New Techniques Based On Odd-Edge Total Colorings In Topological Cryptosystem

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New Techniques Based On Odd-Edge Total Colorings In Topological Cryptosystem

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Abstract: For building up twin-graphic lattices towards topological cryptograph, we define four kinds of new odd-magic-type colorings: odd-edge graceful-difference total coloring, odd-edge edge-difference total coloring, odd-edge edge-magic total coloring, and odd-edge felicitous-difference total coloring in this article. Our RANDOMLY-LEAF-ADDING algorithms are based on adding randomly leaves to graphs for producing continuously graphs admitting our new odd-magic-type colorings. We use complex graphs to make caterpillar-graphic lattices and complementary graphic lattices, such that each graph in these new graphic lattices admits a uniformly $W$-magic total coloring. On the other hands, finding some connections between graphic lattices and integer lattices is an interesting research, also, is important for application in the age of quantum computer. We set up twin-type $W$-magic graphic lattices (as public graphic lattices vs private graphic lattices) and $W$-magic graphic-lattice homomorphism for producing more complex topological number-based strings.

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Keywords: Odd-magic-type colorings; twin-graphic lattices; integer lattice; algorithm; topological coding.

1 Introduction and preliminary

1.1 Researching background

Lattice-based cryptography, as a new cryptosystem, has attracted much attention because of its great potential application value, since it is the use of conjectured hard problems on point lattices in $\mathbb{R}^n$ as the foundation for secure cryptographic systems, including apparent resistance quantum...
attacks, high asymptotic efficiency and parallelism, security under “worst-case” intractability assumptions, and solutions to long-standing open problems in cryptography said by Chris Peikert in [5]. And moreover, Chris Peikert summarized that lattice-based cryptography possesses conjectured security against quantum attacks, algorithmic simplicity, efficiency, and parallelism, and strong security guarantees from worst-case hardness.

In [4] the author pointed: Lattice-based cryptography has been recognized for its many attractive properties, such as strong provable security guarantees and apparent resistance to quantum attacks, flexibility for realizing powerful tools like fully homomorphic encryption, and high asymptotic efficiency. He has given efficient and practical lattice-based protocols for key transport, encryption, and authenticated key exchange that are suitable as “drop-in” components for proposed Internet standards and other open protocols. The security of all proposals is provably based on the well-studied “learning with errors over rings” problem, and hence on the conjectured worst-case hardness of problems on ideal lattices (against quantum algorithms).

The authors in [6] have presented such an alternative - a signature scheme whose security is derived from the hardness of lattice problems; it is based on recent theoretical advances in lattice-based cryptography and is highly optimized for practicability and use in embedded systems. The public and secret keys are roughly 12000 and 2000 bits long, while the signature size is approximately 9000 bits for a security level of around 100 bits.

In the articles [20, 21, 22] the authors have discussed some topics of topological coding, such as topological graph password and twin odd-graceful graphs for matching topological public-keys and topological private-keys in asymmetric cryptography. The authors, in [3], use the topological graph to generate the honeywords, which is the first application of graphic labeling of topological coding in the honeywords generation. They propose a method to protect the hashed passwords by using topological graphic sequences.

1.2 Examples from topological coding

For introducing topological number-based strings since they are made by Topcode-gpws (the abbreviation “graphic passwords in topological coding”), we show an example as:

Example 1. A colored graph (also, Topcode-gpw) $B_1$ shown in Fig.1(a) corresponds a matrix $T_{\text{code}}(B_1)$ shown in Eq.(2), called a Topcode-matrix, and this matrix distributes us (30)! topological number-based strings like $s_1$ and $s_2$ shown in Eq.(1). As each topological number-based string $s_i$ ($i = 1, 2$) is a public-key, so it has its own private-key $s'_i$ shown in Eq.(3), and two topological number-based strings $s'_1$ and $s'_2$ are induced from the Topcode-matrix $T_{\text{code}}(B_{10})$ shown in Eq.(4), where the colored graph $B_{10}$ is shown in Fig.1(j).

\[ s_1 = 719113665411176119152112413151915017191020 \]
\[ s_2 = 0219191701515191513421121591167114566311917 \]  

(1)
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The procure by a fixed rule is for encrypting a digital file $D_{ocu}$ by a topological number-based string $s_i$ implementing to two Topcode-matrices $T_{code}(B_1)$ and $T_{code}(B_{10})$, respectively. The procure

$$B_1 \rightarrow T_{code}(B_1) \rightarrow_{a_i} s_i$$

is for encrypting a digital file $D_{ocu}$ by a topological number-based string $s_i$, get a encrypted file $s_i(D_{ocu})$. Next, one find another topological number-based string $s'_i$ from another procure

$$B_{10} \rightarrow T_{code}(B_{10}) \rightarrow_{a_i} s'_i$$

A topological authentication consists of a topological structure matching $\langle B_1, B_{10} \rangle$ and a Topcode-matrix matching $\langle T_{code}(B_1), T_{code}(B_{10}) \rangle$ and a group of topological number-based string matchings $\langle s_i, s'_i \rangle$ for $i = 1, 2, \ldots, n$, where each topological number-based string matching $\langle s_i, s'_i \rangle$ is obtained by a fixed rule $a_i$ implementing to two Topcode-matrices $T_{code}(B_1)$ and $T_{code}(B_{10})$, respectively.

Figure 1: Ten Topcode-gpws.

$$T_{code}(B_1) = \begin{pmatrix} 6 & 6 & 6 & 11 & 2 & 4 & 15 & 0 & 2 & 0 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 \\ 7 & 9 & 11 & 4 & 11 & 15 & 2 & 15 & 19 & 19 \end{pmatrix}_{3 \times 10} \quad (2)$$

$$s'_1 = 71108287551771312914311831316181516117201911 \quad (3)$$

$$s'_2 = 11911172018151631316311812914177137558381107$$

$$T_{code}(B_{10}) = \begin{pmatrix} 8 & 7 & 13 & 12 & 8 & 3 & 16 & 1 & 1 & 1 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 \\ 7 & 10 & 8 & 5 & 17 & 14 & 3 & 16 & 18 & 20 \end{pmatrix}_{3 \times 10} \quad (4)$$
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and go to the topological authentication

$$T_{\text{authen}} = (\langle B_1, B_{10} \rangle, \langle T_{\text{code}}(B_1), T_{\text{code}}(B_{10}) \rangle, \langle s_i, s'_i \rangle) \quad (7)$$

After finishing the above topological authentication (7), one can use $s'_i$ to decrypt the encrypted file $s_i(D_{\text{ocu}})$, such that $s'_i(s_i(D_{\text{ocu}})) = D_{\text{ocu}}$.

In the above Example 1, there are the following problems:

Pro-1. Notice that two topological structures $B_1$ and $B_{10}$ are not isomorphic from each other, that is, $B_1 \not\equiv B_{10}$; two Topcode-matrices are not equivalent from each other, also $T_{\text{code}}(B_1) \neq T_{\text{code}}(B_{10})$, and $s_i \neq s'_i$ for $i = 1, 2, \ldots, n$. The vertices of the colored graph $B_1$ are colored with the numbers of a set $f(B_1) = \{0, 2, 4, 6, 7, 9, 11, 15, 19\}$ and the vertices of the colored graph $B_{10}$ are colored with the numbers of another set $g(B_{10}) = \{1, 3, 5, 8, 10, 12, 13, 14, 16, 17, 18, 20\}$, such that $f(B_1) \cup g(B_{10}) = \{0, 1, \ldots, 20\}$, which is in the Topcode-matrix matching $(T_{\text{code}}(B_1), T_{\text{code}}(B_{10}))$. This case was introduced in [21], two colored graphs $B_1$ and $B_{10}$ admits a so-called twin odd-graceful labeling.

Pro-2. Each colored graph $B_k$ with $2 \leq k \leq 9$ shown in Fig. 1 (b)-(i) can be used as a private-key corresponding to the public-key $B_1$. Thereby, one public-key corresponds two or more public-keys.

Pro-3. Finding the public-key $B_1$ is impossible from the public-key string $s_i$, that is, no way for $s_i \rightarrow T_{\text{code}}(B_1) \rightarrow B_1$. Also, no way for $s'_i \rightarrow T_{\text{code}}(B_{10}) \rightarrow B_{10}$.

Pro-4. In the topological structure matching $(B_1, B_{10})$, finding a private-key $B_{10}$ will meet the Graph isomorphic Problem, a NP-problem as known, since there are two or more private-keys like $B_{10}$ to match with a public-key like $B_1$.

Pro-5. Finding the coloring for a private-key will be facing thousands of colorings of topological coding. And no algorithm is for finding out all colorings for a graph having huge numbers of vertices and edges.

The above problems tell us that topological number-based strings will have applications in the age of quantum computers.

Example 2. Let $S_{\text{string}}(B_1) = \{s_1(43), s_2(43), \ldots, s_M(43)\}$ with $M = (30)!$ be the set of topological number-based strings generated from the Topcode-matrix $T_{\text{code}}(B_1)$ shown in Eq. (2), where each topological number-based string $s_i$ has 43 bytes.

Then we have compound number-based strings $L_r = s_{r,1}(43)s_{r,2}(43) \cdots s_{r,M}(43)$ for $r \in \{1, 2, \ldots, M!\}$ with longer bytes, such that each string $L_r$ has $43 \cdot M = 43 \cdot (30)!$ bytes, in total. It is a proof for our techniques having broad application potential.

1.3 Main works

Motivated from lattice-based cryptography, the authors in [12] [13] [11] [10] [9] [15] [25] have proposed graphic lattices and shown many researching results on graphic lattices in topological coding.

In this article, we will make new RLA-algorithms for new colorings: odd-edge graceful-difference total coloring, odd-edge edge-difference total coloring, odd-edge edge-magic total coloring, and
odd-edge felicitous-difference total coloring. Moreover, we will design RLA-algorithms for adding randomly leaves to graphs continuously. We will build up caterpillar-graphic lattices and complementary graphic lattices made by uniformly W-magic total colorings, and show some connections between graphic lattices and integer lattices, and analyze the complexity of graph lattices introduced here.

1.4 Basic notations and definitions

For simplicity and accuracy, we will apply the standard terminology and notation in [1] and [2] in this article. All graphs mentioned here are simple, also, they have no loop and multiple-edge. Others are as follows:

- The notation $[a, b]$ indicates an integer set $\{a, a+1, \ldots, b\}$ with integers $a, b$ holding $0 \leq a < b$.
- $[a, b]^o$ denotes an odd-set $\{m, m+2, \ldots, n\}$ with odd integers $m, n$ with respect to $1 \leq m < n$.

- The number of elements of a set $X$ is written as $|X|$.
- $N(u)$ is the set of vertices adjacent with a vertex $u$, and the number $\deg_G(u) = |N(u)|$ is called the degree of the vertex $u$. The maximum degree $\Delta(G) = \max\{\deg_G(u) : u \in V(G)\}$, and the minimum degree $\delta(G) = \min\{\deg_G(u) : u \in V(G)\}$.
- A leaf is a vertex $x$ having its degree $\deg(x) = 1 = |N(x)|$.
- A $(p, q)$-graph having $p$ vertices and $q$ edges.
- The sentence “adding a leaf $w$ to a graph $G$” is an graph operation defined by adding a new vertex $w$ to $G$, and join $w$ with a vertex $x$ of $G$ by an edge $xw$, the resultant graph is denoted as $G + xw$, called leaf-added graph, such that $w$ is a leaf of $G + xw$.
- Let $G$ and $H$ be two graphs. If $H = G + xy - uv$ for edge $uv \in E(G)$ and edge $xy \not\in E(G)$, then we call $H \leq_{\Delta} -$-dual of $G$, also, added-edge-removed graph.

Definition 1. A lattice $L(B)$ defined as

$$L(B) = \{x_1b_1 + x_2b_2 + \cdots + x_nb_n : x_i \in \mathbb{Z}\}$$

is a set of all integer combinations of $n$ linearly independent vectors of a base $B = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^m$ with $n \leq m$, where $Z$ is the integer set, $m$ is the dimension and $n$ is the rank of the lattice, and $B$ is called lattice base. Particularly, if each component $b_{k,j}$ of each vector $b_k = (b_{k,1}, b_{k,2}, \ldots, b_{k,m})$ of the lattice base $L(B)$ is an integer, we get an integer lattice, denoted as $L(ZB)$. □

In the view of geometry, a lattice is a set of discrete points with periodic structure in $\mathbb{R}^m$. For no confusion, we call $L(B)$ defined in Definition 1 traditional lattice in the following discussion.

Definition 2. [17] A Topcode-matrix (or topological coding matrix) is defined as

$$T_{\text{code}} = \begin{pmatrix} x_1 & x_2 & \cdots & x_q \\ e_1 & e_2 & \cdots & e_q \\ y_1 & y_2 & \cdots & y_q \end{pmatrix}_{3 \times q} = \begin{pmatrix} X \\ E \\ Y \end{pmatrix} = (X, E, Y)^T$$

(9)
where \( v \)-vector \( X = (x_1, x_2, \ldots, x_q) \), \( e \)-vector \( E = (e_1, e_2, \ldots, e_q) \), and \( v \)-vector \( Y = (y_1, y_2, \ldots, y_q) \) consist of non-negative integers \( e_i, x_i \) and \( y_i \) for \( i \in [1, q] \). We say \( T_{\text{code}} \) to be evaluated if there exists a function \( \theta \) such that \( e_i = \theta(x_i, y_i) \) for \( i \in [1, q] \), and call \( x_i \) and \( y_i \) to be the ends of \( e_i \), and \( q \) the size of \( T_{\text{code}} \).

1.5 Particular trees and complex graphs

Recall, a tree \( T \) has that any pair of two vertices can be connected by a unique path, each vertex of degree one is called a leaf in \( T \). If removing all leaves of a tree produces a path \( P = u_1u_2 \cdots u_n \) of \( n \) vertices for \( n \geq 1 \), we call this tree caterpillar, and call the path \( P \) spine path of the caterpillar. If removing some leaves of a tree produces a caterpillar, then we call this tree lobster.

**Definition 3.** * There are four caterpillars \( H, T, T^* \) and \( G \) in the following paragraphs:

- Let \( L_{\text{leaf}}(H) \) be the set of all leaves of the caterpillar \( H \). The delation of all leaves of \( H \) makes a graph, denoted by \( H - L_{\text{leaf}}(H) \). By the definition of a caterpillar, we have \( H - L_{\text{leaf}}(H) = P = u_1u_2 \cdots u_n \), where \( P \) is the spine path of \( H \). Let the set of leaves adjacent with a vertex \( u_i \in V(P) \) be denoted as \( L_{\text{leaf}}(u_i) = \{v_{ij} : j \in [1, a_i]\} \) for \( i \in [1, n] \), where integer \( a_i \geq 0 \). So the leaf set \( L_{\text{leaf}}(H) = \bigcup_{i=1}^{n} L_{\text{leaf}}(u_i) \), such that \( V(H) = V(P) \cup L_{\text{leaf}}(H) \), the caterpillar \( H \) has \( m \) leaves, where \( m = \sum_{i=1}^{n} |L_{\text{leaf}}(u_i)| = \sum_{i=1}^{n} a_i \).

- The caterpillar \( T \) has the spine path \( P' = x_1x_2 \cdots x_n \) for \( n \geq 1 \), and each vertex \( x_i \) of the spine path \( P' \) has the leaf set \( L_{\text{leaf}}(x_i) = \{y_{ij} : j \in [1, b_i]\} \) with \( i \in [1, n] \). If the leaf sets of two caterpillars \( H \) and \( T \) hold \( |L_{\text{leaf}}(u_i)| + |L_{\text{leaf}}(x_i)| = M \) for \( i \in [1, n] \), we say two caterpillars \( H \) and \( T \) to be uniform \( M \)-leaf complementary trees.

- Let \( P'' = x'_1x'_2 \cdots x'_{n'} \) be the spine path of the caterpillar \( T^* \), and each vertex \( x'_i \) of the spine path \( P'' \) has the leaf set \( L_{\text{leaf}}(x'_i) = \{y'_{ij} : j \in [1, b'_i]\} \) with \( i \in [1, n'] \). Suppose \( x'_{j_1}, x'_{j_2}, \ldots, x'_{j_n} \) is a permutation of vertices \( x'_1, x'_2, \ldots, x'_{n'} \), if the leaf sets of two caterpillars \( H \) and \( T^* \) hold \( |L_{\text{leaf}}(u_i)| + |L_{\text{leaf}}(x'_{j_i})| = M \) for \( i \in [1, n] \), we call two caterpillars \( H \) and \( T^* \) to be \( M \)-leaf complementary trees.

- Assume that \( P'''' = s_1s_2 \cdots s_n \) is the spine path of the caterpillar \( G \), and each vertex \( s_i \) of the spine path \( G \) has its own leaf set \( L_{\text{leaf}}(s_i) = \{t_{ij} : j \in [1, c_i]\} \) for \( i \in [1, n] \). If the leaf sets of three caterpillars \( H, T \) and \( G \) satisfy \( |L_{\text{leaf}}(u_i)| + |L_{\text{leaf}}(x_i)| = |L_{\text{leaf}}(s_i)| \) for \( i \in [1, n] \), then the caterpillar \( G \) is called universal graph of each of two caterpillars \( H \) and \( T \), and two caterpillars \( H \) and \( T \) are \( G \)-complementary trees about the caterpillar \( G \). By the graph operation of view, we coincide two spine paths of two caterpillars \( H \) and \( T \) into one, and then get the caterpillar \( G \). □

Computing the number of caterpillars obtained by adding \( m \) leaves will meet the Integer Partition Problem, this is not an easy work.

**Definition 4.** [8] A complex graph \( G \) has its own vertex set \( V(G) = X^o \cup Y^\Box \) with \( X^o \cap Y^\Box = \emptyset \), \( X^o \neq \emptyset \) and \( Y^\Box \neq \emptyset \), such that the degree \( \deg_G(u) \geq 0 \) for each vertex \( u \in X^o \), and the image-degree
\( \text{deg}_G(v) < 0 \) for each vertex \( v \in Y \). Moreover, a vertex \( x \) of the complex graph \( G \) is adjacent with \( m \) leaves, then we define the leaf-degree \( l_{\text{leaf}}(x) = m \) if \( x \in X^o \), and the leaf-image-degree \( l_{\text{leaf}}(x) = -m = mi^2 \) if \( x \in Y \), where \( i^2 = -1 \).

In Fig. 2 each vertex in the spine path of the caterpillar \( A_1 \) has its own leaf-degree \( l_{\text{leaf}}(u_1) = 7 \), \( l_{\text{leaf}}(u_2) = 3 \), \( l_{\text{leaf}}(u_3) = 0 \), \( l_{\text{leaf}}(u_4) = 3 \), \( l_{\text{leaf}}(u_5) = 0 \) and \( l_{\text{leaf}}(u_6) = 6 \), so the caterpillar \( A_1 \) has its own leaf-degree sequence \( d_{\text{leaf}}(A_1) = (7, 3, 0, 3, 0, 6) \). Each vertex in the spine path of the caterpillar \( A_2 \) has its own leaf-degree \( l_{\text{leaf}}(x_1) = 7 \), \( l_{\text{leaf}}(x_2) = 3 \), \( l_{\text{leaf}}(x_3) = 0 \), \( l_{\text{leaf}}(x_4) = 3 \), \( l_{\text{leaf}}(x_5) = 0 \) and \( l_{\text{leaf}}(x_6) = 6 \), then the caterpillar \( A_2 \) has its own leaf-degree sequence \( d_{\text{leaf}}(A_2) = (7, 3, 0, 3, 0, 6) \).

In Fig. 3 each vertex in the spine path of the caterpillar \( B_1 \) has its own leaf-degree \( l_{\text{leaf}}(v_1) = 1 \), \( l_{\text{leaf}}(v_2) = 5 \), \( l_{\text{leaf}}(v_3) = 8 \), \( l_{\text{leaf}}(v_4) = 5 \), \( l_{\text{leaf}}(v_5) = 8 \) and \( l_{\text{leaf}}(v_6) = 5 \), so the caterpillar \( B_1 \) has its own leaf-degree sequence \( d_{\text{leaf}}(B_1) = (1, 5, 8, 5, 8, 2) \). Each vertex in the spine path of the caterpillar \( B_2 \) has its own leaf-degree \( l_{\text{leaf}}(y_1) = -2 \), \( l_{\text{leaf}}(y_2) = 8 \), \( l_{\text{leaf}}(y_3) = 5 \), \( l_{\text{leaf}}(y_4) = 8 \), \( l_{\text{leaf}}(y_5) = 5 \) and \( l_{\text{leaf}}(y_6) = -1 \), then the caterpillar \( B_2 \) has its own leaf-degree sequence \( d_{\text{leaf}}(B_2) = (-2, 8, 5, 8, 5, -1) \).

It is noticeable, Fig. 2 and Fig. 3 show us two groups of isomorphic caterpillars, that is, \( A_1 \cong A_2 \) and \( B_1 \cong B_2 \); the caterpillar \( A_1 \) and the caterpillar \( B_1 \) are the uniformly 8-leaf complement trees; and the caterpillar \( A_2 \) and the caterpillar \( B_2 \) are the 8-leaf complement trees, since there are matchings \( y_1 \leftrightarrow x_6, y_2 \leftrightarrow x_3, y_3 \leftrightarrow x_2, y_4 \leftrightarrow x_5, y_5 \leftrightarrow x_4 \) and \( y_6 \leftrightarrow x_1 \).

Figure 2: Two caterpillars \( A_1 \) and \( A_2 \) have the same topological structure, that is, \( A_1 \cong A_2 \).

Figure 3: Two caterpillars \( A_1 \) and \( B_1 \) are the uniformly 8-leaf complement trees, two caterpillars \( A_2 \) and \( B_2 \) are the 8-leaf complement trees, and \( B_1 \equiv B_2 \), where two caterpillars \( A_1 \) and \( A_2 \) are shown in Fig. 2.
Problem 1. Suppose that a caterpillar $H$ (as a public-key) and another caterpillar $T$ (as a private-key) have the spine paths of the same length, find the conditions (as a topological authentication) if these two caterpillars are $M$-leaf complement trees defined in Definition 3.

Problem 2. Suppose that a complex graph $G$ (as a public-key) and another complex graph $J$ (as a private-key) have the same number of vertices. The complex graph $G$ has its own degree sequence $\deg(G) = (\deg_G(u_1), \deg_G(u_2), \ldots, \deg_G(u_n))$, and the complex graph $J$ has its own degree sequence $\deg(J) = (\deg_J(v_1), \deg_J(v_2), \ldots, \deg_J(v_n))$. If there is a constant $M$, such that $\deg_G(u_i) + \deg_J(v_i) = M$ for $i \in [1,n]$, we call two complex graphs $G$ and $J$ uniformly $M$-complement complex graph matching, denoted as $M_{\text{comp}}(G,J)$. Characterize each uniformly $M$-complement complex graph matching $M_{\text{comp}}(G,J)$.

2 New colorings and dual-type colorings

2.1 Basic labelings and colorings

Definition 5. Suppose that a connected $(p,q)$-graph $G$ admits a mapping $\theta : V(G) \rightarrow \{0,1,2,\ldots\}$. For each edge $xy \in E(G)$, the induced edge color is defined as $\theta(xy) = |\theta(x) - \theta(y)|$. Write vertex color set by $\theta(V(G)) = \{\theta(u) : u \in V(G)\}$, and edge color set by $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\}$. There are the following constraint conditions:

C-1. $|\theta(V(G))| = p$;
C-2. $\theta(V(G)) \subseteq [0,q]$, $\min \theta(V(G)) = 0$;
C-3. $\theta(V(G)) \subseteq [0,2q-1]$, $\min \theta(V(G)) = 0$;
C-4. $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\} = [1,q]$;
C-5. $\theta(E(G)) = \{\theta(xy) : xy \in E(G)\} = [1,2q-1]^p$;
C-6. $G$ is a bipartite graph with vertex bipartition $(X,Y)$ such that $\max \{\theta(x) : x \in X\} < \min \{\theta(y) : y \in Y\}$ (max $\theta(X)$ < min $\theta(Y)$ for short);
C-7. $G$ is a tree having a perfect matching $M$ holding $\theta(x) + \theta(y) = q$ for each matching edge $xy \in M$; and
C-8. $G$ is a tree having a perfect matching $M$ holding $\theta(x) + \theta(y) = 2q - 1$ for each matching edge $xy \in M$.

Then:

Lab-1. A graceful labeling $\theta$ satisfies C-1, C-2 and C-4 at the same time.
Lab-2. A set-ordered graceful labeling $\theta$ holds C-1, C-2, C-4 and C-6 true.
Lab-3. A strongly graceful labeling $\theta$ holds C-1, C-2, C-4 and C-7 true.
Lab-4. A set-ordered strongly graceful labeling $\theta$ holds C-1, C-2, C-4, C-6 and C-7 true.
Lab-5. An odd-graceful labeling $\theta$ holds C-1, C-3 and C-5 true.
Lab-6. A set-ordered odd-graceful labeling $\theta$ abides C-1, C-3, C-5 and C-6.
Lab-7. A strongly odd-graceful labeling $\theta$ holds C-1, C-3, C-5 and C-8, simultaneously.
Lab-8. A set-ordered strongly odd-graceful labeling $\theta$ holds C-1, C-3, C-5, C-6 and C-8 true. □
Definition 6. * In Definition 5 if $|\theta(V(G))| < p$ holds true, we get
(i) A graceful coloring $\theta$ satisfies C-2 and C-4 defined in Definition 5.
(ii) A set-ordered graceful coloring $\theta$ satisfies C-2, C-4 and C-6 defined in Definition 5.
(iii) An odd-graceful coloring $\theta$ satisfies C-3 and C-5 defined in Definition 5.
(ii) A set-ordered odd-graceful coloring $\theta$ satisfies C-3, C-5 and C-6 defined in Definition 5. □

Definition 7. [10] Suppose that a connected $(p,q)$-graph $G \neq K_p$ admits a total coloring $f : V(G) \cup E(G) \rightarrow [1,M]$, and there are $f(x) = f(y)$ for some pairs of vertices $x, y \in V(G)$. Write $f(S) = \{f(w) : w \in S\}$ for a non-empty set $S \subseteq V(G) \cup E(G)$ and let $k$ be a fixed positive integer. There are the following constraint conditions:

1. $|f(V(G))| < p$;
2. $|f(E(G))| = q$;
3. $f(V(G)) \subseteq [1,M]$, min $f(V(G)) = 1$;
4. $f(V(G)) \subset [1,2q + 1]$, min $f(V(G)) = 1$;
5. $f(E(G)) = [1,q]$;
6. $f(E(G)) = [0,q - 1]$;
7. $f(E(G)) = [1,2q - 1]^q$;
8. $f(E(G)) = [2,2]^q$;
9. $f(E(G)) = [c,c + q - 1]$;
10. $f(uv) = |f(u) - f(v)|$ for each edge $uv \in E(G)$;
11. $f(uv) = f(u) + f(v)$ for each edge $uv \in E(G)$;
12. For each edge $uv \in E(G)$, $f(uv) = f(u) + f(v)$ when $f(u) + f(v)$ is even, and $f(uv) = f(u) + f(v) + 1$ when $f(u) + f(v)$ is odd;
13. $f(uv) = (f(u) + f(v)) \mod q$ for each edge $uv \in E(G)$;
14. $f(uv) = (f(u) + f(v)) \mod 2q$ for each edge $uv \in E(G)$;
15. $f(uv) + |f(u) - f(v)| = k$ for each edge $uv \in E(G)$;
16. $|f(uv) - |f(u) - f(v)|| = k$ for each edge $uv \in E(G)$;
17. $|f(u) + f(v) - f(uv)| = k$ for each edge $uv \in E(G)$;
18. $f(u) + f(uv) + f(v) = k$ for each edge $uv \in E(G)$;
19. There exists an integer $k$ so that $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$ for each edge $uv \in E(G)$; and
20. $(X,Y)$ is the bipartition of a bipartite graph $G$ such that $\max f(X) < \min f(Y)$.

