Approximate Solutions for Solving Time-Space Fractional Bioheat Equation Based on Fractional Shifted Legendre Polynomials

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Abstract. The aim of this article was employed a fractional-shifted Legendre polynomials (F-SLPs) in a matrix form to approximate the temporal and spatial derivatives of fractional orders for derived an approximate solutions for bioheat problem of a space-time fractional. The spatial-temporal fractional derivatives are described in the formula by the Riesz-Feller and the Caputo fractional derivatives of orders \( \nu \in (1, 2) \) and \( \gamma \in (0, 1) \), respectively. The proposed methodology applied for two examples for demonstrating its usefulness and effectiveness. The numerical results confirmed that the utilized technique is immensely effective, provides high accuracy and good convergence.

Key words. Collocation method, Time-space fractional bioheat equation, Fractional-shifted Legendre polynomials, Accuracy.

1. Introduction
Medical treatments like cryosurgery, cryopreservation, cancer hyperthermia, skin burns and thermal malady diagnostics, require an understanding of thermal phenomena and temperature behavior in living tissues. Therefore, bioheat transport in human tissues is a topic of high theoretical and applied benefit. Biothermal studies can assist the design of clinical thermal treatment equipment, evaluation of thermal treatment’s effects on skin, and establishment of thermal protections for various purposes [1, 6, 18].

The physical phenomenon that explain heat transport in human tissue that includes the influence of blood flux on tissue temperature in a continuum was presented by Pennes [14], Furthermore it suggested a mathematical model to describe heat flux in biological tissue. The model known as the bioheat equation which that is still widely utilize [3].

Many researchers worked on the development Pennes bioheat model and fractional bioheat equation and gave very important analytic and computational solutions, and provided significant approximate solution, (for example, Ng et al. [13] in (2009), used the boundary element method; Lakhssassi et al. [12] in (2010), presented the analytic and numerical solutions by using the Jacobi elliptic functions and...
the Crank-Nicolson method; Singh et al.[17] in (2011), studied the numerical solutions by using finite difference and homotopy perturbation method; Jiang and Qi [10] in (2012), suggested a new fractional thermal wave model; Damor et al. [4] in (2013), used implicit finite difference method; Ezzat et al. [6] in (2014), suggested a new mathematical model; Ferrás et al.[7] in (2015), utilized an implicit finite difference; Qin and Wu [15] in (2016), applied a quadratic spline collocation technique; Kumar and Rai[11] in (2017), used finite element Legendre wavelet Galerkin methodology; Roohi et. al. [16] in (2018), determined the temperature distribution pattern during the hyperthermial therapy computationally by using Galerkin method; Hosseininia et al. [8] and Kumar et al. [19] in (2019); applied Legendre wavelet method, Kirchhoff’s transformation, finite element Legendre wavelet and Galerkin method).

In this article, will introducing the approximate algorithm for solving one dimensional time-space fractional bioheat equation (T-SFBHE) based on F-SLPs.

2. Time-Space Fractional Bioheat Equation

The time-space fractional version of the one-dimensional unsteady state Pennes bioheat equation can be obtain by replacing the first order time derivative by Caputo fractional derivative of order $\gamma \in (0,1]$ and second order space derivative by Riesz-Feller derivative of fractional arbitrary positive real order $\nu \in (1,2]$. The T-SFBHE is given according to [17]

$$
\rho c \frac{\partial^\gamma T(x,t)}{\partial t^\gamma} - k \frac{\partial^\nu T(x,t)}{\partial x^\nu} + W_b c_b (T(x,t) - T_a) = Q_{\text{ext}} + Q_{\text{met}}, \quad t > 0, 0 \leq x \leq R,
$$

where $\rho$, $c$, $k$, $T$, $x$, $T_a$, $W_b = \rho_b w_b$, $Q_{\text{ext}}$ and $Q_{\text{met}}$ symbolizes density, specific heat, thermal conductivity, temperature, time, distance, artillery temperature, blood perfusion rate, metabolic heat generation in skin tissue and external heat exporter in skin tissue respectively. The units and values of the symbols expressed in this equation are mentioned in Table1[5].

| Symbol | $T$ | $T_a$ | $t$ | $x$ | $\rho$ and $c$ and $c_b$ | $K$ | $\omega_b$ | $Q_{\text{met}}$ |
|--------|-----|-------|-----|-----|-------------------------|----|-----------|-------------|
| Unit   | °C  | °C    | s   | m   | kg/m$^3$                | J/kg °C | W/m °C m$^3$ | W/m$^3$     |
| value  | 37  | 1000  | 4000| 0.5 | 0.0005                  | 420  |            |             |

with initial and boundary conditions

$$T(x,0) = T_c,$$

$$-k_e \frac{\partial T}{\partial x}_{x=0} = q_0,$$

$$-k_e \frac{\partial T}{\partial x}_{x=R} = 0,$$

where, $q_0$ is the heat flux on the skin surface.

