Study properties of Eigenvalues, Eigenvectors and Eigenspace of matrices by using MatLab

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Abstract. The aim of this paper is to study the properties of eigenvalues with types (real, Repeate, complex) of matrices (2x2), (3x3), (4x4), eigenvectors and eigenspace of matrices by using Matlab. It also investigates the left and right eigenvectors, characteristic equation, existence and multiplicity of eigenvalues, gives some examples about the computation of eigenvalues, eigenvectors, and solves the Dynamic Systems of First Order Linear Differential Equations of (2 dim and 3 dim) as well as the Numerical solution to Dynamic systems by using program Matlab.

Keyword: Properties, Eigenvalues, Eigenvectors, Matlab, Matrices

1. Introduction

"Eigenvalues" is often presented in context of theory of linear algebra or theory of matrix. Nevertheless, it originated in history in the quadratic shapes and differential equations study. Euler planned the rationing movement of the rigid body and revealed the significance of the main axes. Lagrange recognized that the main axes are the inertia matrix eigenvectors [1]. Cauchy also created the word (Racine Characteristic equation) (Characteristic Root), which is currently named (Eigenvalue); this word lasts in the (Characteristic Equation). Sturm established more (Fourier's ideas) as well as fetched these ideas for the Cauchy courtesy which were unified with the personal thoughts, reaching a truth that the symmetrical matrixes possess actual eigenvalues [2]. That was prolonged in (1855) via Hermite, to that currently termed (Hermitian Matrixes). About the similar period, Brioschi verified that the eigenvalues of orthogonal matrixes locate upon the unit circle. Liouville intentional eigenvalues difficulties alike to those belong to Sturm; the developed discipline from this effort is nowadays named (Theory of Sturm-Liouville). Schwarz investigated the Laplace's formula initial eigenvalue upon the overall fields adjacent to the (19th) century end, whereas the Poincaré intentional Poisson's formula was studied after certain years. At the (20th) century twitch, Hilbert regarded the integral eigenvalues of operator via observing the operators as infinite matrices. Hilbert was the initial to utilize the (Germani) word (eigen) to represent the eigenvalues and eigenvectors in (1904), although he might have been ensuing the associated use via Helmholtz. The chief numerical procedure to compute the eigenvalues and eigenvectors appeared in (1929), when VonMises printed the power procedure [2, 3]. For every linear change eigenvector, there exists a corroborating scalar value so-named (eigenvalue) to such vector that defines the quantity, which the eigenvector is exceeded underneath the linear change. For instance, the (eigenspace) of certain change to a specific eigenvalue is the eigenvectors group (linear span)
associated with such eigenvalue, collected with zero vectors without direction. Within the linear algebra, each linear change among the finite-dimensional spaces of vector is stated as a matrix that is the numbers’ rectangular array prepared into columns as well as rows. The usual ways for the outcomes, the eigenvalues, eigenvectors, and eigenspaces of a certain matrix are deliberated under [4, 2].

2. Formal Definition

When \((T)\) is a linear conversion from the vector space \((V)\) above the field \((F)\), and \((v)\) is the vector in \((V)\), which isn’t a zero vector, then \((v)\) is the eigenvector of \((T)\) when \(T(v)\) is the \((v)\) scalar multiple. Such condition is stated by this expression \((T(v)=\lambda v)\).

Where, \((\lambda)\) represents the scalar in field \((F)\) recognized as an eigenvalue (characteristic value) or (characteristic root) related to an eigenvector \((v)\). When vector space \((V)\) is a limited-dimensional, then linear conversion \((T)\) is denoted as a square matrix \((A)\), and vector \((v)\) via a column vector, and that renders the abovementioned plotting as a multiplication of matrix upon left-hand side and the column vector scaling upon right-hand side in this expression: \((Av=\lambda v\text{ and }v=x)\) [5, 13].

