EIGENVALUE RESOLUTION OF SELF-ADJOINT MATRICES

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Abstract. Resolution of a compact group action in the sense described by Albin and Melrose is applied to the conjugation action by the unitary group on self-adjoint matrices. It is shown that the eigenvalues are smooth on the resolved space and that the trivial bundle smoothly decomposes into the direct sum of global one-dimensional eigenspaces.

For a general compact Lie group G acting on a smooth compact manifold with corners M, Albin and Melrose [AM11] showed that there is a canonical full resolution such that the group action lifts to the blow-up space Y(M) to have a unique isotropy type. This generalized the result of Borel [BJ06] that if all the isotropy groups of a compact group action are conjugate then the orbit space G\M is smooth.

In this paper, we give an explicit construction of such a resolution of the unitary group action on the space of self-adjoint matrices

\[ S = S(n) = \{ X \in M_n(\mathbb{C}) | X^* = X \} \]

with the unitary group U(n) acting by conjugation:

\[ u \in U(n), X \in S, u \cdot X := uXu^{-1}. \]

The orbit of an element X ∈ S, denoted by U(n)·X, consists of the matrices with the same eigenvalues including multiplicities. For a matrix X ∈ S with \( m \) distinct eigenvalues \( \{ \lambda_j \}_{j=1}^m \), each with multiplicity \( i_k, k = 1, 2, .., m \), the isotropy group of X is isomorphic to a direct sum of smaller unitary groups:

\[ U(n)^X := \{ u \in U(n) | u \cdot X = X \} \cong \bigoplus_{k=1}^m U(i_k). \]

Thus the matrices with the same multiplicities \( \{ i_k \} \), have conjugate isotropy groups. The isotropy types are therefore parametrized by the partition of \( n \) into integers. Note here that the partition contains information about ordering, for example, the two partitions of 3, \( \{ i_1 = 1, i_2 = 2 \} \) and \( \{ i_1 = 2, i_2 = 1 \} \), are not the same type.

For \( n > 1 \), the eigenvalues are not smooth functions on \( S \), but are singular where the multiplicities change. We will show that, by making an iterative blow up, the singularities are resolved and the eigenvalues become smooth functions on the resolved space.

Recall the lemma of group action resolution in [AM11]:

**Lemma 1** ([AM11]). A compact manifold (with corners), M, with a smooth, boundary intersection free, action by a compact Lie group, G, has a canonical full resolution, Y(M), obtained by iterative blow-up of minimal isotropy types.

Consider the trivial bundle over \( S \),

\[ M := S \times \mathbb{C}^n, \]

the fiber of which can be decomposed into \( n \) eigenspaces of the self-adjoint matrix at the base point. This decomposition is not unique at matrices with multiple eigenvalues and in general the eigenspaces are not smooth.

There are two basic kinds of real blow up, namely radial and projective, which give different results; projective blow up of a hypersurface is trivial but radial blow up produces a new boundary. As pointed out in [AM11], projective blow up usually requires an extra step of reflection in the iterative scheme in order to obtain smoothness. We will show that, after radial blow up, the trivial bundle \( M \) decomposes

\[ \text{Date: April 29, 2015.} \]
into the direct sum of \( n \) 1-dimensional eigenspaces. By contrast, after the projective blow up, the eigenvalues are still smooth on the resolved space, and locally this is a smooth decomposition into simple eigenspaces, but the trivial bundle doesn’t split into global line bundles.

Next we recall the resolution in the sense of Albin and Melrose.

**Definition 1 (eigenresolution).** By an eigenresolution of \( S \), we mean a manifold with corners \( \hat{S} \), with a surjective smooth map \( \beta : \hat{S} \to S \) such that the self-adjoint matrices have a smooth (local) diagonalization when lifted to \( \hat{S} \), with eigenvalues lifted to smooth functions on \( \hat{S} \).

Note in the definition we only require the the diagonalization exists locally. To encompass the information of global decomposition of eigenvectors, we introduce the full resolution below.