3. Preliminaries and Notations

In this section, remind the principles essentials of the fractional calculus theory that will be used in this article.

**Definition 1.** The Riemann-Liouville fractional integral operator of order $\alpha > 0$ defined as [2]:

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)ds, \quad \alpha > 0,$$

$$I^0 u(t) = u(t)$$

**Definition 2.** The Riemann-Liouville definition of fractional differential operator given as follows [9]:
\[ D^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u(s)}{(t-s)^{\alpha-n+1}} \, ds, & \alpha > 0, n-1 \leq \alpha < n, \\ \frac{d^n u(t)}{dt^n}, & \alpha = n. \end{cases} \]

**Definition 3.** The Caputo definition of fractional differential operator is defined as [10]:

\[ D^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{u^{(n)}(s)}{(t-s)^{\alpha-n+1}} \, ds, & n-1 \leq \alpha < n, \\ \frac{d^n u(t)}{dt^n}, & \alpha = n. \end{cases} \]

The relation between the Riemann-Liouville and Caputo operators given by the expressions [17]:

\[ D^\alpha I^\beta u(t) = u(t), \]

\[ I^\beta D^\alpha u(t) = u(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} \]

For \( \alpha \geq 0, \nu \geq -1, \) and constant \( C, \) Caputo fractional derivative has some fundamental properties which are needed here as follows [9]:

\[ i) \quad D^\alpha C = 0, \]

\[ ii) \quad D^\alpha t^\nu = \begin{cases} 0 & \text{for } \nu \in \mathbb{N}_0 \text{ and } \nu < [\alpha] \\ \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)} t^{\nu-\alpha} & \text{for } \nu \in \mathbb{N}_0 \text{ and } \nu \geq [\alpha] \end{cases} \]

\[ iii) \quad D^\alpha \left( \sum_{i=0}^{n-1} c_i u_i(t) \right) = \sum_{i=0}^{n-1} c_i D^\alpha u_i(t), \text{ where } \{c_i\}_{i=0}^{n-1} \text{ are constant} \]

**Definition 4.** (generalized Taylor’s formula). Suppose that \( D^{i\alpha} u(t) \in C [0, 1] \) for \( i = 0, 1, \ldots, n-1, \) then one has

\[ u(t) = \sum_{i=0}^{n-1} \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} D^{i\alpha} u(0^+) + \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} D^{n\alpha} u(\xi) \]

where \( 0 < \xi \leq t, \forall t \in [0, k]. \) Also, one has

\[ \left| u(t) - \sum_{i=0}^{n-1} \frac{t^{i\alpha}}{\Gamma(i\alpha + 1)} D^{i\alpha} u(0^+) \right| \leq K_\alpha \frac{t^{n\alpha}}{\Gamma(n\alpha + 1)} \]

where \( K_\alpha \geq |D^{n\alpha} u(\xi)|. \)

In case \( \alpha = 1, \) the generalized Taylor’s formula (10) is the classical Taylor’s formula [9].

4. **Fractional Shifted Legendre Polynomials**

Define the F-SLPs by introducing the change of variable \( x = x^\beta \) and \( N - 1 < \beta \leq N \) on shifted Legendre polynomials. The F-SLPs \( L_N(x^\beta) \) is symbolized by \( F^\beta_l(x). \) The F-SLPs are a particular solution of the normalized eigenfunctions of the Sturm-Liouville problem.

\[ (x - x^{1+\beta}) F^\beta_l(x) + \beta^2 i(i+1)x^{\beta-1} F^\beta_l(x) = 0, x \in [0, 1]. \]

Then \( F^\beta_l(x) \) can be obtained as follows:

\[ F^\beta_{l+1}(x) = \frac{(2i + 1)(2i\beta - 1)}{i + 1} F^\beta_l(x) - \frac{i}{i + 1} F^\beta_{l-1}(x), i = 1, 2, \ldots \]

can derive the analytic form of \( F^\beta_l(x) \) of degree \( i \beta \) as follows:

\[ F^\beta_l(x) = \sum_{s=0}^{l} b_{is} x^{s\beta}, \]

where \( b_{is} = (-1)^{i+s}(i + s)!/(i - s)! \) \((s)!^2\) and \( F^\beta_l(0) = (-1)^i, F^\beta_l(1) = 1. \)

