Linear Time Clustering for High Dimensional Mixtures of Gaussian Clouds

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Abstract

Clustering mixtures of Gaussian distributions is a fundamental and challenging problem that is ubiquitous in various high-dimensional data processing tasks. In this paper, we propose a novel and efficient clustering algorithm for \( n \) points drawn from a mixture of two Gaussian distributions in \( \mathbb{R}^p \). The algorithm involves performing random 1-dimensional projections until a direction is found that yields the user-specified clustering error \( e \). For a 1-dimensional separability parameter \( \gamma \) satisfying \( \gamma = \frac{Q^{-1}(e)}{c \log \log p} \), with \( c \) as the separability parameter of the two Gaussians in \( \mathbb{R}^p \). It is shown that the square of the 1-dimensional separability resulting from a random projection is in expectation equal to \( c^2 \), thus guaranteeing a small number of projections in realistic scenarios. Consequently, the expected overall running time of the algorithm is linear in \( n \) and quasi-linear in \( p \). This result stands in contrast to prior works which learn the parameters of the Gaussian mixture model and provide polynomial or at-best quadratic running time in \( p \) and \( n \). The new scheme is particularly appealing in the challenging setup where the ambient dimension of the data, \( p \), is very large and yet the number of sample points, \( n \), is small or of the same order as \( p \). We show that the bound on the expected number of 1-dimensional projections extends to the case of three or more Gaussian mixture distributions. Finally, we validate these results with numerical experiments in which the proposed algorithm is shown to perform within the prescribed accuracy and running time bounds.

I. INTRODUCTION

Data clustering is a key task that is required of many modern data analysis applications, especially those involving high dimensional data. In this paper we study the problem of data clustering, where each cluster is generated from a Gaussian distribution. In fact, many real life scenarios, even though the data is not a priori known to be generated from a mixture of Gaussian distributions, can still be modeled as such. Such modeling allows the use of parametric approaches for clustering data, as well as for performing other inference tasks.

Consider the problem of clustering a mixture of two Gaussian distributions in \( \mathbb{R}^p \), based on \( n \) random drawings \( \mathbf{X}_1, \ldots, \mathbf{X}_n \). Each point \( \mathbf{X}_j \), with probability \( w_1 \), is drawn from \( \mathcal{N}(\mathbf{m}_1, \Sigma_1) \), and with probability \( w_2 = 1 - w_1 \) from \( \mathcal{N}(\mathbf{m}_2, \Sigma_2) \), where \( \mathbf{m}_i \in \mathbb{R}^p \) and \( \Sigma_i \in \mathbb{R}^{p \times p} \), for \( i = 1, 2 \). Given unlabeled observation points \( \mathbf{X}_1, \ldots, \mathbf{X}_n \) in
R^p, the clustering task aims at labeling each point in R^p as either 1 or 2. The goal is to minimize the clustering error probability, which is defined as the probability that the label of the Gaussian that has generated a point disagrees with its assigned label, up to a fixed permutation of all labels.

While an abundance of heuristics exist (e.g. [1], [2]), these methods provide little guarantees about the accuracy of the solution or the convergence of the process. One classical approach to address the described clustering problem is to first learn the parameters of the component Gaussians, and then design a clustering algorithm based on the learned parameters (see [3]). Yet, the problem of learning the parameters of a mixture of Gaussians for a given accuracy is still a challenging problem. In particular, the estimation of the p × p component covariance matrices (especially in high dimensional spaces) is very challenging, both theoretically and algorithmically. When the sample size n is comparable with the ambient dimension p, and not with p^2, while clustering of the points might be feasible, without any further knowledge about the structure of the Gaussians, estimating the covariance matrices becomes almost impossible. To this end not much research has been dedicated to the effect of parameters estimation error on the actual clustering error.

More recent approaches have been suggested which learn the parameters of a Gaussian mixture model (GMM) by projecting the data points to a lower dimensional space to estimate the labels of the points, and then learning the parameters of the Gaussians in the high dimensional space using the estimated labels (e.g [4], [5]). The number of projections as well as the learning of the parameters in the projected space have been shown to be possible in polynomial time in the dimension p and/or number n of samples, and so far has been shown to be quadratic on p or n. However, it is an open question if one can learn clusters accurately enough in linear time in the ambient dimension p or the sample size n, before or after estimating the Gaussian parameters in the ambient or lower dimensional sub-spaces. This question has motivated our work to explore the tradeoff between the accuracy of clustering in collections of 1-dimensional random projections and the running time. Surprisingly enough, we show that when a user-specified clustering error reflects a 1-dimensional separability close to the separation of the Gaussians in R^p our proposed algorithm achieves the desired accuracy by using a small number of random projections. With a 1-dimensional learner (such as one based on the method of moments (MoM) [6] or expectation maximization (EM) [7]) we achieve a bound on the expected running time that is quasi-linear in p at o(np log p), when the prescribed clustering error e corresponds to a 1-dimensional separability γ such that γ ≤ c log log p, where c is defined as

\[ c = \frac{||m_1 - m_2||}{\sqrt{p} \left( \sqrt{\lambda_{\text{max}}(\Sigma_1)} + \sqrt{\lambda_{\text{max}}(\Sigma_2)} \right)}, \]

and \( \lambda_{\text{max}}(\Sigma_i) \) denotes the maximum eigenvalue of \( \Sigma_i \), \( i = 1, 2 \).

Our results are motivated by an initial observation that a random projection into 1-dimension not only preserves separability c in expectation (as one may hope for), but also, with a probability that depends on c, and the ambient dimension p, can be larger than c. This can be demonstrated in the following empirical observation of the distribution of 1-dimensional separability values γ obtained by projecting a mixture of two Gaussians with separability c into multiple 1-dimensional random directions and measuring their separability values. As seen in Fig. [8], a significant fraction of the probability mass lies in values higher than c. Recovering the random directions in which the
separability is higher than \(c\) can be done remarkably fast with an \(O(n)\) running-time learning algorithm in 1-dimension.

The two observations above, namely, non-negligible probability of random directions with \(\gamma > c\) and \(O(n)\) efficiency in 1-dimensional parametric clustering, as explained later in the paper, give rise to running time that is quasi-linear in \(p\) for clustering GMMs via 1-dimensional projections. Our results provide a significant running time improvement over state-of-the-art algorithms that cluster GMMs by learning their parameters. For instance, the number of 1-dimensional projections is at best quadratic in \(p\) \([7], [8]\), or cubic in \(p\) for spectral-projections-based methods \([5], [9]–[11]\). In other cases, where higher order random projection subspaces are used, the running time is quadratic in \(n\) \([4]\). Clearly, the purpose of these methods is to learn the parameters to a high accuracy, while for clustering we show that we can still generate accurate clustering very efficiently. We provide more details in the following related work section.

![Figure 1](image)

Fig. 1. Empirical probability of 1-dimensional separability values \(\gamma\) generated from random projections of 1000 realizations of 10K data points generated by an equal probability spherical Gaussians mixture in \(\mathbb{R}^{1000}\) with separability \(c = 1\).

Finally, we analyze the sample complexity of our proposed algorithm and its impact on the clustering accuracy. Our analysis has contributions in two aspects: First, we show that for a given sample complexity and 1-dimensional separability \(\gamma\), any recovered 1-dimensional mixture parameters with separability below a \(\gamma\)-dependent threshold can be discarded since the parameter estimation will be degenerate and will yield incorrect clustering with high probability. Second, for those cases in which the separability is high enough for accurate parameter estimation, the effect of the error in parameter-estimation on the actual clustering error is quantified.

### A. Related work

Data clustering is an extensive and well-studied research topic with vast existing literature \([12], [13]\). In the context of GMMs, the main focus of the literature has been on using clustering as means for estimating the parameters, where only in few cases the estimated parameters are used as a step towards assigning data points to clusters.
In such cases, once the parameters are learned, as a byproduct, one can cluster points by assigning to each point the Gaussian cloud with highest posterior probability. Of high popularity in this context are the EM and K-means algorithms [1], [2], whose convergence has weak guarantees. These shortcomings are especially critical at high dimensions, where the number of parameters to be estimated is quadratic in the dimension $p$.

Various approaches to learning the parameters of a GMM have been proposed, most of them involve projecting the data onto some lower dimensional subspace, with a dimension that is typically polynomial in the number of mixture parameters [5], [7], [9]-[11]. The mixture components are learned in that subspace, then, typically, the GMM parameters in the original high-dimensional space are reconstructed. At a high level, the goal of this approach is to project the data in such a way that data points generated from different components are separated in some formal sense. Essentially, learning the Gaussians in a lower dimensional subspace is more efficient, since projection preserves Gaussianity, increases the concentration of the points, and reduces the number of unknown parameters. However, the computational complexity of these methods has been a major bottleneck which renders these approaches as mostly theoretical with limited applicability. For example, methods relying on singular value decomposition (SVD) of the data covariance matrix [5], [9]-[11] require the computationally-heavy factorization of a large and dense covariance matrix. Naively done, this would cost $O(p^3)$ operations. In addition, and as mentioned above, the covariance estimation by itself requires a large sample size in high dimensions. Still, the benefit of using spectral methods is that they provide accuracies up to arbitrary separation, while, on the other hand, in high dimensional settings they are simply not applicable due to their high computational complexity. More efficient methods that rely on Euclidean distance and use random projections, however, have an error scaling with the square root of the ambient dimension, and to-date only polynomial running time.

Most notable, in the context of our work, are the Euclidean-based projection methods [4], [8] and [7]. In [8] for arbitrary close spherical $k$-GMMs, the authors provide a polynomial learning algorithm that is based on projecting sequentially into a $2k^2$-dimensional subspace that maximizes separation of the moments of the Gaussians. Then it examines 1-dimensional directions within this subspace to increase separation. We note that [8] provides no efficient recipe for identifying this optimal subspace, except for trying all $\binom{p}{2k^2}$ subspaces.

Another approach based on random projections is the seminal work of [4], whose separation requirement scales as $\sqrt{p}$. The author uses a random subspace to perform basic clustering of points into Gaussians and then learns the parameters of the Gaussians based on the cluster assignments of the points. The algorithm proposed in [4] requires $d = O(\log 1/(\epsilon \delta))$ projections to estimate the means within $\epsilon \sigma_{\text{max}} \sqrt{p}$ with probability larger than $1 - \delta$. Since the dimension of the projected space is larger than 1, the clustering algorithm runs in a quadratic factor in the sample size. Thus, overall the algorithm runs in quadratic time $O(dn^2 + ndp)$. Additionally, the method is restricted to almost-spherical or shared covariances only. Much of these restriction could be alleviated by using 1-dimensional projections for clustering, as we show in our work. In addition the sample complexity required by [4] is $n = k^O(\log^2(1/\epsilon \delta))$ which by far overwhelming compared to our sample complexity.

Finally, we note [7] which extends the work of [14] into the $k$-GMM setting, and employs $p^2$ random 1-dimensional projections. While the 1-dimensional running time of the MoM [6] algorithm is linear in sample size $n$, the number of projections used is again prohibitive in high dimensions. The running time of [7] is polynomial
in the dimension and inverse accuracy among other GMM parameters.

To remediate the above shortcomings, in this paper we propose a new and efficient algorithmic framework for clustering GMMs, where the GMM parameter estimation is done in 1-dimension to learn the cluster assignments. For a prescribed error (that corresponds directly to a 1-dimensional separability \( \gamma \)), our algorithm computes in expected running time of \( o(np \log p) \) the GMM cluster assignments, if \( \gamma \) is within a log-log-factor of \( p \) times \( c \), i.e. \( \gamma \leq c \log \log p \). If the prescribed error does not correspond to a 1-dimensional separability close enough to \( c \), our algorithm returns after it exhausted its budget of projections with the best achievable clustering error. Since the running-time complexity also depends on the sample size \( n \), we provide a sample complexity bound of \( n = O(\frac{1}{\epsilon^2} \log \frac{1}{\delta}) \) \([13]\), for parameter accuracy \( \epsilon \) and probability \( 1 - \delta \).

B. Main contributions

- We present the first fully theoretically-analyzable clustering algorithm for GMM that is efficiently implementable. In some practical settings described in detail later, the proposed algorithm has \( O(np \log (p)) \) running time. These results improve on the most efficient known algorithms whose running time is at best quadratic in \( p \) \([7]\) or in \( n \) \([4]\). The algorithm receives as input the data points, a prescribed clustering error \( \epsilon \) and a projection budget \( M \). It then proceeds as follows. It sequentially projects the data into a direction \( A \), selected uniformly at random. The mixture parameters of \( \langle A, X \rangle \) are estimated, and used to estimate the clustering error \( \bar{\epsilon} \). If \( \bar{\epsilon} < \epsilon \), the algorithm returns the cluster assignments derived in the 1-dimensional space. Otherwise, if the projection budget is not exhausted, a new random projection is performed and the process continues. If the budget is exhausted, the algorithm returns “Error Not Achievable”, and outputs the best achievable clustering assignment.

- We derive theoretical bounds (Theorems\([2]\) and\([3]\)) on the probability of achieving separability \( \gamma \) in a randomly selected 1-dimensional direction. Therefore for any given user-defined error \( \epsilon \) such that \( \epsilon = Q(\gamma) \), Corollary\([3]\) provides the expected number of projections required in order to find at least one direction in which \( \epsilon \) is achieved. For a non-spherical mixture, by Theorems\([3]\) and\([4]\) we map the number of projections to the rank \( r \) and the maximal eigenvalue of the matrix \( \Sigma = \Sigma_1 + \Sigma_2 \).

- We provide conditions on the ratio \( \frac{\gamma}{\gamma_{min}} \), for spherical (Corollaries\([3]\),\([4]\), and\([5]\)) and non-spherical Gaussian point clouds (Corollaries\([7]\),\([8]\), and\([9]\)), for which the number of projections required to achieve the error \( \epsilon \) is sub-linear and even sub-logarithmic in \( p \). The last renders the expected running time of our algorithm as \( O(np \log p) \), for the corresponding clustering error values.

- Theorem\([2]\) is extended to the case of \( k > 2 \) GMM in Lemma\([2]\). We show that in order to attain a prescribed error with only a small number of random projections, the minimal separability \( \gamma_{min} \) between any two Gaussians in \( \mathbb{R}^p \) needs to be proportional to \( k^2 \gamma_{min} \), where \( \gamma_{min} \) is the minimal separability between any pair of Gaussians in 1-dimension (corresponding to the prescribed error).

- Our sample-size complexity analysis provides a theoretically-founded mechanism for discarding directions which yield very poor clustering performance, or equivalently, have too small separability (Lemma\([4]\)). Moreover, we characterize the effect of parameters estimation error on the estimated clustering error (Lemma\([5]\)).
• The reported empirical experiments validate the reliability of our bounds and the performance of our proposed algorithm in light of two pivotal performance indicators: the error and the number of projections. We study the tradeoff between them, which serves as a strong motivation for using our algorithm.

**Paper Organization:** In section II we define the Gaussian mixture clustering problem. In Section III we study the probabilistic conditions under which a high-dimensional γ-separable mixture is mapped to a γ-separable mixture in 1-dimension, and we derive the expected number of projections to achieve γ-separability. Section IV provides the algorithm and its variant for clustering an arbitrary mixture as well as a mixture of spherical Gaussians. Section V describes the sample complexity of our algorithm in light of using the 1-dimensional parameter estimation of [15]. In Section VI we provide experiments for both the spherical and non-spherical cases. The experimental validation demonstrates the error achieved by our algorithm and the efficiency and running time of the algorithm in light of the upper bounds on the number of projections required to achieve a given error. We conclude in Section VII. The proofs of our main results are given in the appendices.

