PAIRS OF DIAGONAL QUARTIC FORMS: THE ASYMPTOTIC FORMULAE

JÖRG BRÜDERN AND TREVOR D. WOOLEY

Abstract. We establish an asymptotic formula for the number of integral solutions of bounded height for pairs of diagonal quartic equations in 26 or more variables. In certain cases, pairs in 25 variables can be handled.

1. Introduction

Once again we are concerned with the pair of Diophantine equations

\[ a_1 x_1^4 + a_2 x_2^4 + \ldots + a_s x_s^4 = b_1 x_1^4 + b_2 x_2^4 + \ldots + b_s x_s^4 = 0, \] (1.1)

wherein the given coefficients \(a_j, b_j\) satisfy \((a_j, b_j) \in \mathbb{Z}^2 \setminus \{(0, 0)\} \ (1 \leq j \leq s)\). While our focus was on the validity of the Hasse principle for such pairs in two precursors of this article \[6, 9\], we now investigate the asymptotic density of integral solutions. Denote by \(N(P)\) the number of solutions in integers \(x_j\) with \(|x_j| \leq P \ (1 \leq j \leq s)\) to this system. Then, subject to a natural rank condition on the coefficient matrix, one expects an asymptotic formula for \(N(P)\) to hold provided that \(s\) is not too small. Indeed, following Hardy and Littlewood \[11\] in spirit, the quantity \(P^{8-s}N(P)\) should tend to a limit that is itself a product of local densities. On a formal level, the densities are readily described. The real density, also known as the singular integral, is defined by

\[ J = \lim_{T \to \infty} \int_{-T}^{T} \int_{-T}^{T} \prod_{j=1}^{s} \int_{-1}^{1} e((a_j \alpha + b_j \beta)t_j^4) \, dt_j \, d\alpha \, d\beta \] (1.2)

whenever the limit exists. Let \(M(q)\) denote the number of solutions \(x\) in \((\mathbb{Z}/q\mathbb{Z})^s\) satisfying (1.1). Then for primes \(p\), the \(p\)-adic density is defined by

\[ s_p = \lim_{h \to \infty} p^{(2-s)h} M(p^h), \] (1.3)

assuming again that this limit exists. In case of convergence, the product \(S = \prod_p s_p\) is referred to as the singular series, and the desired asymptotic relation can be presented as the limit formula

\[ \lim_{P \to \infty} P^{8-s}N(P) = JS. \] (1.4)

Note that (1.4) can hold only when in each of the two equations comprising (1.1) there are sufficiently many non-zero coefficients. Of course one may pass
from (1.1) to an equivalent system obtained by taking linear combinations of the two constituent equations. Thus, the invariant \( q_0 = q_0(\mathbf{a}, \mathbf{b}) \), defined by

\[
q_0(\mathbf{a}, \mathbf{b}) = \min_{(c,d) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \text{card}\{1 \leq j \leq s : ca_j + db_j \neq 0\},
\]

must be reasonably large. Indeed, it follows from Lemmata 3.1, 3.2 and 3.3 in our companion paper [9] that the conditions \( s \geq 16 \) and \( q_0 \geq 12 \) ensure that the limits (1.2) and (1.3) all exist, that the product \( \mathfrak{S} \) is absolutely convergent, and that the existence of non-singular solutions to the system (1.1) in each completion of the rationals implies that \( \mathfrak{I} \mathfrak{S} > 0 \). A first result concerning the limit (1.4) is then obtained by introducing the moment estimate

\[
\int_{0}^{1} \left| \sum_{x \leq P} e(\alpha x^4) \right|^{14} \, d\alpha \ll P^{10+\varepsilon}, \quad (1.5)
\]

derived as the special case \( u = 14 \) of Lemma 5.3 below, to a familiar method of Cook [10] (see also [2]). Here we point out that the estimate (1.5) first occurs implicitly in the proof of [15, Theorem 4.1], conditional on the validity of the (now proven) main conjecture in Vinogradov’s mean value theorem (for which see [1] and [17, Corollary 1.3]). In this way, one routinely confirms (1.4) when \( s \geq 29 \) and \( q_0 \geq 15 \). This result, although not explicitly mentioned in the literature, is certainly familiar to experts in the area, and has to be considered as the state of the art today. It seems worth remarking in this context that, at a time when the estimate (1.5) was not yet available, the authors [3, 5] handled the case \( s \geq 29 \) with more restrictive rank conditions. The main purpose of this memoir is to make three variables redundant.

**Theorem 1.1.** For pairs of equations (1.1) with \( s \geq 26 \) and \( q_0 \geq 15 \), one has \( \mathcal{N}(P) \sim \mathfrak{I} \mathfrak{S} P^{s-8} \).

Relaxing the rank condition \( q_0 \geq 15 \) appears to be a difficult enterprise, as we now explain. Consider a pair of equations (1.1) with \( s \geq 29 \), and suppose that \( b_i = a_j = 0 \) for \( 1 \leq i \leq 14 < j \leq s \). These two equations are independent and thus \( \mathcal{N}(P) \) factorises as \( \mathcal{N}(P) = N_1(P)N_2(P) \), where \( N_1(P) \) and \( N_2(P) \) denote the number of integral solutions of the respective single equations

\[
a_1x_1^4 + a_2x_2^4 + \ldots + a_{14}x_{14}^4 = 0, \quad (1.6)
\]

with \( |x_j| \leq P \) (\( 1 \leq j \leq 14 \)), and

\[
b_{15}y_1^4 + b_{16}y_2^4 + \ldots + b_{s}y_{s-14}^4 = 0, \quad (1.7)
\]

with \( |y_j| \leq P \) (\( 1 \leq j \leq s - 14 \)). The equation (1.7) has at least 15 non-zero coefficients, and so a straightforward application of the Hardy-Littlewood method using the mean value (1.5) shows that \( P^{18-s}N_2(P) \) tends to a limit as \( P \to \infty \), with this limit equal to a product of local densities analogous to \( \mathfrak{I} \) and \( \mathfrak{s}_p \). By choosing \( b_j = (-1)^j \) for \( 15 \leq j \leq s \), we ensure that this limit is positive, and thus \( P^{8-s}\mathcal{N}(P) \) tends to a limit as \( P \to \infty \) if and only if \( P^{-10}N_1(P) \) likewise tends to a limit. From the definitions (1.2) and (1.3), it is apparent that the local densities \( \mathfrak{I} \) and \( \mathfrak{s}_p \) factorise into components stemming from the equations underlying \( N_1 \) and \( N_2 \). The relation (1.4) therefore holds for
this particular pair of equations if and only if $P^{-10}N_1(P)$ tends to the product of local densities associated with the equation (1.6). In particular, were (1.4) known to hold in any case where $q_0 = 14$ and $s$ is large, then it would follow that $P^{-10}N_1(P)$ tends to the limit suggested by a formal application of the circle method, a result that is not yet known. This shows that relaxing the condition on $q_0$ would imply progress with single diagonal quartic equations.

The invariant $q_0$ is a very rough measure for the entanglement of the two equations present in (1.1). This can be refined considerably. The pairs $(a_j, b_j)$ are all non-zero in $\mathbb{Z}^2$, so they define a point $(a_j : b_j) \in \mathbb{P}(\mathbb{Q})$. Suppose that there are $\nu$ equivalence classes with $r_1, \ldots, r_\nu$ elements, respectively, where $r_1 \geq r_2 \geq \ldots \geq r_\nu$. On an earlier occasion [5] we named the tuple $(r_1, \ldots, r_\nu)$ the profile of the equations (1.1). Note that $q_0 = s - r_1$, whence our assumed lower bound $q_0 \geq 15$ implies that $r_1 \leq s - 15$ and $\nu \geq 2$. If more is known about the profile, then we can save yet another variable.

**Theorem 1.2.** Suppose that $s = 25$ and that $(r_1, \ldots, r_\nu)$ is the profile of the pair of equations (1.1). If $q_0 \geq 16$ and $\nu \geq 5$, then $N(P) \sim \mathcal{N}P^{s-8}$.

For a pair (1.1) in “general position” one has $\nu = s$ and $r_1 = 1$, and in a quantitative sense easily made precise, such pairs constitute almost all such Diophantine systems. Hence, the conclusion of Theorem 1.2 applies to almost all pairs of equations of the shape (1.1).

We pointed out long ago [5] that a diffuse profile can be advantageous. However, even with the estimate (1.5) in hand, the method of [5] only handles cases where $s \geq 27$ and $r_1$ and $r_2$ are not too large. Thus our results improve on all previous work on the subject even if the input to the published versions is enhanced by the newer mean value bound (1.5).

It is time to describe the methods, and in particular the new ideas involved in the proofs. Our more recent results specific to systems of diagonal quartic forms [6, 8, 9] all depend on large values estimates for Fourier coefficients of powers of Weyl sums, and the current communication is no exception. The large values estimates provide upper bounds for higher moments of these Fourier coefficients, and these in turn yield mean value bounds for correlations of Weyl sums. We describe this link here in a setting appropriate for application to pairs of equations. Consider a 1-periodic twice differentiable function $h : \mathbb{R} \to \mathbb{R}$. Its Fourier expansion

$$h(\alpha) = \sum_{n \in \mathbb{Z}} \hat{h}(n)e(\alpha n)$$

converges uniformly and absolutely. Hence, by orthogonality, one has

$$\int_0^1 \int_0^1 h(\alpha)h(\beta)h(-\alpha - \beta) \, d\alpha \, d\beta = \sum_{n \in \mathbb{Z}} \hat{h}(n)^3.$$  \hspace{1cm} (1.9)

The methods of [6, 8, 9] rest on this and closely related identities, choosing $h(\alpha) = |g(\alpha)|^u$ with suitable quartic Weyl sums $g$ and a positive real number $u$. 

As a service to future scholars, we analyse in some detail the differentiability properties of functions like $|g(\alpha)|^u$ in §3. It transpires that when $u \geq 2$ then the relation (1.9) holds. We use (1.9) with $h(\alpha) = |f(\alpha)|^u$, where now

$$f(\alpha) = \sum_{x \leq P} e(\alpha x^4)$$

(1.10)

is the ordinary Weyl sum. We then obtain new entangled mean value estimates for smaller values of $u$. This alone is not of strength sufficient to reach the conclusions of Theorem 1.1.

As experts in the field will readily recognise, for larger values of $u$ the quality of the aforementioned mean value estimates is diluted by major arc contributions, and one would therefore like to achieve their removal. Thus, if $n$ is a 1-periodic set of real numbers with $n \cap [0, 1)$ a classical choice of minor arcs and $1_n$ is the indicator function of $n$, then one is tempted to apply the function $h(\alpha) = 1_n(\alpha)|f(\alpha)|^u$ in place of $|f(\alpha)|^u$ within (1.9). However, this function is no longer continuous. We bypass this difficulty by introducing a smoothed Farey dissection in §4. This is achieved by a simple and very familiar convolution technique that should be useful in other contexts, too. In this way, in §5 we obtain a minor arc variant of the cubic moment method developed in our earlier work [6]. Equipped with this and the mean value bounds that follow from it, one reaches the conclusions of Theorem 1.1 in the majority of cases under consideration. Unfortunately, some cases with exceptionally large values of $r_j$ stubbornly deny treatment. To cope with these remaining cases, we develop a mixed moment method in §6.

The point of departure is a generalisation of (1.9). If $h_1, h_2, h_3$ are functions that qualify for the discussion surrounding (1.8) and (1.9), then by invoking orthogonality once again, we see that

$$\int_{0}^{1} \int_{0}^{1} h_1(\alpha)h_2(\beta)h_3(-\alpha - \beta) \, d\beta = \sum_{n \in \mathbb{Z}} \hat{h}_1(n)\hat{h}_2(n)\hat{h}_3(n).$$

(1.11)

By Hölder’s inequality, the right hand side here is bounded in terms of the three moments

$$\sum_{n \in \mathbb{Z}} |\hat{h}_j(n)|^3.$$  

(1.12)

In all cases where $h_j(\alpha) = |f(\alpha)|^{u_j}$ for some even positive integral exponent $u_j$ one has $\hat{h}_j(n) \geq 0$, so (1.9) can be used in reverse to interpret (1.12) in terms of the number of solutions of a pair of Diophantine equations. The purely analytic description of the method has several advantages. First and foremost, one can break away from even numbers $u_j$, and still estimate all three cubic moments (1.12). This paves the way to a complete treatment of pairs of equations (1.1) with $s \geq 26$ and $d_0 \geq 15$. Beyond this, the identity (1.11) offers extra flexibility for the arithmetic harmonic analysis. Instead of the homogeneous passage from (1.11) to (1.12) one could apply Hölder’s inequality with differing weights. As an example of stunning simplicity, we
note that the expression in (1.11) is bounded above by
\[
\left( \sum_{n \in \mathbb{Z}} |\hat{h}_1(n)|^2 \right)^{1/2} \left( \sum_{n \in \mathbb{Z}} |\hat{h}_2(n)|^4 \right)^{1/4} \left( \sum_{n \in \mathbb{Z}} |\hat{h}_3(n)|^4 \right)^{1/4}.
\]
If we apply this idea with \( h_j(\alpha) = |f(\alpha)|^{u_j} \) and \( u_j \) a positive even integer, then the first factor relates to a single diagonal Diophantine equation while the other two factors concern systems consisting of three diagonal Diophantine equations. This argument is dual (in the sense that we work with Fourier coefficients) to a method that we described as \textit{complication} in our work on systems of cubic forms [7]. There is, of course, an obvious generalisation of (1.9) to higher dimensional integrals that has been used here. This points to a complex interplay between systems of diagonal equations in which the size parameters (number of variables and number of equations) vary, and need not be restricted to natural numbers. We have yet to explore the full potential of this observation.

We briefly comment on the role of the Hausdorff-Young inequality [18, Chapter XII, Theorem 2.3] within this circle of ideas. In the notation of (1.11) this asserts that
\[
\sum_{n \in \mathbb{Z}} |\hat{h}_j(n)|^3 \leq \left( \int_0^1 |h_j(\alpha)|^{3/2} \, d\alpha \right)^2.
\]
Passing through (1.11) and (1.12), one then arrives at the estimate
\[
\left| \int_0^1 \int_0^1 h_1(\alpha)h_2(\beta)h_3(-\alpha - \beta) \, d\alpha \, d\beta \right| \leq \prod_{j=1}^3 \left( \int_0^1 |h_j(\alpha)|^{3/2} \, d\alpha \right)^{2/3}.
\] (1.13)
However, by Hölder’s inequality, one finds
\[
\left| \int_0^1 \int_0^1 h_1(\alpha)h_2(\beta)h_3(-\alpha - \beta) \, d\alpha \, d\beta \right| \leq \prod_{1 \leq i < j \leq 3} \left( \int_0^1 |h_i h_j|^{3/2} \, d\alpha \, d\beta \right)^{1/3},
\]
where, on the right hand side, one should read \( h_1 = h_1(\alpha), h_2 = h_2(\beta) \) and \( h_3 = h_3(-\alpha - \beta) \). By means of obvious linear substitutions, this also delivers the bound (1.13). This last method is essentially that of Cook [10].

Our approach is superior because the methods are designed to remember the arithmetic source of the Weyl sums when estimating moments of Fourier coefficients.

