Homogeneous singularities inside collapsing wormholes

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We analyze analytically and numerically the origin of the singularity in the course of the collapse of a wormhole with the exotic scalar field $\Psi$ with negative energy density, and with this field $\Psi$ together with the ordered magnetic field $H$. We do this under the simplifying assumptions of the spherical symmetry and that in the vicinity of the singularity the solution of the Einstein equations depends only on one coordinate (the homogeneous approximation). In the framework of these assumptions we found the principal difference between the case of the collapse of the ordinary scalar field $\Phi$ with the positive energy density together with an ordered magnetic field $H$ and the collapse of the exotic scalar field $\Psi$ together with the magnetic field $H$. The later case is important for the possible astrophysical manifestation of the wormholes.

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I. INTRODUCTION

Wormholes (WHs) are hypothetical short topological tunnels connecting two different distant asymptotically flat regions of the Universe, or such regions belonging to different universes in the model of the Multiverse \cite{1}. They are typical relativistic objects.

The problem of wormholes has a long history. In the framework of General Relativity, first attempts to construct such objects were performed by Flamm \cite{2} and Einstein and Rosen \cite{3}. Later on, many theoretical aspects of the WH problem were investigated, see, for example \cite{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40}. During the last few decades, interest in wormholes increased in connection with the early Universe \cite{41, 42, 43, 44} and various problems in physics and astrophysics \cite{7, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54}. In the works \cite{10, 11, 12} the hypothesis that some known astrophysical objects (e.g. quasars and active nuclei of some galaxies) could be entrances to wormholes was considered. Wormholes may have existed as primordial objects in early stage of the expanding Universe \cite{8, 43}. It is possible that such primordial wormholes could be preserved after the end of the inflation \cite{14, 51}. This hypothesis can explain some observable facts in astrophysics and can predict new phenomena \cite{11}.

In the middle of the last century it was shown in General Relativity that a vacuum WH pinches so quickly that it cannot be traversed even by a test signal moving with the velocity of light (see review in \cite{13}). In order to prevent the shrinking of a WH and to make it traversable, it is necessary to thread its throat with so-called exotic matter which is matter that violates the averaged null energy conditions (see \cite{8, 13, 16, 17}). Different types of wormholes may exist depending on the type of exotic matter in their throats \cite{11, 24, 21}. For example it could be a “magnetic exotic matter” in which the main component is a strong ordered magnetic field plus a some amount of a “true exotic matter” \cite{11}. Another type is a “scalar exotic matter” in the form of a scalar field with a negative energy density \cite{9, 20}. One more type is a mixture; “magnetic-negative dust exotic matter” which is a mixture of an ordered magnetic field and dust (matter with zero pressure) with negative matter density \cite{21}. The physical properties of different types of WHs are different. WHs with scalar exotic matter were the subject of very intense investigations both analytically and numerically. Possible dynamics of such WHs has been analyzed analytically for example in papers \cite{22, 23, 24, 25, 26, 27, 28, 29, 30}. The most important, however, was the numerical analysis \cite{34, 35, 36, 55}. It was shown that the static WHs of this type are unstable. Perturbations trigger the evolution of the WH and the evolution of the exotic scalar field which maintains it. As a result, the WH either collapses or expands. In the case of the collapse - a black hole (BH) arises. If the collapsing WH has both the exotic scalar field and the magnetic field, the structure of the singularity inside the resulting BH requires special investigation. The reason for extra attention is that the magnetic field in WHs or their remnants has special manifestation in the astrophysical observations in the hypothesis of the existence of WHs in the Universe \cite{11}.

The goal of this paper is to analyze (under simplifying assumptions) the nature of the singularity created during the collapse of the WH with the exotic scalar field and with the magnetic field.

The paper is organized as follows. In section II we describe a model in the framework of which we analyze the problem. In the subsection II A the equations are written out. In the subsection II B we use the leading order analysis for the case of a WH without magnetic field. In the subsection II C we use a numerical code to
solve the equations to understand the behavior of the model for several sets of its parameters. In subsection III we use the leading order analysis for the case with the magnetic field. In the subsection IV the numerical analysis is applied to the case with the magnetic field. In the section V we discuss the results.

