On the Cauchy problem of a two-component b-family equation

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Abstract
In this paper, we study the Cauchy problem of a two-component b-family equation. We first establish the local well-posedness for a two-component b-family equation by Kato’s semigroup theory. Then, we derive precise blow-up scenarios for strong solutions to the equation. Moreover, we present several blow-up results for strong solutions to the equation.

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1 Introduction
In this paper we consider the following two-component b-family equation:

\[
\begin{align*}
    m_t &= um_x + k_1u_xm + k_2\rho x, \quad t > 0, \quad x \in \mathbb{R}, \\
    \rho_t &= k_3(u\rho)_x, \quad t > 0, \quad x \in \mathbb{R}, \\
    m(0, x) &= m_0(x), \quad x \in \mathbb{R}, \\
    \rho(0, x) &= \rho_0(x), \quad x \in \mathbb{R},
\end{align*}
\]

where \( m = u - u_{xx} \) and there are two cases about this system: (i) \( k_1 = b, k_2 = 2b \) and \( k_3 = 1 \); (ii) \( k_1 = b + 1, k_2 = 2 \) and \( k_3 = b \) with \( b \in \mathbb{R} \). Eq.(1.1) was recently introduced by Guha in [1]. The two-component b-family equation is defined on a infinite-dimensional Lie group in [2], which is the group of orientation-preserving diffeomorphisms of the circle. The group \( Diff(S^1) \) of smooth orientation-preserving diffeomorphisms of the circle \( S^1 \) is endowed with a smooth manifold structure based on the Fréchet space \( C^\infty(S^1) \). The composition and inverse are both smooth maps so that \( Diff(S^1) \) is a Lie group modeled on Fréchet space, see [1] for details.

For \( \rho \equiv 0 \), Eq.(1.1) becomes the b-family equation

\[
u_t - \alpha^2 u_{txx} + c_0 u_x + (b + 1) uu_x + \Gamma u_{xxx} = \alpha^2(bu_xu_{xx} + uu_{xx}).\]

Eq.(1.2) can be derived as the family of asymptotically equivalent shallow water wave equations that emerge at quadratic order accuracy for any \( b \neq -1 \) by an appropriate Kodama transformation, cf.[3-4]. For the case \( b = -1 \), the corresponding Kodama transformation is singular and the asymptotic ordering is violated, cf.[3-4].

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With $\alpha = 0$ and $b = 2$ in Eq.(1.2), we find the well-known KdV equation which describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity \cite{5}. The Cauchy problem of the KdV equation has been studied by many authors \cite{6-8} and a satisfactory local or global (in time) existence theory is now available (e.g. see \cite{7-8}). For $b = 2$ and $\gamma = 0$, Eq.(1.2) becomes the Camassa-Holm equation, modelling the unidirectional propagation of shallow water waves over a flat bottom. The Cauchy problem of the Camassa-Holm equation has been the subject of a number of studies, for example \cite{9-10}. For $b = 3$ and $c_0 = \gamma = 0$, then we find the Degasperis-Procesi equation \cite{11} from Eq.(1.2), which is regarded as a model for nonlinear shallow water dynamics. There are also many papers involving Degasperis-Procesi equation, e.g.\cite{12-13}. The advantage of the Camassa-Holm equation and the Degasperis-Procesi equation in comparison with the KdV equation lies in the fact that these two equations have peakon solitons and models wave breaking \cite{14-15}.

In \cite{16} and \cite{17}, the authors studied Eq.(1.2) on the line and on the circle respectively for $\alpha > 0$ and $b, c_0, \Gamma \in \mathbb{R}$. In \cite{16} and \cite{17}, the authors established the local well-posedness, described the precise blow-up scenario, proved the equation has strong solutions which exist globally in time and blow up in finite time. Moreover, the authors showed the existence of global weak solution to Eq.(1.2) on the line and on the circle respectively.

For $\rho \neq 0$, if $k_1 = 2$, Eq.(1.1) becomes two-component Camassa-Holm equation. A classical two-component Camassa-Holm equation

$$
\begin{cases}
m_t + um_x + 2u_x m + \sigma \rho x = 0, & t > 0, x \in \mathbb{R}, \\
\rho_t + (u \rho)_x = 0, & t > 0, x \in \mathbb{R},
\end{cases}
$$

where $m = u - u_{xx}$, $\sigma = \pm 1$ was derived by Constantin and Ivanov \cite{18} in the context of shallow water theory. The variable $u(x,t)$ describes the horizontal velocity of the fluid and the variable $\rho(x,t)$ is in connection with the horizontal deviation of the surface from equilibrium, all measured in dimensionless units \cite{18}. The extended $N = 2$ super-symmetric Camassa-Holm equation was presented recently by Popowicz in \cite{19}. The mathematical properties of the two-component Camassa-Holm equation have been studied in many works cf.\cite{18, 20-25}.

For $\rho \neq 0$ and $b \in \mathbb{R}$, the Cauchy problem of Eq.(1.1) has not been studied yet. The aim of this paper is to establish the local well-posedness, to derive precise blow-up scenarios, to prove the existence of strong solutions which blow up in finite time for Eq.(1.1).

Our paper is organized as follows. In Section 2, we establish the local well posedness of Eq.(1.1). In Section 3, we derive two precise blow-up scenarios for Eq.(1.1). In Section 4, we discuss the blow-up phenomena of Eq.(1.1).

**Notation** Given a Banach space $Z$, we denote its norm by $\| \cdot \|_Z$. Since all space of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity. We let $[A, B]$ denote the commutator of linear operator $A$ and $B$. For convenience, we let $(\cdot|\cdot)_{s \times r}$ and $(\cdot|\cdot)_s$ denote the inner products of $H^s \times H^r$, $s, r \in \mathbb{R}_+$ and $H^s$, $s \in \mathbb{R}_+$, respectively.
2 Local well-posedness

In this section, we will apply Kato’s theory to establish the local well-posedness for the Cauchy problem of Eq.(1.1).

For convenience, we state here Kato’s theory in the form suitable for our purpose. Consider the abstract quasi-linear equation:

\[
\frac{dv}{dt} + A(v)v = f(v), \quad t > 0, \quad v(0) = v_0.
\] (2.1)

Let \(X\) and \(Y\) be Hilbert spaces such that \(Y\) is continuously and densely embedded in \(X\) and let \(Q : Y \to X\) be a topological isomorphism. Let \(L(Y, X)\) denote the space of all bounded linear operator from \(Y\) to \(X\) \((L(X), \text{if } X = Y.)\). Assume that:

(i) \(A(y) \in L(Y, X)\) for \(y \in Y\) with

\[
\|(A(y) - A(z))w\|_X \leq \mu_1\|y - z\|_X\|w\|_Y, \quad y, z, w \in Y,
\]

and \(A(y) \in G(X, 1, \beta), \) i.e. \(A(y)\) is quasi-m-accretive), uniformly on bounded sets in \(Y\).

