Discrete fuzzy de Sitter cosmology

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Abstract

We analyze the spectrum of cosmological time in fuzzy de Sitter cosmological model [3]. We find that time is not a self-adjoint operator: its self-adjoint extensions have discrete spectra with eigenvalues which are logarithmically distributed for large $n$, $\tau_n \sim \ell \log \frac{n}{\pi}$, where $\ell$ is a scale of order of the Planck length. In physical states radius of the universe is bounded below by value $\ell \sqrt{W}$, and there is no big bang singularity.

1 Introduction

The expression ‘quantum space’ which is extensively used today in various approaches to quantum gravity, was introduced in the early days of quantum mechanics by Heisenberg, along with the term ‘quantum derivative’ introduced by Dirac, who first observed that the commutator is a derivation. ‘Points’ of the quantum space are q-numbers, that is, operators. These early ideas can be considered as a starting point, as well as physical motivation for noncommutative geometry. They can also be easily extended to curved

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spaces. If the flat quantum space is identified with the Heisenberg algebra,

\[ [i p_i, x^j] = \partial_i x^j = \delta^j_i, \quad (1.1) \]

(\( \hbar = 1, p_i \) hermitian), a curved quantum space can be defined in the same, Schrödinger representation introducing the vielbein,

\[ [i p_\alpha, x^\mu] = e_\alpha x^\mu = e^\mu_\alpha (x), \quad (1.2) \]

as in general relativity, \([1]\). Thereby gravity is intimately related to the algebraic and differential structures of quantum spaces, \([2]\). Neither the algebra of coordinates \( x^\mu \) nor the algebra of momenta \( p_\alpha \) are commutative any more: in fact in the general case, only the full algebra \( \mathcal{A} \) containing both is closed.

Noncommutative geometry is but one of the approaches towards quantum gravity. Other approaches are perhaps more fundamental: in string theory, existence of an elementary substructure, which after quantization gives geometric description of spacetime, is assumed; in loop quantum gravity, gravitational fields are basic variables which are quantized in a background-independent way. In these approaches quantum spacetime and its properties (that is, properties of coordinates, areas, volume) are derived. But in most cases coordinates are operators in the Hilbert space of states and form an algebra. In that sense, even if description of quantum spacetmes through noncommutative geometry is not fundamental but only effective, it is of interest to study its structure and relations to classical geometry, in particular because physical interpretation often reduces to understanding of representations of the underlying algebras. We hope therefore that the technical part of this paper is relevant or interesting independently of the approach.

The adjective ‘fuzzy’ which we use here is in the literature often used to point out finite-dimensional representations. We apply it somewhat differently, to stress the fact that the results are obtained in a concrete (in this case, infinite-dimensional) representation, not generally or formally.

Our task is to study the operator which describes the cosmological time and to determine its spectrum, in the model of fuzzy de Sitter cosmology introduced in \([3, 4]\). The commutative four-dimensional de Sitter space can be defined as an embedding in the five-dimensional flat space \([5]\),

\[ v^2 - w^2 - x^2 - y^2 - z^2 = -\alpha^2, \quad (1.3) \]
with the line element
\[ ds^2 = dv^2 - dw^2 - dx^2 - dy^2 - dz^2, \quad R = 4\Lambda = \frac{12}{\alpha^2}. \] (1.4)

The \( v \in (\infty, \infty) \) is the ‘embedding time’. One of coordinates can be eliminated, for example by introducing
\[ \hat{t}_\alpha = \log \frac{v + w}{\alpha}, \quad \hat{x}_\alpha = \frac{x}{v + w}, \quad \hat{y}_\alpha = \frac{y}{v + w}, \quad \hat{z}_\alpha = \frac{z}{v + w}, \] (1.5)

and then one obtains the line element in the FRW form, or the steady state universe
\[ ds^2 = d\hat{t}_\alpha^2 - e^{2\hat{t}_\alpha} (d\hat{x}_\alpha^2 + d\hat{y}_\alpha^2 + d\hat{z}_\alpha^2). \] (1.6)

Cosmological time \( \hat{t} \in (\infty, \infty) \) is defined only for \( v + w > 0 \), that is, coordinates (1.5) cover only half of the de Sitter hyperboloid (1.3): the main drawback of the steady state model is that it is not complete.

Fuzzy de Sitter space can be defined in an analogous manner. We start with the group \( SO(1,4) \) with generators \( M_{\alpha\beta} \) \((\alpha, \beta = 0, 1, 2, 3, 4)\),
\[ [M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} - \eta_{\beta\gamma}M_{\alpha\delta} + \eta_{\beta\delta}M_{\alpha\gamma}), \] (1.7)

and \( \eta_{\alpha\beta} = \text{diag} (1, -1, -1, -1, -1) \). Embedding coordinates \( x^\alpha \) which are noncommutative extensions \( v, w, x, y, z \) are proportional to the Pauli-Lubanski vector,
\[ x^\alpha = \ell W^\alpha, \quad W^\alpha = \frac{1}{8} \epsilon^{\alpha\beta\gamma\delta\eta} M_{\beta\gamma} M_{\delta\eta}, \] (1.8)

where \( \ell \) is a characteristic length scale. One of the two Casimir relations of the \( SO(1,4) \),
\[ \eta_{\alpha\beta} W^\alpha W^\beta = -\mathcal{W} = \text{const} \] (1.9)
defines the embedding equivalent to (1.3). In fact in definition of fuzzy de Sitter space \[3\], we assumed more: that it is given by a unitary irreducible representation (UIR) of the de Sitter group. That fixes also the second Casimir operator,
\[ Q = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} = \text{const}. \] (1.10)
All UIR’s of the de Sitter group are infinite-dimensional, labelled by two quantum numbers: conformal weight $\rho$ and spin $s$, \[6\]. In the following we use UIR’s of the principal continuous series, $\rho \geq 0$, $s = 0$, $1/2$, $1$, $3/2$ \ldots,

\[ W = s(s + 1) \left( \frac{1}{4} + \rho^2 \right), \quad Q = -s(s + 1) + \frac{9}{4} + \rho^2. \quad (1.11) \]

