On possible skewon effects on light propagation

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Abstract

We start from a local and linear spacetime relation between the electromagnetic excitation and the field strength. Then we study the generally covariant Fresnel surfaces for light rays and light waves. The metric and the connection of spacetime are left unspecified. Accordingly, our framework is ideally suited for a search of possible violations of the Lorentz symmetry in the photon sector of the extended standard model. We discuss how the skewon part of the constitutive tensor, if suitably parametrized, influences the Fresnel surfaces and disturbs the light cones of vacuum electrodynamics. Conditions are specified that yield the reduction of the original quartic Fresnel surface to the double light cone structure (birefringence) and to the single light cone. Qualitatively, the effects of the real skewon field can be compared to those in absorbing material media. In contrast, the imaginary skewon field can be interpreted in terms of non-absorbing media with natural optical activity and Faraday effects. The astrophysical data on gamma-ray bursts are used for deriving an upper limit for the magnitude of the skewon field.

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INTRODUCTION

Classical electrodynamics is fundamentally linked to the concepts of relativity and spacetime symmetry. In particular, many high precision tests of relativity are based on electromagnetic phenomena. At present, there is a growing interest (both theoretically and experimentally) in the search for possible violations of the Lorentz symmetry. The latter may naturally occur, for example, in the framework of unified theories of physical interactions at the Planck length scale, and may then manifest themselves at the scales of usual high-energy particle physics in the form of corrections to the established quantum field-theoretical models. Even though presently the limits of the standard model of elementary particle physics (SM) become visible, see Sec.10.6 of the new Particle Data Report [1], it still remains the basis of our understanding of the strong and the electroweak interactions. At the foundations of the SM lays special relativity with the Poincaré symmetry group and, associated with it, the light cone structure. Although this rigid light cone structure of special relativity is assumed as framework for the SM, we know from general relativity theory (GR) and from experiments that the light cone structure is in fact “flexible”. In other words, in GR the metric becomes a space and time dependent field. This alone demonstrates that the SM cannot be strictly correct.

Numerous attempts have been made to take care of this state of affairs and to build up a more general framework for the SM and perhaps even to include GR. One of the promising approaches is the standard model extension (SME) of Kostelecký and collaborators, see [2, 3, 4], in which by means of a set of parameters certain violations of the SM are put in a quantitative form amenable to experimental tests. The photon sector of the SME has been particularly closely studied in [5], where the propagation of electromagnetic waves in vacuum was investigated with Lorentz violating terms added in the effective constitutive tensor.

In this paper, we continue the study of electrodynamical phenomena in the framework of the so-called premetric approach, i.e., without assuming a special form of the metric structure on a spacetime manifold. Such an approach provides a general technique for an appropriate physical discussion of the experimental tests of possible violations of the Lorentz symmetry and of parity, since at the outset no assumptions were allowed about geometric structures, such as the metric or the connection of spacetime. On the contrary, the analysis of the
dynamics of the electromagnetic field in this approach yields, via the light propagation, the information about the underlying metrical structure of spacetime. Accordingly, this scheme can be viewed as a general test theory of the searches for possible deviations from the Lorentz symmetry in the photon sector of SME. It is worthwhile to note that the multi-parameter approach of Kostelecký and Mewes [5] can be naturally embedded into the framework of general premetric electrodynamics.

Electromagnetic wave propagation is a very important physical phenomenon in classical field theory. In general, the geometrical structure of spacetime as well as the intrinsic properties and the motion of material media can affect the light propagation. Hence, a theoretical analysis of the latter ultimately results in establishing the properties of the (genuine or effective “optical”) metric structure on the manifold.

In the generally covariant premetric approach to electrodynamics [11, 12, 13, 14], the axioms of electric charge and of magnetic flux conservation manifest themselves in the Maxwell equations for the excitation $H = (\mathcal{D}, \mathcal{H})$ and the field strength $F = (E, B)$:

$$dH = J, \quad dF = 0.$$ (1)

These equations should be supplemented by a constitutive law $H = H(F)$. The latter relation contains the crucial information about the underlying physical continuum (i.e., about spacetime and/or about the material medium). Mathematically, this constitutive law arises either from a suitable phenomenological theory of a medium or from the electromagnetic field Lagrangian. It can be a nonlinear or even nonlocal relation between the electromagnetic excitation and the field strength.

Earlier, we have investigated the propagation of waves in the most general local and linear theory. One can conveniently split the constitutive relation into the three irreducible pieces which are known as the principal part, the skewon part and the axion part. It is important to stress that such a decomposition is intrinsically metricfree, and thus it suits nicely for the purpose of the study of the violations of the Lorentz symmetry in the photon sector. More specifically, we investigated the influence of the axion and the skewon fields on the wave propagation by applying a linear response formalism, see [6]. Subsequently, Itin [7] studied birefringence caused by the axion field in the Caroll-Field-Jackiw model of electrodynamics [8]. More recently, we investigated the birefringence of electromagnetic waves and its consequences for the metric of spacetime [9]. The possible limits of the
Lorentz symmetry can be seen in a particularly transparent way, both qualitatively and quantitatively, including also the case of a possible violation of the charge conservation law \[10\].

The axion field and its interference with electrodynamics is a well-studied subject, and the same applies to the influence of the general principal part of the constitutive tensor which was thoroughly analyzed in \[9\]. However, this is not so in the case of the skewon field, a new structure which allows us to exhaust all 36 parameters of a linear spacetime relation (exact definitions are given below). The only relevant earlier results are those of Nieves and Pal \[15, 16\] which were confined to the case of the spatially isotropic and constant scalar skewon field. In contrast, we present here some results which hold true for all electrodynamical models with an arbitrary local and linear spacetime relation that contains a nontrivial skewon part. We use the expression “spacetime relation”, since it applies to spacetime (“the vacuum”) itself, and in order to distinguish it from the constitutive law of a material medium. From a mathematical point of view, the general local and linear spacetime relation is similar to the constitutive relation for matter with sufficiently complicated electric and magnetic properties. Quite naturally, the study and the physical interpretation of the wave propagation in a manifold with a general spacetime relation appears to be very close to the analysis of the corresponding wave effects in crystal optics. The latter subject was extensively studied in the literature, see \[17, 18, 19, 20\], for example. In particular, our analysis reveals the analogy of the effects of the real skewon field to the propagation of waves in absorbing media with circular dichroism, whereas the imaginary skewon field appears to be analogous to the optical activity tensor.

If local coordinates \(x^i\) are given, with \(i, j, ... = 0, 1, 2, 3\), we can decompose the excitation and field strength 2-forms into their components according to

\[
H = \frac{1}{2} H_{ij} \, dx^i \wedge dx^j, \quad F = \frac{1}{2} F_{ij} \, dx^i \wedge dx^j.
\] (2)

**GENERAL LOCAL AND LINEAR CONSTITUTIVE RELATION**

We confine ourselves to the case in which the electromagnetic excitation and the field strength are related by the local and linear constitutive law,

\[
H_{ij} = \frac{1}{2} \kappa_{ij}^{kl} F_{kl}.
\] (3)
The constitutive tensor $\kappa$ has 36 independent components. One can decompose this object into its irreducible pieces. Obviously, contraction is the only tool for such a decomposition. Following Post \[21\], we can define the contracted tensor of type $[1]$

$$\kappa_i^k := \kappa_{i[l}^k l], \quad (4)$$

with 16 independent components. The second contraction yields the pseudo-scalar function

$$\kappa := \kappa_k^k = \kappa_{k[l}^{k ]l}. \quad (5)$$

The traceless piece

$$\kappa_i^k := \kappa_i^k - \frac{1}{4} \kappa \delta_i^k \quad (6)$$

has 15 independent components. These pieces can now be subtracted out from the original constitutive tensor. Then,

$$\kappa_{ij}^{kl} = (1)\kappa_{ij}^{kl} + (2)\kappa_{ij}^{kl} + (3)\kappa_{ij}^{kl} \quad (7)$$

$$= (1)\kappa_{ij}^{kl} + 2 \kappa[i^{[k} l]^{j]} + \frac{1}{6} \kappa \delta_i^k \delta_j^l. \quad (8)$$

By construction, $(1)\kappa_{ij}^{kl}$ is the totally traceless part of the constitutive tensor:

$$(1)\kappa_{kl}^{kl} = 0. \quad (9)$$

Thus, we split $\kappa$ according to $36 = 20 + 15 + 1$, and the $[2]$ tensor $(1)\kappa_{ij}^{kl}$ is subject to the 16 constraints \[9\] and carries $20 = 36 - 16$ components.

