Prelogarithmic operators and Jordan blocks in $SL(2)_k$ affine algebra

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Abstract
The free field description of logarithmic and prelogarithmic operators in non compact Wess-Zumino-Witten model is analysed. We study the structure of the Jordan blocks of the $\hat{SL}(2)_k$ affine algebra and the role of the puncture operator in the theory in relation with the unitarity bound.

1 Introduction
Wess-Zumino-Witten model formulated on $SL(2)$ manifold is a subject of great importance for several reasons. Since this model describes the string dynamics on three dimensional Anti-de Sitter spacetime, its study has received a renewed interest within the context of $AdS/CFT$ correspondence conjecture [1],[2]. On the other hand, the $SL(2)$ WZW represents a particular case of non compact conformal field theory and establishes the discussion about the unitarity and the spectrum of string theory in presence of nontrivial background fields [3]-[5].

More recently, the $SL(2)$ WZW model as an example of logarithmic conformal field theory has been proposed. Indeed, in reference [12]-[13] this $CFT$ was studied within this context, and in reference [14] some exact solutions of the Knizhnik-Zamolodchikov equation exhibiting logarithmic behaviour were found.
Moreover, in [15] it was asserted that logarithmic operators on the boundary of $AdS_3$ are related to operators on the $WZW$ bulk theory which are in indecomposable representations of $SL(2)_k$ affine algebra and several aspects of the logarithmic structure have been analysed in terms of the $AdS_3/CFT_2$ correspondence [15]-[18].

In this paper we are mainly interested in the free field description of logarithmic and prelogarithmic operators in the $WZW$ model formulated on non compact $SL(2)$ manifolds with the intention to present a useful tool to analyse the identity and origin of this logarithmic structure. In section 2 we review the Wakimoto representation of the conformal field theory in terms of free fields. In section 3 we analyse the logarithmic operators $\tilde{\Phi}_j$ introduced in references [16] and we obtain the free field representation of the puncture operator of [12] which is shown to be closed related with the primary field $\Phi_{-1/2}$; in this section we also study the Jordan blocks structure of the $\hat{SL}(2)_k$ Kac-Moody algebra, and some details about the near boundary limit are commented in terms of the indecomposable representations found previously. In section 4 we summarize the results and suggest the connection between the logarithmic structure of $SL(2)$ $WZW$ model and other interesting related subjects.

2 The conformal field theory

2.1 Wess-Zumino-Witten action on $SL(2, C)/SU(2)$

Let us begin with a review of the free field realization of the theory.

By using the Gauss parametrization to describe the group elements, the Wess-Zumino-Witten model formulated on $SL(2, C)/SU(2)$ is given by

$$S = k \int d^2z [\partial \bar{\phi} \partial \phi + \bar{\partial} \gamma \partial \bar{\gamma} e^{2\phi}]$$ (1)

This action describes strings propagating in three dimensional euclidean Anti-de Sitter space with curvature $-\frac{2}{k}$, metric

$$ds^2 = kd\phi^2 + ke^{2\phi}d\gamma d\bar{\gamma}$$ (2)

and background NS-NS field

$$B = ke^{2\phi}d\gamma \wedge d\bar{\gamma}$$ (3)

The boundary of $AdS_3$ is located at $\phi = \infty$. Near this region quantum effects can be treated perturbatively, the exponent in the last term in (1) is renormalized and a linear dilaton in $\phi$ is generated. Adding auxiliary fields $(\beta, \bar{\beta})$ and rescaling, the action becomes

$$S = \frac{1}{4\pi} \int d^2z [\partial \phi \bar{\partial} \phi - \sqrt{\frac{2}{k-2}} R\phi + \beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma} - \beta \bar{\beta} e^{-\sqrt{\frac{2}{k-2}\phi}}]$$ (4)
This theory has a non compact $SL(2) \times SL(2)$ symmetry generated by currents $J^a(z)$ and $\bar{J}^a(\bar{z})$. In the following we discuss the holomorphic part of the theory because the same considerations apply to the antiholomorphic part.

The $SL(2, R)$ current algebra can be expressed in terms of the fields $(\phi, \beta, \gamma)$ using the Wakimoto representation

\begin{align}
J^+(z) &= \beta(z) \\
J^3(z) &= -\beta(z)\gamma(z) - \sqrt{\frac{k-2}{2}} \partial \phi(z) \\
J^-(z) &= \beta(z)\gamma^2(z) + \sqrt{2k-4}\gamma(z)\partial \phi(z) + k\partial \gamma(z)
\end{align}

Indeed, considering the free field propagators

\begin{align}
\langle \phi(z)\phi(w) \rangle &= -\log(z-w) \\
\langle \gamma(z)\beta(w) \rangle &= -\frac{1}{(z-w)}
\end{align}

it is easy to verify that the OPE of these currents satisfy a $SL(2, R)$ level $k$ Kac-Moody algebra, namely

\begin{align}
J^+(z)J^-(w) &= \frac{k}{(z-w)^2} - \frac{2}{(z-w)}J^3(w) + ... \\
J^3(z)J^\pm(w) &= \pm \frac{1}{(z-w)}J^\pm(w) + ... \\
J^3(z)J^3(w) &= -\frac{k/2}{(z-w)^2} + ...
\end{align}

The Sugawara construction leads to the following energy-momentum tensor

\begin{equation}
T_{SL(2,R)} = \beta \partial \gamma - \frac{1}{2}(\partial \phi)^2 - \frac{1}{\sqrt{2(k-2)}} \partial^2 \phi
\end{equation}

and hence the central charge of the theory is

\begin{equation}
c = 3 + \frac{6}{k-2}
\end{equation}

The spectrum of the theory can be obtained analysing the $SL(2, R)$ Kac-Moody primary states. They are labeled by two quantum numbers $|j, m\rangle$, where

\begin{align}
\hat{C} |j, m\rangle &= j(j+1) |j, m\rangle \\
J^3_0 |j, m\rangle &= m |j, m\rangle
\end{align}
and $\hat{C} = -\frac{1}{2}(J_0^+ J_0^- + J_0^- J_0^+) + J_0^3 J_0^3$ is the Casimir (the subindices refer to the zero-modes of the currents). Notice that the Casimir value is invariant under $j \to -j - 1$ and therefore one can consider only states with $j > -\frac{1}{2}$.

