Thermodynamics of de Sitter black holes in massive gravity  
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Abstract. In this paper, by taking de Sitter space-time as a thermodynamic system, we study the effective thermodynamic quantities of de Sitter black holes in massive gravity, and furthermore obtain the effective thermodynamic quantities of the space-time. Our results show that the entropy of this type of space-time takes the same form as that in Reissner-Nordström-de Sitter space-time, which lays a solid foundation for deeply understanding the universal thermodynamic characteristics of de Sitter space-time in the future. Moreover, our analysis indicates that the effective thermodynamic quantities and relevant parameters play a very important role in the investigation of the stability and evolution of de Sitter space-time.

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1 Introduction

The study of the thermodynamic characteristics of de Sitter space-time has aroused extensive attention in the recent years \cite{121}. At the stage of cosmological inflation in the early time, our universe behaves like a quasi-de Sitter space-time, in which the cosmological constant takes the form of vacuum energy. Moreover, if the dark energy is simply a cosmological constant, i.e., a component with constant equation of state, our universe will evolve into a new stage of de Sitter space-time in this simplest scenario. Therefore, a better knowledge of de Sitter space-time (especially its classical and quantum characteristics) is very important to construct the general framework of cosmic evolution. In the previous works, the black hole horizon and the cosmological horizon are always treated as two independent thermodynamic systems \cite{1710,1113}, from which the thermodynamic volume of de Sitter space-time as well as the corresponding thermodynamic quantities satisfying the first thermodynamics law were obtained \cite{13}. It is commonly recognized that the entropy of de Sitter space-time is the sum of that for the two types of horizons \cite{1113}, however, such statement concerning the nature of de Sitter space-time entropy still needs to be checked with adequate physical explanation.

Considering the fact that all thermodynamic quantities related to the black hole horizon and the cosmological horizon in de Sitter space-time can be expressed as a function of mass $M$, electric charge $Q$, and cosmological constant $\Lambda$, it is natural to consider the dependency between the two types of thermodynamic quantities. More specifically, the discussion of the following two problems is very significant to study the stability and evolution of de Sitter space-time: Do the thermodynamic quantities follow the behavior of their counterparts in AdS black holes, especially when the black hole horizon is correlated with that of the cosmological horizon? What is the specific relation between the entropy of de Sitter space-time and that of the two horizons (the black hole horizon and the cosmological horizon)? The above two problems also provide the main motivation of this paper.

Following this direction, in our analysis we obtain the effective thermodynamic quantities of de Sitter black holes in massive gravity (DSBHMG), based on the correlation between the black hole horizon and the cosmological horizon. Our results show that the entropy of this type of space-time takes the same form as that in Reissner-Nordström de Sitter space-time, which lays a solid foundation for deeply understanding the universal thermodynamic characteristics of de Sitter space-time in the future. This paper is organized as follows. In Sec.II, we briefly introduce the thermodynamic quantities of the horizons of black holes and the Universe in DSBHMG, and furthermore obtain the electric charge $Q$ when the two horizons show the same radioactive temperature. In Sec.III, taking the correlation between the two horizons into consideration, we will present the equivalent thermodynamic quantities of DSBHMG satisfying the first thermodynamic law, and perform a quantitative analysis of the corresponding effective temperature and pressure. Finally, the main conclusions are summarized in Sec.IV. Throughout the paper we use the units $G = \hbar = k_B = c = 1$.  

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2 Thermodynamics of black holes in massive gravity

In the framework of (3 + 1)-dimensional massive gravity with a Maxwell field (denoting \( F_{\mu\nu} \) as the Maxwell field-strength tensor), the corresponding action always expresses as \([22, 25]\),

\[
S = \frac{1}{2k^2} \int d^4x \sqrt{-g} \left[ R - 2\Lambda - \frac{1}{4} F^2 + m^2 \sum \frac{c_i}{4} \right],
\]

Here \( k = +1, 0, -1 \) respectively correspond to the sphere, plane, and hyperbola symmetric cases; \( \Lambda \) is the cosmological constant and \( \mu_i \) represents the contribution of the matrix \( \sqrt{g^{\mu\nu}} F_{\mu\nu} \) with fixed symmetric tensor \( f_{\mu\nu} \). Therefore, generated form the above action, the space-time metric of static black holes (denoting \( h_{ij} \) as Einstein space with constant curvature) can be written as

\[
ds^2 = -f(r)dt^2 + f^{-1}dr^2 + r^2 h_{ij}dx^i dx^j,
\]

with the metric function expressed as \([26, 27]\)

\[
f(r) = k - \frac{\Lambda}{3} r^2 - \frac{m_0}{r} + \frac{q^2}{4r^2} + \frac{c_1 m^2}{2} r + m^2 c_2
\]

Note that the positions of black hole horizon \( r_+ \) and cosmological horizon \( r_c \) are determined when \( f(r_{+,c}) = 0 \).

