Spin-Bits and $\mathcal{N} = 4$ SYM

Alessio Marrani

*Istituto Nazionale di Fisica Nucleare (INFN),
Laboratori Nazionali di Frascati (LNF),
Via Enrico Fermi 40, I-00044 Frascati, Italy

ABSTRACT

We briefly review the spin-bit formalism, describing the non-planar dynamics of the $\mathcal{N} = 4, d = 4$ Super Yang-Mills $SU(N)$ gauge theory. After considering its foundations, we apply such a formalism to the $su(2)$ sector of purely scalar operators. In particular, we report an algorithmic formulation of a deplanarizing procedure for local operators in the planar gauge theory, used to obtain planarly-consistent, testable conjectures for the higher-loop $su(2)$ spin-bit Hamiltonians. Finally, we outlook some possible developments and applications.

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1 Introduction

Large $N$ physics \cite{1} gained noticeable interest in the past few years (for a recent review
see e.g. \cite{2}) due to the AdS/CFT conjecture enlightenment \cite{3, 4} and, more recently, to
the consideration of various limits of this correspondence (\cite{5}-\cite{14}). Initially formulated
in the $N \to \infty$ limit, the conjecture in its strong form extends to finite $N$. It relates
the strongly coupled regime of $\mathcal{N} = 4$ SYM to the weakly coupled string theory and
viceversa. This property, which makes out of this correspondence a very strong and
efficient predictive tool, appeared to be an obstacle in proving the duality in itself.

Berenstein, Maldacena and Nastase studied in \cite{5, 6} the correspondence in the neigh-
borood of null geodesics of $\text{AdS}_5 \times S^5$, where the geometry appears to resemble that of
a gravitational wave \cite{15}-\cite{17}. On the CFT side this corresponds to focusing on SYM
operators with a large $R$-charge. The possibility to find a solution of string theory \cite{18, 19}
in such a background allows for a quantitative comparison with predictions coming from
perturbative SYM computations \cite{20} (see \cite{21} for recent reviews on the BMN correspondence and references). This led to an intensive study of the anomalous dimensions of local gauge-invariant (g.i.) composite operators in $\mathcal{N} = 4$, 4-dimensional ($d = 4$) Super Yang–Mills (SYM) theory \cite{22}. A real breakthrough was the discovery of the integrability of the Hamiltonians governing anomalous dimensions in the planar limit $N \to \infty$ \cite{23, 24}. Then, these results were extended to 2 and higher loops \cite{25, 26}. Indeed, the dynamics in the sector of single-trace bosonic operators of SYM can be mapped into that of the Heisenberg SO(6) spin one model, so that the matrix of planar one-loop anomalous dimensions is identified with the spin chain Hamiltonian \cite{23}. The Bethe Ansatz techniques used for diagonalizing the Hamiltonian become then a powerful tool in determining anomalous dimensions in the gauge theory. As it is now clear, there is a one-to-one correspondence between single-trace operators in SYM theory and spin states in spin-chain models. The improved understanding of SYM overshadowed, to some extent, the study of nonplanar contributions. The latter has to correspond through AdS/CFT to considering the string production on the AdS side. String bits \cite{27} were proposed as a model which mimics this feature out of (but not very far from) the BMN limit. Although the string bit model yields a good tool for the computation of the relevant bosonic quantities, it is affected by serious consistency problems related to the fermionic doubling \cite{14, 28}.

On the SYM side the exact one-loop dilation operator was derived in \cite{22, 23, 29}. When non-planarity is taken into account, single and multi-trace operators get mixed. This could still not be tested in the string dual picture. Waiting for a better understanding of string physics on AdS space, one could hope to learn about string interactions there by exploiting the dual gauge theory picture. This was the main motivation of the work carried out by the LNF research group over the past two years. We studied the corresponding spin system which mixes the integrable spin approach and the string bits one. Such a theory can be called a spin bit model. Since it allows for dynamical splitting and joining of chains and its variable content is given by spins, the spin bit model differs from the spin chain and the string bit ones, although it can be considered as a mixture of them. In particular, there is no fermion doubling, and supersymmetry in the spin bit model is consistently implemented.

At $N \to \infty$ the spin Hamiltonian is a local and integrable operator. The Hamiltonian and the total spin generator represent the first two charges, in the tower of commuting ones, predicted by integrability \cite{29}. Higher charges are given in terms of higher powers of next-to-nearest spin generators summed up over the chain. Corrections in $\frac{1}{N}$ spoil locality and integrability. The Hamiltonian and its higher spin analogs, which can be interpreted as broken symmetries of the would-be integrable system, can still be defined in terms of powers of spin generators. However, now there is no more restriction to next-to-nearest interactions and the corresponding charges are no longer commuting among themselves. The role of these broken charges in the theory near the “integrable” point
\(N \to \infty\) remains to be understood.

If the non-planar contributions are considered, the single-trace sector is not conserved anymore, and one ends up with trace splitting and joining in the operator mixing \[30\]. Even in this case one can still consider a one-to-one map, the so-called spin-bit map, between local g.i. operators and a spin system \[31, 32\]. In this case one has to introduce a set of new degrees of freedom, beyond the spin states, which describes the linking structure of the sites in the spin-chain. This can be encoded in a new field, taking values in the spin-bits permutation group and introducing a new gauge degree of freedom \[32\].

In this paper we use conventions and notations of \[31, 32, 33\].

2 The \(\mathcal{N} = 4, d = 4\) SYM \(SU(N)\) gauge theory

In the following we will consider a particular quantum field theory, namely the \(\mathcal{N} = 4, d = 4\) SYM \(SU(N)\) gauge theory. Such a theory has the noteworthy property to be conformally-invariant, due to the vanishing of its beta function (see e.g. \[34\]). It has the following field content:

\[
F_{\mu\nu} \text{ gauge field strength, } \mu, \nu = 0, 1, 2, 3;
\]
\[
\phi^i \text{ real scalars, } i = 1, \ldots, 6 \text{ (vector repr. of } SO(6));
\]
\[
\lambda^A \alpha, \bar{\lambda}_{\dot{\alpha}} \text{ gauginos, } A = 1, \ldots, \mathcal{N} = 4, \alpha, \dot{\alpha} = 1, 2.
\]

All the fields take value in the adjoint representation (repr.) of the gauge group \(SU(N)\), i.e. for example \(\phi^i = \phi^i_a T^a\), where \(T^a (a = 1, \ldots, N^2 - 1)\) are the generators of \(SU(N)\) in the adjoint. The scalars also span the vector repr. of \(SO(6)\), which is the maximal compact bosonic subgroup of the whole \(\mathcal{N} = 4\) supergroup \(SU(2, 2 | 4)\); moreover, the underlying algebra \(so(6) \sim su(4)\) is the automorphism, or \(R\)-symmetry, algebra of the whole \(\mathcal{N} = 4, d = 4\) superconformal algebra (SCA) \(psu(2, 2 | 4)\). In the following we will use a compact notation for the \(SU(N)\)-gauge covariant derivatives of the fields, namely \((s \in N \cup \{0\})\)

\[
\nabla^s \phi \equiv \nabla_{\mu_1 \mu_2 \ldots \mu_s} \phi^i,
\]
\[
\nabla^s \lambda \equiv \nabla_{\mu_1 \mu_2 \ldots \mu_s} \lambda^A \alpha,
\]
and so on. All the elementary fields, as well their derivatives, can be obtained by acting with generators of the \(\mathcal{N} = 4\) SCA on the “primary” fields \(\phi^i\).

By adopting a convenient “philological” nomenclature, we may say that the \(\mathcal{N} = 4, d = 4\) SYM “alphabet” is composed by the set of “letters”

\[
W_A \equiv \{\nabla^s F, \nabla^s \phi, \nabla^s \lambda, \nabla^s \bar{\lambda}\}.
\]

3
The components of $W_A$ transform in the so-called “singleton” (infinite-dimensional) repr. $V_F$ of the $\mathcal{N} = 4$ SCA. Out of the “letters” $W_A$ one can build $SU(N)$ g.i. “words”, i.e. single-trace composite operators given by traces (in the adjoint of $SU(N)$) of a sequence of “letters” $W_A$’s. Examples are given by

$$O_{i_1 i_2 \ldots i_n} \equiv Tr\left(\phi^{i_1} \phi^{i_2} \ldots \phi^{i_n}\right);$$  
$$O_{i_1 i_2 \ldots i_{m+1}}^{a_1 a_2 \ldots a_m} \equiv Tr\left(\phi^{i_1} \phi^{i_2} \ldots \phi^{i_m} \nabla_{\mu_1 \ldots \mu_n} \phi^{i_{m+1}} \nabla_{\mu_1 \ldots \mu_{n-2}} \lambda_a^A \nabla_{\mu_n} \lambda_{a_A}\right),$$  

with

$$\nabla^\mu = \eta^{\mu \nu} \nabla_\nu;$$

where $\eta^{\mu \nu}$ is the 4-dim. Minkowski metric. Moreover, out of “words” one can produce “sentences”, which are sequences of “words”, i.e. products of single-trace composite operators, given by products of traces (in the adjoint of $SU(N)$) of sequences of “letters”. Some examples are

$$O_{i_1 i_2 \ldots i_{n_1} j_1 j_2 \ldots j_{n_2}} \equiv Tr\left(\phi^{i_1} \phi^{i_2} \ldots \phi^{i_{n_1}}\right) Tr\left(\phi^{j_1} \phi^{j_2} \ldots \phi^{j_{n_2}}\right);$$  
$$O_{i_1 \ldots i_{n_2}}^{\alpha_1 \ldots \alpha_{n_2}} \equiv Tr\left(\nabla_{\mu_1 \ldots \mu_{n_1}} \phi^{i_1} \ldots \phi^{i_{n_2}}\right) \cdot Tr\left(\nabla_{\mu_1 \ldots \mu_{n_1-1}} \lambda_{a_A}\right) Tr\left(\lambda_{\beta}^A F_{\mu_{n_1} \nu}\right).$$

The length of a “word” or “sentence” in $\mathcal{N} = 4$ SYM is defined as the number of “letters” $W_A$’s composing the considered trace structure.

