THE GEOMETRY OF FRACTIONAL BRANES

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Abstract

By looking at fractional $D_p$-branes of type IIA on $T_4/Z_2$ as wrapped branes and by using boundary state techniques we construct the effective low-energy action for the fields generated by fractional branes, build their world-volume action and find the corresponding classical geometry. The explicit form of the classical background is consistent only outside an enhançon sphere of radius $r_e$, which encloses a naked singularity of repulsion-type. The perturbative running of the gauge coupling constant, dictated by the NS-NS twisted field that keeps its one-loop expression at any distance, also fails at $r_e$.

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1 Introduction

The duality relation between string and gauge theories [1] has been studied in detail in the AdS/CFT case, but recently more effort has been devoted to study the gauge/gravity correspondence in the case of non conformal gauge theories and Yang-Mills theories with reduced supersymmetry [2]-[16]. Fractional branes [17], that arise for example in orbifold models, provide a natural set-up to probe the Maldacena conjecture when supersymmetry is partially broken with respect to the $\mathcal{N}=4$ case and conformal invariance is lost. The full knowledge of space-time geometry of fractional branes is then not only an interesting problem relative to singular geometries and their resolution but also a physical issue providing a new insight for understanding the stringy aspects of gauge theory in more realistic models.

In this paper we will discuss the case of fractional D$p$-branes of type IIA string theory compactified on $T_4/Z_2$, where $Z_2$ is generated by the parity operator on the four compact spatial coordinates. While the techniques we use to reconstruct the space-time geometry associated to fractional branes are rather general, for the sake of simplicity we will mainly refer to the case of a fractional D0-brane of type IIA theory on $T_4/Z_2$. As it is well known, this is an orbifold limit of the type IIA string theory compactified on the smooth hyperKähler manifold K3. Within this frame a fractional D$p$-brane can be interpreted as a D$(p + 2)$-brane wrapped on one of the supersymmetric two-cycles of K3 that vanish in the orbifold limit. This point of view provides a natural ansatz to rewrite the relevant massless fields of the ten dimensional IIA orbifold theory in terms of fields coupled to the brane. Then one
obtains the truncation of the low energy action of IIA on $T_4/Z_2$ that provides the dynamics of fields coupled to fractional D$p$-branes. In order to reconstruct the full geometry of a fractional brane one also needs the expression of relevant source terms, namely of a boundary action. This information can be obtained by analyzing the structure of the boundary states \[18\]. The use of the boundary state technique \[19\] is rather natural in this context as it translates open string boundary conditions in terms of closed string, introducing automatically into the game gravity together with all other fields interacting with the fractional brane. The analysis of the couplings between the massless fields and the boundary state allows us to infer the structure of the complete world-volume action of fractional branes on the compact orbifold, which turns out to be also an essential tool for analyzing the motion of a probe brane in the classical background.

One expects that the space-time geometry of fractional branes shares a common feature with other classical backgrounds which are dual to non conformal gauge theories, namely the presence of naked singularities of repulson type. This feature has been found in the study of fractional branes on singular spaces \[20, 5, 6, 8, 12\], of stable non-BPS branes \[10\] and of type IIB fractional branes on $R_{1,5} \times R_4/Z_2$ \[13, 15\]. Our detailed investigation on the D0-brane solution and its generalization to the D2-brane, shows that this is the case also for fractional branes of type IIA on $T_4/Z_2$. However, also in our case, like in \[20, 21, 22\] this singularity is actually unphysical as it is outside the region where the simple supergravity approximation is reliable. In fact, at a distance $r_e$ greater than the one where the singularity is located, a probe fractional brane becomes tensionless, and at $r = r_e$ there is a geometric locus, called enhançon, where extra massless degrees of freedom become relevant (for a review on the probe technique see e.g. Ref. \[23\]). In this case, as discussed in Ref. \[20\], the fractional branes building up the classical background are forced to cover uniformly the hypersphere at $r = r_e$ rather than pile-up at $r = 0$. Then at a distance shorter than the enhançon radius the supergravity description is not valid any more and one has to modify the effective theory.

In this paper we do not study in detail the connection between the fractional brane classical solution and the world volume gauge theory and we leave it to a future work. We just mention that in general the running of the coupling constant of the dual gauge theory turns out to be dictated by the behavior of the twisted fields and in particular of the NS-NS twisted field. A remarkable property of our solution, is that twisted fields keep their harmonic asymptotic form at any distance. This appears as a general feature for fractional branes, apparently translating in geometrical classical terms the $\mathcal{N} = 2$ SUSY gauge theory property of allowing only one-loop perturbative corrections. This is in fact the case for type IIB fractional D3-brane \[13, 15\], where the appropriate perturbative logarithmic behavior of Yang-Mills coupling constant in terms of a stringy description has been obtained.

The paper is organized as follows: section 2 is devoted to the reduction of the low energy effective action for type IIA theory in ten dimension to the six dimensional action describing the dynamics of the fields coupled to a fractional D$p$-
brane on $T_4/\mathbb{Z}_2$. The boundary state technique, which is used to obtain a boundary action and the asymptotic behavior of relevant fields, is described in some detail in section 3. In section 4 we give the solution of the equations of motion and discuss their physical implications. The three appendices are devoted respectively to an alternative derivation of the low energy action based on the S-duality between IIA on $T_4/\mathbb{Z}_2$ and heterotic on $T_4$ (appendix A), to the analysis of the massless spectrum of the $\mathbb{Z}_2$ orbifold theory (appendix B) and to the explicit derivation of the equations of motion and their solution (appendix C).

2 The low energy effective action

One of the ingredients necessary to determine the geometry of a fractional brane is the low energy action expressing the dynamics of the relevant fields in the bulk. In this section we start with type IIA supergravity in ten dimensions with a $\mathbb{Z}_2$ orbifold projection acting as a reflection on four coordinates (which we choose to be $x_6, x_7, x_8, x_9$). Introducing a specific ansatz for the fields, we obtain the peculiar truncation of the low energy action which is appropriate to describe the fractional brane in $R^{1,5} \times R^4/\mathbb{Z}_2$. Then, by performing a Kaluza-Klein reduction to six dimensions, we get the low energy action for fractional branes in the orbifold $R^{1,5} \times T_4/\mathbb{Z}_2$. In appendix A we obtain the same action exploiting the S-duality relation between our theory and heterotic theory compactified on $T_4$.

The ten dimensional type IIA supergravity action on the orbifold $R^{1,5} \times R^4/\mathbb{Z}_2$ in the string frame can be written as

$$S = \frac{1}{2\kappa^2} \left\{ \int d^{10}x \sqrt{-G} e^{-2\Phi} R(G) + \int e^{-2\Phi} \left[ 4 d\Phi \wedge* d\Phi - \frac{1}{2} H_3 \wedge* H_3 \right] + \frac{1}{2} \left[ F_2 \wedge* F_2 + \tilde{F}_4 \wedge* \tilde{F}_4 - B_2 \wedge F_4 \wedge F_4 \right] \right\}$$

(2.1)

where

$$H_3 = dB_2 \ , \ F_2 = d\tilde{C}_1 \ , \ F_4 = dC_3$$

(2.2)

are respectively the field strengths of the Kalb-Ramond field, of the R-R vector field and of the R-R 3-form potential and

$$\tilde{F}_4 = F_4 - \tilde{C}_1 \wedge H_3 \ .$$

(2.3)

Moreover $\Phi$ is the 10-dimensional dilaton fluctuation and $\kappa = (2\pi)^{7/2}\alpha'^2g_S$ is the gravitational coupling constant appropriate to our orbifold background.