The proof of Theorem 1.2 requires yet another tool that is a development of our multidimensional version of Hua’s lemma [3]. This somewhat outdated work is based on Weyl differencing. An analysis of the method shows that whenever a new block of differenced Weyl sums enters the recursive process, a new entry \( r_j \) to the profile of the underlying Diophantine system is needed. It is here where one imports undesired constraints on the profile, as in Theorem 1.2. However, powered with the new upper bound (1.5), the method just described yields a bound for a two-dimensional entangled mean value over eighteen Weyl sums that outperforms the cubic moments technique by a factor \( P^{1/6} \) (compare Theorem 6.1 with Theorem 7.2). Within a circle method approach, this mean
value is introduced via Hölder’s inequality. In the complementary factor, we have available an abundance of Weyl sums. Fortunately the cubic moments technique restricted to minor arcs presses the method home. We point out that our proof of Theorem 1.2 constitutes the first instance in which the cubic moments technique is successfully coupled with the differencing techniques derived from \[3\].

One might ask whether more restrictive conditions on the profile allow one to reduce the number of variables even further. As we demonstrate at the very end of this memoir it is indeed possible to accelerate the convergence in (1.4), but even the extreme condition \( r_1 = 1 \) seems insufficient to save a variable without another new idea.

Once the new moment estimates are established, our proofs of Theorems 1.1 and 1.2 are fairly concise. There are two reasons. First, we may import the major arc work, to a large extent, from \[9\]. Second, more importantly, our minor arc treatment rests on a new inequality (Lemma 2.3 below) that entirely avoids combinatorial difficulties associated with exceptional profiles. This allows us to reduce the minor arc work to a single profile with a certain maximality property. We expect this argument to become a standard preparation step in related work, and have therefore presented this material in broad generality. We refer to §2 where the reader will also find comment on previous attempts in this direction.

**Notation.** Our basic parameter is \( P \), a sufficiently large real number. Implicit constants in Vinogradov’s familiar symbols \( \ll \) and \( \gg \) may depend on \( s \) and \( \varepsilon \) as well as ambient coefficients such as those in the system (1.1). Whenever \( \varepsilon \) appears in a statement we assert that the statement holds for each positive real value assigned to \( \varepsilon \). As usual, we write \( e(z) \) for \( e^{2\pi i z} \).

## 2. Some inequalities

This section belongs to real analysis. We discuss a number of inequalities for products. As is familiar for decades, in an attempt to prove results of the type described in Theorems 1.1 and 1.2 via harmonic analysis, it is desirable to simplify to a situation where the profile is extremal relative to the conditions in hand, that is, the multiplicities \( r_1, r_2, \ldots \) are as large as possible, and consequently \( \nu \) is as small as is possible. In the past, most scholars have applied Hölder’s inequality to achieve this objective, often by an *ad hoc* argument that led to the consideration of several cases separately. The purpose of this section is to make available general inequalities that encapsulate the reduction step in a single lemma of generality sufficient to include all situations that one encounters in practice.

The germ of our method is a classical estimate, sometimes referred to as Young’s inequality: if \( p \) and \( q \) are real numbers with \( p > 1 \) and

\[
\frac{1}{p} + \frac{1}{q} = 1,
\]
then for all non-negative real numbers \( u \) and \( v \) one has
\[
uv \leq \frac{u^p}{p} + \frac{v^q}{q}. \tag{2.1}
\]

This includes the case \( r = 2 \) of the bound
\[
|z_1 z_2 \cdots z_r| \leq \frac{1}{r} (|z_1|^r + \cdots + |z_r|^r) \tag{2.2}
\]
which holds for all \( r \in \mathbb{N} \) and all \( z_j \in \mathbb{C} \) \((1 \leq j \leq r)\). Indeed, the general case of (2.2) follows from (2.1) by an easy induction on \( r \).

In the following chain of lemmata we are given a number \( \nu \in \mathbb{N} \) and integral exponents \( m_j, M_j \) \((1 \leq j \leq \nu)\) with
\[
m_1 \geq m_2 \geq \cdots \geq m_{\nu} \geq 0, \quad M_1 \geq M_2 \geq \cdots \geq M_{\nu} \geq 0 \tag{2.3}
\]
and
\[
\sum_{l=1}^{L} m_l \leq \sum_{l=1}^{L} M_l \quad (1 \leq L < \nu), \quad \sum_{l=1}^{\nu} m_l = \sum_{l=1}^{\nu} M_l. \tag{2.4}
\]

We write \( S_{\nu} \) for the group of permutations on \( \nu \) elements. We refer to a function \( w : S_{\nu} \to [0, 1] \) with
\[
\sum_{\sigma \in S_{\nu}} w(\sigma) = 1
\]
as a weight on \( S_{\nu} \).

**Lemma 2.1.** Suppose that the exponents \( m_j, M_j \) \((1 \leq j \leq \nu)\) satisfy (2.3) and (2.4). Then there is a weight \( w \) on \( S_{\nu} \) with the property that for all non-negative real numbers \( u_1, u_2, \ldots, u_{\nu} \) one has
\[
u^{m_1} u_1^{m_2} \cdots u_{\nu}^{m_{\nu}} \leq \sum_{\sigma \in S_{\nu}} w(\sigma) u_{\sigma(1)}^{M_1} u_{\sigma(2)}^{M_2} \cdots u_{\sigma(\nu)}^{M_{\nu}}. \tag{2.5}
\]

**Proof.** We define
\[
D = \sum_{l=1}^{\nu} |M_l - m_l|
\]
and proceed by induction on \( \nu + D \). In the base case of the induction one has \( \nu + D = 1 \). In this situation \( \nu = 1 \) and \( D = 0 \), and the claim of the lemma is trivially true with \( \sigma = \text{id} \) and \( w(\sigma) = 1 \).

Now suppose that \( \nu + D > 1 \). We consider two cases. First we suppose that there is a number \( \nu_1 \) with \( 1 \leq \nu_1 < \nu \) and
\[
\sum_{l=1}^{\nu_1} m_l = \sum_{l=1}^{\nu_1} M_l.
\]

We put
\[
D_1 = \sum_{l=1}^{\nu_1} |M_l - m_l|, \quad D_2 = \sum_{l=\nu_1+1}^{\nu} |M_l - m_l|, \quad \nu_2 = \nu - \nu_1.
\]
Then (2.3) and (2.4) are valid with \( \nu_1 \) in place of \( \nu \), and one has \( D_1 \leq D \). Hence \( \nu_1 + D_1 < \nu + D \) so that we may invoke the inductive hypothesis to find a weight \( w_1 \) on \( S_{\nu_1} \) with

\[
  u_1^{m_1} u_2^{m_2} \cdots u_{\nu_1}^{m_{\nu_1}} \leq \sum_{\sigma \in S_{\nu_1}} w_1(\sigma) u_{\sigma(1)}^{M_1} u_{\sigma(2)}^{M_2} \cdots u_{\sigma(\nu_1)}^{M_{\nu_1}}. \tag{2.6}
\]

Similarly, in the current situation, the numbers \( m_{\nu_1+j}, M_{\nu_1+j} (1 \leq j \leq \nu_2) \) may take the roles of \( m_j, M_j \) in (2.3) and (2.4) with \( \nu_2 \) in place of \( \nu \). Again, we have \( \nu_2 + D_2 < \nu + D \). Now writing \( \tau \) for a permutation in \( S_{\nu_2} \) acting on the set \( \{\nu_1 + 1, \nu_1 + 2, \ldots, \nu\} \), we may invoke the inductive hypothesis again to find a weight \( w_2 \) on \( S_{\nu_2} \) with

\[
  u_{\nu_1+1}^{m_{\nu_1+1}} u_{\nu_1+2}^{m_{\nu_1+2}} \cdots u_\nu^{m_\nu} \leq \sum_{\tau \in S_{\nu_2}} w_2(\tau) u_{\tau(\nu_1+1)}^{M_{\nu_1+1}} u_{\tau(\nu_1+2)}^{M_{\nu_1+2}} \cdots u_{\tau(\nu)}^{M_\nu}. \tag{2.7}
\]

We multiply the inequalities (2.6) and (2.7). It is then convenient to read permutations \( \sigma \) on \( 1, 2, \ldots, \nu_1 \) and \( \tau \) on \( \nu_1 + 1, \nu_1 + 2, \ldots, \nu \) as permutations on \( 1, 2, \ldots, \nu \) with \( \sigma(j) = j \) for \( j > \nu_1 \) and \( \tau(j) = j \) for \( j \leq \nu_1 \). Then, for permutations of the type \( \sigma \tau \) in \( S_{\nu} \) we put \( w(\sigma \tau) = w_1(\sigma)w_2(\tau) \), and we put \( w(\phi) = 0 \) for the remaining permutations \( \phi \in S_{\nu} \). With this function \( w \) the product of (2.6) and (2.7) becomes (2.5), completing the induction in the case under consideration.

In the complementary case we have

\[
  \sum_{l=1}^L m_l < \sum_{l=1}^L M_l \quad (1 \leq L < \nu). \tag{2.8}
\]

In particular, this shows that \( m_1 < M_1 \). Also, by comparing the case \( L = \nu - 1 \) of (2.8) with the equation corresponding to the case \( L = \nu \) in (2.4), we see that \( m_\nu > M_\nu \), as a consequence of which we have \( m_\nu \geq 1 \). We write \( m_1 = m_\nu + r \).

In view of (2.3), we see that \( r \geq 0 \), and so an application of (2.1) with \( q = r + 2 \) leads to the inequality

\[
  u_1^{r+1} u_\nu \leq \frac{r + 1}{r + 2} u_1^{r+2} + \frac{1}{r + 2} u_\nu^{r+2}.
\]

Recall that \( m_\nu \geq 1 \), whence \( m_1 - r - 1 = m_\nu - 1 \geq 0 \). It follows that

\[
  u_1^{m_1} u_\nu^{m_\nu} \leq u_1^{m_1 - r - 1} u_\nu^{m_\nu - 1} \left( \frac{r + 1}{r + 2} u_1^{r+2} + \frac{1}{r + 2} u_\nu^{r+2} \right),
\]

and thus

\[
  u_1^{m_1} \cdots u_\nu^{m_\nu} \leq \frac{r + 1}{r + 2} u_1^{m_1+1} u_2^{m_2} u_3^{m_3} \cdots u_{\nu-1}^{m_{\nu-1}} u_\nu^{m_\nu-1} + \frac{1}{r + 2} u_1^{m_{\nu-1}} u_2^{m_2} u_3^{m_3} \cdots u_{\nu-1}^{m_{\nu-1}} u_\nu^{m_\nu+1}.
\]

The chain of exponents \( m_1 + 1, m_2, m_3, \ldots, m_{\nu-1}, m_\nu - 1 \) is decreasing, and we have \( m_1 + 1 \leq M_1 \) and \( m_\nu - 1 \geq 0 \). Hence, in view of (2.8), the hypotheses (2.3) and (2.4) are still met when we put \( m_1 + 1 \) in place of \( m_1 \) and \( m_\nu - 1 \) in place of \( m_\nu \). However, \( m_1 + 1 \) is closer to \( M_1 \) than is \( m_1 \), and likewise \( m_\nu - 1 \)
is closer to $M_\nu$ than is $m_\nu$. The value of $D$ associated with this new chain of exponents therefore decreases, and so we may apply the inductive hypothesis to find a weight $W$ on $S_\nu$ with
\[
u^{1 \leq m_1 + m_2 + \cdots + m_\nu - 1} \leq \sum_{\sigma \in S_\nu} W(\sigma)u_\sigma^{M_1}u_\sigma^{M_2} \cdots u_\sigma^{M_\nu}.
\]
Interchanging the roles of $u_1$ and $u_\nu$, and denoting by $\tau$ the transposition of 1 and $\nu$, we obtain in like manner the bound
\[
u^{1 \leq m_1 + m_2 + \cdots + m_\nu - 1} \leq \sum_{\sigma \in S_\nu} W(\sigma \circ \tau)u_\sigma^{M_1}u_\sigma^{M_2} \cdots u_\sigma^{M_\nu}.
\]
If we now import the last two inequalities into the inequality preceding them, we find that \(2.5\) holds with
\[w(\sigma) = \frac{r + 1}{r + 2} W(\sigma) + \frac{1}{r + 2} W(\sigma \circ \tau),
\]
and $w$ is a weight on $S_\nu$. This completes the induction in the second case. □

Lemma 2.2. Suppose that $m_j, M_j \ (1 \leq j \leq \nu)$ satisfy \(2.3\) and \(2.4\). For $1 \leq j \leq \nu$ let $h_j : \mathbb{R}^n \to [0, \infty)$ denote a Lebesgue measurable function. Then
\[
\int h_1^{m_1}h_2^{m_2} \cdots h_\nu^{m_\nu} \, dx \leq \max_{\sigma \in S_\nu} \int h_{\sigma(1)}^{M_1}h_{\sigma(2)}^{M_2} \cdots h_{\sigma(\nu)}^{M_\nu} \, dx.
\]
Proof. Choose $u_j = h_j$ in Lemma 2.1 for $1 \leq j \leq \nu$ and integrate. □

For applications to systems of diagonal equations or inequalities, functions $h_j$ come with an equivalence relation between them. This we encode as a partition of the set of indices $j$ in the final lemma of this section.

Lemma 2.3. Suppose that the exponents $m_j, M_j \ (1 \leq j \leq \nu)$ satisfy \(2.3\) and \(2.4\). Let $s = m_1 + m_2 + \cdots + m_\nu$, and for $1 \leq j \leq s$, let $h_j : \mathbb{R}^n \to [0, \infty)$ denote a Lebesgue measurable function. Finally, suppose that $J_1, J_2, \ldots, J_\nu$ are sets with respective cardinalities $m_1, m_2, \ldots, m_\nu$ that partition $\{1, 2, \ldots, s\}$. Then, there exists a tuple $(i_1, \ldots, i_\nu)$ and a permutation $\sigma \in S_\nu$, with $i_l \in J_{\sigma(l)}$ $(1 \leq l \leq \nu)$, having the property that
\[
\int h_1 h_2 \cdots h_s \, dx \leq \int h_{i_1}^{M_1}h_{i_2}^{M_2} \cdots h_{i_\nu}^{M_\nu} \, dx.
\]
Proof. For each suffix $l$ with $1 \leq l \leq \nu$, it follows from \(2.2\) that
\[
\prod_{j \in J_l} h_j \leq \frac{1}{m_j} \sum_{j \in J_l} h_j^{m_j}.
\]
Multiplying these inequalities together yields the bound
\[
h_1 h_2 \cdots h_s \leq \frac{1}{m_1 \cdots m_\nu} \sum_{j_1 \in J_1} \cdots \sum_{j_\nu \in J_\nu} h_{j_1}^{m_1}h_{j_2}^{m_2} \cdots h_{j_\nu}^{m_\nu}.
\]
Now integrate. One then finds that there exists a tuple \((j_1, \ldots, j_\nu)\), with \(j_l \in J_l\) \((1 \leq l \leq \nu)\), for which
\[
\int h_1 h_2 \cdots h_s \, dx \leq \int h_{j_1}^{m_1} h_{j_2}^{m_2} \cdots h_{j_\nu}^{m_\nu} \, dx.
\]
Finally, we apply Lemma 2.2. One then finds that for some \(\sigma \in S_\nu\) the upper bound (2.9) holds with \(i_l = j_{\sigma(l)}\) \((1 \leq l \leq \nu)\). \(\square\)

3. Smooth Farey dissections

In this section we describe a partition of unity that mimics the traditional Farey dissection. With other applications in mind, we work in some generality. Throughout this section we take \(X\) and \(Y\) to be real numbers with \(1 \leq Y \leq \frac{1}{2}\sqrt{X}\), and then let \(\mathcal{N}(q, a)\) denote the interval of all real \(\alpha\) satisfying \(|q\alpha - a| \leq Y X^{-1}\). Define \(\mathcal{N} = \mathcal{N}_{X,Y}\) as the union of all \(\mathcal{N}(q, a)\) with \(1 \leq q \leq Y\), \(a \in \mathbb{Z}\) and \((a, q) = 1\). Note that the intervals \(\mathcal{N}(q, a)\) comprising \(\mathcal{N}\) are pairwise disjoint. We also write \(\mathcal{M} = \mathcal{M}_{X,Y}\) for the set \(\mathcal{N} \cap [0, 1]\). For appropriate choices of the parameter \(Y\), the latter is a typical choice of major arcs in applications of the Hardy-Littlewood method.

