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Abstract. We extend the class of initial conditions for scalar delayed reaction-diffusion equations
\( u_t(t, x) = u_{xx}(t, x) + f(u(t, x), u(t - h, x)) \) which evolve in solutions converging to monostable traveling waves. Our approach allows to compute, in the moving reference frame, the phase distortion \( \alpha \) of the limiting travelling wave with respect to the position of solution at the initial moment \( t = 0 \). In general, \( \alpha \neq 0 \) for the Mackey-Glass type diffusive equation. Nevertheless, \( \alpha = 0 \) for the KPP-Fisher delayed equation: the related theorem also improves existing stability conditions for this model.

1. Introduction: main results and applications

The previous studies (e.g. see [2, 3, 10, 14]) show that both minimal and non-minimal positive traveling waves \( u(t, x) = \phi(x + ct) \) for the monostable delayed reaction-diffusion equation

\[
(1.1) \quad u_t(t, x) = u_{xx}(t, x) + f(u(t, x), u(t - h, x)), \quad t > 0, \ x \in \mathbb{R},
\]

attract solutions \( u(t, x) \) whose initial segments \( u_0(s, x) \) have the same leading asymptotic terms at \( x = -\infty \) as the shifted wave \( \phi(x + cs) \), for all \( s \in [-h, 0] \). The latter assumption implies that, for some positive \( A_0 \),

\[
(1.2) \quad \lim_{x \to -\infty} \frac{u_0(s, x)}{\phi(x + cs)} = A_0, \quad s \in [-h, 0].
\]

This observation concerns so-called pulled waves for equation (1.1) and smooth traveling waves for delayed degenerate reaction-diffusion equations [8]. The pushed and bistable waves have better stability properties [11, 12] and they are not considered in this work.

Condition (1.2) seems to be excessively restrictive: for example, it excludes initial segments asymptotically similar, in the spirit of (1.2), to \( \phi(x + \alpha(s)) \), \( s \in [-h, 0] \), with nonlinear shift \( \alpha(s) \). This circumstance is irrelevant for the non-delayed equations when \( h = 0 \), however, in the delayed case it restricts severely the range of possible applications. Analysing this problem, in [11 Corollary 1] we have...
shown, under a quasi-monotonicity condition on $f$, that the existence of the limit
\begin{equation}
\lim_{x \to -\infty} \frac{u_0(s,x)}{\phi(x+cs)} = A_0(s) > 0, \quad s \in [-h,0],
\end{equation}
with some continuous function $A_0(s)$ implies that solution $u(t,x)$ evolves in the middle of two shifted traveling waves constituting the lower bound $u_-(t,x) = \phi(x+ct+a_+)$, and the upper bound $u_+(t,x) = \phi(x+ct+a_-)$. Condition (1.3) is easily verifiable. Indeed, it is well known that under some natural restrictions (tacitly assumed in this work) so-called non-critical waves have the following asymptotic representation after an appropriate translation of the time variable:
\begin{equation}
\begin{align*}
\phi(t) &= e^{\lambda_1 t} + e^{(\lambda_1 + \sigma) t} r_1(t), \quad \lambda_1 + \sigma < \lambda_2, \\
\phi'(t) &= \lambda_1 e^{\lambda_1 t} + e^{(\lambda_1 + \sigma) t} r_2(t), \quad t \in \mathbb{R}.
\end{align*}
\end{equation}
Here $\sigma$ is a positive number, $r_1$, $r_2$ are smooth bounded functions and $0 < \lambda_1 < \lambda_2$ are zeros of the characteristic function $\chi_0(z) = z^2 - cz + f_1(0,0) + f_2(0,0)e^{-z h}$. In the paper, $f_j(u,v)$ denotes the partial derivative of $f$ with respect to $j$-th argument. We will assume that $f_j(u,v)$ are locally Lipschitz continuous functions.

A potential possibility that solution $u(t,x)$ can develop non-decaying oscillations between the waves $u_+(t,x)$ and $u_-(t,x)$ was not discarded in [11]. Another question left open in [11] is whether such non-critical waves have the following asymptotic behavior.