II. THE MODEL

We consider collapse of a WH with the formation of a BH. The main feature inside a BH is its space-time singularity. Our goal is to investigate the nature of this singularity, using some simplifying assumption. First, we consider the spherical WH and BH nonlinearly perturbed by a minimally coupled and self-gravitating massless exotic scalar field Ψ with the negative energy density ϵ < 0. We will also consider the same case but with an additional radial magnetic field. It was shown (see [56, 57, 58]) that, in the considered model, in the close vicinity of a space-like singularity of a BH, all processes, as a rule, have high temporal gradient (much higher than the spatial gradients along the singularity) and that the processes depend on the properties of a very restricted space region. It follows from here that, for clarification of some physical processes, one can use a homogenous approximation and that all processes and geometry depend on the time coordinate only. We assume also that in the close vicinity of a time-like singularity of a BH one can use a homogenous approximation also, but now all processes and geometry depend on the radial space coordinate only.

A. The equations

We start with the general homogeneous spherically symmetric line element:

\[ ds^2 = g_{tt}(r)dt^2 + g_{rr}(r)dr^2 + r^2 d\Omega^2 \]  

(1)

\[ d\Omega^2 = d\Theta^2 + \sin^2 \Theta d\phi^2 \]  

(2)

Inside a BH the region between the event horizon (EH) and the Cauchy horizon (which exists in the case with the radial magnetic field; we will call it horizon-2 (H-2)) is so-called T-region (see [13] and references therein). In this region r is time-like and t is space-like coordinate. To describe the contraction, we should consider the variation of the time coordinate r from bigger to smaller values. The r − r, t − t and Θ − Θ components of the Einstein equations (with \( c = 1, G = 1 \)) are given by (see [58])

\[ \frac{g_{tt} - g_{rr}g_{tt} + rg''_{tt}}{r^2g_{rr}g_{tt}} = 8\pi(T_r^r + H_r^r) \]  

(3)

\[ \frac{g_{rr} - g_r^2 - rg''_{rr}}{r^2g_{rr}^2} = 8\pi(T_t^t + H_t^t) \]  

(4)

\[
\frac{1}{4rg_r^2g_{tt}^3}\left\{ g_{tt}\left[ 2g_{rr}(g_{tt} + rg''_{tt}) - (rg'_{rr}g_{tt}) \right] - \right. \\
2g_{tt}g_{rr}^2 - rqg''_{rr} \right\} = 8\pi \left( T_\Theta^\Theta + H_\Theta^\Theta \right)
\]  

(5)

where the primes denote differentiation with respect to \( r \). Tensor \( T \) represent here contribution from the \( \Psi \) field and tensor \( H \) represents contribution from a free radial magnetic field. In the WH, \( H \) is a sourceless magnetic field captured by the topological structure of the 3-D geometry. For both the WH and arising from the collapse of it BH the components of the tensor \( H \) are given by:

\[ H_r^r = H_t^t = -H_\Theta^\Theta = -\frac{q^2}{8\pi r^4} \]  

(6)

where the constant \( q \) characterizes the strength of the magnetic field. The exotic scalar field is governed by the Klein-Gordon equation:

\[ \Psi'(r) = \frac{d}{\sqrt{-g}}g_{rr}, \sin \Theta \]  

(7)

where \( d \) is a constant and \( g \) is the metric determinant. For the exotic field, the value of \( d \) is pure imaginary, \( d^2 < 0 \). Of course \( \Phi' \) does not depend on \( \Theta \) because the metric determinant \( g \) has a factor \( \sin^2 \Theta \).

The components of the tensor \( T \) for the exotic scalar field \( \Psi \) are (see [58]):

\[ T_r^r = -\epsilon \]  

(8)

\[ T_t^t = \epsilon \]  

(9)

\[ T_\Theta^\Theta = \frac{\epsilon}{g_{rr}} \]  

(10)

\[ \epsilon = \epsilon_0 \left( \frac{g_{tt,init}}{g_{tt}} \right) \left( \frac{r_{init}}{r} \right)^4 = -\frac{1}{8\pi g_{rr}}(\Psi')^2 \]  

(11)

\[ \epsilon_0 = \frac{d^2}{8\pi \cdot g_{tt,init} \cdot r_{init}^4} \]  

(12)

where \( \epsilon_0, g_{tt,init}, r_{init}, d^2 < 0 \) are all constants. Substitution of [8, 11, 12] into [3] and [4] enables us to find the unknown functions \( g_{rr}(r) \) and \( g_{tt}(r) \). Equation [3] is the consequence of [3], [4] and hence can be used as a control of the calculations.