Summarizing, the above introduced “words” and “sentences” correspond to (possibly multitrace) $SU(N)$ g.i. polynomial composite operators in $\mathcal{N} = 4, d = 4$ SYM. As we will see further below, the spin-bit map gives an one-to-one spin description of such (multi)trace structures, and thus allows one to perturbatively calculate the non-planar ($N < \infty$) anomalous dimensions in (certain sectors of) the considered conformally-invariant gauge theory.

### 3 The spin-bit model

A generic $M$-trace g.i. polynomial composite operator of length $L$ will have the general form

$$O \equiv Tr\left(W_{A_1} \ldots W_{A_{L_1}}\right) Tr\left(W_{A_{L_1+1}} \ldots W_{A_{L_1+L_2}}\right) \ldots \ldots Tr\left(W_{A_{L-L_{M+1}}} \ldots W_{A_L}\right).$$

(3.1)
Let us now consider an element of the permutation group $S_L$ (of rank $L!$) of $L$ elements, namely
\[ \gamma \equiv (\gamma_1 \gamma_2 \ldots \gamma_L) : \begin{pmatrix} a_1 & a_2 & \cdots & a_L \\ a_{\gamma_1} & a_{\gamma_2} & \cdots & a_{\gamma_L} \end{pmatrix}, \quad (3.2) \]
or equivalently
\[ S_L \ni \gamma = (L_1) (L_2) \ldots (L_M) : \sum_{r=1}^{M} L_r = L, \quad (3.3) \]
where $S_{L_r} \ni (L_r)$ is a cyclic permutation of $L_r$ elements ($r = 1, \ldots, M$). Actually, Eq. (3.3) has a deeper meaning, because in general $S_L$ is split in equivalence classes (labelled by $L_1, L_2 \ldots L_k$ such that $\sum_{r=1}^{k} L_r = L$) of permutations consisting of cycles of respective lengths. By reducing to (products of the) minimal, non-trivial permutational “bricks”, that is to (products of the) pair-site permutations $\sigma_{kl}$, $(k, l) \in \{1, \ldots, L\}^2$ (which simply exchange the $k$-th and $l$-th elements), the decomposition expressed by Eq. (3.3) leads to (planar and) non-planar (pair-site) permutational identities, extensively treated in [35].

Thus, by suitably choosing $\gamma$, the operator $\mathcal{O}$ may be rewritten as
\[ \mathcal{O} = (W_{A_1})^{a_1 a_{\gamma_1}} (W_{A_2})^{a_2 a_{\gamma_2}} \ldots (W_{A_L})^{a_L a_{\gamma_L}}, \quad (3.4) \]
where the matrix structure (with values in the adjoint repr. of $SU(N)$) of the “letters” $W_A$’s is manifest (by convention the first upper index is a row index, whereas the second is a column one).

As it may be easily seen, the fundamental features we have to take into account are the length $L$ of the operator, the number $M$ of traces, and the “linking” configuration expressed by $\gamma$. By thinking each “letter” as sitting on a distinct spin-chain site, we may therefore write the following equivalence relation (which we will extensively comment in the following)
\[ \mathcal{O} \equiv |A_1, \ldots, A_L; \gamma\rangle, \quad (3.5) \]
where $A_k$ is the direction in $V_F$ determined by the “letter” $W_{A_k}$, i.e. the direction of the spin state at the $k$-th spin-chain site. The r.h.s. of Eq. (3.5) represents a state in the Hilbert space of a spin-chain model, but with an explicit extra degree of freedom (represented by $\gamma$), properly describing the structure of the interconnections among spin-chain sites: such an “improved” spin-chain model will be called “spin-bit” model. Due to the separating action of the semicolon in Eq. (3.5), the spin-bit Hilbert space $\mathcal{H}_{sb}$ naturally gets divided in an usual spin-part $\mathcal{H}_{sc}$ (the usual spin-chain Hilbert space) and in a so-called “linking” part. As we will see later, this latter will allow one to correctly take into account also the non-planar contributions to anomalous dimensions. Eq. (3.5) defines the so-called spin-bit map in $\mathcal{N} = 4, d = 4$ SYM, whose isomorphicity we are going to discuss.
3.1 Spin part of $\mathcal{H}_{sb}$: the spin-chain picture

As previously mentioned, the spin part of the “improved” spin-chain state, i.e. of the spin-bit state, is given by

$$\mathcal{H}_{sc} \ni \lvert A_1, \ldots, A_L \rangle = \lvert S_1, \ldots, S_L \rangle = \lvert S_1 \rangle \otimes \ldots \otimes \lvert S_L \rangle.$$  \hspace{1cm} (3.1.1)

As it is well known, the spin-chain Hilbert space $\mathcal{H}_{sc}$ is the tensor product of the one-spin (or, equivalently, one-site) Hilbert spaces $\mathcal{H}_k$'s. Consequently, as shown by Eq. (3.1.1), a generic spin-chain state is given by the tensor product of the one-spin states at each spin-chain site. $\mathcal{H}_k$ is given by the representation space of the considered symmetry (which in the case at hand will be described by a subalgebra of $psu(2, 2 | 4)$) at the $k$-th site. Indeed, $A_k \equiv S_k$ is the direction in $\mathcal{H}_k$ determined by the symmetry of the $N = 4, d = 4$ SYM “letter” $W_{A_k}$ in $V_F$

$$\mathcal{H}_{sc} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_L = (V_F)_1 \otimes (V_F)_2 \otimes \ldots \otimes (V_F)_L,$$  \hspace{1cm} (3.1.2)

where $(V_F)_k$ stands for the “singleton” representation space of the (relevant subalgebra of the) SCA $psu(2, 2 | 4)$ at the $k$-th site.

In the following treatment we will assume that the same representation of the same symmetry algebra will hold at each spin-chain site. With such an assumption we get

$$\mathcal{H}_{sc} = (V_F)^L,$$  \hspace{1cm} (3.1.3)

where $V_F$ is the “singleton” repr. space of the (relevant subalgebra of the) SCA $psu(2, 2 | 4)$, which will determine the symmetry of the spin-chain model.

In $N = 4, d = 4$ SYM theory, the most general symmetry is given by the whole SCA $psu(2, 2 | 4)$, implying that the dimension of the one-spin Hilbert space $\mathcal{H}_k$ is the same for every spin-chain site, and it is infinite, because $\dim (V_F) = \infty$

$$\dim (\mathcal{H}_{sc}) = \dim \left( (V_F)^L \right) = \infty^L.$$  \hspace{1cm} (3.1.4)

The situation changes if compact symmetries (having finite-dimensional unitary representations) are considered (an example is the $su(2)$ sector which will be extensively treated later). Also the possibilities to have non-ultralocalizations and/or different representations of symmetries from site to site along the chain could be taken into account, but here we will not deal with such cases (however, see e.g. [36]).

Of course, $\mathcal{H}_k$ can be endowed with a consistent scalar product $\langle \cdot | \cdot \rangle$. Consequently, by choosing an orthonormal basis $\{e_\alpha\}$ (with $\alpha$ ranging in a numerable set)

$$\langle e_\alpha | e_\beta \rangle = \delta_{\alpha\beta},$$  \hspace{1cm} (3.1.5)
we get
\[ S_k = S_k^\alpha e_\alpha, \]  
(3.1.6)
denoting the decomposition of the spin vector \( S_k \) along the orthonormal basis \( \{e_\alpha\} \) of the “singleton” repr. space \( V_F \) of the \( \mathcal{N} = 4 \) SCA \( psu(2, 2 | 4) \) at the spin-chain site labelled by \( k \). Therefore, we may rewrite Eq. (3.1.1) as follows:
\[
|A_1, ..., A_L\rangle = S_1^\alpha_1 ... S_L^\alpha_L |e_{\alpha_1}\rangle \otimes ... \otimes |e_{\alpha_L}\rangle = \prod_{k=1}^L \otimes S_k^\alpha_k |e_{\alpha_k}\rangle \equiv |S\rangle \in \mathcal{H}_{sc}.
\]  
(3.1.7)

### 3.2 The “linking” part and the permutational redundancy in \( \mathcal{H}_{sb} \)

On the other hand, by considering only the “linking” part of the spin-bit state, we obtain
\[
|\gamma\rangle \in \zeta_L,
\]  
(3.2.1)
where \( \zeta_L \) is nothing but the representation space of the permutation group \( S_L \). It may be considered a metrizable space, too, and consequently it may be endowed with a consistent scalar product \( \langle \gamma | \gamma' \rangle = \delta_{\gamma \gamma'} \).

Thus, it would seem reasonable to define the whole Hilbert space of the spin-bit model \( \mathcal{H}_{sb} \) as the tensor product of the spin part \( \mathcal{H}_{sc} \) (given by Eq. (3.1.7)) and of the “linking” part \( \zeta_L \) (given by Eq. (3.2.1))
\[
|A_1, ..., A_L; \gamma\rangle \equiv |A_1, ..., A_L\rangle \otimes |\gamma\rangle \in \mathcal{H}_{sc} \otimes \zeta_L.
\]  
(3.2.2)
But, by simply doing a tensor product, we would then over-estimate the spin-bit Hilbert space \( \mathcal{H}_{sb} \). Indeed, an extra symmetry exists, given by the action of \( S_L \) on the direct tensor product \( \mathcal{H}_{sc} \otimes \zeta_L \) and determining the following equivalence relation of “permutational conjugation”\(^1\)
\[
|A_1, ..., A_L; \gamma\rangle \sim |A_{\gamma_1}, ..., A_{\gamma_L}; \sigma \cdot \gamma' \cdot \sigma^{-1}\rangle,
\]  
(3.2.3)
\[ \forall (\gamma, \sigma) \in (S_L)^2; \]
in particular, for \( \sigma = \gamma \) we obtain the property of cyclicity of the trace:
\[
|A_1, ..., A_L; \gamma\rangle \sim |A_\gamma, ..., A_\gamma; \gamma\rangle, \quad \forall \gamma \in S_L.
\]  
(3.2.4)
\(^1\)Here and below products in \( S_L \) are understood as
\[ \gamma \cdot \sigma \equiv \gamma \sigma = (\gamma_1, ..., \gamma_L) = (\sigma_{\gamma_1}, ..., \sigma_{\gamma_L}). \]
Otherwise speaking, we may define the representation of the action of the permutational symmetry group $S_L$ on the factorized Hilbert space $H_{sc} \otimes \zeta_L$ with the operator
\[
\hat{\Sigma}_\sigma (|A_1, ..., A_L\rangle \otimes |\gamma\rangle) \equiv |A_{\sigma_1}, ..., A_{\sigma_L}\rangle \otimes |\sigma \cdot \gamma \cdot \sigma^{-1}\rangle.
\]
(3.2.5)

Actually, the action of $\hat{\Sigma}_\sigma$ corresponds to nothing but an $S_L$-covariant relabelling of the spin-chain site indices. It is a symmetry of the spin-bit model, in the sense that it can be easily checked that the r.h.s.’s of Eqs. (3.2.2) and (3.2.5) describe the same (multi)trace polynomial composite g.i. operator of length $L$.

