An analysis of inhomogeneous signature-based Gröbner basis computations

Christian Eder
INRIA, Paris-Rocquencourt Center, PolSys Project
UPMC, Univ. Paris 06, LIP6
CNRS, UMR 7606, LIP6
UFR Ingénierie 919, LIP6
Case 169, 4, Place Jussieu, F-75252 Paris
Christian.Eder@inria.fr

Abstract
In this paper we give an insight into the behaviour of signature-based Gröbner basis algorithms, like F5, G2V or SB, for inhomogeneous input. On the one hand, it seems that the restriction to sig-safe reductions puts a penalty on the performance. The lost connection between polynomial degree and signature degree can disallow lots of reductions and can lead to an overhead in the computations. On the other hand, the way critical pairs are sorted and corresponding s-polynomials are handled in signature-based algorithms is a very efficient one, strongly connected to sorting w.r.t. the well-known sugar degree of polynomials.

1 Introduction
Gröbner bases are a fundamental tool in computer algebra. In 1965 Buchberger introduced a first algorithmic attempt for their computation, see [13].

In [21] Faugère introduced the F5 Algorithm which uses the concept of signatures to detect zero reductions efficiently during the computation of Gröbner bases. In the last couple of years, several variants and optimizations in the class of signature-based algorithms, for example, F5C ([17]), G2V ([25]) or SB ([32]), have been developed. Whereas the above mentioned publications focus mainly on the area of optimizing signature-based criteria for detecting useless critical pairs, a close look at the overall behaviour of signature-based computations in general is still missing. Here we want to fill this gap and discuss advantages and disadvantages of the signature-based attempt. Without going into detail about efficient implementations we analyze the underlying characteristics all signature-based algorithms share:

1. Sorting critical pairs by increasing signatures, and
2. processing only so-called sig-safe reduction steps.

By doing this we clear up myth that signature-based algorithms are only applicable for homogeneous input data, but that they are not useful (either in
the sense of being incorrect or in the sense of being under-performing) in the inhomogeneous setting.

In Section 2 we introduce the basic setting for signature-based Gröbner basis algorithms. There we unify the fundamental framework for such algorithms, describing the differences to a pure polynomial approach. Making some smaller changes to the initial presentation of F5 in Section 3, Gröbner bases for inhomogeneous input can be computed, too. Even more, it turns out that with this new description understanding the algorithm’s inner workings is much easier. Following this, we give for the first time a discussion about the strong connections between the sorting of critical pairs by increasing signatures and the corresponding sorting by the so-called sugar degree. The sugar degree, introduced in [27] and further discussed in [10], is known to be a powerful tool optimizing pure polynomial Gröbner basis computations in the inhomogeneous setting. Thus, explaining its relation to the signature-based world, we are able to allow a first estimate for the usefulness of those kind of algorithms beyond the homogeneous case. Even more, in Section 5 we test the differences in the behaviour of sig-safe reductions comparing different implementations of signature-based algorithms for a wide range of examples side-by-side in the respective inhomogeneous and homogenized version. It turns out that, when compared to pure polynomial attempts, there is no built-in disadvantage relying on signatures for computing Gröbner bases of inhomogeneous input.

The main contribution of this paper is to give a deeper insight into the inner workings of signature-based Gröbner basis algorithms with a view towards optimizing the order in which critical pairs are handled. Besides this, we give first ideas for good heuristics to decide when to use which variant in order to benefit from a better performance.

2 Basic setting

Let \( n \in \mathbb{N}, \mathbb{K} \) a field, and \( \mathcal{R} = \mathbb{K}[x_1, \ldots, x_n] \). Furthermore, we denote the monoid of all monomials in \( x_1, \ldots, x_n \) by \( \mathcal{M} := \{ \prod_{i=1}^{n} x_i^{\alpha_i} \mid (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \} \). We mostly use the shorthand notation \( x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i} \). A polynomial \( p \in \mathcal{R} \) is a finite \( \mathbb{K} \)-linear combination of monomials in \( \mathcal{R} \), \( p = \sum_{\alpha \in U} c_{\alpha} x^\alpha, c_{\alpha} \in \mathbb{K}, U \) a finite subset of \( \mathbb{N}^n \). We define the degree of a polynomial \( p \neq 0 \) by \( \deg(p) = \max \{ \sum_{i=1}^{n} \alpha_i \mid c_{\alpha} \neq 0 \} \). We say that a polynomial \( p \) is homogeneous, if all its monomials have the same degree; otherwise we call \( p \) inhomogeneous.

Let \( F = (f_1, \ldots, f_m) \), where each \( f_i \in \mathcal{R} \), and \( I = \langle F \rangle \subset \mathcal{R} \) is the ideal generated by the elements of \( F \).

Moreover, fixing a well-ordering \( < \) on \( \mathcal{M} \) we get a unique representation of the elements in \( \mathcal{R} \): For a polynomial \( p \in \mathcal{R} \), we denote \( p \)’s leading monomial by \( \text{lm}(p) \), its leading coefficient by \( \text{lc}(p) \), and write \( \text{lt}(p) = \text{lc}(p) \text{lm}(p) \) for its leading term. In particular, a well-ordering \( < \) preferring the degree over any other criterion to sort elements is denoted degree compatible ordering.

Let \( e_1, \ldots, e_m \) be the canonical generators of the free \( \mathcal{R} \)-module \( \mathcal{R}^m \). We define a map

\[
\nu : \mathcal{R}^m \to \mathcal{R}, \quad \sum_{i=1}^{m} p_i e_i \mapsto \sum_{i=1}^{m} p_i f_i,
\]

\[1\] For the zero polynomial we set \( \deg(0) = -1 \).
\( p_i \in \mathcal{R} \) for all \( 1 \leq i \leq m \). Thus we extend the ordering \( \prec \) to an admissible ordering \( \prec \) on the set \( \mathcal{M}' := \{te_i \mid t \in \mathcal{M}, 1 \leq i \leq m\} \). Without any restriction, the reader can think of the following two choices for \( \prec \) in the following:

1. preferring the module position over the term \( \prec_{\text{pot}} \):
   \[ te_i \prec t_j e_j \text{ iff } i < j, \text{ or } i = j \text{ and } t_i < t_j. \]

2. Being induced by \( \prec \), the Schreyer ordering \( \prec_s \):
   \[ te_i \prec t_j e_j \text{ iff } t_i \text{lcm}(\nu(e_i)) < t_j \text{lcm}(\nu(e_j)), \text{ or } t_i \text{lcm}(\nu(e_i)) = t_j \text{lcm}(\nu(e_j)) \]
   and \( i < j \).

