Correction: Naudts, J. Quantum Statistical Manifolds. Entropy 2018, 20, 472

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Abstract: Section 4 of “Naudts J. Quantum Statistical Manifolds. Entropy 2018, 20, 472” contains errors. They have limited consequences for the remainder of the paper. A new version of this Section is found here. Some smaller shortcomings of the paper are taken care of as well. In particular, the proof of Theorem 3 was not complete, and is therefore amended. Also, a few missing references are added.

Theorem 1.

Theorem 2.

1. Corrections in Section 3

The display on top of page 5 should read

\[ ||f_{\rho,K}|| = \sup_{A \in A} \{ f_{\rho,K}(A) : ||A|| \leq 1 \} \]
\[ = \sup_{A \in A} \{ (\pi(A)K\Omega_{\rho},\Omega_{\rho}) : ||A|| \leq 1 \} \]
\[ = ||K^{1/2}\Omega_{\rho}||^2 \]
\[ \leq ||K^{1/2}K||^2 = ||K||. \]

The operator \( K \) is replaced by \( |K| \) because \( K \) need not be positive.

The sentence “This is a prerequisite for proving in the next Theorem that this map is the Fréchet derivative of the chart \( \xi_{\rho} \)” should read “This is a prerequisite for proving in the next Theorem that this map is the Fréchet derivative of the inverse of the chart \( \xi_{\rho} \)”.

The proof of the following Theorem is amended.

**Theorem 3.** The inverse of the map \( \xi_{\rho} : M \mapsto B_{\rho}, \) defined in Theorem 2, is Fréchet-differentiable at \( \omega = \omega_{\rho} \). The Fréchet derivative is denoted \( F_{\rho} \). It maps \( K \) to \( f_{\rho,K} \), where the latter is defined by (10).

**Proof.** Let \( K = \xi_{\rho}(\omega_{\rho}) \). One calculates

\[ ||\omega_{\rho} - \omega_{\rho} - F_{\rho}K|| = \sup_{A \in A} \{ ||\omega_{\rho}(A) - \omega_{\rho}(A) - F_{\rho}K(A)|| : ||A|| \leq 1 \} \]
\[ = \sup_{A \in A} \{ ||(\pi(A)K\Omega_{\rho},\Omega_{\rho})|| : ||A|| \leq 1 \} \]
\[ \leq ||e^{K - \alpha(K)} - I - K|| \]
\[ \leq ||K|| + \alpha(||K - \alpha(K)||). \] (11)

Note that

\[ ||\alpha(K)|| \leq \log ||e^K|| \leq ||K|| \]
and

$$||K - a(K)|| \leq 2||K||.$$ 

In addition, if $$||K|| < 1$$ then one has

$$a_\rho(K) \leq \log(1 + ||K\Omega_\rho||^2) \leq ||K\Omega_\rho||^2.$$ 

This holds because $$\lambda \leq 1$$ implies $$\exp(\lambda) \leq 1 + \lambda + \lambda^2$$. One concludes that (11) converges to 0 faster than linearly as $$||K||$$ tends to 0. This proves that $$F_\rho K$$ is the Fréchet derivative of $$\xi_\rho(\omega_\sigma) \mapsto \omega_\sigma$$ at $$\sigma = \rho$$. □

### 2. New Version of Section 4

Propositions 1 and 2 of [1] are not correct. This only has consequences for one sentence in the Introduction of [1] and for the results reported in Section 4 of [1]. The text in the Introduction “Next, an atlas is introduced which contains a multitude of charts, one for each element of the manifold. Theorem 4 proves that the manifold is a Banach manifold and that the cross-over maps are linear operators.” should be changed to “Next, an atlas is introduced which contains a multitude of charts, one for each element of the manifold. Theorem 4 proves that the manifold is a Banach manifold and that the cross-over maps are continuous.”

A new version of Section 4 follows below:

### 4. The Atlas

Following the approach of Pistone and collaborators [1,3,4,24], we build an atlas of charts $$\xi_\rho$$, one for each strictly positive density matrix $$\rho$$. The compatibility of the different charts requires the study of the cross-over map $$\xi_{\rho_1}(\sigma) \mapsto \xi_{\rho_2}(\sigma)$$, where $$\rho_1, \rho_2, \sigma$$ are arbitrary strictly positive density matrices.

Simplify notations by writing $$\xi_1$$ and $$\xi_2$$ instead of $$\xi_{\rho_1}$$, respectively $$\xi_{\rho_2}$$. Similarly, write $$\Omega_1$$ and $$\Omega_2$$ instead of $$\Omega_{\rho_1}$$, respectively $$\Omega_{\rho_2}$$, and $$F_1, F_2$$ instead of $$F_{\rho_1}$$, respectively $$F_{\rho_2}$$.

**Proposition 1. RETRACTED**

Continuity of the cross-over map follows from the continuity of the exponential and logarithmic functions and from the following result.

**Proposition 2.** Fix strictly positive density matrices $$\rho_1$$ and $$\rho_2$$. There exists a linear operator $$Y$$ such that for any strictly positive density matrix $$\sigma$$ and corresponding positive operators $$X_1, X_2$$ in the commutant $$\mathcal{A}'$$ one has $$X_2 = YX_1Y^*$$. 

**Proof.** Using the notations of the Appendix of [1], one has

$$X_i = J_i(\rho_i^{-1/2}r\rho_i^{-1/2} \otimes I)J_i^*, \quad i = 1, 2.$$ 

Note that the isometry $$J$$ depends on the reference state with density matrix $$\rho$$. Therefore, it carries an index $$i$$. The above expression for $$X_i$$ implies that

$$X_2 = YX_1Y^* \quad \text{with} \quad Y = J_2(\rho_2^{-1/2}r\rho_1^{1/2} \otimes I)J_1^*.$$ 

□
**Theorem 4.** The set $\mathcal{M}$ of faithful states on the algebra $\mathcal{A}$ of square matrices, together with the atlas of charts $\xi_\rho$, where $\xi_\rho$ is defined by Theorem 1, is a Banach manifold. For any pair of strictly positive density matrices $\rho_1$ and $\rho_2$, the cross-over map $\xi_2 \circ \xi_1^{-1}$ is continuous.

**Proof.** The continuity of the map $X_1 \mapsto X_2$ follows from the previous Proposition. The continuity of the maps $K_1 \mapsto X_1$ and $X_2 \mapsto K_2$ follows from the continuity of the exponential and logarithmic functions and the continuity of the function $\alpha$. □

### 3. Corrections in Section 9

In the proof of Proposition 4, the symbol $\Omega_\rho$ is missing five times in obvious places. It has been added.

### 4. Added References

In the overview of papers devoted to the study of the quantum statistical manifold in the finite-dimensional case, the references [2,3] should be added. A quantum version of the work of Pistone and Sempi [4], alternative to [5], is found in [6]. Reference [7] to the work of Ciaglia et al. has been updated.

**References**

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