Laplace Eigenfunctions and Damped Wave Equation on Product Manifolds

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The purpose of this article is to study possible concentrations of eigenfunctions of Laplace operators (or more generally quasi-modes) on product manifolds. We show that the approach of Burq and Zworski [13, 14] applies (modulo rescaling) and deduce new stabilization results for weakly damped wave equations which extend to product manifolds previous results by Leautaud and Lerner [16] obtained for products of tori.

1 Notations and Main Results

In this work, we continue our investigation [12] of concentration properties of eigenfunctions (or more generally quasimodes) of the Laplace–Beltrami operator on submanifolds and we study here the very particular setting of product manifolds.

Let \((M_j, g_j), j = 1, 2\) be two compact Riemannian manifolds. We denote by \((M = M_1 \times M_2, g = g_1 \otimes g_2)\) the product, and by \(d_j\) (respectively, \(d\)) the geodesic distance in \(M_j\) (respectively, \(M\)). Moreover, we shall denote by \(\Delta_g\) the (negative) Laplace–Beltrami operator on \((M, g)\). In the sequel, we assume that the metric \(g_1\) (respectively, \(g_2\)) is \(L^\infty\) (respectively, Lipschitz).

\[
M_j \text{ is } W^{j,\infty} \text{ and } g_j \in W^{j-1,\infty}, \quad j = 1, 2.
\]  

(H)

Let \(q_0 \in M_2\) and

\[
\Sigma = M_1 \times \{q_0\}.
\]
For $\beta > 0$, we introduce
\[ N_\beta = \{ m = (p, q) \in M : d(m, \Sigma) < \beta \} = M_1 \times \{ q \in M_2 : d_q(q, q_0) < \beta \}. \tag{1.1} \]

Our first result is the following.

**Theorem 1.1.** Under the smoothness assumption (H), for any $\delta > 0$, there exists $C > 0, h_0 > 0$ such that for every $0 < h \leq h_0$ and every solution $\psi \in H^2(M)$ of the equation
\[ (h^2 \Delta_g + 1)\psi = F \]
we have the estimate
\[ \|\psi\|_{L^2(N_h \delta)} \leq C (\|\psi\|_{L^2(N_{2h^2} \setminus N_h \delta)} + h^{2d-2} \|F\|_{L^2(N_{2h^2})}). \tag{1.2} \]
\[ \square \]

As an application of Theorem 1.1, we consider weakly damped wave equations on a compact Riemannian manifold $(M, g)$ of dimension $d$,
\[ (\partial_t^2 - \Delta_g + b(m)\partial_t)u = 0, \quad (u, \partial_t u)|_{t=0} = (u_0, u_1) \in H^{1+k}(M) \times H^k(M), \tag{1.3} \]
where $0 \leq b \in L^\infty(M)$, for which the energy
\[ E(u)(t) = \int_M (g_m(\nabla_g u(t, m), \nabla_g u(t, m)) + |\partial_t u(t, m)|^2) \, dv_g(m) \]
is decaying since $\frac{d}{dt}E(u)(t) = -\int_M b(m)|\partial_t u(t, m)|^2 \, dv_g(m) \leq 0$. Let
\[ \omega = \bigcup_{\{U \text{ open : ess inf } U b > 0\}} U \]
be the domain where effective damping occurs.

Denoting by $S^*(M)$ (respectively, $S^*(\omega)$) the cosphere bundle (that is the bundle of unit cotangent vectors on $M$ (respectively, on $\omega$)) and by $\Phi(s) : S^*(M) \to S^*(M)$ the geodesic flow on $M$ we introduce
\[ GC = \{ \rho \in S^*M : \exists s \in \mathbb{R} ; \Phi(s)\rho = (m_1, \xi_1) \in S^*\omega \}, \]
the (open) set of geometrically controlled points. Let
\[ T = S^* M \setminus GC, \quad T = \Pi_x T \tag{1.4} \]
where $T$ is the trapped set and $\Pi_x$ the projection on the base manifold $M$. 
Our second result is the following.

**Theorem 1.2.** Assume that

1. there exists a neighborhood $V$ of $T$ in $\mathcal{M}$, a compact Riemannian manifold $(M_1, g_1)$, $g_1 \in L^\infty$, of dimension $k$, a Lipschitz metric $g_2$ on the unit ball $B(0, 1) \subset \mathbb{R}^{d-k}$ and a $W^{2, \infty}$ diffeomorphism

   $$\Theta : (V, g) \to (M_1 \times B(0, 1), \tilde{g} = g_1 \otimes g_2),$$

2. there exists $\gamma > 0, c, C > 0$ such that

   $$c|z|^{2\gamma} \leq b(\Theta^{-1}(p, z)) \leq C|z|^{2\gamma} \quad \forall (p, z) \in M_1 \times B(0, 1).$$

