Quantifying entanglement of a two-qubit system via experimentally-accessible invariant-based moments of partially-transposed density matrix

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I. INTRODUCTION

Quantum entanglement [1, 2], which is an intrinsically and fundamentally nonclassical effect, has attracted an enormous number of works related to quantum information processing and quantum engineering in the last two decades (for reviews see, e.g., Refs. [3–5]). Although, our understanding of quantum entanglement is much deeper now, there are still many open fundamental problems as listed, e.g., in Ref. [6]. Some of these problems address the question how to experimentally detect and estimate entanglement of a given state [7].

One could think that the most natural and simplest way to measure the entanglement of an unknown state of \( \rho \) is to apply quantum state tomography (QST). This approach enables the reconstruction of \( \rho \) by postprocessing experimental data, and, then, the calculation of arbitrary entanglement measures for \( \rho \). Indeed, various effective QST methods have been developed [8], including those for the reconstruction of the polarization states of two photons (for a recent comparison see Ref. [9]). Nevertheless, this complete reconstruction requires to measure also a large number of parameters, which are irrelevant for the determination of entanglement. This number scales with the square of the dimension of a measured state \( \rho \). Moreover, QST based on linear inversion often leads to unphysical reconstructed states. Then, nonlinear methods (based, e.g., on maximum-likelihood estimation) have to be applied to overcome this problem.

Thus, usually, entanglement is detected and quantified by measuring entanglement witnesses [10] (for a review see Ref. [11]). This approach corresponds to testing the violations of classical inequalities. The operational usefulness of entanglement witnesses has been demonstrated in numerous experimental (see, e.g., earlier experiments reported in Refs. [11, 12]) and theoretical works. The latter include approaches based on: polynomial moments [13–17], collective entanglement witnesses [16, 18–20], experimental adaptive witnesses [21, 22], and others (see, e.g., Refs. [23–27]).

Some of such studies of entanglement witnesses were focused on the quantitative description of entanglement (see, e.g., recent Refs. [28–37] and a review [7] for older references). For example, a lower bound on a generic entanglement measure can be derived from the mean value of entanglement witnesses based on the Legendre transform [38–40]. The estimation of the concurrence and/or negativity from entanglement witnesses was studied in, e.g., Refs. [39, 41–47]. In particular, the violation of a Bell inequality, which is also an entanglement witness, can be used to estimate the concurrence [48], the negativity [49], or the relative entropy of entanglement (REE) [50]. A related problem is the estimation of one entanglement measure from another entanglement measure, e.g., the concurrence from negativity [45, 51, 52] and the REE [53], the negativity from the REE [54], or vice versa.

These approaches based on entanglement witnesses are useful and efficient, but still their usage is limited, because some information about the state should be known prior to its measurement.

In this paper, we study a universal entanglement witness (UWE), which can be used as a sufficient and necessary test of the entanglement of a two-qubit system.
The UWE is defined as the determinant of a partially-transformed density matrix, \( \det \hat{\rho}^{\Gamma} \) [1]. This witness can be given as a function of the moments \( \Pi_n = \text{tr}[(\hat{\rho}^{T})^n] \) [55], which are directly measurable, as recently described in Ref. [56] for a linear-optical setup. The proposed setup is based on the experimental methods described and referenced in Refs. [57,59].

Here we address the problem of applying the UWE to quantify two-qubit entanglement. Namely, the question is whether the UWE (or more precisely, its negative expectation value) can be considered a good entanglement measure. We will show that this is not the case for arbitrary two-qubit states. However, we will identify various classes of states for which the UWE is indeed a good entanglement measure. We will show that this is not the case for arbitrary two-qubit states. Moreover, as one of the main results of this paper, we will demonstrate that the negativity is a key requirement of a good entanglement measure (see, e.g., Ref. [61]).

It is worth stressing that the moments \( \Pi_n \), in Eq. (1) for \( n = 2, 3, 4 \), are not independent. To show the connection between these moments, let us analyze them in terms of the correlation matrix \( \hat{\beta} \), with elements \( \beta_{ij} = \text{tr}[(\hat{\sigma}_i \otimes \hat{\sigma}_j)\hat{\rho}] \), and the Bloch vectors \( s \) and \( p \), with elements \( s_i = \text{tr}[(\hat{\sigma}_i \otimes \hat{0})\hat{\rho}] \) and \( p_i = \text{tr}[(\hat{0} \otimes \hat{\sigma}_i)\hat{\rho}] \). The matrices \( \hat{\sigma}_i \) for \( i = 1, 2, 3 \) are the Pauli matrices, and \( \hat{0} \) is the single-qubit identity matrix. As shown in Ref. [60], we can write the first four moments as

\[
\Pi_1 = 1, \\
4\Pi_2 = 1 + x_1, \\
16\Pi_3 = 1 + 3x_1 + 6x_2, \\
64\Pi_4 = 1 + 6x_1 + 24x_2 + x_1^2 + 2x_3 + 4x_4, 
\]

