On parabolic Whittaker functions

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Abstract

We derive a Mellin-Barnes integral representation for solution to generalized (parabolic) quantum Toda lattice introduced in [GLO], which presumably describes the $S^1 \times U_N$-equivariant Gromov-Witten invariants of Grassmann variety.

Introduction

The $\mathfrak{gl}_N$-Whittaker functions, being solutions to the quantum cohomology D-module $QH^\ast(\text{Fl}_N)$ of the complete flag variety $\text{Fl}_N = GL_N(\mathbb{C})/B$, describe the corresponding equivariant Gromov-Witten invariants of $\text{Fl}_N$ (see [Giv1], [Giv2] and references therein). However, the Givental’s approach to representation theory description of quantum cohomology of homogeneous spaces is inapplicable to generic incomplete flag variety $\text{Fl}_{m_1,\ldots,m_k}$ since no relevant Whittaker model (Toda lattice) associated with an incomplete flag variety was known.

From the other hand, in [HV] it was conjectured a description of quantum cohomology of Grassmannians in terms of (non-Abelian) gauged topological theories, together with a period-type integral representation for the corresponding generating function.

Recently, in [GLO] a generalization of the $\mathfrak{gl}_N$-Whittaker function to the case of the Grassmann variety $\text{Gr}_{m,N} = GL_N(\mathbb{C})/P_m$, $1 \leq m < N$ is proposed. Namely, in [GLO] it is defined a Toda-type D-module and its solution $\Psi^{(m,N)}_{\lambda_1,\ldots,\lambda_N}(x_1,\ldots,x_N)$ (referred to as $\text{Gr}_{m,N}$-Whittaker function), such that after specialization $x_2 = \ldots = x_N = 0$ the symbols of this D-module reproduce the small quantum cohomology algebra $qH^\ast(\text{Gr}_{m,N})$ according to [AS] and [K]. Conjecturally, the constructed generalized Whittaker function describe the equivariant Gromov-Witten invariants of $\text{Gr}_{m,N}$, and in [GLO] this conjecture is verified in particular case of projective space $\mathbb{P}^{N-1} = \text{Gr}_{1,N}$.

In this note we construct Mellin-Barnes type integral representation of the specialized $\text{Gr}_{m,N}$-Whittaker function, following an original generalization of Whittaker models to incomplete flag manifolds from [GLO]; this integral formula has been announced in [GLO]. Our derivation involves a generalization of the Gelfand-Zetlin realization to infinite-dimensional $\mathcal{U}(\mathfrak{gl}_N)$-modules introduced in [GKL]. Our main result (Theorem 2.1) generalizes Theorem 1.1 of [GLO] to arbitrary Grassmannian $\text{Gr}_{m,N}$. Moreover, our integral representation verifies the conjectural integral formula from [HV], although we construct another solution to the D-module with a different asymptotic behavior.

The paper is organized as follows. In Section 1 we review on parabolic Whittaker functions introduced in [GLO], and formulate our main results: the Mellin-Barnes integral representation for the specialized $\text{Gr}_{m,N}$-Whittaker function (Theorem 1.1), and its asymptotic behavior (Theorem 1.2). The second part of the text contains a detailed proof of the main results. In particular, we

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recall the generalized Gelfand-Zetlin realization of the universal enveloping algebra $U(\mathfrak{gl}_N)$ from [GKL], and then we find out the Whittaker vectors (Proposition 2.1). In Section 3 we prove Theorems 1.1 and 1.2.

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1 The $Gr_{m,N}$-Whittaker function and its integral representation

Let $\mathfrak{g}l_N$ be the Lie algebra of $(N \times N)$ real matrices with the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}l_N$ of diagonal matrices, and let $\mathfrak{b}_s \subset \mathfrak{g}l_N$ be a pair of opposed Borel subalgebras containing $\mathfrak{h}$. Then one has the triangular decomposition $\mathfrak{g}l_N = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_- \subset \mathfrak{b}_-$ are the nilpotent radicals given by strictly lower- and upper-triangular matrices. In this way, the set of roots $R \subset \mathfrak{h}^*$ decomposes into $R_+ \sqcup R_-$, where $R_+ \subset R \subset \mathfrak{h}^*$ is the set of positive roots. Identifying $\mathfrak{h} \simeq \mathbb{R}^N$ with coordinates $\underline{\mathfrak{x}} = (x_1, \ldots, x_N)$ one may write $R = \{ \alpha \in \mathfrak{h}^* | \alpha(\underline{\mathfrak{x}}) = x_i - x_j, i \neq j \}$ and $R_+ = \{ \alpha \in \mathfrak{h}^* | \alpha(\underline{\mathfrak{x}}) = x_i - x_j, i < j \}$. Clearly, positive roots span the Borel subalgebra $\mathfrak{b}_+$, and $R_-$ span $\mathfrak{b}_- \subset \mathfrak{g}l_N$. Let $\Delta \subset R_+$ be the set of simple roots $\alpha_i(\underline{\mathfrak{x}}) = x_i - x_{i+1} \in \mathfrak{h}^*$, $1 \leq i \leq N - 1$, and let $\{ \omega_m, 1 \leq m \leq N \}$ be the non-reduced set of fundamental weights given by $\omega_m(\underline{\mathfrak{x}}) = x_1 + \ldots + x_m$. The Weyl group $\mathfrak{S}_N$ is generated by simple reflections $s_i = s_{\alpha_i}$, and acts in $\mathfrak{h}^*$ by linear transformations:

$$s_i(\beta) = \beta - (\alpha_i, \beta)\alpha_i, \quad \beta \in \mathfrak{h}^*.$$ 

In particular one has $\mathfrak{S}_N \cdot R_+ = R_-$. Let $I = \{1, 2, \ldots, N - 1\}$ be the set of vertices of Dynkin diagram, then given a subset $J \subseteq I$, let $\overline{J} = I \setminus J$, and let us consider the subgroup $\mathfrak{G}_J \subset \mathfrak{S}_N$ generated by $\{s_j, j \in \overline{J}\}$. Then let $R_J \subseteq R_+$ be a subset of positive roots defined by $\mathfrak{G}_J \cdot R_J = -R_J$, and let $\overline{R}_J = R_+ \setminus R_J$. Then the corresponding parabolic subalgebra is spanned by $R_-$ and $R_J$, and the corresponding parabolic subgroup is denoted by $P_J$. In this paper we restrict ourselves to the case $J = \{m\} \subset \{1, 2, \ldots, N - 1\}$ with $\mathfrak{G}_m = \mathfrak{S}_m \times \mathfrak{S}_{N-m}$, and $GL_N(\mathbb{C})/P_m$ being isomorphic to the Grassmannian $Gr_{m,N}$. In this case we have $I = I' \sqcup I''$ with $I' = \{1, \ldots, m\}$ and $I'' = \{m+1, \ldots, N\}$, then $\overline{R}_m$ is spanned by positive roots $\alpha$ of the form $\alpha(\underline{\mathfrak{x}}) = x_i - x_j; i \in I', j \in I''$.

