Alignment function as a new kind of transverse momentum dependent functions

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Received: 28 February 2022 / Accepted: 14 July 2022
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Communicated by Evgeni Kolomeitsev

Abstract We argue the existence of new $k_\perp$-dependent functions which can be manifested in the Drell-Yan (SIDIS)-like processes. The presented new functions resemble the well-known Boer-Mulders function associated with the quark spin asymmetry, but in contrast they are sensitive to the transverse motion of partons inside the hadron due to the collective alignment of quark spin vectors.

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1 Introduction

Nowadays, the lepton production in nucleon–nucleon collisions in the Drell-Yan (DY) processes attracts still much attentions not only of the experimental collaborations which investigate the composite (spin) structure of hadrons. From the theoretical viewpoint, this kind of processes is interested because it opens the window for the studies of the transverse momentum dependent functions. The most general form of the functions which depend on $k_\perp$ is given by the hadron-hadron matrix element of quark-gluon operators

$$\Phi^{(\pm)}(k) = \int (d^4 z)e^{\mp i(kz)}\langle P, S|\bar{\psi}(0)\{0:z|^{(\pm)}\psi_{\pm}\}(z)|P, S\rangle^H,$$

(1.1)

where $(d^4 z) = d^4 z/(2\pi)^4$ and $H$ indicates the Heisenberg representation (H-representation) used for the Fock states and operators. Its general parametrization involving the hadron spin axial-vector $S$ as one of Lorentz structures plays an important role for the study of different spin characteristics in the Drell-Yan (SIDIS)-like processes (see, for example, [1–6]).

The Heisenberg representation of Eq. (1.1) involves the “dressed” operators and states, i.e. $\psi_H(x) = S(t,0)\psi(x)$ $S(t,0)$ and $|0\rangle^H = S(0,-\infty)|0\rangle$ with $S(t_2,t_1)$ being the $S$-matrix, and it gives a compact form where the interaction is implicitly presented.

The Lorentz parametrization of $\Phi^{[\Gamma]}(\pm)(k)$ with different Fierz projections is an extremely important stage for modern studies [3] because, from the physical viewpoint, it defines the different sorts of parton distributions. Among all parametrizing functions, the $k_\perp$-dependent Boer-Mulders (BM) function introduced in [7] and associated with the $\sigma^{\pm\mp}\gamma_5$-projection of $\Phi(k)$ in Eq. (1.1) can be singled out as a function which describes the transverse spin asymmetry of quarks inside the unpolarized hadron. Indeed, for the BM-function $h_1^\perp(x,k_\perp^2)$ contribution, we have the following representation after the factorization procedure applied

\[ \Phi^{(\pm)}(k) = \int (d^4 z)e^{\mp i(kz)}\langle P, S|\bar{\psi}(0)\{0:z|^{(\pm)}\psi_{\pm}\}(z)|P, S\rangle^H, \]

(1.1)

The underlined Greek indices correspond to the open spinor indices; $[0:z|^{(\pm)}\psi_{\pm}$ stands for the future- and past-pointed Wilson line (WL). Throughout the paper, we use the standard notations for the plus and minus light-cone directions.

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to the corresponding semi-inclusive deep-inelastic scattering (SIDIS) [9]

\[ \Phi^{[\gamma^* g]}(x, k_F^2) \sim \mathcal{P} q^{1/N}(x, k_{-}) - \mathcal{P} q^{1/N}(x, k_{+}) \]

\[ = \frac{|k_{-}|}{m_N} \sin(\phi_\beta - \phi_\delta) h_1^+(x, k_F^2), \] (1.2)

where \( \mathcal{P} q^{1/N}(x, k_{-}) \) denotes the probability to find the transverse polarized quark inside the unpolarized hadron, \( \phi_\beta - \phi_\delta \) defines the angle between \( k_{-} \) and \( s_{-} \) being denoted as the quark transverse momentum and quark transverse spin vector (in two dimensional Euclidian space) respectively. The expression \( |k_{-}| \sin(\phi_\beta - \phi_\delta) \) in Eq. (1.2) stems from the vector product \( k_{-} \wedge s_{-} \) provided \( |s_{-}| = 1 \).

The aim of the paper is to study a new manifestation of \( k_{-} \)-dependent parametrization functions of the matrix element (1.2) which can be associated with the inner transverse quark motion generated by the collective spin alignment rather than the quark spin asymmetry. In other words, the functions we consider are related to the probability \( \mathcal{P} q^{1/N}(x, k_{-}) \) defined by the operator \( [\bar{\psi}(1) \gamma^+ \gamma_5 \gamma_S \psi(1)] \) involving the projections \( \psi^{(1)} = 1/2(1 \pm \gamma_5 \gamma_S) \). Here, \( (i = 1) \leftrightarrow (x) \), \( (i = 2) \leftrightarrow (y) \), and \( x \), \( y \) are the polarization axes.

