String theory in Lorentz-invariant time-like gauge

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Abstract
A theory of closed bosonic string in time-like gauge, related in Lorentz-invariant way with the world sheet, is considered. Absence of quantum anomalies in this theory is shown.
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Introduction

Relativistic string is a curve moving in $d$-dimensional Minkowski space and sweeping by its motion a 2-dimensional surface (world sheet). An area of the world sheet is proposed to be an action of the string.

There are two approaches for the description of string dynamics. Covariant approach keeps the main symmetry of string’s action: group of arbitrary reparametrizations of the world sheet, which should be implemented both in classical and quantum theories. Non-covariant approach eliminates this symmetry, introducing a particular parametrization (gauge) on the world sheet.

In covariant approach the string theory is usually considered in oscillator representation, analogous to the description of field theories by operators of creation and annihilation. It is well known [1], that reparametrization group in this approach has anomalies, violating the parametrical invariance of quantum theory. In special case $d = 26$ the quantum theory has peculiar properties (an existence of properly factorizable null subspace), this is usually considered as an indication of its parametrical invariance.

Covariant string theory was also investigated in non-oscillator representations. Particularly, a consideration, done by Mezincescu and Hennaux in [1] p.157, shows that the theory of closed bosonic string at arbitrary even number of dimensions, quantized in $x,p$-representation, has a solution, possessing quantum parametrical invariance. This fact indicates the absence of anomalies in such representation. Analogous result was obtained in work [2], which shows, that quantum theory of open string in pseudo-Euclidean space with equal number of spatial and temporal directions (particularly, $d = 3 + 3$) can be realized in positively defined extended space of states without anomalies in group of reparametrizations.

Consideration of string theory in covariant approach is related with one difficulty. It is shown in a recent work [3] that the phase space of covariant string theory is much wider than it is usually assumed. It contains infinite regions, filled by classical solutions with negative energies and negative square of mass, related with existence of folds on the world sheets, see fig.1. Conditions, excluding such solutions in classical mechanics, cannot be directly transferred to covariant quantum string theory[3]. Consistent exclusion of tachionic solutions is possible only in non-covariant approach, when one chooses a special parametrization of the world sheet.

1Work [3] shows that standard covariant quantization of string theory in oscillator representation does not exclude classical tachionic solutions, but due to a special effect, caused by indefiniteness of space of states, redefines their square of mass in such a way that $M^2 < 0 \rightarrow M^2 > 0$, and then identifies the tachionic solutions with normal ones.
Standard parametrization, used in non-covariant approach is light cone gauge. This parametrization can be obtained in slicing of the world sheet by a set of parallel planes, where one is tangent to light cone with a center in the origin (fig. 2). Usually the position of such planes is “frozen” in the space-time, i.e. Lorentz transformations change the location of the world sheet with respect to the slicing planes. As a result, Lorentz transformations change the slices on the world sheet, i.e. are followed by reparametrization. In this way the common anomaly in reparametrization group appears also in the group of Lorentz transformations.

A simplest way to avoid this problem was proposed in [4, 5]. In these works Lorentz-invariant light cone gauge was introduced, which relates the slicing planes with the world sheet itself, so that Lorentz transformations move them together the world sheet. In this approach Lorentz transformations are not followed by reparametrization, and on quantum level have no anomalies. Quantum mechanics, based on this idea, was studied by different approaches in [1].

Another Lorentz-invariant parametrization: time-like gauge in center-of-mass frame was introduced by Rohrlich [6]. This gauge leads to a complicated Hamiltonian mechanics. In full generality this mechanics was investigated in [7]. In [8, 9] particular finite-dimensional subsets in the phase space were selected, admitting anomaly-free quantization in $d = 3 + 1$.

In the present work we describe another Lorentz-invariant gauge, which can be considered as weakened variant of the Rohrlich’s one. This gauge leads to a simple Hamiltonian mechanics and gives a possibility to construct anomaly-free quantum theory at arbitrary number of dimensions.

1 Classical mechanics

Theory of closed bosonic string in $d$-dimensional Minkowski space-time is described by canonically conjugated coordinates and momenta:

$$\{x_\mu(\sigma), p_\nu(\bar{\sigma})\} = g_{\mu\nu} \Delta(\sigma - \bar{\sigma}), \quad \mu, \nu = 0...d - 1.$$ 

(1)

Here $x_\mu(\sigma), p_\mu(\sigma)$ are $2\pi$-periodical functions and $\Delta(\sigma)$ is $2\pi$-periodical Dirac’s delta-function. Coordinates and momenta are restricted by constraints:

$$x'p = 0, \quad x'^2 + p'^2 = 0.$$ 

(2)

\footnote{Exception is a case $d = 26$, where the anomalies are absent.}
The constraints belong to the first class in Dirac’s terminology [10]: Poisson brackets of constraints vanish on their surface.

