Gravitational Radiation from Cylindrical Naked Singularity

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Abstract

We construct an approximate solution which describes the gravitational emission from a naked singularity formed by the gravitational collapse of a cylindrical thick shell composed of dust. The assumed situation is that the collapsing speed of the dust is very large. In this situation, the metric variables are obtained approximately by a kind of linear perturbation analysis in the background Morgan solution which describes the motion of cylindrical null dust. The most important problem in this study is what boundary conditions for metric and matter variables should be imposed at the naked singularity. We find a boundary condition that all the metric and matter variables are everywhere finite at least up to the first order approximation. This implies that the spacetime singularity formed by this high-speed dust collapse is very similar to that formed by the null dust and thus the gravitational emission from a naked singularity formed by the cylindrical dust collapse can be gentle.

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I. INTRODUCTION

The cosmic censorship hypothesis is a crucial ansatz for the theorems of black holes. There are two versions for this hypothesis. For spacetimes with physically reasonable matter fields, the weak version claims that a spacetime singularity as a result of generic non-singular initial data is not visible from infinity, while the strong version claims that a spacetime singularity developed from non-singular initial data is invisible for any observer. A singularity censored by the strong version is called a naked singularity, while a singularity censored by the weak version is called a globally naked singularity. However, previous theoretical studies have revealed several candidates for the counter-examples of this hypothesis. Although further detailed and careful studies about these candidates are necessary, these might have some physical importance.

If a globally naked singularity forms, what can we detect from there? In connection with this issue, Nakamura, Shibata and one of the present authors (KN) have proposed a conjecture; if globally naked singularities form, large spacetime curvatures in the neighborhood of these can propagate away to infinity in the form of gravitational radiation, and as a result, almost all of the mass of these naked singularities is lost through this large gravitational emission. If this conjecture is true, the formation processes of the globally naked singularities might be very important for gravitational-wave astronomy. The cylindrically symmetric system will play an important role in getting significant information about this issue by the following reasons; in the asymptotically flat case, the singularities are necessarily naked; there is a degree of gravitational radiation; this system is very simple. There are a few numerical studies on gravitational emissions by cylindrically symmetric gravitational collapse.

Recently, the present authors have investigated the high-speed collapse of cylindrically symmetric thick shell composed of dust and perfect fluid with non-vanishing pressure. In these studies, we use the high-speed approximation which is a kind of the perturbation analysis in the background Morgan solution describing the motion of a cylindrical null dust. In the first paper, it has been revealed that in the gravitational collapse of a cylindrically symmetric thick dust shell, the thinner shell leads to the larger amount of gravitational radiation. This result gives a resolution of an apparent inconsistency between the previous works. In the second paper, we showed that the pressure decelerates the collapsing velocity.
so significantly that the high-speed approximation scheme breaks down before a singularity forms if the equation of state is moderately hard. For example, in the case of mono-atomic ideal gas which seems to be physically reasonable in the high-energy state due to the asymptotic freedom of elementary interactions, the high-speed collapse is prevented by the pressure and thus there is a possibility that the pressure halts the formation of a spacetime singularity in the case of this cylindrical gravitational collapse. However, we should note that if initial collapsing velocity is very large, a region with very large mass concentration can be realized in the neighborhood of the symmetric axis $r = 0$. In this region, the tidal force for free falling observers can be so large that general relativity breaks down and the quantum theory of gravity will be necessary to understand the physical processes realized there. Such a region can be regarded as a singularity for general relativity, called a ‘spacetime border’ [17].

By contrast, in case that the equation of state is sufficiently soft, the high-speed collapse is maintained until a globally naked singularity forms.

In this paper, we investigate the gravitational waves from a naked singularity formed by the high-speed gravitational collapse of a cylindrical thick shell composed of dust. In the previous paper [15], we have investigated the generation process of those in the causal past of the Cauchy horizon associated with the naked singularity, i.e., the region which does not suffer the influence of the naked singularity (for details of the Cauchy horizon, causal future and causal past, see, for example, the textbook by Wald [18]). On the other hand, in this paper, we focus on the generation process of the gravitational waves in the causal future of the naked singularity. In order to know what happens in the causal future of the naked singularity, we need specify the boundary condition at the naked singularity. This is equivalent to defining the naked singularity as a physical entity. We would like to show that in the high-speed approximation scheme, there is a boundary condition that all the metric and matter variables are everywhere finite at least up to the first order. There are several studies on how to fix the boundary condition for test fields or gravitational perturbations in static naked singular spacetimes [19, 20, 21, 22, 23]. By contrast, the present case is dynamical, and as far as we are aware, this is the first example of gravitational emissions from a naked singularity developed from non-singular initial data.

This paper is organized as follows. In Sec.II, we present the basic equations of the cylindrically symmetric dust system. In Sec.III, we derive the basic equations in the high-speed approximation scheme which describe the gravitational collapse with velocity almost equal to...
to the speed of light \( c \). In Sec.IV, we consider the boundary condition at the background naked singularity and then present solutions for the basic equations given in Sec.III. In Sec.V, we study the gravitational emissions from the naked singularity. Finally Sec.VI is devoted to summary and discussion.