2 New transverse momentum dependent functions

In this section, we introduce new transverse momentum dependent functions which are associated with the spin structure of quark content of hadrons.

To begin with, we rewrite Eq. (1.1) in the interaction representation (see Appendix 1 for details) focusing e.g. on the vector projection

\[ \Phi^{[\gamma^\mu]}(k) = \int (d^4z) e^{i(kz)} \left\langle P, S \right| [\bar{\psi}(0) \gamma^\mu \psi(z) \mathbb{S}[\psi, \bar{\psi}, A]|P, S \right\rangle, \] (2.1)

where \( P \) and \( S \) are the momentum vector and spin axial-vector of hadron, respectively, and

\[ \mathbb{S}[\psi, \bar{\psi}, A] = T \exp \left\{ i \int (d^4 \xi) \left[ \mathcal{L}_{QCD}(\xi) + \mathcal{L}_{QED}(\xi) \right] \right\}. \] (2.2)

For brevity, all corresponding normalization and dimensionful pre-factors have been absorbed in the definitions of integration measures or in the definitions of parametrizing functions. Notice that the function \( \Phi^{[\gamma^\mu]}(k) \) is a very typical one which appears to describe the different processes in (semi)inclusive and (semi)exclusive channels.

2.1 Lorentz parametrization of \( \Phi^{[\gamma^\mu]}(k) \): the “standard” way

The Lorentz decomposition (or parameterization) of any relevant correlators is usually based on the following standard scheme:

- the parton operators in correlators are supposed to be considered as free operators, i.e. \( \mathbb{S}[\psi, \bar{\psi}, A] = I \), see Fig. 1, the left panel;
- one forms the orthogonal system of Lorentz tensors;
- with the help of the basis Lorentz tensors, one parametrizes the given Fierz projection of correlators based on the principle of Lorentz covariance;
- including \( \mathbb{S} \)-matrix in the correlator, one studies the corresponding evolution of parametrizing functions.

Within this standard scheme, the most general parametrizations of relevant correlators can be found, for example, in [3,9]. For illustration, let us dwell on the following decomposition which includes the transverse momentum dependence of quarks:

\[ \Phi^{[\gamma^\mu]}(k) \bigg|_{\mathbb{S}[\psi, \bar{\psi}, A] = I} = P^+ f_1 \left( x; k^2_F, (k_\perp P_{\perp}) \right), \] (2.3)

It is important to emphasize that the quark combination \( \langle P, S | [\bar{\psi}\gamma^+ \psi] | P, S \rangle \) of Eq. (2.3) singles out the unpolarized quarks only. In addition to \( P^+ \)-structure, we may construct (without thinking on the orthogonality) the Lorentz vector \( i \varepsilon^{+ - k_\perp S_{\perp}} \) for parametrization of \( \Phi^{[\gamma^\mu]}(k) \) in Eq. (2.3) even for the unpolarized quark combination \( i \). Indeed, in this case, \( S_{\perp} \) plays a role of the exterior axial-vector which is not dictated by the quark combination. It leads to the term with the well-known function \( f^+_1(x, k_\perp) \) in parametrization of \( \Phi^{[\gamma^\mu]}(k) \), i.e. we go over to the following parametrization [3] \( \Phi^{[\gamma^\mu]}(k) \bigg|_{\mathbb{S}[\psi, \bar{\psi}, A] = I} = P^+ f_1 \left( x; k^2_F, (k_\perp P_{\perp}) \right) + i \varepsilon^{+ - k_\perp S_{\perp}} f^+_1 \left( x; k^2_F, (k_\perp P_{\perp}) \right). \] (2.4)

However, the Lorentz vector \( i \varepsilon^{+ - k_\perp S_{\perp}} \) with the quark spin axial-vector \( ^5 \) is absent for the unpolarized quarks in correlator of Eq. (2.4) due to the trace given by

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\(^5\) \( S_{\perp} \) is the covariant spin axial-vector of transversely polarized hadron.

\(^4\) Modulo the prefactors which are irrelevant for our study.

\(^5\) Based on the Lorentz covariance, the combination \( i \varepsilon^{+ - k_\perp S_{\perp}} \) as the Lorentz vector is not formally excluded from the parametrization of the vector correlator.
In this subsection we propose an alternative way to parametrize the correlator: In other words, the substantial difference between the standard and nonstandard ways is the following: in the standard way, we first consider free operators to parametrize the given correlator where \( S \)-matrix has been included in the consideration to derive the evolution of the already-introduced parametrizing functions; while, in the nonstandard way, we parametrize the given correlator where \( S \)-matrix has been presented in Eq. (2.1) from the very beginning.

