Connections between Tomboulis vortices and projection vortices

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By using the freedom of picking a representative we explore connections between the Tomboulis SO(3) × Z(2) form of the partition function and the SU(2) form. We are able to express the monopole and vortex observables of the former in terms of configurations of the latter. Also we can measure Tomboulis and projection vortex counters on the same configuration to search for correlations.

1. Introduction

In 1980, Tomboulis derived an alternative form of the SU(2) partition function which is invariant under sign flips of the links\(^1\). This effectively maps the SU(2) manifold to SO(3). The Z(2) dependence is carried by new plaquette valued variables. By breaking down the SU(2) group into its SO(3) and Z(2) factors, the underlying monopoles and vortices are revealed. This was further developed by Kovacs and Tomboulis\(^2\) [KT] in which thin, thick and hybrid vortex linkages with a Wilson loop could be defined.

In a recent paper\(^3\) we made a more direct contact between this approach and the usual SU(2) formalism. We made use of SO(3) representative invariance by making appropriate sign flips of the links. We point out that there is a representative for which the two formulations are identical. Therefore we can define the KT vortex counters on ordinary SU(2) configurations.

Secondly, by choosing another representative we show how projection vortices\(^4\) arise naturally in this formalism. This also gives an alternative perspective on the projection approximation.

2. SU(2) configurations in SO(3) × Z(2) variables

The Wilson form of the partition function can be recast by introducing Z(2) valued independent variables \(\sigma(p)\) defined on plaquettes\(^5\)

\[
Z_{SO(3)\times Z(2)} = \int [dU(b)] \sum_{\sigma(p)} \left[ \prod_c \delta(\sigma(\partial c)\eta(\partial c)) \right] \exp \left( \beta \sum_p \frac{1}{2} |\text{Tr}[U(\partial p)]\sigma(p)| \right),
\]

where the dependent variables \(\eta(p)\) are defined by \(\text{Tr}[U(\partial p)] \equiv |\text{Tr}[U(\partial p)]\eta(p)|\).

The "cube constraint" factor requires that \(\prod_6 \eta(p)\sigma(p) = +1\) over the six faces of all cubes. Wilson loops have Z(2) valued plaquette tiling factors, \(\sigma\) and \(\eta\) on an arbitrary surface \(S\) bounded by \(C\)

\[
W_{m\times n}(C) = \text{Tr}[U(C)]|\text{Tr}[S\sigma(S)]| = \text{Tr}[U(\partial p)]|\sigma(p)|,
\]

\[
W_{1\times 1} = \text{Tr}[U(\partial p)]|\eta(p)\sigma(p)| = |\text{Tr}[U(\partial p)]|\sigma(p)|.
\]

Properties of this form include:

- Z(2) invariance of \(Z\) and of observables under \(U(b) \rightarrow -U(b)\). There are therefore \(2^N\) representatives of SO(3), where \(N\) is the number of links.

- There exist co-closed \(\sigma(p) - \eta(p)\) vortex sheets due to the cube constraint with patches of either \(\sigma(p) = -1\) or \(\eta(p) = -1\), \(\sigma(p)\eta(p) = -1\). Pure \(\sigma(p)\) or \(\eta(p)\) vortex sheets are limiting cases.
2.2. The representative \( \hat{U}(b) \)

This is defined by the condition

\[
\sigma(p)\eta(p) = +1, \quad \forall p.
\]

In this case the cube constraint is automatically satisfied. There are further simplifications:

\[
|\text{Tr}[\hat{U}(\partial p)]|\sigma(p) = \text{Tr}[\hat{U}(\partial p)]\eta(p)\sigma(p),
\]

\[
= \text{Tr}[\hat{U}(\partial p)],
\]

\[
\tilde{Z} = \int \left[ d\hat{U}(b) \right] \exp \left( \beta \sum_p \frac{1}{2} \text{Tr}[\hat{U}(\partial p)] \right),
\]

\[
W_{m\times n} = \text{Tr}[\hat{U}(C)], \quad W_{1\times 1} = \text{Tr}[\hat{U}(\partial p)].
\]

We showed\(^3\) that starting from a cold configuration, \( U(b) = \sigma(p) = +1 \), we can reach the full configuration space of the independent variables \( \{ U(b), \sigma(p) \} \) through local updates while staying in the representative \( \hat{U}(b) \). In this representative all \( \sigma - \eta \) vortices are absent.

This particular representative provides the connection of this formulation to the SU(2) formalism with the Wilson action. As a consequence, we can define the Tomboulis thin, thick and hybrid vortex counters on ordinary SU(2) configurations as will be given below.

