Testing unitarity of the $3 \times 3$ neutrino mixing matrix in an atomic system

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Abstract

Unitarity of the $3 \times 3$ lepton flavor mixing matrix $V$ is unavoidably violated in a seesaw mechanism if its new heavy degrees of freedom are slightly mixed with the active neutrino flavors. We propose to use the atomic transition process $|e\rangle \rightarrow |g\rangle + \gamma + \nu_i + \nu_j$ (for $i,j = 1,2,3$), where $|e\rangle$ and $|g\rangle$ stand respectively for the excited and ground levels of an atomic system, to probe or constrain the unitarity-violating effects of $V$. We find that the photon spectrum of this transition will be distorted by the effects of $VV^\dagger \neq 1$ and $V^\dagger V \neq 1$ as compared with the $VV^\dagger = V^\dagger V = 1$ case. We locate certain frequencies in the photon spectrum to minimize the degeneracy of effects of the unitarity violation and uncertainties of the flavor mixing parameters themselves. The requirements of a nominal experimental setup to test the unitarity of $V$ are briefly discussed.

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1 Introduction

To naturally interpret the finite but tiny masses of three known neutrinos $\nu_i$ (for $i = 1, 2, 3$) corresponding to their flavor eigenstates $\nu_\alpha$ (for $\alpha = e, \mu, \tau$), the most popular and well-motivated way is to extend the standard electroweak model by introducing three heavy sterile neutrinos and allow for lepton number violation — the canonical seesaw mechanism [1–5]. In this connection the small mixing between light and heavy degrees of freedom unavoidably gives rise to a slight departure of the $3 \times 3$ Pontecorvo-Maki-Nakagawa-Sakata (PMNS) lepton flavor mixing matrix $V$ from unitarity (i.e., $V^\dagger V \neq 1$ and $VV^\dagger \neq 1$) [6], and it is particularly appreciable in some interesting and testable TeV-scale seesaw models [7]. This kind of indirect unitarity violation of $V$ can in principle be probed or constrained at low energies, such as in long-baseline accelerator neutrino oscillation experiments [8–14] and medium-baseline reactor antineutrino oscillation experiments [15–18].

Different from previous works, the present paper aims to put forward a new and interdisciplinary method for testing unitarity of the PMNS matrix $V$ by looking at possible indirect unitarity violation of $V$. Our approach is closely related to atomic physics and has nothing to do with neutrino or antineutrino oscillations. Let us elaborate this novel idea in the following.

In 2006 one of us (M.Y.) proposed to use some fine atomic transitions as a powerful tool to determine the absolute neutrino masses and the nature of massive neutrinos (namely, whether they are the Majorana or Dirac particles) [19]. The relevant transition process in a feasible experimental setup is $|e\rangle \rightarrow |g\rangle + \gamma + \nu_i + \bar{\nu}_j$ (for $i, j = 1, 2, 3$), where $|e\rangle$ is the excited level in an atomic or molecular system, and $|g\rangle$ denotes the ground one. Such a transition can take place via an intermediate state $|\nu\rangle$. The information about neutrino properties is encoded in the spectrum of the emitted photons $\gamma$, just like the spectrum of the emitted electrons in a nuclear $\beta$-decay experiment. Before and after the transition, the total energy of the system is conserved: $E_{eg} = \omega + E_i + E_j$, where $E_{eg}$ represents the energy difference between $|e\rangle$ and $|g\rangle$, $\omega$ stands for the energy of the emitted photon, and $E_i$ (or $E_j$) denotes the energy of the neutrino $\nu_i$ (or $\nu_j$) with the mass $m_i$ (or $m_j$). The Feynman diagrams responsible for the transition under consideration are shown in Fig. 1, where the relevant weak neutral- and charged-current interactions are described by

$$-\mathcal{L}_{nc} = \frac{g_w}{4 \cos \theta_w} \left[ \left( \begin{array}{ccc} \nu_1 & \nu_2 & \nu_3 \end{array} \right) \gamma^\mu (1 - \gamma_5) V^\dagger V \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \nu_3 \end{array} \right) Z_{\mu} + \bar{e} \gamma^\mu \left( 4 \sin^2 \theta_w - 1 + \gamma_5 \right) e Z_{\mu} \right],$$

$$-\mathcal{L}_{cc} = \frac{g_w}{2 \sqrt{2}} \left( e \mu \tau \right) \gamma^\mu (1 - \gamma_5) V \left( \begin{array}{c} \nu_1 \\ \nu_2 \\ \nu_3 \end{array} \right) W^-_{\mu} + \text{h.c.},$$

where $g_w$ denotes the weak-interaction coupling constant, and $\theta_w$ is the weak mixing angle. Integrating out the relevant heavy degrees of freedom (i.e., the massive $W^\pm$ and $Z$ bosons) in Eq. (1) and performing the Fierz transformations, we are left with the following effective
Figure 1: The Feynman diagrams for weak neutral- and charged-current interactions that contribute to the radiative emission of neutrino pairs in an atomic system.