The set \(\mathcal{N}\) has period 1. Its indicator function \(1_{\mathcal{N}}\) has finitely many discontinuities in \([0, 1)\), implying unwanted delicacies concerning the convergence of the Fourier series of \(1_{\mathcal{N}}\). We avoid complications associated with this feature by a familiar convolution trick, which we now describe.

Define the positive real number
\[
\kappa = \int_{-1}^{1} \exp(1/(t^2 - 1)) \, dt,
\]
and the function \(K : \mathbb{R} \to [0, \infty)\) by
\[
K(t) = \begin{cases} \kappa^{-1} \exp(1/(t^2 - 1)) & \text{if } |t| < 1, \\ 0 & \text{if } |t| \geq 1. \end{cases}
\]
As is well known, the function \(K(t)\) is smooth and even. We scale this function with the positive parameter \(X\) in the form
\[
K_X(t) = 4X K(4Xt).
\]
Then \(K_X\) is supported on the interval \(|t| \leq 1/(4X)\) and satisfies the important relation
\[
\int_{-\infty}^{\infty} K_X(t) \, dt = \int_{-\infty}^{\infty} K(t) \, dt = 1. \tag{3.1}
\]
We now define the function \(N_{X,Y} : \mathbb{R} \to [0, 1]\) by
\[
N_{X,Y}(\alpha) = \int_{-\infty}^{\infty} 1_{\mathcal{M}}(\alpha - t) K_X(t) \, dt = \int_{-\infty}^{\infty} 1_{\mathcal{M}}(t) K_X(\alpha - t) \, dt. \tag{3.2}
\]
The main properties of this function \(N = N_{X,Y}\) are listed in the next lemma.
Lemma 3.1. The function $N = N_{X,Y}$ is smooth, and for all $\alpha \in \mathbb{R}$ one has $N(\alpha) \in [0, 1]$. Further, whenever $2 \leq Y \leq \frac{1}{4}\sqrt{X}$, the inequalities

$$1_{\mathfrak{N}_{X,Y/2}}(\alpha) \leq N(\alpha) \leq 1_{\mathfrak{N}_{X,2Y}}(\alpha)$$  \hspace{1cm} (3.3)

and

$$N'(\alpha) \ll X, \quad N''(\alpha) \ll X^2$$  \hspace{1cm} (3.4)

hold uniformly in $\alpha \in \mathbb{R}$.

Proof. The integrands in (3.2) are non-negative, so $N(\alpha) \geq 0$, while (3.1) shows that $N(\alpha) \leq 1$. Since $K$ is smooth and compactly supported, the second integral formulation of $N$ in (3.2) shows that $N$ is smooth, and that the derivative is obtained by differentiating the integrand. Thus, we obtain

$$N'(\alpha) = \int_{\mathfrak{N}} \frac{\partial}{\partial \alpha} K_X(\alpha - t) \, dt,$$

whence

$$|N'(\alpha)| \leq 4X \int_{-1}^{1} |K'(t)| \, dt.$$

This confirms the inequality for the first derivative in (3.4). The bound for the second derivative follows in like manner by differentiating again.

We now turn to the task of establishing (3.3). First suppose that $\alpha \in \mathfrak{N}_{X,Y/2}$. Then, there is a unique pair of integers $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $q \leq \frac{1}{2}Y$ and $|q\alpha - a| \leq \frac{1}{2}Y X^{-1}$. For $|t| \leq (4X)^{-1}$ we then have

$$\left| \left( \alpha - t \right) - \frac{a}{q} \right| \leq \frac{1}{4X} + \frac{Y}{2qX} \leq \frac{Y}{qX}.$$

Thus $\alpha - t \in \mathfrak{N}(q,a) \subseteq \mathfrak{N}_{X,Y}$. Since $K_X$ is supported on $[-1/(4X), 1/(4X)]$, we deduce from (3.1) and (3.2) that

$$N(\alpha) \geq \int_{-\infty}^{\infty} 1_{\mathfrak{N}(q,a)}(\alpha - t)K_X(t) \, dt \geq \int_{-1}^{1} K_X(t) \, dt = 1.$$

It follows that one has $N(\alpha) = 1$ for all $\alpha \in \mathfrak{N}_{X,Y/2}$. However, we know already that $N(\alpha)$ is non-negative for all $\alpha \in \mathbb{R}$, and thus we have proved the first of the two inequalities in (3.3).

We complete the proof of the lemma by addressing the second inequality in (3.3). Suppose that $N(\alpha) > 0$. Then, it follows from (3.2) that for some $t \in \mathbb{R}$ with $|t| \leq (4X)^{-1}$, one has $\alpha - t \in \mathfrak{N}_{X,Y}$. Hence, there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(a, q) = 1$, $q \leq Y$ and $|\alpha - t - a/q| \leq Y/(qX)$. By the triangle inequality,

$$\left| \alpha - \frac{a}{q} \right| \leq \frac{Y}{qX} + \frac{1}{4X} \leq \frac{2Y}{qX}.$$

This shows that $\alpha \in \mathfrak{N}_{X,2Y}$. Since $0 \leq N(\alpha) \leq 1$, the second of the inequalities in (3.3) also follows. \hfill \square
We consider $N = N_{X,Y}$ as a smooth model of the major arcs $\mathcal{M}_{X,Y}$. It is convenient to define corresponding minor arcs $n = n_{X,Y}$, with $n_{X,Y} = \mathbb{R} \setminus \mathcal{M}_{X,Y}$, and to write $m = [0,1] \setminus \mathcal{M}$ for the set of minor arcs complementary to $\mathcal{M}$. The smoothed version of $n_{X,Y}$ is the function $n_{X,Y}: \mathbb{R} \to [0,1]$ defined by

$$n(\alpha) = \int_{-\infty}^{\infty} 1_n(\alpha - t)K_X(t)\,dt.$$ 

We trivially have $1_N(\alpha) + 1_n(\alpha) = 1$ for all $\alpha \in \mathbb{R}$, so it is a consequence of (3.1) and (3.2) that $n = n_{X,Y}$ satisfies the identity

$$N(\alpha) + n(\alpha) = 1.$$ \hspace{1cm} (3.5)

The properties of $n$ can therefore be deduced from the corresponding facts concerning $N$. In particular, Lemma 3.1 translates as follows.

**Lemma 3.2.** The function $n = n_{X,Y}$ is smooth, and for all $\alpha \in \mathbb{R}$ one has $n(\alpha) \in [0,1]$. Further, whenever $2 \leq Y \leq \frac{1}{4}\sqrt{X}$, the inequalities

$$1_{n_{x,y}}(\alpha) \leq n(\alpha) \leq 1_{n_{X,Y}}(\alpha)$$

and

$$n'(\alpha) \ll X, \quad n''(\alpha) \ll X^2$$

hold uniformly in $\alpha \in \mathbb{R}$.

### 4. Fractional powers of Weyl sums

In this section we consider a trigonometric polynomial

$$T(\alpha) = \sum_{M < n \leq M+N} c_n e(\alpha n) \quad (4.1)$$

with complex coefficients $c_n$. The associated ordinary polynomial

$$P(z) = \sum_{n=1}^{N} c_{M+n}z^n \quad (4.2)$$

is related to $T$ via the identity

$$T(\alpha) = e(M\alpha)P(e(\alpha)). \quad (4.3)$$

**Lemma 4.1.** Let $k \in \mathbb{N}$. Then, for any real number $u > k$, the real function $\Omega_u: \mathbb{R} \to \mathbb{R}$, defined by $\Omega_u(\alpha) = |T(\alpha)|^u$, is $k$ times continuously differentiable.

**Proof.** In view of (4.3), we see that it suffices to prove this result in the special case where $M = 0$. This reduction step noted, we proceed by a succession of elementary exercises.

Let $u \in \mathbb{R}$. We begin by considering the function $\theta_u: \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $\theta_u(\alpha) = |\alpha|^u$. This function is differentiable on $\mathbb{R} \setminus \{0\}$, and one has

$$\theta_u'(\alpha) = u|\alpha|^u\alpha^{-1} = u\theta_u(\alpha)\alpha^{-1}. \quad (4.4)$$

By induction, it follows that for any $l \in \mathbb{N}$ the function $\theta_u$ is $l$ times differentiable, and that the $l$-th derivative is

$$\theta_u^{(l)}(\alpha) = u(u-1)\cdots(u-l+1)\theta_u(\alpha)\alpha^{-l}. \quad (4.4)$$
Now suppose that \( u > 0 \). Then, by putting \( \theta_u(0) = 0 \) we extend \( \theta_u \) to a continuous function on \( \mathbb{R} \). More generally, whenever \( u > l \), then
\[
\lim_{\alpha \to 0} \frac{\theta_u(\alpha)}{\alpha^l} = 0.
\]
By (4.4), this shows that whenever \( u > l \) then \( \theta_u^{(l)}(0) = 0 \), and that \( \theta_u^{(l-1)} \) is differentiable at 0 with derivative 0. We summarize this last statement as follows:

(a) Let \( k \in \mathbb{N} \) and \( u > k \). Then \( \theta_u \) is \( k \) times continuously differentiable on \( \mathbb{R} \).

Next, for \( u > 0 \), consider the function \( \rho_u : \mathbb{R} \to \mathbb{R} \) defined by putting \( \rho_u(\alpha) = |\sin \pi\alpha|^u \). For \( \alpha \in (0,1) \) one has \( \sin \pi\alpha > 0 \), whence \( \rho_u(\alpha) = (\sin \pi\alpha)^u \). Thus \( \rho_u \) is smooth on \( (0,1) \). But \( \rho \) has period 1, so it suffices to examine its differentiability properties at \( \alpha = 0 \), a point at which \( \rho_u \) is continuous. For all real \( \alpha \) we have \( \sin \pi\alpha = \pi\alpha E(\alpha) \), where
\[
E(\alpha) = \sum_{j=0}^{\infty} (-1)^j \frac{\pi^j}{(2j+1)!}.
\]
The function \( E \) is smooth on \( \mathbb{R} \) with \( E(0) = 1 \). Hence \( E(\alpha) > 0 \) in a neighbourhood of 0 where we then also have
\[
\rho_u(\alpha) = \pi^u |\alpha|^u E(\alpha)^u.
\]
By applying the product rule in combination with our earlier conclusion (a), we therefore conclude as follows:

(b) Let \( k \in \mathbb{N} \) and \( u > k \). Then \( \rho_u \) is \( k \) times continuously differentiable on \( \mathbb{R} \).

We now turn to the function \( T \) where we suppose that \( M = 0 \), as we may. The sum in (4.1) defines a holomorphic function of the complex variable \( \alpha \), and hence the function \( T : \mathbb{R} \to \mathbb{C} \) is a smooth map of period 1. The sum
\[
\bar{T}(\alpha) = \sum_{1 \leq n \in \mathbb{N}} \bar{c}_n e(-\alpha n)
\]
defines another trigonometric polynomial, and for \( \alpha \in \mathbb{R} \) we have \( \overline{T(\alpha)} = \bar{T}(\alpha) \). Consequently, for real \( \alpha \) we have
\[
|T(\alpha)|^2 = T(\alpha)\bar{T}(\alpha), \quad (4.5)
\]
whence the function \( |T|^2 : \mathbb{R} \to \mathbb{C} \), given by \( \alpha \mapsto |T(\alpha)|^2 \), is smooth on \( \mathbb{R} \) with
\[
\frac{d}{d\alpha} |T(\alpha)|^2 = T'(\alpha)\bar{T}(\alpha) + T(\alpha)\bar{T}'(\alpha). \quad (4.6)
\]
On noting that \( T(\alpha)^j \) is again a trigonometric polynomial for all \( j \in \mathbb{N} \), we see that \( |T(\alpha)|^{2j} \) is smooth. Hence, from now on, we may suppose that \( u \) is a real number but not an even natural number. Also, the conclusion of Lemma 4.1 is
certainly true in the trivial case where \( c_n = 0 \) for all \( n \). In the contrary case, the polynomial in (4.2) has at most finitely many zeros. Therefore, the set
\[
Z = \{ \alpha \in \mathbb{R} : T(\alpha) = 0 \}
\]
is 1-periodic with \( Z \cap [0, 1) \) finite, and consequently \( \mathbb{R} \setminus Z \) is open.

We next examine the function \(|T|^u : \mathbb{R} \setminus Z \rightarrow \mathbb{C} \), given by \( \alpha \mapsto |T(\alpha)|^u \).

(c) When \( u \) is real but not an even natural number, the function \(|T|^u\) is smooth.

In order to confirm this assertion, note that \(|T(\alpha)|^u = \theta_{u/2}(|T(\alpha)|^2)\). By applying the chain rule in combination with the preamble to conclusion (a) and (4.6), we find that \(|T(\alpha)|^u\) is differentiable for \( \alpha \in \mathbb{R} \setminus Z \). Indeed,
\[
\frac{d}{d\alpha} |T(\alpha)|^u = \theta'_{u/2}(|T(\alpha)|^2) \left( T'(\alpha) T(\alpha) + T(\alpha) T'(\alpha) \right) = \frac{u}{2} |T(\alpha)|^{u-2} \left( T'(\alpha) T(\alpha) + T(\alpha) T'(\alpha) \right).
\]
Since the final factor on the right hand side here is smooth, we may repeatedly apply the product rule to conclude that \(|T(\alpha)|^u\) is smooth on \( \mathbb{R} \setminus Z \), as claimed.

