A NOTE ON TYPE 2 CHANGHEE AND DAEHEE POLYNOMIALS

DAE SAN KIM AND TAEKYUN KIM

Abstract. In recent years, many authors have studied Changhee and Daehee polynomials in connection with many special numbers and polynomials. In this paper, we investigate type 2 Changhee and Daehee numbers and polynomials and give some identities for these numbers and polynomials in relation to type 2 Euler and Bernoulli numbers and polynomials. In addition, we express the central factorial numbers of the second kind in terms of type 2 Bernoulli, type 2 Changhee and type 2 Daehee numbers of negative integral orders.

1. Introduction

Let $p$ be a fixed prime number with $p \equiv 1 \pmod{2}$. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of $p$-adic integers, the field of $p$-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. As is known, the Stirling numbers of the second kind are defined by

$$\frac{1}{k!}(e^t - 1)^k = \sum_{n=k}^{\infty} S_2(n, k) \frac{t^n}{n!}, \quad (k \geq 0), \quad (\text{see [1, 2, 4, 15]}). \quad (1.1)$$

For $n \geq 0$, the central factorial is defined as (see [14])

$$x^{[0]} = 1, \quad x^{[n]} = x\left(x + \frac{n}{2} - 1\right)\left(x + \frac{n}{2} - 2\right) \cdots \left(x - \frac{n}{2} + 1\right), \quad (n \geq 1). \quad (1.2)$$

The central factorial numbers of the second kind are given by

$$x^n = \sum_{k=0}^{n} T(n, k)x^{[k]}, \quad (n \geq 0), \quad (\text{see [1, 2, 15, 17]}). \quad (1.3)$$
Let $f(x)$ be a continuous function on $\mathbb{Z}_p$. Then the fermionic $p$-adic integral on $\mathbb{Z}_p$ is defined by Kim as
\[
\int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{ (see [3,5-8]).} \tag{1.4}
\]

Thus, by (1.4), we get
\[
\int_{\mathbb{Z}_p} f(x+1) d\mu_{-1}(x) + \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = 2f(0), \quad \text{ (see [5,6]).} \tag{1.5}
\]

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, the generating function of Changhee polynomials can be expressed in terms of the following fermionic integral on $\mathbb{Z}_p$:
\[
\int_{\mathbb{Z}_p} (1+t)^{x+y} d\mu_{-1}(y) = \frac{2}{2+t} (1+t)^x = \sum_{n=0}^{\infty} Ch_n(x) \frac{t^n}{n!}. \tag{1.6}
\]

In particular, for $x = 0$, $Ch_n = \text{Ch}_n(0)$ are called the Changhee numbers (see [7,11-14]).

From (1.6), we note that
\[
E_n(x) = \sum_{l=0}^{n} S_2(n,l) Ch_l(x), \quad (n \geq 0), \quad \text{ (see [9,12,13,16]),} \tag{1.7}
\]
where $E_n(x)$ are the ordinary Euler polynomials given by
\[
\frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \quad \text{ (see [4-10]).} \tag{1.8}
\]

From (1.8), we note that
\[
\text{Ch}_n(x) = \sum_{l=0}^{n} E_l(x) S_1(n,l), \quad \text{ (see [9,12,16]),} \tag{1.9}
\]
where $S_1(n,l)$ are the Stirling numbers of the first kind.

In this paper, we introduce type 2 Changhee numbers and polynomials which can be expressed in terms of fermionic $p$-adic integrals on $\mathbb{Z}_p$. We derive some identities for these numbers and the polynomials in relation to type 2 Euler and Bernoulli numbers and polynomials. In addition, we express the central factorial numbers of the second kind in terms of type 2 Bernoulli, type 2 Changhee and type 2 Daehee numbers of negative integral orders.
2. Type 2 Changhee and Daehee numbers and polynomials

In this section, we assume that \( t \in \mathbb{C}_p \) with \( |t|_p < p^{-\frac{1}{p}} \). As is well known, the type 2 Euler polynomials are defined by the generating function

\[
\int_{\mathbb{Z}/p} e^{(2y+1+x)t} d\mu_1(y) = \frac{2}{e^t + e^{-t}} e^{xt} = \sum_{n=0}^{\infty} E^*_n(x) \frac{t^n}{n!}.
\]  

(2.1)

Indeed, we note that \( E^*_n(x) = 2^n E_n(\frac{x+1}{2}) \), \( (n \geq 0) \), (see [4−10]). When \( x = 0 \), \( E^*_n = E_n(0) \) will be called the type 2 Euler numbers in this paper.