It should also be noticed that, due its very definition, the operator $\hat{\Sigma}_\sigma$ may be naturally decomposed as the direct product of two independent operators, acting on distinct spaces ($\forall \sigma \in S_L$)
\[
\hat{\Sigma}_\sigma = U_\sigma \otimes \tilde{\Sigma}_\sigma,
\]
(3.2.6)
with the definitions
\[
U_\sigma |A_1, ..., A_L\rangle \equiv
\equiv (P_{1,\sigma_1} \otimes P_{2,\sigma_2} \otimes ... \otimes P_{L,\sigma_L}) |A_1, ..., A_L\rangle = |A_{\sigma_1}, ..., A_{\sigma_L}\rangle,
\]
(3.2.7)
where $P_{k,l} \equiv P_{kl}$ is the pair-site index permutation operator, acting in the spin-chain Hilbert space $H_{sc}$ as follows (upperscripts denote the site positions):
\[
P_{kl} |A_1, ..., A_k, ..., A_l, ..., A_L\rangle \equiv
\equiv |A_1, ..., A_l, ..., A_k, ..., A_L\rangle = |A_1, ..., A_l, ..., A_k, ..., A_L\rangle,
\]
(3.2.8)
and
\[
\tilde{\Sigma}_\sigma |\gamma\rangle \equiv |\sigma \cdot \gamma \cdot \sigma^{-1}\rangle.
\]
(3.2.9)

Thus, in order to make the spin-bit map a one-to one (i.e. isomorphic) map, we have to quotient by this extra symmetry $S_L$, obtaining the following rigorous definition of (state in the) spin-bit Hilbert space:
\[
H_{sb} \equiv \left\{(V_F)^L \otimes \zeta_L\right\} / S_L \ni |A_1, ..., A_L; \gamma\rangle \equiv
\equiv \left\{|A_1, ..., A_L\rangle \otimes |\gamma\rangle\right\} / S_L \equiv |A_1, ..., A_L\rangle \otimes_{S_L} |\gamma\rangle,
\]
(3.2.10)
where $\otimes_{S_L}$ stands for the direct tensor product, modulo the action of $S_L$ represented by $\hat{\Sigma}_\sigma$. 


Therefore, given an arbitrary factorized basis element $|A_1, ..., A_L⟩ \otimes |\gamma⟩$, one can find the corresponding element of the quotient space $\left( (V_F)^{L} \otimes \zeta_L \right) / S_L$, i.e. the corresponding state in the spin-bit Hilbert space $\mathcal{H}_{sb}$, by "averaging" with respect to the action of $S_L$,

$$
|A_1, ..., A_L⟩ \otimes |\gamma⟩ \equiv \frac{1}{|S_L|} \sum_{\sigma \in S_L} \hat{\Sigma}_\sigma (|A_1, ..., A_L⟩ \otimes |\gamma⟩) = \frac{1}{|S_L|} \sum_{\sigma \in S_L} (|A_{\sigma_1}, ..., A_{\sigma_L}⟩ \otimes |\sigma \cdot \gamma \cdot \sigma^{-1}⟩) \equiv \hat{\Pi} (|A_1, ..., A_L⟩ \otimes |\gamma⟩),
$$

where $\hat{\Pi}$ is the cyclic symmetry operator, defined as

$$
\hat{\Pi} \equiv \frac{1}{|S_L|} \sum_{\sigma \in S_L} \hat{\Sigma}_\sigma = \frac{1}{|S_L|} \sum_{\sigma \in S_L} \left( U_\sigma \otimes \hat{\Sigma}_\sigma \right),
$$

and $|S_L| = L!$ is the rank of $S_L$.

From Eqs. (3.2.7), (3.2.9) and (3.2.12), it is not hard to check that $\hat{\Pi}$ is actually a projective operator

$$
\left( \hat{\Pi} \right)^2 = \hat{\Pi},
$$

and that it commutes with permutationally-invariant operators. Therefore the state $|A_1, ..., A_L; \gamma⟩$, defined by Eq. (3.2.10), is $S_L$-invariant, as it has to be in order to correctly estimate the spin-bit Hilbert space $\mathcal{H}_{sb}$, and therefore to make the spin-bit map an isomorphic one.

At 1 loop in SYM perturbation theory, it can be also explicitly shown that the "extra" symmetry $S_L$ of $\mathcal{H}_{sb}$ is nothing but a "gauge" symmetry, in the sense that the spin-bit model may be seen as arising from the corresponding spin-chain model by "gauging" with respect to the permutational symmetry $S_L$ [32], where, as previously mentioned, $L$ is the length of the considered operator, i.e. the total number of spin-chain sites, and also the total length of the spin-chain (if unit distance between neighboring sites is assumed).

### 3.3 Canonical reduction of $S_L$

By recalling Eq. (3.3), a generic element $\gamma \in S_L$ may be decomposed (uniquely, up to some possible pair-site permutational identities [35]) as follows:

$$
S_L \ni \gamma = (L_1) (L_2) \ldots (L_M) : \sum_{r=1}^{M} L_r = L, \ M \leq L,
$$

where $(L_r)$ is a cyclic permutation of $L_r$ elements ($r = 1, ..., M$). Due to the "extra" symmetry $S_L$ determining the equivalence relation (3.2.3), by choosing $\sigma \in S_L$ (in a suitable way, depending on the starting element $\gamma \in S_L$) it is always possible to reduce
γ to its canonical form, i.e. to the form where each spin-chain site index is sent to the immediate next one modulo cyclicity

$$
\gamma = (L_1) (L_2) ... (L_M)
$$

\[\begin{array}{c}
\downarrow \\
\gamma' = \sigma \cdot \gamma \cdot \sigma^{-1} = \sigma (L_1) (L_2) ... (L_M) \sigma^{-1}:
\end{array}\]

\[k_r \mapsto [k_r + 1] \equiv k_r + 1, \text{ mod. } L_r,
\]

(3.3.2)

where \(k_r\) is a spin-chain site index running inside the \(r\)-th trace.

Even though in what follows we will not restrict ourselves to consider (only) the canonical form of the permutations, one should bear in mind that the \(S_L\)-covariant relabelling of site indices corresponding to the permutational conjugation given by Eq. (3.2.3) determines an equivalence relation which makes the switching to canonical permutational forms not implying any loss of generality.

4 The dilatation operator in \(\mathcal{N} = 4, d = 4\) SYM

The anomalous dimensions of g.i. operators in the conformally-invariant \(\mathcal{N} = 4, d = 4\) SYM gauge theory are given by the action of the dilatation operator \(\Delta\).

In perturbation theory, it may be written as

$$\Delta \left(g_{YM}\right) = \sum_{n=0}^{\infty} H_{2n} \lambda^n,$$

(4.1)

where \(g_{YM}\) is the Yang-Mills coupling constant, and

$$\lambda = \lambda \left(g_{YM}\right) \equiv \frac{g_{YM}^2 N}{16\pi},$$

(4.2)

is the ’t Hooft coupling. \(H_{2n}\) is the \(n\)-loop effective vertex, determined by an explicit evaluation of the divergencies of the \(n\)-loop, 2-point function Feynman amplitudes \(\langle \mathcal{O}(0)\mathcal{O}(x)\rangle\) in \(\mathcal{N} = 4, d = 4\) SYM.

The first few effective vertices read [22]

\[n = 0 \text{ (tree level)}: \ H_0 = \Delta_0 A \text{Tr} \left( W_A \tilde{W}^A \right); \]

\[n = 1 \text{ (1-loop level)}:
\]

\[H_2 = -\frac{2}{N} \sum_{j=0}^{\infty} h(j) (P_j)^{AB}_{CD} : \text{Tr} \left[ W_A, \tilde{W}^C \right] \left[ W_B, \tilde{W}^D \right] ;,
\]

(4.4)

where

\[
(\tilde{W}^A)_{ab} = \frac{\partial}{\partial (W_A)^{ba}}
\]

(4.5)
is the “letter” operatorial derivative in $\mathcal{N} = 4, d = 4$ SYM, such that

$$\left(\tilde{W}^A\right)_{ab} (W^A)^{bc} = \delta_a^c,$$  \hspace{1cm} (4.6)$$

and $:\!\colon\!:\!$ denotes the “normal-ordering” of the operators inside, namely the fact that the derivatives $(\tilde{W}^A)_{ab}$ never act on the “letters” from the same group inside the colons. Moreover, $\Delta_{0A}$ stands for the classical (bare) dimension of the “letter” $W_A$. For the elementary fields previously mentioned, it is $\Delta_0 = 1$ for each scalar field $\phi^i$ and each $(SU(N)$-covariant) derivative, $\Delta_0 = \frac{3}{2}$ for the gauginos and $\Delta_0 = 2$ for the gauge field strength.

$(P_j)^{AB}_{CD}$ is the (rank 4) $psu(2, 2|4)$ projector to the irreducible module $V_j$ in the expansion of the tensor product of two $\infty$-dim. “singleton” $V_F$ representations of the $\mathcal{N} = 4, d = 4$ SCA $psu(2, 2|4)$

$$V_F \otimes V_F = \sum_{j=0}^{\infty} V_j.$$  \hspace{1cm} (4.7)$$

In general, the first modules $V_0, V_1$ and $V_2$ contain the symmetric, antisymmetric and trace components in the tensor product of two SYM scalars and their superpartners. Higher modules $V_j, j \geq 3$, contain spin $(j - 2)$ currents and their superpartners. Finally, $h(j)$ is the $j$-th harmonic number, defined as

$$h(j) \equiv \sum_{s=1}^{j} \frac{1}{s}, \quad h(0) \equiv 0.$$  \hspace{1cm} (4.8)$$

5 The spin-bit Hamiltonian at 1 loop

In general, by applying the isomorphic spin-bit map to the dilatation operator of $\mathcal{N} = 4, d = 4$ SYM $SU(N)$ gauge theory, we obtain an operator acting on the spin-bit Hilbert space $H_{sb}$. Such an operator may be identified with the spin-bit Hamiltonian; as we will see, it yields a “deplanarized” form of the related spin-chain Hamiltonian, in the sense that it perfectly reproduces the known results from the theory of spin-chains in the planar limit $N \to \infty$.

