1. INTRODUCTION

It is well known that radio waves propagating in the interstellar medium (ISM) are scattered by the irregularities in the Galactic electron density. The scattering, in turn, gives rise to a number of observable phenomena. Among others, these include angular broadening and intensity fluctuations (both in time and frequency) of compact radio sources. While a nuisance in many radio astronomical observations, these phenomena can be used to investigate the nature of the irregularities in the interstellar plasma density. These density irregularities, in turn, are believed to follow the fluctuations in the interstellar kinetic and magnetic energies. Ideally, one would like to invert observations of radio scintillation and scattering to determine the statistics of the plasma density. As noted by Narayan (1992) this inverse problem is not well posed and one must rely on modeling methods. A complete prediction of scintillation observables requires an a priori knowledge of both the form of the spatial power spectrum of the electron density fluctuations and the distribution of the scattering material along the line of sight. For a given profile of the distribution of the scattering material, one may compare observations and predictions and so constrain the functional form of the spectrum. The power spectrum provides useful insight into the physics of the plasma irregularities. Hence a knowledge of the form of the density spectrum becomes central in both predicting scintillation phenomena and understanding the physics of the interstellar plasma. This paper revisits the investigation of the form of the density spectrum.

A commonly used model for the density spectrum has been based on a power-law model with a large range between "inner" and "outer" scales (e.g., Rickett 1977):

$$P_{\kappa}(q) = \frac{C_{\kappa}(z)}{q^{2 + \kappa_{o}^{2}} \exp \left[ - \frac{q^{2}}{4\kappa_{i}^{2}} \right]}.$$  \hfill (1)

Here $q$ is the magnitude of the three-dimensional wavenumber.

$C_{\kappa}(z)$ denotes the strength of fluctuations (with a weak dependence on distance $z$). $\beta$ is the spectral exponent, and $\kappa_{i}^{-1} = L_{i}$ and $\kappa_{o}^{-1} = L_{o}$ are the inner and outer scales, respectively. For $\kappa_{o} \ll q \ll \kappa_{i}$, we obtain the simple power-law model: $P_{\kappa}(q) = C_{\kappa}^{2} q^{-\beta}$. Armstrong, Rickett, & Spangler (1995) have constructed an empirical density spectrum by combining radio scintillation observations in the local ISM ($\lesssim 1$ kpc) with measurements of the differential Faraday rotation angles and large-scale electron density gradients. They have shown the power spectrum to be consistent with a simple Kolmogorov power-law model ($\beta = 11/3$) over an astronomical 10 orders of magnitude in wavenumber scale ($10^{-18}$ m$^{-1} < q < 10^{-8}$ m$^{-1}$). The Kolmogorov spectrum in density suggests a turbulent cascade in the magnetic and kinetic energies. This has led to several theoretical investigations of the generation and maintenance of hydromagnetic turbulence in the ISM (e.g., Pouquet 1978; Higdon 1984, 1986; Biskamp 1993; Sridhar & Goldreich 1994; Goldreich & Sridhar 1995, 1997; Spangler 1999). However, the Armstrong et al. study combined observations from many lines of sight and the scatter...
among them leaves a substantial uncertainty in the exponent $\beta$. A list of symbols is given in Table 1.

Despite the positive evidence for the simple Kolmogorov spectrum, substantial observational inconsistencies remain. For instance, long-term refractive intensity scintillations of some pulsars have modulation indices as much as a factor of 2 larger than predicted by the simple Kolmogorov model (Gupta, Bhat, & Rao 1999); in addition, sloping bands in the dynamic spectra often persist longer than predicted by the model (cf. Gupta, Rickett, & Coles 1993). Other discrepancies are revealed in the diffractive dynamic spectra of pulsars. On some occasions, periodic fringes not predicted by the simple Kolmogorov model are observed (cf. Roberts & Ables 1982; Cornes & Wolszczan 1986; Rickett, Lyne, & Gupta 1997; Gupta, Bhat, & Rao 1999); in addition, sloping bands in the dynamic spectra often persist longer than predicted by the model (cf. Gupta, Rickett, & Lyne 1994; Bhat, Rao, & Gupta 1999b, 1999c). Furthermore, for some pulsars, the decorrelation bandwidth has larger amplitude variations than predicted for the Kolmogorov spectrum (Bhat et al. 1999c). Such anomalies suggest the presence of large refractive structures giving rise to the focusing and defocusing of the scattered ray bundles. The interference of the ray bundles can also explain the occasional fringes observed in the dynamic spectra of some pulsars. Given the relatively frequent occurrence of such events, one can ask whether they can be considered as mere occasional anomalies or if they should be considered as a widespread phenomenon intrinsic to the spectrum on a grand scale.

The observational inconsistencies suggest the need for an enhancement in the power on the large “diffractive” ($10^{11}$–$10^{12} \text{ m}$) spatial scales relative to the power on the small “refractive” ($10^{7}$–$10^{8} \text{ m}$) scales. There are several means by which this ratio may be enhanced. One is to include the inner scale cutoff in the density spectrum, which reduces the power at small scales (Coles et al. 1987); these authors proposed inner scales of $10^8$–$10^9 \text{ m}$, though this does not correspond to any obvious physical scale. Physically, the inner

| Symbols | Definition |
|---------|------------|
| $a, a(s)$ | Exponent of separation $S$ in structure function |
| $\beta$ | Exponent of wavenumber in density power spectrum |
| $\beta, \beta'$ | Vector spatial offsets in Appendix B |
| $c$ | Speed of light |
| $C_1$ | Constant in uncertainty relation, $2\pi A k_x k_y = C_1$ |
| $C_{i0}$ | Coefficient in electron-density wavenumber spectrum ($m_0^{-20/3}$) |
| $D_{a}D_{o}$ | Wave-/phase-structure function and its longitudinal gradient Eq. (6) |
| $D_{b}$ | Wave-/phase-structure function for a point (spherical wave) source |
| $\Delta z$ | Screen or layer thickness |
| $\Delta k$ | Diffractive decorrelation bandwidth |
| $\epsilon$ | Fractional frequency difference $|k_1 - k_2|/(k_1 + k_2)$ |
| $f(r), f_0$ | Radial profile function for electron density, interior density |
| $F(q)$ | Three-dimensional Fourier transform of $f(r)$ Eq. (3) |
| $g_0(\eta), g_1(\eta)$ | Intermediate functions of $\eta = \kappa^2$ |
| $J_0$ | Zero-order Bessel function of the first kind |
| $\kappa$ | High and low wavenumber cutoffs Eq. (1) |
| $K = 2\pi/(\kappa_1, \kappa_2)$ | Radio wavenumbers |
| $k_m$ | Geometric mean of $k_1, k_2$ |
| $L$ | Pulsar distance |
| $L_0, L_o$ | Inner and outer scales |
| $\lambda$ | Radio wavelength |
| $m$ | Scintillation index (rms/mean) for refractive scintillation Eq. (10) |
| $m_{\text{int}}$ | Scintillation index caused by scales intermediate between diffractive and refractive |
| $n_0$ | Spatial number density of discontinuous objects |
| $v$ | Radio frequency |
| $P_{\kappa}(q)$ | Wavenumber spectrum for the electron density Eq. (1) |
| $q$ | Three-dimensional wavenumber |
| $r_{e}$ | Classic electron radius |
| $r_{F, L}$ | Fresnel scale ($\sqrt{L/\kappa}$) Eq. (11) |
| $r_{F, SC}$ | Fresnel scale for a screen ($\sqrt{z_s/\kappa}$) Eq. (12) |
| $s$ | Transverse separation |
| $S M$ | Scattering measure = line-of-sight integral of $C_{i0}$ |
| $s_0$ | Field coherence scale |
| $s_2$ | Refractive scintillation scale |
| $t_d$ | Pulse decay time to $1/e$ |
| $u$ | Strength of scattering parameter ($= r_{F, L}/s_0$) Eq. (9) |
| $v_s$ | Normalized decorrelation bandwidth Eq. (9) |
| $V_s$ | Sum of wave-structure functions Eq. (B1) |
| $x, y$ | Normalized distances |
| $z, z_p$ | Distance from pulsar along line of sight toward observer |
| $z_0$ | Observer-screen distance |
| $z_s$ | Effective screen observer-distance $= z_o z_p/L$ |
| $\zeta$ | $(s_2 k_m)^2$ |
scale corresponds to the scale at which the turbulent cascade dissipates and becomes a source of heating for the plasma (Spangler 1991). The value of the inner scale is largely unknown. Using different methods, Spangler & Gwinn (1990), Kaspi & Stonebringer (1992), and Gupta et al. (1993) have reported values for the inner scale ranging from \(10^4\) to \(10^9\) m. In proposing the smaller values, Spangler & Gwinn (1990) argued that the inner scale is the larger of the ion inertial length, \(L_i \equiv V_A/\Omega_i\) (where \(V_A\) is the Alfvén speed, and \(\Omega_i\) is the ion cyclotron frequency), and the ion Larmor radius, \(r_i \equiv v_{th}/\Omega_i\) (where \(v_{th}\) is the ion thermal speed); they obtained parameters for the warm ionized medium in reasonable agreement with observations. In a recent discussion, Minter & Spangler (1997) have suggested ion-neutral collisional damping and wave-packet steepening as possible dissipation mechanisms for the turbulence in the diffuse ionized gas, which would make the mean-free path for ion-neutral collisions a possible value for the inner scale. However, this is thought to be larger than the maximum values proposed to explain the observations, making it a less convincing dissipation mechanism. Observationally, the Kolmogorov model with a large inner scale predicts refractive modulation indices consistent with pulsar measurements (Gupta et al. 1993). It has also been proposed to explain the occasional periodic fringes, with an inner scale on the order of the Fresnel scale (Cordes, Pidwerbetsky, & Lovelace 1986; Goodman et al. 1987). However, Rickett et al. (1997) reported a fringe event for the pulsar B0834+06 that could not be explained as the effect of a large inner-scale spectrum. The event requires similar conditions to those needed to explain the extreme scattering events (Fiedler et al. 1987).

