Inflationary power spectra with quantum holonomy corrections

Jakub Mielczarek

Institute of Physics, Jagiellonian University, Reymonta 4, Cracow, 30-059 Poland
Department of Fundamental Research, National Centre for Nuclear Research, Hoża 69, Warsaw, 00-681 Poland

E-mail: jakub.mielczarek@uj.edu.pl

Received November 7, 2013
Revised March 11, 2014
Accepted March 13, 2014
Published March 25, 2014

Abstract. In this paper we study slow-roll inflation with holonomy corrections from loop quantum cosmology. It was previously shown that, in the Planck epoch, these corrections lead to such effects as singularity avoidance, metric signature change and a state of silence. Here, we consider holonomy corrections affecting the phase of cosmic inflation, which takes place away from the Planck epoch. Both tensor and scalar power spectra of primordial inflationary perturbations are computed up to the first order in slow-roll parameters and $V/\rho_c$, where $V$ is a potential of the scalar field and $\rho_c$ is a critical energy density (expected to be of the order of the Planck energy density). Possible normalizations of modes at short scales are discussed. In case the normalization is performed with use of the Wronskian condition applied to adiabatic vacuum, the tensor and scalar spectral indices are not quantum corrected in the leading order. However, by choosing an alternative method of normalization one can obtain quantum corrections in the leading order. Furthermore, we show that the holonomy-corrected equations of motion for tensor and scalar modes can be derived based on effective background metrics. This allows us to show that the classical Wronskian normalization condition is well defined for the cosmological perturbations with holonomy corrections.

Keywords: inflation, primordial gravitational waves (theory), quantum cosmology, quantum gravity phenomenology

ArXiv ePrint: 1311.1344
1 Introduction

Effects of the quantum nature of space at the Planck scale predicted by loop quantum gravity (LQG) [1] can be studied by introducing appropriate modifications at the level of the classical Hamiltonian. This so-called effective approach enables to relate some quantum gravitational phenomena with the realm of classical physics, which proved to be especially fruitful in the cosmological context, known as loop quantum cosmology (LQC) [2, 3].

As discussed in ref. [4], the effective approach in quantum gravity is conceptually similar to the effective approach in solid state physics. Namely, while calculations based on many-body Hamiltonian are extremely difficult to execute, there is a whole range of effective models enabling explanation of macroscopic phenomena in terms of atomic-scale physics. As an example, relevant for our further discussion, let us refer to the nature of refractive index $n$.

In LQG, there exists an analogue of the many-body Hamiltonian in solid state physics. In the effective model of frequency dependence of $n$, one considers a single atomic dipole interacting with electromagnetic plane wave. By virtue of homogeneity of a sample the formula for $n(\omega)$, characterizing macroscopic bulk, can be derived. The formula depends on known microscopic quantities as electron mass and elementary charge, but also contains some characteristic frequencies which cannot be derived from the model. The unknown values can be either fixed experimentally or derived from quantum mechanical computations.
it will be possible to study some macroscopic or mesoscopic gravitational configurations numerically. This is in analogy to computations performed within condensed matter physics or quantum chemistry. Meanwhile, the effective approach, competitive with the first-principle computations, can be utilized.

Construction of effective models is facilitated by certain assumptions regarding symmetries of space. Here, we focus on homogeneous and isotropic background geometry\(^1\) described by the flat Freidmann-Robertson-Walker (FRW) metric on which inhomogeneities are considered perturbatively. Such setup is sufficient to study generation of primordial perturbations during the phase of slow-roll inflation.

Incorporation of LQG effects into cosmological models is performed by taking into account two types of corrections: inverse volume corrections and holonomy corrections. Both corrections reflect discrete nature of space at the Planck scale, however in a different manner. While strength of inverse volume corrections depends on volume element, holonomy corrections are sensitive to energy density. In case of inverse volume corrections, equations of motion for perturbations were derived in ref. \([5]\). Based on this, corrections to the inflationary power spectra were derived in ref. \([6]\). The corrections were shown to be consistent with the 7-year WMAP data \([7, 8]\).

In this paper we focus on derivation of holonomy corrections to inflationary power spectra. As already mentioned, holonomy corrections are sensitive to energy density of matter. The characteristic energy scale, at which holonomy corrections are becoming important is

\[
\rho_c = \frac{3m_{\text{Pl}}^2}{8\pi\gamma^2 L^2},
\]

where the Planck mass \(m_{\text{Pl}} = 1.22 \times 10^{19}\) GeV, \(\gamma \sim \mathcal{O}(1)\) is the so-called Barbero-Immirzi parameter and \(L\) is a characteristic length scale of the quantum gravitational effects. One can expect that \(L\) is of the order of the Planck length. In particular, it is often assumed that the area of elementary loop in LQC is equal to the area gap \(\Delta\) in LQG, leading to \(L = \sqrt{\Delta} = \sqrt{2\sqrt{3\pi\gamma l_{\text{Pl}}}}\) \([3]\). In that case, the critical energy density is of the order of the Planck energy density: \(\rho_c \sim \rho_{\text{Pl}}\), where \(\rho_{\text{Pl}} \equiv m_{\text{Pl}}^4\). However, because of the ambiguity in defining \(L\), in what follows we keep \(\rho_c\) as a free parameter to be fixed observationally.

Holonomy corrections are modifying Friedmann equation into the following form \([9, 10]\)

\[
H^2 = \frac{8\pi}{3m_{\text{Pl}}^2} \rho \left(1 - \frac{\rho}{\rho_c}\right),
\]

where \(H\) is a Hubble factor and \(\rho\) is energy density of the matter content. Positivity of the left hand side of this equation implies that energy density of matter is bounded from above, \(\rho \leq \rho_c\). This leads to resolution of singularity problem of homogeneous cosmological modes. The big bang singularity is replaced by non-singular \textit{bounce}, which merges contracting and expanding phases \([9, 11]\).

Models of inflation are typically constructed with the use of scalar fields. In the simplest case it can be a single scalar field \(\varphi\), the so-called inflaton field \([12]\). The single scalar is sufficient to construct a reliable model of the inflationary phase. Energy density of this field expresses as

\[
\rho = \frac{\dot{\varphi}^2}{2} + V(\varphi),
\]

\(^1\)The same symmetries, but for crystal lattice, were applied in the mentioned effective model of refractive index.
where \(V(\varphi)\) is a potential term. Equation of motion governing evolution of \(\varphi\) is not a subject of holonomy corrections and takes the standard form:

\[
\dddot{\varphi} + 3H \dot{\varphi} + V_{,\varphi} = 0. 
\]

(1.4)

The holonomy corrections can be also introduced into equations governing evolution of cosmological perturbations. In particular, it was found that, while holonomy corrections are present, equation of motion for the Mukhanov variable \(v\) is [13]:

\[
\frac{d^2}{d\tau^2}v - \Omega \nabla^2 v - \frac{z''}{z_S}v = 0. 
\]

(1.5)

Here \(\tau\) is a conformal time defined as \(d\tau = dt/a\). Moreover, the holonomy correction function

\[
\Omega = 1 - 2\frac{\rho}{\rho_c}, 
\]

(1.6)

and

\[
z_S = \frac{a}{H}. 
\]

(1.7)

It is clear that, while \(\rho \ll \rho_c\) the classical expression with \(\Omega = 1\) is correctly recovered.

Based on the Mukhanov variable \(v\), perturbations of curvature \(R = z_S\) can be derived. The quantity \(R\) is a key object characterizing scalar perturbations, allowing for computation of the scalar power spectrum.

The equation (1.5) was originally derived by considering requirements of anomaly freedom for the scalar perturbations [13]. Later, it was shown that this equation can also be obtained from the lattice loop quantum cosmology [14, 15]. The equation (1.5) can be seen as a result of discretization of space for homogeneous cubic cells with the lattice spacing \(L\). Apart from a few exceptions, such as generation of a flat power spectra from a matter bounce [16], cosmological consequences of equation (1.5) have not been studied in details yet.

For tensor modes (gravitational waves), holonomy corrected version of the equation is [17]

\[
\frac{d^2}{d\tau^2}h_i + 2\left(\mathcal{H} - \frac{1}{2\Omega} \frac{d\Omega}{d\tau}\right) \frac{d}{d\tau}h_i - \Omega \nabla^2 h_i = 0, 
\]

(1.8)

where \(i = \otimes, \oplus\) corresponds to two polarizations of gravitational waves. This equation can be rewritten into the form

\[
\frac{d}{d\tau^2}u - \Omega \Delta u - \frac{z''}{z_T}u = 0, 
\]

(1.9)

where \(z_T = a/\sqrt{\Omega}\) and \(u = \frac{a^{\otimes}h^\otimes}{\sqrt{16\pi G \Omega}}\). The equation (1.8) differs from the equation of motion for tensor modes with holonomy corrections originally derived in ref. [18]. This is because the original derivation has not been based on an anomaly-free Hamiltonian. In contrary, equation (1.8) is based on the anomaly-free Hamiltonian, which was obtained thanks to advances achieved in the sector of scalar perturbations with holonomy corrections [13].

