A REMARK ON GROMOV’S CYCLE CONDITIONS

TETSU TOYODA

Abstract. We prove that if a metric space $X$ satisfies Gromov’s Cycl$_4(0)$ condition, or equivalently, if $X$ satisfies the $\Box$-inequalities, then $X$ satisfies Gromov’s Cycl$_k(0)$ condition for every integer $k \geq 4$.

1. Introduction

In [6], Gromov introduced several necessary conditions for metric spaces to admit isometric embeddings into CAT(0) spaces, and studied the relations among them. We first recall the definition of the Cycl$_k(0)$ condition. In this paper, for a positive integer $k$ and an integer $m$, we denote the element of $\mathbb{Z}/k\mathbb{Z}$ represented by $m$ by $[m]_k$.

Definition 1.1 (Gromov [6]). Fix an integer $k \geq 4$. A metric space $(X,d_X)$ is said to satisfy the Cycl$_k(0)$ condition if for any map $f : \mathbb{Z}/k\mathbb{Z} \to X$, there exists a map $g : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2$ such that

$$
\|g(i) - g(i + [1]_k)\| \leq d_X(f(i), f(i + [1]_k)), \quad \|g(i) - g(j)\| \geq d_X(f(i), f(j))
$$

for any $i, j \in \mathbb{Z}/k\mathbb{Z}$ with $j \neq i + [1]_k$ and $i \neq j + [1]_k$.

In [6, §7], Gromov defined the Cycl$_k(\kappa)$ condition for any $\kappa \in (-\infty, 0]$ by replacing $\mathbb{R}^2$ in Definition 1.1 with the 2-dimensional Riemannian model space of sectional curvature $\kappa$. The Cycl$_4(0)$ condition is known to be equivalent to the so-called 4-point condition. Namely, for $k = 4$, the first inequality in Definition 1.1 can be replaced by the equality

$$
\|g(i) - g(i + [1]_4)\| = d_X(f(i), f(i + [1]_4))
$$

(see [6, §7] and [7, Remark 2.7]). It was proved by Bridson and Haefliger [2, Proposition 1.11] that a geodesic space is a CAT(0) space if and only if it satisfies the 4-point condition. Therefore, a geodesic space is a CAT(0) space if and only if it satisfies the Cycl$_4(0)$ condition.

Next, we recall the definition of the Wir$_k$ inequalities.

Definition 1.2 (Gromov [6]). Fix an integer $k \geq 4$. We say that a metric space $(X,d_X)$ satisfies the Wir$_k$ inequalities if any map $f : \mathbb{Z}/k\mathbb{Z} \to X$ satisfies

$$
0 \leq \sin^2 \frac{j\pi}{k} \sum_{i \in \mathbb{Z}/k\mathbb{Z}} d_X(f(i), f(i + [1]_k))^2 - \sin^2 \frac{\pi}{k} \sum_{i \in \mathbb{Z}/k\mathbb{Z}} d_X(f(i), f(i + [j]_k))^2
$$

for every $j \in \mathbb{Z} \cap [2, k - 2]$.

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For each integer $2 \leq m \leq k$, the inequality (1.1) for $j = m$ coincides with that for $j = k - m$. Therefore, the family of Wir$_k$ inequalities consists of $k_0 - 1$ distinct inequalities, where $k_0$ is the greatest integer at most $k/2$. In particular, the family of Wir$_4$ inequalities consists of the single inequality (1.2)

\[ 0 \leq d_X(f([0]_4), f([1]_4))^2 + d_X(f([1]_4), f([2]_4))^2 + d_X(f([2]_4), f([3]_4))^2 + d_X(f([3]_4), f([0]_4))^2 - d_X(f([0]_4), f([2]_4))^2 - d_X(f([1]_4), f([3]_4))^2. \]

This is the inequality called the quadrilateral inequality by Berg and Nikolaev [1], and the roundness 2 inequality by Enflo [4] in connection with the geometry of Banach spaces. It is well-known that the inequality (1.2) holds true whenever the target space $X$ is a CAT(0) space. Furthermore, Berg and Nikolaev [1] proved that a geodesic space $X$ is a CAT(0) space if and only if every map $f : \mathbb{Z}/4\mathbb{Z} \to X$ satisfies the inequality (1.2) (see also Sato [9]). Thus a geodesic space is a CAT(0) space if and only if it satisfies the Wir$_4$ inequalities.

In [6], Gromov proved that for any integer $k \geq 4$ and any metric space $X$, the following implications hold true:

- $X$ is a CAT(0) space $\implies X$ satisfies the Cycl$_k(0)$ condition
- $X$ satisfies the Cycl$_k(0)$ condition $\implies X$ satisfies the Wir$_k$ inequalities.

Combining these implications with the fact that a geodesic space is a CAT(0) space if and only if it satisfies the Wir$_4$ inequalities, we see that if a metric space $X$ is geodesic and satisfies theWir$_4$ inequalities, then $X$ satisfies the Cycl$_4(0)$ condition and the Wir$_k$ inequalities for every integer $k \geq 4$. Because the Cycl$_4(0)$ condition implies the validity of the Wir$_4$ inequalities, we also see that if a metric space $X$ is geodesic and satisfies the Cycl$_4(0)$ condition, then $X$ satisfies the Cycl$_k(0)$ condition and the Wir$_k$ inequalities for every integer $k \geq 4$.

On the other hand, in the study of metric spaces that admit isometric embeddings into CAT(0) spaces, it is worth studying the relations among these conditions without the geodesicness assumption. Omitting the geodesicness assumption seems to change the situation drastically. Kondo, the present author and Uehara [7, Proposition 3.3] presented an example of a five-point metric space that satisfies the Wir$_4$ inequalities but does not satisfy the Wir$_5$ inequalities. The present author [11] presented an example of a four-point metric space that satisfies the Wir$_4$ inequalities but does not satisfy the Cycl$_4(0)$ condition. Thus without the geodesicness assumption, the validity of Wir$_4$ inequalities implies neither the validity of Wir$_5$ inequalities nor the Cycl$_4(0)$ condition.

In [6, §15, Remarks.(b)], Gromov remarked that for a general constant $\kappa \in (-\infty, 0]$, the Cycl$_4(\kappa)$ condition does not imply the Cycl$_k(\kappa)$ conditions for all integers $k \geq 4$ without the geodesicness assumption. For $\kappa = 0$, instead of asking whether the Cycl$_4(0)$ condition implies the Cycl$_k(0)$ condition for every integer $k \geq 4$ without the geodesicness assumption or not, he posed in [6, §25] the question whether the Cycl$_4(0)$ condition implies the validity of the Wir$_4$ inequalities for every integer $k \geq 4$ without the geodesicness assumption or not. Kondo, the present author and Uehara [7] answered his question affirmatively.

**Theorem 1.3 ([7]).** If a metric space $X$ satisfies the Cycl$_4(0)$ condition, then $X$ satisfies the Wir$_k$ inequalities for every integer $k \geq 4$. 
In this paper, we prove that the $\text{Cycl}_4(0)$ condition actually implies the $\text{Cycl}_k(0)$ condition for every integer $k \geq 4$ without the geodesicness assumption.

**Theorem 1.4.** If a metric space $X$ satisfies the $\text{Cycl}_4(0)$ condition, then $X$ satisfies the $\text{Cycl}_k(0)$ condition for every integer $k \geq 4$.

Recently, the present author [11] studied further to what extent the $\text{Cycl}_4(0)$ condition implies the necessary conditions for metric spaces to admit isometric embeddings into $\text{CAT}(0)$ spaces, and proved that a metric space containing at most five points admits an isometric embedding into a $\text{CAT}(0)$ space if and only if it satisfies the $\text{Cycl}_4(0)$ condition, which strengthened the same equivalence result for a metric space containing at most four points remarked by Gromov [6].

Recently, Eskenazis, Mendel and Naor [5] proved that there exists a metric space that does not admit a coarse embedding into any $\text{CAT}(0)$ space. On the other hand, Kondo, the present author and Uehara [7, Proposition 3.1] proved that for any $0 < \alpha \leq 1/2$ and any metric space $(X,d)$, the metric space $(X,d^\alpha)$ satisfies the $\text{Cycl}_4(0)$ condition. Therefore, if we choose a constant $0 < \alpha \leq 1/2$ and a metric space $(X,d)$ that does not admit a coarse embedding into any $\text{CAT}(0)$ space, then the metric space $(X,d^\alpha)$ satisfies the $\text{Cycl}_4(0)$ condition, but does not embed coarsely into any $\text{CAT}(0)$ space. Thus $\text{Cycl}_4(0)$ condition does not imply the isometric embeddability into a $\text{CAT}(0)$ for a general metric space.

