Complete three photon Hong-Ou-Mandel interference at a three port device

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We report the possibility of completely destructive interference of three indistinguishable photons on a three port device providing a generalisation of the well known Hong-Ou-Mandel interference of two indistinguishable photons on a two port device. Our analysis is based on the underlying mathematical framework of SU(3) transformations rather than SU(2) transformations. We show the completely destructive three photon interference for a large range of parameters of the three port device. As each output port can deliver zero to three photons the device generates higher dimensional entanglement. In particular, different forms of entangled states of qudits can be generated depending again on the device parameters. Our system is different from a symmetric three port beam splitter which does not exhibit a three photon Hong-Ou-Mandel interference.

INTRODUCTION

The Hong-Ou-Mandel (HOM) effect [1 2], i.e., the completely destructive interference of two independent but indistinguishable photons, brought a paradigm shift to the field of quantum optics. Until the demonstration of the HOM effect the interference of independent photons was considered to be impossible. Such an interference effect manifests itself in the study of photon correlations rather than in intensity measurements. More specifically, if two single photons are sent from two different ports of a 50/50 beam splitter then the number of coincidence events at the two output ports vanishes. This follows from the fact that, if two photons are indistinguishable with respect to their wavelength and polarisation and their wave packages overlap in time, then the two different quantum paths interfere so that the two photons will never leave the beam splitter at different ports. If one of these parameters is changed, the photons become distinguishable and the dip in the observed coincidence rate starts to disappear.

The effect is quite versatile and has been observed in a very wide class of systems. Besides beam splitters it has been studied in other optical elements such as in integrated devices like evanescently coupled waveguides [3 4] and coupled plasmonic systems [5–8]. It has also been studied in the radiation from two trapped ions [9], atoms [10–12], quantum dots [13 14] and for two different kinds of sources [15 17].

Since the original work of HOM one also has examined the kind of interference that can take place if two single photons are replaced by say two photons on each port or say by one at one port and two at the other port. Hereby one has found very interesting quantum interference effects depending on the beam splitter reflectivity [18–20]. Another interesting possibility occurs if \( n \) photons arrive at each port of a 50/50 beam splitter - in this case the output ports never have odd numbers of photons [21].

In this letter we report a three photon interference effect which is in the original spirit of the HOM effect - we examine the completely destructive interference of three indistinguishable photons on a three port device. We thus shift the focus from a two port device to a three port device. This brings a key change to the underlying mathematical framework as we work with SU(3) transformations rather than SU(2) transformations. We specifically examine a three port integrated device consisting of a small array of three single mode evanescently coupled waveguides as these are relatively easy to fabricate [22]. Although we tailor our discussion to coupled waveguide systems the results will be applicable to a class of wide bosonic systems described by the Hamiltonian [1].

For the three port device we have found an analytical expression for the completely destructive three photon interference. Thereby we produce a variety of two and three photon entanglement at the output ports.

Our three port network is different from the symmetric multport beam splitter which has been extensively studied for the HOM like interferences [23–25]. However for the three port system, such a splitter does not exhibit a perfect three photon HOM interference. On the other hand Campos [26] using the idea of Reck and Zeilinger [26] constructed a SU(3) transformation involving beam splitters and phase shifters which leads to three photon HOM interference. Further Tan et al. [27] showed how the SU(3) transformation involving beam splitters and phase shifters can lead to perfect photon interference depending on specific values of the parameters of SU(2) transformations. Note also that experimental studies of two photon interference in three port and four port devices have been reported in [28–30]. The study of multiport systems is also important in the context of Bosonic sampling [31 33], where the main goal is to use coincidence data to reconstruct the unitary transformation matrix between input and output ports.

While our investigated setup is similar to the experi-
ment studied in [34], which relies on a 3D geometry in order to couple all three modes to each other, we will use a more simple 2D structure, where the outer modes are coupled to the inner mode but not to each other. With the setup of [33] it is possible to suppress states [24] of the form $|2, 1, 0\rangle$, which contain two, one and zero photons in the different output modes, but the output state will still contain a $|1, 1, 1\rangle$-term corresponding to the coincidence detection of all three photon in the three different output modes. We will show that for a whole range of parameters of the waveguide our system can suppress this coincidence event, which corresponds to the original Hong-Ou-Mandel effect [1, 2] extended to three interfering photons.

**MODEL AND TIME EVOLUTION**

The investigated system consists of a $3 \times 3$ waveguide (three input modes and three output modes) with continuous evanescent coupling between the inner mode and the outer modes (see Fig. 1). The coupling strength is given by the coupling coefficients $g_1$ (between the first and second mode) and $g_2$ (between the second and third mode) and is basically determined by the distance between the guides. Note that there is no direct coupling between the two outer modes.

