Governing dynamics by squeezing in a system of cold trapped ions

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We consider a system of laser-cooled ions in a linear harmonic trap and study the phenomenon of squeezing exchange between their internal and motional degrees of freedom. An interesting relation between the quantum noise reduction (squeezing) and the dynamical evolution is found when the internal and motional subsystems are prepared in properly squeezed (intelligent) states. Specifically, the evolution of the system is fully governed by the relative strengths of spectroscopic and motional squeezing, including the phenomenon of total cancellation of the interaction when the initial squeezing parameters are equal.

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In the last few years significant progress has been achieved, both theoretically and experimentally, in the coherent manipulation of quantum states of trapped atomic ions. Attention is mainly devoted to three mutually related problems: (i) generation and analysis of non-classical states of motion, (ii) implementation of quantum logic and computation, and (iii) generation of entangled states which can improve signal-to-noise ratio in spectroscopy. In this work we discuss the exchange of squeezing between the internal and motional degrees of freedom of a system of cold trapped ions. This discussion will reveal a fundamental relation that exists in properly squeezed (intelligent) states, squeezing properties of the system govern its dynamical evolution.

It is known that the quantum noise in spectroscopy can be reduced using the so-called squeezed atomic states. In Ramsey spectroscopy one deals with two-level systems (e.g., trapped atomic ions) interacting with classical radiation fields. One can equivalently describe this physical situation as the interaction of \( N \) spin-half particles with classical magnetic fields. Denoting by \( S_i \) the spin of the \( i \)-th particle, one can use the collective spin operators, \( J = \sum_i S_i \). The basic set of states is \( |j,m\rangle \) (\( m = j, j-1, \ldots, -j \)):

\[
J^2|j,m\rangle = j(j+1)|j,m\rangle, \quad J_z|j,m\rangle = m|j,m\rangle,
\]

and \( j = N/2 \) if only the symmetric Dicke states \( |j,m\rangle \) are considered. A spectroscopic (or interferometric) process can be described mathematically in the Heisenberg picture as a unitary transformation

\[
J_{\text{out}} = U(\phi) J U(\phi)^\dagger = U(\phi) J,
\]

where \( U(\phi) \) is a \( 3 \times 3 \) transformation matrix and \( \phi \) is the parameter to be estimated. The standard transformation consists of three steps: rotation around the \( y \) axis by \( \pi/2 \), rotation around the \( z \) axis by \( \phi \), and rotation around the \( y \) axis by \( -\pi/2 \). In the Ramsey method the rotations are performed through the application of magnetic fields of the type \( B = B_y \sin \omega t \). The phase shift is \( \phi = (\omega - \omega_0)T \), where \( \omega_0 \) is the frequency of the transition between the atomic levels and \( T \) is the duration of the second Ramsey pulse. After the transformation \( U(\phi) \), one measures the population inversion represented by the operator \( J_{z,\text{out}} = |\cos \phi| J_z - |\sin \phi| J_y \), and the information about \( \phi \) is inferred from the value \( \langle J_{z,\text{out}} \rangle \). The uncertainty of the frequency measurement is

\[
\delta \omega_0 = \frac{\Delta J_{z,\text{out}}}{\partial \langle J_{z,\text{out}} \rangle}/(\partial \omega_0).
\]

Taking for simplicity \( \phi = \pi/2 \), this gives

\[
\delta \omega_0 = \frac{\Delta J_y}{T|\langle J_z \rangle|}.
\]

If one uses a Dicke state \( |j,m\rangle \) at the input, then \( \Delta \omega_0 = (T|\langle m \rangle|)^{-1} \left\{ \frac{1}{2} j(j+1) - m^2 \right\}^{1/2} \). This uncertainty is minimized for \( m = \pm j \):

\[
(\delta \omega_0)_{\text{SNL}} = 1/(T\sqrt{N}),
\]

which is the shot-noise limit (SNL). The measure of spectroscopic squeezing is the parameter

\[
\xi_R = \frac{\delta \omega_0}{(\delta \omega_0)_{\text{SNL}}} = \sqrt{N} \frac{\Delta J_y}{|\langle J_z \rangle|}.
\]

