The Standard Model as an extension of the noncommutative algebra of forms

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(Dated: April 16, 2015)

The Standard Model of particle physics can be deduced from a small number of axioms within Connes’ noncommutative geometry (NCG). Boyle and Farnsworth [New J. Phys. 16 (2014) 123027] proposed to interpret Connes’ approach as an algebra extension in the sense of Eilenberg. By doing so, they could deduce three axioms of the NCG Standard Model (i.e. order zero, order one and massless photon) from the single requirement that the extended algebra be associative. However, their approach was only applied to the finite part of the model because it fails for the full model.

By taking into account the differential graded structure of the algebra of noncommutative differential forms, we obtain a formulation where the same three axioms are deduced from the associativity of the extended differential graded algebra, but which is now compatible with the full Standard Model.

PACS numbers: 02.40.Gh, 11.10.Nx, 11.15.-q

I. INTRODUCTION

Noncommutative geometry provides a particularly elegant way to derive and describe the structure and the Lagrangian of the Standard Model in curved spacetime and its coupling to gravitation. The main ingredients of this approach is an algebra \( A = C^\infty(M) \otimes A_F \) (where \( M \) is a Riemann spin manifold and \( A_F = C \oplus G(\mathbb{R}^3) \)) a Hilbert space \( H = L^2(M, S) \otimes H_F \) (where \( H_F \) is 96-dimensional), and a Dirac operator \( D \).

The elements \( a \) of the algebra are represented by bounded operators \( \pi(a) \) over \( H \). In this approach, the gauge bosons are described by gauge potentials (i.e. noncommutative one-forms) in \( \Omega^1_\mathcal{A} \), where \( \Omega^1_\mathcal{A} \) is the differential graded algebra (DGA) constructed from \( \mathcal{A} \), whose differential is calculated by using the commutator with \( D \).

From the physical point of view, a striking success of the noncommutative geometrical approach is that the algebra, the Hilbert space and the Dirac operator of the Standard Model can be derived from a few simple axioms, including the condition of order zero, the condition of order one and the condition of massless photon. This axiom requires the Dirac operator \( D \) to commute with a specific family of elements of \( \mathcal{A} \), whose differential is calculated by using the commutator with \( D \).

However, the approach proposed by Boyle and Farnsworth has two serious drawbacks: i) it is not valid for a spin manifold (i.e. the canonical spectral triple \((C^\infty(M), L^2(M, S), D_M)\) does not satisfy the condition of order two); ii) it uses the DGA algebra \( \Omega \) in which gauge fields with vanishing representation (i.e. \( \pi(A) = 0 \)) can have non-zero field intensity (i.e. \( \pi(dA) \neq 0 \)). This makes the Yang-Mills action ill defined. A consistent substitute for \( \Omega \) is the space \( \Omega_D \) of noncommutative differential forms which is a DGA built as the quotient of \( \Omega \) by a differential ideal \( J \) usually called the junk.

To solve both problems, we define an extension \( \mathcal{E} \) of the physically meaningful algebra \( \Omega_D \) of noncommutative differential forms by a representation space \( M_D \) that we build explicitly. Since the algebra \( \Omega_D \) is a DGA, we require the extension \( \mathcal{E} \) to be also a DGA and we ob-
tain that \( \mathcal{M}_D \) must be a differential graded bimodule over \( \Omega \) (see below). The most conspicuous consequence of this construction is a modification of the condition of order two proposed by Boyle and Farnsworth, which provides exactly the same constraints on the finite part of the spectral triple of the Standard Model, but which is now consistent with the spectral triple of a spin manifold. As a consequence, the full spectral triple of the Standard Model (and not only its finite part) now satisfies the condition of order two and enables us to remove the condition of massless photon. In the next section, we describe the extension of an algebra by a vector space, first proposed by Eilenberg and used by Boyle and Farnsworth. We discuss the modification required to take the differential graded structure into account. Then, we describe the construction by Connes and Lott of noncommutative differential forms and we build a space \( \mathcal{M}_D \) that can be used to extend \( \Omega_D \) into a DGA \( \mathcal{E} \). Finally, we show that the spectral triple of the Standard Model fits into this framework if and only if the condition of massless photon is satisfied.

II. EXTENSION OF ALGEBRAS

The cohomology of Lie algebras plays a crucial role in the modern understanding of classical and quantum gauge field theories. This cohomology theory mixes the gauge Lie algebra and its representation over a vector or spinor bundle. Shortly after the publication of the cohomology of Lie algebras, Eilenberg generalized this idea to the representation of any algebra whose product is defined by a bilinear map with possible linear constraints (Lie, associative, Jordan, commutative, etc.).

A. Eilenberg’s extension

If \( A \) is a (possibly non-associative) algebra with product \( a \cdot b \) and \( V \) is a vector space, then the (possibly non-associative) algebra \( (E, \star) \) is an extension of \( A \) by \( V \) if \( V \subset E \) and there is a linear map \( \varphi : E \to A \) such that \( \varphi(e) = 0 \) iff \( e \in V \), \( \varphi(e \cdot e') = \varphi(e) \cdot \varphi(e') \) for every \( e \) and \( e' \) in \( E \) and \( u \cdot v = 0 \) when \( u \) and \( v \) are in \( V \).