One may call $(1)\kappa_{ij}^{kl}$ the principal, or the metric-dilaton part of the constitutive law. Without such a term, electromagnetic waves are ruled out, see \[22, 23, 24\]. We further identify the two other irreducible parts with a skewon and an axion field, respectively. Conventionally, the skewon and the axion fields are introduced by

$$g_i^j = - \frac{1}{2} \kappa_i^j, \quad \alpha = \frac{1}{12} \kappa. \quad (10)$$

The standard Maxwell-Lorentz electrodynamics arises when both skewon and axion vanish, whereas

$$(1)\kappa_{ij}^{kl} = \lambda_0 \eta_{ij}^{kl}. \quad (11)$$

Here $\lambda_0 = \sqrt{\varepsilon_0/\mu_0}$ is the vacuum impedance. A spacetime metric $g_{ij}$ is assumed on the manifold, and with $g := \det g_{ij}$ one defines $\eta_{ijkl} := \sqrt{-g} \epsilon_{ijkl}$ and $\eta_{ij}^{kl} = \eta_{ijmn} g^{mk} g^{nl}$. It has
been shown recently [9] that taking the linear spacetime relation for granted, one ends up at a Riemannian lightcone provided one forbids birefringence in vacuum, see also [25].

Along with the original $\kappa$-tensor, it is convenient to introduce an alternative representation of the constitutive tensor:

$$\chi_{ijkl} := \frac{1}{2} \epsilon_{ijmn} \kappa_{mn \ kl}. \quad (12)$$

Substituting (8) into (12), we find the corresponding decomposition

$$\chi_{ijkl} = (1) \chi_{ijkl} + (2) \chi_{ijkl} + (3) \chi_{ijkl} \quad (13)$$

with the principal, skewon, and axion pieces defined by

$$\begin{align*}
(1) \chi_{ijkl} &= \frac{1}{2} \epsilon_{ijmn} (1) \kappa_{mn \ kl}, \\
(2) \chi_{ijkl} &= \frac{1}{2} \epsilon_{ijmn} (2) \kappa_{mn \ kl} = -\epsilon_{ijm[k} \kappa_{\ell]n l}, \\
(3) \chi_{ijkl} &= \frac{1}{2} \epsilon_{ijmn} (3) \kappa_{mn \ kl} = \frac{1}{12} \epsilon^{ijkl} \kappa.
\end{align*} \quad (14, 15, 16)$$

Using the S-identity and the K-identity derived in [23], we can verify that $(2)\chi$ is skew-symmetric under the exchange of the first and the second index pair, whereas $(1)\chi$ is symmetric:

$$\begin{align*}
(2) \chi_{ijkl} &= - (2) \chi_{klij}, \\
(1) \chi_{ijkl} &= (1) \chi_{klij}.
\end{align*} \quad (17)$$

**Space-time decomposed constitutive relation**

Making a $(1 + 3)$-decomposition [13] of covariant electrodynamics, we can write $H$ and $F$ as column 6-vectors with the components built from the magnetic and electric excitation 3-vectors $H_a, D^a$ and electric and magnetic field strengths $E_a, B^a$, respectively. Then the linear spacetime relation [3] reads:

$$\begin{pmatrix}
H_a \\
D^a
\end{pmatrix} =
\begin{pmatrix}
C^b_a & B_{ba} \\
A_{ba} & D^b_a
\end{pmatrix}
\begin{pmatrix}
-E_b \\
B^b
\end{pmatrix}.
\quad (18)$$

Here the constitutive tensor is conveniently represented by the 6-matrix

$$\kappa_I^K = \begin{pmatrix} C^b_a & B_{ba} \\ A_{ba} & D^b_a \end{pmatrix}, \quad \chi^{IK} = \begin{pmatrix} B_{ab} & D^b_a \\ C^a_b & A^{ab} \end{pmatrix}. \quad (19)$$
The constitutive $3 \times 3$ matrices $A, B, C, D$ are constructed from the components of the original constitutive tensor as

$$A_{ba} := \chi^{0a0b}, \quad B_{ba} := \frac{1}{4} \hat{e}_{acd} \hat{e}_{bef} \chi^{cdef},$$

$$C^{a}{}_{b} := \frac{1}{2} \hat{e}_{bcd} \chi^{cd}{}^{0a}, \quad D_{a}{}^{b} := \frac{1}{2} \hat{e}_{acd} \chi^{0bcd}. \quad (21)$$

If we resolve with respect to $\chi$, we find the inverse formulas

$$\chi^{000b} = A^{ba}, \quad \chi^{abcd} = \epsilon^{a}{}^{be}{}^{ef} B_{fe},$$

$$\chi^{0abc} = \epsilon^{bcd} D_{a}{}^{a}, \quad \chi^{ab0c} = \epsilon^{abld} C_{d}^{c}. \quad (23)$$

The contributions of the principal, the skewon, and the axion parts to the above constitutive $3$-matrices can be written explicitly as

$$A^{ab} = -\epsilon^{ab} - \epsilon^{abc} \mathcal{S}^{0}{}_{c},$$

$$B_{ab} = \mu^{-1}{}_{ab} + \hat{e}_{abc} \mathcal{S}^{0}{}_{c},$$

$$C^{a}{}_{b} = \gamma^{a}{}_{b} - (\mathcal{S}^{a}{}_{b} - \delta^{a}{}_{b} \mathcal{S}^{c}{}_{c}) + \alpha \delta^{a}{}_{b},$$

$$D_{a}{}^{b} = \gamma^{b}{}_{a} + (\mathcal{S}^{b}{}_{a} - \delta^{b}{}_{a} \mathcal{S}^{c}{}_{c}) + \alpha \delta^{b}{}_{a}. \quad (27)$$

The set of the symmetric matrices $\epsilon^{ab} = \epsilon^{ba}$ and $\mu^{-1}{}_{ab} = \mu^{-1}{}_{ba}$ together with the traceless matrix $\gamma^{a}{}_{b}$ (i.e., $\gamma^{c}{}_{c} = 0$) comprise the principal part $(1)\chi^{ijkl}$ of the constitutive tensor. Usually, $\epsilon^{ab}$ is called permittivity tensor and $\mu^{-1}{}_{ab}$ reciprocal permeability tensor ("impermeability" tensor), since they describe the polarization and the magnetization of a medium, respectively. The magnetolectric cross-term $\gamma^{a}{}_{b}$ is related to the Fresnel-Fizeau effects. The skewon contributions in (24) and (25) are responsible for the electric and magnetic Faraday effects, respectively, whereas skewon terms in (26) and (27) describe optical activity.

**GENERAL FRESNEL EQUATIONS: WAVE AND RAY SURFACES**

Here we briefly summarize the results of previous work [6, 22, 26, 27]. In the Hadamard approach, one studies the propagation of a discontinuity in the first derivative of the electromagnetic field. The basic notions are then the fields of the wave covector and the ray vector that encode the information about the propagation of a wave in a spacetime with a general constitutive relation.
Wave surface

The crucial observation about the surface of discontinuity $S$ (defined locally by a function $\Phi$ such that $\Phi = const$ on $S$) is that across $S$ the geometric Hadamard conditions are satisfied for the components of the electromagnetic field and their derivatives: $[F_{ij}] = 0$, $[\partial_i F_{jk}] = q_i f_{jk}$, $[H_{ij}] = 0$, $[\partial_i H_{jk}] = q_i h_{jk}$. Here $q_i := \partial_i \Phi$ is the wave covector. Then using the Maxwell equations (1) and the constitutive law (3), we find a system of algebraic equations for the jump functions:

$$\chi^{ijkl} q_j f_{kl} = 0, \quad \epsilon^{ijkl} q_j f_{kl} = 0. \quad (28)$$

Solving the last equation in (28) by means of $f_{ij} = q_ia_j - q_ja_i$, we finally reduce (28) to $\chi^{ijkl} q_j q_k a_l = 0$. This algebraic system has a nontrivial solution for $a_i$ only if the determinant of the matrix on the left hand side vanishes. The latter gives rise to our generalized covariant Fresnel equation

$$G^{ijkl}(\chi) q_i q_j q_k q_l = 0, \quad (29)$$

with the fourth order Tamm-Rubilar (TR) tensor density of weight +1 defined by

$$G^{ijkl}(\chi) := \frac{1}{4!} \hat{\epsilon}_{mnpq} \hat{\epsilon}_{rstu} \chi^{mnr} \chi^{ips} \chi^{qkn} \chi^{ltu}. \quad (30)$$

It is totally symmetric, $G^{ijkl}(\chi) = G^{ijkl}(\chi)$, and thus has 35 independent components.