3. Eigenvalue and Eigenvector for Matrices

In the linear algebra, a linear transformation an eigenvector (or characteristic vector) is the non-zero vector, which varies almost via scalar factor if this linear conversion is exerted to it. There exists a straight corresponding between the square matrices of \((n \times n)\) and the linear conversions from the \(n\)-dimensional vector space within it, for every known basis of vector space. Due to such cause, in a finite-dimensional vector space, it’s corresponding to describe the eigenvalues as well as eigenvectors by the language of the matrices or the language of linear conversions. In geometric, an eigenvector, equivalent to an actual nonzero eigenvalue, refers in a way to which it’s hard-pressed via the transformation, and the eigenvalue is the factor, by which it’s extended. When the eigenvalue is non-positive, the route is overturned [6, 11].

3.1 Computation

A linear conversion \((T: R^n \to R^n)\) being known via a matrix \((A)\) of \((n \times n)\). Eigenvalue \((\lambda)\) and \((T)\) eigenvector \((v)\) are described via \(Av=\lambda v\).

Equally, \((v)\) is the non-zero vector in null space \((nul(A-\lambda I))\). Also, vector \((v)\) and number \((\lambda)\) are named eigenvector and eigenvalue of \((A)\), respectively.

The subsequent submits the way to obtain the eigenvalues:

\((\lambda)\): The eigenvalue of \((A)\).
\[\iff (Av=\lambda v)\text{ for certain }v\neq 0\text{ (The eigenvalue description)}\]
\[\iff ((A-\lambda I)x=0)\text{ possesses }x=v\text{ as a non-trivial solution.}\]
\[\iff (A-\lambda I)\text{ isn’t invertible (the invertibility measure for square matrixes)}\]
\[\iff (det(A-\lambda I)=0).	ext{ (Such relationship is between }det\text{ and invertibility)}\]

The square matrix \((A)\) characteristic polynomial is \((det(A-\lambda I))\).

So, eigenvectors and eigenvalues are calculated as tails:

Step (1): Answer the characteristic expression \((det(A-\lambda I)=0)\) and acquire eigenvalues \((\lambda_1), (\lambda_2), \ldots\)
Step (2): To every eigenvalue ($\lambda_i$), resolve the consistent regime ($(A-\lambda_iI)x=0$) and acquire eigenvectors with the ($\lambda_i$) as an eigenvalue.

Within 2nd stage, a reply is really introduced as $\text{nul}(A-\lambda_iI)$ basis, termed the (A) eigen space having the eigenvalue ($\lambda_i$) [7, 14]. The condition, which eigenvector becomes a non-zero, is enforced since the expression $(A0=\lambda0)$ grasps for each (A) and each ($\lambda$). As the expression is continually trivially accurate, it isn’t exciting situation. Contrary, the eigenvalue is (0) in a non-trivial manner. Every eigenvector is related to a particular eigenvalue; see the eigen plane in figure 1.

In geometrical (see Figure 1), the eigenvalue equation is that under the (A) transformation, the eigenvectors appear merely the variations in the value and the sign; ($Ax$) direction ($Ax$) being similar like that of ($x$). Eigenvalue ($\lambda$) is easily a ‘spring’ or ‘shrink’ quantity to that the vector being bared if varied via the (A). Eigenvectors equivalent to various eigenvalues are linearly not dependent importance, particularly those in the n-dimensional space, and (A) linear conversion isn’t able to possess further than (n) eigenvectors having various eigenvalues [13, 14].

**4. Characteristic Polynomial**

A transformation is represented by a square matrix (A), the eigenvalue equation can be expressed as:

$$AX = \lambda X \rightarrow A\lambda Ix = 0.$$  \hspace{1cm} (1)

The needed determinant is named the (characteristic equation) (fewer frequently, secular equations) of (A), and left side is named (characteristic polynomial). If extended, that provides a polynomial expression for ($\lambda$). The eigenvector ($x$) or its constituents don’t exist in the characteristic equation.

From equation (1):  get  \hspace{1cm} (A−\lambda I)x=0 \hspace{1cm} (2)

When it occurs the inverse $(A − \lambda I)^{-1}$, Where, (I) is an (n x n) identity matrix, and (0) is a zero vector.

