Topological Protection of Coherence in Noisy Open Quantum Systems

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(Dated: December 11, 2020)

We consider topological protection mechanisms in dissipative quantum systems in the presence of quenched disorder, with the intent to prolong coherence times of qubits. The physical setting is a network of qubits and dissipative cavities whose coupling parameters are tunable, such that topological edge states can be stabilized. The evolution of a fiducial qubit is entirely determined by a non-Hermitian Hamiltonian which thus emerges from a bona-fide physical process. It is shown how even in the presence of disorder winding numbers can be defined and evaluated in real space, as long as certain symmetries are preserved. Hence we can construct the topological phase diagrams of noisy open quantum models, such as the non-Hermitian disordered Su-Schrieffer-Heeger dimer model and a trimer model that includes longer-range couplings. In the presence of competing disorder parameters, interesting re-entrance phenomena of topologically non-trivial sectors are observed. This means that in certain parameter regions, increasing disorder drastically increases the coherence time of the fiducial qubit.

I. INTRODUCTION

Due to their characteristic protection against environmental noise, topological phases of matter [1–3] are considered to be promising candidates for the realization of noise-resilient quantum computers [4–11]. Furthermore, it was shown in [12] that the presence of topological edge states can preserve quantum mechanical features, e.g. coherence of a fiducial qubit, in the presence of dissipation (see also [13–15] for other works in a similar spirit). In that work, dissipative one-dimensional (1D) quantum optical qubit-cavity architectures were analyzed, where effective non-Hermitian Hamiltonians of the form of a tight-binding chain with diagonal complex entries were derived. The time evolution of the boundary-qubit coherence, driven by such non-Hermitian Hamiltonians, was extensively studied for choices of hopping parameters that admit symmetry-protected topological states localized at the edges of the system. It was found that (quasi-)dark modes, i.e., boundary states with exponentially small (in system size) imaginary parts, protect the edge qubits from decoherence effects.

Moreover, disordered as well as non-Hermitian generalizations of 1D topological insulators, such as the Su-Schrieffer-Heeger (SSH) model [16, 17], have been studied theoretically [18, 19], where the real space winding number was analyzed for different disorder strengths on the hopping parameters. Here, we build on the work presented in [12], addressing the role of quenched disorder in the qubit-cavity arrays. We fully characterize the disordered, non-Hermitian system’s topology by computing the winding number in real space, as long as certain symmetries are preserved. Hence we can construct the topological phase diagrams of noisy open quantum models, such as the non-Hermitian disordered Su-Schrieffer-Heeger dimer model and a trimer model that includes longer-range couplings. In the presence of competing disorder parameters, interesting re-entrance phenomena of topologically non-trivial sectors are observed. This means that in certain parameter regions, increasing disorder drastically increases the coherence time of the fiducial qubit.

The remainder of this paper is organized as follows. In Sec. II, we derive the effective non-Hermitian Hamiltonian describing qubit-cavity arrays using the Lindblad formalism. In Sec. III, the topological characterization of dissipative, disordered systems is illustrated, which is then applied to non-Hermitian dimer and trimer models in Sec IV. Special focus is put on the coherence of the qubit located at the boundary, whose faith can be accurately predicted from the phase diagrams. We then briefly discuss possible applications in quantum computation via dark-state braiding in Sec. V and conclude in Sec. VI.

II. THE SETUP

We consider a network consisting of qubits coupled to dissipative cavities in a Jaynes-Cumming fashion. Specifically, we study networks of $M$ qubits and $K$ dissipative cavities, as illustrated in Fig. 1 for $M = 4$ and $K = 5$. The Hamiltonian of the system has the following form

$$H = \sum_{l,m=1}^{K} J_{l,m}(a_{l}^{\dagger}a_{m} + \text{h.c.}) + \sum_{i=1}^{M} \sum_{l=1}^{K} \tilde{J}_{i,l}(a_{l}^{\dagger}\sigma_{i}^{-} + \text{h.c.}),$$

(1)

where $a_{l}^{\dagger}$ and $a_{l}$ are the bosonic creation and annihilation operators for cavity mode $l$, and $\sigma_{i}^{\pm}$ are the ladder operators for qubit $i$. We consider a Lindblad master equation $\dot{\rho} = \mathcal{L}[\rho]$, where $\mathcal{L} = \mathcal{K} + \mathcal{D}$, the coherent part is $\mathcal{K} = -i[H, \bullet]$, whereas the dissipative part is

$$\mathcal{D}[\rho] = \sum_{l=1}^{K} \Gamma_{l}\left[2a_{l}\rho a_{l}^{\dagger} - \{a_{l}^{\dagger}a_{l}, \rho\}\right].$$

(2)

Such Lindbladian description is accurate at sufficiently low temperature in particular in circuit QED experiments.
The Lindbladian can also be written as $\mathcal{L} = \mathcal{K}' + \mathcal{D}'$, where $\mathcal{K}'(\rho) = -i(H'\rho - \rho H'^\dagger)$ defines the non-Hermitian Hamiltonian $H'$:

$$H' = H - i \sum_{l=1}^{K} \Gamma_{l} a_{l}^\dagger a_{l},$$

and

$$\mathcal{D}'(\rho) = \sum_{l=1}^{K} 2 \Gamma_{l} a_{l} \rho a_{l}^\dagger.$$  

Consider the Fock space of the system $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{n}$ where $\mathcal{H}_{n}$ is the Hilbert space of $n$ particles (at this level the distinction between spins and bosons is unimportant). In the space of operators on $\mathcal{F}$ we define the Hilbert-Schmidt scalar product $\langle \langle x | y \rangle \rangle = \text{Tr}(x^\dagger y)$. Using the isomorphism $B_{HS}(\mathcal{F}) \simeq \mathcal{F} \otimes \mathcal{F}^*$ ($B_{HS}(\mathcal{F})$ is the space of bounded Hilbert-Schmidt operators on $\mathcal{F}$), the space of operators $B_{HS}(\mathcal{F})$ can be identified with

$$B_{HS}(\mathcal{F}) \simeq \bigoplus_{i,j=0}^{\infty} \mathcal{H}_{i} \otimes \mathcal{H}_{j}.$$  

In simpler terms, $B_{HS}(\mathcal{F})$ has a block structure with two labels $(i,j)$ each label being a particle number. The non-Hermitian Hamiltonian $H'$ preserves the number of particles and correspondingly $K'$ is block-diagonal in $(i,j)$. Instead, $\mathcal{D}'$ connects the sector $(i,j)$ with the sector $(i-1,j-1)$, i.e. it decreases the number of particles by one.