The interaction Hamiltonian for this system reads

$$
\hat{H}_{\text{int}} = \hbar \left( g_1 \hat{a}_1 \hat{a}_2^\dagger + g_2 \hat{a}_2 \hat{a}_3^\dagger + g_1^* \hat{a}_2^\dagger \hat{a}_1^\dagger + g_2^* \hat{a}_3^\dagger \hat{a}_2^\dagger \right). 
$$

Each part of Eq. (1) stands for the annihilation of a photon in a certain mode and the creation of a photon in a neighbouring mode associated with the corresponding coupling strength $g_{1/2}$.

In order to analyse the time dependent evolution of the system we switch to the Heisenberg picture. To simplify the calculation we define a vector $\vec{a} = \left( \hat{a}_1^\dagger, \hat{a}_2^\dagger, \hat{a}_3^\dagger \right)^T$ so that the time evolution is governed by

$$
\frac{d}{dt} \vec{a}(t) = i \begin{pmatrix} 0 & g_1 & 0 \\ g_1^* & 0 & g_2 \\ 0 & g_2^* & 0 \end{pmatrix} \vec{a}(t),
$$

where the interaction time $t$ is determined by the length of the waveguide. The equation can easily be solved using the exponential ansatz $V = e^{-iMt}$, which allows to rewrite the creation operators $\hat{a}_j^\dagger(0)$ at time $t = 0$ in terms of creation operators $\hat{a}_j^\dagger(t)$ at time $t$ via $\vec{a}(0) = V \cdot \vec{a}(t)$. The solution of this equation yields the form of the matrix $V$

$$
V = \begin{pmatrix} 
\cos^2(\theta) \cos(G) + \sin^2(\theta) & -i \cos(\theta) \sin(G) e^{i\varphi + \psi} & \cos(\theta) \sin(\theta) e^{i\varphi + \psi} (\cos(G) - 1) \\
-i \cos(\theta) \sin(G) e^{-i\psi} & \cos(G) & -i \sin(\theta) \sin(G) e^{i\varphi} \\
\cos(\theta) \sin(\theta) e^{-i(\varphi + \psi)} (\cos(G) - 1) & -i \sin(\theta) \sin(G) e^{-i\varphi} & \sin^2(\theta) \cos(G) + \cos^2(\theta) 
\end{pmatrix}.
$$

where $g_1 \cdot t = G \cos(\theta) e^{i\psi}$ and $g_2 \cdot t = G \sin(\theta) e^{i\varphi}$. From this expression it is easy to see that the evolution given by $V$ is periodic with respect to $G$ and $\theta$ as these variables only appear as arguments of sine or cosine functions. Note that, as expected, the transformation $V$ is unitary - this is because we are considering a lossless device.

**THREE SINGLE PHOTON INPUT**

Next we focus on an input state where at each input port a single photon is coupled into the waveguide at $t = 0$. The wave function of this state is given by $|\psi_{\text{in}}\rangle = \hat{a}_1^\dagger(0) \hat{a}_2^\dagger(0) \hat{a}_3^\dagger(0) |0\rangle$. Via the transformation matrix $V$ we can easily calculate the general form of the output state

$$
|\psi_{\text{out}}\rangle = \sum_{l=1}^{3} V_{l1} \hat{a}_l^\dagger(t) \cdot \sum_{m=1}^{3} V_{2m} \hat{a}_m^\dagger(t) \cdot \sum_{n=1}^{3} V_{3n} \hat{a}_n^\dagger(t) |0\rangle,
$$

where $V_{mn}$ is the matrix element in the $m$th row and the $n$th column of $V$ in Eq. [3]. As the transformation matrix $V$ only acts on the creation operators of the three different modes in Eq. [4], a vacuum state is not transformed by the time evolution governed by the Hamiltonian of Eq. [1]. The general output state is
a superposition of all possible distributions of the three photons among the three output modes with respective coefficients, which depend on the explicit form of $V$. The general form of the output state reads

$$|\psi_{\text{out}}\rangle = c_{300} |3,0,0\rangle + c_{030} |0,3,0\rangle + \ldots + c_{210} |2,1,0\rangle + \ldots + c_{111} |1,1,1\rangle,$$

where corresponding coefficients can be calculated by explicitly expanding Eq. (3) or alternatively using a formalism for linear optical networks involving permanents \cite{35}.

