HIGHER INDEPENDENCE COMPLEXES OF GRAPHS AND THEIR HOMOTOPY TYPES

PRIYAVRAT DESHPANDE AND ANURAG SINGH

Abstract. For \( r \geq 1 \), the \( r \)-independence complex of a graph \( G \) is a simplicial complex whose faces are subset \( I \subseteq V(G) \) such that each component of the induced subgraph \( G[I] \) has at most \( r \) vertices. In this article, we determine the homotopy type of \( r \)-independence complexes of certain families of graphs including complete \( s \)-partite graphs, fully whiskered graphs, cycle graphs and perfect \( m \)-ary trees. In each case, these complexes are either homotopic to a wedge of equi-dimensional spheres or are contractible. We also give a closed form formula for their homotopy types.

1. Introduction

Let \( G \) be a simple undirected graph. A subset \( I \subseteq V(G) \) of vertex set of \( G \), is called an independent set if the vertices of \( I \) are pairwise non-adjacent in \( G \). The independence complex of \( G \), denoted \( \text{Ind}_1(G) \), is a simplicial complex whose faces are the independent subsets of \( V \).

The study of homotopy type of independence complexes of graphs has received a lot of attention in last two decades. For example, in Babson and Kozlov’s proof of Lovász’s conjecture (in [1]) regarding odd cycles and graph homomorphism complexes the independence complexes of cycle graphs played an important role. In [17], Meshulam related homology groups of \( \text{Ind}_1(G) \) with the domination number of \( G \). The problem of determining a closed form formula for the homotopy type of \( \text{Ind}_1(G) \) for various classes of graphs is also well studied. For instance, see [16] for paths and cycle graphs, [13] for forests, [4, 5] for grid graphs, [14] for chordal graphs and [10] for categorical product of complete graphs and generalized mycielskian of complete graphs. Barmak [2] studied the topology of independence complexes of triangle-free graphs and claw-free graphs. He also gave a lower bound for the chromatic number of \( G \) in terms of the strong Lusternik-Schnirelmann category of \( \text{Ind}_1(G) \).

Recently in [19], Paolini and Salvetti generalized the notion of independence complexes by defining \( r \)-independence complex for any \( r \geq 1 \). For a graph \( G \), a subset \( I \subseteq V(G) \) is called \( r \)-independent if each connected component of the induced subgraph \( G[I] \) has at most \( r \) vertices. For \( r \geq 1 \), the \( r \)-independence complex of \( G \), denoted \( \text{Ind}_r(G) \) is a simplicial complex whose faces are all \( r \)-independent subsets of \( V(G) \). They established a relationship between the twisted homology of the classical braid groups and the homology of higher independence complexes of associated Coxeter graphs. In particular they showed that \( r \)-independence complexes of path graphs are homotopy equivalent to a wedge of spheres (see Theorem [12]).

The aim of this article is to initiate the study of these so-called higher independence complexes of graphs. Our focus is on determining a closed form formula for its homotopy type. In the article we identify several classes of graphs for which these complexes are either homotopic to a wedge of equi-dimensional spheres or are contractible. In each case we also determine the dimension of the spheres and their number; we achieve this using discrete Morse theory.

The paper is organized as follows. In Section 2 we recall all the important definitions and relevant tools from discrete Morse theory. The formal definition and basic properties of higher independence complexes is given in Section 3; here we also look at the complexes associated with...
with complete $s$-partite graphs and show that they are always homotopic to a wedge of spheres. We also show that if a graph is modified by attaching leaves to every vertex then the higher independence complexes of these new graphs are either wedge of spheres or are contractible. In Section 4 we consider the case of cycle graphs and in Section 5 we consider perfect $m$-ary trees; in both the cases the associated complexes are either wedge of spheres or are contractible. Moreover, in both the cases we construct optimal discrete Morse functions on these complexes. As a result all the critical cells are concentrated in a fixed dimension. The construction of these Morse functions as well as the formula for the number of critical cells both are combinatorially involved. Finally in Section 6 we outline some questions and conjectures.

2. Preliminaries

Let $G$ be a simple, undirected graph and $v \in V(G)$ be a vertex of $G$. The total number of vertices adjacent to $v$ is called degree of $v$, denoted $\deg(v)$. If $\deg(v) = 1$, then $v$ is called a leaf vertex. A graph $H$ with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ is called a subgraph of the graph $G$. For a nonempty subset $U$ of $V(G)$, the induced subgraph $G[U]$, is the subgraph of $G$ with vertices $V(G[U]) = U$ and $E(G[U]) = \{(a, b) \in E(G) : a, b \in U\}$. In this article, $G[V(G) \setminus A]$ will be denoted by $G - A$ for $A \subseteq V(G)$.

**Definition 2.1.** An (abstract) simplicial complex $\mathcal{K}$ on a finite set $X$ is a collection of subsets such that

(i) $\emptyset \in \mathcal{K}$, and
(ii) if $\sigma \in \mathcal{K}$ and $\tau \subseteq \sigma$, then $\tau \in \mathcal{K}$.

The elements of $\mathcal{K}$ are called simplices of $\mathcal{K}$. If $\sigma \in \mathcal{K}$ and $|\sigma| = k + 1$, then $\sigma$ is said to be $k$-dimensional (here, $|\sigma|$ denotes the cardinality of $\sigma$ as a set). Further, if $\sigma \in \mathcal{K}$ and $\tau \subseteq \sigma$ then $\tau$ is called a face of $\sigma$ and if $\tau \neq \sigma$ then $\tau$ is called a proper face of $\sigma$. The set of 0-dimensional simplices of $\mathcal{K}$ is denoted by $\mathcal{V}(\mathcal{K})$, and its elements are called vertices of $\mathcal{K}$. A subcomplex of a simplicial complex $\mathcal{K}$ is a simplicial complex whose simplices are contained in $\mathcal{K}$. For $s \geq 0$, the $k$-skeleton of a simplicial complex $\mathcal{K}$, denoted $\mathcal{K}^{(s)}$, is the collection of all those simplices of $\mathcal{K}$ whose dimension is at most $s$. In this article, we do not distinguish between an abstract simplicial complex and its geometric realization. Therefore, a simplicial complex will be considered as a topological space, whenever needed.

Let $S'$ denote a sphere of dimension $r$ and $\ast$ denotes join of two spaces. The following results will be used repeatedly in this article.

**Lemma 2.2 (3 Lemma 2.5).** Suppose that $\mathcal{K}_1$ and $\mathcal{K}_2$ are two finite simplicial complexes.

1. If $\mathcal{K}_1$ and $\mathcal{K}_2$ both have the homotopy type of a wedge of spheres, then so does $\mathcal{K}_1 \ast \mathcal{K}_2$.
2. $\bigcup_i S^{a_i} \ast \bigcup_j S^{b_j} \simeq \bigcup_{i,j} S^{a_i + b_j + 1}$

We now discuss some tools needed from discrete Morse theory. The classical reference for this is [9]. However, here we closely follow [15] for notations and definitions.

**Definition 2.3 (15 Definition 11.1).** A partial matching on a poset $P$ is a subset $\mathcal{M} \subseteq P \times P$ such that

(i) $(a, b) \in \mathcal{M}$ implies $a < b$; i.e., $a < b$ and no $c$ satisfies $a < c < b$, and
(ii) each $a \in P$ belong to at most one element in $\mathcal{M}$.

Note that, $\mathcal{M}$ is a partial matching on a poset $P$ if and only if there exists $A \subseteq P$ and an injective map $\mu : A \to P \setminus A$ such that $\mu(a) > a$ for all $a \in A$.

An acyclic matching is a partial matching $\mathcal{M}$ on the poset $P$ such that there does not exist a cycle

$\mu(a_1) > a_1 < \mu(a_2) > a_2 < \mu(a_3) > a_3 \ldots \mu(a_t) > a_t < \mu(a_1), t \geq 2$.

For an acyclic partial matching on $P$, those elements of $P$ which do not belong to the matching are called critical.
The main result of discrete Morse theory is the following.

**Theorem 2.4** ([13] Theorem 11.13). Let \( K \) be a simplicial complex and \( M \) be an acyclic matching on the face poset of \( K \). Let \( c_i \) denote the number of critical \( i \)-dimensional cells of \( K \) with respect to the matching \( M \). Then \( K \) is homotopy equivalent to a cell complex \( K_c \) with \( c_i \) cells of dimension \( i \) for each \( i \geq 0 \), plus a single 0-dimensional cell in the case where the empty set is also paired in the matching.

Following can be inferred from Theorem 2.4.

**Corollary 2.5.** If an acyclic matching has critical cells only in a fixed dimension \( i \), then \( K \) is homotopy equivalent to a wedge of \( i \)-dimensional spheres.

**Corollary 2.6.** If the critical cells of an acyclic matching on \( K \) form a subcomplex \( K' \) of \( K \), then \( K \) simplicially collapses to \( K' \), implying that \( K' \) is homotopy equivalent to \( K \).

In this article, by matching on a simplicial complex \( K \), we will mean that the matching is on the face poset of \( K \). Let \( K \) be a simplicial complex with vertex set \( X \) and \( N_x = \{ \sigma \in K : \sigma \setminus \{x\}, \sigma \cup \{x\} \in K \} \) be a subcomplex of \( K \), where \( x \in X \). Define a matching on \( K \) using \( x \) as follows:

\[
M_x = \{ (\sigma \setminus \{x\}, \sigma \cup \{x\}) : \sigma \setminus \{x\}, \sigma \cup \{x\} \in K \},
\]

**Definition 2.7.** Matching \( M_x \), as defined above, is called an element matching on \( K \) using vertex \( x \).

The following result tells us that an element matching is always acyclic.

**Lemma 2.8** ([15] Lemma 3.2]). The matching \( M_x \) is an acyclic matching on \( K \) and perfect acyclic matching on \( N_x \).

To obtain an acyclic matching on a simplicial complex \( K \), the next result tells us that one can define a sequence of element matchings on \( K \) using its vertices.

**Proposition 2.9** ([10] Proposition 3.1]). Let \( K_1 \) be a simplicial complex and \( x_1, x_2, \ldots, x_n \) are vertices of \( K_1 \). Then, \( \bigcup_{i=1}^{n} M_{x_i} \) is an acyclic matching on \( K_1 \), where \( M_{x_i} = \{ (\sigma \setminus \{x_i\}, \sigma \cup \{x_i\}) : \sigma \setminus \{x_i\}, \sigma \cup \{x_i\} \in K_i \} \) and \( K_{i+1} = K_i \setminus \{ \sigma : \sigma \in \eta \text{ for some } \eta \in M_{x_i} \} \) for \( i \in \{1, \ldots, n\} \).

**Proposition 2.9** will be used heavily in this article. Another useful way to construct an acyclic matching on a poset \( P \) is to first map \( P \) to some other poset \( Q \), then construct acyclic matchings on the fibers of this map and patch these acyclic matchings together to form an acyclic matching for the whole poset.

**Theorem 2.10** (Patchwork theorem ([15] Theorem 11.10)). If \( \varphi : P \to Q \) is an order-preserving map and for each \( q \in Q \), the subposet \( \varphi^{-1}(q) \) carries an acyclic matching \( M_q \), then \( \bigcup_q M_q \) is an acyclic matching on \( P \).

The following result is a special case of Theorem 2.10.

**Theorem 2.11** ([11] Lemma 4.3]). Let \( K_0 \) and \( K_1 \) be disjoint families of subsets of a finite set such that \( \tau \not\subset \sigma \) if \( \sigma \in K_0 \) and \( \tau \in K_1 \). If \( M_i \) is an acyclic matching on \( K_i \) for \( i = 0, 1 \) then \( M_0 \cup M_1 \) is an acyclic matching on \( K_0 \cup K_1 \).

3. **Basic results for higher independence complex**

We begin this section by exploring some basic results related to the main object of this article, i.e., higher independence complex. Henceforth, unless otherwise mentioned, \( r \geq 1 \) is a natural number and \( [n] \) will denote the set \( \{1, \ldots, n\} \).

**Definition 3.1.** Let \( G \) be a graph and \( A \subseteq V(G) \). Then \( A \) is called \( r \)-independent if connected components of \( G[A] \) have cardinality at most \( r \).
Definition 3.2. Let $G$ be a graph and $r \in \mathbb{N}$. The $r$-independence complex of $G$, denoted $\text{Ind}_r(G)$ has vertex set $V(G)$ and its simplices are all $r$-independent subsets of $V(G)$.

Example 3.3. Fig. 1 shows a graph $G$, its 1-independence complex and 2-independence complex. The 1-independence complex of $G$ consists of 2 maximal simplices, namely $\{v_2, v_3, v_4\}$ and $\{v_1\}$. The complex $\text{Ind}_2(G)$ consists of 4 maximal simplices, namely $\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}$ and $\{v_2, v_3, v_4\}$.

![Figure 1](image-url)

The following are some easy observations from the definition of $r$-independence complex.

Observation 3.4. (i) For any graph $G$, $\text{Ind}_r(G)$ is $(r-2)$-connected. Moreover, if $r \geq |V(G)|$ then $\text{Ind}_r(G) \simeq \{\text{point}\}$.