So far, the equation (1.8) was applied to study generation of tensor perturbation across the cosmic bounce [16, 19]. Nevertheless, there is a whole aggregation of previous analyses performed with the use of the original equation for tensor modes with holonomy corrections (See e.g. [20–23]). There were also earlier attempts to study scalar perturbations with holonomy corrections (See e.g. [24]). However, they were not consistent with the requirement
of anomaly freedom. There is also an alternative approach to incorporate loop quantum corrections to cosmological perturbations developed in refs. \[25–27].

It is worth mentioning at this point that because we consider model with the scalar matter, the vector modes are not activated and are identically equal zero \[28].

For both tensor and scalar perturbations the deformation factor $\Omega$ is placed in front of Laplace operator. Therefore, it can be considered as an effective speed of light squared. Namely, by neglecting the cosmological factor and assuming the plane wave solution $v \propto e^{i(k \cdot x - \omega \tau)}$, we find the following dispersion relation $\omega^2 = \Omega k^2$, based on which the phase velocity $v_{\text{ph}} = \frac{k}{\Omega} = \sqrt{\Omega}$. Therefore, the refractive index

$$n \equiv \frac{1}{v_{\text{ph}}} = \frac{1}{\sqrt{\Omega}}. \quad \text{\(1.10\)}$$

While $\rho \to \rho_c/2$ ($\Omega \to 0$) the refractive index becomes infinite, and speed of propagation tends to zero. As discussed in ref. \[29] this can be associated with the state of asymptotic silence. At the energy densities $\rho \in (\rho_c/2, \rho_c]$ the refractive index is purely imaginary. As discussed in ref. \[19] this not necessarily means that space is opaque for the propagation of waves. The waves are not only evanescent in this region, but can be amplified as well. Behavior observed from the numerical computations differs with the intuition gained from \textit{e.g} analysis of waves in plasma with frequencies lower than the plasma frequency. Furthermore, as discussed in refs. \[30, 31\] the region of negative $\Omega$ can be associated with the change of metric signature from Lorentzian to Euclidean one.

However, in our calculations of inflationary power spectra, we restrict ourselves to the regime where $\Omega > 0$. Therefore, the interesting behavior in vicinity of $\Omega = 0$ and at the negative values of $\Omega$ will not be relevant. We will come back to the issue of evolution of modes in the $\Omega \leq 0$ in our further studies.

## 2 Slow-roll inflation

During the slow-roll roll inflation Universe underwent an almost exponential expansion. The deviation from the exponential (de Sitter) growth of the scale factor is parametrized by the slow-roll parameters, which are much smaller than unity. The slow-roll inflation is characterized by gradual decreasing of $\phi$ in a potential $V(\phi)$. In this regime, energy density of the scalar field is dominated by its potential energy, therefore $\dot{\phi}^2 \ll V(\phi)$. Because of that, the modified Friedmann equation \textcolor{red}{(1.2)} can be approximated by

$$H^2 \simeq \frac{8\pi}{3m^2_{\text{Pl}}} V \left(1 - \frac{V}{\rho_c}\right). \quad \text{\(2.1\)}$$

Furthermore, flatness of the potential implies that the $\ddot{\phi}$ factor in equation \textcolor{red}{(1.4)} can be neglected, such that

$$3H \dot{\phi} + V_{,\phi} \simeq 0. \quad \text{\(2.2\)}$$

Using \textcolor{red}{(2.2)} to eliminate $\dot{\phi}$ from the condition $\phi^2 \ll V(\phi)$ and by using \textcolor{red}{(2.1)}, one can define \textcolor{red}{[24]}

$$\epsilon := \frac{m^2_{\text{Pl}}}{16\pi} \left(\frac{V_{,\phi}}{V}\right)^2 \left(1 - \frac{V}{\rho_c}\right)^{-1} = \frac{\dot{H}}{H^2 \left(1 - V/\rho_c\right)}. \quad \text{\(2.3\)}$$

such that $\epsilon \ll 1$ for the slow-roll inflation.
By differentiating the slow-roll equation
\[ \dot{\phi} \simeq -\frac{V_{\phi \phi}}{3H} + \frac{V_{\phi} H}{3H^2}, \]
we find
\[ \ddot{\phi} = -\frac{V_{\phi \phi \phi}}{3H} + \frac{V_{\phi \phi} \dot{H}}{3H^2} + \frac{V_{\phi \phi}}{3H^3} + \frac{V_{\phi} \dot{H}}{3H^2}. \]
(2.4)

Because \( |\ddot{\phi}| \ll |V_{\phi \phi}| \), the absolute value of
\[ \frac{\ddot{\phi}}{V_{\phi \phi}} \simeq \frac{1}{3} \eta - \frac{1}{3} \epsilon \left( 1 - \frac{V}{\rho_c} \right) \]
has to be much smaller than unity. Following ref. [24] let us introduce the second slow-roll parameter
\[ \eta := \frac{m_{\text{Pl}}^2}{8\pi} \left( \frac{V_{\phi \phi}}{V} \right) \frac{1}{(1 - V/\rho_c)}, \]
(2.6)
satisfying \( |\eta| \ll 1 \) for \( |\ddot{\phi}| \ll |V_{\phi \phi}| \). Based on (2.5) we can also define
\[ \delta := \eta - \epsilon \left( 1 - \frac{V}{\rho_c} \right) = -\frac{\ddot{\phi}}{H \dot{\phi}}, \]
(2.7)
satisfying \( \delta \ll 1 \).

While studying cosmological perturbations it is convenient to work with the conformal time \( \tau \equiv \int \frac{dt}{a} \). Here, it is defined such that \( \tau \in (-\infty, 0) \). Based on the definition of conformal time and integrating by parts, we find
\[ \tau = \int \frac{dt}{a} = \int \frac{da}{a^2 H} = -\frac{1}{H a} \int \frac{\dot{H}}{a^2 H} dt = -\frac{1}{H a} + \tau \epsilon \left( 1 - \frac{V}{\rho_c} \right), \]
(2.8)
where in the last equality we applied (2.3). This enables us to write expression for the time dependence of the scale factor
\[ a = -\frac{1}{H \tau} \frac{1}{\left[ 1 - \epsilon \left( 1 - \frac{V}{\rho_c} \right) \right]}. \]
(2.9)

In the slow-roll regime the \( \Omega \) function, defined in eq. (1.6), is approximated by
\[ \Omega \simeq 1 - 2\delta_H, \]
(2.10)
where for the later convenience we introduced parameter
\[ \delta_H := \frac{V}{\rho_c}. \]
(2.11)
This parameter reflects deviation from the classical slow-roll inflation due to holonomy corrections. In the classical limit, which corresponds to \( \rho_c \to \infty \), \( \delta_H \) goes to zero. In what follows we will consider only linear corrections in \( \delta_H \). This is because \( \delta_H \) is expected to be a very small quantity. The assumption that the slow-roll regime takes place in the Lorentzian regime \( (\Omega > 0) \) implies that \( \delta_H < 1/2 \). One can however motivate that \( \delta_H \ll 1/2 \) unless the critical energy density \( \rho_c \) is not much smaller than the Planck energy density. As an example, let us consider model with a massive potential \( V(\phi) = \frac{1}{2} m^2 \phi^2 \). Taking the inflaton mass \( m \sim 10^{-6} m_{\text{Pl}} \) and value of the scalar field \( \phi \sim m_{\text{Pl}} \) in agreement with cosmological observations one can estimate that \( V(\phi) \sim 10^{-12} \rho_{\text{Pl}} \). Therefore, for \( \rho_c \sim \rho_{\text{Pl}} \) one can expect that \( \delta_H \)}
has the extremely small value $\delta H \sim 10^{-12}$. On the other hand if $\rho_c \sim 10^{-12} \rho_{Pl}$ or smaller, the holonomy corrections are becoming observationally relevant and can be constrained with use of the CMB data.

Here, we keep terms linear in both slow-roll parameters and $\delta H$ as well as the mixed terms $O(\epsilon \delta H)$ and $O(\eta \delta H)$. Contribution from the second order expansion in the slow-roll parameters is not taken into account. However, in case the $\delta H$ is extremely small, as estimated above, the terms $O(\epsilon^2)$ will dominate over the contributions from $O(\epsilon \delta H)$. The derived expressions will therefore have practical application only to the regime with $\rho_c \in (\sim 0.01, \sim 1/2)$,\(^2\) where the lower limit comes from estimating values of the slow-roll parameters. The estimated range overlaps with the domain which can be probed with use of currently available observational data. Furthermore, the theoretical predictions performed here will set a stage for more comprehensive considerations of the second order expansion in the slow-roll parameters. Such analysis could extend a range of testable values of $\rho_c$. However, only if sufficiently precise observational data are available.

3 Normalization of modes

In this section we will present some possible choices of the short scale normalizations for the perturbations with holonomy corrections. In must be stressed that we do not explore any representative class of states. However, the considered normalizations seem to be the most reliable and best physically motivated. Because of this ambiguity, the choice of the normalization is the weakest point of a whole construction of the model of generation of primordial perturbations during the inflationary phase. This concerns also the case without quantum gravitational corrections.\(^3\) Therefore, here we pay a lot of attention to this issue.

