Two partial monoid structures on a set

Rachel A.D. Martins
Centro de Álgebra da Universidade de Lisboa, Av. Prof. Gama Pinto, 2, 1649-003 Lisboa, Portugal
2 February 2015

Abstract

It is well-known that small categories have equivalent descriptions as partial monoids. We provide a formulation of partial monoid and partial monoid homomorphism involving \( s \) and \( t \) instead of identities and then following a recent investigation into involutive double categories, we prove that a double category is equivalent to a set equipped with two partial monoid structures in which all structure maps \((s,t,\circ)\) are partial monoid homomorphisms. We discuss the light this purely algebraic perspective sheds on the symmetries of these structures and its applications. Iteration of the procedure leads also to a pure algebraic formulation of the notion of \( n \)-fold category.

1 Introduction

The partial monoid description of a small category \([16]\) can be thought of as another generalisation of the notion of group. The key idea in translating higher categorical structures into a pure algebraic language, is to afford the \( m < n \) cells the status of \( n \)-cells, so in a purely algebraic double category or 2-category, the objects and 1-morphisms are described as members of the set of 2-morphisms. The author learned about this point of view from discussions with Pedro Resende, Paolo Bertozzini and Roberto Conti and the original theme dates back to some handwritten notes by John Roberts in the 1970s.

Let \( C \) be a category. In this contribution following \([4]\), we replace the identity map \( \text{id}_C : \text{Ob}(C) \rightarrow C \) with source and target maps \( s, t : C \rightarrow C \) leading to an emergence of \( \text{Ob}(C) \) as a (not pre-distinguished) subset of the set of morphisms \( C \). In our construction, the notion of identity morphism does not appear anywhere.
explicitly because it is completely absorbed into the data provided by \(s\) and \(t\). We provide a concise translation of the notion of double category (a category internal in the category of small categories \([9]\)) into a pure algebraic language, in terms of a set with two commuting partial monoid structures. Note that Ehresmann’s definition of double category is very concise since it takes account of all the symmetries involved in the structure. The notion given below of partial monoid homomorphism, which contains 3 conditions (the identity preservation condition of a covariant functor is split into 2 conditions, one of which involves the source map \(s\) and the other, the target map \(t\)) is the key factor that allows our translation to be almost as concise. The technique can be iterated \(n\) times, and leads to a corresponding purely algebraic formulation of \(n\)-fold category \([7]\).

As detailed in \([4]\), the notion of double category together with its 4-fold system of opposite categorical structures, can be illustrated by any one choice of 4 different classes of cubical set, (or square 2-cell), although only one of these choices exactly corresponds to the original Ehresman notion of double category. The particular construction considered here, results in the elimination of all the other three choices of cubical set class. (The term “double partial monoid” has been reserved for the pure algebraic formulation of double categories including the full generality of allowing all such choices. A detailed expanded definition of double partial monoid will appear in a coming preprint \([4]\).)

In this language, the notion of 2-category has a more complicated translation than the notion of double category. This underlines the point of view that double categories are more naturally occurring algebraic objects.

We introduce some of our motivations in this paragraph and discuss some applications in the final section. (i) There are several theories relating inverse semigroups with topological groupoids (including \([18], [13]\)). Viewing groupoids as partial monoids with all inverses, might allow a comparison to be made within a single algebraic language. In contrast to working in \(\text{Top}\) to construct groupoids, where one usually begins with a space \(X\) and next a space of arrows is defined over \(X\), examples of localic groupoids \([18]\) tend to arise from a set of arrows with the structure of a groupoid. Consider for example the groupoid of ultrafilters of a tiling semigroup \([11], [10]\), where the groupoid units are constructed directly and afterwards are interpreted as objects. Other applications to algebra may follow from comparing congruences of higher categories viewed as sub-structures of sets equipped with several commuting partial monoid structures. (ii) Double groupoids have proved to be important tools in homotopy theory \([6]\) and therefore it is worth looking for ways to generalise them so that some of the concepts can be explored in a non-commutative context\([1]\). (iii) In \([19]\) Roberts and Ruzzi

\footnote{For example, Prof. Ronald Brown has opened a popular discussion on the construction of}
repackage the notion of $n$-category in terms of a set of $n$-arrows subject to a list of axioms to clarify their approach to the non-Abelian cohomology of a poset.

