Quantum Contribution to Magnetotransport in Weak Magnetic Fields and Negative Longitudinal Magnetoresistance

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Longitudinal magnetoresistance (LMR) refers to the change in resistance due to a magnetic field when the current and the magnetic field are parallel to each other. For this to be nonzero in weak magnetic fields, kinetic theory stipulates that the electronic dispersion must satisfy certain conditions: it should either be sufficiently anisotropic or have topological features. The former results in a positive LMR while the latter results in a negative LMR. Here, I propose a different mechanism that leads to LMR in any dispersion without a need to satisfy the above requirements. The mechanism is quantum in origin but is applicable in the said regime. It arises due to the change in the density of states with the magnetic field and is not kinetic in origin. Remarkably, LMR is found to be negative even if the dispersion is non-topological, provided it is non-parabolic. An analytical expression is derived for this novel contribution to LMR. It is found to depend on the orbital magnetic susceptibility. The analytical findings are confirmed by numerical calculations.

I. INTRODUCTION

Magnetotransport—the motion of charge carriers in the presence of both an electric and a magnetic field—is one of the most commonly studied phenomena in solids. As a charge accelerates under the electric field, it suffers repeated collisions with scatterers, giving rise to resistance. With the introduction of the magnetic field, the charge now experiences an additional Lorentz force and bends away from its linear path. It spends more time in traversing the direction of the electric field and suffers more collisions resulting in an increased resistance. Thus, magnetoresistance is expected to be positive and arise only when the current and the magnetic field have components perpendicular to each other [1].

Based on the above kinetic picture which is essentially classical, one does not expect longitudinal magnetoresistance (LMR) to exist because it requires the current to be parallel to the magnetic field; and even if it exists, it should not be negative (N). Nevertheless, in solids where electrons do not have a free-particle dispersion, LMR—and in some cases NLMR—can arise within the same basic mechanism at weak fields where kinetic theory is valid. It has been shown that if the dispersion possesses a certain kind of anisotropy such that the velocity of electrons parallel and perpendicular to the magnetic field can not be decoupled, LMR can be nonzero and is necessarily positive [2,11]. On the other hand, if the dispersion features topological properties, along with the Lorentz force the kinetics is influenced by an additional contribution that arises from the Berry curvature. Then anisotropy is not a necessity and a nonzero LMR can arise. In this case, however, LMR is negative [5,9]. Apart from these two mechanisms, others that lead to LMR—and in some cases NLMR—are known to exist. However, these are either extrinsic in origin, such as scenarios that require specific models of scattering [11,14] and inhomogeneities [12,13], or are beyond the semiclassical regime requiring, for example, a very high magnetic field which forces electrons to occupy only the lowest Landau level [15] or low enough temperatures such that quantum interference effects lead to weak (anti)localization [16].

In this work I show that there exists another mechanism, intrinsic in origin and applicable in weak fields but not kinetic in nature, that contributes to magnetotransport. This gives rise to a nonzero value of LMR in cases where kinetic theory predicts a zero value, which can even become negative. The mechanism derives from the change of density of states due to the magnetic field and is quantum in origin in spite of appearing in a classical regime. A simple understanding can be obtained by considering a familiar context in which the same mechanism is at play: Landau diamagnetism. It is well known that classically Landau diamagnetism cannot arise since the magnetic field through its kinetic contribution can not affect the total energy of a system. Quantum mechanically, however, it is allowed since the density of states becomes a function of the magnetic field through the formation of discrete Landau levels. Note that, in spite of being quantum in origin, the effect manifests at weak fields such that \( \omega_c \ll E_F \), where \( \omega_c \) is the cyclotron frequency and \( E_F \) is the Fermi energy (\( \hbar = 1 \)). Extending this mechanism to transport, magnetoresistance should also inherit a similar contribution, irrespective of the orientation of the current and the magnetic field. Importantly, the orbital magnetic susceptibility, while being diamagnetic for a parabolic dispersion (Landau diamagnetism), becomes paramagnetic when nonparabolicity is introduced in the dispersion [17]. The same can arise in the context of magnetotransport with magnetoresistance switching sign from positive to negative as the dispersion acquires nonparabolicity. The different physical origins of the two contributions to magnetoresistance, kinetic and quantum, are expected to show up in their functional dependence on the magnetic field: the former is expected to be a function of \( \omega_c \tau \), where \( \tau \) is the relevant
scattering time, whereas the latter should be a function of \( \omega_c \). Because \( \omega_c \tau \) can reach values much larger than one while satisfying \( \omega_c \ll E_F \), in general the kinetic contribution will dominate over the quantum contribution. However, if the former is identically zero, the latter can become the leading contribution. As discussed earlier, this could happen with LMR making the new quantum contribution relevant in this context.

