Four-tap shift-register-sequence random-number generators

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Abstract

It is shown how correlations in the generalized feedback shift-register (GFSR) random-number generator are greatly diminished when the number of feedback taps is increased from two to four (or more) and the tap offsets are lengthened. Simple formulas for producing maximal-cycle four-tap rules from available primitive trinomials are given, and explicit three- and four-point correlations are found for some of those rules. A number of generators are also tested using a simple but sensitive random-walk simulation that relates to a problem in percolation theory. While virtually all two-tap generators fail this test, four-tap generators with offset greater than about 500 pass it, have passed tests carried out by others, and appear to be good multi-purpose high-quality random-number generators.
I. INTRODUCTION

The generalized feedback shift-register (GFSR) random-number generator \( R(a, b, c, \ldots) \) produces pseudo-random numbers by the linear recursion \([1–3]\)

\[
x_n = x_{n-a} \oplus x_{n-b} \oplus x_{n-c} \oplus \ldots
\]

where \( \oplus \) is the exclusive-or operation (addition modulo 2) and \( a, b, c, \ldots \) are the feedback taps. Here the \( x_n \) are either single bits or multi-bit words, in which case the \( \oplus \) operation is carried out bit-wise. This recursion was first studied extensively by Golomb \([2]\) in the context of computer science, where it has many other applications, including cryptology and error-correcting codes. Its use as a random-number generator was introduced to the computational physics community by Kirkpatrick and Stoll \([4]\), who suggested the two-tap rule \( R(103, 250) \), and became fairly popular due to its simplicity and generally accepted quality.

However, it is now widely known that such generators, in particular with the two-tap rules such as \( R(103, 250) \), have serious deficiencies. Many years ago, Compagner and Hoogland \([5]\) reported irregularities in an Ising model simulation using \( R(15, 127) \). The present author found problems using \( R(103, 250) \) in a hull-walk simulation \([6]\), and switched to an empirical combination generator \([7]\). Marsaglia \([8]\) observed very poor behavior with \( R(24, 55) \) and smaller generators, and advised against using generators of this type altogether.

More recently, Ferrenberg et al. \([9]\) found that \( R(103, 250) \) leads to results being more than 100 standard deviations from the (known) true values, in simulations of the Ising model with the Wolff cluster-flipping Monte-Carlo algorithm. Coddington \([10]\) confirmed this observation with an extensive study involving a large number of various random-number generators. Grassberger found striking errors in an efficient depth-first self-avoiding random-walk algorithm when \( R(103, 250) \) was used \([11]\). Vattulainen et al. \([12]\) devised a number of simple tests that clearly show the effective correlations and deficiencies in two-tap GFSR generators. And very recently, Shchur et al. \([13]\) simplified the one-dimensional Wolff algorithm to
a repeating one-dimensional random walk test, which they showed fails dramatically when \( R(103, 250) \) is used.

The basic problem of two-tap generators \( R(a, b) \) is that they have a built-in three-point correlation between \( x_n, x_{n-a}, \) and \( x_{n-b}, \) simply given by the generator itself, such that if any two of the \( x_n \) are known, the third follows directly from the recursion \( x_n = x_{n-a} \oplus x_{n-b}. \) While these correlations are spread over the size \( p = \max(a, b, c, \ldots) \) of the generator itself, they can evidently still lead to significant errors. These three-point correlations have been recently brought out clearly in a simulation by Schmid and Wilding [14].

Other problems with this generator are also known. Compagner and Hoogland [5] have shown how a pattern of all 1’s in the initialization string leads to complex (and beautiful) pattern of subsequent bits that persists for a surprisingly long time. Shchur et al. [13] showed that, if an event occurs with a probability close to one (such as 31/32), it is not too unlikely for say 249 successive true outcomes to occur, which then leads to a very serious error at the 250-th step, when the \( R(103, 250) \) generator is used.

For reasons like these, many people have, over the years, advocated using larger tap offset values \( a, b, \ldots \) and increasing the number of those taps from two to four or more (they are always even in number for maximal-cycle generators). Compagner and co-workers have considered generators with offsets as large as 132049 [23], and have proposed combining two generators to effectively make multi-tap rules, which possess good behavior [24,25]. The advantages of using larger offsets are well documented; for example, Ferrenberg et al. found the generator \( R(216, 1279) \) to be nearly acceptable for their problem, and Coddington showed that \( R(1393, 4423) \) reduces the error below the measurable limit for the simulation cutoff that he used. Similar trends were seen by by Compagner and Hoogland [5] and Vattulainen et al. [12].

