A Model Counter’s Guide to Probabilistic Systems

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Abstract. In this paper, we systematize the modeling of probabilistic systems for the purpose of analyzing them with model counting techniques. Starting from unbiased coin flips, we show how to model biased coins, correlated coins, and distributions over finite sets. From there, we continue with modeling sequential systems, such as Markov chains, and revisit the relationship between weighted and unweighted model counting. Thereby, this work provides a conceptual framework for deriving \#SAT encodings for probabilistic inference.

Keywords: Model Counting · Markov Chains · Probabilistic Inference

1 Introduction

Model checking of probabilistic systems, such as Markov Chains, as well as probabilistic inference on graphical models, capture a diverse range of applications from biology\textsuperscript{15} to network reliability estimation\textsuperscript{17} to learning from unlabeled demonstrations\textsuperscript{28}. At their core, such problems often rely on Monte Carlo methods\textsuperscript{19}, Belief Propagation\textsuperscript{3}, and (explicit-state/BDD-based) probabilistic model checking\textsuperscript{16,11,21}. While powerful in specific domains, none of these approaches is a panacea. Probabilistic model checking does not scale well to complex systems, and Monte Carlo as well as Belief Propagation requires exponential effort to analyze rare events. Thus, both methods struggle in settings that involve the analysis of rare events in complex systems.

Model counting is a promising alternative algorithmic approach to analyzing probabilistic models and probabilistic inference. The recent rapid improvements in SAT-based model counting, particularly in approximate model counting\textsuperscript{12,6,23,7,1,24}, raise hope that the Herculean improvements in SAT solving could be leveraged for probabilistic inference. In particular, model counting may enable us to transition away from Monte Carlo methods and BDD-based probabilistic model checking, in the same way that SAT solvers resulted in a transition away from explicit-state/BDD-based model checking\textsuperscript{2}.

However, analyzing probabilistic systems with model counting algorithms requires a different approach to modeling than for functional verification. For example, in SAT-based bounded model checking, each satisfying assignment represents a path of the modeled system. Since functional verification only
requires finding a single path, the existence of potentially redundant paths or non-deterministic choices in the encoding is largely irrelevant. Moreover, because introducing non-determinism and redundant models often simplifies system models, such tricks are frequently employed in system encodings. By contrast, model counting concerns itself with the model count, i.e. the number of satisfying assignments, and hence we must be careful when introducing non-deterministic choices in the model, as they can increase the number of satisfying assignments.

**Contributions:** In this paper, we systematically develop a framework for modeling probabilistic systems as model counting problems. The central idea underlying our framework is that feeding random unbiased coin flips into a Boolean predicate simulates a biased coin flip. Based on this deceptively simple observation, the rest of the paper develops numerous gadgets, which, when combined, can encode arbitrary distributions over finite sets, as well as Markov Chains.

1. A framework for deriving unweighted model counting encodings for queries about probabilistic systems such as Markov Chains. Two key features of this framework are (i) the focus on Boolean functions rather than constraints. This alternative focus facilitates composition and preserves model counts. (ii) the use of sequential circuits, in particular the decomposition given in Fig. 6 for encoding Markov Chains.
2. An alternative perspective on the reduction from weighted to unweighted model counting provided by [5], which makes apparent that the central gadget used is the less-than operator on unsigned integers.
3. Three algorithms for encoding queries about distributions over finite sets as model counting problems. The first builds on the weighted to unweighted reduction given in [5] while the other two illustrate how our framework facilitates interfacing with the larger random number generation literature.

The rest of this paper is organized as follows: We begin by establishing the connection between model counting and unbiased coin-flips. Then, in Section 3, we demonstrate how to use unbiased coins to model biased coins and correlated coins. In Section 4, we show how to use these components to model sequential probabilistic systems, such as Markov Chains. We demonstrate how to model arbitrary distributions over finite sets in Section 5 including the binomial distribution, which is fundamental to probabilistic inference. Finally, we revisit the relationship of weighted and unweighted model counting in Section 6.

2 Related Work

This work connects with literature in two primary ways. First and foremost, we provide an encoding of probabilistic systems as sequential circuits, which when unrolled, are suited to be analyzed with model counting algorithms. Numerous encodings of probabilistic systems as weighted model counting problems have been proposed [22, 8, 10], which has then spurred the adaptation of a number of unweighted model counting algorithms to solve weighted model counting problems [22, 11]. Unfortunately, such adaptations require expert knowledge of the inner workings of model counters, making the transfer of advancements from unweighted model counting to weighted model counting difficult. Hence,
techniques for efficient and automated reductions from weighted model counting to unweighted model counting have been proposed [5]. In many ways, this article continues in this direction, by providing a framework for encoding probabilistic systems and inferences on said systems directly into unweighted model counting problems. Such reductions are particularly appealing given that, to our knowledge, there is no major algorithmic advantage in using weighted over unweighted model counting algorithms.

