Representations for the Bloch Type Semi-norm of Fréchet Differentiable Mappings

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Abstract
In this paper we give some results concerning Fréchet differentiable mappings between domains in normed spaces with controlled growth. The results are mainly motivated by Pavlović’s equality for the Bloch semi-norm of continuously differentiable mappings in the Bloch class on the unit ball of the Euclidean space as well as the very recent Jocić’s generalization of this result.

Keywords Bloch functions · Normal functions · Operator monotone functions · Normed spaces · Banach spaces · Fréchet differentiable mappings

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1 Introduction

The starting point of this paper is the Pavlović expression for the Bloch semi-norm of a continuously differentiable function on the unit ball $B^m \subseteq \mathbb{R}^m$, obtained in 2008, and the Jocić’s result concerning Fréchet differentiable mappings that satisfy a Bloch type growth condition obtained in 2019.

The following is a content of [13, Theorem 2].

Proposition 1.1 (see [13,14]) For a complex-valued continuously differentiable function $f$ on the unit ball $B^m$, $m \geq 2$, we have

$$
\|f\|_{\mathcal{B}} = \sup_{(x,y)\in \Delta^c_{B^m}} (1-|x|^2)^{\frac{1}{2}} (1-|y|^2)^{\frac{1}{2}} \frac{|f(x) - f(y)|}{|x - y|},
$$
where

\[ \| f \|_B = \sup_{x \in B^m} (1 - |x|^2) \| df(x) \| \]

is the Bloch semi-norm of the function \( f \).

In the above proposition \( df(x) \) is the differential of \( f \) at \( x \in B^m \), and \( \| df(x) \| \) is the operator norm of the linear mapping \( df(x) : \mathbb{R}^m \to \mathbb{R}^2 \). For a set \( S \) we have denoted

\[ \Delta_S = \{(x, x) : x \in S\} \subseteq S \times S \quad \text{and} \quad \Delta^c_S = S \times S \setminus \Delta_S. \]

In [13, Remark 4] the author conjectures that his theorem remains valid if one assumes that \( f \) is a continuous differentiable mapping on the unit ball of a Hilbert space with values in a Banach space. This is confirmed by Jocić [10, Corollary 3.4] even for Fréchet differentiable mappings on the unit ball of a Banach space that have image in another Banach space.

If \( X \) is a normed space with a norm \( \| \cdot \|_X \), we denote by \( B_X = \{x \in X : \|x\|_X < 1\} \) the unit ball in \( X \), and \( B_X(x, r) = x + rB_X \) is the open ball with centre \( x \) and radius \( r \). The corresponding metric on \( X \) is \( d_X(x, y) = \|x - y\|_X \). For the sake of simplicity we will sometimes drop the indexes.

Assume that \( X \) and \( Y \) are normed spaces. Recall that a mapping \( f : \Omega \to Y \), where \( \Omega \subseteq X \) is a domain, is Fréchet differentiable at \( x \in \Omega \) if there exists a continuous linear mapping \( df(x) : X \to Y \) (called Fréchet differential), such that in an open ball \( B_X(x, r) \) we have

\[ f(y) - f(x) = df(x)(y - x) + \alpha(y - x), \quad y \in B_X(x, r), \]

where \( \alpha : B_X(0, r) \to Y \) satisfies \( \lim_{h \to 0} \alpha(h)/\|h\| = 0 \). We denote by

\[ \| df(x) \| = \sup_{\xi \in B_X} \| df(x)\xi \| \]

the operator norm of \( df(x) : X \to Y \).

For differential calculus of mappings on domains in a normed space we refer to Cartan’s book [3].

Jocić [10] proved a result which is given in the next proposition.

**Proposition 1.2** (see [10]) Let \( X \) and \( Y \) be Banach spaces, and let \( f : B_X \to Y \) be a Fréchet differentiable mapping. For a nonconstant operator monotone function \( \varphi \) on \((-1, 1)\), such that \( \varphi' \) is increasing on \([0, 1)\), the Bloch type semi-norm

\[ \| f \|_{B^m, \varphi'} = \sup_{x \in B_X} \frac{\| df(x) \|}{\varphi'(\|x\|)} \]

may be expressed as

\[ \| f \|_{B^m, \varphi'} = \sup_{(x, y) \in \Delta^c_{B_X}} \frac{\| f(x) - f(y) \|}{\sqrt{\varphi'(\|x\|)\|x - y\| \sqrt{\varphi'(\|y\|)}}}. \]
As in [10] take
\[ \varphi(t) = \tanh^{-1} t = \frac{1}{2} \log \frac{1 + t}{1 - t}, \quad t \in (-1, 1), \]
in Proposition 1.2 in order to recover Proposition 1.1 in this new and more general setting. Indeed, the function \( \varphi(t) \) is operator monotone on \((-1, 1)\) (for this fact see [10]) and we have
\[ \varphi'(t) = (1 - t^2)^{-1}, \quad t \in [0, 1). \]

It follows that the statement of Proposition 1.1 is valid for Fréchet differentiable mappings on the unit ball of a Banach space with values in another Banach space. Note that Jocić has removed the condition concerning continuity of a differential. In this paper we prove a generalization of Proposition 1.2 for Fréchet differentiable mappings on a convex domain in a normed space that have image in another normed space. The proof is based on our approach. Thus, we are able to remove the completeness condition which figures in the proposition above, as well as the condition concerning continuity of a differential on a domain.

Among other results in this paper we obtain the following one. Let \( \omega(x) \) be a continuous and positive function on a domain \( \Omega \) of a normed space \( X \), and let it be a monotone function of \( \|x\| \). If \( f : \Omega \to Y \) is a Fréchet differentiable mapping which has an image in a normed space \( Y \), then the Bloch type semi-norm
\[ \|f\|_{\mathcal{B}, \omega} = \sup_{x \in \Omega} \|d_f(x)\| / \omega(x) \]
is equal to the differential-free semi-norm
\[ \|f\|_{\mathcal{B}, \omega} = \sup_{(x, y) \in \Delta_{\Omega}} \min \left\{ \frac{1}{\omega(x)}, \frac{1}{\omega(y)} \right\} \frac{\|f(x) - f(y)\|}{\|x - y\|}. \]

It seems that we have a better equivalent for the quantity \( \|f\|_{\mathcal{B}, \varphi'} \) then the one given in Proposition 1.2, since
\[ \min \left\{ \frac{1}{\varphi'(\|x\|)}, \frac{1}{\varphi'(\|y\|)} \right\} \leq \frac{1}{\varphi'(\|x\|)^{\frac{1}{2}} \varphi'(\|y\|)^{\frac{1}{2}}}, \quad x \in \Omega, \ y \in \Omega. \]

The above inequality is a consequence of the obvious one \( \min\{a, b\} \leq \sqrt{ab} \) for nonnegative numbers \( a \) and \( b \). Note that we need not the assumption on operator monotonicity of the primitive function for \( \omega \).

The result we have just described is a consequence of a more general one given in Theorem 4.5. This min-max theorem also generalize the author recent result [12, Corollary 3.1] for continuously differentiable mappings on a convex domain in \( \mathbb{R}^m \).