Different irreducible parts of the constitutive tensor (8) contribute differently to the general Fresnel equation. A straightforward analysis [23, 24] shows that the axion piece drops out completely from the TR-tensor, whereas the two remaining irreducible parts of the constitutive tensor contributes to (30) as follows:

$$G^{ijkl}(\chi) = G^{ijkl}(\chi) + (1)^{m(n|j} g^{k} s_{m} s_{n}^{l)}. \quad (31)$$

Ray surface

A ray can be defined by a vector field $s = s^i \partial_i$ that is dual to the wave covector $q = q_i dx^i$ and to the wave front in the following sense [27, 28]:

$$s|h = 0, \quad s|f = 0, \quad s|q = 0. \quad (32)$$

Here the 2-forms $h = \frac{1}{2} h_{ij} dx^i \wedge dx^j$ and $f = \frac{1}{2} f_{ij} dx^i \wedge dx^j$ represent the jumps of the first derivatives of the electromagnetic field. Equations (32) are metric-free. The system of the
first two equations can be analysed in complete analogy to the system (28). As a result, one arrives at the “dual” Fresnel equation imposed on the components $s^i$ of the ray vector:

$$
\hat{G}_{ijkl} s^i s^j s^k s^l = 0.
$$

(33)

Like the TR-tensor density (30), the totally symmetric tensor density of weight $-1$ is constructed in terms of the components of the constitutive tensor:

$$
\hat{\chi}_{ijkl}(\chi) := \frac{1}{4!} \hat{\epsilon}^{pqmn} \hat{\epsilon}^{rstu} \hat{\chi}_{mnr}(i) \hat{\chi}_{jps}(k) \hat{\chi}_{lti} qtu
\begin{equation}
= \frac{1}{4 \cdot 4!} \chi^{pqmn} \hat{\epsilon}^{mnr}(i) \hat{\epsilon}^{jps}(k) \hat{\epsilon}^{tui}(l) \chi^{turs}.
\end{equation}

(34)

As before, a complete symmetrization has to be performed over the four indices $i, j, k, l$, with the vertical lines separating out those indices that are excluded from the symmetrization. Here we introduced a “double dual” of the constitutive tensor density as

$$
\hat{\chi}_{ijkl} = \frac{1}{4} \hat{\epsilon}^{ijkl} \chi^{mnij}.
$$

(35)

The contraction of this new object with the original constitutive tensor yields

$$
\frac{1}{2} \hat{\chi}_{klmn} \chi^{mnij} = \frac{1}{2} \kappa_{kl}^{nn} \kappa_{mn}^{ij}.
$$

(36)

Using (13), we find the corresponding decomposition

$$
\hat{\chi}_{ijkl} = (1) \hat{\chi}_{ijkl} - 2 \hat{\chi}_{jkm[n} \hat{\varphi}_{l]}^m + \hat{\chi}_{ijkl} \alpha,
$$

(37)

Substituting this into (34), we arrive at the skewon contribution to the ray surface:

$$
\hat{G}_{ijkl}(\chi) = \hat{G}_{ijkl}^{(1)}(\chi) + (1) \hat{\chi}_{m(nij} \hat{\varphi}_{k]l}^m \hat{\varphi}_{l}^n.
$$

(38)

SKEWON FIELD: DIFFERENT PARAMETRIZATIONS AND WAVE PROPAGATION EFFECTS IN VACUUM

The main aim of this paper is to investigate the possible influence of the skewon field $\varphi_{i}^j$ on the electromagnetic wave propagation. For simplicity, we assume that the principal part of the constitutive tensor is determined by the Maxwell-Lorentz law (11), that is,

$$
(1) \chi^{mnij} = \lambda_0 \sqrt{-g} \left( g^{mni} g^{j} - g^{mj} g^{ni} \right), \quad (1) \hat{\chi}_{ij} = - \frac{\lambda_0}{\sqrt{-g}} \left( g_{mn} g_{ji} - g_{mj} g_{ni} \right).
$$

(39)

However, the skewon structure will be kept as general as possible. As a preliminary step, we note that direct computations yield

$$
\hat{G}_{ijkl}^{(1)}(\chi) = - \lambda_0^3 \sqrt{-g} g^{(ij} g^{kl)}, \quad \hat{G}_{ijkl}(\chi) = \frac{\lambda_0^3}{\sqrt{-g}} g^{(ij} g^{kl)}.
$$

(40)
‘Double vector’ parametrization

The skewon, as a traceless tensor field of type \([1,1]\), has 15 independent components. Given the spacetimes metric \(g_{ij}\), we can decompose the skewon field into an antisymmetric and a symmetric part. The latter can be conveniently constructed from a pair of arbitrary (co)vector fields \(v^i\) and \(w_i\), so that we find eventually

\[
\delta_{i}^{j} = a_{i}^{j} + w_{i}v^{j} - \frac{1}{4} \delta_{i}^{j} (w v).
\]  

Here \(a_{i}^{j} = a_{i}^{k} g_{kj} = -a_{ji}\) is an arbitrary antisymmetric tensor and \((w v) = w_{i}v^{i}\). Such a parametrization produces almost a general skewon field, since the number of independent components is here \(14 = 6(a_{ij}) + 4(v^{i}) + 4(w_{i})\).

Substituting (41) into (31) and (38), and taking into account (39), we find

\[
(1) \chi_{m}^{(i)} |_{n}^{j} \delta_{k}^{m} \delta_{l}^{n} = \lambda_{0} \sqrt{-g} \left[ \left( a_{m}^{(i} a_{m}^{j)} + 2v^{(i} b^{j)} + w^{2} v^{(i} v^{j)} g^{kl} - v^{(i} v^{j} w^{k} w^{l}) \right) \right], \tag{42}
\]

\[
(1) \hat{\chi}_{m}^{(i) |n} \delta_{k}^{m} \delta_{l}^{n} = - \lambda_{0} \sqrt{-g} \left[ \left( a_{m}^{(i} a_{m}^{j)} + 2w_{i} \hat{b}_{j} + v^{2} w_{i} v_{j} g_{kl} - v_{i} v_{j} w_{k} w_{l} \right) \right]. \tag{43}
\]

Here \(b^{i} := w^{j} a_{j}^{i}, \hat{b}_{i} := a_{i}^{j} v_{j}\) and \(w^{2} = w_{i} w^{i}, v^{2} = v_{i} v^{i}\).

Adding the principal terms (40) to the contributions of the skewon (42) and (43), we see that neither the wave surface nor the ray surface decompose into the product of cones in general. In other words, birefringence is absent. The last terms in (42) and in (43) are responsible for that. Let us investigate those particular cases of the skewon field in the ‘double vector’ representation (41) which do admit the birefringence phenomenon.

Purely antisymmetric skewon

When the skewon field is represented by its antisymmetric part only, i.e., \(\delta_{i}^{j} = a_{i}^{j}\) (which is obtained by putting both \(v^{i} = w_{i} = 0\)), then the analysis of (31) shows that the light propagation is birefringent with one light cone defined by the spacetime metric \(g^{ij}\) and the second “optical” metric

\[
g^{ij}_{(2)} = g^{ij} - \frac{1}{\lambda_{0}^{2}} a_{m}^{mi} a_{m}^{j}. \tag{44}
\]

This result actually holds true for any other parametrizations of the skewon field which will be considered later. At the same time, inspection of (38) yields the decomposition of the
ray surface into the light cone of the spacetime metric $g_{ij}$ and a second cone defined by

$$g_{ij}^{(2)} = g_{ij} - \frac{1}{\lambda_0} a_{mi} a^{mjj}. \quad (45)$$

Although (44) and (45) look very similar, we notice that they are not mutual inverse, and hence, they do not determine a unique geometric structure on spacetime.