![Metric function f(r) varying with r.](image)

In Fig. 1 we display the behavior of the metric function \( f(r) \), where the parameters are chosen as \( \Lambda = 1, m_0 = 30, m = 2.12, c_1 = 2, c_2 = 3.18, q = 1.7 \), while \( k \) is fixed at 1, 0, and 1. It is obvious that there are two intersection points between \( f(r) \) and the axis of \( r \), which respectively correspond to the positions of black hole horizon \( r_+ \) and cosmological horizon \( r_c \). Thus, the mass \( m_0 \) can be expressed in terms of \( r_{+,c} \) as

\[
M = \frac{m_0}{2} = \frac{(k + m^2 c_2) r_c x (1 + x)}{2(1 + x + x^2)} + \frac{q^2 (1 + x^2) (1 + x^2)}{8r_c x (1 + x + x^2)} + \frac{r_c^2 m^2 c_1 x^2}{4(1 + x + x^2)}
\]

and

\[
\frac{\Lambda}{3} r^2 (1 + x + x^2) = k - \frac{q^2}{4r^2} + \frac{c_1}{2} m^2 r_c (1 + x) + m^2 c_2
\]

where \( x = r_+ / r_c \). The temperature of the black hole horizons and cosmic horizon can be written as \([28]\)

\[
T_{+,c} = \frac{\pm f(r_{+,c})}{4\pi} = \frac{1}{4\pi r_{+,c} [k - \Lambda r^2 \pm \frac{q^2}{4r^2} + m^2 c_1 r_{+,c} + m^2 c_2]}
\]

Turning to the contribution of the electrical charge \( q \), it will also generate a chemical potential as

\[
\mu_{+,c} = \frac{q}{r_{+,c}}
\]

According to the Hamiltonian approach, we have the mass \( M \) and electric charge \( Q \) as

\[
M = \frac{\nu_2 m_0}{8\pi}, \quad Q = \frac{\nu_2 q}{4\pi}
\]

and the entropy of the two horizons respectively express as

\[
S_{+,c} = \frac{\nu_2}{4} r^2_{+,c}
\]

where \( \nu_2 \) is the area of a unit volume of constant \((t, r)\) space (which equals to 4\( \pi \) for \( k = 0 \)). It is apparent that the thermodynamic quantities corresponding to the two horizons satisfy the first law of thermodynamics

\[
dM = T_{+,c} dS_{+,c} + V_{+,c} dP + \mu_{+,c} dQ
\]

where

\[
V_{+,c} = \frac{\nu_2}{3} r^3_{+,c}, \quad P = \frac{\Lambda}{8\pi}
\]

When the temperature of the black hole horizon is equal to that of the cosmological horizon, the electric charge \( Q \) and the cosmological constant \( \Lambda \) are related as

\[
\frac{1}{r_+} [k - \Lambda r^2_+ - \frac{q^2}{4r^2_+} + m^2 c_1 r_+ + m^2 c_2] = -\frac{1}{r_c} [k - \Lambda r^2_c - \frac{q^2}{4r^2_c} + m^2 c_1 r_c + m^2 c_2]
\]

As can be seen from Eq. (5) and (12) the electric charge of the system satisfies the following expression

\[
\frac{q^2 (1 + x)^2}{4r^2_{c}x^2} = k + \frac{c_1 m^2 r_c x}{2(1 + x)} + m^2 c_2.
\]

When taking \( T_+ = T_c \), the combination of Eqs. (10), (11) and (13) will lead to the temperature \( T \) as

\[
T = T_+ = T_c = \frac{(1 - x)}{2\pi r_c (1 + x)^2} \left[ k + \frac{m^2 c_1 r_c (1 + 4x + x^2)}{4(1 + x)} + m^2 c_2 \right]
\]
3 Effective thermodynamic quantities

Considering the connection between the black hole horizon and the cosmological horizon, we can derive the effective thermodynamic quantities and corresponding first law of black hole thermodynamics as

$$dM = T_{\text{eff}} dS - P_{\text{eff}} dV + \phi_{\text{eff}} dQ,$$

(15)

where the thermodynamic volume is defined by [3, 5, 6, 29]

$$V = \frac{4\pi}{3} (r_c^3 - r_+^3).$$

(16)

It is obvious that there exit three real roots for the equation $f(r) = 0$: the cosmological horizon (CEH) $r = r_c$, the inner (Cauchy) horizon of black holes, and the outer horizon (BEH) $r = r_+$ of black holes. Moreover, the de Sitter space-time is characterized by $\Lambda > 0$, while $\Lambda < 0$ denotes the anti-de Sitter scenario.

