Yang-Mills Matrix Theory

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Abstract

In this thesis, we discuss bosonic and supersymmetric Yang-Mills matrix models with compact semi-simple gauge group.

We begin by finding convergence properties for the models. In the supersymmetric case, we show that the partition function converges when $D = 4, 6$ and $10$, and that correlation functions of degree $k < k_c = 2(D - 3)$ are convergent independently of the group. In the bosonic case we show that the partition function is convergent when $D \geq D_c$, and that correlation functions of degree $k < k_c$ are convergent, and calculate $D_c$ and $k_c$ for each group.

We move on to consider the supersymmetric theories in both their Yang-Mills and cohomological formulations. Specialising to the case of $SU(N)$ with large $N$, we find all quantities which are invariant under the supercharges.

Finally we apply the deformation method of Moore, Nekrasov and Shatashvili directly to the supersymmetric Yang-Mills model with $D = 4$. We find a deformation of the action which generates mass terms for all the matrix fields whilst preserving some supersymmetry. This allows us to rigorously integrate over a BRST quartet and arrive at the well known formula of MNS.

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Chapter 1

Introduction

In this chapter, we give a brief introduction to Yang-Mills matrix models and discuss some of their applications in modern physics.

1.1 Yang-Mills Gauge Theories

The Yang-Mills matrix models are related to gauge theories by dimensional reduction. In this section, we recall the structure of Yang-Mills gauge theories, and this will allow us to set up the notations used throughout the thesis.

A Yang-Mills gauge theory in $D$ dimensions has Lagrangian

$$
\mathcal{L} = \frac{1}{4} F^2 - \frac{i}{2} \bar{\psi} \gamma \cdot D \psi.
$$

The fields in this theory are the vector potential $X_\mu^a$, and fermions $\psi^a_\alpha$. Here $\mu$ is a spacetime index running from 0 up to $D - 1$, and for the moment we use a Minkowski metric. The $\gamma^\mu_{\alpha\beta}$ are Dirac matrices, and $\bar{\psi}$ is defined

$$
\bar{\psi} = \psi^\dagger \gamma^0.
$$

The fields are in the Lie algebra of a compact semi-simple gauge group $G$ so we can write

$$
X_\mu = X_\mu^a t^a \quad \psi_\alpha = \psi_\alpha^a t^a
$$

where the $t^a$ ($a = 1, \ldots, g$) are the generators of the Lie algebra which we choose such that

$$
\text{Tr} t^a t^b = 2 \delta^{ab}
$$

\footnote{In this section we follow the notation and conventions of \cite{footnote}.}
and
\[ [t^a, t^b] = if^{abc} t^c. \] (1.5)

The gauge field strength \( F \) is defined
\[ F_{\mu\nu}^a = \partial_\mu X_\nu^a - \partial_\nu X_\mu^a + cf^{abc} X_\mu^b X_\nu^c \] (1.6)
and the gauge covariant derivative is
\[ (D_\mu \psi)^a = \partial_\mu \psi^a + cf^{abc} X_\mu^b \psi^c. \] (1.7)

The parameter \( c \) is a coupling constant. The theory is invariant under gauge transformation
\[ \psi \rightarrow U \psi U^{-1}, \quad X_\mu \rightarrow UX_\mu U^{-1} - ic^{-1}(\partial_\mu U)U^{-1} \] (1.8)
where \( U \in G \). The fermions are optional in this model. We can define a purely bosonic gauge theory by simply omitting them.

### 1.2 Spinors and Supersymmetry

Concentrating on those theories that contain fermions, we now recall the spin structure. For the present, we continue to use Minkowski metric \( \eta^{\mu\nu} = \text{diag}(-1, +1, \cdots, +1) \). In any dimension \( D \), we can find an irreducible representation of the Dirac matrices \( \gamma^\mu_{\alpha\beta} \) which satisfy the Clifford algebra
\[ 2 \{ \gamma^\mu_{\alpha\beta}, \gamma^\nu_{\gamma\delta} \} = 2 \eta^{\mu\nu} \delta^\alpha_\gamma \delta^\beta_\delta. \] (1.9)

Then, defining
\[ S^{\rho\sigma} = \frac{1}{4} [\gamma^\rho, \gamma^\sigma] \] (1.10)
one can verify that these matrices satisfy the Lorentz algebra
\[ [S^{\rho\sigma}, S^{\tau\upsilon}] = \eta^{\tau\upsilon} S^{\rho\sigma} - \eta^{\rho\tau} S^{\sigma\upsilon} + \eta^{\rho\upsilon} S^{\sigma\tau} - \eta^{\sigma\upsilon} S^{\rho\tau}. \] (1.11)

On the other hand, we have the Lorentz generators
\[ (M^{\sigma\rho})_\nu^\mu = \eta^{\mu\upsilon} \delta^\sigma_\nu - \eta^{\upsilon\sigma} \delta^\mu_\nu \] (1.12)

\[ ^2 \text{In this section, we follow [3].} \]
1.2 Spinors and Supersymmetry

which act on the ordinary spacetime indices carried by $X^\mu$. These two representations of the Lorentz algebra are inequivalent. We define Dirac spinors $\psi_\alpha$ to transform as

$$\psi \to \psi + \omega_{\rho\sigma} S^{\rho\sigma} \psi$$  \hspace{1cm} (1.13)

whilst $X^\mu$ transforms as

$$X^\mu \to X^\mu + \omega_{\mu\rho}(M^{\rho\sigma})^\mu_\nu X^\nu$$ \hspace{1cm} (1.14)

under an infinitesimal Lorentz transformation. The nice property

$$[\gamma^\mu, S^{\rho\sigma}] = (M^{\rho\sigma})^\mu_\nu \gamma^\nu$$  \hspace{1cm} (1.15)

shows that the Dirac matrices are invariant under Lorentz transformations, and so the Lagrangian 1.1 transforms as a Lorentz scalar.

When $D$ is even, one can define

$$\gamma = (-1)^{\frac{1}{4}(D+2)}\gamma^0\gamma^1\ldots\gamma^{D-1}$$  \hspace{1cm} (1.16)

which has the properties

$$\gamma^2 = 1, \quad \{\gamma, \gamma^\mu\} = 0, \quad [\gamma, S^{\rho\sigma}] = 0. \hspace{1cm} (1.17)$$

Then $\gamma$ is Lorentz invariant, and has eigen-values +1 and −1. In this case, we can define Weyl spinors by projecting onto one of these eigen spaces.

If it is possible to choose the Dirac matrices to be all real or all imaginary, then one can impose the spinors to be real and still preserve Lorentz symmetry. This is called the Majorana condition, and is possible when $D = 0, 1, 2, 3$ or 4 (mod 8). (Of course, it may be more convenient to work in a different basis in which the Dirac matrices are not all real or all imaginary, and in that case one needs to appropriately modify the Majorana condition.) Precisely when $D = 2$ (mod 8), it is possible to apply both the Weyl and Majorana conditions.

We are particularly interested in the possible dimensions and spinor types for which the theory 1.1 may be supersymmetric. The crucial constraint is that there must be the same number of physical bosonic modes (that is $(D - 2)g$) as physical fermions. The size of the minimal representation of the Dirac matrices increases very rapidly with $D$, so this can only be possible for certain small $D$. In fact it is possible in $D = 3$ with a Majorana spinor, $D = 4$ with a Weyl (or Majorana) spinor, $D = 6$ with a Weyl spinor and $D = 10$ with a Majorana-Weyl spinor. All of these theories are invariant
under the supersymmetry
\begin{align*}
\delta X^a_\mu &= \frac{i}{2} \xi \gamma_\mu \psi^a \quad \text{and} \\
\delta \psi^a &= -F^a_\mu S^\mu \xi
\end{align*}
(1.18)
as long as one applies the equations of motion. This may not seem like much help, but it is possible to introduce some auxiliary fields to \(1.1\), and then write down a supersymmetry which is also valid off shell. We shall do this explicitly for \(D = 4\) in chapter \(3\).

Each of these theories has \((D - 2)g\) physical fermionic degrees of freedom. The spinor index \(\alpha\) runs from 1 to \(2(D - 2)\) in the case of a real (Majorana) representation, and from 1 to \(D - 2\) in the case of a complex representation. Then we see from \(1.18\) that there are \(N = 2(D - 2)\) real supercharges.

### 1.3 Yang-Mills Matrix Models

To obtain a Yang-Mills matrix model, we take the Lagrangian \(1.1\) and assume all the fields are independent of space and time. Effectively, this means we drop all the derivative terms from \(1.1\).

At this stage, we also move from Minkowski to Euclidean signature. We do this by setting
\begin{equation}
X_0^a = i X_D^a, \quad (a = 1, \ldots, g)
\end{equation}
(1.19)
and taking the \(X_D^a\) real. We also set
\begin{equation}
\gamma^0 = i \gamma^D.
\end{equation}
(1.20)

We have been careful to leave this manipulation until last because we wish to study the “Wick rotation” of a Minkowski theory\(^3\). This leads to a rather strange effect in the case of \(D = 10\) when the fermions are Majorana. Since the Dirac matrices can no longer all be imaginary, an \(SO(D)\) transformation would break the Majorana condition. However, after integrating out the fermions, full \(SO(D)\) invariance is restored since we can analytically continue in \(X_D\).

We arrive at the matrix model action
\begin{equation}
S_{YM} = -\text{Tr} \left( \frac{1}{4} [X_\mu, X_\nu] [X_\mu, X_\nu] + \frac{1}{2} \overline{\psi} \gamma^\mu [X_\mu, \psi] \right)
\end{equation}
(1.21)
\(^3\)We certainly do not attempt to give any justification for performing a Wick rotation here. However, the reader may take some solace from the fact that, in one case, it is precisely this model that appears physically. This is the Yang-Mills integral as the bulk part of a Witten index, which we shall discuss a little more later.
where we have dropped the coupling constant $c$ since it can be scaled out in a trivial manner. In those cases where the fermions were originally Majorana (before Wick rotation), we may choose the representation in which the $\psi^a_\alpha$ are real. In those cases where the fermions are complex, it will sometimes be convenient to re-write the $(\psi^a_\alpha)$ for each $a$ as a real vector of double the length. We can also absorb the $\gamma^0$ which appears in the definition of $\overline{\psi}$ into the $\gamma^\mu$. Thus we shall sometimes write the action in the form

$$S_{YM} = -\text{Tr} \left( \frac{1}{4} [X_\mu, X_\nu] [X_\mu, X_\nu] + \frac{1}{2} \psi_\alpha \Gamma^\mu_{\alpha\beta} [X_\mu, \psi_\beta] \right)$$

(1.22)

where the $\Gamma^\mu$ are some new matrices defined in terms of the $\gamma^\mu$, and the $\psi^a_\alpha$ are now always real. In the case where the original fermions were complex, the range of the indices $\alpha$ and $\beta$ has been doubled.

We define a partition function

$$Z_{D,G} = \int \prod_{\mu=1}^D dX_\mu \prod_{\alpha=1}^N d\psi_\alpha \exp \left( \frac{1}{4} \sum_{\mu,\nu} \text{Tr} [X_\mu, X_\nu]^2 + \frac{1}{2} \text{Tr} \psi_\alpha \Gamma^\mu_{\alpha\beta} [X_\mu, \psi_\beta] \right)$$

(1.23)

which we shall also sometimes refer to as the Yang-Mills integral. In principle one can integrate out the fermions to obtain

$$Z_{D,G} = \int \prod_{\mu=1}^D dX_\mu \mathcal{P}_{D,G}(X_\mu) \exp \left( \frac{1}{4} \sum_{\mu,\nu} \text{Tr} [X_\mu, X_\nu]^2 \right)$$

(1.24)

where the Pfaffian $\mathcal{P}_{D,G}$ is a homogeneous polynomial of degree $\frac{1}{2}N_g$. In this representation, the gauge symmetry is

$$X_\mu \rightarrow U_\dagger X_\mu U, \quad U \in G$$

(1.25)

and $SO(D)$ symmetry

$$X_\mu \rightarrow \sum_\nu Q_{\mu\nu} X_\nu, \quad Q \in SO(D).$$

(1.26)

In addition, we shall consider simple correlation functions

$$< C_k(X_\sigma) > = \int \prod_{\mu=1}^D dX_\mu C_k(X_\sigma) \mathcal{P}_{D,G}(X_\mu) \exp \left( \frac{1}{4} \sum_{\mu,\nu} \text{Tr} [X_\mu, X_\nu]^2 \right)$$

(1.27)

with $C_k$ a function of the $X_\sigma$ which grows like a polynomial of degree $k$.

We shall study two cases of particular interest. In the first case, we suppress the fermions and consider the purely bosonic model (so that the number of supercharges...
Chapter 1: Introduction

is $\mathcal{N} = 0$ and the Pfaffian is just 1). Since the bosonic action also gives the exponent in all models which include fermions, it is crucial for understanding the behaviour of the flat directions and how they can be suppressed. Secondly, we study the models with supersymmetric action. They can only be written down when $D = 3, 4, 6$ and 10, and have $\mathcal{N} = 2(D - 2)$ real supercharges. In the particular case of $D = 10$, this is the IKKT model of IIB string theory.

The first question one must ask about these models is whether the integrals 1.23 and 1.27 which define the partition function and correlation functions are well defined. Certainly, we must require at least that the partition function is finite for the theory to make any sense. The difficulty here is that the potential $\text{Tr} [X_\mu, X_\nu] [X_\mu, X_\nu]$ has flat directions in which the matrices commute. For example, in the bosonic case, one can move to infinity along one of these directions whilst keeping the integrand constant, and thus it was widely believed that these integrals may be infinite. However, in the case of $SU(2)$ it is possible to perform the integrals for the partition function exactly. This was done originally in the supersymmetric cases \[3–6\] and it was found that the partition function does converge at least for $D = 4, 6, 10$. Subsequently, eigen-value densities and some correlation functions have been calculated in \[7\]. It was believed that the supersymmetric versions should be more convergent than the bosonic because the contributions from the fermionic integrals would be close to zero near the flat directions. However, the $SU(2)$ bosonic partition function was calculated in \[8\], and was found to converge when $D \geq 5$.

The authors of \[8\] were able to use Monte Carlo methods to calculate the supersymmetric integrals numerically for $SU(2)$ and $SU(3)$, and the calculations have been extended to various other gauge groups, and also to the bosonic theories \[9, 10\]. A difficulty with numerical simulations for the supersymmetric integrals is in performing the fermionic integrations to obtain the Pfaffian, and for this reason, the exact model has only been studied for the smaller gauge groups. However, the bosonic models have now been studied for $SU(N)$ with $N$ up to 768 \[11, 12\]. Analytic approximation schemes have also been constructed for the bosonic models in \[13\] and recently for the $D = 4$ supersymmetric model \[14\].

The conclusions of the numerical methods are that the supersymmetric partition function converges when $D = 4, 6, 10$ and that the bosonic partition functions converge at least when $D$ is large enough \[10\]. Chapter 2 of this thesis will be devoted to an analytic study of the convergence properties of these integrals.
1.4 The IKKT Model of the IIB Superstring

The supersymmetric Yang-Mills matrix theory with \( D = 10 \) has been proposed as a constructive definition of IIB superstring theory \([15]\). We give a very brief introduction here, but for a review see \([16]\).

The idea of the IKKT conjecture is to begin with the Green-Schwarz action for the superstring in the Schild gauge:

\[
S_{GS} = \int d^2\sigma \left[ \sqrt{\hat{g}} \alpha \left( \frac{1}{4} \{ x^\mu, x^\nu \}^2 - \frac{i}{2} \bar{\psi} \Gamma^\mu \{ x^\mu, \psi \} \right) + \beta \sqrt{\hat{g}} \right]. \tag{1.28}
\]

Here \( \sigma \) are 2-dimensional world-sheet coordinates, \( \hat{g} = \det(\hat{g}_{ab}) \) is the determinant of the world sheet metric, and \( \alpha, \beta \) are parameters (which could be scaled out). The \( x^\mu \) are target space coordinates, and the Poisson bracket is defined

\[
\{ x, y \} = \frac{1}{\sqrt{\hat{g}}} \epsilon^{ab} \partial_a x \partial_b y. \tag{1.29}
\]

The theory is then regularised essentially following a method of Goldstone and Hoppe (for a review, see \([17]\)). A function \( y \) on the world sheet is replaced by an \( N \times N \) traceless hermitian matrix \( Y \), with a correspondence

\[
\int d^2\sigma \sqrt{\hat{g}} y \leftrightarrow \text{Tr} Y \tag{1.30}
\]

and

\[
\{ x, y \} \leftrightarrow -i [X, Y]. \tag{1.31}
\]

Performing this regularisation, the action \ref{eq:1.28} becomes

\[
S_{IKKT} = -\alpha \left( \frac{1}{4} \text{Tr} [X_\mu, X_\nu] [X_\mu, X_\nu] + \frac{1}{2} \text{Tr} \bar{\psi} \Gamma^\mu [X^\mu, \psi] \right) + \beta N \tag{1.32}
\]

The string partition function

\[
\int D[x] D[\psi] \exp(-S_{GS}) \tag{1.33}
\]

then becomes the matrix integral

\[
\int \prod_{\mu=1}^D dX_\mu \prod_{\alpha=1}^{16} d\psi_\alpha \exp(-S_{IKKT}) \tag{1.34}
\]

which after scaling out \( \alpha \) becomes the Yang-Mills matrix partition function \( Z_{10, SU(N)} \) of equation \ref{eq:1.23} with an additional factor \( e^{-\beta N} \).
In their original proposal, IKKT interpreted the integral over the world sheet metric $\int D[\hat{g}]$ as a requirement to sum over $N$:

$$Z_{IKKT} \sim \sum_N Z_{10, SU(N)} e^{-\beta N}. \quad (1.35)$$

However, in general, the matrix regularisation procedure outlined above is valid in the limit $N \to \infty$, and the partition function is often taken as the large $N$ limit

$$Z_{IKKT} \sim \lim_{N \to \infty} Z_{10, SU(N)} \quad (1.36)$$

which would correspond to a more literal application of the Goldstone Hoppe regularisation. The large $N$ limit is not yet well understood, and it is not clear exactly how to interpret the model. Nevertheless, an argument relating Wilson loops in the matrix model to string field theory in light-cone gauge provides additional evidence for the importance of the IKKT model [18].

In principle, one could use the IKKT model to calculate any quantity in string theory, given enough computer time. In practice though, only small $N$ calculations are accessible numerically because of difficulty in generating the Pfaffian. Although it is the $D = 10$ theory which is relevant for the IKKT model, it is also possible to study the models in $D = 4$ and $D = 6$. For $D = 4$ the Pfaffian is real and non-negative (see [19]), and this has allowed Monte Carlo studies up to $N = 48$ [12, 19, 20]. For $D = 6$ and $D = 10$, the Pfaffian is complex in general and standard lattice methods for dealing with the fermions do not work.

Since the exact IKKT model is difficult to study numerically, a low energy effective theory was derived in [21], and this has been studied for large $N$ in [22] by taking the absolute value of the Pfaffian. An alternative approach has been applied [23, 24], in which configurations which are saddle points of the phase of the Pfaffian have been studied. In these calculations the authors find the intriguing result that the integrals are dominated by regions corresponding to a lower dimension than $D = 10$ (it was suggested in [21] that the dimension 4 might arise as the natural dimension of a branched polymer which describes the model). There is also a random surface approach to the IKKT model which has been studied in [25].

### 1.5 The Matrix Model of M-theory

Shortly before the IKKT model for superstring theory was proposed, a related model was conjectured as a constructive definition of M-theory [26]. The model corresponds to the quantum mechanics that one obtains by dimensionally reducing the $SU(N)$
1.5 The Matrix Model of M-theory

gauge theory \[1.1\] with \( D = 10 \) to one (time) dimension. The proposal is that by taking \( N \to \infty \), this model gives M theory in a light cone coordinate system, and has become known as the BFSS conjecture (for a review, see \[17\]). After fixing the gauge \( X^0 = 0 \), one obtains the Hamiltonian \[27\]

\[
H = \text{Tr} \left( \frac{1}{2} P_i P_i - \frac{1}{4} [X^i, X^j] [X^i, X^j] - \frac{1}{2} \psi^i \psi^i \right) \tag{1.37}
\]

where the indices \( i, j \) run from 1 to 9, and the \( P_i \) are the canonical momenta for the \( X^i \). Physical states must also satisfy the gauge constraint

\[
C^a |\text{phys} \rangle = 0, \quad a = 1, \ldots, \dim[\text{su}(N)] \tag{1.38}
\]

where

\[
C^a = f^{abc}(X^b P^c_i - \frac{i}{2} \psi^b_\alpha \psi^c_\alpha). \tag{1.39}
\]

An important issue in this model is the question of whether there exists a unique normalisable vacuum state. This issue can also be discussed in the \( D = 4 \) and \( D = 6 \) models, and with general gauge group. It is known from early work \[28\] on supersymmetric Yang-Mills quantum mechanics that the models have continuous spectrum of \( H \) arbitrarily close to zero, but the question of an exact vacuum state has remained unresolved. An approach that has been used is to consider the Witten Index\[2\] which is defined

\[
I_W = \lim_{\beta \to \infty} \text{Tr}(-1)^F e^{-\beta H} = n^b_0 - n^f_0 \tag{1.40}
\]

where the trace is over all physical states, and \( F \) is the fermion number operator.