**Mechanics in center-of-mass frame.** Total momentum of the string is given by expression $P_\mu = \oint d\sigma p_\mu(\sigma)$ (here and further $\oint d\sigma$ denotes $\int_{2\pi}^{0} d\sigma$). Following [7], introduce orthonormal basis of vectors, dependent on total momentum: $N_\mu^\alpha(P), N_\mu^\alpha N_\mu^\beta = g^{\alpha\beta} = \text{diag}(+1, -1, .., -1)$, with $N_\mu^0 = P_\mu/\sqrt{P^2}$. This basis defines center-of-mass frame (CMF), where $N_\mu^0$ is temporal axis (directed along total momentum $P_\mu$) and $N_\mu^i, i = 1..d - 1$ are spatial axes (orthogonal to $P_\mu$).

String dynamics in CMF is defined by a set of new canonical variables $(Z_\mu, P_\mu, q^\alpha(\sigma), p^\alpha(\sigma))$ with Poisson brackets

$$\{Z_\mu, P_\nu\} = g_{\mu\nu}, \quad \{Z_\mu, p_\nu^0(\sigma)\} = N_\mu^0 \Delta(\sigma), \quad \{q^\alpha(\sigma), p_\nu^\beta(\bar{\sigma})\} = g^{\alpha\beta}(\Delta(\sigma - \bar{\sigma}) - \Delta(\bar{\sigma})),$$

other Poisson brackets vanish. New variables are related with old ones by expressions (for detailed proofs and derivations look to Appendix 1):

$$q^\alpha(\sigma) = x^\alpha(\sigma) - x^\alpha(0), \quad x^\alpha(\sigma) = N_\mu^\alpha x_\mu(\sigma), \quad p^\alpha(\sigma) = N_\mu^\alpha p_\mu(\sigma),$$

$x^\alpha, p^\alpha$ are projections of coordinates and momenta onto axes of CMF; mean coordinate $Z_\mu$:

$$Z_\mu = N_\mu^0 x^0(0) + \frac{1}{2} N_\nu^\alpha (\partial N_\nu^\beta / \partial P_\mu) M^{\alpha\beta}, \quad M^{\alpha\beta} = \oint d\sigma x^{[\alpha} p^{\beta]}.$$

Here square brackets denote antisymmetrization: $x^{[\alpha} p^{\beta]} = x^\alpha p^\beta - x^\beta p^\alpha$. Variables $M^{\alpha\beta}$ can be obtained from usual Lorentz generators $M_{\mu\nu} = \oint d\sigma x^{[\mu} p^{\nu]}$ by projection to CMF.

New canonical variables are restricted by constraints:

$$q^\alpha(0) = 0, \quad \delta^{\alpha 0} \sqrt{P^2} - \oint d\sigma p^\alpha(\sigma) = 0, \quad q^0 p^0 - \bar{q}^0 \bar{p} = 0, \quad (q^0)^2 + (p^0)^2 - \bar{q}^\alpha \bar{q}^\alpha - \bar{p}^\alpha \bar{p}^\alpha = 0,$$

which as earlier belong to the first class.

**Time-like gauge in CMF.** Let’s impose a condition (gauge): $q^0(\sigma) \equiv 0$. It introduces a particular parametrization on the world sheet, requiring that the string $x_\mu(\sigma)$ should always be an equal-time slice of the world sheet in CMF, see fig. 3. This gauge gives a possibility to exclude $p^0(\sigma)$ from the set of independent variables, defining it identically as $p^0 \equiv \sqrt{\bar{q}^\alpha \bar{q}^\alpha + \bar{p}^\alpha \bar{p}^\alpha}$. Remaining variables have Poisson brackets:

$$\{Z_\mu, P_\nu\} = g_{\mu\nu}, \quad \{q^i(\sigma), p^j(\bar{\sigma})\} = -\delta^{ij}(\Delta(\sigma - \bar{\sigma}) - \Delta(\bar{\sigma})), \quad i, j = 1..d - 1$$

(others are zero). The constraints

$$\bar{q}(0) = 0, \quad \oint d\sigma \bar{p}(\sigma) = 0, \quad \bar{q}^i \bar{p} = 0, \quad \sqrt{P^2} - \oint d\sigma \sqrt{\bar{q}^\alpha \bar{q}^\alpha + \bar{p}^\alpha \bar{p}^\alpha} = 0$$

are again of the first class. In the spirit of Dirac’s theory of constrained systems [11], each constraint, being used as Hamiltonian, generates canonical transformations in the phase space, representing symmetries of the system. In our case the constraints generate reparametrizations of the world sheet, i.e. the transformations, which preserve the action.
In more details: from (3) one can conclude that the first two constraints in (3) have identically vanishing Poisson brackets with all dynamical variables and generate no transformations (they are auxiliary elements of our construction, not related with any symmetry of the system). The third constraint generates reparametrizations of the string \( q^i(\sigma) \rightarrow q^i(\tilde{\sigma}) \), followed by a translation, shifting new \( q^i(\tilde{\sigma} = 0) \) to the origin – see fig.4. This translation is necessary due to the first constraint in (3). The same translation, but in opposite direction, is applied to \( x^i(0) \) (see Appendix 1), resulting to pure reparametrization of equal-time slice \( x^i(\sigma) \). The fourth constraint generates translations of slicing plane in \( P_\mu \) direction, and correspondent evolution of equal-time slice, generated by Hamiltonian \( H = \oint d\sigma \sqrt{\vec{q}'^2 + \vec{p}^2} \). This evolution has a period \( T = \sqrt{P^2} \) (evolution, generated by an equivalent constraint \( (P^2 - H^2)/2\pi = 0 \), is \( 2\pi \)-periodic).