(ii) \(QA(y)Q^{-1} = A(y) + B(y), \) where \(B(y) \in L(X)\) is bounded, uniformly on bounded sets in \(Y\). Moreover,

\[
\|(B(y) - B(z))w\|_X \leq \mu_2\|y - z\|_Y\|w\|_X, \quad y, z \in Y, \ w \in X.
\]

(iii) \(f : Y \to Y\) and extends also to a map from \(X\) to \(X\). \(\ f\) is bounded on bounded sets in \(Y\), and

\[
\|f(y) - f(z)\|_Y \leq \mu_3\|y - z\|_Y, \quad y, z \in Y,
\]

\[
\|f(y) - f(z)\|_X \leq \mu_4\|y - z\|_X, \quad y, z \in Y.
\]

Here \(\mu_1, \mu_2, \mu_3\) and \(\mu_4\) depend only on \(\max\{\|y\|_Y, \|z\|_Y\}\).

Theorem 2.1 (26). Assume that (i), (ii) and (iii) hold. Given \(v_0 \in Y\), there is a maximal \(T > 0\) depending only on \(\|v_0\|_Y\) and a unique solution \(v\) to Eq.(2.1) such that

\[
v = v(\cdot, v_0) \in C([0, T); Y) \cap C^1([0, T); X).
\]

Moreover, the map \(v_0 \to v(\cdot, v_0)\) is continuous from \(Y\) to

\[
C([0, T); Y) \cap C^1([0, T); X).
\]
or the equivalent form:

\[
\begin{aligned}
  u_t - uu_x &= \partial_x(1 - \partial_x^2)^{-1}(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2), \quad t > 0, x \in \mathbb{R}, \\
  \rho_t - k_3u\rho_x &= k_3u_x\rho, \quad t > 0, x \in \mathbb{R}, \\
  u(0, x) &= u_0(x), \quad x \in \mathbb{R}, \\
  \rho(0, x) &= \rho_0(x).
\end{aligned}
\]

(2.3)

The main result in this section is the following theorem.

**Theorem 2.2.** Given \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s \geq 2 \), there exists a maximal \( T = T(\| z_0 \|_{H^s \times H^{s-1}}) > 0 \), and a unique solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to Eq. (2.3) such that

\[
z = z(\cdot, z_0) \in C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2}).
\]

Moreover, the solution depends continuously on the initial data, i.e. the mapping

\[
z_0 \to z(\cdot, z_0) : H^s \times H^{s-1} \to C([0, T); H^s \times H^{s-1}) \cap C^1([0, T); H^{s-1} \times H^{s-2})
\]

is continuous.

The remainder of this section is devoted to the proof of Theorem 2.2.

Let \( z := \begin{pmatrix} u \\ \rho \end{pmatrix} \), \( A(z) = \begin{pmatrix} -u\partial_x & 0 \\ 0 & -k_3u\partial_x \end{pmatrix} \) and

\[
f(z) = \begin{pmatrix} \partial_x(1 - \partial_x^2)^{-1}(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2) \\ k_3u_x\rho \end{pmatrix}.
\]

Set \( Y = H^s \times H^{s-1}, X = H^{s-1} \times H^{s-2}, \Lambda = (1 - \partial_x^2)^{\frac{1}{2}} \) and \( Q = \begin{pmatrix} \Lambda & 0 \\ 0 & \Lambda \end{pmatrix} \). Obviously, \( Q \) is an isomorphism of \( H^s \times H^{s-1} \) onto \( H^{s-1} \times H^{s-2} \). In order to prove Theorem 2.2, in view of Theorem 2.1, we only need to verify \( A(z) \) and \( f(z) \) satisfy the conditions (i)-(iii).

We first recall the following lemma.

**Lemma 2.1** ([27]). Let \( r, t \) be real numbers such that \( -r < t \leq r \). Then

\[
\| fg \|_{H^t} \leq c \| f \|_{H^r} \| g \|_{H^t}, \quad \text{if } r > \frac{1}{2},
\]

\[
\| fg \|_{H^{t+r-rac{1}{2}}} \leq c \| f \|_{H^r} \| g \|_{H^t}, \quad \text{if } r < \frac{1}{2},
\]

where \( c \) is a positive constant depending on \( r, t \).
Similar to the proofs of Lemmas 2.5-2.7 in [21], we get the following three lemmas.

**Lemma 2.2.** The operator $A(z) = \begin{pmatrix} -u\partial_x & 0 \\ 0 & -k_3u\partial_x \end{pmatrix}$ with $z \in H^s \times H^{s-1}$, $s \geq 2$, belongs to $G(H^{s-1} \times H^{s-2}, 1, \beta)$.

**Lemma 2.3.** Let $A(z) = \begin{pmatrix} -u\partial_x & 0 \\ 0 & -k_3u\partial_x \end{pmatrix}$ with $z \in H^s \times H^{s-1}$, $s \geq 2$. Then $A(z) \in L(H^s \times H^{s-1}, H^{s-1} \times H^{s-2})$ and

$$\|(A(z) - A(y))w\|_{H^s \times H^{s-1}} \leq \mu_1\|z - y\|_{H^{s-1} \times H^{s-2}}\|w\|_{H^s \times H^{s-1}},$$

for all $z, y, w \in H^s \times H^{s-1}$.

**Lemma 2.4.** Let $B(z) = QA(z)Q^{-1} - A(z)$ with $z \in H^s \times H^{s-1}$, $s \geq 2$. Then $B(z) \in L(H^{s-1} \times H^{s-2})$ and

$$\|(B(z) - B(y))w\|_{H^{s-1} \times H^{s-2}} \leq \mu_2\|z - y\|_{H^s \times H^{s-1}}\|w\|_{H^{s-1} \times H^{s-2}},$$

for all $z, y \in H^s \times H^{s-1}$ and $w \in H^{s-1} \times H^{s-2}$.

We now prove that $f$ satisfies the condition (iii) in Theorem 2.1.