Equation (2) possesses non-zero solution ($x$) when and only when the determinant of matrix $(A−\lambda I)$ is zero. Consequently, (A) eigenvalues are the ($\lambda$) values that satisfy the equation (2). Then, either side are
left multiplied via a converse for obtaining a trivial solution \((x=0)\). Therefore, one needs that there is no converse via supposing from linear algebra that the determinant is equal to \((0)\):

\[
\det(A-\lambda I) = 0
\]  

(3)

Via the rule of Leibnizs for determinant, left side of equation \((3)\) being the variable \((\lambda)\) polynomial function, and its polynomial degree is \((n)\); the order of matrix \((A)\).

\[
|A-\lambda I| = (\lambda_1-\lambda). (\lambda_2-\lambda) \ldots (\lambda_n-\lambda).
\]

Where, every \((\lambda_i)\) may be the actual, but generally it’s a complex no. The numbers \((\lambda_1, \lambda_2, \ldots \lambda_n)\) that mayn’t entirely possess distinctive magnitudes are the polynomial roots and are the \((A)\) eigenvalues \([8,9]\).

5. Eigenspaces for Matrices

It is assumed that a specific eigenvalue \((\lambda)\) of a \((n \times n)\) matrix \((A)\) defines the set \((S)\) to become the whole vectors \((v)\), which meets the expression \((2)\).

\[
S = \{ v: (A-\lambda I) v= 0 \}
\]

From one side, such a set becomes accurately a kernel or a matrix \((A-\lambda I)\) nullspace. From other side, via the categorization, every non-zero vector that meets such state is the \((A)\) eigenvector related to \((\lambda)\). Thus, the set \((S)\) is the union of zero vectors with a set of whole \((A)\) eigenvectors related to \((\lambda)\), and \((S)\) is equal to the \((A-\lambda I)\) nullspace. \((S)\) is named the \((A)\) (characteristic space) or (eigenspace) related to \((\lambda)\) \([9,12]\).

6. Matrix Examples

6.1 Two-Dimensions Matrix Examples

**Example 1**: Study the matrix \(A = \begin{bmatrix} 13 & -4 \\ -4 & 7 \end{bmatrix}\) of linear convention \(T(x_1, x_2) = (13x_1-4x_2, -4x_1+7x_2)\) on \(\mathbb{R}^2\). Characteristic polynomial is

\[
\det(A-\lambda I)=\begin{vmatrix} 13-\lambda & -4 \\ -4 & 7-\lambda \end{vmatrix}=(13-\lambda)(7-\lambda)-(-4)(-4)=75-20\lambda+\lambda^2
\]

When \(75-20\lambda+\lambda^2 = (5-\lambda)(15-\lambda)\), one finds: \((\lambda_1=5\) and \(\lambda_2=15)\)

When \(\lambda_1=5\), one can have

\[
A-5I = A = \begin{bmatrix} 13-5 & -4 \\ -4 & 7-5 \end{bmatrix} = \begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix}
\]
The answers of \((A-5I)x=0\) have the form \((x=c(1,2))\), where \((c)\) is random. One gets eigenvector \((v_1=(1,2))\) that is in reality the eigen spacee base \((\text{null}(A-5I))\).

When \((\lambda_2=15)\), one can possess:

\[
A-15I = \begin{bmatrix} 13 & -15 & -4 \\ -4 & 7 & -15 \\ \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix}
\]

The solution of \(((A-15I)x=0)\) has the type \((x=c(-2,1))\), where \((c)\) is random. One gets eigenvector \((v_2=(-2,1))\), and the eigenspace being \((\text{null}(A-15I))\).

The difficulty of problem to find the roots / eigenvalues of characteristic polynomial rises fatly with the rising of polynomial degree (vector space dimension). Accurate answers exist for the dimensions \((<5)\), but for those \((\geq5)\), usually no accurate answers exist, and one should re-sort the numerical procedures for finding them nearly.

6.2 Three – Dimensions Matrix Example:

**Example 1:** Study the matrix: \(A = \begin{bmatrix} 1 & 3 & -3 \\ -3 & 7 & -3 \\ -6 & 6 & -2 \end{bmatrix}\)

The characteristic polynomial was calculated previously:

\[
\det(A-\lambda I)=(-\lambda)^2(-2-\lambda).
\]

One can include \(\lambda_1=4, \lambda_2=-2\).