Various possible choices of differential calculi on fuzzy de Sitter space were discussed in \[3\]; there are two that provide metric which in the commutative limit gives the usual de Sitter metric. Simpler seems to be the calculus generated by four momentum operators, translations $M_{i4} + M_{0i}$ and dilatation $M_{04}$. The vielbein which this choice implies suggests to introduce the FRW spatial coordinates as $W^i$ and the cosmological time as $-\log(W^0 + W^4)$, and in fact gives the wrong direction to time. We will in the following assume that cosmological time $\tau$ is given, as in the commutative case, by

\[ \frac{\tau}{\ell} = \log \frac{x^0 + x^4}{\ell} = \log (W_0 - W_4). \quad (1.12) \]

An important observation is that components $W^\alpha$ of the Pauli-Lubanski vector are the Casimir operators of the subgroups of $SO(1,4)$: $W^0$ is one of the Casimirs of the subgroup $SO(4)$, while $W^\alpha$, $\alpha = 1, 2, 3, 4$, are the Casimirs of $SO(1,3)$. Thus, in principle, the eigenvalues of $W^\alpha$ are known once one knows how the given UIR of the $SO(1,4)$ reduces to the sum of UIR’s of the corresponding subgroup. Using this fact (and in parallel, an explicit calculation) we showed in \[4\] that the spectra of $W^\alpha$, $\alpha = 1, 2, 3, 4$, are continuous, in accordance with the group theory result \[7\]. Embedding time $W^0$, on the other hand, is diagonal in the basis of \[6\], so its spectrum is discrete in all UIR’s of the $SO(1,4)$. Similar strategy is possible for the cosmological time $W_0 - W_4$: as

\[ [M_{0i} + M_{i4}, M_{0j} + M_{j4}] = 0, \quad [M_{0i} + M_{i4}, M_{jk}] = \frac{1}{2} \epsilon_{ijk} (M^{0k} + M^{k4}), \quad (1.13) \]

the $M_{0i} + M_{i4}$ and $M_{jk}$ are the generators of the $E(3)$ subgroup. The $E(3)$ has two Casimir operators, $(M_{0i} + M_{i4})(M^{0i} + M^{i4})$ and $\frac{1}{2} \epsilon^{ijk} (M_{0i} + M_{i4})M_{jk}$, and second one coincides with $W_0 - W_4$. Therefore we could in principle find its spectrum by reduction of the principal continuous series representations $(\rho, s)$ with respect to the UIR’s of $E(3)$. We have not managed to find an appropriate reduction formula in the literature, and we will derive it in the particular case $(\rho, \frac{1}{2})$. 

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Properties of the spectrum in many cases can be inferred directly from the algebra. In this case we have relation
\[ [iM_{04}, W_0 - W_4] = W_0 - W_4, \]
which implies that the group action of dilatation \( M_{04} \) is given by
\[ e^{i\alpha M_{04}} (W_0 - W_4) e^{-i\alpha M_{04}} = e^{\alpha} (W_0 - W_4). \]
The last formula apparently means that the spectrum of \( W_0 - W_4 \) is continuous. If there is one nonzero eigenvalue \( \lambda \) and the corresponding eigenvector \( \psi_\lambda \),
\[ (W_0 - W_4) \psi_\lambda = \lambda \psi_\lambda, \]
then \( e^{-i\alpha M_{04}} \psi_\lambda \) is the eigenvector for the eigenvalue \( e^{\alpha} \lambda \), for every \( \alpha \in \mathbb{R} \). For \( \lambda > 0 \), the spectrum consists of all real positive numbers. We shall see in the following that the spectrum of \( W_0 - W_4 \), calculated in the Hilbert space representation \( (\rho, \frac{1}{2}) \), is in fact discrete. Namely, the differential equation corresponding to (1.16) has finite-norm solutions for all positive \( \lambda \in \mathbb{R} \). That in turn implies that \( W_0 - W_4 \) cannot be self-adjoint (hermitian): it is only formally symmetric, and the domains of \( W_0 - W_4 \) and \( (W_0 - W_4)^\dagger \) are not equal. Proper self-adjoint extensions reduce the initial space of states, implying thereby also discrete spectrum of \( W_0 - W_4 \).

