Localized non-Abelian gauge fields in non-compact extra-dimensions

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Dynamical localization of non-Abelian gauge fields in non-compact flat $D$ dimensions is worked out. The localization takes place via a field-dependent gauge kinetic term when a field condenses in a finite region of spacetime. Such a situation typically arises in the presence of topological solitons. We construct four-dimensional low-energy effective Lagrangian up to the quadratic order in a universal manner applicable to any spacetime dimensions. We devise an extension of the $R_\xi$ gauge to separate physical and unphysical modes clearly. Out of the $D$-dimensional non-Abelian gauge fields, the physical massless modes reside only in the four-dimensional components, whereas they are absent in the extra-dimensional components. The universality of non-Abelian gauge charges holds due to the unbroken four-dimensional gauge invariance. We illustrate our methods with models in $D = 5$ (domain walls), in $D = 6$ (vortices), and in $D = 7$. 
I. INTRODUCTION AND CONCLUSIONS

Theories with extra-dimensions give a solution of the gauge hierarchy problem in the framework such as the brane-world scenario [1–5]. One of the most popular models is in five-dimensional spacetime where the fifth dimension is compactified on an orbifold. In this kind of models, several assumptions are made: i) The fifth dimension is compact. ii) Branes exist. iii) Matter fields are localized on the branes with the boundary Lagrangian (proportional to a delta function). iv) Nontrivial $\mathbb{Z}_2$ parity assignments are imposed on fields. This setup provides models akin to the standard model (SM) with several nice solutions to long-standing problems of the SM. However, the origins of these nontrivial assumptions have not been explained.

In contrast, these points can be achieved not as assumptions but as consequences of dynamics in a model with non-compact extra-dimensions. We do not need to prepare a specific geometry for the extra-dimensions. For five-dimensional models, the minimal assumption is the presence of discrete degenerate vacua. Spontaneous symmetry breaking of the discrete symmetry dynamically yields stable domain walls. Thus, our four-dimensional world is dynamically realized on the domain walls. Furthermore, they automatically lead to localization of zero modes of matter fields such as chiral fermions and scalars, out of D-dimensional matter fields in the bulk [6–8]. The extra-dimensional models can give a natural explanation also for the hierarchy among the effective four-dimensional Yukawa couplings [9–13], irrespective of compact or non-compact extra dimensions.

Unfortunately, the localization of gauge fields is quite difficult [14–37] in the brane-world scenario with the topological defects. A popular resolution is the so-called Dvali-Shifman mechanism [38]. However, this mechanism assumes the confinement in higher-dimensional spacetime, whose validity is far from being clear. It was found that gravity can localize gauge fields but it works only for six dimensions [39]. A problem of using gravity is that gravity affects all the fields on an equal footing. While the gauge fields localization may be achieved, the localization of fermions is lost [39].

It has been noted that the localization of gauge fields in flat non-compact spacetime requires the confining phase rather than the Higgs phase in the bulk outside the brane [11–38]. A semi-classical realization of the confinement can be obtained by the position-dependent gauge coupling [40–43], which is achieved by domain walls in five dimensions through the
field-dependent gauge coupling function. This semi-classical mechanism was successfully applied to localize gauge fields on domain walls [44–50]. As an advantage of using this mechanism, we can explicitly determine mode functions of massless gauge bosons. Recently, we proposed a model realizing Grand Unified Theory (GUT) on domain walls in five dimensions [50] via the geometric Higgs mechanism [49] which gives the familiar Higgs mechanism by means of the geometric information (position of walls along the extra-dimension). Another advantage is that our localization mechanism assures charge universality of matter fields by preserving the 3 + 1-dimensional gauge invariance.

The main goal of this paper is to establish a rigorous formulation of the localization mechanism of gauge fields by generalizing the non-trivial gauge kinetic function from five dimensions [44–50] to higher dimensions. With this established formula at hand, one can naturally construct brane-world models in higher dimensions without assuming either a compact geometry or the confinement in higher dimensions. Especially, the models in six dimensions has a nice virtue that a single family in six dimensions automatically generates $k$ copies of massless fermions in four dimensions where $k \in \pi_1(S^1)$ is the topological vortex number, giving an explanation of the three generations in the SM. A similar mechanism has been discussed in models without the localization of gauge fields [51, 52], assuming Dvali-Shifman mechanism [53], and with $S^2$ as the compact extra dimensions [54].

In this work, we study the localization of non-Abelian gauge fields as generically as possible. Our analysis is quite model independent and it is applicable to any number of spacetime dimensions. Since our primary aim here is to clarify the physical mass spectrum appearing in low energy four-dimensional physics, we will analyze the action up to the quadratic order in fields. Then, we can treat Abelian and non-Abelian gauge fields on equal footing. Although we do not consider quantum loop calculations in this paper, we develop an extension of $R_\xi$ gauge appropriate for models in higher dimensions in order to separate physical and unphysical degrees of freedom. In contrast, let us recall our previous studies in five dimensions [44–50] where the axial gauge $A_y = 0$ was chosen. Although the axial gauge is simple, it is inappropriate to establish the possible presence of zero modes of $A_y$ besides being awkward for loop calculations. One should note that the zero mode is gauge invariant. The analysis in our $R_\xi$-like gauge will not only provide clearcut understanding of the physical spectrum but also is applicable to higher dimensions $D \geq 6$ where the axial gauge $A_y = 0$ does not naively make sense. It is gratifying that we do not find any additional
zero modes except for desired four-dimensional gauge fields in low-energy effective theory: a fact that is also insensitive to the details of the model. This is due to the fact that the field dependent-gauge coupling function spontaneously breaks the gauge symmetry in such a way that the gauge symmetry only in the four dimensional sense is preserved. In comparison, the standard compactification of extra-dimensions cannot avoid new zero modes from extra components of the gauge fields, and an additional structure, such as orbifolding, is required to suppress them. This point offers a possibility for our mechanism to become a universal tool for the brane-world model building.

To be concrete, we give two examples: one is a domain wall in five dimensions and the other is a Nielsen-Olsen type local vortex in six dimensions. While so many works have been done to localize gauge fields on domain walls, the number of works are quite a few on the vortices. In particular, if we do not assume the Dvali-Shifman mechanism \[53\], compact extra dimensions \[54\], or do not use gravity \[39\], the example given here is the first model which provides massless non-Abelian gauge fields on the vortices in six dimensions. We also give an example for \( D = 7 \) case. We emphasize that our localization mechanism automatically gives universality of gauge charges in models in any dimensions \[38, 55\].

To analyze the physical spectra in non-compact spacetime, we find a formulation similar to the supersymmetric quantum mechanics quite useful. When we determine mass spectra of Kaluza-Klein (KK) modes, we always end up with a Schrödinger type problem. The corresponding Hamiltonians we will encounter are indeed special ones. In five-dimensional case, they are precisely the Hamiltonians of supersymmetric (SUSY) quantum mechanics (QM). Therefore, the spectra can be analytically obtained in many cases. In the higher-dimensional cases with \( D \geq 6 \), the Hamiltonians are still similar to SUSY QM ones. This structure is very helpful both analytically and numerically.

This paper is organized as follows. In Sec. II we present a general argument of gauge field localization in general \( D \)-dimensions. Furthermore, we separate physical and unphysical modes of massive as well as massless four-dimensional fields, and work out the low-energy effective theory. We extend the \( R_\xi \) gauge and develop a SUSY QM technique. In Sec. III we provide three explicit examples of brane-world scenarios with models of one non-compact extra dimensions (domain walls), two extra dimensions (vortices), and three extra dimensions.

Note added: While finishing this work, a new paper \[56\] appeared that has a partial
overlap with some of our results. Just after this paper was posted on arXiv, another new paper [57] appeared.

II. LOCALIZATION AND HIGSS-LIKE MECHANISMS

A. Generic formula

Let us consider a simple Yang-Mills model in $D$ dimensions

$$\mathcal{L}_A = -\beta^2 \text{Tr} \mathcal{F}_{MN} \mathcal{F}^{MN},$$

(II.1)

where $\mathcal{F}_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]$ is a non-Abelian field strength. Throughout the paper, we use small greek letters for four-dimensional indices $\mu = 0, 1, 2, 3$, small roman letters for extra-dimensional spatial coordinates $a = 4, \cdots, D - 1$ and the capital roman letters for the $D$-dimensional indices $M = 0, 1, \cdots, D - 1$. Mass dimensions of the gauge fields and $\beta$ are $[A_M] = 1$ and $[\beta] = \frac{D-4}{2}$, respectively. We assume that $\beta$ is a Lorentz scalar and a gauge invariant. We denote the four-dimensional coordinates as $x = \{x^\mu\}$ the extra-dimensional coordinates as $y = \{x^a\}$, and the metric as $\eta_{\mu\nu} = (1, -1, \cdots, -1)$. The non-minimal gauge kinetic term of type (II.1) are studied in various contexts though most of them concern four dimensions [58–67].

When $\beta$ is a constant, it is nothing but the inverse gauge coupling constant, i.e. $\beta^{-1} = \sqrt{2g}$. In this work, we will investigate what happens when $\beta$ is not a constant. There are at least three cases where this situation is realized: i) the spacetime geometry is nontrivial [68] with $\sqrt{-g}$ identified as $\beta^2$, ii) $\beta^2$ is identified [15] as $\beta^2 = e^\varphi$ with the dilaton field $\varphi$, iii) the gauge coupling is a function of scalar fields $\varphi_i$ as $\beta = \beta(\varphi_i)$ with $\varphi_i$ acquiring nonvanishing $y$-dependent vacuum expectation values inside a finite region in the extra-dimensions[1]. Each has its own (dis)advantages, but all the technical aspects, which we investigate here, are applicable for all of the cases.

Minimal assumption for us is that $\beta$ depends only on the extra-dimensional coordinates $y$. We further assume the square integrability

$$\int d^{D-4}x \beta(y)^2 < \infty.$$  

(II.2)

1 Thorough out this paper, we assume $\varphi_i$ to be singlet of the gauge group of $A_M$. Therefore, the condensation of $\varphi_i$ does not directly lead to spontaneous breakdown of the gauge symmetry. The singlet scalar $\varphi_i$ interacts with $A_M$ only through Eq. (II.1).
As stated above, the reason why $\beta$ depends on $y$ is not important for our results. For concreteness, however, we will give several examples in later sections.

The square integrability condition implies that $\beta$ approaches zero as $|y| \to \infty$. This means that the gauge coupling become very large in the bulk. This is a semiclassical realization of the confining phase in the bulk, which is necessary to realize localization of the massless gauge fields on branes \[1, 38\].

We first introduce differential operators which will play a central role in what follows:

$$D_a = -\partial_a + (\partial_a \log \beta) = -\beta \partial_a \frac{1}{\beta}, \quad D^\dagger_a = \partial_a + (\partial_a \log \beta) = \frac{1}{\beta} \partial_a \beta.$$  

(II.3)

It is straightforward to verify the following

$$D^\dagger_b D_a = \frac{1}{\beta} \partial_a \beta^2 \partial_b \frac{1}{\beta},$$  

(II.4)

$$D_a D^\dagger_b = -\beta \partial_a \frac{1}{\beta^2} \partial_b \beta,$$  

(II.5)

$$[D_a, D^\dagger_b] = -2 (\partial_a \partial_b \log \beta),$$  

(II.6)

$$[D_a, D_b] = [D^\dagger_a, D^\dagger_b] = 0.$$  

(II.7)

Throughout the paper, we will use the convention that the derivatives acts on everything to the right, unless explicitly delimited by parenthesis as shown in (II.6). Let us define an analog to a superpotential in one-dimensional SUSY quantum mechanics

$$W_a = (\partial_a \log \beta) = \frac{(\partial_a \beta)}{\beta}.$$  

(II.8)

Since $D_a = -\partial_a + W_a$, $D^\dagger_a = \partial_a + W_a$, we define\(^2\)

$$D^2 \equiv D^\dagger_a D_a = -\partial_a^2 + W_a^2 + (\partial_a W_a) = -\partial_a^2 + \frac{(\partial_a^2 \beta)}{\beta},$$  

(II.9)

$$\bar{D}^2 \equiv D_a D^\dagger_a = -\partial_a^2 + W_a^2 - (\partial_a W_a) = -\partial_a^2 + \frac{(\partial_a^2 \beta^{-1})}{\beta^{-1}},$$  

(II.10)

Let $\phi_n$ and $\bar{\phi}_n$ be eigenfunctions of $D^2$ and $\bar{D}^2$, respectively.

$$D^2 \phi_n = m_n^2 \phi_n, \quad \bar{D}^2 \bar{\phi}_n = \bar{m}_n^2 \bar{\phi}_n.$$  

(II.11)

Here $n$ is symbolic index suitably labelling both discrete and continuum parts of the spectrum, including possible degenerate states. Note that $D^2$ and $\bar{D}^2$ are semi-positive definite

\(^{2}\) Here and in the following, we use a convention to sum over repeated indices unless stated otherwise.
operators, so that their eigenvalues are real and nonnegative. We normalize the eigenfunctions by

\[ \int d^{D-4}x \phi_m \phi_n = \delta_{mn}, \quad \int d^{D-4}x \bar{\phi}_m \bar{\phi}_n = \delta_{mn}, \]  

(II.12)

where \( \delta_{mn} \), again, symbolically represent both Kronecker’s delta for discrete modes and delta function for continuum modes. The mass dimension is \( [\phi_n] = [\bar{\phi}_n] = \frac{D-4}{2} \).

