Conditional measure on the Brownian path and other random sets

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Abstract

Let $B$ denote the range of the Brownian motion in $\mathbb{R}^d$ ($d \geq 3$). For a deterministic Borel measure $\nu$ on $\mathbb{R}^d$ we wish to find a random measure $\mu$ such that the support of $\mu$ is contained in $B$ and it is a solution to the equation $E(\mu(A)) = \nu(A)$ for every Borel set $A$. We discuss when exactly we can find such a solution $\mu$. We study several properties of $\mu$ such as the probability of $\mu \neq 0$ and we establish a formula for the expectation of the double integral with respect to $\mu \times \mu$. We calculate $\mu$ in terms of the occupation measure when $\nu$ is the Lebesgue measure. We generalise the theory for more general random sets in complete separable metric spaces.

1 Introduction

Let $\nu$ be a finite Borel probability measure on the unit square of the plane, let $\pi(x, y) = x$ be the projection onto the $x$-axis and $\pi^*\nu = \nu \circ \pi^{-1}$ be the projection measure. Then by the existence of the regular conditional measure [6, Theorem 5.1.9] for $\pi^*\nu$ almost every $x \in [0, 1]$ there exists a Borel probability measure $\nu_x$ on the slice $\{x\} \times [0, 1]$ such that $d\nu(x, y) = d\nu_x(x, y)d\pi^*\nu(x)$. When $\pi^*\nu \ll \lambda$, where $\lambda$ denotes the 1-dimensional Lebesgue measure, the measure $\nu_x$ can be obtained as a weak limit of certain rescaled restrictions of $\nu$. Assume that $\pi^*\nu \ll \lambda$, then for Lebesgue almost every $x \in [0, 1]$ the weak limit of the measures

$$\frac{\nu|_{\pi^{-1}B(x, r)}}{2r}$$

exists as $r$ approaches 0, see [12] Chapter 10, where $\nu|_A$ denotes the restriction of $\nu$ to $A$, that is $\nu|_A(B) = \nu(A \cap B)$ for every Borel sets $A, B \subseteq \mathbb{R}^2$. Let this weak limit be $\mu_x$ for Lebesgue almost every $x \in [0, 1]$, then

$$d\nu(x, y) = d\mu_x(x, y)d\lambda(x),$$

(1.2)

see Mattila [13] Lemma 3.4. Thus by the uniqueness of the conditional measure

$$\nu_x = \left(\frac{d\pi^*\nu}{d\lambda}(x)\right)^{-1} \cdot \mu_x$$

for $\pi^*\nu$ almost every $x \in [0, 1]$. 
It was shown by Mattila [13, Lemma 3.4] that for a Borel function $f : [0, 1]^2 \to \mathbb{R}$ with $\int |f| \, d\nu(z) < \infty$ we have that

$$\int f \, d\mu_x(z) = \lim_{r \to 0} (2r)^{-1} \int_{B(x,r)} f \, d\mu_x(z)$$

(1.3)

for Lebesgue almost every $x \in [0, 1]$. Mattila [12, Theorem 10.7] also discusses the double integral of certain kernels with respect to $\mu_x \times \mu_x$.

One can look at it as we randomly choose a slice $B = \{x\} \times [0, 1]$ where we choose $x$ uniformly in $[0, 1]$. Then $\mu = \mu_x$ is a random measure supported on the random slice, and

$$d\nu(z) = d\mu(z) dP(x)$$

(1.4)

holds by (1.2), i.e. $\nu$ is the expectation of $\mu$. Our main goal in this paper is to construct this kind of slice measures on random slices but instead of taking the random slices to be straight line segments we take the slices to be the Brownian path or other random sets.

Let $Q_k(z)$ be the dyadic cube $[\frac{i_1}{2^k}, \frac{i_1+1}{2^k}] \times [\frac{i_2}{2^k}, \frac{i_2+1}{2^k})$ for $i_1, i_2 \in \mathbb{Z}$ such that $z \in Q_k(z)$ for some $z \in [0, 1]^2$ and let $Q_k = \{Q_k(z) : z \in [0, 1]^2\}$. It can be shown that for Lebesgue almost every $x \in [0, 1]$ we get the same weak limit $\mu_x$ if we consider the sequence

$$\mu_k = \frac{\nu|_{\pi^{-1}(\pi(Q_k(x,0)))}}{2^{-k}} = \sum_{Q \in Q_k} P(Q \cap B \neq \emptyset)^{-1} \cdot I_{Q \cap B \neq \emptyset} \cdot \nu|_Q$$

(1.5)

instead of (1.1), where $I_{Q \cap B \neq \emptyset}$ is the indicator function of the event $Q \cap B \neq \emptyset$. We can obtain from the analogue of (1.3) that for every Borel set $A \subseteq \mathbb{R}^2$

$$\lim_{k \to \infty} \mu_k(A) = \mu(A)$$

(1.6)

almost surely, i.e. for Lebesgue almost every slice.

If $\pi^* \nu$ is singular to the Lebesgue measure than by Lebesgue’s density theorem

$$\lim_{k \to \infty} \mu_k([0, 1]^2) = 0$$

(1.7)

for Lebesgue almost every $x \in [0, 1]$, i.e. $\mu = 0$ almost surely. Hence in general we can decompose $\nu$ into two parts

$$\nu = \nu_R + \nu_\perp$$

(1.8)

such that $\pi^* \nu_R \ll \lambda$ and $\pi^* \nu_\perp \perp \lambda$, one part corresponds to a vanishing limit (1.7), the other part corresponds to an $L^1$ limit (1.4). Thus for the almost sure weak limit $\mu$ of the sequence of random measures $\mu_k$ we obtain the disintegration formula

$$d\nu_R(z) = d\mu(z) dP.$$  (1.9)

Our main goal is to show the existence of the limit of (1.5) in the case when $B$ is the Brownian path and to obtain a disintegration formula as in (1.9).
Theorem 1.1. Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$ and let $\nu$ be a locally finite Borel measure on $\mathbb{R}^d$. Then $\nu = \nu_R + \nu_\perp$ such that there exists a Borel set $A$ such that $\nu_\perp(\mathbb{R}^d \setminus A) = 0$, $P(B \cap A \neq \emptyset) = 0$ and there exists a random, locally finite, Borel measure $\mu$ supported on $B$ such that

$$d\mu(z)dP = d\nu_R(z)$$

and

$$d\mu(x)d\mu(y)dP = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}d\nu_R(x)d\nu_R(y).$$

Note that if $\mu$ is a random measure supported on the Brownian path $B$ and $A$ is a deterministic Borel set such that $P(B \cap A \neq \emptyset) = 0$ then $\mu(A) \leq \mu(\mathbb{R}^d \setminus B) = 0$ almost surely and so

$$\int_A \int d\mu(z)dP = E(\mu(A)) = 0.$$

This means that there is no hope to satisfy (1.4) for $\nu_\perp$ in Theorem 1.1. We restate Theorem 1.1 in a stronger form of Theorem 1.16.

Our construction works for more general random closed sets than the Brownian path and in more general metric spaces than $\mathbb{R}^d$. We use the sum in (1.5) to define the sequence $\mu_k$ in the general case (see Section 1.1).

1.1 Notations

1.1.1 General assumptions throughout the paper

Throughout the paper let $(X, d)$ be a complete, separable metric space. Let $\varphi : [0, \infty) \rightarrow [0, \infty]$ be a nonnegative, continuous, monotone decreasing function with finite values on $(0, \infty)$. We consider the composition kernel $\varphi(d(x, y))$ on $X \times X$ which we denote by $\varphi(x, y)$. We note that we use $\varphi$ to denote both $\varphi(r)$ and $\varphi(x, y)$ but in the context it should be clear depending on what is the domain of $\varphi$.

Example 1.2. Commonly used examples are the harmonic kernel $\varphi(x, y) = \|x - y\|^{-\alpha}$ and the logarithmic kernel $\varphi(r) = \max\{0, \log(1/\|x - y\|)\}$.

Let $Q_k$ be a sequence of countable families of Borel subsets of $X$ such that $Q \cap S = \emptyset$ for $Q, S \in Q_k$, $Q \neq S$ for all $k \in \mathbb{N}$ and

$$\lim_{k \rightarrow \infty} \sup \{\text{diam}(Q) : Q \in Q_k\} = 0,$$

where diam denotes the diameter in $X$. We further assume that for every $Q \in Q_k$, $k > 1$ there exists a unique $D \in Q_{k-1}$ such that

$$Q \subseteq D.$$ 

Define $X_0 := \bigcap_{k=1}^\infty (\bigcup_{Q \in Q_k} Q)$.
Example 1.3. Let $Q_k = \{[\frac{i_k}{2^k}, \frac{i_k+1}{2^k}) \times \cdots \times [\frac{i_d}{2^k}, \frac{i_d+1}{2^k}) \setminus \{0\} : i_1, \ldots, i_d \in \mathbb{Z}\}$ for $k \in \mathbb{N}$, i.e. the dyadic cubes of side length $2^{-k}$ which are left closed, right open and we subtract the origin 0 from the one that contains it. Then $X_0 = \mathbb{R}^d \setminus \{0\}$.

Let $\nu$ be a finite Borel measure on $X$. Throughout most of the paper we make the following assumption in our statements that

$$\nu(X \setminus X_0) = 0. \quad (1.12)$$

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $B = B_\omega \subseteq X$ be a random closed set such that $\{B \cap K \neq \emptyset\} \in \mathcal{A}$ for every compact set $K \subseteq X$ and $\{B \cap Q \neq \emptyset\} \in \mathcal{A}$ for every $Q \in Q_k$, $k \in \mathbb{N}$. We assume that

$$P(Q \cap B \neq \emptyset) > 0 \quad (1.13)$$

for every $Q \in Q_k$.

We write

$$C_k(\nu) = \sum_{Q \in Q_k} P(Q \cap B \neq \emptyset)^{-1} \cdot I_{Q \cap B \neq \emptyset} \cdot \nu|_Q. \quad (1.14)$$

It follows from (1.12), (1.13) and that the elements of $Q_k$ are disjoint that

$$E(C_k(\nu)(A)) = \nu(A) \quad (1.15)$$

for every Borel set $A \subseteq X$.

1.1.2 Special assumptions for our main results

For our main results we make the assumptions of Section 1.1.2 on $\varphi$, $Q_k$ and $B$. We list these assumptions below, however, we will note in the statement of the results and in the text when we make these assumptions.

For some $\delta > 0$ there exist $c_2, c_3 < \infty$ such that

$$\varphi(r) \leq c_2 \varphi(r \cdot (1 + 2\delta)) + c_3 \quad (1.16)$$

for all $r > 0$. We assume that

$$\varphi(0) = \infty, \quad (1.17)$$

which ensures that whenever $\nu(\{x\}) > 0$ for some $x \in X$ and Borel measure $\nu$ then $\iint \varphi(x, y) d\nu(x) d\nu(y) = \infty$. See Example 1.2.

There exists $\delta > 0$ (that is the same $\delta$ as in (1.16)) and $M_\delta < \infty$ independent of $k$ such that for every $Q \in Q_k$

$$\#\{S \in Q_k : \max\{\text{diam}(Q), \text{diam}(S)\} \geq \delta \cdot \text{dist}(Q, S)\} \leq M_\delta, \quad (1.18)$$

where $\text{dist}(Q, S) = \inf_{x \in Q, y \in S} \|x - y\|$. There exists $0 < M < \infty$, independent of $k$, such that

$$0 < \text{diam}(Q)/M \leq \text{diam}(S) \leq \text{diam}(Q) \cdot M < \infty \quad (1.19)$$
for every $Q, S \in \mathcal{Q}_k$, $k \in \mathbb{N}$.

The $\varphi$-energy of a Borel measure $\nu$ on $X$ is

$$I_\varphi(\nu) = \int \int \varphi(x, y) d\nu(x) d\nu(y).$$

(1.20)

The $\varphi$-capacity of a Borel set $K \subseteq X$ is

$$C_\varphi(K) = \sup \{ I_\varphi(\nu)^{-1} : \nu \text{ is a Borel probability measure on } K \}.$$  

(1.21)

When $\varphi(r) = r^{-\alpha}$ for some $\alpha \geq 0$ then we write $I_\alpha(\nu) = I_\varphi(\nu)$ and $C_\alpha(K) = C_\varphi(K)$.

There exists $a > 0$ such that

$$aC_\varphi(Q) \leq P(Q \cap B \neq \emptyset)$$

(1.22)

for every $Q \in \mathcal{Q}_k$. Hence if $C_\varphi(Q) > 0$ for every $Q \in \mathcal{Q}_k$ then (1.13) holds.

There exists $0 < \delta < 1$ and $0 < c < \infty$ such that whenever $Q \in \mathcal{Q}_k$, $S \in \mathcal{Q}_n$ and $\max \{\text{diam}(Q), \text{diam}(S)\} < \delta \cdot \text{dist}(Q, S)$ then

$$P(Q \cap B \neq \emptyset \text{ and } S \cap B \neq \emptyset) \leq c \cdot P(Q \cap B \neq \emptyset) \cdot P(S \cap B \neq \emptyset) \cdot \varphi(\text{dist}(Q, S)).$$

(1.23)

We further assume that $\delta > 0$ is the same value for (1.16), (1.18) and (1.23).

**Example 1.4.** Let $\mathcal{Q}_k^i = \{[\frac{k}{2^i}, \frac{k+1}{2^i}] \times \cdots \times [\frac{k}{2^i}, \frac{k+1}{2^i}] \subseteq [-2^i, 2^i]^d \setminus \left[-\frac{1}{2^i}, \frac{1}{2^i}\right]^d : i_1, \ldots, i_d \in \mathbb{Z}\}$ for some $k, i \in \mathbb{N}, i \leq k$, i.e. the dyadic cubes of side length $2^{-k}$ which are contained in $[-2^i, 2^i]^d \setminus \left[-\frac{1}{2^i}, \frac{1}{2^i}\right]^d$.

We show, in Section 12.2 that for fixed $i \in \mathbb{N}$ for $\mathcal{Q}_k^i$, $(k \geq i)$ in Example 1.4 we have that (1.10), (1.11), (1.13), (1.18), (1.19), (1.22) and (1.23) hold for sufficient constants $0 < \delta^i < 1$, $M_0^i < \infty$, $0 < M^i < \infty$, $a^i < \infty$ and $0 < c^i < \infty$.

### 1.2 Decomposition of measures

In the spirit of (1.8) we would like to decompose $\nu$ into a vanishing part and a part for which we obtain convergence in $\mathcal{L}^1$. Similar results to the following proposition was published by Kahane [5, Section 3]. However, we are not aware that this kind of results appeared in the literature in English.

**Proposition 1.5.** Let $\nu$ be a locally finite Borel measure on $X$. There exist two locally finite, Borel measures $\nu_{\varphi R} = \nu_R$ and $\nu_{\varphi \perp} = \nu_\perp$ with the following properties:

i) $\nu = \nu_R + \nu_\perp$

ii) $\nu_R \perp \nu_\perp$

iii) $\nu_\perp$ is singular to every locally finite Borel measure with finite $\varphi$-energy

iv) there exists a sequence of disjoint Borel sets $\{A_n\}_{n \in \mathbb{N}}$ such that $\nu_R = \nu|_{\bigcup_{n \in \mathbb{N}} A_n} = \sum_{n \in \mathbb{N}} \nu|_{A_n}$ and $I_\varphi(\nu|_{A_n}) < \infty$.

**Notation 1.6.** We call $\nu_R$ the $\varphi$-regular part of $\nu$ and we call $\nu_\perp$ the $\varphi$-singular part of $\nu$. These are uniquely determined by $\nu$ and $\varphi$. When $\varphi(r) = r^{-\alpha}$ for some $\alpha \geq 0$ then we say that $\nu_R$ is the $\alpha$-regular part of $\nu$ and $\nu_\perp$ is the $\alpha$-singular part of $\nu$. 

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Proposition 1.7. If $\nu$ is a locally finite Borel measure that is singular to every finite Borel measure with finite $\varphi$-energy then there exists a Borel set $Z \subseteq X$ such that $\nu(X \setminus Z) = 0$ and $C_\varphi(Z) = 0$.

We prove Proposition 1.5 and Proposition 1.7 in Section 4.

1.3 Summary of the main results

Let $\mu$ and $\mu_k$ be a sequence of random, finite, Borel measures on $X$ (for the definition of random, finite, Borel measures see Definition 3.27). We say that $\mu_k$ weakly converges to $\mu$ subsequentially in probability if for every subsequence $\{\alpha_k\}_{k=1}^\infty$ of $\mathbb{N}$ there exists a subsequence $\{\beta_k\}_{k=1}^\infty$ of $\{\alpha_k\}_{k=1}^\infty$ and an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\mu_{\beta_k}$ converges weakly to $\mu$ on the event $H$. The notion of weak convergence subsequentially in probability was considered independently by Berestycki in [1, Section 6] in the context of Gaussian multiplicative chaos.

Let $\mu$ and $\mu_k$ be a sequence of random, locally finite, Borel measures on $X$ (for the definition of random, locally finite, Borel measures see Definition 3.27). We say that $\mu_k$ vaguely converges to $\mu$ in probability if $\int_X f(x) d\mu_k(x)$ converges to $\int_X f(x) d\mu(x)$ in probability for every compactly supported continuous function $f$.

For a Borel measure $\nu$ on $X$ let $\text{supp}(\nu)$ denote the support of $\nu$, that is the smallest closed set with full measure, i.e.

$$\text{supp}(\nu) = \bigcap \{K : \nu(X \setminus K) = 0, K \subseteq X \text{ is a closed set}\}.$$ 

If $f : X \rightarrow \mathbb{R}$ is a nonnegative Borel function then we denote by

$$f(x) d\nu(x)$$

the measure $\tau$ defined by $\tau(A) = \int_A f(x) d\nu(x)$.

Below we define the main object of the paper.

Definition 1.8. Let $\nu$ be a finite, Borel measure on $X$. If $C(\nu)$ is a random, finite, Borel measure that satisfies the following:

i.) $C_k(\nu)$ weakly converges to $C(\nu)$ subsequentially in probability,

ii.) $\int_X f(x) d\mu_k(x)$ converges to a random variable $S(f)$ in probability with $E(\int_X f(x) dC(\nu)(x)) = E(S(f)) = \int_X f(x) d\nu_R(x)$ for every $f : X \rightarrow \mathbb{R}$ Borel measurable function such that $\int_X |f(x)| d\mu(x) < \infty$,

iii.) for every countable collection of deterministic Borel measurable functions $f_n : X \rightarrow \mathbb{R}$ with $\int_X |f_n(x)| d\nu(x) < \infty$ we have that $\int_X f_n(x) dC(\nu)(x) = S(f_n)$ for every $n \in \mathbb{N}$ almost surely,

iv.) for every countable collection of deterministic, Borel sets $A_n \subseteq X$ with $\nu(A_n) < \infty$ we have that $S(\chi_{A_n}) = C(\nu)(A_n)$ for every $n \in \mathbb{N}$ almost surely,

v.) $E(C(\nu)(A)) = E(S(\chi_A)) = \nu_R(A) \leq \nu(A)$ for every Borel set $A \subseteq X$ with $\nu(A) < \infty$,

vi.) $C_k(\nu_R)(A)$ converges to $C(\nu)(A)$ in $\mathcal{L}^1$ for every Borel set $A \subseteq X$ with $\nu(A) < \infty$ in probability,

vii.) $C(\nu_\perp) = 0$ almost surely,
viii.) $C(\nu) = C(\nu_R)$ almost surely,
ix.) if $\nu = \sum_{i=1}^{\infty} \nu^i$ for a sequence of locally finite, Borel measures $\nu^i$ then $C(\nu) = \sum_{i=1}^{\infty} C(\nu^i)$ almost surely,
i*.) are the same almost surely, 

if $f : X \rightarrow \mathbb{R}$ is a nonnegative Borel function such that $\int_X f(x) d\nu(x) < \infty$ then $C(f(x) d\nu(x)) = f(x) dC(\nu)(x)$, in particular, if $\gamma \in [0, \infty)$ then $C(\gamma \cdot \nu) = \gamma \cdot C(\nu)$ almost surely,

xi.) supp$C(\nu) \subseteq$ supp$\nu \cap B$ almost surely,

then we say that the conditional measure of $\nu$ on $B$ exists with respect to $Q_k$ ($k \geq 1$) with regularity kernel $\varphi$ and it is $C(\nu)$.

If $X$ is locally compact and $\nu$ is a locally finite, Borel measure on $X$ and $C(\nu)$ is a random, locally finite, Borel measure that satisfies ii.)-ix.), xi.) and additionally also satisfies the following:

i*.) $C_k(\nu)$ vaguely converges to $C(\nu)$ in probability,

i*.) if $f : X \rightarrow \mathbb{R}$ is a nonnegative Borel function such that for every $y \in X$ there exists a neighbourhood $U$ of $y$ such that $\int_U f(x) d\nu(x) < \infty$ then $C(f(x) d\nu(x)) = f(x) dC(\nu)(x)$, in particular, if $\gamma \in [0, \infty)$ then $C(\gamma \cdot \nu) = \gamma \cdot C(\nu)$ almost surely,

then we say that the conditional measure of $\nu$ on $B$ exists with respect to $Q_k$ ($k \geq 1$) with regularity kernel $\varphi$ and it is $C(\nu)$.

We defined the conditional measure of finite Borel measures and in locally compact spaces we defined the conditional measure of locally finite Borel measures. To avoid confusion we need to show that the two definitions of the conditional measure of finite measures in locally compact spaces are the same. Proposition 1.9 is proven at the end of Section 3.4.

**Proposition 1.9.** If $X$ is locally compact and $\nu$ is a finite Borel measure then the two definitions of the conditional measure of $\nu$ on $B$ with respect to $Q_k$ ($k \geq 1$) with regularity kernel $\varphi$ in Definition 1.8 are equivalent and the limits $C(\nu)$ in Property i.) and in Property i*.) are the same almost surely.

We develop the theory of weak convergence subsequentially in probability and vague convergence in probability in Section 3 that is essential in showing the existence of the conditional measure. The proofs are based on classical functional analysis and classical probability. The following theorem is proved in Section 3.4.

**Theorem 1.10.** Let $\nu$ be a finite, Borel measure on $X$ or let $X$ be locally compact and $\nu$ be a locally finite, Borel measure on $X$. Assume that $\nu(X \setminus X_0) = 0$. Then the conditional measure $C(\nu)$ of $\nu$ on $B$ exists with respect to $Q_k$ ($k \geq 1$) with regularity kernel $\varphi$ if and only if $C_k(\nu|_D)(X) = C_k(\nu)(D)$ converges in $L^1$ for every compact set $D \subseteq X$ with $I_\nu(\nu|_D) < \infty$ and $C_k(\nu|_D)(D)$ converges to 0 in probability for every compact set $D \subseteq X$.

The sequence $C_k(\nu)$ might give the impression of a $T$-martingale that was introduced by Kahane [7]. However, when $B$ is the Brownian path $C_k(\nu)$ is not a $T$-martingale with respect to the natural filtration $F_k = \sigma\{B \cap Q \neq \emptyset\}_{Q \in \mathcal{Q}_k}$ and we cannot prove almost sure convergence of $C_k(\nu)$, that is why we needed to develop the convergence of random measures in probability. Despite that, $C_k(\nu)$ and its limit $C(\nu)$ if the conditional measure
exists, exhibits many similar properties to Kahane’s $T$-martingales. To get around the trouble that $C_k(\nu)$ is not necessarily a $T$-martingale we define the following kernels that are key objects in showing that the limit of $C_k(\nu)$ exists.

**Notation 1.11.** For $x \in X$ and $k \in \mathbb{N}$ let $Q_k(x) = Q$ if $x \in Q$ for some $Q \in \mathcal{Q}_k$ and $Q_k(x) = \emptyset$ otherwise. There is at most one such $Q$ since elements of $\mathcal{Q}_k$ are disjoint hence $Q_k(x)$ is well-defined.

**Definition 1.12.** For $k, n \in \mathbb{N}$ let $F_{k,n} : X \times X \rightarrow \mathbb{R}$ be the nonnegative function

$$F_{k,n}(x, y) = \begin{cases} \frac{P(Q_k(x) \cap B \neq \emptyset \text{ and } Q_n(y) \cap B \neq \emptyset)}{P(Q_k(x) \cap B \neq \emptyset)P(Q_n(y) \cap B \neq \emptyset)} & \text{if } Q_k(x) \neq \emptyset, Q_n(y) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

**Definition 1.13.** We define the following functions

$$F_N(x, y) = \sup_{n, k \geq N} F_{k,n}(x, y)$$

and

$$F_N(x, y) = \inf_{n, k \geq N} F_{k,n}(x, y)$$

and their limits

$$F(x, y) = \limsup_{N \rightarrow \infty} F_N(x, y) = \lim_{N \rightarrow \infty} F_N(x, y)$$

and

$$F(x, y) = \liminf_{N \rightarrow \infty} F_N(x, y) = \lim_{N \rightarrow \infty} F_N(x, y).$$

**Remark 1.14.** If (1.23) holds for some $\delta > 0$ then

$$F(x, y) \leq F(x, y) \leq c \cdot \varphi(x, y)$$

for $x \neq y$.

In Section 5 we discuss that $C_k(\nu \perp)$ converges to 0. The existence of the kernel $F(x, y) = F(x, y) = F(x, y)$ is the key assumption in order to prove the existence of the conditional measure in the nondegenerate case. Using the observation that $E (C_k(\nu)(A)^2) = \int_A \int_A F_{k,k}(x, y)d\nu(x)d\nu(y)$, we show, throughout Section 6 and 7, that if $I_\varphi(\nu) < \infty$ then $C_k(\nu)$ converges in $\mathcal{L}^2$ and so in $\mathcal{L}^1$. Finally, using the measure decomposition result of Proposition 1.3 we conclude one of our deepest result in Section 8 and Section 9. Theorem 1.15 is proven at the end of Section 9.

**Theorem 1.15.** Assume that (1.17), (1.19) and (1.23) hold and there exists $0 < \delta < 1$ such that (1.17), (1.23) and (1.18) hold. Assume that $F(x, y) = F(x, y) = F(x, y)$ for every $(x, y) \in X \times X$. Assume that if $C_\varphi(D) = 0$ for some compact set $D \subseteq X_0$ then $B \cap D = \emptyset$ almost surely. Let either $\nu$ and $\tau$ be finite Borel measures on $X$ or $X$ be locally compact and $\nu$ and $\tau$ be locally finite Borel measures on $X$. Assume that $\nu(X \setminus X_0) = 0$ and $\tau(X \setminus X_0) = 0$. Then the conditional measure $C(\nu)$ of $\nu$ and $C(\tau)$ of $\tau$ on $B$ exist with respect to $Q_k (k \geq 1)$ with regularity kernel $\varphi$ and

$$E \left( \int \int f(x, y)dC(\nu)(x)dC(\tau)(y) \right) = \int \int F(x, y)f(x, y)d\nu_R(x)d\tau_R(y) \quad (1.24)$$
for every $f : X \times X \rightarrow \mathbb{R}$ Borel function with $\int \int F(x,y) \cdot |f(x,y)| \, d\nu_R(x)d\tau_R(y) < \infty$, in particular if $\int \int \varphi(x,y) \cdot |f(x,y)| \, d\nu(x)d\tau(y) < \infty$.

We study the conditional measure on the Brownian path in Section 12. We show that the conditions of Theorem 1.15 hold for the Brownian path $B \subseteq \mathbb{R}^d$ $(d \geq 3)$ and the sequence $Q_k$, $(k \geq 1)$ in Example 1.3 for sufficient constant that depends on $i$ and $d$. However, our main goal is to show the existence of the conditional measure with respect to $Q_k$, $(k \geq 1)$ in Example 1.3. In Section 10 we discuss how we can extend Theorem 1.16 to a $Q_k$, $(k \geq 1)$ if we know that that conditions of Theorem 1.15 hold for $Q_k^i$, $(k \geq i)$ such that $Q_k^i \subseteq Q_k$ and $Q_k^i$ is approaching $Q_k$ in some sense as $i$ goes to $\infty$. For the exact statement see Theorem 10.4. Applying this result to the Brownian path we conclude in Section 12.3 the following theorem.

**Theorem 1.16.** Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$. Let $\nu$ be a locally finite Borel measure on $\mathbb{R}^d$ such that $\nu\{0\} = 0$. Then the conditional measure $C(\nu)$ of $\nu$ on $B$ exists with respect to $Q_k$ $(k \geq 1)$ (where $Q_k$ is as in Example 1.3) with regularity kernel $\varphi(x,y) = ||x - y||^{2-d}$ (where $Q_k$ is as in Example 1.3). Let $\tau$ also be a locally finite Borel measure on $\mathbb{R}^d$ such that $\tau\{0\} = 0$. Then

$$E \left( \int \int f(x,y) dC(\nu)(x) dC(\tau)(y) \right) = \int \int \frac{||x||^{d-2} + ||y||^{d-2}}{||x - y||^{d-2}} f(x,y) d\nu_R(x) d\tau_R(y)$$

(1.25)

for every Borel function $f(x,y) : X \times X \rightarrow \mathbb{R}$ with $\int \int \frac{||x||^{d-2} + ||y||^{d-2}}{||x - y||^{d-2}} |f(x,y)| \, d\nu_R(x)d\tau_R(y) < \infty$.

Recall, that in (1.25) $\nu_R$ and $\tau_R$ refers to the $(d-2)$-regular part of the measures (see Notation 1.6). Note that if $\nu$ is a point mass on $\{0\}$ then $C_k(\nu) = \nu$ for every $n$.

**Remark 1.17.** Let $\nu$ be a finite Borel measure such that $\text{supp}\nu$ is bounded away from 0 and infinity and $I_{\beta}(\nu) < \infty$ for some $\beta > d-2$. One consequence of the double integration formula (1.25) that $I_{\beta+2-d}(C(\nu)) < \infty$ almost surely. It is a useful tool in the geometric measure theory of the random intersection $B \cap K$ for some fixed deterministic Borel set $K$.

The question naturally rises what happens when $\nu$ is the Lebesgue measure. The answer is given in terms of the occupation measure of the Brownian motion, that counts the amount of time the Brownian motion spends inside a set. Section 13 is dedicated to deal with this question.

**Theorem 1.18.** Let $B_0(t)$ be a Brownian motion in $\mathbb{R}^d$ for $d \geq 3$, let $B$ be the range of the Brownian motion and let $\lambda$ be the Lebesgue measure in $\mathbb{R}^d$. Let

$$\tau(A) = \int_0^\infty I_{B_0(t) \in A} dt$$

be the occupation measure of $B_0$. Then

$$dC(\lambda)(x) = \frac{1}{c(d)} ||x||^{d-2} d\tau(x)$$
\[ C(\|x\|^{2-d} \, d\lambda(x)) = \frac{1}{c(d)} \, d\tau(x) \]

almost surely where

\[ c(d) = \Gamma(d/2 - 1)2^{-d/2}\pi^{-d/2} \quad (1.26) \]

and \( \Gamma(x) = \int_0^\infty s^{x-1} e^{-s} \, ds \) is the Gamma function.

This result gives us a tool to calculate the occupation measure, \( \tau(A) = c(d) \int_A \|x\|^{2-d} \, dC(\lambda)(x) \) can be approximated by \( c(d) \int_A \|x\|^{2-d} \, dC_k(\lambda)(x) \) which converges to \( \tau(A) \) in probability. To calculate the value of \( c(d) \int_A \|x\|^{2-d} \, dC_k(\lambda)(x) \) we only need to know which boxes of \( Q_k \) does \( B \) intersect.

Remark 1.19. The Green’s function of the Brownian motion in \( \mathbb{R}^d \) is \( G(x, y) = c(d) \|x - y\|^{2-d} \) for the constant \( c(d) \) in (1.26), see [14, Theorem 3.33].

As an application of our results, we obtain a formula for the first moment of the occupation measure \( \tau \) is known (Proposition 13.20):

\[ E(\tau(A)) = c(d) \int_A \frac{1}{\|x\|^{d-2}} \, dx. \]

As an application of our results, we obtain a formula for the second moment of the occupation measure. Theorem 1.20 is a special case of Theorem 13.21. We deduce the result from Theorem 1.18 and the double integration formula (1.25). We note however, that Theorem 1.20 could be deduced, via a direct calculation, from the transition probability kernels \( p^*(t, x, y) \). For the definition of \( p^*(t, x, y) \) see [14, Theorem 3.30].

**Theorem 1.20.** Let \( B_0(t) \) be a Brownian motion in \( \mathbb{R}^d \) for \( d \geq 3 \) and let \( \tau \) be the occupation measure of \( B_0 \). For every Borel set \( A \subseteq \mathbb{R}^d \)

\[ E(\tau(A)^2) = c(d)^2 \int_A \int_A \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2} \cdot \|x\|^{d-2} \cdot \|y\|^{d-2}} \, dx \, dy \]

where \( c(d) \) is as in (1.20).

For a wide class of random closed sets we have that the hitting probability of a compact set \( K \) is comparable to the \( \varphi \)-capacity of \( K \) for a sufficient kernel \( \varphi \). See Proposition 14.2 in case of the ‘percolation limit set’ or Proposition 1.23 in case of the Brownian path. In Section 11 we discuss the probability of the nonextinction of the conditional measure and we establish analogous results to the hitting probabilities, namely the probability of the nonextinction of \( C(\nu) \) is comparable to the capacity of the measure \( \nu \). For the definition of \( C(\nu) \) and \( \overline{C}(\nu) \) see Definition 11.1 and Definition 11.2.

**Theorem 1.21.** Assume that the conditions of Theorem 1.15 hold and \( P(D \cap B \neq \emptyset) \leq b \cdot C(\nu)(D) \) for every compact set \( D \subseteq X_0 \). Let \( \nu \) be a finite Borel measure such that \( \nu(X \setminus X_0) = 0 \). Then

\[ c^{-1} \cdot C(\nu)(X) > 0 \leq b \cdot \overline{C}(\nu). \]
We prove Theorem 1.21 in Section 11. The following result gives lower and upper bound for the probability of nonextinction of $C(\nu)$ when $B$ is the Brownian path, that we show in Section 12.4.

**Theorem 1.22.** Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$. Let $\nu$ be a locally finite Borel measure on $\mathbb{R}^d$ such that $\nu(\{0\}) = 0$. Let $C(\nu)$ be the conditional measure of $\nu$ on $B$ with respect to $Q_k$ ($k \geq 1$) with regularity kernel $\varphi(x, y) = \|x - y\|^{2-d}$. Then

$$C_F(\nu) \leq P(C(\nu)(X) > 0) \leq 2C_F(\nu)$$

for

$$F(x, y) = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}. \quad (1.27)$$

The results on the probability of nonextinction suggest an analogy between the random intersection $B \cap K$ for a fixed compact set $K$ and the conditional measure $C(\nu)$ of a fixed measure $\nu$ on $B$. Compare Theorem 1.22 to the following known result on hitting probabilities.

**Proposition 1.23.** Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$. Let $A \subseteq \mathbb{R}^d \setminus \{0\}$ be a compact set. Then

$$C_F(A) \leq P(B \cap A \neq \emptyset) \leq 2C_F(A) \quad (1.28)$$

where $F$ is as in (3.4).

Let $G(x, y) = c(d) \|x - y\|^{2-d}$ be the Green’s function of the Brownian motion in $\mathbb{R}^d$ (see [14, Theorem 3.33]) and let $M(x, y) = G(x, y)/G(0, y)$ be the Martin’s kernel. Then $F(x, y) = M(x, y) + M(y, x)$ and so $2I_M(\nu) = I_F(\nu)$ for every finite Borel measure $\nu$. Thus Proposition 1.23 is a reformulation of [14, Theorem 8.24].

We stated many of our main results in general complete, separable metric space $X$ for a ‘reasonable’ random closed set $B$. In Section 14 we let $X = \partial T$ to be the boundary of an infinite rooted tree $T$. We discuss the theory of the conditional measure when $B$ is a ‘percolation limit set’. We establish many properties of the conditional measure such as the double integration formula and the probability of nonextinction. In the end, we prove that $C(\nu)$ is a certain ‘random multiplicative cascade measure’. For the exact statement of these results and the discussion see Section 14.

## 2 Preliminary remarks

In this section we summarise the background and preliminary lemmas.

For $x \in X$, $r > 0$ let $B(x, r) = \{y \in X : d(x, y) < r\}$ and for a set $A \subseteq X$ let $B(A, r) = \{y \in X : x \in A, d(x, y) < r\}$.

**Notation 2.1.** For $A \subseteq X$ let

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

be the *characteristic function* of $A$.
Notation 2.2. For a probability event \( A \in \mathcal{A} \) let
\[
I_A(\omega) = \begin{cases} 
1 & \omega \in A \\
0 & \omega \notin A 
\end{cases}
\]
be the indicator function of \( A \).

2.1 Convergence in probability

We list below some folklore properties of the convergence in probability that we need throughout the paper.

Definition 2.3. Let \( Y_n (n \in \mathbb{N}) \) and \( Y \) be random variables. We say that \( Y_n \) converges to \( Y \) in probability (as \( n \) goes to \( \infty \)) if
\[
\lim_{n \to \infty} P(|Y_n - Y| > 0) = 0
\]
for every \( \varepsilon > 0 \). If the random variables take values in a metric space than we can replace the difference \(|.|\) by the metric to obtain a definition of convergence in probability for random variables that take values in a metric space.

Remark 2.4. Let \( Y \) and \( Y_n \) be a sequence of random variables that take values in a metric space. Then \( Y_n \) converges to \( Y \) in probability if and only if for every subsequence \( \{n_k\}_{k=1}^\infty \) of \( \mathbb{N} \) there exists a subsequence \( \{j_k\}_{k=1}^\infty \) of \( \{n_k\}_{k=1}^\infty \) such that \( Y_{j_k} \) converges to \( Y \) almost surely (see [6, Theorem 2.3.2]).

Remark 2.5. The convergence in probability is a metric convergence and is completely metrizable by the following metric \( \rho \) (see [6] Exercise 2.3.8 and 2.3.9, note that \( X \) is a complete separable metric space). For random variables \( Y, Z \) let
\[
\rho(Y, Z) = E\left( \frac{|Y - Z|}{1 + |Y - Z|} \right).
\]

Remark 2.6. Note that \( \rho(Y, Z) \leq E(|Y - Z|) \). We use this fact without reference throughout the paper.

Lemma 2.7. Let \( Y_n : \Omega \longrightarrow M \) be a sequence of random variables that take values in a metric space \( M \). If \( Y : \Omega \longrightarrow M \) is a function and there exists an event \( H \) with \( P(H) = 1 \) such that \( Y_n(\omega) \) converges to \( Y(\omega) \) for every \( \omega \in H \) then \( Y \) is a random variable.