### THREE PHOTON HONG-OU-MANDEL INTERFERENCE

A permanent of a matrix is equal to its determinant, but without the sign of the permutation taken into account. For a $N \times N$ matrix $A$ it is given by

$$\text{Perm} A = \sum_{\sigma} \prod_{j=1}^{N} A_{j\sigma(j)},$$

where the sum runs over all possible permutations $\sigma$ of the set $\{1,2,\ldots,N\}$.

The coefficients $c_{klm}$ of Eq. (6) can be expressed by permanents of matrices $V^{(k,l,m)}$, where $k$, $l$ and $m$ are the number of photons in the three output modes so that $k + l + m = 3$. Hereby $V^{(k,l,m)}$ is a $3 \times 3$ matrix and is constructed via the transformation matrix in Eq. (3). It consists of $k$ copies of the first column of $V$, $l$ copies of the second column of $V$ and $m$ copies of the third column of $V$ \cite{35}. Dividing the permanent of $V^{(k,l,m)}$ by a normalisation factor yields the final expression for the coefficients

$$c_{klm} = \frac{\text{Perm} V^{(k,l,m)}}{\sqrt{k!l!m!}}.$$  

One can show that the absolute value of these coefficients depends only on $G$ and $\theta$, but not on the phases $\psi$ and $\varphi$ of $g_{1/2}$. The last two variables are linked to the phases of the coupling coefficients $g_{1/2}$ and will only have an impact on the phases of the coefficients $c_{klm}$.

Note that Eq. (7) only holds true for the input state $|1,1,1\rangle$. However as shown in \cite{35}, the formalism involving permanents can be expanded to arbitrary initial states by additional consideration of the rows of the transformation matrix corresponding to the input state.

In the following we analyse a particular type of states displaying a three-photon Hong-Ou-Mandel (HOM) interference. In analogy to the original two-photon Hong-Ou-Mandel experiment \cite{11,2}, where two photons are never detected simultaneously at the two different output modes of a 50/50 beam splitter, the probability for all three photons leaving the waveguide at different output ports vanishes if

$$c_{111} = 0.$$  

To analyse the conditions for the three-photon HOM interference, we have to calculate $c_{111}$ explicitly. With Eqs. (3) and (7) we find

$$c_{111} = V_{11}V_{22}V_{33} + V_{12}V_{23}V_{31} + V_{13}V_{21}V_{32} + V_{11}V_{23}V_{32} + V_{12}V_{21}V_{33} + V_{13}V_{22}V_{31},$$

where each summand corresponds to a different three-photon quantum path leading to the same final state $|1,1,1\rangle$. For example the first term corresponds to the case, where all three photons exit the waveguide in the same mode they came in, the second term corresponds to the case, where the photon in the first/second/third mode switches to the second/third/first mode etc. As the photons are indistinguishable the coefficient $c_{111}$ is a coherent superposition of the amplitudes of all these quantum paths. By inserting the expression for the various matrix elements $V_{mn}$ and solving Eq. (8) we can find an analytical expression for the HOM contour in the variable space $(G, \theta)$, where all states $|\psi_{\text{out}}(G,\theta)\rangle$ have a vanishing $c_{111}$ coefficient:

$$\theta(G) = n\pi \pm \arccsc \left[ 4 \sqrt{\frac{8 \pm \sqrt{2 \cos^4\left(\frac{\theta}{2}\right) \sin^2\left(\frac{\theta}{2}\right) (20 \cos(G) + 3(8 \cos(2G)) + 4 \cos(3G) + 3 \cos(4G) + 5))}{3 \cos(G) + 2}} \right].$$
As can be seen from Eq. (10) completely destructive three photon HOM interference can take place for a large range of the parameters $g_1$ and $g_2$. Note that for some values of $G$ these equations would result in a complex valued $\theta$ and are therefore not considered as a solution. Fig. 2 shows a plot of this HOM contour, which is (2$\pi$-) periodic in $G$ and $\theta$.

**INTERESTING STATES ON THE HONG-OU-MANDEL CONTOUR**

Finally we investigate the states determined by the HOM contour. In the original HOM experiment a maximally entangled state of the form $|2,0\rangle - |0,2\rangle$ is produced at the output [12]. Similar states can be found in the case of a three-photon interference. Additionally to the condition $c_{111} = 0$ we find that at certain points some coefficients $c_{klm}$ of Eq. (5) will vanish as well, so that further terms are suppressed. Other coefficients will have the same absolute value, so that the states can be written in a compact form. We found three different kinds of states fulfilling this condition, which display entanglement between two and possibly three output modes.

