Construction of a mathematical model of a by-pass turbojet engine as part of a complex aircraft model of an on-board navigation system

V V Markelov\textsuperscript{1}, M O Kostishin\textsuperscript{2,4}, I O Zharinov\textsuperscript{2} and O O Zharinov\textsuperscript{3}

\textsuperscript{1}Scientific and Research Center, Stock Company «Experimental Design Bureau «Electroavtomatika» named after P A Yefimov, 40, Marshala Govorova St., Saint Petersburg, 198095, Russia

\textsuperscript{2}Faculty of Information Security and Computer Technologies, ITMO University, 49, Kronverksky Av., Saint Petersburg, 197101, Russia

\textsuperscript{3}Department of Problem-Oriented Computing Complexes, Saint Petersburg State University of Aerospace Instrumentation, 67, Bolshaya Morskaia str., Saint Petersburg, 190000, Russia

\textsuperscript{4}E-mail: maksim@kostishin.com

Abstract. One of the main trends in the development of modern on-board navigation systems is aimed at increasing the predictability of the aircraft's flight path in space and time. That in its turn should improve the accuracy of evaluation of fuel and operating costs, improving the quality dispatch services and bandwidth airspace. At the same time, the predictability of the trajectory can be an additional factor in ensuring the safety of flight operations. The current level of development of on-board computing systems allows to use the most effective method of forecasting - complex mathematical flight models for solving the problem of predicting a four-dimensional flight path and evaluating fuel costs. At the same time, a feature of the flight models implemented in on-board systems remains the orientation to minimize the computing resources used while ensuring the required accuracy of the estimation of the predicted parameters. One of the components of a complex mathematical model of flight is the engine model, which allows determining the fuel costs and energy capabilities of the aircraft. As one of the options for implementing the engine model as part of a complex mathematical model of the flight of an on-board navigation system, a mathematical description of the functioning model of a two-circuit turbojet engine is presented. For improving the accuracy of the parameter estimation, the model is built on the basis of thermogasdynamic calculation using a limited amount of initial data provided by the engine developer. The presented model of a by-pass turbojet engine provides the calculation of the main parameters of the engine operation, including fuel consumption and available thrust, at the specified values of the required thrust. And also, accordingly, determines the permissible operating modes of the engine when simulating operation as part of an aircraft. The results of the proposed model development are presented.

1. Introduction
In order to solve the problems of information support for the crew, four-dimensional navigation and evaluation of fuel and time costs, the use of a complex mathematical model of the flight of an aircraft as an object of control in on-board computing systems becomes relevant [1, 2, 3].
One of the components of this model is the engine model. The engine model in a complex flight simulation environment is designed to provide the calculation of fuel consumption, as well as the determination of the available thrust values in the current flight mode.

The main requirement for both a complex mathematical model of an aircraft flight and an engine model implemented in on-board computing systems is to ensure that the calculated data corresponds to the actual operational data while minimizing the computing resources used [4, 5, 6].

At the same time, a feature of the model construction is a limited amount of initial data, including on the engine, which can be provided by the developers of individual aircraft systems based on their own commercial considerations [7, 8, 9].

As one of the options for constructing an engine model that meets the requirements, a mathematical description of a by-pass turbojet engine is presented.

The construction of the presented engine model is based on the equations of thermogasdynamics, which describe the actual, physical process of converting fuel combustion energy into engine thrust. This makes it possible to obtain an evaluation of fuel consumption and available thrust with an accuracy sufficient to solve problems as part of a complex mathematical model of an aircraft flight. At the same time, the main technical characteristics of the engine used do not go beyond the known, published data. Additional parameters are determined during the model setup by restoring the characteristics based on the current measurements of the aircraft condition during the flight.

2. Model description of a by-pass turbojet engine

In general, a turbojet engine generates thrust by using the energy of the fuel combustion. The model equations are given for a by-pass version of the engine with separate circuits, while the engine with one circuit is a special case of a by-pass one.

When describing and compiling the engine model, a number of assumptions are used that do not have a noticeable impact on the simulation of the aircraft flight. In particular, the losses in the engine are taken as constant values, and the losses at the input and output of the engine are included in the internal compression and expansion losses. The amount of energy generated by the engine and not used to create thrust is also invested in internal losses. It is assumed that the work of the compressor of the main circuit for air compression includes both the work of the compressor itself and the part of the fan through which the volume of air passes into the circuit [10, 11].

Overall, the described mathematically turbojet engine includes a core, the inner circuit, which compress the incoming volume of air the compressor, the combustion of fuel in the combustion chamber with heat and subsequent expansion of the gas energy is required to drive the turbine and the formation of thrust. In the external circuit of the engine, additional traction is provided by compressing the air entering the circuit with a fan driven by the operation of the turbine of the main circuit.