In this paper we will be mostly interested in the coherence of a fiducial qubit, that, without loss of generality we place at site 1. In the standard basis, the coherence of a qubit in state $\rho_{i}$ can be defined as $\mathcal{C} = 2|\rho_{i,1}|$ [20]. We initialize the system such that all cavities are empty and qubits are in the lowest state ($\downarrow$), while on the fiducial qubit the state is $|\psi\rangle = \alpha |\downarrow\rangle + \beta |\uparrow\rangle$. We further fix $|\alpha\beta| = 1/2$ which means that at the beginning the coherence assumes its maximal value one. We are interested in the evolution of the coherence as a function of time.

Let $|0\rangle$ be the vacuum state with no excitation on any qubit or cavity, while $|(j)\rangle$ denotes a single excitation on the $(j)$th site, describing either an excited qubit or a cavity hosting a photon. We use the notation $|(j)\rangle \leftrightarrow |0\rangle \langle j|$. It can be shown [12] that the evolution of the coherence at later time is given by

$$\mathcal{C}(t) = 2|\rho_{1,1}(t)| = 2|\langle 0|\rho(t)|1\rangle| = 2|\text{Tr}[|1\rangle \langle 0| \rho(t)]| = 2|\langle |1| \rho(t)\rangle| = 2|\langle |1| e^{-i\mathcal{L}t}\rho(0)\rangle|.$$  

(6)

Because of the block structure of the Lindbladian, $\langle 1| e^{-i\mathcal{L}t}\rho(0)\rangle = \langle 1| e^{-i\mathcal{L}t}\rho(0)\rangle_{\text{one particle}}$, where $\mathcal{X}$ is the operator $\mathcal{X}$ restricted to the linear space $\mathcal{V}_{0,1} = \text{Span}(|j\rangle, j = 1, 2, \ldots, N)$. In particular, $\rho(0) = \sigma \rho|0\rangle\langle 1| = (1/2)|0\rangle\langle 1|$. For what regard the restriction of the Lindbladian we have $\mathcal{L} = \mathcal{K}'$. Moreover, in $\mathcal{V}_{0,1}$,

$$\langle 1| e^{-i\mathcal{L}t}\rho(0)\rangle = \text{Tr}[|1\rangle \langle 0| \mathcal{K}'(\rho(0))].$$  

(7)

In other words, the evolution of the coherence of the fiducial qubit is entirely determined by the non-Hermitian Hamiltonian $-\mathcal{H}^T$ in the one-particle sector. Calling $\mathcal{H} := -\mathcal{H}^T$ one particle we have finally

$$\mathcal{C}(t) = \big|\langle 1| e^{-i\mathcal{H}t}|1\rangle\big|.$$  

(10)

We would like to stress that, in this setting, a non-Hermitian Hamiltonian emerges from a genuine bona-fide quantum evolution whereas in most current proposals non-Hermitian Hamiltonians are simulated in classical dissipative waveguides via the analogy between Helmholtz and Schrödinger equation (see e.g. [21]).

In order to prolong the coherence Eq. (10), one seeks a (non-Hermitian) Hamiltonian $\mathcal{H}$ that admits i) long-lived states, i.e. eigenstates of $\mathcal{H}$ with small (negative) imaginary part; and ii) that also have large amplitude on the site $|1\rangle$ (conventionally placed at the beginning of the chain).

Interestingly, both these requirement are satisfied to a large degree in one dimensional topological systems which admit edge states with the required properties in the non-trivial phase. The classification of such dissipative, non-Hermitian, topological chains has been done in [22] and utilized to prolong quantum coherence for the first time in [12]. Here we extend the analysis of [12] to disordered systems where translational invariance is broken. The phase diagrams of topological dissipative chains will tell us which parameter regions and models can be used to prolong the quantum coherence of the fiducial qubit.
We now briefly recall the topological classification of non-Hermitian quantum systems provided by Rudner et al. in Ref. \[22\]. For non-Hermitian quantum systems hosting dissipative sites, the topological invariant can be defined as the winding number around the dark-state manifold in the Hamiltonian parameter space. A non-trivial phase in dissipative systems corresponds to long-lived edge modes with infinite or exponential lifetimes.

In previous work \[12\], the topological classification of non-Hermitian models was formulated within the framework of Bloch theory, which we briefly outline here for comparison with the real-space approach to be introduced. Consider a one-dimensional periodic non-Hermitian chain with \(n\) sites per unit cell. In the thermodynamic limit, the Hamiltonian is given by \(H = \oint dk/(2\pi) \sum_{\alpha,\beta=1}^n H_{\alpha,\beta}(k)|k,\alpha\rangle\langle k,\beta|\). We shall only focus on the cases with one leaky site per unit cell, as the topological characterization is trivial in all other cases if no additional constraints are imposed \[22\]. The Bloch Hamiltonian of any such system is an \(n \times n\) matrix, which can be written as

\[
H(k) = \begin{pmatrix}
h(k) & v_k \\
v_k^\dagger & \Delta(k) - i\Gamma
\end{pmatrix},
\]

where \(h(k)\) is an \((n-1) \times (n-1)\) Hermitian matrix, \(v_k\) is a \((n-1)\)-dimensional vector and \(\Delta(k) - i\Gamma\) is a complex number. The Hamiltonian can be further decomposed in the following manner

\[
H(k) = \begin{pmatrix}
U(k) & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
\tilde{h}(k) & \tilde{v}_k \\
\tilde{v}_k^\dagger & \tilde{\Delta}(k) - i\tilde{\Gamma}
\end{pmatrix} \begin{pmatrix}
U(k)^\dagger & 0 \\
0 & 1
\end{pmatrix},
\]

where \(U(k)\) is a \((n-1) \times (n-1)\) unitary matrix whose columns are the eigenvectors of \(h(k)\), and \(\tilde{h}(k)\) is the \((n-1) \times (n-1)\) diagonal matrix of the corresponding eigenvalues. The phases of the eigenvectors are fixed by making all entries of the \((n-1)\)-dimensional vector \(\tilde{v}_k\) real and positive. Any \(U(k)\) satisfying the above criteria can be chosen without affecting the final result. Since \(U(k)\) is the only component parametrizing the Hamiltonian that can lead to non-trivial topology \[22\], the winding number of \(H(k)\) reduces to the one of \(U(k)\), which is given by

\[
W = \int \frac{dk}{2\pi i} \partial_k \ln \det\{U(k)\}.
\]

We now construct a real-space representation of the winding number that remains well defined when translation invariance is destroyed by e.g. the presence of disorder. Consider a chain with \(n\) sites in each cell and \(M\) number of unit cells. For what we said previously, we consider only one leaky site per unit cell, which, without loss of generality, we place at the final site of the cell.