Proposition 1.1 is motivated by the Holland-Walsh characterisation of analytic Bloch functions on the unit disk obtained at the end of eighties [9, Theorem 3].
characterization says that an analytic function $f$ on the unit disk $U$ is a Bloch function, i.e., \((1 - |z|^2)|f'(z)|\) is bounded on $U$, if and only if
\[
\sqrt{1 - |z|^2} \sqrt{1 - |w|^2} \frac{|f(z) - f(w)|}{|z - w|}
\]
is bounded as a function of $z \in U$ and $w \in U$ for $z \neq w$. This clearly follows from Proposition 1.1 having in mind that $\|d_f(z)\| = |f'(z)|$, $z \in U$.

We provide a proof of Proposition 1.1 in the last section which is based on our approach. In that section we also find a new characterization of normal mappings on the unit disk which states: An analytic function $f$ is normal on $U$ if and only if
\[
\frac{\sqrt{1 - |z|^2} \sqrt{1 - |w|^2}}{\sqrt{1 + |f(z)|^2} \sqrt{1 + |f(w)|^2}} \frac{|f(z) - f(w)|}{|z - w|}
\]
is bounded as a function of $z \in U$ and $w \in U$ for $z \neq w$.

2 Preliminaries

Among other things in this section we will briefly describe the concept of the Riemann integral of a function over a rectifiable path in a metric space. It seems that one cannot find any reference for integration in a such general context. Of course, if one restricts attention to the domains in $\mathbb{R}^n$, this becomes an ordinary material.

A path $\gamma$ in a metric space $X$ is a continuous mapping $\gamma : [a, b] \to X$, where $[a, b] \subseteq \mathbb{R}$. It is convenient to identify the path $\gamma$ with its image $\gamma([a, b]) \subseteq X$.

Let $\mathcal{P}[a, b]$ be the family of all partitions of the segment $[a, b]$, i.e., the family of all finite sequences $T : t_0, t_1, \ldots, t_n$, such that $n \in \mathbb{N}$, and $a = t_0 < t_1 < \cdots < t_n = b$.

For a path $\gamma : [a, b] \to X$ and a partition $T \in \mathcal{P}[a, b]$, $T : t_0, t_1, \ldots, t_n$, denote
\[
\sigma_\gamma(T) = \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i)).
\]
The path $\gamma$ is rectifiable if the set $\{\sigma_\gamma(T) : T \in \mathcal{P}[a, b]\}$ is bounded. The length of the path $\gamma$ is
\[
\ell(\gamma) = \sup_{T \in \mathcal{P}[a, b]} \sigma_\gamma(T).
\]
Note that we obviously have $\ell(\gamma) \geq d(\gamma(a), \gamma(b))$.

It is not hard to prove that if $\gamma : [a, b] \to X$ is a rectifiable path in a metric space $X$, and if $[c, d] \subseteq [a, b]$, then the restricted path $\gamma : [c, d] \to X$ is also rectifiable. This new path is denoted by $\gamma_{[c,d]}$. It is clear that $\ell(\gamma) = \ell(\gamma_{[a,c]}) + \ell(\gamma_{[c,b]})$.

The segment $[x, y]$ in a normed space $X$ with the endpoints $x \in X$ and $y \in X$ is the path $\gamma(t) = (1 - t)x + ty$, $t \in [0, 1]$. It is easy to show that $[x, y]$ is a rectifiable path, and $\ell([x, y]) = \|x - y\|$. 

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Let $X$ and $Y$ be metric spaces. For a mapping $f : X \to Y$ introduce

$$d_f^*(x) = \limsup_{y \to x} \frac{d(f(x), f(y))}{d(x, y)}, \quad x \in X.$$ 

We set $d_f^*(x) = 0$, if $x \in X$ is an isolated point.

The content of the following lemma is maybe well known, however we will provide
a proof since we are not able to find a reference.

**Lemma 2.1** Let $\Omega \subseteq X$ be a domain in a normed space $X$, and $f : \Omega \to Y$ be a Fréchet differentiable mapping at $x \in \Omega$, where $Y$ is an another normed space. For the operator norm of the Frechét differential $d_f(x) : X \to Y$, $\|d_f(x)\| = \sup_{\xi \in \partial B_X} \|d_f(x)\xi\|$ we have $\|d_f(x)\| = d_f^*(x)$, i.e.,

$$\|d_f(x)\| = \limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|}.$$

**Proof** For the sake of simplicity let us denote $d_f(x) = A$. In a sufficiently small open ball $B_X(x, r)$ there exists a mapping $\alpha : B_X(x, r) \to Y$ such that

$$f(y) - f(x) = A(y - x) + \alpha(y - x), \quad \lim_{y \to x} \frac{\alpha(y - x)}{\|y - x\|} = 0.$$

Therefore,

$$\|f(y) - f(x)\| = \|A(y - x) + \alpha(y - x)\| \leq \|A\| \|y - x\| + \|\alpha(y - x)\|.$$

It follows

$$\frac{\|f(y) - f(x)\|}{\|y - x\|} \leq \|A\| + \frac{\|\alpha(y - x)\|}{\|y - x\|},$$

and

$$\limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|} \leq \|A\|.$$

Assume now contrary, that we have the strict inequality above. Then there exists a real number $t$ that satisfies

$$\limsup_{y \to x} \frac{\|f(y) - f(x)\|}{\|y - x\|} < t < \|A\|.$$

This means that there exists an open ball $B_X(x, \rho)$ such that

$$\|f(y) - f(x)\| \leq t\|y - x\|, \quad y \in B_X(x, \rho).$$

For $s \in (0, \rho)$ and $\xi \in \partial B_X$, we have $y = x + s\xi \in B_X(x, \rho)$. Since $f(y) - f(x) = A(y - x) + \alpha(y - x)$, the preceding inequality becomes $\|A(s\xi) + \alpha(s\xi)\| \leq t\|s\xi\|,$
i.e., \(|A\zeta + \alpha(s\zeta)/s| \leq t\). If we let \(s \to 0\) we obtain \(|A\zeta| \leq t\). Since this holds for every \(\zeta \in \partial BX\), we reach a contradiction \(|A| \leq t\). \(\square\)

The following lemma may be found in \([4]\), but for the sake of completeness we will adapt a proof here.

**Lemma 2.2** (see \([4]\)) Let \(X\) and \(Y\) be metric spaces, let \(\gamma\) be a rectifiable path, and let a mapping \(f : X \to Y\) be a such one that there exists \(d_f^*(x)\) for every \(x \in X\). Assume, moreover, that there exists a constant \(m\) such that \(d_f^*(x) \leq m\), \(x \in X\). Then the path \(\delta = f \circ \gamma\) is also rectifiable, and \(\ell(\delta) \leq m\ell(\gamma)\).

