Superhydrophobic Cassie textures with trapped gas bubbles reduce drag, by generating large effective slip, which is important for a variety of applications that involve a manipulation of liquids at the small scale. Here we discuss how the dissipation in the gas phase of textures modifies their friction properties and effective slip. We propose an operator method, which allows us the mapping of the flow in the gas subphase to a local slip boundary condition at the liquid/gas interface. The determined uniquely local slip length depends on the viscosity contrast and underlying topography, and can be immediately used to evaluate an effective slip of the texture. Beside Cassie surfaces our approach is valid for Wenzel textures, where a liquid follows the surface relief, as well as for rough surfaces impregnated by a low-viscosity ‘lubricant’. These results provide a framework for the rational design of textured surfaces for numerous applications.

PACS numbers: 83.50.Rp, 47.61.-k, 68.03.-g

Introduction. – Superhydrophobic (SH) textures have raised a considerable interest and motivated numerous studies during the past decade. Such surfaces in the Cassie state, i.e., where the texture is filled with gas, can induce exceptional wetting properties [1] and, due to their superlubricating potential [2,3], are also extremely important in context of fluid dynamics. To quantify the drag reduction associated with two-component (e.g., local no-slip and slip) SH surfaces with given area fractions it is convenient to construct the effective slip boundary condition (on the scale larger than the pattern characteristic length) applied at the imaginary smooth homogeneous surface [4,5], which mimics the actual one and fully characterizes the flow at the real surface and is generally a tensor [6]. Once eigenvalues of the slip-length tensor are determined, they can be used to solve complex hydrodynamic problems without tedious calculations. The eigenvalues depend on the local slip at the gas sectors. A key difficulty is that there is no general analytical theory that relates it to the relief of the texture, so that prior work often simply neglected the viscous dissipation in the gas phase, by imposing idealized shear-free boundary conditions at the gas sectors [7,8,9].

To account for the extra dissipation within the gas subphase it is necessary to solve Stokes equations by applying conditions

$$z = 0 : \mathbf{u} = \mathbf{u}_g, \quad \frac{\partial \mathbf{u}_r}{\partial z} = \mu_g \frac{\partial \mathbf{u}_g}{\partial z},$$

(1)

where $\mathbf{u}$ and $\mu$ are the velocity and the dynamic viscosity of the liquid, and $\mathbf{u}_g$ and $\mu_g$ are those of the gas, $\mathbf{u}_r = (u_x, u_y)$ is the tangential velocity. This problem has been resolved for rectangular grooves [10,11]. Such a strategy appears rather hopeless in context of exact analytical results, especially for complex configurations, which are typical for many applications. To bypass this problem, it is advantageous to replace the two-phase approach, by a single-phase problem with spatially dependent partial slip boundary condition [2,14], which for grooved surfaces (see Fig. 1) takes a form

$$z = 0 : \mathbf{u}_r - b(y) \frac{\partial \mathbf{u}_r}{\partial z} = 0,$$

(2)

where $b(y)$ is the local slip length at the gas areas, which is assumed to simply conform the texture relief according to predictions of the ‘gas cushion’ model [13]

$$b^{x,y}(y) \simeq k^{x,y} \frac{\mu}{\mu_g} e(y),$$

(3)

where prefactors $k^{x,y} = 1$ can reduce to $1/4$ if the net gas flux becomes zero (due to end walls) [15]. Such an approach, justified for a continuous gas layer at a homogeneous surface [15] and later for shallow grooves [16], is not by means obvious for an arbitrary texture, where the gas subphase can be deep and strongly confined. In such

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**FIG. 1:** Sketch of the grooved SH surface.
a situation it remains largely unknown if the gas flow can be indeed excluded from the analysis being equivalently replaced by \( b(y) \), and how (and whether) this local slip profile is uniquely related to the relief of the texture.

