GENERALIZED HYPERGEOMETRIC ARITHMETIC $\mathcal{D}$-MODULES UNDER A $p$-ADIC NON-LIOUVILLENESS CONDITION.

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Abstract. We prove that the arithmetic $\mathcal{D}$-modules associated with the $p$-adic generalized hypergeometric differential operators, under a $p$-adic non-Liouvillelessness condition on parameters, are described as an iterative multiplicative convolution of (hypergeometric arithmetic) $\mathcal{D}$-modules of rank one. As a corollary, we prove the overholonomicity of hypergeometric arithmetic $\mathcal{D}$-modules under a $p$-adic non-Liouvillelessness condition.

0. Introduction.

N. M. Katz [Kat90] introduces the hypergeometric $\mathcal{D}$-modules $\mathcal{Hyp}(\alpha; \beta)$ over $\mathbb{G}_m$ with complex parameters $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$, and uses them to nicely describe an “inductive” structure of hypergeometric objects. To be precise, Katz proves that $\mathcal{Hyp}(\alpha; \beta)$ is described as iterative multiplicative convolution, denoted by $\ast$, of the hypergeometric $\mathcal{D}$-modules $\mathcal{Hyp}(\alpha_i; \emptyset)$’s and $\mathcal{Hyp}(\emptyset; \beta_j)$’s (i.e. those with only one parameter):

Theorem ([Kat90, 5.3.1]). Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{C}^m$ and $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$ be complex parameters, and assume that for any $i$ and $j$, $\alpha_i - \beta_j$ is not an integer.

Then, there exist isomorphisms

$$\mathcal{Hyp}(\alpha; \beta) \cong \mathcal{Hyp}(\alpha_1, \ldots, \alpha_{m-1}; \beta) \ast \mathcal{Hyp}(\alpha_m; \emptyset),$$

$$\mathcal{Hyp}(\alpha; \beta) \cong \mathcal{Hyp}(\alpha; \beta_1, \ldots, \beta_{n-1}) \ast \mathcal{Hyp}(\emptyset; \beta_n).$$

In the previous article [Miy16], the author introduces the $p$-adic hypergeometric differential operators with $p$-adic parameters $\alpha \in (\mathbb{Z}_p)^m$ and $\beta \in (\mathbb{Z}_p)^n$ (under a choice of Dwork’s $\pi$) and also the arithmetic $\mathcal{D}$-modules $\mathcal{H}_\pi(\alpha; \beta)$ associated with such differential operators. The author then proves that, in the case where all components of $\alpha$ and $\beta$ lie in $\frac{1}{2} \mathbb{Z}$, then $\mathcal{H}_\pi(\alpha; \beta)$ has an analogous description as the theorem above by using the multiplicative convolution of arithmetic $\mathcal{D}$-module on $\mathbb{G}_m$ [Miy16, 3.2.5]. As an application of this theorem, the author proves that a $p$-adic hypergeometric differential operator defines an overconvergent $F$-isocrystal on $\mathbb{G}_m$ if $m \neq n$, and on $\mathbb{G}_m \setminus \{1\}$ if $m = n$ [Miy16, 4.1.3].

The goal of this article is to extend this decomposition of hypergeometric arithmetic $\mathcal{D}$-modules $\mathcal{H}_\pi(\alpha; \beta)$ to more general parameters which are not necessarily rational numbers (thus they do not necessarily come from a multiplicative character on the residue field). In fact, we prove this under a $p$-adic non-Liouvillelessness condition on parameters:
Theorem (Theorem 3.1.1). Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{Z}_p)^m$ and $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{Z}_p)^n$ be parameters in $p$-adic integers. Assume that, for any $i$ and $j$, $\alpha_i - \beta_j$ is not an integer nor a $p$-adic Liouville number.

Then, we have isomorphisms
\[
\mathcal{H}_\pi(\alpha; \beta) \cong \mathcal{H}_\pi(\alpha_1, \ldots, \alpha_{m-1}; \beta) \ast \mathcal{H}_\pi(\alpha_m; \emptyset)[-1].
\]
\[
\mathcal{H}_\pi(\alpha; \beta) \cong \mathcal{H}_\pi(\alpha; \beta_1, \ldots, \beta_{n-1}) \ast \mathcal{H}_\pi(\emptyset; \beta_n)[-1].
\]

Since an algebraic number in $\mathbb{Z}_p$ is not a $p$-adic Liouville number, the theorem above is, in particular, applicable to any algebraic parameters with no integer differences.

As an application of this main theorem, we prove the quasi-$\Sigma$-unipotency in the sense of Caro [Car18], in particular the overholonomicity, of our $\mathcal{H}_\pi(\alpha; \beta)$ under a stronger condition of $p$-adic non-Liouviliness:

Corollary (Proposition 3.2.2). Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{Z}_p)^m$ and $\beta = (\beta_1, \ldots, \beta_n) \in (\mathbb{Z}_p)^n$ be parameters in $p$-adic integers. Assume that $(m, n) \neq (0, 0)$, that $\alpha_i - \beta_j \notin \mathbb{Z}$ for any $(i, j)$, and that the subgroup $\Sigma$ of $\mathbb{Z}_p/\mathbb{Z}$ generated by $\alpha_i$'s and $\beta_j$'s does not have a $p$-adic Liouville number. Then, $\mathcal{H}_\pi(\alpha; \beta)$ is a quasi-$\Sigma$-unipotent $\mathcal{D}_{\mathbb{F}_p}^+_{\mathbb{Q}_{\mathbb{F}_p}}$-module. In particular, it is an overholonomic $\mathcal{D}_{\mathbb{F}_p}^+_{\mathbb{Q}_{\mathbb{F}_p}}$-module.

Contrary to the results in the previous article, our $\mathcal{H}_\pi(\alpha; \beta)$'s do not necessarily have a Frobenius structure (in fact, for example, $\mathcal{H}_\pi(\alpha; \emptyset)$ does not have a Frobenius structure if $\alpha$ is not rational). It is thus worth to remark that the corollary above gives examples of overholonomic $\mathcal{D}_{\mathbb{F}_p}^+$-modules without assuming the existence of a Frobenius structure.

We conclude this introduction by explaining the organization of this article.

In Section 1, after a quick review of the theory of cohomological operations on arithmetic $\mathcal{D}$-modules, we define the multiplicative convolution for arithmetic $\mathcal{D}$-modules and study the relationship with Fourier transform.

In Section 2, we firstly introduce the hypergeometric arithmetic $\mathcal{D}$-modules. Then, after recalling the notion of $p$-adic Liouvilleness, we give a crucial lemma on hypergeometric arithmetic $\mathcal{D}$-modules under a $p$-adic non-Liouville condition, which we will need in proving the main theorem.

In Section 3, we establish the main theorem and give an application to the quasi-$\Sigma$-unipotence.

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Conventions and Notations. In this article, $V$ denotes a complete discrete valuation ring of mixed characteristic $(0, p)$ whose residue field $k$ is a finite field with $q$ elements. The fraction field of $V$ is denoted by $K$. We denote by $| \cdot |$ the norm on $K$ normalized by $|p| = p^{-1}$. Throughout this article, we assume that there exists an element $\pi$ of $K$ that satisfies $\pi^{q-1} + (-p)^{(q-1)/(p-1)} = 0$, and fix such a $\pi$.