In this paper, we adopt \( c = 1 \) unit and basically follow the convention of the Riemann and metric tensors and the notation in the textbook by Wald [18].

**II. CYLINDRICALLY SYMMETRIC DUST SYSTEM**

In this paper, we focus on the spacetime with whole-cylinder symmetry which is defined by the following line element [21],

\[
\begin{aligned}
    ds^2 &= e^{2(\gamma - \psi)} \left(-dt^2 + dr^2\right) + e^{-2\psi} R^2 d\phi^2 + e^{2\psi} dz^2.
\end{aligned}
\] (2.1)

Then Einstein equations are

\[
\begin{aligned}
    \gamma' &= \left(R'^2 - \dot{R}^2\right)^{-1} \left\{ RR' \left(\dot{\psi}^2 + \dot{\psi}'^2\right) - 2R\dot{R}\psi\dot{\psi}' + R'R'' - \dot{R}\dot{R}' \right. \\
    & \quad \left. - 8\pi G \sqrt{-g} \left(R'T'_t + \dot{R}T'_t\right) \right\}, \\
    \dot{\gamma} &= - \left(R'^2 - \dot{R}^2\right)^{-1} \left\{ R\dot{R} \left(\dot{\psi}^2 + \dot{\psi}'^2\right) - 2RR'\dot{\psi}\dot{\psi}' + \dot{R}R'' - R'\dot{R}' \right. \\
    & \quad \left. - 8\pi G \sqrt{-g} \left(\dot{R}T'_t + R'T'_t\right) \right\}, \\
    \ddot{\gamma} - \gamma'' &= \dot{\psi}^2 - \dot{\psi}'^2 - \frac{8\pi G}{R} \sqrt{-g} T^\phi_\phi, \\
    \ddot{\dot{\gamma}} - R'' &= -8\pi G \sqrt{-g} \left(T'_t + T'_r\right), \\
    \ddot{\psi} + \frac{\dot{R}}{R} \dot{\psi} - \psi'' - \frac{R'}{R} \psi' &= -\frac{4\pi G}{R} \sqrt{-g} \left(T'_t + T'_r - T^z_z + T^\phi_\phi\right),
\end{aligned}
\] (2.2) (2.3) (2.4) (2.5) (2.6)

where a dot means the derivative with respect to \( t \) while a prime means the derivative with respect to \( r \).

As mentioned, we consider the dust fluid as a matter field. The stress-energy tensor is

\[
T_{\mu\nu} = \rho u_\mu u_\nu,
\] (2.7)

where \( \rho \) is rest mass density and \( u^\mu \) is a 4-velocity of the fluid element. Due to the assumption of the whole-cylinder symmetry, the components of the 4-velocity \( u^\mu \) are written as

\[
u^\mu = u^t \left(1, -1 + V, 0, 0\right).
\] (2.8)
By the normalization $u^\mu u_\mu = -1$, $u^t$ is expressed as
$$u^t = \frac{e^{-\gamma + \psi}}{\sqrt{V(2 - V)}}. \quad (2.9)$$

Here denoting the determinant of the metric tensor $g_{\mu \nu}$ by $g$, we introduce new density variable $D$ defined by
$$D := \frac{\sqrt{-g}u^t}{\sqrt{V(2 - V)}} = \frac{Re^{\gamma - \psi} \rho}{V(2 - V)}. \quad (2.10)$$

The components of the stress-energy tensor are then expressed as
$$\sqrt{-g}T^t_t = -e^{\gamma - \psi} D, \quad (2.11)$$
$$\sqrt{-g}T^r_t = e^{\gamma - \psi} D(1 - V) = -\sqrt{-g}T^t_r, \quad (2.12)$$
$$\sqrt{-g}T^r_r = e^{\gamma - \psi} (1 - V)^2 D, \quad (2.13)$$

and the other components vanish.

The equation of motion $\nabla^\alpha T^\alpha_\beta = 0$ leads to
$$\partial_u D = -\frac{1}{2} (DV)' + \frac{D}{2} (1 - V) \left\{ 2\partial_u(\psi - \gamma) - V(\psi - \gamma) \right\}, \quad (2.14)$$
$$D\partial_u V = (1 - V)\partial_u D + \frac{1}{2} \left\{ V(1 - V)D \right\}' - \frac{D}{2} \left\{ 2\partial_u(\psi - \gamma) - V(\psi - \gamma) \right\}, \quad (2.15)$$

where $u = t - r$ is the retarded time and $\partial_u$ is the partial derivative of $u$ with fixed advanced time $v = t + r$. The first equation comes from the $t$-component while the second one comes from the $r$-component. The $z$- and $\varphi$-components are trivial.