(ii) If $G$ is connected graph and $|V(G)| = r + 1$, then $\text{Ind}_r(G) \simeq S^{r-1}$.

(iii) Let $K_n$ be the complete graph on $n$ vertices, then $\text{Ind}_r(K_n)$ is equal to $(r-1)^{\text{th}}$ skeleton of an $(n-1)$-simplex, denoted $\Delta^{n-1}$, i.e.,

$$\text{Ind}_r(K_n) = (\Delta^{n-1})^{(r-1)}.$$ 

(iv) If $G$ and $H$ are two disjoint graphs, then

$$\text{Ind}_r(G \sqcup H) \simeq \text{Ind}_r(G) \ast \text{Ind}_r(H).$$

(v) If $G$ has a non-empty connected component of cardinality at most $r$, then $\text{Ind}_r(G)$ is contractible.

In Observation 3.4(iii), we saw that $\text{Ind}_r(K_n)$ is homotopy equivalent to a wedge of spheres of dimension $r-1$. So one would expect a similar result for complete $s$-partite graphs for $s \geq 2$. Where, a complete $s$-partite graph is a graph in which vertex set can be decomposed into $s$ disjoint sets $V_1, V_2, \ldots, V_s$ such that no two vertices within the same set $V_i$ are adjacent and if $v \in V_i$ and $w \in V_j$ for $i \neq j$ then $v$ is adjacent to $w$.

Theorem 3.5. Let $s \geq 2$ and $r \geq 1$. Given $m_1, m_2, \ldots, m_s \geq 1$, the homotopy type of $r^{\text{th}}$ independence complex of the complete $s$-partite graph $K_{m_1, \ldots, m_s}$ is given as follows,

$$\text{Ind}_r(K_{m_1, \ldots, m_s}) \simeq \bigvee_t S^{r-1},$$

where $t = \binom{M-1}{r} - \sum_{i=1}^{s} \binom{m_i-1}{r}$ and $M := \sum_{i=1}^{s} m_i$.

Proof. For simplicity of notations, we denote $K_{m_1, \ldots, m_s}$ by $G$ in this proof. Let $V_1, V_2, \ldots, V_s$ be the partition of vertices of $G$ and $V_i = \{v_{i1}^{m_i}, \ldots, v_{in}^{m_i}\}$ for $i \in [s]$. We now define a sequence of element matching on $\Delta_0 := \text{Ind}_r(G)$ using vertices $v_1^1, v_2^1, \ldots, v_s^1$. For $i \in [s]$, define

$M_i = \{(\sigma, \sigma \cup v_i^1) : v_i^1 \notin \sigma \text{ and } \sigma, \sigma \cup v_i^1 \in \Delta_{i-1}\}$,

$N_i = \{\sigma \in \Delta_{i-1} : \sigma \in \eta \text{ for some } \eta \in M_i\}$, and

$\Delta_i = \Delta_{i-1} \setminus N_i$. 


This completes the proof of Theorem 3.5. □

Using Proposition 2.10, we get that $M = \bigcup_{i=1}^{s} M_i$ is an acyclic matching on Ind$_{r}(G)$ with $\Delta_s$ as the set of the critical cells.

**Claim 1.** The set of critical cells after $s$th element matching is given as follows:

$$\Delta_s = \{ \sigma \in \text{Ind}_r(G) : |\sigma| = r, \ v_1^i \notin \sigma \ \forall i \in [s] \text{ and } \sigma \nsubseteq V_i \text{ for any } i \in [s] \} \bigcup \{ \sigma \in \text{Ind}_r(G) : |\sigma| = r, \ v_1^i \notin \sigma \text{ and } v_i^1 \in \sigma \text{ for some } i \in \{2, \ldots, s\} \}.$$ 

**Proof of Claim 7.** Clearly, if $|\sigma| = r$, $v_1^i \notin \sigma \ \forall i \in [s]$ and $\sigma \nsubseteq V_i$ for any $i \in [s]$ then $G[\sigma \cup v_1^i]$ is a connected graph of cardinality $r + 1$ implying that $\sigma \notin N_i$ for all $i \in [s]$. Therefore, $\{ \sigma \in \text{Ind}_r(G) : |\sigma| = r, \ v_1^i \notin \sigma \ \forall i \in [s] \text{ and } \sigma \nsubseteq V_i \text{ for any } i \in [s] \} \subseteq \Delta_s$. Now, let $|\sigma| = r$ and $v_i^1 \in \sigma$ for some $i \in \{2, \ldots, s\}$. For $i \in \{2, \ldots, s\}$, if $v_i^1 \in \sigma$ then $\sigma \setminus v_i^1 \in N_i$ implies that $\sigma \notin N_i$. If $v_i^1 \notin \sigma$, then $|\sigma| = r$ and $v_i^1 \in \sigma$ for some $i \neq j$ implies that $G[\sigma \cup v_j^1]$ is connected subgraph of cardinality $r + 1$, hence $\sigma \notin N_j$. Thus $\{ \sigma \in \text{Ind}_r(G) : |\sigma| = r, \ v_1^i \notin \sigma \text{ and } v_i^1 \in \sigma \text{ for some } i \in \{2, \ldots, s\} \} \subseteq \Delta_s$.

Now consider $\sigma \in \Delta_s$. If $\sigma \subseteq V_i$ or $|\sigma| < r$ or $v_1^i \notin \sigma$, then $\sigma \in N_i$. If $\sigma \subseteq V_i$ for some $i \in [s]$ and $v_1^i \notin \sigma$. Then $\sigma \cup v_1^i \in \text{Ind}_r(G)$ implying that $\sigma \in N_i$ which is a contradiction to the fact that $\sigma \in \Delta_s$. Thus, either $\sigma \nsubseteq V_i$ for any $i \in [s]$ or if $\sigma \subseteq V_i$ for some $i \in \{2, \ldots, s\}$ then $v_i^1 \in \sigma$. Now, let $|\sigma| > r$. $\sigma \in \text{Ind}_r(G)$ implies that $\sigma \subseteq V_i$ for some $i \in [s]$ but then $\sigma \in N_i$. Therefore, $\sigma = r$. This completes the proof of Claim 1. □

Using Claim 1, we get that $M$ is an acyclic matching on Ind$_r(G)$ with exactly $|\Delta_s|$ critical cells of dimension $(r - 1)$. Therefore, Corollary 2.6 implies that Ind$_r(G)$ is homotopy equivalent to a wedge of $|\Delta_s|$ spheres of dimension $r - 1$. We now compute the cardinality of the set $\Delta_s$.

Using Claim 1, we get

$$|\Delta_s| = \left( \sum_{i=1}^{s} \frac{m_i - 1}{r} \right) - \sum_{i=1}^{s} \left( \frac{m_i - 1}{r} \right) + \sum_{j=2}^{s} \left( \sum_{i=1}^{j-1} \frac{m_i - 1}{r - 1} \right)$$

This completes the proof of Theorem 3.5. □

We now show that adding a whisker (a leaf vertex) at each vertex of $G$ simplifies the homotopy type of higher independence complex. By adding a whisker at vertex $v$ of $G$, we mean a new vertex is attached to $v$ (the induced subgraph $K_2$ is called whisker). We show that the higher independence complex of fully whiskered graphs is homotopy equivalent to a wedge of equi-dimensional spheres.

**Definition 3.6.** Given a graph $G$, a fully whiskered graph of $G$, denoted $W(G)$, is a graph in which a whisker is added to each vertex of $G$.

![Figure 2](image-url)

FIGURE 2

5
Theorem 3.7. Let $G$ be a connected graph and $V(G) = \{a_1, a_2, \ldots, a_n\}$ be the set of vertices of $G$. The homotopy type of $\text{Ind}_r(W(G))$ is given by the following formula:

$$\text{Ind}_r(W(G)) \simeq \begin{cases} \bigvee_{r-1} S^{r-1}, & \text{if } n \leq r \leq 2n - 1, \\ \{\text{point}\}, & \text{otherwise}. \end{cases}$$

Proof. Let $\{b_1, b_2, \ldots, b_n\}$ denote the set of leaves of graph $W(G)$ such that $b_i$ is adjacent to $a_i$ for each $i \in [n]$. Let $\Delta_0 = \text{Ind}_r(W(G))$. We define a sequence of element matching on $\Delta_0$ using the leaf vertices. For $i \in [n]$, define

$$M(b_i) = \{\sigma, \sigma \cup b_i : b_i \notin \sigma, \text{ and } \sigma, \sigma \cup b_i \in \Delta_{i-1}\},$$

and

$$N(b_i) = \{\sigma \in \Delta_{i-1} : \sigma \in m \text{ for some } m \in M(b_i)\} \\text{and} \\Delta_i = \Delta_{i-1} \setminus N(b_i).$$

Claim 2. If $\sigma \in \text{Ind}_r(W(G))$ and $V(G) \not\subset \sigma$ then $\sigma \notin \Delta_n$, i.e. $\sigma$ is not a critical cell.

Let $p = \min\{i : a_i \notin \sigma\}$. From Eq. (1), $\sigma$ belongs to $N(b_p)$, which implies that $\sigma \notin \Delta_n$. This prove Claim 2.

Firstly, let $r < n$. Since $G$ is connected, if $\sigma \in \text{Ind}_r(W(G))$ then $V(G) \not\subset \sigma$. Hence, result follows from Claim 2 and Corollary 2.5.

Secondly, assume that $r \geq n$. From definition of $\text{Ind}_r(G)$, it is easy to see that if $\sigma \in \text{Ind}_r(G)$ and cardinality of $\sigma$ is less than $r$ then $\sigma \in N(b_1)$. Thus, if $\sigma \in \Delta_n$ then cardinality of $\sigma$ is at least $r$ and $b_1 \notin \sigma$. Using Claim 2 we see that if $\sigma \in \Delta_n$ then $V(G) \subset \sigma$. Further, if $\sigma \in \text{Ind}_r(G)$ and $V(G) \subset \sigma$ then $\sigma \notin N(b_i)$ for any $i \in [n]$. Which shows that $\sigma \in \Delta_n$ iff $V(G) \subset \sigma$, $a_1 \notin \sigma$ and $|\sigma| \geq r$. Moreover, $V(G) \subset \sigma$ implies that $G[\sigma]$ is always connected. Therefore, cardinality of $\sigma$ is exactly $r$. Combining all these arguments together, we see that $\Delta_n$ is a set of $\binom{n-1}{r-n}$ cells of dimension $r-1$. Thus the result follows from Corollary 2.5.

We now show that, for a graph $G$, adding more whiskers at non-leaf vertices of $W(G)$ does not affect the connectivity of the higher independence complex. In particular, we give closed form formula for the homotopy type of $r$-independence complexes of these new graphs.

Theorem 3.8. Let $G$ be a connected graph and $W = \{a_1, a_2, \ldots, a_n\}$ be the set of all non-leaf vertices of $G$. For $i \in \{1, \ldots, n\}$, let $l_i$ denote the number of leaves adjacent to vertex $a_i$. If $l_i > 0$ for all $i \in \{1, \ldots, n\}$, then the homotopy type of $\text{Ind}_r(G)$ is given as follows.

$$\text{Ind}_r(G) \simeq \begin{cases} \bigvee_{r-1} S^{r-1}, & \text{if } r \geq n, \\ \{\text{point}\}, & \text{otherwise}, \end{cases} \text{where } t = \frac{\sum_{i=1}^{n} l_i - 1}{r - n}.$$ 

Proof. Arguments in this proof are similar to that of in proof of Theorem 3.7. For $i \in [n]$, let $\{b_{1i}, b_{2i}, \ldots, b_{li}\}$ denote the set of leaves adjacent to $a_i$. Let $\Delta_0 = \text{Ind}_r(G)$. We define a sequence of element matching on $\Delta_0$ using leaf vertices $b_{1i}, b_{2i}, \ldots, b_{li}$. For $i \in [n]$, define

$$M(b_{1i}) = \{\sigma, \sigma \cup b_{1i} : b_{1i} \notin \sigma, \text{ and } \sigma, \sigma \cup b_{1i} \in \Delta_{i-1}\},$$

and

$$N(b_{1i}) = \{\sigma \in \Delta_{i-1} : \sigma \in m \text{ for some } m \in M(b_{1i})\} \\text{and} \\Delta_i = \Delta_{i-1} \setminus N(b_{1i}).$$

Claim 3. If $\sigma \in \text{Ind}_r(G)$ and $W \not\subset \sigma$ then $\sigma \notin \Delta_n$, i.e. $\sigma$ is not a critical cell.

Let $p = \min\{i : a_i \notin \sigma\}$. From Eq. (2), $\sigma$ belongs to $N(b_{p,1})$, which implies that $\sigma \notin \Delta_n$. This prove Claim 3.
Firstly, let $r < n$. Since $G$ is connected and $W$ is collection of all non-leaf vertices, $G[W]$ is connected subgraph of cardinality $n$. Therefore, if $\sigma \in \text{Ind}_r(G)$ then $W \not\subseteq \sigma$. Hence, result follows from Claim 3 and Corollary 2.5.