Actually, assuming (1.3), we prove that the solution $u(t,x)$ converges to a shifted wave $\phi(x+ct+a_*)$, where $a_*$ is completely determined by the function $A_0(s)$:
\begin{equation}
a_* = \frac{1}{\lambda_1} \ln A_\infty, \quad \text{where } A_\infty := \frac{A_0(0) + q \int_{-h}^{0} A_0(s) ds}{1 + q h}, \quad q := \frac{f_2(0,0)e^{-\lambda_1 h}}{\lambda_1}.
\end{equation}
We obtain $A_\infty$ as the limit value at $+\infty$ of the solution $A(t)$, $t \geq 0$, to the initial value problem $A(t) = A_0(s) > 0, s \in [-h,0]$, for the monotone scalar delay differential equation
\begin{equation}
A'(t) = q (A(t) - h) - A(t), \quad t \geq 0.
\end{equation}
Indeed, it is clear that $A(t) > 0$ for all $t \geq -h$. Since the characteristic equation $z + q = q e^{-z h}$ for equation (1.6) with $f_2(0,0) > 0$ has a unique simple real root $z = 0$, other (complex) roots $z_j$ satisfying the inequality $\Re z_j < 0$ (see Appendix), there are real numbers $A_\infty \geq 0$ and $d < 0$ (cf. [11] Theorem 3.2) such that
\begin{equation}
|A(t) - A_\infty| \leq e^{dt}, \quad t \geq 0.
\end{equation}
By integrating (1.6) on $]0, +\infty[$, we find that
\begin{equation}
A_\infty(1 + q h) = A_0(0) + q \int_{-h}^{0} A_0(s) ds > 0.
\end{equation}
Now, (1.3), (1.4) imply that the initial function $u(s,x)$ evaluated at the moment $s = 0$ behaves as $\phi(x+a_0)$, where $a_0 = \ln A_0(0)/\lambda_1$. Therefore the total traveled distance $\delta_a$ between the initial (at the moment $t = 0$) and final (as $t \to +\infty$) positions of the solution in the moving reference frame is
\begin{equation}
\delta_a = a_* - a_0 = \frac{1}{\lambda_1} \ln \frac{1 + q \int_{-h}^{0} A_0(s)/A_0(0) ds}{1 + q h}.
\end{equation}
Note that the function $A(t)$ and $\delta_a$ are completely determined by the speed $c$, the initial values $A_0(s)$ and the partial derivatives $f_1(0,0), f_2(0,0)$. They do not depend on other characteristics of solution $u(t,x)$ and wavefront $\phi(x+ct)$, including their bounds $M_1 \leq M_3 \in \mathbb{R} \cup \{+\infty\}$, $M_2 \leq 0$,

$$0 \leq \phi(x) \leq M_1, \quad M_2 \leq u(t,x) \leq M_3, \quad (t,x) \in [-h, +\infty) \times \mathbb{R},$$

and associated parameters $L_2 \geq f_2(0,0) \geq 0$ and $D \in \mathbb{R}$ chosen to satisfy

$$|f(w,v_1) - f(w,v_2)| \leq L_2 |v_1 - v_2|, \quad (w,v_1,v_2) \in [0,M_1] \times [M_2,M_3]^2,$$

$$D = \inf_{(w_1,w_2,v) \in [M_2,M_3]^3, w_1 \neq w_2} \frac{f(w_1,v) - f(w_2,v)}{w_2 - w_1}.$$

**Remark 1.1.** Clearly, $D = 1$ for the Mackey-Glass type nonlinearity $f(w,v) = -w + b(v)$. Considering monotone wavefronts for the KPP-Fisher delayed equation \[2 \ 3 \ 4 \ 5 \ 6\], when $f(w,v) = w(1-v)$, we find that $L_2 = M_1 = 1$, $M_3 = +\infty$. In the general case of non-monotone waves for the latter equation, we can take $L_2 = M_1 = e^{eh}, M_3 = +\infty$, cf. \[2\]. In both cases (monotone and non-monotone), we have that $D = \inf_{(t,x) \in [0, +\infty) \times \mathbb{R}} u(t,x) - 1$. Hence, if $u_0 \geq 0 = M_2$ then $D = -1$.

First, we consider an easier situation when $f_2(0,0) > 0$.

**Theorem 1.2.** Assume that $f_2(0,0) > 0$ and

$$\gamma^2 - c\lambda - D - \gamma + L_2 e^{-\lambda h} e^{-\gamma h} < 0,$$

for some $\lambda \in (\lambda_1, \min\{2\lambda_1, \lambda_2\})$ and $\gamma \in (d,0)$. If, in addition, $u_0(s,x)$ verifies

$$|u_0(s,x) - \phi(x+c(s+\alpha_0(s)))| \leq K e^{\lambda x}, \quad (s,x) \in [-h,0] \times \mathbb{R},$$

then, for some $K' \geq K$, solution $u(t,x)$ of (1.1) with the initial function $u_0$ satisfies

$$\sup_{x \in \mathbb{R}} \left(e^{-\lambda x} |u(t,x - ct - \alpha(t)) - \phi(x)|\right) \leq K' e^{\gamma t}, \quad t \geq -h.$$

Here $A(t) = e^{\lambda \alpha(t)}$ solves (1.6) with the initial datum $A_0(s) = e^{\lambda_1 \alpha_0(s)}, s \in [-h,0]$ so that $\alpha(+\infty) = \alpha_0 \in \min\{\alpha_0\}_{s=-h}^{\alpha_0\in\mathbb{R}}$, $\max\{\alpha_0\}_{s=-h}^{\alpha_0\in\mathbb{R}}$ is given by (1.3). Finally, $\delta_a = 0$ if and only if $A(0) = (1/h) \int_{-h}^{0} A_0(s) ds$.