The one-particle (non-Hermitian) Hamiltonian can be written as

\[
H = \sum_{i,j=1}^M \sum_{\alpha,\beta=1}^n H_{\alpha,\beta}^{i,j} |i,\alpha\rangle\langle j,\beta|
\]

\[
H = \begin{pmatrix}
H_{1,1} & H_{1,2} & \cdots & H_{1,n} \\
H_{2,1} & H_{2,2} & \cdots & H_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
H_{n,1} & H_{n,2} & \cdots & H_{n,n}
\end{pmatrix},
\]

where each \(H_{\alpha,\beta}\) is a \(M \times M\) matrix. The matrices \(H_{\alpha,\alpha}\) \(\alpha = 1, \ldots, (n-1)\) are diagonal with chemical potentials on the diagonal. Since we put the leaky site at position \(\alpha = n\), the matrix \(H_{n,n} = \epsilon_n - i\Gamma\), where \(\epsilon_n\) is a diagonal matrix of chemical potentials and for simplicity we set the leakage to have value \(\Gamma\) on each site.

Recalling the approach used in \(k\)-space, we first write the real-space Hamiltonian as

\[
\Lambda = \begin{pmatrix}
\Lambda & V \\
V^\dagger & \epsilon_n - i\Gamma
\end{pmatrix}
\]

\[
= \begin{pmatrix}
U & 0 \\
0 & U^\dagger
\end{pmatrix} \begin{pmatrix}
\Lambda & V \\
V^\dagger & \epsilon_n - i\Gamma
\end{pmatrix} \begin{pmatrix}
U^\dagger & 0 \\
0 & U
\end{pmatrix}
\]

\(\Lambda\) is a \((n-1)M \times (n-1)M\) Hermitian matrix, while \(V\) is a \((n-1)M \times M\) real matrix describing the hopping between decaying and non-decaying sites. \(\Lambda\) is a \((n-1)L \times (n-1)L\) diagonal matrix with real eigenvalues of \(\Lambda\), and \(U\) is a \((n-1)L \times (n-1)L\) unitary matrix that diagonalizes \(\Lambda\). The degrees of freedom for the choice of \(U\) are fixed by making each \(L \times L\) submatrix in \(V\) positive-definite, analogous to the procedure in reciprocal space.

With these preparations, the winding number of the unitary matrix \(U\) in real space can be evaluated with the prescription of \[23\] and further elaborations of Refs. \[18, 19, 24\]. In particular, \(\oint_{\Gamma} (dk/2\pi) \times \text{tr}\{\}\) and \(\partial_k\) become trace per volume and the commutator \(-i[X,\cdot]\) (\(X\) being the position operator), respectively. Note that \(X\) is the \(M\)-sized cell position operator, i.e. \(X = \text{diag}(1, 2, \ldots, M, 1, 2, \ldots, M, \ldots, M-1, M)\).

Thus, Eq. (13) in real space can be written as

\[
W = \frac{1}{L} \text{tr}'(U^\dagger[X, U]).
\]

Here, \(\text{tr}'\) stands for trace with truncation. Specifically, we take the trace over the middle interval of length \(M'\) and leave out \(\ell\) sites on each side (total length \(M = M' + 2\ell\)). With Eq. (17), we can explore topological phases in presence of dissipation and disorder. Note that in the model that we will consider, the matrix \(\Lambda\) is not noisy. In general, the model supports a non-trivial topological phase as long as a certain (chiral) symmetry is preserved. Disorder on the elements of \(\Lambda\) destroys the symmetry and consequently the system becomes topologically trivial.
Non-Hermitian Su-Schrieffer-Heeger (SSH) model with open boundary conditions. The qubit-cavity and cavity-qubit couplings are given by $J_1$ and $J_2$ respectively. Off-diagonal disorder is controlled via a uniform distribution function from which the hopping parameters are drawn.

**IV. DISORDERED NON-HERMITIAN SYSTEMS**

We now apply the real space formalism to investigate topological features in two explicit network geometries, namely the disordered non-Hermitian SSH dimer model and a disordered non-Hermitian trimer model.

**A. Disordered non-Hermitian SSH Dimer Model**

This model describes an open quantum system of coupled qubits and optical cavities which are arranged in an alternating manner, as shown in Fig. 2. In the super-one-particle sector, the corresponding restricted Hamiltonian $H$ in the presence of disorder is given by

\[
H = \sum_{j=1}^{M} \epsilon_{A,j}|j,A\rangle\langle j,A| + (\epsilon_{B,j} - i\Gamma)|j,B\rangle\langle j,B|
+ \sum_{j=1}^{M} (J_{1,j}|j,B\rangle\langle j,A| + \text{h.c.})
+ \sum_{j=1}^{M-1} (J_{2,j}|j+1,A\rangle\langle j,B| + \text{h.c.}).
\]

Due to the chiral symmetry and the pseudo-anti-hermiticity of the non-dissipative and dissipative model, respectively [12, 25], the topological states are expected to be robust against the chiral symmetry preserving off-diagonal disorder, i.e., noise in the hopping parameters. In contrast, disorder in the on-site potentials breaks the symmetries and is thus expected to quickly diminish topological features. Indeed, diagonal disorder leads to a unit cell as large as the system, thus having more than one dissipative site per unit cell and hence preventing the existence of topological dark states according to the argument in [22]. We therefore restrict the randomness to act on the hopping parameters, i.e., $J_{1,j} \equiv J_1 + \mu_1 \omega_{1,j}$ and $J_{2,j} \equiv J_2 + \mu_2 \omega_{2,j}$, where $\omega_{\alpha,j}$ are independent random variables with uniform distribution in the range $[-1, +1]$.

The effect of off-diagonal disorder on the spectrum of the restricted Hamiltonian $H$ is illustrated in Figure 3, where the density of states is plotted in the complex plane. In the topologically trivial regime of the clean system, Fig. 3 (a), all eigenvalues have imaginary part $-\Gamma/2$. When disorder is introduced, they mainly wash out on axis $\text{Im}(E) = -\Gamma/2$, as seen in Fig. 3(b). There is, however, a notable non-vanishing density of states emerging in the vicinity of $\text{Re}(E) = 0$. In the topologically non-trivial regime, Figs. 3 (c)&(d), a dark state with corresponding $\text{Im}(E) = 0$ can be found. Its topological protection against off-diagonal disorder manifests itself in its eigenvalue being left almost unchanged when disorder disturbs the system, while the bulk states featuring eigenvalues with imaginary part $-\Gamma/2$ blur out. The protected dark state corresponds to an edge state having support only on the non-dissipative sites, thus not decaying through the cavities. Another state emerging in the non-trivial phase lives on, on the contrary, only on the dissipative sites, with eigenvalue satisfying $\text{Im}(E) = -\Gamma$, as also seen in Fig. 3 (c). The mentioned destructive character of on-site potential disorder is discussed in the Appendix, Sec. C, where the density of states for diagonal disorder is analyzed, see Figs 10 (a)-(d).