A $W$-type coloring $f$ is one of the following colorings:

1. TCL-1. An edge-gracefully total coloring if (1*), (2*) and (3*) hold true.
2. TCL-2. A set-ordered edge-gracefully total coloring if (1*), (2*), (5*) and (20*) hold true.
3. TCL-3. An edge-odd-gracefully total coloring if (1*), (4*) and (7*) hold true.
4. TCL-4. A set-ordered edge-odd-gracefully total coloring if (1*), (4*), (7*) and (20*) hold true.
5. TCL-5. A gracefully total coloring if (1*), (3*), (5*) and (10*) hold true.
6. TCL-6. A set-ordered gracefully total coloring if (1*), (3*), (5*) and (10*) and (20*) hold true.
7. TCL-7. An odd-gracefully total coloring if (1*), (4*), (7*) and (10*) hold true.
TCL-8. A set-ordered odd-gracefully total coloring if $[7^*], [4^*], [7^*], [10^*]$ and $[20^*]$ hold true.
TCL-9. A felicitous total coloring if $[8^*], [13^*]$ and $[6^*]$ hold true.
TCL-10. A set-ordered felicitous total coloring if $[3^*], [13^*], [6^*]$ and $[20^*]$ hold true.
TCL-11. An odd-elegant total coloring if $[4^*], [14^*]$ and $[7^*]$ hold true.
TCL-12. A set-ordered odd-elegant total coloring if $[4^*], [14^*], [7^*]$ and $[20^*]$ hold true.
TCL-13. A harmonious total coloring if $[5^*], [13^*]$ and $[6^*]$ hold true, and when $G$ is a tree, exactly one edge color may be used on two vertices.
TCL-14. A set-ordered harmonious total coloring if $[3^*], [13^*], [6^*]$ and $[20^*]$ hold true.
TCL-15. A strongly harmonious total coloring if $[3^*], [13^*], [6^*]$ and $[19^*]$ hold true.
TCL-16. A properly even harmonious total coloring if $[4^*], [8^*]$ and $[14^*]$ hold true.
TCL-17. A $c$-harmonious total coloring if $[3^*], [11^*]$ and $[9^*]$ hold true.
TCL-18. An even sequential harmonious total coloring if $[4^*], [12^*]$ and $[8^*]$ hold true.
TCL-19. A pan-harmonious total coloring if $[2^*]$ and $[11^*]$ hold true.
TCL-20. An edge-magic total coloring if $[18^*]$ holds true.
TCL-21. A set-ordered edge-magic total coloring if $[18^*]$ and $[20^*]$ hold true.
TCL-22. A gracefully edge-magic total coloring if $[5^*]$ and $[18^*]$ hold true.
TCL-23. A set-ordered graceful edge-magic total coloring if $[5^*], [18^*]$ and $[20^*]$ hold true.
TCL-24. An edge-difference total coloring if $[15^*]$ holds true.
TCL-25. A set-ordered edge-difference total coloring if $[15^*]$ and $[20^*]$ hold true.
TCL-26. A graceful edge-difference total coloring if $[5^*]$ and $[15^*]$ hold true.
TCL-27. A set-ordered graceful edge-difference total coloring if $[5^*], [15^*]$ and $[20^*]$ hold true.
TCL-28. A felicitous-difference total coloring if $[1^*], [2^*]$ and $[17^*]$ hold true.
TCL-29. A set-ordered felicitous-difference total coloring if $[1^*], [2^*], [17^*]$ and $[20^*]$ hold true.
TCL-30. An graceful-difference total coloring if $[16^*]$ holds true.
TCL-31. A set-ordered graceful-difference total coloring if $[16^*]$ and $[20^*]$ hold true.
TCL-32. An edge-graceful graceful-difference total coloring if $[5^*]$ and $[16^*]$ hold true.
TCL-33. A set-ordered edge-graceful graceful-difference total coloring if $[5^*], [16^*]$ and $[20^*]$ hold true.

2.2 New labelings and colorings

For the convenience of statement, the word “magic-type” is as the same as the word “$W$-magic” in the following discussion.

Definition 8. * Let $G$ be a bipartite $(p,q)$-graph with vertex bipartition $(X,Y)$, then $V(G) = X \cup Y$ with $X \cap Y = \emptyset$. There are four labelings defined as follows:

(i) A set-ordered odd-edge edge-magic total labeling is a mapping $f : V(G) \cup E(G) \rightarrow [0, 2q - 1]$, such that $f(x) \neq f(y)$ for $x,y \in V(G)$, and the set-ordered restriction $\max f(X) < \min f(Y)$ holds true, and the edge color set $f(E(G)) = [1, 2q - 1]^o$, as well as each edge $uv \in E(G)$ holds $f(u) + f(uv) + f(v) = k_1$, where $k_1$ is a positive integer.
(ii) A set-ordered odd-edge edge-difference total labeling is a mapping $g : V(G) \cup E(G) \rightarrow [0, 2q - 1]$, such that $g(x) \neq g(y)$ for $x, y \in V(G)$, the set-ordered restriction $\max g(X) < \min g(Y)$ holds true, and the edge color set $g(E(G)) = [1, 2q - 1]^o$, as well as each edge $uv \in E(G)$ satisfies $g(uv) + |g(u) - g(v)| = k_2$, where $k_2$ is a positive integer.

(iii) A set-ordered odd-edge felicitous-difference total labeling is a mapping $h : V(G) \cup E(G) \rightarrow [0, 2q - 1]$, such that $h(x) \neq h(y)$ for $x, y \in V(G)$, the set-ordered restriction $\max h(X) < \min h(Y)$ holds true, and the edge color set $h(E(G)) = [1, 2q - 1]^o$, as well as each edge $uv \in E(G)$ satisfies $|h(u) + h(v) - h(uv)| = k_3$, where $k_3$ is a non-negative integer.

(vi) A set-ordered odd-edge graceful-difference total labeling is a mapping $\alpha : V(G) \cup E(G) \rightarrow [0, 2q - 1]$, such that $\alpha(x) \neq \alpha(y)$ for $x, y \in V(G)$, the set-ordered restriction $\max \alpha(X) < \min \alpha(Y)$ holds true, and the edge color set $\alpha(E(G)) = [1, 2q - 1]^o$, as well as each edge $uv \in E(G)$ satisfies $|\alpha(u) - \alpha(v)| + \alpha(uv) = k_4$, where $k_4$ is a non-negative integer.

Definition 9. * Let “W-magic” be one of edge-magic, edge-difference, felicitous-difference, graceful-difference. We will obtain four odd-edge W-magic total labelings if we remove the restriction “set-ordered” from Definition 8. If we allow that there is at least a pair of vertices colored with the same color in Definition 8, we will obtain four odd-edge W-magic total colorings (see Fig.4). ☐

Example 3. Fig.4 is for illustrating Definition 8 and Definition 9, we can see:

(a) The graph $M_0$ admits a set-ordered odd-graceful coloring $f_0$, since there are two vertices colored with 25. And $f_0(E(M_0)) = \{f_0(xy) = f_0(y) - f_0(x) : xy \in E(M_0)\} = [1, 31]^o$.

(b) The graph $M_1$ admits a set-ordered odd-edge edge-magic total coloring $f_1$, since there are two vertices colored with 25. Each edge $xy \in E(M_1)$ holds $f_1(x) + f_1(xy) + f_1(y) = 42$ true.

(c) The graph $M_2$ admits a set-ordered odd-edge edge-difference total coloring $f_2$, since there are two vertices colored with 6. Each edge $xy \in E(M_2)$ holds $f_2(xy) + |f_2(x) - f_2(y)| = f_2(xy) + f_2(y) - f_2(x) = 32$ true.

(d) The graph $M_3$ admits a set-ordered odd-edge felicitous-difference total coloring $f_3$, since there are two vertices colored with 25. Each edge $xy \in E(M_3)$ holds $|f_3(x) + f_3(y) - f_3(xy)| = 10$ true.

(e) The graph $M_4$ admits a set-ordered odd-edge graceful-difference total coloring $f_4$, since there are two vertices colored with 17. Each edge $xy \in E(M_4)$ holds $|f_4(x) - f_4(y)| = f_4(x) - f_4(y) - f_4(xy) = 0$ true. ☐

Example 4. Fig.5 is for illustrating Definition 5, Definition 8 and Definition 9, there are:

(a) The graph $R_1$ admits a set-ordered graceful labeling $g_1$, such that the edge color set $g_1(E(R_1)) = \{g_1(xy) = g_1(y) - g_1(x) : xy \in E(R_1)\} = [1, 13]$.

(a-1) The graph $O_1$ admits a set-ordered odd-graceful labeling $g_1'$, such that the edge color set $g_1'(E(O_1)) = \{g_1'(xy) = g_1'(y) - g_1'(x) : xy \in E(O_1)\} = [1, 25]^o$.

(b) The graph $R_2$ admits a set-ordered felicitous-difference total labeling $g_2$, such that each edge $xy \in E(R_2)$ holds $|g_2(x) + g_2(y) - g_2(xy)| = 7$ true, and the edge color set $g_2(E(O_2)) = [1, 13]$. 

(b-2) The graph $O_2$ admits a set-ordered odd-edge felicitous-difference total labeling $g'_2$, such that each edge $xy \in E(O_2)$ holds $|g'_2(x) + g'_2(y) - g'_2(xy)| = 7$ true, and the edge color set $g'_2(E(O_2)) = [1,25]^o$.

(c) The graph $R_3$ admits a set-ordered edge-magic total labeling $g_3$, such that each edge $xy \in E(R_3)$ holds $g_3(x) + g_3(xy) + g_3(y) = 21$ true, and the edge color set $g_3(E(O_3)) = [1,13]$.

(c-1) The graph $O_3$ admits a set-ordered odd-edge edge-magic total labeling $g'_3$, such that each edge $xy \in E(O_3)$ holds $g'_3(x) + g'_3(xy) + g'_3(y) = 40$ true, and the edge color set $g'_3(E(O_3)) = [1,25]^o$.

(d) The graph $R_4$ admits a set-ordered odd-edge graceful-difference total labeling $g_4$, such that each edge $xy \in E(R_4)$ holds $|g_4(x) - g_4(y)| = |g_4(xy)| = |g_4(y) - g_4(x)| = |g_4(xy)| - g_4(xy) = 0$ true, and the edge color set $g_4(E(O_4)) = [1,13]$.

(d-1) The graph $O_4$ admits a set-ordered odd-edge graceful-difference total labeling $g'_4$, such that each edge $xy \in E(O_4)$ holds $|g'_4(x) - g'_4(y)| = |g'_4(xy)| = |g'_4(y) - g'_4(x)| - g'_4(xy) = 0$ true, and the edge color set $g'_4(E(O_4)) = [1,25]^o$.

(e) The graph $R_5$ admits a set-ordered edge-difference total labeling $g_5$, such that each edge $xy \in E(R_5)$ holds $g_5(x) + |g_5(x) - g_5(y)| = g_5(xy) + g_5(y) - g_5(x) = 14$ true, and the edge color set $g_5(E(O_5)) = [1,13]$.

(e-1) The graph $O_5$ admits a set-ordered odd-edge edge-difference total labeling $g'_5$, such that each edge $xy \in E(O_5)$ holds $g'_5(x) + |g'_5(x) - g'_5(y)| = g'_5(xy) + g'_5(y) - g'_5(x) = 26$ true, and the edge color set $g'_5(E(O_5)) = [1,25]^o$. □

Example 5. We present Fig.6 for illustrating Definition 5 and Definition 9 there are:

(A) The tree $T_1$ with a perfect matching $M(T_1)$ admits a set-ordered strongly graceful labeling $h_1$, such that each matching edge $uv \in M(T_1)$ holds $h_1(u) + h_1(v) = 11$ true, and the edge color set $h_1(E(T_1)) = \{h_1(xy) = h_1(y) - h_1(x) : xy \in E(T_1)\} = [1,11]$.

(A-1) The tree $P_1$ with a perfect matching $M(P_1)$ admits a set-ordered strongly odd-graceful labeling $h'_1$, such that each matching edge $uv \in M(P_1)$ holds $h'_1(u) + h'_1(v) = 21$ true, and the edge color set $h'_1(E(P_1)) = \{h'_1(xy) = h'_1(y) - h'_1(x) : xy \in E(P_1)\} = [1,21]^o$.

(B) The tree $T_2$ with a perfect matching $M(T_2)$ admits a set-ordered felicitous-difference total labeling $h_2$: (i) each edge $xy \in E(T_2)$ holds $|h_2(x) + h_2(y) - h_2(xy)| = 5$ true; (ii) the edge color
set $h_2(E(T_2)) = [1, 11]$; and (iii) the matching edge set \{h_2(u) + h_2(v) : uv \in M(T_2)\} = \{6, 8, 10, 12, 14, 16\}.

(B) The tree $P_2$ with a perfect matching $M(P_2)$ admits a set-ordered odd-edge felicitous-difference total labeling $h'_2$: (i) each edge $xy \in E(P_2)$ holds $|h'_2(x) + h'_2(y) - h'_2(xy)| = 10$ true; (ii) the edge color set $h'_2(E(P_2)) = [1, 21]$; and (iii) the matching edge set \{h'_2(u) + h'_2(v) : uv \in M(P_2)\} = \{11, 15, 19, 23, 17, 31\}.

(C) The tree $T_3$ with a perfect matching $M(T_3)$ admits a set-ordered edge-magic total coloring $h_3$: (i) each edge $xy \in E(T_3)$ holds $h_3(x) + h_3(xy) + h_3(y) = 17$ true; (ii) the edge color set $h_3(E(T_3)) = [1, 11]$; and (iii) the matching edge set \{h_3(u) + h_3(v) : uv \in M(T_3)\} = \{6, 8, 10, 12, 14, 16\}.

(C-1) The tree $P_3$ with a perfect matching $M(P_3)$ admits a set-ordered odd-edge edge-magic total coloring $h'_3$: (i) each edge $xy \in E(P_3)$ holds $h'_3(x) + h'_3(xy) + h'_3(y) = 32$ true; (ii) the edge color set $h'_3(E(P_3)) = [1, 21]$; and (iii) the matching edge set \{h'_3(u) + h'_3(v) : uv \in M(P_3)\} = \{11, 15, 19, 23, 17, 31\}.

(D) The tree $T_4$ with a perfect matching $M(T_4)$ with a perfect matching $M(T_4)$ admits a set-ordered strongly graceful-difference total labeling $h_4$: (i) each edge $xy \in E(T_4)$ holds $|h_4(x) - h_4(y)| - h_4(xy) = 0$ true; (ii) the edge color set $h_4(E(T_4)) = [1, 11]$; and (iii) each matching edge $uv \in M(T_4)$ holds $h_4(u) + h_4(v) = 11$ true.

(D-1) The tree $P_4$ with a perfect matching $M(P_4)$ with a perfect matching $M(P_4)$ admits a set-ordered strongly odd-edge graceful-difference total labeling $h'_4$: (i) each edge $xy \in E(P_4)$ holds $|h'_4(x) - h'_4(y) - h'_4(xy)| = 0$ true; (ii) the edge color set $h'_4(E(P_4)) = [1, 21]$; and (iii) each
matching edge $uv \in M(P_4)$ holds $h'_4(u) + h'_4(v) = 21$ true.

(E) The tree $T_5$ with a perfect matching $M(T_5)$ admits a set-ordered strongly edge-difference total labeling $h_5$: (i) each edge $xy \in E(T_5)$ holds $h_5(xy) + |h_5(x) - h_5(y)| = 12$ true; (ii) the edge color set $h_5(E(T_5)) = [1,11]$; and (iii) each matching edge $uv \in M(T_5)$ holds $h_5(u) + h_5(v) = 11$ true.

(E-1) The tree $P_5$ with a perfect matching $M(P_5)$ admits a set-ordered strongly odd-edge edge-difference total labeling $h'_5$: (i) each edge $xy \in E(P_5)$ holds $h'_5(xy) + |h'_5(x) - h'_5(y)| = 22$ true; (ii) the edge color set $h'_5(E(P_5)) = [1,21]^o$; and (iii) each matching edge $uv \in M(P_5)$ holds $h'_5(u) + h'_5(v) = 21$ true.

Figure 6: For basic-definition-perfect-matching.

Definition 10. * Let $G$ be a $(p,p-1)$-tree with a perfect matching $M(G)$ and the vertex bipartition $(X,Y)$ holding $V(G) = X \cup Y$ and $X \cap Y = \emptyset$ true. Suppose that $G$ admits a total labeling $f : V(G) \cup E(G) \rightarrow [0,M]$. Let $a,b,c$ be non-negative integers, there are the following restrictions:

\begin{itemize}
  \item [Sc-1. C-1.] $|f(V(G))| = p$;
  \item [Sc-2.] $f(V(G)) \subseteq [0,p-1]$, min $f(V(G)) = 0$;
  \item [Sc-3.] $f(V(G)) \subseteq [0,2p-3]$, min $f(V(G)) = 0$;
  \item [Sc-4.] $f(E(G)) = \{f(xy) : xy \in E(G)\} = [1,p-1]$;
  \item [Sc-5.] $f(E(G)) = \{f(xy) : xy \in E(G)\} = [1,2p-3]^o$;
  \item [Sc-6.] max $f(X) < \min f(Y)$;
  \item [Sc-7.] $|f(x) - f(y)| - f(xy)| = a$ for each edge $xy \in E(G)$;
  \item [Sc-8.] $f(xy) + |f(x) - f(y)| = b$ for each edge $xy \in E(G)$;
  \item [Sc-9.] $|f(x) - f(y)| - f(xy)| = c$ for each edge $xy \in E(G)$;
  \item [Sc-10.] $f(x) + f(y) + f(xy) = d$ for each edge $xy \in E(G)$;
  \item [Sc-11.] each matching edge $uv \in M(G)$ holds $f(u) + f(v) = e$ true.
\end{itemize}

We call $f$: 
Sp-1. A set-ordered strongly graceful-difference total labeling if Sc-1, Sc-2, Sc-4, Sc-6, Sc-7 and Sc-11 hold true.

Sp-2. A set-ordered strongly odd-edge graceful-difference total labeling if Sc-1, Sc-3, Sc-5, Sc-6, Sc-7 and Sc-11 hold true.

Sp-3. A set-ordered strongly edge-difference total labeling if Sc-1, Sc-2, Sc-4, Sc-6, Sc-8 and Sc-11 hold true.

Sp-4. A set-ordered strongly odd-edge edge-difference total labeling if Sc-1, Sc-3, Sc-5, Sc-6, Sc-8 and Sc-11 hold true.

Sp-5. A set-ordered strongly edge-magic total labeling if Sc-1, Sc-2, Sc-4, Sc-6, Sc-10 and Sc-11 hold true.

Sp-6. A set-ordered strongly odd-edge edge-magic total labeling if Sc-1, Sc-3, Sc-5, Sc-6, Sc-10 and Sc-11 hold true.

Sp-7. A set-ordered strongly felicitous-difference total labeling if Sc-1, Sc-2, Sc-4, Sc-6, Sc-9 and Sc-11 hold true.

Sp-8. A set-ordered strongly odd-edge felicitous-difference total labeling if Sc-1, Sc-3, Sc-5, Sc-6, Sc-9 and Sc-11 hold true.

We present the twin odd-edge $W$-magic total labelings as follows:

**Definition 11.** * Let $G$ be a bipartite $(p, q)$-graph having its own vertex set $V(G) = X_G \cup Y_G$ with $X_G \cap Y_G = \emptyset$, and let $T$ be another bipartite $(p', q)$-graph having its own vertex set $V(T) = X_T \cup Y_T$ with $X_T \cap Y_T = \emptyset$. The bipartite $(p, q)$-graph $G$ admits a total labeling $F : V(G) \cup E(G) \rightarrow [0, 2q - 1]$, and the bipartite $(p', q)$-graph $T$ admits a total labeling $F^* : V(T) \cup E(T) \rightarrow [0, 2q]$.

(i) If

(i-1) $F$ is a set-ordered odd-edge edge-magic total labeling of $G$;

(i-2) the set-ordered restriction $F^*_{\text{max}}(X_T) < F^*_{\text{min}}(Y_T)$ holds true;

(i-3) the edge color set $F^*(E(T)) = [1, 2q - 1]^o$;

(i-4) there is a positive integer $c_1$, so that each edge $xy \in E(T)$ holds a magic-type restriction $F^*(x) + F^*(xy) + F^*(y) = c_1$; and

(i-5) $F(V(G)) \cup F^*(V(T)) \subseteq [0, 2q]$,

then, we call $\langle F, F^* \rangle$ a twin set-ordered odd-edge edge-magic total labeling of two graphs $G$ and $T$.

Especially, we call $\langle F, F^* \rangle$ a perfect twin set-ordered odd-edge edge-magic total labeling of $G$ and $T$ if $F(V(G)) \cup F^*(V(T)) = [0, 2q]$.

(ii) If

(ii-1) $F$ is a set-ordered odd-edge edge-difference total labeling of $G$;

(ii-2) the set-ordered restriction $F^*_{\text{max}}(X_T) < F^*_{\text{min}}(Y_T)$ holds true;

(ii-3) the edge color set $F^*(E(T)) = [1, 2q - 1]^o$;

(ii-4) there is a positive integer $c_2$, so that each edge $xy \in E(T)$ holds a magic-type restriction $F^*(xy) + |F^*(y) - F^*(x)| = c_2$; and

(ii-5) $F(V(G)) \cup F^*(V(T)) \subseteq [0, 2q]$,

then, $\langle F, F^* \rangle$ is called a twin set-ordered odd-edge edge-difference total labeling of two graphs $G$ and
Moreover, we call \((F, F^*)\) a perfect twin set-ordered odd-edge edge-difference total labeling of \(G\) and \(T\) if \(F(V(G)) \cup F^*(V(T)) = [0, 2q]\).

(iii) \(F\)

(iii-1) \(F\) is a set-ordered odd-edge felicitous-difference total labeling of \(G\);

(iii-2) the set-ordered restriction \(F^*_{\text{max}}(X_T) < F^*_{\text{min}}(Y_T)\) holds true;

(iii-3) the edge color set \(F^*(E(T)) = [1, 2q - 1]^o\);

(iii-4) there is a non-negative integer \(c_3\), so that each edge \(xy \in E(T)\) holds a magic-type restriction \(|F^*(y) + F^*(x) - F^*(xy)| = c_3\); and

(iii-5) \(F(V(G)) \cup F^*(V(T)) \subseteq [0, 2q]\),

we call \((F, F^*)\) a twin set-ordered odd-edge felicitous-difference total labeling of two graphs \(G\) and \(T\). And we call \((F, F^*)\) a perfect twin set-ordered odd-edge felicitous-difference total labeling of \(G\) and \(T\) if \(F(V(G)) \cup F^*(V(T)) = [0, 2q]\).

(vi) \(F\)

(vi-1) \(F\) is a set-ordered odd-edge graceful-difference total labeling;

(vi-2) the set-ordered restriction \(F^*_{\text{max}}(X_T) < F^*_{\text{min}}(Y_T)\) holds true;

(vi-3) the edge color set \(F^*(E(T)) = [1, 2q - 1]^o\);

(vi-4) there is a non-negative integer \(c_4\), so that each edge \(xy \in E(T)\) holds a magic-type restriction \(||F^*(y) - F^*(x)| - F^*(xy)| = c_4|\); and

(vi-5) \(F(V(G)) \cup F^*(V(T)) \subseteq [0, 2q]\),

then, \((F, F^*)\) is called a twin set-ordered odd-edge graceful-difference total labeling of two graphs \(G\) and \(T\). Furthermore, we call \((F, F^*)\) a perfect twin set-ordered odd-edge graceful-difference total labeling of \(G\) and \(T\) if \(F(V(G)) \cup F^*(V(T)) = [0, 2q]\).

\(\square\)

Example 6. For illustrating Definition [11], we see examples shown in Fig. 7 as follows: Notice that \(A_0 \cong B_0 \cong A_i \cong B_i\) for \(i \in [1, 4]\). Let \(q = |E(A_0)| = 9\).

(a) A graph \(A_0\) admits a set-ordered odd-graceful labeling \(f_0\), and \(f_0(E(A_0)) = [1, 17]^o\).

(a-1) A graph \(B_0\) admits a set-ordered labeling \(g_0\) with \(g_0(E(B_0)) = [1, 17]^o\).

Since \(f_0(V(A_0)) \cup g_0(V(B_0)) = [0, 18] = [0, 2q]\), \((f_0, g_0)\) is a twin set-ordered odd-graceful labeling.

(b) A graph \(A_1\) admits a set-ordered odd-edge felicitous-difference total labeling \(f_1\) holding \(|f_1(x) + f_1(y) - f_1(xy)| = 8\) for each edge \(xy \in E(A_1)\) and \(f_1(E(A_1)) = [1, 17]^o\).

(b-1) A graph \(B_1\) admits a set-ordered odd-edge felicitous-difference total labeling \(g_1\) with \(|g_1(u) + g_1(v) - g_1(uv)| = 8\) for each edge \(uv \in E(B_1)\) and \(g_1(E(B_1)) = [1, 17]^o\).

Since \(f_1(V(A_1)) \cup g_1(V(B_1)) = [0, 18] = [0, 2q]\), \((f_1, g_1)\) is a twin set-ordered felicitous-difference total labeling.

(c) A graph \(A_2\) admits a set-ordered odd-edge edge-magic total labeling \(f_2\) holding \(f_2(x) + f_2(xy) + f_2(y) = 26\) for each edge \(xy \in E(A_2)\) and \(f_2(E(A_2)) = [1, 17]^o\).

(c-1) A graph \(B_2\) admits a set-ordered odd-edge edge-magic total labeling \(g_2\) with \(g_2(u) + g_2(uv) + g_2(v) = 28\) for each edge \(uv \in E(B_2)\) and \(g_2(E(B_2)) = [1, 17]^o\).

