k_i}(x)|}{|P_{E_l}(h(x))|} \leq e^{r_n}.
\]

There is a point in the cylinder \( R_j^\ell \) which belongs to \( \Gamma_n(F) \), so

\[
e^{-\alpha r_n} \leq \frac{|R_j^\ell|^{\alpha}}{|h(R_j^\ell)|^{\alpha}} \leq e^{\alpha r_n}.
\]

Note that \( h(W_j^\ell) = \bigcup_k h(C_j^{\ell k}) \) is a dd-interval for \( G \) and \( h(R_j^\ell) \) is its root cylinder. So, using Eq. (68)

\[
\frac{1}{K} \leq \frac{|h(R_j^\ell)|^{\alpha}}{\sum_i |h(C_j^{\ell k})|^{\alpha}} \leq K
\]
But the union of the cylinders \( h(C^i_j k) \) covers \( \Gamma_n(G) \) up to a countable set and Eq. (68), Eq. (69), Eq. (70) and Eq. (71) gives
\[
\sum_{j,k,\ell} |h(C^i_j k)|^\alpha \leq K^3 e^{\alpha r_n} \epsilon.
\]

Since \( \alpha > HD(\Gamma_n(F)) \) and \( \epsilon \) is arbitrary we obtain that \( HD(\Gamma_n(G)) \leq HD(\Gamma_n(F)) \).

Switching the roles of \( F \) and \( G \) in the above argument gives the opposite inequality. \( \square \)

Lemma 7.8. Let \( G \in On + Ra + Rb \) be a random walk. For every \( \alpha > 0 \) there exist \( \epsilon \) and \( C \) so that
\[
(72) \sum_{P \in P_n^\ell} |P|^{1-\epsilon} \leq C(1 + \alpha)^n,
\]
for all \( n \) and \( \ell \).

Proof. Indeed, denote
\[
(73) P_n^\ell = \{Q_j^i\}_j \text{ and } P_{n+1}^\ell = \{Q_k^i\}_j,k,
\]
in such way that \( Q_k^i \subset Q_j^i \). To avoid cumbersome notation we are omitting explicit indexing on \( n \) and \( \ell \). Since \( G \in BD + Ra + Rb \), it is possible to order \( Q_k^i \) so that there exist \( C \) and \( \lambda < 1 \) satisfying
\[
(74) \frac{|Q_k^i|}{|Q_j^i|} \leq C\lambda^k,
\]
for every \( j,k,n \). As a consequence the family of functions
\[
h_{j,\ell,n}(\epsilon) = \sum_{k} \frac{|Q_k^i|^{1-\epsilon}}{|Q_j^i|^{1-\epsilon}}
\]
is an equicontinuous set of functions in a small neighborhood of 0. In particular, since \( h_{j,\ell,n}(0) = 1 \), there exists \( \epsilon_0 \) so that, for every \( \epsilon < \epsilon_0 \) and every \( j, \ell \) and \( n \)
\[
(75) \sum_{k} \frac{|Q_k^i|^{1-\epsilon}}{|Q_j^i|^{1-\epsilon}} \leq 1 + \alpha.
\]
So
\[
\sum_{P \in P_{n+1}^\ell} |P|^{1-\epsilon} = \sum_{j,k} |Q_k^i|^{1-\epsilon} \leq (1 + \alpha) \sum_{j} |Q_j^i|^{1-\epsilon} = (1 + \alpha) \sum_{P \in P_n^\ell} |P|^{1-\epsilon}.
\]
\( \square \)

From now on we are going to assume that \( F = (f, \psi) \in On \) is a homogeneous random walk with negative mean drift and \( G \) is an asymptotically small perturbation of \( F \).

Lemma 7.9. Let \( G \in On + Ra + Rb \) be a random walk that is an asymptotically small perturbation of a homogeneous random walk \( F \in On + Ra + Rb \) with negative mean drift. Then for every \( \alpha > \int \psi \, d\mu \), there exists \( C > 0 \), \( \sigma < 1 \) so that for any \( n_1 \geq n_0 \), with \( n_0 \) large enough,
\[
(76) m\{p \in I_{n_1} : \pi_2(G^k(p)) \geq n_0, \text{ for } k \leq n, \text{ and } \pi_2(G^n(p)) - n_1 \geq n_0 \} \leq C\alpha^n.
\]
Proof. Denote
[\Lambda_{n_0,n_1}^n (F) := \{ p \in I_n : \pi_2(G^n(p)) \geq n_0 \text{ for all } k \leq n \text{ and } \pi_2(G^n(p)) - n_1 \geq \alpha n \}].

The statement for \( F \) is consequence of the large deviations estimative (see, for instance [B]) for every \( K > 0 \) there exists \( C_K > 0, \gamma_K \in (0,1) \) such that

\[ m\{ p \in I_n : \sum_{k=0}^{n-1} \psi(f^k(p)) - \int \psi \, d\mu \geq K \} \leq C_K \gamma^n \]

Pick \( K = \alpha - \int \psi \, d\mu \) and \( \bar{\sigma} = \gamma_K \). Then for every \( n_1 \)

\[ m\{ p \in I_{n_1} : \sum_{k=0}^{n_1-1} \psi(f^k(\pi_1(p))) - \pi_2(F^n(p)) - n_1 \geq \alpha n_1 \} \leq C \bar{\sigma}^{n_1}, \]

which implies (of course)

\[ m(\Lambda_{n_0,n_1}^n (F)) \leq C \bar{\sigma}^{n}. \]

We are going to use this estimative to obtain Eq. (76) for the perturbation of \( F \).