1. Arithmetic $\mathcal{D}$-modules

1.1. Cohomological operations on $D^b_{\text{coh}}(\mathcal{D}_{\mathbb{F}_p}^+(1T))^\text{'s}$. In this subsection, we recall some notation and fundamental properties concerning cohomological operations on $D^b_{\text{coh}}(\mathcal{D}_{\mathbb{F}_p}^+(1T))$. 
Definition 1.1.1. (i) A d-couple is a pair \((\mathcal{P}, T)\), where \(\mathcal{P}\) is a smooth formal scheme over \(\text{Spf}(V)\) and where \(T\) is a divisor of the special fiber of \(\mathcal{P}\) (an empty set is also a divisor). If a \(k\)-variety \(X\) is the special fiber of \((\mathcal{P}, T)\), we say that \((\mathcal{P}, T)\) realizes \(X\).

(ii) A morphism of d-couples \(f : (\mathcal{P}', T') \to (\mathcal{P}, T)\) is a morphism \(\overline{f} : \mathcal{P}' \to \mathcal{P}\) such that \(\overline{f}(\mathcal{P}' \setminus T') \subset \mathcal{P} \setminus T\) and that \(\overline{f}^{-1}(T)\) is a divisor (or empty). We say that \(f\) realizes the morphism \(f_0 : X' \to X\) of \(k\)-varieties if \((\mathcal{P}', T')\) (resp. \((\mathcal{P}, T)\)) realizes \(X'\) (resp. \(X\)) and if \(f\) induces \(f_0\).

Remark 1.1.2. In the previous article [Miy16], we usually denote a morphism of d-couples by putting a tilde, like \(\tilde{f}\). Then, we have a push-forward functor \(\mathcal{P}\)-couples by putting a tilde, like \(\tilde{f}\). Then, we have the extraordinary pull-back functor \(\mathcal{P}\)-varieties.

1.1.3. For each d-couple \((\mathcal{P}, T)\), we denote by \(\mathcal{O}_{\mathcal{P}, Q}(\mathcal{T})\) the sheaf of functions on \(\mathcal{P}\) with overconvergent singularities along \(T\) [Ber96 4.2.4], and denote by \(\mathcal{D}^+_\mathcal{P}, \mathcal{Q}(\mathcal{T})\) the sheaf of differential operators on \(\mathcal{P}\) with overconvergent singularities along \(T\) [Ber96 4.2.5].

1.1.4. Extraordinary pull-back functors [Car06 1.1.6]. Let \(f : (\mathcal{P}', T') \to (\mathcal{P}, T)\) be a morphism of d-couples. Then, we have the extraordinary pull-back functor \(f^! : D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}') \to D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T})\).

If \(\overline{f}\) is smooth, or if \(\overline{f}\) induces an open immersion \(\overline{f}^{-1}(\mathcal{P} \setminus T) \hookrightarrow \mathcal{P}' \setminus T'\), then the essential image of \(f^!\) lies in \(D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}')\). In the case where \(\overline{f}\) is an isomorphism, we also denote \(f^!\) by \(f^\ast\). In this case, we have \(f^\ast(\mathcal{M}) = \mathcal{D}^+_\mathcal{P}, \mathcal{Q}(\mathcal{T}') \otimes_{\mathcal{O}_{\mathcal{P}', \mathcal{Q}}(\mathcal{T}'(\mathcal{f}^{-1}(\mathcal{T})))} \overline{f}^{-1}(\mathcal{M})\).

Let \(f' : (\mathcal{P}'', T'') \to (\mathcal{P}', T')\) be another morphism of d-couples and let \(\mathcal{M}\) be an object of \(D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}')\). Then, as long as \(f'\mathcal{M}\) belongs to \(D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}')\), we have a natural isomorphism \(f'^\ast f^!(\mathcal{M}) = (f \circ f')^!(\mathcal{M})\) of functors.

1.1.5. Ordinary push-forward functors [Car06 1.1.6]. Let \(f : (\mathcal{P}', T') \to (\mathcal{P}, T)\) be a morphism of d-couples. Then, we have a push-forward functor \(f_* : D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}') \to D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T})\).

If \(\overline{f}\) is proper and if \(T' = \overline{f}^{-1}(T)\), then the essential image of \(f_*\) lies in \(D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}')\).

Let \(f' : (\mathcal{P}'', T'') \to (\mathcal{P}', T')\) be another morphism of d-couples, and let \(\mathcal{M}''\) be an object of \(D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}'')\). Then, as long as \(f'^\ast \mathcal{M}''\) is an object of \(D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}'')\), we have a natural isomorphism \(f_* f'^\ast \mathcal{M}'' = (f \circ f'^\ast) \mathcal{M}''\).

If \(\mathcal{P}' = \mathcal{P}\) and if \(\overline{f}\) is the identity morphism on \(\mathcal{P}\) (thus \(f\) represents an open immersion), then \(f_*\) is obtained by considering the complex of \(D^+_\mathcal{P}, \mathcal{Q}(\mathcal{T}')\)-modules as a complex of \(D^+_\mathcal{P}, \mathcal{Q}(\mathcal{T}')\)-module via the inclusion \(D^+_\mathcal{P}, \mathcal{Q}(\mathcal{T}') \hookrightarrow D^+_\mathcal{P}, \mathcal{Q}(\mathcal{T}')\).

The base change is also available. Suppose that we are given a cartesian diagram of d-couples

\[
\begin{array}{ccc}
(D, D') & \xrightarrow{g'} & (\mathcal{P}', T') \\
\downarrow f & & \downarrow f \\
(D, D) & \xrightarrow{g} & (\mathcal{P}, T),
\end{array}
\]

In this case, we have

\[
\begin{array}{ccc}
D^b_{\mathcal{P}', \mathcal{Q}}(\mathcal{T}') & \xrightarrow{f^!} & D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}) \\
\downarrow g^! & & \downarrow g^! \\
D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}') & \xrightarrow{f^!} & D^b_{\mathcal{P}, \mathcal{Q}}(\mathcal{T}).
\end{array}
\]
and let \( \mathcal{M} \) be an object of \( D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(D) \right) \). If \( f'^*\mathcal{M} \) belongs to \( D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(D') \right) \) and \( g_+\mathcal{M} \) belongs to \( D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \), then we have a natural isomorphism \( g_+f'^*\mathcal{M} \cong f'g_+\mathcal{M} \) by [Abe14] Remark 5.7.

1.1.6. Interior tensor functor. Let \((\mathcal{P}, T)\) be a d-couple. Then, we have an overconvergent tensor functor [Car15 2.1.3]
\[
\tilde{\otimes}_{\mathcal{P}, Q}^{\dagger}(T) : D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \times D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \to D^b\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right).
\]
We define an interior tensor functor
\[
\tilde{\otimes}_{\mathcal{P}, Q}^{\dagger}(T) : D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \times D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \to D^b\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right)
\]
by \( \mathcal{M} \tilde{\otimes}_{\mathcal{P}, Q}^{\dagger}(T) \mathcal{N} := \mathcal{M} \tilde{\otimes}_{\mathcal{P}, Q}^{\dagger}(T) \mathcal{N} [\dim \mathcal{P}] \). If no confusion would occur, we omit the subscript \((\mathcal{P}, T)\).

Let \( f : (\mathcal{P}', T') \to (\mathcal{P}, T) \) be a morphism of d-couples, and let \( \mathcal{M} \) and \( \mathcal{N} \) be objects of \( D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \). Assume that \( \mathcal{M} \tilde{\otimes}_{\mathcal{P}, Q}^{\dagger}(T) \mathcal{N} \) belongs to \( D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \) and that \( f'^*\mathcal{M} \) and \( f'^*\mathcal{N} \) belong to \( D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \). Then, we have an isomorphism \( f'^*\mathcal{M} \tilde{\otimes}_{\mathcal{P}, Q}^{\dagger}(T) f'^*\mathcal{N} \) by [Car15 2.1.9.1].

