Studies on the Rabi Model

H. Q. Lin\textsuperscript{1}, M. X. Liu\textsuperscript{1,2}, Stefano Chesi\textsuperscript{1}, H. G. Luo\textsuperscript{1,3}

\textsuperscript{1} Beijing Computational Science Research Center, Beijing 100193
\textsuperscript{2} State Key Laboratory of Information Photonics and Optical Communications, Beijing University of Posts and Telecommunications, Beijing 100876
\textsuperscript{3} Center for Interdisciplinary Studies and Key Laboratory for Magnetism and Magnetic Materials of the MoE, Lanzhou University, Lanzhou 730000

Abstract. We present our recent studies on the quantum Rabi model (QRM). Firstly, by using a variational wave function, which facilitates to extract physics in entire parameter regime with high accuracy, we unveil a ground-state phase diagram of the QRM and argue that the main constituents are polaron and anti-polaron. Secondly, introducing an anisotropy into the QRM, in which the rotating- and counter-rotating terms are allowed to have different coupling strength, we explore that how the criticality of the model interpolates between two known limits with distinct universal properties. Through a combination of analytic and numerical approaches we compute phase diagram, scaling functions and critical exponents, and establish that the universality class at the finite anisotropy is the same as that of the isotropic limit.

1. Introduction

In this section we review our efforts in studying the Rabi model [1, 2]. A quantum version of the Rabi model could be obtained by quantizing the cavity, with the Hamiltonian

\[ H = \omega a^+ a + \frac{\Omega}{2} \sigma_x + g \sigma_z (a^+ + a), \]  

where \( \omega \) is the oscillator frequency, \( \Omega \) the energy splitting of the qubit and \( g \) is the coupling constant. The model has \( \mathbb{Z}_2 \) symmetry with well-defined parity. It might be one of the simplest models of quantum light interacting with matter, and plays a fundamental role in quantum optics (cavity/circuit QED), quantum information, and condensed matter physics (magnetic resonance, molecular physics), etc. It can be experimentally realized in optical cavity, superconductor circuit, trapped ions, quantum dot with spin-orbital coupling, and so on. The model has attracted great attention in recent years due to the fact that experimentally one can achieve strong or even ultra-strong coupling.

By combing various analytical approaches and numerical methods, we carried systematic studies on the quantum Rabi model. Many of our results were published already [3]. We present two important results in this talk. Firstly we proposed a variational wave function based on the polaron and anti-polaron picture. The wave function has 5 minimum variational parameters, accounting for the shift of well minimum on the left and right, renormalization of oscillator frequencies, and the normalization factor. The proposed variational wave function works outstanding for the ground state and excellent for low lying excited states. We also obtained the ground state phase diagram. Secondly, we study the role of counter-rotating term on the phase diagram and relevant quantum phase transitions.
\[ H = \omega a^+ a + \frac{\Omega}{2} \sigma_x + g[(\sigma^+ a + \sigma^- a^+) + \lambda(\sigma^+ a^+ + \sigma^- a)]. \]  

The anisotropy \( \lambda \) varies from negative value to positive, including \( \lambda = 0 \) (the Jaynes-Cummings model [4]) and \( \lambda = 1 \) (the Rabi model), representing the strength of counter-rotating term. In the limit of \( \eta = \Omega/\omega \to \infty \), we performed Schrieffer-Wolff transformation to obtain an effective Hamiltonian and did scaling analysis. A phase diagram was obtained and we found that all quantum phase transitions along coupling direction are second-order. Along the anisotropy direction, we find a first order phase transition at \( \lambda = 0 \) (JC line) if coupling reaches the superradiant phase. It is reveals that the transitions are universal for all \( \lambda > 0 \). There exists symmetry between x-type super-radiant phase (in \( \lambda > 0 \)) and p-type super-radiant phase (in \( \lambda < 0 \)). Moreover our results also apply to the Dicke model, i.e., to the system contains arbitrary \( N \) spins as well.

2. Entanglement of the oscillator and the qubit

When one takes the frequency ratio limit \( \eta = \Omega/\omega \to \infty \), the model undergoes a quantum phase transition at \( g_c = \sqrt{\omega \Omega}/2 \). For a finite \( \eta \), the scaling form turns out. To address the criticality issue, we take the critical behavior of the entanglement entropy as an example. The entanglement of the oscillator and the qubit is quantified by the entropy \( S \) of the qubit state. It is defined by the trace that

\[ S = -Tr\{\rho_q \log_2 \rho_q\}, \]  

where \( \rho_q \) is the reduced density matrix of the qubit, which satisfies

\[ \rho_q = -Tr\{|G\rangle\langle G|\}, \]  

\( |G\rangle \) is the ground state.

At the critical regime, there should be the scaling form for the entanglement entropy that (we take \( \Omega \) as the unity)

\[ S = \omega^{-\beta_s/\nu_s} \tilde{S}(Z), \]  

where \( Z = g - g_c \omega^{-1/\nu_s} \) is the scaling variable, \( \tilde{S} \) is the universal scaling function, \( \beta_s \) and \( \nu_s \) are two critical exponents. Fig. 1 shows the scaling function of the entanglement entropy calculated by the numerical way. Curves for different scales of \( \omega \) collapse to a single one, to which related the scaling function \( \tilde{S} \). This numerical result confirms the scaling relation in Eq. 4, and thus supports the existence of the critical behavior for the entanglement. The critical exponent \( \nu_s \) is of the value about 0.667, which agrees with 2/3 predicted by the mean-field-like analysis. Interestingly, \( \beta_s \) is about 0.5. It has not been understood by the mean-field-like framework, and requires further exploration. The above-unpublished results are new and will not be published elsewhere.

In summary, we proposed a variational wave function constructed from polaron and antipolaron induced by spin-bosonic mode coupling to study the ground state properties of the quantum Rabi model. This variational wave function provides high accuracy for all coupling strengths. We also obtained a phase diagram as function of coupling strength and oscillator frequency. Moreover, our proposed wave function can also be used to extract excited state information of the quantum Rabi model. We would like to mention that preliminary results suggest that similar wave function could be applied to study the spin-boson model. Adding an anisotropy to the counter-rotating term of the quantum Rabi model, which interpolates between the two known limits with distinct universal properties, we have characterized quantum phase
transitions of the quantum Rabi model and demonstrated its universality. Scaling behaviors were analyzed and we found that except at zero anisotropy, i.e., the JC model, all other anisotropies belong to the same universality class. Our results emphasize the critical role played by counter-rotating terms, and it also applies to multi spin cases. Our findings are relevant to a variety of systems that are able to realize strong coupling between matter and light.

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