RESEARCH ARTICLE

Links all of whose cyclic-branched covers are L-spaces

Ahmad Issa1 | Hannah Turner2

1Department of Mathematics, University of British Columbia, Vancouver, Canada
2Department of Mathematics, University of Texas at Austin, Austin, Texas, USA

Correspondence
Hannah Turner, Department of Mathematics, University of Texas at Austin.
Email: hannahturner@math.utexas.edu

Funding information
NSF, Grant/Award Number: DGE-1610403

Abstract
We show that for the pretzel knots \( K_k = P(3, -3, -2k - 1) \), the \( n \)-fold cyclic-branched covers are L-spaces for all \( n \geq 1 \). In addition, we show that the knots \( K_k \) with \( k \geq 1 \) are quasi-positive and slice, answering a question of Boileau–Boyer–Gordon. We also extend results of Teragaito giving examples of two-bridge knots with all L-space cyclic-branched covers to a family of two-bridge links.

MSC 2020
57K10, 57K18 (primary)

1 | INTRODUCTION

Given a knot \( K \) in \( S^3 \), it is an interesting problem to determine the set

\[
\mathcal{L}_{br}(K) = \{ n \geq 1 : \text{the n-fold cyclic-branched cover } \Sigma_n(K) \text{ is an L-space} \},
\]

where \( \Sigma_n(K) \) denotes the \( n \)-fold cyclic-branched cover of \( S^3 \) over \( K \), and an L-space is a rational homology 3-sphere whose Heegaard Floer homology is as simple as possible. For example, when \( K \) is the trefoil, \( \mathcal{L}_{br}(K) = \{ 1, 2, \ldots, 5 \} \) (see [4]) and when \( K \) is the figure-eight knot, \( \mathcal{L}_{br}(K) = \{ 1, 2, \ldots \} \); see, for example, [21]. Evidence suggests that the set \( \mathcal{L}_{br}(K) \) always takes one of the two forms \( \mathcal{L}_{br}(K) = \{ 1, 2, \ldots \} \) or \( \mathcal{L}_{br}(K) = \{ 1, \ldots, N \} \) for some \( N \), and Boileau–Boyer–Gordon asked whether this holds in general. Some results are known, for example, nonsplit alternating links have L-space double-branched covers, and this is true more generally for Khovanov homology thin links [20]. It is known that the \( n \)-fold cyclic-branched cover of \( K \) is not an L-space for \( n \) sufficiently large, provided that \( K \) is a nonslice quasi-positive knot [2], or \( K \) is fibered with nonzero fractional Dehn twist coefficient [13, 22], see also [7]. This class includes, for example, all L-space knots.

© 2023 The Authors. The publishing rights in this article are licensed to the London Mathematical Society under an exclusive licence.
One motivation for the study of $\mathcal{L}_{br}(K)$ comes from the L-space conjecture that, for an irreducible 3-manifold $M$, asserts that $M$ is an L-space if and only if $M$ has a nonleft orderable fundamental group [3]. Cyclic-branched covers provide a natural class for which to study the L-space conjecture, and various results are known for left orderability of fundamental groups of branched covers; see, for example, [3–5, 8].

In contrast to the case $\mathcal{L}_{br}(K) = \{1, \ldots, N\}$, knots for which $\mathcal{L}_{br}(K) = \{1, 2, \ldots\}$, that is, $\Sigma_n(K)$ is an L-space for all $n \geq 1$, are less well understood. In fact, prior to our work, the only known example was a family of 2-bridge knots [21, 25]. We prove the following theorem, giving a family of pretzel knot examples; see Figure 3 for our notation of pretzel knots.

**Theorem 1.1.** The $n$-fold cyclic-branched cover $\Sigma_n(P(3, -3, -2k - 1))$ is an L-space for all integers $k$ and $n \geq 1$.

Boileau–Boyer–Gordon showed that if a quasi-positive knot $K$ is not smoothly slice, then $\Sigma_n(K)$ is not an L-space for $n$ sufficiently large, and asked the following question.

**Question 1.2** [2]. Does there exist a slice quasi-positive knot all of whose branched cyclic covers are L-spaces?

Let $K_k$ denote the pretzel knot $P(3, -3, -2k - 1)$. These pretzel knots are well known to be slice; see Figure 13 for a ribbon diagram. For $k \geq 1$, we show that $K_k$ is a track knot, a notion introduced by Baader [1]. Since track knots are quasi-positive, we obtain the following proposition, answering the above question in the affirmative.