Next, let us recall an original construction of $Gr_{m,N}$-Whittaker functions from [GLO]. Let $B = B_- \subset GL_N(\mathbb{C})$ be the Borel subgroup of lower-triangular matrices, and let us pick a character $\chi_\Delta: B_- \to \mathbb{C}$ defined by $\chi_\Delta = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$. Then the associated Whittaker function is defined as a certain matrix element of a principle series representation $\mathcal{V}_\Delta = Ind^B_{B_-} \chi_\Delta$.

Let us associate with $P_m$ a decomposition of the Borel subalgebra $\mathfrak{b}_+ \subset \mathfrak{g}l_N$

$$\mathfrak{b}_+ = \mathfrak{h}(m) \oplus \mathfrak{n}_+^{(m)},$$

into the commutative subalgebra $\mathfrak{h}(m) \subset \mathfrak{b}_+$ spanned by

$$H_1 = E_{11} + \ldots + E_{mm} ; \quad H_k = E_{1,k} , \quad 2 \leq k \leq m ;$$

$$H_{m+k} = E_{m+k,m} + \ldots + E_{m+1,m+1} + \ldots + E_{\ell+m,\ell+m} ; \quad (1.1)$$

and the Lie subalgebra $\mathfrak{n}_+^{(m)} \subset \mathfrak{b}_+$ generated by

$$\mathfrak{n}_+^{(m)} = \langle E_{1,\ell+m}; E_{1,m+1}; E_{m,\ell+m}; E_{kk}, 2 \leq k \leq N-1; E_{j,j+1}, 2 \leq j \leq N-2 \rangle . \quad (1.2)$$
Note that dim $\mathfrak{h}^{(m)} = \text{rank } \mathfrak{g}l_N = N$ and dim $\mathfrak{n}_+^{(m)} = N(N - 1)/2$. Let $H^{(m)}$ and $N_+^{(m)}$ be the Lie groups corresponding to the Lie algebras $\mathfrak{h}^{(m)}$ and $\mathfrak{n}_+^{(m)}$. An open part $GL^0_N$ (the big Bruhat cell) of $GL_N$ allows the following analog of the Gauss decomposition:

$$GL^0_N = N_- H^{(m)} N_+^{(m)}. \tag{1.3}$$

Let $\mathcal{U} = \mathcal{U}(\mathfrak{g}l_N)$ be the universal enveloping algebra of $\mathfrak{g}l_N$. The principal series representation $V_{\lambda}$ admits a natural structure of $\mathcal{U}$-module, as well, as a module over the opposite algebra $\mathcal{U}^{opp}$. Let us assume that the action of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}l_N$ in $V_{\lambda}$ is integrable to the action of the Cartan torus $H \subset GL_N(\mathbb{R})$. Below we introduce a pair of elements, $\langle \psi_L |, | \psi_R \rangle \in V_{\lambda}$ generating a pair of dual submodules, $W_L = \langle \psi_L | \mathcal{U}^{opp} \rangle$ and $W_R = \mathcal{U} | \psi_R \rangle$, in $V_{\lambda}$ (we adopt the bra- and ket-vector notations to distinguish $\mathcal{U}$ and $\mathcal{U}^{opp}$-modules structures on $W_R$ and $W_L$ respectively).

**Definition 1.1 [GLO]** The $Gr_{m,N}$-Whittaker vectors $\langle \psi_L | \in V_{\lambda}'$ and $| \psi_R \rangle \in V_{\lambda}$ are defined by the following conditions:

$$\langle \psi_L | E_{n+1,n} = h^{-1}(\psi_L), \quad 1 \leq n \leq N - 1, \tag{1.4}$$

$$\begin{cases}
E_{kk} | \psi_R \rangle = 0, & 2 \leq k \leq N - 1; \\
E_{k,k+1} | \psi_R \rangle = 0, & 2 \leq k \leq N - 2; \\
E_{1,m+1} | \psi_R \rangle = E_{m,N} | \psi_R \rangle = 0; \\
E_{1,N} | \psi_R \rangle = (-1)^{\epsilon(m,N)} \frac{1}{\hbar} | \psi_R \rangle
\end{cases} \tag{1.5}$$

where $\epsilon(m,N)$ is an integer number and $\hbar \in \mathbb{R}$.

Note that the equations (1.4) define a one-dimensional representation $\langle \psi_L |$ of the Lie algebra $\mathfrak{n}_-$ of strictly lower-triangular matrices, and the equations (1.5) define a one-dimensional representation of $\mathfrak{n}_+^{(m)}$.

**Definition 1.2 [GLO]** The $Gr_{m,N}$-Whittaker function associated with the principal series representation $(\pi_{\lambda}, V_{\lambda})$ is defined as the following matrix element:

$$\Psi_{\lambda}^{(m,N)}(\underline{x}) = e^{-x^t \frac{m(N-m)}{2}} \langle \psi_L | \pi_{\lambda} (g(x_1, \ldots, x_N)) | \psi_R \rangle, \tag{1.6}$$

where the left and right vectors solve the equations (1.4) and (1.5) respectively. Here $g(\underline{x})$ is a Cartan group valued function given by

$$g(\underline{x}) = \exp \left\{- \sum_{i=1}^{N} x_i H_i \right\}, \tag{1.7}$$

where $\underline{x} = (x_1, \ldots, x_N)$ and the generators $H_i, i = 1, \ldots, N$ are defined by (1.1).

In [GLO] (Theorem 1.1) an integral representation of the $Gr_{m,N}$-Whittaker function (1.6) was constructed in the case of projective space $\mathbb{P}^{N-1}$, corresponding to $m = 1$. We generalize this construction to generic Grassmannians $Gr_{m,N}$. 

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Theorem 1.1  The specialized $\text{Gr}_{m,N}$-Whittaker function possesses the following integral representation:

$$
\Psi^{(m,N)}_{\lambda}(x,0,\ldots,0) = \int_{\mathcal{C}} d\gamma e^{-\frac{x}{\hbar} \sum_{i=1}^{m} \gamma_i \prod_{i,j=1}^{N} \Gamma(\gamma_i - \lambda_j | \hbar)} \prod_{i,k=1}^{m} \Gamma(\gamma_i - \gamma_k | \hbar),
$$

with $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$. The integration contour is given by $\mathcal{C} = (i\mathbb{R} + \epsilon)^m$, where $\epsilon > \max_{1 \leq j \leq N} \{ \lambda_j \}$.