Taking the limit of large \( \beta \) projects onto the zero energy states so that the index gives the number of bosonic vacuum states minus number of fermionic vacuum states. If it is true that there is a unique bosonic vacuum state, then the Witten index must be 1.

In a theory with supersymmetry, equal positive energy bosons and fermions come in pairs and so we expect

\[
I(\beta) = \text{Tr}(-1)^F e^{-\beta H} \tag{1.41}
\]

to be independent of \( \beta \). If \( H \) has a discrete spectrum, one can prove this very easily, however the proof fails if \( H \) has a continuous spectrum since one would be trying to cancel infinite quantities inside the trace. The approach of \[3\] and \[4\] is to rewrite the index as

\[
I_W = I(0) + I^d \tag{1.42}
\]

where \( I^d \) is the deficit term \( I^d = I(\infty) - I(0) \), and \( I(0) \) is known as the principle or

\[\footnote{For an alternative approach, see [29].}\]
bulk term. In a supersymmetric theory with discrete spectrum \( I^d = 0 \), but we cannot assume this in the Yang-Mills model. In order to calculate \( I(0) \), the expression \( I_{\text{d}} = 0 \) is rewritten as a path integral on the interval \([0, \beta]\) by introducing a projection operator onto the gauge invariant states. When the limit \( \beta \to 0 \) is taken, the path integral becomes an ordinary integral

\[
I(0) = \frac{1}{\mathcal{F}_G} \mathcal{Z}_{D,G}
\]

(1.43)

where \( \mathcal{Z}_{D,G} \) is precisely the supersymmetric matrix partition function of equation \( \text{(1.23)} \) (at least up to some inverse factors of \( 2\pi \)). The constant \( \mathcal{F}_G \) is a group dependent factor and has been calculated for \( SU(N) \) in \[8\], and some other groups in \[10\]. The field \( X^0 \) that is missing from the quantum mechanics has become reinstated by the gauge fixing procedure. Moreover, although the original quantum mechanics has a Minkowski metric, the matrix \( X^0 \) appears in \( \text{(1.43)} \) with precisely the Wick rotated signature that we have been discussing.

The authors \[4,5\] calculated \( I(0) \) for \( SU(2) \) and gave an argument for calculating the deficit term, and reached the conclusion that \( I_W = 1 \) for \( D = 10 \) and \( I_W = 0 \) for \( D = 4, 6 \). Then in \[30\] the argument for calculating the deficit term was extended to all \( SU(N) \) for \( D = 10 \) suggesting

\[
I_{D=10}^d = - \sum_{m|N, m\geq 1} \frac{1}{m^2} \quad (1.44)
\]

and the arguments have subsequently been extended to other dimensions and groups \[31\].

Thus, the onus appeared to be on calculating the principle part of the index which is given by the supersymmetric matrix integral \( \text{(1.23)} \). Also in \[31\], Green and Gutperle made the conjecture based on \( D \)-instanton physics

\[
\mathcal{Z}_{10, SU(N)} = \mathcal{F}_{SU(N)} \sum_{m|N, m>0} \frac{1}{m^2}.
\]

(1.45)

In \[8\], the constant \( \mathcal{F}_{SU(N)} \) was calculated and the conjecture extended to \( D = 4, 6 \)

\[
\mathcal{Z}_{4, SU(N)} = \mathcal{Z}_{6, SU(N)} = \mathcal{F}_{SU(N)} \frac{1}{N^2}.
\]

(1.46)

These conjectures have been confirmed for small values of \( N \) by the Monte Carlo evaluations of \[8,9\]. The values \( \text{(1.43), (1.46)} \) also appeared in a very interesting calculation \[32\] based on deforming a cohomological action. We shall discuss this in detail in chapter \[4\].

However, the issue of the Witten index has not yet been fully resolved. To begin with, there does not yet exist a proof of the values of the bulk part of the index. Also
for at least one exceptional group, inconsistencies between the known bulk contribution and conjectured deficit contribution have been pointed out in 3.

1.6 Additional Motivation

The two models of sections 1.4 and 1.5 give motivation enough for studying Yang-Mills matrix theories. In addition, the $D = 10$ supersymmetric $SU(N)$ model can be thought of as a low energy effective theory for $D$-instantons ($D_{-1}$-branes) 34. Perhaps rather than as a physical theory, the $D$-instanton partition function should be regarded as a quantity which is likely to appear in many stringy calculations. Indeed it was such a calculation that allowed Green and Gutperle to predict its value (see section 1.5).

1.7 Thesis Plan

In chapter 2 of this thesis we shall consider the question of convergence of the partition function 1.23 and correlation functions 1.27. We find convergence conditions for the bosonic and supersymmetric models with any compact semi-simple gauge group. This is work originally published in 35, and builds on a previous paper 36.

In chapter 3 we shall consider the supersymmetric theories both in their original formulation, and as cohomological models. Concentrating on the $SU(N)$ models at large $N$, we completely classify the quantities which are invariant under the supersymmetry. This is work originally published in 37.

In chapter 4 we consider how to apply the deformation method of 32 directly to the supersymmetric Yang-Mills matrix models. The aim is to use the supersymmetry to obtain a rigorous exact calculation of the partition function. We find a deformation of the action that can generate mass terms for all the fields and still preserve some supersymmetry. This allows us to integrate over a BRST quartet rigorously, and confirm the formula that was obtained in 32. We show why this method fails so that an alternative regularisation must be found. However, a proof that the contour prescription of Moore, Nekrasov and Shatashvili is the correct regularisation remains elusive.
Chapter 2

Convergence

In this chapter, we establish the convergence properties of Yang-Mills matrix models. We consider the partition function and simple correlation functions in theories with compact semi-simple gauge group. In the supersymmetric case, we show that the partition function converges when $D = 4, 6$ and $10$, and that correlation functions of degree $k < k_c = 2(D - 3)$ are convergent independently of the group. In the bosonic case, we show that the partition function converges when $D \geq D_c$, and that correlation functions of degree $k < k_c$ are convergent, and calculate $D_c$ and $k_c$ for each group. The special case of $SU(N)$ establishes the convergence of the partition function and a set of correlation functions in the IKKT model of IIB strings.

2.1 Convergent Bosonic Integrals

We consider first the integral (1.23) without fermions so that $\mathcal{N} = 0$ and there is no Pfaffian. The factor $\frac{1}{4}$ in the original action can be scaled out in a trivial manner, so we drop it here, giving

$$Z_{D,G} = \int \prod_{\mu=1}^{D} dX_\mu \exp \left( \sum_{\mu,\nu} \text{Tr} [X_\mu, X_\nu]^2 \right). \quad (2.1)$$

Since the action is built out of commutators, there are flat directions in which the magnitude of the $X_\mu$ can be taken to $\infty$ while keeping the integrand constant. It is these regions which may lead to a divergence. Therefore it is useful to define a radial variable $R$ giving the magnitude of the $X_\mu$. Let

$$X_\mu = R x_\mu, \quad \text{Tr} x_\mu x_\mu = 1 \quad (2.2)$$
where from now on we use the summation convention for repeated indices. Noting that
\[ \text{Tr} x_\mu x_\mu = \sum_{\mu,i,j} |(x_\mu)_{ij}|^2 \]  
we see that the $x_\mu$ lie on a compact manifold. To rewrite the integral over the $X_\mu$ in terms of $R$ and $x_\mu$, we insert unity
\[ 1 = \int_0^\infty dR \, 2R \delta(R^2 - \text{Tr} X_\mu X_\mu) \]  
and scale out $R^2$ from the $\delta$-function. Then
\[ Z_{D,G} = 2 \int_0^\infty dR R^{Dg-1} \mathcal{X}_{D,G}(R) \]  
with
\[ \mathcal{X}_{D,G}(R) = \int \prod_{\nu=1}^D dx_\nu \, \delta(1 - \text{Tr} x_\mu x_\mu) \exp(-R^4 S) \]  
and
\[ S = -\text{Tr} [x_\mu, x_\nu] [x_\mu, x_\nu] \\ = \sum_{i,j,\mu,\nu} |[x_\mu, x_\nu]_{ij}|^2. \]  
We note that for any $R$ the integral $\mathcal{X}_{D,G}(R)$ is bounded by a constant. If for large $R$
\[ |\mathcal{X}_{D,G}(R)| < \frac{\text{const}}{R^\nu}, \quad \text{with } \nu > Dg, \]  
then the partition function $Z_{D,G}$ is finite. Our tactic for proving convergence of $Z_{D,G}$ is to find a bound of the form $2.8$ on $\mathcal{X}_{D,G}(R)$. A sufficient condition for the correlation function $1.27$ to converge is obtained by modifying $2.8$ to require $\nu > Dg + k$.

From now on, we are only interested in large $R$, so we shall always assume $R > 1$. Let us split the integration region in $2.6$ into two
\[ \mathcal{R}_1 : \quad S < (R^{-2-\eta})^2 \]  
\[ \mathcal{R}_2 : \quad S \geq (R^{-2-\eta})^2 \]  
where $\eta$ is small but positive. We see immediately that the contribution to $Z_{D,G}(R)$ from $\mathcal{R}_2$ is bounded by $A_1 \exp(-R^{2\eta})$ (we will use the capital letters $A$, $B$ and $C$ to denote constants throughout this chapter) and thus automatically satisfies $2.8$. Thus we can confine our efforts to the contribution from $\mathcal{R}_1$. In this region, we replace the
exponential function by unity and get the total bound

$$|X_{D,G}(R)| < A_1 \exp(-R^{2g}) + \mathcal{I}_{D,G}(R) \quad (2.10)$$

where

$$\mathcal{I}_{D,G}(R) = \int_{R_1}^R \prod_{\nu=1}^D dx_\nu \delta(1 - \text{Tr} x_\mu x_\mu). \quad (2.11)$$

From now on, we shall work with $\mathcal{I}_{D,G}(R)$, and seek a bound of the form

$$\mathcal{I}_{D,G}(R) < \frac{\text{const.}}{R^\nu}. \quad (2.12)$$

Then a sufficient condition for the partition function $Z_{D,G}$ to converge is

$$\nu > Dg \quad (2.13)$$

and for the correlation function $\mathcal{I}_{D,G}(R)$ to converge,

$$\nu > Dg + k. \quad (2.14)$$

The condition in (2.2) means that at least one of the matrices $x_\mu$ (say $x_1$) must satisfy

$$\text{Tr} x_1 x_1 \geq D^{-1}. \quad (2.15)$$

It is convenient to express the Lie algebra using the Cartan-Weyl basis

$$\{H^i, E^\alpha\} \quad (2.16)$$

where $i$ runs from 1 to the rank $l$ and $\alpha$ denotes a root. In this basis

$$[H^i, H^j] = 0, \quad [H^i, E^\alpha] = \alpha^i E^\alpha \quad (2.17)$$

and

$$[E^\alpha, E^\beta] = N_{\alpha\beta} E^{\alpha+\beta} \quad \text{if } \alpha + \beta \text{ is a root}$$

$$= 2|\alpha|^{-2} \alpha \cdot H \quad \text{if } \alpha = -\beta$$

$$= 0 \quad \text{otherwise.} \quad (2.18)$$

Here $E^{-\alpha} = (E^\alpha)^\dagger$, and the normalisation is chosen such that

$$\text{Tr} H^i H^j = \delta^{ij}, \quad \text{Tr} E^\alpha E^\beta = 2|\alpha|^{-2} \delta^{\alpha+\beta}. \quad (2.19)$$

Since the integrand and measure are gauge invariant, we can always use a group
2.1 Convergent Bosonic Integrals

element to move $x_1$ into the Cartan subalgebra

$$x_1 = x^i H^i$$

and reduce the integral over $x_1$ to an integral over its Cartan modes

$$\prod_{a=1}^{g} dx_1^a \to \text{const.} \left( \prod_{i=1}^{l} dx^i \right) \prod_{\alpha>0} (x \cdot \alpha)^2.$$ (2.21)

Here

$$\Delta^2_G(x) = \prod_{\alpha>0} (x \cdot \alpha)^2$$ (2.22)

is the Weyl measure. We expand the remaining $x_\nu$

$$x_\nu = x^i_\nu H^i + x^\alpha_\nu E^\alpha \quad \nu = 2, \cdots, D$$

with $x^{-\alpha}_\nu = (x^\alpha_\nu)^*$.

Looking back to equation 2.24, we certainly have

$$- \text{Tr} [x_1, x_\nu]^2 = | \text{Tr} [x_1, x_\nu]^2 | < S$$ (2.24)

and so, in the region $\mathcal{R}_1$

$$- \text{Tr} [x_1, x_\nu]^2 < R^{-2(2-\eta)}$$ (2.25)

for $\nu = 2, \cdots, D$. Writing this in terms of the basis 2.16 gives

$$4 \sum_{\alpha>0} \frac{(x \cdot \alpha)^2}{|\alpha|^2} |x^\alpha_\nu|^2 < R^{-2(2-\eta)}.$$ (2.26)

This is the key result because, whenever $(x \cdot \alpha)^2$ is bigger than a constant, it gives us a bound on $x^\alpha_\nu$ and so allows us to bound the integral.

As yet, we have not specified our choice of ordering giving the concept of positivity for roots. Since there are a finite number of roots, there is only a finite number of possible choices. In fact, for any $x$, there is always a choice such that $x \cdot \alpha \geq 0$ whenever $\alpha$ is a positive root. To see this, temporarily fix $x$ and change basis in the Cartan subalgebra so that $x = (1, 0, \cdots, 0)$. Now follow the usual construction, and define $\alpha$

\[ \text{The statement 2.24 due to Weyl is of course non-trivial. It comes from the fact that any } X \text{ in the Lie algebra can be written } X = U C U^\dagger \text{ where } C \text{ is in the Cartan subalgebra and } U \text{ is a group element. The integration measure becomes } dX = dU dC J(C) \text{ where the Jacobian } J \text{ is the Weyl measure. Then since the integrand is gauge invariant, the } U \text{ integration just gives a constant - loosely the volume of the group. In the particular case of } SU(N) \text{ the result is well known and the Weyl measure is the square of the Vandermonde. I am grateful to Dr M. Staudacher for explaining these results.} \]
to be positive if it’s first non-zero element is positive. Then, in particular, there is a set of \( l \) simple roots \( \{s_i\} \) which are positive, and such that any positive root can be be written \( \alpha = \alpha_is_i \) with the \( \{\alpha_i\} \) non-negative integers. Finally move back to the original basis. The property \( x \cdot \alpha \geq 0 \) when \( \alpha > 0 \) is preserved.

We can now split the integration region into a finite number of sub-regions; one for each choice of the positive roots. On each subregion, we have the condition \( x \cdot s_i \geq 0 \) for \( x \).

The \( \{s_i\} \) form a basis for the \( l \)-dimensional vectors. We can define a number \( c \) by

\[
c = \min_{\{\alpha^2 = 1\}} \max_i |a \cdot s_i|
\]  

(2.27)

which must be positive. Then the condition \( 2.13 \) tells us that at least one of the simple roots, \( s_1 \) say, satisfies \( x \cdot s_1 \geq cD^{-\frac{1}{2}} \). In addition, any positive root \( \alpha \) which contains the simple root \( s_1 \) must satisfy the same relation so that

\[
|x \cdot \alpha| \geq cD^{-\frac{1}{2}} \text{ whenever } \alpha \text{ contains } s_1.
\]

(2.28)

Let us now split up the Lie algebra \( \mathcal{G} \) as follows. Define \( \mathcal{G}' \) to be the regularly embedded subalgebra of \( \mathcal{G} \) obtained by omitting the simple root \( s_1 \). Then \( \mathcal{G}' \) has rank one less than \( \mathcal{G} \) and so there is one Cartan generator \( J \) outside \( \mathcal{G}' \). We can always choose \( J \) to commute with \( \mathcal{G}' \). To see this, note that \( s_2, \ldots, s_l \) span an \( l - 1 \) dimensional subspace of the \( l \) dimensional root space, so we can choose a basis in which they all have first component zero. In this basis, choose \( J = H^1 \). Then \( 2.17 \) shows \( [J, E_{\alpha'}^\nu] = 0 \) when \( E_{\alpha'}^\nu \in \mathcal{G}' \) so that \( [J, \mathcal{G}'] = 0 \). Let us rename the remaining generators of \( \mathcal{G} \) as \( \{F^\beta\} \) where \( \beta \) is any root which contains \( s_1 \). We can summarise some of the commutation relations as follows:

\[
[J, \mathcal{G}'] = 0
\]

\[
[F^\beta, \mathcal{G}'] \subset \{F^\gamma\}
\]

\[
[J, F^\beta] \subset \{F^\gamma\}
\]

\[
[\mathcal{G}', \mathcal{G}'] \subset \mathcal{G}'
\]

(2.29)

The first relation of \( 2.29 \) is given by the construction of \( J \). The other three relations follow immediately from \( 2.17 \) and \( 2.18 \).

Expanding

\[
x_\mu = y_\mu + \rho_\mu J + \omega_\mu F^\beta,
\]

(2.30)

\(^2\text{A subalgebra is “regularly embedded” if it is obtained by knocking out some simple roots from the original algebra.}\)
with \( y_\mu \in G' \), the conditions \( 2.26 \) and \( 2.28 \) give us a bound on the \( \omega_\mu \). \[
|\omega_\nu^\beta| < c^{-1}D^\frac{1}{2}\beta^2 R^{-(2-\eta)}, \quad \nu = 2, \ldots, D
\] (2.31)

There are a number of possible choices for \( G' \) depending on which simple root has been removed. The correct choice depends first of all on which of the \( x_\mu \) satisfies \( 2.15 \) (and so is relabeled \( x_1 \)). And then, given \( x_1 \), on which of the simple roots satisfies the condition in \( 2.28 \) (and so is relabeled \( s_1 \)). Thus, we have split the integration region up into a finite number of subregions according to the correct choice of \( G' \). We shall use \( 2.31 \) to bound the integral \( 2.11 \) in each of these regions. The region giving the least inverse power of \( R \) will then give a bound on \( I^D_{D,G} \).

Let us expand the action in terms of the variables \( 2.30 \). Using the commutation relations \( 2.29 \) and inner products \( 2.19 \) we see that the terms linear in \( \omega \) vanish giving

\[
S_G(x_\mu) = S'_{G'}(y_\mu) + 2\text{Tr}[y_\mu, y_\nu][F^3, F^\gamma]\omega_\mu^\beta\omega_\nu^\gamma + \text{Tr}(\omega_\mu^\beta[y_\mu, F^3] - \omega_\mu^\beta[y_\nu, F^3]) + (\rho_\mu\omega_\nu^\beta - \rho_\nu\omega_\mu^\beta)[J, F^3] + \omega_\mu^\beta\omega_\nu^\gamma[F^3, F^\gamma]^2.
\] (2.32)

Here we have added suffices to the actions to emphasise that \( S_G(x_\mu) \) is the original \( G \)-invariant action whilst \( S'_{G'}(y_\mu) \) is the \( G' \)-invariant action. Since the \( y_\mu \) and \( \rho_\mu \) are bounded by a constant, this can be written

\[
S_G(x) = S'_{G'}(y) + O(\omega^2).
\] (2.33)

Then the bound \( 2.31 \) on \( \omega \) shows that (up to a trivial scaling constant)

\[
x \in \mathcal{R}_1(G) \Rightarrow y \in \mathcal{R}_1(G').
\] (2.34)

We shall now take the expression \( 2.11 \) for \( \mathcal{I}_{D,G}(R) \), restrict the integration region to that where the appropriate subalgebra is \( G' \), and use the preceding results to form a bound and integrate out the variables \( \rho \) and \( \omega \). First, using \( 2.21 \) to reduce the \( x_1 \) integral to Cartan modes \( x \) gives

\[
\mathcal{I}_{D,G}(R) = \text{const.} \int_{\mathcal{R}_1(G)} dx \Delta_G^2(x) \prod_{\nu=2}^D dx_\nu \delta(1 - \text{Tr}x_\mu x_\mu).
\] (2.35)

Next note that the Weyl measure \( 2.22 \) for \( G \) can be bounded by that for \( G' \)

\[
\Delta_G^2(x) < \text{const.} \Delta_{G'}^2(y).
\] (2.36)

This is because when \( \alpha \) does not contain \( s_1 \), \( \alpha \cdot x = \alpha \cdot y \). Thus \( \Delta_G^2(x) \) is equal to
$\Delta_{G'}^2(y)$ up to some additional factors which are bounded by a constant. At this stage, we can also use the inner product relations \(2.19\) to decompose

$$\text{Tr} x_\mu x_\mu = \text{Tr} y_\mu y_\mu + \rho_\mu \rho_\mu + \frac{2}{|\beta|^2} |\omega_\mu^\beta|^2.$$  \hfill (2.37)

It is convenient to rescale the $\omega$ variables to get rid of the $2/|\beta|^2$ constants, and to use polar coordinates for both the $\rho$ and $\omega$ variables so that

$$\text{Tr} x_\mu x_\mu = \text{Tr} y_\mu y_\mu + \rho^2 + \omega^2.$$  \hfill (2.38)

In polars, the measures become

$$\prod_{\mu=1}^D d\rho_\mu = d\Omega_\rho d\rho^D d^{-1}$$  \hfill (2.39)

and

$$\prod_{\mu=2, \ldots, D} d\omega_\mu^\beta = \text{const.} d\Omega_\omega d\omega(D-1) (g-g' -1) -1$$  \hfill (2.40)

Counting the number of $\omega_\mu^\beta$ to get the exponent in \(2.40\) is crucial to eventually get the correct bound. There is an $\omega_\mu^\beta$ for each $\mu = 2, \ldots, D$ (but not $\mu = 1$ since $x_1$ was moved into the Cartan subalgebra), and each of the $F^\beta$. The $F^\beta$ are the generators of $G$ which are neither $J$, nor in $G'$, so there are $g - g' - 1$ of them. Then the total number of variables $\omega_\mu^\beta$ is $(D - 1)(g - g' - 1)$.