Indeed, for every \( \delta > 0 \), there is \( n_0 \) so that if \( \pi_2(x) \geq n_0 \) then

\[ 1 - \delta \leq \frac{|DF(x)|}{|DG(H(x))|} \leq 1 + \delta, \]

Here \( H \) is the topological conjugacy between \( F \) and \( G \) which preserves states. Note that \( \Lambda_{n_0,n_1}^n (F) \) is a disjoint union of elements \( Q_i \in \mathcal{P}^n(F) \), so \( \Lambda_{n_0,n_1}^n (G) \) is a disjoint union of the intervals \( H(Q_i) \). Due the property BD of \( F \) and \( G \), Eq. (77) and Eq. (78), we have

\[ m(\Lambda_{n_0,n_1}^n (G)) = \sum_i |H(Q_i)| \leq \sum_i C(1 + \delta)^n |Q_i| \leq C(1 + \delta)^n \bar{\sigma}^n. \]

Choose \( n_0 \) large enough such that \( \bar{\sigma} := (1 + \delta)\bar{\sigma} < 1 \). \( \square \)

We would like to replace \( n_0 \) by an arbitrary state in Eq. (76). The following Lemma will be useful for this task:

**Lemma 7.10.** Let \( p_n \) and \( q_n \) sequences of non-negative real numbers such that

1. \( p_0 + q_0 \leq 1, \)
2. \( \text{There exists } \epsilon > 0 \text{ and } \ell \geq 1 \text{ such that } s_n := p_n + q_n \leq (1 - \epsilon)^\ell p_{n - \ell} + q_{n - \ell} \text{ for every } n \geq \ell \text{ and } q_n \leq C(1 - \epsilon)^n + \sum_{k=1}^n (1 - \epsilon)^k p_{n - k}, \text{ for every } n. \)

Then there exists \( C > 0 \) and \( \delta = \delta(\epsilon) > 0 \) such that \( s_n \leq C(1 - \delta)^n, \) for every \( n \in \mathbb{N} \).

**Proof.** If \( n \geq \ell \), we have \( s_n \leq (1 - \epsilon)^\ell p_{n - \ell} + q_{n - \ell} = (1 - \epsilon)s_{n - \ell} + \epsilon q_{n - \ell}. \) It follows by induction that if \( n = i\ell + r \), with \( r < \ell \), then

\[ s_n \leq (1 - \epsilon)^\ell s_r + \sum_{k=0}^{i-1} \epsilon (1 - \epsilon)^k q_{n - (k+1)\ell} \]

\[ \leq C(1 - \epsilon)^n / \ell s_r + \sum_{k=0}^{n-\ell} \epsilon (1 - \epsilon)^k q_{n - \ell - k} \]
Since $q_{n-\ell} \leq C(1-\epsilon)^{n-\ell} + \sum_{k=1}^{n-\ell} (1-\epsilon)^{k}p_{n-\ell-k}$, we obtain

$$s_n \leq C(1-\epsilon)^{n/\ell} s_r + C\epsilon (1-\epsilon)^{n/\ell} + \sum_{k=1}^{n-\ell} \epsilon (1-\epsilon)^{k} (p_{n-\ell-k} + q_{n-\ell-k}) \leq (1-\epsilon)^{n/\ell} C(s_r + \epsilon) + \sum_{k=1}^{n-\ell} \epsilon (1-\epsilon)^{k} s_{n-\ell-k},$$

for every $n \geq \ell$.

We claim that there exists $\delta < 1$ and $K$ so that $s_n \leq K(1-\delta)^n$, for every $n$. Indeed, fix $\delta < 1$. For each $n$, define $K_n := s_n/(1-\delta)^n$. Note that

$$s_n \leq (1-\epsilon)^{n/\ell} C(s_r + \epsilon) + \sum_{k=1}^{n-\ell} \epsilon (1-\epsilon)^{k} K_{n-\ell-k}(1-\delta)^{n-\ell-k}$$

(80)

$$\leq \left[\frac{(1-\delta)^{1/\ell}}{1-\delta}\right]^n C \max_j s_j + \epsilon \max_{i < n-\ell} K_i \left(\frac{\epsilon}{1-\delta}\right)^\ell \sum_{k=1}^{n-\ell} \left(\frac{1-\epsilon}{1-\delta}\right)^k (1-\delta)^n$$

Choose $\delta > 0$ close enough to 0 so that

$$\sigma_1 := \frac{(1-\epsilon)^{1/\ell}}{1-\delta} < 1,$$

and

$$\sigma_2 := \frac{\epsilon}{1-\delta} \sum_{k=1}^{\infty} \left(\frac{1-\epsilon}{1-\delta}\right)^k < 1.$$

Then by Eq. (7.2) we have $K_n \leq \sigma_2 \max_{i < n-\ell} K_i + C \sigma_1^n$, for every $n > \ell$, which easily implies that max, $K_i < \infty$.

Define

$$\Omega_{+}^{n_1,n} := \{p \in I_{n_1} : \pi_2(G^k(p)) \geq 0, \text{ for } 0 \leq k \leq n\}.$$

Lemma 7.11. Let $G \in On + Ra + Rb$ be a random walk that is an asymptotically small perturbation of a homogeneous random walk $F \in On + Ra + Rb$ with negative mean drift. Then there exists $\delta < 1$ so that for every $n_1 \geq 0$ there exists $C = C(n_1)$ satisfying

$$m(\Omega_{+}^{n_1,n}(G)) \leq C (1-\delta)^n.$$

Proof. Take $n_0$ as in Lemma 7.9 and fix $n_1 \geq 0$. Define the sets and sequences

$$s_n := m(\Omega_{+}^{n_1,n}),$$

$$p_n := m(B^n), \text{ where } B^n := \{p \in \Omega_{+}^{n_1,n} : \pi_2(G^n(p)) \in [0, n_0]\}, \text{ and }$$

$$q_n := m(C^n), \text{ where } C^n := \{p \in \Omega_{+}^{n_1,n} : \pi_2(G^n(p)) > n_0\}.$$

To prove Lemma 7.11, it is enough to verify that these sequences satisfy the assumptions of Lemma 7.10. Indeed, of course $p_0 + q_0 \leq 1$. To prove the other
assumptions, take $i \in [0, n_0]$. Since $G$ is topologically transitive, there are $\ell_i \in \mathbb{N}$ and intervals $J_i \subset I_i$ so that $\pi_2(G^\ell_i(J_i)) < 0$. Denote $\ell = \max_{0 \leq i \leq n_0} \ell_i$ and $r = \min_{0 \leq i \leq n_0} |J_i|/|I_i|$.