**Purely symmetric ‘vector’ skewon**

Assume now that the skew-symmetric skewon part is absent, which is evidently achieved if $a_{ij} = v_i w_j$. Then $b^i = (1/2)[w^i(vw) - v^i w^2]$, $\hat{b}_i = (1/2)[v_i(vw) - w_i v^2]$, and we find the TR-tensor in the birefringent form

$$G^{ijkl} = -\lambda_0^2 \sqrt{-g} g_{ij}^{(45)} g_{kl}^{(45)}. \quad (46)$$

Here the two light cones are defined by the two optical metrics

$$g_{ij}^{\pm} = g_{ij} - \frac{1}{2\lambda_0^2} \left[ (vw) \pm \sqrt{(vw)^2 - 4\lambda_0^2} \right]. \quad (47)$$

Here $(vw) = v_i w^i$. Obviously, in order to have the wave propagation along separate light cones, the skewon field must satisfy the condition

$$(vw)^2 \geq 4\lambda_0^2. \quad (48)$$

When this condition is violated, the Fresnel wave surface is of 4th order and birefringence is absent. For the equality sign in (48), the birefringence disappears and the two light cones coincide. However, one can straightforwardly prove that for $(vw)^2 = 4\lambda_0^2$ the optical metric has an Euclidean signature. Hence there is no wave propagation in this case. The type of the signature correlates with the sign of the determinant of the metric: the latter is negative for the Lorentzian case and positive for the Euclidean one.

A direct computation yields

$$\det \left( g^{ij} - kv^{(i} w^{j)} \right) = (\det g^{ij}) \left\{ 1 - k (vw) + \frac{k^2}{4} \left[ (vw)^2 - v^2 w^2 \right] \right\}. \quad (49)$$

Comparing this with (47), we identify the coefficient as $k_\pm = \left( (vw) \pm \sqrt{(vw)^2 - 4\lambda_0^2} \right) / 2\lambda_0^2$ and finally obtain

$$(\det g^{ij}_\pm) = (\det g^{ij}) \frac{k^2}{4} \left[ (vw)^2 - v^2 w^2 - 4\lambda_0^2 \right]. \quad (50)$$
Since the spacetime metric is Lorentzian, its determinant is negative, \( (\det g^{ij}) < 0 \). As a result, we conclude that the optical metrics will be also Lorentzian only if

\[ (vw)^2 - v^2 w^2 > 4\lambda_0^2. \tag{51} \]

Note that the two conditions (51) and (48) look pretty similar. In fact, we can verify that only one of them is an essential inequality. Indeed, when the two vectors are either both timelike \( (v^2 > 0, w^2 > 0) \) or both spacelike \( (v^2 < 0, w^2 < 0) \), the new condition (51) makes the constraint (48) redundant, whereas when at least one of the vectors is null, (51) and (48) coincide. Finally, when one of the vectors is timelike and another spacelike, the only essential condition is (48) and (51) is redundant.

With these conditions, we can classify the effects of the skewon field on wave propagation. The spatially isotropic skewon field of Nieves & Pal \[15, 16\], for example, belongs to the subcase when the two vectors \( w \) and \( v \) are proportional to each other. Indeed, let us consider the (flat) Minkowski spacetime with the Lorentzian metric \( g_{ij} = \text{diag}(c^2, -1, -1, -1) \), and let us put \( w^i = -v^i = \delta^i_0 \sqrt{2S/c} \) with some scalar function \( S \) of dimension \( [\lambda_0] \). Then, from (41), we recover the isotropic skewon field

\[ S^i^j = \frac{S}{2} \begin{pmatrix} -3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{52} \]

As a result, the spacetime relation becomes

\[ \mathcal{D}^a = \varepsilon_0 \delta^{ab} E_b + (-S + \alpha) B^a, \tag{53} \]
\[ \mathcal{H}_a = (-S - \alpha) E_a + \mu_0^{-1} \delta_{ab} B^b. \tag{54} \]

Accordingly, in the special case when skewon and axion become \textit{constant} fields, one can speak of the 4 electromagnetic constants for a vacuum spacetime with spatial isotropy: The electric constant \( \varepsilon_0 \), the magnetic constant \( \mu_0 \), the spatially isotropic part \( S \) of the skewon \( S^i^j \), and the axion \( \alpha \).

In this special case we evidently find \( (vw)^2 - v^2 w^2 = 0 \). Hence (50) yields that both optical metrics have a wrong Euclidean signature. This is in complete agreement with our earlier observations \[23\] in which the absence of wave propagation was reported on the background of the isotropic skewon field of Nieves & Pal.
However, now we obviously obtain a far more general result: whereas Nieves & Pal have considered only a very special configuration with the spacelike vectors $v$ and $w$ being proportional to each other, the wave propagation is actually also absent for any $w^i \sim v^i$ and any Riemannian spacetime metric $g_{ij}$. Then again $(vw)^2 - v^2w^2 = 0$ and hence $(\text{det } g^{ij}) = -(\text{det } g^{ij})\lambda_0^2 k_\pm^2 > 0$.

**General parametrization**

It is impossible to construct a general parametrization of the skewon field by using covariant 4-dimensional objects. Instead, we have to switch to the space and time decomposition techniques and to allow for the use of 3-dimensional (3D) tensor objects. Then we can write the required parametrization as follows:

$$S^i_j = \left( \begin{array}{l} -s^c_e m^a_b \\ n_b \\ s^a_b \end{array} \right).$$

Here $m^a$ and $n_b$ are 3D (co)vector fields, whereas $s^a_b$ is a 3D tensor of type $[1]$. From now on, the indices from the beginning of the Latin alphabet denote the spatial 3D components, i.e., $a, b, c, \ldots = 1, 2, 3$. The count of the independent components, namely $3 + 3 + 9 = 15$, shows that we indeed have the most general representation of the skewon field. It is straightforward to establish the dimensions of the different pieces of skewon field: We find that $[s^a_b] = [\lambda_0]$, $[n_b] = [\epsilon_0]$ and $[m^a] = 1/|\mu_0|$.

With this general parametrization, the linear spacetime relation (18) can be written symbolically as follows:

$$\left( \begin{array}{l} H_a \\ D^a \end{array} \right) = \kappa \left( \begin{array}{l} -E_b \\ B^b \end{array} \right).$$

Here the linear operator on the right-hand side explicitly reads:

$$\kappa := \left( \begin{array}{cc} \gamma^b_a & \mu^{-1}_{ab} \\ -\epsilon^{ab} & \gamma^a_b \end{array} \right) + \left( \begin{array}{cc} -s^b_a + \delta^b_a s^c_e & -\hat{\epsilon}_{abc} m^c_e \\ \epsilon^{abc} n_c & s^a_b - \delta^a_b s^c_e \end{array} \right) + \alpha \left( \begin{array}{l} \delta^b_a \\ 0 \\ \delta^a_b \end{array} \right).$$

Now we again specialize to the Maxwell-Lorentz principal part (39). Moreover, since we are specifically interested in the skewon effects, we will assume for concreteness that the spacetime metric is the flat Minkowskian one with Lorentzian signature, $g_{ij} = (c^2, -\delta_{ab}) = \text{diag}(c^2, -1, -1, -1)$. The Euclidean 3-metric tensor $\delta_{ab}$ should not be confused with the
The components of the latter are always either 0 or 1, whereas the former has these values in Cartesian coordinates only. Then one can further decompose the 3D skewon tensor into its symmetric and skewsymmetric parts:

\[ s^a_b = u^a_b + \epsilon^{ac}_{\ b} z_c. \] (58)

Here \( u_{ab} = u_{ba} \), and the 3D indices are raised and lowered by means of the Euclidean 3D metric, i.e., \( u^a_b =: u_{bc} \delta^a_c \) and \( \epsilon^{abc}_{\ de} := \delta^{ad} \delta^e_{\ de} \). The covector field \( z_c \) describes the antisymmetric part of the 3D tensor skewon. Obviously, both irreducible parts of the tensor skewon have the same dimension \([u_{ab}] = [z_c] = [\lambda_0] \).

