Differential Geometrical Methods for Deriving Dirac’s Equation in Curved Spacetime to Account for the Presence of Matter

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Abstract

Differential geomtrical methods for deriving the Dirac equation in Curved Spacetime are presented. Einstein’s field equation is applied in a novel manner; in the most current standard reference, Birrell and Davies, 1994 [1], the suggestions for deriving the Dirac equation in Curved Spacetime make no mention of employing Einstein’s field equation. Thus, to date, the literature on the derivation of the Dirac equation could not include an expression for the presence of matter. This lack is consistent with earlier publications, including Lichnerowicz’s well-known 1964 journal article [3], which presented the first such derivation, and Dimock’s 1982 article [2]. The new differential geometrical methods go beyond all previous suggestions, which only apply to cases in the absence of matter. These differential geometrical methods have resulted in derivations of the Dirac equation in Curved Spacetime that apply to either the presence or absence of matter.
General Relativity had been initiated by Einstein through establishing an interdisciplinary contact between differential geometry and Special Relativity, where differential geometry contains the concept of curved space and Special Relativity contains the concept of time. This cross-fertilization dramatically introduced the concept of Curved Spacetime. Einstein’s cross-fertilization brought forward the concept of Curved Spacetime in the juncture of differential geometry and relativistic field theory. The research I present here follows this same combinatorial vein of thought by bringing together differential geometry and relativistic quantum field theory, adding to Curved Spacetime the concept of spin quantum number. This combination of the most advanced tools available, from both General Relativity and quantum field theory, will allow us to perform investigations at the frontiers of science.

In this article, the techniques I employ are a part of my original research; however, I was inspired by the work of Lichnerowicz, and I further developed mathematical leads suggested in his paper [3].

Let us begin by making the transition from Flat to Curved Spacetime. If $\eta_{\alpha\beta}$ denotes the Minkowskian metric tensor (flat spacetime) and if $g_{\mu\nu}$ denotes the Riemannian metric tensor (curved spacetime), then

$$g_{\mu\nu}(x) = V_\alpha^\mu(x) V_\beta^\nu(x) \eta_{\alpha\beta},$$

$\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1)$. If $\gamma^\alpha$ denotes Dirac’s Spinor matrix in flat spacetime [3] therefore $\gamma^\mu = V_\alpha^\mu \gamma^\alpha$ will be the $\gamma^\alpha$ version in curved spacetime. $V_\alpha = V_\beta^\beta \eta_{\alpha\beta}$.

$$\frac{1}{2} V^\alpha V^\beta (\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha) = V^\alpha V^\beta \eta_{\alpha\beta}$$

then $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\eta_{\alpha\beta}$ which shows the anti-commutation relation still hold or $\gamma^\lambda \gamma^\mu + \gamma^\mu \gamma^\lambda = 2\eta^{\lambda\mu}$. Let $\Delta = \gamma^\lambda \gamma^\mu \nabla_\lambda \nabla_\mu$ where $\nabla$ is the $\partial$ operaor version in curved spacetime so that $\partial$ in flat spacetime will transform to $\nabla$ where $\nabla_\beta \omega_\alpha = \partial_\beta \omega_\alpha - \Gamma^d_{\beta \alpha} \omega_\delta$, $\Gamma^d_{\beta \alpha}$ is the Christoffel symbol.

$$\Delta \Psi = \frac{1}{2} \left( (\gamma^\lambda \gamma^\mu + \gamma^\mu \gamma^\lambda) \nabla_\lambda \nabla_\mu \Psi + \frac{1}{2} (\gamma^\lambda \gamma^\mu - \gamma^\mu \gamma^\lambda) \nabla_\lambda \nabla_\mu \Psi \right)$$
so that the above equation can be written as

\[
\Delta \Psi = \eta^{\lambda \mu} \nabla_\lambda \nabla_\mu \Psi + \frac{1}{2} (\gamma^\lambda \gamma^\mu - \gamma^\mu \gamma^\lambda) \nabla_\lambda \nabla_\mu \Psi
\]

\[
\Delta \Psi = \nabla^\lambda \nabla_\lambda \Psi + \frac{1}{2} \gamma^\lambda \gamma^\mu \left( \nabla_\lambda \nabla_\mu - \nabla_\mu \nabla_\lambda \right) \Psi
\]

\[
\left[ \nabla_\lambda, \nabla_\mu \right] \Psi
\]

Thanks are due to Lichnerowicz for having brought us thus far. I will now introduce my own extension of this method by involving Einstein’s field equation in order to account for the presence of matter.

