Cone construction via real intersection theory

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Abstract
We show that the cone construction extends the Lefschetz standard conjecture to the coniveau filtration.

Contents
1 Introduction 2
1.1 Generalization of the Lefschetz standard conjecture . . . . . . . . 2
1.2 Outline of the proof . . . . . . . . . . . . . . . . . . . . . . . . 3
1.2.1 Cone family of cycles of currents . . . . . . . . . . . . . . 4
1.2.2 Two different aspects of end cycles . . . . . . . . . . . . . 4
2 Cone family 11
2.1 Family of algebraic cycles . . . . . . . . . . . . . . . . . . . . . 11
2.2 Family of cycles of currents . . . . . . . . . . . . . . . . . . . . 11
3 End cycles in the cone family 11
3.1 Algebraic part of end cycles . . . . . . . . . . . . . . . . . . . . 11
3.2 Non-algebraic part of end cycles . . . . . . . . . . . . . . . . . 16
4 Algebraicity and cohomologicity 22
4.1 Algebraicity . . . . . . . . . . . . . . . . . . . . . . . . . . . . 22
4.2 Cohomology with support . . . . . . . . . . . . . . . . . . . . . 23
4.3 Cohomology without support . . . . . . . . . . . . . . . . . . . 28
Appendix A Coniveau Filtration of currents’ version 28

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1 Introduction

1.1 Generalization of the Lefschetz standard conjecture

The Lefschetz standard conjecture were proposed by Grothendieck ([3]) in formulating a solution to Weil's conjectures. The conjecture addresses a smooth projective variety $X$ of dimension $n$ over an algebraically closed field of arbitrary characteristic. We step back to assume the ground field is $\mathbb{C}$, and the cohomology is Betti cohomology with rational coefficients. Thus we consider the associated projective manifold which is still denoted by $X$. Let $u$ be the hyperplane section class in $H^2(X; \mathbb{Q})$. For $0 \leq h \leq n$, let $L^h$ denote the homomorphism on the cohomology

$$L^h: \sum_{i=0}^{2n-2h} H^i(X; \mathbb{Q}) \to \sum_{i=0}^{2n-2h} H^{i+2h}(X; \mathbb{Q}) \alpha \to \alpha \cdot u^h.$$  (1.1)

The hard Lefschetz theorem says $L^h$ has a property in topology. Precisely the restriction to $H^{n-h}(X; \mathbb{Q})$ is an isomorphism to $H^{n+h}(X; \mathbb{Q})$. Grothendieck envisioned that $L^h$ also has a property in algebraic geometry. More precisely he proposed the Lefschetz standard conjecture:

**Conjecture 1.1.** Let $A^j(X) \subset H^{2j}(X; \mathbb{Q})$ for any integer $j$ be the subspace spanned by algebraic cycles. Then the restriction $L^{q-p}_0$ of $L^{q-p}$ to $A^p(X)$,

$$L^{q-p}_0: A^p(X) \to A^q(X) \alpha \to \alpha \cdot u^{q-p}.$$  (1.2)

for $p + q = n, q \geq n$ is an isomorphism.

The conjecture has been claimed to be correct ([7]). In this paper, we use the real intersection theory to extend the same proof to the coniveau filtration. This furthers Grothendieck’s vision.

The coniveau filtration is the decreasing filtration on the cohomology $H^j(X; \mathbb{Q})$ over $\mathbb{Q}$:

$$H^j(X; \mathbb{Q}) \supset \cdots \supset N^i H^j(X) \supset \cdots \supset N^0 H^j(X)$$  (1.3)

where $N^i H^j(X)$ is the subspace spanned by

$$Ker\{H^j(X; \mathbb{Q}) \to H^j(X - W; \mathbb{Q})\}$$  (1.4)

for all subvarieties $W$ of codimension at least $i$. The index $i$ is called the coniveau and $j - 2i$ the level.
Theorem 1.2. (Main theorem)

Let \( p, q, k \) be whole numbers satisfying

\[
p + q = n - k, p \leq q.
\]

When \( L^{q-p} \) are restricted to the subgroups of the coniveau filtration, the restricted homomorphism

\[
L_k^{q-p} : N^p H^{2p+k}(X) \to N^q H^{2q+k}(X)
\]

\[\alpha \to \alpha \cdot u^{q-p}.\]  (1.5)

is an isomorphism, where \( k \) is the level of the coniveau filtration.

Remark The special case for \( k = 0 \) is Conjecture 1.1.

1.2 Outline of the proof

The logic path of the proof is identical to that of [7]. But we’ll make a modification along the path to address the currents in real intersection theory ([5], [6]). Because many parts of this paper are duplicates of [7], [5] and [6], to avoid the redundancy, we will only cite the results and proofs from the references without proofs.

We use angle bracket \( \langle \bullet \rangle \) to denote the object in the cohomology.

For the indexes \( p, q, k \) in Theorem 1.2, we construct a linear map

\[
\langle Con_{h,p} \rangle : H^{2q+k}(X; \mathbb{Q}) \to H^{2p+k}(X; \mathbb{Q})
\]

which will be proved to

(1) respect the coniveau structure as it maps the subspace \( N^p H^{2p+k}(X) \) onto \( N^q H^{2q+k}(X) \).

(2) and be the topological inverse of the map (1.2), i.e.

\[
\langle Con_{h,p} \rangle \circ L_k^{q-p} = \text{identity} \quad L_k^{q-p} \circ \langle Con_{h,p} \rangle = \text{identity}. \]  (1.6)

The topological homomorphism \( \langle Con_{h,p} \rangle \) is reduced from an extrinsic operator \( Con_{h,p} \) which is a homomorphism on currents constructed with external data. It contains information that are both topological and algebraic, extrinsic and intrinsic.
1.2.1 Cone family of cycles of currents

The operator $\text{Con}_{n,p}$ is constructed from a family of currents, which begins as a family of algebraic cycles.

Let $\mathbb{C}^{n+2}$ be a linear space over $\mathbb{C}$ with a basis

$$e_0, \cdots, e_{n+1}. \quad (1.7)$$

Let $h$ be a natural number $< n$. Consider two subspaces

$$\mathbb{C}^{n+2-h} = \text{span}(e_0, \cdots, e_{n+1-h}),$$

$$\mathbb{C}^h = \text{span}(e_{n+2-h}, \cdots, e_{n+1}). \quad (1.8)$$

Then

$$\mathbb{C}^{n+2-h} \oplus \mathbb{C}^h = \mathbb{C}^{n+2}. \quad (1.9)$$

Next we consider a variation of $\mathbb{C}^h$. Let $\mathbb{C} \cup \{\propto\} \simeq \mathbb{P}^1$ be the parameter space of the variation, denoted by $\Upsilon$, where $\propto$ is the infinity point of $\mathbb{P}^1$. The variation is defined as

$$\mathbb{C}^h_z = \text{span}(ze_0, e_{n+3-h}, \cdots, e_{n+1}), \quad \text{for } z \in \mathbb{C} \quad (1.10)$$

and $\mathbb{C}^h_0$ is the original $\mathbb{C}^h$ which is the limit of subspaces $\mathbb{C}^h_z$ in Grassmannian as $z \to \propto$. Let $U = \mathbb{C}^* \cup \{\propto\}$ be the affine open set that parametrizes those $\mathbb{C}^h_z$ satisfying

$$\mathbb{C}^{n+2} = \mathbb{C}^{n+2-h} \oplus \mathbb{C}^h_z. \quad (1.11)$$

The only point $z = 0$ not in $U$ corresponds to the plane $\mathbb{C}^h_0$ that fails the decomposition (1.11). We call $z = 0$ the unsteady point, others steady points. Therefore for each steady point $z \in U$, we have the unique decomposition (1.11)

$$x = (x_1(z), x_2(z)) \quad (1.12)$$

for the vector $x \in \mathbb{C}^{n+2}$. This decomposition gives a regular map

$$\mathbb{C} \times U \times (\mathbb{C}^{n+2-h} \oplus \mathbb{C}^h_0) \to \mathbb{C}^{n+2-h} \oplus \mathbb{C}^h_z = \mathbb{C}^{n+2}$$

$$(t, z, [x_1(z), x_2(z)]) \to ([x_1(z), tx_2(z)]. \quad (1.13)$$

After completing $\mathbb{C}, U$ and projectivizing the linear spaces, the regular map determines a rational map of the projective variety

$$\kappa: \mathbb{P}^1 \times \Upsilon \times \mathbb{P}^{n+1} \to \mathbb{P}^{n+1}$$

$$(t, z, [x_1(z), x_2(z)]) \to [x_1(z), tx_2(z)]. \quad (1.14)$$

where $t, z$ are points in the affine open sets $\mathbb{C}, U$. Let

$$\Omega = \text{graph}(\kappa) \subset \mathbb{P}^1 \times \Upsilon \times \mathbb{P}^{n+1} \times \mathbb{P}^{n+1} \quad (1.15)$$

where the graph of a rational map is defined to be the closure of the graph at the regular locus (the same for images and preimages of rational maps).
1 INTRODUCTION