Further, this work is intimately related to the work on simulating discrete distributions using a stream of random bits. This framework, called the random bit model and first introduced by von Neumann [20], has gone on to spawn numerous techniques (for a more detailed survey, we point the reader to [18]). One of the goals in this work has been to illustrate how to draw from this vast literature to create new encodings of probabilistic circuits, e.g., in Section 5, we illustrate how to encode a binomial distribution as well as encode Knuth and Yao’s [13] classic algorithm into a sequential circuit.

3 Circuits, Coin Flips, and Model Counting

In the sequel, we develop a framework for analyzing probabilistic systems via model counting. We begin by defining bit-vectors and bit-vector predicates.

\[ \begin{align*}
010 & , 101 \\
\rightarrow & \\
010 & 101
\end{align*} \]

(a) Example concatenation of 3 and 2

\[ \begin{align*}
\phi \& (x) = 1 \\
\rightarrow & \\
\text{a n-ary bit-vector predicate as a circuit with n inputs.}
\end{align*} \]

Fig. 1: Illustrations to accompany Definitions 1 and 2.

**Definition 1** A \(n\)-bit-vector, \(x \in \{0, 1\}^n\), is a tuple of \(n \in \mathbb{N}\) Boolean values. The concatenation of a \(n\)-bit-vector, \(x\), and a \(m\)-bit-vector, \(x'\), is an \((n+m)\)-bit-vector, \(xx'\), where the first \(n\) bits form \(x\) and the final \(m\) bits form \(x'\) (see Fig. 1a).

To avoid clutter, if \(n = m = 1\), we simply write \(xx'\). Thus, we denote decomposing an \(n\)-bit-vector, \(x\), individual bits via \(x = x_1 x_2 \ldots x_n\). Next, we define a model counter’s main object of study, predicates over bit-vectors.

**Definition 2** A \(n\)-ary bit-vector predicate maps \(n\)-bit vectors to \(\{0, 1\} \subseteq \mathbb{R}\), e.g.
\[ \varphi : \{0, 1\}^n \rightarrow \{0, 1\}. \] (1)

If \(\varphi(x) = 1\), we additionally call \(x\) a model of \(\varphi\). We define the model count of \(\varphi\), denoted \(\#(\varphi)\), as the number of models of \(\varphi\), i.e,
\[ \#(\varphi) \triangleq \sum_{x \in \{0, 1\}^n} \varphi(x). \] (2)
When the number of inputs of a predicate $\varphi$ is unambiguous, we shall write $\varphi(x)$ as a logical sentence over $x$, where True is mapped to 1 and False is mapped to 0.

**Example 1.** Let $\varphi: \{0,1\}^{10} \to \{0,1\}$ denote the 10 input map,

$$\varphi(x) = x_1 \land x_7 \overset{\text{def}}{=} \begin{cases} 1 & \text{if } x_1 \land x_7 \\ 0 & \text{otherwise} \end{cases}.$$ (3)

Thus $\varphi(x) = 1$ iff $x_1$ and $x_7$ are True (i.e., 1). Further observing that there are 8 other “don’t care” inputs, each with two possible values, yields $\#(\varphi) = 2^8$.

**Example 2.** Given $n \in \mathbb{N}$, let $k$ be an integer between 0 and $2^n - 1$ and let $\varphi: \{0,1\}^n \to \{0,1\}$ denote,  

$$\varphi(x) = x < k,$$ (4)

where $x$ is interpreted as an integer between 0 and $2^n - 1$. Observe that $\#(\varphi) = k$ since there are only $k$ unsigned integers less than $k$.

Next, observe that a circuit can be made probabilistic by feeding the results of random coin flips as inputs. To this end, we introduce notation for the process of generating a bit-vector using $n$ unbiased coin flips.

**Definition 3** Denote by $x_1 x_2 \ldots x_n \sim \{0,1\}^n$ the act of creating an $n$-bit-vector by flipping $n$ independent unbiased coins with

$$\Pr_{x \sim \{0,1\}^n}(x_i = 0) = \Pr_{x \sim \{0,1\}^n}(x_i = 1) = \frac{1}{2},$$ (5)

and thus, the probability of drawing any particular bit-vector, $x^*$ is:

$$\Pr_{x \sim \{0,1\}^n}(x = x^*) = \frac{1}{2^n}.$$ (6)

Our framework for studying random inputs to bit-vector functions relies on the following key (though unsurprising) observation.