Here we use the following normalization of classical Gamma-function:

$$
\Gamma_1(z | \hbar) = \hbar^z \Gamma(z \hbar).
$$

We prove Theorem 1.1 in Section 3.

Clearly, the integral (1.8) converges absolutely due to the Stirling formula:

$$
\Gamma(z + \lambda) = \sqrt{2\pi} z^{z+\lambda-\frac{1}{2}} e^{-z} \left[ 1 + O(z^{-1}) \right], \quad z \to \infty,
$$

when $|\arg(z)| < \pi$.

Integral representation (1.8) coincides with the expected one (5.1) in [GLO]. Besides, a similar integral formula was conjectured in [HV] (see formulas (A.1) and (A.2) in Appendix), but with different integration measure $\tilde{\mu} = \prod_{i<j} (\gamma_i - \gamma_j)$. Our choice of measure (2.3) is provided by the generalized Gelfand-Zetlin realization [GKL], and it is crucial in our representation theory framework. Actually, the two solutions given by integral formulas (1.8) and the one from [HV], have different asymptotic behavior, and below we derive asymptotic of our solution.

Theorem 1.2  When $x \to -\infty$, the specialized $\text{Gr}_{m,N}$-Whittaker function has the following asymptotic behavior:

$$
\Psi^{(m,N)}_{\lambda}(x,0,\ldots,0) \sim m! \sum_{\sigma \in S_N / \mathfrak{S}_m} e^{-x(\sigma \cdot \omega_m(\lambda))} (\sigma \cdot c_m)(\lambda).
$$

with $\omega_m(\lambda) = \lambda_1 + \ldots + \lambda_m$, and

$$
c_m(\lambda) = \prod_{\alpha \in \mathbb{R}_m} \Gamma_1(\alpha(\lambda) | \hbar), \quad (\sigma \cdot c_m)(\lambda) = \prod_{\alpha \in \mathbb{R}_m} \Gamma_1(\sigma \cdot \alpha(\lambda) | \hbar).
$$

A proof of Theorem 1.2 is given in Section 3.1.

2  Construction of $\text{Gr}_{m,N}$-Whittaker vectors

In this Section we construct explicit solutions to (1.4) and (1.5) using the generalized Gelfand-Zetlin realization of principal series $U(g_{1N})$-modules from [GKL]. Namely, let $\gamma_1, \ldots, \gamma_m$ be a triangular array consisting of $N(N-1)/2$ variables $\underline{\gamma}_n = (\gamma_{m1}, \ldots, \gamma_{mn}) \in \mathbb{C}^n, n = 1, \ldots, N$. The following
operators define a representation $\pi$ of $\mathcal{U}$ in the space $\mathcal{M}_N$ of meromorphic functions in $N(N - 1)/2$ variables $(\gamma_1, \ldots, \gamma_{N-1})$:

$$E_{kk} = \frac{1}{\hbar} \left( \sum_{j=1}^{n} \gamma_{n,j} - \sum_{i=1}^{n-1} \gamma_{n-1,i} \right), \quad 1 \leq k \leq N;$$

$$E_{n,n+1} = -\frac{1}{\hbar} \sum_{i=1}^{n} \frac{n^{n+1}}{\prod_{s \neq i}^{n} (\gamma_{n,i} - \gamma_{n,s})} e^{-\hbar \partial_{n,i}}, \quad 1 \leq n \leq N - 1;$$

$$E_{n+1,n} = \frac{1}{\hbar} \sum_{i=1}^{n} \frac{n^{n-1}}{\prod_{s \neq i}^{n} (\gamma_{n,i} - \gamma_{n-1,i} + \frac{\hbar}{2})} e^{\hbar \partial_{n,i}}, \quad 1 \leq n \leq N - 1,$$

where $E_{ij} = \pi(e_{ij})$, $1 \leq i, j \leq N$ for the standard elementary matrix units $e_{ij} \in \mathfrak{gl}_N$. This realization of universal enveloping algebra $\mathcal{U}$ is referred to as generalized Gelfand-Zetlin realization.

**Remark 2.1** Evidently the Weyl group $\mathfrak{S}_N$ acts on $E_{ij}$ in (2.1) by permutations of indices $(i, j)$. This provides $N!$ different realizations of $\mathcal{U}$, and we use certain $\mathfrak{S}_N$-twisted generalized Gelfand-Zetlin realizations of $\mathcal{U}$ in the next Section for derivation of Whittaker vectors (1.4), (1.5).

The universal enveloping algebra $\mathcal{U}$ acts in $\mathcal{W}_R \subseteq \mathcal{M}_N$ by differential operators (2.1), and the opposite algebra $\mathcal{U}^{opp}$ acts in $\mathcal{W}_L \subseteq \mathcal{M}_N$ via the adjoint operators:

$$E_{ij}^\dagger = \mu(\gamma)^{-1} E_{ij} \mu(\gamma), \quad 1 \leq i, j \leq N;$$

with

$$\mu(\gamma) = \prod_{n=2}^{N-1} \mu_n(\gamma_n) = \prod_{n=2}^{N-1} \prod_{i,j=1}^{n} \frac{1}{\Gamma\left(\frac{\gamma_{ni} - \gamma_{nj}}{\hbar}\right)}.$$  

(2.3)

It was shown in [GKL] that there exist a non-degenerate pairing between the modules $\mathcal{W}_L$ and $\mathcal{W}_R$ with measure (2.3):

$$\langle \phi_1, \phi_2 \rangle = \int_{\mathbb{R}^{N(N-1)/2}} \phi_1(\gamma) \phi_2(\gamma) \mu(\gamma) \prod_{k,n=1}^{N} d\gamma_{nk},$$

(2.4)

where $\phi_1 \in \mathcal{W}_L$ and $\phi_2 \in \mathcal{W}_R$.

**Proposition 2.1** For $1 < m < N$ the Whittaker vectors have the following expressions.

1. A solution to (1.4) is given by

$$\psi^{(m)}_L = e^{\pi \gamma_{11}} \prod_{i=1}^{m-1} \prod_{j=1}^{m} \frac{1}{\Gamma\left(\frac{\gamma_{m-1,i} - \gamma_{m,j} + \frac{\hbar}{2}}{\hbar}\right)}.$$  

(2.5)
2. A solution to (1.5) is given by
\[ \psi^{(m)}_R = \delta(\gamma_{11}) \prod_{i=1}^{m} \prod_{j=1}^{N} \Gamma_1(\gamma_{N-1,i} - \gamma_{N,j} + \frac{h}{2} | h) \cdot \prod_{a=1}^{m-1} \prod_{b=1}^{m} \Gamma_1(\gamma_{m-1,a} - \gamma_{mb} + \frac{h}{2} | h) \]
\[ \times \prod_{n=2}^{N-1} \left[ \delta(\sum_{j=1}^{n} \gamma_{nj} - \sum_{i=1}^{n-1} \gamma_{n-1,i}) \prod_{k=1}^{n-1} \delta(\gamma_{n-1,k} - \gamma_{nk} + \frac{h}{2}) \prod_{i,j=1}^{n} \Gamma_1(\gamma_{ij} - \gamma_{nj} | h) \right], \]  
(2.6)

Further, substituting the Whittaker vectors (2.5) and (2.6) into the pairing (2.4) we arrive to the following integral representation.