2 Hilbert space representation

We work in the Hilbert space representation of the principal continuous series \( \rho, s = \frac{1}{2} \), [8]. For all values of \( s \) such representation can be constructed using the Bargmann-Wigner UIR’s of the Poincaré group, characterized by mass \( m > 0 \) and spin \( s \), [9]. Generators of the Lorentz subgroup are the same as in the original representation,
\[ M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \]
where \( L_{\mu\nu} \) and \( S_{\mu\nu} \) are orbital and spin generators,
\[ L_{mn} = i \left( p_m \frac{\partial}{\partial p^n} - p_n \frac{\partial}{\partial p^m} \right), \quad L_{0n} = ip_0 \frac{\partial}{\partial p^n}. \]
Generators of the Poincaré translations, multiplication operators \( p_\mu \), are used to define the remaining \( M_4 \) of the de Sitter group by

\[
M_4 = \frac{\rho}{m} p_\mu - \frac{1}{2m} (p^\rho M_{\rho\mu} + M_{\rho\mu} p^\rho),
\]

with \( p_0 = \sqrt{m^2 + (p_i)^2} \). This representation was used in [4] in the simplest nontrivial case, \( s = \frac{1}{2} \), to obtain the spectra of spatial coordinates \( W^i, i = 1, 2, 3 \). We will repeat here some details and derive relations found in the meanwhile, to simplify the calculation.

The Bargmann-Wigner space \( \mathcal{H} \) is for \( s = \frac{1}{2} \) the space of bispinors in the momentum representation, \( \psi(p) \), which are square-integrable solutions to the Dirac equation. Using the Dirac representation of \( \gamma \)-matrices,

\[
\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix},
\]

\( \psi(p) \) can be written in the block form

\[
\psi(p) = \begin{pmatrix} \Phi(p) \\ -\vec{p} \cdot \vec{\sigma} \Phi(p) \end{pmatrix},
\]

where \( \Phi(p) \) is an unconstrained spinor, \( -\vec{p} \cdot \vec{\sigma} = p_k \sigma^k \). The scalar product is given by

\[
(\psi, \psi') = \int \frac{d^3p}{p_0} \psi^\dagger \gamma^0 \psi' = \int \frac{d^3p}{p_0} \frac{2m}{p_0 + m} \Phi^\dagger \Phi'.
\]

Written in blocks of \( 2 \times 2 \) matrices, generators \( M_{\alpha\beta} \) and \( W_\alpha \) have a specific form,

\[
M = \begin{pmatrix} A & B \\ B & A \end{pmatrix},
\]

and the matrix elements of such operator \( M \) are given by

\[
(\psi, M\psi') = \int \frac{d^3p}{p_0} \psi^\dagger \gamma^0 M\psi' = \int \frac{d^3p}{p_0} \Phi^\dagger \left( A - \frac{p_k \sigma^k}{p_0 + m} \frac{A}{p_0 + m} p_i \sigma^i + [B, \frac{p_k \sigma^k}{p_0 + m}] \right) \Phi'.
\]

Eigenvalue problem \( M\psi = \lambda \psi \) can be written as a set of two equations:

\[
\left( A - \frac{p_k \sigma^k}{p_0 + m} \frac{A}{p_0 + m} p_i \sigma^i + [B, \frac{p_k \sigma^k}{p_0 + m}] \right) \Phi = \lambda \frac{2m}{p_0 + m} \Phi,
\]

*We fixed the positions of \( \gamma^0 \) and \( 1/p_0 \) in the scalar product: this ordering is not essential and can be changed, but implies appropriate changes in some of relations which follow.
\[
\left( \left[ A, \frac{p_k \sigma^k}{p_0 + m} \right] + B - \frac{p_k \sigma^k}{p_0 + m} B \frac{p_i \sigma^i}{p_0 + m} \right) \Phi = 0. \tag{2.8}
\]

One can easily check that the second, compatibility equation, is fulfilled for all solutions of the first, \(2.7\).

For \(M = W_0 - W_4\) the blocks \(A\) and \(B\) are

\[
A = -\frac{1}{2m} \left( \rho - \frac{i}{2} \right) p_i \sigma^i - \frac{i}{2m} p_0(p_0 + m) \frac{\partial}{\partial p_i} \sigma_i, \tag{2.9}
\]

\[
B = -\frac{1}{2m} \epsilon^{ijk}(p_0 + m)p_i \frac{\partial}{\partial p_j} \sigma_k - \frac{3i}{4m} (p_0 + m). \tag{2.10}
\]

Eigenvalue equation \(2.7\) then becomes

\[
\left( \frac{1}{2m} \rho (\vec{p} \cdot \vec{\sigma}) - \frac{1}{2} (p_0 + m) (\vec{r} \cdot \vec{\sigma}) - \frac{1}{2m} (\vec{p} \cdot \vec{r})(\vec{p} \cdot \vec{\sigma}) \right) \Phi = \lambda \Phi \tag{2.11}
\]

where, in the signature which we use,

\[
\vec{p} = (p_i), \quad \vec{L} = (L_i), \quad \vec{\sigma} = (\sigma_i), \quad \vec{r} = (x^i) = \left( i \frac{\partial}{\partial p_i} \right),
\]

\[
(\vec{r} \cdot \vec{\sigma})(\vec{p} \cdot \vec{\sigma}) = i \left( 3 + p \frac{\partial}{\partial p} + \vec{L} \cdot \vec{\sigma} \right), \quad \sigma_i \sigma_j = -\eta_{ij} - \epsilon_{ijk} \sigma^k.
\]

As \(W_0 - W_4\) commutes with 3-rotations \(M_{ij}\), we can choose the eigenfunctions in the form

\[
\Phi_{\lambda jm}(\vec{p}) = \frac{f(p)}{p} \phi_{jm}(\theta, \varphi) + \frac{h(p)}{p} \chi_{jm}(\theta, \varphi). \tag{2.12}
\]

