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Abstract. In this paper, we study the nonlinear Schrödinger equation with non-symmetric electromagnetic fields

$$\left(\frac{\nabla}{i} - A_\epsilon(x)\right)^2 u + V_\epsilon(x)u = f(u), \; u \in H^1(\mathbb{R}^N, \mathbb{C}),$$

where $A_\epsilon(x) = (A_{\epsilon,1}(x), A_{\epsilon,2}(x), \ldots, A_{\epsilon,N}(x))$ is a magnetic field satisfying that $A_{\epsilon,j}(x)(j = 1,\ldots,N)$ is a real $C^1$ bounded function on $\mathbb{R}^N$ and $V_\epsilon(x)$ is an electric potential. Both of them satisfy some decay conditions but without any symmetric conditions and $f(u)$ is a superlinear nonlinearity satisfying some non-degeneracy condition. Applying two times finite reduction methods and localized energy method, we prove that there exists some $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the above problem has infinitely many complex-valued solutions.

1. Introduction and main result. In this paper, we investigate the existence of standing waves $\psi(x,t) = e^{-\frac{iEt}{\hbar}}u(x)$, $E \in \mathbb{R}$, $u : \mathbb{R}^N \to \mathbb{C}$ to the time-dependent nonlinear Schrödinger equation with an external electromagnetic field

$$i\hbar \frac{\partial \psi}{\partial t} = \left(\frac{\hbar}{i}\nabla - A(x)\right)^2 \psi + G(x)\psi - f(x,\psi), \; x \in \mathbb{R}^N,$$

which arises in various physical contexts such as nonlinear optics or plasma physics where one simulates the interaction effect among many particles by introducing a nonlinear term (see [29]). The function $\psi(x,t)$ takes on complex values, $\hbar$ is the

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Planck constant, $i$ is the imaginary unit. Here $A$ denotes a magnetic potential and the Schrödinger operator is defined by
\[
\left( \frac{\hbar}{i} \nabla - A(x) \right)^2 \psi := -\hbar^2 \Delta \psi - \frac{2\hbar}{i} A \cdot \nabla \psi + |A|^2 \psi - \frac{\hbar}{i} \psi \text{div} A.
\]

Actually, in general dimension, the magnetic field $B$ is a 2-form where $B_{k,j} = \partial_j A_k - \partial_k A_j$; in the case $N = 3$, $B = \text{curl} A$. The function $G$ represents an electric potential.

Assuming $f(x, e^{i\theta} u) = e^{i\theta} f(x, u)$, $\theta \in \mathbb{R}$ and substituting this ansatz $\psi(x, t) = e^{-\frac{\hbar}{i} t} u(x)$ into (1.1), one is led to solve the complex semilinear elliptic equation
\[
\left( \frac{\hbar}{i} \nabla - A(x) \right)^2 u + (G(x) - E) u = f(x, u), \quad x \in \mathbb{R}^N. \tag{1.2}
\]

For simplicity, let $V(x) = (G(x) - E)$ and assume that $V$ is strictly positive on the whole space $\mathbb{R}^N$. The transition from quantum mechanics to classical mechanics can be formally described by letting $\hbar \to 0$, and thus the existence of solutions for $\hbar$ small has physical interest. Standing waves for $\hbar$ small are usually referred as semi-classical bound states (see [17]).

When $A(x) \equiv 0$, problem (1.2) arises in various applications, such as chemotaxis, population genetics, chemical reactor theory, and the study of standing waves of certain nonlinear Schrödinger equations. In recent years, a considerable amount of work has been devoted to study wave solutions of (1.2) with $A(x) \equiv 0$. Among of them, we refer to [5, 7, 11, 12, 14, 16, 22, 23, 26, 28, 30]. Recently, in [1], Ao and Wei applying localized energy method obtained infinitely many positive solutions for (1.2) with non-symmetric electric potential.

On the contrary, there are still relatively few papers which deal with the case $A(x) \not\equiv 0$, namely when a magnetic field is present. The first result on magnetic nonlinear Schrödinger equation is due to Esteban and Lions in [15]. They obtained the existence of standing waves to (1.2) for $\hbar$ fixed and for special classes of magnetic fields by solving an appropriate minimization problem for the corresponding energy functional in the cases of $N = 2, 3$. In [10], Cao and Tang constructed semiclassical multi-peak solutions for (1.2) with bounded vector potentials. In [9], using a penalization procedure, Cingolani and Secchi extended the result in [8] to the case of a vector potential $A$, possibly unbounded. The penalization approach was also used in [3] by Bartsch, Dancer and Peng to obtain multi-bump semiclassical bound for problem (1.2) with more general nonlinear term $f(x, u)$. In [20], Kurata proved the existence of least energy solution of (1.2) for $\hbar > 0$ under a condition relating $V(x)$ and $A(x)$. In [17, 18], Helffer studied asymptotic behavior of the eigenfunctions of the Schrödinger operators with magnetic fields in the semiclassical limit. See also [2] for generalization of the results and in [19] for potentials which degenerate at infinity. In [21], Li, Peng and Wang applied the finite reduction method to obtain infinitely many non-radial complex valued solutions for (1.2) with radial electromagnetic fields satisfying some algebraic decaying conditions. Liu and Wang in [24] extends the result to some weaker symmetric conditions. In [27], Pi and Wang obtained multi-bump solutions for (1.2) with $\hbar = 1$, $f(x, u) = |u|^{p-2} u$ and an electrical potential satisfying a condition by applying the finite reduction method.

In this paper, inspired by [1, 31], our main idea is to use the Lyapunov-Schmidt reduction method. We want to point out that the only assumption we need is the non-degeneracy of the bump. We have no requirements on the structure of the nonlinearity.
If \( h = 1 \), \( A(x) = A_0 + \epsilon \tilde{A}(x), V(x) = 1 + \epsilon \tilde{V}(x) \) and \( f(x, u) = f(u) \), then (1.2) is reduced to the following complex problem

\[
\left( \frac{\nabla}{i} - A_0 - \epsilon \tilde{A}(x) \right)^2 u + (1 + \epsilon \tilde{V}(x))u = f(u), \ u \in H^1(\mathbb{R}^N, \mathbb{C}).
\]

For simplicity of notations, in the sequel, we denote

\[
A_\epsilon(x) = A_0 + \epsilon \tilde{A}(x) \quad \text{and} \quad V_\epsilon(x) = 1 + \epsilon \tilde{V}(x).
\]

Then we are concerned with the following problem

\[
\left( \frac{\nabla}{i} - A_\epsilon(x) \right)^2 u + V_\epsilon(x)u = f(u), \ u \in H^1(\mathbb{R}^N, \mathbb{C}). \tag{1.3}
\]

And we consider the functional \( J : H^1(\mathbb{R}^N) \to \mathbb{R} \) given by

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left| \left( \frac{\nabla}{i} - A_\epsilon(x) \right)u \right|^2 + V_\epsilon(x)u^2 dx - \int_{\mathbb{R}^N} F(u) dx, \ u \in H^1(\mathbb{R}^N, \mathbb{C}), \tag{1.4}
\]

whose critical points are corresponding to solutions of (1.3).

In order to state our main result, we give the conditions imposed on \( \tilde{A}(x), \tilde{V}(x) \) and \( f \), part of which come from [1]:

- (A1) \( \lim_{|x| \to \infty} |\tilde{A}(x)| = 0; \)
- (A2) \( \exists 0 < \alpha_1 < 1, \lim_{|x| \to \infty} |\tilde{A}(x)|^2 e^{\alpha_1|x|} = +\infty; \)
- (A3) \( \exists 0 < \alpha_2 < 1, \lim_{|x| \to \infty} |\text{div}\tilde{A}(x)|^2 e^{\alpha_2|x|} = +\infty; \)
- (A4) \( \lim_{|x| \to \infty} |\nabla\tilde{A}(x)| = 0; \)
- (V1) \( \tilde{V}(x) \in C(\mathbb{R}^N, \mathbb{R}^+ \) and \( \lim_{|x| \to \infty} \tilde{V}(x) = 0; \)
- (V2) \( \exists 0 < \alpha_3 < 1, \lim_{|x| \to \infty} \tilde{V}(x) e^{\alpha_3|x|} = +\infty; \)
- (f1) \( f : \mathbb{C} \to \mathbb{C} \) is of class \( C^{1+\delta} \) for some \( 0 < \delta \leq 1, f'(0) = 0; \)
- (f2) \( f(e^{i\theta}u) = e^{i\theta}f(u), \theta \in \mathbb{R}; \)
- (f3) The equation

\[
\begin{cases}
-\Delta w + w = f(w), \quad w > 0 \text{ in } \mathbb{R}^N, \\
\lim_{|x| \to \infty} w(x) = 0, \quad w(0) = \max_{x \in \mathbb{R}^N} w(x),
\end{cases} \tag{1.5}
\]

has a non-degenerate solution \( w \), i.e.,

\[
\ker(\Delta - 1 + f'(w)) \cap L^\infty(\mathbb{R}^N) = \text{span} \left\{ \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_N} \right\}.
\]

Particularly, \( f(u) = |u|^{p-1}u(1 < p < 5) \) satisfies (f2).

Under the above assumptions, the spectrum of the linearized operator

\[
\Delta \varphi - \varphi + f'(w)\varphi = \lambda \varphi, \ \varphi \in H^1(\mathbb{R}^N)
\]

admits the following decompositions

\[
\lambda_1 > \lambda_2 > \ldots > \lambda_n > \lambda_{n+1} = 0 > \lambda_{n+2},
\]

where each of the eigenfunction corresponding to the positive eigenvalue \( \lambda_j \) decays exponentially. These eigenfunctions will play an important role in our secondary Lyapunov-Schmidt reduction (see Section 3 below).
Remark 1.1. It is easy to find that $w$ is a solution of (1.5) if and only if $e^{iA_0 \cdot x}w$ is a solution of the following problem

$$
\begin{cases}
\left(\frac{\nabla}{i} - A_0\right)^2 u + u = f(u), \quad x \in \mathbb{R}^N, \\
\lim_{|x| \to \infty} |u(x)| = 0, \quad |u(0)| = \max_{x \in \mathbb{R}^N} |u(x)|, 
\end{cases}
$$

(1.6)

from which and $(f_3)$ we can deduce that (1.6) has a non-degenerate solution $e^{i\sigma + iA_0 \cdot x}w$, i.e.

$$
\ker \left(\left(\frac{\nabla}{i} - A_0\right)^2 + 1 - f'(w)\right) = \text{span} \left\{ \frac{\partial(e^{i\sigma + iA_0 \cdot x}w)}{\partial x_1}, \ldots, \frac{\partial(e^{i\sigma + iA_0 \cdot x}w)}{\partial x_N}, \frac{\partial(e^{i\sigma + iA_0 \cdot x}w)}{\partial \sigma} \right\}.
$$

For the detailed, one can refer to Lemma 2.3 in [6].

In the sequel, the Sobolev space $H^1(\mathbb{R}^N)$ is endowed with the standard norm

$$
\|u\| = \left( \int |\nabla u|^2 + |u|^2 \right)^{\frac{1}{2}},
$$

which is induced by the inner product

$$
\langle \nabla u, \nabla v \rangle = \int (\nabla u \nabla v + uv).
$$

Denote $\alpha = \min\{\alpha_1, \alpha_2, \alpha_3\}$.

Our main result of this paper is as follows:

**Theorem 1.2.** Assume that $(A_1)$-$\cdot$$(A_4)$, $(V_1)$-$\cdot$$(V_2)$ and $(f_1)$-$\cdot$$(f_3)$ hold. Then there exists $\epsilon_0 > 0$ such that $0 < \epsilon < \epsilon_0$, problem (1.3) has infinitely many complex-valued solutions.

In the following, we sketch the main idea in the proof of Theorem 1.2.

We introduce some notations first. Let $\mu > 0$ be a real number such that $w(x) \leq ce^{-|x|}$ for $|x| > \mu$ and some constant $c$ independent of $\mu$ large. Now we define the configuration space

$$
\Omega_1 = \mathbb{R}^N, \Omega_m := \left\{ Q_m = (Q_1, Q_2, \ldots, Q_m) \in \mathbb{R}^{mN} : \min_{k \neq j} |Q_k - Q_j| \geq \mu \right\}, \forall m > 1.
$$

Let $w$ be the non-degenerate solution of (1.5) and $m \geq 1$ be an integer. Define the sum of $m$ spikes as

$$
w_{Q_j} = w(x - Q_j), \quad \xi_j = e^{i\sigma + iA_0 \cdot (x - Q_j)}, \quad z_{Q_j} = \xi_j w(x - Q_j) \text{ and } z_{Q_m} = \sum_{j=1}^{m} z_{Q_j},
$$

where $\sigma \in [0, 2\pi]$.

Let the operator be

$$
S(u) = -\left(\frac{\nabla}{i} - A_\epsilon(x)\right)^2 u - V_\epsilon(x)u + f(u).
$$

Fixing $(\sigma, Q_m) = (\sigma, Q_1, \ldots, Q_m) \in [0, 2\pi] \times \Omega_m$, we define the following functions as the approximate kernels:

$$
D_{j,k} = \frac{\partial(e^{i\sigma + iA_0 \cdot (x - Q_j)}w_{Q_j})}{\partial x_k} \eta_j(x), \text{ for } j = 1, \ldots, m, k = 1, \ldots, N
$$
and
\[ D_{j,N+1} = \frac{\partial (e^{i\sigma + iA_0 \cdot (x-Q_j)} w_{Q_j})}{\partial \sigma} \eta_j(x), \quad j = 1, \ldots, m, \]
where \( \eta_j(x) = \eta(t) \left( \frac{2|x-Q_j|}{\mu - 1} \right) \) and \( \eta(t) \) is a cut off function, such that \( \eta(t) = 1 \) for \( |t| \leq 1 \) and \( \eta(t) = 0 \) for \( |t| \geq \frac{\mu^2}{\mu - 1} \). Note that the support of \( D_{j,k} \) belongs to \( B_{\frac{\mu^2}{\mu + 1}}(Q_j) \).

Applying \( z_{Q_m} \) as the approximate solution and performing the Lyapunov-Schmidt reduction, we can show that there exists a constant \( \mu_0 \), such that for \( \mu \geq \mu_0 \) and \( \epsilon < c \mu \), for some constant \( c \mu \) depending on \( \mu \) but independent of \( m \) and \( Q_m \), we can find a \( \varphi_{\sigma,Q_m} \) such that
\[ S(z_{Q_m} + \varphi_{\sigma,Q_m}) = \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k}, \]
and we can show that \( \varphi_{\sigma,Q_m} \) is \( C^1 \) in \( (\sigma, Q_m) \). This is done in Section 2.

After that, for any \( m \), we define a new function
\[ M(\sigma, Q_m) = J(z_{Q_m} + \varphi_{\sigma,Q_m}), \quad (1.7) \]
we maximize \( M(\sigma, Q_m) \) over \( [0, 2\pi] \times \bar{\Omega}_m \).