We now turn to the computation of the winding number. In absence of disorder we can go to reciprocal space and realize that the unitary $U(k)$ in Eq. (12) is simply given by the phase of $J_1 + J_2 e^{-ik}$. The winding number of the dissipative system is thus the same as the winding number of the closed, Hermitian SSH-chain, resulting in

\[
W = \Theta(|J_2| - |J_1|),
\]

where $\Theta$ is the Heaviside function ($\Theta(x) = 1$ for $x > 0$ and $\Theta(x) = 0$ for $x < 0$). In order to compute the winding number in real space for the non-Hermitian SSH model, we follow the steps described in Sec. III. First, the Hamiltonian is written in the order of sublattices and divided into four blocks, as in Eq. (16). In this case, $\Lambda = H_{1,1} = \epsilon_A \mathbb{I}$, and $V = H_{1,2}$. From Eq. (16), we get $U\hat{V} = V$, where $U$, $\hat{V}$ and $V$ are all of dimension $M \times M$. To determine the unitary matrix...
we need to fulfill two requirements: i) the columns of $U$ need to be eigenvectors of $\Lambda$ and ii) $\tilde{V}$ needs to be positive definite. Since $\Lambda \propto I$, the first requirement is satisfied for any vector. In order to satisfy the second requirement, we recall that the polar decomposition of an invertible square matrix $V$ is a factorization of the form $V = U\tilde{V}$, where $U$ is a unitary matrix and $\tilde{V}$ is a positive-definite Hermitian matrix. $\tilde{V}$ is uniquely determined by $\tilde{V} = (V^\dagger V)^{1/2}$. As a result, $U$ can be written as

$$U = \tilde{V} (V^\dagger V)^{-1/2}.$$  \hspace{1cm} (20)

Finally, the winding number $W$ can be calculated via Eq. (17). From here on, we set the on-site potentials to be zero, i.e., $\epsilon_A = \epsilon_B = 0$.

Fig. 4 presents the phase diagrams of the disordered dissipative SSH model as a function of coupling and disorder strength. In Fig. 4 (a), the disorder is isotropic, i.e. $\mu_1 = \mu_2 = \mu$ and $J_2 = 1$, while in Fig. 4 (b), we consider anisotropic disorder with $\mu_1 = 2\mu_2 = \mu$ and $J_2 = 1$. The exact location of the phase transition, illustrated by the white lines in Fig. 4, can be obtained analytically by studying loci of the divergences in the localization length of the edge modes [18, 26], as elucidated in more detail in the Appendix, Sec. B. In Fig. 4 (a), the phase transition occurs at $J_2/J_1 = 1$ for all disorder strengths as for the clean case. Fig. 4 (b) shows a non-trivial topology by disorder effect. Namely, for fixed value of $|J_1| > 1$ close to one, one enters the topologically non-trivial region by increasing the disorder strength $\mu$, before transitioning into the topologically trivial regime after further increasing the noise. This widening of the topological phase boundary is observed for any kind of anisotropic disorder $\mu_1 \neq \mu_2$.

As already mentioned, the exact phase transition points can be evaluated from the divergence of the localization length. In particular, the phase boundary of the disordered SSH model is given by the equation $\mathbb{E}(\log |J_{1,j}|) = \mathbb{E}(\log |J_{2,j}|)$, where $\mathbb{E}(\bullet)$ denotes average over disorder (see Eq. (B3)). We first discuss the widening at small disorder strengths observed in Fig. 4 (b). The second order Taylor expansion of $\mathbb{E}(\log |X|)$ in $\mu_1/J_1$ reads $\mathbb{E}(\log |X|) \simeq \log(\mathbb{E}[|X|]) - \frac{\log^2 |X|}{2\mathbb{E}[|X|^2]}$ [27], resulting in the following approximation of the phase boundary equation,

$$\log |J_1| - \frac{\mu_1^2}{6J_1^2} \simeq \log |J_2| - \frac{\mu_2^2}{6J_2^2}. \hspace{1cm} (21)$$

Fixing $J_2$ and $\mu_2$ such that the right hand side of Eq. (21) is constant, we see that the function $\log |J_1| - \frac{\mu_1^2}{6J_1^2}$ is monotonically increasing in $J_1$ and decreasing in $\mu_1$. Hence, if $\mu_1$ increases, $J_1$ needs to grow as well in order to compensate. This corresponds to a widening of the topologically non-trivial regime for small increasing noise. In the opposite, strong disorder limit, we can expand $\mathbb{E}(\log |J_{\alpha,j}|) = \mathbb{E}(\log |J_{\alpha,j}|) = \log |\mu_\alpha| - 1 + O(J_\alpha/\mu_\alpha)$. The phase boundary equation in this regime becomes

$$\log |\mu_1| \simeq \log |\mu_2|. \hspace{1cm} (22)$$

Hence, for strong disorder, the phase boundary is roughly independent of $J_1$ accounting for the horizontal boundary in Fig. 4 (b). Similar disorder-induced topological characteristics were also recently discussed in the context of other non-Hermitian models [19, 28].

For each phase diagram, we now fix $J_2 = 1$ and choose four characteristic parameter configurations in order to get representative coherence time evolutions for the different topological sectors, depicted Fig. 5. For isotropic disorder, Fig. 5 (a), we choose three points along the vertical $J_1 = 0$ with $\mu = 0.1, 0.5, 1.0$ as well as the configuration $J_1 = 1.5, \mu = 0.5$, representing the disordered topologically non-trivial and trivial regime, respectively. The coherence decays to a nonzero (respectively zero) value at large times in the topologically non-trivial (respectively trivial) sector, thus matching
the phase diagram Fig. 4 (a). In the topologically non-trivial regime, increasing disorder leads to a smaller asymptotic value of the coherence. Similarly, for anisotropic disorder, Fig. 5 (b), we choose three points along the vertical $J_1 = 1.2$ with $\mu = 0.5, 1.5, 2.5$ as well as $J_1 = 0, \mu = 0.5$. The former three parameter pairs lie on a vertical line cutting through the broadening of the topologically non-trivial regime, thus representing the reentrance phenomenon into a higher topological phase. It can be seen that a finite coherence of the first qubit is present at large times only for $\mu = 1.5$, being in consent with the corresponding phase diagram Fig. 4 (b). For $J_1 = 0$ and $\mu = 0.5$, a similar behavior as for the isotropic disordered chain can be observed, with a large asymptotic coherence value. In previous work [12], it was shown that for large chains in the topologically non-trivial regime, the coherence saturates to approximately

$$C(t \to \infty) \approx 1 - x^2,$$

where $x = J_1/J_2$, with $|x| < 1$. It is thus natural to assume that the expectation value of the asymptotic coherence including disorder is given by

$$E[C(t \to \infty)] \approx E(1 - x^2) = \int_{-\mu_2}^{\mu_2} \int_{-\mu_1}^{\mu_1} 1 - \left(\frac{J_1 + \mu_1}{J_2 + \mu_2}\right)^2 d\mu_1 d\mu_2 \approx 1 - \frac{3J_1^2 + \mu_1^2}{3J_2^2 - 3\mu_2^2}.$$ 

Note that this only holds for weak to moderate disorder such that no change of topological phase can be generated randomly, i.e., $\mu_1 + \mu_2 < |J_2| - |J_1|$. In Fig. 5, the prediction Eq. (23) is illustrated by black solid lines for disorder strengths falling into the discussed regime. For large disorder, random phase changes result in a decrease of the mean coherence in the simulation, and Eq. (23) breaks down.

