TATE CLASSES ON SELF-PRODUCTS OF ABELIAN VARIETIES OVER FINITE FIELDS

YURI G. ZARHIN

Abstract. We deal with $g$-dimensional abelian varieties $X$ over finite fields. We prove that there is an universal constant (positive integer) $N = N(g)$ that depends only on $g$ that enjoys the following property. If a certain self-product of $X$ carries an exotic Tate class then the self-product $X^{2N}$ of $X$ also carries an exotic Tate class. This gives a positive answer to a question of Kiran Kedlaya.

1. Introduction

Let $X$ be an abelian variety of positive dimension $g$ over a finite field $k = \mathbb{F}_q$ of characteristic $p$ (where $q$ is a power of $p$), $F_{X}$ the Frobenius endomorphism of $X$, and $P_{X}[t] \in \mathbb{Z}[t]$ the characteristic polynomial of $F_{X}$, which is a degree $2g$ monic polynomial with integer coefficients. Let $L = L_{X}$ be the splitting field of $P_{X}[t]$ over the field $\mathbb{Q}$ of rational numbers and therefore is a number field. Since $\text{deg}(P_{X}) = 2g$, the degree $[L_{X} : \mathbb{Q}]$ divides $(2g)!$. (In fact, one may prove that $[L_{X} : \mathbb{Q}]$ divides $2^g g!$, see below).

We write $R_{X}$ for the set of eigenvalues of $F_{X}$; clearly, $R_{X}$ coincides with the set of roots of $P_{X}[t] \in \mathbb{Z}[t]$ and is viewed as a certain finite subset of $L_{X}^{\times}$. Clearly, $R_{X}$ consists of algebraic integers and $\#(R_{X}) \leq 2g$. (The equality holds if and only if $P_{X}[t]$ has no repeated roots.) By a classical theorem of A.Weil [8], all algebraic numbers $\alpha \in R_{X}$ have the same archimedean value $\sqrt{q}$. In addition, $\alpha \mapsto q/\alpha$ is a permutation of $R_{X}$. If $\alpha$ is a root of $P_{X}[t]$ (i.e., $\alpha \in R_{X}$) then we write $\text{mult}_{X}(\alpha)$ for its multiplicity. It is well known that if $\alpha \in R_{X}$ then

$$\text{mult}_{X}(\alpha) = \text{mult}_{X}(q/\alpha); \quad \text{if } \alpha = q/\alpha \text{ then } \text{mult}_{X}(\alpha) \text{ is even.} \quad (1)$$

In particular, the constant term $\prod_{\alpha \in R_{X}} \alpha^{\text{mult}_{X}(\alpha)}$ of $P_{X}[t]$ is $q^g$.

The Galois group $\text{Gal}(L_{X}/\mathbb{Q})$ of $L_{X}/\mathbb{Q}$ permutes elements of $R_{X}$ and

$$\text{mult}_{X}(\sigma(\alpha)) = \text{mult}_{X}(\alpha) \forall \sigma \in \text{Gal}(L_{X}/\mathbb{Q}), \alpha \in R_{X}. \quad (2)$$

In this paper we continue our study of multiplicative relations between elements of $R_{X}$ that was started in [22, 6, 23, 24]. (In [22, 6] we concentrated on abelian varieties with rather special type of Newton polygons; in [23, 24]...
we studied abelian varieties of small dimension). In order to state results of the present paper, we need the following definitions.

**Definition 1.1.** An integer-valued function $e: R_X \to \mathbb{Z}$ is called

(i) **admissible** if there exists an integer $d$ such that

\[
\prod_{\alpha \in R_X} \alpha^{e(\alpha)} = q^d,
\]

Such a $d$ is called the **degree** of $e$ and is denoted $\text{deg}(e)$.

The nonnegative integer $\sum_{\alpha \in R_X} |e(\alpha)|$ is called the **weight** of $e$ and denoted $\text{wt}(e)$.

(ii) **trivial** if

\[
e(\alpha) = e(q/\alpha) \ \forall \alpha \in R_X; \ e(\beta) \in 2\mathbb{Z} \ \forall \beta \in R_X \text{ with } \beta^2 = q.
\]

**Definition 1.2.** An **admissible** integer-valued function $e: R_X \to \mathbb{Z}$ is called **reduced** if it enjoys the following properties:

(i) $\text{deg}(e) \geq 1$ and all $e(\alpha) \geq 0$.

(ii) If $\alpha \in R_X$ and $\alpha \neq q/\alpha$ then either $e(\alpha) = 0$ or $e(q/\alpha) = 0$.

(iii) $e(\beta) = 0$ or $1 \ \forall \beta \in R_X$ with $\beta^2 = q$.

**Remarks 1.3.**

(i) It follows from (1) that $\alpha \mapsto \text{mult}_X(\alpha)$ is a trivial admissible function of degree $g$ and weight $2g$.

(ii) Every trivial function is admissible.

(iii) If $e: R_X \to \mathbb{Z}$ is admissible then it follows from Weil’s theorem that

\[
2 \deg(e) = \sum_{\alpha \in R_X} e(\alpha).
\]

(iv) If $e: R_X \to \mathbb{Z}$ is reduced admissible then it follows from (5) that

\[
\text{wt}(e) = 2 \deg(e).
\]

Our first main result is the following assertion.

**Theorem 1.4.** Let $g$ be a positive integer. There exists a positive integer $N = N(g)$ that depends only on $g$ and enjoys the following property.

Let $X$ be a $g$-dimensional abelian variety over a finite field $k$ such that there exists a nontrivial admissible function $R_X \to \mathbb{Z}$. Then there exists a reduced admissible function of degree $\leq N(g)$.

Our main tool in the proof of Theorem 1.4 is the multiplicative (sub)group $\Gamma(X, k) \subset L_X^*$ generated by $R_X$, which was first introduced in [19, 20] (see also [22, 6, 23, 24]).

**Definition 1.5.**

(i) We say that $k = \mathbb{F}_q$ is **small** with respect to $X$ if there exist distinct $\alpha_1, \alpha_2 \in R_X$ such that $\alpha_1/\alpha_2$ is a root of unity.

(ii) We say that $k$ is **sufficiently large** with respect to $X$ if $\Gamma(X, k)$ does not contain roots of unity except 1 (see [23, 24]).
Remarks 1.6.

(i) If \( k \) is not small with respect to \( X \) then there is at most one \( \beta \in R_X \) with \( \beta^2 = q \).

(ii) If \( k \) is sufficiently large with respect to \( X \) then it is not small.

The role of \( \Gamma(X, k) \) is explained by the following statement.

Lemma 1.7. Suppose that \( k \) is not small with respect to \( X \). Then the following three conditions are equivalent.

(i) There exists a nontrivial admissible function \( R_X \to \mathbb{Z} \).

(ii) There exists a reduced admissible function \( R_X \to \mathbb{Z} \).

(iii) The rank of \( \Gamma(X, k) \) does not exceed \( \lfloor \#(R_X)/2 \rfloor \).

Our second main result deals with Tate classes on abelian varieties (see [15, 16, 18, 21, 23] and Section 7.9 below for the definition of these classes and their basic properties). Recall that a Tate class is called exotic if it cannot be presented as a linear combination of products of divisor classes.

Theorem 1.8. Let \( g \) be a positive integer and let \( N = N(g) \) be as in Theorem 1.4.

Let \( X \) be a \( g \)-dimensional abelian variety over a finite field \( k \) of characteristic \( p \). Assume that there exist a positive integer \( n \) and a prime \( \ell \neq p \) such that the self-product \( X^n \) of \( X \) carries an exotic \( \ell \)-adic Tate cohomology class.

Then the self-product \( X^{2N} \) of \( X \) carries an exotic \( \ell \)-adic cohomology Tate class for all primes \( \ell \neq p \).

Remark 1.9. Theorem 1.8 gives a positive answer to a question of Kiran Kedlaya, who pointed out that this result is related to the algorithmic problem of deciding whether or not a given abelian variety (specified by its Weil polynomial) is neat in a sense of [23, Sect. 3], [24].

Is it possible to get all Tate classes on all self-products of \( X \), using only Tate classses of bounded dimension? In order to answer this question, we need the following result about nonnegative admissible functions.

Theorem 1.10. Let \( g \) be a positive integer. Then there exists a positive even integer \( H = H(g) \) that enjoys the following property.

Let \( X \) be a \( g \)-dimensional abelian variety over a finite field \( k \). Then there exist a positive integer \( d \) and \( d \) nonnegative admissible functions \( e_i : R_X \to \mathbb{Z}_+ \) such that:

(i) the weight of each \( e_i \) does not exceed \( H(g) \);

(ii) each nonnegative admissible function \( e : R_X \to \mathbb{Z}_+ \) may be presented as a linear combination of \( e_1, \ldots, e_d \) with nonnegative integer coefficients.

Theorem 1.10 implies the following assertion.

Theorem 1.11. Let \( g \) be a positive integer. Let \( H = H(g) \) be as in Theorem 1.10.
Let $X$ be a $g$-dimensional abelian variety over a finite field $k$. Assume that $k$ is sufficiently large w.r.t $X$. Let $\ell$ be a prime different from $\text{char}(k)$ and $n$ be a positive integer. Then every $\ell$-adic Tate cohomology class on $X^n$ may be presented as a linear combination of products of $\ell$-adic Tate cohomology classes of dimension $\leq H(g)$.

The paper is organized as follows. In Section 2 we discuss basic useful results about $R_X$ and related objects, including the Newton polygons. In addition, we discuss roots of unity in $\Gamma(X,k)$ (Lemma 2.2) and the structure and degree of $L_X$ (Lemma 2.5). In Section 3 we study multiplicative relations between Weil numbers (i.e., admissible functions) and their weights; in particular, we prove Lemma 1.7 (see Lemma 3.10). In Sections 4 and 5 we prove Theorems 1.4 and 1.10 respectively. Section 6 contains certain constructions from multilinear algebra that we use in Section 7 in order to prove Theorems 1.8 and 1.11.

As usual, $\ell$ and $l$ are primes different from $p$, and $\mathbb{N}, \mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Q}_\ell$ stand for the set of positive integers, the rings of integers and $\ell$-adic integers, the fields of rational, real, complex, and $\ell$-adic numbers respectively. We write $\mathbb{Z}_+$ and $\mathbb{R}_+$ for the additive semigroups of nonnegative integers and of nonnegative real numbers respectively. If $z$ is a complex number then we write $\bar{z}$ for its complex-conjugate. Similarly, if $\phi : E \hookrightarrow \mathbb{C}$ is a field embedding then we write $\bar{\phi}$ for the corresponding complex-conjugate field embedding

$$\bar{\phi} : E \hookrightarrow \mathbb{C}, \ x \mapsto \bar{\phi(x)}.$$ 

If $M$ is a positive integer and $v$ and $w$ are two vectors in $\mathbb{R}^M$ then we write $v \cdot w$ for their scalar product. If $A$ is a finite set then we write $\#(A)$ for the number of its elements. We write $\text{rk}(\Delta)$ for the rank of a finitely generated commutative group $\Delta$.

Acknowledgements. I am deeply grateful to Kiran Kedlaya for interesting stimulating questions and to the referee, whose comments helped to improve the exposition.

Part of this work was done during my stay at Centre Émile Borel (Institut Henri Poincaré, Paris) in June-July 2019, whose hospitality and support are gratefully acknowledged.

2. Preliminaries

In this section we discuss basic properties of $L = L_X, R_X, \Gamma(X,k)$. Let us start with the formal definition of $\mathcal{P}_X(t)$.

Throughout this paper $k$ is a finite field of characteristic $p$ that consists of $q$ elements, $\bar{k}$ an algebraic closure of $k$ and $\text{Gal}(k) = \text{Gal}(\bar{k}/k)$ the absolute Galois group of $k$. It is well known that the profinite group $\text{Gal}(k)$ is procyclic and the Frobenius automorphism

$$\sigma_k : \bar{k} \rightarrow \bar{k}, \ x \mapsto x^q.$$
is a topological generator of $\text{Gal}(k)$. If $\ell \neq p$ is a prime then we write

$$\chi_\ell: \text{Gal}(k) \to \mathbb{Z}_\ell^*$$

for the $\ell$-adic cyclotomic character that defines the Galois action on all $\ell$-power roots of unity in $\bar{k}$. By definition,

$$\chi_\ell(\sigma_k) = q \in \mathbb{Z}_\ell^*.$$

Let $X$ be an abelian variety of positive dimension over $k$. We write $\text{End}(X)$ for the ring of its $k$-endomorphisms and $\text{End}^0(X)$ for the corresponding (finite-dimensional semisimple) $\mathbb{Q}$-algebra $\text{End}(X) \otimes \mathbb{Q}$. We write $\text{Fr}_X = \text{Fr}_{X,k}$ for the Frobenius endomorphism of $X$. We have

$$\text{Fr}_X \in \text{End}(X) \subset \text{End}^0(X).$$

It is well known that

$$\sigma_k(x) = \text{Fr}_X(x) \quad \forall x \in X(\bar{k}). \quad (7)$$

By a theorem of Tate [16, Sect. 3, Th. 2 on p. 140], the $\mathbb{Q}$-subalgebra $\mathbb{Q}[\text{Fr}_X]$ of $\text{End}^0(X)$ generated by $\text{Fr}_X$ coincides with the center of $\text{End}^0(X)$. In particular, if $\text{End}^0(X)$ is a field then $\text{End}^0(X) = \mathbb{Q}[\text{Fr}_X]$.

If $\ell$ is a prime different from $p$ then we write $T_\ell(X)$ for the $\mathbb{Z}_\ell$-Tate module of $X$ and $V_\ell(X)$ for the corresponding $\mathbb{Q}_\ell$-vector space $V_\ell(X) = T_\ell(X) \otimes \mathbb{Q}_\ell$.

It is well known [8, Sect. 18] that $T_\ell(X)$ is a free $\mathbb{Z}_\ell$-module of rank $2\dim(X)$ that may be viewed as a $\mathbb{Z}_\ell$-lattice in the $\mathbb{Q}_\ell$-vector space $V_\ell(X)$ of dimension $2\dim(X)$. The Galois action on $X(\bar{k})$ induces the continuous group homomorphism $[11, 12]$

$$\rho_\ell = \rho_{\ell,X}: \text{Gal}(K) \to \text{Aut}_{\mathbb{Z}_\ell}(T_\ell(X)) \subset \text{Aut}_{\mathbb{Q}_\ell}(V_\ell(X)).$$

In addition, there is a canonical isomorphism of $\text{Gal}(k)$-modules $X[\ell] \cong T_\ell(X)/\ell$ where $X[\ell]$ is the kernel of multiplication by $\ell$ in $X(\bar{k})$.

By functoriality, $\text{End}(X)$ and $\text{Fr}_X$ acts on $(T_\ell(X)$ and) $V_\ell(X)$; it is well known that the action of $\text{Fr}_X$ coincides with the action of $\rho_\ell(\sigma_k)$. By a theorem of A. Weil [8, Sect. 19 and Sect. 21], $\text{Fr}_X$ acts on $V_\ell(X)$ as a semisimple linear operator, its characteristic polynomial

$$P_X(t) = P_{X,k}(t) = \det(t\text{Id} - \text{Fr}_X, V_\ell(X)) \in \mathbb{Z}_\ell[t]$$

lies in $\mathbb{Z}[t]$ and does not depend on a choice of $\ell$. In addition, all eigenvalues of $\text{Fr}_X$ (which are algebraic integers) have archimedean absolute value equal to $q^{1/2}$, and if an eigenvalue of $\text{Fr}_X$ is a square root of $q$ then its multiplicity is even (see [22, p. 267]). This implies that the constant term of $P_{X,k}(t)$ is $q^{\dim(X)}$. In particular, $\text{Fr}_X$ acts as an automorphism of the free $\mathbb{Z}_\ell$-module $T_\ell(X)$.

This means that if

$$L = L_X \subset \mathbb{C}$$
is the splitting field of $\mathbb{P}_X(t)$ and

$$R_X = R_{X,k} \subset L$$

is the set of roots of $P(t)$ then $L$ is a finite Galois extension of $\mathbb{Q}$ such that for every field embedding $L \hookrightarrow \mathbb{C}$ we have $|\alpha| = q^{1/2}$ for all $\alpha \in R_X$. Let $\text{Gal}(L/\mathbb{Q})$ be the Galois group of $L/\mathbb{Q}$. Clearly, $R_X$ is a $\text{Gal}(L/\mathbb{Q})$-invariant (finite) subset of $L^\times$. It follows easily that if $\alpha \in R_X$ then $q/\alpha \in R_X$. Indeed, $q/\alpha$ is the complex-conjugate $\bar{\alpha}$ of $\alpha$. We have

$$q^{-1} \alpha^2 = \frac{\alpha}{q/\alpha}.$$

**Definition 2.1.** Let $\ell$ be a prime and $n$ a positive integer. We write $e_\ell(n)$ for the largest order of elements of the general linear group $\text{GL}(n, \mathbb{F}_\ell)$. We write $\exp_\ell(n)$ for the exponent of $\text{GL}(n, \mathbb{F}_\ell)$.

