Simple BRST quantization of general gauge models.

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Abstract

It is shown that the BRST charge $Q$ for any gauge model with a Lie algebra symmetry may be decomposed as

$$Q = \delta + \delta^\dagger, \quad \delta^2 = \delta^\dagger 2 = 0, \quad [\delta, \delta^\dagger]_+ = 0$$

provided dynamical Lagrange multipliers are used but without introducing other matter variables in $\delta$ than the gauge generators in $Q$. Furthermore, $\delta$ is shown to have the form $\delta = c^\dagger a \phi_a$ (or $\phi'_a c^\dagger a$) where $c^a$ are anticommuting expressions in the ghosts and Lagrange multipliers, and where the non-hermitian operators $\phi_a$ satisfy the same Lie algebra as the original gauge generators. By means of a bigrading the BRST condition reduces to $\delta|ph\rangle = \delta^\dagger|ph\rangle = 0$ which is naturally solved by $c^a|ph\rangle = \phi_a|ph\rangle = 0$ (or $c^{\dagger a}|ph\rangle = \phi'_a \dagger|ph\rangle = 0$). The general solutions are shown to have a very simple form.
1 Introduction.

In the general investigation performed in [1] it was noted that a BRST quantization on an inner product space is equivalent to a generalized Gupta-Bleuler quantization. Furthermore, it was shown that in this case the BRST charge may always be written in the form

\[ Q = \delta + \delta^\dagger \]  

(1.1)

where

\[ \delta^2 = \delta^\dagger \delta = 0 \]  

(1.2)

and

\[ [\delta, \delta^\dagger]_+ = 0 \]  

(1.3)

consistent with a nilpotent \( Q \). The considered physical states

\[ Q|\text{ph}\rangle = 0 \]  

(1.4)

were shown to satisfy

\[ \delta|\text{ph}\rangle = \delta^\dagger|\text{ph}\rangle = 0 \]  

(1.5)

This was performed within a particular formalism in which the ghost part of the state space was gauge fixed by an allowed gauge fixing [2]. In order for (1.4) to imply (1.5) we need a bigrading of the state space, namely [3]

\[ F|k, m\rangle = k|k, m\rangle, \quad F^\dagger|k, m\rangle = m|k, m\rangle \]  

(1.6)

where \( F \) satisfies

\[ \delta = [F, Q], \quad N = F - F^\dagger \]  

(1.7)

where \( N \) is the ghost number operator. The decomposition (1.1) follows then from (1.7) since \([N, Q] = Q\). (However, the conditions (1.2) and (1.3) restrict the possible forms of \( F \).) We have now

\[ Q|k, m\rangle = 0 \iff \delta|k, m\rangle = 0, \quad \delta^\dagger|k, m\rangle = 0 \]  

(1.8)
Strictly positive normed physical states can only be obtained for states with ghost number zero, \( i.e. \ |m,m\rangle \) for some \( m \). The above properties were summerized in section 5 in [3].

It is instructive to consider a typical example. Consider therefore a gauge model with \( 2m \) independent gauge generators given by \( \phi_a, \phi_a^{\dagger} \ (a = 1, \ldots, m) \) which satisfy

\[
[\phi_a, \phi_b] = iU_{ab} \phi_c \\
[\phi_a, \phi_b^{\dagger}] = 0
\]  

(1.9)

where \( U_{ab} \) are constants. The BRST charge for this model may be written in the form (1.1) where

\[
\delta = \phi_a c^{\dagger a} - \frac{i}{2} U_{ab} c^{\dagger a} \phi^{\dagger b} c^{\dagger b} k_c
\]

(1.10)

where \( c^a \) and \( k_a \) are complex fermionic ghosts and their conjugate momenta satisfying the algebra (the nonzero part)

\[
[k_a, c^{\dagger b}]_+ = \delta^b_a
\]

(1.11)

Eq (1.10) satisfies (1.2) and (1.3). There are two natural forms in which to write \( \delta \). They are

\[
\delta = c^{\dagger a} \phi'_a
\]

(1.12)

or

\[
\delta = \phi''_a c^{\dagger a}
\]

