Two Families of Hypercyclic Non-Convolution Operators

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Abstract

Let \( H(\mathbb{C}) \) be the set of all entire functions endowed with the topology of uniform convergence on compact sets. Let \( \lambda, b \in \mathbb{C} \), let \( C_\gamma : H(\mathbb{C}) \to H(\mathbb{C}) \) be the composition operator \( C_\gamma f(z) = f(\lambda z + b) \), and let \( D \) be the derivative operator. We extend results on the hypercyclicity of the non-convolution operators \( T_{\lambda,b} = C_\gamma \circ D \) by showing that whenever \( |\lambda| \geq 1 \), the algebra of operators

\[ \{ \psi(T_{\lambda,b}) : \psi(z) \in H(\mathbb{C}), \psi(0) = 0 \text{ and } \psi(T_{\lambda,b}) \text{ is continuous} \} \]

and the family of operators

\[ \{ C_\gamma \circ \varphi(D) : \varphi(z) \text{ is an entire function of exponential type with } \varphi(0) = 0 \} \]

consist entirely of hypercyclic operators (i.e., each operator has a dense orbit).

1 Introduction

Let \( \mathbb{C} \) denote the complex plane and \( H(\mathbb{C}) \) be the set of all entire functions endowed with the topology of uniform convergence on compact sets. This topology makes \( H(\mathbb{C}) \) a separable Fréchet space, which is a locally convex and metrizable topological vector space that is both complete and separable. A continuous linear operator \( T \) defined on a Fréchet space \( F \) is said to be hypercyclic if there exists \( f \in F \) (called a hypercyclic vector for \( T \)) such that the orbit \( \{ T^n f : n \in \mathbb{N} \} \) is dense in \( F \). We refer the reader to the books [7] and [2] for a thorough introduction to the study of hypercyclic operators.

The first example of a hypercyclic operator was given by Birkhoff in 1929, who showed that the translation operator \( T : f(z) \mapsto f(z + 1) \) is hypercyclic [3]. In 1952, MacLane showed that the derivative operator \( D : f(z) \mapsto f'(z) \) is also hypercyclic [10]. Both of these results were unified in a substantial paper by Godefroy and Shapiro in 1991, who proved that every continuous

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linear operator $L : H(\mathbb{C}) \to H(\mathbb{C})$ which commutes with translations (these operators are called convolution operators) and which is not a scalar multiple of the identity is hypercyclic \[6\].

It is then natural to ask for examples of hypercyclic non-convolution operators, that is, operators which are hypercyclic but do not commute with all translations. This study was initiated by Aron and Markose in 2004 in [1], where they introduced the non-convolution operators $T_{\lambda,b}f(z) = f'(\lambda z + b)$, $\lambda, b \in \mathbb{C}$ and proved, along with the authors in [5], that such operators are hypercyclic when $|\lambda| \geq 1$. Their result was extended by León-Saavedra and Romero-de la Rosa to show, among other things, that such operators are not hypercyclic when $|\lambda| < 1$ in [9]. An $N$-dimensional analogue of these operators was studied in [11], and other examples of hypercyclic non-convolution operators can be found in [8, 12]. The purpose of this paper is to extend and complement the examples of hypercyclic non-convolution operators given in [1, 5, 9].

This paper is organized as follows. In Section 2, we establish some lemmas and prove that if $\psi(z)$ is an entire function such that $\psi(0) = 0$ and $\psi(T_{\lambda,b})$ is continuous, then the operator $\psi(T_{\lambda,b})$ is hypercyclic if and only if $|\lambda| \geq 1$. Our other main result is in Section 3, the description of which requires some established notation. Let $D$ be the derivative operator. Godefroy and Shapiro showed that an operator $V$ on $H(\mathbb{C})$ is a convolution operator if and only if $V = \phi(D)$, where $\phi(z)$ is an entire function of exponential type. For $\lambda, b \in \mathbb{C}$, let $\gamma(z) = \lambda z + b$ and define the composition operator $C_\gamma : H(\mathbb{C}) \to H(\mathbb{C})$ by $C_\gamma f(z) = f(\lambda z + b)$. If $\phi(D)$ is a convolution operator, define by $L_{\gamma,\phi}$ the operator $L_{\gamma,\phi} = C_\gamma \circ \phi(D)$. If $\phi(z) = z$, then $L_{\gamma,\phi} = T_{\lambda,b}$, so that these operators $L_{\gamma,\phi}$ generalize those introduced by Aron and Markose. In Section 3, we prove that if $\phi(0) = 0$ and $|\lambda| \geq 1$, then $L_{\gamma,\phi}$ is hypercyclic.

## 2 The hypercyclicity of polynomials of $T_{\lambda,b}$

To show that an operator $T$ on $H(\mathbb{C})$ is hypercyclic, we will employ the well-known Hypercyclicity Criterion. It states, for our purposes, that a continuous linear operator $T$ is hypercyclic on $H(\mathbb{C})$ if there exists a dense set $P \subset H(\mathbb{C})$ and a sequence of mappings $S_n : P \to H(\mathbb{C})$ such that

(a) $T^n f \to 0$ for all $f \in P$,

(b) $S_n f \to 0$ for all $f \in P$, and

(c) $T^n S_n f \to f$ for all $f \in P$.