In [20] it is shown that the above orderings are the most efficient ones for signature-based Gröbner basis computations. The author has made similar experiences in various tests of his implementations, see Section 5 for more details.

Most of the considerations in this paper are independent of the chosen extended ordering, thus we use the notation \( \prec \) and specify to \( \prec_{\text{pot}} \) respectively \( \prec_s \) whenever differences appear. The notions of leading monomial, leading term, and leading coefficient generalize naturally to \( \mathcal{R}^m \) w.r.t. \( \prec \) on \( \mathcal{M}' \). Additionally, for \( 0 \in \mathcal{R}^m \) we define \( \text{lcm}(0) = \text{lcm}(0) = 0 \).

**Notation 2.1.** For an easier description in the following let us agree on the notation \( \mathcal{L} := \mathcal{M}' \times I \).

**Definition 2.2.** Let \( p \) be a polynomial in \( I \).

1. Let \( h = \sum_{i=1}^{m} h_i e_i \in \mathcal{R}^m \) be such that \( p = \nu(h) \). We say that \( \text{lcm}(h) \in \mathcal{M}' \) is a signature of \( p \). Moreover, considering a well-ordering \( \prec \) on \( \mathcal{M}' \) there exists for each \( p \in \mathcal{R} \) a unique, minimal signature.

2. An element \( f = (te_i, p) \in \mathcal{L} \) is called a labeled polynomial, if \( te_i \) is a signature of \( p \). For a labeled polynomial \( f = (te_i, p) \) we define the shorthand notations \( \text{poly}(f) = p, \text{sig}(f) = te_i, \text{ and index}(f) = i \). Talking about the leading monomial, leading term, leading coefficient, degree, and least common multiples of \( f \in \mathcal{L} \) we always assume the corresponding value of \( \text{poly}(f) \). Furthermore, if \( G = \{g_1, \ldots, g_\ell\} \subset \mathcal{L} \), then we define \( \text{poly}(G) := \{\text{poly}(g_1), \ldots, \text{poly}(g_\ell)\} \subset I \).

3. Let \( f \in \mathcal{L} \), let \( t \in \mathcal{M} \), and let \( c \in \mathcal{K} \). We define a multiplication of \( f \) by \( ct \) via \( ct f := (ts(\text{sig}(f)), ct \text{poly}(f)) \in \mathcal{L} \).

4. A critical pair of two labeled polynomials \( f \) and \( g \) is a tuple \( (f, g) \in \mathcal{L}^2 \). \( \deg(f, g) := \deg (\text{lcm}(\text{lm}(f), \text{lm}(g))) \) defines the degree of a critical pair. Moreover, we define the s-polynomial of two labeled polynomials \( f \) and \( g \) in \( \mathcal{L} \) by

\[
S(f, g) = (\omega, \text{lc}(g)u_f \text{poly}(f) - \text{lc}(f)u_g \text{poly}(g))
\]

where \( \omega = \text{lcm}(u_f \text{sig}(f) - u_g \text{sig}(g)) \) and \( u_h = \frac{\text{lcm}(\text{lm}(f), \text{lm}(g))}{\text{lm}(h)} \) for \( h \in \{f, g\} \). \( S(f, g) \) is called non-minimal if \( \omega \prec \max\{u_f \text{sig}(f), u_g \text{sig}(g)\} \).

5. We define the signature degree of a labeled polynomial \( f = (te_i, p) \in \mathcal{L} \) by

\[ \text{sig-deg}(f) := \deg(t) + \deg(\nu(e_i)). \]

Moreover, in the following it makes sense to speak about the signature degree of a critical pair: \( \text{sig-deg}(f, g) := \text{sig-deg}(S(f, g)) \).
Next we extend the notions of reduction and standard representation from the pure polynomial setting to the signature-based one:

**Definition 2.3.** Let $f, g \in \mathcal{L}$ be two labeled polynomials. Moreover, let $G \subset \mathcal{L}$.

1. We say that $f$ reduces sig-safe to $g$ modulo $G$ if there exist $r_0 = f, \ldots, r_k = g \in \mathcal{L}$ such that for all $i \in \{1, \ldots, k\}$ there exist $g_{j_i} \in G$, $t_i \in \mathcal{M}$ and $c_i \in \mathcal{K}$ fulfilling
   
   (a) $r_i = r_{i-1} - c_i t_i g_{j_i}$,
   
   (b) $\text{lm}(r_i) < \text{lm}(r_{i-1})$, and
   
   (c) $t_i \text{sig}(g_{j_i}) \prec \text{sig}(r_{i-1})$.

2. $f$ has a standard representation with respect to $G$ if there exist $h_1, \ldots, h_k \in \mathcal{R}$, $g_1, \ldots, g_k \in G$ such that $\text{poly}(f) = \sum_{i=1}^{k} h_i \text{poly}(g_i)$, and for each $i \in \{1, \ldots, k\}$ either $h_i = 0$, or
   
   (a) $\text{lm}(h_i) \text{lm}(g_i) \leq \text{lm}(f)$, and
   
   (b) $\text{lm}(h_i) \text{sig}(g_i) \preceq \text{sig}(f)$.

3. If there exists $h \in G$ such that $\text{sig}(h) | \text{sig}(f)$ and $\text{lm}(h) \text{lm}(f)$, then we say that $f$ is sig-redundant to $G$.

Clearly, if $f$ reduces sig-safe to 0 modulo $G$, then it has a standard representation w.r.t. $G$.

The restriction of the reducer $g_{j_i}$ by $t_i \text{sig}(g_{j_i}) \prec \text{sig}(r_{i-1})$ in each step of a sig-safe reduction is essential for the correctness of signature-based algorithms. If a labeled polynomial $f$ has a standard representation w.r.t. $G$, then $\text{poly}(f)$ has a standard representation w.r.t. $\text{poly}(G)$. Thus we can give a statement similar to Buchberger’s Criterion, see [13], for the signature-based setting.

**Theorem 2.4.** Let $G = \{g_1, \ldots, g_k\} \subset \mathcal{L}$ such that $\{f_1, \ldots, f_m\} \subset \text{poly}(G)$. If for each pair $(i, j)$ with $i > j$, $1 \leq i, j \leq k$, either

1. $S(g_i, g_j)$ is non-minimal, or

2. $S(g_i, g_j)$ has a standard representation w.r.t. $G$,

then $\text{poly}(G)$ is a Gröbner basis of $I$.

**Proof.** See, for example, [17, 18].

**Remark 2.5.** It is well-known that non-minimal elements are useless for the resulting Gröbner basis as well as for the intermediate computations in signature-based algorithms. We refer to [13] for more details on this fact.