In Fig. 3(a), where the HOM contour is shown, some coordinates are marked by points, where one can find maximally bipartite entangled states. A closer investigation yields that for these all coordinates we find states of the form

$$|\psi_{\text{out}}\rangle = \frac{1}{\sqrt{2}} (|2_j, 0_k\rangle + |0_j, 2_k\rangle) |l_t\rangle$$

(11)

with $j = 1, k = 3$ and $l = 2$ at the red crosses, with $j = 2, k = 3$ and $l = 1$ at the green dots and with $j = 1, k = 2$ and $l = 3$ at the blue diamonds. Note that for simplicity we neglected phase factors in front each state (see appendix for the exact analytical expression for each coefficient and the coordinates). All the three states have a similar form, where one mode containing one photon is separable while the remaining two modes are in a maximally entangled state. Depending on the phase $\psi/\varphi$ of the coupling coefficients $g_{1/2}$ the relative phase between the non-separable states can be varied. Note that these states are created in a deterministic way so that no post selection is necessary.

In Fig. 3(b) and (c) the coordinates of possible tripartite entangled states along the HOM contour are displayed. Note that we have three modes and each mode has four possible states-corresponding to the occupation of 0, 1, 2 and 3 photons. We are thus dealing with higher dimensional entanglement of three qudits ($d = 4$). A complete classification of the classes of entangled states for qudits ($d = 4$) does not exist. However the structure of tripartite states generated for the case of Figs. 3(b) and (c) suggests three qudit ($d = 4$) entanglement. The form of the states of Fig. 3(b) and their coordinates read

$$|\psi_{\text{out}}\rangle = \frac{\sqrt{3}}{4} (|3_j, 0_k\rangle + |0_j, 3_k\rangle) |l_t\rangle$$

$$+ \frac{1}{4} (|2_j, 1_k\rangle + |1_j, 2_k\rangle) |l_t\rangle$$

$$+ \frac{1}{2} (|1_j, 0_k\rangle + |0_j, 1_k\rangle) |2_t\rangle$$

(12)

with $j = 2, k = 3$ and $l = 1$ at the red crosses, with $j = 1, k = 3$ and $l = 2$ at the green dots and with $j = 1, k = 2$ and $l = 3$ at the blue diamonds. The form of the wave function of these states of Fig. 3(c) and their coordinates are given by

$$|\psi_{\text{out}}\rangle = \frac{1}{3\sqrt{2}} (2 |3_j, 0_k, 0_l\rangle + |0_j, 3_k, 0_l\rangle + |0_j, 0_k, 3_l\rangle)$$

$$+ \frac{1}{\sqrt{6}} (|1_j, 0_l\rangle + |0_j, 1_l\rangle) |2_k\rangle$$

$$+ \frac{1}{\sqrt{6}} (|1_j, 0_l\rangle + |0_j, 1_l\rangle) |2_k\rangle$$

(13)

with $j = 1, k = 2$ and $l = 3$ at the red crosses, with $j = 2, k = 1$ and $l = 3$ at the green dots and with $j = 3, k = 1$ and $l = 2$ at the blue diamonds. As before we neglected phase factors in front each state for simplicity (see appendix for tables containing all coefficients and the analytical expressions for coordinates for all states). We have written the states in a way that a certain mode is always factored out in each term so that the entanglement between the two remaining modes is clearly visible; this suggests that the states are not separable and are good candidates for tripartite entanglement.

**CONCLUSION**

In conclusion, we investigated the dynamics of a 3×3 waveguide where the outer modes are coupled to the inner mode by evanescent coupling but not to each other. Beginning with three indistinguishable single photons at the three input ports we showed that for a wide range of waveguide parameters this leads to completely destructive three photon interference, i.e. for these parameters the photons will never leave the waveguide in three separate ports. This is a generalisation of the well known Hong-Ou-Mandel effect from two to three photons. Additionally the produced output states consisting of three qudits ($d = 4$), exhibit highly interesting structures displaying bipartite or possibly tripartite entanglement.

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FIG. 3. Plots of the HOM contour and the coordinates of the investigated states (red crosses, green dots and blue diamonds), which display bipartite entanglement (a) or possibly tripartite entanglement (b)/(c).