(2)

where

\[
x_1 = I_2 + I_4 + I_7, \quad x_2 = I_1 + I_{12}, \quad x_3 = I_2^2 - I_3, \quad x_4 = I_5 + I_8 + I_{14} + I_4I_7, 
\]

(3)

are the functions of nine out of the eighteen Makhlin invariants [62], i.e., \( I_1 = \text{det} \hat{\beta}, \quad I_2 = \text{tr}(\hat{\beta}^{T}\hat{\beta}), \quad I_3 = \text{tr}(\hat{\beta}^{T}\hat{\beta})^2, \quad I_4 = s^2, \quad I_5 = [s\hat{\beta}]^2, \quad I_7 = p^2, \quad I_8 = [\hat{\beta}p]^2, \quad I_{12} = s^4\hat{p}, \quad \text{and} \quad I_{14} = e_{ijk}e_{lmn}s_{ip}p_{j}p_{k}n_{l}n, \text{ where } e_{ijk} \text{ is the Levi-Civita symbol}.\]

The invariants are also a subset of 21 fundamental and independent two-qubit invariants described by King and Welsh in Ref. [63]. This demonstrates explicitly that, in general, in order to measure the UWE one needs to measure these nine fundamental physical quantities (invariants). Any function of invariants is also an invariant. We can, therefore, introduce the following six independent invariants that need to be measured to estimate the values of moments \( \Pi_n \) for \( n = 2, 3, 4 \). These invariants are

\[
y_1 = I_2, \quad y_2 = I_4, \quad y_3 = I_7, \quad y_4 = I_1 + I_{12}, \quad y_5 = I_5 + I_8 + I_{14}, \quad y_6 = I_3, 
\]

(4)

This means that in order to quantify entanglement via the UWE, one needs to measure exactly six instead of nine independent quantities. The number of necessary measurements is by 10 smaller than the number of measurements needed for a full quantum-state tomography. We can conjecture that this is the minimum number of independent measurements needed for estimating the entanglement of an arbitrary two-qubit state.

II. UNIVERSAL ENTANGLEMENT WITNESS AND MAKHLIN INVARIANTS

Arguably, the simplest two-qubit separability condition can be formulated as follows [1]: A two-qubit state \( \hat{\rho} \) is entangled if and only if \( \det \hat{\rho}^{\Gamma} < 0 \), where \( \hat{\rho}^{\Gamma} \) is the partially-transposed (marked by \( \Gamma \)) matrix \( \hat{\rho} \). This theorem can be easily shown by recalling that the partially-transposed matrix of an arbitrary entangled two-qubit state has full rank and has exactly one negative eigenvalue. Thus, one can introduce the UWE \( \hat{W} \) for a two-qubit state \( \hat{\rho} \) defined as an operator for which the expected value is equal to \( \det \hat{\rho}^{\Gamma} \). This witness can also be given in terms of the experimentally-accessible moments \( \Pi_n = \text{tr}[(\hat{\rho}^{T})^n] \) as follows [55]:

\[
\hat{W} \equiv \det \hat{\rho}^{\Gamma} = \langle \hat{W} \rangle := \text{tr}(\hat{W}\hat{\rho}^{\otimes 4}) = \frac{1}{21}(1 - 6\Pi_4 + 8\Pi_3 + 3\Pi_2^2 - 6\Pi_2). 
\]

(1)

For convenience, we call the UWE not only the observable \( \hat{W} \) but also its expectation value \( W \). (This convention is also used in, e.g., Ref. [60] and references therein). In order to directly measure the UWE one could perform joint measurements on the four copies \( \hat{\rho}^{\otimes 4} \) of a two-qubit state \( \hat{\rho} \). A direct and efficient method for the measurement of \( \langle \hat{W} \rangle \) has been recently proposed for polarization qubits in a linear optical setup [56]. The witness \( \hat{W} \), contrary to a typical entanglement witness, is invariant under local unitary operations, which follows from the invariance of the moments of the partially-transposed density matrix that forms the witness. This invariance is a key requirement of a good entanglement measure (see, e.g., Ref. [61]).
III. NEGATIVITY VIA MOMENTS OF $\hat{\rho}^T$

The negativity of a two-qubit state $\hat{\rho}$ is usually defined as

$$N = 2 \max\{0, -\min[eig(\hat{\rho}^T)]\},$$

in terms of the minimum (negative) eigenvalue $\lambda \equiv -\mu = \min[eig(\hat{\rho}^T)]$ ($\mu > 0$) of the partially-transposed density matrix $\hat{\rho}^T$. The task of finding $\lambda$ is usually not easy because the operation of partial transposition is not physical, so this operator can only be implemented approximately.