Since \(f_2(V(A_2)) \cup g_2(V(B_2)) = [0, 18] = [0, 2q]\), \((f_2, g_2)\) is a twin set-ordered edge-magic total labeling.
(d) A graph $A_3$ admits a set-ordered odd-edge edge-difference total labeling $f_3$ holding $f_3(xy) + |f_3(y) - f_3(x)| = 18$ for each edge $xy \in E(A_3)$ and $f_3(E(A_3)) = [1, 17]^0$.

(d-1) A graph $B_3$ admits a set-ordered odd-edge edge-difference total labeling $g_3$ with $g_3(uv) + |g_3(u) - g_3(v)| = 18$ for each edge $uv \in E(B_3)$ and $g_3(E(B_3)) = [1, 17]^0$.

Since $f_3(V(A_3)) \cup g_3(V(B_3)) = [0, 18] = [0, 2q]$, $\langle f_3, g_3 \rangle$ is a twin set-ordered edge-difference total labeling.

(e) A graph $A_4$ admits a set-ordered odd-edge graceful-difference total labeling $f_4$ holding $||f_4(x) - f_4(y)| - f_4(xy)|| = 0$ for each edge $xy \in E(A_4)$ and $f_4(E(A_4)) = [1, 17]^0$.

(e-1) A graph $B_4$ admits a set-ordered odd-edge graceful-difference total labeling $g_4$ with $||g_4(u) - g_4(v)| - g_3(uv)|| = 0$ for each edge $uv \in E(B_4)$ and $g_4(E(B_4)) = [1, 17]^0$.

Since $f_4(V(A_4)) \cup g_4(V(B_4)) = [0, 18] = [0, 2q]$, $\langle f_4, g_4 \rangle$ is a twin set-ordered graceful-difference total labeling.

\[ \square \]

**Example 7.** In Fig 7, each graph $A_i$ admits a labeling $f_i$ and each graph $C_i$ admits a labeling $h_i$, such that $f_i(xy) + h_i(xy) = 18$ for each edge $xy \in E(A_i) = E(C_i)$ with $i \in [0, 4]$. So, we call $\langle f_i, h_i \rangle$ an edge-matching $W$-magic total labeling for $i \in [0, 4]$, where “$W$-magic” is one of edge-magic, edge-difference, felicitous-difference, graceful-difference.

\[ \square \]

**Definition 12.** * Let “$W$-magic” be one of edge-magic, edge-difference, felicitous-difference, graceful-difference. Removing the restriction “set-ordered” in Definition [11] will produce four twin odd-edge
W-magic total labelings. If there is at least a pair of vertices colored with the same color in Definition 11, we obtain four twin odd-edge W-magic total colorings.

Definition 13. * m-tuple (set-ordered) odd-edge W-magic total labeling/coloring. F : V(G_i) → [0,M] with i ∈ [1,m], such that: F is an (set-ordered) odd-edge W-magic total labeling/coloring of G_1; F(E(G_i)) = [1,2q - 1]^0; F(u_{i,s}v_{i,j}) ∈ E(G_i) holds one of four magic-type restrictions defined in Definition 11 and \( \bigcup_{i=1}^{m} F(V(G_i)) \subseteq [0,M] \).

Definition 14. * There are m-tuple odd-edge Topcode-matrix team \( T^i_{code} = (X_i, E_i, Y_i)^T \) with v-vector \( X_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,n}) \), e-vector \( E_i = (e_{i,1}, e_{i,2}, \ldots, e_{i,n}) \) and v-vector \( Y_i = (y_{i,1}, y_{i,2}, \ldots, y_{i,n}) \) for i ∈ [1,m], such that \( \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \{x_{i,j}\} \cup \{y_{i,j}\} \subseteq [0,M] \), and each \( e_{i,j} \) is odd and holds one W-magic restriction of \( |x_{i,j} + e_{i,j} + y_{i,j}| = c_1, |x_{i,j} + y_{i,j} - e_{i,j}| = c_2, |x_{i,j} + y_{i,j} - e_{i,j}| = c_3 \) and \( |x_{i,j} - y_{i,j}| = c_4 \), as well as \( \{e_{i,j} : e_{i,j} ∈ E_i\} = [1,2n - 1]^0 \).

Remark 1. A graphic group \( \{F_m(G,f); \oplus\} \) admits a n-tuple (set-ordered) odd-edge W-magic total labeling/coloring for \( n ≤ m \).

2.3 Dual-type labelings and colorings

Part of the content in this subsection are cited from [7]. Let \( G \) be a connected bipartite (p, q)-graph admitting a set-ordered graceful labeling \( f \), and let \( (X,Y) \) be the bipartition of vertex set \( V(G) \), where \( X = \{x_1, x_2, \ldots, x_s\} \) and \( Y = \{y_1, y_2, \ldots, y_t\} \) with \( s + t = p \). Without loss of generality, there are inequalities

\[
0 = f(x_1) < f(x_2) < \cdots < f(x_s) < f(y_1) < f(y_2) < \cdots < f(y_t) = q
\]

also, \( \max f(X) < \min f(Y) \), and \( f(E(G)) = [1,q] \). See a connected bipartite (8,9)-graph \( G_0 \) admitting a set-ordered graceful labeling shown in Fig.3(a).

We are ready to define the following set-dual type labelings:

Set-Dual-1. The total set-dual labeling \( f_{\text{dual}} \) of \( f \) is defined as:

\[
f_{\text{dual}}(w) = \max f(V(G)) + \min f(V(G)) - f(w) = q - f(w)
\]

for \( w ∈ V(G) \), and the induced edge color of each edge \( x_iy_j \) is

\[
f_{\text{dual}}(x_iy_j) = |f_{\text{dual}}(x_i) - f_{\text{dual}}(y_j)| = |f(x_i) - f(y_j)| = f(y_j) - f(x_i) = f(x_iy_j)
\]

Then \( f_{\text{dual}}(E(G)) = f(E(G)) = [1,q] \) and there are

\[
0 = f_{\text{dual}}(y_t) < f_{\text{dual}}(y_{t-1}) < \cdots < f_{\text{dual}}(y_1) < f_{\text{dual}}(x_s) < f_{\text{dual}}(x_{s-1}) < \cdots < f_{\text{dual}}(x_2) < \cdots < f_{\text{dual}}(x_1) = q
\]

also, the dual labeling \( f_{\text{dual}} \) is a set-ordered graceful labeling of \( G \) too.
Problem 3. Suppose that a connected bipartite \((p,q)\)-graph \(G\) admitting a set-ordered graceful labeling \(f\), and \(f_{\text{dual}}\) is the dual labeling of \(f\). Does \(f(V(G)) \cup f_{\text{dual}}(V(G)) = [0, q]?\)

Another total dual labeling \(f^*_{\text{dual}}\) of \(f\) is defined as

\[
f^*_{\text{dual}}(w) = \max f(V(G)) + \min f(V(G)) - f(w) = q - f(w)
\]

for \(w \in V(G)\), and the induced edge color of each edge \(x_iy_j\) is defined by

\[
f^*_{\text{dual}}(x_iy_j) = \max f(E(G)) + \min f(E(G)) - f(x_iy_j) = q + 1 - f(x_iy_j)
\]

for \(x_iy_j \in E(G)\), then \(f^*_{\text{dual}}(E(G)) = f(E(G)) = [1, q]\). Because of

\[
f^*_{\text{dual}}(x_iy_j) + |f^*_{\text{dual}}(y_j) - f^*_{\text{dual}}(x_i)| = q + 1 - f(x_iy_j) + |f(y_j) - f(x_i)| = q + 1
\]

so \(f^*_{\text{dual}}\) is a set-ordered edge-difference total labeling of \(G\).

Theorem 1. A connected bipartite graph \(G\) admits a set-ordered graceful labeling \(f\) if and only if the dual labeling \(f_{\text{dual}}\) of the labeling \(f\) is a set-ordered graceful labeling and another dual labeling \(f^*_{\text{dual}}\) of the labeling \(f\) is a set-ordered edge-difference total labeling.

Set-Dual-2. The \(XY\)-set-dual labeling \(g_{\text{setXY}}\) of \(f\) is defined as:

\[
g_{\text{setXY}}(x_i) = \max f(X) + \min f(X) - f(x_i), \ x_i \in X
\]

and

\[
g_{\text{setXY}}(y_j) = \max f(Y) + \min f(Y) - f(y_j), \ y_j \in Y
\]

and the induced edge color of each edge \(x_iy_j\) is defined by

\[
g_{\text{setXY}}(x_iy_j) = |g_{\text{setXY}}(x_i) - g_{\text{setXY}}(y_j)|
\]

\[
= |[\max f(X) + \min f(X) - f(x_i)] - [\max f(Y) + \min f(Y) - f(y_j)]|
\]

\[
= |\max f(Y) + \min f(Y) - \max f(X) + \min f(X) - f(x_iy_j)|
\]

\[
= q + \min f(Y) - \max f(X) - f(x_iy_j)
\]

and the edge color set

\[
g_{\text{setXY}}(E(G)) = [q + \min f(Y) - \max f(X) - q, \ q + \min f(Y) - \max f(X) - 1]
\]

\[
= [\min f(Y) - \max f(X), \ \min f(Y) - \max f(X) + (q - 1)]
\]

then the \(XY\)-set-dual labeling \(g_{\text{setXY}}\), when as \(\min f(Y) - \max f(X) = 1\), is a set-ordered graceful labeling of \(G\).

And another \(XY\)-set-dual labeling \(g^*_{\text{setXY}}\) is defined as \(g^*_{\text{setXY}}(w) = g_{\text{setXY}}(w)\) for \(w \in V(G)\), and each edge \(x_iy_j \in E(G)\) is colored with

\[
g^*_{\text{setXY}}(x_iy_j) = \max f(E(G)) + \min f(E(G)) - f(x_iy_j) = q + 1 - f(x_iy_j)
\]
which induces edge color set \( g^*_{setXY}(E(G)) = f(E(G)) = [1, q] \). Since \( g^*_{setXY}(u) \neq g^*_{setXY}(v) \) for distinct vertices \( u, v \in V(G) \), and
\[
\|g^*_{setXY}(y_j) - g^*_{setXY}(x_i)\| - g^*_{setXY}(x_iy_j) = \min f(Y) - \max f(X) - f(x_iy_j) - [q + 1 - f(x_iy_j)]
\]
for each edge \( x_iy_j \in E(G) \), so \( g^*_{setXY} \) is a set-ordered graceful-difference total labeling of \( G \).

Here, \( g^*_{setXY} \) has its own dual labeling \( \alpha_set \) defined by
\[
\alpha_set(w) = \max g^*_{setXY}(V(G)) + \min g^*_{setXY}(V(G)) - g^*_{setXY}(w)
\]
for \( w \in V(G) \), and the edge color of each edge \( x_iy_j \) is \( \alpha_set(x_iy_j) = g^*_{setXY}(x_iy_j) \), so it is not hard to show that \( \alpha_set \) is a set-ordered graceful labeling of \( G \).

See Fig 8 for understanding the labelings introduced in Set-Dual-1 and Set-Dual-2.

Figure 8: (a) \( G_0 \) admits a set-ordered graceful labeling \( f \); (b) \( G_1 \) admits a set-ordered graceful labeling \( f_{dual} \), which is the total set-dual labeling of \( f \); (c) \( G_2 \) admits a set-ordered edge-difference total labeling \( f^*_{dual} \); (d) \( G_3 \) admits a \( XY \)-set-dual labeling \( g_{setXY} \) of \( f \); (e) \( G_4 \) admits a set-ordered graceful-difference total labeling \( g^*_{setXY} \).

**Theorem 2.** A connected bipartite graph \( G \) admits a set-ordered graceful labeling \( f \) if and only if the set-dual labeling \( g_{setXY} \) of the labeling \( f \) is a set-ordered graceful labeling, and \( g^*_{setXY} \) is a set-ordered graceful-difference total labeling of \( G \).

**Set-Dual-3.** The \( X \)-set-dual labeling \( h_{setX} \) of \( f \) is defined as:
\[
h_{setX}(x_i) = \max f(X) + \min f(X) - f(x_i) = \max f(X) - f(x_i), \ x_i \in X
\]
and \( h_{setX}(y_j) = f(y_j) \) for \( y_j \in Y \), and the edge color of each edge \( x_iy_j \) is \( h_{setX}(x_iy_j) = f(x_iy_j) \) for \( x_iy_j \in E(G) \), so \( h_{setX}(E(G)) = f(E(G)) = [1, q] \). Furthermore, we have
\[
h_{setX}(x_i) + h_{setX}(y_j) - h_{setX}(x_iy_j) = \max f(X) + \min f(X) - f(x_i) + f(y_j) - f(x_iy_j)
\]
\[
= \max f(X)
\]
so $h_{setX}$ is a set-ordered felicitous-difference total labeling of $G$.

Moreover, we define another $X$-set-dual labeling $h_{setX}$ by $h_{setX}^*(w) = h_{setX}(w)$ for $w \in V(G)$, and

$$h_{setX}^*(x_iz_j) = \max f(E(G)) + \min f(E(G)) - f(x_iz_j) = q + 1 - f(x_iz_j)$$

for each edge $x_iz_j \in E(G)$, then $h_{setX}^*(E(G)) = f(E(G)) = [1, q]$. Since

$$h_{setX}^*(x_i) + h_{setX}^*(x_iz_j) + h_{setX}^*(y_j) = h_{setX}(x_i) + q + 1 - f(x_iz_j) + h_{setX}(y_j)
= \max f(X) + h_{setX}(x_iz_j) + q + 1 - f(x_iz_j)
= \max f(X) + f(x_iz_j) + q + 1 - f(x_iz_j)$$

(15)

which shows that $h_{setX}^*$ is a set-ordered edge-magic total labeling of $G$.

**Theorem 3.** A connected bipartite graph $G$ admits a set-ordered graceful labeling $f$ if and only if the set-dual labeling $h_{setX}$ of the labeling $f$ is a set-ordered felicitous-difference labeling, and $h_{setX}^*$ is a set-ordered edge-magic total labeling of $G$.

**Set-Dual-4.** The $Y$-set-dual labeling $h_{setY}$ of $f$ is defined as: $h_{setY}(x_i) = f(x_i)$ for $x_i \in X$, $h_{setY}(y_j) = \max f(Y) + \min f(Y) - f(y_j) = q + \min f(Y) - f(y_j)$, $y_j \in Y$

and the edge color of each edge $x_iz_j$ is $h_{setY}(x_iz_j) = f(x_iz_j)$ for $x_iz_j \in E(G)$, immediately, $h_{setY}(E(G)) = f(E(G)) = [1, q]$. Moreover, we confirm that $h_{setY}$ is an edge-magic total labeling of $G$, since

$$h_{setY}(x_i) + h_{setY}(x_iz_j) + h_{setY}(y_j) = f(x_i) + f(x_iz_j) + q + \min f(Y) - f(y_j)
= q + \min f(Y)$$

(16)

for each edge $x_iz_j \in E(G)$.

And another case, we define another $Y$-set-dual labeling $h_{setY}^*$ by $h_{setY}^*(w) = h_{setY}(w)$ for $w \in V(G)$, and

$$h_{setY}^*(x_iz_j) = \max f(E(G)) + \min f(E(G)) - f(x_iz_j) = q + 1 - f(x_iz_j)$$

for $x_iz_j \in E(G)$, then $h_{setY}^*(E(G)) = f(E(G)) = [1, q]$. We omit the proof for $h_{setY}^*$ being a set-ordered felicitous-difference total labeling of $G$.

**Theorem 4.** A connected bipartite graph $G$ admits a set-ordered graceful labeling $f$ if and only if the set-dual labeling $h_{setY}$ of the labeling $f$ is a set-ordered edge-magic labeling, and $h_{setY}^*$ is a set-ordered felicitous-difference total labeling of $G$.

See Fig.9 for understanding the labelings introduced in **Set-Dual-3** and **Set-Dual-4**, although the examples admits set-dual colorings (refer to Definition 9).

The above set-dual type labelings from **Set-Dual-1** to **Set-Dual-4** produce the following coloring matchings:
Figure 9: (a) $H_0$ admits a set-ordered graceful total coloring $h$, there are two vertices colored with the same color; (b) $H_1$ admits a set-ordered felicitous-difference total coloring $h_{setX}$, which is a $X$-set-dual coloring of the set-ordered graceful coloring $h$; (c) $H_2$ admits a set-ordered edge-magic total coloring $h^*_{setX}$; (d) $T_1$ admits a set-ordered edge-magic total coloring $h_{setY}$; (e) $T_2$ admits a set-ordered felicitous-difference total coloring $h^*_{setY}$.

**Matching-1.** The set-ordered graceful matching $\langle f, f^*_{dual}\rangle$ holds $f(w) + f^*_{dual}(w) = q$ for $w \in V(G)$ and $f(x_i y_j) + f^*_{dual}(x_i y_j) = q+1$ for $x_i y_j \in E(G)$.

**Matching-2.** $\langle g_{setXY}, g^*_{setXY}\rangle$ is a matching of a set-ordered graceful labeling and a graceful-difference total labeling, such that $|g_{setXY}(x_i y_j) - g^*_{setXY}(x_i y_j)|$ is equal to a constant for $x_i y_j \in E(G)$.

**Matching-3.** $\langle h_{setX}, h^*_{setY}\rangle$ is a matching of two set-ordered edge-magic total labelings, such that $h_{setY}(x_i y_j) + h^*_{setY}(x_i y_j) = q+1$ for $x_i y_j \in E(G)$.

**Matching-4.** $\langle h^*_{setX}, h_{setY}\rangle$ is a matching of two set-ordered felicitous-difference total labelings, such that $h_{setY}(x_i y_j) + h^*_{setY}(x_i y_j) = q+1$ for $x_i y_j \in E(G)$.

**Definition 15.** * If there is at least a pair of vertices colored with the same color in Set-Dual-$k$ for $k \in [1, 4]$ above, we get: the dual total coloring, the $XY$-set-dual total coloring, the $X$-set-dual total coloring and the $Y$-set-dual total coloring, as well as four (set-ordered) $W$-magic total colorings, where “$W$-magic” is one of edge-magic, edge-difference, felicitous-difference, graceful-difference (see examples shown in Fig.9).

3 Algorithms of adding leaves randomly

In this subsection, the sentence “RANDOMLY-LEAF-adding algorithm” is abbreviated as “RLA-algorithm”. For constructing multiple-operation graphic lattices, we introduce the following RLA-algorithms.

3.1 RLA-algorithm-A of the odd-edge graceful-difference total coloring

RLA-algorithm-A of the odd-edge graceful-difference total coloring.
Input: A connected bipartite \((p, q)\)-graph \(G\) admitting a set-ordered odd-edge graceful-difference total labeling \(f_{\text{grd}}\).

Output: A connected bipartite \((p + m, q + m)\)-graph \(G_A\) admitting an odd-edge graceful-difference coloring \(f^*_{\text{grd}}\), where \(G_A\), called leaf-added graph, is the result of adding randomly \(m\) leaves to \(G\).

Initialization. A connected bipartite \((p, q)\)-graph \(G\) has its own vertex set \(V(G) = X \cup Y\) with \(X \cap Y = \emptyset\), where \(X = \{x_1, x_2, \ldots, x_s\}\) and \(Y = \{y_1, y_2, \ldots, y_t\}\) with \(s + t = p = |V(G)|\). By the definition of a set-ordered odd-edge graceful-difference total labeling, so we have the set-ordered restriction max \(f_{\text{grd}}(X) < \min f_{\text{grd}}(Y)\), without loss of generality,

\[0 = f_{\text{grd}}(x_1) < f_{\text{grd}}(x_2) < \cdots < f_{\text{grd}}(x_s) < f_{\text{grd}}(y_1) < f_{\text{grd}}(y_2) < \cdots < f_{\text{grd}}(y_t) = 2q - 1\]

so each color \(f_{\text{grd}}(x_i)\) for \(i \in [1, s]\) is even, and each color \(f_{\text{grd}}(y_j)\) for \(j \in [1, t]\) is odd, and each edge \(x_i y_j \in E(G)\) satisfies

\[||f_{\text{grd}}(x_i) - f_{\text{grd}}(y_j)|| - f_{\text{grd}}(x_i y_j) = N_A \geq 0\] 

(17)
as well as edge color set \(f_{\text{grd}}(E(G)) = \{f_{\text{grd}}(x_i y_j) : x_i y_j \in E(G)\} = [1, 2q - 1]^o\).

Adding randomly \(a_i\) new leaves \(u_{i,k} \in L(x_i) = \{u_{i,k} : k \in [1, a_i]\}\) to each vertex \(x_i \in X \subset V(G)\) by joining \(u_{i,k}\) with \(x_i\) together by new edges \(x_i u_{i,k}\) for \(k \in [1, a_i]\) and \(i \in [1, s]\), and adding randomly \(b_j\) new leaves \(v_{j,r} \in L(y_j) = \{v_{j,r} : r \in [1, b_j]\}\) to each vertex \(y_j \in Y \subset V(G)\) by joining \(v_{j,r}\) with \(y_j\) together by new edges \(y_j v_{j,r}\) for \(r \in [1, b_j]\) and \(j \in [1, t]\), it may happen some \(a_i = 0\) or some \(b_j = 0\) here. The resultant graph is denoted as \(G_A\).

Let \(m = M_X + M_Y\), where \(M_X = \sum_{c=1}^s a_c\) and \(M_Y = \sum_{c=1}^t b_c\). Suppose that \(f_{\text{grd}}(y_j) - f_{\text{grd}}(x_i) \geq f_{\text{grd}}(x_i y_j) \geq 1\) in Eq.(47), we define a coloring \(f^*_{\text{grd}}\) of the leaf-added graph \(G_A\) in the following steps.

**Step A-1.** Color edges \(y_j v_{j,r}\) for leaves \(v_{j,r} \in L(y_j)\) with \(r \in [1, b_j]\) and \(j \in [1, t]\) as:

(A-11) \(f^*_{\text{grd}}(y_1 v_{1,r}) = 2r - 1\) for \(r \in [1, b_1]\), \(f^*_{\text{grd}}(y_1 v_{t+1}) = 2b_t - 1\);

(A-12) \(f^*_{\text{grd}}(y_{j-1} v_{j-1,r}) = 2b_t + 2r - 1\) for \(r \in [1, b_{j-1}]\), \(f^*_{\text{grd}}(y_{j-1} v_{j-1,b_{j-1}+1}) = 2b_t + 2b_{j-1} - 1\);

(A-13) \(f^*_{\text{grd}}(y_{j-1} v_{j-1,j-1,1}) = 2r - 2 + 2 \sum_{c=t-j+1}^t b_c\) for \(r \in [1, b_{j-1}]\) and \(j \in [1, t-1]\), \(f^*_{\text{grd}}(y_{j-1} v_{j-1,j-1,b_{j-1}+1}) = 2b_{j-1} - 1 + 2 \sum_{c=t-j+1}^t b_c\);

(A-14) \(f^*_{\text{grd}}(y_1 v_{1,1}) = 2r - 2 + 2 \sum_{c=1}^t b_c\) for \(r \in [1, b_1]\), the last edge \(y_1 v_{1,b_1}\) is colored as

\[f^*_{\text{grd}}(y_1 v_{1,b_1}) = 2b_1 - 1 + 2 \sum_{c=2}^t b_c = -1 + 2 \sum_{c=1}^t b_c = 2M_Y - 1\]

**Step A-2.** Color edges \(x_i u_{i,k}\) for leaves \(u_{i,k} \in L(x_i)\) with \(k \in [1, a_i]\) and \(i \in [1, s]\) as:

(A-21) \(f^*_{\text{grd}}(x_s u_{s,k}) = 2M_Y + 2k - 1\) for \(k \in [1, a_s]\), \(f^*_{\text{grd}}(x_s u_{s,1}) = 2M_Y + 2a_s - 1\);

(A-22) \(f^*_{\text{grd}}(x_{s-1} u_{s-1,k}) = 2M_Y + 2a_s + 2k - 1\) for \(k \in [1, a_{s-1}]\), \(f^*_{\text{grd}}(x_{s-1} u_{s-1,1}) = 2M_Y + 2a_s + 2a_{s-1} - 1\);

(A-23) \(f^*_{\text{grd}}(x_{s-i} u_{s-i,k}) = 2k - 1 + 2M_Y + 2 \sum_{c=s-i+1}^s a_c\) for \(k \in [1, a_i]\) and \(i \in [1, s-1]\), \(f^*_{\text{grd}}(x_{s-i} u_{s-i,1}) = 2a_i - 1 + 2M_Y + 2 \sum_{c=s-i+1}^s a_c\);
(A-24) \( f^*_\text{grd}(x_1u_{1,k}) = 2k - 1 + 2M_Y + 2\sum_{c=2}^s a_c \) for \( k \in [1, a_1] \), the last edge \( x_1u_{1,a_1} \) is colored with
\[
f^*_\text{grd}(x_1u_{1,a_1}) = 2a_1 - 1 + 2M_Y + 2\sum_{c=2}^s a_c = 2M_Y - 1 + 2\sum_{c=1}^s a_c = 2(M_Y + M_X) - 1\]

**Step A-3.** Recolor each element of \( V(G) \cup E(G) \) by \( f^*_\text{grd}(w) = f^*_\text{grd}(w) + 2(M_Y + M_X) \) for \( w \in E(G) \), and \( f^*_\text{grd}(z) = f^*_\text{grd}(z) \) for \( z \in V(G) \). So
\[
|f^*_\text{grd}(x_i) - f^*_\text{grd}(y_j)| = |f^*_\text{grd}(x_i) - f^*_\text{grd}(y_j)| - f^*_\text{grd}(x_iy_j) - 2(M_Y + M_X) = N_A - 2(M_Y + M_X)|.\tag{18}
\]
Let \( N^*_A = |N_A - 2(M_Y + M_X)| = |N_A - 2m| \). Thereby, the edge color set \( f^*_\text{grd}(E(G_A)) \) of the leaf-added graph \( G_A \) is as
\[
f^*_\text{grd}(E(G_A)) = [1, 2(M_Y + M_X) + \max f^*_\text{grd}(E(G))]|^o = [1, 2(m + q) - 1]^o \tag{19}
\]

**Step A-4.** Color the added leaves of \( L(y_j) \) and \( L(x_i) \) with \( j \in [1, t] \) and \( i \in [1, s] \).

**Step A-4.1.** Each leaf \( v_{j,r} \in L(y_j) \) with \( r \in [1, b_j] \) and \( j \in [1, t] \) is colored by
\[
f^*_\text{grd}(v_{j,r}) = N^*_A + f^*_\text{grd}(y_j) + f^*_\text{grd}(y_jv_{j,r}) = N^*_A + f^*_\text{grd}(y_j) + f^*_\text{grd}(y_jv_{j,r}) \tag{20}
\]
which induces \( |f^*_\text{grd}(y_j) - f^*_\text{grd}(v_{j,r})| = N^*_A \) for \( r \in [1, b_j] \) and \( j \in [1, t] \).

