Asymptotic Behavior of Spanning Forests and Connected Spanning Subgraphs on Two-Dimensional Lattices

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We calculate exponential growth constants $\phi$ and $\sigma$ describing the asymptotic behavior of spanning forests and connected spanning subgraphs on strip graphs, with arbitrarily great length, of several two-dimensional lattices, including square, triangular, honeycomb, and certain heteropolygonal Archimedean lattices. By studying the limiting values as the strip widths get large, we infer lower and upper bounds on these exponential growth constants for the respective infinite lattices. Since our lower and upper bounds are quite close to each other, we can infer very accurate approximate values for these exponential growth constants, with fractional uncertainties ranging from $O(10^{-4})$ to $O(10^{-2})$. We show that $\phi$ and $\sigma$, are monotonically increasing functions of vertex degree for these lattices.

I. INTRODUCTION

Let $G = (V, E)$ be a graph defined by its vertex and edge sets $V$ and $E$. Let $n(G) = |V|$, $e(G) = |E|$, and $k(G)$ denote the number of vertices (=sites), edges (= bonds), and connected components of $G$, respectively. The degree $\Delta$ of a vertex $v \in V$ is the number of edges that are incident on $v$. A graph with the property that all vertices have the same degree is denoted a $\Delta$-regular graph. A subgraph of $G$ is defined as a graph with vertex and edge sets that are subsets of $V$ and $E$. For a given $G$, it is of interest to enumerate the number of subgraphs of a specific type. A spanning subgraph is defined as a subgraph that contains all of the vertices of $G$ and a subset of the edges of $G$. In the construction of a spanning subgraph there is a two-fold choice for each edge of $G$, namely whether it is present or absent, so the number of spanning subgraphs of $G$ is $N_{SSG}(G) = 2^{e(G)}$. We shall restrict here to initial graphs $G$ that are connected and do not have any loops, i.e., edges that emerge from a given vertex,
loop back, and reconnect to this vertex. Because our focus here is on regular lattice graphs
and their $n \to \infty$ limits, we shall also restrict to graphs that do not have multiple edges
connecting adjacent vertices. In general, a spanning subgraph may contain cycles, i.e., paths
along edges of the subgraph that are circuits. Spanning forests on $G$ (abbreviated as SF) are
defined as spanning subgraphs of $G$ that do not contain any cycles. Note that a spanning
forest may consist of more than one connected component, i.e., are not connected. We denote
the number of spanning forests of a graph $G$ as $N_{SF}(G)$. A second set of subgraphs of $G$
is comprised of connected spanning subgraphs (abbreviated as CSSG). We denote the number
of these connected spanning subgraphs as $N_{CSSG}(G)$ and observe that a member of this set
may contain cycles.

The numbers of spanning forests and connected spanning subgraphs in a graph $G$ can be
calculated as special valuations of the Tutte (also called Tutte-Whitney) polynomial of $G$,
$T(G, x, y)$, defined as

$$T(G, x, y) = \sum_{G' \subseteq G} (x - 1)^{k(G') - k(G)} (y - 1)^{c(G')} , \quad (1.1)$$

where $G'$ is a spanning subgraph of $G$ and $c(G')$ denotes the number of (linearly independent)
cycles on $G'$ [1–3]. Recall that we take $k(G) = 1$. As is evident directly from the definition
(1.1), the number of spanning forests in $G$ is

$$N_{SF}(G) = T(G, 2, 1) \quad (1.2)$$

and the number of connected spanning subgraphs in $G$ is

$$N_{CSSG}(G) = T(G, 1, 2) . \quad (1.3)$$

The Tutte polynomial is equivalent to the Whitney rank polynomial [4] and to the Potts
model partition function (see Appendix A).

An interesting problem in graph theory is to calculate the asymptotic behavior of $N_{SF}(G)$
and $N_{CSSG}(G)$ as $n(G) \to \infty$ for some families of graphs. For a wide class of families of
graphs, $N_{SF}(G)$ and $N_{CSSG}(G)$ grow exponentially rapidly as functions of $n(G)$ for large
$n(G)$. This is true, in particular, for lattice graphs. It is thus natural to define exponential
growth constants describing this asymptotic behavior:

$$\phi(\{G\}) = \lim_{n(G) \to \infty} \left[ N_{SF}(G) \right]^{1/n(G)} \quad (1.4)$$

and

$$\sigma(\{G\}) = \lim_{n(G) \to \infty} \left[ N_{CSSG}(G) \right]^{1/n(G)} , \quad (1.5)$$
where the symbol \( \{G\} \) denotes the \( n(G) \to \infty \) limit of graphs in a given family. Two simple examples of families of graphs for which one has exact expressions for \( T(G, x, y) \) and hence exact values of \( \phi \) and \( \sigma \) are \( n \)-vertex tree graphs, \( T_n \), and circuit graphs, \( C_n \), for which \( T(T_n, x, y) = x^{n-1} \) and \( T(C_n, x, y) = y + \sum_{j=1}^{n-1} x^j \). Hence, in the \( n \to \infty \) limits of the \( T_n \) and \( C_n \) graphs, \( \phi = 2 \) and \( \sigma = 1 \). However, the problem of calculating \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) on infinite lattices \( \Lambda \) with dimension 2 or higher is open.

In this paper we present exact calculations of these exponential growth constants for spanning forests and connected spanning subgraphs on strips, with fixed width and length going to infinity, of several types of two-dimensional lattices, including square, triangular, honeycomb, and certain heteropolygonal Archimedean lattices. By calculating the limiting values of the exponential growth constants as functions of strip width for infinite-length strips, we infer lower and upper bounds on these exponential growth constants for the respective infinite lattices, denoted generically as \( \Lambda \). Our lower and upper bounds are quite close to each other, which enables us to infer very accurate approximate values for these exponential growth constants, with fractional uncertainties ranging from \( \sim O(10^{-4}) \) to \( \sim O(10^{-2}) \).

We show that \( \phi \) and \( \sigma \), are monotonically increasing functions of vertex degree for these lattices. Our methods of obtaining lower and upper bounds on \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) are similar to those that we have used in our earlier works \([5]-[7]\) in which we inferred lower and upper bounds on the exponential growth constants for acyclic orientations, acyclic orientations with a unique source, and totally cyclic orientations of directed graphs. Our results make interesting connections between statistical physics and mathematical graph theory, since the Tutte polynomial is equivalent to the partition function of a classical spin model, namely the Potts model.

Previous studies have focused on lower and upper bounds on \( \phi \) on the square and/or triangular lattices \([8]-[12]\). After the early work \([8]\), Ref. \([9]\) obtained the lower and upper bounds (given, respectively, in Theorem 6.15 and Corollary 5.4 of \([9]\))

\[
3.64497565 \leq \phi(sq) \leq 3.74100178 .
\]

Ref. \([10]\) improved these bounds to

\[
3.698573 \leq \phi(sq) \leq 3.73264 .
\]

or equivalently, \(1.307947 \leq \ln[\phi(sq)] \leq 1.317115\) (from Eqs. (7.32a) and (2.41) in \([10]\)), where the lower bound is inferred from the monotonicity of \( \phi \) values for infinite-length, finite-width lattice strips. Ref. \([11]\) obtained the bounds (given in Theorem 5.3 of \([11]\))

\[
3.65166 \leq \phi(sq) \leq 3.73635 .
\]
A more stringent upper bound was presented in [12], namely
\[
\phi(sq) \leq 3.705603.
\] (1.9)

Our results in this paper include a further improvement of the upper bound to \( \phi(sq) \leq 3.699659 \), as will be discussed below. For the triangular lattice, Ref. [10] obtained the bounds
\[
5.479547 \leq \phi(tri) \leq 5.77546.
\] (1.10)
or equivalently, \( 1.7010224 \leq \ln[\phi(tri)] \leq 1.75362 \) (from Eqs. (7.57a) and (2.43) in [10]). Our results in this paper include an improvement of the upper bound to \( \phi(tri) \leq 5.494840 \). As noted, we also infer lower and upper bounds on \( \phi(\Lambda) \) for a number of other planar lattices, as well as lower and upper bounds on \( \sigma(\Lambda) \) for these lattices. Our analysis is thus complementary to earlier work on \( \phi(\Lambda) \) in that, by studying a substantial variety of Archimedean lattices with widely differing vertex degrees ranging from 3 to 6, we are able to infer the interesting monotonicity relation given below in Eq. (4.2) for \( \phi(\Lambda) \). The exponential growth constants \( \sigma(\Lambda) \) for connected spanning subgraphs on lattices \( \Lambda \) do not seem to have received as much attention as \( \phi(\Lambda) \) for spanning forests. Here again, by obtaining lower and upper bounds, and resultant approximate values of \( \sigma(\Lambda) \) on a variety of Archimedean lattices, we are able to discern the monotonicity relation (4.2) for \( \sigma(\Lambda) \). For a given lattice \( \Lambda \), our upper bounds on \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) approach respective limiting values more rapidly than our lower bounds, so we infer that the exact values of these exponential growth constants are closer to our upper bounds. This was also the case with our earlier calculations of bounds on exponential growth constants for acyclic orientations, acyclic orientations with a unique source, and totally cyclic orientations for directed lattices in Refs. [6, 7].