For the proof of Lemma 2.7 see [1] Theorem 4.2.2

Lemma 2.8. Assume that \( \{f_{i,k}\}_{i,k \in \mathbb{N}} \) is a family of random variables such that \( f_{i,k} \) converges in probability as \( k \) goes to infinity for every \( i \). Then for every subsequence \( \{n_k\}_{k=1}^\infty \) of \( \mathbb{N} \) there exists a subsequence \( \{j_k\}_{k=1}^\infty \) of \( \{n_k\}_{k=1}^\infty \) and there exists a probability event \( H \in \mathcal{A} \) with \( P(H) = 1 \) such that \( f_{j_k} \) converges on the event \( H \) as \( k \) goes to infinity for every \( i \).
Proof. Let \( \{\alpha_k\}_{k=1}^{\infty} \) be a subsequence of \( \{n_k\}_{k=1}^{\infty} \) such that \( f^{1}_{\alpha_k} \) converges almost surely and let \( j_1 = \alpha_1^1 \). If \( \{\alpha_k^i\}_{k=1}^{\infty} \) and \( j_1, \ldots, j_i \) are defined let \( \{\alpha_k^i+1\}_{k=i+1}^{\infty} \) be a subsequence of \( \{\alpha_k\}_{k=i+1}^{\infty} \) such that \( f^{i+1}_{\alpha_k^i+1} \) converges almost surely as \( k \) goes to infinity and let \( j_{i+1} = \alpha_{i+1}^i \). Then \( \{j_k\}_{k=1}^{\infty} \) is a subsequence of \( \{\alpha_k^i\}_{k=i}^{\infty} \) and hence \( f^{i}_{j_k} \) converges almost surely as \( k \) goes to infinity for every \( i \).

\[ \Box \]

**Lemma 2.9.** Let \( Y \) and \( Y_n \) (\( n \in \mathbb{N} \)) be a sequence of random variables. If for every \( \varepsilon > 0 \) and every subsequence \( \{\alpha_k\}_{k=1}^{\infty} \) of \( \mathbb{N} \) we can find a subsequence \( \{\beta_k\}_{k=1}^{\infty} \) of \( \{\alpha_k\}_{k=1}^{\infty} \) such that

\[
\lim_{k \to \infty} P(|Y_{\beta_k} - Y| > \varepsilon) = 0
\]

then \( Y_n \) converges to \( Y \) in probability as \( n \) goes to \( \infty \).

**Proof.** In a topological space \( x_k \) converges to \( x \) if and only if for every subsequence \( \{\alpha_k\}_{k=1}^{\infty} \) of \( \mathbb{N} \) we can find a subsequence \( \{\beta_k\}_{k=1}^{\infty} \) of \( \{\alpha_k\}_{k=1}^{\infty} \) such that \( x_{\beta_k} \) converges to \( x \). Applying this to \( x_k = P(|Y_k - Y| > \varepsilon) \) it follows that \( \lim_{k \to \infty} P(|Y_k - Y| > \varepsilon) = 0 \).

\[ \Box \]

**Lemma 2.10.** Let \( Y_n \) be a sequence of real valued random variables such that \( Y_n \) converges to \( Y \) in probability and \( \lim_{n \to \infty} E|Y_n| = E|Y| < \infty \). Then \( Y_n \) converges to \( Y \) in \( \mathcal{L}^1 \).

For details of the proof see [3, Theorem 5.5.2]

**Lemma 2.11.** If \( Y_n \) (\( n \in \mathbb{N} \)) is a sequence of nonnegative, real valued random variables, \( Y_n \) converges to \( Y \) in probability and there exists \( c < \infty \) such that \( E(Y_n) \leq c \) for every \( n \in \mathbb{N} \) then \( E(Y) \leq c \).

**Proof.** Let \( n_k \) be a sequence such that \( Y_{n_k} \) converges to \( Y \) almost surely. Then by Fatou's lemma

\[
E(Y) = E(\liminf_{k \to \infty} Y_{n_k}) \leq \liminf_{k \to \infty} E(Y_{n_k}) \leq c.
\]

\[ \Box \]

### 2.2 Weak* and Vague convergence of measures

We recall some properties of the Weak* and vague convergences of measures and some related lemmas.

Let \( C_b(X) \) denote the space of bounded continuous functions of \( X \) equipped with the supremum norm and let \( C_c(X) \) denote the space of all compactly supported continuous functions on \( X \) equipped with the supremum norm. We denote by \( \text{supp}(f) \) the support of a function \( f : X \to \mathbb{R} \).

The following lemma states that every finite Borel measure in a Polish space is inner regular. We will use this folklore throughout the paper without referencing every time. For the details of the proof see [9, Theorem 17.11].

**Lemma 2.12.** Let \( \nu \) be a finite, Borel measure on \( X \), let \( \varepsilon > 0 \) and \( A \subseteq X \) be a Borel set. Then there exists compact set \( K \subseteq A \) such that \( \nu(A \setminus K) < \varepsilon \).
Lemma 2.13. Let \( \nu \) be a finite, Borel measure on \( X \) and \( G \subseteq X \) be a Borel set. Then there exists a sequence of disjoint compact subsets \( K_1, K_2, \ldots \) of \( A \) such that \( \nu(A \setminus \bigcup_{i=1}^{\infty} K_i) = 0 \).

Proof. By inner regularity (Lemma 2.12) we can find \( K_1 \subseteq A \) such that \( \nu(A \setminus K_1) < 1 \). Once we have \( K_1, \ldots, K_n \) we can find, by inner regularity, \( K_{n+1} \subseteq A \setminus \bigcup_{i=1}^{n} K_i \) such that \( \nu(A \setminus \bigcup_{i=1}^{n+1} K_i) < 1/n \). After countably many steps we end up with the desired sequence.

Remark 2.14. Lemma 2.13 holds for locally finite Borel measure \( \nu \). It can be deduced from that \( \nu \) is locally finite and \( X \) satisfies the Lindelöf property as it is a separable metric space.

Definition 2.15. Let \( \mu \) and \( \mu_k \) \( (k \in \mathbb{N}) \) be finite Borel measures on \( X \). We say that \( \mu_k \) weakly converges to \( \mu \) (as \( k \) goes to \( \infty \)) if

\[
\lim_{k \to \infty} \int f(x) d\mu_k(x) = \int f(x) d\mu(x)
\]

for every \( f \in C_b(X) \).

Remark 2.16. It is well-known, that \( \mu_k \) converges to \( \mu \) weakly if and only if \( \mu(G) \leq \liminf_{k \to \infty} \mu_k(G) \) for every open set \( G \). See [2] Theorem 2.1.

Lemma 2.17. Let \( \mu \) and \( \nu \) be finite Borel measures on \( X \) such that \( \mu \neq \nu \). Then there exists \( f \in C_b(X) \) such that \( \int_X f(x) d\mu(x) \neq \int_X f(x) d\nu(x) \).

Lemma 2.17 is shown in [2] Theorem 1.2. In locally compact space the same proof results the following lemma because by inner regularity (Lemma 2.12) the measure of compact sets determines the measure.

Lemma 2.18. Let \( X \) be locally compact. Let \( \mu \) and \( \nu \) be locally finite Borel measures on \( X \) such that \( \mu \neq \nu \). Then there exists \( f \in C_b(X) \) such that

\[
\int_X f(x) d\mu(x) \neq \int_X f(x) d\nu(x).
\]

Lemma 2.19. Assume that \( \mu_k \) converges to both \( \mu \) and \( \nu \) weakly then \( \mu = \nu \).

Lemma 2.19 follows from Lemma 2.17.

Lemma 2.20. Let \( f, f_k \in C_b(X) \) \( (k \in \mathbb{N}) \) such that \( \lim_{k \to \infty} \|f - f_k\|_{\infty} = 0 \). Assume that \( \mu \) and \( \nu \) are finite Borel measures on \( X \) such that \( \int_X f_k(x) d\mu(x) = \int_X f(x) d\nu(x) \) for every \( k \in \mathbb{N} \). Then

\[
\int_X f(x) d\mu(x) = \int_X f(x) d\nu(x).
\]

Proof. Let \( \varepsilon > 0 \) and \( k \in \mathbb{N} \) be such that \( \|f - f_k\|_{\infty} < \varepsilon \). Then

\[
\left| \int_X f(x) d\mu(x) - \int_X f(x) d\nu(x) \right| < \varepsilon.
\]
\[ \leq \left| \int_X f(x)d\mu(x) - \int_X f_k(x)d\mu(x) \right| + \left| \int_X f_k(x)d\nu(x) - \int_X f(x)d\nu(x) \right| \leq \varepsilon(\mu(X) + \nu(X)). \]

Hence the statement follows. \qed

**Lemma 2.21.** Let \( \Psi \subseteq C_b(X) \) be a dense subset with respect to the supremum norm. Assume that \( \mu \) and \( \nu \) are finite Borel measures on \( X \) such that \( \int_X f(x)d\mu(x) = \int_X f(x)d\nu(x) \) for every \( f \in \Psi \). Then \( \mu = \nu \).

**Proof.** It follows from Lemma 2.20 that \( \int_X f(x)d\mu(x) = \int_X f(x)d\nu(x) \) for every \( f \in C_b(X) \). Hence by Lemma 2.17 the statement follows. \qed

**Definition 2.22.** The Prohorov distance between two finite Borel measures \( \mu \) and \( \nu \) on \( X \) is

\[ \pi(\mu, \nu) = \inf \{ \varepsilon : \mu(A) \leq \nu(B(A, \varepsilon)) + \varepsilon \text{ and } \nu(A) \leq \mu(B(A, \varepsilon)) + \varepsilon \text{ for } \forall A \in B(X) \} \]

where \( B(X) \) denotes the set of Borel subsets of \( X \).

**Lemma 2.23.** We have that

\[ \pi(\mu, \nu) \leq \mu(X) + \nu(X). \]

**Lemma 2.24.** We have that

\[ \pi(\mu + \nu, \nu) \leq \mu(X). \]

The statement of Lemma 2.23 and Lemma 2.24 follow from the definition of the Prohorov distance.

**Proposition 2.25.** The Prohorov distance is a complete separable metric on the set of all finite Borel measures \( M_+(X) \). We have that \( \mu_k \) converges to \( \mu \) weakly if and only if \( \lim_{k \to \infty} \pi(\mu_k, \mu) = 0 \). (Note that throughout the paper we assume that \( X \) is a complete separable metric space.)

For the proof see [2, Theorem 6.8]

**Lemma 2.26.** If \( K \subseteq X \) is compact then \( C_b(K) \) is separable.

See [5, page 437].

**Lemma 2.27.** Let \( K \subseteq X \) be a compact subset, let \( \Psi \subseteq C_b(K) \) be a dense subset with respect to the supremum norm and let \( \mu_k \) be a sequence of finite, Borel measures on \( K \) such that \( \int_X f(x)d\mu_k(x) \) converges to a limit \( S(f) < \infty \) for every \( f \in \Psi \). Then \( \mu_k(K) \) is bounded and \( \mu_k \) converges weakly to a finite, Borel measure.
Proof. Let \( g \in \Psi \) such that \( \|\chi_K - g\| < 1/2 \). Then \( \chi_K \leq 2g \) on \( K \) thus

\[
\limsup_{k \to \infty} \mu_k(K) \leq 2 \limsup_{k \to \infty} \int_X g(x) d\mu_k(x) = 2S(g) < \infty
\]

and so \( \mu_k(K) \) is bounded.

Since \( \mu_k(K) \) is bounded and \( K \) is a compact metric space it follows, by [9] (17.22) Theorem], that there exists a subsequence \( n_k \) of \( \mathbb{N} \) such that \( \mu_{n_k} \) weakly converges to a Borel measure \( \tau \) of finite total mass. Then

\[
\int_X g(x) d\tau(x) = \lim_{k \to \infty} \int_X g(x) d\mu_{n_k}(x) = S(g)
\]

for every \( g \in \Psi \). Let \( f \) be a bounded continuous function and \( g \in \Psi \) be such that \( \|f - g\|_{\infty} < \varepsilon \). Then

\[
\limsup_{k \to \infty} \left| \int_X f(x) d\tau(x) - \int_X f(x) d\mu_k(x) \right| \leq \limsup_{k \to \infty} \left| \int_X f(x) d\tau(x) - \int_X g(x) d\tau(x) \right|
\]

\[
+ \left| \int_X g(x) d\tau(x) - \int_X g(x) d\mu_k(x) \right| + \left| \int_X g(x) d\mu_k(x) - \int_X f(x) d\mu_k(x) \right|
\]

\[
\leq \int_X |f(x) - g(x)| d\tau(x) + 0 + \limsup_{k \to \infty} \int_X |f(x) - g(x)| d\mu_k(x) \leq \varepsilon \left( \tau(K) + \limsup_{k \to \infty} \mu_k(K) \right).
\]

By taking \( \varepsilon \) goes to 0 it follows that

\[
\int_X f(x) d\tau(x) = \lim_{k \to \infty} \int_X f(x) d\mu_k(x).
\]

Definition 2.28. Let \( \mu \) and \( \mu_k \ (k \in \mathbb{N}) \) be locally finite Borel measures on \( X \). We say that \( \mu_k \) vaguely converges to \( \mu \) (as \( k \) goes to \( \infty \)) if

\[
\lim_{k \to \infty} \int_X f(x) d\mu_k(x) = \int_X f(x) d\mu(x)
\]

for every \( f \in C_c(X) \).

Lemma 2.29. Let \( X \) be locally compact. There exists \( \Psi \subseteq C_c(X) \) countable and dense subset with respect to the supremum norm such that if \( \mu_k \) is a sequence of locally finite Borel measures on \( X \) such that \( \int_X f(x) d\mu_k(x) \) converges to a finite limit \( S(f) \) for every \( f \in \Psi \) then \( \mu_k \) vaguely converges to a locally finite Borel measure.

Proof. Since \( X \) is a separable metric space it satisfies the Lindelöf property. Thus, because \( X \) is locally compact, we can find a sequence of open sets \( G_1 \subseteq G_2 \subseteq \ldots \) such that \( \bigcup_{i=1}^{\infty} G_i = X \) and \( \overline{G_i} \) is compact for every \( i \in \mathbb{N} \). Hence for every compact set \( K \) there exists \( i \in \mathbb{N} \) such that \( K \subseteq G_i \). We can further assume that \( \overline{G_i} \subseteq G_{i+1} \). By Lemma [2.20] we can find a countable and dense subset \( \Psi_i \) of \( \left\{ f \in C_b(X) : \text{supp}(f) \subseteq \overline{G_i} \right\} \). Let \( \Psi = \bigcup_{i=1}^{\infty} \Psi_i \).
Let $\mu_k$ be a sequence of deterministic, locally finite, Borel measures on $X$ such that $\int_X f(x) d\mu_k(x)$ converges to a finite limit $S(f)$ for every $f \in \Psi$. If $\int_X f(x) d\mu_k(x)$ converges to a finite limit $S(f)$ for every $f \in C_c(X)$ then clearly $S$ is a positive linear functional, hence by the Riesz-Markov theorem \cite{15} Theorem 2.14 there exists a locally finite, Borel measures $\mu$ on $X$ such that $S(f) = \int_X f(x) d\mu(x)$ for every $f \in C_c(X)$. Hence the statement would follow. Thus to finish the proof we need to show that $\int_X f(x) d\mu_k(x)$ converges to a finite limit $S(f)$ for every $f \in C_c(X)$.

Let $f \in C_c(X)$, let $K = \text{supp}(f)$. There exists $i \in \mathbb{N}$ such that $K \subseteq G_i \subseteq \overline{G_i} \subseteq G_{i+1}$. By Tietze’s extension theorem we can find $h_0 \in C_c(X)$ such that $h_0(x) = 1$ for $x \in \overline{G_i}$ and $h_0(x) = 0$ for $x \notin G_{i+1}$. There exists $h \in \Psi$ such that $\|h_0 - h\|_\infty < 1/2$. Then $\limsup_{k \to \infty} \mu_k(G_i) \leq 2 \limsup_{k \to \infty} \int_X h(x) d\mu_k(x) = 2S(h)$.

Let $g_n \in \Psi$, be such that $\|f - g_n\|_\infty < 1/2$. Then

$$\limsup_{k \to \infty} \left| \int_X f(x) d\mu_k(x) - S(g_n) \right| \leq \limsup_{k \to \infty} \left| \int_X f(x) d\mu_k(x) - \int_X g_n(x) d\mu_k(x) \right| + \limsup_{k \to \infty} \left| \int_X g_n(x) d\mu_k(x) - S(g_n) \right|$$

$$\leq \limsup_{k \to \infty} \mu_k(G_i)/n + 0 \leq 2S(h)/n.$$

Thus for every $n \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $\left| \int_X f(x) d\mu_k(x) - S(g_n) \right| \leq 3S(h)/n$ for every $k \geq N$ and so $\left| \int_X f(x) d\mu_k(x) - \int_X f(x) d\mu_j(x) \right| \leq 6S(h)/n$ for every $j, k \geq N$. Hence $\int_X f(x) d\mu_k(x)$ is a Cauchy sequence so it has a finite limit $S(f)$ in $\mathbb{R}$.

**Lemma 2.30.** Let $X$ be locally compact, let $\nu$ and $\mu$ be locally finite Borel measures on $X$ and let $\Psi \subseteq C_c(X)$ be as in Lemma 2.29. If $\int_X f(x) d\nu(x) = \int_X f(x) d\mu(x)$ for every $f \in \Psi$ then $\mu = \nu$.

**Proof.** Let $f \in C_c(X)$ be fixed. Similarly to the proof of Lemma 2.29 it can be shown that there exist $h \in \Psi$ and $g_n \in \Psi$ for every $n \in \mathbb{N}$ such that

$$\left| \int_X f(x) d\mu(x) - \int_X f(x) d\nu(x) \right| \leq \left| \int_X f(x) d\mu(x) - \int_X g_n(x) d\mu(x) \right| + \left| \int_X g_n(x) d\nu(x) - \int_X f(x) d\nu(x) \right|$$

$$\leq 4 \int_X h(x) d\mu(x)/n.$$

Thus $\int_X f(x) d\mu(x) = \int_X f(x) d\nu(x)$ for every $f \in C_c(X)$ and the statement of the lemma follows by Lemma 2.18.
Proposition 2.31. Assume that $X$ is locally compact. The set of all locally finite Borel measures $\mathcal{M}_f(X)$ can be equipped with a complete separable metric such that $\mu_k$ converges to $\mu$ in the induced topology if and only if $\mu_k$ converges to $\mu$ vaguely.

Proposition 2.31 is shown for $X = \mathbb{R}^d$ in [12, Chapter 14] in the section called ‘A metric on measures’ starting on page 194. The proof goes similarly for locally compact complete separable metric spaces.

Lemma 2.32. Let $\nu$ and $\tau$ be two locally finite Borel measures on $X$ such that $\tau(G) \leq \nu(G)$ for every open set $G$. Then $\tau(A) \leq \nu(A)$ for every Borel set $A$.

Proof. If $\nu$ and $\tau$ are finite Borel measures then it follows from [9, Theorem 17.10] that $\tau(A) \leq \nu(A)$ holds for every Borel set $A$. Hence the statement holds locally for locally finite Borel measures. Thus the statement follows from the fact that $X$ satisfies the Lindelöf property since it is a separable metric space.

Lemma 2.33. Let $D \subseteq X$ be a countable and dense subset and let

$$\mathcal{G} = \{\bigcup_{i=1}^m B(x_i, r_i) : m \in \mathbb{N}, x_i \in D, r_i \in \mathbb{Q}, r_i > 0, \text{ for } i = 1, \ldots, m\}.$$

Let $\nu$ and $\tau$ be two locally finite Borel measures on $X$ such that $\tau(G) = \nu(G)$ for every $G \in \mathcal{G}$. Then $\tau(A) = \nu(A)$ for every Borel set $A$.

Proof. Let $A \subseteq X$ be an open set. Then

$$\{B(x, r) : x \in D, r \in \mathbb{Q}, r > 0, B(x, r) \subseteq A\}$$

is an open cover of $A$ hence, by the Lindelöf property, there exists a countable subcollection $\{B(x_i, r_i)\}_{i=1}^\infty$ such that $A = \bigcup_{i=1}^\infty B(x_i, r_i)$. Then

$$\tau(A) = \lim_{m \to \infty} \tau(\bigcup_{i=1}^m B(x_i, r_i)) = \lim_{m \to \infty} \nu(\bigcup_{i=1}^m B(x_i, r_i)) = \nu(A).$$

We can conclude that $\tau(A) = \nu(A)$ for every open set $A \subseteq X$ and so for every Borel set $A \subseteq X$ by Lemma 2.32.

Proposition 2.34. Let $(\Omega, \mathcal{B})$ be a measurable space, let $\nu$ and $\tau$ be $\mathcal{B}$-measurable finite measures on $\Omega$. Let $\mathcal{S}$ be a semiring of sets of $\mathcal{B}$ that generates the $\sigma$-algebra $\mathcal{B}$, assume that $\Omega \in \mathcal{S}$ and $\nu(S) \leq \tau(S)$ for every $S \in \mathcal{S}$. Then $\nu(A) \leq \tau(A)$ for every $A \in \mathcal{B}$.

Proof. By [11, Section 1.5.1] we have that $\nu$ and $\tau$ are uniquely determined by their values on $\mathcal{S}$, and $\nu$ and $\tau$ equal to their Charatódrey extension from $\mathcal{S}$. Hence $\nu(A) \leq \tau(A)$ for every $A \in \mathcal{B}$ by the definition of Charatódrey extension [11, Section 1.4.4].

3 Convergence of random measures

We combine the convergence of random variables in probability and the convergence of measures to obtain the convergence of measures in probability. This section includes four subsections. Section 3.1 develops the theory of weak convergence of random measures.
subsequentially in probability. In Section 3.2 we briefly introduce the concept of weak convergence of random measures in probability. In Section 3.3 we discuss the vague convergence of random measures in probability in the situation when \( X \) is a locally compact space. Finally, in Section 3.4 we use the results on the convergence of random measures to obtain some general results about the conditional measure of deterministic measures on random sets, including Theorem 1.10 and Proposition 1.9.

**Definition 3.1.** The set of all finite Borel measures \( \mathcal{M}_+(X) \) on \( X \) equipped with the weak∗-topology on the dual space of \( C_b(X) \) is a Polish space (see Proposition 2.25). A random, finite, Borel measure is an element of \( L^0(\mathcal{M}_+(X)) \), i.e. a finite Borel measure valued random variable.

**Lemma 3.2.** Let \( \mu_k \) be a sequence of random, finite Borel measures. If there exists \( H \in \mathcal{A} \) with \( P(H) = 1 \) such that for every outcome \( \omega \in H \) we have that \( \mu_k \) weakly converges to a finite, Borel measure \( \mu \) (note that \( \mu \) depends on \( \omega \in H \)) then \( \mu \) is a random, finite Borel measure.

The lemma follows from Lemma 2.7.

### 3.1 Weak convergence subsequentially in probability

**Definition 3.3.** Let \( \mu \) and \( \mu_k \) be a sequence of random, finite, Borel measures on \( X \). We say that \( \mu_k \) weakly converges to \( \mu \) subsequentially in probability if for every subsequence \( \{\alpha_k\}_{k=1}^{\infty} \) of \( \mathbb{N} \) there exists a subsequence \( \{\beta_k\}_{k=1}^{\infty} \) of \( \{\alpha_k\}_{k=1}^{\infty} \) and an event \( H \in \mathcal{A} \) with \( P(H) = 1 \) such that \( \mu_{\beta_k} \) converges weakly to \( \mu \) on the event \( H \).

**Remark 3.4.** It follows from the Definition 3.21 that if \( \mu_k \) is a sequence of random, finite, Borel measures on \( X \) such that \( \mu_k \) weakly converges to a random, finite, Borel measure \( \mu \) almost surely then \( \mu_k \) weakly converges to \( \mu \) subsequentially in probability.

**Proposition 3.5.** The limit in Definition 3.3 is unique in \( L^0(\mathcal{M}_+(X)) \) if exists.

**Proof.** Assume that a sequence of random, finite Borel measures converges weakly to both of the random, finite Borel measures \( \mu \) and \( \nu \) subsequentially in probability. Then there exists \( \{\alpha_k\}_{k=1}^{\infty} \) and an event \( H \in \mathcal{A} \) with \( P(H) = 1 \) such that \( \mu_{\alpha_k} \) weakly converges to both \( \mu \) and \( \nu \) on the event \( H \). Hence \( \mu = \nu \) on the event \( H \) by Lemma 2.19.

**Definition 3.6.** Let \( \mu \) and \( \nu \) be two random, finite, Borel measures. We define

\[
\rho_\pi(\mu, \nu) = E \left( \frac{\pi(\mu, \nu)}{1 + \pi(\mu, \nu)} \right)
\]

where \( \pi \) is the Prohorov distance defined in Definition 2.72.

**Proposition 3.7.** We have that \( \rho_\pi \) is a metric and \( \mu_k \) weakly converges to \( \mu \) subsequentially in probability if and only if \( \lim_{k \to \infty} \rho_\pi(\mu_k, \mu) = 0 \).
\textbf{Proposition 3.8.} Let $\mu_k$ be a sequence of random, finite, Borel measures on $X$. If $\mu_k$ weakly converges to a random, finite, Borel measure $\mu$ subsequentially in probability then $\int_X f(x) \, d\mu_k(x)$ converges to $\int_X f(x) \, d\mu(x)$ in probability for every $f \in C_b(X)$.

\textbf{Proof.} Let $f \in C_b(X)$. For every subsequence $\{\alpha_k\}_{k=1}^\infty$ of $\mathbb{N}$ there exists a subsequence $\{\beta_k\}_{k=1}^\infty$ of $\{\alpha_k\}_{k=1}^\infty$ and an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\mu_{\beta_k}$ converges weakly to $\mu$ on the event $H$. Thus $\int_X f(x) \, d\mu_{\beta_k}(x)$ converges to $\int_X f(x) \, d\mu(x)$ almost surely. Hence $\int_X f(x) \, d\mu_k(x)$ converges to $\int_X f(x) \, d\mu(x)$ in probability by Remark 2.21. 

\textbf{Proposition 3.9.} Let $K \subseteq X$ be a compact subset, let $\Psi \subseteq C_b(K)$ be a countable dense subset with respect to the supremum norm and let $\mu$ and $\mu_k$ be a sequence of random, finite Borel measures on $K$. If $\int_X f(x) \, d\mu_k(x)$ converges to $\int_X f(x) \, d\mu(x)$ in probability for every $f \in \Psi$ then $\mu_k$ weakly converges to $\mu$ subsequentially in probability.

\textbf{Proof.} By Lemma 2.28 for every subsequence $\{\alpha_k\}_{k=1}^\infty$ of $\mathbb{N}$ there exists a subsequence $\{\beta_k\}_{k=1}^\infty$ of $\{\alpha_k\}_{k=1}^\infty$ and there exists an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\int_X f(x) \, d\mu_{\beta_k}(x)$ converges to $\int_X f(x) \, d\mu(x)$ for every $f \in \Psi$ on the event $H$. We have that $\mu_{\beta_k}$ weakly converges to a measure $\tau$ on the event $H$ by Lemma 2.27. We have that $\int_X f(x) \, d\mu(x) = \int_X f(x) \, d\tau(x)$ for every $f \in \Psi$ on the event $H$ and hence $\tau = \mu$ on the event $H$ by Lemma 2.21. So $\mu_{\beta_k}$ weakly converges to the measure $\mu$ on the event $H$. 

\textbf{Proposition 3.10.} Let $K \subseteq X$ be a compact subset, let $\Psi \subseteq C_b(K)$ be a countable dense subset with respect to the supremum norm and let $\mu_k$ be a sequence of random Borel measures on $K$ such that $\int_X f(x) \, d\mu_k(x)$ converges in probability to a random limit $S(f)$ for every $f \in \Psi$ and $|S(f)| < \infty$ almost surely. Then $\mu_k$ weakly converges to a random, finite, Borel measure $\mu$ subsequentially in probability.

\textbf{Proof.} By Lemma 2.28 there exists a subsequence $\{\beta_k\}_{k=1}^\infty$ of $\mathbb{N}$ and there exists an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\int_X f(x) \, d\mu_{\beta_k}(x)$ converges to $S(f)$ for every $f \in \Psi$ on the event $H$. Then $\mu_{\beta_k}$ weakly converges to a random, finite, Borel measure $\mu$ on the event $H$ by Lemma 2.27 and Lemma 3.2. Thus $\int_X f(x) \, d\mu_k(x)$ converges to $S(f) = \int_X f(x) \, d\mu(x)$ in probability for every $f \in \Psi$ and so $\mu_k$ weakly converges to $\mu$ subsequentially in probability by Proposition 3.9. 

\textbf{Proposition 3.11.} Let $\nu$ be a deterministic Borel measure on $X$ and $\mu_k$ be a sequence of random, finite, Borel measures on $X$ such that $\mu_k \ll \nu$ almost surely for every $k$, there exists $c > 0$ such that $E \left( \frac{\mu_k}{\nu}(x) \right) \leq c$ for every $k \in \mathbb{N}$, $x \in X$ and $\mu_k(A)$ converges in probability for every compact set $A \subseteq X$. Let $f : X \to \mathbb{R}$ be a Borel measurable function such that $\int_X |f(x)| \, d\nu(x) < \infty$. Then $\int_X f(x) \, d\mu_k(x)$ converges to a random variable $Y$ in probability and $E(|Y|) \leq c \int_X |f(x)| \, d\nu(x)$. 

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Proof. It is enough to prove the statement for a nonnegative $f$. We have that

$$
E \left( \int_X g(x) d\mu_k(x) \right) = E \left( \int_X g(x) \frac{d\mu_k}{d\nu}(x) d\nu(x) \right) \leq c \int_X g(x) d\nu(x) \tag{3.1}
$$

for every nonnegative Borel function $g$ by Fubini’s theorem. Let $g_n(x) = \sum_{i=1}^{N_n} b_{i,n} \cdot \chi_{A_{i,n}}(x)$, where $0 \leq b_{i,n} < \infty$, $N_n \in \mathbb{N}$ and $A_{i,n} \subseteq X$ are compact subsets, such that $g_n \leq f$ on $X$ and $0 \leq \int_X f(x) - g_n(x) d\nu(x) < 1/n$, note that by the definition of Lebesgue integration we can find such $g_n$ with $A_{i,n}$ being Borel sets and by inner regularity (Lemma 2.12) we can further assume the $A_{i,n}$ to be compact. By assumption $\int_X g_n(x) d\mu_k(x)$ converges in probability as $k$ goes to infinity to a random variable $Y_n$, thus

$$
\lim_{k \to \infty} \rho \left( \int_X g_n(x) d\mu_k(x), Y_n \right) = 0.
$$

Hence

$$
\limsup_{k \to \infty} \rho \left( \int_X f(x) d\mu_k(x), Y_n \right) \leq \limsup_{k \to \infty} \rho \left( \int_X f(x) d\mu_k(x), \int_X g_n(x) d\mu_k(x) \right) + \rho \left( \int_X g_n(x) d\mu_k(x), Y_n \right)
$$

$$
\leq \limsup_{k \to \infty} \mathbb{E} \left| \int_X f(x) d\mu_k(x) - \int_X g_n(x) d\mu_k(x) \right| + 0 \leq \limsup_{k \to \infty} \mathbb{E} \left( \int_X |f(x) - g_n(x)| d\mu_k(x) \right)
$$

$$
\leq c \int_X f(x) - g_n(x) d\nu(x) < c/n,
$$

and so there exists $m_n \in \mathbb{N}$ such that $\rho \left( \int_X f(x) d\mu_k(x), Y_n \right) < c/n$ for every $k \geq m_n$. If $k \geq \max \{m_n, m_l\}$ then

$$
\rho(Y_n, Y_l) \leq \rho \left( \int_X f(x) d\mu_k(x), Y_n \right) + \rho \left( \int_X f(x) d\mu_k(x), Y_l \right) \leq c/n + c/l.
$$

It follows that $Y_n$ is a Cauchy sequence for the metric $\rho$, which is a complete metric. Let $Y$ be the limit of $Y_n$ in probability. Then

$$
\limsup_{k \to \infty} \rho \left( \int_X f(x) d\mu_k(x), Y \right) \leq \limsup_{k \to \infty} \rho \left( \int_X f(x) d\mu_k(x), Y_n \right) + \rho(Y_n, Y) \leq c/n + \rho(Y_n, Y).
$$

By taking limit $n$ goes to infinity it follows that $\int_X f(x) d\mu_k(x)$ converges to $Y$ in probability. By applying Lemma 2.11 twice and by (3.1)

$$
E(Y) \leq \liminf_{n \to \infty} E(Y_n) \leq \liminf_{n \to \infty} \liminf_{k \to \infty} \int_X g_n(x) d\mu_k(x) \leq c \liminf_{n \to \infty} \int_X g_n(x) d\nu(x) \leq c \int_X f(x) d\nu(x).
$$

\[\square\]
Corollary 3.12. Let \( \nu \) be a deterministic Borel measure on \( X \) such that \( \text{supp} \nu \) is compact. Let \( \mu_k \) be a sequence of random, finite, Borel measures on \( X \) such that \( \mu_k \ll \nu \) almost surely for every \( k \), there exists \( c > 0 \) such that \( E \left( \frac{d\mu_k}{d\nu}(x) \right) \leq c \) for every \( k \in \mathbb{N}, x \in X \) and \( \mu_k(A) \) converges to a random variable \( \tau(A) \) in probability for every compact set \( A \subseteq X \). Then \( \mu_k \) weakly converges to a random, finite, Borel measure \( \mu \) subsequentially in probability.

Proof. There exists \( \Psi \subseteq C_b(\text{supp} \nu) \) countable and dense subset by Lemma 2.26. The conditions of Proposition 3.10 are satisfied for \( K = \text{supp} \nu \) by Proposition 3.11. Thus there exists a random Borel measure \( \mu \) such that \( \mu_k \) weakly converges to \( \mu \) subsequentially in probability. \( \square \)

Lemma 3.13. Let \( g : X \rightarrow \mathbb{R} \) be a nonnegative Borel function and let \( \nu \) be a deterministic finite Borel measure on \( X \). Let \( \mu_k \) be a sequence of random, finite, Borel measures on \( X \) such that \( \mu_k \ll \nu \) almost surely for every \( k \) and there exists \( c > 0 \) such that \( E \left( \frac{d\mu_k}{d\nu}(x) \right) \leq c \) for every \( k \in \mathbb{N}, x \in X \). If \( \mu_k \) weakly converges to a random, finite, Borel measure \( \mu \) subsequentially in probability then \( E \left( \int_X g(x)d\mu(x) \right) \leq c \cdot \int_X g(x)d\nu(x) \) and \( E \left( \int_X g(x)d\mu_k(x) \right) \leq c \cdot \int_X g(x)d\nu(x) \).

Proof. Let \( G \subseteq X \) be an open set. Then

\[
E \left( \int_X \chi_G(x)d\mu_k(x) \right) = E \left( \int_X \chi_G(x) \frac{d\mu_k}{d\nu}(x) d\nu(x) \right) \leq c \int_X \chi_G(x) d\nu(x)
\]

by Fubini’s theorem. Since it holds for every open set \( G \) it follows by Lemma 2.32 that

\[
E \left( \mu_k(A) \right) \leq c \cdot \nu(A)
\]

(3.2)

for every Borel set \( A \). Hence

\[
E \left( \int_X g(x)d\mu_k(x) \right) \leq c \cdot \int_X g(x)d\nu(x).
\]

Let again \( G \subseteq X \) be an open set. There exists a subsequence \( \{ \beta_k \}_{k=1}^{\infty} \) of \( \mathbb{N} \) and an event \( H \in \mathcal{A} \) with \( P(H) = 1 \) such that \( \mu_{\beta_k} \) weakly converges to \( \mu \) on the event \( H \). Then by Remark 2.16 Fatou’s lemma and (3.2)

\[
E \left( \mu(G) \right) \leq E \left( \liminf_{k \rightarrow \infty} \mu_{\beta_k}(G) \right) \\
\leq \liminf_{k \rightarrow \infty} E \left( \mu_{\beta_k}(G) \right) = c \cdot \nu(G).
\]

Since it holds for every open set \( G \) it follows by Lemma 2.32 that \( E \left( \mu(A) \right) \leq c \cdot \nu(A) \) for every Borel set \( A \) and hence

\[
E \left( \int_X g(x)d\mu(x) \right) \leq c \cdot \int_X g(x)d\nu(x).
\]

\( \square \)
Let \( \mu^i \) and \( \mu^i_k \) be a sequence of random, finite, Borel measures on \( X \) for every \( i \in \mathbb{N} \). Assume that \( \mu^i_k \) weakly converges to \( \mu^i \) subsequentially in probability for every \( i \in \mathbb{N} \) as \( k \) goes to \( \infty \). Assume that for every \( \varepsilon > 0 \) there exist \( N, n_0 \in \mathbb{N} \) such that \( \sum_{i=N}^{\infty} E(\mu^i_k(X)) < \varepsilon \) for every \( k \geq n_0 \). Then there exists \( n_1 \in \mathbb{N} \) such that \( \sum_{i=n_1}^{\infty} \mu^i_k \) is a sequence of random, finite Borel measures, for \( k \geq n_1 \), that weakly converges to the random, finite Borel measure \( \sum_{i \in \mathbb{N}} \mu^i \) subsequentially in probability.

Proof. Since there exists \( N_1, n_1 \in \mathbb{N} \) such that \( \sum_{i=N_1}^{\infty} E(\mu^i_k(X)) \leq 1 \) for every \( k \geq n_1 \) it follows that \( \sum_{i=N_1}^{\infty} \mu^i_k(X) < \infty \) almost surely for \( k \geq n_1 \) and so \( \sum_{i=1}^{\infty} \mu^i_k(X) < \infty \) almost surely for \( k \geq n_1 \). Thus \( \sum_{i=1}^{\infty} \mu^i_k \) is a random, finite, Borel measure.

Since \( \mu^i_k \) weakly converges to \( \mu^i \) subsequentially in probability it follows by \( 3.8 \) that \( \mu^i_k(X) \) converges to \( \mu^i(X) \) in probability as \( k \) goes to \( \infty \). Thus we can find, by Lemma \( 2.8 \) a subsequence \( \{\alpha_k\}_{k=1}^{\infty} \) of \( A \) and an event \( H \in A \) with \( P(H) = 1 \) such that \( \mu^i_{\alpha_k}(X) \) converges to \( \mu^i(X) \) on \( H \) as \( k \) goes to \( \infty \) for every \( i \in \mathbb{N} \). Then by Fatou’s lemma and Fubini’s theorem

\[
E\left(\sum_{i=N_1}^{\infty} \mu^i(X)\right) = E\left(\sum_{i=N_1}^{\infty} \lim_{k \to \infty} \mu^i_{\alpha_k}(X)\right) \leq \liminf_{k \to \infty} \sum_{i=N_1}^{\infty} E(\mu^i_{\alpha_k}(X)) \leq 1. \tag{3.3}
\]

Thus \( \sum_{i=N_1}^{\infty} \mu^i_k(X) < \infty \) almost surely and so \( \sum_{i=1}^{\infty} \mu^i(X) < \infty \) almost surely. Hence \( \sum_{i=1}^{\infty} \mu^i \) is a random, finite, Borel measure.

Let \( \varepsilon > 0 \) be fixed and let \( N, n_0 \in \mathbb{N} \) such that \( \sum_{i=N}^{\infty} E(\mu^i_k(X)) < \varepsilon \) for every \( k \geq n_0 \). Similarly to \( (3.3) \) we have that \( E(\sum_{i=N}^{\infty} \mu^i(X)) \leq \varepsilon \). Thus

\[
\rho(\sum_{i=1}^{\infty} \mu^i_k, \sum_{i=1}^{\infty} \mu^i) \
\leq \rho(\sum_{i=1}^{\infty} \mu^i_k, \sum_{i=1}^{N-1} \mu^i_k) + \rho(\sum_{i=1}^{\infty} \mu^i_k, \sum_{i=1}^{N-1} \mu^i) + \rho(\sum_{i=1}^{\infty} \mu^i, \sum_{i=1}^{N-1} \mu^i) \
\leq \varepsilon + \rho(\sum_{i=1}^{\infty} \mu^i_k, \sum_{i=1}^{N-1} \mu^i) + \varepsilon
\]

where we used the fact that \( \rho(\mu, \nu) \leq E(\pi(\mu, \nu)) \) and Lemma \( 2.21 \). Thus

\[
\limsup_{k \to \infty} \rho(\sum_{i=1}^{\infty} \mu^i_k, \sum_{i=1}^{\infty} \mu^i) \leq 2\varepsilon
\]

since \( \sum_{i=1}^{N-1} \mu^i_k \) weakly converges to \( \sum_{i=1}^{N-1} \mu^i \) subsequentially in probability. By taking limit \( \varepsilon \) goes to 0 it follows by Proposition \( (3.7) \) that \( \sum_{i=1}^{\infty} \mu^i_k \) weakly converges to \( \sum_{i=1}^{\infty} \mu^i \) subsequentially in probability.