Actually, there are more general conditions which ensure the hypercyclicity of an operator, but we will not use them. We refer the reader to [7, Chapter 3] for more details about the Hypercyclicity Criterion.

As seen in condition (c) above, the mappings $S_n$ act almost as right inverses (as $n \to \infty$) for $T^n$ on $P$. Borrowing the ideas in [4, Lemma 1], we begin with a crucial lemma that will help determine these mappings for a large class of
operators. Let \( I \) be the identity operator on \( H(\mathbb{C}) \) and \( \mathcal{P} \) be the collection of complex polynomials in \( H(\mathbb{C}) \), which is a dense subset of \( H(\mathbb{C}) \).

**Lemma 1.** Let \( \psi(z) = \sum_{k=0}^{\infty} w_k z^k \) be an entire function such that \( \psi(0) \neq 0 \). Suppose \( G : H(\mathbb{C}) \to H(\mathbb{C}) \) is an operator such that

(a) \( G(\mathcal{P}) \subseteq \mathcal{P} \),

(b) \( \deg Gp < \deg p \) for all nonzero \( p \in \mathcal{P} \), and

(c) for all \( \lambda \in \mathbb{C} \), \( \psi(\lambda G) = \sum_{k=0}^{\infty} w_k \lambda^k G^k \) is a continuous linear operator on \( H(\mathbb{C}) \).

Then for each nonzero \( \lambda \in \mathbb{C} \), there exists a right-inverse mapping \( S_{\psi(\lambda G)} : \mathcal{P} \to \mathcal{P} \) for \( \psi(\lambda G) \) such that \( \psi(\lambda G) S_{\psi(\lambda G)} p = p \) for all \( p \in \mathcal{P} \). Moreover, for each non-negative integer \( m \), there exists \( C > 0 \) such that for each positive integer \( n \) there exist constants \( a_{i,n} \), \( 1 \leq i \leq m \), such that for each nonzero \( \lambda \in \mathbb{C} \) and each nonzero polynomial \( p \in \mathcal{P} \) of degree \( m \), \( S_{\psi(\lambda G)}^n p \) has the form

\[
S_{\psi(\lambda G)}^n p = w_0^{-n} (I + a_{1,n} G + \cdots + a_{m,n} G^m) p,
\]

and the coefficients \( a_{i,n} \) satisfy \( |a_{i,n}| < C n^m \).

**Proof.** Let \( p \) be a nonzero polynomial, let \( m \) be the degree of \( p \), and let \( \lambda \in \mathbb{C} \) be nonzero. Since \( \deg Gp < \deg p \), we have that \( \psi(\lambda G)p = \sum_{k=0}^{m} w_k \lambda^k G^k p \).

Let \( \alpha_1, \alpha_2, \ldots, \alpha_m \) be the zeros of the polynomial \( q(z) = w_0 + w_1 z + \cdots + w_m z^m \), repeated according to multiplicity. Since \( w_0 \neq 0 \), none of the \( \alpha_i \) equal zero, and thus \( q(z) = a_0 (1 - z/\alpha_1) \cdots (1 - z/\alpha_m) \). Hence we can write

\[
\psi(\lambda G)p = w_0 \left( I - \frac{\lambda G}{\alpha_1} \right) \left( I - \frac{\lambda G}{\alpha_2} \right) \cdots \left( I - \frac{\lambda G}{\alpha_m} \right) p.
\]

We find a right-inverse mapping for each factor \( I - \frac{\lambda G}{\alpha_i} \) as follows. Let \( i \) be an integer satisfying \( 0 \leq i \leq m \). Since \( \deg Gp < \deg p \),

\[
p = \left( I - \left( \frac{\lambda G}{\alpha_i} \right)^{m+1} \right) p = \left( I - \frac{\lambda G}{\alpha_i} \right) \left( I + \frac{\lambda^2 G^2}{\alpha_i^2} + \cdots + \frac{\lambda^m G^m}{\alpha_i^m} \right) p.
\]

Thus define \( S_i p = \left( I + \frac{\lambda^2 G^2}{\alpha_i^2} + \cdots + \frac{\lambda^m G^m}{\alpha_i^m} \right) p \), and observe \( S_i p \) is a polynomial of degree \( m \). We then define \( S_{\psi(\lambda G)}^n p \) as \( S_{\psi(\lambda G)} p = \frac{1}{w_0} S_1 \cdots S_m p \). Thus \( S_{\psi(\lambda G)}^n p \) has degree \( m \) and \( \psi(\lambda G) S_{\psi(\lambda G)}^n p = p \).