Secondly, assume that $r \geq n$. From definition of $\text{Ind}_r(G)$, it is easy to see that if $\sigma \in \text{Ind}_r(G)$ and cardinality of $\sigma$ is less than $r$ then $\sigma \in N(b_{i,1})$. Thus, if $\sigma \in \Delta_n$ then cardinality of $\sigma$ is at least $r$ and $b_{1,1} \notin \sigma$. Using Claim 3 we see that if $\sigma \in \Delta_n$ then $W \subseteq \sigma$. Further, if $\sigma \in \text{Ind}_r(G)$ and $W \subseteq \sigma$ then $\sigma \notin N(b_{i,1})$ for any $i \in [n]$. Which shows that $\sigma \in \Delta_n$ iff $W \subseteq \sigma$, $b_{1,1} \notin \sigma$ and $|\sigma| \geq r$. Moreover, $W \subseteq \sigma$ implies that $G[\sigma]$ is always connected. Therefore, cardinality of $\sigma$ is exactly $r$. Combining all these arguments together, we see that $\Delta_n$ is a set of $\left(\sum_{i=1}^n l_i - 1 \right) / r - 1$ cells of dimension $r-1$. Thus the result follows from Corollary 2.5.

For $n \geq 1$, a path graph of length $n$, denoted $P_n$, is a graph with vertex set $V(P_n) = \{1, \ldots, n\}$ and edge set $E(P_n) = \{(i, i+1) \mid 1 \leq i \leq n-1\}$. For $n \geq 3$, a cycle graph, denoted $C_n$, is a graph with vertex set $V(C_n) = \{1, \ldots, n\}$ and edge set $E(C_n) = \{(i, i+1) \mid 1 \leq i \leq n-1\} \cup \{(1, n)\}$. We can now compute $r$-independence complexes of almost all caterpillar graphs. A caterpillar graph is a path graph with some whiskers on vertices.

**Definition 3.9.** Let $G$ be a graph with $V(G) = \{a_1, \ldots, a_n\}$ and $L = \{l_1, \ldots, l_n\}$ be a set of $n$ non-negative integers. Define a graph $G^L$ with the following data:

$$V(G^L) = V(G) \cup \bigcup_{l_i > 0} \{b_{i,1}, \ldots, b_{i,l_i}\}$$

$$E(G^L) = E(G) \cup \bigcup_{l_i > 0} \{(a_i, b_{i,j}) : 1 \leq j \leq l_i\}$$

See Fig. 3 for examples. Clearly, $P_n^L$ is a caterpillar graph.

![Figure 3](image-url)

**Corollary 3.10.** Given $L = (l_1, l_2, \ldots, l_n)$ with $l_i > 0$ for every $i \in \{0, 1, \ldots, n\}$. Then,

$$\text{Ind}_r(P_n^L) \simeq \text{Ind}_r(C_n^L) \simeq \begin{cases} \bigvee \left(\sum_{i=1}^n l_i - 1 \right) \setminus S^{r-1} & \text{if } r \geq n, \\ \text{point} & \text{otherwise.} \end{cases}$$

4. **Higher Independence Complexes of Cycle Graphs**

Kozlov, in [15], computed the homotopy type of 1-independence complex of cycle graphs using discrete Morse theory. He proved the following result:

**Proposition 4.1** ([15] Proposition 11.17). For any $n \geq 3$, we have

$$\text{Ind}_1(C_n) \simeq \begin{cases} S^{k-1} \setminus S^{k-1} & \text{if } n = 3k, \\ S^{k-1} & \text{if } n = 3k \pm 1. \end{cases}$$
In this section, we generalize this result and compute the homotopy type of \( \text{Ind}_r(C_n) \) for any \( n \geq 3 \) and \( r \geq 1 \). In particular, we define a perfect acyclic matching on \( \text{Ind}_{d-2}(C_n) \). We will use the following result, proved by Paolini and Salvetti in \([19]\).

**Theorem 4.2 ([19] Proposition 3.7).** For \( d \geq 3 \), we have

\[
\text{Ind}_{d-2}(P_n) \cong \begin{cases} 
S^{dk-2k-1}, & \text{if } n = dk \text{ or } n = dk - 1; \\
\{\text{point}\}, & \text{otherwise.}
\end{cases}
\]

To make our computations of \( \text{Ind}_{d-2}(C_n) \) easier, we first improve the acyclic matching defined by Paolini and Salvetti on \( \text{Ind}_r(P_n) \), and get a perfect acyclic matching on \( \text{Ind}_{d-2}(P_n) \).

**Proposition 4.3.** There exists a perfect acyclic matching on \( \text{Ind}_{d-2}(P_n) \). In particular, if \( n = dk \) or \( dk - 1 \) and \( \{1, 2, \ldots, n\} \) is the vertex set of \( P_n \), then the only critical cell is \( \bigsqcup_{i=0}^{k-1} \{di + 2, \ldots, di + d - 1\} \).

**Proof.** Let \( n = dk - t \) for some \( t \in \{0, 1, \ldots, d - 1\} \), let \( \Delta = \{\sigma \in \text{Ind}_{d-2}(P_n) : \sigma \cap \{d, 2d, \ldots, dk\} \neq \emptyset\} \) and let \( \Delta_0 = \text{Ind}_{d-2}(P_n) \setminus \Delta \). In \([19]\) Proposition 3.7, Paolini and Salvetti constructed an acyclic matching \( M \) on \( \text{Ind}_{d-2}(P_n) \) with \( \Delta_0 \) as the set of critical cells. Here, we construct an acyclic matching on \( \Delta_0 \). For \( i \in \{0, \ldots, k - 1\} \), define

\[
M_i = \{\sigma, \sigma \cup \{di + 1\} : di + 1 \notin \sigma \text{ and } \sigma, \sigma \cup \{di + 1\} \in \Delta_i\},
\]

\[
N_i = \{\sigma \in \Delta_i : \sigma \in \eta \text{ for some } \eta \in M_i\},
\]

\[
\Delta_{i+1} = \Delta_i \setminus N_i.
\]

From Proposition 2.9 \( M' = \bigsqcup_{i=0}^{k-1} M_i \) is an acyclic matching on \( \Delta_0 \) with \( \Delta_k \) as the set of critical cells. Clearly, if \( n = dk \) or \( dk - 1 \) then \( \Delta_k = \{\sigma\} \), where \( \sigma = \bigsqcup_{i=0}^{k-1} \{di + 2, \ldots, di + d - 1\} \).

Further, if \( n \neq dk, dk - 1 \) then \( N_{k-1} = \Delta_{k-1} \). Using Theorem 2.11 we get that \( M \sqcup M' \) is an acyclic matching on \( \text{Ind}_{d-2}(P_n) \) with \( \Delta_k \) as set of critical cells. This completes the proof of Proposition 4.3.

Following are some immediate corollaries of Proposition 4.3.

**Corollary 4.4.** Let \( d \geq 3 \) and \( G \) be disjoint union of \( m \) path graphs of lengths \( d \) or \( d - 1 \). Then there exists an acyclic matching on \( \text{Ind}_{d-2}(G) \) with exactly one critical cell of dimension 0 and one of dimension \( (d - 3)m + m - 1 = dm - 2m - 1 \).

**Corollary 4.5.** Let \( d \geq 3 \) and \( G \) be disjoint union of \( m \) path graphs. If any connected component of \( G \) has length less than \( d - 1 \) or greater than \( d \) and less than \( 2d - 2 \), then there exists an acyclic matching on \( \text{Ind}_{d-2}(G) \) with no critical cell.

From Observation 3.3 (i) and (ii), we get that \( \text{Ind}_{d-2}(C_n) \cong \{\text{point}\} \) for all \( n \leq d - 2 \) and \( \text{Ind}_{d-2}(C_{d-1}) \cong S^{d-3} \). We now determine the homotopy type of \( \text{Ind}_{d-2}(C_n) \) for \( n \geq d \). The idea of this proof is to define acyclic matching of subsets of face poset of \( \text{Ind}_r(C_n) \) and then use Theorem 2.10.

**Theorem 4.6.** For \( n \geq d \geq 3 \), we have

\[
\text{Ind}_{d-2}(C_n) \cong \begin{cases}
\bigsqcup_{i=0}^{d-1} S^{dk-2k-1}, & \text{if } n = dk; \\
S^{dk-2k-1}, & \text{if } n = dk + 1; \\
S^{dk-2k}, & \text{if } n = dk + 2; \\
\vdots & \vdots \\
S^{dk-2k+d-3}, & \text{if } n = dk + (d - 1).
\end{cases}
\]
Proof. In this proof, we assume that the vertices of $C_n$ are labeled as $1, 2, \ldots, n$ anti-clockwise. Let $k$ denote the maximal integer such that $dk \leq n$. Furthermore, let $E$ be a chain with $k + 1$ elements labeled as follows:

$$e_d > e_{2d} > \cdots > e_{dk} > e_r.$$  

We define a map

$$\phi : \mathcal{F}(\text{Ind}_{d-2}(C_n)) \to E$$

by the following rule. The simplices that contain the vertex labeled $d$ get mapped to $e_d^\ell$; the simplices that do not contain the vertex labeled $d$, but contain the vertex labeled $2d$ get mapped to $e_{2d}$; the simplices that do not contain the vertices labeled $d$ and $2d$, but contain the vertex labeled $3d$ get mapped to $e_{3d}$; and so on. Finally, the simplices that do not contain any of the vertices labeled $d, 2d, \ldots, dk$ all get mapped to $e_r$.

Clearly, the map $\phi$ is order-preserving, since if one takes a larger simplex, it will have more vertices, and the only way its image may change is to go up when a new element from the set $\{d, 2d, \ldots, dk\}$ is added and is smaller than the previously smallest one.

Let us now define acyclic matchings on the preimages of elements of $E$ under the map $\phi$. We split our argument into cases.

**Case 1:** We first consider the preimages $\phi^{-1}(e_{2d})$ through $\phi^{-1}(e_{dk})$. Let $t$ be an integer such that $2 \leq t \leq k$. The preimage $\phi^{-1}(e_{dt})$ consists of all simplices $\sigma$ such that $d, 2d, \ldots, dt - 1 \notin \sigma$, while $dt \in \sigma$. Since $\sigma \in \text{Ind}_{d-2}(C_n)$, $\{dt - 1, dt - 2, \ldots, dt - (d - 2)\} \notin \sigma$. This means that the pairing $\sigma \leftrightarrow \sigma \cup \{dt - (d - 1)\}$ provides a well-defined matching, which is acyclic from Lemma 2.3.

**Case 2:** Next, we consider the preimage $\phi^{-1}(e_d)$. For $\sigma \in \text{Ind}_{d-2}(C_n)$, let $\text{conn}_d(\sigma)$ be the number of vertices of connected components of $C_n[\sigma]$ containing vertex labeled $d$. We define a map $\psi : \phi^{-1}(e_d) \to \{e_1 < e_2 < \cdots < e_{d-2}\}$

$$\psi(\sigma) = \begin{cases} e_1, & \text{if } \text{conn}_d(G[\sigma]) = 1, \\ e_2, & \text{if } \text{conn}_d(G[\sigma]) = 2, \\ \vdots \\ e_{d-2}, & \text{if } \text{conn}_d(G[\sigma]) = d - 2. \end{cases}$$

Clearly, $\psi$ is a poset map and for $i \in \{1, \ldots, d - 2\}$, if $\sigma \in \psi^{-1}(e_i)$ then cardinality of $\sigma$ is at least $i$.

For $t \geq 1$, let $P_t^{(i+1, \ldots, i+t)}$ denote the path graph of length $t$ whose vertices are labeled as $i + 1, i + 2, \ldots, i + t$ (see Fig. 4).

\begin{figure}[h]
\centering
\begin{tikzpicture}
    \node (i1) at (0,0) {$i+1$};
    \node (i2) at (1,0) {$i+2$};
    \node (it) at (5,0) {$i+t$};
    \node (it1) at (6,0) {$i+t-1$};
    \node (it2) at (7,0) {$i+t-2$};
    \node (i) at (8,0) {$\cdots$};
    \node (i1) at (10,0) {$i+1, \ldots, i+t$};
    \draw (i1) -- (i2) -- (it) -- (it1) -- (it2) -- (i);
\end{tikzpicture}
\caption{$P_t^{(i+1, \ldots, i+t)}$}
\end{figure}

We now define a matching on $\phi^{-1}(e_d)$ if $d - 2$ steps as follows.

**Step 1:** For $p \geq 1$, it is clear that the $p$-cells of $\psi^{-1}(e_1)$ are in 1-1 correspondence with the $p - 1$ cells of $\text{Ind}_{d-2}(P_{n-3}^{(d+2, \ldots, n, 1, \ldots, d-2)})$ with one extra cell of dimension 0, which is $\{d\}$. Using Proposition 4.3, let $M_0$ be a perfect matching on $\text{Ind}_{d-2}(P_{n-3}^{(d+2, \ldots, n, 1, \ldots, d-2)})$. Define a matching $M_1$ on $\psi^{-1}(e_1)$ as follows: $(\sigma, \tau) \in M_0$ iff $(\sigma \cup d, \tau \cup d) \in M_1$. Therefore, we get the following.