Eilenberg showed that, if \( \eta : A \to E \) is a map such that \( \eta(a) \) represents \( a \) in \( E \) (i.e. for every \( a \in A \), \( \varphi(\eta(a)) = a \) ), then \( \eta(a \cdot b) = \eta(a) \star \eta(b) + f(a, b) \), where \( f \) is a bilinear map \( A \times A \to V \). The product \( \star \) in \( E \) induces two bilinear maps \( \star : (a, v) \mapsto a \star v = \eta(a) \star v \) and \( \cdot : (v, a) \mapsto v \cdot a = v \star \eta(a) \) and it can be shown that \( a \star v \) and \( v \cdot a \) are in \( V \) and independent of \( \eta \). Conversely, a product in \( A \) and two bilinear maps \( \cdot \) and \( \star \) determine an extension \( E \) of \( A \) by \( V \) and a product \( \cdot \), which are unique up to an equivalence determined by \( f \).

Noncommutative geometry belongs to this framework if we define \( A = A, V = \mathcal{H}, a \cdot v = \pi(a) v \), where \( \pi(a) \) is the representation of \( a \) in the space \( B(\mathcal{H}) \) of bounded operators on \( \mathcal{H} \) and \( v \cdot b = \pi(b) v \), where \( \pi(b) = J \pi(b) J^{-1} \) and \( J \) is an antilinear isometry called the real structure.

Notice that, in the last expression, the right action \( v \cdot b \) of \( b \) on \( v \) is replaced by the left product by \( \pi(b)^* \) on \( v \). This remark will turn out to be crucial. When there is no ambiguity, we sometimes use a common abuse of notation and write \( a \star (a \cdot b) = a \cdot (a \cdot b) = (a \cdot a) \cdot b \).

Then, Eilenberg showed that the extension \( E \) is associative iff the following conditions are satisfied:

\[
\begin{align*}
1. & \quad a \cdot (b \cdot c) = (a \cdot b) \cdot c, \\
2. & \quad a \cdot (b \cdot v) = (a \cdot b) \cdot v, \\
3. & \quad (v \cdot a) \cdot b = v \cdot (a \cdot b), \\
4. & \quad (a \cdot v) \cdot b = a \cdot (v \cdot b), \\
5. & \quad a \cdot f(b, c) + f(a, b \cdot c) = f(a, b, c) + f(a, b) \cdot c.
\end{align*}
\]

Condition (1) means that \( A \) is an associative algebra, condition (2) that \( \cdot \) is a left action of \( A \) on \( V \), condition (3) that \( \cdot \) is a normal action of \( A \) on \( V \), condition (4) that the right and left actions are compatible (i.e. that \( V \) is a bimodule), the map \( f \) in condition (5) is required for the extension to have better functorial properties but we do not use it here and we consider the case \( E = A \oplus V \), \( \varphi(a + v) = a, \varphi = \text{Id} \) and \( f = 0 \). In the NCG example of the extension of \( A \) by \( H \) that we gave at the beginning of the paragraph, condition (5) becomes \( \pi(b)^* \pi(a) = \pi(a) \pi(b)^* \), which is the condition of order zero of NCG usually written \( [a, b^*] = 0 \).

B. Differential graded-representations of a DGA

We noticed in the introduction that \( \Omega \) and \( \Omega_D \) are DGA. It is now time to explain what that means. A graded vector space is a direct sum \( V = \bigoplus_{n \geq 0} V^n \) of vector spaces. If \( v \in V^n \) we say that the degree of \( v \) is \( |v| = n \). A DGA is a graded vector space \( A \) equipped with an associative product \( \cdot \) and a differential \( \delta \). The product of this algebra satisfies \( [a \cdot b] = [a] + [b] \). The differential satisfies \( \delta A^n = A^{n+1}, \delta^2 = 0 \) and the graded Leibniz rule \( \delta(a \cdot b) = (\delta a) \cdot b + (-1)^{|a|} a \cdot (\delta b) \). Differential graded algebras are a basic tool of cohomological physics.

A graded left-representation of \( A \) is a graded vector space \( \mathcal{M} \) with a left action \( \circ \) of \( A \) over \( \mathcal{M} \) such that \( [a \circ m] = [a] + [m] \), with a similar definition for a graded right-representation. A differential graded left-representation of \( A \) is a graded left-representation \( \mathcal{M} \) furnished with a differential \( \delta : \mathcal{M}^n \to \mathcal{M}^{n+1} \) such that:

\[
\delta(a \circ m) = (\delta a) \circ m + (-1)^{|a|} a \circ (\delta m). \tag{6}
\]

Similarly, in a differential graded right-representation:

\[
\delta(m \circ a) = (\delta m) \circ a + (-1)^{|m|} m \circ (\delta a). \tag{7}
\]

The signs in Eqs. (6) and (7) are imposed by the fact that an algebra is a left and right representation of itself. By a straightforward generalization of Eilenberg’s result, we see that \( E \) is an extension of \( A \) by \( \mathcal{M} \) as a DGA.
iff $\mathcal{M}$ is a differential graded bimodule over $A$ (i.e. $\mathcal{M}$ is a differential graded left-representation, a differential graded right-representation and the left and right actions are compatible in the sense of Eq. (1)).