By writing the formula for \( S_{\psi(\lambda G)}^n p \) as

\[
S_{\psi(\lambda G)}^n p = \frac{1}{w_0} \left( I + \prod_{i=1}^{m} \left( I + \frac{\lambda G}{\alpha_i} + \frac{\lambda^2 G^2}{\alpha_i^2} + \cdots + \frac{\lambda^m G^m}{\alpha_i^m} \right) \right) p,
\]
we obtain the form for \( S_{\psi(LG)}^n p \) by multiplying out the above product and keeping only the terms involving \( G^i \) for \( 0 \leq i \leq m \) to obtain

\[
S_{\psi(LG)}^n p = a_0^n (I + a_{1,n} \lambda G + \cdots + a_{m,n} \lambda^m G^m) p.
\]

Let \( r = \max \{1, |\alpha_1|^{-1}, |\alpha_2|^{-1}, \ldots, |\alpha_m|^{-1} \} \) and let \( C(mn, i) \) be the coefficient of \( y^i \) in the expansion of \((1 + y + y^2 + y^3 + \ldots)^{mn}\). Then \( |a_{i,n}| \leq r^i C(mn, i) \). Since \((1 + y + y^2 + \ldots)^{mn} = 1/(1-y)^{mn}\) for \( y \in (-1, 1) \), and the Taylor Series for \( 1/(1-y)^{mn} \) is

\[
1 + mny + \frac{mn(mn+1)y^2}{2!} + \frac{mn(mn+1)(mn+2)y^2}{3!} + \ldots,
\]

we have that

\[
C(mn, i) = \binom{mn+i-1}{i} = \left( \frac{mn}{1} \right) \left( \frac{mn+1}{2} \right) \cdots \left( \frac{mn+i-1}{i} \right) \leq (mn)^i,
\]

which implies that

\[
|a_{i,n}| \leq r^i C(mn, i) \leq (rmn)^i \leq (rmn)^m. \tag{1}
\]

By taking \( C = (rm)^m \), the proof is complete.

To simplify our presentation, let us recall some notation from above and establish a bit more for the remainder of this paper. Let \( \lambda, b \in \mathbb{C} \), and recall that we define \( \gamma(z) = \lambda z + b \), the operator \( C_\gamma : H(\mathbb{C}) \to H(\mathbb{C}) \) by \( C_\gamma f(z) = f(\lambda z + b) \), and the operator \( T_{\lambda,b} : H(\mathbb{C}) \to H(\mathbb{C}) \) by \( T_{\lambda,b} f(z) = f'(\lambda z + b) \). With this notation, we can view \( T_{\lambda,b} \) as a composition of two operators, namely

\[
T_{\lambda,b} = C_\gamma \circ D.
\]

More generally, if \( \varphi(z) \) is an entire function of exponential type, then we compose \( C_\gamma \) and the convolution operator \( \varphi(D) \) to define the operator \( L_{\gamma,\varphi} \) as

\[
L_{\gamma,\varphi} = C_\gamma \circ \varphi(D).
\]

Observe that if \( \varphi(z) = z \), then \( L_{\gamma,z} = T_{\lambda,b} \). Our immediate focus will turn to the operators \( L_{\gamma,z} = C_\gamma \circ D^m \), where \( m \in \mathbb{N} \).

In this case, one can check that \( C_\gamma^n f(z) = f \left( \lambda^n z + \frac{1-\lambda^n}{1-\lambda} b \right) = f(\lambda^n z - r_n) \), where \( r_n = -b \frac{1-\lambda^n}{1-\lambda} \). A straightforward induction argument then yields that

\[
L_{\gamma,z}^n f(z) = \lambda \frac{mn(n-1)}{2} C_\gamma^n \circ D^m f(z) = \lambda \frac{mn(n-1)}{2} f^m(\lambda^n z - r_n). \tag{2}
\]

We want to use the Hypercyclicity Criterion to show that these operators are hypercyclic when \( |\lambda| \geq 1 \). The bulk of the work, as usual, is to determine the right-inverse mappings. To motivate what follows, let us describe a derivation of these mappings by closely following the work in [5] to determine a sequence of
right-inverse mappings $S_{m,n}$ for $L_{\gamma,z}^n$ defined on the set of complex polynomials $\mathcal{P}$.

Define a “formal operator” $A_m$ on the set $\mathcal{D} = \{1\} \cup \{d(z + c)^k : k \in \mathbb{N}, c, d \in \mathbb{C}\}$ by

\[
A_m(1) = \frac{(z - b)^m}{m!}, \quad \text{and} \quad A_m(d(z + c)^k) = \frac{k!d(z + c)^{k+m}}{(k + m)!}.
\]