**Lemma 2.5.** Let $z \in H^s \times H^{s-1}, s \geq 2$ and let

$$f(z) = \begin{pmatrix} \partial_x(1 - \partial_k^2)^{-1}(\frac{k_1}{2}(y_1^2 - u^2) - \frac{3}{2}k_1(y_1, x - u_x)^2 + \frac{k_2}{2}(y_2 - \rho^2)) \\ k_3u_x\rho \end{pmatrix}.$$

Then $f$ is bounded on bounded sets in $H^s \times H^{s-1}$, and for all $y, z \in H^s \times H^{s-1}$ satisfies

(a) $\|f(y) - f(z)\|_{H^s \times H^{s-1}} \leq \mu_3\|y - z\|_{H^s \times H^{s-1}},$

(b) $\|f(y) - f(z)\|_{H^{s-1} \times H^{s-2}} \leq \mu_4\|y - z\|_{H^{s-1} \times H^{s-2}}.$

**Proof** Let $y, z \in H^s \times H^{s-1}, s \geq 2$. Note that $H^{s-1}$ is a Banach algebra. Then, we have

$$\|f(y) - f(z)\|_{H^s \times H^{s-1}} \leq \|\partial_x(1 - \partial_k^2)^{-1}(\frac{k_1}{2}(y_1^2 - u^2) - \frac{3}{2}k_1(y_1, x - u_x)^2 + \frac{k_2}{2}(y_2 - \rho^2))\|_{H^s} + |k_3|$$

$$\|y_1, x, y_2 - u_x\rho\|_{H^{s-1}}$$

$$\leq \frac{|k_1|}{2}\|\partial_x(y_1 - u)(y_1 + u)\|_{H^{s-1}} + \frac{|3 - k_1|}{2}\|\partial_x(y_1, x - u_x)(y_1, x + u_x)\|_{H^{s-1}} +$$

$$|k_2|\|\partial_x(y_2 - \rho)(y_2 + \rho)\|_{H^{s-1}} + |k_3|\|\partial_x(y_1, x, y_2 - y_1, x, u_x)\|_{H^{s-1}} +$$

$$|y_1, x, y_2 - y_1, x, \rho - u_x\rho|_{H^{s-1}}$$

$$\leq \frac{|k_1|}{2}\|y_1 - u\|_{H^{s-1}}\|y_1 + u\|_{H^{s-1}} + \frac{|3 - k_1|}{2}\|y_1 - u\|_{H^s}\|y_1 + u\|_{H^s} +$$

$$|k_2|\|y_2 - \rho\|_{H^{s-1}}\|y_2 + \rho\|_{H^{s-1}} + |k_3|\|y_1\|_{H^s}\|y_2 - \rho\|_{H^{s-1}} +$$

$$|k_3|\|y_1 - u\|_{H^s}\|\rho\|_{H^{s-1}}$$

$$\leq \left(\frac{|k_1|}{2} + \frac{|3 - k_1|}{2} + \frac{|k_2|}{2} + 2|k_3|\right)\left(\|y\|_{H^s \times H^{s-1}} + \|z\|_{H^s \times H^{s-1}}\right)\|y - z\|_{H^s \times H^{s-1}}.$$
This proves (a). Taking $y = 0$ in the above inequality, we obtain that $f$ is bounded on bounded set in $H^s \times H^{s-1}$.

Next, we prove (b). Note that $H^{s-1}$ is a Banach algebra. Then, we have

$$ \|f(y) - f(z)\|_{H^{s-1} \times H^{s-2}} \leq \|\partial_x (1 - \partial_x^2)^{-1}(k_1(y^2 - u^2) - \frac{3}{2}k_1(y_1^2, x - u_x^2) + \frac{k_2}{2}(y_2^2 - \rho^2))\|_{H^{s-1}} + |k_3| \|y_{1,x}y_2 - u_x\rho\|_{H^{s-2}}$$

where we applied Lemma 2.1 with $r$.

Let $c$ be a constant depending only on $r$.

Then, we have the following useful result.

$$\leq \frac{|k_1|}{2} \|y_1 - u\|_{H^{s-2}} + \frac{|3 - k_1|}{2} \|y_{1,x} - u_x\|_{H^{s-2}} + \frac{|k_2|}{2} \|y_2 - \rho\|_{H^{s-2}} + \frac{|k_3|}{2} \|y_{1,x}y_2 - y_{1,x}\rho\|_{H^{s-2}} + \frac{|k_3|}{2} \|y_{1,x}\rho - u_x\rho\|_{H^{s-2}}$$

where $c$ only depends on $k_1, k_2$ and $k_3$. This proves (b) and completes the proof of the lemma.

**Proof of Theorem 2.2:** Combining Theorem 2.1 and Lemmas 2.2-2.5, we get the statement of Theorem 2.2.

### 3 Precise blow-up scenarios

In this section, we will derive precise blow-up scenarios for strong solutions to Eq.(1.1).

We first recall the following two useful lemmas.

**Lemma 3.1** ([26]). If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover

$$\|fg\|_{H^r} \leq c(\|f\|_{L^\infty}\|g\|_{H^r} + \|f\|_{H^r}\|g\|_{L^\infty}),$$

where $c$ is a constant depending only on $r$.

**Lemma 3.2** ([26]). If $r > 0$, then

$$\|g\|_{L^2} \leq c(\|\partial_x f\|_{L^\infty}\|g\|_{L^2} + \|\partial_x f\|_{L^2}\|g\|_{L^\infty}),$$

where $c$ is a constant depending only on $r$.

Then, we have the following useful result.

**Theorem 3.1.** Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}$, $s \geq 2$, be given and assume that $T$ is
the maximal existence time of the corresponding solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to Eq.(1.1) with the initial data \( z_0 \). If there exists \( M > 0 \) such that
\[
\| u_x(t, \cdot) \|_{L^\infty} + \| \rho(t, \cdot) \|_{L^\infty} + \| \rho_x(t, \cdot) \|_{L^\infty} \leq M, \quad t \in [0, T),
\]
then the \( H^s \times H^{s-1} \)-norm of \( z(t, \cdot) \) does not blow up on \([0, T)\).

**Proof** Let \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) be the solution to Eq.(1.1) with the initial data \( z_0 \in H^s \times H^{s-1}, \ s \geq 2, \) and let \( T \) be the maximal existence time of the corresponding solution \( z \), which is guaranteed by Theorem 2.2.