There is another approach based on measuring moments $\Pi_n = \text{tr}(\hat{\rho}^T)^n$ of the partially-transposed matrix $\hat{\rho}^T$ of a two-qubit state $\hat{\rho}$. It has recently been shown in Ref. [56] that all the four first moments $\Pi_n$ can be measured directly using at most four copies of the investigated two-qubit state. This was shown on the example of the measurement of a two-photon polarization state by using a linear-optical setup. We note that this approach of Ref. [56] can be generalized to other implementations of qubits and various setups.

The first two moments $\Pi_n$ are equivalent to the trace and purity of $\hat{\rho}$, i.e., $\Pi_1 = 1$ and $\Pi_2 = p$ respectively. An efficient method for measuring the purity of an arbitrary polarization state of two photons has been proposed recently in Ref. [59]. The higher-order moments $\Pi_3$ and $\Pi_4$ can be measured as described in Ref. [56]. Let us also mention that, as long as there is some entanglement, $\hat{\rho}^T$ has four nonzero eigenvalues, among which only one is negative and equals to $\lambda$. This property holds for an arbitrary two-qubit state [64].

Let us derive an expression for the negativity in terms of the experimentally-accessible moments $\Pi_n$. As a result we obtain the following expression

$$J_1 = \Pi_1 = \text{tr}\hat{\rho}^T = \lambda_1 + \lambda_2 + \lambda_3 - \mu = 1,$$

$$J_2 = \frac{1}{2}(\Pi_1^2 - \Pi_2) = \frac{1}{2}[(\text{tr}\hat{\rho}^T)^2 - \text{tr}(\hat{\rho}^T)^2] = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 - \mu(1 + \mu),$$

$$J_3 = \text{det}\hat{\rho}^T = -\lambda_1\lambda_2\lambda_3\mu.$$  

After simple algebraic manipulations we derive

$$\sum_{n=1}^{3}[-\lambda_n\mu^3 + \frac{1}{2}\Pi_2\lambda_n\mu + (\lambda_n^2 - \lambda_n)\mu^2]$$

$$-\frac{1}{2}(2\lambda_n^3 - 2\lambda_n^2 + \lambda_n\mu - \text{det}\hat{\rho}^T) = 0.$$  

This sum can be directly calculated using the definition of moments $\Pi_n$. As a result we obtain the following expression

$$-3\mu^4 - 3\mu^3 + \frac{3}{2}(\mu^2 + \mu)\Pi_2 - 3\Pi_3\mu - \frac{3}{2}\mu^2$$

$$-3\text{det}\hat{\rho}^T - \frac{3}{2}\mu = 0,$$

where $N = 2\mu$. Equation (10) has an important consequence, i.e., we can calculate the negativity $N$ after measuring $\Pi_n$ for $n = 1, 2, 3, 4$. As discussed in Ref. [56] measuring these four experimentally-accessible moments can be done more efficiently than performing full quantum state tomography. Equation (10) is a fourth degree polynomial in $N$, which after simplification reads

$$48\text{det}\hat{\rho}^T + 3N^4 + 6N^3 - 6N^2(\Pi_2 - 1)$$

$$-4N(3\Pi_2 - 2\Pi_3 - 1) = 0.$$  

In our opinion this is one of the main results of this work. We can be sure that it has solutions if $\text{det}\hat{\rho}^T < 0$ (i.e., when the state $\hat{\rho}$ is entangled). The solutions can be found analytically by applying the well-known Ferrari and Cardano formulae. Equation (11) has four solutions, however there is only one real solution where $N > 0$. Therefore, the value of the negativity is uniquely defined by Eq. (11). We do not give these solutions explicitly, as they are lengthy and can be easily obtained by using a computer algebra system.

Unfortunately, the value of the negativity calculated from Eq. (11) is very sensitive to the uncertainty of measuring $\Pi_n$. This can be observed in Fig. 1, where the relation between the theoretical and the experimentally measured values of the negativity is depicted for several values of the maximal relative uncertainties in estimating $\Pi_n$. Figure (1b) suggests that if the relative error is close to 10%, the noise level starts to be too high for estimating the negativity with a reasonable precision in its entire range. For uncertainty levels $> 10\%$, the measurement method is not reliable for any value of $N$. Note that, in all cases, the level of the uncertainty in estimating the negativity is the largest for the values of $N \approx 0$.

IV. UNIVERSAL ENTANGLEMENT WITNESS AS AN ENTANGLEMENT MEASURE

As shown in Fig. 2 measuring $\hat{W}$ is less prone to noise than estimating the negativity.

It is convenient to use the rescaled value of $\langle \hat{W} \rangle$ defined as

$$w := \max\{0, -16\langle \hat{W} \rangle\}.$$  

Now we explicitly describe that the UWE $w$ satisfies the following standard criteria for a good entanglement measure (as listed in, e.g., Ref. [4]):

C1. The inequalities hold $0 \leq w(\hat{\rho}) \leq 1$, where $w(\hat{\rho}) = 0$ for any unentangled state and $w(\hat{\rho}) = 1$ for the Bell states.