B. Leading order analysis for the case \( q = 0 \)

We will use the method which has been proposed by Burko [56, 57] and after that used in [58]. This is the
non-linear generalization in the homogenous case of linear analysis of [54]. In this section we consider the case 
$q = 0$. Let us consider the leading order terms in a series expansion for the metric functions and the leading 
order terms in the Einstein equations [3], [4]. $q = 0$ near the singularity $r = 0$. We assume that the leading terms 
for each $g_{tt}$ and $g_{rr}$ can be written in the power form $(\text{const} \cdot r^\mu)$. We demand also that the corresponding 
terms in the Einstein equations tend to zero when $r \to 0$, otherwise this method doesn’t work (see Appendix A). In the vicinity of $r = 0$ the solution of 
[3], [4] can be written, as the first approximation, in the following form

\[ g_{tt}^{(1)} = 2m_0C r^\beta, \]
\[ g_{rr}^{(1)} = - (\beta + 2) \frac{1}{2m_0} r^{\beta + 2}, \]

where $m$, $C$ and $\beta$ are all constants. For the scalar field $\Psi$, the leading order (in $r$) analysis gives:

\[ \Psi_r^{(1)} = \sqrt{\beta + 1} \ln r. \]

We have also for the constant $d^2$ (see [7] and [12]):

\[ d^2 = \frac{(\beta + 1)}{(\beta + 2)} \cdot 4m_0^2 C. \]

Here we use constants $m_0$ and $C$ which were introduced in [58]. These constants have direct physical meaning in 
the case of empty (without $\Psi$ field) BH, when $\beta = -1$. In such a case, $m$ is the BH mass, $C$ is a gauge parameter 
related to the possibility of changing the scale of measurement of the $t$ space coordinate.

In the case of presence of the $\Psi$ field, $\beta$ determines the strength of this field. In contrast to the case analyzed by 
Burko in [56, 57], in our case here $\Psi$ should be imaginary and from [10] we have the restriction:

\[ \beta < -1. \]

On the other hand, for $m_0 > 0$ we obtain from [13] and [14]:

\[ C > 0, \]
\[ \beta + 2 > 0. \]

Thus we have for $\beta$ the restrictions $-2 < \beta < -1$. In the case of violation of the inequality [19] this method does not work as we mentioned above and in Appendix A.

The above solution [13], [14] is generic, because it depends on three arbitrary parameters. In our case $m$ and $\beta$ are essential physical parameters, and $C$ is a gauge parameter. Using the arguments which are analogous to the arguments of Burko [56, 57] one may conclude that the dependence on two physical and one gauge parameter 
means that the solution is generic. If we suppose that in the solution [13], [14] $m_0 < 0$, $C > 0$, then it means 
that $g_{tt}^{(1)} < 0$, $g_{rr}^{(1)} > 0$; $t$ is the time coordinate and $r$ is the space coordinate. In this case $r = 0$ is a time-like 
singularity. Probably it corresponds to negative mass of the object and a naked singularity.

C. Numerical analysis

We use a simple numerical code to solve numerically equations [3], [4] to understand the behavior of model 
[11], [2] as a function of the parameters of the model.

We will represent the results for the case of the exotic scalar field $\Psi$ with $\epsilon < 0$, and compare them with 
the results for the case of the ordinary scalar field $\Phi$ with $\epsilon > 0$. For the latter case $\Phi$-field with $\epsilon > 0$, we consider 
also the case of the presence of the magnetic field $q \neq 0$ to emphasize the essential difference the cases $q = 0$ and $q \neq 0$.

We start the computation from $r_{\text{init}} = 0.95 r_{EH}$, and set $G_{tt, \text{init}}$, $G_{rr, \text{init}}$ equal to their values at $r_{\text{init}}$ for the zero matter content Schwarzschild (or Reisner-Nordström if $q \neq 0$) solution with initial $m_0 = 1$ and $q = 0.95$ if $q \neq 0$. Here $r_{EH}$ is the value of the event horizon for the 
zero matter content BH with the same parameters. We consider several values of $\epsilon_0$, which is the characteristic 
of the initial amplitude of the scalar field.