Next, we consider the ‘degenerate’ situation when $f_2(0,0) = 0$. From (1.6), we can expect that $\alpha(t) \equiv \alpha(0)$ for $t \geq 0$. Below, we prove that this is indeed the case for a class of the KPP-Fisher type nonlinearities.

**Theorem 1.3.** Assume that $f(u,v) = g(u)(\kappa - v)$ with $\kappa > 0$, $g(0) = 0$, that (1.8) holds for some $\lambda \in (\lambda_1, \lambda_2)$ and $\gamma < 0$, and that $u_0(s,x)$ satisfies, for some $\lambda_1^* > \lambda_1$,

$$|u_0(s,x) - \phi(x + cs + \alpha_0(s))| \leq K e^{\lambda_1^* x}, \quad (s,x) \in [-h,0] \times \mathbb{R}.$$

Set $\lambda_* = \min\{\lambda^*, \lambda, 2\lambda_1\}$. If $\lambda_*^2 - c\lambda_* - D - \gamma < 0$, then, for some $K' \geq K$, solution $u(t,x)$ of equation (1.1) with the initial function $u_0(s,x)$ satisfies

$$\sup_{x \in \mathbb{R}} \left(e^{-\lambda_* x} |u(t,x - ct - \alpha(0)) - \phi(x)|\right) \leq K' e^{\gamma t}, \quad t \geq -h.$$
Corollary 1.4. Let propagation in the model. In this way, we obtain the following conclusion: equation. Denote by \( f \) has the reaction term satisfying the equality \( c > c^\ast \) and Remark 1.1, in the general case of non-monotone waves we have to consider is bounded once its initial fragment \( u \) for some \( \gamma_0 \). Then there exist \( \mu \in (\lambda, \lambda_1) \), \( \gamma < 0 \) and \( Q \geq K \) such that \( |u(t, x) - \phi(x + ct + \alpha(t))| \leq Q e^{\mu(x+ct)} e^{\gamma t} \), \( x \in \mathbb{R}, \ t \geq -h \).

The function \( \alpha(t) \) is converging at \( +\infty \) and generically develops ‘rapid’ oscillations around its limiting value \( \alpha(+\infty) \).

Other aforementioned model, the KPP-Fisher delayed equation \( u_t(t, x) = u_{xx}(t, x) + u(t, x)(1 - u(t - h, x)) \), \( u = u_0(s, x), s \in [-h, 0], x \in \mathbb{R}, \) has the reaction term satisfying the equality \( f_2(0, 0) = 0 \). In view of Theorem 1.3 and Remark 1.1 in the general case of non-monotone waves we have to consider oscillations around \( \alpha(+\infty) \) (these oscillations can be significant when \( q \) is relatively large, see Figure 1). More precisely, generically \( \alpha(t) \) crosses two times the level \( \alpha(+\infty) \) on each half-open interval of the length \( h \). Indeed, an application of the Laplace transform to (1.6) yields the following representation

\[
A(t) = A_\infty + 2 \text{Re}(A_1 e^{z_1 t})(1 + o(1)),
\]

where \( A_1(1 + h(z_1 + q)) = A_0(0) + q e^{-z_1 h} \int_{-h}^{0} e^{-z_1 s} A_0(s) ds \)

with \( z_1 = x_1 + iy_1, y_1 h \in (\pi, 2\pi), \) being the leading complex eigenvalue of (1.6).