Thus, inserting \(2.38\) into \(2.33\), and using the bounds \(2.31\), \(2.34\) and \(2.36\) gives

$$I_{D,G}(R) < B_0 \int_{y_\mu \in \mathcal{R}_{1}(G')} dy_\mu \Delta_{G'}^2(y) \prod_{\nu=2}^D dy_\nu \int_0^{A_0 R^{-(2-\eta)}} d\omega \omega^{(D-1)(g-g' -1) -1} \int_0^1 d\rho \rho^{D-1} \delta(1 - \text{Tr} y_\mu y_\mu - \rho^2 - \omega^2).$$  \hfill (2.41)

where $B_0$ and $A_0$ are constants, and we have integrated out the angular variables $\Omega_\rho$ and $\Omega_\omega$.

Considering the inner integral first, we can integrate $\rho$ out immediately to obtain

$$\frac{1}{2} (1 - \text{Tr} y_\mu y_\mu - \omega^2)^{D-2} \theta(1 - \text{Tr} y_\mu y_\mu - \omega^2).$$  \hfill (2.42)

The original Yang-Mills integral \(1.23\) only makes sense when $D \geq 2$, and in this case,
we can bound the leading factor by a constant. In addition, we have the bound

\[ \theta(1 - \text{Tr} y_\mu y_\mu - \omega^2) \leq \theta(1 - \text{Tr} y_\mu y_\mu) \]  

so that (2.41) can be bounded by

\[ B_0 \int_{y_\mu \in \mathcal{R}_1(G')} dy \Delta^2_G(y) \prod_{\nu=2}^D dy_\nu \int_0^{A_0 R^{-(2-\nu)}} d\omega \omega^{(D-1)(g-g'-1)-1} \theta(1 - \text{Tr} y_\mu y_\mu). \]  

Finally integrating out \( \omega \) gives

\[ I_{D,G}(R) < B_1 R^{-(2-\eta)(D-1)(g-g'-1)} \mathcal{F}_{D,G'}(R) \]  

where

\[ \mathcal{F}_{D,G'}(R) = \int_{\mathcal{R}_1(G')} \prod_{\nu=1}^D dy_\nu \theta (1 - \text{Tr} y_\mu y_\mu). \]  

Here, since the integrand is gauge invariant, we have absorbed the \( G' \) Weyl measure and restored the integral to \( G' \) gauge invariant form. Using the identity \( \theta(1 - \text{Tr} y_\mu y_\mu) = \int_0^1 dt \delta(t - \text{Tr} y_\mu y_\mu) \); and then rescaling \( t = [u/R]^{2-\eta} \) and \( y_\mu = \bar{y}_\mu |u/R|^{1-\eta/2} \) gives

\[ \mathcal{F}_{D,G'}(R) = (2-\eta) R^{-(1-\eta/2)Dg'} \int_0^R du u^{(1-\eta/2)Dg'-1} I_{D,G'}(u). \]  

We shall proceed by induction. Our aim is to show that

\[ \int_0^\infty dR R^{Dg-1} I_{D,G}(R) < \text{const}. \]  

If this is true for \( G' \), then the integral in (2.47) is bounded by a constant and so

\[ \mathcal{F}_{D,G'}(R) < B_2 R^{-(1-\eta/2)Dg'} \]  

so that by (2.45)

\[ I_{D,G}(R) < B_3 R^{-(1-\eta/2)[D+2(D-1)(g-g'-1)+Dg']} \]  

when \( R > 1 \) (recall that all of our bounds apply only to \( R > 1 \)). The integral (2.48) is certainly convergent in the region \( 0 \leq R \leq 1 \) since \( I_{D,G} \) is always finite. For \( R > 1 \), we can substitute the bound (2.50) into (2.48) and decide whether (2.48) also converges for \( G \).

Our task then is to find the regularly embedded subalgebras \( G' \) of \( G \) and choose the one which leads to the least inverse power of \( R \) in (2.50). Fortunately, the regularly embedded subalgebras can easily be found using the Dynkin diagram. The Dynkin
diagram for a Lie algebra has a node for each simple root. The nodes are connected by 3, 2, 1 or 0 lines respectively as the angle between the corresponding roots is 150, 135, 120 or 90 degrees. In addition to these restrictions on angles, the simple roots of a compact simple Lie algebra can come in at most two different lengths. The notation in this thesis is that nodes corresponding to shorter roots are coloured black. Knowledge of the Dynkin diagram is enough to reconstruct the entire Lie algebra. Thus we can find the regularly embedded subalgebras $G'$ by removing one node from the Dynkin diagram. For an excellent review of Lie algebra methods including tables of the Dynkin diagrams and dimensions that we shall use in the following, see [38].

Before proceeding to consider each group in turn, we make a final observation. If the regularly embedded subalgebra $G'$ is a direct sum of two (mutually commuting) subalgebras $G' = G'_1 \oplus G'_2$ then we have

$$F_{D,G'}(R) < F_{D,G'_1}(R) F_{D,G'_2}(R)$$ (2.51)

since $\theta(1 - \text{Tr}_{G'} y_\mu y_\mu) \leq \theta(1 - \text{Tr}_{G'_1} y_\mu y_\mu) \theta(1 - \text{Tr}_{G'_2} y_\mu y_\mu)$. The result 2.50 is unaffected, but this will help us to deal with the few $G'$ which have divergent Yang-Mills integrals so that 2.49 and 2.50 are not true.

We shall now consider each group in turn. We only consider dimensions $D \geq 3$ since, as we will see in section 2.2, the partition function is always divergent when $D = 2$. The case of $SU(r + 1)$ is most tricky because $SU(2)$ and $SU(3)$ are rather special having divergent partition function for some low values of $D$. We work through the $SU(r + 1)$ groups in detail to show the method. For the other groups, one can easily follow the same method, and so we give less detail.

**SU(r+1):** The Dynkin diagram for $su(r + 1)$ is

$$\circ \longrightarrow \circ \longrightarrow \circ \cdots \longrightarrow \circ$$ (2.52)

where there are $r$ nodes. To find the regularly embedded subalgebras $G'$ we remove one of the nodes, and discover

$$su(r + 1) \rightarrow G' = su(m) \oplus su(r + 1 - m), \quad 1 \leq m \leq r$$ (2.53)

where we define $su(1) = 0$. The dimension of $su(m)$ is $m^2 - 1$, so that

$$g = (r + 1)^2 - 1, \quad g' = m^2 + (r + 1 - m)^2 - 2.$$ (2.54)

The Lie algebra $su(2)$ has no regularly embedded subalgebra, so $g' = 0$. The arguments leading to 2.50 are all still valid (the only difference is that in this
2.1 Convergent Bosonic Integrals

case there are no variables $y_\mu$ corresponding to $G'$ so we know explicitly that the $F_{D,G'}$ appearing in 2.45 is just a constant). Then setting $g = 3$ and $g' = 0$ in 2.50 gives

$$I_{D,SU(2)} < B_3 R^{-(1-\eta/2)4(D-1)}, \quad R > 1.$$  \hspace{1cm} (2.55)

Referring back to 2.48, we see that we need

$$3D < (1 - \eta/2)4(D - 1)$$  \hspace{1cm} (2.56)

which can be re-written

$$D > 4 + 2\eta(D - 1)$$  \hspace{1cm} (2.57)

for convergence. Of course, this also corresponds to the original condition 2.13 for the partition function to converge, so by choosing $\eta$ sufficiently small, $Z_{D,SU(2)}$ is finite for $D \geq 5$.

In the cases $D = 3$ and $D = 4$, we have failed to show that the desired induction statement 2.48 is true. However, we can substitute 2.55 back into 2.47 to obtain a bound on $F_{D,SU(2)}$ even when $D < 5$:

$$F_{D,SU(2)} < B_4 R^{(1-\eta/2)3D} R^{1-\eta/2} \delta_{D,3} \delta_{D,4} (\log R)^{\delta_{D,4}}, \quad R > 1.$$  \hspace{1cm} (2.58)

In dimensions 3 and 4, the result is at variance with 2.49. However, since $\log R$ tends to $\infty$ more slowly than any positive power of $R$, modification by a $\log R$ factor will not affect any of our conclusions. (We can modify 2.12 to add a $\log R$ factor and still leave the conditions 2.13 and 2.14 unchanged.) Thus it is only for $D = 3$ that we must be careful to use the modified formula.

The Lie algebra $su(3)$ has $su(2)$ as its only regularly embedded subalgebra. Then substituting 2.58 into 2.45 gives the bound

$$I_{D,SU(3)} < B_3 R^{-(1-\eta/2)[11D-8]} R^{(1-\eta/2)\delta_{D,3}} (\log R)^{\delta_{D,4}}$$  \hspace{1cm} (2.59)

and we discover $Z_{D,SU(3)}$ converges for $D \geq 4$. In this case, the formula 2.49 for $F_{D,SU(3)}$ is modified only in the case $D = 3$, and only by a factor of $\log R$ which will not affect our results, and we may proceed as if the induction statement 2.48 were true.

For $SU(r+1)$ with $r \geq 3$, it is a simple exercise to discover which of the possible $G'$ gives the least inverse power of $R$ behaviour in 2.50. We substitute the $g$ and $g'$ of 2.54 into 2.50, and choose the value of $m$ which gives the dominant behaviour. The only point to remember is that we must include an extra $R^{1-\eta/2}$ factor in
the case of $G' = su(2) \oplus su(r - 1)$ when $D = 3$, to allow for the anomalous behaviour of $F_{3,SU(2)}$.

We discover that the dominant behaviour is always obtained when $G' = su(r)$, giving $g' = r^2 - 1$. Then $2.50$ becomes

$$I_{D,SU(r+1)} < B_3 R^{-(1-\eta/2)(D(r^2+4r-1)-4r)}, \quad r \geq 3.$$  \hfill (2.60)

Taking $\eta$ small, the condition $2.13$ is met and so the partition function $Z_{D,SU(r+1)}$ is convergent for $D \geq 3$ when $r \geq 3$ (and of course, crucially, the induction statement $2.48$ is true).

Finally, comparing the bounds $(2.55, 2.59$ and $2.60)$ on $I_{D,G}$ with the condition $2.14$, we see that the correlation function $1.27$ converges when $k < k_c$ with

$$k_c = (2rD - D - 4r - \delta_{D,3}\delta_{r,2}), \quad r \geq 1, \quad D \geq 3.$$ \hfill (2.61)

In this formula, the cases with $k_c \leq 0$ are those for which the method fails to prove convergence even of the partition function. In section 2.2 we shall show that these cases are indeed divergent.

**SO(2r+1), $r \geq 2$:** The Dynkin diagram for $so(2r+1)$ is

\begin{center}
\begin{tikzpicture}
\node[vertex] (1) at (0,0) {\Large $\bullet$};
\node[vertex] (2) at (1,0) {\Large $\bigcirc$};
\node[vertex] (3) at (2,0) {\Large $\bigcirc$};
\node[vertex] (4) at (3,0) {\Large $\cdots$};
\node[vertex] (5) at (4,0) {\Large $\bigcirc$};
\node[vertex] (6) at (5,0) {\Large $\bigcirc$};
\end{tikzpicture}
\end{center}

(2.62)

where there are $r$ nodes, and the dimension is $g = 2r^2 + r$. By removing one node, we see that the possible $G'$ are $so(2m+1) \oplus su(r-m)$ with $0 \leq m \leq r - 1$.

We discover the most important contribution is always from $G' = so(2r - 1)$, and that $Z_{D,SO(2r+1)}$ always converges for $r \geq 2$ and $D \geq 3$. The critical degree $k_c$ for correlation functions is

$$k_c = \begin{cases} 2 & r = 2, D = 3 \\ 4rD - 8r - 3D + 4 & \text{otherwise.} \end{cases}$$ \hfill (2.63)

The exception when $r = 2$ and $D = 3$ occurs because of the anomalous behaviour of $F_{3,SU(2)}$.

**Sp(2r), $r \geq 2$:** The Dynkin diagram for $sp(2r)$ is

\begin{center}
\begin{tikzpicture}
\node[vertex] (1) at (0,0) {\Large $\bigcirc$};
\node[vertex] (2) at (1,0) {\Large $\bullet$};
\node[vertex] (3) at (2,0) {\Large $\bullet$};
\node[vertex] (4) at (3,0) {\Large $\cdots$};
\node[vertex] (5) at (4,0) {\Large $\bullet$};
\node[vertex] (6) at (5,0) {\Large $\bullet$};
\end{tikzpicture}
\end{center}

(2.64)

where there are $r$ nodes, and the dimension is $g = 2r^2 + r$. The possible $G'$ are
Convergent Bosonic Integrals

\( sp(2m) \oplus su(r-m) \) with \( 0 \leq m \leq r-1 \), and the dominant contribution is from \( sp(2r-2) \). The partition function \( Z_{D,Sp(2r)} \) converges for all \( r \geq 2 \) and \( D \geq 3 \) and the critical correlation function is given by

\[
\begin{align*}
    k_c &= 2 & r = 2, D = 3 \\
    k_c &= 4rD - 8r - 3D + 4 & \text{otherwise.}
\end{align*}
\] (2.65)

**SO(2r), \( r \geq 4 \):** The Dynkin diagram for \( so(2r) \) is

\[
\begin{align*}
    \circ & \circ \cdots \circ \circ 
\end{align*}
\] (2.66)

where there are \( r \) nodes, and the dimension is \( g = 2r^2 - r \). The possible \( G' \) are \( so(2m) \oplus su(r-m) \) for \( 4 \leq m \leq r-1 \), \( su(4) \oplus su(r-3) \), \( su(r-2) \oplus su(2) \oplus su(2) \) and \( su(r) \). The dominant contribution always comes from \( so(2r-2) \), and we discover that \( Z_{D,SO(2r)} \) always converges for \( D \geq 3 \) and \( r \geq 4 \). The critical correlation function is given by

\[
k_c = 4rD - 5D - 8r + 8.
\] (2.67)

**G_2:** The Dynkin diagram is

\[
\begin{align*}
    \circ \circ \bullet 
\end{align*}
\] (2.68)

and the dimension is 14. The only regularly embedded subalgebra is \( su(2) \), and we discover \( Z_{D,G_2} \) converges for \( D \geq 3 \) with

\[
k_c = 9D - 20 - \delta_{D,3}.
\] (2.69)

**F_4:** The Dynkin diagram is

\[
\begin{align*}
    \bullet \circ \circ 
\end{align*}
\] (2.70)

and the dimension \( g = 52 \). The dominant contributions come equally from \( G' = so(7) \) and \( G' = sp(6) \), each having \( g' = 21 \). Then \( Z_{D,F_4} \) converges for \( D \geq 3 \) and

\[
k_c = 29D - 60.
\] (2.71)
\textbf{E}_6: The Dynkin diagram is

\[\begin{array}{c}
\bullet \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}\]

and the dimension \(g = 78\). The dominant contribution comes from \(G' = \text{so}(10)\) having \(g' = 45\). Then \(Z_{D,E_6}\) converges for \(D \geq 3\) and

\[k_c = 31D - 64.\] (2.73)

\textbf{E}_7: The Dynkin diagram is

\[\begin{array}{c}
\bullet \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}\]

and the dimension \(g = 133\). The dominant contribution comes from \(G' = e_6\) with \(g' = 78\). Then \(Z_{D,E_7}\) converges for \(D \geq 3\) and

\[k_c = 53D - 108.\] (2.75)

\textbf{E}_8: The Dynkin diagram is

\[\begin{array}{c}
\bullet \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\circ \\
\end{array}\]

with dimension \(g = 248\). The dominant contribution comes from \(G' = e_7\) with \(g' = 133\). Then \(Z_{D,E_8}\) converges for \(D \geq 3\) and

\[k_c = 113D - 228.\] (2.77)

\section{2.2 Divergent Bosonic Integrals}

The lowest \(D\) partition function that we can sensibly write down is for \(D = 2\),

\[Z_{2,G} = \int dX_1 dX_2 \exp \left( \text{Tr} [X_1, X_2]^2 \right).\] (2.78)
We can use the gauge symmetry (by invoking \ref{2.21}) to reduce the $X_1$ integral to Cartan modes, but then the integrand is independent of the Cartan modes of $X_2$. Thus, it is immediate that this integral diverges for every group. From now on in this section, we assume $D \geq 3$.

In the previous section \ref{6} we found an upper bound on $\mathcal{I}_{D,G}(R)$. This equivalently gave us an upper bound on $\mathcal{X}_{D,G}(R)$ (originally defined in equations \ref{2.7} and \ref{2.7}). We used the large $R$ behaviour to show that many of the partition and correlation functions are finite.

We shall now find a lower bound on $\mathcal{X}_{D,G}(R)$. We shall discover that the large $R$ behaviour of this lower bound is almost identical to that of the upper bound. The only difference is that the (arbitrarily small) parameter $\eta$ of the previous section is set to zero.

Since the integrand is positive, it is sufficient to consider a sub-region of the phase space in order to find a lower bound. This time, we consider the region

$$\mathcal{R} : \quad S < R^{-4}$$

Then $\exp(-R^4S) > \exp(-1)$ and so

$$\mathcal{X}_{D,G}(R) > C_1 \mathcal{I}_{D,G}$$

where now

$$\mathcal{I}_{D,G} = \int_{\mathcal{R}} \prod_{\nu=1}^{D} dx_{\nu} \delta (1 - \text{Tr} x_{\mu} x_{\mu})$$

and, moving $x_1$ into the Cartan subalgebra,

$$\mathcal{I}_{D,G}(R) = C_2 \int_{\mathcal{R}} \prod_{i=1}^{l} dx_{1i} \Delta_{G}^2(x_{1i}) \prod_{\nu=2}^{D} dx_{\nu} \delta (1 - \text{Tr} x_{\mu} x_{\mu}).$$

Now pick a regularly embedded sub-algebra $G'$ of $G$ (with rank 1 less than $G$). As before, write $x = y + \rho J + \omega^{\beta} F^{\beta}$ with $y \in G'$. We will again write $s_1$ for the simple root of $G$ which is removed in order to obtain $G'$. As usual, we use a basis in which $s_1$ is the only simple root which has its first element non-zero (indeed, we have already chosen this basis, since we have set $H^1 = J$ with $[J, G'] = 0$).

Define a new region $\mathcal{R}_\epsilon'$ by

$$\mathcal{R}_\epsilon' : \quad ||\omega|| < \epsilon R^{-2}$$

$$S_{G'}(y_{\mu}) < \epsilon R^{-4}.$$
Then by taking $\epsilon$ small enough, we see from (2.33)

$$R'_\epsilon \subset R$$

so that

$$\mathcal{I}_{D,G}(R) > C_2 \int_{R'} \prod_{i=1}^{l-1} dx_i^i \Delta_G^2(x_1^i) \prod_{\nu=2}^D dx_{\nu} \delta (1 - \text{Tr}x_{\mu}x_{\mu}).$$

(2.84)

As in the previous section, it is convenient to rescale the $\omega^\beta$ and write them in polar form (as per equation 2.37). However, we leave the $\rho_{\mu}$ as they are for the moment. Then the integral becomes

$$\mathcal{I}_{D,G}(R) > C_3 \int_{R'} \prod_{i=1}^{l-1} dy_i^i \prod_{\nu=2}^D dy_{\nu} \prod_{\mu=1}^D d\rho_{\mu} \Delta_G^2(y_1^i, \rho_1)
\quad \times d\omega \omega^{(D-1)(g-g'-1)-1} \delta (1 - \text{Tr}y_{\mu}y_{\mu} - \omega^2 - \rho_{\mu}\rho_{\mu})$$

(2.85)

where now the $i$ index runs from 1 to $l-1$, and $\nu$ runs from 2 to $D$.