Now we consider the smooth projective variety $X$ of dimension $n$. Let $X \xrightarrow{\mu} \mathbb{P}^{n+1}$ be a birational morphism to a hypersurface of $\mathbb{P}^{n+1}$ with very ample line bundle $\mu^*(\mathcal{O}_{\mathbb{P}^{n+1}}(1))$, and proper intersections with all the coordinates planes of the basis $e_0, \ldots, e_{n+1}$. Also $\mu(X)$ does not contain lines. The collection of above spaces $\mathbb{C}^{n+2}, \mathbb{C}^{n+2}, \mathbb{C}^\mu$ and $\mu$ is called cone data. Next we construct a family of algebraic cycles with the cone data. Using $\Omega$ we define the subvariety $\Sigma$ to be the image of the rational map $(id, id, \mu^{-1}, \mu^{-1}) = \tau^{-1}$,

$$\Omega \cap (\mathbb{P}^1 \times \Upsilon \times \mu(X) \times \mu(X)) \longrightarrow \mathbb{P}^1 \times \Upsilon \times X \times X,$$

(1.16)

where $\tau = (id, id, \mu, \mu)$ is the birational-to-image map. We have a straightforward assertion for $\Sigma$.

**Proposition 1.3.** $\Sigma$ is reduced with two components of the same dimension $n + 1$, and one of components is

$$(1) \times \Upsilon \times \Delta_X$$

(1.17)

where $\Delta_X$ is the diagonal of $X \times X$.

The proposition gives a definition

**Definition 1.4.**

We define

$$\Theta$$

(1.18)

to be the other component of $\Sigma$, i.e.

$$\Theta = \Sigma - (\{1\} \times \Upsilon \times \Delta_X).$$

(1.19)

We denote the algebraic cycle of $\Theta$ also by $\Theta$, and denote the fibres of $\Theta$ over $t \in \mathbb{P}^1, z \in \Upsilon$ as cycles in $X \times X$ by $\Theta_t^z$. They are not always irreducible. Similarly $\Theta_t$ over each $t \in \mathbb{P}^1$ is a well-defined family of algebraic cycles in $\Upsilon \times X \times X$. Also to avoid the overwhelming notations in the context, the fibres lifted to $\mathbb{P}^1 \times \Upsilon \times X \times X$ are denoted by the same notations $\Theta_t^z, \Theta_t$ respectively. Their corresponding images in $\mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$ and $\Upsilon \times \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}$ are denoted by $\Omega_t^z$ and $\Omega_t$ respectively.

The intersection of the family members with an algebraic cycles led to the proof of Lefschetz standard conjecture in [7]. In this paper, we show that the same intersection with singular cycles yields Main theorem. So next we use a novel tool that allows us to take the intersection with singular cycles. The tool
is the real intersection theory that has been developed in [5], [6]. It is a study of geometry in algebraic varieties without the structure of a manifold through the geometric measure theory. The importance of real intersection theory goes beyond the content of this paper. However we should give a brief introduction in the following. On any manifold $Y$, there is a special type of currents inside of $\mathcal{D}'(Y)$, called Lebesgue currents which include singular chains and $C^\infty$-forms, and they form a subspace denoted by $C(Y)$. For any two Lebesgue currents $T_1, T_2$, we define an extrinsic intersection current as the currents’ limit of the De Rham’s homotopy regularization ([1]), denoted by

\[ [T_1 \wedge T_2]. \tag{1.20} \]

The intersection is also Lebesgue, but it depends on a special type of extrinsic covering on the manifold called De Rham data. This currents’ intersection, even though is extrinsic, but unites all the known products. Real intersection theory is an extension of Fulton’s intersection theory ([2]). For instance the notion of a correspondence is parallel to that in Fulton’s intersection theory, and plays the key role. Let $X \times Y$ be the Cartesian product of compact manifolds equipped with a De Rham data, and $\mathcal{J} \in C(X \times Y)$ a Lebesgue current. Then there is a homomorphism $\mathcal{J}_*$ called the correspondence of currents, defined by

\[
\mathcal{J}_* : C(X) \to C(Y), \quad \sigma \mapsto (P_Y)_*[\mathcal{J} \wedge (\sigma \otimes Y)], \tag{1.21}
\]

where $P_Y : X \times Y \to Y$ is the projection. In algebraic geometry, we assume $\mathcal{J}, \sigma$ are algebraic cycles and $\wedge$ is the well-defined intersection of algebraic cycles. Then $\mathcal{J}_*$ is the usual correspondence. This consistency provides the basis for our study of coniveau filtration.

**Remark** Most of notions for real intersection theory resemble those in Fulton’s intersection theory. However the dependence of the extrinsic De Rham data distinguish them from each other.

Let’s come back to the setting of the cone data. First we let all spaces be equipped with De Rham data where the Cartesian product has product De Rham data, and each factor has projection De Rham data (in order to use the projection formula). Let $\sigma$ be a singular cycle in $X$, which is also a Lebesgue current. Next we work with the real intersection theory. For each $t \in \mathbb{P}^1$, we define $\Psi_t(\sigma)$ to be a family of currents

\[
(\eta_t)_*[\Theta \wedge (\{t\} \otimes Y \otimes \sigma \otimes X)] \tag{1.22}
\]

where $\eta_t : \mathbb{P}^1 \times Y \times X \times X (4th \ factor)$ is the projection to the last factor $X$. Each member $\Psi_t(\sigma)$ is a closed current called $t$-end cycle. The main theorem is a result of careful analysis of three EXTRINSICALLY determined end cycles: $\Psi_0(\sigma), \Psi_1(\sigma), \Psi_\infty(\sigma)$. (However in application they’ll not have a common $\sigma$).

Real intersection theory shows
Proposition 1.5. All end cycles above are homotopic.

This is because the calculation of intersection of currents for any two points \( t_1, t_2 \in \mathbb{P}^1 \) leads to the homotopy

\[ \Psi_{t_1}(\sigma) - \Psi_{t_2}(\sigma) = d\Lambda \]  

where \( \Lambda \) is a current of higher dimension and \( d \) is the differential of currents.

- 1-end cycle

Proposition 1.6. Let \( C_0(X) \) be the subspace consisting of closed Lebesgue currents. There is an algebraic cycle \( \omega \in Z_n(X \times X) \) extrinsically determined such that

\[ \Psi_1(\sigma) = m\sigma + \omega(\sigma) \]  

where \( m \) is some natural number.

- \( \infty \)-end cycle

This end cycle when restricted to a Zariski open set gives the factorization of the identity in (1.6). It however does not have a straightforward process.

Let \( V^h = \text{div}(\mu^*(x_{n+1-h})) \cap \cdots \cap \text{div}(\mu^*(x_{n+1})) \) be the \( h \)-codimensional, smooth, irreducible plane section of \( X \) by the very ample line bundle \( \mu^*(\mathcal{O}_{\mathbb{P}^{n+1}}(1)) \). The decomposition (1.11) has a natural projection,

\[ \mathbb{C}^{n+2-h} \oplus C^h_z \rightarrow \mathbb{C}^{n+2-h}, z \in U \]  

which gives the rational map,

\[ \{\infty\} \times U \times \mathbb{P}^{n+1} \rightarrow \mathbb{P}^{n+1-h}, \quad \{\infty\} \times U \times \mathbb{P}^{n+1} \rightarrow \mathbb{P}^{n+1-h} \]  

\( x = x_1(z) \oplus x_2(z) \) is the unique decomposition (1.11). Let \( G \) be the transpose of its graph (transposed in the order \( \{\infty\} \times U \times \mathbb{P}^{n+1-h} \)). Let \( \hat{I}_h \) be the intersection cycle

\[ (\{\infty\} \times U \times V^h \times X) \cdot G. \]  

We denote the linear operator (for currents),

\[ C_o(V^h) \rightarrow C_o(\mathbb{P}^{n+1}) \]  

\[ \alpha \rightarrow \frac{1}{m+2}(\eta_k)_* [\hat{I}_h \wedge (\{\infty\} \otimes U \otimes \alpha \otimes X)] \]  

(1.28)
by $\text{Con}_h$, where the closure is defined as the convergence of currents (see Definition 3.6), and $d = \deg(\mu^*(\mathcal{O}_{\mathbb{P}^{n+1}}))$

Let $v^h$ be the linear operator

$$
\begin{align*}
\mathcal{C}_o(X) & \xrightarrow{v^h} \mathcal{C}_o(V^h) \\
\sigma & \mapsto [\sigma \wedge V^h]_{V^h},
\end{align*}
$$

where $[\sigma \wedge V^h]_{V^h}$ is the intersection of currents of manifold $V^h$.