To estimate the energy flux of the gravitational radiation, we investigate $C$-energy $E$ and its flux vector proposed by Thorne[24]. $C$-energy $E = E(t, r)$ is the energy per unit coordinate length along $z$-direction within the radius $r$ at time $t$, which is defined by
$$E = \frac{1}{8} \left\{ 1 + e^{-2\gamma} \left( \dot{R}^2 - R'^2 \right) \right\}. \quad (2.16)$$

The energy flux vector $J^\mu$ associated to the $C$-energy is defined by
$$\sqrt{-g}J^\mu = \left( \frac{\partial E}{\partial r}, -\frac{\partial E}{\partial t}, 0, 0 \right). \quad (2.17)$$

By its definition, $J^\mu$ is divergence free. Using the equations of motion for the metric variables, we obtain the following expression of $C$-energy flux vector,
$$\sqrt{-g}J^t = \frac{e^{-2\gamma}}{8\pi G} \left\{ RR' (\dot{\psi}^2 + \psi'^2) - 2RR\dot{\psi} \dot{\psi}' - 8\pi G \sqrt{-g} (R T^t_t + \dot{R} T^t_t) \right\}, \quad (2.18)$$
$$\sqrt{-g}J^r = \frac{e^{-2\gamma}}{8\pi G} \left\{ R\dot{R} (\dot{\psi}^2 + \psi'^2) - 2R R'\dot{\psi} \dot{\psi}' - 8\pi G \sqrt{-g} (R T^r_r - \dot{R} T^r_r) \right\}. \quad (2.19)$$
and the other components vanish.

III. HIGH SPEED APPROXIMATION SCHEME

Let us consider the ingoing null limit of the cylindrical dust fluid. From Eqs. (2.11)-(2.13), the stress-energy tensor is written in the following form,

\[ T_{\mu\nu} = \frac{e^{3(\psi-\gamma)}}{D} k_\mu k_\nu, \]  

(3.1)

where

\[ k^\mu = (1, -1 + V, 0, 0). \]  

(3.2)

The timelike vector \( k^\mu \) becomes the ingoing null vector in the limit of \( V \to 0^+ \). Hence in the limit of \( V \to 0^+ \) with \( D \) fixed, the stress-energy tensor agrees with that of the collapsing null dust. This means that in the case of very large collapsing velocity, i.e., \( 0 < V \ll 1 \), the dust fluid system will be well approximated by the null dust system. Then we treat the “deviation \( V \) of the 4-velocity from null” as a perturbation and perform linear perturbation analyses.

In the case of the complete null dust \( V = 0 \), the solution is easily obtained as

\[ \psi = 0, \]  

(3.3)

\[ \gamma = \gamma_B(v), \]  

(3.4)

\[ R = r, \]  

(3.5)

\[ 8\pi GDe^{\gamma} = \frac{d\gamma_B}{dv}, \]  

(3.6)

where \( \gamma_B(v) \) is an arbitrary function of the advanced time \( v \). This solution was first obtained by Morgan and was studied subsequently by Letelier and Wang and Nolan in detail. Assuming the null energy condition, the density variable \( D \) is non-negative. This means that \( \gamma_B \) is non-decreasing function of \( v \).

The situation of the collapsing null dust solution can be understood by Fig.1. In this paper, we assume that the density variable \( D \) has a compact support \( 0 < v < v_w \) which is depicted by a shaded region in Fig.1. We can find in Eqs. (3.1) and (3.5) that if \( D \) does not vanish at the symmetric axis \( r = 0 \), components of the stress-energy tensor \( T_{\mu\nu} \) with respect to the coordinate basis diverge there, and the same is true for the Ricci tensor by
FIG. 1: Morgan’s cylindrical null dust solution. There is the null dust in the shaded region; $D(v) > 0$ for $0 < v < v_w$. The dashed line on $t$-axis corresponds to the intermediate singularity at which an observer suffers infinite tidal force although any scalar polynomials of the Riemann tensor do not vanish. On the other hand, the dotted line on $t$-axis is the conical singularity. The Cauchy horizon is short dashed line of $t = r$. We estimate $C$-energy transferred through the null hypersurface $v = v_w$ from $u = 0$ to $u = v_w$, which is depicted by dot-dashed line.

Einstein equations. This is a naked singularity which is depicted by the dashed line at $r = 0$ in $0 < t < v_w$ in Fig.1. Although all the scalar polynomials of Riemann tensor vanish there, freely falling observers suffer infinite tidal force at $r = 0$ in $0 < t < v_w$; this type of singularity is called p.p. curvature singularity\[28\]. The Cauchy horizon associated with this intermediate naked singularity is represented by the short dashed line $t = r$ in Fig.1. On the other hand, the region at $r = 0$ in $t \geq v_w$ is conical singularity which is depicted by the thick dotted line in this figure.

We introduce a small parameter $\epsilon$ and assume the order of the variables as $V = O(\epsilon)$ and $\psi = O(\epsilon)$. Further we rewrite the variables $\gamma, R$ and $D$ as

$$e^\gamma = e^{\gamma_0}(1 + \delta_\gamma), \quad (3.7)$$
$$R = r(1 + \delta_R), \quad (3.8)$$
$$D = D_B(1 + \delta_D), \quad (3.9)$$
and assume that $\delta_\gamma$, $\delta_R$ and $\delta_D$ are $O(\epsilon)$, where
\[ D_B := \frac{1}{8\pi G e^{\gamma_B}} \frac{d\gamma_B}{dv}. \] (3.10)

We call the perturbative analysis with respect to this small parameter $\epsilon$ the high speed approximation scheme.