3. Calculation of parameters of the turbojet engine

The main engine parameters used in the integrated aircraft model are engine thrust and fuel consumption.

The thrust of a jet engine, taking into account the full expansion of the gas at the outlet of the engine to atmospheric pressure, is defined as [10, 11]:

\[
P = (G_I + G_{II})P_m; \\
P_m = ((c_I - V) + m_{II}(c_{II} - V))/(1 + m_{II}),
\]

where \( P \) – engine thrust (H); \( P_m \) – specific thrust of the engine (H/kg); \( V \) – flight speed (m/sec); \( G_I \) – gas consumption from the main circuit (kg/sec); \( G_{II} \) – air flow from the external circuit (kg/sec); \( c_I \) – gas outflow rate from the main circuit (m/sec); \( c_{II} \) – the rate of air outflow from the external circuit (m/sec); \( m_{II} \) – engine by-pass ratio.

The degree of dual circuit of the engine characterizes the distribution of air mass between the circuits:

\[ m_{II} = G_{II}/G_I. \]
Fuel consumption is defined as:

\[ G_t = G_t \cdot q_t. \]

where \( G_t \) – fuel consumption (kg/sec); \( q_t \) – relative fuel consumption.

The values of the air or gas flow through the contours are determined based on the flow equation. As in the main circuit, so in the external circuit, the presence of a critical cross-section of some unchangeable area is assumed, in which the Mach number for the calculated operation of the engine has a fixed value corresponding to the critical one.

It is assumed that these critical sections are located: in the main circuit-at the exit from the combustion chamber, in the external circuit – at the exit from the fan. For these critical sections, the flow equations are converted to the values of the blocked flow [10, 11]:

\[ G_I = S_I M_I p_{g0} T_{g0}^{-1/2} \left( k/R_p \right)^{1/2} \left( 1 + (k - 1)/2 M_I^2 \right)^{1/(1+k)} \]

\[ G_{II} = S_{II} M_{II} p_{f0} T_{f0}^{-1/2} \left( k/R_g \right)^{1/2} \left( 1 + (k_g - 1)/2 M_{II}^2 \right)^{1/(1+k)} \]

where \( S_I \) – the area of the critical section of the main contour (m²); \( S_{II} \) – the area of the critical section of the outer contour (m²); \( M_I \) – critical Mach number in the main contour; \( M_{II} \) – critical Mach number in the outer contour; \( p_{g0} \) – pressure deceleration of the gas at the outlet of the combustion chamber (Pa); \( p_{f0} \) – braking pressure of the air at the outlet of the fan (Pa); \( T_{g0} \) – gas deceleration temperature at the combustion chamber outlet (K); \( T_{f0} \) – braking temperature of the air at the fan outlet (K); \( R_p \) – specific gas constant of air (287.05287 J / (K kg)); \( R_g \) – specific gas constant of gas (287.5 J / (K kg)); \( k \) – adiabatic coefficient of air (1.4); \( k_g \) – adiabatic coefficient of gas (1.3).

When calculating the flow through the external circuit, it can be assumed that the stagnation temperature of the air at the fan outlet used in the calculation is determined from the condition that the compression in the fan proceeds without losses and without heat supply, which is permissible for describing the flow rate in the engine model in an aircraft [10, 11].

\[ T_{f0} / T = \pi_f^{(k-1)/k}; \]

\[ \pi_f = p_{f0} / p. \]

where \( T \) – air temperature (K); \( p \) – air pressure (Pa); \( \pi_f \) – degree of pressure increase in the fan.

Thus, taking into account the accepted assumption of the critical section, the flow equations can be represented in the form:

\[ G_I = p_{g0} T_{g0}^{-1/2} k_{G_I}^{1/2}; \]

\[ G_{II} = p T_{f0}^{-1/2} (\pi_f)^{(k+1)/(2k)} k_{G_{II}}^{1/2}; \]

where: \( k_{G_I} \) – air flow coefficient in the main circuit (m³·c²·K); \( k_{G_{II}} \) – air flow coefficient in the external circuit (m³·c²·K).

The value of the relative fuel consumption is determined based on the thermodynamic characteristics of the fuel used, as well as on the amount of heat supplied to heat the air in the combustion chamber.

For an engine model as part of a complex aircraft model, the value of the relative fuel consumption can be determined from the heat balance equation [10,11]:

\[ g_t = k/(k - 1) R_p \left( \theta_g T_{g0} - T_{k0} \right) / (H u \eta_q), \]

where: \( H u \) – specific heat of combustion of fuel (42900000 J/kg); \( \theta_g \) – temperature coefficient of fuel combustion; \( \eta_q \) – coefficient of completeness of combustion; \( T_{k0} \) – braking temperature of the air at the compressor outlet (K).
The temperature coefficient of fuel combustion characterizes the level of temperature decrease from the temperature at which the chemical process of fuel combustion occurs to the temperature at the exit from the combustion chamber.