- Matching $M_1$ is an acyclic matching on $\psi^{-1}(e_1)$ with the following property. If $n - 3 = dk - 1$ or $n - 3 = dk$, i.e., $n = dk + 2$ or $dk + 3$, then there is only one critical cell of
and with no critical cell otherwise.

$$\psi L \in (6)$$

$$\psi$$

Element of the following set:

$$(5)$$

$$\psi$$

merge them together to get an acyclic matching on

$$\psi \Delta = \psi (d-1,d) \cup \psi (d,d+1)$$

Here, $$\Delta$$ is collection of all those cells $$\sigma \in \psi^{-1}(c_2)$$ such that $$\{d-1,d\}$$ is the connected component of $$C_n[\sigma]$$. Similarly, $$\Delta_{\psi^{-1}(c_2)}$$ is collection of all those cells $$\sigma \in \psi^{-1}(c_2)$$ such that $$\{d,d+1\}$$ is the connected component of $$C_n[\sigma]$$. Clearly, $$\psi^{-1}(c_2) = \psi (d-1,d) \cup \psi (d,d+1)$$ and $$\psi (d-1,d) \cap \psi (d,d+1) = \emptyset$$. Now, the idea is to define an acyclic matching on $$\psi (d-1,d)$$, $$\psi (d,d+1)$$ and merge them together to get an acyclic matching on $$\psi^{-1}(c_2)$$.

**Step 2:** Observe that, in $$C_n$$, there are exactly two connected subgraphs of cardinality two containing vertex $$d$$, which are $$C_n[\{d-1,d\}] = P_{2}^{(d-1,d)}$$ and $$C_n[\{d,d+1\}] = P_{2}^{(d,d+1)}$$. Thus, cells of $$\psi^{-1}(c_2)$$ can be partitioned into two smaller disjoint subsets $$\Delta_{\psi^{-1}(c_2)}$$ and $$\Delta_{\psi^{-1}(c_2)}$$. Here, $$\Delta_{\psi^{-1}(c_2)}$$ is collection of all those cells $$\sigma \in \psi^{-1}(c_2)$$ such that $$\{d-1,d\}$$ is the connected component of $$C_n[\sigma]$$. Similarly, $$\Delta_{\psi^{-1}(c_2)}$$ is collection of all those cells $$\sigma \in \psi^{-1}(c_2)$$ such that $$\{d,d+1\}$$ is the connected component of $$C_n[\sigma]$$. Clearly, $$\psi^{-1}(c_2) = \psi (d-1,d) \cup \psi (d,d+1)$$ and $$\psi (d-1,d) \cap \psi (d,d+1) = \emptyset$$. Now, the idea is to define an acyclic matching on $$\psi (d-1,d)$$, $$\psi (d,d+1)$$ and merge them together to get an acyclic matching on $$\psi^{-1}(c_2)$$.

1. Observe that, for $$p \geq 2$$, the $$p$$-cells of $$\Delta_{\psi^{-1}(c_2)}$$ are in 1-1 correspondence with the $$p-2$$ cells of $$\text{Ind}_{d-2}(P_{n-4}^{(d+2,...,n,1,...,d-3)})$$ with one extra cell of dimension 1, which is $$\{d-11,d\}$$. Using Proposition 4.3, let $$M$$ be a perfect matching on $$\text{Ind}_{d-2}(P_{n-4}^{(d+2,...,n,1,...,d-3)})$$. Define a matching $$M$$ on $$\Delta_{\psi^{-1}(c_2)}$$ as follows: $$(\sigma, \tau) \in M$$ iff $$(\sigma \cup \{d-1,d\}, \tau \cup \{d-1,d\}) \in M$$. Therefore, we get the following.

Matching $$M$$ is an acyclic matching on $$\Delta_{\psi^{-1}(c_2)}$$ with the following property. If $$n - 4 = dk - 1$$ or $$dk + 4$$, then there is only one critical cell of dimension $$dk - 2k + 1$$ and that is

$$\{d-1,d\} \cup \{id+3,...,(i+1)d\} \cup \{n,1,...,d-3\}$$, if $$n = dk + 3$$,

$$\{d-1,d\} \cup \{id+3,...,(i+1)d\} \cup \{n-1,1,...,d-4\}$$, if $$n = dk + 4$$.

Otherwise, there is no critical cell.

2. Similar to the case of $$\Delta_{\psi^{-1}(c_2)}$$ and using the matching of $$\text{Ind}_{d-2}(P_{n-4}^{(d+3,...,n,1,...,d-2)})$$, we get an acyclic matching, say $$M$$ on $$\Delta_{\psi^{-1}(c_2)}$$ with the following property.

If $$n - 4 = dk - 1$$ or $$dk$$, then there is only one critical cell of dimension $$dk - 2k + 1$$ and that is

$$\{d,d+1\} \cup \{id+4,...,(i+1)d+1\} \cup \{1,...,d-2\}$$, if $$n = dk + 3$$,

$$\{d,d+1\} \cup \{id+4,...,(i+1)d+1\} \cup \{n,1,...,d-3\}$$, if $$n = dk + 4$$.

Otherwise, there is no critical cell.

Since $$\psi^{-1}(c_2) = \psi (d-1,d) \cup \psi (d,d+1)$$, $$M$$ is an acyclic matching on $$\psi^{-1}(c_2)$$ with exactly two critical cells of dimension $$dk - 2k + 1$$ whenever $$n = dk + 3$$ or $$dk + 4$$ and with no critical cell otherwise.

We now define a matching on $$\psi^{-1}(c_{d-2})$$. Idea here is similar to that of step 2.

**Step 2:** Observe that, in $$C_n$$, there are exactly $$d - 2$$ connected subgraphs of cardinality $$d - 2$$ containing vertex $$d$$, and these subgraphs are path graphs of length $$d - 2$$, i.e., one of the element of the following set: $$\mathcal{L} = \{L_{d-2}^{3,4,...,d-1,d}, L_{d-2}^{4,5,...,d-1,d+1},..., L_{d-2}^{d,d+1,...,2d-4,2d-3}\}$$.

Thus, cells of $$\psi^{-1}(c_{d-2})$$ can be partitioned into $$d - 2$$ smaller disjoint subsets $$\Delta_L$$ for each
$L \in \mathcal{L}$. Here, $\Delta_L$ is collection of all those cells $\sigma \in \psi^{-1}(c_{d-2})$ such that $L$ is the connected component of $C_\nu[\sigma]$. Clearly, $\psi^{-1}(c_{d-2}) = \bigsqcup_{L \in \mathcal{L}} \Delta_L$. Now, the idea is to define acyclic matchings on $\Delta_L$ for each $L \in \mathcal{L}$ and merge them together to get an acyclic matching on $\psi^{-1}(c_{d-2})$.

1. Observe that, for $p \geq d - 2$, the $p$-cells of $\Delta_{f_{d-2}^{(3,4,\ldots,d-1,d)}}$ are in 1-1 correspondence with the $p - (d - 2)$ cells of $\text{Ind}_{d-2}(P_{n-d}^{(d+2,\ldots,n,1)})$ with one extra cell of dimension $d - 3$, which is $\{3,4,\ldots,d-1,1\}$. Using Proposition 4.3 let $M$ be a perfect matching on $\text{Ind}_{d-2}(P_{n-d}^{(d+2,\ldots,n,1)})$. Define a matching $M_{d-2}$ on $\Delta_{f_{d-2}^{(3,4,\ldots,d-1,d)}}$ as follows: $(\sigma, \tau) \in M$ iff $(\sigma \cup \{3,4,\ldots,d-1,d\}, \tau \cup \{3,4,\ldots,d-1,d\}) \in M_{d-2}^3$. Therefore, we get the following.

Matching $M_{d-2}^3$ is an acyclic matching on $\Delta_{f_{d-2}^{(3,4,\ldots,d-1,d)}}$ with the following property.

If $n - d = dk - 1$ or $dk$, i.e., $n = d(k + 1) - 1$ or $d(k + 1) + 1$, then there is only one critical cell of dimension $dk - 2k - 1 + d - 2 = d(k + 1) - 2(k + 1) - 1$ and that is

$$\{3,4,\ldots,d-1,d\} \cup \bigcup_{i=1}^{k-1} \{id + 3,\ldots,(i+1)d\} \cup \{dk + 3,\ldots,n,1\}, \text{ if } n = d(k + 1) - 1,$$

(7)

Otherwise, there is no critical cell.

2. We now define a matching on $\Delta_{f_{d-2}^{(4,5,\ldots,n-d-2)}}$ for each $t \in \{4,5,\ldots,d\}$. Similar to the case of $\Delta_{f_{d-2}^{(3,4,\ldots,d-1,d)}}$, we define an acyclic matching on $\Delta_{f_{d-2}^{(t+1,\ldots,d+1)}}$, say $M_{d-2}^t$ using the perfect matching defined on $\text{Ind}_{d-2}(P_{n-d}^{(d+2,\ldots,n,1)})$. We thus get the following.

If $n - d = dk - 1$ or $dk$, i.e., $n = d(k + 1) - 1$ or $d(k + 1) + 1$, then there is only one critical cell of dimension $dk - 2k - 1 + d - 2 = d(k + 1) - 2(k + 1) - 1$ and that is

$$\{t,t+1,\ldots,d+t-3\} \cup \bigcup_{i=1}^{k-1} \{id + t,\ldots,(i+1)d+t-3\} \cup \{dk + t,\ldots,n,1,\ldots,t-2\},$$

if $n = d(k + 1) - 1$ and

$$\{t,t+1,\ldots,d+t-3\} \cup \bigcup_{i=1}^{k-1} \{id + t,\ldots,(i+1)d+t-3\} \cup \{dk + t,\ldots,n,1,\ldots,t-3\},$$

if $n = d(k + 1) + 1$.

Otherwise, there is no critical cell.

Since $\psi^{-1}(c_{d-2}) = \bigsqcup_{L \in \mathcal{L}} \Delta_L$, $M_{d-2} = \bigsqcup_{t=3}^d M_{d-2}^t$ (defined in step $d - 2$) is an acyclic matching on $\psi^{-1}(c_{d-2})$ with exactly $d - 2$ critical cells of dimension $d(k + 1) - 2(k + 1) - 1$ whenever $n = d(k + 1) - 1$ or $d(k + 1) + 1$ and with no critical cell otherwise.

Using Theorem 2.4, we observe that $M = \bigsqcup_{i=1}^{d-2} M_i$ is an acyclic matching on $\psi^{-1}(c_d)$ with:

- no critical cell if $n = dk + 1$,
- exactly 1 critical cell of dimension $dk - 2k$ if $n = dk + 2$
- exactly $t - 2$ critical cells of dimension $dk - 2k + t - 3$ and $t - 1$ critical cells of dimension $dk - 2k + t - 2$ if $n = dk + t$ for some $t \in \{3,\ldots,d-1\}$
- exactly $d - 2$ critical cells of dimension $d(k + 1) - 2(k + 1) - 1$ if $n = d(k + 1)$.

We now define another matching on the set of critical cells corresponding to matching $M$ on $\psi^{-1}(c_d)$. The idea is the following. If $n = dk + 3$, then observe from step 1 and step 2 that if $\gamma$ is critical of dimension $dk - 2k$ then $\gamma \cup \{d - 1\}$ is critical of dimension $dk - 2k + 1$. So match $\gamma$ with $\gamma \cup \{d - 1\}$. Now, let $n = dk + t$ for some $t \in \{4,\ldots,d - 1\}$. From step $t - 2$ and step
t - 1 we see that, if in step t - 2, \( \gamma = \{d - i, \ldots, d, \ldots, d + t - i - 3\} \cup \{\beta\} \) is a critical cell of dimension \( dk - 2k + t - 3 \) then in step \( t - 1 \), \( \{d - i - 1, d - i, \ldots, d, \ldots, d + t - i - 3\} \cup \{\beta\} \) is critical of dimension \( dk - 2k + t - 2 \). Here, we match \( \gamma \) with \( \gamma \cup \{d - i - 1\} \). Let the matching defined above is \( M' \).

**Claim 4.** Let \( M \) and \( M' \) be matchings on \( \phi^{-1}(e_d) \) as defined above. Then, \( M = M \sqcup M' \) is an acyclic matching on \( \phi^{-1}(e_d) \) with

- no critical cell if \( n = dk + 1 \),
- exactly 1 critical cell of dimension \( dk - 2(k+1) + t \) if \( n = dk + t \) for some \( t \in \{2, \ldots, d-1\} \),
- exactly \( d - 2 \) critical cells of dimension \( d(k + 1) - 2(k + 1) - 1 \) if \( n = d(k + 1) \).