We believe that the nonstandard way is more adequate because, first, \( S \)-matrix is presented in the correlator even before factorization and, second, as shown below, one can discover the new structure functions which are invisible in the standard way.

Using the nonstandard way, we cannot exclude the following new structure:

\[
\Phi^{[\gamma^+]}(k) = i e^{+ - P_\perp s_\perp} f_1^{(1)}(x; k_\perp^2) + \ldots
\]

where \( s_\perp \) stands for the quark spin axial-vector and \( S \)-matrix presents in the correlator. The details of derivation are collected in the next subsection.

2.3 Derivation of the new \( k_\perp \)-dependent functions

We are going over to the technical details of consequences of the nonstandard scheme. Let us focus on the well-know \( k_\perp \)-dependent function \( f_1 \) which parameterizes the unpolarized hadron matrix element of quark operator involving \( \gamma^+ \) as

\[
\Phi^{[\gamma^+]}(k) = P^+ f_1 \left( x; k_\perp^2, (k_\perp P_\perp) \right)
\]

\[
= P^+(k_\perp P_\perp) f_1^{(1)}(x; k_\perp^2) + \left\{ \text{terms of } (k_\perp P_\perp)^n \mid n = 0, n \geq 2 \right\}
\]

where \( k = (x P^+, k^-, k_\perp) \) and \( f_1 \left( x; k_\perp^2, (k_\perp P_\perp) \right) \) has been decomposed into the powers of \( (k_\perp P_\perp) \). Keeping the term of decomposition with \( n = 1 \) represents the minimal necessary requirement for the manifestation of new functions.
Notice that $P_\perp$ is substantially non-zero within the Collins-Soper (CS) frame of Drell-Yan-like (DY) processes. As mentioned, the matrix elements in Eqs. (1.2) and (2.7) have to be treated within the H-representation (see Appendix 1) which provides the compact forms incorporating the interactions implicitly. For the practical reasons, it is convenient, however, to adhere the interaction representation (I-representation). It is worth to notice that the Lorentz structures which parametrize the corresponding correlators are related to the spinor lines appearing in consideration at the given order of the coupling constant, see below Eqs. (2.25)–(2.27).

After performing the Fourier transforms of the quark operators in (2.7), we obtain that

$$\int (d^4 k) e^{-i(kz)} \phi_{[\gamma^+]}(k) = \int d\mu_{\lambda, \lambda'}^{[\gamma^+]}(k_1, k_2) \langle P, S|b_{\lambda}^+(k_1) b_{\lambda'}^-(k_2)|P, S \rangle^H,$$

where $(d^4 k) = d^4 k/(2\pi)^4$,

$$d\mu_{\lambda, \lambda'}^{[\gamma^+]}(k_1, k_2) = (d^4 k_1)(d^4 k_2) \delta(k_1^2) \delta(k_2^2) e^{-i(kz)} \times \left[ \bar{u}^{(k_1)}(\gamma^+ u)^{(k_2)} \right],$$

and

$$\langle P, S|b_{\lambda}^+(k_1) b_{\lambda'}^-(k_2)|P, S \rangle^H = \langle P, S|b_{\lambda}^+(k_1) b_{\lambda'}^-(k_2)|S\psi, \bar{\psi}, A\rangle|P, S \rangle = \delta^{(4)}(k_1 - k_2) \delta_{\lambda, \lambda'} \mathcal{M}_{\lambda, \lambda'}(k_1^2, k_2^2, S\psi, A),$$

where $b^\pm$ are the quark creation and annihilation operators.

In what follows we omit the spin state indices $\lambda$ and $\lambda'$ and we do not pay an attention on the difference between contranda covariant vectors unless it may lead to a confusion.

Taking into account the delta function of (2.10), we can see that the spinor line formed by $[\bar{u}(k)\gamma^+ u(k)]$ results in the appearance of $k^+ \sim P^+$ in the most trivial case without spin polarizations and interactions.

We now consider the second order of strong interactions in the correlator within the interaction representation, see Fig. 1, the right panel. We have the following expression

$$\langle \mathcal{O}_{\gamma^+}^{(2)} \rangle^{(2)} = \langle P, S|T \bar{\psi}(0)\gamma^+ \psi(z) \mathcal{S}^{(2)}_{QCD}[\psi, \bar{\psi}, A]|P, S \rangle = \int (d^4 k) e^{-i(kz)} \Delta(k^2) \int (d^4 \ell) \Delta(\ell^2) \int (d^4 \bar{k})$$

$$\times \mathcal{M}(k^2, \ell^2, \bar{k}^2, ...) \times [\bar{u}(k)\gamma^+ \bar{\gamma}_{\alpha} u(\bar{k} - \ell)] \times [\bar{u}(\bar{k})\gamma^+ \bar{\gamma}_{\alpha} u(\bar{k} + \ell)],$$