**Proposition 1.7.** Then the end cycle at infinity has a factorization,

$$
\Psi_\infty(\sigma) = (m + d)\text{Con}_h \circ v^h(\sigma) + T_\sigma
$$

(1.29)

for $\sigma \in \mathcal{C}_o(X)$ where $T_\sigma \in \mathcal{C}(X)$ is homologous to zero.

**Remark** The factorization in (1.29) has the direct link to the factorization in (1.6). However the formula holds for closed currents, not for the cohomology classes.

- **0-end cycle**

**Proposition 1.8.** Let $\sigma \in \mathcal{C}_o(X)$. If $\dim(\sigma) < 2n - 2h$, then

$$
\Psi_0(\sigma)
$$

(1.30)

is homologous to a current supported on $V^h$.

Let $H_{p,V^h}(X; \mathbb{Q})$ be the homology supported on $V^h$. Then there is a sequence

$$
H_p(V^h; \mathbb{Q}) \xrightarrow{\nu_1} H_{p,V^h}(X; \mathbb{Q}) \xrightarrow{\nu_2} H_p(X; \mathbb{Q})
$$

(1.31)

with the surjective $\nu_1$. Proposition 1.8 leads to a corollary,

**Corollary 1.9.** The map $\nu_2$ is surjective for $p < 2n - 2h$. Therefore the inclusion map which is the composition of (1.31),

$$
H_p(V^h; \mathbb{Q}) \rightarrow H_p(X; \mathbb{Q})
$$

is also surjective.

**Remark** The corollary is a result in cohomology. It requires a cohomological descend, Proposition 1.10. The corollary extends Lefschetz hyperplane theorem to smaller dimensions. However the assertion is weaker on the larger dimensions. The proof of Corollary 1.8 does not resemble the currently known proofs of Lefschetz hyperplane theorem.
1.2.2 Two different aspects of end cycles

End cycles have two contrasted aspects that yield the main theorem. First is the cohomological descend, which is purely topological. The second is algebro-geometric about the preservation of the level of coniveau filtration.

- **Cohomological descend.** First we consider the 1-end cycle. In transcendental geometry, we will have

\[ \langle \omega_*(\sigma) \rangle \cup u = d \langle \sigma \rangle \cup u \]  

where \( u \) is the hyperplane section class \( c_1(\mu^*\langle \mathcal{O}_{P^{n+1}} \rangle) \) and \( d = \text{deg}(\mu^*\langle \mathcal{O}_{P^{n+1}} \rangle) \). The hard Lefschetz theorem implies an equality in cohomology,

\[ \langle \omega_*(\sigma) \rangle = d \langle \sigma \rangle \]  

for \( \text{deg}(\sigma) > n \). By the symmetry, (1.33) is extended to all non-middle dimensional cohomology groups.

**Proposition 1.10.** So for \( \sigma \in C_0(X) \) and \( \text{dim}(\sigma) \neq n \),

\[ \frac{1}{m+d} \langle \Psi_1(\sigma) \rangle = \langle \sigma \rangle \]  

Therefore

\[ \frac{1}{m+d} \langle \Psi_0(\sigma) \rangle = \langle \sigma \rangle. \]  

**Remark** In application, (1.34) and (1.35) may not be applied to a common \( \sigma \).

Now we study the descend of \( \infty \)-end cycle. By the proposition 1.7, it amounts to prove the cohomological descend of \( \text{Con}_h \). Nevertheless this descending in general is false due to the existence of primitive cycles in cohomology. So we show the cohomological descend for a specific case. We restrict \( \text{Con}_h \) to the closed Lebesgue currents of dimension \( n-h \) on \( V^h \). We denote the restriction map by \( \text{Con}_{h,p} \) where \( p = n - h \). Applying the real intersection theory, we obtain that

\[ v^h \circ \text{Con}_{h,p}(\delta) = \delta + J_{\delta}, \]  

for \( \delta \in C_0(V^h) \), where \( J_{\delta} \) is a current homologous to 0. The hard Lefschetz theorem on the equality (1.36) asserts that

\[ \delta \xrightarrow{\text{homological eqv}} 0 \text{ in } X \implies \text{Con}_{h,p}(\delta) \xrightarrow{\text{homological eqv}} 0 \text{ in } X. \]

Hence \( \text{Con}_{h,p} \) descends to the cohomology as an isomorphism on the cohomology

\[ \langle \text{Con}_{h,p} \rangle : H^{n+h}_V(X; \mathbb{Q}) \xrightarrow{\cong} H^{n-h}(X; \mathbb{Q}). \]  

(1.37)
where $H_{V^h}^{n-h}(X; \mathbb{Q})$ denotes the cohomology with the support on $V^h$.

- Level of coniveau filtration

It is obvious that $(\text{Con}_{h,p})$ represents the Poincaré duality which does not contain the algebro-geometric information. But the specific isomorphism $(\text{Con}_{h,p})$ does. In general the intersection of currents is additive on the index - level. Since the $\text{Con}_{h,p}$ is obtained by intersecting with the algebraic cycles which has level 0, $\text{Con}_{h,p}$ preserves the level in algebraic geometry. So does its cohomological descend. More precisely using real intersection theory we obtain that $(\text{Con}_{h,p})$ sends the level $k$ cycles onto the level $k$ cycles for any $k$. So

**Proposition 1.11.**

\[
(\text{Con}_{h,p}) : N^q H^{2q+k}_{V^h}(X; \mathbb{Q}) \rightarrow N^p H^{2p+k}(X; \mathbb{Q})
\]

(1.38)

is an isomorphism where $p + q = n - k$.

This is the second aspect of end cycles – algebraicity.

At last we use the same homotopy for the end cycle at 0, specifically Corollary 1.9, to obtain the isomorphism

\[
H^{2q+k}_{V^h}(X; \mathbb{Q}) \simeq H^{2q+k}(X; \mathbb{Q})
\]

(1.39)

and furthermore the isomorphism preserves the level of cycles as before. Therefore there is an isomorphism

**Proposition 1.12.**

\[
N^q H^{2q+k}_{V^h}(X; \mathbb{Q}) \simeq N^q H^{2q+k}(X; \mathbb{Q}).
\]

So we obtain Main theorem

\[
N^q H^{2q+k}_{V^h}(X; \mathbb{Q}) \simeq N^p H^{2p+k}(X; \mathbb{Q}).
\]

(1.40)

In the following sections we give the details. In section 2, we construct the cone family of currents. It starts in subsection 2.1, where we give a construction of a family of algebraic cycles. In subsection 2.2, we intersect the algebraic cycles with singular cycles to obtain a family of currents. Main theorem comes from the study of three particular members of the family called end cycles. Thus in section 3, we study the end cycles as currents. In sections 4, we address two contrasted aspects of end cycles – algebraicity and cohomologicity that result in Main theorem.
2 Cone family

The cone family of currents is a homotopy deduced from an algebro-geometric deformation with parameter space $\mathbb{P}^1$. So first we introduce the algebraic deformation.

2.1 Family of algebraic cycles

We continue with the cone data established in section 1. In this subsection we prove Propositions 1.3 which is the understanding of the family of algebraic cycles. See [7] for the proof of Proposition 1.3.

2.2 Family of cycles of currents

Proof. of Proposition 1.5: Let $\epsilon, 0 \in \mathbb{P}^1$. Let $S^1 \subset \mathbb{P}^1$ be a real circle that pass through $\epsilon, 0$. Continuing from the setting of Proposition 1.5, we defined the current

$$J = [\Theta \wedge (S^1 \otimes Y \otimes X \otimes X)]$$

which is closed. Now we use the diffeomorphism between a piece of $S^1$ and the closed interval $[0, \epsilon]$. Applying Proposition 4.12, [6], we obtain that

$$[J_\epsilon \wedge (Y \otimes \sigma \otimes X)] - [J_0 \wedge (Y \otimes \sigma \otimes X)] = d(-1)^k p ((P_X)_* J_{I, \epsilon}(T))$$

where $k = \text{deg}(J), p = \text{deg}(T), P_X : \mathbb{P}^1 \times X \to X$ is the projection. Hence

$$[J_\epsilon \wedge (Y \otimes \sigma \otimes X)]$$

is homotopic to

$$[J_0 \wedge (Y \otimes \sigma \otimes X)]$$.

By the associativity of currents intersection, Property 2.6, [6], we have

$$[J_\epsilon \wedge (Y \otimes \sigma \otimes X)] = [\Theta \wedge (\{\epsilon\} \otimes Y \otimes \sigma \otimes X)]$$

and

$$[J_0 \wedge (Y \otimes \sigma \otimes X)] = [\Theta \wedge (\{0\} \otimes Y \otimes \sigma \otimes X)]$$.

This completes the proof.

