Analytical eighth-order light-by-light QED contributions from leptons with heavier masses to the anomalous magnetic moment of electron

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Abstract

The important consequences of the recent results of the numerical evaluations of eighth and tenth order QED contributions to the anomalous magnetic moment of electron are commented. The correctness of the results of the numerical evaluation of new eighth order QED corrections to the electron anomaly are supported by the demonstration of their consistency with the new analytical expressions for the QED contributions to $a_e$ from the diagrams with fourth-order light-by-light scattering muon and tau-lepton loops. The consistency of the similar results are demonstrated in the case of eighth order massive dependent contribution to the muon anomalous magnetic moment,

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One of the most precise at present experimental results in the modern particle physics is the measurement of the electron anomaly \( a_e = (g_e - 2)/2 \) \[1\], \[2\], which gives
\[
a_e = 1159652180.73(0.28) \times 10^{-12} \quad [0.24 \text{ppb}]. \tag{1}
\]
The previous stage of the Standard Model theoretical prediction
\[
a_e^{\text{th}} = a_e^{\text{QED}} + a_e(\text{hadrons}) + a_e(\text{weak}) \tag{2}
\]
was summarized in the most detailed review on the subject \[3\]. The perturbative QED contribution to Eq.(2) is defined as
\[
a_e^{\text{QED}} = A_1 + A_2(m_e/m_\mu) + A_2(m_e/m_\tau) + A_3(m_e/m_\mu, m_e/m_\tau) \tag{3}
\]
Up to recently theoretical and phenomenological applications were based on the following expression for its dominant term \( A_1 \)
\[
A_1 = \sum_{l=1}^{5} A_1^{(2l)}(\frac{\alpha}{\pi})^l = 0.5 \left(\frac{\alpha}{\pi}\right) - 0.32847896557919378 \ldots \left(\frac{\alpha}{\pi}\right)^2
\]
\[
+ 1.181241456587 \ldots \left(\frac{\alpha}{\pi}\right)^3 - 1.9144(35) \left(\frac{\alpha}{\pi}\right)^4 + 0.0(3.8) \left(\frac{\alpha}{\pi}\right)^5 \tag{4}
\]
where \( l \) is the number of loops of the Feynman diagrams, which are contributing to the corresponding perturbative QED expression. Three first coefficients, presented in Eq.(4) in the numerical form, were evaluated analytically. The first term was calculated by Schwinger \[4\], the second correction was evaluated by Petermann \[5\] and Sommerfield \[6\]. This result was confirmed later on by Terentiev \[7\], who used different technique. The project of analytical evaluation of all three-loop QED contributions to \( a_e \) was completed by Laporta and Remiddi \[8\]. The cited value for the four-loop QED correction to \( a_e \) was numerically obtained in Refs.[9], [10] by Kinoshita and collaborators. The rough CODATA estimate of the coefficient of the 5-loop term in Eq.(4), namely \( A_1^{(10)} = \pm 2|A_1^{(8)}| \) \[11\], gave the idea what might be theoretical uncertainties in the value of \( a_e \) due to unknown up to recently total value of the tenth-order QED effects. The project of their direct numerical evaluation, started in 2005 by Kinoshita and Nio in Ref. [12], continued in the series of works of Refs. \[13\]–[21], was successfully completed in May of 2012 by Ayoama, Hayakawa, Kinoshita and Nio \[22\].