The 1st order equations with respect to $\epsilon$ are given as follows; the Einstein equations (2.2)-(2.6) lead to
\[ \delta_\gamma' = 8\pi G D_B e^{\gamma_B} \{ \delta_\gamma - \psi + \delta_D - 2\partial_v (r\delta_R) \} + (r\delta_R)'', \] (3.11)
\[ \dot{\delta}_\gamma = 8\pi G D_B e^{\gamma_B} \{ \delta_\gamma - \psi + \delta_D - 2\partial_v (r\delta_R) - V \} + (r\delta_R)', \] (3.12)
\[ \ddot{\delta}_\gamma - \delta_\gamma'' = 0, \] (3.13)
\[ r\ddot{\delta}_R - (r\delta_R)''' = 16\pi G e^{\gamma_B} D_B V, \] (3.14)
\[ \ddot{\psi} - \psi'' - \frac{1}{r}\psi' = \frac{8\pi G}{r} e^{\gamma_B} D_B V; \] (3.15)

the conservation law (2.14) leads to
\[ \partial_u (\delta_D + \delta_\gamma - \psi) = -\frac{1}{2D_B} \frac{dD_B}{dv} V - \frac{1}{2} \left( V' - \frac{d\gamma_B}{dv} V \right); \] (3.16)

the Euler equation (2.15) becomes as
\[ \partial_u V = 0, \] (3.17)

where we have used Eq.(3.16).

$C$-energy $E$ up to the first order of $\epsilon$ is given by
\[ E = \frac{1}{8} \left[ 1 - e^{-2\gamma_B} + 2e^{-2\gamma_B} \{ \delta_\gamma - (r\delta_R)' \} \right]. \] (3.18)

From Eq.(3.6), we can easily see that $\gamma_B$ is constant in the vacuum region $D_B = 0$. Further, from Eqs.(3.11) and (3.12), we find that $\delta_\gamma - (r\delta_R)'$ is also constant in the vacuum region. Therefore up to the first order, $C$-energy is constant in the vacuum region. This means that in the vacuum region, $C$-energy flux vector $J^\mu$ vanishes up to the first order and thus it is a second order quantity which is given by
\[ \sqrt{-g} J^t = \frac{r}{8\pi G} \left( \dot{\psi}^2 + \psi'^2 \right), \] (3.19)
\[ \sqrt{-g} J^r = -\frac{r}{4\pi G} \dot{\psi} \psi'. \] (3.20)

$C$-energy flux vector takes very similar form to that of massless Klein-Gordon field.
IV. SOLUTIONS FOR DUST COLLAPSE

In this section, we study the behaviors of perturbation variables in the causal future of the background naked singularity. Einstein equations are hyperbolic differential equations which send out the information of the background naked singularity into its causal future. Therefore in order to get the solutions of the basic equations for the perturbation variables in the causal future of the background naked singularity, we need specify the boundary condition for these at the background naked singularity.

In general, it is very difficult to control the solutions by the boundary conditions due to the nonlinearity of Einstein equations. However, fortunately, there is a great advantage in the present situation; the perturbation variables can be regarded as test fields with sources in the fixed background Morgan spacetime. This seems to be similar situation to the problems investigated by Wald and a several researchers, i.e., the behavior of test fields in naked singular static spacetimes [19, 20, 21, 22, 23]. However, there is a significant difference between the previous studies and the present case. In the previous cases, the effect of the naked singularity appears in the differential operator in the equations of motion for the test fields. On the other hand, in the present case, the manifestation of the naked singularity appears as source terms in the differential equations for the perturbation variables. Therefore we have to determine the boundary condition from different point of view from previous works.

We do not know the law of physics at the spacetime singularity, i.e., the quantum theory of gravity and hence, at present, it is difficult to impose a physically satisfactory boundary condition at the naked singularity. There is a possibility that the quantum gravity puts the additional source terms on the right hand side of the differential equations (3.11)-(3.15) at the background naked singularity; the additional source term might be a delta-function like one causing gravitational wave bursts, or by contrast, might completely cancel out the original source term and, as a result, lead no gravitational radiation from the naked singularity. However, at first glance, it is expected that the geometrical structure of the spacetime investigated here is not so different from the background Morgan spacetime even in the neighborhood of the background naked singularity. Thus in order to search for such a solution, we require that the perturbation variables are expressed in the form of the Maclaurin series around the naked singularity. As will be shown below, this condition
guarantees that all the perturbation variables are everywhere finite up to the first order, and this is just what we expect. The cylindrical thick shell composed of dust also forms a naked singularity at the same coordinate position as the background naked singularity. The Cauchy horizon associated with this naked singularity is located also at the same coordinate position as the background one.

A. Solution for $\delta_\gamma$

Equation (3.13) is easily solved. In this equation, there is no singularity even at the background naked singularity. Before the appearance of the background naked singularity, i.e., $t \leq 0$, we impose a regularity condition to guarantee the locally Minkowskian nature at $r = 0$; $\delta_\gamma$ is $C^2$ function with $\delta_\gamma'_{|r=0} = 0$. We also impose the same condition at $r = 0$ after the background naked singularity formation $t > 0$. Then we obtain general solutions for $\delta_\gamma$

$$
\delta_\gamma = \Delta_\gamma(t-r) + \Delta_\gamma(t+r),
$$

where $\Delta_\gamma$ is an arbitrary $C^2$ function. Here we should note that after the appearance of the naked singularity $t \geq 0$, the regularity condition $\delta_\gamma'_{|r=0} = 0$ does no longer guarantee the locally Minkowskian nature since the background spacetime itself does not have the locally Minkowskian nature at the naked singularity.