Motivated by (1.6) and (2.1), we define the type 2 Changhee polynomials as

\[
\int_{\mathbb{Z}/p} (1 + t)^{2y+1+x} du_1(y) = \frac{2}{(1 + t) + (1 + t)^{-1}}(1 + t)^x = \sum_{n=0}^{\infty} c_n(x) \frac{t^n}{n!},
\]  

(2.2)

When \( x = 0 \), \( c_n = c_n(0) \) are called the type 2 Changhee numbers.

By replacing \( t \) by \( \log(1 + t) \) in (2.1), we get

\[
\frac{2}{(1 + t) + (1 + t)^{-1}}(1 + t)^x = \sum_{l=0}^{\infty} E^*_l(x) \frac{1}{l!}(\log(1 + t))^l = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} S_1(n, l) c_l(x) \right) \frac{t^n}{n!},
\]  

(2.3)

where \( S_1(n, l) \) are the Stirling numbers of the first kind.

From (2.2) and (2.3), we have the following theorem.

**Theorem 2.1.** For \( n \geq 0 \), we have

\[
c_n(x) = \sum_{l=0}^{n} E^*_l(x) S_1(n, l).
\]

Replacing \( t \) by \( e^t - 1 \) in (2.2), we have

\[
\frac{2}{e^t + e^{-t}} e^{xt} = \sum_{l=0}^{\infty} c_l(x) \frac{1}{l!}(e^t - 1)^l = \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} S_2(n, l) c_l(x) \right) \frac{t^n}{n!}.
\]  

(2.4)

Therefore, by (2.1) and (2.3), we get the following theorem.

**Theorem 2.2.** For \( n \geq 0 \), we have

\[
E^*_n(x) = \sum_{l=0}^{n} S_2(n, l) c_l(x).
\]
For $\alpha \in \mathbb{R}$, let us define the type 2 Changhee polynomials of order $\alpha$ by
\[
\left(\frac{2}{(1+t) + (1+t)^{-1}}\right)^\alpha (1+t)^x = \sum_{n=0}^{\infty} c_n^{(\alpha)}(x) \frac{t^n}{n!}. \tag{2.5}
\]
When $x = 0$, $C_n^{(\alpha)} = C_n^{(\alpha)}(0)$ are called the type 2 Changhee numbers of order $\alpha$.

For $k \in \mathbb{N} \cup \{0\}$, let us take $\alpha = -k$. Then, by (2.5) with $x = 0$ and $t$ replaced by $e^{t^2} - 1$, we get
\[
\frac{1}{2^k} (e^{t^2} - e^{-t^2})^k = \sum_{l=0}^{\infty} \frac{k!}{k^l} \sum_{l=0}^{n} S_2(n,l)c_l^{(-k)} \frac{t^n}{n!}. \tag{2.6}
\]

On the other hand, by Proposition 1 to be shown below, we get
\[
\frac{1}{2^k} (e^{t^2} - e^{-t^2})^k = \frac{k!}{2^k} \sum_{n=0}^{\infty} T(n,k) \frac{t^n}{n!}. \tag{2.7}
\]

Therefore, by (2.6) and (2.7), we get the following theorem.

**Theorem 2.3.** For $n \geq k$, we have
\[
T(n,k) = \frac{2^{k-n}}{k!} \sum_{l=0}^{n} S_2(n,l)c_l^{(-k)},
\]
and, for $0 \leq n \leq k - 1$,
\[
\sum_{l=0}^{n} S_2(n,l)c_l^{(-k)} = 0.
\]

The central difference operator is given by (see [14,15])
\[
\delta f(x) = f(x + \frac{1}{2}) - f(x - \frac{1}{2}). \tag{2.8}
\]
After applying the operator $\delta$ once more, we obtain
\[
\delta^2 f(x) = f(x + 1) - 2f(x) + f(x - 1) = \sum_{l=0}^{2} \binom{2}{l} (-1)^{2-l} f(x + l - \frac{2}{2}). \tag{2.9}
\]
Continuing this process, we have
\[
\delta^k f(x) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} f(x + l - \frac{k}{2}), \quad (k \in \mathbb{N} \cup \{0\}). \tag{2.10}
\]
Let $f(x)$ be analytic at $x = b$, ($b \in \mathbb{R}$). Then $f(x)$ can be rewritten as

$$f(x) = \sum_{n=0}^{\infty} A_n(x-b)^{[n]}.$$  (2.11)

Now, we observe that

$$\delta x^{[n]} = (x + \frac{1}{2})^{[n]} - (x - \frac{1}{2})^{[n]}$$

$$= (x + \frac{1}{2})(x + \frac{n-1}{2}) \cdots (x - \frac{n-3}{2}) - (x - \frac{1}{2})(x + \frac{n-3}{2}) \cdots (x - \frac{n-1}{2})$$

$$= n(x + \frac{n-3}{2})(x + \frac{n-4}{2}) \cdots (x - \frac{n-3}{2})$$

$$= n x^{[n-1]}.$$  (2.12)