## II. BACKGROUND

### A. Problem statement

Consider the problem of clustering \( n \) points \( \mathbf{X}_1, \ldots, \mathbf{X}_n \) in \( \mathbb{R}^p \) that are generated according to a mixture of two Gaussian distributions as \( w_1 \mathcal{N}(\mathbf{m}_1, \Sigma_1) + w_2 \mathcal{N}(\mathbf{m}_2, \Sigma_2) \). Let \( f_1 \) and \( f_2 \) denote the corresponding probability density distributions (pdf) of \( \mathcal{N}(\mathbf{m}_1, \Sigma_1) \) and \( \mathcal{N}(\mathbf{m}_2, \Sigma_2) \), respectively. A clustering algorithm \( C : \mathbb{R}^p \rightarrow \{1, 2\} \), without having access to the parameters \((\mathbf{m}_1, \mathbf{m}_2, \Sigma_1, \Sigma_2, w_1)\), maps each point in \( \mathbb{R}^p \) into label 1 or label 2. An error occurs whenever a point \( \mathbf{X} \) generated according to \( \mathcal{N}(\mathbf{m}_1, \Sigma_1) \) is assigned to label 2 and vice versa. For \( \mathbf{X} \in \mathbb{R}^p \) generated according to the described distribution, let \( T(\mathbf{X}) \) denote the random variable that represents the underlying correct label. In other words, \( T(\mathbf{X}) = 1 \), if \( \mathbf{X} \) is drawn according to \( \mathcal{N}(\mathbf{m}_1, \Sigma_1) \), and \( T(\mathbf{X}) = 2 \), otherwise. In the **Bayesian classification setting**, where parameters of the Gaussians and their weights are known, an optimal mapping \( C^* \), which minimizes the probability of error, can be derived as the solution of the following optimization

\[
C^*(\mathbf{x}) = \arg \max_{i \in \{1, 2\}} w_i f_i(\mathbf{x}).
\]  

(2)

From this formulation, the decision boundary of an optimal separator can be characterized as the set of points which satisfy \( w_1 f_1(\mathbf{x}) = w_2 f_2(\mathbf{x}) \), or

\[
\log w_1 - \frac{p}{2} \log |\Sigma_1| - \frac{1}{2} (\mathbf{x} - \mathbf{m}_1)^T \Sigma_1^{-1} (\mathbf{x} - \mathbf{m}_1) = \log w_2 - \frac{p}{2} \log |\Sigma_2| - \frac{1}{2} (\mathbf{x} - \mathbf{m}_2)^T \Sigma_2^{-1} (\mathbf{x} - \mathbf{m}_2).
\]

Taking all the terms to one side of the equation, we derive

\[
\frac{1}{2} \mathbf{x}^T (\Sigma_1^{-1} - \Sigma_2^{-1}) \mathbf{x} + \mathbf{m}_2^T \Sigma_2^{-1} - \mathbf{m}_1^T \Sigma_1^{-1} \mathbf{x} + \frac{1}{2} (\mathbf{m}_1)^T \Sigma_1^{-1} \mathbf{m}_1 - \frac{1}{2} (\mathbf{m}_2)^T \Sigma_2^{-1} \mathbf{m}_2 + \log \frac{w_2}{w_1} + \frac{p}{2} \log \frac{|\Sigma_1|}{|\Sigma_2|} = 0.
\]  

(3)
An optimal Bayesian classifier has optimal decision region that is described by the quadratic equation in (3). Let \( \epsilon(\alpha, \mathbf{m}_1, \mathbf{m}_2, \Sigma_1, \Sigma_2) \) denote the error of this optimal Bayesian classifier of a mixture of two Gaussians, then \( \epsilon(\alpha, \mathbf{m}_1, \mathbf{m}_2, \Sigma_1, \Sigma_2) \) is characterized as follows:

**Theorem 1.** Consider points in \( \mathbb{R}^p \) drawn from a mixture of two Gaussian distributions \( \alpha \mathcal{N}(\mathbf{m}_1, \Sigma_1) + (1 - \alpha) \mathcal{N}(\mathbf{m}_2, \Sigma_2) \). Assume that the two components of the mixture are \( c \)-separable. Then,

\[
\epsilon(\alpha, \mathbf{m}_1, \mathbf{m}_2, \Sigma_1, \Sigma_2) \leq Q \left( \frac{c}{\sqrt{p}} \right),
\]

where

\[
Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-\frac{u^2}{2}} du.
\]

In general, (3) describes a quadratic surface in \( \mathbb{R}^p \). However, in the special case, where \( \Sigma_1 = \Sigma_2 = \Sigma \), the second order term cancels out, and the boundary region can be characterized as a hyperplane

\[
(\mathbf{m}_2 - \mathbf{m}_1)^T \Sigma^{-1} \mathbf{x} + \frac{1}{2} (\mathbf{m}_1)^T \Sigma^{-1} \mathbf{m}_1 - \frac{1}{2} (\mathbf{m}_2)^T \Sigma^{-1} \mathbf{m}_2 + \log \frac{w_2}{w_1} = 0.
\]

In this case, the two Gaussians are called **linearly-separable.** While in general, two Gaussian distributions are not linearly separable, it is still appealing, especially from a practical perspective, to focus on linear separators, whose decision boundaries can be described by hyperplanes. The following corollary derives a tighter upper bound on \( \epsilon(\alpha, \mathbf{m}_1, \mathbf{m}_2, \Sigma_1, \Sigma_2) \) for the special case of \( \Sigma_1 = \Sigma_2 \):

**Corollary 1.** Consider points in \( \mathbb{R}^p \) drawn from a mixture of two Gaussian distributions with shared covariance \( \alpha \mathcal{N}(\mathbf{m}_1, \Sigma) + (1 - \alpha) \mathcal{N}(\mathbf{m}_2, \Sigma) \). Assume that the two components of the mixture are \( c \)-separable. Then,

\[
\epsilon(\alpha, \mathbf{m}_1, \mathbf{m}_2, \Sigma, \Sigma) \leq Q \left( c\sqrt{p} \right),
\]

In our setting of data clustering, of course, the parameters are not known and one cannot directly characterize (3). Instead of having access to the parameters \( (\mathbf{m}_1, \mathbf{m}_2, \Sigma_1, \Sigma_2, w_1) \), in the data clustering setup, we just have access to \( X_1, \ldots, X_n \), which are independent and identically distributed (i.i.d.) as \( w_1 \mathcal{N}(\mathbf{m}_1, \Sigma_1) + (1 - w_1) \mathcal{N}(\mathbf{m}_2, \Sigma_2) \). Therefore, in order to perform clustering of mixture of Gaussians, a natural approach is to first learn (estimate) the model parameters based on the observed data as \( (\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2, \hat{\Sigma}_1, \hat{\Sigma}_2, \hat{w}_1) \) and then use the estimated parameters in (3) to approximate the optimal decision boundary. While this approach seems plausible in theory, especially in the setting where \( n \) is not very large compared to \( p \), it has major shortcomings both theoretically and algorithmically.

From a theoretical perspective, in general, to estimate the \( p^2 + 3p + 1 \) unknown parameters of the model, \( n \) should be at least of order \( p^2 \). However, in many scenarios \( n \) is not that large and is even of the same order as \( p \). Moreover, theoretically, it is not even clear whether this method of estimating the parameters and then building a decision boundary based on those estimated values is optimal in general. On the other hand, from a practical perspective, the computational complexity of estimating the \( p \times p \) covariance matrices \( \Sigma_1 \) and \( \Sigma_2 \) is very high.

To address the above-mentioned challenges, in this paper we propose a novel and highly efficient approach for
clustering a mixture of two general Gaussian distributions. The proposed method does not rely on estimating the parameters of the high-dimensional Gaussian distributions.

**B. Linear separators**

Linear separators play a key role in our proposed method. Hence, in the following, we describe two approaches for linear clustering algorithms for points lying in $\mathbb{R}^p$. We then briefly discuss their equivalence.

1) **Linear clustering in $\mathbb{R}^p$**: A linear clustering algorithm for points in $\mathbb{R}^p$ can be described by vector $a \in \mathbb{R}^p$ and scalar $b \in \mathbb{R}$. A separator described by $(a, b)$ labels a point $x \in \mathbb{R}^p$ as 1, if $a^T x > b$, and otherwise as 2.

If $X \in \mathbb{R}^p$ is generated according to a mixture of two Gaussians described as $w_1 N(m_1, \Sigma_1) + w_2 N(m_2, \Sigma_2)$, $a^T X - b$ is also a mixture of two Gaussians in 1-dimension. More precisely, $a^T X - b$ is distributed as $w_1 N(a^T m_1 - b, a^T \Sigma_1 a) + w_2 N(a^T m_2 - b, a^T \Sigma_2 a)$. Therefore, the error probability under a linear clustering algorithm described by $(a, b)$ can be expressed as

$$
\epsilon_{\text{lin}}(a, b) = w_1 Q \left( \frac{b - a^T m_1}{\sqrt{a^T \Sigma_1 a}} \right) + w_2 Q \left( \frac{a^T m_2 - b}{\sqrt{a^T \Sigma_2 a}} \right),
$$

(7)

Hence, the optimal Bayesian linear separator in $\mathbb{R}^p$, $(a^*, b^*)$, is defined as the one that minimizes $\epsilon_{\text{lin}}(a, b)$, or in other words,

$$(a^*, b^*) = \arg \min_{(a, b)} \epsilon_{\text{lin}}(a, b),$$

(Refer to Appendix A and Lemma 6 within that section for some properties of $(a^*, b^*)$.)

2) **Projection-based clustering in 1-dimension**: A 1-dimensional projection-based linear clustering algorithm for points in $\mathbb{R}^p$ is described by a projection direction $c \in \mathbb{R}^p$ and a scalar $d \in \mathbb{R}$. Each point $x \in \mathbb{R}^p$ is first projected into 1-dimension as $c^T x$, and then is labeled as 1 if $c^T x > d$ and is labeled as 2 otherwise.

As in the previous case, if $X \in \mathbb{R}^p$ is generated according to $w_1 N(m_1, \Sigma_1) + w_2 N(m_2, \Sigma_2)$, $c^T X - d$ is distributed as $w_1 N(c^T m_1 - d, c^T \Sigma_1 c) + w_2 N(c^T m_2 - d, c^T \Sigma_2 c)$. Therefore, the clustering error of a 1-dimensional projection-based linear clustering algorithm described by $(c, d)$ can be characterized as

$$
\epsilon_{\text{lin}}(c, d) = w_1 Q \left( \frac{d - c^T m_1}{\sqrt{c^T \Sigma_1 c}} \right) + w_2 Q \left( \frac{c^T m_2 - d}{\sqrt{c^T \Sigma_2 c}} \right),
$$

(8)

3) **Equivalence of the two approach**: Comparing (7) and (8) and setting $a = c$ and $b = d$ reveals the following.

**Claim 1.** There is a one-to-one correspondence between the optimal linear classifier in $\mathbb{R}^p$ and an optimal projection-based classifiers in 1-dimension.

In this paper, we take advantage of this observation and argue that projecting high-dimensional data into different random 1-dimensional directions and selecting the direction that satisfies the prescribed error yields an efficient clustering method for mixture of two Gaussians in its original high dimensional space.

More precisely, we provide a recipe for designing clustering algorithms that are based on linear separators. The new method does not rely on estimating the parameters in the high dimensional space $\mathbb{R}^p$. Instead, it projects the data into 1-dimensional random directions and estimates the parameters of the projected mixture of Gaussians. Unlike
prior results in the literature, we do not use projections to estimate the parameters in \( \mathbb{R}^p \) from the estimated parameters in 1-dimensional directions. Instead, after each random projection, we estimate the corresponding clustering error probability for the high-dimensional data and stop after the desired accuracy is achieved. In the following section we establish the necessary theoretical ingredients to show that the required number of random projections is very small, especially when the desired clustering error corresponds closely with \( c \). In particular, we establish conditions on the achievable error with respect to \( c \) under which it is attained in a number of projections that is sub-linear or even sub-logarithmic in \( p \). When the number of projections is sub-logarithmic in \( p \) we obtain an overall running time that is \( O(np \log p) \).

III. Separability After 1-Dimensional Random Projection

In this section we consider data that is generated according to a mixture of two Gaussian distributions \( \mathcal{N}(m_1, \Sigma_1) \) and \( \mathcal{N}(m_2, \Sigma_2) \), which are \( c \)-separable, where

\[
\frac{\|m_1 - m_2\|}{\sqrt{p \left( \sqrt{\lambda_{\max}(\Sigma_1)} + \sqrt{\lambda_{\max}(\Sigma_2)} \right)}} \geq c.
\]

When understanding the relationship between the clustering error and separability the most basic intuition is to expect the clustering error to be a decreasing function of \( c \). After projecting points into 1-dimension, if \( n \) is sufficiently large, with high probability, the parameters of the two 1-dimensional Gaussians can be well estimated. In particular, if a 1-dimensional projection is such that it preserves the \( c \)-separability of the two Gaussians, then with accurate estimates of the projected GMM parameters the resulting clustering error will correspond nicely to the clustering error bound of an optimal Bayes classifier. In this section, we study the probability that a random projection achieves a 1-dimensional separability \( \gamma \) or higher, which can be directly related to a prescribed clustering error. Moreover, we provide conditions for the number of 1-dimensional projections required to achieve separability \( \gamma \) to be sub-linear or even sub-logarithmic in \( p \) when \( \gamma \) (corresponding to a clustering error in 1-dimension) is similar to \( c \). The last results allows the construction of very efficient clustering algorithms. We divide the discussion into two cases. The first case is when the two Gaussians are spherical balls. The second case is when \( \Sigma_1 \) and \( \Sigma_2 \) are arbitrary semi-positive matrices. We also demonstrate the extension of our theoretical analysis for a mixture of \( k \) Gaussians.

A. Spherical balls

Consider the special case where \( \Sigma_i = \sigma_i^2 I_p \), for \( i = 1, 2 \). Consider projecting the points generated as \( w_1 \mathcal{N}(m_1, \Sigma_1) + w_2 \mathcal{N}(m_2, \Sigma_2) \) using a random vector \( A = (A_1, \ldots, A_p) \), where \( A_1, \ldots, A_p \) are i.i.d. as \( \mathcal{N}(0, 1) \). Using this projection, we derive a mixture of two Gaussians in \( \mathbb{R}^p \). Conditioned on \( A = a \), the two Gaussians \( \mathcal{N}(m_1, \Sigma_1) \) and \( \mathcal{N}(m_2, \Sigma_2) \) in \( \mathbb{R}^p \) are mapped to \( \mathcal{N}(\langle m_1, a \rangle, \sigma_1^2 \|a\|^2) \), and \( \mathcal{N}(\langle m_2, a \rangle, \sigma_2^2 \|a\|^2) \), respectively. Therefore, the two clusters are \( \gamma \)-separable after projection, if \( |\langle m_1, a \rangle - \langle m_2, a \rangle| > \gamma(\sigma_1 + \sigma_2)\|a\| \), or

\[
|\langle m_1 - m_2, a \rangle| > \gamma(\sigma_1 + \sigma_2)\|a\|.
\]
Since $A$ is not a fixed vector, the question is that given the randomness in the generation of the projection vector $A$, what is the probability that condition (10) holds. In other words, given $m_1$, $m_2$, $\sigma_1$ and $\sigma_2$, we are interested in

$$P(|\langle m_1 - m_2, A \rangle| > \gamma(\sigma_1 + \sigma_2)\|A\|),$$

or

$$P \left( \left| \frac{m_1 - m_2}{\|m_1 - m_2\|}, \frac{A}{\|A\|} \right| > \frac{\gamma(\sigma_1 + \sigma_2)}{\|m_1 - m_2\|} \right),$$

where $A_1, \ldots, A_p \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$. The following theorem derives a lower bound on this probability.

**Theorem 2.** Consider $m_1, m_2 \in \mathbb{R}^p$ and $\sigma_1, \sigma_2 \in \mathbb{R}^+$. Assume that $A = (A_1, \ldots, A_p)$ are generated i.i.d. according to $\mathcal{N}(0, 1)$. Given $\gamma > 0$, let

$$\alpha \triangleq \frac{\gamma^2(\sigma_1 + \sigma_2)^2 p}{\|m_1 - m_2\|^2 \tau}.$$

Then, for any $\tau > 0$

$$P \left( |\langle m_1 - m_2, A \rangle| \geq \gamma(\sigma_1 + \sigma_2) \right) \geq P \left( A_1^2 > \alpha \frac{(1 - \frac{1}{p})}{(1 - \frac{2}{p})^2} (1 + \tau) \right) \left( 1 - e^{-\frac{\tau}{2\pi} (\tau - \log(1 + \tau))} \right). \quad (11)$$

As shown by the following lemma, for two spherical $c$-separable distributions in $\mathbb{R}^p$, the expected value of the squared separability of the randomly projected Gaussian distributions in 1-dimension is equal to $c^2$.

**Lemma 1.** Consider $m_1, m_2 \in \mathbb{R}^p$ and $\sigma_1, \sigma_2 \in \mathbb{R}^+$ and let

$$c \triangleq \frac{\|m_1 - m_2\|}{(\sigma_1 + \sigma_2)^\frac{1}{p}}.$$

Then, under a random 1-dimensional projection with $A = (A_1, \ldots, A_p)$, where $A_1, \ldots, A_p \overset{i.i.d.}{\sim} \mathcal{N}(0, 1)$,

$$E \left[ \frac{|\langle A, m_1 - m_2 \rangle|^2}{(\sigma_1 + \sigma_2)^2 \|A\|^2} \right] = c^2.$$

Finally, the following corollary, which is a direct consequence of Theorem 2, characterizes the probability that two Gaussians with $\Sigma_1 = \Sigma_2 = \sigma^2 I_p$ that are $c$-separable in $\mathbb{R}^p$ remain $\gamma$-separable in 1-dimension under a random projection.

**Corollary 2.** Consider two spherical Gaussian distributions in $\mathbb{R}^p$, with means $m_1, m_2 \in \mathbb{R}^p$ and covariance matrices $\Sigma_1 = \Sigma_2 = \sigma^2 I_p$. Let $c = \|m_1 - m_2\|/\sqrt{p \sigma^2}$. Assume that $A = (A_1, \ldots, A_p)$ are generated i.i.d. according to $\mathcal{N}(0, 1)$. Then, the probability that the 1-dimensional projected Gaussian distributions are $\gamma$-separable can be lower bounded as

$$P \left( |\langle m_1 - m_2, A \rangle| \geq 2\gamma \sigma \right) \geq P \left( A_1^2 > \frac{(1 - \frac{1}{p})\gamma^2}{(c^2 - \frac{2}{p})} (1 + \tau) \right) \left( 1 - e^{-\frac{\tau}{2\pi} (\tau - \log(1 + \tau))} \right), \quad (12)$$

where $\tau > 0$ can be selected freely.