A quantum state for which \( \xi_R < 1 \) is spectroscopically squeezed, and the use of such a state will reduce the quantum noise of a spectroscopic measurement below the shot-noise limit. The Heisenberg limit of \( \delta \omega_0 \) is \( (T\sqrt{2j(j+1)})^{-1} \), that is \( \xi_R = (j+1)^{-1/2} \), which is obtained from the uncertainty relation

\[
\Delta J_x \Delta J_y \geq \frac{1}{2} \langle |J_z| \rangle
\]

and the condition \( (\Delta J_x)^2 \leq \frac{1}{2} j(j+1) \).

It can be shown that spectroscopic squeezing is achieved with the so-called intelligent (minimum-uncertainty) states. The \( J_z-J_y \) intelligent
states are defined as states which give the equality in the uncertainty relation \( \Delta x \cdot \Delta p \geq \hbar \). They are determined by the eigenvalue equation

\[
(qJ_x - iJ_y)|\lambda, \eta \rangle = \lambda|\lambda, \eta \rangle,
\]

where \( \lambda \) is a complex eigenvalue and \( \eta \) is a real parameter given by \( |\eta| = \Delta J_y / \Delta J_x \). For \( |\eta| < 1 \), the intelligent states are squeezed in \( J_y \) and anti-squeezed in \( J_x \), which results in spectroscopic squeezing. Equation (7) can be solved using the analytic representation method. The spectrum is discrete: \( \lambda = im_0 \sqrt{1 - \eta^2} \) where \( m_0 = j, j - 1, \ldots, -j \). The intelligent states can be written in the Dicke basis using the Jacobi polynomials \( P_n^{(\alpha, \beta)}(x) \). The expectation value of \( J_z \) and the squeezing parameter for the intelligent states are

\[
\langle J_z \rangle = -j\eta F, \quad \xi_R = F^{-1/2}, \quad F = 1 + \frac{j + |m_0|}{j} (1 - \eta^2)^{1/2},
\]

(8)

In the range \( |\eta| < 1 \) we always find \( \xi_R \leq 1 \). For \( m_0 = \pm j \) the intelligent states are simultaneously the atomic coherent states with \( \xi_R = 1 \). Numerical results show that the minimum value of \( \xi_R \) (the best value of squeezing) for given \( j \) and \( \eta \) is achieved for the intelligent states with \( m_0 = 0 \). For these states the Heisenberg limit is achieved as \( |\eta| \to 0 \).

Wineland et al. have studied the possibility to generate spectroscopic squeezing using an interaction that couples the internal electronic levels of laser-cooled ions and a center-of-mass mode of their quantized oscillatory motion in a linear harmonic trap. The interaction Hamiltonian is \( (\hbar = 1) \)

\[
H = g(a^\dagger a J_- + a J_+),
\]

(10)

where \( g \) is the coupling constant, \( a \) and \( a^\dagger \) are the boson operators of the center-of-mass mode of the oscillatory motion, and \( J_\pm = J_x \pm iJ_y \) are the collective ion raising and lowering operators. The Hamiltonian (10) is the same as in the Tavis-Cummings version of the Dicke model in quantum optics, in which a collection of two-level atoms interacts with a single-mode radiation field inside a cavity. This model can be considered as a multiatom generalisation of the Jaynes-Cummings model. An important feature of the realization with cold trapped ions is that decoherence processes can be made very small during the interaction.

The Hilbert space \( \mathcal{H} \) of the whole system can be decomposed into a direct sum of finite-dimensional invariant subspaces \( \mathcal{H}_L \):

\[
\mathcal{H} = \bigoplus_{L=0}^{\infty} \mathcal{H}_L.
\]

(11)

where \( L = a^\dagger a + J_+ + N/2 \) is the total excitation (a constant of motion). Each subspace \( \mathcal{H}_L \) is spanned by the orthonormal basis \( |j, L - j, \eta \rangle \) \( \bigotimes |n \rangle \) where \( |n \rangle \) are the Fock states of the oscillator. If the oscillator is initially in a Fock state and the ions are in a Dicke state, the system will evolve in one invariant subspace. For the oscillator and/or ions prepared initially in a superposition state, one should take into account contributions from different subspaces. The exact solution of the problem is obtained by diagonalization of the interaction Hamiltonian (10) in each of the invariant subspaces \( \mathcal{H}_L \) involved.