The Kac-Moody primarie states $|j, m\rangle$ are defined by the condition

$$J^a_n |j, m\rangle = 0, \quad n > 0$$

while the descendents are constructed by acting with $J^a_{-n}$, $n > 0$ on $|j, m\rangle$.

In terms of the quantum numbers $j$ and $m$, the vertex operators create states from the $SL(2)$ vacum, namely

$$\lim_{z \to 0} \Phi_{j, m} |0\rangle = |j, m\rangle$$

These operators have the conformal dimension $h(j)$ given by

$$h(j) = -\frac{2}{\alpha_+^2} j(j + 1)$$

where $\alpha_+^2 = 2(k - 2)$.

### 2.2 Near boundary limit of the vertex operator

It is usual to introduce auxiliary coordinates $(x, \bar{x})$ in order to organize the $SL(2)$ representations. In this picture, the vertex operators are given by

$$\Phi_j(x, \bar{x}) = \frac{1}{\pi} \left( |\gamma - x|^2 e^{\frac{j}{\alpha_+} \phi} + e^{-\frac{j}{\alpha_+} \phi} \right)^{-2j - 2}$$

and are related with the picture $\Phi_{j, m, \bar{m}}$ by a Fourier transform defined as follows

$$\Phi_{j, m, \bar{m}} = \int d^2x \Phi_j(x, \bar{x}) x^{j-m} \bar{x}^{j-\bar{m}}$$

In this paper we are mainly interested in the situation when $\phi$ is large since the free field aproximation is trusted in that case; thus it will be useful to take the large $\phi$ limit of the expression (13) to obtain the form of the vertex operators in the near boundary region.

Taking into account the following representation of the two-dimensional Dirac function

$$\delta^{(2)}(\gamma - x) = \delta(\gamma - x) \delta(\bar{\gamma} - \bar{x}) = \frac{n-1}{\pi} \lim_{\varepsilon \to 0} \frac{\varepsilon^{2n-2}}{(\varepsilon^2 + |\gamma - x|^2)^n}$$

, we can relate $n = 2j + 2$ and $\varepsilon = e^{-\frac{\phi}{\alpha_+}}$ in order to write

$$\lim_{\phi \to \infty} \Phi_j(x, \bar{x}) = \frac{e^{\frac{j}{\alpha_+} \phi}}{2j + 1} \delta^{(2)}(\gamma - x) + \text{others}$$
This \( \delta \)-term is only dominant in the region \( \gamma \approx x \); and the complete expression for the leading behaviour in \( \phi \) is given by \( \Phi \)

\[
\Phi_j(x, \bar{x}) \approx \frac{e^{2i\phi}}{2j + 1} \delta^{(2)}(\gamma - x) + \frac{1}{\pi} |\gamma - x|^{-4j - 4} e^{\frac{2i\phi}{\alpha_+}} + O(e^{\frac{1}{\alpha_+} |\phi|})
\]

(15)

In the particular case \( j = -\frac{1}{2} \) both \( e^{\frac{\phi}{\alpha_+}} \) and \( e^{-\frac{\phi}{\alpha_+}} \) are the dominant terms and we are in presence of a resonance. Indeed, it will be very useful to our further analysis to study the expression of the leading term in the large \( \phi \) expansion in that particular case. In order to do that, we can start from the continuous representation \( j = -\frac{1}{2} + i\varepsilon \) and then take the limit \( \varepsilon \to 0 \).

Thus, for large values of \( \phi \) we have

\[
\Phi_{-\frac{1}{2} + i\varepsilon}(x) \approx \frac{e^{\frac{\phi}{\alpha_+}}}{2i\varepsilon} \delta^{(2)}(\gamma - x) + \frac{1}{\pi} |\gamma - x|^{-2 - 4i\varepsilon} e^{-\frac{\phi}{\alpha_+}} + O(e^{\frac{1}{\alpha_+} |\phi|}) =
\]

(16)

\[
\left(1 + 2i\varepsilon \frac{\phi}{\alpha_+} + O(\varepsilon^2 |\phi|^2)\right) \times \frac{1}{2i\varepsilon} e^{-\frac{\phi}{\alpha_+} \delta^{(2)}(\gamma - x)} + \frac{1}{\pi} |\gamma - x|^{-2 - 4i\varepsilon} e^{-\frac{1 + 2i\varepsilon}{\alpha_+} \phi}
\]

Now, it is possible to perform the Fourier transform \((14)\) to obtain the \((m, \bar{m})\) picture of this operator. And it is very interesting to notice that the term \( \frac{1}{2i\varepsilon} \) is canceled with a term \(-\frac{1}{2i\varepsilon}\) coming from the Fourier transform of \( \frac{1}{\pi} |\gamma - x|^2 \). Indeed, by using that

\[
\int d^2x |x|^{-2r} (1 - x)^{-2s} = \frac{\Gamma(1 - r) \Gamma(1 - s) \Gamma(r + s - 1)}{\Gamma(r) \Gamma(s) \Gamma(2 - r - s)}
\]

we can write

\[
\Phi_{-\frac{1}{2} + i\varepsilon, m, \bar{m}} \approx \left(1 + \frac{\phi}{\alpha_+}\right) \gamma^{-\frac{1}{2} + i\varepsilon - m} \gamma^{-\frac{1}{2} + i\varepsilon - \bar{m}} e^{-\phi} +
\]