**Proof** Let \(\varepsilon > 0\) be any number, and let \(\gamma : [a, b] \to X\). We will firstly prove that

\[
d(f(\gamma(t)), f(\gamma(s))) \leq (m + \varepsilon)\ell(\gamma|_{[s,t]}), \quad a \leq s < t \leq b
\]

Indeed, for every \(x \in [s, t]\) we have \(d_f^*(x) \leq m\), so there exists an interval \(I_x = (x - \delta_x, x + \delta_x), \delta_x > 0\), such that

\[
d(f(\gamma(x)), f(\gamma(x'))) \leq (m + \varepsilon)d(\gamma(x), \gamma(x')), \quad x' \in I_x \cap [s, t].
\]

Let \(J\) be a finite subset of \([s, t]\) such that \(\{I_x : x \in J\}\) is a cover of \([s, t]\) containing no proper subcover. Let \(x_1, x_3, \ldots, x_{2n-1}\) be a sequence of elements of \(J\) in the increasing order. Since of minimality of the cover, for every \(i, 0 \leq i \leq n - 1\) there exists \(x_{2i} \in I_{2i-1} \cap I_{2i+1} \cap (x_{2i-1}, x_{2i+1})\). Let, moreover, \(x_0 = s \in I_{x_1}\) and \(x_{2n} = t \in I_{x_{2n-1}}\). In particular, we have \(s = x_0 < x_1 < x_2 < \cdots < x_{2n-1} < x_{2n} = t\) and \(x_{2i}, x_{2i+2} \in I_{2i+1}\) for every \(i \in \{0, 1, \ldots, n - 1\}\). Therefore,

\[
d(f(\gamma(t)), f(\gamma(s))) \leq \sum_{k=0}^{2n-1} d(f(\gamma(x_k)), f(\gamma(x_{k+1})))
\]

\[
\leq \sum_{k=0}^{2n-1} (m + \varepsilon)d(\gamma(x_k), \gamma(x_{k+1}))
\]

\[
\leq (m + \varepsilon)\ell(\gamma|_{[s,t]}),
\]

justifying the inequality stated at the beginning of the proof.

To finish the proof, choose a partition \(T : t_0, t_1, \ldots, t_n\) of \([a, b]\) such that

\[
\ell(f \circ \gamma) \leq \sum_{i=0}^{n-1} d(f(\gamma(t_{i+1})), f(\gamma(t_i))) + \varepsilon.
\]
By the just proved inequality we obtain
\[ \ell(f \circ \gamma) \leq \sum_{i=0}^{n-1} d(f(\gamma(t_{i+1})), f(\gamma(t_i))) + \varepsilon \]
\[ \leq \sum_{i=0}^{n-1} (m + \varepsilon) \ell(\gamma|_{[t_{i-1}, t_i]}) + \varepsilon \]
\[ = (m + \varepsilon) \ell(\gamma) + \varepsilon. \]

Since this holds for every \( \varepsilon > 0 \), the desired inequality, \( \ell(f \circ \gamma) \leq m \ell(\gamma) \), follows. \( \square \)

Let \( \gamma : [a, b] \rightarrow X \) be a rectifiable path in a metric space \( X \), and let \( f \) be a function on \( X \). To a partition \( T : t_0, t_1, \ldots, t_n \) of a segment \( [a, b] \) coordinate another sequence \( S : s_1, s_2, \ldots, s_n \), such that \( s_i \in [t_{i-1}, t_i] \) for every \( i, 1 \leq i \leq n \), and denote
\[ \sigma_{\gamma}(f, T, S) = \sum_{i=1}^{n} f(\gamma(s_i)) \ell(\gamma|_{[t_{i-1}, t_i]}). \]

The Riemann integral \( \int_{\gamma} f \) is a number (if there exists a such number) which satisfies: for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ \left| \int_{\gamma} f - \sigma_{\gamma}(f, T, S) \right| < \varepsilon \]
for every \( T \in \mathcal{P}[a, b] \), \( \text{diam}(T) = \max_{1 \leq i \leq n} |t_i - t_{i-1}| < \delta \) and every sequence \( S \) coordinated to \( T \).

It is not hard to show that instead of \( \sigma_{\gamma}(f, T, S) \) we could consider
\[ \tilde{\sigma}_{\gamma}(f, T, S) = \sum_{i=1}^{n} f(\gamma(s_i)) d(\gamma(t_{i-1}, \gamma(t_i))), \]
in order to define \( \int_{\gamma} f \).

It is a routine to prove that the following statements are equivalent (actually, the proof is almost the same as in the Euclidean case):

(i) There exists \( \int_{\gamma} f \).

(ii) For every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( |\sigma_{\gamma}(f, T, S) - \sigma_{\gamma}(f, T', S')| < \varepsilon \)
for every partitions \( T, T' \) such that \( \text{diam}(T) < \delta \), \( \text{diam}(T') < \delta \), and every sequences \( S \) coordinated to \( T \), and \( S' \) coordinated to \( T' \).

(iii) For every \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that \( |\sigma_{\gamma}(f, T, S) - \sigma_{\gamma}(f, T', S')| < \varepsilon \)
for every partition \( T \), \( \text{diam}(T) < \delta \), and every sequences \( S \) and \( S' \) coordinated to \( T \).

If \( f \) is a continuous function on a metric space \( X \), and if a path \( \gamma : [a, b] \rightarrow X \) is rectifiable, then the Riemann integral \( \int_{\gamma} f \) exists. Indeed, we will use the third criterion.
stated above. Let \( \varepsilon > 0 \) be any number. Since of uniform continuity for \( f \circ \gamma \) on \([a, b]\), there exists \( \delta > 0 \) such that \( |f \circ \gamma (s) - f \circ \gamma (s')| < \varepsilon \) if \( |s - s'| < \delta \). For any partition \( T : t_0, t_1, \ldots, t_n \) of \([a, b]\) with \( \text{diam}(T) < \delta \), and sequences \( S : s_1, s_2, \ldots, s_n \) and \( S' : s'_1, s'_2, \ldots, s'_n \) coordinated to \( T \), we have

\[
|\sigma(f, T, S) - \sigma(f, T, S')| = \left| \sum_{i=1}^{n} f(\gamma(s_i)) \ell(\gamma|_{[t_{i-1}, t_i]}) - \sum_{i=1}^{n} f(\gamma(s'_i)) \ell(\gamma|_{[t_{i-1}, t_i]}) \right|
\leq \sum_{i=1}^{n} |f(\gamma(s_i)) - f(\gamma(s'_i))| \ell(\gamma|_{[t_{i-1}, t_i]})
\leq \varepsilon \sum_{i=1}^{n} \ell(\gamma|_{[t_{i-1}, t_i]}) \leq \varepsilon \ell(\gamma).
\]

It is clear that if \( f_1 \leq f_2 \), then \( \int f_1 \leq \int f_2 \) provided that the both integrals exist.

If there exists \( \int f \), then we have \( \int f = \lim_{n \to \infty} \sigma_\gamma(f, T_n, S_n) \), where \((T_n)_{n \in \mathbb{N}}\) is any sequence of partitions of \([a, b]\) such that \( \text{diam}(T_n) \to 0 \), as \( n \to \infty \), and \( S_n \) is a finite sequence coordinated to \( T_n \) for every \( n \in \mathbb{N} \).

If \([x, y]\) is a segment in a normed space \( X \), then we have

\[
\int_{[x, y]} f = \|x - y\|_X \int_0^1 \tilde{f}(t) dt, \quad \tilde{f}(t) = f((1 - t)x + ty), \quad t \in [0, 1],
\]

where on the right side is the ordinary Riemann integral. Indeed,

\[
\int_{[x, y]} f = \lim_{\text{diam}(T) \to 0} \sum_{i=1}^{n} f((1 - t_{i-1})x + t_{i-1}y)(|((1 - t_i)x + t_iy) - ((1 - t_{i-1})x + t_{i-1}y)|
= \|x - y\|_X \lim_{\text{diam}(T) \to 0} \sum_{i=1}^{n} f((1 - t_{i-1})x + t_{i-1}y)|t_i - t_{i-1}|
= \|x - y\|_X \int_0^1 \tilde{f}(t) dt.
\]

A metric space \( X \) is rectifiable connected if for every \( x \in X \) and \( y \in X \) there exists a rectifiable path \( \gamma : [a, b] \to X \) that connects \( x \) and \( y \), i.e., a such one that \( \gamma(a) = x \) and \( \gamma(b) = y \). Note that any domain in a normed space is rectifiable connected metric space.