Performing the Fourier transform $v(x, \tau) = \int \frac{d^3 k}{(2\pi)^{3/2}} v_k(\tau) e^{i k \cdot x}$ of the equations of modes for tensor and scalar perturbations we find

$$
\frac{d}{d\tau} v_k + \Omega k^2 v_k - \frac{z''}{z} v_k = 0, \quad (3.1)
$$

where $k^2 = k \cdot k$, and expression on $z$ depends on whether scalar or tensor modes are studied. In both cases $\frac{z''}{z} \approx \mathcal{H}^2 \approx \frac{1}{z}$. Based on this, one can define super-horizontal limit when $\sqrt{\Omega} k \ll \mathcal{H}$ and short scale limit when $\sqrt{\Omega} k \gg \mathcal{H}$. In the super-horizontal limit the $\Omega k^2 v_k$ factor in eq. (3.1) can be neglected and an approximate solution $v_k = c_1 z + c_2 \int^\tau d\tau' \frac{z'}{z}$ can be found. Because the physical amplitudes of perturbations are proportional to the ratio $v_k/z$, it is clear that amplitudes are “frozen” at the super-horizontal scales. This process beings when $\sqrt{\Omega} k \approx \mathcal{H}$.\(^4\) In the short scale limit, the factor $\frac{z''}{z} u_k$ in eq. (3.1) can be neglected and the equation for modes reduces to

$$
\frac{d^2}{d\tau^2} v_k + \Omega(t) k^2 v_k \approx 0. \quad (3.2)
$$

\(^2\)Under assumption that the potential of the inflaton field is quadratic.

\(^3\)We mean by this that the initial state not necessary has to be a vacuum state. Usually, the so-called Buch-Davies vacuum is used, which is obtained by minimizing an energy functional. However, the initial state e.g. for the phase of inflation can differ from the vacuum state because of a former cosmological evolution.

\(^4\)It is worth noticing that this condition differs from the classical one $k \approx \mathcal{H}$ due to presence of time dependent function $\Omega$. Furthermore, it is worth to stress that $\frac{z''}{z} \approx \mathcal{H}^2 \approx \frac{1}{z}$. In particular, for the tensor modes $\frac{z''}{z} \approx \mathcal{H}^2 [2 - \epsilon(1 + 5 \delta_H) + \ldots]$. The correction due to $\delta_H$ contributes however together with the $\epsilon$ factor, contrary to the contribution $\Omega = 1 - 2 \delta_H + \ldots$ in front of the $k^2$ factor.
For the slow-roll inflation, the $\Omega$ is only weakly dependent on $\tau$ and solution to equation (3.2) can be found by applying the WKB approximation. We find that

$$v_k = \frac{c_1}{\sqrt{2k\sqrt{\Omega}}} e^{-ik\int^{\tau} \sqrt{\Omega(\tau')} d\tau'} + \frac{c_2}{\sqrt{2k\sqrt{\Omega}}} e^{+ik\int^{\tau} \sqrt{\Omega(\tau')} d\tau'},$$

(3.3)

which is superposition of plane waves traveling forward ($e^{-ik\int^{\tau} \sqrt{\Omega(\tau')} d\tau'}$) and backward ($e^{+ik\int^{\tau} \sqrt{\Omega(\tau')} d\tau'}$) in time. Validity of the WKB approximation requires that

$$\left| \frac{\Omega'}{\Omega} \right| \frac{1}{k\sqrt{\Omega}} \ll 1.$$  

(3.4)

Because we are in the short scale limit $\sqrt{\Omega}k \gg \frac{1}{|\tau|}$ and

$$\frac{\Omega'}{\Omega} = -\frac{4\epsilon}{\tau} \delta H,$$

(3.5)

the condition of validity of the WKB approximation simplifies to $|\epsilon \delta H| < 0$. Because both $\epsilon$ and $\delta H$ are smaller than unity for the considered slow-roll evolution, the WKB approximation (3.3) holds.

Canonical commutation relation between quantum field $\hat{v}$ and its conjugated momenta requires Wronskian condition

$$W(v_k, v'_k) \equiv v_k \frac{dv_k^*}{d\tau} - v_k^* \frac{dv_k}{d\tau} = i$$

(3.6)

to be satisfied. This is the usual way the modes are normalized and also the place where quantum mechanics enters into description of primordial perturbations. The Wronskian condition applied to solution (3.3) leads to relation

$$|c_1|^2 - |c_2|^2 = 1.$$  

(3.7)

The initial four real numbers ($c_1, c_2 \in \mathbb{C}$) parametrizing solution (3.3) are therefore reduced to three. Because the total phase is physically irrelevant, the family of normalized solutions in the short scale limit is characterized by two real numbers. Their values have to be fixed by hand. The obtained solutions are used to normalize general solutions to the equations of motion.

It is worth stressing at this point that while considering the short scale limit $\sqrt{\Omega}k \ll \mathcal{H}$ one has to be cautious about the limit $k \to \infty$. Such limit can be performed only formally because for $k > \frac{a}{l_{Pl}}$ the classical description of space is expected to be no more valid due to quantum gravitational effects.\footnote{The condition $k > \frac{a}{l_{Pl}}$ is equivalent to the statement that physical length scale $\lambda$ of a given mode $k$ has to be greater than the Planck length: $\lambda > l_{Pl}$, where $\lambda = \frac{a}{2}$.} We do not consider such trans-Planckian modes here.

### 3.1 $\Omega$-deformed Minkowski vacuum

For the particular choice $c_1 = 1$, solution (3.3) contains incoming modes only:

$$v_k = \frac{1}{\sqrt{2k\sqrt{\Omega}}} e^{-ik\int^{\tau} \sqrt{\Omega(\tau')} d\tau'}.$$  

(3.8)
This solution reduces to the so-called Minkowski (Bunch-Davies) vacuum

\[ v^M_k \equiv \frac{e^{-ik\tau}}{\sqrt{2k}} \]  

(3.9)
in the classical limit \((\Omega \to 1)\), which has been extensively used to normalize cosmological perturbations.

Based on (3.8) we define \(\Omega\)-deformed Minkowski vacuum to be

\[ v^\Omega_k \equiv \frac{e^{-ik \int^\tau_{\tau'} \sqrt{\Omega(\tau')} d\tau'}}{2\sqrt{\Omega k}} \approx v^M_k \left[ 1 + \left( \frac{1}{2} + ik\tau \right) \delta_H + O(\delta_H^2) \right], \tag{3.10} \]

where in the second equality we neglected time variation of \(\Omega\).

### 3.2 Bojowald-Calcagni normalization

Another possibility of normalizing modes was proposed in ref. [6] for the case of perturbations with inverse volume corrections. The proposal made by Bojowald and Calcagni was that in the short scale limit the solution to (3.2) can be written up to the first order in \(\delta_H\) as follows

\[ v^\text{BC}_k = v^M_k (1 + y(k, \tau)\delta_H), \tag{3.11} \]

where \(y(k, \tau)\) is some unknown function. By plugging (3.11) into (3.2) we find the following equation for the function \(y\):

\[ y'' - 2(2H\epsilon + ik)y' + 4i\mathcal{H}_k\epsilon k - 2\epsilon H^2 k^2 y - 2k^2 = 0, \tag{3.12} \]

where we used relations

\[ \delta'_H = -2\epsilon H \delta_H, \tag{3.13} \]

\[ \delta''_H = -2\epsilon H^2 \delta_H. \tag{3.14} \]

The equation (3.12) requires certain simplifications. Firstly, because we are interested in the first order correction in \(\delta_H\) we can skip the factor \(-2\delta_H k^2 y\) in (3.11), which would generate higher order terms in \(\delta_H\). Secondly, because we are looking for the short scale solution \((\sqrt{\Omega k} \gg \mathcal{H})\) the factor \(-2\epsilon H^2 y\) can be ignored as well. This second approximation turns out to be beneficial while searching for analytic solution to the equation of motion for \(y\). The reduced equation (3.12) is now

\[ y'' - 2(2H\epsilon + ik)y' + 4i\mathcal{H}_k\epsilon k y - 2k^2 = 0. \tag{3.15} \]

For \(\epsilon = 0\), solution to this equation is

\[ y = ik\tau + c_1 + c_2 e^{2ik\tau}, \tag{3.16} \]

where \(c_1\) and \(c_2\) are constants of integration. Because the \(e^{2ik\tau}\) factor would lead to outgoing modes we fix \(c_2 = 0\). Value of the factor \(c_1\) can be determined by considering the case \(\epsilon \neq 0\). With use of \(\mathcal{H} = -\frac{1}{\tau}\) we find that equation (3.15) has special solution \(y = a + b\tau\) with

---

6In the original paper [6] the expansion was performed not in terms of \(\delta_H\) but \(\delta_{Pl}\) relevant for inverse volume corrections.

7Here we use the simplifed de Sitter solution instead of \(\mathcal{H} = -\frac{1}{\tau} \frac{1}{1 + \frac{1}{3}\delta_{Pl}}\), which is sufficient within the considered order of approximation.
\( a = \frac{1}{1+2\epsilon} \) and \( b = \frac{i}{1+2\epsilon} \). Requirement of analytic continuity (with respect to \( \epsilon \)) between the cases \( \epsilon = 0 \) and \( \epsilon \neq 0 \) fixes the value of \( c_1 \). In consequence, we obtain

\[
v_{BC}^k = v_{k}^M\left[1 + \frac{1}{1 + 2\epsilon} (1 + ik\tau) \delta_H + O(\delta_H^2)\right] \simeq v_{k}^M \left[1 + (1 + ik\tau) \delta_H\right].
\]

It is worth noticing a slight difference between this case and predictions of the \( \Omega \)-deformed Minkowski vacuum (3.10). In contrast to that case, the method presented in this subsection does not utilize the Wronskian condition in order to normalize the mode functions. This may have advantages if we have reason to suppose that the Wronskian condition is deformed but the form of deformation is not known.