Another way to enhance the ratio of the power between the refractive and diffractive scales in the spectrum is to steepen the spectrum—with spectral exponents \(\beta > 4\) (Blandford & Narayan 1985; Goodman & Narayan 1985; Romani, Narayan, & Blandford 1986). While power-law spectra with \(\beta \sim 11/3\) have a turbulence connotation, spectra with \(4 < \beta < 6\) might involve some forms of turbulence but are also consistent with a distribution of non-turbulent structures with a range of spatial scales. Such steep spectra with \(4 < \beta < 6\) predict refractive modulation indices close to unity (Goodman & Narayan 1985), which is substantially larger than the range \(30\% - 40\%\) observed from the nearby pulsars. On this basis, Rickett & Lyne (1990) and Armstrong et al. (1995) have rejected spectra steeper than \(4\) for the interstellar plasma. However, the special case of \(\beta = 4\) has been given little attention. We can conceive of a power-law model with spectral exponent \(\beta = 4\) given by

\[
\frac{P_n(q; z)}{C_n^2} = \frac{C_n^2}{(q^2 + \kappa_0^2)^2}. \tag{2}
\]

Hereafter, we refer to this model as the \("\beta = 4\) model." Blandford & Narayan (1985) briefly discussed this special case without including a cutoff at low wavenumbers. It is interesting to note that even though the Kolmogorov spectral exponent \(\beta = 11/3\) is very close to \(4\), the \(\beta = 4\) model has a very different physical implication. Its physical origin has rarely been discussed, and it has not been formally compared with observations. Physically, this spectrum suggests the random distribution in location and orientation of discrete discontinuous objects across the line of sight. An "outer" scale, \(L_o = \kappa_0^{-1}\), is included to account for the typical size of such objects. The "inner" scale here would correspond to the scale of the sharpness of a typical discontinuity. We assume the "inner" scale to be smaller than the diffractive scale of scintillations; hence, it has no significant effect on the scintillations. The \(\beta = 4\) model could characterize stellar wind boundaries, supernova shock fronts, sharp boundaries of \(\text{H II}\) regions at the Strömgren radius, or any plasma "cloud" with sufficiently sharp boundaries (transition regions shorter than the diffractive scintillation scale) that may cross the line of sight. Note that, though turbulence is not necessarily implied by the model, strong turbulence that has steepened to form shocks would also be described by the \(\beta = 4\) model. In these models, a single discrete object crossing the line of sight could also explain the "extreme scattering events" observed in the flux density variations of extragalactic sources (e.g., Fiedler et al. 1987).

We note that the analysis of such events have been based primarily on geometrical optics involving a single "cloud" (Goodman et al. 1987; Cordes et al. 1986; Cordes & Wolszczan 1986; Roberts & Ables 1982; Ewing et al. 1970). Our analysis of the \(\beta = 4\) model includes the "wave optics" effects when the line of sight passes through very many such clouds. The ISM is assumed to consist of a random assembly of discrete structures with abrupt density steps that may be independent of each other. If the \(\beta = 4\) model were compatible with all of the scintillation observations, it would remove the implication of interstellar plasma turbulence with an inertial range spanning as much as 10 orders of magnitude in scale, which has become the canonical model for interstellar scintillation (ISS) phenomena. It would, however, still be consistent with turbulence that has steepened into shocks.

We start with a simple derivation of the \(\beta = 4\) model in § 2. In § 3, we give equations for the wave-structure function for the \(\beta = 4\) model and compare predictions. In § 4, the variation of the diffractive decorrelation bandwidth with frequency is used to test the \(\beta = 4\) model against the simple Kolmogorov model and the Kolmogorov model with an inner-scale (hereafter, the "inner-scale model"). In § 5, the observed variation of the refractive scintillation index with the normalized diffractive decorrelation bandwidth is compared with theoretical predictions for the simple Kolmogorov, inner-scale, and \(\beta = 4\) models. We use theoretical results from a previous paper (Lambert & Rickett 1999, hereafter LR99), in which we developed the theory of diffractive scintillations in a medium modeled by these spectra, and we add details of the theory for refractive scintillation in Appendix A. In Appendix B, we discuss the use of the square of the second moment to approximate the intensity correlation function for diffractive scintillation with \(\beta = 4\). In § 6 we give a discussion and our conclusions.

2. THE SPECTRUM FOR RANDOM DISCONTINUITIES IN DENSITY

This section gives a simple derivation of the \(\beta = 4\) model. Consider first a random distribution in space of identical plasma structures (blobs). The resulting electron density may be described as the convolution of a three-dimensional Poisson point process with the density profile of one individual blob. This is the spatial analog of shot noise, in which each charge carrier contributes the same temporal profile of current arriving at randomly distributed times. In the wave-number domain, the power spectrum of the electron density...
is then given by the spectrum of the Poisson point process multiplied by the squared magnitude of the Fourier transform of one blob. The spectrum of the Poisson point process is a constant equal to the number density, \( n_o \), of the blobs in space (Papoulis 1991). Thus the power spectrum of the medium follows the same shape as that of an individual blob. A similar description applies to water droplets in a fog and many naturally occurring media, as discussed by Ratcliffe (1956). If the blobs are asymmetrical and randomly oriented, one must also average the spectrum over the possible orientations.

The basic feature of structures with discontinuous boundaries is that their power spectrum has an asymptotic behavior at large wavenumbers that varies as \( (\text{wavenumber})^{-4} \). This is the spatial analog of the idea that the temporal spectrum of any pulse shape with an abrupt rise or fall has a high-frequency asymptote as \( (\text{frequency})^{-2} \). The prime example is a rectangular pulse, for which the spectrum is a sinc-squared function, the high-frequency envelope of which follows this law. The simplest spatial example is spherical blobs of radius \( a \) with uniform density \( f_0 \) inside and zero outside; clearly, an added uniform background density does not affect the power spectrum. The three-dimensional spatial Fourier transform of an isotropic function \( f(r) \) of radial distance \( r \) is given by (Tatarskii 1961)

\[
F(q) = \frac{1}{2\pi^2q} \int_0^\infty f(r) \sin (qr)rdr,
\]

where \( q \) is the magnitude of the wavenumber. For any function \( f(r) \) that falls to zero beyond a radius \( a \), the integral will vary as \( 1/q \) in the high-wavenumber limit \( qa \gg 1 \), making \( F(q) \propto 1/q^2 \). For the spherical blobs of size \( a \) we have

\[
F(q) = \frac{f_0}{2\pi^2q} \left\{ -\frac{a \cos (qa)}{q} + \frac{\sin (qa)}{q^2} \right\}.
\]

For the squared magnitude of the Fourier transform, for very small and large values of \( qa \), we have

\[
|F(q)|^2 = \begin{cases} 
\frac{f_0^2 a^2}{36\pi^4}, & qa \ll 1, \\
q^4 a^2 f_0^2 / (8\pi^4), & qa \gg 1.
\end{cases}
\]

The spectrum of the electron density is then \( n_o |F(q)|^2 \). We can approximate this by equation (2), which exactly matches equation (2) at small and large values of \( qa \), with \( C_n = n_o a^2 f_0^2 / (8\pi^4) \) and \( \kappa_0^{-1} = L_o = (2/9)^{0.25} a \).

There are evident generalizations to make the model more like a real medium. The spheres could have a probability distribution for their radii, which would weight the average of the square of equation (4). The spheres could have smoothly varying density inside and an abrupt boundary at radius \( a \). As with the pulse example, the high-wavenumber behavior would not change. More realistic models would include anisotropic structures with random orientations and a distribution of scales. For example, consider a circular cylinder as a model for an anisotropic blob, with a particular radius and thickness. Its Fourier transform can be computed with respect to its major axes. This can then be transformed to tilted axes and the result averaged over an isotropic distribution of angles of tilt. Evidently the result will also be isotropic; we have solved this approximately and find a spectrum with the same high-wavenumber asymptotic form as equation (2), where the “outer scale,” \( L_o \), is the smaller of the thickness and radius. This result could be averaged over a probability distribution of radii and thicknesses and a similar conclusion reached with \( L_o \) as a weighted average of the smallest dimension. Planar sheets at random angles are modeled very flat cylinders and provide a crude representation of a distribution of shocks. Another simplified model would be ellipsoids of different sizes, eccentricities, and orientations; again we expect a similar result.

We conclude that the \( \beta = 4 \) spectrum is an approximate description of the power spectrum for a medium with a random distribution of discontinuous structures, in the limit for wavenumbers greater than the reciprocal of the smallest dimension across the structures. We note that for wavenumbers at or below about \( \kappa_o = L_o^{-1} \), the functional form depends on the detailed shapes, though it will be much less steep than \( \kappa^{-4} \). Thus equation (2) generally will not be applicable at the lowest wavenumbers. Similarly, in practice there is a finite scale over which the density jump occurs (e.g., shock thickness). This can be modeled by convolving the idealized discontinuous structures with a suitable narrow, say, Gaussian function that thus provides a very high-wavenumber cutoff beyond which the spectrum falls rapidly to zero. Consequently, we consider a spectrum with an inverse fourth power law between inner and outer scales, although these do not have the same connotations as inner and outer scales in a turbulent cascade. We do not discuss the physical origin of the supposed discontinuous plasma structures beyond the ideas formulated in § 1, such as supernova shock fronts, stellar wind boundaries, or sharp boundaries of H II regions at the Strömgren radius. There needs to be no physical coupling between the individual blob structures, and there is no turbulent cascade implied by the \( \beta = 4 \) spectrum. However, we note that shocks developing from the steepening of very strong turbulence could also be described by the model; so, if successful, the model would not rule out such strong turbulence, although in this case the \( \beta = 4 \) range does not correspond to an inertial cascade as does the 11/3 spectrum.

In the following sections, we give the necessary scintillation theory needed to develop equations for the scintillation observables corresponding to the \( \beta = 4 \) model, which we then compare with ISS observations. We concentrate on measurements that are sensitive to the form of the density spectrum and are relatively insensitive to the distribution of scattering material along the line of sight.

3. PHASE-STRUCTURE FUNCTION

Central to the description of the second moment and the coherence function for intensity is the phase-structure function. The longitudinal gradient of the phase-structure function is related to the electron density spectrum through (cf. Coles et al. 1987)

\[
D_\phi(s, z; v) = \frac{8\pi^2 r_o^2 c^2}{v^2} \int_0^\infty \int_0^\infty P_N(\kappa, q_z = 0; z) \times [1 - J_0(\kappa s)] d\kappa dk.
\]

Here \( c \) is the speed of light, \( r_o \) is the classical electron radius, and \( v \) is the radio frequency. For a plane wave incident on a scattering medium, the line-of-sight integral of \( D_\phi(s, z; v) \) gives the structure function of the geometric optics phase, also called the wave-structure function, \( D_\phi(s) \). For the spherical wave geometry applicable to a pulsar at distance \( L \), the separation \( s \) at the observer’s plane is projected back to the
Fig. 1.—Local logarithmic slope, $\alpha(s)$, of the phase-structure function (upper panel) and the phase-structure function (lower panel) for the $\beta = 4$ model. We have set $s_0 = 1000$ km for all curves, and the solid lines correspond to different values of the outer scale, $L_o$. In going from the thinnest to thickest solid lines, the outer scales are $10^4$, $10^5$, $10^7$, $10^9$, and $10^{11}$ km, respectively. The thin dashed line corresponds to the simple Kolmogorov model with logarithmic slope of 5/3, and the thin dotted line corresponds to the square-law–structure function. The points in the upper panel correspond to measured local logarithmic slopes of the phase-structure function obtained from VLBI (Spangler & Gwinn 1990).