Theorem 3.15. Let \( \nu \) be a deterministic, finite, Borel measure on \( X \). Let \( \mu_k \) be a sequence of random, finite, Borel measures on \( X \) such that \( \mu_k \ll \nu \) almost surely for every \( k \), there exists \( c > 0 \) such that \( E\left(\frac{d\mu_k}{d\nu}(x)\right) \leq c \) for every \( k \in \mathbb{N} \), \( x \in X \) and \( \mu_k(A) \)
converges to a random variable $\tau(A)$ in probability for every compact set $A \subseteq X$. Then $\mu_k$ weakly converges to a random, finite, Borel measure $\mu$ subsequentially in probability. Furthermore, $\int_X f(x) d\mu_k(x)$ converges to a random variable $S(f)$ in probability with $E(|S(f)|) \leq c \int_X |f(x)| d\nu(x)$ for every $f : X \to \mathbb{R}$ Borel measurable function such that $\int_X |f(x)| d\nu(x) < \infty$. For every countable collection of Borel measurable functions $f_n : X \to \mathbb{R}$ such that $\int_X |f_n(x)| d\nu(x) < \infty$ we have that $\int_X f_n(x) d\mu(x) = S(f_n)$ for every $n$ almost surely.

Proof. Let $K_1, K_2, \ldots$ be a sequence of disjoint compact subsets of $X$ as in Lemma 2.13. Let $\nu^i = \nu|_{K_i}$ and $\mu^i_k = \mu_k|_{K_i}$. Then $\mu_k = \sum_{i=1}^{\infty} \mu^i_k$ and $\mu^i_k$ weakly converges to a random, finite Borel measure $\mu^i$ subsequentially in probability by Corollary 3.12. Since $\sum_{i=N}^{\infty} E(\mu^i(X)) \leq c \sum_{i=N}^{\infty} \nu(K_i) < \infty$ it follows that the conditions of Proposition 3.14 are satisfied and so $\mu = \sum_{i=1}^{\infty} \mu^i$ is a random, finite, Borel measure and $\mu_k$ weakly converges to $\mu$ subsequentially in probability.

By Proposition 3.11 it follows that $\int_X f(x) d\mu_k(x)$ converges to a random variable $S(f)$ in probability with $E(|S(f)|) \leq c \int_X |f(x)| d\nu(x)$ for every $f : X \to \mathbb{R}$ Borel measurable function such that $\int_X |f(x)| d\nu(x) < \infty$.

Let $n \in \mathbb{N}$ be fixed. For every $\varepsilon > 0$ we can find $g \in C_b(X)$ such that $\int_X |f_n(x) - g(x)| d\nu(x) < \varepsilon$ by [10, Proposition 1.3.22]. Then

$$
\rho \left( \int_X f_n(x) d\mu(x), S(f_n) \right)
\leq \rho \left( \int_X f_n(x) d\mu(x), \int_X g(x) d\mu(x) \right) + \rho \left( \int_X g(x) d\mu(x), \int_X g(x) d\mu_k(x) \right)
+ \rho \left( \int_X g(x) d\mu_k(x), \int_X f_n(x) d\mu_k(x) \right) + \rho \left( \int_X f_n(x) d\mu_k(x), S(f_n) \right)
\leq E \left( \int_X |f_n(x) - g(x)| d\mu(x) \right) + \rho \left( \int_X g(x) d\mu(x), \int_X g(x) d\mu_k(x) \right)
+ E \left( \int_X |g(x) - f_n(x)| d\mu(x) \right) + \rho \left( \int_X f_n(x) d\mu_k(x), S(f_n) \right)
\leq c \cdot \int_X |f_n(x) - g(x)| d\nu(x) + \rho \left( \int_X g(x) d\mu(x), \int_X g(x) d\mu_k(x) \right) +
+ c \cdot \int_X |f_n(x) - g(x)| d\nu(x) + \rho \left( \int_X f_n(x) d\mu_k(x), S(f_n) \right)
$$
where we used Lemma 3.13 and Remark 2.6. Hence taking limit $k$ goes to infinity it follows that

$$\rho \left( \int_X f_n(x)d\mu(x), S(f_n) \right) \leq 2c \cdot \varepsilon$$

since $\int_X g(x)d\mu(x)$ converges to $\int_X g(x)d\mu_k(x)$ in probability by Proposition 3.8 and $\int_X f_n(x)d\mu_k(x)$ converges to $S(f_n)$ in probability by the definition of $S(f_n)$. Taking limit $\varepsilon$ goes to 0 it follows that $\int_X f_n(x)d\mu(x) = S(f_n)$ almost surely. \qed

Remark 3.16. In particular, if we take $f_n = \chi_{A_n}$ in Theorem 3.15 for a countable collection of Borel sets $A_n$ then it follows that $\mu(A_n) = S(\chi_{A_n})$ almost surely for every $n$ where $\mu_k(A_n)$ converges to $S(\chi_{A_n})$ in probability as $k$ goes to infinity.

Remark 3.17. In Theorem 3.15 we can relax the condition $E(\frac{d\mu_k}{d\nu}(x)) \leq c$ for every $k \in \mathbb{N}$, $x \in X$. It is enough to assume that there exists a nonnegative Borel function $c : X \rightarrow \mathbb{R}$ such that $\int c(x)d\nu(x) < \infty$ and $E(\frac{d\mu_k}{d\nu}(x)) \leq c(x)$ for $\nu$ almost every $x \in X$ for every $k$. It can be seen by replacing $d\nu(x)$ by $c(x)d\nu(x)$.

Proposition 3.18. Let $\mu_k$ be a sequence of random, finite, Borel measures on $X$ such that there exists a sequence of random closed sets $F_1 \supseteq F_2 \supseteq \ldots$ such that $\text{supp} \mu_k \subseteq F_k$ almost surely. If $\mu_k$ weakly converges to a random Borel measure $\mu$ subsequentially in probability then $\text{supp} \mu \subseteq \bigcap_{n=1}^{\infty} F_n$ almost surely.

Proof. Let $G_n = X \setminus F_n$ be a random open set. Let $\{\alpha_k\}_{k=1}^{\infty}$ be a subsequence of $\mathbb{N}$ such that $\mu_{\alpha_k}$ weakly converges to $\mu$ on the event $H \in \mathcal{A}$ and $P(H) = 1$. We can further assume that $\text{supp} \mu_k \subseteq F_k$ on the event $H$. Then $\mu(G_n) = \liminf_{k \to \infty} \mu_{\alpha_k}(G_n) = 0$ on the event $H$ by Remark 2.16. Hence $\mu(\bigcup_{n=1}^{\infty} G_n) = 0$ on the event $H$ and so $\text{supp} \mu \subseteq \bigcap_{n=1}^{\infty} F_n$ almost surely. \qed

Lemma 3.19. Let $\nu$ be a deterministic, finite, Borel measure on $X$ with compact support. Let $\mu_k$ be a sequence of random, finite, Borel measures on $X$ such that $\mu_k \ll \nu$ almost surely for every $k$, there exists $c > 0$ such that $E(\frac{d\mu_k}{d\nu}(x)) \leq c$ for every $k \in \mathbb{N}$, $x \in X$ and $\mu_k(A)$ converges to a random variable $\tau(A)$ in probability for every compact set $A \subseteq X$. Let $f : X \rightarrow \mathbb{R}$ be a nonnegative Borel measurable function such that $\int_X f(x)d\nu(x) < \infty$. Then the sequence of random, finite, Borel measures $f(x)d\mu_k(x)$ weakly converges to the random, finite, Borel measure $f(x)d\mu(x)$ subsequentially in probability where $\mu$ is the random, finite, Borel measure such that $\mu_k$ weakly converges to $\mu$ subsequentially in probability.

Proof. We can find a countable and dense $\Psi \subseteq C_b(\text{supp}(\nu))$ by Lemma 2.26. By Theorem 3.15 it follows that $\int_X g(x)f(x)d\mu_k(x)$ converges to $\int_X g(x)f(x)d\mu(x)$ in probability for every $g \in \Psi$. Thus the sequence of random, finite, Borel measures $f(x)d\mu_k(x)$ weakly converges to the random, finite, Borel measure $f(x)d\mu(x)$ subsequentially in probability by Proposition 3.9. \qed

Proposition 3.20. Let $\nu$ be a deterministic, finite, Borel measure on $X$. Let $\mu_k$ be a sequence of random, finite, Borel measures on $X$ such that $\mu_k \ll \nu$ almost surely for every
There exists $c > 0$ such that $E \left( \frac{d\mu_k}{d\nu}(x) \right) \leq c$ for every $k \in \mathbb{N}$, $x \in X$ and $\mu_k(A)$ converges to a random variable $\tau(A)$ in probability for every compact set $A \subseteq X$. Let $f : X \to \mathbb{R}$ be a nonnegative Borel measurable function such that $\int_X f(x) d\nu(x) < \infty$. Then the sequence of random, finite, Borel measures $f(x) d\mu_k(x)$ weakly converges to the random, finite, Borel measure $f(x) d\mu(x)$ subsequentially in probability where $\mu$ is the random, finite, Borel measure such that $\mu_k$ weakly converges to $\mu$ subsequentially in probability.

**Proof.** Let $K_1, K_2, \ldots$ be compact sets, $\nu^i = \nu|_{K_i}$, $\mu^i_k = \mu_k|_{K_i}$, $\mu^i$ and $\mu$ as in the proof of Theorem 3.15. Then by Lemma 3.19 it follows that the sequence of measures $f(x) d\mu^i_k(x)$ weakly converges to $f(x) d\mu^i(x)$ subsequentially in probability. Since

$$\sum_{i=N}^\infty E \left( \int f(x) d\mu^i_k(x) \right) \leq c \sum_{i=N}^\infty \int f(x) d\nu(x) = c \int_X f(x) d\nu(x) < \infty,$$

it follows by the application of Proposition 3.14 that $f(x) d\mu(x) = \sum_{i=1}^\infty f(x) d\mu^i(x)$ is a random, finite, Borel measure and $f(x) d\mu_k(x)$ weakly converges to $f(x) d\mu(x)$ subsequentially in probability. □

### 3.2 Weak convergence in probability

**Definition 3.21.** Let $\mu_k$ be a sequence of random, finite, Borel measures on $X$. We say that $\mu_k$ weakly converges to a random, finite, Borel measure $\mu$ in probability if $\int_X f(x) d\mu_k(x)$ converges to $\int_X f(x) d\mu(x)$ in probability for every deterministic, bounded, continuous function $f : X \to \mathbb{R}$.

**Proposition 3.22.** The convergence weakly in probability is induced by the topology which has base elements formed by finite intersection of sets in the form:

$$\left\{ \mu : \rho \left( \int_X f(x) d\mu(x), Y \right) < r \right\}$$

where $f \in C_b(X)$, $r > 0$ and $Y$ is a real-valued random variable with almost surely finite values.

Proposition 3.22 can be verified easily, we leave the details for the reader.

**Proposition 3.23.** Let $\mu$ and $\mu_k$ be a sequence of random, finite, Borel measures on $X$. If $\mu_k$ weakly converges to $\mu$ subsequentially in probability then $\mu_k$ weakly converges to $\mu$ in probability.

Proposition 3.23 is a reformulation of Proposition 3.8.

**Proposition 3.24.** Let $\mu$ and $\mu_k$ be a sequence of random, finite, Borel measures supported on a deterministic compact subset $K \subseteq X$. If $\mu_k$ weakly converges to $\mu$ in probability then $\mu_k$ weakly converges to $\mu$ subsequentially in probability.
Proof. We can find a countable and dense $\Psi \subseteq C_b(K)$ by Lemma 2.26. Hence the statement follows from Proposition 3.9.

Proposition 3.25. Let $\mu_k$ be a sequence of random, finite, Borel measures. Assume that $\tau$ and $\nu$ are random, finite, Borel measures on $X$ such that $\mu_k(G)$ converges to $\tau(G)$ in probability and $\mu_k(G)$ converges to $\nu(G)$ in probability for every open set $G \subseteq X$. Then $\mu = \tau$ almost surely.

Proof. Let $D \subseteq X$ be a countable and dense subset and let $G$ be as in Lemma 2.33. Let $G \in G$. Then $\mu_k(G)$ converges to both $\mu(G)$ and $\tau(G)$ in probability hence $\mu(G) = \tau(G)$ almost surely. Since $G$ is countable it follows that $\mu(G) = \tau(G)$ for every $G \in G$ almost surely, i.e. there exists an event $H \in A$ with $P(H) = 1$ such that $\mu(A) = \tau(A)$ for every Borel set $A \subseteq X$ by Lemma 2.33. Hence $\mu = \tau$ almost surely.

Proposition 3.26. Let $\mu^i$ and $\mu^i_k$ be a sequence of random, finite, Borel measures on $X$ for every $i \in \mathbb{N}$. Assume that $\mu^i_k$ weakly converges to $\mu^i$ in probability for every $i \in \mathbb{N}$ as $k$ goes to $\infty$. Assume that for every $\varepsilon > 0$ there exist $N, n_0 \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} E(\mu^i_k(X)) < \varepsilon$ for every $k \geq n_0$. Then there exists $n_1 \in \mathbb{N}$ such that $\sum_{i \in \mathbb{N}} \mu^i_k$ is a sequence of random, finite Borel measures, for $k \geq n_1$, that weakly converges to the random, finite Borel measure $\sum_{i \in \mathbb{N}} \mu^i$ in probability.

The proof of Proposition 3.26 goes similarly to the proof of Proposition 3.14 with the difference that instead of

$$\rho_n(\sum_{i=1}^{\infty} \mu^i_k(X), \sum_{i=1}^{\infty} \mu^i(X))$$

we need to estimate

$$\rho(\sum_{i=1}^{\infty} \int_X f(x) d\mu^i_k(x), \sum_{i=1}^{\infty} \int_X f(x) d\mu^i(x))$$

for a given $f \in C_b(X)$. We leave for the reader to check the details. We provide a similar proof in the proof of Proposition 3.40.

3.3 Vague convergence in probability

Throughout this section we assume that $X$ is locally compact.

Definition 3.27. The set of all locally finite Borel measures $\mathcal{M}_f(X)$ on $X$ equipped with the weak$^\ast$-topology on the dual space of $C_c(X)$ is a Polish space (see Lemma 2.31). A random, finite, Borel measure is an element of $L^0(\mathcal{M}_f(X))$, i.e. a locally finite Borel measure valued random variable.

Lemma 3.28. Let $\mu_k$ be a sequence of random, locally finite, Borel measures. If there exists $H \in A$ with $P(H) = 1$ such that for every outcome $\omega \in H$ we have that $\mu_k$ vaguely converges to a locally finite, Borel measure $\mu$ (note that $\mu$ depends on $\omega \in H$) then $\mu$ is a random, locally finite Borel measure.
The lemma follows from Lemma 2.7.

**Definition 3.29.** Let $\mu$ and $\mu_k$ be a sequence of random, locally finite, Borel measures on $X$. We say that $\mu_k$ vaguely converges to $\mu$ in probability if $\int_X f(x) d\mu_k(x)$ converges to $\int_X f(x) d\mu(x)$ in probability for every $f \in C_c(X)$.

**Remark 3.30.** It follows from Definition 3.24 that if $\mu_k$ is a sequence of random, locally finite, Borel measures on $X$ such that $\mu_k$ vaguely converges to a random Borel measure $\mu$ almost surely then $\mu_k$ vaguely converges to $\mu$ in probability.

**Remark 3.31.** It follows from Definition 3.24 that if $\mu_k$ is a sequence of random, finite, Borel measures on $X$ such that $\mu_k$ weakly converges to a random, finite, Borel measure $\mu$ in probability then $\mu_k$ vaguely converges to $\mu$ in probability.

**Proposition 3.32.** The convergence vaguely in probability is induced by the topology which has base elements formed by finite intersection of sets in the form:

$$\left\{ \mu : \rho \left( \int_X f(x) d\mu(x), Y \right) < r \right\}$$

where $f \in C_c(X)$, $r > 0$ and $Y$ is a real-valued random variable with almost surely finite values.

Proposition 3.32 can be verified easily, we leave the details for the reader.

**Proposition 3.33.** Let $X$ be locally compact. The limit in Definition 3.24 is unique in $\mathcal{L}^0(\mathcal{M}_t(X))$ if exists.

**Proof.** Let $\mu_k$ be a sequence of random, finite, Borel measures on $X$ such that $\mu_k$ vaguely converges to a random, finite, Borel measure $\mu$ and also to a random, finite, Borel measure $\tau$ in probability. Let $\Psi \subseteq C_c(X)$ as in Lemma 2.29. Then $\int_X f(x) d\mu_k(x)$ converges to $\int_X f(x) d\mu(x)$ in probability for every $f \in \Psi$ and also to $\int_X f(x) d\tau(x)$ in probability. Since $\Psi$ is countable it follows that $\int_X f(x) d\mu(x) = \int_X f(x) d\tau(x)$ for every $f \in \Psi$ almost surely. Thus $\mu = \nu$ almost surely by Lemma 2.30.

**Lemma 3.34.** Let $X$ be locally compact. Let $\mu$ and $\mu_k$ be a sequence of random, locally finite, Borel measures on $X$. Assume that for every subsequence $\{\alpha_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ there exists a subsequence $\{\beta_k\}_{k=1}^{\infty}$ of $\{\alpha_k\}_{k=1}^{\infty}$ and there exists an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\mu_k$ vaguely converges to $\mu$ on $H$ then $\int_X f(x) d\mu_k(x)$ converges to $\int_X f(x) d\mu(x)$ in probability for every $f \in C_c(X)$.

Lemma 3.34 can be proven similarly to the proof of Proposition 3.8.

**Lemma 3.35.** Let $X$ be locally compact, let $\Psi \subseteq C_c(X)$ as in Lemma 2.29 and $\mu$ and $\mu_k$ be a sequence of random, locally finite Borel measures on $X$. If $\int_X f(x) d\mu_k(x)$ converges to $\int_X f(x) d\mu(x)$ in probability for every $f \in \Psi$ then for every subsequence $\{\alpha_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ there exists a subsequence $\{\beta_k\}_{k=1}^{\infty}$ of $\{\alpha_k\}_{k=1}^{\infty}$ and there exists an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\mu_k$ vaguely converges to $\mu$ on the event $H$. 

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Lemma 3.35 can be shown similarly to the proof of Proposition 3.9 by replacing the use of Lemma 2.27 by the use of Lemma 2.29 and the use of Lemma 2.21 by Lemma 2.30.

Lemma 3.36. Let $X$ be locally compact, let $\Psi \subseteq C_c(X)$ as in Lemma 2.29 and $\mu_k$ be a sequence of random, locally finite Borel measures on $X$. Assume that $\int_X f(x) \, d\mu_k(x)$ converges in probability to a random finite limit $S(f)$ for every $f \in \Psi$. Then there exists a random, locally finite Borel measures $\mu$ on $X$ such that for every subsequence $\{\alpha_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ there exists a subsequence $\{\beta_k\}_{k=1}^{\infty}$ of $\{\alpha_k\}_{k=1}^{\infty}$ and there exists an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\mu_k$ vaguely converges to $\mu$ on the event $H$.

Lemma 3.36 can be shown similarly to the proof Proposition 3.10 by replacing the use of Lemma 2.27 by the use of Lemma 2.29, the use of Lemma 2.21 by Lemma 2.30 and the use of Proposition 3.9 by the use of Lemma 3.35.

Theorem 3.37. Let $X$ be locally compact and let $\mu$ and $\mu_k$ be a sequence of random, locally finite Borel measures on $X$. Then $\mu_k$ vaguely converges to $\mu$ in probability if and only if for every subsequence $\{\alpha_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ there exists a subsequence $\{\beta_k\}_{k=1}^{\infty}$ of $\{\alpha_k\}_{k=1}^{\infty}$ and there exists an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\mu_k$ vaguely converges to $\mu$ on the event $H$.

Theorem 3.37 follows from Lemma 3.34 and Lemma 3.35.

Remark 3.38. Let $X$ be locally compact. Proposition 3.32 states that the vague convergence in probability is a topological convergence. Due to Theorem 3.37 the ‘convergence vaguely subsequentially in probability’ and the convergence vaguely in probability are the same convergence. Hence it can be shown similarly to Proposition 3.31 that convergence vaguely in probability is also a metrizable convergence using the fact that the vague convergence of deterministic locally finite measures is a metric convergence, see Proposition 2.31.

Proposition 3.39. Let $X$ be locally compact. Let $\nu$ be a deterministic, locally finite, Borel measure on $X$ and $\mu_k$ be a sequence of random, locally finite, Borel measures on $X$ such that $\mu_k \ll \nu$ almost surely for every $k$, there exists $c > 0$ such that $E\left(\frac{d\mu_k}{d\nu}(x)\right) \leq c$ for every $k \in \mathbb{N}$, $x \in X$ and $\mu_k(A)$ converges in probability for every compact set $A \subseteq X$. Let $f : X \to \mathbb{R}$ be a Borel measurable function such that $\int_X |f(x)| \, d\nu(x) < \infty$. Then $\int_X f(x) \, d\mu_k(x)$ converges to a random variable $Y$ in probability and $E(|Y|) \leq c \int_X |f(x)| \, d\nu(x)$.

The proof of Proposition 3.39 is identical to the proof of Proposition 3.11.

Proposition 3.40. Let $X$ be locally compact. Let $\mu^i$ and $\mu^i_k$ be a sequence of random, locally finite, Borel measures on $X$ for every $i \in \mathbb{N}$. Assume that $\mu_k^i$ vaguely converges to $\mu^i$ in probability for every $i \in \mathbb{N}$ as $k$ goes to $\infty$. Assume that for every compact set $K \subseteq X$ and $\varepsilon > 0$ there exist $N \in \mathbb{N}$ such that $\sum_{i=1}^{\infty} E(\mu_k^i(K)) < \varepsilon$ for every $k \in \mathbb{N}$. Then $\sum_{i \in \mathbb{N}} \mu^i_k$ is a sequence of random, locally finite Borel measures that vaguely converges to the random, locally finite Borel measure $\sum_{i \in \mathbb{N}} \mu^i$ in probability.
Proof. Since for every compact set $K \subseteq X$ there exists $N_1 \in \mathbb{N}$ such that $\sum_{i=N_1}^{\infty} E(\mu_i^k(K)) \leq 1$ for every $k \in \mathbb{N}$ it follows that $\sum_{i=N_1}^{\infty} \mu_i^k(K) < \infty$ almost surely for $k \in \mathbb{N}$ and so $\sum_{i=1}^{\infty} \mu_i^k(K) < \infty$ almost surely. Thus $\sum_{i=1}^{\infty} \mu_i^k$ is a random, locally finite, Borel measure since $X$ is locally compact. Let $h \in C_c(X)$ such that $h(x) = 1$ for every $x \in K$ and $\|h\|_{\infty} \leq 1$, we can find such $h$ by Tietze’s extension theorem and the fact that $X$ is locally compact. We can find $N_2 \in \mathbb{N}$ such that $\sum_{i=N_2}^{\infty} E(\mu_i^k(\text{supp}(h))) \leq 1$ for every $k \in \mathbb{N}$. We can find, by Lemma 2.8, a subsequence $\{\alpha_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ and an event $H \in \mathcal{A}$ with $P(H) = 1$ such that $\int_X h(x) \text{d}\mu_{\alpha_k}(x)$ converges to $\int_X h(x) \text{d}\mu(x)$ on $H$ as $k$ goes to $\infty$ for every $i \in \mathbb{N}$. Then by Fatou’s lemma and Fubini’s theorem

$$E(\sum_{i=N_2}^{\infty} \mu_i^k(K)) \leq E(\sum_{i=N_2}^{\infty} \int_X h(x) \text{d}\mu_i^k(x)) = E(\sum_{i=N_2}^{\infty} \lim_{k \to \infty} \int_X h(x) \text{d}\mu_i^k(x))$$

$$\leq \liminf_{k \to \infty} \sum_{i=N_2}^{\infty} E(\int_X h(x) \text{d}\mu_i^k(x)) \leq \liminf_{k \to \infty} \sum_{i=N_2}^{\infty} E(\mu_i^k(\text{supp}(h))) \leq 1. \quad (3.4)$$

Thus $\sum_{i=N_1}^{\infty} \mu_i^k(K) < \infty$ almost surely and so $\sum_{i=1}^{\infty} \mu_i^k(K) < \infty$ almost surely. Hence $\sum_{i=1}^{\infty} \mu_i^k$ is a random, locally finite, Borel measure.

Let $f \in C_c(X)$ and $K = \text{supp}(f)$. Let $\varepsilon > 0$ be fixed and let $N \in \mathbb{N}$ such that $\sum_{i=N}^{\infty} E(\mu_i^k(K)) < \varepsilon$ for every $k \in \mathbb{N}$. Similarly to (3.1) we can further assume that $E(\sum_{i=N}^{\infty} \mu_i^k(K)) \leq \varepsilon$. Thus

$$\rho(\sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x), \sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x)) \leq \rho(\sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x), \sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x)) + \rho(\sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x), \sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x))$$

$$+ \rho(\sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x), \sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x))$$

$$\leq \varepsilon \|f\|_{\infty} + \rho(\sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x), \sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x)) + \varepsilon \|f\|_{\infty}$$

where we used the fact that $\rho(Y, Z) \leq E(|Y - Z|)$. Thus

$$\limsup_{k \to \infty} \rho(\sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x), \sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x)) \leq 2\varepsilon \|f\|_{\infty}$$

since $\sum_{i=1}^{N-1} \int_X f(x) \text{d}\mu_i^k(x)$ converges to $\sum_{i=1}^{N-1} \int_X f(x) \text{d}\mu_i^k(x)$ in probability. By taking limit $\varepsilon$ goes to 0 it follows that $\sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x)$ converges to $\sum_{i=1}^{\infty} \int_X f(x) \text{d}\mu_i^k(x)$ in probability. \qed
Theorem 3.41. Let $X$ be locally compact. Let $\nu$ be a deterministic, locally finite, Borel measure on $X$. Let $\mu_k$ be a sequence of random, locally finite, Borel measures on $X$ such that $\mu_k \ll \nu$ almost surely for every $k$, there exists $c > 0$ such that $E\left(\frac{d\mu_k}{d\nu}(x)\right) \leq c$ for every $k \in \mathbb{N}$, $x \in X$ and $\mu_k(A)$ converges to a random variable $\tau(A)$ in probability for every compact set $A \subseteq X$. Then $\mu_k$ vaguely converges to a random, locally finite, Borel measure $\mu$ in probability. Furthermore, $\int_X f(x)d\mu_k(x)$ converges to a random variable $S(f)$ in probability with $E(|S(f)|) \leq c\int_X |f(x)|d\nu(x)$ for every $f : X \to \mathbb{R}$ Borel measurable function such that $\int_X |f(x)|d\nu(x) < \infty$. For every countable collection of Borel measurable functions $f_n : X \to \mathbb{R}$ such that $\int_X |f_n(x)|d\nu(x) < \infty$ we have that $\int_X f_n(x)d\mu(x) = S(f_n)$ almost surely for every $n$.

Proof. Due to Remark 2.14 there exists a sequence of disjoint compact sets $A_i$ such that $\nu^i = \nu|_{A_i}$ is a finite measure for every $i \in \mathbb{N}$ and $\nu(X \setminus (\cup_{i \in \mathbb{N}}A_i)) = 0$. Let $\mu^i_k = \mu_k|_{A_i}$. Then $\mu_k = \sum_{i=1}^{\infty} \mu^i_k$ and $\mu^i_k$ vaguely converges to a random, finite Borel measure $\mu^i$ in probability by Remark 3.31 and Theorem 3.15. Since $\sum_{i=1}^{\infty} E(\mu^i_k(K)) \leq c\sum_{i=1}^{\infty} \nu(K) < \infty$ for every compact set $K$ it follows that the conditions of Proposition 3.40 are satisfied and so $\mu = \sum_{i=1}^{\infty} \mu^i$ is a random, locally finite, Borel measure and $\mu_k$ vaguely converges to $\mu$ in probability.

By Proposition 3.39 it follows that $\int_X f(x)d\mu_k(x)$ converges to a random variable $S(f)$ in probability with $E(|S(f)|) \leq c\int_X |f(x)|d\nu(x)$ for every $f : X \to \mathbb{R}$ Borel measurable function such that $\int_X |f(x)|d\nu(x) < \infty$.

Let $n \in \mathbb{N}$ be fixed. For every $\varepsilon > 0$ we can find $g \in C_c(X)$ such that $\int_X |f_n(x) - g(x)|d\nu(x) < \varepsilon$ by [15] Theorem 3.14. The rest of the proof proceeds similarly to the proof of Theorem 3.14.

Remark 3.42. In particular, if we take $f_n = \chi_{A_n}$ in Theorem 3.41 for a countable collection of Borel sets $A_n$ such that $\nu(A_n) < \infty$ then it follows that $\mu(A_n) = S(\chi_{A_n})$ almost surely for every $n$ where $\mu_k(A_n)$ converges to $S(\chi_{A_n})$ in probability as $k$ goes to infinity.

Remark 3.43. In Theorem 3.41 we can relax the condition $E(\frac{d\mu_k}{d\nu}(x)) \leq c$ for every $k$. It is enough to assume that there exists a nonnegative Borel function $c : X \to \mathbb{R}$ such that $\int_K c(x)d\nu(x) < \infty$ for every compact set $K$ and $E\left(\frac{d\mu_k}{d\nu}(x)\right) \leq c(x)$ for $\nu$ almost every $x \in X$. It can be seen by replacing $d\nu(x)$ by $c(x)d\nu(x)$.

Proposition 3.44. Let $\mu_k$ be a sequence of random, locally finite, Borel measures on $X$ such that there exists a sequence of random closed sets $F_1 \supseteq F_2 \supseteq \ldots$ such that $\text{supp} \mu_k \subseteq F_k$ almost surely. If $\mu_k$ vaguely converges to a random, locally finite, Borel measure $\mu$ in probability then $\text{supp} \mu \subseteq \bigcap_{n=1}^{\infty} F_n$ almost surely.

Proposition 3.44 can be shown similarly to the proof Proposition 3.18 due to the equivalence in Theorem 3.37.

Proposition 3.45. Let $X$ be locally compact. Let $\nu$ be a deterministic, locally finite, Borel measure on $X$. Let $\mu_k$ be a sequence of random, locally finite, Borel measures on $X$ such that $\mu_k \ll \nu$ almost surely for every $k$, there exists $c > 0$ such that $E\left(\frac{d\mu_k}{d\nu}(x)\right) \leq c$ for every $k \in \mathbb{N}$, $x \in X$ and $\mu_k(A)$ converges to a random variable $\tau(A)$ in probability for every compact set $A \subseteq X$. Let $f : X \to \mathbb{R}$ be a nonnegative Borel measurable function
such that for every \( y \in X \) there exists a neigbourhood \( U \) of \( y \) such that \( \int_U f(x) \, d\nu(x) < \infty \). Then the sequence of random, locally finite, Borel measures \( f(x) \, d\mu_k(x) \) vaguely converges to the random, locally finite, Borel measure \( f(x) \, d\mu(x) \) in probability where \( \mu \) is the random, locally finite, Borel measure such that \( \mu_k \) weakly converges to \( \mu \) subsequentially in probability.

**Proof.** Let \( U \) be an open set such that \( \int_U f(x) \, d\nu(x) < \infty \). Then \( E(\int_U f(x) \, d\mu_k(x)) \leq c \int_U f(x) \, d\nu(x) < \infty \) and by Theorem 3.41 we have that \( E(\int_U f(x) \, d\mu(x)) \leq c \int_X |f(x)| \, d\nu(x) < \infty \). Hence \( f(x) \, d\mu_k(x) \) and \( f(x) \, d\mu(x) \) are random, locally finite, Borel measures.

Let \( \Psi \subseteq C_c(X) \) as in Lemma 2.29. By Theorem 3.41 it follows that \( \int_X g(x) f(x) \, d\mu_k(x) \) converges to \( \int_X g(x) f(x) \, d\mu(x) \) in probability for every \( g \in \Psi \).

By the application of Theorem 3.41 to the reference measure \( f(x) \, d\nu(x) \) and the sequence \( f(x) \, d\mu_k(x) \) it follows that the sequence of random, locally finite, Borel measures \( f(x) \, d\mu_k(x) \) converges to a random, locally finite, Borel measure \( \tau \) and \( \int_X g(x) f(x) \, d\mu_k(x) \) converges to \( \int_X g(x) \, d\tau(x) \) in probability for every \( g \in \Psi \).

Thus \( \int_X g(x) f(x) \, d\mu(x) = \int_X g(x) \, d\tau(x) \) for every \( g \in \Psi \) almost surely. Hence \( f(x) \, d\mu(x) = \, d\tau(x) \) almost surely by Lemma 2.30.

### 3.4 General existence of the conditional measure

In this section our main goal is to prove Theorem [1.10](#) and Proposition [1.9](#) The proof of Proposition [1.9](#) can be found at the end of the section and Theorem [1.10](#) follows from Theorem 3.49, Theorem 3.52 and Lemma 3.50.

**Lemma 3.46.** Let \( A \subseteq X \) be a Borel set \( \nu = \sum_{i=1}^{\infty} \nu^i \) be a finite Borel measure and \( \mu_k^i \) be random finite Borel measures for every \( i \in \mathbb{N} \) such that \( \mu_k^i(A) \) converges in \( \mathcal{L}^1 \) to a random variable \( \mu^i(A) \) for every \( i \in \mathbb{N} \) and \( E(\mu_k^i(A)) = \nu^i(A) \). Then \( \sum_{i=1}^{\infty} \mu_k^i(A) \) converges in \( \mathcal{L}^1 \) to \( \sum_{i=1}^{\infty} \mu^i(A) \) as \( k \) goes to \( \infty \).

**Proof.** Since \( \mu_k^i(A) \) converges to \( \mu^i(A) \) in \( \mathcal{L}^1 \) it follows that \( E(\mu^i(A)) = E(\mu_k^i(A)) = \nu^i(A) \). Thus by Fubini’s theorem that

\[
E(\sum_{i=n}^{\infty} \mu_k^i(A)) = \sum_{i=n}^{\infty} \nu^i(A) = E(\sum_{i=n}^{\infty} \mu^i(A))
\]

for every \( n \in \mathbb{N} \) and \( \sum_{i=1}^{\infty} \nu^i(A) = \nu(A) < \infty \). Let \( \eta > 0 \) and let \( n \in \mathbb{N} \) be large enough that \( \sum_{i=n}^{\infty} \nu^i(A) < \eta \). By assumption \( \sum_{i=1}^{n} \mu_k^i(A) \) converges in \( \mathcal{L}^1 \) to \( \sum_{i=1}^{n} \mu^i(A) \), hence converges in probability. Thus

\[
\lim_{k \to \infty} \rho \left( \sum_{i=1}^{\infty} \mu_k^i(A), \sum_{i=1}^{\infty} \mu^i(A) \right) \leq
\]

\[
\lim_{k \to \infty} \rho \left( \sum_{i=1}^{\infty} \mu_k^i(A), \sum_{i=1}^{n} \mu_k^i(A) \right) + \lim_{k \to \infty} \rho \left( \sum_{i=1}^{n} \mu_k^i(A), \sum_{i=1}^{\infty} \mu^i(A) \right) + \rho \left( \sum_{i=1}^{n} \mu^i(A), \sum_{i=1}^{\infty} \mu^i(A) \right) \leq
\]

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Hence by taking limit \( \eta \) goes to 0 it follows that \( \sum_{i=1}^{\infty} \mu_k^i(A) \) converges in probability to \( \sum_{i=1}^{\infty} \mu^i(A) \). Thus the statement follows from Lemma 2.10.

\[ \limsup_{k \to \infty} E \left( \sum_{i=1}^{\infty} \mu_k^i(A) \right) + 0 + E \left( \sum_{i=1}^{\infty} \mu^i(A) \right) = 2 \sum_{i=1}^{\infty} \nu^i(A) \leq 2\eta. \]

Lemma 3.47. Let \( \nu \) be a finite Borel measure on \( X \) such that \( \nu(X \setminus X_0) = 0 \) and let \( A \subseteq X \) be a Borel set. Assume that \( C_k(\nu|_D)(X) = \mathcal{C}_k(\nu)(D) \) converges in \( L^1 \) for every compact set \( D \subseteq X_0 \) with \( I_\varphi(\nu|_D) < \infty \). Assume that \( \mathcal{C}_k(\nu_\perp)(X) \) converges to 0 in probability. Then \( C_k(\nu_R)(A) \) converges in \( L^1 \) to a limit \( \mu(A) \) and \( \mathcal{C}_k(\nu)(A) \) converges in probability to the same limit \( \mu(A) \) and \( E(\mu(A)) = \nu_R(A) \leq \nu(A) \).

Proof. Let us take a sequence \( (A_n)_{n \in \mathbb{N}} \) as in Proposition 1.13 for the measure \( \nu|_A \) in place of \( \nu \). Note that we can assume that \( A_n \subseteq A \cap X_0 \) for every \( n \in \mathbb{N} \) because \( \nu_A(X \setminus X_0) = 0 \). By Lemma 2.13 we can further assume that all the \( A_n \) are compact. Then \( \nu|_A = \nu_\perp|_A + \sum_{n \in \mathbb{N}} \nu|_{A_n} \) and \( I_\varphi(\nu|_{A_n}) < \infty \). By the assumption of the statement it follows that \( C_k(\nu|_{A_n})(X) = \mathcal{C}_k(\nu)(A_n) \) converges in \( L^1 \) for every \( n \in \mathbb{N} \) as \( k \) goes to \( \infty \). By (1.13) it follows that \( E(C_k(\nu)(A_n)) = \nu(A_n) \). Thus \( \mathcal{C}_k(\nu_R)(A) = \sum_{n \in \mathbb{N}} \mathcal{C}_k(\nu|_{A_n})(X) \) converges in \( L^1 \) to a limit \( \mu(A) \) by Lemma 3.40. Thus \( \mathcal{C}_k(\nu_R)(A) \) converges in probability to \( \mu(A) \) and \( E(\mu(A)) = \nu_R(A) \). Since \( \mathcal{C}_k(\nu_\perp)(A) \leq \mathcal{C}_k(\nu_\perp)(X) \) we have that \( \mathcal{C}_k(\nu_\perp)(A) \) converges to 0 in probability. Thus \( \mathcal{C}_k(\nu)(A) \) converges in probability to \( \mu(A) \).

Proposition 3.48. Let \( \nu \) be a finite Borel measure on \( X \) such that \( \nu(X \setminus X_0) = 0 \). Assume that \( \mathcal{C}_k(\nu|_D)(X) = \mathcal{C}_k(\nu)(D) \) converges in \( L^1 \) for every compact set \( D \subseteq X_0 \) with \( I_\varphi(\nu|_D) < \infty \). Assume that \( \mathcal{C}_k(\nu_\perp)(X) \) converges to 0 in probability. Then the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( \mathcal{Q}_k \) \( (k \geq 1) \) with regularity kernel \( \varphi \).

