QUANTUM GROUPS OF SPLIT TYPE VIA DERIVED HALL ALGEBRAS

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Dedicated to Professor Jie Xiao on the occasion of his sixtieth birthday

Abstract. A quantum symmetric pair consists of a quantum group $U$ and its coideal subalgebra $U_\xi$ (called an $\xi$-quantum group) with parameters $\xi$. In this note, we use the derived Hall algebras of 1-periodic complexes to realize the $\xi$-quantum groups $U_\xi$ of split type.

1. Introduction

1.1. Backgrounds.

1.1.1. Quantum groups via Hall algebras. Inspired by Gabriel Theorem, Ringel [Rin90] used the Hall algebra of representations of a Dynkin quiver to realize the positive part $U^+$ of the quantum group, which is generalized by Green [Gr95] to Kac-Moody setting. Later, Lusztig gave a geometric counterpart of Ringel’s construction by considering quiver varieties, and constructed the canonical basis of $U^+$.

Since then, many experts tried to realize the whole quantum groups by using Hall algebras. There are three typical constructions during this procedure. The first one is the realization of Kac-Moody algebras via Hall type Lie algebras of root categories given by Peng-Xiao [PX00]; the second one is the Drinfeld double of Hall algebras given by Xiao [X97]; the third one is the derived Hall algebras introduced by Toën [T06] and Xiao-Xu [XX08].

This problem was solved by Bridgeland in 2013 [Br13], who actually gave a categorical realization of the Drinfeld double quantum groups $\tilde{U}$, a variant of $U$ with the Cartan subalgebra doubled (with generators $K_i, K'_i$, for $i \in I$). A reduced version, which is the quotient of $\tilde{U}$ by the ideal generated by the central elements $K_iK'_i - 1$, is then identified with $U$.

Bridgeland’s Hall algebras has found further generalizations and improvements which allow more flexibilities. Gorsky [Gor18] constructed semi-derived Hall algebras for Frobenius categories. More recently, motivated by the works of Bridgeland and Gorsky, Lu-Peng [LP21] formulated the semi-derived (Ringel-)Hall algebras by using 2-periodic complexes of hereditary abelian categories. The semi-derived Ringel-Hall algebras were further constructed for arbitrary 1-Gorenstein algebras by Lu in [LW22, Appendix A].

1.1.2. $\xi$Quantum groups via $\xi$Hall algebras. As a quantization of symmetric pairs $(g, g^\theta)$, the quantum symmetric pairs $(U, U^\prime)$ were formulated by Letzter [Let99, Let02] (also cf. [Ko14]) with Satake diagrams as inputs. The symmetric pairs are in bijection with the real forms of...
complex simple Lie algebras, according to Cartan. By definition, $U^\iota = U^\iota_\xi$ is a coideal subalgebra of $U$ depending on parameters $\xi = (\xi_i)_{i \in I}$ (subject to some compatibility conditions) and will be referred to as an $\iota$-quantum group in this note. As suggested in [BW18a], most of the fundamental constructions in the theory of quantum groups should admit generalizations in the setting of $\iota$-quantum groups; see [BW18a, BK19, BW18b] for generalizations of (quasi) $R$-matrix and canonical bases, and also see [BKLW18] (and [Li19]) for a geometric realization and [BSWW18] for KLR type categorification of a class of (modified) $U^\iota$.

Following terminologies in real group theory, we call an $\iota$-quantum group quasi-split if the underlying Satake diagram does not contain any black node. In other words, the involution $\theta$ on $g$ is given by $\theta = \omega \circ \tau$, where $\omega$ is the Chevalley involution and $\tau$ is a diagram involution which is allowed to be Id. In case $\tau = \text{Id}$, $U^\iota$ is called split. For example, a quantum group is a quasi-split $\iota$-quantum group associated to the symmetric pair of diagonal type, and thus it is instructive to view $\iota$-quantum groups as generalizations of quantum groups which may not admit a triangular decomposition.

In [LW22, LW20], Lu-Wang developed a Hall algebra approach to study the $\iota$-quantum groups. More explicitly, they introduced a new kind of 1-Gorenstein algebras: $\iota$-quiver algebras, and used their $\iota$-Hall algebras (aka twisted semi-derived Hall algebras of $\iota$-quiver algebras) to realize the quantum groups. In fact, analogous to Bridgeland’s result, the $\iota$-Hall algebra construction produces a universal $\iota$-quantum group $\tilde{U}^\iota$. The main difference between the quantum groups $U^\iota$ a la Letzter [Let99] and the universal $\iota$-quantum groups $\tilde{U}^\iota$ (a coideal subalgebra of $\tilde{U}$) in [LW22] is that $U^\iota$ depends on various parameters while $\tilde{U}^\iota$ admits various central elements. A central reduction of $\tilde{U}^\iota$ recovers $U^\iota$.

1.1.3. Derived Hall algebras. Derived Hall algebras were firstly defined by Toën [T06] for DG-enhanced triangulated categories, and then extended to arbitrary triangulated categories satisfying some homological finiteness conditions by Xiao-Xu [XX08]. Unfortunately, so far it seems that derived Hall algebras could not be used to realize quantum groups, since none of these methods can be applied to the periodic triangulated category, especially 2-periodic triangulated category or root category, for it does not satisfy the homological finiteness conditions. However, Xu-Chen [XC13] constructed derived Hall algebras for odd periodic triangulated categories later.

A triangulated category $\mathcal{T}$ is called algebraic if there exists a Frobenius category $\mathcal{C}$ such that $\mathcal{T}$ is equivalent to the stable category $\underline{\mathcal{C}}$ of $\mathcal{C}$. For algebraic odd periodic triangulated category, Sheng-Chen-Xu [SCX18] considered the relation between Gorsky’s semi-derived Hall algebra of $\mathcal{C}$ and Xu-Chen’s derived Hall algebra of $\underline{\mathcal{C}}$: compared with Gorsky’s result [Gor18] for any Frobenius category whose stable category is homologically finite. Lin-Peng [LinP] revisited and consolidated their results by considering Lu-Peng’s semi-derived Ringel-Hall algebras. Roughly speaking, derived Hall algebra of odd periodic triangulated category is a subalgebra of semi-derived Ringel-Hall algebras, and they are isomorphic by twisting certain elements (related to Cartan elements) to both kinds of Hall algebras respectively.

1.2. Main results.

1.2.1. Goal. The $\iota$-Hall algebras constructed in [LW22, LW20] are used to realize the universal $\iota$-quantum groups $\tilde{U}^\iota$ other than $\iota$-quantum groups $U^\iota$. The goal of this note is to find a direct Hall algebra approach to realize Letzter’s $\iota$-quantum groups $U^\iota$. On the other hand, people
also would like to know whether derived Hall algebras could be used to realize any interesting and important quantum algebras.

