Gauge Theory: Instantons, Monopoles, and Moduli Spaces

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March 28, 2022

Abstract

In this expository review we discuss various aspects of gauge theory. While the focus is on mathematics, wherever possible we make contact with theoretical high energy physics. Particular emphasis is placed on instantons and monopoles, which admit physical interpretation, and yield interesting and nontrivial mathematics. We give a clear and essentially self-contained exposition of the mathematical structure of the Seiberg-Witten monopole equations. Other topics include Donaldson’s theorem on moduli spaces of monopoles, compactification of spaces of ASD connections, the Abelian monopole equations, and Abelian Higgs vortices.

1 Introduction

In this section we introduce the basic notation and setup for the later sections, which will discuss instantons and monopoles both from the viewpoint of physics, and from the viewpoint of relevance to Floer homology.

1.1 Gauge Theory

The mathematical scenario in which we work is that of Yang-Mills-Higgs theory, which in $d$ space dimensions is defined by a gauge potential

$$A = A_j(x) \, dx^j$$

and a scalar Higgs field

$$\Phi = \Phi(x)$$

The components $A_j(x)$ take values in the Lie algebra $\mathfrak{g}$ of a finite dimensional Lie group $G$, the gauge group. The field $\Phi$ takes values in a representation space $L$ of $G$, corresponding to a representation $\rho : G \to \text{Aut}(L)$. We regard the book [1] as a fundamental reference for the global analysis of these structures.
The gauge potential defines a curvature

\[ F = dA + A \wedge A = \frac{1}{2} F_{ij}(x) \, dx^i \wedge dx^j \]

where in the last expression, the components \( F_{ij} \) are given by the equations

\[ F_{ij}(x) = \partial_i A_j(x) - \partial_j A_i(x) + [A_i(x), A_j(x)] \]

The connection (Lie algebra-valued 1-form) \( A \) defines a covariant derivative acting on the Higgs field, by the following equation

\[ D_A \Phi = (\nabla_A)_j(\Phi) dx^j \]

where \((\nabla_A)_j(\Phi) = \nabla_j \Phi + \rho(A_j)(\Phi)\). This is a special case of the general possibility of defining an exterior covariant derivative on \( p \)-forms taking values in a representation of a Lie algebra (\( g \)-module). For an \( L \)-valued \( p \)-form \( \omega \), we define

\[ D_A \omega \equiv d\omega + \rho(A) \wedge \omega \]

This is covariant with respect to the natural action of a gauge transformation \( g : \mathbb{R}^d \to G \) in the following way:

\[ D_{Ag}(\rho(g)\omega) = \rho(g) \cdot D_A \omega \]

With these definitions, the Euclidean Yang-Mills-Higgs action is

\[ A(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^d} \left\{ (F_A, F_A) + (D_A \Phi, D_A \Phi) + \frac{\lambda}{4} (|\Phi|^2 - 1)^2 \right\} \]

where \( \lambda \geq 0 \) is a coupling constant and the last term in (1) represents the self-interaction of the Higgs field.

In more mathematical language, \( A \) is a connection in a principal \( G \)-bundle \( P \) over a manifold \( M \). \( \Phi \) is a section of the adjoint bundle \( \text{ad}(P) = P \otimes \text{ad} \mathfrak{g} \). The gauge transformations \( g \) are taken as smooth sections of the bundle \( \text{aut}(P) \). In a fundamental paper [7], Floer studies actions of the form (1) with Higgs coupling \( \lambda = 0 \).

The question is to characterize the field configurations \( c \) which minimize the action functional

\[ A(c) = \int_M \left( |F_A|^2 + |d_A \Phi|^2 \right) d \text{vol}_M \]

It is well known that for \( M = \mathbb{R}^3 \), the absolute minima of (2) are characterized by the Bogomolny equations

\[ d_A \Phi = \pm \ast F_A \]

An interesting aspect of Yang-Mills-Higgs theory is that the moduli space of solutions of (3), modulo gauge equivalence, is a finite dimensional manifold for a large class of interesting examples.
With the insertion of the Higgs self-interaction term $\frac{\lambda}{4} \ast (|\Phi|^2 - 1)^2$, the variational equations for (1) are given by

$$D_A \ast F = \ast J$$  \hspace{1cm} (4)
$$\nabla^2_A \Phi = \frac{\lambda}{2} \Phi (|\Phi|^2 - 1)$$  \hspace{1cm} (5)

where $J$ is known in the physics literature as the current, and is defined for an arbitrary representation $\rho$ of the Gauge group by the equation

$$J = -(\rho(h^a)\Phi, D_A\Phi) h_a$$  \hspace{1cm} (6)

in which $\{h^a : a = 1, \ldots, \dim \mathfrak{g}\} = \{h_a\}$ is a chosen basis of $\mathfrak{g}$. The solutions to equations (4), (5), (6) do not depend on this choice.

1.2 Instantons

In this section we begin our discussion of instantons. Recall that finite-action solutions of the field equations (4), (5), (6) are called solitons. In the physics literature, the term instanton refers to a soliton in $d = 4$ Euclidean “pure Yang-Mills theory” (that is, the theory that results from (1) by setting $\lambda = 0$ and $\Phi \equiv 0$, in other words the variational theory of connections on a principal bundle).