An everywhere positive and continuous function \( \omega \) on a metric space \( X \) will be called a weight function. If \( \gamma \) is a rectifiable path, its \( \omega \)-length is given by the Riemann
integral
\[ \ell_\omega(\gamma) = \int_\gamma \omega. \]

Note that in particular we have
\[ \ell_1(\gamma) = \int_\gamma 1 = \ell(\gamma). \]

The \( \omega \)-distance between \( x \in X \) and \( y \in X \) (on the metric space \( X \)) is
\[ d_\omega(x, y) = \inf_\gamma \ell_\omega(\gamma), \]
where \( \gamma \) is among all rectifiable paths that connect \( x \) and \( y \). It is not hard to check that \( d_\omega \) is a distance function on \( X \).

### 3 The Equality of the Bloch and the Lipschitz Number

The aim of the present section is to put the equality statement of Proposition 1.1 in a very general setting. We will consider two types of growth conditions for mappings between metric spaces, and we will introduce the Bloch number and the Lipschitz number of a mapping. The generalized result of Pavlović, given in Theorem 3.1, says that these numbers are equal.

Let us first introduce the Bloch type growth condition and the Bloch number of a mapping. Let \( \omega \) and \( \tilde{\omega} \) be weight functions on metric spaces \( X \) and \( Y \), respectively. Consider a mapping \( f : X \to Y \) such that \( d^*_f(x) \) is finite for every \( x \in X \), and which satisfies the growth condition of the following form \( \tilde{\omega}(f(x))d^*_f(x) \leq m\omega(x) \) for every \( x \in X \), where \( m \) is a positive number. Introduce the following quantity
\[ \mathfrak{B}_f = \sup_{x \in X} \frac{\tilde{\omega}(f(x))}{\omega(x)}d^*_f(x), \]
which is called the Bloch number of the mapping \( f \) with respect to the weights \( \omega \) and \( \tilde{\omega} \). Note that if \( f \) satisfies the Bloch type growth condition, then \( d^*_f \) is bounded on compact subsets of \( X \). This follows since \( \omega \) and \( \tilde{\omega} \) are continuous.

Introduce now the Lipschitz type growth condition and the Lipschitz number of a mapping between metric spaces. Here we consider mappings \( f : X \to Y \) that satisfy the growth condition of the following form \( d(f(x), f(y)) \leq m(x, y)d(x, y), \) \( x, y \in X \) for a positive function \( m(x, y) \) on \( X \times X \).

For weight functions \( \omega \) on \( X \), and \( \tilde{\omega} \) on \( Y \), and a mapping \( f : X \to Y \) consider an everywhere positive function \( \Psi_f \) on \( X \times X \) such that it satisfies the following conditions:

(1)
\[ \Psi_f(x, y) = \Psi_f(y, x), \quad x, y \in X; \]
\(\Psi_f(x, x) = \frac{\tilde{\omega}(f(x))}{\omega(x)}, \quad x \in X;\)

\(\liminf_{y \to x} \Psi_f(x, y) \geq \Psi_f(x, x), \quad x \in X;\)

\(\Psi_f(x, y) \frac{d(f(x))}{d(x, y)} \leq \frac{d_\tilde{\omega}(f(x), f(y))}{d_\omega(x, y)}, \quad (x, y) \in \Delta^c_X.\)

We say that \(\Psi_f\) is an admissible for the mapping \(f\) with respect to the weight functions \(\omega\) and \(\tilde{\omega}\). Note that if \(\Psi_f(x, y)\) is not symmetric, but satisfies all other conditions stated above, we can replace it by the symmetric function

\[ \tilde{\Psi}_f(x, y) = \max\{\Psi_f(x, y), \Psi_f(y, x)\}, \quad x, y \in X. \]

This new function will be also admissible for \(f\), as it is easy to check.

Let us note that if we set \(\tilde{\omega} \equiv 1\), then the distance \(d_\tilde{\omega}\) is in some cases equal to the metric on \(Y\). For example, this occurs in normed spaces. In this case the fourth condition of admissibility is independent of the function \(f\), and one has to find an universal admissible function (that depend only on the weights) \(\Psi(x, y) = \Psi_f(x, y)\) which satisfies the following simplified condition

\(\Psi_f(x, y) d_\omega(x, y) \leq d(x, y), \quad x, y \in X.\)

Introduce now the following quantity

\[ L_f = \sup_{(x, y) \in \Delta^c_X} \Psi_f(x, y) \frac{d(f(x), f(y))}{d(x, y)}, \]

where \(\Psi_f\) is any admissible for \(f\) with respect to \(\omega\) and \(\tilde{\omega}\). We call it the Lipschitz number of \(f\). The result stated below says that the Lipschitz number does not depend on the choice of an admissible function for mappings \(f\). Therefore, the definition of the Lipschitz number \(L_f\) is correct.

We will now prove our main result in this section.

**Theorem 3.1** Let \(X\) and \(Y\) be rectifiable connected metrics spaces with weight functions \(\omega\) and \(\tilde{\omega}\), respectively. Let \(f : X \to Y\) be a mapping such that for every \(x \in X\) there exists \(d^*_f(x)\), and let \(\Psi_f\) be an admissible function for the mapping \(f\) with respect to the weights \(\omega\) and \(\tilde{\omega}\). If the Bloch number \(\mathcal{B}_f\), or the Lipschitz number \(L_f\), is finite, then the both numbers are finite, and \(\mathcal{B}_f = L_f\). Therefore, the Lipschitz number is independent of the choice of an admissible function \(\Psi_f\), and it may be expressed in the differential-free way

\[ \mathcal{B}_f = \sup_{(x, y) \in \Delta^c_X} \Psi_f(x, y) \frac{d(f(x), f(y))}{d(x, y)}. \]
Proof For one direction, assume that the Lipschitz number of a function $f$ is finite, i.e., assume that

$$\mathcal{L}_f = \sup_{(x,y) \in \Delta_X} \Psi_f(x,y) \frac{d(f(x), f(y))}{d(x,y)}$$

is finite. We are going to show that $\mathfrak{B}_f \leq \mathcal{L}_f$, which implies that the Bloch number $\mathfrak{B}_f$ is also finite.

For every $x, y \in X$ we have

$$\mathcal{L}_f = \sup_{(y,z) \in \Delta_X} \Psi_f(y,z) \frac{d(f(y), f(z))}{d(y,z)} \geq \limsup_{z \to x} \Psi_f(x,z) \frac{d(f(x), f(z))}{d(x,z)}$$

$$\geq \liminf_{z \to x} \Psi_f(x,z) \sup_{z \to x} \frac{d(f(x), f(z))}{d(x,z)} \geq \Psi_f(x,x) d^*_f(x) = \frac{\tilde{\omega}(f(x))}{\omega(x)} d^*_f(x).$$

It follows that

$$\mathcal{L}_f \geq \sup_{x \in X} \frac{\tilde{\omega}(f(x))}{\omega(x)} d^*_f(x) = \mathfrak{B}_f,$$

which we aimed to prove.