We shall use derived Hall algebras of 1-periodic derived category of quiver algebras to realize split quantum groups $U^\lambda$.

1.2.2. Main results. The first main result is to compare derived Hall algebras of 1-periodic derived categories and Lu-Peng’s semi-derived Hall algebras. Let $A$ be a hereditary abelian category linearly over a finite field $F_q$. Let $C_1(A)$ be the category of 1-periodic complexes over $A$, and $D_1(A)$ the 1-periodic derived category. We prove in Theorem 3.4 that the derived Hall algebra $\mathcal{DH}_1(A)$ of $D_1(A)$ is a quotient algebra of Lu-Peng’s semi-derived Hall algebra $\tilde{\mathcal{H}}(A)$ by the ideal generated by the central elements $[K_\alpha] - 1$ for $\alpha \in K_0(A)$; see §2.3 for the definition of $[K_\alpha]$. As a corollary, we give a Hall multiplication formula for $\mathcal{DH}_1(A)$ by using the Hall number of $A$; see Corollary 3.5.

Let $\text{rep}^{nil}(Q)$ be the category of finite-dimensional nilpotent representations of arbitrary quiver $Q$. Let $\mathcal{DH}_1(kQ)$ be the derived Hall algebra of $D_1(\text{rep}^{nil}(kQ))$. The second main result is to realize the split $\lambda$ quantum group $U^\lambda$ by using $\mathcal{DH}_1(kQ)$; see Theorem 4.4. In fact, Theorem 4.4 follows from Theorem 3.4 and the realization of universal $\lambda$ quantum group via $\lambda$ Hall algebras obtained in [LW22, LW20].

The paper is organized as follows. Section 2 is devoted to reviewing the materials on Hall algebras, 1-periodic complexes and semi-derived Hall algebras. In Section 3, we compare derived Hall algebras and semi-derived Hall algebras for the categories of 1-periodic complexes, and Theorem 3.4 is proved. In Section 4, we review the $\lambda$ quantum groups and their categorical realization via semi-derived Hall algebras, and use $\mathcal{DH}_1(kQ)$ to realize the split $\lambda$ quantum group $U^\lambda$.

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2. $\lambda$Hall algebras

In this paper, we take the field $k = F_q$, a finite field of $q$ elements. Let $A$ be a hereditary abelian category over $k$. We define the $\lambda$Hall algebra $\tilde{\mathcal{H}}(A)$ as a twisted semi-derived Ringel-Hall algebra for the category $C_1(A)$ of 1-periodic complexes over $A$.

2.1. Hall algebras. Let $E$ be an essentially small exact category in the sense of Quillen, linearly over $k = F_q$. Assume that $E$ has finite morphism and extension spaces, i.e.,

$$|\text{Hom}(M, N)| < \infty, \quad |\text{Ext}^1(M, N)| < \infty, \quad \forall M, N \in E.$$ 

Given objects $M, N, L \in E$, define $\text{Ext}^1(M, N)_L \subseteq \text{Ext}^1(M, N)$ as the subset parameterizing extensions whose middle term is isomorphic to $L$. We define the Ringel-Hall algebra $\mathcal{H}(E)$ (or Hall algebra for short) to be the $\mathbb{Q}$-vector space whose basis is formed by the isoclasses
of objects $M$ in $\mathcal{E}$, with the multiplication defined by (see [Br13])

$$[M] \diamond [N] = \sum_{[L] \in \text{Iso}(\mathcal{E})} \frac{|\text{Ext}^1(M, N)_L|}{|\text{Hom}(M, N)|} [L]. \quad (2.1)$$

We remark that the Ringel-Hall algebra used here is the dual of the original one defined in [Rin90].

For any three objects $L, M, N$, let

$$F^L_{MN} := \left| \{ X \subseteq L \mid X \cong N, L/X \cong M \} \right|.$$

The Riedtmann-Peng formula states that

$$F^L_{MN} = \frac{|\text{Ext}^1(M, N)_L|}{|\text{Hom}(M, N)|} \cdot \frac{|\text{Aut}(L)|}{|\text{Aut}(M)| |\text{Aut}(N)|},$$

where $\text{Aut}(M)$ denotes the automorphism group of $M$. For any object $M$, let

$$[M] := \frac{|M|}{|\text{Aut}(M)|}.$$

Then the Hall multiplication (2.1) can be reformulated to be

$$[M] \diamond [N] = \sum_{[L]} F^L_{M,N}[L],$$

which is the version of Hall multiplication used in [Rin90].

2.2. The category of 1-periodic complexes. Let $\mathcal{A}$ be a hereditary abelian category which is essentially small with finite-dimensional homomorphism and extension spaces.

A 1-periodic complex $X^\bullet$ in $\mathcal{A}$ is a pair $(X, d)$ with $X \in \mathcal{A}$ and a differential $d : X \to X$ with $d^2 = 0$. A morphism $(X, d) \to (Y, e)$ is given by a morphism $f : X \to Y$ in $\mathcal{A}$ satisfying $f \circ d = e \circ f$. Let $C_1(\mathcal{A})$ be the category of all 1-periodic complexes in $\mathcal{A}$. Then $C_1(\mathcal{A})$ is an abelian category. A 1-periodic complex $X^\bullet = (X, d)$ is called acyclic if $\text{Ker} d = \text{Im} d$. We denote by $C_{1,ac}(\mathcal{A})$ the full subcategory of $C_1(\mathcal{A})$ consisting of acyclic complexes. Denote by $H(X^\bullet) \in \mathcal{A}$ the cohomology group of $X^\bullet$, i.e., $H(X^\bullet) = \text{Ker} d / \text{Im} d$, where $d$ is the differential of $X^\bullet$.

The category $C_1(\mathcal{A})$ is Frobenius with respect to the degree-wise split exact structure. The 1-periodic homotopy category $K_1(\mathcal{A})$ is obtained as the stabilization of $C_1(\mathcal{A})$, and the 1-periodic derived category $D_1(\mathcal{A})$ is the localization of the homotopy category $K_1(\mathcal{A})$ with respect to quasi-isomorphisms. Both $K_1(\mathcal{A})$ and $D_1(\mathcal{A})$ are triangulated categories.

