Using known zeta-series to elucidate the origin of the Dancs-He series for \( \ln 2 \) and \( \zeta(2n+1) \)

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Abstract. In a recent work, Dancs and He found new “Euler-type” formulas for \( \ln 2 \) and \( \zeta(2n+1) \), \( n \) being a positive integer, each containing a series that apparently can not be evaluated in closed form, distinctly from \( \zeta(2n) \), for which the Euler’s formula applies showing that the even zeta-values are rational multiples of even powers of \( \pi \). There, however, the formulas are derived through certain series manipulations, by following Tsumura’s strategy, which makes it curious — in the words of those authors themselves — the appearance of the numbers \( \ln 2 \) and \( \zeta(2n+1) \). In this short paper, I show how some known zeta-series can be used to derive the Dancs-He series in a more straightforward manner.

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1. Introduction

The Riemann zeta function \( \zeta(s) \) is defined, for complex numbers \( s \) with \( \Re(s) > 1 \), by

\[
\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}.
\]

There is also a product representation due to Euler (1749), namely \( \zeta(s) = \prod_p 1/(1 - p^{-s}) \), where the product is taken over all prime numbers \( p \), which is the main reason for the interest of number theorists in this function. As noted by Euler, since the harmonic series diverges, then from Eq. (1) one deduces that \( \lim_{s \to 1^+} \zeta(s) = \infty \), which implies, from the product representation, that there are an infinitude of prime numbers. For real values of \( s, s > 1 \), the series in Eq. (1) converges by the integral test and its sum for integer values of \( s \) has attracted much interest since the times of J. Bernoulli, who proved that \( \sum_{k=1}^{\infty} 1/k^2 \) converges to a number between 1 and 2. Further, Euler (1735) proved that this sum is equal to \( \pi^2/6 \), cracking the so-called Basel problem. He also studied this kind of series for greater integer values, finding, for even values of \( s \), the notable formula

\[
\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} \pi^{2n}}{(2n)!} B_{2n},
\]

where \( n \) is a positive integer and \( B_{2n} \) are Bernoulli numbers\(^1\). For odd values of \( s, s > 1 \), on the other hand, no analogous closed-form expression is known. In fact, except for the Apéry proof (1978) \([2]\), not even an irrationality proof is presently known for \( \zeta(2n+1) \), which makes the things enigmatic.

In trying to find out a closed-form expression for \( \zeta(2n+1) \) similar to that in Eq. (2), Dancs and He found a new “Euler-type” formula containing series involving the numbers \( E_{2n+1}(1) \), where \( E_{2n+1}(x) \) denotes the Euler’s polynomial of degree \( 2n + 1 \) \([3]\). Their main result follows from some intricate series manipulations, in the lines of those found in Tsumura’s proof of Eq. (2) \([1]\). As the Dancs-He series apparently can not be reduced to a finite closed form expression, some insight was given into why the odd case is more difficult. However, the fortuitous appearance of the numbers \( \ln 2 \) and \( \zeta(2n+1) \) in these formulas, which is hard to be explained with usual series expansions, might well remain a mystery.

By noting that the numbers \( E_{2n+1}(1) \) can be written in terms of \( B_{2n+2} \), and then in terms of \( \zeta(2n) \), via Eq. (2), I show here in this work how the

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\(^1\)Since \( B_{2n} \in \mathbb{Q} \) and \( \pi \) is a transcendental number, as proved by Lindemann (1882), then Eq. (2) implies that \( \zeta(2n) \) is a transcendental number.
Dancs-He series for $\ln 2$ and $\zeta(2n+1)$ can be derived from some known zeta-series.

2. The Dancs-He formula for $\ln 2$ from known zeta-series

For $\ln 2$, Dancs and He found that (see Eq. (2.6) of Ref. [3]):

**Theorem 1** (Dancs-He series for $\ln 2$). Let $E_{2m+1}(x)$ denote the Euler’s polynomial of degree $2m+1$, $m$ being a nonnegative integer. Then

\[
\frac{\pi^2}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\pi^{2m}}{(2m+3)!} E_{2m+1}(1) = \ln 2.
\]

**Proof.** Let $L$ be the number at the left-hand side of Eq. (3). By noting that $E_{2m+1}(1) = -E_{2m+1}(0) = 2^{2m+2}/2m+2 - 1/2m+2 B_{2m+2}$, one has

\[
L = \frac{\pi^2}{2} \sum_{m=0}^{\infty} (-1)^m \frac{\pi^{2m}}{(2m+3)!} \left(2^{2m+2} - 1/2m+2 - \frac{1}{2m+2} B_{2m+2}\right).
\]

By substituting $n = m + 1$, one finds that

\[
L = -\sum_{n=1}^{\infty} (-1)^n \frac{B_{2n}}{2n(2n+1)(2n)!} [(2\pi)^{2n} - \pi^{2n}] = -\sum_{n=1}^{\infty} (-1)^n \frac{\pi^{2n} B_{2n}}{(2n)!} \frac{2^{2n} - 1}{2n(2n+1)}.
\]