Below I substantiate the above claims with analytical and numerical calculations. LMR is calculated for a dispersion that is separable in directions parallel and perpendicular to the magnetic field using the Kubo formula. The choice of such a dispersion is not necessary, but is done for two reasons: first, it greatly simplifies the calculation, and second, it is known that the kinetic contribution to LMR for such a choice is identically zero [2]; therefore, any LMR found is necessarily of quantum origin. A general expression for the quantum contribution is derived which is found to be intimately related to the orbital magnetic susceptibility. It is explicitly shown that, in contrast to conventional wisdom, even a parabolic dispersion exhibits LMR, which becomes negative as non-parabolicity is introduced in the dispersion.

II. MODEL

Consider a metallic system with a dispersion

\[
E(k) = \varepsilon_{xy}(k_x, k_y) + \varepsilon_z(k_z).
\]  

Without any loss of generality it is assumed that the minimum value of each term is zero. A magnetic field \( \mathbf{B} \), described by the vector potential \( \mathbf{A} = (0, Bz, 0) \), is applied in the \( z \)-direction. The dispersion becomes (spin is ignored for simplicity)

\[
E(n, k_z) = \varepsilon_{xy}(n) + \varepsilon_z(k_z),
\]  

where \( \varepsilon_{xy}(n) \) denotes the Landau levels in two dimensions. The eigenfunctions are given by

\[
\psi_{k_z, k_y, n} = e^{ik_z z + ik_y y} \phi_n(x - k_y l_B^2),
\]  

where \( \phi_n \) are the Landau levels eigenfunctions corresponding to \( \varepsilon_{xy}(n) \) and \( l_B^2 = \frac{\hbar^2}{eB} \). The corresponding single particle Green’s function is

\[
G(k_z, k_y, x, x', \omega) = \sum_n \frac{\phi_n^*(x - k_y l_B^2) \phi_n(x' - k_y l_B^2)}{\omega - \xi_n(k_z) + \frac{\gamma}{2} \text{sgn}(\omega)},
\]  

where \( \xi_n(k_z) = E(n, k_z) - E_F \). Here, I have included a phenomenological scattering time \( \tau \) without worrying about its microscopic origin and assumed it to be field independent. This will be revisited later. Throughout this work, it will be assumed that scattering is weak so that \( 1/\tau \rightarrow 0 \).

III. LONGITUDINAL MAGNETOCO nductivity

In calculating the longitudinal magnetoconductivity \( \sigma_{zz} \), I closely follow Abrikosov [18] who first calculated the same for a parabolic dispersion. Using the Kubo formula,

\[
\sigma_{zz}(B) = \text{Re} \frac{e^2}{(2\pi)^3} \int d\omega \frac{n_F(\omega) - n_F(\omega + \Omega)}{\Omega} \int dk_z dk_y dx' v_z(k_z)G^R(k_z, k_y, x, x', \omega)v_z(k_z)G^A(k_z, k_y, x, x, \omega + \Omega)\]  

Here, \( n_F \) is the Fermi function, \( v_z = \frac{\partial E}{\partial k_z} = \frac{\partial \varepsilon_z}{\partial k_z} \), \( G^R(A) \) is the retarded (advanced) Green’s function corresponding to Eq. [4], and \( \Omega \) is the external frequency. At \( T = 0, \Omega = 0 \), the frequency integral pins all energies on the Fermi surface. Using Eq. [6] in Eq. [5], I have

\[
\sigma_{zz}(B) = \frac{e^2}{(2\pi)^3} \sum_{n,n'} \int dk_z dk_y dx' \frac{v_z^2(k_z)}{\xi^2_z(k_z) + \frac{1}{\tau^2}} \\
\phi_n^*(x - k_y l_B^2) \phi_n(x' - k_y l_B^2) \phi_n^*(x' - k_y l_B^2) \phi_n(x - k_y l_B^2).
\]  

Using the fact that the Landau level eigenfunctions form an orthonormal complete basis, the integral over \( x' \) gives
\[ \sum_{\nu} \sigma_{\nu}(B) = \frac{e^2}{(2\pi)^3} \int \frac{dk_z}{\xi(k_z) + 4\tau} \]  