**Space-time decomposition and wave and ray surfaces**

The analysis of the wave propagation in the presence of the general skewon field (55) is not possible in the above covariant 4-dimensional framework. Since the space and time components are explicitly separated in the general parametrization (55), we need a corresponding approach to the wave and ray surfaces in which a space and time decomposition has been performed.

Namely, let us denote the independent components of the TR-tensor (30) as follows:

\[ M := G^{0000} = \det A, \] (59)

\[ M^a := 4 G^{0000a} = -\epsilon_{bcd} (A^b_a A^{ce} C^d_e + A^{ab} A^{cc} D^d_e), \] (60)

\[ M^{ab} := 6 G^{000ab} = \frac{1}{2} A^{(ab)} \left[ \left( C^d_c \right)^2 + \left( D^c_d \right)^2 - \left( C^d_c + D^c_d \right)^2 \right] \]

\[ + \left( C^d_c + D^c_d \right) (A^{(a} C^{b)} + D^{(a} A^{b)} - C^d_c A^{(a} C^{b)}_d - C^d_c A^{(a} C^{b)}_{cd} - C^d_c A^{(a} C^{b)}_{cd} - C^d_c A^{(a} C^{b)}_{cd}) \]

\[ - D^{(a} A^{b)c} C^d_d - A^{dc} C^{(a} D^{b)} + (A^{(ab)} A^{dc} - A^{d(a} A^{b)c}) B_{dc}, \] (61)

\[ M^{abc} := 4 G^{000abc} = \epsilon^{de(c)} \left[ B_{df} (A^{ab} D^{d}_e - D^{d}_e A^{b)df}) \right. \]

\[ + B_{fd}(A^{ab}) C^f_e - A^{f(e} C^{b)} + C^a_f D^{a} D^f_d + D^a_f C^{b}) C^f_d \], (62)

\[ M^{abcd} := G^{abcd} = \epsilon^{ef(c} \epsilon^{gh)d} B_{h} \left[ \frac{1}{2} A^{ab} B_{ge} - C^{a}_e D^{b}_g \right]. \] (63)

Then, in decomposed form, the generalized Fresnel equation (29) reads

\[ q_0^4 M + q_0^3 q_a M^a + q_0^2 q_a q_b M^{ab} + q_0 q_a q_b q_c M^{abc} + q_a q_b q_c q_d M^{abcd} = 0. \] (64)

Similarly to (53)-(53), we can decompose the 4-dimensional tensor density (34) into a set
FIG. 1: Fresnel surface for a real skewon of the electric Faraday type. It has the form of a toroid (depicted with a cut). Here $\overline{n}_a = \overline{n} \delta_a^3$ with $\overline{n}^2 = 10$; we use the dimensionless variables $x := cq_1/q_0$, $y := cq_2/q_0$, $z := cq_3/q_0$.

of 3-dimensional objects:

\[
\hat{M} := \hat{G}_{0000} = \det \mathcal{B},
\]

\[
\hat{M}_a := 4 \hat{G}_{000a} = -\epsilon^{bcd} (B_{ba} B_{ce} D_d e + B_{ab} B_{ce} C_d^e) ,
\]

\[
\hat{M}_{ab} := 6 \hat{G}_{00ab} = \frac{1}{2} B_{(ab)} \left[ (C_d^c)^2 + (D_c^d)^2 - (C_d^c + D_d^c)(C_d^e + D_d^e) \right]
\]

\[
+ (C_d^c + D_d^c)(B_{c(a} D_b^d) + C_d^c B_{b)c}) - D_d^e B_{c(a} D_{b)c}^e
\]

\[
- C^c_{(a} B_{b)c} C^d_d - B_{de} D_{(a} C^d_{b)} + \left( B_{(ab)} B_{dc} - B_{d(a} B_{b)c} \right) A^{dc}
\]

\[
\hat{M}_{abc} := 4 \hat{G}_{0abc} = \hat{e}_{de(c} \left[ A^{df} (B_{ab} C_f^c - C_f^a B_{b)f}) \right]
\]

\[
+ A^{f[1}(B_{ab} D_f^c - B_{f[a} D_{b)]^c) + D_{a}^{f} C_{c]^{b} C_{d}^{f} + C_{a}^{f} D_{b)^{e} D_{f]^{d}}}
\]

\[
\hat{M}_{abcd} := \hat{G}_{abcd} = \hat{e}_{ef(c} \hat{e}_{gh)d} A^{hf} \left[ \frac{1}{2} B_{ab} A^{ge} - D_{a}^{e} C_{b}^{g} \right].
\]

Then the Fresnel type equation for the quartic ray surface in decomposed form reads

\[
(s^0)^4 \hat{M} + (s^0)^3 s^a \hat{M}_a + (s^0)^2 s^a s^b \hat{M}_{ab} + s^0 s^a s^b s^c \hat{M}_{abc} + s^a s^b s^c s^d \hat{M}_{abcd} = 0.
\]

Now we are in a position to study the effects induced by the different irreducible parts of the skewon field.

**Skewonic electric Faraday effect**

When only the $n_a$ part of the skewon is nontrivial, whereas $m_a = 0$ and $s_b^a = 0$, the electric constitutive relation reads $D^a = \varepsilon_0 E^a + \epsilon^{abc} n_b E_c$. We will call this the skewonic electric Faraday effect, since it describes the electric Faraday effect in optically active media.
FIG. 2: Fresnel surface for a skewon of the electric Faraday type. It has the form of a spheroid for the large purely imaginary skewon \( n \). Here \( \pi_a = \pi \delta_a^3 \) with \( \pi^2 = -10 \); we use the dimensionless variables \( x := cq_1/q_0, y := cq_2/q_0, z := cq_3/q_0 \).

for purely imaginary \( n_a \). Then, from (59)-(63), we find the coefficients of the Fresnel equation as

\[
M_a = 0, \quad M_{abc} = 0,
\]

whereas the nontrivial components read

\[
M^a = -\varepsilon_0 \left( \varepsilon_0^2 + n^2 \right),
\]

\[
M^{abcd} = -\frac{1}{\mu_0} \lambda_0^2 \delta^{(ab)} \delta^{cd},
\]

\[
M^{ab} = \varepsilon_0 \left[ 2\lambda_0^2 \delta^{ab} + c^2 \left( \delta^{ab} n^2 - n^a n^b \right) \right].
\]

(71)

(72)

Here, as usual, \( n^n = \delta^{ab} n_b \) and \( n^2 = n^a n_a \).

On the other hand, according to (65)-(69), the analysis of the ray surface yields \( \hat{M}_a = 0 \), \( \hat{M}_{abc} = 0 \), and

\[
\hat{M} = \frac{1}{\mu_0}, \quad \hat{M}_{ab} = -\frac{2\lambda_0^2}{\mu_0} \delta_{ab}, \quad \hat{M}_{abcd} = \varepsilon_0 \delta_{(ab)} \left( \lambda_0^2 \delta_{cd} + c^2 n_c n_d \right).
\]

(73)

As a result, we obtain the following Fresnel surfaces for the wave covectors and the ray vectors, respectively:

\[
(q_0/c)^4 (1 + \pi^2) - (q_0/c)^2 \left[ 2q^2 + \pi^2 q^2 - (\pi q)^2 \right] + (q^2)^2 = 0,
\]

(74)

\[
(s_0 c)^4 - 2(s_0 c)^2 s^2 + s^2 \left[ s^2 + (\pi s)^2 \right] = 0.
\]

(75)

Here we denote \( q^2 = q_a q^a \), \( s^2 = s_a s^a \), \( (\pi q) = \pi_a q^a \), \( (\pi s) = \pi_a s^a \). Moreover, we use the dimensionless skewon vector \( \pi_a := n_a/\varepsilon_0 \).