**Theorem 1.** The FLPs are orthogonal with the weight function \( \omega^\beta_l(x) = x^{\beta-1} \) on the interval \([0,1],\) then be orthogonally condition is

\[ \int_0^1 F^\beta_N(x) F^\beta_M(x) \omega^\beta_l(x) \, dx = \frac{1}{(2N+1)^\beta} \delta_{NM} \]
Proof. With \( \int_0^1 L_N(x) L_M(x) \omega_i(x) \, dx = \frac{1}{(2N+1)} \delta_{NM} \), where \( \delta_{NM} \) is the Kronecker function and the weight function \( \omega_i(x) = 1 \), let \( x = x^\beta \) and then have

\[
\int_0^1 L_N(x^\beta) L_M(x^\beta) \, dx = \int_0^1 L_N(x^\beta) L_M(x^\beta) \, dx = \frac{1}{(2N+1)} \delta_{NM},
\]

Then the theorem is proved.

A temperature function \( T(x) \) square integrable on interval \( [0,1] \), may be expressed in order of F-SLPs as

\[
T(x) = \sum_{i=0}^N c_i F_{l_i}(x)
\]

where the coefficients \( c_i \) are obtained by

\[
c_i = \beta(2i + 1) \int_0^1 F_{l_i}(x) T(x) \omega_i(x) \, dx, \quad i = 0, 1, 2, \ldots
\]

If consider truncated series when \( (N + 1) \)-term the F-SLPs in (17), obtain

\[
T(x) \approx T_N(x) = \sum_{i=0}^N c_i F_{l_i}(x) = C \Phi(x)
\]

where the fractional-shifted Legendre coefficient vectors \( C \) and \( \Phi(x) \) are given by

\[
C = [c_0, c_1, \ldots, c_N], \quad \Phi(x) = [F_{l_0}(x), F_{l_1}(x), \ldots, F_{l_N}(x)].
\]

**Theorem 2.** Suppose that \( D_0 T(x) \in C[0,1] \) for \( i = 0, 1, \ldots, N, (2N+1) \beta \geq 1 \) and \( P_N^\beta = \text{span} \{ F_{l_0}(x), F_{l_1}(x), \ldots, F_{l_N}(x) \} \). If \( T_N(x) = C \Phi(x) \) is the best approximation to \( T(x) \) from \( P_N^\beta \) then the error bound is presented as follows:

\[
\| T(x) - T_N(x) \|_\omega \leq \frac{K_\beta}{\Gamma(N\beta + 1)} \left( \frac{1}{(2N+1)\beta} \right)^{1/2},
\]

where \( K_\beta \geq \| D_{N\beta} T(x) \|_\infty, \quad x \in [0,1] \).

Proof. Consider generalized Taylor’s formula

\[
T(x) = \sum_{i=0}^N \frac{x^\beta}{\Gamma(i+1)} D_0^i T(0^+) + \frac{x^N}{\Gamma(N\beta + 1)} D_0^{N\beta} T(\xi),
\]

where \( 0 < \xi \leq x, \forall x \in [0,1] \). Also, one has

\[
\left| T(x) - \sum_{i=0}^N \frac{x^\beta}{\Gamma(i+1)} D_0^i f(0^+) \right| \leq M_\beta \frac{x^N}{\Gamma(N\beta + 1)}
\]

Since \( T_N(x) = C \Phi(x) \) is the best approximation to \( T(x) \) from \( P_N^\beta \) and

\[
\left\| T(x) - T_N(x) \right\|^2_\omega \leq \left\| T(x) - \sum_{i=0}^N \frac{x^\beta}{\Gamma(i+1)} D_0^i T(0^+) \right\|^2_\omega \leq \frac{K_\beta^2}{(\Gamma(N\beta + 1))^2} \int_0^1 x^{2N\beta} x^\beta \, dx,
\]

\[
\left\| T(x) - T_N(x) \right\|^2_\omega \leq \frac{K_\beta^2}{(\Gamma(N\beta + 1))^2} \int_0^1 x^{(2N+1)\beta - 1} \, dx
\]

\[
= \frac{K_\beta^2}{(\Gamma(N\beta + 1)))^2(2N+1)\beta}
\]

Now, take the square root both sides, then the theorem proved.