### 3.3 Deformed Wronskian condition

Let us suppose that indeed the Wronskian condition is deformed due to presence of \( \Omega \). Such deformation can come from the fact that, because \( \Omega \) is present in equations of motion, inner product must differ from the classical one. We will discuss this issue in more details in the next section, while here we assume the classical Wronskian condition (3.6) is deformed to

\[
W_{\Omega}(v_k, v'_k) = v_k \frac{dv_k^*}{d\eta} - v^*_k \frac{dv_k}{d\eta} = if(\Omega),
\]

where \( f(\Omega) \) is some function of \( \Omega \), defined such that \( \lim_{\Omega \to 1} f(\Omega) = 1 \). In case we have no hints what the functional form of \( f(\Omega) \) is we can investigate a power-low parametrization

\[
f(\Omega) = \Omega^n.
\]

In this case, the counterpart of (3.10) is

\[
v_k^{(n)} \equiv \frac{e^{-i\sqrt{\Omega}k\tau}}{\sqrt{2k\Omega^{1/2-n}}} = v_M \left[1 + \left(\frac{1}{2} - n + ik\tau\right) \delta_H + O(\delta_H^2)\right].
\]

As we see, for \( n = -\frac{1}{2} \), the normalization from the deformed Wronskian condition overlaps with Bojowald-Calcagni normalization up to the first order in \( \delta_H \):

\[
v_k^{(-1/2)} \simeq v_{BC}^k.
\]

### 4 Inner product and the Wronskian condition

Let us now address in more details the issue of validity of the Wronskian normalization in presence of holonomy corrections.

For the a pair of fields \( \phi_1, \phi_2 \) satisfying Klein-Gordon equation \((\Box - m^2)\phi = 0\), the inner product is [32]

\[
\langle \phi_1 | \phi_2 \rangle := i \int_{\Sigma} d^3x \sqrt{q} n^\mu \left( \phi_2^* \partial_\mu \phi_1 - \phi_1 \partial_\mu \phi_2^* \right),
\]

where \( n^\mu \) is a future-direction unit vector \((g_{\mu\nu}n^\mu n^\nu = -1)\) normal to Cauchy surface \( \Sigma \) and \( q \) is a determinant of the spatial metric on \( \Sigma \). To remind, the Cauchy surface is a spatial hypersurface at which the initial conditions are imposed. The inner product (4.1) is defined such that it does not depend on the choice of a Cauchy surface:

\[
\langle \phi_1 | \phi_2 \rangle_{\Sigma_1} = \langle \phi_1 | \phi_2 \rangle_{\Sigma_2}.
\]
The proof is direct and employs a Gauss law, Klein-Gordon equation and vanishing of $\phi$ at spatial infinities (See e.g. ref. [33]). It is also worth noticing, that the inner product (4.1) is not positive-definite.

In the case studied in this paper, the Klein-Gordon equations for scalar and tensor perturbations are deformed with respect to the classical one. Therefore, in general, one could expect that (4.1) is not a good scalar product because the condition (4.2), requiring the Klein-Gordon equation to be satisfied, is not fulfilled. This can imply that the Wronskian condition (3.6), resulting from normalization of modes with use of (4.1), is deformed.

However, if we manage to find an effective metric $g_{\mu\nu}^{\text{eff}}$ which leads to holonomy deformations of the equations of perturbations, then proof of the condition (4.2) would remain in force, and the inner product (4.1) can be used. In what follows we show that such construction is possible for tensor and scalar perturbations.

For any component $\phi$ of the tensor perturbations, the equation of motion is

$$\frac{d^2}{d\tau^2}\phi + 2\left(\mathcal{H} - \frac{1}{2\Omega} \frac{d\Omega}{d\tau}\right) \frac{d}{d\tau}\phi - \Omega \nabla^2 \phi = 0. \tag{4.3}$$

In the coordinate time ($dt = a d\tau$) this equation can be written as

$$\ddot{\phi} + 3H \dot{\phi} - \frac{\dot{\Omega}}{\Omega} \dot{\phi} - \frac{\Omega}{a^2} \nabla^2 \phi = 0. \tag{4.4}$$

The classical Klein-Gordon equation $\square \phi = 0$ (the tensor modes are massless) on the FRW background is recovered by taking $\Omega = 1$. It can be proved by direct calculation, that the holonomy-corrected equation for tensor modes can be derived from the wave equation $\square \phi = 0$ at the effective FRW background given by the line element

$$ds^2_{\text{eff}} = g_{\mu\nu}^{\text{eff}} dx^\mu dx^\nu = -\sqrt{\Omega} N^2 dt^2 + \frac{a^2}{\sqrt{\Omega}} \delta_{ab} da^a dx^b, \tag{4.5}$$

where $N$ is a lapse function. In particular, for the coordinate time ($N = 1$) we have

$$\Box \phi = \nabla^\mu \nabla_\mu \phi = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi)
= -\frac{1}{\sqrt{\Omega}} \dot{\phi} + \frac{\sqrt{\Omega}}{a^2} \nabla^2 \phi - \frac{1}{\sqrt{\Omega}} \left(3H - \frac{\dot{\Omega}}{\Omega}\right) \dot{\phi}, \tag{4.6}$$

where we used $g_{\mu\nu} = g_{\mu\nu}^{\text{eff}}$. By equating (4.6) to zero and multiplying by $-\sqrt{\Omega}$, the equation (4.4) is recovered.

It is worth noticing that the effective metric (4.5) is conceptually similar to dressed metric approach to quantum fields on quantum spaces [25, 34]. In our case, quantum gravitational effects are “dressing” the FRW metric leading to the effective metric (affected by $\Omega$ terms), which is felt by the test fields.

Now we can check if the Wronskian condition derived based on (4.1) holds the classical form. We will be interested in the form of the Wronskian condition for the field

$$u = \frac{a}{\sqrt{\Omega}} \phi, \tag{4.7}$$

which, as can be seen by substituting to (4.3), fulfills equation

$$\frac{d^2}{d\tau^2} u - \Omega \nabla^2 u - \frac{z''}{z} u = 0, \tag{4.8}$$
where \( z = a/\sqrt{\Omega} \). Because the Cauchy hypersurface is spatial, based on \( g_{\mu\nu}n^\mu n^\nu = -1 \), we find \( n^0 = \frac{1}{\sqrt{\Omega} a} \) and \( n^a = 0 \), which gives us \( n^\mu \partial_\mu = \frac{1}{\sqrt{\Omega} a} \partial_0 \). Then, for the conformal time \( (N = a) \), the inner product of two fields \( \phi \) is

\[
\langle \phi | \phi \rangle = i \int d^3x \sqrt{q} (\phi^* n^\mu \partial_\mu \phi - \phi n^\mu \partial_\mu \phi^*)
\]

\[
= i \int d^3x \frac{a^3}{\Omega^{3/4}} \frac{1}{a \Omega^{1/4}} \left( u^* \frac{du}{d\tau} - u \frac{du^*}{d\tau} \right)
\]

\[
= i \int d^3x \left( u^* \frac{du}{d\tau} - u \frac{du^*}{d\tau} \right)
\]

\[
= -i \int d^3x W(u, u') = \int d^3x = V_0 = 1,
\]

(4.9)

where the classical Wronskian condition (3.6) was used to get the proper normalization \( \langle \phi | \phi \rangle = 1 \). Here, we have restricted spatial integration to \( V_0 \), which can be conventionally fixed to one. Alternatively, the field \( \phi \) can be rescaled by \( \phi \rightarrow \frac{1}{\sqrt{V_0}} \phi \) in order to compensate contribution from the spatial integration over \( V_0 \).

In summary, for the tensor modes, the inner product (4.1) is properly defined and the normalization condition \( \langle \phi | \phi \rangle = 1 \) leads to the classical Wronskian condition (3.6). The \( \Omega \)–deformed vacuum seems to be therefore the right choice for the tensor modes.