**Proposition 1.3.** The knots $K_k$ are slice and quasi-positive for all $k \geq 1$.

Before discussing these examples in more depth, we first look at a two-bridge example. The figure-eight knot $K$ is the simplest knot for which $\Sigma_n(K)$ is an L-space for all $n \geq 1$. One way to see that this is as follows. First observe that the figure-eight knot $K$ is a 2-periodic knot, with quotient knot $Q$ and axis $A$ in the quotient, as shown in Figure 1. The $n$-fold cyclic-branched cover
\[ \Sigma_n(K) \] can be recovered by taking the twofold branched cover over the axis \( A \) and then taking the \( n \)-fold cyclic-branched cover over the lift of \( Q \). Reversing the order in which we take branched covers, \( \Sigma_n(K) \) can also be obtained by taking the \( n \)-fold cyclic-branched cover over \( Q \) and then the twofold branched cover over the lift of \( A \), which we denote as \( L_n \). Using the fact that there is an ambient isotopy interchanging the two components of \( Q \cup A \), it is not difficult to obtain a diagram for \( L_n \). The link \( L_n \) is alternating and non-split, because it has a non-split alternating diagram \[ [17] \]. Thus, its double-branched cover \( \Sigma_2(L_n) \cong \Sigma_n(K) \) is an L-space for all \( n \geq 1 \) \[ [20] \].

Peters proved that \( \Sigma_n(K) \) is an L-space for two-bridge knots \( K \) with fraction \[ [2a, 2b] + := 2a + \frac{1}{2b} \] for all \( a, b \geq 1 \) and \( n \geq 1 \) \[ [21] \]. Teragaito \[ [25] \] generalized these examples to two-bridge knots with continued fractions of the form \[ [2a_1, \ldots, 2a_{2k}]^+ \], where \( a_1, \ldots, a_{2k} > 0 \). In order to establish that \( \Sigma_n(K) \) is the double-branched cover of an alternating link in these cases, both Peters and Teragaito appeal to work of Mulazzani and Vesnin \[ [19] \], which proceeds by means of certain surgery presentations of these manifolds. We remark that their results can be obtained directly using the 2-periodic nature of two-bridge links, generalizing the figure-eight knot case above. For any two-bridge link \( L \) (with a particular orientation), we have \( \Sigma_n(L) \cong \Sigma_2(L_n) \) for some link \( L_n \); see, for example, \[ [18] \]. In certain cases, this link \( L_n \) is non-split and alternating. This line of argument extends Teragaito’s family to two-bridge links \[ [25] \] as follows. We note that the special case of two-bridge torus links with fraction \[ [2a]^+ = 2a \] and antiparallel string orientations was first shown by Peters \[ [21] \].

**Theorem 1.4.** Let \( L \) be a two-bridge link with fraction of the form

\[ [2a_1, 2a_2, \ldots, 2a_n]^+ := 2a_1 + \frac{1}{2a_2 + \frac{1}{\ddots + \frac{1}{2a_n}}}, \]

where \( a_1, \ldots, a_n \) are all positive integers. In the case that \( L \) is a two-component link, we orient the link as in Figure 14. Then \( \Sigma_n(L) \) is an L-space for all \( n \geq 1 \).

More generally, whenever a link \( L \) is 2-periodic with quotient link the unknot, we can express \( \Sigma_n(L) \) as the double-branched cover over a link \( L_n \). This is the case when \( L \) is an odd pretzel knot with all tassel parameters odd integers. Thus, \( \Sigma_n(P(3, -3, -2k - 1)) \) is the double-branched cover of a link \( L_{n,k} \). However, unlike in the figure-eight knot case, the links \( L_{n,k} \) are not, in general, alternating links. For example, \( L_{2,1} \) is the pretzel knot \( P(3, -3, -3) \) that is not even quasi-alternating \[ [6] \]; see also \[ [9] \]. We instead show that the links \( L_{n,k} \) are twofold quasi-alternating (TQA), a generalization of quasi-alternating links introduced by Scaduto and Stoffregen \[ [24] \]. TQA links are \( \mathbb{Z}/2 \) reduced Khovanov thin \[ [24] \]. Thus, \( \Sigma_2(L_{n,k}) = \Sigma_n(P(3, -3, -2k - 1)) \) is a Heegaard Floer L-space \[ [20] \] and a framed instant on L-space \[ [23] \] with \( \mathbb{Z}/2 \) coefficients. In fact, we prove that the larger family of links \( L(k_1, k_2, \ldots, k_n) \) in Figure 2 are all TQA. The link \( L_{n,k} \) is given by \( L(-k, -k, \ldots, -k) \), where \( -k \) appears \( n \) times.

**Theorem 1.5.** Let \( L(k_1, k_2, \ldots, k_n) \) be the link in Figure 2. Then \( L \) is TQA for all integers \( k_1, \ldots, k_n \).

---

† A two-bridge knot with fraction \( \frac{p}{q} \) has double-branched cover homeomorphic to the lens space \( L(p, q) \). The figure-eight knot has fraction \( \frac{5}{2} = [2, 2]^+ = 2 + \frac{1}{2} \).
The link $L(k_1, k_2, \ldots, k_n)$, where $k_1, \ldots, k_n$ are any integers. A twist box labeled $k_i$ denotes $k_i$ signed half-twists. See Figure 3 for an example of our signed twist conventions.

At the time of this writing, the links in Theorems 1.1 and 1.4 include all currently known examples of knots and links all of whose branched cyclic covers are L-spaces. It is rather curious that in all of these examples, the knots satisfy the following properties. It is an interesting question whether any of these properties hold more generally.

