Weights for finite Clifford groups of odd dimension and odd primes *

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November 18, 2020

Abstract

By reduction theorems of Navarro-Tiep [23] and Späth [26], a way to prove the Alperin weight conjecture and its blockwise version is to verify the inductive Alperin weight condition and the inductive blockwise Alperin weight condition for all finite simple groups respectively. In this paper, we verify the Alperin weight conjecture for finite special Clifford groups of odd dimension and non-defining characteristic odd primes, which is a first step to check the inductive Alperin weight condition for simple groups of Lie type B via a criterion given by J. Brough and B. Späth [4].

2010 Mathematics Subject Classification: 20C20, 20C33.

Keyword: Alperin weight conjecture, inductive condition, finite special Clifford groups, type B.

1 Introduction

For an arbitrary finite group $H$, we denote by $dz(H)$ the set of its all irreducible characters of defect zero. For $K \triangleleft H$ and $\theta \in \text{Irr}(K)$, set $dz(H \mid \theta) = dz(H) \cap \text{Irr}(H \mid \theta)$. Let $G$ be a finite group and $\ell$ be a prime dividing $|G|$. An $\ell$-weight of $G$ means a pair $(R, \varphi)$ with $\varphi \in dz(N_G(R)/R)$, in which case, $R$ is necessarily an $\ell$-radical subgroup of $G$, i.e. $R = O_\ell(N_G(R))$ and $\varphi$ is called a weight character. Following [4], we denote the set of all weights of $G$ by $\text{Alp}^0(G)$ and the set of all $G$-conjugacy classes of weights of $G$ by $\text{Alp}(G)$.

J.L. Alperin proposed an important conjecture in [1], called Alperin weight conjecture (AWC), asserting that the number of irreducible $\ell$-Brauer characters of a finite group $G$ is equal to the number of $G$-conjugacy classes of $\ell$-weights of $G$. This conjecture has a blockwise version, called blockwise Alperin weight conjecture. AWC and its blockwise version have been verified for many cases. Here we just mention the papers of J.L. Alperin and P. Fong [2] and J. An [3], which consider many finite classical groups and odd primes.

The general proof of the AWC seems very difficult at present. An alternative accessible way is to reduce it to simple groups. G. Navarro and P.H. Tiep obtained a reduction for the AWC first in [23], and then B. Späth reduced the blockwise version in [26]. By their results, to prove the AWC and its blockwise version, it suffices to verify the so-called inductive Alperin weight (AW) condition and inductive blockwise Alperin weight (BAW) condition for all finite simple groups respectively. Recently, J. Brough and B. Späth give a new criterion for the inductive condition in [4], adapted for the finite simple groups of Lie type. Using this criterion, Z. Feng, J. Zhang and the author verified the inductive AW condition

*Supported by the NSFC (No. 11631001, No. 11901478).
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for simple groups of type A in [12] and the inductive BAW condition for some cases of finite simple groups of type A in [13]; the author verified the inductive BAW condition for simple groups of type C and odd primes in [19] and [20]; Z. Feng and G. Malle verified the inductive BAW condition for simple groups of type C and the prime 2 in [14]. We remark that the case of finite simple groups of Lie type and defining characteristic has been verified by B. Späth [26].

According to the criterion in [4], to consider the inductive condition for type B and odd primes, one needs to consider the weights of the finite spinor groups of odd dimension and its regular embedding—finite special Clifford groups (note that special Clifford groups of even dimension are not the regular embeddings of spin groups). In this paper we consider the finite special Clifford groups of odd dimension and non-defining characteristic odd primes.

**Theorem 1.** The Alperin weight conjecture holds for finite special Clifford groups of odd dimension and odd primes different from defining characteristic.

This paper is structured as follows. In §2 we fix some notation and give some preliminaries. Then in §3 combinatorial parametrizations of semisimple elements and irreducible characters of relevant classical groups are given. Next, in §4 we recall the results for $\text{SO}_{2e+1}(q)$ in [3] and make a slight modification for our purpose. Finally, we classify the weights and prove the main theorem in §5.

**Acknowledgement** The author is grateful to the fruitful discussions with Dr. Z. Feng and Y. Du and acknowledges the support of Sustech International Center for Mathematics for the stay in the summer of 2020.

## 2 Preliminaries

In this section, we fix notation and give some preliminaries.

2.1. Let $G$ be an arbitrary finite group. For $g \in G$, denote by $\text{Cl}_G(g)$ its $G$-conjugacy class. The cyclic group of order $n$ will be denoted as $\mathbb{Z}_n$.

The notation for representations of finite groups in this paper is standard, which can be found for example in [22] except that we use $\text{Ind}$ and $\text{Res}$ for induction and restriction and use $\chi_0$ for the restriction of an ordinary character $\chi$ of $G$ to the $\ell$-regular elements. We will consider irreducible Brauer characters and weights with respect to a fixed prime $\ell$, thus we will abbreviate $\ell$-Brauer characters, $\ell$-weights, etc. as Brauer characters, weights, etc.

2.2. Let $q$ be a power of an odd prime $p$, $\mathbb{F}_q$ be the finite field of $q$ elements and $\overline{\mathbb{F}}_q$ be the algebraic closure of $\mathbb{F}_q$. Let $\ell$ be an odd prime different from $p$ and denote by $v$ the discrete valuation such that $v(\ell) = 1$. Assume $e$ is the multiplicative order of $q^2$ in $\mathbb{Z}_\ell$ and set $a = v(q^{2e} - 1)$. $\ell$ is called linear or unitary according to $\ell | q^e - 1$ or $\ell | q^e + 1$; set $e = 1$ or $-1$ according to $\ell$ is linear or unitary. When $\ell$ is linear, $e$ is necessarily odd.

2.3. Let $V$ be a symplectic space or an orthogonal space over $k = \mathbb{F}_q$ or $\overline{\mathbb{F}}_q$. When $V$ is an orthogonal space, denote by $\eta(V) = \pm 1$ the type of $V$.

The classical groups $\text{I}(V)$, $\text{I}_0(V)$, $\text{J}(V)$, $\text{J}_0(V)$ are as in [16 §1]. Thus when $V$ is a symplectic space, $\text{I}(V) = \text{I}_0(V) = \text{Sp}(V)$ and $\text{J}_0(V) = \text{J}(V) = \text{CSp}(V)$ are the symplectic group and conformal symplectic group over $V$; while when $V$ is an orthogonal space, $\text{I}(V) =$
GO(V) and I_0(V) = SO(V) are the orthogonal group and special orthogonal group over V, and if furthermore \( \dim_k V \) is even, J(V) = CO(V) and J_0(V) = CSO(V) are the conformal orthogonal group and special conformal orthogonal group over V.

Given a basis of \( V \), these classical groups can be identified with some matrix groups. In the sequel, when an element of these classical groups is represented by a matrix, it is understood that an appropriate basis is chosen.

We recall some basic facts about the Clifford algebra and Clifford group over an orthogonal space as follows; for details, see for example, [8].

2.4. Let \( V \) be an orthogonal space over the field \( k = \mathbb{F}_q \) or \( \overline{\mathbb{F}}_q \) with the quadratic form \( Q \) and bilinear form \( B(\cdot, \cdot) \) such that
\[
B(u, v) = Q(u + v) - Q(u) - Q(v), \quad u, v \in V,
\]
thus \( B(u, u) = 2Q(u) \) for any \( u \in V \) and the quadratic form \( Q \) can be recovered from the bilinear form \( B(\cdot, \cdot) \) since \( q \) is odd. So for any element \( u \) of \( V \), \( u \) is isotropic (i.e. \( B(u, u) = 0 \)) if and only if \( u \) is singular (i.e. \( Q(u) = 0 \)).

The Clifford algebra \( C(V) \) on \( V \) is defined to be the \( k \)-algebra with identity \( e \) generated by the elements of \( V \) modulo the relations
\[
v^2 = Q(v) \cdot e, \quad v \in V.
\]
Note that the symbol \( e \) has been used for other meaning in 2.2, but this can be distinguished from context. Thus an element \( v \in V \) is a unit in \( C(V) \) if and only if \( v \) is non-isotropic, in which case, \( v^{-1} = Q(v)^{-1} \cdot v \). We also have
\[
uv + vu = B(u, v), \quad u, v \in V;
\]
in particular, when \( u, v \) are orthogonal, i.e. \( B(u, v) = 0 \), we have \( uv = -vu \). Given an orthogonal basis \( e_1, e_2, \ldots, e_m \) of \( V \), \( C(V) \) has a basis as a \( k \)-linear space
\[
e_1 e_2 \cdots e_m, \quad e_i \in \{0, 1\}.
\]
The Clifford algebra \( C(V) \) has a \( \mathbb{Z}_2 \)-gradation structure
\[
C(V) = C_0(V) \oplus C_1(V),
\]
where \( C_0(V) \) is called the special Clifford algebra over \( V \), and the elements in \( C_0(V) \) and \( C_1(V) \) are called even and odd respectively. The multiplication of two homogenous elements \( a, b \) in \( C(V) \) satisfies
\[
ab = (-1)^{\deg a \deg b} ba.
\]
Thus if there is a decomposition
\[
V = V_1 \perp V_2,
\]
then for any \( a \in C_0(V_1) \) and \( b \in C(V_2) \), we have \( ab = ba \).

2.5. Keep the notation in 2.4. Then the Clifford group and special Clifford group over \( V \) is defined as
\[
D(V) = \{ g \in C(V)^\times \mid gVg^{-1} = V \},
\]
\[
D_0(V) = \{ g \in C_0(V)^\times \mid gVg^{-1} = V \}.
\]
For any non-isotropic element \( x \) of \( V \), the conjugation of \( x \) induces the orthogonal transformation \( -\varrho_x \) on \( V \), where \( \varrho_x \) is the orthogonal reflection defined by \( x \), i.e.
\[
\varrho_x : \quad V \to V, \quad v \mapsto v - \frac{B(x, v)}{Q(x)} x.
\]
This induces a group homomorphism \( \pi : D(V) \to \text{GO}(V) \), which is surjective if and only if \( \dim_k V \) is even by [8] II.3.1. The restriction of \( \pi \) induces an exact sequence

\[
1 \to k^\times \cdot e \longrightarrow D_0(V) \overset{\pi}{\longrightarrow} \text{SO}(V) \to 1.
\]

When \( \dim_k V \) is odd, \( Z(D_0(V)) = k^\times \cdot e \).

2.6. Keep the notation in 2.4 and 2.5 and assume \( V \) is of plus type and of odd dimension \( 2n + 1 \). Choose a basis of \( V \)

\[
e_1, \ldots, e_n, \eta_0, \eta_1, \ldots, \eta_n
\]

such that for \( v = x_0e_0 + x_1e_1 + \cdots + x_n e_n + y_1\eta_1 + \cdots + y_n\eta_n \), we have

\[
Q(v) = x_0^2 + x_1 y_1 + \cdots + x_n y_n.
\]

Thus the metric matrix of the bilinear form \( B(-, -) \) under the basis \((2.6.1)\) is

\[
\begin{bmatrix}
0 & 0 & I_n \\
0 & 2 & 0 \\
I_n & 0 & 0
\end{bmatrix}.
\]

By [8] II.2.6, there is an odd element \( z_V \) (scalar product of the product of vectors in an orthogonal basis) in \( Z(C(V)) \) such that \( z_V^2 = 1 \). Note that \( z_V \in D(V) \) but \( z_V \notin D_0(V) \). Furthermore, by [8] II.3.4, \( D_0(V) \) is generated by the products \( vz_V \), where \( v \) runs over all non-isotropic elements of \( V \); denote \( (v_1z_V)(v_2z_V) = v_1v_2 \) for any two non-isotropic elements \( v_1, v_2 \in V \).

2.7. Keep the conventions in 2.6 in particular, \( \dim_k V = 2n + 1 \) and a basis \((2.6.1)\) of \( V \) is chosen. The Weyl groups of \( \text{SO}(V) \) and \( D_0(V) \) are both isomorphic to \( \mathbb{Z}_2 \wr \Sigma(n) \), where \( \Sigma(n) \) denotes the symmetric group of \( n \) symbols.

Note that \( e_i - \eta_i \) is a non-isotropic element in \( V \), then \( \xi_{e_i-\eta_i} \) denotes the corresponding orthogonal reflection. Set

\[
W_0 = \langle -\xi_{e_i-\eta_i} \mid i = 1, 2, \ldots, n \rangle,
\]

and

\[
W_1 = \{ \text{diag}(P_\sigma, 1, P_\tau) \mid \sigma \in \Sigma(n) \},
\]

where \( P_\sigma \) is the permutation matrix in \( \text{GL}_n(k) \) representing \( \sigma \). Then the Weyl group of \( \text{SO}(V) \) can be identified with \( W_0 \rtimes W_1 \). We will identify \( W_1 \) with \( \Sigma(n) \). Denote by \( w_{ij} \) the transposition \( (i, j) \) and \( w_i = -\xi_{e_i-\eta_i} \).