Clearly $\Omega_{n_1}^{G,n} = B^n \cup C^n \subset B^{n-\ell} \cup C^{n-\ell}$. Let $J \subset B^{n-\ell}$ be an interval so that $G^{n-\ell}(J) = I_i$, with $0 \leq i \leq n_0$. Note that $B^{n-\ell}$ is a disjoint union of such intervals. By the bounded distortion control for $G$,

$$m(J \cap \Omega_{n_1}^{G,n}) \leq 1 - \frac{m(J \cap G^{-(n-\ell)}J_i)}{m(J)} \leq (1 - \frac{r}{c})$$

(81)

Choose $\epsilon_0$ satisfying $(1 - r/c) \leq (1 - \epsilon_0)^{\frac{1}{\ell}}$. Then Eq. (81) implies

$$m(B^{n-\ell} \cap \Omega_{n_1}^{G,n}) \leq (1 - \epsilon_0)^{\frac{1}{\ell}} m(B^{n-\ell})$$

and we obtain

$$s_n = m(B^{n-\ell} \cap \Omega_{n_1}^{G,n}) + m(C^{n-\ell} \cap \Omega_{n_1}^{G,n}) \leq (1 - \epsilon_0)^{\frac{1}{\ell}} p_{n-\ell} + s_{n-\ell}.$$

It remains to prove that $q_n \leq \sum_{k=1}^{n} (1 - \epsilon)^k p_{n-k}$. There are two kind of points $p$ in $C^n$:

Type 1. For every $j \leq n$ we have $\pi_2(G^j(p)) \geq n_0$ (in particular $n_1 \geq n_0$). We are going to estimate the measure of the set of these points, denoted $\Theta_1^n$. It follows from Lemma 7.9, choosing $\alpha = \int \psi \, d\mu/2 < 0$, that

$$m(\Theta_1^n) \leq C_{n_1} \sigma^n,$$

for some $\sigma < 1$ which does not depend on $n_1$.

Type 2. For some $j \leq n$ we have $\pi_2(G^j(p)) \leq n_0$. Denote by $\Theta_2^n$ the set of points $p$ so that $k \geq 1$ is the smallest natural satisfying $\pi_2(G^{n-k}p) \leq n_0$. Clearly $\Theta_2^n$ is a disjoint union of these sets. We are going to estimate their measure. Note that $\Theta_2^n \subset B^{n-k}$. The set $B^{n-k}$ is a disjoint union of intervals $L$ so that $\pi_2(G^{n-k}L) = I_i$, for some $i \leq n_0$. To estimate

$$\frac{m(\Theta_2^n \cap L)}{|L|},$$

note that $L \subset B^{n-k}$, and $\Theta_2^n \cap L$ is the set of points $p \in L$ so that $\pi_2(G^{n-k+j}p) > n_0$, for every $0 \leq j \leq k$. Define

$L_y := \{p \in L : \psi(G^{n-k}p) = y\}.$

Firstly note that for $y \leq n_0 - i$ we have

$$|L_y \cap \Theta_2^n| = 0,$$

(83)

since $p \in L_y \cap \Theta_2^n$ satisfies $\pi_2(G^{n-k+j}p) = i + \psi(G^{n-k}p) = i + y > n_0$. In particular for $y < 0$ we have $|L_y \cap \Theta_2^n| = 0$, which implies, due the bounded distortion control

$$\frac{m(L \cap \Theta_2^n)}{|L|} \leq \sum_{y \geq 0} |L_y| \leq (1 - \delta),$$

(84)
for some $\delta < 1$ which does not depends on $k$, $L$ or $n_1$, which implies

\[
(84) \quad m(\Theta^n_{2,k}) \leq (1 - \delta)m(B^{n-k}) = (1 - \delta)p_{n-k}.
\]

Furthermore, using again the distortion control and the regularity condition $GD$ (big jumps are rare) we have

\[
\sum_{y > -\alpha(k-1)} |L_y \cap \Theta^n_{2,k}| / |L| \leq C \gamma^k,
\]

for some $C \geq 0$ and $\gamma < 1$.

To estimate $|L_y \cap \Theta^n_{2,k}| / |L|$, in the case $n_0 - i \leq y \leq -\alpha(k-1)$, recall that $G^{n-k+1}L_y = I_{i+y}$, with $i + y > n_0$. By Lemma 7.9, we have

\[
m\{p \in I_{i+y}: \pi_2(G^m(p)) \geq n_0, \text{ for } m \leq k-1, \text{ and } \pi_2(G^{k-1}(p)) \geq i+y+\alpha(k-1)\} \leq C\sigma^k.
\]

Since $i + y + \alpha(k-1) \leq n_0$, this implies that

\[
m\{p \in I_{i+y}: \pi_2(G^m(p)) \geq n_0, \text{ for every } m \leq k-1\} \leq C\sigma^k.
\]

The points in $L_y \cap \Theta^n_{2,k}$ are exactly the points whose $(n - k + 1)$th-iteration belongs to the set in the estimate above. Using the bound distortion control we have

\[
|L_y \cap \Theta^n_{2,k}| / |L_y| \leq C\sigma^k,
\]

so

\[
\frac{\sum_{n_0-i \leq y \leq -\alpha(k-1)} |L_y \cap \Theta^n_{2,k}|}{|L|} \leq C \frac{\sum_{n_0-i \leq y \leq -\alpha(k-1)} |L_y \cap \Theta^n_{2,k}|}{|L_y|} \leq C\sigma^k.
\]

Choose $\epsilon < \epsilon_0$ so that $\min\{\max\{C\sigma^k, C\gamma^k\}, 1 - \delta\} \leq (1 - \epsilon)^k$, for every $k \geq 0$, and put together Eq. (83), Eq. (84), Eq. (85) and Eq. (86), to get $m(L \cap \Theta^n_{2,k}) \leq (1 - \epsilon)^k|L|$. Since $B^{n-k}$ is a disjoint union of such intervals $L$, we obtain

\[
m(\Theta^n_{2,k}) \leq (1 - \epsilon)^k m(B^{n-k}) = (1 - \epsilon)^k p_{n-k}
\]

and now we can conclude with

\[
q_a = m(\Theta^n) + \sum_k m(\Theta^n_{2,k}) \leq C_n \sigma^n + \sum_k (1 - \epsilon)^k p_{n-k}.
\]

\[\square\]

**Proof of Theorem 7.** There are three cases:

* $F$ is transient with $M > 0$. If $M > 0$ then the random walk $F$ is transient and it is easy to see (using for instance Proposition 4.1) that $m(\Omega_+(F)) > 0$. Since the conjugacy with an asymptotically small perturbation $G$ is absolutely continuous (Theorem 2), we conclude that $m(\Omega_+(G)) > 0$.