The thermodynamic characteristics of the engine, necessary for calculating the thrust and fuel consumption and including the velocities of air and gas outflow, the temperature and pressure behind the compressor, fan and combustion chamber, are determined from the equation describing a turbojet engine as a heat engine.

The energy conservation equation of a heat engine determines the equality of the sum of the supplied heat and the work carried out by compressing the air, the sum of the rejected heat and the work done during the expansion of the gas. Accordingly, the available work of the thermodynamic cycle is defined as the difference between the supplied and rejected heat.

Taking into account the losses of the cycle and the complete expansion of the gas at the outlet of the engine to atmospheric pressure, the equation for the available work of the thermodynamic cycle of the main circuit as the difference between the supplied and removed heat is represented in the form [10]:

$$L_e = k_g/(k_g - 1)R_gT_0g \left(1 - 1/\pi_s^{(k_g-1)/k_g}\right)\eta_p - k/(k - 1)R_pT(\pi_s^{(k-1)/k} - 1)/\eta_c,$$

where $L_e$ – available work of the thermodynamic cycle (J / kg); $\pi_s$ – total pressure increase in the cycle; $\eta_p$ – expansion efficiency; $\eta_c$ – compression efficiency.

$$\pi_s = p_g/p.$$

It is assumed that the processes of compression in the compressor and expansion in the turbine proceed without heat supply, and the air pressure does not change when heated in the combustion chamber. Taking into account the accepted assumptions:

$$\pi_s^{k_g-1/k_g} = T_0g/T_n;$$

$$\pi_s^{k-1/k} = T_{ko}/T,$$

where $T_n$ – gas temperature at the outlet of the main circuit nozzle (K).

The braking air temperature at the outlet of the main circuit compressor used to calculate the relative fuel consumption with a certain acceptable degree of accuracy can be calculated as follows [10,11]:

$$T_{ko} = T \left(1 + (\pi_s^{(k-1)/k} - 1)/\eta_c\right).$$

When simulating the operation of the engine, it is assumed that the engine control system at any given thrust provides an optimal ratio between the heat supply in the combustion chamber, which is characterized by the stagnation temperature of the gas at the exit from the combustion chamber, and the degree of pressure increase in the cycle. The optimal ratio is determined from the condition of obtaining the maximum value of the available work of the thermodynamic cycle, that is, equating to zero the derivative of the function of the available work on the total degree of pressure increase. Taking the equality of the thermodynamic characteristics of the gas to the characteristics of the air, the optimal ratio between the temperature of the air at the outlet from the combustion chamber and the degree of pressure increase:

$$\pi_s^{(k-1)/k} = (T_0g/T\eta_p\eta_c)^{1/2}.$$

At the same time, the available work of the thermodynamic cycle of the main loop as the difference between the work on expansion and compression in the absence of losses in the nozzle:

$$L_e = AL + c_i^2/2 - V^2/2,$$

where $AL$ – difference between turbine and compressor operation.

In a single-circuit engine, the difference between the operation of the turbine and the compressor is zero, and in the by-pass engine it is used to drive the external circuit fan. Thus, for the by-pass engine:
\[ \Delta L = m_I L_f, \]

where \( L_f \) — fan operation (\( J / \text{kg} \)).

In this case, the work of the fan drive of the external circuit of a two-circuit engine, spent on air compression, is determined by the temperature at the fan inlet and the degree of pressure increase in the fan [10,11]:

\[ L_f = \left( k/(k - 1)R_p T \left( \pi_s^{(k-1)/k} - 1 \right) - V^2 /2 \right) / \eta_f, \]

where \( \eta_f \) — fan efficiency.

In this equation, the stagnation temperature at the fan inlet is represented in terms of the outside air temperature, taking into account the incoming air velocity.

Applying the equation of the available work of the thermodynamic cycle without heat supply for the external circuit, the available work can be expressed in terms of the compression work and the efficiency of the external circuit, equal to the ratio of the expansion work to the compression operation:

\[ L_{II}(\eta_{II} - 1) = c_i^2/2 - L_f - V^2 /2. \]

Considering the definition of the compression operation:

\[ L_{II} = L_f + V^2 /2; \]
\[ L_f = c_i^2 / (2 \eta_{II}) - V^2 /2, \]

where \( L_{II} \) - compression operation in the second circuit; \( \eta_{II} \) - efficiency of the second circuit.