where $\mathcal{S}^{(2)}_{QCD}$ denotes the $S$-matrix operator at the order of $g^2$. In Eq. (2.11) the quark and gluon propagators read

$$S(k) = \hat{k} \Delta(k^2), \quad D_{\mu\nu}(\ell) = \frac{g_{\mu\nu}}{\ell^2} \Delta(\ell^2),$$

$$\Delta(k^2) = \frac{1}{k^2 + i\epsilon}, \quad \hat{k} = (k\gamma).$$

and the amplitude $\mathcal{M}$ is given by

$$\mathcal{M}(k_1^2, (k_i, \ell), ...) \delta^{(4)}(k_1 + k_3 - k_2 - k_4) = \langle P, S|b^+ \bar{(k_1)} b^- (k_2) b^+(k_3) b^- (k_4)|P, S \rangle.$$ (2.13)

We single out the region where $|\ell| \ll |k|, |\bar{k}|$ and as a result we obtain that

$$\langle \mathcal{O}_{\gamma^+}^{(2)} \rangle = \int (d^4 k) e^{-i(kz)} \Delta(k^2) \bar{u}(k)\gamma^+ \bar{\gamma}_{\alpha} u(k) \times \int (d^4 \bar{k}) [\bar{u}(\bar{k})\gamma^+ \bar{\gamma}_{\alpha} u(\bar{k})$$

$$\times \int (d^4 \ell) \Delta(\ell^2). \mathcal{M}(k^2, \ell^2, \bar{k}^2, ...).$$ (2.14)

The next stage is to transform the spinor lines of this expression. For the first spinor line, we write

$$[\bar{u}(k)\gamma^+ \bar{\gamma}_{\alpha} u(k)] = \mathcal{F}^{\mu\nu\beta\delta}[\bar{u}(k)\gamma^\beta u(k)] + \text{(axial)}$$

$$\Rightarrow k^\alpha \mathcal{F}^{\mu\nu\beta\delta}[\bar{u}(k)\gamma^\beta u(k)] + \text{(other terms)},$$

where the following notation has been used

$$\mathcal{F}^{\mu\nu\beta\delta} = \frac{1}{4} \text{tr}[\gamma^\mu \gamma^\nu \gamma^\beta \gamma^\delta].$$ (2.16)

The second spinor line can be considered with the help of the covariant (invariant) integration given by

$$k^\alpha \int (d^4 \bar{k}) [\bar{u}(\bar{k})\gamma^\alpha u(\bar{k})]. \mathcal{M}(\bar{k}^2, (\bar{k} P), ...) = k^\alpha \int (d^4 \bar{k}) \bar{k}^\alpha \mathcal{M}(\bar{k}^2, (\bar{k} P), ...)$$

$$= (k_\perp P_\perp) \int (d^4 \bar{k}) \frac{(\bar{k} \perp P_\perp)}{P^2_\perp} \mathcal{M}(\bar{k}^2, (\bar{k} P), ...).$$ (2.17)

Using (2.15) and (2.17), one can see that the form of the t.r.h.s. of (2.14) coincides with the parametrization of (2.7) at $g^2$-order. Indeed, we have the following

$$P^+(k_\perp P_\perp) f_1^{(1)}(x; k_\perp^2) \sim$$

$$[\bar{u}(k)\gamma^+ u(k)] \int (d^4 \bar{k}) [\bar{u}(\bar{k})\hat{k} u(\bar{k})]$$

$$\times \int (d^4 \ell) \Delta(\ell^2) \Delta(k_\perp^2) \mathcal{M}(\bar{k}^2, \ell^2, \bar{k}^2, ...).$$ (2.18)

In other words, the parametrization of (2.7) with the Lorentz combination $P^+(k_\perp P_\perp)$ is related to the two spinor lines

$$[\bar{u}(k)\gamma^+ u(k)] \Rightarrow k^+ \sim P^+, \quad [\bar{u}(\bar{k})\hat{k} u(\bar{k})] \Rightarrow (k_\perp P_\perp)$$ (2.19)
at $g^2$-order.