**Step A-4.2.** Each leaf \( u_{i,k} \in L(x_i) \) with \( k \in [1, a_i] \) and \( i \in [1, s] \) is colored by
\[
f^*_\text{grd}(u_{i,k}) = N^*_A + f^*_\text{grd}(x_i) + f^*_\text{grd}(x_iu_{i,k}) = N^*_A + f^*_\text{grd}(x_i) + f^*_\text{grd}(x_iu_{i,k}) \tag{21}
\]
which induces \( |f^*_\text{grd}(x_i) - f^*_\text{grd}(u_{i,k})| = N^*_A \) for \( k \in [1, a_i] \) and \( i \in [1, s] \).

**Step A-5.** Return the odd-edge graceful-difference total coloring \( f^*_\text{grd} \) of the leaf-added graph \( G_A \), and \( f^*_\text{grd}(u_0) = 0 \) for some \( u_0 \in V(G_A) \).

**Example 8.** The examples shown in Fig[10] are for understanding the RLA-algorithm-A of the odd-edge graceful-difference total coloring:

(a) the graph \( G_{XYR-twin} \) admits a twin set-ordered odd-edge graceful-difference total labeling \( \alpha^*_\text{grd} \) holding \( f^*_\text{grd}(V(G_{XYR-twin})) \cup \alpha^*_\text{grd}(V(G_{XYR-twin})) \leq [0, 20] \);

(b) the graph \( G_{XYR} \) admits a set-ordered odd-edge graceful-difference total coloring \( f^*_\text{grd} \) holding \( |f^*_\text{grd}(x) - f^*_\text{grd}(y)| = 0 \);

(c) the graph \( G_{XYR-leaf} \) admits a set-ordered odd-edge graceful-difference total coloring \( f^*_\text{grd} \) holding \( |f^*_\text{grd}(x) - f^*_\text{grd}(y)| = 26 \);

(d) the graph \( H_{XYR-leaf} \) admits a set-ordered odd-edge graceful-difference total labeling \( h^*_\text{grd} \) holding \( |h^*_\text{grd}(x) - h^*_\text{grd}(y)| = 26 \). There is a graph homomorphism \( G_{XYR-leaf} \rightarrow H_{XYR-leaf} \). \( \square \)
Problem 4. In the RLA-algorithm-A of the odd-edge graceful-difference total coloring, there are the following problems:

(i) Integer Partition Problem. We can select \( k \) vertices from a \((p, q)\)-graph \( G \) for adding \( m \) leaves to them, then we have \( A_k^p = p(p - 1) \cdots (p - k + 1) \) selections, rather than \( \binom{p}{k} = \frac{p!}{k!(p-k)!} \). Next, we decompose \( m \) into a group of \( k \) parts \( m_1, m_2, \cdots, m_k \) holding \( m = m_1 + m_2 + \cdots + m_k \) with each \( m_i \neq 0 \). Suppose there is \( n(m,k) \) groups of such \( k \) parts. For a group of \( k \) parts \( m_1, m_2, \cdots, m_k \), let \( m_{i_1}, m_{i_2}, \cdots, m_{i_k} \) be a permutation of \( m_1, m_2, \cdots, m_k \), so we have the number of such permutations is a factorial \( k! \). Since the \((p, q)\)-graph \( G \) is colored well by the \( W \)-magic coloring/labeling \( f \), then we have the number \( A_{\text{leaf}}(G, m) \) of all possible adding \( m \) leaves as follows

\[
A_{\text{leaf}}(G, m) = \sum_{k=1}^{m} A_k^p \cdot n(m,k) \cdot k! = \sum_{k=1}^{m} n(m,k) \cdot p! \tag{22}
\]

where \( n(m,k) = \sum_{r=1}^{k} n(m - k, r) \). Here, computing \( n(m,k) \) can be transformed into finding the number \( A(m,k) \) of solutions of \( \text{Diophantine equation} \ m = \sum_{i=1}^{k} ix_i \). There is a recursive formula

\[
A(m,k) = A(m,k - 1) + A(m - k, k) \tag{23}
\]

with \( 0 \leq k \leq m \). It is not easy to compute the exact value of \( A(m,k) \), for example, the authors in [23] and [24] computed exactly

\[
A(m, 6) = \left\lfloor \frac{1}{1036800} (12m^5 + 270m^4 + 1520m^3 - 1350m^2 - 19190m - 9081) + \frac{(-1)^m (m^2 + 9m + 7)}{768} + \frac{1}{81} \left[ (m+5) \cos \frac{2m\pi}{3} \right] \right\rfloor
\]
For any odd integer \( m \geq 7 \) it was conjectured \( m = p_1 + p_2 + p_3 \) with three primes \( p_1, p_2, p_3 \) from the famous Goldbach's conjecture: "Every even integer, greater than 2, can be expressed as the sum of two primes." In other word, determining \( A(m, 3) \) is difficult, also, it is difficult to express an odd integer \( m = \sum_{k=1}^{3n} p_k' \) with each \( p_k' \) is a prime.

(ii) **Estimate** the extremum number

\[
\min \{ \max f^*_{grd}(V(G_A)) : f^*_{grd} \text{ is an odd-edge graceful-difference total coloring of } G_A \}
\]

(24) over all odd-edge graceful-difference total colorings of the leaf-added graph \( G_A \).

(iii) Notice that there are \( m! \) permutations \( w_{i_1}, w_{i_2}, \ldots, w_{i_m} \) from the added leaves of the leaf-added set \( L^*(G_A) = \left( \bigcup_{i=1}^{s} L(x_i) \right) \cup \left( \bigcup_{j=1}^{t} L(y_j) \right) \) of the leaf-added \( (p + m, q + m) \)-graph \( G_A \), notice that the leaf permutation in the RLA-algorithm-A of the odd-edge graceful-difference total coloring is one of these \( m! \) permutations. We define a new coloring \( F_{grd} \) for the leaf-added graph \( G_A \) as:

Color leaf-edge \( w_{i_j}z_{i_j} \) with \( F_{grd}(w_{i_j}z_{i_j}) = 2j - 1 \) for \( j \in [1, m] \), where \( z_{i_j} \in X \cup Y = V(G) \), and each element \( e \in V(G) \cup E(G) = [V(G_A) \cup E(G_A)] \setminus L^*(G_A) \) is colored as \( F_{grd}(e) = f_{ed}(e) \), as well as color each added leaf \( w_{i_j} \in L^*(G_A) \) by

\[
F_{grd}(w_{i_j}) = N_A^* + F_{grd}(z_{i_j}) + F_{grd}(w_{i_j}z_{i_j}), \quad N_A^* = |N_A - 2m|
\]

By Eq. \([18]\), Eq. \([20]\) and Eq. \([21]\), the coloring \( F_{grd} \) is an odd-edge graceful-difference total coloring based on a permutation \( w_{i_1}, w_{i_2}, \ldots, w_{i_m} \).

(vi) In Fig. \[10\] the Topcode-matrix \( T_{code}(G_{XYR-twin}) \) can be directly obtained from the Topcode-matrix \( T_{code}(G_{XYR}) \), and these two Topcode-matrices induce two graph sets \( S(T_{code}(G_{XYR-twin})) \) and \( S(T_{code}(G_{XYR})) \), we call these two graph sets as twin odd-edge graceful-difference graph sets. Thereby, a public-key graph \( G \in S(T_{code}(G_{XYR})) \) may correspond many private-key graphs \( S(T_{code}(G_{XYR-twin})) \). The Topcode-matrix \( T_{code}(G_{XYR-leaf}) \) can be made by adding leaves to the Topcode-matrix \( T_{code}(G_{XYR}) \).

**Definition 16.** * Suppose that a connected \((p, q)\)-graph \( G \) admits an odd-edge graceful-difference total coloring \( f \), so \(||f(u) - f(v)| - f(uv)| = c\) for each edge \( uv \in E(G) \), where \( c \) is a non-negative integer.

(1) Let \( w_{i_1}, w_{i_2}, \ldots, w_{i_p} \) be a permutation of vertices of \( V(G) \), and \( e_{j_1}, e_{j_2}, \ldots, e_{j_q} \) be a permutation of edges of \( E(G) \), and \( c_{i_1}, c_{i_2}, \ldots, c_{i_p} \) be a permutation of vertex colors of \( f(V(G)) \) and \( c_{j_1}, c_{j_2}, \ldots, c_{j_q} \) be a permutation of edge colors of \( f(E(G)) \). We define a new total coloring \( \theta \) for \( G \) as:

(i) each edge \( uv \in E(G) \) corresponds two different vertices \( x, z \in V(G) \) holding \(||\theta(x) - \theta(z)| - \theta(uv)| = c\); and

(ii) each vertex \( y \in V(G) \) corresponds another vertex \( t \in V(G) \) and an edge \( ab \in E(G) \) holding \(||\theta(y) - \theta(t)| - \theta(ab)| = c\). We call \( \theta \) a ve-separably derived odd-edge graceful-difference total coloring of the total coloring \( f \).

(2) Let \( z_{i_1}, z_{i_2}, \ldots, z_{i_{p+q}} \) be a permutation of vertices and edges of \( V(G) \cup E(G) \), and let \( b_{j_1}, b_{j_2}, \ldots, b_{j_{p+q}} \) be a permutation of elements of \( f(V(G)) \cup f(E(G)) \). We define a new total coloring
ο for G as: \( \varphi(z_i) = b_j \) for \( s \in [1, p + q] \), such that each element \( z \in V(G) \cup E(G) \) corresponds two elements \( x, y \in V(G) \cup E(G) \) holding one of \( |\varphi(z) - \varphi(x)| = c \) and \( |\varphi(y) - \varphi(x)| = c \) true, we call \( \varphi \) a derived odd-edge graceful-difference total coloring of the total coloring \( f \).

**Remark 2.** About Definition [16] we have the following facts:

1. Similarly with Definition [16] there are six derived-type magic-type total colorings: (ve-separably) derived odd-edge edge-difference total coloring, (ve-separably) derived odd-edge felicitous-difference total coloring, and (ve-separably) derived odd-edge edge-magic total coloring.

2. Suppose that the connected \((p, q)\)-graph \( G \) admits \( M \) odd-edge graceful-difference total colorings. For each odd-edge graceful-difference total coloring \( f \) of these \( M \) colorings, we have \((p!)^2(q!)^2\) ve-separably derived odd-edge graceful-difference total colorings of the total coloring \( f \), we put them into a set \( S^{gr}_{ve}(G, f) \), then we get \( M \cdot (p!)^2(q!)^2 \) ve-separably derived odd-edge graceful-difference total colorings in total.

3. Two Topcode-matrices \( T_{code}(G, \theta) \neq T_{code}(G, \varphi) \) for two different odd-edge graceful-difference total colorings \( \theta, \varphi \in S^{gr}_{ve}(G, f) \), in general. And moreover, \( T_{code}(G, \theta) \neq T_{code}(G, \varphi) \) for \( \theta \in S^{gr}_{ve}(G, f) \) and \( \varphi \in S^{gr}_{ve}(G, g) \), where \( g \) is another odd-edge graceful-difference total coloring of \( G \).

4. These number-based strings \( c_{i_1}c_{i_2} \cdots c_{i_p}c_{j_1}c_{j_2} \cdots c_{j_q} \) and \( b_{i_1}b_{i_2} \cdots b_{i_p+q} \) defined in Definition [16] differ from number-based strings \( d_{i_1}d_{i_2} \cdots d_{i_s} \) made by Topcode-matrices like \( T_{code}(G, \theta) \) and \( T_{code}(G, \varphi) \).

### 3.2 RLA-algorithm-B of the odd-edge edge-difference total coloring

**RLA-algorithm-B of the odd-edge edge-difference total coloring**

**Input:** A connected bipartite \((p, q)\)-graph \( G \) admitting a set-ordered odd-edge edge-difference total labeling \( f_{edd} \).

**Output:** A connected bipartite \((p + m, q + m)\)-graph \( G_B \) admitting an odd-edge edge-difference total coloring \( f^*_edd \), where \( G_B \), called leaf-added graph, is the result of adding randomly \( m \) leaves to \( G \).

**Initialization.** A connected bipartite \((p, q)\)-graph \( G \) has its own vertex set \( V(G) = X \cup Y \) with \( X \cap Y = \emptyset \), where \( X = \{x_1, x_2, \ldots, x_s\} \) and \( Y = \{y_1, y_2, \ldots, y_t\} \) with \( s + t = p = |V(G)| \). By the definition of a set-ordered odd-edge edge-difference total labeling, so the set-ordered restriction \( \max f_{edd}(X) < \min f_{edd}(Y) \) holds true, without loss of generality, there are inequalities

\[
0 = f_{edd}(x_1) < f_{edd}(x_2) < \cdots < f_{edd}(x_s) < f_{edd}(y_1) < f_{edd}(y_2) < \cdots < f_{edd}(y_t) = 2q - 1
\]

so each color \( f_{edd}(x_i) \) for \( i \in [1, s] \) is even, and each color \( f_{edd}(y_j) \) for \( j \in [1, t] \) is odd, and each edge \( x_iy_j \in E(G) \) holds

\[
f_{edd}(x_iy_j) + |f_{edd}(x_i) - f_{edd}(y_j)| = N_B > 0
\]

true, as well as \( f_{edd}(E(G)) = \{f_{edd}(x_iy_j) : x_iy_j \in E(G)\} = [1, 2q - 1]^o \).

Adding randomly \( a_i \) new leaves \( u_{i,k} \in L(x_i) = \{u_{i,k} : k \in [1, a_i]\} \) to each vertex \( x_i \in X \subset V(G) \) by joining \( u_{i,k} \) with \( x_i \) together by new edges \( x_iu_{i,k} \) for \( k \in [1, a_i] \) and \( i \in [1, s] \), and adding randomly
b_j new leaves $v_{j,r} \in L(y_j) = \{v_{j,r} : r \in [1,b_j]\}$ to each vertex $y_j \in Y \subset V(G)$ by joining $v_{j,r}$ with $y_j$ together by new edges $y_j v_{j,r}$ for $r \in [1,b_j]$ and $j \in [1,t]$, it may happen some $a_i = 0$ or some $b_j = 0$. The resultant graph is denoted as $G_B$.

Let $M_X = \sum_{c=1}^s a_c$ and $M_Y = \sum_{c=1}^t b_c$, so $m = M_X + M_Y$. Define a coloring $f^*_{edd}$ of the leaf-added graph $G_B$ in the following steps.

**Step B-1.** Color edges $x_i u_{i,k}$ for leaves $u_{i,k} \in L(x_i)$ with $k \in [1,a_i]$ and $i \in [1,s]$ as follows:
(B-11) $f^*_{edd}(x_i u_{1,k}) = 2k - 1$ for $k \in [1,a_1]$, $f^*_{edd}(x_i u_{1,a_1}) = 2a_s - 1$;
(B-12) $f^*_{edd}(x_i u_{i,k}) = 2k - 1 + 2\sum_{c=1}^{i-1} a_c$ for $k \in [1,a_i]$ and $i \in [2,s]$, $f^*_{edd}(x_i u_{i,a_i}) = 2a_i - 1 + 2\sum_{c=1}^{i-1} a_c$; and
(B-13) The last edge $x_s u_{s,a_s}$ is colored by $f^*_{edd}(x_s u_{s,a_s}) = 2a_s - 1 + 2\sum_{c=1}^{s-1} a_c = -1 + 2\sum_{c=1}^s a_c = 2M_X - 1$.

**Step B-2.** Color edges $y_j v_{j,r}$ for leaves $v_{j,r} \in L(y_j)$ with $r \in [1,b_j]$ and $j \in [1,t]$ as follows:
(B-21) $f^*_{edd}(y_j v_{1,r}) = 2M_X + 2r - 1$ for $k \in [1,b_1]$, $f^*_{edd}(y_j v_{1,b_1}) = 2M_X + 2b_1 - 1$;
(B-22) $f^*_{edd}(y_j v_{j,r}) = 2M_X + 2r - 1 + 2\sum_{c=1}^{j-1} b_c$ for $r \in [1,b_j]$ and $j \in [1,t]$, and the last edge $x_s u_{s,a_s}$ is colored by

$$f^*_{edd}(y_j v_{t,b_t}) = 2M_X + 2b_t - 1 + 2\sum_{c=1}^{t-1} b_c = 2(M_X + M_Y) - 1 = 2m - 1$$

The edge color set is

$$f^*_{edd}(E(G_B)) = \{f^*_{edd}(x_i u_{i,k}) : k \in [1,a_i], i \in [1,s]\} \cup \{f^*_{edd}(y_j v_{j,r}) : r \in [1,b_j], j \in [1,t]\} \cup f^*_{edd}(E(G)) \cup [1,2(m + q) - 1]^o,$$

(B-23)

**Step B-3.** Recolor each element of $V(G) \cup E(G)$ as: $f^*_{edd}(w) = f_{edd}(w) + 2m$ for $w \in E(G)$, and $f^*_{edd}(z) = f_{edd}(z)$ for $z \in V(G)$, which induces

$$f^*_{edd}(x_i y_j) + |f^*_{edd}(x_i) - f^*_{edd}(y_j)| = f_{edd}(x_i y_j) + 2m + |f_{edd}(x_i) - f_{edd}(y_j)| = 2m + N_B$$

(B-24) for each edge $x_i y_j \in E(G)$.

**Step B-4.** Let $N^*_B = 2m + N_B$. Color the added leaves of $L(y_j)$ and $L(x_i)$ with $j \in [1,t]$ and $i \in [1,s]$.

**Step B-4.1.** Each leaf $v_{j,r} \in L(y_j)$ with $r \in [1,b_j]$ and $j \in [1,t]$ is colored by

$$f^*_{edd}(v_{j,r}) = N^*_B + f^*_{edd}(y_j) - f^*_{edd}(v_{j,r})$$

(B-25) so $f^*_{edd}(y_j v_{j,r}) + |f^*_{edd}(y_j) - f^*_{edd}(v_{j,r})| = N^*_B$ for $r \in [1,b_j]$ and $j \in [1,t]$.

**Step B-4.2.** Each leaf $u_{i,k} \in L(x_i)$ with $k \in [1,a_i]$ and $i \in [1,s]$ is colored by

$$f^*_{edd}(u_{i,k}) = N^*_B + f^*_{edd}(x_i) - f^*_{edd}(u_{i,k})$$

(B-26) immediately, $f^*_{edd}(x_i u_{i,k}) + |f^*_{edd}(x_i) - f^*_{edd}(u_{i,k})| = N^*_B$ for $k \in [1,a_i]$ and $i \in [1,s]$.

**Step B-5.** Return the odd-edge edge-difference total coloring $f^*_{edd}$ of the leaf-added graph $G_B$. 
Example 9. In Fig[11] for illustrating the RLA-algorithm-B of the odd-edge edge-difference total coloring, we can see the following examples:

(a) the graph $G_{XY}$-twin admits a twin set-ordered odd-edge edge-difference total labeling $\beta_{edd}$ holding $f_{edd}(V(G_{XY})) \cup \beta_{edd}(V(G_{XY}-twin)) \subseteq \{0, 20\}$;

(b) the graph $G_{XY}$ admits a set-ordered odd-edge edge-difference total coloring $f_{edd}$ holding $f_{edd}(xy) + |f_{edd}(x) - f_{edd}(y)| = 20$;

(c) the graph $G_{XY}$-leaf admits a set-ordered odd-edge edge-difference total coloring $f^*_{edd}$ holding $f^*_{edd}(xy) + |f^*_{edd}(x) - f^*_{edd}(y)| = 46$;

(d) the graph $H_{XY}$-leaf admits a set-ordered odd-edge edge-difference total labeling $h^*_{edd}$ holding $h^*_{edd}(xy) + |h^*_{edd}(x) - h^*_{edd}(y)| = 46$. Notice that $G_{XY}$-leaf $\rightarrow$ $H_{XY}$-leaf.

\[\Box\]

Figure 11: Examples for knowing RLA-algorithm-B of the odd-edge edge-difference total coloring.

Problem 5. In the RLA-algorithm-B of the odd-edge edge-difference total coloring, there are the following problems:

(i) Estimate the extremum number

$$\min\{\max f^*_{edd}(V(G_A)) : f^*_{edd} \text{ is an odd-edge edge-difference total coloring of } G_B\}. \quad (30)$$

over all odd-edge edge-difference total colorings of the leaf-added graph $G_B$.

(ii) Since $m = M_X + M_Y$, there are $m!$ permutations $w_{i_1}, w_{i_2}, \ldots, w_{i_m}$ from the added leaves of the leaf-added set $L^*(G_B) = (\bigcup_{i=1}^p L(x_i)) \cup (\bigcup_{j=1}^q L(y_j))$ of the leaf-added $(p + m, q + m)$-graph $G_B$, notice that the leaf permutation in the RLA-algorithm-B of the odd-edge edge-difference total coloring is one of these $m!$ permutations. We define a new coloring $F_{edd}$ for the leaf-added graph $G_B$ as: Color leaf-edge $w_i z_j$ with $F_{edd}(w_i z_j) = 2j - 1$ for $j \in [1, m]$, where $z_{ij} \in X \cup Y = V(G)$,
and each element $e \in V(G) \cup E(G) = [V(G_B) \cup E(G_B)] \setminus L^*(G_B)$ is colored as $F_{edd}(e) = f^*_{edd}(e)$, as well as color each added leaf $w_{ij} \in L^*(G_B)$ by

$$F_{edd}(w_{ij}) = N_B^* + F^*_{edd}(z_{ij}) - F_{edd}(w_{ij}z_{ij}), \quad N_B^* = 2m + N_B$$

By Eq. (26), Eq. (28) and Eq. (29), the coloring $F_{edd}$ is an odd-edge edge-difference total coloring based on a permutation $w_{i_1}, w_{i_2}, \ldots, w_{i_m}$.

### 3.3 RLA-algorithm-C of the odd-edge edge-magic total coloring

**RLA-algorithm-C of the odd-edge edge-magic total coloring.**

**Input:** A connected bipartite $(p, q)$-graph $G$ admitting a set-ordered odd-edge edge-magic total labeling $f_{ema}$.

**Output:** A connected bipartite $(p + m, q + m)$-graph $G_C$ admitting an odd-edge edge-magic total coloring $f^*_{ema}$, where $G_C$, called leaf-added graph, is the result of adding randomly $m$ leaves to $G$.

**Initialization.** A connected bipartite $(p, q)$-graph $G$ has its own vertex set $V(G) = X \cup Y$ with $X \cap Y = \emptyset$, where $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$ with $s + t = p = |V(G)|$. By the definition of a set-ordered odd-edge edge-magic total labeling, so we have the set-ordered restriction $\max f_{ema}(X) < \min f_{ema}(Y)$, without loss of generality, we have

$$0 = f_{ema}(x_1) < f_{ema}(x_2) < \cdots < f_{ema}(x_s) < f_{ema}(y_1) < f_{ema}(y_2) < \cdots < f_{ema}(y_t) = 2q - 1$$

so each color $f_{ema}(x_i)$ for $i \in [1, s]$ is even, and each color $f_{ema}(y_j)$ for $j \in [1, t]$ is odd, and each edge $x_iy_j \in E(G)$ satisfies

$$f_{ema}(x_i) + f_{ema}(x_jy_j) + f_{ema}(y_j) = N_C > 0$$

as well as $f_{ema}(E(G)) = \{f_{ema}(x_iy_j) : x_iy_j \in E(G)\} = [1, 2q - 1]^c$.

Adding randomly $a_i$ new leaves $u_{i,k} \in L(x_i) = \{u_{i,k} : k \in [1, a_i]\}$ to each vertex $x_i \in X \subset V(G)$ by joining $u_{i,k}$ with $x_i$ together by new edges $x_iu_{i,k}$ for $k \in [1, a_i]$ and $i \in [1, s]$, and adding randomly $b_j$ new leaves $v_{j,r} \in L(y_j) = \{v_{j,r} : r \in [1, b_j]\}$ to each vertex $y_j \in Y \subset V(G)$ by joining $v_{j,r}$ with $y_j$ together by new edges $y_jv_{j,r}$ for $r \in [1, b_j]$ and $j \in [1, t]$, it may happen some $a_i = 0$ or some $b_j = 0$. The resultant graph is denoted as $G_C$.

Let $M_X = \sum_{c=1}^{s} a_c$ and $M_Y = \sum_{c=1}^{t} b_c$, so $m = M_X + M_Y$. We define a coloring $f^*_{ema}$ of the leaf-added graph $G_C$ in the following steps:

**Step C-1.** Color edges $y_jv_{j,r}$ for leaves $v_{j,r} \in L(y_j)$ with $r \in [1, b_j]$ and $j \in [1, t]$ as follows:

- (C-11) $f^*_{ema}(y_jv_{j,r}) = 2r - 1$ for $r \in [1, b_j]$.
- (C-12) $f^*_{ema}(y_{t-1}v_{1-1,r}) = 2b_t + 2r - 1$ for $r \in [1, b_{t-1}]$.
- (C-13) $f^*_{ema}(y_{j-1}v_{j-1,1}) = 2b_t - 1 + 2\sum_{c=t-j+1}^{t} b_c$ for $r \in [1, b_{t-1}]$ and $j \in [1, t-1]$.

$$f^*_{ema}(y_{t-1}v_{1-1,b_{t-1}}) = 2b_t - 1 + 2\sum_{c=t-j+1}^{t} b_c.$$
(C-14) \( f_{ema}(y_{1v_1,r}) = 2r - 1 + 2\sum_{c=2}^{t} b_c \) for \( r \in [1, b_1] \), the last edge \( y_{1v_1,b_1} \) is colored by

\[
f_{ema}(y_{1v_1,b_1}) = 2b_1 - 1 + 2\sum_{c=2}^{t} b_c = -1 + 2\sum_{c=1}^{t} b_c = 2My - 1
\]

**Step C-2.** Color edges \( x_iu_{i,k} \) for leaves \( u_{i,k} \in L(x_i) \) with \( k \in [1, a_i] \) and \( i \in [1, s] \) as follows:

(C-21) \( f_{ema}(x_su_{s,k}) = 2MY + 2k - 1 \) for \( k \in [1, a_s] \), \( f_{ema}(x_su_{s,a_s}) = 2MY + 2a_s - 1 \);

(C-22) \( f_{ema}(x_{s-1u_{s-1,k}}) = 2MY + 2a_s + 2k - 1 \) for \( k \in [1, a_{s-1}] \), \( f_{ema}(x_{s-1u_{s-1,a_{s-1}}}) = 2MY + 2a_s + 2a_{s-1} - 1 \);

(C-23) \( f_{ema}(x_{s-iu_{s-i,k}}) = 2k - 1 + 2MY + 2\sum_{c=s-i+1}^{a_s} a_c \) for \( k \in [1, a_i] \) and \( i \in [1, s-1] \), \( f_{ema}(x_{s-iu_{s-i,a_i}}) = 2a_i - 1 + 2MY + 2\sum_{c=s-i+1}^{a_s} a_c \);

(C-24) \( f_{ema}(x_{1u_1,k}) = 2k - 1 + 2MY + 2\sum_{c=2}^{s} a_c \) for \( k \in [1, a_1] \), and the last edge \( x_{1u_1,a_1} \) is colored with

\[
f_{ema}(x_{1u_1,a_1}) = 2a_1 - 1 + 2MY + 2\sum_{c=2}^{s} a_c = 2MY - 1 + 2\sum_{c=1}^{s} a_c = 2(My + Mx) - 1 = 2m - 1 \quad (32)
\]

**Step C-3.** Recolor each element of \( V(G) \cup E(G) \) in the following way: \( f_{ema}^*(w) = f_{ema}(w) + 2m \) for \( w \in E(G) \), and \( f_{ema}^*(z) = f_{ema}(z) \) for \( z \in V(G) \). So, each edge \( x_iy_j \in E(G) \) holds

\[
f_{ema}^*(x_i) + f_{ema}^*(x_iy_j) + f_{ema}^*(y_j) = f_{ema}(x_i) + f_{ema}(x_iy_j) + 2m + f_{ema}(y_j) = 2m + N_C \quad (33)
\]

By Eq.\( 32 \) and Eq.\( 33 \), we have the edge color set \( f_{ema}^*(E(G_C)) \) of the graph \( G_C \) as follows:

\[
f_{ema}^*(E(G_C)) = [1, 2m - 1] \cup [1 + 2m, 2q - 1 + 2m] = [1, 2(q + m) - 1] \quad (34)
\]

**Step C-4.** Let \( N_C^* = 2m + N_C \). Color the added leaves of \( L(y_j) \) and \( L(x_i) \) with \( j \in [1, t] \) and \( i \in [1, s] \).