Then there exists $C > 0$ such that for any $(u_0, u_1) \in H^2(\mathcal{M}) \times H^1(\mathcal{M})$, the solution $u$ to (1.3) satisfies

$$E(u)^{1/2}(t) \leq \frac{C}{t^{1+\frac{1}{2}}} (\|u_0\|_{H^2(\mathcal{M})} + \|u_1\|_{H^1(\mathcal{M})}).$$

**Remark 1.3.** A simpler (but weaker) statement would be to assume

1. $(\mathcal{M}, g) = (M_1 \times M_2, g_1 \otimes g_2)$, $q_0 \in M_2$, $T = \Sigma = M_1 \times \{q_0\}$,
2. $cd(m, \Sigma)^{2\gamma} \leq b(m) \leq C d(m, \Sigma)^{2\gamma} \quad \forall m \in M_1 \times U.$

It is classical that for non trivial dampings $b \geq 0$, the energy of solution to (1.3) converge to 0 as $t$ tend to infinity. The rate of decay is *uniform* (and hence exponential) in energy space if and only if the *geometric control condition* [2, 8, 19] is satisfied. In [12], we explored the question when some trajectories are trapped and exhibited decay rates (assuming more regularity on the initial data). This latter question was previously studied in a general setting in [17] and on tori in [1, 11, 18] (see also [13, 14]) and more recently by Leautaud and Lerner [16]. The geometric assumptions in [12] are much more general than in [16], which is essentially restricted to the case of product of flat tori. On the other hand, due to this more favorable geometry, the decay rate in [16] is better than in [12]. Theorem 1.2 shows that Leautaud–Lerner’s result (the better decay rate) extends straightforwardly to the case of product manifolds $(M_1 \times M_2, g = g_1 \otimes g_2)$.

**Remark 1.4.**

1. On a compact Riemannian manifold endowed with an $L^\infty$ metric, the Laplace operator is naturally defined in the sense of quadratic forms on $H^1$. It is
easy to check that it is self-adjoint with its natural domain and has compact resolvent.

2. According to Theorem 1.6 in [16] the rate of decay in $t$ obtained in Theorem 1.2 above is optimal in general.

3. Theorem 1.1 is a propagation result in the $z$-variable in $B(0,1)$, and since $z$ is actually very close to 0, the relevant object is $g_z(0)$ (constant coefficients) rather than $g_z(z)$. The smaller $\delta$, the further we need to propagate in the $z$ variable and hence the better the quasi modes we need to consider (due to the worse error factor $h^{2\delta-2}$).

4. The case $\delta = \frac{1}{2}$ in Theorem 1.1 is a particular case of our results in [12] (which are actually much more general and hold without the “product” assumption on the geometry). Note that the results in [12] are local, while for $\delta < \frac{1}{2}$, the estimate (1.2) is non local. Indeed, trying to replace $\psi$ by $\chi \psi$ will add to the r.h.s. a term $(h^2\Delta, \chi)\psi$ which is clearly bounded in $L^2$ by $O(h)$, giving an error of order $O(h^{2\delta-1}) \gg 1$ to the final result. On the other hand, as soon as $\delta < \frac{1}{2}$, estimate (1.2) is false without the product structure assumption as can be easily seen on spheres by considering the eigenfunctions $e_n = (x_1 + ix_2)^n$ with eigenvalues $\lambda_n = n(n + d - 1) = h_n^{-2}$ which concentrate in an $h_n^{1/2}$-neighborhood of the equator

$$E = \{x \in \mathbb{R}^{d+1} : |x| = 1, x_3 = \cdots = x_{d+1} = 0\}.$$ 

In this case, we get $(h_n^2\Delta + 1)e_n = 0$, but

$$\|e_n\|_{L^2(N_{h_n^2})} \sim C_1 h_n^{\frac{d-1}{2}}, \quad \|e_n\|_{L^2(N_{h_n^2}\setminus N_{h_n^2})} \leq C_2 e^{-c h_n^{2\delta-1}}, \quad n \to +\infty,$$

contradicting (1.2) since $2\delta - 1 < 0$.

5. The estimates in the case $\delta = \frac{1}{2}$ [12] are related to Strichartz-type estimates [3, 9, 15, 20], while here the result is obtained by propagation estimates [2, 8, 10, 16, 19].

6. No smoothness is assumed on the function $b \in L^\infty(M)$. Note, however, that (contrary to the results in [12]) the lower bound in (1.5) is not sufficient (at least with our approach) and we do need also the upper bound.

7. As will appear clearly in the proof, we could assume that $T$ is isometric to finitely many product manifolds, with possibly different constants $\gamma$, the final decay rate being given by the largest $\gamma$. □
The paper is organized as follows. We first show how to deduce from Theorem 1.1 a resolvent estimate which according to previous works by Borichev–Tomilov imply Theorem 1.2. Then we prove Theorem 1.1 by elementary scaling and propagation arguments.

