Path Integrals on Riemannian Manifolds with Symmetry and Induced Gauge Structure

Shogo Tanimura

Department of Engineering Physics and Mechanics
Kyoto University, Kyoto 606-8501, Japan
E-mail: tanimura@kues.kyoto-u.ac.jp

Abstract

We formulate path integrals on any Riemannian manifold which admits the action of a compact Lie group by isometric transformations. We consider a path integral on a Riemannian manifold $M$ on which a Lie group $G$ acts isometrically. Then we show that the path integral on $M$ is reduced to a family of path integrals on a quotient space $Q = M/G$ and that the reduced path integrals are completely classified by irreducible unitary representations of $G$. It is not necessary to assume that the action of $G$ on $M$ is either free or transitive. Hence our formulation is applicable to a wide class of manifolds, which includes inhomogeneous spaces, and it covers all the inequivalent quantizations. To describe the path integral on inhomogeneous space, stratification geometry, which is a generalization of the concept of principal fiber bundle, is necessarily introduced. Using it we show that the path integral is expressed as a product of three factors; the rotational energy amplitude, the vibrational energy amplitude, and the holonomy factor. When a singular point arises in $Q$, we determine the boundary condition of the path integral kernel for a path which runs through the singularity.

*The title has been changed from the first manuscript, ‘Path Integral Method with Symmetry’.*
1 Introduction

Symmetry has always played an important role in the theory of dynamical systems. In Hamiltonian mechanics, if a dynamical system admits large enough symmetry characterized by an Abelian group, the system is reduced to a dynamical system defined on a torus and becomes completely integrable [1]. Even when the system has non-Abelian symmetry, it is well-known [2] that the system is reduced to a system with a smaller number of degrees of freedom. In quantum mechanics, needless to say, symmetry plays a decisive role in the spectrum analysis of Hamiltonian operators. The Hilbert space of a quantum system is decomposed into a series of subspaces following decomposition of the unitary representation of the symmetry group and each subspace provides a reduced dynamical system. On the other hand, the path integral is a useful formulation to study global and geometric aspects of quantum mechanical systems, and hence it is natural to examine how symmetry helps the analysis of quantum systems in the scheme of path integral.

Our investigation of quantum systems with symmetry is originally motivated by the study of molecular mechanics. A molecular system is a quantum mechanical system which consists of several nuclei and electrons. It has both translational and rotational symmetries. The translational invariance enables us to separate the center-of-mass motion from the relative internal motion of the particles by introducing the center-of-mass coordinate. Thus the relative motion is described by a dynamical system with fewer degrees of freedom. Moreover, we may expect that the rotational invariance enables us to divide the degrees of freedom of the relative motion further into two components; the rigid rotational motion of the whole molecule and the nonrigid vibrational motion. But the separation of rotational and vibrational motions is not so trivial as the separation of center-of-mass and relative motions. Actually it has been proved by Guichardet [3] that there is no coordinate system to separate the rotational motion from the vibrational motion. A more sophisticated method is requested to describe these motions and in fact differential geometry of fiber bundles and connections provides a useful language to describe them as shown by Iwai [4, 5]. He formulated Hamiltonian mechanics and Schrödinger equations of reduced systems by using the theory of principal fiber bundle.

In the words of differential geometry, the original system has a Riemannian manifold $M$ as a configuration space and the symmetry is characterized by a Lie group $G$ of isometric transformations of $M$. The reduced system is defined by a quotient space $Q = M/G$ and if the action of the group $G$ on $M$ is free, then the canonical projection map $p : M \rightarrow Q$ naturally defines a principal fiber bundle and the Riemannian metric of $M$ induces a connection by demanding that the horizontal subspace is orthogonal to the fiber. This kind of geometric setting has been examined...
in detail by Iwai and Montgomery [4, 6].

In the context of molecular mechanics, $M$ is the configuration space of the molecule in the three dimensional space and $G$ is the group of rigid rotations and therein it is natural to call the quotient space $Q = M/G$ a shape space. In this context, the induced connection represents nonholonomic constraint imposed on the molecule by conservation of angular momentum. Non-separability of the rotational and vibrational motions of the molecule is a consequence of the nonvanishing curvature of the connection. Thus the system is neatly described in terms of geometry of fiber bundle.

However, in most of the interesting models for physical application, the action of the group $G$ is not necessarily free but there are some points in $M$ which admit nontrivial isotropy groups. For example, in molecular mechanics, the collinear configuration in which all the particles lie on a line is invariant under rotations around the molecule axis. At this point the configuration space $M$ fails to become a principal fiber bundle and therefore it brings a singular point to the shape space $Q = M/G$, where the ordinary local differential structure is broken. To include such a singular case in consideration, Davis [7] introduced stratification structure and showed that the set of spaces $(M, Q)$ can be regarded as a collection of smooth fiber bundles. In our previous paper [8], by extending the concept of connection to be defined on a stratified manifold, we provided a geometric setting to describe the reduced quantum system on the singular quotient space $Q$. Although Wren [9] also showed that quantum systems on singular quotient spaces can be described by the $C^*$ algebra theory, our geometric method is more concrete and intuitive than his algebraic method.

The study of path integrals on manifolds has a long expanding history, which is impossible to review in this paper. We quote only a few works here; Schulman [10] studied the path integral of spin, which is the path integral on $SO(3) = SU(2)/\mathbb{Z}_2$, and noticed the existence of inequivalent quantizations. Laidlaw and DeWitt [11] showed that inequivalent quantizations are classified by unitary representations of the homotopy group of the configuration space. Mackey [12] elucidated classification of inequivalent quantizations on homogeneous spaces from the point of view of representation theory. When Ohmuki and Kitakado [13] wrote down representations of quantization algebra of spheres concretely, they noticed existence of inequivalent representations and emergence of the gauge potential in the representation of momentum operators. Although Marinov and Terentyev [14] have been studying path integrals on homogeneous space, they did not pay much attention to inequivalent quantizations. Landsman, Linden [15] and we [16, 17] studied path integrals on a homogeneous space $Q = G/H$, where $H$ is a subgroup of the Lie group $G$. In this case, the projection map $p : G \to G/H$ defines a principal fiber bundle with a structure group $H$. They constructed the path integrals on $Q$ by reducing the path
integral on $G$. They also showed that the reduced path integrals are classified by irreducible representations of the group $H$ and that the path integral on $Q$ naturally contains the holonomy of the induced connection. A recent work on the path integral on sphere by Ikemori et. al. is also noticeable for their fine use of spinor structure of the sphere.