Finally, we consider any element \( \alpha_0 \in Z \). Then one has \( P(e(\alpha_0)) = 0 \). Since \( P \) is not the zero polynomial, there exists \( r \in \mathbb{N} \) and a polynomial \( Q \in \mathbb{C}[z] \) with \( Q(e(\alpha_0)) \neq 0 \) such that \( P(z) = (z - e(\alpha_0))^r Q(z) \). Write \( U(\alpha) = Q(e(\alpha)) \) for the trigonometric polynomial associated with \( Q \). Then \( T(\alpha) = (e(\alpha) - e(\alpha_0))^r U(\alpha) \). For \( u > 0 \) and all real \( \alpha \) we then have
\[
|T(\alpha)|^u = |e(\alpha) - e(\alpha_0)|^r u |U(\alpha)|^u = |2 \sin \pi (\alpha - \alpha_0)|^r u |U(\alpha)|^u.
\]
There is an open neighbourhood of \( \alpha_0 \) on which \( U(\alpha) \) does not vanish. By our conclusion (c) it is apparent that \(|U(\alpha)|^u\) is smooth on this neighbourhood. If \( u > k \), then the conclusion (b) implies that the function \(|2 \sin \pi (\alpha - \alpha_0)|^r u |U(\alpha)|^u\) is \( k \) times continuously differentiable. The conclusion of the lemma therefore follows by application of the product rule. \( \square \)

We mention in passing that if more is known about the zeros of \( P \), then the argument that we have presented shows more. For example, if all the zeros in \( Z \) are double zeros and \( u > k \), then \(|T(\alpha)|^u\) is \( 2k \) times differentiable.

**Lemma 4.2.** Let \( W : \mathbb{R} \rightarrow \mathbb{R} \) be a twice continuously differentiable function of period 1, and let \( u \geq 2 \). For \( l \in \mathbb{Z} \) let
\[
b_l = \int_0^1 W(\alpha) |T(\alpha)|^u e(-\alpha l) \, d\alpha.
\]
Then, for all \( l \in \mathbb{Z} \setminus \{0\} \), one has
\[
|b_l| \leq \frac{1}{(2\pi l)^2} \int_0^1 \left| \frac{d^2}{d\alpha^2} W(\alpha) |T(\alpha)|^u \right| \, d\alpha.
\]
Moreover, for all \( \alpha \in \mathbb{R} \) one has the Fourier series expansion
\[
W(\alpha) |T(\alpha)|^u = \sum_{l \in \mathbb{Z}} b_l e(\alpha l),
\]
in which the right hand side converges absolutely and uniformly on \( \mathbb{R} \).

**Proof.** By (4.3) and Lemma 4.1 the condition \( u \geq 2 \) ensures that \( W(\alpha)|T(\alpha)|^u \) is twice continuously differentiable. Hence, the integral on the right hand side of (4.9) exists, and the upper bound (4.9) follows from (4.8) by integrating by parts two times. Furthermore, the upper bound (4.9) ensures that the series in (4.10) converges absolutely and uniformly on \( \mathbb{R} \). Thus, by [18] Chapter II, Theorem 8.14, this Fourier series sums to \( W(\alpha)|T(\alpha)|^u \). \( \square \)

In this paper Lemmata 4.1 and 4.2 will only be used with the quartic Weyl sum \( f \), as defined in (1.10), in the role of \( T \). The weight \( W \) will be either constantly 1 or a smooth minor arc. Let \( u > 0 \) and define the Fourier coefficient

\[
\psi_u(n) = \int_0^1 |f(\alpha)|^u e(-\alpha n) \, d\alpha. \tag{4.11}
\]

Also, with a parameter \( Y \) at our disposal within the range \( 1 \leq Y \leq \frac{1}{4}P^2 \), we consider the smooth minor arcs \( n(\alpha) = n_{p^4,Y}(\alpha) \) and introduce the related Fourier coefficient

\[
\phi_u(n) = \int_0^1 n(\alpha)|f(\alpha)|^u e(-\alpha n) \, d\alpha. \tag{4.12}
\]

**Lemma 4.3.** Suppose that \( u \geq 2 \) and \( 1 \leq Y \leq \frac{1}{4}P^2 \). Then, for all \( n \in \mathbb{Z} \setminus \{0\} \), one has

\[
|\phi_u(n)| + |\psi_u(n)| \ll P^{u+8}n^{-2}.
\]

**Proof.** We first compute the derivatives of \( |f(\alpha)|^u \). Suppose temporarily that \( u \) is not an even natural number. By (4.7), whenever \( f(\alpha) \neq 0 \), we have

\[
\frac{d}{d\alpha} |f(\alpha)|^u = \frac{u}{2} |f(\alpha)|^{u-2} \left( f'(\alpha)f(\alpha) + f(\alpha)f''(\alpha) \right),
\]

and we may differentiate again to confirm the identity

\[
\frac{d^2}{d\alpha^2} |f(\alpha)|^u = \frac{u(u-2)}{4} |f(\alpha)|^{u-4} \left( f'(\alpha)f(\alpha) + f(\alpha)f''(\alpha) \right)^2 + \frac{u}{2} |f(\alpha)|^{u-2} \left( f''(\alpha)f(\alpha) + 2f'(\alpha)f''(\alpha) + f(\alpha)f'''(\alpha) \right).
\]

These formulae hold for all \( \alpha \in \mathbb{R} \) when \( u \) is an even natural number, and thus

\[
\left| \frac{d}{d\alpha} |f(\alpha)|^u \right| \leq u |f(\alpha)|^{u-1} |f'(\alpha)|
\]

and

\[
\left| \frac{d^2}{d\alpha^2} |f(\alpha)|^u \right| \leq u(u-1) |f(\alpha)|^{u-2} |f'(\alpha)|^2 + u |f(\alpha)|^{u-1} |f''(\alpha)|.
\]

Hence, the trivial estimates \( f(\alpha) \ll P \), \( f'(\alpha) \ll P^5 \) and \( f''(\alpha) \ll P^9 \) suffice to conclude that the upper bounds

\[
\frac{d}{d\alpha} |f(\alpha)|^u \ll P^{u+4} \quad \text{and} \quad \frac{d^2}{d\alpha^2} |f(\alpha)|^u \ll P^{u+8} \tag{4.13}
\]
hold for all \( \alpha \in \mathbb{R} \) when either \( u = 2 \) or \( f(\alpha) \neq 0 \). However, when \( u > 2 \) these derivatives will be zero whenever \( f(\alpha) = 0 \), so the inequalities (4.13) hold uniformly in \( \alpha \in \mathbb{R} \). The upper bound \( \psi_u(n) \ll P^{u+8}n^{-2} \) is now immediate from Lemma 4.2. Furthermore, an application of the product rule in combination with Lemma 3.2 and (4.13) shows that

\[
\frac{d}{d\alpha} n(\alpha)|f(\alpha)|^u \ll P^{u+4} \quad \text{and} \quad \frac{d^2}{d\alpha^2} n(\alpha)|f(\alpha)|^u \ll P^{u+8}.
\]

The estimate \( \phi_u(n) \ll P^{u+8}n^{-2} \) therefore follows by invoking Lemma 4.2 once again, and this completes the proof of the lemma. \( \square \)

5. Cubic moments of Fourier coefficients

The principal results in this section are the upper bounds for cubic moments of \( \phi_u(n) \) and \( \psi_u(n) \) embodied in Theorem 5.1 below. The proof of these estimates involves a development of the ideas underpinning the main line of thought in our earlier paper [6]. For \( u > 0 \) it is convenient to define

\[
\delta(u) = \frac{(25 - 3u)}{6}.
\]

In many of the computations later it is useful to note that

\[
3u - 8 + \delta(u) = \frac{5}{2}u - \frac{23}{6}.
\]

**Theorem 5.1.** Let \( u \) be a real number with \( 6 \leq u \leq 25/3 \). Then

\[
\sum_{n \in \mathbb{Z}} |\psi_u(n)|^3 \ll P^{3u-8+\delta(u)+\varepsilon}.
\]

Further, when \( 2P^{4/15} \leq Y \leq P/16 \) and \( 6 \leq u \leq 11 \), one has

\[
\sum_{n \in \mathbb{Z}} |\phi_u(n)|^3 \ll P^{3u-8+\delta(u)+\varepsilon}.
\]

When \( u \geq 6 \), the contribution from the major arcs to the sum in (5.3) is easily seen to be of order \( P^{3u-8} \). Since \( \delta(u) \) is negative for \( u > 25/3 \), we cannot expect that the upper bound (5.3) holds for such \( u \). However, as is evident from (5.4), a minor arcs version remains valid for \( u \leq 11 \). Before we embark on the proof of this theorem, we summarize some mean value estimates related to the Weyl sum (1.10). In the following two lemmata, we assume that \( 1 \leq Y \leq P/8 \) and write \( M = M_{p^4,Y} \) and \( m = m_{p^4,Y} \). It is useful to note that \( m_{p^4,Y} = m_{p^4,p/8} \cup \mathbb{R} \), where \( \mathbb{R} = M_{p^4,p/8} \setminus M_{p^4,Y} \). Then, from [13 Lemma 5.1], we have the bounds

\[
\int_{\mathbb{R}} |f(\alpha)|^6 \, d\alpha \ll P^2 \quad \text{and} \quad \int_{\mathbb{R}} |f(\alpha)|^6 \, d\alpha \ll P^2 Y^{\varepsilon-1/4}.
\]

**Lemma 5.2.** Suppose that \( P^{4/15} \leq Y \leq P/8 \). Then

\[
\int_{m} |f(\alpha)|^{20} \, d\alpha \ll P^{15+\varepsilon}.
\]
Proof. For $Y = P/8$, the desired estimate is the case $k = 4, w = 20$ of Wooley [16, Lemma 3.1]. For smaller values of $Y$, we make use of the case $Y = P/8$ and apply the second bound of (5.5). On combining [14, Theorem 4.1] with [14, Lemma 2.8 and Theorem 4.2], moreover, one readily confirms that the upper bound $f(\alpha) \ll PY^{-1/4}$ holds uniformly for $\alpha \in \mathbb{R}$. Consequently, one has the estimate

$$
\int_{\mathbb{R}} |f(\alpha)|^{20} d\alpha \ll P^{16}Y^{\varepsilon - 15/4},
$$

and the conclusion of the lemma follows.

Lemma 5.3. When $8 \leq u \leq 14$, one has

$$
\int_{0}^{1} |f(\alpha)|^{u} d\alpha \ll P^{\frac{5}{6}u - \frac{4}{3} + \varepsilon}.
$$

(5.6)

Meanwhile, when $8 \leq u \leq 20$, then uniformly in $P^{4/15} \leq Y \leq P/8$, one has

$$
\int_{m} |f(\alpha)|^{u} d\alpha \ll P^{\frac{5}{6}u - \frac{4}{3} + \varepsilon}.
$$

(5.7)

Proof. It is a consequence of Hua’s Lemma [14, Lemma 2.5] that

$$
\int_{m} |f(\alpha)|^{8} d\alpha \leq \int_{0}^{1} |f(\alpha)|^{8} d\alpha \ll P^{5+ \varepsilon}.
$$

(5.8)

One interpolates linearly between this estimate and the bound established in Lemma 5.2 via Hölder’s inequality to confirm the upper bound (5.7) for $8 \leq u \leq 20$. The upper bound (5.6) then follows on noting that for $6 \leq u \leq 14$, it follows from (5.5) that

$$
\int_{\mathbb{N}} |f(\alpha)|^{u} d\alpha \ll P^{u - 4} \ll P^{\frac{5}{6}u - \frac{5}{3}}.
$$

Since $[0, 1] = \mathbb{N} \cup m$, the desired conclusion follows at once.

In the special case $u = 14$, the first conclusion of Lemma 5.3 assumes the simple form already announced in (1.5).

Lemma 5.4. Let $\mathcal{Z}$ be a set of $\mathbb{Z}$ integers. Then

$$
\int_{0}^{1} \left| \sum_{z \in \mathcal{Z}} e(z\alpha) \right|^{2} |f(\alpha)|^{2} d\alpha \ll PZ + P^{1/2 + \varepsilon}Z^{3/2}
$$

and

$$
\int_{0}^{1} \left| \sum_{z \in \mathcal{Z}} e(z\alpha) \right|^{4} |f(\alpha)|^{4} d\alpha \ll P^{3}Z + P^{2+ \varepsilon}Z^{3/2}.
$$

Proof. This is essentially contained in [12, Lemma 6.1], where these estimates are established in the case when $\mathcal{Z}$ is contained in $[0, P^{4}]$. As pointed out in [9, Lemma 2.2] this condition is not required.

We now have available sufficient infrastructure to derive upper bounds for cubic moments of $\phi_u(n)$ and $\psi_u(n)$. 


The proof of Theorem 5.1. Let \( \vartheta(n) \) denote one of \( \psi(n) \), \( \phi(n) \). On examining the statement of the theorem, it is apparent that we may assume that in the former case we have \( 6 \leq u \leq 25/3 \), and in the latter case \( 6 \leq u \leq 11 \) and \( 2P^{4/15} < Y < P/16 \). We begin with the observation that, by Lemma 4.3, one has \( \vartheta(n) \ll P^{-8}n^{-2} \). Consequently, when \( u \geq 6 \), one has

\[
\sum_{|n|>P^7} |\vartheta(n)|^3 + \sum_{|n| \leq P^7} |\vartheta(n)|^3 \ll P^7 + P^{3u+24} \sum_{|n|>P^7} n^{-6} \ll P^{3u-11}.
\]

It remains to consider the contribution of those integers \( n \) with \( |n| \leq P^7 \) and \( |\vartheta(n)| > 1 \). We put \( \Theta(\alpha) = 1 \) when \( \vartheta = \psi_u \), and \( \Theta(\alpha) = n(\alpha) \) when \( \vartheta = \phi_u \). Then the definitions (4.11) and (4.12) take the common form

\[
\vartheta(n) = \int_0^1 \Theta(\alpha)|f(\alpha)|^\psi e(-\alpha n) \, d\alpha.
\]

By Lemma 3.2 it follows that \( \Theta(\alpha) \in [0, 1] \). Thus, by Lemma 5.3, one finds that

\[
|\vartheta(n)| \leq \vartheta(0) \ll \psi_u(0) \ll P^{\frac{2u-\frac{7}{3}+\epsilon}{u}} \quad (8 \leq u \leq 11).
\]

In the missing cases where \( 6 \leq u < 8 \) one interpolates between (5.8) and the elementary inequality

\[
\int_0^1 |f(\alpha)|^4 \, d\alpha \ll P^{2+\epsilon},
\]

also a consequence of Hua’s Lemma [14, Lemma 2.5], to conclude that

\[
|\vartheta(n)| \leq \vartheta(0) \ll \psi_u(0) \ll P^{2+\frac{2}{3}(u-4)+\epsilon}.
\]

Fix a number \( \tau \) with \( 0 < \tau < 10^{-10} \) and define \( T_0 \) by

\[
T_0 = \begin{cases} 
P^{\frac{2u-1+\tau}{u}}, & \text{when } 6 \leq u < 8, \\
P^{\frac{2u-\frac{7}{3}+\tau}{u}}, & \text{when } 8 \leq u \leq 11. 
\end{cases}
\]

Then, on recalling the upper bounds for \( \vartheta(n) \) just derived, a familiar dyadic dissection argument shows that there is a number \( T \in [1, T_0] \) with the property that

\[
\sum_{n \in \mathbb{Z}} |\vartheta(n)|^3 \ll P^{3u-11} + (\log P) \sum_{|n| \leq P^7} |\vartheta(n)|^3 \\
\ll P^{3u-11} + P^\epsilon T^3 \mathcal{Z},
\]

where \( \mathcal{Z} \) denotes the number of elements in the set

\[
\mathcal{Z} = \{ n \in \mathbb{Z} : |n| \leq P^7 \text{ and } T < |\vartheta(n)| \leq 2T \}.
\]