In this letter, we propose a general theoretical method, which allows us to validate the local slip approach and to generalize the ‘gas cushion’ model for any SH surfaces in the Cassie and Wenzel states, or impregnated by a low-viscosity ‘lubricant’.

**Theory.** To illustrate our approach, we consider periodic rectangular grooves of width \( \delta \), depth \( e \) and period \( L \) (see Fig. 1). The fraction of gas area is then \( \phi = \delta / L \). We then assume no-slip at the solid area, i.e. neglect slippage of liquid \([17, 18]\) and gas \([19]\) past hydrophobic surface, which is justified provided the nanometric slip is small compared to parameters of the texture. We make no further assumptions, aside from distinguishing between longitudinal and transverse gas flow, with \( k^x = 1 \) and \( k^y = 1/4 \), to address the most anisotropic case.

The linearity of Stokes equations implies that the boundary condition at the liquid/gas interface for longitudinal and transverse directions can be formulated as:

\[
z = 0 : \frac{\partial u_{x,y}}{\partial z} - P^{x,y}[u_{g}] = 0, \tag{4}
\]

where we introduced the linear operator \( P^{x,y} \) that belongs to a general class of Dirichlet-to-Neumann ones \([20]\). The meaning of Eq. (3) becomes clear when we solve the Stokes equations in the gas subphase numerically by using collocation methods and introducing a grid at the interface. In this case the operator is simply a matrix \( P_{ij} \) that relates the shear rate at a given point \( i \) with velocities in every other points of the interface \( j = 1..M \) (the condition is essentially nonlocal). Unlike the local slip length, the operator depends only on the groove geometry, but not on the solution outside. It is universal and, once calculated for a given groove shape, can be applied for any geometry of the outside flow and viscosity ratio. Then, in view of Eq. (1), the non-local boundary condition for fluid flow over the groove reads:

\[
\frac{\partial u_{x,y}(y_j,0)}{\partial z} - \frac{\mu}{\mu_g} \sum_{j=1}^{M} P^{x,y}_{ij} u_{x,y}(y_j,0) = 0, \quad i = 1..M. \tag{5}
\]

Now we are able to solve the Stokes equations for fluid phase separately, and then determine the local slip length by using Eq. (2). We remark and stress that Eq. (5) does not necessarily imply that \( b^{x,y}(y) \) will be scaled with \( \mu / \mu_g \). To calculate the matrices \( P^{x,y}_{ij} \), we solve the Stokes equations in the gas subphase by using Fourier series \([21]\) similarly to the approach suggested before \([12]\). For the longitudinal flow the gas velocity is represented in terms of a cosine series which satisfies automatically the Stokes equations and the no-slip conditions at the side and at the bottom walls. The normal derivative of the gas velocity is expressed in terms of the same cosine series with the coefficients calculated analytically. Cutting the series to \( M \) terms and evaluating the gas shear rate and gas velocity at \( M \) grid nodes \( y_i = (i-1)/(2M), \quad i = 1..M \) of the interface we obtain an explicit form of the matrix \( P_{ij}^{x} \) (with summation over repeated indices):

\[
P^{x}_{ij} = \frac{1}{\delta} \left( F_{im} \Pi_{ml} F_{ij}^{-1} \right),
F_{im} = \cos(k_m^* y_i / \delta), \quad \Pi_{ml} = k_m^* \coth(k_m^* e / \delta) \delta_{ml}, \tag{6}
\]

where \( k_m^* = (2m-1) \pi \) and \( \delta_{ml} \) is the Kronecker delta.

The non-local boundary condition, Eq. (6), can now easily be implemented to solve the Stokes equations by using the Fourier method to obtain the liquid velocity \( u_x(y, z), \quad u_y(y, z) \) and then to reconstruct the local slip by using Eq. (2).