The \(p\) is radial momentum, \(p^2 = -p_i p^i = p_0^2 - m^2\), and \(\phi_{jm}\) and \(\chi_{jm}\) are the eigenfunctions of angular momentum \(M_{ij}M^{ij}\), \(M_{12}\), which are orthonormal and satisfy

\[
\phi_{jm} = \frac{\vec{p} \cdot \vec{\sigma}}{p} \chi_{jm}, \quad \chi_{jm} = \frac{\vec{p} \cdot \vec{\sigma}}{p} \phi_{jm}, \tag{2.13}
\]

\[
(\vec{L} \cdot \vec{\sigma}) \phi_{jm} = (j - \frac{1}{2}) \phi_{jm}, \quad (\vec{L} \cdot \vec{\sigma}) \chi_{jm} = -(j + \frac{3}{2}) \chi_{jm}.
\]

Using \(2.12\) we can separate the angular variables and obtain radial equations:

\[
(p_0 + 1) \frac{df}{dp_0} + i pf - \frac{j + \frac{1}{2}}{p_0 - 1} f = 2i\lambda \frac{h}{p}, \tag{2.14}
\]

\[
(p_0 + 1) \frac{dh}{dp_0} + i ph + \frac{j + \frac{1}{2}}{p_0 - 1} h = 2i\lambda \frac{f}{p}. \tag{2.15}
\]
In order to simplify the system, we rescale the momentum to be dimensionless, \( p \rightarrow m \), \( p_0 \rightarrow m p_0, \) \((p \in (0, \infty), p_0 \in (1, \infty))\) and introduce the following change

\[
f = (p_0 + 1)^{-i p - \frac{2i+1}{2}} (p_0 - 1)^{\frac{2i+1}{2}} F, \quad (2.16)
\]

\[
h = (p_0 + 1)^{-i p + \frac{2i+1}{2}} (p_0 - 1)^{-\frac{2i+1}{2}} H. \quad (2.17)
\]

We then obtain separate equations for \( F \) and \( H \),

\[
(p_0^2 - 1) \frac{d^2 F}{dp_0^2} + 2(p_0 + j) \frac{dF}{dp_0} + \frac{4\lambda^2}{(p_0 + 1)^2} F = 0, \quad (2.18)
\]

\[
(p_0^2 - 1) \frac{d^2 H}{dp_0^2} + 2(p_0 - j - 1) \frac{dH}{dp_0} + \frac{4\lambda^2}{(p_0 + 1)^2} H = 0, \quad (2.19)
\]

with additional relations

\[
\frac{dF}{dp_0} = 2i\lambda(p_0 + 1)^{j-1}(p_0 - 1)^{-j-1} H, \quad \frac{dH}{dp_0} = 2i\lambda(p_0 + 1)^{-j-2}(p_0 - 1)^j F. \quad (2.20)
\]

Solutions to these equations can be written in terms of the Bessel functions. It is straightforward to check that, introducing variable \( z \),

\[
z = \sqrt{\frac{p_0 - 1}{p_0 + 1}}, \quad z \in (0, 1) \quad (2.21)
\]

both equations reduce to the Bessel equation

\[
\zeta^2 \frac{d^2 Y}{d\zeta^2} + \zeta \frac{dY}{d\zeta} + (\zeta^2 - a^2) Y = 0 \quad (2.22)
\]

for \( \zeta = 2\lambda z \in (0, 2\lambda) \). For equation (2.18), \( a = j \); for (2.19), \( a = -j - 1 \).

Two linearly independent solutions to the Bessel equation are the Bessel functions \( J_a(\zeta), J_{-a}(\zeta) \) or \( J_a(\zeta), Y_a(\zeta) \). Since in our case \( a \) is half-integer (\( j \) is the angular momentum of a spin \( \frac{1}{2} \) particle), \( J_{-j-1}(\zeta) = (-1)^j \frac{1}{2} Y_{j+1}(\zeta) \). Taking into account relations (2.20) and the recurrence relations between the Bessel functions, we find two solutions:

\[
F_{\lambda j} = C z^{-j} J_j(2\lambda z), \quad H_{\lambda j} = i C z^{j+1} J_{j+1}(2\lambda z), \quad (2.23)
\]

\[
\tilde{F}_{\lambda j} = \tilde{C} z^{-j} J_{-j}(2\lambda z), \quad \tilde{H}_{\lambda j} = -i \tilde{C} z^{j+1} J_{-j-1}(2\lambda z). \quad (2.24)
\]
The Bessel functions around zero \( \zeta = 0 \), behave as \( J_a(\zeta) \sim (\frac{\zeta}{2})^a / \Gamma(a + 1) \), therefore the second solution diverges, \( \tilde{\psi}_{\lambda jm} \sim \zeta^{-j-\frac{3}{2}} \). It also diverges in norm\(^1\), and we will discard it. Therefore we obtain one regular solution,

\[
\begin{align*}
 f_{\lambda j} &= C \left( \frac{2}{1 - z^2} \right)^{-i\rho} \sqrt{z} J_j(2\lambda z), \\
 h_{\lambda j} &= iC \left( \frac{2}{1 - z^2} \right)^{-i\rho} \sqrt{z} J_{j+1}(2\lambda z).
\end{align*}
\]

(2.25)

This solution exists for every real \( \lambda \); however, for \( \lambda < 0 \) the solution is proportional to solution for \( \lambda > 0 \), as \( J_a(e^{\pi i}\zeta) = e^{a\pi i}J_a(\zeta) \). Therefore the spectrum can be restricted to the positive real axis, \( \lambda > 0 \).