Recall that $\Gamma(X, k)$ is the multiplicative subgroup of $L$ generated by $R_X$.

**Lemma 2.2.** If $\gamma \in \Gamma(X, k)$ is a root of unity then there is a positive integer $m \leq \max(2e_2(2g), e_3(2g))$ such that $\gamma^m = 1$. In addition, $\gamma^{D(g)} = 1$ where

$$D(g) := \text{LCM}(2\exp_2(2g), \exp_3(2g)),$$

and $m | D(g)$.

**Proof.** In what follows we choose a prime $\ell \neq p$ and view $\text{Fr}_X$ as the automorphism of free $\mathbb{Z}_\ell$-module $T_\ell(X)$ of rank $2g$. Then $\text{Fr}_X$ induces the automorphism $\text{Fr}_X \mod \ell$ of the $2g$-dimensional $\mathbb{F}_\ell$-vector space $T_\ell(X)/\ell = X[\ell]$.

Let $r$ be the order of $\text{Fr}_X \mod \ell \in \text{Aut}_{\mathbb{F}_\ell}(X[\ell]) \cong \text{GL}(2g, \mathbb{F}_\ell)$.

Clearly, $r \leq e_\ell(2g)$ and $r | \exp_\ell(2g)$.

In addition,

$$\text{Fr}_X^r \in \text{Id} + \ell \text{End}_{\mathbb{Z}_\ell}(T_\ell(X)).$$

Let $\Delta$ be the multiplicative group generated by all the eigenvalues of $\text{Fr}_X^r$. Clearly, $\delta = \gamma^r \in \Delta$. Applying a variant of Minkowski’s Lemma [13 Lemma 2.4], we obtain that $\delta = 1$ if $\ell > 2$ and $\delta^2 = 1$ if $\ell = 2$. This implies that $\gamma^r = 1$ if $\ell > 2$ and $\gamma^{2r} = 1$ if $\ell = 2$. Now let us put $\ell = 2$ if $p \neq 2$ and $\ell = 3$ if $p = 2$. The rest is clear.

Let $\mathcal{O}_L$ be the ring of integers in $L$. Clearly, $R_X \subset \mathcal{O}_L$. By a classical theorem of A. Weil (Riemann’s hypothesis) [3], if $j : L_X = L \hookrightarrow \mathbb{C}$ is a field embedding then $j(\alpha)j(\alpha) = q$. This implies that if $\mathfrak{B}$ is a maximal ideal in $\mathcal{O}_L$ such that $\text{char}(\mathcal{O}_L/\mathfrak{B}) \neq p$ then all elements of $R_X$ are $\mathfrak{B}$-adic units. The $p$-adic behaviour of $R_X$ is described in terms of the set $\text{Slp}_X$ of slopes of the Newton polygon of $X$ [9] (see also [21 Sect. 4]). Recall that $\text{Slp}_X$ is a finite nonempty set of rational numbers that enjoys the following properties.
(i) \(0 \leq c \leq 1\) for all \(c \in \text{Slp}_X\).
(ii) \(c \in \text{Slp}_X\) if and only if \(1 - c \in \text{Slp}_X\).
(iii) If \(c \in \text{Slp}_X\) then either \(c = 1/2\) or there is a positive integer \(h \leq g = \dim(X)\) such that \(c \in 1/h\mathbb{Z}\).
(iv) Let \(\mathfrak{p}\) be any maximal ideal in \(\mathcal{O}_L\) such that \(\text{char}(\mathcal{O}_L/\mathfrak{p}) = p\) and let
\[
\text{ord}_\mathfrak{p} : L^* \to \mathbb{Q}
\]
be the discrete valuation map attached to \(\mathfrak{p}\) that is normalized by the condition
\[
\text{ord}_\mathfrak{p}(q) = 1.
\]
Then
\[
\text{ord}_\mathfrak{p}(R_X) = \text{Slp}_X.
\]
(v) If \(\alpha \in R_X\) then
\[
\text{ord}_\mathfrak{p}(q/\alpha) = 1 - \text{ord}_\mathfrak{p}(\alpha).
\]
(vi) Let \(\mu_L\) the multiplicative group of all roots of unity in \(L\). Then its image \(\text{ord}_\mathfrak{p}(\mu_L) = \{0\}\).

Properties (i)-(vi) imply readily the following assertion.

**Lemma 2.3.** Let \(g\) be a positive integer. Let us consider the set \(\text{Slp}(g)\) of all rational numbers \(c\) that enjoy the following properties.

1. \(0 \leq c \leq 1\).
2. Either \(c = 1/2\) or there exists a positive integer \(h \leq g\) such that \(c \in 1/h\mathbb{Z}\).

Then \(\text{Slp}(g)\) is a finite nonempty set that enjoys the following property.

If \(X\) is a \(g\)-dimensional abelian variety over a finite field then \(\text{Slp}_X\) lies in \(\text{Slp}(g)\).

**Remarks 2.4.** Let \(S(p)\) be the set of all maximal ideals in \(\mathcal{O}_L\) such that \(\text{char}(\mathcal{O}_L/\mathfrak{p}) = p\). Since \(L/\mathbb{Q}\) is Galois, \(#(S(p))\) divides \(|L : \mathbb{Q}|\); in particular, \(#(S(p))\) divides \(g!2^g\) (see Lemma 2.5(ii) below). Let us define a group homomorphism
\[
w_X : \Gamma(X, K) \to \mathbb{Q}^{S(p)}, \gamma \mapsto \{\text{ord}_\mathfrak{p}(\gamma)\}_{\mathfrak{p} \in S(p)}.
\]

(a) Clearly, \(w_X(q)\) is the vector \(1 \in \mathbb{Q}^{S(p)}\), all whose coordinates equal 1, hence
\[
w_X \left(\frac{q}{\alpha}\right) = 1 - w_X(\alpha) \quad \forall \alpha \in R_X.
\]
(b) By Property (iv) and Lemma 2.3
\[
w_X(R_X) \subset \text{Slp}_X^{S(p)} \subset \text{Slp}(g)^{S(p)} \subset \mathbb{Q}^{S(p)}.
\]
(c) It follows from Property (v) that a vector \(\tilde{c} \in \mathbb{Q}^{S(p)}\) lies in \(w_X(R_X)\) if and only if \(1 - \tilde{c} \in w_X(R_X)\).
(d) In light of Property (vi), \( w_X(\gamma) = 0 \) if \( \gamma \) is a root of unity. The converse is also true: it is proven in [21 Prop. 2.1 on p. 249] (see also [20 Prop. 3.1.5]) that \( \ker(w_X) \) consists of roots of unity.

(e) It follows readily from (d) that:

1. none of elements in \( R_X \) lies in \( \ker(w_X) \);
2. if \( k \) is not small w.r.t \( X \) and \( \alpha_1, \alpha_2 \) are distinct elements of \( R_X \) then

\[
w_X(\alpha_1) \neq w_X(\alpha_2).
\]

**Lemma 2.5.**

(i) The field \( L_X \) is either \( \mathbb{Q} \) or \( \mathbb{Q}(\sqrt{p}) \) or a CM field.

(ii) The field \( L_X \) is a finite Galois extension of \( \mathbb{Q} \) and its degree \( [L_X : \mathbb{Q}] \) divides \( g!2^g \).

(iii) \( #(S(p)) \) divides \( g!2^g \).

**Proof.** Let us prove (i,ii). By definition of the splitting field, \( L_X/\mathbb{Q} \) is Galois.

Suppose that \( X \) is simple. According to [16, 17], \( \mathcal{P}_X(t) \) is a power \( \mathcal{P}_{\text{irr}}(t)^a \) of a \( \mathbb{Q} \)-irreducible monic polynomial \( \mathcal{P}_{\text{irr}}(t) \) where \( a \) is a positive integer dividing \( 2g \) and

\[
\deg(\mathcal{P}_{\text{irr}}) = \frac{2g}{a}.
\]

Clearly, \( L_X \) is the normal closure of the degree \( 2g/a \) number field \( E_X := \mathbb{Q}[t]/\mathcal{P}_{\text{irr}}(t) \). According to [17, Exemples], \( E_X \) is either \( \mathbb{Q} \) or \( \mathbb{Q}(\sqrt{p}) \) or a CM field.

In the first two cases \( L_X = \mathbb{Q} \) or \( \mathbb{Q}(\sqrt{p}) \); in particular, it is a totally real number field, whose degree divides \( 2g = 2\dim(X) \).

In the third case let \( E_X^+ \) be the maximal totally real subfield of \( E_X \); \( [E_X^+ : \mathbb{Q}] = g/a \) and \( E_X \) is a purely imaginary quadratic extension \( E_X^+/(\sqrt{-}\delta) \) of \( E_X^+ \). Here \( \delta \) is a totally positive element of \( E_X^+ \). Let \( L_X^+ \) be the normal closure of \( E_X^+ \). Since \( E_X^+ \) is totally real, \( L_X^+ \) is totally real as well, and its degree \( [L_X^+ : \mathbb{Q}] \) divides \( [E_X^+ : \mathbb{Q}] = (g/a)! \). Since \( -\delta \in E_X^+ \subset L_X^+ \), its Galois orbit in \( L_X^+ \) consists at most of \( [E_X^+ : \mathbb{Q}] = g/a \) elements. This implies that \( [L_X : L_X^+] \) divides \( 2^{g/a} \), since \( L_X \) is obtained from \( L_X^+ \) by adjoining square roots of all the (totally negative) Galois conjugates \( -\delta \). This implies that \( L_X \) is a CM field and \( [L_X : \mathbb{Q}] \) divides \( (g/a)! \cdot 2^{g/a} \), which in turn, divides \( g! \cdot 2^g \).

Now let us consider the general case when \( X \) is isogenous to a product \( \prod_{i=1}^m X_i \) of \( m \) nonzero simple abelian varieties \( X_i \). It is well known that if we put

\[
g_i := \dim(X_i), \quad \text{and} \quad L_i := L_{X_i}
\]

then

\[
g = \dim(X) = \sum_{i=1}^m g_i, \quad \mathcal{P}_X(t) = \prod_{i=1}^m \mathcal{P}_{X_i}(t).
\]

Let \( \bar{\mathbb{Q}} \) be an algebraic closure of \( \mathbb{Q} \). We may and will view all \( L_i \) as subfields of \( \bar{\mathbb{Q}} \). Then \( L_X \) is the **compositum** of \( m \) number fields \( L_{X_i} = L_i \) in \( \bar{\mathbb{Q}} \).
Applying (i,ii) to simple $X_i$'s, we obtain that $L_X$ is either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{p})$ or a CM field, which proves (i).

In order to prove (ii), recall that all $L_i/\mathbb{Q}$ and $L_X/\mathbb{Q}$ are finite Galois extensions. Let $\text{Gal}(L_i/\mathbb{Q})$ and $\text{Gal}(L_X/\mathbb{Q})$ be the corresponding (finite) Galois groups. Clearly, each $L_i$ is a $\text{Gal}(L_X/\mathbb{Q})$-invariant subfield of $L_X$, and the corresponding restriction map

$$\text{Gal}(L_X/\mathbb{Q}) \to \text{Gal}(L_i/\mathbb{Q}), \ s \mapsto s_i$$

is a surjective group homomorphism. On the other hand, since all the $L_i$'s generate $L_X$ as a field, the product-map

$$\text{Gal}(L_X/\mathbb{Q}) \to \prod_{i=1}^m \text{Gal}(L_i/\mathbb{Q}), \ s \mapsto \{s_i\}_{i=1}^m$$

is a group embedding. By Lagrange's theorem, $\#(\text{Gal}(L_X/\mathbb{Q}))$ divides $\prod_{i=1}^m \#(\text{Gal}(L_i/\mathbb{Q}))$. In other words, $[L_X : \mathbb{Q}]$ divides $\prod_{i=1}^m [L_i : \mathbb{Q}]$, which, in turn, divides

$$\prod_{i=1}^m g_i! 2^{\beta_i} = 2^g \prod_{i=1}^m g_i!.$$ 

Since $\sum_{i=1}^m g_i = g$, the product $\prod_{i=1}^m g_i!$ divides $g!$. This implies that $[L_X : \mathbb{Q}]$ divides $2^g g!$, which ends the proof of (ii).

Let us prove (iii). Since $L_X/\mathbb{Q}$ is Galois, $\#(S(p))$ divides $[L_X : \mathbb{Q}]$. Now (iii) follows readily from (ii).

3. Multiplicative relations between Weil numbers

This section contains auxiliary results that will be used in Section 4 in the proof of Theorem 1.4.

3.1. The involution

Recall that there is an involution map

$$\iota : R_X \to R_X, \ \alpha \mapsto \frac{q}{\alpha} = \bar{\alpha}. \quad (8)$$

Let $R_X^\iota$ be the subset of fixed points of $\iota$. Its elements (if there are any) are square roots of $q$; hence,

$$\#(R_X^\iota) \leq 2. \quad (9)$$

In addition, if $k$ is not small with respect to $X$ then at most one square root of $q$ lies in $R_X$, hence,

$$\#(R_X^\iota) \leq 1. \quad (10)$$

**Remark 3.2.** Suppose that $k$ is not small w.r.t $X$. If $\beta$ is an element of $R_X$ such that $\beta^2/q$ is a root of unity then $q/\beta \in R_X$ and the ratio

$$\frac{\beta}{q/\beta} = \frac{q}{\beta^2}$$

is a root of unity. This implies that $\beta = q/\beta$, i.e., $\beta \in R_X^\iota$. 

Let us consider the free abelian group $\mathbb{Z}^{R_X}$ of functions $e : R_X \to \mathbb{Z}$. The involution $\iota$ induces an automorphism (also an involution)

$$\iota^* : \mathbb{Z}^{R_X} \to \mathbb{Z}^{R_X}, \quad \iota^* e(\alpha) := e(\iota\alpha) = e(q/\alpha).$$

Let us consider the group homomorphism

$$\Pi : \mathbb{Z}^{R_X} \to \Gamma(X, k) \subset L_X^*, \quad e \mapsto \prod_{\alpha \in R_X} \alpha^{e(\alpha)}.$$ 

We have

$$\Pi(\iota^* e) = \Pi(e).$$

Remarks 3.3.

(i) If $\iota^* e = e$ then $\Pi(e) = \Pi(e)$ is (totally) real; it follows from Weil’s theorem that $\Pi(e)^2$ is an integral power of $q$. In particular, if $\sum_{\alpha \in R_X} e(\alpha)$ is even then it follows from Weil’s theorem that $\Pi(e)$ is $\pm$ integral power of $q$.

(ii) If $f$ is a function $R_X \to \mathbb{Z}$ then the function $e := f + \iota^* f$ is obviously trivial.

Conversely, one may easily check that a function $e : R_X \to \mathbb{Z}$ is trivial if and only if there exists $f : R_X \to \mathbb{Z}$ such that $e = f + \iota^* f$.

(iii) Clearly, $e$ is admissible if and only if $\Pi(e)$ lies in the cyclic multiplicative subgroup $q\mathbb{Z}$ generated by $q$.

3.4. Ranks and Orbits. The complement $R_X \setminus R'_X$ splits (if it is not empty) into a disjoint union of 2-element orbits of $\iota$ say $\{\alpha, q/\alpha\}$. Let $r_X$ be the number of such orbits, which is a nonnegative integer that vanishes if and only if $R_X = R'_X$. We have

$$\#(R_X \setminus R'_X) = 2r_X; \quad r_X \leq \frac{\#(R_X)}{2} \leq \frac{2g}{2} = g. \quad (11)$$

If $r_X \geq 1$ (i.e., $R_X \neq R'_X$) then we have $r_X$ 2-elements $\iota$-orbits $O_1, \ldots, O_{r_X}$ in $R_X \setminus R'_X$. By choosing arbitrarily an element $\alpha_i \in O_i$ for all $i = 1, \ldots, r_X$, we get

$$O_i = \{\alpha_i, q/\alpha_i\} \forall i = 1, \ldots, r_X; \quad R_X \setminus R'_X = \{\alpha_1, q/\alpha_1, \ldots, \alpha_{r_X}, q/\alpha_{r_X}\}. \quad (12)$$

Recall [23], that $\Gamma(X, k)$ always contains $q$. Since $\beta^2 = q$ for all $\beta \in R'_X$, the subgroup $\Gamma_1(X, k)$ of $\Gamma(X, k)$ generated by $q$ and all elements of $R_X \setminus R'_X$ has finite index in $\Gamma(X, k)$. In particular,

$$\text{rk}(\Gamma_1(X, k)) = \text{rk}(\Gamma(X, k)). \quad (13)$$

Clearly, if $r_X = 0$ then $\Gamma_1(X, k) = q^{\mathbb{Z}}$ has rank 1, hence $\text{rk}(\Gamma(X, k)) = 1$. It follows from [12] that if $r_X \geq 1$ then $\Gamma_1(X, k)$ is generated by $\{\alpha_1, \ldots, \alpha_{r_X}; q\}$. In particular,

$$\text{rk}(\Gamma_1(X, k)) \leq r_X + 1. \quad (14)$$
Remark 3.5. Suppose that $k$ is not small w.r.t $X$. Then $\#(R_X^e) = 0$ or $1$ and therefore $\#(R_X) = 2r_X$ or $2r_X + 1$ respectively. In both cases

$$r_X = \left\lfloor \frac{\#(R_X)}{2} \right\rfloor.$$  

Combining (15) with (14) and (13), we obtain that

$$\text{rk}(\Gamma(X,k)) = \text{rk}(\Gamma_1(X,k)) \leq r_X + 1 = \left\lfloor \frac{\#(R_X)}{2} \right\rfloor + 1. \quad (16)$$

3.6. Nontrivial and reduced admissible functions. The existence of a nontrivial admissible function implies certain restrictions on $R_X$.