We saw in section [14] that in the NCG framework, the right action of an element $a$ of the algebra is represented as a left product by the operator $a^\circ$. We want to retain this type of representation for $\mathcal{M}$: we want to define a linear map $a \mapsto a^\circ$ such that the right action by $a$ (i.e. $m \circ a$) is represented by the left product with $a^\circ$ (i.e. $a^\circ m$). We do not assume that $a^\circ$ belongs to the algebra $A$ but we require that $|a^\circ| = |a|$. However, the representation of $m \circ a$ by $a^\circ m$ is not as simple as when $A$ is only an algebra because compatibility with the DGA structure imposes the following sign [14]:

$$a^\circ m = (-1)^{|a||m|} m \circ a.$$  \hspace{1cm} (8)

Indeed, $\delta(a^\circ m) = \delta(a^\circ) m + (-1)^{|a||\delta a|} \delta m$ implies

$$(-1)^{|a||m|} \delta(m \circ a) = (-1)^{|a|+|m|} m \circ \delta a$$

$$+ (-1)^{|a|+|\delta a||m|+1} \delta m \circ a,$$

and we recover Eq. (1).

The map $\circ$ is compatible with the differential graded bimodule structure if the following conditions hold for every $a$ and $b$ in $A$:

$$a^\circ b^\circ = (-1)^{|a||b|}(b \cdot a)^\circ,$$  \hspace{1cm} (9)

$$a^\circ b = (-1)^{|a||b|} b^\circ a^\circ, \hspace{1cm} (10)$$

$$\delta(a^\circ) = (\delta a)^\circ.$$  \hspace{1cm} (11)

Equation (9) follows from Eqs. (1) and (8). Eq. (10) follows from Eqs. (1) and (8). To derive Eq. (11), we apply transformation (8) to Eq. (7) to obtain $\delta(a^\circ m) = (-1)^{|a||\delta a|} \delta m + (\delta a)^\circ m$ and we compare with the expression for $\delta(a^\circ m)$ given after Eq. (8).

### III. NONCOMMUTATIVE DIFFERENTIAL FORMS

If $\mathcal{A}$ is an algebra, its associated universal differential graded algebra $\Omega = \bigoplus_{n \geq 0} \Omega^n$ is defined as follows [15,16]. In degree zero $\Omega^0 = \mathcal{A}$. The space $\Omega^n$ is generated by the elements $a_0(a_1) \ldots (a_n)$, where $a_0, \ldots, a_n$ are elements of $\mathcal{A}$ and $\delta$ is a linear operator satisfying $\delta^2 = 0$, $\delta(a) = (\delta a)$, and $\delta(\omega \rho) = (\delta \omega \rho + (-1)^{|\omega|} \omega \delta \rho)$.

In NCG, $\delta a$ is represented over $\mathcal{H}$ as the (bounded) operator $[D,a]$ and the $n$-form $\omega = a_0(a_1) \ldots (a_n)$ is represented by $\pi(\omega) = \pi(a_0)[D,\pi(a_1)] \ldots [D,\pi(a_n)]$ where $\pi$ becomes now a $*$-representation of $\Omega$. However, this representation is not graded (a fact which is sometimes overlooked) because $\pi(\Omega) \subset B(\mathcal{H})$. To obtain a graded representation we replace $\pi$ by $\tilde{\pi}: \Omega \to B^\infty(\mathcal{H}) = \bigoplus_n V^n$ where each $V^n$ is $B(\mathcal{H})$: if $\omega \in \Omega^n$, then $\tilde{\pi}(\omega) \in V^n$ (we shall shortly explain why we cannot take $V^n = \mathcal{H}$). However the difference between $\pi$ and $\tilde{\pi}$ is invisible as long as we only consider homogeneous elements and are always careful about their degree. This is why we will stick to the notation $\pi$ in the sequel when no confusion can arise.

The representation $\pi$ (i.e. $\tilde{\pi}$) is now a graded $*$-representation of $\Omega$. However, it is not a well-defined representation of the differential because there can be $n$-forms $\omega$ such that $\pi(\omega) = 0$ and $\pi(\delta \omega) \neq 0$, as we illustrate now with the spectral triple of a spin manifold.

Let $f$ and $g$ be two functions in $C^\infty(M)$. They are represented over $\mathcal{H} = C^\infty(M,S)$, where $S$ is the spin bundle, by multiplication: $\pi(f) = f \psi$. Then, $\delta f$ is represented by $\pi(\delta f) = [D_M,f] = \sum_{\mu} \gamma^\mu \partial_\mu f$, where $\gamma^\mu$ runs over the $\gamma$-matrices of the spin bundle. If we consider $\omega = g(\delta f) - (\delta f) g$, then $\pi(\omega) = \sum_{\mu}(g \partial_\mu f - \partial_\mu g f)\gamma^\mu = 0$ because the functions $g$ and $\partial_\mu f$ commute. However, $\delta \omega = (\delta g)(\delta f) + (\delta f)(\delta g)$ by the graded Leibniz rule and $\pi(\delta \omega)$ is generally not zero because

$$\pi(\delta \omega) = \sum_{\mu \nu} \partial_\mu f \partial_\nu g(\gamma^\nu \gamma^\mu + \gamma^\mu \gamma^\nu)$$

$$= 2 \sum_{\mu \nu} g^{\mu \nu} \partial_\mu f \partial_\nu g \mathbb{I} = 2(\partial f) \cdot (\partial g) \mathbb{I},$$

where $\mathbb{I}$ is the unit matrix in the spinor fibre.