However, the use of more than two taps has not been common in practice. One reason is undoubtedly that tables of primitive polynomials on \( \text{GF}(2) \) (the Galois field on binary numbers) of order higher than three, which are needed to construct maximal-cycle rules, have been limited (although some have appeared more recently [15,16]), and their direct
determination is a non-trivial exercise in number theory. Golomb has given a prescription for making new generators from existing ones based upon sequence decimation \[17\], which can be used to construct multi-tap rules. In the present paper, I simplify this procedure by giving explicit formulas for 3-, 5-, and 7-decimation of two-tap rules, in which cases four-tap rules always result. These four-tap rules generate, in single calls, the same sequences that come from \(D\)-decimation of the the two-tap generators they derive from.

It turns out that this decimation procedure has been frequently employed in a literal sense: simply by using every \(D\)-th call of a given generator. For example, Ferrenberg et al. considered using every fifth call of the generator \(R(103,250)\), and found that its severe problems seem to disappear. Below I show that this five-call process is equivalent to making a single call of the four-tap generator \(R(50,103,200,250)\), and also discuss the inherent four-point correlations that that generator possesses. Coddington \[10\] and Vattulainen et al. \[12\] also utilized this decimation procedure. From a speed point of view, however, it is clearly advantageous to use the equivalent four-tap rule instead of having to make multiple calls of a two-tap rule for each random number needed.

Recently, some lists of higher-order primitive polynomials have appeared in the literature. Those of André et al. \[18\] concern relatively small offset values \(p\) and have insufficient cycle lengths. Note that these (and other) authors advocate using many more feedback taps — of the order of \(p/2\) — which however would be impractical for the large \(p\) recommended here. Some larger primitive pentanomials have been given by Kurita and Matsumoto \[15\] and more recently by Živković \[16\]; but none of these have been tested here. (The present work was carried out in 1992-94.)

The formulas for constructing new four-tap generators are given in Section II, along with proofs. In Section III, the correlations on smaller generators are found explicitly, and show that four-tap rules are vastly superior to two-tap rules in regards to three- and four-point correlations, except for certain classes of four-tap rules which have strong four-point correlations and probably should not be used. In Section IV, a new test for random-number generators which makes use of a kinetic self-avoiding random walk related to percolation and
the lattice Lorentz gas [19] is introduced. While the two-tap and smaller four-tap generators
badly fail the test, four-tap generators with moderately large offsets pass, and suggest that
with larger offsets, the errors should be nearly unmeasurable. This test is evidently partic-
ularly sensitive to the type of asymmetric correlation that occurs these generators. Some of
our four-tap generators have also been tested by Coddington [10] and Vattulainen et al. [12],
who confirmed the trends seen here.

II. RULES FOR FOUR-TAP GENERATORS.

The taps \((a, b, c, \ldots)\) are chosen so that the corresponding polynomial \(1 + z^a + z^b + z^c + \ldots\) is
primitive over GF(2), guaranteeing that the cycle length will be the maximum possible value
\(2^p - 1\), where \(p = \max(a, b, c, \ldots)\) [2,20]. Besides giving the maximum number of random
numbers before repeating, maximal rules have the advantage that they can be initialized
with any sequence (other than all zeros). For two-tap rules, values of \(a\) and \(b\) can be found
from extensive tables of primitive trinomials [21–23]. Golomb has shown that higher-order
polynomials can be generated from trinomials by using a formal procedure based upon the
concept of decimation [4].

