Locally private online change point detection

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Abstract

We study online change point detection problems under the constraint of local differential privacy (LDP) where, in particular, the statistician does not have access to the raw data. As a concrete problem, we study a multivariate nonparametric regression problem. At each time point $t$, the raw data are assumed to be of the form $(X_t, Y_t)$, where $X_t$ is a $d$-dimensional feature vector and $Y_t$ is a response variable. Our primary aim is to detect changes in the regression function $m_t(x) = \mathbb{E}(Y_t | X_t = x)$ as soon as the change occurs. We provide algorithms which respect the LDP constraint, which control the false alarm probability, and which detect changes with a minimal (minimax rate-optimal) delay. To quantify the cost of privacy, we also present the optimal rate in the benchmark, non-private setting. These non-private results are also new to the literature and thus are interesting per se. In addition, we study the univariate mean online change point detection problem, under privacy constraints. This serves as the blueprint of studying more complicated private change point detection problems.

1 Introduction

Online change point detection has been an active statistical research area for decades, originated from the demand of a reliable quality control mechanism under time and resources constraints (e.g. Wallis [1980]). In recent years, due to the advance of technology, the applications of online change point detection are well beyond quality control and include climatology, speech recognition, imaging processing, among many others. While collecting the data as they are generated, one wishes to detect the underlying distributional changes as soon as the changes occur.

As the ability of collecting and storing data improves exponentially, protecting users’ privacy has become one of the central concerns of data science. In practice, many of the systems that we monitor contain sensitive information. For instance, change point algorithms are used in cyber security to detect attacks, with the ultimate aim often being to protect private information (Lévy-Leduc & Roueff 2009; Tartakovsky 2014; Bagci et al. 2015). Other common application areas include public health (Kass-Hout et al. 2012; Gregori et al. 2020) and finance (Hand & Blunt 2001), in which many of the data involved are highly personal. Given the prevalence of such problems, we address two questions in this paper:

- whether we can detect changes without the need for direct access to the sensitive raw data;
- and what the cost of protecting privacy is in terms of the detection delay and accuracy.

Traditional anonymisation of data has been shown to be an outdated method of privacy protection, particularly in multivariate settings (Sweeney 2002; Rocher et al. 2019). Even when direct identifiers such as names and locations are removed, data sets often contain enough information for researchers to identify individual subjects, and hence there is a need to quantify and control the release of privileged information. Formal privacy constraints provide us with a rigorous framework in which we
may tackle such problems and explore the fundamental limits of private methods. Constraints of this type have been imposed in analyses carried out by Apple \cite{Team2017}, Tang \textit{et al.} \cite{Tang2017}, Google \cite{Erlingsson2014} and Microsoft \cite{Ding2017}, and provides organisations with a way of demonstrating the General Data Protection Regulation compliance \cite{Cohen2020}.

The most popular privacy constraint is that of differential privacy \cite{Dwork2006}, which assumes the existence of a third party who can be trusted to handle all the raw data. In situations where this assumption cannot be made, which are the focus in our work, we strengthen this constraint and require our algorithms to satisfy local differential privacy \cite{Kairouz2014, Duchi2018}, which, in particular, insists that the raw data are not accessed by anyone except its original holder. Solutions to a wide range of statistical problems have been given, and much underlying theory has been developed \cite{Wang2018, Duchi2018, Duchi2018, Rohde2019, Erlingsson2019, Joseph2018}.

In this paper, we are concerned with a multivariate nonparametric regression online change point detection problem, with privatised data. To be specific, we assume that the original data are \((X_i, Y_i)_{i \in N^*} \subset \mathbb{R}^d \times \mathbb{R}\), where the regression functions

\[
m_i(x) = E(Y_i|X_i = x), \quad x \in \mathbb{R}^d, \quad i \in N^*,
\]

satisfy that \(m_{i+1} \neq m_i\) if and only if \(i = \Delta \in N^*\). The goal is to find a stopping time \(\hat{t}\), which minimises the detection delay \((\hat{t} - \Delta)_+\), while controlling the false alarm probability \(P(\hat{t} < \Delta)\).

With the concern of maintaining privacy, we do not directly have access to the original data \((X_i, Y_i)_{i \in N^*}\), but a privatised version. Specifically, the original data are transmitted through an \(\alpha\)-locally differentially private (\(\alpha\)-LDP) channel for some fixed \(\alpha > 0\). The privatised data are denoted by \((W_i, Z_i)_{i \in N^*} \subset \mathbb{R}^d \times \mathbb{R}\). In our upper bounds we restrict attention to non-interactive mechanisms \cite{Duchi2018}, and so a privacy mechanism is given by a sequence \(\{Q_i\}_{i \in N^*}\) of conditional distributions, with the interpretation that \((W_i, Z_i)|(X_i, Y_i) = (x_i, y_i) \sim Q_i(\cdot|(x_i, y_i))\). For \(\{Q_i\}_{i \in N^*}\) to satisfy the \(\alpha\)-LDP constraint we require that

\[
\sup_{i \in N^*} \sup_{A} \sup_{(x,y),(x',y') \in \mathbb{R}^d \times \mathbb{R}} \frac{Q_i(A(x,y))}{Q_i(A(x',y'))} \leq e^\alpha.
\]

In our lower bounds, however, we allow mechanisms to be sequentially interactive \cite{Duchi2018}, so that a privacy mechanism is given by \(\{Q_i\}_{i \in N^*}\), with the interpretation that

\[
(W_i, Z_i)|(X_i, Y_i, W_{i-1}, Z_{i-1}, \ldots, W_1, Z_1) = (x_i, y_i, w_{i-1}, z_{i-1}, \ldots, w_1, z_1) \sim Q_i(\cdot|(x_i, y_i, w_{i-1}, z_{i-1}, \ldots, w_1, z_1)).
\]

Here the \(\alpha\)-LDP constraint requires that

\[
\sup_{i \in N^*} \sup_{A} \sup_{(x,y),(x',y') \in \mathbb{R}^d \times \mathbb{R}} \frac{Q_i(A(x,y, w_{i-1}, z_{i-1}, \ldots, w_1, z_1))}{Q_i(A(x',y', w_{i-1}, z_{i-1}, \ldots, w_1, z_1))} \leq e^\alpha.
\]

Since our upper and lower bounds match, up to a logarithmic factor, we may conclude that simpler non-interactive procedures result in optimal performance for this problem.

We will assume throughout that \(\alpha \leq 1\), though this can be relaxed to \(\alpha \leq C\) for any \(C > 0\). This restricts attention to the strongest constraints, and is often the regime of primary interest \cite{Duchi2018}.

### 1.1 Summary of contributions and related literature

To the best of our knowledge, this is the first work on a few fronts.

Firstly, this is the first paper to consider change point detection under local privacy constraints. Previous work has focused on the central model of differential privacy, where there exists a third party trusted to have access to all of the data. Cummings \textit{et al.} \cite{Cummings2018} use established tools from the central model of differential privacy to detect changes in both the online and online settings. The pre- and post-change distributions are assumed to be known, and a private version of the likelihood ratio statistic is analysed. Further development of these ideas, in particular the extension to detection of multiple changes in the online setting, is given in Zhang \textit{et al.} \cite{Zhang2021}. Canonne \textit{et al.} \cite{Canonne2019} give
differentially private tests of simple hypotheses, shown to be optimal up to constants, which are then applied to the change point detection problem with known pre- and post-change distributions. In a setting in which the distributions are unknown, Verzelen et al. (2020) develop private versions of the Mann–Whitney test to detect a change in location. The problem has also been studied under different notions of central privacy (Lau & Peng Tay, 2020).

Secondly, this is the first paper to study the fundamental limits in multivariate nonparametric regression change point detection problems. We have derived the minimax rate of the detection delay, allowing the jump size $|m_\Delta - m_{\Delta+1}|$, the variance of the additive noise $\sigma^2$ and the privacy constraint $\alpha$ to vary with the location of the change point $\Delta$. There has been a vast body of literature discussing the detection boundary and optimal estimation in the offline change point analysis (e.g. Verzelen et al., 2020; Yu, 2020). Their counterparts in online change point analysis are relatively scarce and existing work includes univariate mean change (e.g. Yu et al., 2020) and dynamic networks (e.g. Yu et al., 2021). On a separate note, multivariate nonparametric regression estimation, under privacy constraints, is studied in Berrett et al. (2021), and classification is studied in Berrett & Butucea (2019).

In addition, we have also provided the analysis and results based on the univariate mean online change point detection problem, with privatised data. This is, arguably, the simplest privatised, online change point detection problem. The analysis and results we shown in this paper enrich statisticians’ toolboxes and serve as a benchmark for more complex problems.

2 Methodology

In this section, we describe our private change point detection algorithm, which takes the privatised data as input. The whole algorithm consists of two key ingredients: (1) the privacy mechanism and (2) the change point detection method.

The privacy mechanism. Throughout this paper, a binned estimator is the core of the analysis. Recall that the raw data at time point $i$ include a $d$-dimensional feature vector $X_i$, which we assume is supported within some bounded set $\mathcal{X} \subset \mathbb{R}^d$, and a univariate response variable $Y_i$. We denote $\{A_h,j\}_{j=1,\ldots,N_h}$ as a set of cubes of volume $h^d$, such that $\{\mathcal{X} \cap A_{h,j}\}_{j=1,\ldots,N_h}$ is a partition of $\mathcal{X}$, and write $x_{h,j}$ for the centre of $A_{h,j}$. The data point $(X_i, Y_i)$ is then randomised by taking

$$W_{i,j} = \mathbb{1}_{\{X_i \in A_{h,j}\}} + 4\alpha^{-1}\epsilon_{i,j} \quad \text{and} \quad Z_{i,j} = [Y_i - M_{\alpha}]A_{h,j} + 4M\alpha^{-1}\zeta_{i,j},$$

where $\{\epsilon_{i,j}, \zeta_{i,j}\}$ are independent and identically distributed standard Laplace random variables, and where $[Y - M_{\alpha}]A_{h,j} = \min(M_{\alpha},\max(Y - M))$ with $M_{\alpha} > 0$ a truncation parameter. It is shown in Proposition 1 in Berrett et al. (2021) (see also Berrett & Butucea 2019) that this non-interactive mechanism is an $\alpha$-LDP channel. We emphasise that privacy is guaranteed without any assumptions on the distribution of $(X, Y)$.

The change point detection method. Given data $\{(W_{i,j}, Z_{i,j})\}$, as for online change point detection, we propose Algorithm 1 with the CUSUM estimator defined in Definition 1 and the nonparametric estimators involved defined in Definition 2.

Algorithm 1 Online change point detection via CUSUM statistics

INPUT: $\{(W_{u,j}, Z_{u,j})\}_{u,j=1,2,\ldots} \subset \mathbb{R}^d \times \mathbb{R}$, $\{b_{s,t}, 1 \leq s < t < \infty, \ldots\} \subset \mathbb{R}$.

$t \leftarrow 1$, FLAG $\leftarrow 0$;

while FLAG $= 0$ do

$t \leftarrow t + 1$; FLAG $= 1 - \prod_{s=1}^{t-1} \mathbb{1}\left\{\hat{D}_{s,t} \leq b_{s,t}\right\}$;

end while

OUTPUT: $t$.

Definition 1. Given a sequence $\{(W_{i,j}, Z_{i,j})\}$ and a pair of integers $1 \leq s < t$, we define the CUSUM statistic

$$\hat{D}_{s,t} = \max_{j=1}^{N_h} \sqrt{s(t-s)/t} \left| \hat{m}_{1:s}(x_{h,j}) - \hat{m}_{(s+1):t}(x_{h,j}) \right|,$$

where $\hat{m}_{\cdot}(\cdot)$ is defined in Definition 2.
We also assume that there exists an absolute constant $c > 0$ such that $\mu(A_{h,j}) \geq c \min h^d$ for all $A_{h,j}$ in the partition of $X$. Assume that the regression functions $m_i(\cdot)$, given by (1), are well defined for $\mu$-almost all $x$ and $i \in \mathbb{N}^*$, such that there exists an absolute constant $C_{\text{Lip}} > 0$ with

$$\sup_{i \in \mathbb{N}^*} |m_i(x_1) - m_i(x_2)| \leq C_{\text{Lip}} \|x_1 - x_2\|, \quad \forall x_1, x_2 \in X.$$ 

We also assume that there exists $\sigma > 0$ such that for all $\lambda \in \mathbb{R}$ we have

$$\sup_{i \in \mathbb{N}^*} \sup_{x \in \mathbb{R}^d} \mathbb{E}(e^{\lambda|Y_i - m_i(x)|} \mid X_i = x) \leq e^{\sigma^2 x^2}$$

and that $\sup_{i \in \mathbb{N}^*} \sup_{x \in \mathbb{R}^d} |m_i(x)| \leq M_0$ for some $M_0 > 0$.

Assumption 1 is the main model assumption. It is a general feature of the local differential privacy constraint that it is not possible to work with unbounded parameter spaces (see e.g. Duchi et al. 2013).
Appendix G). Thus, we assume that the feature vectors are supported in a bounded set \( \mathcal{X} \), and we moreover assume that the regression functions \( m_i \) are bounded, though these bounds are allowed to vary with the pre-change sample size \( \Delta \). In addition, we also assume that the regression functions are Lipschitz with a constant \( C_{\text{Lip}} \). The Lipschitz condition can be easily relaxed to other type of continuity conditions, e.g. Hölder continuity. We assume the additive noise is sub-Gaussian with parameter \( \sigma \), which is allowed to vary with the pre-change sample size. Assumption 1 is required to ensure that our algorithm controls the false alarm probability correctly and to ensure its optimality.

**Assumption 2** (No change point). Assume that \( m_1 = m_2 = \cdots \).

**Assumption 3** (One change point). Assume that there exists a positive integer \( \Delta \geq 1 \) such that

\[
m_1 = \cdots = m_\Delta \neq m_{\Delta+1} = m_{\Delta+2} = \cdots.
\]

In addition, let \( \kappa = \|m_\Delta - m_{\Delta+1}\|_\infty \).

**Assumption 4** (Signal-to-noise ratio). There exists a sufficiently large absolute constant \( C_{\text{SNR}} > 0 \) such that

\[
\kappa^2 h^{2d} \Delta^3 / \max\{\sigma^2, M_0^2\} \geq C_{\text{SNR}} \log \{\Delta / (c_{\min} h^{2d} \gamma)\},
\]

where \( \gamma \in (0, 1) \) is the desired bound on false alarm probability and \( h \) is the bin-width used in constructing the estimators.

Assumptions 2, 3 and 4 describe different scenarios and aspects of the change point assumptions. Assumption 2 is a formal assumption describing when there is no change point and Assumption 3 describes the scenario when there is one change point. Recall that in Definition 1 we proposed the CUSUM statistic which is a sample version of a normalised sup-norm difference between two functions. This is to be consistent with \( \kappa \), the characterisation of the jump, introduced in Assumption 3.

Assumption 4 is the signal-to-noise ratio condition, and is required when we study the optimality of our algorithm to detect changes. Recall that \( \kappa \) is the jump size, \( \sigma \) is the fluctuation size, \( M_0 \) is the upper bound of the mean function \( m_i \), \( \alpha \) is the privacy constraint and \( \Delta \) is the size of the pre-change point sample size. Assumption 4 requires that \( \kappa^2 h^{2d} \Delta \min\{M_0^{-2}, \sigma^{-2}\} \) is larger than a logarithmic factor. This in fact allows \( \kappa, \sigma, M_0, \alpha \) to vary as \( \Delta \) diverges. It may appear to be unnatural to involve tuning parameters in the signal-to-noise ratio condition. We remark that the involvement of \( h \) in Assumption 4 is to provide more flexibility in the tuning parameter selection. We will elaborate this point in Section 3.2.

### 3.2 Optimal online change point detection with privatised data

Theorem 1 below is the main result, which shows that for any \( \gamma \in (0, 1) \), with properly chosen tuning parameters, with probability at least of \( 1 - \gamma \), Algorithm 1 does not have false alarms and has a detection delay upper bounded by \( \epsilon \) in (11).