C2. Any local unitary transformations of the form $U_A \otimes U_B$ do not change $w(\hat{\rho})$ for any state $\hat{\rho}$.

C3. An additional property: The witness $w(|\psi\rangle)$ is simply related to the entropy of entanglement for any pure state $|\psi\rangle$, i.e., by a relation corresponding to the Wooters formula for the entanglement of formation [65].

$$E_F(w) = h \left( \frac{1}{2} \left[ 1 + \sqrt{1 - \sqrt{w}} \right] \right),$$  

where $h(x) = -x \log_x - x \log_{1-x}$.
Unfortunately, in general, the following two important properties do not hold for the witness \( w(\hat{\rho}) \):

**C4.** A good entanglement measure \( E(\hat{\rho}) \) should not increase for any state \( \hat{\rho} \) and any local operations with classical communication (LOCC). This property can be violated for \( w(\hat{\rho}) \) as shown in Appendix B.

**C5.** A good entanglement measure \( E(\hat{\rho}) \) should be convex under discarding information, i.e., \( \sum_i p_i E(\hat{\rho}_i) \geq E(\sum_i p_i \hat{\rho}_i) \). In other words, one cannot increase \( E(\hat{\rho}) \) by mixing states \( \hat{\rho}_i \). An example of the violation of this property for \( w(\hat{\rho}) \) is given in Appendix C.

where \( h(y) = -y \log_2 y - (1-y) \log_2 (1-y) \) is binary entropy. This property follows from the observation that

\[
\begin{align*}
  w(|\psi\rangle) &= N^4(|\psi\rangle) = C^4(|\psi\rangle) \\
  \text{for any state } |\psi\rangle, \text{ which corresponds to case 1 in Table 1.}
\end{align*}
\]

Property C.2 follows from the fact that the UWE can be expressed as a function of local polynomial invariants \([62]\). For pure states, the UWE is equivalent to the so-called \( G \) concurrence \([55, 66]\), which is a monotone under LOCC (C.3). Thus, even if the properties C.4 and C.5 are not satisfied in general, the witness \( w \) for two-qubit states is a useful parameter for quantifying entanglement.

Moreover, the UWE \( w \) provides tight upper and lower bounds for the negativity \( N(\hat{\rho}) \) of an arbitrary two-qubit state \( \hat{\rho} \) \([53]\):

\[
  f(w) \leq N \leq \sqrt{w},
\]

where \( f(w) = \frac{1}{2}(-3 + \sqrt{z} + \sqrt{3-z+\frac{2}{\sqrt{z}}}) \) and \( z = 1-y+x \) with \( y = 36w/x \), and

\[
  x = 3\sqrt{2w^2(16w+1) - 2w}.
\]

We show in Table 4 that the states saturating the upper and lower bounds are pure (case 1) and Werner’s states (case 8) \([67]\), respectively.
The boundary states can be found in the set of the so-called X states. These states can be simply manipulated \([60]\) and are universal in the sense that an arbitrary two-qubit state can be converted, by a unitary transformation, into its X-state counterpart \([88]\). Moreover, the X states appear as solutions in many simple physical models in, e.g., the XYZ Heisenberg model \([69, 70]\) or decaying entangled qubits coupled to a common reservoir exhibiting the effects of sudden death \([71]\) and rebirth \([72]\) of entanglement. The name of these states becomes clear when its density matrix \(\rho\) is given explicitly in the standard computational basis, i.e.,

\[
\hat{\rho} = \begin{pmatrix}
    a & 0 & 0 & b \\
    0 & c & d & 0 \\
    0 & d^* & e & 0 \\
    b^* & 0 & 0 & f
\end{pmatrix}.
\]  

(18)

The partial transpose with respect to the second subsystem of two-qubit density matrix \(\hat{\rho}\) reads

\[
\hat{\rho}^T = \begin{pmatrix}
    a & 0 & 0 & d \\
    0 & c & b & 0 \\
    0 & b^* & e & 0 \\
    d^* & 0 & 0 & f
\end{pmatrix}.
\]  

(19)

Now, it follows from the Laplace expansion that the UWE for the X states can be given as a product of determinants,

\[
W = \det \begin{pmatrix}
    a & d \\
    d^* & f
\end{pmatrix} \det \begin{pmatrix}
    c & b \\
    b^* & e
\end{pmatrix}
= \det \begin{pmatrix}
    a & |d| \\
    |d| & f
\end{pmatrix} \det \begin{pmatrix}
    c & |b| \\
    |b| & e
\end{pmatrix}.
\]  

(20)

This is a four-dimensional volume (a product of two areas). We can expand it further to obtain

\[
W = |\langle d | - \sqrt{af} \rangle (|d| + \sqrt{af})| |\langle b | - \sqrt{ce} \rangle (|b| + \sqrt{ce})|.
\]  

(21)

This corresponds to the volume of a four-dimensional box. Note that the length of its longest negative edge (the longest edge of negative orientation) corresponds to the negativity. However, the expression for the negativity is not simple because it requires finding the smallest eigenvalue of \(\hat{\rho}^T\), i.e., factorizing \(\det \hat{\rho}^T\) in another way.