1. The roots of the Alexander polynomial $\Delta_K(t)$ are all real and positive [14, 15].
2. The Tristram–Levine signature function $\sigma_{\omega}(K)$ is identically zero. This is implied by (1).
3. The $d$-invariant $d(\Sigma_n(K), \mathfrak{s}) = 0$ on the specific spin$^c$ structure $\mathfrak{s}$ described in [16], for all $n \geq 2$.
4. The knot group $\pi_1(X_K)$ is biorderable [10, Corollary 1.11], [11].

We remark that property (3) follows from property (2) together with work of Lin–Ruberman–Saveliev establishing a relationship between the Tristram–Levine signatures and the Heegaard Floer $d$-invariant $d(\Sigma_n(K), \mathfrak{s})$ for a certain spin$^c$ structure when $K$ is a knot such that $\Sigma_n(K)$ is an L-space for all $n \geq 2$ [16]. We remark that the oriented two component links $L$ of Theorem 1.4 all satisfy $|\sigma(L)| = 1$, which is as small as possible among two component links whose double-branched covers are rational homology spheres.

We end the introduction with the following question.

**Question 1.6.** Does there exist a non-2-periodic knot $K$ for which $\Sigma_n(K)$ is an L-space for all $n \geq 1$?

1.1 | Organization

We exhibit $\Sigma_n(K_k)$ as the double-branched cover of a link $L_{n,k}$ in Section 2. In Section 3, we give background on TQA links. We prove Theorems 1.1 and 1.5 in Section 4. We prove Proposition 1.3 in Section 5. We conclude with a discussion of the case of two-bridge links and prove Theorem 1.4.

2 | BRANCHED CYCLIC COVERS OF $P(3, -3, -2k - 1)$

In this section, we discuss 2-periodic symmetries for pretzel links and apply these symmetries to express $\Sigma_n(P(3, -3, -2k - 1))$ as the double-branched cover of a link $L_{n,k}$. Let the pretzel link $P(p_1, p_2, \ldots, p_n)$ be defined by replacing the boxes in Figure 3 with $|p_i|$ half-twists with sign determined by the sign of $p_i$. We remark that cyclic permutations of the parameters do not change the link isotopy class. Then $K_k = P(3, -3, -2k - 1)$, and, in fact, any pretzel link $P(p_1, p_2, \ldots, p_n)$ with each $p_i$ odd, admits an involution whose axis is disjoint from the link which we will now
FIGURE 3  The pretzel link \( P(p_1, p_2, \ldots, p_n) \) and a symmetry of \( P(-3, 3, -3) \).

FIGURE 4  Let \( K_k = P(-3, -2k - 1, 3) \cong P(3, -3, -2k - 1) \). There is an isotopy of \( K_k \cup \alpha \) to a diagram where \( \alpha \) sits inside a solid torus with \( K_k \) its meridian.

describe. Let \( \alpha \) be the horizontal line that passes through each twist box in the standard diagram for \( K_k \) through the central crossing of each twist box. Then \( K_k \) is rotationally symmetric about \( \alpha \); see Figure 3. Rotating about this axis \( \alpha \) by an angle of \( \pi \) defines an involution \( \iota \) that preserves the knot setwise. The fixed set of this involution in \( S^3 \) is the axis \( \alpha \).

We are interested in the quotient by this involution. The quotient 3-manifold is \( S^3 \) and we denote the quotient knots by \( \iota(S^3, K_k, \alpha) = (S^3, \overline{K_k}, \overline{\alpha}) \) pictured in Figure 4. In fact, \( \iota \) defines a branched covering map from \( S^3 \) to itself where the branching set is \( \overline{\alpha} \) and the branching is of index 2.

Taking any partition of the components of an oriented link \( L \) into \( l \) subsets so that \( L = L_1 \cup \cdots \cup L_l \), one can consider different branching indices \( n_1, \ldots, n_l \) for each sublink. To denote the unique \( \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_l \) branched cover obtained by this process, we will write \( \Sigma_{n_1, n_2, \ldots, n_l}(L_1 \cup L_2 \cup \cdots \cup L_l) \). Thus, a feature of links \( L \) that admit such a twofold symmetry with axis \( \alpha \) avoiding the link is that the \( n \)-fold branched cyclic cover \( \Sigma_n(L) \) can be expressed as the \( \mathbb{Z}/n \times \mathbb{Z}/2 \) branched cover of the quotient link \( \overline{L} \cup \overline{\alpha} \). The order of branching does not affect the homeomorphism type of the manifold \( \Sigma_{n_2}(\overline{L} \cup \overline{\alpha}) \). Thus, one can consider first branching (of index \( n \)) over \( \overline{L} \), and then branching (of index 2) over the lift of \( \overline{\alpha} \) which we denote as \( L_n \); see Figure 5. This process and commutative diagram are well known; see, for instance, [18]. From this commuting diagram, we see that if \( L_n = \phi_{n,1}^{-1}(\overline{\alpha}) \), then \( \Sigma_n(L) \cong \Sigma_{n,2}(\overline{L} \cup \overline{\alpha}) \cong \Sigma_2(L_n) \) can be expressed both as an \( n \)-fold cyclic-branched cover over \( L \) and a twofold branched cover over \( L_n \).
For 2-periodic link $L$, the $n$-fold branched cover can be expressed in two ways. The left-hand side of the diagram describes $\Sigma_{n,2}(\overline{L} \cup \overline{a})$ as first branching over $\overline{a}$, and then over $L$ (the lift of $L$). On the right-hand side, we branch first over $L$ and then over the lift of $\overline{a}$ which we denote by $L_n$.