### 2.1 Proof of Proposition

Due to the action of the Weyl group \( S_N \), actually one has \( N! \) realizations of \( U(\mathfrak{g}_N) \) defined by
\[ E^w_{ij} := wE_{ij}w^{-1}, \quad w \in S_N, \quad i, j = 1, \ldots, N. \]  
(2.7)

Let us call these realizations of \( U(\mathfrak{g}_N) \) the \( w \)-twisted Gelfand-Zetlin realizations.

Given the simple reflections \( s_i \in S_N, i \in I \), let us introduce the Coxeter elements \( c_n = s_n \cdots s_1, n \in I \). In particular, one has \( c_1 = s_1 \), and for the longest element \( w_0 \in S_N \) the following decomposition holds:
\[ w_0 = c_1c_2 \cdots c_{N-1}. \]

**Proof of Proposition 2.1.** Given \( m > 1 \) let us solve the defining relations (1.4), (1.5), using the \( c_{m-1} \)-twisted Gelfand-Zetlin realization (2.7).

**2.1.1** Let us start from solving the equations (1.4) for the left \( Gr_{m,N} \)-Whittaker vector. Namely, one has to check that (2.5) satisfies (1.4):
\[ (E^{c_{m-1}}_{k+1,k}) \psi^{(m)}_L = h^{-1}\psi^{(m)}_L, \quad 1 \leq k < N. \]  
(2.8)

Actually, one has to check only two relations:
\[ -(E^{c_{m-1}}_{21}) \psi^{(m)}_L = (E^{c_{m-1}}_{m+1,m}) \psi^{(m)}_L = h^{-1}\psi^{(m)}_L, \]  
(2.9)

and the other relations are evidently true, since the difference operators \( E^{c_{m-1}}_{k+1,k}, k \neq 1, m \) act trivially on \( \psi^{(m)}_L = \psi^{(m)}_L(\gamma_{m-1}, \gamma_m) \). The relations (2.9) can be verified using the following combinatorial formulas.

**Lemma 2.1** Given a set of variables \( \gamma = (\gamma_1, \ldots, \gamma_n) \) the following identities hold:

1. \[ \sum_{i=1}^{n} \gamma_i^m \prod_{i \neq k}^{n} \frac{1}{\gamma_i - \gamma_k} = \delta_{m,n-1}, \quad m < n; \]  
(2.10)

More generally, one has
\[ \sum_{i=1}^{n} \gamma_i^m \prod_{i \neq k}^{n} \frac{1}{\gamma_i - \gamma_k} = \sum_{k_1+\cdots+k_n=n+1-m} \gamma_1^{k_1} \cdots \gamma_n^{k_n}. \]
2. \[
\sum_{i=1}^{n} \frac{1}{\gamma_i - \gamma_k} = 1, \tag{2.11}
\]
for any constant c.

**Proof.** We have
\[
\sum_{i=1}^{n} \gamma_i^m \prod_{i \neq k} \frac{1}{\gamma_i - \gamma_k} = \int d\lambda \frac{\lambda^m}{A_n(\lambda)}, \quad A_n(\lambda) = \prod_{i=1}^{n} (\lambda - \gamma_i).
\]
Taking the residue at infinity we get
\[
\int d\lambda \frac{\lambda^{m-n}}{1 + \sum_{k=1}^{n} (-1)^k \sigma_k(\gamma) \lambda^{-k}} = \chi_{n+1-m}(\gamma) \Theta(m + 1 - n),
\]
where
\[
\sigma_k(\gamma) = \sum_{i_1 < \cdots < i_k} \gamma_{i_1} \cdots \gamma_{i_k}, \quad \chi_k(\gamma) = \sum_{i_1 + \cdots + i_n = k} \gamma_{i_1} \cdots \gamma_{i_k}
\]
are the characters of finite-dimensional representations \(\mathfrak{h}^k \mathbb{C}^n\) and \(\text{Sym}^k \mathbb{C}^n\) respectively.

The other identity can be proved similarly. \(\square\)

Then for the second relation in (2.9) one readily derives the following:
\[
(E_{m+1,m})_{\uparrow} \psi_L^{(m)} = E_{m+1,m-1} \psi_L^{(m)} = \frac{1}{\hbar} \sum_{i=1}^{m} \prod_{k_1 \neq i_1} \frac{1}{\gamma_{m,i_1} - \gamma_{m,k_1}} \times \prod_{i_2=1}^{m-1} \prod_{k_2 \neq i_2} \frac{(\gamma_{m,i_2} - \gamma_{m-1,k_2} - \hbar)}{\gamma_{m-1,i_2} - \gamma_{m-1,k_2}} \prod_{j=1}^{m-2} \left(\gamma_{m-1,i_2} - \gamma_{m-2,j} - \hbar \right) e^{-\hbar(\delta_{m,i_1} + \delta_{m-1,i_2})} \psi_L^{(m)}
\]
\[
= \frac{1}{\hbar} \sum_{i_1=1}^{m} \prod_{k_1 = 1}^{m} \frac{1}{\gamma_{m,i_1} - \gamma_{m,k_1}} \sum_{i_2=1}^{m-1} \prod_{k_2 \neq i_2} \frac{(\gamma_{m-1,i_2} - \gamma_{m,j_2} - \hbar)}{\gamma_{m-1,i_2} - \gamma_{m-1,k_2}} \prod_{j_2=1}^{m-2} \left(\gamma_{m-1,i_2} - \gamma_{m-2,j} - \hbar \right) \psi_L^{(m)} = \ldots
\]
Using (2.10) we have
\[
\ldots = \frac{1}{\hbar} \sum_{i_1=1}^{m} \prod_{k_1 = 1}^{m} \frac{\gamma_{m,k_1}}{\gamma_{m,i_1} - \gamma_{m,k_1}} \sum_{i_2=1}^{m-1} \prod_{k_2 \neq i_2} \frac{-\gamma_{m-1,i_2}}{\gamma_{m-1,i_2} - \gamma_{m-1,k_2}} \psi_L^{(m)} = \ldots
\]
and finally we apply (2.11) and obtain
\[
\frac{1}{\hbar} \sum_{i_1=1}^{m} \prod_{k_1 = 1}^{m} \frac{-\gamma_{m,k_1}}{\gamma_{m,i_1} - \gamma_{m,k_1}} = \frac{1}{\hbar} \psi_L^{(m)}.
\]
The first relation in (2.9) reads as follows:

\[
(E^{c_{m-1}}_{21})^† \psi^{(m)}_L = E^{†}_{1m} \psi^{(m)}_L = -\frac{1}{\hbar} \sum_{i_1=1}^{m-1} \left( \gamma_{m-1,i_1} - \gamma_{m,j_1} + \frac{\hbar}{2} \right) \sum_{j_2=1}^{m-2} \left( \gamma_{m-2,i_2} - \gamma_{m-1,j_2} + \frac{\hbar}{2} \right) \times \ldots
\]