(17)

\[
+ \frac{-1}{2i\varepsilon} \left(1 - \frac{2i\varepsilon \phi}{\alpha_+}\right) \times
\]

\[
\frac{\Gamma\left(\frac{1}{2} + m + i\varepsilon\right) \Gamma\left(\frac{1}{2} - \bar{m} + i\varepsilon\right)}{\Gamma\left(1 + 2i\varepsilon\right) \Gamma\left(\frac{1}{2} + m - i\varepsilon\right) \Gamma\left(\frac{1}{2} - \bar{m} - i\varepsilon\right)} \gamma^{-\frac{1}{2} - i\varepsilon - m} \gamma^{-\frac{1}{2} - i\varepsilon - \bar{m}} e^{-\phi} + O(\varepsilon)
\]

(18)

where \( \zeta \) is the Euler-Mascheroni constant. And taking the limit \( \varepsilon \) going to cero, we obtain

\[
\Phi_{-\frac{1}{2}, m, \bar{m}} = \left(\frac{2\phi}{\alpha_+} + \ln(\gamma) - \zeta\right) \gamma^{-\frac{1}{2} - m} \gamma^{-\frac{1}{2} - \bar{m}} e^{-\frac{\phi}{\alpha_+}}
\]

(19)

The above expression has to be understood as the free field representation of the vertex operator for the particular case \( j = -\frac{1}{2} \). Actually, this free field description of \( \Phi_{-\frac{1}{2}} \) completes the answer to the suggestion effectuated in reference \( [12] \) about the
existence of operators with the form $\Phi \sim \phi e^{-\frac{\phi}{\alpha_+}}$ in the $WZ$ model on non-compact groups.

It could be interesting to mention that a free field description for the puncture operator can be done in terms of the Dotsenko conjugate representation [7] (see also [8, 9, 10]). For instance, it is possible to demonstrate that the functional form for such conjugated vertex operator in the limit $j \to -\frac{1}{2}$ can be given by

$$\left(\ln \beta - \frac{2}{\alpha_+} \phi \right) \beta^{k-2} e^{-\frac{(3-2k)}{\alpha_+} \phi};$$

notice that this field is, in fact, a primary field.

It is also important to comment that if quantum (finite-$k$) effects are taken into account [11] the form for the vertex operator to be considered becomes

$$\Phi_{-\frac{1}{2}+i\epsilon}(x) \approx \frac{e^{\frac{1+2i\epsilon}{\alpha_+}}}{2i\epsilon} \delta^{(2)}(\gamma - x) + R(j) \left|\gamma - x\right|^{-2-4i\epsilon} e^{-\frac{1-2i\epsilon}{\alpha_+} \phi} + O(e^{-\frac{1}{2\alpha_+} \phi})$$

where the reflection factor $R(j)$ is now given by

$$R(j=-\frac{1}{2}+i\epsilon) = \frac{1}{\pi} \left( \frac{\Gamma(1 - \frac{k-2}{2}) \Gamma(1 + \frac{2i\epsilon}{k-2})}{\Gamma(1 + \frac{1}{k-2}) \Gamma(1 + \frac{2i\epsilon}{k-2})} \right)^{2i\epsilon}$$

(20)

And, in this case, we can see from this new expression that the result (19) is recovered in the limit $\epsilon \to 0$.

On the other hand, it is straightforward to see from (13) that the $(m, \bar{m})$ picture for the case $j > -\frac{1}{2}$ is given by the usual expression

$$\Phi_{j > -\frac{1}{2}, m, \bar{m}} = \gamma^{j-m} \gamma^{j-\bar{m}} \phi^{2j} e^{\frac{2j}{\alpha_+} \phi}$$

(21)

up to a factor $2j + 1 > 0$.

It will be convenient for our analysis to introduce new notation to denote the fields of the theory, being

$$\Psi_{j, m, \bar{m}} \equiv \frac{2}{\alpha_+} \phi^{j-m} \gamma^{j-\bar{m}} e^{\frac{2j}{\alpha_+} \phi}$$

$$\Theta_{j, m, \bar{m}} \equiv \gamma^{j-m} \gamma^{j-\bar{m}} e^{\frac{2j}{\alpha_+} \phi}$$

(22)

and hence, we can write $\Phi_{j > -\frac{1}{2}, m, \bar{m}} = \Theta_{j, m, \bar{m}}$ and $\Phi_{-\frac{1}{2}, m, \bar{m}} \sim \Psi_{-\frac{1}{2}, m, \bar{m}}$. Notice that we have taken into account only the leading term $\sim \phi e^{\frac{2j}{\alpha_+} \phi}$ in this definition of nomenclature.

### 3 Logarithmic and prelogarithmic operators
3.1 Logarithmic conformal field theory

A logarithmic conformal field theory is a CFT with not diagonalizable $L_0$ Virasoro generator. In these theories, we have ordinary primaries operators $\Phi$ as well as logarithmic operators $\tilde{\Phi}$ which form Jordan blocks in Virasoro algebra with the following structure [19]

$$T(z)\Phi(w) = \frac{h\Phi}{(z-w)^2} + \frac{\partial \Phi}{(z-w)} + ...$$

$$T(z)\tilde{\Phi}(w) = \frac{h\tilde{\Phi}}{(z-w)^2} + \frac{\xi \Phi}{(z-w)^2} + \frac{\partial \tilde{\Phi}}{(z-w)} + ...$$

(23)

where $\xi$ is a complex number.