The purpose of this paper is to make the reduction method of path integral applicable to more general manifolds, which are not necessarily homogeneous spaces, or principal fiber bundles either. Actually, our method is applicable to any Riemannian manifold $M$ which admits the action of a compact Lie group $G$ of isometric transformations. In such a general case, the singularities are realized as boundary points of the quotient space $Q = M/G$. As a byproduct of our reduction method of path integral, we will obtain a boundary condition of the path integral kernel for a path running through the singularity. This paper is a natural extension of the previous paper by Tanimura and Iwai, in which the Schrödinger formalism for reduced systems was studied.

Our formalism is immediately applicable to quantum molecular dynamics, in which methods to compute rotational and vibrational energy spectra and to compute reaction cross sections are desired. We also hope to apply our method to the gauge field theory. The gauge field is a dynamical system which has gauge symmetry and its physical degrees of freedom are described in terms of the quotient space of the gauge field configuration space modulo gauge transformations as discussed by many authors. We postpone these applications for further investigation.

This paper is organized as follows. in Sec.II we will give a brief review of the general reduction method to make the paper self-contained and will introduce a time-evolution operator for the reduced system, which is to be expressed in terms of the path integral. In Sec.III we will construct an integral kernel which bridges between the abstract time-evolution operator and the more concrete path integral. In Sec.IV we will introduce stratification geometry, which is needed as a proper language to describe path integrals on inhomogeneous spaces. In Sec.V we will use it to write down the path integral explicitly. In Sec.VI behavior of the kernel at a singular point will be examined. To give a simple but nontrivial example, our formalism will be applied to the quantum system on $R^2$ with the $SO(2)$ symmetry in Sec.VII. Section VIII will be devoted to conclusion and discussion of the future work.

2 Reduction of quantum system

In this section we briefly review the method of reduction of quantum dynamical system and make a step toward the definition of path integral of the reduced system. The detailed explanation of the reduction method is given in a previous paper.
Suppose that a quantum system \((\mathcal{H}, H)\) admits a symmetry specified by a set \((G, T)\). Here \(\mathcal{H}\) is a Hilbert space with an inner product denoted by \(\langle \phi, \psi \rangle\). The Hamiltonian \(H\) is a self-adjoint operator acting on \(\mathcal{H}\). Let \(U(\mathcal{H})\) denote a group of the unitary operators on the Hilbert space \(\mathcal{H}\). A compact Lie group \(G\) is equipped with a homomorphism \(T : G \to U(\mathcal{H})\), which is a unitary representation of \(G\) on \(\mathcal{H}\). By symmetry we imply that commutativity \(T(g)H = HT(g)\) holds for any \(g \in G\). The group \(G\) has a normalized invariant measure \(dg\) satisfying \(\int_G dg = 1\).

To construct a reduced system we bring an irreducible unitary representation of \(G\) on another Hilbert space \(\mathcal{H}^\chi\), that is a homomorphism \(\rho^\chi : G \to U(\mathcal{H}^\chi)\). The superscript \(\chi\) labels all the inequivalent representations. It has finite dimensions, \(d^\chi = \text{dim} \mathcal{H}^\chi\). Let a set \(\{e^\chi_1, \ldots, e^\chi_d\}\) be an orthonormal basis of \(\mathcal{H}^\chi\). A tensor product of operators \(\rho^\chi(g) \otimes T(g)\) for \(g \in G\), which acts on \(\mathcal{H}^\chi \otimes \mathcal{H}\), also provides a unitary representation of \(G\). The subspace of the invariant vectors of \(\mathcal{H}^\chi \otimes \mathcal{H}\), which is defined by

\[
(\mathcal{H}^\chi \otimes \mathcal{H})^G := \{ \psi \in \mathcal{H}^\chi \otimes \mathcal{H} \mid (\rho^\chi(h) \otimes T(h)) \psi = \psi, \forall h \in G \},
\]

(2.1)
is called the reduced Hilbert space. Let us define a transformation \(S^\chi_i : \mathcal{H} \to \mathcal{H}^\chi \otimes \mathcal{H}\) by

\[
S^\chi_i \psi := \sqrt{d^\chi} \int_G dg (\rho^\chi(g)e^\chi_i) \otimes (T(g)\psi), \quad \psi \in \mathcal{H},
\]

(2.2)
which then satisfies

\[
(\rho^\chi(h) \otimes T(h))S^\chi_i = S^\chi_i, \quad \forall h \in G,
\]

(2.3)
and therefore it has an image \(\text{Im} S^\chi_i \subset (\mathcal{H}^\chi \otimes \mathcal{H})^G\). Actually, it can be shown that \(\text{Im} S^\chi_i = (\mathcal{H}^\chi \otimes \mathcal{H})^G\). Using the family of transformations \(\{S^\chi_i\}_{(\chi,i)}\) labeled by the inequivalent representations \(\{\rho^\chi\}\) and the basis vectors \(\{e^\chi_i\}\), we obtain an orthogonal decomposition of the Hilbert space

\[
\mathcal{H} \cong \bigoplus_{\chi,i} \text{Im} S^\chi_i = \bigoplus_{\chi,i} (\mathcal{H}^\chi \otimes \mathcal{H})^G.
\]

(2.4)
Since these statements have been already proven in the previous paper [8], we omit the proof here.

Let us introduce another operator \(P^\chi : \mathcal{H}^\chi \otimes \mathcal{H} \to \mathcal{H}^\chi \otimes \mathcal{H}\), which is defined by

\[
P^\chi := \int_G dg \rho^\chi(g) \otimes T(g).
\]

(2.5)
It can be easily seen that

\[
(P^\chi)^\dagger = P^\chi, \quad (P^\chi)^2 = P^\chi, \quad \text{Im} P^\chi = (\mathcal{H}^\chi \otimes \mathcal{H})^G.
\]

(2.6)
Hence $P^x$ is a projection operator $\mathcal{H}^x \otimes \mathcal{H} \to (\mathcal{H}^x \otimes \mathcal{H})^G$. Then an equation

$$P^x S^x_i = S^x_i$$

holds.

The range of the Hamiltonian $H : \mathcal{H} \to \mathcal{H}$ is trivially extended to $(\text{id} \otimes H) : \mathcal{H}^x \otimes \mathcal{H} \to \mathcal{H}^x \otimes \mathcal{H}$. The symmetry $T(g)H = HT(g)$ implies that

$$S^x_i H = (\text{id} \otimes H)S^x_i,$$  \hspace{1cm} (2.9)

$$P^x (\text{id} \otimes H) = (\text{id} \otimes H)P^x.$$  \hspace{1cm} (2.10)

The time-evolution operator $U(t)$ on $\mathcal{H}$ is defined as a unitary operator $U(t) := e^{-iHt}$ in the usual way. Then $(\text{id} \otimes U(t))$ becomes a time-evolution operator on $\mathcal{H}^x \otimes \mathcal{H}$. As a consequence of the invariance (2.10), $U^x(t) := P^x (\text{id} \otimes U(t))$ also provides a unitary time-evolution operator on $(\mathcal{H}^x \otimes \mathcal{H})^G$.