Lemma 3.7. Suppose that there exists a nontrivial admissible function $e: R_X \rightarrow \mathbb{Z}$ of degree, say, $d$.

Then the following conditions hold.

(i) $R_X \neq R_X^e$, i.e., $R_X \setminus R_X^e$ is a nonempty subset of $R_X$.

(ii) For each nonzero integer $m$ the function $m \cdot e: R_X \rightarrow \mathbb{Z}$ is also nontrivial admissible.

(iii) Let us consider the function $e_0: R_X \rightarrow \mathbb{Z}$ that vanishes identically on $R_X^e$ (if this subset is nonempty) and coincides with $e$ on $R_X \setminus R_X^e$. Then $e_0$ is nontrivial. In addition, for each nonzero even integer $m$ the function $m \cdot e_0: R_X \rightarrow \mathbb{Z}$ is nontrivial admissible, and its weight

$$\text{wt}(m \cdot e_0) = |m|\text{wt}(e_0) \leq |m|\text{wt}(e). \quad (17)$$

Proof. If $R_X^e = \emptyset$ then all three assertions of Lemma are obviously true. So, let us assume that $R_X^e \neq \emptyset$.

(i) Suppose that $R_X = R_X^e$. Then

$$\prod_{\alpha \in R_X^e} \alpha^e(\alpha) = q^d \quad \text{and} \quad \sum_{\alpha \in R_X^e} e(\alpha) = 2d$$

is an even integer. Since $R_X^e$ consists of one or two elements and $e$ is nontrivial, there is $\beta \in R_X^e$ such the $e(\beta)$ is odd. This implies that $R_X^e$ consists of two elements, say, $\beta$ and $-\beta$,

$$e(\beta) + e(-\beta) = 2d$$

and both integers $e(\beta)$ and $e(-\beta)$ are odd. This implies that (recall that $\beta^2 = q$)

$$q^d = \beta^e(\beta) \cdot (-\beta)^e(-\beta) = \beta^e(\beta) \cdot (-1) \cdot \beta^e(-\beta) = -\beta^{e(\beta)+e(-\beta)} = -\beta^{2d} = -q^d.$$

So, $q^d = -q^d$, which is absurd. The obtained contradiction proves (i).
(ii) The admissibility of \( m \cdot e \) is obvious. The nontriviality is also clear if there exists \( \alpha \in R_X \setminus R_X^\prime \) with
\[
e(\alpha) \neq e(q/\alpha).
\]
So, we may assume that \( m \cdot e \) is trivial (we are going to arrive to a contradiction), and
\[
e(\alpha) = e(q/\alpha) \ \forall \alpha \in R_X \setminus R_X^\prime.
\tag{18}
\]
This implies that there is an integer \( n \) such that
\[
\prod_{\alpha \in R_X^\prime} \alpha^{e(\alpha)} = q^n.
\]
It follows that the sum
\[
\sum_{\alpha \in R_X^\prime} e(\alpha) = 2n \in 2\mathbb{Z}
\tag{19}
\]
is an even integer. On the other hand, the nontriviality of \( e \) combined with (18) implies that there is \( \beta \in R_X^\prime \) with odd \( e(\beta) \). Since \( \#(R_X^\prime) \leq 2 \), it follows from (19) that integer \( e(\alpha) \) is odd for all \( \alpha \in R_X^\prime \). It follows that \( \prod_{\alpha \in R_X^\prime} \alpha = q^d \) for some integer \( d \). Therefore \( \#(R_X^\prime) = 2d \) is a positive even integer, i.e., \( R_X^\prime \) consists of two elements \( \beta, -\beta \) with \( \beta^2 = q \); in addition, both integers \( e(\beta) \) and \( e(-\beta) \) are odd. The same computations as in the proof of (i) give us that
\[
q^d = \beta^{e(\beta)}(-\beta)^{e(-\beta)} = -q^d,
\]
hence, \( q^d = -q^d \). The obtained contradiction proves the nontriviality of \( m \cdot e \).

(iii) Suppose that \( e_0 \) is trivial, i.e.,
\[
e(\alpha) = e(q/\alpha) \ \forall \alpha \in R_X \setminus R_X^\prime.
\]
Then it is admissible and therefore there is an integer \( h \) such that
\[
\prod_{\alpha \in R_X \setminus R_X^\prime} \alpha^{e_0(\alpha)} = \prod_{\alpha \in R_X \setminus R_X^\prime} \alpha^{e(\alpha)} = q^h.
\]
Since \( e \) is admissible of degree \( d \),
\[
\prod_{\beta \in R_X^\prime} \beta^{e(\beta)} = q^{d-h}.
\]
The nontriviality of \( e \) implies that there is \( \beta \in R_X^\prime \) such that integer \( e(\beta) \) is odd. Now the same computations as in the proof of (i) give us that \( R_X^\prime \) consists of two elements \( \beta \) and \( -\beta \), both integers \( e(\beta) \) and \( e(-\beta) \) are odd and eventually, \( q^{d-h} = -q^{d-h} \). The obtained contradiction proves that \( e \) is nontrivial, which is the first assertion of (iii).
Let us prove the second assertion of (iii). Since \( m \) is even, there is an integer \( n \) such that \( m = 2n \). We have
\[
q^{md} = \left( \prod_{\alpha \in R_X} \alpha^{e(\alpha)} \right)^m = \left( \prod_{\alpha \in R_X} \alpha^{e(\alpha)} \right) \times \left( \prod_{\beta \in R_X} \beta^{e(\beta)} \right)^m \\
= \left( \prod_{\alpha \in R_X} \alpha^{m-e_0(\alpha)} \right) \times \left( \prod_{\beta \in R_X} \beta^{2n-e(\beta)} \right) \\
= \left( \prod_{\alpha \in R_X} \alpha^{m-e_0(\alpha)} \right) \times \left( \prod_{\beta \in R_X} \beta^{n-e(\beta)} \right)
\]
This implies that \( \prod_{\alpha \in R_X} \alpha^{m-e_0(\alpha)} \) is an integral power of \( q \), i.e., \( m \cdot e_0 \) is admissible. The nontriviality of \( m \cdot e_0 \) follows from the nontriviality of \( e_0 \), because \( e_0 \) vanishes identically on \( R_X \). This ends the proof of (iii). The last assertion of (iii) about weights follows readily from obvious inequality \( \text{wt}(e_0) \leq \text{wt}(e) \).

It turns out that one may easily construct a reduced admissible function when \( k \) is small w.r.t \( X \).

**Lemma 3.8.** Assume that there are distinct \( \alpha_1, \alpha_2 \in R_X \) such that \( \gamma := \alpha_2/\alpha_1 \) is a root of unity. Then there is a reduced admissible function \( e : R_X \to \mathbb{Z} \), whose weight \( w \) enjoys the following properties.
\[
w \leq 4e_2(2g) \quad \text{if} \ p \neq 2; \quad w \leq 2e_3(2g) \quad \text{if} \ p = 2.
\]

**Proof.** Clearly, \( \gamma \in \Gamma(X, k) \). By Lemma 2.2 there is a positive integer \( m \) such that
\[
\gamma^m = 1; \ m \leq 2e_2(2g) \quad \text{if} \ p \neq 2; \ m \leq e_3(2g) \quad \text{if} \ p = 2.
\]
Hence, it suffices to produce a reduced multiplicative relation of weight \( 2m \).
To this end, notice that \( q/\alpha_1 \in R_X \) and
\[
\alpha_2^m (q/\alpha_1)^m = q^m.
\]
If \( \alpha_2 \neq q/\alpha_1 \) then we may define
\[
e : R_X \to \mathbb{Z}, \ e(\alpha_2) := m, e(q/\alpha_1) := m; \ e(\alpha) := 0 \quad \text{for all other} \ \alpha.
\]
Clearly, \( e \) is a reduced admissible function of weight \( 2m \).
Suppose that \( \alpha_2 = q/\alpha_1 \). Since \( \alpha_1 \neq \alpha_2 \),
\[
\alpha_1 \neq q/\alpha_1, \ \alpha_1^2 \neq q.
\]
Then we have
\[
q^m = (\alpha_1 \alpha_2)^m = \alpha_1^{2m}, \ \text{i.e.}, \ \alpha_1^{2m} = q^m.
\]
Now let us consider
\[
e : R_X \to \mathbb{Z}, \ e(\alpha_1) := 2m; \ e(\alpha) := 0 \quad \text{for all other} \ \alpha.
\]
Clearly, \( e \) is a reduced admissible function of weight \( 2m \).

The next Lemma asserts that the existence of a nontrivial admissible function implies the existence of a reduced admissible function, whose weight we can control.

**Lemma 3.9.** Let \( w \) be a positive integer. Suppose that \( k \) is not small w.r.t \( X \) and there exists a nontrivial admissible function of weight \( \leq w \).

Then there exist a nonempty subset \( A_1 \subset R_X \), an integer-valued function \( \tilde{e} : A_1 \to \mathbb{Z} \), and a positive integer \( s \leq w \) that enjoy the following properties.

1. \( \forall \alpha \in A_1 \) we have \( \frac{q}{\alpha} \notin A_1 \), \( \tilde{e}(\alpha) > 0 \).
2. \[ \prod_{\alpha \in A_1} \alpha^{\tilde{e}(\alpha)} = q^s. \] (20)

In particular, if we define

\[ f : R_X \to \mathbb{Z}, \quad f(\alpha) := \tilde{e}(\alpha), \quad f(\alpha) := 0 \quad \forall \alpha \notin A_1 \]

then \( f \) is a reduced admissible function of weight \( 2s \leq 2w \) that vanishes identically on \( R^X \).

**Proof.** By Lemma \( 3.7 \), \( R_X \setminus R^X \) is not empty. Let \( e : R_X \to \mathbb{Z} \) be a nontrivial admissible function \( e \) of weight \( \leq w \). Let us consider (in the notation of Lemma \( 3.7 \)) the function

\[ h_2 = 2 \cdot e_0 : R_X \to \mathbb{Z}. \]

It follows from Lemma \( 3.7 \) that \( h_2 \) is nontrivial admissible, it vanishes identically on \( R^X \) and its weight does not exceed \( 2w \). This implies that

\[ \prod_{\alpha \in R_X \setminus R^X} \alpha^{h_2(\alpha)} = q^d, \quad 2w \geq \sum_{\alpha \in R_X} |h_2(\alpha)| \] (21)

where \( d \) is an integer such that

\[ |d| \leq \text{wt}(h_2) \leq 2w. \]

The nontriviality and vanishing everywhere at \( R^X \) of \( h_2 \) imply that the subset \( A \) of \( R_X \) defined by

\[ A := \{ \alpha \in R_X \mid h_2(\alpha) \neq h_2(q/\alpha) \} \]

is nonempty. It follows from the very definition that \( A \) is \( \iota \)-invariant and does not meet \( R^X \). Let us define the subset \( A_1 \) of \( A \) by

\[ A_1 := \{ \alpha \in A \mid h_2(\alpha) > h_2(q/\alpha) \} \subset A \subset R_X. \]

Clearly, if \( \alpha \in A \) then \( \alpha \in A_1 \) if and only if \( \tilde{\alpha} = q/\alpha \notin A_1 \). This implies that \( A_1 \) is nonempty and \( A \) is the disjoint union of \( A_1 \) and \( \iota(A_1) \). In particular, \( \#(A) = 2\#(A_1) \).

On the other hand, if

\[ B := \{ \beta \in R_X \setminus R^X \mid h_2(\beta) = h_2(q/\beta) \} \subset R_X \setminus R^X \]
then $B$ is $\iota$-invariant, $R_X \setminus R_X^\iota$ is a disjoint union of $A$ and $B$, and
\[
\prod_{\beta \in B} \beta^{h_2(\beta)} = q^n
\]
for some integer $n$ with
\[
|n| \leq \wt(h_2) \leq 2w.
\]
Since $R_X \setminus R_X^\iota$ is a disjoint union of $A$ and $B$, it follows from (21) that
\[
\prod_{\alpha \in A} \alpha^{h_2(\alpha)} = q^d = q^{d-n}.
\]
Since $A$ is a disjoint union of $A_1$ and $\iota(A_1)$, we get
\[
q^{d-n} = \left( \prod_{\alpha \in A_1} \alpha^{h_2(\alpha)} \right) \times \left( \prod_{\alpha \in A_1} \iota(\alpha)^{h_2(\alpha)} \right) = \\
\left( \prod_{\alpha \in A_1} \alpha^{h_2(\alpha)} \right) \times \left( \prod_{\alpha \in A_1} (q/\alpha)^{h_2(q/\alpha)} \right) = \\
\left( \prod_{\alpha \in A_1} \alpha^{h_2(\alpha)-h_2(q/\alpha)} \right) \times q^m
\]
where $m := \sum_{\alpha \in A_1} h_2(q/\alpha) \in \Z$. If we define the function
\[\tilde{e} : A_1 \to \Z, \alpha \mapsto h_2(\alpha) - h_2(q/\alpha)\]
then $\tilde{e}(\alpha) > 0 \ \forall \alpha \in A_1$,
\[
\sum_{\alpha \in A_1} \tilde{e}(\alpha) \leq \sum_{\alpha \in A_1} (|h_2(\alpha)| + |h_2(q/\alpha)|) = \sum_{\alpha \in A} |h_2(\alpha)| \leq \wt(h_2) \leq 2w,
\]
and
\[
q^{d-n} = \left( \prod_{\alpha \in A_1} \alpha^{\tilde{e}(\alpha)} \right) \times q^m,
\]
i.e.,
\[
\prod_{\alpha \in A_1} \alpha^{\tilde{e}(\alpha)} = q^{d-n-m}.
\]
It remains to put $s := d - n - m$. This ends the proof. \hfill \Box

The following assertion contains Lemma 1.7. (Recall that $\Gamma_1(X,k)$ is defined in Subsection 3.4)

**Lemma 3.10.** Suppose that $k$ is not small w.r.t $X$.

Then the following conditions are equivalent.

(1a) There is a nontrivial admissible function on $R_X$.

(1b) There is a nontrivial admissible function on $R_X$ that vanishes at $R_X^\iota$.

(2a) There is a reduced admissible function on $R_X$. 

(2b) There is a reduced admissible function on \(R_X\) that vanishes at \(R'_X\).

(3a) \(\text{rk}(\Gamma(X,k)) \leq \lceil \#(R_X)/2 \rceil\).

(3b) \(\text{rk}(\Gamma_1(X,k)) \leq \lceil \#(R_X)/2 \rceil\).

Proof. Obviously, (1b) implies (1a), (2b) implies (2a), (2a) implies (1a), and (2b) implies (1b). By Lemma 3.9, (1a) implies (2b). This implies that (1a), (1b), (2a), (2b) are equivalent.

In light of (i3), conditions (3a) and (3b) are equivalent.

In order to handle conditions (3), let us discuss the parity of \(\#(R_X)\), using the observations and notation of Subsection 3.1.

In order to check the equivalence of (1) and (3), let us start with the “degenerate” case \(r_X = 0\), i.e., \(R_X = R'_X = \{\beta\}\). Then \(\Gamma(X,k)\) is a nontrivial cyclic group generated by \(\beta\) containing the index 2 subgroup generated by \(\beta^2 = q\). Therefore \(\text{rk}(\Gamma(X,k)) = 1 > 0\), i.e., (3a) does not hold. On the other hand, we have already seen (Lemma 3.7) that if \(R_X = \{\beta\} = R'_X\) then (1a) does not hold.

So, we may assume that \(R_X \neq R'_X\). Then the positive integer \(r_X = \lceil \#(R_X)/2 \rceil\) is the number of all \(\iota\)-orbits \(O_1, \ldots, O_{r_X}\) in \(R_X \setminus R'_X\), see Subsection 3.4. If we choose any element \(\alpha_i\) of \(O_i\) for all \(i\) then the \(2r_X\)-element set

\[
R_X \setminus R'_X = \{\alpha_1, q/\alpha_1, \ldots, \alpha_{r_X}, q/\alpha_{r_X}\}
\]

and \(\Gamma_1(X,k)\) is generated by \(q\) and \(\{\alpha_1, \ldots, \alpha_{r_X}\}\), see Subsection 3.4.