[1] C. K. Hong, Z. Y. Ou and L. Mandel, “Measurement of subpicosecond time intervals between two photons by interference”, Phys. Rev. Lett. 59, 2044 (1987).
[2] Y. H. Shih and C. O. Alley, “New Type of Einstein-Podolsky-Rosen-Bohm Experiment Using Pairs of Light Quanta Produced by Optical Parametric Down Conversion”, Phys. Rev. Lett. 61, 2921 (1988).
[3] A. Politi, M. J. Cryan, J. G. Rarity, S. Yu and J. L. O’Brien, “Silicon-on-silicon waveguide quantum circuits”, Science 320, 646 (2008).
[4] A. Rai, G. S. Agarwal and J. H. H. Perk, “Transport and quantum walk of nonclassical light in coupled waveguides”, Phys. Rev. A 78, 042304 (2008).
[5] M. S. Tame, K. R. McEnery, S. K. Ozdemir, J. Lee, S. A. Maier and M. S. Kim, “Quantum plasmonics”, Nature Physics 9, 329 (2013).
[6] S. D. Gupta and G. S. Agarwal, “Two-photon quantum interference in plasmonics: theory and applications”, Opt. Lett. 39, 390 (2014).
[7] G. Di Martino, Y. Sonnefraud, M. S. Tame, S. Kena-Cohen, F. Dielman, S. K. Ozdemir, M. S. Kim and S. A. Maier, “Observation of quantum interference in the plasmonic Hong-Ou-Mandel effect”, Phys. Rev. Applied 1, 034004 (2014).
[8] J. S. Fakonas, H. Lee, Y. A. Kelaita, and H. A. Atwater, “Two-plasmon quantum interference”, Nature Photonics 8, 317 (2014).
[9] P. Maunz, D. L. Moehring, S. Olmschenk, K. C. Younge, D. N. Matsukevich and C. Monroe, “Quantum interference of photon pairs from two remote trapped atomic ions”, Nature Physics 3, 538 (2007).
[10] J. Gillet, G. S. Agarwal, T. Bastin, “Tunable entanglement, antibunching, and saturation effects in dipole blockade”, Phys. Rev. A 81, 013837 (2010).
[11] R. Wiegner, C. Thiel, J. von Zanthier and G. S. Agarwal, “Creating path entanglement and violating Bell inequalities by independent photon sources”, Phys. Rev. A 374, 3405 (2010).
[12] J. Hofmann, M. Krug, N. Ortegel, L. Gerard, M. Weber, W. Rosenfeld, and H. Weinfurter, “Heralded Entanglement Between Widely Separated Atoms”, Science 337, 72 (2012).
[13] R. B. Patel, A. J. Bennett, I. Farrer, C. A. Nicoll, D. A. Ritchie and A. J. Shields, “Two-photon interference of the emission from electrically tunable remote quantum dots”, Nature Photonics 4, 632 (2010).
[14] P. Gold, A. Thoma, S. Maier, S. Reitzenstein, C. Schneider, S. Höfling and M. Kamp, “Two-photon interference from remote quantum dots with inhomogeneously broadened linewidths”, Phys. Rev. B 89, 035313 (2014).
[15] X. Li, L. Yang, L. Cui, Z. Y. Ou and D. Yu, “Observation of quantum interference between a single-photon state and a thermal state generated in optical fibers”, Opt. Express 16(17), 12505 (2008).
[16] K. Laiho, K. N. Cassemiro, Ch. Silberhorn, “Producing high fidelity single photons with optimal brightness via waveguided parametric down-conversion”, Opt. Express 17(25), 22823 (2009).
[17] R. Wiegner, J. von Zanthier, and G. S. Agarwal, “Quantum interference and non-locality of independent photons from disparate sources”, J. Phys. B 44, 055501 (2011).
[18] Z. Y. Ou, J.-K. Rhee and L. J. Wang, “Observation of Four-Photon Interference with a Beam Splitter by Pulsed Parametric Down-Conversion”, Phys. Rev. Lett. 83, 959 (1999).
[19] H. Wang and T. Kobayashi, “Phase measurement at the Heisenberg limit with three photons”, Phys. Rev. A 71, 021802 (2005).
[20] B. H. Liu, F. W. Sun, Y. X. Gong, Y. F. Huang, G. C. Guo and Z. Y. Ou, “Four-photon interference with asymmetric beam splitters”, Opt. Lett. 32(10), 1320 (2007).
[21] G. S. Agarwal, Quantum Optics, (Cambridge University Press, 2012), Sec. (5.7).
[22] S. Tanzilli, A. Martin, F. Kaiser, M. P. De Micheli, O. Alibart and D.B. Ostrowsky, “On the genesis and evolution of Integrated Quantum Optics”, Laser & Photon.
Rev. 6, 115 (2012).

[23] Y. L. Lim and A. Beige, “Generalized Hong-Ou-Mandel experiments with bosons and fermions”, New J. Phys. 7, 155 (2005).