When \(\lambda_1 = 4\), one can have:

\[
A-4I = \begin{bmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ -6 & 6 & -6 \end{bmatrix}
\]

The resolution of \(((A-4I)x=0)\) has the type \((x=x_1(1,1,0) x_3(1,0,-1))\), \((x_2)\) and \((x_3)\) illogical. They have double eigenvectors \(v_1=(1,1,0)\), and \(v_2=(1,0,-1)\), which make the eigenspace base \((\text{null}(A-4I))\).

When \(\lambda_2=-2\), one can possess:

\[
A+2I = \begin{bmatrix} -3 & 3 & -3 \\ -3 & 9 & -3 \\ -6 & 6 & 0 \end{bmatrix}
\]

The answers of \(((A+2I)x=0)\) have the type \((x=c(1,1,2))\), where \((c)\) is random. One gets eigenvector \((v_3=(1,1,2))\), and the eigenspace is \((\text{null}(A+2I))\).

- These might prove that \((v_1), (v_2), \text{ and } (v_3)\) are truly forming the \((\mathbb{R}^3)\) basis. In geometrical, one can have the complete recognizing of the linear transformation provided via \(A\), see figure 2.
Figure 2. Graph of eigenvalues and eigenvectors to matrix (A)

According to the examples, one can notice for the \((n \times n)\) matrix \((A)\), the \((\text{Det}(A-\lambda I))\) being the \((n)\) level polynomial.

When \((A)\) being a linear conversion matrix \((T:V \rightarrow V)\) upon a limited-dimensional vector space corresponding to \((1)\) basis, permitting to debate \textit{now}, matrix \((T)\) corresponding to the other basis being \((B=PAP^{-1})\). Throughout such characteristic, one can possess

\[
\text{Det}(B-\lambda I)=\text{Det}(PAP^{-1}-\lambda I)=\text{Det}(P(A-\lambda I)P^{-1})=\text{Det}(P^{-1}P(A-\lambda I))=\text{Det}(A-\lambda I).
\]

7. Left and Right Eigenvectors
The eigenvector expression properly represents the right eigenvector \((x_R)\). It’s described via the overhead eigenvalue expression \((Ax_R=\lambda_R x_R)\), and it’s the best normally utilized eigenvector. Though, a left eigenvector \((x_L)\) presents, also it’s obvious via \((x_L A=\lambda_L x_L)\).

8. Eigenvalues Further Features:
Let \((A)\) is a random \((n \times n)\) matrix of the complex nos. having eigenvalues \((\lambda_1, \lambda_2, \ldots, \lambda_n)\). Every eigenvalue looks \(\mu_A(\lambda_i)\) times within such list, when \(\mu_A(\lambda_i)\) is an algebraic multiplicity of eigenvalue [11, 14].

The subsequent are the features of such matrix and its eigenvalues:

- The trace is:
  \[
  \text{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n.
  \]
- The trace is described as a summation of its diagonal elements; also it’s a sum of the whole eigenvalues.

The \(A\) determinant is the whole its eigenvalues product.
The (k\textsuperscript{th}) power eigenvalues of (A); that means the (A\textsuperscript{k}) eigenvalues for every positive integer (k) are \( \lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k \).

- When \( A \) is squared, the eigenvectors stay the same. The eigenvalues are squared. \( AX = \lambda X \rightarrow A^2X = \lambda^2X \). The eigenvalues of \( A^2 \) and \( A^{-1} \) are \( \lambda^2 \) and \( \lambda^{-1} \), with the same eigenvectors.
- The matrix \( A \) being invertible when and merely when each eigenvalue being non-zero.
- When \( A \) equals to its conjugate transpose \( A^* \), or equally when \( A \) is Hermitian, at that time, each eigenvalue is actual. The similar is real for every symmetric actual matrix.
- When \( A \) is unitary, every eigenvalue possesses an absolute value \( |\lambda_i| = 1 \).

9. Eigenvalues Actuality and Multiplicity

1. For the conversions upon the actual vector spaces, characteristic polynomial coefficients are totally actual. Nevertheless, the roots aren’t essentially actual; they might contain complex numbers with an imaginary constituent, which is nonzero. For instance, a matrix demonstrating a \( (45^\circ) \) planar rotation shall not departure each nonzero vector, indicating to the similar direction. Above the complex vector space, algebra principle theorem assures that characteristic polynomial possesses at minimum a single root, therefore the linear conversion possesses at minimum a single eigenvalue. Nevertheless, possessing frequent roots doesn’t indicate there are a multiple different (that means straightly not dependent) eigenvectors with such eigenvalue. The eigenvalue geometric multiplicity being described as associated eigenspace dimension, eigenvectors [7, 13], as shown in figure 3.