Set \( \tilde{w}_i = z_V(e_i - \eta_i) \) and

\[
\tilde{w}_{ij} = -\frac{1}{2}(e_i - e_j + \eta_i - \eta_j)(e_i - e_j - \eta_i + \eta_j).
\]

Then \( w_i = \pi(\tilde{w}_i), \tilde{w}_i^2 = -e \) and \( w_{ij} = \pi(\tilde{w}_{ij}), \tilde{w}_{ij}^2 = e \), where \( \pi \) is as in 2.5. Set

\[
\tilde{W}_0 = \langle \tilde{w}_i \mid i = 1, 2, \ldots, n \rangle
\]

and

\[
\tilde{W}_1 = \langle \tilde{w}_{i,i+1} \mid i = 1, 2, \ldots, n - 1 \rangle.
\]

Then the Weyl group of \( D_0(V) \) can be identified with \( (\tilde{W}_0 \rtimes \tilde{W}_1) / (-e) \).

2.8. Denote by \( \text{Irr}(\mathbb{F}_q[X]) \) the set of all monic irreducible polynomials over \( \mathbb{F}_q \). Fix a \( \xi \in \mathbb{F}_q^\times \). For any \( \Gamma \in \text{Irr}(\mathbb{F}_q[X]) - \{ X \} \), denote by \( \Gamma_\xi \) the monic irreducible polynomial whose roots
are $\xi \alpha^{-1}$ with $\alpha$ running over all roots of $\Gamma$. Set
\[
\mathcal{F}_{\xi,0} = \left\{ \{X - \xi_0, X + \xi_0\}, \quad \xi = \xi_0^2 \in (\mathbb{F}_q^*)^2, \right. \\
\left. \{X^2 - \xi\}, \quad \xi \notin (\mathbb{F}_q^*)^2 \right\}.
\]
\[
\mathcal{F}_{\xi,1} = \{ \Gamma \in \text{Irr}(\mathbb{F}_q[X]) \mid \Gamma^\alpha = \Gamma, \Gamma \neq X, \Gamma \neq X^2 - \xi \};
\]
\[
\mathcal{F}_{\xi,2} = \{ \Gamma \Gamma' \mid \Gamma \in \text{Irr}(\mathbb{F}_q[X]), \Gamma' \neq \Gamma \}.
\]
Set $\mathcal{F}_\xi = \mathcal{F}_{\xi,0} \cup \mathcal{F}_{\xi,1} \cup \mathcal{F}_{\xi,2}$. When $\xi = 1$, $\mathcal{F}_1$ is abbreviated as $\mathcal{F}$, which is as in \[16, 3\]. Set $d_\Gamma = \deg \Gamma$ for any polynomial $\Gamma$. Note that for any $\Gamma \in \mathcal{F}_{\xi,1} \cup \mathcal{F}_{\xi,2}$, $d_\Gamma$ is even. Since $\delta_\Gamma = \frac{1}{2} d_\Gamma$ or $d_\Gamma$ if $\Gamma \notin \mathcal{F}_{\xi,0}$ or $\Gamma \in \mathcal{F}_{\xi,0}$. A polynomial $\Gamma \in \mathcal{F}_\xi$ can be identified with the set of its roots. For $\Gamma \in \mathcal{F}_{\xi,2}$, there is $b \in \mathbb{F}_{q^{d_\Gamma}}$ of $\Gamma$ such that all roots of $\Gamma$ are
\[
(2.8.1) \quad b, b^q, \ldots, b^{q^{d_\Gamma-1}}, \xi b^{-q}, \ldots, \xi b^{q^{d_\Gamma-1}}.
\]
For $\Gamma \in \mathcal{F}_{\xi,1}$, there is $b \in \mathbb{F}_{q^{d_\Gamma}}$ of $\Gamma$ such that $b^{q^{d_\Gamma+1}} = \xi$ and all roots of $\Gamma$ are
\[
(2.8.2) \quad b, b^q, \ldots, b^{q^{d_\Gamma-1}}, \xi b^{-q} = b^{q^{d_\Gamma+1}}, \ldots, \xi b^{q^{d_\Gamma-1}} = b^{q^{d_\Gamma+1}}.
\]
Set also
\[
\varepsilon_{\Gamma} = \begin{cases} 
1, & \Gamma \in \mathcal{F}_{\xi,2}; \\
-1, & \Gamma \in \mathcal{F}_{\xi,1}.
\end{cases}
\]
For a polynomial $\Gamma \in \mathcal{F}_\xi$, set $\beta_{\Gamma} = 1$ or 2 according to $\Gamma \notin \mathcal{F}_{\xi,0}$ or $\Gamma \in \mathcal{F}_{\xi,0}$.

Let $a$ be a natural number. For any monic polynomial $\Delta$ in $\mathbb{F}_{q^{d_{\Gamma^\alpha}}} [X]$, let $\hat{\Delta}$ be the monic polynomial in $\mathbb{F}_{q^{d_{\Gamma^\alpha}}} [X]$ whose roots of $\hat{\Delta}$ are $\omega^{-q^{\ell(a)}}$ with $\omega$ running over all roots of $\Delta$. As in \[16\] p.160, set
\[
\mathcal{E}_{a,0} = \{ \Delta \in \text{Irr} \left( \mathbb{F}_{q^{d_{\Gamma^\alpha}}} [X] \right) \mid \Delta \neq X \},
\]
\[
\mathcal{E}_{a,1} = \{ \Delta \in \text{Irr} \left( \mathbb{F}_{q^{d_{\Gamma^\alpha}}} [X] \right) \mid \Delta \neq X, \hat{\Delta} = \Delta \},
\]
\[
\mathcal{E}_{a,2} = \{ \Delta \hat{\Delta} \mid \Delta \in \text{Irr} \left( \mathbb{F}_{q^{d_{\Gamma^\alpha}}} [X] \right), \hat{\Delta} \neq \Delta \},
\]
and
\[
\mathcal{E}_a = \begin{cases} 
\mathcal{E}_{a,0}, & \varepsilon = 1; \\
\mathcal{E}_{a,1} \cup \mathcal{E}_{a,2}, & \varepsilon = -1.
\end{cases}
\]
Let $\mathcal{F}'_\xi$ and $\mathcal{E}'_a$ be the subsets of all polynomials whose roots are of $\ell'$-order in $\mathcal{F}_\xi$ and $\mathcal{E}_a$ respectively.

2.9. For any square $\xi = \xi_0^2 (\xi_0 \in \mathbb{F}_q^*)$, define the map (see \[5, 4\])
\[
\xi_0^\varepsilon : \quad \mathcal{F} \to \mathcal{F}_\xi, \quad \Gamma \mapsto \xi_0^\varepsilon \cdot \Gamma
\]
be such that the roots of $\xi_0 \cdot \Gamma$ are the multiplications of roots of $\Gamma$ by $\xi_0$.

Assume $\xi$ is a non-square. If $\varepsilon = 1$, define the map
\[
\mathcal{N}_{a,\xi} : \quad \mathcal{E}_a \to \mathcal{F}_\xi, \quad \Delta \mapsto \mathcal{N}_{a,\xi}(\Delta)
\]
be such that the set of roots of $\mathcal{N}_{a,\xi}(\Delta)$ is the orbit of the roots of $\Delta$ under the actions of the following operations
\[
b \mapsto b^{q^i}; \quad b \mapsto \xi b^{-1}.
\]
If $\varepsilon = -1$, choose $\xi_1 \in \mathbb{F}_q$ such that $\xi = \xi_1^{q^{d_{\Gamma^\alpha}+1}}$ and define the map
\[
\mathcal{N}_{a,\xi} : \quad \mathcal{E}_a \to \mathcal{F}_\xi, \quad \Delta \mapsto \mathcal{N}_{a,\xi}(\Delta)
\]
be such that the roots of $\mathcal{N}_{a,\xi}(\Delta)$ are
\[
(\xi_1 b)^{q^i}, \quad i = 0, 1, \ldots
\]
where $b$ runs over the set of all roots of $\Delta$. See [5, 6] and Lemma [5, 9] for the above definitions; see also [16] p.161 for a related map $N_{\alpha}$. Denote also by $N_{\alpha, \xi}$ and $\xi_0^\prime$ their restriction to the subsets $E'_\alpha$ and $F'$ respectively.

### 2.10. The concepts and notation concerning partitions, Lusztig symbols and their cores and quotients can be found in [24] Chapter I or [7] §13.8. Here we only make a remark about the Lusztig symbols of odd defect.

Let $e$ be an arbitrary positive integer in this paragraph. For a Lusztig symbol $\lambda$, its $e$-core $\lambda(e)$ is again a Lusztig symbol, but its $e$-quotient $\lambda(e)$, defined as in [24], is an unordered pair of two $e$-tuples of partitions $[\lambda_0, \ldots, \lambda_{e-1}; \mu_0, \ldots, \mu_{e-1}]$, i.e. $[\lambda_0, \ldots, \lambda_{e-1}; \mu_0, \ldots, \mu_{e-1}]$ and $[\mu_0, \ldots, \mu_{e-1}; \lambda_0, \ldots, \lambda_{e-1}]$ are identified. In general, given an $e$-core and an $e$-quotient $[\lambda_0, \ldots, \lambda_{e-1}; \mu_0, \ldots, \mu_{e-1}]$, there may be one or two Lusztig symbols with the given core and quotient; see [24] pp.39-40. When the core is non-degenerate, the number is one or two according to $(\lambda_0, \ldots, \lambda_{e-1}) = (\mu_0, \ldots, \mu_{e-1})$ or $(\lambda_0, \ldots, \lambda_{e-1}) \neq (\mu_0, \ldots, \mu_{e-1})$. Thus Lusztig symbols of odd defect, which are those occurring in this paper, can be determined by its (non-degenerate) $e$-core and $2e$-tuples of partitions $(\lambda_0, \ldots, \lambda_{e-1}; \mu_0, \ldots, \mu_{e-1})$; see also [19] §4.

### 3 Semisimple elements and irreducible characters

In this section, we consider the semisimple elements in conformal symplectic groups and irreducible characters of special Clifford groups of odd dimension.

#### 3.1. Assume $V$ is a symplectic space over $\mathbb{F}_q$. Let $\tilde{s}$ be a semisimple element of $J(V) = J_0(V) = \text{CSp}(V)$ with multiplier $\xi$. By [25] §1, the set $F_\xi$ serves as the set of the elementary divisors of $\tilde{s}$; precisely, there are corresponding decompositions of $\tilde{s}$ and $V$ as follows

$$\tilde{s} = \prod_{\Gamma \in F_\xi} \tilde{s}_\Gamma, \quad V = \bigwedge_\Gamma V_\Gamma$$

where $\tilde{s}_\Gamma = m_\Gamma(\tilde{s})$ is a primary semisimple element having a unique elementary divisor $\Gamma$ with multiplicity $m_\Gamma(\tilde{s})$, and $m_\Gamma(\tilde{s})$ is even for $\Gamma \in F_\xi,0$.

To calculate the centralizers of semisimple elements, we consider the primary component $\tilde{s}_\Gamma$ more carefully. Here we identify the conformal symplectic group with a matrix group via an appropriate basis.

1. Assume $\xi = \xi_0^2 \in (\mathbb{F}_q^\times)^2$ and $\Gamma = \pm \xi_0$. Then $\tilde{s}_\Gamma = \pm \xi_0 I_{m_\Gamma(\tilde{s})}$.