The following one-instanton solution of pure Yang-Mills theory was given by Belavin, Polyakov, Tyupkin, and Schwartz. Consider the gauge group $G = SU(2)$ and let $g : \mathbb{R}^4 \rightarrow SU(2)$ be given by $g = |x|^{-1} (x^0 + ix^k \sigma_k)$, where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices. Then $A = \frac{x^2}{x^2 + \mu^2} g \, dg^{-1}$ is a finite-action field configuration satisfying the pure Yang-Mills equations. This solution was found by explicitly computing the curvature two-form

$$F = \sum_{\ell=1}^3 \tau_\ell \left( \frac{2}{r} \frac{d}{dr} e^4 \wedge e^\ell + \frac{2}{r^2} (f^2 - f) \sum_{j,k=1}^3 \epsilon_{jke} e^j \wedge e^k \right)$$  \hspace{1cm} (7)

with $\tau_j = -\frac{1}{2} \sigma_j$, where $\sigma_j$ are the Pauli matrices. One then explicitly computes the Hodge star of (7) and shows that the self-duality equation $F = \ast F$ leads to the following differential equation for $f$:

$$\frac{df}{dr} = -\frac{2}{r} f (f - 1)$$

which has solution $f(r) = \frac{r^2}{r^2 + c^2}$, $c \in \mathbb{R}$.

1.3 The Instanton Bundle

In Section 1.2 above, we introduced the instanton potential,

$$A_1 = \frac{r^2}{r^2 + c^2} \gamma^{-1} d\gamma$$
as a solution to the Euclidean Yang-Mills equations. The map \( \gamma : \mathbb{R}^4 \setminus \{0\} \rightarrow SU(2) \) is given by

\[
\gamma(x) = \frac{1}{r} \left( x^4 - i \sum_{j=1}^{3} \sigma_j x^j \right),
\]

and \( \sigma_j \) are the Pauli matrices. The potential \( A_1 \) is regular at \( x = 0 \), but decays only as \( O(r^{-1}) \) at infinity. However, we can change these features with a gauge transformation. Explicitly,

\[
A_2 = \gamma A_1 \gamma^{-1} + \gamma d\gamma^{-1} = \frac{e^2}{r^2 + e^2} \gamma d\gamma^{-1}
\]

and note that this \( A_2 \) is now singular at \( x = 0 \), but vanishes as \( O(r^{-3}) \) at infinity. Let \( U_1 = S^4 - \{ \text{south pole} \} \) and \( U_2 = S^4 - \{ \text{north pole} \} \) denote the standard covering of \( S^4 \) by two charts, where each chart is identified with \( \mathbb{R}^4 \) by means of stereographic projection. We can thus consider \( A_j (j = 1, 2) \) as being defined on \( U_j \) via pullback. The slow decay of \( A_1 \) at \( r \rightarrow \infty \) means that \( A_1 \) cannot be extended across the south pole. Similarly, the singularity of \( A_2 \) at the origin prevents its being extended from \( U_2 \) to all of \( S^4 \) (but the rapid decrease at infinity implies that its stereographic projection is well defined across the north pole). It now follows as a special case of a general existence theorem for connections on principal bundles (Theorem 1, reproduced below) that \( A_1 \) and \( A_2 \) are local representatives of a connection on an \( SU(2) \) bundle over \( S^4 \) whose transition function is \( \gamma \). The total space of this bundle is \( S^7 \).

**Theorem 1.** Assume \( P \xrightarrow{\pi} M \) is a principal \( G \)-bundle. Let \( \{U_r\} \) be an open covering of \( M \). Given a family of local \( g \)-valued 1-forms \( A_r \in \Lambda^1(U_r, g) \) which fulfill the compatibility condition:

\[
\text{for } x \in U_r \cap U_s, \quad A_{r,x} = \text{Ad}(g_{sr}^{-1}(x)) A_{s,x} + (g_{sr}^* \zeta)_x
\]

where \( \zeta \) is the Maurer-Cartan form on \( G \), and given a set of local sections \( \sigma_r : U_r \rightarrow \pi^{-1}(U_r) \) satisfying

\[
\sigma_s(x) = \sigma_r(x) \rho_{rs}(x)
\]

there is a unique connection \( A \) on \( P \) such that \( A_r = \sigma_r^* A \).

### 1.4 Monopoles

In this section we begin our discussion of monopoles. We again consider a special case of the Yang-Mills-Higgs variational equations, in which now \( d = 3 \), \( G = U(1) \), the representation is the adjoint representation \( L = g \), \( \rho(x) = \text{ad}(x) \) and \( \lambda = 0 \), so the Higgs field self-interaction is infinitely weak but the gauge potential still couples to the Higgs field through the covariant derivative terms.

This is the mathematical model of the physical phenomena of electromagnetism. In this model, \( A \) corresponds to the “magnetic vector potential” encountered in Maxwell theory, and \( \Phi \) is the scalar electric potential.

Introduce a density \( q_e(x) \) of electric charge, and \( q_m(x) \) a corresponding magnetic charge density. In the presence of these background charges the static Maxwell equations are given
by
\[ d \ast F = 0 \quad (8) \]
\[ dF = (4\pi) \ast q_m \quad (9) \]
\[ -\Delta \Phi = 4\pi q_e \quad (10) \]

One obtains Dirac’s magnetic monopole by considering the case in which \( q_m(x) = \delta(x-x_0) \).
This allows for the possibility that the integral of the closed 2-form \( F \) over a two-sphere in \( \mathbb{R}^3 \setminus \{x_0\} \) is nonzero, which gives in turn a nonzero divergence of the magnetic field (which is, by definition \( \vec{B} = \ast F = \ast dA \)).