We conclude this short section by summarizing the above discussion; the space of the invariant vectors $(\mathcal{H}^x \otimes \mathcal{H})^G$ provides a closed dynamical system governed by the projected Hamiltonian $H^x := P^x (\text{id} \otimes H)$. Thus we obtain decomposition of the original quantum system $(\mathcal{H}, H)$ into a family of quantum systems $\{((\mathcal{H}^x \otimes \mathcal{H})^G, H^x)\}_x$. The operator $S^x_i$, which reduces the original Hilbert space to the invariant subspace, is called the reduction operator. The main purpose of this paper is to develop a path integral expression for the reduced time-evolution operator $U^x(t) = P^x (\text{id} \otimes U(t))$.

### 3 Reduction of kernel

This section is a step toward the definition of the reduced path integral. Let us assume that the Hilbert space $\mathcal{H}$ is the space of square integrable functions $L_2(M)$ on a measured space $(M, dx)$. Assume that the compact Lie group $G$ acts on $M$ by preserving the measure $dx$. The symmetry operation $T(g)$ is represented on $f \in L_2(M)$ by

$$(T(g)f)(x) := f(g^{-1}x).$$ \hspace{1cm} (3.1)

Moreover, the canonical projection map $p : M \to Q = M/G$ induces a measure $dq$ of the quotient space $Q$ as follows: let $\phi(q)$ be a function on $Q$ such that $\phi(p(x))$ is a measurable function on $M$. The measure $dq$ of $Q$ is then defined by

$$\int_Q dq \phi(q) := \int_M dx \phi(p(x)).$$ \hspace{1cm} (3.2)

Suppose that the time-evolution operator $U(t)$ is expressed in terms of an integral kernel $K : M \times M \times \mathbb{R} \to \mathbb{C}$ as

$$(U(t)f)(x) = \int_M dy K(x, y; t)f(y)$$ \hspace{1cm} (3.3)
for any $f(x) \in L_2(M)$. The symmetry $T(g)H = HT(g)$ implies a symmetry of the kernel

$$K(g^{-1}x, y; t) = K(x, gy; t), \quad g \in G, \quad (3.4)$$

or

$$K(gx, gy; t) = K(x, y; t), \quad g \in G. \quad (3.5)$$

On the other hand, a vector $\psi \in \mathcal{H}^x \otimes L_2(M)$ can be identified with a measurable map $\psi : M \to \mathcal{H}^x \otimes C \cong \mathcal{H}^x$. Then with the representation (3.1) the tensor product operator $\rho^x(g) \otimes T(g)$ acts on $\psi$ as

$$((\rho^x(g) \otimes T(g))\psi)(x) = \rho^x(g)\psi(g^{-1}x), \quad g \in G. \quad (3.6)$$

Therefore, following the definition (2.1), an invariant vector $\psi \in (\mathcal{H}^x \otimes L_2(M))^G$ satisfies

$$((\rho^x(g) \otimes T(g))\psi)(x) = \rho^x(g)\psi(g^{-1}x) = \psi(x), \quad (3.7)$$

or equivalently,

$$\psi(gx) = \rho^x(g)\psi(x). \quad (3.8)$$

A function $\psi : M \to \mathcal{H}^x$ satisfying the property (3.8) is called an equivariant function. Hence the reduced Hilbert space is identified with the space of the equivariant functions $L_2(M; \mathcal{H}^x)^G$. The projection operator $P^x : L_2(M; \mathcal{H}^x) \to L_2(M; \mathcal{H}^x)^G$, which was defined at (2.5) in a general form, is now given by

$$(P^x\psi)(x) = \int_G dg \rho^x(g)\psi(g^{-1}x). \quad (3.9)$$

Then the time-evolution operator of the reduced system is expressed as

$$(P^x(id \otimes U(t))\psi)(x) = \int_G dg \int_M dy \rho^x(g)K(g^{-1}x, y; t)\psi(y). \quad (3.10)$$

Therefore we define a map $K^x : M \times M \times \mathbb{R} \to \text{End} \mathcal{H}^x$ by

$$K^x(x, y; t) := \int_G dg \rho^x(g)K(g^{-1}x, y; t), \quad (3.11)$$

which we call an integral kernel of the reduced time-evolution operator $U^x(t) = P^x(id \otimes U(t))$. It is also equivariant; namely, for any $h \in G$, it satisfies

$$K^x(hx, y; t) = \rho^x(h)K^x(x, y; t), \quad (3.12)$$

$$K^x(x, hy; t) = K^x(x, y; t)\rho^x(h^{-1}). \quad (3.13)$$

Now what we want to find is a path integral expression for the reduced kernel $K^x$. 
4 Reduction of path integral

To obtain a more concrete expression of the reduced kernel $K^\chi(x,y;t)$, we assume that $M$ is a Riemannian manifold on which the Lie group $G$ acts with preserving the metric $g_M$. The Hamiltonian is

$$H = \frac{1}{2}\Delta M + V(x), \quad (4.1)$$

where $\Delta M$ is the Laplacian of $M$ and $V(x)$ is an invariant potential function such that $V(gx) = V(x)$ for any $x \in M$ and $g \in G$.

The projection map $p : M \to Q = M/G$ is equipped with stratification structure as we already discussed in the previous paper [8]. Although the theory of stratification structure may look cumbersome at a first glance, it provides a natural language to describe quantization on the inhomogeneous space $M$. Here we review the theory of stratification structure quickly. The group $G$ acts on $M$ by isometry. Let $G_x := \{g \in G \mid gx = x\}$ denote the isotropy group of $x \in M$. The orbit through $x \in M$, $O_x = \{gx \mid g \in G\}$, is diffeomorphic to a homogeneous space $G/G_x$. We call the tangent vector space $V_x = T_xO_x$ a vertical subspace and the orthogonal complement $H_x = (V_x)^\perp$ a horizontal subspace at $x \in M$. The vertical and horizontal projections, $P_V : T_xM \to V_x$ and $P_H : T_xM \to H_x$ respectively, are defined immediately. A curve in $M$ whose tangent vector is always in the horizontal subspace is called a horizontal curve and a map which transfers the beginning point of the horizontal curve to its end point is called parallel transportation. The dual spaces, $V_x^* = \{\phi \in T_x^*M \mid \phi(v) = 0, \forall v \in V_x\}$ and $H_x^* = \{\phi \in T_x^*M \mid \phi(v) = 0, \forall v \in H_x\}$, are also defined. Then the projection operators in the cotangent space, $P_V^* : T_x^*M \to V_x^*$ and $P_H^* : T_x^*M \to H_x^*$, are defined similarly. Let $g$ denote the Lie algebra of the group $G$. Then the subgroup $G_x$ is accompanied with the Lie subalgebra $g_x$. The unitary representation $\rho^\chi$ of the group $G$ on the Hilbert space $\mathcal{H}^\chi$ induces a representation $\rho_x^\chi$ of the Lie algebra $g$ via differentiation. The group action $G \times M \to M$ also induces an infinitesimal transformation $g \times M \to TM$. The tangent vectors generated by the Lie algebra define a linear map $\theta_x : g \to T_xM$ at each point $x \in M$. Thus it is apparent that $\text{Ker} \theta_x = g_x$ and $\text{Im} \theta_x = V_x$. Then a quotient map $\bar{\theta}_x : g/g_x \to V_x$ is an isomorphism. The stratified connection form $\omega$ is defined as a linear map