2. Assume $\xi \notin (\mathbb{F}_q^\times)^2$ and $\Gamma = X^2 - \xi$. Let $\xi_0 \in \mathbb{F}_q^\times$ be such that $\xi_0^2 = \xi$, then $\xi_{0}^{-1} = -1$ and $\xi_{0}^{-1} = -\xi_0$. Let

$$\tilde{s}_{\Gamma,0} = I_{m_\Gamma(\tilde{s})/2} \otimes \text{diagonal}(\xi_0, \xi_0^a, \xi_0, \xi_{0}^{-1})$$

and set

$$v_{\Gamma} = I_{m_\Gamma(\tilde{s})/2} \otimes \text{diagonal}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right).$$

3. Assume $\Gamma \in F_{\xi,2}$ with roots as in (2.8.1). Let

$$\tilde{s}_{\Gamma,0} = I_{m_\Gamma(\tilde{s})} \otimes \text{diagonal}(b, b^q, \ldots, b^{q^{t-1}}, \xi b, \xi^{-1} b^{-1}, \xi b^{-q}, \xi b^{-q^{t-1}})$$

and set

$$v_{\Gamma} = I_{m_\Gamma(\tilde{s})} \otimes \text{diagonal}\left(\begin{bmatrix} 0 & 1 \\ I_{q^{t-1}} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ I_{q^{t-1}} & 0 \end{bmatrix}\right).$$
(4) Assume $\Gamma \in \mathcal{F}_{\xi,1}$ with roots as in (2.8.2). Let
\[ \tilde{\tau}_{\Gamma,0} = I_{m_{(\tilde{\tau})}} \otimes \text{diag}\{b, b', \ldots, b'^{r-1}, b'^r, \ldots, b'^{r+1}\} \]
and set
\[ \nu_{\Gamma} = I_{m_{(\tilde{\tau})}} \otimes \begin{bmatrix} 0 & -1 \\ 1_{d_{r-1}} & 0 \end{bmatrix}. \]
By the construction in (2) ~ (4), $\nu_{\Gamma} \in \text{Sp}_{m_{(\tilde{\tau})} \otimes \mathbb{F}_q}$. By Lang-Steinberg theorem, there is $g_{\Gamma} \in \text{Sp}_{m_{(\tilde{\tau})} \otimes \mathbb{F}_q}$ such that $\tilde{\tau}^{-1} F_q(g_{\Gamma}) = \nu_{\Gamma}$, where $F_q$ is the standard Frobenius map on $\text{Sp}_{m_{(\tilde{\tau})} \otimes \mathbb{F}_q}$. Then it is easy to see that $\tilde{\tau}_{\Gamma,0} \in \text{Sp}_{m_{(\tilde{\tau})} \otimes \mathbb{F}_q \otimes F}$ and we may assume $\tilde{\tau}_{\Gamma} = g_{\Gamma} \tilde{\tau}_{\Gamma,0} \tilde{\tau}^{-1}$ up to conjugacy.

3.2 Lemma. With the conventions and notation in 3.2 we have
\[ C_{\text{hol}(V)}(\tilde{\tau}) = \prod_{\Gamma} C_{\Gamma}(\tilde{\tau}_\Gamma), \quad C_{\Gamma}(\tilde{\tau}_\Gamma) = C_{\text{hol}(V)}(\tilde{\tau}_\Gamma), \]
where
\[ C_{\Gamma}(\tilde{\tau}_\Gamma) \equiv \begin{cases} I_0(\Gamma), & \xi = \xi_0^2 \in (\mathbb{F}_q^\times)^2, \Gamma = \Gamma \pm \xi_0, \\ \text{Sp}_{m_{(\tilde{\tau})}}(q^2), & \xi \notin (\mathbb{F}_q^\times)^2, \Gamma = \Gamma^2 - \xi, \\ \text{GL}_{m_{(\tilde{\tau})}}(q^2), & \Gamma \in \mathcal{F}_{\xi,1} \cup \mathcal{F}_{\xi,2}. \end{cases} \]
Proof. The decomposition is obvious. Since
\[ C_{\Gamma}(\tilde{\tau}_\Gamma) \equiv C_{\text{Sp}_{m_{(\tilde{\tau})} \otimes \mathbb{F}_q} (\tilde{\tau}_{\Gamma,0}) \otimes F}, \]
the assertion for $C_{\Gamma}(\tilde{\tau}_\Gamma)$ follows easily by direct calculations. \qed

3.3. Keep the conventions and notation in 3.1. Let $\zeta$ be a generator of $\mathbb{F}_q^\times$. For $\Gamma \in \mathcal{F}_{\xi,0}$, set
\[ \tau_{\Gamma,0} = \begin{cases} I_{m_{(\tilde{\tau})}/2} \otimes \text{diag}\{1, \zeta\}, & \zeta = \zeta_0^2 \in (\mathbb{F}_q^\times)^2, \Gamma = \Gamma \pm \xi_0, \\ I_{m_{(\tilde{\tau})}/2} \otimes \text{diag}\{1, 1, \zeta, \zeta\}, & \zeta \notin (\mathbb{F}_q^\times)^2, \Gamma = \Gamma^2 - \xi. \end{cases} \]
For $\Gamma \in \mathcal{F}_{\xi,2}$, set
\[ \tau_{\Gamma,0} = I_{m_{(\tilde{\tau})}/2} \otimes \text{diag}\{1, \zeta\}. \]
For $\Gamma \in \mathcal{F}_{\xi,1}$, let $\xi_1 \in \mathbb{F}_q$ be such that $\zeta = \zeta_1^{2r+1}$ and set
\[ \tau_{\Gamma,0} = I_{m_{(\tilde{\tau})}/2} \otimes \text{diag}\{\xi_1, \ldots, \xi_1^{q-1}, \xi_1^{q+1}, \xi_1^{q-1}, \ldots, \xi_1^{q-1}\}; \]
note that $\xi_1^{2r+1} = \xi_1^{-1}, \xi_1^{q-1} = \xi_1^{-q}, \ldots, \xi_1^{q} = \xi_1^{q-1} \xi$. For all cases, set $\tau_{\Gamma} = g_{\Gamma} \tau_{\Gamma,0} \tilde{\tau}^{-1}$.

3.4 Lemma. Keep the conventions and notation in 3.2 and 3.3. and let $\tau = \prod_{\Gamma} \tau_{\Gamma}$.
(1) We have $\tau_{\Gamma}^{q-1} \in Z(C_{\Gamma}(\tilde{\tau}))$ for any $\Gamma \in \mathcal{F}_{\xi}$ and $[\tau_{\Gamma}, C_{\Gamma}(\tilde{\tau})] = 1$ for any $\Gamma \notin \mathcal{F}_{\xi,0}$.
(2) We have $J_0(V) = (\tau, I_0(V)), C_{\text{hol}(V)}(\tilde{\tau}) = (\tau, C_{\text{hol}(V)}(\tilde{\tau}))$ and $\tau_{\Gamma}^{q-1} \in Z(C_{\text{hol}(\tilde{\tau})})$.
(3) We have $|C_{\text{hol}(V)}(\tilde{\tau}) : C_{\text{hol}(V)}(\tilde{\tau})| = |J_0(V) : I_0(V)| = q - 1$.
Proof. All the assertions follow by direct calculations. \qed

3.5 Remark. The above proposition generalize slightly [16] (1A), (1B), but our proof here is different from that in [16].

3.6. Now we fix the notation for some classical groups which will be used from now on. Let $n$ be a natural number and $V$ ($V^\ast$) be the orthogonal (symplectic) space over $\mathbb{F}_q$ of dimension $2n + 1$ ($2n$). Set $G = D_0(V)$, $G = \text{Spin}(V)$ and $H = \text{SO}(V)$. Then $Z(G) = \mathbb{F}_q$.
connected, and $H = \hat{G}/Z(\hat{G})$, $\hat{G} = Z(\hat{G})G$, $G = [\hat{G}, \hat{G}]$. Set $\hat{G}^* = \text{CSp}(V^*)$, $G^* = \text{PCSp}(V^*)$ and $H^* = \text{Sp}(V^*)$, which are dual algebraic groups of $\hat{G}$, $G$ and $H$ respectively. Denote by $F$ both the standard Frobenius maps on $\hat{G}$ and $G^*$ defining an $\mathbb{F}_q$-structure on them. Set $\hat{G} = \hat{G}^F$, $G = \hat{G}^F$, $H = H^F$, $\hat{G}^* = G^*F$, $G^* = G^*F$, $H^* = H^F$; these finite groups can be viewed as classical groups on an orthogonal space $V$ of dimension $2n + 1$ or a symplectic space $V^*$ of dimension $2n$ over $\mathbb{F}_q$. We reserve the notation $G$, $G$ for spin groups and use $H$, $H$ for special orthogonal groups to keep consistent with the notation in the criterion given by \cite{4}.

3.7 Proposition. The irreducible ordinary characters of $G = D_0(V)$ can be labelled as $\check{\chi}_{\lambda, \mu}$, where $\lambda = \prod_{\Gamma \in \mathcal{T}_G} \lambda^\Gamma$ with $\lambda^\Gamma$ a Lusztig symbol of rank $\frac{m(\lambda)}{2}$ and odd defect for $\Gamma \in \mathcal{T}_{G,0}$ and $\lambda^\Gamma$ a partition of $m(\lambda)$ for $\lambda = \prod_{\Gamma \in \mathcal{T}_G} \lambda^\Gamma$ and only if if the two pairs $(\lambda, \lambda^\Gamma)$ are conjugate under $G^*$. The set of characters of the form $\check{\chi}_{\lambda, \mu}$ with $\lambda$ an $\ell^*$-element is a basic set of $G$.

Proof. By Lusztig’s Jordan decomposition of characters, irreducible ordinary characters of $\hat{G}$ can be parametrized by $\hat{G}^*$-conjugacy classes of pairs $(\lambda, \varphi)$, where $\lambda$ is a semisimple element of $\hat{G}^*$ and $\varphi$ is a unipotent character of $C_{\hat{G}^*}(\lambda)$. Since $[C_H(\lambda), C_H(\lambda)] = [C_{\hat{G}^*}(\lambda), C_{\hat{G}^*}(\lambda)]$ by Lemma 3.4, the set of unipotent characters of $C_H(\lambda)$ is the same as that of $C_{\hat{G}^*}(\lambda)$ by \cite{10} Proposition 11.3.8. The set of unipotent characters of each component of $C_H(\lambda)$ is again in bijection with that of the corresponding simple group of adjoint type, which can be found in \cite{7} §13.8. So the assertion concerning the irreducible ordinary characters follows from Lemma 3.2. The last assertion follows from \cite{17}. □

3.8. Define $e_\Gamma$ for each $\Gamma \in \mathcal{T}_G$ as follows.

1. When $e^2 \in (\mathbb{F}_q^*)^2$ and $\Gamma \in \mathcal{T}_{G,0}$, let $e_\Gamma = e$.

2. When $e \notin (\mathbb{F}_q^*)^2$ and $\Gamma \in \mathcal{T}_{G,0}$, let $e_\Gamma = e$ or $\tilde{e}$ according to $e$ is odd or even. Note that $e_\Gamma$ is equal to the multiplicative order of $q^d$ in $\mathbb{Z}_{\ell^*}$; see Lemma 3.2.

3. When $\Gamma \in \mathcal{T}_{G,1} \cup \mathcal{T}_{G,2}$, let $e_\Gamma$ be the multiplicative order of $e_\Gamma q^d$ in $\mathbb{Z}_{\ell^*}$.

For $\Gamma \in \mathcal{T}$, parts (1) and (3) is the same as in [3].

3.9 Lemma. Let $\check{\chi}_{\lambda, \mu}$ be an irreducible ordinary character of $\hat{G}$ as in Proposition 3.7. Then $\check{\chi}_{\lambda, \mu}$ is the $\ell^*$-rational character in a block of $\hat{G}$ with the defect group $Z(\hat{G})_\ell$ if and only if $\lambda$ is a semisimple $\ell^*$-element of $\hat{G}^*$ and $\lambda^\Gamma$ is an $e_\Gamma$-core for each $\Gamma$.

Proof. Similar as \cite{16} (11B), using \cite{10} Proposition 11.5.6 and Lemma 3.4. □

3.10 Lemma. There is an isomorphism

$$\hat{G}^*/G^* \rightarrow \text{Irr}(Z(\hat{G})), \hat{G}^*/G^* \rightarrow \omega(\xi),$$

where $\hat{G}$ is a semisimple element of $\hat{G}^*$ with multiplier $\xi$. Let $\check{\chi}_{\lambda, \mu}$ be an irreducible ordinary character of $\hat{G}$ as in Proposition 3.7, with $\xi$ the multiplier of $\lambda$, then the above isomorphism can be chosen such that the linear character of $Z(\hat{G})$ induced by $\check{\chi}_{\lambda, \mu}$ is $\omega(\xi)$. Thus $(-e) \in \text{Ker} \check{\chi}_{\lambda, \mu}$ if and only if the multiplier $\xi$ is a square in $\mathbb{F}_q^*$. Proof. Let $\mathbf{T}^*$ be a maximal torus of $\hat{G}^*$ containing $\hat{\lambda}$. Let $(\hat{\mathbf{T}}, \hat{\lambda})$ be a pair corresponding to $(\hat{\mathbf{T}}^*, \hat{\lambda})$. Let $\hat{T}^* = \mathbf{T}^*F$ and $\hat{T}^* = \hat{\mathbf{T}}^* \cap H^*$. Via the duality, $\text{Irr}(Z(\hat{G}))$ is isomorphic to $G^*G^* = \hat{T}^*/\hat{T}^*$. So the isomorphism and the next assertion follow. Note that the duality induces isomorphism $\hat{T}^*/\hat{T}^*Z(\hat{G}^*) \cong \hat{G}^*/G^*Z(\hat{G}^*) \cong \text{Irr}(-e)$, which means that $\text{Res}^{\hat{G}}_{\langle -e \rangle} \check{\chi}_{\lambda, \mu} = \check{\chi}_{\lambda, \mu} \cdot \text{Res}^{\hat{G}}_{\langle -e \rangle} \hat{\lambda}$, so the last assertion follows. □
4 Radical subgroups and weights of $\text{SO}_{2n+1}(q)$

Keep the settings in \[3.6\] and use the convention to denote $\text{GU}(n, q)$ as $\text{GL}(n, -q)$.

