Some inequalities for the Fan product of $M$-tensors

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Abstract

In this paper, we investigate some inequalities for the Fan product of $M$-tensors. We propose exact characterizations of $M$-tensors and establish some inequalities on the minimum eigenvalue for the Fan product of two $M$-tensors. Furthermore, the inclusion relations among them are discussed. Numerical examples show the validity of the conclusions.

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1 Introduction

Let $\mathbb{C}(\mathbb{R})$ be the set of all complex (real) numbers, $\mathbb{R}_+(\mathbb{R}_{++})$ be the set of all nonnegative (positive) numbers, $\mathbb{C}^n(\mathbb{R}^n)$ be the set of all dimension $n$ complex (real) vectors, and $\mathbb{R}^n_+(\mathbb{R}^n_{++})$ be the set of all dimension $n$ nonnegative (positive) vectors. An $m$th order $n$-dimensional tensor $A = (a_{i_1i_2...i_m})$ is a higher-order generalization of matrices, which consists of $n^m$ entries:

$$a_{i_1i_2...i_m} \in \mathbb{R}, \quad i_k \in \mathbb{N} = \{1, 2, \ldots, n\}, k = 1, 2, \ldots, m.$$

$A$ is called nonnegative (positive) if $a_{i_1i_2...i_m} \in \mathbb{R}_+$ ($a_{i_1i_2...i_m} \in \mathbb{R}_{++}$).

Tensors have many similarities with matrices and many related results of matrices such as determinant, eigenvalue, and algorithm theory can be extended to higher order tensors [1–3]. Furthermore, structured matrices such as nonnegative matrices, $H$-matrices and $M$-matrices can also be extended to higher order tensors and these are becoming the focus of recent tensor research [4–26]. In particular, $M$-tensors play important roles in the stability study of nonlinear autonomous systems via Lyapunov’s direct method in automatic control [27–29] and spectral hypergraph theory [3, 30, 31].

On the other hand, Fan product of $M$-matrices and Hadamard product of nonnegative matrices are significant for practical problems, such as the weak minimum principle in partial differential equations, products of integral equation kernels, characteristic functions in probability theory, the study of association schemes in combinatorial theory, and so on (see [32]). Some inequalities on the spectral radius for the Hadamard product of two nonnegative matrices and some inequalities on the minimum eigenvalue for the Fan product of two $M$-matrices can be found in [33–37]. Recently, Sun et al. [14] investigated...
some inequalities for the Hadamard product of tensors and obtained some bounds on the spectral radius, and used them to estimate the spectral radius of a directly weighted hypergraph. It is well known that an $M$-tensor is defined based on a $Z$-tensor and its algebra properties can be explored using the spectral theory of nonnegative tensors [23].

Motivated by these observations, we expect to establish sharp lower bounds on the minimal eigenvalue for the Fan product of two $M$-tensors and discuss some inclusion relations among them.

The remaining of this paper is organized as follows. In Sect. 2, we introduce important notation and recall some preliminary results on tensor analysis. In Sect. 3, based on exact characterizations of $M$-tensors, we give a lower bound on the minimum eigenvalue for the Fan product of two $M$-tensors. An improved result is established for irreducible nonnegative tensors by the ratio of the smallest and largest values of a Perron vector. Finally, making use of the information of the absolute maximum in the off-diagonal elements, we obtain a new lower bound on the minimum eigenvalue for the Fan product. With numerical examples, we exhibit the efficiency of the results given in Theorems 1–3.

2 Notation and preliminaries

We start this section with some fundamental notions and properties developed in tensor analysis [1, 3], which are needed in the subsequent analysis.

**Definition 1** Let $\mathcal{A}$ be an $m$-order $n$-dimensional tensor. Assume that $\mathcal{A}x^{m-1}$ is not identical to 0. We say that $(\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ is an eigenvalue–eigenvector of $\mathcal{A}$ if

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m=1}^n a_{i_1 i_2 \ldots i_m} x_{i_2} \cdots x_{i_m}$, $x^{[m-1]} = [x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1}]^T$, and $(\lambda, x)$ is called an $H$-eigenpair if they are both real.

**Definition 2** Let $\mathcal{A}$ and $\mathcal{I}$ be $m$-order $n$-dimensional tensors.

(i) We call $\sigma(\mathcal{A})$ as the set of all eigenvalues of $\mathcal{A}$. Assume $\sigma(\mathcal{A}) \neq \emptyset$. Then the spectral radius of $\mathcal{A}$ is denoted by

$$\rho(\mathcal{A}) = \max\{ |\lambda| : \lambda \in \sigma(\mathcal{A}) \}.$$ 

Meanwhile, we use $\tau(\mathcal{A})$ to denote the minimal value of the real part of eigenvalues of $\mathcal{A}$.

(ii) We call a tensor $\mathcal{A}$ reducible if there exists a nonempty proper index subset $I \subset \{1, 2, \ldots, n\}$ such that

$$a_{i_1 i_2 \ldots i_m} = 0, \quad \forall i_1 \in I, i_2, \ldots, i_m \notin I.$$ 

If $\mathcal{A}$ is not reducible, then we call $\mathcal{A}$ irreducible.

(iii) We call a nonnegative matrix $GM(\mathcal{A})$ the representation associated to a nonnegative tensor $\mathcal{A}$, if the $(i, j)$th entry of $GM(\mathcal{A})$ is defined to be the sum of $a_{i_2 i_3 \ldots i_m}$ with indices $j \in \{i_2, i_3, \ldots, i_m\}$. We call a tensor $\mathcal{A}$ weakly reducible, if its representation $GM(\mathcal{A})$ is reducible. It is weakly irreducible if it is not weakly reducible.
(iv) We call $I$ is a unit tensor whose entries are

$$
\delta_{i_1i_2...i_m} = \begin{cases} 
1, & \text{if } i_1 = i_2 = \cdots = i_m, \\
0, & \text{otherwise.}
\end{cases}
$$

It is noted that the spectral radius $\rho(A)$ is the largest $H$-eigenvalue for the nonnegative tensor [4] and $\tau(A)$ is smallest $H$-eigenvalue for the $M$-tensor [23].

The Perron–Frobenius theorem for nonnegative weakly irreducible tensors has been established in [9, 11, 22].

**Lemma 1** Let $A$ be a weakly irreducible nonnegative tensor of order $m$ and dimension $n$. Then the following results hold:

(i) $A$ has a positive eigenpair $(\lambda, x)$ and $x$ is unique up to a multiplicative constant.