When modeling the operation of the engine, it is assumed that the engine control system provides an optimal distribution of energy between the circuits at any given thrust. The optimal distribution of energy corresponds to the condition of obtaining the maximum value of the specific thrust of the engine, which in turn guarantees a reduction in fuel consumption. The maximum value of the specific thrust is determined by equating to zero the derivative of its function with respect to the air flow rate from the external circuit:

\[ P_m = \left( \left( (2L_e + V^2(1 + m) - m_{II}(c_i^2/\eta_{II})) \right)^{1/2} - V \right) + m_{II}(c_i - V) \right) / (1 + m_{II}). \]

Thus, the optimal distribution of energy between the circuits corresponds to the fulfillment of the condition:

\[ c_{II} = c_i \eta_{II}. \]

The presented mathematical dependencies together provide the implementation of the thermogasdynamic calculation of the turbojet engine.

4. Equations of the turbojet engine model

The model of a turbojet engine is the following system of 12 equations, including the main dependencies from the thermogasdynamic calculation of its parameters.

\[ \pi_s^{(k-1)/k} = \left( T_{g0} / T \eta_p \eta_c \right)^{1/2}; \]
\[ L_e = k g / (k - 1) R_p T_{g0} (1 - 1/\pi_s^{k-1/k g}) \eta_p - k / (k - 1) R_p T \left( \pi_s^{(k-1)/k} - 1 \right) / \eta_c; \]
\[ c_i^2 /2 = (L_e + V^2 /2(1 + m_{II})) / (1 + m_{II} \eta_{II}); \]
\[ G_t = p \pi_s T_{g0}^{-1/2} k_{g1}^{1/2}; \]
\[ P_m = (c_i - V + m_{II}(c_i \eta_{II} - V)) / (1 + m_{II}); \]
The output parameters for solving the system of equations used in the integrated aircraft flight model are fuel consumption and maximum engine thrust.
5. Model results
The presented model of a bypass turbojet engine was tested on a complex model of the Yak-40 aircraft with AI-25 turbojet engines and an Airbus-320 aircraft model with turbojet engines CFM-56-5B4 [12,13,14].

The results of a comparative analysis of the accuracy of calculating fuel consumption when simulating typical flight modes of the Yak-40 aircraft weighing 15.5 tons: the longest flight (LRC), cruising flight for the longest range (CR) and maximum cruise mode when the engines are operating in 0.85 nominal mode (85) are shown in figure 1.

![Figure 1](image1.png)

**Figure 1.** Comparison of actual ($Q$) and calculated ($Q_{model}$) fuel consumption for typical modes of horizontal flight of a Yak-40 aircraft with a mass of 15.5 tons.

The figure shows, depending on the flight altitude ($H$), a comparison of the normative operational published data of the flow rate ($Q$) and the flow rate obtained when testing the described model as part of an integrated aircraft model ($Q_{model}$).

The results of assessing the available thrust for the indicated operating modes are presented in figure 2. Where the current used required thrust of the engines is indicated in percent on the ordinate axis as a percentage of the nominal thrust ($P$, %).

![Figure 2](image2.png)

**Figure 2.** Values of the required thrust as a percentage of the nominal for typical modes of horizontal flight of the Yak-40 aircraft with a mass of 15.5 tons.
The results of a comparative analysis of the accuracy of calculating fuel consumption, obtained when simulating typical flight modes of an Airbus-320 aircraft weighing 60 tons: the longest-range flight (LRC) and cruise flight at longest range (CR), are shown in figure 3.

![Figure 3. Comparison of actual (Q) and calculated (Q model) fuel consumption for typical modes of horizontal flight of an Airbus-320 aircraft with a mass of 60 tons.](image)

The figure shows, depending on the flight altitude (H), the standard operational published flow data (Q) and the flow rate obtained when testing the described model as part of the integrated model of the aircraft (Q model) [15,16,17].

According to the results of a comparative analysis of the accuracy characteristics of calculating fuel consumption using the presented model of a two-circuit turbojet engine, with correctly selected engine parameters, the error of this model does not exceed 2%. This allows using this model when calculating the fuel-time parameters of the aircraft flight.

6. Conclusion
The presented model of a by-pass turbojet engine as part of an integrated aircraft flight model implemented in on-board computational systems for aircraft navigation provides the calculation of the main parameters of the engine operation at given values of the required thrust. These parameters include fuel consumption and available thrust. The calculation error for these parameters does not exceed 2% of the standard data.

Building a model based on thermogasdynamic calculation determines an increase in the accuracy of the calculated parameters. At the same time, the volume of initial data on the engine parameters used allows this model to be applied to a wide range of by-pass turbojet engines and to be guided by the published technical and operational characteristics of the engine.

One of the characteristic features of the presented system of equations of the model is its orientation towards the minimization of computing resources, which is conducive to its adaptation to implementation in on-board computing systems.

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