The Compositions of the Differential Operations and Gateaux Directional Derivative

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Abstract

In this paper we determine the number of the meaningful compositions of higher order of the differential operations and Gateaux directional derivative.

1 The compositions of the differential operations of the space $\mathbb{R}^3$

In the real three-dimensional space $\mathbb{R}^3$ we consider the following sets:

$$A_0 = \{ f : \mathbb{R}^3 \rightarrow \mathbb{R} \mid f \in C^\infty(\mathbb{R}^3) \} \quad \text{and} \quad A_1 = \{ \vec{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \mid \vec{f} \in \vec{C}^\infty(\mathbb{R}^3) \}. \quad (1)$$

Then, over the sets $A_0$ and $A_1$ in the vector analysis, there are $m = 3$ differential operations of the first-order:

$$\text{grad } f = \nabla_1 f = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right) : A_0 \rightarrow A_1,$$

$$\text{curl } \vec{f} = \nabla_2 \vec{f} = \left( \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3}, \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}, \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \right) : A_1 \rightarrow A_1,$$

$$\text{div } \vec{f} = \nabla_3 \vec{f} = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \frac{\partial f_3}{\partial x_3} : A_1 \rightarrow A_0. \quad (2)$$

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Let us present the number of the meaningful compositions of higher order over the set \( A_3 = \{ \nabla_1, \nabla_2, \nabla_3 \} \). As a well-known fact, there are \( m = 5 \) compositions of the second-order:

\[
\Delta f = \text{div grad } f = \nabla_3 \circ \nabla_1 f,
\]
\[
\text{curl curl } \tilde{f} = \nabla_2 \circ \nabla_2 \tilde{f},
\]
\[
\text{grad div } \tilde{f} = \nabla_1 \circ \nabla_3 \tilde{f},
\]
\[
\text{curl grad } f = \nabla_2 \circ \nabla_1 f = \vec{0},
\]
\[
\text{div curl } \tilde{f} = \nabla_3 \circ \nabla_2 \tilde{f} = 0.
\]

(3)

Malešević [2] proved that there are \( m = 8 \) compositions of the third-order:

\[
\text{grad div grad } f = \nabla_1 \circ \nabla_3 \circ \nabla_1 f,
\]
\[
\text{curl curl curl } \vec{f} = \nabla_2 \circ \nabla_2 \circ \nabla_2 \vec{f},
\]
\[
\text{div grad div } \vec{f} = \nabla_3 \circ \nabla_1 \circ \nabla_3 \vec{f},
\]
\[
\text{curl curl grad } f = \nabla_2 \circ \nabla_2 \circ \nabla_1 f = \vec{0},
\]
\[
\text{div curl grad } \vec{f} = \nabla_3 \circ \nabla_2 \circ \nabla_1 f = 0,
\]
\[
\text{div curl curl } \vec{f} = \nabla_3 \circ \nabla_2 \circ \nabla_2 \vec{f} = 0,
\]
\[
\text{grad div curl } \vec{f} = \nabla_1 \circ \nabla_3 \circ \nabla_2 \vec{f} = \vec{0},
\]
\[
\text{curl grad div } \vec{f} = \nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f} = \vec{0}.
\]

(4)

If we denote by \( f(k) \) the number of compositions of the \( k \)-th-order, then Malešević [3] proved:

\[
f(k) = F_{k+3},
\]

(5)

where \( F_k \) is \( k \)-th Fibonacci number.

2 The compositions of the differential operations and Gateaux directional derivative on the space \( \mathbb{R}^3 \)

Let \( f \in A_0 \) be a scalar function and \( \vec{e} = (e_1, e_2, e_3) \in \mathbb{R}^3 \) be a unit vector. Thus, the Gateaux directional derivative in direction \( \vec{e} \) is defined by [1, p. 71]:

\[
\text{dir}_{\vec{e}} f = \nabla_0 f = \nabla_1 f \cdot \vec{e} = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 : A_0 \to A_0.
\]