2 From Concentration to Stabilization Results (Proof of Theorem 1.2)

According to the works by Borichev and Tomilov [4, Theorem 2.4], in an abstract semigroup setting, stabilization results are equivalent to resolvent estimates. As a consequence, for the damped wave equation, to prove Theorem 1.2, it is enough to prove (see [1, Proposition 2.4])

**Proposition 2.1.** We keep the geometric assumptions in Theorem 1.2. Consider for \( h > 0 \) the operator

\[
L_h = -h^2 \Delta_g - 1 + ihb, \quad b \in L^\infty(M).
\]

Then there exist \( C > 0, h_0 > 0 \) such that for all \( 0 < h \leq h_0 \)

\[
\| \varphi \|_{L^2(M)} \leq C h^{1-\frac{1}{\gamma}} \| L_h \varphi \|_{L^2(M)},
\]

for all \( \varphi \in H^2(M) \).

**Proof.** We start with a simple \textit{a priori estimate}. Multiplying both sides of the equation

\[
(-h^2 \Delta_g - 1 + ihb) \varphi = f.
\]

by \( \bar{\varphi} \), integrating by parts on \( M \) and taking real and imaginary parts gives

\[
\int_M b(m) |\varphi(m)|^2 \, dv_g(m) \leq \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)},
\]

\[
\int_M g_m(\nabla_g \varphi(m), \nabla_g \overline{\varphi(m)}) \, dv_g(m) \leq \| \varphi \|_{L^2(M)}^2 + \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)}.
\]

Now, in the neighborhood \( V \) of \( T \) we use our isometry \( \Theta \) and we set

\[
u(p, z) = \varphi(\Theta^{-1}(p, z)), \quad \tilde{b}(p, z) = b(\Theta^{-1}(p, z)), \quad \tilde{f}(p, z) = f(\Theta^{-1}(p, z)).
\]

Then from (2.2) we obtain the equation on \( M_1 \times B(0, 1) \)

\[
(h^2 \Delta_{\tilde{g}} + 1) u = ih\tilde{b}u - \tilde{f}.
\]

We can therefore apply Theorem 1.1 and we obtain

\[
\| u \|_{L^2(M_1 \times \{|z| \leq h^2\})} \leq C \| u \|_{L^2(M_1 \times \{|z| \leq 2h^2\})} + C h^{2\gamma - 2} \| ih\tilde{b}u - \tilde{f} \|_{L^2(M_1 \times \{|z| \leq 2h^2\})}.
\]
On the other hand, from (2.3), and the lower bound in assumption (1.5), we deduce
\[ \| u \|^2_{L^2(M_1 \times \{|x| \leq \frac{1}{2}h\})} \leq C h^{-1-2\delta'} \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)}. \] (2.7)

while from the upperbound in assumption (1.5), we get
\[ \| ihb \varphi \|^2_{L^2(M_1 \times \{|x| \leq \frac{1}{2}h\})} \leq h^2 \left( \sup_{M_1 \times \{|x| \leq \frac{1}{2}h\}} |\tilde{b}| \right) \| \tilde{b}^{1/2} u \|^2_{L^2(M_1 \times \{|x| \leq \frac{1}{2}h\})} \leq Ch^{1+2\delta'} \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)}. \] (2.8)

Gathering (2.6)–(2.8) we obtain
\[ \| u \|^2_{L^2(M_1 \times B(0,1))} \leq Ch^{-1-2\delta'} \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)} + Ch^{4\delta'-4} (h^{1+2\delta'} \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)} + \| f \|^2_{L^2(M)}). \] (2.9)

Optimizing with respect to \( \delta \) leads to the choice \( 2\delta = \frac{1}{1+\gamma} \), which gives
\[ \| u \|^2_{L^2(M_1 \times B(0,1))} \leq Ch^{-1-\frac{\gamma}{1+\gamma}} \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)} + Ch^{2-\frac{2\gamma}{1+\gamma}} \| f \|^2_{L^2(M)}. \]

According to (2.5) this implies
\[ \| \varphi \|_{L^2(V)} \leq Ch^{-1-\frac{\gamma}{1+\gamma}} \| \varphi \|_{L^2(M)} \| f \|_{L^2(M)} + Ch^{-2-\frac{2\gamma}{1+\gamma}} \| f \|^2_{L^2(M)}. \] (2.10)