The scalar product of two eigenfunctions of the form (2.12) is

\[
(\psi_{jm}, \psi'_{jm'}) = 2\delta_{jj'}\delta_{mm'} \int_0^1 dz \left( f^* f' + h^* h' \right) = 2\delta_{jj'}\delta_{mm'} \int_0^1 dz \left( z^{2j+1} F^* F' + z^{-2j-1} H^* H' \right)
\]

\[
= 2\delta_{jj'}\delta_{mm'} C^* C' \int_0^1 dz \left( J_j(2\lambda z) J_j(2\lambda' z) + J_{j+1}(2\lambda z) J_{j+1}(2\lambda' z) \right).
\]

(2.26)

We observe immediately that the norm of each eigenfunction \( \psi_{\lambda jm} \) is finite as the range of integration \( z \in (0, 1) \) is finite, while the values \( J_j(2\lambda z) \) are bounded. That is, all solutions are normalizable. The conclusion is: not all of the formal solutions to equation (2.11) are the eigenvectors of a self-adjoint operator \( W_0 - W_4 \).

3 Self-adjoint extensions

The obtained result is apparently contradictory and requires additional analysis. We started with a unitary representation of the \( SO(1, 4) \) that is, with a set of self-adjoint

\[\text{The corresponding scalar product, similarly to (2.26), is}
\]

\[
(\tilde{\psi}_{jm}, \tilde{\psi}'_{jm'}) = 2\delta_{jj'}\delta_{mm'} C^* C' \int_0^1 dz \left( J_{-j}(2\lambda z) J_{-j'}(2\lambda' z) + J_{-j-1}(2\lambda z) J_{-j'-1}(2\lambda' z) \right)
\]

and it is divergent in the lower limit. This divergence depends on only \( j \) and not on the difference \( \lambda - \lambda' \) i.e. it does not have the form \( \delta(\lambda - \lambda') \) that is required for the continuous spectrum.

\[\text{\footnote{\text{The corresponding scalar product, similarly to (2.26), is}}}
\]

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(hermitian) generators $M_{\alpha\beta}$. We defined $W_\alpha$ by (1.8), as a sum of products of operators which mutually commute. Therefore, formally at least, $W_0 - W_4$ is hermitian and should have an orthonormal eigenbasis (discrete or continuous). But, when we represented $W_0 - W_4$ in concrete representation and solved the eigenvalue differential equation, we obtained a continuous set of eigenfunctions which have finite norm and are not orthogonal.

Therefore, $W_0 - W_4$ cannot be self-adjoint. It can only be formally symmetric, with its domain $\mathcal{D}(W_0 - W_4)$ unequal to the domain of the adjoint operator, $\mathcal{D}((W_0 - W_4)^\dagger)$. To define a self-adjoint extension we have to resolve the issue of domains.

The problem is obviously in the radial equation. Separation of angular variables gives a division of $\mathcal{H}$ into subspaces with fixed $j$ and $m$, in which $W_0 - W_4$ reduces to radial operators $N_{jm}$.

$$
(\psi_{jm}, (W_0 - W_4)\psi'_{j'm'}) \equiv \delta_{jj'}\delta_{mm'} \int_0^1 dz \Phi^\dagger N_{jm} \Phi' 
$$

$$
= \delta_{jj'}\delta_{mm'} \int_0^1 dz \begin{pmatrix} F^* & H^* \end{pmatrix} \begin{pmatrix} 0 & -i \frac{d}{dz} \\ -i \frac{d}{dz} & 0 \end{pmatrix} \begin{pmatrix} F' \\ H' \end{pmatrix}. 
$$

The radial scalar product remaining after the integration of angular variables is

$$
(\psi_{jm}, \psi'_{j'm'}) = 2\delta_{jj'}\delta_{mm'} \int_0^1 dz \begin{pmatrix} F^* & H^* \end{pmatrix} \begin{pmatrix} z^{2j+1} & 0 \\ 0 & z^{-2j-1} \end{pmatrix} \begin{pmatrix} F' \\ H' \end{pmatrix}. 
$$

In order to find $N_{jm}^\dagger$ we have to do the partial integration:

$$
(\psi_{jm}, (W_0 - W_4)\psi'_{j'm'}) = -i \delta_{jj'}\delta_{mm'} \int_0^1 dz \left( \frac{dF^*}{dz} H' + H^* \frac{dF'}{dz} \right)
$$

$$
= i \delta_{jj'}\delta_{mm'} \int_0^1 dz \left( \frac{dF^*}{dz} H' + \frac{dH^*}{dz} F' \right) - i \delta_{jj'}\delta_{mm'} (F^* H' + H^* F') \bigg|_0^1. 
$$

This means that $N_{jm}$ is only formally symmetric: the first term in (3.2) would give self-adjointness, had the boundary term vanished. We can also check the deficiency indices of $N_{jm}$, that is, solve

$$
N_{jm} \Phi = \pm i \Phi. 
$$
Solutions to these equations are the same as solutions to (2.11) for \( \lambda = \pm i \): the Bessel functions of imaginary argument, that is, the modified Bessel functions \( I_a(\zeta) \) and \( K_a(\zeta) \),

\[
I_a(\zeta) = i^{-a} J_a(i\zeta), \quad K_a(\zeta) = \frac{\pi}{2} i^{a+1} (J_a(i\zeta) + iY_a(i\zeta)).
\] (3.4)

