D-brane form factors
at high energy

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We study the high energy, fixed angle, asymptotics of D-brane form factors to all orders in string perturbation theory, using the Gross-Mende saddle point techniques. The effective interaction size of all D-branes grows linearly with the energy as $\alpha' E/n$, where $n$ is the order of perturbation theory, except for the D-instanton, whose form factor is dominated by end-point contributions, and remains point-like at high orders. The qualitative features are independent of the R-R or NS-NS character of the states used to probe the D-brane.
An interesting question about D-branes, the stringy solitons carrying R-R charge in type II string theories [1][2], is their physical size. In particular, it was observed by Shenker in [3] that the physics of the R-R condensates [4] could be a hint at a new dynamical scale in string theory, of order $\lambda \sqrt{\alpha'}$, where $\lambda$ is the string coupling. In a technical sense D-branes are infinitely thin, because they are introduced as a sharp Dirichlet boundary condition for string propagation. This is analogous to the fact that an elementary string has zero thickness, by contrast to regular solitonic solutions of the low energy effective field theory. In practice, interactions set the physical size of elementary strings to be of order $\sqrt{\alpha'}$. Likewise, since D-branes interact via string exchange, we expect their physical size to be of order $\sqrt{\alpha'}$ as well [5]. As a direct check of this one could calculate the form factor of a static D-brane, as probed by string scattering [6]. Interestingly enough, the leading form factor shows stringy features for all D-branes except the D-instanton, which exhibits point-like behavior [7]. This is surprising in the light of the heuristic comments above.

A quick way of searching for stringy form factors is to identify soft high energy behavior. String scattering amplitudes at high center of mass squared energy $s$ and fixed angle $\phi$ have the form

$$A_g \sim e^{-s\alpha' \phi/(g+1)}$$

where $g + 1$ is the order in perturbation theory. This result was derived by Gross and Mende [8] by means of a saddle point evaluation of the worldsheet path integral (see also [9]). The existence of a saddle point in the interior of the moduli space is a signature of stringy behavior. On the other hand, we would interpret a power fall off with energy as a signature of a point-like object, by contrast to the soft high energy dependence displayed in (1).

It is very easy to adapt these techniques to the study of the D-brane form factor at high energy and fixed angle. An interesting aspect of this regime is its universality. If a Gross-Mende saddle exists in the interior of moduli space, then it is universal in the sense that it is independent of the particular string theory we are considering (one can work with the bosonic theory), and furthermore it is independent of the states in external legs. The exponential term (1) only depends on the tachyonic part of the vertex operator, and so the results apply to R-R external particles as well as NS-NS.

We consider the two point function of string states with the appropriate D-brane boundary conditions on $X^\mu$: Neumann for $\mu = 0, ..., p$; Dirichlet for $\mu = p + 1, ..., d - 1$. In the s channel this will be interpreted as the scattering of a string state of momentum
$p_1$ off a heavy stationary p-brane which absorbs $q = -p_2 - p_1$ momentum. Split the d-dimensional vectors $p_i$ into Neumann and Dirichlet components as

$$p_i = (p_i^0, \vec{N}_i, \vec{D}_i) \quad (2)$$

The Neumann components are conserved $p_i^0 + p_j^0 = 0 = \vec{N}_1 + \vec{N}_2$ and the p-brane absorbs the Dirichlet components of momentum $q = (0, 0, -\vec{D}_1 - \vec{D}_2)$. Define the momentum transfer squared

$$t = -q^2 = -\vec{D}_1^2 - \vec{D}_2^2 - 2\vec{D}_1 \cdot \vec{D}_2 \quad (3)$$

and the “transverse” invariant energy squared

$$s = -(p_i^0)^2 + \vec{N}_i^2 = -(p_j^0)^2 + \vec{N}_j^2 \quad (4)$$

These quantities satisfy $\vec{D}_i^2 = s - m_i^2$ and $2\vec{D}_1 \vec{D}_2 = -t - 2s + m_1^2 + m_2^2$. We will consider “frontal” scattering $\vec{N}_i = 0$ for which the following relations hold at high energy:

$$\vec{D}_1 \cdot \vec{D}_2 \sim -s \cos \phi \; , \; \; t \sim -4s \sin^2 \frac{\phi}{2} \quad (5)$$