Next, I make a change of variable: \[ \int dk_z \rightarrow \int \frac{\xi(k_z)}{\xi(k_z) + 4\tau} dk_z, \] where the factor 2 is included since \( E \) is an even function of \( k_z \) (see comment [19]). In the limit \( \frac{1}{\tau} \rightarrow 0, \] \( \frac{\xi(k_z)}{\xi(k_z) + 4\tau} \rightarrow 2\pi\delta(\xi_n). \) Thus, \[ \sigma_{zz}(B) = \frac{e^2 \tau B}{2\pi^2} \sum_n |v_{zn}|, \] where \[ v_{zn} = \left. \frac{\partial \xi_z(k_z)}{\partial k_z} \right|_{k_z = \xi_z^{-1}(E_F - \varepsilon_{xy}(n)) > 0}. \] The summation over \( n \) runs from 0 to \( N \), the maximum value of \( n \) for which \( \varepsilon_{xy}(n) \leq E_F \). Equation (7) has a simple interpretation. The magnetic field has reduced the three-dimensional spectrum into a set of one-dimensional bands dispersing along \( k_z \), each with a degeneracy proportional to \( B \). The total conductivity is the sum of the velocity in the \( z \)-direction at the Fermi energy contributed by all the partially occupied bands. As shown in Fig. 1, the number of such bands is given simply by the number of bands \( E_F \) crosses—this corresponds to \( N \). At very high magnetic fields, only the lowest band is occupied \( (N = 1) \) which contributes to transport. This is a purely quantum regime. As the magnetic field decreases, more bands get populated by going below the Fermi level. When the number of occupied levels is large \( (N \gg 1) \) one enters the semiclassical regime. In this regime, with change in \( B \), the sum in Eq. (7) changes in two ways: a part that evolves smoothly and another that changes abruptly due to a sudden change from \( N \) to \( N + 1 \) each time an extra band gets populated. Together, they give rise to LMR, the former appearing as a smooth background while the latter manifesting as quantum oscillations. Quantum oscillations are vestigial signatures of quantum effects in the semiclassical regime. The fully quantum regime along with quantum oscillations in the semiclassical regime have been extensively studied before by Arbrikosov [18] and others [15, 21] as manifestations of quantum effects. However, the smooth background is considered to be purely classical, described by the kinetic theory, devoid of any quantum effects. Below, through explicit calculations I show that this is not correct: the smooth background contribution to LMR also inherits an intrinsic quantum contribution, hitherto unexplored, with novel consequences.

As a simple example consider first a parabolic spectrum: \( \varepsilon_{xy}(k_x, k_y) = \frac{k_x^2 + k_y^2}{2m} \) giving \( \varepsilon_{xy}(n) = (n + \frac{1}{2})\omega_c \), where \( \omega_c = \frac{eB}{m} \) and \( \varepsilon_z(k_z) = \frac{k_z^2}{2m} \). Then, \[ v_{zn} = \sqrt{\frac{2}{m} E_F - (n + \frac{1}{2}) \omega_c}. \] The summation over \( n \) in Eq. (7) can be converted into an integral by using the Euler-Maclaurin formula (see Supplemental Material). Ignoring the oscillating part and keeping only the smooth part up to \( O(B^2) \), I find \[ \sigma_{zz}(B) \approx \sigma_{zz}(0) \left[ 1 - \frac{1}{32} \frac{\omega_c^2}{E_F^2} \right], \] where \( \sigma_{zz}(0) = n_0 e^2, n_0 = \frac{(2mE_F)^{2/3}}{6\pi^2} \) being the zero-field charge density. Thus, even for a parabolic spectrum the longitudinal conductivity is magnetic field dependent. This should be contrasted with the kinetic theory result which predicts absence of any field-dependence. The field-dependent part scales with \( \omega_c^2/\tau \) instead of \( \omega_c/\tau \), confirming its quantum origin. The negative sign implies that LMR, obtained by taking the inverse, is positive. However, as \( E_F \) is increased, the spectrum becomes nonparabolic and \( \alpha \) becomes negative—in this regime, LMR is negative. I now generalize the above idea to a general spectrum. The Landau levels \( \varepsilon_{xy}(n) \) no longer have a simple analytical form. They are, instead, derived from the semiclassical quantization condition, \[ S_{Lz}^2 = 2\pi(n + \gamma), \] where \( S(\varepsilon) \) is the area of the surface enclosed by the isoenergy contour \( \varepsilon_{xy} = \varepsilon \) in the two-dimensional \( k \)-space and \( \gamma \) is the semiclassical phase. It is easy to check that for a parabolic dispersion, \( S(\varepsilon) = \pi(k_x^2 + k_y^2)|\varepsilon_{xy} = \varepsilon| = 2\pi m\varepsilon \) and \( \gamma = \frac{1}{2} \) reproduce the correct Landau level spectrum \( \varepsilon(n) = (n + \frac{1}{2})\omega_c \). When the dispersion is non-parabolic, two changes arise: \( S(\varepsilon) \) is no longer the area of a circle and, more importantly, \( \gamma \) is no longer a constant but a function of \( \varepsilon \) itself. While \( S(\varepsilon) \) is a simple geometrical quantity, calculation of \( \gamma(\varepsilon) \) requires more care. In the simplest case where singularities in the
is energy contours and interband effects can be ignored, it was shown by Roth [20, 21]

\[ \gamma(\varepsilon) = \frac{1}{2} \frac{eB}{48\pi} \frac{\partial}{\partial \varepsilon} \int \delta(\varepsilon_{xy} - \varepsilon) \left[ m_{xx}^{-1} m_{yy}^{-1} - (m_{xy}^{-1})^2 \right] d^2k, \]  