The following three equations from General Relativity will be used [4]

(i) \[ \left[ \nabla_\lambda, \nabla_\mu \right] V^\ell = R^\ell_{\nu \lambda \mu} V^\nu, \] where \( R^\ell_{\nu \lambda \mu} \) is the Riemann Curvature tensor.

(ii) \[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 0 \] Einstein’s field equation in absence of any matter.

(iii) \[ R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R + k T_{\mu \nu} = 0 \] Einstein’s field equation in presence of matter.

\[
\Delta \Psi = \nabla^\lambda \nabla_\lambda \Psi + \frac{1}{2} \gamma^\lambda \gamma^\mu \left[ \nabla_\lambda, \nabla_\mu \right] \Psi.
\]

We notice that from (i) the last term to the right side of the above equation can be expressed in terms of the Riemann Curvature tensor and thus we can conclude that Dirac’s equation in curved spacetime gave rise to an extra term depends on \( R \).

To find the value of \( \frac{1}{2} \gamma^\lambda \gamma^\mu \left[ \nabla_\lambda, \nabla_\mu \right] \Psi \) in terms of \( R \).

\[
\frac{1}{2} \gamma^\lambda \gamma^\mu \left[ \nabla_\lambda, \nabla_\mu \right] \Psi^\ell = \frac{1}{2} \gamma^\lambda \gamma^\mu R^\ell_{\nu \lambda \mu} \Psi^\nu
\]

\[ = \frac{1}{2} \gamma^\lambda \gamma^\mu R^\ell_{\nu \lambda \mu} \Psi^\ell
\]

\[ = \frac{1}{2} \gamma^\lambda \gamma^\mu R_{\lambda \mu} \Psi^\ell.
\]

Since \( R^\nu_{\nu \lambda \mu} = R_{\lambda \mu} \), the above equation can be written as

\[
\frac{1}{2} \gamma^\lambda \gamma^\mu \left[ \nabla_\lambda, \nabla_\mu \right] \Psi^\ell = \frac{1}{2} \gamma^\lambda \gamma^\mu R_{\lambda \mu} \Psi^\ell
\]

\[
\frac{1}{2} \gamma^\lambda \gamma^\mu \left[ \nabla_\lambda, \nabla_\mu \right] \Psi = \frac{1}{2} \gamma^\lambda \gamma^\mu \left[ R_{\lambda \mu} \Psi \right]_{\frac{1}{2} g_{\lambda \mu} R}
\]

But (ii) is \( R_{\lambda \mu} = \frac{1}{2} g_{\lambda \mu} R \) Einstein’s field equation in absence of any matter.
Therefore \( \frac{1}{2} \gamma^\lambda \gamma^\mu [\nabla_\lambda, \nabla_\mu] \Psi = \frac{1}{4} \gamma^\lambda \gamma^\mu g_{\lambda \mu} R \Psi \). But \( \gamma^\mu g_{\lambda \mu} = \frac{\gamma^\mu \gamma^\lambda}{\gamma_\lambda}, \) \( g_{\lambda \mu} = g_{\lambda \mu} \)

so that

\[
\frac{1}{2} \gamma^\lambda \gamma^\mu [\nabla_\lambda, \nabla_\mu] \Psi = \frac{1}{4} \gamma^\lambda \gamma^\lambda R \Psi
\]

\[
\sum_{\lambda=1}^{4} \gamma^\lambda \gamma^\lambda = 4 \quad \text{each of} \quad \gamma^\lambda \gamma^\lambda = 1
\]