We will use the notation $(\mathcal{C}, C^0, \iota, \circ, s, t)$ to refer to a category $\mathcal{C}$ where $C^0$ denotes the objects $\text{Ob}(\mathcal{C})$ of $\mathcal{C}$, with $\text{id}_{\mathcal{C}}$ or $\iota$ for the unit map $\iota : C^0 \to \mathcal{C}$ and $s$ and $t$ the source and target maps $s, t : \mathcal{C} \to C^0$. For composable arrows $x$ and $y$ in $\mathcal{C}$, we write $(x, y) \in \mathcal{C} \times_{\circ} \mathcal{C}$.

2 Partial monoids

Definition 2.1. A partial monoid $(\mathcal{C}, \circ, s, t)$ is a set with a partially defined associative multiplication $\circ$ together with two idempotent maps $s, t : \mathcal{C} \to \mathcal{C}$ called partial identities such that $x \circ y$ is defined in $\mathcal{C}$ exactly when $s(x) = t(y)$ (and $z \circ x$ is defined in $\mathcal{C}$ exactly when $t(x) = s(z)$), and such that $x \circ s(x) = x$, $t(x) \circ x = x$, for all $x \in \mathcal{C}$.

Remark 2.2. The previous definition is equivalent to that of partial monoid by Mac Lane in [16] because the category units (or identity morphisms) are obtained from the images of the source $s$ and target $t$ partial identity maps whenever $s(x) = t(y)$.

Caveat: Given the partial monoid description of a category $\mathcal{C}$, it is not always possible to fully reconstruct the objects $\text{Ob}(\mathcal{C})$ of the original category $\mathcal{C}$.

In this pure algebraic language, a covariant functor has an equivalent description as a partial monoid homomorphism:

Definition 2.3. Let $(\mathcal{C}_1, \circ_1, s_1, t_1)$ and $(\mathcal{C}_2, \circ_2, s_2, t_2)$ be two partial monoids. A partial monoid homomorphism is a map $F : \mathcal{C}_1 \to \mathcal{C}_2$ such that for each $x \circ_1 y \in \mathcal{C}_1$,

$$F(x \circ_1 y) = F(x) \circ_2 F(y),$$  \hspace{0.5cm} (1)

$$F(s_1(x)) = s_2(F(x)), \hspace{0.5cm} F(t_1(x)) = t_2(F(x)).$$  \hspace{0.5cm} (2)

The above contains 3 explicit conditions, whereas the definition of covariant functor contains only 2 lines.

convolution C*-algebras for double groupoids, which has involved viewing a double groupoid in terms of its underlying set of 2-arrows.

2by associative in this context, we mean whenever one of the two terms $z \circ (x \circ y)$ and $(z \circ x) \circ y$ exists, the other one exists as well, and they coincide.

3If two definitions are equivalent, it does not follow that any example can be precisely reconstructed from the other definition. This point is illustrated by recalling that the real algebras $\mathbb{H}$ and $M_2(\mathbb{R})$ have the same unit, so in a category whose objects are vector spaces there can be an ambiguity.
3 Double categories and partial monoids

Recall that Ehresmann’s construction of a double category, is a category internal in the category Cat of small categories \([\mathcal{C}]\). One transparent albeit simplified way to unpack this definition, is to present a double category as a small category \(\mathcal{D}\) with four categorical structures \((\mathcal{D}, \mathcal{D}_v^1, \mathcal{D}_v^0, \mathcal{D}_h^1, \mathcal{D}_h^0, \mathcal{D}_{v, h}^1, \mathcal{D}_{v, h}^0)\), \((\mathcal{D}, \mathcal{D}_v^1, \mathcal{D}_v^0, \mathcal{D}_h^1, \mathcal{D}_h^0, \mathcal{D}_{v, h}^1, \mathcal{D}_{v, h}^0)\), \((\mathcal{D}, \mathcal{D}_v^1, \mathcal{D}_v^0, \mathcal{D}_h^1, \mathcal{D}_h^0, \mathcal{D}_{v, h}^1, \mathcal{D}_{v, h}^0)\), such that the partially defined associative multiplications are compatible with all units \(\iota\), and with one another (more details below). The definition of double category is expanded in various equivalent presentations in [4] and a good definition also appears in [2].