For a general spectral triple, Connes and Lott [17] remove all the badly-behaving forms by defining the junk $J = J_0 + \delta J_0$, where $J_0 = \bigoplus_{n \geq 0} J^n_0$ and $J^n_0 = \{ \omega \in \Omega^n | \pi(\omega) = 0 \}$. The ideal $J_0$ is the kernel of $\pi$ but not the kernel of $\pi$. The term $\delta J_0$ is needed because $J_0$ is a graded ideal of $\Omega$ but not a differential ideal (i.e. $\delta J_0$ is generally not a subset of $J_0$). But $J$ is a graded differential ideal of $\Omega$ because $\delta^2 = 0$ implies $\delta J = \delta J_0 \subset J$ and $\Omega_D = \Omega/J$ is now a well-defined DGA called the space of noncommutative differential forms of the spectral triple. Moreover [16],

$$\Omega_D = \Omega/J \cong \bigoplus_{n \geq 0} \pi(\Omega^n)/\pi(\delta J_0^{n-1}).$$  \hspace{1cm} (12)

For a spin manifold, $\Omega_D$ is then isomorphic to the usual space $\Gamma(M,\Lambda T^*M)$ of differential forms on $M$.

Why don’t we represent $\Omega$ over $\mathcal{H}$? Indeed, since $\mathcal{A}$ is represented over $\mathcal{H}$, it would be tempting to represent $\Omega$ over a graded version of $\mathcal{H}$ (i.e. a graded vector space $\mathcal{V}$ where every $\mathcal{V}^n = \mathcal{H}$) and to represent $\Omega_D$ as some quotient. However, in such a picture we would have to represent $\Omega_D$ over the graded vector space $W$ where $W^n = \pi(\Omega^n)/\pi(\delta J_0^{n-1})\mathcal{H}$ and this quotient is often trivial. For the example of the spin manifold, we saw that $2(\partial f \cdot \partial g) \mathbb{I}$ belongs to $\pi(\delta J_0^1)$. As a consequence, $\pi(\delta J_0^1)\mathcal{H} = \mathcal{H}$ and $M^2 = \{0\}$.

Our purpose is now to extend $\Omega_D$ by $\mathcal{M}_D$ in the sense of Eilenberg, where $\mathcal{M}_D$ is a differential graded bimodule over $\Omega_D$ naturally defined out of the spectral triple data. This will be done in two steps. We first explain how a left-right graded representation of $\Omega$ can be viewed as a left graded representation of a certain algebra $B$. Then
we take the junk into account, what leads us to quotient $B$ by an ideal $K$. We then obtain a graded algebra $B/K$ which, under some condition, has the ability to produce differential graded $\Omega_D$-bimodules out of graded $B$-bimodules.

A. Left and right representations of $\Omega$ as left representations of $B$

The only mean at our disposal to produce a right action of $\Omega$ is by extending the definition of the map $x \mapsto x^0$ from $\pi(A)$ to $\pi(\Omega)$. We define:

$$\pi(\omega)^0 = (-1)^{|\omega|(|\omega|+1)/2}(\epsilon')|\omega|J\pi(\omega)^1J^{-1},$$

where $\epsilon'$ is such that $JD = \epsilon' DJ$. The sign is uniquely determined by $\pi(\delta a)^0 = [D, a^2] = \delta\pi(a)^0$ and condition (9). This definition is also compatible with the relation $(\delta a)^* = -\delta(a^*)$.

The receptacle for the objects we need to manipulate is the graded $\ast$-algebra generated by all the elements of the form $\pi(\omega)$ or $\pi(\omega)^0$ for $\omega \in \Omega$. We call $B = \bigoplus_{n \geq 0} B^n$ this algebra, where each $B^n \subset B(\mathcal{H})$ and we observe that the grading of $B$ follows from the grading of $\Omega$: $[\pi(a)] = [\pi(\omega)] = 0$, $[\pi(\delta a)] = [\pi(\delta a)^0] = 1$.