**Step C-4.1.** Each leaf \( v_{j,r} \in L(y_j) \) with \( r \in [1, b_j] \) and \( j \in [1, t] \) is colored by

\[
f_{ema}^*(v_{j,r}) = N_C^* - f_{ema}^*(y_{j}) - f_{ema}^*(y_{j}v_{j,r}) \quad (35)
\]

so \( f_{ema}^*(y_{j}) + f_{ema}^*(y_{j}v_{j,r}) + f_{ema}^*(v_{j,r}) = N_C^* \) for \( r \in [1, b_j] \) and \( j \in [1, t] \).

**Step C-4.2.** Each leaf \( u_{i,k} \in L(x_i) \) with \( k \in [1, a_i] \) and \( i \in [1, s] \) is colored by

\[
f_{ema}^*(u_{i,k}) = N_C^* - f_{ema}^*(x_{i}) - f_{ema}^*(x_{i}u_{i,k}) \quad (36)
\]

immediately, \( f_{ema}^*(x_{i}) + f_{ema}^*(x_{i}u_{i,k}) + f_{ema}^*(u_{i,k}) = N_C^* \) for \( k \in [1, a_i] \) and \( i \in [1, s] \).

**Step C-5.** Return the odd-edge edge-magic total coloring \( f_{ema}^* \) of the leaf-added graph \( G_C \).

**Example 10.** For understanding the RLA-algorithm-C of the odd-edge edge-magic total coloring, see Fig.\ref{fig:example10} we have

(a) the graph \( G_{Y\text{-twin}} \) admits a twin set-ordered odd-edge edge-magic total labeling \( \gamma_{ema} \) holding \( f_{ema}^*(V(G_Y)) \cup \gamma_{ema}(V(G_{Y\text{-twin}})) \subseteq [0, 20] \);
(b) the graph $G_Y$ admits a set-ordered odd-edge edge-magic total coloring $f_{ema}$ holding $f_{ema}(x) + f_{ema}(y) = 26$;

(c) the graph $G_{Y\text{-leaf}}$ admits a set-ordered odd-edge edge-magic total coloring $f^*_{ema}$ holding $f^*_{ema}(x) + f^*_{ema}(xy) + f^*_{ema}(y) = 52$;

(d) the graph $H_{Y\text{-leaf}}$ admits a set-ordered odd-edge edge-magic total labeling $h^*_{ema}$ holding $h^*_{ema}(x) + h^*_{ema}(xy) + h^*_{ema}(y) = 52$. There is a graph homomorphism $G_{Y\text{-leaf}} \to H_{Y\text{-leaf}}$. \hfill \qed

Figure 12: A scheme for understanding RLA-algorithm-C of the odd-edge edge-magic total coloring.

Problem 6. In the RLA-algorithm-C of the odd-edge edge-magic total coloring, there are the following problems:

(i) \textbf{Estimate} the extremum number

$$\min \{ \max f_{ema}(V(G_A)) : f_{ema} \text{ is an odd-edge edge-magic total coloring of } G_C \}$$

over all odd-edge edge-magic total colorings of the leaf-added graph $G_C$.

(ii) Notice that there are $m!$ permutations $w_{i_1}, w_{i_2}, \ldots, w_{i_m}$ from the added leaves of the leaf-added set $L^*(G_C) = \bigcup_{i=1}^s L(x_i) \cup \bigcup_{j=1}^t L(y_j)$ of the leaf-added $(p+m, q+m)$-graph $G_C$, notice that the leaf permutation in the RLA-algorithm-C of the odd-edge edge-magic total coloring is one of these $m!$ permutations. We define a new coloring $F_{ema}$ for the leaf-added graph $G_C$ as: Color leaf-edge $w_{ij} z_{ij}$ with $F_{ema}(w_{ij} z_{ij}) = 2j - 1$ for $j \in [1, m]$, where $z_{ij} \in X \cup Y = V(G)$, and each element $e \in V(G) \cup E(G) = [V(G_C) \cup E(G_C)] \setminus L^*(G_C)$ is colored as $F_{ema}(e) = f^*_{ed}(e)$, as well as color each added leaf $w_{ij} \in L^*(G_C)$ by

$$F_{ema}(w_{ij}) = N^*_C - F^*_{ema}(z_{ij}) - F_{ema}(w_{ij} z_{ij}), \quad N^*_C = 2m + N_C$$

By Eq.\,(33), Eq.\,(35) and Eq.\,(36), the coloring $F_{ema}$ is an odd-edge edge-magic total coloring based on a permutation $w_{i_1}, w_{i_2}, \ldots, w_{i_m}$. 

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### 3.4 RLA-algorithm-D of the odd-edge felicitous-difference total coloring.

**RLA-algorithm-D of the odd-edge felicitous-difference total coloring.**

**Input:** A connected bipartite \((p, q)\)-graph \(G\) admitting a set-ordered odd-edge felicitous-difference total labeling \(h_{fed}\).

**Output:** A connected bipartite \((p + m, q + m)\)-graph \(G_D\) admitting an odd-edge felicitous-difference total coloring \(h_{fed}^*\), where \(G_D\), called *leaf-added graph*, is the result of adding randomly \(m\) leaves to \(G\).

**Initialization.** A connected bipartite \((p, q)\)-graph \(G\) has its own vertex set \(V(G) = X \cup Y\) with \(X \cap Y = \emptyset\), where \(X = \{x_1, x_2, \ldots, x_s\}\) and \(Y = \{y_1, y_2, \ldots, y_t\}\) with \(s + t = p = |V(G)|\). By the definition of a set-ordered odd-edge felicitous-difference total coloring, so we have the set-ordered restriction \(\max h_{fed}(X) < \min h_{fed}(Y)\), without loss of generality,

\[
0 = h_{fed}(x_1) < h_{fed}(x_2) < \cdots < h_{fed}(x_s) < h_{fed}(y_1) < h_{fed}(y_2) < \cdots < h_{fed}(y_t) = 2q - 1
\]

so each color \(h_{fed}(x_i)\) for \(i \in [1, s]\) is even, and each color \(h_{fed}(y_j)\) for \(j \in [1, t]\) is odd, and

\[
|h_{fed}(x_i) + h_{fed}(y_j) - h_{fed}(x_i y_j)| = N_D \geq 0
\]  \hspace{1cm} (38)

for each edge \(x_i y_j \in E(G)\), as well as \(h_{fed}(E(G)) = \{h_{fed}(x_i y_j) : x_i y_j \in E(G)\} = [1, 2q - 1]^o\).

Adding randomly \(a_i\) new leaves \(u_{i,k} \in L(x_i) = \{u_{i,k} : k \in [1, a_i]\}\) to each vertex \(x_i \in X \subset V(G)\) by joining \(u_{i,k}\) with \(x_i\) together by new edges \(x_i u_{i,k}\) for \(k \in [1, a_i]\) and \(i \in [1, s]\), and adding randomly \(b_j\) new leaves \(v_{j,r} \in L(y_j) = \{v_{j,r} : r \in [1, b_j]\}\) to each vertex \(y_j \in Y \subset V(G)\) by joining \(v_{j,r}\) with \(y_j\) together by new edges \(y_j v_{j,r}\) for \(r \in [1, b_j]\) and \(j \in [1, t]\), it may happen some \(a_i = 0\) or some \(b_j = 0\).

The resultant graph is denoted as \(G_D\). Let \(A_X = \sum_{i=1}^{s} a_i\) and \(B_X = \sum_{l=1}^{t} b_l\), so \(m = A_X + B_X\).

Define a coloring \(h_{fed}^*\) for \(G_D\) in the following steps:

**Step D-1.** Color edges \(x_i u_{i,k}\) by setting \(h_{fed}^*(x_i u_{i,k}) = 2k - 1\) for \(k \in [1, a_i]\), and

\[
h_{fed}^*(x_{i+1} u_{i+1,k}) = 2r - 1 + 2 \sum_{l=1}^{i} a_l, \ r \in [1, a_{i+1}], \ i \in [1, s - 1]
\]  \hspace{1cm} (39)

and \(h_{fed}^*(x_s u_{s,r}) = 2r - 1 + \sum_{l=1}^{s-1} 2a_l\) for \(s \in [1, a_s]\), so the last edge \(x_s u_{s,a_s}\) is colored with \(h_{fed}^*(x_s u_{s,a_s}) = -1 + \sum_{l=1}^{s-1} 2a_l\).

**Step D-2.** Color edges \(y_j v_{t,k}\) with \(h_{fed}^*(y_j v_{t,k}) = 2A_X + 2k - 1\) for \(k \in [1, b_t]\), and

\[
h_{fed}^*(y_{t+1} v_{t,k}) = 2A_X - 1 + 2b_t, \ h_{fed}^*(y_{t-1} v_{t-1,k}) = 2A_X - 1 + 2b_t + 2k, \ k \in [1, b_{t-1}]
\]

and \(h_{fed}^*(y_{t-1} v_{t-1,b_{t-1}}) = 2A_X - 1 + 2b_t + 2b_{t-1}\), and

\[
h_{fed}^*(y_{t-j} v_{t-j,k}) = 2k + 2A_X - 1 + 2 \sum_{l=t-j+1}^{t} b_l, \ k \in [1, b_{t-j}], \ j \in [1, t - 2]
\]  \hspace{1cm} (40)
the last edge \( y_1v_{1,b_1} \) is colored with \( h^*_f \)(\( y_1v_{1,b_1} \)) = 2A_X + 2B_X - 1 = 2m - 1. Therefore, the edge color set of the leaf-added graph \( G_D \) is as

\[
h^*_f(E(G_D)) = [1, 2(A_X + B_X) + \max h^*_f(E(G))]^o = [1, 2(q + m) - 1]^o. \tag{41}
\]

**Step D-3.** Recolor each element \( w \in V(G) \cup E(G) \) with \( h^*_f(w) = h_f(w) + 2m \). So,

\[
| h^*_f(x_i) + h^*_f(y_j) - h^*_f(x_iy_j) | = 2m + | h_f(x_i) + h_f(y_j) - h_f(x_iy_j) | = N_D + 2m \tag{42}
\]

**Step D-4.** We color added-leaves \( u_{i,k} \in L(x_i) \) with \( h^*_f(u_{i,k}) = N_D + h^*_f(x_iu_{i,k}) - h_f(x_i) \) for \( k \in [1, a_i] \) and \( i \in [1, s] \), since

\[
N_D + 2m = | h^*_f(u_{i,k}) + h^*_f(x_i) - h^*_f(x_iu_{i,k}) | = h^*_f(u_{i,k}) + h_f(x_i) + 2m - h^*_f(x_iu_{i,k}) \tag{43}
\]

Again, we color leaves \( v_{j,r} \in L(y_j) \) with \( h^*_f(v_{j,r}) = N_D + h^*_f(v_{j,r}y_j) - h_f(y_j) \) for \( r \in [1, b_j] \) and \( j \in [1, t] \), because of

\[
N_D + 2m = | h^*_f(v_{j,r}) + h^*_f(y_j) - h^*_f(v_{j,r}y_j) | = h^*_f(v_{j,r}) + h_f(y_j) + 2m - h^*_f(v_{j,r}y_j) \tag{44}
\]

Thereby, \( h^*_f \) is an odd-edge felicitous-difference total coloring of the leaf-added graph \( G_D \).

**Step D-5.** Return the odd-edge felicitous-difference total coloring \( h^*_f \) of the leaf-added graph \( G_D \).

**Example 11.** For understanding the RLA-algorithm-D of the odd-edge felicitous-difference total coloring, Fig.13 shows us the following facts:

(a) the graph \( G_{YR-twin} \) admits a twin set-ordered odd-edge felicitous-difference total labeling \( \theta^*_f \) holding \( h_f(V(G_{YR})) \cup h_f(V(G_{YR-twin})) \subseteq [0, 20] \);

(b) the graph \( G_{YR} \) admits a set-ordered odd-edge felicitous-difference total coloring \( h_f \) holding

\[
|h_f(x) + h_f(y) - h_f(xy)| = 6;
\]

(c) the graph \( G_{YR-leaf} \) admits a set-ordered odd-edge felicitous-difference total coloring \( h^*_f \) holding

\[
|h^*_f(x) + h^*_f(y) - h^*_f(xy)| = 32;
\]

(d) the graph \( H_{YR-leaf} \) admits a set-ordered odd-edge felicitous-difference total labeling \( g^*_f \) holding

\[
|g^*_f(x) + g^*_f(y) - g^*_f(xy)| = 32, \text{ as well as } G_{YR-leaf} \rightarrow H_{YR-leaf}. \]

**Problem 7.** In the RLA-algorithm-D of the odd-edge felicitous-difference total coloring, there are the following problems:

(i) **Estimate** the extremum number

\[
\min \{ \max h^*_f(V(G_A)) : h^*_f \text{ is an odd-edge felicitous-difference total coloring of } G_D \}. \tag{45}
\]

over all odd-edge felicitous-difference total colorings of the leaf-added graph \( G_D \).

(ii) **Find** other ways for constructing graphs admitting odd-edge felicitous-difference total colorings/labelings. Two examples \( G_{X-add} \) and \( G_{YR-add} \) show in Fig.14 are obtained by adding directly
Figure 13: Examples for illustrating RLA-algorithm-D of the odd-edge felicitous-difference total coloring. 

(iii) We have $m!$ permutations $w_{i_1}, w_{i_2}, \ldots, w_{i_m}$ from the added leaves of the leaf-added set $L^*(G_D) = (\bigcup_{i=1}^{s} L(x_i)) \bigcup (\bigcup_{j=1}^{t} L(y_j))$ of the leaf-added $(p + m, q + m)$-graph $G_D$, so the leaf permutation in the RLA-algorithm-D of the odd-edge felicitous-difference total coloring is one of these $m!$ permutations. We define a new coloring $F_{fed}$ for the leaf-added graph $G_D$ as: Color leaf-edge $w_{i_j}z_{i_j}$ with $F_{fed}(w_{i_j}z_{i_j}) = 2j - 1$ for $j \in [1, m]$, where $z_{i_j} \in X \cup Y = V(G)$, and each element $e \in V(G) \cup E(G) = [V(G_D) \cup E(G_D)] \setminus L^*(G_D)$ is colored as $F_{fed}(e) = h_{fed}(e)$, as well as color each added leaf $w_{i_j} \in L^*(G_D)$ by

$$F_{fed}(w_{i_j}) = N_D^* + F_{fed}(w_{i_j}z_{i_j}) - F_{fed}^*(z_{i_j}), \ N_D^* = 2m + N_D$$

By Eq. (42), Eq. (43) and Eq. (44), the coloring $F_{fed}$ is an odd-edge felicitous-difference total coloring based on a permutation $w_{i_1}, w_{i_2}, \ldots, w_{i_m}$.

**Example 12.** In Fig[14] we can see:

(a) The graph $G_X$ admits a set-ordered odd-edge felicitous-difference total labeling $f_{fed}$ holding $|f_{fed}(x) + f_{fed}(y) - f_{fed}(xy)| = 6$;

(b) the graph $G_X$-add obtained from $G_X$ by adding vertices and edges admits a set-ordered odd-edge felicitous-difference total labeling $f_{fed}^*$ holding $|f_{fed}^*(x) + f_{fed}^*(y) - f_{fed}^*(xy)| = 6$;

(c) the graph $G_Y$ admits a set-ordered odd-edge felicitous-difference total labeling $i_{fed}$ holding $|i_{fed}(x) + i_{fed}(y) - i_{fed}(xy)| = 6$;

(d) the graph $G_Y$-add admits a set-ordered odd-edge felicitous-difference total labeling $j_{fed}$ holding $|j_{fed}(x) + j_{fed}(y) - j_{fed}(xy)| = 6$. \qed
Figure 14: The examples for adding no leaf to graphs admitting $W$-magic total colorings.

3.5 RLA-algorithm-E for adding leaves continuously

**RLA-algorithm-E for adding leaves continuously.**

**Input:** A connected bipartite $(p, q)$-graph $G$ admitting an odd-edge graceful-difference total coloring $h_0^{gr}$.

**Output:** A connected bipartite $(p + m, q + m)$-graph $H$ admitting an odd-edge graceful-difference total coloring $h_1^{gr}$, where the $(p + m, q + m)$-graph $H$, called *leaf-added graph*, is the result of adding randomly $m$ leaves to $G$.

**Initialization.** Let $G$ be a bipartite $(p, q)$-graph admitting an odd-edge graceful-difference total coloring $h_0^{gr}$, and let $V(G) = \{u_1, u_2, \ldots, u_p\}$. By the definition of an odd-edge edge-magic total labeling, we have

$$0 \leq h_0^{gr}(u_1) < h_0^{gr}(u_2) < \cdots < h_0^{gr}(u_p) \leq 2q - 1 \quad (46)$$

so that each edge $u_iv_j \in E(G)$ satisfies the following equation

$$\left| |h_0^{gr}(u_i) - h_0^{gr}(v_j)| - h_0^{gr}(u_iv_j) \right| = \lambda_0 \quad (47)$$

where integer $\lambda_0 \geq 0$, as well as $h_0^{gr}(E(G)) = \{h_0^{gr}(u_iv_j) : u_iv_j \in E(G)\} = [1, 2q - 1]^o$.

**Step-E-1.** Adding new leaves $w_{i,1}, w_{i,2}, \ldots, w_{i,n_i}$ to each vertex $u_i$ of $G$ produces a leaf set $L(u_i) = \{w_{i,1}, w_{i,2}, \ldots, w_{i,n_i}\}$ with $i \in [1, p]$, here, it is allowed $n_j = 0$ for some $j \in [1, p]$. The resultant graph is denoted as $H = G + E^*(G)$, where the leaf edge set

$$E^*(G) = \{w_{i,j}u_i : w_{i,j} \in L(u_i), j \in [1, n_i], i \in [1, p]\}$$

and $m = |E^*(G)|$.

**Step-E-2.** Define a coloring $h_1^{gr}$ for the leaf-added graph $H$ as:

**Step-E-2.1. The ascending-order sub-algorithm.**

(1-1) Color $h_1^{gr}(w_{1,j}u_1) = 2j - 1$ with $j \in [1, n_1]$;
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(1-2) Color $h_i^gr(w_{i,j}u_i) = 2j - 1 + 2\sum_{k=1}^{i-1} n_k$ with $j \in [1, n_i]$ and $i \in [2, p]$.

(1-3) Color leaves $w_{s,t} \in \bigcup_{i=1}^{p} L(u_i)$ with $h_i^gr(w_{s,t})$ holding

$$\|h_i^gr(u_s) - h_i^gr(w_{s,t}) - h_i^gr(w_{s,t}u_s)\| = \lambda_0, \ t \in [1, n_s], \ s \in [1, p] \tag{48}$$

(1-4) Color each element $z \in V(G) \cup E(G)$ with $h_i^gr(z) = h_0^gr(z)$.

Thereby, $h_i^gr(u_1) = 0$, and

$$\|h_i^gr(x) - h_i^gr(y)| - h_i^gr(xy)\| = \lambda_0, \ xy \in E(H), \ h_i^gr(E(H)) = \left[1, |E(G)| + \sum_{k=1}^{p} n_k\right] \tag{49}$$

Step-E-2.2. The descending-order sub-algorithm.

(2-1) Color $h_i^gr(w_{p,j}u_1) = 2j - 1$ with $j \in [1, n_p]$;

(2-2) Color $h_i^gr(w_{i,j}u_i) = 2j - 1 + 2\sum_{k=1}^{p-i} n_p-k+1$ with $j \in [1, n_i]$ and $i \in [1, p - 1]$.

(2-3) Color leaves $w_{s,t} \in \bigcup_{i=1}^{p} L(u_i)$ with $h_i^gr(w_{s,t})$ holding Eq.(48) true.

(2-4) Color each element $z \in V(G) \cup E(G)$ with $h_i^gr(z) = h_0^gr(z)$.

We get $h_i^gr(u_1) = 0$ and Eq.(49).

Step-E-2.3. The random-order sub-algorithm.

(3-1) Color $h_i^gr(e_j) = 2j - 1$ with $j \in [1, A]$, where edges $e_1, e_2, \ldots, e_A$ is a permutation of leaf edges $w_{i,j}u_i$ with $j \in [1, n_i]$ and $i \in [1, p]$, and $A = \sum_{i=1}^{p} |L(u_i)|$.

(3-2) Color leaves $w_{s,t} \in \bigcup_{i=1}^{p} L(u_i)$ with $h_i^gr(w_{s,t})$ with $h_i^gr(w_{s,t})$ holding Eq.(48) true.

(3-3) Color each element $z \in V(G) \cup E(G)$ with $h_i^gr(z) = h_0^gr(z)$.

Thereby, $h_i^gr(u_1) = 0$ and Eq.(49) holds true.

Step-E-3. Return an odd-edge graceful-difference total coloring $h_i^gr$ of the leaf-added graph $H$.

Example 13. Fig.15 and Fig.16 show us examples for illustrating the RLA-algorithm for adding leaves continuously under the odd-edge graceful-difference total coloring:

In Fig.15 (a) A graph $H_{1\text{-leaf}}$ admits an odd-edge graceful-difference total coloring $f_1^gr$ holding $\|f_1^gr(x) - f_1^gr(y)| - f_1^gr(xy)\| = 26$ for $xy \in E(H_{1\text{-leaf}})$; (b) a graph $AH_{2\text{-leaf}}$ obtained by adding leaves to the graph $H_{1\text{-leaf}}$; (c) a graph $AH_{2\text{-leaf}}$ based on the graph $AH_{1\text{-leaf}}$ admits an odd-edge graceful-difference total coloring $f_2^gr$ holding $\|f_2^gr(x) - f_2^gr(y)| - f_2^gr(xy)\| = 26$ for $xy \in E(H_{2\text{-leaf}})$.

In Fig.16 (a) A graph $AH_{2\text{-leaf}}$ is obtained by adding leaves to the graph $H_{2\text{-leaf}}$ shown in Fig.15 (c); (b) a graph $AH_{3\text{-leaf}}$ based on the graph $AH_{2\text{-leaf}}$ admits an odd-edge graceful-difference total coloring $f_3^gr$ holding $\|f_2^gr(x) - f_2^gr(y)| - f_3^gr(xy)\| = 26$ for $xy \in E(H_{3\text{-leaf}})$.

By the RLA-algorithm-E for adding leaves continuously, we have

**Theorem 5.** Suppose that a connected bipartite $(p, q)$-graph $G_0$ admits an odd-edge graceful-difference total coloring $h_0^gr$, then there are connected bipartite graph sequence $\{G_k\}_{k=0}^n$ such that each connected bipartite graph $G_k \in \{G_k\}_{k=0}^n$ is obtained by adding randomly $a_k (\geq 1)$ leaves to $G_{k-1}$ and admits an odd-edge graceful-difference total coloring $h_k^gr$ with $k \geq 1$, and moreover $h_i^gr(V(G_i)) \cap h_j^gr(V(G_j)) \neq \emptyset$ for any pair integers $i, j \in [0, n]$.
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Theorem 6. Each tree admits an odd-edge graceful-difference total coloring.

Proof. Let $T$ be a tree of $q$ ($\geq 1$) edges. If $T$ is a star, that is, $T$ has its own vertex set $V(T) = \{u, x_i : i \in [1, q]\}$ and edge set $E(T) = \{ux_i : i \in [1, q]\}$. It is not hard to present an odd-edge graceful-difference total coloring (or labeling) for the star $T$, see six stars shown in Fig[17].

So, the leaf-removed graph $T_1 = T - L(T)$ is a tree still. If the leaf-removed graph $T_1$ is not a star, then we get the leaf-removed tree $T_2 = T_1 - L(T_1)$, go on in this way, we have leaf-removed trees $T_k = T_{k-1} - L(T_{k-1})$ with $k \leq \frac{D(T)}{2}$, where $D(T)$ is the diameter of $T$. Suppose that $T_m = T_{m-1} - L(T_{m-1})$ is a star, and $T_m$ admits an odd-edge graceful-difference total coloring (or labeling). Adding the leaves of $L(T_{k-1})$ to each leaf-removed tree $T_k$ for getting the tree $T_{k-1}$, and the tree $T_{k-1}$ admits an odd-edge graceful-difference total coloring (or labeling) $f_k^{gr}$, since $T_k$ admits an odd-edge graceful-difference total coloring (or labeling) $f_k^{gr}$ by the RLA-algorithm for adding leaves continuously, such that $f_i^{gr}(V(T_i)) \cap f_j^{gr}(V(T_j)) \neq \emptyset$.

The proof is complete. 

3.6 RLA-algorithm-F for adding leaves continuously

RLA-algorithm-F for adding leaves continuously.

Input: A connected bipartite $(p, q)$-graph $G$ admitting an odd-edge edge-magic total coloring $h_{0}^{ma}$.

Output: A connected bipartite $(p + m, q + m)$-graph $H$ admitting an odd-edge edge-magic total coloring $h_{1}^{ma}$, where the $(p + m, q + m)$-graph $H$, called leaf-added graph, is the result of adding randomly $m$ leaves to $G$.

Initialization. Let $G$ be a bipartite $(p, q)$-graph admitting an odd-edge edge-magic total coloring $h_{0}^{ma}$, and let $V(G) = \{u_1, u_2, \ldots, u_p\}$. By the definition of an odd-edge edge-magic total
Figure 16: An example for the RLA-algorithm-E for adding leaves continuously under the odd-edge graceful-difference total coloring.