Clearly, \( D^2 \) has a zero eigenfunction \( \phi_0 \), with eigenvalue \( m_0 = 0 \), given as

\[ \phi_0 = N_0 \beta. \]  

(II.13)

It’s normalizability is ensured by square-integrability of \( \beta \). It will be proven that the zero eigenfunction \( \phi_0 \propto \beta \) is important to assure the universality of non-Abelian gauge charges in four-dimensional effective Lagrangian. Uniqueness of the normalizable zero eigenfunction can be easily shown at least for the case where \( \beta = \beta(r) \) depends only on radial coordinate \( r = \sqrt{x^a x^a} \). Let us first note that \( \phi_0 \) should be a function of \( r \) only because energy inevitably increases if \( \phi_0 \) depends on angular coordinates. Then let us rewrite Eq. (II.11) in terms of \( \varphi_0 = r^{\frac{D-5}{2}} \phi_0 \) as

\[ \left( -\frac{d^2}{dr^2} + V(r) \right) \varphi_0 = 0, \quad V = \frac{\partial^2 \beta}{\beta} + \frac{(D-7)(D-5)}{4r^2}. \]  

(II.14)

Square integrability condition is \( \int dr r^{D-5} \varphi_0^2 = \int dr \varphi_0^2 < \infty \). Since this is nothing but a problem of one-dimensional quantum mechanics, all bound states are nondegenerate. Hence, the normalizable zero eigenfunction (III.13) is unique at least in rotationally invariant backgrounds.

We also see that a solution to the equation \( \bar{D}^2 \bar{\phi}_0 = 0 \) is given as

\[ \bar{\phi}_0 \propto \beta^{-1}. \]  

(II.15)

However, this is not square-integrable and, hence, not a part of a physical spectrum.

Fundamental mass scales involved in the Schrödinger problem are

\[ \Omega_a = \lim_{|y| \to \infty} |\partial_a \log \beta|. \]  

(II.16)

We assume that there is a mass gap of order \( \Omega_a \) between the zero mode \( m_0 = 0 \) and massive modes \( m_n \) \((n \neq 0)\).

Our primary interest is to work out physical spectra in the low-energy four-dimensional physics. Therefore we will consider the action up to the quadratic order in fields. As a
consequence, non-Abelian gauge fields and Abelian gauge fields can be treated on the same footing. For ease of notation, we will concentrate on the Abelian case in what follows.

In order to find physical degrees of freedom and mass spectra, we have to find a suitable gauge fixing condition. Inspired by the usual $R_\xi$ gauge, we choose the gauge-fixing Lagrangian as

$$L_{gf} = -\frac{2\beta^2}{\xi} f^2, \quad f = \partial^\mu A_\mu + \xi \frac{1}{\beta^2} \partial^\mu \beta^2 A_a,$$  \hspace{1cm} (II.17)

where $\xi$ is an arbitrary constant. Note that if $\beta$ is a constant and take $\xi = 1$, this is nothing but the gauge fixing condition of the covariant gauge $f = \partial^M A_M$ in $D$ dimensions. On the other hand, if we replace $\beta^{-2} \partial^\mu \beta^2 A_a$ by $m_h h$ as product of “Higgs” mass $m_h$ and a “Nambu-Goldstone field” $h$, it is almost identical to the gauge fixing functional used in the familiar $R_\xi$ gauge for the Higgs mechanism in four dimensions. The reason for the choice of $f$ in Eq. (II.17) is to eliminate mixing between the four-dimensional gauge fields $A_\mu$ and extra-dimensional gauge fields $A_a$.

Even though our analysis is essentially Abelian, it proves useful to investigate spectra of ghost fields as well, in order to clearly identify unphysical degrees of freedom. By varying the gauge fixing functional $f$ in Eq. (II.17), we find the ghost action as

$$L_{gh} = -\bar{C} \left( \partial^2 + \xi \frac{1}{\beta^2} \partial^\mu \beta^2 \partial_a \right) C,$$ \hspace{1cm} (II.18)

with the mass dimensions $[C] = 1$ and $[\bar{C}] = D - 3$.

For further convenience, let us switch to the canonically normalized fields

$$A_M = \frac{A_M}{2\beta}, \quad C = \frac{c}{\beta}, \quad \bar{C} = \beta \bar{c}.$$ \hspace{1cm} (II.19)

Mass dimensions of these fields are given as $[A_M] = [c] = [\bar{c}] = \frac{D-2}{2}$. In terms of the new fields, after performing integration by parts, the Lagrangians (II.1) can be expressed as

$$L_A = \frac{1}{2} A_\mu \left( \eta^{\mu\nu} \partial^2 - \partial^\mu \partial^\nu + \eta^{\mu\nu} D^2 \right) A_\nu$$
$$- \frac{1}{2} A_a \left( \delta_{ab} D^2 - D_b^a D_a + \delta_{ab} \partial^2 \right) A_a - (D_a^i A_a) \partial^\mu A_\mu,$$ \hspace{1cm} (II.20)

$$L_{gf} = \frac{1}{2\xi} A_\mu \partial^\mu \partial^\nu A_\nu + (D_a^i A_a) \partial^\mu A_\mu - \frac{1}{2} \xi A_a D_a D_b A_b,$$ \hspace{1cm} (II.21)

$$L_{gh} = -\bar{c} \left( \partial^2 + \xi D^2 \right) c,$$ \hspace{1cm} (II.22)
with $\partial^2 = \partial_\mu \partial^\mu$. Collecting all pieces, we find our Lagrangian is of the form

$$\mathcal{L}_\xi = \frac{1}{2} A_\mu \left[ \eta^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} \partial^2 \right] A_\nu$$

$$- \frac{1}{2} A_a \left[ \delta_{ab} D^2 - \left( D_b^a D_a - \xi D_a D_b^a \right) + \delta_{ab} \partial^2 \right] A_b$$

$$- \bar{c} \left( \partial^2 + \xi D^2 \right) c.$$ \hspace{1cm} (II.23)

Interestingly, the four-dimensional part and the extra-dimensional part have similar structure under the exchange of $\partial_\mu$ and $D_a$. The gauge fixing parameter $\xi$ serves as a mark to distinguish physical and unphysical degrees of freedom.

**B. Four-dimensional components of gauge fields $A_\mu$**

Firstly, let us investigate the first line of Eq. (II.23). It is quite similar to the Lagrangian of the gauge theory in four dimensions. The differences are that $A_\mu$ is function of not only $x = \{x^\mu\}$ but also $y = \{x^a\}$, and $D^2$ is not a mass but the differential operator in terms of $\partial_a$.

In order to get the physical spectrum, let us expand $A_\mu$ in terms of the eigenfunctions of $D^2$ defined in Eq. (II.11) as

$$A_\mu = A_\mu^{(0)}(x) \phi_0(y) + \sum_{n \neq 0} A_\mu^{(n)}(x) \phi_n(y).$$ \hspace{1cm} (II.24)

Since $[A_\mu] = D^2 / 2$ and $[\phi_n] = D^2 / 2$, this expansion ensures for the four-dimensional gauge fields $A_\mu^{(n)}(x)$ to have correct mass dimension $[A_\mu^{(n)}] = 1$. Plugging this into the first line of Eq. (II.23) and integrate it over the extra-dimensions, we get

$$\int d^{D-4}x \mathcal{L}_\xi \big|_{1\text{st}} = \frac{1}{2} A_\mu^{(0)} \left[ \eta^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A_\nu^{(0)}$$

$$+ \sum_{n \neq 0} \frac{1}{2} A_\mu^{(n)} \left[ \eta^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} m_n^2 \right] A_\nu^{(n)}.$$ \hspace{1cm} (II.25)

Note that in terms of the original field $A_\mu$ the above expansion is rewritten as

$$A_\mu = \frac{N_0}{2} A_\mu^{(0)}(x) + \sum_{n \neq 0} A_\mu^{(n)}(x) \phi_n(y) \frac{\phi_n(y)}{2\beta}.$$ \hspace{1cm} (II.26)

Remarkably, the zero mode wave function is constant in $y$. This ensures the universality of non-Abelian gauge charges of matter fields, since overlap integral of the wave functions of gauge field and matter fields do not depend on the details of the localization mechanism \cite{49, 50, 55}.  

C. Extra-dimensional components of gauge fields $A_a$

Let us next investigate the physical spectrum of extra-dimensional gauge fields $A_a$ from the second line of Eq. (II.23). From the viewpoint of four dimensions, they are scalar fields.

We first consider the extra-dimensional divergence $K = D^\dagger_a A_a$. By applying $D^\dagger_a$ on the field equation for $A_a$, we obtain the field equation for $K$ as $(\partial^2 + \xi D^2)K = 0$. Hence we expand $K$ in terms of the eigenfunctions of $D^2$ as

$$K(x, y) = -\Omega \phi_0(y) K^{(0)}(x) - \sum_{n \neq 0} m_n \phi_n(y) K^{(n)}(x). \quad \text{(II.27)}$$

Note that the mass dimensions are $[K] = D/2$ and $[K^{(n)}] = 1$ due to the intentional insertion of $\Omega$ and $m_n$. In the following, however, we will demonstrate the absence of zero mode $K^{(0)}(x) = 0$. Let us suppose that there is a zero mode

$$D^\dagger_a A_a = -\Omega \phi_0(y) K^{(0)}(x) \equiv k(x) \beta(y). \quad \text{(II.28)}$$

Multiplying this by $\beta$ we obtain

$$\partial_a (\beta A_a) = k \beta^2. \quad \text{(II.29)}$$

Now we integrate this over extra-dimensions. The right-hand side is non-zero due to our square-integrability condition on $\beta$. However, for regular $A_a$ the left-hand side is

$$\int d^{D-4}x \partial_a (\beta A_a) = \int d^{D-3} S_a (\beta A_a) = 0, \quad \text{(II.30)}$$

since $\beta$ vanishes at the boundary. We arrive at the contradiction, which shows the absence of zero mode: $K^{(0)}(x) = 0$.

Absence of zero mode implies that $D^{-2}$ is well-defined on $K$. Hence we can define a projection operator $P_{ab}$ acting on $A_a$ to obtain the divergence part $A^d_a$

$$A^d_a = P_{ab} A_b = D_a D^{-2} K. \quad \text{(II.31)}$$

$$P_{ab} = D_a D^{-2} D^\dagger_b. \quad \text{(II.32)}$$

The operator $P_{ab}$ enjoys the properties of a projection operator:

$$P_{ab} P_{bc} = P_{ac}, \quad \left(\delta_{ab} D^2 - D^\dagger_b D_a\right) P_{bc} = 0, \quad \left(\delta_{ab} - P_{ab}\right) D_b D^\dagger_c = 0. \quad \text{(II.33)}$$
The remaining part is defined as divergence-free part: 

\[ A^\text{df}_a = A^d_a + A^\text{df}_a \]

\[ A^\text{df}_a = (\delta_{ab} - P_{ab}) A_b, \quad (\text{II.34}) \]

These parts satisfy

\[ D^\dagger_a A^\text{df}_a = K, \quad D^\dagger_a A^d_a = 0. \quad (\text{II.35}) \]

By using the above identities, we can rewrite the second line of Eq. (II.23) as

\[ L_\xi |_{2\text{nd}} = -\frac{1}{2} A^\text{df}_a \left( \delta_{ab} \partial^2 + \delta_{ab} D^2 \right) A^\dagger_b A^\text{df}_b - \frac{1}{2} A^d_a \left( \delta_{ab} \partial^2 + \xi D_a D^\dagger_b \right) A^d_b, \quad (\text{II.36}) \]

Now we see that the divergence-free part \( A^\text{df}_a \) does not contain the gauge-fixing parameter \( \xi \), whereas the divergence part \( A^d_a \) depends on \( \xi \), rendering it an unphysical degree of freedom.