Assume now that the Bloch number $\mathfrak{B}_f$ of a mapping $f$ is finite, and form the sake of simplicity let $\mathfrak{B}_f = m$, i.e., let $\tilde{\omega}(f(x)) d^*_f(x) \leq m \omega(x), x \in X$. We will first prove that in this case we have

$$d^*_\omega(f(x), f(y)) \leq m d^\omega(x, y), \quad x, y \in X.$$  

Let $\gamma : [a, b] \to X$ be any rectifiable path connecting $x \in X$ and $y \in X$, i.e., such that $\gamma(a) = x$ and $\gamma(b) = y$. Consider the path $\delta = f \circ \gamma \subseteq Y$. This path connects $f(x)$ and $f(y)$. Since $d^*_f(x)$ is finite for every $x \in X$, by Lemma 2.2, the path $\delta = f \circ \gamma$ is also a rectifiable one. Therefore, $\int_\delta \tilde{\omega}$ is a finite number.

Let $\varepsilon > 0$ be arbitrary. Since the function

$$g(z) = \frac{m \omega(z)}{\tilde{\omega}(f(z))}, \quad z \in X$$

is continuous on $X$, and since any path is a compact set, we may found a partition $T : t_0, t_1, \ldots, t_n$ of the segment $[a, b]$ such that $\text{diam}(T) < \varepsilon$, and

$$|g(\gamma(t)) - g(\gamma(t_{i-1}))| < \varepsilon, \quad t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n.$$  

Since $g(\gamma(t)) < g(\gamma(t_{i-1})) + \varepsilon$, it follows

$$d^*_f(\gamma(t)) \leq g(\gamma(t)) < g(\gamma(t_{i-1})) + \varepsilon, \quad t \in [t_{i-1}, t_i], \quad 1 \leq i \leq n.$$
Let $S$ be the sequence $t_0, t_1, \ldots, t_{n-1}$. Since $\gamma$ is compact, there exists a constant $C$ such that $\tilde{\omega}(f(x)) \leq C$, $x \in \gamma$. Now, we have

$$
\sigma_{\delta}(\tilde{\omega}, T, S)
= \sum_{i=1}^{n} \tilde{\omega}(f(\gamma(t_{i-1}))) \ell(\delta_{[t_{i-1}, t_i]})
\leq \sum_{i=1}^{n} \tilde{\omega}(f(\gamma(t_{i-1}))) (g(\gamma(t_{i-1})) + \varepsilon) \ell(\gamma_{|t_{i-1}, t_i]})
\leq \sum_{i=1}^{n} \tilde{\omega}(f(\gamma(t_{i-1}))) g(\gamma(t_{i-1})) \ell(\gamma_{|t_{i-1}, t_i]}) + \varepsilon \sum_{i=1}^{n} \tilde{\omega}(f(\gamma(t_{i-1}))) \ell(\gamma_{|t_{i-1}, t_i]})
\leq m \sum_{i=1}^{n} \omega(\gamma(t_{i-1})) \ell(\gamma_{|t_{i-1}, t_i]}) + \varepsilon C \sum_{i=1}^{n} \ell(\gamma_{|t_{i-1}, t_i])
\leq m \sum_{i=1}^{n} \omega(\gamma(t_{i-1})) \ell(\gamma_{|t_{i-1}, t_i]}) + \varepsilon C \ell(\gamma)
$$

If we let $\varepsilon \to 0$ above, we obtain $\int_{\delta} \tilde{\omega} \leq m \int_{\gamma} \omega$. Now, if we take the infimum over all paths $\gamma$, we obtain $\inf_{\gamma} \int_{f_{\gamma}} \tilde{\omega} \leq m d_{\omega}(x, y)$. If $\delta$ is among all paths that connect $f(x)$ and $f(y)$, we have

$$
d_{\tilde{\omega}}(f(x), f(y)) = \inf_{\delta} \int_{\delta} \tilde{\omega} \leq \inf_{\gamma} \int_{f_{\gamma}} \tilde{\omega} \leq m d_{\omega}(x, y).
$$

Therefore, we have proved that aimed inequality.

We will prove now the reverse inequality $L_f \leq B_f$, which in particular implies that the Lipschitz number $L_f$ is also finite. By the above we have $d_{\tilde{\omega}}(f(x), f(y)) \leq B_f d_{\omega}(x, y)$ for every $x \in X$ and $y \in X$. Applying now conditions posed on the admissible function $\Psi_f$, we obtain

$$
\Psi_f(x, y) \frac{d(f(x), f(y))}{d(x, y)} \leq \frac{d_{\tilde{\omega}}(f(x), f(y))}{d_{\omega}(x, y)} \leq B_f, \quad (x, y) \in \Delta_X.
$$

It follows that

$$
L_f = \sup_{(x, y) \in \Delta_X} \Psi_f(x, y) \frac{d(f(x), f(y))}{d(x, y)} \leq B_f,
$$

which we needed to prove.

To apply Theorem 3.1 for normed spaces, we will need the following auxiliary result.
Lemma 3.2 Let $X$ be a normed space, let $\Omega \subseteq X$ be a domain, and let $\omega$ a weight function on $\Omega$. For $\omega$-distance $d_\omega$ on $\Omega$, we have

$$\lim_{y \to x} \frac{d_\omega(x, y)}{\|x - y\|} = \omega(x), \quad x \in \Omega.$$ 

Proof Since $\omega$ is continuous, there exists an open ball $B_X(x, r) \subseteq \Omega$ such that

$$0 < \omega(x) - \varepsilon < \omega(y) < \omega(x) + \varepsilon, \quad y \in B_X(x, r),$$

where $\varepsilon > 0$ is a sufficiently small number. Now, we have

$$d_\omega(x, y) \leq \int_{[x, y]} \omega \leq (\omega(x) + \varepsilon) \ell([x, y]) = (\omega(x) + \varepsilon) \|x - y\|.$$ 

On the other hand, if $\gamma \subseteq \Omega$ is among paths that connect $x$ and $y$, then

$$d_\omega(x, y) = \inf_{\gamma} \int_{\gamma} \omega \geq (\omega(x) - \varepsilon) \|x - y\|.$$ 

Therefore,

$$\omega(x) - \varepsilon \leq \frac{d_\omega(x, y)}{\|x - y\|} \leq \omega(x) + \varepsilon, \quad y \in B_X(x, r).$$

This means that

$$\lim_{y \to x} \frac{d_\omega(x, y)}{d(x, y)} = \omega(x).$$

By Lemma 3.2 we have that the topology on $\Omega$ induced by the $\omega$-distance $d_\omega$ is the same as the topology induced by the norm on $X$. For similar results we refer to [16].