B. Solution for $V$

The solution for the Euler equation (3.17) is given by

$$
V = C_V(v),
$$

where $C_V$ is an arbitrary function which is everywhere finite. Therefore even at the background naked singularity, the velocity perturbation $V$ does not vanish. This fact means that the type of the singularity formed at the symmetric center $r = 0$ is changed from that of the background Morgan spacetime. Up to the first order of $\epsilon$, the Ricci scalar is obtained as

$$
R^\mu_{\mu} = -8\pi G T^\mu_{\mu} = \frac{16\pi Ge^{-\gamma B} D_B V}{r}.
$$

We can easily find in the above equation that the Ricci scalar diverges at the symmetric center $r = 0$ if $D_B$ and $V$ do not vanish there. Thus the background naked p.p. curvature
singularity becomes the s.p. curvature singularity which is defined as the spacetime singularity accompanied by the divergence of a scalar polynomial constructed from the Riemann tensor\textsuperscript{28}.

C. Solution for $\delta_R$

Using the solution (4.2), we can get the solution for $\delta_R$. For notational simplicity, we introduce a function $S$ defined by

$$S(v) = 8\pi G e^{\gamma_B} D_B C_V.$$  \hspace{1cm} (4.4)

Note that two times $S(v)$ is the source term in the equation (3.14) for $\delta_R$.

As mentioned, we search for the solution which can be expressed by Maclaurin series in the neighborhood of the background naked singularity as

$$\delta_R = \delta_R^{(0)}(t) + \delta_R^{(1)}(t) r + \delta_R^{(2)}(t) r^2 + \delta_R^{(3)}(t) r^3 ... $$  \hspace{1cm} (4.5)

The source function $S$ is also assumed to be written by Maclaurin series as

$$S(v) = S^{(0)}(t) + S^{(1)}(t) r + S^{(2)}(t) r^2 + .... $$  \hspace{1cm} (4.6)

Substituting the above expressions into Eq. (3.14), we obtain the following equations

$$-2\delta_R^{(1)} = 2S^{(0)}, \hspace{1cm} \ddot{\delta}_R^{(0)} - 6\delta_R^{(2)} = 2S^{(1)}, \hspace{1cm} \ddot{\delta}_R^{(1)} - 12\delta_R^{(3)} = 2S^{(1)}, ... $$  \hspace{1cm} (4.7)

Thus the boundary condition at $r = 0$ is given by

$$\delta_R'|_{r=0} = -S(t). $$  \hspace{1cm} (4.8)

Since the expansion coefficients $S^{(i)}(t)$ of the source function are finite, the coefficients $\delta_R^{(i)}$ are also finite even in the neighborhood of the naked singularity.

The solution for $\delta_R$ with the boundary condition (4.8) is easily obtained by constructing a Green function. Since $r\delta_R$ should vanish at $r = 0$, the Green function $G_R$ for $r\delta_R$ should also vanish there. Therefore $G_R$ is given by

$$G_R(t, r; \tau, x) = \frac{1}{\pi} \int_0^\infty \frac{dk}{k} \sin(kr) \sin(kx) \sin\{k(t-\tau)\}. $$  \hspace{1cm} (4.9)
Thus the solution for $\delta_R$ is given by

$$
\delta_R(t, r) = \frac{2}{\pi r} \int_{t_i}^t d\tau \int_0^\infty dx G_R(t, r; \tau, x) S(\tau + x)
+ \frac{1}{r} \left\{ \Delta_R(t + r) - \Delta_R(t - r) \right\},
$$

(4.10)

where $t_i$ is some constant, $\Delta_R$ is an arbitrary $C^2$ function.

**D. Solution for $\psi$**

Before the appearance of the background naked singularity $t \leq 0$, we impose the regularity condition on $\psi$ on the symmetric axis $r = 0$; $\psi$ is $C^2$ function satisfying $\psi'\rvert_{r=0} = 0$. However, at the naked singularity $0 < t < v_w$, this condition is not satisfied because the right hand side of Eq.(3.15) diverges at $r = 0$ if $S\rvert_{r=0} \neq 0$. Even in such a situation, we can find solutions everywhere finite.

Here we assume that $\psi$ can be written in the form of the Maclaurin series around $r = 0$,

$$
\psi = \psi^{(0)}(t) + \psi^{(1)}(t)r + \psi^{(2)}(t)r^2 + \ldots
$$

(4.11)

Substituting this expression and Eq.(4.6) into Eq.(3.15), we find

$$
-\psi^{(1)} = S^{(0)}, \quad \ddot{\psi}^{(0)} - 4\psi^{(2)} = S^{(1)}, \quad \ddot{\psi}^{(1)} - 9\psi^{(3)} = S^{(2)}, \ldots
$$

(4.12)

Thus the boundary condition at the background naked singularity is given by

$$
\psi'\rvert_{r=0} = -S(t).
$$

(4.13)

The above condition guarantees that the solution for $\psi$ is everywhere finite. To get the numerical values of $\psi$, we need numerical calculations. We will do it in Sec.V.