(ii) 

$$
\min_{x \in \mathbb{R}^n_+} \max_{1 \leq i \leq n} (A^{m-1})_{ii} \leq \rho(A) = \max_{x \in \mathbb{R}^n \setminus \{0\}, x_i \neq 0, 1 \leq i \leq n} \min_{x \not\equiv 0} (A^{m-1})_{ii} \leq \min_{x \not\equiv 0} (A^{m-1})_{ii}.
$$

The following specially structured tensors are extended from matrices [8, 23].

**Definition 3** Let $A$ and $U$ be $m$-order $n$-dimensional tensors.

(i) We call $A$ is a $Z$-tensor if all its off-diagonal entries are nonpositive.

(ii) We call $A$ is an $M$-tensor if there exist a nonnegative tensor $U$ and a positive real number $\eta \geq \rho(U)$ such that

$$
A = \eta I - U.
$$

If $\eta > \rho(U)$, then $A$ is called a strong $M$-tensor.

(iii) We call $A$ is a weakly irreducible $M$-tensor if $U$ is weakly irreducible.

(iv) Assume $A$ and $B$ are $M$-tensors. The Fan product of $A$ and $B$ is denoted by $A \cdot B = D = (d_{i_1i_2...i_m})$ and defined by

$$
d_{i_1i_2...i_m} = \begin{cases} 
|a_{i_1...i_m}|b_{i_1...i_m}, & \text{if } i_1 = i_2 = \cdots = i_m = i, \\
-|a_{i_1i_2...i_m}b_{i_1i_2...i_m}|, & \text{otherwise.}
\end{cases}
$$

It is easy to see that all the diagonal entries of an $M$-tensor are nonnegative [23], and the (strong) $M$-tensor is closely linked with the diagonal dominance defined below.

**Definition 4** An $m$-order $n$-dimensional tensor $A$ is called diagonally dominant if

$$
|a_{i_1...i_m}| \geq \sum_{a_{i_2...i_m} \neq 0} |a_{i_2...i_m}| \quad \forall i \in N;
$$

$A$ is called strictly diagonally dominant if the strict inequalities hold for all $i \in N$.

Define a positive diagonal matrix $D = \text{diag}(d_1, \ldots, d_n)$ and set

$$
B = \big( b_{i_1i_2...i_m} \big) = A \cdot D^{(m-1)}D \cdots D = \left( a_{i_1...i_m}d_{i_1}^{(m-1)}d_{i_2} \cdots d_{i_m} \right). \quad (1)
$$

We obtain the following necessary and sufficient condition for identifying $M$-tensors.
Lemma 2 ([23]) Suppose $A$ is a weakly irreducible $Z$-tensor and its all diagonal elements are nonnegative. Then $A$ is an (strong) $M$-tensor if and only if there exists a positive diagonal matrix $D$ such that $B$ defined in (1) is (strictly) diagonally dominant.

3 Some inequalities on the minimum eigenvalue for the Fan product

In this section, we shall give lower bounds on the minimum eigenvalue for the Fan product. Firstly, we establish characterizations of $M$-tensors.

Lemma 3 Let $Q$ be a weakly irreducible $M$-tensor of order $m$ and dimension $n$. If $Qz^{m-1} \succeq kz^{m-1}$ for a vector $z \in R^n_{++}$ and a real number $k$, then $k \leq \tau (Q)$.

Proof Since $Q$ is an $M$-tensor, there exists a nonnegative tensor $U$ such that

$$Q = \lambda I - U,$$

where $\lambda$ is a nonnegative real number and $\lambda \geq \rho (U)$. It is easy to see that $\tau (Q) = \lambda - \rho (U)$. Furthermore, $\rho (U) = \lambda - \tau (Q)$. Taking into account that $Q$ is weakly irreducible, we deduce that $U$ is weakly irreducible. From the assumption and (2), we have

$$(\lambda I - U)z^{m-1} \succeq kz^{m-1},$$

that is,

$$(\lambda - k)z^{m-1} \succeq Uz^{m-1}.$$

It follows from Lemma 1 that

$$\lambda - k \geq \rho (U) = \lambda - \tau (Q).$$

So, $\tau (Q) \geq k$. □

Lemma 4 Let $P$, $Q$ be two $M$-tensors of order $m$ and dimension $n$. Then $P \star Q$ is an $M$-tensor. Furthermore, if $P$, $Q$ are strong $M$-tensors, then $P \star Q$ is a strong $M$-tensor.

Proof By the definition of $P \star Q$, it holds that

$$P \star Q = \begin{cases} p_{i_1 \ldots i_m}q_{i_1 \ldots i_m} & \text{if } i_2 = i_3 = \cdots = i_m = i, \\ -|p_{i_2 \ldots i_m}q_{i_2 \ldots i_m}| & \text{otherwise}. \end{cases}$$

Since $P$, $Q$ are $M$-tensors, by Lemma 1, there exist positive diagonal matrices $C$, $D$ such that

$$A = P \cdot \underbrace{C \cdots C}_{m-1}, \quad B = Q \cdot \underbrace{D \cdots D}_{m-1}$$

with

$$a_{i_1 \ldots i_m} = p_{i_1 \ldots i_m}c_{i_1} \cdots c_{i_m}, \quad b_{i_1 \ldots i_m} = q_{i_1 \ldots i_m}d_{i_1} \cdots d_{i_m}.$$
Specifically,

\[ a_{1,i} = p_{1,i}, \quad b_{1,i} = q_{1,i}. \]

Taking into account that \( A \) and \( B \) are diagonally dominant, we conclude that

\[ |p_{1,i}| = |a_{1,i}| \geq \sum_{\delta_{i2...im} = 0} |p_{i2...im}| c_i^{-(m-1)} c_{i2} \cdots c_{im}, \]

\[ |q_{1,i}| = |b_{1,i}| \geq \sum_{\delta_{i2...im} = 0} |q_{i2...im}| d_i^{-(m-1)} d_{i2} \cdots d_{im}. \]

Furthermore, it holds that

\[ |p_{1,i}q_{1,i}| = |a_{1,i}b_{1,i}| \]
\[ \geq \sum_{\delta_{i2...im} = 0} (|p_{i2...im}| c_i^{-(m-1)} c_{i2} \cdots c_{im}) \sum_{\delta_{i2...im} = 0} (|q_{i2...im}| d_i^{-(m-1)} d_{i2} \cdots d_{im}) \]
\[ \geq \sum_{\delta_{i2...im} = 0} |p_{i2...im}q_{i2...im}| d_i^{-(m-1)} c_i^{-(m-1)} c_{i2} \cdots c_{im} d_{i2} \cdots d_{im} \]
\[ = \sum_{\delta_{i2...im} = 0} |p_{i2...im}q_{i2...im}| d_i^{-(m-1)} c_i^{-(m-1)} c_{i2} \cdots c_{im} d_{i2} \cdots d_{im}. \]  

(3)