In \(D\)-decimation, every \(D\)-th term of a given sequence is selected to produce a new
sequence. The resulting sequence also satisfies a recursion like (1), corresponding to a
polynomial of order \(p\), although the number of taps is in general different. For some special
cases of interest here, I have found simple formulas which give four-tap rules directly. Before
deriving them, I first introduce the following alternate notation for the recursion (1): Let
\([a, b, c, \ldots]\) indicate that the \(x_n\) satisfy the relation

\[
x_{n-a} \oplus x_{n-b} \oplus x_{n-c} \oplus \ldots = 0
\]

for all \(n\). Thus, \([0, a, b]\) is an equivalent way to write (1) for \(R(a, b)\). These relations satisfy
some obvious properties: If \([a, b, c, \ldots]\) is satisfied on a given sequence, then \([a + k, b + k, c +
k, \ldots]\) will also be satisfied for any \(k\) on that sequence (shift operation). Furthermore, if
both \([a, b, \ldots]\) and \([a', b', \ldots]\) are satisfied, then their union or sum \([a, a', b, b', \ldots]\) will also be satisfied (addition property). Finally, if an offset occurs twice in the list, then it can be eliminated, because \(x_i \oplus x_i = 0\): \([a, b, b, c, \ldots] = [a, c, \ldots]\).

Now, when a shift-register sequence is decimated by any power of 2, the original sequence is reproduced exactly, only shifted \([2]\). To prove this, consider the sequence generated by \(R(a, b) = [0, a, b]\). By the shift property, \([a, 2a, a + b]\) and \([b, a + b, 2b]\) are also satisfied on this sequence. Adding these three relations together yields \([0, a, a, b, b, a + b, a + b, 2a, 2b] = [0, 2a, 2b]\), which implies that every other term in the original sequence satisfies \([0, a, b]\). Thus, it follows that the original sequence and the two-decimated sequence must be identical. Because the decimation wraps around the entire sequence, which is odd in length, the decimated sequence is of the same maximal length as the original one. This proof can be easily generalized for any (even) number of taps, and decimation by any power of 2.

When decimation by a number that is not a power of 2, a new sequence representing a different rule will, in general, be produced. While in general the number of taps varies and may be large, it turns out that four-tap rules always result when a two-tap rule \(R(a, b)\) is decimated by \(D = 3, 5\) and \(7\). Those four-tap rules are given explicitly by the following formulas:

\[
R(a, b) \times 3 = \begin{cases} 
R(a, a/3, 2a/3, b) & 3|a \\
R(a, (2a + b)/3, (a + 2b)/3, b) & 3|(a - b)
\end{cases} \quad (3)
\]

\[
R(a, b) \times 5 = \begin{cases} 
R(a, a/5, 4a/5, b) & 5|a \\
R(a, (4a + b)/5, (a + 4b)/5, b) & 5|(a - b) \\
R(a, (a + b)/5, 2(a + b)/5, b) & 5|(a + b) \\
R(a, (3a + b)/5, (a + 2b)/5, b) & 5|(2a - b)
\end{cases} \quad (4)
\]

\[
R(a, b) \times 7 = \begin{cases} 
R(a, (a + b)/7, 3(a + b)/7, b) & 7|(a + b) \\
R(a, (5a + b)/7, (a + 3b)/7, b) & 7|(2a - b)
\end{cases} \quad (5)
\]

where \(D|a\) indicates that \(a\) is divisible by \(D\) ("\(D\) divides \(a\)"). The remaining cases follow by switching \(a\) and \(b\) in the various formulas — for example, when \(2b - a\) is divisible by \(D\),
then \(a\) and \(b\) must be switched in (4d) and (5b). Cases for \(7|a\) and \(7|(a - b)\) are not listed because these cases do not occur among the primitive trinomials.

I deduced these decimation formulas by generating specific examples using Golomb’s methods \([2,17]\), and finding generalizations. I then verified the formulas by application of the shift and add properties given above.

For example, consider the case (3a). Shifting \([0, a, b]\) by \(b, 2a, 2b\) and \(a + b\) respectively yields the following five relations,

- \([0, a, b]\) original rule
- \([b, a + b, 2b]\) rule shifted by \(b\)
- \([2a, 3a, 2a + b]\) rule shifted by \(2a\)
- \([2b, 2b + a, 3b]\) rule shifted by \(2b\)
- \([a + b, 2a + b, 2b + a]\) rule shifted by \(a + b\)

Summing these and canceling out common terms, one finds

\([0, a, 2a, 3a, 3b]\)

The final relationship (a five-point correlation) holds for any rule \(R(a, b)\). However, when \(a\) is divisible by 3, then all five elements are divisible by 3, so it follows that the 3-decimated sequence satisfies \([0, a/3, 2a/3, a, b]\) or the rule \(R(a/3, 2a/3, a, b)\) as given in (3a).