In [6, §7], Gromov remarked that the $\text{Cycl}_4(0)$ condition is equivalent to the validity of a certain family of inequalities defined as follows.

**Definition 1.5.** We say that a metric space $(X,d_X)$ satisfies the $\boxplus$-inequalities if for any $t,s \in [0,1]$ and any $x,y,z,w \in X$, we have
\[
0 \leq (1-t)(1-s)d_X(x,y)^2 + t(1-s)d_X(y,z)^2 + tsd_X(z,w)^2 + (1-t)sd_X(w,x)^2 - t(1-t)d_X(x,z)^2 - s(1-s)d_X(y,w)^2.
\]

Gromov [6] and Sturm [10] proved that every $\text{CAT}(0)$ space satisfies the $\boxplus$-inequalities. The name “$\boxplus$-inequalities” is based on a notation used by Gromov [6], and was used in [7] and [11]. Sturm [10] called these inequalities the weighted quadruple inequalities. When $s = t = \frac{1}{2}$, the $\boxplus$-inequality becomes the quadrilateral inequality [1.2]. Gromov [6] established the following fact.

**Theorem 1.6** (Gromov [6]). A metric space satisfies the $\text{Cycl}_4(0)$ condition if and only if it satisfies the $\boxplus$-inequalities.

For a detailed proof of Theorem 1.6 see Section 4 of this paper (see also [7, Lemma 2.6]). In Section 4 we prove the following proposition, which implies Theorem 1.4 by Theorem 1.6.

**Proposition 1.7.** If a metric space $X$ satisfies the $\boxplus$-inequalities, then $X$ satisfies the $\text{Cycl}_k(0)$ condition for all integers $k \geq 4$.

The paper is organized as follows. In Section 2 we recall some definitions and results from metric geometry. In Section 3 we recall some properties of metric spaces that satisfy the $\boxplus$-inequalities, which will be used to prove Proposition 1.7. In Section 4 we prove Proposition 1.7, which implies Theorem 1.4 by Theorem 1.6.

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2. Preliminaries

In this section, we set up some notations, and review some definitions and results in metric geometry. Throughout this paper, for every positive integer \( n \), \( \mathbb{R}^n \) is always equipped with the Euclidean metric. For any distinct points \( x, y \in \mathbb{R}^n \), we denote by \( \overline{xy} \) the straight line through \( x \) and \( y \). For any \( x, y, z \in \mathbb{R}^2 \) with \( x \neq y \) and \( y \neq z \), we denote by \( \angle xyz \in [0, \pi] \) the interior angle measure at \( y \) of the (possibly degenerate) triangle with vertices \( x, y \) and \( z \).

A geodesic in a metric space \( X \) is an isometric embedding of an interval of the real line into \( X \). For any \( x, y \in X \), we call the image of a geodesic \( \gamma : [0, d_X(x, y)] \to X \) with \( \gamma(0) = x \) and \( \gamma(d_X(x, y)) = y \) a geodesic segment with endpoints \( x \) and \( y \). A metric space \( X \) is called geodesic if for any \( x, y \in X \), there exists a geodesic segment with endpoints \( x \) and \( y \).

**Definition 2.1.** A metric space \((X, d_X)\) is called a CAT(0) space if \( X \) is geodesic, and any \( x, y, z \in X \) and any geodesic \( \gamma : [0, d_X(x, y)] \to X \) with \( \gamma(0) = x \) and \( \gamma(d_X(x, y)) = y \) satisfy that

\[
d_X(z, \gamma(td_X(x, y)))^2 \leq (1 - t)d_X(x, z)^2 + td_X(y, z)^2 - t(1 - t)d_X(x, y)^2
\]

for any \( t \in [0, 1] \).

On \( \mathbb{R}^n \), the inequality (2.1) always holds with equality. A subset \( S \) of a geodesic space \( X \) is called convex if any geodesic segment in \( X \) with endpoints \( x \) and \( y \) is contained in \( S \) whenever \( x, y \in S \). Clearly, a convex subset of a CAT(0) space equipped with the induced metric is a CAT(0) space. A geodesic space \( X \) is called uniquely geodesic if for any \( x, y \in X \), a geodesic segment in \( X \) with endpoints \( x \) and \( y \) is unique. It is easily observed that every CAT(0) space is uniquely geodesic. For any points \( x \) and \( y \) in a uniquely geodesic space, we denote the geodesic segment with endpoints \( x \) and \( y \) by \([x, y]\). We also denote the sets \([x, y] \setminus \{x, y\} \), \([x, y] \setminus \{x\}\) and \([x, y] \setminus \{y\}\) by \((x, y)\), \((x, y] \) and \([x, y)\), respectively. For a subset \( S \) of a uniquely geodesic space \( X \), the convex hull of \( S \) is the intersection of all convex subsets of \( X \) containing \( S \), or equivalently, the minimal convex subset of \( X \) containing \( S \). We denote the convex hull of \( S \) by \( \text{conv}(S) \).

Suppose that \((X_1, d_1)\) and \((X_2, d_2)\) are metric spaces, and that \( Z_1 \) and \( Z_2 \) are closed subsets of \( X_1 \) and \( X_2 \), respectively. Suppose further that \( Z_1 \) and \( Z_2 \) are isometric via an isometry \( f : Z_1 \to Z_2 \). We denote by \( X_1 \sqcup X_2 \) the disjoint union of \( X_1 \) and \( X_2 \). For any \( x, y \in X_1 \sqcup X_2 \), define \( d_0(x, y) \in [0, \infty) \) by

\[
d_0(x, y) = \begin{cases} 
  d_1(x, y), & \text{if } x, y \in X_1, \\
  d_2(x, y), & \text{if } x, y \in X_2, \\
  \min_{z \in Z_1} \{d_1(x, z) + d_2(f(z), y)\}, & \text{if } x \in X_1, y \in X_2, \\
  \min_{z \in Z_2} \{d_1(y, z) + d_2(f(z), x)\}, & \text{if } x \in X_2, y \in X_1.
\end{cases}
\]

Then \( d_0 \) is a semi-metric on \( X_1 \sqcup X_2 \). In other words, \( d_0 \) satisfies the axioms for a metric except the requirement that \( d_0(x, y) = 0 \) implies \( x = y \). Define relation \( \sim \) on \( X_1 \sqcup X_2 \) by declaring \( x \sim y \) if and only if \( d_0(x, y) = 0 \). Then \( \sim \) is an equivalence relation on \( X_1 \sqcup X_2 \), and the projection \( \overline{d}_0 \) of \( d_0 \) onto the quotient space \( X_0 = (X_1 \sqcup X_2) / \sim \) is well defined. It is easily observed that \((X_0, \overline{d}_0)\) is a metric space, which is called the gluing of \( X_1 \) and \( X_2 \) along the isometry \( f \). Assume in addition that \( X_1 \) and \( X_2 \) are complete locally compact CAT(0) spaces, and that \( Z_1 \) and \( Z_2 \) are convex subsets of \( X_1 \) and \( X_2 \), respectively. Then
by Reshetnyak’s gluing theorem, the gluing of $X_1$ and $X_2$ along $f$ becomes a CAT(0) space. For a proof of this fact, see [8] or [3, Theorem 9.1.21]. A more general statement is in [2, Chapter II, Theorem 11.1]. When two geodesic segments $[a, b] \subseteq X_1$ and $[c, d] \subseteq X_2$ are isometric, we mean by “the metric space obtained by gluing $X_1$ and $X_2$ by identifying $[a, b]$ with $[c, d]$” the gluing of $X_1$ and $X_2$ along the isometry $f : [a, b] \to [c, d]$ with $f(a) = c$ and $f(b) = d$.

3. Comparison Quadrangles in the Euclidean Plane

In this section, we recall some properties of metric spaces that satisfy the $\boxtimes$-inequalities. First, we recall the following fact, which was established by Sturm when he proved in [10, Theorem 4.9] that a geodesic space is a CAT(0) space whenever it satisfies the $\boxtimes$-inequalities.