Recently, the logarithmic conformal field theories have been extensively studied in relation with several topics in theoretical physics (see [20]); and as we have commented before, it has been proposed that the $SL(2)$ WZW model could be an example of this class of CFT.

Indeed, by exploring the similarity between the non-compact WZW and the $c_{q,p}$ models, references [12] and [16] have mentioned the possibility to define the logarithmic operators in the $SL(2)$ WZW models as

$$\tilde{\Phi}_j(x) = \frac{d}{dj} \Phi_j(x) = \frac{1}{\pi} \left( |\gamma - x|^2 e^{\frac{\phi}{\alpha_+}} + e^{-\frac{\phi}{\alpha_+}} \right)^{-2(j+1)} \ln \left( |\gamma - x|^2 e^{\frac{\phi}{\alpha_+}} + e^{-\frac{\phi}{\alpha_+}} \right)^2$$

(24)

being $\Phi_j(x)$ the ordinary primary field [13]. This operator has the following conformal structure

$$L_0 \tilde{\Phi}_j(z, x) = -\frac{j(j+1)}{k-2} \tilde{\Phi}_j(z, x) - \frac{2j+1}{k-2} \Phi_j(z, x)$$

(25)

where we can clearly see that $\tilde{\Phi}_{-\frac{1}{2}}$ appears as a particular case, being actually an ordinary primary state while $\tilde{\Phi}_{j \neq -\frac{1}{2}}$ presents the quoted logarithmic structure [16]. This class of operators belonging to a logarithmic branch but with the property to be a primary field of the Virasoro algebra are known as prelogarithmic operators (or puncture operators in the Liouville nomenclature).

It was explicitly shown in [12] that the prelogarithmic operator can be in an indecomposable representation of the Kac-Moody algebra even though they are primary fields of the Virasoro algebra.

Now, let us to analyse the free field description of the $\tilde{\Phi}_{j,m}$ operators. To do that, we can derivate the expression (18) with respect to $j$ or directly take the large $\phi$ limit in (24) as it was done for the operator [13]. In both cases, the result is given by

$$\tilde{\Phi}_{j,m} = \gamma^{j-m} e^{\frac{2j}{\alpha_+} \phi} (\ln(\gamma) + \frac{2}{\alpha_+} \phi) = \Theta_{j,m}(\ln(\gamma) + \frac{2}{\alpha_+} \phi)$$

(26)
And now, by using the prescription (see [21] and references therein)

$$\langle \beta(z) f(\gamma(w)) \rangle =: \left( \beta(z) + \frac{1}{(z-w)} \frac{\partial}{\partial \gamma} \right) f(\gamma(w)) : + ...$$

(27)

it is easy to verify from (9) that the $\tilde{\Phi}_{j,m}$ fields exhibit the Jordan block structure of Virasoro algebra given by (25), namely

$$T_{SL(2,R)}(z) \tilde{\Phi}_{j,m}(w) = \left( -\frac{j(j+1)}{k-2} \right) \frac{\tilde{\Phi}_{j,m}}{(z-w)^2} - \left( \frac{2j+1}{k-2} \right) \frac{\Phi_{j,m}}{(z-w)^2} + \frac{\partial \tilde{\Phi}_{j,m}}{(z-w)} + ...$$

The fact that the $(m, \bar{m})$ picture exhibits a similar logarithmic structure to the $(x, \bar{x})$ picture becomes important if it is taken into account that it is general not clear the subtle detail of the correspondence $\Phi_{j,m} \leftrightarrow \Phi_j(x, \bar{x})$, as it was remarked in reference [2].

3.2 The puncture operator and the resonance

Let us present an heuristic argument within the context of logarithmic structure of the operator product expansion. As it was mentioned in the last subsection, the particular case $\tilde{\Phi}_{-\frac{1}{2},m}$ is actually a primary field of Virasoro algebra. This is the puncture operator of the theory and it was extensively studied in reference [16].

On the other hand, since it is our intention to make clear the relation between the operator $\tilde{\Phi}_{-\frac{1}{2}}$ and the prelogarithmic field analysed in reference [12], it is important to remark that the operators $\Psi_{j,m}$ also exhibit the logarithmic structure (25). This fact could induce to interpret the $\Psi_{-\frac{1}{2},m}$ as a puncture operator of the theory, and we have shown before that this field is actually the large $\phi$ limit of $\Phi_{-\frac{1}{2},m}$. This is a very important point, the free field representation of the primary field with $j = -\frac{1}{2}$ presents the functional form of the logarithmic operators $\sim \phi e^{2\pi \phi}$. In fact, before expanding it in power of $\phi$ and performing the Fourier transform, the aspect of the $\Phi_{-\frac{1}{2}}(x)$ was substantially different from $\tilde{\Phi}_{-\frac{1}{2}}(x)$, which shows that the free field description in terms of $\Phi_{j,m}$ suggests more explicitly the particular connection existing between $\tilde{\Phi}_{-\frac{1}{2}}$ and $\Phi_{-\frac{1}{2}}$.

In [13] it was commented that it is not possible to have $\Phi_{-\frac{1}{2}}$ in the spectrum of the $SL(2)$ WZW model without $\tilde{\Phi}_{-\frac{1}{2}}$ also been included. Thus, in order to analyse the possibility of the occurrence of logarithmic operators in the spectrum of the $SL(2)$ model, the operator product expansion of two fields have been studied in the literature (see [16] and [17]). Indeed, taking into account the OPE’s

$$\Theta_{j_1,m_1}(z) \Theta_{j_2,m_2}(w) \sim |z-w|^{-\frac{1}{2}j_1j_2} \Theta_{j_1+j_2, m_1+m_2}(w) + ...$$

(28)
and
\[ \Theta_{j_1,m_1}(z)\Psi_{j_2,m_2}(w) \sim |z-w|^\frac{k}{k-2}j_1j_2 (\Psi_{j_1+j_2,m_1+m_2}(w) + \mathcal{O}(|z-w|^{k-2}) + \cdots) \quad (29) \]
it is possible to observe that the logarithm in the operator product expansion manifestly appears.