This paper is organized as follows. In Section II we discuss our methods for inferring lower and upper bounds on the exponential growth constants \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) for infinite lattices \( \Lambda \) from calculations on infinite-length strip graphs of varying widths. In Sections III and IV we present our results on these lower and upper bounds for \( \phi(\Lambda) \) and \( \sigma(\Lambda) \) and approximate values of these exponential growth constants for various lattices \( \Lambda \). Our conclusions are given in Section V. Some graph theory background is included in Appendix A.

**II. CALCULATIONAL METHODS**

In this section we explain a method that we use to infer lower and upper bounds on the exponential growth constants \( \phi(\Lambda) \) and \( \sigma(\Lambda) \). Our method is the same as the one that we have used in previous work [5–7] to infer lower and upper bounds on exponential growth
constants for other graph-theoretic quantities, such as acyclic orientations, \( \alpha(\Lambda) \), acyclic orientations with a unique source vertex, \( \alpha_0(\Lambda) \), and totally cyclic orientations, \( \beta(\Lambda) \), of directed graphs, and so we refer the reader to these previous works for further details.

We consider a family of strip graphs of a given type of lattice \( \Lambda \) (square, triangular, honeycomb, etc.) of fixed transverse width \( L_y \) and arbitrarily great length \( L_x \) with certain boundary conditions. As indicated, the longitudinal and transverse directions on the strip are taken to be in the \( x \) and \( y \) directions, respectively. (No confusion should result in the use of the symbols \( x \) and \( y \) as arguments of the Tutte polynomial \( T(G, x, y) \); the context will always make the meaning clear.) This is a recursive family, in the sense of Ref. [13]. We shall indicate the infinite-length limit of a width-\( L_y \) strip graph of the lattice \( \Lambda \) with specified transverse boundary conditions as \( \{ \Lambda, (L_y)_{BC_y} \times \infty \} \). We make use of a property of the Tutte polynomials for a strip graph of this type, namely that it is a sum of certain coefficients multiplied by powers of various functions, generically denoted \( \lambda \), depending on \( \Lambda \), \( L_y \), and the boundary conditions, but not on \( L_x \). The powers to which these \( \lambda \) functions are raised are given by the length, \( L_x \), of the strip. As \( L_x \to \infty \), the \( \lambda \) function with the largest magnitude dominates the sum. Henceforth, we will denote this simply as \( \lambda \) for a given strip. From our calculations of Tutte polynomials for strip graphs of various lattices, we know this dominant \( \lambda \) function in each case. Some of our calculations are in [5],[14]-[24]; some others are listed in [25]. Thus, for the infinite-length limit of a given finite-width strip of some lattice \( \Lambda \), to calculate \( \phi \) or \( \sigma \), we only need this dominant \( \lambda \) function. This is a significant simplification, since for a general graph, the calculation of the Tutte polynomials \( T(G, 1, 2) \) and \( T(G, 2, 1) \) are \#P hard [26, 27]. This \( \lambda \) function, and hence the results for \( \phi \) and \( \sigma \), are independent of the boundary condition (free, periodic, or Möbius) in the longitudinal direction, but do depend on the boundary condition in the transverse direction, denoted \( BC_y \). We therefore denote the results as \( \phi(\Lambda, (L_y)_{BC_y} \times \infty) \) and \( \sigma(\Lambda, (L_y)_{BC_y} \times \infty) \). These are given by the following limits:

\[
\phi(\Lambda, (L_y)_{BC_y} \times \infty) = \lim_{n \to \infty} \left[ N_{SF}(\Lambda, (L_y)_{BC_y} \times L_x) \right]^{1/n} = \left[ \lambda(\Lambda, L_y, BC_y)(2, 1) \right]^{1/c_{\Lambda L_y}}, \quad (2.1)
\]

and

\[
\sigma(\Lambda, (L_y)_{BC_y} \times \infty) = \lim_{n \to \infty} \left[ N_{CSSG}(\Lambda, (L_y)_{BC_y} \times L_x) \right]^{1/n} = \left[ \lambda(\Lambda, L_y, BC_y)(1, 2) \right]^{1/c_{\Lambda L_y}}, \quad (2.2)
\]

where the arguments \( (2, 1) \) and \( (1, 2) \) are the arguments of the respective Tutte polynomial; \( c_\Lambda \) is a constant depending on \( \Lambda \), with \( c_{sq} = c_{tri} = 1, c_{hc} = 2 \), etc.; and for brevity of notation, we set \( n(G) \equiv n \).

Next, for each type of lattice \( \Lambda \), we study the dependence of the exponential growth constants for the finite-width, infinite-length strips, \( \phi(\Lambda, (L_y)_{BC_y} \times \infty) \) and \( \sigma(\Lambda, (L_y)_{BC_y} \times \infty) \)
∞), on the strip width. For both types of transverse boundary conditions (free and periodic) and for all lattices Λ considered here, we show that these are monotonically increasing functions of the strip width. This provides strong support for the inference that these are lower bounds on the respective exponential growth constants φ(Λ) and σ(Λ) on the infinite lattices Λ. Furthermore, a consequence of this monotonicity property is that, for a given transverse boundary condition BC_y, the values of φ(Λ, (L_y)BC_y × ∞) and σ(Λ, (L_y)BC_y × ∞) on the strip with the greatest width are the best lower bounds, for this set of transverse boundary conditions, on the respective values of φ(Λ) and σ(Λ). Since for the periodic (P) transverse boundary conditions the strips have no transverse boundary, one expects that these would yield the best lower bounds, and our results are in accord with this expectation. Thus, for the infinite lattice Λ, we infer

$$\xi(\Lambda) > \xi(\Lambda, [(L_y)_{\max}]_{PBC_y} \times \infty) \quad \text{for } \xi = \phi, \sigma$$  \hspace{1cm} (2.3)

A measure of the rapidity with which the values of φ and σ on finite-width, infinite-length strips of a lattice Λ approach their infinite-width limits is provided by the ratio of values of each respective exponential growth constant on strips of width L_y and L_y − 1. We denote this ratio as

$$R_{\xi,\Lambda,(L_y+1)/L_y,BC_y} = \frac{\xi(\Lambda, (L_y + 1)_{BC_y} \times \infty)}{\xi(\Lambda, (L_y)_{BC_y} \times \infty)} \quad \text{for } \xi = \phi, \sigma.$$ \hspace{1cm} (2.4)

As will be evident from our results for various lattices, even for modest values of the strip widths, these ratios approach very close to unity, showing the rapid approach to the L_y → ∞ limit.

We proceed to discuss upper bounds on these exponential growth constants φ and σ. Our analysis here is similar to our earlier analyses in [5]-[7] for the other set of exponential growth constants α, α_0, and β. We first prove a useful inequality. For this purpose, we begin by considering lattice strip graphs with width L_y = 2^p for some (positive) integer power p. This inequality applies to a generic exponential growth constant denoted ξ for the Tutte polynomial of a family of lattice strip graphs for x ≥ 0 and y ≥ 0, where ξ is defined as

$$\xi(\{G\}, x, y) = \lim_{n(G) \to \infty} [T(G, x, y)]^{1/n(G)}.$$ \hspace{1cm} (2.5)

For our applications here, ξ = φ with (x, y) = (2, 1) or ξ = σ with (x, y) = (1, 2). If an edge e ∈ E is not a loop or a bridge (see Appendix A for definitions), then the Tutte polynomial satisfies the deletion-contraction relation

$$T(G, x, y) = T(G - e, x, y) + T(G/e, x, y),$$ \hspace{1cm} (2.6)
where $G - e$ denotes $G$ with the edge $e$ deleted and $G/e$ is the result of deleting the edge $e$ from $G$ and identifying the vertices that had been connected by this edge. Applying this deletion-contraction relation repeatedly yields a set of inequalities for the dominant $\lambda$ function for the two cases of interest here, $(x, y) = (2, 1)$ and $(x, y) = (1, 2)$. If one compares the Tutte polynomial for an $4 \times L_x$ strip graph with the Tutte polynomial for a (disconnected) graph consisting of two copies of an $2 \times L_x$ strip, then the former has $L_x$ more edges, whose deletion produces the the latter two graphs. By iterative application of the deletion-contraction theorem, one can then relate the free strip of width $L_y$ to the graph consisting of two free strips each of width $L_y/2$. Henceforth, for definiteness, we specialize to strips of the square lattice (and use the symbol “F” for free transverse boundary conditions). With appropriate changes, our results apply to strips of other lattices also. We then have the series of inequalities, for $(x, y) = (2, 1)$ or $(x, y) = (1, 2)$:

$$\lambda_{sq,1,F}(x, y) \leq [\lambda_{sq,2,F}(x, y)]^{1/2} \leq [\lambda_{sq,4,F}(x, y)]^{1/4} \leq [\lambda_{sq,8,F}(x, y)]^{1/8} \leq ... \leq \lim_{L_y \to \infty} [\lambda_{sq,L_y,F}(x, y)]^{1/L_y} .$$