**Proposition 3.1.** Let $(X, d_X)$ be a metric space that satisfies the $\boxtimes$-inequalities. Suppose $x, y, z \in X$ are points such that $x \neq z$, and

$$d_X(x, z) = d_X(x, y) + d_X(y, z). \quad (3.1)$$

Set $t = d_X(x, y) / d_X(x, z)$. Then we have

$$d_X(y, w)^2 \leq (1-t)d_X(x, w)^2 + td_X(z, w)^2 - t(1-t)d_X(x, z)^2. \quad (3.2)$$

for any $w \in X$.

**Proof.** By the hypothesis (3.1), we compute

$$(1-t)d_X(x, y)^2 + td_X(y, z)^2 = \frac{d_X(y, z)}{d_X(x, z)} d_X(x, y)^2 + \frac{d_X(x, y)}{d_X(x, z)} d_X(y, z)^2$$

$$= \frac{d_X(x, y) d_X(y, z)}{d_X(x, z)} (d_X(x, y) + d_X(y, z))$$

$$= d_X(x, y) d_X(y, z)$$

$$= t(1-t)d_X(x, z)^2.$$ 

Combining this with the $\boxtimes$-inequality on $X$ yields

$$0 \leq (1-t)(1-s)d_X(x, y)^2 + t(1-s) d_X(y, z)^2 + tsd_X(z, w)^2$$

$$+ s(1-t)d_X(w, x)^2 - t(1-t)d_X(x, z)^2 - s(1-s)d_X(y, w)^2$$

$$= (1-s)((1-t)d_X(x, y)^2 + td_X(y, z)^2) + tsd_X(z, w)^2$$

$$+ s(1-t)d_X(w, x)^2 - t(1-t)d_X(x, z)^2 - s(1-s)d_X(y, w)^2$$

$$= (1-s)t(1-t)d_X(x, z)^2 + tsd_X(z, w)^2$$

$$+ s(1-t)d_X(w, x)^2 - t(1-t)d_X(x, z)^2 - s(1-s)d_X(y, w)^2$$

$$= tsd_X(z, w)^2 + s(1-t)d_X(w, x)^2 - st(1-t)d_X(x, z)^2 - s(1-s)d_X(y, w)^2$$

for every $s \in [0, 1]$. For any $s \in (0, 1]$, dividing this by $s$, we obtain

$$(1-s)d_X(y, w)^2 \leq td_X(z, w)^2 + (1-t)d_X(w, x)^2 - t(1-t)d_X(x, z)^2.$$ 

Letting $s \to 0$ in this inequality yields the desired inequality. \qed

**Remark 3.2.** Suppose that points $x, y, z \in X$ satisfies $x = z$ and (3.1). Then $x = y = z$, and hence the inequality (3.2) clearly holds with equality for any $w \in X$ and any $t \in \mathbb{R}$. 


The following lemma is a key ingredient in the proof of [7, Lemma 2.6], which plays a fundamental role in the study of metric spaces that satisfy the $\boxtimes$-inequalities.

**Lemma 3.3.** Let $(X, d_X)$ be a metric space that satisfies the $\boxtimes$-inequalities. Suppose $x, y, z, w \in X$ and $x', y', z', w' \in \mathbb{R}^2$ are points such that

\[ d_X(x, y) \leq \|x' - y'\|, \quad d_X(y, z) \leq \|y' - z'\|, \quad d_X(z, w) \leq \|z' - w'\|, \]

\[ d_X(w, x) \leq \|w' - x'\|, \quad \|x' - z'\| \leq d_X(x, z), \]

and $[x', z'] \cap [y', w'] \neq \emptyset$. Then $d_X(y, w) \leq \|y' - w'\|.$

**Proof.** We consider three cases.

**Case 1:** $[x', z'] \cap (y', w') \neq \emptyset$. In this case, there exist $s \in (0, 1)$ and $t \in [0, 1]$ such that

\[ (1 - t)x' + tz' = (1 - s)y' + sw'. \]

It follows from this equality that

\[ 0 = \|((1 - t)x' + tz') - ((1 - s)y' + sw')\|^2 \]

\[ = (1 - t)(1 - s)\|x' - y'\|^2 + t(1 - s)\|y' - z'\|^2 + ts\|z' - w'\|^2 + (1 - t)s\|w' - x'\|^2 \]

\[ - t(1 - t)\|x' - z'\|^2 - s(1 - s)\|y' - w'\|^2 \]

\[ \geq (1 - t)(1 - s)d_X(x, y)^2 + t(1 - s)d_X(y, z)^2 + tsd_X(z, w)^2 + (1 - t)sd_X(w, x)^2 \]

\[ - t(1 - t)d_X(x, z)^2 - s(1 - s)d_X(y, w)^2. \]

On the other hand,

\[ 0 \leq (1 - t)(1 - s)d_X(x, y)^2 + t(1 - s)d_X(y, z)^2 + tsd_X(z, w)^2 + (1 - t)sd_X(w, x)^2 \]

\[ - t(1 - t)d_X(x, z)^2 - s(1 - s)d_X(y, w)^2 \]

because $X$ satisfies the $\boxtimes$-inequalities. Comparing these yields

\[ d_X(y, w) \leq \|y' - w'\|. \]

**Case 2:** $[x', z'] \cap (y', w') = \emptyset$ and $x' \neq z'$. In this case, $y' \in [x', z']$ or $w' \in [x', z']$ because $[x', z'] \cap [y', w'] \neq \emptyset$ by the hypothesis. We assume without loss of generality that $y' \in [x', z']$. Then

\[ d_X(x, z) \leq d_X(x, y) + d_X(y, z) \leq \|x' - y'\| + \|y' - z'\| = \|x' - z'\| \leq d_X(x, z), \]

which implies that

\[ d_X(x, z) = d_X(x, y) + d_X(y, z) = \|x' - y'\| + \|y' - z'\| = \|x' - z'\|. \]

The second equality in (3.3) implies that

\[ d_X(x, y) = \|x' - y'\|, \quad d_X(y, z) = \|y' - z'\|. \]

Hence we can write

\[ y' = (1 - c)x' + cz', \]

where

\[ c = \frac{\|x' - y'\|}{\|x' - z'\|} = \frac{d_X(x, y)}{d_X(x, z)} \in [0, 1]. \]

(3.4)
It follows that
\[\|y' - w'\|^2 = \|(1 - c)x' + cz' - w'\|^2\]
\[= (1 - c)\|x' - w'\|^2 + c\|z' - w'\|^2 - c(1 - c)\|x' - z'\|^2\]
\[\geq (1 - c)d_X(x, w)^2 + cd_X(z, w)^2 - c(1 - c)d_X(x, z)^2.\]

On the other hand, Lemma 3.3 and Proposition 3.1 imply that
\[d_X(y, w)^2 \leq (1 - c)d_X(x, w)^2 + cd_X(z, w)^2 - c(1 - c)d_X(x, z)^2.\]
Combining these yields
\[d_X(y, w) \leq \|y' - w'\|.\]

**Case 3:** $x' = z'$. In this case, $x' \in [y', w']$ because $[x', z'] \cap [y', w'] \neq \emptyset$. Therefore,
\[\|y' - w'\| = \|y' - x'\| + \|x' - w'\| \geq d_X(y, x) + d_X(x, w) = d_X(y, w).\]

The above three cases exhaust all possibilities. \qed

Lemma 3.3 implies the following corollary.

**Corollary 3.4.** Let $(X, d_X)$ be a metric space that satisfies the $\Box$-inequalities, and let $(Y, d_Y)$ be a metric space. Suppose $x, y, z, w \in X$ and $x', y', z', w' \in Y$ are points such that
\[d_X(x, y) \leq d_Y(x', y'), \quad d_X(y, z) \leq d_Y(y', z'), \quad d_X(z, w) \leq d_Y(z', w'), \quad d_X(w, x) \leq d_Y(w', x').\]
Assume that there exist subsets $S$ and $T$ of $Y$ that satisfy the following conditions:
1. $S$ and $T$ are isometric to convex subsets of Euclidean spaces.
2. $\{x', y', z'\} \subseteq S$ and $\{x', w', z'\} \subseteq T$.
3. There is a geodesic segment $Γ_1$ in $Y$ with endpoints $x'$ and $z'$ such that $Γ_1 \subseteq S \cap T$.
4. There is a point $p \in Γ_1$ such that $d_Y(x', z') = d_Y(x', p) + d_Y(p, z')$.