**Proof.** Note that in Theorem 2

$$\alpha = \frac{\gamma^2(2\sigma)^2 p}{\|m_1 - m_2\|^2} = \frac{c^2}{\sigma^2}.$$

$\square$
The probability derived in (11) or (12) can be used to calculate the expected number of projection to be examined until \( \gamma \) is attained.

**Corollary 3.** Consider two spherical Gaussian distributions in \( \mathbb{R}^p \), that are \( c \)-separable. Denote \( d(\gamma) \) as the average number of 1-dimensional random projections required to attain \( \gamma \)-separation in 1-dimension. Then as \( p \to \infty \) we obtain \( d(\gamma) \leq \frac{1}{2Q(\frac{\gamma}{c})} \).

**Proof.** We first note that in (12) \( \tau > 0 \), the term \( 1 - e^{-p/2(\tau - \log(1 + \tau))} \to 1 \), as \( p \to \infty \). Next, in the limit we observe

\[
\lim_{p \to \infty} P \left( A_1^2 > \frac{(1 - \frac{1}{2})\gamma^2}{(c^2 - \frac{\gamma^2}{p})}(1 + \tau) \right) = P \left( A_1^2 > \left( \frac{\gamma^2}{c^2} \right)(1 + \tau) \right) .
\]

Therefore, for every \( \tau > 0 \), from (11),

\[
\lim_{p \to \infty} d(\gamma) \leq \frac{1}{P \left( A_1^2 > \left( \frac{\gamma^2}{c^2} \right)(1 + \tau) \right)} .
\]

Since \( \tau \) is a free parameter, choosing it arbitrary close to zero yields the desired result. That is,

\[
\lim_{p \to \infty} d(\gamma) \leq \frac{1}{2Q(\frac{\gamma}{c})} .
\]

In the following corollaries we establish the conditions on \( \gamma \) and \( c \) so that with a number of projections that is sub-linear or even sub-logarithmic in \( p \), \( \gamma \) can be achieved.

**Corollary 4.** Consider two spherical Gaussian distributions in \( \mathbb{R}^p \), with means \( m_1, m_2 \in \mathbb{R}^p \) and covariance matrices \( \Sigma_1 = \Sigma_2 = \sigma^2 I_p \). Let \( d(\gamma) \) denote the expected number of projections required to achieve \( \gamma \)-separability. If \( \gamma \) is such that \( \frac{\gamma}{c} \leq (\ln p)^\frac{1-\eta}{2} \), where \( \eta > 0 \) is a free parameter, then \( d(\gamma) = o(p) \).

**Proof.** By Corollary 2 choosing \( p \) large enough so that \( e^{-\frac{p-1}{2p}(\tau - \log(1 + \tau))} \leq \frac{1}{2} \), it follows that

\[
d(\gamma) \leq P \left( A_1^2 > \frac{(1 - \frac{1}{2})\gamma^2}{(c^2 - \frac{\gamma^2}{p})}(1 + \tau) \right) = \frac{2}{Q \left( \frac{(1 - \frac{1}{2})\gamma^2}{(c^2 - \frac{\gamma^2}{p})}(1 + \tau) \right)} .
\]

On the other hand, for all \( x > 0 \), we have

\[
\frac{x}{1 + x^2} \phi(x) < Q(x),
\]

where \( \phi(x) \) denotes the pdf of a standard normal distribution. Therefore,

\[
d(\gamma) \leq \frac{\sqrt{2\pi}(1 + x^2)}{x} e^{-\frac{x^2}{2}} ,
\]

where \( x = \frac{(1 - \frac{1}{2})\gamma^2}{(c^2 - \frac{\gamma^2}{p})}(1 + \tau) \). The desired result follows by noting that for \( p \) large enough \( \frac{x^2}{p} \) is negligible, and by assumption, \( \gamma \leq c(\ln p)^{\frac{1-\eta}{2}} \), where \( \eta > 0 \).
In a similar manner Corollary 5 captures the tradeoff between the number of projections and the resulting 1-dimensional separability for \( \gamma = (\ln \ln p)^{\frac{1}{1-\eta}} \) with \( d = o(\ln p) \) projections. This result provides a substantially better sub-linear running-time for a tradeoff in the accuracy. The proof follows similarly to the proof of Corollary 4.

**Corollary 5.** Consider two spherical Gaussian distributions in \( \mathbb{R}^p \), with means \( m_1, m_2 \in \mathbb{R}^p \) and covariance matrices \( \Sigma_1 = \Sigma_2 = \sigma^2 I_p \). Let \( d(\gamma) \) denote the expected number of projections required to achieve \( \gamma \)-separability. If \( \gamma \) is such that \( \gamma = (\ln \ln p)^{\frac{1}{1-\eta}} \), then \( d(\gamma) = o(\ln p) \).

To exemplify the tradeoff implications, consider \( \gamma = \sqrt{\ln \ln p} = 1.49 \), \( p = 10^4 \), and \( c = 1 \). According to an optimal Bayes classifier this yields 5% clustering error in 1-dimension. To achieve that error \( d(\gamma) \leq 9.24 \) projections are sufficient to be examined, on average. On the other hand, for \( \gamma = \sqrt{\ln p} = 3.03 \) the clustering error is essentially 0, however, the average number of projections required to achieve this error rate is \( d(\gamma) \leq 10^4 \).

To conclude, Lemma 1 states that the expected separability - \( E[\gamma] \) after a 1-dimensional random projection is \( c \). The conditions provided in corollaries 4 and 5 address the similarity between \( \gamma \) and \( c \) and enable us to construct novel and efficient algorithms employing remarkably small number of projections if \( \gamma \) is close to \( c \) up to a logarithmic or log-logarithmic factor in \( p \).

We extend Theorem 2 to the case of \( k \) Gaussians.

**Lemma 2.** Consider \( m_1, \ldots, m_k \in \mathbb{R}^p \) and \( \sigma_1, \ldots, \sigma_k \in \mathbb{R}^+ \). Assume that \( A = (A_1, \ldots, A_p) \) are generated i.i.d. according to \( N(0, 1) \). Given \( \gamma_{\min} > 0 \), and \( i, j \in \{1, \ldots, k\} \) let

\[
 c_{(i,j)} = \frac{\|m_i - m_j\|}{\sqrt{p} (\sigma_i + \sigma_j)}. 
\]

Let \( c_{\min} \triangleq \min_{i,j} c_{(i,j)} \). Define event \( B \) as having separability larger than \( \gamma_{\min} \) by all pairs of projected Gaussians. That is,

\[
 B \triangleq \left\{ \left\| \frac{m_i - m_j}{\|A\|} \right\| \geq \gamma_{\min} (\sigma_i + \sigma_j) : \forall (i, j) \in \{1, \ldots, k\}^2, i \neq j \right\}. \tag{15}
\]

Then,

\[
P(B^c) \leq \frac{k^2}{2} \left( 1 - 2Q\left( \frac{\gamma_{\min}}{c_{\min} \sqrt{1.1}} \right) \right) \left( 1 - e^{-0.002p} \right). \tag{16}
\]

To understand the implications of Lemma 2 assume that \( p \) is large enough such that the upper bound in (16) can be well-approximated by

\[
 \frac{k^2}{2} \left( 1 - 2Q\left( \frac{\gamma_{\min}}{c_{\min} \sqrt{1.1}} \right) \right). \tag{17}
\]

For this bound to be a non-trivial bound, \( 1 - 2Q\left( \frac{\gamma_{\min}}{c_{\min} \sqrt{1.1}} \right) \) should be smaller than 1/\( k^2 \). This suggests that
the argument of the $Q$ function should be close to zero. For for small values of $x$, where $x > 0$,

$$1 - 2Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-\frac{u^2}{2}} du \approx \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} (1 - \frac{u^2}{2}) du = \sqrt{\frac{2}{\pi}} \left( x - \frac{x^3}{6} \right).$$

(18)

Using this approximation, in summary, for large values of $p$, the upper bound can be approximated as

$$\frac{k^2}{2} \sqrt{\frac{2.2}{\pi} c_{\min}}.$$

Therefore, roughly speaking, in order for the bound to guarantee a small number of projections, $c_{\min}$ should be $O(k^2 \gamma_{\min})$. A better understanding of the running time dependency on the number of components $k$ can be derived in the following Corollary:

**Corollary 6.** Consider a mixture of $k$ Gaussian distributions in $\mathbb{R}^p$, where component $i, i = 1, \ldots, k$, is distributed as $N(m_i, \sigma_i^2 I_p)$. For $i, j \in \{1, \ldots, k\}, i \neq j$, let $c_{\min} \triangleq \min_{i,j} c_{(i,j)}$, where

$$c_{(i,j)} = \|m_i - m_j\| \sqrt{\frac{1}{\sigma_i + \sigma_j}}.$$

At each projection step, assume that all Gaussians are projected using an independently drawn vector $A \in \mathbb{R}^p$, where $A_1, \ldots, A_p$ are i.i.d. $N(0, 1)$. Let $d(\gamma_{\min}, p)$ denote the expected number of projections required to obtain separability $\gamma_{\min}$ between each pair of projected Gaussians. Then, if

$$\gamma_{\min} \leq (1 - \alpha) \sqrt{\frac{2\pi}{1.1 k^2}} c_{\min},$$

for some $\alpha \in (0, 1)$, then

$$\limsup_{p \to \infty} d(\gamma_{\min}, p) \leq \frac{1}{\alpha}.$$

**Proof.** Consider event $\mathcal{B}$ defined in (15), which denotes the desired event where each pair of projected Gaussians satisfy the desired separability. But,

$$d(\gamma_{\min}, p) = \frac{1}{P(\mathcal{B})},$$

(19)

where $P(\mathcal{B}^c)$ is upper-bounded by Lemma 2. Taking the limit as $p$ grows to infinity, it follows that

$$\limsup_{p \to \infty} d(\gamma_{\min}, p) \leq \frac{1}{1 - \frac{k^2}{2} \left( 1 - 2Q \left( \frac{\gamma_{\min} \sqrt{1.1}}{c_{\min}} \right) \right)}$$

(20)

On the other hand, for $x > 0$,

$$1 - 2Q(x) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{x} e^{-\frac{u^2}{2}} du \leq \sqrt{\frac{2}{\pi}} x.$$

(21)
Combining (21) and (20), it follows that
\[
\limsup_p d(\gamma_{\min}, p) \leq \frac{1}{1 - k^2 \left(\sqrt{\frac{1}{2\pi} \gamma_{\min}}\right)} \leq \frac{1}{1 - k^2 \sqrt{\frac{1}{2\pi} \left(1 - \alpha\right)} \sqrt{\frac{2\pi}{1.1}} \frac{1}{k^2}} = \frac{1}{\alpha},
\]
(22)
where the last inequality follows from our assumption about \(\gamma_{\min}\).

B. Mixture of two general Gaussians

In this section, we generalize the results of the previous section to general Gaussians with covariance matrices \(\Sigma_1\) and \(\Sigma_2\). But first we note that the notion of separability prescribed by (9) (or any other distance Euclidean-based metric) in the non-spherical mixture setting is a rather conservative one. In particular, since for non-spherical Gaussians the directions of maximal variance corresponding to \(\lambda_{\max}\) of each of the covariances do not necessarily align with each other in many realistic high dimensional cases which renders \(\lambda_{\max}\) as a rather crude spherical estimate. To this end, better separability prevails between the two Gaussians than what is prescribed by (9), which, in turn, provides lower clustering error.

Conditioned on \(A = a\), projecting points \(X\) drawn from Gaussian distribution \(N(m_i, \Sigma_i)\) as \(X^T a\) are distributed as a Gaussian distribution with mean \(E[\langle X, a \rangle] = \langle m_i, a \rangle\), and variance \(\text{var}(\langle X, a \rangle) = a^T \Sigma_i a\). As argued before, the two projected clusters are separable, if \(|\langle m_1, a \rangle - \langle m_2, a \rangle| > \gamma (\sqrt{a^T \Sigma_1 a} + \sqrt{a^T \Sigma_2 a})\), or
\[
|\langle m_1 - m_2, a \rangle| > \gamma \left(\sqrt{a^T \Sigma_1 a} + \sqrt{a^T \Sigma_2 a}\right),
\]
(23)
for some appropriate \(\gamma > 0\). Unlike the condition stated in (10), both sides of (23) depend on the direction of \(a\). Therefore, analyzing the following probability
\[
P \left( |\langle m_1 - m_2, A \rangle| > \gamma \left(\sqrt{A^T \Sigma_1 A} + \sqrt{A^T \Sigma_2 A}\right) \right),
\]
(24)
is more complicated. The following theorems 3 and 4 provide lower bounds on this probability for the cases of \(\Sigma_1 + \Sigma_2\) having a full rank \(r = p\), and for the case of partial rank \(r < p\), respectively.

**Theorem 3.** Consider \(m_1, m_2 \in \mathbb{R}^p\) and semi-positive definite matrices \(\Sigma_1\) and \(\Sigma_2\). Assume that the entries of \(A = (A_1, \ldots, A_p)\) are generated i.i.d. according to \(N(0, 1)\). Let \(\lambda_{\max}\) denote the maximum eigenvalue of \(\Sigma_1 + \Sigma_2\). Also, given \(\gamma > 0\), let
\[
\beta \triangleq \frac{2\gamma^2 \lambda_{\max} p}{\|m_1 - m_2\|^2}.
\]

Then, for any \(\tau > 0\), the probability that the 1-dimensional projected Gaussians using a uniformly random direction are \(\gamma\)-separated, i.e.,
\[
P \left( |\langle m_1 - m_2, A \rangle| > \gamma \left(\sqrt{A^T \Sigma_1 A} + \sqrt{A^T \Sigma_2 A}\right) \right),
\]
can be lower-bounded by
\[
P \left( A_1^2 > \beta \left(1 - \frac{1}{p}\right) (1 + \tau) \left(1 - e^{-\frac{1}{p}(\tau - \log(1 + \tau))}\right) \right).
\]
(25)
Corollary 7. Consider two $c$-separable Gaussian distributions in $\mathbb{R}^p$ with means $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^p$ and covariance matrices $\Sigma_1$ and $\Sigma_2$. Let $\beta \equiv \frac{2\sqrt{\lambda_{\max}}}{\|\mathbf{m}_1 - \mathbf{m}_2\|}$, where $\lambda_{\max}$ denotes the maximal eigenvalue of the matrix $\Sigma_1 + \Sigma_2$. Denote $d(\gamma)$ as the average number of 1-dimensional random projections required to attain $\gamma$-separation in 1-dimension. Then as $p \to \infty$ we obtain $d(\gamma) \leq \frac{1}{2Q(\sqrt{\beta})}$.

Proof. We note that in (25) for any $\tau > 0$, $\lim_{p \to \infty} (1 - e^{-\frac{\tau}{\sqrt{p}}(\log(1+\tau))}) = 1$. Since

$$c \equiv \frac{\|\mathbf{m}_1 - \mathbf{m}_2\|}{\sqrt{p} \left( \sqrt{\lambda_{\max}(\Sigma_1)} + \sqrt{\lambda_{\max}(\Sigma_2)} \right)},$$

$\beta$ can be expressed as

$$\beta = \frac{2\gamma^2 \lambda_{\max}}{c^2 \left( \sqrt{\lambda_{\max}(\Sigma_1)} + \sqrt{\lambda_{\max}(\Sigma_2)} \right)}.$$

Next, since $\lambda_{\max}(\Sigma_1 + \Sigma_2) \leq \lambda_{\max}(\Sigma_1) + \lambda_{\max}(\Sigma_2)$ we observe that $\beta \leq \frac{2\gamma^2}{c^2}$, for all $p$. Therefore

$$\lim_{p \to \infty} \frac{1 - \frac{1}{p}}{1 - \frac{\beta}{p}} = 1.$$

We obtain that for any $\tau > 0$

$$\lim_{p \to \infty} P \left( A_1^2 > \beta \frac{(1 - \frac{1}{p})}{(1 - \frac{\beta}{p})} (1 + \tau) \right) (1 - e^{-\frac{\tau}{\sqrt{p}}(\log(1+\tau))}) = P \left( A_1^2 > \beta (1 + \tau) \right). \quad (26)$$

Since $\tau$ is a free parameter, we have

$$\lim_{p \to \infty} d(\gamma) \leq \frac{1}{P \left( A_1^2 > \beta \right)} = \frac{1}{2Q(\sqrt{\beta})} \quad (27)$$

Similarly to the spherical case, the number of projections required for attaining a separability $\gamma$ can be derived in the following Corollaries 8 and 9. The proof follows closely to the proofs of Corollaries 4 and 5.
Corollary 8. Consider two Gaussian distributions in $\mathbb{R}^p$, with means $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^p$ and covariance matrices $\Sigma_1, \Sigma_2$. Let $d(\gamma)$ denote the expected number of projections required to achieve $\gamma$-separability. If $\gamma$ is such that $\sqrt{\beta} \leq (\ln p)^{\frac{1-\eta}{2}}$, where $\eta > 0$ is a free parameter, then $d(\gamma) = o(p)$.