A fractional D$p$-brane can be interpreted as a D$(p + 2)$-brane wrapped on the vanishing 2-cycle of the orbifold $[24]$. According to this interpretation we make the following ansatz on the Kalb-Ramond and R-R 3-form potential

$$B = b\omega_2 \ , \ C_3 = A_1 \wedge \omega_2$$

(2.4)
where $\omega_2$ is the closed 2-form dual to the vanishing 2-cycle which we normalize in such a way that
\[
\int \omega_2 \wedge^* \omega_2 = 1 ,
\] (2.5)
and the scalar field $b$ is
\[
b = \frac{1}{2} (2\pi \sqrt{\alpha'})^2 + D .
\] (2.6)

Using the previous equations in Eq.(2.1) we obtain the following expression for the action
\[
S = \frac{1}{2\kappa^2} \left\{ \int d^{10}x \sqrt{-G} e^{-2\Phi} R(G) + \int \left[ 4 e^{-2\Phi} d\Phi \wedge^* d\Phi + \frac{1}{2} F_2 \wedge^* F_2 \right] \\
- \frac{1}{2} \int_6 \left[e^{-2\varphi} dD \wedge^* dD - (dA_1 - \tilde{C}_1 \wedge dD) \wedge^* (dA_1 - \tilde{C}_1 \wedge dD) \right] \right\} \] (2.7)
where the index 6 means that the fields are integrated over the 6-dimensional space-time which is unaffected by the orbifold projection. Notice that under the ansatz (2.4) the Chern-Simon term appearing in eq. (2.1) vanishes, showing that it does not contribute to the dynamics of fractional branes. The action appropriate to the case of the compact orbifold $T_4/Z_2$ can now be obtained from (2.7), by making a simple Kaluza-Klein reduction of the ten dimensional part
\[
S = \frac{\mathcal{V}}{2\kappa^2} \left\{ \int d^6x \sqrt{-g} e^{-2\varphi} R(g) + \int_6 e^{-2\varphi} \left[ 4 d\varphi \wedge^* d\varphi - d\tilde{\eta}_a \wedge^* d\tilde{\eta}_a \right] \right\} - \frac{1}{2} \int_6 \left[F_2^A \wedge^* F_2^A - 2 F_2^A \wedge^* (\tilde{C}_1 \wedge dD) + \tilde{C}_1 \wedge dD \wedge^* (\tilde{C}_1 \wedge dD) \right] \] (2.8)
where $g_{\mu\nu}$ is the six dimensional metric, while the 6-dimensional dilaton field $\varphi$ and the scalar fields $\tilde{\eta}_a$ are defined as follows
\[
\varphi = \Phi - \frac{1}{4} \ln \left( \prod_a G_{aa} \right) , \quad G_{aa} = e^{2\tilde{\eta}_a}
\] (2.9)
with $a = 6, 7, 8, 9$. Moreover we have introduced $F_2^A = dA_1$ and denoted the volume of $T_4$ by
\[
\mathcal{V} = \prod_a (2\pi R_a) .
\] (2.10)
Defining $\kappa_{\text{orb}} = \kappa/\mathcal{V}^{1/2}$ and going to the Einstein frame, we get
\[
S = \frac{1}{2\kappa_{\text{orb}}^2} \left\{ \int d^6x \sqrt{-g} R(g) + \int_6 \left[ -d\varphi \wedge^* d\varphi - d\tilde{\eta}_a \wedge^* d\tilde{\eta}_a \right] \right\} +
\]
\[ + \frac{1}{2} e^{\varphi} \prod_a e^{\bar{\eta}_a} F_2 \wedge^* F_2 - \prod_a e^{-\bar{\eta}_a} d\bar{D} \wedge^* d\bar{D} + \]

\[ + \frac{e^{\varphi}}{2} \left( \bar{F}_2 \wedge^* \bar{F}_2 - 2\sqrt{2}\bar{F}_2 \wedge^* (\bar{C}_1 \wedge d\bar{D}) + 2\bar{C}_1 \wedge d\bar{D} \wedge^* (\bar{C}_1 \wedge d\bar{D}) \right) \]\n
(2.11)

where we have made the following rescaling

\[ \tilde{A}_1 = \frac{A_1}{\sqrt{V}} , \quad \tilde{D} = \frac{D}{\sqrt{2V}} . \]  

(2.12)

The action (2.11) describes the fields that couple to an electric fractional D0-brane, or equivalently to a magnetic fractional D2-brane. In next section, using the boundary state formalism, we will instead obtain the asymptotic behavior of the classical fields generated by a generic electric fractional Dp-brane. In solving the classical equations of motion derived from (2.11) with the boundary conditions dictated by the boundary state, it is then more natural to address the case of the D0 fractional brane. This is what we will actually do in detail in the following. The explicit solution found in that case can be however easily generalized also to case of D2 fractional brane, as we will see in section 4.

3 The boundary state description

The boundary state formalism allows one to connect microscopic and supergravity description of D-branes by using the language of closed string state to evaluate both the couplings of supergravity massless fields to the brane and their asymptotic behavior. The boundary state description of fractional Dp-branes in type II theories on orbifold has been already discussed in Refs. [25, 26, 27] in the case of $Z_N$ projection, and for general discrete groups in Ref. [28]. Here we briefly review the structure of the boundary state in the simple case of $Z_2$, in order to derive the relevant physical information about fractional branes.

The boundary state is a closed string state which embodies the presence of a Dp-brane in space-time and is defined by the overlap equations, which, in the notation and conventions of Ref. [13], read as

\[ \partial_\tau X^\alpha|_{\tau=0}|B\rangle = 0 , \quad \alpha = 0, ..., p , \]  

(3.1)

\[ X^i|_{\tau=0}|B\rangle = y^i|B\rangle , \quad i = p + 1, ..., d - 1 . \]  

(3.2)

and

\[ (\psi^\mu (\sigma, 0) - i\eta S^\mu \nu \psi^\nu (\sigma, 0))|B\psi, \eta\rangle = 0 \]  

(3.3)

where $S^{\mu \nu} = (\eta^{\alpha \beta}, -\delta^{ij})$, and $\eta = \pm 1$. 

5
As it is well known, the spectrum of type II strings on orbifolds consists not only of states of the original theories which are invariant under the orbifold projection, the untwisted sector, but also of states which belong to the so-called twisted sectors living at the orbifold fixed hyperplanes. Therefore in an orbifold theory the complete boundary state is a linear combination of all the Ishibashi states coming from the resolution of eqs. (3.1)-(3.3) in each sector.

In the specific case of the orbifold $T_4/Z_2$, there are 16 distinct twisted sectors, one for each fixed plane. However, when the fractional brane does not wrap in the compact directions, there are only two different coherent states which solve the overlap equations, one for the untwisted and the other for the twisted sector associated to the particular fixed plane on which the brane lives.

In this particular case we can therefore write

$$|B\rangle = \mathcal{N}_U^T \left( |B\rangle_{\text{NS}}^U + \epsilon_1 |B\rangle_{\text{R}}^U \right) + \mathcal{N}_T^T \left( |B\rangle_{\text{NS}}^T + \epsilon_2 |B\rangle_{\text{R}}^T \right)$$

where $\mathcal{N}_U^T$ and $\mathcal{N}_T^T$ are two normalization constant to be fixed, $U$ and $T$ stand for untwisted and twisted respectively and $\epsilon_1, \epsilon_2$ are the two R-R charges in the untwisted and twisted sectors. The states $|B\rangle^U$ are the usual boundary states of a bulk-brane on a compact space with the standard GSO-projection [29], while the states $|B\rangle^T$ will be constructed in the following subsection.

Notice that the boundary state in (3.4) is the one associated to the trivial representation of the $Z_2$ group on Chan-Paton factors of open strings attached to fractional branes [28].

### 3.1 Construction of the boundary state

The Ishibashi states of the twisted sector, satisfying the overlap conditions (3.1)-(3.3), can be written as

$$|B, \eta\rangle_{\text{NS}}^T = |B_{\chi}\rangle_T^T |B_{\psi}, \eta\rangle_{\text{NS}}^T$$

in the NS-NS twisted sector and similarly

$$|B, \eta\rangle_{\text{R}}^T = |B_{\chi}\rangle_T^T |B_{\psi}, \eta\rangle_{\text{R}}^T$$

in the R-R twisted sector [1], where

$$|B_{\chi}\rangle_T = \delta^{5-p}(\vec{q} - y) \prod_{n=1} e^{-\frac{i}{\lambda} a^\mu_n s_{\mu\nu} a^\nu \cdot n} \prod_{r=1} e^{\frac{i}{\lambda} g^a_{\mu - r} \tilde{a}^a_{\nu - r}} \langle 0|_{\alpha} \langle 0|_{\tilde{\alpha}}$$

$$|B_{\psi}, \eta\rangle_{\text{NS}}^T = - \prod_{t=1} e^{i \eta \psi_{\mu - t} s_{\mu\nu} \tilde{\psi}^\nu_{- t}} \prod_{t=1} e^{-i \eta \psi_{\mu - t} \tilde{\psi}^\nu_{- t}} |B_{\psi}, \eta\rangle_{\text{NS}}^{(0)} T$$

$$|B_{\psi}, \eta\rangle_{\text{R}}^T = \prod_{t=1} e^{i \eta \psi_{\mu - t} s_{\mu\nu} \tilde{\psi}^\nu_{- t}} \prod_{t=\frac{1}{2}} e^{-i \eta \psi_{\mu - t} \tilde{\psi}^\nu_{- t}} |B_{\psi}, \eta\rangle_{\text{R}}^{(0)} T$$