Let $\mathcal{C}^b(\mathcal{A})$ be the category of bounded complexes over $\mathcal{A}$ and $D^b(\mathcal{A})$ be the corresponding derived category with the suspension functor $\Sigma$. Then there is a covering functor $\pi : \mathcal{C}^b(\mathcal{A}) \to C_1(\mathcal{A})$, inducing a covering functor $\pi : D^b(\mathcal{A}) \to D_1(\mathcal{A})$ which is dense (see, e.g., [St17, Lemma 5.1]). The orbit category $D^b(\mathcal{A})/\Sigma$ is a triangulated category [Ke05], and we have

$$D_1(\mathcal{A}) \simeq D^b(\mathcal{A})/\Sigma.$$

For any $X \in \mathcal{A}$, denote the stalk complex by

$$C_X = (X, 0).$$
(or just by $X$ when there is no confusion), and denote by $K_X$ the following acyclic complex:

$$K_X := (X \oplus X, d), \quad \text{where } d = \begin{pmatrix} 0 & \text{Id} \\ 0 & 0 \end{pmatrix}.$$  

**Lemma 2.1** ([LRW20, Lemma 2.2]). For any acyclic complex $K^\bullet$ and $p \geq 2$, we have

$$\text{Ext}^p_{C_1(\mathcal{A})}(K^\bullet, -) = 0 = \text{Ext}^p_{C_1(\mathcal{A})}(-, K^\bullet).$$

For any $K^\bullet \in C_{1,ac}(\mathcal{A})$ and $M^\bullet \in C_1(\mathcal{A})$, by [LRW20, Corollary 2.4], define

$$\langle K^\bullet, M^\bullet \rangle = \dim_k \text{Hom}_{C_1(\mathcal{A})}(K^\bullet, M^\bullet) - \dim_k \text{Ext}^1_{C_1(\mathcal{A})}(K^\bullet, M^\bullet),$$

$$\langle M^\bullet, K^\bullet \rangle = \dim_k \text{Hom}_{C_1(\mathcal{A})}(M^\bullet, K^\bullet) - \dim_k \text{Ext}^1_{C_1(\mathcal{A})}(M^\bullet, K^\bullet).$$

These formulas give rise to well-defined bilinear forms (called Euler forms), again denoted by $\langle \cdot , \cdot \rangle$, on the Grothendieck groups $K_0(C_{1,ac}(\mathcal{A}))$ and $K_0(C_1(\mathcal{A}))$.

Denote by $\langle \cdot , \cdot \rangle_\mathcal{A}$ (also denoted by $\langle \cdot , \cdot \rangle$ if no confusion arises) the Euler form of $\mathcal{A}$, i.e.,

$$\langle M, N \rangle_\mathcal{A} = \dim_k \text{Hom}_\mathcal{A}(M, N) - \dim_k \text{Ext}^1_{\mathcal{A}}(M, N).$$

Let $\text{res} : C_1(\mathcal{A}) \to \mathcal{A}$ be the restriction functor by forgetting differentials.

**Lemma 2.2** ([LRW20, Lemma 2.7]). We have

1. $\langle K_X, M^\bullet \rangle = \langle X, \text{res}(M^\bullet) \rangle_\mathcal{A}$, $\langle M^\bullet, K_X \rangle = \langle \text{res}(M^\bullet), X \rangle_\mathcal{A}$, for $X \in \mathcal{A}$, $M^\bullet \in C_1(\mathcal{A})$;
2. $\langle M^\bullet, N^\bullet \rangle = \frac{1}{2} \langle \text{res}(M^\bullet), \text{res}(N^\bullet) \rangle_\mathcal{A}$, for $M^\bullet, N^\bullet \in C_{1,ac}(\mathcal{A})$.

2.3. $i$Hall algebras. Define

$$v := \sqrt{q}.$$

We continue to work with a hereditary abelian category $\mathcal{A}$ as in §2.2. Let $\mathcal{H}(C_1(\mathcal{A}))$ be the Ringel-Hall algebra of $C_1(\mathcal{A})$ over $\mathbb{Q}(v)$, i.e., $\mathcal{H}(C_1(\mathcal{A})) = \bigoplus_{M^\bullet \in \text{Iso}(C_1(\mathcal{A}))} \mathbb{Q}(v)[M^\bullet]$, with multiplication defined by

$$[M^\bullet] \circ [N^\bullet] = \sum_{[L^\bullet] \in \text{Iso}(C_1(\mathcal{A}))} \frac{[\text{Ext}^1(M^\bullet, N^\bullet) \otimes L^\bullet]}{[\text{Hom}(M^\bullet, N^\bullet)]} [L^\bullet].$$

Following [LP21, LW22, LW20], we consider the ideal $\mathcal{I}$ of $\mathcal{H}(C_1(\mathcal{A}))$ generated by

$$\left\{ [M^\bullet] - [N^\bullet] \mid H(M^\bullet) \cong H(N^\bullet), \quad \text{Im} d_{M^\bullet} = \text{Im} d_{N^\bullet} \right\}.$$

Here we use $\hat{X}$ to denote the image of $X \in \mathcal{A}$ in the Grothendieck group $K_0(\mathcal{A})$. We denote

$$S := \{ a[K^\bullet] \in \mathcal{H}(C_1(\mathcal{A}))/\mathcal{I} \mid a \in \mathbb{Q}(v)^\times, K^\bullet \in C_1(\mathcal{A}) \text{ acyclic} \},$$

a multiplicatively closed subset of $\mathcal{H}(C_1(\mathcal{A}))/\mathcal{I}$ with respect to $S$, denoted by $(\mathcal{H}(C_1(\mathcal{A}))/\mathcal{I})[S^{-1}]$.

**Lemma 2.3** ([LW22, Proposition A.5]). The multiplicatively closed subset $S$ is a right Ore, right reversible subset of $\mathcal{H}(C_1(\mathcal{A}))/\mathcal{I}$. Equivalently, there exists the right localization of $\mathcal{H}(C_1(\mathcal{A}))/\mathcal{I}$ with respect to $S$, denoted by $(\mathcal{H}(C_1(\mathcal{A}))/\mathcal{I})[S^{-1}]$.

The algebra $(\mathcal{H}(C_1(\mathcal{A}))/\mathcal{I})[S^{-1}]$ is the semi-derived Ringel-Hall algebra of $C_1(\mathcal{A})$ in the sense of [LP21, LW22] (also cf. [Gor18]), and will be denoted by $SD\mathcal{H}(C_1(\mathcal{A}))$. 