**Definition 2 (full eigenresolution).** A full eigenresolution is an eigenresolution with global eigenbundles. The eigenvalues are lifted to \( n \) smooth functions \( f_i \) on \( \hat{S} \), and \( M \), which is the trivial \( n \)-dimensional complex vector bundle on \( \hat{S} \), is decomposed into \( n \) smooth line bundles:

\[
\hat{S} \times \mathbb{C}^n = \bigoplus_{i=1}^n E_i
\]

such that

\[
\beta(x)v_i = f_i(x)v_i, \forall v_i \in E_i(x), \forall x \in \hat{S}.
\]

We use the blow-up constructions introduced by Melrose in the book [Mel, Chapter 5] and show that we can obtain resolutions in this way and, in particular, full resolution if we use radial blow-up.

**Theorem 1.** The iterative blow up of the isotropy types in \( S \), in an order compatible with inclusion of the conjugation class of the isotropy group, yields an eigenresolution. In particular, radial blow up gives a full eigenresolution.

The proof proceeds through induction on dimension. We begin the proof by discussing the first example which is the \( 2 \times 2 \) matrices.

**Lemma 2 (2 × 2 case).** For the \( 2 \times 2 \) matrices \( S(2) \), the eigenvalues and eigenvectors are smooth except at the scalar matrices. After radial blow up, the singularities are resolved and the trivial 2-dim bundle splits into the direct sum of two line bundles. The projective blow up also gives smooth eigenvalues, but does not give two global line bundles.

**Proof.** In this case \( S = S(2) = \{ \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} | a_{ii} \in \mathbb{R}, z_{12} \in \mathbb{C} \} \cong \mathbb{R}^4 \). Thus \( S \) is isomorphic to the product of \( \mathbb{R} \) and the trace-free subspace

\[
S_0 = \{ \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} | a_{11} + a_{22} = 0 \},
\]

i.e. there is a bijective linear map:

\[
\phi : S \to S_0 \times \mathbb{R}
\]

\[
A = \begin{pmatrix} a_{11} & z_{12} \\ \bar{z}_{12} & a_{22} \end{pmatrix} \mapsto (A_0 := A - (a_{11} + a_{22})I, a_{11} + a_{22})
\]

The eigenvalues \( \lambda_i \) and eigenvectors \( v_i \) of \( A \) are related to those \( A_0 \) by \( \lambda_i(A) = \lambda_i(A_0) + tr(A) \), \( v_i(A) = v_i(A_0), i = 1, 2 \). Therefore, we can restrict the discussion of resolution to the subspace \( S_0 \), since the smoothness of eigenvalues and eigenvectors on \( S \) follows.

Let \( z_{12} = c + di \). The space \( S_0 \) can be identified with \( \mathbb{R}^3 = \{(a_{11}, c, d)\} \). The eigenvalues of this matrix are:

\[
\lambda_{\pm} = \pm \sqrt{a_{11}^2 + c^2 + d^2}.
\]

Hence the only singularity of the eigenvalues on \( S_0 \) is at the point \( a_{11} = c = d = 0 \) which is the zero matrix.
Based on the resolution formula in [Mel], the radial blow up can be realized as

\[ \hat{S}_0 = [S_0, \{0\}] = S^+ N\{0\} \cup (S_0 \setminus \{0\}) \simeq S^2 \times [0, \infty)_+ \]

where the front face \( S^+ N\{0\} \simeq S^2 \). Here the radial variable \( r \) is \( \sqrt{a_{11}^2 + c^2 + d^2} \). The blow-down map is

\[ \beta : [S_0, \{0\}] \to S_0, (r, \theta) \mapsto r \theta, r \in \mathbb{R}_+, \theta \in S^2. \]

The radial variable \( r \) lifts to be a smooth on the blown up space, therefore the two eigenvalues \( \lambda_\pm = \pm r \) become smooth functions.