* $F$ is recurrent ($M = 0$). If $M = 0$ then $F$ is recurent $[G]$ and its asymptotically small perturbations are recurrent by Theorem 4. In particular almost every point visits negative states infinitely many times, so $m(\Omega_+(G)) = 0$. It remains to prove
that $HD \Omega_+(G) = 1$. By Theorem 6 it is enough to verify that $HD \Omega_+(F) = 1$. Indeed, it is easy to show using the Central Limit Theorem that if

$$\int \psi \, d\mu = 0$$

then there exist $C > 0$ and for each $n$, subsets $\mathcal{A}_n \subset \mathcal{P}_0^n$ so that

$$\sum_{i=0}^{n-1} \psi(f^i(x)) > 0$$

for all $x \in J \in \mathcal{A}_n$ and

(87) $$1 \geq m \left( \bigcup_{J \in \mathcal{A}_n} J \right) > C > 0.$$  

here $C$ does not depend on $n$. Of course we can assume that $\mathcal{A}_n$ is finite. Property $Ex$ implies that there exists $\theta \in (0, 1)$ such that

$$\sup_{J \in \mathcal{A}_n} |J| \leq \theta^n.$$  

Consider the function

$$h(\epsilon) := \sum_{J \in \mathcal{A}_n} |J|^{1-\epsilon}.$$  

Then by Eq. (87) if $0 \leq \epsilon < 1$ we have

$$h'(\epsilon) := \sum_{J \in \mathcal{A}_n} -\log |J||J|^{1-\epsilon} \geq -Cn \log \theta.$$  

In particular if

$$\tilde{\epsilon} := \frac{C - 1}{Cn \log \theta}$$

then $h(\tilde{\epsilon}) \geq 1$. Since $h(0) \leq 1$ there exist $\epsilon_n = 1 - O(1/n)$ such that $h(\epsilon_n) = 1$. But $VHD(\mathcal{A}_n) = \epsilon_n$, so

$$|VHD(\mathcal{A}_n) - 1| \leq \frac{C}{n}.$$  

By property $BD$ that there exists $C_1 > 0$ such that for every $n$

$$d_n := \sup_{C \in \mathcal{A}_n} \sup_{x, y \in C} \log \frac{Df_{\mathcal{A}_n}(y)}{Df_{\mathcal{A}_n}(x)} \leq C_1$$

and since $\mathcal{A}_n \subset \mathcal{P}_0^n$, by property $Ex$ we have that there exists $\theta \in (0, 1)$ such that for every $n$

$$\lambda_n := \inf_{C \in I} \inf_{x \in C} |Df(x)| \geq \frac{1}{\theta^n},$$

we can apply Proposition 7.4 to obtain

$$|HD \Lambda(\mathcal{A}_n) - VHD(\mathcal{A}_n)| = O\left(\frac{1}{n}\right).$$

so

$$HD(\mathcal{A}_n) = 1 - O\left(\frac{1}{n}\right).$$

If $\mu_{\mathcal{A}_n}$ is the geometric invariant measure of $f_{\mathcal{A}_n}$ then

$$\int \psi_{\mathcal{A}_n} \, d\mu_{\mathcal{A}_n} > 0.$$  

So by the Birkhoff Ergodic Theorem
\[
\lim_{n \to \infty} \sum_{i=0}^{n-1} \psi(f^i(x)) = +\infty
\]

in a set \( S_n \subset A_n \) satisfying \( \mu_{A_n}(S_n) = 1 \), so \( HD S_n = 1 - O(1/n) \). In particular the set \( S \) of points satisfying Eq. (88) has Hausdorff dimension 1. We can decompose \( S \) in subsets \( B_j \) defined by

\[
B_j := \{ x \in S : \min_{n} \sum_{i=0}^{n-1} \psi(f^i(x)) \geq -j \}.
\]

Clearly \( \sup_j HD B_j = 1 \).

By properties \( GD + On \), for each \( j \) there are \( k_j \) and \( J_j \neq \emptyset \in P^{k_j} \) so that for all \( x \in J_j \) we have

\[
\sum_{i=0}^{k_j} \psi(f^i(x)) \geq j.
\]

Then

\[
(J_j \cap f^{-k_j} B_j) \times \{0\}
\]

belongs to \( \Omega_+(F) \), for every \( j \). This implies \( HD \Omega_+(F) \geq HD B_j \) so

\[
HD \Omega_+(F) \geq \sup_j HD B_j = 1.
\]

\( F \) is transient with \( M < 0 \). By Lemma 7.11, there is some \( \delta \in (0, 1) \), which does not depend on \( n_1 \), so that

\[
m(\Omega_+^{n_1:n}) \leq C(1 - \delta)^n.
\]

By Lemma 7.8, there exists \( \epsilon \) so that

\[
\sum_{P \in P^n, \ P \subset I_k} |P|^{1-\epsilon} \leq C(1 - \delta)^{n/2}.
\]

Denote by \( \{J_i^n\}_i \subset P^n \) the family of disjoint intervals so that \( \Omega_+^{n_1:n} = \bigcup_i J_i^n \). We claim that there exists \( C > 0 \) satisfying

\[
\sum_i |J_i^n|^{1-\epsilon/4} \leq C(1 - \delta)^n.
\]

Since \( \sup_i |J_i^n| \to_n 0 \), this proves that \( HD \Omega_+^{n_1:\infty} \leq 1 - \epsilon/4 \).