**Theorem 1.** Consider the settings described in Assumption 1 with inputs \( \{W_{tj}, Z_{tj}\}_{t=1,j=1}^{\infty} \) and \( \{b_{s,t}\}_{t=2,3, \ldots; s=1, \ldots, t} \), where

\[
b_{s,t} = \begin{cases} 
2 \sqrt{\frac{s(t-s)}{t}} \{2(M - M_0)e^{-\frac{(M-M_0)^2}{2s^2}} - M c_{\min} h^{2d} \alpha \sqrt{\log \left( \frac{72t^3}{c_{\min} h^3} \right)}
& \text{otherwise.} \\
\infty, & \text{otherwise.}
\end{cases}
\]

Assume that the truncation parameter satisfies

\[
M \geq M_1 = M_0 + \sigma \sqrt{2 \log(2 + \sigma / h) + \log \log(2 + \sigma / h)}
\]

and the bandwidth satisfies \( h \leq C \kappa \), where \( C > 0 \) is an absolute constant.

If Assumption 2 holds, then

\[
\mathbb{P}_\infty \left\{ \hat{\tau} < \infty \right\} < \gamma.
\]
Under Assumption \( \text{[3]} \) we have
\[
P_\Delta \{ \hat{t} \leq \Delta \} < \gamma, \tag{6}
\]
for any \( \Delta \geq 1 \). If Assumptions \( \text{[3] and [4]} \) both hold, then
\[
P_\Delta \{ \Delta < \hat{t} \leq \Delta + \epsilon \} \geq 1 - \gamma, \text{ where } \epsilon = C_\varepsilon \frac{M^2}{\kappa^2 h^{2d} \alpha^2} \log \left( \frac{\Delta}{h^{2d} \epsilon_{\min} \gamma} \right), \tag{7}
\]
and \( C_\varepsilon > 0 \) depends only on \( d \).

To better understand Theorem \( \text{[1]} \) we first inspect the sources of the error in the procedure. The estimators of the regression functions \( m_i \) are defined in Definition \( \text{[2]} \), which is a binned estimator averaging over cubes of volume \( h^d \). This is a typical nonparametric estimator, which brings in both bias and variance. On top of this, due to the constraints of privacy, we truncate the responses by \( M \) in the privacy channel. The truncation level \( M \) should be an upper bound on \( M_0 - 1 \) - a large-probability upper bound on the response, consisting of the upper bound on the regression function and the additive noise – so that the truncation bias is no larger than the bias due to smoothing. On the other hand, larger values of \( M \) result in larger variance, due to the need to add more noise in the privacy mechanism. If \( M < M_0 \) then the change point may be undetectable, as the bias could be larger than the signal. The same phenomenon occurs in nonparametric testing and change point detection problems, that the smoothing parameter should not be too large to mask the signal.

Algorithm \( \text{[1]} \) declares the existence of a change point \( t \), if there exists an integer pair \((s,t)\) such that \( D_{s,t} > b_{s,t} \). The threshold sequence is detailed in \( \text{[7]} \). It is separated into two cases: (i) when \( s(t-s)/t \) is large enough for both \( m_{1,s} \) and \( m_{(s+1),t} \) to be estimated accurately, and (ii) otherwise. In view of the sources of errors, we can see that in case (i), with probability at least \( 1 - \gamma \), the threshold \( b_{s,t} \) is set to be the sum of an upper bound of all sources of errors. In case (ii), the threshold is set to be infinity, so that we never declare a change.

When there is no change point, or when there is a change point but \( t \leq \Delta \), the thresholds, with probability at least \( 1 - \gamma \), are upper bounds on the estimators. When there is a change point and \( t > \Delta \), due to Assumption \( \text{[4]} \) one can always let \( s = \Delta \) such that there are enough samples to provide a good estimator of \( m_\Delta \), the pre-change regression function. If \( t - \Delta \) is not large enough such that case (i) holds, then we cannot provide a good estimator of \( m_{\Delta+1} \). Once enough data are collected after the change point, Algorithm \( \text{[1]} \) is able to tell the difference and declare a change point, with delay upper bounded by \( \epsilon \) in \( \text{[7]} \). Note that the conditions of case (i) will be satisfied before time \( \Delta + \epsilon \).

We require the bandwidth \( h \leq C \kappa \) to ensure that the binning will not smooth out the jump, and this condition is necessary. In practice, the bin-width can be chosen in a data-driven way (e.g. Yu et al. 2021). In view of Assumption \( \text{[4]} \) the smaller \( h \) is, the larger \( \kappa^2 \alpha^2 \Delta / \max \{ \sigma^2, M_0^2 \} \) needs to be. In the best case that \( h \approx \kappa \), Assumption \( \text{[4]} \) and \( \epsilon \) in \( \text{[7]} \) read as
\[
\frac{\kappa^{2+2d} \alpha^2 \Delta}{\max \{ \sigma^2, M_0^2 \}} \geq \log \left( \frac{\Delta}{\epsilon_{\min} \kappa^{2d} \gamma} \right) \text{ and } \epsilon = C_\varepsilon \frac{M^2}{\kappa^{2+2d} \alpha^2} \log \left( \frac{\Delta}{\kappa^{2d} \epsilon_{\min} \gamma} \right). \tag{8}
\]

We remark that the detection delay in \( \text{[7]} \) is of order
\[
\epsilon \approx \frac{M^2}{\kappa^2 h^{2d} \alpha^2} \log(\Delta/(h^{2d} \gamma)) \approx \frac{(M - M_0)^2 + M_0^2 + \sigma^2}{\kappa^2 h^{2d} \alpha^2} \log(\Delta/(h^{2d} \gamma)),
\]
which again reflects all three sources of errors. Now the question is whether this rate can be improved.

To answer the question, we consider a simplified scenario, where we assume that \( M_0 \leq \sigma \).

**Theorem 2** (Lower bound). Denote by \( \mathcal{P}_{\kappa,\alpha,\Delta} \) the class of distributions satisfying Assumptions \( \text{[7] and [3]} \) and assume \( M_0 \leq \sigma \). Given \( \alpha > 0 \) let \( \mathcal{Q}_\alpha \) be the collection of all sequentially interactive \( \alpha \)-locally differentially private mechanisms. Given \( \gamma > 0 \) consider the class of change point estimators
\[
\mathcal{D}(\gamma) = \{ T : T \text{ is a stopping time wrt. the natural filtration and satisfies } P_\infty(T < \infty) \leq \gamma \}.
\]

Then for sufficiently small \( \gamma \), it holds that
\[
\inf_{Q \in \mathcal{Q}_\alpha} \inf_{\hat{t} \in \mathcal{D}(\gamma)} \sup_{P \in \mathcal{P}_{\kappa,\alpha,\Delta}} 2 \kappa^{2+2d} \sigma^{-2} \alpha^2 \sigma P \{ (\hat{t} - \Delta)_+ \} \geq \log(1/\gamma).
\]
Theorem 2 studies the private minimax rate of the detection delay in the framework proposed in Duchi et al. (2018). Compared with standard minimax theory, new tools are required in order to deal with an arbitrary privacy mechanism in \( Q \). We use Lemma 1 in (Duchi et al. 2018 Supplementary material), which provides a uniform bound on the log-likelihood ratio of two distributions seen through a private channel. To the best of our knowledge, this is the first time that these tools have been applied to change point problems. Although, in our upper bounds, we only had to consider non-interactive mechanisms, this lower bound also applies to general sequentially interactive mechanisms. In particular, this shows that the use of interactive mechanisms is unnecessary in our problem.

Since in the minimax sense the lower bound is taken to be the infimum over all possible estimators, to make the results comparable in Assumption 4 and Theorem 1, we let \( h \equiv \kappa \) and compare Theorem 2 with (8). Theorem 2 shows that the lower bound on the detection delay is of order \( \sigma^2/(\kappa^{2d-2}/\alpha^2) \log(1/\gamma) \), which compared to (8) is off by a logarithmic factor and therefore shows that Algorithm 1 is nearly minimax rate optimal.

We now provide a sketch of the proof of Theorem 2, focusing on the non-interactive case for simplicity. The proof is conducted on an online change point detection framework used in Lai (1998) and Yu et al. (2018), based on a change of measure argument. The LDP ingredient used in this proof is devised by Duchi et al. (2020), based on a change of measure argument. The LDP ingredient used in this proof is devised by Duchi et al. (2020).

To be specific, we first construct a pair of distributions based on Lipschitz regression functions \( f_1 \) and \( f_2 \) on some ball \( B(0, r_X) \), where \( r_X = O(1) \), \( f_1 \equiv 0 \) and \( f_2(x) = (\kappa - \|x\|) \mathbb{I}_{x \in B(0, r_X)} \). Note that, when \( \kappa \) is small, the regression functions only differ in a small region, and thus this is a difficult change to detect. Given \( f_1 \) and \( f_2 \), the two distributions of \( (X, Y) \) are denoted \( P_1 \) and \( P_2 \), where \( X \) has the same marginal distribution under both while \( Y|X = x \sim \text{Unif}[f_k(x) - \sigma, f_k(x) + \sigma] \) under \( P_k \) for \( k = 1, 2 \). Letting \( Q \) be any \( \alpha \)-LDP non-interactive privacy mechanism, write \( P_{n,\kappa,\sigma,\nu} \) for the joint distribution of the first \( n \) privatised data points when \((X_t, Y_t) \sim P_1 \) for \( i \leq \nu \) and \((X_t, Y_t) \sim P_2 \) for \( i > \nu \). By controlling the size of the log-likelihood ratio \( Z_{\nu, n} = \log(dP_{n,\kappa,\sigma,\nu}/dP_{n,\kappa,\sigma,\infty}) \), we show that any algorithm that has probability of false alarm bounded by \( \gamma \) will require at least \( \sigma^2 \log(1/\gamma)/(\kappa^{2d+2d}/\alpha^2) \) observations after the change point to reliably detect the change from \( P_1 \) to \( P_2 \) at time \( \nu \).

As each observation is independent, we may decompose \( Z_{\nu, n} = \sum_{i=\nu+1}^{n} Z_i \). A straightforward calculation shows that \( d_{TV}(P_1, P_2) \leq \kappa^{d+1}/\sigma \), and then ideas from Duchi et al. (2018) are applied to see that \( Z_i \leq \alpha \kappa^{d+1}/\sigma \) and \( \mathbb{E} Z_i \leq \alpha^2 \kappa^{2d+2}/\sigma^2 \) for any \( \alpha \)-LDP privacy mechanism. We use the Azuma–Hoeffding inequality to extend this to show that

\[
\mathbb{P}_{\kappa,\sigma,\nu} \left\{ \max_{1 \leq s \leq \nu, t \leq \nu+1} Z_{\nu, s+t} \geq \frac{3}{4} \log(1/\gamma) \mid W_1, \ldots, W_{\nu} \right\} \leq \gamma \quad \text{a.s.}
\]

Using the change of measure argument (e.g. Lai 1998, Yu et al. 2020), the above bound leads us to complete the proof.

### 3.3 Optimal online change point detection with non-private data

As a benchmark, we provide the non-private counterpart of Theorem 1 and 2 in this subsection. For completeness, we also detail the counterparts of Definitions 1 and Algorithm 1 in Definitions 3 and Algorithm 2 respectively. We will conclude this subsection with comparisons of results in the private and non-private cases, quantifying the cost of maintaining privacy.

**Definition 3.** Given a sequence \( \{(X_t, Y_t)\} \in \mathbb{N} \) and a pair of integers \( 1 \leq s < t \), we define the CUSUM statistic

\[
\tilde{D}_{s,t} = \max_{i=1}^{t} \sqrt{s(t-s)/t} \left| \tilde{m}_{1,i}(X_i) - \tilde{m}_{(s+1):t}(X_i) \right|
\]

where \( \tilde{m}_{\cdot,\cdot}(\cdot) \) is defined in Definition 4.

**Definition 4.** Given a sequence \( \{(X_t, Y_t)\} \in \mathbb{N} \), a pair of integers \( 1 \leq s < t \) and a tuning parameter \( h > 0 \), we define the regression function estimator as

\[
\tilde{m}_{s,t}(x) = \frac{\nu_{s,t}(A_{h,j})}{\mu_{s,t}(A_{h,j})}, \quad \text{if} \ x \in A_{h,j},
\]

where

\[
\nu_{s,t}(A_{h,j}) = (t-s+1)^{-1} \sum_{s \leq i \leq t} Y_i \mathbb{I}_{\{X_i \in A_{h,j}\}} \quad \text{and} \quad \mu_{s,t}(A_{h,j}) = (t-s+1)^{-1} \sum_{s \leq i \leq t} \mathbb{I}_{\{X_i \in A_{h,j}\}}.
\]
Algorithm 2 Online change point detection via CUSUM statistics

INPUT: \{\{(X_{u,t}, Y_t)\}_{u=1,2,...} \subset \mathbb{R}^p \times \mathbb{R}, \{\hat{b}_{u,t}, t = 2, 3, \ldots; u = 1, \ldots,t) \subset \mathbb{R} \}

\begin{algorithmic}
\STATE $t \leftarrow 1$; \STATE $\text{FLAG} \leftarrow 0$
\WHILE {$\text{FLAG} = 0$}
\STATE $t \leftarrow t + 1$; \STATE $\text{FLAG} = 1 - \prod_{s=1}^{t-1} 1 \left\{ \hat{D}_{s,t} \leq \hat{b}_{s,t} \right\}$
\ENDWHILE
\end{algorithmic}

OUTPUT: $t$.

Assumption 5 (Non-private signal-to-noise ratio). There exists a sufficiently large absolute constant $C_{SNR} > 0$ such that

$$\kappa^2 h^d \Delta \sigma^{-2} \geq C_{SNR} \log(\Delta / (\gamma h^d)).$$

The following two theorems are the non-private version counterparts of Theorems 1 and 2. Theorem 4 shows that the detection delay rate we obtain in Theorem 3 is optimal off by a logarithmic factor.

Theorem 3. Consider the settings described in Assumption 7. Let $\gamma \in (0, 1)$ and $\bar{t}$ be the stopping time returned by Algorithm 2 with inputs \{\{(X_t, Y_t)\}_{t=1,2,...} and \{\hat{b}_{s,t}\}_{t=2,3,\ldots,s=1,\ldots,t}, where

$$\bar{b}_{s,t} = 2 \sqrt{\frac{s(t-s)}{t}} C_{lip} \sqrt{dh} + \frac{4\sigma}{\sqrt{c_{\min} h^d}} \sqrt{5 \log(t) + \log(32/\gamma)}$$

Assume that the bandwidth satisfies $h \leq C\kappa$.

If Assumption 2 holds, then

$$\mathbb{P}_\infty \{\bar{t} < \infty \} < \gamma.$$  \hfill (10)

Under Assumption 3 we have

$$\mathbb{P}_\Delta \{\bar{t} \leq \Delta \} < \gamma,$$

for any $\Delta \geq 1$. If Assumptions 3 and 5 both hold, then

$$\mathbb{P}_\Delta \{\Delta < \bar{t} \leq \Delta + \epsilon \} \geq 1 - \gamma - (c_{\min} h^d)^{-1} \exp(-C_{\min} h^d),$$

where

$$\epsilon = C_{\varepsilon} \frac{\sigma^2}{\kappa^2 h^d} \log(\Delta / \gamma),$$

with $C_{\varepsilon} > 0$ depending only on $d$. 

Theorem 4. Denote by $\mathcal{P}_{\kappa,\sigma,\Delta}$ the class of distributions satisfying Assumptions 7 and 3. For any $\gamma \in (0, 1)$, consider the class of change point estimators

$$D(\gamma) = \{ T: T \text{ is a stopping time wrt. the natural filtration and satisfies } \mathbb{P}_\infty(T < \infty) \leq \gamma \}.$$ 

Then for any sufficiently small $\gamma$, it holds that

$$\inf_{\tilde{t} \in D(\gamma)} \sup_{P \in \mathcal{P}_{\kappa,\sigma,\Delta}} 2\kappa^{2+d} \sigma^{-2} \mathbb{E}_P \{(\tilde{t} - \Delta)_{+}\} \geq \log(1/\gamma).$$

Theorem 1 equation (11) is

$$\inf_{\tilde{t} \in D(\gamma)} \sup_{P \in \mathcal{P}_{\kappa,\sigma,\Delta}} 2\kappa^{2+d} \sigma^{-2} \mathbb{E}_P \{(\tilde{t} - \Delta)_{+}\} \geq \log(1/\gamma).$$

We are now ready to quantify the cost of maintaining the privacy in the multivariate nonparametric regression online change point detection scenario.

• As we have pointed out, in order to maintain the privacy, we require $\mathcal{X}$ to be a bounded set, the regression function to be upper bounded by $M_0$ and an extra tuning parameter $M$ is introduced in truncation. All these are not needed in the non-private case.