Hence, it follows from (3) that there exists a positive diagonal matrix \( U = \text{diag}(c_1d_1, c_2d_2, \ldots, c_nd_n) \) such that

\[ |p_{1,i}q_{1,i}| \geq \sum_{\delta_{i2...im} = 0} p_{i2...im}q_{i2...im} u_i^{-(m-1)} u_{i2} \cdots u_{im}. \]

It follows from Lemma 2 that \( P \ast Q \) is an \( M \)-tensor. By a similar argument as for the first conclusion, we can obtain the second conclusion. \( \square \)

Suppose that \( P = (p_{i2...im}) \) is a strong \( M \)-tensor of order \( m \) and dimension \( n \). Set \( N = D - P \), where \( D \) denotes the diagonal tensor of the same order, dimension and diagonal entries as \( P \). Note that \( p_{ii,i} > 0 \) for \( i \in N \) when \( P \) is a strong \( M \)-tensor. Define \( J_P = D^{-1}N \). Obviously, \( J_P \) is nonnegative. The following result characterizes \( J_P \) in terms of the spectral radius.

**Lemma 5** Suppose that \( P = (p_{i2...im}) \) is a strong \( M \)-tensor of order \( m \) and dimension \( n \). Then

\[ \rho(J_P) \geq 1 - \frac{\tau(P)}{\min_{1 \leq i \leq n} p_{ii,i}}. \]

Furthermore, if \( P \) is weakly irreducible, then

\[ \rho(J_P) \leq 1 - \frac{\tau(P)}{\max_{1 \leq i \leq n} p_{ii,i}}. \]
Proof. Let $\mathcal{P} = (p_{ij_1 \ldots i_m})$ be a strong $M$-tensor. Then there exists a positive vector $u = (u_i)$ such that

$$p_{i \ldots i_i}^{[m-1]} + \sum_{i_2 \ldots i_m = 0} p_{ii_2 \ldots i_m} u_{i_2} \cdots u_{i_m} = \tau(\mathcal{P}) u_{i_i}^{[m-1]},$$

that is,

$$\frac{\sum_{i_2 \ldots i_m = 0} p_{ii_2 \ldots i_m} u_{i_2} \cdots u_{i_m}}{p_{i \ldots i_i}^{[m-1]}} = \tau(\mathcal{P}) - 1.$$  \hspace{1cm} (4)

Since the tensor $J_{\mathcal{P}}$ is nonnegative, by Lemma 1 and (4), we have

$$\rho(J_{\mathcal{P}}) = \max_{x \in \mathbb{R}^n_+ \setminus \{0\}} \min_{1 \leq i \leq n} \frac{(J_{\mathcal{P}} x^{m-1})_i}{x_i^{[m-1]}} \geq \min_{1 \leq i \leq n} \frac{(J_{\mathcal{P}} u^{m-1})_i}{u_i^{[m-1]}} = \min_{1 \leq i \leq n} \left( 1 - \frac{\tau(\mathcal{P})}{p_{i \ldots i_i}} \right) = 1 - \frac{\tau(\mathcal{P})}{\min_{1 \leq i \leq n} p_{i \ldots i_i}}.$$  \hspace{1cm} (5)

Furthermore, $J_{\mathcal{P}}$ is weakly irreducible when $\mathcal{P}$ is weakly irreducible. From Lemma 1 and (4), it holds that

$$\rho(J_{\mathcal{P}}) = \min_{x \in \mathbb{R}^n_+ \setminus \{0\}} \max_{1 \leq i \leq n} \frac{(J_{\mathcal{P}} x^{m-1})_i}{x_i^{[m-1]}} \leq \max_{1 \leq i \leq n} \frac{(J_{\mathcal{P}} u^{m-1})_i}{u_i^{[m-1]}} = \max_{1 \leq i \leq n} \left( 1 - \frac{\tau(\mathcal{P})}{p_{i \ldots i_i}} \right) = 1 - \frac{\tau(\mathcal{P})}{\max_{1 \leq i \leq n} p_{i \ldots i_i}}.$$  \hspace{1cm} (6)

\hfill $\square$

The following example shows that the bound of Lemma 5 is tight.

Example 1. Let $\mathcal{P} = (p_{ijk})$ be a tensor of order 3 and dimension 3 with elements defined as follows:

$$p_{ijk} = \begin{cases} p_{111} = p_{222} = p_{333} = 3, \\ p_{ijk} = -\frac{1}{4}, & \text{otherwise}. \end{cases}$$

By computations, we get $\tau(\mathcal{P}) = 1$ and

$$\rho(J_{\mathcal{P}}) = 1 - \frac{\tau(\mathcal{P})}{\min_{1 \leq i \leq n} p_{i \ldots i_i}} = 1 - \frac{\tau(\mathcal{P})}{\max_{1 \leq i \leq n} p_{i \ldots i_i}} = 1 - \frac{2}{3} = \frac{1}{3}.$$ 

Based on the characterizations of $M$-tensors, we can immediately obtain these bounds from the following result.
Theorem 1 If $P$ and $Q$ are two strong $M$-tensors of order $m$ and dimension $n$, then

$$\tau(P \ast Q) \geq \left(1 - \rho(J_P)\rho(J_Q)\right) \min_{1 \leq i \leq n} (p_{i \ast q_{i \ast}})$$ \tag{7}$$

Proof Let us distinguish two cases.

Case 1. $P$ and $Q$ are both weakly irreducible. It follows from Lemma 4 that $P \ast Q$ is a strong $M$-tensor. Since $J_P$ and $J_Q$ are weakly irreducible nonnegative tensors, from Lemma 1, there exist two positive vectors $u, v$ such that

$$\rho(J_P)u_i^{[m-1]} = J_P u_i^{[m-1]}, \quad \rho(J_Q)v_i^{[m-1]} = J_Q v_i^{[m-1]}$$

equivalently,

$$\frac{\sum_{\delta_{i2 \cdots im} = 0} |p_{i2 \cdots im}| u_{i2} \cdots u_{im}}{p_{i \ast i \ast} u_i^{[m-1]}} = \rho(J_P), \quad \frac{\sum_{\delta_{i2 \cdots im} = 0} |q_{i2 \cdots im}| v_{i2} \cdots v_{im}}{q_{i \ast i \ast} v_i^{[m-1]}} = \rho(J_Q)$$ \tag{8}$$