Now we consider the eigenvectors of the corresponding eigenvalues \( \lambda_\pm \)

\[ v_\pm = (c + di, \pm \sqrt{a_{11}^2 + c^2 + d^2 - a_{11}}) \in \mathbb{C}^2. \]

Similar to the discussion of the eigenvalues, the only singularity is at \( r = 0 \), which becomes a smooth function on \([S_0, \{0\}]\), it follows that \( v_+ \) and \( v_- \) span two smooth line bundles on \([S_0, \{0\}]\).

If we do the projective blow up instead, which identifies the antipodal points in the front face of \( S^2 \), the fiber is \( S \), which hits at two different places thus both \( a_{11} \) are on the curve, and \( (1+\gamma^2) \) are similar. The two eigenvalues we get from here are

\[ v_\pm = \pm \sqrt{a_{11}^2 + c^2 + d^2} = \pm |x_1|\sqrt{(1 + y_1^2 + z_1^2)}. \]

which is smooth at \( \{x_1 > 0\} \). Similar discussions hold for the other two coordinate patches.

However, the trivial bundle does not decompose into two line bundles as in the radial case. The nontriviality of eigenbundles can be seen by taking a loop in \( \mathbb{R}\mathbb{P}^2 \)

\[ l = \beta^{-1}((r = 1)) \subset \hat{S} \]

which is a curve that winds twice around origin. This curve intersects the line \( c = d = 0 \) twice, which hits at two different places thus both \( a_{11} \) are on the curve, and \( (6) \) shows that starting from \( v_- = (0, -2) = (0, -2a_{11}^{-1}) \), this turns into \( v_+ = (0, -2) = (0, 2a_{11}^{-1}) \), which means they are not separated by projective blow up.

Now that we have done the radial resolution for the trace free slice \( S_0 \), the resolution of \( S \) follows. Consider \( S \) as a 3-dim vector bundle on \( \mathbb{R} \) with trace being the projection map, then at each base point \( \lambda \), the fiber is \( S_0 + \lambda I \). The resolution is \([S_0 + \lambda I; \lambda I] \cong [S_0; 0] \). Since the trace direction is transversal to the blow up, and therefore

\[ [S; \mathbb{R}I] = [S_0; \{0\}] \times \mathbb{R}. \]

And because the trace don’t change the eigenvectors, the smoothness follows. \( \square \)

To proceed to higher dimensions, we first discuss the partition of eigenvalues into clusters. The basic case is when the eigenvalues are divided into two clusters, then the \( U(n) \) action of the matrices can be decomposed to two commuting actions.

\textit{Definition 3} (spectral gap). A connected neighborhood \( U \subset S \) has a spectral gap at \( c \in \mathbb{R} \), if \( c \) is not an eigenvalue of \( X \), for any \( X \in U \).

Note here that since \( U \) is connected, the number of eigenvalues less than \( c \) stays the same for all \( X \in U \), denoted by \( k \).
Lemma 3 (local eigenspace decomposition). If a neighborhood $U \subset S(n)$ has a spectral gap at $c$, then the matrices in $U$ can be decomposed into two self-adjoint commuting matrices smoothly:

$$X = L_X + R_X, L_X R_X = R_X L_X.$$ 

with $\text{rank}(L_X) = k$, $\text{rank}(R_X) = n - k$.

Proof. Let $\gamma$ be a simple closed curve on $\mathbb{C}$ such that it intersects with $\mathbb{R}$ only at $-R$ and $c$, where $R$ is a sufficiently large number. In this way, for any matrix $X \in U$, the $k$ smallest eigenvalues are all contained inside $\gamma$. We consider the operator

$$P_X : \mathbb{C}^n \to \mathbb{C}^n$$

(9)

$$P_X := -\frac{1}{2\pi i} \oint_{\gamma} (X - sI)^{-1} ds$$

Since the resolvent is nonsingular on $\gamma$, $P_X$ is a well-defined operator and varies smoothly with $X$. And the integral is independent of choice of $\gamma$ up to homotopy.