For each \( n \in \mathcal{Z} \) there is a complex number \( \eta_n \), with \( |\eta_n| = 1 \), for which \( \eta_n \vartheta(n) \) is a positive real number. Write

\[
K(\alpha) = \sum_{n \in \mathcal{Z}} \eta_n e(-\alpha n).
\]
Then one concludes from (5.9) and orthogonality that
\[ TZ < \sum_{n \in \mathbb{Z}} \eta_n \vartheta (n) = \int_0^1 \Theta (\alpha) K (\alpha) | f (\alpha) |^u \, d\alpha. \tag{5.13} \]

Beyond this point our argument depends on the size of \( T \). Our first argument handles the small values \( T \leq P^{2u - \frac{35}{18}} \). By (5.13) and Hölder’s inequality, we obtain the bound
\[ TZ \leq I^{1/2} \left( \int_0^1 |K (\alpha)^2 f (\alpha)^4| \, d\alpha \right)^{1/3} \left( \int_0^1 |K (\alpha)|^2 \, d\alpha \right)^{1/6}, \tag{5.14} \]
where
\[ I = \int_0^1 \Theta (\alpha)^2 | f (\alpha) |^{2u - \frac{8}{3}} \, d\alpha. \]
By orthogonality, one has
\[ \int_0^1 |K (\alpha)|^2 \, d\alpha = Z, \]
and by a consideration of the underlying Diophantine equations, one deduces via Lemma 5.4 that
\[ \int_0^1 |K (\alpha)^2 f (\alpha)^4| \, d\alpha \ll P^3 Z + P^{2+\varepsilon} Z^{3/2}. \tag{5.15} \]
Next we confirm the bound \( I \ll P^{2u - \frac{35}{18} + \varepsilon} \). Indeed, in the case where \( \Theta = 1 \) we have \( 6 \leq u \leq 25/3 \). In such circumstances \( 8 \leq 2u - 8/3 \leq 14 \), and so (5.6) applies and yields the claimed bound. In the case \( \Theta = n \) we have \( u \leq 11 \), and hence \( 2u - 8/3 < 20 \). Write \( m = m_{P^4, Y/2} \). Then by Lemma 3.2, we have \( 0 \leq n(\alpha) \leq 1_m \). We therefore deduce that in this second case we have
\[ I \leq \int_0^1 n(\alpha) | f (\alpha) |^{2u - \frac{8}{3}} \, d\alpha \leq \int_m | f (\alpha) |^{2u - \frac{8}{3}} \, d\alpha, \]
and (5.7) confirms our claimed bound for \( I \).

Collecting these estimates together within (5.14), we now have
\[ TZ \ll P^\varepsilon \left( P^3 Z + P^{2+\varepsilon} Z^{3/2} \right)^{1/3} Z^{1/6} \left( P^{2u - \frac{35}{18}} \right)^{1/2}. \]
On recalling (5.2), we find that this relation disentangles to yield the bound
\[ T^3 Z \ll P^{2+\frac{8}{3} (\frac{5}{3} u - \frac{35}{18}) + \varepsilon} + T P^{2+\frac{3}{2} u - \frac{35}{18} + \varepsilon} = P^{3u - 8 + \delta (u) + \varepsilon} + T P^{\frac{5}{3} u - \frac{17}{9} + \varepsilon}. \]
It transpires that in the range \( T \leq P^{\frac{5}{3} u - \frac{35}{18}} \) the first term on the right hand side dominates, so that we finally reach the desired conclusion \( T^3 Z \ll P^{3u - 8 + \delta (u) + \varepsilon} \).

In view of (5.11), this is enough to complete the proof of Theorem 5.1 in the case that \( T \) is small.

Our second approach is suitable for \( T \) of medium size, with
\[ P^{\frac{5}{3} u - \frac{35}{18}} \leq T \leq P^{\frac{5}{3} u - \frac{11}{18}}. \tag{5.16} \]
We apply Schwarz’s inequality to (5.13), obtaining the bound
\[ TZ \leq \left( \int_0^1 |K(\alpha)^2 f(\alpha)^4| \, d\alpha \right)^{1/2} \left( \int_0^1 \Theta(\alpha)^2 |f(\alpha)|^{2u-4} \, d\alpha \right)^{1/2}. \]
Note that when $6 \leq u \leq 11$, one has $8 \leq 2u - 4 \leq 18$, and when instead $u \leq 25/3$, we have $2u - 4 < 14$. Hence, as in the proof of our earlier estimate for $I$, it follows from Lemma 5.3 that
\[ \int_0^1 \Theta(\alpha)^2 |f(\alpha)|^{2u-4} \, d\alpha \ll P^{\frac{5}{7}u - 5 + \varepsilon}. \]
Applying this estimate in combination with (5.15), we conclude that
\[ TZ \ll P^\varepsilon (P^3 Z + P^2 Z^{3/2})^{1/2} (P^{\frac{5}{7}u - 5})^{1/2}. \]
This bound disentangles to deliver the relation
\[ T^3 Z \ll TP^{\frac{5}{7}u - 2 + \varepsilon} + T^{-1} P^{\frac{10}{7}u - 6 + \varepsilon}. \]
On recalling (5.2), we find that our present assumptions (5.16) concerning the size of $T$ deliver the estimate
\[ T^3 Z \ll P^{\frac{5}{7}u - 25/6 + \varepsilon} + P^{\frac{5}{7}u - 73/18 + \varepsilon} \ll P^{3u - 8 + \delta(u) + \varepsilon}. \]
The conclusion of Theorem 5.1 again follows in this case, by virtue of (5.11).

The analysis of the large values $T$ satisfying $P^{\frac{5}{7}u - 11/6} < T \leq T_0$ is more subtle. Suppose temporarily that $\vartheta_u = \psi_u$, and hence that $u \leq 25/3$. Then, by (3.5) and (5.13),
\[ TZ \leq \int_0^1 N(\alpha) K(\alpha) |f(\alpha)|^u \, d\alpha + \int_0^1 n(\alpha) K(\alpha) |f(\alpha)|^u \, d\alpha. \]
By hypothesis, we have $u \geq 6$. Also, from Lemma 3.1, we have $N \leq 1_{m_{p^4, P/s}}$, so that (5.5) yields the bound
\[ \int_0^1 N(\alpha) K(\alpha) |f(\alpha)|^u \, d\alpha \leq Z \int_{[m_{p^4, P/s}} |f(\alpha)|^u \, d\alpha \ll Z P^{u-4}. \]
Since $u - 4 < \frac{5}{6} u - \frac{11}{6}$, for large enough $P$ one has $Z P^{u-4} < \frac{1}{2} T Z$. Thus
\[ TZ \ll \int_0^1 n(\alpha) K(\alpha) |f(\alpha)|^u \, d\alpha. \] (5.17)
Note that this is exactly the inequality (5.13) in the case where $\vartheta_u = \phi_u$. Consequently, the upper bound (5.17) holds for the large values of $T$ currently under consideration, irrespective of the choice of $\vartheta_u$. Now apply Schwarz’s inequality to (5.17). Then, by Lemma 3.2 we deduce that
\[ TZ \leq \left( \int_0^1 |K(\alpha)f(\alpha)|^2 \, d\alpha \right)^{1/2} \left( \int_m |f(\alpha)|^{2u-2} \, d\alpha \right)^{1/2}, \]
where again we write $m = m_{p^4, P^2}$. Note here that $u \leq 11$, so that $2u - 2 \leq 20$. Hence, by Lemmata 5.3 and 5.4 we have
\[ TZ \ll P^\varepsilon (P^3 Z + P^2 Z^{3/2})^{1/2} (P^{\frac{5}{7}u - 10})^{1/2}. \]
Consequently, our assumptions concerning the size of $T$ reveal that
\[ T^3 Z \ll TP^5 u^{-7+\varepsilon} + T^{-1}P^{10} u^{-\frac{17}{3}+\varepsilon} \]
\[ \ll T_0 P^5 u^{-\frac{7}{3}+\varepsilon} + P^5 u^{-\frac{17}{3}+\varepsilon}. \]  
(5.18)

When $6 \leq u < 8$, one has
\[ \left(\frac{3}{4}u - 1\right) + \left(\frac{5}{3}u - \frac{7}{3}\right) = \frac{29}{12}u - \frac{10}{3} \leq \frac{5}{2}u - \frac{23}{6}, \]
whilst for $8 \leq u \leq 11$,
\[ \left(\frac{5}{6}u - \frac{5}{3}\right) + \left(\frac{5}{3}u - \frac{7}{3}\right) = \frac{5}{2}u - 4 < \frac{5}{2}u - \frac{23}{6}. \]

Then in either case one finds from (5.18) via (5.2) that $T^3 Z \ll P^{3u-8+\delta(u)+2\tau}$, and the conclusion of Theorem 5.1 follows in this final case, again by (5.11), on taking $\tau$ sufficiently small.

We close this section with a related but simpler result.

**Theorem 5.5.** One has
\[ \sum_{n \in \mathbb{Z}} \psi_4(n)^3 \ll P^{13/2+\varepsilon}. \]

**Proof.** By (4.11) and orthogonality, the Fourier coefficient $\psi_4(n)$ has a Diophantine interpretation that shows on the one hand that $\psi_4(n) \in \mathbb{N}_0$, and on the other that $\psi_4(n) = 0$ for all $n \in \mathbb{Z}$ with $|n| > 2P^4$. By (4.11) and (5.10), we also have the bound $\psi_4(n) \leq \psi_4(0) \ll P^{2+\varepsilon}$. The argument leading to (5.11) now shows that there is a number $T$ with $1 \leq T \leq P^{2+\varepsilon}$ having the property that
\[ \sum_{n \in \mathbb{Z}} \psi_4(n)^3 \ll P^{6+\varepsilon} + P^\varepsilon \sum_{|n| \leq 2P^4} \psi_4(n)^3 \]
\[ \ll P^{6+\varepsilon} + P^\varepsilon T^3 Z, \]  
(5.19)

where $Z$ denotes the number of elements in the set
\[ \mathcal{Z} = \{n \in \mathbb{Z} : |n| \leq 2P^4 \text{ and } T < |\psi_4(n)| \leq 2T\}. \]

As in the corresponding analysis within the proof of Theorem 5.1, we next find that there are unimodular complex numbers $\eta_n$ ($n \in \mathcal{Z}$) having the property that, with $K(\alpha)$ defined via (5.12), one has
\[ TZ < \int_0^1 K(\alpha) |f(\alpha)|^4 \, d\alpha. \]

We first handle small values of $T$. Here, an application of Schwarz’s inequality leads via (5.8) to the bound
\[ TZ \leq \left( \int_0^1 |f(\alpha)|^8 \, d\alpha \right)^{1/2} \left( \int_0^1 |K(\alpha)|^2 \, d\alpha \right)^{1/2} \ll P^{5/2+\varepsilon} Z^{1/2}. \]

This disentangles to yield $T^3 Z \ll TP^{5+\varepsilon}$, proving the theorem for $T \leq P^{3/2}$. 

Next, when $T$ is large, we apply Hölder’s inequality in a manner similar to that employed in the large values analysis of the proof of Theorem 5.1. Thus
\[
TZ \leq \left( \int_0^1 |K(\alpha)^2 f(\alpha)^2| \, d\alpha \right)^{1/2} \left( \int_0^1 |f(\alpha)|^4 \, d\alpha \right)^{1/4} \left( \int_0^1 |f(\alpha)|^8 \, d\alpha \right)^{1/4},
\]
and hence
\[
TZ \ll P^\varepsilon (PZ + P^{1/2}Z^{3/2})^{1/2} P^{7/4}.
\]
We now obtain the bound
\[
T^3Z \ll TP^{9/2+\varepsilon} + T^{-1}P^{8+\varepsilon},
\]
and in view of (5.19), this proves Theorem 5.5 in the complementary case $P^{3/2} \leq T \leq P^{2+\varepsilon}$.

6. Mean values of quartic Weyl sums

In this section we estimate certain entangled moments of quartic Weyl sums, and then apply them to obtain minor arc estimates for use within the proofs of Theorems 1.1 and 1.2. Throughout this section and the next, let the pair of integers $c_i, d_i$ $(1 \leq i \leq 5)$ satisfy the condition that the points $(c_i : d_i) \in \mathbb{P}^1(\mathbb{Q})$ are distinct. Define the linear forms $M_i = M_i(\alpha, \beta) \ (1 \leq i \leq 5)$ by
\[
M_i(\alpha, \beta) = c_i \alpha + d_i \beta.
\]
Let $u > 0$, and recall the definition of the exponent $\delta(u)$ from (5.1). Then, with $2P^{4/15} \leq Y \leq P/16$ and $n = n_{p^u,Y}$, we consider the mean values
\[
I_u = \int_0^1 \int_0^1 |f(M_1)f(M_2)f(M_3)|^u \, d\alpha \, d\beta,
\]
\[
J_u = \int_0^1 \int_0^1 n(M_1)n(M_2)n(M_3)|f(M_1)f(M_2)f(M_3)|^u \, d\alpha \, d\beta.
\]

**Theorem 6.1.** One has $I_4 \ll P^{13/2+\varepsilon}$ and $I_u \ll P^{3u-8+\delta(u)+\varepsilon}$ $(6 \leq u \leq 25/3)$. Also, when $6 \leq u < 11$, one has $J_u \ll P^{3u-8+\delta(u)+\varepsilon}$.