Suppose that (3b) holds. This means that \(\text{rk}(\Gamma_1(X,k)) \leq r_X\). Hence, there are \((r_X + 1)\) integers \(f_1, \ldots, f_{r_X}\) not all zeros, such that

\[
\prod_{i=1}^{r_X} \alpha_i^{f_i} = q^d. \tag{22}
\]

Clearly, not all \(f_1, \ldots, f_{r_X}\) are zeros. Let us define the function

\[
e : R_X \to \mathbb{Z}, \quad e(\alpha_i) = f_i \quad \forall i = 1, \ldots, r_X; \quad f(\alpha) = 0 \quad \text{for all other } \alpha. \tag{23}\]

In light of (22) and (23), \(e\) is a nontrivial admissible function. Hence, (1a) holds.

Now assume that (1a) holds. Then (2b) holds, i.e., there are a nonempty subset \(A_1 \subset R_X\), a function \(\tilde{e} : A_1 \to \mathbb{Z}\) and a positive integer \(s\) that enjoy the following properties.

(i) \(A_1\) and \(\iota(A_1)\) do not meet each other;
(ii) \(\tilde{e}(\alpha) > 0 \quad \forall \alpha \in A_1;\)
(iii) \(\prod_{\alpha \in A_1} \alpha^{\tilde{e}(\alpha)} = q^s\).

Let us put \(n = \#(A_1)\) and let \(A_1 = \{\alpha_1, \ldots, \alpha_n\}\). Then all \(O_i = \{\alpha_i, q/\alpha_i\}\) are disjoint 2-element orbits in \(R_X \setminus R'_X\). In particular, \(n \leq r_X\).

If \(n = r_X\) then \(\{\alpha_1, \ldots, \alpha_{r_X}, q\}\) generate \(\Gamma_1(X,k)\). The property (iii) implies that the rank of this group does not exceed \(r_X\), i.e., (3b) holds.

Now assume that \(n < r_X\). Then there are precisely \((r_X - n)\) other two-element \(\iota\)-orbits \(O_j\) in \(R_X\) \((j = n + 1, \ldots, r_X)\). If we pick for all \(j\) an element \(\delta_j \in O_j\) then \(O_j = \{\delta_j, q/\delta_j\}\) \((n + 1 \leq j \leq r_X)\). Then
\{\alpha_1, \ldots, \alpha_n; \delta_{n+1}, \ldots, \delta_r; q\} generate a subgroup of finite index in \Gamma_1(X,k). The property (iii) implies that the rank of this group does not exceed \(r_X\), i.e., (3b) holds. This ends the proof. \(\square\)

4. Frames and Skeletons of Abelian Varieties over Finite Fields

In the course of our proof of Theorem 1.4 we will need the following notion.

**Definition 4.1.** Let \(g\) be a positive integer. A \(g\)-frame is a triple \((M,r,U)\) that consists of positive integers \(M\) and \(r\), and a finite subset
\[ U \subset \mathbb{Q}^M \]
of nonzero vectors that enjoy the following properties.

(i) \(M\) divides \(2^g g!\), \(r \leq g\), and \(\#(U) = 2r\).
(ii) \(U \subset \text{Slp}(g)^M \subset \mathbb{Q}^M\) (see Lemma 2.3 for the definition of the finite subset \(\text{Slp}(g) \subset \mathbb{Q}\)).
(iii) A vector \(u \in \mathbb{Q}^M\) lies in \(U\) if and only if \(1 - u\) lies in \(U\). Here \(1 = (1, \ldots, 1) \in \mathbb{Q}^M\) is the vector, all whose coordinates are 1.
(iv) Let \(\Delta(U)\) be the additive subgroup of \(\mathbb{Q}^M\) generated by \(1\) and all elements of \(U\). Then the rank of \(\Delta(U)\) does not exceed \(r\).

**Remark 4.2.** The finiteness of \(\text{Slp}(g)\) implies that the set of all frames (for a given \(g\)) is finite.

4.3. The map
\[ \iota_F: \mathbb{Q}^M \to \mathbb{Q}^M, \quad u \mapsto 1 - u \quad (24) \]
is an involution, whose only fixed point is
\[ \frac{1}{2} \cdot 1 = (1/2, \ldots, 1/2). \]

Notice that
\[ \iota_F(U) = U. \]
Since \(\#(U)\) is even, \(U\) does not contain the fixed point \(1/2 \cdot 1\) and therefore splits into a disjoint union of 2-element \(\iota_F\)-orbits \(O_1, \ldots, O_r\). If we choose in each \(O_i\) a vector \(u_i \in O_i\) then
\[ O_i = \{u_i, 1 - u_i\} \forall i = 1, \ldots, r; \quad U = \{u_1, 1 - u_1, \ldots, u_r, 1 - u_r, \ldots, 1 - u_r\} \quad (25) \]

Property (iv) combined with (25) implies that there exist integers \(a_1, \ldots, a_r\) not all zeros and an integer \(d\) such that
\[ \sum_{i=1}^{r} a_i u_i = d \cdot 1 = (d, \ldots, d). \quad (26) \]
Lemma 4.4. Let $g$ be a positive integer. Then there is a positive integer $C(g)$ that depends only on $g$ and enjoys the following property.

Let $(M,r,U)$ be a $g$-frame. Then there are exist $r$ integers $a_1,\ldots,a_r$ not all zeros, an integer $d$, and $r$ distinct vectors $u_1,\ldots,u_r$ in $U$ such that:

(i) the $2r$-element set $U = \{u_1, 1 - u_1, \ldots, u_r, 1 - u_r\}$;
(ii) $\sum_{i=1}^r a_i u_i = d \cdot 1 = (d,\ldots,d)$;
(ii) $\sum_{i=1}^r |a_i| \leq C(g)$.

Proof. The assertions follow readily from the construction of Subsection 4.3 combined with Remark 4.2.

4.5. Let $X$ be a $g$-dimensional abelian variety over a finite field $k$ of characteristic $p$. Suppose that $k$ is not small with respect to $X$ and there exists a nontrivial admissible function $R_X \to \mathbb{Z}$. The aim of this subsection is to assign to $X$ a certain $g$-frame that we call the skeleton of $X$.

First, let us put $r := r_X$ and $M := M_X := \#(S(p))$ where $S(p)$ is the set of maximal ideals in $\mathcal{O}_{L_X}$ that lie above $p$ (see Remark 2.4). It follows from Lemma 2.5 that $M$ divides $2^g \cdot g!$. By Lemma 3.7 the existence of a nontrivial admissible function implies that $R_X \neq R_X^*$ and $r = r_X$ is a positive integer. In addition (see (11)),

$$r \leq g, \quad 2r = \#(R_X \setminus R_X^*).$$

Let us choose an order on the $M$-element set $S(p)$. This allows us to identify $S(p)$ with $\{1,\ldots,M\}$ and $\mathbb{Q}^{S(p)}$ with $\mathbb{Q}^M$. Let us put

$$U = U_X := w_X(R_X \setminus R_X^*) \subset \mathbb{Q}^{S(p)} = \mathbb{Q}^M$$

(where homomorphism $w_X$ is defined in Remark 2.4). It follows from Remark 2.4(d) that the map

$$R_X \setminus R_X^* \to U_X, \quad \alpha \mapsto w_X(\alpha)$$

(27)

is injective; in particular,

$$2r = 2r_X = \#(R_X \setminus R_X^*) = \#(U_X).$$

Since $\ker(w_X)$ consists of roots of unity (see Remark 2.4(d)), the rank of $\Delta(U_X)$ coincides with the rank of multiplicative $\Gamma_1(X,k)$ generated by $R_X \setminus R_X^*$. The existence of a nontrivial admissible function implies (thanks to Theorem 3.10) that

$$\text{rk}(\Delta(U_X)) = \text{rk}(\Gamma_1(X,k)) \leq r_X.$$  \hspace{1cm} (28)

I claim that $(M_X,r_X,U_X)$ is a $g$-frame. Indeed, it follows from Remarks 2.4 that

$$w_X(q) = 1; w_X(\alpha) \neq 0, \quad w_X(q/\alpha) = 1 - w_X(\alpha) \ \forall \alpha \in R_X \setminus R_X^*.$$  \hspace{1cm} (29)

This implies that $(M_X,r_X,U_X)$ enjoys the properties (i)-(iii). As for (iv), its validity follows from (28).
Proof of Theorem 1.4. Let $g$ be a positive integer. In light of Lemma 3.8, we may and will assume that $k$ is not small w.r.t. $X$. In light of Lemma 3.9, it suffices to prove the following assertion.

Claim. There exists a positive integer $E(g)$ that depends only on $g$ and enjoys the following property. Suppose that $X$ is a $g$-dimensional abelian variety over a finite field $k$ such that $k$ is not small w.r.t. $X$ and there exists a nontrivial admissible function $R_X \to \mathbb{Z}$.

Then there exists a nontrivial admissible function $R_X \to \mathbb{Z}$ of weight $\leq E(g)$.

Proof of Claim. Let $X$ be an $g$-dimensional abelian variety over a finite field $k$ such that $k$ is not small w.r.t. $X$ and there exists a nontrivial admissible function $R_X \to \mathbb{Z}$. Let us consider the corresponding $g$-frame $(M_X, r_X, U_X)$.

It follows from the injectiveness of the map (27) combined with Lemma 4.4 that there exist $r_X$ distinct elements $\alpha_1, \ldots, \alpha_{r_X} \in R_X \setminus R_\iota_X$, $r_X$ integers $a_1, \ldots, a_{r_X}$, and an integer $d$ that enjoys the following properties.

1. $R_X \setminus R_\iota_X = \{\alpha_1, q/\alpha_1, \ldots, \alpha_{r_X}, q/\alpha_{r_X}\}$.
2. Not all $a_1, \ldots, a_{r_X}$ are zero.
3. $\sum_{i=1}^{r_X} a_i w_X(\alpha_i) = d \cdot 1 = (d, \ldots, d)$.
4. $\sum_{i=1}^{r_X} |a_i| \leq C(g)$. (Here $C(g)$ is as in Lemma 4.4)

It follows from Remark 2.4(d) that there exists a root of unity $\gamma \in \Gamma(X, k)$ such that

$$\prod_{i=1}^{r_X} \alpha_i^{a_i} = q^d \gamma.\!
$$

According to Lemma 2.2, there exists a positive integer $m \leq D(g)$ such that $\gamma^m = 1$. (See Lemma 2.2 for the explicit formula of $D(g)$.) This implies that

$$\prod_{i=1}^{r_X} \alpha_i^{ma_i} = q^{md}.\!
$$

This implies that the function $e: R_X \to \mathbb{Z}, e(\alpha_i) = m \cdot a_i \forall \alpha_i, e(\alpha) = 0$ for all other $\alpha$ is admissible. On the other hand, it follows from properties (1) and (2) that $e$ is nontrivial. In order to finish the proof of Claim, one has only to notice that

$$\text{wt}(e) = \sum_{i=1}^{r_X} |a_i| = m \sum_{i=1}^{r_X} |a_i| \leq D(g) \cdot C(g) =: E(g).$$

This ends the proof of Theorem 1.4.
5. Applications of Gordan’s Lemma

In order to prove Theorem 1.10, we need the following variant of a classical result of P. Gordan.

Lemma 5.1. Let \( m \) and \( s \) be positive integers and \( v_1, \ldots, v_s \) be elements of \( \mathbb{Q}^m \). Let us consider the additive semigroup

\[
W = \{ u \in \mathbb{Z}^m_+ \mid u \cdot v_j = 0 \ \forall j = 1, \ldots, s \} \subset \mathbb{Z}^m_+.
\]

Then \( W \) is a finitely generated semigroup of \( \mathbb{Z}^m_+ \).

Proof. Replacing all \( v_j \) by \( Nv_j \), where \( N \) is a sufficiently divisible positive integer, we may and will assume that \( v_j \in \mathbb{Z}^m \) for all \( j = 1, \ldots, s \).

Let us consider the rational polyhedral cone \( \sigma \subset \mathbb{R}^m \) that is generated by the standard basis of \( \mathbb{R}^m \) and all the vectors \( \{v_1, \ldots, v_s\} \). Then the dual cone is

\[
\sigma^\vee = \{ u \in \mathbb{R}^m_+ \mid u \cdot v_j \geq 0 \ \forall j = 1, \ldots, s \}.
\]

By Gordan’s Lemma [3, Ch. 1, Prop. 1.2.17], \( \sigma^\vee \cap \mathbb{Z}^m_+ \) is a finitely generated additive semigroup. Let \( G \) be a finite subset of \( \sigma^\vee \cap \mathbb{Z}^m_+ \) that contains 0 and generates \( \sigma^\vee \cap \mathbb{Z}^m_+ \). Then the intersection \( G \cap W \) is a finite subset of \( W \) that contains 0. I claim that \( G \cap W \) generates \( W \) as a semigroup. Indeed, if \( w \in W \) then \( w \in \sigma^\vee \cap \mathbb{Z}^m_+ \) and therefore there exists a positive integer \( r \) and (not necessarily distinct) \( r \) elements \( g_1, \ldots, g_r \in G \) such that \( w = \sum_{i=1}^r g_i \).

We have for all \( j = 1, \ldots, s \)

\[
0 = w \cdot v_j = \sum_{i=1}^r g_i \cdot v_j, \ g_i \cdot v_j \geq 0 \ \forall i = 1, \ldots, s.
\]

This implies that all \( g_i \cdot v_j = 0 \) and therefore all \( g_i \in W \), i.e., \( g_i \in G \cap W \). It follows that \( G \cap W \) generates \( W \) as a semigroup. \( \square \)

We also need the following elementary observation.

Lemma 5.2. Suppose that \( X \) is an abelian variety of positive dimension \( g \) over a finite field \( k \) with \( q \) elements. Suppose that \( k \) is sufficiently large w.r.t. \( X \). Then a nonnegative integer-valued function \( e: R_X \rightarrow \mathbb{Z}_+ \) of even weight is admissible if and only if

\[
w_X \left( \prod_{\alpha \in R_X} \left( \frac{\alpha^2}{q} \right)^{e(\alpha)} \right) = 0 \in \mathbb{Q}^{S(p)}.
\]

(30)

Remark 5.3. Let \( e: R_X \rightarrow \mathbb{Z}_+ \) be an admissible nonnegative integer-valued function. Then its weight is twice its degree and therefore is even.

Proof of Lemma 5.2 Since \( e \) is nonnegative, its weight coincides with

\[
\sum_{\alpha \in R_X} e(\alpha) =: n.
\]

Since this weight is even, there is a nonnegative integer \( d \) such that \( n = 2d \).
Now notice that in light of Remark 2.4(d), (30) holds if and only if \( \prod_{\alpha \in R_X} (\alpha^2/q)^{e(\alpha)} \) is a root of unity. This means that \( \prod_{\alpha \in R_X} (\alpha^2/q)^{e(\alpha)} = 1 \), because \( k \) is sufficiently large w.r.t. \( X \). Hence, (30) means that
\[
\left( \prod_{\alpha \in R_X} \alpha^{e(\alpha)} \right)^2 = q^n \quad \text{with} \quad n = \sum_{\alpha \in R_X} e(\alpha) = 2d.
\]
This means that
\[
\prod_{\alpha \in R_X} \alpha^{e(\alpha)} = \pm q^d. \tag{31}
\]
Since torsion-free \( \Gamma(X,k) \) does not contain \(-1\), (31) is equivalent to
\[
\prod_{\alpha \in R_X} \alpha^{e(\alpha)} = q^d,
\]
i.e., \( e \) is admissible. \( \square \)

**Proof of Theorem 1.11** Let \( X \) be an abelian variety of positive dimension \( g \) over a finite field \( k \) of characteristic \( p \). Suppose that \( k \) is sufficiently large w.r.t. \( X \). Let us put \( s := \#(S(p)) \). By Lemma 2.5, \( s \) divides \( 2^g \cdot g! \). Let us choose an order in \( S(p) \). This allows us to identify \( S(p) \) with \( \{1, \ldots, s\} \) and \( Q_S(p) \) with \( Q^s \). Let us choose an order on \( R_X \): it allows us to list elements of \( R_X \) as \( \{\alpha_1, \ldots, \alpha_m\} \) with \( m = \#(R_X) \). We have \( m \leq 2g \). Let us consider an additive group homomorphism
\[
\tilde{w}_X : \mathbb{Z}^m \to \mathbb{Q}^{S(p)} = \mathbb{Q}^s, \quad u = (a_1, \ldots, a_m) \mapsto w_X \left( \prod_{i=1}^m (\alpha_i^2/q)^{a_i} \right) = 2 \sum_{i=1}^m a_i w_X(\alpha_i) - \left( \sum_{i=1}^m a_i \right) \cdot 1.
\]
Clearly, there is a unique collection of \( s \) vectors \( v_1, \ldots, v_s \in \mathbb{Q}^m \) such that
\[
\tilde{w}_X(u) = (u \cdot v_1, \ldots, u \cdot v_s) \quad \forall u \in \mathbb{Z}^m.
\]
It is also clear that all the coordinates of all \( v_j \)'s lie in the same finite set
\[
2 \cdot S(g) - 1 := \{2c - 1 \mid c \in S(g)\} \subset \mathbb{Q}
\]
that depends only on \( g \). This implies that all the \( v_j \)'s lie in the same finite subset
\[
(2 \cdot S(g) - 1)^m \subset \mathbb{Q}^m
\]
of \( \mathbb{Q}^m \) that depends only on \( g \) and \( m \). Combining this assertion with Lemma 5.11, we obtain that for each positive integers \( m \leq 2g \) and \( s \) dividing \( 2^g \cdot g! \) there is a finite subset \( F_0(g,m,s) \in \mathbb{Z}^m_+ \) that depends only on \( g, m, s \) and enjoys the following property.