In particular, \( \alpha(t) \) is typically oscillating in the case of Nicholson’s diffusive equation \[3, 10, 11, 14\]

(1.13) \( u_t(t, x) = u_{xx}(t, x) - u(t, x) + b(u(t - h, x)), \ x \in \mathbb{R}, \ b(u) = pue^{-u}, \ p > 1, \)

In such a case, \( L_2 = b'(0) = p, \ D = 1, \) and the solution \( u(t, x), t \geq 0, x \in \mathbb{R}, \)

is bounded once its initial fragment \( u_0(s, x), s \in [-h, 0], x \in \mathbb{R}, \) is bounded. In addition, the formulae (1.4) hold for each \( c > c_\ast \), where \( c_\ast \) is the minimal speed of propagation in the model. In this way, we obtain the following conclusion:

**Corollary 1.4.** Let \( u = \phi(x + ct) \) be a non-critical wave for the Nicholson’s diffusive equation. Denote by \( u(t, x) \) solution of the initial problem \( u(s, x) = u_0(s, x), s \in [-h, 0], \) for (1.15) where non-negative function \( u_0 \) satisfies

(1.14) \( |u_0(s, x) - \phi(x + cs + \alpha_0(s))| \leq Ke^{\lambda x}, \ (s, x) \in [-h, 0] \times \mathbb{R}, \)

for some \( \lambda > \lambda_1. \) Then there exist \( \mu \in (\lambda, \lambda_1), \gamma < 0 \) and \( Q \geq K \) such that

\( |u(t, x) - \phi(x + ct + \alpha(t))| \leq Q e^{\mu(x+ct)} e^{\gamma t}, \ x \in \mathbb{R}, \ t \geq -h. \)

Figure 1. On the left: particular solution of (1.6) with \( h = 1, q = 19, A_0(s) = -s. \) Horizontal line is the limit value \( A_\infty = 19/40. \)

On the right, the graph of \( c = c^\ast(h) \) from Corollary 1.5.
Since where with We have that $Ce_\gamma(x) > 0$, $c_\#(0) = 2\sqrt{2}$, $c_\#(+\infty) = 2$, $D = \{(h, c) : \lambda^2 - c\lambda + 1 + e^{-\lambda ch + ch} < 0 \text{ for some } \lambda \} = \{(h, c) : c > c_\#(h), h \geq 0 \}$, where $c = c_\#(h)$, $h \geq 0$, is defined implicitly by

$$-2 + \sqrt{c^2h^2 - 4c^2h^2 + 4 - c^2h^2} \exp\left( c h \left( 1 - \frac{c}{2} + \frac{1}{ch} - \sqrt{\frac{c^2}{4} + \frac{1}{c^2h^2} - 1} \right) \right) = 0.$$ 

Then Theorem 1.3 yields the following conclusion.

**Corollary 1.5.** Let $u = \phi(x + ct)$ be a traveling wave for KPP-Fisher delayed equation (1.15) where $(h, c) \in D$. Denote by $u(t, x)$ solution of the initial problem (1.15) where non-negative function $u_0$, satisfies, for some $\lambda^* > \lambda_1$, $\mu(x, t) = (1.5)$ and (1.7), respectively. Then Lemma 2.1.

$$\{|u_0(s, x) - \phi(x + cs + \alpha_0(s))| \leq Ke^{\lambda^*x}, (s, x) \in [-h, 0] \times \mathbb{R}. \]$$

Then there exist $\mu \in (\lambda_1, 2\lambda_1)$, $\gamma < 0$ and $Q \geq K$ such that

$$|u(t, x) - \phi(x + ct + \alpha_0(0))| \leq Qe^{\mu(x+ct)}e^{\gamma t}, \quad x \in \mathbb{R}, \quad t \geq 0.$$ 

In this way, on the base of an alternative approach, Theorem 1.3 and Corollary 1.5 improve the stability result [2] Theorem 3 in the following two aspects: a) in Corollary 1.5, the initial phase function $\alpha_0(s)$, $s \in [-h, 0]$ is not necessarily constant; b) even if all mentioned results use the same domain for the admissible parameters $(h, c)$, Theorem 3 assumes additionally that the exponent $\lambda^*$ in (1.10) should be larger than some minimal value, specific for each pair $(h, c)$. Observe that for the delayed KPP-Fisher equation it is still not clear whether a) the domain of all admissible parameters can be extended to the quarter-plane $c \geq 2, h \geq 0$; b) the estimate (1.17) with the bounded weight $\min\{e^{\mu x}, 1\}$ is true.

2. **Proof of Theorem 1.2**

The estimation of the auxiliary function

$$P = f(\phi(x+\alpha(t)), \phi(x-ch+\alpha(t))) - f(\phi(x+\alpha(t)), \phi(x-ch+\alpha(t-h))) + \alpha(t)\phi'(x+\alpha(t))$$

is instrumental for proving our first main result.

**Lemma 2.1.** Assume all conditions of Theorem 1.2. Let $q$ and $d$ be defined by (1.2) and (1.7), respectively. Then $|P(t, x)| \leq q_0 e^{\lambda x} e^{dt}$ for some $q_0 \geq 0$ and all $x \in \mathbb{R}, t \geq 0$.