Proof. From the definition of \( \mathcal{C}_k(\nu) \) it follows that that \( E \left( \frac{d\mathcal{C}_k(\nu)}{d\varphi}(x) \right) = 1 \) for every \( x \in X_0 \), i.e. for \( \nu \) almost every \( x \in X \). The assumptions of Theorem 3.15 are satisfied by Lemma 3.47. Hence Property i.), ii.), iv.) of Definition 1.8 hold by Theorem 3.15 and Remark 3.16 Property v.) of Definition 1.8 hold by Remark 3.47 and Remark 3.16 Property vii.), viii.) of Definition 1.8 hold by the fact that \( \mathcal{C}_k(\nu_\perp)(X) \) converges to 0 in probability. Property ix.) of Definition 1.8 holds by Proposition 3.14 and the fact that the limit is unique by Proposition 3.5. Property x.) of Definition 1.8 holds by Proposition 3.20 Property xi.) of Definition 1.8 holds by Proposition 3.18.

Now we show Property ii.) of Definition 1.8. Let \( f : X \to \mathbb{R} \) be a Borel measurable function such that \( \int_X |f(x)| d\nu(x) < \infty \). It follows from Theorem 3.15 that \( \int_X f(x) d\mathcal{C}_k(\nu)(x) \) converges to a random variable \( S(f) \) in probability with \( E(|S(f)|) \leq \int_X |f(x)| d\nu(x) \). Since v.) holds it follows that \( \nu_R(A) = E(\mathcal{C}(\nu)(A)) \) for every Borel set \( A \subseteq X \) and so \( E(\int_X f(x) d\mathcal{C}(\nu)(x)) = \int_X f(x) d\nu_R(x) \). On the other hand \( \int_X f(x) d\mathcal{C}(\nu)(x) = S(f) \) almost surely by iv.). Hence ii.) holds.

Theorem 3.49. Let \( \nu \) be a finite Borel measure on \( X \) such that \( \nu(X \setminus X_0) = 0 \). The conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( \mathcal{Q}_k \) \( (k \geq 1) \) with regularity kernel \( \varphi \) if and only if \( \mathcal{C}_k(\nu|_D)(X) = \mathcal{C}_k(\nu)(D) \) converges in \( L^1 \) for every compact set \( D \subseteq X_0 \) with \( I_\varphi(\nu|_D) < \infty \) and \( \mathcal{C}_k(\nu_\perp)(X) \) converges to 0 in probability.
Proof. Assume that the conditional measure \( C(\nu) \) of \( \nu \) on \( B \) exists with respect to \( Q_k \) \((k \geq 1)\) with regularity kernel \( \varphi \). Then by Property vii.) of Definition 1.8 it follows that \( C_k(\nu_\perp)(X) \) converges to 0 in probability. Let \( D \subseteq X_0 \) be a compact set such that \( I_\varphi(\nu|D) < \infty \). Then \( \nu|D = \nu|D \) and so \( C_k(\nu)(D) = \nu(\nu_R)(D) \). Thus by Property vi.) of Definition 1.8 it follows that \( C_k(\nu)(D) \) converges to \( C(\nu)(D) \) in \( \mathcal{L}^1 \).

The other direction of the equivalence follows from Proposition 3.48 \( \square \)

Lemma 3.50. Let \( \nu \) be a finite Borel measure on \( X \) such that \( \nu(X \setminus X_0) = 0 \). Then \( C_k(\nu_\perp)(X) \) converges to 0 in probability if and only if \( C_k(\nu_\perp)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \).

Proof. If \( C_k(\nu_\perp)(X) \) converges to 0 in probability then of course \( C_k(\nu_\perp)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \).

Assume that \( C_k(\nu_\perp)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \). Let \( \varepsilon > 0 \) be fixed. Let \( D \subseteq X \) be a compact set such that \( \nu(X \setminus D) < \varepsilon \), which we can choose by inner regularity (Lemma 2.12). Then by the fact that \( \rho(Y,Z) \leq E(|Y-Z|) \), by (1.15) and Lemma 3.13 it follows that

\[
\rho(C_k(\nu_\perp)(X),0) \leq \rho(C_k(\nu_\perp)(X),C_k(\nu_\perp)(D)) + \rho(C_k(\nu_\perp)(D),0) \\
E(C_k(\nu_\perp)(X) + C_k(\nu_\perp)(D)) + \rho(C_k(\nu_\perp)(D),0) \leq \nu(X \setminus D) + \rho(C_k(\nu_\perp)(D),0).
\]

Hence

\[\limsup_{k \to \infty} \rho(C_k(\nu_\perp)(X),0) \leq \nu(X \setminus D) \leq \varepsilon \]

because \( C_k(\nu_\perp)(D) \) converges to 0 in probability. It holds for every \( \varepsilon > 0 \) thus \( C_k(\nu_\perp)(X) \) converges to 0 in probability. \( \square \)

Proposition 3.51. Assume that \( C_k(\nu|D)(X) = C_k(\nu)(D) \) converges in \( \mathcal{L}^1 \) for every compact set \( D \subseteq X_0 \) with \( I_\varphi(\nu|D) < \infty \). Assume that \( C_k(\nu_\perp)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \). Then the conditional measure \( C(\nu) \) of \( \nu \) on \( B \) exists with respect to \( Q_k \) \((k \geq 1)\) with regularity kernel \( \varphi \).

Proof. From the definition of \( C_k(\nu) \) it follows that that \( E \left( \frac{dC_k(\nu)}{d\nu}(x) \right) = 1 \) for every \( x \in X_0 \), i.e. for \( \nu \) almost every \( x \in X \). The assumptions of Theorem 3.41 are satisfied by Lemma 3.47 because \( \nu|A \) is a finite Borel measure for every Borel set \( A \subseteq X \) with \( \nu(A) < \infty \), in particular if \( A \) is compact. Hence Property i*.), iii.), iv.) of Definition 1.8 hold by Theorem 3.41 and Remark 3.42 Property v.), vi.) of Definition 1.8 hold by Lemma 3.47 and Remark 3.42 Property vii.), viii.) of Definition 1.8 hold by the fact that

\[0 \leq \left| \int f(x)dC_k(\nu_\perp)(x) \right| \leq \|f\|_\infty C_k(\nu_\perp)(\text{supp}(f)) \]

converges to 0 in probability for every \( f \in C_c(X) \). Property ix.) of Definition 1.8 holds by Proposition 3.40 and the fact that the limit is unique by Proposition 3.33 Property x*.) of Definition 1.8 holds by Proposition 3.45 Property xi.) of Definition 1.8 holds by Proposition 3.44
Now we show Property ii.) of Definition 1.8. Let \( f : X \to \mathbb{R} \) be a Borel measurable function such that \( \int_X |f(x)| \, dv(x) < \infty \). It follows from Theorem 3.41 that \( \int_X f(x) \, d\mathcal{C}_k(\nu)(x) \) converges to a random variable \( S(f) \) in probability with \( E(|S(f)|) \leq \int_X |f(x)| \, dv(x) \). Since \( v. \) holds it follows that \( \nu_R(A) = E(\mathcal{C}(\nu)(A)) \) for every Borel set \( A \subseteq X \) with \( \nu(A) < \infty \) and so \( E(\int_X f(x) \, d\mathcal{C}(\nu)(x)) = \int_X f(x) \, dv_R(x) \). On the other hand \( \int_X f(x) \, d\mathcal{C}(\nu)(x) = S(f) \) almost surely by iii.). Hence ii.) holds.

**Theorem 3.52.** Let \( X \) be locally compact. Let \( \nu \) be a locally finite Borel measure on \( X \) such that \( \nu(X \setminus X_0) = 0 \). Then the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( \mathcal{Q}_k \) \((k \geq 1)\) with regularity kernel \( \varphi \), if and only if \( \mathcal{C}_k(\nu_D) = \mathcal{C}_k(\nu)(D) \) converges in \( \mathcal{L}^1 \) for every compact set \( D \subseteq X_0 \) with \( I_\varphi(\nu|_D) < \infty \) and \( \mathcal{C}_k(\nu_\perp)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \).

**Proof.** Assume that the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( \mathcal{Q}_k \) \((k \geq 1)\) with regularity kernel \( \varphi \). Then by Property ii.), iv.) and vii.) of Definition 1.8 it follows that \( \mathcal{C}_k(\nu_\perp)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \). Let \( D \subseteq X \) be a compact subset such that \( I_\varphi(\nu|_D) < \infty \). Then \( \nu_R|_D = \nu|_D \) and so \( \mathcal{C}_k(\nu)(D) = \mathcal{C}_k(\nu_R)(D) \). Thus by Property vi.) of Definition 1.8 it follows that \( \mathcal{C}_k(\nu)(D) \) converges to \( \mathcal{C}(\nu)(D) \) in \( \mathcal{L}^1 \).

The other direction of the equivalence follows from Proposition 3.51.

**Proof of Proposition 1.9** By Theorem 3.49 Theorem 3.52 and Lemma 3.50 according to both of the two definitions of the conditional measure in Definition 1.8 the conditional measure exists if and only if \( \mathcal{C}_k(\nu|_D)(X) = \mathcal{C}_k(\nu)(D) \) converges in \( \mathcal{L}^1 \) for every compact set \( D \subseteq X_0 \) with \( I_\varphi(\nu|_D) < \infty \) and \( \mathcal{C}_k(\nu_\perp)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \). The limit in Property i.) and in Property i*.) are the same almost surely because if \( \mathcal{C}_k(\nu) \) weakly converges to \( \mathcal{C}(\nu) \) subsequentially in probability then \( \mathcal{C}_k(\nu) \) vaguely converges to \( \mathcal{C}(\nu) \) in probability by Remark 3.31 and Proposition 3.23 and since the limit is unique by Proposition 3.33 the two limits are the same almost surely.

## 4 Decomposition of measure

We prove Proposition 1.5 and Proposition 1.7 in this section.

**Proposition 4.1.** Let \( \nu \) be a finite, Borel measure on \( X \). There exist two finite, Borel measures \( \nu_\varphi_R = \nu_R \) and \( \nu_\varphi_\perp = \nu_\perp \) with the following properties:

i) \( \nu = \nu_R + \nu_\perp \)

ii) \( \nu_R \perp \nu_\perp \)

iii) \( \nu_\perp \) is singular to every finite Borel measure with finite \( \varphi \)-energy

iv) there exists a sequence of disjoint Borel sets \( (A_n)_{n \in \mathbb{N}} \) such that \( \nu_R = \nu_{|\bigcup_{n \in \mathbb{N}} A_n} = \sum_{n \in \mathbb{N}} \nu|_{A_n} \) and \( I_\varphi(\nu|_{A_n}) = I_\varphi(\nu_R|_{A_n}) < \infty \).

**Proof.** Let

\[
c_{\text{max}} = \sup \{ \nu(B) : B \subseteq X \text{ is a Borel set}, B \subseteq \bigcup_{n \in \mathbb{N}} B_n, B_n \subseteq X \text{ is a Borel set with } I_\alpha(\nu|_{B_n}) < \infty \text{ for all } n \}
\]
We can find \( A \) and \( (A_n)_{n \in \mathbb{N}} \) such that \( A = \bigcup A_n \), \( \nu(A) = c_{\text{max}} \) and \( I_\varphi(\nu|_{A_n}) < \infty \) for all \( n \in \mathbb{N} \). Without the loss of generality we can further assume that all the \( A_n \) are disjoint for different \( n \). Let \( \nu_R = \nu|_A \) and \( \nu_{\perp} = \nu|_{\mathcal{X} \setminus A} \). Then \( i), ii) \) and \( iv) \) are satisfied. Assume for a contradiction that \( iii) \) is not satisfied that is there exists a Borel probability measure \( \tau \) with \( I_\varphi(\tau) < \infty \) such that \( \tau \ll \nu_{\perp} \). Then there exists \( N > 0 \) such that \( \nu_{\perp}(C_N) > 0 \) where \( C_N = \{ x : \frac{d\tau}{d\nu_{\perp}}(x) \geq \frac{1}{N} \} \). If \( D \subseteq C_N \) then
\[
\tau(D) = \int_D \frac{d\tau}{d\nu_{\perp}}(x) d\nu_{\perp}(x) \geq \int_D \frac{1}{N} d\nu_{\perp}(x) = \frac{1}{N} \nu_{\perp}(D).
\]
Thus \( I_\varphi(\nu_{\perp}|_{C_N}) \leq N^2 I_\varphi(\tau|_{C_N}) \leq N^2 I_\varphi(\tau) < \infty \). This contradicts with the maximality of \( c_{\text{max}} \). \( \square \)

**Remark 4.2.** Proposition 1.1 holds for locally finite, Borel measures. This can easily be deduced from Proposition 1.3 and the Lindelöf property of \( \mathcal{X} \).

**Proposition 4.3.** Let \( \tau \) be a finite Borel measure on \( \mathcal{X} \) such that \( \int \int \varphi(x, y)d\tau(y)d\tau(x) < \infty \). Then for every \( \varepsilon > 0 \) there exist \( 0 < M < \infty \) and a compact set \( F \subseteq \mathcal{X} \) such that \( \tau(X \setminus F) < \varepsilon \) and \( \int_F \varphi(x, y)d\tau(y) < M \) for every \( x \in F \).

**Proof.** Since \( \int \int \varphi(x, y)d\tau(y)d\tau(x) < \infty \) it follows that \( \int \varphi(x, y)d\tau(y) < \infty \) for \( \tau \) almost every \( x \in X \). Thus there exists a compact set \( F_0 \subseteq X \) and \( M_0 > 0 \) such that \( \tau(X \setminus F_0) < \varepsilon/2 \) and \( \int \varphi(x, y)d\tau(y) < M_0 \) for every \( x \in F_0 \). In the proof of [3, Chapter III, Thm 1, page 15] it is shown that there exists a compact set \( F \subseteq F_0 \) such that \( \tau(X \setminus F) < \varepsilon \) and \( \lim_{x \to x_0} \int_F \varphi(x, y)d\tau(y) = \int_F \varphi(x_0, y)d\tau(y) < M_0 \) for every \( x_0 \in F \). Hence there exists \( r > 0 \) such that if \( \text{dist}(F, x) < r \) then \( \int_F \varphi(x, y)d\tau(y) < M_0 \). Whenever \( \text{dist}(F, x) \geq r \) then \( \int_F \varphi(x, y)d\tau(y) \leq \varphi(r)\tau(X) \) because \( \varphi \) is monotone decreasing. Hence the statement follows. \( \square \)

**Proposition 4.4.** If \( \nu \) is a finite Borel measure that is singular to every finite Borel measure with finite \( \varphi \)-energy then there exists a Borel set \( Z \subseteq X \) such that \( \nu(X \setminus Z) = 0 \) and \( C_\varphi(Z) = 0 \).

**Proof.** Let \( A_n = \{ x \in X : \int \varphi(x, y)d\nu(y) \leq n \} \). Then
\[
I_\varphi(\nu|_{A_n}) = \int_{A_n} \int_{A_n} \varphi(x, y)d\nu(y)d\nu(x) \leq \int_{A_n} \int_X \varphi(x, y)d\nu(y)d\nu(x) \leq n \cdot \nu(A_n) < \infty.
\]
Thus \( \nu(A_n) = 0 \) by the assumption and hence \( \nu(X \setminus Z) = 0 \) for \( Z = \{ x \in X : \int \varphi(x, y)d\nu(y) = \infty \} \).

Assume for a contradiction that \( C_\varphi(Z) > 0 \). Then there exists a probability measure \( \tau \) on \( Z \) such that \( \int \int \varphi(x, y)d\tau(y)d\tau(x) < \infty \). Then by Lemma 3.3 there exist \( 0 < M < \infty \) and \( F \subseteq Z \) such that \( \tau(F) > 0 \) and \( \int_F \varphi(x, y)d\tau(y) < M \) for every \( x \in X \). Thus
\[
\int_X \left( \int_F \varphi(x, y)d\tau(y) \right) d\nu(x) \leq \int_X M d\nu(x) \leq M \cdot \nu(X) < \infty
\]
contradicting with that
\[
\int_X \left( \int_F \varphi(x, y)d\tau(y) \right) d\nu(x) = \int_F \left( \int_X \varphi(x, y)d\nu(x) \right) d\tau(y) = \int_F \infty d\tau(x) = \infty
\]
where we used Fubini’s theorem. Hence \( C_\varphi(Z) = 0 \). \( \square \)
Remark 4.5. Proposition 4.4 holds for locally finite, Borel measures. This can easily be deduced from Proposition 4.4, the Lindelöf property of $X$ and the fact that $C_\varphi(\bigcup_{n=1}^\infty A_n) = 0$ for a sequence of Borel sets $A_n$ with $C_\varphi(A_n) = 0$.

5 Degenerate case

In this section we discuss why the conditional measure of the $\varphi$-singular part $\nu_\perp$ vanishes.

Lemma 5.1. Let $D \subseteq X$ be a compact set and let $B_\omega$ be a closed realisation of the random closed set $B$ such that $D \cap B_\omega = \emptyset$. Let $\nu$ be a finite Borel measure such that supp$\nu \subseteq D$. Then $C_k(\nu)(X) = C_{k,\omega}(\nu)(X)$ converges to 0 for that realisation $B_\omega$.

Proof. Since $B \cap D = \emptyset$ then dist$(B, D) > 0$. Let $k_0$ be such that $\sup \{ \text{diam}(Q) : Q \in Q_k \} < \text{dist}(B, D)$ for ever $k \geq k_0$ (we note that $k_0$ depends on the realisation $B_\omega$ but exists nevertheless). Then $C_{k,\omega}(\nu)(X) = 0$ by the definition of $C_k$ for $k \geq k_0$.

Theorem 5.2. Assume that if $C_\varphi(D) = 0$ for some compact set $D \subseteq X$ then $B \cap D = \emptyset$ almost surely. If $\nu$ is a finite Borel measure that is singular to every finite Borel measure with finite $\varphi$-energy and $\nu(X \setminus X_\emptyset) = 0$ then $C_k(\nu)(X)$ converges to 0 in probability.

Proof. By Proposition 4.4 there exists $Z \subseteq X$ such that $\nu(X \setminus Z) = 0$ and $C_\varphi(Z) = 0$. Let $\eta > 0$ be fixed. There exists a compact set $D \subseteq Z$ such that $\nu(X \setminus D) < \eta$ by inner regularity (Lemma 2.12). Then by assumption $B \cap D = \emptyset$ almost surely. Thus $C_k(\nu|_D)(X)$ converges to 0 almost surely by Lemma 5.1 and hence converges to 0 in probability. Since $C_k(\nu) = C_k(\nu|_D) + C_k(\nu|_{X \setminus D})$ it follows by (1.15) that

$$\limsup_{k \to \infty} \rho(C_k(\nu)(X), 0) \leq \limsup_{k \to \infty} \rho(C_k(\nu)(X), C_k(\nu|_D)(X)) + \rho(C_k(\nu|_D)(X), 0) \leq \limsup_{k \to \infty} E(C_k(\nu|_{X \setminus D})(X)) + 0 \leq \nu(X \setminus D) < \eta.$$

Since we can choose $\eta$ to be arbitrarily small it follows that $C_k(\nu)(X)$ converges to 0 in probability.

Remark 5.3. Let $\nu$ be such that $\nu(X \setminus X_\emptyset) = 0$. Then for the conclusion of Theorem 5.2 it is enough to assume that if $C_\varphi(D) = 0$ for some compact set $D \subseteq X_\emptyset$ (rather than for compact sets $D \subseteq X$) then $B \cap D = \emptyset$ almost surely. In the proof we can choose $D \subseteq Z \cap X_\emptyset$.

Lemma 5.4. Let $\nu$ be a finite Borel measure on $X$ such that $\nu(X_\emptyset) = 0$. Then $C_k(\nu)(X)$ converges to 0 in $L^1$ and so in probability.

Proof. It follows from definition $C_k(\nu)$ that

$$E(C_k(\nu)(X)) = \sum_{Q \in Q_k} \nu(Q) = \nu(\bigcup_{Q_k} Q). \quad (5.1)$$

Since $\nu(X_\emptyset) = 0$ it follows that $\lim_{k \to \infty} \nu(\bigcup_{Q_k} Q) = 0$ and so the statement follows.
6 $\mathcal{L}^2$-boundedness

In this section we show that under the assumptions of Section 1.1.2 if $\nu$ is a finite Borel measure with $I_\varphi(\nu) < \infty$ then the sequence $C_k(\nu)(X)$ is $\mathcal{L}^2$-bounded.

**Lemma 6.1.** Let $Q, S \in \mathcal{Q}_k$, $Q \neq S$ and $\nu_1$ and $\nu_2$ be finite Borel measures on $X$. If (1.23) holds for $Q$ and $S$ then

$$E_\nu \left( C_k(\nu_1)(Q) \cdot C_k(\nu_2)(S) \right) \leq c \cdot \varphi(\text{dist}(Q, S)) \cdot \nu_1(Q) \cdot \nu_2(S).$$

**Proof.** By the definition of $C_k(\nu)$ and (1.23) it follows that

$$E_\nu \left( C_k(\nu_1)(Q) \cdot C_k(\nu_2)(S) \right)$$

$$= E_\nu \left( P(Q \cap B \neq \emptyset)^{-1} \cdot I_{Q \cap B \neq \emptyset} \cdot \nu_1(Q) \cdot P(S \cap B \neq \emptyset)^{-1} \cdot I_{S \cap B \neq \emptyset} \cdot \nu_2(S) \right)$$

$$= P(Q \cap B \neq \emptyset)^{-1} \cdot P(S \cap B \neq \emptyset)^{-1} \cdot P(Q \cap B \neq \emptyset \text{ and } S \cap B \neq \emptyset) \cdot \nu_1(Q) \cdot \nu_2(S)$$

$$\leq c \cdot \varphi(\text{dist}(Q, S)) \cdot \nu_1(Q) \cdot \nu_2(S).$$

$\square$

**Lemma 6.2.** Let $\nu_1$ and $\nu_2$ be finite Borel measures on $X$. Assume that $\delta > 0$ is such that (1.10) holds and $Q, S \in \mathcal{Q}_k$ such that $\max \{\text{diam}(Q), \text{diam}(S)\} < \delta \cdot \text{dist}(Q, S)$. Then

$$\varphi(\text{dist}(Q, S)) \cdot \nu_1(Q) \cdot \nu_2(S) \leq c_2 \int_S \left( \int_Q \varphi(x, y) d\nu_1(x) \right) d\nu_2(y) + c_3 \nu_1(Q) \nu_2(S).$$

**Proof.** If $x \in Q$ and $y \in S$ then

$$d(x, y) \leq \text{dist}(Q, S) + \text{diam}(Q) + \text{diam}(S) < \text{dist}(Q, S) \cdot (1 + 2\delta).$$

Then by (1.16) and the monotonicity of $\varphi$ it follows that

$$\varphi(\text{dist}(Q, S)) \leq c_2 \varphi(\text{dist}(Q, S) \cdot (1 + 2\delta)) + c_3 \leq c_2 \varphi(d(x, y)) + c_3.$$

Integrating over $Q \times S$ with respect to $\nu_1 \times \nu_2$ the statement follows. $\square$

**Lemma 6.3.** Assume that (1.22) holds. Let $\nu_1$ and $\nu_2$ be finite Borel measures on $X$. If $Q, S \in \mathcal{Q}_k$ then

$$E_\nu \left( C_k(\nu_1)(Q) \cdot C_k(\nu_2)(S) \right) \leq a^{-1} \left( \int_Q \int_Q \varphi(x, y) d\nu_1(x) d\nu_1(y) + \int_S \int_S \varphi(x, y) d\nu_2(x) d\nu_2(y) \right).$$

**Proof.** Due to symmetry without the loss of generality we can assume that $\nu_1(Q) \leq \nu_2(S)$. Also we can assume that $0 < \nu_1(Q) \leq \nu_2(S)$ otherwise the proof is trivial. By the definition of $C_k$ and (1.22) it follows that

$$E_\nu \left( C_k(\nu_1)(Q) \cdot C_k(\nu_2)(S) \right)$$

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there exists $\delta > 0$ for every $k$

Proof. We say that a pair $\delta \cdot a \cdot C_\varphi(S)^2 \leq a^{-1} \cdot (I_\varphi(v|s) \cdot \nu(S)^2) \cdot \nu_2(S)^2$

$= P(S \cap B \neq \emptyset)^{-1} \cdot \nu_2(S)^2 \leq a^{-1} \cdot (I_\varphi(v|s) \cdot \nu(S)^2) \cdot \nu_2(S)^2$

$= a^{-1} \int \int \varphi(x, y) d\nu_2(x) d\nu_2(y) \leq a^{-1} \left( \int \int \varphi(x, y) d\nu_1(x) d\nu_1(y) + \int \int \varphi(x, y) d\nu_2(x) d\nu_2(y) \right)$.

Proposition 6.4. Let $\nu$ be a finite Borel measure on $X$. Assume that (1.22) holds and there exists $\delta > 0$ such that (1.10), (1.23) and (1.18) hold. Then

$E(\mathcal{C}_k(\nu)(X) \mathcal{C}_k(\nu)(X)) \leq (cc_2 + 2a^{-1}M_\delta) I_\varphi(\nu) + cc_3\nu(X)^2$

for every $k \in \mathbb{N}$.

Proof. We say that a pair $(Q, S) \in \mathcal{Q}_k \times \mathcal{Q}_k$ is a ‘good’ pair if $\max \{\text{diam}(Q), \text{diam}(S)\} < \delta \cdot \text{dist}(Q, S)$ and is a bad pair if $\max \{\text{diam}(Q), \text{diam}(S)\} \geq \delta \cdot \text{dist}(Q, S)$. Combining Lemma 6.1 and Lemma 6.2 it follows that

$\sum_{(Q,S) \text{is good}} E(\mathcal{C}_k(\nu)(Q) \mathcal{C}_k(\nu)(S)) \leq \sum_{(Q,S) \text{is good}} c \cdot \varphi(\text{dist}(Q, S) \cdot \nu(Q) \cdot \nu(S)$

$\leq c \sum_{(Q,S) \text{is good}} \left( c_2 \int \int \varphi(x, y) d\nu(x) d\nu(y) + c_3\nu(Q)\nu(S) \right)$

$\leq c \sum_{Q, S \in \mathcal{Q}_k} \left( c_2 \int \int \varphi(x, y) d\nu(x) d\nu(y) + c_3\nu(Q)\nu(S) \right) \leq c \left( c_2 I_\varphi(\nu) + c_3\nu(X)^2 \right)$.

By Lemma 6.3 and (1.18)

$\sum_{(Q,S) \text{is bad}} E(\mathcal{C}_k(\nu)(Q) \mathcal{C}_k(\nu)(S))$

$\leq \sum_{(Q,S) \text{is bad}} a^{-1} \left( \int \int \varphi(x, y) d\nu(x) d\nu(y) + \int \int \varphi(x, y) d\nu(x) d\nu(y) \right)$

$= 2a^{-1} \sum_{Q \in \mathcal{Q}_k} \left( \sum_{S \in \mathcal{Q}_k} \int \int \varphi(x, y) d\nu(x) d\nu(y) \right)$

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\[ \leq 2a^{-1} \sum_{Q \in \mathcal{Q}_k} M_S \int_Q \int_Q \varphi(x,y) d\nu(x) d\nu(y) \leq 2a^{-1} M_S I_\varphi(\nu). \]

Hence the statement follows by

\[ E(\mathcal{C}_k(\nu)(X)^2) = \sum_{Q,S \in \mathcal{Q}_k} E(\mathcal{C}_k(\nu)(Q)\mathcal{C}_k(\nu)(S)) \]

\[ = \sum_{(Q,S) \text{ is good}} E(\mathcal{C}_k(\nu)(Q)\mathcal{C}_k(\nu)(S)) + \sum_{(Q,S) \text{ is bad}} E(\mathcal{C}_k(\nu)(Q)\mathcal{C}_k(\nu)(S)) \]

\[ \leq c \left( c_2 I_\varphi(\nu) + c_3 \nu(X)^2 \right) + 2a^{-1} M_S I_\varphi(\nu). \]

\[ \square \]

7 Non-degenerate limit

In this section our main goal is to show that \( \mathcal{C}_k(\nu)(A) \) converges in \( \mathcal{L}^2 \) if \( I_\varphi(\nu) < \infty \) and \( \underline{E}(x,y) = \overline{F}(x,y) \). Recall that the definition of \( F_{k,n} \), \( \underline{E}(x,y) \) and \( \overline{F}(x,y) \) can be found in Section 1.3. A key observation is that \( E(\mathcal{C}_k(\nu)(X)^2) = \int_X \int_X F_{k,k}(x,y) d\nu(x) d\nu(y) \). Using this and the assumption that \( \underline{E}(x,y) = \overline{F}(x,y) \) we prove the \( \mathcal{L}^2 \) convergence in two steps. We divide the double integral into two parts, one part is the double integral on a domain that is bounded away from the diagonal and approximates the double integral uniformly, the other part is around the diagonal that is small. Then from this we deduce the convergence in \( \mathcal{L}^2 \). At the end of the section we show that if \( \mathcal{C}_k(\nu)(A) \) is a martingale then we do not even need the assumption that \( \underline{E}(x,y) = \overline{F}(x,y) \) because then the convergence in \( \mathcal{L}^2 \) is automatic by the \( \mathcal{L}^2 \)-boundedness.

**Lemma 7.1.** Let \( \nu_1 \) and \( \nu_2 \) be finite Borel measures on \( X \). For \( k, n \in \mathbb{N} \)

\[ E(\mathcal{C}_k(\nu_1)(X) \cdot \mathcal{C}_n(\nu_2)(X)) = \int_X \int_X F_{k,n}(x,y) d\nu_1(x) d\nu_2(y). \]

**Proof.** By the definition of \( \mathcal{C}_k \) it follows that

\[ E(\mathcal{C}_k(\nu_1)(X) \cdot \mathcal{C}_n(\nu_2)(X)) = \sum_{Q \in \mathcal{Q}_k} \sum_{S \in \mathcal{Q}_n} \frac{P(Q \cap B \neq \emptyset \text{ and } S \cap B \neq \emptyset)}{P(Q \cap B \neq \emptyset \cdot P(S \cap B \neq \emptyset)} \cdot \nu_1(Q) \cdot \nu_2(S) \]

\[ = \sum_{Q \in \mathcal{Q}_k} \sum_{S \in \mathcal{Q}_n} \int_S \int F_{k,n}(x,y) d\nu_1(x) d\nu_2(y) = \int_X \int F_{k,n}(x,y) d\nu_1(x) d\nu_2(y). \]

\[ \square \]

**Lemma 7.2.** Let \( \nu_1 \) and \( \nu_2 \) be finite Borel measures on \( X \). Let \( Q, S \subseteq X \) be Borel sets. For \( k, n \in \mathbb{N} \), it follows that

\[ E(\mathcal{C}_k(\nu_1)(Q) \cdot \mathcal{C}_n(\nu_2)(S)) = \int_S \left( \int_Q F_{k,n}(x,y) d\nu_1(x) \right) d\nu_2(y). \]

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Lemma 7.2 follows by the application of Lemma 7.1 to the measures $\nu_1|_Q$ and $\nu_2|_S$.

Lemma 7.3. Let $\nu_1$ and $\nu_2$ be finite Borel measures on $X$. Assume that there exists $\delta > 0$ such that (1.23). Let $\varepsilon, \eta > 0$ and let $A_{\varepsilon} = \{(x, y) \in X \times X : d(x, y) > \varepsilon\}$. Then there exists $m \in \mathbb{N}$ such that for every $n, k \geq m$

$$-\eta + \iint_{A_{\varepsilon}} E(x, y) d\nu_1(x)d\nu_2(y) \leq \iint_{A_{\varepsilon}} F_{k,n}(x, y) d\nu_1(x)d\nu_2(y) \leq \eta + \iint_{A_{\varepsilon}} \overline{F}(x, y) d\nu_1(x)d\nu_2(y)$$

Proof. Let $k_0 \in \mathbb{N}$ be large enough that $\sup \{\text{diam}(Q) : Q \subset Q_k\} < \varepsilon/3$ for every $k > k_0$, we can choose such $k_0$ due to (1.10). Whenever $A_{\varepsilon} \cap Q \neq \emptyset$ for $Q \subset Q_k, S \subset Q_n, k, n \geq k_0$ then $\text{dist}(Q, S) > \varepsilon/3$. We can choose $k_1 \geq k_0$ such that $\sup \{\text{diam}(Q) : Q \subset Q_k\} < \delta\varepsilon/3$ for every $k \geq k_1$. Whenever $A_{\varepsilon} \cap Q \neq \emptyset$ for $Q \subset Q_k, S \subset Q_n, k, n \geq k_1$ then $\max(\text{diam}(Q), \text{diam}(S)) < \delta\varepsilon/3 < \delta\text{dist}(Q, S)$. Hence $F_{k,n}(x, y) \leq c\varphi(\text{dist}(Q, S))$ for every $(x, y) \in Q \times S$ by (1.23). Since $\varphi$ is a nonnegative, monotone decreasing, continuous function it follows that $\varphi$ is absolutely continuous on $[\varepsilon/3, \infty)$, hence we can choose $k_2 \geq k_1$, due to (1.10), such that $\varphi(\text{dist}(Q, S)) \leq \varphi(d(x, y)) + \eta/c$ whenever $A_{\varepsilon} \cap Q \neq \emptyset$ for $x \in Q \subset Q_k, y \in S \subset Q_n, k, n \geq k_2$. Hence

$$F_{k,n}(x, y) \leq c\varphi(d(x, y)) + \eta$$

whenever $A_{\varepsilon} \cap Q \neq \emptyset$ for $x \in Q \subset Q_k, y \in S \subset Q_n, k, n \geq k_2$. Thus

$$\underline{F}(x, y) \leq F_{k,n}(x, y) \leq \overline{F}(x, y) \leq c\varphi(d(x, y)) + \eta \leq c\varphi(\varepsilon) + \eta$$

for $(x, y) \in A_{\varepsilon}, N \geq k_2$ and $k, n \geq N$. Since $\underline{F}$ and $\overline{F}$ converge as $N$ goes to $\infty$, due to the dominated convergence theorem, there exists $m \geq k_2$ such that

$$-\eta + \iint_{A_{\varepsilon}} E(x, y) d\nu_1(x)d\nu_2(y) \leq \iint_{A_{\varepsilon}} \underline{F}(x, y) d\nu_1(x)d\nu_2(y)$$

and

$$\iint_{A_{\varepsilon}} \overline{F}(x, y) d\nu_1(x)d\nu_2(y) \leq \eta + \iint_{A_{\varepsilon}} F(x, y) d\nu_1(x)d\nu_2(y)$$

for every $N \geq m$. Thus the statement follows. \hfill \Box

Notation 7.4. Assume that (1.23) holds. For $\varepsilon > 0$ let $k_{\varepsilon}$ be the largest positive integer such that $\inf \{\text{diam}(Q) : Q \subset Q_k\} > \varepsilon$, it is well-defined by (1.23) and (1.10) if $\varepsilon > 0$ is small enough. Let $r_{\varepsilon} = \sup \{\text{diam}(Q) : Q \subset Q_k\}$. Note that $r_{\varepsilon}$ converges to $0$ as $\varepsilon$ approaches $0$ by (1.10) and (1.19).

Lemma 7.5. Let $\nu_1$ and $\nu_2$ be finite Borel measures on $X$ and let $\nu = \nu_1 + \nu_2$. Let $\varepsilon > 0$, let $G_{\varepsilon} = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon\}$ and $H_{\varepsilon} = \{(x, y) \in X \times X : d(x, y) \leq \varepsilon + 2r_{\varepsilon}\}$. Assume that (1.19) and (1.22) hold and there exists $0 < \delta < 1$ such that (1.10), (1.23) and (1.18). Then there exists $c_4 > 0$, depending on $a, c, c_2, c_3$ and $M_{\delta}$, such that

$$\iint_{G_{\varepsilon}} F_{n,n}(x, y) d\nu_1(x)d\nu_2(y) \leq c_4 \iint_{H_{\varepsilon}} \varphi(x, y) d\nu(x)d\nu(y) + c_4 \cdot \nu \times \nu(H_{\varepsilon})$$

for every $n \in \mathbb{N}$.
Proof. Whenever \( Q \times S \cap G_\varepsilon \neq \emptyset \) for \( Q, S \in Q_{k_\varepsilon} \) then \( \text{dist}(Q, S) \leq \varepsilon < \max \{ \text{diam}(Q), \text{diam}(F) \} \). In particular, \( \delta \cdot \text{dist}(Q, S) < \max \{ \text{diam}(Q), \text{diam}(F) \} \). Hence

\[
\# \{ S \in Q_{k_\varepsilon} : Q \times S \cap G_\varepsilon \neq \emptyset \} \leq M_\delta
\]

for every \( Q \in Q_{k_\varepsilon} \) by (1.18). By Lemma 7.2 and Lemma 6.4 it follows that

\[
\int \int_{G_\varepsilon} F_{n,n}(x,y)d\nu_1(x)d\nu_2(y) \leq \int \int_{G_\varepsilon} F_{n,n}(x,y)d\nu(x)d\nu(y)
\]

\[
\leq \sum_{Q,S \in Q_{k_\varepsilon}} \int_{Q \times S} F_{n,n}(x,y)d\nu(x)d\nu(y) = \sum_{Q,S \in Q_{k_\varepsilon}} E (C_n(\nu)(Q) \cdot C_n(\nu)(S))
\]

\[
\leq \sum_{Q,S \in Q_{k_\varepsilon}} (cc_2 + 2a^{-1}M_\delta) I_{\varphi}(\nu|Q + \nu|S) + cc_3\nu(Q \cup S)^2.
\]

(7.2)

Whenever \( Q \times S \cap G_\varepsilon \neq \emptyset \) for \( Q, S \in Q_{k_\varepsilon} \) then \( Q \times S \subseteq H_\varepsilon \). Hence

\[
\sum_{Q,S \in Q_{k_\varepsilon}} \int \int_{Q \times S} \varphi(x,y)d\nu(x)d\nu(y) \leq \int \int_{H_\varepsilon} \varphi(x,y)d\nu(x)d\nu(y).
\]

(7.3)

By (7.1)

\[
\sum_{Q,S \in Q_{k_\varepsilon}} \int \int_{Q \times Q} \varphi(x,y)d\nu(x)d\nu(y) \leq M_\delta \sum_{Q \in Q_{k_\varepsilon}} \int \int_{Q \times Q} \varphi(x,y)d\nu(x)d\nu(y)
\]

\[
\leq M_\delta \int \int_{H_\varepsilon} \varphi(x,y)d\nu(x)d\nu(y)
\]

(7.4)

and similarly

\[
\sum_{Q,S \in Q_{k_\varepsilon}} \nu(Q)^2 \leq M_\delta \sum_{Q \in Q_{k_\varepsilon}} \nu(Q)^2 \leq M_\delta \cdot \nu \times \nu(H_\varepsilon)
\]

(7.5)

It is easy to see that

\[
\sum_{Q,S \in Q_{k_\varepsilon}} \nu(Q)\nu(S) \leq \nu \times \nu(H_\varepsilon).
\]

(7.6)
Since
\[ I_\varphi(\nu|_Q + \nu|_S) = 2 \iint_{Q \times S} \varphi(x,y) \nu(x) \nu(y) + \iint_{Q \times Q} \varphi(x,y) \nu(x) \nu(y) + \iint_{S \times S} \varphi(x,y) \nu(x) \nu(y) \]
and
\[ \nu(Q \cup S)^2 = \nu(Q)^2 + \nu(S)^2 + 2 \nu(Q) \nu(S), \]
the statement follows by combining (7.2), (7.3), (7.4), (7.5) and (7.6). \(\square\)

### 7.1 Limit in the presence of the weighted kernel \(F(x,y)\)

**Lemma 7.6.** Let \(\nu\) be finite Borel measure. Then
\[
\int_{X} \int_{X} F_{n,n}(x,y) + F_{k,k}(x,y) - 2F_{k,n}(x,y) \nu(x) \nu(y) \geq 0 \quad (7.7)
\]
for every \(k, n \in \mathbb{N}\). If
\[
\limsup_{n \to \infty} \limsup_{k \to \infty} \int_{X} \int_{X} F_{n,n}(x,y) + F_{k,k}(x,y) - 2F_{k,n}(x,y) \nu(x) \nu(y) = 0
\]
then \(\mathcal{C}_k(\nu)(X)\) converges in \(L^2\).