In the special case that $\overline{L} \cup \overline{a}$ is the union of two unknotted, it is easy to understand the manifold $M = \Sigma_{n,1}(\overline{L} \cup \overline{a})$. In particular, $M \cong S^3$ and it is possible to obtain a diagram for $(S^3, L_n)$ via the quotient diagram of $\overline{L} \cup \overline{a}$.

Proposition 2.1. $\Sigma_n(P(3, -3, -2k - 1)) \cong \Sigma_2(L_{n,k})$ where $L_{n,k}$ is the link in Figure 6.

Proof. By appealing to the diagram in Figure 5, it suffices to describe the lift $L_{n,k}$ of $\overline{a}$ after branching of index $n$ over $\overline{K}_k$ where $K_k = P(3, -3, -2k - 1)$ and $\overline{K}_k$ denotes its quotient under the involution described above and illustrated by Figure 4.

The manifold $M$ we are interested in understanding is the $n$-fold cover of $\overline{K}_k$ that is unknotted. The complement of $\overline{K}_k$ is a solid torus $T$ whose lift is also a solid torus $\overline{T}$ in $M$. This solid torus $\overline{T}$ has meridian $\overline{\mu}$ and longitude $\overline{\lambda}$. If $\mu$ and $\lambda$ denote the meridian and longitude of $T$, then $\overline{\lambda}$ is the lift of $\lambda$ while $\mu$ lifts to $n$ disjoint translates of $\overline{\mu}$.

Thus, it is easy to obtain a diagram for $L_{n,k}$ the lift of $\overline{a}$. Figure 6 is obtained by cutting open the torus of Figure 4 along $\overline{K}_k$ and “stacking” $n$ copies of the corresponding cylinder and finally identifying the ends by the identity. □

3 | TWOFOLD QUASI-ALTERNATING LINKS

In this section, we recall the definition of TQA links, as well as a few key facts which we will use. All of this material can be found in [24].

A marked link is a pair $(L, \omega)$ where $L \subset S^3$ is a link and $\omega$ is an assignment of an element of $\mathbb{Z}/2$ to each component of $L$. If the number of components marked with $1 \in \mathbb{Z}/2$ is even (resp. odd), we say that $(L, \omega)$ is a twofold marked link (resp. odd marked link). The trivial twofold marking assigns $0 \in \mathbb{Z}/2$ to every component of $L$. 
Let $D$ be a planar diagram representing a link $L$. An arc of $D$ is a strand of $D$ that descends to an edge of the 4-valent graph formed from $D$ upon forgetting its crossings. A marking of $D$ is an assignment $\tilde{\omega} : \Gamma(D) \to \mathbb{Z}/2$, where $\Gamma(D)$ is the set of arcs of $D$. We say that $(D, \tilde{\omega})$ is compatible with or represents $(L, \omega)$ if

$$\sum_{\gamma \in \Gamma(K)} \tilde{\omega}(\gamma) = \omega(K) \pmod{2},$$

for each component $K$ of $L$. This data can be packaged diagrammatically by placing an odd number of dots on each arc $\gamma$ for which $\tilde{\omega}(\gamma) = 1$. Using this diagrammatic interpretation, when we smooth $D$ at a crossing (see Figure 7), we naturally obtain two marked diagrams $(D_0, \tilde{\omega}_0)$ and $(D_1, \tilde{\omega}_1)$ by counting dots mod 2.

**Definition 3.1.** The set of TQA twofold marked links, denoted as $Q$, is the smallest set of twofold marked links satisfying the following.

1. The unknot with its unique trivial twofold marking data is in $Q$.
2. Any twofold marked link that splits into two odd-marked links is in $Q$.
3. Let $(D, \tilde{\omega})$ be a marked diagram representing $(L, \omega)$ such that the two smoothings $(D_0, \tilde{\omega}_0)$ and $(D_1, \tilde{\omega}_1)$ at a crossing represent marked links $(L_0, \omega_0)$ and $(L_1, \omega_1)$, respectively (see Figure 7). If
   - both smoothings $(L_0, \omega_0)$ and $(L_1, \omega_1)$ are in $Q$, and
   - $\text{det}(L) = \text{det}(L_0) + \text{det}(L_1)$
   then $(L, \omega)$ is in $Q$.

We say that a link $L$ is TQA if for the trivial marking $\omega$, we have $(L, \omega) \in Q$.

All nonsplit alternating links are TQA. Finally, the key fact about TQA links which we will use is the following; see [24, Corollary 1].

**Theorem 3.2.** If a link $L$ is TQA and $\text{det}(L) \neq 0$, then $\Sigma_2(L)$ is a Heegaard Floer L-space with $\mathbb{Z}/2$ coefficients.