From Euler’s formula for $\zeta(2n)$ in Eq. (2), one has

\[
L = \sum_{n=1}^{\infty} \frac{1 - 2^{-2n}}{n(2n+1)} \frac{\zeta(2n)}{n(2n+1)}.
\]

Now, let us reduce this latter series to a simple closed-form expression. For this, let us make use of the following series representation for $\zeta(s)$ introduced recently by Tyagi and Holm (see Eq. (3.5) in Ref. [4]):

\[
\frac{\zeta(s) \cdot (1 - 2^{1-s})}{\pi^{s-1} \sin(\pi s/2)} = \sum_{n=1}^{\infty} (2 - 2^{s-2n}) \frac{\Gamma(2n-s+1)}{\Gamma(2n+2)} \zeta(2n-s+1),
\]

where $\Gamma(x)$ is the gamma function. As the series in the right-hand side converges when we make $s = 1$, all we need to do is to take the limit, as $s \to 1^+$, of the factors at the left-hand side. The simple pole of $\zeta(s)$ at $s = 1$ yields $\lim_{s \to 1^+} (s-1) \cdot \zeta(s) = 1$. Also, by applying the l’Hospital

\footnote{Note that $\Gamma(k) = (k-1)!$ for positive integer values of $k.$}
rule it is easy to show that \( \lim_{s \to 1} \left( 1 - 2^{1-s} \right) /(s - 1) = \ln 2. \) The product
of these two limits yields \( \lim_{s \to 1} \zeta(s) \left( 1 - 2^{1-s} \right) = \ln 2, \) thus
\[
(6) \quad \frac{\ln 2}{\pi^0 \sin(\pi/2)} = 2 \sum_{n=1}^{\infty} \left( 1 - 2^{-2n} \right) \frac{(2n-1)!}{(2n+1)!} \zeta(2n),
\]
which simplifies to
\[
(7) \quad \sum_{n=1}^{\infty} \left( 1 - 2^{-2n} \right) \frac{\zeta(2n)}{n \cdot (2n+1)} = \ln 2.
\]
From Eq. (4), one has \( L = \ln 2, \) which completes our proof. \( \square \)

3. The Dancs-He formula for odd zeta-values from known zeta-series

Before presenting a general proof for \( \zeta(2n+1), \) \( n \) being a positive integer, let us tackle the lowest case, i.e. \( \zeta(3), \) for which a number of series were derived since the times of Euler \([6]\). For this number, Dancs and He found the following series representation (see Eq. (3.1) of Ref. \([3]\)).

**Theorem 2** (Dancs-He series for \( \zeta(3) \)). *With the same notations of Theorem 1:*
\[
(8) \quad \frac{\pi^2}{9} \ln 4 - \frac{2}{3} \pi^4 \sum_{m=0}^{\infty} (-1)^m \frac{\pi^{2m}}{(2m+5)!} E_{2m+1}(1) = \zeta(3).
\]

**Proof.** Let \( S \) be the number for which the series at the left-hand side of Eq. (8) converges. By substituting \( E_{2m+1}(1) = 2 \frac{2^{2m+2-1}}{2m+2} B_{2m+2} \) in this series, one has
\[
(9) \quad \pi^2 S = -2 \sum_{m=0}^{\infty} (-1)^{m+1} \frac{(2 \pi)^{2m+2} - \pi^{2m+2}}{(2m+5)!} \frac{B_{2m+2}}{2m+2}.
\]

By substituting \( n = m + 1, \) one finds that
\[
(10) \quad \pi^2 S = -2 \sum_{n=1}^{\infty} \frac{(-1)^n \left( (2 \pi)^{2n} - \pi^{2n} \right)}{2n \cdot (2n+1) \cdot (2n+2) \cdot (2n+3) \cdot (2n)!} B_{2n}.
\]

From the relation between \( B_{2n} \) and \( \zeta(2n) \) in Eq. (2), one has
\[
(11) \quad \frac{\pi^2}{4} S = \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n \cdot (2n+1) \cdot (2n+2) \cdot (2n+3)} - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{2n \cdot (2n+1) \cdot (2n+2) \cdot (2n+3) \cdot 2^{2n}},
\]
which is valid since the series in Eq. (10) converges absolutely. The first series can be easily evaluated from a known summation formula (see Eq. (713) in Ref. [5]), namely

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+1)(2k+1)(2k+2)(2k+3)} t^{2k} = \frac{\zeta(3)}{2\pi^2} t^{-2} + \frac{\ln(2\pi)}{3} - \frac{11}{18} + \frac{t^3}{3} \left[ \zeta'(-3, 1+t) - \zeta'(-3, 1-t) \right],
\]

where \( \zeta(s, a) \) is the Hurwitz (or generalized) zeta function and \( \zeta'(s, a) \) is its derivative with respect to \( s \). As this formula is valid for all \( t \) with \( |t| < 1 \), it is legitimate to take the limit as \( t \to 1^- \) on both sides, which yields