Likewise, for (3b), we sum:

- \([0, 2a, 2b]\) 2-decimated rule
- \([2a, 3a, 2a + b]\) rule shifted by \(2a\)
- \([2b, 2b + a, 3b]\) rule shifted by \(2b\)

to find

\([0, 3a, 2a + b, a + 2b, 3b]\)

which implies (3b) when \(2a + b\) and \(a + 2b\) are both divisible by 3, which occurs when \(a - b\) is divisible by 3. For primitive trinomials, it is always true that either \(a, b,\) or \(a - b\) is divisible by 3 \([4]\), so (3) contains all cases. Proofs for 5- and 7-decimation are similar. Note that decimations by more than 7 (and not a power of 2) do not, in general, give four-tap rules but ones having many more taps. In this regard, \(D = 3, 5,\) and 7 appear to be special cases.
Using the above formulas with $a$ and $b$ taken from existing tables of primitive trinomials\cite{22,23}, numerous four-tap generators can be found. However, some of these generators will not be of maximal cycle length. In order that the cycle of the decimated sequence be the same as that of the original sequence, it is necessary that $D$ and $2^p - 1$ have no common divisors, i.e., the g.c.d.$(D, 2^p - 1) = 1$. This requirement is satisfied for 3-decimation when $p \mod 2 \neq 0$, for 5-decimation when $p \mod 4 \neq 0$, and for 7-decimation when $p \mod 3 \neq 0$. (On the other hand, when these requirements are not satisfied, the cycle length is less than the maximum simply by a factor of 3, 5 or 7, and is therefore still enormous when $p$ is large, so this consideration may not be so important.) An additional criterion for selecting which rules to decimate, concerning four-point correlations, will be discussed below.

**III. CORRELATIONS**

The relation $[a, b, c, \ldots]$ represents a correlation between the points $x_{n-a}$, $x_{n-b}$, $x_{n-c}$, $\ldots$. These are very strong correlations; for example, $[0, a, b]$, implies that if any two of $x_n$, $x_{n-a}$, and $x_{n-b}$ are known, the third is completely determined, as mentioned above. The sequences generated by (1) are literally laced with such correlations. First of all, the basic correlation is given by the defining rule itself, $R(a, b, c, \ldots)$, in that $[0, a, b, c, \ldots]$ is satisfied for each $n$. Then there is also a whole spectrum of three-point correlations in the system: By the so-called “cycle and add” property\cite{2,3}, there exists an $s$ such that $[0, r, s]$ is satisfied for each value of $r = 1, 2, 3, \ldots 2^p - 1$. The value of $\max(r, s)$ is typically on the order of $2^{p/2}$ to $2^p$, when the defining rule is a pentanomial or higher. However, when the defining rule is a trinomial $R(a, b)$, $s$ will be of the order $p$ for $r = a, b, 2a, 2b, 4a, 4b, \ldots$. These closely spaced three-point correlations interact to form numerous closely spaced four-point, five-point, and higher correlations.

For most application, correlations involving the fewest number of points should be the most serious. For example, if a kinetic random walk returns to the same region in space at steps $n$, $n - a$ and $n - b$ for some $n$, then its behavior would undoubtedly be affected.
by the three-point correlation \([0, a, b]\) in the random-number sequence. Higher correlations would correspond to more coincidences in the motion of the walk and should therefore be less likely. I will assume that the reduction of three-point correlations is most important, followed by four-point correlations, and so on.

Using a four-tap rule \(R(a, b, c, d)\) immediately eliminates the overriding three-point correlation \([0, a, b]\) inherent in a two-tap rule \(R(a, b)\), and the remaining three-point correlations are widely spaced as mentioned above. The four-point correlations of a four-tap rule care also generally widely spaced. An exception occurs when the four-tap rule follows from a \(D\)-decimation of a two-tap rule \(R(a, b)\) and \(a, b,\) or \(a - b\) is divisible by \(D\). In this case, the correlation offsets are small and can be derived explicitly. For example, the 3-decimation of \(R(a, b)\) yields \([0, a/3, 2a/3, a, b]\) according to (3a). By shifting this five-point correlation and adding, one finds the four-point correlation

\[
[0, a/3, 2a/3, a, b] + [a/3, 2a/3, a, 4a/3, a/3 + b] = [0, 4a/3, b, a/3 + b]
\]