On the other hand, it is very important to notice that the expression (29) explicitly shows the mixing terms raising from the presence of a field with the form \( \Phi_{-\frac{1}{2},m} \sim \Psi_{-\frac{1}{2},m} \) since that a logarithmic field \( \Phi_{j,\frac{1}{2},m} \rightarrow \Psi_{j,\frac{1}{2},m} \) appears in the operator product expansion if \( j_1 \neq 0 \).

Actually, the logarithmic contributions also appear when the operator product of the fields \( \tilde{\Phi}_j \) are considered, namely
\[ \Theta_{j_1,m_1}(z)\tilde{\Phi}_{j_2,m_2}(w) \sim |z-w|^\frac{k}{k-2}j_1j_2 (\tilde{\Phi}_{j_1+j_2,m_1+m_2}(w) - \frac{2j_1}{k-2} \ln |z-w| \Theta_{j_1+j_2,m_1+m_2}(w)) + \cdots \quad (30) \]

Now, we would like to emphasize the differences and similarities between the operators \( \Psi_{-\frac{1}{2},m} \) and \( \tilde{\Phi}_{-\frac{1}{2},m} \) since these fields play a crucial role in the study of logarithmic structure of \( SL(2)_k \) WZW model \[16\]. It is clear that the linear term in \( \phi \) that it is present in both (18) and (26) dominates near the boundary and thus these fields have the same large \( \phi \) behaviour; however we will see in the next subsection that the presence of the \( \ln(\gamma) \) in (26) makes that the structure of the corresponding Jordan blocks of Kac-Moody algebra are quite different. This fact shows us that the leading terms in the large \( \phi \) expansion are in general not sufficient to analyse the logarithmic structure of the current algebra, and then it would be necessary not to neglect the \( \ln(\gamma) \) term in order to study it.

### 3.3 Jordan blocks of \( \hat{SL}(2)_k \) affine algebra

It was shown in reference \[12\] that the fields \( \Psi_{j,m} \) form Jordan blocks in the Kac-Moody \( \hat{SL}(2)_k \) affine algebra even though \( j = -\frac{1}{2} \). This logarithmic structure of the current algebra is given by
\[
\begin{align*}
J^+(z) \left( \frac{\alpha_+}{2} \Psi_{j,m}(w) \right) &= \frac{j-m}{(z-w)} \left( \frac{\alpha_+}{2} \Psi_{j,m+1}(w) \right) + \cdots \\
J^3(z) \left( \frac{\alpha_+}{2} \Psi_{j,m}(w) \right) &= \frac{m}{(z-w)} \left( \frac{\alpha_+}{2} \Psi_{j,m}(w) + \frac{\alpha_+}{(z-w)} \Theta_{j,m} \right) + \cdots \\
J^-(z) \left( \frac{\alpha_+}{2} \Psi_{j,m}(w) \right) &= -\frac{j+m}{(z-w)} \left( \frac{\alpha_+}{2} \Psi_{j,m-1}(w) \right) - \frac{\alpha_+}{(z-w)} \Theta_{j,m-1} + \cdots
\end{align*}
\]
On the other hand, as it was commented in the previous subsection, from (26) it is easy to verify that the operators \( \tilde{\Phi}_{j,m} \) form a different and more symmetric Jordan block, namely

\[
J^\pm(z)\tilde{\Phi}_{j,m}(w) = \frac{(\pm j - m)}{(z-w)}\tilde{\Phi}_{j,m\pm1}(w) \pm \frac{1}{(z-w)}\Theta_{j,m\pm1}(w) + ... \\
J^3(z)\tilde{\Phi}_{j,m}(w) = \frac{m}{(z-w)}\tilde{\Phi}_{j,m}(w) + ... 
\]

(32)

Thus, the nondiagonal structure (31) must be understood as the Jordan blocks of the large \( \phi \) limit of the primary field \( \Phi_{-\frac{1}{2}} \) rather than the corresponding to the field \( \Phi_j = \frac{d}{dz}\Phi_j \). Then, we have obtained in (32) a new Jordan cell of the affine \( \widetilde{SL}(2)_k \) algebra. It gives us an example to show how the structure of the Jordan blocks of the current algebra depends on the functional form of the field representation beyond the leading terms in large \( \phi \) expansion.

Notice that in the \( \alpha_+ \to 0 \) limit (i.e. \( k \to 2^+ \)) the Jordan structure of the operators \( \frac{\alpha_+}{2}\Psi_{j,m} \) becomes diagonal, while the blocks for the fields \( \tilde{\Phi}_{j,m} \) remain in its Jordan form in that limit. This fact remarks the differences between (31) and (32).