At the maximum point of \( M(\sigma, Q_m) \), we show that \( c_{j,k} = 0 \) for all \( j, k \). Therefore we prove that the corresponding \( z_{Q_m} + \varphi_{\sigma,Q_m} \) is a solution of \( (1.3) \). By the arguments before, we know that there exists \( \mu_0 \) large such that \( \mu \geq \mu_0 \) and \( \epsilon < c \mu \) and for any \( m \), there exists a spike solution to \( (1.3) \) with \( m \) spikes in \( \Omega_m \). Considering that \( m \) is arbitrary, then there exists infinitely many spikes solutions for \( \epsilon < c \mu_0 \) independent of \( m \).

There are three main difficulties in the maximization process. Firstly, we need to show that the maximum points will not go to infinity. Secondly, we have to detect the difference in the energy when the spikes move to the boundary of the configuration space. In the second step, we use the induction method and detect the difference of the \( m \)-th spikes energy and the \( (m+1) \)-th spikes energy. A crucial estimate is Lemma 3.2, where we prove that the accumulated error can be controlled from step \( m \) to step \( m + 1 \). To this end, we make a secondary Lyapunov-Schmidt reduction. This is done in Section 3. Compared with [1], since there is a magnetic field in our problem, we have to overcome some new difficulties which involves many technical estimates.

Our paper is organized as follows. In section 2, we carry out Lyapunov-Schmidt reduction. Then we perform a second Lyapunov-Schmidt reduction in section 3. Finally, we prove our main result in section 4. Throughout the paper, we simply write \( \int f \) to mean the Lebesgue integral of \( f(x) \) in \( \mathbb{R}^N \). The complex conjugate of any number \( z \in \mathbb{C} \) will be denoted by \( \bar{z} \). The real part of a number \( z \in \mathbb{C} \) will be denoted by \( \text{Re} z \). The ordinary inner product between two vectors \( a, b \in \mathbb{R}^N \) will be denoted by \( a \cdot b \).

2. Finite-dimensional reduction. In this section, we perform a finite-dimensional reduction.

Let \( \gamma \in (0, 1) \) and we define
\[ E(\cdot) := \sum_{j=1}^{m} e^{-\gamma|\cdot - Q_j|}, \quad \text{where} \quad Q_m \in \Omega_m. \quad (2.1) \]
Consider the norm
\[
\|f\|_* = \sup_{x \in \mathbb{R}^N} |E(x)^{-1}f(x)|,
\]
which was first introduced in [25] and also used in [1, 31]. Now we investigate
\[
\left\{
\begin{array}{l}
L(\varphi_{\sigma, Q_m}) := -\left(\nabla^2 - A_r(x)\right)\varphi_{\sigma, Q_m} - V_r(x)\varphi_{\sigma, Q_m} + f'(z_{Q_m})\varphi_{\sigma, Q_m} \\
= h + \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k}, \text{ in } \mathbb{R}^N, \\
Re \int \varphi_{\sigma, Q_m} \bar{D}_{j,k} = 0 \text{ for } j = 1, \ldots, m, k = 1, \ldots, N + 1.
\end{array}
\right.
\]
(2.3)

Firstly, we give a result which will be used later.

**Lemma 2.1.** ([13], Lemma 3.4) There exists a constant \(C_N = 6^N\) such that for any \(m \in \mathbb{N}^+\) and any \(Q_m = (Q_1, Q_2, \ldots, Q_m) \in \mathbb{R}^{mN}\),
\[
\sharp\left\{Q_j \left| \frac{l}{2} \leq |x - Q_j| < \frac{l+1}{2} \right. \right\} \leq C_N (l+1)^{N-1}
\]
for all \(x \in \mathbb{R}^N\) and all \(l \in \mathbb{N}\). Particularly, we have
\[
\sharp\left\{Q_j \left| 0 \leq |x - Q_j| < \frac{\mu}{2} \right. \right\} \leq C_N.
\]
(2.4)

**Lemma 2.2.** Let \(h\) with \(\|h\|_*\) bounded and assume that \((\varphi_{\sigma, Q_m}, c_{j,k})\) is a solution to (2.3). Then there exist positive numbers \(\mu_0\) and \(C\), such that for all \(0 < \epsilon < e^{-2\mu}\), \(\mu > \mu_0\) and \((\sigma, Q_m) \in [0, 2\pi] \times \Omega_m\), one has
\[
\|\varphi_{\sigma, Q_m}\|_* \leq C\|h\|_*,
\]
(2.5)

where \(C\) is a positive constant independent of \(\mu, m\) and \(Q_m \in \Omega_m\).

**Proof.** We prove it by contradiction. Assume that there exists a solution \(\varphi_{\sigma, Q_m}\) to (2.3) and \(\|h\|_* \to 0\), \(\|\varphi_{\sigma, Q_m}\|_* = 1\).

Multiplying the equation in (2.3) by \(D_{j,k}\) and integrating in \(\mathbb{R}^N\), we get
\[
Re \int L(\varphi_{\sigma, Q_m}) \bar{D}_{j,k} = Re \int h \bar{D}_{j,k} + c_{j,k} \int |D_{j,k}|^2.
\]
(2.7)

Considering the exponential decay at infinity of \(\frac{\partial w(x)}{\partial x_k}\) and the definition of \(D_{j,k}(k = 1, \ldots, N + 1)\), we have
\[
\int |D_{j,k}|^2 = \int \left| (iA_{0,k} z_{Q_j} + \frac{\partial w_{Q_j}}{\partial x_k} \xi_j) \right|^2 \\
= \int A_{0,k}^2 w_{Q_j}^2 \eta_j^2 \left(\frac{2|x - Q_j|}{\mu - 1}\right) + \int \left( \frac{\partial w_{Q_j}}{\partial x_k} \right)^2 \eta_j^2 \left(\frac{2|x - Q_j|}{\mu - 1}\right) \\
= A_{0,k}^2 \int w^2 + A_{0,k}^2 \int_{B_{\frac{|x|}{\mu - 1}}(0)} \left[ \eta_j^2 \left(\frac{2|x|}{\mu - 1} - 1\right) \right] w^2 \\
+ \int \left( \frac{\partial w}{\partial x_k} \right)^2 + \int_{B_{\frac{|x|}{\mu - 1}}(0)} \left[ \eta_j^2 \left(\frac{2|x|}{\mu - 1} - 1\right) \right] \left( \frac{\partial w}{\partial x_k} \right)^2 \\
= A_{0,k}^2 \int w^2 + \int \left( \frac{\partial w}{\partial x_k} \right)^2 + O(e^{-\mu}), \text{ as } \mu \to +\infty, \ k = 1, 2, \ldots, N
\]
(2.8)
and
\[
\int |D_{j,N+1}|^2 = \int |i\zeta_{Q_j} \eta \left( \frac{2|x - Q_j|}{\mu - 1} \right)|^2 = \int |w_{Q_j} \eta \left( \frac{2|x - Q_j|}{\mu - 1} \right)|^2.
\]
\[
= \int w^2 + \int_{B_{\mu}^{2}(0)} \eta^2 \left( \frac{2|x|}{\mu - 1} - 1 \right) w^2 = \int w^2 + O(e^{-\mu}),
\]
(2.9)
as \(\mu \to +\infty\).

On the other hand, by Lemma A.1 we have
\[
\left| Re \int h \bar{D}_{j,k} \right| \leq \left| Re \int h (-iA_{0,k} \bar{z}_{Q_j} + \frac{\partial w_{Q_j}}{\partial x_k} \xi_j) \eta_j \right|
\]
\[
\leq \int |h||A_{0,k}| w_{Q_j} |\eta_j| + \int |h| \left| \frac{\partial w_{Q_j}}{\partial x_k} \right| |\eta_j|
\]
\[
\leq C||h|| \int_{B_{\mu}^{2}(Q_j)} |A_{0,k}| \sum_{j=1}^{m} e^{-\gamma|x - Q_j|} w(x - Q_j) \left| \eta \left( \frac{2|x - Q_j|}{\mu - 1} \right) \right|
\]
\[
+ C||h|| \int_{B_{\mu}^{2}(Q_j)} e^{-\gamma|x - Q_j|} w(x - Q_j) \left| \frac{\partial w(x - Q_j)}{\partial x_k} \right| \left| \eta \left( \frac{2|x - Q_j|}{\mu - 1} \right) \right|
\]
\[
\leq C||h|| \int_{\tilde{B}_z(Q_j)} e^{-\gamma|x - Q_j|} w(x - Q_j) + C||h|| \int_{\tilde{B}_z(Q_j)} e^{-\gamma|x - Q_j|} \left| \frac{\partial w(x - Q_j)}{\partial x_k} \right|
\]
\[
\leq C||h|| \int_{0}^{\frac{\pi}{2}} e^{-\gamma t} t^{N-1} dt \leq C||h||, \quad k = 1, 2, \ldots, N
\]
(2.10)

and
\[
\left| Re \int h \bar{D}_{j,N+1} \right| \leq \int |h||\bar{D}_{j,N+1}| \leq \int |h||i\zeta_{Q_j} \eta_j|
\]
\[
\leq C||h|| \int_{1 \leq j \leq N} \sum_{j=1}^{m} e^{-\gamma|x - Q_j|} w(x - Q_j) \left| \eta \left( \frac{2|x - Q_j|}{\mu - 1} \right) \right|
\]
\[
\leq C||h|| \int_{\tilde{B}_z(Q_j)} e^{-\gamma|x - Q_j|} w(x - Q_j)
\]
\[
\leq C||h|| \int_{0}^{\frac{\pi}{2}} e^{-\gamma t} t^{N-1} dt \leq C||h||.
\]
(2.11)

Here and in what follows, \(C\) stands for a positive constant independent of \(\epsilon\) and \(\mu\), as \(\epsilon \to 0\). Now if we write \(\bar{D}_{j,k} = \frac{\partial (e^{\sigma + A_{0,j}(x - Q_j) w_{Q_j}})}{\partial x_k} \), then we have
\[
Re \int L(\varphi_{\sigma,Q_m}) \bar{D}_{j,k} = Re \int L(D_{j,k}) \varphi_{\sigma,Q_m}
\]
for some $\beta > 0$.

Moreover, by Lemma A.1 we have

$$\begin{align*}
&\left( \nabla - A_0 \right)^2 D_{j,k}\bar{\phi}_\sigma, Q_m = \nabla A_0 \cdot D_{j,k}\bar{\phi}_\sigma, Q_m + f'(z_{Q_m}) (D_{j,k}\bar{\phi}_\sigma, Q_m) \\
&= \left( \nabla - A_0 \right)^2 D_{j,k}\bar{\phi}_\sigma, Q_m - D_{j,k}\bar{\phi}_\sigma, Q_m + f'(z_{Q_m}) (D_{j,k}\bar{\phi}_\sigma, Q_m) \\
&\quad - \int C \int \left( \frac{\epsilon}{i} \text{div} \tilde{A} - 2\epsilon A_0 \cdot \tilde{A} - \epsilon^2 |\tilde{A}|^2 \right) D_{j,k}\bar{\phi}_\sigma, Q_m + \int \frac{2 \epsilon}{i} \tilde{A}(x) \cdot \nabla D_{j,k}\bar{\phi}_\sigma, Q_m \\
&\leq \int B_{\frac{n^2}{n^2+1}} (Q_j) \left( \nabla - A_0 \right)^2 D_{j,k} - D_{j,k} + f'(z_{Q_m}) (D_{j,k}) \eta_j \bar{\phi}_\sigma, Q_m \\
&\quad + \int B_{\frac{n^2}{n^2+1}} (Q_j) \left( \frac{\epsilon}{i} \text{div} \tilde{A} - 2\epsilon A_0 \cdot \tilde{A} - \epsilon^2 |\tilde{A}|^2 \right) D_{j,k} \eta_j \bar{\phi}_\sigma, Q_m \\
&\quad + \int B_{\frac{n^2}{n^2+1}} (Q_j) \frac{2 \epsilon}{i} \tilde{A}(x) \cdot \nabla D_{j,k} \eta_j \bar{\phi}_\sigma, Q_m. \\
&\leq \int_{\frac{n^2}{n^2+1}} (Q_j) \left[ \left( \nabla - A_0 \right)^2 D_{j,k} - D_{j,k} + f'(z_{Q_m}) (D_{j,k}) \eta_j \bar{\phi}_\sigma, Q_m = 0. \right. (2.12)
\end{align*}$$

Since

$$\begin{align*}
- \left( \nabla - A_0 \right)^2 D_{j,k} - D_{j,k} + f'(z_{Q_m}) (D_{j,k}) \eta_j \bar{\phi}_\sigma, Q_m = 0,
\end{align*}$$

we have

$$\begin{align*}
\int B_{\frac{n^2}{n^2+1}} (Q_j) \left[ \left( \nabla - A_0 \right)^2 D_{j,k} - D_{j,k} + f'(z_{Q_m}) (D_{j,k}) \eta_j \bar{\phi}_\sigma, Q_m = 0. \right. (2.13)
\end{align*}$$

Moreover, by Lemma A.1 we have

$$\begin{align*}
|\text{Re} \int B_{\frac{n^2}{n^2+1}} (Q_j) \sum_{j=1}^{m} e^{-|\gamma|Q_j} \left( \frac{\partial w_{Q_j}}{\partial x_k} \right) |w_{Q_j} | \leq C \| \phi, Q_m \| * \int B_{\frac{n^2}{n^2+1}} (Q_j) \sum_{j=1}^{m} e^{-|\gamma|Q_j} \left( \frac{\partial w_{Q_j}}{\partial x_k} \right) |w_{Q_j} | \\
\int \sum_{j=1}^{m} e^{-|\gamma|Q_j} \left( \frac{\partial w_{Q_j}}{\partial x_k} \right) |w_{Q_j} | \\
\leq C \| \phi, Q_m \| * \int e^{-(1+\gamma)s} s^{N-1} ds \leq C e^{-(1+\beta)^{2}} \| \phi, Q_m \| _* .
\end{align*}$$