In (3.5) and (3.6) we omit the ghost and superghost contribution which is not affected by the orbifold projection.
and $\mu, \nu \in \{0, \ldots, 5\}$ and $a \in \{6, \ldots, 9\}$. The zero modes part of the boundary state has a non trivial structure in both sectors; in the NS-NS case it is given by

$$|B_\psi, \eta\rangle^{(0)}_{\text{NS}} = \left(\hat{C} \frac{1 + i\eta \gamma_5}{1 + i\eta}\right)_{lm} |l\rangle|\tilde{m}\rangle$$

(3.10)

where $\gamma_i$ are the gamma matrices and $\hat{C}$ the charge conjugation matrix of $SO(4)$, $\gamma_5 = \gamma_6 \cdots \gamma_9$ and finally $|l\rangle, |\tilde{m}\rangle$ are spinors of $SO(4)$, while in the R-R case we have

$$|B_\psi, \eta\rangle^{(0)}_{\text{R}} = \left(\bar{\hat{C}} \gamma_0 \cdots \gamma_p \frac{1 + i\eta \gamma_7}{1 + i\eta}\right)_{ab} |a\rangle|\tilde{b}\rangle$$

(3.11)

where analogously $\gamma_i$ are the gamma matrices and $\bar{\hat{C}}$ the charge conjugation matrix of $SO(1, 5)$, $\gamma_7 = \gamma_0 \cdots \gamma_5$ and finally $|a\rangle, |\tilde{b}\rangle$ are spinors of $SO(1, 5)$.

The GSO-projection selects both in the NS-NS and R-R twisted sectors the following combination

$$|B\rangle^T = \frac{1}{2} \left[ |B, +\rangle + |B, -\rangle \right] .$$

(3.12)

The normalization constants $N^U$ and $N^T$ appearing in (3.4) can be fixed by evaluating the interaction amplitude $\langle B|D|B\rangle$ between two fractional $D_p$-branes due to the tree level exchange of closed strings and comparing the result with the corresponding open string channel calculation by means of the world-sheet duality transformation. The closed string amplitude can be written as the sum of two contributions, one for each sector

$$\langle B|D|B\rangle = U \langle B|D|B\rangle^U + T \langle B|D|B\rangle^T$$

(3.13)

An explicit computation easily shows that the untwisted sector contribution vanishes due to the abstruse identity, as in the case of usual bulk brane, while in the twisted sector the NS-NS and R-R contributions cancel each other. Therefore the amplitude eq.(3.13) is vanishing and this is consistent with the fact that fractional branes, being BPS objects, satisfy a no-force condition.

The 1-loop vacuum amplitude which represents the interaction between two fractional $D_p$-brane in the open string channel is also expressed as the sum of two terms, one for each element of the orbifold group $Z_2 = \{1, g\}$. Under world-sheet duality they transform respectively in the two terms of eqs. (3.13) and by comparison one can fix the normalization constants as follows

$$N^U = \frac{T_p}{2\sqrt{2}}, \quad N^T = \frac{T_p}{2\sqrt{2\pi^2\alpha'}} .$$

(3.14)
3.2 Brane couplings to bulk fields and their asymptotic behavior

In order to get information about the geometry of a fractional brane we need to determine its couplings with all the massless fields of the theory. This is achieved by projecting the boundary state onto the massless string states corresponding to the fields of the theory \[18\]. We need therefore a table of correspondence between the classical fields and the string states belonging to the massless spectrum.

As already remarked, the action \( (2.11) \) is a consistent truncation of the full action of type IIA supergravity in six dimensions, and describes the dynamics of the graviton \( h_{\alpha\beta} \), four Kaluza-Klein scalars \( \eta_a \) originating from the ten dimensional metric, a 1-form gauge field \( C_1 \), plus a 1-form gauge field \( A_1 \) and a scalar \( D \), which originates respectively from a three form gauge field and the Kalb-Ramond field with two components along the supersymmetric vanishing cycle of the orbifold. From the string analysis carried out in appendix B, it is easy to realize that the graviton, the Kaluza-Klein scalars and the 1-form \( C_1 \) are represented by the usual massless states of the untwisted sectors, while the 1-form \( A_1 \) and the scalar \( D \) are described by massless states belonging, respectively, to the twisted R-R and NS-NS sector. In the case of the fractional D2-branes the twisted and untwisted 1-forms are obviously replaced by two 3-forms \( (C_3 \) and \( A_3) \).

By projecting the boundary state on the massless states corresponding to these fields (whose explicit expressions are given in appendix B, see also Ref. \[30\]) we find the following couplings

\[
J_h = -\frac{T_p}{\sqrt{2V}} V_{p+1} \sum_{\alpha=0}^{p} h_{\alpha}^\alpha \quad (3.15)
\]

for the graviton,

\[
J_\phi = \frac{T_p}{2\sqrt{2V}} V_{p+1} \phi(1-p) \quad , \quad J_{\eta_a} = \frac{T_p}{2\sqrt{2V}} V_{p+1} \eta_a \quad (3.16)
\]

for the dilaton and the Kaluza-Klein scalars,

\[
J_C = \frac{T_p}{\sqrt{V}} V_{p+1} C_0...p \equiv \mu_U V_{p+1} C_0...p \quad (3.17)
\]

for the R-R untwisted field, and finally

\[
J_D = -\frac{T_p}{2\pi^2 \alpha'} V_{p+1} D \quad , \quad J_{A_{p+1}} = \frac{T_p}{2\pi^2 \alpha'} V_{p+1} A_{0...p} \equiv \mu_T V_{p+1} A_{0...p} \quad (3.18)
\]

for the NS-NS twisted scalar and the R-R twisted field \( \phantom{1} \)

\[2\]Eqs. \( (3.15)-(3.18) \) can be extended to the case of the non-compact orbifold \( C_2/\Gamma \) by dropping the factor \( V \) in all equations.
The previous couplings allow us to infer the structure of the Born-Infeld action for a fractional brane which turns out to be

\[ S_{\text{swv}} = -\frac{T_p}{\sqrt{2}V\kappa_{\text{orb}}} \left\{ \int dp^{1+1} \sqrt{-|g_{\alpha\beta}|} e^{-\kappa_{\text{orb}} \phi (1-p)} \prod_a e^{-\kappa_{\text{orb}} \frac{\phi}{2}} - \sqrt{2}\kappa_{\text{orb}} \int C_{p+1} \right\} \]

\[ -\frac{T_p}{2\pi^2\alpha'} \left\{ \int dp^{1+1} \xi \sqrt{-|g_{\alpha\beta}|} e^{-\kappa_{\text{orb}} \phi (1-p)} \prod_a e^{-\kappa_{\text{orb}} \frac{\phi}{2}} D - \int \left[ A_{p+1} + \sqrt{2}\kappa_{\text{orb}} D C_{p+1} \right] \right\} \]

The presence of the last term is due to the requirement of gauge invariance of the previous action under the gauge transformation

\[ \delta A_{p+1} = \lambda_p \wedge dD \quad , \quad \delta C_{p+1} = \frac{1}{\sqrt{2}\kappa_{\text{orb}}} d\lambda_p \] (3.20)

which for \( p = 0 \) is the gauge transformation that leaves invariant the bulk action \((2.11)\). The structure of the action \((3.19)\) is confirmed also by explicit calculations of closed string scattering amplitudes on a disk with appropriate boundary conditions \([31]\).