where $\phi$ is the scattering angle between the ingoing and outgoing strings, to be held fixed in the high energy limit. This is the situation in which all the products $p_i \cdot p_j$ become large as compared to the masses $m_i^2 = -p_i^2$, and the world sheet path integral can be dominated by a saddle point. The general amplitude has the form

$$A_\chi = \lambda^{g+B+C+N-2} \int d\mu \prod_{i,j}^N e^{-\frac{1}{2} \sum_i p_i^\mu p_j^\nu \langle X_\mu(z_i) X_\nu(z_j) \rangle} \quad (6)$$

where $d\mu$ is some measure in the moduli space of Riemann surfaces with $g$ handles, $B$ boundaries, $C$ cross-caps and $N$ punctures where the vertex operators are inserted. This measure has polynomial dependence on the external momenta and polarizations. The saddle point is thus given by the (unstable) equilibrium momenta and polarizations. The saddle point is thus given by the (unstable) equilibrium configuration of $N$ Minkowskian charges with Coulomb interactions. For closed strings, these saddles where identified by Gross and Mende as the n-fold branched covers of the sphere with a $\mathbb{Z}_n$ automorphism:

$$y^n = \prod_{i=1}^L (z - a_i)^{L_i} \quad (7)$$

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1 In the following, we will focus on orientable world sheets, as appropriate for type II strings.
where \( \sum_i L_i = 0 \mod (n) \), all \( L_i \) are coprime with \( n \), and the operator insertions lie at the branch points. In general, the genus of such a curve is given by the Riemann-Hurwitz formula \( g = 1 - n + B/2 \) where \( B \) is the total branching number, counting multiplicity. For the case above \( (L_i, n) = 1 \) and all branch points have maximum order \( n - 1 \), leading to \( g = (n - 1)(L/2 - 1) \). In particular, with four branch points we have \( g = n - 1 \). The solution to the electrostatic problem is then trivial since the charges are located at the positions of the branch points. A gaussian curve has length proportional to \( g + 1 \) (it must go over each sheet before it closes), so the electric fields scale like \( \frac{1}{g+1} \). The area of the surface is proportional to \( g + 1 \), and the corresponding electrostatic energy is given by

\[
\mathcal{E}_g = \frac{1}{2} \sum_{i,j} p_i^\mu \cdot p_j^\nu \langle X_\mu(z_i)X_\nu(z_j) \rangle_g = \frac{\mathcal{E}_0}{g + 1} \tag{8}
\]

Branch points without associated charges do not contribute to the energy but increase the genus of the surfaces so that, at a given order in perturbation theory, they are subdominant with respect to the \( L = N \) case. In this way one can easily understand the higher genus saddles in terms of the sphere amplitude. It is also possible to deal with open string boundaries and cross-caps [10] by simply using a generalization of the Schwarz reflection principle. The open string saddles are obtained from closed string saddles which are reflection symmetric about the real axis. In the electrostatic problem, one is effectively using the method of images to calculate the potential energy in the presence of a set of conducting boundaries (along the real axis). So, for each open string saddle world sheet, there is a closed string saddle obtained by doubling the previous one. The generalization from Neumann to Dirichlet boundaries is straightforward from this point of view, because they simply correspond to perfectly insulating boundaries\(^2\). We may consider closed string saddles with four branch points located along the imaginary axis, such that the branched cover is symmetric with respect to reflection on the real axis. By an appropriate relabeling of the sheets we can always choose \( L_1 = 1 \) in (7). Thus, we consider curves of the form:

\[
y^{g_0+1} = \left( \frac{z - ia_1}{z + ia_1} \right) \left( \frac{z - ia_2}{z + ia_2} \right)^{L_2} \tag{9}
\]

In general, if we cut a genus \( g_0 \) closed orientable surface through the reflection plane we obtain a surface with \( g_2 + 1 \) boundaries and \( g_1 \) handles, such that \( g_0 = 2g_1 + g_2 \). The

\(^2\) As expected, under T-duality Dirichlet and Neumann conditions are interchanged, as well as momentum and winding modes in external states. So we have an electric-magnetic duality transformation on the world sheet [11].
Euler number of the open Riemann surface is \( \chi = 2 - 2g_1 - (g_2 + 1) = 1 - g_0 \), and we find \( 2 - \chi = 1 + g_0 \). Using (8) we conclude that the exponential term at the saddle point is given by

\[
e^{-\mathcal{E}_x} = e^{-\mathcal{E}_0 / \chi} \tag{10}
\]