Choosing to integrate over just two of the $\rho_{\mu}$, say $\rho_{D-1}$ and $\rho_D$, leads to

$$\int d\rho_{D-1} d\rho_D \delta (1 - \text{Tr}y_{\mu}y_{\mu} - \omega^2 - \rho_{\mu}\rho_{\mu}) = C_4 \theta (1 - \text{Tr}y_{\mu}y_{\mu} - \omega^2 - \sum_{\mu=1}^{D-2} \rho_{\mu}\rho_{\mu})$$

(2.86)

as can quickly be seen by writing $\rho_{D-1}$ and $\rho_D$ in 2-dimensional polars. Now, when $R > 1$, certainly $\omega^2 < \epsilon^2$ by 2.83. In addition, we can restrict the region of integration of each of the remaining $\rho_{\mu}$ to $-\epsilon < \rho_{\mu} < \epsilon$ since we are looking for a lower bound on the integral. Then we have the inequality

$$\theta (1 - \text{Tr}y_{\mu}y_{\mu} - \omega^2 - \sum_{\mu=1}^{D-2} \rho_{\mu}\rho_{\mu}) \geq \theta (1 - \text{Tr}y_{\mu}y_{\mu} - \epsilon^2 - (D-2)\epsilon^2).$$

(2.87)

The integrand is now independent of $\rho_2, \ldots, \rho_{D-2}$, so we can immediately integrate them out to obtain the constant $(2\epsilon)^{D-3}$. The $\theta$-function is also now independent of $\omega$, so we have the factor

$$\int_0^{\epsilon R^{-2}} d\omega \omega^{(D-1)(g-g'-1)-1} = C_5 R^{-2(D-1)(g-g'-1)}$$

(2.88)
Finally, we can scale each of the $y_\mu$ and also $\rho_1$ by a factor $\sqrt{1-(D-1)\epsilon^2}$ to obtain

$$I_{D,G}(R) > C_6 R^{-2(D-1)(g-g'-1)} \int_{\mathcal{R}(G')} \prod_{i=1}^{l-1} dy^i_1 \prod_{\nu=2}^{D} dy_\nu \int_{-\epsilon}^{\epsilon} d\rho_1 \Delta^2_{G'}(y^i_1, \rho_1) \theta(1 - \text{Tr} y_\mu y_\mu).$$

(2.90)

Here, the integration region $\mathcal{R}'_{\epsilon}$ now applies only to the $G'$ variables $y_\mu$. Since $R$ can be trivially scaled by a constant, we have also dropped the subscript $\epsilon$ so that the region has become simply $\mathcal{R}(G')$ in (2.90).

Let's use the definition (2.22) to decompose the Weyl measure for $G$ into the part for $G'$ and additional factors:

$$\Delta^2_{G}(y^i_1, \rho_1) = \Delta^2_{G'}(y^i_1) \prod_{\alpha > 0, s_1 \in \alpha} [\alpha \cdot (\rho_1, y^i_1)]^2,$$

(2.91)

where $(\rho_1, y^i_1)$ represents the vector in $l$-dimensional root space, and the product is over all those positive roots which contain the simple root $s_1$. Since $s_1$ is the only simple root to have its first element non-zero, every factor in the product of (2.91) contains a $\rho_1$. Thus, the integral that we are left with over $\rho_1$ is

$$\int_{-\epsilon}^{\epsilon} d\rho_1 \prod_{\alpha > 0, s_1 \in \alpha} [\alpha \cdot (\rho_1, y^i_1)]^2$$

(2.92)

and can be re-written in a rather more transparent way as

$$C_7 \int_{-\epsilon}^{\epsilon} d\rho_1 \prod_{\alpha > 0, s_1 \in \alpha} (\rho_1 + z_\alpha)^2$$

(2.93)

where the $z_\alpha$ are linear combinations $z_\alpha = \alpha^i y^i_1 / \alpha_0$.

No matter what values the $z_\alpha$ take, the integral (2.93) is always positive. The $y^i_1$ and therefore the $z_\alpha$ lie within a compact set and so we have the bound

$$\int_{-\epsilon}^{\epsilon} d\rho_1 \prod_{\alpha > 0, s_1 \in \alpha} (\rho_1 + z_\alpha)^2 > \min_{\{z_\alpha\}} \int_{-\epsilon}^{\epsilon} d\rho_1 \prod_{\alpha > 0, s_1 \in \alpha} (\rho_1 + z_\alpha)^2 = C_8 > 0.$$

(2.94)

Substituting back into (2.90) gives

$$I_{D,G}(R) > C_3 R^{-2(D-1)(g-g'-1)} \int_{\mathcal{R}(G')} \prod_{\nu=1}^{D} dy_\nu \theta(1 - \text{Tr} y_\mu y_\mu)$$

(2.95)

where since the integrand is now $G'$ gauge invariant we have absorbed the $G'$ Weyl measure and restored the integral to full $G'$-gauge invariant form.
Chapter 2: Convergence

We now follow the method of the previous section (equations 2.46, 2.47) and set

\[ F_{D,G'}(R) = \int_{R(G')} \prod_{\nu=1}^{D} dy_{\nu} \theta (1 - T y_{\mu} y_{\mu}). \quad (2.96) \]

Using the identity \( \theta (1 - T y_{\mu} y_{\mu}) = \int_{0}^{1} dt \delta(t - T y_{\mu} y_{\mu}) \), and then rescaling \( t = \frac{u}{R} \) and \( y_{\mu} = \tilde{y}_{\mu} \frac{u}{R} \) gives

\[ F_{D,G'}(R) = 2R^{-Dg'} \int_{0}^{R} du \ u^{Dg'-1} I_{D,G'}(u). \quad (2.97) \]

Then 2.95 becomes

\[ I_{D,G}(R) > C_{10} R^{-2(D-1)(g-g'-1)-Dg'} \int_{0}^{R} du \ u^{Dg'-1} I_{D,G'}(u). \quad (2.98) \]

Comparing with 2.45 and 2.47 of the previous section, we see that we have proved essentially a converse result. In those majority of cases for which the partition function for \( G' \) is finite, it is sufficient to use the bound

\[ \int_{0}^{R} du \ u^{Dg'-1} I_{D,G'}(u) > \text{const.}, \quad R > 1 \quad (2.99) \]

giving

\[ I_{D,G}(R) > C_{11} R^{-2(D-1)(g-g'-1)-Dg'}. \quad (2.100) \]

In those cases for which the partition function for \( G' \) is not finite, we can find a better bound for \( G \) by inductively substituting the bound found for \( G' \) into 2.98 and performing the integral. For example, as we discovered in the previous section, the crucial exceptional case is when \( G' = SU(2) \). Since \( su(2) \) has no regularly embedded subalgebra, the bound 2.100 holds as it is, with \( g = 3 \) and \( g' = 0 \)

\[ I_{D,SU(2)} > C_{11} R^{-4(D-1)}, \quad R > 1. \quad (2.101) \]

Then

\[ \int_{0}^{R} du \ u^{3D-1} I_{D,SU(2)}(u) > C_{12} \quad D \geq 5 \]
\[ > C_{13} \log R \quad D = 4 \]
\[ > C_{14} R \quad D = 3 \quad (2.102) \]

so that, using 2.98, we obtain a better bound for \( SU(3) \)

\[ I_{D,SU(3)} > C_{15} R^{-8(D-1)-3D} (\log R)^{\delta_{D,4}} R^{\delta_{D,3}}. \quad (2.103) \]
For those simple groups other than $SU(3)$, the bound $2.100$ is enough for our purpose. The final step is that we have to choose the regularly embedded subalgebra $G'$ of $G$ which gives the tightest lower bound. However, we have already performed this task in the previous section since we chose $G'$ to give the least inverse power of $R$ behaviour.

Let's summarise. In the previous section, we found upper bounds on $X_{D,G}(R)$ which allowed us to deduce that certain partition functions and correlation functions are finite. These upper bounds depend on a parameter $\eta$ which can be taken arbitrarily small. In this section, we have found lower bounds on $X_{D,G}(R)$. The large $R$ behaviour for these lower bounds is precisely the limit when $\eta \to 0$ of that for the upper bounds.

Thus, using $X_{D,G} > C_2 I_{D,G} (2.80)$, we can substitute the lower bounds back into the definition $2.5$ and discover that indeed the partition function is divergent for $SU(2)$ when $D = 3, 4$, and for $SU(3)$ when $D = 3$. Further, let's consider the correlation function

$$\langle (\text{Tr} X_\mu X_\mu)^{k/2} \rangle = \int_0^\infty dRR^{Dg+k-1} X_{D,G}(R).$$

(2.104)

Then this integral diverges when $k \geq k_c$ with the values of $k_c$ quoted in the previous section. So, $k_c$ is indeed the critical value for correlation functions. Every correlation function with $k < k_c$ converges, and there is always a correlation function with $k = k_c$ which diverges.

### 2.3 Convergent Supersymmetric Integrals

We now move on to consider the supersymmetric integrals $\mathcal{Z}_{D,G}$ which we recall can be written down in dimensions $D = 3, 4, 6$ and 10 with $\mathcal{N} = 2(D - 2)$. Proceed as for the bosonic integrals to set

$$\mathcal{Z}_{D,G} = \int_0^\infty dRR^{Dg-1} R^{(D-2)g} X_{D,G}(R)$$

(2.105)

where now

$$X_{D,G}(R) = \int \prod_{\nu=1}^D dx_\nu \mathcal{P}_{D,G}(x_\sigma) \delta (1 - \text{Tr} x_\mu x_\mu) \exp (-R^4 S).$$

(2.106)

As before, it is sufficient to consider the region

$$\mathcal{R}_1(G) : \quad S < R^{-2(2-\eta)}$$

(2.107)
We shall again argue by induction, and for the induction step to work, we will need to prove the result for the generalised Pfaffian

\[ [\mathcal{P}_{D,G}^r(x, R)]^{a_1, \ldots, a_{2r}}_{\alpha_1, \ldots, \alpha_{2r}} = R^{-(2-\eta)2r} \int d\psi \exp(\text{Tr} \Gamma^\mu \psi_{\alpha}[x_\mu, \psi_{\beta}]) \psi_{\alpha_1} \cdots \psi_{\alpha_{2r}}. \tag{2.108} \]

The Pfaffian is modified from the usual definition by the inclusion of \(2r\) fermionic insertions, and a factor of \(R^{-(2-\eta)}\) has been included for each insertion. The modified Pfaffian can be written down for any \(r = 0, \ldots, (D-2)g\). If we set \(r = 0\) then the original Pfaffian \(\mathcal{P}_{D,G}\) is recovered (and is of course independent of \(R\)).

The structure of the \(\Gamma\) matrices will be irrelevant from now on (although we shall of course use the fact that their elements are bounded by a constant). For a more compact notation, we shall suppress the dependence on \(\Gamma\), and on the spinor and group indices, and write

\[ \mathcal{P}_{D,G}^r(x, R) = R^{-(2-\eta)2r} \int d\psi \exp(\text{Tr} \psi_{\alpha}[x, \psi]) \psi^1 \cdots \psi^{2r}. \tag{2.109} \]

Then defining

\[ \mathcal{T}_{D,G}^r(R) = \int_{R,(g)} \prod_{\nu=1}^{D} dx_\nu \left| \mathcal{P}_{D,G}^r(x, R) \right| \delta (1 - \text{Tr} x_\mu x_\mu). \tag{2.110} \]

we have

\[ \left| \mathcal{X}_{D,G}^r(R) \right| < A_1 \exp(-R^{2\eta}) + \mathcal{T}_{D,G}^r(R) \tag{2.111} \]

Proceeding as in the bosonic case, we choose the relevant regularly embedded subalgebra \(G'\), expand \(x_\mu = y_\mu + \rho_\mu J + \omega^\beta F^\beta\), and note

\[ |\omega^\beta_\nu| < c^{-1} D^{\frac{3}{2}} R^{-(2-\eta)}, \quad \nu = 2, \ldots, D. \tag{2.112} \]

Further, write

\[ \psi = \phi + \xi + \chi \tag{2.113} \]

with \(\phi \in G'\), \(\xi = \xi J\) and \(\chi = \chi^\beta F^\beta\). Using the relations \(2.29\), we find

\[ \text{Tr} \psi_{\alpha}[x, \psi] = \text{Tr} \phi_{\alpha}[y, \phi] \]

\[ + \text{Tr} \phi_{\alpha}[\omega, \chi] + \text{Tr} \chi_{\alpha}[\omega, \phi] \]

\[ + \text{Tr} \chi_{\alpha}[\omega, \xi] + \text{Tr} \xi_{\alpha}[\omega, \chi] \]

\[ + \text{Tr} \chi_{\alpha}[x, \chi] \tag{2.114} \]

where \(\rho = \rho J\) and \(\omega = \omega^\beta F^\beta\). Inserting this expression into the definition \(2.108\) and
expanding part of the exponential gives

\[ P_{D,G}(x,R) = \int d\phi d\chi d\xi \left( \frac{\xi_1 \cdots \xi_k \phi^1 \cdots \phi^m \chi^1 \cdots \chi^n}{R^k(2-\eta) R^m(2-\eta) R^n(2-\eta)} \right) \]

\[ \times \exp(\text{Tr} \phi [y,\phi] + \text{Tr} \phi [\omega,\chi] + \text{Tr} \chi [\omega,\phi] + \text{Tr} \chi [x,\chi]) \]

\[ \times \frac{1}{(2(D-2) - k)!} \left( \text{Tr} \chi [\omega,\xi] + \text{Tr} \xi [\omega,\chi] \right)^{2(D-2) - k}, \] (2.115)

where \( k + m + n = 2r \). We first perform the integrals over the \( \mathcal{N} = 2(D-2) \) Grassman variables \( \xi_\alpha \) each of which is paired either with an \( \omega \), or with an explicit factor \( R^{2-\eta} \).

Since \( \omega \) is itself bounded by \( R^{2-\eta} \) (2.112), we find

\[ \left| P_{D,G}(x,R) \right| \leq \frac{R^{-2(D-2)(2-\eta)}}{(2(D-2) - k)!} \sum_P \left| \int d\phi d\chi \left( \frac{\phi^1 \cdots \phi^m \chi^1 \cdots \chi^{n+2(D-2) - k}}{R^m(2-\eta) R^n(2-\eta)} \right) \right| \]

\[ \times \exp(\text{Tr} \phi [y,\phi] + \text{Tr} \phi [\omega,\chi] + \text{Tr} \chi [\omega,\phi] + \text{Tr} \chi [x,\chi]) \] (2.116)

where \( P \) indicates all the possible permutations of indices that can be generated. The next step is to expand the \( \phi \omega \chi \) terms to get

\[ \left| P_{D,G}(x,R) \right| \leq \frac{R^{-2(D-2)(2-\eta)}}{(2(D-2) - k)!} \sum_P \left( \sum_l \frac{2^l}{l!(2(D-2) - k)!} \right) \]

\[ \times \left| \int d\phi \left( \frac{\phi^1 \cdots \phi^{m+l}}{R^{m+l}(2-\eta)} \right) \exp(\text{Tr} \phi [y,\phi]) \right| \]

\[ \times \max_x \left| \int d\chi \left( \frac{\chi^1 \cdots \chi^{n+2(D-2) - k+l}}{R^n(2-\eta)} \right) \exp(\text{Tr} \chi [x,\chi]) \right|. \] (2.117)

Finally, integrate out the \( \chi \) fermions and use the fact that \( x \) is bounded to obtain

\[ \left| P_{D,G}(x,R) \right| \leq R^{-2(D-2)(2-\eta)} \sum_{r'} C_{r'} \left| P'_{D,G}(y,R) \right| \] (2.118)

where the \( C_{r'} \) are constants. In the spirit of the notation 2.109, we have suppressed sums over the many possible combinations of indices.

Inserting the bound 2.118 into 2.110 gives

\[ T_{D,G}(R) < R^{-(2-\eta)2(D-2)} \sum_{r'} C_{r'} \int_{R_1}^D dx_\mu \left| P'_{D,G}(y,R) \right| \delta \left( 1 - \text{Tr} x_\mu x_\mu \right) \] (2.119)

and we can now follow the bosonic procedure and integrate out the \( \omega \) and \( \rho \) degrees of freedom to obtain
\[ I_{rD,G}(R) < R^{-(2-\eta)2(D-2)} R^{-(2-\eta)(D-1)(g'-1)} \]
\[ \times \sum_{r'} C_{r'} \int_{R_1(G')}^{D} dy \nu \left| \mathcal{P}'_{D,G'}(y,R) \right| \theta (1 - \text{Try}_y y_{\mu}). \]

(2.120)

As before, replace \( \theta (1 - \text{Try}_y y_{\mu}) = \int_0^1 dt \delta (t - y_{\mu} y_{\mu}) \) and rescale \( t = [u/R]^{2-\eta} \) and \( y_{\mu} = \tilde{y}_{\mu}[u/R]^{1-\eta/2} \) giving

\[ I_{rD,G}(R) < R^{-(2-\eta)2(D-2)} R^{-(2-\eta)(D-1)(g'-1)} \]
\[ \times \sum_{r'} C_{r'} \int_{R_1(G')}^{D} d\tilde{y} \nu \left| \mathcal{P}'_{D,G'}(\tilde{y},u) \right| \delta (1 - \text{Try}_\tilde{y} \tilde{y}_{\mu}). \]

(2.121)

Since \( u/R < 1 \), this can be reduced to

\[ I_{rD,G}(R) < R^{-(2-\eta)2(D-2)+D-1(g'-1)} \sum_{r'} C_{r'} \int_{0}^{R} \frac{du}{u} [u/R]^{(2-\eta)[(D-1)g'+3r'/2]} \]
\[ \times \int_{R_1(G')}^{D} d\tilde{y} \nu \left| \mathcal{P}'_{D,G'}(\tilde{y},u) \right| \delta (1 - \text{Try}_\tilde{y} \tilde{y}_{\mu}). \]

(2.122)

We argue by induction, so assume that, for \( G' \)

\[ \int_0^\infty dR R^{D-1} R^{(D-2)+g'} \mathcal{I}_{rD,G'}(R) \]

(2.123)

converges for \( D > 3 \), and all choices of \( r \). Then (2.122) gives

\[ I_{rD,G}(R) < CR^{-(2-\eta)[2(D-2)+(D-1)(g-1)]} \sum_{r'} C_{r'} \int_{0}^{R} \frac{du}{u} [u/R]^{(2-\eta)(D-1)g'} \mathcal{I}_{rD,G'} \]

(2.124)

and in particular, by the usual power counting argument, the induction statement is true also for \( G \). It remains to check that the induction statement is true for the smallest possible regularly embedded subalgebra, which is \( su(2) \). Since \( su(2) \) has no regularly embedded subalgebra, we can repeat the above arguments with \( G' = 0 \) and find

\[ I_{rD,su(2)}(R) < CR^{-(2-\eta)[2(D-2)+2(D-1)]} \]

(2.125)

so that (2.123) indeed converges for \( D > 3 \).

Taking now \( r = 0 \), we have discovered that, for any compact semi-simple group \( G \),

\[ I_{rD,G}(R) < CR^{-(2-\eta)[2(D-2)+(D-1)(g-1)]} \]

(2.126)

and in particular, the partition function \( Z_{D,G} \) converges for \( D > 3 \). For the correlation
function 1.27 to converge, we require

$$Dg + (D - 2)g + k < 2[2(D - 2) + (D - 1)(g - 1)]$$

(2.127)

and so the critical value is

$$k_c = 2(D - 3)$$

(2.128)

independently of the gauge group.

2.4 Discussion

Bosonic Theory

For the bosonic theories, we have shown that the partition function converges when $D \geq D_c$ and calculated $D_c$ for each of the compact simple groups:

$$D_c = 5, \quad SU(2)$$
$$D_c = 4, \quad SU(3)$$
$$D_c = 3, \quad all\ other\ simple\ groups.$$  

(2.129)

It is a simple exercise to extend the result to the compact semi-simple groups since they are built out of the simple groups. For example, $so(4) = su(2) \oplus su(2)$, so $Z_{D,SO(4)}$ converges when $D \geq D_c = 5$. In addition, we have calculated the critical degree $k_c$ for correlation functions, such that $\langle C_k \rangle$ converges when $k < k_c$. Conversely, we have shown that there always exists a correlation function of degree $k_c$ which diverges.

Restricting ourselves to $D > 2$, it seems rather mysterious that the only divergent partition functions occur for $SU(2)$ with $D = 3, 4$, and for $SU(3)$ with $D = 3$. However, there is an argument which quickly allows us to see why this is so. Begin with the bosonic integral 2.4 and follow the usual procedure to move $X_1$ into the Cartan subalgebra and pick up the Weyl measure (2.21)

$$Z_{D,G} = A_1 \int \prod_{i=1}^{l} dX_i^i \Delta^2_G(X_1^i) \int \prod_{\nu=2}^{D} dX_{\nu} e^{-S(X)}.$$  

(2.130)

We can expand the $X_\nu$ in terms of the basis 2.16

$$X_\nu = X_\nu^i H^i + X_\alpha^\nu E^\alpha, \quad \nu = 2, \ldots, D,$$

(2.131)

and then change variables from the $X_\nu^\alpha$ to $(D - 1)(g - l)$ dimensional polar coordinates
Chapter 2: Convergence

with radial variable $\rho$ and angular variables $\{\theta_a\}$. Then

$$Z_{D,G} = A_2 \int \left( \prod_{\mu=1}^{D} \prod_{i=1}^{l} dX^i_{\mu} \right) \Delta^2_G(X^i_1) \int d\rho \rho^{(D-1)(g-l)-1} d\Omega \exp(-S), \quad (2.132)$$

and the action can be expanded

$$S(X) = \rho^2 X^i_{\mu} X^j_{\nu} Q^i_{\mu \nu}(\theta_a) + \rho^3 X^i_{\mu} F^i_{\mu}(\theta_a) + \rho^4 F_1(\theta_a) \quad (2.133)$$

where $Q$, $F$ and $F_1$ are some functions of the angles $\theta_a$. The action is quadratic in the $X^i_{\mu}$, so we can integrate them out and find

$$Z_{D,G} = A_3 \int_0^{\infty} d\rho \rho^{(D-1)(g-l)-1} \rho^{-Dl-(g-l)} \int d\Omega F_2(\theta_a) \exp(-\rho^4 F_3(\theta_a)) \quad (2.134)$$

where $F_2$ and $F_3$ are some functions of the $\theta_a$, and certainly $F_2$ is positive semi-definite. This integral diverges at $\rho = 0$ when

$$D \leq \frac{2(g-l)}{g-2l}. \quad (2.135)$$

Any group satisfying $2.135$ must have a divergent partition function, and so this gives a quick and illuminating way of seeing that for $SU(2)$ with $D = 3$ and $4$, and for $SU(3)$ with $D = 3$, the partition functions are divergent.

