The $\theta$-Vacua and the Leutwyler–Smilga Scaling Regime

J.T. Lenaghan$^a$*

$^a$The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark
email: lenaghan@alf.nbi.dk

The partition function of QCD is studied in the Leutwyler–Smilga scaling regime for an arbitrary number of quark flavors and masses including the contributions from all winding numbers. For $N_f = 2$ and degenerate quark masses, the partition function becomes independent of the quark masses at $\theta = \pi$ and subsequently the scalar chiral condensate vanishes. There is a discontinuity at $\theta = \pi$ in the first derivative of the energy density with respect to $\theta$ corresponding to the first–order phase transition in which CP is spontaneously broken, known as Dashen’s phenomena. These properties are found to be insensitive to both the pattern of chiral symmetry breaking and the representation of the quark fields.

1. Introduction

The vacuum angle, $\theta$, a fundamental parameter in QCD, is constrained to be zero to within one part in a trillion. Understanding why this parameter is so fine–tuned is a fascinating theoretical problem. Dashen’s phenomenon [1], the first–order phase transition at $\theta = \pi$ in which the discrete CP symmetry is spontaneously broken, is an example of novel physics at nonzero values of $\theta$. Since the physics at nonzero $\theta$ is inherently nonperturbative, the only two theoretical avenues to explore this region of the QCD parameter space are lattice QCD and effective theories. The former approach suffers from a sign–problem which is similar to that which plagues attempts to study QCD at nonzero baryonic chemical potential.

The approach employed here is to study the physics of nonzero $\theta$ using chiral perturbation theory in finite volume [2]. Restricting the Euclidean four-volume, $V = L^4$, to the range $\frac{1}{\Lambda} \ll L \ll \frac{1}{m_\pi}$ where $\Lambda$ is the chiral symmetry breaking scale and $m_\pi$ is the mass of the Goldstone excitations, allows for an exact analytical treatment [3]. This is possible since the lower limit implies that the partition function is dominated by Goldstone modes and the upper limit requires that these modes are constant. As a result, the partition function reduces to a finite–dimensional group integration.

*Work done in collaboration with G. Akemann, K. Splittorff and T. Wilke.
2. Results

The full partition function for \( N_f = 2 \) and \( N_c \geq 3 \) with quarks in the fundamental representation was calculated in Ref. [3] and the necessity of including the contributions from all topological sectors in the partition function was demonstrated in Ref. [4]. For \( N_f \geq 3 \), the problem is technically more difficult. It was shown in Ref. [5] that the full partition function for arbitrary \( N_f, \theta \) and quark masses, \( m_i \), can be expressed as a \( N_f - 2 \) dimensional integral over single Bessel functions.

For simplicity, we focus on \( N_f = 2 \) and \( N_c \geq 3 \) with quarks in the fundamental representation but the following results are generalizable [5,6]. The partition function is

\[
Z(\theta, \mu_1, \mu_2) = I_1(\mu_{12})/\mu_{12},
\]

where \( \mu_{12} = \sqrt{\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2\cos(\theta)} \), \( \mu_i = \Sigma V m_i \) and \( \Sigma \) is the infinite-volume chiral condensate at \( \theta = 0 \) [3]. Taking the two quark masses to be equal and \( \theta \) to be an integer multiple of \( \pi \) leads to the remarkable property that the partition function is independent of the scaling variable with

\[
Z^{(N_f=2)}(\theta = n\pi, \mu) = 1/2.
\]

For arbitrary \( \theta \) and degenerate quark masses, the dependence on the volume, macroscopic chiral condensate, quark masses and \( \theta \) is only through the dimensionless variable \( |\mu|\cos(\theta/2)| \), which as discussed in Ref. [6], has profound implications. On account of this very constrained dependence of the partition function on the parameters of the theory, the chiral and the topological properties are intimately intertwined. In particular, the chiral condensate and the topological density are proportional up to a simple \( \theta \)-dependent function [6].

Since the analytic form of the partition function is known, calculating the vacuum properties is straightforward. Taking the limit of degenerate quark masses, the chiral condensate is

\[
\Sigma(\theta, \mu) = \Sigma |\cos(\theta/2)| I_2(2|\mu|\cos(\theta/2)) / [2I_1(2|\mu|\cos(\theta/2))].
\]

For fixed \( \mu \), the chiral condensate decreases monotonically in the interval \( \theta \in [0, \pi) \) and increases monotonically for \( \theta \in (\pi, 2\pi) \). As \( \mu = m \Sigma V \rightarrow \infty \), the chiral condensate develops a cusp at \( \theta = \pi \). For any value of \( \mu \), the chiral condensate vanishes identically at \( \theta = \pi \). In the limit of degenerate quark masses, the topological density, defined as the first derivative of the logarithm of the partition function with respect to \( \theta \), is

\[
\sigma(\theta, \mu) = m \tan(\theta/2) \Sigma(\theta, \mu).
\]

The most interesting property of this relation is that in the limit of very large scaling variable \( \sigma(\theta, \mu) \) develops a discontinuity at \( \theta = \pi \). This is the first–order phase transition proposed by Dashen [1]. The topological susceptibility can also be calculated with the infinite volume result

\[
\chi(\theta, m, V) = \Sigma \mu |\cos(\theta/2)|.
\]

This is nothing but the flavor singlet Ward–Takahashi identity generalized to nonzero values of \( \theta \) which predicts a linear rise in the topological susceptibility with the quark mass. All of the above results can be shown to be independent of both the pattern of chiral symmetry breaking and the representation of the matter fields [3].

REFERENCES

1. R. Dashen, Phys. Rev. D 3, 1879 (1971).
2. J. Gasser and H. Leutwyler, Phys. Lett. B 188, 477 (1987).
3. H. Leutwyler and A. Smilga, Phys. Rev. D 46, 5607 (1992).
4. P.H. Damgaard, Nucl. Phys. B 556, 327 (1999).
5. J.T. Lenaghan and T. Wilke, hep-th/0108166.
6. G. Akemann, J.T. Lenaghan and K. Splittorff, hep-th/0110157.