Lorentz generators are given by expressions, identical to (8):

\[
M_{\mu\nu} = X_{[\mu} P_{\nu]} + N^i_\mu N^j_\nu M^{ij}, \quad X_\mu = Z_\mu - \frac{1}{2} N^i_\mu (\partial N^j_\nu / \partial P_\mu) M^{ij}. \tag{7}
\]

Here \( M^{ij} \) is the tensor of orbital moment of the string in CMF, which in bosonic string model is identified with the spin of the particle. In the case \( d = 3 + 1 \) it is related with spin vector \( \vec{M} = \oint d\sigma \vec{x} \times \vec{p} \) as \( M^{ij} = \epsilon^{ijk} M^k \).

The clue property: if in quantum mechanics the commutators of \( Z_\mu, P_\mu \) are postulated directly from Poisson brackets (3), and the algebra of CMF-rotations \( SO(d - 1) \) is represented correctly by \( M^{ij} \): e.g. \([M^i, M^j] = i\epsilon^{ijk} M^k\) for \( d = 4 \), then operators, defined by expressions (7), represent correctly the algebra of Lorentz transformations \( SO(d - 1, 1) \). For \( d = 4 \) this was proven in [8] by direct calculation, and for other \( d \) can be proven analogously. Natural explanation of this fact was also given in [8]: variables (7) actually generate Lorentz transformations of the world sheet together with the set of slicing planes. They are not followed by reparametrizations of the world sheet, like in standard light-cone gauge. Namely these auxiliary reparametrizations create problems in standard approach, because they bring anomalies, destroying the Lorentz algebra.

### 2 Quantum mechanics

Canonical commutators

\[
[Z_\mu, P_\nu] = -ig_{\mu\nu}, \quad [q^i(\sigma), p^j(\tilde{\sigma})] = i\delta^{ij}(\Delta(\sigma - \tilde{\sigma}) - \Delta(\tilde{\sigma}))
\]

can be realized in a direct product of space of functions \( \phi(P) \) onto the space of functionals \( \psi[q(\sigma)] \), with definition of operators

\[
Z_\mu = -i \frac{\partial}{\partial P_\mu}; \quad p^i(\sigma) = -i \left( \frac{\delta}{\delta q^i(\sigma)} - \Delta(\sigma) \cdot \oint d\tilde{\sigma} \frac{\delta}{\delta q^i(\tilde{\sigma})} \right). \tag{8}
\]

Constraints:

\[
\oint d\sigma \vec{p}(\sigma) \psi = 0; \quad \vec{q}(0) \psi = 0; \quad \vec{q}' \vec{p} \psi = 0; \quad (\sqrt{P^2} - H) \psi = 0, \quad H = \oint d\sigma \sqrt{\vec{q}'^2 + \vec{p}^2}. \tag{9}
\]

From the definition of \( \vec{p} \) we see that the first constraint is satisfied identically for any \( \psi \). To satisfy the second constraint, we should define \( \psi = \delta(\vec{q}(0)) \Psi \). Then, we see that \( \vec{p} \) commutes...
through $\delta$-factor: $\bar{p}\delta(\bar{q}(0))\Psi = \delta(q(0))\bar{p} \Psi$. The second term in $\bar{p}$ is generator of global translations $\bar{q}(\sigma) \rightarrow \bar{q}(\sigma) + \tilde{\epsilon}$. Requiring that $\Psi$ is translationally invariant $\Psi[q(\sigma) + \epsilon] = \Psi[q(\sigma)]$, we will have $\delta(q(0))\bar{p} \Psi = \delta(q(0))(-i\delta/\delta\bar{q}(\sigma))\Psi$. For the third constraint: considering linear combinations $\oint d\sigma q'(\sigma) \delta/\delta q(\sigma)$, we see that they act on the state $\Psi$ as generators of reparametrizations $\bar{q}(\sigma) \rightarrow \tilde{q}(\sigma) + \epsilon(\sigma)\tilde{q}'(\sigma)$. Requiring additionally that $\Psi$ is parametrical invariant: $\Psi[q(\tilde{\sigma})] = \Psi[q(\sigma)]$, we will satisfy the third constraint. The fourth constraint has a form of mass shell condition. To satisfy it, we should solve eigenvalue problem for operator $H$ (this will automatically determine the spectrum of mass).

**Operator H** can be defined using an expansion of the square root:

$$H = \oint d\sigma \sqrt{q'^2} \left( 1 + \frac{1}{2q'^2\bar{p}^2} - \frac{1}{(4q'^2\bar{p}^2)^2} + \ldots \right).$$

(10)

Our next goal is to show, that each term of this expansion acts in the selected space of states, i.e. functions of the form $\delta(q(0))\Psi$, where $\Psi$ are translationally and parametrically invariant functionals of $\bar{q}(\sigma)$.