Let $z = (z_i)$, where $z_i = u_i v_i \in \mathbb{R}_{++}$ for $i \in N$. Setting $l = l \ast Q$, for $i \in N$, we obtain

$$(lz^{m-1})_i = p_{i \ast l \ast q_{i \ast i \ast} h_i^{[m-1]} v_i^{[m-1]} - \sum_{\delta_{i2 \cdots im} = 0} |p_{i2 \cdots im}| u_{i2} \cdots u_{im} |q_{i2 \cdots im}| v_{i2} \cdots v_{im}$$

$$\geq p_{i \ast l \ast q_{i \ast i \ast} h_i^{[m-1]} v_i^{[m-1]} - \sum_{\delta_{i2 \cdots im} = 0} |p_{i2 \cdots im}| u_{i2} \cdots u_{im} \sum_{\delta_{i2 \cdots im} = 0} |q_{i2 \cdots im}| v_{i2} \cdots v_{im} \quad \sum_{\delta_{i2 \cdots im} = 0} |q_{i2 \cdots im}| v_{i2} \cdots v_{im}$$

$$= p_{i \ast l \ast q_{i \ast i \ast} h_i^{[m-1]} v_i^{[m-1]} - \left(1 - \frac{\sum_{\delta_{i2 \cdots im} = 0} |p_{i2 \cdots im}| u_{i2} \cdots u_{im} \sum_{\delta_{i2 \cdots im} = 0} |q_{i2 \cdots im}| v_{i2} \cdots v_{im}}{q_{i \ast i \ast} v_i^{[m-1]}}\right)$$

$$= p_{i \ast l \ast q_{i \ast i \ast} h_i^{[m-1]} v_i^{[m-1]} (1 - \rho(J_P)\rho(J_Q)) = p_{i \ast l \ast q_{i \ast i \ast} (1 - \rho(J_P)\rho(J_Q)) z_i^{[m-1]}$$ \tag{9}$$

It follows from Lemma 3 and (9) that

$$\tau(P \ast Q) \geq \left(1 - \rho(J_P)\rho(J_Q)\right) \min_{1 \leq i \leq n} (p_{i \ast q_{i \ast}}).$$

Case 2. Either $P$ or $Q$ is weakly reducible. Let $S$ be a tensor of order $m$ and dimension $n$ with

$$s_{i2 \cdots im} = \begin{cases} 1, & \text{if } i_2 = i_3 = \cdots = i_m \neq i, \\ 0, & \text{otherwise.} \end{cases}$$

Then both $P - \epsilon S$ and $Q - \epsilon S$ are weakly irreducible tensors for any $\epsilon > 0$. Now, we claim that $P - \epsilon S$ and $Q - \epsilon S$ are both strong $M$-tensors when $\epsilon > 0$ is sufficiently small. Since $P$ and $Q$ are strong $M$-tensors, there exist positive diagonal matrices $C, D$ such that

$$A = P \ast C^{(m-1)} \cdots C, \quad B = Q \ast D^{(m-1)} \cdots D$$
with

\[
a_{1...im} = \sum_{i=1}^{m} a_{i}^{(m-1)} c_i^{1} \cdots c_i^{m}, \quad b_{1...im} = \sum_{i=1}^{m} b_{i}^{(m-1)} d_i^{1} \cdots d_i^{m}.
\]

In particular,

\[
a_{1...i} = p_{1...i}, \quad b_{1...i} = q_{1...i}.
\]

By Lemma 2, one has

\[
|p_{1...i}| = |a_{1...i}| > \sum_{i=2}^{m} |a_{i}^{(m-1)} c_i^{2} \cdots c_i^{m}|
\]

\[
|q_{1...i}| = |b_{1...i}| > \sum_{i=2}^{m} |b_{i}^{(m-1)} d_i^{2} \cdots d_i^{m}|.
\]

Set

\[
L = \max_{i, j \in N} \left\{ \frac{c_i^{[m-1]} d_j^{[m-1]}}{c_j^{[m-1]} d_i^{[m-1]}} \right\}
\]

and

\[
\epsilon_0 = \min_{i, j \in N} \left\{ \frac{|p_{1...i}| - \sum_{i=2}^{m} |a_{i}^{(m-1)} c_i^{2} \cdots c_i^{m}|}{(n-1)L}, \frac{|q_{1...i}| - \sum_{i=2}^{m} |b_{i}^{(m-1)} d_i^{2} \cdots d_i^{m}|}{(n-1)L} \right\}.
\]

Then for any \(0 < \epsilon < \epsilon_0\), it holds that \(P - \epsilon S\) and \(Q - \epsilon S\) are strong \(M\)-tensors. Substituting \(P - \epsilon S\) and \(Q - \epsilon S\) for \(P\) and \(Q\) and letting \(\epsilon \to 0\), we obtain the desired results by the continuity of \(\tau(P - \epsilon S)\) and \(\tau(Q - \epsilon S)\).

Next, we give a lemma about the ratio of the smallest and largest values of a Perron vector for an irreducible nonnegative tensor.

**Lemma 6** (Lemma 3.2 of [35]) Let \(B\) be a nonnegative irreducible tensor of order \(m \geq 3\) and dimension \(n\) with a Perron vector \(y\). Then we have

\[
\kappa(B) \leq \frac{y_{\min}}{y_{\max}},
\]

where

\[
\kappa(B) = \max_{2 \leq k, k' \leq m} \min_{1 \leq i_1, i_1' \leq n} \sum_{i_2, \ldots, i_m} h_{1i_1-1}^{i_2} \cdots h_{i_m-1}^{i_m}.
\]

Based on the above lemma, we propose the following theorem, which provides a sharp bound under the condition of irreducibility.
Theorem 2 Suppose that $\mathcal{P}$ and $\mathcal{Q}$ are two irreducible strong $M$-tensors of order $m$ and dimension $n$, and $\rho(\mathcal{P})$ and $\rho(\mathcal{Q})$ are their spectral radii with eigenvalue vectors $u$ and $v$, respectively. Then,
\[
\tau(\mathcal{P} \star \mathcal{Q}) \geq \min_{1 \leq j < k \leq p} \left[ 1 - \rho(\mathcal{P}) \rho(\mathcal{Q}) + \frac{\alpha \beta |p_{ij} - q_{ij}|^2}{p_{i...j}} + \frac{\alpha \beta |q_{ij} - r_{ij}|^2}{q_{i...j}} \right],
\]
where $\alpha = \kappa(\mathcal{P})^{\frac{m-1}{m}} \leq \left[ \frac{\kappa_{\max}(\mathcal{P})^{m}}{\kappa(\mathcal{P})^{m-1}} \right]$, $\beta = \kappa(\mathcal{Q})^{\frac{m-1}{m}} \leq \left[ \frac{\kappa_{\max}(\mathcal{Q})^{m}}{\kappa(\mathcal{Q})^{m-1}} \right]$, $r_{ij}(\mathcal{P}) = \sum_{\begin{bmatrix} \delta_{i2}\cdots\delta_{im} \end{bmatrix} = 0} |p_{i2}\cdots im|^{m-1}$, and $r_{ij}(\mathcal{Q}) = \sum_{\begin{bmatrix} \delta_{i2}\cdots\delta_{im} \end{bmatrix} = 0} |q_{i2}\cdots im|^{m-1}$.