(6)
Let us determine the number of the meaningful compositions of higher order over the set $B_3 = \{\nabla_0, \nabla_1, \nabla_2, \nabla_3\}$. There exist $m = 8$ compositions of the second-order:

| Term | Expression |
|------|------------|
| $\text{dir}_e \text{dir}_e f$ | $\nabla_0 \circ \nabla_0 f = \nabla_1 (\nabla_1 f \cdot \vec{e}) \cdot \vec{e}$ |
| $\text{grad} \text{dir}_e f$ | $\nabla_1 \circ \nabla_0 f = \nabla_1 (\nabla_1 f \cdot \vec{e})$ |
| $\Delta f$ | $\text{div} \text{grad} f = \nabla_3 \circ \nabla_1 f$ |
| $\text{curl} \text{curl} \vec{f}$ | $\nabla_2 \circ \nabla_1 \vec{f}$ |
| $\text{dir}_e \text{div} \vec{f}$ | $\nabla_0 \circ \nabla_3 \vec{f} = (\nabla_1 \circ \nabla_3 \vec{f}) \cdot \vec{e}$ |
| $\text{grad} \text{dir}_e \text{div} \vec{f}$ | $\nabla_1 \circ \nabla_3 \vec{f}$ |
| $\text{grad} \text{grad} f$ | $\nabla_2 \circ \nabla_1 f = 0$ |
| $\text{dir}_e \text{div} \text{grad} \vec{f}$ | $\nabla_0 \circ \nabla_2 \circ \nabla_3 \vec{f}$ |
| $\text{curl} \text{grad} \text{div} \vec{f}$ | $\nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f}$ |
| $\text{div} \text{curl} \text{grad} \vec{f}$ | $\nabla_3 \circ \nabla_2 \circ \nabla_1 \vec{f}$ |
| $\text{div} \text{curl} \text{curl} \vec{f}$ | $\nabla_3 \circ \nabla_2 \circ \nabla_1 \vec{f} = 0$ |

(7)

that is, there exist $m = 16$ compositions of the third-order:

| Term | Expression |
|------|------------|
| $\text{dir}_e \text{dir}_e \text{dir}_e f$ | $\nabla_0 \circ \nabla_0 \circ \nabla_0 f$ |
| $\text{grad} \text{dir}_e \text{dir}_e f$ | $\nabla_1 \circ \nabla_0 \circ \nabla_0 f$ |
| $\text{div} \text{grad} \text{dir}_e f$ | $\nabla_3 \circ \nabla_1 \circ \nabla_0 f$ |
| $\text{dir}_e \text{div} \text{grad} \vec{f}$ | $\nabla_0 \circ \nabla_3 \circ \nabla_1 f$ |
| $\text{grad} \text{grad} \text{dir}_e \vec{f}$ | $\nabla_1 \circ \nabla_3 \circ \nabla_1 \vec{f}$ |
| $\text{curl} \text{curl} \text{grad} \text{dir}_e \vec{f}$ | $\nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f}$ |
| $\text{div} \text{curl} \text{grad} \text{dir}_e \vec{f}$ | $\nabla_3 \circ \nabla_2 \circ \nabla_1 \vec{f}$ |
| $\text{curl} \text{grad} \text{div} \text{curl} \vec{f}$ | $\nabla_2 \circ \nabla_1 \circ \nabla_3 \vec{f}$ |

(8)
Using the method from the paper [3] let us define a binary relation $\sigma$ “to be in composition”: $\nabla_i \sigma \nabla_j = \top$ iff the composition $\nabla_j \circ \nabla_i$ is meaningful. Thus, Cayley table of the relation $\sigma$ is determined with

| $\sigma$ | $\nabla_0$ | $\nabla_1$ | $\nabla_2$ | $\nabla_3$ |
|----------|-----------|-----------|-----------|-----------|
| $\nabla_0$ | $\top$ | $\top$ | $\bot$ | $\bot$ |
| $\nabla_1$ | $\bot$ | $\bot$ | $\top$ | $\top$ |
| $\nabla_2$ | $\bot$ | $\bot$ | $\top$ | $\top$ |
| $\nabla_3$ | $\top$ | $\top$ | $\bot$ | $\bot$ |

(9)

Let us form the graph according to the following rule: if $\nabla_i \sigma \nabla_j = \top$ let vertex $\nabla_j$ be under vertex $\nabla_i$ and let there exist an edge from the vertex $\nabla_i$ to the vertex $\nabla_j$.