The formal operator $A_m$ acts as an “$m$th antiderivative” operator, but it is just a formal tool we use to motivate well-defined right-inverse mappings $S_{m,n}$. With a formal antiderivative operator at hand, it is then natural to try and define $S_{m,1}$ on the basis $\{1, z, z^2, \ldots\}$ for $\mathcal{P}$ as

\[S_{m,1}(z^k) = A_m \circ C_{\gamma}^{-1}(z^k),\]

and then take $S_{m,n}$ to be $(S_{m,1})^n$. This would yield

\[S_{m,n}(z^k) = \frac{k!}{(k + mn)!} \lambda^{k/n} \lambda^{mn(n-1)/2} (z + r_n)^{k+mn}, \]

which works well with condition (c) of the Hypercyclicity Criterion since $L_{\gamma,z}^n \circ S_{m,n} = I$ on $\{1, z, z^2, \ldots\}$ for this choice of $S_{m,n}$. However, one would run into issues checking condition (b), as one would find high powers of $\lambda$ paired with low powers of $z$ when expanding $(z + r_n)^{k+mn}$. To deal with this issue, we will add a polynomial of degree less than $mn$ to the definition of $S_{m,n}$ in (4) to “kill off” these high powers of $\lambda$. This new definition of $S_{m,n}$ will then satisfy condition (b) of the Hypercyclicity Criterion, and still satisfy condition (c) because any polynomial of degree less than $mn$ belongs to the kernel of $L_{\gamma,z}^n$. We provide the details in the following lemma.

**Lemma 2.** Let $m \in \mathbb{N}$. There are linear mappings $S_{m,n} : \mathcal{P} \to \mathcal{P}$ defined by

\[S_{m,n}(z^k) = \frac{k!}{(k + mn)!} \lambda^{k/n} \lambda^{mn(n-1)/2} (z + r_n)^{k+mn},\]

such that for all $p \in \mathcal{P}$, $L_{\gamma,z}^n S_{m,n} p = p$.

Furthermore, if $\{\sigma_n\}$ is a sequence of complex numbers for which there exists $t > 0$ such that $|\sigma_n| \leq t^n$ for all $n \in \mathbb{N}$, then for all $p \in \mathcal{P}$ and for all $\ell \in \mathbb{N}$,

\[S_{m,\ell n}(\sigma_n p) \to 0.\]

**Proof.** For each monomial $z^k$, define $S_{m,n}(z^k)$ as above in (4), and extend $S_{m,n}$ linearly to $\mathcal{P}$. Let $\Delta_{k,n} = \frac{k!}{(k + mn)!} \lambda^{k/n} \lambda^{mn(n-1)/2} (z + r_n)^{k+mn} - S_{m,n}(z^k)$. By
expanding \((z + r_n)^{k+mn}\) using the Binomial Theorem and cancelling common terms, one can establish that

\[
\Delta_{k,n} = \frac{k!}{(k + mn)!\lambda^{kn}\lambda^{\frac{mn(n-1)}{2}}} \sum_{j=k+1}^{k+mn} \binom{k+mn}{j} z^{k+mn-j} r_n^j.
\]

Thus the degree of \(\Delta_{k,n}\) is \(mn - 1\), and hence it belongs to the kernel of \(L_{\lambda_n,z}^n\) by equation (2).

Using equations (4) and (2), we compute that

\[
L_{\lambda_n,z}^n S_{m,n}(z^k) = L_{\lambda_n,z}^n (S_{m,n}(z^k) + \Delta_{k,n})
\]

\[
= L_{\lambda_n,z}^n \left( \frac{k!}{(k + mn)!\lambda^{kn}\lambda^{\frac{mn(n-1)}{2}}} (z + r_n)^{k+mn} \right)
\]

\[
= \frac{k!}{(k + mn)!\lambda^{kn}\lambda^{\frac{mn(n-1)}{2}}} \cdot \frac{\lambda^{\frac{mn(n-1)}{2}}}{k!} (k + mn)! (\lambda^n z - r_n + r_n)^k \text{ by (2)}
\]

\[
= z^k.
\]

This verifies that \(S_{m,n}\) is a right inverse for \(L_{\lambda_n,z}^n\) on \(P\).

To verify the limit in (5), let \(p\) be a polynomial and let \(\ell \in \mathbb{N}\). By the linearity of \(S_{m,\ell n}\) it suffices to show that

\[
S_{m,\ell n}(\sigma_n z^k) \to 0
\]

uniformly on compact subsets of \(\mathbb{C}\) for each monomial \(z^k\). Let \(R > 0\), let \(|z| \leq R\), and let \(j\) be an integer satisfying \(0 \leq j \leq k\). As \(S_{m,\ell n}(z^k)\) is a sum of \(k + 1\) terms, to show (4) it suffices to show that each of those terms converges to zero uniformly when \(|z| \leq R\). That is, it suffices to show

\[
\left| \frac{\sigma_n k!}{(k + m\ell n)!\lambda^{kn+\ell n}\lambda^{\frac{m(n-1)}{2}}} \binom{k + m\ell n}{j} z^{k+m\ell n-j} r_n^j \right| \to 0
\]

uniformly. Since \(|\lambda| \geq 1\),

\[
|r_{\ell n}| = |b| \left| \sum_{j=0}^{\ell n-1} \lambda^j \right| \leq \ell n |b| \cdot |\lambda|^{\ell n-1}.
\]

Thus

\[
\left| \frac{\sigma_n k!}{(k + m\ell n)!\lambda^{kn+\ell n}\lambda^{\frac{m(n-1)}{2}}} \binom{k + m\ell n}{j} z^{k+m\ell n-j} r_n^j \right| \leq \frac{t^n k! R^{k+m\ell n-j}(\ell n)^j |b|^j |\lambda|^{(\ell n-1)j}}{(k + m\ell n)!\lambda^{kn+\ell n}\lambda^{\frac{m(n-1)}{2}}} \binom{k + m\ell n}{j} \text{ by (8)}
\]

\[
= \frac{t^n k! R^{k+m\ell n-j}(\ell n)^j |b|^j |\lambda|^{(\ell n-1)j}}{(k + m\ell n)!\lambda^{kn+\ell n}\lambda^{\frac{m(n-1)}{2}}} \frac{(k + m\ell n)!}{j!(k + m\ell n - j)!} \to 0, \quad n \to \infty
\]
which establishes the limit in (7) and completes the proof.

