A QUANTITATIVE OPPENHEIM THEOREM FOR GENERIC TERNARY QUADRATIC FORMS

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Abstract. We prove a quantitative version of Oppenheim’s conjecture for generic ternary indefinite quadratic forms. Our results are inspired by and analogous to recent results for diagonal quadratic forms due to Bourgain [3].

1. Introduction

Let $Q$ be an indefinite quadratic form in $d \geq 3$ variables that is not a multiple of a quadratic form with rational coefficients. The Oppenheim conjecture, proved in a breakthrough paper by Margulis [11], states that $\{Q(n) : n \in \mathbb{Z}^d\}$ is dense in $\mathbb{R}$ (in fact, Oppenheim’s original conjecture is the weaker statement that the values, $Q(n)$, can be arbitrarily close to zero). Margulis’ approach used the dynamics of unipotent flows on homogeneous spaces and was not quantitative or effective, i.e., it did not give any bounds on the size of the integer vector $n$ needed to approximate a real number up to a given accuracy. One of the main difficulties of giving an effective proof is distinguishing between rational forms (for which the statement is false) and irrational forms that are very well approximated by rational forms. Recently, Bourgain investigated a quantitative version of the Oppenheim conjecture for generic diagonal forms. We follow his notation here. Margulis’ theorem implies that there are sequences $N(k) \to \infty$ and $\delta(k) \to 0$ (depending on the diophantine properties of $Q$) such that for all sufficiently large $k$,

$$\sup_{|\xi| \leq N(k)} \min_{\|n\| \leq k} |Q(n) - \xi| \leq \delta(k).$$

(1)

Earlier, in [10], Lindenstrauss and Margulis proved an effective version of Oppenheim’s conjecture, showing that for an irrational indefinite quadratic form in three variables, either it is very well approximated by rational forms, or (1) holds with $N(k)$ and $\delta(k)^{-1}$ growing logarithmically in $k$. One would of course expect that significantly stronger results would hold for generic forms. In [3], Bourgain considered diagonal forms $Q_{\alpha,\beta}(x, y, z) = x^2 + \alpha y^2 - \beta z^2$ with $\alpha, \beta > 0$.

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For these forms, he showed that (1) holds for almost all \( \alpha, \beta \in [\frac{1}{2}, 1] \), as long as \( \frac{N(k)\eta}{k^3\delta(k)\eta^{\frac{1}{2}}} \to 0 \) with \( \eta < 1 \). Moreover, assuming the Lindelöf hypothesis for the Riemann zeta function, (1) holds under the same assumptions for any fixed \( \beta > 0 \) and almost all \( \alpha \in \left[ \frac{1}{2}, 1 \right] \). Unconditionally, (1) holds for any fixed \( \beta > 0 \) and almost all \( \alpha \in \left[ \frac{1}{2}, 1 \right] \) under the stronger assumption that \( \frac{N(k)^3}{k^3\delta(k)\eta^{\frac{1}{2}}} \to 0 \) with \( \eta < 1 \).

In this note we consider the space of all indefinite ternary quadratic forms, not just diagonal, which comes equipped with a natural probability measure. In this space, we prove a quantitative Oppenheim theorem for generic indefinite ternary quadratic forms, similar in spirit to Bourgain’s result.

**Theorem 1.** Let \( N(k), \delta(k) \) be two sequences, with \( \delta(k) \to 0 \). Assume that there is \( \eta < 1 \) such that \( \max\{N(k), \delta(k)\} \to 0 \) as \( k \to \infty \). Then, for almost every indefinite ternary quadratic form \( Q \), there is a constant \( c = c(Q) > 0 \) and \( T_0 = T_0(Q) > 0 \) such that for all \( k \geq T_0 \),

\[
\sup_{|\xi| \leq N(k)} \min_{\|n\| \leq k} |Q(n) - \xi| \leq \delta(k).
\]

**Remark 2.** In fact, our proof implies the stronger result, where in (2) the minimum is taken over primitive integer vectors.