Corollary 9. Consider two spherical Gaussian distributions in $\mathbb{R}^p$, with means $\mathbf{m}_1, \mathbf{m}_2 \in \mathbb{R}^p$ and covariance matrices $\Sigma_1, \Sigma_2$. Let $d(\gamma)$ denote the expected number of projections required to achieve $\gamma$-separability. If $\gamma$ is such that $\sqrt{\beta} = (\ln \ln p)^{\frac{1-\eta}{2}}$, where $\eta > 0$ is a free parameter, then $d(\gamma) = o(\ln p)$.

IV. ALGORITHM

A. Overview

At this stage we are ready to construct an efficient algorithm for clustering a mixture of two Gaussian distributions in high dimensional space $\mathbb{R}^p$. The algorithm receives as input a prescribed error - $\epsilon$, and a maximum number of 1-dimensional projections - $M$. We distinguish between two cases: An arbitrary mixture, for which it is unknown if the GMM is comprised of spherical Gaussians, and the case of known spherical mixture. The last case, in particular, allows an on-the-fly estimation of the expected number of projections required to achieve $\epsilon$ by employing Theorem 2 and possibly its corresponding Corollaries 4 and 5.

The algorithm sequentially performs 1-dimensional projections, where each projection’s direction is chosen uniformly at random. After each random projection, the algorithm estimates the parameters of the projected mixture of Gaussians in 1-dimension and its corresponding clustering error. It proceeds until either the desired accuracy $\epsilon$ is achieved by the current projection, or the maximum number of projections $M$ is reached. The efficiency of our proposed algorithm is justified by the following two fundamental observations:

1) **Minimal number of projections**: Theorems 2 and 3 provide a lower bound on the probability of achieving a separability value $\gamma = Q^{-1}(\epsilon)$, after performing a random projection of $c$-separable (or $\beta$-separable) spherical (or non-spherical) GMM. Theorems 2 and 3 in fact, guarantee that the expected number of random projections required for achieving a prescribed error is bounded, and, in particular, is sub-linear or sub-logarithmic if $\gamma$ corresponds closely to $c$ (see corollaries 4, 5, 8 and 9 for each case).

2) **Efficient parameter estimation in 1-dimension**: The projection into 1-dimension alleviates the need to estimate parameters in high dimension which is known to be difficult. In 1-dimension, we can employ efficient algorithms such as MoM [6] or even EM [1] (which has linear running time per step), and achieve overall running time that is linear in sample size $n$.

Both observations support an overall running time of $O(np \log p)$ for our algorithm.

B. Algorithm flow

The high level flow of the algorithms is shown in Fig. 2 and Fig. 3 for arbitrary Gaussian distributions and for the spherical case, respectively. The algorithm employs an iterative 1-dimensional projection process: At iteration $i, i = 1, 2, \ldots$, both algorithms start by generating a projection vector drawn uniformly at random and projecting the data
into that direction (line 4). The parameters of the projected 1-dimensional mixture are learned as \((\hat{m}_1^i, \hat{m}_2^i, \hat{\sigma}_1^i, \hat{\sigma}_2^i, \hat{w}_1^i)\) and a separator \(C^*\) is constructed in this 1-dimensional space according to Equation (3), i.e., by solving
\[
\left( \frac{1}{\hat{\sigma}_1^i} \right)^2 - \left( \frac{1}{\hat{\sigma}_2^i} \right)^2) x^2 + 2 \left( \frac{\hat{m}_1^i}{\hat{\sigma}_1^i} \right)^2 - \left( \frac{\hat{m}_2^i}{\hat{\sigma}_2^i} \right)^2 + 2 \log \frac{1 - \hat{w}_1^i}{\hat{w}_1^i} + 2 \log \frac{\hat{\sigma}_1^i}{\hat{\sigma}_2^i} = 0,
\]
and deriving the optimal decision boundaries based on the estimated parameters. At this stage the error \(\bar{e}_i\) of the corresponding separator can be easily estimated using the estimated parameters \((\hat{m}_1^i, \hat{m}_2^i, \hat{\sigma}_1^i, \hat{\sigma}_2^i, \hat{w}_1^i)\). Also, at iteration \(i\), the separation of the projected Gaussians is estimated as
\[
\bar{\gamma}_i = \frac{|\hat{m}_1^i - \hat{m}_2^i|}{\hat{\sigma}_1^i + \hat{\sigma}_2^i}.
\]
The clustering error \(\bar{e}_i\) is compared against the user prescribed error \(e\) (line 7). If the desired error \(e\) is achieved, the algorithm outputs the corresponding assignment to clusters as the final output. If the error is not satisfied, the algorithm forks the next step depending on the input information on the shape of the Gaussians. If there is no additional information on the Gaussians’ shapes, or they are known to be non-spherical, the algorithm is prescribed as in Fig. [2] and moves on to iteration \(i + 1\). This process continues until either the error constraint is satisfied or the user prescribed budget \(M\) is exhausted.

If the Gaussians are known to be of spherical shapes, a more efficient stopping criterion can be used (See Fig. [3]). In this special case, in the light of Lemma [1], at iteration \(i\), the separation of the unknown Gaussians in \(\mathbb{R}^p\) can be estimated as
\[
\bar{\gamma}_i = \sqrt{\frac{1}{i} \sum_{j=1}^{i} \bar{\gamma}_j^2}.
\]
Clearly, during the first few steps \(\bar{\gamma}_i\) may not be very accurate. To address this issue in the algorithm, one can either i) set a fixed number of iterations, for instance \(M/2\), after which the estimate is considered accurate enough, or ii) use more rigorous methods from statistics to estimate the confidence in the estimate \(\bar{e}_i\), and only use the estimate once the desired confidence is achieved. Until the confidence in the estimate is not sufficiently high, the algorithm continues with generating another random projection and the process repeats itself. If the confidence is considered high enough Theorem [2] is used with \(\gamma = 2Q^{-1}(e)\) and \(\bar{\gamma}_i\) to compute the expected number of projections \(\bar{M}\) required to attain the error \(e\). If \(\bar{M} < M\) or \(\bar{M} \approx M\), the algorithm continues to iteration \(i + 1\). Otherwise (if \(\bar{M} \gg M\) e.g. \(\bar{M} > 2M\)) the algorithm stops and returns "Error Not Achievable".

C. Learning the parameters of the projected mixture in 1-dimension

To learn the parameters of the GMM in 1-dimension we employ two basic approaches i) employ existing learning algorithms for Gaussian mixture models in 1-dimension, which provably converge to the true parameters as the number of samples grow to infinity [15]. One of them is the based on the Method of Moments (MoM) [6]. Or ii) use efficient algorithms such as EM [1], which search for a maximum likelihood (ML) estimate of the parameters. While there has been much progress on analyzing the first group of algorithms and improving their efficiency, due to practical considerations, in our numerical results presented in Section [VI] we used EM algorithm and observed
function \textsc{ClusterGMM}(X, e, M) \\
i = 1 \bar{e} = \infty \\
while i < M do \\
Project to random direction: \langle X, A^i \rangle \\
Learn 1-dimensional parameters: (\hat{m}_1^i, \hat{m}_2^i, \hat{\sigma}_1^i, \hat{\sigma}_2^i, \hat{w}_1^i) \\
Learn a separator \(C^*\) and compute \(\bar{e}\) \\
if \(\bar{e} < e\) then \\
return(\(C^*\)) \\
end if \\
print("Error not Achievable") \\return(\(C^*\)) \\
end while \\
end function

Fig. 2. Algorithm pseudo-code for the case of GMM with unknown shape or non-spherical shapes.

function \textsc{ClusterSphericalGMM}(X, e, M) \\
i = 1 \bar{e} = \infty \\
while i < M do \\
Project to random direction: \langle X, A^i \rangle \\
Learn 1-dimensional parameters: (\hat{m}_1^i, \hat{m}_2^i, \hat{\sigma}_1^i, \hat{\sigma}_2^i, \hat{w}_1^i) \\
Learn a separator \(C^*\) and compute \(\bar{e}\) and \(\gamma_j\) \\
if \(\bar{e} < e\) then \\
return(\(C^*\)) \\
elseif confident in \(\bar{c}_i\) estimate \\
\(\bar{c}_i = \sqrt{\frac{1}{7} \sum_{j=1}^{i} \bar{\gamma}_j^2}\) \\
Compute \(M\) the number of projections to achieve \(e\) (Theorem 2, Corollaries 4 & 5) \\
\(M = \bar{M}\) \\
end if \\
print("Error not Achievable") \\return(\(C^*\)) \\
end while \\
end function

Fig. 3. Algorithm pseudo-code for the case of spherical Gaussians.

very accurate performance in recovering the parameters and corresponding error probabilities. On the other hand, our sample complexity analysis in section \(\checkmark\) is related to MoM algorithms \([15]\).

D. Running-time complexity

The running time analysis of our algorithm is comprised of a product of the following components:

- Number of projections
- Cost of a projection step
- Cost of 1-dimensional clustering

\textbf{Number of projections}: The expected number of projections to achieve error \(e\) can be derived from Theorems 2 and 4 for the spherical and non-spherical settings respectively. In addition, for the conditions on \(c\) and \(\gamma = Q^{-1}(e)\) (or \(\beta\)) provided in Corollaries 4 5 (or 8 and 9) the number of projection is at sub-linear or sub-logarithmic in \(p\).
**Cost of a projection step:** Naively done the cost of a 1-dimensional projection of a sample of size $n$ with Gaussian iid vector is $np$.

**Cost of 1-dimensional clustering:** While various 1-dimensional clustering algorithms can be employed, we concentrate on two basic approaches which provide linear running time in $n$. The EM algorithm, which is sub-optimal in general, but is an efficient linear time algorithm especially in 1-dimension, where the number of local optimum is small. Every EM step in the iteration employed by this algorithm is comprised of $n$ operations. While the number of iterations required for convergence is theoretically unbounded, in 1-dimension the algorithm typically converges to a local optimum very fast, employing a minimal number of iterations. In our experiments when running the EM algorithm, we fixed the number of iterations to 200 and still attained the theoretical bounds. On the other side, MoM provides a global solution to the problem of estimating the GMM parameters, and requires a solution of a small system of equations comprised of the 6 moments of the mixture. The running time of this algorithm is practically $O(n)$. Its implementation on the other hand is more consuming as one needs to search the grid of GMM parameters for the solution that satisfies the moments equations.

To summarize: the overall expected running time of the projection procedure is $npo(\log p)$ when $\gamma \leq c \log \log p$. The overall running time of the 1-dimensional clustering is $O(n)o(\log P)$. For the setting of interest in which $n$ is of the order of $p$ the expected running time of our algorithms is $npo(\log p)$. Aside from the its asymptotic running-time analysis, the actual running time of our algorithm is largely affected by the sample size complexity - the sample size required to achieve an accuracy in the GMM parameter’s estimation (and therefore in clustering as well). In the next section we elaborate further on the required sample size in light of using the MoM Algorithm 3.3 of [15] and its effect on the clustering error.

**V. Sample complexity**

In this section we study the sample complexity of our algorithm. The study is done at the 1-dimensional setting where the clustering is performed using Algorithm 3.3 of [15] to estimate the parameters of the projected mixture of two Gaussians. Algorithm 3.3 is a variation of the well-known method of moments algorithm proposed by Pearson in [6]. The following result from [15] summarizes the performance of Algorithm 3.3 in estimating the parameters of a mixture of two general Gaussians in 1-dimension.

**Theorem 5 (Theorem 3.10 in [15]).** Consider a mixture of two Gaussian distribution $wN(\mu_1, \sigma_1)+(1-w)N(\mu_2, \sigma_2)$. Let $\sigma^2 = w(1-w)(\mu_1-\mu_2)^2 + w\sigma_1^2 + (1-w)\sigma_2^2$ denote the variance of this distribution. Then, given $n = O(\frac{1}{\gamma^2} \log \frac{1}{\delta})$ samples, Algorithm 3.3, with probability 1 $-$ $\delta$, returns estimates of the parameters as $(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2, \hat{w})$, which under the right permutation of the indices, satisfy the following guarantees, for $i = 1, 2$,

- If $n \geq \left(\frac{\sigma^2}{|\mu_1 - \mu_2|}\right)^6$, then $|\mu_i - \hat{\mu}_i| \leq \epsilon |\mu_1 - \mu_2|$, $|\sigma_i^2 - \hat{\sigma}_i^2| \leq \epsilon |\mu_1 - \mu_2|^2$, and $|w - \hat{w}| \leq \epsilon$.
- If $n \geq \left(\frac{\sigma^2}{|\sigma_1^2 - \sigma_2^2|}\right)^6$, then $|\sigma_i^2 - \hat{\sigma}_i^2| \leq \epsilon |\sigma_1^2 - \sigma_2^2| + |\mu_1 - \mu_2|^2$, and $|w - \hat{w}| \leq \epsilon + \frac{|\mu_1 - \mu_2|^2}{|\sigma_1^2 - \sigma_2^2|}$.
- For any $n \geq 1$, the algorithm performs as well as assuming the mixture is a single Gaussian, and $|\mu_i - \hat{\mu}_i| \leq |\mu_1 - \mu_2| + \epsilon \sigma$, and $|\sigma_i^2 - \hat{\sigma}_i^2| \leq |\mu_1 - \mu_2|^2 + |\sigma_1^2 - \sigma_2^2| + \epsilon \sigma^2$. 

The following corollary is a direct result of Theorem 5. It shows that, if the two components of a Gaussian mixture model are separated enough in 1-dimension, given sufficient sample, Algorithm 3.3 of [15] returns accurate estimates of all parameters.

Corollary 10. Let \((X_1, \ldots, X_n)\) denote \(n\) i.i.d. samples of a mixture of two \(c\)-separable Gaussians \(wN(\mu_1, \sigma_1) + (1-w)N(\mu_2, \sigma_2)\), where \(\mu_1 < \mu_2\) and \(\sigma_1 = \sigma_2\). Further assume that the separability \(c = |\mu_1 - \mu_2|/(\sigma_1 + \sigma_2)\) in 1-dimension is larger than \(\gamma_{\text{min}}\). Let \((\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2, \hat{w})\) denote the estimates of \((\mu_1, \mu_2, \sigma, \sigma, w)\) returned by Algorithm 3.3 of [15]. Then, if \(n = O(\frac{1}{\epsilon_2} \log \frac{1}{\epsilon})\) and \(n \geq \frac{1}{(2\gamma_{\text{min}})^2}\), then \(|\mu_i - \hat{\mu}_i| \leq \epsilon|\mu_1 - \mu_2|\), \(|\sigma_i^2 - \hat{\sigma}_i^2| \leq \epsilon|\mu_1 - \mu_2|^2\), and \(|w - \hat{w}| \leq \epsilon\).

Note that as \(\gamma_{\text{min}}\) converges to zero, the required number of samples for accurate estimation of the parameters grows to infinity. On the other hand, too small separability \(\gamma\) corresponds to large overlap of the two Gaussians. Hence, as confirmed in the following lemma, unless the weights of the two Gaussians are very non-uniform, i.e. \(\min(w, 1-w)\) is far from 0.5, small separability \(\gamma\), corresponds to poor clustering performance.

Lemma 3. Consider i.i.d. points generated as \(wN(\mu_1, \sigma) + (1-w)N(\mu_2, \sigma)\). Without loss of generality, assume that \(\mu_1 \leq \mu_2\) and \(w < 0.5\). Let \(\gamma = (\mu_2 - \mu_1)/(2\sigma)\). Also, let \(e_{\text{opt}}\) denote the error probability of an optimal Bayesian classifier. Then, if \(w \leq 0.1\),

\[
e_{\text{opt}} \geq wQ\left(-\frac{1}{\gamma} + \gamma\right).
\]

For \(w \in (0.1, 0.5]\),

\[
e_{\text{opt}} \geq wQ(\gamma).
\]

Proof. The optimal Bayesian classifier, which has access to the parameters \((\mu_1, \mu_2, \sigma, w)\), divides the real line at

\[
t_{\text{opt}} = \frac{\mu_1 + \mu_2}{2} - \frac{\sigma^2}{(\mu_1 - \mu_2)} \ln \frac{w}{1-w},
\]

and achieves a classification error equal to

\[
e_{\text{opt}} = wP(\mu_1 + \sigma Z \geq t_{\text{opt}}) + (1-w)P(\mu_2 + \sigma Z \leq t_{\text{opt}})
= wQ\left(\frac{\mu_2 - \mu_1}{2\sigma} - \frac{\sigma}{(\mu_1 - \mu_2)} \ln \frac{w}{1-w}\right) + (1-w)Q\left(-\frac{\mu_2 - \mu_1}{2\sigma} + \frac{\sigma}{(\mu_1 - \mu_2)} \ln \frac{w}{1-w}\right)
= wQ\left(\gamma + \frac{1}{2\gamma} \ln \frac{w}{1-w}\right) + (1-w)Q\left(\gamma - \frac{1}{2\gamma} \ln \frac{w}{1-w}\right),
\]

where \(Z \sim \mathcal{N}(0, 1)\). Note that since by assumption \(w < 1 - w, \ln \frac{w}{1-w} \leq 0\). Therefore,

\[
Q\left(\gamma - \frac{1}{2\gamma} \ln \frac{w}{1-w}\right) \leq Q\left(\gamma + \frac{1}{2\gamma} \ln \frac{w}{1-w}\right).
\]
Keeping the larger $Q$ term, it follows from (31) that

$$e_{\text{opt}} \geq wQ \left( -\frac{1}{2\gamma_1} \ln \frac{1-w}{w} + \gamma \right). \quad (32)$$

For $w \leq 0.1$, $0.5 \ln \frac{1-w}{w} \geq 0.5 \ln \frac{1-0.1}{0.1} > 1$. Therefore, since $Q(\cdot)$ is a monotonically decreasing function of its argument, (28) follows. The result for $w \in (0.1, 0.5)$ stated in (28) follows by noting that $-\frac{1}{2\gamma_1} \ln \frac{1-w}{w} + \gamma \leq \gamma$. □

Therefore, if the ultimate goal is to achieve a reasonable clustering error through multiple random projections, for those directions with too small separability $\gamma$, we only need to identify them and discard them. In other words, for such directions, it is not necessary to estimate all the parameters of the projected Gaussians accurately, as they ultimately are not going to be used for clustering. The following lemma provides a mechanism for identifying and discarding all directions that have a separability smaller than some threshold. It states that given $n = O(\frac{1}{\varepsilon} \log \frac{1}{\delta})$ i.i.d. samples of two Gaussians with separability $\gamma$, Algorithm 3.3 of [15] estimates the parameters of the two Gaussians such that the estimated separability is upper-bounded by $\frac{3\gamma + \varepsilon}{1-2\sqrt{\gamma^2 + \varepsilon}}$, with probability larger than $1 - \delta$.