3 End cycles in the cone family

3.1 Algebraic part of end cycles

To sufficiently understand the end cycles, we must have detailed analysis on the family of algebraic cycles $\Theta$. To do that, we use the following coordinates from cone data. Let $x_0, \ldots, x_{n+1}$ be the coefficients of the basis $e_0, \ldots, e_{n+1}$
for $\mathbb{C}^{n+1}$. Then $x_0, \cdots, x_{n+1}$ are homogeneous coordinates for $\mathbb{P}^{n+1}$. Let $z \neq 0$ be complex numbers parametrize the affine neighborhood of $\Upsilon$. Then the homogeneous coordinates for $\mathbb{P}^{n+1-h}$, $\mathbb{P}^{h-1}$ as in the decomposition (1.11) are

\[
\begin{align*}
X_1(z) &= x_0 + \frac{x_{n+2-h}}{z}, x_1, \cdots, x_{n+1-h} \\
X_2(z) &= \frac{x_{n+2-h}}{z}, x_{n+3-h}, \cdots, x_{n+1}.
\end{align*}
\]  (3.1)

In the product

\[
\mathbb{P}^1 \times \Upsilon \times \mathbb{P}^{n+1} \times \mathbb{P}^{n+1},
\]

the coordinates for the third factor $\mathbb{P}^{n+1}$ has $x$-coordinates as above. The same coordinates for the last factor $\mathbb{P}^{n+1}$ in the product will be denoted by the letter $y$,

\[
\begin{align*}
y_1(z) &= y_0 + \frac{y_{n+2-h}}{z}, y_1, \cdots, y_{n+1-h} \\
y_2(z) &= \frac{y_{n+2-h}}{z}, y_{n+3-h}, \cdots, y_{n+1}.
\end{align*}
\]

In the following we focus on the component of its fibre at three points $t = 0, 1, \infty$.

- **Case $t = 1$.**

  Let’s to find the algebraic cycle

  \[
  \Theta^*_1 \subset \{1\} \times U \times X \times X,
  \]

  i.e. $z$ is steady. We push the cycles forward to the projective spaces

  \[
  \{1\} \times U \times \mathbb{P}^{n+1} \times \mathbb{P}^{n+1}
  \]

  to intersect with the subvariety (of codimension 2),

  \[
  \{1\} \times U \times \mu(X) \times \mu(X).
  \]

  Notice $\mu(X)$ is a hypersurface of $\mathbb{P}^{n+1}$. Assume $\mu(X)$ is defined by a polynomial $f$. Then $(\mu^2)_*(X \times X)$ is a complete intersection defined by two polynomials $f(x), f(y)$ in

  \[
  \mathbb{P}^{n+1} \times \mathbb{P}^{n+1},
  \]

  where $x,y$ are coordinates for the two copies of $\mathbb{P}^{n+1}$ in the Cartesian product.

  Then $\Omega^*_t \cap \mu^2(X \times X)$, denoted by $\Theta^*_t$ is explicitly defined by

  \[
  f(x_1, tx_2) = f(x_1, x_2)
  \]

  where $t \in \mathbb{C}$ and $x_i = x_i(z)$ as the coordinates split in (3.1). Observe the expansion

  \[
  f(x_1, tx_2) - f(x_1, x_2) = (t-1)^r q_{r}^x + (t-1)^{r+1} q_{r+2}^x + \cdots.
  \]

  Then the specialization as $t \to 1$ in $\Delta_h$ of coordinates $[x_1, x_2]$ is defined two polynomials

  \[
  f(x_1, x_2) = q_0^x(x_1, x_2) = 0.
  \]
Therefore $\Theta_1^*$ is the specialization which is equal to some hypersurfaces $\{g_1 = 0\}$ of the divisor
$$\{f = 0\} \simeq X \subset \Delta_{P^{n+1}}$$
of the projective space. Thus the projection of $\Theta_1|_{U \times X \times X} \subset U \times X \times X$ to $X \times X$ has the closure that equals to the diagonal $\Delta_X$. If $P : Y \times X \times X \to X \times X$ is the projection, then as an algebraic cycle,
$$P^*(\Theta_1|_{U \times X \times X}) = m\Delta_X,$$
where $m$ is some natural number, and the closure is defined as the cycle projection of the graph of the rational map,
$$P|_{U \times X \times X} : \Theta_1|_{U \times X \times X} \to X \times X.$$

Now we consider the unsteady point $\infty$. The fibre $\Omega_1^\infty$ is an irreducible subvariety of dimension $n + 2$ in $P^{n+1} \times P^{n+1} \times F^{n+1}$ defined by
$$x_i y_j = x_j y_i, 1 \leq i, j \leq n + 1$$
(3.3)
Let’s denote it by $K_1$. So $\omega = K_1 \cdot (X \times X)$ has dimension $n$. This is a part of the cycle $\Theta_1$, originated from the fibre in $\{\infty\} \times \{1\} \times X \times X$. Hence we have

**Proposition 3.1.**
$$P_*(\Theta_1) = m\Delta_{X \times X} + \omega.$$ (3.4)

- Case $t = \infty$.

$\Omega_\infty$ is a subvariety defined by
$$\left\{ \begin{array}{l}
x_i y_j = x_j y_i, 1 \leq i, j \leq n + 1 - h \\
x_i y_j = x_j y_i, n - h + 2 \leq i, j \leq n + 1 \\
(zx_0 + x_{n+2-h})y_j = (zy_0 + y_{n+2-h})x_j, 1 \leq j \leq n + 1 - h \\
x_i y_j = 0, 1 \leq j \leq n + 1 - h, n - h + 2 \leq i \leq n + 1, \\
x_j (zy_0 + y_{n+2-h}) = 0, n - h + 2 \leq j \leq n + 1.
\end{array} \right.$$ (3.5)

First we consider the open component in
$$\{\infty\} \times U \times X \times X,$$
surjective to $U$, i.e., considering generic $z \neq 0$. Observing the fourth equations $x_i y_j = 0$, we obtain that $\Omega_\infty$ over generic for $z \in U$ has two components:
3 END CYCLES IN THE CONE FAMILY

$K_2$ defined by

$$
\begin{aligned}
&x_i = 0, n - h + 2 \leq i \leq n + 1, \\
x_i y_j - x_j y_i = 0, 1 \leq i, j \leq n + 1 - h \\
x_i y_j - x_j y_i = 0, n - h + 2 \leq i, j \leq n + 1 \\
(z x_0 + x_{n+2-h}) y_j - (z y_0 + y_{n+2-h}) x_j = 0, 1 \leq j \leq n + 1 - h.
\end{aligned}
$$

(3.6)

and $K_3$ defined by

$$
\begin{aligned}
y_j = 0, 1 \leq j \leq n + 1 - h, \\
x_i y_j - x_j y_i = 0, 1 \leq i, j \leq n + 1 - h \\
x_i y_j - x_j y_i = 0, n - h + 2 \leq i, j \leq n + 1 \\
(z x_0 + x_{n+2-h}) y_j - (z y_0 + y_{n+2-h}) x_j = 0, 1 \leq j \leq n + 1 - h, \\
x_j (z y_0 + y_{n+2-h}) = 0, n - h + 2 \leq j \leq n + 1.
\end{aligned}
$$

(3.7)

Pulling equations (3.6), (3.7) back to the variety

$$\{\infty\} \times U \times X \times X,$$

and taking the closure, we obtain the cycle $\Phi_1$ of the scheme

$$\tau^{-1}(K_2) \cap (\{\infty\} \times U \times X \times X).$$

The other components from the scheme $\tau^{-1}(K_3) \cap (\{\infty\} \times U \times X \times X)$ lie in

$$\{\infty\} \times \Upsilon \times X \times V^{n+1-h}.$$

At last we consider the components supported in

$$\{\infty\} \times \{0\} \times X \times X.$$

So $\Omega^0_\infty$ is defined by

$$
\begin{aligned}
x_i y_j = x_j y_i, 1 \leq i, j \leq n + 1 - h \\
x_i y_j = x_j y_i, n - h + 2 \leq i, j \leq n + 1 \\
x_{n+2-h} y_j = y_{n+2-h} x_j, 1 \leq j \leq n + 1 - h \\
x_i y_j = 0, 1 \leq j \leq n + 1 - h, n - h + 2 \leq i \leq n + 1, \\
x_j y_{n+2-h} = 0, n - h + 2 \leq j \leq n + 1.
\end{aligned}
$$

(3.8)

Similarly we observe the fourth equations. There are two types of components. The first one denoted by $F_1$ is defined by

$$
\begin{aligned}
x_{n+2-h} = \cdots = x_{n+1} = y_{n+2-h} = 0 \\
x_i y_j = x_j y_i, 1 \leq i, j \leq n + 1 - h
\end{aligned}
$$

and the second $F_2$ (for all $z$) lies in

$$\mathbb{P}^{n+1} \times \mathbb{P}^{h}.$$
where $P^h$ is the $h$-dimensional subspace defined by

$$x_i = 0, \ 1 \leq i \leq n+1-h.$$ 

Then the component of $\Theta_{\infty}$ originated from $F_2$ will be projected to $V^{n+1-h}$.