**Proof.** It follows from Lemmata 3.3 and 4.2 that the function $n(\gamma)|f(\gamma)|^u$ has a uniformly convergent Fourier series with coefficients $\phi_u(n)$. By orthogonality, we conclude that
\[
J_u = \sum_{(n_1, n_2, n_3) \in N} \phi_u(n_1)\phi_u(n_2)\phi_u(n_3),
\]
where $N$ is the set of solutions in integers $n_1, n_2, n_3$ of the linear system
\[
c_1n_1 + c_2n_2 + c_3n_3 = d_1n_1 + d_2n_2 + d_3n_3 = 0.
\]
Since the projective points $(c_i : d_i)$ are distinct, there exist non-zero integers $l_i$, depending only on the $c_i, d_i$, having the property that the solutions of this system are precisely the triples $(n_1, n_2, n_3) = m(l_1, l_2, l_3)$ $(m \in \mathbb{Z})$. It therefore follows from (2.2) that
\[
J_u \leq \frac{1}{3} \sum_{m \in \mathbb{Z}} (|\phi_u(l_1m)|^3 + |\phi_u(l_2m)|^3 + |\phi_u(l_3m)|^3) \leq \sum_{n \in \mathbb{Z}} |\phi_u(n)|^3.
\]
The desired bound for \(J_u\) now follows from Theorem 5.1. The bounds for \(I_4\) and \(I_u\) follow in the same way, but the argument has to be built on the cubic moment estimates for \(\psi_u(n)\) that are provided by Theorems 5.1 and 5.5. \(\square\)

We now turn to related, less balanced mixed moments. With \(u\) and \(Y\) as before, we define

\[
K_u = \int_0^1 \int_0^1 |f(M_1)f(M_2)|^u |f(M_3)|^6 \, d\alpha \, d\beta,
\]

\[
L_u = \int_0^1 \int_0^1 n(M_1)n(M_2)|f(M_1)f(M_2)|^u |f(M_3)|^6 \, d\alpha \, d\beta,
\]

and put

\[
\eta(u) = \frac{19}{6} - \frac{u}{3}.
\]

**Theorem 6.2.** Subject to the hypotheses of this section, one has

\[
K_u \ll P^{2u-2+\eta(u)+\varepsilon} \quad (6 \leq u \leq 19/2),
\]

\[
L_u \ll P^{2u-2+\eta(u)+\varepsilon} \quad (6 \leq u \leq 11).
\]

**Proof.** We proceed as in the initial phase of the proof of Theorem 6.1. Using the same notation, we obtain

\[
L_u = \sum_{(n_1,n_2,n_3)\in N} \phi_u(n_1)\phi_u(n_2)\psi_6(n_3).
\]

Note here that \(\psi_6(m)\) counts solutions of a Diophantine equation, and consequently is a non-negative integer. Hence

\[
L_u \leq \frac{1}{2} \sum_{(n_1,n_2,n_3)\in N} \psi_6(n_3)(|\phi_u(n_2)|^2 + |\phi_u(n_1)|^2).
\]

By symmetry, we may therefore suppose that for appropriate non-zero integers \(l_2\) and \(l_3\), depending at most on \(c\) and \(d\), one has

\[
L_u \leq \sum_{(n_1,n_2,n_3)\in N} \psi_6(n_3)|\phi_u(n_2)|^2 = \sum_{m\in \mathbb{Z}} \psi_6(l_3m)|\phi_u(l_2m)|^2. \quad (6.2)
\]

Next, first applying Hölder’s inequality, and then Theorem 5.1 and (5.2), we obtain the bound

\[
L_u \leq \left( \sum_{n\in \mathbb{Z}} \psi_6(n)^3 \right)^{1/3} \left( \sum_{m\in \mathbb{Z}} |\phi_u(m)|^3 \right)^{2/3}
\]

\[
\ll P^{\varepsilon} \left( P^{15-\frac{23}{6}} \right)^{1/3} \left( P^{2u-\frac{23}{6}} \right)^{2/3}.
\]

The estimate for \(L_u\) recorded in Theorem 6.2 therefore follows on recalling the definition of \(\eta(u)\).

The initial steps in the estimation of \(K_u\) are the same, and one reaches a bound for \(K_u\) identical to (6.2) except that \(\phi_u\) now becomes \(\psi_u\). We split
into major and minor arcs by inserting the relation $1 = N(\alpha) + n(\alpha)$, with parameters $X = P^4$ and $Y = P^{1/3}$, into (4.11). From (5.3) we obtain
\[ \left| \int_0^1 N(\alpha)|f(\alpha)|^u e(-\alpha n) \, d\alpha \right| \leq \int_{\mathbb{R}^4 \times P} |f(\alpha)|^u \, d\alpha \ll P^{u-4}. \]

Hence, we discern from (4.11) and (4.12) that
\[ |\psi_u(n)|^2 \ll |\phi_u(n)|^2 + P^{2u-8}, \]
and so,
\[ K_u \ll \sum_{m \in \mathbb{Z}} \psi_0(l_3m)|\phi_u(l_2m)|^2 + P^{2u-8} \sum_{m \in \mathbb{Z}} \psi_0(l_3m). \]

Here the first sum over $m$ is the same as that occurring in the estimation of $L_u$ in (6.2), and has already been estimated above. Thus, since
\[ \sum_{n \in \mathbb{Z}} \psi_0(n) = |f(0)|^6 \ll P^6, \]
we conclude that
\[ K_u \ll P^{2u-2+\eta(u)+\varepsilon} + P^{2u-8} \sum_{n \in \mathbb{Z}} \psi_0(n) \ll P^{2u-2+\eta(u)+\varepsilon} + P^{2u-2}. \]

Provided that $u \leq 19/2$, which guarantees $\eta(u)$ to be non-negative, this estimate confirms the upper bound for $K_u$ claimed in the theorem. \[ \square \]

Note that the mean values $I_u$ and $J_u$ involve $s = 3u$ Weyl sums, at least for integral values of $u$. By comparison, the number of Weyl sums in $K_u$ and $L_u$ is $s = 2u + 6$. A short calculation shows that when applied with the same value of $s$, with $s \geq 18$, the exponents of $P$ in Theorems 6.1 and 6.2 coincide. Since almost all of Theorem 6.1 may be recovered from Theorem 6.2 via Hölder’s inequality, and since for fixed values of $s$ the exponent $u$ in Theorem 6.2 is at least as large, Theorem 6.2 is morally the stronger result. In our later application of the circle method, this allows for larger values of $r_j$ in the profiles associated to the simultaneous equations (1.1), and this is essential for our method to succeed. Another advantage is that in $L_u$ only two of the forms $M_i$ are on minor arcs, while in the mean value $J_u$ all three are constrained to minor arcs.

We continue with another result in which the profile is even farther out of balance. We consider the integral
\[ M = \int_0^1 \int_0^1 n(M_1)n(M_2)|f(M_1)|^{11} f(M_2)^{11} f(M_3)^4 \, d\alpha \, d\beta. \]

**Theorem 6.3.** Given the hypotheses of this section, one has $M \ll P^{18-1/18+\varepsilon}$.

**Proof.** We again traverse the initial phase of the proof of Theorem 6.1 to confirm the relation
\[ M = \sum_{(n_1, n_2, n_3) \in \mathbb{N}} \phi_{11}(n_1)\phi_{11}(n_2)\psi_4(n_3). \]
Then, just as in the argument of the proof of Theorem 6.2 leading to (6.2), we find that for appropriate non-zero integers \( l_2 \) and \( l_3 \), depending at most on \( c \) and \( d \), one has

\[
M \leq \sum_{m \in \mathbb{Z}} \psi_4(l_3m)|\varphi_{11}(l_2m)|^2.
\]

Thus, an application of Hölder’s inequality in combination with Theorems 5.1 and 5.5 together with (5.2), yields the bound

\[
M \leq \left( \sum_{n \in \mathbb{Z}} \psi_4(n)^3 \right)^{1/3} \left( \sum_{n \in \mathbb{Z}} |\varphi_{11}(n)|^3 \right)^{2/3} \ll P \left( P^{13/2} \right)^{1/3} \left( P^{71/3} \right)^{2/3}.
\]

The desired conclusion follows a rapid computation.

Finally, we transform the estimates for \( L_\mu \) and \( M \) into proper minor arc estimates. In the interest of brevity we write \( \mathfrak{M} = \mathfrak{M}_{p_4, p^{1/3}} \) and put

\[
p = [0,1]^2 \setminus (\mathfrak{M} \times \mathfrak{M}).
\]

**Theorem 6.4.** Suppose that \( 19/2 < u \leq 11 \). Then

\[
\int_{\mathfrak{M}} |f(M_1)f(M_2)|^u |f(M_3)|^6 \, d\alpha \, d\beta \ll P^{2u - 2 + \eta(u) + \varepsilon}.
\]

Furthermore, one has

\[
\int_{\mathfrak{M}} |f(M_1)^{11}f(M_2)^{11}f(M_3)^4| \, d\alpha \, d\beta \ll P^{18 - 1/18 + \varepsilon}.
\]

**Proof.** Let \( N = N_{p_4, p^{2/7}} \) and \( n = 1 - N \). Then

\[
1 = \left( N(M_1) + n(M_1) \right) \left( N(M_2) + n(M_2) \right).
\]

We note at once that whenever \( (\alpha, \beta) \in p \), one has \( N(M_1)N(M_2) = 0 \). The explanation for this observation is that whenever \( N(M_1)N(M_2) > 0 \), then it follows from Lemma 3.1 that \( M_j \in \mathfrak{M}_{p_4, p^{2/7}} \) \( (j = 1, 2) \). By taking suitable linear combinations of \( M_1 \) and \( M_2 \) we find that \( \alpha \) and \( \beta \) lie in \( \mathfrak{M}_{p_4, A p^{2/7}} \), with some \( A \geq 2 \) depending only on the coefficients of \( M_1 \) and \( M_2 \). But \( (\alpha, \beta) \in [0,1]^2 \), and so \( (\alpha, \beta) \in \mathfrak{M} \times \mathfrak{M} \) for large enough \( P \). This is not the case when \( (\alpha, \beta) \in p \), as claimed.

With this observation in hand, we apply (6.6) within the integral on the left hand side of (6.5) to conclude that

\[
\int_{\mathfrak{M}} |f(M_1)^{11}f(M_2)^{11}f(M_3)^4| \, d\alpha \, d\beta \leq M + M_{nN} + M_{nN},
\]

where

\[
M_{nN} = \int_0^1 \int_0^1 N(M_1)n(M_2)|f(M_1)^{11}f(M_2)^{11}f(M_3)^4| \, d\alpha \, d\beta
\]

and \( M_{nN} \) is the integral in (6.8) with \( M_1, M_2 \) interchanged.

By symmetry in \( M_1 \) and \( M_2 \), it now suffices to estimate \( M_{nN} \). Recalling the definition (6.1) of the linear forms \( M_i \), we put \( D = |c_1d_2 - c_2d_1| \) and
note that $D > 0$. Consider the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, with $(\alpha, \beta) \mapsto (\alpha', \beta')$, defined by means of the relation
\[
\begin{pmatrix}
\alpha' \\
\beta'
\end{pmatrix} = D^{-1} \begin{pmatrix}
c_1 & d_1 \\
c_2 & d_2
\end{pmatrix} \begin{pmatrix}
\alpha \\
\beta
\end{pmatrix}.
\] (6.9)

Then $M_1 = D\alpha'$, $M_2 = D\beta'$, and $\alpha$ and $\beta$ are linear forms in $\alpha'$ and $\beta'$ with integer coefficients. By applying the transformation formula as a change of variables, one finds that
\[
M_{Nn} = \int_{\mathfrak{B}} N(D\alpha)n(D\beta)|f(D\alpha)|^{11} f(D\beta)^{11} f(A\alpha' + B\beta')^4| \, d\alpha' d\beta',
\]
wherein $A, B$ are non-zero integers and $\mathfrak{B}$ is the image of $[0, 1]^2$ under the transformation (6.9). The parallelogram $\mathfrak{B}$ is covered by finitely many sets $[0, 1]^2 + t$, with $t \in \mathbb{Z}^2$. Since the integrand in the last expression for $M_{Nn}$ is $\mathbb{Z}^2$-periodic it follows that
\[
M_{Nn} \ll \int_0^1 \int_0^1 N(D\alpha)n(D\beta)|f(D\alpha)|^{11} f(D\beta)^{11} f(A\alpha + B\beta)^4| \, d\alpha d\beta.
\]
Here we have removed decorations from the variables of integration for notational simplicity.

We now inspect all factors of the integrand in the latter upper bound that depend on $\beta$. By Hölder’s inequality, Lemma 5.3 and obvious changes of variable, one obtains the estimate
\[
\int_0^1 n(D\beta)|f(D\beta)|^{11} f(A\alpha + B\beta)^4| \, d\beta \\
\ll \left( \int_0^1 n(D\beta)|f(D\beta)|^{7/5} \, d\beta \right)^{5/7} \left( \int_0^1 |f(A\alpha + B\beta)|^{14} \, d\beta \right)^{2/7} \\
\ll P^\varepsilon (P^{67/6})^{5/7} (P^{10})^{2/7} = P^{65/6+\varepsilon},
\]
uniformly in $\alpha \in \mathbb{R}$. Consequently, applying (5.3) in combination with yet another change of variable, we finally arrive at the bound
\[
M_{Nn} \ll P^{65/6+\varepsilon} \int_0^1 N(D\alpha)|f(D\alpha)|^{11} \, d\alpha \ll P^{18-1/6+\varepsilon}.
\]

We may infer thus far that $M_{Nn} + M_{nN} \ll P^{18-1/6+\varepsilon}$. On substituting this estimate into (6.7), noting also the bound $M \ll P^{18-1/18+\varepsilon}$ supplied by Theorem 6.3, the conclusion (6.5) is confirmed.

The proof of (6.4) is essentially the same, and we economise by making similar notational conventions. The exponents 11 and 4 that occur in (6.5) must now be replaced by $u$ and 6, respectively. The initial phase of the preceding argument then remains valid, and an appeal to Theorem 6.2 delivers the bound
\[
\int\int_p |f(M_1)f(M_2)|^u |f(M_3)|^6 \, d\alpha d\beta \ll L_{Nn} + L_{nN} + P^{2u-2+\eta(u)+\varepsilon},
\] (6.10)
where
\[ L_{Nn} \ll \int_0^1 \int_0^1 N(D\alpha)n(D\beta)|f(D\alpha)f(D\beta)|^u|f(A\alpha + B\beta)|^6 \, d\alpha \, d\beta. \]

Here, we isolate factors of the integrand that depend on \( \beta \) and apply Hölder’s inequality. Note that since \( u \leq 11 \) we have \( 7u/4 < 20 \). Thus, by Lemma 5.3,
\[ \int_0^1 n(D\beta)|f(D\beta)|u f(A\alpha + B\beta)^6 \, d\beta \]
\[ \ll \left( \int_0^1 n(D\beta)|f(D\beta)|^{7u/4} \, d\beta \right)^{4/7} \left( \int_0^1 |f(A\alpha + B\beta)|^{14} \, d\beta \right)^{3/7} \]
\[ \ll P^\varepsilon \left( P^{35u/24} - \frac{4}{5} \right)^{4/7} \left( P^{10} \right)^{3/7}. \]

Applying this bound, which is uniform in \( \alpha \in \mathbb{R} \), together with (5.5), we arrive at the estimate
\[ L_{Nn} \ll P^5 + \varepsilon \int_0^1 N(D\alpha)|f(D\alpha)|^u \, d\alpha \ll P^{11u - \frac{2}{3} + \varepsilon}. \]

When \( u \leq 11 \), the definition of \( \eta(u) \) ensures that \( \frac{11}{6}u - \frac{2}{3} \leq 2u - 2 + \eta(u) \), and hence \( L_{Nn} + L_{nN} \ll P^{2u-2+\eta(u)+\varepsilon} \). The conclusion (6.4) now follows by substituting this estimate into (6.10). \( \square \)

7. Another mean value estimate

This section is an update for quartic Weyl sums of our earlier work [3] on highly entangled mean values. We now attempt to avoid independence conditions on linear forms as far as the argument allows while incorporating the consequences of the recent bound (1.5). We emphasise that throughout this section, we continue to work subject to the overall assumptions made at the outset of the previous section. We begin by examining the mean value
\[ G_1 = \int_0^1 \int_0^1 |f(M_1)^2 f(M_2)^4 f(M_3)^4| \, d\alpha \, d\beta. \quad (7.1) \]