Here we should note that the background conical singularity at $r = 0$ and $t \geq v_w$ remains conical one up to the first order. This is because the perturbation variables $\delta_R$, $\delta_\gamma$ and $\psi$ behave as those of the regular vacuum spacetime at the background conical singularity by the present boundary condition.

**E. Solution for $\delta_D$**

It is easy to integrate Eq.(3.16). Using Eqs.(4.2) and (4.4), the solution is obtained as

$$
\delta_D = \psi - \delta_\gamma + \frac{u}{2} \left( 2S - C_V \frac{d\ln S}{dv} \right) + C_D(v),
$$

(4.14)
where $C_D$ is an arbitrary function which is everywhere finite.

V. GRAVITATIONAL RADIATION FROM NAKED SINGULARITY

In this section, we study the energy carried by gravitational radiation emitted from the naked singularity. Since the energy flux in the vacuum region is given by Eq. (3.20), we numerically calculate $\psi$ in the causal future of the background naked singularity. In order to do so, we adopt the retarded and advanced coordinates $u$ and $v$ instead of $t$ and $r$. Then Eq. (3.15) is rewritten in the form

$$\frac{\partial \psi}{\partial u} = D_u(u, v), \quad (5.1)$$

$$\frac{\partial D_v(u, v)}{\partial u} = \frac{\partial D_u(u, v)}{\partial v} = \frac{1}{2(v-u)} \left\{ D_v(u, v) - D_u(u, v) + S(v) \right\}, \quad (5.2)$$

We consider the one-parameter family of the background null dust solutions. We adopt the thickness $v_w$ of the shell as the parameter and set all the members in this family having the same total $C$-energy $E_B$. Since the total $C$-energy $E_B$ in the background spacetime is given by

$$E_B = \frac{1}{8} \left( 1 - e^{-2\gamma} \right), \quad (5.3)$$

where $\gamma_\infty$ is the asymptotic value of $\gamma_B$ for $v \to \infty$. Thus independence of $C$-energy $E_B$ on the thickness $v_w$ means that $\gamma_\infty$ does not depend on $v_w$. We can see in the above equation that $E_B$ is less than $1/8$. If $E_B$ is equal to $1/8$, $\gamma_\infty$ is infinite, and then the space is closing up in the $r$-direction.

For notational convenience, we introduce the dimensionless null coordinates defined by

$$x = \frac{u}{v_w} \quad \text{and} \quad y = \frac{v}{v_w}. \quad (5.4)$$

Then the asymptotic value $\gamma_\infty$ is written as

$$\gamma_\infty = \int_0^{v_w} dv \frac{d\gamma_B}{dv} = \int_0^1 dy v_w \frac{d\gamma_B}{dv}. \quad (5.5)$$

Since $\gamma_\infty$ does not depend on the thickness $v_w$ of the dust shell, $d\gamma_B/dv$ should be written in the following form,

$$\frac{d\gamma_B}{dv} = \frac{1}{v_w} F_\gamma(y). \quad (5.6)$$
We set the velocity perturbation $V$ being

$$V = C_V(v) = F_V(y), \quad (5.7)$$

so that the maximal value of $V$ does not depend on the thickness $v_w$. Then we find

$$S(v) = V \frac{d\gamma_{NB}}{dv} = \frac{1}{v_w} F_{\gamma} F_V =: \frac{1}{v_w} F_S(y). \quad (5.8)$$

We introduce new dimensionless variables defined by

$$D_x(x,y) = v_w D_u(u,v) \quad \text{and} \quad D_y(x,y) = v_w D_v(u,v). \quad (5.9)$$

Then Eqs. (5.1) and (5.2) take the following forms,

$$\frac{\partial \psi}{\partial x} = D_x, \quad (5.10)$$

$$\frac{\partial D_y}{\partial x} = \frac{\partial D_x}{\partial y} = \frac{1}{2(y-x)} (D_y - D_x + F_S), \quad (5.11)$$

Since $F_S$ is the function of $y$, the thickness $v_w$ does disappear in the above equation. This means that $\psi$, $D_x$ and $D_y$ are the functions of $x$ and $y$ but do not depend on the thickness $v_w$ as long as we use the variables $x$ and $y$ as independent variables.

### A. C-energy

We calculate $C$-energy carried by the gravitational radiation from the naked s.p. curvature singularity formed at $r = 0$. The background singularity in $t \geq v_w$ is a conical singularity at which the source function $S$ vanishes. This means that $\psi$ is not generated there. Thus we focus on the gravitational radiation generated at the intermediate naked singularity $0 < t < v_w$. For this purpose, it is sufficient to investigate $C$-energy flux through the null hypersurface of $v = v_w$ from $u = 0$ to $u = v_w$ (see Fig.1). Using Eq. (3.20), $C$-energy $\Delta E_{NS}$ transferred from the intermediate background p.p. curvature singularity through this null hypersurface is given as

$$\Delta E_{NS} = \frac{1}{4G} \int_0^{v_w} du (v_w - u) D_u^2(u,v_w) = \frac{1}{4G} \int_0^1 dx (1-x) D_x^2(x,1). \quad (5.12)$$