The point of integration, giving $D(s) = \int_D^s D(s/L, z; v)dz$. The integrals are evaluated explicitly in equations (13), (17), and (21) of LR99 for a uniform scattering medium with the simple power-law, inner-scale, and $\beta = 4$ spectral models, respectively. In the extended scattering geometry, the field coherence length at the observer, $s_0$, is defined by $D(s_0) = 1$. For a screen of thickness $\Delta z$ at distance $z_p$ from the pulsar, the wave-structure function at the observer becomes $D(s) = D(s_0 z_p/L, z_p; v)\Delta z$, which has an explicit dependence on the screen location. In LR99, we found it useful to define a coherence scale $s_{0,scr}$ for the screen by its phase-structure function $D(s_{0,scr}) = D(s_{0,scr}, z_p; v)\Delta z = 1$, which is independent of the screen location. Note that in applying equation (6), we are assuming that the screen thickness, $\Delta z$, is much greater than the outer scale $L_o$ and small compared to total distance $L$.

Here, we repeat the equation for the phase-structure function of the $\beta = 4$ model in a thin layer (screen). Substituting...
the $\beta = 4$ model for the density spectrum into equation (6),
integrating, and multiplying by the screen thickness, $\Delta z$, we obtain

$$D_\phi(s) = \frac{4\pi^2 r_c^2 c^2 \mathcal{S} M}{k_z^2 v^2} \left[ 1 - (\kappa_s s)K_1(\kappa_s s) \right],$$

(7)

where $K_1$ is the first-order modified Bessel function of the second
kind, and $\mathcal{S} = C_{6b}^2 \Delta z$ is the scattering measure. Equation (7) may be approximated by the following logarithmic expression,

$$D_\phi(s) = \frac{4\pi^2 r_c^2 c^2 \mathcal{S} M}{v^2} s \ln \left[ 1 + \frac{4}{(\kappa_s s)^2} \right],$$

(8)

which matches the full Bessel expression at both large and small values of the argument. The detailed shape, near where the structure function flattens, is governed by the shape of the low-frequency turnover in our model spectrum, equation (2); however, as noted above, the turnover shape depends on details of the density profiles in the plasma blobs that are not constrained in our density model.

The logarithmic structure function for the $\beta = 4$ model has far-reaching consequences. As the outer scale becomes larger, the structure function approaches a square law. However, because of the presence of the logarithm, this occurs only slowly; that is, only at the limit as $s$ goes to zero does the $\beta = 4$ model structure function become exactly square law. A square-law–structure function for the medium is a convenient mathematical model, but as noted in LR99, in a real medium the structure function must eventually saturate at very large separations. Nevertheless, the square-law–structure function has been widely used since it provides a valid approximation for scales smaller than the reciprocal of the wavenumber at which the spectrum is cut off. That the $\beta = 4$ model structure function deviates from a pure square law, even for very large values of the outer scale, makes this model interesting and an independent investigation worthwhile.

In Figure 1, we have plotted the structure function of the $\beta = 4$ model and its effective local logarithmic slope, $\alpha(s)$, versus the transverse spatial lag, $s$, for various values of the outer scale, $L_0$ (see eq. 20 of LR99). To illustrate the shapes, we have chosen $s_{lo,scr} = 1000$ km for all curves. The thin dashed lines correspond to the simple Kolmogorov model with logarithmic slope of 5/3, and the thin dotted lines correspond to the square-law–structure function. The plot shows how slowly the local slope approaches 2; even for $s/L_0 \sim 10^{-8}$, the slope is 1.95—distinctly less than 2.

The points in the upper panel correspond to the measured local exponent of the wave-structure function obtained from VLBI observations as tabulated by Spangler & Gwinn (1990). In principle, this method of comparison provides a good test for the form of the density spectrum. These authors presented a similar plot corresponding to the inner-scale model, for which $\alpha(s)$ is close to 2.0 for $s$ less than the inner scale and close to 5/3 for $s$ greater than the inner scale, with a transition occurring over about a decade in $s$. They derived estimates of the inner scale ranging from 50,000 to 200,000 m for the highly scattered sources that they studied. Whereas there is some uncertainty in some of the estimates of $\alpha$, there are several examples in which values greater than 5/3 are reliably observed (e.g., Trotter, Moran, & Rodríguez 1998). As can be seen, there is also reasonable agreement with the $\beta = 4$ model. Assuming the model to be correct, a separate model-fitting to each observation yields an outer scale in the range $L_0 \sim 3 \times 10^7$–$10^8$ m. These values are much smaller than the parsec scales of supernova remnants and interstellar clouds. Based on this comparison alone, however, we are unable to discriminate between the inner-scale and $\beta = 4$ models since they both provide equally satisfactory agreement with the observations. Hence, further independent tests are needed, and in the following sections, we present comparisons with a further scintillation observable that, in the end, argues against the $\beta = 4$ model as a universal description of the ionized ISM.

4. DIFFRACTIVE DECORRELATION BANDWIDTH

The diffusive decorrelation bandwidth $\Delta \nu_d$ is perhaps the easiest ISS observable to measure. It is the frequency difference for a decorrelation to 50% of the correlation function for diffusive intensity scintillations. These are usually recorded as a dynamic spectrum centered on a given radio frequency, $\nu$, for a pulsar. Observations of $\Delta \nu_d$ for one pulsar at a wide range of radio frequencies have provided an important test of power-law models for the interstellar density spectrum on that particular line of sight. When the diffusive scale is far from the inner and outer scales, one expects $\Delta \nu_d \propto \nu^{2(\beta-2)}$, which for the simple Kolmogorov model is $\nu^{4/3}$ in both screen and extended medium geometries. In this section, we reexamine the published measurements and compare them with theory for the three different spectral models. Since the scaling laws are all so steep, in Figures 2 and 3, we have plotted the observations logarithmically as $\Delta \nu_d/\nu^4$. Overplotted are lines giving the theoretical predictions for our three spectral models, with the simple Kolmogorov model as a line of slope 0.4. Before giving our conclusions, we must discuss the error bars and various corrections to the data and also the method for deriving the theoretical predictions.

4.1. Observations

Decorrelation bandwidth data, collected together by Cordes, Weisberg, & Boriakoff (1985, hereafter CWB), for pulsars PSR B0834–45 (Vela), PSR B0329+54, PSR B1642–03, PSR B1749–28, and PSR B1933+16 are shown in Figures 2 and 3. We have also included recently measured points by Johnston, Nicastro, & Koribalski (1998) for the Vela pulsar. These are $\nu, \Delta \nu_d = (8.4$ GHz, $13.9$ MHz) and (13.7 GHz, 58.2 MHz). The data are shown as solid circles if derived from dynamic spectra and as solid stars if derived from pulse-broadening measurements. The latter become easier to estimate at low frequencies, at which the resolution bandwidth required for the former becomes too narrow for an adequate signal-to-noise ratio. The conversion of an estimate of a pulse-broadening time to a decorrelation bandwidth relies on the uncertainty relation $2\pi \Delta \nu_d \tau_d = C_1$. Unfortunately, the “constant” $C_1$ takes on different values for different geometries and spectral models. For the $\beta = 4$ model, it is also very weakly dependent on frequency. In Table 1 of LR99, we gave numerical values for $C_1$ for the three spectral models with spherical waves in both a screen and an extended medium geometry. For a given geometry, the smallest $C_1$ is for the Kolmogorov spectrum and the largest (by about 50%) is for the square-law–structure function. Thus, in Figures 2 and 3 we plot, as solid and open stars, the values converted using $C_1$ for the Kolmogorov and square-law–structure function models,
respectively. We display the predictions for extended medium and note that the screen values are only about 20% less for each spectrum. Thus, in comparing data with the theory for the $\beta = 4$ and inner-scale models, the data should lie between the extremes of the open and solid stars, depending on the outer and inner scales, respectively. Note that the value of $C_1$ in equation (7) of the Taylor, Manchester, & Lyne (1993) catalog is more than 50% greater than values computed by LR99, since it concerns the mean pulse delay rather than the 1/e decay time of the pulse.

The second correction to the data concerns the effects of refractive scintillation at the higher radio frequencies, which...
are closer to the transition to weak scintillation. Gupta et al. (1994) gave a heuristic theory of the effect and expressions for the bias to the diffractive $\Delta \nu_d$ caused by refractive shifts; LR99 discussed the effect from the work of Codona et al. (1986, hereafter CCFFH). In Figures 2 and 3, we have plotted as open circles $\Delta \nu_d$ corrected for this effect using equation (D6) of Gupta et al. (1994), who also estimated the variability in estimates of $\Delta \nu_d$ caused by the refractive modulation. We used this to estimate an error bar on the open circles, which is typically larger than the error bars quoted by the original observers, plotted on the solid circles.

In summary, the open circles and their error bars provide the best direct estimates of $\Delta \nu_d$ and the indirect estimates lie between the open and solid stars.

4.2. Theory

The theory of diffractive scintillations has been discussed by many authors; we will use the results of § 6 in LR99 for a point source (spherical waves) either in an extended scattering medium or with a screen at $z_p$ from the pulsar and $z_o$ from the observer (with $z_p + z_o = L$). Following usual practice, we assume the strong scattering limit, in which the frequency decorrelation function for intensity is the squared
magnitude of the diffractive component of the two-frequency second moment of the field. The validity of this approximation is examined in Appendix B, where we find that it introduces a small error for the $\beta = 4$ model, which, however, is negligible compared with the errors in the observations. The decorrelation bandwidth, $\Delta v_d$, can be defined in terms of a normalized bandwidth, $v_d$:

$$\Delta v_d = \frac{v_d v}{u^2}$$

where $u^2 = \frac{z_{\text{scatt}} c}{2\pi v s_0}$. (9)

Here $v$ is the (geometric) mean of the two frequencies, and the parameter $u$ determines the strength of scattering; $c$ is the speed of light and $z_{\text{scatt}}$ is the effective scattering distance. For a uniform scattering medium, $z_{\text{scatt}} = L$, and for a screen geometry, $z_{\text{scatt}} = z_o = z_p / L$. $s_0$ is the field coherence scale where the appropriate phase-structure function equals unity; for the uniform scattering medium, it is defined at the observer, and for a screen, it is $s_{0,\text{scr}}$, which is independent of the distances to the pulsar and screen and leads to $u_{\text{scr}}$ as the appropriate strength of scattering parameter.