V. UNIVERSE NTAL ENTANGLEMENT WITNESS AND CONCURRENCE

Remarkably, the largest negative factor in the expression for the UWE, given by Eq. \((21)\), for the X states, corresponds to another popular entanglement measure. Namely, the Wootters concurrence \([65]\):

\[
C(\hat{\rho}) = \max \left( 0, 2 \lambda_{\text{max}} - \sum_j \lambda_j \right),
\]  

(22)

where \(\lambda_j^2 = \text{eig}[\hat{\rho}(\hat{\sigma}_2 \otimes \hat{\sigma}_2)\hat{\rho}^T(\hat{\sigma}_2 \otimes \hat{\sigma}_2)]_j\) and \(\lambda_{\text{max}} = \max_j \lambda_j\). The witness \(W\) can be interpreted as a geometric mean of all the lengths in Eq. \((21)\). Thus, the UWE is not a good measure of entanglement, because it underestimates the available entanglement. However, the UWE can be used as a measure of entanglement if all the edges have the same length and the volume is negative.

Let us note that there are some constrains on the matrix elements of the X states, e.g., the trace of the partially-transposed matrix equals 1. From this observation follows that \(a + c + e + f = 1\). Other constrains are imposed by the fact that \(\hat{\rho}\) is positive semidefinite, i.e., \(|d| \leq \sqrt{ce}\) and \(|b| \leq \sqrt{af}\). By recalling some properties of density matrices, we can deduce that the UWE is a monotonic function of a proper entanglement measure, i.e., the concurrence. The concurrence is given by the following simple expression for X states \([73]\)

\[
C = 2 \max (0, |d| - \sqrt{af}, |b| - \sqrt{ce}).
\]  

(23)

One can see that the UWE is related to both the negativity and concurrence for the whole class of the X states. For some subclasses of the X states, the negativity and concurrence are equivalent. This happens for pure states, rank-2 Bell-diagonal states, phase-damped states, Bell states with isotropic noise, and the Werner states \([67]\) (see cases 1, 2, 3, and 8 in Table I respectively). For the amplitude-damped states (case 4 in Table I) with the damping parameter \(p = 1 - f\), the relation is also simple as \(C = \sqrt{N^2 + 2fN}[50]\), although it also involves the damping parameter \(p\).

In Table I, we present a survey of the selected subclasses of the X states of various ranks for which the UWE (or a function of only \(\Pi_2\) and \(\Pi_3\) can be considered as an entanglement measure. In each case the UWE is proportional to a fourth-degree (or lower-degree) polynomial of \(N\) or \(C\). For the states given in cases 1,...,4, and 8, the witness \(W\) is a good measure of entanglement because it is a function of \(N\) with constant coefficients. The other states depend on an additional variable. These states include: the degenerate amplitude-damped states (case 5), rank-3 Bell-diagonal states (case 6), and pure states with isotropic noise (case 7). For these states, by measuring \(\hat{W}\) does not provide enough information to determine the entanglement measures. However, for cases 1, 2, 3, and 8 listed in Table I, it is possible to determine \(C\) and \(N\) by measuring solely \(\Pi_2\) and \(\Pi_3\). The states of the largest and smallest ranks are the boundary states for \(N\) versus \(C\). The results are also visualized in Fig. [3].

Note that we focus only on the states that depend on at most three independent variables. This is because, by allowing more freedom we would have to measure all the first four moments of \(\hat{\rho}^T\) to estimate the entanglement. This would give us no benefit with respect to the approach presented in the previous section. The states presented in Table I may appear rather specific. Note that X states must be described, in general, by nine parameters (see, e.g., Ref. [74]). However, these states represent an infinite set of states that can be generated by local
unitary transformations that do not change the entanglement. In other words, by applying local unitary operations, we can always obtain the following rank-specific real $X$ states:

\[
\hat{\rho}_1 \equiv \phi^+(\theta_1),
\]
\[
\hat{\rho}_{2a} \equiv p_1 \phi^+(\theta_1) + p_3 \psi^+(\theta_3),
\]
\[
\hat{\rho}_{2b} \equiv p_1 \phi^+(\theta_1) + p_2 \phi^- (\theta_2),
\]
\[
\hat{\rho}_3 \equiv p_1 \phi^+(\theta_1) + p_2 \phi^- (\theta_2) + p_3 \psi^+ (\theta_3),
\]
\[
\hat{\rho}_4 \equiv p_1 \phi^+(\theta_1) + p_2 \phi^- (\theta_2) + p_3 \phi^+ (\theta_3) + p_4 \psi^- (\theta_4),
\]