**Proof.** Since \(L^2\) is complete it is sufficient to show that \(\mathcal{C}_k(\nu)(X)\) is a Cauchy sequence in \(L^2\). For the integers \(n \leq k\) by Lemma 7.1
\[
E \left( (\mathcal{C}_k(\nu)(X) - \mathcal{C}_n(\nu)(X))^2 \right) = E \left( \mathcal{C}_k(\nu)(X)^2 + \mathcal{C}_n(\nu)(X)^2 - 2\mathcal{C}_k(\nu)(X)\mathcal{C}_n(\nu)(X) \right)
\]
\[
= \int_{X} \int_{X} F_{k,k}(x,y) + F_{n,n}(x,y) - 2F_{k,n}(x,y) \nu(x) \nu(y),
\]

hence (7.7) holds and by the assumption
\[
\limsup_{n \to \infty} \limsup_{k \to \infty} E \left( (\mathcal{C}_k(\nu)(X) - \mathcal{C}_n(\nu)(X))^2 \right) = 0
\]
and so \(\mathcal{C}_k(\nu)(X)\) is a Cauchy sequence in \(L^2\). \(\square\)

**Proposition 7.7.** Let \(\nu\) be a finite Borel measure on \(X\) with \(I_\varphi(\nu) < \infty\). Assume that (1.17), (1.19) and (1.22) hold and there exists \(0 < \delta < 1\) such that (1.16), (1.23) and (1.18). If \(F(x,y) = \overline{F}(x,y)\) for \(\nu \times \nu\) almost every \((x, y)\) then
\[
\limsup_{n \to \infty} \limsup_{k \to \infty} \int_{X} \int_{X} F_{n,n}(x,y) + F_{k,k}(x,y) - 2F_{k,n}(x,y) \nu(x) \nu(y) = 0.
\]
Proof. Let \( \eta, \varepsilon > 0 \). Let \( m \in \mathbb{N} \) as in Lemma 7.3 and \( k, n \geq m \). Let \( A_\varepsilon, G_\varepsilon \) and \( H_\varepsilon \) as in Lemma 7.3 and Lemma 7.5. Then

\[
\int \int F_{n,n}(x, y) + F_{k,k}(x, y) - 2F_{k,n}(x, y) d\nu(x) d\nu(y)
\]

\[
\leq \int \int F_{n,n}(x, y) + F_{k,k}(x, y) d\nu(x) d\nu(y) - 2 \int \int F_{k,n}(x, y) d\nu(x) d\nu(y)
\]

\[
\leq 2\eta + 2 \int_{A_\varepsilon} F(x, y) d\nu(x) d\nu(y) + \int_{G_\varepsilon} F_{n,n}(x, y) + F_{k,k}(x, y) d\nu(x) d\nu(y) + 2\eta - 2 \int_{A_\varepsilon} F(x, y) d\nu(x) d\nu(y)
\]

\[
\leq 4\eta + 2 \cdot c_4 \cdot 4 \left( \int_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + \nu \times \nu(H_\varepsilon) \right)
\]

where we used Lemma 7.5 for \( \nu_1 = \nu_2 = \nu \). Thus

\[
\limsup_{n \to \infty} \limsup_{k \to \infty} \int \int F_{n,n}(x, y) + F_{k,k}(x, y) - 2F_{k,n}(x, y) d\nu(x) d\nu(y)
\]

\[
\leq 4\eta + 2 \cdot c_4 \cdot 4 \left( \int_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + \nu \times \nu(H_\varepsilon) \right)
\]

By \((1.17)\) and Fubini’s theorem \( \nu \times \nu ((x, x) : x \in X) = 0 \). Since \( \nu \times \nu \) and \( \varphi(x, y) d\nu(x) d\nu(y) \) are finite measures it follows that

\[
\int_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + \nu \times \nu(H_\varepsilon)
\]

converges to 0 as \( \varepsilon \) approaches 0 because \( r_\varepsilon \) goes to 0 by Notation 7.4. Taking the limit as \( \eta \) and \( \varepsilon \) go to 0 in \((7.8)\) the statement follows by \((7.7)\) \( \square \).

**Theorem 7.8.** Let \( \nu \) be a finite Borel measure on \( X \) such that \( \nu(X \setminus X_0) = 0 \) and \( I_\nu(\nu) < \infty \). Assume that \((1.17), (1.19)\) and \((1.22)\) hold and there exists \( 0 < \delta < 1 \) such that \((1.16), (1.23)\) and \((1.18)\). If \( F(x, y) = \overline{F}(x, y) \) for \( \nu \times \nu \) almost every \((x, y)\) then for every Borel set \( A \subseteq X \) it follows that \( C_k(\nu)(A) \) converges to a limit \( \mu(A) \) in \( L^2 \), in \( L^1 \) and in probability and \( E(\mu(A)) = E(C_k(\nu)(A)) = \nu(A) \).

**Proof.** By applying Lemma 7.6 and Proposition 7.4 to the measure \( \nu\mid_A \) it follows that \( C_k(\nu)(A) \) converges to a limit \( \mu(A) \) in \( L^2 \) and so in \( L^1 \) and in probability. Thus by \((1.13)\) we have that \( E(\mu(A)) = E(C_k(\nu)(A)) = \nu(A) \). \( \square \)
7.2 Limit in the presence of a martingale filtration

**Theorem 7.9.** Let \( \nu \) be a finite Borel measure on \( X \) such that \( \nu(X \setminus X_0) = 0 \). Assume that for a Borel set \( A \subseteq X \) there exists a filtration \( F_k \) such that \( C_k(\nu)(A) \) is a martingale with respect to the filtration \( F_k \). Then \( C_k(\nu)(A) \) converges to a limit \( \mu(A) \) almost surely and \( E(\mu(A)) \leq E(C_k(\nu)(A)) = \nu(A) \).

If additionally \( I_\varphi(\nu) < \infty \), (1.22) holds and there exists \( \delta > 0 \) such that (1.16), (1.23) and (1.18) hold then \( C_k(\nu)(A) \) converges in \( L^2 \), in \( L^1 \) and \( E(\mu(A)) = E(C_k(\nu)(A)) = \nu(A) \).

**Proof.** Since \( C_k(\nu)(A) \) is a nonnegative martingale it converges almost surely to a random limit \( \nu(A) \) and \( E(\mu(A)) \leq E(C_k(\nu)(A)) = \nu(A) \) by the nonnegative martingale limit theorem [6, Theorem 5.2.9]. If \( I_\varphi(\nu) < \infty \) then \( C_k(\nu)(A) \) is \( L^2 \)-bounded by Proposition 6.4 hence converges in \( L^2 \) and in \( L^1 \) by [6, Theorem 5.4.5]. Thus by (1.15) we have that \( E(\mu(A)) = E(C_k(\nu)(A)) = \nu(A) \). \( \square \)

8 Existence of the conditional measure

We are now prepared to prove the existence of the conditional measure \( \mathcal{C}(\nu) \) when \( F(x, y) = \overline{F}(x, y) = F(x, y) \).

**Theorem 8.1.** Let \( \nu \) be a finite, Borel measure on \( X \) or let \( X \) be locally compact and \( \nu \) be a locally finite, Borel measure on \( X \). Assume that \( \nu(X \setminus X_0) = 0 \). Assume that if \( C_\varphi(D) = 0 \) for some compact set \( D \subseteq X_0 \) then \( B \cap D = \emptyset \) almost surely. Assume that (1.17), (1.19) and (1.22) hold and there exists \( 0 < \delta < 1 \) such that (1.16), (1.23) and (1.18) hold. Assume that at least one of the following conditions hold:

A.) \( F(x, y) = \overline{F}(x, y) \) for \( \nu \times \nu \) almost every \( (x, y) \),

B.) for every Borel set \( A \subseteq X \) there exists a filtration \( F_k \) such that \( C_k(\nu)(A) \) is a martingale with respect to the filtration \( F_k \).

Then the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( Q_k \) \((k \geq 1)\) with regularity kernel \( \varphi \). Moreover, if \( A \subseteq X \) is a Borel set such that \( \nu(A) < \infty \) and \( I_\varphi(\nu \mid A) < \infty \) then \( C_k(\nu\mid A)(A) \) converges to \( \mathcal{C}(\nu)(A) \) in \( L^2 \).

**Proof.** By Theorem 7.8 and Theorem 7.9 if \( D \subseteq X \) is a Borel set such that \( I_\varphi(\nu \mid D) < \infty \) and \( \nu(A) < \infty \) then \( C_k(\nu)(D) \) converges in \( L^2 \) and in \( L^1 \) to a random variable \( \mu(D) \). By Theorem 5.2 and Remark 5.3 we have that \( C_k(\nu \mid D)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \). Thus it follows by Theorem 1.10 that the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( Q_k \) \((k \geq 1)\) with regularity kernel \( \varphi \).

Let \( A \subseteq X \) be a Borel set such that \( \nu(A) < \infty \) and \( I_\varphi(\nu \mid A) < \infty \). As we established at the beginning of the proof \( C_k(\nu)(A) \) converges to \( \mu(A) \) in \( L^2 \). By Property iv.) of Definition 1.8 it follows that \( \mu(A) = \mathcal{C}(\nu)(A) \) almost surely. Hence it follows that \( C_k(\nu)(A) \) converges to \( \mathcal{C}(\nu)(A) \) in \( L^2 \). \( \square \)

**Remark 8.2.** If we assume that \( \nu_R = \nu \) in Theorem 8.1 then for the conclusion to hold we do not even need the assumption that if \( C_\varphi(D) = 0 \) for some compact set \( D \subseteq X_0 \) then \( B \cap D = \emptyset \) almost surely. It is because we only use this assumption to ensure that \( C_k(\nu \mid D)(D) \) converges to 0 in probability. However, if \( \nu_R = \nu \) then \( C_k(\nu \mid D)(X) = 0 \) for every \( k \).
9 Double integration

In this section we prove the double integration formula \((1.24)\).

**Proposition 9.1.** Assume that \((1.17), (1.19)\) and \((1.22)\) hold and there exists \(0 < \delta < 1\) such that \((1.16), (1.23)\) and \((1.18)\) hold. Let \(\nu_1\) and \(\nu_2\) be finite Borel measures on \(X\) with \(I_\varphi(\nu_1 + \nu_2) < \infty\). Assume that the conditional measure \(C(\nu_i)\) of \(\nu_i\) on \(B\) exist with respect to \(Q_k\) \((k \geq 1)\) with regularity kernel \(\varphi\) for \(i = 1, 2\). Assume that if \(A \subseteq X\) is a Borel set then \(C_k(\nu_i)(A)\) converges to \(C(\nu_i)(A)\) in \(L^2\) for \(i = 1, 2\). Then

\[
\int_{A_2} \left( \int_{A_1} F(x, y) d\nu_1(x) \right) d\nu_2(y) \leq E(C(\nu_1)(A_1) \cdot C(\nu_2)(A_2)) \leq \int_{A_2} \left( \int_{A_1} \tilde{F}(x, y) d\nu_1(x) \right) d\nu_2(y)
\]

\[
\leq c \int_{A_2} \left( \int_{A_1} \varphi(x, y) d\nu_1(x) \right) d\nu_2(y) < \infty
\]

for every Borel sets \(A_1, A_2 \subseteq X\).

**Proof.** Without the loss of generality we can assume that \(A_1\) and \(A_2\) are both \(X\) by restricting the measures to \(\nu_1|_{A_1}\) and \(\nu_2|_{A_2}\). We have that \(C_k(\nu_1)(X)\) converges to \(C(\nu_1)(X)\) in \(L^2\) and \(C_k(\nu_2)(X)\) converges to \(C(\nu_2)(X)\) in \(L^2\). Thus \(C_k(\nu_1)(X) \cdot C_k(\nu_2)(X)\) converges to \(C(\nu_1)(X) \cdot C(\nu_2)(X)\) in \(L^1\) and in particular,

\[
E(C(\nu_1)(X) \cdot C(\nu_2)(X)) = \lim_{k \to \infty} E(C_k(\nu_1)(X) \cdot C_k(\nu_2)(X)) = \lim_{k \to \infty} \int_X \int_X F_{k,k}(x, y) d\nu_1(x) d\nu_2(y)
\]

by Lemma 7.1

Let \(\varepsilon, \eta > 0\) be fixed. Then, by Lemma 7.3 and by Lemma 7.5, for large enough \(k \in \mathbb{N}\)

\[
\int_X \int_X F_{k,k}(x, y) d\nu_1(x) d\nu_2(y) \leq \eta + \int_{A_2} \int_{A_1} \tilde{F}(x, y) d\nu_1(x) d\nu_2(y) + c_4 \int_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + c_4 \cdot \nu \times \nu \cdot (H_\varepsilon)
\]

\[
\leq \eta + \int_{A_2} \int_{A_1} \tilde{F}(x, y) d\nu_1(x) d\nu_2(y) + c_4 \int_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) + c_4 \cdot \nu \times \nu \cdot (H_\varepsilon).
\]

Let \(D = \{(x, x) : x \in X\}\). Then \(\nu(D) = 0\) by \((1.17)\) and Fubini’s theorem. Hence by the fact that \(\int_X \int_X \varphi(x, y) d\nu(x) d\nu(y) < \infty\) it follows that

\[
\lim_{\varepsilon \to 0} \int_{H_\varepsilon} \varphi(x, y) d\nu(x) d\nu(y) = 0
\]

by Notation 7.4. Similarly

\[
\lim_{\varepsilon \to 0} \nu \times \nu \cdot (H_\varepsilon) = 0.
\]
Hence by (9.2), (9.3), (9.4), (9.5) and Remark 1.14

\[ E(\mathcal{C}(\nu_1)(X) \cdot \mathcal{C}(\nu_2)(X)) \leq \int_X \int_X F(x,y) d\nu_1(x) d\nu_2(y) \leq c \int_X \int_X \phi(x,y) d\nu_1(x) d\nu_2(y) < \infty. \quad (9.6) \]

By Lemma 7.3, for large enough \( k \in \mathbb{N} \)

\[-\eta + \int_{A_k} F(x,y) d\nu_1(x) d\nu_2(y) \leq \int_{A_k} F_{k,k}(x,y) d\nu_1(x) d\nu_2(y) \leq \int_X \int_{A_k} F_{k,k}(x,y) d\nu_1(x) d\nu_2(y). \quad (9.7)\]

By Remark 1.14 we have that \( E(x,y) \leq c \cdot \phi(x,y) \) for \( \nu \times \nu \) almost every \( (x,y) \). Thus similarly to (9.4)

\[ \lim_{\varepsilon \to 0} \int_{A_k} E(x,y) d\nu_1(x) d\nu_2(y) = \int_X \int_X E(x,y) d\nu_1(x) d\nu_2(y). \quad (9.8) \]

Then it follows from (9.2), (9.7) and (9.8) that

\[ \int_X \int_X E(x,y) d\nu_1(x) d\nu_2(y) \leq E(\mathcal{C}(\nu_1)(X) \cdot \mathcal{C}(\nu_2)(X)). \quad (9.9) \]

So the statement follows from (9.6) and (9.9). \( \square \)

**Proposition 9.2.** Assume that (1.17), (1.19) and (1.22) hold and there exists \( 0 < \delta < 1 \) such that (1.10), (1.23) and (1.18) hold. Let \( \nu_1 \) and \( \nu_2 \) be finite Borel measures on \( X \) with \( I_\phi(\nu_1 + \nu_2) \leq c \). Assume that the conditional measure \( \mathcal{C}(\nu_i) \) of \( \nu_i \) on \( B \) exist with respect to \( Q_k \) \( (k \geq 1) \) with regularity kernel \( \phi \) for \( i = 1,2 \). Assume that if \( A \subseteq X \) is a Borel set then \( C_k(\nu_i)(A) \) converges to \( \mathcal{C}(\nu_i)(A) \) in \( L^2 \) for \( i = 1,2 \). Let \( f : X \times X \longrightarrow \mathbb{R} \) be a nonnegative Borel function. Then

\[ \int \int E(x,y) f(x,y) d\nu_1(x) d\nu_2(y) \leq E \left( \int \int f(x,y) d\mathcal{C}(\nu_1)(x) d\mathcal{C}(\nu_2)(y) \right) \]

\[ \leq \int \int F(x,y) f(x,y) d\nu_1(x) d\nu_2(y). \quad (9.10) \]

**Proof.** It follows from Theorem 9.1 that (9.10) holds for functions of the form \( f(x,y) = \chi_{A_1}(x) \cdot \chi_{A_2}(y) \) for Borel sets \( A_1, A_2 \subseteq X \). Hence, by the fact that the sets of the form \( A_1 \times A_2 \) form a semi-ring generating the Borel \( \sigma \)-algebra of \( X \times X \) we can deduce that (9.10) holds for \( f(x,y) = I_\phi(x,y) \) for Borel sets \( A \subseteq X \times X \) by Proposition 2.34. It follows that (9.10) holds for non-negative simple functions on \( X \times X \) and so we can deduce (9.10) for every nonnegative Borel function on \( X \times X \) using the monotone convergence theorem. \( \square \)

**Lemma 9.3.** Let \( \nu \) and \( \tau \) be locally finite Borel measures on \( X \). Then there exist a sequence of finite measures \( \{ \nu_i \}_{i=1}^\infty \) and another sequence of finite measures \( \{ \tau_i \}_{i=1}^\infty \) such that \( \nu_R = \sum_{i=1}^\infty \nu_i, \tau_R = \sum_{i=1}^\infty \tau_i \) and \( I_\phi(\nu_i + \tau_i) \leq \infty \) for every \( i,j \in \mathbb{N} \).
Proof. By the Lebesgue decomposition [6, Theorem A.4.5] there exist a nonnegative Borel function \( g \) and a locally finite Borel measure \( \nu_s \) such that

\[ \nu_R(A) = \nu_s(A) + \int_A g(x) \, d\tau_R(x) \]

and \( \nu_s \) is singular to \( \tau_R \). Let \( G \subseteq X \) be a Borel set such that \( \tau_R(X \setminus G) = 0 \) and \( \nu_s(G) = 0 \). Since \( \nu_R \) is locally finite and \( X \) is separable metric space it follows that \( g(x) < \infty \) for \( \tau_R \) almost all \( x \). Let \( G_i = \{ x \in G : i - 1 < g(x) < i \} \) for every \( i \in \mathbb{N} \). By Proposition 1.6 for every \( i \in \mathbb{N} \) we can find a sequence \( \{E_{i,j}\}_{j=1}^{\infty} \) of disjoint Borel subsets of \( G_i \) such that

\[ I_\varphi(\tau_R|_{E_{i,j}}) < \infty \]

for every \( j \in \mathbb{N} \) and \( \tau_R|_{G_i} = \sum_{j=1}^{\infty} \tau_R|_{E_{i,j}} \). By Lemma 2.13 and Remark 2.14 we can further assume that \( \{E_{i,j}\}_{j=1}^{\infty} \) is a collection of disjoint compact sets and so \( \tau_R(E_{i,j}) < \infty \) for every \( j \in \mathbb{N} \). Similarly, we can also find a collection of disjoint compact sets \( \{A_j\}_{j=1}^{\infty} \) such that \( I_\varphi(\nu_s|_{A_j}) < \infty \), \( \nu_s(A_j) < \infty \), \( A_j \subseteq X \setminus G \) for every \( j \in \mathbb{N} \) and \( \nu_s = \sum_{j=1}^{\infty} \nu_s|_{A_j} \).

We define the following decomposition of \( \tau_R \) and \( \nu_R \) to obtain the desired decomposition of the statement:

\[ \tau_R = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \tau_R|_{E_{i,j}} \quad (9.11) \]

and

\[ d\nu_R(x) = \sum_{j=1}^{\infty} d\nu_s|_{A_j}(x) + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} g(x) d\tau_R|_{E_{i,j}}(x). \quad (9.12) \]

Then

\[ I_\varphi(g(x) \, d\tau_R|_{E_{i,j}}(x) + d\tau_R|_{E_{k,l}}(x)) \leq (i + 1)^2 I_\varphi(\tau_R|_{E_{i,j}}) < \infty \]

for every \( i, j \in \mathbb{N} \). Let \( i, j, k, l \in \mathbb{N} \) such that \( E_{i,j} \neq E_{k,l} \). Then \( E_{i,j} \) and \( E_{k,l} \) are disjoint compact sets and let \( r = \text{dist}(E_{i,j}, E_{k,l}) \). Then

\[
I_\varphi(g(x) \, d\tau_R|_{E_{i,j}}(x) + d\tau_R|_{E_{k,l}}(x)) \\
= I_\varphi(g(x) \, d\tau_R|_{E_{i,j}}(x)) + I_\varphi(\tau_R|_{E_{k,l}}) + 2 \int_{E_{i,j}} g(x) \left( \int_{E_{k,l}} \varphi(x, y) \, d\tau_R(y) \right) \, d\tau_R(x) \\
\leq i^2 I_\varphi(\tau_R|_{E_{i,j}}) + I_\varphi(\tau_R|_{E_{k,l}}) + 2i \varphi(r) \cdot \tau_R(E_{i,j}) \cdot \tau_R(E_{k,l}) < \infty.
\]

Finally if \( j, k, l \in \mathbb{N} \) then \( A_j \) and \( E_{k,l} \) are disjoint compact sets and let \( r = \text{dist}(A_j, E_{k,l}) \). Then

\[
I_\varphi(\nu_s|_{A_j} + \tau_R|_{E_{k,l}}) = I_\varphi(\nu_s|_{A_j}) + I_\varphi(\tau_R|_{E_{k,l}}) + 2 \int_{A_j} \left( \int_{E_{k,l}} \varphi(x, y) \, d\tau_R(y) \right) \, d\nu_s(x) \\
\leq I_\varphi(\nu_s|_{A_j}) + I_\varphi(\tau_R|_{E_{k,l}}) + 2 \varphi(r) \cdot \nu_s(A_j) \cdot \tau_R(E_{k,l}) < \infty.
\]

Hence decompositions of \( \tau_R \) and \( \nu_R \) in (9.11) and (9.12) satisfy the statement. \( \square \)
Theorem 9.4. Assume that (1.17), (1.19) and (1.22) hold and there exists \( 0 < \delta < 1 \) such that (1.16), (1.23) and (1.18) hold. Let either \( \nu \) and \( \tau \) be finite Borel measures or \( X \) be locally compact and \( \nu \) and \( \tau \) be locally finite Borel measures. Assume that \( \nu(X \setminus X_0) = 0 \) and \( \tau(X \setminus X_0) = 0 \). Assume that the conditional measure \( C(\nu) \) of \( \nu \) and \( C(\tau) \) of \( \tau \) on \( B \) exist with respect to \( Q_k \) \((k \geq 1)\) with regularity kernel \( \varphi \). Assume that if \( A \subseteq X \) is a Borel set such that \( \nu(A) < \infty \) and \( I_{\varphi}(\nu|A) < \infty \) then \( C_k(\nu)(A) \) converges to \( C(\nu)(A) \) in \( L^2 \) and if \( A \subseteq X \) is a Borel set such that \( \tau(A) < \infty \) and \( I_{\varphi}(\tau|A) < \infty \) then \( C_k(\tau)(A) \) converges to \( C(\tau)(A) \) in \( L^2 \). Let \( f : X \times X \to \mathbb{R} \) be a nonnegative Borel function. Then

\[
\int \int E(x,y)f(x,y)d\nu_R(x)d\tau_R(y) \leq E \left( \int \int f(x,y)dC(\nu)(x)dC(\tau)(y) \right)
\]

(9.13)

Proof. By Lemma 9.3 we can decompose \( \nu_R \) and \( \tau_R \) into sum of finite measures such that \( \nu_R = \sum_{i=1}^{\infty} \nu_i \), \( \tau_R = \sum_{i=1}^{\infty} \tau_i \) and \( I_{\varphi}(\nu_i + \tau_i) < \infty \) for every \( i, j \in \mathbb{N} \). By Theorem 9.2

\[
\int \int E(x,y)f(x,y)d\nu_i(x)d\tau_j(y) \leq E \left( \int \int f(x,y)dC(\nu_i)(x)dC(\tau_j)(y) \right)
\]

(9.13)

for every \( i, j \in \mathbb{N} \). By summing over all \( i, j \in \mathbb{N} \) the statement follows by Property vii.), viii.) and ix.) in Definition 1.8.

Proof of Proposition 1.15 By Remark 1.14 \( \int_X \int_X \varphi(x,y)|f(x,y)|d\nu(x)d\tau(y) < \infty \) implies that \( \int F(x,y)|f(x,y)|d\nu_R(x)d\tau_R(y) < \infty \).

The conditional measure \( C(\nu) \) of \( \nu \) and \( C(\tau) \) of \( \tau \) on \( B \) exist with respect to \( Q_k \) \((k \geq 1)\) with regularity kernel \( \varphi \) by Theorem 8.1. The conditions of Theorem 9.4 are satisfied by Theorem 8.1. The statement follows by applying Theorem 9.2 to \( f^+ \) and \( f^- \).

10 Conditional measure on an increasing union

Our aim in this section is to establish the extension of the conditional measure with respect to an increasing union of \( Q_k \).

For every \( i \in \mathbb{N} \) let \( Q_k^i \) be a sequence of countable families of Borel subsets of \( X \) for \( k \geq n_i \), for some \( n_i \in \mathbb{N} \), such that \( Q \cap S = \emptyset \) for \( Q, S \in Q_k^i \), for all \( k \in \mathbb{N} \). Assume that (1.10), (1.11) and (1.13) hold. Assume further that if \( i < j \) and \( k \geq \max\{n_i, n_j\} \) then

\[
Q_k^i \subseteq Q_k^j.
\]

(10.1)

See Example 1.4 Let \( X_0^i = \cap_{k=n_i}^{\infty} \left( \cup_{Q \in Q_k^i} Q \right) \) and let \( X_0^\infty = \cup_{i=1}^{\infty} X_0^i \), note that it is an increasing union. Let either \( \nu \) be a finite Borel measure on \( X \) or \( X \) be locally compact and \( \nu \) be a locally finite Borel measure on \( X \). Assume that

\[
\nu(X \setminus X_0^\infty) = 0
\]

(10.2)
and that the conditional measure $C^i(\nu|_{X^i_0})$ of $\nu|_{X^i_0}$ on $B$ exists with respect to $Q^i_k$ ($k \geq n_i$) with regularity kernel $\varphi$. Let

$$
\mu^\nu := \sum_{i=1}^{\infty} C^i(\nu|_{X^i_0 \setminus X^{i-1}_0})
$$

with the convention that $X^0_0 = \emptyset$.

**Proposition 10.1.** Let $X^i_0$, $X^\infty_0$, $Q^i_k$, $\nu$, $C^i(\nu|_{X^i_0})$ and $\mu^\nu$ be as above. Then the following hold:

1. $E(\int_X f(x)d\mu^\nu(x)) = \int_X f(x)d\nu_R(x)$ for every $f : X \to \mathbb{R}$ Borel measurable function such that $\int_X |f(x)|d\nu(x) < \infty$,
2. $E(\mu^\nu(A)) = \nu_R(A) \leq \nu(A)$ for every Borel set $A \subseteq X$ with $\nu(A) < \infty$,
3. $\mu^{\nu_\infty} = 0$ almost surely,
4. $\mu^\nu = \mu^{\nu_\infty}$ almost surely,
5. if $f : X \to \mathbb{R}$ is a nonnegative Borel function such that $\int_X f(x)d\nu(x) < \infty$ then $\mu^f(x)d\nu(x) = f(x)d\mu^\nu(x)$ almost surely,
6. $\text{supp}\mu^\nu \subseteq \text{supp}\nu \cap B$ almost surely.

**Proof.** Let $f : X \to \mathbb{R}$ be a nonnegative Borel measurable function such that $\int_X |f(x)|d\nu(x) < \infty$, then by Property ii.i) of Definition 1.8 we have that $E(\int_X f(x)dC^i(\nu|_{X^i_0 \setminus X^{i-1}_0})(x)) = \sum_{i=1}^{\infty} \nu_R(x)$ for every $i \in \mathbb{N}$. Then summing over $i \in \mathbb{N}$ and by Fubini's theorem we have that $E(\int_X f(x)d\mu^\nu(x)) = \int_X f(x)d\nu_R(x)$, i.e. 1.) holds for nonnegative $f$. This implies 1.) in the general case since we know it for $f^+$ and $f^-$. Property 2.) is a special case of property 1.).

Property 3.), 4.), 5.) and 6.) follows by the fact that the analogous properties of the conditional measure in Definition 1.8 hold for the summands $C^i(\nu|_{X^i_0 \setminus X^{i-1}_0})$ (for Property 6.) note that $B$ is almost surely closed). \(\square\)

Let $X^i_0$, $X^\infty_0$, $Q^i_k$, $\nu$, $C^i(\nu|_{X^i_0})$ and $\mu^\nu$ be as above. Let $Q_k$ be a sequence of countable families of Borel subsets of $X$ such that $Q \cap S = \emptyset$ for $Q, S \in \mathcal{Q}_k$, for all $k \in \mathbb{N}$. Assume that (1.11), (1.13) hold and assume that

$$
Q^i_k \subseteq Q_k
$$

for $k \geq n_i$ for every $i$. Let $C_k(\nu)$ be as in (1.14).

Recall, that $X_0 = \bigcap_{k=0}^{\infty} (\bigcup_{Q \in Q_k} Q)$. Note that, if

$$
X_0 = X^\infty_0
$$

and $\nu(X \setminus X_0) = 0$ then (10.2) holds.

By (10.2), Property ix.) of Definition 1.8 and (10.1) it follows that

$$
C_k(\nu|_{X^i_0}) = C_k(\nu|_{X^i_0}) = \sum_{j=1}^{i} C_k^j(\nu|_{X^j_0 \setminus X^{j-1}_0}) = \sum_{j=1}^{i} C_k^j(\nu|_{X^j_0 \setminus X^{j-1}_0}).
$$

(10.6)
Proposition 10.2. Let $X_0^i, X_0^\infty, Q_k^i, Q_k, \nu, \mu^\nu$ and $C_k^i(\nu)$ be as above, (note that we assume (10.2)). Then $\int_X f(x) dC_k^i(\nu)(x)$ converges to $\int f(x) d\mu^\nu(x)$ in probability for every Borel measurable function $f : X \rightarrow \mathbb{R}$ such that $\int_X |f(x)| d\nu(x) < \infty$.

Proof. Let $f$ be as in the statement. Let $\varepsilon > 0$ be fixed. Since $\int_X |f(x)| d\nu(x) < \infty$ and we assume (10.2), we can find $i \in \mathbb{N}$ such that $\int_{X \setminus X_0^i} |f(x)| d\nu(x) < \varepsilon$. By (10.4) and (10.6) it follows that

$$E \left| \int f(x) dC_k^i(\nu)(x) - \int f(x) dC_k^i(\nu|_{X_0^i})(x) \right| \leq E \left( \int |f(x)| dC_k^i(\nu|_{X_0^i})(x) \right) = \int_{X \setminus X_0^i} |f(x)| d\nu(x) < \varepsilon.$$  \hfill (10.7)

Since the conditional measure $C^i(\nu|_{X_0^i})$ of $\nu|_{X_0^i}$ on $B$ exists with respect to $Q_k^i$ ($k \geq n_i$) with regularity kernel $\varphi$ it follows by Property $\ddot{\iota}.(\nu|_{X_0^i})$, $x_0$) and $x^*$) of Definition 1.8 that

$$E \left| \int f(x) dC^i(\nu|_{X_0^i \setminus X_0^{i-1}})(x) \right| \leq \int_{X_0^i \setminus X_0^{i-1}} |f(x)| d\nu(x).$$

Thus by (10.3) and (10.6) it follows that

$$E \left| \int f(x) d\mu^\nu(x) - \int f(x) dC^i(\nu|_{X_0^i})(x) \right| = E \left| \sum_{j=i+1}^{\infty} \int f(x) dC^i_0(\nu|_{X_0^i})(x) \right|$$

$$\leq \int_{X \setminus X_0^i} |f(x)| d\nu(x) < \varepsilon. \hfill (10.8)$$

We have that $\int_X f(x) dC_k^i(\nu|_{X_0^i})(x)$ converges to $\int_X f(x) dC^i(\nu|_{X_0^i})(x)$ in probability by Property $\dddot{\iota}.(\nu|_{X_0^i})$ and $\dddot{\iota}_i$) of Definition 1.8. Thus using (10.7) and (10.8) it follows that

$$\rho \left( \int f(x) dC_k^i(\nu)(x), \int f(x) d\mu^\nu(x) \right)$$

$$\leq \rho \left( \int f(x) dC_k^i(\nu)(x), \int f(x) dC_k^i(\nu|_{X_0^i})(x) \right) + \rho \left( \int f(x) dC_k^i(\nu|_{X_0^i})(x), \int f(x) dC^i(\nu|_{X_0^i})(x) \right)$$

$$+ \rho \left( \int f(x) dC^i(\nu|_{X_0^i})(x), \int f(x) d\mu^\nu(x) \right)$$

$$\leq E \left| \int f(x) dC_k^i(\nu)(x) - \int f(x) dC_k^i(\nu|_{X_0^i})(x) \right| + \rho \left( \int f(x) dC_k^i(\nu|_{X_0^i})(x), \int f(x) dC^i(\nu|_{X_0^i})(x) \right)$$

$$+ E \left| \int f(x) d\mu^\nu(x) - \int f(x) dC^i(\nu|_{X_0^i})(x) \right|$$

$$\leq \varepsilon + \rho \left( \int f(x) dC^i(\nu|_{X_0^i})(x), S^i(f) \right) + \varepsilon.$$
It follows that
\[ \limsup_{k \to \infty} \rho \left( \int f(x) dC_k(\nu)(x), \int f(x) d\mu'(x) \right) \leq 2\varepsilon \]
because \( \int_X f(x) dC_k'(\nu|_{\mathcal{B}})(x) \) converges to \( \int_X f(x) dC'(\nu|_{\mathcal{B}})(x) \) in probability. Since \( \varepsilon > 0 \) can be arbitrary the statement follows.

**Theorem 10.3.** Let \( X^0_0, X^\infty_0, \mathcal{Q}_k, \mathcal{Q}_k, \nu, \mu' \) and \( C_k(\nu) \) be as above. Then the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( \mathcal{Q}_k \) \((k \geq 1)\) with regularity kernel \( \varphi \), and \( \mathcal{C}(\nu) = \mu' \) almost surely.

**Proof.** To show the existence of the conditional measure we wish to apply Theorem (1.10).

Hence we need to show that \( C_k(\nu)(D) \) converges in \( L^1 \) for every compact set \( D \subseteq X_0 \) with \( I_\varphi(\nu|_D) < \infty \) and \( C_k(\nu\perp L)(D) \) converges to 0 in probability for every compact set \( D \subseteq X \).

Let \( D \subseteq X_0 \) be a compact set such that \( I_\varphi(\nu|_D) < \infty \) which implies that \( \nu(D) = \nu(D) \). Then \( C_k(\nu)(D) \) converges to \( \mu'(D) \) in probability by Proposition 10.2. By Property ii.) of Theorem 10.1 it follows that \( E(\mu'(D)) = \nu(D) \). It follows from (1.15) that \( E(C_k(\nu)(D)) = \nu(D) \). Thus \( C_k(\nu)(D) \) converges to \( \mu'(D) \) in \( L^1 \) by Lemma 2.10.

Let now \( D \subseteq X \) be an arbitrary compact subset. Then \( C_k(\nu\perp L)(D) \) converges to \( \mu'(D) \) in probability by Proposition 10.2. On the other hand, by Property 3.) of Theorem 10.1 it follows that \( \mu'(D) = 0 \) almost surely. Thus \( C_k(\nu\perp L)(D) \) converges to 0 in probability.

So we can conclude by the application of Theorem 1.10 that the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exist with respect to \( \mathcal{Q}_k \) \((k \geq 1)\) with regularity kernel \( \varphi \). It remains to show that \( \mu' = \mathcal{C}(\nu) \) almost surely.

Assume first that \( \nu \) is a finite Borel measure. Then \( C_k(\nu)(G) \) converges to \( \mathcal{C}(\nu)(G) \) in probability for every open set \( G \subseteq X \) by Property ii.) and iv.) of Definition 1.8. On the other hand, \( C_k(\nu)(G) \) converges to \( \mu'(G) \) in probability for every open set \( G \subseteq X \) by Proposition 10.2. Hence \( \mu' = \mathcal{C}(\nu) \) almost surely.

Now let \( \nu \) be a locally finite Borel measure and let \( X \) be locally compact. Then \( C_k(\nu) \) vaguely converges to \( \mathcal{C}(\nu) \) in probability by Property i*) of Definition 1.8. On the other hand, \( C_k(\nu) \) vaguely converges to \( \mu' \) in probability by Proposition 10.2. Since the limit is unique by Proposition 3.33 it follows that \( \mu' = \mathcal{C}(\nu) \) almost surely.

Let \( F^i(x, y) \) and \( F^j(x, y) \) be defined as in Definition 1.13 for the sequence \( \mathcal{Q}_k \) in place of \( \mathcal{Q}_k \). If \( x \notin X^0_0 \) or \( y \notin X^0_0 \) then \( F^i(x, y) = F^j(x, y) = 0 \). If \( x, y \in X^0_0 \) and \( i < j \) then \( F^i(x, y) = F^j(x, y) \) and \( F^j(x, y) = F^j(x, y) \) by (10.11). Thus

\[ F^\infty(x, y) = \begin{cases} F^i(x, y) & x, y \in X^0_0 \text{ for some } i \\ 0 & x \notin X^\infty_0 \text{ or } y \notin X^\infty_0 \end{cases} \]

and

\[ F^\infty(x, y) = \begin{cases} F^i(x, y) & x, y \in X^0_0 \text{ for some } i \\ 0 & x \notin X^\infty_0 \text{ or } y \notin X^\infty_0 \end{cases} \]

are well-defined on \( X \times X \). It is easy to see, by (10.11), that \( F^\infty(x, y) = F(x, y) \) and \( F^\infty(x, y) = F(x, y) \) for \((x, y) \in X^\infty_0 \times X^\infty_0 \). In particular, if (10.5) holds then \( F^\infty(x, y) = F(x, y) \) and \( F^\infty(x, y) = F(x, y) \) for every \((x, y) \in X \times X \) where \( F \) and \( F \) are defined in (1.13).