Since on the SYM side the dilatation operator is perturbatively known, we will correspondingly obtain a perturbatively expanded expression of the spin-bit Hamiltonian.

At tree level, we trivially get (see Eqs. (4.1) and (4.3))

$$\Delta_{n=0} = H_0 = \Delta_{0A} \text{Tr} \ (W_A \tilde{W}^A);$$  \hspace{1cm} (5.1)$$

by applying the spin-bit map, i.e. by applying $H_0$ on a generic spin-bit state, we get that the tree-level spin-bit Hamiltonian $H_{0, sb}$ is simply proportional to the identity, namely:

$$H_{0, sb} = \Delta_0 1,$$  \hspace{1cm} (5.2)$$

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where, as previously mentioned, $\Delta_0$ is the total classical dimension of the $SU(N)$-g.i. $\mathcal{N} = 4$ SYM (composite) operator uniquely associated to the considered spin-bit state.

Let us now consider $n = 1$, i.e. the 1-loop contribution to $\Delta$; from Eqs. (4.1), (4.2) and (4.4) we obtain

\[
\Delta^{n=1} = -\frac{g^2 Y M}{8\pi^2} \sum_{j=0}^{\infty} h(j) (P_j)_{CD}^{AB}.
\]

(5.3)

in order to obtain the 1-loop spin-bit Hamiltonian, we simply have to apply the 1-loop $\mathcal{N} = 4$ SYM effective vertex $H_2$ (given by Eq. (4.4)) to a generic spin-bit state

\[
H_2 |A_1, ..., A_L; \gamma\rangle.
\]

(5.4)

In such a way we will map (by means of the spin-bit isomorphic correspondence) $H_2$ to the 1-loop spin-bit Hamiltonian $H_{2, sb}$. Clearly, since $H_2$ is a second-order differential operator (it contains two operatorial derivatives), the Leibnitz rule will decompose the result in a sum over all possible couples of spin-chain sites:

\[
H_2 |A_1, ..., A_L; \gamma\rangle = -\frac{2}{N} \sum_{j=0}^{\infty} h(j) (P_j)_{CD}^{AB}.
\]

(5.5)

\[
: Tr [W_A, \tilde{W}^C] [W_B, \tilde{W}^D] : |A_1, ..., A_L; \gamma\rangle = \sum_{k,l=1}^{L} H_{2, kl} |A_1, ..., A_k; A_1, ..., A_l; \gamma\rangle,
\]

where $H_{2, kl}$ is nothing but the restriction of the 1-loop effective vertex to the couple of sites $(k, l) \in \{1, ..., L\}^2$, and it will be later identified, by the spin-bit map, with the two-site 1-loop spin-bit Hamiltonian.

A number of technical, permutational results are used in the explicit calculations; they respectively read:

\[
\text{Fission formula : } Tr (A \tilde{W}^C B W_D) = \delta_D^C Tr (A) Tr (B);
\]

(5.6)

\[
\text{Fusion formula : } Tr (A \tilde{W}^C) Tr (W_D B) = \delta_D^C Tr (A B),
\]

(5.7)

where $A$ and $B$ are supposed not to depend on $W$’s. An useful property (holding true for any permutation $\gamma$ and for any pair-site permutation $\sigma_{kl}$ in $S_L$) is

\[
\sigma_{kl} \gamma = \gamma \sigma_{\gamma_k \gamma_l};
\]

(5.8)
by using it, Eq. (3.2.3) yields
\[
\begin{align*}
\left| A_1, ..., B, ..., A, ..., A_L; \gamma \sigma_{kl} \rightangle &= \\
&= \left| A_1, ..., A, ..., B, ..., A_L; \gamma \sigma_{\gamma k\gamma l} \rightangle
\end{align*}
\]
(5.9)
or, in terms of operators
\[
P_{kl} \Sigma_{kl} = \Sigma_{\gamma k\gamma l},
\]
(5.10)
where \( P_{kl} \) is the \((k, l)\)-site permutation operator acting on \( H_{sc} \) defined by Eq. (3.2.8), and \( \Sigma_{k,l} \equiv \Sigma_{kl} \) is the (1-loop) chain “splitting and joining” (or “twist”) operator, acting on \( \zeta_L \), and defined as
\[
\Sigma_{kl} |\gamma \rangle \equiv \begin{cases} 
|\gamma \sigma_{kl} \rangle, & k \neq l \\
N |\gamma \rangle, & k = l 
\end{cases},
\]
(5.11)
or equivalently
\[
\Sigma_{kl} = N \delta_{kl} + (1 - \delta_{kl}) \overline{\Sigma}_{kl}, \text{ with } \overline{\Sigma}_{kl} |\gamma \rangle = |\gamma \sigma_{kl} \rangle.
\]
(5.12)
The factor \( N \) in the case \( k = l \) appears because splitting and joining a trace/chain at the same site leads to a new chain of length zero, whose corresponding trace is \( Tr (1) = N \), because 1 stands for the identity in the adjoint repr. of the gauge group \( SU(N) \).

The final result is the 1-loop spin-bit Hamiltonian
\[
H_{2, sb} = \frac{1}{2N} \sum_{k,l=1 \atop (k \neq l)}^L H_{kl} \left( \Sigma_{\gamma k\gamma l} + \Sigma_{\gamma k\gamma l} - \Sigma_{kl} - \Sigma_{\gamma k\gamma l} \right)
\]
(5.13)
or, using the canonical form of the permutation \( \gamma \in S_L \),
\[
H_{2, sb} = \\
= \frac{1}{2N} \sum_{k,l=1 \atop (k \neq l)}^L H_{kl} \left( \Sigma_{[k+1],l} + \Sigma_{k,[l+1]} - \Sigma_{k,l} - \Sigma_{[k+1],[l+1]} \right),
\]
(5.14)
where \( H_{k,l} \equiv H_{kl} \equiv H_{kl, sb} \) is the two-site Hamiltonian, acting on \( H_{sc} \), and defined as follows \((k \neq l)\):
\[
H_{kl} |A_1, ..., A_L \rangle \equiv \\
= 4 \sum_{j=0}^{\infty} h(j) (P_j)^{AB} \left| A_1, ..., A, ..., B, ..., A_L \right\rangle.
\]
(5.15)
Notice that \( P_{kl} \) and \( H_{kl} \) act on \( H_{sc} \), whereas \( \Sigma_{kl} \) acts on \( \zeta_L \), and therefore
\[
[P_{kl}, \Sigma_{mn}] = 0 = [H_{kl}, \Sigma_{mn}], \quad \forall (k, l, m, n) \in \{1, ..., L\}^4.
\]
(5.16)
By comparing Eq. (5.5) with Eqs. (5.13) and (5.14), and by disregarding the degenerate case of coinciding sites $k = l$ (this can be shown not implying any loss of generality), we may conclude that

$$H_2 \left| A_1, \ldots, A_k, \ldots, A_l, \ldots, A_L; \gamma \right> =$$

$$= \sum_{(k \neq l)}^{L} H_{2,kl} \left| A_1, \ldots, A_k, \ldots, A_l, \ldots, A_L; \gamma \right> =$$

$$= \frac{1}{2N} \sum_{(k \neq l)}^{L} H_{kl} (\Sigma_{\gamma_k l} + \Sigma_{k \gamma_l} - \Sigma_{kl} - \Sigma_{\gamma_k \gamma_l})$$

(5.17)

$$\left| A_1, \ldots, A_k, \ldots, A_l, \ldots, A_L; \gamma \right> \Leftrightarrow$$

$$\Leftrightarrow H_{2,kl}(\gamma) \equiv H_{2,kl} =$$

$$= \frac{1}{2N} H_{kl} (\Sigma_{\gamma_k l} + \Sigma_{k \gamma_l} - \Sigma_{kl} - \Sigma_{\gamma_k \gamma_l}),$$

$$\forall (k, l) \in \{1, \ldots, L\}^2, k \neq l,$$

where, as previously mentioned, $H_{2,kl}$ is the ($\gamma$-dependent) restriction of the 1-loop spin-bit Hamiltonian to the couple of sites $(k, l) \in \{1, \ldots, L\}^2, k \neq l$.

### 5.1 The planar limit

From Eq. (5.12), the planar limit $N \to \infty$ affects just the (1-loop) “twist” operator in the following way:

$$\lim_{N \to \infty} \frac{1}{N} \Sigma_{kl} = \delta_{kl}. \quad (5.1.1)$$

Therefore the planar contributions to the 1-loop spin-bit Hamiltonian come from terms involving $\Sigma_{kk}$, i.e. from the cases $l = \gamma_k$ and $k = \gamma_l$ (because $k \neq l$); the final result is

$$\lim_{N \to \infty} H_{2,ab} = \sum_{k=1}^{L} H_{k\gamma_k} = \sum_{k=1}^{L} H_{k,[k+1]} = H_{2,sc}. \quad (5.1.2)$$

Otherwise speaking, by construction the planar limit of the 1-loop spin-bit Hamiltonian coincides with the 1-loop spin-chain Hamiltonian $H_{2,sc}$.