(24)

which are incoherent mixtures of pure states

\[
\phi_{\pm} (\theta) = \begin{pmatrix}
\cos^2 \theta & 0 & 0 & \pm \frac{1}{2} \sin (2\theta) \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\pm \frac{1}{2} \sin (2\theta) & 0 & 0 & \sin^2 \theta 
\end{pmatrix}
\]

(25)

and

\[
\psi_{\pm} (\theta) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \cos^2 \theta & \frac{1}{2} \sin (2\theta) & 0 \\
0 & \pm \frac{1}{2} \sin (2\theta) & \sin^2 \theta & 0 \\
0 & 0 & 0 & 0 
\end{pmatrix}
\]

(26)

with weights $p_i > 0$ ($\sum p_i = 1$). Note that the states, given in Eqs. (25) and (26), reduce to the Bell states for $\theta = \pi/4$. One should be careful not to accidentally reduce the rank of a given state by choosing some specific values of $\theta$. The relation between the $X$ states from Table I and the states defined in Eq. (24) is presented in Table II.

VI. CONCLUSIONS

We have described a direct operational method for determining the negativity of an arbitrary two-qubit state. We have derived the method by analyzing the relation between the purity, negativity, and a universal entanglement witness for two-qubit entanglement. In particular, we have expressed the negativity as a function of six invariants which are linear combinations of nine from the complete set 21 fundamental and independent two-qubit invariants listed, e.g., in Ref. [63].

We have demonstrated how to measure the negativity of a two-photon polarization state by measuring three experimentally-accessible moments $\Pi_n$ of the partially-transposed density matrix of a two-photon state. We pointed out that this approach can be more practical than directly estimating the negativity, which is sensitive even to a low-level noise.

We also discussed the possibility of using the universal entanglement witness or lower moments of $\hat{\rho}^T$ as a proper entanglement measure for some classes of states. In particular, we demonstrated their relation to the negativity and concurrence for the $X$ states.

![FIG. 3: Relation between the negativity $N$ and the universal entanglement witness $w$ for various states as defined in Table I. In panel (a) we demonstrate the relations $N_n(w)$ for states, given in the $n$th case in Table I. In panels (b),(c), and (d), the shaded areas depict the relation $N_n(w)$ for two-parameter states given in the cases for $n = 4, 5, 6$, respectively. The covered area lies between the dashed curve corresponding to the lower bound $f(w)$, defined in Eq. (16), and the solid curve corresponding to the upper bound $4 \sqrt{w}$. In all these panels, the shaded areas do not cover the whole space between the boundaries. The whole area is covered only in case 7 which, for brevity, is not presented here.](image)
TABLE I: A survey on the relation between the concurrence $C$, negativity $N$, and witness $W$ for selected subclasses of the $X$ states, where $\hat{x} = (a, b, c, d, e)$ and $g_n = 1 - nf$. Note that $C$ and $W$ can be determined, in some cases, by measuring only $\Pi_3$ and (or) $\Pi_2$. However, in general, these entanglement measures can be obtained by measuring all the four moments of $\hat{\rho}_3$. The presented examples include pure states, which can be affected by the amplitude and phase damping channels, pure state with isotropic noise, and Bell-diagonal states of various ranks $R$. The moments $\Pi_4$ for all these eight subclasses of the $X$ states are given explicitly in Appendix A.