Definition 2.4 ([LR21, Definition 2.5]). The $i$-Hall algebra of a hereditary abelian category $\mathcal{A}$, denoted by $\hat{\mathcal{H}}(\mathcal{A})$, is defined to be the twisted semi-derived Ringel-Hall algebra of $\mathcal{C}_1(\mathcal{A})$, that is, the $\mathbb{Q}((v))$-algebra on the same vector space as $SD\mathcal{H}(\mathcal{C}_1(\mathcal{A})) = (\mathcal{H}(\mathcal{C}_1(\mathcal{A}))/I[S^{-1}]$ equipped with the following modified multiplication (twisted via the restriction functor $\text{res}$: $\mathcal{C}_1(\mathcal{A}) \rightarrow \mathcal{A}$)

$$[M^*] \ast [N^*] = v^{\text{res}(M^*),\text{res}(N^*))}. [M^*] \circ [N^*]. \quad (2.2)$$

For any complex $M^*$ and acyclic complex $K^*$, we have

$$[K^*] \ast [M^*] = [K^* \oplus M^*] = [M^*] \ast [K^*].$$

For any $\alpha \in K_0(\mathcal{A})$, there exist $X,Y \in \mathcal{A}$ such that $\alpha = \hat{X} - \hat{Y}$. Define $[K_\alpha] := [K_X] * [K_Y]^{-1}$. This is well defined, see e.g., [LP21, §3.2]. It follows that $[K_\alpha] (\alpha \in K_0(\mathcal{A}))$ are central in the algebra $\hat{\mathcal{H}}(\mathcal{A})$.

The quantum torus $\hat{T}(\mathcal{A})$ is defined to be the subalgebra of $\hat{\mathcal{H}}(\mathcal{A})$ generated by $[K_\alpha]$ for $\alpha \in K_0(\mathcal{A})$.

Proposition 2.5 ([LRW20, Proposition 2.9]). The following hold in $\hat{\mathcal{H}}(\mathcal{A})$:

1. The quantum torus $\hat{T}(\mathcal{A})$ is a central subalgebra of $\hat{\mathcal{H}}(\mathcal{A})$.
2. The algebra $\hat{T}(\mathcal{A})$ is isomorphic to the group algebra of the abelian group $K_0(\mathcal{A})$.
3. $\hat{\mathcal{H}}(\mathcal{A})$ has an (iHall) basis given by

$$\{[M] \ast [K_\alpha] \mid [M] \in \text{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}.$$

3. Derived Hall algebras vs semi-derived Hall algebras

3.1. 1-Periodic derived Hall algebra. Assume $\mathcal{A}$ is a hereditary abelian category over $k$. Let $\mathcal{C}_1(\mathcal{A})$ be the category of 1-periodic complexes on $\mathcal{A}$, and $\mathcal{D}_1(\mathcal{A})$ be its derived category. Observe that the isoclasses $\text{Iso}(\mathcal{D}_1(\mathcal{A}))$ of $\mathcal{D}_1(\mathcal{A})$ coincides with the isoclasses $\text{Iso}(\mathcal{A})$ of $\mathcal{A}$. In the following we will identify $\text{Iso}(\mathcal{D}_1(\mathcal{A}))$ with $\text{Iso}(\mathcal{A})$.

Lemma 3.1. For any objects $A,B \in \mathcal{A}$, we have

1. $\text{Ext}^1_{\mathcal{C}_1(\mathcal{A})}(A,B) \cong \text{Hom}_{\mathcal{D}_1(\mathcal{A})}(A,B) \cong \text{Hom}_{\mathcal{A}}(A,B) \oplus \text{Ext}^1_{\mathcal{A}}(A,B)$;
2. $|\text{Aut}_{\mathcal{D}_1(\mathcal{A})}(A)| = |\text{Aut}_{\mathcal{A}}(A)| \cdot |\text{Ext}^1_{\mathcal{A}}(A,A)|$.

Proof. (1) The formula $\text{Ext}^1_{\mathcal{C}_1(\mathcal{A})}(A,B) \cong \text{Hom}_{\mathcal{D}_1(\mathcal{A})}(A,B)$ is well known; see e.g. [LinP] for a proof. The formula $\text{Hom}_{\mathcal{D}_1(\mathcal{A})}(A,B) \cong \text{Hom}_{\mathcal{A}}(A,B) \oplus \text{Ext}^1_{\mathcal{A}}(A,B)$ follows from the fact that $\mathcal{D}_1(\mathcal{A}) \simeq \mathcal{D}^b(\mathcal{A})/\Sigma$, where $\Sigma$ is the suspension functor of $\mathcal{D}^b(\mathcal{A})$.

(2) By (1), we have $\text{End}_{\mathcal{D}_1(\mathcal{A})}(A) = \text{End}_{\mathcal{A}}(A) \oplus \text{Ext}^1_{\mathcal{A}}(A,A)$. Since $\mathcal{A}$ is hereditary, it is obvious that any $f \in \text{Ext}^1_{\mathcal{A}}(A,A)$ is nilpotent. Thus $\text{Aut}_{\mathcal{D}_1(\mathcal{A})}(A) = \text{Aut}_{\mathcal{A}}(A) \oplus \text{Ext}^1_{\mathcal{A}}(A,A)$. Then the desired formula follows.

For convenience, we denote by

$$|\text{Aut}_{\mathcal{D}_1(\mathcal{A})}(A)| = \bar{a}_A \quad \text{and} \quad |\text{Aut}_{\mathcal{A}}(A)| = a_A$$

for any $A \in \mathcal{A}$. Hence, we have $\bar{a}_A = a_A \cdot |\text{Ext}^1_{\mathcal{A}}(A,A)|$.

Following [XC13], we use the following notation for any objects $A,B,M \in \mathcal{A}$:

$$(A,B)_M := \{f \in \text{Hom}_{\mathcal{D}_1(\mathcal{A})}(A,B) \mid \text{Cone}(f) \cong M\}.$$
Lemma 3.2. For any objects $A, B, M \in \mathcal{A}$, we have
\[ |(A, B)_M| = \sum_{[X^\bullet] \in \text{Iso}(\mathcal{C}_1(A)), H(X^\bullet) \cong M} |\text{Ext}^1_{\mathcal{C}_1(A)}(A, B)_{X^\bullet}|. \]

Proof. Observe that each exact sequence
\[ 0 \to B \to X^\bullet \to A \to 0 \]
in $\mathcal{C}_1(A)$ lifts to a triangle
\[ A \to B \to X^\bullet \to A \]
in $\mathcal{D}_1(A)$, since we have $\Sigma A \cong A$ in $\mathcal{D}_1(A)$.