(2.14)
Observing that
\[ |f'(z_{Q_m}) - f'(z_{Q_j})| \leq C \left| \sum_{k \neq j} z_{Q_k} \right|^{\delta}, \]
by \((f_1)\) we have
\[
\left| \text{Re} \int_{B_{\mu^2/(\mu+1)}} (f'(z_{Q_m}) - f'(z_{Q_j})) \tilde{D}_{j,k} \eta_j \bar{\phi}_\sigma Q_m \right|
\leq C\|\phi_\sigma Q_m\| \int_{B_{\mu^2/(\mu+1)}} \left| \sum_{k \neq j} z_{Q_k} \right|^{\delta} \left( \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} \left| \frac{\partial w_{Q_j}}{\partial x_k} \right| + w_{Q_j} \right)
\leq C\|\phi_\sigma Q_m\| \int_{B_{\mu^2/(\mu+1)}} \left| \sum_{k \neq j} z_{Q_k} \right|^{\delta} \left( \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} \left| \frac{\partial w_{Q_j}}{\partial x_k} \right| + w_{Q_j} \right) (2.15)
\leq C\|\phi_\sigma Q_m\| \int_{B_{\mu^2/(\mu+1)}} e^{-\frac{\mu}{2}} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} \left( \frac{\partial w_{Q_j}}{\partial x_k} \right) + w_{Q_j}
\leq C\|\phi_\sigma Q_m\| e^{-\frac{\mu}{2}} \int_{0}^{\frac{\mu}{2}} e^{-(1+\gamma)s} s^{N-1} ds \leq C e^{-\frac{\mu}{2}} \|\phi_\sigma Q_m\| .
\]

and
\[
\left| \text{Re} \int_{B_{\mu^2/(\mu+1)}} \epsilon \tilde{V} \tilde{D}_{j,k} \eta_j \bar{\phi}_\sigma Q_m \right| \leq \epsilon \int_{B_{\mu^2/(\mu+1)}} \|\tilde{V}\| \|\tilde{D}_{j,k}\| |\bar{\phi}_\sigma Q_m|
\leq C e^{-2\mu} \|\phi_\sigma Q_m\| \int_{B_{\mu^2/(\mu+1)}} \left| \tilde{V}\|\tilde{D}_{j,k}\| \right| \sum_{j=1}^{m} e^{-\gamma|x-Q_j|}
\leq C e^{-2\mu} \|\phi_\sigma Q_m\| \int_{B_{\mu^2/(\mu+1)}} \left( \frac{\partial w_{Q_j}}{\partial x_k} \right) + w_{Q_j} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|}
\leq C \|\phi_\sigma Q_m\| e^{-\frac{\mu}{2}} \int_{0}^{\frac{\mu}{2}} e^{-(1+\gamma)s} s^{N-1} ds \leq C e^{-\frac{\mu}{2}} \|\phi_\sigma Q_m\| .
\]

Similarly, we can get
\[
\left| \text{Re} \int_{B_{\mu^2/(\mu+1)}} \frac{\epsilon}{i} d i v \tilde{A} \tilde{D}_{j,k} \eta_j \bar{\phi}_\sigma Q_m \right| \leq C \epsilon \int_{B_{\mu^2/(\mu+1)}} \|\tilde{D}_{j,k}\| |\bar{\phi}_\sigma Q_m|
\leq C e^{-\frac{\mu}{2}} \|\phi_\sigma Q_m\| ,
\]
\[
\left| \text{Re} \int_{B_{\mu^2/(\mu+1)}} 2\epsilon A_0 \cdot \tilde{A} \tilde{D}_{j,k} \eta_j \bar{\phi}_\sigma Q_m \right| \leq C \epsilon \int_{B_{\mu^2/(\mu+1)}} \|\tilde{D}_{j,k}\| |\bar{\phi}_\sigma Q_m|
\leq C e^{-\frac{\mu}{2}} \|\phi_\sigma Q_m\| ,
\]
\[
\left| \text{Re} \int_{B_{\mu^2/(\mu+1)}} \epsilon^2 \|\tilde{A}\|^2 \tilde{D}_{j,k} \eta_j \bar{\phi}_\sigma Q_m \right| \leq C \epsilon \int_{B_{\mu^2/(\mu+1)}} \|\tilde{D}_{j,k}\| |\bar{\phi}_\sigma Q_m|
\leq C e^{-\frac{\mu}{2}} \|\phi_\sigma Q_m\| .
\]
\[ \leq C e^{-\beta \frac{\epsilon}{2}} \| \varphi_{\sigma, Q_m} \|_* , \quad (2.19) \]

\[ \left| \text{Re} \int_B \frac{2e}{i} A(x) \cdot \nabla \eta_j \tilde{D}_{j,k} \tilde{\varphi}_{\sigma, Q_m} \right| \leq C e \int_B \frac{2e}{i} |\tilde{D}_{j,k}| |\tilde{\varphi}_{\sigma, Q_m}| \]

\[ \leq C e^{-\beta \frac{\epsilon}{2}} \| \varphi_{\sigma, Q_m} \|_* \quad (2.20) \]

and

\[ \left| \text{Re} \int_B \frac{2e}{i} A(x) \cdot \nabla \tilde{D}_{j,k} \eta_j \tilde{\varphi}_{\sigma, Q_m} \right| \leq C e \int_B \frac{2e}{i} |\nabla \tilde{D}_{j,k}| |\tilde{\varphi}_{\sigma, Q_m}| \]

\[ \leq C e^{-\beta \frac{\epsilon}{2}} \| \varphi_{\sigma, Q_m} \|_* , \quad (2.21) \]

for some \( \beta > 0 \).

It follows from (2.7) to (2.21) that

\[ |c_{j,k}| \leq C (e^{-\beta \frac{\epsilon}{2}} \| \varphi_{\sigma, Q_m} \|_* + \| h \|_*). \quad (2.22) \]

Let now \( \theta \in (0, 1) \). It is easy to check that the function \( E(x) \) in (2.1) satisfies

\[ | - L(E(x)) | \geq \frac{1}{2} (1 - \theta^2) E(x), \text{ in } \mathbb{R}^N \setminus \bigcup_{j=1}^m B_{\bar{\mu}}(Q_j) \quad (2.23) \]

provided \( \bar{\mu} \) is large enough and \( \bar{\mu} \leq \frac{\mu}{2} \). Indeed, by Lemma 2.1 we have

\[ w_{Q_m} \leq \sum_{|x - Q_j| < \frac{\mu}{2}} w(x - Q_j) + \sum_{l=1}^\infty \sum_{\frac{l+1}{2} \leq |x - Q_j| < \frac{l+1}{2} \mu} w(x - Q_j) \]

\[ \leq C w(\bar{\mu}) + C \sum_{l=1}^\infty l^{N-1} e^{-\frac{1}{2} \mu} \leq C w(\bar{\mu}). \]

Then

\[ |f'(z_{Q_m})| \leq C (w_{Q_m})^\delta \leq C w^\delta (\bar{\mu}) \leq \frac{1 - \theta^2}{4}, \text{ in } \mathbb{R}^N \setminus \bigcup_{j=1}^m B_{\bar{\mu}}(Q_j). \quad (2.24) \]

From (2.24) and direct computation, we have

\[ | - L(E(x)) | = \left| \left( \frac{\nabla}{i} - A_e(x) \right)^2 E(x) + V_e E(x) - f'(z_{Q_m}) E(x) \right| \]

\[ = \sum_{j=1}^m \left| - \gamma^2 + 1 + |A_0|^2 + \gamma \frac{(N - 1)}{|x - Q_j|} - 2 \gamma (A_1 \cdot (1, 1, ..., 1)) + \epsilon (\tilde{V}(x) + 2 A_0 \cdot \tilde{A} + i div \tilde{A} + \epsilon |\tilde{A}|^2) - f'(z_{Q_m}) |e^{-\gamma|x - Q_j|} \right| \]

\[ \geq \frac{1}{2} (1 - \theta^2) E(x), \text{ in } \mathbb{R}^N \setminus \bigcup_{j=1}^m B_{\bar{\mu}}(Q_j), \]

which yields that (2.23) is true.

Hence the function \( E(x) \) can be used as a barrier to prove the pointwise estimate

\[ |\varphi_{\sigma, Q_m}(x)| \leq C (|L \varphi_{\sigma, Q_m}| + \sup_j \| \varphi_{Q_m}(x) \|_{L^\infty(Q_j)}) E(x), \quad (2.25) \]

for all \( x \in \mathbb{R}^N \setminus \bigcup_{j=1}^m B_{\bar{\mu}}(Q_j) \).

Now we prove it by contradiction. We assume that there exist a sequence of \( \epsilon \) tending to 0, \( \mu \) tending to \( \infty \) and a sequence of solutions of (2.3) for which the inequality is not true. The problem being linear, we can reduce to the case where we
have a sequence \( \epsilon^{(n)} \) tending to 0, \( \mu^{(n)} \) tending to \( \infty \) and sequences \( h^{(n)}, \varphi^{(n)}, c^{(n)}_{j,k} \) such that
\[
\|h^{(n)}\| \to 0 \quad \text{and} \quad \|\varphi^{(n)}_{\sigma,Q_m}\|_* = 1.
\]

By (2.22), we have
\[
\left\| \sum_{j,k} c_{j,k}^{(n)} D_{j,k} \right\|_* \to 0.
\]

Then (2.25) implies that there exists \( Q_j^{(n)} \in \Omega_m \) such that
\[
\|\varphi_{Q_j^{(n)}}\|_{L^\infty(B_{\mu^2}(Q_j^{(n)}))} \geq C
\]
for some fixed constant \( C > 0 \).

Applying elliptic estimates together with Ascoli-Arzela’s theorem, we can find a sequence \( Q_j^{(n)} \) and we can extract, from the sequence \( \varphi_{Q_j^{(n)}}(\cdot - Q_j^{(n)}) \), a subsequence which will converge to \( \varphi^\infty \) a solution of
\[
\left[ -\left( \frac{\nabla_i}{i} - A_0 \right)^2 - 1 + f'(e^{i\sigma+iA_0 \cdot x}w) \right] \varphi^\infty = 0, \quad x \in \mathbb{R}^N,
\]
which is bounded by a constant times \( e^{-\gamma|x|} \), with \( \gamma > 0 \). Moreover, recall that \( \varphi_{Q_j^{(n)}}^{(n)} \) satisfies the orthogonality conditions in (2.3). Therefore, the limit function \( \varphi^\infty \) also satisfies
\[
Re \int \varphi^\infty \frac{\partial z}{\partial x_j} = 0, \quad j = 1, \ldots, N, \quad \text{and} \quad Re \int \varphi^\infty \frac{\partial z}{\partial \sigma} = 0,
\]
where \( z = e^{i\sigma+iA_0 \cdot x}w(x) \).

Then we have that \( \varphi^\infty \equiv 0 \) which contradicts to (2.26).

From Lemma 2.2, we can obtain the following result

**Proposition 2.3.** Then there exist positive numbers \( \gamma \in (0,1), \mu_0 > 0 \) and \( C > 0 \), such that for all \( 0 < \epsilon < e^{-2\mu}, \mu > \mu_0 \) and for any given \( h \) with \( \|h\|_* \) norm bounded, there is a unique solution \( (\varphi_{\sigma,Q_m}, c_{j,k}) \) to problem (2.3). Moreover,
\[
\|\varphi_{\sigma,Q_m}\|_* \leq C\|h\|_*.
\]

**Proof.** Here we consider the space
\[
\mathcal{H} = \left\{ u \in H^1(\mathbb{R}^N) : Re \int u \overline{D_{j,k}} = 0, (Q_m, \sigma) \in \Omega_m \times [0,2\pi] \right\}.
\]

Problem (2.3) can be rewritten as
\[
\varphi_{\sigma,Q_m} + \mathcal{K}(\varphi_{\sigma,Q_m}) = \tilde{h}, \quad \text{in} \ \mathcal{H},
\]
where \( \tilde{h} \) is defined by duality and \( \mathcal{K} : H \to H \) is a linear compact operator. By Fredholm’s alternative theorem, we know that the equation (2.28) has a unique solution for \( \tilde{h} = 0 \) which in turn follows from Lemma 2.2. The estimate (2.27) follows from directly from (2.6) in Lemma 2.2. The proof is complete.

In the sequel, if \( \varphi_{\sigma,Q_m} \) is the unique solution given by Proposition 2.3, we denote
\[
\varphi_{\sigma,Q_m} = A(h).
\]

By (2.27), we have
\[
\|A(h)\|_* \leq C\|h\|_*.
\]
Now, we consider
\begin{equation}
\begin{aligned}
\left\{ -\left( \frac{\nabla}{i} - A_\epsilon(x) \right)^2 (z_{Q_m} + \varphi_{\sigma, Q_m}) - V_\epsilon(x)(z_{Q_m} + \varphi_{\sigma, Q_m}) + f(z_{Q_m} + \varphi_{\sigma, Q_m}) \\
= \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k}, \text{ in } \mathbb{R}^N, \\
\Re \int \varphi_{\sigma, Q_m} \bar{D}_{j,k} = 0 \text{ for } j = 1, \ldots, m, k = 1, \ldots, N + 1.
\end{aligned}
\tag{2.31}
\end{equation}

We come to the main result in this section.

**Proposition 2.4.** Given $\gamma \in (0, 1)$. There exist positive numbers $\mu_0, C$ and $\eta > 0$ such that for all $\mu > \mu_0$, and for any $(\sigma, Q_m) \in [0, 2\pi] \times \Omega_m$ and $\epsilon < e^{-2\mu}$, there is a unique solution $(\varphi_{\sigma, Q_m}, c_{j,k})$ to problem (2.31). Furthermore, $\varphi_{\sigma, Q_m}$ is $C^1$ in $(\sigma, Q_m)$ and we have
\begin{equation}
\|\varphi_{\sigma, Q_m}\|_* \leq Ce^{-\beta\mu}, |c_{j,k}| \leq Ce^{-\beta\mu}.
\tag{2.32}
\end{equation}

Note that the first equation in (2.31) can be rewritten as
\begin{equation}
L(\varphi_{\sigma, Q_m}) = -S(z_{Q_m}) + \mathcal{N}(\varphi_{\sigma, Q_m}) + \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k},
\tag{2.33}
\end{equation}

where
\begin{equation}
L(\varphi_{\sigma, Q_m}) = -\left( \frac{\nabla}{i} - A_\epsilon(x) \right)^2 \varphi_{\sigma, Q_m} - V_\epsilon(x) \varphi_{\sigma, Q_m} + f'(z_{Q_m}) \varphi_{\sigma, Q_m}
\tag{2.34}
\end{equation}

and
\begin{equation}
\mathcal{N}(\varphi_{\sigma, Q_m}) = -\left[ f(z_{Q_m} + \varphi_{\sigma, Q_m}) - f(z_{Q_m}) - f'(z_{Q_m}) \varphi_{\sigma, Q_m} \right].
\tag{2.35}
\end{equation}

In order to use the contraction mapping theorem to prove that (2.33) is uniquely solvable in the set that $\|\varphi_{\sigma, Q_m}\|_*$ is small, we need to estimate $\|S(z_{Q_m})\|_*$ and $\|\mathcal{N}(\varphi_{\sigma, Q_m})\|_*$ respectively.

**Lemma 2.5.** Given $\gamma \in (0, 1)$. For $\mu$ large enough, and any $(\sigma, Q_m) \in [0, 2\pi] \times \Omega_m$, $\epsilon < e^{-2\mu}$, we have
\begin{equation}
\|S(z_{Q_m})\|_* \leq Ce^{-\beta\mu},
\tag{2.36}
\end{equation}

for some constant $\beta > 0$ and $C$ independent of $\mu, m, Q_m$ and $\sigma$.