**Observation 1** Given an $n$-ary bit-vector predicate, $\varphi$, if one flips $n$ independent unbiased coins, $x \sim \{0,1\}^n$, the probability that $\varphi(x) = 1$ is equal to the fraction of $n$-bit-vectors that are models of $\varphi$, i.e,

$$\Pr_{x \sim \{0,1\}^n}(\varphi(x) = 1) = \sum_{x \in \{0,1\}^n} \frac{1}{2^n} \varphi(x) = \frac{\#(\varphi)}{2^n}.$$ (7)

Therefore, if one wishes to compute $\Pr_{x \sim \{0,1\}^n}(\varphi(x) = 1)$ for some complicated $\varphi$, it suffices to use a model counter to compute (or approximate) $\#(\varphi)$. While straightforward, the power of this observation is only truly realized when one starts composing bit-vector predicates and reusing inputs. We illustrate this through a series of observations.
(a) By sharing inputs, two bit-vector predicates, which model biased coins, can be used to model a pair of correlated coin flips.

(b) Feeding correlated biased coin flips into a bit-vector predicate yields a new bit-vector predicate, and thus models a biased coin flip.

Fig. 2: Illustrations of Observations 2 and 3

Fig. 3: Visualization of $[\varphi \times \varphi'](x)$. The bit sequences have the most significant bit on the left and the least significant bit on the right, e.g., $011 = 3$.

**Observation 2** Using (7), $\varphi$ can be reinterpreted as a process to turn $n$ unbiased coins into a biased coin.

To emphasize Observation 2 we shall denote by $x \sim \varphi$ the process of drawing a biased coin, $x \in \{0, 1\}$, using the distribution given in (7).

**Observation 3** If the results of some coin flips are shared, $F : x \mapsto (\varphi(x), \varphi'(x))$, then $F : \{0, 1\}^n \rightarrow \{0, 1\}^2$ models correlated coin flips.

As above, inspired by Observation 3, given a map between bit-vectors, $F : \{0, 1\}^n \rightarrow \{0, 1\}^m$, we denote by $x \sim F$ the process of drawing $m$ correlated biased coin flips. In particular, if $\varphi_i(x) = F(x)_i$, then $x$ is the concatenation of $m$ bit-vectors such that, $x_i \sim \varphi_i$. Together, Observations 2 and 3 enable studying complex distributions via model counting.

**Example 3.** Let $\varphi$ and $\varphi'$ denote the following 3-bit bit-vector predicates,

$$
\begin{align*}
\varphi & : \{0, 1\}^3 \rightarrow \{0, 1\} \\
\varphi'(x) & \overset{\text{def}}{=} x = 3 \\
\end{align*}
\begin{align*}
\varphi' & : \{0, 1\}^3 \rightarrow \{0, 1\} \\
\varphi'(x) & \overset{\text{def}}{=} x > 3
\end{align*}
$$

where $x$ is interpreted as an unsigned integer.

Next, define $\varphi \times \varphi'$ as the product of $\varphi$ and $\varphi'$, i.e., $[\varphi \times \varphi'] : x \mapsto (\varphi(x), \varphi'(x))$. The resulting map is illustrated in Fig. 3.

Again, note that because $\varphi$ and $\varphi'$ share inputs, then the biased coins they model are correlated (Observation 3). In particular, using Fig. 3 we see that
φ × ϕ' induces the following distribution over 2-bit-vectors:

\[
\Pr_{y \sim \phi \times \phi'}(y = k) = \begin{cases} 
3/8 & \text{if } k = 0 \\
1/8 & \text{if } k = 1 \\
1/2 & \text{if } k = 2 \\
0 & \text{otherwise.}
\end{cases}
\] (9)

Now suppose one wishes to compute the probability that \( k > 0 \) under (9). By (7), it suffices to compute the model count of \([x > 0]\) composed with \(\phi \times \phi'\),

\[
\Pr_{y \sim \phi \times \phi'}(y > 0) = \frac{\#([x > 0] \circ [\phi \times \phi'])}{2^n}.
\] (10)

Of course, in this case, it is easy to look at Fig. 3 to determine that \(\#([x > 0] \circ [\phi \times \phi']) = 5\). However, in general, with bigger circuits and more complicated properties, this explicit reduction to model counting proves incredibly useful.

The framework developed so far has focused on modeling probability distributions where the probability masses are (integer) multiples of \(\frac{1}{2^n}\). Of course, many examples violate this assumption, e.g., a coin with a \(\frac{1}{3}\) bias towards heads. To handle such distributions, we adapt our framework to condition on certain coin flip outcomes not occurring. Note that via the chain rule, any predicate over an input conditioned distribution can be studied using two model counting queries.

