Generalized System of Riccati-Type Equations

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Abstract. A new system of generalized Riccati-type equations is derived. An interconnection between the solutions of n-th order differential equations and the solutions of a generalized system of Riccati-type equations is established. Inverse mapping from the solutions of generalized Riccati-type equations onto the linearly independent solutions of the n-th order differential equation is constructed.

1 Introduction

The structure of the standard Riccati equation is defined in terms of a first order derivative and a second order polynomial. The Riccati equation is associated foremost with differential equation and the Möbius transformation [1]. Analogously, the generalized system of n-th order Riccati-type equations is also associated with the n-th order differential equations. In the present paper this problem is formulated as follows.

Consider the evolution equation with respect to a parameter t generated by the finite dimensional operator H

\[ \frac{d}{dt} \Psi(t) = H \Psi(t), \quad \Psi(0) = \Psi_0. \]  

(1)

the direct closed-form solution of which is given by the formula

\[ \Psi(t) = \exp(tH) \Psi_0. \]

The finite dimensional operator H is represented by an \( n \times n \) matrix which obeys the characteristic polynomial equation

\[ f(H) = 0. \]  

(2)

As a matter of convenience let us suppose that the characteristic polynomial coincides with the minimal polynomial

\[ f(X) = X^n + \sum_{k=1}^{n} (-1)^k a_k X^{n-k}, \quad a_k \in \mathbb{C}. \]  

(3)

Let E be a companion matrix of the polynomial f(X) associated with operator H. The companion matrix obeys the same characteristic equation, namely,

\[ f(E) = 0. \]  

(4)

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Besides the evolution equation generated by the operator $H$ one may consider an evolution generated by the polynomial $f(X)$. This evolution is described by the $n$-th order Riccati equation

$$\frac{d}{dt} U = f(U).$$

(5)

The aim of the present contribution is to establish a mapping between solutions of the equations (1) and (5).

2 Generalized trigonometric functions as solutions of high-order Riccati equation

In the same way as the usual complex algebra induces the trigonometry, the general complex algebra $GC_n$ induces representations of the set of generalized trigonometric functions [2], [3]. A matrix representation of the $GC_n$ algebra is given by the companion matrix. The companion matrix $E$ is the representation of the equivalent class of all $n \times n$ matrices with trace $a_1$, determinant $a_n$ and the sum of corresponding minors $a_i, 2, \ldots, n - 1$. The explicit form of this matrix is defined as follows

$$E = \begin{pmatrix}
0 & 0 & 0 & 0 & (-1)^{n+1}a_n \\
1 & 0 & 0 & 0 & (-1)^{n}a_{n-1} \\
0 & 1 & 0 & 0 & (-1)^{n-1}a_{n-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & -a_2 \\
0 & 0 & \cdots & 0 & 1 & a_1
\end{pmatrix}. \quad (6)
$$

An analogy of the Euler formula is defined by the series

$$\exp \left( \sum_{k=1}^{n-1} E^k \phi_k \right) = Q(E),$$

where the polynomial $Q(U)$ denotes an $(n - 1)$-degree polynomial of the form

$$Q(U) = g_0(\phi) + U g_1(\phi) + U^2 g_2(\phi) + \cdots + U^{n-1}g_{n-1}(\phi).$$

(8)

The parameter $\phi$ stands for the set of $(n - 1)$ parameters $\phi := (\phi_1, \phi_2, \phi_3, \ldots, \phi_{n-1})$.

The structure of the set of differential equations for generalized trigonometric functions $g_0(\phi)$, $g_1(\phi)$, $g_2(\phi)$, $\ldots$, $g_{n-1}(\phi)$ is governed by the matrix $E$ and its degrees $E^k, k = 1, \ldots, n - 1$ formulated in the standard way:

$$\frac{\partial}{\partial \phi_k} \mathbf{v}^\phi(\phi) = E^k \mathbf{v}^\phi(\phi), \quad \phi = (\phi_1, \phi_2, \ldots, \phi_{n-1}), \quad k = 1, \ldots, n - 1;$$

(9)

where $\mathbf{v}^\phi(\phi)$ means a vector of components

$$\mathbf{v}^\phi = [g_0, g_1, g_2, \ldots, g_{n-1}]^T.$$

(10)

As proved in [4], the differential equations (2.9) are reduced to an $n$-th order Riccati equation

$$\frac{d}{d\phi_{n-1}} U = f(U),$$

(11)
under the set of constraints

\[ g_k(\phi) = 0, \quad k = 2, 3, \ldots, n - 1. \]  

(12)

In this approach the solution of the \( n \)-th order Riccati equation is defined as a fraction of two trigonometric functions

\[ U(\phi_{n-1}) = \frac{g_0(\phi_{n-1})}{g_1(\phi_{n-1})}, \]  