The parameters of the model, including the exponent $\beta$, depend on the value of the scalar field $\epsilon_0$. We start 
from the extremely small amplitude $|\epsilon_0|$ and consider the cases (see Figure 1):

\[ \epsilon_0 = +0.0001, \quad q = 0, \]
\[ \epsilon_0 = -0.0001, \quad q = 0, \]
\[ \epsilon_0 = +0.0001, \quad q = 0.95. \]

In Figure 1(a) one can see the propagation ($r$ vs. $t$) of the incoming signal with the velocity $c$ ($c = 1$). Figure 
1(b) shows the mass function (see [58]):

\[ m = \frac{r}{2} \left(1 + \frac{d^2}{r^2} - g_{rr}^{-1}\right). \]

Figure 1(c) presents the evolution of the metric functions, $g_{tt}$ and $g_{rr}$, and Figure 1(d) shows the evolution of the 
values $d \log g_{tt}/d \log r \equiv \beta$ and $d \log |g_{rr}|/d \log r \equiv \alpha$.

It is clear from the results shown in Figures 1(c) and 1(d) that for this extremely small amplitude $|\epsilon_0|$ in the 
case $q = 0$ the propagation of the light-like signal, evolution of $g_{tt}$ and $g_{rr}$ and $d \log g_{tt}/d \log r = \beta$ 
and $d \log |g_{rr}|/d \log r = \alpha$ are practically the same for $\epsilon_0 = \pm 0.0001$ and the difference is not visible in the plots.

On the other hand, the behavior of the mass function for the cases $\epsilon_0 = 0.0001$ and $\epsilon_0 = -0.0001$ (Figure 1(a)) 
is quite different. Of course the reason for this is the opposite signs of $\epsilon$ for these two cases.

The physical processes which can lead to a nonlinear change of the mass function are the following:
The mass $m$ inside a sphere can change because of the work of pressure forces on the surface of the sphere.

The mass can change due to the mass inflation \[58\].

Both processes are described in \[58\]. The process B) is important near the horizon-2 ($H$-2) in the case of presence of the magnetic field $q \neq 0$ and $\epsilon > 0$. In the case $q = 0$ horizon-2 ($H$-2) does not exist. Thus in the case $q = 0$ the process B) is not important, and variations of mass function is small and opposite for the two cases $\epsilon_0 = 0.0001$ and $\epsilon_0 = -0.0001$.

The case of $\epsilon_0 = 0.0001$, $q = 0.95$ is quite different. Figure 1(a) shows that in this case the light-like signal goes close to the initial value of $r$ for the ($H$-2) horizon $r = r_{H-2}$ during a long period, and only after that it comes to $r = 0$ (beyond the framework of the figure). Note that the light-like signal does not cross the $H$-2 horizon. The horizon itself shrinks down to $r = 0$ under the focusing effect, as it is described in \[58\]. The metric functions $g_{tt}$ and $g_{rr}$ go to zero very fast, and very large asymptotic values $d \log g_{tt} / d \log r = \beta$ and $d \log |g_{rr}| / d \log r = \alpha$ are clearly seen in Figure 1(d).

When the solution is close to $r = r_{H-2}$ the mass inflation manifests itself (see Figure 1(b)), the mass function increases dramatically.

We next study the solutions for the larger values of $|\epsilon_0|$. We start with the case $\epsilon_0 > 0$ and consider the following values (see Figure 2):

\begin{align*}
\epsilon_0 &= 0.011, \quad q = 0, \quad \text{(24)} \\
\epsilon_0 &= 0.012, \quad q = 0, \quad \text{(25)} \\
\epsilon_0 &= 0.050, \quad q = 0, \quad \text{(26)}
\end{align*}

The propagation of the light-like signal (Figure 2(a)) qualitatively is the same for all cases but for larger values of $\epsilon_0$ it comes to $r = 0$ later. behavior of the metric functions, Figures 2(b), 2(c) is interesting. The analysis of the leading order terms of the series expansion for the case $\epsilon > 0$ (this means that $d^2 > 0$) leads to the conclusion that $\beta > -1$, but $\beta$ can be positive and negative. Numerical analysis shows that for rather small $\epsilon_0 \leq 0.011$ the function $g_{tt}$ increases when $r \to 0$, but for the larger $\epsilon_0 \geq 0.012$ the function $|g_{tt}|$ decreases when $r \to 0$ (see

FIG. 1: The cases of $\epsilon_0 = 0.0001, q = 0$, $\epsilon_0 = -0.0001, q = 0$, and $\epsilon_0 = 0.0001, q = 0.95$. 

(a) $r$ versus $t$.

(b) Mass function versus $r$.

(c) Metric functions $g_{tt}$ and $|g_{rr}|$ vs $r$.