The other component, determined by $F_1$ has dimension $n-1$. So it can’t be an irreducible component of $\Theta_{\infty}$ (must be the boundary of the component $\Phi_1$).

So overall, we proved that

**Proposition 3.2.**

$$\Theta_{\infty} = \Phi_1 + \Phi_2$$ \tag{3.9}

where $\Phi_2$ is a cycle lying in

$\{\infty\} \times \Upsilon \times X \times V^{n+1-h}$.

Similarly, we’ll denote the projection of $\Phi_i$ to $\Upsilon \times X \times X$ also by $\Phi_i$.

- Case $t = 0$.
    
    Notice that $\Omega_t$ is defined by

    $$\begin{cases}
        x_iy_j = x_jy_i, 1 \leq i, j \leq n+1-h \\
        x_iy_j = x_jy_i, n-h+2 \leq i, j \leq n+1 \\
        y_ix_j = 0, 1 \leq j \leq n+1-h, n-h+2 \leq i \leq n+1, \\
        (zx_0 + x_{n+2-h})y_j = (zy_0 + y_{n+2-h})x_j, 1 \leq j \leq n+1-h \\
        y_j(zx_0 + x_{n+2-h}) = 0, n-h+2 \leq j \leq n+1.
    \end{cases}$$ \tag{3.10}

Then the equations indicate the components of $\Theta_0$ are divided into two types: first type is defined by as the pullback of

$$y_{n+1-h} = \cdots = y_{n+1} = 0.$$ \tag{3.11}

The second type denoted by $W$ is defined by the pullback of

$$x_1 = \cdots = x_{n+1-h} = 0.$$ \tag{3.12}

To summarize it

**Proposition 3.3.**

$$P_*(\Theta_0) = \zeta_1 + \zeta_2,$$ \tag{3.13}

where $\zeta_1$ lies in $W \times X$ and $\zeta_2$ lies in $X \times V^h$. 

3.2 Non-algebraic part of end cycles

In this subsection, we intersect the algebraic end cycles in the previous subsection with currents to obtain the end cycles in the cone family. We’ll focus on the $\infty$-end cycle.

• The 1-end cycle

Proof. of Proposition 1.6: First we recall the proposition. Let $\sigma$ be a singular cycle on $X$. Then we need to prove that with product De Rham data on $X \times X$ and projection De Rham data on $X$,

$$\Psi_1(\sigma) = m\sigma + \omega_*(\sigma) \quad (3.14)$$

where $m$ is a natural number, where $\cdot_*$ denotes the correspondence of currents (see [6]).

We notice that the factorization of the projection yields

$$\Psi_1(\sigma) = (P_* (\Theta_1))_*(\sigma). \quad (3.15)$$

By Proposition 3.1

$$(P_* (\Theta_1))_*(\sigma) = m(\Delta_{X \times X})_*(\sigma) + \omega_*(\sigma) \quad (3.16)$$

By Claim 2.11, [6],

$$m(\Delta_{X \times X})_*(\sigma) = m\sigma. \quad (3.17)$$

This completes the proof.

• $\infty$-end cycle

Proposition 3.4. For any $\sigma \in C(X)$,

$$\Psi_\infty(\sigma) = (\Phi_1)_*(\sigma) + (\Phi_2)_*(\sigma) \quad (3.18)$$

where $(\Phi_1)_*$, similar to (1.22), is

$$(\mu_4)_*[\Phi_1 \wedge \{\infty\} \otimes Y \otimes \sigma \otimes X].$$

Proof. This is the consequence of Proportion 3.2.

In the following we calculate the currents $(\Phi_1)_*(\sigma), (\Phi_2)_*(\sigma)$. 

$\square$
Lemma 3.5.

\[(\Phi_2)_*(\sigma) = 0.\] (3.19)

Proof. If \((\Phi_2)_*(\sigma) \neq 0\), then it has dimension \(\geq 2n - 2h - 1\). However \((\Phi_2)_*(\sigma)\) is the inclusion of a current in \(V^{n+1-h}\). Since

\[\dim(V^{n+1-h}) = 2n - 2h\]

\(V^{n+1-h}\) can’t have a non-zero current of dimension \(> 2n - 2h\). We have a contradiction. We complete the proof. \(\square\)

Our observation to the Lefschetz standard conjecture is that the factorization of the identity (1.6) will occur in an open set of the projective variety, i.e in a quasi-projective variety for non-quotient geometric objects – the \(\infty\)-end cycle. For any algebraic cycle \(A\), its intersection with the open set is denoted by \(\hat{A}\). Then we need a technique to pass on from a quasi-projective to a projective. This is the closure in the following definition.

Definition 3.6. Let \(X\) be a manifold and \(U \subset X\) be an open set such that \(\overline{U} = X\). Let \(T\) be a current on \(U\). For any \(\phi \in \mathcal{D}(X)\), if the evaluation \(\int_U \phi\) as an improper integral is convergent, then we define the current \(\overline{T}\) by the convergent evaluation

\[\int_{\overline{T}} \phi := \int_U \phi.\] (3.20)

(The continuity of the functional also follows from formula (3.20))

Example 3.7. \(X\) is a smooth projective variety over \(\mathbb{C}\), and \(T\) is an irreducible subvariety. Let \(U\) be a non-empty Zariski open set of \(X\). Then currents’ closure \(\overline{T} \cap \overline{U} = T\) provided \(T \cap U \neq \emptyset\).

The following proposition shows that the passing on from quasi-projective to projective is cohomologically trivial.
Proposition 3.8. For \( \sigma \in C_o(X) \),

(1) \[
\left[ \Phi_1 \wedge (U \otimes \sigma \otimes X) \right]
\]

is well-defined in \( C_o(\Upsilon \times X \times X) \).

(2) then the current

\[
\left[ \Phi_1 \wedge (\Upsilon \otimes \sigma \otimes X) \right] - \left[ \Phi_1 \wedge (U \otimes \sigma \otimes X) \right]
\]

is homologous to zero.

Furthermore Proposition holds for arbitrary smooth projective variety \( Y \) in the product

\( \Upsilon \times Y \times X \)

with arbitrary algebraic cycle \( \Phi_1 \).

Proof. Let \( \Upsilon, X \) be equipped with De Rham data, and the subset \( U \subset \Upsilon \) be equipped with the same De Rham data as \( \Upsilon \). Then \( \Upsilon \times X \times X \) and \( U \times X \times X \) are equipped with product De Rham data. Assume \( X \), the second copy of \( X \) in \( U \times X \times X \) is changed to the projection De Rham data with respect to the product De Rham data of \( U \times X \times X \). That projection De Rham data must be the same projection De Rham data with respect to the product De Rham data on \( \Upsilon \times X \times X \). In particular, projection formulas in Proposition 2.8, [6] for currents’ projections

\( P_2 : \Upsilon \times X \times X \to X(2nd \ factor), P_2 : U \times X \times X \to X(2nd \ factor) \)

hold.

Part (1). Let \( \phi \in \mathcal{D}(\Upsilon \times X \times X) \). Let \( \phi^\circ \) be the distribution whose restriction to \( U \times X \times X \) is \( \phi \) and 0 at \( \{\infty\} \times X \times X \). Then there is a family of currents \( \phi_n, n \in \mathbb{N} \), such that \( \phi_n \in \mathcal{D}(\Upsilon \times X \times X) \) are \( C^\infty \), supported in \( U \times X \times X \) and \( \lim_{n \to \infty} \phi_n = \phi \) in \( \mathcal{D}'(\Upsilon \times X \times X) \). Let \( P_2 : U \times X \times X \to X(2nd \ factor) \) be the projection. By the definition, \( \left[ \Phi_1 \wedge (U \otimes \sigma \otimes X) \right] \) evaluated at \( \phi \) is defined to be the limit

\[
\lim_{n \to \infty} \int_{\left[ \Phi_1 \wedge (U \otimes \sigma \otimes X) \right]} \phi_n.
\]
So we calculate
\[
\lim_{n \to \infty} \int_{\Phi_1 \wedge (U \otimes \sigma \otimes X)} \phi_n
\] (3.22)
\[
= \lim_{n \to \infty} \lim_{\epsilon \to 0} \int_{\Phi_1} (P_2)^*(R_\epsilon(\sigma)) \wedge \phi_n
\] (3.23)
(Change the improper integral to proper integral.)
\[
= \lim_{n \to \infty} \lim_{\epsilon \to 0} \int_{\Phi_1} (P_2)^*(R_\epsilon(\sigma)) \wedge \phi_n
\] (3.24)
\[
= \lim_{n \to \infty} \int_{(P_2)_*[\Phi_1 \wedge \phi_n] \wedge \sigma} 1.
\] (3.25)

Next we calculate the family of currents \((P_2)_*[\Phi_1 \wedge \phi_n] \wedge \sigma\). We should note the limit of currents \((P_2)_*[\Phi_1 \wedge \phi_n]\) exists as the functional
\[
f \to \int_{\Phi_1} \phi \wedge (P_2)^* f,
\]
for any test form \(f \in \mathcal{D}(X)\). This functional is equal to
\[
(P_2)_*[\Phi_1 \wedge \phi].
\]
The following is the argument of the assertion. By the same result of Lemma 4.1 in [6], \((P_2)_*[\Phi_1 \wedge \phi]\) is Lebesgue. Then
\[
\int_{(P_2)_*[\Phi_1 \wedge \phi] - (P_2)_*[\Phi_1 \wedge \phi]} f = \int_{\Phi_1} (\phi_n - \phi) \wedge (P_2)^* (f).
\] (3.26)
So (3.26) as \(n \to \infty\) converges to 0 uniformly on bounded set of forms \(f\).