Now we will consider the interaction between the black hole horizon and the cosmological horizon [30, 31]

$$S = \pi br_c^2 \left[ 1 + x^2 + f(x) \right],$$

(17)

Here the undefined function $f(x)$ represents the extra contribution from the correlations of the two horizons. From Eq. (15), we can obtain the effective temperature $T_{\text{eff}}$ and pressure $P_{\text{eff}}$

$$T_{\text{eff}} = \left( \frac{\partial M}{\partial S} \right)_{Q,V} = \left( \frac{\partial M}{\partial S} \right)_{r_c} \left( \frac{\partial V}{\partial r_c} \right)_x - \left( \frac{\partial V}{\partial S} \right)_{r_c} \left( \frac{\partial M}{\partial r_c} \right)_{x}.$$

(18)

$$P_{\text{eff}} = -\left( \frac{\partial M}{\partial V} \right)_{Q,S} = -\left( \frac{\partial M}{\partial V} \right)_{r_c} \left( \frac{\partial S}{\partial r_c} \right)_x - \left( \frac{\partial S}{\partial V} \right)_{r_c} \left( \frac{\partial M}{\partial r_c} \right)_x.$$

(19)

Combining Eqs. (13), (10) and (17), one can obtain

$$T_{\text{eff}} = \frac{B(x,q)}{2\pi r_c [2x^2(1 + x^2 + f(x)) + (1 - x^2)(2x + f'(x))]},$$

(20)

where

$$B(x,q) = \frac{(k + m^2 c_2)}{(1 + x + x^2)} \left[ (1 + x + x^2) + (1 + x + x^2)^2 \right]$$

$$- \frac{q^2}{4r_c^2} \left( 2 + x \right) \left( 1 + x + x^2 \right) \left( 2x + f'(x) \right)$$

$$+ \frac{r_c m^2 c_1 x}{2(1 + x + x^2)}.$$  

(21)

$$P_{\text{eff}} = \frac{D(x,q)}{8\pi r_c^2 [2x^2(1 + x^2 + f(x)) + (1 - x^2)(2x + f'(x))]},$$

(22)

$$D(x,q) = \frac{(k + m^2 c_2)}{(1 + x + x^2)^2} \left\{ (1 + 2x)(1 + x^2 + f(x)) \right\}$$

$$- \frac{q^2}{4r_c^2} \left( 2 + x \right) \left( 1 + x + x^2 \right) \left( 2x + f'(x) \right)$$

$$+ \frac{r_c m^2 c_1 x}{2(1 + x + x^2)}.$$  

(23)

When the temperature of the black hole horizon is equal to that of the cosmological horizon, the effective temperature of the space-time should be

$$T_{\text{eff}} = T = T_c = T_+. $$

(24)

Then substituting Eq. (13) into Eq. (20), we get

$$B(x) = \frac{(1 - x)}{(1 + x)^2} \left[ k + m^2 c_2 + \frac{r_c m^2 c_1 (1 + 4x^2)}{4(1 + x)} \right] (1 + x + x^2).$$

(25)

Then Eq. (25) will transform into

$$f'(x) + \frac{2x^2}{1 - x^3} f(x) = \frac{2x^2(2x^3 + x^2 - 1)}{(1 - x^3)^2}. $$

(26)

with the corresponding solution as

$$f(x) = -\frac{2 \left( 4 - 5x^3 - x^5 \right)}{5(1 - x^3)} + C_1 (1 - x^3)^{2/3}. $$

(27)

Considering the initial condition of $f(0) = 0$, we can obtain $C_1 = 8/5$ and inserting Eq. (25) into Eqs. (20) and (22) will lead to

$$T_{\text{eff}} = \frac{B(x,q)(1 - x^3)}{4\pi r_c x (1 + x^2)}, \quad P_{\text{eff}} = \frac{D(x,q)(1 - x^3)}{16\pi r_c^2 x (1 + x^4)}. $$

(28)