[24] M. C. Tichy, M. Tiersch, F. De Melo, F. Mintert and A. Buchleitner, “Zero-transmission law for multiport beam splitters”, Phys. Rev. Lett. 104, 220405 (2010).

[25] R. Campos, “Three-photon Hong-Ou-Mandel interference at a multiport mixer”, Phys. Rev. A 62, 013809 (2000).

[26] M. Reck and A. Zeilinger, “Experimental realization of any discrete unitary operator”, Phys. Rev. Lett. 73, 58 (1994).

[27] S.-H. Tan, Y. Y. Gao, H. De Guise and B. C. Sanders, “SU(3) quantum interferometry with single-photon input pulses”, Phys. Rev. Lett. 110, 113603 (2013).

[28] G. Weihs, M. Reck, H. Weinfurter and A. Zeilinger, “Two-photon interference in optical fiber multiports”, Phys. Rev. A 54, 893 (1996).

[29] T. Meany, M. Delanty, S. Gross, G. D. Marshall, M. J. Steel and M. J. Withford, “Non-classical interference in integrated 3D multiports”, Opt. Express 20(24), 26895 (2012).

[30] Z. Chaboyer, T. Meany, L. G. Helt, M. J. Withford and M. J. Steel, “Tunable quantum interference in a 3D integrated circuit”, arXiv:1409.4908 [quant-ph] (2014).

[31] M. A. Broome, A. Fedrizzi, S. Rahimi-Keshari, J. Dove, S. Aaronson, T. C. Ralph and A. G. White, “Photonic boson sampling in a tunable circuit”, Science 339, 794 (2013).

[32] J. B. Spring, B. J. Metcalf, P. C. Humphreys, W. S. Kolthammer, X.-M. Jin, M. Barbieri, A. Datta, N. Thomas-Peter, N. K. Langford, D. Kundys, J. C. Gates, B. J. Smith, P. G. R. Smith and I. A. Walmsley, “Boson sampling on a photonic chip”, Science 339, 798 (2013).

[33] M. Tillmann, B. Dakić, R. Heilmann, S. Nolte, A. Szameit and P. Walther, “Experimental boson sampling”, Nature Photonics 7, 540 (2013).

[34] N. Spagnolo, C. Vitelli, L. Aparo, P. Mataloni, F. Sciarrino, A. Crespi, R. Ramponi and R. Osellame, “Three-photon bosonic coalescence in an integrated tritter”, Nature communications 4, 1606 (2013).

[35] S. Scheel, “Permanents in linear optical networks”, arXiv:quant-ph/0406127 (2004).

APPENDIX

For certain waveguide parameters $G$ and $\theta$ we found three interesting sets of states on the HOM contour, on which the coincident event is suppressed and therefore $c_{111}$ vanishes. However only the general form of these states was discussed before but not the explicit values of coefficients. Here we present their analytical values: Tables I to VII contain the exact analytical expressions for the coefficients of Eqs. (11) to (13) of the paper. Note that in Eqs. (12) and (13) of the main paper some signs depend on the value of $n \mod 4$. Therefore in Table III and VI $n$ is replaced by $4\bar{n} + 0, 4\bar{n} + 1, 4\bar{n} + 2$ or $4\bar{n} + 3$, in which case all possible cases are considered.
TABLE I. Table of coefficients for equation (11). Coefficients, which are not listed, are equal to 0.

| $G \theta$ | $\pi(2m+1+\frac{1}{2})$ | $\pi(2m+1+\frac{1}{2})$ | $\frac{\pi}{2}(2m+1)$ | $\frac{\pi}{2}(2m+1)$ |
|------------|----------------|----------------|----------------|----------------|
| $c_{020}$  | $\frac{\pi}{4} \sqrt{2}$ | $\pi(\psi-e)$ | $-\frac{\pi}{8} \sqrt{2}$ | $\pi(2\psi+e)$ |
| $c_{021}$  | $\frac{\pi}{4} \sqrt{2}$ | $\pi(\psi-e)$ | $-\frac{\pi}{8} \sqrt{2}$ | $\pi(2\psi+e)$ |
| $c_{120}$  | $\frac{\pi}{4} \sqrt{2}$ | $\pi(\psi-e)$ | $-\frac{\pi}{8} \sqrt{2}$ | $\pi(2\psi+e)$ |
| $c_{121}$  | $\frac{\pi}{4} \sqrt{2}$ | $\pi(\psi-e)$ | $-\frac{\pi}{8} \sqrt{2}$ | $\pi(2\psi+e)$ |