(d) The $d \log g_{tt}/d \log r$ and $d \log |g_{rr}|/d \log r$ vs $r$. 

A) The mass $m$ inside a sphere can change because of the work of pressure forces on the surface of the sphere.
also Table I.

As it can be seen from Table I, the function \( \lim_{t \to 0} \beta \) where \( \beta = d \log g_{tt} / d \log r \) increases monotonically when \( \epsilon_0 \) varies from \( \beta = -1 \) when \( \epsilon_0 = 0 \), to \( \beta = +51.8 \) when \( \epsilon_0 = 0.05 \). The critical value of \( \epsilon_0 \), when \( \beta \) changes its sign, is \( \epsilon_0, crit \approx 0.0115 \). The value \( \lim_{t \to 0} \alpha \) where \( \alpha = d \log |g_{rr}| / d \log r \) also increases monotonically when \( \epsilon_0 \) varies from \( \alpha = 1 \) when \( \epsilon_0 = 0 \) to \( \alpha = 53.8 \) when \( \epsilon_0 = 0.05 \). The sign of \( \alpha \) never changes.

The difference in behavior of the metric functions leads to difference in the properties of the mass function. As we mentioned above, in the case \( q = 0 \) the horizon \( H-2 \) does not exist and the mass inflation (the process B) is not important for the evolution of the mass function. Thus, the process A), which is related to the deformation of the volume of the reference frame, plays the main role. The longitude deformation is proportional to \( (g_{tt})^{1/2} \), while the transversal deformations are proportional to \( r^2 \). In all cases it leads to the increase of the mass with decrease of \( r \). For large positive \( \beta \) corresponding to large \( \epsilon_0 \) this increase is very fast.

We now turn to the discussion of the case \( \epsilon < 0, q = 0 \) (see Figure 3), which corresponds to the case of the collapse of the WHs analyzed numerically in [55]. Here in the limit \( r \to 0 \), \( \beta \) is always less than -1. With the increase \( \epsilon_0 \) the limit \( \lim_{r \to 0} \beta \) increases insignificantly. Dropping the absolute value, the limit varies from \( \beta = -1.00 \) at \( \epsilon_0 = -0.0001 \) to \( \beta = -1.01 \) at \( \epsilon_0 = -0.05 \). The behavior of the metric functions \( g_{tt} \), \( g_{rr} \) is similar for all cases \( 0.0001 > \epsilon_0 > -0.05 \), see Figures 3(a), 3(b).

FIG. 2: The cases of \( \epsilon_0 = 0.011, q = 0 \), \( \epsilon_0 = 0.012, q = 0 \), and \( \epsilon_0 = 0.050, q = 0 \).

TABLE I: The limit values of \((d \log g_{tt} / d \log r) \equiv \beta \) and \((d \log |g_{rr}| / d \log r) \equiv \alpha \) for \( q = 0 \), for \( r \to 0 \).

| \( \epsilon_0 \) | \( \alpha \) | \( \beta \) |
|---|---|---|
| 0.0001 | 1.00 | -1.00 |
| 0.0010 | 1.02 | -0.99 |
| 0.0025 | 1.02 | -0.98 |
| 0.0050 | 1.06 | -0.94 |
| 0.0100 | 1.52 | -0.48 |
| 0.0110 | 1.79 | -0.21 |
| 0.0120 | 2.16 | +0.16 |
| 0.0200 | 8.95 | +6.95 |
| 0.0500 | 53.8 | +51.8 |
FIG. 3: The cases of $\epsilon_0 = -0.01$, $q = 0$, and $\epsilon_0 = -0.05$, $q = 0$.

The light-like signal comes to $r = 0$ later for the larger values of $|\epsilon_0|$, see Figure 3(c). This property is opposite to the case of $\epsilon_0 > 0$. The changes of the mass function $m$ for the cases $\epsilon_0 = -0.01$ and $\epsilon_0 = -0.05$ are also insignificant (see Figure 3(d)), in sharp contrast with the case $\epsilon_0 > 0$.