Formal calculation, given in Appendix 2, shows that each term in (10) commutes with the first three constraints in [11], and therefore should act in the selected space of states. However, in concrete calculations the definition of $H$ should be refined, because powers of variational derivative $\delta/\delta q^i$ create divergencies. Particularly, action of $\delta/\delta q^i(\sigma)$ on parametrically invariant integrals $\oint d\sigma q^j F^j(q)$ gives local expression $q^j \partial F^j/\partial q^i|_{\sigma}$, and (in the case if $\partial F^j/\partial q^i$ is not constant) the next differentiation $\delta/\delta q^i(\sigma)$ gives a divergence $\Delta(0) \cdot (q^j \partial^2 F^j/\partial q^i \partial q^i)$. Further we will introduce a wide set of translationally and parametrically invariant functionals, for which $H$ can be reasonably defined.

Let’s consider parametrically invariant functionals of the form $A'(k) = \oint d\sigma q^i e^{ikq}$, and took their translationally invariant products: 1, $A^1(k)A^j(-k)$, ..., $A^{i_1}(k_1)\ldots A^{i_n}(k_n)\delta(k_1 + \ldots + k_n)$. Let’s define the powers of variational derivative as $\lim_{\tilde{\sigma} \rightarrow \sigma} p^i(\tilde{\sigma})p^j(\sigma) = p^i(\sigma)^2$. In this definition the divergent terms are omitted (each $A'(k)$ in the product is differentiated once, giving local expression of $\sigma$, and further variational derivatives with respect to $\bar{q}(\tilde{\sigma} \neq \sigma)$ are not applied to it). Note that for each product finitely many variational derivatives can be applied to give non-vanishing result, and expansion (10) is actually truncated to a finite sum. After all differentiations and outer integration $\oint d\sigma$ in (10) we obtain well-defined functional. Particularly,

$$H A'(k)A'(−k) = \oint d\sigma \sqrt{q'^2} \cdot A'(k)A'(−k) = \oint d\sigma \sqrt{q'^2} \left(k^2 + (d-2)(q'^k)^2 \right).$$

Analogous expressions can be written for other products.

**Remarks**

1. The functionals, found in action of $H$ to the products of $A'(k)$, are translationally and parametrically invariant, however, they are not expressed as products of $A'(k)$. If one finds the expansion of resulting functionals by the products of $A'(k)$, i.e. prove that $H$ acts in the space, spanned by these products, this will give a possibility to solve eigenvalue problem for $H$.

2. Another technically difficult task is an introduction of Hermitian structure in this space of states. It is known [11], that in theories with non-compact gauge group the scalar product from extended
space of states is divergent on physical subspace, selected by constraints (it includes an infinite volume of gauge group). Therefore, it is needed to introduce a new scalar product, acting in the physical subspace. Hermitian property is required only for operators, acting in the physical subspace, particularly, each term in expansion \( \langle | \), integrated by \( d \sigma \), should be Hermitian operator (not \( q'(\sigma), p(\sigma) \), from which it is composed – separately these operators do not act in the physical subspace).

3. It was noted in the previous section, that the function \( H^2/2\pi \) is action-type variable, generating \( 2\pi \)-periodical evolution. Quasiclassically this means that \( H^2/2\pi \) takes integer eigenvalues \( (e^{2\pi i(H^2/2\pi)} = 1) \). This property also reflects a general symmetry of mechanics \( \mathbb{R}^2 \): translation by \( P_n \) transforms the world sheet to itself. (However, concrete definitions of \( H \) can violate this property, and the described symmetry can be lost.)

4. Lorentz group in our approach is free of anomalies, because in generator of CMF-rotations \( M = \oint d\sigma \vec{q} \times \vec{p} = -i \oint d\sigma \delta/\delta \vec{q} + i\vec{q}(0) \times \oint d\sigma \delta/\delta \vec{q} \) the second term vanishes on translationally invariant functionals (also due to the presence of \( \delta(\vec{q}(0)) \) in \( \psi \), and the first term defines correct representation of rotation group in considered space of states (acting as rotations of the argument in \( \Psi[\vec{q}(\sigma)] \)). Rotations obviously commute with all constraints and Hamiltonian \( H \), therefore, the eigenspaces of \( H \) should be rotationally invariant, and can be decomposed into a sum of irreducible representations of rotation group, correspondent to definite values of spin.

5. Absence of anomalies in this approach can be also understood in standard oscillator representation of string theory. Here the constraint \( \vec{q}' \vec{p} = 0 \) can be rewritten as \( L_n - \tilde{L}_{-n} = 0 \), where \( L_n, \tilde{L}_n \) are generators of Virasoro algebra for left and right modes. Using a commutator \( [L_n, L_m] = (n - m)L_{n+m} +(d-1)/12n(n^2-1)\delta_{n,-m} \) and the same commutator for \( \tilde{L}_n \) (coefficient \( (d-1) \) is here, because \( L_n, \tilde{L}_n \) include only \( (d-1) \) space-like oscillators from CMF), one can see that \( L_n - \tilde{L}_{-n} \) forms closed Virasoro algebra without central charge. Actually, we have imposed a gauge, eliminating anomalous components in these two Virasoro algebras, and preserving their anomaly-free combinations.