For any two complexes $X^\bullet, Y^\bullet \in \mathcal{C}_1(A)$, by using $\mathcal{D}_1(A) = \mathcal{D}^{\text{b}}(A)/\Sigma$, we have
\[ X^\bullet \cong Y^\bullet \text{ in } \mathcal{D}_1(A) \iff H(X^\bullet) \cong H(Y^\bullet) \text{ in } \mathcal{A}. \]
In particular, $X^\bullet \cong H(X^\bullet)$ in $\mathcal{D}_1(A)$. Then the result follows from Lemma 3.1 (1) immediately. \qed

Following [XC13], we denote by
\[ \{A, B\} := \frac{1}{|\text{Hom}_{\mathcal{D}_1(A)}(A, B)|} \]
for any objects $A, B \in \mathcal{A}$. Then

Lemma 3.3. For any objects $A, B \in \mathcal{A}$, we have
\[ \sqrt{\{A, B\}} = v^{-(A, B)_A} \cdot \frac{1}{|\text{Ext}^1_{\mathcal{A}}(A, B)|}. \quad (3.1) \]
In particular,
\[ \sqrt{\{A, A\}} \cdot \tilde{a}_A = v^{-(A, A)_A} \cdot a_A. \quad (3.2) \]

Proof. By Lemma 3.1, we have
\[ \{A, B\} = \frac{1}{|\text{Hom}_{\mathcal{D}_1(A)}(A, B)|} = \frac{1}{|\text{Hom}_{\mathcal{A}}(A, B)| \cdot |\text{Ext}^1_{\mathcal{A}}(A, B)|} = q^{-(A, B)_A} \cdot \frac{1}{|\text{Ext}^1_{\mathcal{A}}(A, B)|^2}. \]
This proves (3.1). Then (3.2) follows from $\tilde{a}_A = a_A \cdot |\text{Ext}^1_{\mathcal{A}}(A, A)|$ immediately. \qed

For any objects $A, B, M \in \mathcal{A}$, by [XC13, Corollary 2.7] we have
\[ \frac{|(B, M)_A|}{\tilde{a}_B} \sqrt{\frac{\{B, M\}}{\{B, B\}}} = \frac{|(M, A)_B|}{\tilde{a}_A} \sqrt{\frac{\{M, A\}}{\{A, A\}}}. \]
We denote this number by $G^M_{AB}$ in this paper, which satisfies the following derived Riedtmann-Peng formula by [SCX18, Proposition 3.3]:
\[ G^M_{AB} = \frac{\tilde{a}_M \cdot |(A, B)_M|}{\tilde{a}_A \tilde{a}_B} \sqrt{\{A, B\} \{M, M\} / \{A, A\} \{B, B\}}. \]  

(3.3)

Recall that \( \text{Iso}(\mathcal{D}_1(A)) = \text{Iso}(A) \). The \( 1 \)-periodic derived Hall algebra \( \mathcal{D}H_1(A) \) is a \( \mathbb{Q}(v) \)-vector space with the basis \( \{u_A | [A] \in \text{Iso}(\mathcal{D}_1(A))\} \), endowed with the multiplication defined by

\[ u_{[A]} \ast u_{[B]} = \sum_{[M] \in \text{Iso}(\mathcal{D}_1(A))} G^M_{AB} \cdot u_{[M]}. \]

### 3.2. A homomorphism from \( \mathcal{D} \text{Hall algebra to } 1 \)-periodic derived Hall algebra.

In this subsection, we prove our first main result of this note.

Let \( A \) be a hereditary abelian \( k \)-linear category. Recall that \( ^n\mathcal{H}(A) \) has an \( \mathcal{D} \text{Hall basis} \) given by

\[ \{[M] \ast [K_\alpha] | [M] \in \text{Iso}(A), \alpha \in K_0(A)\}, \]

and \([K_\alpha] (\alpha \in K_0(A))\) are central in \( ^n\mathcal{H}(A) \).

**Theorem 3.4.** Let \( A \) be a hereditary abelian \( k \)-linear category. Then there exists an algebra epimorphism

\[ \Phi : ^n\mathcal{H}(A) \twoheadrightarrow \mathcal{D}H_1(A), \]

\[ [M] \ast [K_\alpha] \mapsto \sqrt{\{M, M\} \cdot \tilde{a}_M \cdot u_{[M]}} \]

with \( \text{Ker } \Phi = \langle [K_\alpha] - 1, \alpha \in K_0(A) \rangle \).

**Proof.** For any two objects \( A, B \in A \), in \( ^n\mathcal{H}(A) \) we have

\[ [A] \ast [B] = v^{(A, B), A} \sum_{[X^\bullet] \in \text{Iso}(C(A))} \frac{|\text{Ext}^1_{C(A)}(A, B)_{X^\bullet}|}{|\text{Hom}_{C(A)}(A, B)_{X^\bullet}|} \cdot [X^\bullet] \]

\[ = v^{(A, B), A} \sum_{[X^\bullet] \in \text{Iso}(C(A))} \frac{|\text{Ext}^1_{C(A)}(A, B)_{X^\bullet}|}{|\text{Hom}_{C(A)}(A, B)|} \cdot [H(X^\bullet)] \ast [K_{\text{Im} d_{X^\bullet}}] \]

\[ = v^{(A, B), A} \sum_{[M] \in \text{Iso}(A)} \sum_{[X^\bullet] \in \text{Iso}(C(A))} \frac{|\text{Ext}^1_{C(A)}(A, B)_{X^\bullet}|}{|\text{Hom}_{C(A)}(A, B)|} \cdot [M] \ast [K_{\text{Im} d_{X^\bullet}}], \]

where \( 2 \cdot \text{Im} d_{X^\bullet} = \hat{A} + \hat{B} - \hat{H}(X^\bullet) \).

By the definition of \( \Phi \), we get

\[ \Phi([A] \ast [B]) = v^{(A, B), A} \sum_{[M] \in \text{Iso}(A)} \sum_{[X^\bullet] \in \text{Iso}(C(A)) : H(X^\bullet) \simeq M} \frac{|\text{Ext}^1_{C(A)}(A, B)_{X^\bullet}|}{|\text{Hom}_{C(A)}(A, B)|} \cdot \Phi([M]) \]

\[ = \frac{v^{(A, B), A}}{|\text{Hom}_A(A, B)|} \sum_{[M] \in \text{Iso}(A)} |(A, B)_M| \cdot \Phi([M]), \]

where the second equality follows from Lemma 3.2 and the fact \( \text{Hom}_{C(A)}(A, B) \cong \text{Hom}_A(A, B) \).