![Figure 3. Horizontal shear (angle of shear \((\phi)\) is provided via \((k=\cot \phi)\) [13]](image)

2. The shear within the plane being conversion in which the whole points alongside a certain line stay immovable, where the other points are moved parallel to this line via a distance that is proportional to
their vertical distance from this line. The figure of shearing a plane doesn’t vary in area. The shear is a vertical (alongside the Y-axis) or a horizontal (alongside the X-axis). In horizontal shear (see figure 3) the plane’s point (P) shifts parallel to the (X-axis) for the position P’ in order that the coordinate (y) of it doesn’t vary, whereas the (x) coordinate increases to be \( x' = x + ky \), where (k) is named the factor of shear.

The horizontal shear transformation matrix is 
\[
\begin{bmatrix}
1 & k
0 & 1
\end{bmatrix}
\]

The characteristic equation is \( \lambda^2-2\lambda+1=(1-\lambda)^2=0 \) which possesses one frequent root \( (\lambda=1) \). Thus, eigenvalue \( (\lambda = 1) \) possesses algebraic multiplicity (two). Eigenvector(s) being obtained as answers of 
\[
\begin{bmatrix}
1 -1
0 1 -1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
0
-k
0
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= -ky = 0.
\]

Final equation being equal to \( y = 0 \) that is a line alongside (X-axis). Such line denotes 1D eigen space. In shear situation, eigenvalue algebraic multiplicity (two) is larger than its geometrical multiplicity (one, dimension of eigen space). Eigenvector is the vector along (X-axis) [6, 14].

3. Unequal scaling: figure 4 shows the vertical shrink \((k_2)\) and the horizontal stretch \((k_1)\) of unit square [13].

![Figure 4. The vertical shrink \((k_2)\) is less than one and the horizontal stretch \((k_1)\) is greater than one of unit square [13].](image)

The eigenvector being \((u_1)\) and \((u_2)\), also the eigenvalues being \((\lambda_1=k_1)\) and \((\lambda_2=k_2)\). Such conversion places the whole vectors toward main eigenvector \((u_1)\). To somewhat further intricate instance, deem a leaf that’s extended unevenly within (2) vertical ways alongside coordinate axes, or likewise, extended within (1) way, also contracted in the other way. Within such state, a pair of various factors of scaling are exist; \((k_1)\) for climbing within x-axis direction, and \((k_2)\) for the scaling within y-axis direction.

The conversion matrix is 
\[
\begin{bmatrix}
k_1 & 0 \\
0 & k_2
\end{bmatrix}
\]

and characteristic expression is \( \lambda^2-(k_1\lambda+k_2\lambda)=0 \). Eigenvalues obtained as such expression roots being \((\lambda_1=k_1)\), as well as \((\lambda_2=k_2)\) that are the revenues, as anticipated, that double eigenvalues being factors of topping within double ways (see Figure- 4). Plugging \((k_i)\) rear in the equation of eigenvalue yields (1) of eigenvectors:
\[
\begin{bmatrix}
0 & 0 \\
0 & k_2 - k_1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

or, extra exactly, \( y = 0 \). Eigenvalues and eigenvectors

So, eigenspace is the \( x \)-axis. Correspondingly, replacing \((\lambda = k_2)\) depicts that equivalent eigenspace being the \( y \)-axis. Within such situation, the two eigenvalues possess algebraic and geometrical multiplicities equals to \((1)\) [10, 12].

10. Applications

Table 1 indicates some cases of convention in plane sideways with their (two x two) matrixes and eigenvalues as well as eigenvectors.