From the above equation, we find that $\Delta E_{NS}$ does not depend on the thickness $v_w$ of the dust shell. This fact is different from $C$-energy carried by gravitational waves generated in the causal past of the Cauchy horizon (see also Appendix A).
From Eq.(5.5), we see that $\gamma_{\infty}$ is proportional to $F_{\gamma}$. Since the mean value $V_m$ of the velocity perturbation $V$ is given by

$$V_m = \frac{1}{v_w} \int_0^{v_w} dvV(v) = \int_0^1 dyF_V(y),$$

(5.13)

$V_m$ is proportional to $F_V$. This means that $D_x$ is proportional to $\gamma_{\infty}V_m$ since the source term $F_S$ is the product of $F_{\gamma}$ and $F_V$. Rewriting $\gamma_{\infty}$ by the background total $C$-energy as

$$\gamma_{\infty} = -\frac{1}{2} \ln(1 - 8E_B),$$

(5.14)

we obtain

$$\Delta E_{NS} = \text{Const} \times \{V_m \ln(1 - 8E_B)\}^2.$$  

(5.15)

Since the absolute value of the source term $|F_S|$ should be much smaller than unity so that the high-speed approximation is valid, $|V_m \ln(1 - 8E_B)|$ should be much less than unity. However at present, we can not say whether $\Delta E_{NS}$ is large compared to the total $C$-energy $E_B$ or not. We need focus on a specific model of our interest and then calculate ‘Const’ in Eq. (5.15).

B. Example

Here we consider the following example,

$$S(v) = -\frac{18}{v_w}V_m \ln(1 - 8E_B)y^2(1 - y)^2\Theta(y)\Theta(1 - y),$$

(5.16)

$$V(v) = 6V_m y(1 - y)\Theta(y)\Theta(1 - y),$$

(5.17)

where $\Theta$ is Heaviside’s step function. Note that the above $S$ and $V$ have been chosen so that $E_B$ and $V_m$ are the total $C$-energy of the background spacetime and the mean value of $V$, respectively. We solve Eqs.(5.10) and (5.11) numerically and then estimate Eq.(5.12).

We set the $C$-energy of the present system being the same as that of the background. By Eq.(5.18), this condition leads to the following condition on $u = u_i$,

$$\delta_{\gamma} - (r\delta_R)' = 0.$$  

(5.18)

Eqs.(3.11) and (3.12) lead to

$$\partial_v \{\delta_{\gamma} - (r\delta_R)\}' = 8\pi Ge^{\gamma_0}D_B \left\{\delta_{\gamma} - \psi + \delta_D - 2\partial_v(r\delta_R) - \frac{V}{2}\right\}.$$  

(5.19)
Therefore in the region of $D_B > 0$, the following equation should hold,

$$
\delta \gamma - \psi + \delta D - 2\partial_v (r \delta R) - \frac{V}{2} = 0.
$$

(5.20)

Here we adopt the following initial data so that Eq. (5.18) is satisfied,

$$
\psi = \delta \gamma = \delta R = 0 \quad \text{and} \quad \delta D = \frac{V}{2} \quad \text{on} \quad u = u_i.
$$

(5.21)

It is not so difficult to numerically solve the differential equations (5.10) and (5.11). We set the initial condition $\psi = 0$ and thus $D_y = 0$ at the Cauchy horizon $x = 0$. This initial condition guarantees that there is no ingoing flux across the Cauchy horizon and hence we can see the gravitational radiation generated just at the naked singularity. The result of the three-digit accuracy is obtained as

$$
\Delta E_{NS} = 0.0643 \times \{V_m \ln(1 - 8E_B)\}^2.
$$

(5.22)

Here for notational convenience, we introduce a new parameter defined by

$$
\varepsilon = -V_m \ln(1 - 8E_B).
$$

(5.23)

Since the source function $S$ is proportional to this parameter $\varepsilon$, $\varepsilon$ should be much less than unity so that the high speed approximation is applicable. $E_B$ is expressed by $\varepsilon$ and $V_m$ as $E_B = (1 - e^{-\varepsilon/V_m})/8$ and thus we obtain

$$
\frac{\Delta E_{NS}}{E_B} = 0.514 \times \frac{\varepsilon^2}{1 - e^{-\varepsilon/V_m}}.
$$

(5.24)

We can easily see that the above quantity is monotonically increasing function of both $\varepsilon$ and $V_m$ for $\varepsilon > 0$ and $V_m > 0$. In the parameter space $(\varepsilon, V_m)$, $\varepsilon = V_m = 1$ is the marginal point at which the high speed approximation must break down. The upper bound on $\Delta E_{NS}/E_B$ for the high-speed collapse is realized at this point,

$$
\max \left( \frac{\Delta E_{NS}}{E_B} \right) = 0.813.
$$

(5.25)

The above value is rather large. The large amount of the gravitational radiation seems to be emitted if the collapsing speed $1 - V_m$ is not so large and the background $C$-energy $E_B$ is not so small. On the other hand, in the case of very small $\varepsilon$, the emitted energy is

$$
\frac{\Delta E_{NS}}{E_B} \approx 0.514 \times \varepsilon V_m.
$$

(5.26)

Thus the emitted energy will be at most a few percents of the total energy of this system in the situation that the present approximation scheme describes the system with sufficient accuracy.
VI. SUMMARY AND DISCUSSION

In this paper, we studied the gravitational radiation from the naked singularity formed by high-speed gravitational collapse of a cylindrical thick shell composed of dust. In this issue, it is very important what boundary condition for metric and matter variables should be imposed at the background naked singularity. Here we have adopted the boundary condition that all the perturbation variables are expressed in the form of the Maclaurin series at the background naked singularity. Then we have found a boundary condition that guarantees all the metric and matter variables are finite everywhere at least up to the first order approximation. At first glance, it is expected that the spacetime of the high-speed dust collapse is not so different from the background Morgan spacetime even in the neighborhood of the background naked singularity. The boundary condition adopted in the present paper realize what we expected. Further this implies that the high-speed approximation scheme is everywhere valid.