Proof It follows from Lemma 4 that $\mathcal{P} \star \mathcal{Q}$ is a strong $M$-tensor. Since $\mathcal{P}$ and $\mathcal{Q}$ are strongly irreducible $M$-tensors, $\mathcal{P}$ and $\mathcal{Q}$ are irreducible nonnegative tensors. By the assumption that $\rho(\mathcal{P})$ and $\rho(\mathcal{Q})$ are the spectral radii with eigenvalue vectors $u$ and $v$, we deduce that $u$ and $v$ are positive vectors such that
\[
\rho(\mathcal{P})u_i^{(m-1)} = \mathcal{P}u_i^{(m-1)}, \quad \rho(\mathcal{Q})v_i^{(m-1)} = \mathcal{Q}v_i^{(m-1)},
\]
equivalently,
\[
\sum_{\begin{bmatrix} \delta_{i2}\cdots\delta_{im} \end{bmatrix} = 0} |p_{i2}\cdots im| |u_{i2}\cdots im| = \rho(\mathcal{P}) - \frac{|p_{ij}|u_i^{(m-1)}}{p_{i...j} u_i^{(m-1)}}, \quad (10)
\]
\[
\sum_{\begin{bmatrix} \delta_{i2}\cdots\delta_{im} \end{bmatrix} = 0} |q_{i2}\cdots im| |v_{i2}\cdots im| = \rho(\mathcal{Q}) - \frac{|q_{ij}|v_i^{(m-1)}}{q_{i...j} v_i^{(m-1)}}, \quad (11)
\]
Let $z = (z_i)$, where $z_i = u_i v_i \in \mathbb{R}_{++}$, for $i \in N$. Setting $\mathcal{U} = \mathcal{P} \star \mathcal{Q}$, for $i \in N$, by (10) and (11), we have
\[
(\mathcal{U}z_i^{(m-1)}) = p_{i...j} q_{i...j} z_i^{(m-1)} - |p_{ij} q_{ij}| z_i^{(m-1)} u_j^{(m-1)}
\]
- \sum_{\begin{bmatrix} \delta_{i2}\cdots\delta_{im} \end{bmatrix} = 0} \frac{|p_{i2}\cdots im|}{\begin{bmatrix} \delta_{i2}\cdots\delta_{im} \end{bmatrix} = 0} |u_{i2}\cdots im| z_{i2}\cdots im
\geq p_{i...j} q_{i...j} z_i^{(m-1)} - |p_{ij} q_{ij}| z_i^{(m-1)} u_j^{(m-1)}
- \left( \sum_{\begin{bmatrix} \delta_{i2}\cdots\delta_{im} \end{bmatrix} = 0} |p_{i2}\cdots im| |u_{i2}\cdots im| \right) \left( \sum_{\begin{bmatrix} \delta_{i2}\cdots\delta_{im} \end{bmatrix} = 0} |q_{i2}\cdots im| |v_{i2}\cdots im| \right)
= p_{i...j} q_{i...j} z_i^{(m-1)} \left[ 1 - \frac{|p_{ij} q_{ij}| u_j^{(m-1)} v_j^{(m-1)}}{p_{i...j} u_i^{(m-1)} v_j^{(m-1)}} \right]
- \left( \rho(\mathcal{P}) - \frac{|p_{ij}| u_j^{(m-1)}}{p_{i...j} u_i^{(m-1)}} \right) \left( \rho(\mathcal{Q}) - \frac{|q_{ij}| v_j^{(m-1)}}{q_{i...j} v_i^{(m-1)}} \right)
= p_{i...j} q_{i...j} z_i^{(m-1)} \left[ 1 - \rho(\mathcal{P})\rho(\mathcal{Q}) + \frac{|p_{ij}| u_j^{(m-1)}}{p_{i...j} u_i^{(m-1)}} \left( \rho(\mathcal{Q}) - \frac{|q_{ij}| v_j^{(m-1)}}{q_{i...j} v_i^{(m-1)}} \right) \right]
+ |q_{ij}| v_j^{(m-1)} \left( \rho(\mathcal{P}) - \frac{|p_{ij}| u_j^{(m-1)}}{p_{i...j} u_i^{(m-1)}} \right), \quad (12)
From (10) and Lemma 6, we deduce

$$\rho(J_P) = \frac{|p_{ij}| |u_{m-1}^j|}{p_{i..} |u_{m-1}^j|} \geq \sum_{\delta_{j2-Z_{m-1}}=0}^{\delta_{j2-Z_{m-1}}=0} \frac{|p_{ij}| |u_{m-1}^j|}{p_{i..} |u_{m-1}^j|} = \alpha r_j(J_P). \quad (13)$$

Similarly,

$$\rho(J_Q) = \frac{|q_{ij}| |v_{m-1}^j|}{q_{i..} |v_{m-1}^j|} \geq \sum_{\delta_{j2-Z_{m-1}}=0}^{\delta_{j2-Z_{m-1}}=0} \frac{|q_{ij}| |v_{m-1}^j|}{q_{i..} |v_{m-1}^j|} = \beta r_j(J_Q). \quad (14)$$

Combining (12) with (13) and (14), we have

$$(U z_{m-1}) \geq \left[\left(1 - \rho(J_P) \rho(J_Q) + \frac{\alpha \beta}{p_{i..}} r'_j(J_P) + \frac{\alpha \beta}{q_{i..}} r'_j(J_Q) \right) p_{i..} q_{i..} \right] z_{m-1}. \quad (15)$$

It follows from (15) and Lemma 3 that

$$\tau(P \star Q) \geq \min_{1 \leq i \leq n} \left(1 - \rho(J_P) \rho(J_Q) + \frac{\alpha \beta}{p_{i..}} r'_j(J_P) + \frac{\alpha \beta}{q_{i..}} r'_j(J_Q) \right) \rho(J_P) \rho(J_Q). \quad \square$$

Remark 1 The bound in Theorem 2 is sharper than the result of Theorem 1, since

$$\frac{\alpha \beta}{p_{i..}} r'_j(J_P) + \frac{\alpha \beta}{q_{i..}} r'_j(J_Q) \geq 0.$$

The following example exhibits the efficiency of Theorems 1 and 2.