If, on the other hand, we consider the operator defined as

\[
\Xi_{j,m} \equiv \tilde{\Phi}_{j,m} - \Psi_{j,m} = \ln(\gamma)\Theta_{j,m} 
\]

(33)

it is possible to obtain the following nondiagonal realization

\[
J^\pm(z)\Xi_{j,m}(w) = \frac{(\pm j - m)}{(z-w)}\Xi_{j,m\pm1}(w) + \frac{1}{(z-w)}\Theta_{j,m\pm1}(w) + ... \\
J^3(z)\Xi_{j,m}(w) = \frac{m}{(z-w)}\Xi_{j,m}(w) - \frac{1}{(z-w)}\Theta_{j,m}(w) + ... 
\]

(34)

where (33) is a primary field according to the prescription (27). Then, it would be a new realization of a primary field in Virasoro algebra but belonging to an indecomposable representation of Kac-Moody algebra. These new operators \( \Xi_{j,m} \) form an infinite set of primary states in the Virasoro algebra with logarithmic structure in the \( \widetilde{SL}(2)_k \) affine Kac-Moody algebra. And, on the other hand, it is possible to verify that the operator product expansion of two of these fields does not contain mixing terms with prelogarithmic fields.

3.4 Spacetime stress tensor on a long string

Let us comment in this subsection an important point within the context of \( AdS_3/CFT_2 \) correspondence.
In a recent paper [22] Seiberg and Witten have obtained the spacetime stress tensor of the theory on the single long string solution. This stress tensor $T_{\text{tot}}$ is obtained by twisting the worldsheet tensor (9) to

$$T_{\text{tot}} = T_{SL(2,R)} - \partial J^3$$

(35)

In fact, this result leads to compute the Brown-Henneaux central charge and the corresponding conformal dimension for the primary spacetime fields.

Since it was analysed in [12] that the logarithmic structure can be altered in models with modified stress tensor, it would be interesting to investigate the logarithmic behaviour of the fields (19) and (26) in terms of the theory defined by (35). In order to do so, we consider the following form for the modified stress tensor

$$T_{\text{tot}} = T_{SL(2,R)} - Q \partial J^3 = (1 + Q) \beta \partial \gamma - \frac{1}{2} \left( \frac{\alpha_+}{2} Q - \frac{1}{\alpha_+} \right) \partial^2 \phi$$

(36)

which includes models considered in the study of non compact CFT and two-dimensional gravity.

Thus, by computing the operator product expansion it is possible to verify that the structure of the Virasoro algebra in that case is given by the following expression

$$T_{\text{tot}}(z) \Phi_{j,m}(w) = \left( - \frac{j(j+1)}{k-2} + Qm \right) \frac{\Phi_{j,m}}{z-w}^2 + \frac{\partial \Phi_{j,m}}{z-w} + ...$$

(37)

while for the fields $\tilde{\Phi}_j$ and $\Psi_j$ we obtain

$$T_{\text{tot}}(z) \tilde{\Phi}_{j,m}(w) = \left( - \frac{j(j+1)}{k-2} + Qm \right) \frac{\tilde{\Phi}_{j,m}}{z-w}^2 + \left( \frac{2j+1}{k-2} \right) \frac{\Theta_{j,m}}{z-w}^2 + \frac{\partial \tilde{\Phi}_{j,m}}{z-w} + ...$$

$$T_{\text{tot}}(z) \Psi_{j,m}(w) = \left( - \frac{j(j+1)}{k-2} + Qm \right) \frac{\Psi_{j,m}}{z-w}^2 + \left( \frac{2j+1}{k-2} + Q \right) \frac{\Theta_{j,m}}{z-w}^2 + \frac{\partial \Psi_{j,m}}{z-w} + ...$$

(38)

and thus

$$T_{\text{tot}}(z) \Xi_{j,m}(w) = \left( - \frac{j(j+1)}{k-2} + Qm \right) \frac{\Xi_{j,m}}{z-w}^2 - Q \frac{\Theta_{j,m}}{z-w}^2 + \frac{\partial \Xi_{j,m}}{z-w} + ...$$

(39)

Indeed, from these expressions we can compare the logarithmic structure of both theories (3) and (33); and it is possible to observe that the Jordan blocks of Virasoro algebra for the fields $\tilde{\Phi}_j$ remain unchanged while for the fields $\Phi_{-+}$ do not. This fact makes more evident that the Jordan structure is defined by the different terms in the large $\phi$ expansion.
Notice that since we have shown the relation (13) between the fields $\Psi_{-\frac{1}{2},m}$ and $\Phi_{-\frac{1}{2},m}$, it is possible to affirm that the leading term in the near boundary limit of the primary field $\Phi_{-\frac{1}{2}}$ becomes logarithmic for the theory defined by (35). This is in agreement with one of the results of reference [16], where it was concluded that the worldsheet primary field $\Phi_{-\frac{1}{2}}$ is a logarithmic field in the spacetime theory. It is immediate to see from (37) and (38) that the primary fields $\Xi_{j,m}$ also become logarithmic in the twisted theory $Q \neq 0$.

On the other hand, from the expression (38) we can see that in the spacetime theory on the long string $Q = \pm 1$, as well as in the world-sheet theory $Q = 0$, the logarithmic structure of the Virasoro algebra of fields $\tilde{\Phi}_j$ approximates to the structure of fields $\Psi_j$ in the limit $\alpha_+ \to 0$; and this fact is consistent with the functional forms (13) and (21). However, as we have seen before, the logarithmic structure of the Kac-Moody algebra of these fields remain distinct in that limit.

Notice that in the case $Q = -1$ the prelogarithmic operators are $\tilde{\Phi}_j$ with $j = -\frac{1}{2}$ and $\Psi_j$ with $j = \frac{k-3}{2}$, and these are precisely the minimum and maximum value of $j$ in the discrete series according to satisfy the unitarity bound (to consider the case $Q = +1$ instead $Q = -1$ is consistent with the reflection symmetry $j \leftrightarrow -j - 1$). Moreover, $\Psi_{-\frac{1}{2}}$ appears as the prelogarithmic operator for the worldsheet theory whereas $\Psi_{\frac{k-3}{2}}$ is the prelogarithmic operator in the theory on a single ($\omega = \pm 1$) long string solution.