4 | **$L_{n,k}$ ARE TWOFOLD QUASI-ALTERNATING**

The goal of this section is to show that all the links $L_{n,k}$ in Figure 6 are TQA. This implies that their double-branched covers are L-spaces. We will, in fact, show that a broader family of links is TQA. We first define the links of interests.
Definition 4.1. Let $T(k)$ denote the 3-tangle shown in Figure 8, where $k \in \mathbb{Z} \cup \{\infty\}$. If $T_1$ and $T_2$ are 3-tangles, denote by $T_1 \oplus T_2$ the 3-tangle obtained by stacking the two tangles, with $T_1$ to the left of $T_2$; see, for example, Figure 9. Let $L(k_1, \ldots, k_n) \subset S^3$ denote the link given by the closure of the tangle $T(k_1) \oplus T(k_2) \oplus \cdots \oplus T(k_n)$, where $k_i \in \mathbb{Z} \cup \{\infty\}$ for all $i$.

The link $L_{n,k}$ in Figure 6 is the link $L(-k, -k, \ldots, -k)$. We will show that these links are all TQA.

Lemma 4.2. The link $L(k_1, k_2, \ldots, k_n)$ is a knot provided no $k_i = \infty$.

Proof. The link $L = L(k_1, k_2, \ldots, k_n)$ is the closure of $T(k_1) \oplus \cdots \oplus T(k_n)$. To count the number of components of $L$, we only need to keep track of the way each tangle $T(k_i)$ connects the six endpoints of the tangle, which is determined by the parity of $k_i$. If $T$ and $T'$ are 3-tangles, write $T \sim T'$ if $T$ and $T'$ are homotopic, in other words, they have the same connectivity of endpoints and number of connected components. One can check the four cases: $T(0) \oplus T(1) \sim T(1), T(0) \oplus T(0) \sim T(0), T(1) \oplus T(0) \sim T(0), T(1) \oplus T(1) \sim T(1)$. Thus, $T(k_1) \oplus \cdots \oplus T(k_n) \sim T(k), \text{ where } k \in \{0, 1\}$. Finally, the closure of both $T(0)$ and $T(1)$ are knots; hence, $L$ is a knot.

Lemma 4.3. Let $L = L(k_1, \ldots, k_n)$ with $k_i \in \mathbb{Z} \cup \{\infty\}$ for all $i$. If $k_i = \infty$ for exactly one value of $i \in \{1, \ldots, n\}$, then $L$ is the two component unlink.

Proof. Notice that the link $L = L(k_1, \ldots, k_n)$ is unchanged under any cyclic permutation of the parameters $(k_1, \ldots, k_n)$. Hence, we may assume that $k_n = \infty$. Figure 9 shows that $T(k) \oplus T(\infty)$ is isotopic to $T(\infty)$ as tangles, provided $k \neq \infty$. Applying this $n-1$ times, we see that $T(k_1) \oplus \cdots \oplus T(k_n)$ is the same as $T(\infty)$. Hence, $L$ is the closure of $T(\infty)$ that is the two-component unlink.

Lemma 4.4. The links of the form $L(0, 0, \ldots, 0)$ are all nonsplit alternating knots.
The alternating tangle $T$ is obtained from $T(0)$ by conjugating by a tangle interchanging the top two strands. The middle diagram of $T(0)$ is obtained from the one on the left by an isotopy.

**Proof.** The link $L = L(0, 0, \ldots, 0)$ is the closure of $T(0) \oplus \cdots \oplus T(0)$. It is a knot by Lemma 4.2. If $T$ is a tangle conjugate to $T(0)$, then $L$ is also given by the closure of $T \oplus \cdots \oplus T$. Figure 10 shows that we can conjugate $T(0)$ to an alternating tangle $T$, from which we see that the closure of $T \oplus \cdots \oplus T$ has a nonsplit alternating diagram.

**Theorem 1.5.** The knot $L(k_1, \ldots, k_n)$, where $k_1, \ldots, k_n$ are integers, is TQA.

**Proof.** We prove this by induction on $N = |k_1| + \cdots + |k_n|$. Let $L = L(k_1, \ldots, k_n)$. If $N = 0$, then $k_i = 0$ for all $i$ and $L$ is a nonsplit alternating knot by Lemma 4.4, and hence, is TQA. Assume that $N > 0$ and let $i \in \{1, \ldots, n\}$ be an index such that $|k_i| > 0$.

For convenience, let $\varepsilon = \text{sign}(k_i) \in \{\pm 1\}$. Pick any crossing in the twist box labeled $k_i$. Smoothing $L$ at the crossing results in two links: $L_0 = L(k_1, \ldots, k_{i-1}, \varepsilon(|k_i| - 1), k_{i+1}, \ldots, k_n)$ and $L_1 = L(k_1, \ldots, k_{i-1}, \infty, k_{i+1}, \ldots, k_n)$. By Lemma 4.3, $L_1$ is the two component unlink. Let $D$ denote our diagram for $L$, and take any two arcs of $D$ that, after smoothing the crossing, belong to the two distinct components of $L_1$. Define a marking on $D$ which assigns $1 \in \mathbb{Z}/2$ (a single dot) to each of these two arcs, and $0$ (no dots) to all other arcs. Since $L$ is a knot by Lemma 4.2, this marking represents the trivial marking of $L$ (the two dots cancel mod 2). Smoothing the crossing, with this choice of marking, we obtain marked diagrams representing:

- $L_0$ with its trivial marked diagram since $L_0$ is a knot by Lemma 4.2. This marked link is TQA by the induction hypothesis;
- the two component unlink $L_1$, where each component is assigned 1. This is TQA as it splits into two odd marked unknots.