Let us focus on one of these inequalities, namely $[\lambda_{sq,2,F}(x, y)]^{1/2} \leq [\lambda_{sq,4,F}(x, y)]^{1/4}$ The others can be treated in a similar manner. Here and below, it is understood that $(x, y) = (2, 1)$ for $\phi$ or $(x, y) = (1, 2)$ for $\sigma$. Since $[\lambda_{sq,2,F}(x, y)]^{L_x}$ is the dominant $\lambda$ function for the $2 \times L_x$ strip, it determines the corresponding $\phi$ or $\sigma$ in the limit of infinite strip length, while $[\lambda_{sq,4,F}(x, y)]^{L_x}$ similarly gives the $\phi$ or $\sigma$ function for infinite-length limit of the the $4 \times L_x$ strip. Now compare two $L_y = 2$ strips with a $L_y = 4$ strip. The former strips have $L_x$ fewer edges than the latter, so the Tutte polynomial of the former is smaller than that of the latter, since the coefficients of the Tutte polynomial (in terms of variables $x$ and $y$) are positive. That is, $[\lambda_{sq,2,F}(x, y)]^{2L_x} \leq [\lambda_{sq,4,F}(x, y)]^{L_x}$. This completes the proof of the inequality. By the same type of argument, it follows, for example, that

$$\lambda_{sq,1,F}(x, y) \leq [\lambda_{sq,3,F}(x, y)]^{1/3} \leq [\lambda_{sq,6,F}(x, y)]^{1/6} \leq [\lambda_{sq,12,F}(x, y)]^{1/12} \leq ... \leq \lim_{L_y \to \infty} [\lambda_{sq,L_y,F}(x, y)]^{1/L_y} ,$$

where here $L_y = 3 \cdot 2^s$, where $s$ is a non-negative integer. Other corresponding inequalities with larger values of $L_y$ follow in the same way. By a similar argument, one can prove that, with $(x, y) = (2, 1)$ or $(x, y) = (1, 2),

$$\lambda_{sq,L_y,F}(x, y) \leq \lambda_{sq,L_y,P}(x, y) .$$
Now recall the sequence of inequalities (2.7). The limit as \( L_y \to \infty \) yields the value of the exponential growth constant on the infinite square lattice. Now

\[
[\lambda_{sq,L_y,F}(x, y)]^{1/L_y} < [\lambda_{sq,L_y+1,F}(x, y)]^{1/(L_y+1)},
\]

or equivalently,

\[
\lambda_{sq,L_y,F}(x, y) < [\lambda_{sq,L_y+1,F}(x, y)]^{L_y/(L_y+1)}.
\]

Thus,

\[
[\lambda_{sq,L_y+1,F}(x, y)]^{1/(L_y+1)} < \frac{\lambda_{sq,L_y+1,F}(x, y)}{\lambda_{sq,L_y,F}(x, y)}.
\]

From our explicit calculation, we find that

\[
\frac{\lambda_{sq,8,F}(x, y)}{\lambda_{sq,7,F}(x, y)} < \frac{\lambda_{sq,7,F}(x, y)}{\lambda_{sq,6,F}(x, y)} < \ldots < \frac{\lambda_{sq,3,F}(x, y)}{\lambda_{sq,2,F}(x, y)} < \frac{\lambda_{sq,2,F}(x, y)}{\lambda_{sq,1,F}(x, y)}.
\]

This leads us to infer that the ratio \( \lambda_{sq,L_y+1,F}(x, y)/\lambda_{sq,L_y,F}(x, y) \) serves as an upper bound for \( \phi(sq) \) if \((x, y) = (2, 1)\) and for \( \sigma(sq) \) if \((x, y) = (1, 2)\). We thus infer the inequalities

\[
\phi(\Lambda) < \frac{\lambda_{\Lambda,L_y+1,F}(2, 1)}{\lambda_{\Lambda,L_y,F}(2, 1)}
\]

for the maximal calculated value of \( L_y \)

and

\[
\sigma(\Lambda) < \frac{\lambda_{\Lambda,L_y+1,F}(1, 2)}{\lambda_{\Lambda,L_y,F}(1, 2)}
\]

for the maximal calculated value of \( L_y \).

A useful measure of the approach to the \( L_y \to \infty \) limit is provided by the ratio of upper bounds for adjacent values of \( L_y \), namely

\[
R_{sq,\ell_y} = \frac{\lambda_{sq,L_y,F}(x, y)}{\lambda_{sq,L_y-1,F}(x, y)} = \frac{[\lambda_{sq,L_y,F}(x, y)]^2}{\lambda_{sq,L_y-1,F}(x, y)\lambda_{sq,L_y+1,F}(x, y)}.
\]

This is the ratio of adjacent upper bounds. As our explicit calculations show, this ratio rapidly approaches unity (from above) as the strip width \( L_y \) increases.

Applying the analogous arguments for other lattices, we infer the inequalities corresponding to (2.14) and (2.15) for these other lattices. The ratio of adjacent upper bounds analogous to (2.16) provides a quantitative measure of the rapidity of approach to a limit for these other lattices, as for the square lattice.

### III. Numerical Values of Lower and Upper Bounds for \( \phi(\Lambda) \) and \( \sigma(\Lambda) \)

In this section we present our results for numerical values of lower and upper bounds for \( \phi(\Lambda) \) on various two-dimensional lattices \( \Lambda \). For a given lattice \( \Lambda \), we denote our lower (\( \ell \))
and upper \((u)\) bounds as \(\xi(\Lambda)\) and \(\xi_u(\Lambda)\), where \(\xi = \phi\) or \(\sigma\). With our method, we obtain the lower bounds from strips with periodic transverse boundary conditions and the upper bounds from strips with free boundary conditions.

We recall that an Archimedean lattice is defined as a uniform tiling of the plane with one or more types of regular polygons, such that all vertices are equivalent, and hence is \(\Delta\)-regular. In general, an Archimedean lattice \(\Lambda\) is identified by the ordered sequence of regular polygons traversed in a circuit around any vertex [28, 29]:

\[
\Lambda = \left( \prod p_i^{a_i} \right),
\]

where the \(i\)th polygon has \(p_i\) sides and appears \(a_i\) times contiguously in the sequence (it can also occur non-contiguously). As in [29], we denote the sum of the numbers \(a_i\) in the product (3.1) as \(a_{i,s}\). Of the eleven Archimedean lattices, three are homopolygonal (i.e. each is comprised of only type of regular polygon), namely the square (sq), triangular (tri), and honeycomb (hc) lattices. For a homopolygonal Archimedean lattice composed of \(p\)-gons, the right-hand side of Eq. (3.1) has the simple form \((p^2\Delta)\), where, as above, \(\Delta\) is the vertex degree (i.e., lattice coordination number). Thus, in this notation, the square, triangular, and honeycomb lattices are denoted \((4^4)\), \((3^6)\), and \((6^3)\). The other Archimedean lattices are comprised of more than one type of regular polygon and hence are termed heteropolygonal. The heteropolygonal Archimedean lattices that we will consider here are \((4 \cdot 8^2)\), \((3 \cdot 6 \cdot 3 \cdot 6)\) (kag), \((3^3 \cdot 4^2)\), and \((3^2 \cdot 4 \cdot 3 \cdot 4)\).

In Tables I-XIV and XV-XXVIII we present our results on lower and upper bounds on \(\phi(\Lambda)\) and \(\sigma(\Lambda)\) and relevant ratios for the finite-width, infinite-length strips of the various lattices \(\Lambda\). We include results for both free and periodic transverse boundary conditions on these finite-width, infinite-length strips. In Table XXIX we summarize the best lower and upper bounds that we have obtained for these lattices. To our knowledge, these are the best current lower and upper bounds on \(\phi(\Lambda)\) for the hc, \((4 \cdot 8^2)\), \((3 \cdot 6 \cdot 3 \cdot 6)\) (kag), \((3^3 \cdot 4^2)\), and \((3^2 \cdot 4 \cdot 3 \cdot 4)\) lattices and the best upper bounds on \(\phi(sq)\) and \(\phi(tri)\). As noted, we are not aware of previous published bounds on \(\sigma(\Lambda)\) for these Archimedean lattices.

There are several important features of our bounds. First, for each type of lattice, the lists of ratios of adjacent lower bounds and of adjacent upper bounds, as functions of strip width \(L_y\) show that the lower bounds and the upper bounds rapidly approach a limiting value. Second, the upper and lower bounds are very close to each other. The average of the upper and lower bounds for a given exponential growth constant \(\xi(\Lambda)\) is

\[
\xi_{ave}(\Lambda) = \frac{\xi_u(\Lambda) + \xi(\Lambda)}{2} \quad \text{for} \quad \xi = \phi, \sigma.
\]
The difference between the average and the upper or lower bound is

$$\delta \xi(\Lambda) = \xi_u(\Lambda) - \xi_{ave}(\Lambda) = \xi_{ave}(\Lambda) - \xi_\ell(\Lambda) ,$$  

(3.3)

so the fractional difference is

$$\frac{\xi_u(\Lambda) - \xi_\ell(\Lambda)}{\xi_{ave}(\Lambda)} = \frac{2\delta \xi(\Lambda)}{\xi_{ave}(\Lambda)} \text{ for } \xi = \phi, \sigma .$$  

(3.4)

These fractional differences (3.4) are very small, typically varying from \(O(10^{-4})\) to \(O(10^{-2})\). This is in excellent agreement with the observed rapid approach of each of these bounds to a limiting value and consistent with the inference that, in the \(L_y \to \infty\) limit, this is a common value, describing the exponential growth constant on the infinite two-dimensional lattice.