**Proof of Claim 4.** Let \( \Delta_0 = \{\sigma \in \phi^{-1}(e_d) : \sigma \in \eta \text{ for some } \eta \in M\} \) and \( \Delta_1 = \phi^{-1}(e_d) \setminus \Delta_0 \). Since \( M \) and \( M' \) are union of a sequence of elementary matchings on \( \Delta_0 \) and \( \Delta_1 \) respectively, \( M \) and \( M' \) are acyclic matching from Proposition 2.9.

Further, it is clear from the description of the critical cells given in step-1 to step-(\( d - 2 \)) that if \( \tau \in \Delta_1 \) and \( \sigma \in \Delta_0 \) then \( \tau \not\subseteq \sigma \). Thus, using Theorem 2.11, we get that \( M \) is an acyclic matching on \( \phi^{-1}(e_d) \). Calculation of number of critical cells corresponding to matching \( M \) is straightforward once we fix an \( n \).

**Case 3:** In cases 1 and 2, we defined acyclic matchings on \( \phi^{-1}(e_{id}) \) for \( i \in \{1, \ldots, k\} \). Here, we consider the preimage \( \phi^{-1}(e_r) \) and define a matching \( M' \) on it.

- If \( n = dk \), then \( \phi^{-1}(e_r) \) is isomorphic to \( \text{Ind}_{d-2}(G) \), where \( G \) is isomorphic to the union \( k \) disjoint copies of path graphs of length \( d - 1 \). From Corollary 4.4 there exists an acyclic matching on the face poset of \( \text{Ind}_{d-2}(G) \) with exactly one critical cell of dimension \( dk - 2k - 1 \).
- If \( n = dk + 1 \), then \( \phi^{-1}(e_r) \) is isomorphic to \( \text{Ind}_{d-2}(G_1) \), where \( G_1 \) is isomorphic to the union \( k - 1 \) disjoint copies of \( P_{d-1} \) and one copy of \( P_d \). Again from Corollary 4.4 there exists an acyclic matching on the face poset of \( \text{Ind}_{d-2}(G_1) \) with exactly one critical cell of dimension \( dk - 2k - 1 \).
- If \( n \neq dk, dk + 1 \) then one connected component of \( C_n \setminus \{d, 2d, \ldots, dk\} \) will be a path graph of cardinality either less than \( d - 1 \) or greater than \( d \) and less than \( 2d - 2 \). In both the cases, using Corollary 4.5 there exists a matching on \( \phi^{-1}(e_r) \) with no critical cell.

From Eq. (3), Theorem 2.10, case (1), Claim 4 and case 3, we get that \( M \cup M' \) is an acyclic matching on \( \mathcal{F}(\text{Ind}_{d-2}(C_n)) \) with

- exactly \( d - 1 \) critical cells of dimension \( (dk - 2k - 1) \) if \( n = dk \),
- exactly one critical cell of dimension \( (dk - 2k + t - 2) \) if \( n = dk + t \) for some \( t \in \{1, \ldots, d - 1\} \).

Hence, Theorem 4.6 follows from Corollary 2.5.

### 5. The case of perfect \( m \)-ary trees

For fixed \( m \geq 2 \), an \( m \)-ary tree is a rooted tree in which each node has no more than \( m \) children. A full \( m \)-ary tree is an \( m \)-ary tree where within each level every node has either 0 or \( m \) children. A perfect \( m \)-ary tree is a full \( m \)-ary tree in which all leaf nodes are at the same depth (the depth of a node is the number of edges from the node to the tree’s root node).

Following are some known facts about the perfect \( m \)-ary tree of height \( h \), denoted \( B^m_h \) (see Fig. 5 for example).

1. \( B^m_h \) has \( h \sum_{i=0}^{h} m^i = \frac{m^{h+1} - 1}{m - 1} \) nodes.
2. For \( 0 \leq t \leq h \), the number of nodes of depth \( t \) in \( B^m_h \) is \( m^t \).
3. \( B^m_h \) has \( m^h \) leaf nodes.

Before going into the computations of the homotopy type of \( r \) independence complexes of \( B^m_h \), let us fix some notations.
Remark 5.1. For simplicity of notations, $B^2_h$ will be denoted by $B_h$.

Example 5.2. Here we compute the homotopy type of $\text{Ind}_4(B_2)$. Define an element matching on $\text{Ind}_4(B_2)$ using the vertex $a_{2,1}$ as follows,

$$M(a_{2,1}) = \{(\sigma, \sigma \cup a_{2,1} : a_{2,1} \notin \sigma, \text{ and } \sigma, \sigma \cup a_{2,1} \in \text{Ind}_4(B_2)\}, \text{ and }$$

$$N(a_{2,1}) = \{\sigma \in \text{Ind}_4(B_2) : \sigma \notin \eta \text{ for some } \eta \in M(a_{2,1})\}.$$  

Let $\Delta_1 = \text{Ind}_4(B_2) \setminus N(a_{2,1})$. Observe that, if $\sigma \in \Delta(a_{2,1})$ then $\sigma \cup a_{2,1} \notin \text{Ind}_4(B_2)$. By definition of $\text{Ind}_4(G)$, we observe that either $\{a_{1,1}, a_{0,1}, a_{1,2}, a_{2,2}\} \subseteq \sigma$ or $\{a_{1,1}, a_{1,0}, a_{1,2}, a_{2,3}\} \subseteq \sigma$ or $\{a_{1,1}, a_{0,1}, a_{1,2}, a_{2,4}\} \subseteq \sigma$. Since $\{a_{1,1}, a_{0,1}, a_{1,2}, a_{2,2}\}, \{a_{1,1}, a_{0,1}, a_{2,2}, a_{2,3}\}, \{a_{1,1}, a_{1,0}, a_{1,2}, a_{2,4}\}$ are maximal cells of $\text{Ind}_4(B_2)$, these are the only unmatched cells i.e., $\Delta_1 = \{\{a_{1,1}, a_{0,1}, a_{1,2}, a_{2,2}\}, \{a_{1,1}, a_{0,1}, a_{1,2}, a_{2,3}\}, \{a_{1,1}, a_{0,1}, a_{1,2}, a_{2,4}\}\}$. Therefore, Corollary 2.7 implies that $\text{Ind}_4(B_2) \simeq \bigwedge_3 S^3$.

Example 5.3. Using the homotopy type of $\text{Ind}_4(B_2)$, we compute the homotopy type of $\text{Ind}_4(B_3)$. Here, we show that $\text{Ind}_4(B_3) \simeq \text{Ind}_4(B_3 - \{a_{0,1}\})$. It is easy to see that $B_3 - \{a_{0,1}\} \cong B_2 \cup B_2$. Thus, Observation 3.4 iv implies that $\text{Ind}_4(B_3) \simeq \text{Ind}_4(B_2) * \text{Ind}_4(B_2) \simeq \bigwedge_9 S^7$.

We now prove that $\text{Ind}_4(B_3) \simeq \text{Ind}_4(B_3 - \{a_{0,1}\})$. Let $R(a_{0,1}) = \{\sigma \in \text{Ind}_4(B_3) : a_{0,1} \in \sigma\}$. Clearly, $\text{Ind}_4(B_3) \setminus R(a_{0,1}) = \text{Ind}_4(B_3 - \{a_{0,1}\})$. From Corollary 2.7, it is enough to define a perfect matching on $R(a_{0,1})$. We do so by defining a sequence of elementary matching using vertices $a_{3,1}, a_{3,3}, a_{3,5}, a_{3,7}$ as follows: Let $\Delta_0 = \text{Ind}_4(B_3)$. For $i \in \{1, 2, 3, 4\}$, define

$$M(a_{3,2i-1}) = \{(\sigma, \sigma \cup a_{3,2i-1} : a_{3,2i-1} \notin \sigma \text{ and } \sigma, \sigma \cup a_{3,2i-1} \in \Delta_{i-1}\},$$

$$N(a_{3,2i-1}) = \{\sigma \in \Delta_{i-1} : \sigma \notin \eta \text{ for some } \eta \in M(a_{3,2i-1})\}, \text{ and }$$

$$\Delta_i = \Delta_{i-1} \setminus N(a_{3,2i-1}).$$

Claim 5. $\Delta_4 = \text{Ind}_4(B_3) \setminus R(a_{0,1})$. 

---

**Figure 5**
Since $N(a_{3,2i-1}) \subseteq R(a_{0,1})$ for all $i \in \{1, 2, 3, 4\}$, $\text{Ind}_4(B_3) \setminus R(a_{0,1}) \subseteq \Delta_4$. To show the other way inclusion, it is enough to show that if $\sigma \in \text{Ind}_4(B_3)$ and $a_{0,1} \in \sigma$ then $\sigma \in N(a_{3,2i-1})$ for some $i \in \{1, 2, 3, 4\}$.

Let $\sigma \in \text{Ind}_4(B_3)$ and $a_{0,1} \in \sigma$. Since $a_{0,1} \in \sigma$, it follows from the definition of $\text{Ind}_v(G)$ that $\{a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}\} \not\subseteq \sigma$. If $\{a_{1,1}, a_{2,1}\} \not\subseteq \sigma$, then $\sigma \in N(a_{3,1})$. If $\{a_{1,1}, a_{2,1}\} \subseteq \sigma$ and $a_{2,2} \not\in \sigma$, then $\sigma \in N(a_{3,3})$. If $\{a_{1,1}, a_{2,1}, a_{2,2}\} \subseteq \sigma$ then $a_{1,2} \not\in \sigma$, implying that $\sigma \in N(a_{3,5})$. This completes the proof of Claim 6.

To get the better understanding if the computations, we first prove our results for perfect binary trees. The proof for perfect $m$-ary trees will follows using similar arguments.

**Lemma 5.4.** Let $r \geq 2^h - 1$. Then the homotopy type of $r^{th}$ independence complex of the graph $B_h$ is given as follows,

$$\text{Ind}_r(B_h) \simeq \left\{ \begin{array}{ll}
\bigvee S^{r-1}, & \text{if } r = 2^h - 1 + s \text{ for some } s \in \{0, 1, \ldots, 2^h - 1\}, \\
\{\text{point}\}, & \text{if } r \geq 2^{h+1} - 1.
\end{array} \right.$$  

**Proof.** The idea of the proof here is similar to that of in Example 5.2. If $r \geq 2^{h+1} - 1$, then Observation 5.4(i) implies the result. Let $r = 2^h - 1 + s$ for some fixed $s \in \{0, 1, \ldots, 2^h - 1\}$ and $\Delta_0 = \text{Ind}_r(B_h)$. Define a sequence of elementary matching using the alternate vertices of depth $h$, i.e., $a_{h,1}, a_{h,3}, \ldots, a_{h,2^h-1}$. For $i \in \{1, 2, \ldots, 2^h-1\}$, define

$$M(a_{h,2i-1}) = \{\sigma, \sigma \cup a_{h,2i-1} : a_{h,2i-1} \not\in \sigma \text{ and } \sigma \cup a_{h,2i-1} \in \Delta_{i-1}\},$$

$$N(a_{h,2i-1}) = \{\sigma \in \Delta_{i-1} : \sigma \in \eta \text{ for some } \eta \in M(a_{h,2i-1})\},$$

$$\Delta_i = \Delta_{i-1} \setminus N(a_{h,2i-1}).$$

We now show that the set of critical cells $\Delta_{2^h-1}$, corresponding to the sequence of matching defined in Eq. (10) is a set of $\binom{2^h-1}{s}$ elements of fixed cardinality $r$. Thus, we get the result using Corollary 2.6.

**Claim 6.**

1. If $\sigma \in \Delta_{2^h-1}$, then \( \bigcup_{j=0}^{h-1} V_j(B_h) \subseteq \sigma \).
2. If $\sigma \in \Delta_{2^h-1}$, then $\sigma$ is of cardinality $r$.
3. Cardinality of the set of critical cells $\Delta_{2^h-1}$ is $\binom{2^h-1}{s}$.

**Proof of Claim 6**

To the contrary of Claim 6(1), assume that there exists $\sigma_1 \in \Delta_{2^h-1}$ such that \( \bigcup_{j=0}^{h-1} V_j(B_h) \not\subseteq \sigma_1 \). Let $a_{i,j,1} \in \bigcup_{j=0}^{h-1} V_j(B_h)$ be the smallest element with respect to the given ordering above such that $a_{i,j,1} \not\in \sigma_1$. Since $a_{i,j,1}$ is not a leaf, let $a_{i,j,1}^1$ be the first children of $a_{i,j,1}$. Let $a_{h,\ell}$ be the left most leaf of the sub-tree rooted at $a_{i,j,1}^1$. Further, the number of vertices of sub-tree rooted at $a_{i,j,1}^1$ is not more than $2^h - 1$. Thus, $\sigma_1 \in N(a_{h,\ell})$ (being the left most child of a sub-tree, $\ell$ is an odd number) contradicting the assumption that $\sigma_1 \in \Delta_{2^h-1}$. This proves Claim 6(1).

We now prove the second part of the above claim. Let $\sigma \in \Delta_{2^h-1}$. Clearly, cardinality of $\sigma$ is at least $r$ (because any cell of $\text{Ind}_r(B_h)$ of cardinality less that $r$ is in $N(a_{0,1})$). Using Claim 6(1), we see that $B_h[\sigma]$ is connected graph of cardinality equal to the cardinality of $\sigma$. Therefore, the cardinality of $\sigma$ is at most $r$. This proves Claim 6(2).