All \( U(1) \) monopole field configurations have infinite action; no finite action solutions to (8)-(10) exist when \( q_m(x) \) is a delta function.

### 1.5 The Monopole Bundle

In order to describe a magnetic monopole within the framework of electrodynamics, Dirac used the vector potential
\[ A = i \frac{m}{2r(x^3 - r)} (x^1 dx^2 - x^2 dx^1) \]
in which \( x^0, \ldots, x^4 \) are coordinates in Minkowski space, \( r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 \) denotes the squared distance to the origin in Euclidean space, and \( m \) is an integer constant.

Introducing polar coordinates \((r, \vartheta, \varphi)\) on space, and noting that in the polar expression for \( A \) and its gauge transformations, \( r \) does not appear. Thus we can consider \( A \) (and gauge transformations \( A' = A + \gamma^{-1} d\gamma \)) as living on the 2-sphere. Indeed, on \( U_1 = S^2 - \{(0,0,-1)\} \) we have
\[ A_1 = \frac{1}{2} i m (1 - \cos \vartheta) d\varphi \]
and similarly on \( U_2 = S^2 - \{(0,0,1)\} \) we have
\[ A_2 = \frac{1}{2} i m (-1 - \cos \vartheta) d\varphi \]

On the overlap \( U_1 \cap U_2 \) the connections are related by a gauge transformation \( A_1 = A_2 + \gamma^{-1} d\gamma \), where \( \gamma \) is some function on spacetime taking values in \( U(1) \). An appeal to Theorem \( \mathbb{I} \) shows that \( A_1 \) and \( A_2 \) determine a connection on a \( U(1) \) principal bundle over \( S^2 \) whose transition function \( g_{21} : U_1 \cap U_2 \to U(1) \) is given by \( \gamma \). If \( m \neq 0 \), the bundle is nontrivial, and in general, different values of \( m \in \mathbb{Z} \) do not give rise to equivalent bundles.

We give more details in the case \( m = 1 \). The total space of the bundle is \( S^3 \), and identifying \( S^3 \) with the unit sphere in \( \mathbb{C}^2 \) gives coordinates
\[ z^1 = y^1 + iy^2, \quad z^2 = y^3 + iy^4 \]
which can be used to define a (Lie-algebra valued) connection 1-form
\[ \mathcal{A} := i(y^1 dy^2 - y^2 dy^1 + y^3 dy^4 - y^4 dy^3) \]
which takes values in the Lie algebra \( u(1) \). By direct calculation one verifies that the local representatives of \( \mathcal{A} \) are the potentials \( A_1 \) and \( A_2 \) of the \( m = 1 \) Dirac monopole.
2 Bundles and Covariant Derivatives

A principal $G$-bundle $P$ over a smooth manifold $X$ is a manifold with a smooth right $G$-action, and with $X = \text{the orbit space } P/G$. The action must be locally equivalent to the obvious action on $U \times G$ where $U$ is an open set in $X$. This local product structure defines a fibration $\pi : P \to X$.

We will describe here three equivalent ways to describe a connection on such a bundle:

(i) A smooth distribution of $G$-invariant ‘horizontal subspaces’ $H \subset TP$, where ‘horizontal’ means that for all $p \in P$, we have a decomposition $TP_p = H_p \oplus T(\pi^{-1}(x))$, where $x = \pi(p)$.

(ii) A $G$-invariant 1-form $A$ on $P$ taking values in the Lie algebra $\mathfrak{g} = \text{Lie}(G)$, or in other words a section of the bundle $T^*P \otimes \mathfrak{g}$, where ‘$G$-invariant’ refers to a combination of the given action of $G$ on $P$, and the adjoint action representation of $G$ on $\mathfrak{g}$.

(iii) A covariant derivative $\nabla$ on an associated vector bundle $E$, which is by definition a linear map $\nabla : \Omega^0_X(E) \to \Omega^1_X(E)$ satisfying the Leibniz rule: $\nabla(fs) = f\nabla s + (df)s$ where $f$ is a function on $X$, and $s \in \Gamma(E)$. This gives a connection in the sense of (i), as follows. Let $P$ be the frame bundle of $E$, then a local section $\sigma$ of $P$, which is a collection $(s_1, \ldots, s_n)$ of local sections of $E$, is called ‘horizontal’ if $(\nabla s_i)_x = 0$ for all $i = 1, \ldots, n$. Then define $H_p$ to be the tangent space to (the image of) a horizontal section $\sigma$ through $p$, regarding the latter as a submanifold of $P$.

For the classical groups, these are all equivalent. In working with the Yang-Mills equations on Euclidean spaces, the most useful connection is the standard product connection which can be defined on any trivial bundle $\mathbb{C}^n \times X$ by taking the covariant derivative to be the usual notion of ‘total derivative’ of vector valued functions.

If $X$ is an oriented Riemannian four-manifold, Hodge theory gives a decomposition of the 2-forms on $X$ into self-dual and anti-self-dual (ASD) pieces, which are just the $\pm 1$ eigenspaces of the Hodge $*$ operator:

$$ \Omega^2_X = \Omega^+_X \oplus \Omega^-_X $$

This construction extends naturally to bundle-valued forms, and hence to the curvature form $F_A$ of a connection. The connection is called ASD (self-dual) if $F_A^+(F_A^-)$ is zero.