**Lemma 7.1.** One has \( G_1 \ll P^{5+\varepsilon} \).

**Proof.** This is essentially contained in [4 Section 2], but we give a proof for completeness. Recall the definition (6.1) of the linear forms \( M_i \). By orthogonality, the integral \( G_1 \) is equal to the number of solutions of an associated pair of quartic equations. By taking suitable integral linear combinations of these two equations, we may assume that they take the shape
\[ a(x_1^4 - x_2^4) = b(x_3^4 + x_4^4 - x_5^4 - x_6^4) = c(x_7^4 + x_8^4 - x_9^4 - x_{10}^4), \quad (7.2) \]
for suitable natural numbers \( a, b, c \). Thus, we see that \( G_1 \) is equal to the number of solutions of the Diophantine system (7.2) with \( x_i \leq P \). For each of the \( O(P) \) possible choices for \( x_1 \) and \( x_2 \) with \( x_1 = x_2 \), it follows via orthogonality and
that the number of solutions of this system in the remaining variables 
\(x_3, \ldots, x_{10}\) is equal to 
\[
\left( \int_0^1 |f(\alpha)|^4 \, d\alpha \right)^2 \ll P^{4+\varepsilon}.
\]

Consequently, the contribution to \(G_1\) from this first class of solutions is \(O(P^{5+\varepsilon})\). Now consider solutions of (7.2) in which \(x_1 \neq x_2\). By orthogonality, the total number of choices for \(x_3, \ldots, x_{10}\) satisfying the rightmost equation in (7.2) is 
\[
\int_0^1 |f(b\alpha)f(c\alpha)|^4 \, d\alpha.
\]

Schwarz’s inequality in combination with (5.8) shows this integral to be \(O(P^{5+\varepsilon})\). However, for any fixed choice of \(x_3, \ldots, x_{10}\) in this second class of solutions, one has \(x_1 \neq x_2\), and hence the fixed integer 
\[
N = b(x_1^4 + x_4^4 - x_5^4 - x_6^4)
\]
is non-zero. But it follows from (7.2) that \(x_1^2 - x_2^2\) and \(x_1^2 + x_2^2\) are each divisors of \(N\). Thus, a standard divisor function estimate shows that the number of choices for \(x_1\) and \(x_2\) is \(O(P^{4+\varepsilon})\), and we conclude that the contribution to \(G_1\) from this second class of solutions is \(O(P^{5+\varepsilon})\). Adding these two contributions, we obtain the bound claimed in the statement of the lemma.

We next examine the mean value
\[
G_2 = \int_0^1 \int_0^1 |f(M_1)^2 f(M_2)^4 f(M_3)^4 f(M_4)^4 f(M_5)^4| \, d\alpha \, d\beta.
\]

Theorem 7.2. One has \(G_2 \ll P^{11+\varepsilon}\).

Note that in this result we require the five linear forms \(M_j\) to be pairwise independent. Therefore, the result will be of use only in cases where the profile of (1.1) has \(r_5 \geq 1\). The mean value in Theorem 7.2 involves 18 Weyl sums and should therefore be compared with the bound \(I_6 \ll P^{67/6+\varepsilon}\) provided by Theorem 6.1. The extra savings that we obtain here are the essential stepping stone toward Theorem 1.2.

The proof of Theorem 7.2. As in the proof of Lemma 7.1, it follows from orthogonality that the integral \(G_2\) is equal to the number of solutions of an associated pair of quartic equations. Taking suitable integral linear combinations of these two equations, we reduce to the situation where \(c_4 = d_5 = 0\), and consequently \(M_4 = d_4\beta\) and \(M_5 = c_5\alpha\). Motivated by this observation, we begin our deliberations by estimating the auxiliary mean value
\[
G_3 = \int_0^1 \int_0^1 |f(M_1)^2 f(M_2)^4 f(M_3)^4 f(d_4\beta)^4| \, d\alpha \, d\beta.
\]

The Weyl differencing argument [14, Lemma 2.3] shows that there are real numbers \(u_h\) with \(u_h \ll P^\varepsilon\) for which
\[
|f(\gamma)|^4 \ll P^3 + P \sum_{1 \leq |h| \leq 2P^4} u_h e(\gamma h).
\]

(7.4)
We apply this relation with \( \gamma = M_4 \) to the mean value \( G_3 \) and infer that
\[
G_3 \ll P^3 G_1 + PG_4,
\]
where \( G_1 \) is the mean value defined in (7.1), and
\[
G_4 = \sum_{1 \leq |h| \leq 2P^4} u_h \int_0^1 \int_0^1 |f(M_1)^2 f(M_2)^4 f(M_3)^4| c(d_4 h \beta) \, da \, db.
\]
By orthogonality, the double integral on the right hand side here is equal to the number of solutions of the system of Diophantine equations
\[
c_1(x_1^4 - y_1^4) + c_2(x_2^4 + x_3^4 - y_2^4 - y_3^4) + c_3(x_4^4 + x_5^4 - y_4^4 - y_5^4) = 0 \quad (7.6)
d_1(x_1^4 - y_1^4) + d_2(x_2^4 + x_3^4 - y_2^4 - y_3^4) + d_3(x_4^4 + x_5^4 - y_4^4 - y_5^4) + d_4 h = 0
\]
with \( x_i \leq P \) and \( y_i \leq P \). We may sum over \( h \neq 0 \) and replace \( u_h \) by its upper bound. Then we find that \( G_4 \ll P^{\varepsilon} G_5 \), where \( G_5 \) is the number of solutions of the equation (7.6) with the same conditions on \( x_i \) and \( y_i \). By orthogonality again, we deduce that
\[
G_5 = \int_0^1 |f(c_1 \alpha)^2 f(c_2 \alpha)^4 f(c_3 \alpha)^4| \, d\alpha.
\]
For \( 1 \leq i \leq 3 \) the linear form \( M_i \) is linearly independent of \( M_4 = d_4 \beta \), and thus \( c_1 c_2 c_3 \neq 0 \). The trivial bound \( |f(c_1 \alpha)|^2 \ll P^2 \) therefore combines with Schwarz’s inequality and (5.3) to award us the bound
\[
G_5 \ll P^2 \int_0^1 |f(\gamma)|^8 \, d\gamma \ll P^{7 + \varepsilon}.
\]
We therefore deduce that \( G_4 \ll P^{7 + 2\varepsilon} \). Meanwhile, the estimate \( G_1 \ll P^{5 + \varepsilon} \) is available from Lemma (7.1). On substituting these bounds into (7.5), we conclude thus far that \( G_3 \ll P^{8 + \varepsilon} \).

We now repeat this argument with \( \gamma = M_5 \) in (7.4), applying the resulting inequality within the integral \( G_2 \) defined in (7.3). Thus we obtain
\[
G_2 \ll P^3 G_3 + P^{1 + \varepsilon} G_6, \quad (7.7)
\]
where \( G_6 \) denotes the number of solutions of the Diophantine equation
\[
d_1(x_1^4 - y_1^4) + d_2(x_2^4 + x_3^4 - y_2^4 - y_3^4) + d_3(x_4^4 + x_5^4 - y_4^4 - y_5^4) + d_4(x_6^4 + x_7^4 - y_6^4 - y_7^4) = 0,
\]
with \( x_i \leq P \) and \( y_i \leq P \). By orthogonality,
\[
G_6 = \int_0^1 |f(d_1 \alpha)^2 f(d_2 \alpha)^4 f(d_3 \alpha)^4 f(d_4 \alpha)^4| \, d\alpha.
\]
One may confirm that \( d_1 d_2 d_3 d_4 \neq 0 \) by arguing as above, and so an application of (2.2) in combination with (1.5) reveals that
\[
G_6 \ll \sum_{i=1}^4 \int_0^1 |f(d_i \alpha)|^{14} \, d\alpha = 4 \int_0^1 |f(\gamma)|^{14} \, d\gamma \ll P^{10 + \varepsilon}.
\]
The conclusion of the theorem now follows on substituting this bound together with our earlier estimate for \( G_3 \) into (7.7). \( \square \)
8. The circle method

In this section we prepare the ground to advance to the proofs of Theorems 1.1 and 1.2. A preliminary manoeuvre is in order. Let \( k = 0 \) or \( 1 \), and let \( N_k(P) = N_k \) denote the number of solutions of the system (1.1) with \( k \leq x_j \leq P \) (\( 1 \leq j \leq s \)). Note that the equations (1.1) are invariant under the \( s \) mappings \( x_j \mapsto -x_j \). This observation shows that

\[
2^s N_1(P) \leq N(P) \leq 2^s N_0(P). \tag{8.1}
\]

The goal is then to establish the formulae

\[
\lim_{P \to \infty} 2^s P^8 N_k(P) = IS(k = 0, 1), \tag{8.2}
\]

since then (1.4) follows immediately from (8.1) and the sandwich principle. Thus, we now launch the Hardy-Littlewood method to evaluate the counting functions \( N_k(P) \). This involves the exponential sum

\[
f_k(\alpha) = \sum_{k \leq x \leq P} e(\alpha x^4). \tag{8.3}
\]

This sum is, of course, an instance of the sum (1.10), where we have been deliberately imprecise about the lower end of the interval of summation. The results we have formulated so far are indeed independent of the choice of \( k \), and it is only now and temporarily where this detail matters. We require the linear forms \( \Lambda_j = \Lambda_j(\alpha, \beta) \), defined by

\[
\Lambda_j(\alpha, \beta) = a_j \alpha + b_j \beta \quad (1 \leq j \leq s)
\]

that are associated with the equations (1.1). We then put

\[
\mathcal{F}_k(\alpha, \beta) = f_k(\Lambda_1) f_k(\Lambda_2) \cdots f_k(\Lambda_s), \tag{8.4}
\]

and observe that, by orthogonality, one has

\[
N_k(P) = \int_0^1 \int_0^1 \mathcal{F}_k(\alpha, \beta) \, d\alpha \, d\beta. \tag{8.5}
\]

Subject to conditions milder than those imposed in Theorems 1.1 and 1.2 we reduce the evaluation of the integral (8.5) to the estimation of its minor arc part. With this end in mind we define the major arcs \( \mathfrak{M} \) as the union of the rectangles

\[
\mathfrak{M}(q, a, b) = \{ (\alpha, \beta) \in [0, 1]^2 : |\alpha - a/q| \leq P^{-31/8} \text{ and } |\beta - b/q| \leq P^{-31/8} \},
\]

with \( 0 \leq a, b \leq q, (a, b, q) = 1 \) and \( 1 \leq q \leq P^{1/8} \).

Define the generating functions

\[
S(q, c) = \sum_{x=1}^q e(c x^4 / q) \quad \text{and} \quad v(\gamma) = \int_0^P e(\gamma t^4) \, dt.
\]

Then, given \((\alpha, \beta) \in [0, 1]^2\), if we put \( \gamma = \alpha - a/q \) and \( \delta = \beta - b/q \) for some \( a, b \in \mathbb{Z} \) and \( q \in \mathbb{N} \), one concludes from (8.3) and [14, Theorem 4.1] that

\[
f_k(\Lambda_j) = q^{-1} S(q, \Lambda_j(a, b)) v(\Lambda_j(\gamma, \delta)) + O (q^{1/2 + \varepsilon} (1 + P^4 |\Lambda_j(\gamma, \delta)|)^{1/2}). \tag{8.6}
\]
Note that the right hand side here is independent of $k$. We multiply these approximations for $1 \leq j \leq s$. This brings into play the expressions

$$
\mathcal{I}(q, a, b) = q^{-s} \prod_{j=1}^{s} S(q, \Lambda_j(a, b)) \quad \text{and} \quad \mathcal{V}(\gamma, \delta) = \prod_{j=1}^{s} v(\Lambda_j(\gamma, \delta)).
$$

If $(\alpha, \beta) \in \mathcal{W}(q, a, b) \subseteq \mathcal{W}$ then the error term in (8.6) is $O(P^{1/8+\varepsilon})$, and we infer that

$$
\mathcal{F}_k(\alpha, \beta) = \mathcal{I}(q, a, b) \mathcal{V}(\gamma, \delta) + O(P^{-7/8+\varepsilon}).
$$

Since $\mathcal{W}$ is a set of measure $O(P^{-59/8})$, when we integrate this formula for $\mathcal{F}_k(\alpha, \beta)$ over $\mathcal{W}$, we obtain the asymptotic relation

$$
\int_{\mathcal{W}} \int_{\mathcal{W}} \mathcal{F}_k(\alpha, \beta) \, d\alpha \, d\beta = \mathcal{G}(P^{1/8}) \mathcal{J}^*(P^{1/8}) + O(P^{-33/4+\varepsilon}),
$$

where, for $1 \leq Q \leq P$ we define

$$
\mathcal{G}(Q) = \sum_{q \leq Q} \sum_{a=1}^{q} \sum_{b=1}^{q} \mathcal{I}(q, a, b),
$$

$$
\mathcal{J}^*(Q) = \int_{\mathcal{W}(Q)} \mathcal{V}(\gamma, \delta) \, d\gamma \, d\delta,
$$

and $\mathcal{U}(Q) = [-QP^{-4}, QP^{-4}]^2$.

At this point, we require some more information concerning the matrix of coefficients, and we shall suppose that $q_0 \geq 15$. Then $s \geq 16$, and we may apply [9, Lemma 3.3] to conclude that $\mathcal{G}(Q) = \mathcal{G} + O(Q^{5/8-1})$. Further, we have

$$
\int_{-P}^{P} e(\gamma t^4) \, dt = 2v(\gamma),
$$

and thus [9, Lemma 3.1] shows that the limit (1.2) exists, and that we have $2^s \mathcal{J}^*(Q) = P^{s-8} \mathcal{J} + O(P^{s-8}Q^{-1/4})$. We summarise these deliberations in the following lemma.