4.1. We first recall the radical subgroups of $\text{SO}(V)$ classified in \[3\].

Let $\gamma, \alpha$ be non-negative integers. Denote by $Z_\alpha$ the cyclic group of order $\ell^{\alpha+\gamma}$ and by $E_\gamma$ the extraspecial group of order $\ell^{2\gamma+1}$. An $\ell$-group of symplectic type means the central product $Z_\alpha E_\gamma$ of $Z_\alpha$ and $E_\gamma$. We may assume that $E_\gamma$ is of exponent $\ell$ since the extraspecial group of exponent $\ell^2$ will not appear in radical subgroups by the results in \[3\].

Let $m$ be a positive integer. There is an embedding of $Z_\alpha E_\gamma$ in $\text{GL}(m\ell^\gamma, \ell q^{\ell^\gamma})$ and the images of $Z_\alpha$, $E_\gamma$, $Z_\alpha E_\gamma$ are denoted as $Z_{m,\alpha,\gamma}^{0}$, $E_{m,\alpha,\gamma}^{0}$, $R_{m,\alpha,\gamma}^{0}$ respectively; see for example \[19, \S 6.A\]. The group $\text{GL}(m\ell^\gamma, \ell q^{\ell^\gamma})$ and thus the group $R_{m,\alpha,\gamma}^{0}$ can be embedded in the group $SO^{m}(2m\ell^{\alpha+\gamma}, q)$ (see \[3\] \S 2); denote by $R_{m,\alpha,\gamma}$ the image of $R_{m,\alpha,\gamma}^{0}$ under this embedding. For any sequence $\epsilon = (c_1, c_2, \ldots, c_r)$, let $A_\epsilon = A_{c_1} \amalg A_{c_2} \amalg \cdots \amalg A_{c_r}$, be as in \[2\], where $A_{c_i}$ is an elementary abelian group of order $\ell^{c_i}$. Set $R_{m,\alpha,\gamma,\epsilon} = R_{m,\alpha,\gamma} \amalg A_\epsilon$, then $R_{m,\alpha,\gamma,\epsilon}$ is a subgroup of $SO^{m}(2m\ell^{\alpha+\gamma+|\epsilon|}, q)$, where $|\epsilon| = c_1 + c_2 + \cdots + c_r$. The subgroups of the form $R_{m,\alpha,\gamma,\epsilon}$ are called basic subgroups.

Since we are considering the odd prime $\ell$, the radical subgroups of $\text{GO}(V)$ are the same as those of $\text{SO}(V)$. By \[3\], any radical subgroup $R$ of $\text{GO}(V)$ is conjugate to a subgroup of the following form
\begin{equation}
(4.1.1) \quad R = R_0 \times R_1 \times \cdots \times R_u, \quad V = V_0 \perp V_1 \perp \cdots \perp V_u,
\end{equation}
where $R_0$ is the trivial group on $V_0$ and $R_i = R_{m_i,\alpha_i,\gamma_i,\epsilon_i}$ is a basic subgroup on $V_i$ for each $i > 0$. Note that the type $\eta(V_i) = \ell^{m_i}$ and $\eta(V) = \prod_{i=0}^u \eta(V_i)$ by \[16\] (1.5) since the dimension of each $V_i$ ($i > 0$) is even. Set $V_\pm = V_1 \perp \cdots \perp V_u$ and $R_\pm = R_1 \times \cdots \times R_u$.

To classify the weights, one also needs to know the centralizers and normalizers of radical subgroups. Denote by $C_{m,\alpha,\gamma,\epsilon}$ and $N_{m,\alpha,\gamma,\epsilon}$ the centralizer and normalizer of $R_{m,\alpha,\gamma,\epsilon}$ in $\text{GO}^m(2m\ell^{\alpha+\gamma+|\epsilon|}, q)$ respectively. Then $C_{m,\alpha,\gamma,\epsilon}$ is contained in $SO^m(2m\ell^{\alpha+\gamma+|\epsilon|}, q)$, but $N_{m,\alpha,\gamma,\epsilon}$ may not be contained in $SO^m(2m\ell^{\alpha+\gamma+|\epsilon|}, q)$. Assume
\begin{equation}
(4.1.2) \quad R = R_0 \times \prod_{i=1}^u R_{m_i,\alpha_i,\gamma_i,\epsilon_i}, \quad V = V_0 \perp V_+, \quad V = V_0 \perp V_+,
\end{equation}
is a radical subgroup of $\text{SO}(V)$, where $R_0$ is the trivial subgroup on $V_0$ and $(m_i, \alpha_i, \gamma_i, \epsilon_i) \neq (m_j, \alpha_j, \gamma_j, \epsilon_j)$ for any $i \neq j$, then
\begin{align*}
N_{\text{GO}(V)}(R) &= \text{GO}(V_0) \times \prod_{i=1}^u N_{m_i,\alpha_i,\gamma_i,\epsilon_i} \amalg \Xi(t_i).
\end{align*}
But the normalizer $N_{\text{SO}(V)}(R)$ of $R$ in $\text{SO}(V)$ do not have such a decomposition. To fill the gap, J. An first classified the weights $\text{GO}(V_\pm)$ afforded by $R_\pm$, and the reduces the classification of weights of $\text{SO}(V)$ afforded by $R$ to that of $\text{GO}(V_\pm)$; for details, see \[3\] \S 4.2.

We will make a slight modification to the construction of \[3\] as above, which is more convenient to consider radical subgroups and weights of the special Clifford group over $V$.

4.2. We begin our construction now. For a radical subgroup $R$ of $\text{SO}(V)$ as in \[4.1.1\], we construct a twisted version $R^{\text{tw}}$ of $R$ as follows.

Recall that we keep the settings in \[3.6\]. Decompose $V$ as follows:
\begin{align*}
V &= V_0 \perp V_1 \perp \cdots \perp V_u, \quad V_i = \bigwedge_{j=1}^{q^i} V_{ij},
\end{align*}
where $V_i \perp V_\pm$ (4.1.2), and use the convention to denote $\text{GU}(n, q)$ as $\text{GL}(n, -q)$.

4.1. We first recall the radical subgroups of $\text{SO}(V)$ classified in \[3\].
where \( \dim V_i = 2m_i e^{\ell_i + \gamma_i + |c_i|} \) and \( \dim V_{ij} = 2m_i e^{\ell_i + \gamma_i} \) for \( i > 0 \). Choose a basis for \( V_0 \) and each \( V_{ij} \) such that the union of all these basis is the basis \((2.6.1)\) of \( V \). We also denote \( V_+ = V_1 \perp \cdots \perp V_\mu \).

Assume \( i > 0 \). Let the embedding \( R_{m_0, x, \gamma_i}^0 \) of \( Z_\alpha E_{\gamma_i} \) in \( \text{GL}(m_i e^{\ell_i}, e\ell e^{\ell_i}) \) be as in \([19] \) §6.A. Similar as in \([19] \) §6.A, there is a hyperbolic embedding:

\[
\tilde{h} : \quad \text{GL}(m_i e^{\ell_i}, e\ell e^{\ell_i}) \to \text{SO}(V_i)
\]

\[
A \mapsto \text{diag}[A, F(A), \ldots, F^{e^{\ell_i}-1}(A), A^{-i}, F(A^{-i}), \ldots, F^{e^{\ell_i}-1}(A^{-i})],
\]

where \( A^{-i} \) denotes the inverse of the transposition of \( A \). Note that when \( \varepsilon = -1, A^{-i} = F e^{\ell_i}(A) \). Denote the image of \( \tilde{h}(R_{m_0, x, \gamma_i}^0) \) in \( \text{SO}(V) \) under the natural embedding \( \text{SO}(V_i) \to \text{SO}(V) \) as \( R_{m_0, x, \gamma_i}^{tw} \). For the sequence \( \varepsilon \), we can view \( A_{\varepsilon} \) as a subgroup of \( W_1 \) in \([2.7] \) Set

\[
R_{m_0, x, \gamma_i, \varepsilon}^{tw} = \left( \prod_{j=1}^{\mu} R_{m_0, x, \gamma_i, j}^{tw} \right) \times A_{\varepsilon};
\]

of course, \( R_{m_0, x, \gamma_i, \varepsilon}^{tw} \cong R_{m_0, x, \gamma_i, j} \triangleq A_{\varepsilon} \). Set

\[
(4.2.1) \quad R_i^{tw} = \prod_{i=1}^{\mu} R_i^{tw}, \quad R_i^{tw} = R_{m_0, x, \gamma_i, \varepsilon}^{tw}.
\]

If \( \ell \) is linear, let \( v_{m_0, x, \gamma_i, \varepsilon}^{tw} \) be

\[
\begin{align*}
\text{Id}_{V_0 + \cdots + V_{i-1}} \times \left( I_{m_i e^{\ell_i}} \otimes \text{diag} \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \right) \times \text{Id}_{V_{i+1} + \cdots + V_{\mu}},
\end{align*}
\]

while if \( \ell \) is unitary, let \( v_{m_0, x, \gamma_i, \varepsilon}^{tw} \) be

\[
\begin{align*}
(-1)^{m_i} \text{Id} \cdot \left( I_{m_i e^{\ell_i}} \otimes \left[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right] \right) \times \text{Id}_{V_{i+1} + \cdots + V_{\mu}}.
\end{align*}
\]

Note that \( \det v_{m_0, x, \gamma_i, \varepsilon}^{tw} = 1 \) since \( \dim V \) is odd. Set \( v = \prod_{i=1}^{\mu} v_{m_0, x, \gamma_i, \varepsilon} \), then \( v \in \text{SO}(V) \). It is easy to see that \( R^{tw} \leq \text{SO}(V)^{tw} \).

By Lang-Steinberg theorem, there is \( g \in \text{SO}(V) \) such that \( g^{-1} F(g) = v \). Then there is an isomorphism

\[
\iota : \quad \text{SO}(V)^{tw} \to \text{SO}(V)^{tw} = H
\]

\[
x \mapsto gxg^{-1}.
\]

By the construction and results in \([3] \), \( \iota(R_i^{tw}) \) is conjugate to the radical subgroup \( R \) of \( \text{SO}(V) \) in \((4.1.1)\). By the isomorphism \( \iota \), we can transfer the problems to the twisted group \( \text{SO}(V)^{tw} \). Thus we call \( R_i^{tw} \), a twisted basic subgroup of \( \text{SO}(V) \).

**4.3 Remark.** The point of constructions in \((4.2)\) is that we view each component \( R_i^{tw} \) of \( R^{tw} \) in \((4.2.1)\) as a subgroup of \( \text{SO}(V)^{tw} \) instead of viewing each component \( R_i \) of \( R \) in \((4.1.1)\) to be a subgroup of \( \text{SO}^{tw}(2m_i e^{\ell_i + \gamma_i + |c_i|}, q) \) as in \([3] \). The advantage is that we can multiply some elements involved in the process by \( -\text{Id} \) to adjust its determinant to be 1, so we can avoid to involve the general orthogonal group \( \text{GO}(V_+) \) as in \([3] \). This is convenience when the Clifford group is considered since the map \( \pi : D(V) \to \text{GO}(V) \) is not surjective in our case when \( \dim V \) is odd. See \([3, 5] \) for the consideration of the normalizers of radical subgroups. Also, the naturality of multiplication by \( -\text{Id} \) can be explained by the minus symbol in

\[
\pi(z \psi x) = -\psi x
\]

for any non-isotropic vector \( x \) in \( V \) as in \((2.5) \), where \( z \psi \) is as in \((2.6) \).
We will always work in the twisted group $SO(V)^{\varepsilon}$ and “tw” will be dropped in the sequel unless otherwise stated.