(6)

The spread of this correlation is of the order of \(p\), not \(2^p\). Such a four-point correlation in \(R(38, 89) \times 3 = R(38, 55, 72, 89)\) (where \(89 - 38\) is divisible by 3) was noted in [13]. A similar result holds for the 5-decimation rules (4a,b). Therefore, to avoid these relatively close four-point correlations, all 3-decimations (3) and the 5-decimations (4a,b) should not be used, and will not be considered further below, except for the rule \(R(103, 250) \times 5 = R(50, 103, 200, 250)\) which was considered in [9]. Here, 250 is divisible by 5, and as a consequence the sequence obeys the relatively closely spaced four-point correlation \([0, 309, 359, 800]\).

For generators produced by other rules, it appears that the correlations can only be found by a search procedure, in which a sequence of bits is generated, and different correlations are checked until the sequence is matched. To make this feasible for larger \(p\), I made a list of up to \(2^{21}\) 32-bit sub-sequences, and sorted them with keys pointing to their location in the sequence, in order to be able to quickly find if a sequence generated by a trial correlation occurs. Details will be presented elsewhere. This procedure turned out to be practical for finding three- and four-point correlations for \(p\) up to about 50.
Some representative results from this search are given below. Each line shows respectively the way the rule was generated from the two-tap rules of [22], the equivalent four-tap rule from (3) or (4) (which also represents the smallest five-point correlation \([0, a, b, c, d]\) in the sequence), and the smallest four- and three-point correlations found by our search. These results are:

\[
\begin{align*}
R(5, 17) \times 7 &= R(5, 6, 8, 17) = [0, 77, 79, 101] = [0, 67, 83] \quad (7a) \\
R(5, 23) \times 7 &= R(4, 5, 12, 23) = [0, 13, 50, 421] = [0, 1153, 4933] \quad (7b) \\
R(3, 31) \times 5 &= R(3, 8, 13, 31) = [0, 87, 199, 397] = [0, 30189, 34284] \quad (7c) \\
R(6, 31) \times 7 &= R(6, 7, 23, 31) = [0, 40, 623, 2216] = [0, 14487, 101088] \quad (7d) \\
R(8, 39) \times 7 &= R(8, 9, 29, 39) = [0, 111, 1072, 7006] = [0, 172074, 758257] \quad (7e) \\
R(3, 41) \times 7 &= R(3, 8, 18, 41) = [0, 4280, 6131, 8713] = [0, 351102, 1716109] \quad (7f) \\
R(20, 47) \times 7 &= R(20, 21, 23, 47) = [0, 33579, 138448, 150900] = [0, 8474125, 11136544] \quad (7g) \\
R(21, 47) \times 5 &= R(21, 22, 23, 47) = [0, 63608, 148485, 156350] = [0, 11941097, 13215912] \quad (7h)
\end{align*}
\]

Thus, for example, the four-tap rule \(R(5, 6, 8, 17)\) generates a series that has the three-point correlation \([0, 67, 83]\), four-point correlation \([0, 77, 79, 101]\), as well as the inherent five-point correlation \([0, 5, 6, 8, 17]\) (not shown explicitly). Note that the two-tap rule \(R(67, 83)\) corresponding to this three-tap correlation can only be used to generate the sequence produced by \(R(5, 6, 8, 17)\) if it is started up correctly with the 83 bits from the latter’s sequence, because the sequence generated by \(R(5, 6, 8, 17)\) is only one of many cycles of the non-maximal rule \(R(67, 83)\). Therefore, the correlations in brackets, such as \([0, 67, 83]\), should not be interpreted as suggested rules for random-number generators.