Consider now a graded left-representation $\mathcal{M}$ of $B$. Then $\mathcal{M}$ is automatically a graded left and right representation of $\Omega$ with the following actions for homogeneous elements $\omega \in \Omega$ and $m \in \mathcal{M}$:

$$\omega \triangleright m = \pi(\omega)m$$

$$m \triangleleft \omega = (-1)^{|\omega||m|}\pi(\omega)^0m,$$

Let us check that $\triangleleft$ indeed defines a right action:

$$(m \triangleleft \omega) \triangleleft \omega' = (-1)^{|\omega|(|m|+|\omega|)+|\omega'||m|}\pi(\omega')^0\pi(\omega)^0m$$

$$= (-1)^{|\omega|(|m|+|\omega|)+|\omega'||m|}\pi(\omega\omega')^0m$$

$$= m \triangleleft (\omega \triangleright \omega')$$

B. Bimodule over $\Omega_D$

Thus $\mathcal{M}$ is a graded left and right representation $\Omega$ but it is not a bimodule: the left and right actions are not compatible in general. Moreover, we saw that the elements of $\pi(\Omega)$ cannot be properly identified with differential forms which are given by the quotient of Eq. (12). Because of the isomorphism described by Eq. (12), we can consider an element of $\Omega_D$ from two equivalent points of views: either as a class $[\omega]$ of universal differential forms $\omega$, such that $[\omega] = [\omega']$ iff there is a $\eta$ and a $\rho$ in $J_0$ such that $\omega' = \omega + \rho + \delta\eta$, or as a class $\langle \alpha \rangle$ of elements of $\pi(\Omega)$ such that $\langle \alpha \rangle = \langle \alpha' \rangle$ iff there is an element $\eta$ of $J_0$ such that $\alpha' = \eta + \pi(\delta\eta)$. Since $J$ is an ideal of $\Omega$ and $\pi(\delta J_0)$ is an ideal of $\pi(\Omega)$, the product $[\omega][\omega'] = [\omega\omega']$ or $\langle \alpha \rangle\langle \alpha' \rangle = \langle \alpha\alpha' \rangle$ are well defined and $[\omega\omega'] = \langle \alpha\alpha' \rangle$ if $\alpha = \pi(\omega)$ and $\alpha' = \pi(\omega')$. Moreover, $\delta(\omega) = [\delta\omega] = (\pi(\delta\omega))$ is now a well-defined differential on $\Omega_D$.

Here, $\Omega_D$ was built as the quotient of $\pi(\Omega)$ by the ideal $\pi(\delta J_0)$. Similarly, we can define $\Omega_D^\circ$ as the graded quotient of $\pi(\Omega)^\circ$ by $\pi(\delta J_0)^\circ$. More precisely, we define $\Omega_D^\circ$ as the set of classes $\langle \alpha^\circ \rangle$ where $\langle \alpha^\circ \rangle = \langle \beta^\circ \rangle$ iff there is an $\eta \in J_0$ such that $\beta^\circ = \alpha^\circ + (\delta\eta)^\circ$. This defines a map $\Omega_D \to \Omega_D^\circ$ by $\langle \alpha \rangle^\circ = \langle \alpha \rangle$). Note that the product $\alpha^\circ\beta^\circ$ is well defined as a product in $B^\infty(\mathcal{H})$. Since $(\delta J_0)^\circ$ is an ideal in $\pi(\Omega)^\circ$ we can define similarly

$$\langle \alpha\beta \rangle^\circ = (-1)^{|\alpha||\beta|}(\beta\alpha)^\circ = (-1)^{|\alpha||\beta|}(\beta\alpha)^\circ,$$

where we used the fact that $(\alpha\beta)^\circ = (-1)^{|\alpha||\beta|}\alpha^\circ\beta^\circ$ in $B^\infty(\mathcal{H})$. Finally, the differential on $\Omega_D^\circ$ is compatible with $^\circ$ in the sense that $\delta(\alpha^\circ) = \delta(\alpha)^\circ = \langle \delta \alpha \rangle^\circ$ is well defined. Thus, the compatibility equations (9) and (11) are satisfied.

To complete the conditions on $^\circ$ we still have to satisfy Eq. (10). For this, we first must define the products $\langle \alpha \rangle^\circ(\beta)$ and $\langle \beta \rangle^\circ(\alpha)$. Since $\alpha$ and $\beta$ are elements of $B^\infty(\mathcal{H})$, the product $\alpha\beta^\circ$ is well defined in $B^\infty(\mathcal{H})$. Let us consider $\alpha' = \alpha + \pi(\delta\eta)$ and $\beta' = \beta + \pi(\delta\epsilon)$. Then

$$\alpha'(\beta')^\circ = \alpha\beta^\circ + \pi(\delta\eta)\beta^\circ + \alpha\delta\epsilon^\circ + \pi(\delta\epsilon)(\delta\epsilon)^\circ.$$

Since we need $\langle \alpha(\beta') \rangle^\circ = \langle \alpha(\beta) \rangle^\circ$ to be well defined, all the terms following $\alpha\beta^\circ$ must belong to an ideal $K$. By multiplying with other elements of $\pi(\Omega)$ or $\pi(\Omega)^\circ$, we see that $K$ is the graded ideal generated by $\pi(\delta J_0) + \pi(\delta J_0)^\circ$ in the graded algebra $B$. In $B/K$, the products $(\alpha)\beta^\circ$ and $\langle \beta \rangle^\circ(\alpha)$ are now well defined. Moreover, and this is an important check, if $\beta^\circ \in A$ (more precisely, if, for every $b \in A$, there is a $c \in A$ such that $\pi(c) = \pi(b)^\circ$), then $B = \pi(\Omega)$, $K = J$ and $B/K = \Omega_D$. Note that this is the case of the canonical spectral triple of a spin manifold because $^\circ = f$.