In the sequel, all the subgroups will means subgroups of $SO(V)$ and all the matrices involved mean elements in $SO_{2n+1}(\mathbb{F}_q)$ with an appropriate chosen basis of $V$. For example, when we want to focus on one component, we will write $R_{m,\alpha,\gamma,\varepsilon}$, as $R_{m,\alpha,\gamma,\varepsilon}$, which is still viewed as a subgroup of $SO(V)$. Similar conventions will be used for all related constructions. For example, when $\ell$ is unitary, $v_{m,\alpha,\gamma,\varepsilon}$ in [4.2] will be abbreviated as

$$( -1)^m \text{Id}_V \cdot \begin{pmatrix} I_m & 0 & I \otimes I_{m^\ell} \end{pmatrix}.$$ 

Note that our method here does not apply for the special orthogonal groups of even dimension, in which case, the method in [3] to introduce $GO(V_s)$ seems inevitable.

4.4. Denote by $C^0_{m,\alpha,\gamma}$ and $N^0_{m,\alpha,\gamma}$ the centralizer and normalizer of $R^0_{m,\alpha,\gamma}$ in $GL(m^\ell, \varepsilon q^{e \varepsilon^m})$ respectively. By [12, §3.A], $C^0_{m,\alpha,\gamma}$ and $N^0_{m,\alpha,\gamma}$ are as follows.

1. $C^0_{m,\alpha,\gamma} = GL(m, \varepsilon q^{e \varepsilon^m}) \otimes I_{m^\ell}$;
2. $N^0_{m,\alpha,\gamma} = C^0_{m,\alpha,\gamma}M^0_{m,\alpha,\gamma}$ is the central product of $C^0_{m,\alpha,\gamma}$ and $M^0_{m,\alpha,\gamma}$ over $C^0_{m,\alpha,\gamma} \cap M^0_{m,\alpha,\gamma} = Z(E^0_{m,\alpha,\gamma})$ and $M^0_{m,\alpha,\gamma}/Z(E^0_{m,\alpha,\gamma}) \cong \text{Sp}(2\gamma, \ell)$.

These results are essentially contained in [3] except that part (2) above improves the result in [3] slightly.

4.5. We now consider the centralizers and normalizers of radical subgroups.

Set $C_{m,\alpha,\gamma} = h(C^0_{m,\alpha,\gamma})$ and $C_{m,\alpha,\gamma,\varepsilon} = C_{m,\alpha,\gamma} \otimes I_{\varepsilon \ell^\varepsilon}$. When $\ell$ is linear, set $V_{m,\alpha,\gamma} = \langle v_{m,\alpha,\gamma}, \delta_{m,\alpha,\gamma} \rangle$, where $v_{m,\alpha,\gamma}$ is as in [4.2] and

$$\delta_{m,\alpha,\gamma} = ( -1)^m \text{Id}_V \cdot \begin{pmatrix} I_m & 0 & I \otimes I_{m^\ell} \end{pmatrix}.$$ 

When $\ell$ is unitary, set $V_{m,\alpha,\gamma} = \langle v_{m,\alpha,\gamma} \rangle$, where $v_{m,\alpha,\gamma}$ is again as in [4.2] Note that $V_{m,\alpha,\gamma}$ is cyclic in both cases. Set 

$$N_{m,\alpha,\gamma} = h(N^0_{m,\alpha,\gamma})V_{m,\alpha,\gamma} = C_{m,\alpha,\gamma}M_{m,\alpha,\gamma}V_{m,\alpha,\gamma},$$

where $M_{m,\alpha,\gamma} = h(M^0_{m,\alpha,\gamma})$. Set

$$N_{m,\alpha,\gamma,\varepsilon} = N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma} \otimes N_{\mathbb{Z}(\ell^\varepsilon)}(A_\varepsilon),$$

where $\otimes$ as is in [2, (1.5)], then

$$N_{m,\alpha,\gamma,\varepsilon}/R_{m,\alpha,\gamma,\varepsilon} \cong N_{m,\alpha,\gamma}/R_{m,\alpha,\gamma} \otimes N_{\mathbb{Z}(\ell^\varepsilon)}(A_\varepsilon)/A_\varepsilon,$$

where $N_{\mathbb{Z}(\ell^\varepsilon)}(A_\varepsilon)/A_\varepsilon \cong \text{GL}(c_1, \ell) \times \text{GL}(c_2, \ell) \times \cdots \times \text{GL}(c_r, \ell)$.

We rearrange the components of the (twisted) radical subgroup $R$ of $SO(V)^{\varepsilon}$ in [4.2.1] such that

$$(4.5.1) \quad R = \prod_{i=1}^{u} R^i_{m,\alpha,\gamma,\varepsilon},$$

where $(m_i, \alpha_i, \gamma_i, \varepsilon_i) \neq (m_j, \alpha_j, \gamma_j, \varepsilon_j)$ for any $i \neq j$. Then

$$C := C_{SO(V)}(R)^{\varepsilon} = SO(V_0)^{\varepsilon} \times \prod_{i=1}^{u} C^i_{m,\alpha,\gamma,\varepsilon},$$

where $C^i_{m,\alpha,\gamma,\varepsilon}$ is a centralizer of $R^i_{m,\alpha,\gamma,\varepsilon}$. Note that $C^i_{m,\alpha,\gamma,\varepsilon}$ is not unique.
and

\[ N := N_{\text{SO}(V)}(R)^F = \text{SO}(V_0)^F \times \prod_{i=1}^{a} N_{m,a_i,\gamma_i,e_i} \not\equiv \bar{\Xi}(t_i) \]

as abstract groups.

All the above results follow from results in [3], except that \( \det N_{m,a_i,\gamma_i,e_i} = 1 \) always holds by our construction. Note also that the groups \( N_{\bar{\Xi}(s)}(A_c) \) and \( \bar{\Xi}(t_i) \) above are all subgroups of \( W_1 \) in [2.7]

With the results in [4.2] and [4.5] the weights of \( \text{SO}(V) \) can be classified in the same way as in [3], which we recall as follows using the similar notation as in [21], since this process will be used for special Clifford group later.

4.6. Recall that \( C_{m,a,y}/Z_{m,a,y} \equiv \text{GL}(m, e\:q^{e_0})/Z(\text{GL}(m, e\:q^{e_0})) \), where \( Z_{m,a,y} := h(z_{m,a,y}) = Z(R_{m,a,y}) \). By [15 §4], the set of irreducible characters of defect zero of \( C_{m,a,y}/Z_{m,a,y} \) is not empty if and only if \( (m, \ell) = 1 \), in which case, any irreducible characters of defect zero of \( C_{m,a,y}/Z_{m,a,y} \) is determined by a polynomial \( \Delta \in E' \), and the induced character by \( \theta_{1,\gamma} \), and are denoted as \( \theta_{1,\gamma} \) and \( \alpha_{1,\gamma} \) respectively. Denote \( R_{m,a,\gamma}, C_{m,a,\gamma}, N_{m,a,\gamma} \) as \( R_{\gamma}, C_{\gamma}, N_{\gamma} \) respectively; similar convention will be used for other related constructions. Denote the corresponding character of \( \text{GL}(m, e\:q^{e_0}) \) by \( \theta_{0,\gamma} \) and the induced character of \( C_{\gamma} \) by \( \theta_{1,\gamma} = h(\theta_{0,\gamma} \otimes I_\gamma) \). By [3 §3], \( \Gamma \) determines \( \theta_{1,\gamma} \) up to \( N_{\gamma} \)-conjugacy; in the sequel, for each \( \Gamma \), a representative \( \theta_{1,\gamma} \) of the \( N_{\gamma} \)-conjugacy class is fixed. Obviously, \( h(N_{0,\gamma}^0) = C_{\gamma}M_{\gamma} \) stabilizes \( \theta_{1,\gamma} \), and \( |N_{0,\gamma}^0(\gamma) : h(N_{0,\gamma}^0)| = \beta_{1,\gamma}e^{\ell}_1 \).

Let \( M_{0,\gamma}^0 \) be as in [4.4]. Note that \( M_{0,\gamma}^0/Z(E_0^0) \equiv \text{Sp}(2, \ell) \) and \( \ell \) is bounded by \( m, a_i \) and the induced character of \( M_{\gamma} \); all these characters are denoted by the single notation \( \mathcal{S}_t \). Let \( \delta_{1,\gamma} = \theta_{1,\gamma} \otimes I_\gamma \), then \( dz(C_{\gamma}M_{\gamma}/R_{\gamma} | \theta_{1,\gamma}) = \{ \delta_{1,\gamma} \} \) and \( N_{\gamma}^0(\gamma) = N_{\gamma}^0(\gamma, \theta_{1,\gamma}) \), thus \( |\text{Irr}(N_{\gamma}^0(\gamma) | \theta_{1,\gamma})| = \beta_{1,\gamma}e^{\ell}_1 \).

4.7. Let \( R_{\gamma,\delta} \) be the set of all basic subgroups of the form \( R_{\gamma,\delta} \), with \( \gamma \neq e = \delta \). Label the basic subgroups in \( R_{\gamma,\delta} \) as \( R_{\gamma,\delta,1}, R_{\gamma,\delta,2}, \ldots \) and denote the canonical character \( \theta_{\gamma,\delta,1} \equiv \theta_{1,\gamma} \otimes I_\gamma \). When \( \Gamma, \Gamma' \in F^* \) are such that \( m_{1,\gamma} = m_{1,\gamma'} \) and \( \alpha_{1,\gamma} = \alpha_{1,\gamma'} \), we label the basic subgroups in \( R_{\gamma,\delta} \) and \( R_{\gamma,\delta} \) such that \( R_{\gamma,\delta,1} = R_{\gamma,\delta,1} \). Set

\[ \mathcal{C}_{\gamma,\delta} = \bigcup_{i} \text{dz}(N_{\gamma,\delta,1}(\theta_{\gamma,\delta,1}))/R_{\gamma,\delta,1} | \theta_{\gamma,\delta,1}) \]

By [3 (4A)], \( |\mathcal{C}_{\gamma,\delta}| = \beta_{1,\gamma}e^{\ell}_1 \delta \). Assume \( \psi_{\gamma,\delta,1} = \{ \psi_{\gamma,\delta,1,1} \} \), where \( \psi_{\gamma,\delta,1,1} \) is a character of \( N_{\gamma,\delta,1}(\theta_{\gamma,\delta,1}) \).

Let \( \text{Alp}(H) \) be the \( H^* \)-conjugacy classes of triples \( (s, \kappa, K) \) such that

1. \( s \) is a semisimple \( \ell^* \)-elements of \( H^* \);
2. \( \kappa = \prod_{r \in \Gamma} K_r \), where \( K_r \) is the \( e^r \)-core of some partition of \( m_{1,\gamma} \) for \( \Gamma \neq 0 \) and is the \( e \)-core of some Lusztig symbol of rank \( \frac{m_{1,\gamma}}{2} \) of odd defect for \( \Gamma \in 0 \);
3. \( K = \prod_{r \in \Gamma} K_r \), where \( K_r : \bigcup_{r \in \Gamma} C_{\gamma,\delta} \rightarrow \{ \ell \text{-cores} \} \)} satisfying \( \sum_{j \in \iota, \ell \in \iota} e_{\ell,\gamma,\delta,1,1} = \omega_{\gamma} \) with \( \omega_{\gamma} \) determined by

\( i) \) \( m_{1,\gamma} = |K_1| + e_{\ell,\gamma} \omega_{\gamma} \) if \( \Gamma \neq 0 \);

\( ii) \) \( m_{1,\gamma} = 2r_{K_1} + 2e_{\omega_{\gamma}} \) if \( \Gamma \neq 0 \).

All the above notational conventions are as in [21 §5] and [19 §6.B].
4.8. Let $(\mathcal{R}, \varphi)$ be a (twisted version of) weight of $H$ and assume $\varphi$ lies over a canonical character $\theta$ of $C$. There are corresponding decompositions of $C$ and $N$ as follows

$$C = C_0 \times C_+, \quad N = N_0 \times N_+,$$

where $C_0 = N_0 = \text{SO}(V_0)^F$. Then $\theta$ and $\varphi$ can be decomposed as $\theta = \theta_0 \times \theta_+$ and $\varphi = \varphi_0 \times \varphi_+$, where $\theta_0 = \varphi_0 \in \text{dz}(\text{SO}(V_0)^F)$ and $\varphi_+ \in \text{Irr}(N_+ | \theta_+)$. Assume $\varphi_0 = \chi_{s_0, \kappa}$, where $s_0$ is a semisimple $\ell'$-element of $\text{Sp}(V_0^*)^F$ and $\kappa_\ell'$ is an $e_\ell'$-core for each $\Gamma \in \mathcal{T}'$. Using the notation in [4.7] there is a decomposition

$$\theta_+ = \prod_{\Gamma, \delta, i} \theta_{\Gamma, \delta, i}^{m, \alpha, \iota}.$$

Let $s = s_0 \prod_{\Gamma, \delta, i} s_{\Gamma, \delta, i}$, where $s_{\Gamma, \delta, i}$ is a primary semi-simple element of $\text{Sp}(V_{\Gamma, \delta, i}^*)^F$. Then $N_+(\theta_+)$ has a corresponding decomposition

$$N_+(\theta_+) = \prod_{\Gamma, \delta, i} N_{\Gamma, \delta, i}(\theta_{\Gamma, \delta, i}) \cdot \mathfrak{z}(\Gamma, \delta, i).$$

Then as in [21] p.145, $\varphi_+$ defines a $K$ as in [4.7] and a label $(s, \kappa, K)$ in $i \text{Alp}(H)$ can be given to the weight $(\mathcal{R}, \varphi)$.