We can now conclude the proof of Proposition 2.1 by contradiction. If (2.1) were not true, then there would exists sequences \( \varphi_n \in H^2(M) \), \( f_n \in L^2(M) \), \( 0 < h_n \to 0 \) such that
\[ (-h_n^2 \Delta g - 1 + ih_n b) \varphi_n = f_n, \quad \| \varphi_n \|_{L^2(M)} > \frac{n}{h_n^{1+\frac{\gamma}{1+\gamma}}} \| f_n \|_{L^2(M)}. \]

Dividing \( \varphi_n \) by its \( L^2 \)-norm, we deduce
\[ \| \varphi_n \|_{L^2(M)} = 1, \quad \| f_n \|_{L^2(M)} = o \left( \frac{1}{h_n^{1+\frac{\gamma}{1+\gamma}}} \right), \quad n \to +\infty, \] (2.11)
and from (2.10) we get
\[ \lim_{n \to +\infty} \| \varphi_n \|_{L^2(V)} = 0. \] (2.12)

On the other hand, the sequence \( (\varphi_n) \) is bounded in \( L^2(M) \), and extracting a subsequence, we can assume that it has a semi-classical measure \( \mu \) (see, e.g. [6, Théorème 2]).
We recall that it means that for any symbol $a \in C_0^\infty(S^*\mathcal{M})$,
\[ \lim_{n \to +\infty} (a(x, h_n D_x) \varphi_n, \varphi_n)_{L^2(M)} = \langle \mu, a \rangle. \]
Here, since we work locally, we quantize the symbols $a \in C_0^\infty(T^*\mathbb{R}^d)$ by taking first $\phi \in C_0^\infty(\mathbb{R}^d)$ equal to 1 near the $x$-projection of the support of $a$ and
\[ a(x, hD_x)u = \frac{1}{(2\pi h)^n} \int e^{\frac{i}{h}(x-y)\cdot\xi} a(x, \xi) \phi(y) u(y) \, dy \, d\xi. \]
It is classical that modulo $O(h^\infty)$ smoothing operators, the operator $a(x, hD_x)$ does not depend on the choice of $\phi$.

From (2.4), the sequence $(\varphi_n)$ is $(h_n)$ oscillating and hence any such semi-classical defect measure has total mass $1 = \lim_{n \to +\infty} \|\varphi_n\|_{L^2(M)}$ (see [6, Proposition 4]). From (2.3) and (2.11) we also have (note that $|b| \leq C |b|^{1/2}$)
\[ (-h^2 \Delta - 1) \varphi_n = -ih_n b \varphi_n + f_n = o(h_n)_{L^2}, \]
and consequently (see [7, Proposition 4.4]) the measure $\mu$ is invariant by the bicharacteristic flow. Since from (2.3) it is 0 on $S^*\omega$, we deduce by propagation that it is also 0 on $G\mathcal{C}$, and hence from (2.12) it is identically null, since $S^*(\mathcal{M}) = T \cup G\mathcal{C}$. This gives the contradiction. ■

3 Concentration Properties (Proof of Theorem 1.1)

Recall that we have $(M, g) = (M_1 \times M_2, g_1 \otimes g_2)$. The main idea of the proof is that, since we are dealing with $L^2$-norms, following the strategy in [13, Section 6.3, 14], we can take scalar products with eigenfunctions in $M_1$ and reduce the analysis to a (simpler) lower-dimensional problem on the manifold $M_2$. Here, Theorem 1.1 follows from a rescaling argument and standard non trapping resolvent estimates in $M_2$. When the metric $g_2$ is flat, the scaling argument is straightforward, but it requires a little care in the general case (see Lemma 3.5).

Let $B(q_0, r) \subset M_2$ be the ball (for the metric $d_2$) of radius $r > 0$ centered at $q_0$.

Proposition 3.1. For any $\delta > 0$, there exists $C > 0, h_0 > 0$ such that for every $0 < h \leq h_0$, every $\tau \in \mathbb{R}$, every $U \in H^2(M_2), G \in L^2(M_2)$, solutions of the equation on $M_2$
\[ (-\Delta_{g_2} - \tau)U = G \]
we have the estimate

\[ \| U \|_{L^2(B(q_0, h^n))} \leq C (\| U \|_{L^2(B(q_0, 2h^n) \setminus B(q_0, h^n))} + h^{2\delta} \| G \|_{L^2(B(q_0, 2h^n))}). \]  

(3.1)

3.1 Proof of Theorem 1.1 assuming Proposition 3.1

According to Remark 1.4(1), there exists \((e_n)\), a sequence of eigenfunctions of the Laplace operator on \(M_1\) with eigenvalues \(-\lambda_n^2\) forming an orthonormal basis in \(L^2(M_1)\). For \(\psi \in L^2(M)\), we set \(\hat{\psi}_n(q) = (\psi (\cdot, q), e_n)_{L^2(M)}\). Then we have \(\psi (p, q) = \sum_{n \in \mathbb{N}} \hat{\psi}_n(q) e_n(p)\) and it is easy to see that with the notations in (1.1), for \(r > 0\)

\[ \| \psi \|_{L^2(N_r)}^2 = \| \psi \|_{L^2(M_1 \times B(q_0, r))}^2 = \sum_{n \in \mathbb{N}} \| \hat{\psi}_n \|_{L^2(B(q_0, r))}^2. \]  

