Computing feasible trajectories for an articulated probe in three dimensions

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Problem statement

Articulated probe:
Line segments $ab$ and $bc$, joined at $b$. $ab$ can be arbitrarily long, and $bc$ has a fixed length $r$.

Workspace:
Sphere $S$ containing $n$ triangular obstacles and a target point $t$ in free space.

Probe trajectory:
Straight insertion of $abc$ into $S$, possibly followed by a rotation of $bc$ at $b$ up to $\pi/2$.

Objective:
Find a feasible (obstacle-avoiding) probe trajectory to reach $t$. 
Motivation

Relevance in robotics, particularly in planning for **minimally invasive surgeries**.

- To reach previously unattainable targets by circumventing surrounding critical structures.
- Human body cavity – workspace $S$.
- Critical organ/tissue – triangular mesh.

Modeled instrument already in **clinical use** (e.g., da Vinci EndoWrist by Intuitive Surgical).

Has never been investigated in **three dimensions** from a **theoretical** viewpoint.

- Important to fully explore its rich combinatorial and geometric properties through exact solution approach.
Prior work – two dimensions

**Daescu, Fox, and Teo [2018a]:**

Algorithm for finding a feasible trajectory amidst \( n \) line segment obstacles in the plane.

- \( O(n^2 \log n)\)-time, \( O(n \log n)\)-space.
- Compute **extremal** feasible trajectories tangent to one or two obstacle endpoints.
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- Compute extremal feasible trajectories tangent to one or two obstacle endpoints.

Daescu, Fox, and Teo [2018b]:
Extended algorithm for finding a feasible trajectory of a given clearance $\delta > 0$ from obstacles.

- $O(n^2 \log n)$ time, $O(n^2)$ space.
- Dilate each obstacle by $\delta$ using Minkowski sum, and find a feasible trajectory tangent to one or two (dilated) obstacles.
Prior work – two dimensions

**Daescu and Teo [2019]:**
Characterization of the feasible solution space using a simple-curve arrangement.

- Complexity = $O(k)$, where $k = O(n^2)$ is the number of vertices in the arrangement.
- Constructed in $O(n \log n + k)$ time using $O(n + k)$ space (topological sweep method).
- An $O(\log n)$-factor improvement in time complexity for finding a feasible trajectory.
Current results – three dimensions

Observations:

• If there exists a feasible probe trajectory, then some extremal feasible trajectories must be present.

• An extremal trajectory is characterized by its tangencies to a combination of obstacle vertices, edges, and/or surfaces.

• These extremal trajectories can be represented by $O(n^4)$ combinatorial events.

Solution approach:

• Find an extremal feasible trajectory by enumerating and verifying these combinatorial events for feasibility in $O(n^{4+\varepsilon})$ time using $O(n^{4+\varepsilon})$ space, for any constant $\varepsilon > 0$. 
**Unarticulated vs. articulated**

**Feasible unarticulated trajectory:**
Unarticulated probe can be inserted into $S$ to reach $t$ while avoiding obstacles without a rotation of $bc$.

**Feasible articulated trajectory:**
Unarticulated probe can be inserted into $S$ and a rotation of $bc$ can be performed to reach $t$, all while avoiding obstacles.
Support vertex, edge, and surface

Endpoint of edge $e_1$ is a **support vertex** of line segment $\ell$.

Edge $e_2$ is a **support edge** of line segment $\ell$.

Circular sector $\sigma$ is supported by edge $e_3$ through its endpoint.

Circular arc $\gamma$ is supported by edge $e_4$ through its interior point.

Triangle $\tau$ is a **support surface** of circular arc $\gamma$.

A probe trajectory is **isolated or extremal** with respect to a set of supports if the trajectory cannot be altered without losing its intersections or tangencies with these supports.
Extremal feasible unarticulated trajectories

Extremal feasible unarticulated trajectory:

Isolated by **one support vertex**.

Isolated by **two support edges**.
Compute the set $R$ of rays:

- Originates at $t$.
- Isolated with respect to an obstacle vertex or two obstacle edges.
- Does not intersect any triangular obstacle in its interior.

Each ray of $R$ represents an extremal feasible unarticulated trajectory.

Computing $R$ reduces to:

- Computing the visibility polyhedron from $t$, which takes $O(n^2)$ time and $O(n^2)$ space [McKenna, 1987].
- This yields all feasible unarticulated trajectories.