and

\[
\sum_{\mu} \frac{1}{2} \gamma^\lambda \gamma^\mu [\nabla_\lambda, \nabla_\mu] \Psi = \frac{1}{4} (\gamma^\lambda \gamma^\lambda) R \Psi
\]

becomes,

\[
\sum_{\mu} \frac{1}{2} \gamma^\lambda \gamma^\mu [\nabla_\lambda, \nabla_\mu] \Psi = \frac{1}{4} (\gamma^\lambda \gamma^\lambda) R \Psi, \quad \lambda, \mu = 1, 2, 3, 4 \quad \text{implies that}
\]

\[
\sum_{\mu} \frac{1}{2} \gamma^1 \gamma^\mu [\nabla_1, \nabla_\mu] \Psi = \frac{1}{4} (\gamma^1 \gamma^1) R \Psi,
\]

\[
\sum_{\mu} \frac{1}{2} \gamma^2 \gamma^\mu [\nabla_2, \nabla_\mu] \Psi = \frac{1}{4} (\gamma^2 \gamma^2) R \Psi,
\]

\[
\sum_{\mu} \frac{1}{2} \gamma^3 \gamma^\mu [\nabla_3, \nabla_\mu] \Psi = \frac{1}{4} (\gamma^3 \gamma^3) R \Psi,
\]

\[
\sum_{\mu} \frac{1}{2} \gamma^4 \gamma^\mu [\nabla_4, \nabla_\mu] \Psi = \frac{1}{4} (\gamma^4 \gamma^4) R \Psi
\]

The right side of the above equations are the same, i.e., \( \frac{1}{4} R \Psi \), therefore the left side of the above four equations are equal to each other. Thus, the addition of the four left sides = \( 4 \times \) (anyone of them) = \( 4 \sum_{\mu} \frac{1}{2} \gamma^\lambda \gamma^\mu [\nabla_\lambda, \nabla_\mu] \Psi \), \( \lambda \) can be any of 1, 2, 3, or 4, but the addition of the four left sides = the addition of the four right sides

\[
4 \sum_{\mu} \frac{1}{2} \gamma^\lambda \gamma^\mu [\nabla_\lambda, \nabla_\mu] \Psi = R \Psi,
\]

\[
\sum_{\mu} \frac{1}{2} \gamma^\lambda \gamma^\mu [\nabla_\lambda, \nabla_\mu] \Psi = \frac{1}{4} R \Psi,
\]

or

\[
\sum_{\mu} \frac{1}{2} \gamma^\lambda \gamma^\mu [\nabla_\lambda, \nabla_\mu] \Psi = \frac{1}{4} R \Psi. \quad \text{Therefore,} \quad \Delta \Psi = \nabla^\lambda \nabla_\lambda \Psi + \frac{1}{4} R \Psi.
\]
In absence of any matter, Dirac’s equation in flat spacetime \((\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\Psi = 0\) will become
\[
(\nabla^\lambda\nabla_\lambda + \frac{1}{4}R + m^2)\Psi = 0
\]
in curved spacetime

However, in presence of matter, from (iii) we get for Dirac’s equation in curved spacetime
\[
(\nabla^\lambda\nabla_\lambda + \frac{1}{4}R - \frac{1}{2}K\gamma^\lambda\gamma^\mu T_{\lambda\mu} + m^2)\Psi = 0
\]
Thus we were able to get new information of physical significance, and that is, Einstein’s field equation in presence of matter will lead to Dirac’s equation in curved spacetime in presence of matter which is
\[
(\nabla^\lambda\nabla_\lambda + \frac{1}{4}R - \frac{1}{2}K\gamma^\lambda\gamma^\mu T_{\lambda\mu} + m^2)\Psi = 0.
\]

My method of deriving the above two equations provided new information as far as the Dirac equation in curved spacetime is concerned. This is due to the fact that Einstein’s equations are separated. One due to the absence of matter and the other, due to the presence of matter; however, this has not been presented in books and journals relating to quantum field theory in curved spacetime.

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Bibliography

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