First we show that $P_X$ is a projection operator, i.e.

$$P_X^2 = P_X$$

Let $\gamma_s$ and $\gamma_t$ be two curves satisfying the above condition with $\gamma_s$ completely inside $\gamma_t$, then

$$P_X^2 = -\frac{1}{4\pi^2} \oint_{\gamma_t} (X - tI)^{-1} dt \left( \oint_{\gamma_s} (X - sI)^{-1} ds \right)$$

$$= -\frac{1}{4\pi^2} \oint_{\gamma_t} dt \left[ \oint_{\gamma_s} \frac{1}{s - t} (X - sI)^{-1} ds - \oint_{\gamma_s} \frac{1}{s - t} (X - tI)^{-1} ds \right]$$

$$= I - II$$

where using the fact that $s$ is completely inside $\gamma_t$

$$I = -\frac{1}{4\pi^2} \oint_{\gamma_t} \frac{1}{X - sI} ds \oint_{\gamma_s} \frac{1}{s - t} dt = -\frac{1}{4\pi^2} (-2\pi i) \oint_{\gamma_s} \frac{1}{X - sI} ds = P_X$$

and any $t$ on $\gamma_t$ is outside of the loop $\gamma_s$

$$\oint_{\gamma_s} \frac{1}{s - t} ds = 0$$

we have

$$II = -\frac{1}{4\pi^2} \oint_{\gamma_t} (X - tI)^{-1} dt \oint_{\gamma_s} \frac{1}{s - t} ds = 0$$

Therefore $P_X^2 = P_X$.

Then we show that $P_X$ is self-adjoint. This is because

$$P_X^* = \frac{1}{2\pi i} \int_{\gamma} ((X - sI)^{-1})^* ds = \frac{1}{2\pi i} \int_{\gamma} (X - sI) ds = P_X.$$

$P_X$ maps $\mathbb{R}^n$ to the invariant subspace spanned by the eigenvectors corresponding to eigenvalues that are less than $c$. We denote this invariant subspace by $L$ and its orthogonal complement by $R$. Write $X$ as the diagonalization $X = V\Lambda V^{-1}$ where $\Lambda$ is the eigenvalue matrix and $V$ consists its eigenvectors as columns. Then $L$ is spanned by the first $k$ columns of $V$. Take one of the eigenvectors $v_j \in L, j = 1, 2, ..., k$,

$$P_X v_j = -\frac{1}{2\pi i} \oint_{\gamma} (X - sI)^{-1} v_j ds = -\frac{1}{2\pi i} \oint V(\Lambda - sI)^{-1} V^{-1} v_j = -\frac{1}{2\pi i} v_j \oint \frac{1}{\lambda_j - s} ds = v_j$$

Similarly for $v_j \in R$ that corresponds to an eigenvalue greater than $c$ (therefore $\lambda_j$ is outside the loop),

$$P_X v_j = -\frac{1}{2\pi i} v_j \oint \frac{1}{\lambda_j - s} ds = 0,$$

therefore

$$(I - P_X) v_j = v_j, \forall v_j \in R.$$
Then using the projection $P_X$ we define two operators $L_X$ and $R_X$ as:

\begin{equation}
L_X := P_X X P_X
\end{equation}

and

\begin{equation}
R_X := (I - P_X) X (I - P_X)
\end{equation}

Since $P_X$ is smooth, the two operators are also smooth. Moreover, using the fact that $P_X$ is a projection onto the invariant subspace $L$, we have

\[(I - P_X) X P_X = P_X X (I - P_X) = 0\]

therefore

\[X = L_X + R_X.\]

For an eigenvector $v \in L$,

\begin{equation}
L_X v = X v, R_X v = 0,
\end{equation}

i.e. $L_X$ equals to $X$ when restricted to $L$, similarly $R_X|_L = X$. Since $P_X^* = P_X$, $L_X$ and $R_X$ are also self-adjoint. In this way we get two commuting lower rank matrices $L_X$ and $R_X$. \qed

It is natural to to have a finer decomposition when there is more than one spectral gap in the neighborhood, and we have the following corollary.

**Corollary 1.** If the eigenvalues of matrices in a neighborhood $U$ can be grouped into $k$ clusters, Then the matrices can be decomposed into $k$ lower rank self-adjoint commuting matrices smoothly.