In general, the coefficient \( A_1^{(10)} \) is defined by the contribution of 12672 diagrams, which were classified into 32 gauge-invariant subsets. The result of evaluation of the final, most complicated Set V, together with more detailed numerical calculation of the eighth-order contribution to \( a_e \), was reported recently in Ref. \[22\]. The long-expected expression for the tenth-order massless contribution \[22\] is:
\[
A_1^{(10)} = 9.16(58) \tag{5}
\]
while at the eight-order level the following new results were obtained \[22\]:
\[
A_1^{(8)} = -1.9106(20) \quad A_e^{(8)} = -1.9097(20) \tag{6}.
\]
The \( a_e^{(8)} \)-term differs from Eq.(3) due to the inclusion of complete mass-dependent contributions, numerically evaluated in the process of the works of Refs. \[13\]–[21]. These mass-dependent effects, summarized in the numerical form in Ref. \[22\], read
\[
A_2^{(8)}(m_e/m_\mu) = 9.222(66) \times 10^{-4} \tag{8}
\]
\[
A_2^{(8)}(m_e/m_\tau) = 8.24(12) \times 10^{-6} \tag{9}
\]
\[
A_3^{(8)}(m_e/m_\mu, m_e/m_\tau) = 7.465(18) \times 10^{-7} \tag{10}
\]
While obtaining these results the CODATA-2010 report \[23\] mass ratios \(m_e/m_\mu = 4.83633166(12) \times 10^{-3}, m_e/m_\tau = 2.87592(26) \times 10^{-4}, m_\mu/m_\tau = 5.94649(54) \times 10^{-2}\), with the fixed value of \(\tau\)-lepton pole mass \(m_\tau = 1776.82(16)\) MeV \(^1\), were used. The results of Eq.\((8)\), Eq.\((9)\) and Eq.\((10)\) are presented in more detailed form in Table I of Ref.\[22\] for 12 gauge-invariant groups of 4-loop diagrams, contributing to \(A_2\) and \(A_3\)-terms of Eq.\((3)\). The summary of massless and massive-dependent contributions to \(A_1^{(10)}\) and \(A_2^{(10)}(m_e/m_\mu)\) are presented in Table II of Ref.\[22\]. Note, that the sum of 10-th order massive dependent terms result in the small value of the overall massive correction \(A_2^{(10)}(m_e/m_\mu) = -0.00382(39)\) \[22\].

It should be stressed, that the determination of the concrete value of \(A_1^{(10)}\)- and \(\alpha_\text{e}^{(8)}\)-terms is of real importance. Indeed, prior the work of Ref.\[22\] the scientific community faced with the unique case, when the comparison of the low-energy experimental result of Eq.\((1)\) with the massless perturbative QED predictions of Eq.\((4)\), supplemented with analytically known from the works of Refs.\[25\]- \[31\] massive-dependent forth and sixth-order contributions into \(A_2\) and \(A_3\)-terms and with the well-known values for \(\alpha_\text{e}(\text{hadrons})\) and \(\alpha_\text{e}(\text{weak})\) (see the review \[3\]), namely

\[
\begin{align*}
\alpha_\text{e}(\text{weak}) &= 0.0297(5) \times 10^{-12} \quad , \\
\alpha_\text{e}(\text{hadrons}) &= 1.671(19) \times 10^{-12} \quad ,
\end{align*}
\]

led to the value of the inverse fine coupling constant

\[
\alpha^{-1} = 137.03599084(33)(39) \quad (13)
\]

with theoretical uncertainties \(\pm 39 \times 10^{-8}\), comparable with experimental ones \(\pm 33 \times 10^{-8}\).

The dominant contribution to the theoretical error, namely \(\pm 30 \times 10^{-18}\), was defined by the CODATA estimate of the coefficient of the \(O(\alpha^5)\)-term in Eq.\((4)\). The additional sizable uncertainty in Eq.\((13)\) came from the theoretical error of the numerical evaluation of the eighth-order contributions to Eq.\((4)\) performed in Refs.\[9\], \[10\]. There were no additional theoretical errors in the sum of cited above analytically evaluated forth and sixth-order massive-dependent contributions \(A_2\) and \(A_3\)-terms. Their total contribution can be extracted from the summary part of Ref.\[31\] and reads:

\[
A_2(m_e/m_\mu) = 5.19738667(26) \times 10^{-7}(\frac{\alpha}{\pi})^2 - 7.37394155(27) \times 10^{-6}(\frac{\alpha}{\pi})^3 \quad (14)
\]

\[
A_2(m_e/m_\tau) = 1.83798(34) \times 10^{-9}(\frac{\alpha}{\pi})^2 - 6.5830(11) \times 10^{-8}(\frac{\alpha}{\pi})^3 \quad (15)
\]

\[
A_3(m_e/m_\mu, m_e/m_\tau) = 0.1909(1) \times 10^{-12}(\frac{\alpha}{\pi})^3 \quad . \quad (16)
\]

The errors in Eq.\((14)\)-Eq.\((16)\) are related to the indicated above uncertainties of the CODATA-2010 values of the ratios of leptons masses.