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Theorem 10.4. Let $Q_k$ be a sequence of countable families of Borel subsets of $X$ such that $Q \cap S = \emptyset$ for $Q, S \in Q_k$, for all $k \in \mathbb{N}$. Assume that if $C_\varphi(D) = 0$ for some compact set $D \subseteq X_0$ then $B \cap D = \emptyset$ almost surely. For every $i \in \mathbb{N}$ let $Q^i_k$ be a sequence of countable families of Borel subsets of $X$ for $k \geq n_i$, for some $n_i \in \mathbb{N}$, such that $Q \cap S = \emptyset$ for $Q, S \in Q^i_k$, for all $k \in \mathbb{N}$. Assume that $(1.14), (1.19), (1.22), (1.10), (1.23)$ and $(1.13)$ hold for sufficient constants $0 < \delta' < 1$, $M(i) < \infty$, $0 < M(i) < \infty$, $0 < \alpha < \infty$, $0 < \beta < \infty$, $\alpha_1 < \infty$, $\alpha_2 < \infty$ depending on $i$. Assume further that if $i < j$ and $k \geq \max\{n_i, n_j\}$ then $Q^i_k \subseteq Q^j_k$. Assume that $(1.14), (1.19)$ hold and assume that $Q_k \subseteq Q_k$ for $k \geq n_i$ for every $i$. Let $X^i_0 = \bigcap_{k=n_i}^{\infty} (\bigcup_{Q \in Q_k} Q)$. Assume that $F(x, y) = F(x, y) = F(x, y)$ for every $x, y \in X$. Let either $\nu$ and $\tau$ be finite Borel measures on $X$ or $X$ be locally compact and $\nu$ and $\tau$ be locally finite Borel measures on $X$. Assume that $\nu(X \setminus X^\infty_0) = 0$ and $\tau(X \setminus X^\infty_0) = 0$. Then the conditional measure $C(\nu)$ of $\nu$ and $C(\tau)$ of $\tau$ on $B$ exist with respect to $Q_k (k \geq 1)$ with regularity kernel $\varphi$ and

$$E \left( \int \int f(x, y) dC(\nu)(x) dC(\tau)(y) \right) = \int \int F(x, y) f(x, y) d\nu_R(x) d\tau_R(y)$$

for every $f : X \times X \rightarrow \mathbb{R}$ Borel function with $\int \int |f(x, y)| d\nu_R(x) d\tau_R(y) < \infty$.

Proof. We have that $\overline{F}^\infty(x, y) = \overline{F}^i(x, y)$ and $\overline{F}^\infty(x, y) = \overline{F}^i(x, y)$ for $x, y \in X^i_0$ and that $F(x, y) = F^\infty(x, y) = F^i(x, y)$ for $x, y \in X^\infty_0$. It follows by Theorem 8.1 that the conditional measure $C^i(\nu|X^i_0)$ of $\nu|X^i_0$ on $B$ exists with respect to $Q^i_k (k \geq n_i)$ with regularity kernel $\varphi$ for every $i$. Thus it follows by Theorem 10.3 that the conditional measure $C(\nu)$ of $\nu$ on $B$ exists with respect to $Q_k (k \geq 1)$ with regularity kernel $\varphi$. Similarly, the conditional measure $C(\tau)$ of $\tau$ on $B$ exists with respect to $Q_k (k \geq 1)$ with regularity kernel $\varphi$.

Let $f : X \times X \rightarrow \mathbb{R}$ be a Borel function with $\int \int F(x, y) |f(x, y)| d\nu(x) d\tau(y) < \infty$. By Property ix.) of Definition 1.5 it follows that $C(\nu) = \sum_{i=1}^{\infty} C(\nu|X^i_0 \setminus X^{i-1}_0)$ and $C(\tau) = \sum_{i=1}^{\infty} C(\tau|X^i_0 \setminus X^{i-1}_0)$. Let $i, j \in \mathbb{N}, i \leq j$. By (10.1) we have that $C^i(\nu|X^i_0 \setminus X^{i-1}_0) = C^j(\nu|X^i_0 \setminus X^{i-1}_0)$ and so $C^i(\nu|X^i_0 \setminus X^{i-1}_0) = C^j(\nu|X^i_0 \setminus X^{i-1}_0)$ almost surely. By applying Theorem 1.13 to the measures $\nu|X^i_0 \setminus X^{i-1}_0$ and $\tau|X^i_0 \setminus X^{i-1}_0$ with respect to the sequence $Q^i_k$ it follows that

$$E \left( \int \int f(x, y) dC^i(\nu|X^i_0 \setminus X^{i-1}_0)(x) dC^j(\tau|X^j_0 \setminus X^{j-1}_0)(y) \right)$$

$$= \int \int_{X^i_0 \setminus X^{i-1}_0} \left( \int \int_{X^j_0 \setminus X^{j-1}_0} F(x, y) f(x, y) d\tau_R(y) \right) d\nu_R(x). \quad (10.9)$$

Similarly we can show that (10.9) holds when $i > j$. By (10.4) we have that $C_k(\nu|X^k_0 \setminus X^{k-1}_0) = C_k(\nu|X^k_0 \setminus X^{k-1}_0)$ and so $C(\nu|X^k_0 \setminus X^{k-1}_0) = C^k(\nu|X^k_0 \setminus X^{k-1}_0)$ and also $C(\tau|X^k_0 \setminus X^{k-1}_0) = C^k(\tau|X^k_0 \setminus X^{k-1}_0)$. Then summing (10.9) over all $i, j \in \mathbb{N}$ and by Fubini's theorem we can conclude that

$$E \left( \int \int f(x, y) dC(\nu)(x) dC(\tau)(y) \right) = \int \int F(x, y) f(x, y) d\nu_R(x) d\tau_R(y).$$

□

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11 Probability of non-extinction

In this section we estimate what is the probability that the conditional measure $\mathcal{C}(\nu)$ has positive total mass. Among other estimates we show Theorem 11.21. The upper bound in Theorem 11.21 is Corollary 11.7 and the lower bound follows from Corollary 11.5 Remark 11.14 and the double integration formula 11.24.

Let $K : X \times X \to \mathbb{R}$ be a nonnegative Borel function. Recall the definition of the $K$-energy of a measure (1.20) and the $K$-capacity of a set (1.21).

**Definition 11.1.** The $K$-capacity of a Borel measure $\nu$ is

$$C_K(\nu) = \sup \left\{ \frac{1}{I_K(\tau)} : \tau \ll \nu, \tau(X) = 1 \right\}.$$  

**Definition 11.2.** The upper $K$-capacity of a Borel measure $\nu$ is

$$\overline{C}_K(\nu) = \inf \{ C_K(A) : \nu(X \setminus A) = 0, A \subseteq X \text{ is Borel} \}.$$  

**Theorem 11.3.** Let $\nu$ be a finite Borel measure and $K : X \times X \to \mathbb{R}$ be a nonnegative Borel function such that

$$\tau_{\perp}(X) = \tau_{\varphi \perp}(X) = 0 \quad (11.1)$$

whenever $I_K(\tau) < \infty$ for a finite Borel measure $\tau$ that satisfies $\tau \ll \nu$. Assume that $\mathcal{C}(\tau)$ is a random finite Borel measure on $X$ for every finite Borel measure $\tau$ that satisfies $\tau \ll \nu$ such that the following hold

1.) if $\nu_i$ are finite Borel measures and $\gamma_i \in \mathbb{R}$ are nonnegative such that $\sum_{i=1}^{\infty} \gamma_i \cdot \nu_i \leq \nu$ then $\sum_{i=1}^{\infty} \gamma_i \cdot C(\nu_i) \leq C(\nu)$ almost surely,

2.) $E(\mathcal{C}(\tau)(X)) \geq \tau_R(X) = \tau_{\varphi R}(X),$

3.) $E(\mathcal{C}(\tau)(X)^2) \leq I_K(\tau).$

Then

$$C_K(\nu) \leq P(\mathcal{C}(\nu)(X) > 0).$$

**Proof.** Let $\tau \ll \nu$ with $\tau(X) = 1$ and $I_K(\tau) < \infty$ (if there is no such $\tau$ then the proof is trivial). By 2.) and 11.1 it follows that

$$E(\mathcal{C}(\tau)(X)) \geq \tau_{\varphi R}(X) = \tau(X) = 1. \quad (11.2)$$

Let $A_i = \{ x : i - 1 \leq \frac{d\tau}{d\nu}(x) < i \}$ for every $i \in \mathbb{N}$. Then $\nu = \sum_{i=1}^{\infty} \nu|_{A_i}$ and $\tau = \sum_{i=1}^{\infty} \tau|_{A_i}$. If $D \subseteq A_i$ is a Borel set then

$$\tau(D) = \int_D \frac{d\tau}{d\nu}(x)d\nu(x) \leq \int_D i d\nu(x) = i \cdot \nu(D).$$

Hence $\sum_{i=1}^{\infty} \frac{1}{i} \mathcal{C}(\tau|_{A_i}) \leq \mathcal{C}(\nu)$ almost surely by 1.). It follows that $\mathcal{C}(\nu)(X) = 0$ implies $\mathcal{C}(\tau)(X) = 0$. Thus

$$P(\mathcal{C}(\nu)(X) > 0) \geq P(\mathcal{C}(\tau)(X) > 0).$$

Using Paley-Zygmund inequality [14, Lemma 3.23], (11.2) and 3.) it follows that

$$P(\mathcal{C}(\tau)(X) \geq \theta) \geq P(\mathcal{C}(\tau)(X) \geq \theta E(\mathcal{C}(\tau)(X))) \geq (1-\theta)^2 \frac{E(\mathcal{C}(\tau)(X))^2}{E(\mathcal{C}(\tau)(X)^2)} \geq (1-\theta)^2 \frac{1}{I_K(\tau)}.$$
Hence $P(\mathcal{C}(\nu)(X) > 0) \geq (1 - \theta)^2 \frac{1}{T_K(\tau)}$ for every $0 < \theta < 1$ and $\tau$. Thus $P(\mathcal{C}(\nu)(X) > 0) \geq C_K(\nu)$. \hfill \qed

Remark 11.4. Assume that $K : X \times X \to \mathbb{R}$ is a nonnegative Borel function and $\nu(X \setminus \bigcup_{i=1}^{\infty} A_i) = 0$ for some Borel sets $A_i$ such that there exists $0 < a_i$ such that

$$a_i \cdot \varphi(x, y) \leq K(x, y)$$  \hspace{1cm} (11.3)

for every $x, y \in A_i$. Then $\tau_{\varphi \perp}(A_i) = 0$ whenever $I_K(\tau |_{A_i}) < \infty$ for some finite Borel measure $\tau$ that satisfies $\tau \ll \nu$. If $I_K(\tau) < \infty$ for some $\tau$ then $I_K(\tau |_{A_i}) \leq I_K(\tau) < \infty$ for every $i \in \mathbb{N}$. Hence $\tau_{\varphi \perp}(A_i) = 0$ for every $i \in \mathbb{N}$ and so (11.1) holds.

Corollary 11.5. Let $\nu$ be a finite Borel measure. Assume that the conditional measure $\mathcal{C}(\nu)$ of $\nu$ on $B$ exists with respect to $\mathcal{Q}_k$ ($k \geq 1$) with regularity kernel $\varphi$ and there exists $0 < c < \infty$ such that

$$E(\mathcal{C}(\tau)(X)^2) \leq c \cdot I_\varphi(\tau)$$  \hspace{1cm} (11.4)

for every finite Borel measure $\tau$ that satisfies $\tau \ll \nu$. Then

$$c^{-1} \cdot C_\varphi(\nu) \leq P(\mathcal{C}(\nu)(X) > 0).$$

Proof. We show that the conditions of Theorem 11.3 hold for $K = c \cdot \varphi$ so we can conclude the statement from Theorem 11.3. Clearly (11.1) holds. It follows from Property $x.$ of Definition 11.3 that whenever $\tau \ll \nu$ then the conditional measure $\mathcal{C}(\tau)$ of $\tau$ on $B$ exists with respect to $\mathcal{Q}_k$ ($k \geq 1$) with regularity kernel $\varphi$. Condition 1.) of Theorem 11.3 holds by Property $ix.)$ and $x.)$ of Definition 11.3, Condition 2.) of Theorem 11.3 holds by Property $v.)$ of Definition 11.3, Condition 3.) of Theorem 11.3 holds by (11.4). \hfill \qed

Theorem 11.6. Let $K : X \times X \to \mathbb{R}$ be a nonnegative Borel function and let $\nu$ be a finite Borel measure such that $\nu(X \setminus X_0) = 0$ for some Borel set $X_0 \subseteq X$. Let $B$ be a random closed set, assume that there exists $b > 0$ such that

$$P(D \cap B \neq \emptyset) \leq b \cdot C_K(D)$$ \hspace{1cm} (11.5)

for every compact set $D \subseteq X_0$. Assume that $\mathcal{C}(\nu | A)$ is a random finite Borel measure for every Borel set $A \subseteq X$ such that the following hold

1.) if $\{H_i\}_{i=1}^{\infty}$ is a sequence of disjoint Borel subsets of $X$ such that $\sum_{i=1}^{\infty} \nu |_{H_i} = \nu$ then $\mathcal{C}(\nu) = \sum_{i=1}^{\infty} \mathcal{C}(\nu |_{H_i})$ almost surely.

2.) for a compact set $D \subseteq X_0$ conditional on the event $D \cap B = \emptyset$ we have that $\mathcal{C}(\nu | D)(X) = 0$ almost surely.

Then

$$P(\mathcal{C}(\nu)(X) > 0) \leq b \cdot C_K(\nu).$$

Proof. Let $A_n \subseteq X_0$ be a sequence of Borel sets such that $\nu(X \setminus A_n) = 0$ and $C_K(A_n) \leq C_K(\nu) + 1/n$. Then for $A := \cap_{n=1}^{\infty} A_n$ we have that $\nu(X \setminus A) = 0$ and $C_K(A) = C_K(\nu)$. Let $D_n \subseteq A \subseteq X_0$ be an increasing sequence of compact sets such that $\nu(X \setminus D_n) = 0$ for every $n \geq 1$. Then

$$P(\mathcal{C}(\nu)(X) > 0) \leq b \cdot C_K(\nu).$$
\( \nu(A \setminus D_n) < 1/n \) (we can find such sequence by inner regularity, Lemma \( \text{[2.12]} \)). Conditional on \( D_n \cap B = \emptyset \) we have that \( \mathcal{C}(\nu|_{D_n})(X) = 0 \) almost surely by 2.). Hence by (11.5)

\[
P(\mathcal{C}(\nu|_{D_n})(X) > 0) = P(\mathcal{C}(\nu|_{D_n})(X) > 0) \quad \text{and} \quad D_n \cap B \neq \emptyset \]

\[
\leq P(D_n \cap B \neq \emptyset) \leq b \cdot C_K(D_n) \leq b \cdot C_K(A) = b \cdot C_K(\nu).
\]

Let \( H_1 = D_1 \) and \( H_n = D_n \setminus D_{n-1} \) for \( n \geq 2 \). Then \( \nu|_{D_n} = \sum_{i=1}^n \nu|_{H_i} \) and \( \nu = \sum_{i=1}^\infty \nu|_{H_i} \), since \( \nu(X \setminus D_n) = \nu(A \setminus D_n) < 1/n \). Hence \( \mathcal{C}(\nu|_{D_n}) = \sum_{i=1}^n \mathcal{C}(\nu|_{H_i}) \) and \( \mathcal{C}(\nu) = \sum_{i=1}^\infty \mathcal{C}(\nu|_{H_i}) \) by 1.). Thus

\[
P(\mathcal{C}(\nu)(X) > 0) = \lim_{n \to \infty} P(\mathcal{C}(\nu|_{D_n})(X) > 0) \leq b \cdot C_K(\nu).
\]

\[\square\]

**Corollary 11.7.** Assume that \( P(D \cap B \neq \emptyset) \leq b \cdot \mathcal{C}_\varphi(D) \) for every compact set \( D \subseteq X_0 \).

Let \( \nu \) be a finite Borel measure such that \( \nu(X \setminus X_0) = 0 \) and assume that the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( \mathcal{Q}_k \) \((k \geq 1)\). Then \( P(\mathcal{C}(\nu)(X) > 0) \leq b \cdot \mathcal{C}_\varphi(\nu) \).

**Proof.** We show that conditions 1.)-2.) of Theorem \( \text{[11.6]} \) hold for \( K = \varphi \). Condition 1.) holds by Property \( \text{ix.) of Definition [1.8]} \). For a compact set \( D \subseteq X_0 \) conditional on \( D \cap B = \emptyset \) we have that \( \mathcal{C}(\nu|_{D})(X) = 0 \) by Lemma \( \text{[5.1]} \), hence condition 2.) holds. \( \square \)

**Remark 11.8.** Let \( X \) be locally compact and \( \nu \) be a locally finite Borel measure on \( X \). Then the conclusion of Theorem \( \text{[11.3]} \) Corollary \( \text{[11.5]} \) Theorem \( \text{[11.6]} \) and Corollary \( \text{[11.7]} \) hold for \( \nu \). The proofs are identical to the proofs of the corresponding results.

**Theorem 11.9.** Let \( \mathcal{Q}_k \), \( \mathcal{Q}_k \) and \( X_0^\infty \) be as in Theorem \( \text{[10.4]} \). Assume that if \( \mathcal{C}_\varphi(D) = 0 \) for some compact set \( D \subseteq X_0 \) then \( B \cap D = \emptyset \) almost surely. Assume that \( F(x, y) = \varphi^c(x, y) = \varphi^c(x, y) \) for every \( x, y \in X \) and there exists \( b > 0 \) such that

\[
P(D \cap B \neq \emptyset) \leq b \cdot \mathcal{C}_F(D)
\]

for every compact set \( D \subseteq X_0 \). Assume that \( X_0 = \bigcup_{i=1}^\infty A_i \) such that \( \text{[11.3]} \) holds. Let either \( \nu \) be a finite Borel measure on \( X \) or \( X \) be locally compact and \( \nu \) be a locally finite Borel measure on \( X \). Assume that \( \nu(X \setminus X_0^\infty) = 0 \). Then the conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( \mathcal{Q}_k \) \((k \geq 1)\) with regularity kernel \( \varphi \) and \( \mathcal{C}_F(\nu) \leq \mathcal{C}(\nu)(X) > 0 \leq b \cdot \mathcal{C}_F(\nu) \).

**Proof.** Let \( K(x, y) = F(x, y) \). Since \( \text{[11.3]} \) holds it follows that \( \text{[11.1]} \) holds by Remark \( \text{[11.4]} \) The conditional measure \( \mathcal{C}(\nu) \) of \( \nu \) on \( B \) exists with respect to \( \mathcal{Q}_k \) \((k \geq 1)\) with regularity kernel \( \varphi \) and Condition 3.) of Theorem \( \text{[11.3]} \) holds by Theorem \( \text{[10.4]} \) Conditions 1.)-2.) of Theorem \( \text{[11.3]} \) hold by Property \( \text{v.) of Definition [1.8]} \) Thus it follows from Theorem \( \text{[11.3]} \) and Remark \( \text{[11.8]} \) that \( \mathcal{C}_F(\nu) \leq P(\mathcal{C}(\nu)(X) > 0) \).

To prove the other inequality, we need to check that the conditions of Theorem \( \text{[11.6]} \) are satisfied to conclude from Theorem \( \text{[11.6]} \) and Remark \( \text{[11.8]} \) that \( P(\mathcal{C}(\nu)(X) > 0) \leq b \cdot \mathcal{C}_F(\nu) \). The conditions of Theorem \( \text{[11.6]} \) can be checked similarly to the proof of Corollary \( \text{[11.7]} \). \( \square \)
12 Conditional measure on the Brownian path

Throughout this section let $B$ be the Brownian path in $\mathbb{R}^d$ ($d \geq 3$), i.e. the range of a Brownian motion which is started at the origin unless stated otherwise. We prove Theorem 1.22 and Theorem 1.16 in this section.

12.1 Existence of the weighted kernel $F(x, y)$

This section is dedicated to show that $F(x, y) = F(x, y) = \overline{F}(x, y)$ when $B$ is the Brownian path.

Lemma 12.1. Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$. Let $x \in \mathbb{R}^d$, $x \neq 0$ and $r > 0$ such that $\|x\| > r$. Then

$$P(B \cap B(x, r) \neq \emptyset) = \frac{r^{d-2}}{\|x\|^{d-2}}.$$  

See [14, Corollary 3.19].

Proposition 12.2. Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$. Then for $x, y \in \mathbb{R}^d \setminus \{0\}$, $x \neq y$

$$\lim_{R \to 0} \lim_{r \to 0} \frac{P(B \cap B(x, R) \neq \emptyset, B \cap B(x, r) \neq \emptyset)}{P(B \cap B(x, R) \neq \emptyset) \cdot P(B \cap B(x, r) \neq \emptyset)} = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}. \quad (12.1)$$

Proof. Let $W(t)$ be a Brownian motion in $\mathbb{R}^d$ for some $d \geq 3$, so $B = \{W(t) : t \in [0, \infty)\}$. We denote by $P = P_0$ the probability measure that corresponds to the Brownian motion that is started at the origin and by $P_z$ the probability measure that corresponds to the Brownian motion that is started at $x \in \mathbb{R}^d$. Let

$$T_{x, R} = \inf \{t \in [0, \infty) : W(t) \in \partial B(x, R)\}$$

for $x \in \mathbb{R}^d$ and $r > 0$. Let $x, y \in \mathbb{R}^d$ and $R, r > 0$ such that $r + R < \|x - y\|$, $R < \|x\|$ and $r < \|y\|$. If $z \in \partial B(x, R)$ then

$$P_z(T_{y, r} < \infty) = \frac{r^{d-2}}{\|x - z\|^{d-2}}$$

by Lemma 12.1, thus

$$\frac{r^{d-2}}{(\|x - y\| + R)^{d-2}} \leq P_z(T_{y, r} < \infty) \leq \frac{r^{d-2}}{(\|x - y\| - R)^{d-2}} \quad (12.2)$$

and similarly for $z \in \partial B(y, r)$

$$\frac{R^{d-2}}{(\|x - y\| + r)^{d-2}} \leq P_z(T_{x, R} < \infty) \leq \frac{R^{d-2}}{(\|x - y\| - r)^{d-2}}. \quad (12.3)$$
Let $U = W(T_{x,R})$ and $V = W(T_{y,r})$ be the stopped Brownian motions. Then by Lemma 12.1 and by the Markov property

$$P(W \text{ hits } B(x, R) \text{ and after that } W \text{ hits } B(y, r))$$

$$= P(T_{x,R} < \infty) \cdot E(P_U(T_{y,r} < \infty) \mid T_{x,R} < \infty) \leq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot \frac{r^{d-2}}{\|x-y\|^{d-2} - R)^{d-2}} \tag{12.4}$$

and similarly

$$P(W \text{ hits } B(y, r) \text{ and after that } W \text{ hits } B(x, R))$$

$$= P(T_{y,r} < \infty) \cdot E(P_V(T_{x,R} < \infty) \mid T_{y,r} < \infty) \leq \frac{r^{d-2}}{\|y\|^{d-2}} \cdot \frac{R^{d-2}}{\|x-y\|^{d-2} - r)^{d-2}}.$$  

Hence

$$P(B \cap B(x, R) \neq \emptyset \text{ and } B \cap B(y, r) \neq \emptyset)$$

$$\leq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot \frac{r^{d-2}}{\|x-y\|^{d-2} - R)^{d-2}} + \frac{r^{d-2}}{\|y\|^{d-2}} \cdot \frac{R^{d-2}}{\|x-y\|^{d-2} - r)^{d-2}}$$

$$= r^{d-2}R^{d-2}\|y\|^{d-2}(\|x-y\|^{d-2} - r)^{d-2} + \|x\|^{d-2}(\|x-y\|^{d-2} - R)^{d-2} \cdot \|x-y\|^{d-2} \cdot \|x-y\|^{d-2} - r)^{d-2})$$

$$\leq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot \frac{r^{d-2}}{\|y\|^{d-2}} \cdot \frac{\|x-y\|^{d-2}}{\|x-y\|^{d-2} - r)^{d-2}} \cdot \|x-y\|^{d-2} - R)^{d-2} \cdot \|x-y\|^{d-2} - r)^{d-2}) \tag{12.5}.$$  

Hence by Lemma 12.1 it follows that

$$\limsup_{R \to 0} \sup_{r \to 0} \frac{P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset)}{P(B \cap B(x, R) \neq \emptyset) \cdot P(B \cap B(y, r) \neq \emptyset)}$$

$$\leq \limsup_{R \to 0} \sup_{r \to 0} \frac{\|y\|^{d-2}\|x-y\|^{d-2} + \|x\|^{d-2}\|x-y\|^{d-2}}{(\|x-y\|^{d-2} - r)^{d-2}) \cdot (\|x-y\|^{d-2} - R)^{d-2})} = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x-y\|^{d-2}}. \tag{12.6}$$

By (12.5) and Lemma 12.1 it follows that

$$P(T_{x,R} < \infty, T_{x,R} \leq T_{y,r}) \geq P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) = \emptyset)$$

$$\geq P(B \cap B(x, R) \neq \emptyset) - P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset) \geq \frac{R^{d-2}}{\|x\|^{d-2}}(1 - O(r)) \tag{12.7}$$

and similarly

$$P(T_{y,r} < \infty, T_{y,r} \leq T_{x,R}) \geq \frac{R^{d-2}}{\|y\|^{d-2}} \cdot (1 - O(R)). \tag{12.8}$$

Thus by Lemma 12.1, (12.2), (12.3), (12.7), (12.8) and by the Markov property

$$P(B \cap B(x, R) \neq \emptyset, B \cap B(y, r) \neq \emptyset) \geq P(T_{x,R} \leq T_{y,r} < \infty) + P(T_{y,r} \leq T_{x,R} < \infty)$$

$$\geq \frac{R^{d-2}}{\|x\|^{d-2}} \cdot (1 - O(r)) \cdot \frac{\|y\|^{d-2}}{(\|x-y\| + R)^{d-2})} + \frac{R^{d-2}}{\|y\|^{d-2}} \cdot (1 - O(R)) \cdot \frac{\|y\|^{d-2}}{(\|x-y\| + R)^{d-2})}$$

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So (12.1) holds by (12.6) and (12.9). □

**Lemma 12.3.** Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$, let $0 \in A \subseteq \mathbb{R}^d$ be a compact set with $\text{diam}(A) > 0$. Then for $x \in \mathbb{R}^d \setminus \{0\}$ and $0 < r < 2^{-1}\|x\|/\text{diam}(A)$

$$C_G(A) \frac{r^{d-2}}{\|x\|^{d-2}} (1 - r \cdot \text{diam}(A)/\|x\|)^{d-2} \leq P((r \cdot A + x) \cap B \neq \emptyset)$$

$$\leq C_G(A) \frac{r^{d-2}}{\|x\|^{d-2}} (1 + 2r \cdot \text{diam}(A)/\|x\|)^{d-2},$$

where $G(x, y) = c(d)\|x - y\|^{2-d}$ is the Green's function of the Brownian motion for some constant $c(d) > 0$.

**Proof.** See [14, Theorem 3.33], that $G(x, y)$ is the Green's function of the Brownian motion. By [14] Corollary 8.12 and [14] Theorem 8.27

$$C_G(r \cdot A + x) (\|x\| + r \cdot \text{diam}(A))^{2-d} \leq P((r \cdot A + x) \cap B \neq \emptyset)$$

$$\leq C_G(r \cdot A + x) (\|x\| - r \cdot \text{diam}(A))^{2-d}. \quad (12.10)$$

On the other hand, by the scaling invariance of capacity it follows that

$$C_G(r \cdot A + x) = r^{d-2}C_G(A). \quad (12.11)$$

We have that $\|x\|^{-1} (1 - r \cdot \text{diam}(A)/\|x\|) \leq (\|x\| + r \cdot \text{diam}(A))^{-1}$ and $(\|x\| - r \cdot \text{diam}(A))^{-1} \leq \|x\|^{-1} (1 + 2r \cdot \text{diam}(A)/\|x\|)$ because $0 < r < 2^{-1}\|x\|/\text{diam}(A)$. Thus the statement follows from (12.10) and (12.11). □

**Proposition 12.4.** Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$, let $0 \in A \subseteq \mathbb{R}^d$ be a compact set such that $C_{d-2}(A) > 0$. Let $x, y \in \mathbb{R}^d \setminus \{0\}$, $x \neq y$ and $x_R, y_R \in \mathbb{R}^d$ for every $r, R > 0$ be such that $\lim_{R \to 0} x_R = x$ and $\lim_{R \to 0} y_R = y$. Then

$$\liminf_{R \to 0} \liminf_{r \to 0} \frac{P((R \cdot A + x_R) \cap B \neq \emptyset, (r \cdot A + y_R) \cap B \neq \emptyset)}{P((R \cdot A + x_R) \cap B \neq \emptyset) \cdot P((r \cdot A + y_R) \cap B \neq \emptyset)}$$

$$= \limsup_{R \to 0} \limsup_{r \to 0} \frac{P((R \cdot A + x_R) \cap B \neq \emptyset, (r \cdot A + y_R) \cap B \neq \emptyset)}{P((R \cdot A + x_R) \cap B \neq \emptyset) \cdot P((r \cdot A + y_R) \cap B \neq \emptyset)} = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}.$$
Let $Q_k(x)$ be the dyadic cube $\left[\frac{i_1}{2^k}, \frac{i_1+1}{2^k}\right) \times \cdots \times \left[\frac{i_d}{2^k}, \frac{i_d+1}{2^k}\right)$ for $i_1, \ldots, i_d \in \mathbb{Z}$ such that $x \in Q_k(x)$ and let $Q_k = \{Q_k(x) : x \in \mathbb{R}^d\}$.

**Proposition 12.5.** Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$. Then

$$\lim \inf_{n \to \infty} \lim \inf_{k \to \infty} \frac{P(B \cap Q_n(x) \neq \emptyset, B \cap Q_k(y) \neq \emptyset)}{P(B \cap Q_n(x) \neq \emptyset) \cdot P(B \cap Q_k(y) \neq \emptyset)}$$

$$= \lim \sup_{n \to \infty} \lim \sup_{k \to \infty} \frac{P(B \cap Q_n(x) \neq \emptyset, B \cap Q_k(y) \neq \emptyset)}{P(B \cap Q_n(x) \neq \emptyset) \cdot P(B \cap Q_k(y) \neq \emptyset)} = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x-y\|^{d-2}}.$$

The proof of Proposition 12.4 goes similarly to the proof of Proposition 12.2 replacing the use of Lemma 12.1 by Lemma 12.3. Proposition 12.5 is a special case of Proposition 12.4.

**Theorem 12.6.** Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$. Let $Q_k$ be as in Example 1.3 then

$$F(x, y) = F(x, y) = F(x, y) = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x-y\|^{d-2}}$$

for $x, y \in \mathbb{R}^d \setminus \{0\}$, $x \neq y$.

Theorem 12.6 follows from Proposition 12.5.

### 12.2 Conditions on $Q_k^i$

For the rest of Section 12 let $Q_k$ be as in Example 1.3, $Q_k^i$ be as in Example 1.4 and let $\varphi(r) = r^{-(d-2)}$. In Section 12.2 we show that the assumptions of Section 11.2 hold for the sequence $Q_k^i$ for sufficient constants.

We have that (1.10) holds for every $\delta > 0$ for sufficient $c_2$ depending on $\delta$ and for $c_3 = 0$. We have that (1.11) holds. Clearly (1.10), (1.11), (1.13), (1.19) hold for both $Q_k$ and $Q_k^i$ for every $i \in \mathbb{N}$ with $M=1$. We have that $Q \cap S = \emptyset$ whenever $Q \neq S$ and $Q, S \in Q_k$ or $Q, S \in Q_k^i$. It is easy to see that (1.18) hold for $Q_k^i$ for every $i \in \mathbb{N}$ for every $\delta > 0$ for constant $M_\delta > 0$ that only depends on $\delta$.

It follows from Lemma 12.3 and the scaling invariance of capacity (12.11) that (12.2) holds for $Q_k^i$ for some sufficient constant $a^i < \infty$.

**Lemma 12.7.** Let $B$ be a Brownian path in $\mathbb{R}^d$ for $d \geq 3$ and let $i$ be a positive integer. Then there exist $0 < \delta = \delta^i < 1$ and $0 < c = c^i < \infty$ such that (1.23) holds for $Q_k^i$ ($k \in \mathbb{N}, k \geq i$) and $c$ and $\delta$ depends only on $i$ and $d$.

**Proof.** Let $W(t)$, $P_x$ and $T_{x,r}$ be as in the proof of Proposition 12.2. Let $A = [0, 1)^d$. Let $N \in \mathbb{N}$ be large enough that

$$2^{-N} \leq 2^{-i}/(2\sqrt{d}).$$ (12.12)

Let $\delta \leq 1/(2\sqrt{d})$, $k, n \geq \max\{i, N\}$ and $x \in Q \in Q_k^i$, $y \in S \in Q_n^i$ such that $\max\{\text{diam}(Q), \text{diam}(S)\} < \delta \cdot \text{dist}(Q, S)$. Let $r = 2^{-k}$ and $R = 2^{-n}$. Then

$$\max(r, R) \leq \max\{\text{diam}(Q), \text{diam}(S)\} < (2\sqrt{d})^{-1} \cdot \text{dist}(Q, S) \leq \|z - y\|/(2\sqrt{d})$$
for every $z \in Q$. So
\[ 1 + 2 \frac{R}{\|z - y\|} \text{diam}(A) \leq 2 \]
for every $z \in Q$ and. Thus by Lemma 12.3
\[ P_z(S \cap B \neq \emptyset) \leq a_0 \frac{R^{d-2}}{\|z - y\|^{d-2}} \leq a_0 \frac{R^{d-2}}{\text{dist}(Q, S)^{d-2}} \quad (12.13) \]
for $a_0 = C_G(A) \cdot 2^{d-2}$ for every $z \in Q$.
By (12.12) it follows that
\[ r \leq 2^{-i}/(2\sqrt{d}) \leq \frac{\|x\|}{2\text{diam}(A)} \]
because $x \in Q \in Q_k^i$. Hence by Lemma 12.3
\[ C_G(A) \frac{r^{d-2}}{\|x\|^{d-2}} 2^{d-2} \leq P(Q \cap B \neq \emptyset) \leq C_G(A) \frac{r^{d-2}}{\|x\|^{d-2}} 2^{d-2}. \quad (12.14) \]
Similarly
\[ C_G(A) \frac{R^{d-2}}{\|y\|^{d-2}} 2^{d-2} \leq P(S \cap B \neq \emptyset) \leq C_G(A) \frac{R^{d-2}}{\|y\|^{d-2}} 2^{d-2}. \quad (12.15) \]
Then similarly to (12.14) it follows from (12.14) and (12.13) that
\[ P(W \text{ hits } Q \text{ and after that } W \text{ hits } S) \leq b \cdot \frac{r^{d-2}}{\|x\|^{d-2}} R^{d-2}\text{dist}(Q, S)^{-(d-2)} \]
\[ = \|y\|^{d-2} b \cdot \frac{r^{d-2}}{\|x\|^{d-2}} \frac{R^{d-2}}{\|y\|^{d-2}} \text{dist}(Q, S)^{-(d-2)} \leq (\sqrt{d^2})^{(d-2)} \sqrt{b} \frac{r^{d-2}}{\|x\|^{d-2}} \frac{R^{d-2}}{\|y\|^{d-2}} \text{dist}(Q, S)^{-(d-2)} \]
for $b = C_G(A)2^{d-2}a_0$. Similarly we can show that
\[ P(W \text{ hits } S \text{ and after that } W \text{ hits } Q) \leq (\sqrt{d^2})^{(d-2)} \sqrt{b} \frac{r^{d-2}}{\|x\|^{d-2}} \frac{R^{d-2}}{\|y\|^{d-2}} \cdot \text{dist}(Q, S)^{-(d-2)} \]
Thus
\[ P(Q \cap B \neq \emptyset \text{ and } S \cap B \neq \emptyset) \leq P(W \text{ hits } Q \text{ and after that } W \text{ hits } D) + P(W \text{ hits } D \text{ and after that } W \text{ hits } Q) \]
\[ \leq 2(\sqrt{d^2})^{(d-2)} \sqrt{b} \frac{r^{d-2}}{\|x\|^{d-2}} \frac{R^{d-2}}{\|y\|^{d-2}} \cdot \text{dist}(Q, S)^{-(d-2)}. \]
Hence it follows from (12.14) and (12.15) that
\[ P(Q \cap B \neq \emptyset \text{ and } S \cap B \neq \emptyset) \leq c P(Q \cap B \neq \emptyset) P(S \cap B \neq \emptyset) \text{dist}(Q, S)^{-(d-2)} \]
for $c = 2(\sqrt{d^2})^{(d-2)} \sqrt{b} 2^{d-2}/C_G(A)^2$.
So (12.23) holds if $k, n \geq \max\{i, N\}$ and $\delta \leq 1/(2\sqrt{d})$. We can choose $\delta > 0$ to be small enough such that if either $k < \max\{i, N\}$ or $n < \max\{i, N\}$ then $\max \{\text{diam}(Q), \text{diam}(S)\} < \delta \cdot \text{dist}(Q, S)$ does not hold for every pair of $Q \in Q_k^i$ and $D \in Q_n^i$. □
12.3 Existence of the conditional measure on the Brownian path

We show Theorem 1.16 in this section.

**Lemma 12.8.** If $A \subseteq \mathbb{R}^d \setminus \{0\}$ is a compact set and $C_{d-2}(A) = 0$ then $P(B \cap A \neq \emptyset) = 0$.

**Proof of Theorem 1.16** In Section 12.2 we establish that the assumptions of Section 1.1.2 hold for sufficient constants $c_2 < \infty$, $c_3 = 0$, $0 < \delta_i < 1$, $M_i = 1$, $a_i < \infty$ and $0 < c_i < \infty$ for the sequence $Q^i_k$. We have that $Q^i_k \subseteq Q^j_k \subseteq Q_k$ for $i \leq j \leq k$. It is easy to see that $X_0^\infty = \cup_{i \in \mathbb{N}} X^i_0 = \mathbb{R}^d \setminus \{0\}$. In Theorem 12.6 we prove that

$$F(x, y) = \overline{F}(x, y) = \overline{F}(x, y) = \frac{\|x\|^{d-2} + \|y\|^{d-2}}{\|x - y\|^{d-2}}.$$  

Thus along with Lemma 12.8 the conditions of Theorem 10.4 are satisfied. Hence the statement follows from Theorem 10.4.

12.4 Probability of non-extinction of the conditional measure on the Brownian path

Theorem 1.22 states an analogous result to Proposition 1.23 for measures.

**Proof of Theorem 1.22** We wish to apply Theorem 11.9 for $K(x, y) = F(x, y)$ to conclude 1.28. In the proof of 1.16 we show that the conditions of Theorem 10.4 are satisfied. Obviously we can find a decomposition $\mathbb{R}^d \setminus \{0\} = \cup_{i=1}^\infty A_i$ such that 11.3 holds, namely $A_i = X^i_0$. So, along with Proposition 1.23, the conditions of Theorem 11.9 are satisfied. The explicit formula for $F(x, y)$ is given in Theorem 12.6.

13 Conditional measure of the Lebesgue measure on the Brownian path and the occupation measure

In this Section we prove Theorem 1.18 that is the union of Theorem 13.18 and Theorem 13.19. The proof consists many steps that we sort in three sections. In Section 13.1 we establish an asymptotic result on the number of cubes of side length 1 that is intersected by the Brownian path. We prove this by the application of the ergodic theorem. The major part of the proof is to show the ergodicity of the invariant measure that we define. Then in Section 13.2 we use the scaling invariance of the Brownian motion to deduce an asymptotic result on the number of small cubes that is intersected by the Brownian path. Finally, in Section 13.3 we use the result on the number of small cubes that is intersected by the Brownian path and an approximation argument to finish the proof. In Section 13.4 we conclude the formula for the second moment of the occupation measure, Theorem 1.20.
For a random variable $Y$ and a $\sigma$-algebra $\mathcal{F} \subseteq \mathcal{A}$ we denote the conditional expectation of $Y$ with respect to $\mathcal{F}$ by $E(Y \mid \mathcal{F})$. For random variables $Y$ and $Z$ we write $E(Y \mid Z)$ for $E(Y \mid \sigma(Z))$ where $\sigma(Z)$ is the $\sigma$-algebra generated by $Z$. In that case there exists a deterministic function $f$ such that $E(Y \mid Z) = f(Z)$ and we write $E(Y \mid Z = z)$ for $f(z)$. We write $E(Y \mid Z_1, \ldots, Z_n)$ when $\mathcal{F}$ is the $\sigma$-algebra generated by $Z_1, \ldots, Z_n$. In the rest of the paper we use many basic properties of the conditional expectation without reference. For an overview of the conditional expectation see for example [1, 6].