In the region of $|\vec{k}| \sim |k|$, two spinor lines of (2.18) can be transformed into the other spinor lines with the help of Fierz transformations. Using its general form (see, [8])

\[
\begin{align*}
\left[ \overline{\psi}^{(a)} \gamma_{\mu} \gamma_5 \psi^{(b)} \right] & = \frac{1}{4} \sum_{A, R_1, R_2} \left\{ \frac{1}{4} \text{tr} \left[ \Gamma_A \gamma_5 \Gamma_{R_1} \right] \right\} \left\{ \frac{1}{4} \text{tr} \left[ \Gamma_A \gamma_5 \Gamma_{R_2} \right] \right\} \\
& \times \left[ \overline{u}^{(c)}(\gamma_{\mu} \gamma_5) \right] \left[ u^{(d)} \right]
\end{align*}
\]

(2.20)

with $O_1 = \gamma^+ \gamma^+ \gamma_5 S_1$, $O_2 = 1$, $\Gamma^A = \gamma^+_{R_1}$, $\Gamma^{R_1} = \gamma^+$, $\Gamma^{R_2} = \gamma^+_\perp$, we obtain that

\[
\begin{align*}
\left[ \overline{u}^{(1, x)}(k) \gamma^+ \gamma^+ \gamma_5 u^{(1, x)} \right] & = C_{ij} \left[ \overline{u}^{(1, x)}(k) \gamma^+ \gamma_5 u^{(1, x)} \right]
\end{align*}
\]

(2.21)

Here, we assume that, for the fixed indices, $i \neq j$, the coefficient $C_{ij}$ is given by

\[
C_{ij} = \frac{1}{16} \text{tr} \left[ \gamma^+ \gamma^+ \gamma_5 \gamma_5 S_1 \right] \text{tr} \left[ \gamma^+ \gamma_5 \gamma_5 \right]
\]

(2.22)

Eq. (2.21) can be readily inverted in order to get the following representation of Eq. (2.18)

\[
\begin{align*}
\left[ \overline{u}^{(1, x)}(k) \gamma^+ \gamma^+ \gamma_5 u^{(1, x)} \right] & = \int (d^4 \vec{k}) \int (d^4 \ell) \\
& \times \Delta(\ell^2) \Delta(k^2) \mathcal{M} \left( k^2, \ell^2, \vec{k}_\perp, \ldots \right) \\
& \Rightarrow k^+ i e^{+P.L_{\perp}} \gamma_1^{\perp}(x, k^2_{\perp})
\end{align*}
\]

(2.23)

On the r.h.s. of (2.23), we write down the complex $i$ explicitly in order to stress that it stems from the trace of four $\gamma$-matrices with $\gamma_5$. For the existence of Lorentz vector defined as $e^{+P.L_{\perp}}$, it is necessary to assume that the quark spin $s_{\perp}$ is not a collinear vector to the hadron transverse momentum $P_{\perp}$. Within the CS-frame, the hadron transverse momentum can be naturally presented as $P_{\perp} = (P^+_{\perp}, 0)$. Since the hadron spin vector $S$ can be decomposed on the longitudinal and transverse components as $S^L + S^T = k/P^+ + m_N + S^T$, we get $P \cdot S = P_{\perp} S^L = 0$. Hence, in the CS-frame, it is natural to suppose that quark $s_{\perp}$ and hadron $S^T$ are collinear ones. This is a kinematical constraint (or evidence) for the nonzero Lorentz combination $e^{+P.L_{\perp}}$, and, therefore, for the existence of a new function $f_1^{\perp}(x, k^2_{\perp})$.

We have reached the inference that Eq. (2.23) contains the Lorentz structure with the quark polarization vector $s_{\perp}$. Therefore, within the frame of the “nonstandard way”, the Lorentz vector defined as $e^{+P.L_{\perp}}$ should be included in the general scheme of the Lorentz parametrization applied for the relevant correlators.

Notice that the standard time-reversal transformation and the Hermitian conjugation suggest that the function $f_1^{\perp}(x, k^2_{\perp})$ is a pure imaginary and $T$-odd function. We emphasize that the $k_{\perp}$-dependent parton functions may possess the nontrivial properties under the replacement $k_{\perp} \rightarrow -k_{\perp}$ owing to that the time-reversal transforms convert the future-pointed WL to the past-pointed WL [10].

It is important that the combination $e^{+P.L_{\perp}}$ is not the only new combination. For example, we may construct an analogous combination defined as $e^{+p.L_{\perp}}$. Actually, the “nonstandard” way presented in the paper discovers a new kind of different $k_{\perp}$-dependent functions. The comprehensive analysis of such kind of functions is planned to be done in the forthcoming works.

So, it explicitly shows that the function $f_1^{\perp}(x, k^2_{\perp})$ and its analogues must appear in the parametrization of the hadron matrix element, i.e.

\[
\Phi^{\gamma^+}(k) \\
= \int (d^4 z) e^{i(k \cdot z)} \langle P, S | \phi(0) \gamma^+ \psi(z) S[\phi, \psi, A] | P, S \rangle \\
= i e^{+P.L_{\perp}} f_1^{\perp}(x, k^2_{\perp}) \\
+ i e^{-k_{\perp}} f_2(z; k^2_{\perp}) + \ldots,
\]

(2.24)

where the ellipse denotes the other possible terms of parametrization [13].