Next we present a generic signature-based Gröbner basis algorithm lying an emphasis on the general ideas behind signature-based computations. Proofs of correctness and termination of Algorithm [14] can be found in [13], Theorem 14.

As in the pure polynomial setting a Gröbner basis algorithm without any criteria to detect not necessary computations in advance, like Algorithm [14], represents, is not efficient. In the signature-based world there exist two main criteria to detect useless critical pairs:

---

4Due to signature restrictions the inverse does not necessarily hold.
Algorithm 1 Generic signature-based Gröbner basis algorithm w.r.t. < (SBA)

**Input:** $F = (f_1, \ldots, f_m)$ a finite sequence of elements in $R$

**Ensure:** $\text{poly}(G)$ a Gröbner basis for $\langle F \rangle$ w.r.t. $<$

1. $G \leftarrow \emptyset$, $P \leftarrow \emptyset$
2. for $(i = 1, \ldots, m)$ do
   3. $g_i \leftarrow (e_i, f_i)$
   4. $G \leftarrow G \cup \{g_i\}$
   5. $P \leftarrow P \cup \{(g_i, g_j) \mid g_i, g_j \in G, j < i\}$
3. while ($P \neq \emptyset$) do
   4. Let $(f, g) \in P$ such that $S(f, g)$ has minimal signature w.r.t. $\prec$.
   5. $P \leftarrow P \setminus \{(f, g)\}$
   6. Reduce $S(f, g)$ sig-safe to $r$.
   7. if (poly($r$) $\neq 0$ and $r$ is not sig-redundant to $G$) then
      8. $P \leftarrow P \cup \{(r, h) \mid h \in G, S(r, h) \text{ not non-minimal}\}$
      9. $G \leftarrow G \cup \{r\}$
10. return $\text{poly}(G)$

1. the **non-minimal signature criterion**, based on already known syzygies: It checks if the leading monomial of a syzygy divides the signatures of a critical pair;
2. the **rewritable signature criterion**, based on the fact that for any signature only one polynomial needs to be computed.

The most efficient implementations of signature-based Gröbner basis algorithms nowadays are

1. Faugère’s F5 Algorithm ([21]) and optimizations ([17] [16] [18]),
2. Gao, Guan and Volny’s G2V Algorithm ([25]),
3. Gao, Volny and Wang’s GVW Algorithm ([26] [36]), and
4. Arri and Perry’s algorithm ([3]) respectively Roune and Stillman’s optimized version SB ([32]).

The first two mainly differ in their usage and implementation of the above mentioned signature-based criteria to detect useless critical pairs. They share $\prec_{\text{pot}}$ as ordering used on the signatures which leads to an incremental (w.r.t. the input sequence $F$) computation of Gröbner bases.

However SB as well as GVW are capable of using different orderings on the signatures. In [26] the authors show that the Schreyer ordering $\prec_s$ turns out to be the most efficient one for a wide range of example classes. Due to the fact that $\prec_s$ does not favour the position of the module element over the corresponding term a non-incremental computation is achieved.

For the focus of this paper we are neither interested in the specific variants these criteria can be implemented nor in a comparison of those in terms of efficiency or timings. It is enough to keep in mind that both criteria are based on the signatures of labeled polynomials considered during the algorithm’s workings. Here we focus on the connection of purely polynomial data to the signatures.
Remark 2.6. If we assume $\prec = \prec_{\text{pot}}$, then Algorithm 1 computes a Gröbner basis of $\langle F \rangle$ incrementally, storing the critical pairs of higher index in $P$, but prolonging their reduction until all elements of lower index have been processed.

If there exist several critical pairs in $P$ of the same signature in Line 2 choose the one that entered $P$ first.

Convention 2.7. In the following we often speak about s-polynomials in $P$ meaning the s-polynomial of a corresponding critical pair in $P$. Moreover, for any Gröbner basis algorithm we assume $F$ as input.

Investigating the algorithms’ behaviour for inhomogeneous input data we can focus mainly on the handling of a single s-polynomial: Generate an s-polynomial and compute a sig-safe reduction step of that s-polynomial. Thus we need not specify $\prec$ in the following.

3 Problems of inhomogeneous signature-based computations depending on pure polynomial data

The F5 Algorithm as presented in [21] is restricted to homogeneous input data. None of its successors, like G2V or SB have this restriction. So what is the decisive factor here? Signature-based Gröbner basis computations, in particular the efficiency of the signature-based criteria rely on the fact that s-polynomials are handled by increasing signature.

F5, as presented in [21] chooses s-polynomials differently from Algorithm 1. Instead of picking the next s-polynomial from $P$ w.r.t. minimal signature (Line 7), F5 uses a presorting of $P$ by the degree of the corresponding s-polynomials. To mimic this one needs to change Algorithm 1 beginning in Line 7:

Algorithm 2 Presorting changes for F5

1: . . .
2: $d \leftarrow \min \{\deg(f, g) \mid (f, g) \in P\}$
3: $Q \leftarrow \{(f, g) \mid \deg(f, g) = d\}$
4: $P \leftarrow P \setminus Q$
5: while $(Q \neq \emptyset)$ do
6: Let $(f, g) \in Q$ such that $S(f, g)$ has a minimal signature w.r.t. $\prec$.
7: $Q \leftarrow Q \setminus \{(f, g)\}$
8: . . .

Changing Algorithm 1 as explained above is not enough to ensure the correctness of the resulting algorithm. Replacing the corresponding parts of Algorithm 1 with the pseudo code of Algorithm 2 we need to distinguish where newly generated critical pairs are stored. This postsorting is explained in Algorithm 3.

Note that $\deg(S(r, h)) = d$ in Line 3 of Algorithm 3 is possible due to the restriction to sig-safe reductions. $S(r, h)$ then corresponds to a previously not handled, not sig-safe reduction step of $r$.

Remark 3.1. In [21] a reduction with an element of higher signature is solved in a slightly different way: Once noticed, the corresponding s-polynomial of higher signature is generated. It is clear that for homogeneous input this s-polynomial
Algorithm 3 Postsorting changes for \( F_5 \)

1: 

2: \textbf{if} \ (\text{poly}(r) \neq 0 \text{ and } r \text{ is not sig-redundant to } G) \ \textbf{then}

3: \hspace{1em} \( P \leftarrow P \cup \{ (r, h) \mid h \in G, S(r, h) \text{ not non-minimal, } \deg(r, h) > d \} \)

4: \hspace{1em} \( Q \leftarrow Q \cup \{ (r, h) \mid h \in G, S(r, h) \text{ not non-minimal, } \deg(r, h) = d \} \)

5: \hspace{1em} \( G \leftarrow G \cup \{ r \} \)

6: 

has the same degree as the other elements already in \( Q \). Thus it is directly added to \( Q \), sorted in by increasing signature. See [18] for more information on this.