(1.13)

where

\[
\phi'_a = \phi_a + \frac{i}{2} U_{ab}^{\dagger b} - \frac{i}{2} U_{ab} c^{\dagger b} k_c
\]

\[
\phi''_a = \phi_a - \frac{i}{2} U_{ab}^{\dagger b} + \frac{i}{2} U_{ab}^c k_c c^{\dagger b}
\]

(1.14)

Thus, \( \phi''_a = \phi'_a - \frac{i}{2} U_{ab}^{\dagger b} \). The conditions (1.13) are therefore naturally solved by either

\[
c^a |ph\rangle = 0, \quad \phi'_a |ph\rangle = 0
\]

(1.15)

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or

\[ c^a|ph\rangle = 0, \quad \phi_a^{\dagger}|ph\rangle = 0 \]  

(1.16)

Now consistency requires a closed algebra of the constraints. Although

\[ [c^a, \phi_b']_- = [c^a, c^b]_+ = 0 \]  

(1.17)

we have

\[ [\phi_a', \phi'_b]_- = iU_{ab} c^d \phi'_c - \frac{1}{4} U_{ab} c U_{cd} e c^{d} k_c \]  

(1.18)

Hence, (1.13) requires also

\[ k_a|ph\rangle = 0, \]  

(1.19)

and (1.16) requires

\[ k_a^{\dagger}|ph\rangle = 0, \]  

(1.20)

at least for some \( a \). (One may add a term \( f_{ab} c^b k_c \) to \( \phi'_a \) where \( f_{ab} c \) is symmetric in \( a \) and \( b \) without altering \( \delta \) and such that \( \phi'_a + f_{ab} c^b k_c \) satisfy a closed algebra. However, \( f_{ab} c \) is then model dependent.) The conditions (1.19) and (1.20) are natural additional conditions since only the states with ghost number zero contribute to the inner products. Eqs.(1.13) and (1.19) imply (and similarly for (1.16) and (1.20))

\[ (\phi_a + \frac{i}{2} U_{ab}^b)|ph\rangle = 0 \]  

(1.21)

which is consistent. Which of the formal possibilities (1.13), (1.19) or (1.16),(1.20) or a mixture thereof which actually is possible depends on the explicit form of \( \phi_a \) and the state space at hand.

In the above example as well as in all previous treatments the possibility of a generalized Gupta-Bleuler quantization always restrict the possible form of the gauge model. The gauge generators (constraints) are always required to be possible to write as complex conjugate pairs. In this paper we shall show the existence of another possibility. In fact, it will be shown that any bosonic gauge model with finite
number of degrees of freedom leads to a BRST charge which may be written in the form (1.1) where $\delta$ has the simple form (1.12) or (1.13) and that the conditions (1.5) are naturally solved by (1.15) or (1.16). This will be shown to be possible provided dynamical Lagrange multipliers are introduced. In distinction to the above example it will be shown that $\phi'_a$ and $\phi''_a$ may easily be made to satisfy a closed algebra so that (1.19) and (1.20) are not necessary to impose. Furthermore we give general simple solutions of (1.15) and (1.16) which reveal an interesting general structure. (It was the proposal of [4] and the important observations made in the conclusions of [5] which suggested the existence of this construction.)

2 Decompositions of general BRST charges.

Consider for simplicity a general gauge model with finite number of degrees of freedom in which the gauge group is a Lie group. Within the Hamiltonian formulation of the corresponding BRST invariant model the BRST charge may be chosen to be in the general BFV form (2.1) 

$$Q = \psi_a \eta^a - \frac{1}{2} i U_{bc}^a \eta^b \eta^c - \frac{1}{2} i U_{ab}^b \eta^a + \bar{\eta}^a \pi_a$$

where $\psi_a$ are the bosonic gauge generators (constraint) satisfying

$$[\psi_a, \psi_b] = i U_{ab}^c \psi_c$$

where $U_{ab}^c$ are the structure constants. (We consider only first rank theories.) $\eta^a$ and $\bar{\eta}^a$ are Faddeev-Popov (FP) ghosts and antighosts respectively, and $\eta^a$ and $\bar{\eta}^a$ their conjugate momenta. $\pi_a$ is the conjugate momentum to the Lagrange multiplier $v^a$. Their fundamental algebra is (the nonzero part)