Indeed,

\[
\sum_i |J_i^n|^{1-\epsilon/4} = \sum_{|J_i| > (1 - \delta)^{2n/\epsilon}} |J_i^n|^{1-\epsilon/4} + \sum_{|J_i| \leq (1 - \delta)^{2n/\epsilon}} |J_i^n|^{1-\epsilon/4}
\]
\[ \leq (1 - \delta)^{n/2} \sum_i |J_i^n| + (1 - \delta)^{3n/2} \sum_i |J_i^n|^{1-\epsilon} \]
\[ \leq C(1 - \delta)^{n/2}, \]
where in the last line we made use of Eq. (89) and Eq. (90). The proof is complete. \( \square \)

8. Applications to one-dimensional renormalization theory

8.1. (Classic) infinitely renormalizable maps. Denote \( I = [-1, 1] \). Consider a real analytic unimodal maps \( f: I \to I \), with negative Schwarzian derivative and even order critical point at 0. The map \( f \) is called infinitely renormalizable if there exists an sequence of natural numbers \( n_0 < n_1 < n_2 < \ldots \) and a nested sequence of intervals

\[ I = I_0 \supset I_1 \supset I_2 \supset \cdots \]

so that

- \( f^{n_k} \partial I_k \subset \partial I_k \),
- \( f^{n_k} I_k \subset I_k \),
- \( f^{n_k} : I_k \to I_k \) is a unimodal map.

We say that \( f \) has bounded combinatorics if there exists \( C > 0 \) so that \( n_{k+1}/n_k \leq C \), for all \( k \). Two infinitely renormalizable maps \( f \) and \( g \) have the same combinatorics if there exists a homeomorphism \( h: I \to I \) such that \( f \circ h = h \circ g \).

The following result is a deep result in renormalization theory:

**Proposition 8.1 ([McM96]).** Let \( f \) and \( g \) be two infinitely renormalizable unimodal maps with the same bounded combinatorics and same even order. Then for every \( r > 0 \) there exists \( C > 0 \) and \( \lambda < 1 \) so that

\[ \left\| \frac{1}{|I_k^f|} f^{n_k}(|I_k^f|) - \frac{1}{|I_k^g|} g^{n_k}(|I_k^g|) \right\| \leq C \lambda^k. \]

Here \( |I_k^f| \) denotes the length of \( I_k^f \).

**Proof of Theorem 8.** Let \( f \) be an infinitely renormalizable map with bounded combinatorics. We are going to define an induced map \( F: I \to I \), following Y. Jiang (see [31, 32]): Let \( p_k \) be the periodic point in \( \partial I_k \). Define \( E \) as the set

\[ \{1, -1, -p_k, p_k, f(p_k), -f(p_k), \ldots, f^{n_k-1}(p_k), -f^{n_k-1}(p_k)\} - \{f(p_k), -f(p_k)\}. \]

The set \( E \) cuts \( I_{k-1} \setminus I_k \) in \( m_k \) intervals. Denote these intervals \( M_{k-1, i} \) with \( i = 1, \ldots, m_k \). For each \( x \in M_{k-1, i} \), define \( n(x) \geq 1 \) as the minimal positive integer so that

\[ I_k \subset f^{n(x)n_{k-1}}(M_{k-1, 1}). \]

Note that \( f^{n(x)n_{k-1}} \) does not have critical points on \( M_{k-1, 1} \). Define the induced map \( F \), which is defined everywhere in \( I \), except for a countable set of points:

\[ F(x) := f^{n(x)}(x), \text{ for } x \in I_k \setminus I_{k+1}. \]

See in Fig. 2 the induced map for an infinitely renormalizable maps satisfying \( n_{i+1} = 2n_i \) for all \( i \) (the so called Feigenbaum maps). The map \( F \) is Markovian with respect to the partition

\[ \mathcal{P} := \{M_{k, i}\}_{k \in \mathbb{N}, i \leq m_k}. \]
Figure 2. The "Bat" map: the induced map $F$ for a Feigenbaum
unimodal map

Furthermore, if $f$ and $g$ have the same bounded combinatorics and even order, then
by Proposition 8.1, the corresponding induced maps $F$ and $G$ satisfies

$$|| \frac{1}{|I_k^f|} F(|M_{k,i}^f| \cdot |I_k^f|) - \frac{1}{|I_k^g|} G(|M_{k,i}^g| \cdot |I_k^g|) ||_{C^r([0,1])} \leq C \lambda^k.$$  

Define $L_k$ as, say, the right component of $I_k \setminus I_{k+1}$ and $\gamma_k : I \to L_k$ as the unique
bijective order preserving affine map between this two intervals. We are going to
define a random walk $F : I \times \mathbb{N} \to I \times \mathbb{N}$ from the map $F$ in the following way:

$$F(x,k) := \begin{cases} 
\gamma_i^{-1} \circ F \circ \gamma_k(x), i & \text{if } F \circ \gamma_k(x) \in L_i; \\
\gamma_i^{-1} \circ (-F) \circ \gamma_k(x), i & \text{if } F \circ \gamma_k(x) \in -L_i.
\end{cases}$$

(92)

It is easy to see that we can extend $F : I \times \mathbb{Z} \to I \times \mathbb{Z}$ to a strongly transient
deterministic random walk with non-negative drift. Indeed if $k < 0$ define

$$F(x,k) := \begin{cases} 
\gamma_i^{-1} \circ F \circ \gamma_0(x), k + i & \text{if } F \circ \gamma_k(x) \in L_i; \\
\gamma_i^{-1} \circ (-F) \circ \gamma_0(x), k + i & \text{if } F \circ \gamma_k(x) \in -L_i.
\end{cases}$$

Furthermore if $g$ is another infinitely renormalizable map with the combinatorics
of $f$ then by Proposition 8.1 and Proposition 4.9 we can define the corresponding
random walk $G : I \times \mathbb{N} \to I \times \mathbb{N}$ and extend this to a random walk $G : I \times \mathbb{Z} \to I \times \mathbb{Z}$
defining $G(x,k) = F(x,k)$ if $k < 0$. Then $G$ is an asymptotically small perturbation
of $F$. So we can apply Theorem 3 to conclude that there is a conjugacy between
F and G which is strongly quasisymmetric with respect to the nested sequence of partitions defined by the random walk F. We can now easily translate this result in terms of the original unimodal maps f and g saying that the continuous conjugacy h between f and g is a strongly quasisymmetric mapping with respect to P. □

**Remark 8.2.** An interesting case is when the unimodal map f is a periodic point to the renormalization operator: there exists $n_0$ and $\lambda$, with $|\lambda| < 1$ so that

$$ \frac{1}{\lambda} f^{n_0}(\lambda x) = f(x). $$

In this case, if we take $n_k = kn_0$, then the induced map F will satisfy the functional equation

(93) \[ F(\lambda x) = \lambda F(x). \]

Define the relation $\sim$ in the following way:

$$ x \sim y \text{ iff there exists } i \in \mathbb{Z} \text{ so that } x = \pm \lambda^i y. $$

By Eq. (93), $F$ preserves this relation, so we can take the quotient of $F$ by the relation $\sim$. Note that

$$ L_0 = \mathbb{R}^*/\sim. $$

It is easy to see that $q = F/\sim: L_0 \to L_0$ is a Markov expanding map. Now define $\psi: L_0 \to \mathbb{Z}$ as $\psi(x) = k$, if $f(x) \in I_k \setminus I_{k+1}$. Then $F$ is exactly the homogeneous random walk defined by the pair $(q, \psi)$.