### 6 Spin-bits in the $su(2)$ sector of $\mathcal{N} = 4$ SYM

We will now consider the “minimal” sector of $\mathcal{N} = 4, d = 4$ SYM $SU(N)$ gauge theory, made by “purely-scalar” g.i. (polynomial) operators, i.e. by operators generated only by
two holomorphic combinations of the real SYM scalars, which may be defined as
\[\begin{align*}
\Phi &\equiv \phi^5 + i\phi^6, \\
Z &\equiv \phi^1 + i\phi^2.
\end{align*}\] (6.1)

Thus, \(\Phi\) and \(Z\) will be the only SYM “letters” used to compose “words” and “sentences” in such an operator sector, which may be shown to be closed under the operator mixing due to perturbative renormalization of the theory. \(\Phi\) and \(Z\) will transform in the 2-dim. \(s = 1/2\) fundamental repr. of \(su(2)\), and therefore the whole sector will be \(su(2)\)-symmetric. Notice that \(su(2)\) is the smallest non-trivial bosonic compact subalgebra of the whole \(N = 4\) SCA \(psu(2,2 | 4)\), and the following chain of inclusions holds:
\[su(2) \subset so(6) \sim su(4) \subset so(4,2) \oplus su(4) \subset psu(2,2 | 4)\]. (6.2)

A generic \(M (\leq L)\)-trace g.i. operator in such a \(su(2)\) closed subsector reads
\[O \equiv Tr (\Phi Z \Phi \Phi ...) Tr (\Phi \Phi \Phi Z ...) ... Tr (\Phi Z \Phi Z ...),\] (6.3)
with \(\sum_{r=1}^{M} L_r = L\).

As previously shown, we may use the isomorphic spin-bit map to equivalently represent \(O\) as
\[O \equiv (\phi^{i_1 \alpha_1 \gamma_1}) (\phi^{i_2 \alpha_2 \gamma_2}) ... (\phi^{i_L \alpha_L \gamma_L}) \equiv |S; \gamma\rangle \equiv |S\rangle \otimes_{S_{\gamma}} |\gamma\rangle,\] (6.4)
where now the range of all “\(i\)-indices” of the scalars is \(\{\hat{1}, \hat{2}\}\), with \(\hat{1} \equiv \Phi\) and \(\hat{2} \equiv Z\) by convention. In the case of Eq. (6.3) we have
\[S_{\gamma} \ni \gamma = (L_1) (L_2) ...(L_M),\] (6.5)
with \((L_r)\) denoting a cyclic permutation of \(L_r\) elements, \(r = 1, ..., M \leq L\). The correspondence operated by the spin-bit map is completed by associating to each spin-chain site the spin value \(|-1/2\rangle\) if we find \(\Phi\) there, and \(|1/2\rangle\) if we find \(Z\).

By specializing the general expression (5.15) of the two-site Hamiltonian to the case of 2-dim. \(s = 1/2\) (representation of) \(su(2)\) symmetry, the final result is simply [22]
\[H_{kl, su(2)} = 2 (1 - P_{kl})\], (6.6)
and therefore, by substituting it in the general formulae (5.13) and (5.14), we obtain the 1-loop \(su(2)\) spin-bit Hamiltonian \(H_{2, sb, su(2)} \equiv H_{2, su(2)}\)
\[H_{2, su(2)} = \frac{1}{2N} \sum_{k,l=1}^{L} H_{kl, su(2)} (\Sigma_{\gamma k l} + \Sigma_{k \gamma l} - \Sigma_{k l} - \Sigma_{\gamma k \gamma l}) =
= \frac{2}{N} \sum_{k,l=1}^{L} (1 - P_{kl}) \Sigma_{k \gamma l}.\] (6.7)
In the planar limit, by recalling Eq. (5.1.1) we get
\[
\lim_{N \to \infty} H_{2, su(2)} = 2 \sum_{k,l=1}^{L} (1 - P_{kl}) \lim_{N \to \infty} \frac{1}{N} \sum_{k \gamma l} = \\
= 2 \sum_{k,l=1}^{L} (1 - P_{kl}) \\
\lim_{N \to \infty} \frac{1}{N} \left[ N \delta_{k \gamma l} + (1 - \delta_{k \gamma l}) \sum_{k \gamma l} \right] = \\
= 2 \sum_{k=1}^{L} (1 - P_{k \gamma k}) = \sum_{k=1}^{L} H_{k \gamma k, su(2)} = \\
= H_{2, sc, su(2)}.
\]
(6.8)

In other words, the planar limit of the 1-loop su(2) spin-bit Hamiltonian coincides with the integrable XXX_{s=1/2} Heisenberg su(2) spin-chain Hamiltonian \( H_{2, sc, su(2)} \) [37].

The expression (6.7) of \( H_{2, su(2)} \) might also be obtained by starting from the known combinatorial formula of the 1-loop \( \mathcal{N} = 4, d = 4 \) SYM effective vertex on the su(2)-symmetric closed subsector of “purely scalar” composite (polynomial) g.i. operators, reading [29]
\[
H_{2} = -\frac{4}{N} : Tr ([\Phi, Z] [\tilde{\Phi}, \tilde{Z}]) :,
\]
(6.9)
where
\[
\begin{align*}
(\hat{\Phi})^{ab} & \equiv \frac{\partial}{\partial \Phi^{ca}}, \\
(\hat{Z})^{ab} & \equiv \frac{\partial}{\partial Z^{ca}}.
\end{align*}
\]
(6.10)

7 \( su(2) \) spin-bits at 2 loops

The \( su(2) \) sector of \( \mathcal{N} = 4, d = 4 \) SYM is the only one, as far as we know, for which a detailed treatment of non-planar (\( N < \infty \)) 2-loop anomalous dimensions has been given. This is due to the noteworthy fact that a combinatorial formula of the 2-loop \( \mathcal{N} = 4, d = 4 \) SYM effective vertex on the su(2)-symmetric closed subsector of “purely scalar” composite (polynomial) g.i. operators is known [29]:
\[
H_{4} = \frac{2}{N^2} \left[ : Tr ([Z, \Phi] [Z, [Z, [Z, \Phi]]]) : + \\
+ : Tr ([Z, \Phi] [\Phi, [Z, [Z, \Phi]]]) : + \\
+ 2N : Tr ([\Phi, Z] [\tilde{\Phi}, \tilde{Z}]) : \right].
\]
(7.1)

By applying such a combinatorial formula on a generic \( su(2) \)-symmetric \( \mathcal{N} = 4 \) SYM operator/spin-bit state \( |S; \gamma \rangle \), we obtain, after some permutational algebra and technical
tricks, the following expression for the 2-loop $su(2)$ spin-bit Hamiltonian $H_{4,sb,su(2)} \equiv H_{4,su(2)}$:

$$H_{4,su(2)} = \frac{2}{N^2} \sum_{k,l,m=1}^{L} (2P_{lm} + 2P_{kl} - P_{km} - 3) \Sigma_{klm} (\gamma), \quad (7.2)$$

where $\Sigma_{klm} (\gamma)$ is the $(su(2))$ 2-loop “twist” operator, acting on $\zeta_L$, and defined as

$$\Sigma_{klm} (\gamma) \equiv \Sigma_{k\gamma} \Sigma_{l\gamma m}. \quad (7.3)$$

### 7.1 The planar limit

From Eq. (5.12), the planar limit $N \to \infty$ affects just $\Sigma_{klm} (\gamma)$ in the following way:

$$\lim_{N \to \infty} \frac{1}{N^2} \Sigma_{klm} (\gamma) = \delta_{k\gamma} \delta_{l\gamma m}. \quad (7.1.1)$$

Therefore, we obtain

$$\lim_{N \to \infty} H_{4,su(2)} = \frac{2}{N^2} \sum_{k=1}^{L} \left( 4P_{k\gamma k} - P_{k\gamma 2}^2 - 3 \right) = \sum_{k=1}^{L} \left( 4P_{k,[k+1]} - P_{k,[k+2]} - 3 \right) = H_{4,sc,su(2)}. \quad (7.1.2)$$

Otherwise speaking, by construction the planar limit of the 2-loop $su(2)$ spin-bit Hamiltonian coincides with the integrable $su(2)$ spin-chain Hamiltonian $H_{4,sc,su(2)}$, which in turn corresponds to an integrable (higher-order) deformation of the previously mentioned $XXX_{s=1/2}$ Heisenberg $su(2)$ spin-chain Hamiltonian $H_{2,sc,su(2)}^{29,38}$.

### 8 $su(2)$ spin-bits beyond 2 loops

#### 8.1 The “deplanarizing operator lifts” (d.o.l.) method

$su(2)$-symmetric spin-chain Hamiltonians are known explicitly up to (and including) 5 loops $^{26,29}$. Unfortunately, combinatorial formulae for the SYM effective vertices in the $su(2)$ sector are not known beyond 2 loops, and therefore higher-loop $su(2)$ spin-bit Hamiltonians are not directly obtainable as in the cases of 1 and 2 loops. Thus, other approaches have to be pursued in order to derive them. After the failure of the elegant and geometrically meaningful “spin-edge differences” Ansätze $^{35}$, the only planarly-consistent, fully testable set of conjectures for the higher-loop $su(2)$ spin-bit Hamiltonians are those obtained by applying the recently proposed $^{35,39}$ “deplanarizing operator lifts” (“d.o.l.”) method, eventually with the additional hypothesis of “symmetrization of deplanarizing operator splittings” (hp. “s.d.o.s.”).
In the following we will present such a deplanarizing approach in a sketchy, algorithmic way, addressing the interested reader to the original literature for further elucidations.

The d.o.l. algorithm may be realized through the following steps:

1) we have to start from the known planar results for the Hamiltonian. Since the non-planar 1- and 2-loop orders are already known and have been previously treated, we have to consider the 3-, 4- and 5-loop expressions of the planar $su(2)$ spin-chain Hamiltonian (input of the deplanarizing algorithm);

2) then, we have to perform the (non-reductive) conventional site-index identifications

$$l = k + 1, \ m = k + 2, \ ...;$$

(8.1.1)

3) therefore, we have to consider all possible products of operators $P$’s that, in the planar limit, would give the considered planar permutational term; this will determine some proper “deplanarizing operator lifts”. All such non-planar permutational terms will come with free (real) coefficients, constrained by two requests:

3.1) their algebraic sum must give the right numerical known coefficient of the considered planar permutational term;

3.2) they must make the spin part of the non-planar Hamiltonian completely symmetric under the particular, inverting exchange of site indices

$$(k, l, ..., r, s) \leftrightarrow (s, r, ..., l, k),$$

(8.1.2)

as requested by the site index structure determined by the “linking” part.