6. The gauge introduced here is quite similar to the time-like gauge proposed by Rohrlich \( \mathbb{C} \), used also in works \( [7-9] \). An important difference: the Rohrlhich’s gauge, except of \( q^0 = 0 \), also supposes \( p^0 = \sqrt{\vec{q}^2 + \vec{p}^2} = \text{Const} \). This additional constraint does not commute with \( \vec{q}' \vec{p} = 0 \) (Rohrlhich’s gauge fixes parametrization on equal time slice – our approach preserves this freedom). As a result, the whole set of constraints, appearing instead of \( \mathbb{O} \), is of the second class, and the reduction to these constraints leads to a complicated Hamiltonian mechanics.

There is one more distinctive feature of the approach \( \mathbb{O} \) – in this work the second class constraints of the mechanics were imposed onto state vectors, in the spirit: two real constraints \( x = 0, p = 0 \), \( \{x,p\} = i \rightarrow \) one complex constraint \( (x + ip)\Psi = 0 \). Such interpretation of constraints has some physical ground (explanation was given in \( \mathbb{C} \)), however, it is not equivalent to the standard interpretation \( \mathbb{O} \), where imposition of second class constraints is possible only after reduction on their surface and quantization of the obtained reduced mechanics. The present work and \( [7-9] \) use standard methods of constraints imposition.

**Conclusion**

This work describes a particular gauge in the theory of closed bosonic string, which is applicable in the space-time with any number of dimensions, and on the quantum level guarantees the absence of anomalies in Lorentz group and the rest of reparametrization group. The main problem consists in a suitable definition and determination of spectrum for a single operator: quantum analog of Hamiltonian \( H = \oint d\sigma \sqrt{\vec{q}^2 + \vec{p}^2} \).

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Appendix 1: canonical transformations

1. Canonical variables, defining string dynamics in CMF, were derived in [8], using slightly different representation of string theory. Here we will reproduce this derivation in \( x, p \)-representation. For this purpose we will use a formalism of symplectic forms [8, 13].

Poisson brackets correspond to a closed non-degenerate differential 2-form \( \Omega = \frac{1}{2} \omega_{ij} dx^i \wedge dx^j \), defined on the phase space of the system (which is generally considered as a smooth manifold, endowed by local coordinates \( X^i, i = 1, \ldots, 2n \)). Coefficient matrix of the form \( \omega_{ij} \) is inverse to the matrix of Poisson brackets \( \{ \omega^i_j \} : \omega_{ij} \omega^{jk} = \delta_i^k \).

Let’s consider a surface in the phase space, given by the 2nd class constraints: \( \chi_\alpha(X) = 0 (\alpha = 1, \ldots, r) \), \( det ||\{ \chi_\alpha, \chi_\beta \}|| \neq 0 \). Reduction on this surface consists in the substitution of its explicit parametrization \( X^i = X^i(u^a) (a = 1, \ldots, 2n - r) \) into the form:

\[
\Omega = \frac{1}{2} \Omega_{ab} du^a \wedge du^b, \quad \Omega_{ab} = \frac{\partial X^i}{\partial u^a} \frac{\partial X^j}{\partial u^b} \omega_{ij}, \quad det ||\Omega_{ab}|| \neq 0.
\]

Matrix \( ||\Omega^{ab}|| \), inverse to \( ||\Omega_{ab}|| \), defines Poisson brackets on the surface: \( \{ u^a, u^b \} = \Omega^{ab} \).

This method is equivalent to commonly used Dirac brackets’ formalism. Sometimes it is convenient to combine both methods: some of the constraints \( \chi_\alpha(X) \) are imposed as above, then Dirac brackets on the remaining constraints \( \psi_n(u) \) are calculated by definition:

\[
\{ u^a, u^b \}^D = \{ u^a, u^b \} - \{ u^a, \psi_n \} \Pi^{am} \{ \psi_m, u^b \},
\]

where \( ||\Pi^{nm}|| \) is inverse to \( ||\Pi_{nm}|| : \Pi_{nm} = \{ \psi_n, \psi_m \} \).

In string theory canonical Poisson brackets \( \{ x_\mu, p_\nu (\bar{\sigma}) \} = g_{\mu\nu} \Delta (\sigma - \bar{\sigma}) \) correspond to symplectic form \( \Omega = \frac{1}{2} g_{\mu\nu} d\sigma \eta^\mu \wedge d\sigma \eta^\nu \), which is exact, i.e. can be represented as a differential of 1-form: \( \Omega = d\Psi \), \( \Psi = \int d\sigma \phi_\mu \delta x^\mu \).

Using decomposition \( x_\mu = N^i_\mu x^i \), \( p_\mu = N^i_\mu p^i \), we will have \( \Psi = N^a_\mu dN^\beta_\mu + \int d\sigma \phi_\rho \delta x^\rho \), or after elementary transformations: \( \Psi = -\frac{1}{2} N^i_\mu (\partial N^\beta_\mu / \partial P_\nu - M^{ij}) dP^\nu + \int d\sigma \phi_\nu \delta x^\nu \).