**Proof.** Note that
\begin{align*}
S(z_{Q_m}) &= -\left( \frac{\nabla}{i} - A_\epsilon(x) \right)^2 z_{Q_m} - V_\epsilon(x) z_{Q_m} + f(z_{Q_m}) \\
&= -\left( \frac{\nabla}{i} - A_0 \right)^2 z_{Q_m} - z_{Q_m} - \epsilon \tilde{V}(x) z_{Q_m} + f(z_{Q_m}) \\
&\quad + \frac{\epsilon}{i} \text{div} \tilde{A}(x) z_{Q_m} + 2\epsilon \frac{\tilde{A}(x) \cdot \nabla z_{Q_m} - 2\epsilon A_0 \cdot \tilde{A} z_{Q_m} - \epsilon^2 |\tilde{A}(x)|^2 z_{Q_m}}{i} \\
&= -\epsilon \tilde{V}(x) z_{Q_m} + f(z_{Q_m}) - \sum_{j=1}^{m} f(z_{Q_j}).
\end{align*}
\[
+ \frac{\epsilon}{i} \text{div}\vec{A}(x)zQ_m + 2\epsilon \frac{1}{i} \vec{A}(x) \cdot \nabla zQ_m - 2\epsilon A_0 \cdot \vec{A}zQ_m - \epsilon^2 |\vec{A}(x)|^2 zQ_m
\]
\[
= -\epsilon \hat{V}(x)zQ_m + f(zQ_m) - \sum_{j=1}^{m} f(zQ_j) 
\]
\[
+ \frac{\epsilon}{i} \text{div}\vec{A}(x)zQ_m + 2\epsilon \sum_{j=1}^{m} \xi_j \vec{A}(x) \cdot \nabla w_{Q_j} - \epsilon^2 |\vec{A}(x)|^2 zQ_m.
\]

It follows from (2.5) and (2.6) of section 2.1 in [1] that
\[
|f(zQ_m) - \sum_{j=1}^{m} f(zQ_j)| = |e^{iz}||f(w_{Q_m}) - \sum_{j=1}^{m} f(w_{Q_j})| \leq Ce^{-\beta\mu} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} 
\]
for a proper choice of \(\beta > 0\).
Moreover, by the assumption of \(\epsilon\), we can prove that
\[
|\epsilon \hat{V}(x)zQ_m| \leq Ce^{-\beta\mu} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} 
\]
for some \(\beta > 0\). In fact, on one hand, fix \(j \in \{1, 2, \ldots, m\}\) and consider the region \(|x-Q_j| \leq \frac{\mu}{\pi} \). In this region, we have
\[
|\epsilon \hat{V}(x)zQ_m| \leq Ce^{-2\mu} \leq Ce^{-\mu} e^{-2|x-Q_j|} \leq Ce^{-\beta\mu} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|}. 
\]
On the other hand, considering the region \(|x-Q_j| > \frac{\mu}{\pi} \) for all \(j\), we have
\[
|\epsilon \hat{V}(x)zQ_m| \leq Ce^{-2\mu}|w_{Q_m}| \leq Ce^{-\beta\mu} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|}.
\]

By the same arguments with (2.39), we can prove
\[
\left|\frac{\epsilon}{i} \text{div}\vec{A}(x)zQ_m\right| \leq Ce^{-\beta\mu} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|}, \tag{2.40}
\]
\[
\left|2\epsilon \sum_{j=1}^{m} \xi_j \vec{A}(x) \cdot \nabla w_{Q_j}\right| \leq Ce^{-\beta\mu} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} \tag{2.41}
\]
and
\[
\left|\epsilon^2 |\vec{A}(x)|^2 zQ_m\right| \leq Ce^{-\beta\mu} \sum_{j=1}^{m} e^{-\gamma|x-Q_j|}. \tag{2.42}
\]

It follows from (2.37) to (2.42) that
\[
\|S(zQ_m)\|_* \leq Ce^{-\beta\mu}
\]
for some \(\beta > 0\) independent of \(\mu, m\) and \(Q_m\).

\textbf{Lemma 2.6.} For any \(Q_m \in \Omega_m\) satisfying \(\|\varphi_{\sigma, Q_m}\|_* \leq 1\), we have
\[
\|N(\varphi_{\sigma, Q_m})\|_* \leq C\|\varphi_{\sigma, Q_m}\|_{1+\delta} \tag{2.43}
\]
and
\[
\|N(\varphi_{\sigma, Q_m}^1) - N(\varphi_{\sigma, Q_m}^2)\|_* \leq C(\|\varphi_{\sigma, Q_m}^1\|_{1+\delta} + \|\varphi_{\sigma, Q_m}^2\|_{1+\delta})\|\varphi_{\sigma, Q_m}^1 - \varphi_{\sigma, Q_m}^2\|_*). \tag{2.44}
\]
Proof. By direct computation and applying the mean-value theorem, we have
\[
|N(\varphi_{\sigma,Q_m})| = |f(z_{Q_m} + \varphi_{\sigma,Q_m}) - f(z_{Q_m}) - f'(z_{Q_m})\varphi_{\sigma,Q_m}|
\]
\[
= |f'(z_{Q_m} + \varphi_{\sigma,Q_m})\varphi_{\sigma,Q_m} - f'(z_{Q_m})\varphi_{\sigma,Q_m}|
\]
\[
\leq C|\varphi_{\sigma,Q_m}|^{1+\delta} \leq C\|\varphi_{\sigma,Q_m}\|_s^{1+\delta}(\sum_{j=1}^m e^{-\gamma|x-Q_j|})^{1+\delta}
\]
(2.45)
and
\[
|N(\varphi_{\sigma,Q_m}^1) - N(\varphi_{\sigma,Q_m}^2)|
\]
\[
= |f(z_{Q_m} + \varphi_{\sigma,Q_m}^1) - f(z_{Q_m}) - f'(z_{Q_m})\varphi_{\sigma,Q_m}^1 + f'(z_{Q_m})\varphi_{\sigma,Q_m}^2|
\]
\[
= |f'(z_{Q_m} + \varphi_{\sigma,Q_m}^1 - \varphi_{\sigma,Q_m}^2)(\varphi_{\sigma,Q_m}^1 - \varphi_{\sigma,Q_m}^2) - f'(z_{Q_m})(\varphi_{\sigma,Q_m}^1 - \varphi_{\sigma,Q_m}^2)|
\]
\[
\leq C(|\varphi_{\sigma,Q_m}^1|^{1+\delta} + |\varphi_{\sigma,Q_m}^2|^{1+\delta})(\|\varphi_{\sigma,Q_m}^1 - \varphi_{\sigma,Q_m}^2\|_s)\sum_{j=1}^m e^{-\gamma|x-Q_j|}
\]
(2.46)
From (2.45) and (2.46), we can have
\[
\|N(\varphi_{\sigma,Q_m})\|_s \leq C\|\varphi_{\sigma,Q_m}\|_s^{1+\delta}
\]
and
\[
\|N(\varphi_{\sigma,Q_m}^1) - N(\varphi_{\sigma,Q_m}^2)\|_s \leq C(|\varphi_{\sigma,Q_m}^1|^{1+\delta} + |\varphi_{\sigma,Q_m}^2|^{1+\delta})(\|\varphi_{\sigma,Q_m}^1 - \varphi_{\sigma,Q_m}^2\|_s).
\]

Now, we are ready to prove Proposition 2.4.

Proof of Proposition 2.4. We will use the contraction theorem to prove it. Observe that \(\varphi_{\sigma,Q_m}\) solves (2.31) if and only if
\[
\varphi_{\sigma,Q_m} = A(-S(z_{Q_m}) + N(\varphi_{\sigma,Q_m}))
\]
(2.47)
where \(A\) is the operator introduced in (2.29). In other words, \(\varphi_{\sigma,Q_m}\) solves (2.31) if and only if \(\varphi_{\sigma,Q_m}\) is a fixed point for the operator
\[
T(\varphi_{\sigma,Q_m}) := A(-S(z_{Q_m}) + N(\varphi_{\sigma,Q_m})).
\]

Define
\[
B = \{\varphi_{\sigma,Q_m} \in H^1(\mathbb{R}^N, C) : \|\varphi_{\sigma,Q_m}\|_s \leq e^{-(\beta-\tau)\mu}, \ \text{Re} \int \varphi_{\sigma,Q_m}\overline{D}_{j,k} = 0\},
\]
where \(\tau > 0\) small enough. We will prove that \(T\) is a contraction mapping from \(B\) to itself. On one hand, for any \(\varphi_{\sigma,Q_m} \in B\), it follows from Lemmas 2.5 and 2.6 that
\[
\|T(\varphi_{\sigma,Q_m})\|_s \leq C\|N(\varphi_{\sigma,Q_m})\|_s \leq Ce^{-\beta\mu} + C\|\varphi_{\sigma,Q_m}\|_s^{1+\delta} \leq Ce^{-\beta\mu} + Ce^{-(1+\delta)(\beta-\tau)\mu} \leq e^{-(\beta-\tau)\mu}.
\]
On the other hand, taking $\varphi_{\sigma, Q_m}^1$ and $\varphi_{\sigma, Q_m}^2$ in $B$, by Lemma 2.6 we have
\[
\|\mathcal{T}(\varphi_{\sigma, Q_m}^1) - \mathcal{T}(\varphi_{\sigma, Q_m}^2)\|_* \leq C\|\mathcal{N}(\varphi_{\sigma, Q_m}^1) - \mathcal{N}(\varphi_{\sigma, Q_m}^2)\|_*. 
\]
Hence by the contraction mapping theorem, for any $(\sigma, Q_m) \in [0, 2\pi] \times \Omega_m$, there exists a unique $\varphi_{\sigma, Q_m} \in B$ such that (2.47) holds. So
\[
\|\varphi_{\sigma, Q_m}\|_* = \|\mathcal{T}(\varphi_{\sigma, Q_m})\|_* \leq Ce^{-\beta\mu}.
\]
Now we need to prove that $\varphi_{\sigma, Q_m}$ is $2\pi$-periodic with respect to $\sigma$. Replacing $\sigma$ by $\sigma + 2\pi$ in the above reduction process, we get $\varphi_{\sigma + 2\pi, Q_m}$. Since $z_{Q_m}$ is $2\pi$-periodic, by the uniqueness of $\varphi_{\sigma, Q_m}$, we see $\varphi_{\sigma, Q_m} = \varphi_{\sigma + 2\pi, Q_m}$.

Combining (2.22), (2.36), (2.43) and (2.44) we have
\[
|c_{j,k}| \leq C(e^{-\frac{2\beta}{2}}\|\varphi_{\sigma, Q_m}\|_* + \|\mathcal{S}(\varphi_{\sigma, Q_m})\|_* + \|\mathcal{N}(\varphi_{\sigma, Q_m})\|_*) \leq Ce^{-\beta\mu}.
\]

\[\square\]

3. A secondary Lyapunov-Schmidt reduction. In this section, we present a key estimate on the difference between the solutions in the m-th step and (m+1)-th step. This second Lyapunov-Schmidt reduction has been used in the paper [1,24,30]. For $(\sigma, Q_m) \in [0, 2\pi] \times \Omega_m$, we denote $u_{Q_m}$ as $z_{Q_m} + \varphi_{\sigma, Q_m}$, where $\varphi_{\sigma, Q_m}$ is the unique solution given by Proposition 2.4. The main estimate below states that the difference between $u_{Q_{m+1}}$ and $u_{Q_m} + z_{Q_{m+1}}$ is small globally in $H^1(\mathbb{R}^N, \mathbb{C})$ norm.

For this purpose, we now write
\[
u_{Q_{m+1}} = u_{Q_m} + z_{Q_{m+1}} + \phi_{m+1} =: \bar{u} + \phi_{m+1}.
\]

By Proposition 2.4, we can easily obtain that
\[
\|\phi_{m+1}\|_* \leq Ce^{-\beta\mu}.
\]
However the estimate (3.2) is not sufficient. We need a crucial estimate for $\phi_{m+1}$ which will be given later. (In the following we will always assume that $\gamma > \frac{1}{2}$.) In order to obtain the crucial estimate, we will need the following lemma.

Lemma 3.1. (Lemma 2.3, [4]) For $|Q_j - Q_k| \geq \mu$ large, it holds that
\[
\int f(w(x - Q_j))w(x - Q_k)dx = (\vartheta + e^{-\beta\mu})w(|Q_j - Q_k|)
\]
for some $\beta > 0$ independent of large $\mu$ and
\[
\vartheta = \int f(w)e^{-x_1}dx > 0.
\]

Lemma 3.2. Let $\mu$, $\epsilon$ be as in Proposition 2.4. Then it holds
\[
\|\phi_{m+1}\|_{H^1(\mathbb{R}^N)} \leq C\left[\epsilon \int |\hat{V}(x)||w_{Q_{m+1}}| + 2\epsilon \int |\hat{A}(x)||\nabla w_{Q_{m+1}}| + \epsilon \int |\text{div}A(x)||w_{Q_{m+1}}| \right.
\]
\[
+ \epsilon^2 \int |\hat{A}(x)|^2|w_{Q_{m+1}}| + \epsilon \left(\int |\hat{V}(x)|^2|w_{Q_{m+1}}|^2\right)^{\frac{1}{2}} + \epsilon \left(\int |\hat{A}(x)|^2|\nabla w_{Q_{m+1}}|^2\right)^{\frac{1}{2}}
\]
\[
+ e^{-\beta\mu} \left(\sum_{j=1}^{m} w(|Q_{m+1} - Q_j|)\right)^{\frac{1}{2}} + \epsilon \left(\int |\text{div}A(x)|^2|w_{Q_{m+1}}|^2\right)^{\frac{1}{2}}
\]
for some constant $C > 0$, $\beta > 0$ independent of $\mu, m, \gamma$ and $Q_{m+1} \in \Omega_{m+1}$.

Proof. To prove (3.5), we need to perform a further decomposition.

As we mentioned before, the following eigenvalue problem

$$\Delta \varphi - \varphi + f'(w)\varphi = \lambda \varphi, \quad \varphi \in H^1(\mathbb{R}^N),$$

admits the following set of eigenvalues

$$\lambda_1 > \lambda_2 > \ldots > \lambda_n > \lambda_{n+1} = 0 > \lambda_{n+2} \ldots .$$

We denote the eigenfunctions corresponding to the positive eigenvalues $\lambda_j$ as $\varphi_j$, $j = 1, \ldots, n$.

Now, we have the eigenvalue $\lambda_k (k = 1, \ldots, n)$ with eigenfunction $\tilde{\varphi}_{0,k} = e^{i\sigma + A_0} x \times \varphi_k$ of the following linearized operator

$$-\left(\frac{\nabla}{i} - A_0\right)^2 \varphi - \varphi + f'(w)\varphi = \lambda \varphi.$$

(3.6)

We fix $\tilde{\varphi}_{0,k}$ such that $\max_{x \in \mathbb{R}^N} |\tilde{\varphi}_{0,k}| = 1$. Denote by $\tilde{\varphi}_{j,k} = \eta_j \varphi_{0,k} (x - Q_j)$, where $\eta_j$ is the cut-off function introduced in section 1.