We can rewrite the divergence part of the Lagrangian to make the mass spectra of \( A^d_a \) explicit. Using Eq. (II.31) we obtain from the second term of Eq. (II.36) and the expansion (II.27) without the \( n = 0 \) part

\[ \int d^{D-4} x \, L_\xi |_{2\text{nd}} = -\frac{1}{2} \int d^{D-4} x \, \frac{1}{2} K D^{-2} \left( \partial^2 + \xi D^2 \right) K \]

\[ = -\frac{1}{2} \sum_{n \neq 0} K^{(n)} \left( \partial^2 + \xi m^2_n \right) K^{(n)}. \quad (\text{II.37}) \]

Absence of the massless mode \( (n = 0) \) is physically important in a low energy effective theory, as we will discuss in Sec. III A.

In contrast to the divergence part, the divergence-free part makes sense only for \( D \geq 6 \), since it does not exist in \( D = 5 \). Let us rewrite the first part of Eq. (II.36) as

\[ L_\xi |_{2\text{nd}} = -2 A^\text{df}_a \left( \delta_{ab} \partial^2 + H_{ab} \right) A^\text{df}_b, \quad (\text{II.38}) \]

where we have defined an operator

\[ H_{ab} = \delta_{ab} D^2 - D^\dagger_b D_a. \quad (\text{II.39}) \]

This operator is \( N \times N \) matrix with the rank \( N - 1 \), where we denote \( N \equiv D - 4 \). For two extra-dimensions \( N = 2 \), we have

\[ H = D^\dagger \mathcal{D}, \quad (\text{II.40}) \]

with \( \mathcal{D} = (D_5, -D_4) \). Then we can define a “superpartner" \( \tilde{H} \) as

\[ \tilde{H} = \mathcal{D} D^\dagger = D_5 D^\dagger_5 + D_4 D^\dagger_4. \quad (\text{II.41}) \]
It is easy to verify that $H$ and $\tilde{H}$ have identical spectra except possible zero modes and the construction can be generalized to higher dimensions, as described in App. [A]

Let us parametrize the eigenvectors of $\tilde{H}$ as

$$\vec{A}^{df}(x,y) = \frac{1}{2} \begin{pmatrix} D_5^\dagger \\ -D_4^\dagger \end{pmatrix} (\Omega^{-1}\bar{\phi}_0(y)\tilde{K}^{(0)}(x) + \tilde{D}^{-2}\bar{K}(x,y)) , \quad (II.42)$$

where $\bar{K}(x,y)$ is orthogonal to zero modes $\{\bar{\phi}_0\}$ of $\tilde{D}^2$. In this way, Eq. (II.38) becomes

$$L_{\xi}^{df, D=6} = -\frac{1}{2\Omega^2} \bar{\phi}_0 \tilde{D}^2 (\partial^2 + \tilde{D}^2) \bar{\phi}_0 - \frac{1}{2} \tilde{K} \tilde{D}^{-2} (\partial^2 + \tilde{D}^2) \tilde{K}$$
$$= -\frac{1}{2} \tilde{K} \tilde{D}^{-2} (\partial^2 + \tilde{D}^2) \tilde{K} . \quad (II.43)$$

It is important to realize that the zero modes of $\tilde{D}^2$ always disappear from the physical spectrum. Now, it is natural to expand $\tilde{K}$ in terms of the eigenfunctions of $\tilde{D}^2$ as

$$\tilde{K}(x,y) = \sum_{n\neq 0} \bar{m}_n \bar{\phi}_n(y) \tilde{K}^{(n)}(x), \quad (II.44)$$

with the mass dimensions $[\tilde{K}] = D/2$ and $[\tilde{K}^{(n)}] = 1$. Plugging this into Eq. (II.43) and integrating it over the extra dimensions, we get

$$\int d^2x \ L_{\xi}^{df, D=6} = -\frac{1}{2} \sum_{n\neq 0} \tilde{K}^{(n)} (\partial^2 + \bar{m}_n^2) \tilde{K}^{(n)} . \quad (II.45)$$

This gives us the spectrum of the divergence-free part for two extra-dimensions.

Contrary to $D = 6$, it is not easy to diagonalize $H$ for $D \geq 7$. We will study $D = 7$ in Sec. III C for a spherically symmetric background. We leave analysis of generic $D \geq 7$ as a future problem.

Absence of zero modes in the extra-dimensional components, which we have explicitly shown in $D = 5$ and 6, sounds physically natural in the following sense. The non-trivial $\beta(y)$ implies that the gauge coupling depends on the extra-dimensional coordinate. This seemingly contradicts $D$-dimensional gauge symmetry, and only the four-dimensional gauge symmetry holds. While the four-dimensional gauge symmetry ensures the existence of the massless four-dimensional gauge field as in Eq. (II.25), there is no symmetric reasons explaining zero modes in the extra-dimensional components. This physical intuition makes the absence of zero modes plausible for $A_a$ in generic $D$, although a rigorous proof is lacking. We will partially verify this for $D = 7$ in Sec. III C.
Let us mention, however, that for separable potentials, say \( \beta = \beta_4(x^4)\beta_5(x^5) \cdots \beta_{D-1}(x^{D-1}) \) corresponding to domain wall junctions, we can understand the spectrum completely in a recursive fashion, see Appendix [B] for details. We emphasize that there are no physical zero modes in the divergence-free components in the separable case in generic \( D \) dimensions.

D. Ghosts \( c \) and \( \bar{c} \)

Finally, we are left with the third term of Eq. (II.23) for the ghosts. As before, we expand \( c \) and \( \bar{c} \) in terms of the eigenfunctions \( \phi_n \) of the \( D^2 \) operator as

\[
c(x, y) = \sum_n \phi_n(y) c^{(n)}(x),
\]

and similar for \( \bar{c} \). Plugging these into the third term of Eq. (II.23) and integrate it over the extra-dimensions, we get

\[
\int d^{D-4} L_{\xi}|_{3rd} = -\sum_n \bar{c}^{(n)} \left( \partial^2 + \xi m_n^2 \right) c^{(n)}.
\]

E. Summary of KK decomposition

Let us summarize the four-dimensional effective theory by gathering all the pieces obtained above. The most relevant part for the low energy physics is massless fields. We found them in the four-dimensional gauge fields \( A^{(0)}_\mu \) and the ghosts \( c^{(0)} \) and \( \bar{c}^{(0)} \). Their effective Lagrangian is given by

\[
L_{\text{eff}}^{(n=0)} = \frac{1}{2} A^{(0)}_\mu \left[ \eta^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A^{(0)}_\nu - \bar{c}^{(0)} \partial^2 c^{(0)}.
\]

This is nothing but the ordinary four-dimensional Lagrangian for massless gauge fields in the covariant gauge. There is no other massless fields in our simple model given in Eq. (II.1), which we explicitly showed for \( D = 5, 6 \) above, will show for \( D = 7 \) in Sec. III C and expect for \( D \geq 8 \). This is a virtue of our model which is in sharp contrast to other extra-dimensional models with compact extra-dimensions where \( A_a \) often supplies extra massless scalar fields in the low energy physics. If one wants to avoid such massless scalars, one needs an additional assumption, for example, the \( Z_2 \) parity for \( D = 5 \) model with \( S^1/Z_2 \) extra-dimension.

Next, we describe massive modes. We first collect the relevant pieces to describe the four-dimensional massive gauge fields, \( A^{(n)}_\mu \), the divergence part of extra-dimensional gauge
field $K^{(n)}$ defined in Eq. (II.27), and the ghosts $c^{(n)}$, $\bar{c}^{(n)}$. The effective Lagrangian takes the form

$$
L_{\text{eff};1}^{(n\neq 0)} = \frac{1}{2} A^{(n)}_\mu \left[ \eta^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} m_n^2 \right] A^{(n)}_\nu - \frac{1}{2} K^{(n)} \left( \partial^2 + \xi m_n^2 \right) K^{(n)} - \bar{c}^{(n)} \left( \partial^2 + \xi m_n^2 \right) c^{(n)}.
$$

(II.49)

We find that the four-dimensional divergence part $\partial^\nu A^{(n)}_\nu$, the extra-dimensional divergence part $K^{(n)}$, and the ghosts have the same mass square $\xi m_n^2$. The identical mass spectra assures that the contributions of divergence parts are cancelled by those of ghosts. We note a similarity of the Lagrangian (II.49) to that of four-dimensional $R_\xi$ gauge if we replace $K^{(n)}$ by the Nambu-Goldstone field $^3$. Our gauge-fixing condition (II.17) is designed to exhibit this similarity explicitly. The physical degrees of freedom are the massive gauge field $A^{(n)\text{df}}_\mu$ with mass $m_n$.

For the divergence-free part of extra-dimensional components of gauge fields $A^{\text{df}}_a$, we can explicitly write down effective Lagrangian for $D \leq 6$. For $D = 5$, the divergence free part does not exist. For $D = 6$, we have one scalar field $\tilde{K}$ whose KK modes defined in Eq. (II.42) obey the effective Lagrangian (II.45). With the $\xi$-independent masses, $\tilde{K}^{(n)} (n \neq 0)$ are physical degrees of freedom. For the higher dimensions $D \geq 7$, we anticipate $D - 5$ KK towers of physical scalar fields. We can construct the full spectrum recursively for fully separable $\beta$, see App. B.

III. EXAMPLES

A. Domain walls in $D = 5$

1. A simple gauge kinetic function

In this subsection, we investigate localized modes of gauge fields on a domain wall in $D = 5$. Although, generic results of the previous section are all valid in any dimension, it is worthwhile to illustrate the analysis in $D = 5$ explicitly, as it is the simplest case.

---

3 Similar result was recently reported for $D = 5$ [56].
We begin with a classical Lagrangian
\[ \mathcal{L} = -\alpha^2 \varphi^2 F_{MN} F^{MN} + \mathcal{L}_{\text{kink}}, \quad \text{(III.1)} \]
\[ \mathcal{L}_{\text{DW}} = \partial_M \sigma \partial^M \sigma + \partial_M \varphi \partial^M \varphi - \Omega^2 \varphi^2 - \lambda^2 \left( \sigma^2 + \varphi^2 - v^2 \right)^2. \quad \text{(III.2)} \]

Here, \( \varphi \) and \( \sigma \) are real scalar fields. The scalar field \( \sigma \) is responsible for having a domain wall while \( \varphi \) localizes gauge fields on the domain wall. There are two discrete vacua \( (\sigma, \varphi) = (\pm v, 0) \). If we assume \( \lambda v > \Omega \), the domain wall interpolating between them reads:
\[ \sigma = v \tanh \Omega y, \quad \varphi = \pm \bar{v} \sech \Omega y, \quad \bar{v} \equiv \sqrt{v^2 - \frac{\Omega^2}{\lambda^2}}. \quad \text{(III.3)} \]

With this solution as the background configuration, the relevant part of Lagrangian for the gauge field is given by
\[ \mathcal{L}_A = -\beta(y)^2 F_{MN} F^{MN}, \quad \beta(y) = a \bar{v} \sech \Omega y. \quad \text{(III.4)} \]

The differential operators associated this background solution are
\[ D^2 = D_y^\dagger D_y = -\partial^2 + \Omega^2 \left( 1 - 2 \sech^2 \Omega y \right), \quad \text{(III.5)} \]
\[ \bar{D}^2 = D_y D_y^\dagger = -\partial_y^2 + \Omega^2. \quad \text{(III.6)} \]