As to the uncertainties of the important contributions from \(\alpha_\text{e}(\text{hadrons})\) and \(\alpha_\text{e}(\text{weak})\), they were not sensitive at the previous stage of comparing theoretical and experimental predictions for \(\alpha_\text{e}\).

The results of Ref. \[22\] and Eq.\((5)\) allowed to solve this intriguing problem and to make theoretical uncertainties in the analog of Eq.\((13)\) less important, than experimental ones.

\(^1\) Recently measured value of the \(\tau\)-lepton mass \(m_\tau = 1776.69^{+0.17}_{-0.15} \pm 0.15\) MeV \[24\] should not change a lot the uncertainties of these ratios.
Indeed, substituting new eighth- and tenth-order QED effects of Ref. [22] into the the procedure of the comparison with the precise experimental result of Eq. (17), the authors of Ref. [22] obtained more precise value of the inverse fine coupling constant:

\[ \alpha^{-1} = 137.0359991657(68)(46)(24)(331) \times 0.25 \text{ppb} \] 

(17)

Here the first and second errors are related to the uncertainties of the numerical evaluation of the eighth and tenth-order QED corrections. The third error is determined by the combined uncertainties of the hadronic and electroweak contributions to \( a_e \), which start to manifest themselves at this more precise level of perturbative QED calculations, while the fourth and hugest uncertainty is determined by the experimental error in Eq. (1).

In view of the importance of the results of the complicated eight and tenth-order numerical calculations it is highly desirable to perform the independent cross-checks if not all of them, but at least of some their parts.

In this work this problem is studied by analyzing the consistency of the numerical results for the massive-dependent contributions into \( A_2^{(8)}(m_e/m_\mu) \) and \( A_2^{(8)}(m_e/m_\tau) \)-corrections to the electron anomaly, which are described by the subset of diagrams, formed by external light-by-light scattering muon and \( \tau \)-lepton subgraphs with extra virtual photon, propagating inside this subgraph. The corresponding analytical expressions for the leading term of heavy-mass expansions of these contributions follow from the obtained in Ref. [32] result of analytical QCD calculations of the hadronic light-by-light scattering contribution to the \( \tau \)-lepton anomalous magnetic moments \( a_\tau \) and \( a_\mu \) with hadronic effects, modeled by the internal light-by-light-scattering quark loop, crossed by virtual gluon. Taking into account that in QED one has \( C_F = 1 \) and \( \alpha(m_\tau) = \alpha(1 + O(\alpha^2)) \) (see Ref. [33] for details) we get the corresponding analytical contribution of the leading term in heavy lepton mass expansion of the the eighth-order QED correction to \( a_\tau \) from this subset of massive-dependent light-by-light scattering graphs. The results contain the contributions of the Riemann \( \zeta \)-functions \( \zeta_k = \sum_{n=1}^{\infty} (1/n)^k \) and polylogarithmic functions \( a_k = \text{Li}_k(1/2) = \sum_{n=1}^{\infty} (1/2^n n^k) \) and read:

\[
A_2^{(8)}(X_{1+i}, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = \left( -\frac{473}{180} \zeta_2 \ln^2 2 + \frac{312}{405} \zeta_2 \ln^3 2 - \frac{42853}{2880} \zeta_4 + \frac{5771}{360} \zeta_4 \ln 2 \right. \\
\left. + \frac{473}{1080} \ln^4 2 - \frac{52}{675} \ln^5 2 - \frac{8477}{2700} + \frac{473}{45} a_4 + \frac{416}{45} a_5 + \frac{34727}{2400} \zeta_3 - \frac{23567}{1440} \zeta_5 \right) \sum_{i=1}^{2} X_{1+i} \left( \frac{\alpha}{\pi} \right)^4 .
\]

where \( X_{1+i} = (m_1/m_{1+i}) \) with \( i = 1, 2 \) are the ratios of the electron mass \( m_1 = m_e \) and the muon mass \( m_2 = m_\mu \) or \( \tau \)-lepton mass \( m_3 = m_\tau \). The abbreviations in the parenthesis labels the expressions for the subsets of graphs, which contain the next-to-leading order (NLO) approximation of the light-by-light (lbl) scattering subgraph.