• A more prominent difference roots in the detection delay rate. In the non-private case, the denominator is $\kappa^{2+d}$, which roots in the optimal nonparametric estimation (e.g. Tsybakov 2008). In the private case, the corresponding rate is $\kappa^{2+2d}$. This is in line with the literature in local differential privacy, where the curse of dimensionality is typically worse in nonparametric and high-dimensional problems than in their non-private counterparts (Duchi et al., 2018; Berrett & Butucea, 2019; Butucea et al., 2020; Berrett & Butucea, 2020).

• Another difference is in terms of the privacy parameter $\alpha$, which only shows up in the private case. Recalling that we restrict ourselves to the most interesting regime $\alpha \in (0, 1]$, the effect of $\alpha$ is twofold: (1) Comparing Assumptions 3 and 5, we see that the private case requires a larger signal by a factor of $\alpha^{-2}$; and (2) comparing the detection delay rates in Theorem 1 and 2, we see that the rate in the private case is also inflated by a factor of $\alpha^{-2}$. 


4 Numerical study

In this section we present the results of a numerical study of our locally private method’s performance. We consider raw data \((X_1, Y_1), \ldots, (X_n, Y_n)\) with \(n = 10000, \Delta = 5000, X_i \sim \text{Unif}[0, 1] \) and \(Y_i \sim \text{Unif}[m_i(x) - 1/2, m_i(x) + 1/2]\), where

\[
m_\Delta \equiv 0 \quad \text{and} \quad m_{\Delta+1}(x) = \left(\frac{1}{2}\right) \min(1, \max(5 - 10x, -1)).
\]

To speed up computation, we modify Algorithm 1 so that \(\hat{D}_{s,t}\) is only calculated and compared to its threshold when \(t \in \{n/100, 2n/100, \ldots, n\}\). We privatise the data using \(\alpha\) in the range \(\{1, 1.5, 2, \ldots, 6\}\). For each value of \(\alpha\), we assume that we have a privatised sample of size \(n\) from the pre-change distribution, here \(\text{Unif}[0, 1] \times \text{Unif}[-1/2, 1/2]\). The choices \(M = 1\) and \(h = 0.2\) are used to privatise the data and calculate the test statistics. We permute this sample \(B = 1000\) times to choose our thresholds, as follows: for a range of values of \(C\) we run the modified Algorithm 1 on each permutation of the privatised data with the choice

\[
b_{s,t} = \begin{cases} C h \alpha \sqrt{\log \left( \frac{t}{\gamma h} \right)} & \text{if } \frac{s(t-s)}{t^2} h^2 \alpha^2 \geq C^2 \log \left( \frac{t}{\gamma h} \right); \\ \infty, & \text{otherwise} \end{cases}
\]

and we choose the minimal value of \(C\) for which the overall false alarm probability is bounded above by \(\gamma = 0.1\). With the thresholds chosen we ran the experiment over 1000 repetitions, and results are presented in Figures 1 and 2. The false detection probability was bounded by \(\gamma\) for all values of \(\alpha\). Full details of the implementation and simulation study can be found in the code available online.

5 Conclusions

We studied a multivariate, nonparametric regression, online, privatised change point detection problem. The method we proposed is shown to be minimax optimal in terms of its detection delay, with a theory-guided tuning parameter. As a benchmark result, we have also provided its counterpart for non-private data. The comparisons enable us to understand the cost of maintaining privacy.

In addition to the main results in the paper in Appendix A we investigate an online univariate mean change point detection problem with privatised data. It includes a minimax lower bound on the detection delay and a polynomial-time algorithm which provides a matching upper bound, saving for a logarithmic factor. The framework we set up in Appendix A serves as a blueprint to study private online change detection with more complex data types.

Comparing the results in Appendix A to those in the non-private setting (e.g. Yu et al., 2020) and comparing these differences with those we examined at the end of Section 3.3 one can see that we pay different costs for privacy in different data types. This leads to our future work, studying private change point detection in high-dimensional, functional or other nonparametric data, and understanding the tradeoff between accuracy and privacy in these more challenging situations.

Regarding the setup we have in this paper, it can be easily adjusted to allow for multiple change points. This is because, with large probability, our algorithm will not declare false alarms and will correctly detect a change point with a delay \(\epsilon\) being a small fraction of \(\Delta\). Therefore, provided that
two consecutive change points are at least $\Delta$ apart, refreshing the algorithm whenever a change point is declared can enable us to detect multiple change points accurately.

Another interesting but challenge future work direction is to allow temporal dependence, say weakly dependent time series. It is not even clear how to define valid privacy mechanisms in this setting: if the time points were strongly dependent then our mechanisms would give more information as time progressed. In fact, this is even an open problem without the presence of change points. We would also need to adjust the concentration inequalities we are currently using to those which are suitable for weakly dependent data.

**Supplementary material**

The supplementary material contains all the technical details of this paper.

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## A privatised univariate mean online change point detection

This is a self-contained section and the notation used in this section is independent from the notation in the rest of the paper. We start by considering a simple Laplace privacy mechanism and CUSUM statistics.

**Assumption 6.** Assume the privatised data are $\{Z_1, Z_2, \ldots\}$ satisfying $Z_i = X_i + \epsilon_i/\alpha$, where (i) $\{X_1, X_2, \ldots\}$ is the original data and is a sequence of independent random variables with unknown means $\mathbb{E}(X_i) = f_i$, $i = 1, 2, \ldots$ and such that $\sup_{i=1,2,\ldots} ||X_i||_{\psi_2} \leq \sigma$; (ii) $\{\epsilon_1, \epsilon_2, \ldots\}$ is a sequence of independent and identically distributed standard Laplace random variables, i.e. each $\epsilon_i$ has density $z \mapsto \exp(-|z|)/2$ on $\mathbb{R}$; and (iii) $\alpha > 0$ is a pre-specified value.

Note that when $X_i$ takes values in an interval of length one $[a, a + 1]$ then the privacy mechanism given by $Z_i$ is $\alpha$-LDP. This can be generalised to intervals of any fixed length by increasing the scale of the Laplace noise appropriately. To extend to unbounded variables we may truncate as in our main methodology in Section 2.

**Assumption 7.** Assume that there exists a positive integer $\Delta \geq 1$ such that $f_1 = \cdots = f_\Delta = f_{\Delta+1} = f_{\Delta+2} = \cdots$. Let $\kappa = |f_\Delta - f_{\Delta+1}|$.

**Assumption 8.** There exists a sufficiently large absolute constant $C_{\text{SNR}} > 0$ such that $\Delta \kappa^2(\sigma^2 + 4\alpha^{-2})^{-1} \geq C_{\text{SNR}} \log(\Delta/\gamma)$.

We consider the CUSUM statistics $\hat{D}_{s,t} = |\{(t-s)/(ts)\}^{1/2} \sum_{l=1}^{\infty} Z_l - \lfloor s/\{t-(t-s)\}\rfloor^{1/2} \sum_{l=s+1}^{t} Z_l|$ and run Algorithm 1.

**Theorem 5.** Consider the settings described in Assumption 6. Let $\gamma \in (0, 1)$ and $\hat{t}$ be the stopping time returned by Algorithm 1, with inputs $\{Z_t\}_{t=1,2,\ldots}$ and $\{b_t\}_{t=2,3,\ldots}$, where

$$b_t = 2^{3/2} \sqrt{\sigma^2 + 4\alpha^{-2}} \log^{1/2}(t/\gamma).$$  \hfill (12)

If $\Delta = \infty$, then

$$P_{\infty} \{\hat{t} < \infty\} < \gamma.$$  \hfill (13)

Under Assumption 7, we have

$$P_{\Delta} \{\hat{t} \leq \Delta\} < \gamma,$$  \hfill (14)

for any $\Delta \geq 1$. If Assumptions 7 and 8 both hold, then

$$P_{\Delta} \left\{ \Delta < \hat{t} \leq \Delta + C_d \left( \sigma^2 + 4\alpha^{-2} \right) \log(\Delta/\gamma) \right\} \geq 1 - \gamma,$$  \hfill (15)

where $C_d > 0$ depends only on $d$ and satisfies $C_d < C_{\text{SNR}}$. 

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Theorem 6. Denote $\mathcal{P}_{\kappa,\sigma,\Delta}$ be the class of distributions satisfying Assumptions 6 and 7 supported on an interval of fixed finite length. Denote $\mathcal{Q}_\alpha$ be the collection of all sequentially interactive $\alpha$-locally differentially private mechanisms, $\alpha > 0$. Consider the class of change point estimators

$$D(\gamma) = \{T : T is a stopping time with respect to the natural filtration and satisfies $P_T(T < \infty) \leq \gamma\}.$$ 

Then for small enough $\gamma$, and for $\kappa, \sigma > 0$ with $\kappa < 2\sigma$ and $\kappa + 2\sigma < 1$, it holds that

$$\inf_{Q \in \mathcal{Q}_\alpha} \inf_{T \in D(\gamma)} \sup_{P \in \mathcal{P}_{\kappa,\sigma,\Delta}} \mathbb{E}_{P}\{(\hat{T} - \Delta)_{+}\} \geq \frac{c\sigma^2 \alpha^{-2} \log(1/\gamma)}{\kappa^2},$$

where $c > 0$ is an absolute constant.

B Proofs of main results

We first introduce some additional notation. For any integer pair $1 \leq s < t$, denote the un-smoothed and smoothed population CUSUM statistics as

$$D_{s,t} = \sup_{x \in \mathbb{R}} D_{s,t}(x) = \sqrt{\frac{s(t-s)}{t}} \sup_{x \in \mathbb{R}} |m_{1:s}(x) - m_{(s+1):t}(x)|$$

and

$$D^{(h)}_{s,t} = \sup_{x \in \mathbb{R}} D^{(h)}_{s,t}(x) = \sqrt{\frac{s(t-s)}{t}} \sup_{x \in \mathbb{R}} |m^{(h)}_{1:s}(x) - m^{(h)}_{(s+1):t}(x)|,$$

where

$$m_{s:t}(x) = \frac{1}{t-s+1} \sum_{i=s}^{t} m_i(x)$$

and

$$m^{(h)}_{s:t}(x) = \frac{1}{t-s+1} \sum_{i=s}^{t} \mathbb{E}\{Y_i | X_i \in A_{h,j}\}.$$ 

Proof of Theorem

For any integer pair $(s, t)$, with $1 \leq s < t$, note that

$$|\hat{D}_{s,t} - D_{s,t}| = \left| \sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} |\tilde{m}_{1:s}(X_i) - \tilde{m}_{(s+1):t}(X_i)| - D_{s,t} \right|$$

\begin{align*}
\leq & \sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} |\tilde{m}_{1:s}(X_i) - \tilde{m}_{(s+1):t}(X_i)| - \max_{i=1}^{t} |m^{(h)}_{1:s}(X_i) - m^{(h)}_{(s+1):t}(X_i)| \\
& + \sqrt{\frac{s(t-s)}{t}} \sup_{x \in \mathbb{R}} |m^{(h)}_{1:s}(x) - m^{(h)}_{(s+1):t}(x)| - \sup_{x \in \mathbb{R}} |m^{(h)}_{1:s}(x) - m^{(h)}_{(s+1):t}(x)| \\
& + \sqrt{\frac{s(t-s)}{t}} \sup_{x \in \mathbb{R}} |m^{(h)}_{1:s}(x) - m^{(h)}_{(s+1):t}(x)| - D_{s,t} \\
= & (I) + (II) + (III). \tag{17}
\end{align*}

Step 1. In order to show (10) and (11), we focus on the integer pairs $(s, t)$, with $1 \leq s < t \leq \Delta \leq \infty$. In this case, it follows from Lemma 7 that $D_{s,t} = 0$, from Lemma 8 that $(II) = 0$, from Lemma 9 that $(III) = 0$, from Lemma 11 that with probability $1 - \gamma$,

$$(I) \leq 2 \sqrt{\frac{s(t-s)}{t}} \sqrt{\log h} + \frac{4\sigma}{\sqrt{\min h^2}} \sqrt{5 \log(t) + \log(16/\gamma)}.$$

Therefore, the results (10) and (11) hold due to the choice of $b_{s,t}$ in (9).
Step 2. In order to show (11), due to (17), it suffices to show that
\[ D_{\Delta,\Delta+\epsilon} - (I) - (II) - (III) > \tilde{b}_{\Delta,\Delta+\epsilon} \]
with probability at least \(1 - \gamma\). In this case, it follows from Lemma 7 that
\[ D_{\Delta,\Delta+\epsilon} = \kappa \sqrt{\Delta} \sqrt{\frac{\epsilon}{\Delta + \epsilon}}. \]
from Lemma 8 that
\[ \mathbb{P}\{(II) = 0\} \geq 1 - \frac{1}{c_{\text{min}} h^d} \exp(-\Delta c_{\text{min}} h^d), \]
for \(C_{\text{SNR}}\) large enough, from Lemma 9 that
\[ (III) \leq 2 \sqrt{\frac{\Delta}{\Delta + \epsilon}} C_{\text{Lip}} \sqrt{dh} \]
and from Lemma 11 that
\[ (I) \leq 2 \sqrt{\frac{\Delta}{\Delta + \epsilon}} C_{\text{Lip}} \sqrt{dh} + \frac{4\sigma \sqrt{c_{\text{min}} h^d}}{\sqrt{\epsilon}} \sqrt{5 \log(\Delta + \epsilon) + \log(32/\gamma)} \]
with probability at least \(1 - \gamma/2\). Note that, due to Assumption 5 \(\Delta \geq \epsilon\). Combining the above statements, we thus have with probability at least \(1 - \gamma\) that
\[
\begin{align*}
D_{\Delta,\Delta+\epsilon} - (I) - (II) - (III) - \tilde{b}_{\Delta,\Delta+\epsilon} \\
\geq \kappa \sqrt{\Delta} \sqrt{\frac{\epsilon}{\Delta + \epsilon}} - 2 \sqrt{\frac{\Delta}{\Delta + \epsilon}} C_{\text{Lip}} \sqrt{dh} - \frac{4\sigma \sqrt{c_{\text{min}} h^d}}{\sqrt{\epsilon}} \sqrt{5 \log(\Delta + \epsilon) + \log(32/\gamma)} \\
- 2 \sqrt{\frac{\Delta}{\Delta + \epsilon}} C_{\text{Lip}} \sqrt{dh} - 2 \sqrt{\frac{\Delta}{\Delta + \epsilon}} C_{\text{Lip}} \sqrt{dh} - \frac{8\sigma \sqrt{c_{\text{min}} h^d}}{\sqrt{\epsilon}} \sqrt{5 \log(64\Delta/\gamma)} > 0,
\end{align*}
\]
where the second inequality follows from \(\epsilon \leq \Delta\), and the last inequality holds with a large enough \(C_{\text{SNR}}\).

Proof of Theorem 2 Throughout the proof, we will omit the use \(\lceil \cdot \rceil\) or \(\lfloor \cdot \rfloor\) notation, for the sake of simplicity.

Step 1. We let \(f_1(\cdot) = m_\Delta\) and \(f_2(\cdot) = m_{\Delta+1}\), which are the before and after change point mean functions respectively. To be specific, we let \(X = B(0, r_X)\), with
\[
\begin{align*}
r_X = \max \left\{ \left( \frac{8\sigma^2 \log(1/\gamma)}{\kappa^2} \right)^{1/d}, 2\kappa \right\}.
\end{align*}
\]
We let
\[
\begin{align*}
f_2(x) &= f_1(x) + \begin{cases} 
\kappa - \|x\|, & x \in B(0, \kappa) \\
0, & x \in X \setminus B(0, \kappa),
\end{cases} \quad \text{and} \quad f_1(x) = 0, \forall x \in X.
\end{align*}
\]
We let the distribution of \(X\) be uniform on \(X\) with density \(p_X(x) = u = V_d^{-1} r_X^{-d}\), for any \(x \in X\). We have that
\[
\|f_2 - f_1\|_{\infty} = \kappa
\]
and both \(f_j, j = 1, 2\), are Lipschitz with constant upper bounded by 1.