The projection formula is also available. Namely, let \( f : (\mathcal{P}', T') \to (\mathcal{P}, T) \) be a morphism of d-couples, let \( \mathcal{M} \) be an object of \( D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T') \right) \), and let \( \mathcal{N} \) be an object of \( D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \). Assume that \( f'^*\mathcal{N} \), \( \mathcal{M} \tilde{\otimes}_{\mathcal{P}, Q}^{\dagger}(T) f'^*\mathcal{N} \) and \( f_+\mathcal{M} \) are all coherent objects. Then, we have an isomorphism \( f_+\mathcal{M} \tilde{\otimes}_{\mathcal{P}, Q}^{\dagger}(T) f_+\mathcal{N} \) by [Car15 2.1.6].

1.1.7. Exterior tensor functors [Car15 2.3.3]. At last, we discuss the exterior tensor functor. Let \((\mathcal{P}_1, T_1)\) and \((\mathcal{P}_2, T_2)\) be two d-couples, and let \((\mathcal{P}, T) := (\mathcal{P}_1, T_1) \times (\mathcal{P}_2, T_2)\) be the product of them, that is, \( \mathcal{P} := \mathcal{P}_1 \times \mathcal{P}_2 \) and \( T := (T_1 \times T_2) \cup (T_1 \times T_2) \). Then, we have an exterior tensor functor
\[
\otimes : D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \times D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right) \to D^b_{\text{coh}}\left( \mathcal{D}_{\mathcal{P}, Q}^{\dagger}(T) \right).
\]
As usual, this functor can be described as follows. In the situation above, let \( \text{pr}_i : (\mathcal{P}, T) \to (\mathcal{P}_i, T_i) \) be projections for \( i = 1, 2 \). Then, we have an isomorphism [Car15 2.3.5.2]
\[
\mathcal{E} \otimes \mathcal{F} \cong \text{pr}_1^* \mathcal{E} \tilde{\otimes}_{(\mathcal{P}, T)} \text{pr}_2^* \mathcal{F}.
\]

The Künneth formula is also available for this exterior tensor functor [Car15 2.3.7.2].

1.2. Fourier transform. In this subsection, we recall the notion of Fourier transform for arithmetic \( \mathcal{D} \)-modules [NHU].

1.2.1. Recall from Conventions and Notations that, in this article, we fix an element \( \pi \) in \( K \) that satisfies \( \pi^{p-1} + (-p)^{(q-1)/(p-1)} = 0 \). Let \( \mathbb{L}_\pi \) denote the Dwork isocrystal associated with \( \pi \).

1.2.2. Let us introduce notations which we need to define the Fourier transform.
\[
p_1, p_2 : (\mathcal{P}, T) := \left( \mathbb{F}_V^1, \{\infty\} \right) \times \left( \mathbb{F}_V^1, \{\infty\} \right) \supset \left( \mathbb{F}_V^1, \{\infty\} \right)
\]
be the first and the second projection, respectively. There exists a smooth formal scheme \( \mathcal{P} \) and a projective morphism \( \overline{f} : \mathcal{P} \to \mathcal{P} = \mathbb{F}_V^1 \times \mathbb{F}_V^1 \) such that \( \overline{f} \) induces an isomorphism
Finally, we put $N_\pi$ an object of $D^b_c$. This object $FT_\pi$ is called the geometric Fourier transform.

Definition 1.2.3. The functor

$$FT_\pi: D^b_c(\mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\}) \rightarrow D^b_c(\mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\})$$

is defined by sending $\mathcal{M}$ in $D^b_c(\mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\})$ to

$$FT_\pi(\mathcal{M}) = p_{2*}(p_1^!\mathcal{M}^{\dagger}(\mathcal{D}_{\mathbb{F}_V}^\dagger)|_{N_\pi}).$$

This object $FT_\pi(\mathcal{M})$ is called the geometric Fourier transform of $\mathcal{M}$.

Remark 1.2.4. It is a central result of [NH04] that $FT_\pi$ sends $D^b_c(\mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\})$.

The argument in loc. cit. also shows that, if $\mathcal{M}$ belongs to $D^b_c(\mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\})$, then $p_{1*}\mathcal{M}^{\dagger}(\mathcal{D}_{\mathbb{F}_V}^\dagger)|_{N_\pi}$ is also an object of $D^b_c(\mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\})$. In fact, we may assume that $\mathcal{M}$ is a (single) coherent $\mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\})$-module placed at degree zero, and since such a coherent module has a free resolution [Huy98 5.3.3, (ii)], we may assume that $\mathcal{M} = \mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\})$. The claim follows from the calculation in [NH04 4.2.2].

1.2.5. The geometric Fourier transform has another important description after passing to the global sections. Let $A_1(K)^\dagger$ be the ring defined by

$$A_1(K)^\dagger := \biggl\{ \sum_{l,k \in \mathbb{N}} a_{l,k} x^l \partial^{[k]} \biggm| a_{l,k} \in K, \exists C > 0, \forall \eta < 1, |a_{l,k}|_p < C\eta^{l+k} \biggr\}.$$

Then, by the $\mathcal{D}_{\mathbb{F}_V}^\dagger$-affinity [Huy98 5.3.3], the functor $\Gamma(\mathbb{F}_V^\dagger, -)$ on the category of coherent $\mathcal{D}_{\mathbb{F}_V}^\dagger,\{1,\infty\})$-modules is exact and gives an equivalence of this category with the category of coherent $A_1(K)^\dagger$-modules (cf. [Huy98 p.915]). Under this identification, the geometric Fourier transform is described as follows.

Proposition 1.2.6 ([NH04 5.3.1]). Let $\varphi_\pi: A_1(K)^\dagger \rightarrow A_1(K)^\dagger$ be the ring automorphism defined by $\varphi_\pi(x) = -\partial/\pi$ and $\varphi_\pi(\partial) = \pi x$. Let $\mathcal{M}$ be a coherent $A_1(K)^\dagger$-module and denote by $\varphi_\pi._*\mathcal{M}$ the coherent $A_1(K)^\dagger$-module obtained by letting $A_1(K)^\dagger$ act on $\mathcal{M}$ via $\varphi_\pi$. Then, we have a natural isomorphism $FT_\pi(\mathcal{M}) \cong \varphi_\pi._*\mathcal{M}[1]$.

1.3. Multiplicative Convolutions. In this subsection, we define the notion of multiplicative convolution and study how it is related with Fourier transform.

1.3.1. We follow the notation in the previous subsection. We put $(\mathcal{D}, T') := (\varprojlim_{\mathbb{F}_V^d}, \{0, \infty\}) \times (\varprojlim_{\mathbb{F}_V^d}, \{0, \infty\}) \times (\varprojlim_{\mathbb{F}_V^d}, \{0, \infty\}) = \varprojlim_{\mathbb{F}_V^d} \times \varprojlim_{\mathbb{F}_V^d}$ (which is compatible with the notation in 1.2.2) and $T' := \{(0, \infty) \times \mathbb{F}_V^d \cup (\mathbb{F}_V^d \times \{0, \infty\}) \}$. Let $pr_1, pr_2: (\mathcal{D}, T') \rightarrow (\varprojlim_{\mathbb{F}_V^d}, \{0, \infty\})$ denote the first
and the second projection, respectively. We denote by $f^*: (\mathcal{F}, T^{-1}(T')) \to (\mathcal{P}, T')$ (resp. $\lambda^*: (\mathcal{F}, T^{-1}(T')) \to (\mathcal{P}, \{0, \infty\})$) the morphism of d-couples defined by $f$ (resp. $T$).