**Table 1.** Eigenvalues, and eigenvectors of (2×2) matrices [14]

| Illustration | Characteristic polynomial | Eigenvalues, \( \lambda \) | Algebraic mult., \( \mu = \mu(\lambda) \) | Geometric mult., \( \gamma = \gamma(\lambda) \) | Eigenvectors |
|---------------|---------------------------|-----------------------------|---------------------------------|-----------------|----------------|
| Scaling       | \((\lambda - k)^2\)       | \(\lambda_1 = \lambda_2 = k\) | \(\mu_1 = 2\)                  | \(\gamma_1 = 2\)  | \(u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\) |
| Unequal scaling | \((\lambda - k_1)(\lambda - k_2)\) | \(\lambda_1 = k_1\), \(\lambda_2 = k_2\), \(\mu_1 = 1\), \(\mu_2 = 1\) | \(\gamma_1 = \gamma_2 = 1\), \(\gamma_2 = 1\) | \(\gamma_1 = 1\), \(\gamma_2 = 1\) | \(u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\), \(u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\) |
| Rotation      | \(\lambda^2 - 2\lambda + 1\) | \(\lambda_1 = \lambda_2 = 1\) | \(\mu_1 = 2\)                  | \(\gamma_1 = \gamma_2 = 1\) | \(u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\), \(u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\) |
| Horizontal shear | \((\lambda - 1)^2\)     | \(\lambda_1 = \lambda_2 = 1\), \(\lambda_2 = e^{2\pi}\) | \(\mu_1 = 1\), \(\mu_2 = 1\) | \(\gamma_1 = \gamma_2 = 1\), \(\gamma_2 = 1\) | \(u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\), \(u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\) |
| Hyperbolic rotation | \(\lambda^2 - 2\lambda + 1\) | \(\lambda_1 = \lambda_2 = 1\), \(\lambda_2 = e^{2\pi}\) | \(\mu_1 = 1\), \(\mu_2 = 1\) | \(\gamma_1 = \gamma_2 = 1\), \(\gamma_2 = 1\) | \(u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\), \(u_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}\) |

11. Using MATLAB for Eigenvalues and Eigenvectors Finding

They tell Eigen command that one wants two matrices as an output by using the square brackets with two arguments. The output consists of one matrix (V) containing the eigenvectors in each column and (D) consists of the matrix with the eigenvalues in the diagonal elements. Since the eigenvalue -2 is the 3rd eigenvalue given in the D matrix, the third column is the eigenvector associated with the eigenvalue. Note that the eigenvectors are scaled so that they all have length 1 [4, 7].
11.1 Solutions Some Examples by using Matlab:

**Example 1:** Consider the system to find the Eigen value and the eigenvectors is \( X' = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} X \) by using Matlab.

The solution in Program 1: The corresponding Eigen system is as follows:

**Program 1:**
\[
\begin{align*}
A &= \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix} \\
E &= \text{eig}(A) \\
\{V, D\} &= \text{eig}(A)
\end{align*}
\]

\[
A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}
\]

\[
V = \begin{bmatrix} 0.9487 & -0.7071 \\ 0.3162 & 0.7071 \end{bmatrix}
\]

\[
D = \begin{bmatrix} 2.0000 & 0 \\ 0 & -2.0000 \end{bmatrix}
\]

**Example 2:** Consider the system to find the eigenvalue and the eigenvectors:

\[
X' = \begin{bmatrix} 5 & -2 & -2 \\ 7 & -4 & -2 \\ 3 & 1 & -1 \end{bmatrix} X
\]

The solution by Matlab in Program 2;

**Program 2:**
\[
\begin{align*}
A &= \begin{bmatrix} 5 & -2 & -2 \\ 7 & -4 & -2 \\ 3 & 1 & -1 \end{bmatrix} \\
e &= \text{eig}(A) \\
\{V, D\} &= \text{eig}(A)
\end{align*}
\]

\[
A = \begin{bmatrix} 5 & -2 & -2 \\ 7 & -4 & -2 \\ 3 & 1 & -1 \end{bmatrix}
\]

\[
e = \begin{bmatrix} 1.0000 + 2.0000i \\ 1.0000 - 2.0000i \\ -2.0000 + 0.0000i \end{bmatrix}
\]

\[
V = \begin{bmatrix} 0.3536 + 0.3536i & 0.3536 - 0.3536i & -0.0000 - 0.0000i \\ 0.3536 + 0.3536i & 0.3536 - 0.3536i & -0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.7071 + 0.0000i & 0.7071 + 0.0000i \end{bmatrix}
\]

\[
D = \begin{bmatrix} 1.0000 + 2.0000i & 0.0000 + 0.0000i & 0.0000 + 0.0000i \\ 0.0000 + 0.0000i & 0.7071 + 0.0000i & 0.7071 + 0.0000i \end{bmatrix}
\]
Example 3: Find the eigenvalue and the eigenvectors to the system:

\[
X' = \begin{bmatrix}
3 & -2 & 0 \\
2 & -2 & 0 \\
0 & 1 & 1 \\
\end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