Then $d_X(y, w) \leq d_Y(y', w')$.

**Proof.** Define points $\tilde{x}, \tilde{z} \in \mathbb{R}^2$ by
\[\tilde{x} = (0, 0), \quad \tilde{z} = (d_Y(x', z'), 0).\]
Suppose $\tilde{y} = (y^{(1)}, y^{(2)})$ and $\tilde{w} = (w^{(1)}, w^{(2)})$ are points in $\mathbb{R}^2$ such that
\[\|\tilde{x} - \tilde{y}\| = d_Y(x', y'), \quad \|\tilde{y} - \tilde{z}\| = d_Y(y', z'), \quad y^{(2)} \geq 0,\]
\[\|\tilde{x} - \tilde{w}\| = d_Y(x', w'), \quad \|\tilde{w} - \tilde{z}\| = d_Y(w', z'), \quad w^{(2)} \leq 0.\]
Clearly, such $\tilde{y}$ and $\tilde{w}$ exist uniquely. By the conditions (1) and (2), there exist isometric embeddings $f_1 : \text{conv}(\{\tilde{x}, \tilde{y}, \tilde{z}\}) \to S$ and $f_2 : \text{conv}(\{\tilde{x}, \tilde{w}, \tilde{z}\}) \to T$ such that
\[f_1(\tilde{x}) = x', \quad f_1(\tilde{y}) = y', \quad f_1(\tilde{z}) = z', \quad f_2(\tilde{x}) = x', \quad f_2(\tilde{w}) = w', \quad f_2(\tilde{z}) = z'.\]
Then $f_1([\tilde{x}, \tilde{z}])$ and $f_2([\tilde{x}, \tilde{z}])$ are geodesic segments contained in $S$ and $T$, respectively. Because both $S$ and $T$ are uniquely geodesic by the condition (1), $f_1([\tilde{x}, \tilde{z}]) = Γ_1$ and $f_2([\tilde{x}, \tilde{z}]) = Γ_1$. Therefore, the maps $f_1$ and $f_2$ agree on $[\tilde{x}, \tilde{z}]$. Let $\tilde{p} \in [\tilde{x}, \tilde{z}]$ be the point such that $f_1(\tilde{p}) = f_2(\tilde{p}) = p$. We consider two cases.

**Case 1:** $[\tilde{x}, \tilde{z}] \cap [\tilde{y}, \tilde{w}] \neq \emptyset$. In this case, Lemma 3.3 implies that
\[d_X(y, w) \leq \|\tilde{y} - \tilde{w}\|.\]
Therefore,
\[ d_X(y, w) \leq \|\tilde{y} - \tilde{w}\| \leq \|\tilde{y} - \tilde{p}\| + \|\tilde{p} - \tilde{w}\| = d_Y(y', p) + d_Y(p, w') = d_Y(y', w'). \]

**Case 2:** \([\tilde{x}, \tilde{z}] \cap [\tilde{y}, \tilde{w}] = \emptyset\). It follows from the definitions of \(\tilde{y}\) and \(\tilde{w}\) that there exists \(a \in \mathbb{R}\) such that \((a, 0) \in [\tilde{y}, \tilde{w}]\). By the assumption of **Case 2**, we have \(a < 0\) or \(d_Y(x', z') < a\). We may assume without loss of generality that \(a < 0\). Then we have \(\tilde{x} \in \text{conv}([\tilde{p}, \tilde{y}, \tilde{w}])\), and therefore
\[ \|\tilde{y} - \tilde{x}\| + \|\tilde{x} - \tilde{w}\| \leq \|\tilde{y} - \tilde{p}\| + \|\tilde{p} - \tilde{w}\|. \]

Hence
\[ d_X(y, w) \leq d_X(y, x) + d_X(x, w) \leq d_Y(y', x') + d_Y(x', w') \]
\[ = \|\tilde{y} - \tilde{x}\| + \|\tilde{x} - \tilde{w}\| \leq \|\tilde{y} - \tilde{p}\| + \|\tilde{p} - \tilde{w}\| \]
\[ = d_Y(y', p) + d_Y(p, w') = d_Y(y', w'), \]
which completes the proof.

\[ \square \]

**Remark 3.5.** Clearly the statement of Corollary 3.4 still holds true if we replace the condition (4) with the following condition:

\((4')\) There is a geodesic segment \(\Gamma_2\) in \(Y\) with endpoints \(y'\) and \(w'\) such that \(\Gamma_1 \cap \Gamma_2 \neq \emptyset\).

4. **The Cycl\(_k\)(0) Condition**

In this section, we prove Theorem 1.4. First, we set up some notations and terminologies. Let \(k \geq 3\) be an integer. For a map \(f : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2\), we define
\[ \mathcal{P}(f) = \bigcup_{i \in \mathbb{Z}/k\mathbb{Z}} [f(i), f(i + [1]_k)]. \]

We say that a map \(f : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2\) generates a simple polygon if \(f\) is nonconstant, and for any \(x \in \mathbb{R}^2\), \(i \in \mathbb{Z}/k\mathbb{Z}\) and \(j \in \mathbb{Z} \cap [1, k - 1]\),
\[ x \in [f(i), f(i + [1]_k)] \cap [f(i + [j]_k), f(i + [j + 1]_k)] \]
holds if and only if \(f(i + [m]_k) = x\) for every \(m \in \mathbb{Z} \cap [1, j]\), or \(f(i + [j + n]_k) = x\) for every \(n \in \mathbb{Z} \cap [1, k - j]\).

Suppose \(g : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2\) is a map that generates a simple polygon. Then \(\mathcal{P}(g)\) clearly forms a Jordan curve, and therefore the Jordan curve theorem implies that \(\mathbb{R}^2 \setminus \mathcal{P}(g)\) consists of exactly two connected components, one of which is bounded, and the other unbounded. We call the union of \(\mathcal{P}(g)\) and the bounded connected component of \(\mathbb{R}^2 \setminus \mathcal{P}(g)\) the closed region bounded by \(\mathcal{P}(g)\). If in addition \(g(i) \neq g(i + [1]_k)\) for every \(i \in \mathbb{Z}/k\mathbb{Z}\), then the interior angle measure of \(\mathcal{P}(g)\) at \(g(j)\) for each \(j \in \mathbb{Z}/k\mathbb{Z}\) is defined as usual, which takes a value in \([0, \pi]\). The following fact is well-known.

**Lemma 4.1.** Fix an integer \(k \geq 3\). Suppose \(f : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2\) is a map that generates a simple polygon. Assume that \(f(i) \neq f(i + [1]_k)\) for every \(i \in \mathbb{Z}/k\mathbb{Z}\). Then the following are equivalent:

1. The closed region bounded by \(\mathcal{P}(f)\) is a convex subset of \(\mathbb{R}^2\).
2. For any \(i, j, l \in \mathbb{Z}/k\mathbb{Z}\), \(f(j)\) and \(f(l)\) do not lie on opposite sides of the straight line \(f(i)f(i + [1]_k)\).
Lemma 4.2. Fix an integer $k \geq 3$. Let $f : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2$ be a map such that $f(\mathbb{Z}/k\mathbb{Z}) \subseteq L$ for some straight line $L$, and let $\varphi : L \to \mathbb{R}$ be an isometry. Then the following two conditions are equivalent:

1. For every $x \in L$, the number of elements $i \in \mathbb{Z}/k\mathbb{Z}$ that satisfy $x \in (f(i), f(i + [1]_k))$ is at most two.
2. There exist $i_0 \in \mathbb{Z}/k\mathbb{Z}$ and $j_0 \in \mathbb{Z} \cap [1, k - 1]$ such that the functions $m \mapsto \varphi(f(i_0 + [m]_k))$, $m \mapsto \varphi(f(i_0 - [m]_k))$ are nondecreasing on $\mathbb{Z} \cap [0, j_0]$ and on $\mathbb{Z} \cap [0, k - j_0]$, respectively.