### 3.5 Zero mode contribution and logarithmic operators

Another interesting aspects of the free field representation is the fact that it leads to draw a rare relation existing between the near boundary behaviour of the logarithmic operators (24) and the $r$-string states introduced by Bars, Deliduman and Minic in reference [3].

Bars et al. have proposed that the inclusion of the zero mode contribution plays a crucial role in the study of the spectrum of the $SL(2) WZW$ model. Actually, it was argued that this contribution is important to describe the winding states in the $AdS_3$ spacetime.

Indeed, the inclusion of the zero mode in reference [3] leads to generalize the vertex operator (13) in order to incorporate the description of the $\omega$-winding fields $\Phi_{j,m\omega}^\omega(x)$, namely

$$
\Phi_{j=-\frac{1}{2}+i\lambda}(x) = \Phi_{j=-\frac{1}{2}+i\lambda}(x) \times \left( e^{\frac{\phi_+}{\alpha_+}} |\gamma - x| \right)^{-2(k-2)m\omega - \lambda^2 - i\lambda} \times 
$$

$$
\times _2 F_1 \left( (k-2)m\omega - \lambda^2 - i\lambda, (k-2)m\omega - \lambda^2 - i\lambda; 1 + 2\sqrt{(k-2)m\omega - \lambda^2 - i\lambda}; -e^{\frac{-2}{\alpha_+}} |\gamma - x|^{-2} \right) 
$$

(40)
which coincides with the usual vertex operator $\Phi_j(x)$ in the particular case $m\omega = 0$, namely

$$\Phi^0_{j=-\frac{1}{2}+i\lambda}(x) = \Phi_{j=-\frac{1}{2}+i\lambda}(x)$$ (41)

These $SL(2)$ states are primaries and have a conformal dimension given by

$$h(j) = -\frac{2}{\alpha_+^2}j(j+1) - m\omega = \frac{1}{\alpha_+^2} \left( \frac{1}{4} + \lambda^2 \right) - m\omega$$ (42)

with a new $m\omega$ contribution that becomes important within the context of the unitarity problem \[5\],[6].

And taking the large $\phi$ limit we can see from (40) that the following behaviour is obtained near the boundary \[3\]

$$\Phi_{j=-\frac{1}{2}+i\lambda}(x) \sim N_{(\lambda,m\omega,k)} \left( \ln |\gamma - x|^2 + \frac{2}{\alpha_+} \phi \right)$$ (43)

where

$$N_{(\lambda,m\omega,k)} = \frac{\Gamma(1+2\sqrt{km\omega-\lambda^2})}{\Gamma(\sqrt{km\omega-\lambda^2-i\lambda})\Gamma(\sqrt{km\omega-\lambda^2+i\lambda})}.$$ Hence, from (26) it is possible to observe that the large $\phi$ behaviour of (43) coincide with the power expansion in $\phi$ of the logarithmic operators $\tilde{\Phi}_j(x)$, including a term with $\ln |\gamma - x|^2$.

Notice that the presence of terms of the form $\delta^{(2)}(\gamma - x)$ in the large $\phi$ expansion of $\Phi_{j=-\frac{1}{2}+i\lambda}$ makes it impossible to neglect the term $\ln |\gamma - x|^2$ in general. Thus, we see from (13) that the vertex operators introduced in \[3\] form Jordan blocks of the free field realization of the $SL(2)$ current algebra with the form of the Jordan cells analysed in the previous sections of this note.

Moreover, taking into account that it is usual to assume that the logarithmic sectors in string theory are though to generate additional symmetries \[16\], this fact suggests that it could be interesting to mention a connection between the zero mode contribution remarked by Bars et al., the indecomposable operators $\Xi_{j,m}$ and the logarithmic fields $\tilde{\Phi}_{j,m}$. Actually, this is for further investigation (recently, the winding modes have been successfully described in terms of spectral flow symmetry \[4\]).

### 4 Conclusions

We have studied the free field representation of the logarithmic and prelogarithmic operators in the Wess-Zumino-Witten model formulated on $SL(2)$. We have shown that the $\Phi_{j,m,\bar{m}}$ picture is useful to analyse the logarithmic structure of the Virasoro and Kac-Moody algebra.

The similarities and differences existing between the puncture operator of reference \[16\] and the large $\phi$ behaviour of the resonance $j = -\frac{1}{2}$ were pointed out. We have
shown that the leading terms in the near boundary region of both operators coincide even though the description in terms of the auxiliary variables \((x, \bar{x})\) does not manifest this relation explicitly. This fact shows how the \((m, \bar{m})\) picture results useful to analyse the particular connection between \(\Phi_{-\frac{1}{2}}\) and \(\Phi_{-\frac{1}{2}}^-\) (while the Fourier conjugate picture \((x, \bar{x})\) has shown to be useful to explore the logarithmic structure of the boundary CFT in previous works \([15]-[18]\).

The Jordan blocks of the Kac-Moody algebra were written down for operators of the form \(\Phi_j = \frac{d}{dj} \Phi_j\) and \(\Phi_{-\frac{1}{2}}^-\). These results show how the different terms in the large \(\phi\) expansion determinate the logarithmic structure of the current algebra. Actually, we have argued that the non diagonal structure found in reference \([12]\) must be understood as the Jordan blocks of the large \(\phi\) limit of the primary field \(\Phi_{-\frac{1}{2}}^-\) rather than the corresponding to the operators \(\frac{d}{dj} \Phi_j\).

On the other hand, new free field realization of primary operators \(\Xi_{j,m}\) belonging to indecomposable representations of Kac-Moody algebra have been obtained and we have also analysed the operator product expansion of two fields in order to make explicit the occurrence of the logarithmic structure in terms of the free field description.