| Case | $\hat{\rho}_R = \hat{\rho}(\bar{x})$ | $R$ | $\Pi_2$ | $\Pi_3$ | $W$ | $C$ |
|------|----------------------------------|----|--------|--------|-----|-----|
| 1    | $a = b = f = 0$                 | 1  | 1      | $1 - \frac{1}{2}N^2$ | $\frac{N^4}{16}$ | 2$|d| = N$ |
| 2    | $a = b = f < \frac{1}{2}$      | $\frac{1}{2}$ | $\frac{1}{2}(N^2 + 1)$ | $\frac{1}{4}$ | $\frac{N^2}{16}$ | $|g_4| = N$ |
| 3    | $c = d = e = \frac{2a}{2}$    | $\frac{1}{2}$ | $\frac{1}{2}(N^2 + 1)$ | $\frac{1}{4}$ | $\frac{N^2}{16}$ | 2$|d| = N$ |
| 4    | $e = c = \frac{1}{2}$       | 2$\bar{b}$ | $\frac{1}{2}(N^2 + 1)$ | $\frac{1}{4}$ | $\frac{N^2}{16}$ | 2$|d| = N$ |
| 5    | $a = b = f = 0$                 | 3  | $g_2 + 2f^2$ | $1 - 3\left(1 + \frac{C_2}{2}\right)g_4 + 3g_2^2 + 3\frac{C_2^2}{4} = \frac{C_4}{16}$ | $\sqrt{N^2 + 2fN}$ | 2$|d| = 2f$ |
| 6    | $a = f < \frac{1}{2}d$        | 4  | $\frac{C}{2}(3C + 2)$ | $\frac{1}{2}g_2^2 + 2f^2 - \frac{3}{4}g_4(C + 2f)^2$ | $\frac{C(2C^2 + 2C + 1 - 16d^2)}{64}$ | $|g_4|$ |
| 7    | $c = e = \frac{2a}{2}$       | 4  | $\frac{1}{2}(1 - N)^2$ | $\frac{1}{2}g_2^2 + g_4f + \frac{3}{2}f^2$ | $\frac{1}{2}(1 - N) - 1)^3N$ | $|g_6| = N$ |
| 8    | $b = 0$                        | 2  | $n$ | $\frac{1}{2}(1 - N)^2$ | $\frac{1}{2}g_2^2 + g_4f + \frac{3}{2}f^2$ | $\frac{1}{2}(1 - N) - 1)^3N$ | $|g_6| = N$ |

TABLE II: The relation between the states from Table I and the $X$ states defined in Eq. (23). The correspondence is valid up to local unitary transformations on two qubits.

| Case | $p_1$ | $p_2$ | $p_3$ | $p_4$ | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ |
|------|------|------|------|------|---------|---------|---------|---------|
| 1    | 0    | 0    | 1    | 0    | 0       | 0       | acos $\sqrt{c}$ | 0       |
| 2    | 2    | 0    | $g_2$ | 0    | $\frac{\pi}{2}$ | 0       | $\frac{\pi}{2}$ | 0       |
| 3    | 0    | 0    | \(\frac{1}{2} + |d|\) | \(\frac{1}{2} - |d|\) | 0    | $\frac{\pi}{2}$ | 0       | $\frac{\pi}{2}$ |
| 4    | 2    | 0    | $g_2$ | 0    | $\frac{\pi}{2}$ | 0       | acos $\sqrt{\delta_2}$ | 0       |
| 5    | 2    | 0    | $g_2$ | 0    | $\frac{\pi}{2}$ | 0       | acos $\sqrt{\delta_2}$ | 0       |
| 6    | $g_2(\frac{1}{2} + |d|)\) | $g_2(\frac{1}{2} - |d|)\) | $\frac{\pi}{2}$ | 0    | $\frac{\pi}{2}$ | 0       | acos $\sqrt{\delta_2}$ | 0       |
| 7    | $g_3$ | $f$ | $\frac{\pi}{2}$ | $\frac{\pi}{2}$ | 0       | $\frac{\pi}{2}$ | 0       | acos $\sqrt{\delta_2}$ | 0       |
| 8    | $g_3$ | $f$ | $\frac{\pi}{2}$ | $\frac{\pi}{2}$ | 0       | $\frac{\pi}{2}$ | 0       | acos $\sqrt{\delta_2}$ | 0       |

It is worth noting that the UWE is not necessarily the least-error sensitive entanglement measure, which can be constructed from the moments of the partially-transposed density matrix of a given state. It is possible that a better two-qubit entanglement measure exists that can be measured as a function of $\Pi_n$ for $n = 2, 3, 4$.

We hope that these results can pave the way for direct and efficient methods for measuring two-qubit quantum entanglement.

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Appendix A: Moments $\Pi_\alpha$ and $\Pi_4$ for some states in Table I

Here we show explicitly $\det \hat{\rho}^\Gamma$ and the moments $\Pi_\alpha$ and $\Pi_4$ of the partially-transposed density matrix $\hat{\rho}^\Gamma$ for the selected subclasses of the X states given in Table I. These moments are given as a function of either the concurrence $C$ or the negativity $N$.

The moments $\Pi^{(n)}_\alpha$ for the $n$th case (subclass) of the X states analyzed in Table I are the following:

\[
\begin{align*}
\Pi^{(1)}_4 &= \left( 1 - \frac{N^2}{2} \right)^2, \\
\Pi^{(2)}_4 &= \Pi^{(3)}_4 = \frac{1}{8} (N^4 + 1), \\
\Pi^{(4)}_4 &= \frac{C^4}{4} - g_2 C^2 + g_4 f^4, \\
\Pi^{(5)}_4 &= \left[ g_2^2 - \frac{1}{2} (C + 2 f) C^2 + 3 (C + 2 f)^2 \right] f^4, \\
\Pi^{(6)}_4 &= 2^{-7} \left( 9 C^4 + 4 C^3 + 6 (24 d^2 + 1) C^2 + 4 (48 d^2 + 1) C + 96 d^2 \right), \\
\Pi^{(7)}_4 &= \frac{1}{16} \left( 7 N^4 + 2 N^3 + 6 N^2 + 8 N + 4 \right), \\
\Pi^{(8)}_4 &= \frac{1}{16} \left( 7 N^4 + 2 N^3 + 6 N^2 + 8 N + 4 \right).
\end{align*}
\]

where $g_n = 1 - n f$, while $d$ and $f$ are the elements of $\hat{\rho}$, given in Eq. (18). The moment $\Pi_4$ for the X state in case 6 reads

\[
\Pi_4 = \frac{1}{32} \left[ 3 C (1 - C^2 + C + 16 |d|^2) + 48 |d|^2 + 5 \right].
\]