Applying the operator \( \Lambda^s \) to the first equation in (2.3), multiplying by \( \Lambda^s u \), and integrating over \( \mathbb{R} \), we obtain
\[
\frac{d}{dt} \| u \|_{H^s}^2 = 2(uu_x, u)_s + 2(u, f_{11}(u))_s + 2(u, f_{12}(u))_s,
\]
where
\[
f_{11}(u) = \partial_x(1 - \partial_x^2)^{-1} \frac{k_1}{2} u^2 = (1 - \partial_x^2)^{-1}(k_1 uu_x),
\]
and
\[
f_{12}(u) = \partial_x(1 - \partial_x^2)^{-1} \left( \frac{3 - k_1}{2} u_x^2 + \frac{k_2}{2} \rho^2 \right).
\]
From the proof of Theorem 3.1 in [21], we have
\[
|(uu_x, u)_s| \leq c \| u_x \|_{L^\infty} \| u \|_{H^s}^2,
\]
and
\[
|(f_{11}(u), u)_s| \leq c \| u_x \|_{L^\infty} \| u \|_{H^s}^2.
\]
Furthermore, we estimate the third term of the right hand side of Eq.(3.1) in the following way:
\[
|(f_{12}(u), u)_s| \leq \| f_{12}(u) \|_{H^s} \| u \|_{H^s}
\]
\[
= \| \partial_x(1 - \partial_x^2)^{-1} \left( \frac{3 - k_1}{2} u_x^2 + \frac{k_2}{2} \rho^2 \right) \|_{H^s} \| u \|_{H^s}
\]
\[
\leq \left( \frac{|3 - k_1|}{2} \| u_x^2 \|_{H^{s-1}} + \frac{|k_2|}{2} \| \rho^2 \|_{H^{s-1}} \right) \| u \|_{H^s}
\]
\[
\leq (|3 - k_1| c \| u_x \|_{L^\infty} \| u_x \|_{H^{s-1}} + |k_2| c \| \rho \|_{L^\infty} \| \rho \|_{H^{s-1}}) \| u \|_{H^s}
\]
\[
\leq c (\| u_x \|_{L^\infty} + \| \rho \|_{L^\infty}) (\| u \|_{H^{s-1}}^2 + \| u \|_{H^s}^2).
\]
Here, we applied Lemma 3.1 with \( r = s - 1 \). Combining the above three inequalities with (3.1), we get
\[
\frac{d}{dt} \| u \|_{H^s}^2 \leq c (\| u_x \|_{L^\infty} + \| \rho \|_{L^\infty}) (\| u \|_{H^{s-1}}^2 + \| u \|_{H^s}^2).
\]
In order to derive a similar estimate for the second component \( \rho \), we apply the operator \( \Lambda^{s-1} \) to the second equation in (2.3), multiply by \( \Lambda^{s-1} \rho \), and integrate over \( \mathbb{R} \) we obtain

\[
\frac{d}{dt}\|\rho\|_{H^{s-1}}^2 = 2k_3(u\rho_x, \rho)_{s-1} + 2k_3(u_x \rho, \rho)_{s-1}.
\]

Following the similar argument in the proof of Theorem 3.1 in [21], we have

\[
| (u\rho_x, \rho)_{s-1} | \leq c(\|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty})(\|\rho\|_{H^{s-1}}^2 + \|u\|_{H^s}^2)
\]

and

\[
| (u_x \rho, \rho)_{s-1} | \leq c(\|u_x\|_{L^\infty} + \|\rho\|_{L^\infty})(\|\rho\|_{H^{s-1}}^2 + \|u\|_{H^s}^2).
\]

Then, it follows that

\[
\frac{d}{dt}\|\rho\|_{H^{s-1}}^2 \leq c(\|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty} + \|\rho\|_{L^\infty})(\|\rho\|_{H^{s-1}}^2 + \|u\|_{H^s}^2).
\]

(3.3)

By (3.2) and (3.3), we obtain

\[
\frac{d}{dt}(\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2) \leq c(\|u_x\|_{L^\infty} + \|\rho_x\|_{L^\infty} + \|\rho\|_{L^\infty})(\|\rho\|_{H^{s-1}}^2 + \|u\|_{H^s}^2).
\]

An application of Gronwall’s inequality and the assumption of the theorem yield

\[
\|u\|_{H^s}^2 + \|\rho\|_{H^{s-1}}^2 \leq \exp(cMt)(\|u_0\|_{H^s}^2 + \|\rho_0\|_{H^{s-1}}^2).
\]

This completes the proof of the theorem.

Consider now the following initial value problem

\[
\begin{align*}
q_t &= u(t, -k_3q), \quad t \in [0, T),
q(0, x) &= x, \quad x \in \mathbb{R},
\end{align*}
\]

(3.4)

where \( u \) denotes the first component of the solution \( z \) to Eq.(1.1). Applying classical results in the theory of ordinary differential equations, one can obtain two results on \( q \) which are crucial in studying blow-up phenomena.

**Lemma 3.3.** Let \( u \in C([0, T); H^s) \cap C^1([0, T); H^{s-1}), s \geq 2 \). Then Eq.(3.4) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}) \). Moreover, the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with

\[
q_t(t, x) = \exp \left( \int_0^t -k_3u_x(s, -k_3q(s, x))ds \right) > 0, \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\]

**Proof** Since \( u \in C([0, T); H^s) \cap C^1([0, T]; H^{s-1}), s \geq 2 \) and \( H^s \subset C^1 \), we see that both functions \( u(t, x) \) and \( u_x(t, x) \) are bounded, Lipschitz in the space variable \( x \), and of class \( C^1 \) in time. Therefore, for fixed \( x \in \mathbb{R} \), equation (3.4) is an ordinary differential equation. Then well-known classical results in the theory of ordinary differential equations yield that equation (3.4) has a unique solution \( q \in C^1([0, T) \times \mathbb{R}; \mathbb{R}) \).
Differentiation of equation (3.4) with respect to $x$ yields

\[
\begin{aligned}
\left\{ \begin{array}{l}
\frac{d}{dx}q_x = -k_3u_x(t, -k_3q)q_x, & t \in [0, T), \\
q(0, x) = x, & x \in \mathbb{R}.
\end{array} \right.
\end{aligned}
\tag{3.5}
\]

The solution to equation (3.5) is given by

\[
q_x(t, x) = \exp \left( \int_0^t -k_3u_x(s, -k_3q(s, x))ds \right), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\tag{3.6}
\]

For every $T' < T$, by Sobolev’s imbedding theorem, we get

\[
 \sup_{(s, x) \in [0, T') \times \mathbb{R}} |u_x(s, x)| < \infty.
\]

Thus, we infer from Eq.(3.6) that there exists a constant $K > 0$ such that $q_x(t, x) \geq e^{-|k_3| K t} > 0$ for $(t, x) \in (0, T') \times \mathbb{R}$. This completes the proof of the lemma.