Throughout this subsection let $Q_k$ be as in Example 1.8 and let

$$Q_k^* = \left\{ \left[ \frac{i_1}{2^k}, \frac{i_1+1}{2^k} \right) \times \cdots \times \left[ \frac{i_d}{2^k}, \frac{i_d+1}{2^k} \right) : i_1, \ldots, i_d \in \mathbb{Z} \right\}$$

for $k \in \mathbb{N}$. Recall that we denote the Lebesgue measure by $\lambda$.

### 13.1 Application of the ergodic theorem

Throughout this subsection let $B_i$ ($i \in \mathbb{Z}$) be an i.i.d. sequence of Brownian motions started at $0 \in \mathbb{R}^d$ with domain $[0,1]$. Let $X$ be a random variable uniformly distributed on $[0,1]^d$ such that $X$ and $B_i$ ($i \in \mathbb{Z}$) are mutually independent. Let $W_0 = W_0^\omega$ be the random function defined by

$$W_0(t) = \begin{cases} B_0(t) & \text{if } 0 \leq t < 1 \\ B_n(t-n) + \sum_{i=0}^{n-1} B_i(1) & \text{if } 1 \leq n \leq t < n+1 \text{ for some } n \in \mathbb{Z} \\ B_n(t-n) - \sum_{i=n}^{n+1} B_i(1) & \text{if } n \leq t < n+1 \leq 0 \text{ for some } n \in \mathbb{Z} \end{cases}$$

and let $W(t) = X + W_0(t)$, i.e. $W$ is a two sided Brownian motion started at $X$ (it is due to the independent increments of the Brownian motion). Note, that $B_i(\cdot) = (W(\cdot + i) - W(\cdot))_{[0,1]}$. Thus $W$ determines $X$ and the sequence $B_i$ and vice versa.

For a vector $v = (v_1, \ldots, v_d) \in \mathbb{R}^d$ we denote by $\{v\}$ the equivalence class of $v$ in $\mathbb{R}^d/\mathbb{Z}^d$ and we denote by $\{v\}_0$ the element of the equivalence class of $v$ in $\mathbb{R}^d/\mathbb{Z}^d$ that is contained in $[0,1)^d$. For an equivalence class $w \in \mathbb{R}^d/\mathbb{Z}^d$ we also denote by $\{w\}_0$ the element of the equivalence class $w$ that is contained in $[0,1)^d$. Let $X_n = \{W(n)\}$.

Let $S$ be the right shift map on $C([0,1],d)^\mathbb{Z} \times (\mathbb{R}^d/\mathbb{Z}^d)^\mathbb{Z}$, where $C(K,d)$ denotes the space of continuous functions from the compact set $K$ to $\mathbb{R}^d$ equipped with the supremum norm. So $S( ((f_i)_{i=-\infty}^\infty, (x_i)_{i=-\infty}^\infty) ) = ((f_{i+1})_{i=-\infty}^\infty, (x_{i+1})_{i=-\infty}^\infty) $ for $((f_i)_{i=-\infty}^\infty, (x_i)_{i=-\infty}^\infty) \in C([0,1],d)^\mathbb{Z} \times (\mathbb{R}^d/\mathbb{Z}^d)^\mathbb{Z}$. Let $R$ be the probability distribution of $((B_i)_{i=-\infty}^\infty, (X_i)_{i=-\infty}^\infty)$. Then $R$ is a Borel measure. We show that $R$ is a shift invariant ergodic measure.

**Lemma 13.1.** We have that $R$ is a shift invariant measure, i.e. $R(\cdot) = R(S^{-1}(\cdot))$.

**Proof.** Since $X_i = \{W(i)\}$ and $B_i(\cdot) = (W(\cdot + i) - W(\cdot))_{[0,1]}$ we need to show that the random functions $W(\cdot)$ and $W(\cdot - 1) - [W(-1)]$ have the same distribution, where $[\cdot]$ denotes the floor function. We have that

$$E \left( I_{X_0 \in A + \{B_{-1}(1)\}} \cdot I_{B_{-1} \in H} \mid B_{-1} \right) = I_{B_{-1} \in H} \cdot E \left( I_{X_0 \in A + \{B_{-1}(1)\}} \mid B_{-1} \right) = I_{B_{-1} \in H} \cdot P(X_0 \in A)$$
for every Borel set $A \subseteq \mathbb{R}^d/\mathbb{Z}^d$ and $H \subseteq C([0, 1], d)$ because $X_0$ is uniformly distributed on $\mathbb{R}^d/\mathbb{Z}^d$. Hence

$$P(X_0 - \{B_{-1}(1)\} \in A, B_{-1} \in H) = E(I_{B_{-1} \in H} \cdot P(X_0 \in A)) =$$

$$P(B_{-1} \in H)P(X_0 \in A) = P(B_{-1} \in H)P(X_0 - \{B_{-1}(1)\} \in A)$$

Since the collection of sets of the form $A \times H$ is a semi-ring generating the Borel $\sigma$-algebra it follows from Proposition 2.34 that $B_{-1}$ and $\{W(-1)\} = X_0 - \{B_{-1}(1)\}$ are independent. Obviously we have that $B_i (i \in \mathbb{Z} \setminus \{-1\})$ are mutually independent of $B_{-1}$ and $\{W(-1)\} = X_0 - \{B_{-1}(1)\}$ hence $W_0(.-1)$ is a two sided Brownian motion started at 0 such that $\{W(-1)\}$ and $W_0(.-1)$ are independent. Hence $W(.)$ and $W(.-1) - [W(-1)] = \{W(-1)\}_0 + W_0(.-1)$ have the same distribution. \hfill \Box

**Lemma 13.2.** Let $Y$ be a random variable that takes values in $[0, 1)$ almost surely, the distribution of $Y$ is absolutely continuous with respect to the Lebesgue measure and the density function $f$ is a bounded Riemann-integrable function. Then for every Borel set $A \subseteq [0, 1)$

$$\lim_{k \to \infty} P(\{kY\}_0 \in A) = P(U \in A)$$

where $U$ is a random variable uniformly distributed on $[0, 1)$. Let $M < \infty$ be such that $0 \leq f(x) \leq M$ for Lebesgue almost every $x \in \mathbb{R}$. Then the density function of $\{kY\}_0$ is bounded by $M$ for Lebesgue almost every $x \in \mathbb{R}$.

**Proof.** The density function of $kY$ is $k^{-1}f(x/k)$ for Lebesgue almost every $x$ and so the density function of $\{kY\}_0$ is $k^{-1}\sum_{i=0}^{k-1} f((x+i)/k)$ for almost Lebesgue almost every $x \in [0, 1)$ where we used the fact that $f(x) = 0$ for $x \notin [0, 1)$. For every $x \in [0, 1)$ we have that $k^{-1}\sum_{i=0}^{k} f((x+i)/k)$ converges to 1 since $f$ is Riemann integrable. The sequence $k^{-1}\sum_{i=0}^{k} f((x+i)/k)$ is uniformly bounded by $M$ for Lebesgue almost every $x \in \mathbb{R}$. Hence by the dominated convergence theorem $\lim_{k \to \infty} P(\{kY\}_0 \in A) = P(U \in A)$ for every Borel set $A \subseteq [0, 1)$.

**Lemma 13.3.** Let $Y$ be a standard normally distributed vector in $\mathbb{R}^d$. Then for Borel sets $A_1, \ldots, A_d \subseteq \mathbb{R}/\mathbb{Z}$

$$\lim_{k \to \infty} P(\{kY\} \in A_1 \times \cdots \times A_d) = P(U \in A_1 \times \cdots \times A_d)$$

where $U$ is a random variable uniformly distributed on $\mathbb{R}^d/\mathbb{Z}^d$. Additionally, the density functions of $\{kY\}_0$ are uniformly bounded.

**Proof.** Let $Y = (Y_1, \ldots, Y_d)$ and $U = (U_1, \ldots, U_d)$. The sequence of functions $\sum_{i=-n}^{n} e^{-(x+i)^2/2}$ locally uniformly converges to

$$f(x) = \sum_{i=-\infty}^{\infty} e^{-(x+i)^2/2}$$

hence $f$ is a continuous function and so

$$g(x) = \begin{cases} f(x) & \text{if } x \in [0, 1) \\ 0 & \text{otherwise} \end{cases}$$

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is a bounded Riemann-integrable function. Since \( Y_j \) is standard normally distributed in \( \mathbb{R} \) it follows that \( \frac{1}{\sqrt{2\pi}} g \) is the density function of \( \{Y_j\}_0 \), thus

\[
\lim_{k \to \infty} P \left( \{kY_j\} \in A_j \right) = P(U_j \in A_j)
\]

and the density function of \( \{kY_j\}_0 \) is bounded by \( \sup_{x \in [0,1]} g(x) \) by Lemma 13.2. Since \( Y_j \) are independent for \( j = 1, \ldots, d \) it follows that

\[
\lim_{k \to \infty} P \left( \{kY\} \in A_1 \times \cdots \times A_d \right) = \lim_{k \to \infty} P \left( \{kY_1\} \in A_1 \right) \cdots P \left( \{kY_d\} \in A_d \right) = P(U_1 \in A_1) \cdots P(U_d \in A_d) = P(U \in A_1 \times \cdots \times A_d)
\]

and the density function of \( \{kY\}_0 \) is uniformly bounded by \( \left( \sup_{x \in [0,1]} g(x) \right)^d \).

We say that \( C \subseteq \mathbb{R}^d / \mathbb{Z}^d \) is a box in \( \mathbb{R}^d / \mathbb{Z}^d \) if there exists \( 0 \leq a_j \leq b_j \leq 1 \) (\( j = 1, \ldots, d \)) such that \( C = \{ A \} = \{ \{a\} : a \in A \} \) for \( A = [a_1, b_1) \times \cdots \times [a_d, b_d) \).

**Lemma 13.4.** We have that \( R \) is mixing, i.e.

\[
\lim_{k \to \infty} R(A_1 \cap S^{-k}(A_2)) = R(A_1)R(A_2)
\]

for Borel sets \( A_1, A_2 \subseteq C([0, 1], d) / \mathbb{Z} \times (\mathbb{R}^d / \mathbb{Z}^d) / \mathbb{Z} \).

**Proof.** Since \( R \) is shift invariant by Lemma 13.1 due to [16, Theorem 1.17] it is enough to show that the statement holds for every \( A_1 \) and \( A_2 \) taken from a semi-ring of Borel sets that generates the Borel \( \sigma \)-algebra. Hence it is enough to show the statement for Borel sets of the form \( (H_i)_{i=-\infty}^{\infty} \times (C_i)_{i=-\infty}^{\infty} \) where \( H_i \subseteq C([0, 1], d) \) is a Borel set, \( C_i \) is a box in \( \mathbb{R}^d / \mathbb{Z}^d \) for every \( i \in \mathbb{Z} \) and there exists \( n \in \mathbb{N} \) such that \( H_i = C([0, 1], d) \) and \( C_i = \mathbb{R}^d / \mathbb{Z}^d \) for \( i \in \mathbb{Z} \setminus [-n, n] \).

We say that \( C = \{ A \} = \{ \{a\} : a \in A \} \) for \( A = [a_1, b_1) \times \cdots \times [a_d, b_d) \).

Let \( A_1 = (H_1)_{i=-\infty}^{\infty} \times (C_1)_{i=-\infty}^{\infty} \) and \( A_2 = (G_1)_{i=-\infty}^{\infty} \times (D_1)_{i=-\infty}^{\infty} \) where \( H_i, G_i \subseteq [0, 1]^d \) are Borel sets, \( C_i, D_i \) are boxes in \( \mathbb{R}^d / \mathbb{Z}^d \) for every \( i \in \mathbb{Z} \) and there exists \( n \in \mathbb{N} \) such that \( H_i = G_i = C([0, 1], d) \) and \( C_i = D_i = [0, 1]^d \) for \( i \in \mathbb{Z} \setminus [-n, n] \) (note, that without the loss of generality we can assume that \( n \) is the same for both \( A_1 \) and \( A_2 \)). Then

\[
R(A_1 \cap S^{-k}(A_2)) = E \left( \prod_{i=-n}^{n} I_{B_i \in H_i, I_{X_i \in C_i, I_{B_{i-k} \in G_i, I_{X_{i-k} \in D_i}}} \big| X_{-n-k}, (B_i)_{i=-n}^{n}, (B_i)_{i=-n-k}^{n-k} \right).
\]

\[
(13.1)
\]

Let \( k > 2n \). Then

\[
E \left( \prod_{i=-n}^{n} I_{B_i \in H_i, I_{X_i \in C_i, I_{B_{i-k} \in G_i, I_{X_{i-k} \in D_i}}} \big| X_{-n-k}, (B_i)_{i=-n}^{n}, (B_i)_{i=-n-k}^{n-k} \right) =
\]

\[
\left( \prod_{i=-n}^{n} I_{B_i \in H_i, I_{B_{i-k} \in G_i, I_{X_{i-k} \in D_i}}} \right) E \left( \prod_{i=-n}^{n} I_{X_i \in C_i} \big| X_{-n-k}, (B_i)_{i=-n}^{n}, (B_i)_{i=-n-k}^{n-k} \right)
\]

\[
(13.2)
\]

because \( X_{i-k} \) depends only on \( (X_{-n-k}, (B_i)_{i=-n-k}^{n-k}) \). We have that

\[
X_i = \{ W(i) \} = \{ W(i) - W(-n) \} + \{ W(-n) - W(n-k) \} + \{ W(n-k) \}
\]
where $X_{n-k} = \{W(n-k)\}$ depends only on $(X_{n-k}, (B_i)_{i=n-k}^n)$, $Z_i := \{W(i) - W(-i)\} = \{W_0(i) - W_0(-i)\}$ depends only on $(B_i)_{i=n}^n$, $(W(n) - W(n-k)) = \{W_0(n) - W_0(n-k)\}$ is independent of $(X_{n-k}, (B_i)_{i=n-k}^n)$, and $(W(n) - W(n-k))$ and $\{(k-2n)Y\}$ have the same distribution for a standard normally distributed vector $Y$ that is independent of $(X_{n-k}, (B_i)_{i=n-k}^n)$. Thus

$$E\left(\prod_{i=0}^{n} I_{X_i \in C_i} \mid X_{n-k}, (B_i)_{i=n-k}^n \right) =$$

$$E\left(\prod_{i=0}^{n} I_{(k-2n)Y \in C_i - Z_i - X_{n-k}} \mid X_{n-k}, (B_i)_{i=n-k}^n \right) =$$

$$E\left(I_{(k-2n)Y \in \bigcap_{i=0}^{n}(C_i - Z_i - X_{n-k})} \mid X_{n-k}, (B_i)_{i=n-k}^n \right)$$

and so it follows from (13.2) that

$$E\left(\prod_{i=0}^{n} I_{B_i \in H_i, I_{X_i \in C_i}, I_{B_{i-k} \in G_i}, I_{X_{i-k} \in D_i}} \mid X_{n-k}, (B_i)_{i=n-k}^n \right) =$$

$$\left(\prod_{i=0}^{n} I_{B_i \in H_i, I_{B_{i-k} \in G_i}, I_{X_{i-k} \in D_i}}\right) E\left(I_{(k-2n)Y \in \bigcap_{i=0}^{n}(C_i - Z_i - X_{n-k})} \mid X_{n-k}, (B_i)_{i=n-k}^n \right).$$

(13.3)

Let $(\widetilde{B}_i, \widetilde{X}_i)^n_{i=n-k}$ be a random variable with the same distribution as $(B_{i-k}, X_{i-k})_{i=n-k}$ such that $(\widetilde{B}_i, \widetilde{X}_i)^n_{i=n-k}$ is independent of $(Y, X_{n-k}, (B_i)_{i=n-k}^n, (B_i)_{i=n-k}^n)$. Then the shift invariance (Lemma 13.1) $(\widetilde{B}_i, \widetilde{X}_i)^n_{i=n-k}$ and $(B_i, X_i)^n_{i=n-k}$ have the same distribution. Then

$$\left(\prod_{i=0}^{n} I_{B_i \in H_i, I_{B_{i-k} \in G_i}, I_{X_{i-k} \in D_i}}\right) E\left(I_{(k-2n)Y \in \bigcap_{i=0}^{n}(C_i - Z_i - X_{n-k})} \mid X_{n-k}, (B_i)_{i=n-k}^n, (B_i)_{i=n-k}^n \right)$$

and

$$\left(\prod_{i=0}^{n} I_{B_i \in H_i, I_{\widetilde{B}_i \in G_i}, I_{\widetilde{X}_i \in D_i}}\right) E\left(I_{(k-2n)Y \in \bigcap_{i=0}^{n}(C_i - Z_i - \widetilde{X}_n)} \mid \widetilde{X}_{n-k}, (B_i)_{i=n-k}^n, (\widetilde{B}_i)_{i=n-k}^n \right)$$

have the same distribution. Hence by (13.1) and (13.3) it follows that

$$\lim_{k \to \infty} R(A_1 \cap S^{-k}(A_2)) =$$

$$\lim_{k \to \infty} E\left(\left(\prod_{i=0}^{n} I_{B_i \in H_i, I_{\widetilde{B}_i \in G_i}, I_{\widetilde{X}_i \in D_i}}\right) E\left(I_{(k-2n)Y \in \bigcap_{i=0}^{n}(C_i - Z_i - \widetilde{X}_n)} \mid \widetilde{X}_{n-k}, (B_i)_{i=n-k}^n, (\widetilde{B}_i)_{i=n-k}^n \right)\right).$$

(13.4)
It follows from Lemma 13.3 that
\[
\lim_{k \to \infty} E \left( I_{\{k(2n)\}} \in \cap_i \{ \tilde{X}_i - n \} \mid \tilde{X}_n, (B_i)_i, (\tilde{B}_i)_i \right) =
\]
\[
E \left( I_{U \in \cap_i \{ \tilde{X}_i - n \}} \mid \tilde{X}_n, (B_i)_i, (\tilde{B}_i)_i \right) =
\]
\[
E \left( I_{U \in \cap_i \{ c_i - z_i \}} \mid \tilde{X}_n, (B_i)_i, (\tilde{B}_i)_i \right) =
\]
almost surely where \( U \) is a random variable that is independent of \( (\tilde{X}_n, (B_i)_i, (\tilde{B}_i)_i) \) and is uniformly distributed in \( \mathbb{R}^d / \mathbb{Z}^d \). Thus by the dominated convergence theorem and (13.4) it follows that
\[
\lim_{k \to \infty} R(A_1 \cap S^{-k}(A_2)) =
\]
\[
E \left( \left( \prod_{i=n}^n I_{B_i \in H_i} I_{\tilde{B}_i \in G_i} I_{\tilde{X}_i \in D_i} \right) \mid \tilde{X}_n, (B_i)_i, (\tilde{B}_i)_i \right) =
\]
\[
E \left( \left( \prod_{i=n}^n I_{U \in \{ c_i - z_i \}} \mid (B_i)_i \right) =
\]
Thus by (13.5)
\[
\lim_{k \to \infty} R(A_1 \cap S^{-k}(A_2)) =
\]
\[
E \left( \left( \prod_{i=n}^n I_{B_i \in H_i} I_{\tilde{B}_i \in G_i} I_{\tilde{X}_i \in D_i} \right) \mid (B_i)_i \right) =
\]
\[
R(A_2) E \left( \left( \prod_{i=n}^n I_{B_i \in H_i} I_{U \in \{ c_i \}} \mid (B_i)_i \right) \right)
\]
since \( \prod_{i=n}^n I_{B_i \in G_i} I_{\tilde{X}_i \in D_i} \) and \( E \left( \prod_{i=n}^n I_{B_i \in H_i} I_{U \in \{ c_i \}} \mid (B_i)_i \right) \) are independent and that \( (B_i, X_i)_i \) and \( (B_i, \tilde{X}_i)_i \) have the same distribution.

Similarly to the proof of Lemma 13.1 it can be shown that \( U - \{ W_0(-n) \} \) and \( W_0 \) are independent and \( W = X + W_0 \) and \( \{ U - \{ W_0(-n) \} \} \) and \( W_0 \) have the same distribution. Since \( Z_i = \{ W_0(i) \} - \{ W_0(-n) \} \) it follows that
\[
E \left( \left( \prod_{i=n}^n I_{B_i \in H_i} I_{U \in \{ c_i \}} \right) = E \left( \left( \prod_{i=n}^n I_{B_i \in H_i} I_{\{ W(i) \}} \right) = R(A_1)
\]
and this finishes the proof combined with (13.6).  \[\square\]

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Lemma 13.5. Let
\[ f(B_i, X_i)_{i=\infty}^{\infty} = \# \{ Q \in \mathcal{Q}_0^* : W([0, 1]) \cap Q \neq \emptyset, W((-\infty, 0)) \cap Q = \emptyset \}, \]
i.e. the number of dyadic cubes of side length 1 that is visited by \( W \) in the time interval \([0, 1] \) but never visited before. Let
\[ g(B_i, X_i)_{i=\infty}^{\infty} = \# \{ Q \in \mathcal{Q}_0^* : W([0, 1]) \cap Q \neq \emptyset \}, \]
i.e. the number of dyadic cubes of side length 1 that is visited by \( W \) in the time interval \([0, 1] \). Then \( f \) and \( g \) are \( \mathcal{L}^1 \) functions with respect to the probability measure \( R \).

Proof. It is easy to see that \( f \) and \( g \) are measurable and that \( 0 \leq f \leq g \). Thus to prove the statement it is enough to show that
\[ E \left( g (B_i, X_i)_{i=\infty}^{\infty} \right) < \infty. \]

Let \( Y_1(t) \) be a Brownian motion in \( \mathbb{R} \) started at 0 and let \( x_1 \in [0, 1) \). Then
\[ \# \{ Q \in \mathcal{Q}_0^* : x_1 + Y_1([0, 1]) \cap Q \neq \emptyset \} \leq 2 + 2 \max_{t \in [0, 1]} |x_1 + Y_1(t)| \leq 2 + 2x_1 + 2 \max_{t \in [0, 1]} |Y_1(t)|. \]
(13.7)

Note that \( \mathcal{Q}_0^* \) consists dyadic intervals of the line in (13.7) because \( Y_1 \) is a Brownian motion in \( \mathbb{R} \), however, later on \( Y \) is a Brownian path in \( \mathbb{R}^d \) and so \( \mathcal{Q}_0^* \) consists dyadic cubes of \( \mathbb{R}^d \). Let \( Y(t) = (Y_1(t), \ldots, Y_d(t)) \) be a Brownian motion in \( \mathbb{R}^d \) started at 0 and let \( (x_1, \ldots, x_d) \in [0, 1]^d \). Then by (13.7)
\[ \# \{ Q \in \mathcal{Q}_0^* : x + Y([0, 1]) \cap Q \neq \emptyset \} \leq \prod_{i=1}^{d} \left( 4 + 2 \max_{t \in [0, 1]} |Y_i(t)| \right) . \]

Since \( Y_1, \ldots, Y_d \) are mutually independent it follows that
\[ E \left( g (B_i, X_i)_{i=\infty}^{\infty} \right) \leq E \left( \prod_{i=1}^{d} \left( 4 + 2 \max_{t \in [0, 1]} |Y_i(t)| \right) \right) \leq \prod_{i=1}^{d} \left( 4 + 2 E \left( \max_{t \in [0, 1]} |Y_i(t)| \right) \right) < \infty \]
because \( E \left( \max_{t \in [0, 1]} |Y_i(t)| \right) < \infty \) by [14, Theorem 2.21].

\[ \Box \]

Lemma 13.6. There exist \( 0 \leq \alpha \leq \beta < \infty \) such that
\[ \lim_{N \to \infty} N^{-1} \# \{ Q \in \mathcal{Q}_0^* : W([0, N]) \cap Q \neq \emptyset, W((-\infty, 0)) \cap Q = \emptyset \} = \alpha \quad (13.8) \]

\( R \) almost surely and
\[ \lim_{N \to \infty} N^{-1} \sum_{i=0}^{N-1} \# \{ Q \in \mathcal{Q}_0^* : W([i, i+1]) \cap Q \neq \emptyset \} = \beta \quad (13.9) \]

\( R \) almost surely.
Proof. Let $f$ and $g$ be as in Lemma 13.5. Then
\[
\# \{Q \in Q_0^*: W([0, N]) \cap Q \neq \emptyset, W((-\infty, 0)) \cap Q = \emptyset\} = \sum_{i=0}^{N-1} f \left( S^i (B)_i \right)_{i=-\infty}^\infty
\]
(note that $W(N) \notin \partial Q$ almost surely for every $N \in \mathbb{Z}$ and $Q \in Q_0^*$) and
\[
\sum_{i=0}^{N-1} \# \{Q \in Q_0^*: W([i, i+1]) \cap Q \neq \emptyset\} = \sum_{i=0}^{N-1} g \left( S^i (B)_i \right)_{i=-\infty}^\infty.
\]
By Lemma 13.5 the functions $f$ and $g$ are $L^1$ functions with respect to the probability measure $\mathbb{R}$, by Lemma 13.1 we have that $\mathbb{R}$ is a shift invariant measure and by Lemma 13.4 we have that $\mathbb{R}$ is an ergodic measure since mixing implies ergodicity. Hence by Birkhoff’s ergodic theorem ([16, Theorem 1.14] and the Remark after the theorem) the statement follows for
\[
\alpha = E \left( f (B)_i \right)_{i=-\infty}^\infty
\]
and
\[
\beta = E \left( g (B)_i \right)_{i=-\infty}^\infty.
\]

Proposition 13.7. Let $0 \leq \alpha \leq \beta < \infty$ be as in Lemma 13.6. Let $\tilde{W}_n(t)$ be two-sided Brownian motions in $\mathbb{R}^d$ started at 0 for every $n \in \mathbb{N}$. Let $Y_n$ be a random variable for every $n \in \mathbb{N}$, that is independent of $\tilde{W}_n$ (not necessarily i.i.d.), takes values in $\mathbb{R}^d$, the distributions of $Y_n$ are absolutely continuous with respect to the Lebesgue measure and the density functions of $\{Y_n\}_0$ are uniformly bounded. Let $\tilde{W}_n = \tilde{W}_n + Y_n$. Then
\[
f_N(\omega) = N^{-1} \# \left\{ Q \in Q_0^*: \tilde{W}_N([0, N]) \cap Q \neq \emptyset, \tilde{W}_N((-\infty, 0)) \cap Q = \emptyset \right\}
\]
converges to $\alpha$ in probability as $N$ goes to $\infty$ and
\[
g_N(\omega) = N^{-1} \sum_{i=0}^{N-1} \# \left\{ Q \in Q_0^*: \tilde{W}_N([i, i+1]) \cap Q \neq \emptyset \right\}
\]
converges to $\beta$ in probability as $N$ goes to $\infty$.

Proof. The quantity in (13.10) does not change if we replace $Y_n$ by $\{Y_n\}_0$, hence without the loss of generality we assume that $Y_n = \{Y_n\}_0$. Let $\varepsilon > 0$ be fixed. Let
\[
h_N(\omega) = N^{-1} \# \left\{ Q \in Q_0^*: W([0, N]) \cap Q \neq \emptyset, W((-\infty, 0)) \cap Q = \emptyset \right\},
\]
then
\[
P(h_N < \alpha - \varepsilon) = \int_{\mathbb{R}^d} E (I_{f_N < \alpha - \varepsilon} | Y_N = x) \, dx
\]
(13.12)
because \( \hat{W}_n \) and \( W_0 \) have the same distribution. Let \( D_n(x) \) be the density function of \( \{Y_n\}_0 \) and let \( M < \infty \) be the uniform bound that \( D_n(x) \leq M \) for every \( n \in \mathbb{N} \) and Lebesgue almost every \( x \in [0,1]^d \). Then

\[
P(f_N < \alpha - \varepsilon) = \int_{[0,1]^d} E(I_{f_N < \alpha - \varepsilon} | Y_N = x) D_N(x) dx \leq M \cdot P(h_N < \alpha - \varepsilon)
\]

by (13.12). We have that \( h_N \) converges to \( \alpha \) almost surely by Lemma 13.6 and so in probability. Hence

\[
\lim_{N \to \infty} P(f_N < \alpha - \varepsilon) = 0
\]

by (13.13). Similarly we can show that

\[
\lim_{N \to \infty} P(f_N > \alpha + \varepsilon) = 0,
\]

and hence

\[
\lim_{N \to \infty} P(|f_N - \alpha| > \varepsilon) = 0,
\]

(13.14)
i.e. \( f_n \) converges to \( \alpha \) in probability as \( N \) goes to \( \infty \).

The proof of

\[
\lim_{N \to \infty} P(|g_N - \beta| > \varepsilon) = 0
\]

for every \( \varepsilon > 0 \) is similar to the proof of (13.14), we omit the details. \( \square \)

### 13.2 Number of small cubes intersected by the Brownian path

We say that a number \( a \in \mathbb{R} \) is a dyadic number if \( a = n \cdot 2^{-k} \) for some \( n \in \mathbb{Z}, k \in \mathbb{N} \). We say that an interval \( J \subseteq \mathbb{R} \) is an interval with dyadic endpoints if there exist dyadic numbers \( a < b \) such that \( J \) is one of the following intervals \( (a,b), [a,b], [a,b), (a,b] \).

Throughout this subsection let \( W_0(t) \) be a two-sided Brownian motion in \( \mathbb{R}^d \) started at \( 0 \). For a compact interval \( I = [a,b] \subseteq \mathbb{R} \) with dyadic endpoints we define the following random variables:

\[
f_I^k = 2^{-2k} \# \{Q \in \mathcal{Q}_k^* : W_0(I) \cap Q \neq \emptyset, W_0((-\infty,a)) \cap Q = \emptyset\}
\]

and

\[
g_I^k = 2^{-2k} \sum_{i=0}^{N_k-1} \# \{Q \in \mathcal{Q}_k^* : W_0([a+i2^{-2k}, a+(i+1)2^{-2k}]) \cap Q \neq \emptyset\}
\]

where \( N_k = (b-a)2^{2k} \) (note, that \( N_k \) is an integer for large enough \( k \) because \( b-a \) is also a dyadic number).

**Lemma 13.8.** Let \( r > 0 \) be fixed. Then \( (W_0(t) : t \in \mathbb{R}) \) and \( (r^{-1}W_0(r^2t) : t \in \mathbb{R}) \) have the same distribution.

Lemma 13.8 is a folklore in the theory of Brownian motions, see for example [14, Lemma 1.7].
Lemma 13.9. Let $0 \leq \alpha \leq \beta < \infty$ be as in Lemma 13.4. Let $W_0(t)$ be a two-sided Brownian motion in $\mathbb{R}^d$ ($d \geq 3$) started at 0. Then for every $0 < a < b < \infty$ dyadic numbers and for $I = [a, b]$ we have that $f_k^I$ converges to $\alpha(b - a)$ in probability as $k$ goes to $\infty$ and $g_k^I$ converges to $\beta(b - a)$ in probability as $k$ goes to $\infty$.

Proof. Let $0 < a < b < \infty$ be fixed dyadic numbers. Assume that $k$ is large enough that $N_k$ is an integer. Let $\tilde{W}_k(t) = 2^kW_0(2^{-2k}t + a)$ for every $t \in \mathbb{R}$. Then

$$\# \left\{ Q \in \mathcal{Q}_k^* : W_0(I) \cap Q \neq \emptyset, W_0((-\infty, a)) \cap Q = \emptyset \right\} = \# \left\{ Q \in \mathcal{Q}_k^* : \tilde{W}_k([0, N_k]) \cap Q \neq \emptyset, \tilde{W}_k((-\infty, 0)) \cap Q = \emptyset \right\}$$

and

$$\# \left\{ Q \in \mathcal{Q}_k^* : W_0([a + i2^{-2k}, a + (i + 1)2^{-2k}]) \cap Q \neq \emptyset \right\} = \# \left\{ Q \in \mathcal{Q}_k^* : \tilde{W}_k([i, i + 1]) \cap Q \neq \emptyset \right\}.$$

Hence, to complete the proof, we need to show that

$$N_k^{-1} \# \left\{ Q \in \mathcal{Q}_k^* : \tilde{W}_k([0, N_k]) \cap Q \neq \emptyset, \tilde{W}_k((-\infty, a)) \cap Q = \emptyset \right\}$$

converges to $\alpha$ in probability and

$$N_k^{-1} \sum_{i=0}^{N_k-1} \# \left\{ Q \in \mathcal{Q}_k^* : \tilde{W}_k([i, i + 1]) \cap Q \neq \emptyset \right\}$$

converges to $\beta$ in probability (note that $2^{-2k} = (b - a) \cdot N_k^{-1}$).

We have that $(W_0(t) : t \in \mathbb{R})$ and $(W_0(t + a) - W_0(a) : t \in \mathbb{R})$ have the same distribution. Then for $\tilde{W}_k(t) = 2^kW_0(2^{-2k}t + a) - 2^kW_0(a)$ we get that $(\tilde{W}_k(t) : t \in \mathbb{R})$ and $(W_0(t) : t \in \mathbb{R})$ have the same distribution by Lemma 13.8 for $r = 2^{-k}$. Let $Y_k = 2^kW_0(a)$. Then the distributions of $Y_k$ are absolutely continuous with respect to the Lebesgue measure and the density functions of $\{Y_k\}_0$ are uniformly bounded by Lemma 13.3. Thus the quantities in (13.15) and (13.16) are converging in probability to the desired limit by Proposition 13.7. \qed

Lemma 13.10. Let $W_0(t)$ be a two-sided Brownian motion in $\mathbb{R}^d$ ($d \geq 3$) started at 0 and let $I \subseteq \mathbb{R}$ be a compact interval. Let

$$D = \left\{ t \in \mathbb{R} \setminus \text{int}I : \exists s \in I, W_0(t) = W_0(s) \right\}.$$

Then $D$ is a random compact set and

$$\dim_H D \leq 1/2$$

almost surely, where $\dim_H$ denotes the Hausdorff dimension. In particular, $\lambda(D) = 0$ almost surely.
Proof. We have that
\[ W_0(D) = W_0(I) \cap W_0(\mathbb{R} \setminus \text{int} I) \]
hence almost surely the intersection of a compact set and a closed set and so \( W_0(D) \) is almost surely a compact set. It follows from [14, Lemma 9.4] and the Markov property that \( \dim_H W_0(D) \leq 1 \) almost surely for \( d = 3 \). It follows from [14, Theorem 9.1] and the Markov property that \( D = \partial I \) almost surely for \( d \geq 4 \). So \( \dim_H W_0(D) \leq 1 \) almost surely.

Since \( W_0 \) is almost surely a continuous function it follows that
\[ \partial I = W_0^{-1}(W_0(D)) \cap (\mathbb{R} \setminus \text{int} I) \]
is a closed set and bounded because of the transience of the Brownian motion [14, Theorem 3.20], thus \( D \) is compact. It follows from the fact that \( \dim_H W_0(D) \leq 1 \) almost surely and from Kaufman’s dimension doubling theorem [14, Theorem 9.28] that
\[ \dim_H D \leq 1/2 \]
almost surely.

Lemma 13.11. Let \( W_0(t) \) be a two-sided Brownian motion in \( \mathbb{R}^d \) \( (d \geq 3) \) started at 0, let \( I \subseteq \mathbb{R} \) be a compact interval and let \( \varepsilon > 0 \). Then almost surely there exist finitely many random open intervals \( J_1, \ldots, J_m \) with dyadic endpoints such that
\[ \text{dist} \left( W_0(I), W_0(\mathbb{R} \setminus \left( \text{int} I \cup \left( \bigcup_{i=1}^m J_i \right) \right) \right) > 0 \]
and
\[ \sum_{i=1}^m \lambda(J_i) < \varepsilon. \]

Proof. Let \( D \) be as in Lemma 13.10. Since \( \lambda(D) = 0 \) and \( D \) is compact almost surely we can cover \( D \) with finitely many random open intervals \( J_1, \ldots, J_m \) with dyadic endpoints such that \( \sum_{i=1}^m \lambda(J_i) < \varepsilon \). It is not hard to see that we can choose the open intervals \( J_1, \ldots, J_m \) on a Borel measurable way, so we legitimately say random open intervals.

We have that \( W_0(I) \) and \( W_0(\mathbb{R} \setminus (\text{int} I \cup (\bigcup_{i=1}^m J_i)) \) are disjoint almost surely due to the definition of \( D \) (note that the endpoints of \( I \) are contained in \( D \)). Thus
\[ \text{dist} \left( W_0(I), W_0(\mathbb{R} \setminus (\text{int} I \cup (\bigcup_{i=1}^m J_i)) \right) > 0 \]
almost surely because \( W_0(I) \) is almost surely compact and \( W_0(\mathbb{R} \setminus (\text{int} I \cup (\bigcup_{i=1}^m J_i)) \) is almost surely closed.

Remark 13.12. It follows from [14, Corollary 3.19] that \( W(\mathbb{R}) \cap \{0\} = \emptyset \) almost surely. Hence for every deterministic interval \( I \subseteq \mathbb{R} \) we have that
\[ \{Q \in \mathcal{Q}_k : W(I) \cap Q \neq \emptyset \} = \{Q \in \mathcal{Q}_k^* : W(I) \cap Q \neq \emptyset \} \]
almost surely.
Proposition 13.13. Let $0 \leq \alpha < \infty$ be as in Lemma 13.9. Let $W_0(t)$ be a two-sided Brownian motion in $\mathbb{R}^d$ ($d \geq 3$) started at 0. Let $I = \bigcup_{n=1}^{N} I_n$ where $I_1, \ldots, I_n$ are disjoint compact intervals with positive dyadic endpoints. Then

$$h_{\beta_k}^I := 2^{-2k} \# \{Q \in Q_k : W_0(I) \cap Q \neq \emptyset\}$$

converges to $\alpha \lambda(I)$ in probability.

Proof. To prove the statement of the proposition, by Lemma 2.9, it is enough to show that for every $\varepsilon > 0$ and for every subsequence $\{\alpha_k\}_{k=1}^\infty$ of $\mathbb{N}$ we can find a subsequence $\{\beta_k\}_{k=1}^\infty$ of $\{\alpha_k\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} P \left( |h_{\beta_k}^I - \alpha \lambda(I)| > \varepsilon \right) = 0.$$  \hfill (13.18)

It follows from Remark 13.12 that

$$h_{\beta_k}^I = 2^{-2k} \# \{Q \in Q_k^* : W_0(I) \cap Q \neq \emptyset\}$$

which we use throughout the proof instead of (13.17).