**Supersymmetric Theory**

In the supersymmetric case, we have shown that the partition function converges in $D = 4, 6$ and $10$ with any compact semi-simple gauge group, and that correlation functions of degree

$$k < k_c = 2(D - 3) \quad (2.136)$$

are convergent independent of the gauge group. In the case of $SU(r+1)$, this result corresponds to the conjecture of [39] based on Monte Carlo evaluation of the integrals for small $r$. 
Chapter 3

The Supercharge

We now turn our attention specifically to the supersymmetric Yang-Mills matrix models, and address the question of which quantities are invariant under the supercharges. As we shall see, the supercharges take on a particularly simple form if we reformulate the theory as a cohomological matrix model. We shall give a brief introduction to these models in section 3.1, and then prove our result for these models in the following sections. Finally, in section 3.6, we show that the result can also be applied to the Yang-Mills matrix theories.

3.1 Introduction to Cohomological Matrix Models

A deep relation between the Yang-Mills matrix models and so called cohomological models was uncovered by Moore, Nekrasov and Shatashvili. In a remarkable paper [32], they were able to predict the value of the Yang-Mills partition function by using the cohomological theory. We shall discuss this in detail in chapter 4 but, for now, give a brief introduction to the cohomological model. To illustrate, we shall consider the $D = 4$ model, although the techniques which follow can be applied immediately to the cases of six and ten dimensions by following [32]. The action is

$$S_{YM} = -\text{Tr} \left( \frac{1}{4}[X_\mu, X_\nu]^2 + \bar{\lambda} \sigma^\mu [X_\mu, \lambda] \right).$$

(3.1)

In this chapter we shall mainly be concerned with the gauge group $G = SU(N)$ model so that all fields are $N \times N$ matrices. The gauge fields $X_\mu$ ($\mu = 1, \ldots, 4$) are restricted to the Lie algebra of $G$ which is the set of traceless hermitian matrices. The fermions $\lambda$, which are in the Weyl representation, are complex traceless Grassman matrices. We

\footnote{Note that we have changed the notation from the previous chapters in order to agree with the literature on this subject. We have used “$\lambda$” for the fermions, since the Greek letter $\psi$ will shortly be introduced for a slightly different purpose.}
follow [40] to give an explicit representation $\mathbf{\sigma}^{\mu}$ for the $D = 4$ Dirac matrices projected to a Weyl representation. Define $\sigma^i$ to be the Pauli matrices

\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

and $\sigma^4 = -i I_2$. Then the $\mathbf{\sigma}^{\mu}$ are defined by

\[
\mathbf{\sigma}^{\mu a} = \epsilon^{ab} e^{ab} \mathbf{\sigma}^\mu
\]

and we define

\[
\mathbf{\lambda} = -i \sigma_4 \lambda = -\lambda^\dagger.
\]

The partition function can be written

\[
Z_{4,N} = \int dX d\lambda dD \exp \text{Tr} \left( \frac{1}{4} [X_\mu, X_\nu]^2 + \mathbf{\lambda} \mathbf{\sigma}^\mu [X_\mu, \lambda] - 2D^2 \right)
\]

where an auxiliary field $D$ with appropriate integration measure has been added. The auxiliary field allows us to write down the supersymmetry of this model in a nice linear form,

\[
\delta_\xi X_\mu = -i \mathbf{\lambda} \mathbf{\sigma}^\mu \xi + i \mathbf{\sigma}^\mu \lambda
\]

\[
\delta_\xi \lambda = i \sigma^{\mu \nu} [X_\mu, X_\nu] - 2 \xi D
\]

\[
\delta_\xi D = \frac{1}{2} [X_\mu, \lambda] \mathbf{\sigma}^\mu \xi + \frac{1}{2} \mathbf{\lambda} \mathbf{\sigma}^\mu [X_\mu, \lambda]
\]

where $\sigma^{\mu \nu} = \frac{1}{4} (\sigma^\mu \mathbf{\sigma}^\nu - \sigma^\nu \mathbf{\sigma}^\mu)$. This formula was obtained simply by taking the supersymmetry transformation laws for the four dimensional Yang-Mills theory (for example from [40]) and dimensionally reducing to zero dimensions.

It is important to note that the expression (3.6) is rather formal. If our original $D = 4$ space were Minkowski, then we would have $\mathbf{\sigma}^4 = I_2$. However, since we are working with a Euclidean metric, in fact $\sigma^4 = -i I_2$. This means that, whilst $\sigma^i$ ($i = 1, 2, 3$) are hermitian, $\sigma^4$ is antihermitean. Then $\delta_\xi$ does not preserve hermiticity of the matrix fields so that, for example, $\delta_\xi X^4$ is antihermitean rather than hermitian. Nevertheless, the transformation (3.6) is formally a symmetry of the action, and we shall make rigorous use of this in chapter 4.

The approach of [32] is to make the following field replacements. First rewrite the fermions in terms of their hermitian and antihermitean parts

\[
\lambda_1 = (\eta_2 + i \eta_1)
\]

\[
\lambda_2 = (\psi_1 + i \psi_2).
\]
We also re-write the auxiliary field
\[ D = H + \frac{1}{2}[X_1, X_2] \] (3.8)

By the usual contour shifting argument for a Gaussian integral, \( H \) can be taken hermitian. This is true for the partition function (3.5) and also the correlation functions (1.27) and so makes sense throughout the theory [4]. To obtain the related cohomological action, make the replacement
\[
\phi = \frac{1}{2}(X_3 + iX_4) \\
\phi = -\frac{1}{2}(X_3 - iX_4)
\] (3.9)

and take \( \phi \) hermitian and \( \phi \) antihermitian. This gives
\[
S_0^E \to S_{\text{coh}} = \text{Tr} (H^2 + H[X_1, X_2] - \epsilon^{ab}\eta_1[\psi_a, X_b] + \eta_2[\psi_a, X_a] \\
- \eta_1[\phi, \eta_2] - \psi_a[\phi, \psi_a] + [X_a, \phi][X_a, \phi] + [\phi, \phi]^2)
\] (3.10)

The key point is that \( \phi \) and \( \phi \) are taken to be independent. This is clearly not true in the Yang-Mills model, and so we have defined an entirely new theory.

The supersymmetry (3.4) depends on two complex Grassman parameters \( \xi_a \), and so one can break it down to four linearly independent real supercharges. One can easily write four linearly independent supercharges of \( SYM \) in terms of the new variables, and one of these is
\[
\delta X_a = \psi_a \\
\delta \psi_a = [\phi, X_a] \\
\delta \phi = -\eta_2 \\
\delta \eta_2 = -[\phi, \phi] \\
\delta \eta_1 = H \\
\delta H = [\phi, \eta_1] \\
\delta \phi = 0.
\] (3.11)

Since \( \delta^2 = [\phi, \phi] \), \( \delta \) is nilpotent on gauge invariant quantities. For interest, a representation of all four supercharges is given in appendix A, together with some relations between them.

The action is \( \delta \)-exact. \( S = \delta Q \), where
\[
Q = \text{Tr}(\eta_1[X_1, X_2] + \eta_1 H - \psi_a[X_a, \phi] - \eta_2[\phi, \phi])
\] (3.12)

as can readily be checked. So the symmetry \( \delta S = 0 \) is manifest. The term cohomological to describe the theory is arrived at by analogy of \( \delta \) with an exterior derivative.

In this chapter, we study the supersymmetry operator \( \delta \), and in particular we address the question of which quantities are supersymmetric under \( \delta \). Gauge invariant

\[\text{[4.17]}\]
quantities are formed from traces, and so we seek the general solution to the equation

$$\delta \text{Tr} P = 0$$

(3.13)

where $P$ is a polynomial in the matrix fields. In the analogy of $\delta$ with an exterior derivative, this is the question of finding the cohomology. An important example of the use is to find the possible supersymmetric deformations of a given action. One usually requires a result valid for any gauge group SU($N$), so we shall allow ourselves to make the assumption that $N$ is suitably large. We shall show that

$$\delta \text{Tr} P = 0 \iff \text{Tr} P = \delta \text{Tr} Q + \text{Tr} R(\phi)$$

(3.14)

as long as the degree of $P$ is less than $\frac{2N}{3}$.

The proof requires a number of steps. We form a vector space from the polynomials, and deal with issues of linear dependence in section 3.2. A major technical difficulty is that linearly independent polynomials become dependent after applying a trace. This is overcome in section 3.3 by forming a suitable quotient space. Then in section 3.4 the result is proved for a simplified version of $\delta$ in which the $[\phi, \phi]$ terms are absent. Finally, in section 3.5 the strands are drawn together to prove the result.

### 3.2 Polynomials

We wish to form a vector space from the polynomials, and eventually argue by induction on degree. However, there is a technical difficulty. Two polynomials which look different, because they contain different strings of matrices multiplied together, can turn out to be identical. At this stage, let us be definite and make some careful definitions.

**String** A string of length $l$ is a map from $\{1, \cdots, l\}$ into the set of matrix fields. For example, a typical string of length 5 might be

$$s = X_1 \eta_2 \overline{\phi} X_1 \phi$$

(1) (2) (3) (4) (5) (3.15)

**Monomial** A monomial of degree $d > 0$ is the matrix product of $d$ matrix fields. The monomial of degree 0 is the identity matrix. For example, a typical monomial of degree 5 might be

$$m = X_1 \cdot \eta_2 \cdot \overline{\phi} \cdot X_1 \cdot \phi$$

(3.16)

where $\cdot$ indicates matrix multiplication.
Each string of length \( l \) is naturally associated to a monomial of degree \( l \) by applying matrix multiplication between adjacent fields in the string.

**Polynomial** A polynomial of degree \( d \) is a linear combination of a finite number of monomials whose highest degree is \( d \).

One can form an abstract vector space \( V_s \) over \( \mathbb{C} \) by taking the strings as the basis. In \( V_s \), the strings are linearly independent. However, as polynomials, the strings are not necessarily linearly independent. This is most easily seen when the matrix size \( N \) is 1 so that bosonic matrices commute. Then the two independent strings \( X_1 X_2 \) and \( X_2 X_1 \) are identical as polynomials. Even when \( N > 1 \) so that matrices do not commute, it is possible for independent strings to be linearly dependent as polynomials.

A trivial example is that \( \psi^{N^2} \equiv 0 \) when \( \psi \) is a traceless hermitian \( N \times N \) fermion.

This problem can be overcome by considering only polynomials of degree smaller than the matrix size. Assume that the matrix fields are \( N \times N \) and hermitian. They may also have the constraint of tracelessness, but no other constraints. Then the strings of length less than \( N \) are linearly independent as polynomials.

To see this, denote the strings of length less than \( N \) by \( \{ s^b \} \) and the corresponding monomials \( \{ m^b \} \).

Suppose

\[
\lambda^b m^b \equiv 0 \quad (3.17)
\]

for some \( \lambda^b \in \mathbb{C} \) and (without loss of generality) \( \lambda^1 \neq 0 \). Write \( m^1 = Y^1 \cdots Y^d \) where the \( Y^i \) are matrix fields and the degree of \( m^1 \) is \( d < N \). Then, in particular, the term \( Y^1_{12} Y^2_{23} \cdots Y^d_{d,d+1} \) is absent from 3.17. But the only monomial which gives rise to this term is \( m^1 = Y^1 \cdots Y^d \). Therefore \( \lambda^1 = 0 \), and this is a contradiction.

Note that no assumptions are made about which of the matrix fields are fermionic and which bosonic.

### 3.3 The trace

It will be convenient to work with polynomials rather than traces of polynomials. Unfortunately, two independent polynomials can have identical trace. Defining an equivalence relation \( P \sim Q \Leftrightarrow \text{Tr}P = \text{Tr}Q \), we would like to form the quotient space \( V_p/\sim \), where \( V_p \) is the vector space of polynomials.

Consider a polynomial \( P(A^a) \), where \( \{ A^a : a = 1, \cdots, M \} \) are the matrix fields. Assume that the only constraints which may be applied to the fields are hermiticity and tracelessness. Define an ordering \( O \) such that

- \( O \) acts individually on each monomial term in \( P \)
• $O$ cyclically permutes each monomial in $P$ into a preferred form with a sign to respect fermion statistics

An example of such an ordering would be to define $A^1 > A^2 > A^3 > \cdots$. In this case, for example, $O(A^1 A^2 A^5) = (-1)^{F_4(F_2+F_5)} A^2 A^5 A^4$ where $F_a$ is the fermion number of $A^a$. Then, for $\deg(P) < N$

$$\text{Tr} P = 0 \iff O(P) = 0 \quad (3.18)$$

so that $O$ gives a mapping to the quotient space.

To see that (3.18) is true, first note that $O P = 0 \Rightarrow \text{Tr} O P = 0 \Rightarrow \text{Tr} P = 0$ by the cyclic property of trace. Conversely, suppose $O P \neq 0$. Consider a particular monomial term in $O P$:

$$O P = \lambda Y^1 \cdots Y^M + \cdots \quad (3.19)$$

where $\lambda$ is some non-zero coefficient. Then $\text{Tr} P = \text{Tr} O P$ contains the term:

$$\text{Tr} P = \lambda Y^1_2 Y^2_{23} \cdots Y^M_{M1} + \cdots \quad (3.20)$$

Since $O P$ is ordered, the only monomial that can give such a term is $Y^1 Y^2 \cdots Y^M$. Therefore this term cannot be cancelled and so $\text{Tr} P \neq 0$. Deduce $\text{Tr} P = 0 \Rightarrow O P = 0$.

An ordering operator $O$ is not the most useful way of dealing with the trace. Since there is no way that $O$ will commute with any form of supersymmetry operator, it is more helpful to use the following:

Let $P_M$ be a polynomial in which all of the terms are of degree $M$. Then for $M < N$

$$\text{Tr} P_M = 0 \iff \sum_{\text{cyclic perms}} P_M = 0 \quad (3.21)$$

The cyclic permutations act on the matrix fields in the monomials, and include a sign to respect fermion statistics. For example, suppose $P = \lambda F_1 F_2 B_3 + \mu B_1 B_2 B_4$ where the $F_i$ are fermionic and the $B_i$ bosonic matrix fields. Then the cyclic permutation $\sigma = (123)$ acts as

$$\sigma P = -\lambda F_2 B_3 F_1 + \mu B_2 B_4 B_1 \quad (3.22)$$

Equation (3.21) follows easily from (3.18). First note that

$$\sum_{\text{cyclic perms}} P_M = \sum_{\text{cyclic perms}} O P_M \quad (3.23)$$
Then
\[ \text{Tr} P_M = 0 \Rightarrow O P_M = 0 \Rightarrow \sum_{\text{cyclic perms}} O P_M = 0 \Rightarrow \sum_{\text{cyclic perms}} P_M = 0 \] (3.24)

Conversely
\[ \text{Tr} \sum_{\text{cyclic perms}} P_M = M \text{Tr} P_M \] (3.25)

and so
\[ \sum_{\text{cyclic perms}} P_M = 0 \Rightarrow \text{Tr} P_M = 0 \] (3.26)

### 3.4 Decomposition of the supercharge

If the expression for \( \delta \) (3.11) did not contain the commutator terms, our task to classify the supersymmetric quantities would be much simpler. In this section, we decompose the supercharge into two parts, and prove our result for the simpler part. This will allow us to tackle the full supercharge in the next section.

Returning to the specific theory under discussion, let us write
\[
A^1 = X_1, \quad A^2 = X_2, \quad A^3 = \overline{\phi}, \quad A^4 = \eta_1
\]
\[
B^1 = \psi_1, \quad B^2 = \psi_2, \quad B^3 = -\eta_2, \quad B^4 = H
\]

Then the supersymmetry \( \delta \) can be written as
\[
\delta A^i = B^i, \quad \delta B^i = [\phi, A^i]
\]
\[
\delta \phi = 0
\] (3.29)

If one considers also the six- and ten-dimensional cohomological matrix models, one finds an identical form for \( \delta \) [32], and so the results from this point on apply equally to all three theories.

Define two new operators \( d, \Delta \) by
\[
d A^i = B^i, \quad d B^i = 0
\]
\[
d \phi = 0
\]
\[
\Delta A^i = 0, \quad \Delta B^i = [\phi, A^i]
\]
\[
\Delta \phi = 0
\] (3.30)
One can very easily check that the following relations hold on polynomials

\begin{align*}
  i) & \quad \delta^2 = [\phi, ] \\
  ii) & \quad d^2 = 0 \\
  iii) & \quad \Delta^2 = 0 \\
  iv) & \quad \delta = d + \Delta \\
  v) & \quad \{d, \Delta\} = \delta^2 = [\phi, ]
\end{align*}

(3.32)

and that the operator $d$ has the useful property

\[ d \sum_{\text{cyclic perms}} P_M = \sum_{\text{cyclic perms}} dP_M \]  

(3.33)

which will allow us to deal with the trace.

This nice property (3.33) makes the operator $d$ much simpler to deal with. We begin by forgetting the trace and proving the following result for a polynomial. Let $P$ be a polynomial of degree less than $N$, and suppose $dP = 0$. Then

\[ dP = 0 \Rightarrow P = dQ + R(\phi) \]  

(3.34)

for some polynomials $Q$ and $R$.

To show (3.34), we use induction on the degree of $P$. The case $\deg(P) = 0$ is simple since then $P = \lambda I = R(\phi)$. When $\deg(P) > 0$, expand

\[ P = A^iS^i + B^iT^i + \phi U + \lambda I \]  

(3.35)

for some polynomials $S^i, T^i$ and $U$, and a constant $\lambda$. Applying $d$,

\[ 0 = dP = B^iS^i + (-1)^A^i A^i dS^i + (-1)^B^i B^i dT^i + \phi dU \]  

(3.36)

where the notation $(-1)^{A^i}$ is shorthand for $\pm 1$ respectively as $A^i$ is bosonic or fermionic. Then in particular, since $d$ maps bosons to fermions, $(-1)^{B^i} = (-1)^{A^i+1}$. Since $\deg(P) < N$, the strings are linearly independent (section 3.2), and we deduce

\[ S^i + (-1)^{A^i+1} dT^i = 0 \]  

(3.37)

\[ dS^i = 0 \]  

(3.38)

\[ dU = 0. \]  

(3.39)

(and note that since $d^2 = 0$, (3.38) is implied by (3.37)).
By induction, \( dU = 0 \) implies

\[
U = dV + W(\phi)
\]  
(3.40)

where \( V \) and \( W \) are polynomials. Then substituting \[3.37\] and \[3.40\] back into \[3.35\],

\[
P = A^i(-1)^iT^i + B^iT^i + \phi (dV + W(\phi)) + \lambda I
\]

(3.41)

and the result \[3.34\] follows.

Finally in this section, we introduce the trace. Let \( P \) be a polynomial of degree less than \( N \), and suppose \( d\text{Tr}P = 0 \). Then

\[
d\text{Tr}P = 0 \Rightarrow \text{Tr}P = d\text{Tr}Q + \text{Tr}R(\phi)
\]

(3.42)

for some polynomials \( Q \) and \( R \).

To see this, write \( P = P_0 + \cdots + P_M \) where each \( P_i \) contains only monomials of degree \( i \). Then, since \( d \) preserves the degree of monomials,

\[
d\text{Tr}P = 0 \Rightarrow d\text{Tr}P_i = 0
\]

\[
\Rightarrow \text{Tr}dP_i = 0 \quad (i = 0, \cdots, M).
\]

(3.43)

The case of \( i = 0 \) is simple. For \( i > 0 \), using \[3.21\] implies

\[
\sum_{\text{cyclic perms}} dP_i = 0
\]

(3.44)

and using \[3.33\]

\[
d \sum_{\text{cyclic perms}} P_i = 0.
\]

(3.45)

Then \[3.34\] gives

\[
\sum_{\text{cyclic perms}} P_i = dQ_i + R_i(\phi)
\]

(3.46)

for some polynomials \( Q_i \) and \( R_i \), so that

\[
\text{Tr}P_i = \frac{1}{i} \text{Tr}(dQ_i + R_i(\phi))
\]

(3.47)

by the cyclic property of trace. Then summing over \( i \) gives the result.
3.5 Extension to the full supercharge

The task now is to extend the result from \( d \) to \( \delta \). The commutator terms in the definition of \( \delta \) make it much harder to deal with the trace. Specifically, \( \delta \) does not commute with the sum over cyclic permutations. Instead, we proceed with a less direct approach, and make use of the result for \( d \).