It is convenient to use cubical set diagrams [14] to illustrate this. According to [4], there exist four equivalent classes of such square cell diagrams, each of which captures a notion of double category, and each of which belongs to a system of 4 mutually opposite categorical structures. Only one of these cubical set classes represents the categorical compositions in an intuitive way, and this corresponds to the original Ehresmann double category (recalled above).

**Class one:** 
\[ s_h(s_v) = s_v(s_h), \quad t_h(t_v) = t_v(t_h), \quad t_h(s_v) = s_v(t_h), \quad s_h(t_v) = t_v(s_h). \]

**Class two:** 
\[ s_h(s_v) = t_v(s_h), \quad t_h(t_v) = s_v(t_h), \quad t_h(s_v) = t_v(s_h), \quad s_h(t_v) = s_v(t_h). \]

**Class three:** 
\[ s_h(s_v) = s_v(t_h), \quad t_h(t_v) = t_v(s_h), \quad t_h(s_v) = s_v(s_h), \quad t_v(t_h) = s_h(t_v). \]

**Class four:** 
\[ s_h(s_v) = t_v(t_h), \quad t_h(t_v) = t_v(t_h), \quad t_h(s_v) = t_v(s_h), \quad s_v(t_h) = s_v(t_v). \]

In the context of cubical set class one, the compatibility rules for the multiplications \(\circ_v^1\) and \(\circ_v^2\) are expressed in symbols as follows:

- \((x \circ_v^1 y) \circ_v^2 (z \circ_v^1 w) = (x \circ_v^2 z) \circ_v^2 (y \circ_v^2 w)\) whenever both sides are defined,
- \(s_v(x) \circ_v^2 s_v(z) = s_v(x \circ_h^2 z), \quad t_v(x) \circ_v^2 t_v(z) = t_v(x \circ_h^2 z),\)
- \(s_h(x) \circ_v^2 s_h(y) = s_h(x \circ_v^2 y), \quad t_h(x) \circ_v^2 t_h(y) = t_h(x \circ_v^2 y),\)

for all \((x, z) \in \mathcal{D} \times \mathcal{D}_v^1 \mathcal{D}\) and all \((x, y) \in \mathcal{D} \times \mathcal{D}_v^2 \mathcal{D}\).

The theorem below shows that in the pure algebraic language, a double category can be formulated concisely as a set of 2-morphisms with two commuting partial monoid structures. Note that although Ehresmann’s definition already implies that the structure maps are functors, the following is not a straightforward translation, especially as the identity assigning map \(\text{id}_C : \text{Ob}(\mathcal{C}) \to \mathcal{C}\) has no explicit presence here.

**Theorem 3.1.** A double category (cubical set class one), can be given an equivalent description as a set \(\mathcal{D}\) with two partial monoid structures \(\mathcal{D}_1 = (\mathcal{D}, \circ_1, s_1, t_1)\) and \(\mathcal{D}_2 = (\mathcal{D}, \circ_2, s_2, t_2)\) such that the multiplication \(\circ_i : \mathcal{D}_j \times \mathcal{D}_j \to \mathcal{D}_j, i \neq j \in \{1, 2\}\) and the source and target partial identity maps \(s_i, t_i : \mathcal{D}_j \to \mathcal{D}_j, i \neq j \in \{1, 2\}\) are all partial monoid homomorphisms.
The non-trivial part of the proof is to unpack the construction to obtain the required four categorical structures together with all the appropriate compatibility conditions as recalled above. In other words, we only need to demonstrate that we can construct a double category from the data given in the statement because it will then be clear that any double category will fit this new description. See also the diagrams (a), (b) and (c) below.

**Proof.** Let $\mathcal{D}$ be a set equipped with two partial monoid structures, $\mathcal{D}_h = (\mathcal{D}, \circ_h, s_h, t_h)$ and $\mathcal{D}_v = (\mathcal{D}, \circ_v, s_v, t_v)$.

From $\circ_h : \mathcal{D}_v \times \mathcal{D}_v \to \mathcal{D}_v$ and equation (1) we find that the two composition laws $\circ_h$ and $\circ_v$ are compatible with one another but it still remains to derive the conditions that select the cubical set class before we can unpack the exchange law below in equation (11) because the cubical set class determines the arrangement of the 2-arrows when forming their compositions.