The modified Bessel functions have similar behavior around zero as the Bessel functions and the solution including \( K_a(\zeta) \) is not normalizable. Therefore equation \( N_{jm}\Phi = i\Phi \) has only one regular solution,

\[
F_+ = C z^{-j} I_j(2z), \quad H_+ = -C z^{j+1} I_{j+1}(2z),
\] (3.5)

and similarly, there is one regular solution \( F_-, H_- \) to equation \( N_{jm}\Phi = -i\Phi \). The deficiency indices of \( N_{jm} \) are \((n_+, n_-) = (1, 1)\), hence \( N_{jm} \) is not a self-adjoint operator.

There is a systematic way of extending formally symmetric operators to the self-adjoint\[10, 11]. When the range of independent variable is finite like in our case, \( z \in (0, 1) \), one usually assumes, in the first step, that the domain \( D(N_{jm}) \) is given by normalizable functions that satisfy \( F(0) = H(0) = 0, \ F(1) = H(1) = 0 \). Then \( D(N_{jm}^\dagger) = \mathcal{H} \). To achieve self-adjointness, one then redefines the domains by imposing additional constraints, thereby extending \( D(N_{jm}) \) and restricting \( D(N_{jm}^\dagger) \). We recall the procedure in the notation of \[11\] very briefly. Denoting the boundary term by

\[
B(\Phi, \Phi') = (F^* H' + H^* F')|_0^1,
\] (3.6)

we effectively need to find \( n_+ = n_- \) linearly independent functions \( \Phi_k \), in our case \( \Phi_1 \), such that

\[
B(\Phi_1, \Phi_1) = 0.
\] (3.7)

The domain of a self-adjoint extension \( N_{jm} \) is then defined as a set of functions \( \Phi \)

\[
D(N_{jm}) = D(N_{jm}^\dagger) = \{ \Phi \mid B(\Phi, \Phi_1) = 0 \}.
\] (3.8)

In our case constraint can be imposed separately at two boundary points. It is easy to see that (3.7) is satisfied by

\[
F(0) = H(0) = 0, \quad F_1(1) = \sigma e^{i\beta}, \quad H_1(1) = i\sigma' e^{i\beta + n\pi},
\] (3.9)

\[\dagger\]A necessary condition for it is that the deficiency indices be equal.
where $\beta$, $\sigma$ and $\sigma'$ are real numbers, that is by

$$F(0) = H(0) = 0, \quad H(1) = \pm i \frac{\sigma}{\sigma'} F(1) = i c F(1). \quad (3.10)$$

Let us check first whether eigenfunctions (2.23) satisfy (3.10). The first relation is obviously true, the second gives

$$\frac{J_{j+1}(2\lambda)}{J_j(2\lambda)} = c = \text{const}, \quad (3.11)$$

that is, an equation for $\lambda$. This equation has infinitely many solutions for every real $c$, and the set of solutions is discrete: the easiest way to see this is for large values of $\lambda$ as, asymptotically,

$$\frac{J_{j+1}(2\lambda)}{J_j(2\lambda)} \sim -\tan \left(2\lambda - \frac{(2j + 1)\pi}{4}\right), \quad \lambda \to \infty. \quad (3.12)$$

The eigenvalues can be labelled by a natural number $n$: for large $\lambda$ they are equidistant with period $\frac{\pi}{2}$, as (3.12) becomes periodic. By a choice of constant $c$ we can fix the value of one of the $\lambda$’s, the other eigenvalues are then determined by equation (3.11). This means that, for every $c$, we obtain a different self-adjoint extension $N_{jm}$: we have a one-parameter family. This is in accordance with the fact that the deficiency indices of $N_{jm}$ are (1,1).

Finally, we check orthogonality. Using the recurrence relations between the Bessel func-
tions, we have

\[
(\psi_{\lambda jm}, \psi_{\lambda' j' m'}) = 2C^*C \delta_{jj'} \delta_{mm'} \int_0^1 z \, dz \left( J_j(2\lambda z)J_j(2\lambda' z) + J_{j+1}(2\lambda z)J_{j+1}(2\lambda' z) \right)
\]

\[
= \delta_{jj'} \delta_{mm'} \left| C \right|^2 \frac{\lambda'}{\lambda^2 - \lambda'^2} \left( \lambda' J_j(2\lambda)J'_j(2\lambda') + \lambda' J_{j+1}(2\lambda)J'_{j+1}(2\lambda') 
- \lambda J_j(2\lambda)J'_j(2\lambda) - \lambda J_{j+1}(2\lambda)J'_{j+1}(2\lambda') \right)
\]

\[
= -\frac{\delta_{jj'} \delta_{mm'}}{\lambda - \lambda'} \frac{\left| C \right|^2}{J_{j+1}(2\lambda)J_{j+1}(2\lambda')} \left( \frac{J_j(2\lambda)}{J_{j+1}(2\lambda)} - \frac{J_j(2\lambda')}{J_{j+1}(2\lambda')} \right),
\]

where in the second line \( J'_a(\zeta) \) denotes the first derivative of \( J_a(\zeta) \). The last expression is zero for \( \lambda \neq \lambda' \) for discrete set of eigenfunctions which satisfy (3.11): the corresponding basis is orthogonal.