Since junk forms act on the right as well as on the left, $K$ is a graded ideal of $B$ and $B/K$ is a graded algebra. In the following, we shall use the representation $\mathcal{M}_D = B/K$ but more generally, any left representation $\mathcal{M}$ of $B$ gives rise to a left representation $\mathcal{M}_D = (B/K) \otimes_B \mathcal{M}$ of $B/K$ by extension of scalars, and what is more, $\mathcal{M}_D$ is automatically a left and right representation of $\Omega_D$. The left and right actions of $\Omega_D$ on $\mathcal{M}_D$ are explicitly given by:

$$[\omega] \triangleright m = (\pi(\omega))m,$$

$$m \triangleleft [\omega] = (-1)^{|\omega||m|}(\pi(\omega)^0)m.$$
for all homogeneous $\omega, \omega' \in \Omega$. Since $\Omega$ is generated as an algebra by elements of degree 0 and 1, it is equivalent to require that the usual order 0 and order 1 condition of spectral triples hold modulo $K$, which they obviously do, and that moreover:

$$\pi(\delta a)\pi(\delta b)^o + \pi(\delta b)^o\pi(\delta a) = 0 \mod K. \quad (14)$$

**IV. APPLICATION TO THE STANDARD MODEL**

We saw that condition $[10]$: $(\alpha)^o(\beta) = (-1)^{\alpha || \beta}(\beta)(\alpha)^o$ is equivalent to the four equations

$$\pi(a)b(a) - (b)\pi(a) = 0,$$
$$[D, \pi(a)](b) - (b)[D, \pi(a)] = 0,$$
$$[D, \pi(a)^o](b) - (b)[D, \pi(a)^o] = 0,$$
$$[D, \pi(a)\pi(b)] + [D, (b)\pi(a)] = 0 \mod K.$$  

The first equation is satisfied because it is the condition of order zero, the second equation is the condition of order one, the third equation is a consequence of the condition of order one, the fourth equation is called the condition of order two. It is new and we investigate it for the spectral triple $(A, H, D, J, \gamma)$ of the Standard Model, which is the tensor product of $C^\infty(M, L^2(M, S), D_M, J_M, J_M, \gamma_M)$ and $(A_F, H_F, D_F, J_F, \gamma_F)$ and where $D = D_M \otimes \text{Id} + \gamma_5 \otimes D_F$. Let us first consider these four conditions in a tensor product of general even spectral triples.

**A. Tensor product of even spectral triples**

If $(A_1, H_1, D_1, J_1, \gamma_1)$ and $(A_2, H_2, D_2, J_2, \gamma_2)$ are even spectral triples with representation maps $\pi_1$ and $\pi_2$, then their tensor product is defined by $\mathcal{A} = A_1 \otimes A_2$, $\mathcal{H} = H_1 \otimes H_2$, $D = D_1 \otimes \text{Id}_2 + \gamma_1 \otimes D_2$, $J = J_1 \otimes J_2$ and $\gamma = \gamma_1 \otimes \gamma_2$. The conditions of order zero and one hold for this tensor product, but we must investigate the condition of order two. Let $a = a_1 \otimes a_2$ and $b = b_1 \otimes b_2$, by using the fact that $\gamma_1$ is unitary, self-adjoint, commutes with all elements of $A_1$ and anticommutes with $D_1$ we obtain

$$\{[D, a], [D, b]\} = \{[D_1, a_1], [D_1, b_1]\} \otimes a_2 b_2^* + a_1 b_1^* \otimes \{[D_2, a_2], [D_2, b_2]\}$$

and the condition of order two means that this anticommutator must belong to the junk $\mathcal{K}$ of the tensor product.

In general, the universal DGA $\Omega$ built from $\mathcal{A} = (A_1 \otimes A_2)$ is different from the DGA $\Omega_1 \otimes \Omega_2$ and the expression of $\Omega_D$ in terms of $\Omega_{D1}$ and $\Omega_{D2}$ is rather intricate.

However, when $A_1 = C^\infty(M)$ and $A_2 = A_F$ the situation is simpler and it can be shown that

$$\pi(\delta J_0^1) = \pi_M(\delta J_0^1_M) \otimes \pi(A_F) + C^\infty(M) \otimes \pi(\delta J_0^1_F),$$

where $\pi_\otimes = \pi_M \otimes \pi$.

The space $K^2$ of elements of degree two in the junk of the tensor product is

$$K^2 = \pi(\delta J_0^1) \otimes \pi(A_F) + \pi(\delta J_0^1) \otimes \pi(A_F) + C^\infty(M) \otimes \pi(\delta J_0^1_F).$$

More precisely,

$$K^2 = \pi_M(\delta J_0^1_M) \otimes \pi(A_F) \otimes \pi(A_F) + C^\infty(M) \otimes \pi(\delta J_0^1_F) \otimes \pi(A_F),$$

which must be completed by $\pi_M(\delta J_0^1_M) = C^\infty(M)[],$ where $[]$ is the identity of the spinor bundle.