Example 2 Let $P = (p_{ijk})$, $Q = (q_{ijk})$ be two tensors of order 3 and dimension 3 with elements defined as follows:

$$P = [P(1,:,:), P(2,:,:), P(3,:,:)], \quad Q = [Q(1,:,:), Q(2,:,:), Q(3,:,:)],$$

where

$$P(1,:,:) = \begin{pmatrix} 3 & 0 & -\frac{1}{3} \\ 0 & -1 & 0 \\ -\frac{1}{3} & 0 & -\frac{1}{2} \end{pmatrix}, \quad P(2,:,:) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}, \quad P(3,:,:) = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 5 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{5}{2} \end{pmatrix},$$

$$Q(1,:,:) = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}, \quad Q(2,:,:) = \begin{pmatrix} 0 & 0 & -\frac{1}{3} \\ 0 & -\frac{1}{2} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & 2 \end{pmatrix}. \quad Q(3,:,:) =$$

It is clear that $\min_{1 \leq i \leq n} (p_{i..} q_{i..}) = 9$. By computations, we get

$$\rho(J_P) = 0.6842, \quad \rho(J_Q) = 0.7328, \quad \alpha = \kappa(J_P) = 0.3, \quad \beta = \kappa(J_Q) = 0.3.$$
Without loss of generality, assume that

According to Theorem 2, we obtain

\[
\tau(P \star Q) \geq \min_{1 \leq i \leq n} \left(1 - \rho(J_P) \rho(J_Q)\right) \min_{1 \leq i \leq n} (p_{i..q_{i..}}) = 4.4876.
\]

According to Theorem 2, we obtain

\[
\tau(P \star Q) \geq \min_{1 \leq i \leq n} \left(1 - \rho(J_P) \rho(J_Q) + \frac{\alpha \beta \|p_{i..q_{i..}}\|_1}{\rho(J_P) \rho(J_Q)} + \frac{\alpha \beta \|q_{i..q_{i..}}\|_1}{\rho(J_P) \rho(J_Q)}\right) \min_{1 \leq i \leq n} (p_{i..q_{i..}})
\]

= 4.9074.

By making use of the information of the absolute maximum in the off-diagonal elements, we are at the position to establish the following theorem.

**Theorem 3** Suppose that \( P \) and \( Q \) are two strong M-tensors of order \( m \) and dimension \( n \) and assume that \( \rho(J_P) \) and \( \rho(J_Q) \) are the corresponding spectral radii. Then

\[
\tau(P \star Q) \geq \min_{i \in \mathbb{N}} \left\{ p_{i..q_{i..}} - (\alpha_i \beta_i p_{i..q_{i..}} \rho(J_P) \rho(J_Q))^{\frac{1}{2}} \right\},
\]

where \( \alpha_i = \max_{i_2 \neq i_0} |p_{i_2..i_0}| \) and \( \beta_i = \max_{i_2 \neq i_0} |q_{i_2..i_0}|. \)

**Proof** The proof is broken into two cases.

Case 1. \( P \) and \( Q \) are both weakly irreducible. It follows from Lemma 4 that \( P \star Q \) is a strong M-tensor. Since \( J_P \) and \( J_Q \) are weakly irreducible nonnegative tensors, by Lemma 1, there exist two positive eigenvectors \( u = (u^2) > 0, v = (v^2) > 0 \) such that

\[
\sum_{i_2 \neq i_0} p_{i_2..i_0} u^2_{i_2} \cdots u^2_{i_m} = \rho(J_P),
\]

\[
\sum_{i_2 \neq i_0} q_{i_2..i_0} v^2_{i_2} \cdots v^2_{i_m} = \rho(J_Q).
\]

Without loss of generality, assume that \( u_i, v_i \in \mathbb{R}_{+} \). Let \( z = (z_i) \) with \( z_i = u_i v_i \in \mathbb{R}_{+} \) and \( U = P \star Q \). By Cauchy–Schwartz inequality, for \( 1 \leq i \leq n \), we have

\[
(Uz^{m-1})_i = p_{i..q_{i..}} z_i^{m-1} = \sum_{i_2 \neq i_0} |p_{i_2..i_0}| |q_{i_2..i_0}| |u_{i_2} v_{i_2} | \cdots |u_{i_m} v_{i_m} |
\]

\[
\geq p_{i..q_{i..}} z_i^{m-1} = \sum_{i_2 \neq i_0} |p_{i_2..i_0}| |u_{i_2} \cdots u_{i_m} | \sum_{i_2 \neq i_0} |q_{i_2..i_0}| |v_{i_2} \cdots v_{i_m} |
\]

\[
\geq p_{i..q_{i..}} z_i^{m-1} = \left( \sum_{i_2 \neq i_0} |p_{i_2..i_0}|^{2} |u_{i_2} \cdots u_{i_m} |^{2} \right)^{\frac{1}{2}}
\]

\[
\times \left( \sum_{i_2 \neq i_0} |q_{i_2..i_0}|^{2} |v_{i_2} \cdots v_{i_m} |^{2} \right)^{\frac{1}{2}}.
\]
It follows from the definitions of $\alpha_i$, $\beta_i$ and (18) that
\[
(Uz^{m-1})_i \geq p_{i...q_{i...}}^{(m-1)} - (\alpha_i p_{i...q_{i...}}^2)^{\frac{1}{2}} (\beta_i q_{i...}^2)^{\frac{1}{2}} = \left[p_{i...q_{i...}} - (\alpha_i \beta_i p_{i...q_{i...}} \rho(J_P) \rho(J_Q))^\frac{1}{2}\right]^{m-1}.
\]

Furthermore, using Lemma 3 and (19), one has
\[
\tau(P \ast Q) \geq \min_{i \in N} \left\{p_{i...q_{i...}} - (\alpha_i \beta_i p_{i...q_{i...}} \rho(J_P) \rho(J_Q))^\frac{1}{2}\right\}.
\]

Case 2. Either $P$ or $Q$ is weakly reducible. Similar to the proof of Theorem 1, we obtain the desired result.

In what follows, we give inclusion relations between Theorems 1 and 3.