**Proposition 1.** Let \(\varphi : \{0,1\}^n \to \{0,1\}\) and \(\psi : \{0,1\}^n \to \{0,1\}\) denote any two bit-vector predicates. Then,

\[
\Pr_{x \sim \{0,1\}^n}(\varphi(x) = 1 \mid \psi(x) = 1) = \frac{\#(\varphi \land \psi)}{\#(\psi)}.
\] (11)

**Proof.** By the chain rule,

\[
\Pr_{x \sim \{0,1\}^n}(\varphi(x) = 1 \mid \psi(x) = 1) \cdot \Pr_{x \sim \{0,1\}^n}(\psi(x) = 1) = \Pr_{x \sim \{0,1\}^n}(\varphi \land \psi(x) = 1). \] (12)

Replacing the unconditioned probabilities using (7) gives,

\[
\Pr_{x \sim \{0,1\}^n}(\varphi(x) = 1 \mid \psi(x) = 1) \cdot \frac{\#(\psi)}{2^n} = \frac{\#(\varphi \land \psi)}{2^n}.
\] (13)

Multiplying both sides by \(2^n\) and rearranging yields (11). \(\square\)

To avoid notational clutter, we shall frequently write (11) using the sampling notation, \(y \sim \varphi\), previously introduced, but additionally condition on \(\psi\),
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\[ \Pr_{y \sim \varphi}(y \mid \psi) \overset{\text{def}}{=} \Pr_{x \sim \{0,1\}^n}(\varphi(x) = 1 \mid \psi(x) = 1) \]  

(14)

**Example 4.** Again, suppose we seek to find a pair \( \varphi, \psi \) that encodes a biased coin with probability \( \frac{1}{3} \) of coming up 1. Observe that this can be accomplished by letting \( \varphi(x) \overset{\text{def}}{=} (x = 0), \psi(x) \overset{\text{def}}{=} (x < 3) \), such that,

\[ \Pr_{y \sim \varphi}(y_i \mid \psi) = \frac{\#(x = 0 \land x < 3)}{\#(x < 3)} = \frac{1}{3}, \]  

(15)

where \( x \in \{0,1\}^2 \) is encoded as an unsigned integer.

Note that in many contexts, \(#(\psi)\) can be precomputed, sometimes even without the use of a model counting algorithm.

**Encoding Rational Coins.** Example 4 can be generalized to encode an arbitrary coin with a rational bias. Namely, consider a coin, \( y \), such that \( \Pr(y = 1) = k/m \), for some \( k,m \in \mathbb{N} \). Letting \( n \) be the smallest integer such that \( m \leq 2^n \), and recalling that \( x < k \) has exactly \( k \) models (Ex. 2), observe that \( y \) corresponds to feeding \( n \) unbiased coins into \( \varphi(x) \overset{\text{def}}{=} x < k \) and conditioning on \( \psi(x) \overset{\text{def}}{=} x < m \). Finally, observing that \( x < k \) implies that \( x < m \) yields,

\[ \Pr_{y \sim \varphi}(y = 1 \mid \psi) = \frac{\#(x < k)}{\#(x < m)} = \frac{k}{m}. \]  

(16)

Finally, observe since that \( \Pr_{y \sim \varphi}(\varphi(y) \mid \psi, \psi') = \Pr_{y \sim \varphi}(\varphi(y) = 1 \mid \psi \land \psi') \), Eq. (16) naturally extends to modeling multiple input conditioned coin flips. Of course, biased coins are not very interesting by themselves. Nevertheless, as illustrated in Ex. 3, feeding multiple correlated coin flips into another circuit enables studying more sophisticated objects. Further, as the next section illustrates, by incorporating a notion of state, the framework developed above enables answering non-trivial queries about probabilistic systems via model counting.

### 4 Sequential Circuits

Ultimately, we want study sequential probabilistic systems, such as Markov Chains, probabilistic regular languages, and random walks. While the processes we studied in the previous section involved only an a-priori fixed number of coin flips, sequential systems in general may consume an arbitrary number of bits. We can thus not anymore rely on Boolean predicates, but need to extend our framework. In the following, we thus introduce sequential circuits and show how to employ them for modeling sequential probabilistic systems.
Definition 4 Let $n, m,$ and $p$ denote natural numbers. A sequential circuit is a tuple, $C = (s_0, F)$, where $F : \{0, 1\}^{p+n} \rightarrow \{0, 1\}^{p+m}$ is the transition function, and $s_0 \in \{0, 1\}^p$ is the initial state.

Further, to every sequence of inputs $a_1, a_2, \ldots \in \{0, 1\}^n$, we associate a sequence of states $s_1, s_2, \ldots \in \{0, 1\}^p$ and outputs $y_1, y_2, \ldots \in \{0, 1\}^m$ by:

$$s_i.y_i = F(s_{i-1}, a_i) \quad (17)$$

Finally, if $m = 1$, we refer to $C$ as a monitor.