If \( \#(R_X) = m \) and \( \#(S(p)) = s \) then the additive semigroup \( \ker(\tilde{w}_X) \cap \mathbb{Z}^m_+ \) of \( \mathbb{Z}^m_+ \) is generated by a certain subset of \( F_0(g,m,s) \).
Now let us define the weight \( \text{wt}(u) \) of any \( u = (a_1, \ldots, a_m) \in \mathbb{Z}_+^m \) as \( \sum_{i=1}^m a_i \). It follows from Remark 5.3 combined with Lemma 5.2 that an integer-valued nonnegative function 
\[ b_u : R_X \to \mathbb{Z}_+^m, \quad \alpha_i \mapsto a_i \]
is admissible if and only if \( u \in \ker(\tilde{w}_X) \cap \mathbb{Z}_+^m \) and \( \text{wt}(u) \) is even. (It is also clear that each admissible nonnegative function \( e : R_X \to \mathbb{Z}_+^m \), \( \alpha_i \mapsto a_i \) is admissible if and only if \( u \in \ker(\tilde{w}_X) \cap \mathbb{Z}_+^m \) and \( \text{wt}(u) \) is even. Thus, each \( b_u \) for exactly one vector \( u \in \mathbb{Z}_+^m \).

Then such \( u \) may be presented as a sum of (not necessarily distinct) elements of \( F_0(g,m,s) \). It may happen that some elements of \( F_0(g,m,s) \) in this sum have odd weight. Since the weight of \( u \) is even, the number of such summands is even. By grouping them in pairs, we obtain that \( u \) is a finite sum of some even weight elements from \( F_0(g,m,s) \) and even weight elements from \( F_0(g,m,s) + F_0(g,m,s) \subset \mathbb{Z}_+^m \).

Now let \( F(g,m,s) \subset \mathbb{Z}_+^m \) be the (finite) set of all even weight vectors from \( F_0(g,m,s) \) and from \( F_0(g,m,s) + F_0(g,m,s) \). Clearly, each \( u \in F(g,m,s) \) gives rise to nonnegative admissible \( b_u : R_X \to \mathbb{Z}_+^m \) and each nonnegative admissible \( e : R_X \to \mathbb{Z}_+^m \) may be presented as a linear combination of such \( b_u \)’s with nonnegative integer coefficients. Now one only has to choose as \( H(g) \) the largest of the weights of \( b_u \) among all \( u \) (with even weight) in the union of all \( F(g,m,s) \) where \( 1 \leq m \leq 2g \) and \( s \mid 2^g \cdot g! \). \( \square \)

Theorem 1.10 implies readily the following assertion.

**Corollary 5.4.** Let \( g \) be a positive integer and \( H(g) \) be as in Theorem 1.10. Let \( X \) an abelian variety of positive dimension \( g \) over a finite field \( k \). Let \( e : R_X \to \mathbb{Z}_+^m \) be a nonnegative admissible function. If \( \text{wt}(e) > H(g) \) then \( e \) may be presented as a sum \( e = f_1 + f_2 \) of two nonnegative admissible functions

\[ f_1 : R_X \to \mathbb{Z}_+^m, \quad f_2 : R_X \to \mathbb{Z}_+^m \]

such that \( 2 \leq \text{wt}(f_2) \leq H(g) \).

---

6. **Linear Algebra**

Throughout this section, \( V \) is a nonzero vector space of finite dimension \( n \) over a field \( K \) of characteristic 0, and \( E \) is an overfield of \( K \). We write \( V_E \) for the \( E \)-vector space \( V \otimes_K E \) of the \( E \)-dimension \( n \). Let us put

\[ V^* = \text{Hom}_K(V,K), \quad V_E^* = \text{Hom}_E(V_E,E). \]

Let \( A : V \to V \) be a \( K \)-linear operator and

\[ A^* : V^* \to V^*, \quad \phi \mapsto \phi \circ A \quad \forall \phi \in V^*. \]

As usual, let us define

\[ A_E \in \text{End}(V_E), \quad A_E(v \otimes e) = Av \otimes e \quad \forall v \in V, e \in E. \]

Clearly,

\[ (A_E)^* = (A^*)_E : V_E^* \to V_E^*. \]
Remark 6.1. Let \( a \in K \subset E \) and \( V(a) \) (resp. \( V_E(a) \)) be the eigenspace of \( A \) (resp. of \( A_E \)) attached to eigenvalue \( a \). It is well known that the natural \( E \)-linear map

\[
V(a) \otimes_K E \to V_E(a)
\]

is an isomorphism of \( E \)-vector spaces; in particular,

\[
\dim_K(V(a)) = \dim_E(V_E(a)) \quad \forall a \in K \subset E.
\]

There are well known natural isomorphisms \cite{2} Ch. III, Sect. 7, Prop. 7 of graded \( K \)-algebras

\[
\wedge(V^*) = \bigoplus_{j=0}^n \wedge^j_K (V^*) = \bigoplus_{j=0}^n \text{Hom}_K(\wedge^j_K(V), K)
\]

and of graded \( E \)-algebras \cite{2} Ch. III, Sect. 7, Prop. 8

\[
\wedge(V_E^*) = \bigoplus_{j=0}^n \wedge^j_E (V_E^*) = \bigoplus_{j=0}^n \text{Hom}_E(\wedge^j_E(V_E), E) = \bigoplus_{j=0}^n \text{Hom}_K(\wedge^j_K(V), K) \otimes_K E,
\]

which give rise to the natural isomorphisms of \( E \)-vector spaces

\[
\wedge^j_K(V^*_E) \cong \wedge^j_E(V^*_E). \quad (32)
\]

6.2. Let \( i \) and \( j \) be nonnegative integers. The multiplication in \( \wedge(V^*) \) (resp. in \( \wedge(V^*_E) \)) gives rise to the surjective \( K \)-linear map

\[
\Lambda_{i,j,K} \colon \wedge^i_K (V^*) \otimes_K \wedge^j_K (V^*) \to \wedge^{i+j}_K (V^*), \quad \psi_i \otimes \psi_j \mapsto \psi_i \wedge \psi_j \quad (33)
\]

and to the surjective \( E \)-linear map

\[
\Lambda_{i,j,E} \colon \wedge^i_E (V_E^*) \otimes_E \wedge^j_E (V_E^*) \to \wedge^{i+j}_E (V_E^*), \quad \psi_i \otimes \psi_j \mapsto \psi_i \wedge \psi_j. \quad (34)
\]

Let \( U \) be a \( K \)-vector subspace in \( \wedge^i_K (V^*) \) and \( W \) be a \( K \)-vector subspace in \( \wedge^j_K (V^*) \). Then obviously the images \( \Lambda_{i,j,K}(U \otimes_K W) \subset \wedge^{i+j}_K (V^*) \) and \( \Lambda_{i,j,E}(U_E \otimes_E W_E \subset \wedge^{i+j}_E (V_E^*) \) are related by

\[
\Lambda_{i,j,E}(U_E \otimes_E W_E) = \Lambda_{i,j,K}(U \otimes_K W)_E. \quad (35)
\]

Here we identify \( U_E \) (resp. \( W_E \)) with its isomorphic image in \( \wedge^i_K (V^*)_E = \wedge^i_E (V_E^*) \) (resp. in \( \wedge^j_K (V^*)_E = \wedge^j_E (V_E^*) \)).

The equality \cite{35} implies readily its own generalization. Namely, let \( n \) be a positive integer and suppose that for each positive integer \( r \leq n \) we are given \( K \)-vector subspaces

\[
U_r \subset \wedge^i_K (V^*), \ W_r \subset \wedge^j_K (V^*). \]

Then

\[
\sum_{r=1}^n \Lambda_{i,j,E}(U_{r,E} \otimes_E W_{r,E}) = \left( \sum_{r=1}^n \Lambda_{i,j,K}(U_r \otimes_K W_r) \right)_E. \quad (36)
\]

Here

\[
U_{r,E} = U_r \otimes_K E, \ W_{r,E} = W_r \otimes_K E.
\]
6.3. The operators $A^*$ and $A^*_E$ give rise to the graded $K$-algebra and graded $E$-algebra endomorphisms

$$\wedge(A^*): \wedge (V^*) \to \wedge (V^*)$$

[2] Ch. III, Sect. 7, Prop. 2, whose homogeneous components are $K$-linear and $E$-linear operators

$$\wedge^j(A^*): \wedge^j_K(V^*) \to \wedge^j_E(V^*)$$

respectively, such that

$$\wedge^j(A^*) = \wedge^j(A^*)_E. \quad (37)$$

Since $\wedge(A)$ and $\wedge(A^*_E)$ respect the multiplication in $\wedge(V^*)$ and $\wedge(V^*_E)$ respectively,

$$\wedge^i(A^*)(\psi_i) \wedge \wedge^j(A^*)(\psi_j) = \wedge^{i+j}(A^*)(\psi_i \wedge \psi_j) \in \wedge^{i+j}(V^*) \forall \psi_i \in \wedge^i(V^*), \psi_j \in \wedge^j(V^*); \quad (38)$$

$$\wedge^i(A^*_E)(\psi_{i,E}) \wedge \wedge^j(A^*_E)(\psi_{j,E}) = \wedge^{i+j}(A^*_E)(\psi_{i,E} \wedge \psi_{j,E}) \in \wedge^{i+j}(V^*_E)$$

$$\forall \psi_{i,E} \in \wedge^i(V^*_E), \psi_{j,E} \in \wedge^j(V^*_E).$$

The following assertion is an immediate corollary of (38) and (37).

**Lemma 6.4.** Let $j_1, j_2$ be positive integers such that $j_1 + j_2 \leq \dim(V)$. Let $\lambda_1, \lambda_2$ be elements of $K$. Let $\wedge_{K}^{j_1}(V^*)(\lambda_1) \subset \wedge_{K}^{j_2}(V^*)$ be the eigenspace of $\wedge_{K}^j(A^*)$ attached to $\lambda_r$ ($r = 1, 2$). Then the image of the $K$-linear map

$$\wedge_{K}^{j_1}(V^*)(\lambda_1) \otimes_K \wedge_{K}^{j_2}(V^*)(\lambda_2) \to \wedge_{K}^{j_1+j_2}(V^*), \psi_{j_1} \otimes \psi_{j_2} \mapsto \psi_{j_1} \wedge \psi_{j_2}$$

lies in the eigenspace $\wedge_{K}^{j_1+j_2}(V^*)(\lambda_1 \lambda_2)$ of $\wedge_{K}^{j_1+j_2}(A^*)$ attached to $\lambda_1 \lambda_2$.

**Remark 6.5.** The $K$-linear map in Lemma 6.4 is the restriction of $\Lambda_{j_1,j_2,K}$ defined in Subsection 6.2.

**Remark 6.6.** Applying Remark 6.1 to $\wedge^j(A^*)$: $\wedge^j_K(V^*) \to \wedge^j_K(V^*)$ (instead of $A: V \to V$), we obtain that if $\lambda \in K \subset E$ and $\wedge^j_K(V^*)(\lambda)$ (resp. $\wedge^j(V^*_E)(\lambda)$) is the attached to $\lambda$ eigenspace of $\wedge^j(V^*)$ (resp. of $\wedge^j(V^*_E)$) then the natural $E$-linear map

$$\wedge^j_K(V^*)(\lambda) \otimes K E \to \wedge^j(V^*_E)(\lambda)$$

induced by (32) is an isomorphism. Combining this assertion with Lemma 6.4 applied twice (over $K$ and over $E$), we get immediately the following assertion.

**Lemma 6.7.** Let $j_1, j_2$ be positive integers such that $j_1 + j_2 \leq \dim(V)$. Let $\lambda_1, \lambda_2 \in K \subset E$. We keep the notation and assumptions of Lemma 6.3.

The $E$-linear map

$$\wedge^j(V^*_E)(\lambda_1) \otimes_E \wedge^j(V^*_E)(\lambda_2) \to \wedge^j_{E}(V^*)(\lambda_1 \lambda_2), \psi_1 \otimes \psi_2 \mapsto \psi_1 \wedge \psi_2$$

is not surjective if and only if the $K$-linear map

$$\wedge^j_K(V^*)(\lambda_1) \otimes_K \wedge^j_K(V^*)(\lambda_2) \to \wedge^j_K(V^*)(\lambda_1 \lambda_2), \psi_1 \otimes \psi_2 \mapsto \psi_1 \wedge \psi_2$$
is not surjective. Here $\wedge^j_E(V^*_E)(\lambda) \subset \wedge^j_E(V^*_E)$ is the eigensubspace of $\wedge^j(A^*_E)$ attached to $\lambda$.

### 6.8. Main construction.

We keep the notation of Remark 6.6. Suppose that $A_E : V_E \to V_E$ is diagonalizable, $\text{spec}(A) \subset E$ is the set of its eigenvalues, and $\text{mult}_A : \text{spec}(A) \to \mathbb{Z}_+$ is the integer-valued function that assigns to each eigenvalue of $A_E$ its multiplicity.

Let $\lambda \in K$ and $j \leq \dim(V)$ be a positive integer. Let us consider an integer-valued function $e : \text{spec}(A) \to \mathbb{Z}_+$ that enjoys the following properties.

(i) $e(\alpha) \leq \text{mult}_A(\alpha) \quad \forall \alpha \in \text{spec}(A)$;
(ii) $\sum_{\alpha \in \text{spec}(A)} e(\alpha) = j$;
(iii) $\prod_{\alpha \in \text{spec}(A)} \alpha^{e(\alpha)} = \lambda$.

Let us choose an eigenbasis $B$ of $E$-vector space $V_E$ w.r.t. $A_E$ and let $\pi : B \mapsto \text{spec}(A)$ be the surjective map that assigns to each eigenvector $x \in B$ the corresponding eigenvalue of $A_E$. Clearly, for every eigenvalue $\alpha \in \text{spec}(A)$ the preimage $\pi^{-1}(\alpha)$ consists of $\text{mult}_A(\alpha)$ elements of $B$.

Let $B^* = \{ x^* \mid x \in B \}$ be the basis of $V^*_E$ that is dual to $B$. Let us choose an order on $B$ and define for each $j$-element subset $C \subset B$ an element $y_C := \wedge^j_{x \in C} x^* \in \wedge^j(V^*_E)$.

Clearly, all $y_C$'s constitute an eigenbasis of $\wedge^j(V^*_E)$ w.r.t. $\wedge^j(A^*)$. Actually, $\wedge^j(A^*)(y_C) = \left( \prod_{x \in C} \pi(x) \right) y_C$.

Let us assign to $C$ the integer-valued function $e = e_C : \text{spec}(A) \to \mathbb{Z}_+$, $\alpha \mapsto \#(\{ x \in C \mid \pi(x) = \alpha \})$. (39)

Clearly, $y_C$ is an eigenvector of $\wedge^j(A^*)$ with eigenvalue $\lambda$ if and only if $e_C$ enjoys the properties (i)-(iii). This implies that the set of $y_C$'s such that $e_C$ satisfies (i)-(iii) is a $E$-basis of the eigenspace $\wedge^j(V^*_E)(\lambda)$.

Conversely, suppose that $e : \text{spec}(A) \to \mathbb{Z}_+$ is an integer-valued function that enjoys the properties (i)-(iii). I claim that there exists a $j$-element subset $C \subset B$ such that $e = e_C$. Indeed, let us choose a $e(\alpha)$-element subset $C_\alpha \subset \pi^{-1}(\alpha) \subset B$ for all $\alpha \in \text{spec}(A)$ with $e(\alpha) > 0$. The property (i) guarantees that such a choice is possible (but not necessarily unique). Now define $C$ as the (disjoint) union of all these $C_\alpha$'s. Property (ii) implies that $B$ is a $j$-element subset of $B$. It follows from (iii) that $y_C \in \wedge^j(V^*_E)(\lambda)$.

The following assertion will be used in the proof of Theorem 1.8 (with $K = \mathbb{Q}_\ell, V = V_\ell(X^n), A = \text{Fr}_{X^n}$).
**Proposition 6.9.** We keep the notation and assumptions of Subsection 6.8. Assume additionally that $A: V \to V$ is diagonalizable. In particular, $A_E: V_E \to V_E$ is diagonalizable. Suppose that $\lambda_1$ and $\lambda_2$ are nonzero elements of $K$ and $j > 2$, i.e., $j_1 \geq 1$. Then the following conditions are equivalent.