four-fermion interactions at low energies:

$$-\mathcal{L}_{\text{eff}} = \frac{G_F}{2\sqrt{2}} \sum_{i=1}^{3} \sum_{j=1}^{3} \left[ \bar{\nu}_i \gamma^\mu (1 - \gamma_5) (V^\dagger V)_{ij} \nu_j \right] \cdot \left[ \bar{e} \gamma_\mu \left( 4 \sin^2 \theta_W - 1 + \gamma_5 \right) e \right]$$

$$+ \frac{G_F}{\sqrt{2}} \sum_{i=1}^{3} \sum_{j=1}^{3} \left[ \bar{\nu}_i \gamma^\mu (1 - \gamma_5) V^\ast_{ei} V_{ej} \nu_j \right] \cdot \left[ \bar{e} \gamma_\mu (1 - \gamma_5) e \right],$$

where $G_F = g_w^2 / (4\sqrt{2}M_W^2) \simeq 1.166 \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi constant. We assume that the level transition associated with the neutrino-pair emission is of the M1 type via the electron spin flipping. The axial current of the electron field will therefore dominate the transition, as the contribution of the vector current in this case is suppressed by the velocity of the nonrelativistic electron. As a result, Eq. (2) is simplified to

$$\mathcal{L}_{\text{eff}} \rightarrow \mathcal{L}_{\text{eff}}^{(A)} = \frac{G_F}{\sqrt{2}} \sum_{i=1}^{3} \sum_{j=1}^{3} \left\{ \bar{\nu}_i \gamma^\mu (1 - \gamma_5) \left[ V^\ast_{ei} V_{ej} - \frac{1}{2} (V^\dagger V)_{ij} \right] \nu_j \cdot \bar{e} \gamma_\mu \gamma_5 e \right\}.$$  

The axial electron current projected into the atomic levels $\langle v | \bar{e} \gamma^\mu \gamma_5 e | e \rangle$ can be reduced to $\langle v | 2S | e \rangle$ in the nonrelativistic limit, where $S$ denotes the spin operator. Note that the PMNS matrix $V$ is involved into the transition process as a single factor $a_{ij} \equiv V^\ast_{ei} V_{ej} - (V^\dagger V)_{ij} / 2$, which can be simplified to $V^\ast_{ei} V_{ej} - \delta_{ij} / 2$ if $V$ is exactly unitary.

A rough but instructive estimate based on the naive dimensional analysis yields the transition rate $\Gamma \sim N_{\text{tar}} G_F^2 E^5 \sim 10^{-8} \text{ s}^{-1}$ for a macroscopic atomic ensemble with the target atom number $N_{\text{tar}} \sim \mathcal{O}(10^{23})$, where $E \sim \mathcal{O}(1 \text{ eV})$ stands for a typical energy transfer of the atomic level. That is why the transition rate demands some magnification mechanisms for a realistic measurement. In Refs. [20] and [21] it was proposed to utilize the super-radiance (SR) phenomenon in quantum optics to enhance the rate. The total transition rate for a macroscopic ensemble in the stochastic case is just proportional to the number of total target atoms $N_{\text{tar}}$. If atoms in the ensemble are arranged to behave collectively, however, the final rate will be instead proportional to $N_{\text{tar}}^2$. This coherence enhancement makes a realistic
observation possible. In this connection a review of the radiative emission of neutrino pairs (RENP) has been done by the SPAN (SPectroscopy with Atomic Neutrino) group in Ref. [23].

To achieve a coherence among the macroscopic target atoms, the momenta of the outgoing particles must follow the relation \( p_{eg} = k + p_i + p_j \), where \( p_{eg} \) denotes the initial phase imprinted on the medium which can be manufactured to be nonzero by the coherence-establishing procedure in the scenario of the boosted RENP [24], \( k \) is the momentum of the photon, and \( p_i \) (or \( p_j \)) represents the momentum of the neutrino \( \nu_i \) (or \( \bar{\nu}_j \)). Therefore, both energy and momentum conservations should be imposed on the system to have a successful coherent enhancement. Instead of going into the detail in this aspect, we subsequently focus on the particle-physics part of the RENP and illustrate how the effects of indirect unitarity violation of \( V \) can manifest them in this interesting process.