Further on, let us denote by $\nabla_{-1}$ nowhere-defined function $\vartheta$, where domain and range are the empty sets [2]. We shall define $\nabla_{-1} \sigma \nabla_i = \top$ ($i = 0, 1, 2, 3, 4$). For the set $\mathcal{B}_3 \cup \{\nabla_{-1}\}$ the graph of the walks, determined previously, is a tree with the root in the vertex $\nabla_{-1}$.

Fig. 1

Let $g(k)$ be the number of the meaningful compositions of the $k^{th}$-order of the functions from $\mathcal{B}_3$. Let $g_i(k)$ be the number of the meaningful compositions of the $k^{th}$-order beginning from the left by $\nabla_i$. Then $g(k) = g_0(k) + g_1(k) + g_2(k) + g_3(k)$. Based on the partial self similarity of the tree (Fig. 1) we get equalities

$$
\begin{align*}
g_0(k) &= g_0(k - 1) + g_1(k - 1), \\
g_1(k) &= g_2(k - 1) + g_3(k - 1), \\
g_2(k) &= g_2(k - 1) + g_3(k - 1), \\
g_3(k) &= g_0(k - 1) + g_1(k - 1).
\end{align*}
$$

(10)

Hence, a recurrence for $g(k)$ can be derived as follows:

$$
g(k) = 2g(k - 1).
$$

(11)

Based on the initial value $g(1) = 4$, we can conclude:

$$
g(k) = 2^{k+1}.
$$

(12)
3 The compositions of the differential operations of the space \( \mathbb{R}^n \)

Let us present the number of the meaningful compositions of differential operations in the vector analysis of the space \( \mathbb{R}^n \), where differential operations \( \nabla_r \) \((r = 1, \ldots, n)\) are defined over non-empty corresponding sets \( A_s \) \((s = 1, \ldots, m \) and \( m = \lfloor n/2 \rfloor, n \geq 3)\) according to the papers \([3], [4]\):

\[
\begin{align*}
\mathcal{A}_n \ (n = 2m): & \quad \nabla_1 : A_0 \to A_1 \\
& \quad \nabla_2 : A_1 \to A_2 \\
& \quad \vdots \\
& \quad \nabla_i : A_{i-1} \to A_i \\
& \quad \vdots \\
& \quad \nabla_m : A_{m-1} \to A_m \\
& \quad \nabla_{m+1} : A_m \to A_{m-1} \\
& \quad \vdots \\
& \quad \nabla_{n-j} : A_{j+1} \to A_j \\
& \quad \vdots \\
& \quad \nabla_{n-1} : A_2 \to A_1 \\
& \quad \nabla_n : A_1 \to A_0, \\
\end{align*}
\]

\[
\mathcal{A}_n \ (n = 2m+1): \quad \nabla_1 : A_0 \to A_1 \\
& \quad \nabla_2 : A_1 \to A_2 \\
& \quad \vdots \\
& \quad \nabla_i : A_{i-1} \to A_i \\
& \quad \vdots \\
& \quad \nabla_m : A_{m-1} \to A_m \\
& \quad \nabla_{m+1} : A_m \to A_{m-1} \\
& \quad \nabla_{m+2} : A_m \to A_{m-1} \\
& \quad \nabla_{n-1} : A_2 \to A_1 \\
& \quad \nabla_n : A_1 \to A_0. \\
\]

Let us define higher order differential operations as the meaningful compositions of higher order of differential operations from the set \( \mathcal{A}_n = \{ \nabla_1, \ldots, \nabla_n \} \). The number of the higher order differential operations is given according to the paper \([3]\). Let us define a binary relation \( \rho \) “to be in composition”: \( \nabla_i \rho \nabla_j = \top \) iff the composition \( \nabla_j \circ \nabla_i \) is meaningful. Thus, Cayley table of the relation \( \rho \) is determined with

\[
\nabla_i \rho \nabla_j = \begin{cases} 
\top, & (j = i + 1) \lor (i + j = n + 1); \\
\bot, & \text{otherwise.}
\end{cases}
\]

Let us form the adjacency matrix \( \mathbf{A} = [a_{ij}] \in \{0, 1\}^{n \times n} \) associated with the graph, which is determined by the relation \( \rho \). Thus, according to the paper \([4]\), the following statement is true.