Computing a Gröbner basis for an inhomogeneous ideal with \( F_5 \) the idea of homogenization can be used: One homogenizes the elements of \( F \) w.r.t. some new variable, call this \( F^h \). Then a Gröbner basis \( G^h \) for \( \langle F^h \rangle \) is computed w.r.t. a monomial ordering for which the homogenization variable is smaller than all the other ones. Then one can receive a Gröbner basis \( G \) for \( \langle F \rangle \) from \( G^h \).

This attempt has the advantage to compute step-by-step intermediate Gröbner bases up to a given degree \( d \), the degree of generated s-polynomials never drops. Thus all possible reducers are available when they are needed. In our ongoing discussion of \( F_5 \) this means that it is impossible that an element in \( P \) will later on transform to a new labeled polynomial in \( G \) that could be useful for a reduction of an element currently in \( Q \). On the other hand, the problem of this approach is that computing a Gröbner basis for \( \langle F^h \rangle \) can be much harder than the computations for the initial problem by adding solutions at infinity.

Efficiency of signature-based algorithms is based on handling critical pairs by increasing signatures. So in order to understand \( F_5 \)'s restriction to homogeneous input we need to answer the following two questions:

1. Does \( F_5 \) compute new elements by increasing signatures throughout the algorithm's working assuming homogeneous input?

2. If so, does this property get lost when applying \( F_5 \) to inhomogeneous input?

To answer these questions we need to find a connection between the degree of a labeled polynomial, that means, the degree of the polynomial part of it, and its signature.

Looking at the homogeneous situation first, constructing s-polynomials has nice properties: Let \( f \) and \( g \in L \) such that \( \text{poly}(f) \) as well as \( \text{poly}(g) \) are homogeneous. Computing corresponding multipliers \( u \) and \( v \) such that \( ult(f) = vlt(g) \) we can construct their s-polynomial \( S(f, g) \). Clearly, \( u \text{poly}(f) \) and \( v \text{poly}(g) \) are homogeneous, too. It follows that

\[
\deg(S(f, g)) = \deg(u \lt(f)) = \deg(v \lt(g)) = \deg(f, g).
\]

For inhomogeneous \( \text{poly}(f) \) and \( \text{poly}(g) \), the situation is different as, for example, \( \deg(u(f - \lt(f))) \) might be smaller than \( \deg(u(f)) \). So building the s-polynomial of \( f \) and \( g \) a drop for the polynomial degree can happen:

\[
\deg(S(f, g)) \leq \deg(u \lt(f)) = \deg(v \lt(g)) = \deg(f, g).
\]
Next, let us see how F5 handles the signatures and the corresponding degrees:

Input elements of $F$ are initialized to labeled polynomials $g_i = (e_i, f_i)$, by definition it holds that

$$\text{sig-deg} (g_i) = \text{deg} (g_i),$$

regardless of whether $\text{poly}(g_i)$ is homogeneous or not. Generating an s-polynomial of two elements $f$ and $g$ a drop in the degree of the corresponding signature could only happen in the following situations:

1. $\text{sig}(uf) = \text{sig}(vg)$, which would mean that $S(f, g)$ is non-minimal, and thus it would not be computed in F5.
2. Once $S(f, g)$ is built, the signature drops due to some ongoing reduction of the polynomial part. This would not be a sig-safe reduction at this degree step as well as at any upcoming higher degree. Thus such a reduction is not processed.

It follows that

$$\text{sig-deg}(f, g) = \text{sig-deg} (S(f, g)) \geq \text{deg}(f, g) \geq \text{deg}(S(f, g)).$$

Thus new labeled polynomials $h$ are added to $G$ for which $\text{sig-deg}(h) \geq \text{deg}(h)$ holds. Generating new critical pairs and s-polynomials from this point on we come to the following relation for arbitrary $f$ and $g$ in $G$ computed by F5 (or any other signature-based algorithm related to Algorithm 1):

$$\text{sig-deg}(f, g) = \text{sig-deg} (S(f, g)) \geq \text{deg}(f, g) \geq \text{deg}(S(f, g)). \quad (1)$$

Assuming homogeneous input to the algorithm Relation 1 becomes an equation.

So we conclude this discussion with the following facts:

1. In a signature-based Gröbner basis algorithm with homogeneous input the degree of the critical pair (respectively the corresponding s-polynomial) and its signature degree coincide. Therefore it is useless to presort the pair set $P$ by increasing degrees of the s-polynomials and later on sort $Q$ by increasing signatures: The signature of an s-polynomial in $Q$ is always smaller than the signature of an element in $P$.

2. In the inhomogeneous situation the equality between the degree and the signature degree of a labeled polynomial need not hold any longer. Thus presorting critical pairs by the polynomial degree can have bad influence on the signature-based algorithms’ inner working: A processing of the elements by increasing signature can no longer be guaranteed.

Think about the following quite likely situation. Let $(f, g)$ and $(f', g')$ be two critical pairs in $P$, and let $u, v$ respectively $u', v' \in \mathcal{R}$ be the multipliers for $S(f, g)$ respectively $S(f', g')$. Assume that $\text{usig}(f) > \text{usig}(g)$ and $u'\text{sig}(f') \succ v'\text{sig}(g')$. Moreover, assume that $\deg(uf) < \deg(u'f')$. In F5, once all critical pairs of degree smaller than $\deg(uf)$ have been processed, $(f, g)$ is added to $Q$, whereas $(f', g')$ stays in $P$ and its further computation is postponed to a later point. In the situation of inhomogeneous polynomial data it is possible that $\text{usig}(f) \succ u'\text{sig}(f')$ as we just have seen. Thus an element of higher signature is computed before an element of lower signature.

---

3 For example, in Eco-11 such a situation happens hundreds of times.
The main problem is that efficiency of signature-based algorithms which use variants of the non-minimal signature criterion and the rewritable signature criterion together with sig-safe reductions are based on this fact. So F5’s pre-sorting of critical pairs by their polynomial degree can lead to way less efficient computations if the input data is inhomogeneous.

Moreover, we have seen that for homogeneous input F5’s pre-sorting of critical pairs is useless and does not change anything w.r.t. the order in which the algorithm handles its critical pairs: Those are still processed by increasing signatures. On the other hand, exactly this pre-sorting interferes F5 once it comes to inhomogeneous input data.

