THE COMPLEXITY OF COMPUTING THE SIGN OF THE TUTTE POLYNOMIAL (AND CONSEQUENT \#P-HARDNESS OF APPROXIMATION)

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Abstract. We study the complexity of computing the sign of the Tutte polynomial of a graph. As there are only three possible outcomes (positive, negative, and zero), this seems at first sight more like a decision problem than a counting problem. Surprisingly, however, there are large regions of the parameter space for which computing the sign of the Tutte polynomial is actually \#P-hard. As a trivial consequence, approximating the polynomial is also \#P-hard in this case. Thus, approximately evaluating the Tutte polynomial in these regions is as hard as exactly counting the satisfying assignments to a CNF Boolean formula. For most other points in the parameter space, we show that computing the sign of the polynomial is in FP, whereas approximating the polynomial can be done in polynomial time with an NP oracle. As a special case, we completely resolve the complexity of computing the sign of the chromatic polynomial — this is easily computable at \( q = 2 \) and when \( q \leq 32/27 \), and is NP-hard to compute for all other values of the parameter \( q \).

1. Introduction

The Tutte polynomial of a graph is two-variable polynomial that captures many interesting properties of the graph such as (by making appropriate choices of the two variables) the number of \( q \)-colourings, the number of nowhere-zero \( q \)-flows, the number of acyclic orientations, and the probability that the graph remains connected when edges are deleted at random.

Much work \[2, 3, 5, 10, 18\] has studied the difficulty of evaluating the polynomial (exactly or approximately) when the values of the variables are fixed, and a graph is given as input.

Our early paper \[4\] identified a large region of points where the approximate evaluation of the polynomial is NP-hard, and a short hyperbola segment along which approximate evaluation is even \#P-hard. Thus, an approximation of the polynomial at a point on this short hyperbola segment would enable one to exactly solve a problem in \#P. In this paper, we show that, in fact, for most of these NP-hard points (and more), approximation is \#P-hard. Moreover, it is \#P-hard for a very simple reason: determining
the sign of the polynomial — i.e., whether the evaluation of the polynomial is positive, negative or zero — is \#P-hard. This seems surprising since determining the sign of the polynomial is nearly a decision problem (there are only three possible outcomes) but it is \#P-hard nearly everywhere (at all of the red points in the plane in Figure 1).

Past work \cite{8} has studied the sign of the Tutte polynomial — in particular, Jackson and Sokal sought to determine for which choices of the two variables the sign is “trivial” in the sense that it does not depend on the input graph (or it depends only very weakly on the input graph, for example when it depends only on the number of vertices in the graph).

To illustrate how our work fits in with the work of Jackson and Sokal, we start with an important univariate case. The chromatic polynomial $P(G; q)$ of an $n$-vertex graph $G$ is the unique degree-$n$ polynomial in the variable $q$ such that $P(G; q)$ is the number of proper $q$-colourings of $G$. Jackson \cite[Theorem 5]{7} showed that for $q \in (1, 32/27]$ the sign of $P(G; q)$ depends upon $G$ in an essentially trivial way. In particular, for every connected simple graph with $n \geq 2$ vertices and $b$ blocks, $P(G; q)$ is non-zero with sign $(-1)^{n+b-1}$. The sign of $P(G; q)$ is also known to be a trivial function of $G$ for $q \leq 1$. (See, for example, \cite[Theorem 1.1]{8}.) Jackson \cite[Theorem 12]{7} demonstrated the significance of the value $32/27$ by constructing an infinite family of graphs such that $P(G; q) = 0$ at a value of $q$ which is arbitrarily close to $32/27$. In fact, Jackson and Sokal conjectured \cite[Conjecture 10.3(e)]{8} that the value $32/27$ is a phase transition in the sense that, for every $q$ above this critical value, the sign of $P(G; q)$ is a non-trivial function of $G$. In particular, they conjectured that for any fixed $q > 32/27$, and all sufficiently large $n$ and $m$, there are 2-connected graphs $G$ with $n$ vertices and $m$ edges that make $P(G; q)$ non-zero with either sign.

It turns out that this intuition is correct (see Corollary \ref{cor:trivial}) and that $q = 32/27$ is, in some sense, a phase transition for the complexity of computing the sign of $P(G; q)$:

- As was known, for $q \leq 32/27$, the sign of $P(G; q)$ is a trivial function of $G$, which is easily computed.
- At $q = 2$, $P(G; q)$ is the number of 2-colourings of $G$. The sign of $P(G; q)$ is positive if $G$ is bipartite, and is 0 otherwise. Thus, the sign of $P(G; q)$ is not a trivial function of $G$, but $P(G; q)$ is still easily computed in polynomial time.
- However, for every other fixed $q > 32/27$, computing the sign of $P(G; q)$ is NP-hard.

However, the full version of Jackson and Sokal’s conjecture turns out to be incorrect. See Observations \ref{obs:counter} and \ref{obs:counter2} for counter-examples.

While computing the sign of $P(G; q)$ is NP-hard for every $q \neq 2$ which is greater than $32/27$, the precise complexity of computing the sign does actually depend upon $q$. We show (see Corollary \ref{cor:trivial}) that for each fixed non-integer $q > 32/27$, the complexity of computing the sign of $P(G; q)$ is
#P-hard. This means that a polynomial-time algorithm for computing the sign of \( P(G; q) \), given \( G \), would give a polynomial-time algorithm for exactly solving every problem in \( \#P \). On the other hand, for integers \( q > 2 \), the problem of computing the sign of \( P(G; q) \) is merely NP-complete.

As one would expect, both of these results have ramifications for the complexity of approximating \( P(G; q) \). A fully polynomial approximation scheme (FPRAS) for evaluating \( P(G; q) \), given \( G \), can be used as a polynomial-time randomised algorithm for computing the sign of \( P(G; q) \). Thus, we can immediately deduce that if \( q \) is a non-integer which is greater than \( 32/27 \), then there is no FPRAS for \( P(G; q) \) unless there is a randomised polynomial-time algorithm for exactly solving every problem in \( \#P \). See Section 2.4 for a more thorough discussion of this claim.

On the other hand, for integer values \( q > 32/27 \), we show that the problem of evaluating \( P(G; q) \) is in the complexity class \( \#P_Q \), which is defined as follows.

**Definition 1.** FP is the class of functions computable by polynomial-time algorithms. We say that a function \( f : \Sigma^* \to \mathbb{Q} \) is in the class \( \#P_Q \) if \( f(x) = a(x)/b(x) \), where \( a, b : \Sigma^* \to \mathbb{N} \), and \( a \in \#P \) and \( b \in \text{FP} \).

If \( f \) is in \( \#P_Q \) then there is an approximation scheme for \( f \) that runs in polynomial time, using an oracle for an NP predicate (for a more detailed discussion, see [4, Section 2.2]). Thus, it is presumably much easier to approximate \( P(G; q) \) when \( q \) is an integer greater than \( 32/27 \), as compared to a non-integer.

All of these considerations generalise smoothly to the Tutte polynomial, which we now define. Since we will later need the multivariate generalisation [14] of the polynomial, we use the “random cluster” formulation of the Tutte polynomial, which for a graph \( G = (V, E) \), is defined as a polynomial in indeterminates \( q \) and \( \gamma \) as follows,

\[
Z(G; q, \gamma) = \sum_{A \subseteq E} q^{\kappa(V, A)} \gamma^{|A|},
\]

where \( \kappa(V, A) \) denotes the number of connected components in the graph \((V, A)\).

The chromatic polynomial studied earlier is related to the Tutte polynomial via the identity [8, (2.15)] \( P(G; q) = Z(G; q, -1) \).

In fact, Tutte defined the Tutte polynomial using a different, two-variable parameterisation, in terms of variables \( x \) and \( y \). This polynomial is defined for a graph \( G = (V, E) \) by

\[
T(G; x, y) = \sum_{A \subseteq E} (x - 1)^{\kappa(V, A) - \kappa(V, E)} (y - 1)^{|A| - |V| + \kappa(V, A)}.
\]

It is well known (see, for example, [14] (2.26)) that when \( q = (x - 1)(y - 1) \) and \( \gamma = y - 1 \) we have

\[
T(G; x, y) = (y - 1)^{-|V|} (x - 1)^{-\kappa(V, E)} Z(G; q, \gamma).
\]
This paper studies the complexity of computing the sign of the (random cluster) Tutte polynomial. Figure 1 gives a map of the $(x,y)$ plane, illustrating our results.\footnote{For convenience, our proofs use the random cluster formulation of the Tutte polynomial \cite{1}. However, in order to make our results easily comparable to other results in the literature such as \cite{4} and \cite{10}, we classify points using the $(x,y)$-coordinatisation...} The colours depict the complexity of computing the Tutte polynomial is \#P-hard at red points, and is in FP at green points. It is NP-complete at blue points. We have not resolved the complexity at white points. At red points, approximating the Tutte polynomial is also \#P-hard. At blue and green points, it can be done in polynomial time with an NP oracle. Guide for the greyscale version: The red points appear as a darker grey in regions B, C, D, E, F, G, H and I. The green points appear as a lighter grey in regions A, J, K, L and M and also as dashed hyperbola segments and at the points $(-1,0)$, $(-1,-1)$, $(0,-1)$ and $(0,-5)$. The blue points are $(-2,0)$, $(-3,0)$, $(-4,0)$, $(-5,0)$ and $(0,-2)$.
sign of the polynomial for a fixed point \((x, y)\). If the point \((x, y)\) is coloured red, then the problem of computing the sign is \#P-hard. If the point \((x, y)\) is coloured green, then the problem of computing the sign is in \text{FP}. Finally, if the point \((x, y)\) is coloured blue, the the problem of computing the sign is \text{NP}-complete. (There are still some points for which we have not resolved the complexity — these are coloured white.)

Once again, there are ramifications for the complexity of approximating the Tutte polynomial. Since an \text{FPRAS} for \(Z(G; q, \gamma)\) gives a randomised algorithm for computing its sign, we can again deduce that there is no \text{FPRAS} for points that are coloured red (unless there is a randomised polynomial-time algorithm for exactly solving every problem in \#\text{P}). By contrast, for all of the points that are coloured green or blue, we also show that the problem of computing \(Z(G; q, \gamma)\) is in the complexity class \#\text{P}^Q. Thus, the polynomial can be approximated in polynomial-time using an \text{NP} oracle.

In order to reach into some of the regions, for example \(F\), it has been necessary to use gadgets that go beyond the series-parallel graphs that have so-far proved adequate in this area. For example, in exploring region \(F\) has necessitated the use of a gadget based on the Petersen graph.

2. Preliminaries

2.1. The Tutte polynomial. It will be helpful to define the multivariate version of the random cluster formulation of the Tutte polynomial. Let \(\gamma\) be a function that assigns a (rational) weight \(\gamma_e\) to every edge \(e \in E\). We refer to \(\gamma\) as a “weight function”. We define

\[
Z(G; q, \gamma) = \sum_{A \subseteq E} q^{\kappa(A)} \prod_{e \in A} \gamma_e.
\]

Given a graph \(G = (V, E)\) with distinguished nodes \(s\) and \(t\), \(Z_{st}(G; q, \gamma)\) denotes the contribution to \(Z(G; q, \gamma)\) arising from edge-sets \(A\) in which \(s\) and \(t\) are in the same component of \((V, A)\). That is,

\[
Z_{st}(G; q, \gamma) = \sum_{A \subseteq E: s \text{ and } t \text{ in same component}} q^{\kappa(A)} \prod_{e \in A} \gamma_e.
\]

Similarly, \(Z_{st}(G; q, \gamma)\) denotes the contribution arising from edge-sets \(A\) in which \(s\) and \(t\) are in different components, so \(Z(G; q, \gamma) = Z_{st}(G; q, \gamma) + Z_{st}(G; q, \gamma)\).

2.2. Implementing new edge weights, series compositions and parallel compositions. Our treatment of implementations, series compositions and parallel compositions is completely standard and is taken from [5, Section 2.1]. The reader who is familiar with this material can skip this section (which is included here for completeness).

of [2]. This is without loss of generality, since it is easy to go from one coordinate system to the other using [3]. However, the reader should note that if \(y = 1\) then \(\gamma = 0\) and \(q = (x - 1)(y - 1) = 0\) so computing \(Z(G; q, \gamma)\) is trivial, whereas the complexity of computing \(T(G; x, y)\) is unclear. In general, any two-parameter version of the Tutte polynomial will omit some points. This issue is discussed further in [2, Section 1].
Let $W$ be a set of (rational) edge weights and fix a value $q$. Let $w^*$ be a weight (which may not be in $W$) which we want to “implement”. Suppose that there is a graph $\Upsilon$, with distinguished vertices $s$ and $t$ and a weight function $\hat{\gamma} : E(\Upsilon) \rightarrow W$ such that

$$w^* = qZ_{st}(\Upsilon; q, \hat{\gamma})/Z_{st}(\Upsilon; q, \hat{\gamma}).$$

(4)

In this case, we say that $\Upsilon$ and $\hat{\gamma}$ implement $w^*$ (or even that $W$ implements $w^*$).

The purpose of “implementing” edge weights is this. Let $G$ be a graph with weight function $\gamma$. Let $f$ be some edge of $G$ with weight $\gamma_f = w^*$. Suppose that $W$ implements $w^*$. Let $\Upsilon$ be a graph with distinguished vertices $s$ and $t$ with a weight function $\hat{\gamma}$ satisfying (4). Construct the weighted graph $G'$ by replacing edge $f$ with a copy of $\Upsilon$ (identify $s$ with either endpoint of $f$ (it doesn’t matter which one) and identify $t$ with the other endpoint of $f$ and remove edge $f$). Let the weight function $\gamma'$ of $G'$ inherit weights from $\gamma$ and $\hat{\gamma}$ (so $\gamma'_e = \hat{\gamma}_e$ if $e \in E(\Upsilon)$ and $\gamma'_e = \gamma_e$ otherwise). Then the definition of the multivariate Tutte polynomial gives

$$Z(G'; q, \gamma') = \frac{Z_{\gamma'}(\Upsilon; q, \hat{\gamma})}{q^2}Z(G; q, \gamma).$$

(5)

So, as long as $q \neq 0$ and $Z_{\gamma'}(\Upsilon; q, \hat{\gamma})$ is easy to evaluate, evaluating the multivariate Tutte polynomial of $G'$ with weight function $\gamma'$ is essentially the same as evaluating the multivariate Tutte polynomial of $G$ with weight function $\gamma$.