For connections with $SU(r)$ structure group on a 4-manifold $X$, we have

$$ c_2(E) = \frac{1}{8\pi^2} \int_X \text{Tr}(F_A^2) \in \mathbb{Z} $$

and a connection is ASD if and only if we have $\text{Tr}(F_A^2) = |F_A|^2 \ d\mu$, where $d\mu$ is the Riemannian volume element. We now make connection with Yang-Mills theory: the functional

$$ \|F_A\|^2 = \int_A \text{Tr}(F_A^2) \ d\mu $$

is a special case of the Yang-Mills functional, and these observations suffice to show that $|8\pi^2 c_2(E)|$ gives a lower bound on the Yang-Mills functional, which (when $c_2 > 0$) is achieved precisely in case that the connection $A$ is anti-self-dual.
3 Spin, Spin$^c$, and Dirac

The material in this section owes much to [9].

Let $E \rightarrow X$ denote a real $n$-dimensional vector bundle over the smooth manifold $X$. We assume:

1. We have a positive definite inner product $(\ ,\ )$ defined continuously in the fibers.
2. The bundle is oriented, so there is a choice of orientation for the vector space which forms each fiber, chosen in a continuous way.

We will investigate the second condition more closely. Note that in general, we have an isomorphism

\[ H^1(X; G) \cong \left\{ \text{equivalence classes of principal } G\text{-bundles on } X \right\} \]  

This implies that the set $\text{Cov}_2(X)$ of equivalence classes of 2-sheeted coverings of $X$ is isomorphic to $H^1(X; \mathbb{Z}_2)$. Let $P_0 \rightarrow X$ be the principal $O(n)$ bundle whose fiber at $x \in X$ is the space of ON bases of $E_x$. The bundle of orientations $O_E$ is the quotient $P_0/SO(n)$, which is a 2-sheeted covering of $X$. In fact, $E \rightarrow X$ is orientable iff $w_1(E) = 0$, where $w_1$ denotes the 1st Stiefel-Whitney class, which is the element of $H^1(X; \mathbb{Z}_2)$ defined by the covering $O_E \rightarrow X$ via the isomorphism (11).

Thus choosing an orientation is equivalent to simplifying the structure group of the bundle from $O(n)$ to its subgroup $SO(n)$, which is connected. We now ask whether it is possible to obtain a simply connected structure group. Let $P_{SO} \rightarrow X$ denote the bundle of oriented orthonormal frames in $E$, and for $n \geq 3$ let $\xi_0 : \text{Spin}(n) \rightarrow SO(n)$ denote the universal covering group homomorphism.

**Definition 1.** A spin structure on $E$ is a principal Spin($n$) bundle $P_{\text{Spin}}(E)$ together with a 2-sheeted covering $\xi : P_{\text{Spin}}(E) \rightarrow P_{SO}(E)$ such that

\[ \xi(pg) = \xi(p)\xi_0(g) \quad \text{for all } p \in P_{\text{Spin}}(E), g \in \text{Spin}(n) \]

Spin structures on $E \xrightarrow{\pi} X$ are in natural 1-1 correspondence with the 2-sheeted coverings of $P_{SO}(E)$ which are non-trivial on fibers of $\pi$. Also, there is an existence-uniqueness result for spin structures similar to that which was noted above for orientations. A spin structure exists $\iff$ the 2nd Stiefel-Whitney class $w_2(E)$ is zero, and in this situation the distinct spin structures lie in 1-1 correspondence with the elements of $H^1(X; \mathbb{Z}_2)$. In summary, if a Riemannian vector bundle $E$ over $X$ is equivalent to a vector bundle with connected, simply connected structure group, then it is orientable and spin. A manifold $X$ is said to be spin if $TX \rightarrow X$ is a spin bundle.

We now discuss briefly the associated bundle construction. Given a principal $G$-bundle $P \rightarrow X$, $G$ a lie group, and given a continuous homomorphism $\rho : G \rightarrow \text{Homeo}(F)$ from $G$ into the group of homeomorphisms of a space $F$, then consider the free left action of $G$ on $P \times F$ given by

\[ g \cdot (p, f) := (pg^{-1}, \rho(g)f) \]  

(12)
The projection $P \times F \xrightarrow{\pi} X$ is constant on orbits under $\mathcal{G}$, hence it induces a mapping $P \times_\rho F \xrightarrow{\pi_\rho} X$ where $P \times_\rho F$ is the orbit space of the action. If $\rho : G \to GL(V)$ is a linear representation, then $(P \times_\rho F, \pi_\rho)$ is a vector bundle.

Now let $M$ be a left module over the Clifford Algebra $C\ell(\mathbb{R}^n)$, and let $\xi : P_{Spin}(E) \to P_{SO}(E)$ be a spin structure. A real spinor bundle for $E$ is a bundle of the form $S(E) = P_{Spin}(E) \times_\mu M$ where $\mu : Spin(n) \to SO(M)$ is the representation given by left multiplication by elements of $Spin(n) \subset C\ell^0(\mathbb{R}^n)$. Similarly, one can consider complex spinor bundles by considering $M_C = \text{a complex representation for } C\ell(\mathbb{C}^n)$.

Let $n = 2m$ be even and let $S_C(E) = \text{the irreducible complex spinor bundle of } E$. Then $S_C(E)$ splits naturally into a direct sum

$$S_C(E) = S_C^+(E) \oplus S_C^-(E)$$

of $C\ell^0(E)$-modules, which are defined to be the $\pm 1$ eigenspaces of Clifford multiplication by the element defined at each point $x \in X$ by the following equation:

$$\omega_{C,x} := i^m e_1 \cdots e_{2m}$$

where $e_1, \ldots, e_{2m}$ is a positively oriented basis of $E_x$.