(3.2)

Now taking the scalar product of equation (1.2) with \(e_n\) we see easily that

\[ (-\Delta g_\tau - \tau) \hat{\psi}_n = h^{-2} \hat{F}_n, \quad \tau = h^{-2} - \lambda_n^2. \]

Applying Proposition 3.1 to this equation yields

\[ \| \hat{\psi}_n \|_{L^2(B(q_0, h^n))}^2 \leq C (\| \hat{\psi}_n \|_{L^2(B(q_0, 2h^n) \setminus B(q_0, h^n))}^2 + h^{4\delta} \| h^{-2} \hat{F}_n \|_{L^2(B(q_0, 2h^n))}^2). \]

Taking the sum in \(n\) and using (3.2) we obtain the estimate (1.2).

3.2 Proof of Proposition 3.1

Since the problem is local near \(q_0\), after diffeomorphism we can work in a neighborhood of the origin in \(\mathbb{R}^k\) and we may assume that the new metric \(g\) satisfies \(g|_{x=0} = \text{Id}\). Then, we make the change of variables \(z \mapsto x = \frac{z}{h^\delta}\) and we set \(u(x) = U(h^\delta x), F(x) = G(h^\delta x)\). We obtain the equation on \(u\)

\[ (-\Delta g^h - h^{2\delta} \tau) u = h^{2\delta} F, \]

where \(g^h\) is the metric obtained by dilation \(g^h(x) = g(h^\delta x)\). The family \((g^h)\) converges in Lipschitz topology to the flat metric \(g_0 = \text{Id}\) as \(h \to 0^+\). Proposition 3.1 will follow easily from

**Proposition 3.2.** Consider a family \((g_n)\) of metrics on \(B(0, 2) \subset \mathbb{R}^k\), which converges in Lipschitz topology to the flat metric when \(n \to +\infty\). Then there exists \(C > 0, N_0 > 0\) such
that for every \( n \geq N_0, \tau \in \mathbb{R}, u \in H^2(B(0,2)), f \in L^2(B(0,2)) \) solutions of the equation on \( B(0,2) \)

\((−Δ_{g_n} − \tau)u = f\)

we have the estimate

\[ \|u\|_{L^2(B(0,1))} \leq C \left( \|u\|_{L^2(B(0,2)\setminus B(0,1))} + \frac{1}{1 + |τ|^1/2} \|f\|_{L^2(B(0,2))} \right) \tag{3.3}\]

(note that since \( g_n \) converges to the flat metric the choice of the metric to define the \( L^2 \)-norms above is of no importance).

Remark 3.3. Proposition 3.2 is standard for the fixed metric \( g_0 = \text{Id} \) (see, e.g. [7, Section 3]), as the annulus \( \{x: 1 < |x| < 2\} \) controls geometrically the ball \( B(0,1) \). As a consequence, in the special case of [16] when \( g = g_0 \) (and hence \( g_n \) is also the standard flat metric), the proof of Theorem 1.1 is completed. In the general case, we only have to verify that the usual proof can handle the varying metric through a perturbation argument, which is precisely what we do below. It is worth noticing that the proof below implies that the propagation estimates involved in exact controllability results which are known to hold for \( C^2 \) metrics, see [5], are actually stable by small Lipschitz perturbations of the metric. □

For \( r > 0 \) we shall set \( B_r = B(0, r) \subset \mathbb{R}^k \).

To prove Proposition 3.2 we argue by contradiction. Otherwise, there would exist sequences, \( (τ_n) \subset \mathbb{R}, (u_n) \subset H^2(B_2), (f_n) \subset L^2(B_2) \) and a subsequence of the original sequence \( (g_n) \) (still denoted by \( (g_n) \)) such that

\[ (−Δ_{g_n} − τ_n)u_n = f_n, \tag{3.4}\]

\[ 1 = \|u_n\|_{L^2(B_1)} > n \left( \|u_n\|_{L^2(B_2\setminus B_1)} + \frac{1}{1 + |τ_n|^1/2} \|f_n\|_{L^2(B_2)} \right) \tag{3.5}\]

We now distinguish three cases

1. \( \lim \inf_{n \to +\infty} τ_n = −\infty \) (elliptic case)
2. \( (τ_n)_{n \in \mathbb{N}} \) bounded (low frequency case)
3. \( \lim \sup_{n \to +\infty} τ_n = +\infty \) (hyperbolic case)