Proof. The implication $(2) \implies (1)$ is trivial, so we only prove the converse implication. Suppose the condition (1) holds. Choose $i_0 \in \mathbb{Z}/k\mathbb{Z}$ and $j_0 \in \mathbb{Z} \cap [1, k - 1]$ such that

$$\varphi(f(i_0)) = \min\{\varphi(f(i)) \mid i \in \mathbb{Z}/k\mathbb{Z}\}, \quad \varphi(f(i_0 + [j_0]_k)) = \max\{\varphi(f(i)) \mid i \in \mathbb{Z}/k\mathbb{Z}\}.$$ 

To prove that the functions $m \mapsto \varphi(f(i_0 + [m]_k))$ and $m \mapsto \varphi(f(i_0 - [m]_k))$ are nondecreasing on $\mathbb{Z} \cap [0, j_0]$ and on $\mathbb{Z} \cap [0, k - j_0]$, respectively, suppose to the contrary that one of them is not nondecreasing. If $m \mapsto \varphi(f(i_0 + [m]_k))$ is not nondecreasing on $\mathbb{Z} \cap [0, j_0]$, then there exists an integer $m_0 \in \mathbb{Z} \cap [1, j_0 - 2]$ such that

$$\varphi(f(i_0)) \leq \varphi(f(i_0 + [m_0 + 1]_k)) < \varphi(f(i_0 + [m_0]_k)) \leq \varphi(f(i_0 + [j_0]_k)).$$

This requires in particular that $j_0 \geq 3$. We can choose a point

$$x \in (f(i_0 + [m_0 + 1]_k), f(i_0 + [m_0]_k)) \setminus f(\mathbb{Z}/k\mathbb{Z}),$$

and the point $x$ is contained in

$$\bigcup_{m \in \mathbb{Z} \cap [0, m_0 - 1]} (f(i_0 + [m]_k), f(i_0 + [m + 1]_k)),$$

and $(f(i_0 + [m_0]_k), f(i_0 + [m_0 + 1]_k))$. It clearly follows from this observation that there exist at least three elements $i \in \mathbb{Z}/k\mathbb{Z}$ with $x \in (f(i), f(i + [1]_k))$. Thus the condition (1) does not hold. If $m \mapsto \varphi(f(i_0 - [m]_k))$ is not nondecreasing on $\mathbb{Z} \cap [0, k - j_0]$, then we can prove that the condition (1) does not hold in exactly the same way, which completes the proof. \qed

Suppose $f : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2$ is a map such that $f(\mathbb{Z}/k\mathbb{Z}) \subseteq L$ for some straight line $L$, and $f$ satisfies the condition (1) (or (2)) in the statement of Lemma 4.2. Then, we mean by the closed region bounded by $\mathcal{P}(f)$ the line segment $[f(i_0), f(i_0 + [j_0]_k)]$, where $i_0 \in \mathbb{Z}/k\mathbb{Z}$ and $j_0 \in \mathbb{Z} \cap [1, k - 1]$ are as in the statement of Lemma 4.2. In other words, in this case, the closed region bounded by $\mathcal{P}(f)$ is $\mathcal{P}(f)$ itself. To prove Theorem 1.4, we define a stronger version of the Cycl$_k(0)$ condition.

Definition 4.3. Fix an integer $k \geq 3$. We say that a metric space $(X, d_X)$ satisfies the strongly Cycl$_k(0)$ condition if for any map $f : \mathbb{Z}/k\mathbb{Z} \to X$, there exists a map $g : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2$ that satisfies the following conditions:
(1) For any $i, j \in \mathbb{Z}/k\mathbb{Z}$,
\[ \|g(i) - g(i + [1]_k)\| = d_X(f(i), f(i + [1]_k)), \quad \|g(i) - g(j)\| \geq d_X(f(i), f(j)). \]

(2) If $g(\mathbb{Z}/k\mathbb{Z})$ does not contained in any straight line, then $g$ generates a simple polygon, and the closed region bounded by $\mathcal{P}(g)$ is convex.

(3) If $g(\mathbb{Z}/k\mathbb{Z}) \subseteq L$ for some straight line $L$, then for every $x \in L$, the number of elements $i \in \mathbb{Z}/k\mathbb{Z}$ that satisfy $x \in (g(i), g(i + [1]_k))$ is at most two.

We call such a map $g : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2$ that satisfies the above three conditions a *comparison map* of $f$.

**Remark 4.4.** It is known that a map $g : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2$ satisfies the conditions (2) and (3) in Definition 4.3 if and only if $g$ is monotone with respect to any straight line in $\mathbb{R}^2$, i.e., for any straight line $L'$ in $\mathbb{R}^2$ and $x \in L'$, the number of elements $i \in \mathbb{Z}/k\mathbb{Z}$ that satisfy $x \in (p(f(i)), p(f(i + [1]_k)))$ is at most two, where $p : \mathbb{R}^2 \to L'$ is the orthogonal projection. Therefore, we can replace the conditions (2) and (3) with the following condition:

(4) $g$ is monotone with respect to any straight line in $\mathbb{R}^2$.

Because every three-point metric space admits an isometric embedding into $\mathbb{R}^2$, every metric space clearly satisfies the strongly $\text{Cycl}_3(0)$ condition. It is also clear that the strongly $\text{Cycl}_k(0)$ condition implies the $\text{Cycl}_k(0)$ condition for every integer $k \geq 4$. The convexity assumptions (2) and (3) in Definition 4.3 allow us to prove the following theorem by induction. A very similar argument was used by Reshetnyak [8] when he proved the famous majorization theorem.

**Theorem 4.5.** If a metric space $X$ satisfies the $\mathfrak{X}$-inequalities, then $X$ satisfies the strongly $\text{Cycl}_k(0)$ condition for every integer $k \geq 3$.

**Proof.** Suppose a metric space $(X, d_X)$ satisfies the $\mathfrak{X}$-inequalities. We will prove that $X$ satisfies the strongly $\text{Cycl}_k(0)$ condition for every integer $k \geq 3$ by induction on $k$. As we mentioned above, $X$ satisfies the strongly $\text{Cycl}_3(0)$ condition. Fix an integer $k \geq 3$, and assume that $X$ satisfies the strongly $\text{Cycl}_k(0)$ condition. Fix a map $f : \mathbb{Z}/(k + 1)\mathbb{Z} \to X$. We will prove that there exists a comparison map of $f$. If $f(i) = f(i + [1]_{k + 1})$ for some $i \in \mathbb{Z}/(k + 1)\mathbb{Z}$, then the existence of a comparison map of $f$ clearly follows from the inductive hypothesis. Hence we assume that

\[ f(i) \neq f(i + [1]_{k + 1}) \]

for every $i \in \mathbb{Z}/(k + 1)\mathbb{Z}$.

Define a map $f_0 : \mathbb{Z}/k\mathbb{Z} \to X$ by $f_0([m]_k) = f([m]_{k + 1})$ for each $m \in \mathbb{Z} \cap [0, k - 1]$. Then there exists a comparison map $g_0 : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2$ of $f_0$ by the inductive hypothesis. It follows from the assumption (4.1) and the definition of comparison map that

\[ f_0([m]_k) \neq f_0([m + 1]_k), \quad g_0([m]_k) \neq g_0([m + 1]_k) \]

for every $m \in \mathbb{Z} \cap [0, k - 2]$. Since

\[ \|g_0([k - 1]_k) - g_0([0]_k)\| = d_X(f_0([k - 1]_k), f_0([0]_k)) \]
\[ = d_X(f([k - 1]_{k + 1}), f([0]_{k + 1})), \]

there exists a point $p \in \mathbb{R}^2$ such that

\[ \|g_0([k - 1]_k) - p\| = d_X(f([k - 1]_{k + 1}), f([k]_{k + 1})), \]
\[ \|p - g_0([0]_k)\| = d_X(f([k]_{k + 1}), f([0]_{k + 1})). \]
Because \( g_0 \) is a comparison map of \( f_0 \), if \( g_0(\mathbb{Z}/k\mathbb{Z}) \) does not contained in any straight line, then \( g_0 \) generates a simple polygon, and the closed region bounded by \( \mathcal{P}(g_0) \) is convex. Hence by Lemma 4.1, for any \( i, j \in \mathbb{Z}/k\mathbb{Z} \), \( g_0(i) \) and \( g_0(j) \) are not on opposite sides of \( g_0([k-1]_k) \) whenever \( g_0([k-1]_k) \neq g_0([0]_k) \). Therefore, we may assume in addition that \( p \) is not on the same side of \( g_0([k-1]_k) \) as \( g_0(i) \) for every \( i \in \mathbb{Z}/k\mathbb{Z} \) whenever \( g_0([k-1]_k) \neq g_0([0]_k) \). Define a map \( g : \mathbb{Z}/(k+1)\mathbb{Z} \rightarrow \mathbb{R}^2 \) by

\[
g([m]_{k+1}) = \begin{cases} 
g_0([m]_k), & \text{if } m \in \mathbb{Z} \cap [0, k-1], \\
p, & \text{if } m = k. \end{cases}
\]