Two especially useful implementations are series and parallel compositions. These are explained in detail in [8 Section 2.3]. So we will be brief here. Parallel composition is the case in which $\Upsilon$ consists of two parallel edges $e_1$ and $e_2$ with endpoints $s$ and $t$ and $\hat{\gamma}_{e_1} = w_1$ and $\hat{\gamma}_{e_2} = w_2$. It is easily checked from Equation (4) that $w^* = (1 + w_1)(1 + w_2) - 1$. Also, the extra factor in Equation (5) cancels, so in this case $Z(G'; q, \gamma') = Z(G; q, \gamma)$.

Series composition is the case in which $\Upsilon$ is a length-2 path from $s$ to $t$ consisting of edges $e_1$ and $e_2$ with $\hat{\gamma}_{e_1} = w_1$ and $\hat{\gamma}_{e_2} = w_2$. It is easily checked from Equation (4) that $w^* = w_1w_2/(q + w_1 + w_2)$. Also, the extra factor in Equation (5) is $q + w_1 + w_2$, so in this case $Z(G'; q, \gamma') = (q + w_1 + w_2)Z(G; q, \gamma)$. It is helpful to note that $w^*$ satisfies

$$(1 + \frac{q}{w^*}) = \left(1 + \frac{q}{w_1}\right) \left(1 + \frac{q}{w_2}\right).$$

We say that there is a “shift” from $(q, \alpha)$ to $(q, \alpha')$ if there is an implementation of $\alpha'$ consisting of some $\Upsilon$ and $\hat{\alpha} : E(\Upsilon) \rightarrow W$ where $W$ is the singleton set $W = \{\alpha\}$. This is the same notion of “shift” that we used in [4]. Taking $y = \alpha + 1$ and $y' = \alpha' + 1$ and defining $x$ and $x'$ by $q = (x - 1)(y - 1) = (x' - 1)(y' - 1)$ we equivalently refer to this as a shift from $(x, y)$ to $(x', y')$. It is an easy, but important observation that shifts may be composed to obtain new shifts. So, if we have shifts from $(x, y)$ to
(x', y') and from (x', y') to (x'', y''), then we also have a shift from (x, y) to (x'', y'').

The k-thickening of \[ \text{[10]} \] is the parallel composition of k edges of weight \( \alpha \). It implements \( \alpha' = (1 + \alpha)^k - 1 \) and is a shift from (x, y) to (x', y') where \( y' = y^k \) (and \( x' \) is given by \( (x' - 1)(y' - 1) = q \)). Similarly, the k-stretch is the series composition of k edges of weight \( \alpha \). It implements an \( \alpha' \) satisfying

\[
1 + \frac{q}{\alpha'} = \left(1 + \frac{q}{\alpha}\right)^k,
\]

It is a shift from (x, y) to (x', y') where \( x' = x^k \). (In the classical bivariate (x, y) parameterisation, there is effectively one edge weight, so the stretching or thickening is applied uniformly to every edge of the graph.)

Since it is useful to switch freely between \((q, \alpha)\) coordinates and \((x, y)\) coordinates we also refer to the implementation in Equation (11) as an implementation of the point \((x, y) = (q/w^* + 1, w^* + 1)\) using the points

\[
\{(x, y) = (q/w + 1, w + 1) \mid w \in W\}.
\]

Thus, if \( q = (x_1 - 1)(y_1 - 1) = (x_2 - 1)(y_2 - 1) \) then the series composition of \((x_1, y_1)\) and \((x_2, y_2)\) implements the point

\[
\left(\frac{q}{y_1 y_2 - 1} + 1, y_1 y_2\right),
\]

and the parallel composition of these implements the point

\[
\left(x_1 x_2, \frac{q}{x_1 x_2 - 1} + 1\right).
\]

We make extensive use of series and parallel composition, and the above identities will be employed without comment.

### 2.3. Computational Problems

For fixed rational numbers \( q, \gamma \) and \( \gamma_1, \ldots, \gamma_k \), we consider the following computational problems\(^2\) from [4].

- **Name:** Tutte\((q, \gamma)\).
  - **Instance:** A graph \( G = (V, E) \).
  - **Output:** The rational number \( Z(G; q, \gamma) \).

- **Name:** Tutte\((q; \gamma_1, \ldots, \gamma_k)\).
  - **Instance:** A graph \( G = (V, E) \) and a weight function \( \gamma : E \to \{\gamma_1, \ldots, \gamma_k\} \).
  - **Output:** The rational number \( Z(G; q, \gamma) \).

We also consider variants in which the goal is to compute the sign of the Tutte polynomial.

- **Name:** SignTutte\((q, \gamma)\).
  - **Instance:** A graph \( G = (V, E) \).
  - **Output:** Determine whether the sign of \( Z(G; q, \gamma) \) is positive, negative, or 0.

\(^2\)In [4] we referred to these as MultiTutte\((q, \gamma)\) and MultiTutte\((q; \gamma_1, \ldots, \gamma_k)\) respectively, but we use the shorter names here since there is no confusion.
Name: \text{Tutte}(q; \gamma_1, \ldots, \gamma_k).

Instance: A graph $G = (V, E)$ and a weight function $\gamma : E \to \{\gamma_1, \ldots, \gamma_k\}$.

Output: Determine whether the sign of $Z(G; q, \gamma)$ is positive, negative, or 0.

2.4. Randomised algorithms and approximate counting. A randomized algorithm for a computational problem takes as an instance of the problem, and returns a result. We require that for each instance, and each run of the algorithm, the probability that the result is equal to the correct output for the given instance is at least $\frac{3}{4}$.

A randomized approximation scheme is an algorithm for approximately computing the value of a function $f : \Sigma^* \to \mathbb{R}$. The approximation scheme has a parameter $\epsilon > 0$ which specifies the error tolerance. A randomized approximation scheme for $f$ is a randomized algorithm that takes as input an instance $x \in \Sigma^*$ (e.g., an encoding of a graph $G$) and an error tolerance $\epsilon > 0$, and outputs a number $z \in \mathbb{Q}$ (a random variable of the “coin tosses” made by the algorithm) such that, for every instance $x$,

$$\Pr[\epsilon^{-\epsilon} \leq z/f(x) \leq \epsilon^{\epsilon}] \geq \frac{3}{4},$$

where, by convention, $0/0 = 1$. (The slight modification of the more familiar definition is to ensure that functions $f$ taking negative values are dealt with correctly.)

The randomized approximation scheme is said to be a fully polynomial randomized approximation scheme, or FPRAS, if it runs in time bounded by a polynomial in $|x|$ and $\epsilon^{-1}$.

Completeness of a problem in $\#P$ is defined with respect to polynomial-time Turing reduction. Suppose $\text{Tutte}(q, \gamma)$ is $\#P$-hard for some setting of the parameters $q, \gamma$. Then, clearly, $\text{Tutte}(q, \gamma) \in \text{FP}$ would imply $\#P = \text{FP}$. In addition, the existence of a polynomial-time randomized algorithm for $\text{Tutte}(q, \gamma)$ would imply the existence of a polynomial-time randomized algorithm for every problem in $\#P$. The reasoning is as follows. Suppose the randomized algorithm for $\text{Tutte}(q, \gamma)$ has failure probability at most $\frac{1}{4}$. By a standard powering argument, the failure probability can be reduced so that it is exponentially small in the input size. But polynomial-time Turing reduction makes only polynomially many oracle calls, so the probability that even a single one produces the wrong answer is polynomially small, and certainly less than $\frac{1}{4}$. As an immediate consequence, an FPRAS for $\text{Tutte}(q, \gamma)$ would again imply the existence of a polynomial-time randomized (but exact in the event of success) algorithm for the whole of $\#P$.

3. $\#P$-hardness of computing the sign of the Tutte polynomial — the multivariate case

We use the fact that the following problem is $\#P$-complete. This was shown by Provan and Ball [13].
Name: \#Minimum Cardinality \((s,t)\)-Cut.

Instance: A graph \(G = (V,E)\) and distinguished vertices \(s, t \in V\).

Output: \(|\{C \subseteq E : C\} is a minimum cardinality \((s,t)\)-cut in \(G\}|\).

**Lemma 1.** Suppose \(q > 1\) and that \(\gamma_1 \in (-2, -1)\) and \(\gamma_2 \notin [-2, 0]\). Then \(\text{SIGN}\text{Tutte}(q; \gamma_1, \gamma_2)\) is \#P-hard.

**Proof.** We will give a Turing reduction from \#Minimum Cardinality \((s,t)\)-Cut to \text{SIGN}\text{Tutte}(q; \gamma_1, \gamma_2).

Let \(G, s, t\) be an instance of \#Minimum Cardinality \((s,t)\)-Cut. Assume without loss of generality that \(G\) has no edge from \(s\) to \(t\). Let \(n = |V(G)|\) and \(m = |E(G)|\). Assume without loss of generality that \(G\) is connected and that \(m \geq n\) is sufficiently large. Let \(k\) be the size of a minimum cardinality \((s,t)\)-cut in \(G\) and let \(C\) be the number of size-\(k\) \((s,t)\)-cuts.

The following calculations are more general than necessary so that we can re-use them in the proof of Lemma 2 (where \(q < 1\) and \(q\) may even be negative). Let

\[M^* = \max \left( \left( 8 \max \left( \frac{|q|}{|q|}, 1 \right) \right)^m \frac{2}{|q| - 1} \right)\]

Let \(h\) be the smallest integer such that \((\gamma_2 + 1)^h - 1 > M^*\) and let \(M = (\gamma_2 + 1)^h - 1\). Note that we can implement \(M\) from \(\gamma_2\) via an \(h\)-thickening, and \(h\) is at most a polynomial in \(m\).

Let \(\delta = (2 \max(|q|, |q|^{-1}))^m / M\). Let \(M\) be the constant weight function which gives every edge weight \(M\). We will use the following facts:

\[(6) \quad M^m q - \delta M^m |q| \leq Z_{st}(G; q, M) \leq M^m q + \delta M^m |q|\]

and

\[(7) \quad CM^{m-k} q^2 (1 - \delta) \leq Z_{st}^{(1)}(G; q, M) \leq CM^{m-k} q^2 (1 + \delta).\]

Fact (6) follows from the fact that each of the (at most \(2^m\)) terms in \(Z_{st}(G; q, M)\), other than the term with all edges in \(A\), has absolute value at most \(M^{m-1} \max(|q|, 1)^n\) and \(2^m M^{m-1} \max(|q|, 1)^n \leq \delta M^m |q|\). Fact (7) follows from the fact that all terms in \(Z_{st}^{(1)}(G; q, M)\) are \((s,t)\)-cuts. Each term that is not a size-\(k\) \((s,t)\)-cut has absolute value at most \(M^{m-k-1} q^2 \max(|q|, 1)^n\) and

\[2^m M^{m-k-1} q^2 (\max(|q|, 1))^n \leq \delta CM^{m-k} q^2.\]

For a parameter \(\varepsilon\) in the open interval \((0, 1)\) which we will tune below, let \(\gamma' = -1 - \varepsilon \in (-2, -1)\). We will discuss the implementation of \(\gamma'\) below.

Let \(G'\) be the graph formed from \(G\) by adding an edge from \(s\) to \(t\). Let \(\gamma\) be the edge-weight function for \(G'\) that assigns weight \(M\) to every edge of \(G\) and assigns weight \(\gamma'\) to the new edge. Then, using the definition of
the Tutte polynomial,

\[
Z(G'; q, \gamma) = Z_{st}(G; q, M)(1 + \gamma') + Z_{st}(G; q, M) \left( 1 + \frac{\gamma'}{q} \right) = -\epsilon Z_{st}(G; q, M) + Z_{st}(G; q, M) \left( 1 - \frac{1 + \epsilon}{q} \right).
\]

(8)

Now suppose \( \epsilon = M^{-2m} \). Then

\[
Z(G'; q, \gamma) = -M^{-2m}Z_{st}(G; q, M) + Z_{st}(G; q, M) \left( 1 - \frac{1 + M^{-2m}}{q} \right).
\]

Now since \( M > 2/(q-1) \) and \( M \geq 1 \), we have \( 1-(1+M^{-2m})/q \geq (1-1/q)/2 \). So, using (6) and (7),

\[
Z(G'; q, \gamma) \geq ((1-1/q)/2)CM^{m-k}q^2(1-\delta) - M^{-2m}M^m q(1+\delta),
\]

which is positive since \( k \leq m \). On the other hand, using the definition of \( M \) and Facts (6) and (7) above, we can confirm that when \( \epsilon = 1 \), \( Z(G'; q, \gamma) \) is negative. Also, when \( \epsilon = q-1 \) we have \( Z(G'; q, \gamma) = -(q-1)Z_{st}(G; q, M) \), which again is negative.

Thus we have a range from \( \epsilon = M^{-2m} \) to \( \epsilon = \min(1,q-1) \) of length at most 1 in which \( Z(G'; q, \gamma) \) changes sign. The idea is to perform binary search on this range to find an \( \epsilon \) where \( Z(G'; q, \gamma) = 0 \). For this value of \( \epsilon \), we have \( \epsilon Z_{st}(G; q, M) = Z_{st}(G; q, M) \left( 1 - \frac{1 + \epsilon}{q} \right) \). It turns out that, given this identity, estimates (6) and (7) above will give us enough information to calculate \( C \).

As one would expect, there are small technical complications. Since we are somewhat constrained in what values \( \epsilon \) we can implement, we won’t be able to discover the exact value of \( \epsilon \) that we need, but we will be able to approximate it sufficiently closely to compute \( C \) exactly from (6) and (7). Suppose for a moment that we are able, for a given \( \epsilon \in (M^{-2m}, \min(1,q-1)) \), to compute the sign of \( Z(G'; q, \gamma) \). Our basic strategy will be binary search, sub-dividing the initial interval \([m^2 \log M]\) times, so eventually we’ll get an interval of width at most \( M^{-m^2} \) which contains an \( \epsilon \) where \( Z(G'; q, \gamma) = 0 \).

To do this, we need to address the issue of computing the sign of \( Z(G'; q, \gamma) \) using an oracle for \( \text{SIGNTUTTE}(q; \gamma_1, \gamma_2) \). We have already seen above that is easy to implement the weight \( M \) using \( \gamma_2 \) (and that the implementation has polynomial size) — we now need to consider the implementation of \( \gamma' = -1 - \epsilon \) (where \( \epsilon \in (M^{-2m}, \min(1,q-1)) \) is the particular value that is being queried).