Then plugging the multiplication maps $\circ_h$ and $\circ_v$ into equations (2) give compatibility conditions:

\[ s_v(x) \circ_h s_v(z) = s_v(x \circ_h z), \quad t_v(x) \circ_h t_v(z) = t_v(x \circ_h z) \quad (3) \]
\[ s_h(x) \circ_v s_h(y) = s_h(x \circ_v y), \quad t_h(x) \circ_v t_h(y) = t_h(x \circ_v y) \quad (4) \]

for all $(x, z) \in \mathcal{D} \times s_h$ and all $(x, y) \in \mathcal{D} \times t_h$.

Secondly, the partial identities $s_v, t_v : \mathcal{D}_h \to \mathcal{D}_h$, $s_h, t_h : \mathcal{D}_v \to \mathcal{D}_v$ satisfying equations (2) involves

\[ \mathcal{F}(s_1(x)) = s_2(\mathcal{F}(x)) \implies s_v(s_h) = s_h(s_v) \quad (5) \]

and altogether we have, $\forall x \in \mathcal{D}$,

\[ s_h(s_v)(x) = s_v(s_h)(x), \quad t_h(t_v)(x) = t_v(t_h)(x) \quad (6) \]
\[ t_h(s_v)(x) = s_v(t_h)(x), \quad s_h(t_v)(x) = t_v(s_h)(x). \quad (7) \]

and then from the partial identity maps and equation (1), the same set of compatibilities as before (3), (4) are repeated.

The previous equations establish that the cubical set class is class one, and this means that the compositions are understood (now that we know which cubical set class the 2-arrows belong to, we can represent elements using the square diagrams below and hence form their compositions) and we may now plug $\circ_h : \mathcal{D}_v \times \mathcal{D}_v \to \mathcal{D}_v$ into equation (1) and retrieve the exchange law:

\[ (x, y), (w, z) \in \mathcal{D}_v \times \mathcal{D}_v \]
\[ \circ_h((x, y)(w, z)) = (x \circ_v w) \circ_h (y \circ_v z) \quad (8) \]
\[ \circ_h(x, y) \circ_h (w, z) = (x \circ_h y) \circ_v (w \circ_h z) \quad (9) \]
\[ (x \circ_v w) \circ_h (y \circ_v z) = (x \circ_h y) \circ_v (w \circ_h z) \quad (10) \]
(whenever both sides of (11) exist).
Next we show the emergence of the subsets $D_v^1, D_h^1, D^0 \subset D$.
The images of the partial identity maps define two distinct subsets $D_v^1$ and $D_h^1$ of 2-arrows, which due to (6),(7) are partial monoids in their own right,
\begin{align*}
\text{im}(t_h) &= \text{im}(s_h) \Rightarrow (D_h^1, \circ_h, s_h, t_h) \\
\text{im}(s_v) &= \text{im}(t_v) \Rightarrow (D_v^1, \circ_v, s_v, t_v)
\end{align*}
(12)
(13)
By iterating the application of the partial identity maps, one additionally finds partial monoid homomorphisms:
\begin{align*}
s_v : (D_v^1, \circ_v, s_v, t_v) &\rightarrow (D_v^1, \circ_v, s_v, t_v), & t_v : (D_v^1, \circ_v, s_v, t_v) &\rightarrow (D_v^1, \circ_v, s_v, t_v), \\
s_h : (D_h^1, \circ_h, s_h, t_h) &\rightarrow (D_h^1, \circ_h, s_h, t_h), & t_h : (D_h^1, \circ_h, s_h, t_h) &\rightarrow (D_h^1, \circ_h, s_h, t_h)
\end{align*}
(14)
(15)
whose image sets coincide by equations (6), (7). Since all partial identity maps commute and are idempotent, this iterative procedure can only be performed twice before reaching the situation where a subset $D^0 \subset D$ is constructed as $D^0 = \{x \in D : s_h(x) = t_h(x) = s_v(x) = t_v(x)\}$. For clarity and completeness, we state that the elements of $D^0$ give the identity morphisms, or category units in the double category we constructed and that these are compatible with both partial multiplication rules. Now we see that $D$ is a double category with Ob$(D) = D^0$.