Step 2. For any \(n \in \mathbb{N}^*\), let \(P^n\) be the restriction of a distribution \(P\) to \(\mathcal{F}_n\), i.e. the \(\sigma\)-field generated by the observations \(\{(X_i, Y_i)\}_{i=1}^{n}\). For any \(\nu \geq 1\) and \(n \geq \nu\), we let
\[
Z_{\nu,n} = \log \left( \frac{dP_{\nu,n}}{dP_n} \right) = \sum_{i=\nu+1}^{n} Z_i,
\]
for any \(\nu \geq 1\) and \(n \geq \nu\), we let
\[
Z_{\nu,n} = \log \left( \frac{dP_{\nu,n}}{dP_n} \right) = \sum_{i=\nu+1}^{n} Z_i,
\]
where \( P_{\kappa, \sigma, \infty} \) indicates the joint distribution under which there is no change point and

\[
Z_i = \log \left\{ \frac{dP_{\kappa, \sigma, \nu}(X_i, Y_i)}{dP_{\kappa, \sigma, \nu}(X_i, Y_i)} \right\} = \frac{f_2(X_i) - f_1(X_i)}{\sigma^2} \left\{ Y_i - \frac{f_1(X_i) + f_2(X_i)}{2} \right\}.
\]

**Step 2.1.** For any \( \nu \geq 1 \), define the event

\[
E_{\nu} = \left\{ \nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}} \log(1/\gamma), Z_{\nu, T} < \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right\}.
\]

Then we have

\[
\mathbb{P}_{\kappa, \sigma, \nu}(E_{\nu}) = \int_{E_{\nu}} \exp(Z_{\nu, T}) \, dP_{\kappa, \sigma, \infty} \leq \gamma^{-3/4} \mathbb{P}_{\kappa, \sigma, \infty}(E_{\nu})
\]

\[
\leq \gamma^{-3/4} \mathbb{P}_{\kappa, \sigma, \infty} \left\{ \nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}} \log(1/\gamma) \right\} \leq \gamma^{-3/4} \gamma = \gamma^{1/4},
\]

where the last inequality follows from the definition of \( D(\gamma) \).

**Step 2.2.** For any \( \nu \geq 1 \) and \( T \in D(\gamma) \), since \( \{ T \geq \nu \} \in \mathcal{F}_{\nu-1} \), we have that

\[
\mathbb{P}_{\kappa, \sigma, \nu} \left\{ \nu < T \leq \nu + \frac{\sigma^2}{\kappa^{2+d}} \log(1/\gamma), Z_{\nu, T} \geq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right\} \leq \text{ess sup} \mathbb{P}_{\kappa, \sigma, \nu} \left\{ \max_{1 \leq t \leq \frac{\sigma^2}{\kappa^{2+d}} \log(\frac{1}{\gamma})} Z_{\nu, \nu + t} \geq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right\} (X_i, Y_i)_{i=1}^{\nu}
\]

\[
= \frac{\sigma^2}{\kappa^{2+d}} \log \left( \frac{1}{\gamma} \right) \left\{ \max_{1 \leq t \leq \frac{\sigma^2}{\kappa^{2+d}} \log(\frac{1}{\gamma})} g(t) + \max_{1 \leq t \leq \frac{\sigma^2}{\kappa^{2+d}} \log(\frac{1}{\gamma})} h(t) \right\} = (I) + (II),
\]

where

\[
g(t) = \mathbb{E}_X \mathbb{P}_\nu \left\{ \left[ Z_{\nu, \nu + t} \geq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right] \cap \left[ \frac{\nu + t}{\nu + 1} \sum_{i=\nu + 1}^{\nu + t} \frac{f_2(X_i) - f_1(X_i)}{2\sigma^2} \leq \frac{1}{4} \log \left( \frac{1}{\gamma} \right) \right] \right\}
\]

and

\[
h(t) = \mathbb{E}_X \mathbb{P}_\nu \left\{ \left[ Z_{\nu, \nu + t} \geq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right] \cap \left[ \frac{\nu + t}{\nu + 1} \sum_{i=\nu + 1}^{\nu + t} \frac{f_2(X_i) - f_1(X_i)}{2\sigma^2} > \frac{1}{4} \log \left( \frac{1}{\gamma} \right) \right] \right\}.
\]

**Step 2.2.1.** We first deal with (II). As for \( h(t) \), we have that

\[
h(t) \leq \mathbb{P}_X \left\{ \frac{\nu + t}{\nu + 1} \sum_{i=\nu + 1}^{\nu + t} \frac{f_2(X_i) - f_1(X_i)}{2\sigma^2} > \frac{1}{4} \log \left( \frac{1}{\gamma} \right) \right\}
\]

\[
\leq \mathbb{P}_X \left\{ \frac{\nu + t}{\nu + 1} \sum_{i=\nu + 1}^{\nu + t} \frac{f_2(X_i) - f_1(X_i)}{2\sigma^2} > \frac{1}{4} \log \left( \frac{1}{\gamma} \right) \right\}.
\]

If

\[
\frac{1}{2\kappa^d} \log \left( \frac{1}{\gamma} \right) \leq \frac{1}{4} \log \left( \frac{1}{\gamma} \right),
\]

i.e. \( \kappa \geq 2^{1/d} \), then \( h(t) = 0 \). Otherwise,

\[
h(t) \leq V_d \kappa^d u^t \leq V_u \kappa^d u,
\]

13
where the last inequality is due to the fact that $u V_d N^d \leq 1$.

Then we have that
\[
(II) = \frac{\sigma^2}{\kappa^2 + d} \log \left( \frac{1}{\gamma} \right) \leq \frac{\max_{1 \leq t \leq \log(1/\bar{\gamma})} h(t)}{\sigma^2} \log \left( \frac{1}{\gamma} \right) V_d N^d u
\]
\[
= \frac{\sigma^2}{\kappa^2} \log \left( \frac{1}{\gamma} \right) \min \left\{ \frac{\kappa^2}{8\sigma^2 \log(1/\gamma)}, \frac{1}{2^d \kappa^d} \right\} \leq 1/8. \quad (19)
\]

**Step 2.2.2.** We then deal with $(I)$. As for $g(t)$, we have that
\[
g(t) = \mathbb{E}_X \mathbb{P}_\nu \left\{ \left[ Z_{\nu, \nu+t} \geq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right] \cap \left[ \sum_{i=\nu+1}^{\nu+t} \frac{f_2(X_i) - f_1(X_i)}{2\sigma^2} \leq \frac{1}{4} \log \left( \frac{1}{\gamma} \right) \right] \right\}
\]
\[
\leq \mathbb{E}_X \mathbb{P}_\nu \left\{ Z_{\nu, \nu+t} - \sum_{i=\nu+1}^{\nu+t} \frac{f_2(X_i) - f_1(X_i)}{2\sigma^2} \geq \frac{1}{2} \log \left( \frac{1}{\gamma} \right) \right\}
\]
\[
\leq \mathbb{E}_x \left\{ \exp \left\{ -\frac{(1/2)^2 \log^2(1/\gamma)}{2 \sum_{i=\nu+1}^{\nu+t} \frac{(f_2(x_i) - f_1(x_i))^2}{\sigma^4}} \right\} \right\}
\]
\[
= \int \cdots \int_{B(0, \sigma^2)} \exp \left\{ -\frac{(1/2)^2 \log^2(1/\gamma)}{2 \sum_{i=1}^{\nu+t} \frac{(f_2(x_i) - f_1(x_i))^2}{\sigma^2}} \right\} u^t \, dx_1 \cdots dx_t
\]
\[
= \int \cdots \int_{B(0, \sigma^2)} \exp \left\{ -\frac{(1/2)^2 \log^2(1/\gamma)}{2 \sum_{i=1}^{\nu+t} \frac{(f_2(x_i) - f_1(x_i))^2}{\sigma^2}} \right\} u^t \, dx_1 \cdots dx_t
\]
\[
= \mathbb{E}_x \left\{ \exp \left\{ -\frac{(1/2)^2 \log^2(1/\gamma)}{2t \nu^2 \sigma^2} \right\} (u V_d N^d)^t \right\}
\]
Therefore
\[
(1) = \frac{\sigma^2}{\kappa^2 + d} \log \left( \frac{1}{\gamma} \right) \leq \frac{1}{\sigma^2} \log(1/\bar{\gamma}) g(t)
\]
\[
\leq \frac{\sigma^2}{\kappa^2 + d} \log \left( \frac{1}{\gamma} \right) \exp \left\{ -\frac{(1/2)^2 \log^2(1/\gamma)}{2 \sum_{i=1}^{\nu+t} \frac{(f_2(x_i) - f_1(x_i))^2}{\sigma^2}} \right\} u V_d N^d
\]
\[
\leq (1/8) \exp \left\{ \frac{-\kappa^d \log(1/\gamma)}{8} \right\} = (1/8) \gamma^{-d} < 1/8,
\]
where the second inequality is due to (19).

**Step 2.2.2.** Combining the previous two steps, we have that
\[
\mathbb{P}_{\kappa, \sigma, \nu} \left\{ \nu \leq T \leq \nu + \frac{\sigma^2}{\kappa^2 + d} \log(1/\gamma), \; Z_{\nu, T} \geq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right\} < 1/4. \quad (20)
\]

**Step 3.** Combining (18) and (20), we have that
\[
\mathbb{P}_{\kappa, \sigma, \nu} \left\{ \nu \leq T \leq \nu + \frac{\sigma^2}{\kappa^2 + d} \log(1/\gamma) \right\} \leq \gamma^{1/4} + 1/4.
\]
Since the upper bound in the above display is independent of $\nu$, we have that
\[
\sup_{\nu \geq 1} \mathbb{P}_{\kappa, \sigma, \nu} \left\{ \nu \leq T \leq \nu + \frac{\sigma^2}{\kappa^2 + d} \log(1/\gamma) \right\} \leq \gamma^{1/4} + 1/4.
\]
Therefore, for any change point time $\Delta$, we have that

$$
\mathbb{E}_{\kappa,\sigma,\Delta}\{ (T - \Delta)_+ \} \geq \sigma^2 \log(1/\gamma) \mathbb{P}_{\kappa,\sigma,\nu} \left\{ T - \nu > \sigma^2 \log(1/\gamma) \right\} 
$$

$$
= \sigma^2 \log(1/\gamma) \left[ \mathbb{P}_{\kappa,\sigma,\nu} \{ T > \nu \} - \mathbb{P}_{\kappa,\sigma,\nu} \left\{ \nu < T \leq \nu + \frac{\sigma^2}{\kappa^2 + d} \log(1/\gamma) \right\} \right] 
$$

$$
\geq \frac{\sigma^2}{\kappa^2 + d} \log(1/\gamma)(1 - \gamma - \gamma^{1/4} - 1/4) \geq \frac{\sigma^2}{2\kappa^2 + d} \log(1/\gamma),
$$

where the last inequality holds when $\gamma + \gamma^{1/4} < 1/4$. \hfill \Box

**Proof of Theorem 1.** In the case that $\nu \leq \Delta$, we must bound the truncation error roughly corresponds to the non-private $\mathbb{I}$ and $\mathbb{II}$, but new techniques must be applied. Instead of using Lemma 11, we use Lemma 13, and we also need to bound a truncation error, but otherwise the proof is very similar.

For regression functions $m$ we must bound the truncation error

$$
\max_j \frac{1}{\mu(A_{h,j})} \left| \int_{A_{h,j}} \{ m(x) - \mathbb{E}([Y]_{M-M} | X = x) \} \mu(dx) \right|
$$

and this changes the overall proof by just adding some bias. We use the facts that $\sup_x |m(x)| \leq M_0$ and $Y | X = x$ is $\sigma$-subgaussian for all $x$. For a $\sigma$-subgaussian random variable $Z$ and $t \geq \sigma \sqrt{\log(4)}$ we have that (using Exercise 2.3 of Wainwright (2019))

$$
\mathbb{E}(|Z| \mathbb{I}_{|Z| \geq t}) \leq t \inf_{k \in \mathbb{N}} \mathbb{E}(|Z|^k) \frac{t^k}{k^k} \leq t \inf_{\lambda > 0} \mathbb{E}(e^{\lambda |Z|}) \frac{t \inf_{\lambda \in \mathbb{R}} \mathbb{E}(e^{\lambda Z})}{e^{\lambda t}} \leq 2t \inf_{\lambda \in \mathbb{R}} e^{-\lambda M + \lambda^2 \sigma^2/2} = 2t e^{\frac{-\lambda^2}{2\sigma^2}}.
$$

When $M \geq M_0 + \sigma \sqrt{2 \log(2 + \sigma/h)} + \log(2 + \sigma/h)$ we therefore have

$$
\max_j \frac{1}{\mu(A_{h,j})} \left| \int_{A_{h,j}} \{ m(x) - \mathbb{E}([Y]_{M-M} | X = x) \} \mu(dx) \right|
$$

$$
\leq \max_j \frac{1}{\mu(A_{h,j})} \int_{A_{h,j}} \mathbb{E}([Y] - M) \mathbb{I}_{|Y| \geq M} | X = x \} \mu(dx)
$$

$$
\leq \max_j \frac{1}{\mu(A_{h,j})} \int_{A_{h,j}} \mathbb{E}([Y] - m(x)) \mathbb{I}_{|Y - m(x)| \geq M - M_0} | X = x \} \mu(dx)
$$

$$
\leq 2(M - M_0) \exp \left( - \frac{(M - M_0)^2}{2\sigma^2} \right)
$$

$$
\leq 2 \sigma \sqrt{2 \log(2 + \sigma/h) + \log(2 + \sigma/h)} \sqrt{1 + \log(2 + \sigma/h)} \leq 2 \sqrt{2} h,
$$

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which is the same order as the bias. We see that we have
\[ \mathbb{P} \left( | \bar{D}_{s,t} - D_{s,t} | \geq b_{s,t} \right) \leq \gamma / (2t^4). \]  
(21)

On the other hand, in the case that
\[ \frac{s(t-s)}{t} c_{\text{min}}^2 h^{2d} \alpha^2 < 64 \log \left( \frac{72 t^3}{\gamma c_{\text{min}} h^d} \right), \]
we will never flag a change, and there is nothing to prove.

In the setting of Assumption 2, we see from (21) and a union bound argument that we have
\[ \mathbb{P} \left( \max_{t \in \mathbb{N}} \max_{1 \leq s \leq t} (\bar{D}_{s,t} - b_{s,t}) \geq 0 \right) \leq \sum_{t=1}^{\infty} t \frac{\gamma}{2t^3} = \frac{\pi^2}{12} \leq \gamma. \]

Similarly, in the setting of Assumption 3, we have
\[ \mathbb{P} \left( \max_{1 \leq s \leq \Delta} \max_{t \leq t} (\bar{D}_{s,t} - b_{s,t}) \geq 0 \right) \leq \sum_{t=1}^{\Delta} t \frac{\gamma}{2t^3} \leq \gamma. \]

We have that
\[ \frac{\Delta \epsilon}{\Delta + \epsilon} c_{\text{min}}^2 h^{2d} \alpha^2 \geq 64 \log \left( \frac{72 t^3}{\gamma c_{\text{min}} h^d} \right) \]
when \( C_{\epsilon} \) and \( C_{\text{SNR}} \) are chosen large enough. Thus, by (21) and Lemma 7 with probability at least \( 1 - \gamma \), we have
\[ \bar{D}_{\Delta, \Delta+} - b_{\Delta, \Delta+} \geq D_{\Delta, \Delta+} - 2b_{\Delta, \Delta+} \]
\[ \geq \kappa \sqrt{\frac{\Delta \epsilon}{\Delta + \epsilon}} - 4 \sqrt{\frac{\Delta \epsilon}{\Delta + \epsilon}} (C_{\text{Lip}} \sqrt{d} + 2\sqrt{2}) h - \frac{2M}{\epsilon_{\text{min}} h^d} \sqrt{\log \left( \frac{72 t^3}{\gamma c_{\text{min}} h^d} \right)} > 0, \]
where the final inequality holds when \( \kappa / h \) is above a constant threshold and \( C_{\text{SNR}} \) is large enough.

\[ \Box \]

Proof of Theorem 2. Throughout the proof, we will omit the use \( [\cdot] \) or \( \lfloor \cdot \rfloor \) notation, for the sake of simplicity.

Step 1. Let \( Q \) be any sequentially interactive privacy mechanism, with output in some space \( W \), whose density satisfies
\[ q_i(w|x, y, w_{-1}, \ldots, w_1) \leq e^\alpha, \quad \forall w, w_1, \ldots, w_{i-1} \in W, x, x' \in \mathbb{R}^d, y, y' \in \mathbb{R}. \]

Let \( f_1(\cdot) = m_\Delta \) and \( f_2(\cdot) = m_{\Delta+1} \), which are the before and after change point mean functions respectively. Let \( P_X(\cdot) \) be the distribution of \( X \), staying the same before and after the change point. Let \( Y_i = m_i(X) + \text{Unif}[\sigma - \sigma, \sigma] \), for any \( i \in \mathbb{N}^* \). Let \( P_{k|X}, k = 1, 2 \), denote the distribution of \( Y \) given \( X \), before and after the change point. Let \( h_{i,k}(w|w_{i-1}, \ldots, w_1) = \int q(w|x, y, w_{i-1}, \ldots, w_1) dP_X(x) dP_{k|X}(y|x) \). Let \( \mathcal{X} = B(0, r_X) \), with
\[ r_X = \left( \frac{2(e^\alpha - 1)}{\alpha} \right)^{1/d}. \]  
(22)

We let
\[ f_2(x) = f_1(x) + \begin{cases} \kappa - \|x\|, & x \in B(0, \kappa) \\ 0, & x \in \mathcal{X} \setminus B(0, \kappa), \end{cases} \quad \text{and} \quad f_1(x) = 0, \quad \forall x \in \mathcal{X}. \]

We let the distribution of \( X \) be uniform on \( \mathcal{X} \) with density \( p_X(x) = u = V_d^{-1} r_X^{-d} \), for any \( x \in \mathcal{X} \).