Substituting \( x^\nu (\sigma) = x^\nu (0) + (\sigma) \), and taking into account the identity \( \int d\sigma \phi_\rho = 0 \), we can rewrite the second term in \( \Psi \) as \( \sqrt{P^2} dx^0 (0) + \int d\sigma \phi_\nu \delta q^\nu \).

Representing the first term in this expression as \( -x^0 (0) (P_\nu / \sqrt{P^2}) dP^\nu + d(\sqrt{P^2} dx^0 (0)) \), we will have

\[
\Psi = -Z_\nu dP_\nu + \int d\sigma \phi_\nu \delta q^\nu \text{ complete differential, where } Z_\nu = x^0 (0) (P_\nu / \sqrt{P^2}) + \frac{1}{2} N^\nu_\alpha (\partial N^\beta_\nu / \partial P_\rho) M^{\alpha\beta}.
\]

We have constructed new variables in terms of old ones: \( (x_\mu, p_\mu) \rightarrow (Z_\mu, p_\mu, q^\alpha, p^\alpha) \). It is necessary to show that old variables can be reexpressed in terms of new ones, i.e. this mapping is invertible and we actually consider two equivalent bases in the phase space. First of all, it is needed to reconstruct variable \( x(0) \). Considering \( M^{\alpha\beta} \), we see that \( x^\alpha (0) \) enters only in \( M^{\alpha0} \), and other components are expressed completely in terms of new variables:

\[
M^{\alpha0} = x^\alpha (0) \sqrt{P^2} + \int d\sigma q^\alpha |p^\alpha|, \quad M^{ij} = \int d\sigma q^\alpha |p^\alpha|.
\]

Then we are able to obtain the required expression for \( x(0) \):

\[
x_\nu (0) = Z_\nu + (P^2)^{-1/2} N^i_\nu \int d\sigma q^i |p^i| - \frac{1}{2} N^i_\nu (\partial N^i_\nu / \partial P_\rho) M^{ij}.
\]

Finally, old variables are reconstructed by relations \( x_\mu (\sigma) = x_\mu (0) + N^\nu_\mu q^\nu (\sigma) \), \( p_\mu (\sigma) = N^\nu_\mu p^\nu (\sigma) \).

Now we can find symplectic form \( \Omega \) (using antisymmetry of \( \wedge \)-operation and property \( d^2 = 0 \)):

\[
\Omega = d\Psi = dP_\mu \wedge dZ_\mu + \int d\sigma \phi_\nu \delta p^\nu \wedge \delta q^\alpha,
\]

and inverting its coefficient matrix, find Poisson brackets:

\[
\{ Z_\mu, P_\nu \} = g_{\mu\nu}, \quad \{ q^\alpha (\sigma), p^\beta (\bar{\sigma}) \} = g^{\alpha\beta} \Delta (\sigma - \bar{\sigma}),
\]

Variables are restricted by constraints:

\[
q^\alpha (0) = 0, \quad g^\alpha\beta \sqrt{P^2} - \int d\sigma q^\alpha (\sigma) = 0, \quad q^0 p = 0, \quad q^2 + p^2 = 0,
\]

The first two constraints have non-zero Poisson brackets, proportional to \( g^{\alpha\beta} \), and belong to the second class. Calculating new Poisson brackets (= Dirac’s brackets on the surface of second class constraints), we obtain expressions [8].
After this procedure the first two constraints have vanishing Poisson brackets (are in involution) with all dynamical variables\(^4\) and as a result, with all constraints. The Poisson brackets of derivatives \(q^\alpha\) and momenta \(p^\beta\) have the same structure, as those for \(x^\mu(0)\) and \(p_\nu\), therefore, the second pair of constraints (14) obeys the same algebra as (3). Finally, the whole set of constraints belong to the first class.

2. Imposing the gauge \(q^0 = 0\) in symplectic form (13), we see that conjugated variable \(p^0\) drops out. It can be expressed from the fourth constraint in (14) as \(p^0 = \sqrt{q^{12} + \tilde{p}^2}\). Calculating Poisson brackets for the obtained form, we have

\[
\{Z_\mu, P_\nu\} = g_{\mu\nu}, \quad \{q^i(\sigma), p^j(\tilde{\sigma})\} = -\delta^{ij} \Delta(\sigma - \tilde{\sigma}).
\]

The constraints are:

\[
q^i(0) = 0, \quad \oint d\sigma q^i(\sigma) = 0, \quad \dot{q}^i = 0, \quad \sqrt{P^2} - \oint d\sigma \sqrt{q^{12} + \tilde{p}^2} = 0. \tag{15}
\]