We note that the determinant condition $\det(L) = \det(L_0) + \det(L_1)$ is automatically satisfied since $\det(L_1) = 0$. Hence, $L$ (with its trivial marking) is TQA.

Since the links $L_{n,k}$ in Figure 6 are the links of the form $L(-k, -k, \ldots, -k)$, the above theorem implies that they are all TQA. Combining this with Proposition 2.1, we obtain the following.

**Theorem 1.1.** The branched cover $\Sigma_n(P(3, -3, -2k - 1))$ is an L-space for all integers $k$ and $n \geq 1$.

### 5 QUASI-POSITIVITY AND SLICENESS OF THE KNOTS $P(3, -3, -2k - 1)$

In this section, we show that the knots $P(3, -3, -2k - 1)$ are quasi-positive. The argument used also shows the well-known result that these pretzels knots are slice. Let $B_n$ denote the Artin braid group on $n$ strands and let $\sigma_i$ be the standard Artin generators for $1 \leq i \leq n$. 



**Figure 10** The alternating tangle $T$ is obtained from $T(0)$ by conjugating by a tangle interchanging the top two strands. The middle diagram of $T(0)$ is obtained from the one on the left by an isotopy.
FIGURE 11  An immersed labeled interval and the corresponding track knot, which happens to be isotopic to \( P(3, -3, -3) \).

We now recall the definition of a quasi-positive link. A braid \( \beta \in B_n \) is called quasi-positive if it can be written as a product of conjugates of the Artin generators: \( \beta = \prod_k w_k \sigma_i w^{-1}_k \). A link in \( L \) in \( S^3 \) is quasi-positive if it is the closure of a quasi-positive braid. Quasi-positivity is useful in determining properties of the branched cyclic covers of the knots in \( S^3 \) as the following theorem shows.

**Theorem 5.1** [2]. *Suppose that \( K \) is a nonslice quasi-positive knot. Then there is an \( N = N(K) \) so that \( \Sigma_n(K) \) is not an L-space for all \( n \geq N \).*

The following proposition shows that the pretzel knots that were shown in Section 4 to have all L-space branched covers are also quasi-positive. Note that the knots in the proposition are slice; see Figure 13 for a ribbon diagram. This answers Question 1.2.

**Proposition 1.3.** *The knots \( P(3, -3, -2k - 1) \) are slice and quasi-positive for \( k \geq 1 \).*

In order to prove this, we use the notion of a track knot defined by Baader [1].

**Definition 5.2.** Let \( C \) be the image of a generic immersion \( i : [0,1] \to \mathbb{R}^2 \) with a labeling at each double point by a letter in \( \{a, b, c, d\} \), and an angle in \( \{0, \pi/2, \pi, 3\pi/2\} \); here, we assume that the diagram has been isotoped so that double points locally look like \( \times \) in the plane (not some arbitrary rotation of this picture). Finally, specify points \( p_1, p_2, \ldots, p_r \) on the connected components of \( C - \{ \text{double points} \} \) such that each connected component of \( C - \{ p_1, p_2, \ldots, p_r \} \) is contractible. We will call \( C \) a labeled immersed interval.

From such an interval, one can associate a knot as follows (see Figure 11 for an example). Draw an immersed interval parallel to \( C \) and join the two intervals by an arc at each end of \( C \), this forms an immersed band following \( C \) which we will call \( B_C \). Orient \( \partial B_C \) counterclockwise. Each double point of \( C \) corresponds to four self-intersections of \( \partial B_C \), which we replace with over- and undercrossings according to the labeling and angle of \( C \) at that double point, as shown in Figure 12. Replace each point \( p_i \) with a full-twist oriented to introduce positive crossings.

**Definition 5.3.** A knot obtained from a labeled immersed interval by the above procedure is called a track knot.
The process defining a knot from a labeled immersed curve replaces a double point labeled $(a, 0), (b, 0), (c, 0)$, and $(d, 0)$ with the corresponding crossing patterns above. If the angle at the crossing is not 0, rotate the corresponding diagram above by the angle specified in the counterclockwise direction.

An isotopy of the pretzel knot $P(3, -3, -2k - 1)$ that, in particular, exhibits ribbon disks for each knot. After redistributing a half twist from the twist box in the final pictured knot diagram by isotopy, we obtain the bottom right picture that gives a labeled immersed interval for $P(3, -3, -2k - 1)$ realizing these knots as track knots.

**Theorem 5.4** [1]. Track knots are quasi-positive.

**Remark 5.5.** Baader’s proof that track knots are quasi-positive describes an algorithm to obtain a quasi-positive braid word for a track knot, though, in general, it will not be of minimal braid index.