With the previous lemmas at hand, we can now use the Hypercyclicity Criterion to show that there are many hypercyclic non-convolution operators that can be generated by \( T_{\lambda,b} \).

**Theorem 3.** Let \( \lambda, b \in \mathbb{C} \) with \( |\lambda| \geq 1 \), and let \( T_{\lambda,b} : H(\mathbb{C}) \to H(\mathbb{C}) \) be the operator defined by \( T_{\lambda,b} : f(z) \mapsto f(\lambda z + b) \). If \( \psi \) is an entire function such that \( \psi(0) = 0 \) and \( \psi(T_{\lambda,b}) \) is a continuous linear operator, then \( \psi(T_{\lambda,b}) \) is hypercyclic.

**Proof.** Let \( \psi(z) = \xi(z)z^\ell \), where \( \ell \in \mathbb{N} \) and \( \xi(z) \) is an entire function with \( \xi(0) \neq 0 \). Then the operator \( \psi(T_{\lambda,b}) = \xi(T_{\lambda,b})T_{\lambda,b}^{\ell} \), and \( (\psi(T_{\lambda,b}))^n = (\xi(T_{\lambda,b}))^nT_{\lambda,b}^{\ell n} \).

Let \( \mathcal{P} \) be the set of complex polynomials in \( H(\mathbb{C}) \), which is a dense subset of \( H(\mathbb{C}) \), and let \( g \in \mathcal{P} \). Since \( T_{\lambda,b}(g) \in \mathcal{P} \) and \( \deg T_{\lambda,b}(g) < \deg g \), by Lemma 1 there is a mapping \( S_{\xi(T_{\lambda,b})} : \mathcal{P} \to \mathcal{P} \) such that \( \xi(T_{\lambda,b})^nS_{\xi(T_{\lambda,b})}(g) = g \) for all \( g \in \mathcal{P} \). Since \( T_{\lambda,b} = L_{\gamma,z} \), by Lemma 2 there exist linear mappings \( S_{1,\ell n} : \mathcal{P} \to \mathcal{P} \) such that \( T_{\lambda,b}S_{1,\ell n}g = g \) for all \( g \in \mathcal{P} \). Thus for all \( g \in \mathcal{P} \),

\[
\psi(T_{\lambda,b})^nS_{1,\ell n}S_{\xi(T_{\lambda,b})}^n(g) = \xi(T_{\lambda,b})^nT_{\lambda,b}^\ell S_{1,\ell n}S_{\xi(T_{\lambda,b})}^n(g) = g,
\]

so the mapping \( S_{1,\ell n}S_{\xi(T_{\lambda,b})}^n \) is a right inverse for \( \psi(T_{\lambda,b})^n \) on \( \mathcal{P} \).

We check the three conditions of the Hypercyclicity Criterion. As just mentioned, the third condition is satisfied. Let \( p \) be a polynomial of degree \( d \). Since \( T_{\lambda,b}^np = 0 \) whenever \( n > d \), and since \( \psi(0) = 0 \), we have that \( (\psi(T_{\lambda,b}))^np = 0 \) whenever \( n > d \). Thus the first condition of the Hypercyclicity Criterion is satisfied.

What remains to check is the second condition. Let \( \xi(0) = w_0 \). By Lemma 1 there is a constant \( C > 0 \) and are constants \( a_{i,n} \) for \( 1 \leq i \leq d \) such that

\[
S_{\xi(T_{\lambda,b})}^n = w_0^{-n}(I + a_{1,n}T_{\lambda,b} + \cdots + a_{d,n}T_{\lambda,b}^d) p,
\]

and \( |a_{i,n}| < Cn^d \). For each positive integer \( n \), let \( a_{0,n} = 1 \). Let \( i \) be an integer such that \( 0 \leq i \leq d \).

By the linearity of \( S_{1,\ell n} \) and equation (9), to show \( S_{1,\ell n}S_{\xi(T_{\lambda,b})}^n \) uniformly on compact subsets of \( \mathbb{C} \), it suffices to show that

\[
S_{1,\ell n}(w_0^{-n}a_{i,n}T_{\lambda,b}^d) p \xrightarrow{n \to \infty} 0
\]

uniformly on compact subsets of \( \mathbb{C} \). If \( i = 0 \), let \( \sigma_n = w_0^{-n} \) and \( t = |w_0|^{-1} \). Then the limit [3] in Lemma 2 implies [10]. If \( i > 0 \), let \( \sigma_n = w_0^{-n}a_{i,n} \). Since \( |w_0^{-n}a_{i,n}| \leq |w_0|^{-n}Cn^d < t^n \) for some \( t > 0 \), limit [3] in Lemma 2 again yields [10], as desired. This shows that the second condition of the HypercyclicityCriterion is satisfied. Hence \( \psi(T_{\lambda,b}) \) is hypercyclic.