On the other hand, we have

\[ \Phi([A]) \ast \Phi([B]) = \sqrt{\{A, A\} \cdot \tilde{a}_A \cdot u_{[A]} \ast \sqrt{\{B, B\} \cdot \tilde{a}_B \cdot u_{[B]}} \]

\[ = \sqrt{\{A, B\}_M} \cdot \tilde{a}_{AB} \cdot u_{[AB]}, \]
In particular, by Riedtmann-Peng formula we have

\[
\sqrt{\{A, A\} \{B, B\} \cdot \tilde{a}_A \tilde{a}_B \cdot \sum_{[M] \in \text{Iso}(A)} G_{AB}^M \cdot u_{[M]}}
\]

\[
= \sum_{[M] \in \text{Iso}(A)} |(A, B)_M| \sqrt{\{A, B\} \{M, M\} \cdot \tilde{a}_M \cdot u_{[M]}}
\]

\[
= \sqrt{\{A, B\}} \cdot \sum_{[M] \in \text{Iso}(A)} |(A, B)_M| \cdot \Phi([M]).
\]

By (3.1) we have

\[
\sqrt{\{A, B\}} = \frac{v^{-\langle A, B \rangle}_A}{|\text{Ext}^1_A(A, B)|} = \frac{v^{\langle A, B \rangle}_A}{|\text{Hom}_A(A, B)|}.
\]

It follows that

\[
\Phi([A] * [B]) = \Phi([A] * [B]) = 1.
\]

Recall that \(\hat{\mathcal{H}}(A)\) has an Hall basis given by \([M] \cdot [K_\alpha] \mid [M] \in \text{Iso}(A), \alpha \in K_0(A)\), and \([K_\alpha] (\alpha \in K_0(A))\) are central in \(\hat{\mathcal{H}}(A)\). Hence \(\Phi\) is an algebra homomorphism, since \(\Phi([K_\alpha]) = 1\) for any \(\alpha \in K_0(A)\).

Recall that \(\mathcal{DH}_1(A)\) has a basis \(\{u_A \mid [A] \in \text{Iso}(\mathcal{D}_1(A))\}\), and \(\text{Iso}(\mathcal{D}_1(A))\) coincides with \(\text{Iso}(A)\). It follows that \(\Phi\) is surjective and \(\text{Ker} \Phi = ([K_\alpha] - 1, \alpha \in K_0(A))\).

As an application, we obtain the following multiplication formula for the 1-periodic derived Hall algebra \(\mathcal{DH}_1(A)\).

**Proposition 3.5.** For any objects \(A, B, M \in \mathcal{A}\), we have

\[
\sum_{[M] \in \text{Iso}(A)} \sum_{[L], [I], [N] \in \text{Iso}(A)} v^{\langle I, N \rangle}_A + \langle I, I \rangle_A + \langle L, I \rangle_A - \langle L, N \rangle_A \cdot \frac{a_{L} a_{I} a_{N}}{a_A a_B} \cdot F_{NL}^M F_{IN}^A F_{LI}^B \cdot u_{[M]}.
\]

In particular,

\[
G_{AB}^M = \sum_{[L], [I], [N] \in \text{Iso}(A)} v^{\langle I, N \rangle}_A + \langle I, I \rangle_A + \langle L, I \rangle_A - \langle L, N \rangle_A \cdot \frac{a_{L} a_{I} a_{N}}{a_A a_B} \cdot F_{NL}^M F_{IN}^A F_{LI}^B.
\]

**Proof.** By [LW20, Proposition 3.10], we know that the following formula holds in \(\hat{\mathcal{H}}(A)\):

\[
\sum_{[L], [M], [N] \in \text{Iso}(A)} v^{\langle A, B \rangle}_A q^{\langle N, L \rangle} \cdot \frac{\text{Ext}^1_A(N, L)_M}{|\text{Hom}_A(N, L)|} \cdot |\{s \in \text{Hom}_A(A, B) \mid \text{Ker} s \cong N, \text{Coker} s \cong L\}| \cdot [M] * [K_{\alpha} - N].
\]

By Riedtmann-Peng formula we have

\[
\frac{|\text{Ext}^1_A(N, L)_M|}{|\text{Hom}_A(N, L)|} = F_{NL}^M \cdot \frac{a_N a_L}{a_M}.
\]

Moreover, it is well-known that

\[
|\{s \in \text{Hom}_A(A, B) \mid \text{Ker} s \cong N, \text{Coker} s \cong L\}| = \sum_{[I] \in \text{Iso}(A)} F_{IN}^A F_{LI}^B \cdot a_I.
\]
It follows that
\[ [A] \ast [B] = \sum_{[M] \in \text{Iso}(A)} \sum_{[L],[I],[N] \in \text{Iso}(A)} v^{-\langle A,B \rangle_A q^{(N,L)}_A} \cdot F^M_{NL} F^A_{IN} F^B_{Ll} \cdot \frac{a_N a_I a_I}{a_M} \cdot [M] \ast [K_{\hat{A} - \hat{N}}]. \]

By using Theorem 3.4 and (3.2), we get
\[ \Phi([M] \ast [K_{\hat{A} - \hat{N}}]) = \sqrt{\{M,M\}} \cdot \bar{a}_M \cdot u_M = v^{-\langle M,M \rangle_A} \cdot a_M \cdot u_M. \]

Hence
\[ \Phi([A] \ast [B]) = \Phi([A]) \ast \Phi([B]) \]
\[ = \sum_{[M] \in \text{Iso}(A)} \sum_{[L],[I],[N] \in \text{Iso}(A)} v^{-\langle A,B \rangle_A - \langle M,M \rangle_A q^{(N,L)}_A} \cdot F^M_{NL} F^A_{IN} F^B_{Ll} \cdot a_N a_I a_I \cdot u_M. \] 

(3.4)

On the other hand, we have
\[ \Phi([A] \ast [B]) = \Phi([A]) \ast \Phi([B]) \]
\[ = \sqrt{\{A,A\}} \cdot \bar{a}_A \cdot u_A \ast \sqrt{\{B,B\}} \cdot \bar{a}_B \cdot u_B \]
\[ = v^{-\langle A,A \rangle_A} \cdot a_A \cdot u_A \ast v^{-\langle B,B \rangle_A} \cdot a_B \cdot u_B \]
\[ = v^{-\langle A,A \rangle_A - \langle B,B \rangle_A} \cdot a_A a_B \cdot \sum_{[M] \in \text{Iso}(A)} G^M_{AB} \cdot u_M. \] 

(3.5)

By comparing the coefficient of \( u_M \) in (3.4) and (3.5), we have
\[ G^M_{AB} = \sum_{[L],[I],[N] \in \text{Iso}(A)} v^{-\langle A,B \rangle_A + \langle A,A \rangle_A + \langle B,B \rangle_A - \langle M,M \rangle_A q^{(N,L)}_A} \cdot F^M_{NL} F^A_{IN} F^B_{Ll} \cdot \frac{a_N a_I a_I}{a_A a_B}. \] 

(3.6)