**Lemma 3.4.** Let $z_0 \in H^s \times H^{s-1}, s \geq 2$, and let $T > 0$ be the maximal existence time of the corresponding solution $z = \left( \begin{array}{c} u(t) \\ \rho(t) \end{array} \right)$ to Eq.(2.3). Then we have

\[
\rho(t, -k_3q(t, x))q_x(t, x) = \rho_0(-k_3x), \quad \forall (t, x) \in [0, T) \times \mathbb{R}.
\tag{3.7}
\]

Moreover, if $k_3 \leq 0$ and there exists $M > 0$ such that $u_x(t, x) \geq -M$ for all $(t, x) \in [0, T) \times \mathbb{R}$, then

\[
\|\rho(t, \cdot)\|_{L^\infty} = \|\rho(t, -k_3q(t, \cdot))\|_{L^\infty} \leq e^{-k_3MT}\|\rho_0(\cdot)\|_{L^\infty}, \quad \forall t \in [0, T);
\]

if $k_3 \geq 0$ and there exists $M > 0$ such that $u_x(t, x) \leq M$ for all $(t, x) \in [0, T) \times \mathbb{R}$, then

\[
\|\rho(t, \cdot)\|_{L^\infty} = \|\rho(t, -k_3q(t, \cdot))\|_{L^\infty} \leq e^{k_3MT}\|\rho_0(\cdot)\|_{L^\infty}, \quad \forall t \in [0, T).
\]

Furthermore, if there exists $M > 0$ such that $|u_x(t, x)| \leq M$ for all $(t, x) \in [0, T) \times \mathbb{R}$, then

\[
\|\rho(t, \cdot)\|_{L^\infty} = \|\rho(t, -k_3q(t, \cdot))\|_{L^\infty} \leq e^{k_3|MT|}\|\rho_0(\cdot)\|_{L^\infty}, \quad \forall t \in [0, T).
\]

**Proof** Differentiating the left-hand side of Eq.(3.7) with respect to $t$, in view of (3.4) and Eq.(2.3), we obtain

\[
\begin{aligned}
\frac{d}{dt}(\rho(t, -k_3q(t, x))q_x(t, x))
&= (\rho_t + \rho_x \cdot (-k_3q(t, x)))q_x(t, x) + \rho \cdot q_{xt}(t, x) \\
&= (\rho_t - k_3\rho_xu)q_x(t, x) - k_3\rho u_xq_x(t, x) \\
&= (\rho_t - k_3\rho_xu - k_3\rho u_x)q_x(t, x) = 0
\end{aligned}
\]

This proves (3.7). By Lemma 3.3, in view of (3.7), we obtain

\[
\|\rho(t, \cdot)\|_{L^\infty} = \|\rho(t, -k_3q(t, \cdot))\|_{L^\infty} \leq \|\exp \left( k_3 \int_0^t u_x(s, -k_3q(s, \cdot))ds \right) \rho_0(-k_3x)\|_{L^\infty}. \quad \forall t \in [0, T).
\]

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The left proof is obvious, so we omit it here.

By Theorem 3.1 and Lemma 3.4 we have the following corollary.

Corollary 3.1. Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, \ s \geq 2, \) be given and assume that \( T \) is the maximal existence time of the corresponding solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to Eq.(1.1) with the initial data \( z_0 \). If there exists \( M > 0 \) such that
\[
\|u_x(t,\cdot)\|_{L^\infty} + \|\rho_x(t,\cdot)\|_{L^\infty} \leq M, \ t \in [0,T),
\]
then the \( H^s \times H^{s-1} \)-norm of \( z(t,\cdot) \) does not blow up on \([0,T)\).

Our next result describes the precise blow-up scenario for sufficiently regular solutions to Eq.(1.1).

Theorem 3.2. Let \( z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, \ s > \frac{5}{2}, \) and let \( T \) be the maximal existence time of the corresponding solution \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) to Eq.(1.1). If \( k_1 \leq -\frac{1}{2}, \ k_3 \leq \min\{0,k_2\} \), then \( z \) blows up in finite time if and only if
\[
\liminf_{t \to T} \inf_{x \in \mathbb{R}} \{u_x(t,x)\} = -\infty \quad \text{or} \quad \limsup_{t \to T} \|\rho_x(t,\cdot)\|_{L^\infty} = +\infty.
\]
If \( k_1 \geq 1, \ k_3 \geq \max\{0,k_2\} \), then \( z \) blows up in finite time if and only if
\[
\limsup_{t \to T} \sup_{x \in \mathbb{R}} \{u_x(t,x)\} = +\infty \quad \text{or} \quad \limsup_{t \to T} \|\rho_x(t,\cdot)\|_{L^\infty} = +\infty.
\]
Otherwise, \( z \) blows up in finite time if and only if
\[
\limsup_{t \to T} \|u_x(t,\cdot)\|_{L^\infty} = +\infty \quad \text{or} \quad \limsup_{t \to T} \|\rho_x(t,\cdot)\|_{L^\infty} = +\infty.
\]

Proof Let \( z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \) be the solution to Eq.(1.1) for the initial \( z_0 \in H^s \times H^{s-1}, \ s > \frac{5}{2}, \) and let \( T \) be the maximal existence time of the solution \( z \), which is guaranteed by Theorem 2.2.

Multiplying the first equation in (1.1) by \( m = u - u_{xx} \) and integrating by parts, we get
\[
\frac{d}{dt} \int_{\mathbb{R}} m^2 dx = (2k_1 - 1) \int_{\mathbb{R}} m^2 u_x dx + 2k_2 \int_{\mathbb{R}} u \rho_x - 2k_2 \int_{\mathbb{R}} u_{xx} \rho_{xx} \]
\[
= (2k_1 - 1) \int_{\mathbb{R}} m^2 u_x dx - k_2 \int_{\mathbb{R}} u_x \rho^2 + k_2 \int_{\mathbb{R}} u_{xxx} \rho^2.
\]
Differentiating the first equation in (1.1) with respect to $x$, multiplying the obtained equation by $m_x = u_x - u_{xxx}$, and integrating by parts, we get
\[
\frac{d}{dt} \int_{\mathbb{R}} m_x^2 dx = 2(1 + k_1) \int_{\mathbb{R}} u_x m_x^2 dx + 2 \int_{\mathbb{R}} m_x m_{xx} + 2k_1 \int_{\mathbb{R}} u_{xx} mm_x \\
+ 2k_2 \int_{\mathbb{R}} m_x \rho_x^2 + 2k_2 \int_{\mathbb{R}} m_x \rho_{xx} \\
= (1 + 2k_1) \int_{\mathbb{R}} u_x m_x^2 - k_1 \int_{\mathbb{R}} u_x m_x^2 + k_2 \int_{\mathbb{R}} u_{xxx}(\rho^2 - 2\rho \rho_{xx} - 2\rho_x^2).
\]

Here we used the relation $\int_{\mathbb{R}} m^2 m_x dx = 0$.