We have also shown that the primary states of the original theory become logarithmic when a twisted stress tensor is considered and this fact is consistent with results of previous works in this matter. The minimum and maximum values of \(j\) admissible by the unitarity bound were seen as prelogarithmic states for the worldsheet theory and for the spacetime theory on a single long string solution respectively.

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**References**

[1] A. Giveon, D. Kutasov and N. Seiberg, Adv.Theor.Math.Phys. 2 (1998) 733. J. de Boer, H. Ooguri, H. Robins and J. Temenhauser, JHEP 9812 (1998) 026, hep-th/9812046

[2] D. Kutasov and N. Seiberg, JHEP 9904 (1999) 08, hep-th/9903219

[3] J. Evans, M. Gaberdiel and M. Perry, Nucl.Phys. B535 (1998) 152, hep-th/9806024. P. Petropoulos, *String theory on AdS\(_3\): some open questions*, hep-th/9908189. G. Giribet and C. Núñez, JHEP 9909, 031 (1999), hep-th/9909149

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[4] A. Giveon and D. Kutasov, JHEP 9910 (1999) 034, hep-th/9909110. J. Maldacena and H. Ooguri, Strings in AdS\(_3\) and SL(2, R) WZW Model: 1, hep-th/0001053. J. Maldacena, H. Ooguri and J. Son, Strings in AdS\(_3\) and SL(2, R) WZW Model: 2, hep-th/0005183. G. Giribet and C. Núñez, JHEP 0006, 033 (2000), hep-th/0006070. Y. Hikida, K. Hosomichi and Y. Sugawara, String theory on AdS\(_3\) as discrete light-cone Liouville theory, hep-th/0005065. R. Argurio, A. Giveon and A. Shomer, Superstrings on AdS\(_3\) and symmetric products, hep-th/0009242.

[5] I. Bars, C. Deliduman and D. Minic, String theory on AdS\(_3\) revisited, hep-th/9907087.

[6] I. Bars, Phys.Rev. D53 (1996) 3308.

[7] V. S. Dotsenko, Nucl. Phys. B338 (1990) 747.

[8] V. S. Dotsenko, Nucl. Phys. B358 (1991) 547.

[9] I. I. Kogan and A. Nichols, SU(2)\(_0\) and OSp(2\(|\_\_\_2\) WZW models: Two currents algebras, one Logarithmic CFT, hep-th/0107160.

[10] G. Giribet and C. Núñez, JHEP 0106 (2001), hep-th/0105200.

[11] J. Teschner, Nucl.Phys. B546 (1999) 369. J. Teschner, Nucl.Phys. B546 (1999) 390.

[12] I.I. Kogan and A. Lewis, Nucl.Phys. B509 (1998) 687-704, hep-th/9705240

[13] A. Bilal and I.I. Kogan, Nucl.Phys. B449 (1995) 569, hep-th/9503209. A. Bilal and I.I. Kogan, Gravitationally dressed conformal field theory and emergence of logarithmic operators, hep-th/9407151

[14] A. Nichols and Sanjay, Logarithmic operators in the SL(2, R) WZNW model, hep-th/0007007. I.I. Kogan and A. Tsvelik, Mod.Phys.Lett. A15 (2000) 931, hep-th/9912143. I.I. Kogan, A. Lewis and O. Soloviev, Int.J.Mod.Phys. A13 (1998) 1345, hep-th/9703028. M. Bhaseen, I.I. Kogan, O. Soloviev, N. Taniguchi and A. Tsvelik, Nucl.Phys. B580 (2000) 688, cond-mat/9912060.

[15] A. Lewis, Phys.Lett. B480 (2000) 348-354, hep-th/9911163

[16] A. Lewis, Logarithmic CFT on the boundary and the world-sheet, hep-th/0009096

[17] M.R. Rahimi Tabar, A.Aghamohammadi and M. Khorrami, Nucl.Phys.B497 (1997) 555, hep-th/9610168.

[18] I.I. Kogan and J.F. Wheater, Phys.Lett. B486 (2000) 353-361, hep-th/0003184
[19] V. Gurarie, Nucl.Phys. B410 (1993) 535, hep-th/9303160.

[20] S. Moghimi-Araghi and S. Rouhani, Lett.Math.Phys. 53 (2000) 49-57, hep-th/0002142. H. Hata and S. Yamaguchi, Phys.Lett. B482 (2000) 283-286, hep-th/0004189. M.A.I. Flohr, Phys.Lett. B444 (1998) 179, hep-th/9808169. M.R. Gaberdiel and H.G. Kausch, Nucl.Phys. B538 (1999) 631, hep-th/9807091. N.E. Mavromatos and R.J. Szabo, Phys.Lett. B430 (1998) 94, hep-th/9803092. S. Skoulakis and S. Thomas, Phys.Lett. B438 (1998) 301, cond-mat/9802040. I.I. Kogan and N. Mavromatos, Phys.Lett. B375 (1996) 111, hep-th/9512210. J. Caux, I.I. Kogan and A. Tsvelik, Nucl.Phys. B466 (1996) 444, hep-th/9511134. I.I. Kogan, C. Mudry and A. Tsvelik, Phys.Rev.Lett. 77 (1996) 707. I.I. Kogan, Phys.Lett. B458 (1999) 66, hep-th/9903162.

[21] J. Rasmussen, Applications of free field in 2D current algebra, PhD Thesis, Niels Bohr Institute at the University of Copenhagen, hep-th/9610167.

[22] N. Seiberg and E. Witten, JHEP 9904 (1999) 017, hep-th/9903224.