$$[\eta^a, \pi_b]_+ = [\bar{\eta}^a, \bar{\pi}_b]_+ = \delta^a_b, \quad [\pi_b, v^a]_- = -i \delta^a_b$$

which when combined with (2.2) makes $Q$ nilpotent. Since $Q$ is required to have ghost number one, $\eta^a$ and $\bar{\eta}^a$ have ghost number one while $\pi_a$ and $\bar{\pi}_a$ have ghost number minus one. The ghost number operator is

$$N = \eta^a \pi_a - \bar{\eta}^a \bar{\pi}_a$$
Since $Q$ is required to be hermitian, $\psi_a$ must either be hermitian or contain hermitian conjugate pairs as in the example given in the introduction. Here we choose $\psi_a$ as well as all the other variables to be hermitian a choice one always may do.

In a first effort to decompose (2.1) according to (1.1) satisfying (1.2) and (1.3) we introduce the following non-hermitian ghosts

$$c^a \equiv \eta^a - i\bar{\mathcal{P}}^a$$
$$k_a \equiv \frac{1}{2}(\mathcal{P}_a - i\bar{\eta}_a)$$ (2.5)

which due to (2.3) satisfy (the nonzero part)

$$[k_a, c^{\dagger}_b]_+ = \delta^b_a$$ (2.6)

In terms of these ghosts $Q$ becomes

$$Q = \frac{1}{2}c^a(\psi_a + i\pi_a) + \frac{1}{2}c^{\dagger}a(\psi_a - i\pi_a) - \frac{1}{4}iU_{bc}^a(c^{\dagger}b)c^c k_a + k^{\dagger}_a c^{\dagger}b c^c$$
$$- \frac{1}{8}iU_{bc}^a(k^{\dagger}_a c^{\dagger}b c^c + k_a c^b c^c c^c + k^{\dagger}_a c^{\dagger}b c^c c^c + c^{\dagger}b c^c k_a)$$ (2.7)

Since the ghost number operator (2.4) now may be written as

$$N = c^{\dagger}a k_a - k^{\dagger}_a c^a$$ (2.8)

it is natural to try to define $\delta$ by

$$\delta = [c^{\dagger}a k_a, Q]$$ (2.9)

However, although this definition together with (2.8) imply $Q = \delta + \delta^{\dagger}$ (2.9) is not nilpotent when $U_{ab}^c \neq 0$. This problem is caused by the terms containing $k_a c^b c^c$ and $k^{\dagger}_a c^{\dagger}b c^c$ in $Q$. They imply

$$[k^{\dagger}_a c^a, \delta] \neq 0$$ (2.10)

We have therefore to get rid of these terms. For this purpose we consider a unitary transformation which only involve the ghosts and the Lagrange multipliers. It turns out that what we need is a transformation of $\bar{\mathcal{P}}^a$ of the following form

$$\bar{\mathcal{P}}^a = M^a_b \bar{\mathcal{P}}^b + K^a_b \eta^b$$ (2.11)
where $M^a_b$ and $K^a_b$ are hermitian matrix elements depending on the Lagrange multipliers $v^a$. Hermiticity and ghost numbers are preserved by (2.11). Since it requires

$$\tilde{\eta}_a = (M^{-1})^b_a \eta'_b$$

(2.12)

the matrix $M^a_b$ must be nonsingular for all $v^a$. If we fix the ghosts and Lagrange multipliers, i.e.

$$\eta^a = \eta'^a, \quad v^a = v'^a$$

(2.13)

then (2.11) and (2.12) require

$$\mathcal{P}_a = \mathcal{P}'_a - (M^{-1})^b_c K^c_b \tilde{\eta}'_b$$

$$\pi_a = \pi'_a + iH_{ab} \tilde{\eta}'_a \eta'^b + iG_{ab}^c (\tilde{\eta}'_c \tilde{\mathcal{P}}'^b - \tilde{\mathcal{P}}'^b \tilde{\eta}'_c)$$

(2.14)

where

$$H_{ab}^c = -(M^{-1})^c_d \partial_a K^d_b$$

$$G_{ab}^c = -\frac{1}{2} (M^{-1})^c_d \partial_a M^d_b$$

(2.15)