(a) The $K$-linear map
\[
\wedge^j_K(V^*)(\lambda_1) \otimes_K \wedge^2_K(V^*)(\lambda_2) \to \wedge^j_K(V^*)(\lambda_1\lambda_2), \psi \otimes \phi \mapsto \psi \wedge \phi \quad (40)
\]

is not surjective.

(b) There exists a function $e: \text{spec}(A) \to \mathbb{Z}_+$ that enjoys the following properties.

(i) $e(\alpha) \leq \text{mult}_A(\alpha) \ \forall \alpha \in \text{spec}(A)$;

(ii) $\sum_{\alpha \in \text{spec}(A)} e(\alpha) = j$;

(iii) $\prod_{\alpha \in \text{spec}(A)} \phi^e(\alpha) = \lambda_1\lambda_2$.

(iv) If $\alpha \in \text{spec}(A)$ and $e(\alpha) \neq 0$ then $e(\alpha) \geq 1$ and one of the following conditions holds.

1. $\lambda_2/\alpha \notin \text{spec}(A)$;
2. $\lambda_2/\alpha \in \text{spec}(A)$ but $e(\lambda_2/\alpha) = 0$.
3. $\alpha = \lambda_2/\alpha$ (i.e., $\alpha^2 = \lambda_2$) and $e(\alpha) = 1$.

**Remark 6.10.** In light of Lemma 6.7, it suffices to check that condition (b) is equivalent (in the obvious notation) to the non-surjectiveness of the $E$-linear map
\[
\wedge^j_E(V^+_E)(\lambda_1) \otimes_Z \wedge^2_E(V^+_E)(\lambda_2) \to \wedge^j_E(V^+_E)(\lambda_1\lambda_2), \psi \otimes \phi \mapsto \psi \wedge \phi. \quad (41)
\]

**Proof of Proposition 6.9.** We start with the following lemma that describes the image of map (41).

**Lemma 6.12.** The image of the map (41) is generated by all $y_C$’s where $C$ is any $j$-element subset of $B$ that enjoys the following properties.

The set $C$ is a disjoint union of a $(j-2)$-element subset $S$ and a 2-element subset $T$ such that the corresponding functions
\[
eq_S: \text{spec}(A) \to \mathbb{Z}_+, \quad e_T: \text{spec}(A) \to \mathbb{Z}_+
\]
defined as in (39) enjoy the following properties.
\[
\prod_{\alpha \in \text{spec}(A)} \alpha^{e_S(\alpha)} = \lambda_1, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{e_T(\alpha)} = \lambda_2. \quad (42)
\]

**Proof of Lemma 6.12.** It follows from arguments of Subsection 6.8 that all the $y_S$’s (resp. all the $y_T$’s) where $S$ is any $(j-2)$-element subset of $B$ (resp. where $T$ is any 2-element subset of $B$) that satisfies (42) constitute a basis of $\wedge^j_E(V^+_E)(\lambda_1)$ (resp. a basis of $\wedge^2_E(V^+_E)(\lambda_2)$). This implies that the image of map (41) is generated by all $y_S \wedge y_T$. If $S$ meets $T$ then it follows from the very definition of $y_S$ and $y_T$ and basic properties of wedge product.
products that \( y_S \wedge y_T = 0 \). On the other hand, if \( S \) does not meet \( T \) then \( C := S \cup T = S \sqcup T \) is a \( j \)-element subset of \( B \) and \( y_S \wedge y_T = \pm y_C \). This ends the proof. \( \square \)

Now let us start to prove Proposition 6.9. Suppose that (b) holds. In light of Remark 6.11 it suffices to check that map (41) is not surjective. To this end, choose an eigenbasis \( B \) of \( V_E \) w.r.t. \( A_E \), and choose an order on \( B \). Using arguments of Subsection 6.8 choose a \( j \)-element subset \( \tilde{C} \subset B \) such that the function

\[
\tilde{C} : \text{spec}(A) \to \mathbb{Z}_+ 
\]

coincides with \( e \) and therefore enjoys properties (i)-(iv). Then \( y_{\tilde{C}} \in \land^j(V^*)(\lambda_1\lambda_2) \).

I claim that \( y_{\tilde{C}} \) does not lie in the image of map (11). Indeed, \( \land^{j-2}_E(V^*_E)(\lambda_1) \) is generated as the \( E \)-vector space by elements of the form \( y_S \), where \( S \) are \((j-2)\)-element subsets of \( B \) such that \( \prod_{b \in S} \pi(b) = \lambda_1 \). On the other hand, \( \land^2_E(V^*_E)(\lambda_2) \) is generated as the \( E \)-vector space by elements of the form \( y_T \), where \( T \) are 2-element subsets of \( B \) such that \( \prod_{b \in T} \pi(b) = \lambda_2 \). This implies that the image of map (11) is generated as the \( E \)-vector space by all \( y_B \wedge y_T \). If \( S \) meets \( T \) then (as we have already seen) \( y_S \wedge y_T = 0 \). If \( S \) does not meet \( T \) then \( S \cup T = S \sqcup T \) is a \( j \)-element subset of \( B \) and \( y_S \wedge y_T = \pm y_{S \cup T} \).

**Lemma 6.13.** The \( j \)-element \( \tilde{C} \) does not coincide with any of \( S \cup T \).

**Proof of Lemma 6.13** Suppose \( \tilde{C} = S \cup T \). This implies that \( \tilde{C} \) contains a subset \( T \) that consists of two distinct elements say \( x_1, x_2 \subset B \) with \( \pi(x_1)\pi(x_2) = \lambda_2 \). So, \( \tilde{C} \) contains these \( x_1, x_2 \). It follows from the definition of \( e_{\tilde{C}} \) (39) that if we put

\[
\alpha_1 = \pi(x_1), \quad \alpha_2 = \pi(x_2)
\]

then \( \alpha_1, \alpha_2 \in \text{spec}(A) \) and

\[
\alpha_1\alpha_2 = \lambda_2, \quad e(\alpha_1) \geq 1, e(\alpha_2) \geq 1, \quad e(\alpha_1) + e(\alpha_2) \geq 2.
\]

If \( \alpha_1 \neq \alpha_2 = \lambda_2/\alpha_1 \) then \( e(\alpha_1) \geq 1, e(\lambda_2/\alpha_1) \geq 1 \), which violates property (iv). If \( \alpha_1 = \alpha_2 \) then \( \alpha_1 = \lambda_2/\alpha_1 \). It follows that \( \alpha_1 = \pi(x_1) = \pi(x_2) \) and therefore \( e(\alpha_1) \geq 2 \), which also contradicts property (iv). This ends the proof. \( \square \)

**End of Proof of Proposition 6.9** Taking into account that the set of all \( y_C \)'s where \( C \) runs through all \( j \)-element subsets of \( B \) is linearly independent, we conclude that \( y_{\tilde{C}} \) cannot be presented as a \( E \)-linear combination of \( y_{S \cup T} \)'s and therefore does not lie in the image of map (11). Hence, (a) holds. We proved that (b) implies (a).

Conversely, suppose that (a) holds, i.e., the map (11) is not surjective. Then there is a basis vector \( y_C \in \land^j(V^*)(\lambda_1\lambda_2) \) that does not belong to the
image where $C$ is a certain $j$-element subset of $B$ such that
\[ \prod_{x \in C} \pi(x) = \lambda_1 \lambda_2. \]

I claim that the function $e := e_C$ enjoys the properties (b). By definition, $e_C$ enjoys the properties (bi), (bii), (biii). In particular,
\[ \prod_{\alpha \in C} \alpha^{e_C(\alpha)} = \prod_{x \in C} \pi(x) = \lambda_1 \lambda_2. \]

Suppose that $e_C$ does not enjoy the property (biv). This means that there exists $\alpha \in \text{spec}(A)$ such that $\alpha \not\in \pi(C)$ such that
\[ \lambda_2 / \alpha \in \text{spec}(A), \quad e_C \left( \frac{\lambda_2}{\alpha} \right) \neq 0 \]
(i.e.,
\[ \lambda_2 / \alpha \in \pi(C) \subset \text{spec}(A). \]
In addition, if $\lambda_2 / \alpha = \alpha$ then $e_C(\alpha) \geq 2$ (i.e., $\pi^{-1}(\alpha)$ contains at least two elements of $C$).

This implies that there are distinct elements $x_1$ and $x_2$ of $C$ such that
\[ \pi(x_1) = \alpha, \quad \pi(x_2) = \lambda_2 / \alpha. \]

Then $C$ coincides with the disjoint union of the 2-element subset $T = \{x_1, x_2\}$ and the $(j-2)$-element subset $S = C \setminus T$. By definition of $T$,
\[ \prod_{\alpha \in T} \alpha^{e_T(\alpha)} = \prod_{x \in T} \pi(x) = \pi(x_1) \cdot \pi(x_2) = \alpha \cdot \frac{\lambda_2}{\alpha} = \lambda_2. \]
Hence,
\[ \prod_{\alpha \in T} \alpha^{e_T(\alpha)} = \lambda_2. \]

Since $C$ is the disjoint union of $S$ and $T$, we get
\[ \prod_{\alpha \in S} \alpha^{e_S(\alpha)} = \frac{\prod_{\alpha \in C} \alpha^{e_C(\alpha)}}{\prod_{\alpha \in T} \alpha^{e_T(\alpha)}} = \frac{\lambda_1 \lambda_2}{\lambda_2} = \lambda_1. \]
Hence,
\[ \prod_{\alpha \in S} \alpha^{e_S(\alpha)} = \lambda_1. \]

It follows from Lemma 6.12 that $y_C$ lies in the image of the map (II), which is not true. The obtained contradiction implies that $e_C$ enjoys the property (biv).

This ends the proof of Proposition 6.9.

The following assertion will be used in the proof of Theorem 1.11 (with $K = \mathbb{Q}_\ell$, $V = V_\ell(X^n)$, $A = \text{Fr}_{X^n}$).
Proposition 6.14. We keep the notation and assumption of Subsection 6.8. Assume additionally that \( \text{char}(K) = 0 \), \( \dim_K(V) \) is even, and \( A : V \to V \) is invertible. Let \( q \) be a nonzero element of \( K \) that is not a root of unity. Let \( h \) and \( m \) be positive integers that enjoy the following properties.

(i) \( h < m \leq \dim_K(V)/2 \).

(ii) If \( e : \text{spec}(A) \to \mathbb{Z}_+ \) is any nonnegative integer-valued function such that

\[
\sum_{\alpha \in \text{spec}(A)} e(\alpha) = 2m, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{e(\alpha)} = q^m
\]

then there exist positive integers \( j_1, j_2 \) and nonnegative integer-valued functions

\( f_1 : \text{spec}(A) \to \mathbb{Z}_+ \), \( f_2 : \text{spec}(A) \to \mathbb{Z}_+ \)

such that

\[
m = j_1 + j_2, \quad j_2 \leq h;
\]

\[
e(\alpha) = f_1(\alpha) + f_2(\alpha) \quad \forall \alpha \in \text{spec}(A);
\]

\[
\sum_{\alpha \in \text{spec}(A)} f_1(\alpha) = 2j_1, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{f_1(\alpha)} = q^{j_1},
\]

\[
\sum_{\alpha \in \text{spec}(A)} f_2(\alpha) = 2j_2, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{f_2(\alpha)} = q^{j_2}.
\]

Then

\[
\sum_{j=1}^{h} \Lambda_{2(m-j),2j,K} \left( \bigwedge^{2(m-j)}_K (V^*)(q^{m-j}) \otimes K \bigwedge^{2j}_K (V^*)(q^j) \right) = \bigwedge^{2m}_K (V^*)(q^m).
\]

(43)

Proof. In light of Remark 6.5 and arguments of Subsection 6.2, it suffices to check that

\[
\sum_{j=1}^{h} \Lambda_{2(m-j),2j,E} \left( \bigwedge^{2(m-j)}_E (V^*_E)(q^{m-j}) \otimes E \bigwedge^{2j}_E (V^*_E)(q^j) \right) = \bigwedge^{2m}_E (V^*_E)(q^m).
\]

(44)

Recall that (in the notation of Subsection 6.8) that \( B \) is an (ordered) eigenbasis of \( V_E \) and all the \( 2m \)-element subsets \( C \subset \text{spec}(A) \) with \( \prod_{\alpha \in C} \alpha = q^m \) give rise to the base \( \{ y_C = \bigwedge_{x \in C} x^* \} \) of \( \bigwedge^m_E (V^*_E)(q^m) \). So, it suffices to prove that each such \( y_C \) lies in one of the summands in LHS of (44). To this end, let us consider the nonnegative integer-valued function

\( e_C : \text{spec}(A) \to \mathbb{Z}_+ \), \( e(\alpha) = \#(C(\alpha)) \) where \( C(\alpha) := \{ x \in C \subset B \mid \pi(x) = \alpha \} \).

(see (39)). Clearly,

\[
\sum_{\alpha \in \text{spec}(A)} e_C(\alpha) = \#(C) = 2m, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{e_C(\alpha)} = \prod_{\alpha \in C} \alpha = q^m.
\]
By property (ii), there exist positive integers \( j_1, j_2 \) and nonnegative integer-valued functions

\[
f_1: \text{spec}(A) \rightarrow \mathbb{Z}_+, f_2: \text{spec}(A) \rightarrow \mathbb{Z}_+
\]

such that

\[
m = j_1 + j_2, \quad j_2 \leq h;
\]

\[
e(\alpha) = f_1(\alpha) + f_2(\alpha) \quad \forall \alpha \in \text{spec}(A);
\]

\[
\sum_{\alpha \in \text{spec}(A)} f_1(\alpha) = 2j_1, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{f_1(\alpha)} = q^{j_1},
\]

\[
\sum_{\alpha \in \text{spec}(A)} f_2(\alpha) = 2j_2, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{f_2(\alpha)} = q^{j_2}.
\]

Let us partition each \( C(\alpha) \) into a disjoint union of two sets

\[
C(\alpha) = C(\alpha)_1 \cup C(\alpha)_2 \text{ with } C(\alpha)_1 \cap C(\alpha)_2 = \emptyset, \quad \#(C(\alpha)_1) = f_1(\alpha), \quad \#(C(\alpha)_2) = f_2(\alpha)
\]

and define \( C_1 \) (resp. \( C_2 \)) as the (disjoint) union of all \( C(\alpha)_1 \) (resp. of all \( C(\alpha)_2 \)). Then \( C \) becomes a disjoint union of \( C_1 \) and \( C_2 \), and

\[
f_1 = e_{C_1}, f_2 = e_{C_2}.
\]

It follows that

\[
\sum_{\alpha \in \text{spec}(A)} e_{C_1}(\alpha) = 2j_1, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{e_{C_1}(\alpha)} = q^{j_1},
\]

\[
\sum_{\alpha \in \text{spec}(A)} e_{C_2}(\alpha) = 2j_2, \quad \prod_{\alpha \in \text{spec}(A)} \alpha^{e_{C_2}(\alpha)} = q^{j_2}.
\]

This implies that

\[
y_{C_1} \in \wedge^{2(j_1)}(V_E^*)(|q^{j_1}|), \quad y_{C_2} \in \wedge^{2(j_2)}(V_E^*)(|q^{j_2}|).
\]

Since \( C \) is a disjoint union of \( C_1 \) and \( C_2 \),

\[
y_C = \pm y_{C_1} \wedge y_{C_2} \in \Lambda^{2(m-j_2)}(V_E^*)(|q^{m-j_2}|) \otimes E \wedge^{2j_2}(V_E^*)(|q^{j_2}|).
\]

In order to finish the proof, one has only to recall that \( j \leq h_2 \). \qed

### 7. Tate forms

7.1. Recall that \( X \) is an abelian variety of positive dimension \( g \) over a finite field \( k \) with \( \text{char}(k) = p \) and \( \#(k) = q \). Let \( \ell \neq p \) be a prime and \( T_\ell(X) \) the \( \ell \)-adic Tate module of \( X \). Let us consider the corresponding \( \mathbb{Q}_\ell \)-vector space

\[
V_\ell(X) = T_\ell(X) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell,
\]

which is a \( 2g \)-dimensional vector space over \( \mathbb{Q}_\ell \). The action of \( \text{Fr}_X \) extends by \( \mathbb{Q}_\ell \)-linearity to \( V_\ell(X) \). So, we may view \( \text{Fr}_X \) as a \( \mathbb{Q}_\ell \)-linear automorphism of \( V_\ell(X) \), whose characteristic polynomial coincides with \( P_X(t) \). A theorem of Weil [3, 16] asserts that \( \text{Fr}_X \) acts as a semisimple linear operator in \( V_\ell(X) \). Let \( \bar{\mathbb{Q}}_\ell \) be an algebraic closure of \( \mathbb{Q}_\ell \). Let us choose a field embedding

\[
L_X = \mathbb{Q}(R_X) \hookrightarrow \bar{\mathbb{Q}}_\ell.
\]
Further we will identify $L_X$ with its image in $\bar{Q}_\ell$. We have

$$R_X \subset L_X \subset \bar{Q}_\ell.$$  

Let us consider the $2\dim(X)$-dimensional $\bar{Q}_\ell$-vector space

$$V_\ell(X) := V_\ell(X) \otimes_{\bar{Q}_\ell} \bar{Q}_\ell.$$

Extending the action of $\text{Fr}_X$ by $\bar{Q}_\ell$-linearity, we get a $\bar{Q}_\ell$-linear operator

$$\overline{\text{Fr}}_X : V_\ell(X) \rightarrow V_\ell(X), \ v \otimes \lambda \mapsto \text{Fr}_X(v) \otimes \lambda \ \forall v \in V_\ell(X), \lambda \in \bar{Q}_\ell.$$