Again, the first two constraints belong the second class. Calculation of Dirac’s brackets on their surface gives expressions (5). After that the first two constraints are in involution with all variables; the third constraint defines the same closed algebra, as \(\{x'(\sigma)p(\sigma), x'(\tilde{\sigma})p(\tilde{\sigma})\}\); involution of the third and the fourth constraints can be shown in direct calculation:

\[
\{q^i p^j, \oint d\sigma \sqrt{q^{12} + \tilde{p}^2}\} = \oint d\sigma \frac{q^i q^j + \tilde{p}^2}{\sqrt{q^{12} + \tilde{p}^2}} \Delta'(\sigma - \tilde{\sigma}) = -\frac{d}{d\sigma} \oint d\sigma \frac{q^i q^j + \tilde{p}^2}{\sqrt{q^{12} + \tilde{p}^2}} \Delta(\sigma - \tilde{\sigma}) + \oint d\sigma \frac{q^i q^j + \tilde{p}^2}{\sqrt{q^{12} + \tilde{p}^2}} \Delta(\sigma - \tilde{\sigma}) = -\frac{d}{d\sigma} \sqrt{q^{12} + \tilde{p}^2} + \frac{q^i q^j + \tilde{p}^2}{\sqrt{q^{12} + \tilde{p}^2}} = 0.
\]

Thus, the whole set of constraints (5) is of the first class.

The third constraint generates the following evolution:

\[
\delta q(\sigma) = \{\oint d\tilde{\sigma} \tilde{\epsilon} \cdot (\tilde{q} \tilde{p}), q(\sigma)\} = \epsilon(\sigma) q^i(\sigma) - \epsilon(0) q^i(0).
\]

Here the first term corresponds to infinitesimal reparametrization \(q(\sigma) \rightarrow q(\sigma + \epsilon(\sigma))\), and the second one – to a global translation. Because \(\delta q(0) = 0\), this translation keeps \(q(0)\) in the origin. Variables \(p\) are transformed by reparametrizations correctly – as density:

\[
\delta p(\sigma) = \{\oint d\tilde{\sigma} \tilde{\epsilon} \cdot (\tilde{q} \tilde{p}), p(\sigma)\} = (\epsilon(\sigma) p(\sigma))^\prime,
\]

so that cumulative momentum, contained in interval \(\sigma \in [a, b]: P_{ab} = \int_a^b d\sigma p(s)\), has correct infinitesimal change \(\delta P_{ab} = \epsilon(\sigma) p(\sigma)^\prime_{\mid a} = P_{b, b' - \epsilon(b)} - P_{a, a + \epsilon(a)}\), see fig. 5.

![Fig. 5. Transformations of segment [ab].](image)

Let’s find the change of variable \(x^i(0)\). Components \(M^{ij}\) are parametrically and translationally invariant, and the change of \(x^i(0)\) is caused by the second term in (12). The second term is parametrically invariant and is changed in translation \(q^i \rightarrow q^i - \epsilon(0) q^i(0) N_{i\nu}\). Thus, the infinitesimal change of \(x^i(0)\) equals \(\delta x^i(0) = N_{i\nu} \delta x_\nu(0) = \epsilon(0) q^i(0)\): for \(x^i(\sigma) = x^i(0) + q^i(\sigma)\) the translation terms are compensated, and it is subjected to pure reparametrization.

Evolution, generated by Hamiltonian \(H\), is described by equations:

\[
\dot{q}^i(\sigma) = \{H, q^i(\sigma)\} = p^i(\sigma)/p^0(\sigma) - p^i(0)/p^0(0), \quad \dot{p}^i(\sigma) = \{H, p^i(\sigma)\} = (q^0(\sigma)/p^0(\sigma))^\prime, \quad p^0 = \sqrt{q^{12} + \tilde{p}^2}.
\]

\(^4\)This is guaranteed by a structure of Dirac’s brackets and also evident from definition (3).
It’s easy to check, that in such evolution $p^0(\sigma)$ is constant in time: $\dot{p}^0(\sigma) = 0$. This fact gives a possibility to find from (12) that $\dot{x}^i(0) = p^i(0)/p^0(0)$, and reformulate equations in terms of $(x, p)$:

$$\dot{x}^i = p^i/p^0, \quad \dot{p}^i = (x^i/p^0)'.$$

Simplest way to solve these equations is to introduce a special parametrization on the string: $\dot{\sigma} = (2\pi/\sqrt{P^2}) \int_0^\sigma d\tilde{\sigma} \tilde{p}^0$. Using the fact, that $x', p$ are transformed under reparametrizations as densities, we have:

$$\frac{dx}{d\tilde{\sigma}} = x \frac{dx'}{d\tilde{\sigma}} = \frac{\sqrt{P^2} x'}{2\pi} p^0, \quad \dot{p} = \frac{\dot{x}}{\sqrt{P^2} p^0},$$

and see that in selected parametrization $\tilde{p}^0 = \sqrt{(dx/d\sigma)^2 + \dot{\sigma}^2}$ is constant in $\dot{\sigma}$: $\tilde{p}^0 = \sqrt{P^2}/2\pi$. Then it’s possible to rewrite the equations of motion to a simple form:

$$\frac{dx}{d\tau} = \frac{2\pi}{\sqrt{P^2}} \dot{p}, \quad \frac{d\dot{p}}{d\tau} = \frac{2\pi}{\sqrt{P^2}} \frac{d^2x}{d\sigma^2} \Rightarrow \frac{d^2x}{d\tau^2} = \frac{(2\pi)^2}{P^2} \frac{d^2x}{d\sigma^2}.$$