Thus signature-based Gröbner basis algorithms should always be implemented without a purely polynomial degree preselection in order to achieve a better efficiency.

Convention 3.2. In the following we can assume F5 without polynomial degree preselection, hence as a variant of Algorithm 4.

Remark 3.3. Due to the equality between the polynomial degree and the signature degree of a labeled polynomial in the homogeneous situation, signature-based algorithms are designed to handle Gröbner basis computations very well in this setting: Discarding efficiently useless critical pairs and sorting them by polynomial degree is, in general, the best possible selection strategy in this case.

4 The connection between the signature degree and the sugar degree

In the last section we have seen that in the homogeneous case signature-based algorithms handle critical pairs in an optimal order. Naturally, the question about the algorithms’ usefulness in the inhomogeneous comes to one’s mind.

In [27] Giovini, Mora, Niesi, Robbiano, and Traverso introduce the notion of the sugar degree of a polynomial. Later on, Bigatti, Caboara, and Robbiano describe the idea of a self-saturating variant of Buchberger’s algorithm in [10]. There, an in-depth discussion on the theoretical background of the sugar degree is given. The idea behind this kind of degree is to improve a Gröbner basis computation for inhomogeneous input by giving it the flavour of a homogeneous one. Note that there exist other concepts for optimizations in the inhomogeneous setting, see for example [35] or the idea of self-saturation given in [10].

Still, the approach using the sugar degree is so far the most popular one due to its simple implementation.

Definition 4.1. Let $F$ be the input of a purely polynomial Gröbner basis algorithm computing $G$, let $f$ and $g$ be two elements in $G$, and let $t \in \mathcal{M}$. The sugar degree is defined in the following way:

1. $\text{s-deg}(f_i) := \deg(f_i)$ for all $i \in \{1, \ldots, m\}$,
2. $\text{s-deg}(tf) := \deg(t) + \text{s-deg}(f)$, and
3. $\text{s-deg}(f + g) := \max\{\text{s-deg}(f), \text{s-deg}(g)\}$.

\footnote{See also Section 6.}
For a critical pair \((f, g)\) we define the sugar degree by the sugar degree of the corresponding \(s\)-polynomial, 
\[
\text{s-deg}(f, g) := \text{s-deg}(S(f, g)).
\]

Clearly, if \(F\) consists of homogeneous polynomials the sugar degree coincides with degree throughout the Gröbner basis computation. When computing with inhomogeneous data, the sugar degree becomes a useful tool: It mimics the degree the elements would have, if the input sequence would have been homogenized before starting the computations. Thus using the sugar degree the following threepartite sorting of critical pairs emerges to be very efficient in a wide class of example sets tested (see [27] for more information on this):

1. increasing sugar degree,
2. increasing degree,
3. increasing w.r.t. \(<\).

Critical pairs are then sorted as in the homogeneous situation without the overhead of homogenizing at all. Also this sorting needs not to be optimal, it has a positive influence on the efficiency of Gröbner basis algorithms in general.

Next we discuss how sorting critical pairs by increasing signatures is related to the “sugared” ordering.

**Theorem 4.2.** Let \(f\) be a labeled polynomial appearing during a signature-based Gröbner basis computation, then \(\text{sig-deg}(f) = \text{s-deg}(\text{poly}(f))\).

**Proof.** For each \(f_i \in F\) it holds that the initial labeled polynomial \(g_i = (e_i, f_i)\) fulfills that \(\text{sig-deg}(g_i) = \text{deg}(g_i) = \text{deg}(f_i) = \text{s-deg}(f_i)\).

Let \(f\) and \(g\) be two labeled polynomials in a signature-based Gröbner basis algorithm. For \(t \in M\) it holds that \(\text{sig-deg}(tf) = \text{deg}(t) + \text{sig-deg}(f)\).

Let \(u\) and \(v\) be multipliers in \(R\) such that \(u\text{lt}(f) = v\text{lt}(g)\). Note that we can assume \(S(f, g)\) to be not non-minimal. W.l.o.g. let \(\text{sig}(uf) > \text{sig}(vg)\). Then it holds that \(\text{sig-deg}(f, g) = \text{sig-deg}(S(f, g)) = \text{sig-deg}(uf)\).

Thus the signature degree of a labeled polynomial \(f\) in a signature-based Gröbner basis algorithm coincides with the sugar degree of the corresponding polynomial part \(\text{poly}(f)\).

At this point of our discussion we need to distinguish possible choices for \(<\) on the signatures. Let us have a closer look at different situations assuming a degree compatible monomial ordering:

1. Using \(<_s\) on the signatures, results in a non-incremental signature-based algorithm choosing critical pairs by increasing sugar degree. This is based on the fact that
   \[
   \text{sig}(f) <_s \text{sig}(g) \iff \text{sig-deg}(f) < \text{sig-deg}(g).
   \]
2. Choosing \(<_{\text{pot}}\) the situation gets more complicated: The algorithm prefers the signatures of higher module position. Assuming a critical pair being generated by two elements with signatures of different module positions it is not clear that the signature of higher module position also has a higher degree. One can see that ordering by increasing sugar degree in such a setting even breaks the incremental structure of the algorithm, for example in \textit{Cyclic-5}. 

10
If a not degree compatible monomial ordering \(<\) is given then sorting critical pairs by increasing signature need not lead to a pair set sorted by increasing sugar degree. Even if we use \(\prec_s\) on the signatures we cannot guarantee such a behaviour.

As yet, relaxing the restriction of sorting critical pairs by increasing signatures does not make any sense. For example, ordering critical pairs by sugar degree signature-based computations slow down by a large factor. The strengths of signature-based criteria detecting useless critical pairs and sig-safe reductions are based on ordering by increasing signatures. Disrespecting this fact a less efficient algorithm results. The signature ordering \(<\) has to be preferred towards the polynomial ordering \(<\).

Nevertheless we can conclude that signature-based Gröbner basis algorithms choose by default a good selection strategy if a degree compatible monomial ordering is given, whether or not the input data is homogeneous. It coincides with the sugar degree strategy if \(\prec_s\) is used on the signatures.

5 Still, there is a sour taste left

The discovery of the last section seems to be compatible with published experimental results, for example, see [18]: Sometimes signature-based algorithms have problems computing Gröbner bases of inhomogeneous ideals, for example computing Eco-11 is 9 times slower than computing Eco-11-h in our implementation of F5 w.r.t. the graded reverse-lexicographical ordering. For Eco-11 switching from \(\prec_{pot}\) to \(\prec_s\) can improve timings (see Table 2), still such an approach does not always work. As we have seen, the selection strategy is efficient in the given example. Also the signature-based criteria detecting useless critical pairs work good in the inhomogeneous setting, discarding a lot more elements than a Gebauer–Möller implementation. So the only situation where problems can occur is the reduction process.