In the first case, working with a subsequence we may assume that \( \lim_{n \to +\infty} τ_n = −\infty \). Let \( ζ \in C_0^∞(B_2) \) equal to 1 on \( B_{3/2} \). Multiplying (3.4) by \( ζ \hat{u}_n \), integrating by parts and taking
the real part gives
\[
\left| \int (g_n(\nabla g_n u_n, \nabla g_n (\zeta u_n))) - \zeta \tau_n |u_n|^2 \, dv_{g_n} \right| \leq \|u_n\|_{L^2(B_2)} \|f_n\|_{L^2(B_2)}
\] (3.6)
which implies (after another integration by parts)
\[
\left| \int \zeta g_n(\nabla g_n u_n, \nabla g_n (\zeta u_n))) - \left( \tau_n \zeta + \frac{\Delta g_n(\zeta)}{2} \right) |u_n|^2 \, dv_{g_n} \right| \leq \|u_n\|_{L^2(B_2)} \|f_n\|_{L^2(B_2)} = o(|\tau_n|^{1/2}), \quad n \to +\infty.
\] (3.7)

Since \(\Delta g_n \zeta\) is supported in \(1 \leq |x| \leq 2\) and \(\|u_n\|_{L^2(1<|x|<2)} = o(1)\), we deduce if \(\tau_n \to -\infty\)
\[
\lim_{n \to +\infty} \int \zeta |u_n|^2 \, dx = 0,
\]
which contradicts (3.5).

In the second case (low frequency), we can assume (after extracting a subsequence) that \(\tau_n \to \tau\) and (3.7) shows that the sequence \((u_n|_{B_{3/2}})\) is bounded in \(H^1(B_{3/2})\). Hence (after taking a subsequence), we can assume that it converges weakly in \(H^1(B_{3/2})\) (and hence strongly in \(L^2(B_{3/2})\). Due to the convergence of the family of metrics, we get
\[
-\Delta g_n u_n = -\Delta_0 u_n + o(1)_{H^{-1}}, \quad \left( \Delta_0 = \sum_{i=1}^{k} \partial_i^2 \right),
\]
and according to (3.5) this implies that the limit \(u\) satisfies
\[
(-\Delta_0 - \tau) u = 0 \quad \text{in } D'(B_{3/2}), \quad u|_{1<|x|<3/2} = 0.
\]
The analyticity of the solution in \(B_{3/2}\) implies that \(u = 0\). This contradicts the strong convergence of \((u_n)\) in \(L^2(B_{3/2})\) and (3.5).

Finally, it remains to study the last case (hyperbolic). Taking a subsequence, we can assume \(\tau_n \to +\infty\). Moreover, dividing both members of (3.4) by \(\tau_n\) we see that \(u_n\) is solution of an equation of type \((P(x, \tau_n^{-1/2} D_x) - 1) u_n = \tau_n^{-1} f_n \to 0\) in \(L^2(B(0,2))\). The sequence \((u_n|_{|x|<3/2})\) has a semi-classical measure \(\nu\) with scale
\[
\tilde{h}_n = \tau_n^{-1/2},
\]
(see the end of Section 2 for a few fact about these measures). Note that this new semi-classical parameter \(\tilde{h}_n\) has no relationship with the parameter \(h\) in Theorem 1.1. First of all multiplying both sides of (3.7) by \(\tilde{h}_n^2 = \tau_n^{-1}\) and using the fact that \(\|u_n\|_{L^2(B_2)}\) is
uniformly bounded we deduce that there exists $C > 0$ such that
\[
\tilde{h}_n \| \nabla_x u_n \|_{L^2(B_{3/2})} \leq C \quad \forall n \in \mathbb{N}.
\] (3.8)

Using again (3.7) shows that the sequence $u_n|_{|x|<3/2}$ is $\tilde{h}_n$-oscillatory (and hence the measure $\nu$ has total mass $1 = \lim_{n \to +\infty} \| u_n \|_{L^2(B_{3/2})}^2$). Now setting $D_n = \det((g_n)_{ij})$ we can write
\[
\Delta g_n = \Delta_0 + \sum_{i,j=1}^k \partial_i ((g_n^{ij} - \delta_{ij}) \partial_j) + \frac{1}{2D_n} \sum_{i,j=1}^k g_n^{ij} (\partial_i D_n) \partial_j. \tag{3.9}
\]

The only point of importance below will be that
\[
\lim_{n \to +\infty} \| g_n^{ij} - \delta_{ij} \|_{W^{1,\infty}(B_2)} = 0, \quad \lim_{n \to +\infty} \| D_n - 1 \|_{W^{1,\infty}(B_2)} = 0. \tag{3.10}
\]

**Proposition 3.4.** The measure $\nu$ is supported in the set $\{(x, \zeta) : |\zeta| = 1\}$ and is invariant by the bicharacteristic flow associated to the metric $g_0$:
\[
2\xi \cdot \nabla_x \nu = 0. \quad \Box
\]

The contradiction now follows since by (3.5) we have $\| u_n \|_{L^2(1<|x|<2)} \to 0$ which implies that $\nu|_{1<|x|<1.9} = 0$ and by propagation that $\nu|_{|x|<3/2} = 0$. It remains to prove Proposition 3.4.