4 Summary and outlook

In this paper we studied properties of the cosmological time in fuzzy de Sitter cosmological model [3]. We found that time is discrete. The model also implies that the radius of the universe is bounded below by a constant, \( \ell \sqrt{\frac{\Lambda}{W}} \), and there is no big bang singularity.

Fuzzy de Sitter space is defined as a UIR of the de Sitter group, in which the 5-dimensional flat coordinates are components of the Pauli-Lubanski vector, \( x^\alpha = \ell W^\alpha, \alpha = 0, 1, 2, 3, 4 \). As we discussed before constant \( \ell \), though often identified with the Planck length, can in the effective picture be different, related to the cosmological constant \( \Lambda \) and the scale of noncommutativity \( \bar{k} \), for example as \( \ell \sim \bar{k}^{\sqrt{\Lambda}} \). Our choice of the noncommutative frame suggests that 4-dimensional coordinates are \( \tau = \ell \log(W_0 - W_4), x^i = \ell W^i, i = 1, 2, 3 \). The spectra of \( x^i \) are continuous in all UIR’s of the principal continuous series, [4]. Here we obtain that the spectrum of \( \tau \) is discrete, assuming that the condition of self-duality is imposed. The calculation is done in the Hilbert space representation \( (\rho, \frac{1}{2}) \) of the principal continuous series.

Let us check that fuzzy de Sitter space corresponds to an expanding cosmology, and discuss
the absence of the big bang singularity in the model. The squared radius of the universe is
\[(x^i)^2 = -\ell^2 W_i W^i, \tag{4.1}\]
its evolution can be traced by the expectation value \(\langle (x^i)^2 \rangle\) in the eigenstates of the cosmological time. Eigenvalue \(\lambda\) of \(W_0 - W_4\) which used in calculation is related to \(\tau\) as
\[\langle \tau \rangle = (\psi_{\lambda jm}, \tau \psi_{\lambda jm}) = \ell \log \lambda. \tag{4.2}\]
Using Casimir relation (1.11)
\[-W_i W^i = W_0^2 - W_4^2 + \mathcal{W}, \tag{4.3}\]
and assuming that eigenstates \(\psi_{\lambda jm}\) are normalized,
\[\langle W_i W^i \rangle = \mathcal{W} + \langle (W_0 + W_4)(W_0 - W_4) \rangle = \mathcal{W} + \lambda^2 + 2\lambda \langle W_4 \rangle. \tag{4.4}\]
The expectation value \(\langle W_4 \rangle\) can be estimated. As \(W_4 = -\frac{1}{2} \left( \frac{p_0 \vec{r} \cdot \vec{\sigma}}{2(p_0 + m)^2} \frac{i \vec{L} \cdot \vec{\sigma}}{p_0 + m} - \frac{m^2 \vec{r} \cdot \vec{\sigma}}{p_0 + m} - \frac{im(p \cdot \nabla)(p \cdot \vec{\sigma})}{(p_0 + m)^2} \right) \phi_{\lambda jm}\), we have
\[
\langle W_4 \rangle = \int \frac{d^3 p}{p_0} \Phi_{\lambda jm}^\dagger \left( \frac{im p \cdot \vec{\sigma}}{2(p_0 + m)^2} - \frac{m^2 \vec{r} \cdot \vec{\sigma}}{p_0 + m} - \frac{im(p \cdot \nabla)(p \cdot \vec{\sigma})}{(p_0 + m)^2} \right) \Psi_{\lambda jm} \]
\[= -\frac{i}{2} \int_0^1 dz (1 - z^2) \left( F_{\lambda j}^* \frac{dH_{\lambda j}}{dz} + H_{\lambda j}^* \frac{dF_{\lambda j}}{dz} \right) \]
\[= \lambda C^* C \int_0^1 dz z(1 - z^2) \left( J^2_j(2\lambda z) + J^2_{j+1}(2\lambda z) \right). \tag{4.5}\]
Comparing the last integral with (4.4),
\[0 \leq \int_0^1 dz z(1 - z^2) \left( J^2_j(2\lambda z) + J^2_{j+1}(2\lambda z) \right) \leq \int_0^1 dz z \left( J^2_j(2\lambda z) + J^2_{j+1}(2\lambda z) \right) \tag{4.6}\]
we obtain that \( 0 \leq (\psi_{\lambda jm}, W_4 \psi_{\lambda jm}) \leq \frac{1}{2} \), and therefore

\[
W + \lambda^2 \leq (\psi_{\lambda jm}, -W_i W^i \psi_{\lambda jm}) \leq W + 2\lambda^2.
\] (4.7)

We find that the expectation value of the radius of the universe is bounded below by \( \ell \sqrt{W} \): it never vanishes. This is true for all states that lie in the domain of \( W_0 - W_4 \), that is, which can be expanded in its eigenbasis \( \psi_{\lambda jm} \). The radius, on the other hand, grows with time exponentially: for late times we have \( \sqrt{\langle -W_i W^i \rangle} \sim \lambda = e^{(\tau)/\ell} \).

The second important point is discreteness of time \( \tau \). We have a unitary representation, that is, a Hilbert space \( \mathcal{H} \) and the self-adjoint generators \( M_{\alpha \beta} \). But \( W_0 - W_4 \), being quadratic in \( M_{\alpha \beta} \), is not self-adjoint. However it is essentially self-adjoint, as radial operators \( N_{jm} \) have equal deficiency indices, \( n_+ = n_- = 1 \). This means that \( \tau \) can be extended to a one-parameter family of self-adjoint operators, \( \tau_c \). Extensions are obtained by defining the appropriate domains \( \mathcal{D}(\tau_c) \) by additional restriction \( (3.10) \). The \( \tau_c \) have discrete spectra.