Wineland et al. considered the situation when the ions are prepared initially in the internal ground state, \( |j, -j, \eta \rangle \) \( \bigotimes |n \rangle \) while the oscillator is prepared in the squeezed vacuum state \( |\xi_q \rangle \) \( \bigotimes |n \rangle \) where

\[
b_n = \frac{\sqrt{n!}}{(n/2)!} \sqrt{\frac{2\xi_q}{1 + \xi_q^2}} \left( \frac{1 - \xi_q^2}{2} \right)^{n/2} \quad \text{for } n \text{ even},
\]

\[
b_n = 0 \quad \text{for } n \text{ odd}.
\]

(12)

Here, \( q = (a + a^\dagger) / \sqrt{2} \) is the position and

\[
\xi_q = \frac{\Delta q}{\Delta q_{\text{vac}}} = \sqrt{2} \Delta q
\]

(13)

is the squeezing parameter of the oscillator. Squeezing in \( q \) is obtained if \( \xi_q < 1 \). Numerical studies show that motional squeezing can be transferred during the interaction into spectroscopic squeezing of the ions. The minimal value \( (\xi_R)_m \) achieved during the interaction depends on the initial value \( \xi_q(0) \) of motional squeezing.

Let us address the following question: How will the system evolve in the Dicke-model interaction if the ions are prepared in a correlated state with initial spectroscopic squeezing? We start by considering an even number of ions prepared in the spectroscopic intelligent state \( |\eta \rangle \) \( \bigotimes |m \rangle \) with \( m_0 = 0 \). Here

\[
c_{-j+2r} = \left( \frac{j}{r} \right)^{j/2} \left( \frac{2j}{2r} \right)^{-1/2} \left( \frac{1 - \eta}{1 + \eta} \right)^{-r} c_{-j},
\]

(14)

\( r = 0, 1, \ldots, j \), and \( c_{-j} \) is determined by the normalization. The oscillator is prepared in the squeezed vacuum state, so the initial wave function of the whole system is

\[
|\psi(0)\rangle = |\eta \rangle \bigotimes |\xi_q \rangle \bigotimes |m \rangle \bigotimes |n \rangle.
\]

(15)

Letting the state (13) evolve in time, governed by the interaction Hamiltonian (10), one can calculate the time-dependent reduced density matrix of the ion subsystem by tracing out the motional (oscillator) degree of freedom:

\[
\rho_{\text{ion}}(\tau) = Tr_{\text{osc}} (|\psi(\tau)\rangle \langle \psi(\tau)|) .
\]

(16)

Here \( \tau = gt \) is the scaled time. Then one can calculate the von Neumann entropy \( S = -Tr (\rho \ln \rho) \) for the internal subsystem and evaluate from it the degree of entanglement between the internal and motional subsystems.
Since both subsystems are initially in pure states, the entropy of the whole system is always zero. Therefore, the marginal entropies of the two subsystems are equal, \( S_{\text{osc}}(\tau) = S_{\text{ion}}(\tau) \).

As the system evolves, the marginal entropy \( S_{\text{ion}}(\tau) \) rapidly oscillates; the time average \( S_{\text{ion}} \) and the oscillation amplitude \( (S_{\text{ion}})_{\text{max}} - (S_{\text{ion}})_{\text{min}} \) depend on the initial state parameters \( \eta \) and \( \xi_q(0) \). Figure 1 shows \( S_{\text{ion}} \) and \( (S_{\text{ion}})_{\text{max}} - (S_{\text{ion}})_{\text{min}} \) as functions of \( \eta \), for two ions and \( \xi_q(0) = 0.6 \). One can see that as \( \eta \) approaches the value 0.36, both the average and the oscillation amplitude of \( S_{\text{ion}} \) rapidly decrease. At the point \( \eta = 0.36 \) the entropy \( S_{\text{ion}}(\tau) \) remains zero for any time, implying that the two subsystems are permanently disentangled. Actually, we find that for \( \eta = \xi_q^2(0) \) the initial state of the system remains unchanged in time, as can be verified by examining the time evolution of the density matrix. So, an equilibrium point exists in the parameter space, for which the interaction between the subsystems is effectively cancelled. Of course, this situation is possible only if the initial state is an eigenstate of the interaction Hamiltonian.