Third, for a given lattice \(\Lambda\) and exponential growth constant \(\phi(\Lambda)\) or \(\sigma(\Lambda)\), the upper bounds approach a limit more rapidly than the lower bounds, leading one to infer that the actual value on the infinite lattice is closer to the upper bound than to the lower bound. For example, for the \((4 \cdot 8^2)\) lattice, the ratio of adjacent upper bounds \(R_{tri, (L_y-1)(L_y+1)}^{L_y} F(2, 1)\) for the greatest widths in Table VIII is extremely close to 1, being only \(1.6 \times 10^{-7}\) greater than 1, while the corresponding ratio of lower bounds is approximately \(1 \times 10^{-3}\) above 1. Fourth, our numerical results are in agreement with the three exact duality relations \(\phi(sq) = \sigma(sq)\) in Eq. (A10), \(\phi(hc) = [\sigma(tri)]^{1/2}\) in Eq. (A11), and \(\sigma(hc) = [\phi(tri)]^{1/2}\) in Eq. (A12) for the infinite lattices. Accordingly, we have made use of these duality relations in Table XXIX. Specifically, we have used our upper bound on \(\phi(sq)\), namely \(\phi_u(sq) = 3.699659\), as an improvement on the upper bound \(\sigma_u(sq) = 3.751149\) obtained directly from the infinite-length, finite-width strips. Further, we have used the duality relation \(\sigma(hc) = [\phi(tri)]^{1/2}\) together with our upper bound on \(\phi(tri)\), namely \(\phi_u(tri) = 5.494840085\), to compute an upper bound \(\sigma_u(hc) = 2.3441075\), which is more stringent than the upper bound \(\sigma_u(hc) = 2.3601982\) obtained directly from infinite-length, finite-width strips of the honeycomb lattice. Similarly, from duality, we obtain a lower limit \(\phi_\ell(tri) = 5.39333314\), which is more stringent than the lower bound \(\phi_\ell(tri) = 5.3848542\) obtained directly from the analysis of strips of the triangular lattice. Moreover, \([\phi_\ell(hc)]^2 = 7.861223392\) and \([\phi_u(hc)]^2 = 7.866798814\), which are better lower and upper bounds than the respective values \(\sigma_\ell(tri) = 7.859929\) and \(\sigma_u(tri) = 7.933005\) obtained directly from the analysis of infinite-length, finite-width strips of the triangular lattice. We thus use these improved limits in Table XXIX. An important fourth feature of our results will be presented as the relation (4.2) in the next section.
IV. APPROXIMATE VALUES OF $\phi(\Lambda)$ AND $\sigma(\Lambda)$

Since the fractional differences (3.4) are so small, we can infer very accurate approximate values $\xi_{app}(\Lambda)$ for these exponential growth constants on the given lattices. A simple way to do this is use the averages, $\xi_{ave}(\Lambda)$ together with the differences $\delta\xi(\Lambda)$ as a measure of the uncertainty:

$$\xi_{app}(\Lambda) = \xi_{ave}(\Lambda) \pm \delta\xi(\Lambda) \quad \text{for } \xi = \phi, \sigma .$$

(4.1)

This is the procedure that we used in Refs. [6, 7] for certain exponential growth constants describing acyclic and cyclic orientations of edge arrows on directed lattice graphs. We list these approximate values in Table XXX. More complicated analytical methods could also be applied, but this simple procedure is sufficient as a basis for one of our most important results, namely that we find that, for all of those lattices considered here,

$$\phi(\Lambda) \text{ and } \sigma(\Lambda) \text{ are monotonically increasing functions of } \Delta ,$$

(4.2)

where $\Delta$ is the vertex degree (i.e., coordination number in physics terminology) of the lattice $\Lambda$. If we write this $\Delta$ dependence as an empirical power law, then we find, roughly, that $\phi(\Lambda) \sim 3.7(\Delta/4)$ while $\sigma(\Lambda) \sim 3.7(\Delta/4)^{1.8}$. By fitting our upper and lower bounds on the exponential growth constants for infinite-length, finite-width strips to some assumed functional forms for the approach to the infinite-width limit (as in [10] for $\phi(sq)$ and $\phi(tri)$), we could infer corresponding estimates for the values for the exponential growth constants, but this is not necessary for our monotonicity result (4.2).

The dependences of $\phi(\Lambda)$ and $\sigma(\Lambda)$ on $\Delta$ that we have found may be compared and contrasted with the $\Delta$-dependence of the exponential growth constant for the total number of spanning subgraphs of a lattice $\Lambda$, $N_{SSG}(G) = 2^{e(G)}$. Since $e(G) = n\Delta/2$ for a $\Delta$-regular lattice graph $G$, it follows that for such graphs

$$\lim_{n(G)\to\infty} [N_{SSG}(G)]^{1/n(G)} = 2^{\Delta/2} .$$

(4.3)

This is again a monotonically increasing function of $\Delta$, and the property that the right-hand side of Eq. (4.3) increases more rapidly than a power law as a function of $\Delta$ is consistent with the fact that the numbers of spanning forests and connected spanning subgraphs are subsets of the total number of spanning subgraphs.

Another interesting property that we find is that for the homopolygonal lattices $\Lambda = (p^\Delta)$, the relation $p > \Delta \iff \phi(\Lambda) > \sigma(\Lambda)$ holds. We recall that the case $p = \Delta = 4$ is realized for the square lattice, the self duality of which implies that $\phi(sq) = \sigma(sq)$. Given the connection between $\Delta$ and $p$ for the homopolygonal Archimedean lattices, this relation is implied by our monotonicity result (4.2), but it is of interest in its own right.
The analogous relations also hold for exponential growth constants that we calculated in Refs. [6, 7]. Recall that the number of acyclic orientations and totally cyclic orientations of a directed graph $G$ are given by $T(G, 2, 0)$ and $T(G, 0, 2)$, respectively, with the corresponding exponential growth constants

$$\alpha(\{G\}) = \lim_{n(G) \to \infty} \left[ T(G, 2, 0) \right]^{1/n(G)}$$

and

$$\beta(\{G\}) = \lim_{n(G) \to \infty} \left[ T(G, 0, 2) \right]^{1/n(G)}.$$  

As we noted in [6, 7], we found that for the Archimedean lattices that we considered there, $\alpha(\Lambda)$ and $\beta(\Lambda)$ are monotonically increasing functions of $\Delta$.

Furthermore, the relation $p > \Delta \iff \alpha(\Lambda) > \beta(\Lambda)$ holds for the homopolygonoal Archimedean lattices, and the self-duality of the square lattice yields the relation $\alpha(sq) = \beta(sq)$.

V. CONCLUSIONS

In this paper we have calculated the exponential growth constants $\phi$ and $\sigma$ describing the asymptotic growth of the numbers of spanning forests and of connected spanning subgraphs, respectively, for finite-width, infinite-length strips of several different two-dimensional lattices $\Lambda$. From our calculations, we have inferred lower and upper bounds on these exponential growth constants $\phi(\Lambda)$ and $\sigma(\Lambda)$ for the respective infinite lattices $\Lambda$. Our bounds from calculations on infinite-length, finite-width lattice strips converge rapidly even for modest values of strip widths. Since our lower and upper bounds are quite close to each other, we can infer obtain quite accurate approximate values for these exponential growth constants. Our results show that $\phi(\Lambda)$ and $\sigma(\Lambda)$ are monotonically increasing functions of vertex degree for these lattices. An interesting aspect of our work is the connection that is makes between statistical mechanics and mathematical graph theory, reflecting the fact the Tutte polynomial is equivalent to the partition function of a classical spin model, namely the Potts model.

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Appendix A: Some Graph Theory Background

In this appendix we include some graph theory background relevant for our analysis in the paper (for further details, see, e.g., [1]). As in the text, let $G = (V, E)$ be a graph defined by its vertex and edge sets $V$ and $E$. Let $n(G) = |V|$, $e(G) = |E|$, and $k(G)$ denote the number of vertices (=sites), edges (= bonds), and connected components of $G$. We restrict to connected $G$. A loop is defined as an edge that connects a vertex to itself, and a bridge (co-loop) is defined as an edge that has the property that if it is deleted, then this increases the number of components in the resultant graph, relative to the number of components in the initial graph that contained the bridge. As noted in the text, since our primary application is to regular lattices, we restrict to graphs $G$ without loops. A spanning subgraph of $G$, denoted $G'$, is a graph with the same vertex set $V$ and a subset of the edge set $E$, i.e., $G' = G'(V, E')$ with $E' \subseteq E$. The Tutte polynomial $T(G, x, y)$ [2] is defined in Eq. (1.1) in the text. The numbers of spanning forests and connected spanning subgraphs in $G$, denoted $N_{SF}(G)$ and $N_{CSSB}(G)$ respectively, are valuations of $T(G, x, y)$ given by Eqs. (1.2) and (1.3). The corresponding exponential growth constants describing the asymptotic behavior of $N_{SF}(G)$ and $N_{CSSG}(G)$ are given in Eqs. (1.4) and (1.5). From the definition (1.1), it is clear that $T(G, x, y)$ is a polynomial in the two variables $x$ and $y$, so one can write it as

$$T(G, x, y) = \sum_{i,j} t_{ij} x^i y^j ,$$

where the $t_{ij}$ can be determined from (1.1). A basic property of $T(G, x, y)$ that we use in the text is that the nonzero $t_{ij}$ are positive (integers) [1, 2].