Figure 17: Each tree $T_i$ admits an odd-edge graceful-difference total coloring (or labeling) $h_{gr}^*$ holding $|h_{gr}^*(x) - h_{gr}^*(y)| - h_{gr}^*(xy)| = k_i$ for $xy \in E(T_i)$ and $i \in [1, 6]$, where $k_1 = k_2 = k_5 = k_6 = 2, k_3 = k_4 = 0$.

labeling, we have

$$0 \leq h_{0}^{ma}(u_1) < h_{0}^{ma}(u_2) < \cdots < h_{0}^{ma}(u_p) \leq 2q - 1 \quad (50)$$

so that each edge $u_iv_j \in E(G)$ satisfies the following equation

$$h_{0}^{ma}(u_i) + h_{0}^{ma}(v_j) + h_{0}^{ma}(u_iv_j) = \mu_0 \quad (51)$$

where integer $\mu_0 \geq 0$, as well as $h_{0}^{ma}(E(G)) = \{h_{0}^{ma}(u_iv_j) : u_iv_j \in E(G)\} = [1, 2q - 1]^\circ$.

**Step-F-1.** Adding new leaves $w_{i,1}, w_{i,2}, \ldots, w_{i,n_i}$ to each vertex $u_i$ of $G$ produces a leaf set $L(u_i) = \{w_{i,1}, w_{i,2}, \ldots, w_{i,n_i}\}$ with $i \in [1, p]$, here, it is allowed $n_j = 0$ for some $j \in [1, p]$. The resultant graph is denoted as $H = G + E^*(G)$, where the leaf edge set

$$E^*(G) = \{w_{i,j}u_i : w_{i,j} \in L(u_i), j \in [1, n_i], i \in [1, p]\} \quad (52)$$
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and $m = |E'(G)|$.

**Step-F-2.** Define a coloring $h_{1}^{ma}$ for the leaf-added graph $H$ in the following:

**Step-F-2.1. The ascending-order sub-algorithm.**

(1-1) Color $h_{1}^{ma}(w_{1,j}u_{1}) = 2j - 1$ with $j \in [1, n_{1}]$;
(1-2) Color $h_{1}^{ma}(w_{i,j}u_{i}) = 2j - 1 + 2\sum_{k=1}^{i-1} n_{k}$ with $j \in [1, n_{i}]$ and $i \in [2, p]$.
(1-3) Color leaves $w_{s,t} \in \bigcup_{i=1}^{p} L(u_{i})$ with $h_{1}^{ma}(w_{s,t})$ holding

$$h_{1}^{ma}(u_{s}) + h_{1}^{ma}(w_{s,t}) + h_{1}^{ma}(w_{s,t}u_{s}) = \mu_{0}^{*}, \ t \in [1, n_{s}], \ s \in [1, p]$$

where $\mu_{0}^{*} = \mu_{0} + 2m$.
(1-4) Color each edge $uv \in E(G)$ with $h_{1}^{ma}(uv) = h_{0}^{ma}(uv) + 2m$, and color each vertex $w \in V(G)$ with $h_{1}^{ma}(w) = h_{0}^{ma}(w)$.

Thereby, $h_{1}^{ma}(x) + h_{1}^{ma}(y) + h_{1}^{ma}(xy) = \mu_{0}^{*}$ for each edge $xy \in E(H)$, and $h_{1}^{ma}(u_{1}) = 0$, as well as the edge color set

$$h_{1}^{ma}(E(H)) = \left[1, |E(G)| + \sum_{k=1}^{p} n_{k}\right]^{o}$$

(2-1) Color $h_{1}^{ma}(w_{p,j}u_{1}) = 2j - 1$ with $j \in [1, n_{p}]$;
(2-2) Color $h_{1}^{ma}(w_{i,j}u_{i}) = 2j - 1 + 2\sum_{k=1}^{p-i} n_{p-k+1}$ with $j \in [1, n_{i}]$ and $i \in [1, p-1]$.
(2-3) Color leaves $w_{s,t} \in \bigcup_{i=1}^{p} L(u_{i})$ holding Eq. (53) true.
(2-4) Color each edge $uv \in E(G)$ with $h_{1}^{ma}(uv) = h_{0}^{ma}(uv) + 2m$, and color each vertex $w \in V(G)$ with $h_{1}^{ma}(w) = h_{0}^{ma}(w)$.

We get $h_{1}^{ma}(u_{1}) = 0$ and Eq. (54).

**Step-F-2.2. The descending-order sub-algorithm.**

(3-1) Color $h_{1}^{ma}(e_{j}) = 2j - 1$ with $j \in [1, A]$, where edges $e_{1}, e_{2}, \ldots, e_{A}$ is a permutation of leaf edges $w_{i,j}u_{i}$ with $j \in [1, n_{i}]$ and $i \in [1, p]$, and $A = \sum_{i=1}^{p} |L(u_{i})|$.
(3-2) Color leaves $w_{s,t} \in \bigcup_{i=1}^{p} L(u_{i})$ with $h_{1}^{ma}(w_{s,t})$ with $h_{1}^{ma}(w_{s,t})$ holding Eq. (53) true.
(3-3) Color each edge $uv \in E(G)$ with $h_{1}^{ma}(uv) = h_{0}^{ma}(uv) + 2m$, and color each vertex $w \in V(G)$ with $h_{1}^{ma}(w) = h_{0}^{ma}(w)$.

Thereby, $h_{1}^{ma}(u_{1}) = 0$ and Eq. (54) holds true.

**Step-F-3.** Return an odd-edge edge-magic total coloring $h_{1}^{ma}$ of the leaf-added graph $H$.

**Example 14.** Fig. [18] and Fig. [19] show us examples for illustrating the RLA-algorithm-F for adding leaves continuously under the odd-edge edge-magic total coloring:

In Fig. [18] (a) A graph $T_{1}$-leaf admits an odd-edge edge-magic total coloring $h_{1}^{ma}$ holding $h_{1}^{ma}(x) + h_{1}^{ma}(y) + h_{1}^{ma}(xy) = 52$ for $xy \in E(T_{1}$-leaf); (b) a graph $AT_{1}$-leaf obtained by adding leaves to the graph $T_{1}$-leaf; (c) a graph $T_{2}$-leaf based on the graph $AT_{1}$-leaf admits an odd-edge edge-magic total coloring $h_{2}^{ma}$ holding $h_{2}^{ma}(x) + h_{2}^{ma}(y) + h_{2}^{ma}(xy) = 78$ for $xy \in E(T_{2}$-leaf).

In Fig. [19] (a) A graph $AT_{2}$-leaf obtained by adding leaves to the graph $T_{2}$-leaf shown in Fig. [18] (c); (b) a graph $T_{3}$-leaf based on the graph $AT_{2}$-leaf admits an odd-edge edge-magic total coloring $h_{3}^{ma}$ holding $h_{3}^{ma}(x) + h_{3}^{ma}(y) + h_{3}^{ma}(xy) = 102$ for $xy \in E(T_{3}$-leaf).
Theorem 7. Suppose that a connected bipartite \((p, q)\)-graph \(G_0\) admits an odd-edge edge-magic total coloring \(h_0^{gr}\), then there are connected bipartite graph sequence \(\{G_k\}_{k=0}^n\) such that each connected bipartite graph \(G_k \in \{G_k\}_{k=0}^n\) is obtained by adding randomly \(a_k \geq 1\) leaves to \(G_{k-1}\) and admits an odd-edge edge-magic total coloring \(h_k^{gr}\) with \(k \geq 1\), and moreover \(h_i^{gr}(V(G_i)) \cap h_j^{gr}(V(G_j)) \neq \emptyset\) for any pair integers \(i, j \in [0, n]\).

Theorem 8. Each tree admits an odd-edge edge-magic total coloring.

3.7 RLA-algorithm-G for adding leaves continuously

RLA-algorithm-G for adding leaves continuously.

Input: A connected bipartite \((p, q)\)-graph \(G\) admitting an odd-edge edge-difference total coloring \(h_0^{ed}\).

Output: A connected bipartite \((p + m, q + m)\)-graph \(H\) admitting an odd-edge edge-difference total coloring \(h_1^{ed}\), where the \((p + m, q + m)\)-graph \(H\), called leaf-added graph, is the result of adding randomly \(m\) leaves to \(G\).

Initialization. Let \(G\) be a bipartite \((p, q)\)-graph admitting an odd-edge edge-difference total coloring \(h_0^{ed}\), and let \(V(G) = \{u_1, u_2, \ldots, u_p\}\). By the definition of an odd-edge edge-difference total labeling, we have

\[
0 \leq h_0^{ed}(u_1) < h_0^{ed}(u_2) < \cdots < h_0^{ed}(u_p) \leq 2q - 1
\]  

so that each edge \(u_iv_j \in E(G)\) satisfies the following equation

\[
h_0^{ed}(u_iv_j) + |h_0^{ed}(u_i) - h_0^{ed}(v_j)| = \rho_0
\]  

where integer \(\rho_0 \geq 0\), as well as \(h_0^{ed}(E(G)) = \{h_0^{ed}(u_iv_j) : u_iv_j \in E(G)\} = [1, 2q - 1]^o\).
Figure 19: An example for the RLA-algorithm-F for adding leaves continuously under the odd-edge edge-magic total coloring.

Step-G-1. Adding new leaves \( w_{i,1}, w_{i,2}, \ldots, w_{i,n_i} \) to each vertex \( u_i \) of \( G \) produces a leaf set \( L(u_i) = \{ w_{i,1}, w_{i,2}, \ldots, w_{i,n_i} \} \) with \( i \in [1, p] \), here, it is allowed \( n_j = 0 \) for some \( j \in [1, p] \). The resultant graph is denoted as \( H = G + E^*(G) \), where the leaf edge set

\[
E^*(G) = \{ w_{i,j} : w_{i,j} \in L(u_i), j \in [1, n_i], i \in [1, p] \}
\]

having \( m = |E^*(G)| \) edges.

Step-G-2. Define a coloring \( h_1^{ed} \) for the leaf-added graph \( H \) in the following steps:

**Step-G-2.1. The ascending-order sub-algorithm.**

1-1 Color \( h_1^{ed}(w_{1,j}u_1) = 2j - 1 \) with \( j \in [1, n_1] \);
1-2 Color \( h_1^{ed}(w_{i,j}u_i) = 2j - 1 + 2 \sum_{k=1}^{i-1} n_k \) with \( j \in [1, n_i] \) and \( i \in [2, p] \).
1-3 Color leaves \( w_{s,t} \in \bigcup_{i=1}^{p} L(u_i) \) with \( h_1^{ed}(w_{s,t}) \) holding

\[
h_1^{ed}(w_{s,t}u_s) + |h_1^{ed}(u_s) - h_1^{ed}(w_{s,t})| = \rho_0^s, \ t \in [1, n_s], \ s \in [1, p]
\]

with \( \rho_0^s = \rho_0 + 2m \).
1-4 Color each edge \( uv \in E(G) \) with \( h_1^{ed}(uv) = h_0^{ed}(uv) + 2m \), and color each vertex \( w \in V(G) \) with \( h_1^{ed}(w) = h_0^{ed}(w) \).

Thereby, \( h_1^{ed}(xy) + |h_1^{ed}(x) - h_1^{ed}(y)| = \rho_0^y \) for each edge \( xy \in E(H) \), \( h_1^{ed}(u_1) = 0 \), and

\[
h_1^{ed}(E(H)) = \left[ 1, \ |E(G)| + \sum_{k=1}^{p} n_k \right]^o
\]
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Figure 20:  (a) The graph $H_{\text{v-con}}$ is obtained by vertex-coinciding vertices colored with the same colors in the graph $H_{1\text{-leaf}}$ shown in Fig.16(c); (b) the graph $T_{\text{v-con}}$ is obtained by vertex-coinciding vertices colored with the same colors in the graph $T_{3\text{-leaf}}$ shown in Fig.19(c).

**Step-G-2.2. The descending-order sub-algorithm.**

1. Color $h_i^{ed}(w_{p,j}u_1) = 2j - 1$ with $j \in [1, n_p]$;
2. Color $h_i^{ed}(w_{i,j}u_i) = 2j - 1 + 2\sum_{k=1}^{p-i} n_{p-k+1}$ with $j \in [1, n_i]$ and $i \in [1, p - 1]$.
3. Color leaves $w_{s,t} \in \bigcup_{i=1}^{p} L(u_i)$ holding Eq.(58) true.
4. Color each edge $uv \in E(G)$ with $h_1^{ed}(uv) = h_0^{ed}(uv) + 2m$, and color each vertex $w \in V(G)$ with $h_1^{ed}(w) = h_0^{ed}(w)$.

We get $h_1^{ed}(u_1) = 0$ and Eq.(59).

**Step-G-2.3. The random-order sub-algorithm.**

1. Color $h_i^{ed}(e_j) = 2j - 1$ with $j \in [1, A]$, where edges $e_1, e_2, \ldots, e_A$ is a permutation of leaf edges $w_{i,j}u_i$ with $j \in [1, n_i]$ and $i \in [1, p]$, and $A = \sum_{i=1}^{p} |L(u_i)|$.
2. Color leaves $w_{s,t} \in \bigcup_{i=1}^{p} L(u_i)$ with $h_1^{ed}(w_{s,t})$ holding Eq.(58) true.
3. Color each edge $uv \in E(G)$ with $h_1^{ed}(uv) = h_0^{ed}(uv) + 2m$, and color each vertex $w \in V(G)$ with $h_1^{ed}(w) = h_0^{ed}(w)$.

Thereby, $h_1^{ed}(u_1) = 0$ and Eq.(59) holds true.

**Step-G-3.** Return an odd-edge edge-difference total coloring $h_1^{ed}$ of the leaf-added graph $H$.

**Example 15.** Fig.21 and Fig.22 show us examples for illustrating the RLA-algorithm-G for adding leaves continuously under odd-edge edge-difference total coloring:

In Fig.21 (a) a graph $G_{1\text{-leaf}}$ admits an odd-edge edge-difference total coloring $h_1^{ed}$ holding $h_1^{ed}(xy) + |h_1^{ed}(x) - h_1^{ed}(y)| = 46$ for $xy \in E(G_{1\text{-leaf}})$; (b) a graph $AG_{1\text{-leaf}}$ is obtained by adding
leaves to the graph $G_{1\text{-leaf}}$; (c) a graph $G_{2\text{-leaf}}$ based on the graph $AG_{1\text{-leaf}}$ admits an odd-edge edge-difference total coloring $h_{2\text{ed}}$ holding $h_{2\text{ed}}(xy) + |h_{2\text{ed}}(x) - h_{2\text{ed}}(y)| = 70$ for $xy \in E(G_{2\text{-leaf}})$.

In Fig. 22 (a) a graph $AG_{2\text{-leaf}}$ is obtained by adding leaves to the graph $G_{2\text{-leaf}}$ shown in Fig. 21 (c); (b) a graph $G_{3\text{-leaf}}$ based on the graph $AG_{2\text{-leaf}}$ admits an odd-edge edge-difference total coloring $h_{3\text{ed}}$ holding $h_{3\text{ed}}(xy) + |h_{3\text{ed}}(x) - h_{3\text{ed}}(y)| = 92$ for $xy \in E(G_{3\text{-leaf}})$.

\[\text{Figure 21: Examples for the RLA-algorithm-G for adding leaves continuously under odd-edge edge-difference total coloring.}\]

**Theorem 9.** Suppose that a connected bipartite $(p,q)$-graph $G_0$ admits an odd-edge edge-difference total coloring $h_{0\text{ed}}$, then there are connected bipartite graph sequence $\{G_k\}_{k=0}^n$ such that each connected bipartite graph $G_k \in \{G_k\}_{k=0}^n$ is obtained by adding randomly $a_k (\geq 1)$ leaves to $G_{k-1}$ and admits an odd-edge edge-difference total coloring $h_{k\text{ed}}$ with $k \geq 1$, and moreover $h_{i\text{ed}}(V(G_i)) \cap h_{j\text{ed}}(V(G_j)) = \emptyset$ for any pair integers $i,j \in [0,n]$.

**Theorem 10.** Each tree admits an odd-edge edge-difference total coloring.

### 3.8 RLA-algorithm-H for adding leaves continuously

RLA-algorithm-H for adding leaves continuously.

**Input:** A connected bipartite $(p,q)$-graph $G$ admitting an odd-edge felicitous-difference total coloring $h_{0\text{fe}}$.

**Output:** A connected bipartite $(p + m,q + m)$-graph $H$ admitting an odd-edge felicitous-difference total coloring $h_{1\text{fe}}$, where the $(p + m,q + m)$-graph $H$, called leaf-added graph, is the result of adding randomly $m$ leaves to $G$. 

\[\text{Figure 21: Examples for the RLA-algorithm-G for adding leaves continuously under odd-edge edge-difference total coloring.}\]
Figure 22: An example for the RLA-algorithm-G for adding leaves continuously under odd-edge edge-difference total coloring.

**Initialization.** Let $G$ be a bipartite $(p,q)$-graph admitting an odd-edge felicitous-difference total coloring $h^f_0$, and let $V(G) = \{u_1, u_2, \ldots, u_p\}$. By the definition of an odd-edge felicitous-difference total labeling, we have

$$0 \leq h^f_0(u_1) < h^f_0(u_2) < \cdots < h^f_0(u_p) \leq 2q - 1$$

so that each edge $u_iv_j \in E(G)$ satisfies the following equation

$$|h^f_0(u_i) + h^f_0(v_j) - h^f_0(u_iv_j)| = \xi_0$$

where integer $\xi_0 \geq 0$, as well as $h^f_0(E(G)) = \{h^f_0(u_iv_j) : u_iv_j \in E(G)\} = [1, 2q - 1]^o$.

**Step-H-1.** Adding new leaves $w_{i,1}, w_{i,2}, \ldots, w_{i,n_i}$ to each vertex $u_i$ of $G$ produces a leaf set $L(u_i) = \{w_{i,1}, w_{i,2}, \ldots, w_{i,n_i}\}$ with $i \in [1,p]$, here, it is allowed $n_j = 0$ for some $j \in [1,p]$. The resultant graph is denoted as $H = G + E^*(G)$, where the leaf edge set

$$E^*(G) = \{w_{i,j}u_i : w_{i,j} \in L(u_i), j \in [1,n_i], i \in [1,p]\}$$

having $m = |E^*(G)|$ edges.

**Step-H-2.** Define a coloring $h^f_1$ for the leaf-added graph $H$ in the following steps:

**Step-H-2.1. The ascending-order sub-algorithm.**

(1-1) Color $h^f_1(w_{1,j}u_1) = (2q - 1) + 2j$ with $j \in [1,n_1]$;
(1-2) Color \( h_1^{fe}(w_{i,j}u_i) = (2q - 1) + 2j + 2\sum_{k=1}^{i-1} n_k \) with \( j \in [1, n_i] \) and \( i \in [2, p] \).

(1-3) Color leaves \( w_{s,t} \in \bigcup_{i=1}^{p} L(u_i) \) with \( h_1^{fe}(w_{s,t}) \) holding
\[
|h_1^{fe}(u_s) + h_1^{fe}(w_{s,t}) - h_1^{fe}(w_{s,t}u_s)| = \xi_0, \quad t \in [1, n_s], \quad s \in [1, p]
\] (63)

(1-4) Color each edge \( w \in V(G) \cup E(G) \) with \( h_1^{fe}(w) = h_0^{fe}(w) \).

Thereby, we get \( h_1^{fe}(u_1) = 0 \) and
\[
|h_1^{fe}(x) + h_1^{fe}(y) - h_1^{fe}(xy)| = \xi_0, \quad x, y \in E(H); \quad h_1^{fe}(E(H)) = \left[ 1, |E(G)| + \sum_{k=1}^{p} n_k \right]^0
\] (64)

**Step-H-2.2. The descending-order sub-algorithm.**

(2-1) Color \( h_1^{fe}(w_{p,j}u_1) = (2q - 1) + 2j \) with \( j \in [1, n_p] \);

(2-2) Color \( h_1^{fe}(w_{i,j}u_i) = (2q - 1) + 2j + 2\sum_{k=1}^{p} n_{p-k+1} \) with \( j \in [1, n_i] \) and \( i \in [1, p - 1] \).

(2-3) Color leaves \( w_{s,t} \in \bigcup_{i=1}^{p} L(u_i) \) holding Eq.(63) true.

(2-4) Color each edge \( w \in V(G) \cup E(G) \) with \( h_1^{fe}(w) = h_0^{fe}(w) \).

We get \( h_1^{fe}(u_1) = 0 \) and Eq.(64) holds true.

**Step-H-2.3. The random-order sub-algorithm.**

(3-1) Color \( h_1^{fe}(e_j) = (2q - 1) + 2j \) with \( j \in [1, A] \), where edges \( e_1, e_2, \ldots, e_A \) is a permutation of leaf edges \( w_{i,j}u_i \) with \( j \in [1, n_i] \) and \( i \in [1, p] \), and \( A = \sum_{i=1}^{p} |L(u_i)| \).

(3-2) Color leaves \( w_{s,t} \in \bigcup_{i=1}^{p} L(u_i) \) with \( h_1^{fe}(w_{s,t}) \) with \( h_1^{fe}(w_{s,t}) \) holding Eq.(63) true.

(3-3) Color each edge \( w \in V(G) \cup E(G) \) with \( h_1^{fe}(w) = h_0^{fe}(w) \).

Thereby, \( h_1^{fe}(u_1) = 0 \) and Eq.(64) holds true.

**Step-H-3.** Return an odd-edge felicitous-difference total coloring \( h_1^{fe} \) of the leaf-added graph \( H \).

**Example 16.** Fig.23 and Fig.24 show us examples for illustrating the RLA-algorithm-H for adding leaves continuously under the odd-edge felicitous-difference total coloring:

In Fig.23 (a) A graph \( J_{1-leaf} \) admits an odd-edge felicitous-difference total coloring \( h_1^{fe} \) holding
\[
|h_1^{fe}(x) + h_1^{fe}(y) - h_1^{fe}(xy)| = 20 \quad \text{for } x, y \in E(J_{1-leaf});
\] (b) a graph \( AJ_{1-leaf} \) obtained by adding leaves to the graph \( J_{1-leaf} \); (c) a graph \( J_{2-leaf} \) based on the graph \( AJ_{1-leaf} \) admits an odd-edge felicitous-difference total coloring \( h_2^{fe} \) holding
\[
|h_2^{fe}(x) + h_2^{fe}(y) - h_2^{fe}(xy)| = 20 \quad \text{for } x, y \in E(J_{2-leaf}).
\]

In Fig.23 (a) a graph \( AJ_{2-leaf} \) obtained by adding leaves to the graph \( J_{2-leaf} \) shown in Fig.23 (c); (b) a graph \( J_{3-leaf} \) based on the graph \( AJ_{2-leaf} \) admits an odd-edge felicitous-difference total coloring \( h_3^{fe} \) holding
\[
|h_3^{fe}(x) + h_3^{fe}(y) - h_3^{fe}(xy)| = 20 \quad \text{for } x, y \in E(J_{3-leaf}).
\]

**Theorem 11.** Suppose that a connected bipartite \((p,q)\)-graph \( G_0 \) admits an odd-edge felicitous-difference total coloring \( h_0^{gr} \), then there are connected bipartite graph sequence \( \{G_k\}_{k=0}^\infty \) such that each connected bipartite graph \( G_k \in \{G_k\}_{k=0}^\infty \) is obtained by adding randomly \( a_k \geq 1 \) leaves to \( G_{k-1} \) and admits an odd-edge felicitous-difference total coloring \( h_k^{gr} \) with \( k \geq 1 \), and moreover \( h_i^{gr}(V(G_i)) \cap h_j^{gr}(V(G_j)) \neq \emptyset \) for any pair integers \( i, j \in [0, n] \).

**Theorem 12.** Each tree admits an odd-edge felicitous-difference total coloring.
3 ALGORITHMS OF ADDING LEAVES RANDOMLY

(a) $J_1$-leaf  (b) $AJ_2$-leaf  (c) $J_2$-leaf

Figure 23: Examples for the RLA-algorithm-H for adding leaves continuously under the odd-edge felicitous-difference total coloring.

3.9 Theorems based on the $W$-magic total colorings

If a graph admits a set-ordered graceful total coloring, then it admits a set-ordered odd-edge $W$-magic total coloring, where $W$-magic $\in \{\text{edge-magic, edge-difference, felicitous-difference, graceful-difference}\}$.

Theorem 13. If a connected graph admits a set-ordered odd-graceful total labeling, then it admits the following total colorings:

1. a set-ordered odd-edge edge-magic total labeling;
2. a set-ordered odd-edge edge-difference total labeling;
3. a set-ordered odd-edge felicitous-difference total labeling;
4. a set-ordered odd-edge graceful-difference total labeling,
which are equivalent to each other.

Proof. Suppose that a connected bipartite $(p, q)$-graph $G$ admits a set-ordered odd-graceful total labeling $f$. Let $(X, Y)$ be the bipartition of $V(G)$, where $X = \{x_1, x_2, \ldots, x_s\}$ and $Y = \{y_1, y_2, \ldots, y_t\}$ ($s + t = p$). Since $f$ is a set-ordered odd-graceful total labeling, without loss of generality, the vertex colors can be arranged into

$$0 = f(x_1) < f(x_2) < \cdots < f(x_s) < f(y_1) < f(y_2) < \cdots < f(y_t) = 2q - 1$$

also $\max f(X) < \min f(Y)$, and each edge $x_iy_j \in E(G)$ holds

$$f(x_iy_j) = |f(y_j) - f(x_i)| = f(y_j) - f(x_i) \in [1, 2q - 1]^o$$
and the edge color set \( f(E(G)) = [1, 2q - 1]^o \).

**Algorithm-1.** We define a dual total labeling \( f_{oed}^* \) by means of the set-ordered odd-graceful total labeling \( f \) of the graph \( G \) as follows: Each vertex \( w \in V(G) \) is colored as

\[
f_{oed}^*(w) = \max f(V(G)) + \min f(V(G)) - f(w) = 2q - 1 - f(w)
\]

and each edge \( x_iy_j \in E(G) \) is colored as

\[
f_{oed}^*(x_iy_j) = \max f(E(G)) + \min f(E(G)) - f(x_iy_j) = 2q - f(x_iy_j)
\]

Then the edge color set \( f_{oed}^*(E(G)) = f(E(G)) = [1, 2q - 1]^o \). Since

\[
f_{oed}^*(x_iy_j) + |f_{oed}^*(y_j) - f_{oed}^*(x_i)| = 2q - f(x_iy_j) + |f(y_j) - f(x_i)| = 2q
\]

and

\[
0 = f_{oed}^*(y_t) < f_{oed}^*(y_{t-1}) < \cdots < f_{oed}^*(y_1) < f_{oed}^*(x_s) < f_{oed}^*(x_{s-1}) < \cdots < f_{oed}^*(x_1) = 2q - 1
\]

By Definition 8 we claim that the dual total labeling \( f_{oed}^* \) is really a set-ordered odd-edge edge-difference total labeling of \( G \).