By the equations satisfied by $\phi_{m+1}$, we have

$$\bar{L} \phi_{m+1} = -\bar{S} + \sum_{j=1}^{m+1} \sum_{k=1}^{N+1} c_{j,k} D_{j,k}$$

(3.7)

for some constants $c_{j,k}$, where

$$\bar{L} = -\left(\frac{\nabla}{i} - A_0(x)\right)^2 - V(x) + f'(\bar{u}),$$

where

$$f'(\bar{u}) = \begin{cases} \frac{f(\bar{u} + \phi_{m+1}) - f(\bar{u})}{\phi_{m+1}}, & \text{if } \phi_{m+1} \neq 0, \\ f'(\bar{u}), & \text{if } \phi_{m+1} = 0, \end{cases}$$

and

$$\bar{S} = f(u_{Q_m} - z_{Q_{m+1}}) - f(u_{Q_m}) - (1 + i\bar{V}(x)) z_{Q_{m+1}} - \left(\frac{\nabla}{i} - A_0(x)\right)^2 z_{Q_{m+1}}$$

$$= f(u_{Q_m} - z_{Q_{m+1}}) - f(u_{Q_m}) - f(z_{Q_{m+1}}) - \epsilon \bar{V}(x) z_{Q_{m+1}} + \frac{\epsilon}{i} \text{div}(\bar{A}(x)) z_{Q_{m+1}}$$

$$+ 2 \frac{\epsilon}{i} \bar{A}(x) \cdot \nabla z_{Q_{m+1}} - 2 \epsilon A_0 \cdot \bar{A} z_{Q_{m+1}} - \epsilon^2 |\bar{A}(x)|^2 z_{Q_{m+1}}$$

$$= f(u_{Q_m} - z_{Q_{m+1}}) - f(u_{Q_m}) - f(z_{Q_{m+1}}) - \epsilon \bar{V}(x) z_{Q_{m+1}} + \frac{\epsilon}{i} \text{div}(\bar{A}(x)) z_{Q_{m+1}}$$

$$+ 2 \frac{\epsilon}{i} \bar{A}(x) \cdot \nabla w_{Q_{m+1}} - \epsilon^2 |\bar{A}(x)|^2 z_{Q_{m+1}}.$$

(3.8)

Now we proceed the proof into a few steps.

First we estimate the $L^2$-norm of $\bar{S}$. By the estimate in Proposition 2.4, we have the following estimate

$$\int |f(u_{Q_m} + z_{Q_{m+1}}) - f(u_{Q_m}) - f(z_{Q_{m+1}})|^2 \leq C \epsilon^{-\beta} \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|).$$

(3.9)
We also have
\[ \int |\epsilon \tilde{V}(x)z_{Q_{m+1}}|^2 \leq C\epsilon^2 \int \tilde{V}(x)^2 w_{Q_{m+1}}^2, \]
\[ \int \left| \frac{\epsilon}{t} \text{div} \tilde{A} z_{Q_{m+1}} \right|^2 \leq C\epsilon^2 \int |\text{div} \tilde{A}|^2 w_{Q_{m+1}}^2, \]
\[ \int \left| 2\frac{\epsilon}{t} \xi_j \tilde{A}(x) \cdot \nabla w_{Q_{m+1}} \right|^2 \leq C\epsilon^2 \int |\tilde{A}(x)|^2 |\nabla w_{Q_{m+1}}|^2 \]
and
\[ \int \left| \epsilon^2 |\tilde{A}(x)|^2 z_{Q_{m+1}} \right|^2 \leq C\epsilon^2 \int |\tilde{A}(x)|^4 w_{Q_{m+1}}^2. \] (3.10)

It follows from (3.8) to (3.10) that
\[ \|\tilde{S}\|_{L^2}^2 \leq Ce^{-\beta \mu} \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|) + C\epsilon^2 \int \tilde{V}(x)^2 w_{Q_{m+1}}^2 + C\epsilon^4 \int |\tilde{A}|^4 w_{Q_{m+1}}^2 + C\epsilon^2 \int |\text{div} \tilde{A}|^2 w_{Q_{m+1}}^2. \] (3.11)

By the estimate (3.2), we have the following estimate
\[ \left| \tilde{u} - \sum_{j=1}^{m+1} z(x - Q_j) \right| = O(e^{-\beta \mu}). \] (3.12)

Decompose \( \phi_{m+1} \) as
\[ \phi_{m+1} = \psi + \sum_{j=1}^{m+1} \sum_{l=1}^{n} g_{j,l} \tilde{\varphi}_{j,l} + \sum_{j=1}^{m+1} \sum_{k=1}^{N+1} d_{j,k} D_{j,k} \] (3.13)
for some \( g_{j,l} \), \( d_{j,k} \) such that
\[ \text{Re} \int \psi \bar{\varphi}_{j,l} = \text{Re} \int \psi \bar{D}_{j,k} = 0, j = 1, \ldots, m+1, k = 1, \ldots, N+1, l = 1, \ldots, n. \] (3.14)

Since
\[ \phi_{m+1} = \varphi_{\sigma, Q_{m+1}} - \varphi_{\sigma, Q_m}, \] (3.15)
we have for \( j = 1, \ldots, m, \)
\[ d_{j,k} = \langle \phi_{m+1}, D_{j,k} \rangle - \sum_{l=1}^{n} g_{j,l} \langle \tilde{\varphi}_{j,l}, D_{j,k} \rangle \]
\[ = \text{Re} \int \phi_{m+1} \bar{D}_{j,k} - \text{Re} \sum_{l=1}^{n} \int g_{j,l} \bar{\varphi}_{j,l}, \tilde{D}_{j,k} \]
\[ = \text{Re} \int (\varphi_{\sigma, Q_{m+1}} - \varphi_{\sigma, Q_m}) \bar{D}_{j,k} - \text{Re} \sum_{l=1}^{n} \int g_{j,l} \bar{\varphi}_{j,l}, \tilde{D}_{j,k} \] (3.16)
\[ = -\text{Re} \sum_{l=1}^{n} \int g_{j,l} \bar{\varphi}_{j,l}, \tilde{D}_{j,k} \]
\[ = e^{-\beta \mu} \sum_{l=1}^{n} g_{j,l}, \]
\[ d_{m+1,k} = \langle \phi_{m+1}, D_{m+1,k} \rangle - \sum_{l=1}^{m} n_{m+1,l} \langle \bar{\varphi}_{m+1,l}, D_{m+1,k} \rangle \]
\[ = \text{Re} \int \phi_{m+1} \bar{D}_{m+1,k} - \text{Re} \sum_{l=1}^{m} \int g_{m+1,l} \varphi_{m+1,l} \, \bar{D}_{m+1,k} \]
\[ = \text{Re} \int (\varphi_{\sigma, Q_{m+1}} - \varphi_{\sigma, Q_{m}}) \bar{D}_{m+1,k} - \text{Re} \sum_{l=1}^{m} \int g_{m+1,l} \varphi_{m+1,l} \, \bar{D}_{m+1,k} \]
\[ = \text{Re} \int \varphi_{\sigma, Q_{m}} \bar{D}_{m+1,k} - \text{Re} \sum_{l=1}^{m} \int g_{m+1,l} \varphi_{m+1,l} \, \bar{D}_{m+1,k} \]
\[ = -\text{Re} \int \varphi_{\sigma, Q_{m}} \bar{D}_{m+1,k} + e^{-\beta \mu} \sum_{l=1}^{m} n_{m+1,l}, \]

where we use the orthogonality conditions satisfied by \( \varphi_{\sigma, Q_{m}} \) and \( \varphi_{\sigma, Q_{m+1}} \). Hence by Proposition 2.4, we have

\[
\begin{align*}
|d_{j,l}| &\leq C e^{-\beta \mu} \sum_{l=1}^{m} n_{j,l}, \text{ for } j = 1, \ldots, m, \\
|d_{m+1,k}| &\leq C e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |Q_j - Q_{m+1}|} + C e^{-\beta \mu} \sum_{l=1}^{m} n_{m+1,l}.
\end{align*}
\]

By (3.13), we can rewrite (3.7) as

\[
\bar{L}(\psi) + \sum_{j=1}^{m+1} n \sum_{l=1}^{m+1} g_{j,l} \bar{L}(\bar{\varphi}_{j,l}) + \sum_{j=1}^{m+1} n \sum_{k=1}^{m+1} d_{j,k} \bar{L}(D_{j,k}) = -\bar{S} + \sum_{j=1}^{m+1} n \sum_{k=1}^{m+1} c_{j,k} D_{j,k}.
\]

In order to estimate the coefficients \( g_{j,l} \), we use the equation (3.19). First, multiplying (3.19) by \( \bar{\varphi}_{j,l} \) and integrating over \( \mathbb{R}^N \), we have

\[
\text{Re} \int g_{j,l} \bar{L}(\bar{\varphi}_{j,l}) \bar{\varphi}_{j,l} = -\sum_{j=1}^{m+1} n \sum_{k=1}^{m+1} \text{Re} \, d_{j,k} \int \bar{L}(D_{j,k}) \bar{\varphi}_{j,l} - \text{Re} \int \bar{S} \bar{\varphi}_{j,l} - \sum_{j=1}^{m+1} n \sum_{k=1}^{m+1} d_{j,k} \int \bar{L}(\bar{\varphi}_{j,k}) \bar{\varphi}_{j,l} - \text{Re} \int \bar{L}(\psi) \bar{\varphi}_{j,l},
\]

where

\[
\begin{align*}
\text{Re} \int \bar{S} \bar{\varphi}_{j,l} &\leq C e^{-\beta \mu} e^{-\gamma |Q_j - Q_{m+1}|} + 2e \left| \text{Re} \int \frac{1}{i} \xi_j \tilde{A}(x) \cdot \nabla w_{Q_{m+1}} \bar{\varphi}_{j,l} \right| \\
&\quad + e \left| \text{Re} \int \tilde{V}(x) z_{Q_{m+1}} \bar{\varphi}_{j,l} \right| + e \left| \text{Re} \int \frac{1}{i} \text{div} \tilde{A}(x) z_{Q_{m+1}} \bar{\varphi}_{j,l} \right| \\
&\quad + e^2 \left| \text{Re} \int |\tilde{A}(x)|^2 z_{Q_{m+1}} \bar{\varphi}_{j,l} \right|, \quad j = 1, \ldots, m, \\
\text{Re} \int \bar{S} \bar{\varphi}_{m+1,l} &\leq C e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |Q_j - Q_{m+1}|} + e \left| \text{Re} \int \tilde{V}(x) z_{Q_{m+1}} \bar{\varphi}_{m+1,l} \right| \\
&\quad + e \left| \text{Re} \int \frac{1}{i} \text{div} \tilde{A}(x) z_{Q_{m+1}} \bar{\varphi}_{m+1,l} \right| + e^2 \left| \text{Re} \int |\tilde{A}(x)|^2 z_{Q_{m+1}} \bar{\varphi}_{m+1,l} \right| \\
&\quad + 2e \left| \text{Re} \int \frac{1}{i} \xi_j \tilde{A}(x) \cdot \nabla w_{Q_{m+1}} \bar{\varphi}_{m+1,l} \right|.
\end{align*}
\]
By the definition of $\tilde{\varphi}_{j,l}$, we have

$$L(\tilde{\varphi}_{j,k}) = \lambda_k \tilde{\varphi}_{j,k} + O(e^{-\beta \mu}),$$

thus one has

$$Re \int L(\tilde{\varphi}_{j,k}) \tilde{\varphi}_{j,l} = -\delta_{k,l} \lambda_k \int \tilde{\varphi}_{0,l} \tilde{\varphi}_{0,k} + O(e^{-\beta \mu}). \quad (3.22)$$

Recall the definition of $\varphi$, we have

$$Re \int \tilde{L}(\psi) \tilde{\varphi}_{j,l} = -Re \int \psi \tilde{L}(\tilde{\varphi}_{j,l}) = -\lambda_l \int \tilde{\varphi}_{j,l} \psi + O(e^{-\beta \mu}) \| \psi \|_{H^1(B_{\bar{\varpi}}(Q_i))}$$

$$= O(e^{-\beta \mu}) \| \psi \|_{H^1(B_{\bar{\varpi}}(Q_i))}.$$

Combining (3.18), (3.20), (3.21) and (3.22), and the orthogonal conditions satisfied by $\psi$

\[
\begin{align*}
|g_{j,l}| & \leq C e^{-\beta \mu} e^{-\gamma |Q_j - Q_{m+1}|} + \varepsilon \left| Re \int \tilde{V}(x) z_{Q_{m+1}} \tilde{\varphi}_{j,l} \right| \\
+ \varepsilon \left| Re \int \frac{1}{i} \text{div} \tilde{A}(x) z_{Q_{m+1}} \tilde{\varphi}_{j,l} \right| + 2 \varepsilon \left| Re \int \frac{1}{i} \xi_j \tilde{A}(x) \cdot \nabla w_{Q_{m+1}} \tilde{\varphi}_{j,l} \right| \\
+ \varepsilon^2 \left| Re \int \tilde{A}(x)^2 z_{Q_{m+1}} \tilde{\varphi}_{j,l} \right| + e^{-\beta \mu} \| \psi \|_{H^1(B_{\bar{\varpi}}(Q_i))},
\end{align*}
\]

and

\[
\begin{align*}
|d_{k,l}| & \leq C e^{-\beta \mu} e^{-\gamma |Q_j - Q_{m+1}|} + \varepsilon \left| Re \int \tilde{V}(x) z_{Q_{m+1}} \tilde{\varphi}_{j,l} \right| \\
+ \varepsilon \left| Re \int \frac{1}{i} \text{div} \tilde{A}(x) z_{Q_{m+1}} \tilde{\varphi}_{j,l} \right| + 2 \varepsilon \left| Re \int \frac{1}{i} \xi_j \tilde{A}(x) \cdot \nabla w_{Q_{m+1}} \tilde{\varphi}_{j,l} \right| \\
+ \varepsilon^2 \left| Re \int \tilde{A}(x)^2 z_{Q_{m+1}} \tilde{\varphi}_{j,l} \right| + e^{-\beta \mu} \| \psi \|_{H^1(B_{\bar{\varpi}}(Q_i))},
\end{align*}
\]

Next, we estimate $\psi$. Multiplying (3.19) by $\psi$ and integrating over $\mathbb{R}^N$, we find

$$Re \int \tilde{L}(\psi) \tilde{\varphi}_{j,l}$$
\[ = -\text{Re} \int \tilde{S} \psi - \sum_{j=1}^{m+1} \sum_{k=1}^{N+1} d_{j,k} \text{Re} \int \bar{L}(D_{j,k}) \bar{\psi} - \sum_{j=1}^{m+1} \sum_{l=1}^{n} g_{j,l} \text{Re} \int \bar{L}(\varphi_{j,l}) \bar{\psi}. \quad (3.25) \]

We claim that
\[ \text{Re} \int (-\bar{L}(\psi) \bar{\psi}) \geq c_0 \|\psi\|_{H^1}^2 \quad (3.26) \]
for some constant \( c_0 > 0. \)