With the help of the nontrivial vector \( \pi \), we can split the wave covector \( q_a = q_a^\perp + q_a^\parallel \) into its transversal and longitudinal parts, with \( q_a^\perp = q_a - \pi^{-2} \pi_a \pi^\jmath q_\jmath \) and \( q_a^\parallel = \pi^{-2} \pi_a \pi^\jmath q_\jmath \). Then the Fresnel equation (74) is rewritten in the form

\[
\left[ (q_0/c)^2 - q_\perp^2 \right] \left[ (q_0/c)^2 (1 + \pi^2) - q_\perp^2 \right] + q_\parallel^2 \left[ -2(q_0/c)^2 + 2q_\perp^2 + q_\parallel^2 \right] = 0.
\]

(76)
Accordingly, in the plane transversal to the skewon vector $n_a$, we find birefringence with the Minkowski metric and the second optical metric $g^{ij} = \text{diag}((1 + \pi^2)/c^2, -1, -1, -1)$.

However, the Fresnel surface does not decompose into a product of the two light cones. Although, for $n^2 = 0$, the wave covector surface is a sphere, it becomes a toroidal already for a small positive $n^2 > 0$. The latter looks like a very thin and narrow belt for small values of $n^2$. However, it becomes a thick toroid with the inner radius decreasing to zero for large values of $n^2$. A typical Fresnel surface for this case is depicted in Fig. [1]. Physically, the hole of this toroid corresponds to the directions in which waves cannot propagate: In spherical coordinates, the wave propagation occurs for $\theta_0 \leq \theta \leq \pi - \theta_0$, where the spherical angle $\theta$ is counted from the direction of the skewon vector $n_a$ and the limiting angle is determined from $\sin^2 \theta_0 = 2/(1 + \sqrt{1 + n^2})$. As we see, the hole disappears when $n^2 \to \infty$.

This qualitative characterization of the wave propagation can be completed by a numerical estimate for the velocity of light. The phase velocity $v$ of the wave propagation is defined by the components of the wave covector according to $q_a = q_0 k_a/v$, where $k_a$ are the components of the 3D unit covector. Usually, the mean value of the velocity is of interest that is obtained after averaging over the directions of propagation and the polarizations. For the case under consideration, a direct calculation yields

$$<v^2> = c^2 \left( \frac{1 + \pi^2/3}{1 + \pi^2} \right).$$

(77)

Thus, since $\pi^2$ is positive, the velocity of the wave propagation turns out to be less than $c$. Making use of the data on the gamma-ray bursts and following [3], we derive as an upper limit for the skewonic electric Faraday effect $n^2 < 3 \times 10^{-27} \varepsilon_0^2$.

The existence of a hole in the wave covector surface confirms our earlier conclusion about the dissipative nature of the skewon. In particular, when we continue to the purely imaginary values of $n$ so that the square becomes negative, $n^2 < 0$, the topology of the Fresnel surface changes drastically. The hole disappears and the wave covector surface assumes the form of a spheroid around the origin. More exactly, there is a single spheroid elongated in the direction of the skewon vector $n_a$ for $\pi^2 \geq -1$, and two (embedded one into another) spheroids when $-1 < \pi^2 < 0$. The average radius of the inner spheroid in the latter case is smaller than 1 which, in physical terms, means that the velocity of wave propagation is greater than $c$. Although the negative $n^2 < 0$ corresponds to the well known electric Faraday effect in crystals, such a superluminal behavior of waves for $-1 < \pi^2 < 0$ was never reported in the
FIG. 3: Fresnel surface for a skewon of the electric Faraday type. It has the form of two concentric spheroids (depicted here with cuts) for a small purely imaginary skewon $n$. Here $\pi_a = \pi \delta_a^3$ with $\pi^2 = -0.5$; we use the dimensionless variables $x := cq_1/q_0$, $y := cq_2/q_0$, $z := cq_3/q_0$.

literature, to the best of our knowledge. In Fig. 2 and Fig. 3, we depict typical Fresnel wave covector surfaces for $n^2 = -10$ and $n^2 = -0.5$, respectively. The wave surface for $n^2 = -1$ looks qualitatively like Fig. 2 with the top and the bottom of the spheroid at $z = \pm 1$.

**Skewonic magnetic Faraday effect**

In the opposite case, when the $m^a$ part of the skewon is nontrivial, whereas $n_a = 0$ and $s_b = 0$, the magnetic constitutive relation reads $H_a = \mu_0^{-1} B_a + \epsilon_{abc} m^b B^c$. We call this case the skewonic magnetic Faraday effect, since it corresponds to the magnetic Faraday effect when $m^a$ is purely imaginary [12, 20, 32]. From (59)-(63), in full analogy to the previous subsection, we find that $M^a = 0, M^{abc} = 0$, whereas the nontrivial components are

$$M = -\varepsilon_0^3, \quad M^{ab} = 2\varepsilon_0^3 \lambda_0^0 \delta^{ab}, \quad M^{abcd} = -\frac{1}{\mu_0} \delta^{(ab}(\lambda_0^0 \delta^{cd)} + c^2 m^c m^d).$$

(78)

Analogously, from (65)-(69), we find for the ray surface that $\widehat{M}_a = 0, \widehat{M}_{abc} = 0$, and

$$\widehat{M} = \frac{1}{\mu_0} \left( \frac{1}{\mu_0^0} + m^2 \right), \quad \widehat{M}_{abcd} = \varepsilon_0^0 \lambda_0^0 \delta_{(ab} \delta_{cd)},$$

$$\widehat{M}_{ab} = -\frac{1}{\mu_0} \left[ 2\lambda_0^0 \delta_{ab} + c^{-2} \left( \delta_{ab} m^2 - m_a m_b \right) \right].$$

(79)

(80)

Here again $m_a = \delta_{ab} m^b$ and $m^2 = m^a m_a$. If, similarly to the previous subsection, we introduce the dimensionless skewon vector by $m^a := \sqrt{\mu} m_a$, we end up with the following Fresnel wave and ray surfaces, respectively:

$$(q_0/c)^4 - 2(q_0/c)^2 q^2 + q^2 \left[ q^2 + (mq)^2 \right] = 0,$$

(81)
FIG. 4: Fresnel surface for a skewon of the magnetic Faraday type. It has two branches that are both hyperboloids for the large purely imaginary skewon \( m \) (depicted with a cut into half). Here \( \overline{\pi}^a = \overline{m} \delta_a^3 \) with \( \overline{m}^2 = -10 \); the dimensionless variables are \( x := c q_1 / q_0 \), \( y := c q_2 / q_0 \), \( z := c q_3 / q_0 \).

\[
(s_0 c)^4 (1 + \overline{m}^2) - (s_0 c)^2 \left[ 2 s^2 + \overline{m}^2 s^2 - (\overline{m} s)^2 \right] + (s^2)^2 = 0. \tag{82}
\]

Similarly as before, \( \overline{mq} = \overline{m}_a q^a \) and \( \overline{ms} = \overline{m}_a s^a \). Repeating the calculation of the velocity of the wave propagation for this case, we find the mean velocity \( < v^2 > = c^2 \). As a result, the analysis of the gamma-ray bursts data does not impose any limit on the pure skewon field of the magnetic Faraday type.

As one can see immediately, the wave covectors are only trivial in the plane transverse to the direction of the skewon vector, i.e., when \( \overline{mq} = 0 \). The waves propagate in that plane without birefringence along the light cone defined by the standard Minkowski metric, since then \( [(q_0 / c)^2 - q^2]^2 = 0 \) is fulfilled. In contrast, a purely imaginary skewon vector, with \( m^2 < 0 \), yields more interesting results. In this case, the mentioned circle represents a particular configuration that turns out to be the intersection curve of the two branches of the Fresnel surface. The typical Fresnel wave covector surfaces for the cases \( \overline{m}^2 = -10 \) and \( \overline{m}^2 = -0.5 \) are depicted in Fig. 4 and Fig. 5, respectively: Both branches are hyperboloids or spheroids, intersecting in the \( z = 0 \) plane. The wave surface for \( \overline{m}^2 = -1 \) is qualitatively different: It is represented by paraboloids as shown in Fig. 6.