**Proof.** We have that

$$P(t, x) = (\alpha'(t)\phi'(x+\alpha(t)) + f_2(0, 0)[\phi(x-ch+\alpha(t)) - \phi(x-ch+\alpha(t-h))] + \rho[\phi(x-ch+\alpha(t)) - \phi(x-ch+\alpha(t-h))] =: P_1 + P_2,$$

where

$$\rho = f_2(\phi(x+\alpha(t)), \theta(x, t)) - f_2(0, 0)$$

with $\theta(x, t)$ being some point between $\phi(x-ch+\alpha(t))$ and $\phi(x-ch+\alpha(t-h))$. Since $|f_2(u, v) - f_2(0, 0)| \leq C(|u| + |v|)$ on the bounded subset $[0, M_1]^2 \subset \mathbb{R}_+^2$ (in this proof, we are using $C$ as a generic positive constant), we conclude that

$$|P_2| = |\rho[\phi(x-ch+\alpha(t)) - \phi(x-ch+\alpha(t-h))]| \leq Ce^{\lambda x} |\phi(x-ch+\alpha(t)) - \phi(x-ch+\alpha(t-h))| \leq Ce^{\lambda x} e^{\lambda x} e^{dt} \leq Ce^{\lambda x + dt}, \quad x \in \mathbb{R}, \quad t \geq 0.$$
Next, consider \( B(z) = \phi(z) - e^{\lambda_1 z} \), clearly \( B(z) = O(e^{\lambda_2 z}) \) at \( z = -\infty \). Then (1.4) and (1.6) imply that

\[
P_2 = f_2(0,0)[\phi(x - ch + \alpha(t)) - \phi(x - ch + \alpha(t-h))] + \alpha'(t)\phi'(x + \alpha(t)) = e^{\lambda_1 x}[f_2(0,0)(e^{\lambda_1 (\alpha(t)-ch)} - e^{\lambda_1 (\alpha(t-h)-ch)}) + \lambda_1 \alpha'(t)e^{\lambda_1 \alpha(t)}] + f_2(0,0)[B(x - ch + \alpha(t)) - B(x - ch + \alpha(t-h))] + \alpha'(t)r_2(x + \alpha(t))e^{(\lambda_1 + \sigma)(x + \alpha(t))} = f_2(0,0)[B(x - ch + \alpha(t)) - B(x - ch + \alpha(t-h))] + \alpha'(t)r_2(x + \alpha(t))e^{(\lambda_1 + \sigma)(x + \alpha(t))}.
\]

Next, we have that

\[
|\alpha(t) - \alpha(t-h)| = \lambda_1^{-1} \left| \ln \frac{A(t)}{A(t-h)} \right| \leq C|A(t) - A(t-h)| \leq Ce^{dt}, t \geq 0.
\]

As a consequence,

\[
|B(x - ch + \alpha(t)) - B(x - ch + \alpha(t-h))| \leq Ce^{\lambda_3 dt}, \quad x \in \mathbb{R}, \ t \geq 0.
\]

In this way, since \( \alpha'(t) = \lambda_1^{-1} A'(t)/A(t) = O(e^{dt}), \ t \to +\infty \), we find that \( |P_2| \leq Ce^{\lambda_3 dt}, \ x \in \mathbb{R}, \ t \geq 0 \). The obtained estimates for \( |P_1| \) and \( |P_2| \) show that, for some positive constant \( q_0 \),

\[
|P(t,x)| \leq q_0 e^{\lambda_3 dt}, \ x \in \mathbb{R}, \ t > 0.
\]

This completes the proof of Lemma 2.1. \( \square \)

**Proof of Theorem 1.2.** Set \( v(t, x) = u(t, x - ct) \). Then the problem (1.4), (1.6) takes the form

\[
0 = v_{xx}(t, x) - cv_x(t, x) - v_1(t, x) + f(v(t, x), v(t-h, x-ch)), \quad t > 0, \ x \in \mathbb{R},
\]

\[
|v_0(s, x) - \phi(x + \alpha_0(s))| \leq K e^{\lambda x}, \quad (s, x) \in [-h, 0] \times \mathbb{R}.
\]

Take \( q_0 \) as in Lemma 2.1 and let \( Q \geq K \) be sufficiently large to satisfy

\[
-\lambda^2 + c\lambda + D - L_2 e^{-\lambda ch} e^{-\gamma h} - \frac{q_0}{Q} + \gamma > 0.
\]

For \( \phi(x + \alpha(t)) \neq v(t,x) \), set

\[
d(t, x) := \frac{f(\phi(x + \alpha(t)), v(t-h, x-ch)) - f(v(t, x), v(t-h, x-ch))}{\phi(x + \alpha(t)) - v(t, x)},
\]

and for \( \phi(x + \alpha(t)) = v(t, x) \), set \( d(t, x) := f_1(\phi(x + \alpha(t)), v(t-h, x-ch)) \).