**Algorithm-2.** We use the set-ordered odd-graceful total labeling \( f \) of the graph \( G \) to define a total labeling \( g_{oed}^* \) as : \( g_{oed}^*(x_i) = \max f(X) + \min f(X) - f(x_i) = \max f(X) - f(x_i) \) for \( x_i \in X \) and

\[
g_{oed}^*(y_j) = \max f(Y) + \min f(Y) - f(y_j) = 2q - 1 + \min f(Y) - f(y_j), \ y_j \in Y
\]

Figure 24: An examples for the RLA-algorithm-H for adding leaves continuously under the odd-edge felicitous-difference total coloring.
and each edge \( x_iy_j \in E(G) \) is recolored as
\[
g^*_ogd(x_iy_j) = \max f(E(G)) + \min f(E(G)) - f(x_iy_j) = 2q - f(x_iy_j),
\]
so the edge color set \( g^*_ogd(E(G)) = f(E(G)) = [1, 2q - 1]^o \). Because of
\[
\begin{align*}
|g^*_ogd(x_i) - g^*_ogd(y_j)| &= g^*_ogd(y_j) - g^*_ogd(x_i) \\
&= [\max f(Y) + \min f(Y) - f(y_j)] - [\max f(X) + \min f(X) - f(x_i)] \\
&= [\max f(Y) + \min f(Y)] - [\max f(X) + \min f(X)] - f(x_iy_j) \\
&= 2q - 1 + \min f(Y) - \max f(X) - f(x_iy_j)
\end{align*}
\] (70)
and \( g^*_ogd(u) \neq g^*_ogd(v) \) for any pair of two vertices \( u, v \in V(G) \), we can compute
\[
\begin{align*}
|g^*_ogd(y_j) - g^*_ogd(x_i) - g^*_ogd(x_iy_j)| &= |f(x_iy_j) - [2q - f(x_iy_j)]| \\
&= \min f(Y) - \max f(X) - 1,
\end{align*}
\] (71)
for each edge \( x_iy_j \in E(G) \), notice that \( \min f(Y) - \max f(X) - 1 \) is a constant, and \( \max g^*_ogd(X) < \min g^*_ogd(Y) \), thereby we say that the total labeling \( g^*_ogd \) is a set-ordered odd-edge graceful-difference total labeling of \( G \) according to Definition [8].

**Algorithm-3.** The set-ordered odd-graceful total labeling \( f \) of the graph \( G \) can induce a total labeling \( h^*_ofd \) as: \( h^*_ofd(x_i) = \max f(X) + \min f(X) - f(x_i) = \max f(X) - f(x_i) \) for \( x_i \in X \), \( h^*_ofd(y_j) = f(y_j) \) for \( y_j \in Y \), and each edge \( x_iy_j \) is recolored by \( h^*_ofd(x_iy_j) = f(x_iy_j) \) \( x_iy_j \in E(G) \), clearly, \( h^*_ofd(E(G)) = f(E(G)) = [1, 2q - 1]^o \). Since
\[
\begin{align*}
h^*_ofd(x_i) + h^*_ofd(y_j) - h^*_ofd(x_iy_j) &= \max f(X) + \min f(X) - f(x_i) + f(y_j) - f(x_iy_j) \\
&= \max f(X)
\end{align*}
\] (72)
Definition [8] shows that the total labeling \( h^*_ofd \) is a set-ordered odd-edge felicitous-difference total labeling of \( G \).

Again we define a total labeling \( h^*_ofd \) in the following way: \( h^*_ofd(w) = h^*_ofd(w) \) for \( w \in V(G) \), and each edge \( x_iy_j \in E(G) \) is recolored as
\[
h^*_ofd(x_iy_j) = \max f(E(G)) + \min f(E(G)) - f(x_iy_j) = 2q - f(x_iy_j)
\]
so the edge color set \( h^*_ofd(E(G)) = f(E(G)) = [1, 2q - 1]^o \). Notice that
\[
\begin{align*}
h^*_ofd(x_i) + h^*_ofd(x_iy_j) + h^*_ofd(y_j) &= h^*_ofd(x_i) + 2q - f(x_iy_j) + h^*_ofd(y_j) \\
&= \max f(X) + h^*_ofd(x_iy_j) + 2q - f(x_iy_j) \\
&= \max f(X) + f(x_iy_j) + 2q - f(x_iy_j) \\
&= 2q + \max f(X)
\end{align*}
\] (73)
we claim that the total labeling \( h^*_ofd \) is a set-ordered odd-edge edge-magic total labeling of \( G \) from Definition [8]
Algorithm-4. Using the set-ordered odd-graceful total labeling $f$, a total labeling $h^*_{setY}$ of $G$ can be defined as: $h^*_{setY}(x_i) = f(x_i)$ for $x_i \in X$, $h^*_{setY}(y_j) = \max f(Y) + \min f(Y) - f(y_j) = 2q - 1 + \min f(Y) - f(y_j)$ for $y_j \in Y$, as well as each edge $x_iy_j \in E(G)$ is colored by
\[
h^*_{setY}(x_iy_j) = \max f(E(G)) + \min f(E(G)) - f(x_iy_j) = 2q - f(x_iy_j)
\]
we get the edge color set $h^*_{setY}(E(G)) = f(E(G)) = [1, 2q - 1]$. Moreover, we have
\[
h^*_{setY}(x_i) + h^*_{setY}(y_j) - h^*_{setY}(x_iy_j) = f(x_i) + 2q - 1 + \min f(Y) - f(y_j) - [2q - f(x_iy_j)] \]
\[
= \min f(Y) - 1.
\]
Definition enables us to claim that the total labeling $h^*_{setY}$ is a set-ordered odd-edge felicitous-difference total labeling of $G$.

Notice that all transformations in the above four algorithms are linear transformations, thereby, we have completed the necessity and sufficient proof of the theorem.

\[\square\]

Theorem 14. If a graph admits a set-ordered graceful total coloring, then it admits a set-ordered odd-edge $W$-magic total coloring, where $W$-magic is one of edge-magic, edge-difference, felicitous-difference and graceful-difference.

4 Graph lattices based on uniformly-$k^*$ $W$-magic total colorings

An odd-edge $W$-magic total labeling (or coloring) is one of odd-edge felicitous-difference total labeling (or coloring), odd-edge edge-difference total labeling (or coloring), odd-edge graceful-difference total labeling (or coloring) and odd-edge edge-magic total labeling (or coloring) in this section.

Graph lattices include: linear-graphic lattices and non-linear-graphic lattices. Simply, a linear-graph lattice is a set of trees obtained from a tree-base $T^c = \{T_1^c, T_2^c, \ldots, T_m^c\}$ and graph operations, where each $T_i$ is a tree; and a non-linear-graphic lattice is the set of graphs obtained from a non-tree base $H^c = \{H_1^c, H_2^c, \ldots, H_m^c\}$ and graph operations, where there is at least one colored graph $H_i^c$ to be a non-tree graph admitting an odd-edge $W$-magic total labeling.

4.1 Uniformly-$k^*$ $W$-magic graphic lattices

4.1.1 Uniformly-$k^*$ graceful-difference graphic lattices

Let $G_{k^*}$-magic $= \{G_1^c, G_2^c, \ldots, G_m^c\}$ be a uniformly-$k^*$ graceful-difference base with each $G_i^c$ is a connected bipartite graph and admits an odd-edge graceful-difference total labeling (or coloring) $h^{gr}_{i}$ holding
\[
|h^{gr}_{i}(x) - h^{gr}_{i}(y) - h^{gr}_{i}(xy)| = k^* \geq 0, \quad xy \in E(G_i^c), \quad i \in [1, m]
\]
as well as $h^{gr}_{i}(u_i) = 0$ for some $u_i \in E(G_i^c)$, and $G_i^c \neq G_j^c$ for $i \neq j$.

We abbreviate “Leaf-addling-randomly vertex-coinciding” as “LARVC” in the following argument.
LARVC uniformly-$k^*$ graceful-difference algorithm.

**Larvc-Step-1.** Let $T_{i_1}, T_{i_2}, \ldots, T_{i_M}$ be a permutation of graphs $\alpha_1 G_1, \alpha_2 G_2, \ldots, \alpha_m G_m$ based on a uniformly-$k^*$ graceful-difference base $G_{k^*}$-magic, where $M = \sum_{k=1}^{m} \alpha_k \geq 1$, and each graph $T_{i_s}$ is connected and admits an odd-edge graceful-difference total labeling (or coloring) $h_{i_s}^{gr}$.

**Larvc-Step-2.** Adding $m_{ij}$ ($\geq 1$) leaves to some vertices of each connected graph $T_{i_j}$ for $j \in [1, M]$ produces a connected graph $H_i$, denoted as $H_i = \langle m_{ij} [\odot_e] T_{i_j} \rangle$, admitting an odd-edge graceful-difference total labeling (or coloring) $f_{i_j}^{gr}$ induced from the odd-edge graceful-difference total labeling (or coloring) of the connected graph $T_{i_j}$, such that

$$
\begin{align*}
||f_{i_j}^{gr}(x) - f_{i_j}^{gr}(y)| - f_{i_j}^{gr}(xy)| &= k^*, xy \in E(H_i); \\
h_{i_s}^{gr}(V(H_{i_s})) \cap h_{i_j}^{gr}(V(H_{i_j})) \neq \emptyset, 1 \leq s, t \leq M
\end{align*}
$$

(76)

by the RLA-algorithms for adding leaves continuously. Notice that RLA-algorithm of the odd-edge graceful-difference total coloring tells us: The leaf-added graph $G_A$ admits an odd-edge graceful-difference total coloring $f_{gr}^{*}$, and $f_{gr}^{*}(u_0) = 0$ for some $u_0 \in V(G_A)$, thereby, $f_{i_j}^{gr}(w_0) = 0$ for some $w_0 \in V(H_{i_j})$.

**Larvc-Step-3.** We vertex-coincide a vertex $u \in V(H_{i_s})$ with a vertex $v \in V(H_{i_j})$ into one vertex $u \odot v$ if $h_{i_s}^{gr}(u) = h_{i_j}^{gr}(v)$ for $s \neq j$ and $1 \leq s, t \leq M$, such that the resultant graphs are connected and has no multiple edges, called simple vertex-coincided graphs, and we denote these simple vertex-coincided graphs by the following form

$$
\odot_{j=1}^{M} \langle H_{i_j} \rangle = \odot_{j=1}^{M} \langle m_{ij} [\odot_e] T_{i_j} \rangle = \{ \odot_{k=1}^{m} \odot_e \langle \alpha_k G_k \rangle : \alpha_k \in Z^0, G_k \in G_{k^*} \}.
$$

(77)

By the LARVC uniformly-$k^*$ graceful-difference algorithm introduced above, we get a uniformly-$k^*$ magic-type graphic lattice

$$
L_{grd}(Z^0[\odot_e] \langle \odot_e \rangle G_{k^*}-magic) = \{ \odot_{k=1}^{m} \odot_e \langle \alpha_k G_k \rangle : \alpha_k \in Z^0, G_k \in G_{k^*} \}.
$$

(78)

with $\sum_{k=1}^{m} \alpha_k \geq 1$.

**Remark 3.** Each graph $T^* \in L_{grd}(Z^0[\odot_e] \langle \odot_e \rangle G_{k^*}-magic)$ is a connected bipartite graph and admits a compound odd-edge graceful-difference total labeling (or coloring) $F$ holding $|F(x) - F(y)| - F(xy)| = k^*$ for $xy \in E(T^*)$, where $F(z) = f_{i_j}^{gr}(z)$ for $z \in V(H_{i_j}) \cup E(H_{i_j}) \subset V(T^*) \cup E(T^*)$. Each uniformly-$k^*$ magic-type graphic lattice was made by two graph operations: one is the vertex-coinciding operation “$\odot$” and, another is the leaf-adding operation “$\odot_e$”, so the lattice $L_{grd}(Z^0[\odot_e] \langle \odot_e \rangle G_{k^*}-magic)$ is, also, a multiple-operation lattice.

We do the vertex-coinciding operation to each graph $T^* \in L_{grd}(Z^0[\odot_e] \langle \odot_e \rangle G_{k^*}-magic)$ by vertex-coinciding those vertices of $T^*$ colored the same color, and avoiding that case of multiple-edges, the resultant graph is denoted as $T^*_{v-coin}$; then we get another uniformly-$k^*$ magic-type graphic lattice as follows:

$$
L_{grd}(Z^0[\odot_e]^2 \langle \rightarrow \rangle G_{k^*}-magic) = \{ T^*_{v-coin} : T^* \rightarrow_{v-coin} T^*_{v-coin} \}
$$

(79)
so we call the following one graph set being homomorphism to another graph set

$$L_{grad}(Z^0[\odot v] (\odot e) G_{k*\text{magic}}^c) \rightarrow_v \text{coin} L_{grad}(Z^0[\odot v]^2 (\rightarrow) G_{k*\text{magic}}^c)$$ (80)

a uniformly-$k^*$ magic-type graphic-lattice homomorphism.

**Problem 8.** Does there is a group of graphs $T_1^c, T_2^c, \ldots, T_m^c$, such that $T_{k*\text{magic}}^c = \{T_1^c, T_2^c, \ldots, T_m^c\}$ forms a uniformly-$k^*$ graceful-difference base, and two uniformly-$k^*$ graceful-difference graphic lattices hold

$$L_{grad}(Z^0[\odot v] (\odot e) T_{k*\text{magic}}^c) = L_{grad}(Z^0[\odot v] (\odot e) G_{k*\text{magic}}^c)?$$

### 4.1.2 Complexity of uniformly-$k^*$ graceful-difference graphic lattices

According to the LARVC uniformly-$k^*$ graceful-difference algorithm, we have the following complexity analysis:

**Case-1.** In Larvc-Step-1 of the LARVC uniformly-$k^*$ graceful-difference algorithm, there are $M!$ permutations $T_{i_1}, T_{i_2}, \ldots, T_{iT}$ obtained from graphs $\alpha_1 G_1^c, \alpha_2 G_2^c, \ldots, \alpha_m G_m^c$ based on a uniformly-$k^*$ graceful-difference base $G_{k*\text{magic}}$, where $M = \sum_{k=1}^m \alpha_k \geq 1$.

**Case-2.** In Larvc-Step-2 of the LARVC uniformly-$k^*$ graceful-difference algorithm, adding $m_{ij}$ ($\geq 1$) leaves to some vertices of each connected graph $T_{ij}$ with $j \in [1, M]$, we will meet:

(i) **Integer Partition Problem:** $m_{ij} = a_{i,j,1} + a_{i,j,2} + \cdots + a_{i,j,b_{ij}}$ with integers $a_{i,j,k} \geq 1$ and $b_{ij} \geq 2$. This problem is related with an odd integer $m = p_1 + p_2 + p_3$ for primes $p_1, p_2, p_3$, also, the famous Goldbach’s Conjecture. Suppose that we have $P(m_{ij}, b_{ij})$ different ways.

(ii) Selecting randomly $b_{ij}$ vertices of $T_{ij}$ produces $P(T_{ij})$ methods for adding $m_{ij}$ leaves, where vertex number $p(T_{ij}) = |V(T_{ij})|$. So, we have $b_{ij} \cdot P(T_{ij})$ different methods in total.

Summing up the above works, then we have $\prod_{i,j=1}^M P(m_{ij}, b_{ij}) (b_{ij}!) (P(T_{ij}))$ different methods for adding leaves to a permutation $T_{i_1}, T_{i_2}, \ldots, T_{iT}$.

**Case-3.** In Larvc-Step-3 of the LARVC uniformly-$k^*$ graceful-difference algorithm, computing the number $n(\odot_{j=1}^A T_{ij})$ of graphs in the form $[\odot_{k=1}^m] \odot_e \alpha_k G_k^c$ defined in Eq.(77) is extremely difficult.

By Case-1, Case-2 and Case-3, we can say that for each $M = \sum_{k=1}^m \alpha_k \geq 1$, there are at least

$$n(M, G_{k*\text{magic}}^c) = M! \cdot n(\odot_{j=1}^A T_{ij}) \cdot \prod_{i,j=1}^A P(m_{ij}, b_{ij}) (b_{ij}!) (P(T_{ij}))$$ (81)

simple vertex-coincided graphs in the lattice $L_{grad}(Z^0[\odot v] (\odot e) G_{k*\text{magic}}^c)$ defined in Eq.(79).

In a simple case, we vertex-coincide a vertex $u \in V(T_{ij})$ with a vertex $v \in V(T_{ij+1})$ into one vertex $u \odot v$ as $h_{ij}^r(u) = h_{ij+1}^r(v)$ for $s \in [1, M - 1]$, the resultant graph is called a string-form graph,
so we have $M!$ string-form graphs in the form $[\bigcirc_{k=1}^m] \in_e \langle \alpha_k G_k \rangle$, and for each $M = \sum_{k=1}^m \alpha_k \geq 1$, there are at least

$$n_{\text{string}}(M, G_{k*}^{e-magic}) = \frac{n(M, G_{k*}^{e-magic})}{n(\bigcirc_{j=1}^m T_{ij})} = M! \cdot \prod_{ij=1}^A P(m_{ij}, b_{ij})(b_{ij}) \left( p(T_{ij}) \right)$$

(82)

string-form graphs in the lattice $L_{\text{grad}}(Z^0[\bigcirc_m](\bigcirc_e)G_{k*}^{e-magic})$ defined in Eq.(79).

Remark 4. In the application of topological authentication, a uniformly-$k*$ graceful-difference base $G_{k*}^{e-magic} = \{G_1^*, G_2^*, \ldots, G_m^*\}$ can be considered as a public-key base, given a fixed group of non-negative integers $\alpha_1, \alpha_2, \ldots, \alpha_m$ with $\sum_{k=1}^m \alpha_k \geq 1$, the simple vertex-coincided graphs $[\bigcirc_{k=1}^m] \in_e \langle \alpha_k G_k \rangle$ admitting odd-edge graceful-difference total labelings (or colorings) in the lattice $L_{\text{grad}}(Z^0[\bigcirc_m](\bigcirc_e)G_{k*}^{e-magic})$ are as private-keys.

However, finding a particular private-key from those simple vertex-coincided graphs in the lattice $L_{\text{grad}}(Z^0[\bigcirc_m](\bigcirc_e)G_{k*}^{e-magic})$ is not relaxed, since it will meet the Graph Isomorphic Problem, and it will be encountered with exponential level calculations, refer to Eq.(81) and Eq.(82).

The sentence “LARVC uniformly-$k*$ magic-type” is one of LARVC uniformly-$k*$ edge-magic, LARVC uniformly-$k*$ edge-difference, LARVC uniformly-$k*$ graceful-difference and LARVC uniformly-$k*$ felicitous-difference, and each LARVC uniformly-$k*$ magic-type algorithm is exactly like the LARVC uniformly-$k*$ graceful-difference algorithm.

Since the complexities of uniformly-$k*$ magic-type graphic lattices for other uniformly-$k*$ magic-types are like the complexity analysis of uniformly-$k*$ graceful-difference graphic lattices, we omit them here.

4.1.3 Twin uniformly $k*$-magic-graphic lattices

Theorem 15. If a connected bipartite $(p, q)$-graph $G$ admits an odd-edge graceful-difference total coloring $f^*$, then there exists a bipartite graph $G^*$ admitting an odd-edge graceful-difference total coloring $g^*$, such that $\langle f^*, g^* \rangle$ is a twin set-ordered odd-edge graceful-difference total coloring of the $(p, q)$-graph $G$ and the graph $G^*$.

Proof. Suppose the connected bipartite $(p, q)$-graph $G$ admits an odd-edge graceful-difference total coloring $f^* : V(G) \cup E(G) \rightarrow [0, 2q - 1]$, then there exists a bipartite graph $G^*$ admitting an odd-edge graceful-difference total coloring $g^* : V(G^*) \cup E(G^*) \rightarrow [1, 2q]$, such that $G \cong G^*$, and $g^*(w) = f^*(w) + 1$ for $w \in V(G) = V(G^*)$, as well as $g^*(e) = f^*(e)$ for $w \in E(G) = E(G^*)$. By Definition $\big| f^*, g^* \big|$ is a twin set-ordered odd-edge graceful-difference total coloring of the $(p, q)$-graph $G$ and the graph $G^*$.

Let $G_{k*}^{twin} = \{G_1^{twin}, G_2^{twin}, \ldots, G_m^{twin}\}$ be a uniformly-$n*$ graceful-difference graphic base, each graph $G_i^{twin}$ is connected and admits an odd-edge graceful-difference total coloring $\alpha_i^{gr}$, and holds

$$\big| \alpha_i^{gr}(x) - \alpha_i^{gr}(y) - \alpha_i^{gr}(xy) \big| = n^* \geq 0, \ xy \in E(G_i^{twin}), \ i \in [1, m]$$

(83)
and \( \alpha_i^{gr}(u_i) = 1 \) for \( u_i \in E(G_i^{twin}) \), as well as \( G_i^{twin} \neq G_j^{twin} \) if \( i \neq j \).

From \( i = 1 \) to \( i = m \), if \( (h_i^{gr}, \alpha_i^{gr}) \) is the twin odd-edge graceful-difference total labeling/total coloring of the uniformly-\( k^* \) graceful-difference graphic base \( G_i^{grd} \) and the uniformly-\( n^* \) graceful-difference graphic base \( G_i^{grd} \) (refer to Definition 11), then two graceful-difference graphic base \( G_i^{grd} \) and \( G_j^{grd} \) form a twin uniformly-\( (k^*, n^*) \) graceful-difference graphic base.

By the above LARVC-algorithm, we have a uniformly-\( n^* \) graceful-difference graphic lattice based on the uniformly-\( n^* \) graceful-difference graphic base \( G_n^{grd} \) as follows

\[
L_{grd}(Z^0[\oplus_e]G_n^{grd}) = \{[\oplus_{k=1}^m] \oplus \langle \beta_kG_k^{twin} \rangle : \beta_k \in Z^0, G_k^{twin} \in G_n^{grd} \} \quad (84)
\]

where \( \sum_{k=1}^m \beta_k \geq 1 \), such that each graph \( J^* \in L_{grd}(Z^0[\oplus_e]G_n^{grd}) \) is connected and admits an odd-edge graceful-difference total coloring \( g \) holding \( |g(x) - g(y)| = n^* \) for \( x, y \in E(J^*) \).

We call two graphic lattices \( L_{grd}(Z^0[\oplus_e]G_n^{grd}) \) and \( L_{grd}(Z^0[\oplus_e]G_n^{grd}) \) twin uniformly-\( (k^*, n^*) \) graceful-difference graphic lattice. In real application, the uniformly-\( k^* \) graceful-difference graphic lattice is as a public-key lattice and the uniformly-\( n^* \) graceful-difference graphic lattice is as a private-key lattice.

### 4.2 Realization of uniformly-\( n^* \) \( W \)-magic graphic lattices

#### 4.2.1 Connection between complex graphs and integer lattices

1. **Complex graphs and integer lattices.** Let \( F^*(n) \) be the set of complex graphs of \( n \) vertices (refer to Definition 1). A graph base \( H = (H_1, H_2, \ldots, H_n) \) consists of \( n \) vertex-disjoint connected complex graphs, each connected complex graph \( H_k \) has just \( m \) vertices and its own degree sequence \( d_k = (d_{k,1}, d_{k,2}, \ldots, d_{k,m}) = (d_{k,i})_{i=1}^m \), where \( d_{k,i} \geq d_{k,i+1} \) for \( i \in [1, m-1] \). We vertex-coincide a vertex \( x_k \) of a connected complex graph \( G \in F^* \) with a vertex \( w_{k,j} \) of the connected complex graph \( H_k \) into a vertex \( x_k \oplus w_{k,j} \) for \( k \in [1, n] \), the resultant graph is denoted as \( F \oplus_{k=1}^n H_k \), and we get a degree sequence \( d'_k = (d'_{k,1}, d'_{k,2}, \ldots, d'_{k,m}) = (d'_{k,i})_{i=1}^m \), where only one \( d'_{k,j} = d_{k,j} + \deg_G(x_k) \), here \( \deg_G(x_k) \) is the degree of vertex \( x_k \) of the connected complex graph \( G \). Thereby, the graph \( L = F \oplus_{k=1}^n H_k \) has its own degree sequence

\[
d_L = \sum_{k=1}^n d'_k = \left( \sum_{k=1}^n d'_{k,1}, \ldots, \sum_{k=1}^n d'_{k,m} \right) \quad (85)
\]

and these degree sequences forms an integer lattice

\[
L(d) = \left\{ \sum_{k=1}^n \lambda_k d'_k : \lambda_k \in Z, d'_k = (d'_{k,i})_{i=1}^m, H_k \in H, F \in F^*(n) \right\} . \quad (86)
\]

2. **A connection between integer lattices and complex graphs.** An integer lattice \( L(ZB) \) is defined in Definition 1. By the lattice base \( B = (b_1, b_2, \ldots, b_m) \) with \( m \leq n \), we have a vector

\[
\sum_{i=1}^m x_i b_i = \left( \sum_{i=1}^m b_{i,1}, \ldots, \sum_{i=1}^m b_{i,m} \right) = (\alpha_1, \alpha_2, \ldots, \alpha_m) , \quad (87)
\]
which corresponds to a complex graph $C_{gra}$, such that the degree sequence of the complex graph $C_{gra}$ is just $\text{deg}(C_{gra}) = (\alpha_1, \alpha_2, \ldots, \alpha_m)$, we call the set $A_{com}(L(ZB) \to C_{gra})$ of these complex graphs like $C_{gra}$ as complement complex-graphic lattice of the integer lattice $L(ZB)$.

Problem 9. Partition a degree sequence $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ (as a public-key) into $\sum_{i=1}^{m} x_i b_i$ defined in Eq. (87) (as a private-key). Clearly, the number of ways of partitioning a degree sequence is not unique and difficult to estimate, see the Integer Partition Problem.

### 4.2.2 Caterpillar-graphic lattices, Complementary graphic lattices

The authors in [19] present the following ODD-GRACEFUL subdivision-algorithm.

**ODD-GRACEFUL subdivision-algorithm**

**Input:** A caterpillar $H$ having $m = \sum_{i=1}^{n} |L(u_i)| = \sum_{i=1}^{n} m_i$ leaves, where $P = u_1 u_2 \cdots u_n$ is the spine path of the caterpillar $H$.

**Output:** A set-ordered odd-graceful labeling of the caterpillar $H$.

**Step 1.** Notice that the caterpillar $H$ has $m = \sum_{i=1}^{n} |L(u_i)| = \sum_{i=1}^{n} m_i$ leaves. We take a star tree $K_{1,m}$ with vertex set $V(K_{1,m}) = \{u_1, u_1, j \mid j \in [1,m]\}$ and edge set $E(K_{1,m}) = \{u_1 v_{1,1}, u_1 v_{1,2}, \ldots, u_1 v_{1,m}\}$. Let $H_1 = K_{1,m}$, and $L(H_1) = V(H_1) \setminus \{u_1\}$ is the set of leaves of $K_{1,m}$. We define a labeling $\alpha_1$ for the star tree $H_1$ as: $\alpha_1(u_1) = 0$, $\alpha_1(v_{1,j}) = 2j - 1$. So, $H_1$ has its own vertex color set $\alpha_1(V(H_1)) = \{0, 2m - 1\}$ and edge color set $\alpha_1(E(H_1)) = \{1, 2m - 1\}$. Obviously, this labeling $\alpha_1$ is just a set-ordered odd-graceful labeling of $H_1$.