Let $G_{pl}$ be a planar graph. Recall that the planar dual, $G_{pl}^*$, of $G_{pl}$ is defined by a 1-1 correspondence between the vertices (resp. faces) of $G_{pl}$ and the faces (resp. vertices) of $G_{pl}^*$. The Tutte polynomial satisfies the duality relation

$$T(G_{pl}, x, y) = T(G_{pl}^*, y, x) .$$

It follows from this duality relation (A2) and the relations (1.2) and (1.3) that

$$N_{SF}(G_{pl}) = N_{CSSG}(G_{pl}^*) .$$

Let us denote the number of faces of a planar graph $G_{pl}$ as $f(G_{pl})$ and recall the Euler relation for a planar graph $G_{pl}$,

$$f(G_{pl}) - e(G_{pl}) + n(G_{pl}) = 2 .$$

13
From the duality relation, it follows that \( n(G^*_p) = f(G_{pl}) \). For \( \Delta \)-regular graphs \( G \),

\[
e(G) = \frac{\Delta(G) n(G)}{2}.
\] (A5)

For a \( \Delta \)-regular planar graph \( G_{pl} \) we define the ratio

\[
\nu(G_{pl}) \equiv \lim_{n(G_{pl}) \to \infty} \frac{n(G^*_p)}{n(G_{pl})} = \frac{\Delta(G_{pl})}{2} - 1,
\] (A6)

where we have used Eq. (A5) in the last equality in (A6). Note that

\[
\nu\{G_{pl}\} = \frac{1}{\nu\{G^*_p\}}.
\] (A7)

Specifically, \( \nu(sq) = 1 \) and \( \nu(tri) = 1/\nu(hc) = 2 \). The results \( \nu(sq) = 1 \) and that \( \nu(tri) = 1/\nu(hc) \) follow from property that the square lattice is self-dual and the triangular and honeycomb lattices are planar duals of each other. From Eq. (A3), it follows that if a planar graph is self-dual, indicated as \( G_{pl.,sd.} \), then

\[
N_{SF}(G_{pl.,sd.}) = N_{CSSG}(G_{pl.,sd.}),
\] (A8)

and hence

\[
\phi\{G_{pl.,sd.}\} = \sigma\{G_{pl.,sd.}\}.
\] (A9)

In particular, since the square lattice is planar and self-dual, we have

\[
\phi(sq) = \sigma(sq),
\] (A10)

so that the lower and upper bounds that we infer below for \( \phi(sq) \) also hold for \( \sigma(sq) \). For the triangular and honeycomb lattices, we obtain the relations

\[
\phi(hc) = [\sigma(tri)]^{\nu(hc)} = [\sigma(tri)]^{1/2}
\] (A11)

and

\[
\sigma(hc) = [\phi(tri)]^{\nu(hc)} = [\phi(tri)]^{1/2}.
\] (A12)

The Tutte polynomial is equivalent to the Whitney rank polynomial [4],

\[
R(G, \xi, \eta) = \sum_{G' \subseteq G} \xi^{n(G') - k(G')} \eta^{c(G')}
\] (A13)

where \( G' \) is a spanning subgraph of \( G \) and \( c(G') \) is the number of (linearly independent) circuits on \( G' \). Recall that \( c(G') = c(G') + k(G') - n(G') \) and \( n(G') = n(G) \). The equivalence is given by

\[
T(G, x, y) = (x - 1)^{n(G) - k(G)} R(G, \xi, \eta),
\] (A14)
where
\[ \xi = \frac{1}{x - 1}, \quad \eta = y - 1. \] \hspace{1cm} (A15)

(The variable \( \xi \) in Eqs. (A14) and (A15) should not be confused with the symbol used for the generic exponential growth constant in Eq. (2.3).) The Tutte polynomial of a graph \( G \) is also equivalent to a function of interest in statistical physics, namely the Potts model partition function, denoted \( Z(G, q, v) \), which may be expressed as \[ Z(G, q, v) = \sum_{G' \subseteq G} q^{k(G')} v^{e(G')} , \] \hspace{1cm} (A16)

where again, \( G' \) is a spanning subgraph of \( G \). The equivalence is given by
\[ Z(G, q, v) = (x - 1)^{k(G)} (y - 1)^{n(G)} T(G, x, y) , \] \hspace{1cm} (A17)

where
\[ x = 1 + \frac{q}{v}, \quad y = v + 1. \] \hspace{1cm} (A18)

so that \( q = (x - 1)(y - 1) \). Thus, one also has the equivalence
\[ Z(G, q, v) = q^{n(G)} R(G, \xi, \eta) , \] \hspace{1cm} (A19)

where
\[ \xi = \frac{v}{q}, \quad \eta = v. \] \hspace{1cm} (A20)

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As discussed in the text, \( \phi(sq,(L_y)_{BC_y} \times \infty) = [\lambda_{sq,(L_y)_{BC_y}}(2,1)]^{1/L_y} \), and these values are inferred to be lower bounds on \( \phi(sq) \), with the values for periodic \( BC_y \) and the maximal \( L_y \) being the most restrictive. As defined in Eq. (2.4), \( R_{\phi,sq,BC_y,\frac{L_y}{L_y-1}} = \phi(sq,(L_y)_{BC_y} \times \infty)/\phi(sq,(L_y-1)_{BC_y} \times \infty) \). Here and in subsequent tables, a blank entry means that the evaluation is not applicable.

| \( BC_y \) | \( L_y \) | \( \phi(sq,(L_y)_{BC_y} \times \infty) \) | \( R_{\phi,sq,BC_y,\frac{L_y}{L_y-1}} \) |
| --- | --- | --- | --- |
| F | 1 | 2 |  |
| F | 2 | 1 + \( \sqrt{3} = 2.73205081 \ldots \) | 1.36602540 |
| F | 3 | 3.02428923 | 1.10696669 |
| F | 4 | 3.18094706 | 1.05179988 |
| F | 5 | 3.27859286 | 1.03069709 |
| F | 6 | 3.34528558 | 1.02034187 |
| P | 2 | \( \sqrt{15}+\sqrt{7} = 3.25936733 \ldots \) |  |
| P | 3 | 3.53705348 | 1.08519634 |
| P | 4 | 3.62352967 | 1.0244865 |
| P | 5 | 3.65845648 | 1.00963889 |
| P | 6 | 3.67518338 | 1.00457212 |
**TABLE II:** Upper bounds and their ratios for $\phi(sq)$ as functions of strip width $L_y$. The ratio $R_{sq,(L_y-1)(L_y+1)}^{\frac{L_y^2}{2}}(2,1)$ is defined in Eq. (2.10), where F denotes free transverse boundary conditions.

| $\frac{L_y+1}{L_y}$ | $\frac{\lambda_{sq,L_y+1,F}(2,1)}{\lambda_{sq,L_y,F}(2,1)}$ | $R_{sq,(L_y-1)(L_y+1),F}^{\frac{L_y^2}{2}}(2,1)$ |
|----------------------|-----------------------------------------------|-----------------------------------------------|
| 2/1                  | $2 + \sqrt{3} = 3.73205081...$               |                                               |
| 3/2                  | 3.70588916                                   | 1.00705948                                   |
| 4/3                  | 3.70131286                                   | 1.00123640                                   |
| 5/4                  | 3.70008482                                   | 1.00033189                                   |
| 6/5                  | 3.69965942                                   | 1.00011498                                   |

**TABLE III:** Lower bounds and their ratios for $\phi(tri)$ as functions of strip width $L_y$.

| BC$_y$ | $L_y$ | $\phi(tri, (L_y)_{BC_y} \times \infty)$ | $R_{\phi,tri,BC_y,L_y}^{\frac{L_y}{2}}(T_y=y)$ |
|--------|------|----------------------------------------|-----------------------------------------------|
| F      | 2    | $\sqrt{2(3+2\sqrt{2})} = 3.41421356...$ |                                               |
| F      | 3    | 4.01637573                               | 1.17636921                                   |
| F      | 4    | 4.34758961                               | 1.08246586                                   |
| F      | 5    | 4.55702010                               | 1.04817163                                   |
| F      | 6    | 4.70139379                               | 1.03168160                                   |
| P      | 2    | $\frac{46+2\sqrt{505}}{2} = 4.76823893...$ |                                               |
| P      | 3    | 5.17697865                               | 1.08572132                                   |
| P      | 4    | 5.32006369                               | 1.02763872                                   |
| P      | 5    | 5.38485420                               | 1.01217852                                   |
TABLE IV: Upper bounds and their ratios for $\phi(tri)$ as functions of strip width $L_y$.

| $\frac{L_y + 1}{L_y}$ | $\frac{\lambda_{tri,L_y+1,F}(2,1)}{\lambda_{tri,L_y,F}(2,1)}$ | $R_{tri,\frac{L_y+1}{L_y}}\frac{L_y^2}{L_y^2}F(2,1)$ |
|------------------------|-------------------------------------------------|-----------------------------------------------|
| 2/1                    | $3 + 2\sqrt{2} = 5.82842712..$                |                                               |
| 3/2                    | 5.55803958                                     | 1.04864801                                   |
| 4/3                    | 5.51430988                                     | 1.00793022                                   |
| 5/4                    | 5.50060617                                     | 1.00249131                                   |
| 6/5                    | 5.49484009                                     | 1.00104936                                   |

TABLE V: Lower bounds and their ratios for $\phi(hc)$ as functions of strip width $L_y$.