In the case at hand, \( \mathcal{E}_0 \) is the electrostatic energy of the upper half plane with two charges \( p_1 \) and \( p_2 \). The relevant Green function is

\[
\langle X^\mu(z)X^\nu(w) \rangle = -\alpha' \eta^{\mu\nu} (\log|z - w| \pm \log|z - \bar{w}|) \tag{11}
\]

where the \( \pm \) corresponds to the Neumann or Dirichlet components respectively. The resulting electrostatic energy uses image charges \( p_i' = (p_i^0, \vec{N}_i, -\vec{D}_i) \) and has the form

\[
\mathcal{E}_0 = -\frac{\alpha'}{2} \sum_{i \neq j} p_i \cdot p_j \log|z_i - z_j| - \frac{\alpha'}{2} \sum_{i, j} p_i \cdot p_j' \log|z_i - \bar{z}_j| \tag{12}
\]

where singular self-energy terms have been subtracted. We can now use the \( SL(2, \mathbb{R}) \) symmetry to fix the second charge at \( z_2 = i \) and constrain \( z_1 = iy \) to lie in the positive imaginary axis. In the high energy limit \( s \to \infty \) with fixed angle \( \phi \),

\[
\frac{\mathcal{E}_0}{2\alpha'} \to s \log 2 + \frac{t}{4} \log|y - 1| - (s + t/4) \log|1 + y| + \frac{1}{2} s \log y \tag{13}
\]

The saddle point is located at \( \hat{y} = e^{\pm i\phi} \), outside the integration domain for \( y \), and we need to distort the contour. The real part of the energy at the saddle is

\[
\hat{\mathcal{E}}_0 = -\alpha' s \left( \sin^2 \frac{\phi}{2} \log \frac{\sin^2 \phi}{2} + \cos^2 \frac{\phi}{2} \log \cos^2 \frac{\phi}{2} \right) \tag{14}
\]

There is also a phase which combines with others coming from the fluctuation determinant around the saddle point, producing the complete phase following from the Stirling approximation of the expression:

\[
\frac{\Gamma(-\alpha' s)\Gamma(-\frac{\alpha' t}{4})}{\Gamma(-\alpha' s - \frac{\alpha' t}{4})}
\]

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\(^3\) The electrostatic problem has \( SL(2, \mathbb{R}) \) symmetry precisely in the case of Minkowskian charges, where we can neglect \( p_i^2 \) as compared to \( p_i \cdot p_j \).
which occurs in the exact evaluation of the disk form factor \[^{3}\]. Finally, putting everything together we arrive at the final result for the leading exponential dependence of the form factor at order $2 - \chi$ in perturbation theory:

$$A_\chi \sim \lambda^{2-\chi} \exp \frac{\alpha' s}{2 - \chi} \left( \sin^2 \frac{\phi}{2} \log \sin^2 \frac{\phi}{2} + \cos^2 \frac{\phi}{2} \log \cos^2 \frac{\phi}{2} \right)$$

(15)

We see that the form factor is characterized by the string scale $\sqrt{\alpha'}$ independently of the particular states (NS-NS or R-R) we use to probe the D-brane. The genus dependence in (15) is precisely the same that one finds in high energy open string scattering \[^{10}\]. This is hardly a surprise, given the relation between open strings and D-branes via T-duality.

It is interesting to study the structure of the classical saddle world sheets:

$$X^\mu_{c\ell}(z) = \frac{i\alpha'}{2 - \chi} \sum_{j=1}^{N} p^\mu_j \left( \log(|z - a_j| \pm \log|z - \bar{a}_j|) \right)$$

(16)