**Remark 3.** This result is more interesting when \( N(k) \to \infty \), in which case the term \( \max\{N(k), \delta(k)\} \) in the numerator can be replaced by \( N(k) \) and we can retrieve the condition from [3]. However, the result also holds when \( N(k) \to 0 \), in which case we really need to consider \( \max\{N(k), \delta(k)\} \) in case \( N(k)/\delta(k) \to 0 \).

We note that in this case, when \( N(k) \leq \delta(k) \), the conclusion (2) is equivalent (perhaps after replacing \( \delta(k) \) with \( 2\delta(k) \)) to the condition that \( \min_{\|n\| \leq k} |Q(n)| \leq \delta(k) \), which already follows from [8, Theorem 5], at least for \( \delta(k) = k^{-a} \) with \( a < 1 \). This consequence of the theorem also follows from the results in [7, 2], where more effective results for Oppenheim-type problems for generic forms are studied.

Our methods involve an effective mean ergodic theorem for semisimple groups and some results from the geometry of numbers. They are very different from those in [3]. In an earlier work, we studied the shrinking target problem for actions of semisimple groups on homogeneous spaces and as a consequence, obtained a similar result for approximating the single value \( \xi = 0 \). Such a single value result was first established in [7]. The main point of this work is to demonstrate that these methods are powerful enough to handle simultaneous approximation of all values \( \xi \) in a growing interval. Several important papers have previously studied the distribution of values of a quadratic form [4, 5, 14] and sometimes the word “quantitative” is used in the literature in this context.

2. Setup

2.1. Space of forms. In order to make the notion of a generic ternary quadratic form explicit we use the following parametrization for the space of forms. Recall

\[
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\]

\[
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\]
that the action of $G = \text{SL}_3(\mathbb{R})$ on ternary quadratic forms is given by

$$Q^g(v) = Q(\nu g),$$

with $g \in \text{SL}_3(\mathbb{R})$ acting on a row vector $v \in \mathbb{R}^3$ linearly from the right (we note that, in spite of the notation, the action on forms is a left action in the sense that $(Q^g)^f = Q^{g \cdot f}$). We say that two forms are equivalent if $Q_1 = \lambda Q_2^y$ with $\lambda \in \mathbb{R}$ and $\gamma \in \Gamma = \text{SL}_3(\mathbb{Z})$. Note that equivalent forms take the same values on $\mathbb{Z}^3$ after scaling by a constant. To avoid the scaling ambiguity, we restrict to forms of determinant one. To parametrize the space of determinant one forms up to equivalence, fix a form $Q_0(v)$ given by

$$Q_0(x, y, z) = z^2 - x^2 - y^2$$

and note that any determinant one indefinite ternary quadratic form is given by $Q = Q_0^g$ for some $g \in \text{SL}_3(\mathbb{R})$. Moreover, two such forms $Q_0^g, Q_0^{g'}$ are equivalent if and only if $g' = \gamma g h$ with $\gamma \in \Gamma$ and $h \in \text{SO}_3(\mathbb{R})$. We can thus parametrize the space of determinant one indefinite ternary quadratic forms up to equivalence by $\text{SO}_3(\mathbb{R})$ orbits in the space

$$X_3 = \text{SL}_3(\mathbb{Z}) \setminus \text{SL}_3(\mathbb{R}).$$

We recall that $X_3$ also parametrizes the space of unimodular lattices in $\mathbb{R}^3$, explicitly, a point $x = \Gamma g \in X_3$ corresponds to the lattice $\Lambda = \mathbb{Z}^3g$ and its orbit corresponds to the form $Q = Q_0^g$. The probability measure $\mu$ on $\text{SL}_3(\mathbb{R})$ gives us a natural way to characterize generic forms. Explicitly, we say that a property holds for almost all forms if it holds for $\mu$-almost every $\Gamma g \in X_3$.