**Lemma 4.** Let $(X_1, \ldots, X_n)$ denote $n$ i.i.d. samples of a mixture of two separable Gaussians $w \mathcal{N}(\mu_1, \sigma_1) + (1-w)\mathcal{N}(\mu_2, \sigma_2)$, where $\sigma_1 = \sigma_2$, $\gamma = (\mu_2 - \mu_1)/(\sigma_1 + \sigma_2) < 1/2$ and $\mu_1 < \mu_2$. Let $(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2, \hat{w})$ denote the estimates of $(\mu_1, \mu_2, \sigma, \sigma, w)$ returned by Algorithm 3.3 of [15]. Then, if $n = O(\frac{1}{\varepsilon} \log \frac{1}{\delta})$, with probability larger than $1 - \delta$,

$$\frac{|\hat{\mu}_1 - \hat{\mu}_2|}{\hat{\sigma}_1 + \hat{\sigma}_2} \leq \frac{3\gamma + \varepsilon}{1-2\sqrt{\gamma^2 + \varepsilon}}.$$

**Proof of Lemma 4** By Theorem 8 for $n = O(\varepsilon^2 \log \frac{1}{\delta})$, with probability $1 - \delta$, there exists a permutation of indices, such that $|\mu_i - \hat{\mu}_i| \leq |\mu_1 - \mu_2| + \varepsilon \sigma$ and $|\sigma_i^2 - \hat{\sigma}_i^2| \leq |\sigma_1^2 - \sigma_2^2| + \varepsilon \sigma^2 = |\mu_1 - \mu_2|^2 + \varepsilon \sigma^2$. Therefore, by the triangle inequality,

$$|\hat{\mu}_1 - \hat{\mu}_2| \leq \sum_{i=1}^{2} |\mu_i - \hat{\mu}_i| + |\mu_1 - \mu_2| \leq 3|\mu_1 - \mu_2| + \varepsilon \sigma.$$

Hence, since $\sigma^2 = w(1-w)(\mu_1 - \mu_2)^2 + \sigma_1^2$,

$$\frac{|\hat{\mu}_1 - \hat{\mu}_2|}{\hat{\sigma}_1 + \hat{\sigma}_2} \leq \frac{3|\mu_1 - \mu_2| + \varepsilon \sigma}{2\sigma_1 - 2\sqrt{|\mu_1 - \mu_2|^2 + \varepsilon \sigma^2}} \leq \frac{3|\mu_1 - \mu_2| + \varepsilon \sqrt{w(1-w)(\mu_1 - \mu_2)^2 + \sigma_1^2}}{2\sigma_1 - 2\sqrt{|\mu_1 - \mu_2|^2 + \varepsilon \sqrt{w(1-w)(\mu_1 - \mu_2)^2 + \sigma_1^2}}}
= \frac{3\gamma + \varepsilon \sqrt{w(1-w)\gamma^2 + 0.25}}{1 - \sqrt{4\gamma^2 + 4w(1-w)\gamma^2} + \varepsilon}
\leq \frac{3\gamma + 0.5\varepsilon \sqrt{\gamma^2 + 1}}{1 - \sqrt{4\gamma^2 + \varepsilon (1 + \gamma^2)}}
= \frac{3\gamma + \varepsilon}{1 - 2\sqrt{\gamma^2 + \varepsilon}}, \quad (33)$$

where (a) follows by dividing the nominator and denominator by $2\sigma_1$ and (b) holds because $w(1-w) \leq 0.25$. Finally (c) holds, since by assumption $\gamma^2 < 1$. □
To shed more light on the implications of Lemma 4, consider a mixture of two 1-dimensional Gaussians with equal variance and separability $\gamma$ smaller than $\frac{1}{8}$. Then, given $n = O\left(\frac{1}{\gamma^2} \log \frac{1}{\delta} \right)$ i.i.d. samples, with probability larger than $1 - \delta$, the estimated separability (using parameters derived from Algorithm 3.3 of \cite{15}) is smaller than

$$\frac{3\gamma + \epsilon}{1 - 2\sqrt{\frac{\gamma^2}{2}} + \epsilon} = \frac{1}{2} + o(\epsilon).$$

Therefore, if after performing each random projection, we estimate the parameters of the two Gaussians using Algorithm 3.3 of \cite{15} and then estimate the separability of the two Gaussians as $\frac{\|\hat{\mu}_1 - \hat{\mu}_2\|}{\sigma_1 + \sigma_2}$ and discard all those directions that have estimated separability smaller than $\frac{1}{8}$, we would, with high probability, discard all directions with separability smaller than $1/8$. For directions with separability larger than $1/8$, we need to have enough samples to estimate the parameters accurately. The required number of samples for achieving this goal is shown in Corollary 10 which follows directly from Theorem 3.10 of \cite{15}. Note that using this procedure, directions with estimated separabilities smaller than $0.5$ include those directions with separabilities in $(\frac{1}{8}, \frac{1}{2})$, for which, with high probability, we have estimated the parameters accurately, and those directions with separabilities smaller than $\frac{1}{8}$, for which we a crude estimate of the parameters.

Finally, for directions for which we accurately estimate the parameters of the projected Gaussians, the following lemma, connects the error in estimating the parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, w)$ to the error in estimating the clustering error. Since the ultimate goal of our algorithm is to find a direction which yields a desired clustering error, it is important to establish such a connection, which, given the desired clustering error, characterizes some sufficient accuracy in estimating the parameters.

**Lemma 5.** Consider $(X_1, \ldots, X_n)$ that are generated i.i.d. according to a mixture of two $\gamma$-separable Gaussians $w\mathcal{N}(\mu_1, \sigma_1) + (1 - w)\mathcal{N}(\mu_2, \sigma_2)$, where $\sigma_1 = \sigma_2$, $w \in [w_{\min}, 0.5]$, $\mu_1 < \mu_2$ and $\gamma \in [\gamma_{\min}, \gamma_{\max}]$. Let $(\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_2, \hat{w})$ denote the estimate of the unknown parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, w)$. Let $e_{\text{opt}}$ and $\hat{e}$ denote the minimum achievable classification error and the achieved clustering error based on the estimated parameters, respectively. Then, if $|\mu_1 - \hat{\mu}_1| \leq \epsilon |\mu_1 - \mu_2|$, $|\sigma^2_1 - \hat{\sigma}^2_1| \leq \epsilon |\mu_1 - \mu_2|^2$, $|w - \hat{w}| \leq \epsilon$, and

$$(16\gamma^2_{\max} + 8\gamma_{\max} \ln \frac{1 - w_{\min}}{w_{\min}} + 2\gamma_{\max} \epsilon) \epsilon < \frac{1}{2},$$

we have

$$|\hat{e} - e_{\text{opt}}| \leq \left(2\gamma + \frac{1}{w_{\min} \gamma} + \left(1 + 2\gamma\right) \ln \frac{1 - w_{\min}}{w_{\min}} + 8\gamma^2_{\max} \gamma + 2\gamma \left(4\gamma + 2 \ln \frac{1 - w_{\min}}{w_{\min}}\right)^2\right) \epsilon + Q\left(\frac{1}{4\gamma \epsilon} + \epsilon_1\right) + \epsilon_2,$$

where $\epsilon_1 = o(1/\epsilon)$ and $\epsilon_2 = o(\epsilon)$.

**VI. EXPERIMENTS**

Our experimental validation focuses on the performance of the algorithm in light of the theoretical bounds on the clustering error and the number of projections required to attain a given error. These empirical studies suggest that the algorithm requires a very small number of projections in order to achieve competitive clustering accuracy for
a range of separability values underlying the difficulty of the high dimensional problem. We present our empirical results for the two cases of spherical and non-spherical Gaussians.

A. Experiments with spherical Gaussians

In the spherical Gaussians case, we examine 3 trade-offs:

1) accuracy vs. separability, where we validate our algorithms error rate by comparing it against the bounds provided by a Bayes classifier that uses the true parameters (see Corollary 1 - a special case of Theorem 1),

2) number of projections vs. separability, where we show that the bound provided by Theorem 2 is a tight upper bound on the expected value of our algorithms’ required number of projections, and

3) accuracy vs. number of projections, where we show the fast convergence of our algorithm to the optimal error that can be achieved for a given separability.

Accuracy vs. separability: In this experiment we fix the number of projections to 50, and report the true minimum clustering error achieved after performing these many random projections. We compare this error with the theoretical bound of the Bayes classifier error in the high dimension $Q(\sqrt{pc})$ (see Corollary 1), and for the 1-dimensional Bayes classifier $Q(c)$. The experiment is repeated for a range of separability values, and the result is reported in Fig. 4 for mixture realizations of size 4K data points in $\mathbb{R}^3$, $\mathbb{R}^{100}$ and $\mathbb{R}^{1000}$ with equal probabilities, i.e., $w_1 = w_2 = 0.5$. It is observed that the error rates achieved by our algorithm are always better than the error bound in 1-dimension, however, as expected, it does not outperform the optimal Bayes classifier error in the high dimension which has access to the underlying parameters. Also, as expected, the gap between the algorithm’s performance and that of an optimal Bayes classifier grows as the dimension $p$ increases.

Number of projections vs. separability: In this experiment, 10K points are generated from a mixture of two spherical Gaussian distributions in $\mathbb{R}^{100}$, with $w_1 = w_2$ and $\sigma_1 = \sigma_2$. The user’s desired error is fixed at $\epsilon = 20\%$. For different high-dimensional separabilities $c$, we measure the number of projections used until both the true error $e_i$ and the estimated error $\hat{e}_i$ of one of the random 1-dimensional projections are less than 20%. Fig. 5-Left plots the number of projections scanned until the prescribed accuracy is attained for various $c$ values. We also plot the
lower bound provided by Corollary 2 as the inverse of the probability bound defined there for the given dimension $p$, separability $c$, and $\gamma = Q^{-1}(\epsilon)$. It can be observed that the mean number of projections to achieve the prescribed error is tightly bounded by the upper bound provided by Corollary 2.

**Error vs. number of projections:** In this experiment the separability $c$ is fixed at values 0.1, 0.5, 1 and 2. For each values we increase the number of projections scanned and generate a classification based on the best predicted error for each projection and select the minimal error classification. Fig. 5 Right reports the accuracy values for increasing number of projections and for various separability values. A saturation point is observed for each $c$ value at a different location agreeing with the error values reported in Fig. 4 middle for the respective separability values therein. The experiment points to the accurate the number of projections that achieves the minimal possible error for our algorithm. As seen the minimal error is approached closely after just a few projections, suggesting the speed and efficiency in which the algorithm can cluster the data to a prescribed error that reasonably corresponds to the high dimensional separability.

---

**B. Non-spherical Gaussians**

**Accuracy vs. rank:** In the non-spherical case, we consider a mixture of Gaussians in $R^{1000}$ and generate 10K points. The respective covariance matrices are generated by setting a fraction $\zeta$ of the dimension to be populated by points of both Gaussians. Then the same fraction $\zeta$ is used to uniformly sample another subspace for each Gaussian to be populated as well. In this experiment, we fix the high-dimensional separability as $c = 0.5$, but change $\zeta$ so that the summation of the covariance matrices rank $r = \text{rank}(\Sigma_1 + \Sigma_2)$ is changed accordingly. For each value of $r$, we generate the same number of random projections and record the best error attained in 1-dimension by exploring 50 projections at most.

First, we examine the error as function of the rank of the covariance summation matrix $(\Sigma_1 + \Sigma_2)$. Fig. 6 Left demonstrates this error. We note that as the matrix approaches the full rank (with equal variance in the populated
dimensions) the error of our algorithm approaches the error 0.1 attained at the case of spherical Gaussians for $c = 0.5$.

**Number of projections vs. rank:** Next we examine the number of projections needed to achieve a prescribed error as a function of the rank of the summation matrix for mixtures with the same parameters. We validate the bound provided by Theorem 4. In Fig. 6 Right we report results for a 4% error prescribed.

![Graph](image)

**Fig. 6.** Analysis of non-spherical mixtures with unequal covariances as a function of the rank of $(\Sigma_1 + \Sigma_2)$. The data contains realization of mixtures of 10K points in $\mathbb{R}^{1000}$ for $c = 0.5$. **Left:** Algorithm’s error vs. rank. **Right:** number of projections vs. rank compared with the theoretical bound provided by Theorem 4.

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**VII. Conclusion**

We presented a randomized algorithmic framework for clustering $n$ points generated by a mixture of Gaussian distributions in $\mathbb{R}^p$, where the ambient dimension $p$ is very large. In contrast to state-of-the-art, where the mixture parameters are learnt in order to cluster the points, we introduce a significant running-time improvement based on clustering in 1-dimensional projections. In particular, we show a novel tradeoff between accuracy of the clustering and running-time demonstrating that sub-logarithmic number of random projections are sufficient to find a direction in which the separability of the mixture is preserved up to a log-logarithmic factor. This improves on quadratic running time in $n$ or $p$ currently prevailing state-of-the-art methods for learning parameters of GMMs. Our numerical results, while following our theoretical bounds, showed the empirical accuracy of our new method in terms of error and efficiency. We provide sample complexity bounds that improves over state-of-the-art, and provides an enhancement to our algorithm - rejecting mechanism for projection directions in which the estimated separability is below a threshold. We validate our results in light of the optimal Bayes classifier (which uses the true parameters) and in light of our theoretical complexity bounds.

**References**

[1] A. P. Dempster, N. M. Laird, and D. B. Rubin. Maximum likelihood from incomplete data via the EM algorithm. *J. of Roy. Stat. Soc., Ser. B*, 39(1):1–38, 1977.
Lemma 6. Consider a mixture of two Gaussians distributed as \( w_1 N(m_1, \Sigma_1) + (1 - w_1) N(m_2, \Sigma_2) \). Then, the optimal linear separator \((a^*, b^*)\) for these two Gaussians satisfies

\[
\begin{align*}
\frac{w_1}{\sqrt{a^T \Sigma_1 a}} Q\left( \frac{b - a^T m_1}{\sqrt{a^T \Sigma_1 a}} \right) &= \frac{w_2}{\sqrt{a^T \Sigma_2 a}} Q\left( \frac{a^T m_2 - b}{\sqrt{a^T \Sigma_2 a}} \right),
\end{align*}
\]

and

\[
\begin{align*}
m_1 + \frac{(b - a^T m_1) \Sigma_1 a}{a^T \Sigma_1 a} &= m_2 + \frac{(b - a^T m_2) \Sigma_2 a}{a^T \Sigma_2 a}.
\end{align*}
\]

Proof. Since \( \epsilon_{\text{lin}}(a, b) \) is a differentiable function, its gradient should be zero at \((a^*, b^*)\). That is,

\[
\nabla \epsilon_{\text{lin}}(a, b) \bigg|_{(a^*, b^*)} = 0.
\]

But

\[
\begin{align*}
\frac{\partial \epsilon_{\text{lin}}(a, b)}{\partial b} &= \frac{w_1}{\sqrt{a^T \Sigma_1 a}} Q\left( \frac{b - a^T m_1}{\sqrt{a^T \Sigma_1 a}} \right) - \frac{w_2}{\sqrt{a^T \Sigma_2 a}} Q\left( \frac{a^T m_2 - b}{\sqrt{a^T \Sigma_2 a}} \right),
\end{align*}
\]

where

\[
Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.
\]
Therefore, at \((a^*, b)\) we should have

\[
\frac{w_1}{\sqrt{a^T \Sigma_1 a}} Q'(\frac{b - a^T m_1}{\sqrt{a^T \Sigma_1 a}}) = -\frac{w_2}{\sqrt{a^T \Sigma_2 a}} Q'(\frac{a^T m_2 - b}{\sqrt{a^T \Sigma_2 a}}) \tag{A.2}
\]

Also, by the chain rule, taking the partial derivative with respect to \(a\), we derive

\[
-w_1 \left( \frac{m_1}{\sqrt{a^T \Sigma_1 a}} + \frac{(b - a^T m_1) \Sigma_1 a}{(a^T \Sigma_1 a)^{3/2}} \right) Q'(\frac{b - a^T m_1}{\sqrt{a^T \Sigma_1 a}})
\]

\[
w_2 \left( \frac{m_2}{\sqrt{a^T \Sigma_2 a}} + \frac{(b - a^T m_2) \Sigma_2 a}{(a^T \Sigma_2 a)^{3/2}} \right) Q'(\frac{a^T m_2 - b}{\sqrt{a^T \Sigma_2 a}}) = 0, \tag{A.3}
\]

or

\[
\frac{w_1}{\sqrt{a^T \Sigma_2 a}} \left( m_1 + \frac{(b - a^T m_1) \Sigma_1 a}{a^T \Sigma_1 a} \right) Q'(\frac{b - a^T m_1}{\sqrt{a^T \Sigma_1 a}})
\]

\[
= -\frac{w_2}{\sqrt{a^T \Sigma_2 a}} \left( m_2 + \frac{(b - a^T m_2) \Sigma_2 a}{a^T \Sigma_2 a} \right) Q'(\frac{a^T m_2 - b}{\sqrt{a^T \Sigma_2 a}}). \tag{A.4}
\]

Combing (A.2) and (A.4) yields

\[
m_1 + \frac{(b - a^T m_1) \Sigma_1 a}{a^T \Sigma_1 a} = m_2 + \frac{(b - a^T m_2) \Sigma_2 a}{a^T \Sigma_2 a}. \tag{A.5}
\]

Hence, in summary, the optimal linear classifier \((a^*, b^*)\) is described by (A.2) and (A.5).