Figure 13 shows that the knots $P(3, -3, -2k - 1)$ are track knots and exhibit a slice disk for them. We then obtain the following proposition.

**Proposition 5.6.** The pretzel knots $P(3, -3, -2k - 1)$ are slice track knots for $k \geq 1$. 
Figure 14 The top picture depicts a diagram for the two-bridge knot $K$ with fraction $[2a_1, \ldots, 2a_n]^+$ where $n$ is even. On bottom, we have a diagram for the link $L$ with fraction $[2a_1, \ldots, 2a_n]^+$ where $n$ is odd, with a preferred orientation. In both diagrams, the boxes should be replaced with the corresponding number of signed half-twists.

Proposition 1.3 now follows from Theorem 5.4.

6 | BRANCHED CYCLIC COVERS OF TWO-BRIDGE LINKS

In this section, we prove Theorem 1.4 giving a family of two-bridge links, all of whose cyclic-branched covers are L-spaces.

We first briefly recall some background on two-bridge links. The two-bridge link associated with the fraction $\frac{p}{q} \in \mathbb{Q}$, where $p > q > 0$ are coprime, is the unique link $L_{p/q}$ in $S^3$ with double-branched cover the lens space $L(p, q)$. The links $L_{p/q}$ and $L_{p/(p-q)}$ are mirror images, and hence, up to taking mirror images, we may assume that precisely, one of $p$ or $q$ is even. Then, following, for example, [12], one can write $p/q$ as a (positive) even continued fraction expansion

$$
\frac{p}{q} = [2a_1, 2a_2, \ldots, 2a_n]^+ := 2a_1 + \frac{1}{2a_2 + \frac{1}{2a_3 + \cdots + \frac{1}{2a_n}}}.
$$

where $a_1, \ldots, a_n$ are integers. Then $L_{p/q}$ is the link shown in Figure 14. If $n$ is even, $L_{p/q}$ is a knot; otherwise, it is a two-component link.

We are now ready to state a theorem that classifies all examples known to the authors of two-bridge links, all of whose branched covers are L-spaces. The statement restricted to two-bridge
knots with fraction \([2a_1, 2a_2]^+\) and links with fraction \([2a_1]^+\) is due to Peters [21]. The case for two-bridge knots with fraction of the form \([2a_1, 2a_2, \ldots, 2a_n]^+\) was shown by Teragaito [25]. Again, two-bridge knots are known to be 2-periodic with quotient the unknot; this implies that their \(n\)-fold cyclic cover is the twofold cover of a link \(L_n\). This line of reasoning is known to experts; see, for example, [18]. We observe that for a certain family of two-bridge links, \(L_n\) is nonsplit and altering and conclude that \(\Sigma_2(L_n)\) is an L-space.

**Theorem 1.4.** Let \(L\) be the two-bridge link with fraction \([2a_1, 2a_2, \ldots, 2a_n]^+\) where \(a_i > 0\) for all \(i = 1, \ldots, n\). In the case that \(L\) is a two-component link, we consider the link oriented as in Figure 14. Then \(\Sigma_n(L)\) is an L-space for all \(n \geq 2\).

**Proof.** Figure 15 shows a diagram for the link \(L\), where \(n\) is odd, which is rotationally symmetric about the origin (thinking about the diagram on the plane). An analogous process yields a symmetric diagram for the two-bridge knot with fraction \([2a_1, \ldots, 2a_n]^+\) where \(n\) is even.

We sketch a proof that the diagrams in Figures 14 and 15 are isotopic. Using the Conway sphere indicated in Figure 15, all of the half-twists of the left-hand twist box labeled \(a_n\) can be untwisted at the expense of adding \(a_n\) half-twists to the box labeled \(a_n\) on the right.

There is a corresponding Conway sphere \(S_i\) for each \(2 \leq i \leq n\), which contains all twist boxes labeled \(a_j\) for \(j < i\). Twisting along each \(S_i\), either about its horizontal or vertical axis depending on the parity of \(i\), in decreasing order from \(S_n\) to \(S_2\) yields the corresponding link in Figure 14.

Since this diagram is symmetric about the origin, we can take the quotient of \((S^3, L \cup a)\) under the rotation about \(a\) through angle \(\pi\) where \(a\) is the axis in \(S^3\) perpendicular to the origin of the diagram, yielding \((S^3, \overline{L} \cup \overline{a})\); see Figure 16. It is not hard to see that \(\overline{L}\) is unknotted. If we now assume that for each \(a_i > 0\), we also see that the quotient diagram of \(\overline{L}\) is alternating and, in fact, the diagram can be modified by only Reidemeister 1 moves to obtain a diagram for \(\overline{L}\) as the standard unknot. It follows that the link \(\overline{L} \cup \overline{a}\) is symmetric, meaning that for any diagram, there is an isotopy that exchanges the roles of \(\overline{L}\) and \(\overline{a}\), but otherwise leaves the picture unchanged.
The link $\overline{L} \cup a$ on the left. This link is symmetric; exchanging the roles of $\overline{L}$ and $a$, and then cutting $a$ along the obvious disk bounded by $\overline{L}$ yields the picture on the right.