Comparing the “standard” (the Lorentz parametrization with $S = I$ in the correlator) and the “nonstandard” (the Lorentz parametrization with $S$-matrix in the correlator) ways, we see that the quark spinor lines play a crucial role in the construction of possible Lorentz combination. Indeed, it can be demonstrated by the following reasoning. Let us return to Eq. (2.4), we are able to specify the corresponding correlator as

\[
\Phi^{\gamma^+}(k) \big| S[\phi, \psi, A] = I \\
= \langle P, S | \phi(0) \gamma^+ \psi(z) | P, S \rangle \\
= \int (d^4 k_1)(d^4 k_2) e^{-ik_2 z} L^{\gamma^+}(k_1, k_2) \\
\times \langle P, S | b^+(k_1) b^-(k_2) | P, S \rangle.
\]

(2.25)

where the spinor line function $L^{\gamma^+}(k_1, k_2)$ reads

\[
L^{\gamma^+}(k_1, k_2) = \bar{u}(k_1) \gamma^+ u(k_2).
\]

(2.26)

It is clear that if the correlator involves the interactions generated by the corresponding higher orders of $S$-matrix, the spinor line function $L^{\gamma^+}(k_1, k_2)$ becomes more complicated in comparison with Eq. (2.26). That is, we have

\footnote{The taking into account of the final(initial) state interaction in the corresponding in(out)-states may change the conclusion of the functional complexity. This subtlety is not being discussed in the present study.}
\[ \Phi^{[\gamma^+]}(k) \Rightarrow \langle P, S | \bar{\psi}(0) \gamma^+ \psi(z) S^{(\alpha)}[\psi, \bar{\psi}, A] | P, S \rangle \]
\[ = \int d^4\mu(k_1, \ldots, k_m) L^{[\Gamma_1 \ldots \Gamma_N]}(k_1, \ldots k_N) \times \langle P, S | \bar{b}^+(k_1) b^+(k) b^-(k_{i+1}) \ldots b^-(k_N) | P, S \rangle, \]
\[ (2.27) \]

where the spinor line function \( L^{[\Gamma_1 \ldots \Gamma_N]}(k_1, \ldots k_N) \) is determined by the \( N \) spinor lines. As a result, we have a possibility to introduce the new Lorentz combination for parametrization.

Equation (2.24) represents our principal result which reveals the existence of a new transverse-momentum dependent function \( f_1^{(1)}(x; k_1^2) \) and its analogue \( f_2^{(2)}(x; k_1^2) \). We also observe that Lorentz structure tensor, \( \epsilon^{+}P_{\perp s_{\perp}} \), associated with our function (see (2.24)) resembles the Sivers function, \( \epsilon^{+}P_{\perp s_{\perp}} \) in which the nucleon spin vector \( S_{\perp} \) is replaced by the quark spin vector \( s_{\perp} \). However, despite this similarity the Sivers function and the introduced function \( f_1^{(1)}(x; k_1^2) \) have totally different physical meaning.

Last but not least, the structure function \( f_1^{(1)}(x; k_1^2) \) of Eq. (2.24), roughly speaking, coincides with the function \( f_1^{(1)}(x; k_1^2) \) of Eq. (2.18) provided (a) we decipher the hadron matrix element of quark(-gluon) operators at least up where the spinor line function \( L^{[\Gamma_1 \ldots \Gamma_N]}(k_1, \ldots k_N) \) is determined by the \( N \) spinor lines. As a result, we have a possibility to introduce the new Lorentz combination for parametrization.

3 The Drell-Yan process and the new functions

We are now in a position to discuss the possible contribution of new functions to different observables. The simplest example of application is related to the well-known unpolarized Drell-Yan (DY) process, i.e. the lepton-production in nucleon-nucleon collision:

\[ N(P_1) + N(P_2) \rightarrow \gamma^*(q) + X(P_X) \]
\[ \rightarrow \ell(l_1) + \bar{\ell}(l_2) + X(P_X), \]
\[ (3.1) \]

with the initial unpolarized nucleons \( N \). The importance of the unpolarized DY differential cross section is due to the fact that it has been involved in the denominators of any spin asymmetries.