The computations presented in LR99 include a tabulation of the normalized bandwidth $v_d$ for the various models under discussion and associated plots of the intensity decorrelation function itself. For the simple Kolmogorov model, a constant value for $v_d$ is obtained regardless of the frequency or distance (0.773 and 0.654 for the extended medium and screen geometries, respectively). Thus the frequency dependence of $\Delta v_d$ reflects the frequency dependence of $s_0$, giving $\Delta v_d \propto v^{2\beta/\beta - 2}$. CWB used this to estimate $\beta$ from observations of $\Delta v_d$. We now compare the observations with theoretical scaling laws for the two other spectral models. There are, however, some complications. The quantity $v_d$ is no longer exactly independent of frequency, and $s_0$ depends on both frequency and the outer scale, $L_o$, or the inner scale, $L_i$, that parameterize the other two models.

Consider the details for the discontinuity spectrum. In Figure 2, we plot $\Delta v_d / v_{\text{GHz}}$ versus $v_{\text{GHz}}$ as observed for the Vela pulsar. In Figure 2a, we have plotted theoretical curves for the $\beta = 4$ model with various outer scales. We use equation (9) for the models and so need the variation of both $s_0$ and $v_d$ with $v$.

For a screen, we determine $s_{0,\text{scr}}$ from equation (8) for the $\beta = 4$ model and eliminate the scattering measure $SM$ using the same equation for $s_{0,1\text{GHz}}$ at a reference frequency of 1 GHz. This gives a relation between $s_{0,\text{scr}} / s_{0,1\text{GHz}}$ and $v_{\text{GHz}}$, with $L_o / s_{0,1\text{GHz}}$ as a parameter. For an extended scattering medium, we use equation (21) of LR99 to determine $s_0$ at the observer and the same method as for the screen to eliminate $SM$. In Figure 4, we show the variation of $v_d$ with $L_o / s_0$ at a fixed frequency (computed by Lambert 1998 and described in LR99). This is combined with the $s_0 - v_{\text{GHz}}$ rel-

![Figure 4](image-url)

**Figure 4.** Variation of the normalized diffractive decorrelation bandwidth, $v_d$, (a) $\beta = 4$ model, upper curve for an extended medium geometry with parameter $\kappa_o s_0$ and lower curve for a screen geometry vs. $\kappa_{\text{scr}} s_{0,\text{scr}}$. (b) Same plots for the inner-scale model.
tion to obtain the relation of \(v_d\) to \(v_{\text{GHz}}\). For the curves in Figure 2, \(s_{0,1\text{GHz}}\) is determined so that, for each value of \(L_o\), the model \(\Delta v_d\) fits the measured values near 1 GHz. For most of the models of interest, \(L_o \gg s_0\), and then the logarithm functions vary much more slowly than the \(s_0^2\) term in equation (8). In these cases there is nearly a linear relation \(s_0/s_{0,\text{ref}} = v/v_{\text{ref}}\). Furthermore, Figure 4 shows that as \(s_0\) changes, the parameter \(v_d\) changes very slowly. Consequently, for a very wide range of the outer scale, a good approximation is \(\Delta v_d \propto v^4\), which reflects the fact that the underlying structure function then approaches a square-law behavior.

Consider the plot for the Vela pulsar PSR B0833–45 (Fig. 2), for which the data are most extensive. Although we have computed curves and data corrections for both the screen and extended scattering geometries, we show only the extended medium results in Figure 2 because the two plots are so similar. In the comparison with the screen model, the star points are lowered about 0.08 vertical units (20%), and the \(\beta = 4\) model curves are very slightly steeper functions of frequency. However, the scatter among the observations is greater than the differences in the models between screen and extended geometries. In the Vela plot, we see that the \(\beta = 4\) model agrees somewhat better with the observations than does the simple Kolmogorov model. This pulsar is known to lie in a highly scattered region; hence, the presence of discontinuities, as incorporated in the \(\beta = 4\) model, is perhaps reasonable. However, the conclusion is not strong, since the data are also reasonably fitted by an inner scale in the range 100–500 km, as can be found from Figure 2b.

For the other pulsars (Fig. 3), the data show a stronger frequency dependence than predicted by both the simple Kolmogorov and \(\beta = 4\) models. However, the inconsistencies between the measurements at different frequencies are even greater, and better observations are needed before such comparisons could discriminate between the models. A recent series of measurements at 327 MHz (Bhat, Rao, & Gupta 1999a, 1999b, 1999c) documents the variability of \(\Delta v_d\), and a similar long sequence of such measurements at other frequencies is needed on the same pulsars before reliable conclusions can be reached from the frequency scaling observations.

5. REFRACTIVE SCINTILLATION INDEX

Slow variations in the flux of pulsars are caused by refractive interstellar scintillation (RISS) and can be characterized by the rms deviation in flux density normalized by its mean, or scintillation, index \(m_R\). RISS is caused by inhomogeneities in the interstellar electron density on scales much larger than those responsible for diffractive scintillation, as characterized by the decorrelation bandwidth. There are now measurements of both phenomena on a substantial number of pulsars, from which we can constrain the density spectrum on each line of sight. The relation of the refractive scintillation index, \(m_R\), to the normalized diffractive decorrelation bandwidth, \(\Delta v/v\), depends on the ratio of the power in the density spectrum at the large refractive scales (10\(^{-11}\)–10\(^{-12}\) m) to the power at the smaller diffractive scales (10\(^{-7}\)–10\(^{-9}\) m).

We compare pulsar measurements gathered from the literature with theoretical predictions by plotting \(m_R\) versus

### TABLE 2

| Reference          | Pulsar     | Distance (kpc) | \(v\) (MHz) | \(\Delta v\) (MHz) | \(m_R\) | \(\delta m_R\) |
|--------------------|------------|----------------|------------|-------------------|--------|--------------|
| Rickett & Lyne 1990| B0531 + 21 | 2.0            | 610.0      | ~2.96 \times 10^{-2} | 0.327  | 0.052        |
|                    | B0531 + 21 | 2.0            | 610.0      | ~2.96 \times 10^{-2} | 0.402  | 0.095        |
| Kaspi & Stinebring 1992 | B0329 + 54 | 1.4            | 610.0      | 0.44              | 0.39   | 0.07         |
|                    | B0833–45   | 0.55           | 610.0      | ~1.29 \times 10^{-4} | 0.11   | 0.01         |
|                    | B1749–28   | 1.2            | 610.0      | ~5.13 \times 10^{-2} | 0.26   | 0.05         |
|                    | B1911–04   | 2.29           | 610.0      | ~1.91 \times 10^{-3} | 0.20   | 0.03         |
|                    | B1933+16   | 7.8            | 610.0      | ~1.82 \times 10^{-3} | 0.18   | 0.04         |
|                    | B2111+46   | 5.22           | 610.0      | ~2.69 \times 10^{-3} | 0.16   | 0.05         |
|                    | B2217+47   | 2.31           | 610.0      | 0.26              | 0.21   | 0.02         |
| Smirnova et al. 1998 | B0136 + 57 | 2.9            | 610.0      | ~5.7 \times 10^{-3} | 0.15   | 0.02         |
|                    | B0329 + 54 | 1.4            | 610.0      | 0.350             | 0.37   | 0.02         |
|                    | B0525 + 21 | 2.3            | 610.0      | 0.30              | 0.31   | 0.01         |
|                    | B0531 + 21 | 2.0            | 610.0      | ~4.0 \times 10^{-2} | 0.32   | 0.01         |
|                    | B0736–40   | 2.1            | 610.0      | ~9.0 \times 10^{-6} | 0.03   | 0.01         |
|                    | B0740–28   | 1.9            | 610.0      | ~3.5 \times 10^{-3} | 0.13   | 0.01         |
|                    | B0818–13   | 2.5            | 610.0      | 0.061             | 0.23   | 0.01         |
|                    | B0833–45   | 0.5            | 610.0      | ~1.50 \times 10^{-4} | 0.24   | 0.01         |
|                    | B0835–41   | 4.2            | 610.0      | ~7.0 \times 10^{-4} | 0.21   | 0.02         |
|                    | B1641–45   | 4.6            | 610.0      | ~7.0 \times 10^{-7} | ~0.1   | 0.05         |
|                    | B1642–03   | 0.5            | 610.0      | 0.770             | 0.46   | 0.04         |
|                    | B1749–28   | 1.5            | 610.0      | ~6.0 \times 10^{-2} | 0.25   | 0.02         |
|                    | B1818–04   | 1.6            | 610.0      | ~4.0 \times 10^{-4} | ~0.1   | 0.05         |
|                    | B1859–03   | 8.1            | 610.0      | ~5.0 \times 10^{-6} | ~0.05  | 0.03         |
|                    | B1911–04   | 3.2            | 610.0      | ~8.0 \times 10^{-4} | 0.22   | 0.01         |
|                    | B1933+16   | 3.5            | 610.0      | ~2.0 \times 10^{-3} | 0.24   | 0.02         |
|                    | B1946+35   | 7.9            | 610.0      | ~3.0 \times 10^{-5} | ~0.05  | 0.03         |
|                    | B2111+46   | 5.0            | 610.0      | ~3.0 \times 10^{-3} | 0.15   | 0.01         |
|                    | B2217+47   | 2.5            | 610.0      | 0.20              | 0.30   | 0.01         |
\( \Delta v_d/v \) (cf. Rickett & Lyne 1990 and Gupta et al. 1993). Related tests have been made by Armstrong et al. (1995), Bhat et al. (1999a), and Smirnova, Shishov, & Stinebring (1998). \( m_R \) measurements at 610.0 MHz are listed in Table 2 and include the results of long-term monitoring observations by Smirnova et al. (1998); older measurements near 100 MHz are listed in Table 3. The tables also include the decorrelation bandwidth obtained from diffractive scintillation, typically observed at a different radio frequency, scaled to the observing frequency for \( m_R \). We used the Kolmogorov scaling law \( \Delta v_d \propto v^{4/3} \), and we note that the minor differences that result from changes in the scaling law for other spectral models are insignificant in the comparison. The 100 MHz data consist of \( m_R \) measurements at 73.8 MHz, 81.5 MHz, and 156 MHz. The pulsars observed at 610.0 MHz are primarily located at distances \( \geq 1 \) kpc, whereas the pulsars observed at 100 MHz are nearer at distances \( \leq 1 \) kpc. Figures 5 and 6 show plots of the theoretical and measured refractive scintillation index versus the normalized decorrelation bandwidth in separate panels for the two sets of measurements. Theoretical curves are also shown in the figures for different spectral models in both the screen and extended medium geometries. The dashed line corresponds to the simple Kolmogorov model (\( \alpha = 5/3 \)), and the solid lines in Figures 5 and 6 correspond to the \( \beta = 4 \) and inner-scale models, respectively.