**Theorem 3.1.** Let \( P_n(\lambda) = |\mathbf{A} - \lambda \mathbf{I}| = \alpha_0 \lambda^n + \alpha_1 \lambda^{n-1} + \cdots + \alpha_n \) be the characteristic polynomial of the matrix \( \mathbf{A} \) and \( v_n = [1 \ldots 1]_{1 \times n} \). If we denote by \( \mathbf{f}(k) \) the number of the \( k \)th-order differential operations, then the following formulas are true:

\[
\mathbf{f}(k) = v_n \cdot \mathbf{A}^{k-1} \cdot v_n^T
\]

and

\[
\alpha_0 \mathbf{f}(k) + \alpha_1 \mathbf{f}(k-1) + \cdots + \alpha_n \mathbf{f}(k-n) = 0 \quad (k > n).
\]
Lemma 3.2. Let $P_n(\lambda)$ be the characteristic polynomial of the matrix $A$. Then the following recurrence is true:

$$P_n(\lambda) = \lambda^2 (P_{n-2}(\lambda) - P_{n-4}(\lambda)).$$

(17)

Lemma 3.3. Let $P_n(\lambda)$ be the characteristic polynomial of the matrix $A$. Then it has the following explicit representation:

$$P_n(\lambda) = \begin{cases} 
\sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor + 1} (-1)^{k-1} \left( \binom{n-2}{k-1} \right) \lambda^{n-2k+2}, & n = 2m; \\
\sum_{k=1}^{\lfloor \frac{n+2}{2} \rfloor + 2} (-1)^{k-1} \left( \binom{n+3}{2k-2} - \binom{n+3}{2k-4} \right) \lambda^{n-2k+2}, & n = 2m+1.
\end{cases}$$

(18)

The number of the higher order differential operations is determined by corresponding recurrence, which for dimension $n = 3, 4, 5, \ldots, 10$, we refer according to [3]:

| Dimension | Recurrence for the number of the $k^{th}$-order differential operations |
|-----------|---------------------------------------------------------------------|
| $n = 3$   | $f(k) = f(k-1) + f(k-2)$                                           |
| $n = 4$   | $f(k) = 2f(k-2)$                                                   |
| $n = 5$   | $f(k) = f(k-1) + 2f(k-2) - f(k-3)$                                  |
| $n = 6$   | $f(k) = 3f(k-2) - f(k-4)$                                           |
| $n = 7$   | $f(k) = f(k-1) + 3f(k-2) - 2f(k-3) - f(k-4)$                        |
| $n = 8$   | $f(k) = 4f(k-2) - 3f(k-4)$                                         |
| $n = 9$   | $f(k) = f(k-1) + 4f(k-2) - 3f(k-3) - 3f(k-4) + f(k-5)$             |
| $n = 10$  | $f(k) = 5f(k-2) - 6f(k-4) + f(k-6)$                                 |

For considered dimensions $n = 3, 4, 5, \ldots, 10$, the values of the function $f(k)$, for small values of the argument $k$, are given in the database of integer sequences [6] as sequences $A020701 \ (n = 3), A090989 \ (n = 4), A090990 \ (n = 5), A090991 \ (n = 6), A090992 \ (n = 7), A090993 \ (n = 8), A090994 \ (n = 9), A090995 \ (n = 10)$, respectively.

4 The compositions of the differential operations and Gateaux directional derivative of the space $\mathbb{R}^n$

Let $f \in A_0$ be a scalar function and $\vec{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. Thus, the Gateaux directional derivative in direction $\vec{e}$ is defined by [1, p. 71]:

$$\text{dir}_{\vec{e}} f = \nabla_0 f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k} e_k : A_0 \longrightarrow A_0.$$ 