Example 5. Figure 4b illustrates a sequential circuit that checks if $x$ has been constantly 1. Formally if $\varphi(x) \equiv x_0 \land x_1$, then $F(x) \equiv \varphi(x), \varphi(x)$ and $s_0 = 1$. Note that as circuits can reuse outputs, in Fig. 4b $\varphi(x)$ is only computed once.

Now observe that we can reduce the execution of a fixed number of steps of a sequential circuit, $C = (s_0, F)$, back to a bit-vector function simply by composing $F$ with itself, akin to bounded model checking [2].

Fig. 5: The sequential circuit of Fig. 4a unrolled for 3 steps. As in Fig. 4a, the first two inputs and outputs of each copy of $F$ denote the state. Note that the first copy of $F$ has its state inputs grounded to denote that $s_0 = (0, 0)$.

Definition 5 Let $C = (s_0, F)$ denote a sequential circuit with $n$ inputs, $p$ states, and $m$ outputs and let $\text{last}_m : \{0, 1\}^{p+m} \rightarrow \{0, 1\}^m$ denote the bit-vector function that returns the last $m$ bits of input. For all times $\tau \in \mathbb{N}$,
Define the $\tau$-unrolling of $C$, to be the map:

$$U^\tau_C : \{0, 1\}^{\tau \cdot n} \rightarrow \{0, 1\}^m$$

$$U^\tau_C(a_1, a_2, \ldots, a_\tau) \overset{\text{def}}{=} \text{last}_m \circ F(\ldots F(F(F(x_0, a_1), a_2), a_3), \ldots, a_\tau)$$  \hspace{1cm} (18)

where each $a_i$ denotes a bit-vector in $\{0, 1\}^n$.

Note that since unrolling a monitor results in a Boolean predicate, Observations 1, 2, and 3 naturally extend to the sequential circuits. This suggests extending our notation for sampling a coin to sequential circuits. Namely, given a sequential circuit $C$ and a monitor $\psi$ to condition on, we define $\mathbf{x} \sim C$ so that:

$$\Pr_{\mathbf{x} \sim C} (\mathbf{x} \mid \psi) \overset{\text{def}}{=} \Pr_{\mathbf{x} \sim U^\tau_C} (\mathbf{x} \mid U^\tau_\psi)$$  \hspace{1cm} (19)

The key utility of Eq. (19), is how it enables studying probabilistic transition systems via model counting. In particular, observe that queries about Finite Markov Chains decompose into four cascading sequential circuits modeling a control policy, a transition relation, a property monitor, and a validity checker (see Fig. 6). The dynamics circuit corresponds to the Markov Chains underlying discrete automaton, the policy governs the transition probabilities, the valid monitor checks that each coin flip satisfies the conditioning property $\psi$ and the property monitor encodes which property, $\varphi$, about the Markov Chain is being tested.

**Example 6.** Consider the 1-d variant of the classic drunken sailor random walk. A sailor walks along a pier, where with each step, the sailor either stumbles forward by one plank, backward by one plank, or remains on the same plank. Further, suppose the pier is 11 planks long and that if the sailor visits the central plank more than 3 times, the plank will break and the sailor will fall into the water. If the sailor starts on the middle plank and the probability of moving forward is $2/6$, moving backward is $1/6$, and not moving is $3/6$, what is the probability the sailor breaks a plank after 10 steps?

Within the above framework, the dynamics corresponds the a 1-d finite chain (see Fig. 7). An example encoding of such a chain as a sequential circuit is given...
Fig. 7: Illustration of “1-hot” encoding of a chain graph. The right and left arrows represent arithmetic right ($\gg$) and left ($\ll$) shifts of the state respectively.

Fig. 8: “1-hot” encoding of a chain graph as a sequential circuit. The MUX gates use their top input to select which of their two other inputs to output.

in Fig. 8. Similarly, the monitor is a sequential circuit for the regular language (\texttt{.s_0.s_0.s_0.s_0}) which can be efficiently compiled into a sequential circuit \[\text{26}\]. Finally, the policy corresponds to some circuit that models the probability distribution over actions which, using the inputs in Fig. 8, corresponds to modeling two independent biased coin flips, namely, $\Pr(\text{enable} = 0) = \frac{1}{2}$ and $\Pr(\text{direction} = 0) = \frac{2}{3}$. As shown in the previous section, namely \[\text{16}\], we can model the direction coin by feeding the output of $\varphi_{\text{direction}}(x) = x < 2$ into the direction input shown in Fig. 8 and conditioning on $x < 3$, where $x \in \{0, 1\}^2$. Similarly, using a disjoint set of inputs, we can encode the enable coin by feeding the output of $\varphi_{\text{enable}}(x') = x' < 1$ into the enable input of Fig. 8. The resulting sequential circuit, $C_{ds}$, is summarized in Fig. 9. Letting $C_{valid}$ denote the monitor that for all time steps, $x < 3$, then the probability of the sailor falling into the water within the first 50 steps is given via:

$$\Pr(\text{sailor falls into water}) = \Pr(x = 1 \mid C_{valid}) = \frac{25398396}{6^{10}} \approx 0.42 \quad (20)$$