4) Indeed, for what concerns the “linking” part of the Hamiltonian, i.e. the “twisting” operators $\Sigma$’s, we may generalize the “linking” part of Eqs. (6.7) and (7.2) by introducing the (spin-dependent) “splitting and joining chain operator of order $n$” as

$$\Sigma_{k_1k_2...k_{n+1}}(\gamma) \equiv \Sigma_{k_1\gamma k_2} \Sigma_{k_2\gamma k_3} ... \Sigma_{k_{n-1}\gamma k_n} \Sigma_{k_n\gamma k_{n+1}}.$$  

(8.1.3)

5) Generally, at higher-loop orders, some (real) parameters still remain undetermined at this step. In order to obtain a completely determined expression of the higher-loop $su(2)$ spin-bit Hamiltonian as output of the proposed deplanarization procedure, we may proceed as follows.

As a reasonable conjecture, we may formulate an additional assumption, that we are going to call hypothesis of “symmetrization of deplanarizing operator splittings” (hp. “s.d.o.s.”). This conjecture has to be applied after the symmetrization of the non-planar terms with respect to the peculiar renaming of spin-chain site indices given by (8.1.2); it amounts to say that each of the sets of non-planar terms arising from a considered planar term in the deplanarization procedure will equally contribute in the planar limit $N \to \infty$. 

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For example, if, after the symmetrization with respect to (8.1.2), a planar term $\Im$ is deplanarized by the 3-fold splitting

$$\Im \rightarrow a_1 \Im_{M_1} + a_2 \Im_{M_2} + a_3 \Im_{M_3},$$

(8.1.4)

where $\Im_{M_1}$, $\Im_{M_2}$ and $\Im_{M_3}$ are respectively sets consisting of $M_1$, $M_2$ and $M_3$ non-planar terms made by permutation operators, then we will assume that

$$M_1 a_1 = M_2 a_2 = M_3 a_3.$$ 

(8.1.5)

Hence the contribution of $\Im_{M_1}$, $\Im_{M_2}$ and $\Im_{M_3}$ to the planar limit $\Im$ is the same, and therefore the operator splitting given by (8.1.4) may be considered symmetric.

As it will be seen explicitly further below at 3-loops, this additional hypothesis will allow to fix all the free real parameters, otherwise necessarily introduced by the deplanarizing operator lifts acting at the considered higher-loop order. Notice that the constraints 3.i and 3.ii are implied by the hp. s.d.o.s.

Thus, a complete Ansatz for the $su(2)$ spin-bit Hamiltonian at the considered loop order is obtained$^2$ (output of the deplanarizing algorithm).

### 8.2 Application at 3 loops

Let us consider an explicit example of application of the described method, in order to build a consistent Ansatz for the expression of the 3-loop $su(2)$ spin-bit Hamiltonian. Let us follow the previously mentioned steps:

1) we start from the known expression of $H_{6,sc,su(2)}$, namely the 3-loop, integrable, perturbative deformation of the $XXX_{s=1/2}$ Heisenberg $su(2)$ spin-chain Hamiltonian $H_{2,sc,su(2)}$ [26, 29, 38], given by (input of the deplanarizing algorithm for $n = 3$-loop order)

$$H_{6,sc,su(2)} =$$

$$= 4 \sum_{k=1}^{L} \left[ 15 - 26P_{k,k+1} + 6(P_{k,k+1}P_{k+1,k+2} + P_{k+1,k+2}P_{k,k+1}) + P_{k,k+1}P_{k+2,k+3} - (P_{k,k+1}P_{k+1,k+2}P_{k+2,k+3} + P_{k+2,k+3}P_{k+1,k+2}P_{k,k+1}) \right];$$

(8.2.1)

$^2$It should be noticed that here we assume that (eventually rather structurally complicated) non-planar permutational terms, such that their planar limit is zero, do not exist; indeed, for the time being, their existence may not be guessed by an inferring approach starting from the planar level, such as the one adopted in this paper.
2) we conventionally identify (without loss of generality) the spin-chain site indices in the following way:

\[ l \equiv k + 1, \quad m \equiv k + 2, \quad n \equiv k + 3; \quad (8.2.2) \]

3) therefore, we have to find all possible non-planar permutational terms giving rise, in the planar limit \( N \to \infty \), to each of the permutational terms of \( H_{6,sc,su(2)} \) given by Eq. (8.2.1).

3.i) We have that:

3.i.a) the planar term \( P_{k,k+1} \) receives three contributions from the non-planar level, respectively from \( P_{k,k+1} = P_{kl}, \quad P_{k+1,k+2} = P_{lm} \) and \( P_{k+2,k+3} = P_{mn} \), whence the proper “deplanarizing operator lift” of \( P_{k\gamma_k} \) reads \((\xi_1, \xi_2 \in R)\)

\[-26P_{k,k+1} \to \xi_1 P_{kl} + \xi_2 P_{lm} - (26 + \xi_1 + \xi_2)P_{mn}; \quad (8.2.3)\]

3.i.b) the planar-level product \( P_{k,k+1}P_{k+1,k+2} \) instead receives contribution just from two non-planar terms, i.e. \( P_{k,k+1}P_{k+1,k+2} = P_{kl}P_{lm} \) and \( P_{k+1,k+2}P_{k+2,k+3} = P_{lm}P_{mn} \), whence the proper “deplanarizing operator lift” of the term \( P_{k,k+1}P_{k+1,k+2} \) reads \((\xi_3 \in R)\)

\[ 6P_{k,k+1}P_{k+1,k+2} \to \xi_3 P_{kl}P_{lm} + (6 - \xi_3)P_{lm}P_{mn}; \quad (8.2.4) \]

3.i.c) analogously, for the other terms of \( H_{6,sc,su(2)} \) we obtain the following proper “deplanarizing operator lifts” \((\xi_4 \in R)\):

\[ 6P_{k+1,k+2}P_{k,k+1} \to \xi_4 P_{lm}P_{kl} + (6 - \xi_4)P_{mn}P_{lm}; \]
\[ P_{k,k+1}P_{k+2,k+3} \to P_{kl}P_{mn}; \quad (8.2.5) \]
\[ P_{k,k+1}P_{k+1,k+2}P_{k+2,k+3} \to P_{kl}P_{lm}P_{mn}; \]
\[ P_{k+2,k+3}P_{k+1,k+2}P_{k,k+1} \to P_{mn}P_{lm}P_{kl}; \]

3.ii) thence, we impose the symmetry of the spin part under the site index exchange

\[(k, l, m, n) \leftrightarrow (n, m, l, k) ; \quad (8.2.6)\]

the imposition of such a condition on the spin part decreases the number of free (real) parameters from four to two, thence renamed \( \eta_1 \) and \( \eta_2 \);
4) finally, we put
\[
\frac{1}{N^3} \sum_{k\ell mn} (\gamma) = \frac{1}{N^3} \sum_{k\ell} \sum_{\ell m} \sum_{m\gamma_n} (8.2.7)
\]
as the linking variable part.

Thus, we may finally write the most general expression of the 3-loop \(su(2)\) spin-bit Hamiltonian \((\eta_1, \eta_2 \in R)\):
\[
H_{6, su(2)} (\eta_1, \eta_2) = \frac{4}{N^3} \sum_{k,l,m,n=1}^L \sum_{k\ell} \sum_{\ell m} \sum_{m\gamma_n} (8.2.8)
\]

Formulating the hp. s.d.o.s. we get \((\eta_1, \eta_2) = (-\frac{13}{2}, 3)\), and therefore we obtain a completely fixed expression for the 3-loop \(su(2)\) spin-bit Hamiltonian (output of the deplanarizing algorithm for \(n = 3\)-loop order):
\[
H_{6, su(2)} = \frac{4}{N^3} \sum_{k,l,m,n=1}^L \sum_{k\ell} \sum_{\ell m} \sum_{m\gamma_n} (8.2.9)
\]

Analogous, more and more involved, expressions for the 4- and 5-loops \(su(2)\) spin-bit Hamiltonian (with or without the additional hp. s.d.o.s.) have been obtained by applying the d.o.l. method \[35, 39\].

9 Outlook and further developments

The d.o.l. method \[39\] is fully compatible with (independently obtained) known results at the 1- and 2-loop, non-planar level \[31, 32, 33\], and it allows one to obtain explicit formulae for the 3-, 4- and 5-loop \(su(2)\) spin-bit Hamiltonians. By construction, such
expressions are planarly-consistent, i.e. they have the correct planar limit, matching the known results reported in the literature (see e.g. [26, 29, 38]).

It is also worth noticing that, by construction, all the higher-loop $su(2)$ spin-bit Hamiltonians (for 3-loops, see Eqs. (8.2.8) and (8.2.9)) show an explicit full factorization in the spin and chain-splitting parts; as already pointed out in [33], such a property is expected to hold at every loop order, since the Hilbert space of the spin-bit model $\mathcal{H}_{sb}$ is given by the direct product (modulo the action of the permutation group $S_L$) of the spin-chain Hilbert space $\mathcal{H}_{sc}$ and of the linking space $\zeta_L$.

Attention must also be paid to the fact that, while (both at non-planar and planar level) the 1- and 2-loop formulae for the $su(2)$ Hamiltonians are linear in the site permutation operators $P$’s, the 3-, 4- and 5-loop level expressions, both at non-planar and planar level, show a non-linearity (and non-linearizability) in $P$’s. For example, the non-linearizability of the 3-loop $su(2)$ spin-chain Hamiltonian (8.2.1) caused the failure of the elegant and geometrically meaningful “spin edge-differences” [35] approach to higher-loop Ansätze.

Thus, the non-linearity (and non-linearizability) in site permutation operators seems to be a crucial and fundamental feature, starting to hold at the 3-loop order, of the spin part of the Hamiltonian of the $su(2)$ spin-bit model, underlying the non-planar dynamics of the $su(2)$ sector of the $\mathcal{N} = 4, d = 4$ SYM theory. Reasonably, one would expect that such a breakdown of “permutational linearizability” at 3 loops (for the first evidences from 3-loop calculations, see e.g. [26, 29]; for further subsequent developments see e.g. [2, 8, 40]) gives rise, by means of the AdS/CFT correspondence [3, 4], to some “new” features in the dynamics of the (closed) superstrings in the bulk of $AdS_5 \times S^5$. Actually, in the AdS/CFT correspondence framework, there is a problem of discrepancy between the calculations made with fast spinning, semiclassical strings (i.e. in the so-called Frolov-Tseytlin limit) and the calculations made in the thermodynamical limit of long spin-chains with a large number of excitations (i.e. the so-called Berenstein-Maldacena-Nastase limit) (see e.g. [38] and Refs. therein). Such a disagreement starts to hold at 3 loops, and it is one of the most intriguing “mysteries” of the AdS/CFT conjecture. Recently, an explanation for such 3- and higher-loop disagreement has been proposed: it should be related to an “order-of-limit” non-commutation problem in the perturbative expansions and thermodynamical asymptotical regimes or, equivalently, to the presence of operational “wrapping” interactions (see e.g. [38, 41]). Conjecturally, we may here put forward the suggestion that the breakdown of “permutational linearizability” of $su(2)$ spin-chain/spin-bit Hamiltonians, which starts to hold at 3 loops, could be related to such a “3-loop discrepancy mystery” in AdS/CFT, and possibly it could be extended also to larger symmetries inside $psu(2, 2|4)$.