### 2.2. The $\text{SL}_2(\mathbb{R})$ action

Let $H = \text{SL}_2(\mathbb{R})$, $G = \text{SL}_3(\mathbb{R})$, and $\Gamma = \text{SL}_3(\mathbb{Z})$. Let $Q_0$ be as in (3) and consider the double spin cover map $\iota: H \to \text{SO}_3(\mathbb{R})$ given by

$$\iota(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} \frac{a^2 + b^2 - c^2 - d^2}{2} & ac - bd & \frac{a^2 - b^2 + c^2 - d^2}{2} \\ \frac{ab - cd}{2} & bc + ad & \frac{ab + cd}{2} \\ \frac{a^2 + b^2 - c^2 - d^2}{2} & ac + bd & \frac{a^2 + b^2 + c^2 - d^2}{2} \end{pmatrix}.$$

This gives a right action of $H$ on $G$ and hence also on $X_3 = \Gamma \backslash G$.

We give a norm on $H$ using the spin cover map by defining

$$\|h\| := \|\iota(h)^{-1}\|_2,$$

where $\|\|_2$ denotes the the Hilbert-Schmidt norm on $G$ given by $\|g\|^2 = \text{tr}(g^t g)$. With this norm we consider the norm balls

$$H_t = \{ h \in H : \|h\| \leq t \}$$

and note for future reference (see [8, Lemma 9]) that for $t \geq 1$ we have

$$m(H_t) \approx t,$$

where $m$ denotes Haar measure for $H$ and we denote $F(t) \asymp G(t)$ if there is some constant $c > 1$ such that

$$c^{-1}F(t) \leq G(t) \leq cF(t).$$
We will also use the notation \( G(t) \ll F(t) \) to denote that \( G(t) \leq cF(t) \) for some positive constant \( c \), and we will use subscripts to indicate that this constant depends on additional parameters.

2.3. **Mean ergodic theorem.** Using the explicit embedding \( i : \text{SL}_2(\mathbb{R}) \to \text{SL}_3(\mathbb{R}) \) given in (4) together with (effective) property (T) for \( \text{SL}_3(\mathbb{R}) \) it is not hard to see that the representation \( \pi \) of \( H \) on \( L^2(X_3) \) given by \( \pi(h)f(x) = f(xt(h)) \) is tempered on the space orthogonal to the constant functions, and hence the \( H \) action on \( X_3 \) satisfies an effective mean ergodic theorem (see [6, 9]) with best possible rate. That is, for \( f \in L^2(X_3) \), let

\[
\pi_t(f)(x) := \frac{1}{m(H_t)} \int_{H_t} f(xt(h)) \, dm(h),
\]

then we have the following.

**Theorem 4.** For any \( 0 < \kappa < 1/2 \), any \( f \in L^2(X_3) \), and any \( t \geq 1 \),

\[
\|\pi_t f - \int_{X_3} f \, d\mu\|_2 \ll \kappa \, m(H_t)^{-\kappa} \|f\|_2.
\]

**Proof.** This is implicit in [9] and was explicitly used in [6]. However, since we could not find a reference to the precise statement, we include a short proof below.

Let \( L^2_0(X_3) \subseteq L^2(X_3) \) denote the subspace orthogonal to the constant functions, and let \( \pi \) denote the representation of \( H \) on \( L^2_0(X_3) \) given above. Using the bounds on matrix coefficients coming from (effective) property (T) for \( \text{SL}_3(\mathbb{R}) \) [12, Theorem A], we get that for any \( K \)-finite vector \( f \in L^2_0(X_3) \) for all sufficiently small \( \epsilon > 0 \) the matrix coefficient

\[
|\langle \pi(h)f, f \rangle| \ll f, \epsilon \frac{1}{\|h\|^{1-\epsilon}}
\]