Solving the obtained 2-dimensional wave equation, we have

$$x = f \left( \frac{2\pi}{\sqrt{P^2}} \tau + \sigma \right) + g \left( \frac{2\pi}{\sqrt{P^2}} \tau - \sigma \right),$$

$$\dot{p} = f' \left( \frac{2\pi}{\sqrt{P^2}} \tau + \sigma \right) + g' \left( \frac{2\pi}{\sqrt{P^2}} \tau - \sigma \right),$$

where $f, g$ are $2\pi$-periodical functions. From the constraints $x'p = 0$, $\tilde{p}^0 = \sqrt{P^2}/2\pi$ we also have $f'^2 = g'^2 = P^2/(4\pi)^2$. Finally, we see that the resolved evolution has period $\Delta\tau = \sqrt{P^2}$.

**Note:** Periodicity of string dynamics has been established by many methods. The work [12] gives purely geometrical explanation of this property.

3. To obtain the expression (7) for Lorentz generators, we write:

$$M_{\mu\nu} = N^\alpha_\mu N^\beta_\nu M^{\alpha\beta} = -N^j_\mu N^0_\nu M^{00} + N^i_\mu N^i_\nu M^{ij}.$$  

Then, using (11), we rewrite the first term as

$$-x^i(0)N^j_\mu P_\nu - N^i_\mu N^0_\nu \oint d\sigma q^{ij} \tilde{p}^0 = \left( x^i(0) - N^i_\mu \frac{1}{\sqrt{P^2}} \oint d\sigma q^{ij} \tilde{p}^0 \right) P_\nu - (\mu \leftrightarrow \nu)$$

and using (13):

$$= (Z_\mu - \frac{1}{2} N^i_\mu (\partial N^j_\rho / \partial P_\mu) M^{ij}) P_\nu - (\mu \leftrightarrow \nu).$$

Then, defining $X_\mu = Z_\mu - \frac{1}{2} N^i_\mu (\partial N^j_\rho / \partial P_\mu) M^{ij}$, we obtain (7).

**Note:** Mean coordinate $Z_\mu$ is not Lorentz vector, but has more complicated law of transformation [8]. The reason is that variables $N^j_\mu$, being functions of $P_\mu$ only, cannot be Lorentz vectors [5]. However, variables $M_{\mu\nu}$, composed from these Lorentz non-covariant objects, define Lorentz tensor and generate correct Lorentz algebra both on classical and quantum levels. Proof of this fact can be found in [8].

### Appendix 2: commutators of H with constraints

The first constraint in (9) is satisfied identically and commutes with all operators. The second constraint commutes with operator $\tilde{p}$, defined by (8), and as a result, with each term in expansion (13). Let’s show that the third constraint commutes with each term in (14). From the explicit definition of operator $\tilde{p}$ it’s easy to obtain the following commutation relations:

5The condition $p^0 = Const$ actually gives Rohrlich’s gauge [9]. In our case this condition helps to solve the equations of motion. But being used as a constraint in the phase space, it leads to complicated redefinition of Poisson brackets, see the discussion in section [9].

6They should be transformed by a subgroup of Lorentz transformations, not changing $P_\mu$, and this contradicts to the fact, that they are functions of $P_\mu$. 


\[ [q', \tilde{p}] = i\Delta' (\sigma - \tilde{\sigma}) p, \quad [q', q] = i\Delta' (\sigma - \tilde{\sigma}) q', \]
\[ [q', \tilde{p}]^2 = 2i\Delta' (\sigma - \tilde{\sigma}) (p\tilde{p}), \quad [q', q^2] = 2i\Delta' (\sigma - \tilde{\sigma}) (q q'). \]
\[ [q', (\tilde{p}^2)^n] = 2i\Delta' (\sigma - \tilde{\sigma}) (p\tilde{p})(\tilde{p}^2)^{n-1}, \quad [q', (q^2)^{-n+1/2}(\tilde{p}^2)^{n}] = 2i\Delta' (\sigma - \tilde{\sigma}) ((-n + 1/2)(q q')(q^2)^{-n+1/2}(\tilde{p}^2)^{n} + n(q^2)^{-n+1/2}(p\tilde{p})(\tilde{p}^2)^{n-1}) = \]
\[ = 2i \left( \frac{d}{d\sigma} \Delta (\sigma - \tilde{\sigma})((-n + 1/2)(q q')(q^2)^{-n+1/2}(\tilde{p}^2)^{n} + n(q^2)^{-n+1/2}(p\tilde{p})(\tilde{p}^2)^{n-1}) \right). \]

Integrating the result by \( \oint d\tilde{\sigma} \), we have
\[ 2i \left( \frac{d}{d\sigma} \Delta (\sigma - \tilde{\sigma})((-n + 1/2)(q q')(q^2)^{-n+1/2}(\tilde{p}^2)^{n} + n(q^2)^{-n+1/2}(p\tilde{p})(\tilde{p}^2)^{n-1}) \right) = 0. \]

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