The primed variables satisfy the same algebra as the original unprimed ones. The inverse transformation is given by (2.13) together with

$$\tilde{\mathcal{P}}'^a = (M^{-1})^a_b (\tilde{\mathcal{P}}'^b - K^b_c \eta'^c); \quad \tilde{\eta}'_a = M^b_a \tilde{\eta}_b, \quad \mathcal{P}'_a = \mathcal{P}_a + K^b_a \tilde{\eta}_b$$

$$\pi'_a = \pi_a + iH'_{ab} \tilde{\eta}_c \eta'^b + iG'_{ab}^c (\tilde{\eta}_c \tilde{\mathcal{P}}'^b - \tilde{\mathcal{P}}'^b \tilde{\eta}_c)$$

(2.16)

where

$$H'_{ab}^c = \partial_a K^c_b - \partial_a M^c_d (M^{-1})^d_b K^c_b$$

$$G'_{ab}^c = \frac{1}{2} \partial_a M'^d_b (M^{-1})^d_b$$

(2.17)

We insert now the transformation (2.11)-(2.14) into the BRST charge (2.1). We find then (the primes are suppressed in the following)

$$Q = \psi_a \eta^a - \frac{1}{2} i U_{ab}^c \eta^b \eta'^c + M^a_b \pi_a \tilde{\mathcal{P}}^b + K^b_a \pi_a \eta^b + iM^b_a G^c_{bc} \tilde{\mathcal{P}}^b + iK^b_a G^c_{ac} \eta^b -$$

$$\frac{1}{2} i U_{bc}^a \mathcal{P}_a \eta^b \eta'^c - iT_{bc}^d \tilde{\eta}_d \eta^b \eta'^c + iR_{bc}^d \tilde{\mathcal{P}}'^d \tilde{\eta}_c \eta'^c - iV_{bd}^c \tilde{\mathcal{P}}'^b \tilde{\eta}_d \eta'^c$$

(2.18)
where $G_{ab}^c$ is given by (2.15) and
\begin{align*}
T_{bc}^d &\equiv -\frac{1}{2}(M^{-1})_e^d \{ U_{bc}^a K_e^e + K^e_b \partial_a K_c^e - K^e_c \partial_a K_b^e \} \\
V_{bc}^d &\equiv -\frac{1}{2}(M^{-1})_e^d \{ M_b^a \partial_a M_c^e - M_c^a \partial_a M_b^e \} \\
R_{bc}^d &\equiv (M^{-1})_e^d \{ K^a_c \partial_a M_b^e - M_b^a \partial_a K_c^e \} 
\end{align*}

If we now express $Q$ in terms of non-hermitian ghosts $c^a$ and $k_a$ defined as in (2.5) and satisfying (2.6) we find
\begin{align*}
Q = \frac{1}{2}(c^a \psi'_a + c^a \psi'^*_a) + \frac{1}{4} \left( -\frac{1}{2} i U_{bc}^a - i R_{bc}^a + T_{bc}^a - V_{bc}^a \right) c^b c^c k_a \\
+ \frac{1}{4} \left( -\frac{1}{2} i U_{bc}^a - i R_{bc}^a - T_{bc}^a + V_{bc}^a \right) k_a^b k_a^c \\
+ \frac{1}{4} (i U_{bc}^a + i R_{bc}^a + i R_{cb}^a - 2 T_{bc}^a - 2 V_{bc}^a) c^b k_a^c \\
+ \frac{1}{4} (i U_{bc}^a - i R_{bc}^a - i R_{cb}^a + 2 T_{bc}^a + 2 V_{bc}^a) c^b k_a^c \\
+ \frac{1}{4} \left( -\frac{1}{2} i U_{bc}^a + i R_{bc}^a + T_{bc}^a - V_{bc}^a \right) k_a^b c^c \\
+ \frac{1}{4} \left( -\frac{1}{2} i U_{bc}^a + i R_{bc}^a - T_{bc}^a + V_{bc}^a \right) k_a^b c^c \\
+ \frac{1}{4} \left( -\frac{1}{2} i U_{bc}^a + i R_{bc}^a + T_{bc}^a - V_{bc}^a \right) k_a^b c^c \\
+ \frac{1}{4} \left( -\frac{1}{2} i U_{bc}^a + i R_{bc}^a - T_{bc}^a + V_{bc}^a \right) k_a^b c^c 
\end{align*}

where
\begin{align*}
\psi'_a &\equiv \psi_a + K_a^b \pi_b - i M_a^b \pi_b + i K_a^b G_{bc}^c + M_a^b G_{bc}^c + \\
&+ T_{ba}^b - V_{ab}^b + \frac{1}{2} i R_{ab}^b + \frac{1}{2} i R_{ba}^b 
\end{align*}