\( \square \)
For arbitrary a temperature function $T(x, t) \in L^2([0,1] \times [0,1])$, they can be expanded as the following formula:

$$T(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} F_l^\beta_i(x) F_l^\alpha_j(t)$$

where

$$c_{ij} = (2i + 1)(2j + 1)\beta \alpha \int_0^1 \int_0^1 T(x, t) F_l^\beta_i(x) F_l^\alpha_j(t) \omega^\beta_i(x) \omega^\alpha_j(t) dx \, dt.$$  \quad i, j = 0, 1, ...

**Theorem 3.** If the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} F_l^\beta_i(x) F_l^\alpha_j(t)$ converges uniformly to $T(x, t)$ on the square $[0,1] \times [0,1]$, then equation (25) can be proof as following.

**Proof.** By multiplying $\omega^\beta_i(x) \omega^\alpha_j(t)$ both sides of (24), where $i$ and $j$ are fixed and integrating term wise with regard to $x$ and $t$ on $[0,1] \times [0,1]$, then

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \int_0^1 \int_0^1 T(x, t) F_l^\beta_i(x) F_l^\alpha_j(t) dx \, dt = 0 \quad i, j = 0, 1, ...$$

Theorem 4. If the function $T(x, t)$ is a continuous function on $[0,1] \times [0,1]$ and the series $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} F_l^\beta_i(x) F_l^\alpha_j(t)$ converges uniformly to $T(x, t)$, then $\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} F_l^\beta_i(x) F_l^\alpha_j(t)$ is the F-SLPs expansion of $T(x, t)$.

**Proof.** Using contradiction, let

$$T(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} F_l^\beta_i(x) F_l^\alpha_j(t) \neq g_{NM}$$

Then there is at least one coefficient such that $c_{NM} \neq g_{NM}$ however

$$c_{NM} = (2N + 1)(2M + 1)\alpha \beta \int_0^1 \int_0^1 T(x, t) F_l^\beta_N(x) F_l^\alpha_M(t) \omega^\beta_i(x) \omega^\alpha_j(t) dx \, dt = g_{NM}$$

**Theorem 5.** If two continuous functions defined on $[0,1] \times [0,1]$ have the identical F-SLPs expansions, then these two function are identical.

**Proof.** Suppose that $T(x, t)$ and $f(x, t)$ can be expanded by F-SLPs as follows:

$$T(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} F_l^\beta_i(x) F_l^\alpha_j(t),$$

$$f(x, t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} g_{ij} F_l^\beta_i(x) F_l^\alpha_j(t).$$

By subtracting equation (30) from (29), have

$$T(x, t) - f(x, t) \approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (c_{ij} - g_{ij}) F_l^\beta_i(x) F_l^\alpha_j(t)$$

$$= 0 \approx \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} 0 F_l^\beta_i(x) F_l^\alpha_j(t).$$

**Theorem 6.** If the sum of the absolute value of the F-SLPs coefficients of a continuous function $T(x, t)$ forms a convergent series, then the F-SLPs expansion is absolutely uniformly convergent and converges to the function $T(x, t)$.

**Proof.** Consider

$$\left| \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} c_{ij} F_l^\beta_i(x) F_l^\alpha_j(t) \right| \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |c_{ij}| \left| F_l^\beta_i(x) \right| \left| F_l^\alpha_j(t) \right| \leq \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} |c_{ij}|$$

If consider truncated series in (24), satisfy
\( T(x, t) = \sum_{i=0}^{N} \sum_{j=0}^{M} c_{ij} \mathcal{F} l_{i}^{\beta}(x) \mathcal{F} l_{j}^{\beta}(t) = \Phi'(x) C \Phi(t), \)

where \( C = \{ c_{ij}\}_{i,j=0}^{N,M} \), \( \Phi(x) = \left[ \mathcal{F} l_{1}^{\beta}(x), \mathcal{F} l_{2}^{\beta}(x), \ldots, \mathcal{F} l_{N}^{\beta}(x) \right]' \) and \( \Phi(t) = \left[ \mathcal{F} l_{0}^{\beta}(t), \mathcal{F} l_{1}^{\beta}(t), \ldots, \mathcal{F} l_{M}^{\beta}(t) \right]' \).

\[ \text{5. Two Dimensional Fractional-Shifted Legendre Operational Matrix of Fractional Differentiation} \]

The derivative of the \( \Phi(x) \) can be approximated as follows

\[ \Phi'(x) = D \Phi(x), \]

where \( D \) and \( D_{\theta} \) are called the F-SLPs operational matrix of space and time derivatives