In § 4, we noted that variable refraction tends to decrease the measured decorrelation bandwidth and used an expression from Gupta et al. (1994) to apply a nominal correction to measured values. In the table of data from near 100 MHz, we have made use of the recently published measurements of Bhat et al. (1999a). They monitored the apparent decorrelation bandwidth near 327 MHz for 20 nearby pulsars, many of which have \( m_R \) measurements in Table 3. They found the bandwidth to vary by factors 3–5 and discussed the influence of varying refraction as a refractive bias. From their measurements, they derived a corrected decorrelation bandwidth, which we have scaled (using the \( v^{4/3} \) scaling law) to the frequency at which \( m_R \) was observed. These gave values typically 2–5 times bigger than obtained from earlier measurements (e.g., Cordes 1986). The pulsar B0809+74 was not observed by Bhat et al. (1999a); however, the recent weak scintillation observations of this pulsar by Rickett et al. (1999) similarly suggests that earlier decorrelation bandwidth measurements overestimated the strength of scattering. The effect of these changes is to shift the plots to the right in Figures 5 and 6 by about half a decade, compared to Figure 5 of Gupta et al. (1993). There is only one pulsar (B0329+54) in Table 2 common to the Bhat et al. (1999a) observations, and its decorrelation bandwidth has also been corrected for the refractive bias. Since the pulsars observed at 610.0 MHz were mostly more heavily scattered, the refractive bias correction is substantially smaller (see Gupta et al. 1994) and has been ignored.

5.1. Theory

The theory of refractive scintillations has been known since the 1970s and was applied to pulsar flux variations by Rickett, Coles, & Bourgois (1984). For example, Prokhorov et al. (1975) described how the modulation index for intensity can exceed unity in strong scattering in their equations (4.53) et seq. We use the notation of Coles et al. (1987) and confine our discussion to spherical wave sources that are propagating in a scattering plasma that is either concentrated in a screen or extended uniformly between source and observer. The “low-frequency” approximation for the intensity covariance function is given by equation (10) of Coles et al. (1987), from which we obtain the normalized refractive variance, \( m_R^2 \), by setting the spatial offset equal to zero.

\[
m_R^2 = \frac{16\pi^2 c^2 L}{v^2} \int_0^\infty \int_0^\infty P_N(k, q = 0; z) \times \exp \left\{ -L \int_0^1 D_{\phi}[\kappa r_{\phi,L}h(x, y)]dy \right\} \times \sin^2 \left[ 0.5x^2 r_{F,L}(1 - x) \right] dx dy,
\]

(10)

where

\[
h(x, y) = \min[y(1 - x), x(1 - y)]
\]

\[
r_{F,L} = \sqrt{Lc/2\pi v}.
\]

Here the transverse wavenumber is \( \kappa = (q_x^2 + q_y^2)^{1/2} \), and \( x = z/L \), where \( z \) is the distance along the line of sight measured from the source. For the screen geometry, both the density spectrum and the gradient in the phase-structure function \( D_{\phi} \) are concentrated in a thin layer of thickness \( \Delta z \) at distance \( z_p \) from the source and \( z_o \) from the observer. Thus the \( x \)- and \( y \)-integrations in equation (10) give \( x = y = z_p/L \), and we find it useful to define an equivalent Fresnel scale \( r_{F,scr} \).

\[
r_{F,scr} = \sqrt{\Delta z c/2\pi v} \quad \text{where} \quad z_o = z_p z_o/L.
\]

(12)

The \( \kappa \) integration in equation (10) involves the product of the density spectrum, decreasing as a steep power of \( \kappa \), times the high-pass Fresnel filter \( \propto \kappa^4 \) up to \( r_{F,scr}^{-1} \), times the low-pass exponential term that cuts off wavenumbers above \( s_R^{-1} \), where \( s_R = u_{\text{scr}}r_{F,scr} \) is the radius of the scattering disk. Hence, in strong scattering (\( u_{\text{scr}} > 1 \)), we consider only the \( \kappa^4 \) part of the Fresnel filter. Consequently, the integration depends on the ratio \( L_{u/R} \) or \( L_{u/s_{R}} \). For an extended scattering medium, the line-of-sight integration softens the exponential cutoff in the \( \kappa \) integration, but the basic relationships remain the same. As others have noted, the level of refractive scintillation is greater in an extended medium than in a screen with the same observed diffractive scintillation.

The canonical spectral model in ISS studies of pulsars has been the simple Kolmogorov spectrum \( P_N(q, z) = C_N(z)q^{-11/3} \). By fitting this model to diffractive scintillation observations, observers have estimated the scattering measure, \( SM = \int_0^\infty C_N(z)dz \), toward many pulsars. When divided by the pulsar distance, this gives a line-of-sight average of \( C_N(z) \), which is found to vary greatly from one direction to another and to increase dramatically for distant pulsars seen at low Galactic latitudes (e.g., CBW). Taylor & Cordes (1993) have developed a smoothed model for the Galactic plasma density distribution, which includes enhanced scattering in spiral arms and toward the Galactic center. However, there are also large random variations on much finer spatial scales, which would produce a scatter in a plot of \( m_R \) against distance of dispersion measure. For a simple power-law spectrum, both \( m_R \) and \( \Delta v_d/v \) depend on \( SM \) through a single strength of scattering parameter. Thus, for the Kolmogorov spectrum, the variation of \( m_R \) with \( \Delta v_d/v \) is independent of distance or frequency and is given by a single dashed line in each plot in Figures 5 and 6. If the
scattering medium is uniform, the strength of scattering is 
\[ u = \frac{r_{FL}}{s_0} \]. For the screen, it becomes 
\[ u_{\text{scr}} = \frac{r_{FS,\text{scr}}}{s_{0,\text{scr}}} \], and in that case, the dashed theoretical line is independent of the location of the screen between the source and the observer. This point is substantiated in Appendix A, where the theory is laid out in more detail. According to this screen model, along each line of sight there is a scattering layer with a certain \( SM \), which determines both \( m_R \) and \( \Delta \nu / \nu \), but \( SM \) is not necessarily related to the pulse distance. The theoretical curve with \( SM \) as the variable is independent of where the layer is located along the line of sight.

For the \( \beta = 4 \) and inner-scale models, the theoretical curves for \( m_R \) versus \( \Delta \nu / \nu \) depend on the extra parameter \( L_o \) or \( L_i \), respectively. Details of the theory are given in Appendix A, where the relevant parameters are shown to be \( L_o/s_{0,\text{scr}} \) and \( L_i/s_{0,\text{scr}} \). Since \( s_{0,\text{scr}} \) depends on frequency, \( SM \), and distance, then \( m_R \) also depends somewhat on frequency and distance. In order to fix the frequency dependence of the theoretical \( m_R \) values, we have separated the measurements into two groups (at 610.0 and near 100 MHz). The distance dependence is dealt with by assuming that \( SM \propto d \). Whereas this is clearly appropriate for the extended scattering medium, it is less clear for the screen model, since the screen model supposes that \( SM \) is not necessarily related to distance. However, it is reasonable to say, that if on a long line of sight there is a single region that dominates the scattering, its \( SM \) value will statistically increase with line-of-sight distance. Indeed, the experimentally derived scattering measure increases faster with distance than if the medium were uniform (see Figure 1 of Cordes et al. 1991).

In Figure 5 we show the theoretical curves for \( m_R \) in the \( \beta = 4 \) model, which are relatively flat where \( L_o > s_R \). As \( u \) increases (\( \Delta \nu / \nu \) decreases), \( s_R \) increases, and when \( s_R > L_o \), the scintillations are suppressed. In a screen, this occurs when \( \Delta \nu / \nu < (r_{FS,\text{scr}}/L_o)^2 \).

Inspection of Figure 5 shows that the measured \( m_R \) values at 100 MHz are above the prediction of the simple Kolmogorov model for the extended medium geometry. These measurements can, however, be modeled with the \( \beta = 4 \) spectrum with suitable specific choices of outer scale, \( L_o \). The screen geometry is inconsistent with the measurements at 100 MHz for both the simple Kolmogorov and \( \beta = 4 \) models. For the 610.0 MHz data, the measured \( m_R \) values are mostly in agreement with the prediction of the simple Kolmogorov model for the extended medium geometry; however, they lie below the predictions of the \( \beta = 4 \) model. For the screen geometry, the measured \( m_R \) values are above the simple Kolmogorov model curve. However, for the \( \beta = 4 \) model, though less convincingly, an agreement with the measurements can be found by suitable choices of the outer scale \( L_o \). The most striking feature of the results is the good agreement with the uniform extended Kolmogorov model for most of the pulsars. This result relies most heavily on the excellent data from Smirnova et al. (1998), who also note this agreement. We are persuaded by this figure that the \( \beta = 4 \) model cannot be viewed as a global alternative to the Kolmogorov spectrum. We also conclude that for the medium-distance pulsars measured at 610.0 MHz, \( m_R \) does not agree with the screen geometry. It appears that even if the scattering medium is not uniform on scales of a few kpc, there is not a single region that dominates the scattering. Computations of \( m_R \) owing to a patchy distribution along the line of sight could test what distribution would start to approximate the uniformly extended medium.