In order for $\delta = [c^a k_a, Q]$ to be nilpotent the coefficients of the terms involving $k_a c^b c^c$ and $k_a^b c^b c^c$ must vanish. This leads to the conditions
\begin{align*}
T_{bc}^a - V_{bc}^a &= 0 \\
R_{bc}^a - R_{cb}^a - U_{bc}^a &= 0 
\end{align*}

which may be simplified to
\begin{align*}
L_c^a \partial_c L_b^d - L_b^c \partial_c L_a^d &= -i U_{ab}^c L_c^d 
\end{align*}

where we have introduced the complex matrix
\begin{align*}
L_a^b &= M_a^b + i K_a^b 
\end{align*}
The meaning of the conditions (2.23) may be obtained by considering the \( \pi \)-term in the expression (2.21) of \( \psi'_a \) and \( \psi'^*_a \). They are

\[
\phi_a \equiv -iL_{ab}^b \pi_b, \quad \phi'_a \equiv iL_{ab}^b \pi_b
\]  

(2.25)

Conditions (2.23) imply then

\[
[\phi_a, \phi_b] = iU_{ab}^c \phi_c \\
[\phi'_a, \phi'_b] = iU_{ab}^c \phi'_c
\]

(2.26)

The Jacobi identities

\[
\sum_{(abc)} [[\phi_a, \phi'_b], \phi_c] = 0
\]

(2.27)

require furthermore

\[
[\phi_a, \phi'_b] = 0
\]

(2.28)

or equivalently

\[
L_{a}^{c}\partial_{c} L_{b}^{d} - L_{b}^{c}\partial_{c} L_{a}^{d} = 0
\]

(2.29)

\( L_{a}^{c} \) may then be recognized as the left vielbein field on the group manifold with imaginary group coordinates \( \theta^a = iv^a \). \( \phi_a \) and \( \phi'_a \) in (2.23) may be interpreted as generators of left and right translations on the group manifold. One may use (2.23) and (2.29) to solve for \( L_{a}^{b} \) assuming that \( L_{a}^{b} \) is analytic in \( v^a \) and that \( L_{b}^{a}(v^a = 0) = \delta_{b}^{a} \). Since

\[
M_{b}^{a} = \frac{1}{2}(L_{b}^{a} + L_{b}^{b}) \\
K_{b}^{a} = -\frac{1}{2}i(L_{b}^{a} - L_{b}^{b})
\]

(2.30)

one may notice that \( M_{b}^{a}(-v) = M_{b}^{a}(v), M_{b}^{a}(v = 0) = \delta_{b}^{a} \) and \( K_{b}^{a}(-v) = -K_{b}^{a}(v) \).

The conditions (2.23) and (2.29) reduce now the BRST charge (2.20) to the following form

\[
Q = \frac{1}{2} c_i \Phi_i^a + \frac{1}{2} \Phi^i_{a} c^a
\]

(3.31)
where

\[ \Phi_a = \psi_a - iL^b_a \pi_b - \frac{1}{2} \partial_b L^b_a + g^b_{ab} k^c b + 2g_{ab} c^b c^b \phi - \frac{1}{2} iU_{ab} c^i b k_c \]  

(2.32)

where in turn

\[ g^c_{ab} \equiv \frac{1}{4} iU_{ad} L^e_d (M^{-1})^e_c \]  

(2.33)

Notice that

\[ \delta \equiv [c^{ia} k_a, Q] = \frac{1}{2} c^{ia} \Phi_a \]  

(2.34)

Since (cf (1.18)!)

\[ [\Phi_a, \Phi_b] = iU_{ab} c^e \Phi_c - \frac{1}{4} U_{ab} c^e U_{ce} d^e k_d \]  

\[ [\Phi_a, c^{ib}] = -\frac{1}{2} iU_{ac} b^e c^{ie} \]  

(2.35)

one easily verifies that \( \delta^2 = 0 \). \([\delta, \delta^\dagger]_+ = 0\) follows then automatically since \( Q^2 = 0 \) by construction. (However, we have also checked it explicitly.) Thus, we have achieved our goal to prove the existence of the decomposition (1.1) with the properties (1.2) and (1.3) for a general gauge model.