Since the approximate solution is exponentially decay away from the points \( Q_j, \) we have
\[ \text{Re} \int_{\mathbb{R}^N \setminus \bigcup_j B_{\frac{Q_j}{2}}(Q_j)} (-\bar{L}(\psi) \bar{\psi}) \geq \frac{1}{2} \int_{\mathbb{R}^N \setminus \bigcup_j B_{\frac{Q_j}{2}}(Q_j)} (|\nabla \psi|^2 + |\psi|^2). \quad (3.27) \]

Now we only need to prove the above estimates in the domain \( \bigcup_j B_{\frac{Q_j}{2}}(Q_j). \) We prove it by contradiction. Otherwise, there exists a sequence \( \mu_n \to \infty, \) and \( Q_j^{(n)} \) such that
\[ \int_{B_{\frac{\mu_n}{n}}(Q_j^{(n)})} (|\nabla \psi_n|^2 + |\psi_n|^2) = 1, \quad \text{Re} \int_{B_{\frac{\mu_n}{n}}(Q_j^{(n)})} (-\bar{L}(\psi_n) \bar{\psi}_n) \to 0, \text{ as } n \to \infty. \]

Then we can extract from the sequence \( \psi_n(-Q_j^{(n)}) \) a subsequence which will converge weakly in \( H^1(\mathbb{R}^N) \) to \( \psi_\infty, \) and \( \mu_n \to \infty, \) we have
\[ \int \left| \left( \frac{\nabla}{i} - A_0 \right) \psi_\infty \right|^2 + |\psi_\infty|^2 - f'(e^{i\sigma + iA_0 \cdot x} w) \psi_\infty^2 = 0 \quad (3.28) \]
and
\[ \text{Re} \int \psi_\infty \bar{\psi}_{0,l} = \text{Re} \int \psi_\infty \frac{\partial (e^{i\sigma + iA_0 \cdot x} w)}{\partial x_j} = 0, \quad j = 1, \ldots, N, l = 1, \ldots, n. \quad (3.29) \]

It follows from (3.28) and (3.29) that \( \psi_\infty = 0. \) Therefore
\[ \psi_n \rightharpoonup 0 \text{ weakly in } H^1(\mathbb{R}^N). \quad (3.30) \]

Hence, we have
\[ \int_{B_{\frac{\mu_n}{n}}(Q_j^{(n)})} f'(\bar{u}) \psi_n^2 \to 0, \text{ as } n \to \infty. \quad (3.31) \]

Then
\[ \|\psi_n\|_{H^1(\mathbb{R}^N)} \to 0, \text{ as } n \to \infty, \]
which contradicts to the assumption \( \|\psi_n\|_{H^1} = 1. \) Therefore (3.26) holds.

It follows from (3.25) and (3.26) that
\[ \|\psi\|_{H^1(\mathbb{R}^N)}^2 \leq C \left( \sum_{j,k} |d_{j,k}| \left| \text{Re} \int \bar{L}(D_{j,k}) \bar{\psi} \right| + \sum_{j,l} |g_{j,l}| \left| \text{Re} \int \bar{L}(\varphi_{j,l}) \bar{\psi} \right| + \left| \text{Re} \int \bar{S} \bar{\psi} \right| \right) \]
\[ \leq C \left( \sum_{j,k} |d_{j,k}| \|\psi\|_{H^1} + \sum_{j,l} |g_{j,l}| \|\psi\|_{H^1} \right) + \|S\|_{L^2} \|\psi\|_{H^1}, \quad (3.32) \]
By (3.24) and (3.32), we have
\[ \| \psi \|_{H^1(\mathbb{R}^N)} \leq C \left( \sum_{j,k} |d_{jk}| + e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma |Q_j - Q_{m+1}|} + \| \tilde{S} \|_{L^2} + \epsilon \int |\tilde{V}(x)||w_{Q_{m+1}}| \right. \\
+ 2\epsilon \int |\tilde{A}(x)||\nabla w_{Q_{m+1}}| + \epsilon \int |\text{div} \tilde{A}(x)||w_{Q_{m+1}}| + \epsilon^2 \int |\tilde{A}(x)|^2 |w_{Q_{m+1}}| \left. \right) . \]

From (3.11), (3.18) and (3.33), recalling that \( \gamma > \frac{1}{2} \), we get
\[ \| \phi_{m+1} \|_{H^1(\mathbb{R}^N)} \leq C \left[ e^{-\beta \mu} \sum_{j=1}^{m} e^{-\gamma (Q_j - Q_{m+1})} + \epsilon \int |\tilde{V}(x)||w_{Q_{m+1}}| + 2\epsilon \int |\tilde{A}(x)||\nabla w_{Q_{m+1}}| \\
+ \epsilon \int |\text{div} \tilde{A}(x)||w_{Q_{m+1}}| + \epsilon^2 \int |\tilde{A}(x)|^2 |w_{Q_{m+1}}| + \epsilon \left( \int |\tilde{V}(x)|^2 |w_{Q_{m+1}}|^2 \right)^{\frac{1}{2}} \right] . \]

Since we choose \( \gamma > \frac{1}{2} \), by the definition of the configuration space, we have
\[ \left( \sum_{j=1}^{m} e^{-\gamma (Q_j - Q_{m+1})} \right)^2 \leq C \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|) . \]

It follows from (3.34) and (3.35) that
\[ \| \phi_{m+1} \|_{H^1(\mathbb{R}^N)} \leq C \left[ e \int |\tilde{V}(x)||w_{Q_{m+1}}| + 2\epsilon \int |\tilde{A}(x)||\nabla w_{Q_{m+1}}| + e^{-\beta \mu} \left( \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|) \right)^{\frac{1}{2}} \right. \\
+ \epsilon^2 \int |\tilde{A}(x)|^2 |w_{Q_{m+1}}| + \epsilon \int |\text{div} \tilde{A}(x)||w_{Q_{m+1}}| + \epsilon \left( \int |\tilde{V}(x)|^2 |w_{Q_{m+1}}|^2 \right)^{\frac{1}{2}} \right] . \]

Hence (3.5) holds.

Moreover, from the estimates (3.18) and (3.24), and taking into consideration that \( \eta_j \) is supposed in \( B_{\bar{z}}(Q_j) \), using the Hölder inequality, we can get a more accurate estimate on \( \phi_{m+1} \),
\[ \| \phi_{m+1} \|_{H^1(\mathbb{R}^N)} \leq C \left[ \epsilon \sum_{j=1}^{m+1} \left( \int_{B_{\bar{z}}(Q_j)} |\tilde{V}(x)|^2 |w_{Q_{m+1}}|^2 \right)^{\frac{1}{2}} + 2\epsilon \sum_{j=1}^{m+1} \left( \int_{B_{\bar{z}}(Q_j)} |\tilde{A}(x)|^2 |\nabla w_{Q_{m+1}}|^2 \right)^{\frac{1}{2}} \right] . \]
4. Proof of the main result. In this section, first we study a maximization problem. Then we prove our main result.

Fix \( \sigma, Q_m \in [0, 2\pi] \times \Omega_m \), we define a new functional
\[
\mathcal{M}(\sigma, Q_m) = J(u_{Q_m}) = J(z_{Q_m} + \varphi_{\sigma, Q_m}): [0, 2\pi] \times \Omega_m \to \mathbb{R}.
\]
(4.1)
Since both \( z_{Q_m} \) and \( \varphi_{\sigma, Q_m} \) are both 2\( \pi \)-periodic respect to \( \sigma \), we only need to consider the maximum problem of \( \mathcal{M}(\sigma, Q_m) \) respect to \( Q_m \) in \( \Omega_m \). So in the sequel, for simplicity we denote \( \mathcal{M}(\sigma, Q_m) \) as \( \mathcal{M}(Q_m) \).

Define
\[
\mathcal{C}_m = \sup_{Q_m \in \Omega_m} \mathcal{M}(Q_m).
\]
(4.2)
Note that \( \mathcal{M}(Q_m) \) is continuous in \( Q_m \). We will show below that the maximization problem has a solution. Let \( \mathcal{M}(Q_m) \) be the maximum, where \( Q_m = (Q_1, \ldots, Q_m) \in \bar{\Omega}_m \) that is
\[
\mathcal{M}(Q_m) = \max_{Q_m \in \Omega_m} \mathcal{M}(Q_m)
\]
(4.3)
and we denote the solution by \( u_{Q_m} \).

First we prove that the maximum can be attained at finite points for each \( \mathcal{C}_m \).

**Lemma 4.1.** Let assumptions (A1) – (A4), (V1) – (V2) and the assumptions in Proposition 2.4 be satisfied. Then, for all \( m \):
(i) There exists \( Q_m \in \Omega_m \) such that
\[
\mathcal{C}_m = \mathcal{M}(Q_m);
\]
(ii) There holds
\[
\mathcal{C}_{m+1} > \mathcal{C}_m + I(z),
\]
where \( I(z) \) is the energy of the solution \( z \) of (1.6):
\[
I(z) = \frac{1}{2} \int \left| \left( \frac{\nabla}{i} - A_0 \right) z \right|^2 + |z|^2 - \int F(z).
\]
(4.6)
**Proof.** We divide the proof into the following two steps.
**Step 1.** \( C_1 > I(z) \), and \( C_1 \) can be attained at a finite point. First applying standard Lyapunov-Schmidt reduction, we have
\[
\| \varphi_{\sigma, Q} \|_{H^1} \leq C \| \tilde{\varphi} z_Q \|_{L^2} + C \| \tilde{\varphi} |z_Q| \|_{L^2}.
\]
(4.7)
Assuming that \( |Q| \) is large enough, then we have
\[
J(u_Q) = \frac{1}{2} \int \left| \left( \frac{\nabla}{i} - A_0(x) \right) u_Q \right|^2 + V_\epsilon(x) |u_Q|^2 - \int F(u_Q)
\]
(3.37)
\[\begin{align*}
&= \frac{1}{2} \int \left| \left( \frac{\nabla}{i} - A_0 \right) z_Q \right|^2 + \frac{1}{2} \int \left| \left( \frac{\nabla}{i} - A_0 \right) \varphi_{\sigma, Q} \right|^2 + \text{Re} \int \epsilon^2 |\tilde{A}(x)|^2 z_Q \varphi_{\sigma, Q} \\
&\quad + \text{Re} \int \left( \frac{\nabla}{i} - A_0 \right) z_Q \left( \frac{\nabla}{i} - A_0 \right) \varphi_{\sigma, Q} + \frac{1}{2} \int \epsilon^2 |\tilde{A}(x)|^2 |z_Q|^2 \\
&\quad + \frac{1}{2} \int \epsilon^2 |\tilde{A}(x)|^2 |\varphi_{\sigma, Q}|^2 - \text{Re} \int \epsilon \tilde{A}(x) \left( \frac{\nabla w_Q}{i} \sigma_{Q} \varphi_{\sigma, Q} + \frac{\nabla \varphi_{\sigma, Q}}{i} z_Q + \frac{\nabla \varphi_{\sigma, Q}}{i} \varphi_{\sigma, Q} \right) \\
&\quad - A_0 \varphi_{\sigma, Q} \overline{z_Q} - A_0 \varphi_{\sigma, Q} \overline{\varphi_{\sigma, Q}} + \frac{1}{2} \int |z_Q|^2 + \frac{1}{2} \int |\varphi_{\sigma, Q}|^2 + \text{Re} \int z_Q \overline{\varphi_{\sigma, Q}} \\
&\quad + \frac{1}{2} \int \epsilon \tilde{V}(x) (|z_Q|^2 + |\varphi_{\sigma, Q}|^2 + 2 \text{Re} z_Q \overline{\varphi_{\sigma, Q}} - \int F(u_Q) \\
&\geq I(z) + \frac{\epsilon^2}{4} \int \tilde{V}(x)|w_Q|^2 + \frac{\epsilon^2}{4} \int |\tilde{A}(x)|^2 |w_Q|^2 - C\|\varphi_{\sigma, Q}\|^2_{H^1} \\
&\quad - \delta \epsilon \int |\text{div} \tilde{A}(x)|^2 |w_Q|^2 \\
&\geq I(z) + \frac{\epsilon^2}{4} \int \tilde{V}(x)|w_Q|^2 + \frac{\epsilon^2}{4} \int |\tilde{A}(x)|^2 |w_Q|^2 \\
&\quad - \delta \epsilon \int |\text{div} \tilde{A}(x)|^2 |w_Q|^2 - \int \epsilon^2 \tilde{V}(x)|w_Q|^2 - \int \epsilon^4 |\tilde{A}(x)|^2 |w_Q|^2 \\
&\geq I(z) + \frac{1}{8} \left( \int_{B_{\frac{1}{2}}(Q)} \epsilon \tilde{V}(x)|w_Q|^2 - \sup_{B_{\frac{1}{4}}(0)} |w_Q|^2 \int_{\text{supp} \tilde{V}} \epsilon |\tilde{V}(x)| \frac{|w_Q|^2}{2} \right) \\
&\quad + \frac{1}{8} \left( \int_{B_{\frac{1}{2}}(Q)} \epsilon^2 |\tilde{A}|^2 |w_Q|^2 - \sup_{B_{\frac{1}{4}}(0)} |w_Q|^2 \int_{\text{supp} \tilde{A}} \epsilon^2 |\tilde{A}|^2 \frac{|w_Q|^2}{2} \right) \\
&\quad - \sup_{B_{\frac{1}{4}}(0)} |w_Q|^2 \int_{\text{supp} \tilde{A}} \epsilon^2 |\text{div} \tilde{A}(x)|^2 \frac{|w_Q|^2}{2} \\
&\geq I(z) + \frac{1}{8} \int_{B_{\frac{1}{2}}(Q)} \epsilon \tilde{V}(x)|w_Q|^2 + \frac{1}{8} \int_{B_{\frac{1}{2}}(Q)} \epsilon^2 |\tilde{A}|^2 |w_Q|^2 - O(\epsilon^{-2}|Q|),
\end{align*}\]
we have

\[ \text{Step 2. Assume that there exists } \tilde{Q}_m = (\tilde{Q}_1, \ldots, \tilde{Q}_m) \in \Omega_m \text{ such that } C_m = M(Q_m) \text{ and we denote the solution by } u_{Q_m}. \text{ Next we prove that there exists } Q_{m+1} \in \Omega_{m+1} \text{ such that } C_{m+1} \text{ can be attained. Let } Q_{m+1}^{(n)} \text{ be a sequence such that} \]

\[ C_{m+1} = \lim_{n \to \infty} M(Q_{m+1}^{(n)}). \]
We claim that $Q^{(n)}_{m+1}$ is bounded. We prove it by contradiction. In the following we omit index $n$ for simplicity. By direct computation, we have

\[
J(u_{Q_{m+1}}) = J(u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1})
\]

\[
= \frac{1}{2} \int \left| \left( \frac{\nabla}{i} - A(x) \right) \left( u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1} \right) \right|^2 + V_c(x) |u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1}|^2
- \int F(u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1})
\]

\[
= J(u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1})
\]