Let $\varepsilon > 0$ be fixed and $\{\alpha_k\}_{k=1}^\infty$ be a subsequence of $\mathbb{N}$. Let $0 \leq \beta < \infty$ be as in Lemma 13.9. For every compact interval $J$ with positive dyadic endpoints we have that $f_{\beta_k}^I$ converges to $\alpha \lambda(J)$ in probability and $g_{\beta_k}^I$ converges to $\beta \lambda(J)$ in probability by Lemma 13.9. Hence, by Lemma 2.8, we can find an event $H$ with $P(H) = 1$ and a subsequence $\{\beta_k\}_{k=1}^\infty$ of $\{\alpha_k\}_{k=1}^\infty$ such that $f_{\beta_k}^I(\omega)$ converges to $\alpha \lambda(J)$ and $g_{\beta_k}^I(\omega)$ converges to $\beta \lambda(J)$ for every interval $J$ with positive dyadic endpoints for every outcome $\omega \in H$. We can further assume, by Lemma 13.11, that there exist $m_j(\omega) \in \mathbb{N}$ and finitely many open intervals $J_i^J(\omega), \ldots, J_{m_j}^J(\omega)$ with dyadic endpoints for $j = 1, \ldots, N$ such that

$$r_j := \text{dist} \left( W_0^\omega(I_j), W_0^\omega(\mathbb{R} \setminus \left( \text{int} I_j \bigcup \bigcup_{i=1}^{m_j} J_i^J(\omega) \right) ) \right) > 0$$

and

$$\sum_{i=1}^{m_j(\omega)} \lambda(J_i^J(\omega)) < \varepsilon(2N\beta)^{-1}$$

for every $\omega \in H$.

Let $\omega \in H$ be fixed. If $k$ is large enough that $\text{diam}(Q) < 2^{-1}r_j$ for some $j$ and $Q \in Q_k^*$ then whenever $Q \cap W_0^\omega(I_j) \neq \emptyset$ then, by (13.19), either $Q \cap W_0^\omega(I_j) \neq \emptyset$ for $I_j = [a_j, b_j]$ and $Q \cap W_0^\omega(-\infty, a_j) = \emptyset$ or $Q \cap W_0^\omega(I_j^J(\omega)) \neq \emptyset$ for some $i$. Thus

$$\sum_{j=1}^{N} f_{\beta_k}^I(\omega) \leq h_{\beta_k}^I(\omega) \leq \left( \sum_{j=1}^{N} f_{\beta_k}^J(\omega) \right) + \left( \sum_{j=1}^{N} \sum_{i=1}^{m_j(\omega)} g_{\beta_k}^J(\omega) \right)$$

for large enough $k$, where $\overline{J_i}(\omega)$ is the closure of $J_i(\omega)$. Hence

$$\limsup_{k \to \infty} \left| h_{\beta_k}^I(\omega) - \sum_{j=1}^{N} f_{\beta_k}^J(\omega) \right| \leq \limsup_{k \to \infty} \sum_{j=1}^{N} \sum_{i=1}^{m_j(\omega)} g_{\beta_k}^J(\omega) \leq \sum_{j=1}^{N} \sum_{i=1}^{m_j(\omega)} \beta \lambda(\overline{J_i}(\omega)) < \varepsilon/2$$
by (13.20). Since this holds for every $\omega \in H$ it follows that

$$\lim_{k \to \infty} P \left( \left| h_{\beta k}^I - \sum_{j=1}^N f_{\beta k}^I(\omega) \right| > \frac{\varepsilon}{2} \right) = 0. \quad (13.21)$$

We have that

$$P \left( \left| h_{\beta k}^I - \alpha \lambda(I) \right| > \varepsilon \right) \leq P \left( \left| h_{\beta k}^I - \sum_{j=1}^N f_{\beta k}^I(\omega) \right| > \frac{\varepsilon}{2} \right) + P \left( \left| \sum_{j=1}^N f_{\beta k}^I(\omega) - \alpha \lambda(I) \right| > \frac{\varepsilon}{2} \right),$$

and hence (13.18) follows from (13.21) and the fact that $\sum_{j=1}^N f_{\beta k}^I(\omega)$ converges to $\alpha \lambda(I)$ on $H$.

### 13.3 Occupation measure and limit measure

Throughout this subsection let $B_0(.)$ be a standard Brownian motion in $\mathbb{R}^d$ ($d \geq 3$) started at 0, let $B = \{B_0(t) : t \in [0, \infty)\}$ be the range of $B_0$. Let $\tau$ be the occupation measure of $B_0(.)$, that is

$$\tau(A) = \int_0^\infty 1_{B_0(t) \in A} dt \quad (13.22)$$

for every Borel set $A \subseteq \mathbb{R}^d$, i.e. the amount of time that the Brownian motion spends in $A$.

**Lemma 13.14.** Let $Q \in \mathcal{Q}_n$ for some $n \in \mathbb{N}$. Then $\lambda(\partial B_0^{-1}(Q)) = 0$ almost surely.

**Proof.** Since $B_0$ is almost surely continuous it follows that $\partial B_0^{-1}(Q) \subseteq B_0^{-1}(\partial Q)$ almost surely. Hence it is enough to prove that $\lambda \left( B_0^{-1}(\partial Q) \right) = 0$. Let $B_0(t) = (B_1(t), \ldots, B_d(t))$ for every $t \in [0, \infty)$. Then

$$\dim_H(B_i^{-1}(a)) = 1/2$$

almost surely for every $i = 1, \ldots, d$ and $a \in \mathbb{R}$ by [14, Theorem 9.34]. For every side of $Q$ there exists $i \in \{1, \ldots, d\}$ and $a \in \mathbb{R}$ such that the preimage of that side of $Q$ is contained in $B_i^{-1}(a)$. Thus it follows that

$$\lambda \left( B_0^{-1}(\partial Q) \right) = 0$$

because $Q$ has finitely many sides. \qed

**Lemma 13.15.** Let $Q \in \mathcal{Q}_n$ for some $n \in \mathbb{N}$ such that $\text{dist}(Q, 0) > 0$ and let $\varepsilon > 0$ be fixed. Then almost surely there exist random sets $I^- = \bigcup_{n=1}^N I_n^-$ where $I_1^-, \ldots, I_N^-$ are random disjoint compact intervals with positive dyadic endpoints and $I^+ = \bigcup_{n=1}^N I_n^+$ where $I_1^+, \ldots, I_N^+$ are random disjoint compact intervals with positive dyadic endpoints such that

$$I^- \subseteq B_0^{-1}(Q) \subseteq I^+$$

and $\lambda(I^+ \setminus I^-) < \varepsilon$. 

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Proof. We have that $\lambda(\partial B_0^{-1}(Q)) = 0$ almost surely by Lemma 13.14. Hence almost surely there exist random sets $I^- = \bigcup_{n=1}^{N^-} I_n^-$ where $I_1^-, \ldots, I_{N^-}^-$ are random disjoint compact intervals with positive dyadic endpoints and $I^+ = \bigcup_{n=1}^{N^+} I_n^+$ where $I_1^+, \ldots, I_{N^+}^+$ are random disjoint compact intervals with positive dyadic endpoints such that

$$I^- \subseteq B_0^{-1}(Q) \subseteq I^+$$

and $\lambda(I^+ \setminus I^-) < \varepsilon$. It is easy to see that we can choose the intervals on a measurable way. \qed

Lemma 13.16. Let $Q \in Q_n$ for some $n \in \mathbb{N}$ such that $\text{dist}(Q, 0) > 0$. Then there exists $0 < \gamma_0 = \gamma_0(d) < \infty$ that depends only on $d$ such that

$$\lim_{k \to \infty} \sup_{x \in Q} \frac{2^{k(2-d)} \|x\|^{2-d}}{P(Q_k(x) \cap B \neq \emptyset)} = \gamma_0$$

(13.23)

and

$$\lim_{k \to \infty} \inf_{x \in Q} \frac{2^{k(2-d)} \|x\|^{2-d}}{P(Q_k(x) \cap B \neq \emptyset)} = \gamma_0.$$  

(13.24)

Proof. For every $Q = [a_1, b_1] \times \cdots \times [a_d, b_d] \in Q_k$ let $x_Q = (a_1, \ldots, a_d) \in Q$. By applying Lemma 12.3 to $A = [0, 1)^d$ it follows that there exists $\gamma_0$ such that

$$\lim_{k \to \infty} \sup_{x \in Q} \frac{2^{k(2-d)} \|x_Q(x)\|^{2-d}}{P(Q_k(x) \cap B \neq \emptyset)} = \gamma_0$$

(13.25)

and

$$\lim_{k \to \infty} \inf_{x \in Q} \frac{2^{k(2-d)} \|x_Q(x)\|^{2-d}}{P(Q_k(x) \cap B \neq \emptyset)} = \gamma_0.$$  

(13.26)

It is easy to show that

$$\frac{\|x\|^{2-d}}{\|x_Q(x)\|^{2-d}}$$

converges to 1 uniformly on $Q$. Hence (13.23) follows from (13.25) and (13.24) follows from (13.26). \qed

Lemma 13.17. There exists $0 \leq \gamma < \infty$ such that for every $n \in \mathbb{N}$ for every $Q \in Q_n$ such that $\text{dist}(Q, 0) > 0$ we have that

$$\int_Q \|x\|^{2-d} d\mathcal{C}_k(\lambda)$$

converges to $\gamma \cdot \tau(Q)$ in probability as $k$ goes to $\infty$.

Proof. Let

$$M_k = \sup_{x \in Q} \frac{2^{k(2-d)} \|x\|^{2-d}}{P(Q_k(x) \cap B \neq \emptyset)}.$$  

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then
\[
\frac{\int_S \|x\|^{2-d} \, dx}{P(S \cap B \neq \emptyset)} \leq 2^{-2k} M_k
\]
for every \( S \in Q_k, S \subseteq Q \) for every \( k \geq n \). Thus
\[
\int_Q \|x\|^{2-d} \, d\mathcal{C}_k(\lambda) = \sum_{S \in Q_k} \frac{I_{S \cap B \neq \emptyset}}{P(S \cap B \neq \emptyset)} \int_S \|x\|^{2-d} \, dx \leq \sum_{S \in Q_k} I_{S \cap B \neq \emptyset} 2^{-2k} M_k = M_k 2^{-2k} \# \{ S \in Q_k : S \subseteq Q, B \cap S \neq \emptyset \}. \tag{13.27}
\]
Similarly it can be shown that
\[
\int_Q \|x\|^{2-d} \, d\mathcal{C}_k(\lambda) \geq m_k 2^{-2k} \# \{ S \in Q_k : S \subseteq Q, B \cap S \neq \emptyset \} \tag{13.28}
\]
for
\[
m_k = \inf_{x \in Q} \frac{2^{k(2-d)} \|x\|^{2-d}}{P(Q(x) \cap B \neq \emptyset)}. \]

Let \( 0 \leq \alpha < \infty \) be as in Proposition \([13.13]\) let \( \gamma_0 \) be as in Lemma \([13.16]\) and let
\[
\gamma = \alpha \cdot \gamma_0. \tag{13.29}
\]
To prove the statement of the lemma, by Lemma \([2.9]\) it is enough to show that for every \( \varepsilon > 0 \) and for every subsequence \( \{\alpha_k\}_{k=1}^\infty \) of \( \mathbb{N} \) we can find a subsequence \( \{\beta_k\}_{k=1}^\infty \) of \( \{\alpha_k\}_{k=1}^\infty \) such that
\[
\lim_{k \to \infty} P \left( \left| \int_Q \|x\|^{2-d} \, d\mathcal{C}_{\beta_k}(\lambda) - \gamma \cdot \tau(Q) \right| > \varepsilon \right) = 0. \tag{13.30}
\]
Let \( \varepsilon > 0 \) be fixed and \( \{\alpha_k\}_{k=1}^\infty \) be a subsequence of \( \mathbb{N} \). For every \( I = \cup_{n=1}^N I_n \) where \( I_1, \ldots, I_n \) are disjoint compact intervals with positive dyadic endpoints we have that
\[
h_k^I := 2^{-2k} \# \{ S \in Q_k : B_0(I) \cap S \neq \emptyset \}
\]
converges to \( \alpha \lambda(I) \) in probability by Proposition \([13.13]\). Hence, by Lemma \([2.8]\) we can find an event \( H \) with \( P(H) = 1 \) and a subsequence \( \{\beta_k\}_{k=1}^\infty \) of \( \{\alpha_k\}_{k=1}^\infty \) such that \( h_k^{\beta_k}(\omega) \) converges to \( \alpha \lambda(I) \) for every \( I = \cup_{n=1}^N I_n \) where \( I_1, \ldots, I_n \) are disjoint compact intervals with positive dyadic endpoints for every outcome \( \omega \in H \). We can further assume, by Lemma \([13.15]\) that for every \( \omega \in H \) there exist \( N^- (\omega), N^+ (\omega) \in \mathbb{N} \) and sets \( I^- (\omega) = \cup_{n=1}^{N^- (\omega)} I_n^- (\omega) \) where \( I_1^- (\omega), \ldots, I_{N^- (\omega)}^- (\omega) \) are disjoint compact intervals with positive dyadic endpoints.

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and \( I^+(\omega) = \bigcup_{n=1}^{N^+} I_n^+(\omega) \) where \( I_1^+(\omega), \ldots, I_N^+(\omega) \) are disjoint compact intervals with positive dyadic endpoints such that

\[
I^- \subseteq B_0^{-1}(Q) \subseteq I^+
\]

and

\[
\lambda(I^+ \setminus I^-) < \varepsilon/2 \gamma.
\]

Let \( \omega \in H \) be fixed. Then

\[
\hat{\nu}(Q) \geq \frac{\tau(Q) - \varepsilon}{2} = \gamma \tau(Q) - \varepsilon/2.
\]

Hence

\[
\limsup_{k \to \infty} \int_Q \|x\|^{2-d} dC_{\beta_k, \omega}(\lambda) \leq \lim_{k \to \infty} M_{\beta_k} \cdot h_{\beta_k}^+(\omega) = \gamma_0 \cdot \alpha \cdot \lambda(I^+)
\]

by (13.33) and (13.34). Thus

\[
\limsup_{k \to \infty} \int_Q \|x\|^{2-d} dC_{\beta_k, \omega}(\lambda) \leq \gamma_0 \cdot \alpha \cdot \lambda(I^+)
\]

by Lemma 13.16, the fact that \( h_{\beta_k}^+(\omega) \) converges to \( \lambda(I^+) \), (13.29) and that \( \lambda(I^+) \leq \lambda(B_0^{-1}(Q)) + \varepsilon/2 \gamma = \tau(Q) + \varepsilon/2 \gamma \) by (13.32) and the definition of \( \tau \), (13.22). Similarly

\[
\liminf_{k \to \infty} \int_Q \|x\|^{2-d} dC_{\beta_k, \omega}(\lambda) \geq \gamma_0 \cdot \alpha \cdot \lambda(I^+) - \varepsilon/2
\]

by (13.33) and (13.34) hold for every \( \omega \in H \) and \( P(H) = 1 \). Thus (13.30) follows.

**Theorem 13.18.** Let \( \nu(A) = \int_A \|x\|^{2-d} dx \) for every Borel set \( A \subseteq \mathbb{R}^d \). Then \( \nu \) is a locally finite Borel measure, \( \nu = \nu_R \) and \( C(\nu) = \frac{1}{c(d)} \tau \) almost surely where \( c(d) \) is as in 1.26.

**Proof.** Clearly \( \nu \) is a measure. For every \( y \in \mathbb{R}^d \) such that \( \|y\| > r > 0 \) for some \( r \) we have that \( \nu(B(y, r)) \leq (\|y\| - r)^{2-d} \lambda(B(y, r)) \). By the argument in the last paragraph of [12] page 109 it follows that

\[
\nu(B(0, 1)) = \int_{B(0, 1)} \|x\|^{2-d} dx < \infty
\]

and

\[
\int_{B(0, 1)} \|x-y\|^{2-d} dxdy < \infty.
\]
Hence $\nu$ is a locally finite measure and $\nu_R = \nu$.

The conditional measure $C(\nu)$ of $\nu$ on $B$ exists with respect to $Q_k$ ($k \geq 1$) with regularity kernel $\varphi(x, y) = \|x - y\|^{2-d}$ by Theorem 1.16. Let $\gamma$ be as in Lemma 13.17. It follows from Lemma 13.17 that $C_\varphi(\nu)(Q)$ converges to $\gamma \cdot \tau(Q)$ in probability for every $n \in \mathbb{N}$ for every $Q \in Q_n$ such that $\text{dist}(Q, 0) > 0$. By Property ii.) and iv.) of Definition 1.8 we have that $C_\varphi(\nu)(Q)$ converges to $\gamma \cdot \tau(Q)$ almost surely for every $n \in \mathbb{N}$ for every $Q \in Q_n$ such that $\text{dist}(Q, 0) > 0$. Hence it follows that

$$C(\nu)(Q) = \gamma \cdot \tau(Q)$$

almost surely for every $n \in \mathbb{N}$ for every $Q \in Q_n$ because $Q$ can be written as a countable union of boxes $Q_i \in Q_n$ ($i \in \mathbb{N}$) such that $\text{dist}(Q_i, 0) > 0$. By Property v.) of Definition 1.8 we have that $C(\nu)(\{0\}) = 0$ almost surely. The set $\{0\} \cup \bigcup_{n=1}^{\infty} Q_n$ forms a semiring that generates the Borel $\sigma$-algebra hence it follows by Proposition (2.34) that $C(\nu)(\{0\}) = 0$ almost surely. The set $\{0\} \cup \bigcup_{n=1}^{\infty} Q_n$ forms a semiring that generates the Borel $\sigma$-algebra hence it follows by Proposition (2.34) that $C(\nu)(\{0\}) = 0$ almost surely. The set $\{0\} \cup \bigcup_{n=1}^{\infty} Q_n$ forms a semiring that generates the Borel $\sigma$-algebra hence it follows by Proposition (2.34) that $C(\nu)(\{0\}) = 0$ almost surely. The set $\{0\} \cup \bigcup_{n=1}^{\infty} Q_n$ forms a semiring that generates the Borel $\sigma$-algebra hence it follows by Proposition (2.34) that $C(\nu)(\{0\}) = 0$ almost surely.

By Property ii.) of Definition 1.8 it follows that

$$E \left( \gamma \cdot \tau([0, 1]^d) \right) = E \left( C(\nu)([0, 1]^d) \right) = \nu([0, 1]^d) = \int_{[0,1]^d} \|x\|^{2-d} \, dx.$$

On the other hand

$$E \left( \gamma \cdot \tau([0, 1]^d) \right) = \gamma \cdot E \left( \int_0^\infty I_{B_0(t) \in [0,1]^d} \, dt \right) = \gamma \cdot c(d) \cdot \int_{[0,1]^d} \|x\|^{2-d} \, dx$$

by [14, Theorem 3.32] and [14, Theorem 3.33]. Thus $\gamma = c(d)^{-1}$ because the integral is positive and finite, (13.35). \hfill \Box

**Theorem 13.19.** We have that $dC(\lambda) = \frac{1}{c(d)} \|x\|^{d-2} \, d\tau$ almost surely where $c(d)$ is as in (1.26).

**Proof.** The conditional measure $C(\lambda)$ of $\lambda$ on $B$ exists with respect to $Q_k$ ($k \geq 1$) with regularity kernel $\varphi(x, y) = \|x - y\|^{2-d}$ by Theorem 1.16. The statement follows from Theorem 13.18 and Property x*) of Definition 1.8. \hfill \Box

### 13.4 Second moment of the occupation measure

Let $B_0(.)$ be a standard Brownian motion in $\mathbb{R}^d$ ($d \geq 3$) started at 0. Let $\tau$ be the occupation measure of $B_0(.)$, see (13.22). Let $c(d)$ be as in (1.26).

The following proposition is known, see [14, Theorem 3.32].

**Proposition 13.20.** For every Borel set $A \subseteq \mathbb{R}^d$ ($d \geq 3$)

$$E(\tau(A)) = c(d) \int_A \frac{1}{\|x\|^{d-2}} \, dx.$$
The following theorem calculates the second moment of the occupation measure.

**Theorem 13.21.** We have that

\[
E \left( \left( \int \int f(x, y) \tau(x) \tau(y) \right)^2 \right) = c(d)^2 \int \int \frac{||x||^{d-2} + ||y||^{d-2}}{||x-y||^{d-2} \cdot ||x||^{d-2} \cdot ||y||^{d-2}} \, dx \, dy
\]

for every Borel function \( f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) such that \( \int \int |f(x, y)| \frac{||x||^{d-2} + ||y||^{d-2}}{||x-y||^{d-2} \cdot ||x||^{d-2} \cdot ||y||^{d-2}} \, dx \, dy < \infty \).

Theorem 13.21 follows from Theorem 13.18 and Theorem 1.16.

### 14 Conditional measure on the percolation limit sets of trees

In this section we move to other metric spaces than \( \mathbb{R}^d \) to study the conditional measure of measures on the boundary of a tree when \( B \) is a percolation limit set.

**Definition 14.1.** Let \( T = (V, H, \zeta) \) be a countable graph with vertex set \( V \), edge set \( H \) and a special vertex \( \{\zeta\} \), that we call the root of \( T \). We say that \( T \) is a **rooted tree with root** \( \zeta \) if the following hold:

i.) for every \( v, w \in V \) there exists a unique self-avoiding finite path from vertex \( v \) to vertex \( w \),

ii.) the degree of every vertex \( v \in V \) is finite,

iii.) let \( T_0 = \{\zeta\} \) and for every positive integer \( n \) let \( T_n \) be the collection of vertices \( v \in V \) such that there exists a unique self-avoiding path of length \( n \) from \( \zeta \) to \( v \). Then \( \{(v, w) \in V : w \in T_{n+1}\} \neq \emptyset \) for every \( v \in T_n \) \( (n = 0, 1, \ldots) \).

We call an infinite self-avoiding path starting at \( \zeta \) a **ray**. We denote the set of rays by \( \partial T \). For every vertex \( v \in V \) let \(|v|\) be the unique \( n \in \mathbb{N} \) such that \( v \in T_n \). For two rays \( x, y \in \partial T \) let \( x \wedge y \in V \) be the unique vertex such that both \( x \) and \( y \) visit \( x \wedge y \) and for every \( v \in T_n \) for every \( n > |x \wedge y| \) at most one of \( x \) and \( y \) visits \( v \). We define a metric on \( \partial T \) by

\[
d(x, y) := 2^{-|x \wedge y|}.
\]

Then \( X = \partial T \) is a compact separable metric space that is homeomorphic to the Cantor set. For two vertices \( v, w \in V \) let \( v \wedge w \in V \) be the unique vertex such that the unique self-avoiding paths from \( \zeta \) to \( v \) and \( \zeta \) to \( w \) both visit \( v \wedge w \) and for every \( z \in T_n \) for every \( n > |x \wedge y| \) at most one of the unique self-avoiding paths from \( \zeta \) to \( v \) and \( \zeta \) to \( w \) visits \( v \).

For the rest of this section let \( \alpha > 0 \) be fixed. Let \( \varphi_\alpha(r) = r^{-\alpha} \) for \( r > 0 \) and so

\[
\varphi_\alpha(x, y) = d(x, y)^{-\alpha} = 2^{-\alpha|x \wedge y|}
\]

for \( x, y \in \partial T \). Then for every \( \delta > 0 \) we have that (1.16) holds for \( c_2 = (1 + 2\delta)^\alpha \) and \( c_3 = 0 \). Also (1.17) holds.
For a vertex $v \in V$ we denote by $[v]$ the set of rays of $\partial T$ that goes through vertex $v$. Let
\[ Q_k = \{ [v] : v \in T_k, C_\alpha([v]) > 0 \} \tag{14.1} \]
Then $Q_k$ is a sequence of finite families of Borel subsets of $\partial T$ such that $Q \cap S = \emptyset$ for $Q, S \in Q_k$, for all $k \in \mathbb{N}$. It is easy to see that (1.11) holds. For every $v \in T_k$ we have that $\text{diam}([v]) = 2^{-k}$, hence (1.10) and (1.19) hold. For $v, w \in T_k$, $v \neq w$ we have that $\text{dist}([v], [w]) \geq 2^{-(k-1)}$, and so
\[ \{ S \in Q_k : \max \{ \text{diam}(Q), \text{diam}(S) \} \geq \delta \cdot \text{dist}(Q, S) \} = \{ Q \} \]
for every $Q \in Q_k$ for $1/2 < \delta < 1$. Thus (1.18) holds for such $\delta$ and $M_\delta = 1$.

Let
\[ p = 2^{-\alpha} \]
be a probability parameter. For every vertex $v \in V$ let $Y(v)$ be a Bernoulli variable with parameter $p$, such that $Y(v)$ ($v \in T$) are mutually independent. Let
\[ B_N = \bigcap_{k=1}^{N} \bigcup_{v \in T_k, Y(v) = 1} [v] \]
be a random compact set. Let
\[ B = \bigcap_{N=1}^{\infty} B_N \]
be a random compact set which we call the percolation limit set.

The following result is due to Lyons, see for example [14, Theorem 9.17]

**Proposition 14.2.** For every compact set $K \subseteq \partial T$
\[ C_\alpha(K) \leq P(B \cap K \neq \emptyset) \leq 2C_\alpha(K). \]
In particular, $P(B \cap K \neq \emptyset) = 0$ if $C_\alpha(K) = 0$.

By Proposition 14.2 and (14.1) it follows that (1.13) and (1.22) holds for $a = 1$. For $v, w \in V$ such that $[v]$ and $[w]$ are disjoint, we have that
\[ \text{dist}([v], [w]) = 2^{-k} \]
for $k = \text{diam}([v], [w])$, hence if further $C_\alpha([v]) > 0$ and $C_\alpha([w]) > 0$ then
\[ \frac{P(B \cap [v] \neq \emptyset, B \cap [w] \neq \emptyset)}{P(B \cap [v] \neq \emptyset) \cdot P(B \cap [w] \neq \emptyset)} = \frac{p^{-k} \cdot P(B \cap [v] \neq \emptyset \mid B_k \cap [v \wedge w] \neq \emptyset) \cdot P(B \cap [w] \neq \emptyset \mid B_k \cap [v \wedge w] \neq \emptyset)}{p^{-k} \cdot P(B \cap [v] \neq \emptyset \mid B_k \cap [v \wedge w] \neq \emptyset) \cdot p^{-k} \cdot P(B \cap [w] \neq \emptyset \mid B_k \cap [v \wedge w] \neq \emptyset)} \]
Thus (1.23) holds for $c = 1$ for every disjoint $Q$ and $S$ and so for every $\delta > 0$. We have discussed above that the assumptions of Section 1.1.2 are satisfied.

It follows from (14.2) that

$$F(x, y) = F(x, y) = \mathcal{F}(x, y) = \varphi_{\alpha}(x, y),$$

(14.3)

for every $x, y \in \partial T_0 = \bigcap_{k=1}^{\infty} (\bigcup_{Q \in \mathcal{Q}_k} Q), x \neq y$ and $F(x, y) = 0$ otherwise, see (11.13). As we established above the conditions of Theorem 1.15 and Theorem 1.21 are satisfied and hence we can conclude Theorem 14.4.

**Remark 14.3.** If $I_\alpha(\nu) < \infty$ and $\nu(Q) > 0$ then $C_\alpha(Q) > 0$ because $I_\alpha(\nu|_Q) < \infty$ and so supp$\nu \subseteq \partial T_0$. Hence if $\nu(\partial T_0) = 0$ then $\nu_R = \nu_{\varphi_{\alpha}R} = 0$. On the other hand if $\nu$ is a finite Borel measure such that $\nu(\partial T_0) = 0$ then $C_k(\nu)$ converges to 0 in $\mathcal{L}^1$ by Lemma 5.4. Hence for such $\nu$ the conditional measure $C(\nu)$ of $\nu$ on $B$ exists with respect to $\mathcal{Q}_k$ ($k \geq 1$) with regularity kernel $\varphi_{\alpha}$ and $C(\nu) = 0$ almost surely. Thus the assumption that $\nu(\partial T \setminus \partial T_0) = 0$ is eliminated in the following Theorem.

**Theorem 14.4.** Let $\nu$ be a finite, Borel measure on $\partial T$. Then the conditional measure $C(\nu)$ of $\nu$ on $B$ exists with respect to $\mathcal{Q}_k$ ($k \geq 1$) with regularity kernel $\varphi_{\alpha}$ and if $\tau$ is a finite, Borel measure on $\partial T$ then

$$E \left( \int \int f(x, y)dC(\nu)(x)dC(\tau)(y) \right) = \int \int \varphi_{\alpha}(x, y)f(x, y)d\nu_R(x)d\tau_R(y)$$

for every $f : X \times X \rightarrow \mathbb{R}$ Borel function with $\int F(x, y)|f(x, y)|d\nu_R(x)d\tau_R(y) < \infty$. Moreover,

$$C_{\varphi_{\alpha}}(\nu) \leq P(C(\nu)(X) > 0) \leq 2C_{\varphi_{\alpha}}(\nu).$$

**Remark 14.5.** By Theorem 8.1 if $A \subseteq X$ is a Borel set such that $I_\alpha(\nu|_A) < \infty$ then $C_k(\nu)(A)$ converges to $C(\nu)(A)$ in $\mathcal{L}^2$ (recall that $\nu|_A(T \setminus \partial T_0) = 0$ by Remark 14.3).

### 14.1 Random multiplicative cascade measure as conditional measure

Throughout this section for a finite Borel measure $\nu$ on $\partial T$ let $\mathcal{C}(\nu)$ be the conditional measure of $\nu$ on $B$ with respect to $\mathcal{Q}_k$ ($k \geq 1$) with regularity kernel $\varphi_{\alpha}$ (which exists by Theorem 14.4). Let

$$S_k = \{[v] : v \in T_k\},$$

let $\mathcal{F}_k$ be the $\sigma$-algebra generated by the events $\{Q \cap B_k \neq \emptyset\}_{Q \in S_k}$ and for a finite Borel measure $\nu$ on $\partial T$ let

$$\mu_k^\nu = p^{-k}\nu|_{B_k} = \sum_{Q \in S_k} P(Q \cap B_k \neq \emptyset)^{-1} \cdot I_{Q \cap B_k \neq \emptyset} \cdot \nu|_Q.$$  

Note that $\mu_n$ is the conditional measure of $\nu$ on $B_n$ with respect to $S_k$ ($k \geq 1$) with regularity kernel $\varphi(x, y) = 1$.  

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Proposition 14.6. Let $\nu$ be a finite Borel measure on $\partial T$. There exists a random, finite Borel measure $\mu^\nu$ on $\partial T$ with the following properties:

1.) $\mu_k^\nu$ weakly converges to $\mu^\nu$ almost surely,
2.) for a countable collection of Borel sets $A_n \subseteq \partial T$ ($n \in \mathbb{N}$) we have that $\mu_k(A_n)$ converges to $\mu(A_n)$ for every $n \in \mathbb{N}$ as $k$ goes to $\infty$ almost surely,
3.) $\mu^\nu = \mu^{\nu_R}$ almost surely for $\nu_R = \nu_{\varphi_R}$,
4.) $\mu^{\nu_\perp} = 0$ almost surely for $\nu_\perp = \nu_{\varphi_\perp}$,
5.) $\mu^\nu = \sum_{i=1}^{\infty} \mu^\nu_i$ if $\nu = \sum \nu_i$,
6.) $E(\mu^\nu(A)) \leq \nu(A)$ for every Borel set $A \subseteq \partial T$.

Proof. It is easy to check that the sequence of random measures $\mu_k^\nu$ is a $T$-martingale with respect to the filtration $\mathcal{F}_k$. Hence Property 1.) and 2.) follows from [7, Theorem 1]. Property 6.) follows from Property 2.) and the nonnegative martingale convergence theorem [6, Theorem 5.2.9]. Property 5.) follows from Proposition 3.14.

If $C_{\alpha}(K) = 0$ for a compact set $K \subseteq \partial T$ then $P(B \cap K \neq \emptyset) = 0$ by Proposition 14.2. Hence, via an argument that is similar to the proof of Theorem 5.2, it can be shown that $\mu_k^{\nu_\perp}(\partial T)$ converges to 0 in probability. It implies that Property 3.) and 4.) hold.

Remark 14.7. When $T$ is an $m$-ary tree for some $m \in \mathbb{N}$ and $\nu$ is the uniform measure on $\partial T$ then $\mu^\nu$ is the random multiplicative cascade measure with weight variables $Y(v)/p$ for $v \in T$.

Lemma 14.8. Let $\nu$ be a finite Borel measure on $\partial T$ such that $I_\alpha(\nu) < \infty$. Then

$$\lim_{k \to \infty} \sum_{Q \in \mathcal{Q}_k} \frac{1}{P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset)} p^{-k} \nu(Q)^2 = 0.$$ 

Proof. We have that if $\nu(Q) > 0$ then $\nu(Q)^2 / I_\alpha(\nu|Q) \leq C_{\alpha}(Q)$. It follows from Proposition 14.2 that $C_{\alpha}(Q) \cdot p^{-k} \leq P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset)$ for every $Q \in \mathcal{Q}_k$. Hence

$$\sum_{Q \in \mathcal{Q}_k} \frac{1}{P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset)} p^{-k} \nu(Q)^2 \leq \sum_{Q \in \mathcal{Q}_k} \frac{1}{C_{\alpha}(Q)} \nu(Q)^2 \leq \sum_{Q \in \mathcal{Q}_k} \frac{I_\alpha(\nu|Q)}{\nu(Q)^2} \nu(Q)^2 \leq \sum_{Q \in \mathcal{Q}_k} I_\alpha(\nu|Q) \leq \iint_{d(x,y) \leq 2^{-k}} \varphi_\alpha(x,y) d\nu(x) d\nu(y).$$

Since $I_\alpha(\nu) < \infty$ the statement follows because

$$\iint_{d(x,y) = 0} \varphi_\alpha(x,y) d\nu(x) d\nu(y) = 0$$

by Fubini’s theorem. \qed
Lemma 14.9. Let 
\[ f(Q,S) = \left( \frac{I_{Q \cap B_k \neq \emptyset}}{P(Q \cap B_k \neq \emptyset)} - \frac{I_{Q \cap B \neq \emptyset}}{P(Q \cap B \neq \emptyset)} \right) \left( \frac{I_{S \cap B_k \neq \emptyset}}{P(S \cap B_k \neq \emptyset)} - \frac{I_{S \cap B \neq \emptyset}}{P(S \cap B \neq \emptyset)} \right) \]
for \( Q, S \in \mathcal{Q}_k \). Then
\[ E(f(Q,S)) = \begin{cases} 0 & \text{if } Q \neq S \\ p^{-k}(1/P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset) - 1) & \text{if } Q = S \end{cases} \]

Proof. We have that \( P(Q \cap B_k \neq \emptyset) = P(S \cap B_k \neq \emptyset) = p^k \). Since \( Q \in \mathcal{Q}_k \) it follows that
\[ P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset) = \frac{P(Q \cap B \neq \emptyset)}{p^k} > 0 \]
and similarly for \( S \). Then
\[ E(f(Q,S) | \mathcal{F}_k) = \frac{I_{Q \cap B_k \neq \emptyset} \cdot I_{S \cap B \neq \emptyset}}{p^{2k}} \cdot E\left(\left(1 - \frac{I_{Q \cap B \neq \emptyset}}{P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset)}\right) \left(1 - \frac{I_{S \cap B \neq \emptyset}}{P(S \cap B \neq \emptyset | S \cap B_k \neq \emptyset)}\right) | \mathcal{F}_k\right). \]
Given \( \mathcal{F}_k \) if \( Q \neq S \) then we have that \( I_{Q \cap B \neq \emptyset} \) and \( I_{S \cap B \neq \emptyset} \) are independent, hence
\[ E(f(Q,S) | \mathcal{F}_k) = 0. \]
Let \( q = P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset) \). If \( Q = S \) then
\[ E(f(Q,Q) | \mathcal{F}_k) = \frac{I_{Q \cap B_k \neq \emptyset}}{p^{2k}} E\left(\left(1 - \frac{I_{Q \cap B \neq \emptyset}}{P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset)}\right)^2 | \mathcal{F}_k\right) \]
\[ = \frac{I_{Q \cap B_k \neq \emptyset}}{p^{2k}} \left((1 - 1/q)^2q + (1 - q)\right) = \frac{I_{Q \cap B_k \neq \emptyset}}{p^{2k}} \cdot \frac{1 - q}{q}. \]
Hence
\[ E(f(Q,Q)) = p^{-k}(1/q - 1) = p^{-k}(1/P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset) - 1). \]

Proposition 14.10. Let \( \nu \) be a finite Borel measure on \( \partial T \) such that \( I_\alpha(\nu) < \infty \). Then
\[ \lim_{k \to \infty} E\left(\left(\mu_k^\nu(A) - C_k(\nu)(A)\right)^2\right) = 0 \]
for every Borel set \( A \subseteq \partial T \).

Proof. If \( \nu(A) = 0 \) then the proof is trivial so we assume that \( \nu(A) > 0 \). Without the loss of generality we can assume that \( \nu(A) = 1 \) otherwise we replace \( \nu \) by \( \nu \big|_A \). By Remark 14.3 we have that \( \nu(Q) = 0 \) for every \( Q \in \mathcal{S}_k \setminus \mathcal{Q}_k \). Hence by Lemma 14.9,
\[ E\left(\left(\mu_k^\nu(A) - C_k(\nu)(A)\right)^2\right) = \sum_{Q \in \mathcal{Q}_k} \sum_{S \in \mathcal{Q}_k} E(f(Q,S)) \nu(Q) \nu(S) \]
\[ = \sum_{Q \in \mathcal{Q}_k} p^{-k}(1/P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset) - 1) \nu(Q)^2 \leq \sum_{Q \in \mathcal{Q}_k} \frac{1}{P(Q \cap B \neq \emptyset | Q \cap B_k \neq \emptyset)} p^{-k} \nu(Q)^2. \]
Thus the statement follows from Lemma 14.8. □
Proposition 14.11. Let \( \nu \) be a finite Borel measure on \( \partial T \) such that \( I_\alpha(\nu) < \infty \). Then \( \mu^\nu = \mathcal{C}(\nu) \) almost surely.

Proof. We have that \( C_k(\nu)(Q) \) converges to \( \mathcal{C}(\nu)(Q) \) in \( L^2 \) for every \( Q \in S_k \) by Remark 14.5. Hence it follows by Proposition 14.10 that \( \mu^\nu_i(Q) \) converges to \( \mathcal{C}(\nu)(Q) \) in \( L^2 \). So by Property 2.) of Proposition 14.6 it follows that \( \mu^\nu(Q) = \mathcal{C}(\nu)(Q) \) for every \( Q \in S_k \) almost surely. Since \( \bigcup_{k=1}^\infty S_k \) is a semiring that generates the Borel \( \sigma \)-algebra of \( \partial T \) it follows by Proposition 2.34 that \( \mu^\nu = \mathcal{C}(\nu) \) almost surely.

Theorem 14.12. Let \( \nu \) be a finite Borel measure on \( \partial T \). Then \( \mu^\nu = \mathcal{C}(\nu) \) almost surely.

Proof. By Proposition 1.5 there exists a sequence of finite Borel measures \( \nu_i \) such that \( \nu = \nu_{\phi_\alpha \perp} + \sum_{i=1}^\infty \nu_i \) and \( I_\alpha(\nu_i) < \infty \) for every \( i \in \mathbb{N} \). It follows from Proposition 14.11 that \( \mu^\nu_i = \mathcal{C}(\nu_i) \) for every \( i \in \mathbb{N} \) almost surely. It follows from Property 4.) of Proposition 14.6 that \( \mu^\nu_{\phi_\alpha \perp} = 0 \) almost surely and it follows from Property vii.) of Definition 1.8 that \( \mathcal{C}(\nu_{\phi_\alpha \perp}) = 0 \) almost surely. Thus it follows from Property 5.) of Proposition 14.6 and Property ix.) of Definition 1.8 that

\[
\mu^\nu = \mu^\nu_{\phi_\alpha \perp} + \sum_{i=1}^\infty \mu^\nu_i = \mathcal{C}(\nu_{\phi_\alpha \perp}) + \sum_{i=1}^\infty \mathcal{C}(\nu_i) = \mathcal{C}(\nu)
\]

almost surely.

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