$$\omega_x = (\bar{\theta}_x)^{-1} \circ P_V : T_xM \to g/g_x. \quad (4.2)$$

We can also explain physical meanings of these geometric objects. In the context of molecular mechanics [8], the configuration space of the molecule is taken as the space $M$ and the group of rotations $SO(3)$ is taken as the group $G$. The vertical and horizontal spaces are called the rotational and vibrational directions, respectively.
In this context the connection form represents the angular velocity of the molecule and the horizontal curve represents a vibrational motion with zero angular velocity.

The path integral is usually introduced through the composition property of the kernel

\[ K(x'', x; t + t') = \int_M dx' K(x'', x'; t')K(x', x; t). \]  

(4.3)

Repeating insertion of intermediate points, we obtain

\[ K(x', x; t) = \int_M dx_1 \cdots dx_{n-1} K(x', x_{n-1}; \frac{t}{n}) \cdots K(x_2, x_1; \frac{t}{n})K(x_1, x; \frac{t}{n}). \]

(4.4)

For a short distance and a short time interval, the kernel behaves asymptotically as

\[ K(x_{i+1}, x_i; \Delta t) \sim \exp \left[ i \Delta t \left( \frac{\text{dist}^2(x_{i+1}, x_i)}{2(\Delta t)^2} - V(x_i) \right) \right] \]

(4.5)

and therefore a formal limit \( n \to \infty \) leads to the path integral expression

\[ K(x', x; t) = \int_{x}^{x'} [dx] \exp \left[ i \int_{0}^{t} ds \left( \frac{1}{2} ||\dot{x}(s)||^2 - V(x(s)) \right) \right]. \]

(4.6)

Although it is known [23, 24] that on a curved Riemannian manifold the scalar curvature is added to the potential, here we assume that it has already been included in the potential \( V \).

Let us \( \sigma : U \to M \) denote a local section over an open set \( U \subset Q \). For each path \( x(s) \) (\( 0 \leq s \leq t \)) in \( M \), a projected path \( q(s) := p(x(s)) \) in \( Q \) is defined. If the path \( q(s) \) is contained in \( U \), a path \( g(s) \) in \( G \) which satisfies

\[ x(s) = g(s)\sigma(q(s)), \quad 0 \leq s \leq t, \]

(4.7)

also exists. We put \( x(0) = x = g\sigma(q) \) and \( x(t) = x' = g'\sigma(q') \). The path integral (4.6) is formally rewritten as

\[ K(x', x; t) = \int_{q}^{q'} [dq] \int_{g}^{g'} [dg] e^{iI[g(\sigma(q(s)))]}, \]

(4.8)

where \( I \) denotes the action integral. For a generic path \( x(s) \), the projected path \( q(s) = p(x(s)) \) is not contained in the single open set \( U \subset Q \). Thus we need to introduce a family of open covering sets \( \{ U_\alpha \} \) and a family of transformation functions \( \{ \phi_{\alpha\beta} \} \) to make the above equation (4.8) meaningful but they bring unnecessary cumbersomeness. Hence we do not use them explicitly in this paper. More careful treatment including the covering has been discussed in another paper [16].

When we insert an arbitrary smooth function \( \phi(s) \) taking values in \( G \) into the expression

\[ x(s) = g(s)\phi(s)^{-1}\sigma(q(s)), \quad 0 \leq s \leq t, \]

(4.9)
the path integral is left invariant;

\[ K(x', x; t) = \int_q q'[dq] \int_{g\phi(0)} g'[dg] e^{iI[g(\cdot)\phi(\cdot)^{-1}\sigma(q(\cdot))]} . \quad (4.10) \]

Then we will choose the function \( \phi(s) \) to make a physical meaning of the path integral expression of the reduced kernel \( K \chi \) transparent. We choose the function \( \phi(s) \) that makes the curve \( \tilde{x}(s) := \phi(s)^{-1}\sigma(q(s)) \) a horizontal curve starting from \( \tilde{x}(0) = x \). Namely, we demand that a tangent vector

\[ \frac{d}{ds} \tilde{x}(s) = \frac{d}{ds} \left( \phi(s)^{-1}\sigma(q(s)) \right) = \phi^{-1}(-\dot{\phi}\phi^{-1}\sigma + \dot{\sigma}) \quad (4.11) \]

has a vanishing vertical component. This requirement is equivalent to a differential equation

\[ -\dot{\phi}\phi^{-1}\sigma + PV(\dot{\sigma}) = (-\dot{\phi}\phi^{-1} + \omega(\dot{\sigma}))\sigma = 0 \quad (4.12) \]

for \( \phi(s) \) with an initial condition \( \phi(0)^{-1} = g \). Here we have used the stratified connection form \( \omega \), which was defined in (4.2). Equation (4.12) is rewritten as

\[ \dot{\phi}(s) = \omega(\dot{\sigma}(s)) \phi(s) \quad (4.13) \]

and its solution is formally written as

\[ \phi(t) = W_\sigma(t)\phi(0) = W_\sigma(t)g^{-1}, \quad W_\sigma(t) = P e^{\int_0^t ds \omega(\dot{\sigma})}, \quad (4.14) \]

where \( P \) is a symbol indicating the path-ordered product.