and hence lies in \( L^2(H) \). By [9, Theorem 5.3], this implies that \( \pi \) is weakly contained in the regular representation \( \lambda \) of \( H \). From this, it follows that the operator norm of \( \pi_t = \frac{1}{m(H_t)} \int_{H_t} \pi(h) \, dm(h) \), acting on \( L^2_0(X_3) \), is bounded by the operator norm of \( \lambda_t = \frac{1}{m(H_t)} \int_{H_t} \lambda(h) \, dm(h) \). Now by [9, Proposition 5.9], for the regular representation, \( \|\lambda_t\| = \frac{1}{m(H_t)} \int_{H_t} \|\Xi(h)\| \, dm(h) \) where \( \Xi(h) \) is the Harish-Chandra \( \Xi \)-function for \( H \). Since the Harish-Chandra \( \Xi \)-function is in \( L^{2+\epsilon}(H) \) for any \( \epsilon > 0 \), we can use Cauchy-Schwarz to bound \( \|\lambda_t\| \ll \kappa \frac{1}{m(H_t)^{\epsilon}} \) with \( \kappa = \frac{1}{2+\epsilon} \).

Applying this to \( f_0 = f - \int_{X_3} f \, d\mu \in L^2_0(X_3) \) we can write \( \|\pi_t f - \int_{X_3} f \, d\mu\|_2 = \|\pi_t f_0\|_2 \) and bounding \( \|\pi_t f_0\|_2 \leq \|\pi_t\| \|f_0\|_2 \leq \|\lambda_t\| \|f\|_2 \) concludes the proof. \( \square \)

As a direct consequence, for any \( t \geq 1 \) and measurable set \( B \) we can estimate the measure of the set

\[
\mathcal{E}_{t,B} = \{ x \in X_3 : xH_t \cap B = \emptyset \},
\]

where \( xH_t = \{ xt(h) : \|h\| \leq t \} \). Explicitly, we have the following.
**Corollary 5.** Fix \( \eta < 1 \). Then for any measurable \( B \subseteq X_3 \) for any \( t \geq 1 \),

\[
\mu(\mathcal{E}_{t,B}) \ll \eta \frac{1}{t^{\eta} \mu(B)},
\]

and for sets with \( \mu(B) \geq \frac{1}{2} \),

\[
\mu(\mathcal{E}_{t,B}) \ll \eta \frac{1 - \mu(B)}{t^{\eta}}.
\]

**Proof.** Let \( f \) denote the indicator function of \( B \), and note that if \( x \in \mathcal{E}_{t,B} \) then \( \pi_t(f)(x) = 0 \). Applying the mean ergodic theorem with \( \eta = 2 \kappa \), we get

\[
\mu(B)^2 \mu(\mathcal{E}_{t,B}) = \int_{\mathcal{E}_{t,B}} |\pi_t(f)(x) - \mu(B)|^2 \, d\mu(x)
\]

\[
\leq \|\pi_t(f)(x) - \mu(B)\|^2 \ll \eta \frac{\|f\|^2}{m(H_t)^\eta} \ll \frac{\mu(B)}{t^\eta},
\]

and dividing both sides by \( \mu(B)^2 \) gives (11).

Next apply the mean ergodic theorem to \( f_1 = 1 - f \) (the indicator function of the complement of \( B \)). Note that if \( x \in \mathcal{E}_{t,B} \) then \( \pi_t(f_1) = 1 \) and hence

\[
\mu(B)^2 \mu(\mathcal{E}_{t,B}) = \int_{\mathcal{E}_{t,B}} |\pi_t(f_1)(x) - (1 - \mu(B))|^2 \, d\mu(x)
\]

\[
\leq \|\pi_t(f_1)(x) - \int_X f_1 \, d\mu\|^2 \ll \eta \frac{\|f_1\|^2}{m(H_t)^\eta} \ll \frac{1 - \mu(B)}{t^\eta}.
\]

Since we assume that \( \mu(B) \geq 1/2 \), we can divide by it to get (12). \( \square \)

**2.4. Unfolding.** We need the following result relating small sets in \( \mathbb{R}^3 \) to corresponding sets on \( X_3 \). Explicitly, thinking of \( X_3 \) as the space of unimodular lattices, for any set \( \Omega \subseteq \mathbb{R}^3 \) consider the sets \( B_\Omega^\# \subseteq B_\Omega \subseteq X_3 \) be defined as

\[
B_\Omega = \{ \Lambda \in X_3 : \Lambda \cap \Omega \neq \emptyset \}, \quad B_\Omega^\# = \{ \Lambda \in X_3 : \Lambda^\# \cap \Omega \neq \emptyset \},
\]

where \( \Lambda^\# \) denotes the set of primitive vectors in \( \Lambda \). We then have the following.