In the notation of Section 6, let us put

$$K = \bar{Q}_\ell, V = V_\ell(X), A = \text{Fr}_X : V_\ell(X) \rightarrow V_\ell(X), E = \bar{Q}_\ell.$$  

Then

$$V_E = \bar{V}_\ell(X), A_E = \overline{\text{Fr}}_X; \ \text{spec}(A) = R_X, \text{mult}_A = \text{mult}_X : R_X \rightarrow \mathbb{Z}_+. \ (46)$$

**Remark 7.2.** If $m$ is a positive integer and $Y = X^m$ then it is well known that there is a canonical isomorphism of $\bar{Q}_\ell$-vector spaces

$$V_\ell(Y) = \oplus_{i=1}^m V_\ell(X)$$

such that $\text{Fr}_Y$ acts on $V_\ell(Y)$ as

$$\text{Fr}_Y(x_1, \ldots, x_m) = (\text{Fr}_X x_1, \ldots, \text{Fr}_X x_m) \ \forall (x_1, \ldots, x_m) \in \oplus_{i=1}^m V_\ell(X) = V_\ell(Y).$$

This implies that

$$P_Y(t) = P_X(t)^m, \ R_Y = R_X, \ L_X = L_Y, \ \text{mult}_Y(\alpha) = m \cdot \text{mult}_X(\alpha) \ \forall \alpha \in R_X = R_Y. \ (47)$$

In particular,

$$\text{mult}_Y(\alpha) \geq m \ \forall \alpha \in R_Y = R_X. \ (48)$$

Any invertible sheaf/divisor class $\mathcal{L}$ on $X$ gives rise to (defined up to multiplication by an element of $\bar{Q}_\ell^*$) a $\bar{Q}_\ell$-bilinear alternating *Riemann form* (the first $\ell$-adic Chern class of $\mathcal{L}$) \[8\]

$$\phi = \phi_{\mathcal{L}} : V_\ell(X) \times V_\ell(X) \rightarrow \bar{Q}_\ell$$

such that

$$\phi(\text{Fr}_X(x), \text{Fr}_X(y)) = q \cdot \phi(x, y) \ \forall x, y \in V_\ell(X). \ (49)$$

**7.3.** A theorem of Tate \[16\] asserts that every alternating $\bar{Q}_\ell$-bilinear form $\phi$ on $V_\ell(X)$ that satisfies \[19\] is a $\bar{Q}_\ell$-linear combination of forms of type $\phi_{\mathcal{L}}$. We call such a form an *\(\ell\)-adic Tate form of degree 2* and denote by $\text{tate}_2(X, \ell)$ the subspace of all such forms in $\text{Hom}_{\bar{Q}_\ell}(\Lambda^2 V_\ell(X), \bar{Q}_\ell)$. In other words,

$$\text{tate}_2(X, \ell) := \{ \phi \in \text{Hom}_{\bar{Q}_\ell}(\Lambda^2 V_\ell(X), \bar{Q}_\ell) \mid \phi(\text{Fr}_X(x), \text{Fr}_X(y)) = q \cdot \phi(x, y) \ \forall x, y \in V_\ell(X) \}. \ (50)$$

More generally, let us define for each nonnegative integer $d \leq \dim(X) = g$

the subspace $\text{tate}_{2d}(X, \ell)$ of all alternating $2d$-forms $\psi \in \text{Hom}_{\bar{Q}_\ell}(\Lambda^{2d} V_\ell(X), \bar{Q}_\ell)$ such that

$$\psi(\text{Fr}_X(v_1), \ldots, \text{Fr}_X(v_{2d})) = q^d \cdot \psi(v_1, \ldots, v_{2d}) \ \forall x_1, \ldots, x_{2d} \in V_\ell(X). \ (51)$$
We call elements of \(\text{tate}_{2d}(X, \ell)\) Tate forms of degree \(2d\).\(^1\)

**Remark 7.4.** Clearly, \(\text{tate}_{2d}(X, \ell)\) consists of all \(\psi \in \text{Hom}_{\mathbb{Q}_\ell}(\Lambda^{2d}V_\ell(X), \mathbb{Q}_\ell)\) such that

\[
\psi(\text{Fr}_X^{-1}(v_1), \ldots, \text{Fr}_X^{-1}(v_{2d})) = q^{-d} \cdot \psi(v_1, \ldots, v_{2d}) \quad \forall v_1, \ldots, v_{2d} \in V_\ell(X).
\]

Since \(\text{Fr}_X\) acts on \(V_\ell(X)\) as \(\rho_\ell(\sigma_k)\), the subspace \(\text{tate}_{2d}(X, l)\) consists of all \(\psi \in \text{Hom}_{\mathbb{Q}_\ell}(\Lambda^{2d}V_\ell(X), \mathbb{Q}_\ell)\) such that

\[
\psi(\rho_\ell(\sigma_k)^{-1}(v_1), \ldots, \rho_\ell(\sigma_k)^{-1}(v_{2d})) = \chi_\ell(\sigma_k)^{-d} \cdot \psi(v_1, \ldots, v_{2d}) \quad \forall v_1, \ldots, v_{2d} \in V_\ell(X).
\]

Since \(\sigma_k\) is a topological generator of \(\text{Gal}(k)\), the subspace \(\text{tate}_{2d}(X, l)\) consists of all \(\psi \in \text{Hom}_{\mathbb{Q}_\ell}(\Lambda^{2d}V_\ell(X), \mathbb{Q}_\ell)\) such that

\[
\psi(\rho_\ell(\sigma)^{-1}(v_1), \ldots, \rho_\ell(\sigma)^{-1}(v_{2d})) = \chi_\ell(\sigma)^{-d} \cdot \psi(v_1, \ldots, v_{2d}) \quad \forall \sigma \in \text{Gal}(k), v_1, \ldots, v_{2d} \in V_\ell(X).
\]

**Remark 7.5.** In the notation of Section \(6\) \(45\) and \(46\),

\[
\text{tate}_{2d}(X, \ell) = \wedge^d_\ell(V^*) (q^d)
\]

is the eigenspace of \(K\)-linear operator \(\wedge^d_\ell(A^*)\): \(\wedge^d_\ell(V^*) \to \wedge^d_\ell(V^*)\) attached to the eigenvalue \(q^d\).

**7.6.** For each integer \(d \geq 3\) the exterior product map

\[
\text{Hom}_{\mathbb{Q}_\ell}(\Lambda^{2(d-1)}V_\ell(X), \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} \text{Hom}_{\mathbb{Q}_\ell}(\Lambda^2V_\ell(X), \mathbb{Q}_\ell) \to \text{Hom}_{\mathbb{Q}_\ell}(\Lambda^{2d}V_\ell(X), \mathbb{Q}_\ell),
\]

\[
\phi \otimes \psi \mapsto \phi \wedge \psi
\]

induces the \(\mathbb{Q}_\ell\)-linear map

\[
\text{tate}_{2(d-1)}(X, \ell) \otimes_{\mathbb{Q}_\ell} \text{tate}_2(X, \ell) \to \text{tate}_{2d}(X, \ell), \quad \phi \otimes \psi \mapsto \phi \wedge \psi. \tag{50}
\]

**Definition 7.7.** Let \(d > 1\) be an integer. An \(\ell\)-adic Tate form of degree \(2d\) is called exceptional if it does not lie in the image of map \(50\).

**Lemma 7.8.** Let \(d\) be a positive integer such that

\[
2 \leq d \leq \dim(X).
\]

Let \(\ell \neq p\) be a prime. Then the following conditions are equivalent.

(a) There exists an exceptional \(\ell\)-adic Tate form on \(X\) of degree \(2d\).

(b) There exists an admissible reduced function \(e: R_X \to \mathbb{Z}\) of weight \(2d\) such that

\[
0 \leq e(\alpha) \leq \text{mult}_X(\alpha) \quad \forall \alpha \in R_X.
\]

**Proof.** In the notation of \(40\), \(45\) and \(46\), it follows from Remark \(7.5\) that property (b) is equivalent to the non-surjectiveness of

\[
\wedge^d_\ell(V^*) (\lambda_1) \otimes_\ell \wedge^d_\ell(V^*) (\lambda_2) \to \wedge^d_\ell(V^*) (\lambda_1 \lambda_2), \quad \psi \otimes \phi \mapsto \psi \wedge \phi
\]

with

\[
j = 2d, \lambda_1 = q^{d-1}, \lambda_2 = q, \lambda_1 \lambda_2 = q^d.
\]

\(^1\)In \(23\) we called them admissible forms.
By Proposition 6.9, the non-surjectiveness of this map is equivalent to the existence of a function $e: \text{spec}(A) \to \mathbb{Z}_+$ that enjoys properties (i)-(iv) of Proposition 6.9. Since $\text{spec}(A) = R_X$, we may view $e$ as a function $e: R_X \to \mathbb{Z}_+$. Now property (ii) means that $e$ has weight $2d$, property (iii) that $e$ is admissible, and property (iv) that $e$ is reduced. As for property (i), it means that

$$e(\alpha) \leq \text{mult}_A(\alpha) = \text{mult}_X(\alpha) \quad \forall \alpha \in \text{spec}(A) = R_X.$$ 

This implies that properties (i)-(iv) of Theorem 6.9 are equivalent to property (b) of Lemma 7.8. It follows that properties (a) and (b) of Lemma 7.8 are equivalent. □

7.9. Twists and Tate classes. Let us consider an abelian variety $\bar{X} = X \times_k \bar{k}$ over the algebraic closure $\bar{k}$ of $k$ and its étale $\ell$-adic cohomology groups $H^j(\bar{X}, \mathbb{Q}_\ell)$ [15, 5, 4, 1]. Here $\ell$ is any prime different from char($k$), $j$ any nonnegative integer, and $H^j(\bar{X}, \mathbb{Q}_\ell)$ is a certain finite-dimensional $\mathbb{Q}_\ell$-vector space endowed with a continuous linear action of the absolute Galois group $\text{Gal}(k) := \text{Gal}(\bar{k}/k)$ of $k$. We write

$$\chi_\ell: \text{Gal}(k) \to \mathbb{Z}^{\ast}_\ell \subset \mathbb{Q}^{\ast}_\ell$$

for the $\ell$-adic cyclotomic character (see Section 2 above). There exists a certain “naturally defined” one-dimensional $\mathbb{Q}_\ell$-vector space $\mathbb{Q}_\ell(1)$ (that was denoted by $W$ in [15, Sect. 2]) endowed with the natural continuous linear action of the absolute Galois group $\text{Gal}(k) := \text{Gal}(\bar{k}/k)$ of $k$. We write

$$\chi_\ell: \text{Gal}(k) \to \mathbb{Q}^{\ast}_\ell = \text{Aut}_{\mathbb{Q}_\ell}(\mathbb{Q}_\ell(1))$$

(see [15, 5, 4, 7]). Namely, $\mathbb{Q}_\ell(1) = \mathbb{Z}_\ell(1) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ where $\mathbb{Z}_\ell(1)$ is the projective limit of multiplicative groups (finite Galois modules) $\mu_{\ell^n}$ of $\ell^n$th roots of unity in $\bar{k}$.

Let us fix once and for all an $\ell$-adic orientation, i.e., an isomorphism of $\mathbb{Q}_\ell$-vector spaces

$$\mathbb{Q}_\ell(1) \cong \mathbb{Q}_\ell,$$

which allows us to identify $\mathbb{Q}_\ell$ not only with $\mathbb{Q}_\ell(1)$ but also with all tensor powers $\mathbb{Q}_\ell(i)$ [15, 5, 4, 1, 18] of $\mathbb{Q}_\ell(1)$.

Let $i$ be an integer. Let us consider the twist $H^j(\bar{X}, \mathbb{Q}_\ell)(i)$ of the Galois module $H^j(\bar{X}, \mathbb{Q}_\ell)$ by character $\chi_\ell^i$ of $\text{Gal}(k)$ [15, 5, 4, 18]. In other words, $H^j(\bar{X}, \mathbb{Q}_\ell)(i)$ coincides with $H^j(\bar{X}, \mathbb{Q}_\ell)$ as the $\mathbb{Q}_\ell$-vector space but if

$$\sigma, c \mapsto \sigma(c) \quad \forall \sigma \in \text{Gal}(k), \quad c \in H^j(\bar{X}, \mathbb{Q}_\ell)$$

is the Galois action on $H^j(\bar{X}, \mathbb{Q}_\ell)$ then in $H^j(\bar{X}, \mathbb{Q}_\ell)(i)$ a Galois automorphism $\sigma$ sends $c$ to $\chi_\ell(\sigma)^i \sigma(c)$.

If $W$ is a Galois-invariant $\mathbb{Q}_\ell$-vector subspace in $H^j(\bar{X}, \mathbb{Q}_\ell)$ then we write $W(i)$ for the same $\mathbb{Q}_\ell$-vector subspace in $H^j(\bar{X}, \mathbb{Q}_\ell)(i)$. Clearly, $W(i)$ is a Galois-invariant subspace of $H^j(\bar{X}, \mathbb{Q}_\ell)(i)$ but not necessarily isomorphic to $W$ as Galois module.
Definition 7.13. \( W \) lies in \( \text{Galois equivariance of maps (51) and (52) combined with Remark 7.10(ii)} \)

\[
\text{Remarks 7.10. (i) If } j_1, j_2 \text{ are any nonnegative integers then the Galois-equivariant } \mathbb{Q}_\ell \text{-bilinear cup product in the cohomology of } \bar{X} \text{ leads to a Galois-equivariant } \mathbb{Q}_\ell \text{-linear map}
\]

\[
H^{j_1}(\bar{X}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^{j_2}(\bar{X}, \mathbb{Q}_\ell) \to H^{j_1+j_2}(\bar{X}, \mathbb{Q}_\ell), \quad c_1 \otimes c_2 \mapsto c_1 \cup c_2,
\]

which, in turn, gives rise to the natural Galois-equivariant \( \mathbb{Q}_\ell \)-linear map \[51\] \[
H^{j_1}(\bar{X}, \mathbb{Q}_\ell)(i_1) \otimes_{\mathbb{Q}_\ell} H^{j_2}(\bar{X}, \mathbb{Q}_\ell)(i_2) \to H^{j_1+j_2}(\bar{X}, \mathbb{Q}_\ell)(i_1+i_2), \quad c_1 \otimes c_2 \mapsto c_1 \cup c_2.
\]

(ii) Let \( W_1 \) (resp. \( W_2 \)) is a Galois-invariant \( \mathbb{Q}_\ell \)-vector subspace in \( H^{j_1}(\bar{X}, \mathbb{Q}_\ell) \) (resp. in \( H^{j_2}(\bar{X}, \mathbb{Q}_\ell) \)) and \( W \subset H^{j_1+j_2}(\bar{X}, \mathbb{Q}_\ell) \) be the image of subspace

\[
W_1 \otimes_{\mathbb{Q}_\ell} W_2 \subset H^{j_1}(\bar{X}, \mathbb{Q}_\ell) \otimes_{\mathbb{Q}_\ell} H^{j_2}(\bar{X}, \mathbb{Q}_\ell)
\]

under the map \[51\]. It follows readily that the twist

\[
W(i_1 + i_2) \subset H^{j_1+j_2}(\bar{X}, \mathbb{Q}_\ell)(i_1+i_2)
\]

coincides with the image of subspace

\[
W_1(i_1) \otimes_{\mathbb{Q}_\ell} W_2(i_2) \subset H^{j_1}(\bar{X}, \mathbb{Q}_\ell)(i_1) \otimes_{\mathbb{Q}_\ell} H^{j_2}(\bar{X}, \mathbb{Q}_\ell)(i_2)
\]

under the map \[52\].