In the tangent vector of the curve (4.13)

\[ \frac{d}{ds}x = \frac{d}{ds} \tilde{x} + \frac{d}{ds}g \frac{d}{ds} \tilde{x}, \quad (4.15) \]

the first term is a vertical vector and the second term is a horizontal one according to the definition of \( \phi(s) \). Then the norm of the tangent vector (4.13) with respect to the metric \( g_M \) is given as

\[ ||\dot{x}||^2 = ||PV(\dot{x})||^2 + ||PH(\dot{x})||^2 = ||\dot{g}||^2 + ||\dot{q}||^2. \quad (4.16) \]

The last line should be understood as the definitions of ||\dot{g}||^2 and ||\dot{q}||^2. Putting Eqs. (3.11), (4.6), (4.8) and (4.10) together we obtain

\[ K^\chi(x', x; t) = \int_G dh \rho^\chi(h) K(h^{-1}x', x; t) \]

\[ = \int_G dh \rho^\chi(h) \int_x^{h^{-1}x'} [dx] e^{iI[x(\cdot)]} \]

\[ = \int_G dh \rho^\chi(h) \int_q [dq] \int_{g\phi(0)}^{h^{-1}g'} [dg] e^{iI[g(\cdot)\phi(\cdot)^{-1}\sigma(q(\cdot))]} \]

\[ = \int_q [dq] \int_G dh \rho^\chi(h) \int_{g\phi(0)}^{h^{-1}g'} [dg] e^{iI[g(\cdot)\phi(\cdot)^{-1}\sigma(q(\cdot))]} . \quad (4.17) \]
By changing the integral variable \( h \) to \( g'\phi(t)h \) and using the decomposition (4.16) of the kinetic energy \( ||\dot{x}||^2 \), and furthermore by substituting the solution (4.14) for \( \phi(t) \) with \( \phi(0) = g^{-1} \), we obtain

\[
K^x(x',x;t) = \int_q^{q'} [dq] \int_G dh \rho^x(g'\phi(t)h) \int_e^{h^{-1}} [dg] e^{i \int ds \left( \frac{1}{2} ||\dot{q}||^2 + \frac{1}{2} ||q||^2 - V(q) \right)}
\]

\[
= \int_q^{q'} [dq] \rho^x(g'W_\sigma(t)g^{-1}) \left\{ \int_G dh \rho^\chi(h) \int_e^{h^{-1}} [dg] e^{i \int ds ||\dot{q}||^2} \right\} e^{i \int ds \left( \frac{1}{2} ||\dot{q}||^2 - V(q) \right)}. \tag{4.18}
\]

We call the element \( \rho^x(g'W_\sigma(t)g^{-1}) \in \text{End } (\mathcal{H}^x) \) a holonomy factor for the following reason.

A geometrical meaning of the holonomy factor \( \rho^x(g'W_\sigma(t)g^{-1}) \) is easy to understand. Two curves \( x(s) \) and \( \tilde{x}(s) \) have the end points

\[
x(t) = x' = g'\sigma(q(t)), \tag{4.19}
\]

\[
\tilde{x}(t) = \phi(t)^{-1}\sigma(q(t)) = gW_\sigma(t)^{-1}\sigma(q(t)). \tag{4.20}
\]

Then they are related as

\[
x(t) = (g'W_\sigma(t)g^{-1}) \cdot \tilde{x}(t). \tag{4.21}
\]

Hence the group element \( g'W_\sigma(t)g^{-1} \) indicates how much the end point \( x(t) \) differs from the parallelly transported point \( \tilde{x}(t) \). Even if the projected curve \( q(s) \) forms a closed loop as \( q(t) = q(0) \),

\[
\tilde{x}(t) = gW_\sigma(t)^{-1}\sigma(q(t)) = gW_\sigma(t)^{-1}\sigma(q(0)) = (gW_\sigma(t)^{-1}g^{-1}) \cdot \tilde{x}(0), \tag{4.22}
\]

thus the horizontal curve does not close in general. The group element \( gW_\sigma(t)^{-1}g^{-1} \) is called a holonomy associated with the loop \( q(s) \) in proper words of differential geometry of connection. However we also loosely call the element \( \rho^x(g'W_\sigma(t)g^{-1}) \) the holonomy factor.

A physical meaning of the holonomy factor is interesting; if the group \( G \) is the Abelian group \( U(1) \), we can write the connection form as \( \omega = iA \) with a real valued one-form \( A \). The holonomy factor in (4.14) is then written as

\[
W_\sigma(t) = \exp \left[ i \int A_j dx^j \right], \tag{4.23}
\]

thus it brings the gauge potential \( A \) to the Lagrangian effectively. For a general non-Abelian group \( G \) the holonomy factor is to be understood as a coupling of the system with the induced gauge field \( \omega \).

The factor in the brace of (4.18),

\[
\int_G dh \rho^\chi(h) \int_e^{h^{-1}} [dg] e^{i \int ds ||\dot{g}||^2}, \tag{4.24}
\]

is still left less understood. Then we examine it carefully in the subsequent section.
5 Rotational energy amplitude

To explain the above path integral expression \(4.24\) we have to prepare more notations. The Riemannian manifold \(M\) is equipped with a volume form \(v_M\) associated with the metric \(g_M\). The metric \(g_M = g_V + g_H\) is decomposed according to the decomposition of the tangent space \(T_x M = V_x \oplus H_x\). Then the Laplacian \(\Delta_M\) is also decomposed in a similar manner; suppose that \(\psi(x) \in \mathcal{C}_c^\infty(M; \mathcal{H}^\times)\) is a \(\mathcal{C}^\infty\) function taking values in \(\mathcal{H}^\times\) with a compact support. Then with the use of projectors \(P^*_V\) and \(P^*_H\) on \(T^*_x M\), we define vertical and horizontal components of the Laplacian through

\[
\begin{align*}
\int_M |d\psi|^2 v_M &= \int_M \langle \psi, \Delta_M \psi \rangle v_M, \\
\int_M |P^*_V (d\psi)|^2 v_M &= \int_M \langle \psi, \Delta_V \psi \rangle v_M, \\
\int_M |P^*_H (d\psi)|^2 v_M &= \int_M \langle \psi, \Delta_H \psi \rangle v_M.
\end{align*}
\]

Notice that in our convention the Laplacians \(\Delta_M, \Delta_V, \Delta_H\), are nonnegative operators.