2 Methodology

A slight departure of the PMNS neutrino mixing matrix \( V \) from unitarity can in general be parametrized in the following way:

\[
VV^\dagger - 1 = \begin{pmatrix}
\epsilon_{ee} & \epsilon_{e\mu} & \epsilon_{e\tau} \\
\epsilon_{e\mu} & \epsilon_{\mu\mu} & \epsilon_{\mu\tau} \\
\epsilon_{e\tau} & \epsilon_{\tau\mu} & \epsilon_{\tau\tau}
\end{pmatrix}, \quad V^\dagger V - 1 = \begin{pmatrix}
\tilde{\epsilon}_{11} & \tilde{\epsilon}_{12} & \tilde{\epsilon}_{13} \\
\tilde{\epsilon}_{21} & \tilde{\epsilon}_{22} & \tilde{\epsilon}_{23} \\
\tilde{\epsilon}_{31} & \tilde{\epsilon}_{32} & \tilde{\epsilon}_{33}
\end{pmatrix},
\]

(4)

where \( \epsilon_{\alpha\beta} = \epsilon_{\beta\alpha}^* \) (for \( \alpha, \beta = e, \mu, \tau \)) and \( \tilde{\epsilon}_{ij} = \tilde{\epsilon}_{ji}^* \) (for \( i, j = 1, 2, 3 \)) are two sets of small unitarity-violating parameters, and their indirect correlation can be established when a full parametrization of the 6 \( \times \) 6 flavor mixing matrix between three species of light Majorana neutrinos and three species of heavy Majorana neutrinos is made [25]. To see this point in an empirical way, one may simply parametrize \( V \) as \( V = (1 - \eta)U \), where \( U \) is exactly unitary and \( \eta \) is Hermitian and its matrix elements are all small in magnitude. Then \( \epsilon_{\alpha\beta} \simeq -2\eta_{\alpha\beta} \) and \( \tilde{\epsilon}_{ij} \simeq -2(U^\dagger\eta U)_{ij} \) hold in a good approximation. The magnitude of the deviation from unitarity is connected with the mass scale of heavy Majorana neutrinos by the approximate relation \( \epsilon_{\alpha\beta} \sim \tilde{\epsilon}_{ij} \sim \mathcal{O}(M^2_D/M^2_R) \) [26]. In some viable TeV-scale seesaw models (see, e.g., Ref. [27]) one may arrange \( M_D \sim \mathcal{O}(10^2) \) GeV and \( M_R \sim \mathcal{O}(10^3) \) GeV to achieve a percent level of unitarity-violating effect, although some significant structural cancellations in \( M_\nu \simeq -M_D M_R^{-1} M^T_D \) are unavoidable in this case. The mass scale of heavy sterile neutrinos in this work is much larger than the atomic energy transfer, and thus only three light neutrinos can be produced. In other words, the signature of those heavy degrees of freedom at low energies is indirectly reflected by the slight departure of the 3 \( \times \) 3 PMNS matrix \( V \) from unitarity.\(^1\)

Note that the unitarity-violating parameters \( \epsilon_{\alpha\beta} \) and \( \tilde{\epsilon}_{ij} \) are not fully independent. They are

\(^1\)Note that our work is apparently different from the one done in Ref. [28], where a light sterile neutrino species of the \( \mathcal{O}(1) \) eV mass scale as indicated by the short-baseline neutrino oscillation anomaly has been considered. Such a light sterile neutrino can be directly generated in the atomic system via its mixing with the active neutrinos, and hence it violates unitarity of the 3 \( \times \) 3 PMNS matrix in a direct way.
connected with each other via the relation
\[ \epsilon_{\alpha \beta} = \sum_{i=1}^{3} \sum_{j=1}^{3} U_{\alpha i}^* U_{\beta j} \tilde{\epsilon}_{ij}. \] (5)

Given the currently available neutrino oscillation data, precision measurements of electroweak interactions, and stringent constraints on lepton universality and lepton flavor violation, it is found that the upper bounds of \(|\epsilon_{\alpha \beta}|\) and \(|\tilde{\epsilon}_{ij}|\) are at most of order \(5 \times 10^{-3}\) at the 90% confidence level [6, 29–31]. If only the neutrino oscillation data are taken into account, then much looser upper bounds \(|\epsilon_{\alpha \beta}| \ll \mathcal{O}(0.1)\) and \(|\tilde{\epsilon}_{ij}| \ll \mathcal{O}(0.1)\) can be achieved [32]. The subsequent part of this paper will be devoted to illustrating the effects of indirect unitarity violation of \(V\), as described by \(\epsilon_{\alpha \beta}\) and \(\tilde{\epsilon}_{ij}\), on the RENP process in an atomic system.