Operators \( D^2 \) and \( \bar{D}^2 \) can be regarded as components of the Hamiltonians of supersymmetric quantum mechanics in one dimension. Therefore, the energy eigenvalues are identical except for zero eigenvalue. Since \( \bar{D}^2 \) has no zero modes, the general solution corresponding to the zero eigenvalue
\[ \bar{\phi}_0 = Ae^{\Omega y} + Be^{-\Omega y}, \quad m_0 = 0 \quad \text{(III.7)} \]

is not normalizable. The eigenfunctions of physical states in the continuum are
\[ \tilde{\phi}(y; k) = \frac{e^{iky}}{\sqrt{2\pi \Omega}}, \quad m(k) = \sqrt{k^2 + \Omega^2}, \quad \text{(III.8)} \]
where the normalization is chosen as
\[ \int_{-\infty}^{\infty} dy \tilde{\phi}(y; k)^* \tilde{\phi}(y; k') = \frac{1}{\Omega} \delta(k - k'). \quad \text{(III.9)} \]

On the other hand, general solution for zero eigenvalue of \( D^2 \) is given by
\[ \phi_0(y) = \sqrt{\frac{\Omega}{2}} \sech \Omega y + B \left( \sinh \Omega y + y \Omega \sech \Omega y \right), \quad m_0 = 0. \quad \text{(III.10)} \]
Since the second term diverges at $y = \pm \infty$, we take $B = 0$. Fig. 1 shows the Schrödinger potential given in Eq. (III.5) and the zero mode $\phi_0$. The physical continuum of $D^2$ eigenstates are obtained by supersymmetry between $D^2$ and $\bar{D}^2$ which relates

$$D_y^i \tilde{\phi}(y; k) = -m(k)\phi(y; k), \quad D_y \phi(y; k) = -m(k)\bar{\phi}(y; k).$$

(III.11)

From this, we have

$$\phi(y; k) = -\frac{1}{m(k)}D_y^i \tilde{\phi}(y; k) = -i m(k)^{-1} \frac{e^{i k y}}{m(k)} (k + i \Omega \tanh \Omega y), \quad m(k) = \sqrt{k^2 + \Omega^2}. \quad \text{(III.12)}$$

The threshold is $\phi(y; k = 0) = \tanh \Omega y$. The normalization reads

$$\int_{-\infty}^{\infty} dy \phi(y; k)^* \phi(y; k') = \int_{-\infty}^{\infty} dy \frac{1}{m(k)^2} \tilde{\phi}(y; k)^* \bar{D}^2 \phi(y; k') = \frac{1}{\Omega} \delta(k - k'). \quad \text{(III.13)}$$

The quantum Lagrangian given in Eq. (II.23) takes the form

$$\mathcal{L}_\xi = \frac{1}{2} A_\mu \left[ \eta^{\mu \nu} \partial^2 - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu \nu} D^2 \right] A_\nu$$

$$- \frac{1}{2} A_y \left( \partial^2 - \xi \bar{D}^2 \right) A_y - \bar{c} \left( \partial^2 + \xi D^2 \right) c. \quad \text{(III.14)}$$

We expand $A_\nu$ and $c$ in terms of $\phi_0(y)$ and $\phi(y; k)$ as

$$A_\nu(x, y) = \phi_0(y) A_\nu^{(0)}(x) + \int_{-\infty}^{\infty} dk A_\nu(x; k) \phi(y, k), \quad \text{(III.15)}$$

$$c(x, y) = \phi_0(y) c^{(0)}(x) + \int_{-\infty}^{\infty} dk c(x; k) \phi(y, k), \quad \text{(III.16)}$$
and similarly for $\bar{c}$. Here, $A_\nu(x; k)^* = A_\nu(x; -k)$ is imposed. On the other hand, we expand $A_y$ in terms of eigenfunctions of $\bar{D}^2$ operator:

$$A_y(x, y) = \int_{-\infty}^{\infty} dk A_y(x; k) \tilde{\phi}(y, k), \quad \text{(III.17)}$$

with $A_y(x; k)^* = A_y(x; -k)$. Therefore, the absence of massless modes in $A_y$ is a direct consequence of absence of physical zero modes in $\bar{D}^2$. For illustration, let us compare the simple expansion here and one based on the generic arguments around Eq. (II.31). We first write $A_y = D_y D^{-2} K$ and expand $K$ in terms of $\phi(y; k)$. The basis for the expansions are different from Eq. (III.17), nevertheless, we get the same four-dimensional Lagrangian by using (III.11). We can express divergence of Eq. (III.17) as

$$D^\dagger_y A_y(x, y) = -\int_{-\infty}^{\infty} dk m(k) A_y(x; k) \phi(y; k). \quad \text{(III.18)}$$

This is nothing but the counterpart of (II.27).

Plugging the expansions above into Eq. (III.14) and integrating it over $y$, we get

$$L_{\text{eff}}^{\xi} = L^{(0)}_{\xi} + \int_{-\infty}^{\infty} \frac{dk}{\Omega} L^{(k)}_{\xi}, \quad \text{(III.19)}$$

where we have the massless part

$$L^{(0)}_{\xi} = \frac{1}{2} A_{\mu}^{(0)} \left[ g^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A_{\nu}^{(0)} - \bar{c}^{(0)} \partial^2 c^{(0)}, \quad \text{(III.20)}$$

and the massive parts

$$L^{(k)}_{\xi} = \frac{1}{2} A_{\mu}(x; k)^* \left[ g^{\mu\nu} \partial^2 - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu + \eta^{\mu\nu} m(k)^2 \right] A_{\nu}(x; k)$$

$$- \frac{1}{2} A_y(x; k)^* \left( \partial^2 + \xi m(k)^2 \right) A_y(x; k) - \bar{c}(x; k) \left( \partial^2 + \xi m(k)^2 \right) c(x; k). \quad \text{(III.21)}$$

Thus, we conclude that the low energy effective theory on the domain wall in $D = 5$ includes one massless gauge field $A_{\mu}^{(0)}(x)$ and the continuum KK towers of massive vector fields with the mass gap $\Omega$. We emphasise that the absence of other massless modes is not an assumption but a logical consequence.

It would be useful to rewrite the above effective Lagrangians into the standard form. For the massless fields, our model in Eq. (III.20) can be expressed as

$$L^{(0)}_{\xi} = -\frac{1}{4} F_{\mu\nu}^{(0)} F^{(0)\mu\nu} - \frac{1}{2\xi} f^{(0)2} - \bar{c}^{(0)} \partial^2 c^{(0)}, \quad \text{(III.22)}$$

with

$$f^{(0)} = \partial^\mu A_{\mu}^{(0)}. \quad \text{(III.23)}$$
Similarly, (III.21) can be expressed as

\[
\mathcal{L}_\xi(k) = -\frac{1}{4} F_{\mu\nu}(k)^* F^{\mu\nu}(k) + \frac{1}{2} |\partial_\mu A_y(k) - m(k) A_\mu(k)|^2 \\
- \frac{1}{2\xi} |f(k)|^2 - \bar{c}(k) \left( \partial^2 + \xi m(k)^2 \right) c(k),
\]

(III.24)

where we abbreviated \( A_\mu(k) \equiv A_\mu(x; k) \). The gauge fixing function is given by

\[
f(k) = \partial^\mu A_\mu(k) + \xi m(k) A_y(k).
\]

(III.25)

It is now quite clear that \( A_y(k) \) is a St"uckelberg-like field which pretends to be a Nambu-Goldstone field absorbed by the gauge field via the Higgs mechanism. The effective Lagrangians in (III.22) and (III.24) are the result of dynamical compactification of the infinitely large fifth dimension by the domain wall.

Let us now compare this with the model, where the extra-dimension is compactified by hand to a circle \( S^1 \) of the radius \( R \). The most important difference is in the massless fields. Domain-wall compactification produces only massless four-dimensional gauge fields, whereas the \( S^1 \) model has in addition a massless scalar field originating from \( A_y \). One cannot avoid this scalar because all modes are normalizable when the extra-dimension is compact. In order to suppress it, one need additional instruments such as \( Z_2 \) orbifolding and parity conditions, and so on. In contrast, the massive modes of both models are quite similar. In both models there exists a mass gap, the inverse width \( \Omega \) of the domain wall and the inverse radius \( 1/R \), respectively. Above the mass gap the domain wall model has a continuum spectrum, while in \( S^1 \) model there is an equidistant discrete tower of massive modes, which is the compact version of the continuum. A more important difference is that in our model the massless

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**FIG. 2.** KK mass spectra of physical fields for (a) our model and (b) \( S^1 \) extra-dimension.
gauge fields are localized on the domain wall, and all the other massive fields are bulk fields. On the other hand, all fields (both massless and massive) are spread uniformly across the entire extra-dimension in the $S^1$ model. Fig. 2 summarizes the differences.

Before closing this section, let us make a comment on the gauge transformation. Note that the gauge transformation for the original gauge field, $A_M \rightarrow A'_M = A_M + \partial_M \Lambda$ is translated for the canonically normalized fields as $A'_M = A_M + 2\beta \partial_M \Lambda$. Expanding the gauge transformation parameter as

$$\Lambda = \frac{\phi_0(y)}{2\beta} \Lambda^{(0)}(x) + \int_{-\infty}^{\infty} dk \frac{1}{2\beta} \phi(y;k)\Lambda(x;k),$$

(III.26)

with $\Lambda(x;k)^* = \Lambda(x;-k)$, we find

$$A^{(0)}_\mu \rightarrow A^{(0)'}_\mu = A^{(0)}_\mu + \partial_\mu \Lambda^{(0)}$$

(III.27)

$$A_\mu(k) \rightarrow A_\mu(k)' = A_\mu(k) + \partial_\mu \Lambda(k),$$

(III.28)

$$A^{(0)}_y \rightarrow A^{(0)'}_y = A^{(0)}_y$$

(III.29)

$$A_y(k) \rightarrow A_y(k)' = A_y(k) + m(k)\Lambda(k).$$

(III.30)

For example, the gauge transformation of $A_y(x,y)$ can be obtained as follows

$$A'_y = \bar{\phi}_0 A^{(0)}_y + \int_{-\infty}^{\infty} dk A_y(x;k)\bar{\phi}(y;k)$$

$$+ 2\beta \partial_y \left[ \frac{\phi_0(y)}{2\beta} \Lambda^{(0)}(x) + \int_{-\infty}^{\infty} dk \frac{1}{2\beta} \phi(y;k)\Lambda(x;k) \right]$$

$$= \bar{\phi}_0 A^{(0)}_y - \int_{-\infty}^{\infty} dk \frac{1}{m(k)} D_y \phi(y;k) (A_y(k) + m(k)\Lambda(x;k)),$$

(III.31)

where we have kept $\bar{\phi}_0$, although it is unphysical, and we have used the fact $\phi_0 = \beta$, $D_y = -\beta \partial_y \beta^{-1}$ and (III.11).

Note that the gauge transformation law (III.30) correctly derives the ghost Lagrangian in Eq. (III.24) as variations of the gauge fixing function given in Eq. (III.25).

Remark: All the results obtained in this subsection are consistent with our previous works on domain walls in flat 5 dimensions [44–50]. In particular, the absence of $A^{(0)}_y$ is one of the important physical results. However, the previous analysis in [44–50] were carried out in the axial gauge $A_y = 0$. Although the axial gauge is useful at least in classical analysis, $A^{(0)}_y$ is not transformed by any gauge transformation as explicitly shown in Eq. (III.29). Therefore, the axial gauge $A_y = 0$ cannot exclude $A^{(0)}_y$. Therefore, the analysis in this work justifies the absence of the zero mode $A^{(0)}_y$ in our previous works.
2. More general gauge kinetic functions

In the previous section, the only localized field is the massless four-dimensional gauge fields. All the massive modes are continuum bulk modes. When we want several massive bound states, it can be realized, for example, as follows. We do not change the domain wall Lagrangian $\mathcal{L}_{DW}$ in Eq. (III.2). Instead, we modify $\beta$ as

$$\mathcal{L}_A = - (\beta^{(n)})^2 F_{MN} F^{MN}, \quad \beta^{(n)}(\varphi) \equiv a \varphi^n. \quad \text{(III.32)}$$

All the formulae given in Sec. III A 1 remain the same if we replace $\beta$ by $\beta^{(n)}$. The Hamiltonians are given by

$$ (D^{(n)})^2 = - \partial_y^2 + V^{(n)}, \quad V^{(n)} = n \Omega^2 \left( -1 + (n+1) \tanh^2 \Omega y \right), \quad \text{(III.33)}$$

$$ (\bar{D}^{(n)})^2 = - \partial_y^2 + \bar{V}^{(n)}, \quad \bar{V}^{(n)} = n \Omega^2 \left( 1 + (n-1) \tanh^2 \Omega y \right). \quad \text{(III.34)}$$

The following descent relation holds

$$ \bar{V}^{(n)} = V^{(n-1)} + (2n-1)\Omega^2. \quad \text{(III.35)}$$

For any $n$, the physical modes reside only in the four-dimensional part $A_\mu$. As can be seen from Eq. (III.14), the physical mass spectrum is determined by the $(D^{(n)})^2$ operator. The zero mode is immediately found as $\beta^{(n)}$. On the other hand, the descent relation (III.35) implies that the eigenfunction of $\bar{V}^{(n)}$ is in one-to-one correspondence with that of $V^{(n-1)}$, whose eigenvalue is shifted by $(2n-1)\Omega^2$. We can find excited modes of $(D^{(n)})^2$ from eigenfunctions of the superpartners $(\bar{D}^{(n)})^2$, which share the same non-zero eigenvalues, by applying $D^{(n)\dagger}$. Thus we can recursively construct all the discrete modes of $(D^{(n)})^2$ starting from the zero mode. For example, the first excited mode of $\bar{V}^{(n)}$ is $\beta^{(n-1)}$ which is the zero mode of $V^{(n-1)}$ and the mass squared is $(2n-1)\Omega^2$. The first excited state of $V^{(n)}$ can be obtained by multiplying $D^{(n)\dagger}$ on $\beta^{(n-1)}$.