Hence \((P_2)_*[\Phi_1 \wedge \phi_n]\) converges to \((P_2)_*[\Phi_1 \wedge \phi]\) in the space of currents. By the particular type of continuity \(*)\) of the intersection – Proposition 4.12, [6], \((P_2)_*[\Phi_1 \wedge \phi_n] \wedge \sigma\) converges to \((P_2)_*[\Phi_1 \wedge \phi] \wedge \sigma\). Hence the number (3.25) converges to
\[
\int_{(P_2)_*[\Phi_1 \wedge \phi] \wedge \sigma} 1.
\] (3.27)

Next we show it is closed. We have
\[
d[\bar{\Phi}_1 \wedge (U \otimes \sigma \otimes X)] = [d\bar{\Phi}_1 \wedge (U \otimes \sigma \otimes X)] \pm [\bar{\Phi}_1 \wedge (dU \otimes \sigma \otimes X)].
\] (3.28)

Since \(\bar{\Phi}_1, U\) both are quasi-projective, the currents of integration over them are closed. So
\[
d[\bar{\Phi}_1 \wedge (U \otimes \sigma \otimes X)] = 0.
\]

\(^*\) There is no general claim for the continuity of intersection. For instance it requires flatness in case of algebraic geometry.
This proves part (1).

Part (2). Since \( \circ \Phi_1 \wedge (U \otimes \sigma \otimes X) \) is well-defined, it suffices to evaluate the current of (3.21) at the closed forms. By Künneth decomposition it suffices to evaluate at two \( C^\infty \) forms,

\[ 1 \otimes \omega, \psi \otimes \omega \]

where \( \omega \) is a closed form on \( X \times X \) and \( \psi \) is a \( C^\infty \) 2-form supported in a neighborhood of a generic point in \( Y \simeq \mathbb{P}^1 \) and Poincaré dual to the point. Then

\[
\int_{[\Phi_1 \wedge (U \otimes \sigma \otimes X)]} \psi \otimes \omega \\
= \int_{[\Phi_1 \wedge (U \otimes \sigma \otimes X)]} \psi \otimes \omega \\
= \int_{[\Phi_1 \wedge (U \otimes \sigma \otimes X)]} \psi \otimes \omega. \tag{3.31}
\]

For the form \( 1 \otimes \omega \), we calculate

\[
\int_{\Phi_1 \wedge (Y \otimes \sigma \otimes X)} 1 \otimes \omega \\
= \lim_{\epsilon \to 0} \int_{\Phi_1} (P_2)^* (R_\epsilon (\sigma)) \wedge (1 \otimes \omega) \tag{3.32}
\]

( (3.32) only holds for the particular type of forms in \( 1 \otimes \omega \).) \tag{3.33}

\[
= \int_{[\Phi_1 \wedge (Y \otimes \sigma \otimes X)]} 1 \otimes \omega \tag{3.34}
\]

where (3.34) is an improper integral. But in part (1), we have shown

\[
\lim_{n \to \infty} \int_{[\Phi_1 \wedge (U \otimes \sigma \otimes X)]} (\bullet) \tag{3.35}
\]

is convergent. So we obtain that

\[
\int_{[\Phi_1 \wedge (Y \otimes \sigma \otimes X)] - [\Phi_1 \wedge (U \otimes \sigma \otimes X)]} \alpha = 0 \tag{3.36}
\]

for any closed form \( \alpha \) on \( Y \times X \times X \). So we complete the proof of part (2). Furthermore the argument still holds when the second factor \( X \) in \( Y \times X \times X \) is replaced by arbitrary smooth projective variety \( Y \) and algebraic cycle \( \Phi_1 \).

**Proof.** of Proposition 1.7: By Proposition 3.2, \( \Phi_1 \) is the projection of the closure of the zero locus

\[
\Gamma \tag{3.37}
\]
on quasi-projective variety
\[ \infty \times U \times X \times X \]  
(3.38)
of restricted sections
\[
\begin{cases}
\alpha_{ij} = \tau^* (x_i y_j - x_j y_i), 1 \leq i, j \leq n + 1 - h, \\
\alpha_{ij} = \tau^* (x_i y_j - x_j y_i), n + 2 - h \leq i, j \leq n + 1 \\
\alpha_j = \tau^* \left( (z x_0 + x_{n+2-h}) y_j - (z y_0 + y_{n+2-h}) x_j \right), 1 \leq j \leq n + 1 - h \\
\tau^* (x_i), n + 2 - h \leq i \leq n + 1.
\end{cases}
\]  
(3.39)
where \( \tau^*(\bullet) \) are regarded as sections of the line bundle over \( \infty \times U \times X \times X \).

The following is the key observation of equations (3.39) for \( \Gamma \): \( \Gamma \) is a local complete intersection of codimension \( n + 1 + p \). Let’s see this. We divide the collection of equations (3.39) into two parts. First let
\[ \bar{I}_h \subset \{\infty\} \times U \times X \times X \]
be the zero locus of
\[
\begin{cases}
\alpha_{ij}, 1 \leq i, j \leq n + 1 - h, \\
\alpha_{ij}, n + 2 - h \leq i, j \leq n + 1 \\
\alpha_j, 1 \leq j \leq n + 1 - h.
\end{cases}
\]  
(3.40)
This portion defines an l.c.i. on an open set of codimension \( n + 1 \). Notice that \( \bar{I}_h \) has been defined in introduction but through a different expression. Secondly we also notice
\[ \{\infty\} \times U \times V^h \times X \]
is the zero locus of
\[ \tau^* (x_i), n + 2 - h \leq i \leq n + 1 \]  
(3.41)
where \( x_i \) are the \( \tau^* (\mathcal{O}_{\mathbb{P}^{n+1}}(1)) \) sections over the first copy of \( \mathbb{P}^{n+1} \). This portion defines a complete intersection of codimension \( h \). Hence the grouping of the equations (3.39) determines that the cycle \( \Phi_1 \) is the cycle of the scheme
\[ \bar{I}_h \cap \{\infty\} \times U \times V^h \times X \cap \{\infty\} \times U \times X \times X. \]  
(3.42)
Now we observe the intersection \( \{\infty\} \times U \times V^h \times X \cap \{\infty\} \times U \times X \times X \) is proper in \( \{\infty\} \times U \times X \times X \), and the intersection
\[ \bar{I}_h \cap \left( \{\infty\} \times U \times V^h \times X \cap \{\infty\} \times U \times X \times X \right) \]
is also proper in \( \{\infty\} \times U \times V^h \times X \). Then we change the scheme intersection to cycle intersection to have the triple intersection
\[ \bar{I}_h \cap \left( \{\infty\} \times U \times V^h \times X \cap \{\infty\} \times U \times X \times X \right) \]  
(3.43)
where the two static intersections occur in two different quasi-projective varieties: \( \{\infty\} \times U \times X \times X \) and \( \{\infty\} \times U \times V^h \times X \). Applying the associativity in the intersection of currents, we obtain that the equality from the quasi-projective variety

\[
\infty \times U \times X \times X,
\]

\[
(\Phi_1)_*(\sigma) = \tilde{I}_h \cdot \left( \{\infty\} \otimes U \otimes (\sigma \wedge V^h) \otimes X \right)
\]

(3.44)

Therefore by Proposition 3.8, there is homologically trivial cycle \( T_\sigma \) such that

\[
\Psi_\infty(\sigma) = (\eta_4)_* \left( \tilde{I}_h \cdot \left( \{\infty\} \times U \times (\sigma \cdot V^h) \times X \right) \right) + T_\sigma
\]

\[
= (m + d) \text{Con} \circ v^h(\sigma) + T_\sigma.
\]

(3.45)

We complete the proof of Proposition 1.7.

\[ \square \]

- The 0-end cycle

\[ \text{Proof. of Proposition 1.8: First we know} \]

\[
P_*(\Theta_0) = \zeta_1 + \zeta_2.
\]

Let \( \sigma \in C_0(X) \). If \( \text{dim}(\sigma) < 2n - 2h \), then \( W \cap \text{supp}(\sigma) = \emptyset \). Therefore \( (\zeta_1)_*(\sigma) = 0 \). Applying Proposition 4.5, [6] concerning the support of intersection, we obtain that \( \text{supp}((\zeta_2)_*(\sigma)) \subseteq V^h \).

we complete the proof.

\[ \square \]

4 Algebraicity and cohomologicity

4.1 Algebraicity

\[ \text{Definition 4.1. Let } X \text{ be a smooth projective variety of dimension } n \text{ over } \mathbb{C}. \]

\[ \text{Let } T \in (\mathcal{D}')^{2p+k}(X) \text{ be a real current of degree } 2p + k. \text{ If the support of } T \text{ lies in an algebraic set of codimension } p, \text{ we say } T \text{ has an algebraic level } k. \]

\[ \text{Remark The algebraic level of a current } T \text{ is not a unique number. But there is the minimum.} \]


Proposition 4.2. Let $\sigma \in C_o(V^h)$. If $\sigma$ has an algebraic level $k$, so does $\text{Con}_h(\sigma)$.