Then consider the linear differential operator

\[
\mathcal{L}v = v_{xx} - cv_x + d(t, x)v - v_t
\]

and the functions

\[
\delta_{\pm}(t, x) = \pm[v(t, x) - \phi(x + \alpha(t))] - Q e^{\gamma t} e^{\lambda x}.
\]

By our assumptions \( \delta_{\pm}(t, x) \leq 0 \) for \( (t, x) \in [-h,0] \times \mathbb{R} \). Let \( \Pi = [-h, T] \times \mathbb{R} \), \( T \in \mathbb{R}_+ \cup \{+\infty\} \), be the maximal strip where \( \delta_{\pm}(t, x) \leq 0 \). Clearly, inequality (1.10) is satisfied for all \( (t, x) \in \Pi \). Theorem 1.2 will be proved if we establish that \( T = +\infty \). Suppose for a moment that \( T \) is finite. Then we find that, for all \( t \in [T, T+h], \ x \in \mathbb{R} \),

\[
(\mathcal{L} \delta_{\pm})(t, x) = \{ (\mathcal{L} \delta_{\pm})(t, x) \} - Q (e^{\gamma t} e^{\lambda x})(t, x)
\]

\[
= \pm \left\{ f(\phi(x + \alpha(t)), \phi(x - ch + \alpha(t))) - f(v(t, x), v(t-h, x-ch)) + \alpha'(t)\phi'(x + \alpha(t))
\right.
\]

\[
\left. - d(t, x)[\phi(x + \alpha(t)) - v(t, x)] \right\} + Q e^{\lambda x} e^{\gamma t} [\lambda^2 + c\lambda - d(t, x) + \gamma] =
\]

\[
= \pm \left\{ f(\phi(x + \alpha(t)), \phi(x - ch + \alpha(t))) - f(v(t, x), v(t-h, x-ch)) + \alpha'(t)\phi'(x + \alpha(t))
\right.
\]

\[
\left. - d(t, x)[\phi(x + \alpha(t)) - v(t, x)] \right\} + Q e^{\lambda x} e^{\gamma t} [\lambda^2 + c\lambda - d(t, x) + \gamma] =
\]
±\{f(φ(x+α(t)), φ(x-ch+α(t)))−f(φ(x+α(t)), φ(x-ch+α(t−h)))\} + α′(t)φ′(x+α(t))
±\{f(φ(x+α(t)), φ(x-ch+α(t−h)))−f(v(t, x), v(t−h, x−ch))−d(t, x)[φ(x+α(t))−v(t, x)]\}
+Qe^{λx}e^{γt}[−λ^2 + cλ − d(t, x) + γ] ≥
−q_0 e^{λx}e^{γt} ± \{f(φ(x+α(t)), φ(x-ch+α(t−h)))−f(φ(x+α(t)), v(t−h, x−ch))\} +
Qe^{λx}e^{γt}[−λ^2 + cλ − d(t, x) + γ] ≥
−q_0 e^{λx}e^{γt} − L_2 [φ(x-ch+α(t−h))−v(t−h, x−ch)]+ Qe^{λx}e^{γt}[−λ^2 + cλ − d(t, x) + γ] =
Qe^{λx}e^{γt}[\frac{q_0}{Q} − L_2 e^{−γh}e^{−λh} − λ^2 + cλ + D + γ] ≥ 0.

Invoking the Phragmèn-Lindelöf principle at this stage, we conclude that also
δ_±(t, x) ≤ 0 for all t ∈ [T, T+h], x ∈ R. This contradicts the maximality of the strip Π and completes the proof of the theorem. □

3. Proof of Theorem 1.3

The change of variables v(t, x) = u(t, x − ct) transforms (1.1), (1.11) into
0 = v_{xx}(t, x) − cv_x(t, x) − v_t(t, x) + f(v(t, x), v(t−h, x−ch)), \ t > 0, x ∈ R,
|v_0(s, x) − φ(x + α_0(s))| ≤ Ke^{λx}, \ (s, x) ∈ [-h, 0] × R.

Without loss of generality, we can assume that α_0(0) = 0. Our first goal is to obtain a similar estimate for t ∈ [0, h]: we will prove that, for some K_1 ≥ K,
|v(t, x) − φ(x)| ≤ K_1 e^{λx}, \ (t, x) ∈ [0, h] × R.