As for the RENP process \(|e\rangle \rightarrow |g\rangle + \gamma + \nu_i + \nu_j\), there totally exist six thresholds in the fine structure of the outgoing photon energy spectrum due to the finite neutrino masses which are located in the case of vanishing boost \((\mathbf{p}_{eg} = 0)\) at the frequencies \(^2\)
\[ \omega_{ij} = \frac{E_{eg}}{2} - \frac{(m_i + m_j)^2}{2E_{eg}}. \] (6)

One may calculate the rate of such a RENP process with the help of Eq. (3). An external laser with the frequency \(\omega\) can be used to trigger the transition, and the result for its rate can be factorized into the expression [33–35]
\[ \frac{dN_{\gamma}(\omega)}{dt} = 6G_F^2 |V_{\text{tar}}|^3 (2J_p + 1) C_{ep} \frac{E_{eg}}{E_{vg}} I(\omega) \eta_{\omega}(t), \] (7)
where \(V_{\text{tar}}\) represents the target volume, \(n\) stands for the number density of target atoms, \(2J_p + 1\)\(C_{ep}\) denotes the spin factor of the transition, \(E_{vg}\) (or \(\gamma_{vg}\)) is the energy difference (or the dipole strength) between the atomic levels \(|v\rangle\) and \(|g\rangle\), and \(\eta_{\omega}(t)\) is the dynamical factor which quantifies the level of coherence of the medium. The detailed values of these atomic parameters can be found in Table 9 of Ref. [34]. In the following we shall take the ytterbium (Yb) atomic levels, for which \(E_{eg} = 2.14349\) eV and \(E_{vg} = 2.23072\) eV, as an example to show the unitarity-violating effect. Information about the neutrino properties is hidden in the spectrum function
\[ I(\omega) = \frac{1}{(\omega - E_{vg})^2} \sum_{i=1}^{3} \sum_{j=1}^{3} \Delta_{ij}(\omega) \left[ |a_{ij}|^2 I_{ij}(\omega) - m_i m_j \text{Re}(a_{ij}^2) \right] \Theta(\omega_{ij} - \omega), \] (8)
in which the PMNS coefficients \(a_{ij} \equiv V_{ei}^* V_{ej} - (V^\dagger V)_{ij}/2\) (for \(i, j = 1, 2, 3\)) have been defined below Eq. (3). \(\Theta(\omega_{ij} - \omega)\) denotes the Heaviside function which signifies the kinematic

\(^2\)Note that the threshold frequencies will be altered in the boosted RENP scenario [24], which is very interesting for a further study. In the present work we focus our attention on putting forward and illustrating our particle-physics idea by assuming a vanishing boost \(\mathbf{p}_{eg} = 0\).
threshold under consideration, and

\[
\Delta_{ij}(\omega) = \frac{\sqrt{[E_{eg} (E_{eg} - 2\omega) - (m_i + m_j)^2] [E_{eg} (E_{eg} - 2\omega) - (m_i - m_j)^2]} }{E_{eg} (E_{eg} - 2\omega)},
\]

\[
I_{ij}(\omega) = \frac{1}{3} \left[ E_{eg} (E_{eg} - 2\omega) + \frac{1}{2} \omega^2 - \frac{1}{6} \omega^2 \Delta_{ij}^2(\omega) - \frac{1}{2} (m_i^2 + m_j^2) - \frac{1}{2} \frac{E_{eg}^2 - \omega^2}{E_{eg} (E_{eg} - 2\omega)^2} (m_i^2 - m_j^2)^2 \right].
\]  

Note that the terms proportional to \(m_i m_j\) in Eq. (8) exist only for the Majorana neutrinos, but they are strong suppressed by the smallness of \(m_i\) and \(m_j\). These tiny terms will be neglected in the subsequent discussions, because we are mainly concerned about how the spectrum function gets distorted under the unitarity violation of \(V\) in this work.