We investigated $C$-energy sent out by gravitational emission from the naked singularity. In the situation that the present approximation scheme is valid, every perturbation variable is small and as a result, the emitted energy of gravitational radiation is small. We assumed a simple model and found that the upper bound of the emitted energy by the high-speed collapse is almost the same as the total $C$-energy of the system. However, in the case that the high-speed approximation describes the system with sufficient accuracy, the emitted energy might be at most a few percents of the total $C$-energy of the system. Thus the emission of gravitational waves from cylindrical naked singularity can be gentle process. This result seems to imply that the conjecture by Nakamura, Shibata and KN is not necessarily valid. However the dust might be unrealistic assumption and hence further investigation is necessary. On the other hand, the rather large value of the upper bound on the energy of gravitational radiation implies that there remains possibility of the large gravitational radiation by the gravitational collapse of a cylindrical thick dust shell with not so large collapsing velocity. This is also a future problem.
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APPENDIX A

Here we consider the energy sent out by the gravitational radiation generated in the causal past of the Cauchy horizon \( u = 0 \). The emitted energy \( \Delta E_R \) from \( u = u_i(< 0) \) to \( u = 0 \) is given by

\[
\Delta E_R(v) = \frac{1}{4G} \int_{u_i}^{0} du (v-u) D_u^2(u, v) = \frac{1}{4G} \int_{x_i}^{0} dx (y-x) D_x^2(x, y),
\]

where \( x_i = u_i/v_w \) and \( y = v/v_w \) is larger than unity so that the estimation of \( \Delta E_R \) is done outside of the shell. In contrast to the case of the gravitational waves from the naked singularity in Sec.V, the emitted energy \( \Delta E_R \) depends on the thickness \( v_w \) of the shell. In the thin shell limit, \( v_w \to 0 \), the region of integration in Eq.(A1) becomes \((-\infty, 0]\) since \( x_i \to -\infty \). Thus if \( D_x(x, y) \) does not vanish sufficiently rapidly in the limit of \( x \to -\infty \), this integral diverges.

Let us consider the initial data of which \( \psi = 0 \) at \( u = u_i \) (thus \( D_v = 0 \)), or equivalently \( \psi = D_y = 0 \) at \( x = x_i \). This initial condition guarantees that there is no ingoing energy flux across \( x = x_i \). Here we assume \( x_i \ll -1 \) and then consider the case of \( |x| \gg y \). In this case, Eq.(5.11) becomes as

\[
\frac{\partial D_x}{\partial y} \sim -\frac{1}{2x} (D_y - D_x + F_S).
\]

Substituting the expression

\[
D_x = A(x, y)e^{\frac{y}{x}}
\]

into the above equation, we obtain

\[
\frac{\partial A}{\partial y} \sim -\frac{1}{2x} (D_y + F_S) e^{-\frac{y}{x}} \sim -\frac{1}{2x} \left( \frac{\partial \psi}{\partial y} + F_S \right).
\]
Integrating the above equation with respect to $y$, we obtain

$$A \sim -\frac{1}{2x} \left( \psi + \int_0^y F_S(z)dz \right), \quad (A5)$$

where the integration constant has been set so that $A$ vanishes in the vacuum region inside the shell $y < 0$ at $x = x_i$. Therefore we find for $x \sim x_i$ and $y > 1$,

$$D_x \sim -\frac{M}{x}, \quad (A6)$$

where

$$M = \frac{1}{2} \int_0^1 F_S(z)dz \quad (A7)$$

is a positive constant.

The above result means that if we set $x_i$ being $-\infty$, $D_x(x,y) \to -M/x$ for $x \to -\infty$. Hence introducing some constant $\lambda \ll -1$, we see that in the large $x_i$ limit,

$$\Delta E_R = \frac{1}{4G} \int_{x_i}^{0} dx(y-x)D_x^2(x,y)$$

$$= \frac{1}{4G} \left( \int_{x_i}^{\lambda} + \int_{\lambda}^{0} \right) dx(y-x)D_x^2(x,y)$$

$$\simeq \frac{1}{4G} \int_{x_i}^{\lambda} dx \frac{M^2}{x} + F(y; \lambda) \to \infty \quad \text{for} \quad x_i \to -\infty, \quad (A8)$$

where $F(y; \lambda)$ is some function of $y$ and the parameter $\lambda$. This results means that in the thin-shell limit with the total $C$-energy fixed, the emitted energy diverges. This is consistent with the result in Ref. [15].

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