Let \( y' = -\epsilon \) be the point that we desire to implement. Let \( y_1 = \gamma_1 + 1 \). Note that \( y_1 \in (-1,0) \). Let \( j \) be the smallest odd integer so that \( |y_1|^j < \epsilon \). Let \( T^- = |y_1|^{-2} \) and \( T^+ = |y_1|^{-3} \). Let \( T = -\epsilon/y_1^{j+2} \). Note that \( 1 < T^- \leq T \leq T^+ \).

Let \( (x_2, y_2) = (q/\gamma_2 + 1, \gamma_2 + 1) \). Note that \( y_2 \notin [-1,1] \). We will define a small quantity \( \pi \) below. Looking ahead to Lemma 4 we see that, from
the point \((x_2, y_2)\) we can implement a point \((x''', y''')\) with \(T - \pi \leq y''' \leq T\). The size of the graph used to implement \((x''', y''')\) is at most a polynomial in \(\log(\pi^{-1})\). It does not depend upon \(T\), though it does depend on the fixed bounds \(T^-\) and \(T^+\). Now implement \(y'\) by a parallel composition of \(y''\) and \(j + 2\) copies of \(y_1\). (We can do this parallel composition because \(j\) is only polynomially big in \(m\).) Note that \(-\varepsilon \leq y' \leq -\varepsilon + \pi y_1\). So, since \(\delta\), \(\rho\) and \(\varepsilon\) are sufficiently large, we have \(\varepsilon \leq -\varepsilon\).

Thus, in the binary search, we may not be able to query the exact value of \(\varepsilon\) that we want to, but we can query a value that is between \(\varepsilon - \pi\) and \(\varepsilon\).

Recall that our goal is to end up with a sub-interval of the initial interval \((M^{-2m}, \min(1, q - 1))\) such that the subinterval has width at most \(M^{-m^2}\) and contains an \(\varepsilon\) where \(Z(G'; q, \gamma) = 0\). We do this by setting \(\pi = M^{-m^2}/3\) so that \(\pi\) is only a third as large as the smallest subinterval width (where we stop the binary search). We also adjust the binary search, sub-dividing the original interval up to \([m^2 \log_3/2 M]\) times rather than \([m^2 \log_2 M]\) times, to make up for the fact that we might end up with (crudely) at most two-thirds of the interval after one iteration, rather than half. The result, then, is that we can find a subinterval of width at most \(M^{-m^2}\) which contains an \(\varepsilon\) where \(Z(G'; q, \gamma) = 0\).

Now let \(\varepsilon\) be an endpoint of this subinterval. Let

\[\rho = 2^n \max(|q|, 1)^m M^m M^{-m^2}.\]

From the definition of the Tutte polynomial, we will have \(|Z(G'; q, \gamma)| \leq \rho\). Also, since \(\varepsilon \geq M^{-2m}\) and \(m\) is sufficiently large, we have \(\rho \leq \varepsilon M^m |q| 4^{-m}\).

Now using (8), (6) and (7), we have

\[\frac{-\rho + \varepsilon M^m q(1 - \delta)}{\left(1 - \frac{1 + \varepsilon}{q}\right) M^m k q^2(1 + \delta)} \leq C \leq \frac{\rho + \varepsilon M^m q(1 + \delta)}{\left(1 - \frac{1 + \varepsilon}{q}\right) M^m k q^2(1 - \delta)}\]

so, since \(\delta \leq 4^{-m}\),

\[\frac{(1 - 2 \cdot 4^{-m}) \varepsilon M^m q}{\left(1 - \frac{1 + \varepsilon}{q}\right) M^m k q^2(1 + 4^{-m})} \leq C \leq \frac{\varepsilon M^m q(1 + 2 \cdot 4^{-m})}{\left(1 - \frac{1 + \varepsilon}{q}\right) M^m k q^2(1 - 4^{-m})}\]

Now the point is that \(C\) is an integer between 1 and \(2^m\). Even though the value of \(k\) is not known, the fact that \(M > 4^m\) means that there can only be one integer \(k\) such that the above interval contains an integer between 1 and \(2^m\) (so \(k\) can easily be deduced). All of the other quantities in the lower and upper bounds in (9) are known. Now let \(R = \frac{\varepsilon M^k}{q(1 + \varepsilon)}\), so (9) becomes

\[\left(1 - \frac{1 + 2 \cdot 4^{-m}}{1 + 4^{-m}}\right) R \leq C \leq R \left(\frac{1 + 2 \cdot 4^{-m}}{1 - 4^{-m}}\right)\]

Now, \(R < 2^{m+1}\), since otherwise the left-hand-side of (10) is greater than \(2^m\). Also, multiplying through by \((1 + 4^{-m})(1 - 4^{-m})\), the width of the interval is at most \(6 \cdot 4^{-m} R < 1\) so the width of the interval in (10) is less than 1, so the (integral) value of \(C\) can be calculated exactly. \(\square\)
We have a similar lemma for $q < 1$.

**Lemma 2.** Suppose $q < 1$ and $q \neq 0$ and that $\gamma_1 \in (-1, 0)$ and $\gamma_2 \notin [-2, 0]$. Then $\text{SIGN}\text{TUTTE}(q; \gamma_1, \gamma_2)$ is $\#P$-hard.

**Proof.** The situation is very similar to that of Lemma 1.

We start with the situation $0 < q < 1$. In this case, we follow the proof of Lemma 1. Then Facts (6) and (7) hold, as before. For the tuneable parameter $\varepsilon \in (0, 1)$, we let $\gamma' = -1 + \varepsilon \in (-1, 0)$. Implementing $G'$ as in the proof of Lemma 1, we have

\begin{equation}
Z(G'; q, \gamma) = \varepsilon Z_{st}(G; q, M) + Z_{sl}(G; q, M) \left(1 - \frac{1 - \varepsilon}{q}\right).
\end{equation}

Now, suppose $\varepsilon = M^{-2m}$. Then since $M > 2/(1-q)$ and $M \geq 1$, we have

$$1 - (1 - M^{-2m})/q \leq \frac{1}{2}(1 - 1/q) < 0.$$

Using Facts (6) and (7), we find that $Z(G'; q, \gamma)$ is negative. On the other hand, at $\varepsilon = 1 - q$, $Z(G'; q, \gamma)$ is positive.

To implement $\gamma'$, let $y' = \varepsilon$ be the point that we desire to implement. Let $y_1 = \gamma_1 + 1$. Note that $y_1 \in (0, 1)$. Now proceed as in the proof of Lemma 1 with $T = \varepsilon/y_1^{1+2}$, and $T^-$ and $T^+$ as before. Once again we find a subinterval of $(M^{-2m}, 1-q)$ of width at most $M^{-m^2}$ which contains an $\varepsilon$ where $Z(G'; q, \gamma) = 0$, so we let $\varepsilon$ be an endpoint of this subinterval and we conclude that $|Z(G'; q, \gamma)| \leq \rho$. Now we finish as in the proof of Lemma 1.

The argument for $q < 0$ also follows the proof of Lemma 1. Here, $Z_{st}(G; q, M)$ is negative and $Z_{sl}(G; q, M)$ is positive. Taking $\gamma' = -1 + \varepsilon$, as above, we still have (11). Now, suppose $\varepsilon = M^{-2m}$. Then by (11), $Z(G'; q, \gamma) \geq M^{-2m}Z_{st}(G; q, M) + Z_{sl}(G'; q, M)$, which is positive. On the other hand, at $\varepsilon = 1$, $Z(G'; q, \gamma)$ is negative. Now the implementation of $\gamma'$ proceeds as above, except that we use Lemma 1 (working from points $(x_1, y_1)$ and $(x_2, y_2)$) instead of Lemma 1.

So we find a subinterval of $(M^{-2m}, 1)$ of width at most $M^{-m^2}$ which contains an $\varepsilon$ where $Z(G'; q, \gamma) = 0$. Letting $\varepsilon$ be an endpoint of this subinterval, we conclude that $|Z(G'; q, \gamma)| \leq \rho$. Now we finish as in the proof of Lemma 1. \hfill \square

### 4. Implementing new edge weights

In this section, we collect the information that we need about implementing edge weights within various regions of the Tutte Plane. The following straightforward lemmas are useful.

**Lemma 3.** Suppose $q > 0$ and that $(x, y)$ is a point with $x < -1$. Then $(x, y)$ can be used to implement a point $(x', y')$ with $y' > 1$.

**Proof.** A 2-stretch from $(x, y)$ suffices since it implements the point $(x', y') = (x^2, (x+y)/(1+x))$. If $x < -1$ and $q = (x-1)(y-1)$ is positive then $y < 1$. 

so \( x + y \) and \( 1 + x \) are both negative. Since \( y < 1 \) we conclude that \(-y > -1\) so \(-x - y > -1 - x \) and \( y' > 1 \). \( \square \)

We will use the following Lemma, which is [5, Lemma 3.26]. The lemma in [5] was stated for \( q > 5 \) (which was all that was needed in that paper) but the proof only uses \( q > 0 \). The statement in [5] was in terms of the coordinates \( q \) and \( \gamma \) but we have translated it to \((x,y)\) coordinates here, since that is how it will be used here. Finally, the statement of the lemma in [5] allowed the implementation to use two additional points \((x'_2, y'_2)\) and \((x'_3, y'_3)\) (this was to make the statement of the lemma match other lemmas in that paper). However, these additional points were not used in the proof, so we don’t include them here.

Lemma 4. ([5, Lemma 3.26]) Suppose that \((x_1, y_1)\) is a point with \( y_1 \notin [-1, 1] \) and that \( q = (x_1 - 1)(y_1 - 1) > 0 \). Suppose that \( T^- \) and \( T^+ \) satisfy \( 1 < T^- \leq T^+ \). Given a target edge-weight \( T \in [T^-, T^+] \) and a positive value \( \pi \) which is sufficiently small with respect to \( x_1, y_1, T^- \) and \( T^+ \), a point \((x, y)\) with \( T - \pi \leq y \leq T \) can be implemented using the point \((x_1, y_1)\). The size of the graph \( \Upsilon \) used to implement \((x, y)\) is at most a polynomial in \( \log(\pi^{-1}) \). (This upper bound on the size of \( \Upsilon \) does not depend on \( T \), though it does depend on the fixed bounds \( T^- \) and \( T^+ \).)

By duality of \( x \) and \( y \), we have the following corollary.

Corollary 5. Suppose that \((x_1, y_1)\) is a point with \( x_1 \notin [-1, 1] \) and that \( q = (x_1 - 1)(y_1 - 1) > 0 \). Suppose that \( T^- \) and \( T^+ \) satisfy \( 1 < T^- \leq T^+ \). Given a target edge-weight \( T \in [T^-, T^+] \) and a positive value \( \pi \) which is sufficiently small with respect to \( x'_1, y'_1, T^- \) and \( T^+ \), a point \((x, y)\) with \( T - \pi \leq x \leq T \) can be implemented using the point \((x_1, y_1)\). The size of the graph \( \Upsilon \) used to implement \((x, y)\) is at most a polynomial in \( \log(\pi^{-1}) \). (This upper bound on the size of \( \Upsilon \) does not depend on \( T \), though it does depend on the fixed bounds \( T^- \) and \( T^+ \).)

We will also use the following related lemma, which is [5, Lemma 3.27]. Once again, we translated to \((x,y)\) coordinates and eliminated unused points.

Lemma 6. ([5, Lemma 3.27]) Suppose that \((x_1, y_1)\) is a point with \( y_1 \notin [-1, 1] \) and \((x_2, y_2)\) is a point with \( y_2 \in (-1, 1) \). Suppose that \( q = (x_1 - 1)(y_1 - 1) = (x_2 - 1)(y_2 - 1) < 0 \). Suppose that \( T^- \) and \( T^+ \) satisfy \( 1 < T^- \leq T^+ \). Given a target edge-weight \( T \in [T^-, T^+] \) and a positive value \( \pi \) which is sufficiently small with respect to \( x_1, y_1, x_2, y_2, T^- \) and \( T^+ \), a point \((x, y)\) with \( T - \pi \leq y \leq T \) can be implemented using the points \((x_1, y_1)\) and \((x_2, y_2)\). The size of the graph \( \Upsilon \) used to implement \((x, y)\) is at most a polynomial in \( \log(\pi^{-1}) \). (This upper bound on the size of \( \Upsilon \) does not depend on \( T \), though it does depend on the fixed bounds \( T^- \) and \( T^+ \).)

The reader may find it useful to consult Figure 1 and the formal definitions listed early in Section 5 to see the relevant regions of the \((x,y)\) plane that we consider.
4.1. Region B. The following four lemmas prepare the conditions for applying Lemma 1 to points in Region B. Note that the value \( q = (x-1)\) \( (y-1) \) exceeds 1 in this region.

**Lemma 7.** Suppose \((x, y)\) is a point with \(x < -1\) and \(y < -1\). Then we can use \((x, y)\) to implement a point \((x_1, y_1)\) with \(y_1 \in (-1, 0)\) and a point \((x_2, y_2)\) with \(y_2 \notin [-1, 1]\).

**Proof.** Let \( q = (x-1)\) \((y-1) \). Let \( j \) be an odd positive integer which is sufficiently large that \( |x|^2 + 1 > q \). Implement \((x', y') = (x^j, q/(x^j - 1) + 1)\) from \((x, y)\) with a \(j\)-stretch. Note that \( y' \in (0, 1) \). Now, for a sufficiently large positive integer \( k \), implement \((x_1, y_1)\) using the parallel composition of \((x, y)\) with \( k \) copies of \((x', y')\) so \( y_1 = y'^k y \in (-1, 0) \). Finally, let \((x_2, y_2) = (x, y)\).

**Lemma 8.** Suppose \((x, y)\) is a point with \(x < -1\) and \(y = -1\). Then we can use \((x, y)\) to implement a point \((x_1, y_1)\) with \(y_1 \in (-1, 0)\) and a point \((x_2, y_2)\) with \(y_2 \notin [-1, 1]\).

**Proof.** Let \( j \) be a sufficiently large odd integer such that \( q/\) \( (|x|^2 + 1) < 1 \). Implement \((x', y')\) using a \(j\)-stretch from \((x, y)\) so that \( y' = q/(x^j - 1) + 1 \in (0, 1) \). Implement \((x_1, y_1)\) by taking a parallel composition of \((x', y')\) and \((x, y)\) so \( y_1 = -y' \). Finally, implement \((x_2, y_2)\) from \((x, y)\) using Lemma 3.

**Lemma 9.** Suppose \((x, y)\) is a point with \(x < -1\) and \(-1 < y < 0\). Then we can use \((x, y)\) to implement a point \((x_1, y_1)\) with \(y_1 \in (-1, 0)\) and a point \((x_2, y_2)\) with \(y_2 \notin [-1, 1]\).