Now consider the nonzero term on the right-hand side of (3.6). For \( F^A_{IN} F^B_{Ll} \neq 0 \), we know that \( A, B \) fit into the following exact sequences:
\[ 0 \to N \to A \to I \to 0; \quad 0 \to I \to B \to L \to 0. \]

Hence
\[ -\langle A, B \rangle_A + \langle A, A \rangle_A + \langle B, B \rangle_A \]
\[ = -\langle N + I, I + L \rangle_A + \langle N + I, N + I \rangle_A + \langle I + L, I + L \rangle_A \]
\[ = \langle I, N \rangle_A + \langle I, I \rangle_A + \langle L, I \rangle_A + \langle N, N \rangle_A + \langle L, L \rangle_A - \langle N, L \rangle_A. \]

(3.7)

For \( F^M_{NL} \neq 0 \), \( M \) fits into the following exact sequence
\[ 0 \to L \to M \to N \to 0. \]

Hence
\[ -\langle M, M \rangle_A + 2\langle N, L \rangle_A = -\langle N + L, N + L \rangle_A + 2\langle N, L \rangle_A \]
\[ = \langle N, L \rangle_A - \langle N, N \rangle_A - \langle L, L \rangle_A - \langle L, N \rangle_A. \] 

(3.8)

To sum up (3.7) with (3.8), we have
\[ -\langle A, B \rangle_A + \langle A, A \rangle_A + \langle B, B \rangle_A - \langle M, M \rangle_A + 2\langle N, L \rangle_A \]
\[ = \langle I, N \rangle_A + \langle I, I \rangle_A + \langle L, I \rangle_A - \langle L, N \rangle_A. \]
Therefore,
\[ G^M_{AB} = \sum_{[L],[I],[N] \in \text{Iso}(A)} v^{(I,N)A + (I,I)A + (L,I)A - (L,N)A} F^M_{NL} F^A_{I1} F^B_{LI} \cdot \frac{a_{NL}a_{AI}}{a_{LA}a_{LB}}. \]

Then we finish the proof. \( \square \)

4. Quantum Groups of Split Type via Derived Hall Algebras

4.1. Quantum groups of split type. Let \( Q \) be a quiver (without loops) with vertex set \( Q_0 = \mathbb{I} \). Let \( n_{ij} \) be the number of edges connecting vertex \( i \) and \( j \). Let \( C = (c_{ij})_{i,j \in \mathbb{I}} \) be the symmetric generalized Cartan matrix of the underlying graph of \( Q \), defined by \( c_{ij} = 2\delta_{ij} - n_{ij} \).

Let \( \mathfrak{g} \) be the corresponding Kac-Moody Lie algebra. Let \( \alpha_i \ (i \in \mathbb{I}) \) be the simple roots of \( \mathfrak{g} \).

Let \( v \) be an indeterminant. Write \([A,B] = AB - BA\). For \( r, m \in \mathbb{N} \), denote by
\[ [r] = \frac{v^r - v^{-r}}{v - v^{-1}}, \quad [r]! = \prod_{i=1}^r [i], \quad \left[ \frac{m}{r} \right] = \frac{[m][m-1] \ldots [m-r+1]}{[r]!}. \]

Then \( \mathcal{U} := \mathcal{U}_v(\mathfrak{g}) \) is defined to be the \( \mathbb{Q}(v) \)-algebra generated by \( E_i, F_i, \tilde{K}_i, \tilde{K}'_i, \ i \in \mathbb{I} \), where \( \tilde{K}_i, \tilde{K}'_i \) are invertible, subject to the following relations for \( i, j \in \mathbb{I} \):
\[ [E_i, F_j] = \delta_{ij} \frac{\tilde{K}_i - \tilde{K}'_i}{v - v^{-1}}, \quad [\tilde{K}_i, \tilde{K}_j] = [\tilde{K}_i, \tilde{K}'_j] = [\tilde{K}'_i, \tilde{K}_j] = 0, \quad (4.1) \]
\[ \tilde{K}_i E_j = v^{c_{ij}} E_j \tilde{K}_i, \quad \tilde{K}_i F_j = v^{-c_{ij}} F_j \tilde{K}_i, \quad (4.2) \]
\[ \tilde{K}'_i E_j = v^{-c_{ij}} E_j \tilde{K}'_i, \quad \tilde{K}'_i F_j = v^{c_{ij}} F_j \tilde{K}'_i, \quad (4.3) \]
and the quantum Serre relations for \( i \neq j \in \mathbb{I} \),
\[ \sum_{r=0}^{1-c_{ij}} (-1)^r E_i^{(r)} F_j E_i^{(1-c_{ij}-r)} = 0, \quad (4.4) \]
\[ \sum_{r=0}^{1-c_{ij}} (-1)^r F_i^{(r)} F_j E_i^{(1-c_{ij}-r)} = 0. \quad (4.5) \]

Here
\[ F_i^{(n)} = F_i^n/[n]!, \quad E_i^{(n)} = E_i^n/[n]!, \quad \text{for } n \geq 1, \ i \in \mathbb{I}. \]

Note that \( \tilde{K}_i \tilde{K}'_i \) are central in \( \mathcal{U} \) for all \( i \).

Analogously as for \( \mathcal{U} \), the quantum group \( \mathbf{U} := \mathbf{U}_v(\mathfrak{g}) \) is defined to be the \( \mathbb{Q}(v) \)-algebra generated by \( E_i, F_i, K_i, K_i^{-1}, \ i \in \mathbb{I} \), subject to the relations modified from (4.1)–(4.5) with \( \tilde{K}_i \) and \( \tilde{K}'_i \) replaced by \( K_i \) and \( K_i^{-1} \), respectively.

We define \( \mathbf{U}^i \) to be the \( \mathbb{Q}(v) \)-subalgebra of \( \mathbf{U} \) generated by
\[ B_i = F_i + E_i \tilde{K}'_i, \quad k_i = \tilde{K}_i \tilde{K}'_i, \quad \forall i \in \mathbb{I}. \]

Let \( \mathbf{U}^{i0} \) be the \( \mathbb{Q}(v) \)-subalgebra of \( \mathbf{U}^i \) generated by \( k_i \), for \( i \in \mathbb{I} \). The elements \( k_i \) are central in \( \mathbf{U}^i \).

Let \( \varsigma = (\zeta_i) \in (\mathbb{Q}(v)^+)^\mathbb{I} \). Let \( \mathbf{U}^\varsigma := \mathbf{U}^\varsigma \) be the \( \mathbb{Q}(v) \)-subalgebra of \( \mathbf{U} \) generated by
\[ B_i = F_i + \varsigma_i E_i K_i^{-1}, \quad \forall i \in \mathbb{I}. \]
It is known [Let99, Ko14] that \( U^* \) is a right coideal subalgebra of \( U \), i.e., \( \Delta(U^*) \subset U^* \otimes U \); and \((U, U^*)\) is called a quasi-split quantum symmetric pair, as they specialize at \( v = 1 \) to \((U(\mathfrak{g}), U(\mathfrak{g}^\omega))\), where \( \omega \) is the Chevalley involution of \( \mathfrak{g} \).