Now we can write the path integral in \(4.24\) as

\[
K_V(g\bar{x}(t), \bar{x}(0); t) = \langle g\bar{x}(t)|e^{-\frac{i}{\hbar}\int^{t}_0 \Delta_V |\bar{x}(0)\rangle} = \int^0_{\epsilon} [dg] e^{\frac{i}{\hbar}\int^{t}_0 |g|^2},
\]

which has a subtle meaning. At each point \(q \in Q\), a fiber \(F = p^{-1}(q)\) exists. If any point \(x \in p^{-1}(q)\) is chosen as a reference point, a map

\[
\varepsilon(x) : G/G_x \rightarrow O_x = F; \quad [g] \mapsto gx
\]

is defined and turns out to be a diffeomorphism. Thus we have a family of parametrized fibers \(\{F_s = p^{-1}(q(s))\}_{0 \leq s \leq t}\) over the projected curve \(q(s)\). Now a diffeomorphism between fibers \(F_s\) and \(F_0\) is defined by a map \(\varepsilon(\bar{x}(0)) \circ \varepsilon(\bar{x}(s))^{-1} : F_s \rightarrow F_0\) using the points \(\bar{x}(0)\) and \(\bar{x}(s)\) as references. Hence the family of fibers \(\{F_s\}_{0 \leq s \leq t}\) can be identified with the direct product space \(F_0 \times [0, t]\). Moreover, each fiber \(F_s\) is equipped with a Riemannian metric \(g_s\) which is defined by restricting the original metric \(g_M\) to the fiber. The metric \(g_s\) coincides also with the vertical metric \(g_V^s\) restricted to \(F_s\). It is to be noted that the different fibers \((F_s, g_s)\) and \((F_{s'}, g_{s'})\) are not necessarily isometric for \(s \neq s'\). The metric \(g_s\) defines also the Laplacian \(\Delta_s\) of each fiber \(F_s\). Then the kernel \(K_V(\cdot, \cdot, s) : F_s \times F_0 \rightarrow \mathbb{C}\) in \(5.4\) is the fundamental solution of the Schrödinger equation

\[
\frac{\partial}{\partial s} K_V(x, x_0; s) = -\frac{i}{2} \Delta_s K_V(x, x_0; s)
\]

with the initial condition \(K_V(x, x_0; 0) = \delta_F(x, x_0)\). This argument explains a meaning of the first two terms of Eq.\((5.4)\). Since in the context of molecular mechanics the
fiberwise directions represent rotational motions, it seems suitable to call the kernel $K_V(x; x_0; s)$ the rotational energy amplitude.

Next we turn to the third term of (5.4),

$$
\int_{e}^{g} [dg] e^{\frac{s}{2} \int ds \|\dot{\gamma}\|^2}. \quad (5.7)
$$

It should be understood as an integration over paths $g(s)\bar{x}(s)$, which vary by multiplying various $g(s)$ on the fixed reference path $\bar{x}(s)$. By repeating the usual argument which leads to the path integral (4.6), we arrive at the expression (5.7) for the kernel $K_V(x; x_0; s)$.

Moreover, the rotational energy amplitude in (1.24),

$$
K^\chi_V(\bar{x}(t), \bar{x}(0); t) := \int_{G} \rho^\chi(h) K_V(h^{-1}\bar{x}(t), \bar{x}(0); t)
$$

$$
= \int_{G} \rho^\chi(h) \int_{e}^{h^{-1}} [dg] e^{\frac{s}{2} \int ds \|\dot{\gamma}\|^2} \quad (5.8)
$$

is to be understood as the kernel restricted to equivariant functions as stated in (3.11). Accordingly we will show that the equivariant kernel (5.8) can be reduced to a much simpler expression below.

We now examine the action of the vertical Laplacian $\Delta_V$ on an equivariant function $\psi(x)$. When $\psi(x)$ is equivariant, for any $X \in g$ we have

$$
\theta(X)\psi = \rho^\chi(X)\psi, \quad (5.9)
$$

which is derived from (5.8) by differentiation. If we put $X = \omega(v)$ for some $v \in T_x M$, we have $\theta(\omega(v)) = P_V(v)$ according to (1.2). Hence we obtain

$$
P^*_V(d\psi) = \rho^\chi_{\omega}(v)\psi. \quad (5.10)
$$

Therefore, Eq.(5.2) is rewritten as

$$
\int_{M} \|P^*_V(d\psi)\|^2_{v_M} = \int_{M} \|\rho^\chi_{\omega}(\psi)\|^2_{v_M}
$$

$$
= \int_{M} g_M^{-1}\langle \rho^\chi_{\omega}(\psi), \rho^\chi_{\omega}(\psi) \rangle_{v_M}
$$

$$
= - \int_{M} g_M^{-1}\langle \psi, \rho^\chi_{\omega}(\psi) \rangle_{v_M}, \quad (5.11)
$$

where $g_M^{-1}$ is a metric on the cotangent bundle. Here notice that $\rho^\chi_{\omega}$ takes values in anti-hermitian elements of $\text{End}(\mathcal{H}^\chi)$. Combining the metric $g_M^{-1}(x) : T^*_x M \otimes T^*_x M \to \mathbb{R}$ with the connection form $\omega_x \in (g_x \otimes g_x) \otimes T_x M$, let us define an element of the tensor product of the Lie algebra

$$
A_x := -g_M^{-1}(x) \circ (\omega_x \otimes \omega_x) \in (g_x \otimes g_x) \otimes (g_x \otimes g_x), \quad (5.12)
$$
which we call the rotational energy operator. The representation of the Lie algebra \( \rho^x : g \rightarrow \text{End}(H^x) \) can be extended to a representation of the universal envelop algebra \( \rho^x : U(g) \rightarrow \text{End}(H^x) \). Thus the value of \( \Lambda_x \in g \otimes g \) is mapped to \( \rho^x(\Lambda_x) \in \text{End}(H^x) \). It is also apparent that \( \rho^x(\Lambda_x) \) is a nonnegative operator from the definition. Then from (5.2), (5.11) and (5.12), we conclude that the vertical Laplacian \( \Delta_V \) is reduced to an algebraic operator when it acts on equivariant functions as

\[
\Delta_V \psi(x) = \rho^x(\Lambda_x) \psi(x). \tag{5.13}
\]

Then we can write the equivariant vertical kernel (5.8) as

\[
\int_G \rho^x(h) (h^{-1} \bar{x}(t)|e^{-\frac{i}{2} t \Delta_V} |x(0)) = \mathcal{P} \exp \left[ -\frac{i}{2} \int_0^t ds (\rho^x \circ \Lambda)(\bar{x}(s)) \right]. \tag{5.14}
\]

If we put

\[
R(t) := \mathcal{P} \exp \left[ -\frac{i}{2} \int_0^t ds \Lambda(\bar{x}(s)) \right], \tag{5.15}
\]

which takes values in the universal envelop algebra \( U(g) \), then the equivariant kernel (5.14) is represented as \( \rho^x(R(t)) \). Finally, by substituting this into (4.18), we arrive at the expression of the reduced kernel

\[
K^x(x', x; t) = \int_q [dq] \rho^x(g' W_\sigma(t) g^{-1}) \rho^x(R(t)) e^{i \int ds \{ \frac{1}{2} ||\dot{q}||^2 - V(q) \}}
\]

\[
= \int_q [dq] \rho^x \left( g' \mathcal{P} \exp \left[ \int_0^t ds \omega(\dot{\sigma}) \right] g^{-1} \right)
\]

\[
\rho^x \left( \mathcal{P} \exp \left[ -\frac{i}{2} \int_0^t ds \Lambda(\bar{x}(s)) \right] \right)
\]

\[
\exp \left[ i \int_0^t ds \left\{ \frac{1}{2} ||\dot{q}||^2 - V(q) \right\} \right], \tag{5.16}
\]

which is a main result of this paper. The remaining phase factor \( \exp[i \int ds \{ \frac{1}{2} ||\dot{q}||^2 - V(q) \}] \) is called the vibrational energy amplitude in the context of molecular mechanics.