To illustrate the recursive procedure, let us consider $n = 2$ with $\beta^{(2)} = a (\bar{v} \sech \Omega y)^2$. We find two bound states

$$ \phi_0^{(2)} \propto \beta^{(2)} \propto \sech^2 \Omega y, \quad m_0^2 = 0, \quad \text{(III.36)}$$

$$ \phi_1^{(2)} \propto D^{(2)\dagger} \phi_0^{(1)} \propto \sech^2 \Omega y \sinh \Omega y, \quad m_1^2 = 3\Omega^2, \quad \text{(III.37)}$$
with $\phi_0^{(1)} \propto \beta^{(1)}$. The continuum bulk modes follow and their mass squares are given by $m(k)^2 = k^2 + \Omega^2 + 3\Omega^2 = k^2 + 4\Omega^2$. Similarly we can understand $n = 3$:

$$
\phi_0^{(3)} \propto \beta^{(3)} \propto \text{sech}^3 \Omega y, \quad m_0^2 = 0, \quad (\text{III}.38)
$$

$$
\phi_1^{(3)} \propto D^{(3)\dagger} \phi_0^{(2)} \propto \text{sech}^3 \Omega y \sinh \Omega y, \quad m_1^2 = 5\Omega^2, \quad (\text{III}.39)
$$

$$
\phi_2^{(3)} \propto D^{(3)\dagger} \phi_1^{(2)} \propto \text{sech}^3 \Omega y \left(-1 + 4 \sinh^2 \Omega y\right), \quad m_2^2 = 8\Omega^2. \quad (\text{III}.40)
$$

The threshold mass squared for the continuum modes is $\Omega^2 + 3\Omega^2 + 5\Omega^2 = 9\Omega^2$. The Fig. 3 shows the bound states for $n = 1, 2, 3$ cases. The spectrum for generic $n$ is straightforwardly obtained. Having the analytic solutions for the bound KK modes is useful for model building\footnote{We thank to Nobuchika Okada for this point. See also recent paper [56, 57].}

**B. Vortex in $D = 6$**

Compared to the wealth of models in five non-compact dimensions with domain walls, there has been a very few six-dimensional models, where localization of massless gauge fields on a topological soliton is realized. To the best of our knowledge, we believe that the model presented in this section is the first successful model in flat spacetime (without gravity), where this is achieved.
1. ANO vortex string in $D = 6$

According to the general strategy of Sec. II, the field dependent kinetic term (II.1) with nontrivial $\beta$ in any dimensions generates massless gauge fields in four-dimensional low-energy effective theory on a topological soliton. Here, we will give a concrete model in flat six dimensions, where an Abrikosov-Nielsen-Olsen (ANO) vortex is used to localize massless gauge fields.

We consider the following $U(1) \times \tilde{U}(1)$ model

$$
\mathcal{L} = -a^2 \varphi^2 \mathcal{F}_{MN} \mathcal{F}^{MN} + \mathcal{L}_{\text{vortex}},
$$

$$
\mathcal{L}_{\text{vortex}} = -\frac{1}{4e^2} \tilde{F}_{MN} \tilde{F}^{MN} + |D_M \sigma|^2 + (\partial_M \varphi)^2 - V,
$$

$$
V = \frac{\lambda_1}{4} (|\sigma|^2 - v_1^2)^2 + \frac{\lambda_2}{4} (\varphi^2 - v_2^2)^2 + \lambda_3 (|\sigma|^2 - v_1^2)(\varphi^2 - v_2^2).
$$

The field strengths are given as $\mathcal{F}_{MN} = \partial_M A_N - \partial_N A_M$ and $\tilde{F}_{MN} = \partial_M \tilde{A}_N - \partial_N \tilde{A}_M$. The complex scalar field $\sigma$ is charged under $\tilde{U}(1)$, i.e. $D_M \sigma = (\partial_M + i\tilde{A}_M) \sigma$, while $\varphi$ is a real scalar field.

When $\sigma$ develops a non zero expectation value, $\tilde{U}(1)$ gauge symmetry is broken. As a consequence, an ANO-type vortex is formed. The parameters of the potential $V$ are chosen such that $\varphi$ condenses only inside the ANO vortex, which leads to the localization of massless gauge fields. This is similar to superconducting cosmic strings [52, 69], where $\varphi$ is complex and is charged under $U(1)$, so that the ANO string becomes superconducting. In our model, the vortex is not superconducting because $\varphi$ is neutral. Instead, it couples to the $U(1)$ gauge field $A_M$ via the nontrivial field dependent gauge kinetic term.

To find the background vortex solution let us make an Ansatz

$$
\sigma = s(r)e^{i\theta}, \quad \varphi = \varphi(r), \quad \tilde{A}_a = \epsilon_{ab} \frac{x^b}{r^2} \tilde{a}(r),
$$

where $r = \sqrt{x_4^2 + x_5^2}$ and $\theta = \arctan x_5/x_4$. A suitable boundary condition for the vortex is $s = \varphi' = \tilde{a} = 0$ at $r = 0$ and $s = \sqrt{v_1^2 + \frac{2\lambda_3}{\lambda_1} v_2^2}$, $\varphi = 0$ and $\tilde{a} = 1$ at $r = \infty$. A typical solution (obtained by numerical integration of equations of motion) is shown in Fig. 4 for $(\lambda_1, \lambda_2, \lambda_3) = (1, 3, 1)/v_1$, $v_2 = v_1$, $\tilde{e} = v_1^{-1/2}$. As desired, the real scalar field $\varphi$ condenses around the vortex. Therefore $\beta^2 = a^2 \varphi^2$ is square integrable, which ensures localization of the massless $U(1)$ gauge field.
2. The physical spectrum

Let us next study the KK spectrum for the vortex background obtained above. Following the generic arguments in Sec. II, the relevant equations can be read from Eq. (II.23) as

\[
\begin{align*}
[\eta^{\mu\nu} \partial^2 - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu + \eta^{\mu\nu} D^2] A_\nu & = 0, \\
[\delta_{ab} (\partial^2 + D^2) - \left(D_b^\dagger D_a - \xi D_a D_b^\dagger\right)] A_b & = 0, \\
(\partial^2 + \xi D^2) c & = 0.
\end{align*}
\]

In what follows, we will concentrate on the physical modes: the transverse modes of $A_\mu$ and the divergence-free part of $A_a$. The transverse condition $\partial^\mu A^T_\mu = 0$ and the divergence-free condition $P_{ab} A^df_b = 0$ give us the following equations

\[
\begin{align*}
(\partial^2 + D^2) A^T_\mu & = 0, \\
[(\partial^2 + D^2) \delta_{ab} - D_b^\dagger D_a] A^df_b & = 0.
\end{align*}
\]

Since the divergence-free part in $D = 6$ can be expressed by means of $\bar{K}$ (see Eq. (II.42)) as $A^d_a = \frac{1}{2} \epsilon_{ab} D_b^\dagger \bar{D}^{-2} \bar{K}$ we find

\[
(\partial^2 \delta_{ab} - D_b^\dagger D_a) A^df_b = \frac{1}{2} \epsilon_{ab} D^\dagger_b \bar{K},
\]

FIG. 4. The panel (a) shows the profile functions for a typical numerical solution of the single ANO vortex. The panel (b) shows the zero mode wave function of the gauge fields and the trapping Schrödinger type potential.
where we have omitted the zero mode for $A_a^{\text{df}}$ since it does not appear in a physical spectrum. Plugging this into Eq. (III.49), we find the following equation for the divergence-free part

$$\epsilon_{ab} D_b^{\dagger} D^{-2} \left( \partial^2 + \bar{D}^2 \right) \bar{K} = 0.$$  

(III.51)

In short, we just need to find the eigenvalues of the operator $D^2$ for $A^{T}_\mu$ and $\bar{D}^2$ for $A_a^{\text{df}}$.

Let us next consider axially symmetric background with $\beta = \beta(r)$. We expand a function of $x^\mu$, $x^4 = r \cos \theta$ and $x^5 = r \sin \theta$ as

$$f(x, r, \theta) = \sum_{n, l} f_{n, l}(x) \frac{\phi_{n, l}(r) e^{i l \theta}}{\sqrt{r}}, \quad l \in \mathbb{Z}.$$  

(III.52)

Then we find the eigenvalue equations

$$D^2 : \left( - \frac{d^2}{dr^2} + V_l \right) \phi_{n, l} = m_{n, l}^2 \phi_{n, l},$$  

(III.53)

$$\bar{D}^2 : \left( - \frac{d^2}{dr^2} + \bar{V}_l \right) \bar{\phi}_{n, l} = \bar{m}_{n, l}^2 \bar{\phi}_{n, l},$$  

(III.54)

with

$$V_l = \frac{1}{\sqrt{r} \beta} \left( \sqrt{r} \beta \right)^{\prime \prime} + \frac{l^2}{r^2},$$  

(III.55)

$$\bar{V}_l = \frac{1}{\sqrt{r} \beta^{-1}} \left( \sqrt{r} \beta^{-1} \right)^{\prime \prime} + \frac{l^2}{r^2}.$$  

(III.56)

The zero modes of both operators are $\phi_{0, 0} = \sqrt{r} \beta$ and $\bar{\phi}_{0, 0} = \sqrt{r} \beta^{-1}$. Note that we can again rewrite the above equations in the SUSY QM fashion as

$$\left( Q_r^1 \bar{Q}_r + \frac{l^2}{r^2} \right) \phi_{n, l} = m_{n, l}^2 \phi_{n, l},$$  

(III.57)

$$\left( Q_r^1 \bar{Q}_r + \frac{l^2}{r^2} \right) \bar{\phi}_{n, l} = \bar{m}_{n, l}^2 \bar{\phi}_{n, l},$$  

(III.58)

where we introduce

$$Q_r = - \partial_r + \left( \partial_r \log \sqrt{r} \beta \right), \quad Q_r^1 = \partial_r + \left( \partial_r \log \sqrt{r} \beta \right),$$  

(III.59)

$$\bar{Q}_r = - \partial_r + \left( \partial_r \log \sqrt{r} \beta^{-1} \right), \quad \bar{Q}_r^1 = \partial_r + \left( \partial_r \log \sqrt{r} \beta^{-1} \right).$$  

(III.60)

The term $l^2/r^2$ is nothing but the centrifugal potential for the mode of angular momentum $l$. Fig. 5 shows $V_{l=0,1}$ and $\bar{V}_{l=0,1}$ for the numerical vortex solution given in Fig. 4. For modes with $l > 0$, the centrifugal force significantly lifts the potential near origin. Therefore, bound states, if exist, are pushed away from the origin. We will work out analytic solutions for a typical gauge kinetic function $\beta$ in Sec. III B 3.
FIG. 5. The left panel shows $V_0$ (black solid curve) and $\bar{V}_0$ (red solid curve). The broken lines correspond to the zero modes $\phi_{0,0}$ (black) and $\bar{\phi}_{0,0}$ (red), respectively. Note that $\phi_{0,0}$ is related to $\phi_0$ in Fig. 4 by $\phi_{0,0} = \sqrt{r}\phi_0$. The right panel shows $V_1$ and $\bar{V}_1$.