**Definition 1.3.2.** We define a multiplicative convolution functor

$$* : \mathcal{D}_{coh}^b(\mathcal{P}_V^1, \{0, \infty\}) \times \mathcal{D}_{coh}^b(\mathcal{P}_V^1, \{0, \infty\}) \to \mathcal{D}_{coh}^b(\mathcal{P}_V^1, \{0, \infty\})$$

by $\mathcal{E} * \mathcal{F} := \lambda'_* f''(\mathcal{E}, \mathcal{F}) = \lambda'_* f''(\text{pr}_1^* \mathcal{E} \otimes \mathcal{F})$.

In the following, we let $\text{inv}: (\mathcal{P}_V^1, \{0, \infty\}) \to (\mathcal{P}_V^1, \{0, \infty\})$ denote the morphism of d-couples defined by $\text{inv}: \mathcal{P}_V^1 \to \mathcal{P}_V^1; x \mapsto x^{-1}$.

**Lemma 1.3.3.** Let $\mathcal{E}$ be an object of $\mathcal{D}_{coh}^b(\mathcal{P}_V^1, \{0, \infty\})$ and let $\mathcal{F}$ be an overconvergent isocrystal on $\mathbb{G}_{m,k}$ considered as an object of $\mathcal{D}_{coh}^b(\mathcal{P}_V^1, \{0, \infty\})$. Then, we have a natural isomorphism

$$\mathcal{E} * \mathcal{F} \cong \text{pr}_{2*} \left( \text{pr}_1^* \text{inv}^* \mathcal{E} \otimes \mathcal{F} \right)$$

in $\mathcal{D}_{coh}^b(\mathcal{P}_V^1, \{0, \infty\})$.

**Proof.** Let $\sigma: (\mathcal{F}, T^{-1}(T')) \to (\mathcal{P}, T')$ denote the morphism defined by $\mathcal{F} = (\text{inv} \circ \text{pr} \circ \mathcal{F})$. Note that $\sigma$ represents the isomorphism $\mathbb{G}_{m,k} \times \mathbb{G}_{m,k} \to \mathbb{G}_{m,k} \times \mathbb{G}_{m,k}; (x, y) \mapsto (x^{-1}, y)$. Since $\lambda' = \text{pr}_2 \circ \sigma$ and since $\sigma$ preserves coherence, we have an identification $\lambda'_* = \text{pr}_{2*} \circ \sigma_*$. By using this fact, we have

$$\mathcal{E} * \mathcal{F} = \lambda'_* f''(\text{pr}_1^* \mathcal{E} \otimes \mathcal{F}) \equiv \text{pr}_{2*} \sigma_* \left( f'' \text{pr}_1^* \mathcal{E} \otimes (\mathcal{F}, T^{-1}(T')) f'' \text{pr}_1^* \mathcal{F} \right).$$

Moreover, since $\text{pr}_1 \circ f'' = \text{inv} \circ \text{pr}_1 \circ \sigma$, and since each of $f''$, $\text{inv}$ and $\text{pr}_1^*$ preserves coherence, we have an identification $f'' \text{pr}_1^* = \sigma^* \text{pr}_1^* \text{inv}^*$ and therefore

$$\text{pr}_{2*} \sigma_* \left( f'' \text{pr}_1^* \mathcal{E} \otimes f'' \text{pr}_1^* \mathcal{F} \right) \equiv \text{pr}_{2*} \left( \sigma^* \text{pr}_1^* \mathcal{E} \otimes f'' \text{pr}_1^* \mathcal{F} \right) \equiv \text{pr}_{2*} \left( \text{pr}_1^* \mathcal{E} \otimes \sigma_* f'' \text{pr}_1^* \mathcal{F} \right).$$

Since $\sigma$ represents an involution on $\mathbb{G}_{m,k} \times \mathbb{G}_{m,k}$ and since $\mathcal{F}$ is an overconvergent isocrystal, we have $\sigma_* f'' \text{pr}_1^* \mathcal{F} = f'' \sigma^* \text{pr}_1^* \mathcal{F} = f'' \sigma^* \mathcal{F}$, which completes the proof. \qed

**Proposition 1.3.4.** We denote by $j: (\mathcal{P}_V^1, \{0, \infty\}) \to (\mathcal{P}_V^1, \{0, \infty\})$ the morphism of d-couples such that $T = \text{id}_{\mathcal{P}_V^1}$. (Thus, $j$ realizes the inclusion $\mathbb{G}_{m,k} \hookrightarrow k^\times$.) Let $\mathcal{M}$ be an object of $\mathcal{D}_{coh}^b(\mathcal{P}_V^1, \{0, \infty\})$, and assume that $j_* \text{inv}^* \mathcal{M}$ belongs to $\mathcal{D}_{coh}^b(\mathcal{P}_V^1, \{0, \infty\})$. Then, we have a natural isomorphism

$$j^* \left( FT_a(j_* \text{inv}^* \mathcal{M}) \right) \cong \mathcal{M} * (j^* \mathcal{L}_a)[−1]$$

**Proof.** Put $(\mathcal{P}, T_a) := (\mathcal{P}_V^1, \{0, \infty\}) \times (\mathcal{P}_V^1, \{0, \infty\})$. Let $\text{pr}_{1A}: (\mathcal{P}, T_a) \to (\mathcal{P}_V^1, \{0, \infty\})$ (resp. $\text{pr}_{2A}: (\mathcal{P}, T_a) \to (\mathcal{P}_V^1, \{0, \infty\})$) denote the first projection (resp. the second projection), the identity morphism on
\(\mathcal{D} = \widehat{\mathcal{V}}_V \times \widehat{\mathcal{V}}_V\). This morphism represents the first projection \(A_k^1 \times G_{m,k} \to A_k^1\) (resp. the second projection \(A_k^1 \times G_{m,k} \to G_{m,k}\), and the inclusion \(G_{m,k} \times G_{m,k} \hookrightarrow A_k^1 \times G_{m,k}\).

Then, the definition of Fourier transform, we obtain a natural identification

\[j^* (\text{FT}_A(j^* \text{inv}^* \mathcal{M})) = pr_{2,A^+} \left[pr_{1,A^+} (j^* \text{inv}^* \mathcal{M} \widehat{\otimes} j''^* f_\lambda \mathcal{L}_\pi) \right] \{[-1]\],

where \(j'' : (\mathcal{P}, T_A) \to (\mathcal{D}, T)\) is the morphism of d-couples defined by \(j'' = \text{id}_{\mathcal{D}}\), thus represents the inclusion \(A_k^1 \times G_{m,k} \hookrightarrow A_k^1 \times A_k^1\). Here, in the right-hand side, by the coherence assumption and Remark 1.2.4,

\[pr_{1,A^+} j^* \text{inv}^* \mathcal{M} \widehat{\otimes} j''^* f_\lambda \mathcal{L}_\pi \equiv j''^* \left(j^* \text{inv}^* \mathcal{M} \widehat{\otimes} f_\lambda \mathcal{L}_\pi\right)
\]

belongs to \(D^b_{\text{coh}} (\mathcal{D}_{\mathcal{V},\mathcal{Q}} (\mathcal{V} T_A))\).