**Proof.** We let \((x_1, y_1) = (x, y)\). We implement \((x_2, y_2)\) from \((x, y)\) using Lemma 3.

**Lemma 10.** Suppose \((x, y)\) is a point with \(-1 \leq x < 0\) and \(y < -1\). Then we can use \((x, y)\) to implement a point \((x_1, y_1)\) with \(y_1 \in (-1, 0)\) and a point \((x_2, y_2)\) with \(y_2 \notin [-1, 1]\).

**Proof.** Implement \((x_a, y_a)\) by a 2-thickening of \((x, y)\). Note that \( y_a = y^2 > 1 \), and therefore, since \( q > 0 \), \( x_a > 1 \) as well. Let \( j \) be an integer that is sufficiently large that \( |x| \cdot x_a^j + 1 > q \). Implement \((x_b, y_b)\) by a series composition of \((x, y)\) with \( j \) copies of \((x_a, y_a)\) so that \( y_b = q/(x_a^j - 1) + 1 \in (0, 1) \).

Let \( k \) be a sufficiently large integer that \( |y| y_b^k \in (0, 1) \). Implement \((x_1, y_1)\) by a parallel composition of \((x, y)\) and \( k \) copies of \((x_b, y_b)\) so \( y_1 = y y_b^k \). Finally, let \((x_2, y_2) = (x, y)\).

4.2. Regions G, H and I. We next consider the problem of implementing edge weights starting from a point in the “vicinity of the origin”, which corresponds to points with \( |x| < 1 \) and \( |y| < 1 \). In the vicinity of the
origin, we have \(0 < q < 4\). As noted in the introduction, there is a “phase transition” at \(q = 32/27\), so we start by considering \(q > 32/27\).

**Lemma 11.** Suppose \((x, y)\) is a point with \(|x| < 1\) and \(|y| < 1\) and \(q = (x - 1)(y - 1) > 32/27\). Then \((x, y)\) can be used to implement a point \((x', y')\) with \(y' > 1\).

**Proof.** We will use the “diamond operation” of Jackson and Sokal [8, Section 8]. This corresponds to choosing the graph \(\Upsilon\) with vertex set \(\{s, t, u, v\}\) and edge set \(\{(s, u), (u, t), (s, v), (v, t)\}\). \(\Upsilon\) is a parallel composition of two paths from \(s\) to \(t\), each of which is formed from the series composition of two edges. If we start with the weight function \(\gamma\) that assigns weight \(\gamma\) to every edge of \(\Upsilon\), then it is easy to check (see [8, (8.1)]) that the implemented weight \(w^*\) from Equation (4) is \(\frac{2^2(2^2 + 1)}{(q + 2)^2}\). Equivalently, the point implemented from \((x, y)\) (which we denote as \((\Diamond_{q,1}(x, y), \Diamond_{q,2}(x, y))\)) is given by

\[
(\Diamond_{q,1}(x, y), \Diamond_{q,2}(x, y)) = \left( \frac{x + x^2 + x^3 + y}{1 + 2x + y}, \frac{(x + y)^2}{(1 + x)^2} \right).
\]

The diamond operation is well-defined as long as \(x \neq -1\) and \(y \neq -1 - 2x\). Jackson and Sokal [8, Lemma 8.5(c)] prove that if you start from a point \((x_1, y_1)\) with \(y_1 < 1\) and \(q > 32/27\) and apply a sequence of diamond operations for \(j = 1, 2, \ldots\) with \((x_{j+1}, y_{j+1}) = (\Diamond_{q,1}(x_j, y_j), \Diamond_{q,2}(x_j, y_j))\) then for each \(j\), we have \(y_{j+1} > y_j\) and there is a \(k\) such that \(y_k \geq 1\). Their analysis allows the situation \(x_{k-1} = -1\), so the terminating point has \(y_k = \infty\) (which would not give an implementation of a finite \(y_k > 1\), which we require) and it also allows \(y_{k-1} = -1 - 2x_{k-1}\) which gives \(y_k = 1\) (whereas we require \(y_k > 1\)).

We start with \((x_1, y_1) = (x, y)\) and apply the sequence of diamond operations until we reach a point \((x_j, y_j)\) with \(y_j > 1\). However, there are two exceptions.

First, suppose, for some \(j\), that \(y_j = -1 - 2x_j\). then instead of taking \((x_{j+1}, y_{j+1}) = (\Diamond_{q,1}(x_j, y_j), \Diamond_{q,2}(x_j, y_j))\) we define \((x_{j+1}, y_{j+1})\) as follows: We let \((x'_1, y'_1) = (x_j^2, -1)\) be the point implemented by a series composition of two copies of \((x_j, y_j)\). We then let \((x'_2, y'_2) = (x_j^4, (x_j^2 - 1)/(x_j + 1))\) be the point implemented by a series composition of two copies of \((x'_1, y'_1)\). Finally, we let \((x_{j+1}, y_{j+1}) = (1 - x_j - 2x_j, (1 - x_j^2)/(1 + x_j^2))\) be the point implemented by a parallel composition of \((x'_1, y'_1)\) and \((x'_2, y'_2)\). Note that \(y_{j+1} - y_j = 2(x_j^3 + x_j + 1)/(x_j + 1)\). Now note that \(q = 2 - 2x_j^2\) so, since \(q \geq 32/27\), we have \(x_j > -0.64\). Thus, \(y_{j+1} - y_j\) is positive, as required (the denominator is always positive, and the numerator is positive for \(x_j \geq -0.68\)). Note that exceptional points \((x_j, y_j)\) where \(y_j = -1 - 2x_j\) arise at most twice during the sequence of points \((x_1, y_1), (x_2, y_2), \ldots\) since the hyperbola \((x - 1)(y - 1) = q\) only intersects the line \(y = -1 - 2x\) in at most two places. Also, \(y_{j+1} \neq 1\), so the sequence does not terminate incorrectly at \((x_{j+1}, y_{j+1})\).
For the second exception, suppose that we get to a point \((x_j, y_j)\) with \(x_j = -1\). Then \((x_j, y_j) = (-1, -q/2 + 1)\). Now, \(j \neq 1\) since we start in the vicinity of the origin (so we don’t have \(x_1 = -1\)). If \((x_j, y_j)\) was obtained as a result of the exceptional case above, then \(q < 2\) (since then \(q = 2 - 2x_{j-1}^2\) and \(x_{j-1} \neq 0\) since that would imply \(y_{j-1} = -1\), contrary to the fact that the \(y\)’s are all above \(-1\)). Otherwise, \((x_j, y_j)\) was obtained as the result of a diamond operation. It is not possible that \(x = (\cdots, x_j, y_j, \cdots, 0, -1, 0, \cdots)\) and \(y = (\cdots, y_j, \cdots, 0, -1, 0, \cdots)\).

Proof. Let \((x, y) = (\cdots, x_j, y_j, \cdots, 0, -1, 0, \cdots)\). Then \((x_j, y_j) = (\cdots, x_j-1, y_j-1, \cdots, 0, -1, 0, \cdots)\). Suppose that \((x_j, y_j) = (\cdots, x_j-1, y_j-1, \cdots, 0, -1, 0, \cdots)\). Then use Lemma 3. □

**Lemma 12.** Consider a point \((x, y)\) such that \(y < -1 - 2x\) and \(x > -1\). Then \((x, y)\) can be used to implement a point \((x', y')\) with \(y' > 1\).

**Proof.** Let \((x'', y'') = (x^2, \frac{x + y}{1 + x})\) be the point implemented by a 2-stretch from \((x, y)\). Note that \(y'' < -1\). Now implement \((x', y')\) by a 2-thickening of \((x'', y'')\). □

**Lemma 13.** Consider a point \((x, y)\) such that \(x < -1 - 2y\) and \(y > -1\) and \(q = (x-1)(y-1) > 0\). Then \((x, y)\) can be used to implement a point \((x', y')\) with \(y' > 1\).

**Proof.** Let \((x', y') = (\frac{x + y}{1 + y}, y^2)\) be the point implemented by a 2-thickening. Note that \(x' < -1\). Then use Lemma 3. □

**Lemma 14.** Suppose that \((x, y)\) is a point satisfying \(\max(|x|, |y|) < 1\) and \(q = (x-1)(y-1) > 1\). Suppose that \((x, y)\) also satisfies at least one of the following conditions.

- \(q > 32/27\), or
- \(y < -1 - 2x\), or
- \(x < -1 - 2y\).

Then \((x, y)\) can be used to implement a point \((x_1, y_1)\) with \(-1 < y_1 < 0\).

**Proof.** If \(-1 < y < 0\) then we simply take \((x_1, y_1) = (x, y)\). Thus, we can assume \(0 \leq y < 1\). This implies \(-1 < x < 0\), and \(q > 32/27\) or \(y < -1 - 2x\).

By Lemmas 11 and 12 we can implement a point \((x'_1, y'_1)\) with \(y'_1 > 1\). Since \((x'_1 - 1)(y'_1 - 1) = q\), we also have \(x'_1 > 1\).

Note that the restrictions on \(x\) and \(y\) imply \(1 < q < 2\). Choose an even integer \(j\) so that \(x^j < 1 - q/4\). By Corollary 5 (taking \(T = (1 - q/4)/x^j\)}
and \( \pi = q/(8x^j) \), say) the point \((x', y')\) can be used to implement a point \((x'', y'')\) with

\[
\frac{1 - q/2}{x^j} < x'' < \frac{1}{x^j}.
\]

Implement \((x^*, y^*)\) by taking the series composition of \((x'', y'')\) with \(j\) copies of \((x, y)\). Note that \(y^* = \frac{q-1}{y^j} + 1 < -1\).

Now implement \((x_1, y_1)\) by choosing a sufficiently large integer \(\ell\) and taking the parallel composition of \((x^*, y^*)\) with \(\ell\) copies of \((x, y)\) so that \(y_1 = y^*y^\ell\).

\[\square\]

4.3. Regions \(C\) and \(D\).

Lemma 15. Suppose \((x, y)\) is a point satisfying one of the following.

- \(y > 1\) and \(x < -1\), or
- \(x > 1\) and \(y < -1\).

Then \((x, y)\) can be used to implement a point \((x_1, y_1)\) with \(y_1 \in (0, 1)\).

Proof. Note that \(q < 0\). Choose an even number \(j\) such that \(x^j - 1 > |q|\). Implement \((x_1, y_1)\) by taking a \(j\)-stretch of \((x, y)\) so \(y_1 = q/(x^j - 1) + 1\). \[\square\]

4.4. Region \(E\).

Lemma 16. Suppose \((x, y)\) is a point satisfying \(x < -1\) and \(0 < y < 1\) and \(1 < (x - 1)(y - 1) < 2\). Then \((x, y)\) can be used to implement a point \((x_1, y_1)\) with \(-1 < y_1 < 0\).

Proof. Let \(q = (x - 1)(y - 1)\). Note that \(1 - q/2 > 0\) since \(q < 2\). Let \(j\) be a sufficiently large integer that \(0 < y^j < 1 - q/2\). Note that \(1 - q < 0\) so \(1 - q < y^j < 1 - q/2\). Implement \((x', y')\) by \(j\)-thickening from the point \((x, y)\) so \(x' = q/(y^j - 1) + 1\). Note that \(-1 < x' < 0\). Now let \(k\) be an odd integer which is sufficiently large that \(0 < x(x')^k < 1 - q/2\) so \(1 - q < x(x')^k < 1 - q/2\). Implement \((x_1, y_1)\) by taking a series composition of \((x, y)\) with \(k\) copies of \((x', y')\) so \(x_1 = x(x')^k\). Then \(y_1 = q/(x(x')^k - 1) + 1\) so \(-1 < y_1 < 0\), as required. \[\square\]

Lemma 17. Suppose \((x, y)\) is a point satisfying \(x < -1\) and \(0 < y < 1\).

Suppose that \(q = (x - 1)(y - 1) > 2\) is not an integer. Then \((x, y)\) can be used to implement a point \((x', y')\) with \(y' < 0\).

Proof. Let \(q = (x - 1)(y - 1)\). Let us first examine what points we can implement from the point \((x_1, y_1) = (1 - q, 0)\) and from points nearby. We will later show how to implement points near \((x_1, y_1)\) from the given point \((x, y)\). Let \(n = \lfloor q \rfloor + 2\). Note that \(n \geq 4\) and that \(n - 2 < q < n - 1\). Let \(\Gamma_n\) be the graph \(K_n - \{(s, t)\}\). Let \(\gamma\) be the weight function that gives every edge of \(\Gamma_n\) weight \(y_1 - 1 = -1\). From Section 2.2 the graph \(\Gamma_n\) and the weight function \(\gamma\) implement the weight

\[
w(q, n) = \frac{q\mathcal{Z}_{st}(\Gamma_n; q, -1)}{\mathcal{Z}_{st}(\Gamma_n; q, -1)}.
\]
We wish to calculate some properties of \( w(q, n) \). Recall from the introduction that \( Z(G; q, -1) \) is equal to the chromatic polynomial \( P(G; q) \). We will next calculate \( Z_{st}(\Gamma_n; q, -1) \) and \( Z_{st|t}(\Gamma_n; q, -1) \) as polynomials in \( q \) using known facts about the chromatic polynomial. In particular, when \( q \) is a positive integer, \( P(G; q, -1) \) gives the number of proper \( q \)-colourings of \( G \).

Now, let \( V \) denote the vertex set of \( K_n \). We can expand the definition of \( Z(K_n; q, -1) \) as

\[
Z(K_n; q, -1) = \sum_{A \subseteq E - \{s, t\}} \left( q^{\kappa(V, A \cup \{(s, t)\})}(-1)^{|A|+1} + q^{\kappa(V, A)}(-1)^{|A|} \right),
\]

If a subset \( A \) connects \( s \) and \( t \) then \( \kappa(V, A \cup \{(s, t)\}) = \kappa(V, A) \) so the contribution from this \( A \) is zero. On the other hand, if a subset \( A \) does not connect \( s \) and \( t \) then \( \kappa(V, A \cup \{(s, t)\}) = 1 + \kappa(V, A) \). Thus,

\[
Z(K_n; q, -1) = Z_{st}(\Gamma_n; q, -1)(1 - 1) + Z_{st|t}(\Gamma_n; q, -1)(1 - \frac{1}{q})
\]

\[
= Z_{st|t}(\Gamma_n; q, -1)(1 - \frac{1}{q}).
\]

Note that the factor \((1 - \frac{1}{q})\) is positive.