Turning to a comparison with the alternative inner-scale spectrum in Figure 6, the 100 MHz observations could be explained by very large values of \( L_i \) (10\(^{10} - 10^{11} \) m) in a screen configuration or by more modest, but still large, values of \( L_i \) (10\(^8 \) m) in an extended medium. The latter was essentially the proposal made by Coles et al. (1987) for the nearby lines of sight. The 610.0 MHz points are relatively lower than those at 100 MHz. For an inner-scale spectrum in a screen, large values of \( L_i \) (10\(^8 - 10^{10} \) m) would be necessary; and in an extended medium, 60% of the points lie near the simple Kolmogorov model, with the rest requiring inner-scale values (10\(^7 - 10^8 \) m). The 610.0 MHz data in our analysis come from RISS observations of 21 pulsars made by Stinebring and are described in greater detail by Smirnova et al. (1998), who also compare the results with various models for the spectrum and spatial distribution of the electron density. They note that four pulsars that are seen

### Table 3

**Refractive Modulation Index Data at 100 MHz from the Literature**

| Reference          | Pulsar       | Distance | \( \nu \) (MHz) | \( \Delta \nu \) (MHz) | \( m_R \) | \( \delta m_R \) |
|--------------------|--------------|----------|----------------|-----------------------|----------|-----------------|
| Cole et al. 1970   | B0809 + 74   | 0.31     | 81.5           | 1.38 \( \times 10^{-3} \) | 0.45     | 0.18            |
|                    | B0834 + 06   | 0.72     | 81.5           | 1.3 \( \times 10^{-3} \) | 0.39     | 0.11            |
|                    | B1191 + 21   | 0.66     | 81.5           | 1.2 \( \times 10^{-3} \) | 0.53     | 0.18            |
| Helfand, Fowler, & Kuhlman 1977 | B0329 + 54   | 1.4      | 156.0          | 5.7 \( \times 10^{-3} \) | 0.35     | 0.08            |
|                    | B0823 + 26   | 0.38     | 156.0          | 1.0 \( \times 10^{-2} \) | 0.46     | 0.11            |
|                    | B1133 + 16   | 0.27     | 156.0          | 2.9 \( \times 10^{-2} \) | 0.50     | 0.08            |
|                    | B1508 + 55   | 1.93     | 156.0          | 7.0 \( \times 10^{-3} \) | 0.41     | 0.07            |
|                    | B1191 + 21   | 0.66     | 156.0          | 1.4 \( \times 10^{-2} \) | 0.50     | 0.07            |
|                    | B2217 + 47   | 2.31     | 156.0          | 6.45 \( \times 10^{-4} \) | 0.41     | 0.11            |
| Gupta et al. 1993  | B0329 + 54   | 1.4      | 73.8           | 3.1 \( \times 10^{-4} \) | >0.15    | ...             |
|                    | B0809 + 74   | 0.31     | 73.8           | 8.9 \( \times 10^{-4} \) | 0.34     | 0.06            |
|                    | B0834 + 06   | 0.72     | 73.8           | 8.6 \( \times 10^{-4} \) | 0.16     | 0.03            |
|                    | B0950 + 08   | 0.12     | 73.8           | 0.17                | 0.45     | 0.05            |
|                    | B1133 + 16   | 0.27     | 73.8           | 1.6 \( \times 10^{-3} \) | 0.18     | 0.02            |
|                    | B1237 + 25   | 0.56     | 73.8           | 1.8 \( \times 10^{-3} \) | 0.30     | 0.06            |
|                    | B1508 + 55   | 1.93     | 73.8           | 3.8 \( \times 10^{-4} \) | 0.28     | 0.09            |
|                    | B1919 + 21   | 0.66     | 73.8           | 7.5 \( \times 10^{-4} \) | >0.21    | ...             |
through known H II regions or supernovae remnants show relatively elevated values for $m_R$ and find 3 other pulsars with similar behavior. These are the points that lie above the Kolmogorov line in Figure 6b. Their interpretation is that for these objects the scattering is concentrated into regions either near the pulsar or near the Earth and that these regions are characterized by an inner scale near $3 \times 10^8$ m. Their spectrum model is very similar to our inner-scale model, except that the cutoff is characterized by a steep power law rather than by an exponential function. We find a somewhat larger numerical value for the inner scale required to match those objects. We also note that the theoretical $m_R$ values are consistently higher for an extended medium and so require a more modest inner scale.

However, an important result of our analysis is an alternative explanation for the relatively high $m_R$ values seen for some pulsars. We suggest that pulsar lines of sight can pass through discrete clouds with increased plasma density on large scales that steepen the low-wavenumber spectrum as opposed to cutting off the high wavenumbers. A discontinuity spectrum ($\beta = 4$) is one way that the spectrum can be steepened, but smoother structures on scales larger than

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**Fig. 5.**—Theoretical and measured refractive scintillation indices, $m_R$, vs. the normalized diffractive decorrelation bandwidth, $\Delta \nu_d / \nu$. (a) and (b) show measurements at 610.0 MHz; (c) and (d) correspond to measurements near 100 MHz. Solid curves give the theoretical results for the $\beta = 4$ model with a range of outer scales from $10^{10}$ m to $10^{16}$ m, going from right to left in equal logarithmic steps. The dashed line corresponds to the simple Kolmogorov model in the strong scattering limit. (a) and (e) give theory for a screen geometry; (b) and (d) for a uniform scattering medium.
10^{11} \text{ m} (\sim 1 \text{ AU}) would also boost the low wavenumbers and could cause similar enhancements of refractive compared to diffractive scintillations. Such enhancements are likely to be associated with H II regions or supernova remnants. A similar idea also was discussed by Lestrade, Rickett, & Cognard (1998) in the context of extreme scattering events in pulsar timing measurements. To take this idea further, a composite between the Kolmogorov and the $\beta = 4$ or steeper spectra should be investigated, particularly one in which the line of sight is not uniformly weighted. Smirnova et al. (1998) also note that the strength of scattering increases very much faster with distance and dispersion measure than if the medium were statistically uniform. The distant pulsars in their sample are mostly observed at low Galactic latitudes and so are subject to the enhanced density and turbulence described by the Taylor & Cordes (1993) model. It appears that most distant pulsars follow the uniform extended Kolmogorov model with inner scale smaller than 10^{6} \text{ m}. Thus, although these lines of sight are subject to enhanced scattering in the inner Galactic plane, the spectrum effectively follows the Kolmogorov law and the plasma is dispersed enough to approximate a uniform
scattering structures present along these lines of sight, but their contribution to the density spectrum is masked by the higher densities of the inner Galaxy, which still follow an apparently turbulent spectral form.

6. SUMMARY AND CONCLUSION

In this paper, the theory of the $\beta = 4$ model for the electron density spectrum was derived for discontinuous density structures and compared with pulsar observations. A new feature of our analysis is the inclusion of an "outer scale" needed in any realistic model. The model is characterized by an effective exponent $\alpha(s)$ of the structure function, which remains between 1.95 and 1.6 over a very wide range of $L_o$ values (cf. Fig. 1). This at first seems a promising explanation for the spread in the estimates of $\alpha$ derived from VLBI observations of the angular broadening profile, as observed on heavily scattered lines of sight.

As discussed in § 4, from Figures 2 and 3, we find that the $\beta = 4$ model provides a somewhat better agreement with the measurements of the diffractive decorrelation bandwidth versus frequency for pulsar PSR B0833–45 (Vela) than does the simple Kolmogorov model. This might arise from refractive scattering effects caused in the supernova remnant associated with the Vela pulsar. Four other pulsars with decorrelation bandwidths measured against frequency show an appreciably stronger frequency dependence than the predictions of both the simple Kolmogorov and $\beta = 4$ models. However, there are substantial inconsistencies among the measurements and better observations are clearly needed, especially in view of the variability in $\Delta \nu_d$ documented by Bhat et al. (1999c).

The predictions of the $\beta = 4$ model for the variation of the refractive scintillation index with the diffractive decorrelation bandwidth are in partial agreement with the observations. As discussed in § 5, the values of $m_R$ measured near 100 MHz are above the prediction of the simple Kolmogorov model and previously had been explained as the effect of an inner scale, that is, substantially larger than the values invoked by Spangler & Gwinn 1990. However, the measurements are also consistent with the $\beta = 4$ model with suitable choices of the outer scale. For the 610.0 MHz data, most of the measured $m_R$ values are in good agreement with the prediction of the simple Kolmogorov model for the extended medium, and they lie below the curves of the $\beta = 4$ model. However, there are a significant number of good-quality observations, which lie somewhat above the Kolmogorov line. We suggest an alternative to a large inner scale on those lines of sight, namely that they pass through regions of enhanced density, which causes enhanced refractive scattering; these regions must have less small-scale substructure than in a turbulent medium and could include discontinuities.

Based on the above considerations, we reject the $\beta = 4$ model as a universal spectral model for the interstellar electron density fluctuations. The corollary is to strengthen the evidence for the Kolmogorov density spectrum, which in turn suggests a turbulent process in the interstellar plasma. However, the simple Kolmogorov spectrum is not a universal model either, since it disagrees with several of the $m_R$ observations. Since the $\beta = 4$ model provides reasonable agreement for many of these discrepant observations, we propose that enhancements in the large-scale part of the spectrum (which need not be described by discontinuities in density) occur on these lines of sight. With such enhancements causing the increase in refractive scintillation, there is no need to invoke the relatively large inner scales proposed by Coles et al. (1987). As proposed by Spangler & Gwinn (1990), a relatively small inner scale is then likely, controlled by the ion inertial length or Larmor radius.

It appears that different spectral models need to be considered for different lines of sight. A widely distributed turbulent plasma with occasional large ionized structures that increase the effective average power density, $P_{N_e}$, at low wavenumbers (large scales: $10^{11}–10^{14}$ m) is thus a model that needs further formal investigation. Very similar conclusions have been reached by Lestrade et al. (1998) and by Bhat et al. (1999b, 1999c). This model could also explain the occasional "extreme scattering events" and episodes of fringes in dynamic spectra when a line of sight passes through a particular discrete density enhancement. New theoretical work is needed to quantify the expected statistics of these propagation events. It is likely that numerical modeling will be necessary to model nonstationary scattering media.

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APPENDIX A

THEORY FOR REFRACTIVE SCINTILLATION INDEX

A1. SCREEN GEOMETRY

Here we derive expressions for the refractive modulation index, $m_R$, as a function of the normalized diffractive decorrelation bandwidth, $\Delta \nu_d/\nu$, for the inner-scale and $\beta = 4$ models. The simple Kolmogorov model is included as a special case where the inner scale goes to zero. A general integral expression for the refractive scintillation index is given by equation (10). With the medium concentrated into a thin layer (thickness $\Delta z$ at $z_p$ from the source) and an assumption of isotropy in the density spectrum, the equation becomes

$$m_R = \frac{16\pi^2 r_0^2 c^2 \Delta z}{v^2} \int_{0}^{\infty} P_{N_e}(\kappa) \exp \left[ -D_{\phi}(r_{F,sec}^2 \kappa) \right] \sin^2 \left( \frac{r_{F,sec}^2 \kappa^2}{2} \right) d\kappa.$$  \hspace{1cm} (A1)

For both the inner-scale and $\beta = 4$ models, the exponential function in equation (A1) may be approximated by the Gaussian (Coles et al. 1987):

$$\exp \left[ -D_{\phi}(r_{F,sec}^2 \kappa) \right] \approx \exp \left[ -(r_{F,sec}^2 \kappa^2) \right],$$  \hspace{1cm} (A2)
where \( r_{F,scr} \) is the Fresnel scale and \( u_{scr} \) is the strength of scattering, as defined in § 5. We now substitute models for the density spectrum \( P_\kappa (\kappa) \), given by equation (1) (with \( \kappa_s = 0 \)) and equation (2) and obtain relations between \( m_R^2 \) and the scattering measure \( SM = C_2 \Delta z \). To these we add the connections between \( SM \) and \( s_{o,scr} \) for each spectral model from the equations for the phase-structure functions (17), (11), and (21) of LR99.