Remark This is the algebraicity of $\text{Con}_h$.

Proof. Let $\sigma \in C_o^{2n-i}$ of dimension $i$. As a current,

$$\text{Con}_h(\sigma) = \frac{1}{m+d}(\eta_4)_* \left[ \tilde{I}_h \wedge (\{\infty\} \times U \times \gamma \times X) \right], \quad (4.1)$$

has dimension $i + 2h$. So if $\sigma$ has the algebraic level $k$, we need to find an algebraic set of dimension $h + \frac{k+i}{2}$ containing $\text{Con}_h(\sigma)$. By the assumption, there is an algebraic cycle $B$ of dimension $\frac{i+k}{2}$ in $X$ such that

$$\text{supp}(\sigma) \subset B.$$

If $\Upsilon$ is generic with respect to $B$, then the algebraic cycle

$$I_h \cdot (\{\infty\} \times U \times B \times X) \quad (4.2)$$

is well defined, of dimension $h + \frac{k+i}{2}$. Applying Proposition 4.5, [6] concerning the support of intersection, we obtain that

$$(\eta_4)_*(I_h \cdot (\{\infty\} \times U \times B \times X)) \quad (4.3)$$

is an $h + \frac{k+i}{2}$ dimensional algebraic cycle containing the support of $\text{Con}_h(\sigma)$. Since the cohomology has a finite dimension, generic and fixed curve $\Upsilon$ can be selected for all $\text{Con}_h(\sigma)$ to have level $k$.

\[\square\]

4.2 Cohomology with support

- 1-end cycle

Proposition 4.3. For any natural number $i \neq n$, the currents’ correspondence $\omega_*$ descends to the isomorphism on the cohomology

$$\begin{align*}
H^i(X; \mathbb{Q}) & \xrightarrow{\theta} H^i(X; \mathbb{Q}) \\
\theta & \xrightarrow{d\theta}
\end{align*} \quad (4.4)$$

where $d = \text{deg}(\mu^*(\mathcal{O}_{\mathbb{P}^n;1}(1)))$. 
Proof. Applying the cohomology of currents’ intersection, Property 2.6, [6],
For a singular cycle \( \sigma \) of codimension \( i \),
\[
\langle \omega \ast (\sigma) \rangle = \langle \omega \rangle \ast \langle \sigma \rangle.
\] (4.5)
Hence it suffices to prove Proposition 4.3 for the cohomological correspondence
\( \langle \omega \rangle \ast \). First we consider the case \( i < n \). Let \( P^n \) be a hyperplane of \( P^{n+1} \) defined
by \( x_0 = 0 \). Let \( \theta \in H^i(X; \mathbb{Q}) \). \( \langle \omega \rangle \ast (\theta) \) is well-defined cycle class in \( H^i(X; \mathbb{Q}) \).
Let \( \phi \in H^{2n-2-i}(X; \mathbb{Q}) \). Let’s calculate
\[
\left( \langle \omega \rangle \ast (\theta) \cup \langle V^1 \rangle, \phi \right)
\] (4.6)
where \( V^1 \) is the hyperplane section \( \text{div}(\mu^*(x_0)) \), and \((\bullet \bullet)\) is the intersection
number. We use De Rham’s theorem to compute the intersection in the coho-
mology of real coefficients. So we assume \( \theta, \phi \) are closed smooth forms on \( X \). Let
\[
\begin{array}{c}
P_{r_1} \\
\uparrow \quad \uparrow \\
X \times X \\
\downarrow \quad \downarrow \\
X \\
\end{array}
\] (4.6)
be the two projections. Then by De Rham’s theorem
\[
\left( \langle \omega \rangle \ast (\theta) \cup \langle V^1 \rangle, \phi \right) = \int_{\omega \ast (X \times V^1)} P_{r_1}^* (\theta) \wedge P_{r_2}^* (\phi).
\] (4.7)
The right hand side of (4.7) is the integral over the algebraic cycle \( \omega \cdot (X \times V^1) \). There is covering map of degree \( d \) in \( P^{n+1} \)
\[
K : X \rightarrow P^n.
\] (4.8)
By the general position assumption on \( X \), the restriction
\[
K_{|V-1} : V^{-1} \rightarrow V^1
\] (4.9)
is also a covering map of degree \( d \), where \( V^{-1} \subset K^{-1}(V^1) \) is the component
dominating \( V^1 \). We found that
\[
\omega \cdot (X \times V^1)
\] is the cycle of the scheme
\[
\omega \cap (X \times V^1).
\]
Thus the covering map \( K \) gives the isomorphic diagram
\[
\begin{array}{c}
\omega \cap (X \times V^1) \\
\downarrow \rho \\
V^1
\end{array} \cong \begin{array}{c}
V^{-1} \\
\uparrow K
\end{array}
\] (4.10)
where $\omega \cap (X \times V^1)$ is isomorphic to the graph of $K_{V^{-1}}$, and $\rho$ is the projection from the graph of the map to the image of the map. Then we can use the computation in currents’ evaluation where algebraic varieties $\omega, V^1$ and the differential form $\theta$ are all currents.

$$\int_{\omega \cap (X \times V^1)} \text{Pr}_1^*(\theta) \wedge \phi = d \int_{V^1} \theta \wedge \phi = d \int_{V^1 \wedge \theta} \phi$$  \hspace{1cm} (4.11)

where $V^1 \wedge \theta$ is the intersection of currents. The computation (4.11) shows a cohomological equality

$$\langle \omega \rangle \ast (\theta) \cup \langle V^1 \rangle = k \langle \theta \rangle \cup \langle V^1 \rangle.$$  \hspace{1cm} (4.12)

Therefore

$$L(\langle \omega \rangle \ast (\theta)) = L(k \theta),$$  \hspace{1cm} (4.13)

where $L$ is the Lefschetz operator defined in (1.1). Applying the hard Lefschetz theorem, we obtain that

$$\langle \omega \rangle \ast (\theta) = k \theta.$$  \hspace{1cm} (4.14)

Next we assume $i > n$. Because $\omega$ is symmetric, the transpose $\langle \omega \rangle^T$ is equal to $\langle \omega \rangle$. Now we let $\theta, \alpha$ be cohomological classes of degrees $i$ and $2n - i$. We calculate the intersection number,

$$((\langle \omega \rangle \ast \theta), \alpha) = (\theta, (\langle \omega \rangle^T \alpha)$$

$$= (\theta, \langle \omega \rangle \ast \alpha)$$

$$= (\theta, d\alpha)$$

$$= (d\theta, \alpha).$$  \hspace{1cm} (4.15)

Therefore

$$\left(\langle \omega \rangle \ast \theta - d\theta, \alpha \right) = 0$$  \hspace{1cm} (4.16)

holds for all $\theta, \alpha$. We obtain that

$$\langle \omega \rangle \ast \theta = d\theta.$$  \hspace{1cm}

We complete the proof for all cases.

- ∞-end cycle

**Lemma 4.4.** Assume $V^h \times X$ is equipped with a product De Rham data. When restricted to the closed Lebesgue currents

$$C_{o}^{n+h}(V^h)$$

the operator $Con_{h,p}$ is cohomological for the homological equivalence of $X$, i.e. it sends an $n - h$ dimensional current of $V^h$, exact in $X$ to an $n + h$ dimensional current of $X$, exact in $X$. 


Proof. Let $\lambda \in (C_0)(X)$ be the pushforward of current of $\lambda_h \in C_0(V^h)$ under the inclusion map.

Claim 4.5.

$$v^h \circ \text{Con}_{h,p} (\lambda_h) = l\lambda_h + J_{\lambda_h},$$
\hspace{0.5cm} (4.17)

where $l$ is a natural number and $J_{\lambda_h}$ is homologous to zero.