Similarly,

\[
Z_{st}(\Gamma_n; q, -1) = Z(\Gamma_n; q, -1) - Z_{st|t}(\Gamma_n; q, -1),
\]

so we have

\[
Z_{st}(\Gamma_n; q, -1) = Z(\Gamma_n; q, -1) - \frac{Z(K_n; q, -1)}{1 - \frac{1}{q}},
\]

and

\[
Z_{st|t}(\Gamma_n; q, -1) = \frac{Z(K_n; q, -1)}{1 - \frac{1}{q}}.
\]

We wish to calculate some properties of \( w(q, n) \) using the definition (12) and Equations (14) and (15).

Clearly, \( Z(K_n; q, -1) \) is the number of \( q \)-colourings of an \( n \)-clique, which is \( \prod_{i=0}^{n-1} (q - i) \), since we can colour the vertices in order, starting with the \( 0 \)th, and there are \( n - i \) choices for the \( i \)th vertex. Let \( N_{q, n} = \prod_{i=0}^{n-2} (q - i) \), so \( Z(K_n; q, -1) = N_{q, n}(q - n + 1) \). Then \( Z(\Gamma_n; q, -1) = N_{q, n}(q - n + 2) \), since if you colour the vertices in order, colouring \( s \) last, there are \( q - (n - 2) \) choices for \( s \), rather than \( q - (n - 1) \) in \( K_n \). Then, from (12), (14) and (15),
\begin{equation}
    w(q, n) = \frac{qZ_{st}(\Gamma_n; q, -1)}{Z_{s|t}(\Gamma_n; q, -1)} = \frac{qZ(\Gamma_n; q, -1) - qZ(K_n; q, -1)}{Z(K_n; q, -1)} = \frac{(q - 1)(q - n + 2) - q(q - n + 1)}{q - n + 1} = \frac{n - 2}{q - n + 1},
\end{equation}

where we use the fact that \( q \) is not integral, so \( Z(K_n; q, -1) \neq 0 \).

Now since \( n > 2 \) and \( 1 < q < n - 1 \) we can see that the numerator \( q - n + 1 \) is positive and \( n - 2 > n - q - 1 \), and hence \( w(q, n) < -1 \).

We now have

\begin{equation}
    \frac{qZ_{st}(\Gamma_n; q, -1)}{Z_{s|t}(\Gamma_n; q, -1)} < -1.
\end{equation}

Unfortunately, we are not finished, because we cannot necessarily implement the weight \(-1\) exactly from the given point \((x, y)\). However, by continuity, Equation (10) implies that there is a small positive \( \varepsilon \) (depending on \( q \) and \( n \)) such that, if \( |z - Z_{st}(\Gamma_n; q, -1)| \leq \varepsilon \) and \( |z' - Z_{s|t}(\Gamma_n; q, -1)| \leq \varepsilon \), then we have \( \frac{qZ}{Z_{s|t}} < -1 \).

To finish, we will show that we can implement an edge weight \(-1 + \delta\) from \((x, y)\) so that \( |Z_{st}(\Gamma_n; q, -1 + \delta) - Z_{st}(\Gamma_n; q, -1)| \leq \varepsilon \) and \( |Z_{s|t}(\Gamma_n; q, -1 + \delta) - Z_{s|t}(\Gamma_n; q, -1)| \leq \varepsilon \). Thus, we can implement an edge-weight less than \(-1\) by using \( \Gamma_n \) with all edge weights equal to \(-1 + \delta\).

We finish with the relevant technical details. First, let \( V_n \) be the vertex set of \( \Gamma_n \). For any \( \delta \in (0, \varepsilon/(2^n q^n m)) \), note that

\[
Z_{st}(\Gamma_n; q, -1 + \delta) - Z_{st}(\Gamma_n; q, -1) = \sum_A q^\kappa(V_n, A)(-1)^{|A|+1} \left(1 - (1 - \delta)^{|A|}\right) \\
\leq \sum_A q^\kappa(V_n, A) \left(1 - (1 - \delta)^{|A|}\right) \\
\leq 2^n q^n m \delta \\
< \varepsilon,
\]

where the sum is over edge subsets \( A \) with \( s \) and \( t \) in the same component. Similarly, \( Z_{st}(\Gamma_n; q, -1) - Z_{st}(\Gamma_n; q, -1 + \delta) < \varepsilon \) and \( |Z_{s|t}(\Gamma_n; q, -1 + \delta) - Z_{s|t}(\Gamma_n; q, -1)| \leq \varepsilon \).

It remains to show that we can implement weight \(-1 + \delta\) from the given \((x, y)\). Using \((x, y)\) coordinates, the point that we wish to implement is \((x'', y'') = (1 + q/(\delta - 1), \delta)\). This can be done using a \( k\)-thickening from \((x, y)\), choosing \( k \) to be sufficiently large that \( y^k \leq \varepsilon/(2^n q^n m) \).

\[\Box\]
As we shall see shortly, Region B consists of those points \((x, y)\) for which 
\[
\min(x, y) \leq -1 \text{ and } \max(x, y) < 0.
\]
Also, Region G consists of points \((x, y)\) with 
\[
\max(|x|, |y|) < 1 \text{ and } q = (x - 1)(y - 1) > 32/27.
\]
We use these definitions in the following lemma.

**Lemma 18.** Suppose \((x, y)\) is a point satisfying \(x < -1\) and \(0 < y < 1\). Suppose that \(q = (x - 1)(y - 1) > 2\) is not an integer. Then \((x, y)\) can be used to implement a point \((x', y')\) apart from the special point \((-1, -1)\) which is either in Region B or in Region G.

**Proof.** By Lemma 17, the point \((x, y)\) can be used to implement a point \((x', y')\) with 
\[
y' < 0.
\]
We know that \((x', y')\) is not the special point \((-1, -1)\) since \(q\) is not an integer. If \((x', y')\) is in Region B or Region G, then we are finished. Otherwise, the point \((x', y')\) satisfies 
\[
0 \leq x' < 1 \text{ and } y' \leq -1.
\]
Let \(j\) be a sufficiently large integer so that 
\[
|y'|/j < 1.
\]
Then implement the point \((x'', y'')\) by taking the parallel composition of \((x', y')\) with \(j\) copies of \((x, y)\) so 
\[
y'' = y'j.
\]
Note that \(-1 < y'' < 0\) so the point \((x'', y'')\) is in Regions B or G, as required.

4.5. **The Flow Polynomial.** In order to implement new edge weights from Region F (and also to show tractability results and NP-completeness results for Region F in Section 6.5) we must introduce a specialisation of the Tutte polynomial called the flow polynomial.

A **q-flow** of an undirected graph \(G = (V, E)\) is defined as follows [14, Section 2.4]. Choose an arbitrary direction for each edge. Let \(H\) be any Abelian group of order \(q\). A q-flow is a mapping \(\psi : E \rightarrow H\) such that the flow into each vertex is equal to the flow out (doing arithmetic in \(H\)).

Consider the following polynomial, where the sum is over q-flows of \(G\) (see [14, (2.21)])

\[
F(G; q, u) = \sum_{\psi} \prod_{e \in E} (1 + u\delta(\psi(e), 0)),
\]

where \(\delta\) is the Kronecker delta function defined by \(\delta(a, b) = 1\) if \(a = b\) and \(\delta(a, b) = 0\) otherwise. This polynomial is related to the Tutte polynomial via the following identity [14, (2.22)].

**Fact 19.** If \(q\) is a positive integer then 
\[
F(G; q, q/\gamma) = q^{-|V|}(|q| |E|) Z(G; q, \gamma).
\]

The **flow polynomial** of \(G\), which we write as \(F(G; q)\), is given by \(F(G; q, -1)\). A q-flow \(\psi\) of a graph \(G = (V, E)\) is said to be nowhere-zero if, for every \(e \in E\), \(\psi(e) \neq 0\). From Fact 19 it is easy to see that if \(q\) is a positive integer then \(F(G; q) = q^{-|V|}(-1)^{|E|} Z(G; q, -q)\) is the number of nowhere-zero q-flows of \(G\).

---

\(^3\)It is a non-trivial fact that \(F(G; q, u)\) depends only on \(q\), the size of \(H\), and not on \(H\) itself.
4.6. Region F.

Lemma 20. Suppose \((x, y)\) is a point satisfying \(0 < x < 1\) and \(y < -1\) and \(0 < (x-1)(y-1) < 1\). Then \((x, y)\) can be used to implement a point \((x_1, y_1)\) with \(0 < y_1 < 1\).

Proof. Let \(j\) be a sufficiently large positive integer such that \(x^j < 1 - q\). Implement \((x_1, y_1)\) by a \(j\)-stretch of \((x, y)\) so that \(y_1 = q/(x^j - 1) + 1\). □

Lemma 21. Suppose \((x, y)\) is a point satisfying \(0 < x < 1\) and \(y < -1\) and \(1 < (x-1)(y-1) < 2\). Then \((x, y)\) can be used to implement a point \((x_1, y_1)\) with \(-1 < y_1 < 0\).

Proof. Let \(q = (x-1)(y-1)\). Note that \(1 - q/2 > 0\) since \(q < 2\). Let \(j\) be a sufficiently large integer that \(0 < x^j < 1 - q/2\). Note that \(1 - q < 0\) so \(1 - q < x^j < 1 - q/2\). Implement \((x_1, y_1)\) by \(j\)-stretch from the point \((x, y)\) so that \(y_1 = q/(x^j - 1) + 1\). Note that \(-1 < y_1 < 0\). □

Lemma 22. Suppose \((x, y)\) is a point satisfying \(0 < x < 1\) and \(y < -1\) for which \(q = (x-1)(y-1)\) is not an integer. Suppose \(2 < q < 4\). Then \((x, y)\) can be used to implement a point \((x', y')\) with \(x' < 0\).

Proof. Let \(q = (x-1)(y-1)\). As in the proof of Lemma 17 we start by examining what points we can implement from the point \((x', y') = (0, 1 - q)\) and from points nearby.

Suppose that \(G\) is a graph which contains the edge \((s, t)\). Let \(\Gamma = G - (s, t)\). Following the approach of Lemma 17 let

\[
(17)\quad w(q) = \frac{qZ_{st}(\Gamma; q, -q)}{Z_{s|-t}(\Gamma; q, -q)},
\]

which is the weight implemented by \(\Gamma\) with edge weight \(-q\).

Then, using similar reasoning to the derivation of \(13\),

\[
Z(G; q, -q) = Z_{st}(\Gamma; q, -q)(1 - q) + \frac{1}{q}Z_{s|-t}(\Gamma; q, -q)(q - q)
= Z_{st}(\Gamma; q, -q)(1 - q) + Z(G; q, -q)/(q - 1).
\]

(18)

Also,

\[
Z_{s|-t}(\Gamma; q, -q) = Z(\Gamma; q, -q) - Z_{st}(\Gamma; q, -q)
= Z(\Gamma; q, -q) + Z(G; q, -q)/(q - 1).
\]
Thus, we can use Fact [19] to see that
\[
    w(q) = \frac{qZ_{st}(\Gamma; q, -q)}{Z_{sl}(\Gamma; q, -q)}
\]
\[
    = -q \left( \frac{Z(G; q, -q)/(q - 1)}{Z(\Gamma; q, -q) + Z(G; q, -q)/(q - 1)} \right)
\]
\[
    = -q \left( \frac{Z(G; q, -q)}{(q - 1)Z(\Gamma; q, -q) + Z(G; q, -q)} \right)
\]
\[
    = -q \left( \frac{F(G; q)}{F(\Gamma; q) - (q - 1)F(\Gamma; q)} \right).
\]

First, suppose 2 < q < 3. Following the reasoning in Lemma [17], we will show below that, for a suitable G, F(G; q) > 0 and F(\Gamma; q) < 0. Together, these imply that the denominator \(F(G; q) - (q - 1)F(\Gamma; q)\) is positive and also that it is larger than the numerator \(F(G; q)\). Thus, \(w(q) < 0\) and \(w(q) > -q\). (It is our goal to implement a \(\gamma'\) in the range \(-q < \gamma' < 0\) since, for this \(\gamma'\), \(q/\gamma' + 1 < 0\), so the corresponding x-coordinate is less than 0.)

By continuity, there is a positive \(\varepsilon\) (which depends upon \(q\) and \(G\)) such that, if \(|z - Z_{st}(\Gamma; q, -q)| \leq \varepsilon\) and \(|z' - Z_{sl}(\Gamma; q, -q)| \leq \varepsilon\), then \(-q < \frac{\varepsilon}{\gamma'} < 0\). As in the proof of Lemma [17] we can show that, for a sufficiently small \(\delta \in (0, 1)\), \(|Z_{st}(\Gamma; q, -q - \delta) - Z_{st}(\Gamma; q, -q)| \leq \varepsilon\) and \(|Z_{st}(\Gamma; q, -q - \delta) - Z_{sl}(\Gamma; q, -q)| \leq \varepsilon\). Then we finish by implementing the weight \(-q - \delta\) from the given \((x, y)\) using a large stretch so
\[
    (x', y') = (x, q/(x - 1) + 1) = ((\delta / (q + \delta)) + 1 - q - \delta).
\]

For 3 < q < 4 the proof will be similar except that we will establish \(F(G; q) < 0\) and \(F(\Gamma; q) > 0\) so that the denominator of the final expression for \(w(q)\) is negative and is larger in absolute value than the numerator.

To complete the proof, we must establish that \(F(G; q)\) and \(F(\Gamma; q)\) have different signs. Let \(G\) be the Petersen graph. It can be verified, e.g., using Maple, that
\[
    F(G; q) = q^6 - 15q^5 + 95q^4 - 325q^3 + 624q^2 - 620q + 240.
\]
and
\[
    F(\Gamma; q) = q^5 - 12q^4 + 58q^3 - 138q^2 + 157q - 66.
\]
Now we note that \(F(G; q)\) has four real zeroes at \(q = 1, 2, 3, 4\) and two complex zeroes, and \(F(G; 2.5) > 0\). Also, \(F(\Gamma; q)\) has three real zeroes at \(q = 1, 2, 3\) and two complex zeroes, and \(F(\Gamma; 2.5) < 0\).

\[\square\]

Remark 23. The construction used in the proof of Lemma [22] breaks down for \(q > 4\) because \(F(G; q)\) and \(F(\Gamma; q)\) have the same signs. It is conceivable that the lemma could be proved for non-integer \(q\) in the range \(4 < q < 6\) by using a generalised Petersen graph rather than a Petersen graph in the construction. Indeed, Jacobsen and Salas have shown [9] that there are generalised Petersen graphs whose flow polynomials have roots between 5 and 6. Given the current state of knowledge, we are pessimistic about the
prospects of proving the lemma for all $q > 4$. Currently, it is an open question \cite{9} whether there is a some uniform upper bound $Q$ for real zeros of arbitrary bridgeless graphs (so that every bridgeless graph $G$ would have $F(G; q) > 0$ for all $q > Q$). If so, then computing the sign of the flow polynomial will be trivial for $q > Q$, so computing the sign of the Tutte polynomial will also be trivial for $y < -Q + 1$ along the $y$-axis. If not, then it seems likely that the hardness construction can be extended. (Thus, it doesn’t seem to be possible to resolve all of the unresolved points in Region $F$ without solving the open problem about flow polynomials.)