TABLE II. Table of coefficients for equation (12). Coefficients, which are not listed, are equal to 0.

| $G \theta$ | $\frac{\pi}{4}(2(4n+0)+1)$ | $\frac{\pi}{4}(2(4n+1)+1)$ | $\frac{\pi}{4}(2(4n+2)+1)$ | $\frac{\pi}{4}(2(4n+3)+1)$ |
|------------|----------------|----------------|----------------|----------------|
| $c_{020}$  | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ |
| $c_{021}$  | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ |
| $c_{120}$  | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ |
| $c_{121}$  | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ |

TABLE III. Table of coefficients for equation (12). Coefficients, which are not listed, are equal to 0.

| $G \theta$ | $\pi(2m+1+\frac{1}{2})$ | $\pi(2m+1+\frac{1}{2})$ | $\frac{\pi}{2}(2m+1+\frac{1}{2})$ | $\frac{\pi}{2}(2m+1+\frac{1}{2})$ |
|------------|----------------|----------------|----------------|----------------|
| $c_{030}$  | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ |
| $c_{031}$  | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ |
| $c_{130}$  | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ |
| $c_{131}$  | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ | $\frac{\pi}{8} \cdot (e^{-i(2\psi+e)})$ |

TABLE IV. Table of coefficients for equation (12). Coefficients, which are not listed, are equal to 0.
TABLE V. Table of coefficients for equation 13. Coefficients, which are not listed, are equal to 0.
### TABLE VI. Table of coefficients for equation (13). Coefficients, which are not listed, are equal to 0.

| \( \theta \) | \( \frac{2\pi m}{\pi} \) | \( 2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right) \) |
|-------------|----------------|----------------------------------|
| \( \frac{\pi}{4} \) | \( \frac{2\pi m}{\pi} \) | \( \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \) |
| \( \frac{\pi}{8} \) | \( \frac{2\pi m}{\pi} \) | \( \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \) |
| \( \frac{3\pi}{8} \) | \( \frac{2\pi m}{\pi} \) | \( \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \) |
| \( \frac{5\pi}{8} \) | \( \frac{2\pi m}{\pi} \) | \( \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \) |
| \( \frac{7\pi}{8} \) | \( \frac{2\pi m}{\pi} \) | \( \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \) |

\[
G \quad \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \\
\theta \quad \frac{2\pi m}{\pi} \quad \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \\
\frac{\pi}{4} \quad \frac{2\pi m}{\pi} \quad \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \\
\frac{\pi}{8} \quad \frac{2\pi m}{\pi} \quad \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \\
\frac{3\pi}{8} \quad \frac{2\pi m}{\pi} \quad \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \\
\frac{5\pi}{8} \quad \frac{2\pi m}{\pi} \quad \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi} \\
\frac{7\pi}{8} \quad \frac{2\pi m}{\pi} \quad \frac{2\pi m + \arctan\left(\sqrt{2 + \sqrt{3}}\right)}{\pi}
\]
\[ G \quad 2\pi \nu + 2 \arctan \left( \sqrt{5 + 2\sqrt{3}} \right) \]
\[ \theta \quad \pi \nu + \arctan \left( 1 + \sqrt{3} \right) \quad \pi \nu - \arctan \left( 1 + \sqrt{3} \right) \]

\[ \begin{array}{c|c|c}
\text{Index} & \text{Coefficient} & \text{Coefficient} \\
\hline
\text{c300} & 1 \frac{-i (-1)^n e^{-i(2\nu+\phi)}}{\sqrt{2}} & -1 \frac{-i (-1)^n e^{-i(2\nu+\phi)}}{\sqrt{2}} \\
\text{c300} & \frac{1}{3\sqrt{2} i(\psi - \nu)} & -\frac{1}{3\sqrt{2} i(\psi - \nu)} \\
\text{c300} & -i \frac{(-1)^n \sqrt{2} e^{\psi+2\phi}}{3} & -i \frac{(-1)^n \sqrt{2} e^{\psi+2\phi}}{3} \\
\text{c210} & \frac{1}{\sqrt{2} e^{-i(\psi+\phi)}} & -\frac{1}{\sqrt{2} e^{-i(\psi+\phi)}} \\
\text{c201} & -1 \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} & -1 \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} \\
\text{c120} & i \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} & i \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} \\
\text{c021} & -i \frac{(-1)^n \phi}{\sqrt{2}} & -i \frac{(-1)^n \phi}{\sqrt{2}} \\
\end{array} \]