The total spectrum in Eq. (8) is linearly composed of the sub-spectra with six different endpoints, denoted as \(\omega_{11}, \omega_{12}, \omega_{22}, \omega_{13}, \omega_{23}\) and \(\omega_{33}\). The location of the stimulating trigger frequency with respect to those thresholds will be found to be very important for us to obtain a high sensitivity to the unitarity violation of \(V\). The six thresholds can be classified into three major categories: \(\omega_1 = (\omega_{11}, \omega_{12}, \omega_{22})\), \(\omega_II = (\omega_{13}, \omega_{23})\) and \(\omega_III = \omega_{33}\) due to the fact of \(\Delta m_{21}^2 \simeq 7.39 \times 10^{-5} \text{ eV}^2 \ll |\Delta m_{31}^2| \simeq |\Delta m_{32}^2| \simeq 2.525 \times 10^{-3} \text{ eV}^2\) extracted from a global analysis of current neutrino oscillation data [36,38]. Given the normal ordering (NO) of three neutrino masses with \(m_1 = 0.05 \text{ eV}\), for example, the energy gap between two different categories of the thresholds is \(\omega_III - \omega_II \simeq \omega_II - \omega_1 \sim 10^{-3} \text{ eV}\), but the one within the same category (e.g., \(\omega_{11} - \omega_{22}\)) is of order \(\lesssim 10^{-4} \text{ eV}\).

The first step of a RENP experiment might be to pin down the absolute neutrino mass
scale — the value of $m_1$ to a reasonable degree of accuracy. After $m_1$ is measured, all the thresholds $\omega_{ij}$ can then be located in the photon spectrum by inputting the known values of $\Delta m^2_{21}$ and $\Delta m^2_{31}$. On the other hand, the approximate locations of the thresholds might be directly determined by a rough scanning of the *kink* structure of the photon spectrum [35]. To resolve the gap between any two different categories of $\omega_{ij}$, the precision of the laser frequency should be better than the energy gap $\sim 10^{-3}$ eV. Meanwhile, it is more challenging to resolve the tiny gaps within the same category. We are going to show that one can have a much better sensitivity to the unitarity violation of $V$ if the laser frequency is chosen to be well separated from the thresholds inside $\omega_1$ or $\omega_{1i}$, such as $\Delta \omega_i/(\omega_{1i} - \omega) \ll 1$, where $\Delta \omega_i$ denotes the energy gap inside $\omega_1$, and $\omega_{1i}$ is just one of the thresholds within category I. Under the above condition one may follow a perturbative analysis to verify the approximate equalities $\Delta_{11} \simeq \Delta_{12} \simeq \Delta_{22}$ and $I_{11} \simeq I_{12} \simeq I_{22}$ up to a relative difference of the same order as the small quantity $\Delta \omega_i/(\omega_{1i} - \omega)$. A similar observation can be achieved for the quantities associated with $\omega_{1i}$. In Fig. 2 we plot the functions $\Delta_{ij}$ and $I_{ij}$ for the NO case with $m_1 = 0.05$ eV, from which one can see the accuracy of the above approximations. Since the NO seems to be favored over the inverted ordering (IO) of three neutrino masses at the $3\sigma$ level [36][38], we only consider the NO case to numerically illustrate our idea and method in this work [3].

Given the above conditions for the three thresholds of category $\omega_{1i}$, the emission channels of all the $\nu_1$ and $\nu_2$ combinations contribute to the total photon spectrum by an amount of

$$I_1 \approx \frac{1}{(\omega - E_{\text{vg}})^2} \Delta_{11}(\omega) I_{11}(\omega) (|a_{11}|^2 + 2|a_{12}|^2 + |a_{22}|^2)$$

$$= \frac{1}{(\omega - E_{\text{vg}})^2} \Delta_{11}(\omega) I_{11}(\omega) \left[ \frac{1}{2} - |V_{e3}|^2 + |V_{e3}|^4 + (1 - 2|V_{e3}|^2) \epsilon_{ee} + \left( \frac{1}{2} - |V_{e1}|^2 \right) \bar{\epsilon}_{11} \right]$$

$$+ \left( \frac{1}{2} - |V_{e2}|^2 \right) \bar{\epsilon}_{22} - 2\text{Re}(V_{e1}^* V_{e2} \bar{\epsilon}_{12}) + \epsilon_{ee} + \frac{\bar{\epsilon}_{11}^2}{4} + \frac{\bar{\epsilon}_{12}^2}{2} + \frac{\bar{\epsilon}_{22}^2}{4} \right].$$

(10)