It remains to show if the similar construction can be performed for the scalar modes. Here, it is relevant to consider equation of motion for the scalar curvature \( \mathcal{R} = \frac{1}{N^2} S \). By applying this definition to the Mukhanov equation (1.5) and introducing coordinate time we obtain the following equation:

\[
\dddot{\mathcal{R}} + 3H \dot{\mathcal{R}} + 2 \left( \frac{\ddot{\mathcal{R}}}{\dot{\phi}} - \frac{\dot{H}}{H} \right) \dddot{\mathcal{R}} - \frac{\Omega}{a^2} \nabla^2 \mathcal{R} = 0.
\]

(4.10)

The equation (4.10) plays the same role as equation (4.4) for the tensor modes. It can be shown that equation (4.10) can be obtained by considering Laplace-Beltrami operator on the effective FRW line element

\[
ds^2_{\text{eff}} = g^\text{eff}_{\mu\nu} dx^\mu dx^\nu = -N^2 Q_1 dt^2 + a^2 Q_2 \delta_{ab} da^a da^b,
\]

(4.11)

where \( Q_1 \) and \( Q_2 \) are quantum corrections defined as follows:

\[
Q_1 = \Omega Q_2,
\]

(4.12)

\[
Q_2 = C \sqrt{\Omega} \frac{\Omega^{-2}}{a^2},
\]

(4.13)

where \( C \) is a constant of integration. The inner product (4.1) on the effective metric (4.11) is therefore well defined for the scalar perturbations. However, there is a subtle difference with the previously considered case of the tensor modes. Namely, for the tensor modes, the effective metric (4.5) converged to the FRW metric, in the classical limit \( \rho_c \rightarrow 1 \). Therefore, it was possible to consider it as a classical FRW metric “dressed” in the quantum gravitational corrections. The effective metric for the scalar perturbations (4.11) is different because, even in absence of the quantum corrections, it does not agree with the FRW metric. This comes from the fact that the scalar perturbations are not the vacuum degrees of freedom of the
gravitational field. However, this does not prevent us to consider the \((4.1)\) defined on the effective metric \((4.5)\), which leads to the following normalization condition

\[
\langle R|R\rangle = \int_\Sigma d^3x \sqrt{q} \left( R^* n^\mu \partial_\mu R - R n^\mu \partial_\mu R^* \right) = -i \frac{a^2 Q^3}{2 \sqrt{Q_1 z_5^2}} W(v, v'). \tag{4.14}
\]

By equating \((4.14)\) to unity, we find the following expression for the Wronskian

\[
W(v, v') = \frac{i \sqrt{Q_1 z_5^2}}{a^2 Q^3/2} = \frac{i}{C}, \tag{4.15}
\]

where in the second equality we have employed expressions \((4.12)\) and \((4.13)\). The classical Wronskian condition \((3.6)\) is recovered by taking \(C = 1\).

The considerations performed in this section do not support deformations of the Wronskian normalization condition. In case of the tensor modes, the classical form of the Wronskian condition was obtained unambiguously. In case of the scalar modes, a freedom of choosing constant deformation factor appears. The factor is present also in the classical case, therefore there is no reason to take its value to be other than \(C = 1\), which corresponds to the classical Wronskian condition.

While this section shows that it is possible to normalize modes in the standard way, one cannot automatically reject other possible normalization schemes. In particular, such as those considered in subsections 3.2 and 3.3. The differences with the conditions obtained in this section may have its roots in more subtle issues, such as possible deformations of the quantum mechanical rules due to the presence of the \(\Omega\) factor. For example, deformation of the scalar product \(\langle \phi|\phi\rangle = 1\) to \(\langle \phi|\phi\rangle = 1/\sqrt{\Omega}\) leads to the case expressed in \((3.21)\). Therefore, in what follows we do not limit ourselves to the classical normalization condition only.

5 Tensor power spectrum

In this section we will compute inflationary tensor power spectrum with holonomy corrections. Starting point for our considerations is the equation of modes

\[
\frac{d^2}{d\tau^2} u_k + \Omega k^2 u_k - \frac{z_T''}{z_T} u_k = 0, \tag{5.1}
\]

where \(k^2 = k \cdot k\). Having solutions for \(u_T\) and \(z_T\), tensor power spectrum can be found from the definition

\[
\mathcal{P}_T(k) = 64 \pi G \frac{k^3}{2 \pi^2} \left| \frac{u_k}{z_T} \right|^2. \tag{5.2}
\]

With use of \(z_T = a/\sqrt{\Omega}\), expression for the effective mass term can be written as

\[
m^2_{\text{eff}} \equiv -\frac{z_T''}{z_T} = -\frac{a''}{a} + \frac{a'}{a} \frac{\Omega'}{\Omega} + \frac{1}{2} \frac{\Omega''}{\Omega} - \frac{3}{4} \left( \frac{\Omega'}{\Omega} \right)^2. \tag{5.3}
\]
All the factors contributing to $m_{\text{eff}}^2$ can be expressed in terms of conformal time $\tau$ as well as $\epsilon, \eta$ and $\delta_H$. With use of the slow-roll conditions, these terms are:

\begin{align*}
\frac{a'}{a} &= -\frac{1}{\tau} [1 + \epsilon (1 - \delta_H)], \\
\frac{a''}{a} &= \frac{1}{\tau^2} [2 + 3 \epsilon (1 - \delta_H)], \\
\frac{\Omega'}{\Omega} &= \frac{4 \epsilon}{\tau^2} \delta_H.
\end{align*}

The expression for $\frac{\Omega'}{\Omega}$ is given in eq. (3.5). Plugin it into expression for $m_{\text{eff}}^2$ and keeping terms up to the first order in $\epsilon$ and $\delta_H$ we obtain

\[ m_{\text{eff}}^2 = -\frac{1}{\tau^2} [2 + 3 \epsilon (1 - 3\delta_H)]. \tag{5.7} \]

The equation for tensor modes can be therefore expressed as

\[ \frac{d^2 u_k}{d \tau^2} + \left[ \Omega k^2 - \left( \nu_T^2 - \frac{1}{4} \right) \frac{1}{\tau^2} \right] u_k = 0, \tag{5.8} \]

where

\[ |\nu_T| = \sqrt{\frac{9}{4} + 3 \epsilon (1 - 3\delta_H)} \simeq \frac{3}{2} + \epsilon (1 - 3\delta_H). \tag{5.9} \]

Equation (5.8) reminds the standard equation for inflationary modes and it is tempting to find its analytic solution in terms of Hankel functions. This is however not consistent because requires assumption of the constancy of $\Omega$. The slow variation of $\Omega$ cannot be neglected if variation of $\Omega$ in the expression for $m_{\text{eff}}^2$ has already been included. To see it clearly, let us perform the following change of variables:

\begin{align*}
x &:= -k \tau \sqrt{\Omega}, \\
f(x) &:= \frac{u}{\sqrt{x}},
\end{align*}

which transforms (5.8) into

\[ (1 - 4\epsilon \delta_H)x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - \nu_T^2) f = 0. \tag{5.12} \]

While the dependence on $\delta_H$ in $m_{\text{eff}}^2$ is contributing to $\nu_T^2$, the time dependence of $\Omega$ in front of $k^2$ generates factor $-4\epsilon \delta_H$. Because of this factor, solutions to equation (5.12) are not Bessel (or equivalently Hankel) functions, what would be the case if the factor $-4\epsilon \delta_H$ is absent. Nevertheless, solutions to equation (5.8) can be studied numerically.

Because analytic solution to equation (5.8) cannot be easily found, we are forced to use another approach to find the tensor power spectrum. Namely, we will determine amplitude of the perturbations at the Hubble radius with use of the short scale solutions. However first, in order to approve consistency of normalization in case on non-vanishing and time dependent $\Omega$ function we will consider tensor power spectrum for $\Omega = \text{const.}$. 

\[ \text{JCAP03(2014)048} \]
5.1 $\Omega = \text{const}$ case

In this section we derive inflationary tensor power spectrum under the assumption that $\Omega = \text{const}$. As it was explicitly shown in the previous subsection, this assumption violates the slow-roll approximation. However, for $\Omega = \text{const}$, equation of modes can be solved analytically and expression for the power spectra can be found. Such result, despite of limited physical relevance, will be used to validate our further calculations on tensor and scalar perturbations, where a less direct method of determining the power spectra is applied.

For $\Omega = \text{const}$, the effective mass term is

$$m_{\text{eff}}^2 \equiv -\frac{z''}{z_T} = -\frac{a''}{a} = -\frac{1}{\tau^2} [2 + 3\epsilon (1 - \delta_H)]. \quad (5.13)$$

The equation of motion takes the form (5.8) with $\Omega = \text{const}$ and

$$|\nu_T| = \sqrt{\frac{9}{4} + 3\epsilon (1 - \delta_H)} \simeq \frac{3}{2} + \epsilon (1 - \delta_H). \quad (5.14)$$

In this case, exact solution to equation (5.8) can be expressed in terms of Hankel functions:

$$u_k = \sqrt{-\tau} \sqrt{\frac{\pi}{4}} [D_1 H^{(1)}_{|\nu|}(-\sqrt{\Omega}k\tau) + D_2 H^{(2)}_{|\nu|}(-\sqrt{\Omega}k\tau)] . \quad (5.15)$$

The constants $D_1$ and $D_2$ were introduced such chosen such that the Wronskian condition (3.6) leads to relation

$$|D_1|^2 - |D_2|^2 = 1. \quad (5.16)$$

The $\Omega$–deformed Minkowski vacuum normalization is chosen by taking $D_2 = 0$ and $D_1 = e^{i\pi (2|\nu|+1)/4}$. This can be verified by considering asymptotic behavior of the Hankel functions. Namely, for $x \ll 1$ we have $H^{(1)}_{|\nu|}(x) \simeq \sqrt{\frac{2}{\pi x}} \exp (i(x - |\nu|\pi/2 - \pi/4))$, which leads to

$$u_k = \sqrt{-\tau} \sqrt{\frac{\pi}{4}} e^{i\pi (2|\nu|+1)/4} H^{(1)}_{|\nu|}(-\sqrt{\Omega}k\tau) \approx \frac{e^{-i\sqrt{\Omega}k\tau}}{\sqrt{2\sqrt{\Omega}k}} = u_k^{\Omega\text{M}}, \quad (5.17)$$

for $-\sqrt{\Omega}k\tau \gg 1$.