Corollary 3.3 Let $X$ and $Y$ be normed spaces, and let $\omega$ and $\tilde{\omega}$ be weight functions on domains $\Omega \subseteq X$ and $\tilde{\Omega} \subseteq Y$, respectively. The Bloch number

$$\mathfrak{B}_f = \sup_{x \in \Omega} \frac{\tilde{\omega}(f(x))}{\omega(x)} d_f^x(x)$$

of a mapping $f : \Omega \to \tilde{\Omega}$ for which $d_f^x(x)$ is finite for every $x \in \Omega$, is equal to

$$\mathfrak{B}_f = \sup_{(x, y) \in \Delta^1_\Omega} \frac{d_\omega(f(x), f(y))}{d_\omega(x, y)}.$$ 

Proof Indeed, an admissible function $\Psi_f$ is given by

$$\Psi_f(x, y) = \begin{cases} \frac{d_\omega(f(x), f(y))}{d_\omega(x, y)} / \frac{d_\omega(x, y)}{d(x, y)}, & \text{if } x \neq y, \ f(x) \neq f(y); \\ \frac{\tilde{\omega}(f(x))}{\omega(x)} / \frac{\tilde{\omega}(x)}{\omega(x)}, & \text{if } x \neq y, \ f(x) = f(y); \\ \frac{\tilde{\omega}(f(x))}{\omega(x)}, & \text{if } x = y, \ f(x) = f(y). \end{cases}$$
Having in mind Lemma 3.2 it follows that
\[ \liminf_{y \to x} \Psi_f(x, y) = \lim_{y \to x} \Psi_f(x, y) = \frac{\omega(f(x))}{\omega(x)} = \Psi_f(x, x). \]

Other three admissability conditions for \( \Psi_f \) are obviously satisfied. \( \square \)

In other words, the corollary says that a mapping \( f : X \to Y \) satisfies \( \tilde{\omega}(f(x)) \frac{d_f^*(x)}{\omega(x)} \leq m \omega(x) \) \( x \in X \), where \( m \) is a constant, if and only if \( d_\omega(f(x), f(y)) \leq m d_\omega(x, y), x \in X, y \in X. \)

### 4 Admissible Functions in Some Special Cases

Having in mind Lemma 2.1, as an immediate consequence of Theorem 3.1 (and special case of Corollary 3.3) we have

**Proposition 4.1** Let \( \omega \) be a weight function on a domain \( \Omega \) in the normed space \( X \) and let \( d_\omega \) be the \( \omega \)-distance on \( \Omega \). For a Fréchet differentiable mappings \( f : \Omega \to Y \). The Bloch type semi-norm

\[ \| f \|_{B, \omega} = \sup_{x \in \Omega} \frac{\| d_f(x) \|}{\omega(x)} \]

is equal to

\[ \| f \|_{B, \omega} = \sup_{(x, y) \in \Delta_\Omega} \frac{\| f(x) - f(y) \|_Y}{d_\omega(x, y)} \]

Let us note that an admissible function here is independent on \( f \), and is given by

\[ \Psi(x, y) = \begin{cases} \| x - y \| / d_\omega(x, y), & \text{if } x \neq y; \\ 1 / \omega(x), & \text{if } x = y. \end{cases} \]

Since the explicit expression for the distance function \( d_\omega \) is difficult to find in general, the aim of the rest of this section is to find simple admissible functions in some special cases. We do that for weights that are derivative of an operator monotone function on an interval and in the case when weights are monotone in norm.

#### 4.1 The Case of Weights that are Derivatives of Operator Monotone Functions

We will start this subsection with preliminaries on operator monotone function on an open interval \( I \subseteq \mathbb{R} \). The class of operator monotone functions on the interval \( I \), denoted by \( OM(I) \), contains all real-valued functions \( \varphi \) on \( I \) that preserve the Hilbert space operator ordering in the following sense: let \( A \) and \( B \) be self-adjoint operators on a Hilbert space with their spectra contained in the interval \( I \), and satisfying \( A \leq B \), then \( \varphi(A) \leq \varphi(B) \). For a more in operator monotone functions we refer to the classical reference book [7], and to the recent one [8]. We point out that an operator monotone
function on $I$ is continuously differentiable on $I$, and $\varphi'(t) > 0$ for $t \in I$ unless it is a constant.

An analytic function $\varphi : \{z \in \mathbb{C} : \Re z > 0\} \to \{z \in \mathbb{C} : \Re z \geq 0\}$, belongs to the Pick class $P(I)$ if it may be analytically extended across the interval $I$ by the Schwarz reflection principle, i.e., $\varphi(z) = \overline{\varphi(\overline{z})}$. It is clear that $\varphi(t) \in \mathbb{R}$ if $t \in \mathbb{R}$. By the Nevanlinna theorem a function $\varphi \in P(I)$ may be represented as

$$
\varphi(z) = \alpha + \beta z + \int_{\mathbb{R} \setminus I} \frac{1 + \lambda z}{\lambda - z} \, d\mu(\lambda), \quad z \in \mathbb{C} \setminus \mathbb{R} \cup I,
$$

where $\mu$ is finite positive Borel measure on $\mathbb{R} \setminus I$, and $\alpha, \beta \in \mathbb{R}$, $\beta \geq 0$. Moreover, it may be showed that a function $\varphi \in P(-1, 1)$ may be represented in the way

$$
\varphi(z) = \varphi(0) + \varphi'(0) \int_{[-1, 1]} \frac{z}{1 - tz} \, d\mu(t), \quad z \in \mathbb{C} \setminus \mathbb{R} \cup I,
$$

where $\mu$ is a probability measure on $[-1, 1]$. For the first derivative we have

$$
\varphi'(z) = \varphi'(0) \int_{[-1, 1]} \frac{1}{(1 - tz)^2} \, d\mu(t), \quad z \in \mathbb{C} \setminus \mathbb{R} \cup I.
$$

Let us note a simple fact: If $\mu$-measure of the interval $(-1, 0)$ is equal to 0, then $\varphi'(t)$ is increasing on $[0, 1)$.

Using the mentioned integral representation of a function in the Pick class $P(-1, 1)$, and the Löwner’s theorem [8, Theorem 2.7.7] which says that if $\varphi \in P(I)$, then $\varphi|_{I} \in \text{OM}(I)$, and every $\varphi \in \text{OM}(I)$ is a restriction of a function in $P(I)$, Jocić [10, Lemma 2.1] proved the following lemma for $I = (-1, 1)$.

**Lemma 4.2** An operator monotone function $\varphi$ on an open interval $I$ satisfies

$$
\varphi(t) - \varphi(s) \leq \sqrt{\varphi'(t)\varphi'(s)}(t - s), \quad t \in I, \ s \in I, \ s < t.
$$

A proof of Lemma 4.2 is similar as in the particular case $I = (-1, 1)$. However, it may be deduced from this special case. Indeed, it is enough to note that if $\varphi$ belongs to the class $\text{OM}(I)$, where $I$ is a finite interval, then $\varphi \circ g$ belongs to the class $\text{OM}(-1, 1)$, where $g$ is an increasing linear mapping such that $g(-1, 1) = I$. If $I$ is infinite interval, and if $\varphi \in \text{OM}(I)$ then we have $\varphi \in \text{OM}(I')$ for any finite subinterval $I' \subseteq I$.