If the unitarity-violating parameters $\epsilon_{\alpha\beta}$ and $\bar{\epsilon}_{ij}$ are switched off (i.e., $V \rightarrow U$), then $I_1$ will be only dependent on the most accurately measured PMNS matrix element $|U_{e3}|$. This makes it easier to pin down the unitarity-violating contribution to $I_1$, because this kind of new-physics effect is expected to be very small and hence easily contaminated by the uncertainties associated with the PMNS matrix elements. When the channels with the thresholds $\omega_{13}$ and $\omega_{23}$ are concerned, the spectrum function receives additional contributions of the form

$$I_{\Pi} \approx \frac{2}{(\omega - E_{\text{vg}})^2} \Delta_{13}(\omega) I_{13}(\omega) (|a_{13}|^2 + |a_{23}|^2)$$

$$= \frac{2}{(\omega - E_{\text{vg}})^2} \Delta_{13}(\omega) I_{13}(\omega) \left[ |V_{e3}|^2 - |V_{e3}|^4 + |V_{e3}|^2 \epsilon_{ee} - \text{Re}(V_{e1}^* V_{e3} \epsilon_{13}) - \text{Re}(V_{e2}^* V_{e3} \epsilon_{23}) \right]$$

$$+ \frac{|\epsilon_{13}|^2}{4} + \frac{|\epsilon_{23}|^2}{4} \right].$$

(11)

\[3\] In fact, we find that the photon spectrum function has a very similar behavior in the IO case, and thus we shall not discuss this case in detail for the sake of simplicity.
Once again $I_{II}$ will depend only on the matrix element $|V_{e3}| = |U_{e3}|$ in the unitarity limit. In particular, all the terms of $I_{II}$, except for the $O(\epsilon_{ij}^2)$ terms, are suppressed by the smallness of $|V_{e3}|$. That is why the emission rates of the neutrino pairs $\nu_1 + \nu_3$ and $\nu_2 + \nu_3$ are insignificant as compared with the other channels. When the photon energy becomes smaller than $\omega_{33}$, a contribution of the $\nu_3 + \bar{\nu}_3$ emission to the photon spectrum reads

$$I_{III} \approx \frac{1}{(\omega - E_{vg})^2} \Delta_{33}(\omega) I_{33}(\omega) \left[ \frac{3}{4} + \epsilon_{ee} + \frac{1}{2} \sum_{i=1}^{3} \bar{\epsilon}_{ii} - \sum_{i=1}^{3} \sum_{j=1}^{3} \text{Re}(V_{ei}^* V_{ej} \bar{\epsilon}_{ij}) + \epsilon_{ee}^2 + \frac{1}{4} \sum_{i=1}^{3} \sum_{j=1}^{3} |\bar{\epsilon}_{ij}|^2 \right],$$

(12)

Furthermore, if the photon frequency is chosen to be far away from all the six thresholds (e.g., $\omega \to 0$ eV) such that $\Delta \omega / (\omega_{ij} - \omega) \ll 1$ with $\Delta \omega \lesssim 10^{-3}$ eV denoting the energy difference within the six thresholds, one will be left with the approximate equalities $\Delta_{ij} \approx \Delta_{11}$ and $I_{ij} \approx I_{11}$ (for $i, j = 1, 2, 3$). In this case the contributions of all the six thresholds can be summed up as follows:

$$I_{\text{tot}} \approx \frac{1}{(\omega - E_{vg})^2} \Delta_{11}(\omega) I_{11}(\omega) \left[ \frac{3}{4} + \epsilon_{ee} + \frac{1}{2} \sum_{i=1}^{3} \bar{\epsilon}_{ii} - \sum_{i=1}^{3} \sum_{j=1}^{3} \text{Re}(V_{ei}^* V_{ej} \bar{\epsilon}_{ij}) \right],$$

(13)

in which the leading term is simply a constant, corrected by small unitarity-violating terms.

The above analytical results tell us that the RENP process is sensitive to the unitarity-violating parameters $\epsilon_{ee}$, $\bar{\epsilon}_{11}$, $\bar{\epsilon}_{22}$, $\bar{\epsilon}_{12}$, $\bar{\epsilon}_{13}$, $\bar{\epsilon}_{23}$ and $\bar{\epsilon}_{33}$. Taking account of Eq. (5), we find that $\epsilon_{ee}$ can actually be expressed as a linear combination of $\bar{\epsilon}_{ij}$ (for $i, j = 1, 2, 3$):

$$\epsilon_{ee} = \sum_{i=1}^{3} \sum_{j=1}^{3} |U_{ei} U_{ej}^* \bar{\epsilon}_{ij}| \cos (\phi_i - \phi_j + \phi_{ij}),$$

(14)

where $\phi_i \equiv \text{arg}(U_{ei})$ (for $i = 1, 2, 3$) and $\phi_{ij} \equiv \text{arg}(\bar{\epsilon}_{ij})$ (for $i, j = 1, 2, 3$). In the subsequent numerical analysis we shall take $\bar{\epsilon}_{ij}$ as the original unitarity-violating parameters, and determine the value of $\epsilon_{ee}$ by specifying the relevant matrix elements of $U$ and $\bar{\epsilon}$ including their phases. Note that $V = U \sqrt{1 + \bar{\epsilon}} \simeq U (1 + \bar{\epsilon}/2 - \bar{\epsilon}^2/8)$ holds. So the parameters of $U$ should also be input when calculating the photon spectrum of a RENP process.