Having the modes normalized correctly we can study the super-horizontal limit $-\sqrt{\Omega}k\tau \ll 1$. With use of approximation $H^{(1)}_{|\nu|}(x) \simeq -\frac{1}{\pi} \Gamma(|\nu|) \left(\frac{x}{2}\right)^{-|\nu|}$, which holds for $x \ll 1$, we have

$$|u_k|^2 \simeq \frac{1}{2\alpha H} \left(\frac{k\sqrt{\Omega}}{\alpha H}\right)^{-2|\nu|}, \quad (5.18)$$

where $-\tau \simeq \frac{1}{\alpha H}$ has been used. Applying (5.18) to definition (5.2), expression for the tensor power spectrum is obtained:

$$P_T(k) = A_T \left(\frac{k}{\alpha H}\right)^{\nu_T}, \quad (5.19)$$

where the amplitude

$$A_T = \frac{16}{\pi} \left(\frac{H}{m_{\text{Pl}}}\right)^2 \frac{1}{\sqrt{\Omega}} = \frac{16}{\pi} \left(\frac{H}{m_{\text{Pl}}}\right)^2 (1 + \delta_H + \mathcal{O}(\delta_H^2)), \quad (5.20)$$
and the tensor spectral index

\[ n_T = -2\epsilon (1 - \delta_H). \]  \hspace{1cm} (5.21)

With use of the modified Friedmann equation (1.2) in the slow-roll regime (\(\rho \approx V\)), one can rewrite (5.20) into the following form

\[ A_T = \frac{128}{3} \frac{V}{\rho_{Pl}} (1 - \delta_H) (1 + \delta_H) = \frac{128}{3} \frac{V}{\rho_{Pl}} + \mathcal{O}(\delta_H^2). \] \hspace{1cm} (5.22)

This expression is not subject of holonomy corrections in the leading order.

5.2 \(\Omega\)-deformed Minkowski vacuum

Let us now proceed to the proper calculations. The strategy is the following: we will use a given short scale solution and extrapolate it up to the horizontal scale. A mode characterized by \(k\) crosses the horizon scale when \(\sqrt{\Omega} k = aH \simeq -\frac{1}{2}\). Above the horizon scale the modes “freeze out” and the power spectrum remains unchanged. The spectral index can be computed from the horizontal spectrum based on the formula

\[ n_T \equiv \frac{d\ln A_T}{d\ln k}. \] \hspace{1cm} (5.23)

Modulus square of the \(\Omega\)-deformed Minkowski vacuum (3.8) is

\[ |v_{\Omega M}^k|^2 = \frac{1}{2\sqrt{\Omega} k}. \] \hspace{1cm} (5.24)

By inserting (5.24) into the definition (5.2) and calculating the value at \(k\sqrt{\Omega} = aH\) we find

\[ A_T \equiv P_T(k\sqrt{\Omega} = aH) = \frac{16}{\pi} \left( \frac{H}{m_{Pl}} \right)^2 \frac{1}{\sqrt{\Omega}} = \frac{16}{\pi} \left( \frac{H}{m_{Pl}} \right)^2 (1 + \delta_H) + \mathcal{O}(\delta_H^2). \] \hspace{1cm} (5.25)

The obtained amplitude agrees with equation (5.20), where \(\Omega = \text{const}\) was assumed. This approves that the horizon-crossing condition \(\sqrt{\Omega} k = aH\) for the short scale solutions remains in force.

Let us now determine the value of the tensor spectral index. By substituting (5.25) to (5.23), we find

\[ n_T = \frac{2}{H} \frac{dH}{dk} - \frac{1}{2} \frac{d\Omega}{dk} = -2\epsilon + \mathcal{O}(\epsilon^2 \delta_H), \] \hspace{1cm} (5.26)

where we used \(k = -\frac{1}{\tau\sqrt{\Omega}}\) (2.3) and (3.5). Here, correction from \(\delta_H\) contributes together with \(\epsilon^2\) terms. Therefore, in the leading order, the tensor spectral index holds its classical form.

5.3 Bojowald-Calcagni normalization

Let us now find the power spectrum with use of the Bojowald-Calcagni normalization. With use of condition \(\sqrt{\Omega} k = aH \simeq -\frac{1}{2}\), the modulus square of \(v_{\Omega}^k\) is

\[ |v_{\Omega}^k|^2 \simeq \frac{1}{2k} (1 + 2\delta_H). \] \hspace{1cm} (5.27)
It is worth mentioning that this value does not depend on the fact that the amplitude is computed at the horizon. This is because the $i k \tau \delta_H$ term contributes in the second order, which is neglected here. By substituting \((5.27)\) into the definition \((5.2)\) we find

$$A_T \equiv P_T(k \sqrt{\Omega} = aH) = \frac{16G}{\pi} \frac{k^2}{a^2} \Omega (1 + 2\delta_H) = \frac{16}{\pi} \left( \frac{H}{m_{Pl}} \right)^2 (1 + 2\delta_H) + O(\delta_H^2). \quad (5.28)$$

Having amplitude of spectrum computed at the horizon scale, the spectral index is computed from the relation

$$n_T \equiv \frac{d \ln A_T}{d \ln k} = 2 \frac{kHa}{H^2} + 2 \frac{d\delta_H}{d \ln k} = - 2\epsilon (1 + \delta_H), \quad (5.29)$$

where we have used expression \((2.3)\) and the fact that at the horizon $\sqrt{\Omega} k \tau = -1$.

Summing up, the tensor power spectrum with the Bojowald-Calzagni normalization can be expressed as follows

$$P_T(k) = A_T \left( \frac{k}{aH} \right)^{n_T}, \quad (5.30)$$

where the amplitude

$$A_T = \frac{16}{\pi} \left( \frac{H}{m_{Pl}} \right)^2 (1 + 2\delta_H) \quad (5.31)$$

and the tensor spectral index

$$n_T = - 2\epsilon (1 + \delta_H). \quad (5.32)$$

With use of the modified Friedmann equation \((1.2)\) in the slow-roll regime ($\rho \approx V$), one can rewrite \((5.31)\) into the following form

$$A_T = \frac{128}{3} \frac{V}{\rho_{Pl}} (1 + \delta_H). \quad (5.33)$$

### 6 Scalar power spectrum

For the sake of completeness, let us start with deriving equation of motion for the scalar modes in the slow-roll approximation. Because of the same reason as in the case of tensor modes, this equation will not be used to derive spectrum of the scalar inflationary perturbations.

Amplitude of the scalar power spectrum

$$P_S(k) = \frac{k^3}{2\pi^2} \frac{|v_k|^2}{|z_S|^2} \quad (6.1)$$

will be calculated using the short scale solution extrapolated to the horizontal scale. While the spectral scalar is found, the spectral index will be determined by virtue of

$$n_S \equiv \frac{d}{d \ln k} \ln P_S(k \sqrt{\Omega} = aH). \quad (6.2)$$

Similarly as for tensor modes, the Fourier transform of the scalar perturbations fulfills equation

$$\frac{d}{d\tau^2} v_k + \Omega k^2 v_k - \frac{z_S}{z_S} v_k = 0, \quad (6.3)$$
where \( z_S = a \frac{\dot{\varphi}}{H} \). The task now is to determine time dependance of \( \frac{z_S''}{z_S} \) in the slow-roll approximation.

By differentiating \( z_S = a \frac{\dot{\varphi}}{H} \) with respect to conformal time and by using relation (2.3) we obtain

\[
\frac{z_S'}{z_S} = \epsilon (1 - \delta_H) H + \frac{\varphi''}{\varphi'}. \tag{6.4}
\]

With use this, the expression (2.7) for the parameter \( \delta \) can be written as

\[
\delta = 1 - \frac{\varphi''}{\varphi'} H = 1 + \epsilon (1 - \delta_H) - \frac{z_S'}{z_S H}. \tag{6.5}
\]

By differentiating this equality with respect to conformal time and neglecting all the non-leading contributions (i.e. \( \dot{\delta}, \dot{\epsilon}, \epsilon^2, \eta' \)) we obtain the following equality

\[
\frac{z_S''}{z_S} = \left( \frac{z_S'}{z_S} \right)^2 + \frac{\mathcal{H}' z_S'}{\mathcal{H} z_S}. \tag{6.6}
\]

Combining (6.4) together with (6.5) and (2.7) we find

\[
\frac{z_S'}{z_S} = \left[ 1 - \eta + 2 \epsilon (1 - \delta_H) \right] H. \tag{6.7}
\]

Furthermore

\[
\frac{\mathcal{H}'}{\mathcal{H}} = \mathcal{H} \left( 1 + \frac{\dot{\mathcal{H}}}{H^2} \right) = \mathcal{H} \left( 1 - \epsilon (1 - \delta_H) \right), \tag{6.8}
\]

where in the second equality we used (2.3). Plugging (6.7) and (6.8) to (6.6) we obtain

\[
\frac{z_S''}{z_S} = \mathcal{H}^2 \left[ 2 + 5 \epsilon (1 - \delta_H) - 3 \eta \right] = \frac{1}{\tau^2 \left[ 1 - \epsilon (1 - \delta_H) \right]^2} \left[ 2 + 5 \epsilon (1 - \delta_H) - 3 \eta \right] = \frac{1}{\tau^2} \left[ 2 + 9 \epsilon (1 - \delta_H) - 3 \eta \right]. \tag{6.9}
\]

Equation for scalar modes with holonomy correction in the (first order) slow-roll approximation can be therefore written as

\[
\frac{d^2 v_k}{d\tau^2} + \left[ \Omega k^2 - \left( \nu_S^2 - \frac{1}{4} \right) \frac{1}{\tau^2} \right] v_k = 0, \tag{6.10}
\]

where

\[
|\nu_S| = \sqrt{\frac{9}{4} + 9 \epsilon (1 - 3 \delta_H) - 3 \eta} \simeq \frac{3}{2} + 3 \epsilon (1 - 3 \delta_H) - \eta. \tag{6.11}
\]

The classical case is correctly recovered for \( \delta_H \to 0 \).