Finally, let us examine the behavior of the potential for models with higher power of $\varphi$ as the gauge kinetic function $\beta$: namely, we modify the model as

$$\mathcal{L} = -\beta^2 F_{MN}F^{MN} + \mathcal{L}_{\text{vortex}}, \quad \beta = a\varphi^n. \quad \text{(III.61)}$$

We plot the effective potentials for $n = 2$ case in Fig. 6. Compared to the case $n = 1$ given in Fig. 5, the potentials are deeper. Therefore, we expect several excited discrete bound states for higher $n$.

FIG. 6. The effective potentials for $n = 2$. See the caption of Fig. 5 for details.
3. Analytic example of the mass spectrum

Let us illustrate the results of the previous subsection on a concrete example $\beta = e^{-\Omega r}$. The relevant eigenvalue equations read

$$D^2 : \left( -\frac{d^2}{dr^2} + V_l \right) \phi_{n,l} = m_{n,l}^2 \phi_{n,l}, \quad V_l = \Omega^2 - \frac{\Omega}{r} + \frac{l^2 - 1/4}{r^2},$$

(III.62)

$$\bar{D}^2 : \left( -\frac{d^2}{dr^2} + \bar{V}_l \right) \bar{\phi}_{n,l} = \bar{m}_{n,l}^2 \bar{\phi}_{n,l}, \quad \bar{V}_l = \Omega^2 + \frac{\Omega}{r} + \frac{l^2 - 1/4}{r^2}.$$  

(III.63)

The difference between $V_l$ and $\bar{V}_l$ is just $\Omega$ and $-\Omega$. Fig. 7 shows the potentials which are quite similar to those obtained numerically in Fig. 5. The potential $V_l$ has an extremum at $r^* = (4l^2 - 1)/(2\Omega)$ with the value

$$V_l(r^*) = \Omega^2 \left( 1 - \frac{1}{4l^2 - 1} \right).$$

(III.64)

While $V_0(r)$ is infinitely deep at the origin and $r^*$ is its global maximum, $V_{l\neq0}$ is unbounded at $r = 0$ and has a global minimum at $r^*$. Notice that $V_{l\neq0}(r^*)$ is always lower than the asymptotic value $V_l \to \Omega^2$ as $r \to \infty$. Thus, we expect for both $V_0$ and $V_{l\neq0}$ a tower of discrete states.

Indeed, for each $l = 0, \pm1, \pm2, \ldots$ there is an infinite tower of bound states $n = 0, 1, 2, \ldots$
with radial wave functions (up to normalization constant) and eigenvalues given as

\[ \phi_{n,l}(r) = \exp \left( \frac{-\Omega r}{1 + 2|l| + 2n} \right) r^{|l| + \frac{1}{2}} \sum_{k=0}^{n} \frac{(2|l|)!}{(2|l| + k)!} \left( \frac{1}{1 + 2|l| + 2n} \right)^k \left( n \right) \left( \frac{-2\Omega r}{1 + 2|l| + 2n} \right)^k, \quad (III.65) \]

\[ m_{n,l}^2 = \Omega^2 \left( 1 - \frac{1}{(1 + 2|l| + 2n)^2} \right). \quad (III.66) \]

The discrete modes are cumulating at the threshold \( m_{n,l} \to \Omega \) as \( n \to \infty \), above which there is a continuum labelled by a radial momentum \( q \): \( m_{q,l} = \sqrt{\Omega^2 + q^2} \) with eigenfunctions

\[ \phi_{q,l}(r) = c_1 M \left( -i\Omega/(2q), |l|; 2iqr \right) + c_2 W \left( -i\Omega/(2q), |l|; 2iqr \right), \quad (III.67) \]

where \( M(k, m; z) \) and \( W(k, m; z) \) are the Whittaker functions. For illustration, we show several wave functions in Fig. 8.

![Wave Functions](image)

FIG. 8. The analytic wave functions \( \phi_{n,l} \) for \( n = 0, 1, 2 \) and \( l = 0, 1 \) are shown for \( \Omega = 1 \).

On the other hand, the potential \( \bar{V}_l \) has no minimum for \( l \neq 0 \) and, in fact, \( \bar{V}_{l \neq 0} > \Omega^2 \equiv \bar{V}_l(\infty) \). Hence, we cannot expect bound states for \( V_{l \neq 0} \). However, there is an infinite tower of discrete states for \( l = 0 \) tower with radial wave functions (up to normalization constant) and eigenvalues given as

\[ \bar{\phi}_{n,0}(r) = \exp \left( \frac{\Omega r}{1 + 2n} \right) \sqrt{r} \sum_{k=0}^{n} \left( \frac{n}{k} \right) (-1)^k E_{k+1} \left( \frac{2\Omega r}{1 + 2n} \right), \quad (III.68) \]

\[ \bar{m}_{n,0}^2 = \Omega^2 \left( 1 - \frac{1}{(1 + 2n)^2} \right). \quad (III.69) \]
where $E_n(x) = \int_1^\infty e^{-xt}t^{-n}dt$ is the Exponential integral. Note that, even though the zero mode $\bar{\phi}_{0,0}$ is normalizable, it does contribute nothing to the physical spectrum as explained around Eq. (II.43). The eigenfunctions of the continuum part of the spectrum parametrized by a radial momentum $q$ as $\bar{m}_{q,t} = \sqrt{\Omega^2 + q^2}$ can be expressed in of the Whittaker functions as

$$\bar{\phi}_{q,t}(r) = c_1 M\left(i\Omega/(2q), 0; 2iqr\right) + c_2 W\left(i\Omega/(2q), 0; 2iqr\right).$$  \hspace{1cm} (III.70)

Fig. 9 shows first few wave functions $\bar{\phi}_{n,0}$.

![Fig. 9. The analytic wave functions $\bar{\phi}_{n,0}$ for $n = 0, 1, 2$ are shown for $\Omega = 1$.](image)

Thus, the analytic example here explicitly demonstrates the infinite number of bound states, which is in contrast to the domain wall case.

C. Spherically symmetric background in $D = 7$

In this section, we will investigate physical spectrum for spherically symmetric background in $D = 7$. Although we do not specify the background solution, we have a codimension three soliton like a monopole or skyrmion in mind.

1. Analysis of a spherically symmetric background

Let us now investigate the spectrum of gauge fields in arbitrary spherically symmetric background, defined by $\beta(r)$, with $r = \sqrt{x_1^2 + x_5^2 + x_6^2}$. Following the general discussion of Sec. II, the KK spectrum is determined via equations

$$\left(\partial^2 + D^2\right)A^T_\mu = 0,$$  \hspace{1cm} (III.71)

$$\left[\left(\partial^2 + D^2\right)\delta_{ab} - D^d_b D_a\right]A^d_\mu = 0.$$  \hspace{1cm} (III.72)
where $A_T^\mu$ are the transverse four-dimensional gauge fields, i.e. $\partial^\mu A_T^\mu = 0$, and $A_{a}^{\text{df}}$ are divergence-free extra-dimensional gauge fields, i.e. $P_{ab} A_{b}^{\text{df}} = 0$.

To fully utilize the spherical symmetry, let us switch to spherical coordinates $x^4 = r \cos \phi \sin \theta$, $x^5 = r \sin \phi \sin \theta$, $x^6 = r \cos \theta$. The four-dimensional gauge fields satisfy the equation

$$\left( \partial^2 - \Delta + \frac{1}{\beta} \Delta_r \beta \right) A_T^\mu = 0,$$

(III.73)

with $\Delta_r \equiv \partial_r^2 + \frac{2}{r} \partial_r$ and $\Delta \equiv \Delta_r - \frac{L_2^2}{r^2}$, and $L^2 \equiv \partial_\theta^2 + \frac{\cos \theta}{\sin \theta} \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\phi^2$. Let us expand the fields into a common set of four-dimensional zero modes and spherical harmonics

$$A_T^\mu(x, r, \theta, \phi) = \sum_n \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u^{(n,l,m)}(x) A_{nl}(r) Y_l^m(\theta, \phi),$$

(III.74)

where $\partial^\mu u^{(n,l,m)} = 0$, $\partial^2 u^{(n,l,m)} = -\mu^2_{nl} u^{(n,l,m)}$ and $L^2 Y_l^m = l(l+1) Y_l^m$. The Schrödinger equation for the radial wave functions reads

$$\left( -\Delta_r + \frac{l(l+1)}{r^2} + \frac{1}{r \beta} (r \beta)'' \right) A_{nl} = \mu^2_{nl} A_{nl}.$$

(III.75)

As in other examples, there is a unique normalizable zero mode $\mu_{00} = 0$:

$$A_{00}(r) = \beta(r),$$

(III.76)

which exists for arbitrary $\beta$.

In order to tackle Eq. (III.72), we will use the machinery of vector spherical harmonics as it is the most convenient tool for separating radial and angular coordinates for vector-valued equations. In general, any vector $\vec{X} \equiv \vec{X}(r, \theta, \phi)$ can be expanded into the basis of three independent spherical harmonics as

$$\vec{X} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( X^{(r)}_{lm}(r) \hat{Y}^m_l + X^{(1)}_{lm}(r) \hat{\Psi}^m_l + X^{(2)}_{lm}(r) \hat{\Phi}^m_l \right),$$

(III.77)

where $X^{(r)}_{lm}$, $X^{(1)}_{lm}$, $X^{(2)}_{lm}$ are the expansion coefficients and the spherical harmonics $\hat{Y}^m_l$, $\hat{\Psi}^m_l$, $\hat{\Phi}^m_l$ are defined as:

$$\hat{Y}^m_l = \hat{r} Y_l^m,$$

(III.78)

$$\hat{\Psi}^m_l = r \hat{\nabla} Y_l^m = \hat{\theta} \partial_\theta Y_l^m + \hat{\phi} \frac{1}{\sin \theta} \partial_\phi Y_l^m,$$

(III.79)

$$\hat{\Phi}^m_l = \hat{r} \times \hat{\nabla} Y_l^m = \hat{\phi} \partial_\theta Y_l^m - \hat{\theta} \frac{1}{\sin \theta} \partial_\phi Y_l^m,$$

(III.80)

\footnote{We follow mostly the conventions and notation of \cite{70}.}
where \( \hat{r}, \hat{\theta}, \hat{\phi} \) denotes unit vectors in the radial and angular directions. Vector spherical harmonics have various nice properties. Of particular use for us are the following

\[
\Delta \tilde{Y}_l^m = -\frac{l^2 + l + 2}{r^2} \tilde{Y}_l^m + \frac{2}{r^2} \tilde{\Psi}_l^m, \tag{III.81}
\]

\[
\Delta \tilde{\Psi}_l^m = \frac{l(l+1)}{r^2} \left( 2 \tilde{Y}_l^m - \tilde{\Psi}_l^m \right), \tag{III.82}
\]

\[
\Delta \tilde{\Phi}_l^m = -\frac{l(l+1)}{r^2} \tilde{\Phi}_l^m, \tag{III.83}
\]

\[
(\hat{r} \cdot \nabla) \tilde{Y}_l^m = (\hat{r} \cdot \nabla) \tilde{\Psi}_l^m = (\hat{r} \cdot \nabla) \tilde{\Phi}_l^m = 0. \tag{III.84}
\]

These allow us to establish the key identities for tackling the Eq. (III.72), namely

\[
\Delta \vec{X} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ \left( \Delta X_{lm}^{r} - \frac{l^2 + l + 2}{r^2} X_{lm}^{r} + \frac{2}{r^2} \right) X_{lm}^{(1)} \right] \tilde{Y}_l^m + \left( \Delta X_{lm}^{(1)} - \frac{l(l+1)}{r^2} \right) \tilde{\Psi}_l^m + \left( \Delta X_{lm}^{(2)} - \frac{l(l+1)}{r^2} \right) \tilde{\Phi}_l^m \right], \tag{III.85}
\]

\[
\left( \vec{X} \cdot \nabla \right) (\hat{r} f(r)) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( X_{lm}^{r} f(r) \tilde{Y}_l^m + \frac{1}{r} X_{lm}^{(1)} f(r) \tilde{\Psi}_l^m + \frac{1}{r} X_{lm}^{(2)} f(r) \tilde{\Phi}_l^m \right), \tag{III.86}
\]

where \((\cdot)\)' denotes derivative with respect to \(r\).