**Lemma 6.** For any \( \Omega \subseteq \mathbb{R}^3 \) with finite volume,

\[
\frac{\text{vol}(\Omega)}{\zeta(3)} \geq \mu(B_\Omega^\#) \geq \frac{\text{vol}(\Omega)}{\text{vol}(\Omega) + 2\zeta(3)},
\]

where \( \zeta(3) = \sum \frac{1}{n^3} = 1.20205 \ldots \).

**Remark 7.** This inequality implies in particular that

\[
\mu(B_\Omega) \geq \mu(B_\Omega^\#) \geq 1 - \frac{2\zeta(3)}{\text{vol}(\Omega)},
\]

which is non-trivial when \( \text{vol}(\Omega) \) is large, and also that

\[
\mu(B_\Omega) \geq \mu(B_\Omega^\#) \geq \min \left\{ \frac{1}{1 + 2\zeta(3)}, \frac{\text{vol}(\Omega)}{1 + 2\zeta(3)} \right\},
\]

which is useful when \( \text{vol}(\Omega) \) is small. We remark that the inequality \( \mu(B_\Omega) \geq 1 - \frac{\zeta(3)}{\text{vol}(\Omega)} \) (for sets with large volume) was proved in [1, Theorem 2.2]. We include
a short proof, along the same lines, treating primitive vectors and sets with both large and small volume simultaneously.

Proof. Let \( f \) denote the indicator function of \( \Omega \) and denote by \( \hat{f} \) be its modified Siegel transform defined by

\[
\hat{f}(\Lambda) = \sum_{\nu \in \Lambda^\circ} f(\nu).
\]

By [13, Theorem 5] we have the identity

\[
\int_{X_3} |\hat{f}(\Lambda)|^2 \, d\mu = \left( \int_{R^3} f(x) \, dx / \zeta(3) \right)^2 + \frac{1}{\zeta(3)} \left( \int_{R^3} f(x) f(x) \, dx + \int_{R^3} f(x) f(-x) \, dx \right).
\]

For \( f \) an indicator function we can bound \( f(x) f(-x) \leq f(x) \), hence

\[
\int_{X_3} |\hat{f}(\Lambda)|^2 \, d\mu \leq \frac{\text{vol}(\Omega) (\text{vol}(\Omega) + 2 \zeta(3))}{\zeta(3)^2}.
\]

Now by Siegel’s mean value theorem

\[
\int_{X_3} \hat{f}(\Lambda) \, d\mu = \frac{1}{\zeta(3)} \int_{R^3} f(x) \, dx = \frac{\text{vol}(\Omega)}{\zeta(3)}.
\]

Since \( \hat{f}(\Lambda) \geq 1 \) on \( B_{\Omega}^\# \) we get that \( \mu(B_{\Omega}^\#) \leq \frac{\text{vol}(\Omega)}{\zeta(3)} \) and since \( \hat{f}(\Lambda) = 0 \) for \( \Lambda \not\in B_{\Omega}^\# \) we get

\[
\frac{\text{vol}(\Omega)^2}{\zeta(3)^2} = \left| \int_{B_{\Omega}^\#} \hat{f}(\Lambda) \, d\mu \right|^2 \leq \mu(B_{\Omega}^\#) \int_{X_3} |\hat{f}(\Lambda)|^2 \, d\mu \leq \mu(B_{\Omega}^\#) \frac{\text{vol}(\Omega) (\text{vol}(\Omega) + 2 \zeta(3))}{\zeta(3)^2},
\]

implying that \( \mu(B_{\Omega}^\#) \geq \frac{\text{vol}(\Omega)^2}{\text{vol}(\Omega) + 2 \zeta(3)} \) as claimed.