To summarize this discussion, the condition of order two is satisfied for the tensor product $\mathcal{A} = C^\infty(M) \otimes A_F$ if and only if it is satisfied for $C^\infty(M)$ and the anticommutator $\{[D_F, a], [D_F, b]\}$ belongs to $\pi(A_F) \otimes \pi(A_F) + \pi(\delta J_0^1_F) \otimes \pi(A_F) + \pi(\delta J_0^1_F) \otimes \pi(A_F).$ This is what we are going to check in the next sections.

**B. Spin manifold**

For the spectral triple of a spin manifold, $\pi(f)^o = J f^* J^{-1} = J f J^{-1} = f$. Thus, $\pi(A)^o = \pi(A)$ and the right action of $C^\infty(M)$ over $C^\infty(M, S)$ is the same as the left action. As a consequence, $\mathcal{M}_{D_M} = \Omega_{D_M}$ is obviously a differential graded bimodule over itself and we do not need to check the condition of order two. Let us do it anyway by calculating

$$[D_M, f][D_M, g] + [D_M, g][D_M, f] = \pi(\delta \omega),$$

where $\omega = g(\delta f) - (\delta f) g$ was defined in section [11]. Since $\pi(\omega) = 0$, then $\pi(\delta \omega) \in \pi(\delta J_0)$ and $\{[D_M, f], [D_M, g]\}$ indeed belongs to the junk.

**C. The finite spectral triple**

An element of the finite algebra $A_F$ is parametrized by a complex number $\lambda$, a quaternion written as a pair of complex numbers $(\alpha, \beta)$ and a $3 \times 3$ matrix $\mu$. Since we consider only one generation, its representation over the $32$-dimensional Hilbert space $H_F$ is:

$$\pi(a) = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{pmatrix},$$

and

$$\pi(a)^o = \begin{pmatrix} C^T & 0 & 0 & 0 \\ 0 & C^T & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & B^T \end{pmatrix}.$$
The relations with the Yukawa matrix for the Dirac operator:

\[
C = 32 \begin{pmatrix}
\lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda
\end{pmatrix},
\]

and the condition of order one imply the following form:

\[
D = \begin{pmatrix}
\alpha & \beta & 0 & 0 & 0 & 0 & 0 & 0 \\
-\bar{\beta} & \bar{\alpha} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 & \beta & 0 & 0 & 0 \\
0 & 0 & \alpha & 0 & 0 & \beta & 0 & 0 \\
0 & 0 & 0 & \alpha & 0 & 0 & \beta & 0 \\
0 & 0 & -\bar{\beta} & 0 & 0 & \bar{\alpha} & 0 & 0 \\
0 & 0 & 0 & -\bar{\beta} & 0 & \bar{\alpha} & 0 & 0 \\
0 & 0 & 0 & 0 & -\bar{\beta} & 0 & 0 & \bar{\alpha}
\end{pmatrix},
\]

The antilinear real structure \( J \) acts by

\[ J \left( \sum_{i=1}^{32} v_i e_i \right) = \sum_{i=1}^{16} v_{i+16} e_i + \sum_{i=17}^{32} v_{i-16} e_i, \]

where \((e_1, \ldots, e_{32})\) is a basis of \( \mathcal{H}_F \).

\[
\gamma = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

The relations \( D_F^\dagger = D_F, D_F J = J D_F, D_F \gamma = -\gamma D_F \) and the condition of order one imply the following form for the Dirac operator:

\[
D_F = \begin{pmatrix}
0 & \gamma^T & M^\dagger & 0 \\
Y & 0 & 0 & 0 \\
0 & Y^T & M & 0 \\
0 & 0 & 0 & \gamma^T
\end{pmatrix},
\]

with the Yukawa matrix

\[
Y = \begin{pmatrix}
l_{11} & l_{12} & 0 & 0 & 0 & 0 & 0 & 0 \\
l_{21} & l_{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & q_{11} & 0 & 0 & q_{12} & 0 & 0 & 0 \\
0 & 0 & q_{11} & 0 & 0 & q_{12} & 0 & 0 \\
0 & 0 & 0 & q_{11} & 0 & 0 & q_{12} & 0 \\
0 & 0 & q_{21} & 0 & 0 & q_{22} & 0 & 0 \\
0 & 0 & 0 & q_{21} & 0 & 0 & q_{22} & 0 \\
0 & 0 & 0 & 0 & q_{21} & 0 & 0 & q_{22}
\end{pmatrix},
\]

where \( l_{ij} \) stands for \( y_{l,ij} \) and \( q_{ij} \) for \( y_{q,ij} \) in the notation used by Boyle and Farnsworth and the mass matrix

\[
M = \begin{pmatrix}
a & b & c_1 & c_2 & c_3 & 0 & 0 & 0 \\
b & 0 & d_1 & d_2 & d_3 & 0 & 0 & 0 \\
c_1 & d_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_2 & d_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
c_3 & d_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where \( a \) corresponds to the mass of the right-handed neutrino and the other parameters are eliminated by the condition of massless photon. For notational convenience we write \( c = (c_1, c_2, c_3) \) and \( d = (d_1, d_2, d_3) \).