From Eq. (10), it is clear that, if $\sigma \in \text{Ind}_r(B_h)$ and $a_{0,1} \in \sigma$ then $\sigma \in N(a_{h,1})$ implying that $\sigma \not\in \Delta_{2^h-1}$. Hence, using Claim 6(1) and (2), we get that the cardinality of the set $\Delta_{2^h-1}$ is equal to number of $s$-subsets of the set $V_h(B_h) \setminus \{a_{h,1}\}$. Which is equal to $\binom{2^h-1}{s}$. This completes the proof of Claim 6. \(\square\)
From Claim 4, we see that the matching on $\text{Ind}_r(B_h)$ defined in Eq. (11) has $(2^{h-1})$ critical cells of fixed dimension $r - 1$. Therefore, Lemma 5.3 follows from Corollary 2.3. □

We are now ready to present the computation of homotopy type of $\text{Ind}_r(B_h)$ for any $r$.

**Theorem 5.5.** For a fixed $t \geq 1$, let $r = 2^t - 1 + s$ for some $s \in \{0, 1, \ldots, 2^t - 1\}$. Then the $r^{th}$ independence complex of the graph $B_h$ is given as follows,

$$
\text{Ind}_r(B_h) \simeq \begin{cases}
\bigvee_{p_1} S^{q_1}, \quad &\text{if } h = (k-1)(t+2) + t + 1 \text{ for some } k \geq 1, \\
\bigvee_{p_2} S^{q_2}, \quad &\text{if } h = k(t+2) + t \text{ for some } k \geq 0, \\
\{\text{point}\}, &\text{otherwise},
\end{cases}
$$

where,

$$p_1 = \binom{2^t - 1}{s} 2^{(2^t+2^t+2^{t-1}+\cdots+2^{k-1}(t+2))}$$

and

$$q_1 = 2r(2^0 + 2^{t+2} + \cdots + 2^{(k-1)(t+2)}) - 1,$$

$$p_2 = \binom{2^t - 1}{s} 2^{2^t+2^{t-1}+\cdots+2^{k(t+2)}},$$

$$q_2 = r(2^0 + 2^{t+2} + \cdots + 2^{k(t+2)}) - 1.$$ 

**Proof.** The idea here is similar to that of Example 5.3. If $h \leq t$, then the result follows from Lemma 5.4. Let $h > t$. Here, we show that $\text{Ind}_r(B_h) \simeq \text{Ind}_r(G)$, where $G$ is disjoint union of perfect binary trees of height at most $t$. Recall that $V_j(B_h)$ denotes the set of vertices of $B_h$ of depth $j$.

**Claim 7.** $\text{Ind}_r(B_h) \simeq \text{Ind}_r(B_h - V_{h-(t+1)}(B_h)).$

**Proof of Claim 7.** Let $R(V_{h-(t+1)}(B_h)) = \{\sigma \in \text{Ind}_r(B_h) : \sigma \cap V_{h-(t+1)}(B_h) \neq \emptyset\}$. Clearly, $\text{Ind}_r(B_h) \setminus R(V_{h-(t+1)}(B_h)) = \text{Ind}_r(B_h - V_{h-(t+1)}(B_h))$. To prove Claim 6 from Corollary 2.6, it is enough to define a perfect matching on $R(V_{h-(t+1)}(B_h))$. We do so by defining a sequence of elementary matching on $\text{Ind}_r(B_h)$ using vertices $a_{h,1}, a_{h,2}, \ldots, a_{h,2^h-1}$ as follows: Let $\Delta_0 = \text{Ind}_r(B_h)$. For $i \in \{1, 2, \ldots, 2^h-1\}$, define

$$M(a_{h,2i-1}) = \{(\sigma, \sigma \cup a_{h,2i-1}) : \sigma \cap V_{h-(t+1)}(B_h) \neq \emptyset, a_{h,2i-1} \notin \sigma \text{ and } \sigma, \sigma \cup a_{h,2i-1} \in \Delta_i-1\},$$

$$N(a_{h,2i-1}) = \{\sigma \in \Delta_i-1 : \sigma \in \eta \text{ for some } \eta \in M(a_{h,2i-1})\},$$

$$\Delta_i = \Delta_{i-1} \setminus N(a_{h,2i-1}).$$

We now prove that $\Delta_{2^h-1} = \text{Ind}_r(B_h) \setminus R(V_{h-(t+1)}(B_h))$. Which, along with Corollary 2.3, will imply Claim 6. Since $N(a_{h,2i-1}) \subseteq R(V_{h-(t+1)}(B_h))$ for all $i \in \{1, 2, \ldots, 2^h-1\}$, $\text{Ind}_r(B_h) \setminus R(V_{h-(t+1)}(B_h)) \subseteq \Delta_{2^h-1}$. To show that $\Delta_{2^h-1} \subseteq \text{Ind}_r(B_h) \setminus R(V_{h-(t+1)}(B_h))$, it is enough to show that if $\sigma \in \text{Ind}_r(B_h)$ and $\sigma \cap V_{h-(t+1)}(B_h) \neq \emptyset$ then $\sigma \in N(a_{h,2i-1})$ for some $i \in \{1, 2, \ldots, 2^h-1\}$. Let $\sigma_1 \in \text{Ind}_r(B_h)$ such that $\sigma_1 \cap V_{h-(t+1)}(B_h) \neq \emptyset$. Without loss of generality, assume that $a_{h-(t+1),i}$ be the smallest vertex of $V_{h-(t+1)}(B_h)$ such that $a_{h-(t+1),i} \in \sigma_1$. Let $B(a_{h-(t+1),i},B_h)$ be the sub-tree of $B_h$ rooted at $a_{h-(t+1),i}$. Let $S$ denotes the set of all non-leaf vertices of $B(a_{h-(t+1),i},B_h)$, i.e., $S = \bigcup_{j=1}^{t+1} (V_{j} - B_h) \cap V(B(a_{h-(t+1),i},B_h))$. Clearly, $B(a_{h-(t+1),i},B_h)$ is a perfect binary tree of height $t+1$ and the cardinality of $S$ is $2^{t+1} - 1$. Since $B_h[S]$ is a connected graph and $r < 2^{t+1} - 1$, $S \notin \sigma_1$. Let $a_{i_1,j_1}$ be the smallest element of $S$ such that $a_{i+1,j_1} \notin \sigma_1$. Since $a_{i_1,j_1} \in S$ and $a_{i_1+1,j_1} \in \sigma_1$, we get that $i_1 \in \{h-t, h-t+1, \ldots, h-1\}$. Let $a_{i_1+1,j_2}$ be the left children of $a_{i_1,j_1}$ and $a_{h,i_1}$ be the left most leaf of perfect binary sub-tree rooted
at \(a_{i_1+1,j_2}\). Observe that the cardinality of the sub-tree rooted at \(a_{i_1+1,j_2}\) is at most \(2^t - 1\). Therefore, \(\sigma_1 \in N(a_{h,\ell_1})\) (here \(\ell_1\) is an odd number because it is the leftmost leaf of a perfect binary sub-tree of perfect binary tree). This completes the proof of Claim 7. \(\square\)

We prove Theorem 5.5 using induction on \(h\).

**Step 1:** In this step, we prove the result for \(h \in \{t+1, t+2, \ldots, (t+2) + t\}\).

From Claim 7 we see that \(\text{Ind}_r(B_h) \cong \text{Ind}_r(B_h - V_{h-(t+1)}(B_h))\). Observe that \(B_h - V_{h-(t+1)}(B_h)\) is disjoint union of \(2(2^{h-(t+1)})\) copies of perfect binary trees of height \(t\) and one perfect binary tree of height \(h - (t+2)\) (here, by \(B_{-1}\) we mean empty graph). Therefore, using Observation 3.4(iv) and Lemma 5.4, we get the following equivalence.

\[
\text{Ind}_r(B_h) \cong \text{Ind}_r(B_t \sqcup \cdots \sqcup B_t \sqcup B_{h-(t+2)}) \\
\cong \text{Ind}_r(B_t) \ast \cdots \ast \text{Ind}_r(B_t) \ast \text{Ind}_r(B_{h-(t+2)}) \\
\cong \begin{cases} 
\text{Ind}_r(B_t) \ast \text{Ind}_r(B_t) \ast \text{Ind}_r(B_{-1}), & \text{if } h = t + 1, \\
\text{Ind}_r(B_t) \ast \cdots \ast \text{Ind}_r(B_t) \ast \text{Ind}_r(B_t), & \text{if } h = (t+2) + t, \\
\text{Ind}_r(B_t) \ast \cdots \ast \text{Ind}_r(B_t) \ast \{\text{point}\}, & \text{if } t + 1 < h < (t+2) + t.
\end{cases}
\]

(11)

Thus, Lemma 5.4 and Lemma 2.2 implies the result, i.e.,

\[
\text{Ind}_r(B_h) \cong \begin{cases} 
\bigvee \limits_{(2^{t+1})} S^{2r(2^0) - 1}, & \text{if } h = t + 1, \\
\bigvee \limits_{(2^{t+2})} S^{r(2^0+2^{t+2}) - 1}, & \text{if } h = (t+2) + t, \\
\{\text{point}\}, & \text{if } t + 1 < h < (t+2) + t.
\end{cases}
\]

**Step 2:** In this step, we prove the result for \(h \in \{(t+2) + t + 1, \ldots, 2(t+2) + t\}\).

Following similar method as in step 1, we get the following equivalence,

\[
\text{Ind}_r(B_h) \cong \text{Ind}_r(B_t) \ast \cdots \ast \text{Ind}_r(B_t) \ast \text{Ind}_r(B_{h-(t+2)}) \\
\ast \text{Ind}_r(B_i \sqcup \cdots \sqcup B_i \sqcup B_{h-(t+2)}) \\
\cong \begin{cases} 
\text{Ind}_r(B_t) \ast \cdots \ast \text{Ind}_r(B_t) \ast \text{Ind}_r(B_i \sqcup \cdots \sqcup B_i \sqcup B_{h-(t+2)}), & \text{if } h = (t+2) + t, \\
\{\text{point}\}, & \text{if } t + 1 < h < (t+2) + t.
\end{cases}
\]

Observe that \(h - (t+2)\) is in \(\{t+1, t+2, \ldots, (t+2) + t\}\). Thus, result of Step 1 implies the following.
In this step, we prove the result for $B_i$, i.e.,

$$h \simeq \begin{cases} 
\text{ind}_r(B_i) \cdots \text{ind}_r(B_t) \text{ind}_r(B_{-1}), & \text{if } h = (t + 2) + t + 1, \\
\text{2(2}^t + 2^{i+2})\text{-copies}, & \\
\text{ind}_r(B_i) \cdots \text{ind}_r(B_t) \text{ind}_r(B_{-1}), & \text{if } h = 2(t + 2) + t, \\
\text{(2}^{t+2} + 2^{i(t+2)})\text{-copies}, & \\
\text{ind}_r(B_i) \cdots \text{ind}_r(B_{-1}) \{\text{point}\}, & \text{if } (t + 2) + t + 1 < h < 2(t + 2) + t.
\end{cases}$$

Using Lemma 5.4 and Lemma 2.2, we get the result, i.e.,

$$\text{ind}_r(B_h) \simeq \begin{cases} 
\bigvee_{\binom{(2^t)}{2}^t} S^{2r((2^t) + 2^{i+2})}-1, & \text{if } h = (t + 2) + t + 1, \\
\bigvee_{\binom{(2^t)}{2}^t} S^{r((2^t) + 2^{i+2} + 2^t)}-1, & \text{if } h = 2(t + 2) + t, \\
\{\text{point}\}, & \text{if } (t + 2) + t + 1 < h < 2(t + 2) + t.
\end{cases}$$

**Step $k$:** In this step, we prove the result for $h \in \{(k - 1)(t + 2) + t + 1, \ldots, k(t + 2) + t\}$ where $k \geq 3$.