We now remark on the important special case $E = TX$. We set $P_{SO}(X) = P_{SO}(TX)$, $C\ell(X) = C\ell(TX)$. Then there exists a unique connection on $P_{SO}(X)$ with identically vanishing torsion. If $X$ has a spin structure $\xi : P_{Spin}(X) \to P_{SO}(X)$ then we can lift this canonical Riemannian connection to a connection on $P_{Spin}(X)$. Thus all spin bundles inherit this connection.

In general, let $S$ be any bundle of left modules over $C\ell(X)$. Assume $S$ is Riemannian and has the canonical connection. In this scenario, there is a canonical first order differential operator called the Dirac operator

$$D : \Gamma(S) \to \Gamma(S)$$

by using the following local formula: at $x \in X$ we define

$$D\sigma := e^j \cdot \nabla_{e_j} \sigma$$

where $\{e^i\}$ is an ON basis of $TX_x$ and $\cdot$ denotes Clifford multiplication. By direct computation of symbols, one checks that both $D$ and $D^2$ (the Dirac laplacian) are elliptic operators. The Dirac operator in this form was first written down by Atiyah and Singer in the course of their work on the index theorem.

Finally, we remark that one can develop much of the theory of spin structures in the parallel case of Spin$^c$ structures, in which (roughly speaking) many of the important structural elements are complexified. By definition the group Spin$^c(V)$ is the subgroup of the multiplicative group of units of $C\ell(V)\otimes_{\mathbb{R}}\mathbb{C}$ generated by Spin$(V)$ and the unit circle of complex scalars. Then there is an isomorphism Spin$^c(V) \cong \text{Spin}(V) \times_{\mathbb{Z}_2} U(1)$. [Proof: since the circle of unit scalars commutes with Spin$(V)$, it follows that we have a natural surjective map Spin$(V) \times U(1) \to \text{Spin}^c(V)$. The kernel of this map is $\{(\alpha, \alpha^{-1}) \mid \alpha \in \text{Spin}(V) \cap U(1)\}$, but the intersection of Spin$(V)$ with the scalars is $\{\pm 1\} = \mathbb{Z}_2$.]

Thus, Spin$^c(V)$ is the double covering group of $SO(V) \times U(1)$ which is nontrivial on each factor.
Definition 2. Let $P_{SO} \to X$ be a principal $SO(n)$-bundle on $X$. A Spin$^c$ structure on $P_{SO}$ consists of a principal $U(1)$ bundle $P_{U(1)}$ and a principal Spin$^c(n)$-bundle $P_{Spin^c}$ together with a Spin$^c$-equivariant bundle map

$$P_{Spin^c} \to P_{SO} \times P_{U(1)}$$

The integral class $c \in H^2(X; \mathbb{Z})$ corresponding to $P_{U(1)}$ under the isomorphism $H^2(X; \mathbb{Z}) \cong \text{Prin}_{U(1)}(X)$ is called the canonical class of the Spin$^c$ structure.

4 Moduli Spaces of Monopoles: Donaldson’s Theorem

A monopole on $\mathbb{R}^3$ is defined by Atiyah [4] to consist of a gauge field (connection) $A_\mu(x), \mu = 1, 2, 3$, and a Higgs field $\psi(x)$ (all smooth functions of $x \in \mathbb{R}^3$ and take values in the Lie algebra of $SU(2)$) which satisfy the so-called Bogomolny equations

$$D\phi = \ast F$$

where $D_\mu \phi = \partial_\mu \phi + [A_\mu, \phi]$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$. Moreover, we require the energy ($L^2$-norm of $F$) to be finite. The Bogomolny equations are in fact equivalent to the self-dual Yang-Mills equations in Euclidean 4-space. The general structure of the moduli space of solutions to these equations was given by Donaldson in [6], in which he gives an identification of a circle bundle over the moduli space with the space of basepoint preserving rational maps $\mathbb{C}P^1 \to \mathbb{C}P^1$ of degree $k$. To mention a few of the basic details which are used in Donaldson’s proof fits well into this survey. The starting point is a theorem of Hitchin [5]

Theorem 2 (Donaldson). There is a natural equivalence between the following two structures:

1. monopoles for the group $SU(2)$ with charge $k$, up to gauge transformation.
2. conjugacy classes under $O(k, \mathbb{R})$ of matrix-valued functions $f_1(s), f_2(s), f_3(s)$ of one real variable $s \in (0, 2)$ satisfying

$$\frac{dT_i}{ds} + \sum_{j,k} \epsilon_{ijk} T_j T_k = 0$$

$$T_i^*(s) = -T_1(s)$$

$$T_i(2-s) = T_i(s)^t$$

In addition, we require that $T_i$ will extend to a meromorphic function on a neighborhood of $[0, 2]$ with simple poles at $s = 0, 2$ but otherwise analytic, and the residues of the matrices $T_i$ at the poles $s = 0, 2$ define an irreducible representation of $SU(2)$.

Donaldson has observed that the conditions $\frac{dT_i}{ds} + \sum \epsilon_{ijk} T_j T_k = 0$ are equivalent to the ASD equations for the connection

$$A = T_1(s) dx_1 + T_2(x) dx_2 + T_3(s) dx_3$$

on $\mathbb{R} \times \mathbb{R}^3$ with coordinates $s, x_1, x_2, x_3$. 