5. \#P-hardness

In the next two sections, we consider the computational difficulty of evaluating the sign of the Tutte polynomial for points in the $(x, y)$ plane. We say that a point $(x, y)$ is \#P-hard, NP-complete, or in FP, if, for $\gamma = y - 1$ and $q = (x - 1)(y - 1)$, the corresponding problem $\text{SIGNTUTTE}(q; \gamma)$ is \#P-hard, NP-complete, or in FP, respectively. \#P-hardness is defined with respect to polynomial-time Turing reductions. NP-hardness is defined by a many-one reduction from an NP-complete decision problem, whose instance is a “yes instance” if the corresponding instance of $\text{SIGNTUTTE}(q; \gamma)$ has a positive sign, and a “no instance” otherwise. In Figure 4, \#P-hard points are depicted in red, NP-complete points are depicted in blue, and FP points are depicted in green. Points depicted in white are unresolved.

All \#P-hardness results are proved in this section. Tractability results and NP-completeness results are proved in Section 6 where we also show that $\text{TUTTE}(q, \gamma)$ is in \#P$_Q$ for these points.

For convenience, here is a formal description of the Regions in Figure 4. In each case, we use $q$ to denote $(x - 1)(y - 1)$.

- **Region A**: Points $(x, y)$ with $x \geq 0$ and $y \geq 0$. These points are in FP.
- **Region B**: Points $(x, y)$ with $\min(x, y) \leq -1$ and $\max(x, y) < 0$. These points are all \#P-hard, apart from the special point $(x, y) = (-1, -1)$, where $\text{TUTTE}(q, \gamma)$ can be solved in polynomial time.
- **Region C**: Points $(x, y)$ with $x < -1$ and $y > 1$. These points are \#P-hard.
- **Region D**: Points $(x, y)$ with $x > 1$ and $y < -1$. These points are \#P-hard.
- **Region E**: Points $(x, y)$ with $x \leq -1$ and $0 < y \leq 1$. Once again, these points have $q \geq 0$. When $q$ is an integer, they are in FP. When $q$ is a non-integer, they are \#P-hard, apart from the line segment with $x = -1$ and $11/27 \leq y < 1$, which is unresolved.
- **Region F**: Points $(x, y)$ with $0 < x \leq 1$ and $y \leq -1$. Note that these points have $q \geq 0$. When $q$ is an integer, they are in FP. When $q$ is a non-integer satisfying $0 < q < 4$, they are \#P-hard, apart from the
line segment with $y = -1$ and $11/27 \leq x < 1$, which is unresolved. Points with non-integer $q > 4$ are unresolved.

- The boundary between regions B and E: Points $(x, y)$ with $x \leq -1$ and $11/27 \leq x < 1$, which is unresolved.
- The boundary between regions B and F: Points $(x, y)$ with $x = 0$ and $y \leq -1$. When $1 < q < 4$ is not an integer, these points are #P-hard. The point $(0, -1)$ is in FP, and the rest of the points $(0, y)$, where $y$ is a positive integer, are NP-complete.
- Region G: Points $(x, y)$ with $\max(|x|, |y|) < 1$ and $q > 32/27$. These points are #P-hard.
- Region H: Points $(x, y)$ with $\max(|x|, |y|) < 1$ and $q \leq 32/27$ and $y < -2x - 1$. These points are #P-hard, apart from points with $q = 1$, where there is a polynomial-time algorithm for $\text{TUTTE}(q; \gamma)$.
- Region I: Points $(x, y)$ with $\max(|x|, |y|) < 1$ and $q \leq 32/27$ and $x < -2y + 1$. These points are #P-hard, apart from points with $q = 1$, where there is a polynomial-time algorithm for $\text{TUTTE}(q; \gamma)$.
- Region J: Points $(x, y)$ with $-1 \leq x < 0$ and $y \geq 1$. These points are in FP.
- Region K: Points $(x, y)$ with $x \geq 1$ and $-1 \leq y < 0$. These points are in FP.
- Region L: Points $(x, y)$ with $0 < x < 1$ and $-x < y < 0$. These points are in FP.
- Region M: Points $(x, y)$ with $0 < y < 1$ and $-y < x < 0$. These points are in FP.
- The rest: There are some remaining unresolved points. These points (simultaneously) satisfy all of the following inequalities: $\max(|x|, |y|) < 1$, $y < -x$, $q \leq 32/27$, $y \geq -2x - 1$, $x \geq -2y - 1$, and and $q \neq 1$.

5.1. Points in Region B.

**Corollary 24.** Suppose that $(x, y)$ is a point such that $\min(x, y) < -1$ and $\max(x, y) < 0$. Then $(x, y)$ is #P-hard.

**Proof.** Note that $q = (x - 1)(y - 1) > 1$. The corollary follows from Lemmas 11 and 14. □

**Corollary 25.** Suppose that $(x, y)$ is a point satisfying $x = -1$ and $-1 < y < 0$. Then $(x, y)$ is #P-hard.

**Proof.** A 3-thickening from $(x, y)$ implements the point

$$(x', y') = \left(\frac{-1 + y + y^2}{1 + y + y^2}, y^3\right).$$
Now \( x' < -1 \) and \( -1 < y' < 0 \) so \((x', y')\) was already shown to be \#P-hard by Corollary 24.

Similarly, we have the following.

**Corollary 26.** Suppose that \((x, y)\) is a point satisfying \(y = -1\) and \(-1 < x < 0\). Then \((x, y)\) is \#P-hard.

### 5.2. Points in Regions G, H and I.

**Corollary 27.** Suppose that \((x, y)\) is a point satisfying \(\max(\{|x|, |y|\}) < 1\) and \(q = (x - 1)(y - 1) > 1\). Suppose that \((x, y)\) also satisfies at least one of the following conditions.

- \(q > \frac{32}{27}\), or
- \(y < -1 - 2x\), or
- \(x < -1 - 2y\).

Then \((x, y)\) is \#P-hard.

**Proof.** The corollary follows from Lemmas 1, 11, 12, 13, and 14.

**Corollary 28.** Suppose that \((x, y)\) is a point satisfying \(\max(\{|x|, |y|\}) < 1\) and \(q = (x - 1)(y - 1) < 1\). Suppose that \((x, y)\) also satisfies at least one of the following conditions.

- \(y < -1 - 2x\), or
- \(x < -1 - 2y\).

Then \((x, y)\) is \#P-hard.

**Proof.** Note that \(q > 0\). The corollary follows from Lemmas 2, 12, and 13. We implement the point \((x_1, y_1)\) required by Lemma 2 by taking a 2-thickening of \((x, y)\) so \(y_1 = y^2 \in (0, 1)\).

### 5.3. Points in Regions C and D.

**Corollary 29.** Suppose \((x, y)\) is a point satisfying one of the following.

- \(y > 1\) and \(x < -1\), or
- \(x > 1\) and \(y < -1\).

Then \((x, y)\) is \#P-hard.

**Proof.** Note that \(q < 0\). The corollary follows from Lemmas 2 and 15. The point \((x_2, y_2)\) required by Lemma 2 is just \((x, y)\) itself.

### 5.4. Points with non-integer \(q\) in Region E and on the boundary between Regions B and E.

Note that \(q\) is an integer when \((x, y) = (-1, 0)\) and when \(y = 1\). We will discuss these points in Section 6.

**Corollary 30.** Suppose \((x, y)\) is a point satisfying \(x < -1\) and \(0 < y < 1\). Suppose that \(q = (x - 1)(y - 1) > 0\) is not an integer. Then \((x, y)\) is \#P-hard.
Proof. If $0 < q < 1$ then the result follows from Lemmas 24 and 3. If $1 < q < 2$ then the result follows from Lemmas 11, 15, and 3. So suppose $q > 2$. By Lemma 18, the point $(x, y)$ can be used to implement a point, other than the special point $(-1, -1)$ that is in Regions B or G. All of these points are known to be \#P-hard by Corollaries 24, 25, 26, and 27. □

Corollary 31. Consider a point $(x, y)$ satisfying $x < -1$ and $y = 0$. Suppose that $q = (x - 1)(y - 1)$ is not an integer. Then $(x, y)$ is \#P-hard.

Proof. Note that $q = (x - 1)(0 - 1) = 1 - x > 0$. Let

$$(x', y') = \left( x^3, \frac{x + x^2}{1 + x + x^2} \right)$$

be the point implemented by a 3-stretch from $(x, y)$. Note that $x + x^2 > 0$ so $0 < y' < 1$. Also, $x' < -1$. Thus, $(x', y')$ is \#P-hard by Corollary 30. □

Corollary 32. Suppose that $(x, y)$ is a point satisfying $x = -1$ and $0 < y < 11/27$. Then $(x, y)$ is \#P-hard.

Proof. Note that $q = (x - 1)(y - 1) > 32/27$. Implement $(x', y')$ by a 2-thickening from $(x, y)$ so $(x', y') = (\frac{1 + y}{1 + y}, y^2)$. Note that $-1 < x' < 0$ and $0 < y' < 1$ so $(x', y')$ is in Region G, and is \#P-hard by Corollary 27. □

5.5. Points with non-integer $q$ in Region F and on the boundary between regions B and F. Note that $q$ is an integer when $(x, y) = (0, -1)$ and when $x = 1$. We will discuss these points in Section 6.

Corollary 33. Suppose $(x, y)$ is a point satisfying $0 < x < 1$ and $y < -1$. Suppose that $q = (x - 1)(y - 1)$ is not an integer. Suppose $0 < q < 4$. Then $(x, y)$ is \#P-hard.

Proof. If $0 < q < 1$ then the result follows from Lemmas 2 and 20. If $1 < q < 2$ then the result follows from Lemmas 11 and 24. So suppose $2 < q < 4$. By Lemma 22, $(x, y)$ can be used to implement a point $(x', y')$ with $x' < 0$. The point $(x', y')$ is in one of regions E, B, or G. It is not the special point $(-1, -1)$ from Region B, since $q$ is not an integer. It is not the unresolved line segment from Region E, since $q > 2$. Thus, $(x', y')$ is \#P-hard by Corollaries 24, 25, 26, 27, 30, 31, and 32. □

As we explained in Remark 23, it seems possible that Corollary 33 could be extended, say up to $q = 6$, by doing more complicated calculations in the proof of Lemma 22 analysing the flow polynomial of generalised Petersen graphs, rather than just the flow polynomial of the Petersen graph. However, our lack of knowledge about the zeroes of the flow polynomial seems to be a barrier to extending the lemma to cover all $q$.

Corollary 34. Consider a point $(x, y)$ satisfying $x = 0$ and $y < -1$. Suppose that $q = (x - 1)(y - 1)$ is not an integer and that $q < 4$. Then $(x, y)$ is \#P-hard.
Proof. Note that $2 < q < 4$. Let

$$(x', y') = \left( \frac{y + y^2}{1 + y + y^2}, y^3 \right)$$

be the point implemented by a 3-thickening from $(x, y)$. Note that $y + y^2 > 0$ so $0 < x' < 1$. Also, $y' < -1$. Thus, $(x', y')$ is #P-hard by Corollary 33.

**Corollary 35.** Suppose that $(x, y)$ is a point satisfying $0 < x < 11/27$ and $y = -1$. Then $(x, y)$ is #P-hard.

Proof. Note that $q = (x-1)(y-1) > 32/27$. Implement $(x', y')$ by a 2-stretch from $(x, y)$ so $(x', y') = \left( x^2, \frac{1+x}{1+x} \right)$. Note that $0 < x' < 1$ and $-1 < y' < 0$ so $(x', y')$ is in Region G, and is #P-hard by Corollary 27.

6. Tractability results and NP-completeness results

As we mentioned earlier, we say that a point $(x, y)$ is in FP if $\text{SignTutte}(q; \gamma)$ can be solved in polynomial time, where $q = (x-1)(y-1)$ and $\gamma = y - 1$. These points are depicted in green in Figure 1. For each point in FP, and also for the points that are NP-complete (depicted in blue), we show that $\text{Tutte}(q, \gamma)$ is in $\text{#P}_Q$. Thus, $\text{Tutte}(q, \gamma)$ can be efficiently approximated using an NP oracle.

6.1. Points in Region A. The following lemma is implicit in the work of Tutte [16, 17]. The connection is explained explicitly in [4, Section 2.3].

**Lemma 36.** Suppose $(x, y)$ is a point satisfying $\min(x, y) \geq 0$. Let $q = (x-1)(y-1)$ and $\gamma = y - 1$. Then for every graph $G$, $Z(G; q, \gamma) > 0$ so $\text{SignTutte}(q, \gamma)$ is in FP. Furthermore, $\text{Tutte}(q, \gamma)$ is in $\text{#P}_Q$.

6.2. Points in Region B. It is known [10] that $\text{Tutte}(4, -2)$ is in FP (so it is certainly in $\text{#P}_Q$). Thus, the point $(x, y) = (-1, -1)$ is in FP.

6.3. Points with Integer $q$ in Region E. The points in Region E have $x \leq -1$ and $0 < y \leq 1$. Thus, they have $q = (x-1)(y-1) \geq 0$ and $\gamma = y - 1$.

First, if $y = 1$ then $q = 0$. We will handle this easy case below. So, suppose $y < 1$ so $-1 < \gamma < 0$. Note that $q > 0$ so, since we restrict attention to integer $q$, $q \geq 1$. Consider the Potts-model partition function for $G$ (see [14, (2.7)]):

$$Z_{\text{Potts}}(G; q, \gamma) = \sum_{\sigma: V \to [q]} \prod_{e = (u, v) \in E} \left( 1 + \gamma \delta(\sigma(u), \sigma(v)) \right),$$

where $\delta$ is the Kronecker delta function defined by $\delta(a, b) = 1$ if $a = b$ and $\delta(a, b) = 0$ otherwise. The following well-known fact is due to Fortuin and Kasteleyn (see [14, Theorem 2.3]).