Note that, from (A.5) for the case where \(\Sigma_1 = \Sigma_2 = \Sigma\), the optimal direction \(a^*\) is equal to a constant times \(\Sigma^{-1}(m_1 - m_2)\), which confirms that in this special case the two Gaussians are linearly separable.

**Appendix B**

**Proof of Theorem 1**

**Proof.** Since \(\epsilon(\alpha, m_1, m_2, \Sigma_1, \Sigma_2)\) corresponds to the error probability of an optimal classifier, the classification error of any (sub-optimal) classifier serves as an upper bound on \(\epsilon(\alpha, m_1, m_2, \Sigma_1, \Sigma_2)\). In particular, consider the hyperplane classifier that is orthogonal to \(m_1 - m_2\) and passes through \(m_1 + m_2\). Characterizing the error probability of this specific classifier will yield our desired result. This hyperplane, because it is orthogonal to \(m_1 - m_2\) can be described as \((m_1 - m_2)^T x = \beta\). Since it passes through \((m_1 + m_2)/2\), we have

\[
\beta = (m_1 - m_2)^T \frac{m_1 + m_2}{2},
\]

or

\[
\beta = \frac{||m_1||^2 - ||m_2||^2}{2}.
\]

This hyperplane labels every point \(x \in \mathbb{R}^p\) as follows:

1) If

\[
(m_1 - m_2)^T x \geq \frac{||m_1||^2 - ||m_2||^2}{2},
\]

label \(x\) as 1.
2) If
\[(m_1 - m_2)^T x < \frac{\|m_1\|^2 - \|m_2\|^2}{2},\]

label x as 2.

Let \(\epsilon_h(\alpha, m_1, m_2, \Sigma_1, \Sigma_2)\) denote the classification error of the described classifier. Clearly,
\[\epsilon(\alpha, m_1, m_2, \Sigma_1, \Sigma_2) \leq \epsilon_h(\alpha, m_1, m_2, \Sigma_1, \Sigma_2).\] (B.6)

To upper bound \(\epsilon_h(\alpha, m_1, m_2, \Sigma_1, \Sigma_2)\), note that under \((m_1 - m_2)^T X - \beta\), the first component of the original mixture of Gaussians in \(\mathbb{R}^p\) is mapped into a Gaussian distribution in \(\mathbb{R}\) with mean
\[(m_1 - m_2)^T m_1 - \frac{\|m_1\|^2 - \|m_2\|^2}{2} = \frac{\|m_1 - m_2\|^2}{2},\]
and variance
\[(m_1 - m_2)^T \Sigma_1 (m_1 - m_2).\]

Similarly, the second Gaussian is mapped to
\[\mathcal{N}\left(-\frac{\|m_1 - m_2\|^2}{2}, (m_1 - m_2)^T \Sigma_2 (m_1 - m_2)\right).\]

Clearly the weights \((\alpha, 1 - \alpha)\) of the two Gaussian are preserved under this mapping. Therefore, in summary, the classification error \(\epsilon_h(\alpha, m_1, m_2, \Sigma_1, \Sigma_2)\) can be written as
\[\alpha Q\left(\frac{\|m_1 - m_2\|^2}{2\sqrt{(m_1 - m_2)^T \Sigma_1 (m_1 - m_2)}}\right) + (1 - \alpha) Q\left(\frac{\|m_1 - m_2\|^2}{2\sqrt{(m_1 - m_2)^T \Sigma_2 (m_1 - m_2)}}\right).\] (B.7)

Note that, for \(i = 1, 2,\)
\[(m_1 - m_2)^T \Sigma_i (m_1 - m_2) \leq \lambda_{\text{max}}(\Sigma_i) \|m_1 - m_2\|^2,\]
or
\[\frac{\|m_1 - m_2\|^2}{2\sqrt{(m_1 - m_2)^T \Sigma_i (m_1 - m_2)}} \geq \frac{\|m_1 - m_2\|}{2\sqrt{\lambda_{\text{max}}(\Sigma_i)}} \geq \frac{\|m_1 - m_2\|}{2\sqrt{\lambda_{\text{max}}(\Sigma_1)} + 2\sqrt{\lambda_{\text{max}}(\Sigma_2)}}.\] (B.8)

On the other hand, since by assumption the two components were \(c\)-separable in \(\mathbb{R}^p\),
\[\frac{\|m_1 - m_2\|}{\sqrt{\lambda_{\text{max}}(\Sigma_1)} + \sqrt{\lambda_{\text{max}}(\Sigma_2)}} \geq c\sqrt{p}.\] (B.9)

Therefore, since \(Q(x)\) is a decreasing function of \(x\), combining (B.7), (B.8) and (B.9) shows that
\[\epsilon_h(\alpha, m_1, m_2, \Sigma_1, \Sigma_2) \leq \alpha Q\left(\frac{c}{2}\sqrt{p}\right) + (1 - \alpha) Q\left(\frac{c}{2}\sqrt{p}\right) = Q\left(\frac{c}{2}\sqrt{p}\right).\]

Finally, combining this upper bound with (B.6) yields the desired result.

\[\square\]
APPENDIX C
PROOF OF COROLLARY \[1\]

Proof. The proof follows very similar to the proof of Theorem \[1\]. The difference is in \[B.8\] and \[B.9\]. In this case, since $\Sigma_1 = \Sigma_2$,

$$\frac{\|m_1 - m_2\|}{2\sqrt{\lambda_{\text{max}}(\Sigma_i)}} = \frac{\|m_1 - m_2\|}{2\sqrt{\lambda_{\text{max}}(\Sigma)}} = c.$$ \hfill (C.10)

Therefore,

$$\epsilon_h(\alpha, m_1, m_2, \Sigma, \Sigma) \leq \alpha Q(c\sqrt{p}) + (1 - \alpha) Q(c\sqrt{p}) = Q(c\sqrt{p}).$$

\[\square\]

APPENDIX D
PROOF OF THEOREM \[2\]

Proof. Since the unitary vector $A/\|A\|$ is uniformly distributed over the unit sphere, we have

$$P\left(\left|\frac{m_1 - m_2}{\|m_1 - m_2\|} \cdot \frac{A}{\|A\|}\right| > \frac{\gamma(\sigma_1 + \sigma_2)}{\|m_1 - m_2\|}\right) = P\left(\left|(1, 0, \ldots, 0)^T, \frac{A}{\|A\|}\right| > \frac{\gamma(\sigma_1 + \sigma_2)}{\|m_1 - m_2\|}\right)$$

$$= P\left(\frac{|A_1|}{\|A\|} > \frac{\gamma(\sigma_1 + \sigma_2)}{\|m_1 - m_2\|}\right).$$ \hfill (D.11)

Therefore, we are interested in deriving a lower bound on

$$P\left(\frac{|A_1|}{\|A\|} > \frac{\gamma(\sigma_1 + \sigma_2)}{\|m_1 - m_2\|}\right).$$ \hfill (D.12)

Note that, due to symmetry, we have

$$E\left[\frac{A_1^2}{\|A\|^2}\right] = E\left[\frac{A_2^2}{\|A\|^2}\right] = \ldots = E\left[\frac{A_p^2}{\|A\|^2}\right].$$ \hfill (D.13)

Moreover,

$$\sum_{i=1}^p E\left[\frac{A_i^2}{\|A\|^2}\right] = E\left[\frac{\sum_{i=1}^p A_i^2}{\|A\|^2}\right] = 1.$$ \hfill (D.14)

Therefore, combining \[D.13\] and \[D.14\], we have

$$E\left[\frac{A_1^2}{\|A\|^2}\right] = \frac{1}{p}.$$ 

So, intuitively, this suggests that, if

$$\frac{\gamma(\sigma_1 + \sigma_2)}{\|m_1 - m_2\|}$$

is much smaller than $\frac{1}{p}$, the probability mentioned in \[D.12\], which as we discussed earlier is an indicator of separability, is not small.
On the other hand, replacing $\gamma^2(\sigma_1 + \sigma_2)^2/\|m_1 - m_2\|^2$ by $\alpha/n$ in (D.12), we have

$$P \left( \frac{A_1^2}{\|A\|^2} > \frac{\gamma^2(\sigma_1 + \sigma_2)^2}{\|m_1 - m_2\|^2} \right) = P \left( A_1^2 > \frac{\alpha}{\|A\|^2} \sum_{i=1}^{p} A_i^2 \right)$$

$$= P \left( (1 - \frac{\alpha}{\|A\|^2})A_1^2 > \frac{\alpha}{\|A\|^2} \sum_{i=2}^{p} A_i^2 \right). \tag{D.15}$$

But since $A_1, \ldots, A_p \overset{i.d.}{\sim} \mathcal{N}(0, 1)$, $\sum_{i=2}^{p} A_i^2$ has a chi-square distribution of order $n$. Then, for any $\tau > 0$, by Lemma 2 in [?],

$$P \left( \frac{1}{p-1} \sum_{i=2}^{p} A_i^2 \geq 1 + \tau \right) \leq e^{-\frac{\tau}{p-1}(\tau - \log(1+\tau))}. \tag{D.16}$$

Given $\tau > 0$, define event $\mathcal{E}$ as

$$\mathcal{E} \triangleq \{ \frac{1}{p-1} \sum_{i=2}^{p} A_i^2 \leq 1 + \tau \}.$$

By the law of total probability,

$$P \left( (1 - \frac{\alpha}{\|A\|^2})A_1^2 > \frac{\alpha}{\|A\|^2} \sum_{i=2}^{p} A_i^2 \right) = P \left( (1 - \frac{\alpha}{\|A\|^2})A_1^2 > \frac{\alpha}{\|A\|^2} \sum_{i=2}^{p} A_i^2, \mathcal{E} \right) + P \left( (1 - \frac{\alpha}{\|A\|^2})A_1^2 > \frac{\alpha}{\|A\|^2} \sum_{i=2}^{p} A_i^2, \mathcal{E}^c \right)$$

$$\geq P \left( (1 - \frac{\alpha}{\|A\|^2})A_1^2 > \frac{\alpha}{\|A\|^2} \sum_{i=2}^{p} A_i^2, \mathcal{E} \right)$$

$$\geq P \left( (1 - \frac{\alpha}{\|A\|^2})A_1^2 > \frac{\alpha(n-1)}{p} (1 + \tau) > \frac{\alpha}{\|A\|^2} \sum_{i=2}^{p} A_i^2 \right)$$

$$\geq P \left( (1 - \frac{\alpha}{\|A\|^2})A_1^2 > \frac{\alpha(n-1)}{p} (1 + \tau), 1 + \tau > \frac{1}{p-1} \sum_{i=2}^{p} A_i^2 \right)$$

$$= P \left( (1 - \frac{\alpha}{\|A\|^2})A_1^2 > \alpha(1 - \frac{1}{p})(1 + \tau) \right) P \left( \frac{1}{p-1} \sum_{i=2}^{p} A_i^2 < 1 + \tau \right),$$

where the last line follows from the independence of $A_1$ and $(A_2, \ldots, A_p)$.

\[\square\]

**APPENDIX E**

**Proof of Lemma [I]**

**Proof.** Let $A = (A_1, \ldots, A_p)$ be generated i.i.d. according to $\mathcal{N}(0, 1)$. Then the separability of the two projected Gaussians under $A$ is equal to

$$\gamma = \frac{|\langle A, m_1 \rangle - \langle A, m_2 \rangle|}{(\sigma_1 + \sigma_2)\|A\|}.$$

Therefore,

$$\gamma^2 = \frac{|\langle A, m_1 - m_2 \rangle|^2}{(\sigma_1 + \sigma_2)^2\|A\|^2}$$

$$= \frac{\|m_1 - m_2\|^2}{(\sigma_1 + \sigma_2)^2} \left| \frac{A}{\|A\|} \cdot \frac{m_1 - m_2}{\|m_1 - m_2\|} \right|^2.$$
Since \( \frac{A}{\|A\|} \) is uniformly distributed under the unit sphere in \( \mathbb{R}^p \), in evaluating \( E \gamma^2 \), without loss of generality we can assume that \( \frac{m_1 - m_2}{\|m_1 - m_2\|} = (1, 0, \ldots, 0)^T \). Therefore,

\[
E[\gamma^2] = \frac{\|m_1 - m_2\|^2}{(\sigma_1 + \sigma_2)^2} E \left[ \left( \frac{A}{\|A\|}, \frac{m_1 - m_2}{\|m_1 - m_2\|} \right)^2 \right] = \frac{\|m_1 - m_2\|^2}{(\sigma_1 + \sigma_2)^2} E \left[ \frac{A^2}{\|A\|^2} \right].
\]

But, as we showed in the proof of Theorem 2,

\[
E \left[ \frac{A^2}{\|A\|^2} \right] = \frac{1}{p}.
\]

Therefore, in summary,

\[
E[\gamma^2] = \frac{\|m_1 - m_2\|^2}{(\sigma_1 + \sigma_2)^2p} = c^2.
\]

\[\Box\]

**APPENDIX F**

**PROOF OF THEOREM 3**

*Proof.* Note that since \( A^T \Sigma_1 A \geq 0 \) and \( A^T \Sigma_2 A \geq 0 \), we always have

\[
\sqrt{A^T \Sigma_1 A} + \sqrt{A^T \Sigma_2 A} \leq \sqrt{2A^T (\Sigma_1 + \Sigma_2) A}.
\]

Therefore,

\[
P \left( |\langle m_1 - m_2, A \rangle| > \gamma (\sqrt{A^T \Sigma_1 A} + \sqrt{A^T \Sigma_2 A}) \right)
\geq P \left( |\langle m_1 - m_2, A \rangle| > \gamma \sqrt{2A^T (\Sigma_1 + \Sigma_2) A} \right)
\geq P \left( |\langle m_1 - m_2, A \rangle| > \gamma \sqrt{2\lambda_{\max} \|A\|^2} \right),
\]

where the last line follows because, for every \( A \), \( A^T (\Sigma_1 + \Sigma_2) A \leq \lambda_{\max} \|A\|^2 \). Therefore, comparing (F.17) with (D.11) reveals that the desired result follows similar to Theorem 2 by replacing \( \sigma_1 + \sigma_2 \) with \( \sqrt{2\lambda_{\max}} \). \[\Box\]

**APPENDIX G**

**PROOF OF LEMMA 2**

*Proof.* Define

\[
\phi_{(i,j)} = 2Q \left( \frac{\gamma_{\min}}{c_{(i,j)}} \left[ \frac{1 - \frac{1}{p}}{1 - \frac{\gamma_{\min}}{c_{(i,j)}p}} \right] (1 + \tau) \right) \left( 1 - e^{-\frac{\gamma_{\min}}{c_{(i,j)}p}(\tau - \log(1 + \tau))} \right).
\]
By the union bound,
\[ P(B^c) \leq \sum_{i=1}^{k} \sum_{j=i+1}^{k} P \left( \left| \langle m_i - m_j, \frac{A}{\|A\|} \rangle \right| \leq \gamma_{\min}(\sigma_i + \sigma_j) \right) \]
\[ \leq \sum_{i=1}^{k} \sum_{j=i+1}^{k} (1 - \phi(i,j)) \]
\[ \leq k^2 \left( 1 - \max_{(i,j)} \{\phi(i,j)\} \right), \tag{G.18} \]
where (a) follows from Theorem 2 and the fact that for \( i, j \in \{1, \ldots, k\} \)
\[ \frac{\gamma_{\min}^2(\sigma_i + \sigma_j)^2 p}{\|m_i - m_j\|^2} = \frac{\gamma_{\min}^2}{c_{(i,j)}} \]