5  Weights of special Clifford groups of odd dimension

Keep the settings in 4.6. We begin with the following obvious observation.

**5.1 Lemma.** Let $\pi$ be the surjective homomorphism in 2.5. Then the map

$$\text{Rad}(\hat{G}) \to \text{Rad}(H), \quad \hat{R} \mapsto \pi(\hat{R})$$

is a bijection with the inverse

$$\text{Rad}(H) \to \text{Rad}(\hat{G}), \quad R \mapsto \text{O}_\ell(\pi^{-1}(R)),$$

which induces a bijection between conjugacy classes of radical subgroups. Furthermore, if $\hat{R} \in \text{Rad}(\hat{G})$ and $R = \pi(\hat{R})$, then $N_+(R) = \pi(N_+(\hat{R}))$ and $N_+(\hat{R}) = \pi^{-1}(N_+(R))$.

5.2. Let $R_{\ell} \subsetneq \text{SO}(V)^F$ be a twisted radical subgroup of $H$ as in (4.2.1). When $\ell$ is linear, let $\gamma_{m, \alpha, \gamma, \iota}$ be the unique element in $W_1$ such that $\pi(\gamma_{m, \alpha, \gamma, \iota}) = \gamma_{m, \alpha, \gamma, \iota}$. When $\ell$ is unitary, let $\gamma_{m, \alpha, \gamma, \iota}$ be any one of the (two) elements in $W_0 \times W_1$ such that $\pi(\gamma_{m, \alpha, \gamma, \iota}) = \gamma_{m, \alpha, \gamma, \iota}$ (note that

$$\begin{pmatrix} 0 & 1 \\ I_{2e\ell^m_{\iota} - 1} & 0 \end{pmatrix} = \text{diag} \left\{ \begin{pmatrix} 0 & 1 \\ I_{e\ell^m_{\iota} - 1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ I_{e\ell^m_{\iota} - 1} & 0 \end{pmatrix} \right\}.$$

Thus $\gamma_{m, \alpha, \gamma, \iota} \in W_0 \times W_1$). Set $\gamma = \prod_{\iota} \gamma_{m, \alpha, \gamma, \iota}$, then $\pi : D_0\ell(V) \to \text{SO}(V)$ induces a surjective map $D_0\ell(V)^F \to \text{SO}(V)^F$ denoted also by $\pi$ and an exact sequence

$$1 \to \mathbb{F}_q \cdot e \to D_0\ell(V)^F \xrightarrow{\pi} \text{SO}(V)^F \to 1.$$

Let $R_{\ell} = \text{O}_\ell(\pi^{-1}(R_{\ell}))$, then $R_{\ell}$ is the twisted version of the radical subgroup $\hat{R}$ of $D_0\ell(V)$ corresponding to $R$ by Lemma 5.1. As in §4 we make the following convention.

We will always work in the twisted groups $D_0\ell(V)^F$ and $\text{SO}(V)^F$ and “tw” will be dropped in the sequel unless otherwise stated.
Recall that each component $R_{m,a,y,G,e}$ is viewed as a subgroup of $SO(V)^F$. Let $\tilde{R}_{m,a,y,G,e} = O_t(T_m(a,y,G,e))$, then $\tilde{R}_{m,a,y,G,e} = \tilde{R}_{m,a,y} A_e$, where $A_e$ is a subgroup of $\tilde{W}_1$. Then $\tilde{R} = \prod_i \tilde{R}_i$ is a central product over $O_t(F_q \cdot e)$, where $\tilde{R}_i = \tilde{R}_{m,a,y,G,e}$, and there is an exact sequence for each $i$

$$1 \to O_t(F_q \cdot e) \longrightarrow \tilde{R}_i \overset{\pi}{\longrightarrow} R_i \to 1,$$

Set $Z_i = Z_{m,a,y,G}$, then $Z = Z(R) = \prod_i Z_i$. Set $\tilde{Z}_i = \tilde{Z}_{m,a,y} = O_t(T_m(a,y,G,e))$, then $\tilde{Z} = O_t(T_m(a,y,G,e)) = \prod_i \tilde{Z}_i$ is a central product over $O_t(F_q \cdot e)$.

5.3. Keep the conventions and notation in [5.3]. Set $\bar{C}_i := \bar{C}_{m,a,y,G,e} = \pi^{-1}(C_{m,a,y,G,e})$ and $\bar{C} := \pi^{-1}(C)$. Then $\bar{C} = \bar{C}_0 \prod_i \bar{C}_i$ is a central product over $F_q \cdot e$, where $\bar{C}_0 = D_0(V_0)^F$. We also have an exact sequence for each $i > 0$ as follows

$$1 \to F_q \cdot e \longrightarrow \bar{C}_i \overset{\pi}{\longrightarrow} C_i \to 1.$$ 

Let $\bar{\theta} \in \text{dz}(\bar{C}/\bar{Z})$, then $\bar{\theta} = \bar{\theta}_0 \prod_i \bar{\theta}_i$, where $\bar{\theta}_0 \in \text{dz}(\bar{C}_0/ O_t(Z(\bar{C}_0)))$ and $\bar{\theta}_i \in \text{dz}(\bar{C}_i/\bar{Z}_i)$. Thus $\bar{\theta}_0 = \tilde{\theta}_0 \frac{\ell}{\ell_0}$, where $\tilde{\theta}_0$ is a semisimple $\ell^r$-element of $\text{CSp}(V_0)^F$ with multiplier $\xi$ and $\kappa$ satisfies the conditions in Lemma 3.9. Note that $\bar{\theta}_0$, $\bar{\theta}_0$ and $\bar{\theta}_i$ induce the same linear character of $F_q \cdot e$, which is $\omega(\xi)$ by Lemma 3.10. The key step in our construction is to describe $\bar{\theta}_0$. We will focus on one component $\bar{C}_{m,a,y,G}$ in the following several paragraphs and thus drop the subscript $i$.

5.4. The duality induces an isomorphism of abelian groups

$$Z(\tilde{G}^*) \longrightarrow \text{Irr}(\tilde{G}/G), \quad \tilde{z} \mapsto \hat{\tilde{z}}.$$ 

If $\tilde{z}$ is the scalar multiplication in $Z(G^*)$ by $\tilde{\xi}_0 \in F_q \cdot e$, we denote $\tilde{\xi}_0$ as $\hat{\tilde{\xi}}_0$.

Keep the conventions in 5.2, 5.3, and assume $(-e) \leq \text{Ker} \omega(\xi)$, then $\xi$ is a square in $F_q \cdot e$ by Lemma 3.10. As in 2.9 we fix $\tilde{\xi}_0 \in F_q$ such that $\xi = \tilde{\xi}_0$. We can choose the above isomorphism such that $\text{Res}_{F_q \cdot e}^{\tilde{G}^*} \hat{\tilde{\xi}}_0 = \omega(\xi)$. Since there is an exact sequence

$$1 \to (-e) \longrightarrow F_q \cdot e \longrightarrow \tilde{G}/G \longrightarrow \mathbb{Z}/2 \to 1,$$

$\text{Res}_{F_q \cdot e}^{\tilde{G}^*} \hat{\tilde{\xi}}_0$ is independent of the choice of $\tilde{\xi}_0$, which is the reason why we can choose an arbitrary $\tilde{\xi}_0$ such that $\xi = \tilde{\xi}_0$, but once we make such a choice, we should fix it for all the subsequent circumstances. We denote in the sequel also by $\hat{\tilde{\xi}}_0$ its restriction to any subgroups. For $\tilde{\theta} \in \text{dz}(\bar{C}_{m,a,y}/\bar{Z}_{m,a,y})$ inducing $\omega(\xi)$ on $F_q \cdot e$, we have $\bar{\theta} = \hat{\tilde{\xi}}_0 \tilde{\theta}$ with $\tilde{\theta} \in \text{dz}(C/Z)$. There is a $\Gamma_0 \in \mathcal{F}'$ such that $\tilde{\theta} = \tilde{\theta}_{\Gamma_0 \gamma}$, and we denote $\tilde{\theta} = \tilde{\theta}_{\Gamma_0 \gamma}$, where $\Gamma = \xi_0 \cdot \Gamma_0 \in \mathcal{F}'$ as in 2.9. It is easy to see that $\text{dz}(\bar{C}_{\Gamma_0 \gamma}/\bar{Z}_{\Gamma_0 \gamma}) = \{ \tilde{\theta}_{\Gamma_0 \gamma} \}$, where $\tilde{\theta}_{\Gamma_0 \gamma} = \tilde{\theta}_{\Gamma_0 \gamma} S_{\gamma y}$.

5.5. Let $L_{m,a,y} = h(\text{GL}(m\ell', e\ell'\ell)), L_{m,a,y} = \pi^{-1}(L_{m,a,y})$. Note that $C_{m,a,y} M_{m,a,y} \leq L_{m,a,y}$, thus $\bar{C}_{m,a,y} \bar{M}_{m,a,y} \leq \bar{L}_{m,a,y}$. The restriction of $\pi$ induces an exact sequence

$$(5.5.1) \quad 1 \to [L_{m,a,y}, L_{m,a,y}] \cap (F_q \cdot e) \longrightarrow [\bar{L}_{m,a,y}, \bar{L}_{m,a,y}] \overset{\pi}{\longrightarrow} [L_{m,a,y}, L_{m,a,y}] \to 1.$$ 

The map

$$h^* : A \mapsto \text{diag}(A, F(A), \ldots, F^{\ell'-1}(A), A^{-t}, F(A^{-t}), \ldots, F^{\ell'-1}(A^{-t}))$$

from $\text{GL}(m\ell', e\ell'\ell)$ to $\text{Sp}(2m\ell', \mathbb{F}_q)$ is also called a hyperbolic embedding. Set $L_{m,a,y}^* = h^*(\text{GL}(m\ell', e\ell'\ell))$, then $L_{m,a,y}^*$ is in dual with $L_{m,a,y}$. Let $\xi$ be a generator of $F_q$ and $\xi_1 \in F_q$.
satisfying that \( o(\xi_1) = (q - 1)(q^{e\sigma} + 1) \) and \( \zeta = \epsilon_1^{q^{e\sigma} + 1} \). Set

\[
(5.5.2) \quad \tau_{m,\alpha,\gamma} = \begin{cases} 
I_{m^\epsilon} \otimes \text{diag}[1, \zeta], \\
I_{m^\epsilon} \otimes \text{diag}[\zeta_1, \zeta_1^{e\sigma}, \ldots, \zeta_1^{(q^{e\sigma} - 1)/e}] \end{cases} \quad \epsilon = 1; \\
I_{m^\epsilon} \otimes \text{diag}[\zeta_1, \ldots, \zeta_1^{e\sigma}, \ldots, \zeta_1^{q^{e\sigma} - 1}], \quad \epsilon = -1.
\]

Then \( \tilde{L}^*_{m,\alpha,\gamma} := (\tau_{m,\alpha,\gamma}, \tilde{L}^*_{m,\alpha,\gamma}) \) is in dual with \( L_{m,\alpha,\gamma} \). Direct calculation shows that

\[
Z(\tilde{L}^*_{m,\alpha,\gamma}) = \begin{cases} 
(\tau_{m,\alpha,\gamma}) \times Z(L^*_{m,\alpha,\gamma}), \\
(\tau_{m,\alpha,\gamma}) \quad \epsilon = 1; \\
(\tau_{m,\alpha,\gamma}) \quad \epsilon = -1.
\end{cases}
\]

Thus \( |Z(\tilde{L}^*_{m,\alpha,\gamma})| = (q - 1)(q^{e\sigma} - \epsilon) \). The duality gives an isomorphism

\[
(5.5.3) \quad Z(L^*_{m,\alpha,\gamma}) \cong \text{Irr}(L_{m,\alpha,\gamma}/[L_{m,\alpha,\gamma}, L_{m,\alpha,\gamma}]).
\]