**Fact 37.** If $q \geq 1$ is an integer then $Z_{\text{Potts}}(G; q, \gamma) = Z(G; q, \gamma)$.

The following observation now follows from Fact 37.
Observation 38. Let \((x, y)\) be a point with \(x \leq -1\) and \(0 < y \leq 1\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Suppose that \(q\) is an integer.

- If \(y = 1\) then \(Z(G; q, \gamma) = 0\) so \(\text{SignTutte}(q, \gamma)\) and \(\text{Tutte}(q, \gamma)\) are both in \(\mathsf{FP}\).
- Otherwise, \(Z(G; q, \gamma) > 0\) so \(\text{SignTutte}(q, \gamma)\) is in \(\mathsf{FP}\). Also, \(\text{Tutte}(q, \gamma)\) is in \(\mathsf{FP}^{Q}\).

Note that Observation 38 disproves [8, Conjecture 10.3(e)]. Jackson and Sokal conjectured that for every fixed \(0 < x < 1\) satisfying \(q = (x - 1)(y - 1) > 32/27\), for all sufficiently large \(n\) an \(m\), there are 2-connected graphs with \(n\) vertices and \(m\) edges that make \(Z(G; q, y - 1)\) non-zero with either sign, but this is clearly false when \(q\) is an integer.

6.4 Points with Integer \(q\) on the boundary between Regions B and E. These points have \(x \leq -1\) and \(y = 0\). Since \(q = (x - 1)(y - 1) = 1 - x\) is an integer, we conclude that \(x\) is an integer. From Fact 37, \(Z(G; q, -1)\) is the number of proper \(q\)-colourings of \(G\).

Observation 39. The point \((-1, 0)\) is in \(\mathsf{FP}\) since \(Z(G; 2, -1)\) is equal to the number of \(2\)-colourings of \(G\), and this can be computed in polynomial time. For integer \(x < -1\), the point \((x, 0)\) is \(\mathsf{NP}\)-complete. \(Z(G; 1 - x, -1)\) is positive if \(G\) has a proper \((1-x)\)-colouring, and is 0 otherwise. \(\text{Tutte}(1-x, -1)\) is in \(\mathsf{FP}\) so it is in \(\mathsf{FP}^{Q}\).

6.5 Points with Integer \(q\) in Region F. The points in Region F have \(0 < x \leq 1\) and \(y \leq -1\). They have \(q = (x - 1)(y - 1) \geq 0\) and \(\gamma = y - 1\).

First, if \(x = 1\) then \(q = 0\). We will handle this easy case below. So, let us restrict attention to the range \(0 \leq x < 1\). This corresponds to \(\gamma \leq -2\) and \(q/\gamma \in (-1, 0)\). Recall the definition of the flow polynomial from Section 4.5. Using Fact 19 we obtain the following observation.

Observation 40. Let \((x, y)\) be a point with \(0 < x \leq 1\) and \(y \leq -1\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Suppose that \(q\) is an integer.

- If \(x = 1\) then \(Z(G; q, \gamma) = 0\) so \(\text{SignTutte}(q, \gamma)\) and \(\text{Tutte}(q, \gamma)\) are both in \(\mathsf{FP}\).
- Otherwise, \(q^{-|V|}\left(\frac{2}{Z}\right)^{|E|}Z(G; q, \gamma) > 0\) so \(\text{SignTutte}(q, \gamma)\) is in \(\mathsf{FP}\). Also, \(\text{Tutte}(q, \gamma)\) is in \(\mathsf{FP}^{Q}\).

Like Observation 38, Observation 40 provides counter-examples to [8, Conjecture 10.3(3)]. They conjectured that for every fixed \(0 < x \leq 1\) and \(y \leq -1\) satisfying \(q = (x - 1)(y - 1) > 32/27\), for all sufficiently large \(n\) an \(m\) (including even \(m\)), there are 2-connected graphs with \(n\) vertices and \(m\) edges that make \(Z(G; q, y - 1)\) non-zero with either sign, but this is clearly false when \(q\) is an integer.\footnote{The case \(y = 1\) is trivial for us, because we are using the \((q, \gamma)\) parameterisation, where a single point \((q, \gamma) = (0, 0)\) corresponds to the a line \((x, 1)\) in the \((x, y)\) parameterisation. This issue is touched on in the Introduction.}
6.6. Points with Integer $q$ on the boundary between Regions B and F. These points have $x = 0$ and $y \leq -1$. Since $q = (x-1)(y-1) = 1-y$ is an integer, we conclude that $y$ is an integer.

Recall from Section 6.5 that if $q$ is a positive integer then $q^{-|V|}(-1)^{|E|}Z(G; q, -q)$ is the number of nowhere-zero $q$-flows of $G$. A graph has a nowhere-zero 2-flow iff it is Eulerian \cite{1} Theorem 11.21. Thus, this can be tested in polynomial time. On the other hand, it is NP-complete to test whether a graph has a nowhere-zero 3-flow, even if the graph is planar. To see this, note that a planar graph has a nowhere-zero 3-flow iff its dual has a proper 3-colouring, and it is NP-complete to determine whether a planar graph is 3-colourable. A “bridge” (or cut-edge) of a graph is an edge whose deletion increases the number of connected components. It is known \cite{1} Corollary 11.26], that no graph with a bridge has a nowhere-zero $q$-flow for any integer $q \geq 2$. However, Seymour has shown \cite{1} Theorem 11.32] that every bridgeless graph has a nowhere-zero 6-flow. Thus, determining whether a graph has a nowhere-zero $q$-flow is in FP for $q \geq 6$. We do not know the complexity of determining whether a graph has a nowhere-zero 4-flow or 5-flow. Indeed, it is currently an open question whether there exists a bridgeless graph without a nowhere zero 5-flow. It is also an open question whether there exists a bridgeless graph without a Petersen minor that does not have a nowhere zero 4-flow.

Observation 41. The point $(0, -1)$ is in FP since $Z(G; 2, -2)$ is computable from the number of nowhere-zero 2-flows of $G$, and this can be computed in polynomial time. The point $(0, -2)$ is NP-complete since $Z(G; 3, -3)$ allows one to determine the number of nowhere-zero 3-flows of $G$. For integer $y \leq -5$, the point $(0, y)$ is in FP since $Z(G; 1 + y, y - 1)$ is computable from the number of nowhere-zero $(1 - y)$-flows of $G$. This quantity is positive iff $G$ has no bridge. $\text{TUTTE}(1 - x, -1)$ is in #P so it is in #P$_Q$.

6.7. Points in Regions H and I. It is known \cite{10} that points $(x, y)$ with $(x-1)(y-1) = 1$ are in FP since $\text{TUTTE}(1, \gamma)$ is in FP so $\text{SIGNTUTTE}(1, \gamma)$ is also in FP.

6.8. Matroids. The definitions from Section 2 can be generalised from graphs to matroids. To deal with Regions J and K (and also with regions L and M in future sections), it is advantageous to work with matroids, rather than with graphs, because we can then exploit a duality between the variables $x$ and $y$. In order to avoid difficulties over how matroids should be presented, we will work with the class of binary matroids. This is a more general class than the class of graphs — every graph corresponds to a binary matroid, but there are binary matroids that do not correspond to graphical matroids.

A matroid $\mathcal{M}$ is a combinatorial structure defined by a set $E$ (the “ground set”) together with a “rank function” $r_\mathcal{M} : E \rightarrow \mathbb{N}$ which must satisfy the following conditions (see \cite{12} for details).
(1) \(0 \leq r_M(A) \leq |A|\),
(2) \(A \subseteq B\) implies \(r_M(A) \leq r_M(B)\) (monotonicity), and
(3) \(r_M(A \cup B) + r_M(A \cap B) \leq r_M(A) + r_M(B)\) (submodularity).

A subset \(A \subseteq E\) satisfying \(r_M(A) = |A|\) is said to be independent. Every other subset \(A \subseteq E\) is said to be dependent. A maximal (with respect to inclusion) independent set is a basis, and a minimal dependent set is a circuit. A circuit with one element is a loop.

The multivariate Tutte polynomial of a matroid \(M\) with ground set \(E\) and rank function \(r_M\) is defined as follows (see [14, (1.3)]), where the weight function \(\gamma\) assigns weights to elements of the ground set.

(19) \[\tilde{Z}(M; q, \gamma) = \sum_{A \subseteq E} q^{-r_M(A)} \prod_{e \in A} \gamma_e.\]

If \(\gamma\) assigns weight \(\gamma\) to every element of \(E\) then we use \(\tilde{Z}(M; q, \gamma)\) as shorthand for \(\tilde{Z}(M; q, \gamma)\).

Let \(M\) be a matrix over a field \(F\) with row set \(V\) and column set \(E\). \(M\) is said to “represent” a matroid \(M\) with ground set \(E\). The rank \(r_M(A)\) of a set of columns \(A\) in this matroid is defined to be the rank of the submatrix consisting of those columns. A matroid is said to be representable over the field \(F\) if it can be represented in this way. It is said to be binary if it is representable over the two-element field \(F_2\).

The cycle matroid of an undirected graph \(G = (V, E)\) is the binary matroid \(M(G)\) represented by the vertex-edge incidence matrix \(M\) of \(G\) (in which rows are vertices and columns are edges). It can be deduced from the definition above that \(r_M(G)(A) = |V| - \kappa(V, A)\). The Tutte polynomial of a cycle matroid \(M(G)\) is very closely connected to the Tutte polynomial of the underlying graph \(G\). In particular, (see [14, (1.2) and (1.3)]),

(20) \[Z(G; q, \gamma) = q^{|V|} \tilde{Z}(M(G); q, \gamma).\]

Every matroid \(M\) has a dual matroid \(M^*\) with the same ground set. Furthermore, \(M^*\) is binary if and only if \(M\) is (see [12]), and a binary matrix representing \(M^*\) can be efficiently computed from a representation of \(M\) [15, p.63]. A cocircuit in \(M\) is a set that is a circuit in \(M^*\); equivalently, a cocircuit is a minimal set that intersects every basis. A cocircuit with one element is a coloop. We use the following fact [14, (4.14a)].

**Fact 42.** Suppose that \(M\) is a matroid with ground set \(E\) and that \(\gamma\) is a weight function assigning weights to elements in \(E\). Let \(M^*\) be the dual of \(M\) and let \(\gamma^*\) be the weight function that assigns weight \(q/\gamma_e\) to every ground set element \(e \in E\). Then

\[
\tilde{Z}(M^*; q, \gamma) = q^{-r_{M^*}(E)} \left( \prod_{e \in E} \gamma_e \right) \tilde{Z}(M; q, \gamma^*).\]

Two important matroid operations are deletion and contraction. Suppose \(e \in E\) is a member of the ground set of matroid \(M\). The contraction \(M/e\)
of $e$ from $\mathcal{M}$ is the matroid on ground set $E - \{e\}$ with rank function given by $r_{\mathcal{M}/e}(A) = r_\mathcal{M}(A \cup \{e\}) - r_\mathcal{M}(\{e\})$, for all $A \subseteq E - \{e\}$. The deletion $\mathcal{M}\setminus e$ of $\{e\}$ from $\mathcal{M}$ is the matroid on ground set $E - \{e\}$ with rank function given by $r_{\mathcal{M}\setminus e}(A) = r_\mathcal{M}(A)$, for all $A \subseteq E - \{e\}$. Given a matrix representing a matroid $\mathcal{M}$, there are efficient algorithms for constructing matrices representing contractions and deletions of $\mathcal{M}$ [15, Chapter 3].

Fact 43. If $\mathcal{M}$ is a matroid with a loop $e$ then
$$\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma e)\tilde{Z}(\mathcal{M}\setminus e; q, \gamma).$$

We also use a related fact about coloops (see, for example [14, (4.18b)]).

Fact 44. If $\mathcal{M}$ is a matroid with a coloop $e$ then
$$\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma e/q)\tilde{Z}(\mathcal{M}/e; q, \gamma).$$

We introduce two computational problems for binary matroids.

Name: MatroidSignTutte($q, \gamma$).

Instance: A matrix representing a binary matroid $\mathcal{M}$ and an edge weight $\gamma$.

Output: Determine whether the sign of $\tilde{Z}(\mathcal{M}; q, \gamma)$ is positive, negative, or 0.

Name: MatroidTutte($q, \gamma$).

Instance: A matrix representing a binary matroid $\mathcal{M}$ and an edge weight $\gamma$.

Output: $\tilde{Z}(\mathcal{M}; q, \gamma)$.

6.9. Points in Regions J and K. The points in Regions J and K satisfy $-1 \leq \min(x, y) < 0$ and $\max(x, y) \geq 1$. Let $q = (x - 1)(y - 1)$ and $\gamma = y - 1$. Note that $q \leq 0$. It is known (see [8, Theorem 4.1]) that in these regions, the sign of $Z(G; q, \gamma)$ is essentially a trivial function of $G$, apart from some factors arising from loops in the matroid associated with $G$ and in its dual matroid. We will show that, for all of these points, Tutte($q, \gamma$) is in #P$_Q$.

In fact, we will show that MatroidTutte($q, \gamma$) is in #P$_Q$. Working with matroids, instead of with graphs, will enable us to prove the results for one region (Region K) and immediately to deduce the same results for the other region (Region J) (by duality of the variables $x$ and $y$).

6.9.1. Points in Region K. Points in Region K have $x \geq 1$ and $-1 \leq y < 0$. Let $q = (x - 1)(y - 1)$ and $\gamma = y - 1$.

First, if $x = 1$ then $q = 0$. We will handle this easy case below. So, let us restrict attention to the range $x > 1$. Then $q < 0$ and $-2 \leq \gamma < -1$. We will use the following lemma, which is similar in spirit to [8, Theorem 4.1].

---

5 We need to repeat the steps of their proof here because we want to extract computational information in addition to the sign.
Lemma 45. Suppose that $q < 0$ and $\mathcal{M}$ is a loopless matroid. Suppose that $\gamma$ is a weight function in which every weight $\gamma_e$ satisfies $-2 \leq \gamma_e \leq 0$ then $\tilde{Z}(\mathcal{M}; q, \gamma) > 0$ and the problem of computing $\tilde{Z}(\mathcal{M}; q, \gamma)$ is in $\#P_Q$.