### 6.1 \( \Omega \)-deformed Minkowski vacuum

With use of the WKB approximation (3.8) applied to definition (6.1) we find that

\[
A_S \equiv \mathcal{P}_S(k \sqrt{\Omega} = a H) \frac{1}{\pi \epsilon} \left( \frac{H}{\Lambda_{\text{Pl}}} \right)^2 \frac{(1 - \delta_H)}{\Omega^{3/2}} = \frac{1}{\pi \epsilon} \left( \frac{H}{\Lambda_{\text{Pl}}} \right)^2 (1 + 2 \delta_H) + \mathcal{O}(\delta_H^2). \tag{6.12}
\]
Based on this, the inflationary scalar power spectrum:

\[ P_S(k) = A_S \left( \frac{k}{aH} \right)^{n_S - 1}, \tag{6.13} \]

where amplitude of the scalar perturbations

\[ A_S = \frac{1}{\pi \epsilon} \left( \frac{H}{m_{Pl}} \right)^2 (1 + 2\delta_H) + O(\delta_H^2), \tag{6.14} \]

and the scalar spectra index

\[ n_S = \frac{d \ln A_S}{d \ln k} = 1 + 2\eta - 6\epsilon + O(\epsilon^2 \delta_H, \epsilon \eta \delta_H). \tag{6.15} \]

As in case of the tensor modes, the \( \delta_H \) correction to the spectral index is multiplied by the \( \epsilon^2 \) and \( \epsilon \eta \) factors which are negligible in the considered order. In order to see it explicitly let us consider the case of massive scalar field \( (V = \frac{1}{2}m^2 \varphi^2) \), for which \( \epsilon = \eta \). Using (6.15), we get \( n_S = 1 - 4\epsilon + O(\epsilon^2 \delta_H) \). Applying the recent Planck fit \( n_S = 0.9603 \pm 0.0073 \) [35], we obtain \( \epsilon \approx \frac{1}{4} (1 - n_S) \approx 0.01 \). The higher order corrections are, therefore, of the order \( O(\epsilon^2 \delta_H, \epsilon \eta \delta_H) \sim 10^{-4} \delta_H \), with \( |\delta_H| < \frac{1}{2} \). These terms are typically smaller than contributions from the classical second order slow-roll expansion. With use of the present observational precision is impossible to put constrains on such effects.

Moreover, with use of the modified Friedmann equation (1.2) in the slow-roll regime \( (\rho \approx V) \), one can rewrite (6.14) into the following form

\[ A_S = \frac{8}{3} \frac{V}{\epsilon \rho_{Pl}} (1 - \delta_H) (1 + 2\delta_H) \approx \frac{8}{3} \frac{V}{\epsilon \rho_{Pl}} (1 + \delta_H). \tag{6.16} \]

### 6.2 Bojowald-Calcagni normalization

The calculations can be now repeated for the case of Bojowald-Calcagni normalization. The obtained inflationary scalar power spectrum is

\[ P_S(k) = A_S \left( \frac{k}{aH} \right)^{n_S - 1}, \tag{6.17} \]

where amplitude of the scalar perturbations

\[ A_S = \frac{1}{\pi \epsilon} \left( \frac{H}{m_{Pl}} \right)^2 (1 + 3\delta_H) + O(\delta_H^2), \tag{6.18} \]

and the spectral index

\[ n_S = 1 + 2\eta - 6\epsilon (1 + \frac{1}{3} \delta_H). \tag{6.19} \]

In contrary to the previous case, the spectral index is holonomy-corrected in the leading order for the Bojowald-Calcagni normalization.

For completeness, with use of the modified Friedmann equation (1.2) in the slow-roll regime \( (\rho \approx V) \), one can rewrite (6.18) into the following form

\[ A_S = \frac{8}{3} \frac{V}{\epsilon \rho_{Pl}} (1 - \delta_H) (1 + 3\delta_H) \approx \frac{8}{3} \frac{V}{\epsilon \rho_{Pl}} (1 + 2\delta_H). \tag{6.20} \]
7 Tensor-to-scalar ratio

In theoretical studies of inflation as well in confronting theoretical predictions with observations it is often useful to work with tensor-to-scalar ratio $r$. This dimensionless quantity, defined as

$$r := \frac{A_T}{A_S},$$

measures ratio between amplitudes of tensor and scalar perturbations. There is a huge effort to detect B-type polarization of the CMB radiation which would make determination of the amplitude of the tensor perturbations $A_T$ possible. At present, knowing the value of $A_S$ and having observational constraint on $A_T$, upper bound on the value of $r$ can be found. The strongest constraint comes from observations of the Planck satellite: $r < 0.11$ (95\% CL) [35]. The theoretically predicted values of $r$ can be confronted with this bound allowing for elimination of some possible inflationary scenarios. In particular, the massive model of inflation is no more preferred in the light of the new Planck constraint [35, 37].

Let us calculate the tensor-to-scalar ratio $r$ for the models studied in this paper. For the case with $\Omega^-$deformed Minkowski vacuum normalization we obtain

$$r = 16\epsilon \left(\frac{1 + \delta_H}{1 + 2\delta_H}\right) = 16\epsilon (1 - \delta_H) + O(\delta_H^2).$$

Based on this and equation (5.26) the expression for the tensor spectral index

$$r \approx -8n_T (1 - \delta_H).$$

As we have shown in the previous section, for the massive scalar field $\epsilon \approx 0.01$. For the classical case ($\delta_H = 0$) this would give us $r = 16\epsilon = 0.16$, which is in contradiction with the Planck constraint $r < 0.11$. This reflects the mentioned disagreement between the massive scalar field model of inflation and the new Planck data. In the past, when the observational bound on the value of $r$ was weaker, the massive scalar field model of inflation was favored by the data. It is worth noticing that, by applying $\epsilon \approx 0.01$ to (7.2), together with the Planck constraint on $r$, we find that $\delta_H \gtrsim 0.3$. Therefore, presence of the quantum holonomy corrections helps to fulfill the observational bound. However, this would require the critical energy density $\rho_c$ to be much smaller than the Planck energy density.

For the Bojowald-Calcagni normalization we obtain

$$r = 16\epsilon \left(\frac{1 + 2\delta_H}{1 + 3\delta_H}\right) = 16\epsilon (1 - \delta_H) + O(\delta_H^2),$$

which is the same as for the $\Omega^-$deformed Minkowski vacuum normalization. Finally, based on this and (5.32) the expression for the tensor spectral index

$$r \approx -8n_T (1 - 2\delta_H).$$

\(^8\)To be precise, the B-type polarization of the primordial origin was not detected yet. The B-type polarization due to gravitational lensing was recently observed for the first time by the SPTpol observatory [36].
8 Summary

In this paper we found holonomy corrections to inflationary power spectra. Such corrections reflect a discrete nature of space at the Planck scale predicted by loop quantum gravity. Calculations were performed for the slow-roll type inflation driven by a single self-interacting scalar field. The derivations were done up to the first order in the slow-roll parameters $\epsilon$ and $\eta$ as well as in the leading order in the parameter $\delta_H$, characterizing holonomy corrections.

An important issue while considering quantum fields on expanding backgrounds is a proper normalization of the modes. In our calculations we assumed that only ingoing modes are present. Short scale normalization of these modes is a subject of ambiguity due to presence of the quantum holonomy effects. We considered two, best motivated, types of normalization. The first one was based on adiabatic vacuum (WKB) approximation, while the second one was based on the method proposed by Bojowald and Calcagni in ref. [6].

For the first type of normalization, spectral indices are not quantum corrected in the leading order. To be precise, linear corrections in $\delta_H$ are expected. However, they are multiplied not by $\epsilon$ or $\eta$ but $\epsilon^2$ or $\eta^2$ terms. These higher order contributions were not studied systematically in this paper. Nevertheless, calculation of the holonomy-corrected inflationary spectrum including $O(\epsilon^2, \eta \epsilon, \eta^2)$ terms is a natural generalization of the results presented here. This would help constraining $\delta_H$ if sufficiently accurate observational data are available. Investigation of the higher order corrections in the light of the present CMB data is, however, not possible.