**Proof.** We have for $a$ with compact support (in the $x$ variable) in $B(0,2)$,
\[
(a(x, \tilde{h}_n D_x) (\tilde{h}_n^2 \Delta g_n - 1) u_n, u_n)_{L^2} = (1) + (2) + (3),
\]
\[
(1) = (a(x, \tilde{h}_n D_x) (\tilde{h}_n^2 \Delta_0 - 1) u_n, u_n)_{L^2},
\]
\[
(2) = \sum_{i,j} ((g_n^{ij} - \delta_{ij}) \tilde{h}_n \partial_j u_n, \tilde{h}_n \partial_i a^*(x, \tilde{h}_n D_x) u_n)_{L^2}, \tag{3.11}
\]
\[
(3) = \tilde{h}_n \sum_{ij} \left( \frac{1}{2D_n} (g_n^{ij} \partial_i D_n \tilde{h}_n \partial_j u_n, a^*(x, \tilde{h}_n D_x) u_n)_{L^2} \right).
\]

On one hand, using the symbolic calculus, the term (1) tends to
\[
(\nu, (|\zeta|^2 - 1) a(x, \zeta)).
\]

Now using (3.8) and (3.10) we see easily that the terms (2) and (3) tend to zero when $n \to +\infty$. On the other hand, the l.h.s. in (3.11) is equal to
\[
\tilde{h}_n^2 (a(x, \tilde{h}_n D_x) f_n, u_n)_{L^2}.
\]
and according to (3.5) tends to 0. We deduce
\[ \forall a \in C_0^\infty(\mathbb{R}^2), \langle \nu, \|\zeta\|^2 - 1 \rangle a(x, \zeta) \Rightarrow \text{supp}(\nu) \subset \{(x, \zeta); \|\zeta\|^2 = 1\}. \]

To prove the second part in Proposition 3.4, we shall use the following lemma.

**Lemma 3.5.** Let \( a \in C_0^\infty(\mathbb{R}^2) \), and \( b \in W^{1,\infty}(\mathbb{R}^2) \). Then
\[ \|a(x, \tilde{h}_n D_x), b\|_{L^2} \leq C \tilde{h}_n \|\nabla_x b\|_{L^\infty}. \tag{3.12} \]

**Proof.** The kernel of the operator \([a(x, \tilde{h}_n D_x), b]\) is equal to (here \( \phi \in C_0^\infty(\mathbb{R}^2) \) is equal to 1 on the \( x \)-projection of the support of \( a \))
\[ K(x, x') = \frac{1}{(2\pi \tilde{h}_n)^k} \int_{\zeta \in \mathbb{R}^k} e^{\frac{i}{\tilde{h}_n}(x-x') \cdot a(x, \zeta)(b(x) - b(x')) \phi(x')} d\zeta, \]
which is for \( |x - x'| \leq \tilde{h}_n \) (since the support of \( a \) is compact) bounded by
\[ C \tilde{h}_n^{-k} \|\nabla_x b\|_{L^\infty} |x - x'|, \tag{3.13} \]
while for \( |x - x'| \geq \tilde{h}_n \), we can integrate by parts using the identity
\[ \frac{\tilde{h}_n(x - x')}{i|x - x'|^2} \cdot \nabla_\zeta \left( e^{\frac{i}{\tilde{h}_n}(x-x')} \right) = e^{\frac{i}{\tilde{h}_n}(x-x')}, \]
which gives
\[ K(x, x') = \frac{1}{(2\pi \tilde{h}_n)^k} \int_{\zeta \in \mathbb{R}^k} e^{\frac{i}{\tilde{h}_n}(x-x')} \left( \frac{\tilde{h}_n(x - x') \cdot \nabla_\zeta}{i|x - x'|^2} \right)^N a(x, \zeta)(b(x) - b(x')) \phi(x') d\zeta, \]
and hence gives the bound for any \( N \in \mathbb{N}, \)
\[ |K(x, x')| \leq \frac{C N \tilde{h}_n^{N-k}}{|x - x'|^{N-1}} \|\nabla_x b\|_{L^\infty}. \tag{3.14} \]

It follows from (3.13) and (3.14) that
\[ \int_{\mathbb{R}^k} |K(x, x')| dx + \int_{\mathbb{R}^k} |K(x, x')| dx' \leq C \tilde{h}_n \|\nabla_x b\|_{L^\infty}. \]