We call \( U^* \) an quantum group and \( \tilde{U}^* \) a universal quantum group of split type. The algebras \( U^* := U^*_\varsigma \), for \( \varsigma \in (Q(v)^*)^I \), are obtained from \( \tilde{U}^* \) by central reductions.

**Proposition 4.1** ([LW22, Proposition 6.2]). (1) The \( Q(v) \)-algebra \( U^* \) is isomorphic to the quotient of \( \tilde{U}^* \) by the ideal generated by \( k_i - \varsigma_i \) for all \( i \in I \).

(2) The algebra \( \tilde{U}^* \) is a right coideal subalgebra of \( \tilde{U} \).

It is well known that the \( Q(v) \)-algebra \( U^*_\varsigma \) (up to some field extension) is isomorphic to \( U^*_\varsigma \) for some distinguished parameters \( \varsigma^\circ \) (cf. [Let02], [Ko14, Proposition 9.2]; also see [CLW21, Proposition 5]). Throughout this paper, we always assume the distinguished parameters to be

\[
\varsigma^\circ_i = -v^{-2}, \quad \forall i \in I,
\]

and \( U^* = U^*_\varsigma^\circ \); see [LW21, Eq. (7.1)].

For the convenience of readers, we give a presentation of \( U^* \) in the following. For \( i \in I \), generalizing the constructions in [BW18a, BeW18], we define the divided powers of \( B_i \) to be (see also [CLW21])

\[
B_{i,1}^{(m)} = \frac{1}{[m]!} \left\{ \begin{array}{ll}
B_i \prod_{s=1}^k (B_i^2 + v^{-1}[2s-1]^2) & \text{if } m = 2k + 1, \\
\prod_{s=1}^k (B_i^2 + v^{-1}[2s-1]^2) & \text{if } m = 2k;
\end{array} \right.
\]

\[
B_{i,0}^{(m)} = \frac{1}{[m]!} \left\{ \begin{array}{ll}
B_i \prod_{s=1}^k (B_i^2 + v^{-1}[2s]^2) & \text{if } m = 2k + 1, \\
\prod_{s=1}^k (B_i^2 + v^{-1}[2s-2]^2) & \text{if } m = 2k.
\end{array} \right.
\]

The following theorem is an upgrade of (and can be derived from) [CLW21, Theorem 3.1] for \( U^* \) to the setting of a universal quantum group \( \tilde{U}^* \); it generalizes [LW22, Proposition 6.4] for \( \tilde{U}^* \) of ADE type.

**Theorem 4.2** ([CLW21, Theorem 3.1]). Fix \( \bar{\omega}_i \in \mathbb{Z}/2\mathbb{Z} \) for each \( i \in I \). The \( Q(v) \)-algebra \( U^* \) has a presentation with generators \( B_i \) (\( i \in I \)) and the relations

\[
\sum_{n=0}^{1-c_{ij}} (-1)^n B_{i,\bar{\omega}_i}^{(n)} B_j B_{i,\bar{\omega}_i}^{(1-c_{ij}-n)} = 0, \quad \text{if } j \neq i.
\]

(This presentation is called a Serre presentation of \( U^* \).)

### 4.2. Quantum groups and derived Hall algebras

Let \( Q = (Q_0, Q_1) \) be a quiver (not necessarily acyclic), and we sometimes write \( I = Q_0 \). A representation \( V = (V_i, V(\alpha))_{\alpha \in Q_0, \alpha \in Q_1} \) of \( Q \) is called nilpotent if for each oriented cycle \( \alpha_m \cdots \alpha_1 \) at a vertex \( i \), the \( k \)-linear map \( V(\alpha_m) \cdots V(\alpha_1) : V_i \to V_i \) is nilpotent. Let \( \text{rep}_{\text{nil}}^k(Q) \) be the category formed by finite-dimensional nilpotent representations of \( Q \). Let \( S_i \) be the simple modules supported at \( i \in I \).

We denote by \( \text{T} \tilde{H}(kQ) \) the twisted semi-derived Ringel-Hall algebra of \( C_1(\text{rep}_{\text{nil}}^k(Q)) \). Then we have the following result.
Lemma 4.3 ([LW20]). Let $Q$ be an arbitrary quiver. Then there exists a $\mathbb{Q}(\mathbf{v})$-algebra embedding

$$\tilde{\psi}_Q : \tilde{U}_{\mathbf{v}=\mathbf{v}} \longrightarrow \tilde{\mathcal{H}}(kQ),$$

which sends

$$B_i \mapsto -\frac{1}{q-1}[S_i], \quad k_i \mapsto -q^{-1}[K_S], \quad \text{for } i \in \mathbb{I}.$$

Now we can give the second main result of this paper.

Theorem 4.4. Let $Q$ be an arbitrary quiver. Then there exists a $\mathbb{Q}(\mathbf{v})$-algebra embedding

$$\Psi_Q : U_{\mathbf{v}=\mathbf{v}}^t \longrightarrow \mathcal{D}H_1(kQ),$$

which sends

$$B_i \mapsto -\mathbf{v}^{-1}u_{[S_i]}, \quad \text{for } i \in \mathbb{I}.$$

Proof. From Theorem 3.4 and Lemma 4.3, we have an algebra morphism:

$$\Phi \circ \tilde{\psi}_Q : \tilde{U}_{\mathbf{v}=\mathbf{v}}^t \longrightarrow \mathcal{D}H_1(kQ),$$

and its kernel is $\langle k_i + \mathbf{v}^{-2} \mid i \in \mathbb{I} \rangle$. Note that $\Phi \circ \tilde{\psi}_Q(B_i) = -\mathbf{v}^{-1}u_{[S_i]}$ for any $i \in \mathbb{I}$.

Furthermore, it follows from Proposition 4.1 that $U^t = U_{\mathbf{v}^0}$ is isomorphic to the quotient of $\tilde{U}^t$ by the ideal generated by $k_i + \mathbf{v}^{-2}$ for all $i \in \mathbb{I}$. Then $\Phi \circ \tilde{\psi}_Q$ induces the desired $\mathbb{Q}(\mathbf{v})$-algebra embedding

$$\Psi_Q : U_{\mathbf{v}=\mathbf{v}}^t \longrightarrow \mathcal{D}H_1(kQ).$$

□

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