(11)

where \( m_{\alpha\beta}^{-1} = \frac{\partial^2 \varepsilon_{xy}}{\partial k_{\alpha} \partial k_{\beta}} \). This can be written in the terms of the two-dimensional orbital magnetic susceptibility \( \chi \). According to the Landau-Peierl's formula [20],

\[ \chi(\varepsilon) = -\frac{e^2}{24\pi^2} \int \delta(\varepsilon_{xy} - \varepsilon) \left[ m_{xx}^{-1} m_{yy}^{-1} - (m_{xy}^{-1})^2 \right] d^2k. \]  

(12)

Combining the two,

\[ \gamma(\varepsilon) = \frac{1}{2} \frac{eB}{26\pi} \frac{\partial \chi}{\partial \varepsilon}. \]  

(13)

Going back to Eq. (7), the sum is once again computed using the Euler-Maclaurin formula, but keeping in mind that now a change in \( n \) is accompanied by changes in both \( S \) and \( \gamma \) [21]. Ignoring the oscillating part and keeping only the smooth part up to \( O(B^2) \) as before, I find

\[ \sigma_{zz}(B) \approx \sigma_{zz}(0) - \frac{\partial |v_x|}{\partial \varepsilon} \chi \bigg|_{\varepsilon=0} + \frac{1}{6} \int \frac{\partial |v_z|}{\partial \varepsilon} \frac{\partial \chi}{\partial \varepsilon} d\varepsilon \] 

\[ = \sigma_{zz}(0) - \alpha B^2. \]  

(14)

Here, \( \frac{\partial |v_x|}{\partial \varepsilon} \bigg|_{\varepsilon=\varepsilon_x} = \frac{\partial |v_y|}{\partial \varepsilon} \bigg|_{\varepsilon=\varepsilon_y} \) where \( v_x \) is evaluated from Eq. (2) and expressed in terms of \( \varepsilon_{xy} \) [similar to Eq. (9) but now in \((k_x, k_y)\) space]. The expression for LMR is obtained by inverting Eq. (14): \( \rho_{zz}(B) \approx \rho_{zz}(0) + \alpha B^2 \), where \( \rho_{zz} = \frac{1}{\sigma_{zz}} \). Equation (14) clearly shows that the quantum contribution to LMR in a three-dimensional system is intimately related to the orbital magnetic susceptibility of the corresponding two-dimensional spectrum, confirming their common origin.

A remarkable feature of Eq. (14) is that the two terms constituting the coefficient \( \alpha \) need not be of the same sign; therefore, \( \alpha \) can pick a sign depending on which term wins. In the parabolic case, \( v_z = \frac{k^2}{m} = \sqrt{\frac{2}{m} E_F - \varepsilon} \) and \( \chi = -\frac{e^2}{12 \pi m} \) [from Eq. (12)]. The latter is independent of energy, so the second term constituting \( \alpha \) drops out and the expression in Eq. (9) is recovered with \( \alpha \) positive. However, once the dispersions becomes nonparabolic, the second term becomes nonzero and opposite in sign to the first term. For a sufficiently nonparabolic dispersion, \( \alpha \) becomes negative resulting in NLMR. Note that for this to happen, it is sufficient to have only the two-dimensional spectrum \( \varepsilon_{xy} \) nonparabolic, the dispersion along the magnetic field, \( \varepsilon_z \) can still be parabolic. To illustrate this, consider the spectrum \( \varepsilon_{xy} = 4t - 2t |\cos(k_x a) + \cos(k_y a)| \) and \( \varepsilon_z = \frac{k^2}{8m} \), where \( t \) is the nearest neighbor hopping parameter on a square lattice of lattice constant \( a \). Using Eq. (12), one finds

\[ \chi(\varepsilon) = \frac{e^2}{12 \pi a^2} \frac{Q_1}{2} \cdot \frac{1}{8} \left( 1 - \left( \frac{e^{-4}}{8} \right) \right), \]

where \( Q_1 \) is the Legendre function of the second kind and \( \varepsilon \) is in units of \( t \). Using this in Eq. (14), the integral is calculated to compute \( \alpha \). In Fig. (2) the dependence of \( \alpha \) on \( E_F \) is plotted. At small \( E_F \), the spectrum is close to parabolic, and \( \alpha \) is positive. With increase in \( E_F \), nonparabolicity becomes more pronounced and at some value \( \alpha \) switches sign and becomes negative, resulting in NLMR. Equation (14) along with its consequences form the main result of this paper.