Let us study the phenomenon of the interaction cancellation in a more general way. Any product state of the form

\[
|\psi\rangle = \sum_{m=\pm j}^{j} \tilde{c}_m |j, m\rangle_{\text{ion}} \otimes \sum_{n=0}^{\infty} \tilde{b}_n |n\rangle_{\text{osc}} \tag{17}
\]

can be decomposed into a sum over the subspaces \( \mathcal{H}_L \):

\[
|\psi\rangle = \sum_{L=0}^{\infty} |\psi_L\rangle,
\]

\[
|\psi_L\rangle = \sum_{N,L=\text{min}(N,L)}^{\infty} \tilde{c}_{L+s} |L-s\rangle_{\text{ion}} \otimes |L-s\rangle_{\text{osc}}.
\]

The state \( |\psi\rangle \) can be an eigenstate of the Hamiltonian if and only if all \( |\psi_L\rangle \) are eigenstates with the same eigenvalue independent of \( L \). The only eigenvalue of the Hamiltonian \( \tilde{H} \) that appears in different subspaces \( \mathcal{H}_L \) is zero (this eigenvalue actually appears for any odd-dimensional subspace \( \mathcal{H}_L \)). Then we are able to prove that the product state of the form \( \text{(17)} \) is an eigenstate of the Hamiltonian \( \tilde{H} \), if and only if it is the state \( |\eta\rangle_{\text{ion}} \otimes |\xi_q\rangle_{\text{osc}} \) of Eq. \( \text{(15)} \) with

\[
\eta = \tilde{\eta}, \tag{18}
\]

where \( \tilde{\eta} = \xi_q^2 \). At this point it is worth noting that the squeezed vacuum state is an intelligent state of the Weyl-Heisenberg group, obeying the eigenvalue equation

\[
(\tilde{\eta} p - i q)|\xi_q\rangle = 0. \tag{19}
\]

Here \( p = (a - a^\dagger)/\sqrt{2} \) is the momentum of the oscillator, and \( \tilde{\eta} \) is a positive parameter given by \( \tilde{\eta} = \xi_q^2 = \Delta q/\Delta p \).

Equation \( \text{(19)} \) is similar to the equation \( (\eta J_z - i J_y)|\eta\rangle = 0 \) satisfied by the spectroscopic intelligent state \( |\eta\rangle \) with \( m_0 = 0 \). For this reason we call the product state \( \text{(15)} \) the “double intelligent” state. It is interesting that the condition \( \text{(18)} \) can be rewritten in the form

\[
\frac{\Delta q}{\Delta p} = \frac{\Delta J_y}{\Delta J_x} \tag{20}
\]

which implies that the interaction between the internal and motional subsystems is cancelled when they are equally squeezed (or “equally intelligent”). In other words, the dynamical equilibrium of the system is determined by the squeezing equilibrium between its subsystems.

Next we consider what happens if one prepares the ions in the “double intelligent” state of Eq. \( \text{(15)} \) with the squeezing parameters slightly detuned from the equilibrium point. For example, we can make a perturbative expansion of the time-dependent state around the equilibrium point, with \( \eta = \xi_q^2(0) \) being the small parameter. Then it becomes apparent that as \( \eta \) gets closer to \( \xi_q^2(0) \), the state becomes more similar to the Hamiltonian eigenstate and its change with time is less pronounced. Also, the deviation of the marginal entropy \( S_{\text{ion}}(\tau) \) from its initial value (i.e., from zero) rapidly decreases as the squeezing parameters approach the equilibrium point (see Fig. 1). Therefore, we can conclude that the equilibrium point is stable.