**Lemma 8.1.** Suppose that $q_0 \geq 15$ and that $k \in \{0, 1\}$. Then

$$
\int_{\mathcal{W}} \int_{\mathcal{W}} \mathcal{F}_k(\alpha, \beta) \, d\alpha \, d\beta = 2^{-s} P^{-8} \mathcal{G} \mathcal{J} + O(P^{s-8-1/32}).
$$

The major arcs in Lemma 8.1 are certainly too slim for efficient use of Weyl type inequalities on the complementary set. A pruning argument allows us to enlarge the major arcs considerably. Let $\mathcal{W}$ denote the union of the rectangles

$$
\mathcal{W}(q, a, b) = \{(\alpha, \beta) \in [0, 1]^2 : |q\alpha - a| \leq P^{-3} \text{ and } |q\beta - b| \leq P^{-3}\},
$$

with $1 \leq q \leq P$, $0 \leq a, b \leq q$ and $(a, b, q) = 1$. Then $\mathcal{W} \subset \mathcal{W}$, and we proceed to estimate the contribution from $\mathcal{W} \setminus \mathcal{W}$ to the integral (8.5). A careful application of [14, Theorem 4.2] shows that $S(q, c) \ll q^{3/4}(q, c)^{1/4}$. Further, if $V(\gamma) = P(1 + P^4|\gamma|)^{-1/4}$, then by [14, Theorem 7.3], one has $v(\gamma) \ll V(\gamma)$. Hence, whenever $(\alpha, \beta) \in \mathcal{W}(q, a, b)$ with $q \leq P$, one deduces from (8.6) that

$$
f_k(\Lambda_j) \ll q^{-1/4} \left(q, \Lambda_j(a, b)\right)^{1/4} V(\Lambda_j(\alpha - a/q, \beta - b/q)) + P^{1/2+\varepsilon}.
$$
It is immediate that the first term on the right hand side here always dominates the second, and therefore,

\[ \mathcal{F}_k(\alpha, \beta) \ll q^{-s/4} \prod_{j=1}^{s} (q, \Lambda_j(a, b))^{1/4} V(\Lambda_j(\alpha - a/q, \beta - b/q)). \]

We integrate over \( \mathfrak{M} \setminus \mathfrak{M} \). The result is a sum over \( q \leq P \) in which we consider the portion \( q \leq P^{1/8} \) separately. This yields the bound

\[ \int \int_{\mathfrak{M} \setminus \mathfrak{M}} \mathcal{F}_k(\alpha, \beta) \, d\alpha \, d\beta \ll K_1(P^{1/8}) + K_2(P^{1/8}), \quad (8.7) \]

where for \( 1 \leq Q \leq P \), we write

\[ K_1(Q) = \sum_{q \leq Q} \sum_{a=1}^{q} \sum_{b=1}^{q} q^{-s/4} \prod_{j=1}^{s} (q, \Lambda_j(a, b))^{1/4} \int \int_{\mathfrak{M}(Q)} \prod_{j=1}^{s} V(\Lambda_j) \, d\alpha \, d\beta, \]

with \( \mathfrak{M}(Q) = [-1,1]^2 \setminus \mathfrak{M}(Q) \), and

\[ K_2(Q) = \sum_{Q < q \leq P} \sum_{a=1}^{q} \sum_{b=1}^{q} q^{-s/4} \prod_{j=1}^{s} (q, \Lambda_j(a, b))^{1/4} \int \int_{[-1,1]^2} \prod_{j=1}^{s} V(\Lambda_j) \, d\alpha \, d\beta. \]

Still subject to the condition \( q_0 \geq 15 \), the proof of [9, Lemma 3.2] shows that

\[ \sum_{q > Q} \sum_{a=1}^{q} \sum_{b=1}^{q} q^{-s/4} \prod_{j=1}^{s} (q, \Lambda_j(a, b))^{1/4} \ll \sum_{q > Q} q^{s-2} \ll Q^{s-1}, \]

and similarly, the proof of [9, Lemma 3.1] delivers the bound

\[ \int \int_{\mathfrak{M}(Q)} \prod_{j=1}^{s} V(\Lambda_j) \, d\alpha \, d\beta \ll P^{-s} Q^{-1/4}. \]

Thus we deduce that \( K_1(P^{1/8}) + K_2(P^{1/8}) \ll P^{s-8-1/32} \). Substituting this estimate into (8.7), and then recalling Lemma 8.1, we see that in the latter lemma we may replace \( \mathfrak{M} \) by \( \mathfrak{M} \). This establishes the following theorem.

**Theorem 8.2.** Suppose that \( q_0 \geq 15 \) and that \( k \in \{0, 1\} \). Then

\[ \int \int_{\mathfrak{M}} \mathcal{F}_k(\alpha, \beta) \, d\alpha \, d\beta = 2^{-s} P^{-s} \mathfrak{S} + O(P^{s-8-1/32}). \]

Let \( \mathfrak{w} = [0,1]^2 \setminus \mathfrak{M} \) denote the minor arcs. Then, in view of (8.2), (8.5) and Theorem 8.2, whenever \( q_0 \geq 15 \), the asymptotic relation (1.4) is equivalent to the minor arc estimate

\[ \int \int_{\mathfrak{w}} \mathcal{F}_k(\alpha, \beta) \, d\alpha \, d\beta = o(P^{s-8}), \quad (8.8) \]

as \( P \to \infty \), and in the next two sections we shall confirm this subject to the hypotheses imposed in Theorems 1.1 and 1.2.
9. The proof of Theorem 1.1

At the core of the proof of Theorem 1.1 we require two minor arc estimates.

**Lemma 9.1.** Let \(c_1, c_2, d_1, d_2 \in \mathbb{Z}\), and suppose that \(M_j = c_j \alpha + d_j \beta\ (j = 1, 2)\) are linearly independent. Then

\[
\iint |f(M_1) f(M_2)|^{15} \, d\alpha \, d\beta \ll P^{22 - 1/6 + \varepsilon}.
\]

**Proof.** It is immediate from (6.3) that \(w \subset p\). Recall the initial argument within the proof of Theorem 6.4. This shows that for \((\alpha, \beta) \in p\), the forms \(M_1\) and \(M_2\) cannot be in \(M_{p^4, p^{2/7}}\) simultaneously. By symmetry we may therefore suppose that \(M_1 \in n_{p^4, p^{2/7}}\). Now apply the transformation formula as in (6.9). One finds that for an appropriate non-zero integer \(D\), depending at most on \(c\) and \(d\), one has

\[
\iint w |f(M_1) f(M_2)|^{15} \, d\alpha \, d\beta \ll \int_0^1 \int_w |f(\alpha f(D) |^{15} \, d\alpha \, d\beta,
\]

where \(m = m_{p^4, p^{2/7}}\). Thus, applying a trivial estimate for one factor \(f(D\beta)\), we deduce via Lemma 5.3 that

\[
\iint w |f(M_1) f(M_2)|^{15} \, d\alpha \, d\beta \ll P^{\varepsilon} \left( P^{65/6} \right) \left( P^{11} \right) \ll P^{22 - 1/6 + \varepsilon}.
\]

This completes the proof of the lemma. \(\square\)

**Lemma 9.2.** Suppose that any two of the binary linear forms \(M_1, M_2, M_3\) are linearly independent. Then

\[
\iint w |f(M_1)^{11} f(M_2)^{11} f(M_3)^4| \, d\alpha \, d\beta \ll P^{18 - 1/18 + \varepsilon}.
\]

**Proof.** On recalling that \(w \subset p\), the lemma is immediate from Theorem 6.4 \(\square\)

We are now fully equipped to complete the proof of Theorem 1.1. Suppose that we are given a pair of equations (1.1) with \(s \geq 26\), \(q_0 \geq 15\) and profile \((r_1, r_2, \ldots, r_\nu)\). The parameter \(l = s - r_1 - r_2\) determines our argument. In the notation of Section 7, we let \(F = F_k\) with \(k = 0\) or \(1\) be the generating function defined in (8.4).

Small values of \(l\) call for special attention. Initially, we consider the situation with \(0 \leq l \leq 3\). We apply Lemma 2.3 with \(J_1\) and \(J_2\) the subsets of the set of indices \(\{1, 2, \ldots, s\}\) counted by \(r_1\) and \(r_2\), respectively, and with \(J_3\) the subset consisting of the remaining indices. Then \(\text{card}(J_3) = l\). We also choose

\[
M_\nu = \ldots = M_4 = 0, \quad M_3 = l, \quad M_2 = 15 - l \quad \text{and} \quad M_1 = s - 15.
\]

The condition \(q_0 \geq 15\) ensures that \(r_1 \leq s - 15\), and \(r_1 + r_2 = s - l = M_1 + M_2\). Also, we have \(M_1 = s - 15 \geq 15 - l = M_2\) because \(r_1 \geq r_2 \geq 15 - l\) and \(s = r_1 + r_2 + l \geq 2r_2 + l \geq 30 - l\). Finally, since \(0 \leq l \leq 3\) it is apparent
that $M_2 = 15 - l \geq l = M_3$. Therefore, Lemma 2.3 is indeed applicable and delivers the bound
\[ \int \int \mathcal{F}(\alpha, \beta) \, d\alpha \, d\beta \ll \int \int |f(M_1)^{s-15} f(M_2) f(M_3)^{l}| \, d\alpha \, d\beta, \]
where each of the $M_j$ is one of the linear forms $\Lambda_i$, and any two of the $M_j$ are linearly independent. We now reduce the exponent $s - 15$ to $15 - l$ and then apply Hölder’s inequality. Thus
\[ \int \int \mathcal{F}(\alpha, \beta) \, d\alpha \, d\beta \ll P^{s-30+\varepsilon} \int \int |f(M_1)^{s-15} f(M_2) f(M_3)^{l}| \, d\alpha \, d\beta \]
\[ \ll \Upsilon_1^{1/4} \Upsilon_2^{1-1/4}, \]
where
\[ \Upsilon_1 = \int \int |f(M_1)^{s-15} f(M_2) f(M_3)^{l}| \, d\alpha \, d\beta, \]
\[ \Upsilon_2 = \int \int |f(M_1) f(M_2)|^{15} \, d\alpha \, d\beta. \]
In this scenario, therefore, we deduce from Lemmata 9.1 and 9.2 that
\[ \int \int \mathcal{F}(\alpha, \beta) \, d\alpha \, d\beta \ll P^{s-30+\varepsilon} \left( P^{18-1/18} \right)^{1/4} \left( P^{22-1/6} \right)^{1-1/4} \]
\[ \ll P^{s-8-1/18+\varepsilon}. \quad (9.1) \]

We may now suppose that $l \geq 4$. Then $r_1 \leq s - 15$ and $r_1 + r_2 \leq s - 4$. In Lemma 2.3 we now take $J_j$ to be the subset of the set of indices $\{1, 2, \ldots, s\}$ counted by $r_j$. We also choose
\[ M_4 = \ldots = M_4 = 0, \quad M_3 = 4, \quad M_2 = 11 \quad \text{and} \quad M_1 = s - 15, \]
and note that the hypothesis $s \geq 26$ ensures that $M_1 \geq M_2$. The conditions required to apply Lemma 2.3 are consequently in play, and we deduce that
\[ \int \int \mathcal{F}(\alpha, \beta) \, d\alpha \, d\beta \ll \int \int |f(M_1)^{s-15} f(M_2)|^{11} |f(M_3)|^{l} \, d\alpha \, d\beta, \]
where again each of the $M_j$ is one of the linear forms $\Lambda_i$, and any two of the $M_j$ are linearly independent. Here $s - 15 \geq 11$ by the hypothesis $s \geq 26$, and we may estimate excessive copies of $f(M_1)$ trivially and apply Lemma 9.2. This confirms that (9.1) also holds for $l \geq 4$. In particular, we have (8.8) subject to the hypotheses of Theorem 1.1. This completes the proof of Theorem 1.1.

10. THE PROOF OF THEOREM 1.2

We continue to use the notation introduced in §§8 and 9, but now suppose that the hypotheses of Theorem 1.2 are met. Hence $s = 25$ and $r_1 \leq s - q_0 \leq 9$. We also assume that $r_5 \geq 1$. Our goal on this occasion is the estimate
\[ \int \int \mathcal{F}(\alpha, \beta) \, d\alpha \, d\beta \ll P^{17-1/24+\varepsilon}. \quad (10.1) \]
Once this is established, Theorem 1.2 follows in the same way as Theorem 1.1 was deduced from (9.1).

We apply Lemma 2.3 with $J_j$ the subset of the set of indices $\{1, 2, \ldots, s\}$ counted by $r_j$ for $1 \leq j \leq \nu$. Also, we put $m_j = r_j$ for each $j$ and

$$M_\nu = \ldots = M_6 = 0, \quad M_5 = M_4 = 1, \quad M_3 = 5 \quad \text{and} \quad M_2 = M_1 = 9.$$  

On recalling that $r_1 \leq 9$, it is immediate that (2.3) and (2.4) hold. Hence, Lemma 2.3 is applicable, and yields linear forms $M_1, \ldots, M_5$ that are linearly independent in pairs, where each $M_j$ is one of the $\Lambda_i$, and where

$$\int \int_{w} F(\alpha, \beta) \, d\alpha \, d\beta \leq \int \int_{w} |f(M_1)^9 f(M_2)^9 f(M_3)^5 f(M_4) f(M_5)| \, d\alpha \, d\beta.$$  

By Hölder’s inequality, we find that

$$\int \int_{w} F(\alpha, \beta) \, d\alpha \, d\beta \leq \Upsilon_{3}^{1/4} \Upsilon_{4}^{3/4},$$  

where

$$\Upsilon_{3} = \int_{0}^{1} \int_{0}^{1} |f(M_1) f(M_2) f(M_4) f(M_5)| \cdot |f(M_3)|^2 \, d\alpha \, d\beta,$$

$$\Upsilon_{4} = \int \int_{w} |f(M_1) f(M_2)|^{32/3} |f(M_3)|^6 \, d\alpha \, d\beta.$$  

Making use of the bounds supplied by Theorem 7.2 and Theorem 6.4 with $u = 32/3$, we therefore infer that

$$\int \int_{w} F(\alpha, \beta) \, d\alpha \, d\beta \ll P^{\varepsilon} \left( P^{11} \right)^{1/4} \left( P^{19 - 1/18} \right)^{3/4} \ll P^{17 - 1/24 + \varepsilon}.$$  

Thus the bound (10.1) is confirmed, and the proof of Theorem 1.2 is complete.

Finally, we briefly comment on the prospects of reducing the number of variables further. Note that the estimates for the minor arcs and for the whole unit square in Theorem 6.1 coincide for $u = 25/3$. Since $\delta(25/3) = 0$, therefore, when $s = 25$ our basic method narrowly fails to be applicable to the system of equations (1.1). Further, it transpires that each additional variable contributes a factor $P$ to the major arc contribution, but only $P^{5/6}$ to the minor arc versions of Theorems 6.1 and 6.2. As indicated in §1 already, it is worth comparing the 18th moment ($u = 6$) in Theorem 6.1 with that in Theorem 7.2, the latter being superior by a factor $P^{1/6}$. It transpires that even if it were possible to propagate this saving through the moment method, then we would still fail to handle cases of (1.1) with $s = 24$, but only by a factor $P^{\varepsilon}$. However, at this stage, the only workable compromise seems to be to apply Theorem 7.2 in conjunction with Theorems 6.1 or 6.4 via Hölder’s inequality. If the profile of the equations (1.1) is even more illustrious than in Theorem 1.2, then one can put more weight on the bound stemming from Theorem 7.2. For example, if we suppose that $s = 24$ and $r_1 \leq 5$, then $\nu \geq 5$ and $r_5 \leq 4$, so
that in hopefully self-explanatory notation, the minor arc contribution can be reduced to something of the shape
\[
\iint \mathcal{F}(\alpha, \beta) \, d\alpha \, d\beta \ll \iint |f(M_1)^5 f(M_2)^5 f(M_3)^5 f(M_4)^5 f(M_5)^4| \, d\alpha \, d\beta.
\]
One may then introduce the identity (3.5) with \(\alpha = M_j\) for all \(1 \leq j \leq 5\) simultaneously. The most difficult term that then arises is that weighted with 
\[
n(M_1) \cdots n(M_5).
\]
A cascade of applications of Hölder’s inequality together with Theorem 6.1 shows this term to be bounded by
\[
(Y_3)^{3/5}(J_{11})^{2/5} \ll P^{16+1/15+\varepsilon},
\]
which is quite far from saving another variable.

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Mathematisches Institut, Bunsenstrasse 3–5, D-37073 Göttingen, Germany
Email address: jbruede@gwdg.de

Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907-2067, USA
Email address: twooley@purdue.edu