Combining (3.8) with (3.9) and integrating by parts, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}} m^2 + m_x^2 dx = \left( k_1 - 1 \right) \int_{\mathbb{R}} u_x m^2 + (1 + 2k_1) \int_{\mathbb{R}} u_x m_x^2 - k_2 \int_{\mathbb{R}} u_x \rho^2 \\
+ 2k_2 \int_{\mathbb{R}} u_{xxx}(\rho^2 - \rho \rho_{xx} - \rho_x^2) dx.
\]

Multiplying the second equation in (1.1) by $\rho$ and integrating by parts, we deduce
\[
\frac{d}{dt} \int_{\mathbb{R}} \rho^2 dx = k_3 \int_{\mathbb{R}} u_x \rho^2 dx.
\]

Differentiating the second equation in (1.1) with respect to $x$, multiplying the obtained equation by $\rho_x$, and integrating by parts, we find
\[
\frac{d}{dt} \int_{\mathbb{R}} \rho_x^2 dx = 4k_3 \int_{\mathbb{R}} u_x \rho_x^2 + 2k_3 \int_{\mathbb{R}} u \rho_x \rho_{xx} + 2k_3 \int_{\mathbb{R}} u_{xx} \rho_x \\
= 3k_3 \int_{\mathbb{R}} u_x \rho_x^2 - k_3 \int_{\mathbb{R}} u_{xxx} \rho^2.
\]

Differentiating the second equation in (1.1) with respect to $x$ twice, multiplying the obtained equation by $\rho_{xx}$, and integrating by parts, we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}} \rho_{xx}^2 dx = 5k_3 \int_{\mathbb{R}} u_x \rho_{xx}^2 dx + k_3 \int_{\mathbb{R}} u_{xxx}(2\rho \rho_{xx} - 3\rho_x^2) dx.
\]
Thus, in view of (3.10)-(3.13), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx
\]
\[
= (k_1 - 1) \int_{\mathbb{R}} u_x m^2 dx + (1 + 2k_1) \int_{\mathbb{R}} u_x m_x^2 dx
\]
\[+ (k_3 - k_2) \int_{\mathbb{R}} u_x \rho^2 dx + 3k_3 \int_{\mathbb{R}} u_x \rho_x^2 dx + 5k_3 \int_{\mathbb{R}} u_x \rho_{xx}^2 dx
\]
\[+ \int_{\mathbb{R}} u_{xxx} ((2k_2 - k_3) \rho^2 + 2(k_3 - k_2) \rho \rho_{xx} - (2k_2 + 3k_3) \rho_x^2) dx.
\]

If \( k_1 \leq -\frac{1}{2}, \ k_3 \leq \min\{0, k_2\}, \) assume that there exist \( M_1 > 0 \) and \( M_2 > 0 \) such that
\[u_x(t, x) \geq -M_1, \quad \forall \ (t, x) \in [0, T) \times \mathbb{R},\]
and
\[\|\rho_x(t, \cdot)\|_{L^\infty} \leq M_2, \quad \forall \ t \in [0, T).
\]
By Lemma 3.4, we deduce
\[\|\rho(t, \cdot)\|_{L^\infty} \leq e^{-k_3 M_1 T} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall \ t \in [0, T).
\]
It then follows from (3.14) that
\[
\frac{d}{dt} \int_{\mathbb{R}} (m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx
\]
\[\leq (-3k_1 + k_2 - 9k_3) M_1 \int_{\mathbb{R}} (m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx
\]
\[+ (2k_2 - k_3) \|\rho\|_{L^\infty(\mathbb{R})} + 2|k_3 - k_2| \|\rho\|_{L^\infty(\mathbb{R})} + |2k_2 + 3k_3|M_2
\]
\[\int_{\mathbb{R}} (3u_{xxx} + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx
\]
\[\leq c \int_{\mathbb{R}} (m^2 + m_x^2 + \rho^2 + \rho_x^2 + \rho_{xx}^2) dx,
\]
where
\[c = (-3k_1 + k_2 - 9k_3) M_1 + 3((2k_2 - k_3) + 2|k_3 - k_2|) e^{-k_3 M_1 T} \|\rho_0(\cdot)\|_{L^\infty} + |2k_2 + 3k_3|M_2).
\]
By means of Gronwall’s inequality, we obtain
\[\|u(t, \cdot)\|_{H^3}^2 + \|\rho(t, \cdot)\|_{H^2}^2 \leq \|m(t, \cdot)\|_{H^1}^2 + \|\rho(t, \cdot)\|_{H^2}^2
\]
\[\leq (\|m(0, \cdot)\|_{H^1}^2 + \|\rho(0, \cdot)\|_{H^2}^2) e^{ct}, \quad \forall \ t \in [0, T).
\]
The above inequality, Sobolev’s imbedding theorem and Corollary 3.1 ensure that the solution \( z \) does not blow up in finite time.

If \( k_1 \geq 1, \ k_3 \geq \max\{0, k_2\}, \) assume that there exist \( M_1 > 0 \) and \( M_2 > 0 \) such that
\[u_x(t, x) \leq M_1, \quad \forall \ (t, x) \in [0, T) \times \mathbb{R},\]
and
\[ \| \rho_x(t, \cdot) \|_{L^\infty} \leq M_2, \quad \forall t \in [0, T). \]

By Lemma 3.4, we deduce
\[ \| \rho(t, \cdot) \|_{L^\infty} \leq e^{k_1 M_1 T} \| \rho_0(\cdot) \|_{L^\infty}, \quad \forall t \in [0, T). \]

The left proof is similar to the proof above, so we omit it here.

Otherwise, let \( T < \infty \). Assume that there exists \( M_1 > 0 \) and \( M_2 > 0 \) such that
\[ \| u_x(t, \cdot) \|_{L^\infty} \leq M_1, \quad \forall t \in [0, T), \]
and
\[ \| \rho_x(t, \cdot) \|_{L^\infty} \leq M_2, \quad \forall t \in [0, T). \]

We can get
\[ \| u_x(t, \cdot) \|_{L^\infty} + \| \rho_x(t, \cdot) \|_{L^\infty} \leq M, \quad t \in [0, T). \]

Then Corollary 3.1 implies that the solution \( z \) does not blow up in finite time.

On the other hand, by Sobolev’s imbedding theorem, we see that if one of the conditions in the theorem holds, then the solution will blow up in finite time. This completes the proof of the theorem.

For initial data \( z_0 \in H^2 \times H^1 \), we have the following precise blow-up scenario.