We decompose the contributions of different neutrino-pair emissions to the total photon spectrum with unitarity violation in Fig. 3 where the solid black curve signifies the total photon spectrum, and the other curves represent the contributions from explicit neutrino-pair combinations. The best-fit values of two neutrino mass-squared differences and three flavor mixing angles of $U$ in the NO case have been taken as the inputs [38]: $\theta_{12} = 33.8^\circ$, $\theta_{13} = 8.61^\circ$, $\theta_{23} = 49.6^\circ$, $\Delta m_{21}^2 = 7.39 \times 10^{-5}$ eV$^2$ and $\Delta m_{31}^2 = 2.525 \times 10^{-3}$ eV$^2$. The vertical dash-dotted lines in Fig. 3 correspond to all the thresholds in the spectrum function. It is clear that the emissions of $\nu_1 + \bar{\nu}_2$, $\nu_2 + \bar{\nu}_1$ and $\nu_3 + \bar{\nu}_3$ dominate the total spectrum, and this observation has already been noticed in Ref. [33]. Such a result can be obtained for two
Figure 3: The photon spectrum function $I(\omega)$ (black solid curves) and the contributions of different neutrino pairs (gray dashed curves) for $m_1 = 0.05\,\text{eV}$ (left panel) or $m_1 = 0\,\text{eV}$ (right panel), where the normal ordering (NO) of three neutrino masses with $\theta_{12} = 33.82^\circ$, $\theta_{13} = 8.61^\circ$ and $\theta_{23} = 49.6^\circ$ of $U$ has been taken. The vertical dash-dotted lines simply signify all the thresholds in the spectrum function.

Simple reasons: (i) the contributions from $\nu_1 + \bar{\nu}_3$ (or $\nu_3 + \bar{\nu}_1$) and $\nu_2 + \bar{\nu}_3$ (or $\nu_3 + \bar{\nu}_2$) are suppressed by the smallness of $|V_{e3}|$, as shown in Eq. (11); (ii) the emissions of $\nu_1 + \bar{\nu}_1$ and $\nu_2 + \bar{\nu}_2$ are suppressed by the small factors $(|U_{e1}|^2 - 1/2)^2 \simeq 0.03$ and $(|U_{e2}|^2 - 1/2)^2 \simeq 0.04$.

Now let us illustrate the overall unitarity-violating effects without assuming any special values of the relevant parameters. In Fig. 4 we require that $|\tilde{\epsilon}_{ij}|$ (for $i, j = 1, 2, 3$) vary in the range of $[0 \cdots 0.05]$ (orange bands) or $[0 \cdots 0.01]$ (red bands), all the relevant phases of $\tilde{\epsilon}$ and $U$ vary in the range of $[0 \cdots 2\pi]$, and the mixing angles of $U$ vary in their $3\sigma$ ranges as indicated by the global-fit results (i.e., $\theta_{12} \in [31.61^\circ \cdots 36.27^\circ]$, $\theta_{13} \in [8.22^\circ \cdots 8.99^\circ]$ and $\theta_{23} \in [40.3^\circ \cdots 52.4^\circ]$). In the left panels of Fig. 4 the photon spectra with respect to the whole range of $\omega$ (from 0 eV to its largest threshold) have been shown. The upper-left panel stands for the case with $m_1 = 0.05\,\text{eV}$, and the lower-left panel corresponds to the case with $m_1 = 0\,\text{eV}$. One can see that these two cases are almost indistinguishable for very small values of $\omega$. This observation is consistent with Eqs. (8) and (9). As $\omega \to 0$, the effect of neutrino masses becomes negligible in comparison with the atomic energy scale $E_{eg} \simeq 2\,\text{eV} \gg m_i$ for Yb. In the right panel of Fig. 4 we zoom into the energy region near the kinematical thresholds, around which one may see some more details. The standard case without unitarity violation is shown as the much thinner black band. Appreciable unitarity-violating effects can be observed even if the uncertainties of all the PMNS neutrino mixing matrix elements are taken into account.