(19)
Let us extend the set of differential operations $A_n = \{\nabla_1, \ldots, \nabla_n\}$ with Gateaux directional derivational to the set $B_n = A_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$:

$$B_n (n=2m): \begin{array}{l}
\nabla_0 : A_0 \to A_0 \\
\nabla_1 : A_0 \to A_1 \\
\nabla_2 : A_1 \to A_2 \\
\vdots \\
\nabla_i : A_{i-1} \to A_i \\
\vdots \\
\nabla_m : A_{m-1} \to A_m \\
\nabla_{m+1} : A_m \to A_{m-1} \\
\vdots \\
\nabla_{n-j} : A_{j+1} \to A_j \\
\vdots \\
\nabla_{n-1} : A_2 \to A_1 \\
\nabla_n : A_1 \to A_0,
\end{array}$$

$$B_n (n=2m+1): \begin{array}{l}
\nabla_0 : A_0 \to A_0 \\
\nabla_1 : A_0 \to A_1 \\
\nabla_2 : A_1 \to A_2 \\
\vdots \\
\nabla_i : A_{i-1} \to A_i \\
\vdots \\
\nabla_m : A_{m-1} \to A_m \\
\nabla_{m+1} : A_m \to A_{m-1} \\
\vdots \\
\nabla_{n-j} : A_{j+1} \to A_j \\
\vdots \\
\nabla_{n-1} : A_2 \to A_1 \\
\nabla_n : A_1 \to A_0.
\end{array}$$

Let us define higher order differential operations with Gateaux derivative as the meaningful compositions of higher order of the functions from the set $B_n = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. We determine the number of the higher order differential operations with Gateaux derivative by defining a binary relation $\sigma$ “to be in composition”:

$$\nabla_i \sigma \nabla_j = \begin{cases} 
\top, & (i=0 \land j=0) \lor (i=n \land j=0) \lor (j=i+1) \lor (i+j=n+1); \\
\bot, & \text{otherwise.}
\end{cases}$$

Let us form the adjacency matrix $B = [b_{ij}] \in \{0, 1\}^{(n+1)\times n}$ associated with the graph, which is determined by relation $\sigma$. Thus, analogously to the paper [4], the following statement is true.

**Theorem 4.1.** Let $Q_n(\lambda) = |B - \lambda I|$ be the characteristic polynomial of the matrix $B$ and $\nu_{n+1} = [1 \ldots 1]_{1 \times (n+1)}$. If we denote by $g(k)$ the number of the $k$th-order differential operations with Gateaux derivative, then the following formulas are true:

$$g(k) = \nu_{n+1} \cdot B^{k-1} \cdot \nu_{n+1}^T$$

and

$$\beta_0 g(k) + \beta_1 g(k-1) + \cdots + \beta_{n+1} g(k-(n+1)) = 0 \quad (k > n+1).$$

**Lemma 4.2.** Let $Q_n(\lambda)$ and $P_n(\lambda)$ be the characteristic polynomials of the matrices $B$ and $A$ respectively. Then the following equality is true:

$$Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda).$$

7
Proof. Let us determine the characteristic polynomial \( Q_n(\lambda) = |B - \lambda I| \) by

\[
Q_n(\lambda) = \begin{vmatrix}
1 - \lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & 0 & -\lambda & 1 & \ldots & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & -\lambda & 1 & 0 \\
1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda
\end{vmatrix}.
\] (25)

Expanding the determinant \( Q_n(\lambda) \) by the first column we have

\[
Q_n(\lambda) = (1 - \lambda)P_n(\lambda) + (-1)^{n+2}D_n(\lambda),
\] (26)

where is

\[
D_n(\lambda) = \begin{vmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -\lambda & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 & -\lambda & 1 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda
\end{vmatrix}.
\] (27)

Let us expand the determinant \( D_n(\lambda) \) by the first row and then, in the next step, let us multiply the first row by \(-1\) and add it to the last row. Then, we obtain the determinant of order \( n - 1 \):

\[
D_n(\lambda) = \begin{vmatrix}
1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
-\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 1 & 0 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -\lambda & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & \ldots & 0 & -\lambda & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda
\end{vmatrix}.
\] (28)