### 3 The implied general Gupta-Bleuler quantization.

If we impose the bigrading considered in the introduction then we may solve \( Q|ph\rangle = 0 \) by

\[ \delta|ph\rangle = 0, \quad \delta^\dagger|ph\rangle = 0 \]  

(3.1)

In view of the general result that \( \delta = \frac{1}{2} c^{ia} \Phi_a \) these condition are naturally solved by the conditions

\[ c^a|ph\rangle = 0, \quad \Phi_a|ph\rangle = 0 \]  

(3.2)
However, the algebra in (2.35) requires

\[ k_a |ph\rangle = 0 \quad (3.3) \]

at least for some \( a \). Although this condition only eliminates solutions with zero inner products, it may be avoided if we change the last conditions in (3.2). We notice then that \( \delta \) is unaffected if we replace \( \Phi_a \) by \( \Phi_a + f_{ab} c^{ib} k_c \) where \( f_{ab} \) is symmetric in \( a \) and \( b \). In particular we may choose

\[
\Phi'_a \equiv \Phi_a + \frac{1}{2}(L_a^d \partial_d L_b^e + L_b^d \partial_d L_a^e)(L^{-1})^c_d c^{ib} k_c
\]

which satisfies

\[
[\Phi'_a, \Phi'_b] = iU_{ab} c^c \Phi'_c
\]

Since

\[
[\Phi'_a, c^b] = -2g_{ac} b^c
\]

we may then consistently solve (3.1) by

\[
c^a |ph\rangle = 0, \quad \Phi'_a |ph\rangle = 0
\]

Another possibility is

\[
c^{ta} |ph\rangle = 0, \quad \Phi''_{a} |ph\rangle = 0
\]

where \( \Phi''_a \) is defined by \( \delta = \frac{1}{2} \Phi''_a c^{ta} \) and which may be chosen to be

\[
\Phi''_a = \Phi'_a - \partial_b L_a^b
\]

which also satisfies the closed algebra (3.3).

4 Solving general Gupta-Bleuler quantization.

Consider the general BRST charge (2.1) in terms of the original ghost variables. Consider then in particular the BRST invariant operator

\[
A \equiv [\rho, Q]_+
\]

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where

\[ \rho \equiv v^a P_a \]  

(4.2)

It is explicitly

\[ A \equiv (\psi_a + \psi_{a}^{gh})v^a - i P_a \bar{P}^a \]  

(4.3)

where

\[ \psi_{a}^{gh} \equiv \frac{1}{2}i U^{c}_{ab} (P_c \eta^b - \eta^b P_c) \]  

(4.4)

is hermitian and satisfies the same algebra as \( \psi_a \). We notice then that

\[ e^A \eta^a e^{-A} = (L^{-1})^a_b (L^{-1})^c_b \eta^c - i \bar{P}^c \]  

(4.5)

These expressions may be compared to \( c^a \) and \( c^a \) in terms of the original ghost variables (use (2.16)):

\[ c^a = \eta^a - i \bar{P}^a = (M^{-1})^a_b (L^{-1})^c_b \eta^c - i \bar{P}^c \]  

(4.6)

Obviously

\[ c^a |ph\rangle = 0 \iff e^A \eta^a e^{-A} |ph\rangle = 0 \]  

(4.7)

So far nothing is achieved. However, consider now the same transformations of the conjugate momentum to \( v^a \). We find

\[ e^A \phi_a e^{-A} = \psi_a + \psi_{a}^{gh} - i L^{-1}_{a} \pi_b - i (L^{-1})^c_e \partial_d L^e_a P_c \bar{P}^d \]  

(4.8)

where

\[ \phi_a = -i L^{-1}_{a} \pi_b \]  

(4.9)