Definition 7.11. Let \( d \) be a nonnegative integer. Let us consider the \( \mathbb{Q}_\ell \)-vector subspace

\[
\mathcal{T}_{\ell,d}(X) := H^{2d}(\bar{X}, \mathbb{Q}_\ell)(d)^{\text{Gal}(k)}
\]

of Galois invariants in \( H^{2d}(\bar{X}, \mathbb{Q}_\ell)(d) \) and a weight \( \mathbb{Q}_\ell \)-vector subspace

\[
W_{\ell,d}(X) := \{ c \in H^{2d}(\bar{X}, \mathbb{Q}_\ell) \mid \sigma(x) = \chi_\ell(\sigma)^{-d}c \quad \forall \sigma \in \text{Gal}(k) \}
\]

in \( H^{2d}(\bar{X}, \mathbb{Q}_\ell) \). It follows from the very definitions that

\[
\mathcal{T}_{\ell,d}(X) = W_{\ell,d}(X)(d).
\]

Remark 7.12. Let \( d_1 \) and \( d_2 \) be nonnegative integers. It follows from the Galois equivariance of maps \[51\] and \[52\] combined with Remark 7.10(ii) that the image \( W_{\ell,d_1,d_2}(X) \) of

\[
W_{\ell,d_1}(X) \otimes W_{\ell,d_2}(X) \to H^{2(d_1+d_2)}(\bar{X}, \mathbb{Q}_\ell), \quad c_1 \otimes c_2 \mapsto c_1 \cup c_2
\]

lies in \( W_{\ell,d_1+d_2}(X) \). Similarly, the image \( \mathcal{T}_{\ell,d_1,d_2}(X) \) of

\[
\mathcal{T}_{\ell,d_1}(X) \otimes \mathcal{T}_{\ell,d_2}(X) \to H^{2(d_1+d_2)}(\bar{X}, \mathbb{Q}_\ell)(d_1+d_2), \quad c_1 \otimes c_2 \mapsto c_1 \cup c_2
\]

lies in \( \mathcal{T}_{\ell,d_1+d_2}(X) \). In addition, it follows from \[53\] that

\[
\mathcal{T}_{\ell,d_1,d_2}(X) = W_{\ell,d_1,d_2}(X)(d_1+d_2).
\]

Definition 7.13. Let \( \ell \neq \text{char}(k) \) be a prime and \( d \) a nonnegative integer.

(i) Elements of \( \mathcal{T}_{\ell,d}(X) \) are called \( \ell \)-adic Tate classes of dimension \( 2d \) on \( X \).
(ii) A nonzero $2d$-dimensional Tate class $c$ is called **exotic** if $d \geq 1$ and $c$ cannot be presented as a linear combination of products of $d$ Tate classes of dimension 2 with coefficients in $\mathbb{Q}_\ell$.

(iii) A Tate class $c$ of dimension $2d$ is called **very exotic** if $d \geq 1$ and $c$ cannot be presented as a linear combination with coefficients in $\mathbb{Q}_\ell$ of products of Tate classes of dimension $2d-2$ and 2, i.e., $c$ does not belong to $\mathcal{T}_{\ell,d-1,1}(X)$.

**Remarks 7.14.**

(i) Let $Z$ be a closed irreducible subvariety of codimension $d$ in $X$. The choice of the $\ell$-adic orientation allows us to define the $\ell$-adic class $\text{cl}(Z) \in \mathcal{T}_{\ell,d}(X) \subset H^{2d}(\overline{X}, \mathbb{Q}_\ell)(d)$ of $Z$ \cite{15,18}. Tate \cite{15,18} conjectured that for all nonnegative integer $d$ the subspace $\mathcal{T}_{\ell,d}(X)$ is spanned by all $\text{cl}(Z)$ and proved it for $d = 1$ \cite{16}.

(ii) If $d$ is a nonnegative integer and $d \leq g$ then it is known \cite{15,18} that $\mathcal{T}_{\ell,d}(X) \neq \{0\}$.

(iii) Clearly, all 2-dimensional Tate classes are neither exotic nor very exotic. Hence, the existence of an exotic (or very exotic) Tate class of dimension $d$ implies readily that

$$2 \leq d \leq g = \dim(X),$$

because $H^{2d}(\overline{X}, \mathbb{Q}_\ell) = \{0\}$ for all $d > g$, and, therefore, all 2-dimensional Tate classes are just zero. (Actually, it is known that $H^{2g}(\overline{X}, \mathbb{Q}_\ell)(g)$ is a one-dimensional $\mathbb{Q}_\ell$-vector space generated by the $g$th self-product of the class of a hyperplane section of $X$ \cite{15}. Therefore, there are no non-exotic Tate classes of dimension $2g$.)

(iv) Clearly, every very exotic Tate class is exotic.

Conversely, suppose that there exists an exotic $2d$-dimensional $\ell$-adic Tate class on $X$. I claim that there is a positive integer $d' \leq d$ such that there exists a very exotic $2d'$-dimensional $\ell$-adic Tate class on $X$. Indeed, decreasing $d$ if necessary, we may and will assume that $d$ is the smallest positive integer such that there is an exotic $\ell$-adic $2d$-dimensional Tate class on $X$. Let $c$ be such a class. Then $d > 1$.

Assume that $c$ is not very exotic. Then $c$ is a linear combination of cup products $h_i \cup c_i$ where all $c_i$ are nonzero Tate classes of dimension 2 and all $h_i$ are nonzero Tate classes of dimension $2(d-1)$. Since $c$ is exotic, there is an index $i$ such that $h_i$ is exotic. But exotic $h_i$ is $2(d-1)$-dimensional, which contradicts the minimality of $d$. The obtained contradiction implies that $c$ itself is very exotic, which ends the proof.

It follows that $X$ carries an $\ell$-adic exotic Tate class if and only if it carries a very exotic $\ell$-adic Tate class (may be, of different dimension).
7.15. Étale cohomology of abelian varieties. Let us consider the abelian variety \( \bar{X} = X \times_k \bar{k} \) over \( \bar{k} \). Let \( j \) be a nonnegative integer and let \( H_j(\bar{X}, \mathbb{Q}_\ell) \) be the \( j \)th étale \( \ell \)-adic cohomology group of \( \bar{X} \), which is a finite-dimensional \( \mathbb{Q}_\ell \)-vector space endowed with the canonical continuous linear action of \( \text{Gal}(\bar{k}) \) [15, 5, 7]. There is a canonical \( \text{Gal}(\bar{k}) \)-equivariant isomorphism of graded \( \mathbb{Q}_\ell \)-algebras ([15, 5, 1], [7, Sect. 12])

\[
\bigoplus_{j=0}^{2\dim(X)} H_j(\bar{X}, \mathbb{Q}_\ell) \cong \bigoplus_{j=0}^{2\dim(X)} \text{Hom}_{\mathbb{Q}_\ell}(\Lambda^j_{\mathbb{Q}_\ell} V_\ell(X), \mathbb{Q}_\ell).
\]

Its Galois equivariance combined with Remark 7.4 imply that (in the notation of Definition 7.11) the map (54) induces for all nonnegative integers \( d \leq \dim(X) \) a \( \mathbb{Q}_\ell \)-linear isomorphism between

\[
W_{\ell,d}(X) = \{ c \in H^{2d}(\bar{X}, \mathbb{Q}_\ell) \mid \sigma(c) = \chi_\ell(\sigma)^{-d}c \forall \sigma \in \text{Gal}(\bar{k}) \} = \\
\{ c \in H^{2d}(\bar{X}, \mathbb{Q}_\ell) \mid \sigma_k(c) = \chi_\ell(\sigma_k)^{-d}c \} \subset H^{2d}(\bar{X}, \mathbb{Q}_\ell)
\]

and the subspace of \( \ell \)-adic Tate forms of degree \( 2d \) on \( X \). Recall (see Definitions 7.11 and 7.13) that the twist \( W_{\ell,d}(X) \) coincides with the subspace \( T_{\ell,d}(X) \) of \( 2d \)-dimensional \( \ell \)-adic Tate classes on \( X \). Recall that map (54) is a \( \mathbb{Q}_\ell \)-algebra isomorphism. Applying Remark 7.12 to \( d_1 = d - 1 \) and \( d_2 = 1 \), we obtain that the existence of a very exotic \( \ell \)-adic Tate class of dimension \( 2d \) on \( X \) is equivalent to the existence of an exceptional \( \ell \)-adic Tate form of degree \( 2d \) on \( X \).

We will need to state explicitly the following useful assertion.

**Lemma 7.16.** Let \( X \) be an abelian variety over \( k \). Let \( \ell \neq \text{char}(k) \) be a prime. Then the following three conditions are equivalent.

(i) \( X \) carries an exotic \( \ell \)-adic Tate class.

(ii) \( X \) carries a very exotic \( \ell \)-adic Tate class.

(iii) There exists an exceptional \( \ell \)-adic Tate form on \( X \).

In addition, the validity of equivalent conditions (i)-(iii) does not depend on a choice of \( \ell \).

**Proof of Lemma 7.16.** The equivalence of (ii) and (iii) follows readily from the arguments at the end of Subsection 7.15. The equivalence of (i) and (ii) was already proven in Remark 7.14(iv).

Notice that property (b) of Lemma 7.8 does not depend on the choice of \( \ell \). Now Lemma 7.8 implies that the validity of (iii) does not depend on the choice of \( \ell \). \( \square \)

**Proof of Theorem 1.8.** By a theorem of Tate [16], if \( Y \) is any abelian variety over \( k \) (e.g., \( Y = X^n \)) then every element of \( T_{\ell,1}(Y) \) is a linear combination of divisor classes on \( Y \) with coefficients in \( \mathbb{Q}_\ell \).

Suppose that there is an exotic \( \ell \)-adic Tate class on \( X^n \) for some positive integer \( n \). It follows from Lemma 7.16 (applied to \( Y = X^n \) instead of \( X \))
that there is an exceptional $\ell$-adic Tate form on $X^n$. In light of Lemma 7.8 (applied to $Y$ instead of $X$), there exists an admissible reduced function $R_X = R_Y \to \mathbb{Z}_+$. In light of Theorem 1.14, there exists an admissible reduced function $e: R_X \to \mathbb{Z}_+$ of weight $\leq N(g)$. This means that

$$\text{wt}(e) = \sum_{\alpha \in R_X} e(\alpha) \leq 2N(g);$$

in particular,

$$0 \leq e(\alpha) \leq 2N(g) \forall \alpha \in R_X.$$

Let us put $Z = X^{2N(g)}$ and consider $e$ as the reduced admissible function $R_Z = R_X \to \mathbb{Z}_+$, $\alpha \mapsto e(\alpha)$.

In light of Remark 7.2 applied to $m = 2N(g),

$$0 \leq 2N(g) \leq \text{mult}_Z(\alpha) \forall \alpha \in R_Z = R_X.$$ 

It follows from Lemma 7.8 that there is an exceptional $\ell$-adic Tate form on $X^{2N(g)} = Z$. Applying Lemma 7.16 to $Z$, we obtain that there is an exotic $\ell$-adic Tate class on $Z = X^{2N(g)}$. Now the last assertion of Lemma 7.16 implies that there is an exotic $\ell$-adic Tate class on $Z = X^{2N(g)}$ for all primes $l \neq \text{char}(k)$. This ends the proof.

\begin{proof}[Proof of Theorem 1.14] In light of arguments of Subsection 7.15 combined with Remark 7.12, it suffices to check that each $\ell$-adic Tate form of any even degree $2m$ on $X^n$ can be presented as a linear combination of exterior products of $\ell$-adic Tate forms of degree at most $H(g)$. Let us prove it, using induction by $m$.

The assertion is obviously true for all $m \leq H(g)/2$. Suppose that $m > H(g)/2$. First, notice that

$$R_{X^n} = R_X \forall n.$$ 

Applying Theorem 1.10, we conclude that the conditions of Proposition 6.14 are fulfilled for

$$K = \mathbb{Q}_\ell, V = V_\ell(X^n), A = \text{Fr}_{X^n} : V_\ell(X^n) \to V_\ell(X^n),$$

$$\text{spec}(A) = R_{X^n} = R_X, h = H(g)/2.$$ 

Applying Proposition 6.14, we conclude that each $\ell$-adic Tate form of degree $2m$ on $X^n$ can be presented as a linear combination of wedge products

$$\psi_{m-j} \wedge \phi_j \ (j = 1, \ldots, H(g)/2))$$ 

where $\psi_{m-j}$ is an $\ell$-adic Tate form of degree $2(m - j)$ on $X^n$ and $\phi_j$ is an $\ell$-adic Tate form of degree $2j \leq H(g)$ on $X^n$. Applying the induction assumption to all $\psi_{m-j}$'s, we conclude that each $\ell$-adic Tate form of degree $2m$ on $X^n$ can be presented as a linear combination of exterior products of Tate forms of degree at most $H(g)$. This ends the proof.
\end{proof}
References

[1] V.G. Berkovich, The Brauer group of abelian varieties. Funktsional. Anal. i Prilozhen. 6:3 (1972), 10–15; Funkt. Anal. Appl. 6:3 (1972), 180–184.
[2] N. Bourbaki, Algebra I, Chapters 1-3. Springer Verlag, Berlin Heidelberg New York, 1989.
[3] D.A. Cox, J.B. Little, H.K. Schenck, Toric varieties. Graduate Studies in Mathematics 124, Amer. Math. Soc., Providence, RI, 2011.
[4] N.M. Katz, Review of ℓ-adic cohomology. In: Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.). Proceedings of Symposia in Pure Mathematics, 55:1 (1994), pp. 21–30, American Mathematical Society, Providence, RI.
[5] S. Kleiman, Algebraic cycles and the Weil conjecture. In; Dix Exposés sur la Cohomologie des Schémas, pp. 359–386. North Holland, Amsterdam, 1968.
[6] H.W. Lenstra, Jr and Yu.G. Zarhin, The Tate conjecture for almost ordinary abelian varieties over finite fields. In: Advances in Number Theory, CNTA 91 Conference Proceedings (F. Gouvea and N. Yui, eds.), pp.179–194. Oxford University Press, 1993.
[7] J.S. Milne, Abelian Varieties. https://www.jmilne.org/math/CourseNotes/AV.pdf
[8] D. Mumford, Abelian varieties, 2nd edition. Oxford University Press, 1974.
[9] F. Oort, Abelian varieties over finite fields, pp. 123–188. In: Higher-dimensional varieties over finite fields (D. Kaledin, Yu. Tschinkel, eds.), IOS Press, Amsterdam, 2008.
[10] J.-P. Serre, J. Tate, Good reduction of abelian varieties. Ann. Math. 88 (1968), 492–517.
[11] J.-P. Serre, Abelian ℓ-adic representations and elliptic curves. 2nd edition, Addison-Wesley, 1989.
[12] J.-P. Serre, Lettres à Ken Ribet du 1/1/1981 et du 29/1/1981, pp. 1–20. (Œuvres, IV, 1985–1998. Springer–Verlag, Berlin, 2000.
[13] A. Silverberg, Yu.G. Zarhin, Connectedness results for ℓ-adic representations associated to abelian varieties. Compositio Mathematica 97, no 1-2 (1995), 273–284.
[14] A. Silverberg, Yu.G. Zarhin, Variations on a theme of Minkowski and Serre. J. Pure Applied Algebra 111 (1996), 285–302.
[15] J.T. Tate, Algebraic cycles and poles of zeta functions. In: Arithmetical Algebraic Geometry (O.F.G. Schilling, ed.), pp. 93–110. Harper and Row, New York, 1965.
[16] J.T. Tate, Endomorphisms of abelian varieties over finite fields. Invent. Math. 2 (1966), 134–144.
[17] J.T. Tate, Classes d’isogénie de variétés abéliennes sur un corps fini (d’après T. Honda), in Sém. Bourbaki 21, 1968/69, Exp. 352, Lecture Notes Math. 179 (1971), pp. 95–110.
[18] J.T. Tate, Conjectures on algebraic cycles in ℓ-adic cohomology. In: Motives (U. Jannsen, S. Kleiman, J.-P. Serre, eds.). Proceedings of Symposia in Pure Mathematics, 55:1 (1994), pp. 71–83, American Mathematical Society, Providence, RI.
[19] Yu.G. Zarhin, Abelian varieties, ℓ-adic representations and SL₂, Izvestiya AN SSSR ser. matem. 43 (1979), 294–308; Math. USSR Izvestiya 14 (1980), 275–288.
[20] Yu.G. Zarhin, Abelian varieties, ℓ-adic representations and Lie algebras. Rank independence on ℓ. Invent. Math. 55 (1979), 165–176.
[21] Yu.G. Zarhin, Abelian varieties of K3 type and ℓ-adic representations. In: Algebraic Geometry and Analytic Geometry (A. Fujiki, K. Kato, T. Katsura, Y. Kawamata, Y. Miyaoka, eds.). ICM-90 Satellite Conference Proceedings, pp. 231–255. Springer-Verlag, Tokyo, 1991.
[22] Yu.G. Zarhin, Abelian varieties of K3 type, pp. 263–279. In: Séminaire de Théorie des Nombres, Paris, 1990-91 (S. David, ed.), Progress in Math. 108 (1993), Birkh’suser, Boston Basel Stuttgart.
[23] Yu.G. Zarhin, *The Tate conjecture for non-simple abelian varieties over finite fields.* In: Algebra and Number Theory (G. Frey and J. Ritter, eds.), pp. 267–296. Walter de Gruyter, Berlin New York, 1994.

[24] Yu.G. Zarhin, *Eigenvalues of Frobenius Endomorphisms of Abelian varieties of low dimension.* J. Pure Applied Algebra 219:6 (2015), 2076–2098.

Department of Mathematics, Pennsylvania State University, University Park, PA 16802, USA

*Email address:* zarhin@math.psu.edu