Indeed, the difference w(t, x) = v(t, x) − φ(x) solves the following linear inhomogeneous equation

\begin{align*}
w_t(t, x) &= w_{xx}(t, x) − cw_x(t, x) + a(t, x)w(t, x) + b(t, x), \ t ∈ [0, h], x ∈ R, \\
w(s, x) &= w_0(s, x) := v_0(s, x) − φ(x), \ (s, x) ∈ [-h, 0] × R,
\end{align*}

where
\[a(t, x) = \int_0^1 f_1(sv(t, x) + (1−s)φ(x), sv(t−h, x−ch) + (1−s)φ(x−ch))ds\]
\[b(t, x) = −w(t−h, x−ch) \int_0^1 g(sv(t, x) + (1−s)φ(x))ds\]
are Lipschitz continuous functions. Invoking the standard representation formula for the solution of the above Cauchy problem (see [7] Theorem 12), we find that, for (t, x) ∈ [0, h] × R it holds
\[w(t, x) = \int_\mathbb{R} \Gamma(t, x; 0, ξ)w(0, ξ)dξ + \int_0^t \int_\mathbb{R} \Gamma(t, x; τ, ξ)b(τ, ξ)dxdt,\]
where Γ(t, x; τ, ξ) is the fundamental solution for the respective homogeneous equation. Using the estimates (for the first one, see inequality (6.12) on p. 24 of [7])
\[|Γ(t, x; τ, ξ)| ≤ \frac{C}{\sqrt{t−τ}}e^{−\frac{k(τ−ξ)^2}{4(t−τ)}}, \ x, ξ ∈ \mathbb{R}, \ t > τ, t, τ ∈ [0, h],\]
\[|b(τ, ξ)| + |w(0, ξ)| ≤ Ce^{λξ}, \ (τ, ξ) ∈ [0, h] × \mathbb{R},\]
where C > 0 and k ∈ (0, 1) are some constants, we obtain, with some C′ > 0, that
\[e^{−λx} \int_\mathbb{R} \Gamma(t, x; 0, ξ)w(0, ξ)dξ ≤ \int_\mathbb{R} C^2 e^{−\frac{k(ξ−x)^2}{4t}}e^{−λ(ξ−x)}dξ =\]
the functions

\[ e^{-\lambda x} \int_0^t \int_\mathbb{R} \Gamma(t, x; \tau, \xi) h(\tau, \xi) d\xi d\tau \leq e^{-\lambda x} \int_0^t \int_\mathbb{R} \frac{C^2}{|t - \tau|} e^{-\frac{\lambda|x|^2}{|t - \tau|}} e^{\lambda x} d\xi d\tau = \]

\[ |\int_0^t \int_\mathbb{R} \frac{C^2}{|t - \tau|} e^{-\frac{\lambda|x|^2}{|t - \tau|}} e^{\lambda x} d\xi d\tau| < C', \quad t \in [0, h], \quad x \in \mathbb{R}. \]

Then (3.1) follows from these inequalities.

Next, take a sufficiently large negative number \( x_* \) to have

\[ |g(\phi(x))| < -0.25 \gamma e^{0.5 \gamma h} \text{ for all } x \leq x_*. \]

Consider a \( C^\infty \)-smooth non-decreasing function \( \lambda : \mathbb{R} \to \mathbb{R} \) defined, for some appropriate \( \theta > ch \), as \( \lambda(x) = \lambda_* x \) for \( x \leq x_* - \theta \) and \( \lambda(x) = \lambda x \) for \( x \geq x_* - ch \) and \( \lambda'(x) \in [\lambda_*, \lambda] \), \( \lambda''(x) < -\gamma/4 \). Clearly, we can choose \( K_2 > K_1 \) in such a way that the functions

\[ \rho_{\pm}(t, x) = \pm|v(t, x) - \phi(x)| - K_2 e^{0.5 \gamma (t-h)} e^{\lambda(x)} \]

satisfy \( \rho_{\pm}(t, x) \leq 0 \) for \( (t, x) \in [0, h] \times \mathbb{R} \).