As mentioned above, the naive time-reversal transformation together with the Hermitian conjugation imply that the new functions are T-odd functions. Therefore, at the leading order, the hadron tensor which describes the unpolarized DY-process takes the following form:

\[ \psi_{\mu \nu}^{(0)} = \delta(\alpha) \left( \delta(x) dx dy \delta(x P_1^+ - q^+) \delta(y P_2^- - q^-) \times \text{tr}[\gamma_\nu \gamma_\mu \gamma_-] \Phi^{[\gamma^-]}(y) \right) \]
\[ \times \left\{ \int (d^2k_1^+ ) \Phi^{[\gamma^+]}(x, k_1^+ \omega) \right\}, \]
\[ (3.2) \]

where

\[ \Phi^{[\gamma^-]}(y) = P_2^- f(y), \quad \Phi^{[\gamma^+]}(x, k_1^\perp) \]
\[ = ie^{+k_1^\perp \hat{z}} f_2(x; k_1^\perp). \]
\[ (3.3) \]

This expression is derived within the factorization procedure described in detail in [12]. We stress that we adhere the CS-kinematics [9] in which the factorization procedure has the most archetypal form [13].

Calculating the contraction of Eq. (3.2) with the unpolarized lepton tensor \( L_{\mu \nu}^{(U)} \), we derive that

\[ d\sigma^{unpol.} \sim \int (d^2q_\perp) L_{\mu \nu}^{(U)} \psi_{\mu \nu}^{(0)} \]
\[ = \int (dx) (dy) \delta(x P_1^+ - q^+) \delta(y P_2^- - q^-) \]
\[ \times (1 + \cos^2 \theta) f(y) \int (d^2k_1^+) e^{+k_1^\perp \cdot \hat{z}} \]
\[ \times 3mf_2(x; k_1^\perp), \]
\[ (3.4) \]

where

\[ e^{+k_1^\perp \cdot \hat{z}} = k_1^\perp \times s_\perp \sim \sin(\phi_k - \phi_s) \]
\[ (3.5) \]

with \( \phi_A \), for \( A = (k, s) \), denoting the angles between \( A_\perp \) and \( O\hat{x}-\text{axis} \) in the CS-frame.

Notice that the angle \( \phi_s \) cannot explicitly be measured in the experiment. However, the implementation of the covariant (invariant) integration of \( f_2^{(2)}(x; k_1^\perp) \) gives the kinematical constraints on this angle relating the quark spin angle to the corresponding hadron angle. Indeed, let us consider the integration of Eq. (3.4) given by

\[ \gamma_{\perp}^{\perp} \]
\[ = \int (d^2k_1^+) f_2(x, k_1^\perp, s_\perp; P_1) k_1^\perp. \]
\[ (3.6) \]

Based on the Lorentz covariance, the integration \( \gamma_{\perp}^{\perp} \) can be presented as

\[ \gamma_{\perp}^{\perp} = P_1^\alpha \gamma_\alpha + i \varepsilon_{\alpha s_\perp +} \gamma_\alpha \]
\[ (3.7) \]

where \( P_1^\alpha \) and \( \varepsilon_{\alpha s_\perp +} \) form the orthogonal system and, hence, we have

\[ \gamma_\alpha = \int (d^2k_1^+) f_2(x, k_1^\perp, s_\perp; P_1) \left( k_1^\perp \cdot P_1^\perp \right) \frac{k_1^\perp \cdot P_1^\perp}{P_1^\perp \cdot P_1^\perp}, \]
\[ (3.8) \]

\[ \varepsilon_{\alpha s_\perp +} = \int (d^2k_1^+) f_2(x, k_1^\perp, s_\perp; P_1) \frac{1}{s_\perp^\perp} \]
\[ (3.9) \]

Since \( P_1^\perp \) and \( \varepsilon_{\alpha s_\perp +} \) are orthogonal each other by construction, we have that

\[ \varepsilon_{\alpha s_\perp +} \sim \sin \phi_{P_4} = 0 \Rightarrow \phi_P = \phi_s \pm n\pi. \]
\[ (3.10) \]
In other words, one can see that the orthogonality condition required by the covariant integration leads to the (anti)collinearity of $P_1^\perp$ and $s^\perp$. Eq. (3.10) relates the hadron momentum with the quark spin vector and it can be treated as another condition for the existence of this new function.

Thus, the new $k_\perp$-dependent function $f_2(x; k_1^2)$ gives the additional and additive contribution to the depolarization factor $D_{1[1+\cos^2 \theta]}$ appeared in the differential cross section of unpolarized DY process.

4 Conclusions

To conclude, in the paper we have introduced the new $k_\perp$-dependent function $f_1^{(1)}(x; k_1^2)$ and $f_2(x; k_1^2)$ of Eq. (2.24) which describe the transverse quark motion by the quark alignment along the fixed transverse direction. The introduced functions can be considered as a “in-between” functions of the Sivers and Boer-Mulders functions. Indeed, the Lorentz tensors accompanying our functions are quite similar to the analogous tensor at the Sivers function, however they reduced functions can be considered as a “in-between” function like the Boer-Mulders function.