(2.13)

\[
\times \sum_{i_2=1}^{2} \sum_{k_1 \neq i_1}^{m-1} \sum_{k_2 \neq i_2}^{m-2} \left( \gamma_{i_1,i_2} - \gamma_{j_1,j_2} + \frac{\hbar}{2} \right) \prod_{j_1=1}^{m-1} \prod_{k_1 \neq i_1}^{m-2} \left( \gamma_{i_1,j_1} - \gamma_{j_1,k_1} \right) \prod_{j_2=1}^{m-2} \prod_{k_2 \neq i_2}^{m-2} \left( \gamma_{i_2,j_2} - \gamma_{j_2,k_2} \right)
\]

Thus we have to check the following identity:

\[
\frac{1}{\hbar} \sum_{i_1=1}^{m-1} \sum_{i_2=1}^{m-2} \prod_{j_1=1}^{m-1} \prod_{k_1 \neq i_1}^{m-2} \left( \gamma_{i_1,i_2} - \gamma_{j_1,j_2} + \frac{\hbar}{2} \right) \prod_{j_2=1}^{m-2} \prod_{k_2 \neq i_2}^{m-2} \left( \gamma_{i_2,j_2} - \gamma_{j_2,k_2} \right)
\]

(2.14)

\[
\times \sum_{i_2=1}^{2} \sum_{k_1 \neq i_1}^{m-1} \sum_{k_2 \neq i_2}^{m-2} \left( \gamma_{i_1,i_2} - \gamma_{j_1,j_2} + \frac{\hbar}{2} \right) \prod_{j_1=1}^{m-1} \prod_{k_1 \neq i_1}^{m-2} \left( \gamma_{i_1,j_1} - \gamma_{j_1,k_1} \right) \prod_{j_2=1}^{m-2} \prod_{k_2 \neq i_2}^{m-2} \left( \gamma_{i_2,j_2} - \gamma_{j_2,k_2} \right)
\]

This identity can be verified by induction over \( m \). Indeed, for \( m = 2 \) (2.13) reads

\[
(E^{c_{m}}_{21})^† \psi^{(2)}_L = E^{†}_{12} \psi^{(2)}_L
\]

(2.15)

\[
= -\frac{1}{\hbar} \left( \gamma_{11} - \gamma_{21} + \frac{\hbar}{2} \right) \left( \gamma_{11} - \gamma_{22} + \frac{\hbar}{2} \right) e^{\frac{\hbar}{2} \gamma_{11}} \cdot e^{\textrm{i} \pi \gamma_{11}} \prod_{i=1}^{2} \frac{1}{\hbar} \left( \gamma_{11} - \gamma_{2i} + \frac{\hbar}{2} \right)
\]

\[
= \frac{1}{\hbar} \psi^{(2)}_L.
\]

The inductive step directly follows the reasoning from (2.12), using the combinatorial identities (2.14) and (2.15).

2.1.2 Let us check that the expression (2.6) satisfies the relations the \( c_{m-1} \)-twisted relations (1.5):

\[
E^{c_{m-1}}_{nm} \psi^{(m)}_R = 0, \quad n = 2, \ldots, N - 1; \quad E^{c_{m-1}}_{k,k+1} \psi^{(m)}_R = 0, \quad k = 2, \ldots, N - 2;
\]

\[
E^{c_{m-1}}_{1,m+1} \psi^{(m)}_R = E^{c_{m-1}}_{m,1} \psi^{(m)}_R = 0; \quad E^{c_{m-1}}_{1,N} \psi^{(m)}_R = (-1)^{\epsilon(m,N)} \psi^{(m)}_R.
\]

The relations corresponding to Cartan generators \( E^{c_{m-1}}_{nm} \psi^{(m)}_R = 0, n = 2, \ldots, N - 1 \) hold due to the delta-factors

\[
\delta(\gamma_{11}) \prod_{n=2}^{N-1} \delta \left( \sum_{j=1}^{n} \gamma_{n,j} - \sum_{i=1}^{n-1} \gamma_{n-1,i} \right)
\]

in (2.16), and since

\[
E^{c_{m}}_{kk} = E_{k-1,k-1}, \quad k = 2, \ldots, m; \quad E^{c_{m}}_{kk} = E_{kk}, \quad k = m + 1, \ldots, N - 1.
\]
Similarly, the relations $E^{m-1}_{k,k+1} \psi_R^{(m)} = 0$, $k = 2, \ldots, N - 2$ hold due to the delta-factors

$$\prod_{n=2}^{N-1} \prod_{k=1}^{n-1} \delta \left( \gamma_{n-1,k} - \gamma_{nk} + \frac{\hbar}{2} \right)$$

(2.17)

in (2.6), and since

$$E^{m-1}_{k,k+1} = E_{k-1,k}, \quad k = 2, \ldots, m; \quad E^{m-1}_{k,k+1} = E_{k,k+1}, \quad k = m + 1, \ldots, N - 2.$$

Due to the same delta-factor (2.17), one has $E^{m-1}_{1,m+1} \psi_R^{(m)} = E_{m,m+1} \psi_R^{(m)} = 0$.