### 8.2. Fibonacci maps

The Fibonacci renormalization is the simplest way to generalize the concept of classical renormalization as described in Section 8.1. Actually we could prove all the results stated for Fibonacci maps to a wider class of maps: maps which are infinitely renormalizable in the generalized sense and with periodic combinatorics and bounded geometry, but we will keep ourselves in the simplest case to avoid more technical definitions and auxiliary results with its long proofs.

Consider the class of real analytic maps $f$ with $Sf < 0$ and defined in a disjoint union of intervals $I^0_1 \sqcup I^1_1$, where $-I^0_1 = I^1_0$, so that

- The map $f: I^0_1 \to I^0_1 := f(I^1_1)$ is a diffeomorphism. Furthermore $I^1_1$ is compactly contained in $I^0_1$.

- The map $f: I^0_1 \to I^0_0$ is an even map which has as 0 as its unique critical point of even order.

We say that $f$ is **Fibonacci renormalizable** if

$$ f(0) \in I^1_1, \quad f^2(0) \in I^0_0 \text{ and } f^3(0) \in I^1_0. $$

In this case, the Fibonacci renormalization of $f$ is defined as the first return map to the interval $I^0_1$ restricted to the connected components of its domain which contain the points $f(0)$ and $f^2(0)$. This new map is denoted $Rf$: it could be Fibonacci renormalizable again and so on, obtaining an infinite sequence of renormalizations $Rf, R^2f, R^3f, \ldots$.

We will denote the set of infinitely renormalizable maps in the Fibonacci sense with a critical point of order $d$ by $F_d$. A map $f \in F_d$ will be called a **Fibonacci map**.
Figure 3. On the left figure the solid curves represents the part of the $f^S_n$ used in the definition of the induced map. On the right figure the solid curve is the part of $f^S_n$ which coincides with the $n$-th Fibonacci renormalization on its central domain.

As in the original map $f$, the $n$-th renormalization $f_n := R^n f$ of $f$ is a map defined in two disjoint intervals, denoted $I^0_n$ and $I^1_n$, where $-I^0_n = I^1_n$. Indeed $f_n$ on $I^0_n$ is a unimodal restriction of the $S_n$-th iteration of $f$, where $\{S_n\}$ is the Fibonacci sequence

$S_0 = 1$, $S_1 = 2$, $S_2 = 3$, $S_3 = 5$, $\ldots$, $S_{k+2} = S_{k+1} + S_k$, $\ldots$

and $f_n$ on $I^1_n$ is the restriction of the $S_{n-1}$-th iteration of $f$.

Denote by $p_k$ the sequence of points $p_k \in \partial I^k_0$ so that

$$f_k(p_{k+1}) = p_k$$

and denote $I^k_0 = [p_k, p'_k]$.

It is possible to define a sequence $u_k$ of points satisfying

1. $\cdots < p_{k+1} < u_k < p_k < \cdots < p_0$,
2. $f^S_k$ is monotone on $[0, u_k]$,
3. $f^S_k(u_{k+1}) = u_k$,
4. $f^S_k(u_k) = u_{k-2}$.

We are going to define an induced map for an infinitely renormalizable map in the Fibonacci sense in the following way: Firstly, define $f_{-1}: I^0_0 \setminus I^1_0$ as an $C^3$ monotone extension of $f_0$ on $I^1_0$ which has negative Schwarzian derivative and bounded distortion. Define $F: I^0_0 \to \mathbb{R}$ as

$$F(x) := f^S_i(x) \text{ if } x \in [u_i, -u_i] \setminus [u_{i+1}, -u_{i+1}]$$
for each $i \geq 0$.

Define $L_i$ as, say, the right component of $[u_i, -u_i] \setminus [u_{i+1}, -u_{i+1}]$ and $\gamma_i : I \to L_i$ as the unique bijective order preserving affine map between these two intervals.

We are ready to define the map $\mathcal{F} : I \times (\mathbb{N} \setminus \{0\}) \to I \times \mathbb{N}$ as

$$\mathcal{F}(x, k) := \begin{cases} \gamma_i^{-1} \circ F \circ \gamma_k(x), i & \text{if } F \circ \gamma_k(x) \in L_i, \\ \gamma_i^{-1} \circ (-F) \circ \gamma_k(x), i & \text{if } F \circ \gamma_k(x) \in -L_i. \end{cases}$$

If the order of the critical point is even and larger than two then there is a very special Fibonacci map $F^*$, called the Fibonacci fixed point (see, for instance [Smi]), whose induced map $F^*$ satisfies (choosing a good $u_0$)

$$F^*(\lambda x) = \pm \lambda F^*(x)$$

(94)

for some $\lambda \in (0, 1)$. In this case we can use the argument in Remark 8.2 to conclude that the corresponding map $\mathcal{F}^* : I \times (\mathbb{N} \setminus \{0\}) \to I \times \mathbb{N}$ can be extended to a
homogeneous random walk $\mathcal{F}^*: I \times \mathbb{Z} \to I \times \mathbb{Z}$. For an arbitrary Fibonacci map $f$, we can extend $\mathcal{F}: I \times (\mathbb{N} \setminus \{0\}) \to I \times \mathbb{N}$ to a random walk $\mathcal{F}: I \times \mathbb{Z} \to I \times \mathbb{Z}$ defining $\mathcal{F}(x, k) = \mathcal{F}^*(x, k)$ for $k \leq 0$. Then $\mathcal{F}$ is not homogeneous, however due Proposition 4.9 and the following result $\mathcal{F}$ is an asymptotically small perturbation of $\mathcal{F}^*$:

**Proposition 8.3** (see [Sm]). For each even integer larger than two the following holds: for every Fibonacci map $f$, denote

$$g_i = \alpha_i^{-1} \circ f^s_i \circ \alpha_i + 1: I \to I,$$

where $\alpha_i: I \to [u_i^f, -u_i^f]$ is an bijective affine map so that $\alpha_i^{-1}(f_{i+1}(0)) > 0$ and consider the correspondent maps $g_i^* \in \mathcal{F}^*$. Then

$$||g_i - g_i^*||_{C^r} \leq K_r \rho$$

for some $\rho < 1$ and every $r \in \mathbb{N}$.