2.2. The representative \( \hat{U}(b) \)

This is defined by the condition

\[
\text{Tr}[\hat{U}(b)] \geq 0.
\]

This can be obtained by a single sweep. The interest in this is to connect with projection vortices which are defined as follows: One first fixes the gauge, for example the maximal center gauge and then

In \textbf{an arbitrary representative}

\begin{itemize}
  \item Project: \( \text{sign}\text{Tr}[\hat{U}(b)] \rightarrow u(b), \ u(b) = \pm 1 \).
  \item \( P \) vortex: \( u(p) = u(\partial p)\eta(p)\sigma(p) = -1 \)
  \item Proj. approx.: \( W(C) \approx u(S)|_{C=\partial S} \).
\end{itemize}

\textbf{In the \( \hat{U}(b) \) representative}

\begin{itemize}
  \item Project: \( \text{sign}\text{Tr}[\hat{U}(b)] \rightarrow \tilde{u}(b), \ \tilde{u}(b) = +1 \).
  \item \( P \) vortex: \( \tilde{u}(p) = \eta(p)\sigma(p) = -1 \), which is identical to \( \sigma - \eta \) vortex.
  \item Proj. approx.: \( \text{Tr}[\hat{U}(C)] \approx 1 \),
\end{itemize}

where we have used Eqn.(\( \ddot{\text{b}} \)). These two procedures give identical \( P \) vortices.

However in the \( \hat{U}(b) \) representative the \( P \) vortices are identical to the \( \sigma - \eta \) vortices which are a tiling factor in the exact definition of the Wilson loop. The success or failure of a projection approximation depends on whether one can find a gauge such that the sign fluctuations of the perimeter factor in Eqn.(\( \ddot{\text{b}} \)) can be transferred to the tiling factors arising from \( \sigma - \eta \) linkages. If so then one argues that the area law of a Wilson loop arises from \( P \) vortex linkages in that gauge.

3. Kovacs-Tomboulis vortex counters

Kovacs and Tomboulis\(^3\) gave representative independent definitions of three vortex counters based on \( SO(3) \times Z(2) \) configurations.

\[
N_{\text{thin}}(S) = \prod_{p \in S} \sigma(p),
\]

\[
N_{\text{thick}}(S) = \text{sign \text{tr}[U(C)]} \times \prod_{p \in S} \eta(p),
\]

\[
N_{\text{hybrid}} = N_{\text{thin}}(S) \times N_{\text{thick}}(S) = \text{sign } W.
\]

The hybrid counter is necessarily independent of surface. \( N_{\text{thin}}(S) \) and \( N_{\text{thick}}(S) \) count the corresponding vortices only if the value is independent of surface \( S \).

We can express these counters in terms of \( SU(2) \) configurations by evaluating the above expressions in the \( \hat{U}(b) \) representative.

\[
N_{\text{thin}}(S) = \prod_{p \in S} \text{tr}[\partial \hat{U}(p)],
\]

\[
N_{\text{hybrid}} = \text{sign } \text{tr}[\hat{U}(C)],
\]

\[
N_{\text{thick}}(S) = \prod_{p \in S} \text{sign } \text{tr}[\partial \hat{U}(p)] \times \text{sign } \text{tr}[\hat{U}(C)].
\]
4. Numerical Results

It is not feasible to measure these counters on all possible surfaces. We made measurements only on the minimal surface\[^3\]. As a consequence, a measurement giving for example \(N_{\text{thin}}(S) = -1\) indicates only the occurrence of an odd number of \(\sigma\) patches which could be part of thin or hybrid vortices. And similarly for the thick case.

The contribution to the potential from the three types of vortex counters is

\[
V(R) = -\lim_{T \to \infty} \frac{1}{T} \ln(N(W(R, T))),
\]

where \(N(W(R, T))\) is the thin, thick or hybrid counter signal for that particular Wilson loop (taking values \(\pm 1\)).

Fig. 1 shows that the string tension in \(V_{\text{thin}}\) in physical units increases in the continuum limit. Although this is perhaps surprising, we showed that this is canceled by an increasing string tension in the thick potential\[^3\].

The K-T definition for vortices\[^2\] is appealing since it is gauge invariant but they are hard to localize on a lattice. P vortices\[^4\], on the other hand, are easy to localize but are not gauge invariant. It is interesting to see if these two definitions agree. We now have the tools to compare these definitions of vortex counters on the same configuration.

Fig. 2 shows plots of the average of the fraction odd/(odd+even) hybrid and projection vortices linking a Wilson loop as a function of area. The average of the product compared to the product of the average shows that there is essentially no correlation. The corresponding plots for thin vortex fractions and thick ones gives essentially the same result. In Ref.\[^5\] we examine more sensitive signals of correlations but without a definitive result.

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