Based on the above equations, the $P_{\text{eff}} - x$ and $T_{\text{eff}} - x$ diagrams could be derived by taking different value of $k$, $q$, $m$, $c_1$ and $c_2$ (when taking $r_c = 1$).
In Fig. 2 and 3, we illustrate an example of the $P_{\text{eff}} - x$ diagram with different value of relevant parameters, from which one could clearly see the effect of these parameters on the effective pressure of RN-dSQ space-time. Following the same procedure by inserting Eq. (28) into Eq. (17), we can also obtain the $S(x) - x$ and $f(x) - x$ diagrams with $r_c = 1$, which are explicitly shown in Fig. 4. Similarly, in Fig. 5 and 6, we show the evolution of the $T_{\text{eff}} - x$ diagram with different value of relevant parameters, from which one could perceive the effect of these parameters on the effective temperature of RN-dSQ space-time.

Moreover, it is shown that the change of these related parameters may also significantly affect the position of the stability and phase-transition points, which can be clearly seen from the results presented in Table 1 and 2.

| Parametric | $x_0$ | $T_{\text{eff}}$ | $x_0$ |
|------------|-------|-----------------|-------|
| $k = 1$    | 0.3374| 1.7554          | 0.2173|
| $k = 0$    | 0.3468| 1.5799          | 0.2243|
| $k = -1$   | 0.3571| 1.4121          | 0.2321|

Table 1. Summary of the highest effective temperature $T_{\text{eff}}$ and the corresponding $x_0$ for different curves in Fig. 5. The value of $x_0$ when the effective temperature reaches zero is also listed.

4 Conclusion and discussion

In this paper, by taking de Sitter space-time as a thermodynamic system, we study the effective thermodynamic quantities of de Sitter black holes in massive gravity, and
| Parametric | $x_c$ | $T^\text{eff}_c$ | $x_0$ |
|-----------|------|----------------|------|
| $k = -1$  | $q = 1.0$ | 0.22269 | 2.92226 | 0.13498 |
|           | $q = 1.7$ | 0.35710 | 1.41210 | 0.23210 |
|           | $q = 2.7$ | 0.53108 | 0.56870 | 0.38043 |
| $k = -1$  | $m = 1.6$ | 0.46546 | 0.33960 | 0.32108 |
|           | $m = 2.12$ | 0.35710 | 1.41210 | 0.22310 |
|           | $m = 3.12$ | 0.24946 | 5.61153 | 0.15324 |
| $k = -1$  | $c_1 = 1.0$ | 0.36373 | 1.23166 | 0.23898 |
|           | $c_1 = 2.0$ | 0.35710 | 1.41210 | 0.22310 |
|           | $c_1 = 3.0$ | 0.35114 | 1.59004 | 0.22599 |
| $k = -1$  | $c_2 = 2.18$ | 0.41792 | 0.75768 | 0.37926 |
|           | $c_2 = 3.18$ | 0.35710 | 1.41210 | 0.23210 |
|           | $c_2 = 4.18$ | 0.41677 | 2.22455 | 0.20218 |

Table 2. Summary of the highest effective temperature $T^\text{eff}_c$ and the corresponding $x_c$ for different curves in Fig. 6. The value of $x_0$ when the effective temperature reaches zero is also listed.

furthermore obtain the effective thermodynamic quantities of the space-time. Here we summarize our main conclusions in more detail:

- In the previous analysis without considering the correlation between the black hole horizon and the cosmological horizon, i.e., the two horizons are always treated as independent thermodynamic systems with different temperature, the space-time does not satisfy the requirement of thermodynamic stability. In this paper, we find that the establishment of the correlation between the two horizons will generate the common effective temperature $T^\text{eff}$, which may represent the most typical thermodynamic feature of RN-dSQ space-time.

- As can be clearly seen from the $S(x) - x$ and $T^\text{eff} - x$ diagrams, RN-dSQ space-time in unstable under the condition of $x > x_c$ and $x < x_0$. This result indicates that there exist only the RN-dSQ black holes satisfying the condition of $x_0 < x < x_c$, which lays a solid theoretical foundation for the search of black holes in the Universe.

- We find that the interaction term $f(x)$ in the entropy of RN-dSQ space-time takes the same form of that in RN-dS space-time. Considering that the entropy in the two types of space-time is the function of the position of the horizon, which has no relation with other parameters including the electric charge ($Q$) and the constant ($\Lambda$), the entropy in the two types of space-time should take the same form. This finding may contribute to the deep understanding the universal thermodynamic characteristics of de Sitter space-time in the future.

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