(13)

where \( \phi_{n-1} \) depends of \( (n - 2) \) parameters \( \phi_{n-1}(\phi_1, \phi_2, \ldots, \phi_{n-2}) \), this dependence in a implicit way is defined by the constraints (12). The transformation of the linear system of evolution equations into the canonical form of the \( n \)-th order Riccati equation requires the \( n - 2 \) constraints (12). Under these constraints the polynomial \( Q(U) \) of order \( (n - 1) \) is reduced to a linear function of the form

\[ Q(U) = g_0 + U g_1. \]  

(14)

Then, the solution of the equation \( Q(U) = 0 \) turns out to the solution to the \( n \)-th order Riccati equation (11). This observation prompts us the idea to seek differential equations for the roots of the polynomial \( Q(U) \) of order \( (n - 1) \). As a result, we get a system of Riccati-type equations for the functions

\[ u_k = u_k(\phi), \quad k = 1, 2, 3, \ldots, n - 1, \quad \phi = (\phi_1, \phi_2, \ldots, \phi_{n-1}), \]  

(15)

where \( u_k \) are roots of the polynomial \( Q(u_k) \).

### 3 System of Riccati-type equations

Consider the polynomial of the \( (n - 1) \)-th order,

\[ Q(U) = \sum_{j=0}^{n-1} U^j g_j(\phi), \]  

(16)

with roots \( u_k(\phi), k = 1, 2, 3, \ldots, n - 1; \phi = (\phi_1, \phi_2, \ldots, \phi_{n-1}) \). The coefficients of the polynomial \( g_j(\phi), j = 0, 1, 2, \ldots, n - 1 \) are solutions of the system of evolution equations

\[ \partial_i g_j = \sum_{m=1}^{n} (E^i)^m g_{m-1}, \quad i = 1, \ldots, n - 1. \]  

(17)

The main result is given by the following

**Theorem.** The functions \( u_k(\phi), k = 1, \ldots, n - 1 \) obey the following system of nonlinear equations

\[ F(u_m) \sum_{k=1}^{n-p} a_{n-k-p} \partial_k u_m = A_p f(u_m), \quad m = 1, \ldots, n - 1. \]  

(18)

where \( F(u_m) \) is the \( (n - 2) \)-degree truncated polynomial

\[ F(u_m) = \frac{dQ(U)}{dU} \big|_{U=u_m} = u_m^{n-2} + \sum_{k=0}^{n-3} u_m^k A_k(m) = \prod_{k=1, k \neq m}^{n-1} (u_m - u_k), \]  

(19)

and \( A_p(m) \) is the \( p \)-th coefficient of the polynomial \( F(u_m) \).
The explicit form of the system of equations (19) is presented as follows

\[ F(U) (\partial_n - a_1 \partial_{n-2} + a_2 \partial_{n-3} - a_3 \partial_{n-4} + \cdots + (-1)^{n-1} a_{n-2} \partial_1) U = A_{n-1} f(U), \]

\[ \cdots \]

\[ F(U) (\partial_k - a_1 \partial_{k-1} + a_2 \partial_{k-2} - a_3 \partial_{k-3} + \cdots + (-1)^k a_{k-1} \partial_1) U = A_k f(U), \]

\[ \cdots \]

\[ F(U) (\partial_5 - a_1 \partial_4 + a_2 \partial_3 - a_3 \partial_2 + a_4 \partial_1) U = A_5 f(U), \]

\[ F(U) (\partial_4 - a_1 \partial_3 + a_2 \partial_2 - a_3 \partial_1) U = A_4 f(U), \]

\[ F(U) (\partial_3 - a_1 \partial_2 + a_2 \partial_1) U = A_3 f(U), \]

\[ F(U) (\partial_2 - a_1 \partial_1) U = A_2 f(U), \]

\[ F(U) \partial_1 U = A_1 f(U). \]  \hspace{1cm} (20)

If the basic polynomial \( f(X) \) is a polynomial of the \( n \)-th order then the functions \( u_k, k = 1, \ldots, n - 1 \) are roots of the following polynomial of the order \( (n - 1) \):

\[ Q(U) = g_{n-1} U^n + \cdots + g_3 U^3 + g_2 U^2 + g_1 U + g_0. \]  \hspace{1cm} (21)

Each of the roots \( u_k \) obeys the system of differential equations (20), where

\[ F(U) = A_1 U^{n-2} + A_2 U^{n-3} + \cdots + A_{n-2} U + A_{n-1}, \]  \hspace{1cm} (22)

and

\[ A_1 = 1, \quad -A_2 = V + W + Y + Z, \quad A_3 = VW + VY + VZ + WY + WZ + YZ + \cdots. \]  \hspace{1cm} (23)

### 4 Conclusion

We have derived the system of Riccati-type equations from the linear system of evolution equations. The method can be applied in the problem of transformation of the \( n \)-th order differential equation into a Riccati-type equation. It is expected, the present method will be useful in the theory of finite dimensional quantum mechanics.

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