Now, again by the coherence assumption, we have a base change isomorphism

\[pr_{1,A^+} j^* \text{inv}^* \mathcal{M} \equiv j_{A^+} \text{inv}^* \mathcal{M}.
\]

Moreover, since \(j_{A^+} j''^* f_\lambda \mathcal{L}_\pi \equiv f_\lambda' \mathcal{L}_{\psi}\), we see that

\[pr_{1,A^+} j^* \text{inv}^* \mathcal{M} \widehat{\otimes} j''^* f_\lambda \mathcal{L}_\pi \equiv j_{A^+} (pr_{1} j^* \text{inv}^* \mathcal{M} \widehat{\otimes} j''^* f_\lambda \mathcal{L}_\pi)
\]

\[\equiv j_{A^+} (pr_{1} j^* \text{inv}^* \mathcal{M} \widehat{\otimes} f_\lambda' \mathcal{L}_{\psi})
\]

Since this object belongs to \(D^b_{\text{coh}} (\mathcal{D}_{\mathcal{V},\mathcal{Q}} (\mathcal{V} T_A))\) and \(pr_{2} = pr_{2,A} \circ j_A\), we see that

\[pr_{2,A^+} j_{A^+} (pr_{1} j^* \text{inv}^* \mathcal{M} \widehat{\otimes} f_\lambda' \mathcal{L}_{\psi}) \equiv pr_{2,A} \left(pr_{1} j^* \text{inv}^* \mathcal{M} \widehat{\otimes} f_\lambda' \mathcal{L}_{\psi}\right).
\]

By Lemma 1.3.3 this is isomorphic to the right-hand side of (1) as desired. \(\square\)

2. Hypergeometric arithmetic \(\mathcal{D}\)-modules.

2.1. Definitions and fundamental properties.

2.1.1. Firstly, let us define a hypergeometric arithmetic \(\mathcal{D}\)-module on \(G_{m,k}\) as a coherent \(\mathcal{D}^T_{\mathcal{V},\mathcal{Q}} (\{0, \infty\})\)-module. Note that the category of coherent \(\mathcal{D}^T_{\mathcal{V},\mathcal{Q}} (\{0, \infty\})\)-modules is identified with the category of coherent \(B_1(K)^T\)-modules [Huy98 5.3.3 and p.915], where

\[B_1(K)^T := \left\{ \sum_{l \in \mathbb{Z}, k \in \mathbb{N}} a_{l,k} \lambda^l \lambda^k \mid a_{l,k} \in K, \exists C > 0, \exists \eta < 1, |a_{l,k}| < C \eta^{\max(l, -l+k)} \right\}.
\]

Definition 2.1.2. Let \(a_1, \ldots, a_m, b_1, \ldots, b_n\) be elements of \(K\). We write the sequence \(a_1, \ldots, a_m\) by \(\alpha\) and \(b_1, \ldots, b_n\) by \(\beta\).

(i) We define the hypergeometric operator \(\text{Hyp}_\pi (\alpha; \beta) = \text{Hyp}_\pi (a_1, \ldots, a_m; b_1, \ldots, b_n)\) to be

\[\text{Hyp}_\pi (\alpha; \beta) := \prod_{i=1}^{m} (x \partial - a_i) - (-1)^{m+n} n \eta^{m-n} \prod_{j=1}^{n} (x \partial - b_j).
\]

(ii) We define a \(B_1(K)^T\)-module \(\mathcal{H}_\pi (\alpha; \beta) = \mathcal{H}_\pi (a_1, \ldots, a_m; b_1, \ldots, b_n)\) by

\[\mathcal{H}_\pi (\alpha; \beta) := B_1(K)^T \otimes B_1(K)^T \text{Hyp}_\pi (\alpha; \beta).
\]

This is also considered as an object of \(D^b_{\text{coh}} (\mathcal{D}^T_{\mathcal{V},\mathcal{Q}} (\{0, \infty\}))\) by putting it on degree zero.
**Remark 2.1.3.** By definition, $\mathcal{H}_π(\emptyset; \emptyset)$ is the delta module at 1.

If $(m, n) = (1, 0)$, we may immediately check the isomorphism $\mathcal{H}_π(\emptyset; \emptyset) \cong j^* L_{p} \circ \delta^* \mathcal{K}_π$, where $\mathcal{K}_π$ is the Kummer isocrystal associated with $\pi$. Similarly, if $(m, n) = (1, 0)$, we get $\mathcal{H}_π(\emptyset; \beta) \cong \text{inv}^* (j^* L_{(-1)^{m} p} \circ \delta^* \mathcal{H}_β)$. (Recall that $\text{inv} : (\mathcal{H}_π^1, (0, \infty)) \to (\mathcal{H}_π^0, (0, \infty))$ denotes the morphism of d-couples defined by $\text{inv} : x \mapsto x^{-1}$.)

2.1.4. The goal of this article is to prove, under a $p$-adic non-Liouville condition, that $\mathcal{H}_π(\alpha; \beta)$ can be obtained inductively in terms of multiplicative convolution.

2.1.5. The following lemma is obtained by a straight-forward calculation as in [Miy16, Lemma 3.1.3]. (In loc. cit., (ii) is stated in the case where $\gamma \in \frac{1}{p-1}\mathbb{Z}$, but this condition is not necessary.)

**Lemma 2.1.6 ([Miy16 Lemma 3.1.3]).** Under the notation in Definition 2.2.2, $\mathcal{H}_π(\alpha; \beta)$ has the following properties.

(i) $\text{inv}^* \mathcal{H}_π(\alpha; \beta)$ is isomorphic to $\mathcal{H}_π(L_{-1}p, -\alpha, -\beta)$, where $-\alpha$ (resp. $-\beta$) denotes the sequence $-\alpha_1, \ldots, -\alpha_m$ (resp. $-\beta_1, \ldots, -\beta_n$).

(ii) Let $\gamma$ be an element of $\mathbb{Z}_p$. Then, $\mathcal{H}_π(\alpha; \beta) \otimes_{\mathbb{Z}_p} \mathcal{H}_π(\gamma, \beta + \gamma)$, where $\alpha + \gamma$ (resp. $\beta + \gamma$) denotes the sequence $\alpha_1 + \gamma, \ldots, \alpha_m + \gamma$ (resp. $\beta_1 + \gamma, \ldots, \beta_n + \gamma$).

2.2. $p$-adic Liouville numbers. In this subsection, we recall the notion of $p$-adic Liouville numbers and give a lemma which we need later.

**Definition 2.2.1.** Let $\alpha$ be an element of $\mathbb{Z}_p$. We say that $\alpha$ is a $p$-adic Liouville number if one of the two power series,

$$
\sum_{k \geq 0, k \neq 0} \frac{t^k}{\alpha - k} \quad \text{or} \quad \sum_{k \geq 0, k \neq 0} \frac{t^k}{\alpha + k}
$$

has radius of convergence strictly less than 1.

**Proposition 2.2.2 ([Ked10] 13.1.7).** Let $\alpha$ be an element of $\mathbb{Z}_p \setminus \mathbb{Z}$ which is not a $p$-adic Liouville number. Then, the power series

$$
\sum_{k=0}^{\infty} \alpha(1-\alpha)(2-\alpha)\ldots(k-\alpha)
$$

has radius of convergence greater than or equal to $p^{-1/(p-1)}$.

**Lemma 2.2.3.** Let $l$ be a non-negative integer and let $\alpha$ be an element of $\mathbb{Z}_p$.

(i) For any non-negative integer $N \geq l$, the following inequality holds:

$$
\left| \prod_{s=l}^{N} (s - \alpha) \right| \leq p^{-(N-l+1)/(p-1)+1}(N - l + 1).
$$

(ii) Assume that $\alpha$ is neither an integer nor a $p$-adic Liouville number. Then, for all positive real number $r$ with $r < p^{-\frac{1}{p-1}}$, we have

$$
\lim_{k \to \infty} \left| \prod_{s=l}^{N} (s - \alpha) \right| r^{-k} = \infty.
$$
Proposition 3.1.4. This lemma plays a central role in proving the main theorem in this article.