The above results clearly show that the separation in the three-and four-point correlations increases rapidly as \(p\) increases. In fact, the extent of the smallest three-point correlation grows roughly as \(2^{p/2}\), and the extent of the smallest four-point correlations as \(2^{p/3}\). Clearly, for larger \(p\), such correlations will be irrelevant, and the most important correlations in four-tap rules will be the five-point ones generated by the rule itself.
Additional maximal length rules can be generated by Golomb’s method of repeated 3-decimation \[17\]. (For some cases of \( p \), repeated 3-decimation of a single maximal-length rule yields the complete cycle of all possible maximal-length rules.) For comparison, I have studied the behavior of some of these other rules. I found that, for a given \( p \), the three- and four-point correlations have roughly the same separation as found for the rules that follow from simple 5- and 7-decimation. For example, for the four-tap rule \( R(23, 27, 40, 41) \), found by successively 3-decimating \( R(3, 41) \) \( 107005025 \) times — equivalent to decimating once by \( 3^{107005025} \mod (2^{41} - 1) = 1962142349662 \) — I find
\[
R(3, 41) \times 1962142349662 = R(23, 27, 40, 41)
\]
\[
= [0, 20573, 22443, 25575] = [0, 429959, 1013792]
\]
(8)
which may be compared with (7f). Six-tap rules with \( p = 41 \) were found to possess similar three- and four-point correlations.

Thus, for useful generators, we turn to rules with larger \( p \). Following are some larger four-tap rules generated by (4b,c) and (5):

\[
R(38, 89) \times 5 = R(33, 38, 61, 89)
\]
(9a)

\[
R(11, 218) \times 7 = R(11, 39, 95, 218)
\]
(9b)

\[
R(216, 1279) \times 5 = R(216, 299, 598, 1279)
\]
(9c)

\[
R(216, 1279) \times 7 = R(216, 337, 579, 1279)
\]
(9d)

\[
R(471, 9689) \times 5 = R(471, 2032, 4064, 9689)
\]
(9e)

\[
R(471, 9689) \times 7 = R(471, 1586, 6988, 9689)
\]
(9f)

\[
R(33912, 132049) \times 5 = R(33912, 46757, 59602, 132049)
\]
(9g)

\[
R(33912, 132049) \times 7 = R(33912, 43087, 61437, 132049)
\]
(9h)

The three- and four-point correlations for these rules are undoubtedly much larger than can be found by my search program. To assess the quality of these generators, I turn to a test based upon a problem from percolation theory.
IV. TEST ON RANDOM WALK PROBLEM

The test I use is shown in Fig. 1. A walker starts at the lower left-hand corner of a square lattice, and heads in the diagonal direction toward the opposite corner. At each step it turns at a right angle either clockwise or counter-clockwise. When it encounters a site it had never visited before, the walker chooses which direction to turn with a 50-50 probability, while at a site that has been previously visited, it always turns so as not to retrace its path (a so-called kinetic self-avoiding trail on a square lattice). The boundary of the lattice is a square; the lower and left-hand sides are reflecting, while the upper and right-hand side are adsorbing. Clearly, by the perfect symmetry of the problem, the walker should first reach either the top or the right-hand sides with equal probability. We shall see, however, that not all these random-number generators yield this simple result.

It turns out that this walk is precisely the kinetic self-avoiding walk that generates the hull of a bond percolation cluster at criticality. The lattice vertices visited by the walk are located at the centers of the bonds, and the two choices correspond to placing either a bond on the lattice or one on the dual lattice across that vertex point. The 1/2 probability of reaching the upper side first corresponds to a spanning or crossing probability of exactly 1/2 for this system [26–28]. The walk is also identical to a lattice-Lorentz gas introduced by Ruijgrok and Cohen [19] with randomly oriented mirrors, to motion through a system of rotators as introduced by Gunn and Ortuno [29], and to paths on the random tiling of Roux et al. [30]. Note that this test is an actual algorithm that has been used in percolation studies [28,33,35]; it is not a “cooked-up” problem designed specifically to reveal flaws in a specific random-number generator.

Using this procedure, I tested a variety of generators, including the two-tap generators \( R(11,218) \), \( R(103,250) \), \( R(216,1279) \), \( R(576,3217) \), and \( R(471,9689) \), the four-tap generators \( R(20,27,34,41) \), \( R(3,26,40,41) \), \( R(1,15,38,41) \), \( R(1,3,4,64) \), \( R(33,38,61,89) \), \( R(11,39,95,218) \), \( R(50,103,200,250) \), \( R(216,337,579,1279) \), and \( R(471,1586,6988,9689) \), and the six-tap generators (determined through successive 3-decimation [17]) \( R(1,5,8,30,35,41) \),
R(5,14,20,36,37,41), and R(18,36,37,71,89,124). Between 100,000 and 2,000,000 trials were simulated with each generator, yielding an error of about $\pm 0.001$. The lattice was of size 4096×4096, and intermediate results for squares of side $L = 64, 128, 192, \ldots, 4032$ were also recorded. Figs. 2 and 3 show the fraction of walks that first arrived at the upper boundary in each of these runs as a function of $L$. Clearly, some generators are very bad; for example, with the notorious R(103,250), the top of the 4096×4096 square was reached only 32% of the time! This error clearly cannot be statistical in origin; in fact, it is about 180 times the standard deviation $\sigma = 0.001$. All of the smaller two-tap generators are clearly quite poor, but even the largest one with $p = 9689$ is barely within two standard deviations at $L = 4096$.