Thus we have to check the remaining two relations:

$$E^{m-1}_{m,N} \psi_R^{(m)} = E_{m-1,N} \psi_R^{(m)} = 0, \quad E^{m-1}_{1,N} \psi_R^{(m)} = E_{m,N} \psi_R^{(m)} = \frac{(-1)^m}{\hbar} \psi_R^{(m)}.$$  (2.18)

**Lemma 2.2** For $n = 1, \ldots, N - 1$ the following holds:

$$E_{n,N} = -\frac{1}{\hbar} \sum_{i_1=1}^{N-1} \prod_{j_1=1}^{N-1} \left( \gamma_{N-1,i_1} - \gamma_{N,j_1} - \frac{\hbar}{2} \right) \prod_{k_1=i_1}^{N-1} \left( \gamma_{N-1,i_1,k_1} - \gamma_{N-1,j_1,k_1} - \frac{\hbar}{2} \right) \prod_{i_2=1}^{N-2} \prod_{k_2=i_2}^{N-2} \left( \gamma_{N-2,i_2,j_2} - \gamma_{N-2,k_2} - \frac{\hbar}{2} \right) \times \ldots$$

$$\times \sum_{i_{N-n}=1}^{N+1-n} \prod_{j_{N-n}=1}^{N+1-n} \left( \gamma_{n,i_{N-n}} - \gamma_{n+1,j_{N-n}} - \frac{\hbar}{2} \right) \prod_{k_{N-n}=i_{N-n}}^{N+1-n} \left( \gamma_{n,i_{N-n}} - \gamma_{n,k_{N-n}} - \frac{\hbar}{2} \right) e^{-\hbar \sum_{a=n}^{N-1} \partial_{a,i_{N-a}}}.$$  (2.19)

**Proof.** Direct calculation using (2.1). $\square$

At first let us note that due to the delta-factors

$$\prod_{n=m+1}^{N-1} \prod_{k=1}^{n} \delta \left( \gamma_{n-1,k} - \gamma_{nk} + \frac{\hbar}{2} \right)$$

...
in (2.6) one gets only \( m \) non-vanishing terms in (2.19) (for \( n = m \)):

\[
E_{1,N}^{c_{m-1}} \psi_R^{(m)} = E_{m,N} \psi_R^{(m)}
\]

\[
= -\frac{1}{\hbar} \left\{ \sum_{i_1=1}^{N-1} \prod_{j_1=1}^{N} \left( \gamma_{N-1,i_1} - \gamma_{N,j_1} - \frac{\hbar}{2} \right) \prod_{k_1 \neq i_1}^{N-1} \left( \gamma_{N-1,i_1} - \gamma_{N-1,k_1} \right) \prod_{i_2=1}^{N-2} \prod_{j_2 \neq i_1}^{N-1} \left( \gamma_{N-2,i_1} - \gamma_{N-1,j_2} - \frac{\hbar}{2} \right) \prod_{j_2 \neq i_1}^{N-1} \left( \gamma_{N-2,i_1} - \gamma_{N-2,k_2} \right) \right\} \times \ldots
\]

\[
\times \sum_{i_{N-m}=1}^{m} \prod_{j_{N-m}=1}^{N-m} \prod_{k_{N-m} \neq i_{N-m}}^{N-m} \left( \gamma_{m,i_{N-m}} - \gamma_{m+1,j_{N-m}} - \frac{\hbar}{2} \right) \left( \gamma_{m,i_{N-m}} - \gamma_{m,i_{N-m}} - \frac{\hbar}{2} \right) \ldots \times \prod_{k_{N-m} \neq i_{N-m}}^{m} \prod_{i_{N-m}}^{N} \left( \gamma_{m,i_{N-m}} - \gamma_{m,k_{N-m}} \right) \right\} \psi_R^{(m)}
\]

(2.20)

Secondly, taking into account that

\[
e^{-\hbar \partial_{N-1,i}} \cdot m \prod_{a=1}^{N} \prod_{b=1}^{N} \Gamma_{1} \left( \gamma_{N-1,a} - \gamma_{N,b} + \frac{\hbar}{2} \right)
\]

\[
= \prod_{j=1}^{N} \frac{1}{\gamma_{N-1,i} - \gamma_{N,j}} \frac{1}{\frac{\hbar}{2}} \prod_{a=1}^{N} \prod_{b=1}^{N} \Gamma_{1} \left( \gamma_{N-1,a} - \gamma_{N,b} + \frac{\hbar}{2} \right) \cdot e^{-\hbar \partial_{N-1,i}}
\]

(2.21)

and due to the factors \( m \prod_{a=1}^{N} \prod_{b=1}^{N} \Gamma_{1} \left( \gamma_{N-1,i} - \gamma_{N,j} \right) \) in (2.6) one has

\[
\ldots = -\frac{1}{\hbar} \left\{ \sum_{i_1=1}^{N-1} \prod_{j_1=1}^{N} \left( \gamma_{N-1,i_1} - \gamma_{N-1,j_1} - \frac{\hbar}{2} \right) \prod_{k_1 \neq i_1}^{N-1} \left( \gamma_{N-1,i_1} - \gamma_{N-1,k_1} \right) \prod_{i_2=1}^{N-2} \prod_{j_2 \neq i_1}^{N-1} \left( \gamma_{N-2,i_1} - \gamma_{N-1,j_2} - \frac{\hbar}{2} \right) \prod_{j_2 \neq i_1}^{N-1} \left( \gamma_{N-2,i_1} - \gamma_{N-2,k_2} \right) \right\} \times \ldots
\]

\[
\times \sum_{i_{N-m}=1}^{m} \prod_{j_{N-m}=1}^{N-m} \prod_{k_{N-m} \neq i_{N-m}}^{N-m} \left( \gamma_{m,i_{N-m}} - \gamma_{m+2,j_{N-m}} - \frac{\hbar}{2} \right) \prod_{j_{N-m} = 1}^{m+2} \prod_{j_{N-m} \neq i_{N-m}}^{N-m} \left( \gamma_{m,i_{N-m}} - \gamma_{m+1,j_{N-m}} - \frac{\hbar}{2} \right) \ldots \times \prod_{k_{N-m} \neq i_{N-m}}^{m} \prod_{i_{N-m}}^{N} \left( \gamma_{m,i_{N-m}} - \gamma_{m,k_{N-m}} \right) \right\} \psi_R^{(m)}
\]

(2.22)
Next, since
\[
\prod_{j_{a+1} = 1, \quad i_{a+1} \neq a+1}^{N-a} \left( \gamma_{N-a-1, i_{a+1}} - \gamma_{N-a, j_{a+1}} - \frac{\hbar}{2} \right) \equiv \prod_{k_a \neq i_a} (\gamma_{N-a, i_a} - \gamma_{N-a, k_a} - \hbar) \quad \text{mod} \prod_{n=m+1}^{N-1} \prod_{k=1}^{n-1} \delta \left( \gamma_{n-1, k} - \gamma_{nk} - \frac{\hbar}{2} \right)
\]
and since the factor \( \prod_{a=1}^{m-1} \prod_{b=1}^{m} \Gamma_1 \left( \gamma_{m-1, a} - \gamma_{m, b} + \frac{\hbar}{2} \right) \) in (2.6) produces
\[
eq \prod_{r=1}^{m-1} \prod_{a=1}^{m-1} \prod_{b=1}^{m} \Gamma_1 \left( \gamma_{m-1, r} - \gamma_{m, i} - \frac{\hbar}{2} \right) \prod_{a=1}^{m-1} \prod_{b=1}^{m} \Gamma_1 \left( \gamma_{m-1, a} - \gamma_{m, b} + \frac{\hbar}{2} \right) \cdot e^{-\hbar \partial_{m, i}},
\]
one arrives to the following:
\[
\ldots = \frac{1}{\hbar} \sum_{i_1 = 1}^{m} \prod_{k_1 = 1}^{N-1} \left( \gamma_{N-1, i_1} - \gamma_{N-1, k_1} - \hbar \right) \psi^{(m)}_R = \frac{(-1)^m}{\hbar} \psi^{(m)}_R,
\]
where the last equality follows from (2.10).