IV. NUMERICAL CALCULATION

As further proof I now present an exact numerical evaluation of Eq. (7), which is then compared with the analytical result in Eq. (14). The Landau level spectrum \( \varepsilon_{xy}(n) \) corresponding to \( \varepsilon_{xy}(k_x, k_y) = 4t - 2t |\cos(k_x a) + \cos(k_y a)| \) is calculated numerically on a lattice model (see Supplemental Material for details). Using Eq. (8), \( v_z = \sqrt{\frac{2}{m} E_F - \varepsilon_{xy}(n)} \) is computed. This is inserted in Eq. (7) and the sum is evaluated numerically as a function of the field. This yields the total \( \sigma_{zz}(B) \) which includes both the smooth as well as the oscillating parts. To remove the oscillating part, a small temperature is introduced. Temperature influences the two contributions differently: it introduces a negligible correction \( \sim \left( \frac{T}{E_F} \right)^2 \) (Sommerfeld correction) in the smooth part, but reduces the oscillating part exponentially as \( \sim e^{-T/\omega_c} \) for \( T \gg \omega_c \). This is exploited to suppress the oscillating part and reveal the smooth part of \( \sigma_{zz}(B) \). Note that, this is not just a theoretical trick, but also has experimental relevance: to observe the predicted behavior in the smooth part of LMR, one needs to be in the regime \( \omega_c \lesssim T \ll E_F \). The
The effect of temperature is included by using the formula

\[ \sigma_{zz}(E_F, T) = \int \left( -\frac{\partial n_F(E-E_F)}{\partial E} \right) \sigma_{zz}(E, 0) dE. \]

The results are presented in Fig. 3. As expected, \( \sigma_{zz}(B) \) varies quadratically with the field. It is found so far that this is not the case. This can be shown explicitly by considering a simple model where delta-function impurities are scattered randomly in a system with a parabolic spectrum. Assuming weak and dilute impurities, within the first Born approximation one finds (see Supplemental Material)

\[ \tau^{-1} = n_i \int \frac{dE}{\pi} \sum_{n=0}^{N} \frac{1}{|v_{zn}|^2}, \]

where \( n_i \) is the density of impurities. Inserting this in Eq. (7) it is clear that a cancellation does not occur. Carrying out the summation over the Landau levels as before (see Supplemental Material), I find \( \tau \approx \tau_0 \left[ 1 - \frac{\omega^2}{\omega_0^2} \right] \), where \( \tau_0 \) is the scattering time in the absence of the field. Using this in Eq. (9), I get \( \sigma_{zz}(B) \approx \sigma_{zz}(0) \left[ 1 - \frac{1}{24 F^2} \right] \). The field dependence in \( \tau \), instead of destroying LMR, accentuates it.

VI. CONCLUDING REMARKS

To summarize, I have shown that a nonzero LMR can arise in any dispersion in weak magnetic fields, in contrast to the prediction of kinetic theory which states that LMR is nonzero only for dispersions of certain kinds. This arises because a magnetic field affects electronic transport not only kinetically, but also by modifying the density of states. The mechanism is inherently quantum in spite of manifesting in the classically weak-field regime. Importantly, the quantum contribution to LMR can become negative if the dispersion is sufficiently non-parabolic, even if the latter has no topological features. It is found that it is related to the orbital magnetic susceptibility. While the theory presented here considered the simplest case of a single isolated band, it can be extended to include coupled bands. Such extensions are important in the context of topological systems and will be investigated in future.

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SUPPLEMENTAL MATERIAL

A. Calculation of $\sigma_{zz}$ for a parabolic spectrum

1. Constant $\tau$

For the separable energy spectrum in a magnetic field,

$$E(n, k_z) = \varepsilon_{xy}(n) + \varepsilon_z(k_z),$$

(15)

the longitudinal magnetoconductivity is given by

$$\sigma_{zz}(B) = \frac{e^2 \tau eB}{2\pi^2} \sum_n |v_{zn}|,$$

(16)

where

$$v_{zn} = \frac{\partial \varepsilon_z(k_z)}{\partial k_z} \bigg|_{k_z \to k_z + \varepsilon_{xy}(n)}.$$  

(17)

I first assume the dispersion to be parabolic: $\varepsilon_{xy}(n) = (n + 1/2)\omega_c$ and $\varepsilon_z(k_z) = k_z^2/2m$, with $\omega_c = eB/m$. This gives $v_{zn} = \sqrt{2/m} \sqrt{E_F - (n + 1/2)\omega_c}$, and Eq. (16) becomes

$$\sigma_{zz}(B) = \frac{e^2 \tau \omega_c}{2\pi^2 \sqrt{2m}} \sum_{n=0}^N \sqrt{E_F - (n + 1/2)\omega_c}.$$  