Substituting the known values for the transcendental and polylogarithmic functions into Eq. (18) we get:

\[
A_2^{(8)}(X_{1+i}, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = 1.77831 \ldots \sum_{i=1}^{2} X_{1+i}^2 \left( \frac{\alpha}{\pi} \right)^4 .
\]

(18)

The CODATA-2010 values and errors of \( X_1 = m_e/m_\mu \) and \( X_2 = m_e/m_\tau \) are fixing the numerical expressions for the terms we are interested in :

\[
A_2^{(8)}(X_2, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = 4.15948(21) \times 10^{-5} \left( \frac{\alpha}{\pi} \right)^4
\]

(19)

\[
A_2^{(8)}(X_3, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = 1.47082(26) \times 10^{-7} \left( \frac{\alpha}{\pi} \right)^4
\]

(20)
The results of the numerical calculations for the similar contributions are presented in Table 1 of Ref. [22] and read

\[ A_2^{(8)}(X_2, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = 4.105(93) \times 10^{-5} \left( \frac{\alpha}{\pi} \right)^4 \quad (21) \]

\[ A_2^{(8)}(X_3, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = 1.431(95) \times 10^{-7} \left( \frac{\alpha}{\pi} \right)^4 \quad . \quad (22) \]

Taking into account the related uncertainties one can observe good agreement with the numbers of Eq.(19) and Eq.(20), which follow from analytical expression of Eq.(18). This can be considered as the strong check of the results of computer numerical calculations of eight-order contributions to \( a_e \), summarized in Ref.[22]. Moreover, minor discrepancy between Eq.(20) and Eq.(22) may indicate, that in the case of larger lepton mass \( m_\tau \) the exact numerical results of Ref.[22] may be sensitive to taking into account higher terms of large mass expansion of the corresponding eight-order light-by-light-type diagrams, contributing to \( a_e \).

Let us have a look whether the similar feature is manifesting itself in the case of the comparison of the results of analytical and numerical calculation of the similar eight-order light-by-light scattering corrections to the anomalous magnetic moment of muon \( a_\mu \). In this case the expression, analogous to Eq.(18), reads

\[ A_2^{(8)}(X_4, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = 1.77831 \ldots X_4^2 \left( \frac{\alpha}{\pi} \right)^4 \quad . \quad (23) \]

Using the CODATA-10 value for \( X_4 = m_\mu/m_\tau \) we get

\[ A_2^{(8)}(X_4, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = 6.2888(11) \times 10^{-3} \left( \frac{\alpha}{\pi} \right)^4 \quad . \quad (24) \]

This number is in good agreement with the result of the numerical eighth-order computer calculations, obtained in Ref. [34] in the process of completing numerical evaluation of tenth-order QED contributions to the muon anomaly \( a_\mu \), namely with the result

\[ A_2^{(8)}(X_4, \text{lbl, NLO}) \left( \frac{\alpha}{\pi} \right)^4 = 6.106(31) \times 10^{-3} \left( \frac{\alpha}{\pi} \right)^4 \quad . \quad (25) \]

However, as in the case of \( \tau \)-lepton light-by-light-scattering eighth order contributions to \( a_e \), the numerical comparison of the analytically-based result of Eq.(24) with the result of numerical calculations of Eq.(25) seem to indicate the sensitivity to still unknown higher terms of large mass expansions of the analytically evaluated massive-dependent Feynman graphs.

In any case our considerations demonstrate the reliability of the definite results of the complicated important numerical QED calculations from Ref.[22] and Ref.[34]. More detailed considerations of new 10-th QED order results for \( a_\mu \) using the renormalization-group inspired studies of Ref.[35] may be done in future.

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