5 Compactification of Moduli Spaces of ASD Connections

We wish now to introduce moduli spaces of ASD Yang-Mills connections. Let $E$ be a bundle over a compact oriented Riemannian $X^4$. The set $M_E$ is defined to be the set of gauge equivalence classes of ASD connections on $E$. More explicitly, ‘gauge equivalence’ refers to the action of the bundle automorphism group $G$.

We recall briefly how the gauge group acts on the space of connections. For a given trivialization $\tau$ of $E$, we let $A^{\tau}$ denote the connection matrices of $A$ in this trivialization. (By “connection matrices” we will always be implicitly referring to the fact that the connection can be viewed as a one-form taking values in a finite-dimensional semisimple Lie algebra, i.e. a matrix Lie algebra.)

Suppose $u \in G$, the group of bundle automorphisms of $E$. We define the action of $u$ on the connection $A$ by how it transforms the corresponding covariant derivatives:

$$\nabla_{u(A)} s = u \nabla_A (u^{-1} s)$$

Using the fact that $u$ is a section of the vector bundle $\text{End}(E)$, we can take the covariant derivative of it. In this language, we have $u(A) = A - (\nabla_A u) u^{-1}$.

In terms of the connection matrices in some trivialization we have

$$A^{u \tau} = u A^{\tau} u^{-1} - (du) u^{-1}$$

The space $A$ of connections has the structure of an affine space, and the topological and metric structure of an infinite-dimensional Banach manifold, however the moduli space $M_E$ with its inherited topology, is finite dimensional.

The construction of this moduli space can be broken naturally into two steps.

(i) Find the solutions to the ASD equation. $F_A^+ = 0$, where the $+, -$ indices refer to the natural splitting of $\Omega^2_X$ into $\pm 1$ eigenspaces of the Hodge $*$ operator.

(ii) Quotient by the action of the gauge group.

We will now discuss the usefulness of Sobolev spaces in Part (ii) of this program.

In four dimensions, the Sobolev embedding theorem states that $k > 2 \implies W^{k,2}$ embeds into the space of continuous functions. One can then define the notion of a $W^{k,2}$ map from a domain in $X$ to the structure group $G$ of the bundle. In many important examples, $G$ can be identified with a group of unitary matrices, and then one can consider matrix-valued maps which are of class $L^2_k$ with respect to the norm on $U(n)$. One then defines an $L^2_k G$-bundle to be a bundle in which the transition functions are of class $L^2_k$. In a similar line of thought, one defines Sobolev spaces of connections on a $G$-bundle by demanding that in any local trivialization the connections are given by $L^2_k$ connection matrices. Such connections have curvature in $L^2_{k-2}$, Here we have used the fact that pointwise multiplication induces a continuous map

$$L^2_{k-1} \times L^2_{k-1} \longrightarrow L^2_{k-2}$$

for $k > 2$. 

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We write $\mathcal{A}$ for the space of $L^2_{k-1}$ connections on an $L^2_k G$-bundle, and $\mathcal{B}$ for the quotient $\mathcal{A}/\mathcal{G}$, where of course $\mathcal{G}$ denotes automorphisms in the category of $L^2_k$ bundles.

The important point is that the $L^2$ metric on $\mathcal{A}$ is preserved by the action of $\mathcal{G}$ so that

$$d([A], [B]) := \inf_{g \in \mathcal{G}} \| A - g(B) \|$$

gives a metric on $\mathcal{B}$. In particular $\mathcal{B}$ is Hausdorff in the $L^2_{k-1}$ topology, which is finer than the $L^2$ topology.

We now comment briefly on the local structure of the moduli space $\mathcal{B}$. In light of the topological isomorphism

$$\Omega^1(g) = \text{im} d_A \oplus \ker d_A^*$$

that is given by elliptic theory, it holds that a neighborhood of $[A]$ in $\mathcal{B}$ can be described as a quotient of $T_{A,\epsilon}$ for small $\epsilon$, where

$$T_{A,\epsilon} = \{ a \in \Omega^1(g) \mid d_A^* = 0, \|a\|_{L^2_{k-1}} < \epsilon \}$$

For $X$ a compact orientable manifold with Riemannian metric $g$, and $SU(2)$-bundle $E$ with second Chern class $c_2(E) = k$. If $b^+ > 0$, the moduli space $M_k$ of ASD connections on $E$ is, for a generic metric, an orientable smooth manifold of dimension $8k - 3(1 - b_1 + b^+)$, where $b^+$ is the number of positive eigenvalues of the intersection form and $b_1$ is the first Betti number. The manifold $M_k$ is not necessarily compact, but it can be compactified (as shown by Uhlenbeck) Further information on these compactifications can be obtained from [8] or from [10].

As a final remark on the moduli space for the ASD equations, we note that it is not empty. A theorem of Clifford Taubes [12] establishes the existence of self-dual connections on a 4-manifold $M$ whose intersection form is positive definite, using analytic techniques to build the connections on $M$ from those on $S^4$. Atiyah, Hitchin, and Singer were able also to construct these connections, under the additional assumption that $M$ is “half-conformally flat,” using twistor theory to convert Yang-Mills into a problem in algebraic geometry.

### 6 The Abelian Monopole Equations and Seiberg-Witten Theory

Let $(X, g)$ be a closed oriented Riemannian four-manifold. Choose a Spin$^c$ structure on $X$, $c \in \text{Spin}^c(X)$, let $L = L_c$ be the corresponding Hermitian line bundle, and let $S^2_L^+$ denote the corresponding spinor bundles.