On the other hand, the four-tap generator with $p = 89$ begins to show deviations only at the largest $L$, and the generator with $p = 218$ shows no visible deviations at all in this work. (However, in more recent tests of $10^8$ runs on a lattice of size 256×256, I found some error creeping in for R(11,39,95,218), with the crossing probability at $L = 256$ given by $0.50030 \pm 0.00005$.) Clearly, as $p$ is increased, more random numbers need to be generated before the errors can be seen. For four-tap rules with $p$ larger than about 500, it appears that deviations in this test would be nearly impossible to uncover with present-day computers.

There are a number of interesting and puzzling aspects of these results. Evidently, two-tap generators were give low results, four-tap generators give high results, and six-tap ones again give low ones. The supposedly bad generator R(50,103,200,250), with its strong four-point correlations mentioned above, actually yields excellent results. Finally, the generators R(3,26,40,41) and R(1,15,38,41) are mirrors of each other, and so have identical (but mirrored) correlations of all points, and yet give noticeably different behavior. The explanation of these intriguing properties is a subject for future research. One might also investigate whether the choice to grow a new hull immediately after the previous has completed, without any gap the random number sequence, has any bearing on the results.

Note that the plots in Figs. 2 and 3 are nearly, but not quite, linear. In fact, one can
argue that the behavior must grow with a power of $L$ that is less than or equal to $7/8$. For, say that the error grows as $L^x$ with increasing $L$. This error will first be discernible when the number of runs $N_{\text{runs}}$ satisfies $N_{\text{runs}}^{-1/2} \sim L^x$ or $N_{\text{runs}} \sim L^{-2x}$. The number of random numbers generated per run grows as $L^{7/4}$, where $7/4$ is the fractal dimension of the hull. Thus, the total number of random numbers generated grows as $\sim L^{7/4-2x}$. Now, the exponent in the latter expression cannot be negative, since that would imply that going to an infinite system would allow the error to be found with no work. So we deduce $x \leq 0.875$.

Numerically, a value of about $x = 0.7$ seems to give the best fit to the data in Fig. 2. That $x$ is less than 0.875 implies that doing more runs on a smaller lattice, rather than fewer runs on a larger lattice, is actually a more efficient way to uncover the errors in these generators, assuming the same power-law behavior of the error holds for small $L$.

Because this test is completely symmetric, the errors seen here highlight the fundamental asymmetry of the GFSR generator. Indeed, the basic exclusive-or operation has an asymmetry to it, as two 0’s or two 1’s both result in a 0. For a correlation or generator $[0, a, b]$, the three points $x_n$, $x_{n-a}$, and $x_{n-b}$ can have only the values (0,0,0) and (0,1,1) (and permutations) which is clearly not symmetric. (This asymmetry is not in the total abundance of 0’s and 1’s, which are equally probable, but in their correlations.) Another way of demonstrating this asymmetry is to note that changing 1’s to 0’s and 0’s to 1’s in the initial seed sequence does not result in the complementary sequence being generated. That is, complementary sub-sequences are not equally likely.

We also carried out test with 31- and 48-bit linear congruence generators, and no errors were found. Evidently, these generators have a symmetry such that complementary sequences are generated with equal probability, which leads to a probability of reaching the top of exactly $1/2$. This result underscores the proviso that the test used here is not relevant for all random-number generators — as, indeed, no test is.
V. CONCLUSIONS

Clearly, all GFSR random-number generators will eventually show some detectable errors if a sufficiently long run is made. However, when the four-tap generators with \( p \) greater than about 500 is used, the amount of computer time needed to uncover those errors will be prohibitive. Three- and four-point correlations of these generators are projected to be enormously spread apart. Thus, such large four-tap generators appear useful as a practical, high quality pseudorandom-number generator.