D. Leading order analysis for the case $q \neq 0$.

We consider the leading order terms in the series expansion for the metric functions and the leading order terms in the Einstein equations (3), (4), $d \neq 0$ near the singularity $r = 0$. An assumption of $\beta > 0$ makes the term with the magnetic field $q^2$ to be negligible, and the analysis near $r \to 0$ becomes the same as for the cases considered in the section II B. However, one of our conclusions in section II B was that $\beta < -1$ (17). Thus our current assumption of $\beta > 0$ contradicts the result expressed in (17). The case $\beta = 0$ also leads to contradictions. We come to a conclusion that the only allowed values for $\beta$ are negative values, $\beta < 0$, which means that in the vicinity of $r = 0$ we can neglect the scalar field and consider $d^2 = 0$, $q^2 \neq 0$.

To the leading order (3), (4), $d = 0$ assume the forms:

$$g_{tt} = g_{rr} = 0,$$  \hspace{2cm} (27)

$$g_{tt} + g_{tt} + (g_{rr} g_{tt} q^2) / r^2 = 0.$$  \hspace{2cm} (28)

We are looking for solutions to (27), (28) in the vicinity $r = 0$ in the form

$$g_{tt} = B r^\beta,$$  \hspace{2cm} (29)

$$g_{rr} = -A r^\alpha.$$  \hspace{2cm} (30)

Substituting (29), (30) into (27), (28) we have

$$\alpha = 2, \hspace{0.5cm} \beta = -2, \hspace{0.5cm} A = -q^{-2},$$  \hspace{2cm} (31)

$B$ is arbitrary, but for the correct signature $B < 0$.

Of course this is well known the Reissner-Nordström solution in the vicinity of $r = 0$. Arbitrariness of the
coefficient $B$ is related to the arbitrariness of the choice of the $t$ coordinate. Note that now $g_{tt} < 0$ and $g_{rr} > 0$, this means that $t$ is the time coordinate, $r$ is the space coordinate, we are in the $R$-region, see [13], and $r = 0$ is a time-like singularity.

E. Numerical analysis $\epsilon < 0$, $q \neq 0$.

We start with the case $q = 0.95$, $\epsilon_0 = -0.0001$ and $r_{init} = 0.95r_{EH}$. This case differs from those discussed in Section II C by the sign of $\epsilon_0$ only. The change of sign on $\epsilon_0$ greatly influences the physics of the processes taking place. In the case with $\epsilon_0 > 0$ the nonlinear effects of mass inflation and focusing of the $H$-2 horizon down to $r = 0$ arized. As a result, both functions $|q_{rr}|$ and $g_{tt}$ finally come to zero at $r = 0$, and the space-like singularity arized (see Figure 1 in the T-region (see [13]). In our current case of $\epsilon < 0$ there are no mass inflation and other such effects, and one can observe quite different behavior (see Figure 1). The question is, can the collapsing configuration cross the border between the $T$-region (as in Figure 1) and $R$-region and come to an internal $R$-region? To arrive to an answer, one needs to perform a deeper analysis and numerical calculations for the general case when $g_{rr}$ and $g_{tt}$ depend on both $r$ and $t$ coordinates. For now, we see that here the origin of the $r = 0$ singularity in the $T$-region is not possible.

Thus we need to analyze here numerically the case when we are in the $R$-region and finally come to the time-like singularity $r = 0$ as it was discussed in Section II D using the leading order analysis. To accomplish that, we consider the cases (Figure 5):

$$\epsilon_0 = -0.0001, \quad q = 0.95, \quad r_{init} = 0.95r_{H-2}, \quad (32)$$

$$\epsilon_0 = -0.0500, \quad q = 0.95, \quad r_{init} = 0.95r_{H-2}. \quad (33)$$

As we emphasized in Section III D, the asymptotic values of $g_{tt}$ and $g_{rr}$ do not depend on the scalar field. Thus the dependence of the solutions on $\epsilon_0$ is rather weak. Note that $\alpha$ and $\beta$ come to their asymptotic values $\alpha = 2$ and $\beta = -2$ which do not depend on $\epsilon_0$. For larger values of $|\epsilon_0|$ the light-like signal comes to $r = 0$ at slightly
larger $t$, and the dependence of the mass function $m$ on $r$ is stronger (but still rather weak).

III. CONCLUSIONS

The evolution of the WHs can lead to their collapse and the origin of the singularity. In this paper, we investigate the structure of the singularity arising as a result of the collapse of the WH with the exotic scalar field and the ordered magnetic field. We consider the spherical WHs.