Then Lemma 3.5 follows from Schur’s lemma. ■
Denoting by \([A, B]\) the commutator of the operators \(A\) and \(B\) let us set
\[
C = \frac{i}{\hbar_n} ([a(x, \tilde{\hbar}_n D_x), \tilde{\hbar}_n^2 f_n], u_n)_{L^2}.
\]
Then we can write using (3.4) and (3.9),
\[
C = \frac{i}{\hbar_n} ([a(x, \tilde{\hbar}_n D_x), \tilde{\hbar}_n^2 f_n], u_n)_{L^2} = (1) + (2) + (3)
\]
\[
(1) = \frac{i}{\hbar_n} ([a(x, \tilde{\hbar}_n D_x), (\tilde{\hbar}_n^2 \Delta_0 - 1)] u_n, u_n)_{L^2},
\]
\[
(2) = \frac{i}{\hbar_n} \sum_{j,l=1}^{k} ([a(x, \tilde{\hbar}_n D_x), \tilde{\hbar}_n \partial_j ((g_n^{jl} - \delta_{jl}) \partial_l)] u_n, u_n)_{L^2},
\]
\[
(3) = \frac{i}{\hbar_n} \sum_{j,l=1}^{k} \tilde{\hbar}_n \left( [a(x, \tilde{\hbar}_n D_x), \frac{1}{2D_n} g_n^{jl} (\partial_j D_n \tilde{\hbar}_n \partial_l)] u_n, u_n \right)_{L^2}.
\]

By symbolic calculus, the term (1) is modulo an \(O(\hbar_n)\) term equal to
\[
\langle \nu, \{a(x, \zeta), |\zeta|^2\} u_n, u_n \rangle_{L^2},
\]
where \(\{,\}\) denotes the Poisson bracket, and hence tends to
\[
\langle \nu, \{a(x, \zeta), |\zeta|^2\} \rangle = (2\zeta \cdot \nabla_x v, a).
\]

Let us look to (2). Each term in the sum can be bounded by
\[
\frac{1}{\hbar_n} |([a(x, \tilde{\hbar}_n D_x), \tilde{\hbar}_n \partial_j] (g_n^{jl} - \delta_{jl}) \partial_l u_n, u_n)|_{L^2} + \frac{1}{\hbar_n} |([a(x, \tilde{\hbar}_n D_x), g_n^{jl} - \delta_{jl}] \partial_j \tilde{\hbar}_n \partial_l u_n, \tilde{\hbar}_n \partial_j u_n)|_{L^2}
\]
\[
+ \frac{1}{\hbar_n} |((g_n^{jl} - \delta_{jl}) [a(x, \tilde{\hbar}_n D_x), \tilde{\hbar}_n \partial_l] u_n, \tilde{\hbar}_n \partial_j u_n)|_{L^2}.
\]

By the semiclassical symbolic calculus and Lemma 3.5 the norms in \(L(L^2)\) of the operators \([a(x, \tilde{\hbar}_n D_x), \tilde{\hbar}_n \partial_j]\) and \([a(x, \tilde{\hbar}_n D_x), g_n^{jl} - \delta_{jl}]\) are bounded, respectively, by \(C \hbar_n\) and \(C \hbar_n\|\nabla_x g_n^{jl}\|_{L^\infty}\) where \(C\) is independent of \(n\). Therefore using (3.10) and (3.8) we deduce that (2) tends to zero when \(n\) goes to \(+\infty\).

Unfolding the commutator and using (3.5), (3.8) we see that the third term in (3.15) is a finite sum of terms which are bounded by \(C \|\partial_j D_n\|_{L^\infty}\). We deduce from (3.10) that (3) tends to zero when \(n\) goes to \(+\infty\).

Now, opening the commutator we see that the r.h.s. in the first equation in (3.15) is equal to
\[
\frac{i}{\hbar_n} ([a(x, \tilde{\hbar}_n D_x), \tilde{\hbar}_n^2 f_n], u_n)_{L^2} - \frac{i}{\hbar_n} ([\tilde{\hbar}_n^2 f_n, a^*(x, \tilde{\hbar}_n D_x)] u_n)_{L^2}.
\]
These terms are bounded by $C \tilde{h}_n \| f_n \|_{L^2(B_2)} \| u_n \|_{L^2(B_2)}$ and tend to zero when goes to $+\infty$ since, according to (3.5), $\| u_n \|_{L^2(B_2)}$ is uniformly bounded and $\| f_n \|_{L^2(B_2)} = o(\tau_n^{\frac{1}{2}}) = o(\tilde{h}_n^{-1})$.

This ends the proof of Lemma 3.4, and hence of Proposition 3.2.

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