A field configuration in this setup is a pair $(A, \psi)$ where $A$ is a unitary connection on $L$ and $\psi$ is a smooth section of $S^2_L^+$. The Seiberg-Witten equations, which one should think of as abelian monopole equations, are differential equations of these field configurations:

\begin{align}
\vartheta_A \psi &= 0 \\
F_A^+ &= q(\psi) = \psi \psi^* - \frac{|\psi|^2}{2} I
\end{align}
The remainder of this section will be devoted to the structure of these equations. $\mathcal{D}_A$ is the Dirac operator associated to the Levi-Civita connection on the frame bundle of the tangent bundle, and the connection $A$ on the determinant line bundle of the Spin$^c$ structure. The first equation (13) just says that $\psi \in \ker \mathcal{D}_A$. For the first equation, note that $S_L^+$ has a hermitian metric, hence we can identify this bundle with its dual via an anti-hermitian isomorphism. So $\psi^*$ denotes the image of $\psi$ under this isomorphism. Thus,

$$\psi \psi^* \in S_L^+ \mathcal{D}(S_L^+)^* = \text{End}_C \left(S_L^+\right)$$

Now recall that for a positive definite real oriented inner product space $V$, with oriented ON basis $\{e_1, \ldots, e_n\}$, we define $\omega_C = i^{(n+1)/2} e_1 \cdot \ldots \cdot e_n$

This element $\omega_C$ squares to 1 and does not depend on the choice of basis. Moreover, its $\pm 1$ eigenspaces define a canonical splitting of the complexified Clifford algebra $\mathcal{C}(V)\otimes \mathbb{C}$. We write these eigenspaces as $(\mathcal{C}(V)\otimes \mathbb{C})^\pm$. Then Clifford multiplication induces an isomorphism

$$(\mathcal{C}(V)\otimes \mathbb{C})^+ \cong \text{End}_C \left(S_L^+\right),$$

and it is readily seen that

$$(\mathcal{C}(V)\otimes \mathbb{C})^+ \cong \mathbb{C} \left(\overline{1+\omega_C}/2\right) \oplus (\Lambda^2_+(TX)\otimes \mathbb{C}) \quad (15)$$

Under the isomorphism $\mathbb{C} \left(\overline{1+\omega_C}/2\right) \oplus (\Lambda^2_+(TX)\otimes \mathbb{C}) \longrightarrow \text{End}_C \left(S_L^+\right)$ implied by eqns. (15)-(16), $(1+\omega_C)/2$ acts as the identity and the traceless endomorphisms of $S_L^+$ come from elements of $\Lambda^2_+(TX)\otimes \mathbb{C}$. The trace of $\psi \psi^*$ is $|\psi|^2$, so $q(\psi) = \psi \psi^* - \frac{|\psi|^2}{2} I$ is traceless and can therefore be identified with a section of $\Lambda^2_+(TX)\otimes \mathbb{C}$. Using the metric to identify the tangent bundle $TX$ with the cotangent bundle $T^*X$, $q(\psi)$ can be viewed as a complex valued self-dual 2-form. The other Seiberg-Witten equation (14) just says that $q(\psi)$ is the self-dual part of the curvature form.

An excellent reference for further study is [11].

### 7 The Stability of Magnetic, or Abelian Higgs Vortices

In this section, we discuss results on the stability of magnetic (or Abelian Higgs) vortices. These are certain critical points of the energy functional

$$E(\psi, A) = \frac{1}{2} \int_{\mathbb{R}^2} \left\{ |\nabla_A \psi|^2 + (\nabla \times A)^2 + \frac{\lambda}{4} (|\psi|^2 - 1)^2 \right\}$$

for the fields

$$A : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{and} \quad \psi : \mathbb{R}^2 \to \mathbb{C}.$$ 

Here $\nabla_A = \nabla - iA$ is the covariant gradient, and $\lambda > 0$ is a coupling constant. For a vector, $A$, $\nabla \times A$ is the scalar $\partial_1 A_2 - \partial_2 A_1$, and for a scalar $\xi$, $\nabla \times \xi$ is the vector $(-\partial_2 \xi, \partial_1 \xi)$. Here $\nabla$
Critical points of $E(\psi, A)$ satisfy the *Ginzburg-Landau* (GL) equations

$$-\Delta_A \psi + \frac{\lambda}{2}(|\psi|^2 - 1)\psi = 0$$  \hspace{1cm} (18)

$$\nabla \times \nabla \times A - \Im(\bar{\psi} \nabla A \psi) = 0$$  \hspace{1cm} (19)

where $\Delta_A = \nabla_A \cdot \nabla_A$.

Physically, the functional $E(\psi, A)$ gives the difference in free energy between the superconducting and normal states near the transition temperature in the Ginzburg-Landau theory. $A$ is the vector potential ($\nabla \times A$ is the induced magnetic field), and $\psi$ is an *order parameter*. The modulus of $\psi$ is interpreted as describing the local density of superconducting Cooper pairs of electrons.

The functional $E(\psi, A)$ also gives the energy of a static configuration in the Yang-Mills-Higgs classical gauge theory on $\mathbb{R}^2$, with abelian gauge group $U(1)$. In this case $A$ is a connection on the principal $U(1)$-bundle $\mathbb{R}^2 \times U(1)$, and $\psi$ is the *Higgs field* (see [1] for details).