\[
+ R \int \left( \frac{\nabla}{i} - A(x) \right) u_{Q_{m}} \phi_{m+1} - 2 \int f(u_{Q_{m}}) \phi_{m+1} + \int F(u_{Q_{m}} + z_{Q_{m+1}})
- \int f(u_{Q_{m}} + z_{Q_{m+1}}) \phi_{m+1} - \int F(u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1})
+ R \int \left( \frac{\nabla}{i} - A(x) \right) z_{Q_{m+1}} \left( \frac{\nabla}{i} - A(x) \right) \phi_{m+1}
\]

\[
= J(u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1}) - \int \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k} \phi_{m+1}
+ R \int f(u_{Q_{m}}) \phi_{m+1} - 2 \int f(u_{Q_{m}} + z_{Q_{m+1}}) \phi_{m+1} + \int F(z_{Q_{m+1}}) \phi_{m+1}
- \int f'(u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1}) |\phi_{m+1}|^2 + \frac{1}{2} \int \left| \left( \frac{\nabla}{i} - A(x) \right) \phi_{m+1} \right|^2
+ \frac{1}{2} \int V_c(x) |\phi_{m+1}|^2 + \int V_c(x) z_{Q_{m+1}} \phi_{m+1} - \int f(z_{Q_{m+1}}) \phi_{m+1}
\]

\[
= J(u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1}) + O(\|\phi_{m+1}\|^2 + \|S(u_{Q_{m}} + z_{Q_{m+1}})\| \|\phi_{m+1}\|)
\]

\[
- \int \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k} \phi_{m+1}
\]

\[
= J(u_{Q_{m}} + z_{Q_{m+1}} + \phi_{m+1}) + O \left[ e^{-\beta_0} \sum_{j=1}^{m} \left( |Q_{m+1} - Q_j| \right) + \epsilon^2 \left( \int \left| \tilde{V}(x) \right| |w_{Q_{m+1}}| \right)^2 \right]
+ \epsilon^4 \left( \int \left| \tilde{A}(x) \right|^2 |w_{Q_{m+1}}| \right)^2 + \epsilon^2 \left( \int \left| \tilde{A}(x) \right| |\nabla w_{Q_{m+1}}| \right)^2 + \epsilon^2 \left( \int \left| \text{div} \tilde{A}(x) \right| |w_{Q_{m+1}}| \right)^2
+ \epsilon^2 \left( \int \left| \tilde{V}(x) \right|^2 |w_{Q_{m+1}}| \right) + \epsilon^2 \left( \int \left| \tilde{A}(x) \right|^2 |\nabla w_{Q_{m+1}}| \right)^2 + \epsilon^2 \left( \int \text{div} \tilde{A}(x) |w_{Q_{m+1}}|^2 \right)
+ \epsilon^4 \left( \int \left| \tilde{A}(x) \right|^4 |w_{Q_{m+1}}|^2 \right).
\]

(4.13)
Moreover, we have
\[ J(u_{Q_m} + z_{Q_{m+1}}) \]
\[ = \frac{1}{2} \int \left| \left( \sum_{i} - A_r(x) \right)(u_{Q_m} + z_{Q_{m+1}}) \right|^2 + V_r(x)|u_{Q_m} + z_{Q_{m+1}}|^2 \]
\[ - \int F(u_{Q_m} + z_{Q_{m+1}}) \]
\[ \leq C_m + \frac{1}{2} \int |z_{Q_{m+1}}|^2 + \frac{1}{2} \int \left| \left( \sum_{i} - A_0 \right)z_{Q_{m+1}} \right|^2 - \int F(z_{Q_{m+1}}) \]
\[ + \text{Re} \int (1 + \epsilon \tilde{V}(x))u_{Q_m} \tilde{z}_{Q_{m+1}} - \int F(u_{Q_m} + z_{Q_{m+1}}) + \int F(u_{Q_m}) \]
\[ + \text{Re} \int \left( \sum_{i} - A_r(x) \right)u_{Q_m} \left( \sum_{i} - A_r(x) \right)z_{Q_{m+1}} + \int F(z_{Q_{m+1}}) \]
\[ + \frac{1}{2} \int \epsilon \tilde{V}(x)|z_{Q_{m+1}}|^2 - \text{Re} \int \epsilon \tilde{A}(x) \left( \sum_{i} - A_0 \right)z_{Q_{m+1}} \tilde{z}_{Q_{m+1}} \]
\[ + \frac{1}{2} \int \epsilon^2 \tilde{A}(x)^2 |z_{Q_{m+1}}|^2 \]
\[ \leq C_m + I(z) + \frac{1}{2} \int \epsilon \tilde{V}(x)|z_{Q_{m+1}}|^2 + \frac{1}{2} \int \epsilon^2 \tilde{A}(x)^2 |z_{Q_{m+1}}|^2 \]
\[ + \text{Re} \int \left( f(u_{Q_m}) - \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k} \right) \tilde{z}_{Q_{m+1}} - \text{Re} \int f(u_{Q_m}) \tilde{z}_{Q_{m+1}} \]
\[ - \text{Re} \int f(z_{Q_{m+1}}) \bar{u}_{Q_m} + O \left( e^{-\beta \mu} \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|) \right) \]
\[ \leq C_m + I(z) + \frac{1}{2} \int \epsilon \tilde{V}(x)|z_{Q_{m+1}}|^2 - \text{Re} \int \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k} \tilde{z}_{Q_{m+1}} \]
\[ - \text{Re} \int f(z_{Q_{m+1}}) \bar{u}_{Q_m} + \frac{1}{2} \int \epsilon^2 \tilde{A}(x)^2 |z_{Q_{m+1}}|^2 \]
\[ + O \left( e^{-\beta \mu} \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|) \right) \].

By estimate (2.32) in Proposition 2.4, and that the definition of \( D_{j,k} \), we have
\[ \left| \text{Re} \int \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k} \tilde{z}_{Q_{m+1}} \right| \leq C e^{-\beta \mu} \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|). \] (4.15)

By the equation satisfied by \( \varphi_{\sigma, Q_m} \)
\[ L(\varphi_{\sigma, Q_m}) = -S(z_{Q_m}) + N(\varphi_{\sigma, Q_m}) + \sum_{j=1}^{m} \sum_{k=1}^{N+1} c_{j,k} D_{j,k}, \] (4.16)
we have
\[ \text{Re} \int f(z_{Q_{m+1}}) \bar{\varphi}_{\sigma, Q_m} \]
Moreover, we can choose \( \gamma > 1 \), \((1 + \delta)\gamma > 1 \). Then we can easily get

\[
|\text{Re} \left( \mathcal{N}(\varphi_{\sigma,Q_m}) - f'(z_{Q_m})\varphi_{\sigma,Q_m} \right)z_{Q_m+1}| \leq C e^{-\beta \mu} \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|) \tag{4.18}
\]

and

\[
|\text{Re} \left( \mathcal{S}(z_{Q_m}) - \tilde{\mathcal{N}}(\varphi_{\sigma,Q_m})\tilde{z}_{Q_m+1} + \text{Re} \left( \epsilon A \left( \frac{\nabla}{\ell} - A_0 \right) \varphi_{\sigma,Q_m} \tilde{z}_{Q_m+1} + \epsilon \tilde{A} \varphi_{\sigma,Q_m} z_{Q_m+1} \right) - \epsilon^2 |\tilde{A}|^2 \varphi_{\sigma,Q_m} z_{Q_m+1} + \epsilon^2 |\tilde{A}|^2 \varphi_{\sigma,Q_m} \tilde{z}_{Q_m+1} \right) |
\]

\[
= \text{Re} \left( \mathcal{S}(z_{Q_m}) - \tilde{\mathcal{N}}(\varphi_{\sigma,Q_m})\tilde{z}_{Q_m+1} + \text{Re} \left( \epsilon A \left( \frac{\nabla}{\ell} - A_0 \right) \varphi_{\sigma,Q_m} \tilde{z}_{Q_m+1} + \epsilon \tilde{A} \varphi_{\sigma,Q_m} z_{Q_m+1} \right) - \epsilon^2 |\tilde{A}|^2 \varphi_{\sigma,Q_m} z_{Q_m+1} + \epsilon^2 |\tilde{A}|^2 \varphi_{\sigma,Q_m} \tilde{z}_{Q_m+1} \right) \tag{4.19}
\]

\[
\leq C \left( \epsilon \int \tilde{V} w_{Q_m} w_{Q_{m+1}} + \epsilon e^{-\beta \mu} \int \sum_{j=1}^{m} e^{-\gamma |z_{Q_j}|} \tilde{V} w_{Q_{m+1}} + \epsilon \int |\text{div} \tilde{A}(x)| w_{Q_m} w_{Q_{m+1}} + \epsilon \int |\tilde{A}(x)| |\nabla w_{Q_m}| w_{Q_{m+1}} \right)
\]
+ e^{-\beta \mu} \sum_{j=1}^{m} w((Q_{m+1} - Q_{j})) + \epsilon^2 \int |\tilde{A}(x)|^2 w_{Q_{m}} w_{Q_{m+1}}

+ e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} w_{Q_{m+1}} + e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} w_{Q_{m+1}}

+ e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} |\nabla w_{Q_{m+1}}|

+ e^2 e^{-\beta \mu} \int |\tilde{A}(x)|^2 \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} w_{Q_{m+1}}).

From (4.16) to (4.19), we obtain

\[ |\text{Re} \int f(z_{Q_{m+1}}) \tilde{\varphi}_{\sigma, Q_{m}} | \]

\[ \leq C \left( \epsilon \int \tilde{V} w_{Q_{m}} w_{Q_{m+1}} + e^{-\beta \mu} \int \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} \tilde{V} w_{Q_{m+1}} \right) + \epsilon \int |\text{div} \tilde{A}(x)| w_{Q_{m}} w_{Q_{m+1}} + e^{-\beta \mu} \int |\tilde{A}(x)| |\nabla w_{Q_{m+1}}|

\[ + e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} w_{Q_{m+1}} + e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} w_{Q_{m+1}}

\[ + e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} |\nabla w_{Q_{m+1}}|

\[ + e^2 e^{-\beta \mu} \int |\tilde{A}(x)|^2 \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} w_{Q_{m+1}} \right).

Hence by Lemma 3.1, we have

\[ \text{Re} \int f(z_{Q_{m+1}}) \tilde{u}_{Q_{m}} = \text{Re} \int f(z_{Q_{m+1}}) (z_{Q_{m}} + \tilde{\varphi}_{\sigma, Q_{m}}) \]

\[ \geq \frac{1}{4} \epsilon^2 \sum_{j=1}^{m} w((Q_{m+1} - Q_{j})) + O \left( e^{-\beta \mu} \int \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} \tilde{V} w_{Q_{m+1}} \right) \]

\[ + \epsilon \int \tilde{V} w_{Q_{m}} w_{Q_{m+1}} + e^{-\beta \mu} \sum_{j=1}^{m} w((Q_{m+1} - Q_{j})) + \epsilon \int |\text{div} \tilde{A}(x)| w_{Q_{m}} w_{Q_{m+1}} \]

\[ + \epsilon \int |\tilde{A}(x)| |\nabla w_{Q_{m}}| w_{Q_{m+1}} + e^2 \int |\tilde{A}(x)|^2 w_{Q_{m}} w_{Q_{m+1}} \]

\[ + e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} w_{Q_{m+1}} + e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma |x - Q_{j}|} w_{Q_{m+1}} \]

(4.20)

(4.21)
+εe^{-βμ} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-γ|x-Q_j|} |\nabla w_{Q_{m+1}}| \\
+ε^2 e^{-βμ} \int |\tilde{A}(x)|^2 \sum_{j=1}^{m} e^{-γ|x-Q_j|} w_{Q_{m+1}}}

Combining (4.13), (4.14), (4.15) and (4.21), we obtain

\begin{align*}
J(u_{Q_{m+1}}) &= J(u_{Q_m} + z_{Q_{m+1}} + φ_{m+1}) \\
&\leq C_m + I(z) + \frac{1}{2} \int e\tilde{V}(x)|z_{Q_{m+1}}|^2 + \frac{1}{2} \int ε^2|\tilde{A}(x)|^2 |z_{Q_{m+1}}|^2 \\
&- \frac{1}{4} \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|) + O\left[εe^{-βμ} \int \sum_{j=1}^{m} e^{-γ|x-Q_j|} \tilde{V} w_{Q_{m+1}} \right] \\
&+ ε \int \tilde{V} w_{Q_m} w_{Q_{m+1}} + ε^{-βμ} \sum_{j=1}^{m} w(|Q_{m+1} - Q_j|) + ε \int |\text{div}\tilde{A}(x)| w_{Q_m} w_{Q_{m+1}} \\
&+ ε \int |\tilde{A}(x)||\nabla w_{Q_m}| w_{Q_{m+1}} + ε^2 \int |\tilde{A}(x)|^2 w_{Q_m} w_{Q_{m+1}} \\
&+εe^{-βμ} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-γ|x-Q_j|} w_{Q_{m+1}} + εe^{-βμ} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-γ|x-Q_j|} w_{Q_{m+1}} \\
&+ ε^{-βμ} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-γ|x-Q_j|} |\nabla w_{Q_m}| + ε^4 \left( \int |\tilde{A}(x)|^2 |w_{Q_{m+1}}| \right)^2 \\
&+ ε^2 \left( \int |\tilde{A}(x)||\nabla w_{Q_{m+1}}| \right)^2 + ε^2 \left( \int |\text{div}\tilde{A}(x)||w_{Q_{m+1}}| \right)^2 \\
&+ ε^2 \int |\tilde{V}(x)|^2 |w_{Q_{m+1}}|^2 + ε^2 \int |\tilde{A}(x)|^2 |\nabla w_{Q_{m+1}}|^2 + ε^2 \int |\text{div}\tilde{A}(x)|^2 |w_{Q_{m+1}}|^2 \\
&+ ε^2 \left( \int |\tilde{V}(x)||w_{Q_{m+1}}| \right)^2 + ε^2 e^{-βμ} \int |\tilde{A}(x)|^2 \sum_{j=1}^{m} e^{-γ|x-Q_j|} w_{Q_{m+1}} \\
&+ ε^4 \int |\tilde{A}(x)|^4 |w_{Q_{m+1}}|^2 \right].
\end{align*}

By the assumption that $|Q_{m+1}^{(n)}| \to +\infty$, there hold

\begin{align*}
ε \int \tilde{V} w_{Q_m} w_{Q_{m+1}}^{(n)} + εe^{-βμ} \int \sum_{j=1}^{m} e^{-γ|x-Q_j|} \tilde{V} w_{Q_{m+1}}^{(n)} \\
+ ε \int |\text{div}\tilde{A}(x)| w_{Q_m} w_{Q_{m+1}}^{(n)} + ε \int |\tilde{A}(x)||\nabla w_{Q_m}| w_{Q_{m+1}}^{(n)} + ε^2 \int |\tilde{A}|^2 w_{Q_m} w_{Q_{m+1}}^{(n)} \\
+ εe^{-βμ} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-γ|x-Q_j|} w_{Q_{m+1}}^{(n)} + εe^{-βμ} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-γ|x-Q_j|} w_{Q_{m+1}}^{(n)}
\end{align*}