Also, the application of the d.o.l. method, originally introduced for the Hamiltonian, to the higher-order charges of the $su(2)$ spin-chain model [39] raises some interesting and
intriguing questions, such as:

1) the d.o.l. appears to be consistent only for odd higher-order charges, sharing the symmetry properties of the Hamiltonian. Thus, the extension of the deplanarization procedure to even higher-order charges, and in general to antisymmetric operators, should be needed, in order to have a complete deplanarizing algorithm for the local operators in $\mathcal{N} = 4, d = 4$ SYM theory;

2) we know that the spectrum of the (perturbatively expanded) Hamiltonian of the spin-chain/spin-bit model is related to the spectrum of the (perturbatively expanded) anomalous dimensions and mixing of local operators on the $\mathcal{N} = 4, d = 4$ SU($N$) SYM gauge theory side of AdS/CFT. Then a natural question \[29\] to ask is: do the spectra of the higher-order planar charges (and of their deplanarized counterparts) have a physical meaning in $\mathcal{N} = 4$ SYM?

The deep question is evidently: why does exact integrability seem to hold for every loop-order in the planar $\mathcal{N} = 4, d = 4$ SYM \[25, 38, 43, 42\], and why and how is it lost at the non-planar level?

By the way, even if the exact (classical) integrability is lost when deplanarizing, i.e. when passing from the (all-loop) $su(2)$ Heisenberg spin-chain to the (all-loop) $su(2)$ spin-bit model, nevertheless we may put forward the following intriguing suggestion: could the $su(2)$ spin-bit model still be an integrable model, but in a sort of generalized, broader sense, e.g. in the sense of quasi-integrability and quasi-exact solvability (see e.g. \[44\]) or quantum-integrability (see e.g. \[45\])?

If yes, then in general the deplanarization procedure here presented might algebraically correspond to some kind of “deformation” of the (dynamical) symmetries of the system being considered, and thence of the structure and properties of its (eventually conserved) charges.

In our opinion, this is an interesting problem, strictly related to the consistent definition of the higher-order charges of the spin-bits as the deplanarization of the infinitely many conserved charges of the spin-chains, and it is currently under study.

Moreover, we notice that it would be interesting, following recent research directions, to extend the considered deplanarizing method to other operatorial sectors of the $\mathcal{N} = 4, d = 4$ SYM \[46\]. Indeed, sectors with non-compact symmetries have been shown to be relevant, in order to describe the renormalization in the large $N_c$ (non-)SUSY QCD, also in relation with the attempts to construct a string description of QCD (see e.g. \[47\] and Refs. therein).

Finally, some possible additional directions for further research are briefly summarized as follows:

- All the spin-chain/spin-bit models treated so far are characterized by periodic boundary conditions, corresponding to closed chains. The possibility to modify the boundary conditions yields open spin-chain models, corresponding to a dynamical discretization of
the open strings [48]. The issue of deplanarizability, and the related problem of integrability, of such models remains to be discussed.

- In general, the spin-bits may be considered as a dynamical polymer model with decaying and fusing chains; potential applications to relativity theory [49], field theory [50] and biophysics could be addressed.

- Interesting analogies could be explored with the spin-network approach to discrete quantum gravity [51].

- The above mentioned non-planar permutational identities could be linked to the random graph theory on a lattice [35].

- The spin representation of the permutation operators in the $su(2)$ sector leads to some “generalized” Fierz identities for Pauli $\sigma$ matrices [35], deserving a more detailed analysis.

- An alternative approach to the calculation of anomalous dimensions in $\mathcal{N} = 4$ SYM is based on matrix models [12], which can be also formulated in terms of a non-commutative field theory on a torus [13, 52], and whose equivalence with spin-chain/spin-bit models in the $N \to \infty$ (planar) and $N \to 0^+$ limits is still under study. An interesting direction of research to be pursued would be the formulation of such matrix models on other “fuzzy” manifolds, such as the “fuzzy sphere” [53], and the study of the relation of such non-toroidal “fuzzy” matrix models with the initial non-commutative torus representation.

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References

[1] G. ’t Hooft, “A planar diagram theory for strong interactions”, *Nucl. Phys.* **B72** (1974) 461.

[2] A. A. Tseytlin, “Semiclassical strings and AdS/CFT”, contribution to the Proceedings of Cargese Summer School, June 7-19, 2004, *hep-th/0409296*.

[3] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity”, *Adv. Theor. Math. Phys.* **2** (1998) 231, *hep-th/9711200*.

[4] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, “Gauge theory correlators from non-critical string theory”, *Phys. Lett.* **B428** (1998) 105, *hep-th/9802109*. 

24
[5] D. Berenstein, J. M. Maldacena, and H. Nastase, “Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang Mills”, *JHEP* **04** (2002) 013, [hep-th/0202021](http://arxiv.org/abs/hep-th/0202021).

[6] D. Berenstein and H. Nastase, “On lightcone string field theory from super Yang–Mills and holography”, [hep-th/0205048](http://arxiv.org/abs/hep-th/0205048).

[7] D. Berenstein, E. Gava, J. M. Maldacena, K. S. Narain, and H. Nastase, “Open strings on plane waves and their Yang–Mills duals”, [hep-th/0203249](http://arxiv.org/abs/hep-th/0203249).

[8] A. A. Tseytlin, “On semiclassical approximation and spinning string vertex operators in $AdS_5 \times S^5$”, *Nucl. Phys.* **B664** (2003) 247, [hep-th/0304139](http://arxiv.org/abs/hep-th/0304139).

[9] S. Frolov and A. A. Tseytlin, “Multi-spin string solutions in $AdS_5 \times S^5$”, *Nucl. Phys.* **B668** (2003) 77, [hep-th/0304255](http://arxiv.org/abs/hep-th/0304255).

[10] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors”, *Phys. Lett.* **B570** (2003) 96, [hep-th/0306143](http://arxiv.org/abs/hep-th/0306143).

[11] N. Beisert, S. Frolov, M. Staudacher, and A. A. Tseytlin, “Precision spectroscopy of AdS/CFT”, *JHEP* **10** (2003) 037, [hep-th/0308117](http://arxiv.org/abs/hep-th/0308117).

[12] A. Agarwal and S. G. Rajeev, “The Dilatation Operator of $\mathcal{N} = 4$ SYM and Classical Limits of Spin Chains and Matrix Models”, *Mod. Phys. Lett.* **A19** (2004) 2549, [hep-th/0405116](http://arxiv.org/abs/hep-th/0405116) • A. Agarwal and S. G. Rajeev, “Yangian Symmetries of Matrix Models and Spin Chains : The Dilatation Operator of $\mathcal{N} = 4$ SYM”, *Int. J. Mod. Phys.* **A20** (2005) 5453-5490, [hep-th/0409180](http://arxiv.org/abs/hep-th/0409180).

[13] S. Bellucci and C. Sochichiu, “On matrix models for anomalous dimensions of super Yang-Mills theory”, *Nucl. Phys.* **B726** (2005) 233-251, [hep-th/0410010](http://arxiv.org/abs/hep-th/0410010).

[14] S. Bellucci and C. Sochichiu, “On the dynamics of BMN operators of finite size and the model of string bits”, Contribution to the BW2003 Workshop, 29 August - 02 September, 2003 Vrnjacka Banja, Serbia, [hep-th/0404143](http://arxiv.org/abs/hep-th/0404143) • S. Bellucci and C. Sochichiu, “Can string bits be supersymmetric?”, *Phys. Lett.* **B571** (2003) 92, [hep-th/0307253](http://arxiv.org/abs/hep-th/0307253) • S. Bellucci and C. Sochichiu, “Fermion Doubling and Berenstein–Maldacena–Nastase Correspondence”, *Phys. Lett.* **B564** (2003) 115, [hep-th/0302104](http://arxiv.org/abs/hep-th/0302104).

[15] M. Blau, J. Figueroa-O’Farrill, C. Hull, and G. Papadopoulos, “A new maximally supersymmetric background of IIB superstring theory”, *JHEP* **01** (2002) 047, [hep-th/0110242](http://arxiv.org/abs/hep-th/0110242).

[16] M. Blau, J. Figueroa-O’Farrill, C. Hull, and G. Papadopoulos, “Penrose limits and maximal supersymmetry”, *Class. Quant. Grav.* **19** (2002) L87, [hep-th/0201081](http://arxiv.org/abs/hep-th/0201081).
[17] M. Blau, J. Figueroa-O’Farrill, and G. Papadopoulos, “Penrose limits, supergravity and brane dynamics”, Class. Quant. Grav. 19 (2002) 4753, hep-th/0202111

[18] R. R. Metsaev, “Type IIB Green-Schwarz superstring in plane wave Ramond–Ramond background”, Nucl. Phys. B625 (2002) 70, hep-th/0112044.

[19] R. R. Metsaev and A. A. Tseytlin, “Exactly solvable model of superstring in plane wave Ramond–Ramond background”, Phys. Rev. D65 (2002) 126004, hep-th/0202109.

[20] C. Kristjansen, J. Plefka, G. W. Semenoff, and M. Staudacher, “A new double-scaling limit of $N = 4$ super Yang–Mills theory and pp-wave strings”, Nucl. Phys. B643 (2002) 3, hep-th/0205033 • D. J. Gross, A. Mikhailov, and R. Roiban, “Operators with large $R$ charge in $N = 4$ Yang–Mills theory”, Annals Phys. 301 (2002) 31, hep-th/0205066 • N. R. Constable et. al., “pp-wave string interactions from perturbative Yang–Mills theory”, JHEP 07 (2002) 017, hep-th/0205089 • N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff, and M. Staudacher, “BMN correlators and operator mixing in $N = 4$ super Yang–Mills theory”, Nucl. Phys. B650 (2003) 125, hep-th/0208178 • N. R. Constable, D. Z. Freedman, M. Headrick, and S. Minwalla, “Operator mixing and the BMN correspondence”, JHEP 10 (2002) 068, hep-th/0209002 • M. Spradlin and A. Volovich, “Note on plane wave quantum mechanics”, Phys. Lett. B565 (2003) 253, hep-th/0303220.