To quantify the experimental requirement for reaching a given sensitivity of the unitarity violation of $V$ in measuring the RENP process for an atomic system, let us follow Refs. [34]...
Figure 4: An illustration of sensitivities of the photon spectrum function $I(\omega)$ to the unitarity-violating effects of $V$ in the photon energy ranges of $\omega \in [0 \cdots 1.2]$ eV (upper-left and lower-left panels) and $\omega \in [1.06 \cdots 1.075]$ eV (upper-right and lower-right panels). The very thin black bands represent the spectra assuming $V$ to be unitary, while the much wider orange (or red) bands are produced by allowing $\tilde{\epsilon}_{ij}$ (for $i,j = 1,2,3$) to vary in the range of $[0 \cdots 0.05]$ (or $[0 \cdots 0.01]$) with arbitrary phases and by inputting the $3\sigma$ ranges of three flavor mixing angles of $U$ taken from Ref. [38]. The vertical dash-dotted lines signify all the thresholds in the spectrum function.

and [35] to define the rate normalization factor

$$N_{\text{norm}} = \left( \frac{T}{s} \right) \left( \frac{V_{\text{tar}}}{10^2 \text{ cm}^3} \right) \left( \frac{n}{10^{21} \text{ cm}^{-3}} \right)^3 \eta_{\omega} \cdot$$

(15)

The event number can then be determined by using Eq. (7) as $N_{\text{event}} \approx 0.002 \times N_{\text{norm}} \times I(\omega)$ for any given observation time $T$ at the frequency $\omega$ and values of the target volume $V_{\text{tar}}$, target number density $n$ and dynamical factor $\eta_{\omega}$. To break the degeneracy of effects of the
unitarity violation, the uncertainties of the PMNS matrix elements and the uncertainty of the experimental parameter $N_{\text{norm}}$, the trigger laser may be set to scan the following frequencies: $\omega = \{0.1 \, \text{eV}, (\omega_I + \omega_{II})/2, (\omega_{II} + \omega_{III})/2\}$. The medians of different categories of the thresholds have been chosen to minimize the uncertainties from the PMNS matrix elements. An experimental sensitivity to the unitarity-violating parameters $\tilde{\epsilon}_{ij}$ can be obtained by minimizing the chi-square function

$$
\chi^2(\tilde{\epsilon}_{ij}, \theta_{ij}, N_{\text{norm}}) = \chi^2_{\text{osc}}(\theta_{ij}) + \sum_\omega \frac{[N_{\text{event}}(\tilde{\epsilon}_{ij}) - N_{\text{event}}(0)]^2}{N_{\text{event}}(0)} \tag{16}
$$

with respect to $\theta_{ij}$ and $N_{\text{norm}}$. Here $N_{\text{event}}(\tilde{\epsilon}_{ij})$ (or $N_{\text{event}}(0)$) stands for the event number with (or without) unitarity violation, and $\chi^2_{\text{osc}}(\theta_{ij})$ includes the experimental information about the neutrino mixing angles taken from the global-fit results [38]. To roughly reach a $3\sigma$ sensitivity to $|\tilde{\epsilon}_{ij}| \lesssim \mathcal{O}(0.01)$ (for $i, j = 1, 2, 3$), which is equivalent to $\Delta \chi^2 = 9$, we find that $N_{\text{norm}} \gtrsim \mathcal{O}(10^9)$ is required. Similarly, $N_{\text{norm}} \gtrsim \mathcal{O}(10^{11})$ is needed in order to reach the $3\sigma$ sensitivity to $|\tilde{\epsilon}_{ij}| \lesssim \mathcal{O}(10^{-3})$.

3 Summary

We have studied the possibility of testing unitarity of the $3 \times 3$ PMNS lepton flavor mixing matrix $V$ in an atomic system with the intriguing RENP process. The spectrum of the emitted photons will be distorted by the unitarity-violating effects of $V$. We find that in certain regions of the trigger frequency only the smallest and best-measured PMNS matrix element $|V_{e3}|$ contributes to the leading-order term of the transition rate, and in some regions the leading-order term is even independent of the PMNS matrix elements. This observation is greatly helpful to enhance the sensitivity of the RENP process to indirect unitarity violation of $V$. The distortion of the photon spectrum for the Yb atomic levels has been illustrated by taking into account a reasonable parameter space.

As the SPAN group is gradually making progress in an experimental realization of the RENP process [39–42], the potential to probe or constrain possible unitarity violation of the PMNS matrix in the atomic system may be very promising in the foreseeable future. The idea and methodology described here can also be applied to the Raman-stimulated neutrino pair emission [43] in a similar atomic system.

We stress that the interplay between atomic physics and particle physics provides us with a new opportunity to explore new physics hidden at a high energy scale by using some new techniques at low energies, although this kind of endeavor is always challenging. Our present work has added a new example in this connection, to illustrate how to implement an indirect test of the canonical seesaw mechanism by probing possible unitarity violation of the PMNS matrix in an atomic system. Some further and more systematic studies along this line of thought will be carried out later on.
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