Finally, it is worth noting that anecdotally, this problem took $\approx 1$ minute using a BDD and $\approx 1$ sec using the SAT based approximate model counter ApproxMC3 \[\text{25}\].
5 Distributions over Finite Sets

We now return to the topic of modeling distributions over finite sets by feeding coin flips into circuits. The first two techniques can be used to model arbitrary rational-valued distributions over finite sets. Then, for variety, we illustrate how to encode a Binomial distribution as a sequential circuit.

Formally, we first seek to systematically solve the following problem:

**Problem 1.** Let $Y$ be a finite set whose elements, $\hat{y}_i$, are numbered from 1 to $|Y|$, and associate to $Y$ the following rational valued probability distribution:

$$\Pr(\hat{y}_i) = \frac{a_i}{m}, \quad (21)$$

where $a_i, m \in \mathbb{N}$ such that $\sum_{i=1}^{|Y|} a_i = m > 0$. Further, denote by $y \in \{0, 1\}^{|Y|}$ the 1-hot encoding of elements of $Y$, e.g. $y_i = 1$ iff $y$ corresponds to $\hat{y}_i$. Find an $n \in \mathbb{N}$, an $n$-bit-vector function $F : \{0, 1\}^n \rightarrow \{0, 1\}^{|Y|}$, and a $n$-bit-vector predicate $\psi$, such that:

$$\Pr(\hat{y}_i) = \Pr_{y \sim F}(y_i = 1 \mid \psi) \quad (22)$$

Note that the use of a 1-hot encoding is without loss of generality, since one can always feed this encoding into a circuit that transforms it into another encoding.

**Common Denominator Method.** Our first technique is a straightforward generalization of encoding a biased coin [16]. The key idea is to encode $|Y|$ mutually exclusive biased coins, which together, form a 1-hot encoding of $\hat{y}_i$. To begin, let $n$ be the smallest integer such that $m \leq 2^n$. For convenience, define

$$b_0 \overset{\text{def}}{=} 0 \quad b_{i+1} \overset{\text{def}}{=} b_i + a_i. \quad (23)$$

Now, let $\varphi_i : \{0, 1\}^n \rightarrow \{0, 1\}$ denote the circuit,

$$\varphi_i(x) \overset{\text{def}}{=} b_i \leq x < b_i + a_i \quad (24)$$

where $x$ is interpreted as an unsigned integer. Further, note that by construction, $\#(\varphi_i) = a_i$ and the $\varphi_i$ are mutually exclusive. Thus, the product of all $\varphi_i$ results in a 1-hot encoding. Namely, letting $F : \{0, 1\}^n \rightarrow \{0, 1\}^{|Y|}$ denote $\varphi_1 \times \ldots \times \varphi_{|Y|}$ and $\psi(x) \overset{\text{def}}{=} x < m$ yields,

$$\Pr_{y \sim F}(y_i = 1 \mid \psi) = \frac{a_i}{m}, \quad (25)$$

as desired. Finally, before discussing our second technique, we briefly remark that it was using this technique the encodings seen so far have been generated. For example, the circuit seen in Ex. 3 is a straightforward simplification of the circuit formed by feeding the circuit created using, $a_0 = 3, a_1 = 1, a_2 = 4, m = 2^3$, and $n = 3$ into a circuit which transforms the 1-hot encoding of unsigned integers into the base 2 encoding.
Knuth and Yao Random Number Generators. As the reader may be aware, simulating arbitrary discrete distributions using coin flips is well-trodden ground. For example, Knuth and Yao famously provided a systematic technique for simulating arbitrary discrete distributions in a manner that (in expectation) is optimal with respect the number of coin flips required \[13\]. As an example of the flexibility of the above framework, we shall sketch how to embed the Knuth and Yao’s scheme as a sequential circuit with coin flip inputs.

As before, we assume the set-up given in Problem 1 and define \( n \) to be the smallest integer such that \( m \leq 2^n \). Next, write each probability mass in its binary expansion,

\[
\Pr(\hat{y}_i) = 0.p_1 p_2 p_3 \ldots
\]  

(26)

For example, if \( \Pr(\hat{y}_i) = 2^{-3} \), then the corresponding decimal expansion is 0.001. Similarly, \( \Pr(\hat{y}_i) = 1/3 \) yields 0.(01)\( ^\omega \), where (\( \cdot \)\( ^\omega \)) represents an infinite repetition. Knuth and Yao’s key idea is to then construct a (potentially infinite) binary tree where if the \( j \)th bit of the expansion of \( \Pr(\hat{y}_i) \) is 1, then \( \hat{y}_i \) appears as a leaf at depth \( j \). Such a tree is guaranteed to exist due to the Kraft inequality \[14\].