We have that
\[ \| f_2 - f_1 \|_\infty = \kappa. \]
and both \( f_j, j = 1, 2 \), are Lipschitz with constant upper bounded by 1.

**Step 2.** For any \( n \in \mathbb{N}^* \), let \( P^n \) be the restriction of a distribution \( P \) to \( \mathcal{F}_n \), i.e. the \( \sigma \)-field generated by the observations \( \{W_i\}_{i=1}^n \subset \mathcal{W} \). For any \( \nu \geq 1 \) and \( n \geq \nu \), we let

\[
Z_{\nu,n} = \log \left( \frac{dP^n_{\kappa,\sigma,\nu}}{dP^n_{\kappa,\sigma,\infty}} \right) = \sum_{i=\nu+1}^n Z_i = \sum_{i=\nu+1}^n \log \left( \frac{h_{i,2}(W_i|W_{i-1},\ldots,W_1)}{h_{i,1}(W_i|W_{i-1},\ldots,W_1)} \right),
\]

where \( P^n_{\kappa,\sigma,\infty} \) indicates the joint distribution under which there is no change point.

It follows from Lemma 1 in [Duchi et al. 2018] (Supplementary material), that we have

\[
|Z_i| \leq \min\{2, e^\alpha\}(e^\alpha - 1)d_{TV}(P_1, P_2), \quad i \in \{\nu + 1, \ldots, n\}.
\]

where \( P_1 \) and \( P_2 \) are the joint distributions of \((X,Y)\) before and after the change point. Moreover, by calculations around Lemma 1 in [Duchi et al. 2018] (Supplementary material) we have

\[
0 \leq \int h_{i,2}(w|w_{i-1},\ldots,w_1) \log \frac{h_{i,2}(w|w_{i-1},\ldots,w_1)}{h_{i,1}(w|w_{i-1},\ldots,w_1)} \, du \leq \min\{4, e^\alpha\}(e^\alpha - 1)^2d_{TV}(P_1, P_2)^2
\]

for all \( w_1, \ldots, w_{i-1} \). Since

\[
d_{TV}(P_1, P_2) = \frac{1}{2\sigma} \int_{B(0,\kappa)} (\kappa - ||x||) p(x) \, dx \leq \frac{\kappa}{2\sigma} V_d \kappa^d u,
\]

we have that for any \( i > \nu, |Z_i| \leq \min\{2, e^\alpha\}(e^\alpha - 1)\frac{\kappa}{2\sigma} V_d \kappa^d u \leq \alpha \kappa^{d+1}/(2\sigma) \), and \( E(Z_i|W_1,\ldots,W_n) \leq \alpha^2 \kappa^{d+2}/(4\sigma^2) \) almost surely.

**Step 2.1.** For any \( \nu \geq 1 \), define the event

\[
\mathcal{E}_\nu = \left\{ \nu < T \leq \nu + \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2}, Z_{\nu,T} < \frac{3}{4} \log\left(\frac{1}{\gamma}\right) \right\}.
\]

Then we have

\[
\mathbb{P}_{\kappa,\sigma,\nu}(\mathcal{E}_\nu) = \int_{\mathcal{E}_\nu} \exp \left( Z_{\nu,T} \right) dP_{\kappa,\sigma,\infty} \leq \gamma^{-3/4} \mathbb{P}_{\kappa,\sigma,\infty}(\mathcal{E}_\nu)
\]

\[
\leq \gamma^{-3/4} \mathbb{P}_{\kappa,\sigma,\infty} \left\{ \nu < T \leq \nu + \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2} \right\} \leq \gamma^{-3/4} = \gamma^{1/4},
\]

where the last inequality follows from the definition of \( \mathcal{D}(\gamma) \).

**Step 2.2.** For any \( \nu \geq 1 \) and \( T \in \mathcal{D}(\gamma) \), since \( \{T \geq \nu\} \in \mathcal{F}_{\nu-1} \), we have that

\[
\mathbb{P}_{\kappa,\sigma,\nu} \left\{ \nu < T \leq \nu + \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2}, Z_{\nu,T} \geq \frac{3}{4} \log\left(\frac{1}{\gamma}\right) \big| T > \nu \right\}
\]

\[
\leq \text{ess sup} \mathbb{P}_{\kappa,\sigma,\nu} \left\{ \max_{1 \leq t \leq \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2}} Z_{\nu,T+t} \geq \frac{3}{4} \log\left(\frac{1}{\gamma}\right) \big| (W_i)_{i=1}^\nu \right\}
\]

\[
\leq \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2} \max_{1 \leq t \leq \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2}} \mathbb{P}_{\kappa,\sigma,\nu} \left\{ Z_{\nu,T+t} \geq \frac{3}{4} \log\left(\frac{1}{\gamma}\right) \big| (W_i)_{i=1}^\nu \right\}.
\]

\[
= \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2} \max_{1 \leq t \leq \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2}} g(t),
\]

where

\[
g(t) = \mathbb{P}_{\kappa,\sigma,\nu} \left\{ Z_{\nu,T+t} \geq \frac{3}{4} \log\left(\frac{1}{\gamma}\right) \right\}.
\]

As for \( g(t) \), we have that

\[
g(t) \leq \mathbb{P}_{\kappa,\sigma,\nu} \left\{ \sum_{i=\nu+1}^{\nu+t} \{ Z_i - E(Z_i|W_{i-1},\ldots,W_1) \} \geq \frac{3}{4} \log\left(\frac{1}{\gamma}\right) - \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d}\alpha^2} \frac{\alpha^2 \kappa^{d+2}}{4\sigma^2} \right\}
\]
\[
\begin{align*}
\leq & \, P_{\kappa, \sigma, \nu} \left\{ \sum_{i=\nu+1}^{\nu+t} \left\{ Z_i - E(Z_i|W_{i-1}, \ldots, W_1) \right\} \geq \frac{1}{2} \log \left( \frac{1}{\gamma} \right) \right\} \\
\leq & \, \exp \left\{ -\frac{\log^2(1/\gamma)}{\kappa^2 + 2d} \alpha^2 \frac{k^{2d+2}}{\sigma^2} \right\} = \gamma,
\end{align*}
\]
where the second inequality is due the Azuma–Hoeffding inequality (e.g., Wainwright [2019] Corollary 2.20). Therefore
\[
P_{\kappa, \sigma, \nu} \left\{ \nu < T \leq \nu + \frac{\sigma^2}{\kappa^2 + d} \log(1/\gamma), Z_{\nu, T} \geq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right\} \leq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \leq \gamma^{1/4}.
\]

**Step 3.** Combining the above we have that
\[
P_{\kappa, \sigma, \nu} \left\{ \nu < T \leq \nu + \frac{\sigma^2}{\kappa^2 + d} \log(1/\gamma), Z_{\nu, T} \geq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \right\} \leq \frac{3}{4} \log \left( \frac{1}{\gamma} \right) \leq 2 \gamma^{1/4}.
\]
Since the upper bound in the above display is independent of \( \nu \), we have that
\[
\sup_{\nu \geq 1} P_{\kappa, \sigma, \nu} \left\{ \nu < T \leq \nu + \frac{\sigma^2}{\kappa^2 + d} \log(1/\gamma) \right\} \leq 2 \gamma^{1/4}.
\]
Therefore, for any change point time \( \Delta \), we have that
\[
E_{\kappa, \sigma, \Delta} \left\{ (T - \Delta)_+ \right\} \geq \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d} \alpha^2} P_{\kappa, \sigma, \nu} \left\{ T - \nu > \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d} \alpha^2} \right\}
\]
\[
= \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d} \alpha^2} \left[ P_{\kappa, \sigma, \nu} \left\{ T > \nu \right\} - P_{\kappa, \sigma, \nu} \left\{ \nu < T \leq \nu + \frac{\sigma^2 \log(1/\gamma)}{\kappa^{2+2d} \alpha^2} \right\} \right]
\]
\[
\geq \frac{\sigma^2 \log(1/\gamma)}{2 \kappa^{2+2d} \alpha^2} (1 - \gamma - 2 \gamma^{1/4}) \geq \frac{\sigma^2 \log(1/\gamma)}{2 \kappa^{2+2d} \alpha^2}.
\]

\[\square\]

### C Auxiliary lemmas

#### C.1 Population quantities

**Lemma 7.** For \( D_{s, t} \) defined in (16), it holds that
\[
D_{s, t} = \begin{cases} 0, & 1 \leq s < t \leq \Delta, \\ \kappa \sqrt{\Delta} \sqrt{\frac{t - \Delta}{t}}, & s = \Delta < t. \end{cases}
\]

**Proof.** When \( 1 \leq s < t \leq \Delta \), by definition, we have that \( D_{s, t} = 0 \).

When \( s = \Delta < t \), let \( m_{\Delta} = f_1 \) and \( m_{\Delta+1} = f_2 \). Note that
\[
D_{\Delta, t} = \sqrt{\Delta(t - \Delta)} \left\| f_1 - f_2 \right\|_{\infty} = \kappa \sqrt{\Delta} \sqrt{\frac{t - \Delta}{t}},
\]
where the last identity follows from Assumption 3. \[\square\]

**Lemma 8.** When \( 1 \leq s < t \leq \Delta \), it holds that
\[
P \left\{ \max_{i=1}^{t} D_{s, t}^{(h)}(X_i) = 0 \right\} = 1.
\]

When \( s = \Delta < t \), it holds that
\[
P \left\{ \max_{i=1}^{t} D_{s, t}^{(h)}(X_i) > 0 \right\} \leq \frac{1}{c_{\min} h^d} \exp(-\Delta c_{\min} h^d)
\]

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Proof. Case 1. When $1 \leq s < t \leq \Delta$, by definition, we have that
\[ D_{s,t}^{(h)}(x) = 0, \quad x \in \mathbb{R}^d. \]
Then we have that
\[ \mathbb{P} \left\{ \left| \max_{i=1}^{t} D_{s,t}^{(h)}(X_i) - D_{s,t}^{(h)} \right| = 0 \right\} = 1. \]

Case 2. The only way that we can have $\max_{i=1}^{t} D_{s,t}^{(h)}(X_i) \neq D_{s,t}^{(h)}$ is if there exists some $A_{h,j}$ with no observations falling within it. We have that
\[ 1 = \sum_{j} \mu(A_{h,j}) \geq c_{\min} h^d N_{\delta}. \]
Thus, when $s = \Delta < t$, using a union bound, it holds that
\[ \mathbb{P} \left\{ \left| \max_{i=1}^{t} D_{s,t}^{(h)}(X_i) - D_{s,t}^{(h)} \right| > 0 \right\} \leq \mathbb{P} \left\{ \sum_{i=1}^{t} \mathbb{I}_{\{X_i \in A_{h,j}\}} = 0 \text{ for some } j \right\} \leq \frac{1}{c_{\min} h^d} \max_{j} \{1 - \mu(A_{h,j})\}^t \leq \frac{1}{c_{\min} h^d} \exp(-\Delta c_{\min} h^d), \]
as required.

Lemma 9. For any integer pairs $(s, t), 1 \leq s < t \leq \Delta$, it holds that $|D_{s,t}^{(h)} - D_{s,t}| = 0$. When $s = \Delta < t$, it holds that
\[ |D_{s,t}^{(h)} - D_{s,t}| \leq 2 \sqrt{\frac{s(t-s)}{t}} C_{\text{Lip}} \sqrt{d} h. \]

Proof. For any integer pairs $(s, t), 1 \leq s < t \leq \Delta$, it holds that
\[ m_{(s+1):t}^{(h)}(x) = m_{(s+1):t}^{(h)}(x) \quad \text{and} \quad m_{1:s}^{(h)}(x) = m_{(s+1):t}^{(h)}(x), \quad \forall x \in \mathbb{R}^d, \]
which implies that $|D_{s,t}^{(h)} - D_{s,t}| = 0$.

When $s = \Delta < t$, it holds that
\[ \frac{D_{\Delta,t}^{(h)}}{t} = \sqrt{\frac{\Delta(t-\Delta)}{t}} \sup_{x \in \mathbb{R}^p} \left| m_{\Delta}^{(h)}(x) - m_{\Delta+1}^{(h)}(x) \right| \leq \sqrt{\frac{\Delta(t-\Delta)}{t}} \sup_{x \in \mathbb{R}^p} \left| m_{\Delta}^{(h)}(x) - m_{\Delta+1}^{(h)}(x) \right| \leq 2 \sqrt{\frac{\Delta(t-\Delta)}{t}} \max \left\{ \sup_{x \in \mathbb{R}^p} \left| m_{\Delta}^{(h)}(x) - m_{\Delta}^{(h)}(x) \right|, \sup_{x \in \mathbb{R}^p} \left| m_{\Delta+1}^{(h)}(x) - m_{\Delta+1}^{(h)}(x) \right| \right\} \leq 2 \sqrt{\frac{\Delta(t-\Delta)}{t}} C_{\text{Lip}} \sqrt{d} h, \]
where the last inequality follows Assumption 1.

C.2 Sample terms

Lemma 10. Let $n \in \mathbb{N}, p \in (0, 1)$ and $x \in (0, 1)$. Let $(B_1, \ldots, B_n)$ be an independent sequence of $\text{Ber}(p)$ random variables, and let $(\epsilon_1, \ldots, \epsilon_n)$ be a sequence of real-valued random variables such that, conditionally on $(B_1, \ldots, B_n)$, it holds
(i) \( \epsilon_1, \ldots, \epsilon_n \) are independent; and

(ii) \( \mathbb{E}(e^{\lambda \epsilon_i}) \leq e^{\lambda^2/2} \) for all \( \lambda \in \mathbb{R} \).

Then we have

\[
P\left( \left| \sum_{i=1}^{n} \epsilon_i B_i \right| \geq x \sum_{i=1}^{n} B_i \right) \leq 2e^{-npx^2/4}.
\]

**Proof.** Writing \( N = \sum_{i=1}^{n} B_i \), we may condition on \((B_1, \ldots, B_n)\) to see that

\[
P\left( \frac{1}{N} \left| \sum_{i=1}^{n} \epsilon_i B_i \right| \geq x \right) \leq 2 \mathbb{E} \left[ 1_{\{N \geq 1\}} e^{-Nx^2/2} \right]
\]

\[
= 2 \left\{ \left( 1 - p + pe^{-x^2/2} \right)^n - (1-p)^n \right\}
\]

\[
= 2 \left\{ \left( 1 - p(1 - e^{-x^2/2}) \right)^n - (1-p)^n \right\}.
\]

For \( x \in (0, 1) \) we have \( 1 - e^{-x^2/2} \geq x^2/4 \), so that

\[
P\left( \frac{1}{N} \left| \sum_{i=1}^{n} \epsilon_i B_i \right| \geq x \right) \leq 2 \left\{ \left( 1 - \frac{px^2}{4} \right)^n - (1-p)^n \right\} \leq 2e^{-npx^2/4}.
\]

\[\square\]

**Lemma 11.** Under Assumption 1, for any integer pair \((s, t)\), \(1 \leq s < t\), we let

\[
W_{s,t} = \sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} \left| \bar{m}_{1,z}(X_i) - \bar{m}_{(s+1):z}(X_i) \right| - \max_{i=1}^{t} \left| m_{1,z}^{(h)}(X_i) - m_{(s+1):z}^{(h)}(X_i) \right|.
\]

Define the following three scenarios:

(i) There exists an integer pair \((s, t)\), \(1 \leq s < t\), such that

\[
W_{s,t} > 2 \sqrt{\frac{s(t-s)}{t}} C_{\text{Lip}} \sqrt{d h} + \frac{4\sigma}{\sqrt{c_{\text{min}} h d}} \sqrt{5 \log(t) + \log(16/\gamma)}.
\]

(ii) There exists an integer pair \((s, t)\), \(1 \leq s < t \leq \Delta\), such that \(24\) holds.