**Theorem 3.3.** Let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^2 \times H^1 \), and let \( T \) be the maximal existence time of the corresponding solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to Eq. (1.1). If \( k_1 \leq \frac{1}{2} \), \( k_3 \leq \min\{k_2, 0\} \), then the corresponding solution blows up in finite time if and only if
\[ \liminf_{t \to T} \inf_{x \in \mathbb{R}} \{ u_x(t, x) \} = -\infty. \]
If \( k_1 \geq \frac{1}{2} \), \( k_3 \geq \max\{k_2, 0\} \), then the corresponding solution blows up in finite time if and only if
\[ \limsup_{t \to T} \sup_{x \in \mathbb{R}} \{ u_x(t, x) \} = +\infty. \]
Otherwise, the corresponding solution blows up in finite time if and only if
\[ \limsup_{t \to T} \| u_x(t, \cdot) \|_{L^\infty} = +\infty. \]

**Proof** Let \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) be the solution to Eq. (1.1) for the initial \( z_0 \in H^2 \times H^1 \), and let \( T \) be the maximal existence time of the solution \( z \), which is guaranteed by Theorem 2.2.
Combining (3.8), (3.11) and (3.12) we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}} (m^2 + \rho^2 + \rho_x^2) \, dx = (2k_1 - 1) \int_{\mathbb{R}} m^2 u_x \, dx + (k_3 - k_2) \int_{\mathbb{R}} u_x \rho^2 \, dx + 3k_3 \int_{\mathbb{R}} u_x \rho_x^2 \, dx + 2(k_3 - k_2) \int_{\mathbb{R}} u_{xxx} \rho \rho_x \, dx.
\]

(3.16)

If \( k_1 \leq \frac{1}{2}, k_3 \leq \min\{k_2, 0\} \), assume that there exist \( M_1 > 0 \) such that

\[ u_x(t, x) \geq -M_1, \quad \forall \, (t, x) \in [0, T) \times \mathbb{R}. \]

By Lemma 3.4, we have

\[ \|\rho(t, \cdot)\|_{L^\infty} \leq e^{-k_3 M_1 T} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall \, t \in [0, T). \]

It then follows from (3.16) that

\[
\frac{d}{dt} \int_{\mathbb{R}} (m^2 + \rho^2 + \rho_x^2) \, dx \leq (-2k_1 + k_2 - 4k_3 + 1)M_1 \int_{\mathbb{R}} (m^2 + \rho^2 + \rho_x^2) \, dx + 2(k_2 - k_3)\|\rho(t, \cdot)\|_{L^\infty} \int_{\mathbb{R}} (u_x^2 + \rho_x^2) \, dx \leq \ c \int_{\mathbb{R}} (m^2 + \rho^2 + \rho_x^2) \, dx,
\]

where

\[ c = (-2k_1 + k_2 - 4k_3 + 1)M_1 + 2(k_2 - k_3)e^{-k_3 M_1 T} \|\rho_0(\cdot)\|_{L^\infty}. \]

By means of Gronwall’s inequality, we obtain \( \forall \, t \in [0, T) \)

\[ \|u(t, \cdot)\|_{H^2}^2 + \|\rho(t, \cdot)\|_{H^1}^2 \leq \|m(t, \cdot)\|_{L^2}^2 + \|\rho(t, \cdot)\|_{H^1}^2 \leq (\|m(0, \cdot)\|_{L^2}^2 + \|\rho(0, \cdot)\|_{H^1}^2) e^{ct}. \]

The above inequality ensures that the solution \( z \) does not blow up in finite time.

If \( k_1 \geq \frac{1}{2}, k_3 \geq \max\{k_2, 0\} \), assume that there exist \( M_1 > 0 \) such that

\[ u_x(t, x) \leq M_1, \quad \forall \, (t, x) \in [0, T) \times \mathbb{R}. \]

By Lemma 3.4, we have

\[ \|\rho(t, \cdot)\|_{L^\infty} \leq e^{k_3 M_1 T} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall \, t \in [0, T). \]

The left proof is similar to the proof above, so we omit it here.
Otherwise, let $T < \infty$. Assume that there exists $M_1 > 0$ such that
\[ \|u_x(t, \cdot)\|_{L^\infty} \leq M_1, \quad \forall \ t \in [0, T). \]

By Lemma 3.4, we have
\[ \|\rho(t, \cdot)\|_{L^\infty} \leq e^{t k_3 |M_1 T|} \|\rho_0(\cdot)\|_{L^\infty}, \quad \forall \ t \in [0, T). \]

It then follows from (3.16) that
\[
\frac{d}{dt} \int_{\mathbb{R}} (m^2 + \rho^2 + \rho_x^2) \, dx \\
\leq (|2k_1 - 1| + |k_3 - k_2| + 3|k_3|) M_1 \int_{\mathbb{R}} (m^2 + \rho^2 + \rho_x^2) \, dx \\
+ 2|k_2 - k_3| \|\rho(t, \cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} (u_x^2 + \rho_x^2) \, dx \\
\leq (|2k_1 - 1| + |k_3 - k_2| + 3|k_3|) M_1 \int_{\mathbb{R}} (m^2 + \rho^2 + \rho_x^2) \, dx \\
+ 2|k_2 - k_3| e^{t k_3 |M_1 T|} \|\rho_0(\cdot)\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} (m^2 + \rho_x^2) \, dx \\
\leq c \int_{\mathbb{R}} (m^2 + \rho^2 + \rho_x^2) \, dx,
\]
where $c = (|2k_1 - 1| + |k_3 - k_2| + 3|k_3|) M_1 + 2|k_2 - k_3| e^{t k_3 |M_1 T|} \|\rho_0(\cdot)\|_{L^\infty}$.

By means of Gronwall’s inequality, we obtain $\forall \ t \in [0, T)$
\[
\|u(t, \cdot)\|_{H^2}^2 + \|\rho(t, \cdot)\|_{H^1}^2 \leq \|m(t, \cdot)\|_{L^2}^2 + \|\rho(t, \cdot)\|_{H^1}^2 \\
\leq \left(\|m(0, \cdot)\|_{L^2}^2 + \|\rho(0, \cdot)\|_{H^1}^2\right) e^{ct}.
\]

The above inequality ensures that the solution $z$ does not blow up in finite time.

On the other hand, by Sobolev’s imbedding theorem, we see that if one of the conditions in the theorem holds, then the solution will blow up in finite time. This completes the proof of the theorem.

**Remark 3.1.** Note that Theorem 3.2 shows that
\[
T(\|z_0\|_{H^s \times H^{s-1}}) = T(\|z_0\|_{H^{s'} \times H^{s'-1}}), \quad \forall \ s, s' > \frac{5}{2},
\]
while Theorem 3.3 implies that
\[
T(\|z_0\|_{H^s \times H^{s-1}}) \leq T(\|z_0\|_{H^2 \times H^1}), \quad \forall \ s \geq 2.
\]

## 4 Blow up

In this section, we discuss the blow-up phenomena of the system (1.1) and prove that there exist strong solutions to (1.1) which do not exist globally in time.
Theorem 4.1. Let $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, s > \frac{5}{2}$, and $T$ be the maximal time of the solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to (1.1) with the initial data $z_0$. If $1 < k_1 \leq 3$, $k_2 \geq 0$, $u_0$ is odd, $\rho_0$ is even, $\rho_0(0) = 0$, $u_0'(0) > 0$, then $T$ is bounded above by $\frac{2}{k_1 - 1} u_0(0)$ and $u_x(t, 0)$ tends to positive infinite as $t$ goes to $T$.