We begin with a technical result. Suppose \( P_k \) is a polynomial of degree \( k \) satisfying \( d\Delta \text{Tr} P_k = 0 \). Then there exists a polynomial \( T \) such that

\[
d\Delta \text{Tr} P_k = 0 \Rightarrow d\text{Tr} P_k = (d + \Delta)\text{Tr} T, \tag{3.48}
\]

as long as \( N > \frac{2k}{3} \).

The proof follows an inductive argument. By 3.42,

\[
d\Delta \text{Tr} P_k = 0 \Rightarrow \Delta \text{Tr} P_k = -d\text{Tr} P_{k+1} + R_{k+1}(\phi) \tag{3.49}
\]

for some polynomials \( P_{k+1} \) and \( R_{k+1} \) of degree \( k+1 \). Since neither \( d \) nor \( \Delta \) can produce monomials only in \( \phi \), \( R_{k+1}(\phi) = 0 \). Then

\[
d\text{Tr} P_k = (d + \Delta)\text{Tr} P_k + d\text{Tr} P_{k+1} \tag{3.50}
\]

On any monomial, \( d \) acts to increase the number of fields of type \( B^i \) by 1, whilst \( \Delta \) acts to decrease the number of \( B^i \) by 1.

Let \( M_k \) be the maximum number of \( B^i \) occurring in any term of \( P_k \). Then since \( \Delta \text{Tr} P_k = d\text{Tr} P_{k+1} \), we have

\[
M_{k+1} = M_k - 2 \tag{3.51}
\]

Proceed inductively to find

\[
d\text{Tr} P_k = (d + \Delta)\text{Tr}(P_k + P_{k+1} + \cdots + P_{k+q}) + d\text{Tr} P_{k+q+1} \tag{3.52}
\]

where \( P_{k+q+1} \) contains no \( B^i \) fields at all. Then \( \Delta \text{Tr} P_{k+q+1} = 0 \) and so

\[
d\text{Tr} P_k = (d + \Delta)\text{Tr}(P_k + \cdots + P_{k+q+1}) \tag{3.53}
\]

which proves the result as long as \( N > k + q + 1 \) so that each inductive step is valid.

Noting that the case \( M_k = k \) is special and can be reduced to the case \( M_k = k - 1 \), one finds that \( N > \frac{3k}{2} \) is a sufficient condition.

We are now ready to prove the main result of this chapter. Suppose the matrix
fields are of size $N$, and $P$ is a polynomial in the matrix fields. Then for $\text{deg}(P) < \frac{2N}{3}$,

$$\delta \text{Tr} P = 0 \Leftrightarrow \text{Tr} P = \delta \text{Tr} Q + \text{Tr} R(\phi) \quad (3.54)$$

where $Q$ and $R$ are polynomials.

To show this, write $P = P_0 + \cdots + P_M$ where $P_i$ contains monomials only of degree $i$. Then

$$\delta \text{Tr} P = 0 \Rightarrow (d + \Delta) \text{Tr} P = 0 \quad (3.55)$$

and since $d$ preserves degree whilst $\Delta$ increases degree by 1, we have

$$\Delta \text{Tr} P_M = 0$$
$$\Delta \text{Tr} P_i + d \text{Tr} P_{i+1} = 0, \quad i = 0, \cdots, M - 1 \quad (3.56)$$
$$d \text{Tr} P_0 = 0$$

By $(3.32)$,

$$d \text{Tr} P_0 = 0 \Rightarrow \text{Tr} P_0 = d \text{Tr} Q_0 + \text{Tr} R_0(\phi) \quad (3.57)$$

and

$$\Delta \text{Tr} P_0 + d \text{Tr} P_1 = 0 \Rightarrow \Delta (d \text{Tr} Q_0 + \text{Tr} R_0(\phi)) + d \text{Tr} P_1 = 0$$
$$\Rightarrow d(-\Delta \text{Tr} Q_0 + \text{Tr} P_1) = 0$$
$$\Rightarrow \text{Tr} P_1 = \Delta \text{Tr} Q_0 + d \text{Tr} Q_1 + \text{Tr} R_1(\phi). \quad (3.58)$$

Repeating the same argument inductively gives

$$\text{Tr} P_i = \Delta \text{Tr} Q_{i-1} + d \text{Tr} Q_i + \text{Tr} R_i(\phi), \quad i = 1, \cdots, M \quad (3.59)$$

for some polynomials $Q_i$ and $R_i(\phi)$, so that

$$\text{Tr} P = d \text{Tr} Q_0 + \text{Tr} R_0(\phi) + \sum_{i=1}^{M} \Delta \text{Tr} Q_{i-1} + d \text{Tr} Q_i + \text{Tr} R_i(\phi)$$
$$= (d + \Delta) \sum_{i=0}^{M-1} \text{Tr} Q_i + \sum_{i=0}^{M} \text{Tr} R_i(\phi) + d \text{Tr} Q_M. \quad (3.60)$$

If we now apply $\delta$ to $(3.60)$ we find

$$\delta \text{Tr} P = 0 \quad \Rightarrow \quad \Delta d \text{Tr} Q_M = 0 \quad (3.61)$$

and so the technical result $(3.48)$ gives

$$d \text{Tr} Q_M = (d + \Delta) \text{Tr} S \quad (3.62)$$
for some polynomial $S$. Then, since $\delta = d + \Delta$, substituting $3.62$ back into $3.60$ proves the result.

### 3.6 The Yang-Mills Supercharge

The only non-trivial change in moving from the Yang-Mills model to the Cohomological model was the change of variables $\phi_{a/b}$:

$$
\begin{align*}
\phi &= \frac{1}{2}(X_3 + iX_4) \\
\bar{\phi} &= -\frac{1}{2}(X_3 - iX_4)
\end{align*}
$$

(3.63)

since in the Cohomological formulation, we took $\phi$ and $\bar{\phi}$ independent and respectively antihermitian and hermitian. Thus we can write the supercharge $3.11$ in the Yang-

Mills formulation simply by making the replacement $3.63$.

$$
\begin{align*}
\delta X_a &= \psi_a \\
\delta \psi_a &= \frac{1}{2}(X_3 + iX_4), X_a \\
\delta \eta_1 &= H \\
\delta H &= \frac{1}{2}(X_3 + iX_4), \eta_1 \\
\delta X_3 &= \eta_2 \\
\delta \eta_2 &= \frac{i}{2}[X_4, X_3]
\end{align*}
$$

(3.64)

where now $X_3$ and $X_4$ are independent Hermitian matrices$^3$. If we label the matrix fields $A^a$, with $A^1 = X_1$, $A^2 = \psi_1$, etc, then $3.64$ can be rewritten as a first order differential operator

$$
\delta = (\delta A^a) \frac{\partial}{\partial A^a}
$$

(3.65)

where we sum over the repeated index $a$ and, for example, $\delta A^1 = \delta X_1 = \psi_1$.

In an identical way, we can write the supercharge of the cohomological theory as a first order differential operator. Thus, the result of the previous sections amounts to finding the general solution of the first order differential equation

$$
\delta_{\text{CoHo}} f = 0
$$

(3.66)

with the constraint that the function of matrix fields $f$ is of the form $f = \text{Tr}P$ with $P$ a polynomial.

Lets now take a solution $f$ to this equation. Up to this point, $\phi$ and $\bar{\phi}$ have been respectively antihermitian and hermitian matrix fields (and of course, independent). However, since $f$ is a polynomial, it is a trivial matter to analytically continue $f$ so that $\phi$ and $\bar{\phi}$ become general complex matrices. Since this is an analytic continuation,
in particular the differential equation (3.66) still holds.

Now change variables from \( \phi \) and \( \overline{\phi} \) to \( X_3 \) and \( X_4 \) using equation (3.9). Then, using the chain rule, (3.66) becomes
\[
\delta_{YM} f = 0 \tag{3.67}
\]
where \( \delta_{YM} \) is of course precisely the differential operator which is the Yang-Mills supercharge (3.64). Of course, \( X_3 \) and \( X_4 \) are general complex matrices, but we can now restrict their domain to the hermitian matrices to arrive back at the Yang-Mills theory.

Thus we have discovered that, if we have a solution \( \delta_{CoHo} \text{Tr} P = 0 \) in the cohomological theory, then we can take \( P \) and simply make the replacement (3.63) to obtain a solution in the Yang-Mills theory. Conversely, if we have a solution \( \delta_{YM} \text{Tr} P = 0 \) in the Yang-Mills theory, this gives us a solution in the cohomological theory. Then the result of the previous section extends immediately to the Yang-Mills theory,
\[
\delta_{YM} \text{Tr} P = 0 \iff \text{Tr} P = \delta_{YM} \text{Tr} Q + \text{Tr} R(X_3 + iX_4) \tag{3.68}
\]
as long as the degree of \( P \) is less than \( \frac{2N}{3} \).

### 3.7 Concluding Remarks

We have considered the \( SU(N) \) Yang-Mills and cohomological matrix models in four, six and ten dimensions, and shown that
\[
\delta_{CoHo} \text{Tr} P = 0 \iff \text{Tr} P = \delta_{CoHo} \text{Tr} Q + \text{Tr} R(\phi) \tag{3.69}
\]
\[
\delta_{YM} \text{Tr} P = 0 \iff \text{Tr} P = \delta_{YM} \text{Tr} Q + \text{Tr} R(X_3 + iX_4) \tag{3.70}
\]
as long as the degree of \( P \) is less than \( \frac{2N}{3} \).

Although the large \( N \) limit is a case of particular interest, it would also be interesting to understand what happens when \( N \) is small, or the gauge group is not \( SU(N) \). At present, we do not know of any counter examples to the general formulae (3.69) (3.70) in these cases. It would also be interesting to understand whether the result can be extended to a general gauge invariant quantity consisting of an arbitrary function of traces.
Chapter 4

The Deformation Approach

We now return to the approach of Moore, Nekrasov and Shatashvili \[32\]. These authors begin with the cohomological action \[3.10\] with gauge group \(SU(N)\), and consider the partition function

\[
\int d\phi dX_1dX_2d\psi_1d\psi_2d\eta_2d\eta_1d\bar{\phi}dHe^{-S_{coh}}.\tag{4.1}
\]

They add additional terms to the action \(S_{coh} \to S_{coh} + \epsilon \Delta S_{coh}\) in such a way as to preserve some supersymmetry. They are then able to use Witten’s localisation principle \[41\] to integrate out the fields \(H, \bar{\phi}, \eta, \psi, X\), leaving an integral over just \(\phi\), and use the gauge symmetry to diagonalise \(\phi\) in the usual way. The result is an integral of the form

\[
\int d\phi_1 \cdots d\phi_{N-1}z(\phi_1, \cdots, \phi_{N-1}) \tag{4.2}
\]

where the \(\phi_i\) are the eigen-values of \(\phi\). With the form of \(z\) which is obtained from these manipulations, the integral is divergent. However, MNS complete the contours of integration in either the upper or lower half plane following a certain prescription, and perform the contour integrals. In dimensions \(D = 4, 6, 10\), the results are identical to the value of the Yang-Mills partition function \[1.23\] in every case that it is known either numerically or exactly. Furthermore, recently the result has been extended to some groups other than \(SU(N)\) and compared to numerical calculations, and again the results agree \[10, 33\].

There are two puzzles in this calculation. The first is the question of why it should be allowed to replace the Yang-Mills theory with the cohomological theory, and the second is why following the MNS contour prescription gives the correct result. It was hoped that finding the answer to the first question would naturally answer the second (see for example \[42\] in which these methods are applied to the one-dimensional theory).

In this chapter, we carefully apply the deformation method of MNS directly to the
Yang-Mills model. This involves finding a deformation of the Yang-Mills action for which we can be sure that the integrals converge at every step. The final result is closely related to the formula of MNS, and we indicate how the result as it stands comes to be divergent. However, sadly the MNS contour prescription does not arise naturally and we must again impose it by hand.

### 4.1 Yang-Mills Integral

We begin by recalling the model. We shall discuss the $D = 4$, $SU(N)$ model in detail. The action is (3.1)

\[ S_{YM} = -\text{Tr} \left( \frac{1}{4} [X_\mu, X_\nu]^2 + \lambda \sigma^\mu [X_\mu, \lambda] \right) \]  

(4.3)

All fields are $N \times N$ matrices and transform in the adjoint representation of the gauge group $G = SU(N)$. The gauge fields $X_\mu$ ($\mu = 1, \cdots, 4$) are restricted to the Lie algebra of $G$ which is the set of traceless hermitian matrices. The fermions $\lambda_a$ ($a = 1, 2$) are complex traceless Grassman matrices. The $\sigma^\mu$ are the $2 \times 2$ matrices defined in 3.3.

The matrix integral giving the partition function is

\[ Z = \int dX d\lambda dD \exp \text{Tr} \left( \frac{1}{4} [X_\mu, X_\nu]^2 + \lambda \sigma^\mu [X_\mu, \lambda] - 2D^2 \right) \]  

(4.4)

As usual, an auxiliary field $D$ has been added. So that the integral over $D$ does not affect the result, we must fix the measure for $D$,

\[ \int dDe^{-2\text{Tr}D^2} = 1. \]  

(4.5)

For completeness let us fix the integration measure of the other fields now. Write any hermitian matrix $Q$ in terms of its real and imaginary parts

\[ Q = Q^S + iQ^A \]  

(4.6)

so that $Q^S$ and $Q^A$ are respectively symmetric and antisymmetric real $N \times N$ matrices. We define

\[ dQ = 2^{\frac{N(N-1)}{2}} \prod_{i \geq j} dQ^S_{ij} \prod_{i > j} dQ^A_{ij} \]  

(4.7)

taking $-$ when $Q$ is bosonic, and $+$ when $Q$ is Grassmann. The leading powers of 2 may seem rather cumbersome, and they have an advantage as we shall see in equations 4.11 and 4.12. Since there are equal numbers of bosons and fermions, these powers
cancel in any case.

It is often inconvenient to integrate hermitian matrices directly, so we define for a general complex matrix $M$

$$\tilde{M} = \frac{1}{2}(M + M^T) + \frac{1}{2i}(M - M^T)$$

(4.8)

and conversely

$$M = \frac{1}{2}(\tilde{M} + \tilde{M}^T) + \frac{i}{2}(\tilde{M} - \tilde{M}^T)$$

(4.9)

For $Q$ hermitian, observe

$$\tilde{Q} = Q^S + Q^A$$

(4.10)

so that $Q \leftrightarrow \tilde{Q}$ gives a 1-1 correspondence between the real and the hermitian $N \times N$ matrices. With respect to the new variables, the measure is

$$dQ = d\tilde{Q}$$

(4.11)

where

$$d\tilde{Q} = \prod_{i,j} d\tilde{Q}_{ij}$$

(4.12)

is the natural measure on $\mathbb{R}^{N^2}$.

This scheme has the advantage that $\text{Tr}Q^2 = \text{Tr}\tilde{Q}\tilde{Q}^T$ so that a hermitian Gaussian integral is

$$\int dQ e^{-\text{Tr}Q^2} = \int d\tilde{Q} e^{-\sum_{i,j}(\tilde{Q}_{ij})^2} = \pi^{N^2/2}$$

(4.13)

To integrate over the traceless hermitian matrices, insert $\delta(\text{Tr}Q)$. One finds

$$\int_{\text{Tr}Q=0} dQ e^{-\text{Tr}Q^2} = \frac{\pi^{N^2-1}}{\sqrt{N}}.$$

(4.14)

A typical fermionic integral over $\chi$ and $\xi$ traceless hermitian grassman matrices is

$$\int_{\text{Tr}\chi=\text{Tr}\xi=0} d\chi d\xi e^{i\text{Tr}\chi} = \int_{\text{Tr}\tilde{\chi}=\text{Tr}\tilde{\xi}=0} d\tilde{\chi} d\tilde{\xi} e^{i\text{Tr} \tilde{\chi}^T \tilde{\xi}} = i^{N^2-1}N$$

(4.15)

as can readily be checked.

The first step is to replace $D$ in 4.4 with

$$D = H + \frac{1}{2} [X_1, X_2].$$

(4.16)
Then the auxiliary part of the integral becomes

$$\int dH \exp \left\{-2\text{Tr}(H + \frac{1}{2}[X_1, X_2])^2\right\} = \int d\tilde{H} \exp \left\{-2 \sum_{i,j}(\tilde{H}_{ij} + i\tilde{C}_{ij})^2\right\}$$

(4.17)

where $C = \frac{-i}{2}[X_1, X_2]$. The contour of integration of each of the $\tilde{H}_{ij}$ is displaced from the real axis by $-i\tilde{C}_{ij}$. However, we can use the usual argument for Gaussian integrals to shift the contours down onto the real axis. This means we can take the $H$ defined in 4.16 to be hermitian.

At present, $H$ has the normalisation 4.5 which is inherited from $D$, whilst the $X_\mu$ have measure normalised by 4.14. For later convenience, we now exchange the normalisations of the measures of $X_4$ and $H$. This of course leaves the matrix integral 4.4 unaffected. Thus the measure of $X_4$ is now normalised so that

$$\int dX_4 \delta(\text{Tr}X_4) e^{-2\text{Tr}X_4^2} = 1$$

(4.18)

whilst $X_1$, $X_2$, $X_3$ and $H$ are all normalised according to 4.14.

We also follow the notation of the previous chapter (3.7) and split the fermions into hermitian and antihermitian parts

$$\lambda_1 = (\eta_2 + i\eta_1)$$
$$\lambda_2 = (\psi_1 + i\psi_2)$$

(4.19)

so that one of the supercharges is (3.64)

$$\delta X_a = \psi_a$$
$$\delta \psi_a = \left[\frac{1}{2}(X_3 + iX_4), X_a\right]$$
$$\delta \eta_1 = H$$
$$\delta H = \left[\frac{1}{2}(X_3 + iX_4), \eta_1\right]$$
$$\delta X_3 = \eta_2$$
$$\delta \eta_2 = \frac{i}{2}[X_4, X_3]$$
$$\delta X_4 = i\eta_2.$$  

(4.20)

We note that we can scale the exponent in 4.4 by a constant and leave the integral invariant. Then, for later convenience, we shall include include an extra factor $\frac{1}{2}$ so that, in terms of the new variables, the action becomes

$$S = \text{Tr} \left( (H + \frac{1}{2}[X_1, X_2])^2 - \frac{1}{4} \sum_{\mu>\nu} [X_\mu, X_\nu]^2 \right. - \epsilon_{ab} \eta_1 [\psi_a, X_b] - \eta_a \frac{1}{2} [(X_3 + iX_4), \eta_a] - \psi_a \frac{1}{2} [(-X_3 + iX_4), \psi_a] + \eta_2 [\psi_a, X_a] \right).$$

(4.21)
The action is $\delta$-exact, $S = \delta \text{Tr} Q$, where
\[ Q = \left( \eta_1[X_1, X_2] + \eta H + \frac{1}{2} \psi_a [X_a, X_3 - iX_4] - \frac{i}{2} \eta_2 [X_3, X_4] \right) \] (4.22)
as can readily be checked. So the symmetry $\delta S = 0$ is manifest.

### 4.2 Deformation

Our tactic for calculating the partition function is to add small mass terms to the action in a prescribed way that will make the integrals easy, and then send the masses to zero afterwards. Of course, we shall eventually have to worry about whether the integration commutes with the limit of masses going to zero, but for the moment we concentrate on finding a suitable deformation of the action.

Ultimately, we wish to use the supersymmetry to perform the integrals, so we must find a deformation of the action which preserves some supersymmetry. Our aim is to include a mass term for each field. Our first attempt would be to try to preserve $\delta$ exactly. However, the result of chapter 3 shows that then the action must take the form $S = \delta \text{Tr} Q + \epsilon \delta \text{Tr} R + \mu \text{Tr} W (X_3 + iX_4)$. One can quickly discover by playing with $\delta$ that it is not possible to generate mass terms in this way. Therefore, we must actually deform the supercharge itself.