[21] M. Spradlin and A. Volovich, “Light-cone string field theory in a plane wave”, Lectures at the ICTP Spring School on Superstring Theory and Related Topics, Trieste, 31 March–8 April 2003, hep-th/0310033 • R. Russo and A. Tanzini, “The duality between IIB string theory on pp-wave and $N = 4$ SYM: A status report”, Class. Quant. Grav. 21 (2004) S1265, hep-th/0401155.

[22] N. Beisert, “The complete one-loop dilatation operator of $N = 4$ super Yang-Mills theory”, Nucl. Phys. B676 (2004) 3, hep-th/0307015.

[23] J. A. Minahan and K. Zarembo, “The Bethe–Ansatz for $N = 4$ super Yang–Mills”, JHEP 03 (2003) 013, hep-th/0212208.

[24] N. Beisert, “The dilatation operator of $N = 4$ super Yang–Mills theory and integrability”, Phys. Rept. 405 (2005) 1, hep-th/0407277.

[25] D. Serban and M. Staudacher, “Planar $N = 4$ gauge theory and the Inozemtsev long range spin chain”, JHEP 0406 (2004) 001, hep-th/0401057.

[26] N. Beisert, “Higher loops, integrability and the near BMN limit”, JHEP 0309 (2003) 062, hep-th/0308074.
[27] H. Verlinde, “Bits, matrices and 1/N”, *JHEP* **0312** (2003) 052, hep-th/0206059 • J.-G. Zhou, “pp-wave string interactions from string bit model”, *Phys. Rev.* **D67** (2003) 026010, hep-th/0208232 • D. Vaman and H. Verlinde, “Bit strings from N = 4 gauge theory”, *JHEP* **0311** (2003) 041, hep-th/0209215

[28] U. Danielsson, F. Kristiansson, M. Lubcke, and K. Zarembo, “String bits without doubling”, *JHEP* **0310** (2003) 026, hep-th/0306147.

[29] N. Beisert, C. Kristjansen, and M. Staudacher, “The dilatation operator of \( \mathcal{N} = 4 \) super Yang–Mills theory”, *Nucl. Phys.* **B664** (2003) 131, hep-th/0303060.

[30] N. Beisert, C. Kristjansen, J. Plefka, and M. Staudacher, “BMN gauge theory as a quantum mechanical system”, *Phys. Lett.* **B558** (2003) 229, hep-th/0212269.

[31] S. Bellucci, P. Y. Casteill, J. F. Morales, and C. Sochichiu, “Spin bit models from non-planar \( \mathcal{N} = 4 \) SYM”, *Nucl. Phys.* **B699** (2004) 151, hep-th/0404066.

[32] S. Bellucci, P. Y. Casteill, J. F. Morales, and C. Sochichiu, “Chaining spins from (super)Yang–Mills”, contribution to the XI International Conference on Symmetry Methods in Physics (SYMPHYS-11) (2004), hep-th/0408102.

[33] S. Bellucci, P. Y. Casteill, A. Marrani and C. Sochichiu, “Spin bits at two loops”, *Phys. Lett.* **B607** (2005) 180, hep-th/0411261.

[34] V. E. R. Lemes, M. S. Sarandy, S. P. Sorella, O. S. Ventura and L. C. Q. Vilar, “An algebraic criterion for the ultraviolet finiteness of quantum field theories”, *J. Phys. A: Math. Gen.* **34** (2001) 9485, hep-th/0103110.

[35] S. Bellucci and A. Marrani, “Non-Planar Spin Bits beyond two loops”, hep-th/0505106.

[36] A. Kundu, “New nonultralocal quantum integrable models through gauge transformation”, hep-th/0207036 • A. Kundu, “Ultralocal solutions for quantum integrable nonultralocal models”, *Phys. Lett.* **B550** (2002) 128, hep-th/0208147.

[37] L. D. Faddeev, “How algebraic Bethe Ansatz works for integrable models”, in "Les Houches 1995, Relativistic gravitation and gravitational radiation", 149 (1971) hep-th/9605187.

[38] N. Beisert, V. Dippel and M. Staudacher, “A Novel Long-Range Spin Chain and Planar \( \mathcal{N} = 4 \) super Yang-Mills”, *JHEP* **0407** (2004) 075, hep-th/0405001.

[39] S. Bellucci and A. Marrani, “Deplanarization methods for Hamiltonian and higher-order charges in spin-bit models”, in preparation.
[40] E. D’Hoker, P. Heslop, P. Howe and A. V. Ryzhov, “Systematics of quarter BPS operators in $\mathcal{N}=4$ SYM”, JHEP 0304 (2003) 038, hep-th/0301104 • A. V. Ryzhov and A. A. Tseytlin, “Towards the exact dilatation operator of $\mathcal{N}=4$ super Yang-Mills theory”, Nucl. Phys. B698 (2004) 132, hep-th/0404215 • M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, “Large spin limit of $AdS_5 \times S^5$ string theory and low energy expansion of ferromagnetic spin-chains”, Nucl. Phys. B692 (2004) 3, hep-th/0403120.

[41] N. Beisert, “Higher-Loop Integrability in $\mathcal{N}=4$ Gauge Theory”, Comptes Rendus Physique 5 (2004) 1039, hep-th/0409147.

[42] N. Beisert and M. Staudacher, “Long Range $psu(2,2|4)$ Bethe Ansätze for Gauge Theory and Strings”, hep-th/0504190.

[43] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, “Spinning strings in $AdS_5 \times S^5$ and integrable systems,” Nucl. Phys. B671 (2003) 3, hep-th/0307191 • G. Arutyunov and M. Staudacher, “Matching higher conserved charges for strings and spins,” JHEP 0403 (2004) 004, hep-th/0310182 • G. Arutyunov, S. Frolov and M. Staudacher, “Bethe ansatz for quantum strings,” JHEP 0410 (2004) 016, hep-th/0406256.

[44] G. Falqui, C.-M. Viallet, “Singularity, complexity, and quasi-integrability of rational mappings”, Commun. Math. Phys. 154 (1993), 111, hep-th/9212105 • S. Klishchевич, “Quasi-exact solvability and intertwining relations”, hep-th/0410064 • D. Gomez-Ullate, N. Kamran and R. Milson, “Quasi-exact solvability and the direct approach to invariant subspaces”, J. Phys. A38 (2005), 2005, nlin.SI/0401030.

[45] V. A. Kazakov and K. Zarembo, “Classical/quantum integrability in non-compact sector of AdS/CFT”, JHEP 0410 (2004) 060, hep-th/0410015 • V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, “Classical/quantum integrability in AdS/CFT”, JHEP 0405 (2004) 024, hep-th/0402207 • O. Babelon and M. Talon, “Riemann surfaces, separation of variables and classical and quantum integrability”, Phys. Lett. A312 (2003), 71, hep-th/0209071.

[46] S. Bellucci, P. Y. Casteill, J. F. Morales, and C. Sochichiu, “$SL(2)$ spin chain and spinning strings on $AdS_5 \times S^5$”, Nucl. Phys. B707 (2005) 303, hep-th/0409086 • S. Bellucci, P. Y. Casteill, and J. F. Morales, “Superstring sigma models from spin chains: the $SU(1, 1|1)$ case”, Nucl. Phys. B (in press), hep-th/0503159.

[47] G. Ferretti, R. Heise and K. Zarembo, “New integrable structures in Large-N QCD”, Phys. Rev. D70 (2004) 074024, hep-th/0404187 • N. Beisert, G. Ferretti, R. Heise and K. Zarembo, “One Loop QCD Spin Chain and its Spectrum”, Nucl. Phys. B717 (2005) 137, hep-th/0412029.
[48] B. Chen, X.-J. Wang and Y.-S. Wu, “Integrable Open Spin Chain in Super Yang-Mills and the Plane-wave/SYM duality”, *JHEP* **0402** (2004) 029, hep-th/0401016
  • B. Chen, X.-J. Wang and Y.-S. Wu, “Open Spin Chain and Open Spinning String”, *Phys. Lett.* **B591** (2004), 170, hep-th/0403004.

[49] A. Ashtekar, “Polymer geometry at Planck scale and quantum Einstein equations”, *Int. J. Mod. Phys.* **D5**, 629 (1996), hep-th/9601054 • S. Kalyana Rama, “Size of black holes through polymer scaling”, *Phys. Lett.* **B424** (1998), 39, hep-th/9710035
  • R. R. Khuri, “Black holes and strings: the polymer link”, *Mod. Phys. Lett.* **A13** : 1407 (1998), gr-qc/9803095.

[50] O. Bergman and C. B. Thorn, “The size of a polymer of string bits: a numerical investigation”, *Nucl. Phys.* **B502** (1997) 309, hep-th/9702068 • A. Ashtekar, J. Lewandowski and H. Sahlmann, “Polymer and Fock representations for a scalar field”, *Class. Quant. Grav.* **20**: L11-1 (2003), gr-qc/0211012 • F. Ferrari and I. Lazzizzera, “Polymer topology and Chern-Simons field theory”, *Nucl. Phys.* **B559** (1999) 673.

[51] R. Penrose, “Angular momentum: an approach to combinatorial space-time”, in "Quantum Theory and Beyond", Cambridge University Press, 1971 • C. Rovelli and L. Smolin, “Spin networks and quantum gravity”, *Phys. Rev.* **D52** (1995), 5743, gr-qc/9505006

[52] C. Sochichiu, “Continuum limit(s) of Berenstein-Maldacena-Nastase matrix theory: Where is the (nonabelian) gauge group?”, *Phys. Lett.* **B574** (2003) 105, hep-th/0206239

[53] P. Valtancoli, “Stability of the fuzzy sphere solution from matrix model”, *Int. J. Mod. Phys.* **A18** (2003) 967, hep-th/0206075.