**Proof.** Do the decomposition inductively. If $k=2$, then it is the case in lemma 3. Suppose the decomposition for $k = l-1$ is defined. Then for $k = l$, since the eigenvalues can also be divided into 2 clusters (by combining the smallest l-1 groups of eigenvalues together), then $X = L_X + R_X$, with $L_X$ and $R_X$ corresponding to the two intervals. Then $L_X$ satisfies the separation condition for $l-1$ clusters, so by induction, $L_X = L_1 + \ldots + L_{l-1}$. Therefore, $X = L_1 + L_2 + \ldots + L_{l+1} + R_X$ is the desired division. \qed

Using the above lemma 3 of decomposition of matrices in a neighborhood, we can now show that locally the trivial bundle $S \times \mathbb{C}^n$ decomposes into two subspaces if there is a spectral gap. And moreover, locally there is a product structure of two lower order matrices. In order to see this, we need to introduce the Grassmanian. Let $Gr_C(n,k)$ denote the Grassmannian, i.e. the set of k-dim subspace in $\mathbb{C}^n$. Consider the tautological vector bundle over Grassmanian:

\[\pi : T_k \to Gr_C(n,k), \pi^{-1}(p) = V(p),\]

where each fibre is a k-dimensional subspace in $\mathbb{C}^n$, with self-adjoint operators acting on it. Similarly, we define $T_{n-k}$ to be the orthogonal complement of $T_k$:

\[\pi : T_{n-k} \to Gr_C(n,k), \pi^{-1}(p) = V(p)^\perp.\]

**Definition 4** (operator bundle). Let $P_k$ (resp. $P_{n-k}$) be the bundles over $Gr_C(n,k)$ of the fibre-wise self-adjoint operators on the tautological bundle $T_k$ (resp. $T_{n-k}$).

Let $\pi : P_k \oplus P_{n-k} \to Gr_C(n,k)$ be the whitney sum of the two bundles. Each of its fiber is isomorphic to $S(k) \oplus S(n-k)$ when we pick a basis. There is a $U(n)$ action on this bundle:

\begin{equation}
g \cdot (p, (p_k, p_{n-k})) = (g \cdot p, (g \circ p_k \circ g^{-1}, g \circ p_{n-k} \circ g^{-1})), p \in Gr_C(n,k), p_k \in P_k(p), p_{n-k} \in P_{n-k}(p).
\end{equation}

Suppose an open neighborhood $U \subset S$ satisfies the spectral gap condition. Let $U(n) \cdot U$ be the group invariant neighborhood generated by $U$, that is,

\begin{equation}
U(n) \cdot U := \cup_{g \in U(n)} g \cdot U.
\end{equation}

Then $U(n) \cdot U$ is open and connected, and also satisfies the spectral gap condition as $U$ does, since $U(n)$ action preserves the eigenvalues. From the proof of the lemma 3, it is shown that in the neighborhood, the trivial $\mathbb{C}^n$ bundle over $U$ naturally splits into two subbundles $E_k \oplus E_{n-k}$. And this gives a local product structure. We will prove that, for a $U(n)$-invariant neighborhood, there is actually a group equivariant isomorphism with the operator bundles defined above.
Lemma 4 (bundle map). If a point $X_0 \in S$ satisfies the spectral gap condition, then there is a neighborhood $X_0 \in V \subset S$ such that it is isomorphic to a neighborhood in the product of lower rank matrices and Grassmanian, i.e.

$$
\phi : V \cong V(k) \times V(n-k) \times V_{Gr} \subset S(k) \times S(n-k) \times Gr_C(n,k) \subset P_n \oplus P_{n-k}.
$$

Moreover, if we take the neighborhood $U(n) \cdot V$, it is isomorphic to a neighborhood $W \subset P_k \oplus P_{n-k}$ such that $n(W) = Gr_C(n,k)$ and the isomorphism $\phi$ is $U(n)$-invariant.