For the closed string scattering with world sheet $y^{g_0+1} = \prod_{i=1}^{4} (z-a_i)L_i$, two strings wound $g_0 + 1$ times interact in a one-string irreducible vertex and propagate an intermediate state of several closed strings \[^{8}\]. The effective vertices in a particular channel $i + j$ can be obtained from the degeneration in which the branch points $a_i$ and $a_j$ coincide. The branching number of the resulting branch point $a_{i+j}$ depends on whether $g_0 + 1$ and $L_i + L_j$ have a common factor, and it is given by $B_{i+j} = g_0 + 1 - (g_0 + 1, L_i + L_j)$. Applying the Riemann-Hurwitz formula we end up with an effective vertex of genus $g_{i+j} = B_{i+j}/2$, and $g_0 - 2g_{i+j} + 1 = (g_0 + 1, L_i + L_j)$ intermediate propagating strings. For the curves above (9), we get genus zero effective vertices in the $s$ channel, with $M = (2 - \chi, 1 + L_2)$ intermediate open strings propagating along the D-brane, and $M' = (2 - \chi - M)/2$ closed strings exchanged. In the $t$ channel we have a genus $M'$ effective vertex, and $M$ closed strings exchanged with the D-brane. The important feature from the physical point of view is that, according to the formula (16), the classical world sheet saturating the form factor has a scale $\langle X \rangle \sim (\text{energy})/(2 - \chi)$, whereas the Schwarzschild radius for that energy scales like $\lambda^2(\text{energy})$. So, no matter how small we take the string coupling, there is a potential breakdown of perturbation theory at order $2 - \chi \sim \lambda^{-2}$, due to strong gravity effects.

We may also compare formula (15) with the Regge approximation. In the limit of small scattering angles $-t << s$ it is reasonable to regard the higher genus amplitude as a multi-scattering process where $2 - \chi$ closed strings are exchanged with the D-brane. In
fact, this is the structure of the dominant saddle world sheets without effective vertices in the $t$ channel. To lowest order

$$A_0 \sim \lambda s^{\alpha' t/4}$$

We then estimate the amplitude as

$$A_\chi \sim \text{Max} \lambda^{2-\chi} s^{(\alpha' \sum_i t_i)/4}$$

subject to the constraint $\sum_{i=1}^{2-\chi} \sqrt{-t_i} \leq \sqrt{-t}$. The solution is $t_i = t/(2 - \chi)^2$ and we find

$$A_\chi \sim \lambda^{2-\chi} s^\frac{\alpha' t}{4(2-\chi)}$$

in perfect agreement with \(13\). It is amusing to notice that the effective Regge slope for the interaction with the D-brane is $\alpha'_D = \alpha'/2$, precisely the T-dual of the open string Regge slope $\alpha'_{\text{open}} = 2\alpha' = \alpha'^2/\alpha'_D$.

One could easily generalize these results to non-orientable world-sheets (along the lines of ref. [10]) with similar qualitative behavior. Also, more general amplitudes can be considered. For example, “deep inelastic” processes in which we have many strings in the final state. Unfortunately, as for the case of multi-string scattering, the corresponding saddle points are difficult to characterize analytically. One could also discuss amplitudes with Dirichlet boundaries located at different points in the target space (several D-branes). However, such calculations have a limited physical interest, because they do not really capture D-brane dynamics. The saddle point techniques regard the D-branes as static objects capable of absorbing any amount of energy-momentum. This picture is precisely wrong in the high energy limit. Here, we have used the high energy limit only as a device to extract some general features of the form factors at high orders in perturbation theory.

The methods used imply that the D-instanton remains point-like (power behaved form factor at high energy) to all orders in perturbation theory. In this case we have Dirichlet boundary conditions in all coordinates and the high energy limit of the electrostatic energy degenerates to

$$\mathcal{E}_\chi \rightarrow -\frac{\alpha' t}{2(2-\chi)} \log \left| \frac{1+y}{1-y} \right|$$

The integral over $y$ is dominated by the end-point contribution at $y = 0$ and the saddle point at $y = \infty$, both of zero energy, which explains the power-like behavior of the amplitude. In the fully Dirichlet case, the image charge is completely inverted $p' = -p$ and it is thus natural that no equilibrium configuration of separate charges exists: the vertex
operators tend to collapse on the Dirichlet boundary. For p-branes with $p > -1$, some of the components of the image charges have the same sign, and there is a compromise between repulsion and attraction leading to a non-trivial saddle point in the interior of moduli space.

1. Acknowledgements

It is a pleasure to thank D. Gross, I. Klebanov and S. Ramgoolam for useful discussions. This work was supported by NSF PHY90-21984 grant.
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