**Corollary 1** Let $P$ and $Q$ be strong M-tensors of order m and dimension n. If $p_{i...q_{i...}} \rho(J_P) \rho(J_Q) \leq \alpha_i \beta_i$ for $i \in N$, then
\[
\min_{i \in N} (1 - \rho(J_P) \rho(J_Q)) p_{i...q_{i...}} \geq \min_{i \in N} \left\{p_{i...q_{i...}} - (\alpha_i \beta_i p_{i...q_{i...}} \rho(J_P) \rho(J_Q))^\frac{1}{2}\right\};
\]

if $p_{i...q_{i...}} \rho(J_P) \rho(J_Q) \geq \alpha_i \beta_i$ for $i \in N$, then
\[
\min_{i \in N} (1 - \rho(J_P) \rho(J_Q)) p_{i...q_{i...}} \leq \min_{i \in N} \left\{p_{i...q_{i...}} - (\alpha_i \beta_i p_{i...q_{i...}} \rho(J_P) \rho(J_Q))^\frac{1}{2}\right\}.
\]

**Proof** Observe that
\[
(1 - \rho(J_P) \rho(J_Q)) p_{i...q_{i...}} = p_{i...q_{i...}} - p_{i...q_{i...}} \rho(J_P) \rho(J_Q).
\]
When $p_{i...q_{i...}} \rho(J_P) \rho(J_Q) \leq \alpha_i \beta_i$, from (22), we see
\[
(1 - \rho(J_P) \rho(J_Q)) p_{i...q_{i...}} = p_{i...q_{i...}} - \left(p_{i...q_{i...}} \rho(J_P) \rho(J_Q)\right)^\frac{1}{2} (p_{i...q_{i...}} \rho(J_P) \rho(J_Q))^{\frac{1}{2}}
\]
\[
\geq p_{i...q_{i...}} - (\alpha \beta)^\frac{1}{2} (p_{i...q_{i...}} \rho(J_P) \rho(J_Q))^\frac{1}{2}
\]
\[
= p_{i...q_{i...}} - (\alpha \beta p_{i...q_{i...}} \rho(J_P) \rho(J_Q))^\frac{1}{2},
\]
which implies
\[
\min_{i \in N} \left\{p_{i...q_{i...}} (1 - \rho(J_P) \rho(J_Q))\right\} \geq \min_{i \in N} \left\{p_{i...q_{i...}} - (\alpha \beta) p_{i...q_{i...}} \rho(J_P) \rho(J_Q))^\frac{1}{2}\right\}.
\]

So, (20) holds.
If $p_{i...q_{i...}} \rho(J_P) \rho(J_Q) \geq \alpha_i \beta_i$ for $i \in N$, similar to the proof of (20), we obtain (21).

**Remark 2** If $p_{i...q_{i...}} \rho(J_P) \rho(J_Q) \leq \alpha_i \beta_i$ for all $1 \leq i \leq n$, from (20), we verify that the bound of Theorem 1 is sharper than that of Theorem 3. When $p_{i...q_{i...}} \rho(J_P) \rho(J_Q) \geq \alpha_i \beta_i$ for $i \in N$, from (21), we deduce that the bound of Theorem 3 is tighter than that of Theorem 1.
The following examples give numerical comparisons between Theorems 1 and 3.

**Example 3** Let $\mathcal{P} = (p_{ijk})$, $\mathcal{Q} = (q_{ijk})$ be defined in Example 2.

It is clear that $\min_{1 \leq i \leq n} (p_{i \ldots q_{i \ldots}}) = 9$. By computations, we get

\[ \rho(J_P) = 0.6842, \quad \rho(J_Q) = 0.7328, \quad \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1, \quad \alpha_3 = \beta_3 = 1/2. \]

Obviously, $p_{i \ldots q_{i \ldots}} \rho(J_P) \rho(J_Q) \geq \alpha_i \beta_i$ for $i = 1, 2, 3$. From Theorem 1, we have

\[ \tau(\mathcal{P} \star \mathcal{Q}) \geq (1 - \rho(J_P) \rho(J_Q)) \min_{1 \leq i \leq n} (p_{i \ldots q_{i \ldots}}) = 4.4876. \]

From Theorem 3, we have

\[ \tau(\mathcal{P} \star \mathcal{Q}) \geq \min_{i \in \mathbb{N}} \left\{ p_{i \ldots q_{i \ldots}} - (\alpha_i \beta_i p_{i \ldots q_{i \ldots}} \rho(J_P) \rho(J_Q))^{\frac{1}{2}} \right\} = 6.8758, \]

So, the bound of Theorem 3 is tighter than that of Theorem 1.

**Example 4** Let $\mathcal{P} = (p_{ijk})$, $\mathcal{Q} = (q_{ijk})$ be two tensors of order 3 and dimension 3 with elements defined as follows:

\[
\mathcal{P} = [P(1,:,:), P(2,:,:), P(3,:,:)] , \quad \mathcal{Q} = [Q(1,:,:), Q(2,:,:), Q(3,:,:)],
\]

where

\[
P(1,:,:) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & -\frac{11}{4} \\ 0 & 0 & 0 \end{pmatrix}, \quad P(2,:,:) = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -\frac{1}{4} \end{pmatrix},
\]

\[
P(3,:,:) = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & 5 \end{pmatrix}, \quad Q(1,:,:) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -2 \end{pmatrix},
\]

\[
P(2,:,:) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad Q(3,:,:) = \begin{pmatrix} -\frac{1}{4} & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.
\]

By computations, we get

\[ \rho(J_P) = 0.7036, \quad \rho(J_Q) = 0.6458, \quad \alpha_1 = \frac{11}{4}, \]

\[ \beta_1 = 2, \quad \alpha_2 = \beta_2 = 2, \quad \alpha_3 = 3, \quad \beta_3 = 2. \]

From Theorem 1, one has

\[ \tau(\mathcal{P} \star \mathcal{Q}) \geq (1 - \rho(J_P) \rho(J_Q)) \min_{1 \leq i \leq n} (p_{i \ldots q_{i \ldots}}) = (1 - \rho(J_P) \rho(J_Q)) p_{1 \ldots q_{1 \ldots}} = 4.9104. \]
According to Theorem 3, we obtain

$$\tau(P \ast Q) \geq \min_{i \in \mathbb{N}} \left\{ p_{i} \cdots q_{i + 1} \cdots \left( \alpha_{i} \beta_{i} p_{i} \cdots q_{i + 1} \cdots \rho(J_{P}) \rho(J_{Q}) \right)^{1/2} \right\}$$

$$= p_{1} \cdots q_{1 + 1} \cdots \left( \alpha_{1} \beta_{1} p_{1} \cdots q_{1 + 1} \cdots \rho(J_{P}) \rho(J_{Q}) \right)^{1/2} = 4.2674.$$ 

Thus, the bound of Theorem 1 is tighter than that of Theorem 3.

4 Conclusions

In this paper, we generalized important inequalities on the minimum eigenvalue for the Fan product from matrices to tensors. Based on characterizations of $M$-tensors, we proposed lower bound estimates on the minimum eigenvalue for the Fan product of two $M$-tensors. Finally, we gave some sufficient conditions to establish when particular inclusion relations hold.

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The authors declare that they have no competing interests.