**Theorem 7.** Suppose \( D_{\theta} \) is \( (N + 1) \times (N + 1) \) operational matrix of Caputo fractional derivatives of order \( \nu > 0, \beta > \frac{\nu}{2} \), when \( \beta \in \mathbb{N} \), the elements of \( D_{\theta} \) obtained as

\[ \{ d_{ij} \}_{i,j=0}^{N,N} = (2j + 1) \beta \sum_{i=0}^{i} \sum_{r=0}^{j} b_{ij} b_{i}^{\prime} \frac{\Gamma(s + 1)}{\Gamma(s - \nu + 1)} \frac{1}{(s + \nu + 1) \beta - \nu}, \]

where

\[ b_{ij}^{\prime} = \begin{cases} b_{ij}, & s \beta \in \mathbb{N}, s \beta < \nu, \\ 0, & s \beta \notin \mathbb{N}, s \beta \geq \nu \text{ or } s \beta \in \mathbb{N}, s \beta \geq \nu. \end{cases} \]

**Proof.** With the properties of the derivative (iii) and the orthogonality of FLPs, have

\[ D_{\theta} \mathcal{F} l_{i}^{\beta}(x) = \sum_{s=0}^{j} b_{i}^{\prime} x^{s \beta - \nu}. \]

Let \( x^{s \beta - \nu} = \sum_{j=0}^{N} d_{ij} \mathcal{F} l_{i}^{\beta}(x) \)

multiplying both sides of the equation (38) by \( \omega_{i}(x) \mathcal{F} l_{j}^{\beta}(x), \)

\[ d_{ij} = (2j + 1) \beta \sum_{i=0}^{i} \sum_{r=0}^{j} b_{ij} b_{i}^{\prime} \frac{\Gamma(s + 1)}{\Gamma(s - \nu + 1)} \frac{1}{(s + \nu + 1) \beta - \nu}, \]

substituting the equations (38) and (39) into equation (37), yields

\[ D_{\theta} \mathcal{F} l_{i}^{\beta}(x) = (2j + 1) \beta \sum_{i=0}^{i} \sum_{r=0}^{j} b_{ij} b_{i}^{\prime} \frac{\Gamma(s + 1)}{\Gamma(s - \nu + 1)} \frac{1}{(s + \nu + 1) \beta - \nu} \mathcal{F} l_{j}^{\beta}(x), \]

Hence,

\[ d_{ij} = (2j + 1) \beta \sum_{i=0}^{i} \sum_{r=0}^{j} b_{ij} b_{i}^{\prime} \frac{\Gamma(s + 1)}{\Gamma(s - \nu + 1)} \frac{1}{(s + \nu + 1) \beta - \nu}. \]

\[ i, j = 0, 1, \ldots, N \]

\[ \square \]

**6. Method for Solution**

Now will structure the approximate solution of equation (1), under given conditions, as the following series form

\[ T(x, t) = \sum_{i=0}^{N} \sum_{j=0}^{M} t_{ij} \mathcal{F} l_{i}^{\beta}(x) \mathcal{F} l_{j}^{\beta}(t), \]

which equivalent the matrix form

\[ T(x, t) = \Phi(x) T \Phi(t), \]

where \( T = \{ t_{ij}\}_{i,j=0}^{N,M} \), \( \Phi(t) = \left[ \mathcal{F} l_{0}^{\beta}(t), \mathcal{F} l_{1}^{\beta}(t), \ldots, \mathcal{F} l_{M}^{\beta}(t) \right]' \) and

\[ \Phi(x) = \left[ \mathcal{F} l_{0}^{\beta}(x), \mathcal{F} l_{1}^{\beta}(x), \ldots, \mathcal{F} l_{N}^{\beta}(x) \right]' \]

The approximate of the first spatial derivative as

\[ \frac{\partial T(x,t)}{\partial x} = \Phi'(x)(D_{\theta})' T \Phi(t), \]

and the fractional temporal and spatial derivatives as

\[ \frac{\partial^{\nu} T(x,t)}{\partial t^\nu} = \Phi'(x) D_{\theta}^{\nu} \Phi(t), \]
\[
\frac{\partial^\nu T(x,t)}{\partial x^\nu} = \varphi'(x)(D_x^\nu)T(x,t),
\]

applying the solution method for T-SFBHE in (1), have
\[
\rho c \varphi'(x)T_D^\nu \varphi(t) - k_s \varphi'(x)D_x^\nu T(x,t) + W_b c_b \varphi'(x)T(x,t) = \varphi'(x)Q_{ext}(t) + \\
\varphi'(x)Q_{net}(t) + \varphi'(x)W_b c_b T_a \varphi(t),
\]

where \( \varphi'(x)G \varphi(t) = \delta'(x)Q_1 \varphi(t) + \delta'(x)Q_2 \varphi(t) + \delta'(x)Q_3 \varphi(t) \)

where
\[
\begin{align*}
G &= \{g_{ij}\}_{i,j=0}^{NM} \\
g_{ij} &= \alpha \beta (2i + 1)(2j + 1) \int_0^1 \int_0^1 g(x,t) F(x,t) F(t) \omega_j^\nu (x) \omega_i^\nu (x) \, dx \, dt.
\end{align*}
\]