**Results and discussion.** Figs. (a) and (b) show profiles of the longitudinal, \( b^{x}(y) \), and transverse, \( b^{y}(y) \), local slip lengths at fixed groove width \( \delta / L = 0.75 \) and aspect ratio, \( e / \delta \), varying from 0.1 to infinity. The calculations are made using \( \mu / \mu_g = 50 \), which corresponds to a SH texture filled with gas. It can be seen that for shallow grooves, \( e / \delta \ll 1 \), local slip lengths saturate to constant values predicted by Eqs. (3) at the central part of the gas sector, but \( b^{x,y}(y) \) vanish at the edge of the groove. Thus the local slip profiles can be roughly approximated by a trapezoid \([22]\). For deeper grooves the local slip curves look more as parabolic. At \( e / \delta \geq 1 \) the local slip curves converge to a single curve suggesting that \( b^{x,y}(y) \) of deep grooves are controlled by the value of \( \delta \) only, being independent on texture depth. This result does not support Eq. (3), which predicts that \( b^{x,y}(y) \) are growing infinitely with \( e \).

![FIG. 2. Polar coordinates used to evaluate the flow near the edge of the groove.](image-url)
ution of the Stokes equations imply a power-law dependence of velocities on the distance, \( u \propto r^\lambda \):

\[
u_x = r^\lambda \sin(\theta\partial), \quad u_{yy} = r^\lambda \left[ c \sin(\lambda\theta) + \cos(\lambda\theta) \right]. \tag{7}
\]

Here \( u_x \) satisfies the no-slip condition at the liquid/solid interface, \( \theta = 0 \). Eq. (7) should also satisfy the interface boundary conditions, Eq. (1), \( \theta = \pi \) and the no-slip condition at the side wall of the groove \( \theta = 3\pi/2 \). These conditions yield an equation for the exponent \( \lambda \) \cite{21}, which depends on the viscosity ratio only. We then conclude that the local slip length (which is the ratio of \( u_x \) to \( \partial u_x / \partial z \)) is linear in \( r \), \( b^\perp = b_g^\perp \delta \) with \( b_g^\perp = -\tan(\lambda\pi)/\lambda \).

For a large viscosity contrast, \( \mu_g/\mu \ll 1 \), we derive \( \lambda \approx 1/2 - \mu_g/\mu \pi \) and \( b_g^\perp \approx 2\mu/\mu_g \). Applying similar asymptotic analysis to the flow transverse to grooves yields \( b_g^\parallel = b_g^\perp \delta \) \cite{21} with \( b_g^\parallel = \tan(\lambda\pi)/(2\lambda - 2) \), and the asymptotic angle is \( b_g^\parallel \approx \mu/(2\mu_g) \). Therefore, the slopes of the profiles \( b^{x,y}(y) \) at the edge are proportional to \( \mu/\mu_g \), provided this ratio is large enough.

\[ b^{x,y}(y) = \delta \mu/\mu_g \beta^{x,y}(e/\delta, y/\delta). \tag{8} \]

Here we ascribe rescaled dimensionless local slip lengths, \( \beta^{x,y} \), which become linear in \( e/\delta \) when \( e/\delta \) is small, and we recover Eq. (3). At the other extreme, when \( e/\delta \) is large, \( \beta^{x,y} \) saturate to provide an upper limit for local slip lengths. To verify this Anzats in Fig. 3(a) and (b) we plot \( \beta^{x,y} \) as a function of \( y/\delta \) at different \( \phi \) and \( e/\delta \). Here we use a viscosity ratio of the Cassie state as in Figs. 3(a) and (b). Also included are results calculated for the Wenzel state, where the liquid follows the topological variations of the texture, and \( \mu/\mu_g \approx 1 \). The results are somewhat remarkable. We see that for relatively deep grooves, \( e/\delta \geq 1 \), \( \beta^{x,y} \) profiles computed for different \( \mu/\mu_g \), \( e/\delta \) and even \( \phi \), practically converge into a single curve, which can be fitted by simple fourth order polynomials \cite{21}. The conclusion about insensitivity to \( \phi \) is actually counter-intuitive: it shows that a flow in the groove is unaffected by hydrodynamic interactions with others grooves. Note, however, that in case of two-dimensional textures (e.g., pillars) the dependence of local slip on \( \phi \) is expected \cite{25}.