For \( \tau = 0.1 \), \( (\tau - \log(1 + \tau))/2 \geq 0.002 \). Therefore, setting \( \tau = 0.1 \) in (2) and noting that \( Q \) function is a monotonically decreasing function of its argument, it follows that
\[ \phi(i,j) \geq 2Q \left( \frac{\gamma_{\min}}{c_{(i,j)}} \sqrt{1 - \frac{1.1(1 - \frac{1}{p})}{\gamma_{\min}^2 c_{(i,j)}^2 p}} \right) (1 - e^{-0.002p}) \]
\[ \geq 2Q \left( \frac{\gamma_{\min}}{c_{\min}} \sqrt{1 - \frac{1.1}{1 - \frac{\gamma_{\min}^2}{c_{\min}^2 p}}} \right) (1 - e^{-0.002p}), \tag{G.19} \]
where the last inequality holds because
\[ \frac{(1 - \frac{1}{p})}{(1 - \frac{\gamma_{\min}^2}{c_{(i,j)}^2 p})} \leq \frac{1}{1 - \frac{\gamma_{\min}^2}{c_{\min}^2 p}}. \]
Therefore, taking the maximum of the both sides of (G.19), it follows that
\[ \max_{i,j} \phi(i,j) \geq 2Q \left( \frac{\gamma_{\min}}{c_{\min}} \sqrt{1 - \frac{1.1}{1 - \frac{\gamma_{\min}^2}{c_{\min}^2 p}}} \right) (1 - e^{-0.002p}). \]

\[ \square \]

**Appendix H**

**Proof of Theorem 3**

Proof. Note that since \( A^T \Sigma_1 A \geq 0 \) and \( A^T \Sigma_2 A \geq 0 \), we always have
\[ \sqrt{A^T \Sigma_1 A} + \sqrt{A^T \Sigma_2 A} \leq \sqrt{2A^T (\Sigma_1 + \Sigma_2) A}. \]
Therefore,
\[ P \left( |\langle m_1 - m_2, A \rangle| > \gamma (\sqrt{A^T \Sigma_1 A} + \sqrt{A^T \Sigma_2 A}) \right) \]
\[ \geq P \left( |\langle m_1 - m_2, A \rangle| > \gamma \sqrt{2A^T (\Sigma_1 + \Sigma_2) A} \right) \tag{H.20} \]
Since $\Sigma_1 + \Sigma_2$ is always a semi-positive definite matrix, it can be decomposed as

$$\Sigma_1 + \Sigma_2 = P^TDP,$$

where $P \in \mathbb{R}^{p \times p}$ is an orthogonal matrix ($P^T P = I_p$), and $D \in \mathbb{R}^{p \times p}$ is a diagonal matrix whose diagonal entries are non-negative. Let

$$D = \text{diag}(l_1, \ldots, l_p),$$

where $l_i \geq 0$, for all $i$. Using this decomposition, $A^T(\Sigma_1 + \Sigma_2)A$ can be written as

$$A^T(\Sigma_1 + \Sigma_2)A = (PA)^TDPA.$$

Let $B \triangleq PA$. Since $P$ is an orthogonal matrix, $B$ is still distributed as $A$, i.e., $B_1, \ldots, B_p$ are i.i.d. $\mathcal{N}(0, 1)$. By this change of variable, the probability mentioned in (24) can be written as

$$P\left( |\langle m_1 - m_2, A \rangle| \geq \gamma \left( \sqrt{A^T \Sigma_1 A} + \sqrt{A^T \Sigma_2 A} \right) \right) \geq P \left( |\langle m_1 - m_2, P^{-1}B \rangle| > \gamma \sqrt{2B^TDB} \right)$$

$$= P \left( |\langle P(m_1 - m_2), B \rangle| > \sqrt{2\gamma^2 \lambda_{\text{max}} B^2} \right). \quad (H.21)$$

Note that since by assumption $\text{rank}(\Sigma_1 + \Sigma_2) = r$, $\Sigma_1 + \Sigma_2$ has only $r$ non-zero eigenvalues. Define $C \in \mathbb{R}^p$ such that, for $i = 1, \ldots, p,$

$$C_i = B_i \mathbb{1}_{\lambda_i \neq 0}.$$

That is, for every $\lambda_i \neq 0$, $C_i$ is equal to $B_i$. For every $\lambda_i = 0$, $C_i = 0$. Using this definition, $D^\frac{1}{2}B = D^\frac{1}{2}C$. Note that

$$\|D^\frac{1}{2}C\| \leq \sqrt{\lambda_{\text{max}}} \|C\|. \quad (H.22)$$

Combining (H.21) and (H.22), it follows that

$$P\left( |\langle m_1 - m_2, A \rangle| \geq \gamma \left( \sqrt{A^T \Sigma_1 A} + \sqrt{A^T \Sigma_2 A} \right) \right) \geq P \left( |\langle P(m_1 - m_2), B \rangle| > \sqrt{2\gamma^2 \lambda_{\text{max}} \|C\|} \right)$$

$$= P \left( \frac{|\langle P(m_1 - m_2), B \rangle|}{\|B\|} > \sqrt{2\gamma^2 \lambda_{\text{max}} \|C\|} \right). \quad (H.23)$$

Given $\tau_1 > 0$ and $\tau_2 > 0$, define events $\mathcal{E}_1$ and $\mathcal{E}_2$ as

$$\mathcal{E}_1 \triangleq \{ \|B\|^2 \geq p(1 - \tau_1) \},$$

and

$$\mathcal{E}_2 \triangleq \{ \|C\|^2 \leq r(1 + \tau_2) \},$$
respectively. Note that, conditioned on \( \mathcal{E}_1 \cap \mathcal{E}_2 \),

\[
\frac{\|C\|}{\|B\|} \leq \sqrt{\frac{r(1 + \tau_2)}{p(1 - \tau_1)}}.
\]

Therefore,

\[
P \left( \left| \left( P(m_1 - m_2), \frac{B}{\|B\|} \right) \right| \leq \sqrt{2\gamma^2 \lambda_{\text{max}} \|C\|/\|B\|} \right) = P \left( \left| \left( P(m_1 - m_2), \frac{B}{\|B\|} \right) \right| \leq \sqrt{2\gamma^2 \lambda_{\text{max}} \|C\|/\|B\|}, \mathcal{E}_1 \cap \mathcal{E}_2 \right)
\]

\[
+ P \left( \left| \left( P(m_1 - m_2), \frac{B}{\|B\|} \right) \right| \leq \sqrt{2\gamma^2 \lambda_{\text{max}} (1 + \tau_2)r/(1 - \tau_1)p}, (\mathcal{E}_1 \cap \mathcal{E}_2)^c \right)
\]

\[
\leq P \left( \left| \left( P(m_1 - m_2), \frac{B}{\|B\|} \right) \right| \leq \sqrt{2\gamma^2 \lambda_{\text{max}} (1 + \tau_2)r/(1 - \tau_1)p} \right)
\]

\[
+ P(\mathcal{E}_1^c) + P(\mathcal{E}_2^c). \tag{H.24}
\]

But, from Lemma 2 in [?],

\[
P(\mathcal{E}_1^c) \leq e^{\frac{1}{2}(\tau_1 + \log(1 - \tau_1))}, \tag{H.25}
\]

and

\[
P(\mathcal{E}_2^c) \leq e^{\frac{1}{2}(\tau_2 - \log(1 + \tau_2))}.
\]

Also, note that since \( P \) is an orthogonal matrix, \( \|P(m_1 - m_2)\| = \|m_1 - m_2\| \). Therefore, the desired result follows by comparing \( P(\left| \left( P(m_1 - m_2), \frac{B}{\|B\|} \right) \right| \leq \sqrt{2\gamma^2 \lambda_{\text{max}} (1 + \tau_2)r/(1 - \tau_1)p}) \) with (D.11) and using the result of Theorem 2.

\[
\square
\]

APPENDIX I

PROOF OF LEMMA 5

Proof. As shown in the proof of Lemma 3, the optimal Bayesian classifier breaks the real line at \( t_{\text{opt}} = \frac{\mu_1 + \mu_2}{2} - \frac{\sigma^2}{(\mu_1 - \mu_2)} \ln \frac{w}{1 - w} \), and achieves a classification error equal to

\[
e_{\text{opt}} = wQ \left( \frac{t_{\text{opt}} - \mu_1}{\sigma_1} \right) + (1 - w)Q \left( \frac{\mu_2 - t_{\text{opt}}}{\sigma_2} \right)
\]

\[
= wQ \left( \gamma + \frac{1}{2\gamma} \ln \frac{w}{1 - w} \right) + (1 - w)Q \left( \gamma - \frac{1}{2\gamma} \ln \frac{w}{1 - w} \right).
\]

On the other hand, without having access to the exact parameters, a clustering algorithm that operates based on the estimated values \((\hat{\mu}_1, \hat{\mu}_2, \hat{\sigma}_1, \hat{\sigma}_1, \hat{w})\) finds \( \hat{t}_1 \) and \( \hat{t}_2 \), which are the solutions of

\[
\hat{w} = \sqrt{2\pi \hat{\sigma}_1^2} e^{-\frac{(\hat{t}_1 - \hat{\mu}_1)^2}{2\hat{\sigma}_1^2}} = \frac{1 - w}{\sqrt{2\pi \hat{\sigma}_2^2}} e^{-\frac{(\hat{t}_2 - \hat{\mu}_2)^2}{2\hat{\sigma}_2^2}},
\]

and puts the decision boundary points at these two points. For \( i = 1, 2 \), let

\[
\hat{t}_i \triangleq \hat{t}_i - \hat{\mu}_1.
\]

and

\[
(s_i, \hat{s}_i) \triangleq \left( \frac{1}{\hat{\sigma}_i^2}, \frac{1}{\hat{\sigma}_i^2} \right).
\]
Note that $\sigma_1 = \sigma_2$ by assumptions. Therefore $s_1 = s_2$. Let

$$s \triangleq s_1 = s_2,$$

and

$$\hat{\delta}_\mu \triangleq \hat{\mu}_2 - \hat{\mu}_1.$$

Using the mentioned change of variable, $(\hat{t}_1, \hat{t}_2)$ are the solutions of the following second order equation

$$(\hat{s}_1 - \hat{s}_2)x^2 + 2\hat{\delta}_\mu \hat{s}_2 x - \hat{\delta}_\mu^2 \hat{s}_2 + 2 \ln \frac{\hat{s}_2}{\hat{s}_1} - 2 \ln \frac{\hat{w}}{1 - \hat{w}} = 0,$$  \hspace{1cm} (I.26)

Assume that $\hat{t}_1$ denotes the point that approximates $t_{\text{opt}} - \mu_1$. A clustering algorithm that decides based on these estimated boundary points estimates its achieved error as $\hat{e}_{\text{opt}}$, where, if $\hat{\sigma}_1 \leq \hat{\sigma}_2$,

$$\hat{e}_{\text{opt}} = \hat{w} \left( Q \left( \frac{\hat{t}_1}{\hat{\sigma}_1} \right) + Q \left( -\frac{\hat{t}_2}{\hat{\sigma}_1} \right) \right) + (1 - \hat{w}) \left( Q \left( \frac{\hat{\delta}_\mu - \hat{t}_1}{\hat{\sigma}_2} \right) - Q \left( \frac{\hat{\delta}_\mu - \hat{t}_2}{\hat{\sigma}_2} \right) \right),$$

and if $\hat{\sigma}_1 > \hat{\sigma}_2$,

$$\hat{e}_{\text{opt}} = \hat{w} \left( Q \left( \frac{\hat{t}_1}{\hat{\sigma}_1} \right) - Q \left( \frac{\hat{t}_2}{\hat{\sigma}_1} \right) \right) + (1 - \hat{w}) \left( Q \left( \frac{\hat{\delta}_\mu - \hat{t}_1}{\hat{\sigma}_2} \right) + Q \left( \frac{\hat{\delta}_\mu - \hat{t}_2}{\hat{\sigma}_2} \right) \right).$$

Since for all $x$ and $x'$, $|Q(x) - Q(x')| \leq |x - x'|$, if $\hat{\sigma}_1 \leq \hat{\sigma}_2$,

$$|e_{\text{opt}} - \hat{e}_{\text{opt}}| \leq |w - \hat{w}| + \left| \frac{\hat{t}_1}{\hat{\sigma}_1} - \frac{t_{\text{opt}} - \mu_1}{\sigma_1} \right| + \left| \frac{\hat{\delta}_\mu - \hat{t}_1}{\hat{\sigma}_2} - \frac{\mu_2 - t_{\text{opt}}}{\sigma_2} \right| + Q \left( \frac{\hat{t}_2}{\hat{\sigma}_1} \right),$$

and if $\hat{\sigma}_1 > \hat{\sigma}_2$,

$$|e_{\text{opt}} - \hat{e}_{\text{opt}}| \leq |w - \hat{w}| + \left| \frac{\hat{t}_1}{\hat{\sigma}_1} - \frac{t_{\text{opt}} - \mu_1}{\sigma_1} \right| + \left| \frac{\hat{\delta}_\mu - \hat{t}_1}{\hat{\sigma}_2} - \frac{\mu_2 - t_{\text{opt}}}{\sigma_2} \right| + Q \left( \frac{\hat{t}_2}{\hat{\sigma}_2} \right).$$

Note that, by the triangle inequality,

$$\left| \frac{\hat{t}_1}{\hat{\sigma}_1} - \frac{t_{\text{opt}} - \mu_1}{\sigma_1} \right| \leq \left| \hat{t}_1 - t_{\text{opt}} + \mu_1 \right| + \left| t_{\text{opt}} - \mu_1 \right| \frac{1}{\sigma_1} - \frac{1}{\hat{\sigma}_1}. \hspace{1cm} (I.27)$$

Similarly,

$$\left| \frac{\hat{\delta}_\mu - \hat{t}_1}{\hat{\sigma}_2} - \frac{\mu_2 - t_{\text{opt}}}{\sigma_2} \right| \leq \left| \hat{\delta}_\mu - \hat{t}_1 + \mu_2 - t_{\text{opt}} \right| + \left| t_{\text{opt}} - \mu_2 \right| \frac{1}{\sigma_2} - \frac{1}{\hat{\sigma}_2}. \hspace{1cm} (I.28)$$

But, by assumption, $|\sigma_1^2 - \hat{\sigma}_1^2| \leq \epsilon \theta^2_\mu$. Therefore, $\hat{\sigma}_1 \leq \sigma_i \sqrt{1 + \epsilon \theta^2_\mu / \sigma_i^2} = \sigma_i \sqrt{1 + 4\gamma^2\epsilon} \leq \sigma_i (1 + 2\gamma^2\epsilon)$. Similarly, $\hat{\sigma}_i \geq \sigma_i \sqrt{1 - 4\gamma^2\epsilon} \geq \sigma_i (1 - 4\gamma^2\epsilon)$. Hence, $|\sigma_i - \hat{\sigma}_i| \leq 4\gamma^2\epsilon$ and

$$\left| \frac{1}{\hat{\sigma}_i} - \frac{1}{\sigma_i} \right| \leq \frac{4\gamma^2\epsilon}{(1 - 4\gamma^2\epsilon)\sigma_i}.$$
Also, note that since \( t_{\text{opt}} = \frac{\mu_1 + \mu_2}{2} - \frac{\sigma^2}{\mu_1 - \mu_2} \ln \frac{w}{1 - w} \), for \( i = 1, 2 \),
\[
\frac{|t_{\text{opt}} - \mu_i|}{\sigma_1} \leq \gamma + \frac{1}{2\gamma} \ln \frac{1 - w_{\text{min}}}{w_{\text{min}}}.
\]

In summary, if \( \hat{\sigma_1} \leq \hat{\sigma_2} \),
\[
|e_{\text{opt}} - \hat{e}_{\text{opt}}| \leq \left( 1 + 4\gamma^3 + 2\gamma \ln \frac{1 - w_{\text{min}}}{w_{\text{min}}} \right) \epsilon + \frac{1}{\sigma_1} |\hat{t}_1 - t_{\text{opt}} + \mu_1| + Q \left( \frac{\hat{t}_2}{\sigma_2} \right) + o(\epsilon), \quad (I.29)
\]
and if \( \hat{\sigma_1} > \hat{\sigma_2} \),
\[
|e_{\text{opt}} - \hat{e}_{\text{opt}}| \leq \left( 1 + 4\gamma^3 + 2\gamma \ln \frac{1 - w_{\text{min}}}{w_{\text{min}}} \right) \epsilon + \frac{1}{\sigma_1} |\hat{t}_1 - t_{\text{opt}} + \mu_1| + Q \left( \frac{\hat{t}_2 - \hat{\delta}_\mu}{\sigma_2} \right) + o(\epsilon). \quad (I.30)
\]

In the rest of the proof, we mainly focus on bounding \( |\hat{t}_1 - t_{\text{opt}} + \mu_1| \). Since \( \hat{t}_1 \) and \( \hat{t}_2 \) are the solutions of \( I.26 \),
\[
\hat{t}_1, \hat{t}_2 = \frac{-\hat{\delta}_\mu \hat{s}_2 \pm \sqrt{\Delta}}{(\hat{s}_1 - \hat{s}_2)},
\]
where
\[
\Delta = \left( \frac{\hat{\delta}_\mu \hat{s}_2}{\hat{s}_1 - \hat{s}_2} \right)^2 - \left( \hat{s}_1 - \hat{s}_2 \right) \left( -\frac{\hat{\delta}_\mu^2 \hat{s}_2}{\hat{s}_1} + 2 \ln \frac{\hat{s}_2}{\hat{s}_1} - 2 \ln \frac{\hat{w}}{1 - \hat{w}} \right).
\]

Define \( \upsilon \) as
\[
\upsilon \triangleq \hat{\mu}_2 - \hat{\mu}_1 - (\mu_2 - \mu_1). \quad (I.31)
\]

Note that since by assumption \( |\hat{\mu}_i - \mu_i| \leq \epsilon \delta_\mu \), where
\[
\delta_\mu \triangleq |\mu_2 - \mu_1|,
\]
we have
\[
|\upsilon| \leq 2\epsilon \delta_\mu.
\]

Define \( \tau_1 \) and \( \tau_2 \) as
\[
\tau_i \triangleq \hat{s}_i - s.
\]

Note that
\[
|\tau| \leq \frac{\delta_\mu^2 \epsilon}{\sigma_1^2 \sigma_2^2} \leq \frac{\delta_\mu^2 \epsilon}{\sigma_i^2 (\sigma_i^2 - \epsilon \delta_\mu^2)} = \frac{4\gamma^2 s \epsilon}{1 - 4\epsilon \gamma^2} \leq \frac{4\gamma^2 s \epsilon}{1 - 4\epsilon \gamma_{\text{max}}^2} \leq 8\gamma^2 s \epsilon, \quad (I.32)
\]
where the last inequality holds as long as \( 4\gamma_{\text{max}}^2 \epsilon \leq \frac{1}{2} \).