Figure 2 reveals another interesting feature of the “double intelligent” initial state \( \text{(15)} \), namely, the exchange of squeezing between the motional and internal degrees of freedom that occurs during the interaction. For values of \( \xi_q(0) \) below the equilibrium point, spectroscopic squeezing deteriorates [the time average and minimal values \( \xi_q(0) \) are greater than the initial value \( \xi_q(0) \)], while motional squeezing is improved [the time average and minimal values \( \xi_q \) and \( \xi_q \) are less than the initial value \( \xi_q(0) \)]. Correspondingly, for \( \xi_q(0) \) below the equilibrium point, spectroscopic squeezing is improved [except for very low values of \( \xi_q(0) \)], while motional squeezing deteriorates. Thus the exchange of spectroscopic and motional squeezing occurs, depending on their initial strengths. Figure 2 also shows the deviation of the time average \( \langle J_z \rangle \) of the population inversion from its initial value \( \langle J_z \rangle(0) \). Similarly to the behavior of the squeezing parameters, the deviation \( \langle J_z \rangle - \langle J_z \rangle(0) \) changes its sign at the equilibrium point. Thus the behavior of the system prepared in the “double intelligent” state is governed by the relative strengths of spectroscopic and motional squeezing.

The phenomenon in which the dynamics of the system is governed by its squeezing properties appears exclusively for the intelligent spectroscopic state with \( m_0 = 0 \) interacting with the squeezed vacuum. (Note that the condition \( m_0 = 0 \) can be satisfied only for integer \( j \), i.e.,
for even numbers of ions.) If one prepares, for example, the internal subsystem in a spectroscopic intelligent state with \( m_0 \neq 0 \), no equilibrium point will appear since such a state cannot be an eigenstate of the interaction Hamiltonian. Moreover, numerical calculations show that no exchange of squeezing between the internal and motional subsystems occurs in the entire parameter space. Instead, squeezing deteriorates with time for both subsystems.

The generation of the spectroscopic intelligent states is still an open question. A number of schemes were proposed that employ the interaction of atoms (ions) with the broadband squeezed vacuum \([15]\). Alternatively, it is known \([1, 4]\) that quantum logic schemes can be used to generate various correlated states of the cold trapped ions. In particular, the quantum logic methods are suitable for producing the spectroscopic intelligent state for a pair of ions. In this case we obtain \( |\eta\rangle_{\text{ion}} = \sin \theta |1, -1\rangle + \cos \theta |1, 1\rangle \) and \( \xi_R = \left\{ 1 + \eta^2 \right\}^{1/2} = (1 + \sin 2\theta)^{-1/2} \). The equilibrium condition \((20)\) can be written as \( \xi_R^2 = \tan(\theta - \pi/4) \).

In conclusion, we discussed interesting quantum phenomena that take place for a system of cold trapped ions prepared in the “double intelligent” state. The uniqueness of this state is that squeezing properties of the system govern its dynamical evolution. In particular, when the squeezing strengths of the internal and motional subsystems are equal, the system is in the state of stable dynamical equilibrium. Around the equilibrium point exchange of squeezing between the subsystems occurs.

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FIG. 1. Two ions prepared in the state $|\eta\rangle_{\text{ion}} \otimes |\xi_q\rangle_{\text{osc}}$ with $\xi_q(0) = 0.6$. (a) The time average $S_{\text{ion}}$ and (b) the oscillation amplitude $(S_{\text{ion}})_{\text{max}} - (S_{\text{ion}})_{\text{min}}$ of the marginal entropy of the internal subsystem versus $\eta$.

FIG. 2. Two ions prepared in the state $|\eta\rangle_{\text{ion}} \otimes |\xi_q\rangle_{\text{osc}}$ with $\eta = 0.3254$. The time average deviations of (a) the spectroscopic squeezing parameter, $\xi_R - \xi_R(0)$, (b) the motional squeezing parameter, $\xi_q - \xi_q(0)$, and (c) the population inversion, $\langle J_z \rangle - \langle J_z \rangle(0)$, plotted versus $\xi_q(0)$. 