We can conclude that typical elements of $D$ are represented by (a) a square 2-cell diagram from cubical set class one,

\begin{align*}
(a) \quad & A \arrow[1-2]{a}{n} B \\
& c \arrow[2-1]{a}{n} d \\
& C \arrow[3-2]{a}{n} D
\end{align*}
(16)
(17)
(18)
(b) a typical element of $D_h^1$, and (c) by iteration, an object $A \in D^0$. The diagrams also illustrate how the data of the maps $\text{id}_v^1 : \text{Ob}(D) \rightarrow D_v^1$ and $\text{id}_h^1 : \text{Ob}(D) \rightarrow D_h^1$ (or $\iota : C^0 \rightarrow C$) have been absorbed into the source and target maps.

### 3.1 Other higher structures

Observe that since the procedure used in the theorem proof is iterative, if $C$ is a set with $n$ partial monoid structures $(C, \circ_i, s_i, t_i)_{i=1..n}$, we may apply the same
procedure $n$ times, until $s_i(x) = t_j(x)$ for all $i, j \leq n$. We propose therefore a purely algebraic formulation of the notion of (strict) $n$-fold category, as a set $C$, with $n$ partial monoid structures $(C, o_i, s_i, t_i)$, for $i \in I = \{1..n\}$, in which all multiplication $o_i$ and partial identity maps $s_i, t_i$ are partial monoid homomorphisms with respect to each $j \in I$ such that $j \neq i$. (See [7] for the definition of an $n$-fold category.)

The following shows that a double category is a more naturally occurring algebraic object than a 2-category. A 2-category can be given as a double category in which all horizontal units $\iota_h$ are also vertical units $\iota_v$ [19]. Thus a 2-category can be equivalently described as a set $C$ equipped with two partial monoid structures $(C, \circ_1, s_1, t_1)$ and $(C, \circ_2, s_2, t_2)$ such that all structure maps $(s_i, t_i, \circ_i)$ are partial monoid homomorphisms and in which $s_1 = s_1(s_2) = s_1(t_2)$ and $t_1 = t_1(t_2) = t_1(s_2)$ and $s_1 : C \to C^0$ and $t_1 : C \to C^0$ are surjective maps.

A double category $D$ with only one object $A \in D^0$ is equivalent to a set with two commuting partial monoid structures (in the sense studied above) in which all compositions of the partial identity maps are constant maps. Thus, $s_h(s_v)(x) = s_v(t_v)(x) = t_v(t_h)(x) = t_h(s_v)(x) = s_v(t_h)(x) = s_h(t_v)(x) = t_v(s_h)(x) = A$ for all $x \in D$.

4 Applications and discussion

Longo and Roberts [15] introduced the notion of 2-C*-category in terms of an additional structure on the set of 2-arrows in a 2-category. In particular, their definition of 2-C*-category begins with an underlying set with two compositions, each making it into a category. A double category approach to these and other involutive categories in this language of partial monoids is further motivated in a coming preprint [4]. In the simplest terms, an involutive double category can be thought of as a set with two commuting partial monoid structures $v$ and $h$ (in the precise sense studied above), each equipped with an involution $*_{v}$ and $*_{h}$ such that $*_{h}*_{v} = *_{v}*_{h}$.

The following describes a motivating example of an involutive double category. Let $(\mathcal{E}, \pi, \mathcal{G})$ be a complex line bundle over a discrete double pair groupoid $\mathcal{G}$ (that is, $\mathcal{G}$ is a double groupoid in which each of the four underlying categorical structures is a discrete pair groupoid). Modulo a choice of strictification, the set of fibres of $\mathcal{E}$ has the structure of a double groupoid. Consider now a section $\sigma$ of $\pi$ over a 2-arrow $g \in \mathcal{G}$ in the sense that its components consist of an element of each of the 9 fibres over the 2-arrows defined by $g$ (that is, $g, s_h(g), s_v(s_h)(g)$ and so on). We hint at a potential connection with the non-commutative standard model [8] as follows. Let $u_L \in A$, $d_L \in C$, $u_R \in B$, $d_R \in D$ labelling components
of $\sigma$ in terms of diagram (a), and let $M$ be the restriction of $\sigma$ to the groupoid $G_v$. Note that $M + M^*$ provides an automorphism in a category of Fell bundles over groupoids [3], (for Fell bundles see [12]) and a self-adjoint linear operator $\mathcal{D} = M + M^*$ on the Hilbert space $\mathcal{H} = \mathbb{C}^2 \oplus \mathbb{C}^2$.