\[ G \quad 2\pi \nu - \arctan \left( 1 + \sqrt{3} \right) \quad \pi \nu - \arctan \left( 1 + \sqrt{3} \right) \]

\[ \begin{array}{c|c|c}
\text{Index} & \text{Coefficient} & \text{Coefficient} \\
\hline
\text{c300} & 1 \frac{-i (-1)^n e^{-i(2\nu+\phi)}}{\sqrt{2}} & -1 \frac{-i (-1)^n e^{-i(2\nu+\phi)}}{\sqrt{2}} \\
\text{c300} & \frac{1}{3\sqrt{2} i(\psi - \nu)} & -\frac{1}{3\sqrt{2} i(\psi - \nu)} \\
\text{c300} & i \frac{(-1)^n \sqrt{2} e^{\psi+2\phi}}{3} & i \frac{(-1)^n \sqrt{2} e^{\psi+2\phi}}{3} \\
\text{c210} & \frac{1}{\sqrt{2} e^{-i(\psi+\phi)}} & -\frac{1}{\sqrt{2} e^{-i(\psi+\phi)}} \\
\text{c201} & -1 \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} & -1 \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} \\
\text{c120} & i \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} & i \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} \\
\text{c021} & -i \frac{(-1)^n \phi}{\sqrt{2}} & -i \frac{(-1)^n \phi}{\sqrt{2}} \\
\end{array} \]

\[ G \quad 2\pi \nu + 2 \arctan \left( \sqrt{5 - 2\sqrt{3}} \right) \]
\[ \theta \quad \pi \nu + \arctan \left( 1 - \sqrt{3} \right) \quad \pi \nu - \arctan \left( 1 - \sqrt{3} \right) \]

\[ \begin{array}{c|c|c}
\text{Index} & \text{Coefficient} & \text{Coefficient} \\
\hline
\text{c300} & 1 \frac{-i (-1)^n e^{-i(2\nu+\phi)}}{\sqrt{2}} & -1 \frac{-i (-1)^n e^{-i(2\nu+\phi)}}{\sqrt{2}} \\
\text{c300} & \frac{1}{3\sqrt{2} i(\psi - \nu)} & -\frac{1}{3\sqrt{2} i(\psi - \nu)} \\
\text{c300} & i \frac{(-1)^n \sqrt{2} e^{\psi+2\phi}}{3} & i \frac{(-1)^n \sqrt{2} e^{\psi+2\phi}}{3} \\
\text{c210} & \frac{1}{\sqrt{2} e^{-i(\psi+\phi)}} & -\frac{1}{\sqrt{2} e^{-i(\psi+\phi)}} \\
\text{c201} & -1 \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} & -1 \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} \\
\text{c120} & i \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} & i \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} \\
\text{c021} & -i \frac{(-1)^n \phi}{\sqrt{2}} & -i \frac{(-1)^n \phi}{\sqrt{2}} \\
\end{array} \]

\[ G \quad 2\pi \nu - 2 \arctan \left( \sqrt{5 - 2\sqrt{3}} \right) \]
\[ \theta \quad \pi \nu + \arctan \left( 1 - \sqrt{3} \right) \quad \pi \nu - \arctan \left( 1 - \sqrt{3} \right) \]

\[ \begin{array}{c|c|c}
\text{Index} & \text{Coefficient} & \text{Coefficient} \\
\hline
\text{c300} & 1 \frac{-i (-1)^n e^{-i(2\nu+\phi)}}{\sqrt{2}} & -1 \frac{-i (-1)^n e^{-i(2\nu+\phi)}}{\sqrt{2}} \\
\text{c300} & \frac{1}{3\sqrt{2} i(\psi - \nu)} & -\frac{1}{3\sqrt{2} i(\psi - \nu)} \\
\text{c300} & -i \frac{(-1)^n \sqrt{2} e^{\psi+2\phi}}{3} & -i \frac{(-1)^n \sqrt{2} e^{\psi+2\phi}}{3} \\
\text{c210} & \frac{1}{\sqrt{2} e^{-i(\psi+\phi)}} & -\frac{1}{\sqrt{2} e^{-i(\psi+\phi)}} \\
\text{c201} & -1 \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} & -1 \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} \\
\text{c120} & i \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} & i \frac{(-1)^n e^{-i \psi}}{\sqrt{2}} \\
\text{c021} & -i \frac{(-1)^n \phi}{\sqrt{2}} & -i \frac{(-1)^n \phi}{\sqrt{2}} \\
\end{array} \]

**TABLE VII.** Table of coefficients for equation (13). Coefficients, which are not listed, are equal to 0.