As in the proof of Proposition 2.1, we see that we can now express $\Sigma_n(L)$ as $\Sigma_2(L_n)$ where $L_n$ is the lift of $a$ under the $n$-fold branched cover of the unknotted $\overline{L}$. Since $a$ has an alternating diagram as a pattern in the complement of $L$, its lift will also be alternating. Thus, each $\Sigma_n(L)$ is homeomorphic to the double-branched cover of a nonsplit alternating link, and hence, is an L-space.

ACKNOWLEDGMENTS
Conversations with Jonathan Johnson compelled us to study this family of pretzel knots. We would like to thank Cameron Gordon and Liam Watson for helpful feedback on an earlier draft. In addition, we thank the anonymous referees whose comments improved the quality of the paper. Finally, the second author thanks Cameron Gordon for many helpful conversations as well as his advice and support. The second author was supported by an NSF graduate research fellowship under grant no. DGE-1610403.

JOURNAL INFORMATION
The Bulletin of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID
Hannah Turner https://orcid.org/0000-0002-9026-029X

REFERENCES
1. S. Baader, Slice and Gordian numbers of track knots, Osaka J. Math. 42 (2005), no. 1, 257–271.
2. M. Boileau, S. Boyer, and C. McA. Gordon, Branched covers of quasi-positive links and L-spaces, J. Topol. 12 (2019), no. 2, 536–576.
3. S. Boyer, C. McA. Gordon, and L. Watson, On L-spaces and left-orderable fundamental groups, Math. Ann. 356 (2013), no. 4, 1213–1245.
4. C. McA. Gordon and T. Lidman, Taut foliations, left-orderability, and cyclic branched covers, Acta Math. Vietnam. 39 (2014), no. 4, 599–635.
5. C. McA. Gordon, Riley’s conjecture on $\text{SL}(2, \mathbb{R})$ representations of 2-bridge knots, J. Knot Theory Ramifications 26 (2017), no. 2, 1740003, 6.
6. J. Greene, Homologically thin, non-quasi-alternating links, Math. Res. Lett. 17 (2010), no. 1, 39–49.
7. M. Hedden and T. E. Mark, Floer homology and fractional Dehn twists, Adv. Math. 324 (2018), 1–39.
8. Y. Hu, *Left-orderability and cyclic branched coverings*, Algebr. Geom. Topol. **15** (2015), no. 1, 399–413.
9. A. Issa, *The classification of quasi-alternating Montesinos links*, Proc. Amer. Math. Soc. **146** (2018), no. 9, 4047–4057.
10. J. Johnson, *Two-bridge knots and residual torsion-free nilpotence*, Can. J. Math., to appear.
11. J. Johnson, *Residual torsion-free nilpotence, biorderability and pretzel knots*, Algebr. Geom. Topol. **23** (2023), no. 4, 1787–1830.
12. A. Kawauchi, *A survey of knot theory*, Birkhäuser Verlag, Basel, 1996. Translated and revised from the 1990 Japanese original by the author.
13. W. H. Kazez and R. Roberts, *Approximating $C^{1,0}$-foliations*, Interactions between low-dimensional topology and mapping class groups, Geom. Topol. Monogr., vol. 19, Geom. Topol. Publ., Coventry, 2015, pp. 21–72.
14. W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer, New York, 1997.
15. L. Lyubich and K. Murasugi, *On zeros of the Alexander polynomial of an alternating knot*, Topology Appl. **159** (2012), no. 1, 290–303.
16. J. Lin, D. Ruberman, and N. Saveliev, *On the monopole Lefschetz number of finite-order diffeomorphisms*, Geom. Topol. **25** (2021), no. 7, 3591–3628.
17. W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1984), no. 1, 37–44.
18. M. Mulazzani and A. Vesnin, *The many faces of cyclic branched coverings of 2-bridge knots and links*, Atti Sem. Mat. Fis. Univ. Modena **49** (2001), 177–215.
19. M. Mulazzani and A. Vesnin, *Generalized Takahashi manifolds*, Osaka J. Math. **39** (2002), no. 3, 705–721.
20. P. Ozsváth and Z. Szabó, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. **194** (2005), no. 1, 1–33.
21. T. Peters, *On $L$-spaces and non left-orderable 3-manifold groups*, arXiv: 0903.4495, 2009.
22. R. Roberts, *Taut foliations in punctured surface bundles*, II, Proc. London Math. Soc. (3) **83** (2001), no. 2, 443–471.
23. C. W. Scaduto, *Instantons and odd Khovanov homology*, J. Topol. **8** (2015), no. 3, 744–810.
24. C. Scaduto and M. Stoffregen, *Two-fold quasi-alternating links, Khovanov homology and instant on homology*, Quantum Topol. **9** (2018), no. 1, 167–205.
25. M. Teragaito, *Fourfold cyclic branched covers of genus one two-bridge knots are $L$-spaces*, Bol. Soc. Mat. Mexicana (3) **20** (2014), no. 2, 391–403.