2.5. Shrinking targets. Next we define the shrinking targets. For parameters \( \delta \in (0, 1) \) and \( \xi \in \mathbb{R} \), let \( L_\xi = \max\{1, \sqrt{1/|\xi|} \} \) and define

\[
\Omega_{\xi, \delta} = \left\{ (x, y, z) \in \mathbb{R}^3 : |x| \leq L_\xi, |y| \leq L_\xi, \sqrt{|x^2 + y^2| - \xi} \leq \delta \right\}.
\]

We note that \( \Omega_{\xi, \delta} \) has volume \( 4\delta L_\xi \) and that \( |Q_0(v) - \xi| \leq \delta \) for any \( v \in \Omega_{\xi, \delta} \). We now define our shrinking targets as

\[
B_{\xi, \delta}^\# = \left\{ \Lambda \in X_3 : \Lambda^\circ \cap \Omega_{\xi, \delta} \neq \emptyset \right\}.
\]

Then by Lemma 6 (and the following remark) we have that \( \mu(B_{\xi, \delta}^\#) \geq \delta L_\xi \) when \( \delta L_\xi \leq 2 \) is small and that \( \mu(B_{\xi, \delta}^\#) \geq 1 - \frac{1}{\delta L_\xi} \) when \( \delta L_\xi > 2 \) is large.

Remark 8. When \( \xi \) is fixed and \( \delta \to 0 \), the sets \( B_{\xi, \delta}^\# \) are indeed shrinking targets in \( X_3 \) with measure going to zero. We also consider the case when \( \xi \) (and hence \( L_\xi \)) grows as \( \delta \to 0 \), in which case the sets \( B_{\xi, \delta} \) are no longer necessarily shrinking. In fact, if \( L_\xi \delta \to \infty \) then the complement of \( B_{\xi, \delta} \) will have measure going to zero.
3. Proofs

The proof of Theorem 1 follows from the following effective result, bounding the measure of the set of forms that do not well approximate a given point.

**Proposition 9.** For any $\varepsilon > 0$ sufficiently small, for any $\delta \in (0, 1)$ for any $\xi \in \mathbb{R}$ and any $k \geq L_\xi$, there is a set $\mathcal{C}_{k, \delta, \xi} \subseteq X_3$ of measure at most $O\left(\frac{1}{k^{1+\delta}}\right)$ such that for any $\Gamma g \in X_3$ such that $\mathcal{C}_{k, \delta, \xi}$ we have

$$\min_{\|n\| \leq \|g\| k} |Q_0^g(n) - \xi| \leq \delta,$$

where the minimum is taken over primitive vectors in $\mathbb{Z}^3$.

**Proof.** Fix $\eta = 1 - \varepsilon < 1$, then Corollary 5 implies that for any $k \geq 1$ we have

$$\mu\left(\{x \in X_3 : xH_k \cap B_{\delta, \xi}^g = \emptyset\}\right) \ll \frac{1}{k^\eta} \delta L_\xi.$$

Indeed, if $\delta L_\xi \leq 2$ the first part of Corollary 5 gives

$$\mu\left(\{x \in X_3 : xH_k \cap B_{\delta, \xi}^g = \emptyset\}\right) \ll \frac{1}{k^\eta} \mu(B_{\delta, \xi}^g) \ll \frac{1}{k^\eta} \delta L_\xi,$$

while when $\delta L_\xi > 2$ we have that $\mu(B_{\delta, \xi}^g) \geq 1 - \frac{(3)}{2\delta L_\xi} \geq \frac{1}{2}$ and the second part of Corollary 5 gives

$$\mu\left(\{x \in X_3 : xH_k \cap B_{\delta, \xi}^g = \emptyset\}\right) \ll \frac{1 - \mu(B_{\delta, \xi}^g)}{k^\eta} \ll \frac{1}{k^\eta} \delta L_\xi.$$