**Theorem 4.3** Let $\Omega \subseteq X$ be a convex domain in a normed space $X$. Assume that $\varphi$ is a non-constant operator monotone function on $(-\delta, M)$, $\delta > 0$, $M = \sup_{x \in \Omega} \|x\|$, and let $\varphi'$ be increasing on $[0, M)$. For a Fréchet differentiable mapping $f : \Omega \to Y$, the Bloch type semi-norm

$$
\|f\|_{\mathcal{B}, \varphi'} = \sup_{x \in \Omega} \frac{\|d_{f}(x)\|}{\varphi'(\|x\|)}
$$

\[ Springer \]
may be expressed as

\[ \| f \|_{\mathcal{B}, \varphi'} = \sup_{(x, y) \in \Delta_\Omega^c} \frac{\| f(x) - f(y) \|}{\sqrt[\varphi'(|x|)}|x - y| \sqrt[\varphi'(|y|)}}. \]

**Proof** By Theorem 3.1 it is enough to prove that

\[ \Psi(x, y) = \varphi'(|x|)^{-\frac{1}{2}} \varphi'(|y|)^{-\frac{1}{2}}, \quad x \in \Omega, \; y \in \Omega, \]

is an admissible function for the mapping \( f \) with respect to the weight functions \( \omega = \varphi' \) on \( \Omega \) and \( \tilde{\omega} \equiv 1 \) on \( Y \). Obviously, \( \Psi \) is continuous and symmetric on \( \Omega \times \Omega \), and \( \Psi(x, x) = \frac{1}{\varphi'(|x|)} \) for \( x \in \Omega \). It remains to prove the inequality

\[ \Psi(x, y) d\varphi'(x, y) \leq \| x - y \|, \quad x, \; y \in \Omega. \]

Indeed, if \( \gamma \subseteq \Omega \) is among paths that connect \( x \) and \( y \), and if (for example) \( \| x \| < \| y \| \) (if \( \| x \| = \| y \| \) the proof below is trivial), we have

\[
\begin{align*}
d\varphi'(x, y) &= \inf_{\gamma} \int_{\gamma} \varphi' \leq \int_{[x, y]} \varphi' \\
&= \| x - y \| \int_{0}^{1} \varphi'((1 - t)x + ty) \, dt \\
&\leq \| x - y \| \int_{0}^{1} \varphi'((1 - t)\| x \| + t\| y \|) \, dt \\
&= \frac{\| x - y \|}{\| y \| - \| x \|} \int_{0}^{1} \frac{d}{dt}(\varphi(\| x \| + t(\| y \| - \| x \|))) \\
&= \| x - y \| \frac{\varphi(\| y \|) - \varphi(\| x \|)}{\| y \| - \| x \|} \\
&\leq \| x - y \| \sqrt{\varphi(\| x \|)\varphi(\| y \|)} \\
&= \| x - y \| \Psi(x, y)^{-1},
\end{align*}
\]

where we have used Lemma 4.2 in the last inequality.

The following corollary is generalization of Proposition 1.1 for normed spaces and Fréchet differentiable mappings. Bloch spaces on domains in a Banach space have been recently considered in [2,5].

**Corollary 4.4** Let \( X \) and \( Y \) be normed spaces. For a Fréchet differentiable mapping \( f : B_X \to Y \) the Bloch semi-norm

\[ \| f \|_{\mathcal{B}} = \sup_{x \in B_X} (1 - \| x \|^2) \| df(x) \|. \]
is equal to

$$\|f\|_B = \sup_{(x,y) \in \Delta_{\|x\|}^c} (1 - \|x\|)^{\frac{1}{2}} (1 - \|y\|)^{\frac{1}{2}} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$ 

This corollary follows from the Theorem 4.3 if we take \(\phi = \tanh^{-1} \in \text{OM}(-1, 1)\).

### 4.2 The Case of Weights Monotone in Norm

In Theorem 4.5, which particular case (for \(\tilde{\Omega} = Y\) and \(\tilde{\omega} \equiv 1\)) is mentioned in the Introduction, we have found a very simple admissible function for weight functions monotone in norm on domains of normed spaces. It would be of some interest to find a such simple admissible function for other domains and/or for other weight functions not necessary monotone in norm.

**Theorem 4.5** Let \(\omega(x)\) be a weight function on a convex domain \(\Omega\) in a normed space \(X\) which is monotone in \(\|x\|\), and let \(\tilde{\omega}(y)\) be a weight function on a domain \(\tilde{\Omega}\) in a normed space \(Y\) which is monotone in \(\|y\|\). The Bloch number

$$B_f = \sup_{x \in \Omega} \frac{\tilde{\omega}(f(x))}{\omega(x)} \|d_f(x)\|$$

of a Fréchet differentiable mapping \(f : \Omega \to \tilde{\Omega}\) is equal to

$$B_f = \sup_{(x,y) \in \Delta_{\|\|}^c} \frac{\min\{\tilde{\omega}(f(x)), \tilde{\omega}(f(y))\}}{\max\{\omega(x), \omega(y)\}} \frac{\|f(x) - f(y)\|}{\|x - y\|}.$$ 

**Proof** In order to apply Theorem 3.1 for \(\Omega \subseteq X\) and \(\tilde{\Omega} \subseteq Y\) with the metrics induces by the norms on \(X\) and \(Y\), respectively, we have to prove that

$$\Psi_f(x, y) = \frac{\min\{\tilde{\omega}(f(x)), \tilde{\omega}(f(y))\}}{\max\{\omega(x), \omega(y)\}}, \quad x, y \in \Omega$$

is an admissible function for the mapping \(f : \Omega \to \tilde{\Omega}\) with respect to \(\omega\) and \(\tilde{\omega}\). Since it is obviously symmetric and continuous on \(\Omega \times \Omega\), and \(\Psi_f(x, x) = \frac{\tilde{\omega}(f(x))}{\omega(x)}\) for \(x \in \Omega\), we have only to show that \(\Psi_f\) satisfies

$$\Psi_f(x, y) \frac{\|f(x) - f(y)\|}{\|x - y\|} \leq \frac{d_{\tilde{\omega}}(f(x), f(y))}{d_{\omega}(x, y)}, \quad x, y \in \Omega.$$ 

Since \(\omega(z)\) is monotone in \(\|z\|\), and since \(\Omega\) is convex, we have

$$d_{\omega}(x, y) \leq \int_{[x, y]} \omega \leq \max_{z \in [x, y]} \omega(z) \|x - y\| = \max\{\omega(x), \omega(y)\} \|x - y\|.$$
for \( x \in \Omega \) and \( y \in \Omega \).

If \( \tilde{\omega}(z) \) is increasing in \( \|z\| \), let \( \tilde{\gamma} : [a, b] \to Y \) be among paths in \( \Omega \) that connect \( x \in \Omega \) and \( y \in \Omega \) and satisfy \( \|\tilde{\gamma}(t)\| \leq \max\{\|x\|, \|y\|\}, t \in [a, b] \). On the other hand, if \( \omega(z) \) is a decreasing function of \( \|z\| \), then we should consider only the paths \( \tilde{\gamma} \) such that \( \|\tilde{\gamma}(t)\| \geq \min\{\|x\|, \|y\|\} \). Let \( \gamma \subseteq \Omega \) be among paths that connect \( x \) and \( y \) without any restriction. Now, we have

\[
d_{\omega}(x, y) = \inf_{\gamma} \int_{\gamma} \tilde{\omega} = \inf_{\tilde{\gamma}} \int_{\tilde{\gamma}} \tilde{\omega} \geq \inf_{\tilde{\gamma}} \{ \min_{z \in \tilde{\gamma}} \tilde{\omega}(z) \|x - y\| \}
\]

\[
\geq \min\{\tilde{\omega}(x), \tilde{\omega}(y)\} \|x - y\|.
\]