In the very vicinity of the singularity $r = 0$ one can use an approximation where the solution depends only on one coordinate $r$ (the so-called uniform approximation). In this region, it is possible to consider the leading order terms in a series expansion for the metric functions and the leading order terms in the Einstein equations. We have demonstrated that in the case of the presence of the exotic scalar field only (without a magnetic field) the metric functions are

$$g^{(1)}_{tt} \sim r^\beta, \quad g_{rr} \sim r^{\beta+2}, \quad g_{22} \sim r^2,$$

at $r \to 0$. Here $\beta$ is a constant

$$-2 < \beta < -1.$$  \hspace{1cm} (35)

We performed the numerical analysis of the uniform approximation to understand the behavior of the model as a function of its parameters. The results are described in Section II B. For the comparison we also investigated the case of the collapse with the ordinary scalar field $\Phi$ with $\epsilon > 0$ instead of the exotic scalar field $\Psi$ with $\epsilon < 0$. In this case instead of inequality (35) we have

$$\beta > -1.$$  \hspace{1cm} (36)

and the numerical analysis shows that $\beta$ changes its sign at some critical initial $\epsilon_0$, see Table I.

There is the special interest in the case of the exotic scalar field $\Psi$ together with the magnetic field $H$. As it was mentioned in the Introduction I, the magnetic field in the WHs or their remnants has a special manifestation in astrophysical observations if the WHs really exist in the Universe.

From the leading order analysis for this case we concluded that in the vicinity of $r = 0$, $\beta$ is negative and the
magnetic field dominates the scalar field, and the metric functions correspond to the Reissner-Nordström solution. Thus the singularity at $r = 0$ is in the $R$-region and is a time-like singularity in the case of our assumptions. Numerical estimates demonstrate that the asymptotic behavior of the metric functions practically do not depend on the initial value of the exotic scalar field. We have investigated also the properties of the mass-function for all cases. We want to emphasize the principal difference between the collapse of the ordinary scalar field $\Phi$ with $\epsilon_0 > 0$ and the magnetic field $H$, and the collapse of the exotic scalar field $\Psi$ with $\epsilon_0 < 0$ and the magnetic field $H$. In the case of the $\Phi$-field the collapse leads to the formation of the space-like singularity $r = 0$ in the $T$-region, but in the case of the $\Psi$-field the singularity $r = 0$ is a time-like and it is in the $R$-region. This is correct under the assumption which we have made.

At the end we want to note the following. The consideration of this paper can be useful also for the analysis of the processes inside a BH when it is being irradiated by an exotic scalar radiation.

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APPENDIX A: ANALYSIS OF EQS. (3), (4)

We are looking for the solutions to (3), (4), $q = 0$ in the vicinity $r = 0$ in the form

$$g_{tt}^{(1)} = Br^\beta,$$

(A.1)

$$g_{rr}^{(1)} = -Ar^\alpha,$$

(A.2)

Let us consider the case $\alpha = 0$. To the leading order (3), (4), $q = 0$ assume the form

$$-A + A^2 + A^2 d^2 B^{-1} r^{-\beta - 2} = 0$$

(A.3)

$$\beta Br^\beta - Br^\beta (-A) + Br^\beta - A \frac{d^2}{r^2} = 0$$

(A.4)

Formula (A.3) gives

$$\beta = -2,$$

(A.5)

$$-1 + A(1 + \frac{d^2}{B}) = 0,$$

(A.6)

From (A.4) and (A.5) one has

$$A - 1 = \frac{d^2}{B},$$

(A.7)

Formulas (A.6) and (A.7) give

$$A = 1,$$

(A.8)

$$\frac{d^2}{B} = 0.$$  

(A.9)

Expression (A.9) contradicts the conditions $d^2 \neq 0$. Thus the case $\alpha = 0$ is not possible.

Now let us consider the case $\alpha < 0$. To the leading order Eqs (3), (4), $q = 0$ assume the form

$$g_{tt}^{(1)} + g_{rr}^{(1)} \frac{d^2}{g_{tt}^{(1)} r^2} = 0,$$

(A.10)

$$-g_{tt}^{(1)} g_{rr}^{(1)} + g_{rr}^{(1)} \frac{d^2}{r^2} = 0.$$  

(A.11)

From (A.10) one has

$$\beta = -2,$$

(A.12)

$$1 + \frac{d^2}{13} = 0.$$  

(A.13)

From (A.11) and (A.12) we obtain

$$-1 + \frac{d^2}{B} = 0.$$  

(A.14)

Expression (A.13) contradicts (A.14). Thus the case $\alpha < 0$ is not possible.

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