Theorem 2 really provides an algebra of hypercyclic non-convolution operators when \( |\lambda| \geq 1 \) (except when \( \lambda = 1 \), in which case the operators are indeed convolution operators). To see this, let \( \psi_1(z), \psi_2(z) \in H(\mathbb{C}) \) such that \( \psi_1(0) = 0 = \psi_2(0) \) and \( \psi_1(T_{\lambda,b}), \psi_2(T_{\lambda,b}) \) are continuous. Then \( \psi_1(0) + \psi(0) = \psi_2(0) = 0 \).
0, \psi(0) = 0, and \psi_1(T_{\lambda,b}) \psi_2(T_{\lambda,b}) is continuous. Hence \psi_1(T_{\lambda,b}) + \psi_2(T_{\lambda,b}) and \psi_1(T_{\lambda,b}) \psi_2(T_{\lambda,b}) are hypercyclic by Theorem 2, and any non-zero scalar multiple of them is as well. We next show that |\lambda| \geq 1 is necessary for the hypercyclicity of these types of operators.

**Theorem 4.** Let \lambda, b \in \mathbb{C} with |\lambda| < 1, and let T_{\lambda,b} : H(\mathbb{C}) \to H(\mathbb{C}) be the operator defined by T_{\lambda,b} : f(z) \mapsto f(\lambda z + b). If \psi(z) is an entire function such that \psi(0) = 0 and \psi(T_{\lambda,b}) is a continuous linear operator, then \psi(T_{\lambda,b}) is not hypercyclic.

**Proof.** We will show that \( (\psi(T_{\lambda,b})) f \overset{n \to \infty}{\longrightarrow} 0 \) for every \( f \in H(\mathbb{C}) \). Since \( \psi(0) = 0 \), we may write \( \psi(z) = \alpha z^j \xi(z) \), where \( \alpha \in \mathbb{C} \setminus \{0\} \) and \( \xi(z) = \sum_{k=0}^{\infty} w_k z^k \) is an entire function such that \( \xi(0) = 0 = w_0 \). Let \( c_{j,n} \) be the \( j \)-th Taylor coefficient of the Taylor series of \( (\xi(z))^n \) centered at zero, so that

\[
(\xi(z))^n = (1 + w_1 z + w_2 z^2 + \cdots)^n := 1 + c_{1,n} z + c_{2,n} z^2 + \cdots,
\]

and let \( r = \sup \{ |w_k| : k \in \mathbb{N} \cup \{0\} \} \). Then by the same type of argument used in Lemma 1 to obtain the estimate (1), we have that

\[
|c_{j,n}| \leq r^j \left( \frac{n + j - 1}{j} \right) = r^j \left( \frac{n}{1} \right) \left( \frac{n + 1}{2} \right) \cdots \left( \frac{n + j - 1}{j} \right) \leq r^j n^j. \tag{11}
\]

Now let \( f \in H(\mathbb{C}) \), let \( C = \max \{|f(z)| : |z| \leq 1\} \), let \( R > 0 \) be given, and let \( |z| \leq R \). As shown in the proof of [9, Theorem 2.1], there exists an \( n_0 \in \mathbb{N} \) such that for all \( n \geq n_0 \),

\[
|T_{\lambda,b}^n f(z)| \leq C n! 2^n |\lambda|^\frac{n(n-1)}{2}. \tag{12}
\]

Furthermore, since \( \lim_{n \to \infty} 2r^2 |\lambda|^n = 0 \), there exists \( n_1 \in \mathbb{N} \) such that \( n \geq n_1 \) implies

\[
2r^2 |\lambda|^n < 1. \tag{13}
\]

Now let \( n \geq \max\{n_0, n_1\} \), and let \( c_{0,n} = 1 \). Then

\[
|\psi(T_{\lambda,b}) f(z)| = |\alpha^n T^{n\ell} (I + c_{1,n} T + c_{2,n} T^2 + \cdots) f(z)| \leq |\alpha^n| \sum_{j=0}^{\infty} |c_{j,n} T^{n\ell+j} f(z)| \leq |\alpha^n| \sum_{j=0}^{\infty} r^j n^j C(n\ell + j)! 2^{n\ell+j} |\lambda|^{\frac{(n+1)(n+2)(n+3)}{2}} \text{ by (11) and (12)}. \tag{14}
\]