At last we have to verify the remaining first relation in (2.18). Following the same reasoning as in (2.20)-(2.22) above, we obtain the following:
\[
E_{m, N}^{c_{m-1, \psi}}(m) = E_{m, N}^{c_{m-1, \psi}}(m)
\]
\[
= \frac{1}{\hbar} \left\{ \sum_{i_1 = 1}^{m} \prod_{k_1 = 1}^{N-1} \left( \gamma_{N-1, i_1} - \gamma_{N-1, k_1} - \hbar \right) \prod_{j_2 \neq i_1}^{N-1} \left( \gamma_{N-2, i_1} - \gamma_{N-2, j_2} - \frac{\hbar}{2} \right) \times \ldots \right. \]
\[
\left. \prod_{j_{m-1} = 1}^{m+2} \left( \gamma_{m+1, i_1} - \gamma_{m+2, j_{m-1}} - \frac{\hbar}{2} \right) \prod_{j_{m-1} = 1}^{m+1} \left( \gamma_{m, i_1} - \gamma_{m+1, j_{m-1}} - \frac{\hbar}{2} \right) \times \ldots \times \prod_{k_{m-1} = 1}^{m} \left( \gamma_{m+1, i_1} - \gamma_{m+1, k_{m-1}} - \hbar \right) \prod_{k_{m-1} = 1}^{m} \left( \gamma_{m, i_1} - \gamma_{m, k_{m-1}} \right) \right. \]
\[
\left. \times \prod_{j_{m-1} = 1}^{m+1} \left( \gamma_{m+1, i_1} - \gamma_{m+1, j_{m-1}} - \frac{\hbar}{2} \right) \prod_{k_{m-1} = 1}^{m} \left( \gamma_{m+1, i_1} - \gamma_{m+1, k_{m-1}} - \hbar \right) \right. \]
\[
\left. \times \prod_{j_{m-1} = 1}^{m+1} \left( \gamma_{m+1, i_1} - \gamma_{m+1, j_{m-1}} - \frac{\hbar}{2} \right) \prod_{k_{m-1} = 1}^{m} \left( \gamma_{m+1, i_1} - \gamma_{m+1, k_{m-1}} - \hbar \right) \left. \right\} \cdot \psi_R^{(m)} = \ldots
\]
Then we make cancelations due to the factors \( \prod_{n=m+1}^{N-1} \prod_{i \neq j}^{m} \Gamma_1 (\gamma_{ni} - \gamma_{nj} | \hbar) \) in (2.6) and relation (2.23),
take into account the factor \( \prod_{a=1}^{m-1} \prod_{b=1}^{m} \Gamma_1 (\gamma_{m-1,a} - \gamma_{mb} + \frac{\hbar}{2} | \hbar) \) in (2.6) satisfying the relation:

\[
e^{-b \left( \partial_{m-1,j} + \partial_{m,i} \right)} \cdot \prod_{a=1}^{m-1} \prod_{b=1}^{m} \Gamma_1 (\gamma_{m-1,a} - \gamma_{m,b} + \frac{\hbar}{2} | \hbar) \cdot e^{-h \left( \partial_{m-1,j} + \partial_{m,i} \right)},
\]

and then arrive to the following:

\[
\ldots = -\frac{1}{\hbar} \sum_{i_1=1}^{m} \prod_{k_{N-m} = 1}^{m} \frac{1}{\gamma_{m,i_1} - \gamma_{m,k_{N-m}}}
\]

\[
\times \sum_{i_{N+1-m} = 1}^{m-1} \prod_{j \neq i_{N+1-m}}^{m-1} \left( \gamma_{m-1,j} - \gamma_{m,i_1} - \frac{\hbar}{2} \right) \left( \gamma_{m-1,i_{N+1-m}} - \gamma_{m-1,k_{N+1-m}} - \hbar \right) e^{-h \sum_{a=m}^{N-1} \partial_{a,i_1} - \partial_{m-1,i_{N+1-m}} \cdot \psi^{(m)}} \psi_R
\]

\[
= -\frac{1}{\hbar} \sum_{i_{N+1-m} = 1}^{m-1} \prod_{k_{N+1-m} = 1}^{m} \frac{1}{\gamma_{m-1,i_{N+1-m}} - \gamma_{m-1,k_{N+1-m}} - \hbar}
\]

\[
\times \sum_{i_1=1}^{m} \prod_{k_{N-m} = 1}^{m} \frac{1}{\gamma_{m,i_1} - \gamma_{m,k_{N-m}}} \left( \gamma_{m-1,j} - \gamma_{m,i_1} - \frac{\hbar}{2} \right) e^{-h \sum_{a=m}^{N-1} \partial_{a,i_1} - \partial_{m-1,i_{N+1-m}} \cdot \psi^{(m)}} = 0,
\]

where the sum above vanishes due to (2.10). This completes the proof of Proposition.
3 Mellin-Barnes integral and its asymptotic