From the couplings in eqs.\((3.15)-(3.18)\) one can also determine the contributions to the interaction between two fractional branes due to the exchange of each massless state. In particular, looking first at the untwisted states one finds the contribution of graviton

\[ U_h = \left( \frac{T_p}{2} \right)^2 \frac{V_{p+1} (p+1)(3-p)}{V 2k_\perp^2} , \] (3.21)

of the dilaton and Kaluza-Klein scalars

\[ U_\phi = \left( \frac{T_p}{2} \right)^2 \frac{V_{p+1} (1-p)^2}{V 2k_\perp^2} , \quad U_{\eta_a} = \left( \frac{T_p}{2} \right)^2 \frac{V_{p+1} 1}{V 2k_\perp^2} , \] (3.22)

and of the R-R untwisted field

\[ U_C = -T_p^2 \frac{V_{p+1} 1}{V k_\perp^2} . \] (3.23)

In the twisted sector instead, we find that the NS-NS scalar and the R-R gauge field contribute respectively as

\[ U_D = \left( \frac{T_p}{2\pi^2\alpha'} \right)^2 \frac{V_{p+1} 1}{k_\perp^2} , \quad U_{A(p+1)} = -\left( \frac{T_p}{2\pi^2\alpha'} \right)^2 \frac{V_{p+1} 1}{k_\perp^2} . \] (3.24)

The potential energies due to the exchange of the various fields satisfy the conditions

\[ U_h + U_\phi + \sum_a U_{\eta_a} + U_C = 0 \quad , \quad U_{A(p+1)} + U_D = 0 \] (3.25)

which are consistent with the fact that the no-force condition is due to the separate cancellation of the untwisted and twisted contributions to the interaction energy.
The knowledge of the contribution to the interaction of each massless state is also useful to determine the mass of the fractional brane. Indeed in order to evaluate the mass we have to sum up all attractive contributions to the potential energy and compare it with Newton’s law in six dimensions. By making a Fourier transformation to configuration space, we obtain that the attractive energy is

\[ U_{\text{attr}} = \frac{T_p^2}{(3-p)r^{3-p}\Omega_{4-p}} \left[ \frac{1}{V} + \frac{1}{(2\pi^2\alpha')^2} \right] \] (3.26)

where \( \Omega_q = \frac{2\pi^{(q+1)/2}}{\Gamma((q+1)/2)} \) is the area of a unit \( q \)-dimensional sphere. Comparing eq.(3.26) with Newton’s law

\[ U_{\text{Newt}} = \frac{2\kappa^2_{\text{orb}}M_p^2}{(3-p)r^{3-p}\Omega_{4-p}} \] (3.27)

we get

\[ 2\kappa^2_{\text{orb}}M_p^2 = T_p^2 \left[ \frac{1}{V} + \frac{1}{(2\pi^2\alpha')^2} \right] = (\mu_U)^2 + (\mu_T)^2, \] (3.28)

which is the usual relation between mass and charges of a BPS object charged with respect to two different gauge fields.

Another important information provided by the boundary state analysis is the behavior of all classical fields generated by a fractional brane at large distance. This can be obtained by saturating the boundary state with the various states of the theory after inserting a closed string propagator. The asymptotic behavior for the various fields in our case is

\[ h^\infty = \frac{T_p}{4\sqrt{2}V (3-p)r^{3-p}\Omega_{4-p}} \left( \eta_{\alpha\beta}(p-3), \delta_{ij}(p+1) \right) \] (3.29)

for the graviton,

\[ \phi^\infty = \frac{T_p}{2\sqrt{2}V (3-p)r^{3-p}\Omega_{4-p}} (1-p), \eta_\alpha^\infty = \frac{T_p}{2\sqrt{2}V (3-p)r^{3-p}\Omega_{4-p}} \] (3.30)

for the dilaton and the scalars,

\[ C_{0...p}^\infty = -\frac{T_p}{\sqrt{V} (3-p)r^{3-p}\Omega_{4-p}} \] (3.31)

for the untwisted R-R field, and finally

\[ D^\infty = -\frac{T_p}{2\pi^2\alpha' (3-p)r^{3-p}\Omega_{4-p}}, \quad A_{0...p}^\infty = -\frac{T_p}{2\pi^2\alpha' (3-p)r^{3-p}\Omega_{4-p}} \] (3.32)

for the twisted NS-NS scalar and the twisted R-R form. We remind that these asymptotic fields obtained with boundary state techniques have canonical normalization; therefore they do not coincide with the corresponding fields appearing in the bulk action given in eq.(2.11). The relations between the two set of fields are

\[ \phi = \frac{\varphi}{\kappa_{\text{orb}}}, \eta_\alpha = \frac{\bar{\eta}_\alpha}{\kappa_{\text{orb}}}, \quad C_{(p+1)} = \frac{\bar{C}_{(p+1)}}{\sqrt{2}\kappa_{\text{orb}}}, \quad D = \frac{\bar{D}}{\kappa_{\text{orb}}}, \quad A_{(p+1)} = \frac{\bar{A}_{(p+1)}}{\sqrt{2}\kappa_{\text{orb}}}. \] (3.33)
Using these new fields, the world-volume action of eq.(3.19) becomes

\[ S_{wv} = -\frac{T_p}{\sqrt{2\kappa_{\text{orb}}}} \left\{ \int d^{p+1} \xi \sqrt{-|g_{\alpha \beta}|} e^{-\frac{\phi(1-p)}{2}} \prod_a e^{-\frac{\eta_a}{2}} - \int \tilde{C}_{p+1} \right\} \]

(3.34)

This action will generate the source terms in the equations of motion for the bulk fields.

4 Classical solution of fractional D-branes

In order to determine the classical solution of a fractional Dp-brane we can either solve the inhomogeneous field equations obtained by varying the total action i.e. the sum of eqs.(2.11) and (3.34), or solve the homogeneous field equations supplemented by the asymptotic behavior of fields, determined by the boundary state analysis, eqs. (3.29)-(3.32). In appendix C we find the classical solution associated to the fractional D0-brane following the latter procedure: we solve the homogeneous field equations under the assumption that all the fields are functions only of the radial transverse coordinate of the six-dimensional space and that the metric obeys the standard extremal black 0-brane ansatz.

The first corrections to the asymptotic behavior of the fields can be easily obtained by solving iteratively the equations of motion. From this analysis one can learn that, up to second order, the twisted fields do not get corrections with respect to their harmonic asymptotic behavior. This appears to be an important physical feature of the twisted fields. Therefore we take it as an ansatz for the full solution. Once made this assumption the equations of motion are easily solved in terms of a single function \( H \), which is a very simple generalization of the harmonic function appearing in the classical solution of ordinary bulk branes

\[ H = 1 + \frac{1}{2} \frac{Q_0}{r^3} - \frac{1}{2} \frac{\mathcal{V}}{(2\pi)^4 \alpha'^2} \frac{Q_0^2}{r^6} , \]

(4.1)

where

\[ Q_0 = \frac{2\sqrt{2} T_0 \kappa_{\text{orb}}}{3 \Omega_4 \mathcal{V}^{1/2}} . \]

(4.2)

More explicitly we have

\[ ds^2 = -H^{-\frac{2}{7}} dt^2 + H^{\frac{1}{7}} (dr^2 + r^2 d\Omega^2) \]

(4.3)

for the metric,

\[ e^\varphi = H^{\frac{1}{7}} , \quad e^{\eta_0} = H^{\frac{1}{7}} \]

(4.4)
for the dilaton and for the scalar fields,

\[ C_0 = (H^{-1} - 1) \]  

(4.5)

for the untwisted R-R vector, and finally

\[ D = -\frac{1}{\sqrt{2}} \frac{\mathcal{V}^{1/2}}{4\pi^2\alpha'} \frac{Q_0}{r^3}, \quad A_0 = -\frac{\mathcal{V}^{1/2}}{4\pi^2\alpha'} \frac{Q_0}{r^3} \]  

(4.6)

for the twisted NS-NS scalar and the twisted R-R vector, respectively.

The structure of the D0-brane solution is very simple: for the twisted fields the first order correction to the background value is exact, while the untwisted fields have the same expression of the fields associated to a bulk brane in terms of the function \( H \). This fact suggests a natural generalization of our solution to the case of the fractional D2-brane, namely

\[ ds^2 = H^{-\frac{1}{4}} (-dt^2 + dx_1^2 + dx_2^2) + H^{\frac{3}{4}} (dr^2 + r^2 d\Omega^2) \]  

(4.7)

for the metric,

\[ e^{\varphi} = H^{-\frac{1}{4}}, \quad e^{\eta_a} = H^{\frac{3}{4}} \]  

(4.8)

for the dilaton and for the scalar fields,

\[ C_{012} = (H^{-1} - 1) \]  

(4.9)

for the untwisted R-R vector and

\[ D = -\frac{1}{\sqrt{2}} \frac{\mathcal{V}^{1/2}}{4\pi^2\alpha'} \frac{Q_2}{r}, \quad A_{012} = -\frac{\mathcal{V}^{1/2}}{4\pi^2\alpha'} \frac{Q_2}{r} \]  

(4.10)

for the twisted NS-NS scalar and for the twisted R-R vector respectively, where

\[ H = 1 + \frac{1}{2} \frac{Q_2}{r} - \frac{1}{2} \frac{\mathcal{V}}{(2\pi)^4\alpha'^2} \frac{Q_2^2}{r^2} \]  