Expanding the previous determinant by the last column we have

\[
D_n(\lambda) = (-1)^n
\]

\[
\begin{vmatrix}
-\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\
0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 1 & 0 \\
0 & 0 & -\lambda & 1 & \ldots & 0 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & -\lambda & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -\lambda & 1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda
\end{vmatrix}.
\] (29)
If we expand the previous determinant by the last row, and if we expand the obtained determinant by the first column, we have the determinant of order $n - 4$:

$$D_n(\lambda) = (-1)^n \lambda^2 \begin{vmatrix} -\lambda & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 1 & 0 & \ldots & 0 & 0 & 1 & 0 \\ 0 & 0 & -\lambda & 1 & \ldots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & \ldots & 0 & -\lambda & 1 & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & -\lambda \end{vmatrix}. \quad (30)$$

In other words

$$D_n(\lambda) = (-1)^n \lambda^2 P_{n-4}(\lambda). \quad (31)$$

From equalities (31) and (26) there follows:

$$Q_n(\lambda) = (1 - \lambda)P_n(\lambda) + \lambda^2 P_{n-4}(\lambda). \quad (32)$$

On the basis of Lemma 3.2, the following equality is true:

$$Q_n(\lambda) = \lambda^2 P_{n-2}(\lambda) - \lambda P_n(\lambda). \quad (33)$$

**Lemma 4.3.** Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix $B$. Then the following recurrence is true:

$$Q_n(\lambda) = \lambda^2 (Q_{n-2}(\lambda) - Q_{n-4}(\lambda)). \quad (34)$$

**Proof.** On the basis of Lemma 3.2. and Lemma 4.2. there follows the statement. \[ \Box \]

**Lemma 4.4.** Let $Q_n(\lambda)$ be the characteristic polynomial of the matrix $B$. Then it has the following explicit representation:

$$Q_n(\lambda) = \left\{ \begin{array}{ll}
(\lambda - 2) \sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor+1} (-1)^{k-1} \left( \frac{n+1}{2} - \frac{k}{k-1} \right) \lambda^{n-2k+2}, & n = 2m+1; \\
\sum_{k=1}^{\lfloor \frac{n+3}{2} \rfloor+2} (-1)^{k-1} \left( \left( \frac{n^2-k+2}{k-1} \right) + \left( \frac{n^2-k+2}{k-2} \right) \lambda \right) \lambda^{n-2k+3}, & n = 2m.\
\end{array} \right. \quad (35)$$

**Proof.** On the basis of Lemma 3.3 and Lemma 4.2. there follows the statement. \[ \Box \]
The number of the higher order differential operations with Gateaux derivative is determined by corresponding recurrences, which for dimension $n = 3, 4, 5, \ldots, 10$, we can get by the means of [5]:

| Dimension | Recurrence for the num. of the $k^{th}$-order diff. operations with Gateaux derivative: |
|-----------|--------------------------------------------------------------------------------------------------|
| $n = 3$   | $g(k) = 2g(k - 1)$                                                                               |
| $n = 4$   | $g(k) = g(k - 1) + 2g(k - 2) - g(k - 3)$                                                          |
| $n = 5$   | $g(k) = 2g(k - 1) + g(k - 2) - 2g(k - 3)$                                                         |
| $n = 6$   | $g(k) = g(k - 1) + 3g(k - 2) - 2g(k - 3) - g(k - 4)$                                              |
| $n = 7$   | $g(k) = 2g(k - 1) + 2g(k - 2) - 4g(k - 3)$                                                       |
| $n = 8$   | $g(k) = g(k - 1) + 4g(k - 2) - 3g(k - 3) - 3g(k - 4) + g(k - 5)$                                 |
| $n = 9$   | $g(k) = 2g(k - 1) + 3g(k - 2) - 6g(k - 3) - g(k - 4) + 2g(k - 5)$                                |
| $n = 10$  | $g(k) = g(k - 1) + 5g(k - 2) - 4g(k - 3) - 6g(k - 4) + 3g(k - 5) + g(k - 6)$                     |

For considered dimensions $n = 3, 4, 5, \ldots, 10$, the values of the function $g(k)$, for small values of the argument $k$, are given in the database of integer sequences [6] as sequences [A000079] ($n = 3$), [A090990] ($n = 4$), [A007283] ($n = 5$), [A090992] ($n = 6$), [A000079] ($n = 7$), [A090994] ($n = 8$), [A020714] ($n = 9$), [A129638] ($n = 10$), respectively.

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