| BC$_y$ | $L_y$ | $\phi(hc, (L_y)_{BC_y} \times \infty)$ | $R_{\phi,hc,1,\frac{L_y+1}{L_y},\frac{L_y+2}{L_y},BC_y}$ |
|--------|------|----------------------------------------|------------------------------------------------------|
| F      | 2    | $(16 + 4\sqrt{15})^{1/4} = 2.36891693..$|                                                      |
| F      | 3    | 2.50613944                               | 1.05792627                                           |
| F      | 4    | 2.57768156                               | 1.02854674                                           |
| F      | 5    | 2.62158102                               | 1.01703060                                           |
| F      | 6    | 2.65126155                               | 1.01132162                                           |
| P      | 2    | $1 + \sqrt{3} = 2.73205081$             |                                                      |
| P      | 4    | 2.79825703                               | 1.02423316                                           |
| P      | 6    | 2.80378733                               | 1.00197634                                           |
### TABLE VI: Upper bounds and their ratios for $\phi(hc)$ as functions of strip width $L_y$.

| $(L_y + 1)/L_y$ | $[\lambda_{hc,L_y+1,F}(2,1)/\lambda_{hc,L_y,F}(2,1)]^{1/2}$ | $R^{L_y^2}_{hc,(L_y-1)(L_y+1),F}(2,1)$ |
|-----------------|-------------------------------------------------|-----------------------------------|
| 2/1             | $\frac{\sqrt{6}+\sqrt{10}}{2} = 2.80588370..$ | 1.00035381                        |
| 3/2             | 2.80489129                                     | 1.00003378                        |
| 4/3             | 2.80479655                                     | 1.00000378                        |
| 5/4             | 2.80478358                                     | 1.00000042                        |
| 6/5             | 2.80478142                                     | 1.00000077                        |

### TABLE VII: Lower bounds on $\phi((4 \cdot 8^2))$ and their ratios, as functions of strip width $L_y$.

| BCy | $L_y$ | $\phi((4 \cdot 8^2),(L_y)_{BCy} \times \infty)$ | $R_{\phi((4 \cdot 8^2),L_y+1/L_y,BCy)}^{(L_y+2/L_y),BCy}$ |
|-----|------|-----------------------------------------------|----------------------------------------------------------|
| F   | 2    | $(478 + 2\sqrt{57057})^{1/8} = 2.35799035..$   | 1.056354974237...                                         |
|     | 3    | 2.49087484                                    | 1.027787462852...                                         |
|     | 4    | 2.56008993                                    | 1.016580737054...                                         |
|     | 5    | 2.60253811                                    | 1.011023473885...                                         |
|     | 6    | 2.63122712                                    | 1.011023473885...                                         |
| P   | 2    | $1 + \sqrt{3} = 2.73205081$                   | 1.01622617                                                 |
|     | 4    | 2.77638152                                    | 1.00099181                                                 |
|     | 6    | 2.77913516                                    | 1.00099181                                                 |
### TABLE VIII: Upper bounds on $\phi((4 \cdot 8^2))$ and their ratios, as functions of strip width $L_y$.

| $(L_y + 1)/L_y$ | $\left[\lambda_{(4,8^2), L_y+1,F(2,1)}/\lambda_{(4,8^2), L_y,F(2,1)}\right]^{1/4} R_{\lambda_{(4,8^2), L_y,F(2,1)}}^{L_y^2}$ | $R_{\phi_{(4,8^2), (L_y-1)/(L_y+1), F(2,1)}}$ |
|-----------------|---------------------------------------------------------------------|-----------------------------|
| $2/1$           | $\left(\frac{487 + 2\sqrt{57057}}{2}\right)^{1/4} = 2.78005925..$        |                             |
| $3/2$           | 2.77953194                                                           | 1.00018971                  |
| $4/3$           | 2.77949034                                                           | 1.00001497                  |
| $5/4$           | 2.77948671                                                           | 1.00000131                  |
| $6/5$           | 2.77948627                                                           | 1.00000016                  |

### TABLE IX: Lower bounds on $\phi(kag)$ and their ratios, as functions of strip width $L_y$.

| BC $L_y$ | $\phi(kag, (L_y)_{BC} \times \infty$ | $R_{\phi, kag, (L_y+1)/L_y, BC_y}$ |
|----------|-------------------------------------|------------------------------------|
| $F 2$    | $(97 + \sqrt{8777})^{1/5} = 2.8580905..$ |                                    |
| $F 3$    | 3.12095363                           | 1.09200271                         |
| $P 1$    | $33^{1/3} = 3.20753433..$            |                                    |
| $P 2$    | $\left(\frac{1991 + 19\sqrt{10545}}{2}\right)^{1/6} = 3.54091952..$ | 1.10393815                         |
| $P 3$    | 3.59048515                           | 1.01399796                         |

### TABLE X: Upper bounds on $\phi(kag)$ and their ratios, as functions of strip width $L_y$.

| $(L_y + 1)/L_y$ | $\left[\lambda_{kag, L_y+1,F(2,1)}/\lambda_{kag, L_y,F(2,1)}\right]^{1/3} R_{\lambda_{kag, L_y,F(2,1)}}^{L_y^2}$ | $R_{\phi_{kag, (L_y-1)/(L_y+1), F(2,1)}}$ |
|-----------------|---------------------------------------------------------------------|-----------------------------|
| $2/1$           | $(\frac{97 + \sqrt{8777}}{4})^{1/3} = 3.62592933..$               |                             |
| $3/2$           | 3.6140446                                                           | 1.00328848                  |
TABLE XI: Lower bounds on $\phi((3^3 \cdot 4^2))$ and their ratios, as functions of strip width $L_y$. One can define different paths transverse to the longitudinal direction on a strip of this lattice (see Fig. 1(a) in [30]). We list results for both choices.

| BC_y | L_y | $\phi((3^3 \cdot 4^2), (L_y)_{BC_y} \times \infty)$ | $R_{\phi((3^3 \cdot 4^2), (L_y)_{BC_y}^{L_y+2} / L_y^{L_y+1}, BC_y}$ |
|------|-----|---------------------------------|---------------------------------|
| F    | 3   | 3.49582205                      |                                 |
| F    | 5   | 3.8871789                       | 1.1194985                       |
| P    | 2   | $\sqrt{5} + \sqrt{3} = 3.96811879$. |                                 |
| P    | 4   | 4.42938725                      | 1.11624361                       |
| P    | 6   | 4.50622854                      | 1.01734807                       |
| F    | 2   | $(44 + 8\sqrt{30})^{1/4} = 3.06122777$. |                                 |
| F    | 3   | 3.49986242                      | 1.143287167                      |
| F    | 4   | 3.73916108                      | 1.068373734                      |
| F    | 5   | 3.88977485                      | 1.040280095                      |
| F    | 6   | 3.99328734                      | 1.026611435                      |
| P    | 2   | $\sqrt{123} + \sqrt{15105} = 3.95995902$. |                                 |
| P    | 3   | 4.30996446                      | 1.088386127                      |
| P    | 4   | 4.42859682                      | 1.027525136                      |
| P    | 5   | 4.48019516                      | 1.011651170                      |
TABLE XII: Upper bounds on $\phi((3^3 \cdot 4^2))$ and their ratios, as functions of strip width $L_y$. See caption to Table XI.

| $\frac{L_y+2}{L_y}$ or $\frac{L_y+1}{L_y}$ | $\left[\frac{\lambda((3^3 \cdot 4^2),L_y+2/1,F(2,1))}{\lambda((3^3 \cdot 4^2),L_y,F(2,1))}\right]^{1/2}$ | $R_{(3^3 \cdot 4^2),F(2,1)}^{[\frac{L_y^2}{L_y^2-2}]}$ $\frac{F(2,1)}{F(3,1)}$ |
|-----------------------------------------|-------------------------------------------------|--------------------------------------------------|
| 3/1                                    | 4.62177690                                      | 1.01402033                                       |
| 5/3                                    | 4.55787399                                      | 1.01402033                                       |
| 2/1                                    | $\sqrt{5} + \sqrt{6} = 4.68555772...$         |                                                  |
| 3/2                                    | 4.57468959                                      | 1.02423512                                       |
| 4/3                                    | 4.55977887                                      | 1.00327005                                       |
| 5/4                                    | 4.55539056                                      | 1.00096332                                       |
| 6/5                                    | 4.55366469                                      | 1.00037901                                       |

TABLE XIII: Lower bounds and their ratios for $\phi((3^2 \cdot 4 \cdot 3 \cdot 4))$ as functions of strip width $L_y$.

| $BC_y$ | $L_y$ | $\phi((3^2 \cdot 4 \cdot 3 \cdot 4),L_y)_{BC_y \times \infty}$ | $R_{\phi((3^2 \cdot 4 \cdot 3 \cdot 4),L_y+1/L_y+2,BC_y}$ |
|--------|------|-------------------------------------------------|--------------------------------------------------|
| F      | 2    | $(44 + 8\sqrt{30})^{1/4} = 3.06122777...$       |                                                  |
| F      | 3    | 3.50500542                                      | 1.14496721                                       |
| F      | 4    | 3.74646778                                      | 1.06889072                                       |
| F      | 5    | 3.89838787                                      | 1.04055022                                       |
| F      | 6    | 4.00278463                                      | 1.02677947                                       |
| P      | 2    | $\sqrt{5} + \sqrt{3} = 3.968118785...$         |                                                  |
| P      | 4    | 4.43763851                                      | 1.11832300                                       |