(4.11)

and

\[ Q_2 = \frac{2\sqrt{2} T_2 \kappa_{\text{orb}}}{\Omega_2 \mathcal{V}^{1/2}} \]  

(4.12)

Let us make some comments about the solution both for \( p = 0 \) and \( p = 2 \). First of all one can notice that this solution is consistent with the BPS nature of fractional D\( p \)-branes, as seen from the mass-charge relation (3.28) and from the one loop no-force condition, verified by the boundary state technique. In fact, by computing the world-volume action (3.34) for a probe fractional brane in the background of Eqs. (4.3-4.6) or Eqs.(4.7-4.10), one can check that the distance dependent part identically vanishes, and therefore there is no static force acting on the probe.
Another important observation is that the untwisted fields, and the metric in particular, have a naked singularity at \( r = r_+ \) where
\[
(r_+)^{3-p} = \frac{Q_p}{4} \left( -1 + \sqrt{1 + \frac{V}{2\pi^4\alpha'^2}} \right), \quad p = 0, 2. \tag{4.13}
\]

However the metric we have found is strictly analogous to the one studied in Ref. [20], so we expect that this singularity is of repulsion type and one may cure it by means of an enhançon mechanism first proposed therein. To see this, one notices that at a distance \( r_e \), where
\[
(r_e)^{3-p} = 2Q_p \frac{V}{(2\pi)^4\alpha'^2} > (r_+)^{3-p} \tag{4.14}
\]
the derivative of \( H \) vanishes and both gravitational and gauge forces change sign. Therefore, even if a fractional Dp-brane probe feels no net force at any distance, it becomes tensionless at \( r = r_+ \) and acquires a negative tension at shorter distances. Indeed expanding the DBI part of the world-volume action in the velocities and keeping only the lowest order terms one gets
\[
- \frac{T_p}{\sqrt{2}V\kappa_{\text{orb}}} \int d\xi^{p+1} \delta_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} \left( 1 + \frac{\sqrt{2V}}{2\pi^2\alpha'} D \right) = \]
\[
- \frac{T_p}{\sqrt{2}V\kappa_{\text{orb}}} \int d\xi^{p+1} \delta_{ij} \frac{\partial x^i}{\partial \tau} \frac{\partial x^j}{\partial \tau} \left( 1 - \frac{r_e^{3-p}}{r^{3-p}} \right). \tag{4.15}
\]

We see then that if \( D \) has the form of eq.(4.6) or (4.10) the Dp-brane becomes tensionless exactly at \( r = r_e \). It is then clear that the classical solution cannot be trusted at \( r < r_e \). This may correspond to the fact that, as shown in Ref. [20], the Dp-branes building up the classical background, rather than piling up at \( r = 0 \), are forced to cover uniformly the hypersphere at \( r = r_e \). If one identifies the (properly rescaled) transverse coordinates with the Higgs fields, one may use the procedure just outlined to derive their kinetic term. Then, for the \((2+1)\) gauge theory dual to the D2 fractional brane, the factor \( \delta_{ij}(1 - r_e/r) \) can be interpreted as the metric in moduli space, which turns out to be rather independent from the detailed geometry of D2-brane. Actually from this point of view the fact that NS-NS twisted field keeps its harmonic asymptotic form may be seen as expressing in geometrical classical terms the quantum property of \( \mathcal{N} = 2 \) SUSY theory allowing only one-loop perturbative corrections. It is then worth investigating more closely the correspondence between the detailed structure of fractional Dp-brane geometry and the associated world-volume gauge theory.
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A Alternative derivation of the bulk action

In this appendix we give an alternative derivation of the truncated bulk action necessary to reconstruct the geometry of a fractional brane. This alternative strategy consists in exploiting the $S$-duality relating type IIA on $T_4/I_4$ to heterotic theory compactified on $T_4$. Furthermore our task is made easier by turning on only the fields which couple to fractional $D_p$-branes at tree level. For the sake of definiteness we refer here to the case of $D_0$-brane where the fields are the graviton $g_{\mu\nu}$, 4 scalar fields $\eta_a$, the dilaton $\phi$, 1 untwisted R-R vector $C_0$, 16 twisted R-R vectors $A^I_0$ ($I = 1, \ldots, 16$) and 16 twisted NS-NS scalars $D^I$.

One may easily check that under the chain of dualities (implementing the 6-dimensional $S$-duality from IIA on $T_4/I_4$ to the heterotic string on $T_4$)

$$IIA \xrightarrow{\frak{T}_4} IIB \xrightarrow{\frac{T_4}{(-1)^{F_L}I_4}} S \xrightarrow{\frak{T}_4} IIB \xrightarrow{\frac{T_4}{\Omega I_4}} \text{Type I} \xrightarrow{S} \text{heterotic T}_4,$$

the previous fields transform as follows

$$\varphi = -\varphi_{he}, \quad g_{\mu\nu} = e^{-2\varphi_{he}}g_{\mu\nu}^{he}, \quad (A.2)$$

$$G_{aa} = \left(\prod_b G_{bb}^{he}\right)^{1/2} \left(G_{aa}^{he}\right)^{-1} \quad \text{for } a \neq 9, \quad \text{and} \quad G_{99} = \left(\prod_b G_{bb}^{he}\right)^{-1/2} G_{99}^{he}, \quad (A.3)$$

$$C_\mu = G_{\mu9}^{he} G_{he}^{99} \equiv A_\mu^{he}, \quad (A^I_\mu) = \frac{A^I_\mu^{he} + A^{I+1he}}{\sqrt{2}}, \quad (A.4)$$

$$D^I = \frac{A^I_9 + A^{I+1}_9}{\sqrt{2}}, \quad (A.5)$$

where the label “he” refers to heterotic fields.

The effective action of the heterotic string compactified on a 4-torus of volume $\mu^4$ is given by

$$S^{he} = \frac{\mu^4}{2\kappa_{10}^2} \int d^6x \sqrt{-g^{he}} e^{-2\varphi^{he}} \left[ R(g^{he}) + 4\partial_\mu \varphi^{he} \partial^\mu \varphi^{he} + \frac{1}{4} (\partial_\mu G_{ab} \partial^\mu G^{ab})^{he} \right]$$
\[-\frac{1}{4} \left( G_{ab} F^{a}_{\mu} F^{\mu b} \right)^{\text{he}} - \frac{1}{12} \left( h_{\mu \nu \rho} h^{\mu \nu \rho} \right)^{\text{he}} - \frac{1}{4} \left( G_{ab} h^{a}_{\mu} h^{b}_{\mu} \right)^{\text{he}} - \frac{1}{4} \left( G_{ab} G_{cd} h^{ac} h^{bd} \right)^{\text{he}} \]

\[-\frac{\mu^4}{4 g_{10}^2} \int d^6 x \sqrt{-g} e^{-2 \phi} \sum_{I=1}^{16} \left( \tilde{F}^{I}_{\mu \nu} \tilde{F}^{I \mu \nu} + 2 G_{ab} \tilde{F}^{a}_{\mu} \tilde{F}^{b \mu \nu I} \right)^{\text{he}} \quad (A.6)\]

where we have assumed that the heterotic gauge group is broken to $U(1)^{16}$ and have defined the field $\tilde{F}^{I}_{\mu \nu}^{\text{he}}$ follows

\[\tilde{F}^{I}_{\mu \nu}^{\text{he}} = \left( F^{I}_{\mu \nu} + A^{a}_{\mu} \partial_{\nu} A^{I}_{a} \right)^{\text{he}}. \quad (A.7)\]

To make contact with the case considered in the previous sections, we first neglect in the action (A.9) all terms that contain fields not dual to those of the truncated type IIA theory we are interested in, which contains only one pair of twisted fields (say $(A^J, D^J)$ for a given $J$). Doing this, we get

\[S^{\text{he}} = \frac{\mu^4}{2 k_{10}^2} \int d^6 x \sqrt{-g} e^{-2 \phi} \left[ R(g^{\text{he}}) + 4 \partial_{\mu} \varphi^{\text{he}} \partial^{\mu} \varphi^{\text{he}} \right. \]
\[\left. + \frac{1}{4} \left( \partial_{\mu} G_{aa} \partial^{\mu} G^{aa} \right)^{\text{he}} - \frac{1}{4} \left( G^{9 g}_{\mu \nu} F^{9 \mu \nu g} \right)^{\text{he}} \right] \]
\[-\frac{\mu^4}{4 g_{10}^2} \int d^6 x \sqrt{-g} e^{-2 \phi} \sum_{I=J, J+1} \left( \tilde{F}^{I}_{\mu \nu} \tilde{F}^{I \mu \nu} + 2 G_{99} \tilde{F}^{9}_{\mu} \tilde{F}^{9 \mu \nu I} \right)^{\text{he}}. \quad (A.8)\]