At this point, let us expand the extra-dimensional three-vector \((\vec{A})_a \equiv A_a\) in terms of vector spherical harmonics with the four-dimensional effective fields \(u_{1}^{(n,l,m)}(x)\) and \(u_{2}^{(n,l,m)}(x)\) as coefficients

\[
\vec{A} = \sum_n \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left[ u_{1}^{(n,l,m)}(x) (A_{nl}^{r}(r) \tilde{Y}_l^m + A_{nl}^{(1)}(r) \tilde{\Psi}_l^m) + u_{2}^{(n,l,m)}(x) A_{nl}^{(2)}(r) \tilde{\Phi}_l^m \right], \tag{III.87}
\]

where

\[
\partial^2 u_{1}^{(n,l,m)}(x) = -m_{n,l}^2 u_{1}^{(n,l,m)}(x), \quad \partial^2 u_{2}^{(n,l,m)}(x) = -m_{n,l}^2 u_{2}^{(n,l,m)}(x). \tag{III.88}
\]

Now, the divergence part \(K = D_{a}^\dagger A_a\) is expanded as

\[
K = \sum_n \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{1}^{(n,l,m)}(x) \left( \partial_r A_{nl}^{r} + \frac{2}{r} A_{nl}^{r} - \frac{l(l+1)}{r} A_{nl}^{(1)} + (\log \beta)' A_{nl}^{r} \right) Y_l^m. \tag{III.89}
\]

In particular, we see that \(K\) is independent of \(A_{nl}^{(2)}\). Thus, \(A_{nl}^{(2)}\) contains only physical degrees of freedom. Since we are interested in physical degrees of freedom, we set \(K = 0\). Therefore, for \(l = 0\) we set \(A_{n0}^{r} = 0\) and for \(l \neq 0\), we eliminate \(A_{nl}^{(1)}\) as

\[
A_{nl}^{(1)} = \frac{r}{l(l+1)} \left( \partial_r A_{nl}^{r} + \frac{2}{r} A_{nl}^{r} + (\log \beta)' A_{nl}^{r} \right). \tag{III.90}
\]
Plugging these into Eq. (III.72), we find for $l = 0$

\[
\left( -\Delta_r + \frac{\beta''}{\beta} \right) A^{(1)}_{n0} = m_{n,0}^2 A^{(1)}_{n0}, \quad (\text{III.91})
\]

\[
\left( -\Delta_r + \frac{\beta''}{\beta} \right) A^{(2)}_{n0} = \tilde{m}_{n,0}^2 A^{(2)}_{n0}, \quad (\text{III.92})
\]

and for $l \neq 0$

\[
\left( -\partial_r^2 + \frac{4}{r} \partial_r \right) + \frac{l(l+1)}{r^2} + \beta \left( \frac{1}{\beta} \right)'' \right) \quad A_r^{nl} = m_{n,l}^2 A_r^{nl}, \quad (\text{III.93})
\]

\[
\left( -\Delta_r + \frac{\beta''}{\beta} \right) A^{(2)}_{nl} = \tilde{m}_{n,l}^2 A^{(2)}_{nl}. \quad (\text{III.94})
\]

It is clear that $m_{n,0} = \tilde{m}_{n,0}$ holds. Notice that for $l = m = 0$ vector spherical harmonics takes the values

\[
Y^0_0 = \frac{\hat{r}}{\sqrt{4\pi}}, \quad \Psi^0_0 = 0, \quad \Phi^0_0 = 0. \quad (\text{III.95})
\]

As a consequence, we can freely set $A_a^{(1,2)} = 0$. Therefore, we are guaranteed that there are no zero modes for $A_{00}^{df}$.

For simplicity, let us introduce

\[
A_{nl} = \frac{B_{nl}}{r}, \quad A_r^{nl} = \frac{B_r^{nl}}{r^2}, \quad A^{(1)}_{n0} = \frac{B^{(1)}_{n0}}{r}, \quad A^{(2)}_{nl} = \frac{B^{(2)}_{nl}}{r}. \quad (\text{III.96})
\]

Then we finally obtain the following set of one-dimensional Schrödinger equations from Eqs. (III.75), (III.91), (III.93) and (III.94) as

\[
\left( -\partial_r^2 + \frac{l(l+1)}{r^2} + \frac{1}{r} \left( r\beta \right)'' \right) B_{nl} = \mu_{nl}^2 B_{nl}, \quad (\text{III.97})
\]

\[
\left( -\partial_r^2 + \frac{l(l+1)}{r^2} + \frac{\beta''}{\beta} \right) B_{nl}^{(2)} = \tilde{m}_{n,l}^2 B_{nl}^{(2)}, \quad (\text{III.98})
\]

\[
\left( -\partial_r^2 + \frac{\beta''}{\beta} \right) B_{n0}^{(1)} = m_{n,0}^2 B_{n0}^{(1)}, \quad (\text{III.99})
\]

\[
\left( -\partial_r^2 + \frac{l(l+1)}{r^2} + \beta \left( \frac{1}{\beta} \right)'' \right) B_r^{nl} = m_{n,l}^2 B_r^{nl}. \quad (\text{III.100})
\]
2. **Analytic example of the mass spectrum**

Let us investigate the spectrum on a concrete background \( \beta(r) = e^{-\Omega r} \) which substantially simplify the relevant equations as

\[
-\partial_r^2 + \frac{l(l+1)}{r^2} + \Omega^2 - \frac{2\Omega}{r} B_{nl} = \mu_{nl}^2 B_{nl}, \tag{III.101}
\]

\[
-\partial_r^2 + \frac{l(l+1)}{r^2} + \Omega^2 B^{(2)}_{nl} = \tilde{m}_{n,l}^2 B^{(2)}_{nl}, \tag{III.102}
\]

\[
-\partial_r^2 + \Omega^2 B^{(1)}_{n0} = m_{n,0}^2 B^{(1)}_{n0}, \tag{III.103}
\]

\[
-\partial_r^2 + \frac{l(l+1)}{r^2} + \Omega^2 B^r_{nl} = m_{n,l}^2 B^r_{nl}. \tag{III.104}
\]

Eq. (III.101) is nothing but the hydrogen atom with the Coulomb potential. Therefore, \( B_{nl} \) for bound states is the Laguerre polynomials with discrete mass \( \mu_{n,l}^2 = \Omega^2 \left(1 - \frac{1}{(l+l+n)^2}\right) \). Continuum modes are labeled by \( q \): \( \mu_l(q) = \sqrt{\Omega^2 + q^2} \). On the other hand, the Schrödinger potentials for the fields \( B^{(2)}_{nl}, B^{(1)}_{n0} \) and \( B^r_{nl} \) are constants. Therefore, no localized modes exist. Hence, in this example, only four-dimensional components of gauge fields have a discrete tower of localized states, while for the extra-dimensional components there is only a continuum of bulk modes.

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Appendix A: Generalization of the analysis of $A_a$ to $D \geq 7$

The operator $H_{ab}$ in Eq. (II.39) in $N \equiv D - 4$ Euclidean dimensions can be factorized as

$$H_{ab} = \frac{1}{(N-2)!} \varepsilon_{i_1 \ldots i_{N-2}ad} \varepsilon_{i_1 \ldots i_{N-2}bc} D_d^\dagger D_c,$$

where $\varepsilon_{i_1 \ldots i_N}$ is a completely anti-symmetric symbol. We can rewrite the operator as a product of a $(N/2) \times N$ matrix $\mathcal{D}$ and its Hermitian conjugate as

$$H = \mathcal{D}^\dagger \mathcal{D},$$

(A.2)

The row index of $\mathcal{D}$ spans all $(N/2)$ inequivalent values of the first $N-2$ indices of $\varepsilon_{i_1 \ldots i_{N-2}ab}$. In this way, the degeneracies are taken care of so that the numerical factor in (A.1) does not appear in (A.2). In particular, we have

$$\mathcal{D}^{N=2} = (D_5, -D_4),$$

(A.3)

$$\mathcal{D}^{N=3} = \begin{pmatrix} 0 & D_6 & -D_5 \\ -D_6 & 0 & D_4 \\ D_5 & -D_4 & 0 \end{pmatrix},$$

(A.4)

$$\mathcal{D}^{N=4} = \begin{pmatrix} 0 & 0 & D_7 & -D_6 \\ 0 & -D_7 & 0 & D_5 \\ 0 & D_6 & -D_5 & 0 \\ D_7 & 0 & 0 & -D_4 \\ -D_6 & 0 & D_4 & 0 \\ D_5 & -D_4 & 0 & 0 \end{pmatrix},$$

(A.5)

and so on.

The zero modes of $H$ are annihilated by $\mathcal{D}$ as

$$\mathcal{D} \begin{pmatrix} \psi_4 \\ \vdots \\ \psi_D \end{pmatrix} = 0.$$

(A.6)

This has an obvious solution with arbitrary function $f$

$$\psi_a = D_a f.$$

(A.7)

---

6 Here the Hermitian conjugation acts both on matrix space and on operator space.
which is valid for arbitrary $\beta$.

However, this is not suitable for the divergence free part since $(\delta_{ab} - P_{ab})\psi_b = 0$. The examples in subsequent section indicate that for certain $\beta$ zero mode in $A^\text{df}_a$ is possible.

Beyond these observations it is difficult to establish the spectrum of $A^\text{df}_a$ in arbitrary dimensions. However, the $N = 2$ case is analyzed completely as follows. Let us introduce a single component “superpartner” to $H$ as

$$
\tilde{H} = DD^\dagger = (D_5, -D_4) \begin{pmatrix} D_5^\dagger \\ -D_4^\dagger \end{pmatrix} = \bar{D}^2.
$$

(A.8)

It is well known that the pair of operators $\{H, \tilde{H}\}$ share the same spectrum except for the possible zero modes. Indeed, if we denote eigenvectors of $H$ as $\psi_\lambda$, i.e. $H\psi_\lambda = \lambda\psi_\lambda$, then $D\psi_\lambda$ is an eigenvector of $\tilde{H}$ with exactly the same eigenvalue:

$$
\tilde{H}D\psi_\lambda = DD^\dagger D\psi_\lambda = DH\psi_\lambda = \lambda D\psi_\lambda.
$$

(A.9)

Similarly, denoting the eigenvectors of $\tilde{H}$ as $\tilde{\psi}_\lambda$, that is $\tilde{H}\tilde{\psi}_\lambda = \lambda\tilde{\psi}_\lambda$, we see that $D^\dagger \tilde{\psi}_\lambda$ is an eigenvector of $H$:

$$
H D^\dagger \tilde{\psi}_\lambda = D^\dagger DD^\dagger \tilde{\psi}_\lambda = D^\dagger \tilde{H}\tilde{\psi}_\lambda = \lambda D^\dagger \tilde{\psi}_\lambda.
$$

(A.10)

Note that the zero mode $\psi_0$ of $H$ does not give a zero mode of $\tilde{H}$.

In the $N = 3$ case the superpartner reads

$$
\tilde{H}^{N=3} = \begin{pmatrix} D_5D_5^\dagger + D_6D_6^\dagger & -D_5D_4^\dagger & -D_6D_4^\dagger \\ -D_4D_5^\dagger & D_4D_4^\dagger + D_6D_6^\dagger & -D_6D_5^\dagger \\ -D_4D_6^\dagger & -D_5D_6^\dagger & D_4D_4^\dagger + D_5D_5^\dagger \end{pmatrix}.
$$

(A.11)

Since the level of complexity in finding the spectrum of this operator is about the same as for $H_{ab}^{N=3}$, we gain little advantage by switching to the superpartner. We will give a concrete analysis specialized for $N = 3$ $(D = 7)$ in Sec. III C. For $N > 3$ cases the situation gets even worse as the superpartner is $(N^2)$-dimensional operator, which is a much larger matrix then the original $H_{ab}$. We leave as a future problem to derive general results about the spectrum of extra-dimensional gauge fields.