It is important to note that, even though the phase diagram is the same as those found in previous works [18, 19], the physical interpretation is different, as our models include dissipation. Edge states do not correspond to actual electronic states located at one of the boundaries of the chain, but rather describe the physics of the projected density matrix introduced in Sec. II. A non-trivial topological phase, resulting in quasi-dark states of the restricted Hamiltonian, leads to having an exponentially long (in system size) coherence time of the edge qubit. In the topologically trivial regime, the decoherence of the edge qubit is governed by dissipation, leading to a finite coherence time.

B. Disordered non-Hermitian Trimer Model

Next, we consider a trimer chain with nearest-neighbor as well as next-nearest-neighbor couplings, as depicted in Fig. 6. The corresponding non-Hermitian Hamiltonian, derived from the restricted Lindbladian, is given by

$$H = \sum_{j=1}^{M} (J_{1,j}|j,B\rangle\langle j,A| + \text{h.c.}) + \sum_{j=1}^{M-1} (J_{2,j}|j,C\rangle\langle j,B| + \text{h.c.}) + \sum_{j=1}^{M} (J_{j}|j|,A\rangle\langle j,A| + \text{h.c.}) + \sum_{j=1}^{M} (J_{3,j}|j+1,A\rangle\langle j,C| + \text{h.c.}) + \sum_{j=1}^{M} \epsilon_{A}|j,A\rangle\langle j,A| + \sum_{j=1}^{N} \epsilon_{B}|j,B\rangle\langle j,B| + \sum_{j=1}^{M} (\epsilon_{C,j} - i\Gamma)|j,C\rangle\langle j,C|.$$ 

It has been demonstrated that robust chiral edge modes exist in non-dissipative trimer chains, even in the absence of inversion symmetry [29]. It has been argued that their topological character is inherited through a mapping of a higher-dimensional model, namely the commensurate off-diagonal Aubry-André-Harper model, which is topologically equivalent to a two-dimensional tight-binding lattice pierced by a magnetic flux [30]. The topological classification by Rudner et al. including dissipation, however, imposes only translational symmetry. In fact, it turns out that the winding number in Eq. (17) can be used as a reliable predictor for the number of (quasi)-dark states located on the edge of the trimer chain with open boundary conditions. In previous work [12], it was found that in the clean case, the presence of next-nearest-neighbor couplings enable winding numbers $W = 0, 1, 2$. Concretely, $W$ is given by

$$W = \Theta(|J_3| - |J + J_2 \tan(\theta/2)|) + \Theta(|J_3| - |J - J_2 \cot(\theta/2)|),$$ 

where $\theta = \arccos \left(\frac{\epsilon_A - \epsilon_B}{\sqrt{4J_1^2 + (\epsilon_A - \epsilon_B)^2}}\right)$. We further verify the above equation in the Appendix, Sec. A, by solving the system analytically for a convenient system size.
Recall that we must fix the gauge freedom in $U$ as $t \to \infty$. Dissipation is set to $\Gamma = 0.5$ and a chain with $N = 300$, $J_1 = 1$, $J_2 = 2$, and $J = 1$ is considered. The three time evolutions of the coherence in the topological sectors $W = 0, 1, 2$ correspond to parameter choices $J_3 = 0.5, 2.0, 3.5$, respectively.

and counting the number of dark states localized on one edge of the chain. In order to calculate the winding number using the real-space approach, we first rewrite the Hamiltonian with respect to its sublattices and decompose it as in Eq. (17). In this case, the matrices $\Lambda$ and respectively $V$ with dimensions $2M \times 2M$ (respectively $2M \times M$) are given by

$$\Lambda = \begin{pmatrix} \epsilon_A 1 & H_{AB} \\ H_{BA} & \epsilon_B 1 \end{pmatrix}; \quad V = \begin{pmatrix} H_{AC} \\ H_{BC} \end{pmatrix}. \quad (26)$$

Here, $H_{AB} = J_1 1$. Due to the symmetry of $\Lambda$, $U$ from Eq. (16) can be written as

$$U = \begin{pmatrix} -\cos(\theta/2)U_- & \sin(\theta/2)U_+ \\ \sin(\theta/2)U_- & \cos(\theta/2)U_+ \end{pmatrix}, \quad (27)$$

where $U_\pm$ are two $M \times M$ so far unspecified unitaries and $\theta$ has been given above. From Eq. (16), we further get $U V = V$, which gives

$$\begin{pmatrix} -\cos(\theta/2)U_- & \sin(\theta/2)U_+ \\ \sin(\theta/2)U_- & \cos(\theta/2)U_+ \end{pmatrix} \begin{pmatrix} V_- \\ V_+ \end{pmatrix} = \begin{pmatrix} H_{AC} \\ H_{BC} \end{pmatrix}, \quad (28)$$

where $\hat{V} := (\hat{V}_-, \hat{V}_+)^T$. From the above equation we find

$$U_+ \hat{V}_+ = \frac{1}{2}(\cos(\theta/2)H_{AC} + \sin(\theta/2)H_{BC}),$$

$$U_- \hat{V}_- = \frac{1}{2}(-\sin(\theta/2)H_{AC} + \cos(\theta/2)H_{BC}). \quad (29)$$