For example, for a three sided die, \( \Pr(\square) = \Pr(\triangle) = \Pr(\odot) = 0.(01)\( ^\omega \) \) with the corresponding of the infinite binary tree shown in Fig. 10. Note that if a sub-tree is self similar to an ancestor node, we draw a back edge. We refer to the tree with back edges as a parse tree. Knuth and Yao’s algorithm then performs a depth first search from the root to a leaf where at each node, one flips a coin and takes the left branch if the coin comes up tails and take the right branch otherwise. Once a leaf is reached, the algorithm then outputs the leaf’s value.

The central idea in porting this algorithm to our framework is to encode the transition system given by the depth first search on the parse tree into a sequential circuit. This is done by viewing the parse tree as \(|Y|\) monitors, \( C_i \), each accepting iff the corresponding element has been reached. For example, for the three sided dice, the parse tree given in Fig. 10 results in three monitors corresponding to recognizing \( \square, \triangle, \) and \( \odot \) respectively. These \(|Y|\) monitors are then fed the same stream of random coin flips and have their outputs concatenated to form the 1-hot encoding of \( Y \); however, note that until a leaf is reached, the resulting circuit, \( C \), will output the all zeros bit-vector, 0\( |Y| \).

Finally, to create a model counting problem, one observes that (asymptotically) the probability of not having reached a leaf state exponentially decreases with the number of coins flipped. Thus, if \( \tau \in \mathbb{N} \) is sufficiently large and \( \psi \) denotes the monitor checking if the last state is 0\( |Y| \) then,

\[
\Pr(\hat{y}_i) \approx \Pr_{y^\sim C}(y_i = 1 \mid \psi)
\]  

(27)

While the utility of this method may seem suspect, we note that (i.) For many cases, \( \tau \) simply needs to be the height of the tree. For example, if \( m \) is a power of 2,
then no back edges will exist. Similarly, if all back edges occur at leaves and go to the root, as in Fig. 10, then the back edges can be safely removed by conditioning on $\psi$. In fact, many times, such as Fig. 10, this encoding is equivalent to technique 1! (ii.) The parse tree automatically takes into account the particularities of the distribution and as previously stated, is known to be optimal in expectation, which translates to optimality when no back edges are present. (iii.) Finally, and most importantly, this example illustrates how the literature on transforming discrete distributions could shape the design of model counting encodings of probabilistic systems. With this connection to prior literature explored, we now evaluate how the framework developed relates to prior work on reducing weighted model counting to unweighted model counting.

**Binomial Distribution.** For our final technique, we illustrate how to encode a Binomial Distribution as a probabilistic circuit. Formally, let $X$ be the number of successes after $n$ independent trials, the Binomial Distribution with bias $p$ is defined by:

$$\Pr(X = k) = \binom{n}{k} p(1 - p). \quad (28)$$

Aside from being an interesting example, Binomial distributions are an important building block for many probabilistic systems. Of particular interest is the common use of (28) to approximate Gaussian distributions when discretizing continuous domains. The key idea in encoding (28) is to feed biased coins into a circuit that counts the number of successful trials (out of $n$). An example encoding is given in Fig. 12. Intuitively, this circuit can be visualized as a modeling the transitions of pascals triangle (Fig. 11). Given such a sequential circuit, $C$, the encoding for a $p = \frac{1}{2}$ Binomial Distribution of $n$ trials is simply the $\tau = n$ unrolling of $C$. Therefore, if $\varphi_k(x) \overset{\text{def}}{=} (x = k)$ and $F_n(x) \overset{\text{def}}{=} U^n_C(x)$ then:

$$\Pr(X = k) = \Pr_{y \sim \varphi_k \circ F_n}(y = 1) = \frac{\#(f_k \circ F_n)}{2^n}. \quad (29)$$

Finally, observe that by feeding the output of the circuit encoding of a biased coin (10) into the trial input of $C$ enables encoding any rational $p$ Binomial Distribution through appropriate unrolling.

## 6 Relationship to Weighted Model Counting

In the preceding sections, we have developed a modeling framework for probabilistic systems based on feeding unbiased coins into Boolean predicates or sequential circuits. Our encodings require only unweighted model counting algorithms for their analysis, and thus directly benefit from the recent dramatic performance
Fig. 12: Sequential Circuit for counting trials. The latch is initialized with 1 and every time a successful trial occurs (trial = 1) the state is left shifted. For example, if \( n = 3 \), then \( s_0 = 001 \). If a successful trial occurs then \( s_1 = 010 \).