As we have shown, equation of motion for tensor and scalar modes with holonomy corrections can be derived from wave equations defined on effective metrics. This observation allowed us to define a proper inner product and to show that normalization of modes with use of the classical Wronskian condition is well defined.

In this paper we focused on the region where $\Omega > 0$. Much more interesting is behavior of modes in the vicinity of $\Omega = 0$ and for $\Omega < 0$ where the equations of modes become elliptic. The issue of imposing initial conditions at $\Omega = 0$ will be a subject of the forthcoming paper [38]. Evolution of tensor modes across the region with negative $\Omega$ was addressed in ref. [19]. As it was shown there, tensor power spectrum is enormously amplified in the UV regime. This new behavior certainly deserves more detailed studies. Furthermore, investigation of simultaneous effects of holonomy and inverse volume corrections is now possible thanks to new results presented in ref. [39].

Acknowledgments

I would like to thank Gianluca Calcagni for useful discussion.

References

[1] A. Ashtekar and J. Lewandowski, Background independent quantum gravity: A Status report, Class. Quant. Grav. 21 (2004) R53 [gr-qc/0404018] [INSPIRE].
[2] M. Bojowald, Loop quantum cosmology, Living Rev. Rel. 8 (2005) 11 [gr-qc/0601085] [INSPIRE].
[3] A. Ashtekar and P. Singh, Loop Quantum Cosmology: A Status Report, Class. Quant. Grav. 28 (2011) 213001 [arXiv:1108.0893] [INSPIRE].
[4] M. Bojowald, Back to the beginning of quantum spacetime, Phys. Today 66 (2012) 35.
[5] M. Bojowald, G.M. Hossain, M. Kagan and S. Shankaranarayanan, *Gauge invariant cosmological perturbation equations with corrections from loop quantum gravity*, Phys. Rev. D 79 (2009) 043505 [Erratum ibid. D 82 (2010) 109903] [arXiv:0811.1572 [INSPIRE]].

[6] M. Bojowald and G. Calcagni, *Inflationary observables in loop quantum cosmology*, JCAP 03 (2011) 032 [arXiv:1011.2779 [INSPIRE]].

[7] M. Bojowald, G. Calcagni and S. Tsujikawa, *Observational constraints on loop quantum cosmology*, Phys. Rev. Lett. 107 (2011) 211302 [arXiv:1011.2779 [INSPIRE]].

[8] M. Bojowald, G. Calcagni and S. Tsujikawa, *Observational test of inflation in loop quantum cosmology*, JCAP 11 (2011) 046 [arXiv:1107.1540 [INSPIRE]].

[9] A. Ashtekar, T. Pawlowski and P. Singh, *Quantum nature of the big bang*, Phys. Rev. Lett. 96 (2006) 141301 [gr-qc/0602086 [INSPIRE]].

[10] A. Ashtekar, T. Pawlowski and P. Singh, *Quantum Nature of the Big Bang: Improved dynamics*, Phys. Rev. D 74 (2006) 084003 [gr-qc/0607039 [INSPIRE]].

[11] M. Bojowald, *What happened before the Big Bang?*, Nature Phys. 3 (2007) 523 [INSPIRE].

[12] A.D. Linde, *Chaotic Inflation*, Phys. Lett. B 129 (1983) 177 [INSPIRE].

[13] T. Cailleteau, J. Mielczarek, A. Barrau and J. Grain, *Anomaly-free scalar perturbations with holonomy corrections in loop quantum cosmology*, Class. Quant. Grav. 29 (2012) 095010 [arXiv:1111.3535 [INSPIRE]].

[14] E. Wilson-Ewing, *Holonomy Corrections in the Effective Equations for Scalar Mode Perturbations in Loop Quantum Cosmology*, Class. Quant. Grav. 29 (2012) 085005 [arXiv:1108.6265 [INSPIRE]].

[15] E. Wilson-Ewing, *Lattice loop quantum cosmology: scalar perturbations*, Class. Quant. Grav. 29 (2012) 215013 [arXiv:1205.3370 [INSPIRE]].

[16] E. Wilson-Ewing, *The Matter Bounce Scenario in Loop Quantum Cosmology*, JCAP 03 (2013) 026 [arXiv:1211.6269 [INSPIRE]].

[17] T. Cailleteau, A. Barrau, J. Grain and F. Vidotto, *Consistency of holonomy-corrected scalar, vector and tensor perturbations in Loop Quantum Cosmology*, Phys. Rev. D 86 (2012) 087301 [arXiv:1206.6736 [INSPIRE]].

[18] M. Bojowald and G.M. Hossain, *Loop quantum gravity corrections to gravitational wave dispersion*, Phys. Rev. D 77 (2008) 023508 [arXiv:0709.2365 [INSPIRE]].

[19] L. Linsefors, T. Cailleteau, A. Barrau and J. Grain, *Primordial tensor power spectrum in holonomy corrected \( \Omega \) loop quantum cosmology*, Phys. Rev. D 87 (2013) 107503 [arXiv:1212.2852 [INSPIRE]].

[20] E.J. Copeland, D.J. Mulryne, N.J. Nunes and M. Shaeri, *The gravitational wave background from super-inflation in Loop Quantum Cosmology*, Phys. Rev. D 79 (2009) 023508 [arXiv:0810.0104 [INSPIRE]].

[21] J. Grain and A. Barrau, *Cosmological footprints of loop quantum gravity*, Phys. Rev. Lett. 102 (2009) 081301 [arXiv:0902.0145 [INSPIRE]].

[22] J. Mielczarek, T. Cailleteau, J. Grain and A. Barrau, *Inflation in loop quantum cosmology: dynamics and spectrum of gravitational waves*, Phys. Rev. D 81 (2010) 104049 [arXiv:1003.4660 [INSPIRE]].

[23] J. Mielczarek, *Gravitational waves from the Big Bounce*, JCAP 11 (2008) 011 [arXiv:0807.0712 [INSPIRE]].

[24] M. Artymowski, Z. Lalak and L. Szulc, *Loop Quantum Cosmology: holonomy corrections to inflationary models*, JCAP 01 (2009) 004 [arXiv:0807.0160 [INSPIRE]].
[25] I. Agullo, A. Ashtekar and W. Nelson, A Quantum Gravity Extension of the Inflationary Scenario, Phys. Rev. Lett. 109 (2012) 251301 [arXiv:1209.1609] [inSPIRE].

[26] I. Agullo, A. Ashtekar and W. Nelson, Extension of the quantum theory of cosmological perturbations to the Planck era, Phys. Rev. D 87 (2013) 043507 [arXiv:1211.1354] [inSPIRE].

[27] I. Agullo, A. Ashtekar and W. Nelson, The pre-inflationary dynamics of loop quantum cosmology: Confronting quantum gravity with observations, Class. Quant. Grav. 30 (2013) 085014 [arXiv:1302.0254] [inSPIRE].

[28] J. Mielczarek, T. Cailleteau, A. Barrau and J. Grain, Anomaly-free vector perturbations with holonomy corrections in loop quantum cosmology, Class. Quant. Grav. 29 (2012) 085009 [arXiv:1106.3744] [inSPIRE].

[29] J. Mielczarek, Asymptotic silence in loop quantum cosmology, AIP Conf. Proc. 1514 (2012) 81 [arXiv:1212.3527] [inSPIRE].

[30] M. Bojowald and G.M. Paily, Deformed General Relativity and Effective Actions from Loop Quantum Gravity, Phys. Rev. D 86 (2012) 104018 [arXiv:1112.1899] [inSPIRE].

[31] J. Mielczarek, Signature change in loop quantum cosmology, arXiv:1207.4657 [inSPIRE].

[32] R.M. Wald, Quantum field theory in curved spacetime and black hole dynamics, The University of Chicago Press, Chicago (1994).

[33] L.H. Ford, Quantum field theory in curved space-time, in Particles and Fields: Proceedings of the XI Jorge Andre Swieca Summer School, Campos dos Jordao, SP, Brazil, February 1997, pg. 345 [gr-qc/9707062] [inSPIRE].

[34] A. Ashtekar, W. Kaminski and J. Lewandowski, Quantum field theory on a cosmological, quantum space-time, Phys. Rev. D 79 (2009) 064030 [arXiv:0901.0933] [inSPIRE].

[35] PLANCK collaboration, P.A.R. Ade et al., Planck 2013 results. XXII. Constraints on inflation, arXiv:1303.5082 [inSPIRE].

[36] SPTpol collaboration, D. Hanson et al., Detection of B-mode Polarization in the Cosmic Microwave Background with Data from the South Pole Telescope, Phys. Rev. Lett. 111 (2013) 141301 [arXiv:1307.5830] [inSPIRE].

[37] A. Ijjas, P.J. Steinhardt and A. Loeb, Inflationary paradigm in trouble after PlanckK2013, Phys. Lett. B 723 (2013) 261 [arXiv:1304.2785] [inSPIRE].

[38] J. Mielczarek, L. Linsefors and A. Barrau, Silent initial conditions in loop quantum cosmology, in preparation.

[39] T. Cailleteau, L. Linsefors and A. Barrau, Anomaly-free perturbations with inverse-volume and holonomy corrections in Loop Quantum Cosmology, arXiv:1307.5238 [inSPIRE].