Can find a feasible unarticulated trajectory in $O(n^2)$ time using $O(n^2)$ space.
Extremal feasible articulated trajectories

Extremal feasible articulated trajectory:
• Isolated by at most four supports (vertices, edges, surfaces, or a combination of three).
• Twenty some distinct cases – for instance:
Extremal articulated trajectories:

- Characterized by $O(n^4)$ combinatorial events.
- Can be enumerated in $O(n^4)$ time using geometric operations or algebraic-geometric approach.

An extremal articulated trajectory is **feasible** if and only if none of the obstacles intersects

i) segment $ab$

  $\rightarrow$ **ray shooting query**

  $O(n^{4+\varepsilon})$ preprocessing time/space,
  $O(\log n)$ query time

  [de Berg et al., 1994; Pellegrini, 1993].

ii) circular sector $\sigma$ (i.e., area swept by $bc$)

  $\rightarrow$ **circular sector emptiness query**
A feasible articulated trajectory always lies in a plane $\Pi$ passing through $t$.

- $\Pi$ can be parameterized using two variables $I$ and $\Omega$.
- Let $\tau$ be a triangle that intersects $\Pi$ in a line segment $s$.
- $bc'$ is the farthest radius from $bt$ before the minor circular sector bounded by $bt$ and $bc'$ intersects $s$.
- $\rho_s$ is the angle of $bc'$ with respect to $bt$.

**Fix $\Pi$ and characterize $\rho_s$ as a function of $\theta$.**

- Each curve $\rho_s(\theta)$ is partially defined, continuous, and monotone over $\theta$.
- Any two curves $\rho_s(\theta)$ can only intersect at most once.
Circular sector emptiness queries

New data structure for circular sector emptiness queries in **two dimensions**:

- Let $V$ be the **lower envelope** of $\rho_s$ for all given line segments $s$.
- Only require an implicit representation of $V$:
  - Vertices of $V$ and segments that induce various pieces of $V$.
- Complexity of $V = O(n \alpha(n))$, 3rd-order Davenport-Schinzel sequence.
- $V$ is computable in $O(n \log n)$ time.
- Can determine if a query sector $\sigma$ intersects any segment in $O(\log n)$ time:
  - Retrieve the segment $s$ that induces the curve $\rho_s$ defining $V$ at query $\theta$, and check if $bc$ of $\sigma$ intersects $s$. 
Circular sector emptiness queries

For each \( \tau \), define a trivariate function \( \rho_\tau(I, \Omega, \theta) \) so that \( \rho_s(\theta) \) is characterized with respect to all planes \( \Pi(I, \Omega) \).

\( \rho_\tau(I, \Omega, \theta) \) is an inverse trigonometric function, based on which we can define an algebraic function \( f_\tau = \sin(\rho_\tau/2) \) of low constant degree.

Data structure for circular sector emptiness queries in three dimensions:

- Let \( V \) be the lower envelope of trivariate piecewise algebraic functions \( f_\tau \)
- Complexity of \( V = O(n^{3+\varepsilon}) \) [Sharir, 1994].
- \( V \) is computable in \( O(n^{4+\varepsilon}) \) time using \( O(n^{4+\varepsilon}) \) space [Koltun, 2004].
- Can determine if a query sector intersects any triangular obstacle in \( O(\log n) \) time.
Extremal feasible articulated trajectories

$O(n^4)$ queries are to be processed in the worst case.

Can determine an **extremal feasible articulated trajectory** in time

\[
O(n^4) + O(n^{4+\varepsilon}) + O(n^4 \log n) + O(n^{4+\varepsilon}) + O(n^4 \log n) = O(n^{4+\varepsilon})
\]

using $O(n^{4+\varepsilon})$ space.
Conclusion

A **feasible probe trajectory** in three dimensions can be determined in $O(n^{4+\varepsilon})$ time using $O(n^{4+\varepsilon})$ space.

An $O(n^5)$-time algorithm with linear space usage is achievable by performing a simple $O(n)$-check on each of the $O(n^4)$ events.

Algorithm is **highly parallel**, considering that each combinatorial event can be generated and verified for feasibility independently of the others.

Addressed a special instance of the **circular sector emptiness** query problem in **three dimensions**.

New $\mathbb{R}^2$ query data structure simplifies the two-part approach formerly proposed by Daescu, Fox, and Teo [2018a] while maintaining the same time and space complexity.
Open problems (in progress)

• Compute a feasible probe trajectory of a given (or maximum) clearance in three dimensions.

• Assuming there is no unarticulated feasible probe trajectory, find the minimum length $r > 0$ of segment $bc$ such that a feasible articulated probe trajectory exists, and report (at least) one such trajectory.
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Thank you.