(18)

To calculate the discrete sum, I use the Euler-Maclaurin formula

$$\sum_{r=0}^R f(r) = \int_0^R f(r)dr + \frac{1}{2} f(0) + \frac{1}{12} [f(R) + f(0)] + \cdots.$$  

(19)

For convenience, define $n + 1/2 = x$ and $X = E_F/\omega_c$, i.e., the value $x$ takes at $E_F$, and $\delta = X - (N + 1/2)$. Then,

$$\sum_{n=0}^N \sqrt{E_F - (n + 1/2)\omega_c} \omega_c = \int_{X-\delta}^{X+\delta} \sqrt{E_F - x\omega_c} dx + \frac{1}{2} \left[ \sqrt{E_F - (X-\delta)\omega_c} + \sqrt{E_F - (X+\delta)\omega_c} \right]$$

$$+ \frac{1}{12} \left[ \left( \sqrt{E_F - x\omega_c} \right)'_{x=X-\delta} - \left( \sqrt{E_F - x\omega_c} \right)'_{x=X+\delta/2} \right] + \cdots$$

$$= -\frac{2}{3\omega_c} \left[ (\delta\omega_c)^{3/2} - (E_F - \omega_c/2)^{3/2} \right] + \frac{1}{2} \left[ (\delta\omega_c)^{1/2} + (E_F - \omega_c/2)^{1/2} \right]$$

$$- \frac{\omega_c}{24} \left[ (\delta\omega_c)^{1/2} - (E_F - \omega_c/2)^{1/2} \right] + \cdots.$$  

(20)

Expanding in $\omega_c/E_F$,

$$\sum_{n=0}^N \sqrt{E_F - (n + 1/2)\omega_c} \omega_c \approx \left[ \frac{2 E_F^{3/2}}{3 \omega_c} \right] + \left[ -\frac{1}{48} \omega_c \frac{E_F^{3/2}}{E_F^{1/2}} \right] + \left[ \frac{\omega_c^{1/2}}{\delta^{1/2}} \left( -\frac{3}{4} \delta^2 + \frac{1}{2} \delta - \frac{1}{24} \right) \right].$$  

(21)

Using this in Eq. (18), I finally have

$$\sigma_{zz}(B) \approx \frac{e^2 \tau 2^{3/2} m^{1/2}}{3 \pi^2} E_F^{3/2} \left[ 1 - \frac{1}{32} \frac{\omega_c^2}{E_F^{3/2}} - \omega_c^{3/2} \frac{1}{E_F^{3/2} \delta^{1/2}} \left( \delta^2 - \frac{3}{4} \delta + \frac{1}{16} \right) \right].$$  

(22)

The first term is the zero-field contribution, the second is the smooth part that varies quadratically with the field, and the third leads to oscillations. I discard the oscillating part and take the first two terms, which appear in the main text.
In the previous calculation $\tau$ was considered to be some phenomenological constant. Here, I derive it in the simple case of weak and dilute disorder with delta-function impurities scattered randomly. The self-energy $\Sigma$ considering the simplest diagram within the first Born approximation (Fig. 4) is given by

$$\Sigma = n_l U_0^2 \frac{1}{2(2\pi)^2} \sum_{n=0}^{N} \int \frac{\phi_n^*(x - k_y l^2_B) \phi_n(x - k_y l^2_B)}{\omega - \xi_n(k_z) + i\eta \text{sgn}(\omega)} dk_y dk_z,$$

where $U_0$ is the Born scattering amplitude and $n_l$ is the density of impurities. They are, instead, derived from the semiclassical quantization condition leads to oscillations. I discard the oscillating part and take the first two terms, which appear in the main text. The first term is the zero field contribution, the second is the smooth contribution quadratic in field, and the third

$$\text{Expanding in } \omega \text{, I finally have}$$

$$\Sigma = -i\eta \text{sgn}(\omega)n_l U_0^2 \frac{eB}{2\pi} \sum_{n=0}^{N} \frac{1}{|v_{zn}|},$$

which is quoted in the main text. The discrete sum is evaluated as before:

$$\sum_{n=0}^{N} \frac{1}{\sqrt{E_F - (n + 1/2)|\omega_c|}} = \sum_{n=0}^{N} \frac{1}{\sqrt{E_F - (n + 1/2)|\omega_c|}} = \begin{array}{c}
\int_{1/2}^{X-\delta} \frac{1}{\sqrt{E_F - x\omega_c}} dx + \frac{1}{2} \left[ \frac{1}{\sqrt{E_F - (X-\delta)|\omega_c|}} + \frac{1}{\sqrt{E_F - \omega_c/2}} \right] \\
+ \frac{1}{12} \left[ \left( \frac{1}{\sqrt{E_F - \omega_c}} \right)_{x \to X-\delta} - \left( \frac{1}{\sqrt{E_F - 3\omega_c}} \right)_{x \to 1/2} \right] + \cdots
\end{array} \begin{array}{c}
= -\frac{2}{\omega_c} \left( \delta\omega_c \right)^{1/2} (E_F - \omega_c/2)^{1/2} + \frac{1}{2} \left[ \frac{1}{(\delta\omega_c)^{1/2}} + \frac{1}{(E_F - \omega_c/2)^{1/2}} \right] \\
+ \frac{\omega_c}{24} \left( \delta\omega_c)^{3/2} - \frac{1}{(E_F - \omega_c/2)^{3/2}} \right) + \cdots
\end{array}$$

Expanding in $\omega_c/E_F$,

$$\sum_{n=0}^{N} \frac{1}{\sqrt{E_F - (n + 1/2)|\omega_c|}} \approx \left[ 2 E_F^{1/2}/\omega_c \right] + \left[ \frac{1}{48} E_F^{3/2}/\omega_c \right] + \left[ \omega_c^{-1/2} \delta^{-3/2} \left( -2\delta^2 + \frac{1}{2} \delta + \frac{1}{24} \right) \right].$$

Inserting this in Eq. (25), I finally have

$$\tau^{-1} \approx \frac{n_l U_0^2}{2\pi} \frac{1}{2^{1/2} m^{3/2} E_F^{1/2}} \left[ 1 + \frac{1}{96} E_F - \frac{\omega_c^{1/2}}{E_F^{1/2} \delta^{3/2}} \left( \delta^2 - \frac{1}{4} \delta - \frac{1}{48} \right) \right].$$

The first term is the zero field contribution, the second is the smooth contribution quadratic in field, and the third leads to oscillations. I discard the oscillating part and take the first two terms, which appear in the main text.

### B. Calculation of $\sigma_{zz}$ for a general spectrum

I now calculate $\sigma_{zz}(B)$ for an arbitrary choice of $\varepsilon_{xy}$ and $\varepsilon_z$. The Landau levels $\varepsilon_{xy}(n)$ no longer have a simple analytical form. They are, instead, derived from the semiclassical quantization condition

$$S_{E_B}^2 = 2\pi(n + \gamma),$$

(29)
where $S(\varepsilon)$ is the area of the surface enclosed by the isoenergy contour $\varepsilon_{xy} = \varepsilon$ in the two-dimensional $k$–space, $l_B = \frac{\sqrt{\pi \hbar}}{eB}$ is the magnetic length, and $\gamma$ is the semiclassical phase. In the general case, when the dispersion is non-parabolic, two changes arise: $S$ is no longer the area of a circle and, more importantly, $\gamma$ is no longer a constant but a function of $\varepsilon$ itself. In the simplest case where singularities in the isoenergy contours and interband effects can be ignored, it can be shown that

$$
\gamma(\varepsilon) = \frac{1}{2} = -\frac{\pi B}{2e} \frac{\partial \chi}{\partial \varepsilon},
$$

(30)

where $\chi$ is the two-dimensional orbital magnetic susceptibility corresponding to $\varepsilon_{xy} = \varepsilon$. Here, it has been assumed that the band-bottom is parabolic so that $\gamma_0 = \gamma(0) = 1/2$. To compute the discrete sum in Eq. (15) with the help of Euler-Maclaurin formula as before, once again define $n + \gamma = x$ with $X$ as the value $x$ takes at $E_F$, and $\delta = X - (N + \gamma)$. The key point is, unlike in the parabolic case, now $\partial\varepsilon/\partial x$ may diverge, as happens for example in the parabolic case. This arise because the oscillating part need not be an analytic function of $\omega_c$ or $\delta$, as evidenced in Eq. (22). Since this is not the focus of the calculation and will be ignored anyway, I do not discuss it further. Next, I consider term T2. Integrating by parts and keeping terms up to linear order in $B$, I get

$$
T2 = -|v_z(E_F)| \left( \gamma(E_F) - \frac{1}{2} \right) + \int_{0}^{E_F} \frac{\partial |v_z|}{\partial \varepsilon} \left( \gamma(\varepsilon) - \frac{1}{2} \right) d\varepsilon.
$$