Proof Let $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ be the solution to Eq.(1.1) for the initial $z_0 \in H^s \times H^{s-1}, s > \frac{5}{2}$, and let $T$ be the maximal existence time of the solution $z$, which is guaranteed by Theorem 2.2.

Note that $\partial^2_x p \ast f = p \ast f - f$. Differentiating the first equation in (2.2) with respect to $x$, then we have

$$u_{tx} - uu_{xx} = -\frac{k_1}{2}u^2 + \frac{k_1 - 1}{2}u_x^2 - \frac{k_2}{2}\rho^2 + p \ast (\frac{k_1}{2}u^2 + \frac{3 - k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2). \quad (4.1)$$

Note that Eq.(1.1) is invariant under the transformation $(u, x) \rightarrow (-u, -x)$ and $(\rho, x) \rightarrow (\rho, -x)$. Thus we deduce that if $u_0(x)$ is odd and $\rho_0(x)$ is even, then $u(t, x)$ is odd and $\rho(t, x)$ is even with respect to $x$ for any $t \in [0, T)$. By continuity with respect to $x$ of $u$ and $u_{xx}$, we have

$$u(t, 0) = u_{xx}(t, 0) = 0, \quad \forall \ t \in [0, T). \quad (4.2)$$

From Eq.(3.4) and $u$ being odd with respect to $x$, we infer that $q(t, x)$ is odd with respect to $x$. Then we have that $q(t, 0) = 0$ for all $t \in [0, T)$. Since $\rho_0(0) = 0$, it follows from Lemmas 3.3-3.4 that $\rho(t, 0) = \rho(t, -k_3q(t, 0)) = 0$ for all $t \in [0, T)$. Hence, in view of (4.1) and (4.2), we obtain

$$u_{tx}(t, 0) = \frac{k_1 - 1}{2}u_x^2(t, 0) + p \ast (\frac{k_1}{2}u^2 + \frac{3 - k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2)(t, 0). \quad (4.3)$$

By $p \ast (\frac{k_1}{2}u^2 + \frac{3 - k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2) \geq 0$ and (4.3), we get

$$u_{tx}(t, 0) \geq \frac{k_1 - 1}{2}u_x^2(t, 0), \quad t \in [0, T).$$

Set $h(t) = u_x(t, 0)$. Since $h(0) > 0$, in view of $u_0'(0) > 0$, it follows that

$$0 < \frac{1}{h(t)} \leq \frac{1}{h(0)} - \frac{k_1 - 1}{2}t. \quad (4.4)$$

The above inequality implies that $T < \frac{2}{k_1 - 1} u_0(0)$ and $u_x(t, 0)$ tends to positive infinite as $t$ goes to $T$. This completes the proof of the theorem.

Theorem 4.2. Let $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s \times H^{s-1}, s > \frac{5}{2}$, and $T$ be the maximal time of the solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to (1.1) with the initial data $z_0$. If $1 < k_1 \leq 3$, $k_2 \geq 0$, $u_0$ is odd, $\rho_0$ is
even, $\rho_0(0) = 0, u_0'(0) = 0$, then $T$ is finite.

**Proof** Let $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ be the solution to Eq.(1.1) for the initial $z_0 \in H^s \times H^{s-1}, s > \frac{5}{2}$, and let $T$ be the maximal existence time of the solution $z$, which is guaranteed by Theorem 2.2.

Following the similar proof in Theorem 4.1, we have

$$u_{tx}(t, 0) = \frac{k_1 - 1}{2} u_x^2(t, 0) + p^* \left( \frac{k_1}{2} u^2 + \frac{3 - k_1}{2} u^2_x + \frac{k_2}{2} \rho^2 \right)(t, 0). \quad (4.5)$$

By $u_0'(0) = 0$, the continuity of the ordinary differential equation and the uniqueness, we have

$$\frac{d}{dt} u_x(t, 0) \geq p^* \left( \frac{k_1}{2} u^2 + \frac{3 - k_1}{2} u^2_x + \frac{k_2}{2} \rho^2 \right)(t, 0) > 0.$$

Therefore, $h(t)$ is strictly increasing on $[0, T)$. Since $h(0) = 0$, it follows that $h(t_0) > 0$ for some $t_0 \in (0, T)$. Solving the following inequality

$$\frac{d}{dt} h(t) > \frac{k_1 - 1}{2} h(t)^2,$$

we obtain

$$0 < \frac{1}{h(t)} \leq \frac{1}{h(t_0)} - \frac{k_1 - 1}{2} (t - t_0), \quad t \in [t_0, T).$$

Consequently, we get $T < t_0 + \frac{2}{k_1 - 1} \frac{1}{h(t_0)}$. This completes the proof of the theorem.

**Remark 4.1.** Note that Eq.(1.1) is also invariant under the transformation $(u, x) \to (-u, -x)$ and $(\rho, x) \to (-\rho, -x)$. Thus if the condition "$\rho_0$ is even, $\rho_0(0) = 0$" in Theorems 4.1-4.2 and the following Corollary 4.1 is replaced by "$\rho_0$ is odd" the conclusions also hold true.

Then we give a corollary about Theorems 4.1-4.2.

**Corollary 4.1.** Let $z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s \times H^{s-1}, s > \frac{5}{2}$, and $T$ be the maximal time of the solution $z = \begin{pmatrix} u \\ \rho \end{pmatrix}$ to (1.1) with the initial data $z_0$. If $1 < k_1 \leq 3, k_2 \geq 0, m_0$ is odd, $\rho_0$ is even, $\rho_0(0) = 0, \int_0^{+\infty} e^{-y} m_0(y) dy \geq 0$, then $T$ is finite.

**Proof** Note that $p(-x) = p(x)$, if $m_0$ is odd, then

$$u_0(x) = \int_{\mathbb{R}} p(x - y)m_0(y) dy = \int_{\mathbb{R}} p(-x + y)(-m_0(-y)) dy$$

$$= - \int_{\mathbb{R}} p(-x - y)m_0(y) dy = -u_0(-x),$$

from which we know that $u_0(x)$ is odd as well. Since

$$u_0'(x) = -\frac{1}{2} e^{-x} \int_{-\infty}^{x} e^y m_0(y) dy + \frac{1}{2} e^{x} \int_{x}^{+\infty} e^{-y} m_0(y) dy,$$
we get
\[ u'_0(0) = \int_{0}^{+\infty} e^{-y}m_0(y)dy. \]

It follows from Theorems 4.1-4.2 that the corresponding solution to Eq.(1.1) blows up infinite time.

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