Comparison of the results of the two last subsections demonstrates that the electric and magnetic Faraday skewon effects are dual to each other in the following sense: When we interchange the wave and the ray vectors \( q \leftrightarrow s \) and simultaneously the skewon vectors \( \overline{\pi} \leftrightarrow \overline{m} \), then the wave surface \( (74) \) has the same form as the ray surface \( (82) \) and vice versa: The ray surface \( (75) \) has the same form as the wave surface \( (81) \). In this sense, Figs. 4, 6 show the ray (wave) surfaces dual to the respective wave (ray) surfaces depicted
FIG. 5: Fresnel surface for a skewon of the magnetic Faraday type. It has two branches that are both spheroids for a small purely imaginary skewon \( m \) (depicted with a cut into half). Here \( \overline{\pi}^a = \overline{\pi} \delta^a_3 \) with \( \overline{\pi}^2 = -0.5 \); the dimensionless variables are \( x := cq_1/q_0 \), \( y := cq_2/q_0 \), \( z := cq_3/q_0 \).

in Figs. 1 and 2 with the mentioned replacement of \( n \leftrightarrow m \) and with the interchange between the dimensionless wave covector variables \( x = cq_1/q_0 \), \( y = cq_2/q_0 \), \( z = cq_3/q_0 \) and the dimensionless ray vector variables \( x = s^1/s^0c \), \( y = s^2/s^0c \), \( z = s^3/s^0c \).

Skewonic magneto-electric optical activity

Let us now assume that only the tensor part of the skewon is present in (55), whereas \( m^a = 0 \) and \( n^a = 0 \). We call this a skewon of the magneto-electric type, since it corresponds to the case of the natural optical activity in matter for the purely imaginary \( s^a_b \), see \([12, 20, 33, 34]\).

Furthermore, we consider the effects of the symmetric and skew-symmetric parts (58). In the absence of the skew-symmetric tensor part (\( z_c = 0 \)), we find \( M^a = 0 \), \( M^{abc} = 0 \) and

\[
M = -\varepsilon_0^3, \quad M^{abcd} = \frac{1}{\mu_0} \left[ -\lambda_0^2 \delta^{(ab}\delta^{cd)} + \delta^{(ab}u_e^{c}\delta^{|e|d)} - u^{(ab}u^{c|d)} \right], \quad (83)
\]

\[
M^{ab} = \varepsilon_0 \left[ 2\lambda_0^2 \delta^{ab} + \left( 2u^{ab} - \delta^{ab}u^{d}d \right) u^{c}c - u^{a}a u^{b|b} \right]. \quad (84)
\]

As a check of the consistency of our formalism, we can verify that the isotropic case \( u^{ab} = -S\delta^{ab} \) reduces to the birefringent case \((46)\) and \((47)\) with \( u^i = -v^i = \delta^i_0 \sqrt{2S}/c \).

As to the ray surface, a direct computation yields \( \tilde{M}_a = 0 \), \( \tilde{M}_{abc} = 0 \) and

\[
\tilde{M} = \mu_0^3, \quad \tilde{M}_{ab} = -c^2 \delta_{aa'} \delta_{bb'} M^{a'b'}, \quad \tilde{M}_{abcd} = -c^{-2} \delta_{aa'} \delta_{bb'} \delta_{cc'} \delta_{dd'} M^{a'b'c'd'}. \quad (85)
\]

Accordingly, the form of the Fresnel ray and wave surfaces turns out to be exactly the same.
FIG. 6: Fresnel surface for a skewon of the magnetic Faraday type. It has two branches that are both paraboloids for the purely imaginary skewon with \( m^2 = -1 \) (depicted with a cut into half). Here \( m^2 = \pi \delta_a^a \); we use the dimensionless variables \( x := cq_1/q_0 \), \( y := cq_2/q_0 \), \( z := cq_3/q_0 \).

A direct calculation of the mean velocity of the light propagation now yields

\[
\langle v^2 \rangle = c^2 \left( 1 - \frac{\pi_a^a \pi_b^b + \pi_a^a \pi_b^b}{6} \right). \tag{86}
\]

As usual, we again introduce the dimensionless skewon variable \( \pi_a^b = u_a^b / \lambda_0 \). Then, similarly to the above cases, we find as upper value for the symmetric tensor skewon field from gamma-ray burst data \( \pi_a^b u_a^a < 7 \times 10^{-27} \lambda_0^2 \).

The Fresnel wave covector surface has now quite a different form as compared to the two cases above. For concreteness, let us take a tensor skewon with only one nontrivial component, \( u_a^b = u (\delta_1^a \delta_2^b + \delta_2^a \delta_1^b) \). Then, for small values of the skewon \( \pi < 1 \), the Fresnel wave surface is depicted in Fig. 7. As a comment to this figure, let us recall that for a vanishing skewon \( \pi = 0 \) we have a pure vacuum spacetime relation, and the Fresnel wave surface is then a sphere. With \( \pi \neq 0 \), this sphere degenerates to a pair of highly deformed intersecting toroids that, for extremely small values of \( \pi \), are nearly covering the sphere. But with growing \( \pi \), the toroids becomes thin and thinner. The intermediate situation is actually depicted in Fig. 7. Since we do not expect large skewon fields in general, we here limit ourselves to the case of a small \( \pi \).

In the complementary situation, when \( m^a = 0 \), \( n_a = 0 \) and the symmetric part is absent in (38), \( u_a^b = 0 \), we obtain a particular case described in Sec. 4: Birefringence with the Minkowski light cone and the second optical metric

\[
g^{(2)}_{ij} = \begin{pmatrix}
  c^2 & 0 \\
  0 & (1 - z^2 / \lambda_0^2)^{-1} (-\delta_{ab} + z_a z_b / \lambda_0^2)
\end{pmatrix}. \tag{87}
\]
FIG. 7: Fresnel surface for a skewon of the magneto-electric optical activity type. It has two intersecting toroidal branches for the real skewon $\pi = 0.8$. We use the dimensionless variables $x := cq_1/q_0$, $y := cq_2/q_0$, $z := cq_3/q_0$.

Unlike as with the symmetric tensor skewon, here the optical metric is always Lorentzian with the determinant $(\det g_{ij}^{(2)}) = -c^2/(1 - z^2/\lambda_0^2)^2$. This means that the waves propagates along both light cones (except for a skewon satisfying $z^2 = \delta^{ab}z_az_b = \lambda_0^2$ when the optical metric becomes degenerate). A straightforward computation of the mean speed of the wave propagation now yields

$$<v^2> = c^2 \left(1 - \frac{z^2}{3\lambda_0^2}\right),$$

and, accordingly, the upper limit for the skewon field from the gamma-ray data is again $z^2 < 7 \times 10^{-27}\lambda_0^2$.

**SKEWON EFFECTS IN MATTER**

In the previous section we studied the wave propagation on a vacuum spacetime described by the principal part $^{(1)}\chi_{ijkl}$ of the constitutive tensor $^{\text{[39]}}$. We found that the characteristic effect of the skewon field in that case was the emergence of holes in the wave covector surface. In physical terms this means the complete damping of the wave propagation in certain directions. Here we will briefly study the wave propagation in an anisotropic dielectric medium and demonstrate that a similar effect occurs in the presence of a skewon field.

The constitutive relation for anisotropic matter is most easily formulated in terms of the $3 \times 3$-matrix parametrization $^{[19]}-^{[21]}$. In the absence of a skewon field, an arbitrary
dielectric medium is described by the spacetime relation
\[ \mathcal{A}^{ab} = -\varepsilon_0 \varepsilon^{ab}, \quad \text{with} \quad \varepsilon^{ab} = \begin{pmatrix} \varepsilon_1 & 0 & 0 \\ 0 & \varepsilon_2 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix}, \quad \mathcal{B}_{ab} = \frac{1}{\mu_0} \delta_{ab}. \quad (89) \]

The matrices \( \mathcal{C}^{ab} = 0 \) and \( \mathcal{D}^{ab} = 0 \). Then, from (59)-(63), we find \( M^a = 0 \), \( M^{abc} = 0 \), and
\[ M^{ab} = \varepsilon_0 \lambda_0^2 \begin{pmatrix} \varepsilon_1 (\varepsilon_2 + \varepsilon_3) & 0 & 0 \\ 0 & \varepsilon_2 (\varepsilon_1 + \varepsilon_3) & 0 \\ 0 & 0 & \varepsilon_3 (\varepsilon_1 + \varepsilon_2) \end{pmatrix}. \quad (90) \]

As a result, the Fresnel equation for the wave surface (64) can be recast into the simple form
\[ \frac{\varepsilon_1 q_1^2}{c^2 q^2 - \varepsilon_1 q_0^2} + \frac{\varepsilon_2 q_2^2}{c^2 q^2 - \varepsilon_2 q_0^2} + \frac{\varepsilon_3 q_3^2}{c^2 q^2 - \varepsilon_3 q_0^2} = 0; \quad (91) \]
as in the previous section, \( q^2 = q_a q^a \).