Recall that we must fix the gauge freedom in $U$ by requiring the submatrices $\hat{V}_\pm$ to be positive definite. Consequently, $U_\pm$ can be determined by polar decomposition of the right hand side of Eq. (29), after which the unitary matrix $U$ is obtained using Eq. (27). Finally, the winding number is computed via Eq. (17). Interestingly, it can be shown that the winding number of $U$ is nothing more than the sum of the winding numbers of $U_+$ and $U_-$. For simplicity, we again limit our considerations to the case of vanishing the on-site chemical potentials, i.e., $\epsilon_A = \epsilon_B = \epsilon_C = 0$. Using the real-space winding number approach for the clean trimer model results in Fig. 7 (a), matching Eq. (25). Figs 7 (b)-(d) show the typical behavior of the coherence in the three distinct topological sectors $W = 0, 1, 2$ in the clean trimer model, respectively. In the topologically trivial regime, no dark states are present, driving decoherence of the first qubit. For $W = 2$, the existence of two dark states result in Rabi like oscillations of the first qubit’s coherence. The asymptotic solution, Eq. (A5), is also featured in Figs. 7 (b)-(d). Because of the $J_1$ dependence of the dark states, disorder in $J_1$ is expected to quickly destroy the topological features of the system. This is further suggested by the degree of freedom of the matrix $U$, Eq. (16), which collapses as soon as $J_1$ becomes disordered, leading to an immediate collapse of a well-defined winding number. Therefore, we shall from now on focus on the analysis of the disordered regime where only $J_2, J_3, J_4$ are exposed to noise, which we control via additive random noise drawn from a uniform distribution. Concretely, if $j$ labels the unit cell and $\{\omega_1, \omega_2, \omega\}$ are sets of independent, uniformly distributed random variables $\in [-1, 1]$, $J_{1,i,j} = J_1 + \mu_i \omega_{i,j}$ for $i = 2, 3, J_{j,j} = J + \mu_j \omega_{j,j}$ and $J_{1,j} = J_1$ for all $j$. Looking at the density of states for the different disorder types, depicted in Figure 11, the selection rules for the type of disorder under which topological dark states are stable is further underlined.

As for the disordered non-Hermitian SSH model, the full phase diagram for different disorder strengths can be constructed, shown in Fig. 8. Again, the exact phase transition points in the thermodynamic limit are depicted by white lines, which are derived via the dark state localization length con-

FIG. 7. Topology and coherence for the clean, non-disordered trimer model. (a) Winding number and (b)-(d) time dependent coherence for three parameter configurations corresponding to the three topological sectors. The dotted lines indicate the theoretically predicted asymptotic coherence as $t \to \infty$. Dissipation is set to $\Gamma = 0.5$ and a chain with $N = 300$, $J_1 = 1$, $J_2 = 2$, and $J = 1$ is considered. The three time evolutions of the coherence in the topological sectors $W = 0, 1, 2$ correspond to parameter choices $J_3 = 0.5, 2.0, 3.5$, respectively.

FIG. 8. Phase diagram of the disordered, dissipative trimer model for $N = 1500$, $\Gamma = 0.5$, $J_1 = 2$, $J_2 = 2$, $J_3 = 3$. Results are averaged over 40 random realizations. In (a), $\mu_2 = \mu_3 = \mu$, whereas (b) describes disorder with $2\mu_2 = 2\mu_3 = \mu$. White lines indicate the loci of diverging localization lengths in the thermodynamic limit.
Results are averaged over 40 random realizations. (a) For $\mu_J = \mu_2 = \mu_3 = \mu$, we show the coherence for $(J, \mu) = (0, 1)$ (orange), $(J, \mu) = (3, 1)$ (purple), and $(J, \mu) = (0, 7)$ (black), corresponding to $W = 2.1.0$, respectively. (b) For $2\mu_J = 2\mu_2 = \mu_3 = \mu$, we highlight the reentrance into a higher topological phase along the vertical line $J = 1.2$, with $\mu = 1$ (purple) and $\mu = 7$ (orange), corresponding to $W = 1, 2$, respectively. We further show the trivial regime by evaluating the coherence for $(J, \mu) = (6, 1)$ (black). The observable broadening of the curves is due to the error of the mean, pictured by error bars for every data point.

FIG. 9. Coherence of the first qubit in the disordered, dissipative trimer model for $N = 300$, $\Gamma = 0.5$, $J_1 = 1$, $J_2 = 2$ and $J_3 = 3$. Results are averaged over 40 random realizations. (a) For $\mu_J = \mu_2 = \mu_3 = \mu$, we show the coherence for $(J, \mu) = (0, 1)$ (orange), $(J, \mu) = (3, 1)$ (purple), and $(J, \mu) = (0, 7)$ (black), corresponding to $W = 2.1.0$, respectively. (b) For $2\mu_J = 2\mu_2 = \mu_3 = \mu$, we highlight the reentrance into a higher topological phase along the vertical line $J = 1.2$, with $\mu = 1$ (purple) and $\mu = 7$ (orange), corresponding to $W = 1, 2$, respectively. We further show the trivial regime by evaluating the coherence for $(J, \mu) = (6, 1)$ (black). The observable broadening of the curves is due to the error of the mean, pictured by error bars for every data point.

We shall again pick three points in each phase diagram and illustrate the corresponding time evolution of the first qubits coherence, seen in Fig. 9. For $\mu_J = \mu_2 = \mu_3 = \mu$, Fig. 9 (a), we choose the parameter pairs $(J, \mu) = (0, 1), (3, 1), (0, 7)$, belonging to winding numbers $W = 2, 1, 0$, respectively (cf. Fig. 8). For all configurations, we find that the asymptotic behavior of the coherence the one of the clean case, namely a decrease to zero for $W = 0$, a convergence to a constant larger than zero for $W = 1$, and an oscillation for $W = 2$. For different disorder strengths $2\mu_J = 2\mu_2 = \mu_3 = \mu$, we focus on the reentrance phenomenon $W = 1 \to 2$ by computing the coherence for $(J, \mu) = (1.2, 1), (1.2, 7)$. Indeed, we find that for large enough disorder, an oscillating behavior emerges, signaling the change of topological phase. For completeness, we also include $(J, \mu) = (6, 1)$ representing the trivial sector, where a vanishing coherence can be observed at large times.

VI. CONCLUSIONS

Ever since Kitaev’s proposal [23] to braid anyons in order to realize non-trivial quantum gates, the field of topological quantum computation has been an exceptionally active field of research [4–11]. This is mainly due to the promising protection against environmental noise governed by the non-locality of the state manifold used for braiding [31]. Spinless p-wave superconductor wires hosting non-Abelian Majorana fermions bound to topological defects have been of particular interest [32], as the intrinsic particle-hole symmetry of the BdG-Hamiltonian promises a realizable topological protection. Recently, the SSH model has been analyzed in terms of its applicability to quantum computation [33], where it was found that the non-trivial braiding statistics of the topological edge modes can be used to build quantum gates via Y-junctions. However, as for all quantum gates based on symmetry protected topological states, the set of quantum gates is not universal [31]. Nevertheless, studying the braiding statistics for our concrete open disordered models seems like an exciting and promising work for future projects.
L.C.V. acknowledges partial support from the Air Force Research Laboratory award no. FA8750-18-1-0041. The research is based upon work (partially) supported by the Office of the Director of National Intelligence (ODNI), Intelligence Advanced Research Projects Activity (IARPA), via the U.S. Army Research Office contract W911NF-17-C-0050. The views and conclusions contained herein are those of the authors and should not be interpreted as necessarily representing the official policies or endorsements, either expressed or implied, of the ODNI, IARPA, or the U.S. Government. The U.S. Government is authorized to reproduce and distribute reprints for Governmental purposes notwithstanding any copyright annotation thereon.