24
TABLE XIV: Upper bounds and their ratios for $\phi((3^2 \cdot 4 \cdot 3 \cdot 4))$ as functions of strip width $L_y$.

| $\frac{(L_y + 1)}{L_y}$ | $\sqrt[\lambda]{\frac{\lambda(3^2 \cdot 4 \cdot 3 \cdot 4)}{3^2 \cdot 4 \cdot 3 \cdot 4} \cdot \frac{L_y + 1}{L_y} \cdot F(2, 1)}$ | $R_{\frac{L_y^2}{(3^2 \cdot 4 \cdot 3 \cdot 4) \cdot (L_y - 1) \cdot F(2, 1)}}$ |
|------------------------|-----------------------------------------------------------------|-----------------------------|
| 2/1                    | $\sqrt{5 + \sqrt{6}} = 4.68555772..$                           |                             |
| 3/2                    | 4.59488654                                                      | 1.01973306                  |
| 4/3                    | 4.57532478                                                      | 1.00427549                  |
| 5/4                    | 4.57022128                                                      | 1.00111668                  |
| 6/5                    | 4.56823149                                                      | 1.00043557                  |

TABLE XV: Lower bounds and their ratios for $\sigma(sq)$ as functions of strip width $L_y$.

| BC$_y$ | $L_y$ | $\sigma(sq, (L_y)_{BC_y} \times \infty)$ | $R_{\frac{L_y}{\sigma(sq, (L_y)_{BC_y}}} \times \frac{1}{BC_y}}$ |
|--------|-------|------------------------------------------|-----------------------------------------------------------|
| F      | 1     | 1                                        |                                                           |
| F      | 2     | $\sqrt{\frac{10 + 2\sqrt{17}}{2}} = 2.13577921..$ | 2.13577921..                                              |
| F      | 3     | 2.62742787                                | 1.23019639                                                |
| F      | 4     | 2.88792764                                | 1.09914631                                                |
| F      | 5     | 3.04750858                                | 1.05525794                                                |
| F      | 6     | 3.15487018                                | 1.03522930                                                |
| P      | 2     | $\sqrt{\frac{15 + \sqrt{7}}{2}} = 3.25936733..$ | 3.25936733..                                              |
| P      | 3     | 3.53705348                                | 1.08519634                                                |
| P      | 4     | 3.62352967                                | 1.02444865                                                |
| P      | 5     | 3.65845648                                | 1.00963889                                                |
| P      | 6     | 3.67518338                                | 1.00457212                                                |
### TABLE XVI: Upper bounds and their ratios for $\sigma(sq)$ as functions of strip width $L_y$.  

| $(L_y + 1)/L_y$ | $\lambda_{sq,L_y+1,F}(1,2)/\lambda_{sq,L_y,F}(1,2)$ | $R_{sq,L_y+1,F}(1,2)/R_{sq,L_y,F}(1,2)$ |
|----------------|---------------------------------|---------------------------------|
| 2/1            | $\frac{5+\sqrt{17}}{2} = 4.56155281$.. |                                  |
| 3/2            | 3.97630508                      | 1.14713881                      |
| 4/3            | 3.83488921                      | 1.03687613                      |
| 5/4            | 3.77902232                      | 1.01478342                      |
| 6/5            | 3.75114866                      | 1.00743070                      |

### TABLE XVII: Lower bounds and their ratios for $\sigma(tri)$ as functions of strip width $L_y$.  

| BC $L_y$ | $\sigma(tri,(L_y)_{BC_y} \times \infty)$ | $R_{\sigma,tri,(L_y+1)/L_y,BC_y}$ |
|----------|----------------------------------------|----------------------------------|
| F 2      | $\sqrt{6 + 4\sqrt{2}} = 3.41421356$.. |                                  |
| F 3      | 4.65472093                              | 1.36333620                      |
| F 4      | 5.35640463                              | 1.15074668                      |
| P 5      | 5.80398594                              | 1.08356003                      |
| P 6      | 6.11427423                              | 1.05346124                      |
| P 2      | $\sqrt{29 + \sqrt{817}} = 7.58836029$.. |                                  |
| P 3      | 7.80037170                              | 1.02793903                      |
| P 4      | 7.84674402                              | 1.00594489                      |
| P 5      | 7.85992934                              | 1.00168036                      |
TABLE XVIII: Upper bounds, their ratios relative to the exact $\sigma(\text{tri})$, and ratios of adjacent bounds, as functions of strip width $L_y$.

| $(L_y + 1)/L_y$ | $\lambda_{\text{tri},L_y+1,F}(1,2)/\lambda_{\text{tri},L_y,F}(1,2)$ | $R_{\text{tri},L_y,L_y+2,F}(1,2)/\lambda_{\text{tri},L_y,L_y+1,F}(1,2)$ |
|-----------------|-------------------------------------------------|-------------------------------------------------|
| 2/1             | $6 + 4\sqrt{2} = 11.65685425..$                |                                                 |
| 3/2             | 8.65166268                                    | 1.34735422                                      |
| 4/3             | 8.16230020                                    | 1.05995399                                      |
| 5/4             | 8.00088909                                    | 1.02017415                                      |
| 6/5             | 7.93300485                                    | 1.00855719                                      |

TABLE XIX: Lower bounds and their ratios for $\sigma(\text{hc})$ as functions of strip width $L_y$.

| BC | $L_y$ | $\sigma(\text{hc},(L_y)_{BC_y} \times \infty)$ | $R_{\sigma,\text{hc},L_y+1,L_y+2,BC_y}$ |
|----|------|-----------------------------------------------|----------------------------------------|
| F  | 2    | $(\frac{7+\sqrt{41}}{2})^{1/4} = 1.60895542..$ |                                        |
| F  | 3    | 1.84524123                                    | 1.14685665                             |
| F  | 4    | 1.96759470                                    | 1.06630758                             |
| F  | 5    | 2.04197649                                    | 1.03780341                             |
| F  | 6    | 2.09186520                                    | 1.02443158                             |
| P  | 2    | $\frac{10+2\sqrt{17}}{2} = 2.13577921..$     |                                        |
| P  | 4    | 2.29347361                                    | 1.07383460                             |
| P  | 6    | 2.32235509                                    | 1.01259290                             |
TABLE XX: Upper bounds and their ratios for $\sigma(hc)$ as functions of strip width $L_y$.

| $(L_y + 1)/L_y$ | $[\lambda_{hc,L_y+1,F}(1,2)/\lambda_{hc,L_y,F}(1,2)]^{1/2}$ | $R_{hc,(Ly+1)/(Ly+1)}(1, 2)$ |
|-----------------|---------------------------------|----------------------------------|
| 2/1             | $\sqrt{\frac{14+2\sqrt{41}}{2}} = 2.588737553078...$ |                                  |
| 3/2             | 2.42700921                       | 1.06663689                       |
| 4/3             | 2.38552036                       | 1.01739195                       |
| 5/4             | 2.36870574                       | 1.00709866                       |
| 6/5             | 2.36019825                       | 1.00360457                       |

TABLE XXI: Lower bounds and their ratios for $\sigma((4\cdot 8^2))$ as functions of strip width $L_y$.

| $BC_y$ | $L_y$ | $\sigma((4\cdot 8^2), (L_y)_{BC_y} \times \infty)$ | $R_{\sigma((4\cdot 8^2), \frac{L_y+1}{L_y}}(\frac{L_y+2}{L_y}, BC_y)$ |
|--------|------|--------------------------------------------------|----------------------------------|
| F      | 2    | $\frac{41+3\sqrt{415}}{2}^{1/8} = 1.59026075...$ |                                  |
| F      | 3    | 1.82207863                                      | 1.14577350                       |
| F      | 4    | 1.94327804                                      | 1.06651711                       |
| F      | 5    | 2.01743612                                      | 1.03816133                       |
| F      | 6    | 2.06740176                                      | 1.02476690                       |
| P      | 2    | $\frac{10+2\sqrt{17}}{2} = 2.13577921...$      |                                  |
| P      | 4    | 2.27644959                                      | 1.06586373                       |
| P      | 6    | 2.30261139                                      | 1.01149237                       |
**TABLE XXII**: Upper bounds and their ratios for $\sigma((4 \cdot 8^2))$ as functions of strip width $L_y$.

| $(L_y + 1)/L_y$ | $\left[\frac{\lambda(4.8^2),L_y+1,F(1,2)}{\lambda(4.8^2),L_y,F(1,2)}\right]^{1/4}$ | $R_{\sigma(4.8^2),\frac{L_y^2}{(L_y-1)(L_y+1)},F}(1,2)$ |
|-----------------|---------------------------------|---------------------------------|
| 2/1             | $(\frac{41+3\sqrt{185}}{2})^{1/4} = 2.52892927..$ |                                               |
| 3/2             | 2.39201919                                      | 1.05723620                                      |
| 4/3             | 2.35742796                                      | 1.01467329                                      |
| 5/4             | 2.34346889                                      | 1.00595658                                      |
| 6/5             | 2.33641686                                      | 1.00301831                                      |