Now let $x = \Gamma g$ be such that $xH_k \cap B_{\delta, \xi}^g \neq \emptyset$; then there is some $h \in H_k$ and a primitive $n \in \mathbb{Z}^3$ such that $nh \in \Omega_{\xi, \delta}$, so that $|Q_0(n) - \xi| \leq \delta$ and also $\|nh\| \leq L_\xi$. We can then bound

$$\|n\| = \|nhh^{-1}g^{-1}\| \leq \|nh\|\|h\|\|g\| \leq \|g\| L_\xi k.$$

Let $\tilde{k} = L_\xi k$ then $\|n\| \leq \|g\| \|\tilde{k}\| |Q_0^g(n) - \xi| \leq \delta$ and this holds for all but $O\left(\frac{1}{k^{1+\delta}}\right)$ of the points $\Gamma g \in X_3$. □

**Proof of Theorem 1.** For each $k \in \mathbb{N}$ let $-N(k) < \xi_{k,1} < \cdots < \xi_{k,M(k)} < N(k)$ be dense (so $M(k) = 1 + O(N(k)/\delta(k))$). Fix a fundamental domain $\mathcal{F} \subseteq \text{SL}_3(\mathbb{R})$ for $X_3$ and let $\mathcal{C} \subseteq \mathcal{F}$ denote the set of all points $g \in \mathcal{F}$ such that for any $T > 0$ there is $k > T$ and $\xi_{k,i}$ such that $\min_{|n| \leq \|g\| k} |Q_0^g(n) - \xi_{k,i}| > \delta(k)/2$, that is,

$$\mathcal{C} = \bigcap_{T > 0} \bigcup_{k \geq T} \bigcup_{i = 1}^{M(k)} \mathcal{C}_{k,i},$$

and $\mathcal{C}_{k,i}$ is the set of points such that $\min_{|n| \leq \|g\| k} |Q_0^g(n) - \xi_{k,i}| > \delta(k)/2$.

Now splitting into dyadic intervals, note that for any $l = 2^l$ the set

$$\mathcal{C}_l = \bigcup_{k \geq l} \bigcup_{i = 1}^{M(k)} \mathcal{C}_{k,i} \subseteq \bigcup_{l = 1}^{M(2l)} \left\{ g \in \mathcal{F} : \min_{|n| \leq \|g\| l} |Q_0^g(n) - \xi_{2l,i}| > \delta(2l) \right\},$$
and hence, by Proposition 9, with $\varepsilon$ sufficiently small we have for $l \geq \sqrt{N(k)}$ that
\[
\mu(\mathcal{E}_1) \ll \frac{M(2l)}{l^{1-c}\delta(2l)} \ll \frac{1}{l^{1-c}\delta(2l)} + \frac{N(2l)}{l^{1-c}\delta(2l)^2} \ll l^{-c}.
\]
Since we can write $\mathcal{E} = \bigcap_{T > 0} \bigcup_{j \geq \log(T)} \mathcal{E}_2^j$, for all $T \geq \sqrt{N(k)}$ we can bound
\[
\mu(\mathcal{E}) \leq \sum_{j \geq \log(T)} \mu(\mathcal{E}_2^j) \ll \sum_{j \geq \log(T)} 2^{-jc} \ll T^{-c}.
\]
This holds for all sufficiently large $T$, hence $\mu(\mathcal{E}) = 0$.

Now for any $g \in \mathcal{F} \sim \mathcal{E}$ there is $T_0 = T_0(g)$ such that for any $k \geq T_0$, for any $\xi \in \mathbb{R}$ with $\xi \leq N(k)$ there is $\xi_{k,i} \in (\xi - \frac{\delta(k)}{2}, \xi + \frac{\delta(k)}{2})$ and primitive $n \in \mathbb{Z}^3$ with $\|n\| \leq \|g\|k$ such that $|Q^k_0(n) - \xi_{k,i}| \leq \frac{\delta(k)}{2}$. Hence $\min_{\|n\| \leq \|g\|k} |Q^k_0(n) - \xi| \leq \delta(k)$ as claimed.

\[\square\]

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