Proof. From the proof of Lemma 3, there is a neighborhood $U$ of $X_0$, such that each element $X \in U$ are decomposed into $L_X + R_X$. Moreover, it induces a decomposition of the trivial bundle $U \times \mathbb{C}^n$ into two subbundles:

$$(15) \quad U \times \mathbb{C}^n = E_k \oplus E_{n-k}$$

where $E_k(X)$ and $E_{n-k}(X)$ are determined by the projection operator $P_X$ defined in equation (9):

$$(16) \quad E_k(X) = Im(P_X), E_{n-k}(X) = Im(P_X)^\perp$$

Let $(\xi_1, \ldots, \xi_k)$ be the basis for $E_k(X_0)$. $E_k$ over $U$ is an open neighborhood in $Gr_C(n,k)$. We can find a neighborhood $V$ of $X_0$ (possibly smaller than $U$) such that, for every point in $V$, the $k$-dimensional space $E_k$ projects onto $E_k(X_0)$. And an orthonormal basis of $E_k(X)$ is uniquely determined by requiring the projection of the first $j$ vectors to $E_k(X_0)$ spans $(\xi_1, \ldots, \xi_j)$ for every $j$ smaller than $k$. In this way, we picked a basis for each fiber of $E_k$ and $E_k$ is trivialized to be a $k$-dimensional vector bundle on $V$. Since the action of $X$ on $\mathbb{C}^n$ has been decomposed to $L_X$ and $R_X$, then with the choice of basis, the action of $L_X$ on $E_k(X)$ gives a $k \times k$ self-adjoint matrix, and by continuity, these matrices form a neighborhood $V_k$ in $S(k)$. And the same argument works for $R_X$.

Therefore, we have the following map $\phi$:

$$(17) \quad \phi : V \rightarrow P_k \oplus P_{n-k} \quad X \mapsto (E_k(X), (L_X|_{E_k(X)}, R_X|_{E_{n-k}(X)}))$$

This map is an isomorphism between $V$ and $\phi(V)$. It’s injective, since the action of the two invariant subspaces uniquely determines the action on $\mathbb{C}^n$, therefore gives the unique operator $X$. The continuity of $\phi$ and $\phi^{-1}$ comes from the continuity of the projection operator defined in Lemma 1 therefore the continuity of $E_k, L_X$ and $R_X$ are continuous.

Now take $U(n) \cdot V$, since $E_k$ takes every possible $k$-subspace of $\mathbb{C}^n$ under the action of $U(n)$, we know that the first entry of $\phi(U(n) \cdot V)$ maps onto $Gr_C(n,k)$. Moreover, since the decomposition respects the action of $U(n)$, it is easily seen that, for $g \in U(n), X \in G \cdot V$,

$$
(18) \quad \phi(g \cdot X) = (g \cdot E_k(X), (g \circ L_X \circ g^{-1}, g \circ R_X \circ g^{-1})) = g \cdot (\phi(X))
$$

which means the isomorphism is group invariant. \qed

To do the induction, we will need to define an index on the inclusion isotropy types, so the blow up procedure could be done in the partial order given by the index. Recall that two matrices have the same isotropy type if they have the same “clustering” of eigenvalues. Now we define the isotropy index of a matrix $X$ as follows.

Definition 5 (Isotropy index). Suppose the eigenvalues of a matrix $X$ are

$$
\lambda_1 = \ldots = \lambda_i_1 < \lambda_{i_1+1} = \ldots = \lambda_i_2 < \lambda_{i_2+1} = \ldots < \lambda_{i_{k-1}+1} = \ldots = \lambda_n
$$

then the isotropy index of $X$ is defined as the set

$$
I(X) = \{i_0 = 0, i_1, i_2, \ldots, i_{k-1}, i_k = n\}.
$$

There is a partial order of this index on $S$, given by the inclusion of isotropy types. That is, if for matrix $X$ and $Y$ we have $I(X) \subset I(Y)$ then we say that the order is $X \leq Y$. Note there is an inverse inclusion to isotropy group. The smallest isotropy index is $I(\lambda I) = \{0, n\}$ while the isotropy group is $U(n)$ which is the largest. And the largest index is $\{0, 1, 2, \ldots, n-1, n\}$ which correspond to $n$ distinct eigenvalues, where the isotropy group contains only identity.
The last lemma we need before the induction is the comparability of conjugacy class inclusion and the decomposition to two submatrices, which shows the order of resolution in lemma 1 is comparable with the decomposition.