(33)
Indeed, carrying out the sum for $\Omega$, one finds
\[
\sum_{n=0}^{N} |v_{zn}| = \left[ \int_{0}^{X} |v_{z}(x)| dx \right] + \left[ \frac{\omega_{c}(0) \partial |v_{z}|}{2\pi} \right]_{0}^{E_{F}} + \left[ \int_{0}^{E_{F}} \frac{\partial |v_{z}|}{\partial \varepsilon} \left( \gamma(\varepsilon) - \frac{1}{2} \right) d\varepsilon \right].
\]
(34)

Both terms $S1$ and $S2$ can be related to the two-dimensional orbital magnetic susceptibility $\chi$. For term $S2$, it is obvious: using Eq. (30), $S2$ becomes $-\frac{eB}{2\pi} \int_{0}^{E_{F}} \frac{\partial |v_{z}|}{\partial \varepsilon} d\varepsilon$. For $S1$, recall the thermodynamic definition, $\chi = -\frac{\partial^{2}\Omega}{\partial B^{2}}$, where $\Omega$ is the grand potential defined at $T = 0$ as $\Omega = \frac{eB}{2\pi} \sum_{n=0}^{N} [\varepsilon_{xy}(n) - E_{F}]$. Comparing it with Eq. (16), it is obvious that the calculation of $\Omega$ is identical as $\sigma_{zz}(B)$ if one identifies $|v_{zn}| \to |E_{F} - \varepsilon_{xy}(n)|$ so that $\frac{\partial |v_{z}|}{\partial \varepsilon} \to -1$.

Indeed, carrying out the sum for $\Omega$, one finds $\chi = -\frac{e^{2}}{12\pi m}$ [21]. Thus, $S1$ is simply $-\frac{eB}{2\pi} \frac{\partial |v_{z}|}{\partial \varepsilon} \chi$, where $\chi$ is $\omega_{c}(0) = eB/m$. Rewriting $S1$ and $S2$ in terms of $\chi$ and plugging Eq. (34) back into Eq. (16), the final expression is
\[
\sigma_{zz}(B) \approx \sigma_{zz}(0) - \left[ \frac{\partial |v_{z}|}{\partial \varepsilon} \right]_{\varepsilon=0}^{E_{F}} + \int_{0}^{E_{F}} \frac{\partial |v_{z}|}{\partial \varepsilon} d\varepsilon \right].
\]
(35)

This expression is quoted in the main text.

C. Numerical calculation of Landau levels for a square lattice spectrum

To calculate $\sigma_{zz}(B)$ numerically using Eq. (16), one needs to calculate the Landau level spectrum $\varepsilon_{xy}(n)$ corresponding to $\varepsilon_{xy}(k_x, k_y)$ in Eq. (15). In the main text, I considered the dispersion $\varepsilon_{xy}(k_x, k_y) = 4t - 2t[\cos(k_x a) + \cos(k_y a)]$. The corresponding lattice Hamiltonian is that of a square lattice with nearest neighbor interaction:
\[
H = 4t \sum_{<i,j>} c_{i}^{\dagger} c_{j} + h.c.,
\]
(36)

with $t_{ij} = t$. Magnetic field is introduced via Peierls substitution for the hopping parameters as $t_{ij} = t_{e^{i\phi_{0}} A_{ij}}$, where $A$ is the magnetic vector potential, and $d\Omega$ denotes an infinitesimal line element from points $i$ to $j$ on the lattice. I use the gauge $A = (0, Bx, 0)$. Writing $x$ as $la$, where $l$ is an integer, the phase in the hopping parameter becomes $e^{iA_{ij}dy} = 2\pi l \phi / \phi_{0}$, with $\phi$ being the magnetic flux and $\phi_{0}$ being the flux quantum. It is seen that for $\phi / \phi_{0} = p/q$, where $p$ and $q$ are integers, a periodicity of $qa$ in the $x$-direction is restored. In my calculations I take $p = 1$. Going to the Fourier space, Eq. (36) can be cast in terms of a $q$-component basis $C = [c_{1}^{\dagger}, \cdots, c_{q}^{\dagger}]$ as
\[
-t_{k_x,k_y}^{n+1} e^{ik_{x}a} - t_{k_x,k_y}^{n-1} e^{-ik_{x}a} - 2tc_{k_x,k_y}^{n} [\cos(k_y b - 2\pi n\phi) + 4t], \quad n = 1, \cdots, q.
\]
(37)
Thus, Eq. (36) becomes
\[
H = \sum_{k} C_{k}^{\dagger} \mathcal{H}_{k} C_{k}
\]
(38)
with $\mathcal{H}$ a $q \times q$ matrix given by (37). The problem is thus reduced to an eigenvalue problem for a $q \times q$ matrix. Solving the eigenvalue problem numerically for $k_{x} = k_{y} = 0$, I get the discrete energy values for each value of $\phi = n/q$, $n = 1, \cdots, q$, which gives us the Landau level spectrum $\varepsilon_{xy}(n)$.