We must check that \([D_F, a], [D_F, b^c]\) = 0 up to the junk. Since the condition of order two is of degree two, we need the junk of degree two, which was determined in section \( [V A] \). We do not try to determine \( K^2 \) more explicitly. We calculate the \( 32 \times 32 \) matrices \( m = \pi(\delta a')\pi(\delta b')\pi(c^e) \), \( n = \pi(\delta a)^e\pi(\delta b)^e\pi(c') \) and \( p = e' e^c \) for generic \( a, b, c, a', b', c', e \) and \( e' \) in \( A_F \) and we notice that there are 820 pairs of indices \((k, l)\) such that \( m_{kl} = n_{kl} = p_{kl} = 0 \) (see Fig. [A]).

Among the pairs of indices where the junk is zero, 68 of them correspond to matrix elements of \(([D_F, a], [D_F, b^c])\) that are not generically zero (see Fig. [B]). Since they cannot be compensated by the junk, the condition of order two implies that these 68 matrix elements must be equal to zero. Because of the symmetry generated by the adjoint \( \dagger \) and the \( ^o \) operations, this gives 17 different equations that can be grouped into three systems. Let the elements \( a \) and \( b \) of the algebra be parametrized by
λ, α, β, µ and X', α', β', µ', respectively. The first system (of two equations):

\[(α - λ)y_{11} + βy_{21})b(λ' - X') = 0,\]

\[(α - λ)y_{21} - βy_{11})b(λ' - X') = 0.\]

is solved by either \(b = 0\) or \(y_{11} = y_{21} = 0\) in the lepton Yukawa matrix. The second system (of 12 equations):

\[(α - λ)y_{11} + βy_{21})\left(\sum_{i=1}^{3} \hat{d}_i μ_{ij} - λ\hat{d}_j\right) = 0,\]

\[(α - λ)y_{21} - βy_{11})\left(\sum_{i=1}^{3} \hat{d}_i μ_{ij} - λ\hat{d}_j\right) = 0,\]

\[(α - λ)y_{11} + βy_{21})\left(\sum_{i=1}^{3} \tilde{c}_i μ_{ij} - λ\tilde{c}_j\right) = 0,\]

\[(α - λ)y_{21} - βy_{11})\left(\sum_{i=1}^{3} \tilde{c}_i μ_{ij} - λ\tilde{c}_j\right) = 0,\]

where \(j = 1, 2, 3\), is solved by \(c = d = 0\) or \(y_{q11} = y_{q21} = 0\) in the quark Yukawa matrix. The third system (of three equations):

\[b(λ' - X')\left(\sum_{i=1}^{3} c_i μ_{ij} - λc_j\right) = 0,\]

where \(j = 1, 2, 3\), is solved by either \(b = 0\) or \(c = 0\). By putting these solutions together we recover exactly the four solutions found by Boyle and Farnsworth: (i) \(b = c = d = 0\); (ii) \(b = y_{q11} = y_{q21} = 0\); (iii) \(y_{l11} = y_{l21} = c = d = 0\); (iv) \(y_{l11} = y_{l21} = y_{q11} = y_{q21} = c = 0\). Three of these four solutions are not physically acceptable because they correspond to vanishing matrix elements of the Yukawa matrices that are experimentally non-zero. The remaining solution (i) is precisely the result of the condition of zero photon mass.

Note that we solved the anticommutator equation \(\{[DF, a], [DF, b]\} = 0\) up to the junk while Boyle and Farnsworth solve the commutator equation \([DF, a], [DF, b]\] = 0 without junk condition and it may seem surprising that we find the same solutions. This is explained by the fact that the four solutions give in fact \([DF, a][DF, b]\) = \([DF, b]\) \([DF, a]\] = 0. As a consequence, both the commutator and anticommutator equations are satisfied. Moreover, we do not need to determine the junk more precisely since we already have \(\{[DF, a], [DF, b]\} = 0\) without junk condition.

V. CONCLUSION

Chamseddine and Connes based a derivation of the Standard Model on a bimodule over an algebra \(A\). Boyle and Farnsworth proposed to use a bimodule over the universal differential algebra \(Ω\) which is physically more satisfactory because it contains (up to the junk) the gauge fields, the field intensities, the curvature and the Lagrangian densities. But their approach was not compatible with the manifold part of the Standard Model.

To take into account the differential graded structure of \(Ω_D\), we built a differential graded bimodule that takes the junk into account. The grading transforms the Boyle and Farnsworth condition on the commutator \([π(δa), π(δb)]\) = 0 into a condition on the anticommutator \(\{π(δa), π(δb)\}\) \(K\), which is now satisfied for the full Standard Model and not only for its finite part.

This indicates that, in a reinterpretation of the noncommutative geometric approach to field theory, the differential graded structure of the boson fields must be accounted for. This is a good news for any future quantization and renormalization of NCG because the differential graded structure is also an essential ingredient of the Becchi-Rouet-Stora-Tyutin and Batalin-Vilkovisky approaches.

Our differential graded bimodule retains the advantages of the Boyle and Farnsworth approach: (i) it unifies the conditions of order zero and one and the condition of massless photon into a single bimodule condition; (ii) it can be adapted to non-associative or Lie algebras.

We hope to use our construction for the quantization of a noncommutative geometric description of the Standard Model coupled with gravity.

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