A localic groupoid is defined in [18] as a groupoid internal in Loc (recall that a topological groupoid is a groupoid internal in Top). Examples of localic groupoids tend to be constructed from a set of arrows with the structure of a groupoid (that is, a partial monoid with all inverses). One might suggest a definition for a localic double groupoid $L$ as a set with two commuting partial monoid structures (again, in the precise sense studied above) such that each partial monoid contained in $L$ has the structure of a locale.

We add the suggestion that the symmetries of a Penrose tiling can be described by a partial action $\alpha$ of the group $G = SO(2) \ltimes \mathbb{R}$ on the underlying inverse category defined by the patterns in the tiling (this inverse category is detailed in [11]). Since $G$ is a subgroup of the Poincaré group, which Majard [17] describes as a double group, it would be interesting to investigate the structure of set of ultrafilters of the inverse semigroup arising from the partial action $\alpha$.

### 5 Acknowledgements

Thank you, grazie and obrigada to Paolo Bertozzini, Pedro Resende and Roberto Conti for discussions and especially to Paolo Bertozzini for the very significant contributions to this paper.

This work was supported by Fundação para as Ciências e a Tecnologia (FCT) through the following projects: PEst-OE/EEI/LA0009/2013, SFRH/BPD/32331/2006, POCI 2010/FEDER (CAMGSD), EXCL/MAT-GEO/0222/2012 (IST Geometry and mathematical physics project) and PEst-OE/MAT/UIO143/2014 (CAUL).

### References

[1] Baez J, An Introduction to n-Categories, Arviv: 9705009.

[2] Baez J, Crans A (2004), Higher-dimensional Algebra VI: Lie 2-algebras, *Theory and Applications of Categories* 12 n.15:492-538.

[3] P. Bertozzini, R. Conti R, W. Lewkeeratiyutkul, (2009), Enriched Fell Bundles and Spaceoids, (“proceedings of the 2010 RIMS thematic year on perspectives on deformation quantization and noncommutative geometry”), ArXiv: 1112.5999
[4] Bertozzini B, Conti R, Martins R, Involutive double categories. In preparation.

[5] Brown R, Mosa G H (1999) Double Categories, 2-categories, Thin Structures and Connections Theory and Applications of Categories 5 n.7:163-175

[6] Brown R, Higgins PJ (1978), On the connection between the second relative homotopy groups of some related spaces, Proc. London Math. Soc. 3,36,1978,193-212.

[7] Brown R, Higgins PJ (1981), The equivalence of $\infty$-groupoids and crossed complexes, Cahiers Top Géom Diff 22:371-386.

[8] A. Chamseddine, A. Connes, The spectral action principle. Comm. Math. Phys. Vol.186 (1997), N.3, 731-750.

[9] Ehresmann C (1963), Catégories Structurées III: Quintettes et Applications Covariantes, Cahiers Top Géom Diff 5:1-22.

[10] J. Kellendonk, Topological equivalence of tilings, J. Math. Phys. 38, No. 4 (1997).

[11] J. Kellendonk, M.V. Lawson, Tiling semigroups, Journal of Algebra, 224, 1,2000,140-150.

[12] A. Kumjian, Fell bundles over groupoids, Arxiv: math.oa/607230, Proceedings of the American mathematical society, Vol. 126, No. 4 (Apr., 1998) pp. 1115–1125.

[13] Lawson M V, Non-commutative Stone duality: inverse semigroups, topological groupoids and C*-algebras. International Journal of Algebra and Computation Vol. 22, No. 6 (2012) 1250058.

[14] Leinster T (2004), Higher Operads, Higher Categories, Cambridge University Press, Arxiv: 0305049.

[15] Longo R, Roberts J (1997) A Theory of Dimension K-Theory 11:133-159.

[16] Mac Lane S (1997), Categories for the Working Mathematician, Graduate Texts in Mathematics, Springer.

[17] D. Majard, On Double Groups and the Poincaré group, Arxiv: 1112.6208.
[18] Resende P, Étale groupoids and their quantales, *Adv. Math.* 208 (2007) 147-209, Arxiv: math/0412478.

[19] Roberts J E, Ruzzi G (2006), A Cohomological Description of Connections and Curvature Over Posets, *Theory Appl Categ* 16(30):855-895, ArXiv: 0604173.