### 6 Singular points and boundary condition

Here let us mention characteristic behavior of the equivariant kernel (3.11) at a singularity. Suppose that a point \( x \in M \) admits a nontrivial isotropy group \( G_x \neq \{e\} \). Then the equivariance of the kernel (3.12) leads to the invariance

\[
K^x(x, y; t) = K^x(hx, y; t) = \rho^x(h) K^x(x, y; t), \quad h \in G_x. \tag{6.1}
\]

Then the value of \( K^x(x, y; t) \) is restricted to the subspace of invariant vectors

\[
(\mathcal{H}^x)^{G_x} := \{ v \in \mathcal{H}^x | \rho^x(h)v = v, \forall h \in G_x \}. \tag{6.2}
\]
Let \( dh \) denote an normalized invariant measure of \( G_x \). Then let us introduce an operator

\[
B^\chi(x) := \int_{G_x} dh \rho^\chi(h),
\]

which is an element of \( \text{End}(\mathcal{H}^\chi) \). It is easily verified that \( B^\chi(x) \) is a projection operator onto \( (\mathcal{H}^\chi)^{G_x} \). Thus the value of the kernel \( K^\chi(x, y; t) \) automatically satisfies

\[
B^\chi(x)K^\chi(x, y; t) = K^\chi(x, y; t)
\]

and in a composition relation similar to (6.3)

\[
K^\chi(x'', x; t + t') = \int_M dx' K^\chi(x'', x'; t')K^\chi(x', x; t),
\]

we can freely insert the projection operator as

\[
K^\chi(x'', x; t + t') = \int_M dx' K^\chi(x'', x'; t')B^\chi(x')K^\chi(x', x; t).
\]

When the path runs through a singular point \( x' \), at which the dimension of \( G_{x'} \) abruptly increases, the rank of the projection operator \( B^\chi(x') \) decreases. This gives a kind of boundary condition to the kernel when the path hits a singular point.

### 7 Example

Let us explore a simple application of the above developed formalism. As an example we take the Euclidean space \( \mathbb{R}^2 \) with the standard metric 

\[
\mathcal{G} = dx^2 + dy^2 = dr^2 + r^2d\theta^2
\]

as \( M \). The group \( G = SO(2) \) acts on \( \mathbb{R}^2 \) and defines a quotient space \( Q = \mathbb{R}^2/\text{SO}(2) = \mathbb{R}_{\geq 0} \). The group action

\[
\text{SO}(2) \times \mathbb{R}^2 \rightarrow \mathbb{R}^2,
\]

induces the action of the Lie algebra

\[
\mathfrak{so}(2) \times \mathbb{R}^2 \rightarrow T\mathbb{R}^2;
\]

which also defines a map

\[
\theta : \mathfrak{so}(2) \times \mathbb{R}^2 \rightarrow T\mathbb{R}^2;
\]

Then the stratified connection (4.2) now becomes a one-form taking values in \( \mathfrak{so}(2) \),

\[
\omega = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) d\theta.
\]
The vertical and horizontal components of the metric are given as \( g_V = r^2 d\theta^2 \) and \( g_H = dr^2 \), respectively. In the dual space the metric is given as

\[
(g_M)^{-1} = \frac{\partial}{\partial r} \otimes \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \otimes \frac{\partial}{\partial \theta}.
\]  

(7.5)

The rotational energy operator [(5.12)] is now calculated as

\[
\Lambda = -(g_M)^{-1} \circ (\omega \otimes \omega) = -\frac{1}{r^2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]  

(7.6)

Any irreducible unitary representation of \( SO(2) \) is one-dimensional. It is labeled by an integer \( n \in \mathbb{Z} \) and defined by

\[
\rho_n : SO(2) \to U(1); \quad \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \mapsto e^{i n \phi}.
\]  

(7.7)

Thus the differential representation of the Lie algebra of \( SO(2) \) is defined by

\[
(\rho_n)_* : \mathfrak{so}(2) \to \mathfrak{u}(1); \quad \begin{pmatrix} 0 & -\phi \\ \phi & 0 \end{pmatrix} \mapsto i n \phi.
\]  

(7.8)

The rotational energy operator is then represented as

\[
(\rho_n)_*(\Lambda) = -\frac{1}{r^2} (i n)^2 = \frac{n^2}{r^2}.
\]  

(7.9)

The origin \( r = 0 \) admits a nontrivial isotropy group \( G_0 = SO(2) \). The boundary projection operator [(6.3)] becomes

\[
B_n(0) = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i n \phi} = \delta_{n0}.
\]  

(7.10)

If we take the map \( r \mapsto (r, 0) \) as a section \( \sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}^2 \) then the pullback of the connection identically vanishes, \( \sigma^* \omega = 0 \), and the reduced path integral [(5.16)] is now given by

\[
K_n(r', r; t) = \int_r^{r'} [r \ dr] B_n[r] \exp \left[ i \int_0^t ds \left\{ \frac{1}{2} r^2 - \frac{n^2}{2r^2} - V(r) \right\} \right],
\]  

(7.11)

where the factor \( B_n[r] \) represents the boundary condition at \( r = 0 \). If \( n \neq 0 \) and \( r(s) = 0 \) at some time \( s \) between \( 0 \leq s \leq t \), then \( B_n[r] = 0 \). Otherwise, \( B_n[r] = 1 \).