Let's introduce a deformation parameter $\epsilon$. The deformed action will be
\[ S_{\text{def}} = S + \epsilon S_1 + \epsilon^2 S_2 + \cdots \] (4.23)
where the $S_i$ are all gauge invariant. The simplest possible modification of $\delta$ is
\[ \delta = \delta + \epsilon T \] (4.24)
where $T$ is some operator. Then
\[ \delta^2 = \delta^2 + \epsilon \{\delta, T\} + \epsilon^2 T^2 \] (4.25)
Our aim is to preserve the supersymmetry, so that
\[ \delta S_{\text{def}} = 0 \] (4.26)
Since $\delta^2 = 0$ on gauge invariant quantities,
\[ \delta S_{\text{def}} = 0 \Rightarrow \delta^2 S_{\text{def}} = 0 \Rightarrow \{\delta, T\} S = 0 \] (4.27)
and so \( \{\delta, T\} \) generates one of the continuous bosonic symmetries of \( S \). These are the
gauge transformations and \( SO(D = 4) \) rotations. We follow \[32,43\] and use an \( SO(2) \)
subgroup of the \( SO(4) \)

\[
U : \quad X_a \rightarrow \epsilon_{ab} X_b \quad \psi_a \rightarrow \epsilon_{ab} \psi_b
\]

(4.28)
to set

\[
\{\delta, T\} = U. \quad (4.29)
\]

Now \( U \) is a compact symmetry which we would like to preserve, so we impose

\[
\{\delta, T\} S_i = 0, \quad i = 1, \cdots 
\]

(4.30)
Then (4.23) and (4.25) imply

\[
T^2 S = T^2 S_i = 0, \quad i = 1, \cdots
\]

(4.31)
and so we shall require

\[
T^2 = 0. \quad (4.32)
\]
The simplest possible form for \( T \) is linear

\[
TA^a = \alpha^{ab} A^b
\]

(4.33)
where \( A^a \) are the matrix fields and \( \alpha^{ab} \) are some parameters. Imposing \(4.29\) and \(4.32\)
on (4.33) gives two possibilities

(i)

\[
TX_a = 0 \quad TX_3 = -i\nu \eta_1 \quad T\eta_1 = 0 \quad TX_4 = \nu\eta_1
\]

(4.34)

(ii)

\[
TX_a = 0 \quad TX_3 = 0 \quad T\psi_a = i\epsilon_{ab} X_b \quad T\eta_1 = -i\gamma \frac{1}{2}(X_3 - iX_4) - i\lambda \frac{1}{2}(X_3 + iX_4) \quad T\eta_2 = 0 \quad TX_4 = 0 \quad TH = i\gamma \eta_2
\]

(4.35)
where \( \nu, \gamma \) and \( \lambda \) are complex parameters.

\[1\] These authors apply the localisation method to the cohomological theory where, although \( SO(4) \)
is broken, this \( SO(2) \) is still a symmetry.
The first possibility is not helpful for generating mass terms in \( X_3 \) and \( X_4 \), so we consider only the second possibility. Then we have arrived at an altered supercharge \( \delta = \delta + \epsilon T \) which we hope to be able to preserve as a symmetry of a regularised action.

\[
\begin{align*}
\delta X_a &= \psi_a, \quad \delta \psi_a = \frac{i}{2} [X_3 + iX_4, X_a] + i \epsilon_{ab} X_b \\
\delta X_3 &= \eta_2, \quad \delta \eta_2 = \frac{i}{2} [X_4, X_3] \\
\delta \eta_1 &= H - i \frac{\bar{\epsilon}}{2} \{ \gamma (X_3 - iX_4) + \lambda (X_3 + iX_4) \} \\
\delta H &= \frac{i}{2} [X_3 + iX_4, \eta_1] + i \epsilon \eta_2 \\
\delta X_4 &= -i \eta_2
\end{align*}
\] (4.36)

This modified supercharge \( \delta \) has the property

\[
\delta^2 = \delta^2 + \epsilon \mathcal{U}
\] (4.37)

The first row of (4.36) is the part which generates the rotation \( \mathcal{U} \). This corresponds to the deformation used in [32] and [43] for the cohomological theory. However, we have also added the terms in the third row of (4.36) in order to generate some useful mass terms for \( X_3 \) and \( X_4 \).

On \( \mathcal{U} \)-invariant quantities, \( \delta^2 = \delta^2 \). This gives three particularly useful identities:

(i) \( \delta^2 = \delta^2 \) on quantities independent of \( X_\alpha, \psi_\alpha \)

(ii) \( \delta^2 (\psi_\alpha X_\alpha) = \delta^2 (\psi_\alpha X_\alpha) \)

(iii) \( \delta^2 (\epsilon^{\alpha \beta} \psi_\alpha X_\beta) = \delta^2 (\epsilon^{\alpha \beta} \psi_\alpha X_\beta) \)

We are now ready to define the deformed action. Recall that the original action is \( S = \delta \text{Tr} Q \) where \( Q \) is defined in equation (4.22). We define

\[
S_{\text{def}} = \delta \text{Tr} Q - i \kappa_1 \delta \text{Tr} R_1 - i \kappa_2 \delta \text{Tr} R_2 - \frac{\mu^2}{4} \text{Tr} (X_3 + iX_4)^2.
\] (4.38)

Using the identities above, we note that \( Q \) is \( \mathcal{U} \)-invariant. If we also choose \( R_1 \) and \( R_2 \) to be \( \mathcal{U} \)-invariant then \( S_{\text{def}} \) will satisfy

\[
\delta S_{\text{def}} = 0
\] (4.39)

so that \( \delta \) is a supersymmetry of the deformed action. Further, the original action \( S \) will be recovered in the limit \( \epsilon, \kappa_1, \kappa_2, \mu \to 0 \). Specifically, we choose

\[
R_1 = \frac{1}{2} \epsilon_{ab} \psi_a X_b
\] (4.40)
and
\[ R_2 = \frac{1}{2} \eta_1 (X_3 - iX_4) \] (4.41)
which are again both \( \mathcal{U} \)-invariant by the above identities.

For reference, the complete deformed action given by this prescription is given in appendix \[3\]. However, for our present purpose, we shall find it useful to choose parameters
\[
\begin{align*}
\lambda &= 0 \\
\gamma &= -3 \\
\kappa_1 &= \epsilon.
\end{align*}
\] (4.42)
and we also impose
\[ \mu^2 < 3\epsilon \kappa_2. \] (4.43)

Then the deformed action is
\begin{align*}
S_{\text{def}} &= S \\
&+ \epsilon \text{Tr} \left( X_4 [X_1, X_2] + 3i \frac{1}{2} (X_3 - iX_4) H + 3i \eta_1 \eta_2 + i \psi_1 \psi_2 + \frac{\epsilon}{2} (X_1^2 + X_2^2) \right) \\
&+ \kappa_2 \text{Tr} \left( -\frac{i}{2} H (X_3 - iX_4) + \frac{3\epsilon}{4} (X_3 - iX_4)^2 + i \eta_1 \eta_2 \right) \\
&- \mu^2 \text{Tr} \left( \frac{1}{2} (X_3 + iX_4) \right)^2.
\end{align*}
(4.44)

We now consider the deformed partition function
\begin{equation}
Z_{\text{def}} = \int dX_4 dX_1 dX_2 dH dX_3 d\psi_1 d\psi_2 d\eta_1 e^{-S_{\text{def}}}. \tag{4.45}
\end{equation}
We must first check that this is a convergent integral. Performing the integrals over the fermions generates a polynomial which is the deformed version of the Pfaffian, so we write
\begin{equation}
Z_{\text{def}} = \int dX_4 dX_1 dX_2 dH dX_3 \mathcal{P}_{\text{def}}(X_0) \exp (-S_{\text{def}}|_{\psi=\eta=0}). \tag{4.46}
\end{equation}
We shall always perform the integrals in the order indicated by the measures of 4.43 and 4.46, so we begin by considering the integral over \( X_3 \). We see from 4.44 that the integrand is exponentially damped in \( X_3 \) as long as the condition 4.43 holds. This means we can change variables from \( X_3 \) to
\begin{equation}
\bar{\phi} = -\frac{1}{2} (X_3 - iX_4) \tag{4.47}
\end{equation}
and then follow the same procedure as before\[4\] to shift the contours of integration so
\[2\]That is, the procedure we used to change variables from \( D \) to \( H \) and then take \( H \) hermitian.
that $\overline{\phi}$ becomes *hermitian*, and the measure becomes

$$dX_3 = 2^{N^2-1}d\overline{\phi}.$$  \hfill (4.48)

One might worry that after doing some of the integrations there may be some poles that we should pick up. However, since the integrand is continuous, we can always change the order of integration so that the contour we are shifting is the first, and in that case we do not have to worry about any poles since the integrand is analytic.

In this new picture, the deformed action is

$$S_{\text{def}} = S + \epsilon \text{Tr} \left( X_4 [X_1, X_2] - 3i\overline{\phi}H + 3i\eta_1\eta_2 + i\psi_1\psi_2 + \frac{\epsilon}{2}(X_1^2 + X_2^2) \right)$$

$$+ \kappa_2 \text{Tr} \left( i\overline{\phi}H + 3\epsilon\overline{\phi}^2 + i\eta_1\eta_2 \right)$$

$$- \mu^2 \text{Tr} \left( iX_4 - \overline{\phi} \right)^2$$ \hfill (4.49)

where

$$S = \text{Tr}\{H^2 + H [X_1, X_2] + i[X_a, X_4][X_a, \overline{\phi}] - [X_a, \overline{\phi}]^2 - [X_4, \overline{\phi}]^2$$

$$- \epsilon^{ab}\eta_1[\psi_a, X_b] + \eta_2[\psi_a, X_a] - \eta_a \left[ iX_4 - \overline{\phi}, \eta_a \right] - \psi_a \left[ \overline{\phi}, \psi_a \right] \}$$ \hfill (4.50)

and the deformed partition function

$$Z_{\text{def}} = \int dX_4dX_1dX_2dHd\overline{\phi} \mathcal{P}_{\text{def}} \exp \left( -S_{\text{def}}|_{\psi=\eta=0} \right).$$ \hfill (4.51)

It is now easy to see that $Z_{\text{def}}$ is convergent, since \ref{4.51} is absolutely convergent. To check this, we examine the real part of the exponent:

$$\text{Re}(S_{\text{def}}|_{\psi=\eta=0}) > \text{Tr} \left[ H^2 + \frac{\epsilon^2}{2}(X_1^2 + X_2^2) + 3\epsilon\kappa_2\overline{\phi}^2 + \mu^2X_4^2 - \mu^2\overline{\phi}^2 \right],$$ \hfill (4.52)

where the inequality is obtained by dropping some positive terms from $S$. Then since we assumed $3\epsilon\kappa_2 > \mu^2$, we see that the deformed partition function \ref{4.45} is a manifestly convergent integral.
4.3 Integration by Parts

In the current scheme, the deformed supercharge (4.36) is

\[
\begin{align*}
\delta X_a &= \psi_a \\
\delta \phi &= -\eta_2 \\
\delta \eta_1 &= H - 3i\epsilon \phi \\
\delta X_4 &= i\eta_2
\end{align*}
\]

As short hand, lets write \( A^a \) for the matrix fields so that \( A^1 = X_4, \ A^2 = X_1, \) and so on. We write \( dA \) for the measure \( dA = dA^1 \cdots dA^9 \). Then the deformed matrix integral (4.43) has become

\[
Z_{\text{def}}(\epsilon, \kappa_2, \mu) = \int dA e^{-S_{\text{def}}}. \tag{4.54}
\]

Differentiating with respect to \( \kappa_2 \) gives

\[
\frac{\partial}{\partial \kappa_2} Z_{\text{def}}(\epsilon, \kappa_2, \mu) = -i \int dA \overline{\delta} (\text{Tr} R_2) e^{-S_{\text{def}}} \tag{4.55}
\]

where

\[
R_2 = -\eta_1 \overline{\phi} \tag{4.56}
\]

as we see from the definitions (4.38) and (4.41). This integral is also manifestly convergent because of the exponential vanishing of \( e^{-S_{\text{def}}} \). Since \( \overline{\delta} S_{\text{def}} = 0 \) we have

\[
\frac{\partial}{\partial \kappa_2} Z_{\text{def}}(\epsilon, \kappa_2, \mu) = -i \int dA \overline{\delta} (\text{Tr} R_2 e^{-S_{\text{def}}}) \tag{4.57}
\]

As before, we write the supercharge (4.53) as a differential operator

\[
\overline{\delta} = (\overline{\delta} A^a) \frac{\partial}{\partial A^a}. \tag{4.58}
\]

We note from (4.53) that each \( \overline{\delta} A^a \) is independent of the matrix field \( A^a \). Thus we can write

\[
\overline{\delta} \cdot = (\overline{\delta} A^a) \frac{\partial}{\partial A^a} = \frac{\partial}{\partial A^a} (\overline{\delta} A^a \cdot) \tag{4.59}
\]

and so

\[
\frac{\partial}{\partial \kappa_2} Z_{\text{def}}(\epsilon, \kappa_2, \mu) = -i \int dA \frac{\partial}{\partial A^a} (\overline{\delta} A^a \text{Tr} R_2 e^{-S_{\text{def}}}) \tag{4.60}
\]

We consider each term in the sum over \( a \) in this expression separately.

If \( A^a \) is fermionic, then each \( A^a_{ij} \) is Grassman. Differentiating \( \partial/\partial A^a_{ij} \) removes \( A^a_{ij} \) from the integrand, and so the integral is identically zero.
If $A^a$ is bosonic, then since the integrand vanishes exponentially, we also get zero by the divergence theorem. So we have found
\[ \frac{\partial}{\partial \kappa_2} Z_{\text{def}}(\epsilon, \kappa_2, \mu) = 0; \] (4.61)
the deformed partition function is independent of $\kappa_2$. Our tactic will be to evaluate the integral when $\kappa_2$ is large.

The part of the deformed action (4.49) which depends on $\kappa_2$ is
\[ \kappa_2 \text{Tr}(i\tilde{\phi}H + 3\epsilon \tilde{\phi}^2 + i\eta_1\eta_2) = \kappa_2 \text{Tr}(3\epsilon (\tilde{\phi} + \frac{i}{6\epsilon} H)^2 + \frac{1}{12\epsilon} H^2 + i\eta_1\eta_2). \] (4.62)

We shall take $\kappa_2$ large, and use a saddle point method to integrate out $\tilde{\phi}$. To avoid breaking the flow of argument here, a full description of the relevant saddle point method is included in appendix C. The first step is to follow the by now familiar technique, and set $\tilde{\phi} = \phi + \frac{i}{6\epsilon} H$, and shift the contours so that $\tilde{\phi}$ becomes hermitian. In order to apply the saddle point method, we use equations 4.8-4.14 to change to tilde type variables, and then apply the method to each of these real integration variables. Then we find $\tilde{\phi}$ localises at $-\frac{i}{6\epsilon} H$, and $H$ localises at 0. Thus
\[ \int d\tilde{\phi} dH g(\tilde{\phi}, H) \exp \left\{ -\kappa_2 \text{Tr}(i\tilde{\phi}H + 3\epsilon \tilde{\phi}^2) \right\} = \frac{2^{N^2-1}}{N} \frac{\pi^{N^2-1}}{(\kappa_2)^{N^2-1}} g(0, 0) + O\left(\kappa_2^{-N^2}\right) \] (4.63)
where the $2^{N^2-1}$ factor comes from the factors $3\epsilon$ and $(12\epsilon)^{-1}$ which appear in equation 4.62.

We would now like to integrate out $\eta_1$ and $\eta_2$. Consider
\[ I(\kappa_2) = \int d\eta_2 d\eta_1 f(\eta_1, \eta_2) \exp(-i\kappa_2 \text{Tr}{\eta_1}{\eta_2}) \] (4.64)
for some $f$. Changing variables to $\tilde{\eta}_1, \tilde{\eta}_2$, this is
\[ I(\kappa_2) = \int d\tilde{\eta}_2 d\tilde{\eta}_1 f(\tilde{\eta}_1, \tilde{\eta}_2) \exp(-i\kappa_2 \text{Tr}{\tilde{\eta}_1}{T}{\tilde{\eta}_2}) \] (4.65)
and using $(\text{Tr}{\tilde{\eta}_1}{T}{\tilde{\eta}_2})^{N^2} = 0$ (since $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are traceless real $N\times N$ Grassman matrices),
\[ I(\kappa_2) = i^{N^2-1} \int d\tilde{\eta}_1 d\tilde{\eta}_2 f(0, 0) \frac{\kappa_2^{N^2-1}}{(N^2 - 1)!} (\text{Tr}{\tilde{\eta}_1}{T}{\tilde{\eta}_2})^{N^2-1} + O(\kappa_2^{-N^2-2}). \] (4.66)
Note that we also switched the order of integration of $\eta_1$ and $\eta_2$ in order to cancel the $(-1)^{N^2-1}$ which came from expanding the exponential. At present, we are integrating
over traceless $\tilde{\eta}_1, \tilde{\eta}_2$. It is simpler to do the integration if we insert
\[
\delta(\text{Tr}\tilde{\eta}_1) = (\tilde{\eta}_1)_{11} + \cdots + (\tilde{\eta}_1)_{NN}
\]
and similar for $\eta_2$, and integrate over the full matrices. If we do this, and also expand the trace in \ref{4.66}, we find
\[
I(\kappa_2) = i^{N^2-1} \int d\tilde{\eta}_1 d\tilde{\eta}_2 f(0,0) \kappa_2^{N^2-1} N \prod_{i,j} \tilde{\eta}_{1ij} \tilde{\eta}_{2ij} + O(\kappa_2^{N^2-2})
\]
and
\[
\kappa_2^{N^2-1} i^{N^2-1} N f(0,0) + O(\kappa_2^{N^2-2})
\]
We can now use these results to integrate out $\phi, H, \eta_1$ and $\eta_2$ from the deformed partition function
\[
Z_{\text{def}} = \int dX_4 dX_1 dX_2 d\psi_1 d\psi_2 d\eta_1 d\eta_2 dH \exp(-S_{\text{def}})
\]
\[
= (4i\pi)^{N^2-1} \int dX_4 dX_1 dX_2 d\psi_1 d\psi_2 \exp(-S_{\text{def}})|_{\eta_1=\eta_2=0, \phi=H=0} + O(\kappa_2^{-1}).
\]
However, since we know $Z_{\text{def}}$ is independent of $\kappa_2$, the $O(\kappa_2^{-1})$ terms must actually be zero. Writing this out in full gives
\[
Z_{\text{def}} = (4i\pi)^{N^2-1} \int dX_4 dX_1 dX_2 d\psi_1 d\psi_2 \\
\exp \text{Tr} \left( -\epsilon X_4 [X_1, X_2] - \epsilon i \psi_1 \psi_2 - \frac{\epsilon^2}{2} (X_1^2 + X_2^2) - \mu^2 X_4^2 \right)
\]
and we note here that we can scale out a factor of $\epsilon$ from the first three terms in the exponential. This process of integrating out $\phi, H, \eta_1, \eta_2$ in the large $\kappa_2$ limit has sometimes been known as integrating over a BRST quartet.

The fermions $\psi_1, \psi_2$ can be integrated out immediately using \ref{4.13} (but note that the sign difference gives an extra factor $(-1)^{N^2-1}$). We can also integrate out $X_2$ by completing the square and using \ref{4.14} to give
\[
Z_{\text{def}} = \sqrt{N} \left( \frac{2\pi}{\epsilon} \right)^{N^2-1} (4\pi)^{N^2-1} \int dX_4 dX_1 \exp \text{Tr} \left( \frac{1}{2\epsilon} [X_4, X_1]^2 - \frac{\epsilon}{2} X_1^2 - \mu^2 X_4^2 \right).
\]
We now follow the usual procedure to reduce the integral over $X_4$ to an integral over its eigenvalues (see for example \ref{44} for an account of this method applied to the $SU(N)$ groups). For a function $f$ which depends only on the eigen values $x_1, \cdots, x_N$ of $X_4$, \ref{4.67}
we have
\[
\int dX_4 f(x_1, \ldots, x_N) = c_N \int dx_1 \cdots dx_N \prod_{j<k} (x_j - x_k)^2 f(x_1, \ldots, x_N). \tag{4.73}
\]

Recall that for \(X_4\) only, we are using the normalisation \(\int dX_4 \delta(\text{Tr}X_4) e^{-2\text{Tr}X_4^2} = 1\) (equation 4.18). In this case, the constant is
\[
c_N = \frac{2^{N^2} \sqrt{N\pi/2}}{(2\pi)^{N/2} 1!2! \cdots N!} \tag{4.74}
\]
as can most easily be seen by adjusting the value given in [44] to our conventions. It is now convenient to use the tilde notation of equations 4.8-4.14, \(X_1 \to \tilde{X}\), since then
\[
\text{Tr} [X_1, X_4]^2 = -\sum_{i,j} (\tilde{X}_{ij})^2 (x_i - x_j)^2. \tag{4.75}
\]

Then 4.72 becomes
\[
Z_{\text{def}} = \sqrt{N} \left(\frac{2\pi}{\epsilon}\right)^{\frac{N^2-1}{2}} (4\pi)^{N^2-1} c_N \int dx_1 \cdots dx_N \prod_{j<k} (x_j - x_k)^2 d\tilde{X} \delta(x_1 + \cdots + x_N)
\]
\[
\times \exp \left(-\frac{1}{2\epsilon} \sum_{i,j} (\tilde{X}_{ij})^2 \left\{ (x_i - x_j)^2 + \epsilon^2 \right\} - \mu^2 \sum_i x_i^2 \right). \tag{4.76}
\]

It is convenient to define
\[
F_N = \frac{2^{\frac{N(N+1)}{2}} \pi^{\frac{N-1}{2}}}{2\sqrt{N} \prod_{i=1}^{N-1} i!} \tag{4.77}
\]
since this is the normalisation constant that emerged in the numerical calculations of the partition function of [8, 9]. Then, integrating out \(\tilde{X}\), we are left with
\[
Z_{\text{def}} = (2\pi)^{2(N^2-1)} 2^{\frac{N}{2}(N^2-1)} \frac{F_N}{(2\epsilon)^{N-1}(N-1)!}
\]
\[
\times \int dx_1 \cdots dx_N \delta(x_1 + \cdots + x_N) \prod_{i>j} \frac{(x_i - x_j)^2}{(x_i - x_j)^2 + \epsilon^2} e^{-\mu^2 \sum_i x_i^2}. \tag{4.78}
\]

This result is almost identical to the formula of MNS up to normalisation. To get back to their formula, we would send \(\mu \to 0\), and change the sign of the \(\epsilon^2\) which appears in the denominator of the product (this latter difference is an advantage since MNS added an imaginary part to \(\epsilon\) by hand as part of their prescription for performing the contour integrals). As it stands, the formula 4.78 diverges in the limit \(\epsilon \to 0\) or \(\mu \to 0\).
4.4 Dangerous Fermion Masses

Let's summarise. We began by considering the supersymmetric Yang-Mills integral \[ 4.4 \]. We added small mass terms (and a small cubic term) to the action leading to the deformed action \[ 4.44 \]. After checking that the resulting deformed partition function \[ Z_{\text{def}} \] is convergent, we observed that it is independent of one of the small parameters \( \kappa_2 \). Then we followed the localisation method, and calculated the integral by taking \( \kappa_2 \) large.