For the inner-scale model, we obtain
\[
\frac{8\pi^2 r_s^2 c^2 SM}{v^2} = \frac{\alpha 2^s}{s_{o,scr}} \frac{1 + (L_i / s_{o,scr})}{\Gamma(1 - \alpha/2)} \left[ 1 + \left( \frac{L_i}{s_{o,scr}} \right)^{\alpha} \right]^{(2 - \alpha)/\alpha} \mu_y \, ,
\]
where
\[
\mu_y = \left[ \frac{\alpha}{2} \Gamma(1 + \alpha/2) \right]^{1/(\alpha - 2)} \, .
\]

Then, substituting for \( SM \), we obtain an integral for \( m_R^2 \), which can be solved analytically using standard techniques (see, e.g., Appendix 2 of Rickett 1973):
\[
m_R^2 = 2^\Gamma \left( 1 + \frac{\alpha}{2} \right) \left[ 1 + \left( \frac{L_i}{s_{o,scr}} \right)^{\alpha} \right]^{(2 - \alpha)/\alpha} \mu_y^2 \, ,
\]
where \( u_{scr} = r_{F,scr}/s_{o,scr} \), and \( \psi \) is given by
\[
\psi = \cot^{-1} \left[ \left( \frac{L_i}{s_{o,scr} u_{scr}} \right)^2 + u_{scr}^2 \right] \, .
\]

We see that \( m_R \) depends on \( u_{scr} \) with \( L_i/s_{o,scr} \) as a parameter. Similarly, the diffractive decorrelation bandwidth, \( \Delta \nu_d/v \), depends on \( u_{scr} \) through equation (9) with \( u_y \) as a parameter, which is a very slow function of \( L_i/s_{o,scr} \), as discussed in § 4.

For the simple power-law model, we set \( L_i = 0 \), and in the strong scattering limit \( (u_{scr} \gg 1) \), equation (A6) gives \( \psi \approx u_{scr}^{-2} \), and we obtain the asymptotic expression
\[
m_R^2 = \frac{\alpha 2^s}{4} \left( 1 - \frac{\alpha}{2} \right) \Gamma \left( 1 + \frac{\alpha}{2} \right) \left( \frac{1}{u_{scr}} \right)^{2 - \alpha} \, .
\]

Putting this together for the simple Kolmogorov model with \( \alpha = 5/3 \), we have the simple relation \( m_R = 0.459(\Delta \nu_d/v)^{0.167} \), plotted as the thick dashed lines in the screen plots in Figures 5 and 6.

For the general inner-scale model, with \( L_i \) nonzero, we show computed curves in Figure 6. We can recognize two regimes. Consider first the case of small \( L_i \). With sufficiently strong scattering, \( s_{o,scr} < L_i < r_{F,scr} \), \( m_R \propto (L_i/s_{o,scr})^{2-x} \). However, there is a complication in portraying this behavior in a plot versus \( \Delta \nu_d/v \) for fixed inner scale because of the variation of \( r_{F,scr} \) with distance, which is not specified by the horizontal variable \( \Delta \nu_d/v \). We deal with this by obtaining an approximate scaling of \( r_{F,scr} \) with \( \Delta \nu_d/v \). When \( s_{o,scr} < L_i \), equation (A3) relates \( SM \propto s_{o,scr} \), or \( SM r_{F,scr}^2 \propto u_{scr}^2 \). We now argue that, on average, \( SM \propto \) pulsar distance \( \propto r_{F,scr}^2 \), and eliminating \( SM \), we see that \( r_{F,scr} \propto u_{scr}^{0.5} \). With these scalings and \( s_{o,scr} < L_i < r_{F,scr} \), we see
\[
m_R^2 \propto (L_i/r_{F,scr} u_{scr})^{2-x} \propto (L_i/r_{F,scr} u_{scr}^{1.5})^{2-x} \\
\propto u_{scr}^{-0.5} \propto \left( \Delta \nu_d/v \right)^{0.25} \, .
\]

Here \( r_{F,ref} \) is the Fresnel scale for a “reference” pulsar, and the last version in equation (A8) comes from using the Kolmogorov exponent \( \alpha = 5/3 \), which gives the asymptotic slope of 1/8 for curves at the lower left of Figure 6. Now with \( L_i \) and \( r_{F,scr} \) fixed, let \( s_{o,scr} \) increase (i.e., less scattering) until \( L_i < s_{o,scr} < r_{F,scr} \). The inner scale is no longer important, and we get the same relation as for the simple Kolmogorov spectrum: \( m_R \propto u_{scr}^{2-x} \propto (\Delta \nu_d/v)^{1/3} \); this is visible where the curves steepen with increasing \( \Delta \nu_d/v \).

For larger inner scales, the curves in Figure 6 show a pronounced peak. These occur for inner scales greater than the Fresnel scale and will be associated with focusing and caustics. Since our treatment includes only the first-order term in the low wavenumber expansion, it is not reliable in the region of the peak. Goodman et al. (1987) have discussed caustics at length for the same inner-scale spectrum. They note that when \( s_{o,scr} < r_{F,scr} < L_i < s_R \), the scintillation power spectrum starts to fill in at scales intermediate between the diffractive and refractive wavenumbers. Their equation (2.57) gives an estimate of the variance in this extra term as
\[
m_{int}^2 \sim 2(L_i/s_R)^2 \ln (L_i/s_{o,scr}) \\
\sim 2(L_i/u_{scr} r_{F,scr})^2 \ln (L_i u_{scr} / r_{F,scr}) \, .
\]

With fixed \( L_i \), \( m_{int}^2 \) increases much more steeply with decreasing \( u_{scr} \) than does \( m_R^2 \). Thus, as \( u_{scr} \) decreases, there are fewer and fewer independent phase perturbations across the scattering disk. When \( u_{scr} r_{F,scr} \sim L_i \), focusing represented by \( m_{int}^2 \sim 4 \ln (L_i/r_{F,scr}) > 1 \) occurs. We note that the theoretical \( m_R \) versus \( \nu_d/v \) plots in Gupta et al. (1993) omit the focusing condition and are wrong for inner scales greater than the Fresnel scale. The effect of higher order terms in the low-wavenumber expansion has been studied by Dashen & Wang (1993). They obtain a more efficient expansion scheme that gives improved
accuracy near the peak in the scintillation index. Nevertheless, it seems that a reliable prediction for the behavior near the peak in scintillations requires numerical evaluation. This becomes even more necessary in treating an extended scattering medium.

We also note that the drop in $m_R$ as scattering gets weaker past the peak in Figure 6 is real. It represents the fact that when a square-law–structure function applies for scales from $s_0$, up to the scattering disk size, there is insufficient phase curvature and the scintillations remain weak even though $u_{s0} > 1$. In such circumstances the “scattering disk” is a misnomer, since an observer would see only a single angle of arrival that could wander over a region of scale $u_{s0} F,scr$.

Turning to the screen analysis of the $\beta = 4$ model, we relate $s_0$ to $SM$ using equation (21) from LR99, where the phase-structure function equals one. This gives the following analog of equation (A3) for the inner-scale model:

$$m_R^2 = \frac{8\pi^2 r_s^2 c^2 SM}{v^2} \frac{\Gamma(\infty)}{\sqrt{\eta + \kappa_s^2}} \int_0^\infty \frac{1}{(\eta + \kappa_s^2)^{\frac{1}{2}}} \sin^2 \left( -\frac{r_F,scr u_{scr}}{u_{scr}} \eta \right) d\eta .$$

(A10)

Substituting the $\beta = 4$ model for the density spectrum in equation (A1) and letting $\eta = \kappa_s^2$, we obtain

$$m_R^2 = \frac{8\pi^2 r_s^2 c^2 SM}{v^2} \frac{\Gamma(\infty)}{\sqrt{\eta + \kappa_s^2}} \int_0^\infty \frac{1}{(\eta + \kappa_s^2)^{\frac{1}{2}}} \exp \left[ -\left( r_F,scr u_{scr} \right)^2 \eta \right] \sin^2 \left( \frac{r_F,scr \eta}{2} \right) d\eta .$$

(A11)

To evaluate this integral, we let $g(\eta)$ represent the integrand, and we let $g_1(\eta)$ be the same as $g(\eta)$ but with $\kappa_s$ set equal to zero. We can then rewrite the integral in equation (A11) as

$$m_R^2 = 8 \left[ s^2_{0,scr} \ln \left[ 1 + \left( 2L_w/s^2_{0,scr} \right)^2 \right] \right]^{-1} \left\{ \int_0^\infty g_1(\eta) d\eta - \int_0^\infty \left[ g_1(\eta) - g(\eta) \right] d\eta \right\} .$$

(A12)

The first integral can be evaluated analytically (see, e.g., Gradshteyn & Ryzhnik 1965). In strong scattering, the exponential term cuts off the oscillations of the sine term, which can be approximated by its argument, and the $g_1(\eta) - g(\eta)$ becomes negligible for values of $\eta$ larger than $\kappa_s^2$. With these approximations, we can also do the second integral. Putting these together, we obtain

$$m_R^2 = \left[ \ln \left[ 1 + \left( 2L_w/s^2_{0,scr} \right)^2 \right] \right]^{-1} \left\{ 4u_{scr} \tan^{-1} \left( u_{scr} \right) - 2u_{scr}^2 \ln \left( 1 + u_{scr}^{-2} \right) - 2\zeta(2 + \zeta) E_1(\zeta) \right\} ,$$

(A13)

where $E_1$ is the exponential integral, $\zeta = (s_0 \kappa_s^2)^{u_{scr}} = (s_R \kappa_s^2)^{u_{scr}}$. In the latter form, $m_R^2 = r_F,scr u_{scr}$ is the refractive scale (equal to the scattering disk radius). Again, $m_R$ is related to $\Delta v_F/\nu$ through $\Delta v_F/\nu = v_F/u_{scr}$. The resulting curves are shown in the screen panels of Figure 5. Consider the asymptotic behavior for $m_R^2$ in equation (A13) as a function of $u_{scr}$. As the strength of scattering $u_{scr}$ increases, $\Delta v_F$ decreases ($\propto u_{scr}^{-2}$). In equation (A11), the exponential term cuts off the integral at $1/s_R^2$ before the oscillations of the $\sin^2$ Fresnel filter, which then approximates $\eta^2 F,scr/4$. If also $\zeta = (s_R \kappa_s^2)^{u_{scr}} \ll 1$, we can ignore $\kappa_s^2$ in the denominator and the remaining $\eta^2$ cancels the $\eta^2$ from the Fresnel filter, and the integral depends only on $1/s_R$. In this approximation, $m_R^2$ is then simply proportional to the slowly varying logarithmic term, which explains the relatively flat part of the curves in Figure 5; under these conditions, in equation (A13) the first two terms in the curly brackets sum to 2 and the last term is negligible. With $L_w$ fixed, now let $u_{scr}$ increase, making $s_R$ increase. Eventually $\zeta$ becomes greater than one when the scattering disk becomes greater than the outer scale. At this point, the exponential term cuts off the integral below $\kappa_s^2$, where the spectrum flattens. As $u_{scr}$ increases even further, the integral decreases steeply, causing the downturn at very small $\Delta v_F/\nu$. We again note that our expressions rely on the first order of an expansion and will not be reliable near the peak in the scintillation index. However, there is not the same focusing condition that applied for very large inner scales.