For \( \phi(x) \neq v(t, x) \), set

\[ m(t, x) := \frac{f(\phi(x), v(t-h, x-ch)) - f(v(t, x), v(t-h, x-ch))}{\phi(x) - v(t, x)}, \]

and for \( \phi(x) = v(t, x) \), set \( m(t, x) := f_1(\phi(x), v(t-h, x-ch)). \)

Then consider the linear differential operator

\[ \mathcal{L}v = v_{xx} - cv_x + m(t, x)v - v_t \]

and let \( \Pi = [0, T] \times \mathbb{R}, \quad T \in [h, +\infty] \) be the maximal strip where \( \rho_{\pm}(t, x) \leq 0 \). Suppose for a moment that \( T \) is finite. Then we find that, for all \( t \in [T, T+h], \quad x \in \mathbb{R}, \)

\[ (\mathcal{L} \rho_{\pm})(t, x) = \{ \pm(\mathcal{L} v)(t, x) \mp (\mathcal{L} \phi(\cdot))(t, x) \} - K_2 e^{-0.5 \gamma h} (\mathcal{L} e^{0.5 \gamma e^{\lambda(x)}})(t, x) = \]

\[ \pm\{f(\phi(x), \phi(x-ch)) - f(v(t, x), v(t-h, x-ch)) - m(t, x)[\phi(x) - v(t, x)]\} \]

\[ + K_2 e^{\lambda(x)} e^{0.5 \gamma (t-h)} [-\lambda''(x) - (\lambda'(x))^2 + c\lambda'(x) - m(t, x) + 0.5 \gamma] \geq \]

\[ -|g(\phi(x))||\phi(x-ch) - v(t-h, x-ch)| + \]

\[ K_2 e^{\lambda(x)} e^{0.5 \gamma (t-h)} [-\lambda''(x) - (\lambda'(x))^2 + c\lambda'(x) + D + 0.5 \gamma] =: \mathcal{E}(t, x). \]

Now, if \( x \leq x_* \) then

\[ \mathcal{E}(t, x) \geq -|g(\phi(x))|K_2 e^{0.5 \gamma (t-2h)} e^{\lambda(x-ch)} + \]

\[ K_2 e^{\lambda(x)} e^{0.5 \gamma (t-h)} [-\lambda''(x) - (\lambda'(x))^2 + c\lambda'(x) + D + 0.5 \gamma] \geq \]

\[ K_2 e^{\lambda(x)} e^{0.5 \gamma (t-h)} [\gamma - (\lambda'(x))^2 + c\lambda'(x) + D] > 0. \]

On the other hand, if \( x \geq x_* \) then

\[ \mathcal{E}(t, x) \geq K_2 e^{\lambda(x)} e^{0.5 \gamma (t-h)} [-L_2 e^{-0.5 \gamma h} e^{-\lambda ch} - \lambda^2 + c\lambda + D + \gamma] > 0. \]

Invoking the Phragmén-Lindelöf principle at this stage, we conclude that also \( \delta_{\pm}(t, x) \leq 0 \) for all \( t \in [T, T+h], \quad x \in \mathbb{R} \). This contradicts the maximality of the strip \( \Pi \) and completes the proof of the theorem.
Here we analyse the zeros of the entire function $z + q - qe^{-zh}$, where $q, h$ are positive parameters. It is convenient to include the case $q = +\infty$ by introducing $\epsilon = 1/q \geq 0$ and analysing $\chi(z) = \epsilon z + 1 - e^{-zh}$. Clearly, $\chi$ has only one real zero $z = 0$. Thus $\chi'(z_j) = \epsilon + h(z_j + 1) \neq 0$ at each zero $z_j$ of $\chi(z)$ so that $z_j = z_j(\epsilon)$ is a smooth function of $\epsilon \geq 0$. Set $z_j = x + iy$ with $y > 0$, then $\epsilon x + 1 = e^{-zh} \cos(yh), \epsilon y = -e^{-zh} \sin(yh)$ and therefore the unique zero of $\chi(z)$ with non-negative real part is $z = 0$. Moreover, the equality $\epsilon y = -e^{-zh} \sin(yh)$ shows that $yh \in (\pi + 2\pi k, 2\pi + 2\pi k), k \in \mathbb{N} \cup \{0\}$, whenever $\epsilon > 0$. Next, $1 = e^{-z_j(0)h}$ implies that $z_j(0)h = i(\pi + 2\pi k)$. Since the relation $z_j(\epsilon, -) = \infty$ cannot happen for a finite $\epsilon > 0$, we conclude that $z_j(\epsilon) \in \{z : h \Im z \in (\pi + 2\pi k, 2\pi + 2\pi k), \Re z < 0\}$ is well defined for every $\epsilon > 0$. Consequently, the original function $z + q - qe^{-zh}$ has a unique zero $z_j$ at each horizontal strip $(\pi + 2\pi k)/h < \Im z < (2\pi + 2\pi k)/h$ while its complete list of zeros is given by $\{z_0 = 0, z_k, \bar{z}_k, k \in \mathbb{N}\}$. Since $|z_j + q| = q e^{-\Re z_j h}$ we conclude that $\Re z_j$ is a strictly decreasing sequence converging to $-\infty$.

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