Let $Pr_3 : \Upsilon \times V^h \times X \to X$ be the projection. Let the Cartesian product $\Upsilon \times X \times X$ have a product De Rham data and and the last factor $X$ have projection De Rham data. By the projection formula (Proposition 2.8, [6]),

$$V^h \wedge (Pr_3)_* [I^h \wedge (U \otimes \lambda_h \otimes X)]$$
\hspace{0.5cm} (4.18)

$$= (Pr_3)_* [(\Upsilon \otimes V^h \otimes V^h) \wedge [I^h \wedge (U \otimes \lambda_h \otimes X)]].$$
\hspace{0.5cm} (4.19)

By Proposition 3.8, which is also valid in this case,

$$[I^h \wedge (U \otimes \lambda_h \otimes X)]$$

is homologous to

$$[I_h \wedge (\Upsilon \otimes \lambda_h \otimes X)].$$

Hence (4.19) is equal to

$$(Pr_3)_* \left[ (\Upsilon \otimes V^h \otimes V^h) \wedge I_h \right] + J_{\lambda_h},$$
\hspace{0.5cm} (4.20)

where $J_{\lambda_h}$ is homologous to zero. Then the associativity says

$$(Pr_3)_* \left[ (\Upsilon \otimes V^h \otimes V^h) \wedge I_h \right] + J_{\lambda_h} = (Pr_3)_* \left[ (\Upsilon \otimes V^h \otimes V^h) \wedge I_h \right]$$
\hspace{0.5cm} (4.21)

We first consider the current $(\Upsilon \otimes V^h \otimes V^h) \wedge I_h$ which is the current of integration over the algebraic cycle

$$(\Upsilon \times V^h \times V^h) \cdot I_h.$$

By the definition of the variety $I_h$, the projection

$$Pr_{23} : (\Upsilon \times V^h \times V^h) \cdot I_h \to V^h \times V^h$$
\hspace{0.5cm} (4.23)

is a multiple cover of the diagonal $\Delta_{V^h}$ whose multiplicity is denoted by $l$. Let

$$Pr : V^h \times V^h \to V^h(2\text{nd factor})$$
be the projection. Using the calculation of the intersection of currents, specifically, projection formula in Proposition 2.8, and claim 2.11 in [6], we obtain that
\[
(Pr_3)_* \left( [\Upsilon \otimes V^h \otimes V^h] \right.
\left. \wedge \left[ I^h \wedge (\Upsilon \otimes \lambda^h \otimes X) \right] \right)
\]
\[= (Pr)_* \left[ \Delta V^h \wedge (\lambda^h \otimes X) \right] \] (4.25)
\[= l\lambda^h. \] (4.26)
Combining with (4.20), we complete the proof of Claim 4.5. Now we assume \( \lambda \) has degree \( n + h \), i.e. restrict \( \text{Con}^h \) to \( n - h \) dimensional cycles. Then we can apply hard Lefschetz theorem to obtain that if \( \lambda \) is cohomologically trivial in \( X \), so is \( \text{Con}^h, p(\lambda^h) \) in \( X \). This shows that \( \text{Con}^h, p \) is cohomological. We complete the proof.

**Proof.** of Proposition 1.11: Lemma 4.4 shows that \( \text{Con}^h, p \) is cohomological (with respect to the homological equivalence of \( X \)), i.e. \( \text{Coh}^h, p \) is reduced to a homomorphism \( \langle \text{Coh}^h, p \rangle \) on the homogeneous part of cohomology,
\[
\langle \text{Coh}^h, p \rangle : N^p H^{2p+k}_{V^h}(X) \rightarrow N^q H^{2q+k}(X)
\] (4.27)
where \( N^p H^{2p+k}_{V^h}(X) \) is the cohomology supported on \( V^h \). Furthermore (1.29) becomes
\[
\langle \text{Coh}^h, p \rangle \circ L^k_{\lambda} = \text{identity}.
\] (4.28)
Also by Claim 4.5 for \( \delta \in C_\sigma(X) \), we have the cohomological version
\[
u^h \cup \langle \text{Con}^h, p(\delta) \rangle = l_\delta(\delta).
\] (4.29)
for some rational number \( l_\delta \). In the following we prove that \( l_\delta = 1 \). Let \( \sigma \in C_\sigma(X) \).
We have
\[
u^h \cup \langle \text{Con}^h, p \circ \nu^h(\sigma) \rangle = l_{\nu^h(\sigma)} \nu^h \cup \langle \sigma \rangle
\] (4.30)
and on the other hand by (4.28)
\[
u^h \cup \langle \text{Con}^h, p \circ \nu^h(\sigma) \rangle = \langle \nu^h(\sigma) \rangle, \quad \text{in } X
\] (4.31)
We obtain that
\[
l_{\nu^h(\sigma)} \nu^h \cup \langle \sigma \rangle = \langle \nu^h(\sigma) \rangle.
\]
By the hard Lefschetz theorem for \( L^h \), \( l_{\nu^h(\sigma)} = 1 \), and \( l_\delta = 1 \) holds for all closed \( \delta \). This result has two implications:
\[
\langle \text{Con}^h, p \rangle \circ L^k_{\lambda} = \text{identity}, \quad \text{by (4.28)}
\]
\[
L^h_{\lambda} \circ \langle \text{Con}^h, p \rangle = \text{identity}, \quad \text{by (4.29)}.
\] (4.32)
Hence \langle \text{Coh}_{h,p} \rangle \text{ is an isomorphism}

\[ N^p H^2_{Vh}(X) \simeq N^q H^2_{Vh}(X). \quad (4.33) \]

\section*{4.3 Cohomology without support}

\begin{proof}

of Proposition 1.12: At \( t = 0 \), we consider \( \sigma \) having dimension \( 2p+k > 0 \).

By Proposition 1.8,

\[ \Psi_0(\sigma), \quad (4.34) \]

must lie in \( V^h \). By (1.35), a rational multiple of \( \sigma \) is homologous to

\[ \Psi_0(\sigma), \quad (4.35) \]

which a cycle of the same dimension, lying in \( V^h \). To consider the level, we let

\( A \) be an algebraic cycle containing \( \sigma \). The same operation \( \Psi_0(\cdot) \) takes \( A \) to an

algebraic cycle \( \Psi_0(A) \) of the same dimension, lying in \( V^h \). Therefore the level

is not changed. We have

Therefore

\[ N^q H^2_{Vh}(X) \simeq N^q H^2_{Vh}(X). \quad (4.36) \]

We complete the proof of Main theorem.
\end{proof}

\section*{A Coniveau Filtration of currents’ version}

While we review the definition of coniveau filtration, we’ll give another description using currents. Recall that in [4], Grothendieck created a filtration \( \text{Filt}^{p} \), which he called “Arithmetic filtration, as it embodies deep arithmetic properties of the scheme”. This later was referred to as the coniveau filtration, denoted by

\[ N^p H^{2p+k}(X) \]

where \( p \) is the coniveau and \( k \) is the level. It is defined as a linear span of kernels of the linear maps

\[ H^{2p+k}(X; \mathbb{Q}) \rightarrow H^{2p+k}(X - W; \mathbb{Q}) \quad (A.1) \]
for all subvarieties $W$ of codimension at least $p$. This is the original definition. In the same paper, Grothendieck immediately interpreted it as a linear span of images of Gysin homomorphisms

$$H^{\dim(W)+2p+k-2n}(\tilde{W};\mathbb{Q}) \rightarrow H^{2p+k}(X;\mathbb{Q}) \quad (A.2)$$

for all subvarieties $W$ of codimension at least $p$ with the smooth resolution $\tilde{W}$. Now it is known that the proof of the description requires Deligne’s mixed Hodge structures. We introduce another interpretation. It is through currents, which are known to unite both homology and cohomology. Let $D'(X)$ be the space of currents over $\mathbb{R}$ on $X$. Let $CD'(X)$ be its subset of closed currents and $ED'(X)$ be its subset of exact currents. Then

$$\frac{CD'(X)}{ED'(X)} = \sum H^\bullet(X;\mathbb{R}). \quad (A.3)$$

There is a restriction map on currents

$$\mathcal{R} : D'(X) \rightarrow D'(X - W) \quad (A.4)$$

for a subvariety $W$.

Using (A.3) and (A.4), we define

$$\mathcal{D}^pH^{2p+k}(X)$$

to be the linear span of classes in $H^{2p+k}(X;\mathbb{Q})$ such that they lie in

$$\frac{CD'(X) \cap \ker(\mathcal{R})}{ED'(X) \cap \ker(\mathcal{R})}. \quad (A.5)$$

for some $W$ of codimension at least $p$.

**Proposition A.1.** Let $X$ be a smooth projective variety over $\mathbb{C}$. Then

$$\mathcal{D}^pH^{2p+k}(X) = N^pH^{2p+k}(X). \quad (A.6)$$

It says that the cohomology class $\alpha$ lies in

$$N^pH^{2p+k}(X) \quad (A.7)$$

if and only if it is represented by a current whose support is contained in an algebraic set of codimension at least $p$.

**Proof.** See [8].
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