(iii) Under Assumption 2 there exists an integer \( t > \Delta \), such that

\[
W_{\Delta, t} > 2 \sqrt{\frac{\Delta(t-\Delta)}{t}} C_{\text{Lip}} \sqrt{d h} + \frac{4\sigma}{\sqrt{c_{\text{min}} h d}} \sqrt{5 \log(t) + \log(16/\gamma)}.
\]

We have that

- under Assumption 2 (i) holds with probability at most \( \gamma \); and
- under Assumption 3 (ii) and (iii) hold with probability at most \( \gamma \).

**Proof.** For any integer pairs \(1 \leq s < t\) and any \( \eta, \eta', \eta'' > 0, \eta' + \eta'' = \eta \), it holds that

\[
P\left( \sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} \left| \bar{m}_{1,z}(X_i) - \bar{m}_{(s+1):z}(X_i) \right| - \max_{i=1}^{t} \left| m_{1,z}^{(h)}(X_i) - m_{(s+1):z}^{(h)}(X_i) \right| \geq \eta \right)
\]

\[
\leq P\left( \sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} \left| \bar{m}_{1,z}(X_i) - \bar{m}_{(s+1):z}(X_i) \right| - \max_{i=1}^{t} \left| m_{1,z}^{(h)}(X_i) - m_{(s+1):z}^{(h)}(X_i) \right| \geq \eta \right)
\]

\[
\leq P\left( \sqrt{\frac{s(t-s)}{t}} \left\{ \max_{i=1}^{t} \left| \bar{m}_{1,z}(X_i) - m_{1,z}^{(h)}(X_i) \right| + \max_{i=1}^{t} \left| \bar{m}_{(s+1):z}(X_i) - m_{(s+1):z}^{(h)}(X_i) \right| \right\} \geq \eta \right)
\]
Combining (26), (27) and (28), with a union bound argument, we have that, for any \( \eta_1 \in (0, \sigma] \)

\[
\Pr \left\{ \frac{\sqrt{s(t-s)}}{t} \max_{i=1}^{t} |\tilde{m}_{1:s}(X_i)| \geq \eta_1 \right\} \leq 2t \exp \left( -\frac{s\mu(A_{h,j})\eta_1^2}{4\sigma^2} \right).
\]  

As for the term \((I.1)\), we apply Lemma 10 and take \( B_i = \mathbb{1}_{\{X_i \in A_{h,j}\}} \). For any \( \eta_1 \in (0, \sigma] \) we have

\[
\Pr \left\{ \frac{1}{\mu_{1:s}(A_{h,j})} \sum_{i=1}^{s} \{Y_i - m_i(X_i)\} \mathbb{1}_{\{X_i \in A_{h,j}\}} \geq \eta_1 \right\} \leq 2 \exp \left( -\frac{s\mu(A_{h,j})\eta_1^2}{4\sigma^2} \right). \tag{27}
\]

As for the term \((I.2)\), when \( x \in A_{h,j} \), due to Assumption 1 we have

\[
|m_i(x) - \int_{A_{h,j}} m_{i}(x') \mu(dx')| \leq C_{\text{Lip diam}}(A_{h,j}) = C_{\text{Lip}} \sqrt{dh},
\]

hence with probability one, it holds that

\[
\frac{1}{\mu_{1:s}(A_{h,j})} \sum_{i=1}^{s} \left( m_i(X_i) - \int_{A_{h,j}} m_{i}(x') \mu(dx') \right) \mathbb{1}_{\{X_i \in A_{h,j}\}} \leq C_{\text{Lip}} \sqrt{dh}. \tag{28}
\]

Combining (26), (27) and (28), with a union bound argument, we have that, for any \( \eta_1 \in (0, \sigma] \)

\[
\Pr \left\{ \frac{\sqrt{s(t-s)}}{t} \max_{i=1}^{t} |\tilde{m}_{1:s}(X_i) - m_{1:s}(X_i)| \geq \eta_1 \right\} \leq 2t \exp \left( -\frac{s\mu(A_{h,j})\eta_1^2}{4\sigma^2} \right). \tag{29}
\]
For any \( \eta_2 \in (0, \sigma] \), almost identical arguments lead to
\[
\mathbb{P} \left\{ \sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} |\bar{m}_{1:s}(X_t) - \bar{m}_{(s+1):t}(X_t)| \geq \sqrt{\frac{s(t-s)}{t}} (\eta_2 + C_{\text{Lip}} \sqrt{\delta}h) \right\} \\
\leq 2t \exp \left( -\frac{(t-s)\mu(A_{\text{hyp}})\eta_2^2}{4\sigma^2} \right)
\]
(30)

Combining (25), (29) and (30), we have that
\[
\mathbb{P} \left\{ \sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} |\bar{m}_{1:s}(X_t) - \bar{m}_{(s+1):t}(X_t)| \right. \\
\geq \sqrt{\frac{s(t-s)}{t}} (\eta_1 + \eta_2) + 2 \sqrt{\frac{s(t-s)}{t}} C_{\text{Lip}} \sqrt{\delta}h \} \\
\leq 2t \exp \left( -\frac{4\sigma^2}{s\min h^d} \right) + 2t \exp \left( -\frac{(t-s)\mu(A_{\text{hyp}})\eta_2^2}{4\sigma^2} \right)
\]
where the last inequality follows from Assumption 1.

We let
\[
Q_{s,t} = \sqrt{\frac{s(t-s)}{t}} \max_{i=1}^{t} |\bar{m}_{1:s}(X_t) - \bar{m}_{(s+1):t}(X_t)| - \frac{t}{s} \max_{i=1}^{t} m_{1:s}^{(h)}(X_t) - m_{(s+1):t}^{(h)}(X_t) \\
- 2 \sqrt{\frac{s(t-s)}{t}} C_{\text{Lip}} \sqrt{\delta}h,
\]
\begin{align*}
\eta_1 &= \sqrt{\frac{4\sigma^2}{s\min h^d}} \varepsilon_t \\
\eta_2 &= \sqrt{\frac{4\sigma^2}{(t-s)\min h^d}} \varepsilon_t,
\end{align*}
with \( \varepsilon_t > 0 \) to be specified. Therefore
\[
\mathbb{P} \left\{ Q_{s,t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{s\min h^d}} \right\} \leq \mathbb{P} \left\{ Q_{s,t} \geq \sqrt{\frac{s(t-s)}{t}} (\eta_1 + \eta_2) \right\} \leq 4t \exp(-\varepsilon_t^2).
\]

**Case 1.1** When \( \Delta = \infty \), we have that
\[
\mathbb{P} \left\{ \exists s, t \in \mathbb{N}^*, t > 1, s \in [1, t): Q_{s,t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{s\min h^d}} \right\}
\leq \sum_{j=1}^{\infty} \mathbb{P} \left\{ \max_{1 \leq t < 2^{j+1}} \max_{1 \leq s < t} Q_{s,t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{s\min h^d}} \right\} \leq \sum_{j=1}^{\infty} 2^j \max_{1 \leq t < 2^{j+1}} \mathbb{P} \left\{ \max_{1 \leq s < t} Q_{s,t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{s\min h^d}} \right\}
\leq \sum_{j=1}^{\infty} 2^j \max_{1 \leq t < 2^{j+1}} 4t \mathbb{P} \left\{ Q_{s,t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{s\min h^d}} \right\} \leq 4 \sum_{j=1}^{\infty} 2^j \max_{1 \leq t < 2^{j+1}} t^2 \exp(-\varepsilon_t^2)
\]
\[
\leq 4 \sum_{j=1}^{\infty} 2^{3j+2} \exp(-\varepsilon_t^2) \leq \gamma,
\]
where the last inequality holds by taking
\[
\varepsilon_t = \sqrt{5 \log(t) + \log(16/\gamma)}.
\]
(31)

**Case 1.2** When \( \Delta < \infty \), with \( \varepsilon_t \) defined in (31), it holds that
\[
\mathbb{P}_\Delta \left\{ \exists s, t \in \mathbb{N}^*, 1 \leq s < t \leq \Delta : Q_{s,t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{s\min h^d}} \right\}
\]
\[ \mathbb{P}_{\infty} \left\{ \exists s, t \in \mathbb{N}^*, t > 1, s \in [1, t) : Q_{s, t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{c_{\min} h^d}} \right\} \leq \gamma. \]

**Case 2.** In this case, we consider \( s = \Delta < t \). Note that for any such integer pair \((s, t)\), within both intervals \([1 : s]\) and \([(s + 1) : t]\), there is one and only one underlying distribution. Therefore, based on identical arguments as those in **Case 1.**, we have that
\[
\mathbb{P} \left\{ Q_{\Delta, t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{c_{\min} h^d}} \right\} \leq 4t \exp(-\varepsilon_t^2).
\]

Then we have
\[
\mathbb{P}_{\Delta} \left\{ \exists t \in \mathbb{N}^*, t > \Delta : Q_{\Delta, t} \geq \frac{4\sigma \varepsilon_t}{\sqrt{c_{\min} h^d}} \right\} \leq \gamma.
\]

We then complete the proof. \( \square \)

**Lemma 12** (Laplace concentration). Let \( \varepsilon_1, \ldots, \varepsilon_n \) be independent standard Laplace-distributed random variables (mean zero, variance 2). Then for all \( x > 0 \) we have
\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \geq x \right) \leq \exp \left( -\frac{3nx^2}{4 + 3x} \right).
\]

Note that this implies Lemma 1 of [Berrett et al. 2021].

**Proof.** Using a Chernoff bound and taking \( \lambda = \frac{n}{x}(\sqrt{1 + x^2} - 1) \) we have
\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \geq x \right) \leq \inf_{\lambda \in (0, n)} e^{-\lambda x} \left( 1 - \frac{\lambda^2}{2n^2} \right)^{-n} \\
\leq \exp \left( -n(\sqrt{1 + x^2} - 1) - n \log \left( 1 - \frac{2 + x^2 - 2\sqrt{1 + x^2}}{x^2} \right) \right) \\
= \exp \left( -n \left\{ \frac{x^2}{1 + \sqrt{1 + x^2}} - \log \left( \frac{2}{1 + \sqrt{1 + x^2}} \right) \right\} \right).
\]

It is a fact (checked numerically) that
\[
\frac{x^2}{1 + \sqrt{1 + x^2}} - \log \left( \frac{2}{1 + \sqrt{1 + x^2}} \right) \geq \frac{3x^2}{4 + 3x}
\]
for all \( x \geq 0 \), and the result follows. \( \square \)

**Lemma 13** (Private version of Lemma 11). Suppose that
\[
\frac{s(t - s)}{t} c_{\min}^2 h^{2d} \alpha^2 \geq 64 \log \left( \frac{72t^3}{\gamma c_{\min} h^d} \right).
\]

Then, if either \( 1 \leq s < t \leq \Delta \) or \( \Delta = s < t \), with probability at least \( 1 - \gamma/(2t^3) \) we have
\[
\sqrt{\frac{s(t - s)}{t}} \sup_{x \in \mathbb{R}^d} |\hat{m}_{1:s}(x) - \hat{m}_{(s+1):t}(x)| = \sup_{x \in \mathbb{R}^d} |m_{1:s}^{(h)}(x) - m_{(s+1):t}^{(h)}(x)| \\
\leq 2 \sqrt{\frac{s(t - s)}{t}} C_{\text{Lip}} \sqrt{dh} + 2^9 M c_{\min}^2 h^{2d} \alpha \sqrt{\log \left( \frac{72t^3}{\gamma c_{\min} h^d} \right)}.
\]

**Proof.** We start by writing
\[
\sup_{x \in \mathbb{R}^d} |\hat{m}_{1:s}(x) - \hat{m}_{(s+1):t}(x)| = \sup_{x \in \mathbb{R}^d} |m_{1:s}^{(h)}(x) - m_{(s+1):t}^{(h)}(x)|
\]
We have thus reduced our problem to two standard concentration inequalities. By Lemma 12 we have
\[
\leq \sup_{x \in \mathbb{R}^d} |\tilde{m}_{1:s}(x) - m_{1,s}(x)| + \sup_{x \in \mathbb{R}^d} |\tilde{m}_{(s+1):t}(x) - m_{(s+1):t}(x)|
\]
As in the proof of Lemma 11, it suffices to consider the first term in the case that \( s \leq \Delta \). This is given by
\[
\max_j \left| \tilde{p}_{1:s}(A_{h,j}) \right| \frac{\mu_1(A_{h,j}) \mathbb{1}(\tilde{p}_{1:s}(A_{h,j}) \geq \log(n)/n)}{\mu(A_{h,j})} - \frac{\int_{A_{h,j}} m_{\Delta}(x) \mu(dx)}{\mu(A_{h,j})}.
\]
For any \( j \) with \( \tilde{p}_{1:s}(A_{h,j}) \geq \log(n)/n \) we have
\[
\left| \tilde{p}_{1:s}(A_{h,j}) \right| \frac{\mu_1(A_{h,j}) \mathbb{1}(\tilde{p}_{1:s}(A_{h,j}) \geq \log(n)/n)}{\mu(A_{h,j})} \leq \left| \tilde{p}_{1:s}(A_{h,j}) \right| \frac{1}{\mu_1(A_{h,j})} \mathbb{1}(\tilde{p}_{1:s}(A_{h,j}) \geq \log(n)/n) \leq \left| \tilde{p}_{1:s}(A_{h,j}) \right| \frac{1}{\mu_1(A_{h,j})} \mu_1(A_{h,j}) \leq \frac{\mu_1(A_{h,j})}{\mu(A_{h,j})} \left| \tilde{p}_{1:s}(A_{h,j}) - \mu_1(A_{h,j}) \right| - \frac{\int_{A_{h,j}} m_{\Delta}(x) \mu(dx)}{\mu(A_{h,j})}
\]
The final term can be bounded exactly as in the proof of Lemma 11. For the first two terms
\[
P \left( \left| \frac{\tilde{p}_{1:s}(A_{h,j})}{\mu_1(A_{h,j})} \right| \frac{(4/\alpha) \sum_{i=1}^s \mathbb{1}_{\{X_i \in A_{h,j}\}}}{\sum_{i=1}^s \mathbb{1}_{\{X_i \in A_{h,j}\}}} \right) \geq \frac{\alpha}{2} + \left \{ \begin{array}{ll}
\frac{(4/\alpha) \sum_{i=1}^s \mathbb{1}_{\{X_i \in A_{h,j}\}}}{\sum_{i=1}^s \mathbb{1}_{\{X_i \in A_{h,j}\}}} \geq \eta \end{array} \right \}
\]
We have thus reduced our problem to two standard concentration inequalities. By Lemma 12 we have
\[
P \left( \sum_{i=1}^s \mathbb{1}_{\{X_i \in A_{h,j}\}} \right) \geq \alpha \min \left \{ \eta/(32M), 1/8 \right \} \sum_{i=1}^s \mathbb{1}_{\{X_i \in A_{h,j}\}} \right \}
\]
Using the fact that \( e^{-x} \leq 1 - x + x^2/2 \) for all \( x > 0 \) we have the Chernoff bound
\[
P \left( \sum_{i=1}^s \mathbb{1}_{\{X_i \in A_{h,j}\}} < s \mu(A_{h,j})/2 \right) \leq \inf_{\lambda > 0} \left \{ \begin{array}{ll}
\lambda \mu(A_{h,j})^2/2 \{ 1 - \mu(A_{h,j}) + \mu(A_{h,j}) e^{-\lambda/s} \}^s
\end{array} \right \}
\]
\[
\leq \inf_{\lambda > 0} \left \{ \begin{array}{ll}
\lambda \mu(A_{h,j})^2/2 \{ 1 - \lambda \mu(A_{h,j}) + \lambda^2 \mu(A_{h,j})/(2s^2) \}^s
\end{array} \right \}
\]
\[
\begin{align*}
\leq \inf_{\lambda > 0} \exp \left( -\lambda \mu(A_{h,j})/2 + \lambda^2 \mu(A_{h,j})/(2s) \right) \\
= \exp \left( -\frac{sp(A_{h,j})}{8} \right).
\end{align*}
\]

Hence,
\[
\mathbb{P} \left( \left| \frac{\hat{P}_{1:t}(A_{h,j})}{\hat{P}_{1:t}(A_{h,j})} \right| \left( \sum_{i=1}^{s} \epsilon_{i,j} \right) + \left( \sum_{i=1}^{s} 1 \{ X_i \in A_{h,j} \} \right) \geq \epsilon \right) 
\leq 16 \exp \left( -s \alpha^2 \mu(A_{h,j})^2 \min \{ \epsilon^2/(2^{14} M^2), 2^{-6} \} \right).
\]