As to the ray surface, we can immediately verify from (65)-(69) that \( \tilde{M}_a = 0 \) and \( \tilde{M}^{abc} = 0 \), whereas \( \tilde{M} = \mu_0^{-3}, \tilde{M}^{abcd} = \varepsilon_0 \lambda_0^2 \delta^{(ab} \varepsilon^{cd)} \varepsilon_1 \varepsilon_2 \varepsilon_3 \), and
\[ \tilde{M}^{ab} = -\frac{\lambda_0^2}{\mu_0} \begin{pmatrix} \varepsilon_2 + \varepsilon_3 & 0 & 0 \\ 0 & \varepsilon_1 + \varepsilon_3 & 0 \\ 0 & 0 & \varepsilon_1 + \varepsilon_2 \end{pmatrix}. \quad (92) \]

Then, from (70), one can straightforwardly derive the dual equation of the Fresnel ray surface for the components of the ray vector \( s = (s^0, s^1, s^2, s^3) \):
\[ \frac{(s^1)^2}{\varepsilon_1 s^2 - c^2(s^0)^2} + \frac{(s^2)^2}{\varepsilon_2 s^2 - c^2(s^0)^2} + \frac{(s^3)^2}{\varepsilon_3 s^2 - c^2(s^0)^2} = 0. \quad (93) \]

Typical wave and ray surfaces are depicted in Figs. 8 and 9 respectively. Both surfaces consist of two non-intersecting branches touching each other exactly in 4 points.

When the skewon field is present, the mentioned picture becomes more nontrivial. In particular, for the case of the isotropic skewon (52), the constitutive relation is modified by adding to (89) the nonvanishing \( 3 \times 3 \) matrices
\[ \mathcal{C}^{ab} = -S \delta^{ab}_b, \quad \mathcal{D}^{ab} = S \delta^{ab}_a. \quad (94) \]
FIG. 8: Fresnel wave covector surface for an anisotropic dielectric medium with \( \varepsilon_1 = 39.7, \varepsilon_2 = 15.4, \varepsilon_3 = 2.3 \). There are two branches, the outer part of the surface is cut into half in order to show the inner branch; we use the dimensionless variables \( x := cq_1/q_0, y := cq_2/q_0, z := cq_3/q_0 \).

Then, a quick computation shows that all the \( M \)-coefficients remain the same except that (90) is replaced with

\[
M^{ab} = \varepsilon_0 \lambda_0^2 \begin{pmatrix}
\varepsilon_1 (\varepsilon_2 + \varepsilon_3 - 4S^2) & 0 & 0 \\
0 & \varepsilon_2 (\varepsilon_1 + \varepsilon_3 - 4S^2) & 0 \\
0 & 0 & \varepsilon_3 (\varepsilon_1 + \varepsilon_2 - 4S^2)
\end{pmatrix}.
\] (95)

Here we introduced the dimensionless skewon variable \( S := S/\lambda_0 \).

Analogously we can verify that the dual \( \tilde{M} \)-coefficients are the same as in the skewonless case except that (92) is changed to

\[
\tilde{M}_{ab} = -\frac{\lambda_0^2}{\mu_0} \begin{pmatrix}
\varepsilon_2 + \varepsilon_3 - 4\overline{S}^2 & 0 & 0 \\
0 & \varepsilon_1 + \varepsilon_3 - 4\overline{S}^2 & 0 \\
0 & 0 & \varepsilon_1 + \varepsilon_2 - 4\overline{S}^2
\end{pmatrix}.
\] (96)

Then the equations of wave and ray surfaces cannot be represented any longer in the compact form (91) and (93). The skewon affects the form of the surfaces by creating typical holes that correspond to the directions of wave covectors along which no wave propagation occurs. More specifically, the original two branches of the wave and ray surfaces, depicted in Figs. 8, 9 merge and form a single non-simply-connected surface. In the process of such a merging, the four points where the original two branches touched each other become the wormholes through which one can move from the outer surface into the inner one. Such a typical skewon effect is shown in Fig. 10.
FIG. 9: Fresnel ray vector surface for an anisotropic dielectric medium with $\varepsilon_1 = 39.7$, $\varepsilon_2 = 15.4$, $\varepsilon_3 = 2.3$. There are two branches, 1/4 part of the outer surface is cut off in order to show the second inner branch; we use the dimensionless variables $x := s^1/s^0 c$, $y := s^2/s^0 c$, $z := s^3/s^0 c$.

FIG. 10: Fresnel wave covector surface for an anisotropic dielectric medium with $\varepsilon_1 = 39.7$, $\varepsilon_2 = 15.4$, $\varepsilon_3 = 2.3$ in presence of a skewon. The two original branches (cf. with Fig. 8) are now merged into a single surface with the wormholes replacing the original four points where the branches touched. The surface is cut into half; we use the dimensionless variables $x := cq_1/q_0$, $y := cq_2/q_0$, $z := cq_3/q_0$.

DISCUSSION AND CONCLUSION

The violation of the Lorentz symmetry of spacetime results in “spoiling” the ordinary light cone and, in particular, in the emergence of birefringence in light propagation. Thus, the investigation of electromagnetic wave phenomena provides an understanding of the Lorentz violation in the photon sector of SME. The birefringence type effects represent the observable consequences of the model and the corresponding measurements impose limits on the Lorentz-violating parameters in the general linear spacetime relation. It was shown previously by using cosmological and laboratory observations that the birefringence-related parameters of the principal part of the constitutive tensor must be smaller than $10^{-32} \lambda_0$.

In this paper we investigated the general covariant Fresnel equation for the wave covectors.
as well as for the ray vectors in the case of a linear spacetime relation (3) that connects the electromagnetic excitation with the field strength. Strictly speaking, the physical information contained in the ray vector surface is the same as that in the wave covector surface. The reason why we still analyzed the ray along with the wave surfaces is twofold. Firstly, we developed the four-dimensional covariant approach to the ray surfaces in complete analogy with the earlier results obtained for the Fresnel wave surfaces. Such a formalism might be helpful in the study of the various aspects of the wave propagation in media, since the construction of the ray surfaces is quite a common tool in crystal optics, see [17, 18, 19, 20, 28, 31]. Secondly, we demonstrated explicitly that although the ray surface is dual to the wave surface (which is also clearly seen from the figures above), the optical metric, which is naturally derived in the case of birefringence, is different for the wave and for the ray surfaces and the two metrics derived are not dual to each other.

The skewon part (11)\textsuperscript{1}, (15) of the constitutive tensor was in the center of our study. We introduced its general parametrization (55). This enabled us to distinguish between three different types of effects: the electric and magnetic Faraday effects and the (magneto-electric) optical activity. The influence of a skewon field on the Fresnel wave and ray surfaces is qualitatively different in these three case. This is illustrated in the figures in the corresponding subsections of Sec.\textsuperscript{a}. However, the characteristic sign of the skewon is the emergence of the specific holes in the Fresnel surfaces that correspond to the directions in space along which the wave propagation is damped out completely. This effect is in complete agreement with our earlier conclusion on the dissipative nature of the skewon field [13, 22, 23].

Besides the qualitative results, one may derive numerical limits on the magnitude of the skewon components from the experimental search for the anisotropy of the velocity of light. Using the data for the gamma-ray bursts and following [5], we then obtain the estimates $s_a^b s_b^a \sim a_a^b a_b^a < 7 \times 10^{-27} \lambda_0^2$ and $n_a n^a < 3 \times 10^{-27} \varepsilon_0^2$. This shows that if the skewon indeed spoils the light cone structure (yielding a violation of the Lorentz symmetry and an anisotropy of the propagation of light), its influence should be extremely small. As a final remark we note that the experimental laboratory data [35] imposes much stronger constraints on the principal part of the constitutive tensor (see the corresponding analysis in [5]), and we can expect that the same will be true for the skewon too since the theoretical formalisms are pretty much the same in both cases.
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