This is generate \( NM + N + M + 1 \) algebraic equations by multiplying \( \omega_j^\nu (x) \omega_i^\nu (t) F_j^\nu (x) F_i^\nu (t) \)

for \( i = 0, 1, 2, ..., N; j = 0, 1, 2, ..., M \), integrating from 0 to 1 and using the orthogonal property, to get
\[
T(\varphi'(x)D_x^\nu - \omega_b \rho c_b I) - k_s \left(D_x^\nu\right)T = G,
\]

with the initial condition from equation (2) in matrix form
\[
T(0) \equiv F
\]

where \( f_i, f_j = \beta (2j + 1) \int_0^1 T(x,0) F(x,t) \omega_i^\nu (x) \, dx \)

and boundary conditions respectively from equations (3) and (4) in matrix form, have
\[
\begin{align*}
-k_s \varphi'(0)D_x^\nu T &\approx K' \\
-k_s \varphi'(R)D_x^\nu T &\approx H'
\end{align*}
\]

where \( K = [k_0 ... k_N]' \) and \( H = [h_0 ... h_N]' \)

\[
\begin{align*}
k_i &= \alpha (2i + 1) \int_0^1 T(R,t) F(t) \omega_i^\nu (t) \, dt \\
h_i &= \alpha (2i + 1) \int_0^1 T(x,0) F(t) \omega_i^\nu (t) \, dt
\end{align*}
\]

which generate \( NM + N + M + 1 \) linear algebraic equations by equation (49) together with equations (50),(52) and (53). These unknown coefficients \( T \) can be solve by solving Sylvester system.

7. Error Analysis
Consider \( e(x,t) = T(x,t) - T_{NM}(x,t) \) as the error function where \( T_{NM}(x,t) \) and \( T(x,t) \) are the approximate and exact solutions of equation (1).

Therefore, \( T_{NM}(x,t) \) satisfies the following problem
\[
\rho c \frac{\partial^\nu T_{NM}(x,t)}{\partial x^\nu} - k_s \frac{\partial^\nu T_{NM}(x,t)}{\partial x^\nu} + W_b c_b T_{NM}(x,t) = R_{NM}(x,t) = g(x,t),
\]

where \( R_{NM}(x,t) \) is the residual function,
\[
R_{NM}(x,t) = \rho c \frac{\partial^\nu T_{NM}(x,t)}{\partial x^\nu} - k_s \frac{\partial^\nu T_{NM}(x,t)}{\partial x^\nu} + W_b c_b T_{NM}(x,t) - g(x,t).
\]

find an approximation \( \tilde{e}_{nm}(x,t) \) to the error function \( e_{nm}(x,t) \) in the same previous procedure, so the solution of the problem, the error function satisfies the problem
\[
\rho c \frac{\partial^\nu e_{nm}(x,t)}{\partial x^\nu} - k_s \frac{\partial^\nu e_{nm}(x,t)}{\partial x^\nu} + W_b c_b e_{nm}(x,t) = R_{NM}(x,t)
\]

should note that in order to construct the approximate \( \tilde{e}_{nm}(x,t) \) to the error function \( e_{nm}(x,t) \), only equation (58) needs to be recomputed in the same procedure as doing before for the solution of equation (1).

8. Numerical Examples
In this section, apply the algorithm, which presented in section 6 for solving the T-SFBHE in the two examples based on F-SLPs. In order to showing a capability of the collocation method for achieving the
high accuracy. In these examples, the solution obtained from the approximate technique is synonymous with the accurate solution.