Combining the previous bound with bounds from the proof of Lemma 11 when \( \eta \in (0, 2^6 M \sqrt{s(t - s)/t}] \) we have
\[
\mathbb{P} \left( \sqrt{\frac{t(s - t)}{t}} \sup_{x \in \mathbb{R}^d} |\hat{m}_{1:t}(x) - \hat{m}_{(s+1):t}(x)| - \sup_{x \in \mathbb{R}^d} |m^{(h)}_{1:t}(x) - m^{(h)}_{(s+1):t}(x)| \geq \eta/4 + \sqrt{\frac{s(t - s)}{t} \text{Cov} \sqrt{d}h} \right)
\leq \mathbb{P} \left( \sqrt{\frac{t(s - t)}{t}} \sup_{x} \left| \hat{m}_{1:t}(x) - m^{(h)}_{1:t}(x) \right| \geq \eta/4 + \sqrt{\frac{s(t - s)}{t} \text{Cov} \sqrt{d}h} \right)
+ \mathbb{P} \left( \frac{t(s - t)}{t} \max_{j} \left\{ \frac{\hat{P}_{1:t}(A_{h,j})}{\hat{P}_{1:t}(A_{h,j})} \left( \sum_{i=1}^{s} \epsilon_{i,j} \right) + \left( \sum_{i=1}^{s} 1 \{ X_i \in A_{h,j} \} \right) \right\} \geq \eta/4 \right)
+ \mathbb{P} \left( \frac{t(s - t)}{t} \max_{j} \left\{ \frac{\hat{P}_{(s+1):t}(A_{h,j})}{\hat{P}_{(s+1):t}(A_{h,j})} \left( \sum_{i=1}^{s} \epsilon_{i,j} \right) + \left( \sum_{i=1}^{s} 1 \{ X_i \in A_{h,j} \} \right) \right\} \leq \eta/4 \right)
\leq \frac{2}{\epsilon \min h^2} \left\{ \exp \left( -t/s \frac{c_{\min} h^d\eta^2}{64 M^2} \right) + \exp \left( -t/s \frac{c_{\min} h^d\eta^2}{64 M^2} \right) + 8 \exp \left( -\alpha^2 c_{\min} h^d \min \{ t \eta^2/(s(t - s)2^{18} M^2), 2^{-6} \} \right) \\
+ 8 \exp \left( -(t-s)\alpha^2 c_{\min} h^d \min \{ t \eta^2/(s(t - s)2^{18} M^2), 2^{-6} \} \right) \right\}
\leq \frac{4}{\epsilon \min h^d} \left\{ \exp \left( -\frac{c_{\min} h^d\eta^2}{64 M^2} \right) + 8 \exp \left( -\frac{\alpha^2 c_{\min} h^d\eta^2}{2^{18} M^2} \right) \right\} \leq \frac{36}{\epsilon \min h^d} \exp \left( -\frac{\alpha^2 c_{\min} h^d\eta^2}{2^{18} M^2} \right) .
\]

The result follows on taking
\[
\eta = \frac{2^9 M}{\epsilon \min h^d \alpha} \sqrt{\log(72t^3) + \log(1/\epsilon_{\min}) + \log(h^{-d}) + \log(1/\gamma)}.
\]

\[\square\]

## D Proofs of the results in Appendix A

**Lemma 14.** For any \( \gamma > 0 \), it holds that
\[
\mathbb{P} \left\{ \exists s, t \in \mathbb{N}, t > 1, s \in [1, t] : \left| \frac{t-s}{ts} \sum_{i=1}^{s} (Z_i - f_i) \right| - \left\{ \frac{s}{t(t-s)} \sum_{i=s+1}^{t} (Z_i - f_i) \right\} \geq \frac{t}{2} \sqrt{\sigma^2 + 4\alpha^{-2} \log^2(t)} \right\} \leq \gamma.
\]

**Proof.** It holds that for any sequence \( \{ \epsilon_t > 0 \} \),
\[
\mathbb{P} \left\{ \exists s, t \in \mathbb{N}, t > 1, s \in [1, t] : \left| \frac{t-s}{ts} \sum_{i=1}^{s} (Z_i - f_i) - \left\{ \frac{s}{t(t-s)} \sum_{i=s+1}^{t} (Z_i - f_i) \right\} \geq \epsilon_t \right\}
\]
We therefore complete the proof.

We remark that (32) is due to Assumption 6, which implies that
\[ \epsilon_t \text{ satisfies that for any } (e.g. (2.9) in Wainwright, 2019). \]

It follows from Lemma 14 that
\[ \max_{1 \leq i < 2^{j+1}} \frac{1}{2} \sum_{l=1}^{s} (Z_l - f_l) - \left\{ \frac{s}{t(t-s)} \right\} \sum_{l=s+1}^{t} (Z_l - f_l) \geq \varepsilon_t \]

\[ \leq \sum_{j=1}^{\infty} 2^j \max_{1 \leq i < 2^{j+1}} \frac{1}{2} \sum_{l=1}^{s} (Z_l - f_l) - \left\{ \frac{s}{t(t-s)} \right\} \sum_{l=s+1}^{t} (Z_l - f_l) \geq \varepsilon_t \]

where \( \varepsilon_t = \varepsilon_t' \), for any \( t \in \{2^j, \ldots, 2^{j+1} - 1\} \), \( j = 1, 2, \ldots \) and \( W \) is a mean zero sub-Gaussian random variable with
\[ ||W||_{\psi_2} \leq \sqrt{\sigma^2 + 4\alpha^{-2}}. \] (32)

We remark that (32) is due to Assumption 6 which implies that \( \mathbb{E}(Z_t) = f_t \) and
\[ ||Z_t - f_t||_{\psi_2} \leq \sqrt{||X_t - f_t||_{\psi_2}^2 + ||\epsilon||_{\psi_2}^2} = \sqrt{\sigma^2 + 4\alpha^{-2}}, \]
where the last identity follows from Lemma 1 in [Berrett et al. (2021)].

For any \( t = 1, 2, \ldots \), let
\[ \varepsilon_t = \sqrt{2\sigma^2 + 8\alpha^{-2}} \left[ 2 \log(t) + \log \log(t) + \log \{ \log(t) + \log(2) \} - 2 \log \log(2) - \log(\gamma) \right]^{1/2}. \]

Due to the sub-Gaussianity, we have that for any \( \zeta > 0 \), \( \mathbb{P} \{ |W| \geq \zeta \} < 2 \exp(-2\zeta^2/(\sigma^2 + 4\alpha^{-2})) \) (e.g. (2.9) in Wainwright, 2019).

\[ \mathbb{P} \left\{ \exists s, t \in N, t > 1, s \in [1, t) : \left( \frac{t-s}{ts} \right) \sum_{l=1}^{s} (Z_l - f_l) - \left\{ \frac{s}{t(t-s)} \right\} \sum_{l=s+1}^{t} (Z_l - f_l) \geq \varepsilon_t \right\} \]

\[ \leq \sum_{j=1}^{\infty} \max_{1 \leq i < 2^{j+1}} \exp \left[ (2j + 2) \log(2) - 2 \log(t) - \log \log(t) - \log \{ \log(t) + \log(2) \} + 2 \log \log(2) + \log(\gamma) \right] \]

\[ \leq \sum_{j=1}^{\infty} \exp \left\{ (2j + 2) \log(2) - 2j \log(2) - 2 \log(j) - \log \{ (j + 1) \log(2) \} + 2 \log \log(2) + \log(\gamma) \right\} \]

\[ \leq \gamma \sum_{j=1}^{\infty} \frac{1}{j(j+1)} \leq \gamma. \]

For simplicity, we let
\[ \varepsilon_t = 2^{3/2} \sqrt{\sigma^2 + 4\alpha^{-2}} \log^{1/2}(t/\gamma), \]
which satisfies that for any \( t \geq 2 \) and \( \gamma \in (0, 1) \),
\[ 2^{3/2} \log^{1/2}(t/\gamma) \geq \sqrt{2} \left[ 2 \log(t) + \log \log(t) + \log \{ \log(t) + \log(2) \} - 2 \log \log(2) - \log(\gamma) \right]^{1/2}. \] (33)

We therefore complete the proof.

**Proof of Theorem 5**

**Step 1.** Define the event
\[ \mathcal{B} = \left\{ \forall s, t \in N, t > 1, s \in [1, t) : \left( \frac{t-s}{ts} \right) \sum_{l=1}^{s} (Z_l - f_l) - \left\{ \frac{s}{t(t-s)} \right\} \sum_{l=s+1}^{t} (Z_l - f_l) < b_t \right\}. \] (34)

It follows from Lemma 14 that \( \mathbb{P} \{ \mathcal{B} \} > 1 - \gamma \). Throughout the proof we assume that the event \( \mathcal{B} \) holds.

For any \( s, t \in \mathbb{N}, 1 \leq s < t \), it holds that
\[ |\tilde{D}_{s,t} - D_{s,t}| < b_t, \]
which implies that
\[ D_{s,t} + b_t > \tilde{D}_{s,t} > D_{s,t} - b_t. \] (35)
Step 2. For any \( t \leq \Delta \), we have that \( D_{s,t} = 0 \), for all \( s \in [1, t) \). Thus, using (35), we conclude that, \( \bar{t} > t \) and, therefore that \( \bar{t} > \Delta \).

Step 3. Now we consider any \( t > \Delta \). If there exists \( s \in [1, t) \) such that \( \hat{D}_{s,t} > b_t \), then \( d \leq t - \Delta \).

Step 4. Write for convenience \( m = t^* - \Delta \), so that \( \bar{t} - \Delta \leq m \). Recalling that

\[ b_t = 2^{3/2} \sqrt{\sigma^2 + 4\alpha^{-2} \log^{1/2}(t/\gamma)} \]

we seek the smallest integer \( m \) such that

\[ \max_{s \in [\Delta, m+\Delta]} \left[ \Delta \kappa \left( \frac{m + \Delta - s}{m + \Delta} \right)^{1/2} - 2^{5/2} \sqrt{\sigma^2 + 4\alpha^{-2} \log^{1/2} ((m + \Delta)/\gamma)} \right] > 0 \]

which is equivalent to finding the smallest integer \( m \) such that

\[ \max_{s \in [\Delta, m+\Delta]} \left[ \Delta^2 \kappa^2 - 32(\sigma^2 + 4\alpha^{-2}) \frac{s(m + \Delta)}{m + \Delta - s} \log \left( (m + \Delta)/\gamma \right) \right] > 0 \]

In turn, the above task corresponds to that of computing the smallest integer \( m \) such that

\[ \frac{\Delta^2 \kappa^2}{\sigma^2 + 4\alpha^{-2}} > \min_{s \in [\Delta, m+\Delta]} \left[ 32 \frac{s(m + \Delta)}{m + \Delta - s} \log \left( (m + \Delta)/\gamma \right) \right] \]

or, equivalently, such that

\[ m \left[ \frac{\Delta \kappa^2}{32(\sigma^2 + 4\alpha^{-2})} - \log \left( (m + \Delta)/\gamma \right) \right] > \Delta \log \left( (m + \Delta)/\gamma \right) \]

under Assumption[8]

Let \( C_d \) be an absolute constant large enough and also upper bounded by \( C_{SNR} \). The claimed result now follows once we show that the value

\[ m^* = \left[ C_d \log(\Delta/\gamma) (\sigma^2 + 4\alpha^{-2}) \kappa^{-2} \right] \]

satisfies (37). To see this, assume for simplicity that \( C_d \log(\Delta/\gamma) (\sigma^2 + 4\alpha^{-2}) \kappa^{-2} \) is an integer; if not, the proof only requires trivial modifications. We first point out that \( m^* \leq \Delta \) because of Assumption[8] and the fact that \( C_d \leq C_{SNR} \). Now, the left hand side of inequality (37) is equal, for this choice of \( m \), to

\[ C_d \log(\Delta/\gamma) \frac{\Delta}{32} - C_d \frac{\sigma^2 + 4\alpha^{-2}}{\kappa^2} \log(\Delta/\gamma) \log \left( \frac{C_d \log(\Delta/\gamma) (\sigma^2 + 4\alpha^{-2})/\kappa^2 + \Delta}{\gamma} \right) \]

(38)
Using again Assumption 8 and the fact that \( m^* \leq \Delta \), the second term in the previous expression is upper bounded by
\[
\frac{2C_d}{C_{\text{SNR}}} \Delta \log(\Delta / \gamma),
\]
due to the fact that \( 2 \log(x) \geq \log(2x), x \geq 2 \). Thus, the quantity in (38) is lower bounded by
\[
\Delta \log(\Delta / \gamma) \left( \frac{C_d}{32} - \frac{2C_d}{C_{\text{SNR}}} \right) \geq 2 \Delta \log(\Delta / \gamma) \geq \Delta \log(2\Delta / \gamma) \geq \Delta \log ((m^* + \Delta) / \gamma),
\]
where the first inequality is justified by first choosing a large enough \( C_d \) and then choosing \( C_{\text{SNR}} \) larger than \( C_d \), and the second and third inequalities follow from \( \log(\Delta / \gamma) \geq 0 \) and \( m^* \leq \Delta \), respectively. Thus, combining the last display with (37) and (38) yields (15). Finally, (13) and (14) follow immediately from Steps 1 and 2.

**Proof of Theorem 6.** Throughout the proof we will assume for simplicity that \( \alpha^2 \sigma^{-2} \log(1/\gamma) \kappa^{-2} \) is an integer.

**Step 1.** For any \( n \), let \( P^n \) be the restrictions of a distribution \( P \) to \( \mathcal{F}_n \), i.e. the \( \sigma \)-field generated by the observations \( \{Z_i\}_{i=1}^n \).

Let \( P_1 \) be the distribution of the original \( X_t \) before the change and let \( P_2 \) be the distribution after. Write
\[
m_{i,k}(z|z_1, \ldots, z_{i-1}) = \int q_i(z|x, z_1, \ldots, z_{i-1}) dP_k(x)
\]
for the conditional density of the \( Z_t \) before \( (k = 1) \) and after \( (k = 2) \) the change point. To be specific, we let
\[
P_1 = \text{Unif}[0, 2\sigma], \quad P_2 = \kappa + \text{Unif}[0, 2\sigma],
\]
which satisfy that
\[
\|P_1 - P_2\|_{\text{TV}} = \frac{\kappa}{2\sigma}, \tag{39}
\]
assuming that \( \kappa < 2\sigma \).

For any \( \nu \geq 1 \) and \( n \geq \nu \), we have that for any \( n \geq \Delta \), it holds that
\[
\frac{dP_{\nu,\sigma}}{dP_{\nu,\kappa,\sigma,\infty}} = \exp \left( \sum_{i=\nu+1}^{n} W_i \right),
\]
where \( P_{\nu,\sigma,\infty} \) indicates the joint distribution under which there is no change point and
\[
W_i = \log \frac{m_{i,2}(Z_i|Z_{i-1}, \ldots, Z_1)}{m_{i,1}(Z_i|Z_{i-1}, \ldots, Z_1)}.
\]

For any \( \nu \geq 1 \), define the event
\[
\mathcal{E}_\nu = \left\{ \nu < T < \nu + \frac{\alpha^2 \sigma^{-2} \log(1/\gamma)}{\kappa^2}, \sum_{i=\nu+1}^{T} W_i < \frac{3}{4} \log \left( \frac{1}{\alpha} \right) \right\}.
\]

Then we have
\[
\mathbb{P}_{\nu,\sigma,\infty}(\mathcal{E}_\nu) = \int_{\mathcal{E}_\nu} \exp \left( \sum_{i=\nu+1}^{T} W_i \right) dP_{\nu,\sigma,\infty} \leq \exp \{ (3/4) \log(1/\gamma) \} \mathbb{P}_{\nu,\sigma,\infty}(\mathcal{E}_\nu) \leq \exp \{ (3/4) \log(1/\gamma) \} \mathbb{P}_{\nu,\sigma,\infty} \left\{ \nu < T < \nu + \frac{\alpha^2 \sigma^{-2} \log(1/\gamma)}{\kappa^2} \right\} \leq \gamma^{-3/4} \gamma = \gamma^{1/4}, \tag{40}
\]
where the first two inequalities follow from the definition of \( \mathcal{E}_\nu \), and the last inequality follows from the definition of \( D(\gamma) \).