By performing the $S$-duality transformations (eqs. (A.2)-(A.5)) on the previous action, we get the following truncated low energy action for IIA on $T_4/I_4$:

\[S = \frac{\mu^4}{2 k_{10}^2} \int d^6 x \sqrt{-g} e^{-2 \phi} \left[ R(g) + 4 \partial_{\mu} \varphi \partial^{\mu} \varphi - \partial_{\mu} \tilde{\eta}^{a} \partial^{\mu} \tilde{\eta}^{a} \right. \]
\[\left. - \frac{1}{4} \prod_{a} e^{\tilde{\eta}^{a}} F^{a}_{\mu \nu} F^{\mu \nu} \right] - \frac{\mu^4}{4 g_{10}^2} \int d^6 x \sqrt{-g} \left[ F_{\mu \nu}^{J} F^{J \mu \nu} \right. \]
\[+ 2 F_{\mu \nu}^{J} C^{[\mu \partial^{\nu}] D^{J}} + C_{[\mu \partial_{\nu}] D^{J} C^{[\mu \partial^{\nu}] D^{J}} + 2 e^{-2 \phi} \partial_{\mu} D^{J} \partial^{\mu} D^{J} \prod_{a} e^{-\tilde{\eta}^{a}} \right] \quad (A.9)\]

where $F_{\mu \nu} = \partial_{[\mu} C_{\nu]}$ and $F_{\mu \nu}^{J} = \partial_{[\mu} A_{\nu]}^{J}$. After extracting the dilaton vacuum expectation value, one may rewrite the previous action in the Einstein frame as follows:

\[S = \frac{1}{2 \kappa_{2}^2} \int d^6 x \sqrt{-g} \left[ R(g) - \partial_{\mu} \varphi \partial^{\mu} \varphi - \partial_{\mu} \tilde{\eta}^{a} \partial^{\mu} \tilde{\eta}^{a} - \frac{1}{4} e^{\phi} \prod_{a} e^{\tilde{\eta}^{a}} \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} \right. \]
\[\left. - \frac{1}{4} e^{\phi} \tilde{F}_{\mu \nu} \tilde{F}^{J \mu \nu} - \frac{1}{\sqrt{2}} e^{\phi} \tilde{F}_{\mu \nu} \tilde{C}^{[\mu \partial^{\nu}] \tilde{D}^{J}} - \frac{1}{2} e^{\phi} \tilde{C}_{[\mu \partial_{\nu}] \tilde{D}^{J} \tilde{C}^{[\mu \partial^{\nu}] \tilde{D}^{J}} \right. \]
\[\left. - \partial_{\mu} \tilde{D}^{J} \partial^{\mu} \tilde{D}^{J} \prod_{a} e^{-\tilde{\eta}^{a}} \right] \quad (A.10)\]
where $\kappa_{\text{orb}}$ has been defined in section 2 and $g_{\text{orb}}^2 = g_{10}^2/\mu^4$. Moreover the following rescalings on the fields in eq. (A.9) have been made:

$$
\tilde{C}_\mu = \frac{g_s\mu^2}{\prod_a (2\pi R_a)^{1/2}} C_\mu \quad , \quad \tilde{A}_\mu^J = \frac{\sqrt{2g_{\text{orb}}}}{g_{\text{orb}}} A_\mu^J \quad , \quad \tilde{D}^J = \frac{\kappa_{\text{orb}} \prod_a (2\pi R_a)^{1/2}}{g_{\text{orb}} g_s \mu^2} D^J .
$$

Under the identifications

$$
\tilde{D} = -\tilde{D}^J \quad , \quad \tilde{A}_\mu = \tilde{A}_\mu^J
$$

the action (A.10) coincides with the one obtained in section 2 with a different procedure.

**B Untwisted and twisted states**

In this appendix we briefly review the description of the orbifold theory we are interested in, namely type IIA compactified on $T_4/Z_2$, and its spectrum.

In a generic orbifold theory the periodicity conditions of the closed string are satisfied up to the action of a generic element of the orbifold group. Thus in the case of the discrete group $Z_N$ which consists of $N$ elements \{1, $g$, ..., $g^{N-1}$\} we may have

$$
X(\tau, \sigma + \pi) = g^m X(\tau, \sigma) g^{-m} \quad \text{with} \quad m \in \{0, ..., N - 1\} \quad (B.1)
$$

For $m = 0$ one recovers the boundary conditions of closed string without orbifold projection, called untwisted boundary conditions, while for $m \neq 0$ one has $N - 1$ different twisted boundary conditions. In the case of a $Z_2$ projection, acting as a reflection over four space coordinates, the ten-dimensional Lorentz group $SO(1,9)$ decomposes in $SO(1,5) \times SO(4)$. We fix our convention as follows: $M, P, Q$ are the ten-dimensional indices; $a, b \in \{6, 7, 8, 9\}$ are the indices corresponding to the coordinates which are reflected by $I_4$ and $\mu, \nu \in \{0, ..., 5\}$ are the directions transverse to the orbifold projection. The theory has a unique twisted sector for each one of the sixteen orbifold fixed planes, characterized by the following boundary conditions

$$
X^\mu(\tau, \sigma + \pi) = X^\mu(\tau, \sigma) \quad \text{and} \quad X^a(\tau, \sigma + \pi) = -X^a(\tau, \sigma) \quad (B.2)
$$

for the bosonic coordinates and

$$
\psi^\mu_\pm(0, \tau) = \psi^\mu_\pm(\pi, \tau) \quad ; \quad \psi^a_\pm(0, \tau) = -\psi^a_\pm(\pi, \tau) \quad (B.3)
$$

for the R-R twisted sector and

$$
\psi^\mu_\pm(0, \tau) = -\psi^\mu_\pm(\pi, \tau) \quad ; \quad \psi^a_\pm(0, \tau) = \psi^a_\pm(\pi, \tau) \quad (B.4)
$$
for the NS-NS twisted sector. Obviously different boundary conditions correspond to different mode expansions of the string coordinates. The bosonic string mode expansion of the twisted sector is given by

\[ X^\mu(\tau, \sigma) = q^\mu + 2\alpha' p^\mu \tau + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left( \frac{\alpha^\mu_n}{n} e^{-2in(\tau - \sigma)} + \frac{\tilde{\alpha}^\mu_n}{n} e^{-2in(\tau + \sigma)} \right) \]  

(B.5)

for each \( \mu \in \{0, \ldots, 5\} \) and

\[ X^a(\tau, \sigma) = i \sqrt{\frac{\alpha'}{2}} \sum_{r=1/2}^\infty \left( \frac{\alpha^a_r}{r} e^{-2ir(\tau - \sigma)} + \frac{\tilde{\alpha}^a_r}{r} e^{-2ir(\tau + \sigma)} \right) \]  

(B.6)

for each \( a \in \{6, \ldots, 9\} \). For the fermionic coordinates one finds

\[ \psi^M_\tau = \sqrt{\alpha'} \sum_t \psi^M_t e^{-2it(\tau - \sigma)} \quad \psi^M_\tau = \sqrt{\alpha'} \sum_t \tilde{\psi}^M_t e^{-2it(\tau + \sigma)} \]  

(B.7)

where

\[ \begin{cases} \psi^\mu_t \text{ and } \tilde{\psi}^\mu_t \quad t \in Z \\ \psi^0_t \text{ and } \psi^0_t \quad t \in Z + \frac{1}{2} \end{cases}, \rightarrow \text{R - R twisted sector} \]  

(B.8)

and

\[ \begin{cases} \psi^\mu_t \text{ and } \tilde{\psi}^\mu_t \quad t \in Z + \frac{1}{2} \\ \psi^a_t \text{ and } \psi^a_t \quad t \in Z \end{cases}, \rightarrow \text{NS - NS twisted sector} \]  

(B.9)

To construct the massless spectrum of the theory one has to impose the mass-shell condition separately in each sector. For the sake of simplicity we consider a 4-torus of the type \( T_4 = T_1 \times T_1 \times T_1 \times T_1 \); in this case the mass-shell condition in the twisted NS-NS sector reads as follows

\[ \left\{ \frac{2}{\alpha'} \left( \tilde{N}_\alpha + \tilde{N}_\psi + N_\alpha + N_\psi - 1 \right) + \sum_a \left[ \left( \frac{n_a}{R_a} \right)^2 + \left( \frac{w_a R_a}{\alpha'} \right)^2 \right] \right\} |\text{state}\rangle = 0 \]  