Let \( \beta_{j,n} \) be the \( j \)-th term in the sum in the previous line. Then (13) implies

\[
\left| \frac{\beta_{j+1,n}}{\beta_{j,n}} \right| = r(n\ell + j + 1)2 |\lambda|^{n+1} \leq r(n\ell(j+2))2 |\lambda|^{n+1} < (j+2)|\lambda|^2.
\]
One could prove by induction that the inequality $|\beta_{j+1,n}| < (j+2)|\lambda|^2|\beta_{j,n}|$ implies $|\beta_j| < (j+1)!|\lambda|^2|\beta_0|$ for each $j \in \mathbb{N} \cup \{0\}$. Hence the sum in the inequality is less than the sum $|\beta_0| \sum_{j=0}^{\infty} (j+1)!|\lambda|^2$, which converges to zero as $n \to \infty$. This proves that $|\beta_{j,n}| < (j+1)!|\lambda|^j|\beta_0|$, which converges to zero as $n \to \infty$. This proves that $(\psi(T_{\lambda,b}))^n f \underset{n \to \infty}{\to} 0$ uniformly on compact subsets of $\mathbb{C}$, so $\psi(T_{\lambda,b})$ cannot be hypercyclic.

We summarize the previous two theorems in the following characterization.

**Theorem 5.** Let $\lambda, b \in \mathbb{C}$ and let $T_{\lambda,b} : H(\mathbb{C}) \to H(\mathbb{C})$ be the operator defined by $T_{\lambda,b} : f(z) \mapsto f'(\lambda z + b)$. The algebra of operators

$$\{ \psi(T_{\lambda,b}) : \psi(z) \in H(\mathbb{C}), \psi(0) = 0 \text{ and } \psi(T_{\lambda,b}) \text{ is continuous} \}$$

consists entirely of hypercyclic operators if and only if $|\lambda| \geq 1$.

### 3 Hypercyclicity of $C_\gamma \circ \varphi(D)$

We now look at another generalization of the operators $T_{\lambda,b} = C_\gamma \circ D$. Let $\varphi(z)$ be an entire function of exponential type, so that the operator $\varphi(D)$ is a convolution operator. We consider in this section the operators $L_{\gamma,\varphi} = C_\gamma \circ \varphi(D)$, each of which is a non-convolution operator whenever $\lambda \neq 1$. We first prove a type of commutation relation between $C_\gamma$ and $\varphi(D)$.

**Lemma 6.** Suppose $\varphi(D)$ is a convolution operator for some non-constant entire function $\varphi$ of exponential type. Let $\lambda, b \in \mathbb{C}$ with $\lambda \neq 0$, and let $C_\gamma : H(\mathbb{C}) \to H(\mathbb{C})$ be the composition operator $C_\gamma : f(z) \mapsto f(\lambda z + b)$. Then $C_\gamma \circ \varphi(D) = \varphi(\lambda^{-1}D) \circ C_\gamma$.

**Proof.** Let $\varphi(z) = \sum_{k=0}^{\infty} w_k z^k$ and let $f(z) \in H(\mathbb{C})$. Then

$$\varphi(\lambda^{-1}D)C_\gamma f(z) = \varphi(\lambda^{-1}D)f(\lambda z + b) = \sum_{k=0}^{\infty} w_k \lambda^{-k} D^k(f(\lambda z + b)) = C_\gamma \left( \sum_{k=0}^{\infty} w_k D^k f(z) \right) = C_\gamma \varphi(D)f(z).$$

We now provide yet another family of hypercyclic non-convolution operators.

**Theorem 7.** Suppose $\varphi(D)$ is a convolution operator for some non-constant entire function $\varphi$ of exponential type with $\varphi(0) = 0$. Let $\lambda, b \in \mathbb{C}$ and let $C_\gamma : H(\mathbb{C}) \to H(\mathbb{C})$ be the composition operator $C_\gamma : f(z) \mapsto f(\lambda z + b)$. If $|\lambda| \geq 1$, then the operator $L_{\gamma,\varphi} = C_\gamma \circ \varphi(D)$ is hypercyclic.
Proof. We first write \( \varphi(z) = z^m \psi(z) \), where \( \psi(z) = \sum_{k=0}^{\infty} w_k z^k \) is an entire function of exponential type with \( \psi(0) \neq 0 \). By repeatedly applying Lemma \ref{Lemma6} we have that

\[
L_{\gamma,\varphi}^n = C^{\gamma}_{\varphi} (D) C^{\gamma}_{\varphi} (D) \cdots C^{\gamma}_{\varphi} (D)
\]

\[
= \varphi (\lambda^{-n} D) \varphi (\lambda^{1-n} D) \cdots \varphi (\lambda^{-1} D) C^n_{\gamma},
\]

\[
= \frac{D^n}{\lambda^n} \varphi (\lambda^{-n} D) \frac{D^n}{\lambda^{(n-1)m}} \varphi (\lambda^{1-n} D) \cdots \frac{1}{\lambda^m} \varphi (\lambda^{-1} D) C^n_{\gamma}
\]

\[
= \psi (\lambda^{-n} D) \psi (\lambda^{1-n} D) \cdots \psi (\lambda^{-1} D) \psi (\lambda^{-n} D) \psi (\lambda^{1-n} D) \cdots \psi (\lambda^{-1} D) L_{\gamma,\varphi}^n \text{ by } \ref{Lemma2},
\]

where \( L_{\gamma,\varphi} : H(\mathbb{C}) \to H(\mathbb{C}) \) is the operator \( L_{\gamma,\varphi} : f(z) \mapsto f^{(m)}(\lambda z + b) \) considered in Lemma \ref{Lemma2}.