We have shown that, to the second order of strong interactions, the new parametrizing function $f_1^{(1)}(x; k_1^2)$ can be related to the function $f_1^{(1)}(x; k_1^2)$ of (2.7) imposing the condition $\ell \ll \hat{k} \sim \hat{k}$ which corresponds to the regime where the appeared two spinor lines are interacting by exchanging of soft gluon. Moreover, the occurred four spinors generated by two spinor lines have the polarizations aligned along the same transverse direction. In physical terms, the $k_\perp$-dependent function $f_1^{(1)}(x; k_1^2)$ which describes the regime where $k_\perp$-dependence (or the transverse motion of quarks inside the hadron) has been entirely generated by the quark spin alignment.

As a practical application of the new functions, we have illustrated that the function $f_2(x; k_1^2)$ provides the additional contribution to the depolarization factor $D_{1[1+\cos^2 \theta]}$ which is associated with the differential cross section of unpolarized DY process.

Acknowledgements We thank colleagues from the Theoretical Physics Division of NCBJ (Warsaw) for useful and stimulating discussions. The work of L.Sz. is supported by the grant 2019/33/B/ST2/02588 of the National Science Center in Poland. This work is also supported by the Ulam Program of NAWA No. PPN/ULM/2020/1/00019.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: This is a theoretical study and it does not contain any experimental data].

Appendix

A. The amplitude and the interaction (or Heisenberg) representation

In this appendix we remind a role of the interaction representation in the parametrization of relevant nonperturbative correlators, despite the subject is the well-known textbook.

For example, we begin with the forward Compton scattering (CS) amplitude which is related to the deep inelastic scattering. It reads

$$\mathcal{A}_{\mu\nu} = \langle P | a_{\nu}^\perp(q) \mathbb{S}[\bar{\psi}, \psi, A] a_{\mu}^\perp(q) | P \rangle,$$

and we recall that

$$\mathbb{S}[\bar{\psi}, \psi, A] = T \exp \left\{ i \int (d^4 z) \big[ \mathcal{L}_{QCD}(z) + \mathcal{L}_{QED}(z) \big] \right\}.$$  

The commutation relations of creation (or annihilation) operators with $S$-matrix are given by

$$[a_{\mu}^\perp(q), \mathbb{S}[\bar{\psi}, \psi, A]] = \int (d^4 \xi) e^{\pm iq \xi} \frac{\delta \mathbb{S}[\bar{\psi}, \psi, A]}{\delta A^\mu(\xi)},$$

where

$$\frac{\delta \mathbb{S}[\bar{\psi}, \psi, A]}{\delta A^\mu(\xi)} = T \left\{ \int (d^4 z) \frac{\delta \mathcal{L}_{QED}(z)}{\delta A^\mu(\xi)} \mathbb{S}[\bar{\psi}, \psi, A] \right\}.$$  

Having used Eqn. (A.2) and the translation invariance, the CS-amplitude takes the form of

$$\mathcal{A}_{\mu\nu} = \int (d^4 \xi_1) (d^4 \xi_2) e^{-iq(\xi_1 - \xi_2)} \langle P | \frac{\delta^2 \mathbb{S}[\bar{\psi}, \psi, A]}{\delta A^\mu(\xi_1) \delta A^\nu(\xi_2)} | P \rangle \Rightarrow \int (d^4 z) e^{-iqz} \langle P | T \big( J_{\nu}(0) J_{\mu}(z) \mathbb{S}[\bar{\psi}, \psi, A] \big) | P \rangle.$$

From Eq. (A.4), using Wick’s theorem we can readily derive the “hand-bag” diagram contribution which has a form of (in the momentum representation before the factorization procedure applied)

$$\mathcal{A}_{\mu\nu} = \int (d^4 k) \text{tr} \big[ E_{\mu\nu}(k) \Phi(k) \big],$$

where

$$E_{\mu\nu}(k) = \gamma_\mu S(k + q) \gamma_\nu + \gamma_\nu S(k - q) \gamma_\mu,$$

$$\Phi(k) = \int (d^4 z) e^{iqz} \langle P | T \bar{\psi}(0) \psi(z) \mathbb{S}[\bar{\psi}, \psi, A] \big) | P \rangle_c.$$
where the subscript \( c \) denotes the connected diagram contributions only. Notice that after the factorization procedure applied to the CS-amplitude, \( S \)-matrix has to be included in the corresponding correlator forming the soft part of amplitude.

It is worth to notice that in Eq. (A.7) the nonperturbative correlator has been written in the interaction representation. In the literature, the wide-used Heisenberg representation of this correlator, i.e.

\[
\Phi(k) = \int (d^4 z) e^{ikz} \langle P | \bar{\psi}(0) \psi(z) | P \rangle^H, \tag{A.8}
\]
gives a very compact form but it may mislead, however, in many cases. Indeed, the neglecting \( H \) in Eq. (A.8) results in the wrong impression about the absence of interaction in the correlator.

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