Two-tap generators, in contrast, do not pass the test carried out here, except perhaps those with the largest tap offsets. Thus, for critical applications, it appears that all two-tap generators, not just \( R(103,250) \), should be excluded.

If a problem is sensitive to the built-in five-point correlations of a four-tap generator, then a higher number of taps should be used. For this, the combination generator discussed by Compagner [24,25] is useful.

In spite of their known problems, there are many reasons that GFSR random-number generators remain of interest. In contrast to some combination generators, they are clean and well-characterized; a large body of fundamental theory on their properties has been produced (i.e., [34]). Even with four taps, they remain fast and easy to program. Each bit is entirely independent, which is not the case for linear congruence generators or “lagged-Fibonacci” generators with addition or multiplication. Although they require storing a long list to exhibit good behavior, the memory requirements are not a problem for present-day computers.

Over the last 10 years, we have carried out numerous extensive simulations on a variety of problems in percolation and interacting particle models using the four-tap generators derived here. Our earlier work (i.e., [31]) made use of \( R(157,314,471,9689) \) which derives from \( R(471,9689) \times 3 \); more recently (i.e., [28,33,32]) we switched to the 7-decimation generator \((9f)\) given above, because of the inherent four-point correlations in a 3-decimation rule as discussed in this paper (although we never observed any problem with the former,
presumably because of its large $p$). In all this work, in which we often made checks with exact results when available, we never found any indication of error. In a recent paper determining the bond percolation threshold for the Kagomé lattice [32], we also checked the results of using $R(471,1586,6988,9689)$ against runs using a 64-bit congruential generator, as well as the 3-decimation of $R(471,1586,6988,9689)$ (thus equivalent to $R(471,9689) \times 21$), and found complete consistency throughout.

In closing, I give an explicit example to of the generator, written in a single line of the C programming language. It makes use of the `define` statement, which results in in-text substitution during the pre-compiling stage, so that no time is lost in a function call:

```c
#define RandomInteger (++nd, ra[nd & M] = ra[(nd-A) & M] \n    ^ ra[(nd-B) & M] ^ ra[(nd-C) & M] ^ ra[(nd-D) & M])
```

The generator is called simply as follows

```c
if (RandomInteger < prob) ...
```

where, for rule (9f) for example, $A=471$, $B=1586$, $C=6988$, $D=9689$; and $M = 16383$ (defined as constants), $ra$ is an integer array over $0..M$ that is typically initialized using a standard congruential random-number, $nd$ is its index (an integer), $\&$ is the bitwise “and” operation, and $\^$ is the bitwise “xor” operation. “Anding” with $M$ effectively causes the numbers to cycle endlessly around the list, when $M+1$ is chosen to be a power of two as above. The list in this example requires 64 kilobytes of memory ($16384 \cdot 4$), if 32-bit (4-byte) integers are used. Here, $prob$ is the probability of the event occurring, converted to an integer in the range of 0 to the maximum integer. A floating-point number can also be produced by dividing $RandomInteger$ by the maximum integer (which depends upon the number of bits in the generator), but this added step consumes additional time. Using the above program, an HP 9000/780 workstation computer generates a random number in about 50 nanoseconds, or one billion ($10^9$) in less than a minute.

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FIGURES

FIG. 1. The random walk algorithm used to test the random numbers, shown for a system of size $L = 4$. The test is whether the walker, which turns $90^\circ$ to the left or right with equal probability at each newly visited site, first reaches the right or top with equal probability. The left and bottom sides are reflecting.

FIG. 2. Plot of the probability of the walk reaching the top of an $L \times L$ system, vs. $L$, showing large deviations from the expected value of 1/2 for many of the generators.

FIG. 3. Central portion of Fig. 1 expanded vertically.
Probability of hitting the top
Probability of hitting the top

L

0 500 1000 1500 2000 2500 3000 3500 4000 4500

- 20,27,34,41
- 1,3,4,64
- 3,26,40,41
- 1,15,38,41
- 33,38,61,89
- 11,39,95,218
- 216,337,579,1279
- 50,103,200,250
- 471,1586,6988,9689
- 1,5,8,30,35,41
- 18,36,37,71,89,124
- 471,9689
- 5,14,20,36,37,41
- 576,3217
- 216,1279
- 11,218
- 103,250