Then

\[
g([m]_{k+1}) - g([n]_{k+1}) = \|g_0([m]_k) - g_0([n]_k)\| \geq \|f_0([m]_k) - f_0([n]_k)\| = d_X(f([m]_{k+1}), f([n]_{k+1}))
\]

for any \( m, n \in \mathbb{Z} \cap [0, k-1] \), and

\[
g([l]_{k+1}) - g([l+1]_{k+1}) = \|g_0([l]_k) - g_0([l+1]_k)\| = \|f_0([l]_k) - f_0([l+1]_k)\| = d_X(f([l]_{k+1}), f([l+1]_{k+1}))
\]

for any \( l \in \mathbb{Z} \cap [0, k-2] \). Furthermore,

\[
g([k-1]_{k+1}) - g([k]_{k+1}) = \|g_0([k-1]_k) - p\| = d_X(f([k-1]_{k+1}), f([k]_{k+1})),
\]

\[
g([k]_{k+1}) - g([0]_{k+1}) = \|p - g_0([0]_k)\| = d_X(f([k]_{k+1}), f([0]_{k+1})),
\]

\[
g([k-1]_{k+1}) - g([0]_{k+1}) = \|g_0([k-1]_k) - g_0([0]_k)\| = d_X(f_0([k-1]_k), f_0([0]_k)) = d_X(f([k-1]_{k+1}), f([0]_{k+1})).
\]

We consider five cases. To describe those cases, we define the following two conditions:

(A) g generates a simple polygon, and the closed region bounded by \( \mathcal{P}(g) \) is convex.

(B) \( g(\mathbb{Z}/(k+1)\mathbb{Z}) \subseteq L \) for some straight line \( L \), and for every \( x \in L \), the number of elements \( i \in \mathbb{Z}/(k+1)\mathbb{Z} \) that satisfy \( x \in (g(i), g(i + [1]_{k+1})) \) is at most two.

**CASE 1: g satisfies (A).** In this case,

\[
[g([k-1]_{k+1}), g([0]_{k+1})] \cap [g([m]_{k+1}), g([k]_{k+1})] \neq \emptyset
\]

for every \( m \in \mathbb{Z} \cap [0, k-1] \). Therefore, for every \( m \in \mathbb{Z} \cap [0, k-1] \), Lemma 3.3 implies that

\[
g([m]_{k+1}) - g([k]_{k+1}) \geq d_X(f([m]_{k+1}), f([k]_{k+1}))
\]
because we have
\[
d_X(f([0]_{k+1}, [m]_{k+1}) \leq \|g([0]_{k+1}) - g([m]_{k+1})\|,
\]
\[
d_X(f([m]_{k+1}, [k-1]_{k+1}) \leq \|g([m]_{k+1}) - g([k-1]_{k+1})\|,
\]
\[
d_X(f([k-1]_{k+1}, [k]_{k+1}) = \|g([k-1]_{k+1}) - g([k]_{k+1})\|,
\]
\[
d_X(f([k]_{k+1}, [0]_{k+1}) = \|g([k]_{k+1}) - g([0]_{k+1})\|,
\]
\[
d_X(f([0]_{k+1}, [k-1]_{k+1}) = \|g([0]_{k+1}) - g([k-1]_{k+1})\|
\]
by (4.3), (4.5), (4.6) and (4.7). By (4.3), (4.4), (4.5), (4.6) and (4.9),
\[
\|g(i) - g(i + [1]_{k+1})\| = d_X(f(i), f(i + [1]_{k+1})), \quad \|g(i) - g(j)\| \geq d_X(f(i), f(j))
\]
for any \(i, j \in \mathbb{Z}/(k+1)\mathbb{Z}\). Thus \(g\) is a comparison map of \(f\) in Case 1.

**Case 2:** \(g\) satisfies (B). We first prove that (4.8) holds for every \(m \in \mathbb{Z} \cap [0, k-1]\) in this case as well. To prove this, suppose to the contrary that
\[
[g([k-1]_{k+1}), g([0]_{k+1})] \cap [g([m_0]_{k+1}), g([k]_{k+1})] = \emptyset
\]
for some \(m_0 \in \mathbb{Z} \cap [1, k-2]\). Then the set
\[
M = L \setminus ([g([k-1]_{k+1}), g([0]_{k+1})] \cup [g([m_0]_{k+1}), g([k]_{k+1})])
\]
consists of two unbounded connected components and one bounded connected component. The bounded connected component \(I\) of \(M\) is isometric to an open interval in \(\mathbb{R}\) of positive length, and therefore we can choose a point \(x_0 \in I \setminus g(\mathbb{Z}/(k+1)\mathbb{Z})\). Then \(x_0\) is contained in \((g([k-1]_{k+1}), g([k]_{k+1})), (g([k]_{k+1}), g([0]_{k+1}))\) and
\[
\bigcup_{m \in \mathbb{Z} \cap [0, m_0-1]} (g([m]_{k+1}), g([m+1]_{k+1})).
\]
It clearly follows that there exist at least three elements \(i \in \mathbb{Z}/(k+1)\mathbb{Z}\) that satisfy \(x_0 \in (g(i), g(i + [1]_{k+1})),\) contradicting the assumption that \(g\) satisfies (B). Thus (4.8) holds for every \(m \in \mathbb{Z} \cap [0, k-1]\), and therefore the same argument as in Case 1 implies that (4.9) holds in Case 2 as well. By (4.3), (4.4), (4.5), (4.6) and (4.9), \(g\) becomes a comparison map of \(f\) in Case 2 as well.

**Case 3:** \(g\) satisfies neither (A) nor (B), and neither \(\{g_0([k-1]_k), p, g_0([0]_k)\}\) nor \(g_0(\mathbb{Z}/k\mathbb{Z})\) is contained in a straight line. The assumption that \(\{g_0([k-1]_k), p, g_0([0]_k)\}\) is not contained in a straight line implies in particular that
\[
g_0([k-1]_k) \neq g_0([0]_k), \quad p \notin g_0([k-1]_k)g_0([0]_k).
\]
The assumption that \(g_0(\mathbb{Z}/k\mathbb{Z})\) is not contained in a straight line implies that \(g_0\) generates a simple polygon, and the closed region bounded by \(P(g_0)\) is convex because \(g_0\) is a comparison map of \(f_0\). Therefore, it clearly follows from the definition of \(g\) that \(g\) generates a simple polygon. Moreover, by Lemma 4.1, the interior angle measure of \(P(g_0)\) at \(g_0(i)\) is at most \(\pi\) for every \(i \in \mathbb{Z}/k\mathbb{Z}\), and therefore by definition of \(g\), the interior angle measure of \(P(g)\) at \(g([m]_{k+1})\) is at most \(\pi\) for every \(m \in \mathbb{Z} \cap [1, k-2]\). Furthermore, the interior angle measure of \(P(g)\) at \(g([k]_{k+1})\) is equal to \(\angle g_0([k-1]_k) g_0([0]_k) < \pi\). Therefore, the interior angle measure of \(P(g)\) at \(g([k-1]_{k+1})\) or at \(g([0]_{k+1})\) is greater than \(\pi\) because otherwise the closed region bounded by \(P(g)\) would be convex by Lemma 4.1, contradicting the assumption that \(g\) does not satisfy (A). We may assume without loss of generality that the interior angle measure of \(P(g)\) at \(g([k-1]_{k+1})\) is greater than \(\pi\).
Let $S$ be the closed region bounded by $\mathcal{P}(g_0)$, and let $T = \text{conv}(\{g_0([k-1]_k), p, g_0([0]_k)\})$. Equip the subsets $S$ and $T$ of $\mathbb{R}^2$ with the induced metrics, and regard them as disjoint metric spaces. Define $(R, d_R)$ to be the metric space obtained by gluing $S$ and $T$ by identifying $[g_0([k-1]_k), g_0([0]_k)] \subseteq S$ with $[g_0([k-1]_k), g_0([0]_k)] \subseteq T$. Then $R$ is a CAT(0) space by Reshetnyak’s gluing theorem. We denote by $g_m$ the point in $R$ represented by $g_0([m]_k) \in S$ for each $m \in \mathbb{R} \cap [0, k - 1]$, and by $r_k$ the point in $R$ represented by $p \in T$. In CASE 3, $R$ is isometric to the closed region bounded by $\mathcal{P}(g)$ equipped with the induced length metric, which is pictured in FIGURE 4.1.