Authors’ contributions

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References

1. Qi, L.Q.: Eigenvalues of a real supersymmetric tensor. J. Symb. Comput. 40(6), 1302–1324 (2005)
2. Qi, L.Q., Luo, Z.Y.: Tensor Analysis: Spectral Theory and Special Tensors. SIAM, Philadelphia (2017)
3. Lim, L.H.: Singular values and eigenvalues of tensors: a variational approach. In: Proceedings of the IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP'05), pp. 129–132 (2005)
4. Chang, K.C., Pearson, K., Zhang, T.: Perron–Frobenius theorem for nonnegative tensors. Commun. Math. Sci. 6, 507–520 (2008)
5. Chen, H.B., Wang, Y.J.: On computing the minimal H-eigenvalue of sign-structured tensors. Front. Math. China 12, 1289–1302 (2017)
6. Chen, H.B., Qi, L.Q., Song, Y.S.: Column sufficient tensors and tensor complementarity problems. Front. Math. China 13, 255–276 (2018)
7. Chen, H.B., Chen, Y.N., Li, G.Y., Qi, L.Q.: A semi-definite program approach for computing the maximum eigenvalue of a class of structured tensors and its applications in hypergraphs and copositivity test. Numer. Linear Algebra Appl. (2018). https://doi.org/10.1002/nla.2125
8. Ding, W.Y., Qi, L.Q., Wei, Y.M.: M-Tensors and nonsingular M-tensors. Linear Algebra Appl. 439, 3264–3278 (2013)
9. Friedland, S., Gaubert, S., Han, L.X.: Perron–Frobenius theorem for nonnegative multilinear forms and extensions. Linear Algebra Appl. 438, 738–749 (2013)
10. Guo, G.B., Shao, W., Lin, L., Zhu, X.H.: Parallel tempering for dynamic generalized linear models. Commun. Stat. Theory Methods 45, 6299–6310 (2016)
11. Hu, S.R., Huang, Z.H., Qi, L.Q.: Strictly nonnegative tensors and nonnegative tensor partition. Sci. China Math. 57, 181–195 (2014)
12. Kannan, M., Monderer, N., Berman, A.: On weakly irreducible nonnegative tensors and interval hull of some classes of tensors. Linear Multilinear Algebra 64, 667–679 (2016)
13. Shao, W., Guo, G.B., Zhao, G.Q., Meng, F.Y.: Simulated annealing for the bounds of Kendall’s $\tau$ and Spearman’s $\rho$. J. Stat. Comput. Simul. 84, 2688–2699 (2014)
14. Sun, L.Z., Zheng, B.D., Zhou, J., Yan, H.: Some inequalities for the Hadamard product of tensors. Linear Multilinear Algebra 66, 1199–1214 (2018)
15. Wang, Y.J., Qi, L.Q., Zhang, X.Z.: A practical method for computing the largest $M$-eigenvalue of a fourth-order partially symmetric tensor. Numer. Linear Algebra Appl. 16, 589–601 (2009)
16. Wang, Y.J., Zhou, G.L., Caccetta, L.: Convergence analysis of a block improvement method for polynomial optimization over unit spheres. Numer. Linear Algebra Appl. 22, 1059–1076 (2015)
17. Wang, Y.J., Zhang, K.L., Sun, H.C.: Criteria for strong $H$-tensors. Front. Math. China 11, 577–592 (2016)
18. Wang, G., Zhou, G.L., Caccetta, L.: Z-Eigenvalue inclusion theorems for tensors. Discrete Contin. Dyn. Syst., Ser. B 22, 187–198 (2017)
19. Wang, Y.X., Wang, G.: Two S-type Z-eigenvalue inclusion sets for tensors. J. Inequal. Appl. 2017, Article ID 152 (2017)
20. Wang, G., Zhou, G.L., Caccetta, L.: Sharp Brauer-type eigenvalue inclusion theorems for tensors. Pac. J. Optim. 14, 227–244 (2018)
21. Wang, Y.N., Wang, G.: Solution structures of tensor complementarity problem. Front. Math. China 13(4), 935–945 (2018)
22. Yang, Y.N., Yang, Q.Z.: Further results for Perron–Frobenius theorem for nonnegative tensors I. SIAM J. Matrix Anal. Appl. 31, 2517–2530 (2010)
23. Zhang, L.P., Qi, L.Q., Zhou, G.L.: M-Tensors and some applications. SIAM J. Matrix Anal. Appl. 35, 437–452 (2014)
24. Zhang, K.L., Wang, Y.J.: An $H$-tensor based iterative scheme for identifying the positive definiteness of multivariate homogeneous forms. J. Comput. Appl. Math. 305, 1–10 (2016)
25. Zhao, J., Sang, C.L.: Two new lower bounds for the minimum eigenvalue of $M$-tensors. J. Inequal. Appl. 2016, Article ID 268 (2016)
26. Zhou, G.L., Wang, G., Qi, L.Q., Alqahtani, A.: A fast algorithm for the spectral radii of weakly reducible nonnegative tensors. Numer. Linear Algebra Appl. 2018, https://doi.org/10.1002/nla.2134
27. Ni, Q., Qi, L.Q., Wang, F.: An eigenvalue method for testing the positive definiteness of a multivariate form. IEEE Trans. Autom. Control 53, 1096–1107 (2008)
28. Gao, L.J., Wang, D.D., Wang, G.: Further results on exponential stability for impulsive switched nonlinear time-delay systems with delayed impulse effects. Appl. Math. Comput. 268, 186–200 (2015)
29. Gao, L.J., Wang, D.D.: Input-to-state stability and integral input-to-state stability for impulsive switched systems with time-delay under asynchronous switching. Nonlinear Anal. Hybrid Syst. 20, 55–71 (2016)
30. Cai, J.Q., Li, H., Sun, Q.: Longest cycles in 4-connected graphs. Discrete Math. 340, 2955–2966 (2017)
31. Zhou, J., Sun, L.Z., Wei, Y.P., Bu, C.J.: Some characterizations of $M$-tensors via digraphs. Linear Algebra Appl. 495, 190–198 (2016)
32. Horn, R., Johnson, C.: Topics in Matrix Analysis. Cambridge University Press, Cambridge (1985)
33. Fang, F.: Bounds on eigenvalues of Hadamard product and the Fan product of matrices. Linear Algebra Appl. 425, 1–15 (2007)
34. Huang, R.: Some inequalities for the Hadamard product and the Fan product of matrices. Linear Algebra Appl. 428, 1551–1559 (2008)
35. Li, Y.T., Li, Y.Y., Wang, R.W., Wang, Y.Q.: Some new bounds on eigenvalues of the Hadamard product and the Fan product of matrices. Linear Algebra Appl. 432, 536–545 (2010)
36. Chen, H.B., Wang, Y.J.: A family of higher-order convergent iterative methods for computing the Moore–Penrose inverse. Appl. Math. Comput. 218, 4012–4016 (2011)
37. Zhou, D.M., Chen, G.L., Wu, G.X., Zhang, X.Y.: On some new bounds for eigenvalues of the Hadamard product and the Fan product of matrices. Linear Algebra Appl. 438, 1415–1426 (2013)