Since we lose the connection \(\deg(f) = \text{sig-deg}(f)\) for all labeled polynomials \(f\) computed in signature-based algorithms for inhomogeneous input, forcing the reduction to be sig-safe can have a bad impact on the algorithms’ behaviour.

In the homogeneous case the signature of the multiplied reducer can only be greater due to either its signature index (considering \(\prec_{pot}\)) or lexicographic considerations (depending on the underlying monomial ordering \(<\)). It always holds that \(\text{sig-deg}(f) = \text{sig-deg}(tg)\) for \(tg\) being a reducer of \(f\), possibly sig-safe. Assuming the input to be inhomogeneous it is even possible that \(\text{sig-deg}(tg) > \text{sig-deg}(f)\). The number of not sig-safe reductions could increase compared to the homogeneous setting. This again means that a bunch of new critical pairs is generated, decreasing the algorithms’ efficiency. Assume in the above setting that the reduction itself is not allowed as \(t\text{sig}(g) \succ \text{sig}(f)\). So in the following a new critical pair \((g, f)\) with signature \(t\text{sig}(g)\) is generated and later on computed. The problem is the “later on”: Whereas \((g, f)\) has the same signature degree as \(f\) in the homogeneous setting, assuming polynomials to be inhomogeneous it is possible that

\[
\text{sig-deg}(g, f) = \text{sig-deg}(tg) > \text{sig-deg}(f).
\]

This means that the corresponding data needed from the reduction step of \(f\) and \(tg\) cannot be used in the algorithm at the time it really is needed. This triggers other reductions that would be helpful to take place at an earlier point of the
algorithm to be delayed. Correctness is still ensured, the corresponding reduction steps needed for a Gröbner basis computation are executed nevertheless later on. As we are assuming an underlying well-ordering on the polynomials the delay due to introducing useless data has to be overruled after finitely many computational steps.\footnote{For more details on termination of signature-based algorithms we refer to \cite{16}.} Notwithstanding, the overhead that is computed due to these postponed reduction steps has a penalty on the performance of signature-based Gröbner basis algorithms.

We have tested a wide class of benchmarks from \cite{12} and \cite{28}, covering different admissible orderings, finite and infinite ground fields, including parameters, etc. We have implemented four different variants of signature-based Gröbner basis algorithms (SBA) in the computer algebra system SINGULAR. Since the implementations are still experimental, undergoing further development, they are currently not part of the stable SINGULAR repository. Still, they are publicly available in the branch \texttt{sba} at

\url{https://github.com/ederc/Sources}

In order not to overcharge the reader, we present in this paper results only for a part of our test suite. Those benchmarks represent the algorithms’ overall behaviour quite accurately. The reader interested in the complete data set can get it at

\url{https://github.com/ederc/benchmarks}

The 4 variants of SBA are tested for homogeneous and for inhomogeneous input each, thus we must represent 8 different values per benchmark. We decided to visualize our results by colorized bars. See Table \ref{tab:variants} for an overview of the implemented variants and the color scheme chosen.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
Alg \ Ord & \texttt{\textless}pot & \texttt{\textless}s \\
\hline
F5 & blue & orange \\
AP & green & red \\
\hline
\end{tabular}
\caption{Implemented signature-based Gröbner basis algorithms and color scheme for Figure \ref{fig:overview}}
\end{table}

Warm colors represent computations done in a non-incremental way, cold ones stand for incremental variants. In Figure \ref{fig:overview} the darker variation illustrates results for the respective inhomogeneous example, whereas the lighter variation stands for results achieved computing the corresponding homogenized example.

In Figure \ref{fig:overview} we give an overview of the behaviour of the 4 variants of SBA in several different benchmark sets, both homogeneous and inhomogeneous. We lay our focus on the differences in the reduction process with a look at the ratio between the number of higher signature detections and the number of reduction steps in total. We give these ratios in percentage, represented by the height of the respective bars in the diagrams. With this we would like to get a better feeling for the influence of losing the connection between $\text{deg}(f)$ and $\text{sig-deg}(f)$

\footnote{In this paper we used commit 291021c19066befbcdd8a7d7626e5ddc26d421db4.}

\footnote{Presented in \cite{21}, including the optimizations mentioned in \cite{17} \cite{18} and Section \ref{sec:optimizations}.}

\footnote{Presented in \cite{5}, including the optimizations mentioned in \cite{17} \cite{18}.}

\footnote{For more details on termination of signature-based algorithms we refer to \cite{16}.}
Table 2: Timings in seconds for the computation of a Gröbner basis for the given test case.

| Test case       | F5,≺_{pot} | AP,≺_{pot} | F5,≺_{s}  | AP,≺_{s} |
|-----------------|-------------|-------------|------------|----------|
| Cyclic-7        | 1.330       | 1.260       | 1.840      | 2.660    |
| Cyclic-7-h      | 1.180       | 1.140       | 1.820      | 2.630    |
| Cyclic-8        | 468.260     | 442.870     | 314.970    | 184.900  |
| Cyclic-8-h      | 387.890     | 382.230     | 307.990    | 186.780  |
| Ext-Cyclic-6    | 157.590     | 129.340     | 13.420     | 16.770   |
| Ext-Cyclic-6-h  | 662.380     | 569.880     | 10.260     | 14.090   |
| Ilias-12        | 3,447.510   | 458.480     | 639.870    | 283.960  |
| Ilias-12-h      | 4,381.080   | 2,240.890   | 553.180    | 239.490  |
| Eco-10          | 45.610      | 2.780       | 7.990      | 7.190    |
| Eco-10-h        | 14.990      | 13.280      | 3.660      | 4.290    |
| Eco-11          | 2,398.970   | 29.810      | 163.830    | 125.340  |
| Eco-11-h        | 372.090     | 319.710     | 48.710     | 56.680   |
| Red-Eco-11      | 2.620       | 2.430       | 14.530     | 14.580   |
| Red-Eco-11-h    | 2.600       | 2.460       | 19.290     | 18.760   |
| Red-Eco-12      | 22.010      | 20.330      | 161.530    | 158.430  |
| Red-Eco-12-h    | 21.390      | 18.610      | 246.470    | 241.590  |
| F-744           | 1.550       | 0.740       | 0.430      | 0.450    |
| F-744-h         | 2.090       | 1.470       | 0.360      | 0.380    |
| F-855           | 50.670      | 27.200      | 122.670    | 96.080   |
| F-855-h         | 133.470     | 65.980      | 48.600     | 48.930   |
| Fabrice-24      | 101.900     | 72.250      | 113.710    | 108.300  |
| Fabrice-24-h    | 121.900     | 101.190     | 361.570    | 326.040  |
| Katsura-12      | 111.690     | 61.490      | 1,287.250  | 1,303.360|
| Katsura-12-h    | 109.970     | 54.510      | 1,260.380  | 1,223.710|

in the inhomogeneous setting. For an even better estimate we combine the ratios of Figure 1 with the timings given in Table 2.