**Step 2.** Add a new vertex $u_2$ to the star tree $H_1$, and join the vertex $u_1$ with the vertex $u_2$ by a new edge $u_1 u_2$; and partition the leaf set $L(H_1)$ into two subsets $L(u_1)$ and $L(u_2)$, where $L(u_1) = \{v_{1,m-1}, v_{1,m-2}, \ldots, v_{1,m-m_1+1}\}$ and $L(u_2) = \{v_{2,m-m_1}, v_{2,m-m_1-1}, \ldots, v_{2,1}\}$, we have $v_{2,m-m_1-k} = v_{1,m-m_1+1-k}$ for $k \in [1, m - m_1]$. Thereby, we get a caterpillar, denoted as $H_2$, having its spine path $P_2 = u_1 u_2$.

**Step 3.** Notice that the set-ordered odd-graceful labeling $\alpha_1$ of the star tree $H_1$ holds $\alpha_1(u_1) = 0$, $\alpha_1(v_{1,m-j+1}) = 2(m - j + 1) - 1$ for $j \in [0, m_1 - 1]$, $\alpha_1(v_{2,m-m_1+1-k}) = 2(m - m_1 + 1 - k) - 1$ for $k \in [1, m - m_1]$. We define a labeling $\alpha_2$ for the caterpillar $H_2$ in the following way: $\alpha_2(u_1) = 0$, $\alpha_2(v_{1,m-j+1}) = \alpha_1(v_{1,m-j+1}) + 2$ for $j \in [0, m_1 - 1]$; $\alpha_2(u_2) = 2(m - m_1 + 1)$, $\alpha_2(v_{2,m-m_1+1-k}) = \alpha_2(v_{2,m-m_1+1-k}) - (2k - 1)$ for $k \in [1, m - m_1]$. It is easy to verify that $\alpha_2$ is just a set-ordered odd-graceful labeling of $H_2$.

**Step 4.** Add a new vertex $u_{k+1}$ to the caterpillar $H_k$, and join the vertex $u_k$ of the spine path of the caterpillar $H_k$ with new vertex $u_{k+1}$ by a new edge $u_k u_{k+1}$, and partition the leaf set $L(u_k)$ of $H_k$ into two leaf subsets, then join the leaves of two leaf subsets with $u_k$ and $u_{k+1}$ respectively, the resultant graph is just a new caterpillar $H_{k+1}$, and the caterpillar $H_{k+1}$ has its own spine path $P_{k+1} = u_1 u_2 \cdots u_{k+1}$. Repeat the works in Step 2 and Step 3, until we get a set-ordered odd-graceful labeling of the caterpillar $H$.

See an example for the ODD-GRACEFUL subdivision-algorithm shown in Fig[25]. Notice that a set-ordered odd-graceful labeling of a caterpillar $H$ is just a set-ordered odd-graceful total labeling.
By the ODD-GRACEFUL subdivision-algorithm and Theorem 13 each caterpillar admits one of the labelings and colorings defined in Definition 5 and Definition 8, and there are algorithms that can be effectively and quickly apply these labelings and colorings to practice.

1. **Caterpillar-graphic lattices.** A caterpillar base $T_{\text{cater}} = (T_1, T_2, \ldots, T_m)$ consists of $m$ vertex-disjoint caterpillars $T_1, T_2, \ldots, T_m$. Under a graph operation “($\ast$)”, we call the following set

$$L(G(\ast)T_{\text{cater}}) = \{ J(\ast)_{k=1}^n \beta_k T_k : \beta_k \in \mathbb{Z}^0, T_k \in T_{\text{cater}}, J \in G \}$$  \hspace{1cm} (88)

as ($\ast$)-operation caterpillar-graphic lattice, where $\sum_{k=1}^n \beta_k \geq 1$.

2. **A connection between integer lattices and caterpillar-graphic lattices.** A caterpillar $T_i$ has its own spine path $P_{i,n} = x_{i,1}x_{i,2} \cdots x_{i,p}$ with $p \geq 2$, and each vertex $x_{i,k}$ of the spine path $P_{i,p}$ has its own leaf set $L(x_{i,k}) = \{ y_{i,k,j} : j \in [1, b_{i,k}] \}$ for $k \in [1, p]$. We define the leaf topological vector of the caterpillar $T_i$ by $V_{ec}(T_i) = (c_{i,1}, c_{i,2}, \ldots, c_{i,p})$, where $|c_{i,k}| = |L(x_{i,k})|$ is the number of leaves adjacent with the vertex $x_{i,k}$, also, $c_{i,k}$ is a leaf-degree or a leaf-image-degree. In Fig.2 a caterpillar $A_1$ has its own leaf topological vector $V_{ec}(A_1) = (7, 3, 0, 3, 0, 6)$, and another caterpillar $A_2$ has its own leaf topological vector $V_{ec}(A_2) = (7, -3, 0, -3, 0, 6)$.

A complex caterpillar base $T_{\text{cater}} = (T_1, T_2, \ldots, T_m)$ has its own leaf topological vector base $V_{ec}(T) = (V_{ec}(T_1), V_{ec}(T_2), \ldots, V_{ec}(T_m))$, immediately, we get an integer lattice

$$L(Z(\Sigma) V_{ec}(T)) = \left\{ \sum_{k=1}^m \lambda_k V_{ec}(T_k) : \lambda_k \in Z, V_{ec}(T_k) \in V_{ec}(T) \right\}.$$  \hspace{1cm} (89)
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where \( \sum_{k=1}^{m} \lambda_k \geq 1. \)

3. Complement caterpillar-graphic lattices. For two caterpillar bases \( T_{\text{cater}} = (T_1, T_2, \ldots, T_m) \) and \( T^*_{\text{cater}} = (T^*_1, T^*_2, \ldots, T^*_m) \), each caterpillar \( T_j \) has its own spine path \( P_j = x_{j,1} x_{j,2} \cdots x_{j,n} \) with \( n \geq 1 \), where each vertex of \( P_j \) has its own leaf set \( L_{\text{leaf}}(x_{j,i}) = \{ y_{j,i,s} : s \in [1, b_{j,i}] \} \) for \( i \in [1, n] \) and \( j \in [1, m] \); and each caterpillar \( T^*_j \) has its own spine path \( P^*_j = x^*_{j,1} x^*_{j,2} \cdots x^*_{j,n} \) with \( n \geq 1 \), where each vertex of \( P^*_j \) has its own leaf set \( L_{\text{leaf}}(x^*_{j,i}) = \{ y^*_{j,i,s} : s \in [1, b^*_{j,i}] \} \) for \( i \in [1, n] \) and \( j \in [1, m] \). If \( |L_{\text{leaf}}(x_{j,i})| + |L_{\text{leaf}}(x^*_{j,i})| = M \) for two caterpillars \( T_j \) and \( T^*_j \), we call two caterpillar bases \( T_{\text{cater}} \) and \( T^*_{\text{cater}} \) as \( M \)-leaf complement caterpillar base matching, denoted as \( M(T_{\text{cater}}, T^*_{\text{cater}}) \). By Eq. \( (88) \), two graph lattices \( \mathbf{L}(G(\bullet)T_{\text{cater}}) \) and \( \mathbf{L}(G(\bullet)T^*_{\text{cater}}) \) form a complement caterpillar-graphic lattice matching.

**Problem 10.** A uniformly \( M \)-complement sequence \( \{(a_{i,1}, a_{i,2}, \ldots, a_{i,n})\}_{i=1}^{p} \) holds \( \sum_{i=1}^{p} a_{i,j} = M \) for \( j \in [1, n] \), \( p \geq 2 \) and \( n \geq 2 \). **Generalize** the complement caterpillar-graphic lattice matching to general graphs.

4.2.3 Application examples

**Example 17. (Lobsters)** By Theorem \( 13 \) and Theorem \( 14 \) each caterpillar \( T \) admits each set-ordered odd-edge \( W \)-magic total labeling defined in Definition \( 5 \) and Definition \( 8 \). So, we have lobsters obtained by adding leaves to caterpillars, by the methods introduced in \( [27] \) and \( [28] \), these lobsters admit odd-graceful labelings and odd-edge \( W \)-magic total labelings.

**Example 18. (Complement caterpillar-graphic lattice)** As a graph operation \( "(\bullet)" \) guarantees that each graph \( G \) in the \( (\bullet) \)-operation caterpillar-graphic lattice \( \mathbf{L}(G(\bullet)T_{\text{cater}}) \) defined in \( [88] \) is a caterpillar, then there are lobsters obtained by adding leaves to each caterpillar \( G \in \mathbf{L}(G(\bullet)T_{\text{cater}}) \), we put these lobsters into a set, which is just a complement caterpillar-graphic lattice of the caterpillar-graphic lattice \( \mathbf{L}(G(\bullet)T_{\text{cater}}) \).

**Example 19. (Topological authentication)** The ODD-GRACEFUL subdivision-algorithm shows that each caterpillar \( T \) admits a set-ordered odd-edge graceful total labeling \( f_1 \). By Theorem \( 13 \), the caterpillar \( T \) admits a set-ordered odd-edge edge-magic total labeling \( f_2 \), a set-ordered odd-edge edge-difference total labeling \( f_3 \), a set-ordered odd-edge felicitous-difference total labeling \( f_4 \), as well as a set-ordered odd-edge graceful-difference total labeling \( f_5 \). We define a set-coloring \( F \) of a caterpillar \( T \) as:

\[
F(u) = \{ f_1(u), f_2(u), f_3(u), f_4(u), f_5(u) \} \text{ for each vertex } u \in V(T),
\]

\[
F(xy) = \{ f_1(xy), f_2(xy), f_3(xy), f_4(xy), f_5(xy) \} \text{ for each edge } xy \in E(T),
\]

such that each edge \( xy \) holds \( f_i(xy) = \theta_i(f_i(x), f_i(y)) \) for \( i \in [1, 5] \). Write the caterpillar admitting the set-coloring \( F \) by \( T^* \), we get a more complex Topcode-matrix \( T_{\text{code}}(T^*) \). Furthermore, it makes people to get more complicated number-based string public-keys and number-based string private-keys, and brings more convenience and options for users.
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Theorem 16. (Coloring closure) Since each tree admits a W-magic total coloring with \( W \)-magic \( \in \{ \text{graceful-difference, edge-magic, edge-difference, felicitous-difference} \} \), so a uniformly-\( n^* \) \( W \)-magic graphic lattice is closure to the \( W \)-magic total coloring.

5 Graphic lattices based on \( W \)-magic total colorings

We call some graphic lattices as linear-graphic lattices if they were made by tree-bases and each element in them is a tree, otherwise non-linear graphic lattices.

5.1 Linear-graphic lattices

CONSTRUCTION algorithm-I.

Step-11. Let \( J_{i_1}, J_{i_2}, \ldots, J_{i_A} \) be a permutation of trees \( a_1T_1^c, a_2T_2^c, \ldots, a_mT_m^c \) based on a tree-base \( T^c \), where \( A = \sum_{k=1}^{m} a_k \geq 1 \), and each \( T_i^c \) admits an odd-edge \( W \)-magic total labeling (or coloring) \( f_i \).

Step-12. Adding \( n_{ij} \) leaves to each tree \( J_{ij} \) for \( j \in [1, A] \) produces a tree \( H_{ij} \), denoted as \( H_{ij} = \langle n_{ij}, [\circ_e]J_{ij} \rangle \), admitting an odd-edge \( W \)-magic total labeling (or coloring) \( f_{ij} \) induced from the odd-edge \( W \)-magic total labeling (or coloring) of the tree \( J_{ij} \), such that \( f_{is}(V(H_{is})) \cap f_{is+1}(V(H_{is+1})) \neq \emptyset \) for \( s \in [1, A-1] \).

Step-13. Vertex-coincide a vertex \( x \in V(H_{is}) \) with a vertex \( y \in V(H_{is+1}) \) into one vertex \( x \circ y \), where \( f_{is}(x) = f_{is+1}(y) \) for \( s \in [1, A-1] \), the resultant tree is denoted as

\[
\circ_{j=1}^A H_{ij} = \circ_{j=1}^A \langle n_{ij}, [\circ_e]J_{ij} \rangle = \left[ \circ_{k=1}^m \circ_{j=1}^A a_k T_k^c \right] \tag{90}
\]

since each tree \( H_{ij} = \langle n_{ij}, [\circ_e]J_{ij} \rangle \) is the result of adding \( n_{ij} \) leaves to each tree \( J_{ij} \) for \( j \in [1, A] \).

By the CONSTRUCTION algorithm-I above, we get a linear-graphic lattice

\[
L_W(Z^0[\circ_v \circ_e]T^c) = \left\{ \left[ \circ_{k=1}^m \circ_{j=1}^A a_k T_k^c \right] : \ a_k \in Z^0, \ T_k^c \in T^c \right\} \tag{91}
\]

with \( A = \sum_{k=1}^{m} a_k \geq 1 \).

There are the following facts:

(I-1) Each tree \( T \in L_W(Z^0[\circ_v \circ_e]T^c) \) admits a coloring \( \lambda \) with each edge \( uv \in E(T) \) holding \( \lambda(uv) = f_{ij}(uv) \) if this edge \( uv \) is an edge of some tree \( H_{ij} \) defined in Step-12 of the CONSTRUCTION algorithm-I, that is, \( \lambda(uv) \) is an odd integer. Because of \( f_{ij}(uv) \) satisfies one of the following forms:

(I-1-i) \( f_{ij}(uv) + |f_{ij}(u) - f_{ij}(v)| = a_{ij} \);

(I-1-ii) \( |f_{ij}(u) - f_{ij}(v)| - f_{ij}(uv) = b_{ij} \);

(I-1-iii) \( |f_{ij}(u) + f_{ij}(v) - f_{ij}(uv)| = c_{ij} \); and

(I-1-vi) \( f_{ij}(u) + f_{ij}(uv) + f_{ij}(v) = d_{ij} \), where \( a_{ij}, b_{ij}, c_{ij} \) and \( d_{ij} \) are non-negative integers.

Then the coloring \( \lambda \) is called an odd-edge compound multiple-magic total coloring of the tree \( T \).
(I-2) Each tree $T \in \mathbf{L}_W(Z^0[\circ_v \oplus_e]T^c)$ defined in Eq. (91) corresponds another graph $T^*$ admitting a twin odd-edge compound multiple-magic total coloring $\mu$ such that $\mu(V(T^*)) \cup \lambda(V(G)) \subseteq [0,M]$, and we call the graph $T^*$ a twin odd-edge graph of the tree $T$. All twin odd-edge graphs of trees of $\mathbf{L}_W(Z^0[\circ_v \oplus_e]T^c)$ form a set $\mathbf{L}_{\text{twin}}$ and we call it twin odd-edge linear-graphic lattice of the linear-graphic lattice $\mathbf{L}_W(Z^0[\circ_v \oplus_e]T^c)$.

(13) Each tree $H \in \mathbf{L}_W(Z^0[\circ_v \oplus_e]T^c)$ defined in Eq. (91) corresponds another graph $H^*$ forming a uniformly $M$-complement complex graphic matching $M_{\text{comp}}(H, H^*)$ (refer to Problem 2). All such graphs $H^*$ form a $M$-uniform matching odd-edge graphic lattice

\[
\mathbf{L}(M_{\text{comp}}) = \{M_{\text{comp}}(H, H^*) : H \in \mathbf{L}_W(Z^0[\circ_v \oplus_e]T^c)\}
\]  

with $A = \sum_{k=1}^{m} a_k \geq 1$.

5.2 Complexity of linear-graphic lattices

The Topcode-matrix $T_{\text{code}}(G)$ can distribute us $3q(G)!$ number-based strings, where $q(G) = |E(G)|$. Clearly, rewriting the Topcode-matrix $T_{\text{code}}(G)$ from one of these $3q(G)!$ number-based strings is quite difficult, and reconstructing the tree $G$ from $T_{\text{code}}(G)$ is impossible since

\[
G = [\circ_k^m A_j^{A}]a_k T_k^c = \circ_j^A (n_{ij} [\circ_e] J_{ij}) = \circ_j^A H_{ij}
\]  

according to Eq. (90).

The above works are related with two unsolved problems: One is the Subgraph Isomorphic Problem, and another one is the Integer Partition Problem: $n_{ij} = m_{ij,1} + m_{ij,2} + \cdots + m_{ij,p_{ij}}$ with with integers $m_{ij,k} \geq 1$ and $p_{ij} \geq 2$, and there are $(p_{ij}!)$ methods to adding these $n_{ij}$ leaves to a group of $p_{ij}$ vertices selected from $(p_{ij}!)$ groups of vertices of tree $J_{ij}$ in Step-12 of the CONSTRUCTION algorithm-I, where $p(J_{ij})$ is the number of vertices of tree $J_{ij}$. Suppose that there are $S(n_{ij})$ ways of partitioning integer $n_{ij}$, then we have $\prod_{j=1}^{A} S(n_{ij})(p_{ij}!)(p(J_{ij}))$ methods to add leaves.

There are $A!$ permutations $J_{i_1}, J_{i_2}, \ldots, J_{i_A}$ of trees $a_1T_1^c, a_2T_2^c, \ldots, a_mT_m^c$ based on a tree-base $T^c$, such that each permutation produces at least one tree $G \in \mathbf{L}_W(Z^0[\circ_v \oplus_e]T^c)$, since $\circ_j^A H_{ij}$ defined in Eq. (90) may make two or more trees of $\mathbf{L}_W(Z^0[\circ_v \oplus_e]T^c)$. The notation $n(\circ_j^A H_{ij})$ is the number of different trees made by $\circ_j^A H_{ij}$. So, we claim that there are at least

\[
N(A, T^c) = A! \cdot n(\circ_j^A H_{ij}) \cdot \prod_{i=1}^{A} S(n_{ij})(p_{ij}!)(p(J_{ij}))
\]  

trees of the linear-graphic lattice $\mathbf{L}_W(Z^0[\circ_v \oplus_e]T^c)$ for each $A = \sum_{k=1}^{m} a_k \geq 1$.

5.3 Non-linear graphic lattices

Let $\mathbf{H}^c = \{H^1, H^2, \ldots, H^m\}$ be a non-tree base, where there is at least one colored graph $H^i$ to be a non-tree graph, and each $H^i$ is connected and admits an odd-edge $W$-magic total labeling.
CONSTRUCTION algorithm-II.

**Step-21.** Let \( T_{i_1}, T_{i_2}, \ldots, T_{i_B} \) be a permutation of graphs \( b_1H^c_1, b_2H^c_2, \ldots, b_mH^c_m \) based on a non-tree base \( H^c \), where \( B = \sum_{k=1}^{m} b_k \geq 1 \), and each graph \( H^c_i \) is connected and admits an odd-edge \( W \)-magic total labeling \( (or coloring) \) \( g_i \). Here, there is at least one connected graph \( H^c_i \) to be not a tree with \( b_i \neq 0 \).

**Step-22.** Adding \( m_{ij} \geq 1 \) leaves to each connected graph \( T_{i_j} \) for \( j \in [1, B] \) produces a connected graph \( G_{i_j} \), denoted as \( G_{i_j} = \langle m_{ij} \cup e \mid T_{i_j} \rangle \), admitting an odd-edge \( W \)-magic total labeling \( (or coloring) \) \( g_{ij} \) induced from the odd-edge \( W \)-magic total labeling \( (or coloring) \) of the connected graph \( T_{i_j} \), such that \( g_{i_j}(V(G_{i_j})) \cap g_{i_{j+1}}(V(G_{i_{j+1}})) \neq \emptyset \) for \( s \in [1, B-1] \).

**Step-23.** Vertex-coincide a vertex \( w \in V(G_{i_s}) \) with a vertex \( z \in V(G_{i_{s+1}}) \) into one vertex \( w \odot z \), where \( g_{i_s}(w) = g_{i_{s+1}}(z) \) for \( s \in [1, B-1] \), the resultant graph is denoted as

\[
\odot_{j=1}^{B} G_{i_j} = \odot_{j=1}^{B} (m_{ij} \cup e \mid T_{i_j}) = \left[ \odot_{k=1}^{m} \odot_{j=1}^{B} b_k H^c_k \right] \tag{95}
\]

By the CONSTRUCTION algorithm-II above, we get a graphic lattice

\[
L_W(Z^0[\odot_v \odot_c] H^c) = \left\{ \left[ \odot_{k=1}^{m} \odot_{j=1}^{B} b_k H^c_k : b_k \in Z^0, H^c_k \in H^c \right] \right\} \tag{96}
\]

with \( B = \sum_{k=1}^{m} b_k \geq 1 \), and there is at least one connected graph \( H^c_i \) to be not a tree with \( b_i \neq 0 \).

Thereby, we have the following facts:

(II-1) Each connected graph \( G \in L_W(Z^0[\odot_v \odot_c] H^c) \) admits a coloring \( \theta \) with each edge \( uv \in E(G) \) holding \( \theta(uv) = g_{ij}(uv) \) if the edge \( uv \) is an edge of some connected graph \( T_{i_j} \) defined in Step-22 of the CONSTRUCTION algorithm-II, also, \( \theta(uv) \) is an odd integer. Notice that the edge color \( g_{ij}(uv) \) satisfies one of the following forms:

- (II-1-i) \( g_{ij}(uv) + |g_{ij}(u) - g_{ij}(v)| = \alpha_{ij} \);
- (II-1-ii) \( |g_{ij}(u) - g_{ij}(v)| = \beta_{ij} \);
- (II-1-iii) \( |g_{ij}(u) + g_{ij}(v) \ | = \gamma_{ij} \); and
- (II-1-iv) \( g_{ij}(u) + g_{ij}(uv) + g_{ij}(v) = \delta_{ij} \), where \( \alpha_{ij}, \beta_{ij}, \gamma_{ij} \) and \( \delta_{ij} \) are non-negative integers.

So we call the coloring \( \theta \) an odd-edge compound multiple-magic total coloring of \( G \).

(II-2) Each connected graph \( G \in L_W(Z^0[\odot_v \odot_c] H^c) \) corresponds a graph \( G^* \) admitting a twin odd-edge compound multiple-magic total coloring \( \varphi \) such that \( \varphi(V(G^*)) \cup \theta(V(G)) \subseteq [0, M] \), and we call the graph \( G^* \) a twin graph of the connected graph \( G \). Such twin graphs \( G^* \) form a set \( L_{twin} \), called the twin-graphic lattice of the graphic lattice \( L_W(Z^0[\odot_v \odot_c] H^c) \).

6 Concluding remarks

Four new colorings, called odd-edge graceful-difference total coloring, odd-edge edge-difference total coloring, odd-edge edge-magic total coloring, and odd-edge felicitous-difference total coloring, are introduced for producing twin-type \( W \)-magic graphic lattices. The randomly growing graphs admitting four new colorings can be constructed by the so-called RANDOMLY-LEAF-ADDING algorithms and techniques of adding leaves to continuously.
6 CONCLUDING REMARKS

The uniformly $W$-magic total colorings help us to build up caterpillar-graphic lattices and complementary graphic lattices. One important work is to show a connection between complex graphs and integer lattices, which establishes indirectly some connections between our graphic lattices and some integer lattices of traditional lattices.

Some research topics in this article can be further developed. For example, suppose that a tree $H$ has its own vertex set $V(H) = \{x_1, x_2, \ldots, x_n\}$, we add a leaf set $L_{eaf}(x_i) = \{y_{i,j} : j \in [1, M_i]\}$ to each vertex $x_i \in V(H)$ for $i \in [1, n]$, the resultant tree is denoted as $H^* = H \cup L_{eaf}$, where $L_{eaf} = \bigcup_{i=1}^n L_{eaf}(x_i)$. Notice that the tree $H^*$ has its own vertex set $V(H^*) = V(H) \cup L_{eaf}$ and its own edge set $E(H^*) = E(H) \cup E(L_{eaf})$, where $E(L_{eaf}) = \bigcup_{i=1}^n E(L_{eaf}(x_i))$ with $E(L_{eaf}(x_i)) = \{x_iy_{i,j} : j \in [1, M_i]\}$.

There is a group of trees $T_1, T_2, \ldots, T_A$, in which each tree $T_k$ with $k \in [1, A]$ has its own leaf sets $L_{eaf}(x_{k,i}) = \{u_{k,i,j} : j \in [1, m_{k,i}]\}$ for $x_{k,i} \in V(T_k)$ with $i \in [1, n]$, such that removing all leaves from the tree $T_k$ produces a tree $T_k^* = T_k - \bigcup_{i=1}^n L_{eaf}(x_{k,i}) = H$, so

$$V(H) = \{x_1, x_2, \ldots, x_n\} = V(T_k^*) = \{x_{k,1}, x_{k,2}, \ldots, x_{k,n}\}$$ (97)

We call $H$ the leaf-core of each tree $T_k$ for $k \in [1, A]$.

Notice that it is allowed for $L_{eaf}(x_i) = \emptyset$ for some vertex $x_i \in V(H)$, or $L_{eaf}(x_{k,i}) = \emptyset$ for some vertex $x_{k,i} \in V(T_k)$.

- For two trees $T_k$ and $T_r$, if there exists a positive integer $B_{k,r}$, such that $|L_{eaf}(x_{k,i})| + |L_{eaf}(x_{r,i})| = B_{k,r}$ for $i \in [1, n]$, we call two trees $T_k$ and $T_r$ to be uniform $B_{k,r}$-leaf complementary trees.

- For two trees $T_s$ and $T_j$, if there exists a positive integer $C_{s,j}$, such that $|L_{eaf}(x_{s,i})| + |L_{eaf}(x_{j,i})| = C_{s,j}$ for $i \in [1, n]$, we call two trees $T_k$ and $T_r$ to be $C_{s,j}$-leaf complementary trees, where $x_{j,1}', x_{j,2}', \ldots, x_{j,n}'$ is a permutation of vertices of the tree $T_j^* = T_j - \bigcup_{i=1}^n L_{eaf}(x_{j,i})$.

- Since

$$|L_{eaf}(x_i)| = M_i = \sum_{k=1}^A m_{k,i} = \sum_{k=1}^A |L_{eaf}(x_{k,i})|$$ (98)

we call the tree $H^*$ universal tree of the group of trees $T_1, T_2, \ldots, T_A$.

From graph operation of view, the universal tree $H^*$ is the result of coinciding the leaf-cores of trees $T_1, T_2, \ldots, T_A$ into one.

Since each tree admits an odd-edge $W$-magic total coloring for each $W$-magic $\in \{edge-magic, edge-difference, felicitous-difference, graceful-difference\}$, let $f$ be an odd-edge $W$-magic total coloring of the universal tree $H^*$, then each tree $T_k$ admits a total coloring $f_k$ induced by $f$ such that $f_k(w) \in f(V(H^*) \cup E(H^*))$ for $w \in V(T_k) \cup E(T_k)$, that is, $f_k(V(T_k) \cup E(T_k)) \subset f(V(H^*) \cup E(H^*))$.

Therefore, the above tree $H^*$ is the topological authentication of the trees $T_1, T_2, \ldots, T_A$ if some tree $T_k$ is considered as a public-key, and others are private-keys in applications of blockchain, financial networks, digital currency.
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