Proof. We start with some pre-processing. Before trying to compute $\tilde{Z}(\mathcal{M}; q, \gamma)$, we first modify $\mathcal{M}$, without changing its Tutte polynomial, to get rid of any size-2 circuits. We do this by parallel composition. So if we have a size-2 circuit containing elements $e_1$ and $e_2$, we replace it with a new element $e$ which is the parallel composition of the two elements in the circuit. In the matrix representing $\mathcal{M}$, the size-2 circuit arises as a pair of identical columns. In the representation of the new matroid, the columns corresponding to elements $e_1$ and $e_2$ are deleted and the new element $e$ corresponds to one of these columns. The new weight $\gamma_e$ is given by $\gamma_{e_1} + \gamma_{e_2} + \gamma_{e_1} \gamma_{e_2}$ (see [8, 2.34]). The reason that we want to do this pre-processing is that, in the recursive step, we will want to be able to contract an element of a circuit without creating a loop. The reason that we can do the pre-processing without falsifying the conditions in the statement of the lemma is that the region $-2 \leq \gamma \leq 0$ maintains itself for parallel composition: If $-2 \leq \gamma_{e_1} \leq 0$ and $-2 \leq \gamma_{e_2} \leq 0$ then $-2 \leq \gamma_e \leq 0$.

Now suppose that $\mathcal{M}$ has no size-2 circuit. Let $r = r(\mathcal{M})$ and $E = E(\mathcal{M})$. Then

$$\tilde{Z}(\mathcal{M}; q, \gamma) = \sum_{A \subseteq E} q^{-r(A)} \prod_{e \in A} \gamma_e.$$ 

**Base Case:** If $r(E) = |E|$ then, from the axioms of rank functions of matroids, for every $S \subseteq E$, $r(S) = |S|$, so

$$\tilde{Z}(\mathcal{M}; q, \gamma) = \sum_{A \subseteq E} q^{-|A|} \prod_{e \in A} \gamma_e = \sum_{A \subseteq E} \prod_{e \in A} \gamma_e / q.$$

The contribution from $A = \emptyset$ is 1 and the contribution from each other $A$ is non-negative. Also, $\tilde{Z}(\mathcal{M}; q, \gamma)$ can be computed by summing over the sets $A$.

**Recursive Step:** Pick any $e$ in a circuit. Then from [14] (4.18a),

$$\tilde{Z}(\mathcal{M}; q, \gamma) = \tilde{Z}(\mathcal{M} \setminus e; q, \gamma) + \frac{\gamma_e}{q} \tilde{Z}(\mathcal{M}/e; q, \gamma).$$

Now the point is that the fraction $\gamma_e / q$ doesn’t change the sign, and is easy to compute. Also, the two minors $\mathcal{M} \setminus e$ and $\mathcal{M}/e$ both satisfy the conditions of the theorem.

Both minors are matroids on ground set $E \setminus e$. The rank functions are given by $r_{\mathcal{M}\setminus e}(A) = r(A)$ and $r_{\mathcal{M}/e}(A) = r(A \cup e) - 1$.

To see that $\mathcal{M}/e$ has no loop, note that $r_{\mathcal{M}/e}({\{e\}'}) = r(\{e, e\}') - 1$ and since $\{e, e\}'$ is not a circuit, by the pre-processing step, $r(\{e, e\}') = 2$.  

We can now classify the points in Region K. See also [8] Theorem 4.1 which shows that the sign is trivial in this region.
Lemma 46. Let \((x, y)\) be a point with \(x \geq 1\) and \(-1 \leq y < 0\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Then \(\text{MatroidSignTutte}(q, \gamma)\) is in \(\text{FP}\) and \(\text{MatroidTutte}(q, \gamma)\) is in \(\#P_Q\).

Proof. If \(\mathcal{M}\) has \(k\) loops then, by Fact \[43\], \(\widetilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma)^k \widetilde{Z}(\mathcal{M}'; q, \gamma)\), where \(\mathcal{M}'\) is the matrix formed from \(\mathcal{M}\) by deleting these loops. If \(q = 0\) then \(\widetilde{Z}(\mathcal{M}'; q, \gamma) = 1\). Otherwise, \(q < 0\). Now Lemma \[45\] shows that \(\widetilde{Z}(\mathcal{M}'; q, \gamma) > 0\) and can be computed in \(\#P_Q\). □

The following corollary follows immediately using Equation \[20\].

Corollary 47. Let \((x, y)\) be a point with \(x \geq 1\) and \(-1 \leq y < 0\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Then \(\text{SignTutte}(q, \gamma)\) is in \(\text{FP}\) and \(\text{Tutte}(q, \gamma)\) is in \(\#P_Q\).

6.9.2. Points in Region J. The following lemma classifies points in Region J. See also [8, Theorem 4.4].

Lemma 48. Let \((x, y)\) be a point with \(-1 \leq x \leq 0\) and \(y \geq 1\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Then \(\text{MatroidSignTutte}(q, \gamma)\) is in \(\text{FP}\) and \(\text{MatroidTutte}(q, \gamma)\) is in \(\#P_Q\).

Proof. This follows from Fact \[42\] and from Lemma \[46\]. □

The following corollary follows immediately using Equation \[20\].

Corollary 49. Let \((x, y)\) be a point with \(-1 \leq x \leq 0\) and \(y \geq 1\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Then \(\text{SignTutte}(q, \gamma)\) is in \(\text{FP}\) and \(\text{Tutte}(q, \gamma)\) is in \(\#P_Q\).

6.10. Points in Regions L and M. We use the following Lemma. The statement is a slight generalisation of [8, Theorem 5.4]. However, their proof (a straightforward generalisation of their proof of [8, Theorem 5.1]) suffices.

Lemma 50. (Jackson and Sokal) Let \(\mathcal{M}\) be a matroid with ground set \(E\) and let \(q \in (0, 1)\). Suppose that \(\gamma\) is a weight function such that

1. \(\gamma_e > -1\) for every loop \(e\);
2. \(\gamma_e < -q\) for every coloop \(e\); and
3. \(-1 - \sqrt{1 - q} < \gamma_e < -1 + \sqrt{1 - q}\) for every normal (i.e., non-loop and non-coloop) element \(e\).

Then

\[(-1)^{r_{\mathcal{M}}(E)} \widetilde{Z}(\mathcal{M}; q, \gamma) > 0\]

and the problem of computing \(\widetilde{Z}(\mathcal{M}; q, \gamma)\), given such a matroid \(\mathcal{M}\) is in \(\#P_Q\).

Proof. We follow the inductive argument that Jackson and Sokal use to prove \[21\] for the graphical case. This is the proof of [8, Theorem 5.1]. The induction is on \(m\), the number of elements in the ground set of \(\mathcal{M}\). If \(m = 0\), then \(r_{\mathcal{M}}(E) = 0\) so \(\widetilde{Z}(\mathcal{M}; q, \gamma) = 1\), so the lemma is true. For \(m > 0\), there
are five cases. We apply these in order, so in each case we assume that the previous cases don’t apply.

(1) If \( \mathcal{M} \) has a loop \( e \) then by Fact 143
\[
\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma_e)\tilde{Z}(\mathcal{M} \setminus e; q, \gamma).
\]
Note that \( 1 + \gamma_e > 0 \) and \( r_{\mathcal{M} \setminus e}(E \setminus e) = r_{\mathcal{M}}(E \setminus e) = r_{\mathcal{M}}(E) \). Thus, the result follows by induction.

(2) If \( \mathcal{M} \) has a coloop \( e \) then by Fact 144
\[
\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma_e/q)\tilde{Z}(\mathcal{M}/e; q, \gamma).
\]
Note that \( 1 + \gamma_e/q < 0 \) and \( r_{\mathcal{M}/e}(E \setminus e) = r_{\mathcal{M}}(E) - r_{\mathcal{M}}(e) = r_{\mathcal{M}}(E) - 1 \). Thus, the result follows by induction.

(3) If \( \mathcal{M} \) has a size-2 circuit consisting of edges \( e_1 \) and \( e_2 \). Let \( \mathcal{M}' \) be the matroid formed from \( \mathcal{M} \) by deleting \( e_2 \) and let \( \gamma' \) be the weight function that is the same as \( \gamma \) except that \( \gamma'_{e_1} \) is the effective weight from the parallel composition of \( e_1 \) and \( e_2 \)
\[
\gamma'_{e_1} = \gamma_{e_1} + \gamma_{e_2} + \gamma_{e_1 \gamma_{e_2}}.
\]
Then, as in the proof of Lemma 145 (see [8, (2.34)]), \( \tilde{Z}(\mathcal{M}; q, \gamma) = \tilde{Z}(\mathcal{M}'; q, \gamma') \). Also, \( r_{\mathcal{M}'}(E \setminus e_2) = r_{\mathcal{M}}(E \setminus e_2) = r_{\mathcal{M}}(E) \). Finally, Jackson and Sokal show that \( \mathcal{M}' \) and \( \gamma' \) satisfy the conditions of the lemma (so \( \tilde{Z}(\mathcal{M}; q, \gamma) \) can be computed by induction).

(4) If \( \mathcal{M} \) has a size-2 cocircuit consisting of edges \( e_1 \) and \( e_2 \). Let \( \mathcal{M}' \) be the matroid formed from \( \mathcal{M} \) by contracting \( e_2 \) and let \( \gamma' \) be the weight function that is the same as \( \gamma \) except that \( \gamma'_{e_1} \) is the effective weight from the series composition of \( e_1 \) and \( e_2 \)
\[
\gamma'_{e_1} = \gamma_{e_1} \gamma_{e_2} / (q + \gamma_{e_1} + \gamma_{e_2}).
\]
Then from [8, (2.40)] \( \tilde{Z}(\mathcal{M}; q, \gamma) = \left( \frac{q + \gamma_{e_1} + \gamma_{e_2}}{q} \right) \tilde{Z}(\mathcal{M}'; q, \gamma') \). Also, Jackson and Sokal show that
\[
\left( \frac{q + \gamma_{e_1} + \gamma_{e_2}}{q} \right) < 0.
\]
This is what we require, since \( r_{\mathcal{M}'}(E \setminus e_2) = r_{\mathcal{M}}(E) - r_{\mathcal{M}}(e_2) = r_{\mathcal{M}}(E) - 1 \). Finally, Jackson and Sokal show that \( \mathcal{M}' \) and \( \gamma' \) satisfy the conditions of the lemma (so \( \tilde{Z}(\mathcal{M}; q, \gamma) \) can be computed by induction).

(5) Otherwise, pick any ground set element \( e \) and apply the deletion-contraction identity [8, (2.29a)]
\[
\tilde{Z}(\mathcal{M}; q, \gamma) = \tilde{Z}(\mathcal{M} \setminus e; q, \gamma) + \frac{\gamma_e}{q} \tilde{Z}((\mathcal{M}/e; q, \gamma).
\]
Since \( e \) is not a cocircuit, \( r_{\mathcal{M}\setminus e}(E \setminus e) = r_{\mathcal{M}}(E) \). As Jackson, and Sokal argue, \( \mathcal{M} \setminus e \) and \( \gamma \) satisfy the conditions of the lemma. Also, \( \gamma_e/q < 0 \) and \( r_{\mathcal{M}/e}(E \setminus e) = r_{\mathcal{M}}(E) - 1 \). Again, Jackson and Sokal argue that \( \mathcal{M}/e \) and \( \gamma \) satisfy the conditions of the lemma, so the result follows by induction.

\[\square\]
6.11. Points in Region L.

Lemma 51. Let \((x, y)\) be a point with \(0 < x < 1\) and \(-x < y < 0\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Then MatroidSignTutte\((q, \gamma)\) is in FP and MatroidTutte\((q, \gamma)\) is in \#P\(Q\).

Proof. Note that \(q = (1 - x)(1 - y) < (1 - x)(1 + x) = 1 - x^2 < 1\). Also, \(q > (1 - x) > 0\). Thus, \(q \in (0, 1)\).

Now since \(y > -x\) we have \(y(y - 1) < (-x)(y - 1)\) so \(y^2 - y < x - xy\) which implies \(y^2 < x + y - xy = 1 - q\). This implies that \(y < \sqrt{1 - q}\) so \(y > -\sqrt{1 - q}\). Thus, \(-1 - \sqrt{1 - q} < \gamma < -1 + \sqrt{1 - q}\).

Finally, since \(0 < x(1 - y)\), we have \(y < y + x(1 - y) = 1 - q\) so \(\gamma > -q\).

Now let \(\mathcal{M}\) be a matroid and let \(\gamma\) be a weight function assigning weight \(\gamma\) to every element the ground-set of \(\mathcal{M}\). If \(\mathcal{M}\) has \(k\) loops then by Fact 43, \(\tilde{Z}(\mathcal{M}; q, \gamma) = (1 + \gamma)^k \tilde{Z}(\mathcal{M'}; q, \gamma)\), where \(\mathcal{M'}\) is the matroid formed from \(\mathcal{M}\) by deleting these loops. Note that \(\mathcal{M'}\) and \(\gamma\) satisfy the hypotheses of Lemma 50.

The following corollary follows immediately using Equation 20.

Corollary 52. Let \((x, y)\) be a point with \(0 < x < 1\) and \(-x < y < 0\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Then SignTutte\((q, \gamma)\) is in FP and Tutte\((q, \gamma)\) is in \#P\(Q\).

6.12. Points in Region M.

Lemma 53. Let \((x, y)\) be a point with \(0 < y < 1\) and \(-y < x < 0\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Then MatroidSignTutte\((q, \gamma)\) is in FP and MatroidTutte\((q, \gamma)\) is in \#P\(Q\).

Proof. This follows from Fact 42 and from Lemma 51.

The following corollary follows immediately using Equation 20.

Corollary 54. Let \((x, y)\) be a point with \(0 < y < 1\) and \(-y < x < 0\). Let \(q = (x - 1)(y - 1)\) and \(\gamma = y - 1\). Then SignTutte\((q, \gamma)\) is in FP and Tutte\((q, \gamma)\) is in \#P\(Q\).

7. Putting things together for points with \(|y| < 1\)

Collecting Observations 39 and 38 and Corollaries 24, 25, 27, 30, 31 and 32.

Corollary 55. Suppose \((x, y)\) is a point satisfying \(|y| < 1\) such that \(q = (x - 1)(y - 1) \geq 32/27\). Let \(\gamma = y - 1\).

- If \((x, y) = (-1, 0)\) then SignTutte\((q, \gamma)\) and Tutte\((q, \gamma)\) are in FP.
- If \((x, y) = (x, 0)\) for any integer \(x < -1\) then SignTutte\((q, \gamma)\) is NP-complete. Tutte\((q, \gamma)\) is in \#P\(Q\).
- If \(x \leq -1\) and \(0 < y < 1\) and \(q \) is an integer then \(Z(G; q, \gamma) > 0\) so SignTutte\((q, \gamma)\) is in FP. Also, Tutte\((q, \gamma)\) is in \#P\(Q\).
• Otherwise, \textsc{SignTutte}(q, \gamma) is \#P-hard.

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