We can compare this result [(7.11)] with a reduced kernel obtained from an exact kernel for the specific case of a free particle with the Hamiltonian

\[
H = \frac{1}{2} \Delta = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).
\]  

(7.12)
The exact kernel on $\mathbb{R}^2$ is

$$K((x_2, y_2), (x_1, y_1); t) = \frac{1}{2\pi it} \exp \left[ \frac{i}{2t} \left\{ (x_2 - x_1)^2 + (y_2 - y_1)^2 \right\} \right]. \quad (7.13)$$

If we put $z_j = x_j + iy_j = r_j e^{i\theta_j} (j = 1, 2)$, it is rewritten as

$$K(z_2, z_1; t) = \frac{1}{2\pi it} \exp \left[ \frac{i}{2t} |z_2 - z_1|^2 \right] = \frac{1}{2\pi it} \exp \left[ \frac{i}{2t} \left\{ r_2^2 + r_1^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1) \right\} \right]. \quad (7.14)$$

Then the reduced kernel is explicitly calculated as

$$K_n(z_2, z_1; t) = \int_0^{2\pi} \frac{d\phi}{2\pi} \rho_n(e^{i\phi}) K(e^{-i\phi} z_2, z_1; t)$$

$$= \frac{1}{2\pi it} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{in\phi} \exp \left[ \frac{i}{2t} \left\{ r_2^2 + r_1^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1 - \phi) \right\} \right]$$

$$= \frac{1}{2\pi it} e^{i(r_2^2+r_1^2)/(2t)} e^{in(\theta_2-\theta_1)} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{in\phi} \exp \left[ \frac{i}{t} r_1 r_2 \cos \phi \right]$$

$$= \frac{1}{2\pi it} e^{i(r_2^2+r_1^2)/(2t)} e^{in(\theta_2-\theta_1-\pi/2)} J_n \left( \frac{1}{t} r_1 r_2 \right), \quad (7.15)$$

where $J_n(x)$ denotes the $n$-th Bessel function. Asymptotic behavior of the Bessel function for a short time interval $t$ such that $t/r_1 r_2 \ll 1$ is given by

$$J_n \left( \frac{r_1 r_2}{t} \right) \sim \sqrt{\frac{2t}{\pi r_1 r_2}} \exp \left[ -i \left( \frac{r_1 r_2}{t} + \frac{n^2 t}{2r_1 r_2} - \frac{n\pi}{2} \right) \right]. \quad (7.16)$$

The derivation of this formula is shown in the appendix. Thus for the short interval the reduced kernel behaves as

$$K_n(z_2, z_1; t) \sim \frac{1}{\pi i \sqrt{2\pi tr_1 r_2}} e^{in(\theta_2-\theta_1)} \exp \left[ it \left\{ \frac{(r_2 - r_1)^2}{2 t^2} - \frac{n^2}{2 r_1 r_2} \right\} \right], \quad (7.17)$$

which reproduces the integrand of the path integral $(7.11)$.

### 8 Conclusion

Now let us summarize our discussion. We began with a review of the method of reduction of quantum systems. We assumed that the original system was defined in terms of the Riemannian manifold $M$ and that the isometric transformations of $M$ formed the compact Lie group $G$. Then the projection map $p : M \to Q = M/G$ was equipped with stratification structure, which is a generalization of principal fiber bundle structure. Our main purpose was to give a path integral expression on the quotient space $Q$ to the time-evolution operator of the reduced system. The reduced path integral $(5.16)$ was factorized into three parts; the rotational energy amplitude,
the vibrational energy amplitude, and the holonomy factor. The rotational energy amplitude represented the integration over the fiber directions, while the vibrational energy amplitude represented the integration over the directions perpendicular to the fiber. These perpendicular directions defined the connection, and when they were non-integrable, the holonomy factor arose.

At the singular point which was characterized by larger symmetry than the neighboring points, the amplitude was restricted to values invariant under the symmetry operations, and therefore the boundary value projection operator was automatically introduced. As a simple example, we applied our formalism to the quantum system on $\mathbb{R}^2$ with symmetry specified by $SO(2)$. For the case of a free particle, our result was compared with the exact result, and then we confirmed their agreement.

Finally, we would like to mention remaining problems. For further application, the quantum system on the sphere $SU(2)/U(1)$ and the one on the meridian line $U(1)\backslash SU(2)/U(1)$ are also interesting to study the role of the boundary conditions. The quantum system on the adjoint orbit space $G/\text{Ad}G$ of a Lie group $G$ is also interesting for physical application since it has a role as a toy model of the gauge field theory as other authors [19, 20] discussed. Hence it is expected that our method of reduction of quantum systems is useful for gauge invariant analysis of the field theory, which is a subject attracting attention recently in high energy physics [21, 22].

Although the path integral formalism presented here is a method to treat quantum systems, it is naturally applicable to stochastic processes, too. Statistical problems concerning the shape space $Q = M/G$ have been studied by Kendall [25] and other people. The Schrödinger equation is replaced by the diffusion equation in the context of stochastic problems, and the Wiener integral could be also reduced in a similar way if it admits symmetry.

For a strong attractive potential, for example, the inverse-square potential $V = -kr^{-2}$, the particle falls into the origin within a finite time in the context of classical mechanics. It is an interesting question [23, 24] to ask what kind of phenomenon in quantum mechanics is corresponding to the finite-falling particle. Facing such a problem, it is expected that we need to modify the boundary reflection condition but we do not have a definite answer yet.

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Appendix A. Asymptotic formula of Hankel

The asymptotic behavior of the Bessel function $J_n(x)$ is given by the following equations: if two functions

$$A_N(x) = 1 + \sum_{r=1}^{[N/2]} (-1)^r \frac{(4n^2 - 1^2)(4n^2 - 3^2) \cdots [4n^2 - (4r - 1)^2]}{(2r)! (8x)^{2r}},$$

$$B_N(x) = \sum_{r=0}^{[(N-1)/2]} (-1)^r \frac{(4n^2 - 1^2)(4n^2 - 3^2) \cdots [4n^2 - (4r + 1)^2]}{(2r + 1)! (8x)^{2r+1}},$$

are introduced, for $|x| >> n, 1$ the Bessel function is expanded as

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \left\{ A_0(x) \cos \left( x - \frac{(2n + 1)\pi}{4} \right) - B_0(x) \sin \left( x - \frac{(2n + 1)\pi}{4} \right) \right\},$$

which is called Hankel’s asymptotic expansion formula of the Bessel function. Now we put

$$\omega := x - \frac{(2n + 1)\pi}{4}, \quad \varepsilon := B_0(x) = \frac{4n^2 - 1^2}{8x} \sim \frac{n^2}{2x},$$

then the Bessel function is approximated as

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \{ A_0(x) \cos \omega - B_0(x) \sin \omega \}
= \sqrt{\frac{2}{\pi x}} \{ \cos \omega - \varepsilon \sin \omega \}
\sim \sqrt{\frac{2}{\pi x}} \cos(\omega + \varepsilon).$$

Thus the Bessel function that we encountered in this paper is approximated as

$$J_n \left( \frac{r_1 r_2}{t} \right) \sim \sqrt{\frac{2t}{\pi r_1 r_2}} \Re \exp \left[ -i \left( \frac{r_1 r_2}{2t} + \frac{n^2 t}{2r_1 r_2} - \frac{(2n + 1)\pi}{4} \right) \right],$$

which justifies Eq. (7.16) in the body of this paper.

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