**A2. EXTENDED SCATTERING MEDIUM**

In order to obtain expressions for $m_R$ for the two spectrum models in the *uniform* extended medium geometry, we must complete the line-of-sight integrals in equation (10) in addition to following the steps used in the screen geometry. For each distance $x = z/L$ in the line of sight, there is also an integration over variable $y$ in the exponential cutoff. If $D(s) \propto s^2$, this $y$-integration yields $L_D \left[ k r_{F,L} x (1 - x) \right] / (\alpha + 1)$, where $r_{F,L} = \sqrt{L_D / (2\pi v)}$. This again provides a low-pass cutoff at the reciprocal of the radius of the effective scattering disk, where $\kappa \sim (u_{F,L} s_R)^{-1}$. We define the scattering strength by $u_S = r_{F,L} / s_0$, with the field coherence scale $s_0$, as in LR99, defined where the spherical wave-structure function equals unity, measured in the observing plane. For the other spectrum models there is not such a simple relation for the $y$-integration, but there is still an effective cutoff given by a similar equation. When the $x$-integration is completed, the effective Fresnel scale is actually smaller than $r_{F,L}$ because of averaging over the $\sqrt{x(1 - x)}$.

In analogy with the screen geometry, we make use of identities similar to those given by equations (A3) and (A10). For the extended medium, these identities are derived, in turn, from the wave-structure functions for the inner-scale and $\beta = 4$ models (cf. LR99). The identities obtained thus are similar for the screen geometry, except that for the inner-scale model, (1) there will be a factor of 3 on the right side of equation (A3), and (2) the 1 in square brackets is replaced by $\left[ 3(1 + \alpha) \right]^{1/2} \approx 1.8$ for the Kolmogorov exponent. For the $\beta = 4$ model, the only change will be a factor of 3 on the right side of equation (A10). We use all of the aforementioned identities for the inner-scale and $\beta = 4$ models and compute the $x$-integral numerically since it cannot be carried out analytically.

The shapes of the curves bear a close relationship to the screen results, although the extended medium values generally lie above the associated screen values at the same $\Delta v_F/\nu$. 
APPENDIX B  

DIFFRACTIVE INTENSITY CORRELATION FUNCTION  

The second moment of intensity is needed to describe the fluctuations of intensity. Under strong scintillation conditions, separate forms can be used for refractive and diffractive fluctuations since their spatial scales differ by several orders of magnitude. In LR99, as elsewhere in the ISS literature, the correlation of diffractive scintillations is approximated by the squared magnitude of the second moment of the field, leading to the simple result that the spatial scale of the diffractive scintillations is equal to the scale where the phase-structure function equals unity ($s_0$). However, Goodman & Narayan (1985) showed that for steep spectra ($\beta > 4$) this is no longer the case and the diffractive scale can be larger than $s_0$. Here we examine this question for the $\beta = 4$ spectrum. We give the details for a phase screen with plane wave source, which are readily generalized to a spherical wave source.

The two-frequency intensity cross-spectrum at wavenumber $\kappa$ for a screen at distance $L$ is given by the Fourier-like integral equation (17) of CCFFH. This depends on the combination of structure functions $V_4$, which for a plasma screen can be written as

$$V_4 = \frac{k_m^2}{k_1^2} D_\phi \left( \kappa \frac{L}{k_1} \right) + \frac{k_m^2}{k_2^2} D_\phi \left( \kappa \frac{L}{k_2} \right) - D_\phi \left( \frac{\kappa L}{k_o} + \beta' \right)$$

$$- D_\phi \left( \frac{\kappa L}{k_o} - \beta' \right) + D_\phi \left( \beta' + \frac{\kappa L}{k_o} \right) + D_\phi \left( \beta' - \frac{\kappa L}{k_o} \right),$$

where $k_1$ and $k_2$ are the two radio wavenumbers, $k_m$ is their geometric mean, $\bar{k}$ is their arithmetic mean, $k_o = k_m/\bar{k}$, $\epsilon = |k_1 - k_2|/2\bar{k}$, and $D_\phi$ is evaluated at $k_o$; $\beta'$ is a spatial offset that is the variable of integration.

Consider first the single frequency case $k_1 = k_2(\epsilon = 0)$. In the limit of very large $\kappa$, the first four structure functions saturate and sum to zero. The last two are equal and $V_4 \approx 2D_\phi(\beta')$, which gives the simple diffractive limit mentioned above. This is the zero-order term of an expansion, which is obtained in terms of the sum of the first four terms as a small quantity. The zero-order result requires full saturation, which requires $\kappa L/k_o \gtrsim L_o$. In diffractive scintillation $\kappa \sim 1/s_0$; hence, the condition becomes that the refractive scale $s_r = L/(k_m s_0) \gtrsim L_o$. Our concern here is to consider what happens when the diffractive $\kappa$ is not large enough for saturation of $D_\phi$. For shallow density spectra ($\beta < 4$), small argument approximations to the structure function follow an exponent $\beta - 2 < 2$, and the zero-order term gives a good approximation even when $s_r < L_o$. However, for steep spectra, Goodman & Narayan (1985) showed that the leading term in the structure function follows a square law, which exactly cancels in the result is that the high wavenumber limit depends on the next term in the structure function expansion, which yields a diffractive scale greater than the scale $s_0$ (defined by the square-law term).

Now we consider the case for the $\beta = 4$ model, when the scattering disk $s_R$ is smaller than the outer scale $L_o$. Here we can approximate equation (8) by

$$D_\phi(s) = \frac{s^2}{s_0^2} - \frac{s^2 \ln(s^2/s_0^2)}{s_0^2 \ln(4/s_0^2 \kappa_2^2)}.$$  

(B2)

As for the steep spectra, the leading term in the structure function follows a square law, which cancels when substituted into equation (B1). $V_4$ can then be approximated for large $\kappa$ by expanding in $\beta'k_m/(\kappa L)$. The result is

$$V_4 \approx \frac{2\beta'^2[\ln(u^4) + 1 + 2 \cos^2 \theta - \ln(\beta'^2/s_0^4)]}{s_0^2 \ln(4/s_0^2 \kappa_2^2)}.$$  

Here $u$ is the strength of scattering defined in equation (9); $\theta$ is the angle between vectors $\kappa$ and $\beta'$. We note that for a Kolmogorov spectrum in the high-wavenumber limit, $V_4$ also includes terms in $\cos^2 \theta$, which would be accounted for in the higher order terms of the expansion. A result similar to equation (B3) is given by Dashen & Wang (1993), though it is for a one-dimensional phase screen. In considering the spectrum of intensity fluctuations (eq. 17 of CCFFH), one can show that the dominant wavenumber is approximately $\kappa \sim \beta'^{-1}$. With these substitutions in $V_4$, which we then set $\beta' = 0$, we solve for $s_R$; this gives an approximate equation for the diffractive spatial scale $s_d$:

$$s_d^2 \sim \frac{s_0^2}{\ln(4/s_0^2 \kappa_2^2)} \ln(u^4 + 2).$$  

(B4)

Under the condition assumed in this approximation, $s_R \ll L_o$, we find $s_d > s_0$. However, in practice the ratio $s_d/s_0$ never becomes large. With a large outer scale, say $L_o = 3$ pc, and typical observing conditions $s_0 \sim 10^8$ m and $u \sim 100$, we find $s_d \lesssim 1.7s_0$. Thus, the diffractive scale could be 70% greater than $s_0$ and would slowly approach $s_0$ for smaller outer scales.

Turning to the two-frequency intensity correlation ($0 < \epsilon < 1$) in the diffractive limit of large $\kappa$, the results of CCFFH still apply. Namely, the last two terms of equation (B1) largely control the decorrelation versus frequency. They group the remaining terms into a filter that depends only on $\kappa$ and a smaller term that becomes the basis of the expansion. The filter term was discussed by LR99 and shown to be important only as the strength of scattering decreases. It is the last two terms in $V_4$ that determine the zero-order result, so we looked at the effect of the higher order terms. The quantity we are ultimately concerned with is the cross-correlation of intensity at offset frequencies at the same observing point. This comes from the integral of the cross-spectrum. Equations (31) through (34) of CCFFH give the zero- and first-order terms of the cross
spectrum in terms of the spectrum of refractive index fluctuations in the layer. For the $\beta = 4$ spectrum we reduced these to a sum of confluent hypergeometric functions, which can be explicitly computed. For a sample observing condition we found that the higher order terms for the cross-spectrum itself were significant compared to the zero-order term; however, when integrated to give intensity cross-correlation, they had only a minor effect on the decorrelation bandwidth itself ($\leq 5\%$ increase). The reason for this appears to be the dominant effect of the last two terms in the $\chi_4$ summation with unequal frequencies.

To summarize, we find a modest (logarithmic) increase in the diffractive scale relative to the field coherence scale but this remains less than a factor of 1.7 for the likely ISS parameters. This is accompanied by a smaller increase in the decorrelation bandwidth relative to the calculations of LR99, which relied on the normal zero-order expansion at high wavenumbers. This small offset in the decorrelation bandwidth is negligible compared with the measurement errors for the observations under consideration. We assume that the conclusions reached here for a screen would also apply for an extended scattering medium.

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