Proof. (i) The proof is the same as that of the first inequality of [Miy16, 3.1.5]. We include a proof here for the convenience for the reader.

Since the inequality is trivial if \( \alpha \in \{1, \ldots, N\} \), we assume that this is not the case. For each positive integer \( m \), let \( t_m \) denote the number of \((s - \alpha)\)'s for \( s = l, \ldots, N \) that belongs to \( p^m \mathbb{Z}_p \):

\[
t_m := \# \{ s \in \{l, \ldots, N\} \mid s - \alpha \in p^m \mathbb{Z}_p \}.
\]

Then, we have \( v_p \left( \prod_{s=1}^{N} (s - \alpha) \right) = \sum_{m=1}^{\infty} t_m \) (note that the right-hand side is essentially a finite sum). Now, since there is exactly one multiple of \( p^m \) in every \( p^m \) successive \((s - \alpha)\)'s, we have \( t_m \geq \left\lfloor \frac{N - l + 1}{p^m} \right\rfloor \). This shows that

\[
v_p \left( \prod_{s=1}^{N} (s - \alpha) \right) = \sum_{m=1}^{\infty} t_m \geq \sum_{m=1}^{\infty} \left\lfloor \frac{N - l + 1}{p^m} \right\rfloor.
\]

The right-hand side equals \( v_p ((N - l + 1)!) \) and it is well-known that, for any positive integer \( M \) we have \( v_p(M!) \geq \frac{M}{\log_{p} M} - M.1. Therefore, we have \( v_p \left( \prod_{s=1}^{N} (s - \alpha) \right) \geq \frac{N - l + 1}{p^{-l}} - \log_{p} (N - l + 1) \), from which the assertion follows.

(ii) Since \( l - \alpha \) is neither an integer nor a \( p \)-adic Liouville number, Proposition 2.2.2 shows that the power series

\[
\sum_{k=0}^{\infty} \frac{x^{k}}{(l - \alpha)(l + 1 - \alpha) \ldots (l + k - \alpha)}
\]

has radius of convergence greater than or equal to \( p^{-1} \). This means that for all \( r \in (0, p^{-1}) \), we have

\[
\lim_{k \to \infty} \left| \prod_{s=1}^{k} (s - \alpha) \right|^{-1} r^{k} = 0,
\]

which shows the claim. \( \square \)

2.3. A lemma on hypergeometric arithmetic \( \mathcal{D} \)-modules under a \( p \)-adic non-Liouville condition. In this subsection, we establish the following lemma that generalizes [Miy16, Proposition 3.1.4]. This lemma plays a central role in proving the main theorem in this article.

Lemma 2.3.1. Let \( \alpha_1, \ldots, \alpha_m \) and \( \beta_1, \ldots, \beta_n \) be elements of \( \mathbb{Z}_p \), and assume that \( \alpha_i \)'s does not have an integer nor have a \( p \)-adic Liouville numbers. Let \( j : (\overline{\mathbb{D} \mathbb{V}_p}, \{0, \infty\}) \to (\overline{\mathbb{D} \mathbb{V}_p}, \{0, \infty\}) \) be the morphism of d-couples defined by \( \overline{j} = \text{id}_{\overline{\mathbb{D} \mathbb{V}_p}} \). Then, the following assertions hold.

(i) \( j^{*} (A_1(K)^{\dagger} / A_1(K)^{\dagger} \text{Hyp}_{\pi}(\alpha; \beta)) \) is isomorphic to \( H_{\pi}(\alpha; \beta) \).

(ii) The natural morphism

\[
A_1(K)^{\dagger} / A_1(K)^{\dagger} \text{Hyp}_{\pi}(\alpha; \beta) \longrightarrow j_{*} j^{*} (A_1(K)^{\dagger} / A_1(K)^{\dagger} \text{Hyp}_{\pi}(\alpha; \beta))
\]

is an isomorphism.

Proof. (i) follows from the exactness of \( j^{*} \) on the category of coherent \( A_1(K)^{\dagger} \)-modules. The proof of (ii) is, as in the proof of [Miy16, Proposition 3.1.4], reduced to the following Lemma. \( \square \)
Lemma 2.3.2. Let $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ be elements of $\mathbb{Z}_p$. Assume that $\alpha_i$’s does not have an integer nor have a $p$-adic Liouville number. Then, on $A_1(K)^\dagger / A_1(K)^\dagger \text{Hyp}_n(\alpha; \beta)$, the multiplication by $x$ from the left is bijective.

Proof. Firstly, we prove the injectivity.

To prove this, it suffices to show that if $P, Q \in A_1(K)^\dagger$ satisfy $xP = Q\text{Hyp}_n(\alpha; \beta)$ then $Q \in xA_1(K)^\dagger$. In fact, then since $x$ is not a zero-divisor in $A_1(K)^\dagger$, we get that $P \in A_1(K)^\dagger \text{Hyp}_n(\alpha; \beta)$ and the injectivity follows.

In order to show that $Q \in xA_1(K)^\dagger$, we may assume that $Q$ is of the form $Q = \sum_{i=0}^\infty c_i \partial[i]$, where $c_i$’s are elements of $K$ satisfying $\exists C > 0, \exists \eta < 1, \forall l, |c_l| < C\eta^l$. Then, by using the congruence $\partial[l]\partial[x] \equiv \partial[l-1] (\text{mod } xA_1(K)^\dagger)$, we have

$$Q \text{Hyp}_n(\alpha; \beta) \equiv \sum_{i=0}^\infty c_i \prod_{k=1}^m (l - \alpha_i) \partial[l] - (-1)^{m+n} \pi^{m-n} \sum_{i=1}^\infty c_i \prod_{j=1}^n (l - 1 - \beta_j) \partial[l-1]$$

equivalently modulo $xA_1(K)^\dagger$. By assumption, the left-hand side belongs to $xA_1(K)^\dagger$, which shows the recurrence relation

$$c_i \prod_{k=1}^m (l - \alpha_i) = (-1)^{m+n} \pi^{m-n} c_{i+1} \prod_{j=1}^n (l - \beta_j).$$

Now, fix a non-negative integer $l$ that exceeds all $\beta_j$’s which are integers. Then, by the recurrence relation, we have

$$c_{i+k} = (-1)^{k(m+n)} \pi^{-k(m-n)} \prod_{k=1}^m (l + k - 1 - \alpha_i)(l + k - 2 - \alpha_i) \cdots (l - \alpha_i) \prod_{j=1}^n (l + k - 1 - \beta_j)(l + k - 2 - \beta_j) \cdots (l - \beta_j) c_l.$$

Let us choose $C > 0$ and $\eta < 1$ such that $\forall l, |c_l| < C\eta^l$. The series $\{|\eta^{-k} |c_{i+k}|\}_{k=0}^\infty$ is then bounded.