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k(2k+1)(2k+2)(2k+3)} = \frac{\zeta(3)}{8\pi^2} + \frac{\ln(2\pi)}{12} - \frac{11}{72},
\]

and

\[
\frac{1}{12} \lim_{t \to 1^-} \left[ \zeta'(-3, 1+t) - \zeta'(-3, 1-t) \right].
\]

The remaining limit is null, since \( \lim_{t \to 1^-} \zeta'(-3, 1+t) = \zeta'(-3, 2) = \zeta'(-3) = \lim_{t \to 1^-} \zeta'(-3, 1-t) \), which reduces Eq. (13) to

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k(2k+1)(2k+2)(2k+3)} \cdot 2^{2k} = \frac{\zeta(3)}{8\pi^2} + \frac{\ln(\pi)}{3} - \frac{11}{18}.
\]

For the second series in Eq. (11), let us make use of the Wilton’s formula (Eq. (38) at p. 303, in Ref. [6]; also Eq. (54) in Ref. [7]):

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(k+1)(2k+1)(2k+3)} = \frac{2\zeta(3)}{\pi^2} + \frac{\ln(\pi)}{3} - \frac{11}{18},
\]

which can be written as

\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{2k(2k+1)(2k+2)(2k+3)} \cdot 2^{2k} = \frac{2\zeta(3)}{\pi^2} + \frac{\ln(\pi)}{12} - \frac{11}{72}.
\]

Now, by doing a member-to-member subtraction of Eqs. (14) and (16) and putting the result in Eq. (11), one has the desired result:

\[
\frac{2\pi^2}{9} \ln 2 - \frac{2\pi^4}{3} S = \zeta(3).
\]

Now, let us generalize the above result for all \( \zeta(2m+1), m \) being a positive integer. See Eq. (3.1) of Ref. [3].
Theorem 3 (Dancs–He series for odd zeta-values). For any integer \( m \), \( m > 0 \),

\[
(1 - 2^{-2m}) \zeta(2m + 1) = \sum_{j=1}^{m-1} \frac{(-1)^j \pi^{2j}}{(2j + 1)!} (2^{2j-2m} - 1) \zeta(2m - 2j + 1) - \frac{(-1)^m \pi^{2m} \ln 2}{(2m + 1)!}
+ \frac{(-1)^m \pi^{2m+2}}{2} \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k} E_{2k+1}(1)}{(2k + 2m + 3)!}.
\]

(18)

Proof. Let \( \tilde{S} \) be the number for which the series at the right-hand side of Eq. (18) converges, i.e.

\[
\tilde{S} := \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k} E_{2k+1}(1)}{(2k + 2m + 3)!}.
\]

(19)

By substituting \( E_{2k+1}(1) = 2 \frac{2^{2k+2} - 1}{2k+2} B_{2k+2} \) in this series, one has

\[
\pi^2 \tilde{S} = \sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k+2}}{(2k + 2m + 3)!} 2 \left( 2^{2k+2} - 1 \right) \frac{B_{2k+2}}{2k + 2}.
\]

(20)

By putting \( n = k + 1 \), one finds that

\[
\pi^2 \tilde{S} = -2 \sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n}}{(2n + 2m + 1)!} \frac{2(2^{2n} - 1)}{2n} B_{2n}.
\]

(21)

From Eq. (2), one has

\[
\frac{\pi^2}{2} \tilde{S} = 2 \sum_{n=1}^{\infty} (1 - 2^{-2n}) \frac{(2n - 1)!}{(2n + 2m + 1)!} \zeta(2n)
= \sum_{n=1}^{\infty} (2 - 2^{1-2n}) \frac{\Gamma(2n)}{\Gamma(2n + 2m + 2)} \zeta(2n).
\]

(22)

This latter series is just the one that appears in a formula for odd zeta-values derived recently by Milgran (see Eq. (13) in Ref. [8]), namely

\[
\zeta(2m + 1) = \frac{(-1)^m \pi^{2m}}{1 - 2^{-2m}} \left[ -\frac{\ln 2}{(2m + 1)!} + \frac{\pi^2}{2} \tilde{S} \right] + \frac{1}{1 - 2^{-2m}}
\times \sum_{n=1}^{m-1} (2^{2n-2m} - 1) (-\pi^2)^n \frac{\zeta(2m - 2n + 1)}{(2n + 1)!}.
\]

(23)
By multiplying both sides by $1 - 2^{-2m}$, one has

\[
(1 - 2^{-2m}) \zeta(2m + 1) = (-1)^m \pi^{2m} \left[ -\frac{\ln 2}{(2m + 1)!} + \frac{\pi^2}{2} \tilde{S} \right] + \sum_{n=1}^{m-1} (-1)^n \frac{(2^{2n-2m} - 1) \pi^{2n} \zeta(2m - 2n + 1)}{(2n + 1)!} \]

\[
= -(-1)^m \pi^{2m} \frac{\ln 2}{(2m + 1)!} + (-1)^m \frac{\pi^{2m+2}}{2} \tilde{S} + \sum_{n=1}^{m-1} \frac{(-1)^n \pi^{2n}}{(2n + 1)!} (2^{2n-2m} - 1) \zeta(2m - 2n + 1),
\]

which completes our proof. □

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