Consequently, \( ||\tilde{L}^*_{m,\alpha,\gamma}, L_{m,\alpha,\gamma}|| = ||L_{m,\alpha,\gamma}, L_{m,\alpha,\gamma}|| \), thus \( [\tilde{L}^*_{m,\alpha,\gamma}, L_{m,\alpha,\gamma}] \cap (\mathbb{F}_q^\times, e) = \{ e \} \) from (5.5.1). So there is an injective map

\[
(5.5.4) \quad \mathbb{F}_q^\times \cdot e \hookrightarrow L_{m,\alpha,\gamma}/[\tilde{L}^*_{m,\alpha,\gamma}, L_{m,\alpha,\gamma}].
\]

More precisely, by the duality, we have when \( \epsilon = 1 \)

\[
\tilde{L}_{m,\alpha,\gamma} \cong (\mathbb{F}_q^\times \cdot e) \times L_{m,\alpha,\gamma}
\]

(in fact, by a result in [11], the Schur multiplier of any finite general group is trivial; see also [18] p.26)) and

\[
(5.5.5) \quad \tilde{L}_{m,\alpha,\gamma}/[\tilde{L}^*_{m,\alpha,\gamma}, L_{m,\alpha,\gamma}] \cong (\mathbb{F}_q^\times \cdot e) \times L_{m,\alpha,\gamma}/[L_{m,\alpha,\gamma}, L_{m,\alpha,\gamma}] \cong \mathbb{Z}_{q-1} \times \mathbb{Z}_{q^{e\sigma}-1};
\]

while when \( \epsilon = -1 \), we have

\[
\tilde{L}_{m,\alpha,\gamma}/[\tilde{L}^*_{m,\alpha,\gamma}, L_{m,\alpha,\gamma}] \cong \mathbb{Z}_{(q-1)(q^{e\sigma}+1)}.
\]

5.6. Keep the conventions in [5.3] and assume \((-e) \notin \text{Ker} \omega(\xi)\), then \( \xi \) is a non-square. Use also the conventions and notation in [5.5] in particular, \( \zeta, \xi_1 \) are as in [5.5], and \( \tau_{m,\alpha,\gamma} \) is as in (5.5.2).

Assume first that \( \epsilon = 1 \). Let \( k \) be the natural number such that \( \xi = \zeta^k \) and set \( \tau(\xi) = \tau_{m,\alpha,\gamma} \). Denote by \( \tau(\xi) \) the image of \( \tau(\xi) \) under the isomorphism (5.5.3), which can be chosen such that \( \text{Res}_{\mathbb{F}_q^\times \cdot e}^\tau(\xi) = \omega(\xi) \) and \( L_{m,\alpha,\gamma} \leq \text{Ker} \tau(\xi) \). We denote also by \( \tau(\xi) \) its restriction to subgroups of \( \tilde{L}_{m,\alpha,\gamma} \). For \( \tilde{\theta} \in \text{dz}(\tilde{C}_{m,\alpha,\gamma}/Z_{m,\alpha,\gamma}) \) inducing \( \omega(\xi) \) on \( \mathbb{F}_q^\times \cdot e \), we have \( \tilde{\theta} = \tau(\xi)\tilde{\theta} \) with \( \tilde{\theta} \in \text{dz}(C_{m,\alpha,\gamma}/Z_{m,\alpha,\gamma}) \). Recall that \( C_{m,\alpha,\gamma}/Z_{m,\alpha,\gamma} \cong \text{GL}(m, \mathbb{F}_q^{e\sigma})/Z(\text{GL}(m, \mathbb{F}_q^{e\sigma})) \).

When viewed as a character of \( \text{GL}(m, \mathbb{F}_q^{e\sigma}) \), \( \theta = \chi_\Delta \), where \( \chi_\Delta \) is the irreducible character of \( \text{GL}(m, \mathbb{F}_q^{e\sigma}) \) labelled by the semi-simple element of \( \text{GL}(m, \mathbb{F}_q^{e\sigma}) \) with unique elementary divisor \( \Delta \in \mathcal{E}_q^\alpha \) of multiplicity one. Let \( \Gamma = N_{\alpha,\xi}(\Delta) \) with \( N_{\alpha,\xi} \) as in (2.9) then \( \Gamma \in \mathcal{T}_\theta' \). We denote \( \tilde{\theta} \) as \( \tilde{\theta}_{1,\gamma} \).

Assume then \( \epsilon = -1 \). Choose \( \xi_1 \in \mathbb{F}_q^\times \) such that \( \xi = \xi_1^{q^{e\sigma}+1} \) as in (2.9) Let \( k \) be the natural number such that \( \xi_1 = \zeta_1^k \) and set \( \tau(\xi_1) = \tau_{m,\alpha,\gamma} \). Denote by \( \tau(\xi_1) \) the image of \( \tau(\xi_1) \) under the isomorphism (5.5.3), which can be chosen such that \( \text{Res}_{\mathbb{F}_q^\times \cdot e}^\tau(\xi_1) = \omega(\xi) \). Note \( \text{Res}_{\mathbb{F}_q^\times \cdot e}^\tau(\xi_1) \) is independent of the choice of \( \xi_1 \), which is the reason why we can choose an arbitrary \( \xi_1 \) such that \( \xi = \xi_1^{q^{e\sigma}+1} \), but once we make such a choice, we should fix it for all the subsequent circumstances. As above, we denote also by \( \tau(\xi_1) \) its restriction to subgroups of \( \tilde{L}_{m,\alpha,\gamma} \). For \( \tilde{\theta} \in \text{dz}(\tilde{C}_{m,\alpha,\gamma}/Z_{m,\alpha,\gamma}) \) inducing \( \omega(\xi) \) on \( \mathbb{F}_q^\times \cdot e \), we have \( \tilde{\theta} = \tau(\xi_1)\tilde{\theta} \) with \( \tilde{\theta} = \chi_\Delta \in \text{dz}(C_{m,\alpha,\gamma}/Z_{m,\alpha,\gamma}) \), where \( \Delta \in \mathcal{E}_q^\alpha \) is as in the above case. Let \( \Gamma = N_{\alpha,\xi}(\Delta) \) with \( N_{\alpha,\xi} \) as in (2.9) then \( \Gamma \in \mathcal{T}_\theta' \). Again, we denote \( \tilde{\theta} \) as \( \tilde{\theta}_{1,\gamma} \).
When the canonical character considered is \( \tilde{\theta}_{1,\gamma} \), we will replace the subscripts \( m, \alpha \) in all the relevant constructions by \( \Gamma \) as before. Since \( \tau(\xi) \) or \( \tau(\xi^\ell) \) is a linear character of \( \tilde{\Gamma}_{1,\gamma} \) which contains \( \tilde{C}_{1,\gamma} \tilde{M}_{1,\gamma} \), it is easy to see that \( dz(\tilde{C}_{1,\gamma} \tilde{M}_{1,\gamma} / \tilde{R}_{1,\gamma} \mid \tilde{\theta}_{1,\gamma}) = \{ \tilde{\theta}_{1,\gamma} \} \), where \( \tilde{\theta}_{1,\gamma} = \tilde{\theta}_{1,\gamma} \text{ St}_{\gamma} \).

5.7. Set \( \tilde{N}_{m,a,y,e} = \pi^{-1}(N_{m,a,y,e}) \). Assume first that \( c = (0) \) and keep the notation in 5.5. When \( \ell \) is linear, we have that \( v_{m,a,y} \in \tilde{W}_1 \) and \( \delta_{m,a,y} \in \tilde{W}_0 \), where \( \tilde{W}_1, \tilde{W}_0 \) are as in [2.7] let \( \tilde{v}_{m,a,y} \) be the unique pre-image of \( v_{m,a,y} \) in \( \tilde{W}_1 \) and \( \delta_{m,a,y} \) be any one of the (two) pre-images of \( \delta_{m,a,y} \) in \( \tilde{W}_0 \). When \( \ell \) is unitary, let \( \tilde{v}_{m,a,y} \) be one of the (two) pre-images of \( v_{m,a,y} \) in \( \tilde{W}_0 \times \tilde{W}_1 \) as in [5.2]. Then we have that

\[
\tilde{N}_{m,a,y,e} = \begin{cases} (\tilde{C}_{m,a,y} \tilde{M}_{m,a,y}, \tilde{v}_{m,a,y}, \delta_{m,a,y}), & \varepsilon = 1, \\ (\tilde{C}_{m,a,y} \tilde{M}_{m,a,y}, \tilde{v}_{m,a,y}), & \varepsilon = -1, \end{cases}
\]

where, \( \tilde{M}_{m,a,y} = \pi^{-1}(M_{m,a,y}) \) and \( \tilde{C}_{m,a,y} \tilde{M}_{m,a,y} \) is a central product over \( \mathbb{F}_q^* \cdot e \). Now,

\[
(5.7.1) \quad \tilde{N}_{m,a,y,e} / \tilde{R}_{m,a,y,e} \cong N_{m,a,y} / \tilde{R}_{m,a,y,e} \otimes N_{\mathbb{Z}(\psi)(e)}(A_c)/A_c,
\]

where \( N_{\mathbb{Z}(\psi)(e)}(A_c) / A_c \cong \text{GL}(e_1, \ell) \times \text{GL}(e_2, \ell) \times \cdots \times \text{GL}(e_r, \ell) \).

5.8 Lemma. Keep the conventions and notation in [5.5] and 5.7. For \( A \in \text{GL}(m\ell^r, \mathbb{F}_{q^e}^\ell) \), denote by \( |A| \) its determinant in \( \text{GL}(m\ell^r, \mathbb{F}_{q^e}^\ell) \) and by \( \tilde{A} \in L_{m,a,y} \) any one pre-image of \( h(A) \in L_{m,a,y} \) under \( \pi \).

(1) When \( \varepsilon = 1 \), \( \tilde{v}_{m,a,y} \) stabilizes the two components in (5.5, 5.5) and

\[
\tilde{\delta}_{m,a,y}^{-1} \tilde{A} \tilde{\delta}_{m,a,y} \equiv |A|^{q^e-1} \tilde{A}^{-1} \text{ mod } [L_{m,a,y}, L_{m,a,y}].
\]

(2) When \( \varepsilon = -1 \), we have

\[
F_q(\tilde{A}) \equiv \tilde{A}^q \text{ mod } [L_{m,a,y}, L_{m,a,y}].
\]

Proof. Modulo \( [L_{m,a,y}, L_{m,a,y}] \), it suffices to assume \( A = \text{diag}(b, 1, \ldots, 1) \). Furthermore there is no loss of generality to assume that \( m = 1, \gamma = 0 \). Set then \( \delta := e^{\ell^0} \) and choose a basis of \( V_{1,a,0} \)

\[
e_1, e_2, \ldots, e_\delta, \eta_1, \eta_2, \ldots, \eta_\delta
\]

such that \( B(e_i, e_j) = B(\eta_i, \eta_j) = 0 \) and \( B(e_i, \eta_j) = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker symbol. Thus

\[
h(A) = \text{diag}(b, b^q, \ldots, b^{e\ell^1}, b^{e^{-\ell}}, b^q, \ldots, b^{-q\ell^1}).
\]

Assume \( \varepsilon = 1 \). We can write \( \tilde{v}_{1,a} \) and \( \tilde{\delta}_{1,a} \) explicitly:

\[
\tilde{v}_{1,a} = \prod_{i=1}^{\delta} \tilde{w}_{i+1}, \quad \tilde{\delta}_{1,a} = \prod_{i=1}^{\delta} \tilde{w}_i,
\]

where \( \tilde{w}_{i+1}, \tilde{w}_i \) are as in [2.7]. Note that \( \delta \in \mathbb{F}_q^\times \) and set

\[
\tilde{A} = \prod_{i=1}^{\delta} (e_i + \eta_i) (e_i + b^{q\ell^1} \eta_i) = \prod_{i=1}^{\delta} (e_i + \eta_i) (e_i + b^{-q\ell^1} \eta_i).
\]

Then direct calculation shows that \( \tilde{v}_{1,a} F(\tilde{A}) \tilde{v}_{1,a}^{-1} = \tilde{A} \), so \( \tilde{A} \in D_0(V)^{\mathbb{F}} \). The action of \( \tilde{\delta}_{1,a} \) follows also from direct calculation.

Assume \( \varepsilon = -1 \). We can write \( \tilde{v}_{1,a} \) explicitly (see [2.2, 1]):

\[
\tilde{v}_{1,a} = \left( \prod_{i=1}^{\delta} \tilde{w}_{i+1} \right) \tilde{w}_\delta.
\]
Note that in this case $b^{q+1} = 1$ and set

$$
\hat{A} = b_0 \prod_{i=1}^{\delta} (e_i + \eta_i)(e_i + b^{q-1} \eta_i),
$$

where $b_0 \in \mathbb{F}_q^*$ is such that $b^{q-1} = b$, then direct calculation shows that $\tau_{1, e} F(\hat{A})^{-1}_{1, e} = \hat{A}$, so $\hat{A} \in D_0(V)^{e_F}$. The assertion in this case also follows from direct calculation.