**Step 2.** Note that for any \( i, z, z_{i-1}, \ldots, z_1 \) and an arbitrary \( x_0 \) we have
\[
\frac{m_{i,2}(z|z_{i-1}, \ldots, z_1)}{m_{i,1}(z|z_{i-1}, \ldots, z_1)} = \frac{\int q_{i,1}(z|x, z_{i-1}, \ldots, z_1) dP_2(x)}{\int q_{i,2}(z|x, z_{i-1}, \ldots, z_1) dP_1(x)} \leq \frac{e^{\alpha} q(z|x_0, z_{i-1}, \ldots, z_1) \int dP_2(x)}{e^{-\alpha} q(z|x_0, z_{i-1}, \ldots, z_1) \int dP_1(x)} = e^{2\alpha},
\]
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and we can similarly see that $m_{i,2}(z|z_{i-1}, \ldots, z_1)/m_{i,1}(z|z_{i-1}, \ldots, z_1) \geq e^{-2\alpha}$. When $\alpha$ is small, therefore, (say $\alpha \leq 0.5 \log(0.5)$), we will have for any $z_1, \ldots, z_{i-1}$ that
\[
\int m_{i,2}(z|z_{i-1}, \ldots, z_1) \log \frac{m_{i,2}(z|z_{i-1}, \ldots, z_1)}{m_{i,1}(z|z_{i-1}, \ldots, z_1)} \, dz \\
\leq - \int m_{i,2}(z|z_{i-1}, \ldots, z_1) \left\{ \frac{m_{i,1}(z|z_{i-1}, \ldots, z_1)}{m_{i,2}(z|z_{i-1}, \ldots, z_1)} - 1 - \left( \frac{m_{i,1}(z|z_{i-1}, \ldots, z_1)}{m_{i,2}(z|z_{i-1}, \ldots, z_1)} - 1 \right)^2 \right\} \, dz \\
= \int \frac{\{m_{i,2}(z|z_{i-1}, \ldots, z_1) - m_{i,1}(z|z_{i-1}, \ldots, z_1)\}^2}{m_{i,2}(z|z_{i-1}, \ldots, z_1)} \, dz \\
\leq \min(4, e^{2\alpha})(e^\alpha - 1)^2 \|P_1 - P_2\|_{TV}^2,
\]
where the final inequality is Lemma 1 in [Duchi et al. 2018 Supplementary material]. Calculations around Lemma 1 in [Duchi et al. 2018 Supplementary material] also reveal that
\[
\left| \log \frac{m_{i,2}(z|z_{i-1}, \ldots, z_1)}{m_{i,1}(z|z_{i-1}, \ldots, z_1)} \right| \leq \min(2, e^\alpha)(e^\alpha - 1)\|P_1 - P_2\|_{TV}.
\]
It follows from the Azuma–Hoeffding inequality (e.g. [Wainwright 2019 Corollary 2.20]) that for any $x > 0$ and $t \in \mathbb{N}$ we have
\[
P\left( \sum_{i=\nu+1}^{\nu+t} W_i \geq x + t \min(4, e^{2\alpha})(e^\alpha - 1)^2 \|P_1 - P_2\|_{TV} \ | Z_{\nu}, \ldots, Z_1 \right) \\
\leq \mathbb{P} \left( \sum_{i=\nu+1}^{\nu+t} \{W_i - \mathbb{E}(W_i|Z_{i-1}, \ldots, Z_1)\} \geq x \ \bigg| \ Z_{\nu}, \ldots, Z_1 \right) \\
\leq \exp\left( - \frac{2tx^2}{t \min(4, e^{2\alpha})(e^\alpha - 1)^2 \|P_1 - P_2\|_{TV}^2} \right),
\]
Due to (39) and our assumption that $\alpha \leq 1$, for small enough $c > 0$ we have that
\[
\frac{c\sigma^2\alpha^{-2} \log(1/\gamma)}{\kappa^2} \times \min(4, e^{2\alpha})(e^\alpha - 1)^2 \|P_1 - P_2\|_{TV}^2 \leq \frac{1}{4} \log(1/\gamma).
\]
For any $\nu \geq 1$ and $T \in D(\gamma)$, since $\{T \geq \nu\} \in \mathcal{F}_{\nu-1}$, it therefore follows that for such $c$ we have
\[
P_{\kappa, \sigma, \nu} \left\{ \nu < T < \nu + \frac{c\sigma^2\alpha^{-2} \log(1/\gamma)}{\kappa^2}, \sum_{i=\nu+1}^{T} W_i \geq (3/4) \log(1/\gamma) \ \bigg| \ T > \nu \right\} \\
\leq \text{ess sup} \sup_{\kappa, \sigma, \nu} \left\{ \max_{1 \leq t \leq \frac{\log(1/\gamma)}{c\sigma^2\alpha^{-2}}} \sum_{i=\nu+1}^{\nu+t} W_i \geq (3/4) \log(1/\gamma) \ \bigg| \ Z_1, \ldots, Z_{\nu} \right\} \\
\leq \text{ess sup} \sup_{\kappa, \sigma, \nu} \left\{ \max_{1 \leq t \leq \frac{\log(1/\gamma)}{c\sigma^2\alpha^{-2}}} \sum_{i=\nu+1}^{\nu+t} \{W_i - \mathbb{E}(W_i|Z_{i-1}, \ldots, Z_1)\} \geq (1/2) \log(1/\gamma) \ \bigg| \ Z_1, \ldots, Z_{\nu} \right\} \\
\leq \frac{c\sigma^2\alpha^{-2} \log(1/\gamma)}{\kappa^2} \exp \left\{ - \frac{(1/2) \log^2(1/\gamma)}{\min(4, e^{2\alpha})(e^\alpha - 1)^2 \|P_1 - P_2\|_{TV}^2} \right\} \\
\leq \frac{c\sigma^2\alpha^{-2} \log(1/\gamma)}{\kappa^2} \exp \left\{ - \log(1/\gamma) \right\} \leq \gamma^{1/4},
\]
where the third inequality follows by a union bound argument, the fourth inequality holds for small enough $c > 0$, and the last inequality holds for small enough $\gamma$. Since the upper bound is independent of $\nu$, it holds that
\[
\sup_{\nu \geq 1} P_{\kappa, \sigma, \nu} \left\{ \nu < T < \nu + \frac{c\sigma^2\alpha^{-2} \log(1/\gamma)}{\kappa^2}, \sum_{i=\nu+1}^{T} W_i \geq (3/4) \log(1/\gamma) \ \bigg| \ T \geq \nu \right\} \leq \gamma^{1/4},
\]
which leads to
\[
\sup_{\nu \geq 1} P_{\kappa, \sigma, \nu} \left\{ \nu < T < \nu + \frac{c\sigma^2\alpha^{-2} \log(1/\gamma)}{\kappa^2}, \sum_{i=\nu+1}^{T} W_i \geq (3/4) \log(1/\gamma) \right\} \leq \gamma^{1/4}. \quad (41)
\]
Combining (40) and (41), we have
\[
\sup_{\nu \geq 1} \mathbb{P}_{\kappa, \sigma, \nu} \left\{ \nu < T < \nu + \frac{c \sigma^2 \alpha^{-2} \log(1/\gamma)}{\kappa^2} \right\} \leq 2\alpha^{1/4}. \tag{42}
\]

**Step 3.** We now have, for any change point time $\Delta$,
\[
\mathbb{E}_{\kappa, \sigma, \Delta} \{ (T - \Delta)_+ \} \geq \frac{c \sigma^2 \alpha^{-2} \log(1/\gamma)}{\kappa^2} \mathbb{P}_{\kappa, \sigma, \Delta} \left\{ T - \Delta \geq \frac{c \sigma^2 \alpha^{-2} \log(1/\gamma)}{\kappa^2} \right\}
\]
\[
= \frac{c \sigma^2 \alpha^{-2} \log(1/\gamma)}{\kappa^2} \left[ \mathbb{P}_{\kappa, \sigma, \Delta} \{ T > \Delta \} - \mathbb{P}_{\kappa, \sigma, \Delta} \left\{ \Delta < T < \Delta + \frac{c \sigma^2 \alpha^{-2} \log(1/\gamma)}{\kappa^2} \right\} \right]
\]
\[
\geq \frac{c \sigma^2 \alpha^{-2} \log(1/\gamma)}{2\kappa^2},
\]
where the first inequality is due to Markov’s inequality, the second is due to (42) and the definition of the class of $\mathcal{D}(\gamma)$ of stopping times. 
\[\square\]
References

Bagci, Ibrahim Ethem, Roedig, Utz, Martinovic, Ivan, Schulz, Matthias, & Hollick, Matthias. 2015. Using channel state information for tamper detection in the internet of things. Pages 131–140 of: Proceedings of the 31st Annual Computer Security Applications Conference.

Berrett, Thomas, & Butucea, Cristina. 2019. Classification under local differential privacy. Annales de l’ISUP, 63 – 80 ans de Denis Bosq.

Berrett, Thomas B, & Butucea, Cristina. 2020. Locally private non-asymptotic testing of discrete distributions is faster using interactive mechanisms. Advances in Neural Information Processing Systems 34.

Berrett, Thomas B., Györfi, László, & Walk, Harro. 2021. Strongly universally consistent nonparametric regression and classification with privatised data. Electron. J. Statist., 15(1), 2430–2453.

Butucea, Cristina, Dubois, Amandine, Kroll, Martin, & Saumard, Adrien. 2020. Local differential privacy: Elbow effect in optimal density estimation and adaptation over Besov ellipsoids. Bernoulli, 26(3), 1727–1764.

Canonne, Clément L, Kamath, Gautam, McMillan, Audra, Smith, Adam, & Ullman, Jonathan. 2019. The structure of optimal private tests for simple hypotheses. Pages 310–321 of: Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing.

Cohen, Aloni, & Nissim, Kobbi. 2020. Towards formalizing the GDPR’s notion of singling out. Proceedings of the National Academy of Sciences, 117(15), 8344–8352.

Cummings, Rachel, Krehbiel, Sara, Mei, Yajun, Tuo, Rui, & Zhang, Wanrong. 2018. Differentially Private Change-Point Detection. In: Advances in Neural Information Processing Systems, vol. 31.

Cummings, Rachel, Krehbiel, Sara, Lat, Yulia, & Zhang, Wanrong. 2020. Privately detecting changes in unknown distributions. Pages 2227–2237 of: Proceedings of the 37th International Conference on Machine Learning, vol. 119.

Ding, Bolin, Kulkarni, Janardhan, & Yekhanin, Sergey. 2017. Collecting telemetry data privately. Pages 3571–3580 of: Advances in Neural Information Processing Systems.

Duchi, John C, & Ruan, Feng. 2018. The right complexity measure in locally private estimation: It is not the Fisher information. arXiv preprint arXiv:1806.05756.

Duchi, John C, Jordan, Michael I, & Wainwright, Martin J. 2013. Local privacy, data processing inequalities, and minimax rates. arXiv preprint arXiv:1302.3203.

Duchi, John C, Jordan, Michael I, & Wainwright, Martin J. 2018. Minimax optimal procedures for locally private estimation. Journal of the American Statistical Association, 113(521), 182–201.

Dwork, Cynthia, McSherry, Frank, Nissim, Kobbi, & Smith, Adam. 2006. Calibrating noise to sensitivity in private data analysis. Pages 265–284 of: Theory of Cryptography Conference. Springer.

Erlingsson, Úlfar, Pihur, Vasyl, & Korolova, Aleksandra. 2014. Rappor: Randomized aggregatable privacy-preserving ordinal response. Pages 1054–1067 of: Proceedings of the 2014 ACM SIGSAC conference on computer and communications security.

Erlingsson, Úlfar, Feldman, Vitaly, Mironov, Ilya, Raghu Nathan, Ananth, Talwar, Kunal, & Thakurta, Abhradeep. 2019. Amplification by shuffling: From local to central differential privacy via anonymity. Pages 2468–2479 of: Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms. SIAM.

Gregori, Dario, Azzolina, Danila, Lanera, Corrado, Prosepe, Ilaria, Destro, Nicolas, Lorenzoni, Giulia, & Berchialla, Paola. 2020. A first estimation of the impact of public health actions against COVID-19 in Veneto (Italy). J Epidemiol Community Health, 74(10), 858–860.

Hand, David J, & Blunt, Gordon. 2001. Prospecting for gems in credit card data. IMA Journal of management Mathematics, 12(2), 173–200.
Joseph, Matthew, Roth, Aaron, Ullman, Jonathan, & Waggoner, Bo. 2018. Local differential privacy for evolving data. *arXiv preprint arXiv:1802.07128*.

Kairouz, Peter, Oh, Sewoong, & Viswanath, Pramod. 2014. Extremal Mechanisms for Local Differential Privacy. *Pages 2879–2887 of: Advances in Neural Information Processing Systems*, vol. 27.

Kass-Hout, Taha A, Xu, Zhiheng, McMurray, Paul, Park, Soyoung, Buckeridge, David L, Brownstein, John S, Finelli, Lyn, & Groseclose, Samuel L. 2012. Application of change point analysis to daily influenza-like illness emergency department visits. *Journal of the American Medical Informatics Association, 19*(6), 1075–1081.

Lai, Tze Leung. 1998. Information bounds and quick detection of parameter changes in stochastic systems. *IEEE Transactions on Information Theory, 44*(7), 2917–2929.

Lau, T. S., & Peng Tay, W. 2020. Privacy-Aware Quickest Change Detection. *Pages 5999–6003 of: ICASSP 2020 - 2020 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*.

Lévy-Leduc, Céline, & Roueff, François. 2009. Detection and localization of change-points in high-dimensional network traffic data. *Annals of Applied Statistics, 3*(2), 637–662.

Padilla, Oscar Hernan Madrid, Yu, Yi, Wang, Daren, & Rinaldo, Alessandro. 2019. Optimal nonparametric multivariate change point detection and localization. *arXiv preprint arXiv:1910.13289*.

Rocher, Luc, Hendrickx, Julien M, & De Montjoye, Yves-Alexandre. 2019. Estimating the success of re-identifications in incomplete datasets using generative models. *Nature Communications, 10*(1), 1–9.

Rohde, Angelika, & Steinberger, Lukas. 2020. Geometrizing rates of convergence under local differential privacy constraints. *Annals of Statistics, 48*(5), 2646–2670.

Sweeney, Latanya. 2002. k-anonymity: A model for protecting privacy. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 10*(05), 557–570.

Tang, Jun, Korolova, Aleksandra, Bai, Xiaolong, Wang, Xueqiang, & Wang, Xiaofeng. 2017. Privacy loss in Apple’s implementation of differential privacy on MacOS 10.12. *arXiv preprint arXiv:1709.02753*.

Tartakovsky, Alexander G. 2014. Rapid detection of attacks in computer networks by quickest changepoint detection methods. *Pages 33–70 of: Data Analysis for Network Cyber-Security*. World Scientific.

Team, Apple Differential Privacy. 2017. *Learning with privacy at scale*.

Tsybakov, Alexandre B. 2008. *Introduction to nonparametric estimation*. Springer Science & Business Media.

Verzelen, Nicolas, Fromont, Magalie, Larasle, Matthieu, & Reynaud-Bouret, Patricia. 2020. Optimal Change-Point Detection and Localization. *arXiv preprint arXiv:2010.11470*.

Wainwright, Martin J. 2019. *High-dimensional statistics: A non-asymptotic viewpoint*. Vol. 48. Cambridge University Press.

Wallis, W Allen. 1980. The statistical research group, 1942–1945. *Journal of the American Statistical Association, 75*(370), 320–330.

Wang, Di, Gaboardi, Marco, & Xu, Jinhui. 2018. Empirical risk minimization in non-interactive local differential privacy revisited. *Pages 965–974 of: Advances in Neural Information Processing Systems*.

Yu, Yi. 2020. A review on minimax rates in change point detection and localisation. *arXiv preprint arXiv:2011.01857*.
Yu, Yi, Padilla, Oscar Hernan Madrid, Wang, Daren, & Rinaldo, Alessandro. 2020. A note on online change point detection. *arXiv preprint arXiv:2006.03283*.

Yu, Yi, Padilla, Oscar Hernan Madrid, Wang, Daren, & Rinaldo, Alessandro. 2021. Optimal network online change point localisation. *arXiv preprint arXiv:2101.05477*.

Zhang, Wanrong, Krehbiel, Sara, Tuo, Rui, Mei, Yajun, & Cummings, Rachel. 2021. Single and Multiple Change-Point Detection with Differential Privacy. *Journal of Machine Learning Research, 22*(29), 1–36.