**Appendix B: The divergence free part in the separable $\beta$**

The $(\binom{N+1}{2}) \times (\binom{N+1}{2})$ matrix-valued operator $H^{N+1}$ in Eq. (A.8) for $N + 1$ extra-dimensions can be decomposed into the $(\binom{N}{2}) \times (\binom{N}{2})$ matrix-valued operator $H^N$ for $N$ extra-dimension
and one-dimensional subspaces as

\[
H^{N+1} = \begin{pmatrix}
H^N + D_{N+4}^\dagger D_{N+4} & -D_{N+4}^\dagger \vec{D} \\
-D_{N+4}^\dagger D_N & D_N^2
\end{pmatrix},
\]

(B.1)

where we denoted an \(N\)-dimensional vector \(\vec{D}^\dagger \equiv (D_4^\dagger, \ldots, D_{N+3}^\dagger)\) and \(D_N^2 \equiv \vec{D}^\dagger \cdot \vec{D} = D_4^\dagger D_4 + \ldots + D_{N+3}^\dagger D_{N+3}\). Let us further decompose the wave-function as

\[
\psi^{(N+1)} = \begin{pmatrix}
\psi^{(N)} \\
\phi_N
\end{pmatrix}.
\]

Here, \(\psi^{(N)}\) denotes an \(N\)-dimensional vector and \(\phi_N\) a scalar. Suppressing all labelling of eigenfunctions, the eigenvalue problem \(H^{N+1} \psi^{(N+1)} = \lambda^{(N+1)} \psi^{(N+1)}\) is rewritten as

\[
H^N \psi^{(N)} + D_{N+4}^\dagger D_{N+4} \psi^{(N)} - D_{N+4}^\dagger \vec{D} \phi_N = \lambda^{(N+1)} \psi^{(N)},
\]

\[
-\vec{D}^\dagger D_{N+4} \psi^{(N)} + D_N^2 \phi_N = \lambda^{(N+1)} \phi_N.
\]

Together with these, we also impose divergence-free condition

\[
\vec{D}^\dagger \psi^{(N)} + D_{N+4}^\dagger \phi_N = 0.
\]

Due to this condition, we expect that \(\psi^{(N+1)}\) contains \(N\) independent degrees of freedom, all of which has its own tower of eigenmodes.

At this point, let us assume that \(\beta\) is separable in at least one direction, say, \(x^{N+4}\)-th.

\[
\beta(y_{N+1}) \equiv \beta(y_N) b(x^{N+4}).
\]

(B.6)

Here, we used \(y_N \equiv \{x^4, \ldots, x^{N+3}\}\) to denote remaining directions. Notice that \(\beta(x^{N+4})\) must be normalizable. Since \(\beta\) appears in \(D_a\) only as \(\partial_a \log \beta\), the above condition implies that \(D_{N+4}\) commutes with all other operators. Hence, we can also separate the variables in wave-functions as

\[
\psi^{(N)}(y_{N+1}) \equiv \psi^{(N)}(y_N) S(x^{N+4}), \quad \phi_N(y_{N+1}) \equiv \phi_N(y_N) F(x^{N+4}).
\]

(B.7)

First, let us consider the case where \(\psi^{(N)}(y_N)\) is an eigenvector of \(H^N\) with eigenvalue \(\lambda^{(N)}\) and it is divergence-free, that is \(\vec{D}^\dagger \psi^{(N)} = 0\). Eq. (B.5) implies \(D_{N+4}^\dagger F(x^{N+4}) = 0\). However, solving this condition as \(F \propto 1/b\) we obtain non-normalizable wave-function and,
hence, we must set \( F = 0 \). Thus, Eq. (B.4) is solved trivially. Eq. (B.3) reduces to the
eigenproblem for \( S(x^{N+4}) \) in the form
\[
D_{N+4}^\dagger D_{N+4} S = \left( \lambda^{(N+1)} - \lambda^{(N)} \right) S.
\]  
(B.8)

Denoting the eigenvalues of \( D_{N+4}^\dagger D_{N+4} \) as \( \lambda_{N+4} \) we arrive at the solution
\[
\psi^{(N+1)}(y_{N+1}) = \begin{pmatrix} \psi^{(N)}(y_N)S(x^{N+4}) \\ 0 \end{pmatrix}, \quad \lambda^{(N+1)} = \lambda^{(N)} + \lambda_{N+4}.
\]  
(B.9)

In other words, we see that for a separable direction, the Hilbert space is a direct product
of Hilbert spaces generated by \( H^N \) and \( D_{N+4}^\dagger D_{N+4} \). Notice that zero mode \( \lambda^{(N+1)} = 0 \) can
exist only if \( \lambda^{(N)} = 0 \) does. As a consequence, for a fully separable \( \beta \) there is no zero mode
in any number of extra-dimensions, as we can recursively apply this argument down to the
\( N = 2 \) case, where we establish that zero mode does not exists.

The above solution contains only \( N - 1 \) independent degrees of freedom, which are con-
tained in \( \psi^{(N)} \). One remaining solution can be found by taking \( H^N \psi^{(N)} = 0 \). In other words,
we set
\[
\psi^{(N+1)}(y_{N+1}) = \begin{pmatrix} \bar{D}D_N^{-2}K^{(N)}(y_N)S(x^{N+4}) \\ K^{(N)}(y_N)F(x^{N+4}) \end{pmatrix}.
\]  
(B.10)

Here, \( K^{(N)} \) stands for divergence part of extra-dimensional gauge fields. Moreover, the
divergence-free condition (B.5) implies \( S = -D_{N+4}^\dagger F \). Plugging this into Eqs. (B.3)-(B.4)
we ultimately obtain two eigenproblems
\[
D_{N+4}^\dagger D_{N+4} F = \lambda_{N+4} F, \quad D_N^2 K^{(N)} = \left( \lambda^{(N+1)} - \lambda_{N+4}' \right) K^{(N)}.
\]  
(B.11)

As we see, the solution space is again furnished by a direct product of solution spaces of two
operators. One is \( D_N^2 \), which gives the spectrum to four-dimensional gauge fields. However,
we know that the divergence part \( K^{(N)} \) has no zero mode and hence \( D_N^2 K^{(N)} = \lambda'_4 K^{(N)} \),
where prime signals the absence of zero mode in an otherwise identical spectrum. The second
operator is \( D_{N+4}^\dagger D_{N+4} \), which is just a superpartner to \( D_{N+4}^\dagger D_{N+4} \). Thus, its eigenvalues are
\( \lambda_{N+4}' \). Putting these observation together, we see that \( \lambda^{(N+1)} = \lambda'_4 + \lambda_{N+4}' \). It is obvious
that for this degree of freedom zero mode cannot exits.

In summary, for a separable \( \beta \), we find that the spectrum of \( N \) independent divergence-
free eigenvectors of \( H^{N+1} \) can be constructed out of \( N - 1 \) divergence-free eigenvectors of
\( H^N \) and \( N \)-dimensional divergence part \( K^{(N)} \) combined with eigenfunctions of \( D_{N+4}^\dagger D_{N+4} \)
and its superpartner.
Appendix C: \( D = 5 \) with \( S^1 \) extra-dimension

Let us give pedagogical derivation of a low energy effective action from the Abelian gauge theory without matters in the \( M^{1,3} \times S^1 \) spacetime. The five-dimensional Lagrangian in the \( R_\xi \) gauge is given by

\[
\mathcal{L}_\xi = -\frac{1}{4} F_{MN} F^{MN} - \frac{1}{2} f^2 - \bar{c} \left( \partial_\mu \partial^\mu - \xi \partial_y^2 \right) c, \tag{C.1}
\]

with the gauge fixing functional

\[
f = \partial_\mu A_\mu - \xi \partial_y A_y. \tag{C.2}
\]

The gauge-fixing condition eliminates the mixing between \( A_\mu \) and \( A_y \)

\[
\mathcal{L}_\xi = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} \partial^\nu A_\mu \partial^\nu A_\mu - \left( 1 - \frac{1}{\xi} \right) \partial^\nu A_\mu \partial^\nu A_\mu + \frac{1}{2} \partial_y A_\mu \partial_y A^\mu \\
+ \frac{1}{2} \partial_\mu A_y \partial_\mu A_y - \frac{\xi}{2} (\partial_y A_y)^2 - \bar{c} \left( \partial_\mu \partial^\mu - \xi \partial_y^2 \right) c. \tag{C.3}
\]

We expand the gauge field and the ghosts as

\[
A_M = \sum_n A_M^{(n)}(x) e^{i \frac{n}{R} y}, \quad c = \sum_n c^{(n)}(x) e^{i \frac{n}{R} y}, \quad \bar{c} = \sum_n \bar{c}^{(n)}(x) e^{i \frac{n}{R} y}, \tag{C.4}
\]

for \( n \in \mathbb{Z} \) with \( A_M^{(-n)} = A_M^{(n)*} \). Then we find

\[
F_{\mu \nu} = \frac{F_{\mu \nu}^{(0)}}{\sqrt{2 \pi R}} + \sum_{n \neq 0} \frac{e^{i \frac{n}{R} y}}{\sqrt{2 \pi R}} F_{\mu \nu}^{(n)}, \tag{C.5}
\]

\[
F_{\mu y} = \frac{\partial_\mu A_y^{(0)}}{\sqrt{2 \pi R}} + \sum_{n \neq 0} \frac{e^{i \frac{n}{R} y}}{\sqrt{2 \pi R}} \left( \partial_\mu A_y^{(n)} - i \mu_n A_\mu \right), \tag{C.6}
\]

\[
f = \frac{\partial^\mu A_y^{(0)}}{\sqrt{2 \pi R}} + \sum_{n \neq 0} \frac{e^{i \frac{n}{R} y}}{\sqrt{2 \pi R}} \left( \partial^\mu A_y^{(n)} - i \xi \mu_n A_y^{(n)} \right), \tag{C.7}
\]

with

\[
\mu_n = \frac{n}{R}. \tag{C.8}
\]

After integrating the five-dimensional Lagrangian over \( y \), we obtain a sum of four-dimensional Lagrangians for KK modes as

\[
\mathcal{L}^{\text{eff}}_\xi = \mathcal{L}^{(0)}_\xi + \sum_{n=1}^{\infty} \mathcal{L}^{(n)}_\xi, \tag{C.9}
\]
with
\[ \mathcal{L}^{(n=0)}_{\xi} = -\frac{1}{4} F^{(0)\mu\nu} F^{(0)\mu\nu} - \frac{1}{2 \xi} \left( \partial^\mu A^{(0)}_\mu \right)^2 - \bar{c}^{(0)} (\partial^2 c^{(0)}) + \frac{1}{2} \partial^\mu A^{(0)}_y \partial^\mu A^{(0)}_y, \] (C.10)

and
\[ \mathcal{L}^{(n\neq 0)}_{\xi} = A^{(-n)}_{\mu} \left[ (\partial^2 + \mu^2_n) \eta^{\mu\nu} - \left( 1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right] A^{(n)}_{\mu} \\
- A^{(-n)}_{y} (\partial^2 + \xi \mu^2_n) A^{(n)}_{y} \\
- \bar{c}^{(-n)} (\partial^2 + \xi \mu^2_n) c^{(n)} - \bar{c}^{(n)} (\partial^2 + \xi \mu^2_n) c^{(-n)}. \] (C.11)

The effective four-dimensional Lagrangian \( \mathcal{L}^{(0)}_{\xi} \) for massless modes is identical to the ordinary \( U(1) \) gauge theory, a massless scalar field \( A^{(0)}_y \) and the ghost field associated to the covariant gauge fixing condition. Thus we find single massless scalar field as an additional physical degree of freedom. The effective Lagrangian for massive modes contains massive complex vector fields \( A^{(n)}_{\mu} \) with mass \( \mu_n \) besides ghost fields \( c^{(n)}, \bar{c}^{(-n)}, \bar{c}^{(n)}, \bar{c}^{(-n)} \) with common gauge-dependent mass squared \( \xi \mu^2_n \). Their contributions in physical processes cancel each other, and have no physical effect. This is consistent with the possibility to choose the axial gauge where we can eliminate \( A_y \) by gauge transformations. This gauge choice is possible only as a five-dimensional field, and does not exclude possible zero modes, as we find explicitly here.

The presence of the physical massless scalar field \( A^{(0)}_y \) is a common property of extra-dimensional models with compact extra-dimension. The mass gap between massless and massive modes is proportional to \( 1/R \). Presence of massless scalar fields is phenomenologically not preferable. In order to forbid them, it is commonly introduced \( Z_2 \) parity by considering \( S^1/Z_2 \) orbifold compactification
\[ A_\mu(x, y) \rightarrow A_\mu(x, -y) = A_\mu(x, y), \] (C.12)
\[ A_y(x, y) \rightarrow A_y(x, -y) = -A_y(x, y). \] (C.13)

Since the zero mode of \( A^{(0)}_y \) is \( Z_2 \) odd, it is eliminated from the physical spectrum.

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