The proof here is exactly similar to that of Step 2. Therefore,

$$\text{ind}_r(B_h) \simeq \text{ind}_r(B_i) \cdots \text{ind}_r(B_t) \text{ind}_r(B_{-1} \cdots \text{ind}_r(B_{h-(t+2)})$$

Thus, result of Step $k - 1$ implies the following equivalence.

$$\text{ind}_r(B_h) \simeq \begin{cases} 
\text{ind}_r(B_i) \cdots \text{ind}_r(B_{-1}), & \text{if } h = (k - 1)(t + 2) + t + 1, \\
\text{2(2}^{k+2} + \cdots + 2^{(k-1)(t+2)})\text{-copies}, & \\
\text{ind}_r(B_i) \cdots \text{ind}_r(B_{-1}) \text{ind}_r(B_{-1}), & \text{if } h = k(t + 2) + t, \\
\text{(2}^{k+2} + \cdots + 2^{k(t+2)})\text{-copies}, & \\
\text{ind}_r(B_i) \cdots \text{ind}_r(B_{-1}) \{\text{point}\}, & \text{if } (k - 1)(t + 2) + t + 1 < h < k(t + 2) + t.
\end{cases}$$

$$\simeq \begin{cases} 
\text{ind}_r(B_i) \cdots \text{ind}_r(B_{-1}), & \text{if } h = (k - 1)(t + 2) + t + 1, \\
\text{2(2}^{k+2} + \cdots + 2^{(k-1)(t+2)})\text{-copies}, & \\
\text{ind}_r(B_i) \cdots \text{ind}_r(B_{-1}), & \text{if } h = k(t + 2) + t, \\
\text{(2}^{k+2} + \cdots + 2^{k(t+2)})\text{-copies}, & \\
\{\text{point}\}, & \text{if } (k - 1)(t + 2) + t + 1 < h < k(t + 2) + t.
\end{cases}$$
Hence, using Lemma 5.4 and Lemma 2.2, we get the result, i.e.,

\[ S^{2r(2^0+2^1+2^2+\ldots+2^{k-1}(t+2))} \]

\[ \text{Ind}_r(B_h) \simeq \begin{cases} 
V \{ \sigma \} & \text{if } h = (k-1)(t+2) + t + 1, \\
S^{2r(2^0+2^1+2^2+\ldots+2^{k-1}(t+2))} \cup \{ \text{point} \} & \text{otherwise.}
\end{cases} \]

This completes the proof of Theorem 5.5.

We are now ready to generalize Lemma 5.4 and Theorem 5.5 for perfect \( m \)-ary trees. Henceforth, \( m \geq 3 \) will be a fixed integer.

**Lemma 5.6.** Let \( r \geq \frac{m^h-1}{m-1} \). Then the homotopy type of \( r \)-th independence complex of the graph \( B_h^m \) is given as follows,

\[ \text{Ind}_r(B_h^m) \simeq \begin{cases} 
V \{ \sigma \} & \text{if } r = \frac{m^h-1}{m-1} + s \text{ for some } s \in \{0,1,\ldots,m^h-1\}, \\
S^{2r-1} & \text{if } r = \frac{m^h-1}{m-1}, \\
\{ \text{point} \} & \text{if } r \geq \frac{m^{h+1}-1}{m-1}.
\end{cases} \]

**Proof.** The proof here is exactly similar to the proof of Lemma 5.4 but we explain some part here as well for completeness. If \( r \geq \frac{m^h-1}{m-1} \), then Observation 5.4(i) implies the result. Let \( r = \frac{m^h-1}{m-1} + s \) for some fixed \( s \in \{0,1,\ldots,m^h-1\} \) and \( \Delta_0 = \text{Ind}_r(B_h^m) \). Define a sequence of elementary matching using the following vertices of depth \( h \): \( a_{h,1}, a_{h,m+1}, \ldots, a_{h,m(m^h-1)+1} \).

For \( i \in \{1,2,\ldots,m^h-1\} \), define

\[ (12) \]

\[ M(a_{h,mi-(m-1)}) = \{ \sigma, \sigma \cup a_{h,mi-(m-1)} : a_{h,mi-(m-1)} \notin \sigma \text{ and } \sigma, \sigma \cup a_{h,mi-(m-1)} \in \Delta_i \}, \]

\[ N(a_{h,mi-(m-1)}) = \{ \sigma \in \Delta_i-1 : \sigma \in \eta \text{ for some } \eta \in M(a_{h,mi-(m-1)}) \}, \] and

\[ \Delta_i = \Delta_{i-1} \setminus N(a_{h,mi-(m-1)}) \].

We now show that the set of critical cells \( \Delta_{m^h-1} \), corresponding to the sequence of matching defined in Eq. (12) is a set of \( \binom{m^h-1}{s} \) cells of fixed dimension \( r-1 \).

**Claim 8.**

1. If \( \sigma \in \Delta_{m^h-1} \), then \( \bigcup_{j=0}^{h-1} V_j(B_h^m) \subseteq \sigma \).
2. If \( \sigma \in \Delta_{m^h-1} \), then \( \sigma \) is of cardinality \( r \).
3. Cardinality of the set of critical cells \( \Delta_{m^h-1} \) is \( \binom{m^h-1}{s} \).

Using exactly similar arguments as in the proof of Claim 5, we get the proof of Claim 8.

From Claim 8 we see that the matching on \( \text{Ind}_r(B_h^m) \) defined in Eq. (12) has \( \binom{m^h-1}{s} \) critical cells of fixed dimension \( r-1 \). Therefore, Lemma 5.6 follows from Corollary 2.4.

We are now ready to present the main result of this section.

**Theorem 5.7.** For a fixed \( t \geq 1 \), let \( r = \left( \sum_{i=0}^{t-1} m^i \right) + s = \frac{m^t-1}{m-1} + s \) for some \( s \in \{0,1,\ldots,m^t-1\} \).

Then the \( r \)-th independence complex of the graph \( B_h^m \) is given as follows,

\[ \text{Ind}_r(B_h^m) \simeq \begin{cases} 
V S^{p_1} & \text{if } h = (k-1)(t+2) + t + 1 \text{ for some } k \geq 1, \\
V S^{p_2} & \text{if } h = k(t+2) + t \text{ for some } k \geq 0, \\
\{ \text{point} \} & \text{otherwise,}
\end{cases} \]
where,

\[ p_1 = \binom{m^t - 1}{s}^{m(m^0 + m^{t+2} + \cdots + m^{(k-1)(t+2)}) + 1} \]

\[ q_1 = ms^0(m^0 + m^{t+2} + \cdots + m^{(k-1)(t+2)}) - 1, \]

\[ p_2 = \binom{m^t - 1}{s}^{m^0 + m^{t+2} + \cdots + m^{k(t+2)}}, \]

\[ q_2 = r(m^0 + m^{t+2} + \cdots + m^{k(t+2)}) - 1. \]

**Proof.** If \( h \leq t \), then the result follows from Lemma 5.1. Let \( h > t \). Here, we show that \( \text{Ind}_r(B^m_h) \simeq \text{Ind}_r(G) \), where \( G \) is disjoint union of perfect \( m \)-ary trees of height at most \( t \). Recall that \( \text{Ind}_r(G) \) denotes the set of vertices of \( B^m_h \) of depth \( j \).

**Claim 9.** \( \text{Ind}_r(B^m_h) \simeq \text{Ind}_r(B^m_h - V_{h-(t+1)}(B^m_h)) \).

**Proof of Claim 9** Let \( R(V_{h-(t+1)}(B^m_h)) = \{ \sigma \in \text{Ind}_r(B^m_h) : \sigma \cap V_{h-(t+1)}(B^m_h) \neq \emptyset \} \). Clearly, \( \text{Ind}_r(B^m_h) \setminus R(V_{h-(t+1)}(B^m_h)) = \text{Ind}_r(B^m_h - V_{h-(t+1)}(B^m_h)) \). Thus, it is enough to define a perfect matching on \( R(V_{h-(t+1)}(B^m_h)) \). We do so by defining a sequence of elementary matching on \( \text{Ind}_r(B^m_h) \) using vertices \( a_{h,1}, a_{h,m+1}, \ldots, a_{h,m^h-(m-1)} \) as follows: Let \( \Delta_0 = \text{Ind}_r(B^m_h) \). For \( i \in \{1, 2, \ldots, m^h-1\} \), define

\[ M(a_{h,mi-(m-1)}) = \{(\sigma, \sigma \cup a_{h,mi-(m-1)}) : \sigma \cap V_{h-(t+1)}(B^m_h) \neq \emptyset, a_{h,mi-(m-1)} \notin \sigma \text{ and } \sigma, \sigma \cup a_{h,mi-(m-1)} \in \Delta_{i-1}\}; \]

\[ N(a_{h,mi-(m-1)}) = \{\sigma \in \Delta_{i-1} : \sigma \in \eta \text{ for some } \eta \in M(a_{h,mi-(m-1)})\}, \]

\[ \Delta_i = \Delta_{i-1} \setminus N(a_{h,mi-(m-1)}). \]

Using similar arguments as in the proof of Claim 7, we get that \( \Delta_{m^h-1} = \text{Ind}_r(B^m_h) \setminus R(V_{h-(t+1)}(B^m_h)) \). This completes the proof of Claim 9. \( \square \)

We prove Theorem 5.7 using induction on \( h \).

**Step 1:** In this step, we prove the result for \( h \in \{t+1, t+2, \ldots, (t+2) + t\} \). From Claim 9, we see that \( \text{Ind}_r(B^m_h) \simeq \text{Ind}_r(B^m_h - V_{h-(t+1)}(B^m_h)) \). Observe that \( B^m_h - V_{h-(t+1)}(B^m_h) \) is disjoint union of \( m(m^h-(t+1)) \) copies of perfect \( m \)-ary trees of height \( t \) and one perfect \( m \)-ary tree of height \( h - (t+2) \) (here, by \( B^1_m \) we mean empty graph). Therefore, using Observation 5.3(iv) and Lemma 5.6 we get the following equivalence.

\[ \text{Ind}_r(B^m_h) \simeq \text{Ind}_r(B^m_{t+1} \sqcup \cdots \sqcup B^m_t \sqcup B^m_{h-(t+2)}) \]

\[ \simeq \text{Ind}_r(B^m_{t+1}) * \cdots * \text{Ind}_r(B^m_t) * \text{Ind}_r(B^m_{h-(t+2)}) \]

\[ \simeq \begin{cases} 
\text{Ind}_r(B^m_{t+1}) * \cdots * \text{Ind}_r(B^m_t) * \text{Ind}_r(B^m_{h-(t+2)}), & \text{if } h = t+1, \\
\text{Ind}_r(B^m_{t+2}) * \cdots * \text{Ind}_r(B^m_{t+1}) * \text{Ind}_r(B^m_{h-(t+2)}), & \text{if } h = (t+2) + t, \\
\text{Ind}_r(B^m_{t+2}) * \cdots * \text{Ind}_r(B^m_{t+1}) * \{\text{point}\}, & \text{if } t + 1 < h < (t+2) + t.
\end{cases} \]
Thus, Lemma 5.6 and Lemma 2.2 implies the result, i.e.,

\[
\text{Ind}_r(B^m_h) \simeq \begin{cases} 
\bigvee_{(m^t-1)} S^{mr-1}, & \text{if } h = t + 1, \\
\bigvee_{(m^t-1)(m^0+m^t+2)} S^{r(m^0+m^t+2)-1}, & \text{if } h = (t + 2) + t, \\
\{\text{point}\}, & \text{if } t + 1 < h < (t + 2) + t.
\end{cases}
\]

**Step 2:** In this step, we prove the result for \( h \in \{(t + 2) + t + 1, \ldots, 2(t + 2) + t\} \).

Following similar method as in step 1, we get the following equivalence,

\[
\text{Ind}_r(B^m_h) \simeq \frac{\text{Ind}_r(B^m_1) \ast \cdots \ast \text{Ind}_r(B^m_t) \ast \text{Ind}_r(B^m_{h-(t+2)})}{m(m^h-(t+1))-\text{copies}}
\]

Observe that \( h - (t + 2) \) is in \( \{t + 1, t + 2, \ldots, (t + 2) + t\} \). Thus, result of Step 1 implies the following.

\[
\begin{align*}
\text{Ind}_r(B^m_h) & \simeq \begin{cases} 
\text{Ind}_r(B^m_1) \ast \cdots \ast \text{Ind}_r(B^m_t) \ast \text{Ind}_r(B^m_{h-(t+2)}), & \text{if } h = (t + 2) + t + 1, \\
\text{Ind}_r(B^m_1) \ast \cdots \ast \text{Ind}_r(B^m_t) \ast \text{Ind}_r(B^m_h), & \text{if } h = 2(t + 2) + t, \\
\text{Ind}_r(B^m_1) \ast \cdots \ast \text{Ind}_r(B^m_t) \ast \{\text{point}\}, & \text{if } (t + 2) + t + 1 < h < (t + 2) + t.
\end{cases}
\end{align*}
\]

Using Lemma 5.6 and Lemma 2.2, we get the result, i.e.,

\[
\text{Ind}_r(B^m_h) \simeq \begin{cases} 
\bigvee_{(m^t-1)} S^{mr(m^0+m^t+2)-1}, & \text{if } h = (t + 2) + t + 1, \\
\bigvee_{(m^t-1)(m^0+m^t+2+m^2(t+2))} S^{r(m^0+m^t+2+m^2(t+2))-1}, & \text{if } h = 2(t + 2) + t, \\
\{\text{point}\}, & \text{if } (t + 2) + t + 1 < h < 2(t + 2) + t.
\end{cases}
\]

**Step k:** In this step, we prove the result for \( h \in \{(k - 1)(t + 2) + t + 1, \ldots, k(t + 2) + t\} \) where \( k \geq 3 \).