For shallow grooves \( e/\delta \leq 0.1 \) for a longitudinal and \( e/\delta \leq 0.25 \) for a transverse case) the \( \beta^{x,y} \) profiles depend only on the depth of the groove, and can be approximated by trapezoids with the central region of a constant slip given by Eq. (3), and linear edge regions where the local slip length is described by our asymptotic model.

\[ b^{x,y} = \delta \mu/\mu_g \beta^{x,y}(e/\delta, y/\delta). \tag{8} \]

FIG. 3: Longitudinal (a) and transverse (b) local slip lengths computed with \( \phi = 0.75 \), \( \mu/\mu_g = 50 \). Dashed lines show the predictions of Eq. (3).

The evaluated asymptotic slopes indicate that an upper bound for the maximal local slip of deep grooves is below \( \delta\mu/\mu_g \). It is natural then to propose the generalization of Eq. (3) for a textured surface, where we scale with \( \delta \) instead of \( L \):

\[ b^{x,y}(y) = \delta \mu/\mu_g \beta^{x,y}(e/\delta, y/\delta). \tag{8} \]

Here we ascribe rescaled dimensionless local slip lengths, \( \beta^{x,y} \), which become linear in \( e/\delta \) when \( e/\delta \) is small, and we recover Eq. (3). At the other extreme, when \( e/\delta \) is large, \( \beta^{x,y} \) saturate to provide an upper limit for local slip lengths. To verify this Anzats in Fig. 3(a) and (b) we plot \( \beta^{x,y} \) as a function of \( y/\delta \) at different \( \phi \) and \( e/\delta \). Here we use a viscosity ratio of the Cassie state as in Figs. 3(a) and (b). Also included are results calculated for the Wenzel state, where the liquid follows the topological variations of the texture, and \( \mu/\mu_g = 1 \). The results are somewhat remarkable. We see that for relatively deep grooves, \( e/\delta \geq 1 \), \( \beta^{x,y} \) profiles computed for different \( \mu/\mu_g \), \( e/\delta \) and even \( \phi \), practically converge into a single curve, which can be fitted by simple fourth order polynomials \cite{21}. The conclusion about insensitivity to \( \phi \) is actually counter-intuitive: it shows that a flow in the groove is unaffected by hydrodynamic interactions with others grooves. Note, however, that in case of two-dimensional textures (e.g., pillars) the dependence of local slip on \( \phi \) is expected \cite{25}.

For shallow grooves \( e/\delta \leq 0.1 \) for a longitudinal and \( e/\delta \leq 0.25 \) for a transverse case) the \( \beta^{x,y} \) profiles depend only on the depth of the groove, and can be approximated by trapezoids with the central region of a constant slip given by Eq. (3), and linear edge regions where the local slip length is described by our asymptotic model.

\[ b^{x,y} = \delta \mu/\mu_g \beta^{x,y}(e/\delta, y/\delta). \tag{8} \]

FIG. 4: Rescaled longitudinal (a) and transverse (b) local slip-length for deep and shallow grooves. Solid and dotted curves correspond to the Cassie, \( \mu/\mu_g = 50 \), and Wenzel, \( \mu/\mu_g = 1 \), states. From top to bottom \( \phi = 0.1, 0.5, 0.9 \). Dash-dotted lines show asymptotic solutions near the edges of the groove.

\[ b^{x,y} = \delta \mu/\mu_g \beta^{x,y}(e/\delta, y/\delta). \tag{8} \]

FIG. 5: Longitudinal (a,b) and transverse (c,d) effective slip lengths for textures with shallow (a,c) and deep (b,d) grooves. From top to bottom \( \mu/\mu_g = 50, 5, 1 \). Exact theoretical results are shown by circles, analytical results [Eq. (3)] with local slip given by Eq. (3) are plotted by solid curves. Filled symbols show earlier data \cite{22}.