(B.10)

and must be imposed together with the level matching condition

\[ (\tilde{N}_\alpha + \tilde{N}_\psi - N_\alpha - N_\psi) |\text{state}\rangle = \sum_a n_a w_a |\text{state}\rangle . \]  

(B.11)

with \( N_\alpha \) and \( N_\psi \) being the bosonic and fermionic occupation numbers, \( R_a \) the compactification radius of the coordinate \( X^a \) and finally \( n_a \) and \( w_a \) respectively Kaluza-Klein and winding modes along the compact direction \( a \). For \( R \neq \sqrt{\alpha'} \) it turns out that the massless states in the NS-NS untwisted sector are the graviton which transforms as \((3, 3)\) under the action of the little group \( SO(4) \), the Kalb-Ramond field \(((1, 3) + (3, 1))\), the dilaton \(((1, 1))\), 10 Kaluza-Klein scalars coming from the compactification of the ten-dimensional graviton \((10(1, 1))\), and 6 scalars...
coming from the compactification of the Kalb-Ramond field \((6(1,1))\). The explicit expressions of these states are

\[
\begin{align*}
|\Psi_h\rangle &= h_{\mu\nu} \bar{\psi}_{\mu}^{\alpha} \psi_{\nu}^{\alpha} \ |0_\psi\rangle_{-1} |0_{\bar{\psi}}\rangle_{-1} \prod_a \frac{1}{\sqrt{\Phi_a}} |n_a = w_a = 0\rangle, \\
|\Psi_B\rangle &= \frac{B_{\mu\nu}}{\sqrt{2}} \bar{\psi}_{\mu}^{\alpha} \psi_{\nu}^{\alpha} \ |0_\psi\rangle_{-1} |0_{\bar{\psi}}\rangle_{-1} \prod_a \frac{1}{\sqrt{\Phi_a}} |n_a = w_a = 0\rangle, \\
|\Psi_\phi\rangle &= \frac{\phi}{\sqrt{8}} (\eta_{\mu\nu} - k_{\mu} \ell_{\nu} - k_{\nu} \ell_{\mu}) \bar{\psi}_{\mu}^{\alpha} \psi_{\nu}^{\alpha} \ |0_\psi\rangle_{-1} |0_{\bar{\psi}}\rangle_{-1} \prod_a \frac{1}{\sqrt{\Phi_a}} |n_a = w_a = 0\rangle, \\
|\Psi_{R_a}\rangle &= \eta_a \bar{\psi}_{\mu}^{\alpha} \psi_{\mu}^{\alpha} \ |0_\psi\rangle_{-1} |0_{\bar{\psi}}\rangle_{-1} \prod_a \frac{1}{\sqrt{\Phi_a}} |n_a = w_a = 0\rangle, \\
|\Psi_{B_{ab}}\rangle &= \frac{B_{ab}}{\sqrt{2}} \bar{\psi}_{\mu}^{\alpha} \psi_{\alpha}^{\beta} \ |0_\psi\rangle_{-1} |0_{\bar{\psi}}\rangle_{-1} \prod_a \frac{1}{\sqrt{\Phi_a}} |n_a = w_a = 0\rangle,
\end{align*}
\]

where \(\Phi_a\) is the self-dual volume of the compact direction \(x_a\) \([22, 29]\). Moreover \(h_{\mu\nu}, B_{\mu\nu}\) and \(\phi\) are graviton, Kalb-Ramond and dilaton field respectively, while \(B_{ab}\) and \(\eta_a\) are the scalars corresponding to the compact components of Kalb-Ramond field and to the compact diagonal components of the metric. In the R-R untwisted sector instead the mass-shell condition reads as follows

\[
\begin{align*}
\left\{ \frac{2}{\alpha} \left( \tilde{N}_\alpha + \tilde{N}_\psi + N_\alpha + N_\psi \right) + \sum_a \left[ \left( \frac{n_a}{R_a} \right)^2 + \left( \frac{w_a R_a}{\alpha'} \right)^2 \right] \right\} |\text{state}\rangle &= 0
\end{align*}
\]

and it should be imposed together with the level-matching condition given in eq.\((3.11)\). In type IIA theory compactified on \(T_4/Z_2\) the massless states turn out to be 8 vectors transforming as \((2,2)\) under the action of the little group \(SO(4)\), whereas in type IIB theory on \(T_3/Z_2\) one find 8 scalars, and 3 2-forms potential, transforming respectively as \((1,1)\), and \(((3,1)+(1,3))\) under the little group. Their explicit expression is given by

\[
|C_{(n)}\rangle = \frac{1}{2\sqrt{2} n!} C_{\mu_1 \ldots \mu_n} \left[ (CT^{\mu_1 \ldots \mu_n} \Pi^+_{AB}) \cos(\gamma_0 \beta_0) \right.
\]

\[
+ \left. (CT^{\mu_1 \ldots \mu_n} \Pi^-_{AB}) \sin(\gamma_0 \beta_0) \right] |A\rangle_{-1/2} \ |\tilde{B}\rangle_{-3/2} |k\rangle
\]

where \(\Pi^\pm = (1 \pm \Gamma_{11})/2\), and \(\beta_0\) and \(\gamma_0\) are the superghost zero-modes.

Let us examine now explicitly the twisted mass spectrum. In this case the mass-shell condition is independent of the Kaluza-Klein and winding modes and it reduces to

\[
\frac{2}{\alpha} \left( \tilde{N}_\alpha + \tilde{N}_\psi + N_\alpha + N_\psi \right) |\text{state}\rangle = 0
\]

both in the NS-NS and R-R twisted sectors. The NS-NS twisted states are a product of two spinors of the internal \(SO(4)\) having the same chirality (this is true both
in type IIA and type IIB theory on $T_4/Z_2$). Therefore, from the point of view of the internal space, these states transform as $(2, 1) \otimes (2, 1) = (3, 1) + (1, 1)$ which corresponds to the transformation properties of a self dual 2-form that we denote by $D^I_{ab}$ and a scalar that we denote by $D_I$, where the index $I$ runs from 1 to 16. All these states transform as scalars under the little group, thus there are $4 \times 16$ scalars more in the theory.

Following the same notations of sect. 3 the explicit expression of these states is

$$|D^I_{(n)}\rangle = \frac{1}{\sqrt{2^n n!}} D^I_{a_1...a_n} \left( \hat{C} \hat{\gamma}^{a_1...a_n} \hat{\Pi}_+ \right)_{lm} |l\rangle_{-1} |\bar{m}\rangle_{-1} |k\rangle$$

Finally, in the R-R twisted sectors, one finds only one massless state for each fixed point which is a vector in type IIA and an antiself-dual 2 form plus a scalar in type IIB theory. These states have the following expression

$$|A^I_{\mu_0,...,\mu_m}\rangle = \frac{1}{\sqrt{2}} \frac{A^I_{\mu_0,...,\mu_m}}{m!} \left[ \left( \hat{C} \hat{\gamma}^{\mu_0,...,\mu_m} \hat{\Pi}_+ \right)_{ab} \cos(\gamma_0 \tilde{\beta}_0) + \left( \hat{C} \hat{\gamma}^{\mu_0,...,\mu_m} \hat{\Pi}_- \right)_{ab} \sin(\gamma_0 \tilde{\beta}_0) \right] |a\rangle_{-1/2} |\bar{b}\rangle_{-3/2} |k\rangle$$