The moment $\Pi_3$ and $\det \hat{\rho}^\Gamma$ for the X states in case 7 read

\[
\begin{align*}
\Pi_3 &= \frac{1}{4} C^4 + 2 f C^3 + \frac{3}{4} (5 f^2 - 6 f - 1) C^2 + \frac{289}{8} f^4 \\
&\quad + 4 (f^2 - 18 f + 4) C - \frac{89}{2} f^3 + \frac{67}{2} f^2 + g_{10}, \\
\det \hat{\rho}^\Gamma &= -\frac{1}{16} C^4 - \frac{1}{2} f C^3 - \frac{1}{16} f (15 f + 2) C^2 \\
&\quad - \frac{1}{4} f^2 (2 - f) C.
\end{align*}
\]

Appendix B: Violation of the LOCC condition

Here we show that the LOCC criterion C4., characterizing a good entanglement measure, can be violated for the UWE. Thus, we analyze the following two-qubit Bell-diagonal state

\[
\hat{\rho} = p \psi_- (\frac{\pi}{4}) + (1 - p) \phi_+ (\frac{\pi}{4}),
\]

for which $w(\hat{\rho})$ can increase under some local operations, as shown explicitly below.

As an example of an LOCC operation, we apply the “twirling” operation [73], where a random SU(2) rotation is performed on each qubit. This twirling changes $\hat{\rho}$ into the Werner state

\[
\begin{align*}
\hat{\rho}' &= p \psi_- (\frac{\pi}{4}) + \frac{1}{2} (1 - p) \left[ \phi_+ (\frac{\pi}{4}) + \phi_- (\frac{\pi}{4}) + \psi_+ \left( \frac{\pi}{4} \right) \right] \\
&= q \psi_- (\frac{\pi}{4}) + \frac{1}{2} (1 - q) I,
\end{align*}
\]

which is a mixture of the singlet state $\psi_- (\frac{\pi}{4})$, with the weight $q = (4p - 1)/3$, and the maximally-mixed state as given by the four-dimensional identity operator $I$. Consequently, for $p = (3\sqrt{17} - 7)/8$, we observe the largest violation of the LOCC condition for this particular state $\hat{\rho}$. This is because, $w(\hat{\rho}) = 0.11719$ and $w(\hat{\rho}') = 0.16294$, hence $w(\hat{\rho}) < w(\hat{\rho}')$.

It is worth noting that if these twirling operations are applied to the concurrence, negativity, or the REE, then property C.4 is always satisfied. Anyway, the twirling operations can be used to show that the Werner states determine the lower bounds of the concurrence for a given value of the negativity [51], the REE vs negativity [53, 76], or the REE vs the Bell nonlocality [50].

Appendix C: Violation of the convexity condition

Here we show that the convexity criterion C5., which is another important condition for a good entanglement measure, can also be violated for the UWE and some states.

Thus, let us consider a mixture $\hat{\rho} = (\hat{\rho}_1 + \hat{\rho}_2)/2$ of the following two-qubit density matrices

\[
\begin{align*}
\hat{\rho}_1 &= \frac{1}{2} \left[ \phi_+ (0) + \psi_+ \left( \frac{\pi}{4} \right) \right], \\
\hat{\rho}_2 &= \frac{1}{2} \left[ \phi_+ (0) + \psi_- \left( \frac{\pi}{4} \right) \right].
\end{align*}
\]

For these states, the convexity condition should imply that

\[
w(\hat{\rho}) \leq \frac{1}{2} w(\hat{\rho}_1) + \frac{1}{2} w(\hat{\rho}_2).
\]

However, the relevant values of the UWE read $w(\hat{\rho}_1) = 2^{-6}$, $w(\hat{\rho}_2) = 2^{-6}$, and $w(\hat{\rho}) = 2^{-5}$. It is seen that $w(\hat{\rho}_1) + w(\hat{\rho}_2) \neq w(\hat{\rho})$. Thus, the convexity condition [C5] is clearly violated because $w(\hat{\rho}) \neq \frac{1}{2} w(\hat{\rho})$ for $w(\hat{\rho}) = \frac{555}{32}$. 

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