5.9 Lemma. Keep the conventions and notation above and assume $\hat{\theta}_{1, e}$ be as in 5.4 and 5.6. Then $m_1 e_F^{e_F} = e_F \delta_F$ and $|dz(\hat{N}_{1, e} / \hat{R}_{1, e} | \hat{\theta}_{1, e})| = \beta_1 e_F$.

Proof. By the construction, we have that $\hat{N}_{1, e}(\hat{\theta}_{1, e}) = \hat{N}_{1, e}(\hat{\theta}_{1, e})$, where $\hat{\theta}_{1, e} = \hat{\theta}_{1, e} S_{1, e}$ as in 5.4 and 5.6. Noting also that $\hat{N}_{1, e} / \hat{C}_{1, e} M_{1, e} \equiv \mathbb{Z}_{2e}$ is cyclic, we may assume that $\gamma = 0$ and in particular $\hat{C}_{1, e} = \hat{L}_e$.

Assume $\hat{\xi} = \xi_0^2 (\xi_0 \in \mathbb{F}_q^*)$ is a square. Then by 5.4 we may assume $\hat{\theta}_e = \hat{\xi}_0 \theta_0$ with $\theta \in \mathcal{F}'$ and $\Gamma = \xi_0 \cdot \Gamma_0$. Since $\hat{\xi}_0$ is a linear character of $D_0(V)^{e_F}$, we have that $\hat{N}_1(\hat{\theta}_e) = \hat{N}_1(\theta_0)$. From $\Gamma = \xi_0 \cdot \Gamma_0$, we have that $e_\Gamma = e_\Gamma_0$. Then the result in this case follows from the result in [3, §3] for $SO(V)^{e_F}$.

Assume $\hat{\xi}$ is a non-square and $\ell$ is linear. Assume $\hat{\theta}_e = \hat{\xi} \epsilon_0 \theta_0$ with $\epsilon_0 \in \mathbb{Z}_{2e}$. Note that $\hat{N}_e / \hat{C}_e M_{e} \equiv \mathbb{Z}_{2e}$ is cyclic, we may assume that $\Gamma = 0$ and $\hat{C}_e = \hat{L}_e$. Then $e_\Gamma = e_\Gamma_0$. Then the result in this case follows from the result in [3, §3] for $SO(V)^{e_F}$.

5.10 Remark. We list some special cases of the above lemma. Assume $\hat{\xi}$ is a non-square and $\ell$ is unitary; recall that $e$ is odd in this case.

1. Assume $\Delta = X^2 - \xi$, then $\Gamma = X^2 - \xi$. Thus $\delta_\Gamma = \beta_\Gamma = 2$, $e_\Gamma = e$, $m_\Gamma = 2$, $\alpha_\Gamma = 0$. Note that $\hat{\xi} = \xi_0^2$ for some $\xi_0 \in \mathbb{F}_q^*$ and $\xi_0 = -\xi_0$. From the action of $\hat{N}_1$ in the proof of Lemma 5.9 we have $\hat{N}_1(\hat{\theta}_e) = \hat{N}_1$.

2. Assume $\Delta = X^2 + 1$, then $\Gamma = (X \pm 1)(X \pm \xi) \in \mathbb{F}_{q^2}$. Thus $\delta_\Gamma = \beta_\Gamma = 1$, $e_\Gamma = 1$, $m_\Gamma = 1$, $\alpha_\Gamma = 0$. From the action of $\hat{N}_1$ in the proof of Lemma 5.9 we have that $\hat{N}_1(\hat{\theta}_e) = \hat{C}_e$.

3. Assume $\Delta = (X - b)(X - b')$ with $b \in \mathbb{F}_q$ and $b^{q+1} = \xi$, then $\Gamma = \prod_{i=0}^{q+1} (X - b^i) \in \mathbb{F}_{q^2}$ (note that $b^{q+1} = \xi b^{-1}$). Thus $\beta_\Gamma = 1$, $\delta_\Gamma = e$, $e_\Gamma = -1$, $m_\Gamma = 2$, $\alpha_\Gamma = 0$. From the action of $\hat{N}_1$ in the proof of Lemma 5.9 we have that $\hat{N}_1(\hat{\theta}_e) = (\hat{C}_e, \hat{\theta}_e)$.

Assume now $\hat{\xi}$ is a non-square and $\ell$ is unitary.

1. Assume $\Gamma = X^2 - \xi$ with $\delta_\Gamma = \beta_\Gamma = 2$. Note that $\hat{\xi} = \xi_0^2$ for some $\xi_0 \in \mathbb{F}_q^*$ and $\xi_0 = -\xi_0$. We may assume $\xi_0 = \xi_1^{-1}$. Note that $\xi_1^{-1} = \frac{\xi_0^{\delta_\Gamma} + \xi_0^{e_\Gamma} + \xi_0}{\xi_1^{\alpha_\Gamma}} = -1$ and $\xi_1^{-1} = -\xi_1^{-1}$.
Then using (5.7.1), we can see similarly as in [3, (4A)] that
\\( \tilde{\chi}(1) \) is the basic subgroups in \( \Delta = \langle \tilde{\theta} \rangle \) according to e is odd or even.

(2) Assume \( \Delta = X \pm 1 \), then \( \Gamma = (X \pm \xi_1)(X \pm \xi_1^2) \cdots (X \pm \xi_1^{2e-1}) \in T\xi_1. \) Thus \( \beta_{\tilde{\Gamma}} = 1, e_1 = 1, e_1 = 1, m_1 = 1, \alpha_1 = 0. \) Note that \( \xi_1^{2e} = \xi_1^{-1}. \) From the action of \( \tilde{\Delta} \) in the proof of Lemma 5.9 we have that \( \tilde{\Delta}(\tilde{\theta}) = \langle \tilde{C}, \tilde{\psi} \rangle. \)

(3) Assume \( \Delta = (X-c)(X-c^{-e}) \) with \( c \in F_{q_r}^\times \) such that \( \Gamma = \prod_{i=0}^{e-1}(X-b^i)(X-\xi b^{-i}) \in T\xi_2. \) Thus \( \beta_{\tilde{\Gamma}} = 1, e_1 = 1, e_1 = 1, m_1 = 2, m_1 = 2, \alpha_1 = 0. \) From the action of \( \tilde{\Delta} \) in the proof of Lemma 5.9 we have that \( \tilde{\Delta}(\tilde{\theta}) = \langle \tilde{C}, \tilde{\psi} \rangle. \)

5.11. As in [4, 6] we denote \( \hat{R}_{m_1, \alpha, \gamma, c} \) as \( \hat{R}_{\Gamma, \gamma, c} \) when the corresponding canonical character is \( \hat{\Delta}_{\Gamma, \gamma, c}. \) Let \( \hat{R}_{\Gamma, \gamma, c} \) be the set of all the (twisted) basic subgroups of the form \( \hat{R}_{\Gamma, \gamma, c} \) with \( \gamma + |\gamma| \delta. \) Label the basic subgroups in \( \hat{R}_{\Gamma, \gamma, c} \) as \( \hat{R}_{\Gamma, \gamma, c}, \hat{R}_{\Gamma, \gamma, c}, \ldots \) and denote the canonical character \( \hat{\Delta}_{\Gamma, \gamma, c} \). When \( \Gamma, \Gamma' \in T\xi_2 \) are such that \( m_1 = m_1' \) and \( \alpha_1 = \alpha_1', \) we label the basic subgroups in \( \hat{R}_{\Gamma, \gamma, c} \) and \( \hat{R}_{\Gamma', \gamma, c} \) such that \( \hat{R}_{\Gamma, \gamma, c} = \hat{R}_{\Gamma', \gamma, c}. \) Set

\[ \hat{\eta}_{\Gamma, \gamma, c} \triangleq \int \text{d}z \left( \tilde{\Delta}_{\Gamma, \gamma, c}(\tilde{\theta}_{\Gamma, \gamma, c})/R_{\Gamma, \gamma, c}^{\text{ev}} | \tilde{\theta}_{\Gamma, \gamma, c} \right), \]

then using (5.7.1), we can see similarly as in [3] (4A]) that \( \hat{\eta}_{\Gamma, \gamma, c} = \beta_{\Gamma} e_1 \ell_\gamma. \) Assume \( \tilde{\psi}_{\Gamma, \gamma, c} = \{ \tilde{\psi}_{\Gamma, \gamma, c} \}, \) where \( \tilde{\psi}_{\Gamma, \gamma, c} \in \text{Irr}(\tilde{\Delta}_{\Gamma, \gamma, c}(\tilde{\theta}_{\Gamma, \gamma, c})). \)

Let \( i \text{Alp}(\tilde{G}) \) be the \( \tilde{G}^* \)-conjugacy classes of triples \( (\tilde{s}, \kappa, \lambda) \) such that

1. \( \tilde{s} \) is an \( \ell' \) semisimple elements of \( \tilde{G}^* \) with multiplier \( \xi; \)
2. \( \kappa = \prod_{i=1}^{r} \kappa_i, \) where \( \kappa_i \) is the \( e_i \)-core of some partition of \( m_i(\tilde{s}) \) for \( \Gamma \notin T\xi_0 \), and is the \( e \)-core of some Lustzig symbol of rank \( m_i(\tilde{s}) \) \( 2 \) with odd defect for \( \Gamma \in T\xi_0; \)
3. \( K = \prod_{i=1}^{s} K_i \), where \( K_i : \bigcup_{j} \tilde{\psi}_{\Gamma, j} \rightarrow \{ \ell \text{-cores} \} \) satisfying \( \sum_{i,j} \ell_\gamma |K_i(\tilde{\psi}_{\Gamma, i,j})| = \omega_\gamma \) with \( \omega_\gamma \) determined by
   - (i) \( m_i(s) = k_i + e_1 \omega_\gamma \) if \( \Gamma \notin T\xi_0; \)
   - (ii) \( m_i(s) = 2 \text{rk } k_i + 2 e_1 \omega_\gamma \) if \( \Gamma \in T\xi_0. \)

5.12. (Proof of the main theorem). Let \( (\tilde{R}, \tilde{\varphi}) \) be a (twisted version of) weight of \( \tilde{G} \) and assume \( \tilde{\varphi} \) lies over a canonical character \( \tilde{\theta} \) of \( \tilde{C}. \) Note that there are decompositions \( \tilde{C} = \tilde{C}_{\tilde{\theta}}, \tilde{N} = \tilde{N}_{\tilde{\theta}}, \tilde{\theta} = \tilde{\theta}_{\tilde{\theta}} \) and

\[ \tilde{\theta}_{\tilde{\theta}} = \prod_{\Gamma, \gamma, c} \tilde{\theta}_{\Gamma, \gamma, c}, \quad \tilde{N}_{\Gamma, \gamma, c}(\tilde{\theta}_{\Gamma, \gamma, c}) = \prod_{\Gamma, \gamma, c} \tilde{N}_{\Gamma, \gamma, c}(\tilde{\theta}_{\Gamma, \gamma, c}) = \Xi(\Gamma, \gamma, c). \]

Here, note that \( \tilde{N}_{\Gamma, \gamma, c}(\tilde{\theta}_{\Gamma, \gamma, c}) \) is a central product. With the above informations, we can give a label \( (\tilde{s}, \kappa, K) \) in \( i \text{ Alp}(\tilde{G}) \) to \( (\tilde{R}, \tilde{\varphi}) \) in the same way as in [4, 8]. Then for each \( \Gamma; \) \( K \) corresponds to a \( \beta_1 e_1 \) tuples of partitions using the method in [2] (1A), and by 2.10 the pair \( (\kappa, K) \) corresponds to a \( \lambda \) as in Proposition 3.7

5.13 Remark. Let \( (\tilde{R}, \tilde{\varphi}) \) be a weight of \( \tilde{G} \) with the linear character of \( F \) \( e \) induced by \( \tilde{\varphi} \) being \( \omega_\xi(\xi) \). When \( \xi = \xi_0^2 \) is a square, let \( \xi_0 \) be the linear character of \( \tilde{G} \) as in 5.4. Then \( (\tilde{R}, \tilde{\xi}^2_0 \tilde{\varphi}) \) is a weight of \( H = \text{SO}(V) \). This gives another method for weights when \( \xi \) is a
square. But the arguments we use above can give a uniform treatment for both the cases when \( \xi \) is a square or not.

5.14 Remark. With the construction of weights given as above, the action of field automorphisms on weights can be calculated as in [19, §6.C], which turns out to be induced by the action of field automorphisms on elementary divisors.

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