Let \( \mathcal{P} \) be the set of complex polynomials in \( H(\mathbb{C}) \), which is a dense subset of \( H(\mathbb{C}) \). Let \( p \) be a nonzero polynomial of degree \( d \). We define a right-inverse \( F_n : \mathcal{P} \to \mathcal{P} \) for \( L_{\gamma,\varphi}^n \) on \( \mathcal{P} \) as follows. By Lemma \ref{Lemma1} there exist \( C > 0 \) and constants \( a_i \in \mathbb{C}, 1 \leq i \leq d \), such that for each positive integer \( j \), the mapping \( S_{\psi(\lambda^{-j} D)} : \mathcal{P} \to \mathcal{P} \) defined by

\[
S_{\psi(\lambda^{-j} D)} p = w_0^{-1} (I + a_1 \lambda^{-j} D + \cdots + a_d (\lambda^{-j})^d D^d) p
\]

is a right-inverse for \( \psi(\lambda^{-j} D) \) on \( \mathcal{P} \), and \( |a_i| < C \).

Let \( S_{m,n} : \mathcal{P} \to \mathcal{P} \) be the linear right inverse of \( L_{\gamma,\varphi}^n \) as defined in Lemma \ref{Lemma2}. We then define the mapping \( F_n : \mathcal{P} \to \mathcal{P} \) by

\[
F_n p = S_{m,n} S_{\psi(\lambda^{-d} D)} \cdots S_{\psi(\lambda^{-2} D)} S_{\psi(\lambda^{-1} D)} p,
\]

which satisfies \( L_{\gamma,\varphi}^n F_n p = p \).

The condition \( \varphi(0) = 0 \) implies \( \deg L_{\gamma,\varphi} < \deg p \), which implies \( L_{\gamma,\varphi}^n p = 0 \) whenever \( n > d \). What remains to show for the Hypercyclicity Criterion is that

\[
F_n p \xrightarrow{n \to \infty} 0
\]

uniformly on compact subsets of \( \mathbb{C} \). By multiplying out the product \( S_{\psi(\lambda^{-d} D)} \cdots S_{\psi(\lambda^{-2} D)} S_{\psi(\lambda^{-1} D)} \) using equation \ref{Equation16}, we have that

\[
S_{\psi(\lambda^{-d} D)} \cdots S_{\psi(\lambda^{-2} D)} S_{\psi(\lambda^{-1} D)} p
\]

\[
= w_0^{-n} [I + a_1 D + \cdots + a_d D^d] \cdots [I + a_1 \lambda^{-d} D + \cdots + a_d (\lambda^{-d})^d D^d] p
\]

\[
= w_0^{-n} [I + c_1 D + \cdots + c_d D^d] p,
\]

where the coefficients \( c_{j,n} \) for \( 1 \leq j \leq d \) satisfy

\[
c_{j,n} = \sum_{j_1 + \cdots + j_n = j} \frac{a_{j_1}}{(\lambda^{j_1})^0} \frac{a_{j_2}}{(\lambda^{j_2})^1} \cdots \frac{a_{j_n}}{(\lambda^{j_n})^{n-1}},
\]

which implies

\[
F_n p \xrightarrow{n \to \infty} 0
\]

uniformly on compact subsets of \( \mathbb{C} \).
where each $j_k$ is a non-negative integer.

For each positive integer $n$, let $c_{0,n} = 1$. Since $S_{m,n}$ is linear, to show \[17\] it suffices to show that
\[
S_{m,n}(w_0^{-n}c_{j,n}p) \xrightarrow{n \to \infty} 0. \tag{19}
\]
uniformly on compact subsets of $\mathbb{C}$ for each integer $j$ such that $0 \leq j \leq d$.

Let $j$ be an integer satisfying $0 \leq j \leq d$. The number of terms in the sum \[18\] is equal to the number of multinomial coefficients in a multinomial sum, which is \[
\binom{j+n-1}{n-1}.
\]
Let $\alpha = \max\{1, |a_1|, |a_2|, \cdots, |a_d|\}$. Since $|\lambda| \geq 1$, by \[18\] we have that
\[
|c_{j,n}| \leq \binom{j+n-1}{n-1} \alpha^j \leq n^j \alpha^j \leq n^d \alpha^d < e^{nd} \alpha^d. \tag{20}
\]
Now let $t = |w_0|^{-1}e^{d\alpha d}$. Then $|w_0^{-n}c_{j,n}| \leq t^n$, and thus $S_{m,n}(w_0^{-n}c_{j,n}p) \xrightarrow{n \to \infty} 0$ uniformly on compact subsets of $\mathbb{C}$ by limit \[5\] in Lemma \[2\] which shows that \[19\] holds and completes the proof.

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