All examples were computed on an INTEL® XEON® X5460 @ 3.16GHz processor with 64 GB of RAM and 120 GB of swap space running a 2.6.31–gentoo–r6 GNU/Linux 64-bit operating system.

Note that all of the examples presented in this paper are computed w.r.t. the graded reverse-lexicographical ordering. The complete benchmark set available online also includes computations w.r.t. lexicographical orderings. Due to our discussion in Section 3 signature-based computations w.r.t. not degree compatible monomial orderings are rather inefficient in terms of sorting critical pairs. In such cases we found that it is more efficient to compute a Gröbner basis w.r.t. the graded reverse-lexicographical ordering and then to use a Gröbner conversion via FGLM or even a Gröbner walk.

The results in Figure 4 are rather ambiguous: Sometimes the ratio is several times greater in the inhomogeneous setting than in the corresponding homogeneous one (see, for example, Cyclic-8 and Ext-Cyclic-6 for F5 and AP using ≺_{pot}). Whereas in examples like Ilias-12 it is just the other way around.

In various examples the number of sig-safe reduction steps is a factor of 1000 greater than the number of higher signature detections, for example, see Noon-n
or Katsura-n. Not depending on whether the input is homogeneous or not, the influence of not sig-safe data is not even measurable in these cases.

Talking about incremental versus non-incremental computations there is an inclination that the ratio of the number of higher signature detections and the number of reduction steps done is mostly smaller in the non-incremental setting. Still one needs to keep in mind that Figure 4 presents only the ratios: For example, in Katsura-12 the non-incremental variants of Sba are multiple times slower than the incremental ones (see Table 2), they do approximately 50 times more reduction steps. Due to this high amount of reductions the ratio gets lower. Furthermore, finding a heuristic when to prefer incremental computations over non-incremental ones, for example, see Katsura-n, is of great importance.

Remark 5.1.

1. A discussion on the differences of the implementations of the non-minimal signature criterion and the rewritable signature criterion those 4 algorithms use is not in the focus of this paper. We refer to the corresponding papers for more details. [18] and [19] give an overview on how the 4 variants are related to each other. Note that combining F5 with $\prec_s$ does not introduce any theoretical problems for correctness of the algorithm.

Note that in various low-level implementations in SINGULAR G2V respectively GVW were not competitive to the 4 signature-based algorithms presented here. The lack of a real implementation of the rewritable signature criterion seems to be the reason for this, we refer to [13]. In [30] GVWHS is presented, a variant of GVW using the rewritable criterion of AP. This algorithm as well as the recently by Roune and Stillman in [32] presented SB algorithm coincide with our AP implementation.

2. The number of higher signature detections also depends on the order in which the list of possible reducers is searched through. We can state that in most benchmarks, again independent of the homogeneity of the input polynomials, using the settings and heuristics of SINGULAR’s internal, Gebauer-Möller-like Gröbner basis algorithm groebner is a good choice. Of course there are examples like Fabrice-24 where adjusting the search by hand leads to an improvement in timings of a factor of 10, but in other examples exactly this choice slows down computations by a factor of 100 and even more. Finding good heuristics for searching in the set of reducers is an open problem; doing this by increasing respectively decreasing signature is not a good choice in a wide range of example classes.

6 Conclusion and further research

We have given an in-depth discussion about the behaviour of signature-based Gröbner basis algorithms in the inhomogeneous case.

Explaining, why F5, as initially presented in [21] is restricted to homogeneous input data, we found a solution for relaxing this condition. Moreover, by doing this the presentation of the algorithm simplifies. This makes it easier for a reader without prior knowledge of signature-based algorithms to get access to this area of Gröbner basis theory.
Figure 1: Ratio of higher-signature detections to reduction steps for various benchmarks
Furthermore, we have presented for the first time the strong connection between the signature degree and the sugar degree of the corresponding polynomial parts. It is a delightful discovery that signature-based algorithms sort the corresponding pair set in a nearly optimal order from the polynomial point of view when assuming a degree compatible monomial ordering. Reordering critical pairs is bounded by the condition of computing by increasing signatures. The question if we can find more efficient orderings on the signatures in these situations remains unanswered and needs further investigation.

Investigating the suspicion that the lost connection between polynomial degree and signature degree in the inhomogeneous setting can affect the sig-safe reduction process negatively cannot be confirmed. There are specific examples where the number of higher signature detections increase strongly in the inhomogeneous setting (compared to the homogeneous one), but there are also examples behaving just the other way around.

Further investigations might be done in the direction of combining sig-safe reduction steps with the idea of self-saturation given in [10]. The overall idea of self-saturation is to use special kinds of reduction steps to achieve so-called (weak) saturating remainders. Thereby the Gröbner basis algorithm starts with the homogenized set of generators, but instead of plainly computing the homogeneous Gröbner basis of the homogenized input data, reducers respectively the remainders of reductions are exchanged by saturated pendants. The process of self-saturation has a positive effect on Buchberger-like Gröbner basis algorithms as shown in [10]. Being restricted to sig-safe reductions in signature-based algorithms the freedom of choice for the saturated elements is limited and might break its positive effects on the computations.

Acknowledgments. The author would like to thank the SINGULAR team at the University of Kaiserslautern for their support. Moreover, the author especially wishes to thank the anonymous referees whose comments improved the paper.

References

[1] Albrecht, M., Cid, C., Faugère, J.-C., and Perret, L. On the relation between the MXL family of algorithms and Gröbner basis algorithms. [http://www-salsa.lip6.fr/~jcf/Papers/ACFP12.pdf](http://www-salsa.lip6.fr/~jcf/Papers/ACFP12.pdf) 2012. (in press).

[2] Albrecht, M. and Perry, J. F4/5. [http://arxiv.org/abs/1006.4933](http://arxiv.org/abs/1006.4933), 2010.

[3] Arri, A. and Perry, J. The F5 Criterion revised. *Journal of Symbolic Computation*, 46(2):1017–1029, June 2011. Preprint online at [arxiv.org/abs/1012.3664](http://arxiv.org/abs/1012.3664).