Now, put $r := \eta^{1/2m}p^{-1/(p-1)}$; if $m = 0$, we interpret $\eta^{1/2m} = 1$. Lemma 2.2.3 (i) shows that $\eta^{-k} |c_{i+k}| \leq p^{k/(p-1)} - k^{-1} - 1$ for each $j = 1, \ldots, n$. Moreover, Lemma 2.2.3 (ii) shows that $|l + k - 1 - \alpha_i| \cdots |l - \alpha_i| p^{-k} \to \infty$ as $k \to \infty$ for each $i = 1, \ldots, m$. We therefore have, since $|\pi| = p^{-1/(p-1)}$,

$$\eta^{-k} |c_{i+k}| = \eta^{-k} p^{k(m-n)/p} p^{-k} \prod_{k=1}^m \{|l + k - 1 - \alpha_i| \cdots |l - \alpha_i| p^{-k}\} |c_l| \geq p^{-n} \left(\eta^{-k/2} \pi^{-k}\right) \prod_{i=1}^m \{|l + k - 1 - \alpha_i| \cdots |l - \alpha_i| p^{-k}\} |c_l|.$$
number in \( \{ \beta_j + 1 \mid j \in \{1, \ldots, n\} \} \cap \mathbb{Z}_{>0} \) if this set is not empty; we set \( l_0 = 0 \) if it is empty.

To prove the existence of \( R \in A_1(K)^v \) as above, we may assume that \( c_l = 0 \) if \( l < l_0 \) by the following reason. If \( A_1(K) \) denotes the usual Weyl algebra with coefficients in \( K \), then since \( \alpha_i \)'s are not integers, the right multiplication by \( \text{Hyp}_n(\alpha; \beta) \) is bijective on \( A_1(K)/xA_1(K) \) \([\text{Kat90} \, 2.9.4, (3) \Rightarrow (2)]\). This shows that there exists \( R' \in A_1(K) \) such that \( \sum_{l=0}^{l_0-1} c_l \partial_1[l] + R' \text{Hyp}_n(\alpha; \beta) \in xA_1(K) \) (The proof in the reference \([\text{Kat90}]\) is given over \( \mathbb{C} \), but it remains valid for all field of characteristic 0). Now, we assume that \( c_l = 0 \) if \( l < l_0 \).

We put \( d_l = 0 \) if \( l < l_0 \), and for each \( s \geq 0 \) we put

\[
(3) \quad d_{l_0+s} = \sum_{s \geq 0} (-1)^{(t-s)(m+np+1)} \frac{\prod_{i=1}^{m} (l_0 + t - \beta_j) \cdots (l_0 + s - \beta_j)}{\prod_{i=1}^{m} (l_0 + t - \alpha_i) \cdots (l_0 + s - \alpha_i)} c_{l_0+s};
\]

let us firstly check that this infinite series actually converges. Lemma \([\text{Kat90}] \, 2.2.3 \) (i) shows that the norm of each summand in the right-hand side of (3) is bounded by \( \eta^{l_0} \). Nextly, we put \( R := \sum_{l=0}^{l_0} d_l \partial_1[l] \) and prove that \( R \in A_1(K)^v \). By the bound of each of the summands of (3) given above, we have

\[
(4) \quad |d_{l_0+s}| < Cp_{\max_{t \geq s}} \frac{(t-s)^{n+2}}{\prod_{i=1}^{m} (l_0 + t - \alpha_i) \cdots (l_0 + s - \alpha_i)} \eta^{l_0+s/2}.
\]

If \( m = 0 \), then it is easy to check that there exists a constant \( C' > 0 \) such that \( |d_{l_0+s}| \leq C' \eta^{s/2} \). We thus assume that \( m > 0 \).

For each \( i = 1, \ldots, m \), Lemma \([\text{Kat90}] \, 2.2.3 \) (i) shows the inequality

\[
\frac{1}{(l_0 + t - \alpha_i) \cdots (l_0 + s - \alpha_i)} \leq \frac{1}{(l_0 + t - \alpha_i) \cdots (l_0 + s - \alpha_i)} \leq \frac{1}{(l_0 + t - \alpha_i) \cdots (l_0 - \alpha_i)} \leq \frac{1}{(l_0 + t - \alpha_i) \cdots (l_0 - \alpha_i)} \leq \frac{1}{(l_0 + t - \alpha_i) \cdots (l_0 - \alpha_i)} \eta^{l_0+s/2m}.
\]
By Lemma 2.2.3 (ii), the fraction \( \frac{r^t}{(l_0 + t - \alpha_i) \ldots (l_0 - \alpha_i)} \) is bounded by a constant independent of \( t \). Therefore, by looking at (4), there exists a constant \( C_1 > 0 \) such that

\[
|d_{l_0+s}| < C_1 \max_{t \geq s} \left\{ (t-s)^n \eta^{(t-s)/2} s^n \eta^{-s/2} \right\}^{2^n} = C_1 \max_{t \geq s} \left\{ (t-s)^n \eta^{(t-s)/2} \right\} s^n \eta^{-s/2} = C_1 \max_{t \geq 0} \left\{ r^n \eta^{t/2} \right\} \left( s^n \eta^{t/4} \right)^{2^n}.
\]

Now, \( C_1 \max_{t \geq 0} \left\{ r^n \eta^{t/2} \right\} \left( s^n \eta^{t/4} \right)^{2^n} \) is bounded by a constant \( C_2 \) independent of \( s \) and we have \( |d_{l_0+s}| \leq C_2 \eta^{t/4} \) for all \( s \geq 0 \). This proves that \( R = \sum_{i=0}^{\infty} d_i \phi^{(i)} \) belongs to \( A_1(K)^{\dagger} \).

It remains to prove that \( R \) satisfies \( P + R \text{Hyp}_n(\alpha; \beta) \in x A_1(K)^{\dagger} \), and this is just a formal calculation. In fact, it is equivalent to showing that

\[
d_l = \prod_{i=1}^m (l - \alpha_i) - (-1)^{m+n} \pi^{m-n} dl_{l+1} \prod_{j=1}^n (l - \beta_j) + c_l = 0
\]

for all \( l \geq 0 \). It trivially holds if \( l < l_0 - 1 \) because \( d_l = d_{l+1} = c_l = 0 \) in this case; it also holds if \( l = l_0 - 1 \) because \( d_l = c_l = 0 \) and \( l - \beta_{j_0} = 0 \) for some \( j_0 \); otherwise, we may check it directly by using (3). \( \square \)

3. Hypergeometric Arithmetic \( \mathcal{D} \)-modules and Multiplicative Convolution.

3.1. Main Theorem. Now, we are ready to state and prove the main theorem of this article.

**Theorem 3.1.1.** Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) be sequences of elements of \( \mathbb{Z}_p \). Assume that, for any \( i \) and \( j \), \( \alpha_i - \beta_j \) is not an integer nor a \( p \)-adic Liouville number.

(i) Assume that \( m \geq 1 \) and put \( \alpha' = (\alpha_2, \ldots, \alpha_m) \). Then, we have an isomorphism

\[
\mathcal{H}_n(\alpha'; \beta) * \mathcal{H}_n(\alpha_1; \emptyset)[-1] \cong \mathcal{H}_n(\alpha; \beta).
\]

(ii) Assume that \( n \geq 1 \) and put \( \beta' = (\beta_1, \ldots, \beta_n) \). Then, we have an isomorphism

\[
\mathcal{H}_n(\alpha; \beta') * \mathcal{H}_n(\emptyset; \beta_1)[-1] \cong \mathcal{H}_n(\alpha; \beta).
\]

**Proof.** We prove (i) and (ii) by induction on \( m + n \). If \( (m, n) = (1, 0) \) (resp. \( (m, n) = (0, 1) \)), then (i) (resp. (ii)) follows from the fact that \( \mathcal{H}_n(\emptyset; \emptyset) \) is a unit object for the multiplicative convolution. The latter fact can be checked as in the proof of [Miy16, 2.1.2].