In this section, we take a second look at previously proposed encoding of weighted model counting \[7,1,25\]. In contrast, many previous works on probabilistic inference using model counting algorithms have built on algorithms for weighted model counting. But adapting advances in unweighted model counting to weighted model counting can be quite challenging, and we are not aware of clear performance benefits that can be gained by considering weighted model counting in the algorithm itself. Instead, it appears to be easier to reduce weighted model counting to unweighted model counting \[5\].

In this section, we focus on literal-weighted model counting, which extends unweighted model counting by a function \( W \) mapping each pair of input coin and Boolean outcome to a real value. The weight of a model is then defined as the product of the weights of its components. For example, consider the weight function assigning \( W(x_1) = 0.2 \), \( W(x_2) = 3 \), and \( W(\neg x_2) = 15 \) the model \( x_1 \land \neg x_2 \) has thus weight \( W(x_1) \cdot W(\neg x_2) = 3 \).

Chavira and Darwiche have shown that this general setting can be efficiently reduced to the case that \( W(x) \in [0,1] \) and \( W(x) = 1 - W(\neg x) \) \[8\]. In particular, this means that we can simply model weighted literals by biased coins (Section 3).

The curious reader may ask how this compares to the encodings proposed in prior work \[5\]. There, the authors build on the following gadget:

\[
\text{Definition 6} \quad \text{Let } k, n, k_{n-1} \ldots k_1 \text{ be the standard base } 2 \text{ representation of } k \text{ as an unsigned integer with } k_1 \text{ being the least significant bit. We define}
\]

\[
H^k_n(x) \overset{\text{def}}{=} x_n \square_n H^k_{n-1}(x),
\]

where \( H^k_0(x) = 0, \square_n \overset{\text{def}}{=} \land \text{ if } k_n = 0 \text{ and } \lor \text{ otherwise.} \)

The key utility of \( H^k_n \) is that \( \#(H^k_n) = k \) (see \[5\]). The reduction from weighted to unweighted model counting then straightforwardly follows from,

\[
\#(H^k_n(y) \land \phi(x)) = k \cdot \#(\phi(x)).
\]

In particular, if \( \phi(x) \) is given as a CNF formula, then applying the above reduction to each clause is structurally equivalent to the reduction given in \[5\].
We simply want to observe here that $H^k_n(\overline{x})$ satisfies the same set of models as the bit-vector comparison $x < k$.

**Proposition 2.**

$$H^k_n(\overline{x}) \equiv x < k,$$

where $\overline{x}$ indicates the bit-wise negation of $x$.

**Proof.** Observe that one can test if $x < k$ by recursively testing if the most significant bit of $x$ is less than the most significant bit of $k$ in base 2 representation, i.e,

$$\hat{H}^k_n(x) \overset{\text{def}}{=} (x_n < k_n) \lor (x_n = k_n \land \hat{H}^k_{n-1}(x))$$

where $\hat{H}^k_0(0) \overset{\text{def}}{=} \emptyset$ and $(x_n < k_n) \equiv (\neg x_n \land k_n)$. Observe that if $k_n = 0$, then that (33) reduces to, $\hat{H}^k_n(x) = \neg x_n \land \hat{H}^k_{n-1}(x)$. Similarly, if $k_n = 1$ then,

$$\hat{H}^k_n(x) = \neg x_n \lor \left( x_n \land \hat{H}^k_{n-1}(x) \right) = \neg x_n \lor \hat{H}^k_{n-1}(x).$$

Defining, $\Box_n \overset{\text{def}}{=} \lor$ if $k_n = 1$ and $\Box_n \overset{\text{def}}{=} \land$ if $k_n = 0$, then (33) becomes, $\hat{H}^k_n(x) = \neg x_n \Box_n \hat{H}^k_{n-1}(x)$, which is equivalent to $H_n(\overline{x})$.

In particular, this shows that the basis of this work, the encoding of biased coins via unbiased coins, is equivalent to prior encodings.

**7 Conclusion**

In this paper, we systematically developed a framework for modeling probabilistic systems as model counting problems. Starting from unbiased coins, we construct biased coins, correlated coins, and conditional probabilities. We discuss how to model arbitrary distributions over finite sets and how to combine our building blocks into sequential systems. While the building blocks discussed in this work have been used in previous works (e.g. [28,21]), we believe that the explicit discussion of the modeling techniques in this work will enable future case studies on probabilistic systems with SAT-based model counting algorithms. For example, as illustrated by the Knuth Yao and binomial encodings, probabilistic sequential circuits an excellent mechanism to adapt techniques from the larger random bit model literature into #SAT encodings. Finally, implementations for the common denominator method and the binomial distribution method can by found at [https://github.com/mvcisback/py-aiger-coins](https://github.com/mvcisback/py-aiger-coins) and are implemented using the py-aiger library [27].

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