Expressing the right hand side of (4.8) in terms of the new ghost variables by means of (2.11)-(2.14) we find

\[ e^A \phi_a e^{-A} = \Phi^f_a + (L^{-1})^d_b \partial_d L^e_a (L^{-1})^c_e - g_{ab}^c k_c e^b \]  

(4.10)
Hence, $e^a|ph⟩ = 0$, $\Phi'_a|ph⟩ = 0$ has the solution

$$|ph⟩ = e^A|\phi⟩$$  \hspace{1cm} (4.11)

where $|\phi⟩$ satisfies

$$\eta^a|\phi⟩ = \pi_a|\phi⟩ = 0$$  \hspace{1cm} (4.12)

which is trivially solved. Similarly

$$e_{-A}\phi'_a e^A = (e^A\phi_a e^{-A})^\dagger - \partial_b L_b^\dagger =$$

$$= (\Phi''_a)^\dagger - (L_d^q \partial_d L_e^r c (L^\dagger_{c,a} c^\dagger e_a) - g_{ab}) k_c^\dagger c^b$$  \hspace{1cm} (4.13)

where

$$\phi'_a = i L_a^b \pi_b$$  \hspace{1cm} (4.14)

Hence,

$$e^{iA}|ph⟩ = 0, \Phi^{ra}_{a}|ph⟩ = 0$$  \hspace{1cm} (4.15)

is solved by

$$|ph⟩ = e^{-A}|\phi⟩$$  \hspace{1cm} (4.16)

where $|\phi⟩$ satisfies (4.12). Since $e^{\pm A}$ is BRST invariant also $|\phi⟩$ must be BRST invariant. In fact, $Q|\phi⟩ = 0$ follows from (4.12). Notice that $|\phi⟩$ does not belong to an inner product space although $|ph⟩$ must do.

5 Further properties of the general solutions.

We have found that the general Gupta-Bleuler quantization implied by $\delta|ph⟩ = \delta^\dagger|ph⟩ = 0$ lead to two different solutions

$$|ph⟩_\pm = e^{\pm [\rho, Q]}|\phi⟩$$  \hspace{1cm} (5.1)
We must therefore require that these two sets of solutions yield equivalent physical results. Now, there are even more solutions. In fact

$$|ph\rangle_\alpha = e^{\alpha [\rho, Q]} |\phi\rangle$$

(5.2)
is a satisfactory solution for any real \(\alpha \neq 0\). The reason for this is that the BRST charge \(Q\) is invariant under the unitary transformation

$$\left(\pi_a, v^a\right) \rightarrow \left(\frac{\pi_a}{\alpha}, \alpha v^a\right), \quad \left(\mathcal{P}^a, \bar{\eta}_a\right) \rightarrow \left(\alpha \mathcal{P}^a, \frac{\bar{\eta}_a}{\alpha}\right)$$

(5.3)
where \(\alpha\) is a real positive constant. However, \(\delta\) and \(\rho\) are not invariant under (5.3). Instead we have

$$\rho \rightarrow \alpha \rho, \quad \delta \rightarrow \delta' = U\delta U^\dagger, \quad |ph\rangle_\alpha = U|ph\rangle_\pm$$

(5.4)
where \(e^{a[\rho, Q]} = Ue^{\pm[a, Q]}U^\dagger\). Notice that \(|\phi\rangle' = U|\phi\rangle\) satisfies (4.12) for any \(\alpha \neq 0\) in (5.2). Thus, we have proved that (5.2) reduces to the two possible solutions (5.1). (Notice that the solution (5.2) corresponds to a different bigrading than (5.1).)

Consider now the inner product of (5.2). We find

$$\langle \phi | e^{2\alpha[a, Q]} |\phi\rangle = '\langle \phi | e^{\pm[a, Q]} |\phi\rangle'$$

(5.5)
where we have performed a unitary scale transformation of the type (5.3). Since only the ghost number zero part of \(|\phi\rangle'\) contribute in the inner product we have

$$\alpha \langle ph | ph\rangle_\alpha = '\langle \phi | e^{\pm[a, Q]} |\phi\rangle' = '\langle \phi | e^{\pm[a, Q]} |\phi\rangle'_0$$