We hoped that we could then take the small parameters to zero, and arrive back at the original Partition function. However, this is clearly not the case since if we do this in equation \[ 4.78 \], we get \( \infty \), whilst we have shown in chapter 2 that the result should be finite.

A rough calculation indicates why this may be so. We have attempted to deform the action by adding mass terms for the bosons and fermions

\[
S_{\text{def}} \sim S + i\epsilon \text{Tr}(\psi_1\psi_2 + \eta_1\eta_2) + \epsilon^2 \text{Tr}(X_\mu X_\mu). \tag{4.79}
\]

We did not quite achieve this in equation \[ 4.44 \] since the mass term for \( X_4 \) has the wrong sign. However, rewriting \[ 4.44 \] in the form \[ 4.49 \] shows that \[ 4.79 \] is a fair model. Let’s go back to the convergence argument of section 2.3. The new fermion mass terms introduce extra insertions into the modified pfaffian \[ 2.108 \]. However, these insertions do not come with associated powers of \( R^{-(2-n)} \) as they do in the convergence proof, but they do each have a power of \( \epsilon \). In particular, consider the term in which all the \( 2(D-2) \) \( J \)-type insertions are present (the \( \xi_{\alpha} \)). Then we loose a factor \( R^{-(2-n)(D-2)} \sim R^{-4(D-2)} \) from the convergence proof, and also an \( R^{D-2} \) from the measure, but gain \( \epsilon^{D-2} \). So, without the \( \epsilon^2 R^2 \text{Tr} x_\mu x_\mu \) mass terms, we would have

\[
\int \frac{dR}{R} R^D g R^{(D-2)g} R^{-(D-2)} I_{D,G} \sim \epsilon^{D-2} \int \frac{dR}{R} R^D. \tag{4.80}
\]

We have not yet considered the bosonic mass terms which appear in the exponential

\[
\exp(-\epsilon^2 R^2 \text{Tr} x_\mu x_\mu). \tag{4.81}
\]

We would like to find the best possible bound on the \( \epsilon \)-behaviour of the partition function. Thus, we would like to use the bosonic mass terms just enough to make \[ 4.80 \] convergent. We can bound

\[
\exp(-\epsilon^2 R^2 \text{Tr} x_\mu x_\mu) < \exp(-\epsilon^2 R^2 \text{Tr} \rho_\mu \rho_\mu) \tag{4.82}
\]
and then, for fixed \( \epsilon \) and large \( R \), the integrations over the \( \rho_\mu \) in the convergence proof lead to an additional \( (R\epsilon)^{-D} \). If we add this into the rhs of (4.80), we see that this sets the \( R \)-integration on the threshold of convergence. Thus, the best bound we can get on \( Z_{\text{def}} \) using the methods we have developed is

\[
Z_{\text{def}} \sim \epsilon^{-2}.
\]  (4.83)

Thus, the convergence proof (which we found gave perfect predictions for convergence of the undeformed partition function) fails for the deformed partition function when \( \epsilon \to 0 \). Of course, this does not prove anything since this only gives a crude upper bound on certain terms of \( Z_{\text{def}} \). However, it does indicate very clearly how after adding fermion mass terms to an action, we cannot expect to regain the partition function by sending the masses back to zero. In the case of \( SU(2) \) where no iteration is involved, there is a decent chance for this basic argument to give the correct power of \( \epsilon \). Indeed, setting \( \mu = \epsilon \) in (4.78) and sending \( \epsilon \to 0 \), we obtain \( Z_{\text{def}} \sim \epsilon^{-2(N-1)} \) for \( SU(N) \), and so \( \epsilon^{-2} \) for \( SU(2) \).

Since the limit of masses going to zero is dangerous, one might also worry about taking the limit \( \kappa_2 \to \infty \) in the arguments of the previous section. We know that the integral is independent of \( \kappa_2 \) after doing the integration, but we don’t know whether we can take \( \kappa_2 \to \infty \) before integrating. However in reality, we do not take the limit as \( \kappa_2 \to \infty \) first, but rather find the first term in an asymptotic expansion, and use the result of appendix C to show that the correction term is finite, and vanishes as \( \kappa_2 \to \infty \) and so is zero.

### 4.5 An Unresolved Issue

We recall that in [32], the deformation method was applied to the cohomological theory. The result is almost identical to the equation (4.78) that we found by applying the method to the Yang-Mills theory. In [32], this result is found as an integral over the eigenvalues of the cohomological theory field \( \phi \), whilst we have obtained it as an integral over the eigenvalues of \( X_4 \). The authors of [32] integrated out the \( \delta \)-function, and then set a prescription for completing the contours of the \( N-1 \) remaining integrals around either the upper or lower half plane. They were able to perform the contour integrals, and found

\[
\frac{1}{(2\pi\epsilon)^{N-1}(N-1)!} \int dx_1 \cdots dx_{N-1} \int dx_N \delta(x_1 + \cdots + x_N) \prod_{i \neq j} \frac{x_i - x_j}{x_i - x_j + i\epsilon} = \frac{1}{N^2}.
\]  (4.84)
They also applied the same method to \( D = 6 \) and \( D = 10 \) theories, and noticed that for \( D = 10 \) the result corresponds to the conjecture of Green and Gutperle \[30\]. At the same time, Krauth, Nicolai and Staudacher \[8\] were able to apply Monte Carlo techniques to evaluate the partition function for some values of \( N \) and obtained the same result for \( D = 4, 6, 10 \), and also calculated the normalisation factor \( \mathcal{F}_N \) as a group volume.

It is interesting to understand exactly why the group volume \( \mathcal{F}_N \) appears as the normalisation factor for the MNS formula. Therefore, it is worthwhile to adapt our conventions to those that were used for the numerical calculations \[8, 9\], and check agreement. For \( D = 4 \), these authors calculate the partition function defined as

\[
Z_{4,N}' = \int \frac{dX_\mu}{\sqrt{2\pi}} \left( \prod_{\alpha=1}^{4} \frac{d\Psi_\alpha}{\sqrt{2\pi}} \right) \exp \left[ \frac{1}{2} \sum_{\mu,\nu} \text{Tr} [X_\mu, X_\nu]^2 + \text{Tr} \Psi_\alpha \left[ \Gamma_{\alpha\beta} X_\mu, \Psi_\beta \right] \right]
\]

where the fermions are written in the real representation of equation 1.23. Comparing to 4.4, this differs from our definition by a factor \((2\pi)^{-2(N^2-1)}\) in the measure, and a factor \(\frac{1}{2}\) from our potential \(\frac{1}{4} [X_\mu, X_\nu] [X_\mu, X_\nu]\). We can get back to our action 4.3 by scaling \(X_\mu \to 2^{-\frac{1}{4}} X_\mu\) and \(\Psi_\alpha \to 2^\frac{1}{8} \Psi_\alpha\). Then we pick up an additional factor \((2^{-\frac{1}{4}})^{4(N^2-1)} (2^\frac{1}{8})^{-4(N^2-1)} = 2^{\frac{1}{2}(N^2-1)}\) in the measure. Then adding these factors into the result 4.78, and sending \(\mu \to 0\), leads to

\[
\mathcal{F}_N \left( \frac{1}{(2\pi\epsilon)^{N-1}(N-1)!} \int dx_1 \cdots dx_N \delta(x_1 + \cdots + x_N) \prod_{i>j} \frac{(x_i - x_j)^2}{(x_i - x_j)^2 + \epsilon^2} \right)
\]

so that applying the MNS contour prescription gives

\[
Z_{4,N}' = \mathcal{F}_N \frac{1}{N^2}
\]

in exact agreement with the numerical calculations.

It was hoped that once the relation between the cohomological calculation of \[32\] and the Yang-Mills model was understood, the reason for requiring the contour integrals would become readily apparent. Sadly, although through the work of this chapter we can now understand the calculation entirely from the Yang-Mills perspective, this has not come to pass. It is clear from section 4.4 that the problem with the calculation as it stands is in the deformation of the action. Therefore, the task ahead is to find a new deformation that will still allow all the various steps of the previous calculation, but not suffer the problems of the current deformation. For the present, we must leave this as an unresolved issue.
Summary of Results

In chapter 2 we considered the convergence properties of Yang-Mills matrix models. For the bosonic theories, we showed that the partition function converges when $D \geq D_c$ and calculated $D_c$ for each group. We also calculated a critical value $k_c$ for each group with the property that any correlation function of degree $k < k_c$ is convergent. Conversely, we showed that there is always a correlation function of degree $k_c$ which is divergent.

For the supersymmetric models, we showed that the partition function converges when $D = 4, 6$ and $10$, and that correlation functions of degree $k < 2(D - 3)$ are convergent. This result applies for any compact semi-simple gauge group.

In chapter 3 we considered the supersymmetric models. With particular reference to the $D = 4$ model, we found all quantities invariant under the supercharge. We also indicated how the result extends immediately to $D = 6$ and $D = 10$. In appendix A, we point out that all four supercharges in the $D = 4$ model are related by permutations of the fields. Thus the result can immediately be applied to any of the supercharges.

In chapter 4 we considered how to apply the deformation method of [32] directly to the supersymmetric Yang-Mills model with $D = 4$. We found a deformation of the action that can generate mass terms for all the fields and still preserve some supersymmetry. This allowed us to integrate over a BRST quartet rigorously, and confirm the formula of [32]. We showed why this method fails to reproduce exact values for the partition function so that an alternative regularisation must be found. However, a proof that the contour prescription of Moore, Nekrasov and Shatashvili is the correct regularisation remains elusive.
Appendix A

Supercharges for $D = 4$

In this appendix we give an explicit representation of the four linearly independent supercharges of the $D = 4$ supersymmetric Yang-Mills matrix theory. They are obtained directly from the supersymmetry 3.6. We use the notation of chapter 3, and in addition, define $-H' = H + [X_1, X_2]$. Although they are written in terms of fields $\phi$ and $\overline{\phi}$ from the Cohomological theory, one can simply make the replacement 3.9 to return explicitly to the Yang-Mills theory.

\[
\begin{align*}
\delta_1 X_a & = \psi_a & \delta_1 \psi_a & = [\phi, X_a] \\
\delta_1 \overline{\phi} & = -\eta_2 & \delta_1 \eta_2 & = -[\phi, \overline{\phi}] \\
\delta_1 \eta_1 & = H & \delta_1 H & = [\phi, \eta_1] \\
\delta_1 \phi & = 0
\end{align*}
\]

\[
\begin{align*}
\delta_2 X_a & = -\epsilon_{ab} \psi_b & \delta_2 \psi_a & = \epsilon_{ab} [\phi, X_b] \\
\delta_2 \overline{\phi} & = \eta_1 & \delta_2 \eta_1 & = [\phi, \overline{\phi}] \\
\delta_2 \eta_2 & = H & \delta_2 H & = [\phi, \eta_2] \\
\delta_2 \phi & = 0
\end{align*}
\]

\[
\begin{align*}
\delta_3 X_a & = \eta_a & \delta_3 \eta_a & = [\overline{\phi}, X_a] \\
\delta_3 \phi & = -\psi_2 & \delta_3 \psi_2 & = -[\overline{\phi}, \phi] \\
\delta_3 \psi_1 & = H' & \delta_3 H' & = [\overline{\phi}, \psi_1] \\
\delta_3 \overline{\phi} & = 0
\end{align*}
\]

\[
\begin{align*}
\delta_4 X_a & = -\epsilon_{ab} \eta_b & \delta_4 \eta_a & = \epsilon_{ab} [\overline{\phi}, X_b] \\
\delta_4 \phi & = \psi_1 & \delta_4 \psi_1 & = [\overline{\phi}, \phi] \\
\delta_4 \psi_2 & = H' & \delta_4 H' & = [\overline{\phi}, \psi_2] \\
\delta_4 \overline{\phi} & = 0
\end{align*}
\]
These obey the algebra
\[
\begin{aligned}
\delta_1^2 &= [\phi, \ ] \\
\delta_2^2 &= [\phi, \ ] \\
\delta_3^2 &= [\bar{\phi}, \ ] \\
\delta_4^2 &= [\bar{\phi}, \ ] \\
\{\delta_1, \delta_2\} &= 0 \\
\{\delta_1, \delta_4\} &= [X_1, \ ] \\
\{\delta_2, \delta_3\} &= [X_2, \ ] \\
\{\delta_3, \delta_4\} &= 0
\end{aligned}
\]  
\hspace{1cm} (A.1)

and so
\[
\{\delta_i, \delta_j\} = 0 \hspace{1cm} (A.2)
\]
on gauge invariant quantities.

As we have pointed out, none of the supercharges in the Yang-Mills theory can preserve hermiticity. A nice feature of the Cohomological theory, in which \(\phi\) and \(\bar{\phi}\) become independent and respectively antihermitian and hermitian, is that then \(\delta_1\) and \(\delta_2\) become truly real. However, it is interesting that, unlike the Minkowski theory, only two of the supercharges preserve hermiticity in this way in the Cohomological theory.

The action is invariant under certain permutations of the matrix fields. Defining

\[
\begin{aligned}
\Pi & : \psi_a \leftrightarrow \eta_a \\
H & \leftrightarrow H' \\
\phi & \leftrightarrow \bar{\phi}
\end{aligned}
\]
\hspace{1cm} (A.3)

\[
\begin{aligned}
\Omega & : \psi_a \rightarrow \epsilon_{ab}\psi_b \\
\eta_a & \rightarrow -\epsilon_{ab}\eta_b
\end{aligned}
\]
\hspace{1cm} (A.4)

\[
\begin{aligned}
\Sigma & : X_a \rightarrow -\epsilon_{ab}X_b \\
\eta_a & \rightarrow -\epsilon_{ab}\eta_b
\end{aligned}
\]
\hspace{1cm} (A.5)

one finds \(\Pi S = \Omega S = \Sigma S = S\).

The supersymmetries are related to each other by the permutation symmetries. For example
\[
\begin{aligned}
\Omega^{-1}\delta_1\Omega &= \delta_2 \\
\Sigma^{-1}\delta_1\Sigma &= \delta_2 \\
\Pi^{-1}\delta_1\Pi &= \delta_3 \\
\Pi^{-1}\delta_2\Pi &= \delta_4
\end{aligned}
\]
\hspace{1cm} (A.6) \hspace{1cm} (A.7) \hspace{1cm} (A.8) \hspace{1cm} (A.9)

Each supersymmetry can be related to \(\delta_1\) by a permutation symmetry. Rather than a model with four supersymmetries, one could think of the theory as a model with one supersymmetry together with some permutation symmetries.
For interest, it is possible to represent the action 3.10 in terms of all four supercharges. There are three ways of doing this:

\[
S = \frac{1}{2} \delta_1 \delta_2 \delta_3 \delta_4 \text{Tr}X_1^2
\]
\[
= \frac{1}{2} \delta_1 \delta_2 \delta_3 \delta_4 \text{Tr}X_2^2
\]
\[
= -\delta_1 \delta_2 \delta_3 \delta_4 \text{Tr}\phi\bar{\phi}
\]

(A.10)

Since the \( \delta_i \) anticommute, these representations render \( S \) manifestly invariant under all four supercharges.
Appendix B

Deformed Action

The deformed action 4.38 written out in full is given by

\[ S_{\text{def}} = S + \epsilon \text{Tr} \left( \frac{i}{2} (\gamma + 2) (X_3 + iX_4) [X_1, X_2] - \frac{i}{2} \gamma (X_3 - iX_4) H \right. \\
\left. - \frac{i}{2} \lambda (X_3 + iX_4) (H + [X_1, X_2]) - i \gamma \eta_1 \eta_2 \right) \\
+ \kappa_1 \text{Tr} \left( i \psi_1 \psi_2 - \frac{i}{2} (X_3 + iX_4) [X_1, X_2] + \frac{\epsilon^2}{2} (X_3^2 + X_4^2) \right) \\
+ \kappa_2 \text{Tr} \left( - \frac{i}{2} H (X_3 - iX_4) - \frac{\epsilon \gamma}{4} (X_3 - iX_4)^2 - \frac{\epsilon \lambda}{4} (X_3^2 + X_4^2) + i \eta_1 \eta_2 \right) \\
- \mu^2 \text{Tr} \left( \frac{1}{2} (X_3 + iX_4) \right)^2 \] (B.1)

where \( S \) is the original action

\[ S = \text{Tr} \left( (H + \frac{1}{2} [X_1, X_2])^2 - \frac{1}{4} \sum_{\mu > \nu} [X_\mu, X_\nu]^2 \right. \\
\left. - \epsilon_{ab} \eta_1 [\psi_a, X_b] - \eta_a \frac{1}{2} [(X_3 + iX_4), \eta_a] - \psi_a \frac{1}{2} [(-X_3 + iX_4), \psi_a] + \eta_2 [\psi_a, X_a] \right) . \] (B.2)

It is interesting to know how much supersymmetry is preserved by this deformation. Certainly, \( S_{\text{def}} \) is invariant under the deformed supercharge \( \overline{\delta} \) given in equation 4.36.

Since, \( \overline{\delta} \) was deformed from the original supercharge \( \delta_1 \) of appendix A, lets write \( \overline{\delta} \equiv \delta_1 \). We note that \( S_{\text{def}} \) is also invariant under the permutation symmetry \( \Omega \) defined in equation A.4. Then, following equation A.6, we can define

\[ \overline{\delta}_2 = \Omega^{-1} \overline{\delta}_1 \Omega \] (B.3)
so that
\[
\tilde{\delta}_2 S_{\text{def}} = 0. \tag{B.4}
\]
Since \( S_{\text{def}} \) is not invariant under the permutation \( \Pi \), it is not possible to define deformed versions of \( \delta_3 \) and \( \delta_4 \) in this way. Thus \( S_{\text{def}} \) is invariant under only two supercharges compared to the four of the original action \( S \).
Appendix C

Saddle Point Method

In this appendix, we prove the precise form of the saddle point method that we use in chapter 4. The proof follows the usual argument of integration by parts; see for example [45].

Let $g(t)$ be continuous, twice differentiable and

$$\int_0^\infty dt |g^{(n)}(t)|$$

converge for $n = 0, 1, 2$. Then

$$\int_0^\infty dtg(t) \exp(-kt^2) = g(0)\frac{\sqrt{\pi}}{2\sqrt{k}} + R(k)$$

where

$$|R(k)| < \frac{\text{const.}}{k}. \quad (C.3)$$

To see this, we use integration by parts,

$$I(k) = \int_0^\infty dtg(t) \exp(-kt^2) \quad (C.4)$$

$$= \left[ -g(t) \int_t^\infty \exp(-ku^2)du \right]_0^\infty + \int_0^\infty dtg'(t) \int_t^\infty du \exp(-ku^2) \quad (C.5)$$

$$= g(0) \int_0^\infty du \exp(-ku^2) + \int_0^\infty dtg'(t) \int_1^\infty dvt \exp(-kt^2v^2) \quad (C.6)$$

$$= g(0)\frac{\sqrt{\pi}}{2\sqrt{k}} + R(k) \quad (C.7)$$

where

$$R(k) = \int_0^\infty dtg'(t) \int_1^\infty dvt \exp(-kt^2v^2). \quad (C.8)$$
Integrating by parts again,

\[
R(k) = \int_{1}^{\infty} dv \left( \left[ g'(t) \frac{-1}{2kv^2} \exp(-kt^2v^2) \right]_0^\infty - \int_0^\infty dt g''(t) \frac{-1}{2kv^2} \exp(-kt^2v^2) \right)
\]

\[
= \int_{1}^{\infty} dv \frac{g'(0)}{2kv^2} + \int_{1}^{\infty} dv \int_0^\infty dt g''(t) \frac{\exp(-kt^2v^2)}{2kv^2}
\]

so that

\[
|R(k)| \leq \frac{1}{k} \int_{1}^{\infty} dv \frac{|g'(0)|}{2v^2} + \frac{1}{k} \left( \int_{1}^{\infty} dv \frac{1}{2v^2} \right) \int_0^\infty dt |g''(t)| = \text{const.} \quad (C.9)
\]

There is an issue we must address specifically in relation to the saddle point calculation of chapter 4. In this case we must repeat the saddle point process many times as we integrate out each variable in the matrices \( \overline{\phi} \) and \( H \). Properly, we should write \( \kappa_2 = k + 1 \). Then setting \( k = 0 \) (i.e. \( \kappa_2 = 1 \)) gives us the integrand which corresponds to \( g(t) \). This integrand is exponentially damped, and so the properties \( \text{(C.1)} \) always hold. In particular the integrals which define the remainder term always converge.
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