[4] Ars, G. *Applications des bases de Gröbner à la cryptographie*. PhD thesis, Université de Rennes I, 2005.

[5] Ars, G. and Hashemi, A. Extended F5 Criteria. *Journal of Symbolic Computation, MEGA 2009 special issue*, 45(12):1330–1340, 2010.

[6] M. Bardet. *Étude des systèmes algébriques surdéterminés. Applications aux codes correcteurs et à la cryptographie*. PhD thesis, Université Paris 6, 2004.
[7] Bardet, M. On the Complexity of a Gröbner Basis Algorithm. INRIA Algorithms seminar 2002–2004, 2004.

[8] Bardet, M., Faugère, J.-C., and Salvy, B. On the complexity of Gröbner basis computation of semi-regular overdetermined algebraic equations. http://www-salsa.lip6.fr/~jcf/Papers/43BF.pdf, November 2004.

[9] Bardet, M., Faugère, J.-C., Salvy, B., and Yang, B.Y. Asymptotic expansion of the degree of regularity for semi-regular systems of equations. http://www-salsa.lip6.fr/~jcf/Papers/BFS05.pdf, May 2005.

[10] Bigatti, A. M., Caboara, M., and Robbiano, L. Computing Inhomogeneous Gröbner Bases. Journal of Symbolic Computation, 46:498–510, 2011.

[11] Bigatti, A. M., La Scala, R., and Robbiano, L. Computing toric ideals. Journal of Symbolic Computation, 27:351–365, 1999.

[12] Bini, D. A. and Mourrain, B. Polynomial Test Suite. http://www-sop.inria.fr/saga/POL/, 2012.

[13] Buchberger, B. Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal. PhD thesis, University of Innsbruck, 1965.

[14] Collart, S., Kalkbrener, M., and Mall, D. Converting Bases with the Groebner Walk. Journal of Symbolic Computation, 24:265–469, 1997.

[15] Decker, W., Greuel, G.-M., Pfister, G., and Schönemann, H. SINGULAR 3-1-5 — A computer algebra system for polynomial computations, 2012. http://www.singular.uni-kl.de.

[16] Eder, C., Gash, J., and Perry, J. Modifying Faugère’s F5 Algorithm to ensure termination. ACM SIGSAM Communications in Computer Algebra, 45(2):70–89, 2011. [http://arxiv.org/abs/1006.0318]

[17] Eder, C. and Perry, J. F5C: A Variant of Faugère’s F5 Algorithm with reduced Gröbner bases. Journal of Symbolic Computation, MEGA 2009 special issue, 45(12):1442–1458, 2010. dx.doi.org/10.1016/j.jsc.2010.06.019.

[18] Eder, C. and Perry, J. Signature-based Algorithms to Compute Gröbner Bases. In ISSAC 2011: Proceedings of the 2011 international symposium on Symbolic and algebraic computation, pages 99–106, 2011.

[19] Eder, C. and Roune, B. H. Signature Rewriting in Gröbner Basis Computation. In ISSAC 2013: Proceedings of the 2013 international symposium on Symbolic and algebraic computation, page tba, 2013.

[20] Faugère, J.-C. A new efficient algorithm for computing Gröbner bases (F4). Journal of Pure and Applied Algebra, 139(1–3):61–88, June 1999. http://www-salsa.lip6.fr/~jcf/Papers/F99a.pdf” http://www-salsa.lip6.fr/~jcf/Papers/F99.pdf.

[21] Faugère, J.-C. A new efficient algorithm for computing Gröbner bases without reduction to zero F5. In ISSAC’02, Villeneuve d’Ascq, France, pages 75–82, July 2002. Revised version from http://fgbbrs.lip6.fr/jcf/Publications/index.html.
[22] Faugère, J.-C., Gianni, P. M., Lazard, D., and Mora, T. Efficient Computation of Zero-Dimensional Gröbner Bases by Change of Ordering. *Journal of Symbolic Computation*, 16(4):329–344, 1993.

[23] Galkin, V. Simple signature-based Groebner basis algorithm. *http://arxiv.org/abs/1205.6050*, 2012.

[24] Galkin, V. Termination of original F5. *http://arxiv.org/abs/1203.2402*, 2012.

[25] Gao, S., Guan, Y., and Volny IV, F. A New Incremental Algorithm for Computing Groebner Bases. *Journal of Symbolic Computation – ISSAC 2010 Special Issue*, 1:13–19, 2010.

[26] Gao, S., Volny IV, F., and Wang, D. A new algorithm for computing Groebner bases. *http://eprint.iacr.org/2010/641*, 2010.

[27] Giovini, A., Mora, T., Niesi, G., Robbiano, L., and Traverso, C. “One sugar cube, please” or selection strategies in the Buchberger algorithm. In *ISSAC’91*, pages 49–54, 1991.

[28] Gräbe, H.-G. *The SymbolicData Project – Tools and Data for Testing Computer Algebra Software*, 2011. *http://www.symbolicdata.org*.

[29] Greuel, G.-M. and Pfister, G. *A Singular Introduction to Commutative Algebra*. Springer Verlag, 2nd edition, 2007.

[30] Huang, L. A new conception for computing Gröbner basis and its applications. *http://arxiv.org/abs/1012.5425*, 2010.

[31] Pan, S., Hu, Y., and Wang, B. The Termination of Algorithms for Computing Gröbner Bases. *http://arxiv.org/abs/1202.3524*, 2012.

[32] Roune, B. H. and Stillman, M. Practical Gröbner Basis Computation. In *ISSAC 2012: Proceedings of the 2012 international symposium on Symbolic and algebraic computation*, 2012.

[33] Sun, Y. and Wang, D. A New Proof of the F5 Algorithm. *http://arxiv.org/abs/1004.0084*, 2010.

[34] Sun, Y. and Wang, D. A generalized criterion for signature related Gröbner basis algorithms. In *ISSAC 2011: Proceedings of the 2011 international symposium on Symbolic and algebraic computation*, pages 337–344, 2011.

[35] Ufnarovski, V. On the Cancellation Rule in the Homogenization. *Computer Science Journal of Moldova*, 16(1):133–145, 2008.

[36] Volny, F. *New algorithms for computing Gröbner bases*. PhD thesis, Clemson University, 2011.

[37] Wichmann, T. Der FGLM-Algorithmus: verallgemeinert und implementiert in SINGULAR. *Diploma thesis at the university of Kaiserslautern*, 1997.

[38] Zobnin, A. I. Generalization of the F5 algorithm for calculating Gröbner bases for polynomial ideals. *Programming and Computer Software*, 36:75–82, 2010. *http://dx.doi.org/10.1134/S0361768810020040*.