Now we are ready to construct the integral representation for the Gr\(_{m,N}\)-Whittaker function (1.6). Let us substitute (2.5) and (2.6) into (1.6) and then obtain:

\[
\Psi^{(m,N)}(x) = e^{-x\frac{m(N-m)}{2}} \int S \prod_{n=1}^{N-1} d\gamma_n \prod_{i,j=1}^{m} \prod_{i \neq j}^{N} \frac{1}{\Gamma(\gamma_{mj}-\gamma_{nj}|h)} e^{-\frac{m}{\pi} \sum_{k=1}^{m} \gamma_{mk}} \\
\times \delta(\gamma_{11}) \prod_{i=1}^{m} \Gamma\left(\gamma_{N-1,i} - \gamma_{Nj} + \frac{\hbar}{2}|h\right) \cdot \prod_{a=1}^{m-1} \prod_{b=1}^{m} \Gamma\left(\gamma_{m-1,a} - \gamma_{mb} + \frac{\hbar}{2}|h\right) \\
\times \prod_{n=2}^{N-1} \left[ \delta\left(\sum_{j=1}^{n} \gamma_{nj} - \sum_{i=1}^{n-1} \gamma_{n-1,i}\right) \prod_{k=1}^{n-1} \delta(\gamma_{n-1,k} - \gamma_{nk} + \frac{\hbar}{2}) \cdot \prod_{i,j=1}^{n} \Gamma\left(\gamma_{mi} - \gamma_{nj}|h\right) \right] \\
\times e^{i\pi \gamma_{11}} \prod_{n=2}^{N} \prod_{i \neq j} \Gamma\left(\gamma_{m-1,i} - \gamma_{mj} + \frac{\hbar}{2}|h\right) \\
= e^{-x\frac{m(N-m)}{2}} \int S \prod_{n=1}^{N-1} d\gamma_n \prod_{k=m+1}^{N} d\gamma_{N-1,k} e^{-\frac{m}{\pi} \sum_{k=1}^{m} \gamma_{mk}} \prod_{i,j=1}^{m} \Gamma\left(\gamma_{N-1,i} - \gamma_{Nj} + \frac{\hbar}{2}|h\right) \\
\times \delta(\gamma_{11}) e^{i\pi \gamma_{11}} \prod_{n=2}^{N-1} \left[ \delta\left(\sum_{j=1}^{n} \gamma_{nj} - \sum_{i=1}^{n-1} \gamma_{n-1,i}\right) \prod_{k=1}^{n-1} \delta(\gamma_{n-1,k} - \gamma_{nk} + \frac{\hbar}{2}) \right] = \ldots
\]

Making integration over \(\prod_{n=1}^{N-2} d\gamma_n \prod_{k=m+1}^{N} d\gamma_{N-1,k}\) we integrate out the delta-functions and arrive to

\[
\ldots = e^{-x\frac{m(N-m)}{2}} \int S \prod_{n=1}^{N-1} d\gamma_n e^{-\frac{m}{\pi} \sum_{k=1}^{m} \left(\gamma_{N-1,k} - \frac{N-1-m}{N} \gamma_{Nj}\right)} \prod_{i,j=1}^{m} \Gamma\left(\gamma_{N-1,i} - \gamma_{Nj} + \frac{\hbar}{2}|h\right) / \prod_{i,j=1}^{m} \Gamma\left(\gamma_{mi} - \gamma_{nj}|h\right). \hspace{1cm} (3.2)
\]

Finally, we shift the integration contour by \(\gamma_k = \gamma_{N-1,k} + \frac{\hbar}{2}, k = 1, \ldots, m\), and readily get (1.8).

\[\square\]

3.1 Asymptotic of \(\Psi^{(m,N)}(x,0,\ldots,0)\)

To complete the analysis of integral representation (1.8) let us derive its asymptotic when \(x \to -\infty\).

Proof of Theorem 2.2. The contour of integration \(\mathcal{C} = \mathcal{C}_m\) is a product of \(m\) copies of contour \(\mathcal{C}_1\), corresponding to integration over \(\gamma_k, k = 1, \ldots, m\), going from \(\epsilon - i\infty\) to \(\epsilon + i\infty\) with \(\epsilon > \max\{\lambda_1, \ldots, \lambda_N\}\). Let us enclose \(\mathcal{C}_1\) by a half-circle of infinitely large radius in the left half-plane, then the closed contour \(\mathcal{C}_1\) embraces all the poles of gamma-factors

\[
\prod_{j=1}^{N} \Gamma\left(\gamma_k - \lambda_j|\hbar\right)
\]
in (1.8). Actually, we can replace the integration over $C_1$ by integration over $C_1$, since the contribution over the half-circle of infinitely large radius vanishes due to the exponent $e^{-\hbar^{-1}x\gamma_k}$ in the integrand of (1.8). Then this transformation of integration contour allows to calculate the integral as the sum over the residues at poles of Gamma-factors. Namely, each Gamma-function $\Gamma_1(\gamma_k - \lambda_j | h)$ has the poles at $\gamma_k = \lambda_j - n\hbar$, $n = 0, 1, 2, \ldots$, and therefore integration over $\gamma_k$ implies the following:

$$e^{-x\gamma_k} \prod_{j=1}^{N} \Gamma_1(\gamma_k - \lambda_j | h) = \sum_{i=1}^{N} \sum_{n=0}^{\infty} n! \ e^{-x(\lambda_j - n\hbar)} \prod_{j=1}^{N} \Gamma_1(\lambda_i - \lambda_k - n\hbar | h).$$

When $x \to -\infty$ only the terms with $n = 0$ give contributions into the asymptotic:

$$e^{-x\gamma_k} \prod_{j=1}^{N} \Gamma_1(\gamma_k - \lambda_j | h) \sim \sum_{i=1}^{N} \prod_{j=1}^{N} \Gamma_1(\lambda_i - \lambda_j | h), \quad x \to -\infty.$$

At the next step we obtain:

$$e^{-x(\gamma_1 + \gamma_2)} \prod_{j=1}^{N} \Gamma_1(\gamma_1 - \lambda_j | h) \Gamma_1(\gamma_2 - \lambda_j | h)$$

$$\sim \sum_{i_1=1}^{N} \prod_{j=1}^{N} \Gamma_1(\lambda_{i_1} - \lambda_j | h) \sum_{i_2=1}^{N} \prod_{j=1}^{N} \Gamma_1(\lambda_{i_2} - \lambda_j | h) \prod_{j \neq i_1}^{N} \Gamma_1(\lambda_{i_1} - \lambda_{i_2} | h) \prod_{j \neq i_1}^{N} \Gamma_1(\lambda_{i_2} - \lambda_i | h)$$

$$= \sum_{i_1=1}^{N} \prod_{j=1}^{N} \Gamma_1(\lambda_{i_1} - \lambda_j | h) \sum_{i_2=1}^{N} \prod_{j \neq i_1}^{N} \Gamma_1(\lambda_{i_2} - \lambda_j | h) \prod_{j \neq i_1}^{N} \Gamma_1(\lambda_{i_1} - \lambda_{i_2} | h).$$

In this way we proceed step by step over $k$, making cancelations of Gamma-factors in measure $\mu_m(\gamma)$, and finally we arrive to $N!$ terms (with multiplicities $m!$) that can be arranged into the $S_N/\mathfrak{W}_m$-orbit of the term

$$m! e^{-x(\gamma_1 + \ldots + \gamma_m)} \prod_{i=1}^{m} \prod_{k=1}^{N-m} \Gamma_1(\lambda_i - \lambda_{m+k} | h).$$

Thus we obtain (1.9). □

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