Where the parameters \( \rho, c, k, T, t, x, T_a, W_b = \rho_b w_b \) and \( Q_{met} \) are obtained from Table 1

**Example1:** Consider the T-SFBHE (1) where by choosing \( Q_{ext} \) so the exact solution is:

\[
T(x, t) = xt^2(2-x) + 37
\]

with the initial condition

\[ T(x, 0) = 37, \ x \in [0, R] \]

and boundary conditions

\[
-k_x \frac{\partial T(0,t)}{\partial x} = 2t^2, \ t > 0
\]

\[
-k_x \frac{\partial T(R,t)}{\partial x} = 2t^2(1 - R), \ t > 0
\]

**Table2.** Absolut errors obtained for Example 1 with \( R = 1 \) and \( N = M = 12 \).

| (\( x, t \)) | Absolute error | Absolute error | Absolute error |
|-------------|----------------|----------------|----------------|
| \( \alpha = 0.5, \beta = 1.5 \) | \( \alpha = 0.75, \beta = 1.75 \) | \( \alpha = 0.95, \beta = 1.95 \) |
| (0.0) | 1.1373e-08 | 1.9178e-07 | 1.1509e-07 |
| (0.1,0.1) | 1.6373e-05 | 7.5360e-05 | 9.8106e-05 |
| (0.2,0.2) | 4.5590e-05 | 1.3012e-04 | 5.5986e-05 |
| (0.3,0.3) | 9.7485e-05 | 9.4045e-05 | 1.5015e-04 |
| (0.4,0.4) | 1.2577e-04 | 5.6854e-05 | 3.7275e-04 |
| (0.5,0.5) | 8.1411e-05 | 2.4769e-04 | 4.6486e-04 |
| (0.6,0.6) | 2.7426e-05 | 3.8733e-04 | 4.2448e-04 |
| (0.7,0.7) | 1.4291e-04 | 4.6677e-04 | 3.8853e-04 |
| (0.8,0.8) | 2.0478e-04 | 5.6629e-04 | 5.7097e-04 |
| (0.9,0.9) | 1.4727e-05 | 4.7077e-04 | 8.9061e-04 |
| (1,1) | 1.7356e-03 | 2.9567e-03 | 3.9126e-03 |

**Figure1.** Comparison between the numerical solutions for Example 1 at \( t = 1, R = 1, N = M = 12 \).
Figure 2. Comparison between the numerical solutions for Example 1 at $t = 1.3$, $R = 1$, $N = M = 12$.

**Example 2:** Consider the T-SFBHE (1) where by choosing $Q_{\text{ext}}$ so the exact solution is:

$$ T(x, t) = x^3 e^{-t} + 37 $$

with the initial condition

$$ T(x, 0) = x^3 + 37, \ x \in [0, R] $$

and boundary conditions

$$ -k_\nu \frac{\partial T(0, t)}{\partial x} = 0, \ t > 0 $$

$$ -k_\nu \frac{\partial T(R, t)}{\partial x} = \frac{3}{2} (R)^{\frac{1}{2}} e^{-t}, \ t > 0 $$

**Table 3.** Absolut errors obtained for Example 2 with $R = 1$ and $N = M = 12$.

| $(x, t)$ | $x = 0.5, \beta = 1.5$ | $x = 0.75, \beta = 1.75$ | $x = 0.95, \beta = 1.95$ |
|--------|-------------------------|---------------------------|--------------------------|
| (0.0)  | 1.1620e-08              | 1.7895e-03                | 4.5399e-03               |
| (0.1,0.1)| 8.5099e-08              | 2.0225e-04                | 6.9776e-04               |
| (0.2,0.2)| 1.8188e-08              | 7.8602e-05                | 3.1626e-04               |
| (0.3,0.3)| 1.2060e-07              | 4.7694e-05                | 1.8943e-04               |
| (0.4,0.4)| 2.9296e-08              | 3.9389e-05                | 1.7286e-04               |
| (0.5,0.5)| 1.2435e-07              | 4.9954e-05                | 1.2595e-04               |
| (0.6,0.6)| 5.2647e-08              | 1.0175e-04                | 9.7641e-05               |
| (0.7,0.7)| 1.1404e-07              | 1.3178e-04                | 7.7762e-05               |
| (0.8,0.8)| 4.4359e-08              | 1.3219e-04                | 1.1677e-04               |
| (0.9,0.9)| 1.4109e-07              | 6.2032e-05                | 3.8348e-05               |
| (1,1)   | 1.4402e-07              | 3.5154e-05                | 2.7918e-07               |
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Figure 3. Comparison between the numerical solutions for Example 2 at \( t = 1, R = 1, N = M = 12 \).

Figure 4. Comparison between the numerical solutions for Example 2 at \( t = 1.15, R = 1, N = M = 12 \).

9. Conclusions
In this work, the approximate algorithm structured on the F-SLPs in the matrix form to estimate the fractional derivatives to found the numerical solutions of the T-SFBHE. The Caputo formula utilized into approximate the fractional derivatives. Figs. 1-4 and Tables 2-3 indicated that the numerical results for Example 1 and 2 of a present technicality has a higher accuracy, good convergence, reasonable stability as well as a minimal computational effort by utilizing a few mesh grid. Concluded that the target numerical approach can be solve a various kinds of models of any fractional orders. In addition expected that the present methodology may present a more exact estimate by employing some other families based on orthogonal polynomials.