The real Julia set of $f$, denoted $J_\mathbb{R}(f)$, is the maximal invariant of the map

$$f: I_0^+ \cup I_1^+ \to I_0^+,$$

in other words,

$$J_\mathbb{R}(f_j) := \cap_i f_j^{-i}I_0^+.$$

Denote

$$\Omega_+^j(F) := \{(x, i) \text{ s.t. } \pi_2(F^n(x, i)) \geq j \text{ for all } n \geq 0\}.$$

**Proposition 8.4.** There exists some $k_0$ so that

$$\Omega_+^{j+1}(F) \subseteq J_\mathbb{R}(f_j) \subseteq \Omega_+^{j-1}(F).$$

In particular

$$HD \ \Omega_+^{j+1}(F) \leq HD \ J_\mathbb{R}(f_j) \leq HD \ \Omega_+^{j-1}(F),$$

and, for the Fibonacci fixed point, since $\Omega_+^{j+1}(F)$ is an affine copy of $\Omega_+^{j-1}(F)$ we have

$$HD \ \Omega_+^j(F) = HD \ J_\mathbb{R}(f).$$

for all $j \geq 0$.

**Proof.** Denote by $F_\ell$ the restriction of $F$ to $\cup_{i \geq \ell}L_i$. Then the maximal invariant set of $F_\ell$

$$\Lambda(F_\ell) := \cap_{i \in \mathbb{N}} F_\ell^{-i}\mathbb{R}$$

is $\Omega_+^\ell(F)$. Consider the extension of $f_j$ described in Fig. (4). Let’s call this extension $\tilde{f}_j$. An easy analysis of its graph shows that $f_j$ and $\tilde{f}_j$ have the same maximal invariant set. We claim that $\tilde{f}_{j+1}$ is just a map induced by $\tilde{f}_j$. Indeed, the restriction of $\tilde{f}_{j+1}$ to $[u_{j+1}, u_{j+1}']$ coincides with $\tilde{f}_j^2$ on the same interval. On the rest of $\tilde{f}_{j+1}$-domain $\tilde{f}_{j+1}$ coincides with $\tilde{f}_j$.

By consequence, for $i \geq j$ the map $\tilde{f}_i$ is induced by $\tilde{f}_j$ and, since $F_{j+1}$ restricted to $L_i$ is equal to $\tilde{f}_i$, we obtain that $F_{j+1}$ is a map induced by $\tilde{f}_j$. In particular

$$\Lambda(F_{j+1}) \subseteq \Lambda(\tilde{f}_j) = J_\mathbb{R}(f_j).$$

To prove that $\Lambda(\tilde{f}_j) \subseteq \Lambda(F_{j-1})$, we are going to prove that

$$x \in \Lambda(\tilde{f}_j) \implies F_{j-1}(x) \in \Lambda(\tilde{f}_j).$$

(97)
If \( x \) belongs to the interval \( I_j^1 \subset L_{j-1} \), where \( \tilde{f}_j \) coincides with \( F_{j-1} \), then \( F_{j-1}(x) \in \Lambda(\tilde{f}_j) \). Otherwise \( x \in \mathcal{F}_j^0 \subset \bigcup_{i \geq j} L_i \), so \( x \in \Lambda(\tilde{f}_j) \cap L_i \), for some \( i \geq j \), then \( F_{j-1} \) is an iteration of \( \tilde{f}_j \) on \( L_i \), so \( F_{j-1}(x) \in \Lambda(\tilde{f}_j) \). This finishes the proof of Eq. (97). Since \( \Lambda(\tilde{f}_j) \) is invariant by the action of \( F_{j-1} \) we have \( \Lambda(\tilde{f}_j) \subset \Lambda(F_{j-1}) \). \( \square \)

**Proof of Theorem 9.** Consider the homogeneous random walk \( F^* = (g, \psi) \) induced by \( f^* \). Denote

\[
M = \int \psi \, d\mu,
\]

where \( \mu \) is the absolutely continuous invariant measure of \( g \). Using Theorem 7, there are three cases:

1. **\( M < 0 \).** In this case \( F^* \) is transient and we have that \( HD \, \Omega_+(F) < 1 \) for every asymptotically small perturbation of \( F^* \), in particular when \( F \) is a random walk induced by a Fibonacci map \( f \). By Proposition 8.4, \( HD \, J_R(f) < 1 \).

2. **\( M = 0 \).** Then \( F^* \) is recurrent \([G]\) so every asymptotically small perturbation \( G \) of \( F^* \) is recurrent and \( m(\Omega_+(G)) = 0 \) but \( HD \, \Omega_+(G) = 1 \). By Proposition 8.4 we obtain \( m(J_R(f)) = 0 \) and \( HD \, J_R(f) = 1 \).

3. **\( M > 0 \).** In this case \( F^* \) is transient with \( m(\Omega_+(F^*)) > 0 \) and the conjugacy between \( F^* \) and any asymptotically small perturbation of it is absolutely continuous.
on $\Omega_1^d(F^*)$. In particular $m(\Omega_+(F)) > 0$ for every random walk $F$ induced by a Fibonacci map $f$ so $m(J_R(f)) > 0$ by Proposition 8.4.