Solution by Matlab in Program 3:

Program 3:

Clear all
clc
A=[3 -2 0;2 -2 0;0 1 1]
[xi,R]=eig(sym(A))
b=[3;5;0];
c=xi
b
plot(diag(R))
plot(diag(A))
Run is: into figure 5 and figure 6,(a & b).

Example 4: find the numerical solution of two linear differential equations by Matlab. 

\[
X' = x - 2y, \quad Y' = -x, \quad x(0) = 2, \quad y(0) = -2
\]

The solution of example is program 4
**Program 4:**

Clear all

clc

\[
f = \text{inline}([x(1) - 2x(2); -x(1)])', 't', 'x');
\]

\[
[t, xa] = \text{ode45}(f, [0:0.2:2], [2 -2]);
\]

disp(' time x1(t) x2(t) ')

\[
[t xa]
\]

run of program is:

| time  | x1(t)   | x2(t)   |
|-------|---------|---------|
| 0     | 2.0000  | -2.0000 |
| 0.2000| 1.4780  | -2.3519 |
| 0.4000| 0.7105  | -2.5748 |
| 0.6000| -0.2895 | -2.6205 |
| 0.8000| -1.4854 | -2.4459 |
| 1.0000| -2.8155 | -2.0174 |
| 1.2000| -4.1923 | -1.3165 |
| 1.4000| -5.5051 | -0.3447 |
| 1.6000| -6.6244 | 0.8727  |
| 1.8000| -7.4086 | 2.2827  |
| 2.0000| -7.7147 | 3.8042  |

**Example 5:** Find the eigenvalue and the eigenvectors to the system:

\[
\begin{bmatrix}
X' \\
= \begin{bmatrix}
2 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 1 & 3
\end{bmatrix}X
\end{bmatrix}
\]

Solution by Matlab in program5:
The eigenvalues and eigenvectors of A are:

**Program 5:**

```matlab
>> A=[2 0 0 0;1 2 0 0;0 1 3 0;0 0 1 3]

>> e=eig(A)

>> [V,D]=eig(sym(A))

V =
[  0, 0]
[  1, 0]
[ -1, 0]
[  1, 1]

D =
[ 2, 0, 0, 0]
[ 0, 2, 0, 0]
[ 0, 0, 3, 0]
[ 0, 0, 0, 3]
```

12. **Conclusions**

1- When a matrix being diagonal, the eigenvalues of such matrix are nos. upon diagonal, also the eigenvectors of such matrix being the basis vectors for which these nos. provide in paper programs.

2- The algebra fundamental theorem indicates that (n x n) matrix A, as polynomial of level (n), could be factored in product of (n), assuming matrix (A) has a measurement (n), and the (d≤n) distinct eigenvalues are generalized to the Jordan normal form.

3- Matlab was used to calculate or find the true, different, repeated and complex eigenvalues of a system of linear equations. And, the calculation of the results was conducted theoretically and practically in the dynamic system analysis.

4- Any polynomial of degree (n) always has (n) roots (real and/or complex; not of necessity distinct), any (n x n) matrix for all time has at least one, and up to (n) different eigenvalues. Each particular eigenvalue was put back into the linear system \((A - \lambda I) x = 0\) to get the related eigenvectors.

5- The eigenvalues of \(A^2\), \(A^4\) and \(A^5\) are \(\lambda^2\), \(\lambda^4\) and \(\lambda^5\), respectively with the same eigenvectors.

6- There exist accurate results for the dimensions (< 5), but for those (≥ 5), usually no exact answers are exist, also one should decide the numerical ways to obtain them virtually.
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