Discreteness obtained by requiring self-adjointness in known in other cases of quantum spaces. One prominent example is the \( q \)-deformed Heisenberg algebra,

\[
[p, x] = -i + (q - 1)xp,
\] (4.8)

and its unitary representations. An explicit abstract construction of the space of states with an analysis of self-adjointness and deficiency indices of the generators was done in \( [12, 13] \), and shows that \( x \) is essentially self-adjoint; a one-parameter family of self-adjoint extensions can be defined; both \( x \) and \( p \) have discrete spectra.

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5One can exercise the formalism and check this explicitly. Dilatation \( M_{04} \) for example in the basis of spinor spherical harmonics has matrix elements

\[
(\psi_{jm}, M_{04} \psi'_{j' m'}) \equiv \delta_{jj'} \delta_{mm'} \int_0^1 dz \Phi \Phi^\dagger (\Phi'(0) \Phi'(1)) \int_0^{8z^2} dz \frac{8z^2 dz}{(1-z^2)^2} \frac{1}{1-z^2} + i \frac{3 + z^2}{2(1-z^2)} + i z \frac{d}{dz} \Phi^\dagger \Phi' + 8i z^3 \frac{1}{(1-z^2)^2} \Phi^\dagger \Phi'_{j,0}.
\]

In the radial space it is defined by operator \( D \) which is formally symmetric, with nonvanishing boundary term \( B(\Phi, \Phi') = \frac{8i z^3}{(1-z^2)^2} \Phi^\dagger \Phi'_{j,0} \). \( B(\Phi, \Phi') \) however can be set to zero in the usual way, by choosing the appropriate type of decay of functions in \( \mathcal{D}(D) \): \( \Phi(z)/(1-z^2) \rightarrow 0 |_{z \rightarrow 1} \), that is \( \rho_0 \Phi(p_0) \rightarrow 0 |_{p_0 \rightarrow \infty} \).

One can determine the spectrum of \( D \): it is continuous.
Another interesting case is the minimal-length Heisenberg algebra,
\[ [p, x] = -i β \beta p^2. \] (4.9)

It was represented in [14] in the Schrödinger representation, which allowed a more concrete discussion of measure, coherent states, uncertainty relations and necessary conditions to define physical states. Again it was found that \( x \) has a one-parameter family of self-adjoint extensions which put its spectrum on a lattice. Let us note that our relation (1.14) also can be expressed as a deformation of the Heisenberg algebra. Denoting \( M_{04} = p, W_0 - W_4 = 1 + x \), we have
\[ [p, x] = -i - ix. \] (4.10)

A further important instance of obtaining a discrete spectrum of time is the loop quantum cosmology. However, the general physical framework as well as the mathematical details are quite different to be compared directly. Basic variables, flux \( x \) and holonomy \( V(\mu) = e^{iμp} \) [15, 16], satisfy the Heisenberg-Weyl algebra
\[ [x, V(\mu)] = -\mu V(\mu). \] (4.11)

But holonomy is not weakly continuous in \( μ \) and no self-adjoint generator \( p \) exists. A concrete representation of the corresponding kinematical Hilbert space \( H_{poly} \) is the space of square-integrable functions on the Bohr-compactified real line; introduction of the Hamiltonian constraint, which is a difference equation, reduces \( H_{poly} \) to the space of physical states in which time and the time evolution are discrete.

To conclude: representation of fuzzy de Sitter space discussed here provides a quantization mechanism, not new, to obtain discrete spectrum of time. It goes with a reduction of the space of states \( H \) to the domain of the cosmological time, \( H \rightarrow H_c = D(τ_c) \). It is yet not completely clear whether \( H_c \) is to be identified with the space of physical states or not. Another natural question is, whether observables with eigenvalues that depend on the choice of an extension, are physical. Further, whether perhaps such choice can be interpreted as a sort of branching (of the universe) which happens at the instance of measurement.

This brings us to another point, already discussed in [17] in the context of quantum groups. The choice of an extension \( τ_c \) reminds of spontaneous symmetry breaking. That symmetry...
is broken can indeed be seen in our example: the dilatation subgroup of the $SO(1, 4)$, $G = \{ e^{i\alpha M_0} | \alpha \in \mathbb{R} \}$, does not preserve $\mathcal{H}_c$ because of (1.15). If we proclaim $\mathcal{H}_c$ to the space of physical states, the initial $SO(1, 4)$ symmetry breaks to the $SO(3) \times G_c^\perp$, where $G_c$ denotes the subgroup of dilatations preserving condition (3.10). This subgroup is represented nonlinearly. But for large eigenvalues, (3.10) becomes periodic and $\lambda$ are equally spaced: the dilatation subgroup reduces to the additive group of integers. On the macroscopic scale on the other hand one can use continuum approximation $\ell \to 0$, and the complete symmetry is recovered.

In any case, what we learn is that changes of coordinates on quantum space, although formally allowed, are very subtle, not at all straightforward as on commutative manifolds. Therefore, identification of physical observables is a very important issue which can bring additional requirements on the set of physical states, along with deformation or breaking of classical symmetries.

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