We finally turn to the effective slip lengths. The cal-
Calculations are made using the viscosity ratio of the Cassie and Wenzel states. For completeness we include the data for $\mu/\mu_g = 5$, which correspond to oil-impregnated textures. Fig. 4 shows longitudinal (a,b) and transverse (c,d) effective slip lengths (circles) as a function of solid fraction, $1-\phi$, for shallow (a,c) and deep (b,d) grooves. The eigenvalues of the effective lengths of a striped surface with a piecewise constant local slip, $b_{\text{eff}}^{x,y}$, have been calculated analytically [14]:

$$b_{\text{eff}}^x \approx \frac{L}{\pi} \frac{\ln \left( \sec \left( \frac{\pi \phi}{2} \right) \right)}{1 + \frac{L}{\pi b_{\text{c}}^x} \ln \left( \sec \left( \frac{\pi \phi}{2} \right) + \tan \left( \frac{\pi \phi}{2} \right) \right)},$$

$$b_{\text{eff}}^y \approx \frac{L}{2\pi} \frac{\ln \left( \sec \left( \frac{\pi \phi}{2} \right) \right)}{1 + \frac{L}{2\pi b_{\text{c}}^y} \ln \left( \sec \left( \frac{\pi \phi}{2} \right) + \tan \left( \frac{\pi \phi}{2} \right) \right)}.$$  

Let us now try to define average local slip lengths at the gas sectors. Eq. (8) suggests the following definition

$$\langle b^{x,y} \rangle = \delta \frac{\mu}{\mu_g} \langle \beta^{x,y} \rangle,$$  

(10)

where dimensionless average slip lengths, $\langle \beta^{x,y} \rangle$, depend only on the aspect ratio of the texture, $c/d$. We can now obtain $\langle \beta^{x,y} \rangle$ by fitting of our theoretical results for $b_{\text{eff}}^{x,y}$, and calculate the effective slip lengths predicted by Eq. (9). In doing so, we substitute $b_{\text{c}}^{x,y}$ by $\langle b^{x,y} \rangle$. The results of calculations are included in Fig. 4.

A general conclusion is that the predictions of Eq. (9) with average local slip defined by Eq. (10) are in excellent agreement with exact theoretical results in the whole range of parameters: $0.1 \leq \phi \leq 0.9$ and $\mu/\mu_g \geq 1$. Note that included in Fig. 4 effective slip lengths for perfectly smooth surfaces [8, 11], practically coincide with our results for $\mu/\mu_g = 50$. We can then conclude that SH surfaces in the Cassie state provide the very general upper bound for effective slip of textured surfaces, valid for whatever large viscosity contrast (e.g. polymer melts [26]). Finally, we observe an excellent agreement of our results with earlier data for the Wenzel state obtained by using a different approach [27].

**Concluding remarks.** We have proposed an operator method, which allowed us the mapping of the flow in the gas subphase to a local slip boundary condition at the gas area of SH surfaces. The determined slip length is shown to be a unique function of the viscosity contrast and topography of the underlying texture. Our main results, Eqs. (8) and (10), can be thus viewed as a general ‘gas cushion’ model for textured surfaces, which transforms to the standard model, Eq. (3), in case of shallow textures. We have proven that beside Cassie surfaces our approach is valid for Wenzel textures, as well as rough surfaces impregnated by a ‘lubricant’ with lower viscosity. Thus, our results may guide the design of textured surfaces with superlubricating potential in microfluidic devices, tribology, polymer science, and more.

We checked the validity of our approach by studying a flow past canonical rectangular grooves, but our strategy can be immediately applied for grooves with different cross-sections or extended to more complex textures. Another fruitful direction could be to apply our method to calculations of an electro-osmotic flow past textured surfaces [27, 28].

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