**Lemma 5** (Compatibility with conjugacy class). *The partial order of conjugacy class inclusion is comparable with the decomposition in lemma 3.*

**Proof.** Suppose a neighborhood $V \subset S(n)$ has a decomposition in lemma 3. We need to show that, if $S(n)^{I^k}$ is the stratum of minimal isotropy type in $V$, then the decomposition of this stratum corresponds to the minimal isotropy type in $U(k)$ and $U(n-k)$.

Since $V$ satisfies the spectral gap condition, all the isotropy types in $V$ would be subgroups of $U(k) \oplus U(n-k)$. Suppose the minimal stratum corresponds to the index $I = \{0, i_1, ..., i_m\}$ which must contain $k$ as one element because of the spectral gap condition. Then the isotropy type of two subgroups are $\{0, i_1, ..., k\}$ and $\{i_j - k = 0, j_{j+1}, ..., n - k\}$. They would still be the minimal in each subgroup, otherwise when the two smallest elements combined it’ll give a smaller index than $I$ which is a contradiction.

Now we can finally prove theorem 1 using the above lemmas.

**Proof of Theorem 1.** We prove the theorem by induction of the matrix size. The $2 \times 2$ case is shown in lemma 2. Suppose the claim holds for all the cases up to $n - 1$. Now we claim that, by an iterative blow up, we can get $\hat{S}(n)$ for dimension=$n$, with eigenvalues and eigenbundles lifted to satisfy the full eigenresolution properties.

As in the $2 \times 2$ example, we shall first consider the trace free slice $S_0(n)$ since other slices have the same behavior in terms of smoothness of eigenvalues and eigenbundles. Take the smallest index $I = \{0, n\}$ with the largest possible isotropy group $U(n)$, and the stratum in $S_0(n)$ with such an isotropy group is the zero matrix. After blowing up, we get $[S_0;0]$ as the first step.

The next smallest index is $\{0,k,n\}$ where $1 < k < n$. And the strata corresponding to different $k$ become disjoint in $[S_0;0]$ because if the eigenvalues of a matrix $X \in S_0$ satisfy $k_1 \lambda_1 + k_2 \lambda_2 = 0, k_1' \lambda_1 + k_2' \lambda_2 = 0$, then $\lambda_1 = \lambda_2 = 0$, which has been blown up in the previous step. Therefore we can blow up those strata separately. For any point $X \in S_0(n)$ with $I(X) = \{0, k, n\}$, we can generate a neighborhood $U(n) \cdot V$ as in lemma 4, which is isomorphic to a neighborhood in the bundle $P_k \oplus P_{n-k}$.

Locally there is a product structure $V \cong V_k \times V_{n-k} \times Gr \subset S(k) \times S(n-k) \times Gr(n,k)$. For every base point $p \in V_{Gr}$, since the fibre is isomorphic to a neighborhood in $S(k) \times S(n-k)$, the resolution can be done separately to $V_k$ and $V_{n-k}$. And according to lemma 5 the index order is preserved when decomposed into two subspaces, so the blow up construction indexed by isotropy type inclusion can be done on $V_k$ and $V_{n-k}$. By induction, after the full resolution of the two subspaces, $E_k$ and $E_{n-k}$ both split into line bundles, and eigenvalues also extend to the front face smoothly. And since this local product structure is $U(n)$-invariant on $U(n) \cdot V$, the splitting of eigenbundles are actually global.

Therefore, after the resolution, we have iteratively blown up the stratum according to isotropy indices to get

\[
\hat{S} = [S;\{0\}; S^{I_1}; S^{I_2}; ..., S^{I_n}],
\]

above which there are $n$ line bundles as eigenbundles and the corresponding eigenvalues are also smooth.

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