(5.6)
where \(|\phi\rangle'_0\) satisfies

$$\bar{\eta}_a |\phi\rangle'_0 = 0$$

(5.7)
apart from (4.12). (Notice that \([\rho, Q]\) has ghost number zero.) Eq.(5.7) implies then that

$$|ph\rangle_\pm = e^{\pm[a, Q]} |\phi\rangle'_0$$

(5.8)
satisfies

$$k_a |ph\rangle_\pm = 0, \quad k_a^\dagger |ph\rangle_\pm = 0$$

(5.9)
apart from (3.7) and (3.8). These conditions imply that the solutions (5.8) have
ghost number zero. Eq.(5.9) could also follow from antiBRST invariance. (Notice,
however, that \([\rho, Q]\) is only antiBRST invariant for totally antisymmetric structure
constants \(U_{abc}\).) In \(|\phi\rangle_0\) in (5.8) the ghost as well as the Lagrange multiplier
dependence is completely fixed. This means that all dependence on ghosts and
Lagrange multipliers in (5.8) is through the factor \(e^{\pm[\rho, Q]}\). In the inner products
all ghost dependence disappears after a reduction of the factor \(e^{\pm[\rho, Q]}\) and only an
integration over the Lagrange multiplier \(v^a\) remains. However, this is the tricky
part. Firstly, there is an ambiguity in how the Lagrange multipliers should be
quantized. In [5] the following general rule was extracted from [4]: The Lagrange
multipliers must be quantized with opposite metric states to the variables which
the corresponding gauge generators eliminate. Thus, one has to specify exactly how
the original matter variables should be quantized. Secondly, one has to specify the
range of the Lagrange multipliers. Typically one may find that the range must be
restricted in order for \(|ph\rangle\) to belong to an inner product space. In such a case they
should probably at most be restricted to the group manifold. In the case when the
matter variables are quantized with positive metric states we find formally

\[ a\langle ph|ph\rangle_a \propto \int dx|\delta(\psi'_a)|2 \]  

(5.10)

since when \(v^a\) is quantized with negative metric states the spectral representation
requires imaginary eigenvalues [3], which means that we obtain

\[ \int d^mve^{\pm iv^a\psi'_a} \propto \delta(\psi'_a) \]  

(5.11)

in (5.6). We have also introduced a complete set of states \(|x\rangle\) in the matter space
in (5.10), i.e. \(\int dx|x\rangle \langle x| = 1\).

6 Final remarks.

We have proved that the BRST charge \(Q\) may be decomposed as

\[ Q = \delta + \delta^\dagger \]  

(6.1)
where $\delta$ satisfies (1.2) and (1.3) for any gauge model with finite number of degrees of freedom and whose gauge symmetry is a Lie group. Furthermore, we have shown that the general solutions are very simple to

$$\delta|ph\rangle = \delta^\dagger|ph\rangle = 0,$$

which is equivalent to $Q|ph\rangle = 0$ when the bigrading (1.6) is used. However, the solutions could not be made completely explicit since it remains to specify the properties of the Lagrange multipliers which depend on the more detailed structure of the matter states. Each model has therefore to be investigated separately in order to determine these properties exactly. Some examples are given in [8].

The general decomposition (6.1) found in section 2 and defined up to the scale transformations (5.4) are not the only possible ones. However, other decompositions will involve other matter variables than the gauge generators $\psi_a$ which will make $\delta$ less attractive and the corresponding solutions not so useful as those found in this paper. Now we have only proved (6.1) for bosonic gauge models with finite number of degrees of freedom. However, we believe that the generalization to graded gauge groups is straight-forward and that most formulas are trivially generalized to this case. Examples of this type are given in [8]. Also theories with second class constraints or anomalous gauge theories should be possible to treat by this method since the bigrading leading to (6.2) were shown to be essential for such models in [3]. However, this application involves additional features since the property (1.3) is no longer valid. The generalization to gauge theories of higher rank than the first is of course highly nontrivial and to which the present method is not immediately generalizable. (However, one could try transformations of the type (4.5).) The method is also not trivially generalizable to gauge models with infinite degrees of freedom. However, for strings the oscillator part may be decomposed like in the introduction as was first made by Banks and Peskin [1]. Our method suggests then that if one also introduces dynamical Lagrange multiplier fields also the zero mode part will be possible to decompose as in (6.1) making the complete BRST charge in the form (6.1).
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