The proof here is exactly similar to that of Step 2. Therefore,

\[
\text{Ind}_r(B^m_h) \simeq \frac{\text{Ind}_r(B^m_1) \ast \cdots \ast \text{Ind}_r(B^m_t) \ast \text{Ind}_r(B^m_{h-(t+2)})}{m(m^h-(t+1))-\text{copies}}
\]
Thus, result of Step $k - 1$ implies the following equivalence.

$$\text{Ind}_r(B^m_h) \simeq \begin{cases} \text{Ind}_r(B^m_1) \ast \cdots \ast \text{Ind}_r(B^m_t) \ast \text{Ind}_r(B^m_{m-1}), & \text{if } h = (k-1)(t+2) + t + 1, \\ m(m^0 + m^1 + \cdots + m^{k-1})-\text{copies} & \\ \text{Ind}_r(B^m_1) \ast \cdots \ast \text{Ind}_r(B^m_t) \ast \text{Ind}_r(B^m_{m-1}), & \text{if } h = k(t+2) + t, \\ m(m^i + \cdots + m^{k-1})-\text{copies} & \\ \text{Ind}_r(B^m_1) \ast \cdots \ast \text{Ind}_r(B^m_t) \ast \{\text{point}\}, & \text{if } (k-1)(t+2) + t + 1 < h < k(t+2) + t. \\ m(m^i + \cdots + m^{k-1})-\text{copies} & \\ \{\text{point}\}, & \text{if } (k-1)(t+2) + t + 1 < h < k(t+2) + t. \\ \end{cases}$$

Hence, using Lemma 5.6 and Lemma 2.2, we get the result (recall that $t$ is fixed), i.e.,

$$\text{Ind}_r(B^m_h) \simeq \begin{cases} \bigvee_{(m-1)}^{m^0 + m^1 + \cdots + m^{k-1}} S^{\text{mr}(m^0 + m^1 + \cdots + m^{k-1})-1}, & \text{if } h = (k-1)(t+2) + t + 1, \\ m(m^i + \cdots + m^{k-1})-\text{copies} & \\ \bigvee_{(m-1)}^{m^i + \cdots + m^{k-1}} S^{\text{mr}(m^0 + m^1 + \cdots + m^{k})-1}, & \text{if } h = k(t+2) + t, \\ \{\text{point}\}, & \text{otherwise.} \\ \end{cases}$$

This completes the proof of Theorem 5.7.

\section*{6. Concluding Remarks}

In this section, we list a few interesting questions and conjectures.

\subsection*{6.1. Universality of higher independence complexes.} It was shown in [8] that every simplicial complex arising as the barycentric subdivision of a CW complex may be represented as the 1-independence complex of a graph. One can investigate whether a similar statement holds for all $r$-independence complexes. From the definition it is clear that $\text{Ind}_r(G)$ contains all subsets of $V(G)$ of cardinality at most $r+1$ implying that $\text{Ind}_r(G)$ is always $(r-2)$-connected. Moreover, the following example (which was done using SAGE) tells us that the homology groups of $r$-independence complexes of graphs may have torsion. Let $M_s(G)$ denotes the $s^{th}$ generalised mycielskian of a graph $G$. Then,

$$\tilde{H}_i(\text{Ind}_2(M_4(C_4))) = \begin{cases} \mathbb{Z}_2 & \text{if } i = 3, \\ \mathbb{Z}^{45} & \text{if } i = 5, \\ 0 & \text{otherwise.} \end{cases}$$

One can now ask the following question.

\textbf{Question 1.} \emph{Given $r \geq 2$ and an $(r-2)$-connected simplicial complex $X$, does there exists a graph $G$ such that $\text{Ind}_r(G)$ is homeomorphic to $X$?}

\subsection*{6.2. Trees.} Kawamura [13] computed the exact homotopy of 1-independence complexes of trees and showed that they are either contractible or homotopy equivalent to a sphere. In Section 5, it was shown that the homotopy type of higher independence complexes of $m$-ary trees is also that wedge of spheres. So, one might hope for a similar result for the class of all trees as well.

In another project [7] with Samir Shukla, authors have determined the homotopy type of $\text{Ind}_r(G)$ for chordal graphs $G$ (note that class of tress is a subclass of chordal graphs). A \textit{chordal graph} is a graph in which every cycle on more than 3 vertices has a chord. Homotopy
type of 1-independence complexes of chordal graphs was studied by Kawamura in [14]. Here, we only announce our result, without proving it.

**Theorem 6.1 (7).** The higher independence complexes of chordal graphs are either contractible or homotopy equivalent to a wedge of spheres.

However, the following question is still unanswered.

**Question 2.** Given \( r \geq 2 \) and a trees \( T \), find a formula for the number of spheres in the homotopy decomposition of \( \text{Ind}_r(T) \)?

6.3. Shellable higher independence complexes. In [22], Woodroofe showed that 1-independence complexes of chordal graphs are vertex-decomposable (hence shellable [21, Theorem 1.2]). In a joint work [6] with Manikandan, we have indentified a few classes of graphs whose complexes of chordal graphs are vertex-decomposable (hence shellable [21, Theorem 1.2]).

**Question 3.** For which classes of graphs, the higher independence complexes are shellable?

One might expect a positive answer to the following question.

**Question 4.** Whether \( \text{Ind}_r(G) \) is vertex-decomposable for each \( r \geq 2 \) and chordal graph \( G \)?

There is also the case of chordal graphs.

**Conjecture 6.2.** If \( G \) is a chordal graph then \( \text{Ind}_r(G) \) is shellable for all \( r \).

6.4. Grid graphs. For \( m, n \geq 2 \), a rectangular grid graph, denoted \( G_{m,n} \) is a graph with \( V(G_{m,n}) = \{ (i,j) : i \in \{ m \}, j \in \{ n \} \} \) as its vertex set and \( (i,j) \) is adjacent to \( (i_1,j_1) \) in \( G_{m,n} \) if and only if either \( 'i_1 = i \) and \( j_1 = j + 1 \) or \( 'j_1 = j \) and \( i_1 = i + 1 \). In the last decade, 1-independence complexes of grid graphs have studied in details (see [4, 5, 12] for more details). We have analysed the complex \( \text{Ind}_r(G_{2,n}) \) (for small values of \( n \)) and also computed homology their of using SageMATH [20] (see Table 1 below). Based on our calculations, we make the following conjecture.

**Conjecture 6.3.** For all \( r \geq n \), \( \text{Ind}_r(G_{2,n}) \) is either contractible or homotopy equivalent to a wedge of spheres of dimension \( r - 1 \).

From Table 1, we also see that \( \tilde{H}_i(G_{2,9}) \) is non-trivial in two different dimensions (the notation \( i : \mathbb{Z}^p \) means \( \tilde{H}_i(\text{Ind}_r(G_{2,n}))) = \mathbb{Z}^p \)). This raises the following question.

**Question 5.** What is the homotopy type of higher independence complexes of grid graphs \( G_{m,n} \)?

| \( r \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 : \mathbb{Z}^1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 : \mathbb{Z}^1 | 1 : \mathbb{Z}^4 | 2 : \mathbb{Z}^3 | 3 : \mathbb{Z}^5 | 4 : \mathbb{Z}^1 | 5 : \mathbb{Z}^7 | 6 : \mathbb{Z}^1 | 0 | 0 |
| 3 | 1 : \mathbb{Z}^1 | 1 : \mathbb{Z}^2 | 2 : \mathbb{Z}^5 | 3 : \mathbb{Z}^5 | 4 : \mathbb{Z}^1 | 0 | 0 | 0 | 0 |
| 4 | 1 : \mathbb{Z}^1 | 3 : \mathbb{Z}^2 | 0 | 3 : \mathbb{Z}^7 | 4 : \mathbb{Z}^{13} | 5 : \mathbb{Z}^7 | 6 : \mathbb{Z}^1 | 0 | 0 |
| 5 | 2 : \mathbb{Z}^1 | 3 : \mathbb{Z}^2 | 5 : \mathbb{Z}^1 | 4 : \mathbb{Z}^5 | 5 : \mathbb{Z}^{25} | 6 : \mathbb{Z}^{25} | 7 : \mathbb{Z}^9 | 8 : \mathbb{Z}^8 |
| 6 | 2 : \mathbb{Z}^1 | 3 : \mathbb{Z}^2 | 5 : \mathbb{Z}^{12} | 7 : \mathbb{Z}^2 | 0 | 5 : \mathbb{Z}^2 | 6 : \mathbb{Z}^{40} | 7 : \mathbb{Z}^{12} | 8 : \mathbb{Z}^{14} |
| 7 | 3 : \mathbb{Z}^1 | 5 : \mathbb{Z}^{10} | 5 : \mathbb{Z}^8 | 7 : \mathbb{Z}^{11} | 9 : \mathbb{Z}^1 | 0 | 6 : \mathbb{Z}^8 | 7 : \mathbb{Z}^{56} | 8 : \mathbb{Z}^{128} |
| 8 | 3 : \mathbb{Z}^1 | 5 : \mathbb{Z}^{13} | 8 : \mathbb{Z}^7 | 7 : \mathbb{Z}^{19} | 9 : \mathbb{Z}^{57} | 11 : \mathbb{Z}^2 | 0 | 7 : \mathbb{Z}^8 | 8 : \mathbb{Z}^{12} |
| 9 | 4 : \mathbb{Z}^1 | 5 : \mathbb{Z}^7 | 7 : \mathbb{Z}^4 | 8 : \mathbb{Z}^{45} | 7 : \mathbb{Z}^9 | 9 : \mathbb{Z}^{160} | 11 : \mathbb{Z}^{79} | 13 : \mathbb{Z} | 0 | 8 : \mathbb{Z}^8 |

**Table 1.** Reduced homology groups of \( r \)-independence complexes of grid graphs \( G_{2,n} \). For all \( n \leq 9 \) and \( r \leq 9 \), \( i : 0 \) (i.e. \( \tilde{H}_i(\text{Ind}_r(G_{2,n}))) = 0 \) for all \( i \) not mentioned in the table.
References

[1] E. Babson and D. N. Kozlov. Proof of the Lovász conjecture. *Annals of Mathematics*, 165(3):965–1007, 2007.
[2] J. A. Barmak. Star clusters in independence complexes of graphs. *Advances in Mathematics*, 241:33–57, 2013.
[3] A. Björner and V. Welker. The homology of "k-equal" manifolds and related partition lattices. *Advances in mathematics*, 110(2):277–313, 1995.
[4] M. Bousquet-Mélou, S. Linusson, and E. Nevo. On the independence complex of square grids. *Journal of Algebraic combinatorics*, 27(4):423–450, 2008.
[5] B. Braun and W. K. Hough. Matching and independence complexes related to small grids. *The Electronic Journal of Combinatorics*, 24(4), 2017.
[6] P. Deshpande, N. Manikandan, and A. Singh. Shelling in higher independence complexes. *in preparation*, 2019.
[7] P. Deshpande, S. Shukla, and A. Singh. Distance r-domination number and r-independence complexes of graphs. *in preparation*, 2019.
[8] R. Ehrenborg and G. Hetyei. The topology of the independence complex. *European Journal of Combinatorics*, 110(2):277–313, 1995.
[9] R. Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998.
[10] S. Goyal, S. Shukla, and A. Singh. Homotopy type of independence complexes of certain families of graphs. *arXiv preprint arXiv:1905.06926*, 2019.
[11] J. Jonsson. *Simplicial complexes of graphs*, volume 3. Springer, 2008.
[12] J. Jonsson. Certain homology cycles of the independence complex of grids. *Discrete & Computational Geometry*, 43(4):927–950, 2010.
[13] K. Kawamura. Homotopy types of independence complexes of forests. *Contributions to Discrete Mathematics*, 5(2), 2010.
[14] K. Kawamura. Independence complexes of chordal graphs. *Discrete Mathematics*, 310(15-16):2204–2211, 2010.
[15] D. Kozlov. *Combinatorial algebraic topology*, volume 21. Springer Science & Business Media, 2007.
[16] D. N. Kozlov. Complexes of directed trees. *Journal of Combinatorial Theory, Series A*, 88(1):112–122, 1999.
[17] R. Meshulam. Domination numbers and homology. *Journal of Combinatorial Theory, Series A*, 102(2):321–330, 2003.
[18] N. Nilakantan and A. Singh. Homotopy type of neighborhood complexes of kneser graphs, $kg_{2,k}$. *Proceedings Mathematical Sciences*, 128(5):53, 2018.
[19] G. Paolini and M. Salvetti. Weighted sheaves and homology of artin groups. *Algebraic & Geometric Topology*, 18(7):3943–4000, 2018.
[20] Sage developers. Sagemath, 2016.
[21] A. Van Tuyl and R. H. Villarreal. Shellable graphs and sequentially cohen–macaulay bipartite graphs. *Journal of Combinatorial Theory, Series A*, 115(5):799–814, 2008.
[22] R. Woodroofe. Vertex decomposable graphs and obstructions to shellability. *Proceedings of the American Mathematical Society*, 137(10):3235–3246, 2009.

Chennai Mathematical Institute, India
E-mail address: pdeshpande@cmi.ac.in

Chennai Mathematical Institute, India
E-mail address: anuragsingh@cmi.ac.in