**TABLE XXIII**: Lower bounds and their ratios for $\sigma(kag)$ as functions of strip width $L_y$.

| BC_y | $L_y$ | $\sigma(kag,(L_y)BC_y \times \infty)$ | $R_{\sigma(kag,(L_y+1)/L_y,BC_y}$ |
|------|------|--------------------------------------|-----------------------------------|
| F    | 2    | $(40 + 12\sqrt{10})^{1/5} = 2.38979281..$ | 1.19496602                                      |
| F    | 3    | 2.85572120                                      |                                               |
| P    | 1    | $32^{1/3} = 3.17480210..$                      | 1.12679342                                      |
| P    | 2    | $(1056 + 128\sqrt{66})^{1/6} = 3.57734613..$ | 1.01882858                                      |
| P    | 3    | 3.64470247                                      |                                               |

**TABLE XXIV**: Upper bounds and their ratios for $\sigma(kag)$ as functions of strip width $L_y$.

| $(L_y + 1)/L_y$ | $\left[\frac{\lambda_{kag,L_y+1,F}(1,2)}{\lambda_{kag,L_y,F}(1,2)}\right]^{1/3}$ | $R_{\sigma_{kag,\frac{L_y^2}{(L_y-1)(L_y+1)},F}(1,2)$ |
|-----------------|---------------------------------|---------------------------------|
| 2/1             | $(40 + 12\sqrt{10})^{1/3} = 4.27169679..$ |                                               |
| 3/2             | 3.84274644                                      | 1.11162598                                      |
TABLE XXV: Lower bounds and their ratios for $\sigma((3^3 \cdot 4^2))$ as functions of strip width $L_y$.

| $BC_y$ | $L_y$ | $\sigma((3^3 \cdot 4^2))$, $(L_y)_{BC_y} \times \infty$ | $R_{\sigma((3^3 \cdot 4^2))}$, $\frac{L_y+2}{L_y}$, $\frac{L_y+1}{L_y}$, $BC_y$ |
|--------|------|------------------------------------------------|------------------------------------------------|
| F 3    | 3.51850132 |                                      |                                             |
| F 5    | 4.24280788 | 1.20585656                                   |                                             |
| P 2    | $\sqrt{13 + \sqrt{161}} = 5.06839003..$ |                                      |                                             |
| P 4    | 5.40602876 | 1.06661656                                   |                                             |
| P 6    | 5.44463590 | 1.00714150                                   |                                             |
| F 2    | $(27 + \sqrt{721})^{1/4} = 2.70893969..$ |                                      |                                             |
| F 3    | 3.51703426 | 1.29830659                                   |                                             |
| F 4    | 3.96327800 | 1.12688069                                   |                                             |
| F 5    | 4.24306553 | 1.07059498                                   |                                             |
| F 6    | 4.43422383 | 1.04505193                                   |                                             |
| P 2    | $\sqrt{313 + \sqrt{97873}} = 5.00169236..$ |                                      |                                             |
| P 3    | 5.30268205 | 1.06017757                                   |                                             |
| P 4    | 5.39237466 | 1.01691457                                   |                                             |
| P 5    | 5.42627704 | 1.00628710                                   |                                             |
TABLE XXVI: Upper bounds and their ratios for $\sigma((3^3 \cdot 4^2))$ as functions of strip width $L_y$.

| $\frac{L_y+2}{L_y}$ or $\frac{L_y+1}{L_y}$ | \[ \frac{\lambda_{(3^3 \cdot 4^2), L_y+2/1, F(1,2)}}{\lambda_{(3^3 \cdot 4^2), L_y, F(1,2)}} \] $^{1/2}$ | $R_{(3^3 \cdot 4^2), L_y, F(1,2)}$ | $R_{(3^3 \cdot 4^2), L_y+2/1, F(1,2)}$ |
|----------------------------------------|----------------------------------------|----------------------------------------|----------------------------------------|
| 3/1                                   | $6.59988817$                          |                                        |                                        |
| 5/3                                   | $5.61819539$                          |                                        |                                        |
| 2/1                                   | $\sqrt{27 + \sqrt{721}} \approx 7.33835425$ |                                        |                                        |
| 3/2                                   | $5.92831296$                          | $1.23784866$                          |                                        |
| 4/3                                   | $5.67137481$                          | $1.04530439$                          |                                        |
| 5/4                                   | $5.57417435$                          | $1.01743764$                          |                                        |
| 6/5                                   | $5.52722284$                          | $1.00849459$                          |                                        |

TABLE XXVII: Lower bounds and their ratios for $\sigma((3^2 \cdot 4 \cdot 3 \cdot 4))$ as functions of strip width $L_y$.

| BC_y | L_y | $\sigma((3^2 \cdot 4 \cdot 3 \cdot 4), (L_y)_{BC_y} \times \infty)$ | $R_{\sigma(3^2 \cdot 4 \cdot 3 \cdot 4), L_y+1/\infty, BC_y}$ |
|------|-----|-----------------------------------------------------------------|-------------------------------------------------|
| F    | 2   | $(27 + \sqrt{721})^{1/4} = 2.70893969$                           | 1.30200118                                       |
| F    | 3   | 3.52704267                                                        |                                                 |
| F    | 4   | 3.97204751                                                        | 1.12616940                                       |
| F    | 5   | 4.25003524                                                        | 1.06998600                                       |
| F    | 6   | 4.43971476                                                        | 1.04463010                                       |
| P    | 2   | $\sqrt{13 + \sqrt{161}} = 5.06839003$                           |                                                 |
| P    | 4   | 5.40726946                                                        | 1.06686136                                       |
TABLE XXVIII: Upper bounds and their ratios for $\sigma((3^2 \cdot 4 \cdot 3 \cdot 4))$ as functions of strip width $L_y$.

| $L_y+1/L_y$ | $\left[\frac{\lambda((3^2 \cdot 4 \cdot 3 \cdot 4), L_y+1,F(1,2))}{\lambda((3^2 \cdot 4 \cdot 3 \cdot 4), L_y,F(1,2))}\right]^{1/2}$ | $R_{(3^2 \cdot 4 \cdot 3 \cdot 4), L_y,F(1,2)}^{L_y^2}$ |
|-------------|-------------------------------------------------------------------------------------------------------------|-------------------------------------|
| 2/1         | $\sqrt{27 + \sqrt{721}} = 7.33835425$..                                                                           | 1.22734089                           |
| 3/2         | 5.97906767                                         | 1.22734089                           |
| 4/3         | 5.67316738                                         | 1.05392055                           |
| 5/4         | 5.57063774                                         | 1.01840537                           |
| 6/5         | 5.5290732                                          | 1.00864226                           |

TABLE XXIX: Lower and upper bounds on $\phi(\Lambda)$, denoted $\phi^\ell(\Lambda)$ and $\phi^u(\Lambda)$, and on $\sigma(\Lambda)$, denoted $\sigma^\ell(\Lambda)$ and $\sigma^u(\Lambda)$, for the lattices $\Lambda$ analyzed here. The lattices are listed in order of increasing vertex degree $\Delta(\Lambda)$. See text for further discussion.

| $\Lambda$ | $\Delta(\Lambda)$ | $\phi^\ell(\Lambda)$ | $\phi^u(\Lambda)$ | $\sigma^\ell(\Lambda)$ | $\sigma^u(\Lambda)$ |
|-----------|---------------------|------------------------|---------------------|------------------------|---------------------|
| $(4 \cdot 8^2)$ | 3                  | 2.779135               | 2.779486            | 2.302611               | 2.336417            |
| $(6^3) = hc$  | 3                  | 2.803787               | 2.804781            | 2.322355               | 2.344107            |
| $(4^4) = sq$  | 4                  | 3.675183               | 3.699659            | 3.675183               | 3.699659            |
| $(3 \cdot 6 \cdot 3 \cdot 6)$ | 4                  | 3.590485               | 3.614045            | 3.644702               | 3.842746            |
| $(3^3 \cdot 4^2)$ | 5                  | 4.506228               | 4.553665            | 5.444636               | 5.527223            |
| $(3^2 \cdot 4 \cdot 3 \cdot 4)$ | 5                  | 4.437638               | 4.568231            | 5.407269               | 5.522907            |
| $(3^6) = tri$ | 6                  | 5.393333               | 5.494840            | 7.861223               | 7.866799            |
TABLE XXX: Approximate values $\phi_{app}(\Lambda)$ and $\sigma_{app}(\Lambda)$, as defined in Eq. (4.1), for the lattices $\Lambda$ analyzed here.

| $\Lambda$ | $\Delta(\Lambda)$ | $\phi_{app}(\Lambda)$          | $\sigma_{app}(\Lambda)$          |
|-----------|---------------------|---------------------------------|-----------------------------------|
| $(4 \cdot 8^2)$ | 3                   | $2.77931 \pm 0.00018$           | $2.3195 \pm 0.017$               |
| $(6^3) = hc$ | 3                   | $2.80428 \pm 0.00050$           | $2.333 \pm 0.011$               |
| $(4^4) = sq$ | 4                   | $3.687 \pm 0.012$              | $3.687 \pm 0.012$               |
| $(3 \cdot 6 \cdot 3 \cdot 6)$ | 4                   | $3.602 \pm 0.012$              | $3.74 \pm 0.10$                 |
| $(3^3 \cdot 4^2)$ | 5                   | $4.530 \pm 0.024$              | $5.486 \pm 0.041$               |
| $(3^2 \cdot 4 \cdot 3 \cdot 4)$ | 5                   | $4.503 \pm 0.065$              | $5.465 \pm 0.058$               |
| $(3^6) = tri$ | 6                   | $5.444 \pm 0.051$              | $7.864 \pm 0.0028$               |