(4.23)
and

\[-\frac{1}{4} \theta \sum_{j=1}^{m} |Q_{m+1} - Q_j| + O\left(e^{-\beta \mu} \sum_{j=1}^{m} |Q_{m+1} - Q_j| \right) < 0.\quad (4.24)\]

Combining (4.12), (4.22), (4.23) and (4.24), we have

\[C_{m+1} \leq C_{m} + I(z).\quad (4.25)\]

On the other hand, since by the assumption, \(C_{m}\) can be attained at \((\bar{Q}_1, \ldots, \bar{Q}_m)\), so there exists other point \(Q_{m+1}\) which is far away from the \(m\) points which will be determined later. Next let’s consider the solution concentrated at the points \((\bar{Q}_1, \ldots, \bar{Q}_m, Q_{m+1})\), and we denote the solution by \(u_{Q_{m+1}}\), then similar with the above argument, applying the estimate (3.37) of \(\phi_{m+1}\) instead of (3.5), we have the following estimate:

\[
J(u_{Q_{m+1}}) \\
= J(u_{Q_m}) + I(z) + \frac{1}{2} \epsilon \int \tilde{V}(x)|z_{m+1}|^2 + \frac{1}{2} \epsilon^2 |\tilde{A}(x)|^2 |z_{m+1}|^2 \\
- O\left(\sum_{j=1}^{m} |Q_{m+1} - Q_j| \right) + O\left(\epsilon \int \tilde{V}w_{Q_m}w_{Q_{m+1}} + \epsilon \int |\tilde{A}||\nabla w_{Q_m}w_{Q_{m+1}}| \\
+ \epsilon \int |\text{div} \tilde{A}(x)|w_{Q_m}w_{Q_{m+1}} + \epsilon e^{-\beta \mu} \int \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} \tilde{V}w_{Q_{m+1}} \\
+ \epsilon^2 \int |\tilde{A}(x)|^2 w_{Q_m}w_{Q_{m+1}} + \epsilon e^{-\beta \mu} \int |\text{div} \tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} w_{Q_{m+1}} \\
+ \epsilon e^{-\beta \mu} \int |\tilde{A}(x)| \sum_{j=1}^{m} e^{-\gamma|x-Q_j|} w_{Q_{m+1}} + \epsilon^2 \int |\tilde{A}(x)|^2 w_{Q_{m+1}} \\
+ \epsilon^4 \int |\tilde{A}(x)|^4 w_{Q_{m+1}} + \epsilon^2 \int |\tilde{V}(x)|^2 |w_{Q_{m+1}}|^2 + \epsilon^2 \int |\tilde{A}(x)|^2 |\nabla w_{Q_{m+1}}|^2 \\
+ \epsilon^2 \int |\text{div} \tilde{A}(x)|^2 |z_{m+1}|^2 + O\left(\frac{1}{2} \left(\sum_{j=1}^{m+1} \left(\int_{B_{\mu/2}(Q_j)} |\tilde{A}(x)|^2 |\nabla w_{Q_{m+1}}|^2 \right) \right)^{\frac{1}{2}} \right)^2 \quad (4.26)\]
By the asymptotic behavior of $V$, $A$ and $\nabla A$ at infinity, for some $\alpha < 1$, we choose $\gamma > \alpha$, then we can choose $Q_{m+1}$ such that

$$|Q_{m+1}| \gg \frac{\max_{j=1}^m |Q_j| - \ln \epsilon}{\gamma - \alpha},$$

(4.27)

then we can get

$$\frac{1}{2} \epsilon^2 \sum_{j=1}^{m+1} \left( \int_{B_{\epsilon j}^j(Q_j)} |\hat{A}(x)|^2 |w_{Q_{m+1}}|^2 \right)^{\frac{1}{2}} + \epsilon^2 \sum_{j=1}^{m+1} \left( \int_{B_{\epsilon j}^j(Q_j)} |\hat{V}(x)|^2 |w_{Q_{m+1}}|^2 \right)^{\frac{1}{2}} + \epsilon^2 \sum_{j=1}^{m+1} \left( \int_{B_{\epsilon j}^j(Q_j)} |\nabla \hat{A}(x)|^2 |w_{Q_{m+1}}|^2 \right)^{\frac{1}{2}}.$$
Then there exists \((j, k)\) such that \(|\bar{Q}_j - \bar{Q}_k| = \mu\). Without loss of generality, we assume \((j, k) = (j, m)\). Then following the estimates (4.13), (4.14), (4.15) and (4.31), we have
\[
\mathcal{C}_m = J(u_{\bar{Q}_m}) 
\leq \mathcal{C}_{m-1} + I(z) + \frac{\varepsilon}{2} \int \nabla(x)|w_{Q_{m+1}}|^2 + \frac{1}{2} \int \nabla(x)|\bar{A}(x)|^2|w_{Q_{m+1}}|^2 
- \frac{\varepsilon}{4} \sum_{j=1}^{m-1} w(|\bar{Q}_m - \bar{Q}_j|) + O\left(e^{-\beta \mu} \sum_{j=1}^{m-1} e^{-|\bar{Q}_m - \bar{Q}_j|}\right) + O(\varepsilon) 
\leq \mathcal{C}_{m-1} + I(z) - \frac{\varepsilon}{4} \sum_{j=1}^{m-1} w(|\bar{Q}_m - \bar{Q}_j|) + O\left(e^{-\beta \mu} \sum_{j=1}^{m-1} e^{-|\bar{Q}_m - \bar{Q}_j|}\right) + O(\varepsilon). 
\] (4.32)

By the definition of the configuration set, we observe that given a ball of size \(\mu\), there are at most \(C_N := 6^N\) number of non-overlapping ball of size \(\mu\) surrounding this ball. Since \(|\bar{Q}_j - \bar{Q}_k| = \mu\), we have
\[
\sum_{j=1}^{m-1} w(|\bar{Q}_m - \bar{Q}_j|) = w(|\bar{Q}_m - \bar{Q}_j|) + \sum_{k \neq j} w(|\bar{Q}_m - \bar{Q}_k|) 
\] and
\[
\sum_{k \neq j} w(|\bar{Q}_m - \bar{Q}_k|) \leq C e^{-\mu} + C_N e^{-\mu - \frac{\mu}{2}} + \ldots + C_N^k e^{-\mu - \frac{k\mu}{2}} 
\leq C e^{-\mu} \sum_{j=1}^{\infty} c_j^{(\ln C_N - \frac{\mu}{2})} \leq C e^{-\mu}, \tag{4.33}
\]
if \(C_N < e^{\frac{\mu}{2}}\), which is true for \(\mu\) large enough.

So
\[
\mathcal{C}_{m+1} \geq J(u_{\bar{Q}_{m+1}}) > \mathcal{C}_m + I(z). \tag{4.29}
\]

It follows from (4.25) and (4.29) that
\[
\mathcal{C}_m + I(z) < \mathcal{C}_{m+1} \leq \mathcal{C}_m + I(z), \tag{4.30}
\]
which is impossible. Hence we prove that \(\mathcal{C}_{m+1}\) can be attained at finite points in \(\Omega_{m+1}\).

Now we are in position to prove our main result.

**Proof of Theorem 1.2.** In order to prove our main result, we only need to prove that the maximization problem
\[
\max_{Q_m \in \Omega_m} M(Q_m) \tag{4.31}
\]
has a solution \(Q_m \in \Omega_m\), i.e., the interior of \(\Omega_m\).

We prove it by an indirect method. Assume that \(Q_m = (\bar{Q}_1, \ldots, \bar{Q}_m) \in \partial \Omega_m\). Then there exists \((j, k)\) such that \(|\bar{Q}_j - \bar{Q}_k| = \mu\). Without loss of generality, we assume \((j, k) = (j, m)\). Then following the estimates (4.13), (4.14), (4.15) and (4.21), we have
\[
\mathcal{C}_m = J(u_{\bar{Q}_m}) 
\leq \mathcal{C}_{m-1} + I(z) + \frac{\varepsilon}{2} \int \nabla(x)|w_{Q_{m+1}}|^2 + \frac{1}{2} \int \nabla(x)|\bar{A}(x)|^2|w_{Q_{m+1}}|^2 
- \frac{\varepsilon}{4} \sum_{j=1}^{m-1} w(|\bar{Q}_m - \bar{Q}_j|) + O\left(e^{-\beta \mu} \sum_{j=1}^{m-1} e^{-|\bar{Q}_m - \bar{Q}_j|}\right) + O(\varepsilon) 
\leq \mathcal{C}_{m-1} + I(z) - \frac{\varepsilon}{4} \sum_{j=1}^{m-1} w(|\bar{Q}_m - \bar{Q}_j|) + O\left(e^{-\beta \mu} \sum_{j=1}^{m-1} e^{-|\bar{Q}_m - \bar{Q}_j|}\right) + O(\varepsilon). 
\] (4.32)
Hence, we have
\[ C_m \leq C_{m-1} + I(z) + C\epsilon - \frac{\partial}{4}w(\mu) + O(e^{-(1+\beta)\mu}) < C_{m-1} + I(z), \] (4.34)
which contradicts to (4.5) in Lemma 4.1. \(\square\)

Appendix A. Some technical estimates. In this section, we give some technical estimates which are used before.

Denote
\[ \Lambda_j := \{x \mid |x - Q_j| \leq \frac{\mu}{2}\}, \quad \Lambda = \bigcup_{j=1}^m \Lambda_j \text{ and } \Lambda^C = \mathbb{R}^N \setminus \Lambda. \]

Lemma A.1. For any \(x \in \Lambda_j (j = 1, \ldots, m)\) and \(\gamma \in (0, 1)\), we have
\[ \sum_{k=1}^m e^{-\gamma|x - Q_k|} \leq e^{-\gamma|x - Q_j|} + Ce^{-\frac{\mu}{2}}. \] (A.1)

For any \(x \in \Lambda^C\), we have
\[ \sum_{k=1}^m e^{-\gamma|x - Q_k|} \leq Ce^{-\frac{\mu}{2}}. \] (A.2)

Proof. Note that given a ball of size \(\mu\), there are at most \(C_N := 6^N\) number of non-overlapping ball of size \(\mu\) surrounding this ball. Since \(|x - Q_j| \leq \frac{\mu}{2}\), we have
\[ |x - Q_k| \geq |Q_k - Q_j| - |x - Q_j| \geq \frac{\mu}{2} \text{ for all } k \neq j. \]

Then we have
\[ \sum_{k=1}^m e^{-\gamma|x - Q_k|} = e^{-\gamma|x - Q_j|} + \sum_{k \neq j} e^{-\gamma|x - Q_k|} \leq e^{-\gamma|x - Q_j|} + \sum_{k=1}^\infty C_N e^{-\frac{k\mu}{2}} \]
\[ \leq e^{-\gamma|x - Q_j|} + \sum_{k=1}^\infty e^{k(\ln C_N - \frac{\mu}{2})} \leq e^{-\gamma|x - Q_j|} + O(e^{-(\frac{\mu}{2} + \ln C_N)}) \]
\[ \leq e^{-\gamma|x - Q_j|} + Ce^{-\frac{\mu}{2}}, \]
if \(\ln C_N < \frac{\mu}{2}\), which is true for \(\mu\) large enough.

The proof of (A.2) is similar. \(\square\)

Proof of (4.9). By direct computation, we have
\[ |Re \int \epsilon^2 |\tilde{A}(x)|^2 z_Q \tilde{\varphi}_{\sigma,Q}| \leq \epsilon^2 \left( \int |\tilde{A}(x)|^2 |z_Q|^2 \right)^\frac{1}{2} \left( \int |\tilde{A}(x)|^2 |\tilde{\varphi}_{\sigma,Q}|^2 \right)^\frac{1}{2} \]
\[ \leq \delta \epsilon^2 \int |\tilde{A}(x)|^2 w_Q^2 + C_\delta \epsilon^2 \int |\tilde{A}(x)|^2 |\varphi_{\sigma,Q}|^2 \]
\[ \leq \delta \epsilon^2 \int |\tilde{A}(x)|^2 w_Q^2 + C \|\varphi_{\sigma,Q}\|_{H^1}^2, \]
\[ |Re \int \epsilon \tilde{V}(x) z_Q \tilde{\varphi}_{\sigma,Q}| \leq \epsilon \left( \int |\tilde{V}(x)|^2 |z_Q|^2 \right)^\frac{1}{2} \left( \int |\tilde{V}(x)|^2 |\tilde{\varphi}_{\sigma,Q}|^2 \right)^\frac{1}{2} \]
\[ \leq \delta \epsilon \int \tilde{V}(x) w_Q^2 + C_\delta \epsilon \int \tilde{V}(x) |\varphi_{\sigma,Q}|^2 \]
\[ \leq \delta \epsilon \int \tilde{V}(x) w_Q^2 + C |\varphi_{\sigma,Q}|_{H^1}^2. \]
Moreover, we have
\[\delta > 0\]
where we can choose
\[\epsilon \in (0, 1)\] and
\[\epsilon \in (0, 1)\] as well.

Similarly, we can prove
\[|Re \int \epsilon A(x) (\nabla \phi_{\sigma, Q} - A_0 \phi_{\sigma, Q})| \leq \delta \epsilon \int |A|^2 w_Q^2 + C \|\phi_{\sigma, Q}\|_{H^1}^2.\]

Moreover, we have
\[|Re \int \frac{\epsilon}{i} iA(x) \cdot \nabla w_Q \phi_{\sigma, Q}| \leq | -Re \int \frac{\epsilon}{i} w_Q (div A\xi_Q \phi_{\sigma, Q} + iA \cdot A_0 \xi_Q \phi_{\sigma, Q} + \xi_Q A \cdot \nabla \phi_{\sigma, Q})| \leq \epsilon \int |div A| w_Q |\phi_{\sigma, Q}| + \epsilon \int |A||A_0| w_Q |\phi_{\sigma, Q}| + \epsilon \int |A| w_Q |\nabla \phi_{\sigma, Q}| \leq \epsilon \left( \int |div A|^2 w_Q^2 \right)^{\frac{1}{2}} \left( \int \phi_{\sigma, Q}^2 \right)^{\frac{1}{2}} + \epsilon \left( \int |A|^2 w_Q^2 \right)^{\frac{1}{2}} \left( \int |A_0|^2 \phi_{\sigma, Q}^2 \right)^{\frac{1}{2}} + \epsilon \left( \int |A|^2 w_Q^2 \right)^{\frac{1}{2}} \left( \int |\nabla \phi_{\sigma, Q}|^2 \right)^{\frac{1}{2}} \leq \delta \epsilon \int |div A|^2 w_Q^2 + C \delta \epsilon \int \phi_{\sigma, Q}^2 + \delta \epsilon \int |A|^2 w_Q^2 + C \delta \epsilon \int \phi_{\sigma, Q}^2 + \delta \epsilon \int |A|^2 w_Q^2 + C \|\phi_{\sigma, Q}\|_{H^1}^2,\]

where we can choose \(\delta > 0\) small enough. From all the estimates above, then (4.9) holds.

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