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Appendix A: Dark states in the dissipative trimer model

We here derive an exact form of the asymptotic coherence dynamics and the topological phase transition in the trimer model by studying the dark states, i.e., by finding all states that obey $H|\psi\rangle = E|\psi\rangle$ with $E \in \mathbb{R}$. For the sake of convenience, the following considerations assume chain lengths $N \mod 3 = 2$, as the system then hosts exact dark states with vanishing imaginary part. For all other system sizes the states are quasi-dark, as they have an imaginary part exponentially small in the system size. Of course, in the thermodynamic limit, these differences vanish, and the dynamics is exactly described by the result below. The ansatz is to look for possible dark states with energies $E = \pm J_1$, i.e., to find the kernel of the matrix

$$
H \equiv \Pi J_1 = \begin{pmatrix}
\mp J_1 & J_1 & J \\
J_1 & \mp J_1 & J_2 \\
J & J_2 & \mp J_1 - i\Gamma \\
0 & 0 & J_3 \\
0 & 0 & J_1 & \mp J_1 - i\Gamma \\
0 & 0 & 0 & J \\
0 & 0 & 0 & J & \ddots \\
0 & 0 & 0 & 0 & \ddots
\end{pmatrix}.
$$

(A1)

For $N \mod 3 = 2$, solutions of (A1) are of the form

$$
v_+ = (1, 1, 0, -\delta_-, \delta_+, 0, (\delta_+)^2, 0, ..., (-\delta_+)^{N-2}, (-\delta_+)^{N-2})^T,
$$

$$
v_- = (1, -1, 0, \delta_-, -\delta_+, 0, \delta_-^2, 0, ..., \delta_-^2, 0, ..., \delta_-^2, 0, ..., \delta_-^2)^T.
$$

(A2)

These solutions are intuitive and analogous to the open SSH model [12], in the sense that they disappear on all dissipative sites. The condition $E = \pm J_1$ signals the equivalence of the first two sites of each unit cell up to a sign factor. Eq. (A2) leads to

$$
\delta_{\pm} = \frac{|J \mp J_2|}{J_3},
$$

(A3)

where the sign of the solution is fixed without loss of generality by assuming $\delta_+$ to be positive. The winding number classification is illustrated in the corresponding vectors, as we find zero, one, or two dark states localized at the outer left qubit for different topological sectors, i.e., $W = \Theta(J_3 > |J - J_2|) + \Theta(J_3 > J + J_2)$. Taking into account the normalization factor of the solutions,

$$
A_{\pm}^{-2} = 2 \sum_{k=0}^{N-2} \delta_{\pm}^{2k} = 2 \frac{1 - \delta_{\pm}^{2N-4}}{1 - \delta_{\pm}^2},
$$

(A4)

the time dependent coherence can be approximated for large times $t \gg 1/\Gamma$,

$$
C(t) = |\langle \langle 1 | e^{-iHt} | 1 \rangle \rangle| \approx |e^{-ij_1t} A_{\pm}^2 + e^{ij_1t} A_{\mp}^2 |
$$

$$
= |A_{\pm}^4 + A_{\pm}^4 + 2A_{\pm}^2 A_{\mp}^2 \cos 2J_1 t|.
$$

(A5)

Appendix B: Analytical Determination of Critical Phase Transition Contours

In [18], the critical phase transition surface was derived for the Hermitian SSH model, using the numerical transfer matrix method and level-spacing statistics analysis. The analytical critical phase transition contour for non-Hermitian models can be calculated in a similar manner. To see this, consider the non-Hermitian SSH model. Here, the dark edge state is exactly at zero energy and only lives on the non-decaying sublattice. We consider the critical phase transition in the thermodynamic limit, such that the results for the linear chain of odd length coincide with the results of even length. Now recall that the edge state of the disordered Hermitian SSH model is also supported entirely by one sublattice or the other. Its zero energy edge state on sublattice A ($\psi_{n,B} = 0$) can be written as

$$
\psi_{n,A} = i^{n-1} \prod_{j=1}^{n} \left| J_{3j} \right| J_{2j} \psi_{1,A},
$$

(B1)

where $J_{1j}$ and $J_{2j}$ are the two perturbed hopping parameters in the $j$th unit cell. The edge states in the two systems share an identical distribution in the clean limit. Consequently, in this case, the non-Hermitian problem follows the same localization length and phase transition as a one-dimensional Hermitian SSH model.
With the exact wave function distribution as in Eq. (B1), the inverse localization length of an edge mode can be obtained by

\[
\Lambda^{-1} = - \lim_{n \to \infty} \frac{1}{n} \log |\psi_{n,A}|
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \left( \ln |J_{1j}| - \ln |J_{2j}| \right)
\]

(B2)

An analytical result can be obtained by taking the ensemble average of the last expression. The limit of the sum turns into an integration for independent and identically distributed disorder,

\[
\Lambda^{-1} = \frac{1}{4} \int_{-1}^{1} d\omega \int_{-1}^{1} d\omega' \left( \ln |J_1 + \mu_1 \omega| - \ln |J_2 + \mu_2 \omega'| \right)
\]

(B3)

where \(J_1\) and \(J_2\) are the unperturbed hopping parameters, \(\mu_1\) and \(\mu_2\) control the strength of disorder in \(J_1\) and \(J_2\) respectively. The random variables \(\omega\) and \(\omega'\) are both drawn from a uniform distribution in the range \([-1, 1]\), leading to a normalization prefactor \(1/4\). The analytic solution to this integral has been obtained in [18],

\[
\Lambda^{-1} = \frac{1}{4\mu_1} [(J_1 + \mu_1) \log |J_1 + \mu_1| - (J_1 - \mu_1) \log |J_1 - \mu_1|]
\]

\[
- \frac{1}{4\mu_2} [(J_2 + \mu_2) \log |J_2 + \mu_2| - (J_2 - \mu_2) \log |J_2 - \mu_2|]
\]

(B4)