Reference
[1] F. Ahmad, A. Alomari, A. Bataineh, J. Sulaiman and I. Hashim, “On the approximate solutions of systems of ODEs by Legendre operational matrix of differentiation”, *Italian Journal of Pure and Applied Mathematics* 36, 483–494 (2016).
[2] Y. Chen, Y. Sun and L. Liu, “Numerical solution of fractional partial differential equations with variable coefficients using generalized fractional-order Legendre functions”, *Applied Mathematics and Computation* 244, 847–858(2014).
[3] Z. Cui, G. Chen and R. Zhang, “Analytical solution for the time-fractional Pennes bioheat transfer equation on skin tissue”, *Advanced Materials Research* 1049-1050, 1471-1474(2014).
[4] R. Damor, S. Kumar and A. Shukla, “Numerical Solution of Fractional Bioheat Equation with Constant and Sinusoidal Heat Flux Condition on Skin Tissue”, *American Journal of Mathematical Analysis*, 1(2), 20-24(2013).
[5] M. Dehghan and M. Sabouri, “A spectral element method for solving the Pennes bioheat transfer
equation by using triangular and quadrilateral elements”, *Applied Mathematical Modelling*, 36, 6031–6049(2012).

[6] M. Ezzat, N. AlSowayan, Z. Al-Muhiameed and S. Ezzat, “Fractional modelling of Pennes’ bioheat transfer equation”, *Heat and Mass Transfer* 50(7), 907–914(2014).

[7] L. Ferrás, N. Ford, M. Morgado, J. Nobrega and M. Rebelo, “Fractional Pennes’ bioheat equation: Theoretical and numerical studies”, *Fractional Calculus and Applied Analysis* 18(4), 1080–1106 (2015).

[8] M. Hosseinzia, M. Heydari, R. Roohi and Z. Avazzadeh, “A computational wavelet method for variable-order fractional model of dual phase lag bioheat equation”, *Journal Computational Physics*, 395, 1-18(2019).

[9] Q. Huang, F. Zhao, J. Xie, L. Ma, J. Wang and Y. Li, “Numerical approach based on two-dimensional fractional-order Legendre functions for solving fractional differential equations”, *Discrete Dynamics in Nature and Society* 1-12(2017).

[10] X. Jiang and H. Qi, “Thermal wave model of bioheat transfer with modified Riemann–Liouville fractional derivative”, *Journal of Physics A: Mathematical and Theoretical* 45, 1-11(2012).

[11] D. Kumar and K. Rai,” Numerical simulation of time fractional dual-phase-lag model of heat transfer within skin tissue during thermal therapy”, *Journal of Thermal Biology*, 67, 49-58(2017).

[12] A. Lakhssassi, E. Kengne and H. Semmaoui, “Modified pennes’ equation modelling bio-heat transfer in living tissues: analytical and numerical analysis”, *Natural Science* 2(12), 1375-1385(2010).

[13] E. Ng , H. Tan and E. Ooi,” Boundary element method with bioheat equation for skin burn injury”, *Burns* 35, 987–997(2009).

[14] H. Pennes, “Analysis of tissue and arterial blood temperature in the resting forearm”, *Journal of applied physiology* 1, 93–122(1948).

[15] Y. Qin and K. Wu, “Numerical solution of fractional bioheat equation by quadratic spline collocation method”, *Journal of Nonlinear Science and Applications* 9, 5061-5072(2016).

[16] R. Roohi, M. Heydari, M. Aslami, and M. Mahmoudi, “A comprehensive numerical study of space-time fractional bioheat equation using fractional-order Legendre functions”, *The European Physical Journal Plus* 133(412),1-15(2018).

[17] J. Singh, P. Gupta and K. Rai, “Solution of fractional bioheat equations by finite difference method and HPM”, *Mathematical and Computer Modelling*, 54, 2316-2325(2011).

[18] E. Tohidi, “Legendre approximation for solving linear HPDEs and comparison with Taylor and Bernoulli matrix methods”, *Applied Mathematics*, 3, 410-416(2012).

[19] Kumar R., Vashisht A., Ghangas S., Variable thermal conductivity approach for bioheat transfer during thermal ablation. *Arab Journal of Basic and Applied Sciences*, 26(1), 78–88(2019).