Define \( \epsilon \) as
\[
\epsilon \triangleq \frac{\delta_\mu \hat{s}_2 - \hat{s}_1}{(\hat{\delta}_\mu \hat{s}_2)^2} \left( -\hat{\delta}_\mu^2 \hat{s}_2 + 2 \ln \frac{\hat{s}_2}{\hat{s}_1} - 2 \ln \frac{\hat{w}}{1 - \hat{w}} \right). \quad (I.33)
\]
Then, using this definition, it follows from (I.26) that

$$\tilde{t}_1 = \frac{\hat{\delta}_{\mu} s_2}{s_2 - s_1} (1 - \sqrt{1 + \epsilon}),$$  (I.34)

and

$$\tilde{t}_2 = \frac{\hat{\delta}_{\mu} s_2}{s_2 - s_1} (1 + \sqrt{1 + \epsilon}).$$  (I.35)

Define function $f$ as $f(x) = \sqrt{1 + x}$. Then, using the Taylor expansion of function $f$ around zero,

$$f(\epsilon) = 1 + \frac{1}{2} \epsilon + \frac{f''(r)}{2} \epsilon^2,$$  (I.36)

where $|r| \leq |\epsilon|$. Note that

$$\frac{\hat{\delta}_{\mu} s_2}{s_2 - s_1} \epsilon = \frac{1}{\hat{\delta}_{\mu} s_2} \left( -\hat{\delta}_{\mu}^2 s_2 + 2 \ln \frac{s_2}{s_1} - 2 \ln \frac{s_1}{1 - w} \right).$$

$$= -\hat{\delta}_{\mu} + \frac{2}{\hat{\delta}_{\mu} s_2} \left( \ln \frac{s_2}{s_1} - \ln \frac{s_1}{1 - w} \right).$$  (I.37)

Therefore, we have

$$\tilde{t}_1 = \frac{\hat{\delta}_{\mu}}{2} + \frac{1}{\hat{\delta}_{\mu} s_2} \left( \ln \frac{s_1}{1 - w} - \ln \frac{s_2}{s_1} \right) - \frac{f''(r)}{2} \left( \frac{\hat{\delta}_{\mu} s_2}{s_2 - s_1} \right) \epsilon^2.$$  (I.38)

and

$$\tilde{t}_2 = \frac{2\hat{\delta}_{\mu} s_2}{s_2 - s_1} - \tilde{t}_1.$$  (I.39)

As a reminder $t_{opt} = \mu_1 + \mu_2 - \frac{1}{(\mu_1 - \mu_2)} \ln \frac{\tilde{w}}{1 - \tilde{w}}$. Therefore, from (I.38), we have

$$|\tilde{t}_1 - t_{opt} + \mu_1| = \left| \frac{\hat{\delta}_{\mu}}{2} + \frac{1}{\hat{\delta}_{\mu} s_2} \left( \ln \frac{s_1}{1 - w} - \ln \frac{s_2}{s_1} \right) - \frac{f''(r)}{2} \left( \frac{\hat{\delta}_{\mu} s_2}{s_2 - s_1} \right) \epsilon^2 - t_{opt} + \mu_1 \right|$$

$$\leq \hat{\delta}_{\mu} \epsilon + \left| \frac{1}{\hat{\delta}_{\mu} s_2} \ln \frac{s_1}{1 - w} - \frac{1}{s(\mu_2 - \mu_1)} \ln \frac{w}{1 - w} \right| + \left| \frac{1}{\hat{\delta}_{\mu} s_2} \ln \frac{s_2}{s_1} \right| + \frac{f''(r)}{2} \left( \frac{\hat{\delta}_{\mu} s_2}{s_2 - s_1} \right) \epsilon^2 \right|.$$  (I.40)

We next bound the error terms in (I.40). Note that, by the triangle inequality,

$$\left| \frac{1}{\hat{\delta}_{\mu} s_2} \ln \frac{s_1}{1 - w} - \frac{1}{s(\mu_2 - \mu_1)} \ln \frac{w}{1 - w} \right| = \left| \frac{1}{\hat{\delta}_{\mu} s_2} \left( \ln \frac{s_1}{1 - w} - \ln \frac{w}{1 - w} \right) + \frac{1}{s(\mu_2 - \mu_1)} \ln \frac{w}{1 - w} \right|$$

$$\leq \frac{1}{|\hat{\delta}_{\mu}|} \left| \ln \frac{s_1}{1 - w} - \ln \frac{w}{1 - w} \right| + \left| \ln \frac{w}{1 - w} \right| \left| \frac{1}{\hat{\delta}_{\mu} s_2} \right| + \frac{1}{|s(\mu_2 - \mu_1)|}$$  (I.41)

Since $|\mu_i - \hat{\mu}_i| \leq \epsilon \hat{\delta}_{\mu}$, $|\hat{\delta}_{\mu}| = |\hat{\mu}_1 - \hat{\mu}_2| \geq \hat{\delta}_{\mu}(1 - 2\epsilon)$. Therefore, we have

$$\frac{1}{|\hat{\delta}_{\mu}|} \leq \frac{1}{\hat{\delta}_{\mu}(1 - 2\epsilon)}.$$  (I.42)
Let \( g(w) = \ln \frac{w}{1-w} \). Then, \( g'(w) = \frac{1}{w} + \frac{1}{1-w} \). Therefore, since by assumption, \( |w - \hat{w}| \leq \epsilon \), we have

\[
\left| \ln \frac{\hat{w}}{1-\hat{w}} - \ln \frac{w}{1-w} \right| \leq \left( \frac{1}{w_{\text{min}}} + \frac{1}{1-w_{\text{min}}} \right) \epsilon \leq \frac{2\epsilon}{w_{\text{min}}}. \tag{I.43}
\]

Note that since \( w \in (w_{\text{min}}, 0.5) \), and since \( \frac{\hat{w}}{1-\hat{w}} \) is an increasing function of \( w \) in this interval, we have

\[
|\ln \frac{\hat{w}}{1-\hat{w}}| \leq \ln \frac{1-w_{\text{min}}}{w_{\text{min}}}. \tag{I.44}
\]

Note that

\[
\ln \frac{\hat{s}_2}{\hat{s}_1} = \ln \frac{s + \tau_1}{s + \tau_2} = \ln \frac{1 + \tau_1/s}{1 + \tau_2/s}.
\]

Hence,

\[
\left| \ln \frac{\hat{s}_2}{\hat{s}_1} \right| \leq \ln \frac{1 + \frac{\tau_1}{s}}{1 - \frac{\tau_2}{s}} \leq \ln \frac{1 + \frac{4\epsilon^2}{1-4\epsilon^2\gamma_{\text{max}}} \mu_2 s}{1 - \frac{4\epsilon^2}{1-4\epsilon^2\gamma_{\text{max}}} \mu_2 s} = \ln \frac{1}{1 - 8\epsilon^2\gamma_{\text{max}}} \leq \frac{8\epsilon^2\gamma_{\text{max}}}{1 - 8\epsilon^2\gamma_{\text{max}}} \leq 16\epsilon^2\gamma_{\text{max}}. \tag{I.45}
\]

where the last line holds if \( 8\epsilon^2\gamma_{\text{max}} < \frac{1}{2} \). Combining (I.41), (I.42), (I.43), (I.44) and (I.45) with (I.40), it follows that

\[
|\hat{t}_1 - t_{\text{opt}} + \mu_1| \leq \delta_\mu \epsilon + \frac{2\epsilon}{(1-2\epsilon)(1-8\gamma s)w_{\text{min}}\delta_\mu s} + \frac{1}{\hat{s}_2} \ln \frac{1-w_{\text{min}}}{w_{\text{min}}} \left| \frac{1}{\delta_\mu} - \frac{1}{(\mu_2 - \mu_1)} \right| + \frac{1}{\delta_\mu} \ln \frac{1-w_{\text{min}}}{w_{\text{min}}} \left| \frac{1}{\hat{s}_2} - \frac{1}{s} \right| + \frac{16\gamma_{\text{max}}^2 \epsilon}{(1-2\epsilon)(1-8\gamma^2 s)\delta_\mu s} + \frac{2\epsilon}{(1-2\epsilon)(1-8\gamma^2 s)\delta_\mu s} \left| f''(r) \left( \frac{\hat{s}_2}{\hat{s}_2 - \hat{s}_1} \right) \epsilon^2 \right|
\]

\[
\leq \delta_\mu \epsilon + \frac{2\epsilon}{(1-2\epsilon)(1-8\gamma s)w_{\text{min}}\delta_\mu s} + \frac{16\gamma_{\text{max}}^2 \epsilon}{(1-2\epsilon)(1-8\gamma^2 s)\delta_\mu s} + \frac{2\epsilon}{(1-2\epsilon)(1-8\gamma^2 s)\delta_\mu s} \left| f''(r) \left( \frac{\hat{s}_2}{\hat{s}_2 - \hat{s}_1} \right) \epsilon^2 \right|
\]

\[
= \left( \delta_\mu + \frac{2}{w_{\text{min}}\delta_\mu s} + \frac{2}{\delta_\mu s + \delta_\mu} \ln \frac{1-w_{\text{min}}}{w_{\text{min}}} + \frac{16\gamma_{\text{max}}^2 \epsilon}{\delta_\mu s} \right) \epsilon + \frac{f''(r)}{2} \left| \hat{s}_2 \right| \epsilon^2 + o(\epsilon).
\]

Finally, we need to bound \( \epsilon \), defined in (I.33). By the triangle inequality and (I.42) it follows that

\[
|\epsilon| \leq \frac{|\hat{s}_2 - \hat{s}_1|}{\delta_\mu^2 (1-2\epsilon)^2 (\hat{s}_2)^2} \left( \delta_\mu^2 (1+2\epsilon)^2 \hat{s}_2^2 + 2 \left| \ln \frac{\hat{s}_2}{\hat{s}_1} \right| + 2 \left| \ln \frac{\hat{w}}{1-\hat{w}} \right| \right)
\]

\[
= \frac{|\hat{s}_2 - \hat{s}_1|}{\delta_\mu^2 (1-2\epsilon)^2 (\hat{s}_2)^2} \left( \delta_\mu^2 (1+2\epsilon)^2 \hat{s}_2^2 + 2 \left| \ln \frac{\hat{s}_2}{\hat{s}_1} \right| + 2 \left| \ln \frac{\hat{w}}{1-\hat{w}} - \ln \frac{w}{1-w} + \ln \frac{w}{1-w} \right| \right)
\]

\[
\leq \frac{|\hat{s}_2 - \hat{s}_1|}{\delta_\mu^2 (1-2\epsilon)^2 (1-8\gamma^2 s)^2} \left( \delta_\mu^2 (1+2\epsilon)^2 (1+8\gamma^2 s) s + 32\gamma_{\text{max}}^2 \epsilon + \frac{4\epsilon}{w_{\text{min}}} \right) + 2 \ln \frac{1-w_{\text{min}}}{w_{\text{min}}}
\]

\[
\leq \frac{|\hat{s}_2 - \hat{s}_1|}{\delta_\mu^2 (1-2\epsilon)^2 (1-8\gamma^2 s)^2} \left( 1 + \frac{1}{2\gamma} \ln \frac{1-w_{\text{min}}}{w_{\text{min}}} + O(\epsilon) \right), \tag{I.47}
\]

where (a) follows from (I.32), (I.43), (I.44) and (I.45), and (b) holds because \( \delta_\mu^2 s = 4\gamma \). Also, note that, from
and $|f''(r)| = \frac{1}{2}(1 + r)^{-\frac{3}{2}} \leq \frac{1}{\sqrt{2}} < 1$. Combining (I.47) with (I.46), it follows that

$$|\tilde{t}_1 - t_{opt} + \mu_1| \leq \left( \delta_\mu + \frac{2}{\min \sigma_1 \delta \mu s} + \left( \frac{2}{\delta \mu s} + \delta \mu \right) \ln \frac{1 - \min \sigma_{\min}}{\min \sigma_{\min}} + \frac{16\gamma^2}{\delta \mu s} \right) \epsilon \quad (I.48)$$

$$+ |f''(r)||\delta_\mu \delta_2|\tilde{\delta}_2 - \tilde{\delta}_1| \left( 1 + \frac{1}{2\gamma} \ln \frac{1 - \min \sigma_{\min}}{\min \sigma_{\min}} + O(\epsilon) \right)^2 + o(\epsilon)$$

$$\leq \left( \delta_\mu + \frac{2}{\min \sigma_1 \delta \mu s} + \left( \frac{2}{\delta \mu s} + \delta \mu \right) \ln \frac{1 - \min \sigma_{\min}}{\min \sigma_{\min}} + \frac{16\gamma^2}{\delta \mu s} \right) \epsilon$$

$$+ |f''(r)||\delta_\mu \sigma_2| (1 + \epsilon) (16\gamma^2) \left( 1 + \frac{1}{2\gamma} \ln \frac{1 - \min \sigma_{\min}}{\min \sigma_{\min}} + O(\epsilon) \right)^2 + o(\epsilon)$$

$$\leq \left( \delta_\mu + \frac{2}{\min \sigma_1 \delta \mu s} + \left( \frac{2}{\delta \mu s} + \delta \mu \right) \ln \frac{1 - \min \sigma_{\min}}{\min \sigma_{\min}} + \frac{16\gamma^2}{\delta \mu s} \right) + |f''(r)||\delta_\mu \sigma_2| \left( 4\gamma + 2 \ln \frac{1 - \min \sigma_{\min}}{\min \sigma_{\min}} \right) \epsilon + o(\epsilon). \quad (I.49)$$

Dividing both sides of (I.49) by $\sigma_1$, and noting that $\delta_\mu / (2\sigma_1) = \gamma$ and $|f''(r)| \leq 1$, we derive

$$\frac{|\tilde{t}_1 - t_{opt} + \mu_1|}{\sigma_1} \leq \left( 2\gamma + \frac{1}{\min \gamma} + \left( \frac{1}{\gamma} + 2\gamma \right) \ln \frac{1 - \min \sigma_{\min}}{\min \sigma_{\min}} + \frac{8\gamma^2}{\gamma} \right) \epsilon + o(1) \quad (I.50)$$

Finally, as a reminder, from (I.39), $\tilde{t}_2 = \frac{2\delta_\mu \sigma_1}{\tilde{\sigma}_2 - \tilde{\sigma}_1} - \tilde{t}_1$. From (I.32), $|\tilde{\sigma}_2 - \tilde{\sigma}_1| \leq 16\gamma^2 \sigma_\epsilon$. Hence, if $\tilde{\sigma}_1 \leq \tilde{\sigma}_2$,

$$-\frac{\tilde{t}_2}{\tilde{\sigma}_1} = \frac{1}{\tilde{\sigma}_1} \left( \frac{2\delta_\mu \sigma_2}{\tilde{\sigma}_2 - \tilde{\sigma}_1} - \tilde{t}_1 \right) \geq \frac{1}{4\gamma \epsilon} + o(1) \epsilon.$$

Similarly, if $\tilde{\sigma}_2 \leq \tilde{\sigma}_1$,

$$\frac{\tilde{t}_2 - \tilde{\sigma}_\mu}{\tilde{\sigma}_2} = \frac{1}{\tilde{\sigma}_2} \left( \frac{2\delta_\mu \sigma_2}{\tilde{\sigma}_2 - \tilde{\sigma}_1} - \tilde{t}_1 - \tilde{\sigma}_\mu \right) \geq \frac{1}{4\gamma \epsilon} + o(1) \epsilon.$$

Combining (I.50) and the above equations with (I.29) and (I.30) yields the desired result. \qed