C Field equations

In this appendix we write down and solve the homogeneous equations of motion and fix all the integration constants in order to obtain a solution consistent with the large distance behavior of all the fields. For the sake of simplicity let us start by rewriting in components the low-energy action obtained in section 2

$$S = \frac{1}{2\kappa_{\text{orb}}^2} \int d^6x \sqrt{-g} \left\{ R(g) - \partial_{\mu} \varphi \partial^{\mu} \varphi - \sum_{a=6}^9 \partial_{\mu} \eta_a \partial^{\mu} \eta_a - \frac{1}{4} \prod_a e^{\eta_a} F_{\mu_1 \mu_2} F^{\mu_1 \mu_2} A^A_{\mu_1 \mu_2} - 2\sqrt{2} A^A_{\mu_1 \mu_2} C^{[\mu_1 \partial^{\mu_2]} D + 2C_{[\mu_1 \partial_{\mu_2]} D C^{[\mu_1 \partial^{\mu_2]} \partial_{\mu_2]} D - \partial_{\mu} \partial^{\mu} D \prod_a e^{-\eta_a} \right\} \right.$$}

where we have dropped all tildas from the fields, to simplify the notation. The corresponding equations of motion read as follows:

$$\frac{2}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} \partial^{\mu} \varphi \right) = \frac{1}{4} e^{\varphi} \prod_a e^{\eta_a} F_{\mu_1 \mu_2} F^{\mu_1 \mu_2} +$$

$$+ F_{\mu_1 \mu_2} A^A_{\mu_1 \mu_2} - 2\sqrt{2} A^A_{\mu_1 \mu_2} C^{[\mu_1 \partial^{\mu_2]} \partial_{\mu_2]} D + 2C_{[\mu_1 \partial_{\mu_2]} D C^{[\mu_1 \partial^{\mu_2]} \partial_{\mu_2]} D \right)$$
for the dilaton;

$$\frac{2}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} g^{\mu \nu} \eta_a \right) = \frac{\epsilon^\phi}{4} \prod_a e^{\eta_a} F_{\mu_1 \mu_2} F_{\mu_1 \mu_2} - \prod_a e^{-\eta_a} \partial_\mu D \partial^\mu D$$  \hspace{1cm} (C.3)

for the scalar fields;

$$\frac{1}{\sqrt{-g}} \partial_\nu \left( \sqrt{-g} e^\phi \prod_a e^{\eta_a} F^{\nu \mu} \right) = e^\phi \partial_\nu D \left[ 2 C^{[\nu} \partial^{\mu]} D - \sqrt{2} F_A^{\nu \mu} \right]$$  \hspace{1cm} (C.4)

for R-R untwisted field;

$$\partial_\nu \left[ \sqrt{-g} e^\phi F_{A}^{\nu \mu} - \sqrt{2} e^\phi C^{[\nu} \partial^{\mu]} D \right] = 0$$  \hspace{1cm} (C.5)

for R-R twisted field;

$$\partial_\mu \left\{ \sqrt{-g} \left[ \prod_a e^{-\eta_a} \partial^\mu D + e^\phi C_\nu \left( C^{[\nu} \partial^{\mu]} D - \frac{1}{\sqrt{2}} F_{A}^{\nu \mu} \right) \right] \right\} = 0$$  \hspace{1cm} (C.6)

for the NS-NS twisted field. Finally the Einstein equations are:

$$\left( R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R \right) - \left( \partial^\mu \varphi \partial^\nu \varphi - \frac{1}{2} g^{\mu \nu} \partial^\gamma \varphi \partial_\gamma \varphi \right) - \sum_a \left( \partial^\mu \eta_a \partial^\nu \eta_a - \frac{1}{2} g^{\mu \nu} \partial^\gamma \eta_a \partial_\gamma \eta_a \right)$$

$$- \frac{\epsilon^\phi}{4} \left\{ \prod_a e^{\eta_a} \left[ 2 F_{a}^{\mu \nu} F^{\mu \nu} - \frac{g^{\mu \nu}}{2} F_{a \alpha \beta} F^{a \alpha \beta} \right] + \left[ 2 F_{a}^{A} F_{a}^{A} - \frac{g^{\mu \nu}}{2} F_{a \alpha \beta} F_{a \alpha \beta} \right] \right\}$$

$$- 2 \sqrt{2} \left( 2 F_{a}^{A} C^{[\alpha \beta]} D - \frac{g^{\mu \nu}}{2} F_{a \alpha \beta} C^{[\alpha \beta] D} \right) + 2 \left[ 2 C_{[\alpha \beta]} D C^{[\alpha \beta]] D} \right.$$

$$- \frac{g^{\mu \nu}}{2} C^{[\alpha \beta]} D C_{[\alpha \beta]} D A \right\} - \prod_a e^{-\eta_a} \left[ \partial^\mu \partial^\nu D \partial^\gamma D \partial_\gamma D \right]$$

\hspace{1cm} (C.7)

To obtain the fractional D0-brane solution, we make the following ansatz for the metric

$$ds^2 = -B^2(r) dt^2 + F^2(r) \left( dr^2 + r^2 d\Omega^2 \right)$$  \hspace{1cm} (C.8)

and take all the fields to be function only of $r$. Under this assumption, introducing the quantity $\xi \equiv \ln B + 3 \ln F$ the previous equations become

$$\frac{e^{-\xi}}{r^4} \partial_r \left( e^{\xi} r^4 \partial_r \varphi \right) = -B^{-2} e^\phi \left[ \frac{1}{2} \left( \partial_r A_0 \right)^2 \right.$$

$$+ \sqrt{2} (\partial_r A_0) C_0 \partial_r D + (C_0 \partial_r D)^2 + \frac{1}{2} \prod_a e^{\eta_a} \left( \partial_r C_0 \right)^2 \right]$$

\hspace{1cm} (C.9)

for the dilaton fields.

$$\frac{e^{-\xi}}{r^4} \partial_r \left( e^{\xi} r^4 \partial_r \eta_a \right) = -\frac{1}{4} B^{-2} e^\phi \prod_a e^{\eta_a} \left( \partial_r C_0 \right)^2 - \frac{1}{2} \prod_a e^{-\eta_a} \left( \partial_r D \right)^2$$  \hspace{1cm} (C.10)
for the scalar fields

\[
\frac{e^{-\xi}}{r^4} \partial_r \left( e^{\xi} B^{-2} r^4 e^\varphi \prod_a e^{\eta_a} \partial_r C_0 \right) = e^\varphi B^{-2} \partial_r D \left[ 2C_0 \partial_r D + \sqrt{2} \partial_r A_0 \right]
\]

(C.11)

for the R-R untwisted gauge field,

\[
\partial_r \left[ e^{\xi} r^4 B^{-2} e^\varphi \left( \partial_r A_0 + \sqrt{2} C_0 \partial_r D \right) \right] = 0
\]

(C.12)

for the R-R twisted vector field

\[
\partial_r \left\{ e^{\xi} r^4 \left[ 2 \prod_a e^{-\eta_a} \partial_r D - B^{-2} e^\varphi C_0 \left( \sqrt{2} \partial_r A_0 + 2C_0^2 \partial_r D \right) \right] \right\} = 0
\]

(C.13)

for the NS-NS twisted scalar field. And finally

\[
R_{rr} = F^{-2} \left[ (\partial_r \varphi)^2 + \sum_a (\partial_r \eta_a)^2 + \prod_a e^{-\eta_a} (\partial_r D)^2 \right] +
\]

\[
-\frac{3}{8} (BF)^{-2} e^\varphi \left\{ \prod_a e^{\eta_a} (\partial_r C_0)^2 + (\partial_r A_0)^2 + 2\sqrt{2}(\partial_r D)C_0 \partial_r A_0 + 2(C_0 \partial_r D)^2 \right\}
\]

(C.14)

\[
R_{00} = -\frac{3}{8} (BF)^{-2} e^\varphi \left\{ \prod_a e^{\eta_a} (\partial_r C_0)^2 + (\partial_r A_0)^2 + 2\sqrt{2}(\partial_r D)C_0 \partial_r A_0 + 2(C_0 \partial_r D)^2 \right\}
\]

(C.15)

\[
R_\theta = \frac{1}{8} (BF)^{-2} e^\varphi \left\{ \prod_a e^{\eta_a} (\partial_r C_0)^2 + (\partial_r A_0)^2 + 2\sqrt{2}(\partial_r D)C_0 \partial_r A_0 + 2(C_0 \partial_r D)^2 \right\}
\]

(C.16)

for the Einstein equations. To simplify the set of eqs.\((C.9)-(C.16)\) it is convenient to obtain an equation for the quantity \(\xi\). This is achieved by combining the Einstein equations for the \(R_{00}\) and \(R_\theta\) components, obtaining:

\[
\partial_r^2 \xi + \frac{7 \partial_r \xi}{r} + (\partial_r \xi)^2 = 0
\]

(C.17)

As asymptotically \(\xi = 0\) we take this to be the solution everywhere. Then by using the dilaton equation and the one for the \(R_{rr}\) component, taking into accounts the asymptotic behavior, one gets:

\[
B = F^{-3} = e^{-\frac{1}{2} \varphi}
\]

(C.18)

Expanding the fields up to the second order around their asymptotic values, one sees that the twisted fields do not get corrections with respect to their harmonic asymptotic behavior. Taking this as an \textit{ansatz} for the full solution, one can easily solve the field equations. The solution is written in section 4.
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