A map $f : I \to I$ is called a unimodal map if $f$ has a unique critical point, with even order $d$, which is a maximum, and $f(\partial I) \subset \partial I$. We will assume that $f$ is real analytic, symmetric with respect to the critical point and $Sf < 0$. If the critical value is high enough, then $f$ has a reversing fixed point $p$. Let $I_0^p := [-p, p]$. Consider the map of first return $R$ to $f$: if $x \in I_0^p$ and $f^r(x) \in I_0^p$, but $f^n(x) \not\in I_0^p$ for $i < r$, define

$$R(x) := f^r(x).$$

If there exists exactly two connected components $I_0^p$ and $I_1^p$ of the domain of $R$ containing points in the orbit of the critical point, and furthermore the map

$$R: I_0^p \cup I_1^p \to I_0^p$$

is a Fibonacci map, then we will call $f$ an unimodal Fibonacci map. The class of all unimodal Fibonacci maps will be denoted $\mathcal{F}_d^{uni}$.

**Proof of Theorem 10.** We will use the notation in the proof of Theorem 9. Since $m(J_R(f)) > 0$, we conclude that the mean drift $M$ of $F^*$ is positive. By Proposition 5.1 any asymptotically small perturbation $G$ of $F^*$ has the following property: there exists $\lambda \in [0, 1)$, $C > 0$ and $K > 0$ so that for every $P \in \mathcal{P}^0(G)$

$$m(p \in P : \sum_{i=0}^{n-1} \psi(G^i(p)) < Kn) \leq C\lambda^n|P|.$$  

This implies that

$$m(p \in I_j : \sum_{i=0}^{\ell} \psi(G^i(p)) \geq K\ell \text{ for every } \ell \geq n) \geq (1 - C\lambda^n).$$

So if $j = n|\min \psi|$ we obtain

$$m(\Omega_j^d(G)) \geq 1 - C\lambda^{C_1j}.$$ 

here $C_1 > 0$. If $G$ is a random walk induced by a Fibonacci map $g$ then this implies that for $j$ large

$$m(L_j \setminus J_R(g)) = m((-L_j) \setminus J_R(g)) \leq C\lambda^{C_1j}|L_j|.$$ 

Since

$$[-u_{j+1}, u_{j+1}] = \bigcup_{i \geq j} L_i \cup (-L_i),$$

we conclude that

$$m([u_{j+1}, -u_{j+1}] \setminus J_R(g)) \leq C\lambda^{C_1j}|u_{j+1}|.$$ 

For every $\delta$, choose $j$ so that $|u_{j+2}| \leq \delta \leq |u_{j+1}|$. Because $|u_{j+2}| > \theta|u_{j+1}|$, where $\theta \in (0, 1)$ does not depend on $j$, we have that $|u_j| \geq C\theta^j$. Together with Eq. (98) this implies

$$m([-\delta, \delta] \setminus J_R(g)) \leq C\lambda^{C_1j}|u_{j+1}| \leq C|u_{j+1}|^{1+\alpha} \leq C|\delta|^{1+\alpha}.$$  

□
Proof of Theorem 11. We will prove each one of the following implications:

(1) implies (2): From the proof of Theorem 9, if \( m(J_R(f)) > 0 \) for some \( f \in \mathcal{F}_d \) the mean drift \( M \) of the homogeneous random walk \( \mathcal{F}^* \) of \( f^* \) is positive. So \( \mathcal{F}^* \) (and all its asymptotically small perturbations) is transient (to \( +\infty \)). In terms of the original Fibonacci map \( f \), this means that almost every orbit in \( J_R(f) \) accumulates in the post-critical set: So \( f \) has a wild attractor.

(2) implies (3): if there exists a wild attractor for \( f \) then \( m(J_R(f)) > 0 \). From the proof of Theorem 9 we obtain that the mean drift \( M \) of \( \mathcal{F}^* \) is positive. So there exists an absolutely continuous conjugacy between \( \mathcal{F}^* \) and any asymptotically small perturbation of \( \mathcal{F}^* \). This implies that any two maps \( f_1, f_2 \in \mathcal{F}_d \) admits a continuous and absolutely continuous conjugacy

\[
h : J_R(f_1) \to J_R(f_2).
\]

Now consider two arbitrary maps \( g_1, g_2 \in \mathcal{F}^{uni}_d \). Then we already know that there exists an absolutely continuous conjugacy

\[
h : J_R(R_{g_1}) \to J_R(R_{g_2})
\]

between the induced Fibonacci maps \( R_{g_1} \) and \( R_{g_2} \) associated to \( g_1 \) and \( g_2 \). Of course \( h \) is just the restriction of a topological conjugacy between \( g_1 \) and \( g_2 \). By a Blokh and Lyubich result [BL] (see also page 332 in [dMvS]), every map of \( \mathcal{F}^{uni}_d \) is ergodic with respect the Lebesgue measure. Since \( g_1 \) and \( g_2 \) have wild attractors, this implies that the orbit of almost every point \( x \in I \) hits \( J_R(R_{g_1}) \) at least once. Let \( n(x) \) be a time when this happens.

So consider a arbitrary measurable set \( B \subset I \) so that \( m(B) > 0 \). Then for at least one \( n_0 \in \mathbb{N} \) the set

\[
B_{n_0} := \{ x \in B : n(x) = n_0 \}
\]

has positive Lebesgue measure. This implies that \( f^{n_0}B_{n_0} \) has positive Lebesgue measure, so \( m(f^{n_0}B_{n_0}) > 0 \). Now it is easy to conclude that \( m(h(B_{n_0})) > 0 \) and \( h(B) > 0 \). Switching the places of \( g_1 \) and \( g_2 \) in this argument we can conclude that \( h \) is absolutely continuous on \( I \).

Finally note that the eigenvalues of the periodic points are not constant on the class \( \mathcal{F}^{uni}_d \).

(3) implies (4): By the argument in Martens and de Melo [MdM], if a Fibonacci map does not have a wild attractor then any continuous absolutely continuous conjugacy with other Fibonacci map is \( C^1 \); in particular the conjugacy preserves the eigenvalues of the periodic points. So if (3) holds then we can use the same argument in the proof of the previous implication to conclude that every Fibonacci map has a wild attractor.

(4) implies (5): The proof goes exactly as the proof of (2) \( \Rightarrow \) (3).

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E-mail address: gugu@impa.br
URL: *www.impa.br/~gugu/*

E-mail address: smania@icmc.usp.br
URL: *www.icmc.usp.br/~smania/*