![Figure 4.1. The metric space R in Case 3.](image)

Define a map $f_1 : \mathbb{Z}/k\mathbb{Z} \to R$ by

$$f_1([m]_k) = \begin{cases} r_m, & \text{if } m \in \mathbb{Z} \cap [0, k - 2], \\ r_k, & \text{if } m = k - 1. \end{cases}$$

Since $R$ is a CAT(0) space, $R$ satisfies the $\mathbb{R}$-inequalities. Hence there exists a comparison map $g_1 : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2$ of $f_1$ by the inductive hypothesis. Because the interior angle measure of $\mathcal{P}(g)$ at $g([k-1]_{k+1})$ is greater than $\pi$, we clearly have

$$\|g_1([k-2]_k) - g_1([k-1]_k)\| = d_R(r_{k-2}, r_k) = d_R(r_{k-2}, r_{k-1}) + d_R(r_{k-1}, r_k).$$

Therefore, there exists a point $q \in [g_1([k-2]_k), g_1([k-1]_k)]$ such that

$$\|g_1([k-2]_k) - q\| = d_R(r_{k-2}, r_{k-1}), \quad \|q - g_1([k-1]_k)\| = d_R(r_{k-1}, r_k).$$

Define a map $g_2 : \mathbb{Z}/(k+1)\mathbb{Z} \to \mathbb{R}^2$ by

$$g_2([m]_{k+1}) = \begin{cases} g_1([m]_k), & \text{if } m \in \mathbb{Z} \cap [0, k - 2], \\ q, & \text{if } m = k - 1, \\ g_1([k-1]_k), & \text{if } m = k. \end{cases}$$

Then

$$\|g_2([m]_{k+1}) - g_2([n]_{k+1})\| = \|g_1([m]_k) - g_1([n]_k)\|$$

$$\geq d_R(f_1([m]_k), f_1([n]_k)) = d_R(r_m, r_n) = \|g_0([m]_k) - g_0([n]_k)\|$$

$$\geq d_X(f_0([m]_k), f_0([n]_k)) = d_X(f([m]_{k+1}), f([n]_{k+1}))$$
for any $m, n \in \mathbb{Z} \cap [0, k - 2]$, and

$$
(4.14) \quad \|g_2([l]_{k+1}) - g_2([l + 1]_{k+1})\| = \|g_1([l]_k) - g_1([l + 1]_k)\|
$$

$$
= d_R(f_1([l]_k), f_1([l + 1]_k)) = d_R(r_l, r_{l+1}) = \|g_0([l]_k) - g_0([l + 1]_k)\|
$$

$$
= d_X(f_0([l]_k), f_0([l + 1]_k)) = d_X(f([l]_{k+1}), f([l + 1]_{k+1}))
$$

for any $l \in \mathbb{Z} \cap [0, k - 3]$. Furthermore,

$$
(4.15) \quad \|g_2([k - 2]_{k+1}) - g_2([k - 1]_{k+1})\| = \|g_1([k - 2]_k) - q\|
$$

$$
= d_R(r_{k-2}, r_{k-1}) = \|g_0([k - 2]_k) - g_0([k - 1]_k)\|
$$

$$
= d_X(f_0([k - 2]_k), f_0([k - 1]_k)) = d_X(f([k - 2]_{k+1}), f([k - 1]_{k+1})),
$$

$$
(4.16) \quad \|g_2([k - 1]_{k+1}) - g_2([k]_{k+1})\| = \|q - g_1([k - 1]_k)\| = d_R(r_{k-1}, r_k)
$$

$$
= \|g_0(k - 1) - p\| = d_X(f([k - 1]_{k+1}), f([k]_{k+1})),
$$

$$
(4.17) \quad \|g_2([k]_{k+1}) - g_2([0]_{k+1})\| = \|g_1([k - 1]_k) - g_1([0]_k)\|
$$

$$
= d_R(f_1([k - 1]_k), f_1([0]_k)) = d_R(r_k, r_0)
$$

$$
= \|p - g_0([0]_k)\| = d_X(f([k]_{k+1}), f([0]_{k+1})).
$$

Because $q \in g_1([k - 2]_k), g_1([k - 1]_k)$, for every $m \in \mathbb{Z} \cap [0, k - 2]$, Lemma 3.3 implies that

$$
(4.18) \quad \|g_1([m]_k) - q\| \geq d_R(r_m, r_{k-1})
$$

because

$$
\|g_1([k - 1]_k) - g_1([m]_k)\| \geq d_R(f_1([k - 1]_k), f_1([m]_k)) = d_R(r_k, r_m),
$$

$$
\|g_1([m]_k) - g_1([k - 2]_k)\| \geq d_R(f_1([m]_k), f_1([k - 2]_k)) = d_R(r_m, r_{k-2}),
$$

$$
\|g_1([k - 2]_k) - q\| = d_R(r_{k-2}, r_{k-1}), \quad \|q - g_1([k - 1]_k)\| = d_R(r_{k-1}, r_k),
$$

$$
\|g_1([k - 2]_k) - g_1([k - 1]_k)\| = d_R(f_1([k - 2]_k), f_1([k - 1]_k)) = d_R(r_{k-2}, r_k).
$$

Hence

$$
(4.19) \quad \|g_2([m]_{k+1}) - g_2([k - 1]_{k+1})\| = \|g_1([m]_k) - q\|
$$

$$
\geq d_R(r_m, r_{k-1}) = \|g_0([m]_k) - g_0([k - 1]_k)\|
$$

$$
\geq d_X(f_0([m]_k), f_0([k - 1]_k)) = d_X(f([m]_{k+1}), f([k - 1]_{k+1})).
$$

for every $m \in \mathbb{Z} \cap [0, k - 2]$. Let $S'$ and $T'$ be the images of $S$ and $T$, respectively under the natural inclusions into $R$. Then clearly

$$
S' \cap T' = [r_0, r_{k-1}], \quad [r_0, r_{k-1}] \cap [r_m, r_k] \neq \emptyset
$$

for every $m \in \mathbb{Z} \cap [0, k - 1]$. Therefore, for each $m \in \mathbb{Z} \cap [0, k - 1]$, Corollary 3.4 implies that

$$
(4.20) \quad d_R(r_m, r_k) \geq d_X(f([m]_{k+1}), f([k]_{k+1}))
$$
because
\[ d_R(r_0, r_m) = \|g_0([0]_k) - g_0([m]_k)\| \geq d_X(f_0([0]_k), f_0([m]_k)) \]
\[ = d_X(f([0]_{k+1}), f([m]_{k+1})), \]
\[ d_R(r_m, r_{k-1}) = \|g_0([m]_k) - g_0([k - 1]_k)\| \geq d_X(f_0([m]_k), f_0([k - 1]_k)) \]
\[ = d_X(f([m]_{k+1}), f([k - 1]_{k+1})), \]
\[ d_R(r_{k-1}, r_k) = \|g_0([k - 1]_k) - p\| = d_X(f([k - 1]_{k+1}), f([k]_{k+1})), \]
\[ d_R(r_k, r_0) = \|p - g_0([0]_k)\| = d_X(f([k]_{k+1}), f([0]_{k+1})), \]
\[ d_R(r_0, r_{k-1}) = \|g_0([0]_k) - g_0([k - 1]_k)\| = d_X(f_0([0]_k), f_0([k - 1]_k)) \]
\[ = d_X(f([0]_{k+1}), f([k - 1]_{k+1})). \]