It follows

\[
\frac{d_{\omega}(f(x), f(y))}{d_{\omega}(x, y)} \geq \frac{\min\{\tilde{\omega}(f(x)), \tilde{\omega}(f(y))\}}{\max\{\omega(x), \omega(y)\}} \frac{\|f(x) - f(y)\|}{\|x - y\|},
\]

which proves the fourth condition of admissibility. \( \square \)

### 4.3 Characterizations of Bloch and Normal Mappings

Recall that an analytic function \( f \) on the unit disc is a Bloch function if it satisfies the growth condition

\[
|f'(z)| \leq \frac{C}{1 - |z|^2}, \quad z \in U,
\]

where \( C \) is a constant. For equivalent definitions we refer to [1]. On the other hand, an analytic function \( f \) is normal on the unit disc if the family

\[
\{f \circ \varphi : \varphi \text{ is a Moebius transform of } U\}
\]

is a normal family [11]. However, it is known that this condition is equivalent to the growth condition

\[
\frac{|f'(z)|}{1 + |f(z)|^2} \leq \frac{C}{1 - |z|^2}, \quad z \in U,
\]

where \( C \) is a constant. For similarities between analytic Bloch and normal functions we refer to [1,6].

Let \( \rho \) be the hyperbolic distance on the unit disc \( U \), and let \( \sigma \) be the spherical distance on \( C \). It is very well known that an analytic function is Bloch function if and only if it is Lipschitz continuous with respect to the Euclidean distance and the hyperbolic distance, respectively, i.e.,

\[
|f(z) - f(w)| \leq C \rho(z, w), \quad z, w \in U,
\]

as well as that the function \( f \) is normal if and only if it satisfies

\[
\sigma(f(z), f(w)) \leq C \rho(z, w), \quad z, w \in U.
\]
It is also well known that these conditions are equivalent to the corresponding growth conditions. This follows also from the remark after Corollary 3.3.

The aim of this last subsection is to give a more direct proof of Proposition 1.1 following and simplifying the proof form [12], and to consider the Bloch type growth condition which satisfies a normal function. We derive a new criteria for normality of analytic mapping on the unit disc.

In Theorem 3.1 let us take $X = B^m$ with the standard distance, and let $Y$ be a normed space. Moreover, let $\omega(x) = (1 - |x|^2)^{-1}, x \in B^m$ and $\tilde{\omega} \equiv 1$ be the weight functions. Then $d_\omega$ is the hyperbolic distance on the unit ball $B^m$, which in the sequel we denote by $\rho$, and $d_{\tilde{\omega}}$ is the distance on $Y$ produced by the norm $\| \cdot \|_Y$. The hyperbolic distance on $B^m$ is given explicitly by

$$\rho(x, y) = \sinh \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}, \quad x, y \in B^m$$

(see [15]). We will prove that an admissible function in this setting is

$$\Psi(x, y) = \sqrt{1 - |x|^2} \sqrt{1 - |y|^2}, \quad x, y \in B^m.$$ 

We only have to show that $\rho(x, y) \psi(x, y) \leq |x - y|, x, y \in B^m$. By the inequality $\sinh t \leq t$ for $t \geq 0$ (indeed, let $\phi(t) = \sinh(t) - t$; then we have $\phi(0) = 0$, and $\phi'(t) = \frac{1}{1 + t^2} - 1 < 0$ for $t > 0$, so our inequality follows since $\phi(t) \leq \phi(0) = 0$), we deduce

$$\frac{|x - y|}{\rho(x, y)} = |x - y| : \sinh \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}} \geq |x - y| : \sqrt{1 - |x|^2} \sqrt{1 - |y|^2}$$

$$= \sqrt{1 - |x|^2} \sqrt{1 - |y|^2} = \Psi(x, y), \quad x, y \in B^m.$$

Proposition 1.1 now follows from Theorem 3.1.

As we have said the growth condition given in the corollary below is satisfied by the class of normal functions on the unit disc.

**Corollary 4.6** A differentiable mapping $f : B^m \rightarrow \mathbb{R}^2$ satisfies the growth condition

$$\frac{\|d_f(x)\|}{1 - |x|^2} \leq \frac{C}{1 + |f(x)|}, \quad x \in B^m,$$

where $C$ is a constant, if and only if

$$|f(x) - f(y)| \leq C|x - y| \frac{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}, \quad x, y \in B^m.$$
Proof In Theorem 3.1 let us take $X = B^m$ and $Y = \mathbb{R}^2$ with Euclidean distances, and let 
\[ \omega(x) = (1 - |x|^2)^{-1}, \quad x \in B^m, \quad \tilde{\omega}(z) = (1 + |z|^2)^{-1}, \quad z \in \mathbb{R}^2 \]
be the weight functions on these domains. As we have already said, $d_\omega$ is the hyperbolic distance $\rho$ on the unit ball $B^m$. On the other hand, $d_{\tilde{\omega}}$ is the spherical distance $\sigma$ on $\mathbb{R}^2$. It is well known that 
\[ \sigma(z, w) = \frac{|z - w|}{\sqrt{1 + |z|^2} \sqrt{1 + |w|^2}}, \quad z, w \in \mathbb{R}^2. \]

We are going to show that 
\[ \Psi_f(x, y) = \frac{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}}, \quad x, y \in B^m \]
is an admissible function for the mapping $f$ with respect to the hyperbolic and spherical weights. Since 
\[ \Psi_f(x, x) = \frac{1 - |x|^2}{1 + |f(x)|^2} = \frac{1}{1 + |f(x)|^2} : \frac{1}{1 - |x|^2}, \quad x \in B^m, \]
and since $\Psi_f(x, y)$ is obviously symmetric and continuous, it remains only to prove that $\Psi_f(x, y)$ satisfies 
\[ \Psi_f(x, y) \frac{|f(x) - f(y)|}{|x - y|} \leq \sigma(f(x), f(y)) \frac{|f(x) - f(y)|}{\rho(x, y)}, \quad x, y \in B^m. \]

Having on mind the inequality \( \text{asinh} \ t \leq t \) for $t \geq 0$, we obtain 
\[
\begin{align*}
\frac{\sigma(f(x), f(y))}{\rho(x, y)} &= \frac{|f(x) - f(y)|}{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}} : \frac{\text{asinh} \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}}}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}} \\
&\geq \frac{|f(x) - f(y)|}{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}} : \frac{|x - y|}{\sqrt{1 - |x|^2} \sqrt{1 - |y|^2}} \\
&= \frac{|f(x) - f(y)|}{\sqrt{1 + |f(x)|^2} \sqrt{1 + |f(y)|^2}} \frac{|x - y|}{|x - y|} \\
&= \Psi_f(x, y) \frac{|f(x) - f(y)|}{|x - y|},
\end{align*}
\]
which is the inequality we aimed to prove. \Box

Based on the preceding results, we now state the characterisations of Bloch and normal functions on the unit disc.
Proposition 4.7  Let \( f : U \to \mathbb{C} \) be an analytic function.

\[
\text{f is a Bloch if and only if } \sqrt{1 - |z|^2} \sqrt{1 - |w|^2} \left| \frac{f(z) - f(w)}{|z - w|} \right| \text{ is bounded for } z \neq w.
\]

\[
\text{f is a normal iff } \sqrt{1 - |z|^2} \sqrt{1 - |w|^2} \left| \frac{f(z) - f(w)}{|z - w|} \right| \text{ is bounded for } z \neq w.
\]

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