A central feature of the functional $E(\psi, A)$ (and the GL equations) is its infinite-dimensional symmetry group. Specifically, $E(\psi, A)$ is invariant under $U(1)$ gauge transformations,

$$\psi \mapsto e^{i\gamma} \psi$$  \hspace{1cm} (20)

$$A \mapsto A + \nabla \gamma$$  \hspace{1cm} (21)

for any smooth $\gamma : \mathbb{R}^2 \to \mathbb{R}$. In addition, $E(\psi, A)$ is invariant under coordinate translations, and under the coordinate rotation transformation

$$\psi(x) \mapsto \psi(g^{-1}x) \hspace{1cm} A(x) \mapsto gA(g^{-1}x)$$  \hspace{1cm} (22)

for $g \in SO(2)$.

Finite energy field configurations satisfy

$$|\psi| \to 1 \hspace{1cm} \text{as} \hspace{1cm} |x| \to \infty$$  \hspace{1cm} (23)

which leads to the definition of the *topological degree*, $\text{deg}(\psi)$, of such a configuration:

$$\text{deg}(\psi) = \text{deg} \left( \frac{\psi}{|\psi|_{|x|=R}} : S^1 \to S^1 \right)$$

($R$ sufficiently large). The degree is related to the phenomenon of flux quantization. Indeed, an application of Stokes’ theorem shows that a finite-energy configuration satisfies

$$\text{deg}(\psi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} (\nabla \times A).$$

We study, in particular, “radially-symmetric” or “equivariant” fields of the form

$$\psi^{(n)}(x) = f_n(r)e^{in\theta} \hspace{1cm} A^{(n)}(x) = n \frac{a_n(r)}{r} \hat{x}^\perp$$  \hspace{1cm} (24)
where \((r, \theta)\) are polar coordinates on \(R^2\), \(\hat{x}^\perp = \frac{1}{r}(-x_2, x_1)^t\), \(n\) is an integer, and

\[
f_n, a_n : [0, \infty) \to R.
\]

It is easily checked that such configurations, if they satisfy \([23]\), have degree \(n\). The existence of critical points of this form is well-known; they are called \(n\)-vortices.

The main results cited in this section concern the stability of these \(n\)-vortex solutions. Let

\[
L^{(n)} = \text{Hess } E(\psi^{(n)}, A^{(n)})
\]

be the linearized operator for GL around the \(n\)-vortex, acting on the space

\[
X = L^2(R^2, C) \oplus L^2(R^2, R^2).
\]

The symmetry group of \(E(\psi, A)\) gives rise to an infinite-dimensional subspace of \(\ker(L^{(n)}) \subset X\), which we denote here by \(Z_{\text{sym}}\). We say the \(n\)-vortex is (linearly) stable if for some \(c > 0,\)

\[
L^{(n)}|_{Z_{\text{sym}}} \geq c,
\]

and unstable if \(L^{(n)}\) has a negative eigenvalue. A basic result due to Gustafson and Sigal is the following linearized stability statement:

**Theorem 3.**

1. **(Stability of fundamental vortices)**
   For all \(\lambda > 0\), the \(\pm 1\)-vortex is stable.

2. **(Stability/instability of higher-degree vortices)**
   For \(|n| \geq 2\), the \(n\)-vortex is

   \[
   \begin{cases}
   \text{stable} & \text{for } \lambda < 1 \\
   \text{unstable} & \text{for } \lambda > 1.
   \end{cases}
   \]

Theorem 3 is the basic ingredient in a proof of the nonlinear dynamical stability/instability of the \(n\)-vortex for certain dynamical versions of the GL equations. These include the GL gradient flow equations, the Abelian Higgs (Lorentz-invariant) equations, and the Maxwell equations coupled to a nonlinear Schrödinger equation.

**Theorem 4.** As a solution of either \((GF)\) or \((AH)\), the \(n\)-vortex is stable (in the sense of Lyapunov) for \(\lambda > 1\) or for \(\lambda < 1\) and \(n = \pm 1\), and unstable for \(\lambda > 1\) and \(|n| \geq 2\).

The statement of theorem 3 was conjectured in \([11]\) on the basis of numerical observations (see \([3]\)). Bogomolnyi \((2)\) gave an argument for instability of vortices for \(\lambda > 1, |n| \geq 2\).

The solutions of \([15, 19]\) are well-understood in the case of critical coupling, \(\lambda = 1\). In this case, the Bogomolnyi method \((2)\) gives a pair of first-order equations whose solutions are global minimizers of \(E(\psi, A)\) among fields of fixed degree (and hence solutions of the GL equations). Taubes \((13, 14)\) has shown that all solutions of GL with \(\lambda = 1\) are solutions of these first-order equations, and that for a given degree \(n\), the gauge-inequivalent solutions form a \(2|n|\)-parameter family. The \(2|n|\) parameters describe the locations of the zeros of the
scalar field. This is discussed in more detail in [1]. We remark that for $\lambda = 1$, an $n$-vortex solution (24) corresponds to the case when all $|n|$ zeros of the scalar field lie at the origin.

It is observed numerically (eg [3]) that in fields containing localized vortices of like-signed winding number, the vortices attract each other when $\lambda < 1$ (bringing vortex centres closer together lowers the energy) and repel each other when $\lambda > 1$ (separating vortex centres lowers the energy). When $\lambda = 1$ there is no interaction (which allows for the existence of the stable multi-vortex solutions of Taubes, described above).

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