Now, assume that \( m + n \geq 2 \) and let us prove the assertions (i) and (ii). In fact, Lemma 2.1.6 (i) and the isomorphism \( \text{inv}^*(\mathcal{M} * \mathcal{N}) \cong (\text{inv}^* \mathcal{M}) * (\text{inv}^* \mathcal{N}) \), whose proof is straightforward and left to the reader, show that (ii) is deduced from (i).
The proof of (i) is reduced to the case where \( \alpha_1 = 0 \) as follows. Because of the isomorphism \( \lambda'^t \mathcal{K} \cong f'^t (\mathcal{P}_1 \mathcal{K} \otimes \mathcal{P}_2 \mathcal{K}) \) [1], we have
\[
(\mathcal{M} \ast \mathcal{H}_\pi(0, \emptyset)) \otimes \mathcal{K} \cong \lambda'^t \left( f'^t (\mathcal{P}_1 \mathcal{M} \otimes \mathcal{P}_2 \mathcal{H}_\pi(0, \emptyset)) \otimes \lambda'^t \mathcal{K} \right)
\]
\[
\cong \lambda'^t \left( f'^t \left( \mathcal{P}_1 \mathcal{M} \otimes \mathcal{P}_2 \mathcal{H}_\pi(0, \emptyset) \otimes \mathcal{P}_1 \mathcal{K} \otimes \mathcal{P}_2 \mathcal{K} \right) \right) \] [1]
\[
= \lambda'^t \left( \left( \mathcal{M} \otimes \mathcal{K} \right) \mathcal{H}_\pi(\alpha_1; 0) \right)
\]
\[
= (\mathcal{M} \otimes \mathcal{K}) \ast \mathcal{H}_\pi(\alpha_1; 0).
\]

Therefore, if the assertion (i) is proved for \( \alpha_1 = 0 \), then we get the desired theorem for general \( \alpha_1 \) by tensoring \( \mathcal{K} \), with the aid of Lemma [2.1.6(ii)].

In the case where \( \alpha_1 = 0 \), we may prove the assertion in the same way as [Miy16, Theorem 3.2.5]. We include here a sketch of the proof.

By the induction hypothesis, \( \mathcal{H}_\pi(\alpha'; \beta) \cong \mathcal{H}_\pi(\alpha' + 1; \beta + 1) \) because for the Kummer isocrystals \( \mathcal{K} \), we have an isomorphism \( \mathcal{K} \cong \mathcal{K} + 1 \). Therefore, since \( (-\beta_j - 1)' \)’s do not have a \( p \)-adic Liouville number, we see by Lemma [2.1.6(i)] and Lemma [2.3.1] that
\[
j_+ \text{inv}^* \mathcal{H}_\pi(\alpha'; \beta) \cong j_+ \text{inv}^* \mathcal{H}_\pi(\alpha' + 1; \beta + 1)
\]
\[
= j_+ \mathcal{H}_\pi(-1)^\dagger (\mathcal{N}_\pi(-\beta - 1; -\alpha' - 1))
\]
\[
= A_1(K)^\dagger / A_1(K)^\dagger \text{Hyp}_{-1}(\mathcal{N}_\pi(-\beta - 1; -\alpha' - 1)).
\]

Because this is a coherent \( A_1(K)^\dagger \)-module, Proposition [1.3.4] shows that
\[
\mathcal{H}_\pi(\alpha'; \beta) \ast (j_+ \mathcal{N}_\pi[-1]) \cong j^* \left( \text{FT}_\pi \left( j_+ \text{inv}^* \mathcal{H}_\pi(\alpha'; \beta) \right) \right).
\]

Finally, by a direct calculation using Proposition [1.2.6] we may prove the isomorphism
\[
\text{FT}_\pi \left( A_1(K)^\dagger / A_1(K)^\dagger \text{Hyp}_{-1}(\mathcal{N}_\pi(-\beta - 1; -\alpha' - 1)) \right) \cong A_1(K)^\dagger / A_1(K)^\dagger \text{Hyp}_\pi(\alpha; \beta)
\]
(ce the proof of [Miy16, 3.2.5]). Now the assertion follows by Lemma [2.3.1(i)].

3.2. Quasi-\( \Sigma \)-unipotence. In this last subsection, we discuss the quasi-\( \Sigma \)-unipotence of arithmetic hypergeometric \( \mathcal{D} \)-modules.

3.2.1. Let \( \Sigma \) be the subgroup of \( \mathbb{Z}_p / \mathbb{Z} \) that does not contain a \( p \)-adic Liouville number. Caro [Car18, 3.3.5] defines, for each smooth formal scheme \( \mathcal{P} \) over \( V \), the subcategory \( D^b_{\text{q-cit}}(\mathcal{P}_{\mathcal{V}, \mathcal{Q}}) \) of \( D^b_{\text{coh}}(\mathcal{P}_{\mathcal{V}, \mathcal{Q}}) \) consisting of “quasi-\( \Sigma \)-unipotent” objects. These categories are stable under Grothendieck’s six operations.

**Proposition 3.2.2.** Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \) and \( \beta = (\beta_1, \ldots, \beta_n) \) be sequences of elements of \( \mathbb{Z}_p \), and assume that \( (m, n) \neq (0, 0) \). Let \( \Sigma \) be the subgroup of \( \mathbb{Z}_p / \mathbb{Z} \) generated by the canonical images of \( \alpha_i \)’s and \( \beta_j \)’s. Assume that \( \alpha_i - \beta_j \notin \mathbb{Z} \) for any \( i, j \), and that \( \Sigma \) does not contain the canonical image of a \( p \)-adic Liouville number.

Then, \( \mathcal{H}_\pi(\alpha; \beta) \) is an object of \( D^b_{\text{q-cit}}(\mathcal{P}_{\mathcal{V}, \mathcal{Q}}) \). In particular, it is an overholonomic \( \mathcal{D}_\mathcal{V} \)-module.
Proof. If the canonical image of a $p$-adic number $\gamma \in \mathbb{Z}_p$ in $\mathbb{Z}_p/\mathbb{Z}$ belongs to $\Sigma$, then $\mathcal{X}_\gamma$ is an object of $\mathcal{D}^b_{q \Sigma}\left(\mathcal{O}_{P,1}^{-\hat{1}}\right)$ because it is (the realization on $(\mathcal{G}_{m,k}, \mathbb{P}^1_V)$ of) an overconvergent isocrystal on $\mathbb{G}_{m,k}$ whose exponent is $\gamma \in \Sigma$ (resp. $-\gamma \in \Sigma$) at $0$ (resp. at $\infty$), and because $\mathcal{D}^b_{q \Sigma}\left(\mathcal{O}_{P,1}^{-\hat{1}}\right)$ contains all such objects by construction.

We may also show that $\mathcal{L}_\gamma$ is also an object of $\mathcal{D}^b_{q \Sigma}\left(\mathcal{O}_{P,1}^{-\hat{1}}\right)$. In fact, it is a direct factor of the push-forward of the trivial isocrystal on $\mathbb{A}^1_k$ along the Artin–Schreier morphism. Now, the trivial isocrystal on $\mathbb{A}^1_k$ is an object of $\mathcal{D}^b_{q \Sigma}\left(\mathcal{O}_{P,1}^{-\hat{1}}\right)$ (the exponent at $\infty$ is $0 \in \Sigma$).

Since $\mathcal{D}^b_{q \Sigma}$ is stable under push-forward and direct factor, the claim follows.

Now, by Remark 2.1.3, the corollary holds for $(m,n) = (1,0), (0,1)$. For general $(m,n)$, Theorem 3.1.1 and the stability of $\mathcal{D}^b_{q \Sigma}$ under Grothendieck’s six functors show the assertion. $\square$

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