Hence
\[ g_2([m]_{k+1}) - g_2([k]_{k+1}) = g_1([m]_k) - g_1([k - 1]_k) \geq d_R(f_1([m]_k), f_1([k - 1]_k)) \]
\[ = d_R(r_m, r_k) \geq d_X(f([m]_{k+1}), f([k]_{k+1})), \]

for each \( m \in \mathbb{Z} \cap [0, k - 2]. \) By \( (4.13), (4.14), (4.15), (4.16), (4.17), (4.19) \) and \( (4.21), \)
\[ \|g_2(i) - g_2(i + [1]_{k+1})\| = d_X(f(i), f(i + [1]_{k+1})), \]
\[ \|g_2(i) - g_2(j)\| \geq d_X(f(i), f(j)) \]
for any \( i, j \in \mathbb{Z}/(k + 1)\mathbb{Z}. \) Moreover, because \( g_1 \) is a comparison map of \( f_1, \) \( g_2 \) clearly satisfies the properties corresponding to (2) and (3) in Definition 4.3 by definition of \( g_2. \)
Thus \( g_2 \) is a comparison map of \( f. \)

**Case 4:** \( g \) satisfies neither (A) nor (B), and \( \{g_0([k - 1]_k), p, g_0([0]_k)\} \) is contained in a straight line. In this case, \( p \notin \{g_0([k - 1]_k), g_0([0]_k)\} \) since otherwise \( g \) would clearly satisfy (A) or (B) because \( g_0 \) is a comparison map of \( f_0. \) Hence \( g_0([k - 1]_k) \in \{p, g_0([0]_k)\} \) or \( g_0([0]_k) \in \{g_0([k - 1]_k), p\}. \) We may assume without loss of generality that
\[ (4.22) \quad g_0([k - 1]_k) \in \{p, g_0([0]_k)\}. \]

Define \( R, f_1 : \mathbb{Z}/k\mathbb{Z} \to R \) and \( g_1 : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2 \) as in Case 3. Then it clearly follows from \( (4.22) \) and the definition of \( R \) that \( (4.10) \) holds in this case as well. Hence there exists a point \( q \in [g_1([k - 2]_k), g_1([k - 1]_k)] \) that satisfies \( (4.11) \) in this case as well. Define a map \( g_2 : \mathbb{Z}/(k + 1)\mathbb{Z} \to \mathbb{R}^2 \) by \( (4.12) \) again. Then the same argument as in Case 3 implies that \( g_2 \) is a comparison map of \( f \) in Case 4 as well.

**Case 5:** \( g \) satisfies neither (A) nor (B), and \( g_0(\mathbb{Z}/k\mathbb{Z}) \) is contained in a straight line. Let \( L_1 \) be the straight line such that \( g_0(\mathbb{Z}/k\mathbb{Z}) \subseteq L_1, \) and let \( \varphi : L_1 \to \mathbb{R} \) be an isometry. Because \( g_0 \) is a comparison map of \( f_0, \) for every \( x \in L_1, \) the number of elements \( i \in \mathbb{Z}/k\mathbb{Z} \) that satisfy \( x \in (g_0(i), g_0(i + [1]_k)) \) is at most two. Hence Lemma 4.2 implies that there exist \( i_0 \in \mathbb{Z}/k\mathbb{Z} \) and \( j_0 \in \mathbb{Z} \cap [1, k - 1] \) such that the functions \( m \mapsto \varphi(f(i_0 + [m]_k)) \) and \( m \mapsto \varphi(f(i_0 - [m]_k)) \) are nondecreasing on \( \mathbb{Z} \cap [0, j_0] \) and on \( \mathbb{Z} \cap [0, k - j_0], \) respectively. Here, we have \( \{i_0, i_0 + [j_0]_k\} \neq \{[k - 1]_k, [0]_k\} \) since otherwise \( g \) would clearly satisfy (A) or (B). Therefore, \( [k - 1]_k \notin \{i_0, i_0 + [j_0]_k\} \) or \( [0]_k \notin \{i_0, i_0 + [j_0]_k\}. \) We may assume without loss of generality that \( [k - 1]_k \notin \{i_0, i_0 + [j_0]_k\}. \) Then we have
\[ \varphi(g_0([k - 2]_k)) \leq \varphi(g_0([k - 1]_k)) \leq \varphi(g_0([0]_k)) \]
or
\[ \varphi(g_0([0]_k)) \leq \varphi(g_0([k - 1]_k)) \leq \varphi(g_0([k - 2]_k)). \]
It follows that
\[(4.23) \quad g_0([k-1]_k) \in [g_0([k-2]_k), g_0([0]_k)].\]
Define \( R, f_1 : \mathbb{Z}/k\mathbb{Z} \to R \) and \( g_1 : \mathbb{Z}/k\mathbb{Z} \to \mathbb{R}^2 \) as in Case 3. Then it clearly follows from (4.23) and the definition of \( R \) that \((4.10)\) holds in this case as well. Hence there exists a point \( q \in [g_1([k-2]_k), g_1([k-1]_k)] \) that satisfies \((4.11)\) in this case as well. Define a map \( g_2 : \mathbb{Z}/(k+1)\mathbb{Z} \to \mathbb{R}^2 \) by \((4.12)\) again. Then the same argument as in Case 3 implies that \( g_2 \) is a comparison map of \( f \) in Case 5 as well.

The above five cases exhaust all possibilities. \( \square \)

Theorem 4.5 implies in particular that the validity of the \( \bar{\Xi} \)-inequalities implies the strongly \( \text{Cycl}_4(0) \) condition, which gives a proof of Theorem 1.6 as follows.

**Proof of Theorem 1.6.** If a metric space \( X \) satisfies the \( \bar{\Xi} \)-inequalities, then \( X \) satisfies the strongly \( \text{Cycl}_4(0) \) condition by Theorem 4.5 and therefore \( X \) satisfies the \( \text{Cycl}_4(0) \) condition because the strongly \( \text{Cycl}_4(0) \) condition implies the \( \text{Cycl}_4(0) \) condition.

For the converse, assume that \( X \) satisfies the \( \text{Cycl}_4(0) \) condition. Fix \( x, y, z, w \in X \). Since \( X \) satisfies the \( \text{Cycl}_4(0) \) condition, there exist points \( x', y', z', w' \in \mathbb{R}^2 \) such that
\[
\|x' - y'\| \geq d_X(x, y), \quad \|y' - z'\| \geq d_X(y, z), \quad \|z' - w'\| \geq d_X(z, w),
\|
\|w' - x'\| \geq d_X(w, x), \quad \|x' - z'\| \leq d_X(x, z), \quad \|y' - w'\| \leq d_X(y, w).
\]

Then for any \( s, t \in [0, 1] \), we have
\[
(1-t)(1-s)d_X(x, y)^2 + t(1-s)d_X(y, z)^2 + tsd_X(z, w)^2 + (1-t)sd_X(w, x)^2
\]
\[
- t(1-t)d_X(x, z)^2 + s(1-s)d(y, w)^2
\]
\[
\geq (1-t)(1-s)\|x' - y'\|^2 + t(1-s)\|y' - z'\|^2 + ts\|z' - w'\|^2 + (1-t)s\|w' - x'\|^2
\]
\[
- t(1-t)\|x' - z'\|^2 + s(1-s)\|y' - w'\|^2
\]
\[
= \|(1-t)x' + tz') - ((1-s)y' + sw')\|^2 \geq 0,
\]
which proves that \( X \) satisfies the \( \bar{\Xi} \)-inequalities. \( \square \)

Proposition 1.7 follows immediately from Theorem 4.5.

**Proof of Proposition 1.7.** Suppose \( X \) is a metric space that satisfies the \( \bar{\Xi} \)-inequalities. Then Theorem 4.5 implies that \( X \) satisfies the strongly \( \text{Cycl}_k(0) \) condition for every integer \( k \geq 3 \), which clearly implies that \( X \) satisfies the \( \text{Cycl}_k(0) \) condition for every integer \( k \geq 4 \). \( \square \)

Theorem 1.4 follows immediately from Theorem 1.6 and Proposition 1.7.

**Proof of Theorem 1.4.** Suppose \( X \) is a metric space that satisfies the \( \text{Cycl}_4(0) \) condition. Then \( X \) satisfies the \( \bar{\Xi} \)-inequalities by Theorem 1.6. Therefore, Proposition 1.7 implies that \( X \) satisfies the \( \text{Cycl}_k(0) \) condition for every integer \( k \geq 4 \). \( \square \)

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E-mail address, Tetsu Toyoda: toyoda@cc.kogakuin.ac.jp

(Tetsu Toyoda)
KOGAKUIN UNIVERSITY,
2665-1, NAKANO, HACHIOJI, TOKYO, 192-0015 JAPAN