For small disorder, \(\mu_1, \mu_2 \ll J_2, J_1\), the localization length Eq. (B3) can be approximated by

\[
\Lambda^{-1} \propto \int_{-1}^{1} d\omega_1 d\omega_2 \ln |J_1 + \omega_1 \mu_1| - \ln |J_2 + \omega_2 \mu_2|
\]

\[
= \int_{-1}^{1} d\omega_1 d\omega_2 \ln |J_1| + \omega_1 \mu_1 \frac{1}{J_1} - \frac{1}{2} \left( \frac{\omega_1 \mu_1}{J_1} \right)^2 - \left[ \ln |J_2| + \frac{\omega_2 \mu_2}{J_2} - \frac{1}{2} \left( \frac{\omega_2 \mu_2}{J_2} \right)^2 \right]
\]

\[
+ O\left( \left( \frac{\mu_1}{J_1} \right)^3 \right) + O\left( \left( \frac{\mu_2}{J_2} \right)^3 \right)
\]

(B5)

Performing the integration up to order \(O\left( \left( \frac{\mu_1}{J_1} \right)^3 \right)\) and \(O\left( \left( \frac{\mu_2}{J_2} \right)^3 \right)\), one finds that the localization length diverges for

\[
|J_1| (\mu_1, \mu_2) = |J_2| \exp \left( \frac{J_2^2 \mu_1^2 - J_1^2 \mu_2^2}{J_1^2 J_2^2} \right)
\]

(B6)

which, up to leading order in the expansion of the exponential function, reduces to

\[
|J_1| (\mu_1, \mu_2) = |J_2| \exp \left( \frac{\mu_1^2 - \mu_2^2}{J_2^2} \right)
\]

(B7)

We thus arrive at the conclusion that the value of \(J_1\) where the non-trivial\(\leftrightarrow\)trivial transition occurs increases (decreases) compared to the clean case for small disorder strengths if \(\mu_2 < \mu_1\) (\(\mu_2 > \mu_1\)). This corresponds to the topology by disorder effect discussed in the main text and can be nicely seen in Fig. 4(b). For \(\mu_1 = \mu_2\), the phase transition always occurs at \(J_1 = J_2\), as observed in Fig. 4(a). Now we continue to generalize the result to the non-Hermitian trimer model. In Sec. A, we have shown that the trimer model of length \(N \mod 3\) can host two dark edge modes with energies \(E = \pm J_1\). These edge modes are supported purely by non-decaying sublattices. The wave functions of the two dark states for disordered three-site model is given by

\[
\psi_{n,d_s \pm} = (-1)^{n-1} \prod_{j=1}^{n} |J_{2j} \pm J_1 \over J_{3j}| \psi_{1,d_s \pm},
\]

(B8)

where \(J_1, J_{2j}\) and \(J_{3j}\) are perturbed hopping parameters in the \(j\)th unit cell. Since there exist two different edge modes, we would expect two disjoint localization lengths,

\[
\Lambda_{d_s \pm}^{-1} = - \lim_{n \to \infty} \frac{1}{n} \log |\psi_{n,d_s \pm}|
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \left( \ln |J_{2j} \pm J_1| - \ln |J_{3j}| \right)
\]

(B9)
Again, we take the ensemble average, and the summation turns into an integration, which gives
\[ \Lambda_{ds}^{-1} = \frac{1}{8} \left| \int_{-1}^{1} d\omega \int_{-1}^{1} d\omega' \int_{-1}^{1} d\omega'' (\ln |(J + \mu \omega) \pm (J_2 + \mu_2 \omega')| - \ln |J_3 + \mu_3 \omega''|) \right|. \] (B11)

Here \( J, J_2 \) and \( J_3 \) are unperturbed hopping parameters. \( \mu, \mu_2 \) and \( \mu_3 \) define the amplitudes of disorder. \( \omega, \omega' \) and \( \omega'' \) are three independent and identically distributed random variables in the range of \([-1, 1]\). After performing the integration explicitly, we arrive at
\begin{align*}
\Lambda_{ds}^{-1} &= 2\mu\mu_2 \left\{ (J \pm J_2 - \mu - \mu_2)^2 \log |(J \pm J_2 - \mu - \mu_2)| - (J \pm J_2 + \mu - \mu_2)^2 \log |J \pm J_2 + \mu - \mu_2| \\
&- (J \pm J_2 - \mu + \mu_2)^2 \log |(J \pm J_2 - \mu + \mu_2)| + (J \pm J_2 + \mu + \mu_2)^2 \log |J \pm J_2 + \mu + \mu_2| \\
&- \frac{1}{\mu_3} \left\{ (J_3 + \mu_3) \log |J_3 + \mu_3| - (J_3 - \mu_3) \log |J_3 - \mu_3| \right\} \right\} - 4. \quad \text{(B12)}
\end{align*}

Eq. (B4) and Eq. (B12) allow us to trace the exact critical phase transition contours in the non-Hermitian SSH dimer and trimer models.

**Appendix C: Diagonal Disorder**

As argued in the main text, diagonal disorder destroys the protective chiral symmetry of the SSH model, making it collapse to a topologically trivial phase. This effect can be nicely seen when considering the eigenspectrum density of states of the restricted Hamiltonian, as already introduced in the main text for symmetry conserving disorder. In analogy to off-diagonal disorder, the coherence time is not infinite anymore, but it is reduced to a finite value \( \tau \). By assuming an exponential decay in time, i.e., \( \mathcal{C}(t) = \mathcal{C}(t_0)e^{-(t-t_0)/\tau} \) for some \( t_0 \gg 1/\Gamma \), we can extract \( \tau \) by integrating over the time evolution of the coherence, i.e.,
\[ I := \int_{t_0}^{t_1} \mathcal{C}(t)dt = \int_{t_0}^{t_1} \mathcal{C}(t_0)e^{-(t-t_0)/\tau}dt = \tau(\mathcal{C}(t_1) - \mathcal{C}(t_0)). \] (C1)

Numerical integration leads to the results depicted in Fig. 10(e), where a sharp drop of the coherence time away from the fully dimerized, clean limit can be observed (notice the logarithmic scaling on the \( z \)-axis). For the trimer model, very similar behavior is being observed, for disorder acting on either on-site potentials or the coupling parameter \( J_1 \), see Fig. 11 for the DOS.
FIG. 11. Density of states for the trimer model. (a)-(c) Clean density of states for $W = 2, 1, 0$, respectively. (d)-(f) Diagonal disorder $\mu = 1$. (g)-(i) Off diagonal disorder on $J_2, J_3, J$ with $\mu = 1$. Here, $N = 21, J_1 = J_2 = 2, J_3 = 3,$ and $J = 0, 3, 6$ for the topological phases $W = 2, 1, 0$, respectively. It is seen how $W$ quasi-dark states with energies $E = \pm J_1$ exist in the clean system, being unstable (stable) for the considered diagonal (off-diagonal) disorder.