From (2.11) and (2.12), we have

$$\delta^k f(b) = \delta^k f(x)|_{x=b} = A_k k!.$$  (2.13)

By (2.11) and (2.12), we get

$$f(x) = \sum_{k=0}^{\infty} \frac{\delta^k f(b)}{k!} (x-b)^{[k]}.$$  (2.14)

In particular, if we take $b = 0$, then we have

$$f(x) = \sum_{k=0}^{\infty} \frac{\delta^k f(0)}{k!} x^{[k]}.$$  (2.15)

Let us take $f(x) = e^t$. Then, by (2.10), we get

$$\delta^k f(0) = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} t^{(l-\frac{1}{2})}$$

$$= e^{-\frac{1}{2}t} (e^{t} - 1)^k = (e^{\frac{1}{2}} - e^{-\frac{1}{2}})^k.$$  (2.16)

From (2.15) and (2.16), we have

$$e^t = \sum_{k=0}^{\infty} \frac{1}{k!} (e^{\frac{1}{2}} - e^{-\frac{1}{2}})^k x^{[k]}.$$  (2.17)
On the other hand, by (1.3), we get
\[ e^{xt} = \sum_{n=0}^{\infty} x^n t^n n! = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} T(n, k) x^k \right) \frac{t^n}{n!} \]
\[ = \sum_{k=0}^{\infty} \left( \sum_{n=k}^{\infty} T(n, k) x^n \frac{t^n}{n!} \right) x^k. \] (2.18)

Therefore, by (2.17) and (2.18), we obtain the following proposition.

**Proposition 1.** For \( k \geq 0 \), we have
\[ \frac{1}{k!} (e^t - e^{-t})^k = \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!}. \]

It is well known that Daehee polynomials are defined by
\[ \log(1 + t) \frac{(1 + t)^x}{t} = \sum_{n=0}^{\infty} D_n(x) t^n n!, \] (see [4, 10]). (2.19)

When \( x = 0 \), \( D_n = D_n(0) \) are called the Daehee numbers.

Now, we consider the type 2 Bernoulli polynomials given by
\[ \frac{t}{e^t - e^{-t}} e^{xt} = \sum_{n=0}^{\infty} b_n(x) \frac{t^n}{n!}. \] (2.20)

When \( x = 0 \), \( b_n = b_n(0) \) are called the type 2 Bernoulli numbers.

Indeed, we note that
\[ b_n(x) = 2^{n-1} B_n \left( \frac{x + 1}{2} \right), \quad (n \geq 0), \] (2.21)
where \( B_n(x) \) are the ordinary Bernoulli polynomials defined by
\[ \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \] (see [5, 15]). (2.22)

In the view of (2.19) and (2.20), we define the type 2 Daehee polynomials by
\[ \log(1 + t) (1 + t)^x (1 + t)^{-1} (1 + t)^x = \sum_{n=0}^{\infty} d_n(x) \frac{t^n}{n!}. \] (2.23)

When \( x = 0 \), \( d_n = d_n(0) \) are called the type 2 Daehee numbers.
Replacing $t$ by $e^t - 1$ in (2.23), we get

$$\frac{t}{e^t - e^{-t}} e^{xt} = \sum_{l=0}^{\infty} \frac{d_l(x)}{l!} (e^t - 1)^l$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} S_2(n,l) b_l(x) \right) \frac{t^n}{n!},$$

where $S_2(n,l)$ are the Stirling numbers of the second kind.

From (2.20) and (2.24), we have the following theorem.

**Theorem 2.4.** For $n \geq 0$, we have

$$b_n(x) = \sum_{l=0}^{n} S_2(n,l) b_l(x).$$

In particular,

$$b_n = \sum_{l=0}^{n} S_2(n,l) b_l.$$

From (2.20), we can derive the following equation.

$$\log(1 + \frac{t}{1 + t}) (1 + t)^x = \sum_{l=0}^{\infty} b_l(x) \frac{1}{l!} (\log(1 + t))^l$$

$$= \sum_{n=0}^{\infty} \left( \sum_{l=0}^{n} S_1(n,l) b_l(x) \right) \frac{t^n}{n!},$$

From (2.20) and (2.25), we obtain the following theorem.

**Theorem 2.5.** For $n \geq 0$, we have

$$d_n(x) = \sum_{l=0}^{n} S_1(n,l) b_l(x).$$

In particular,

$$d_n = \sum_{l=0}^{n} S_1(n,l) b_l.$$

For $\alpha \in \mathbb{R}$, let us define the type 2 Daehee polynomials of order $\alpha$ by

$$\left( \frac{\log(1 + t)}{(1 + t) - (1 + t)^{-1}} \right)^\alpha (1 + t)^x = \sum_{n=0}^{\infty} d_n^{(\alpha)}(x) \frac{t^n}{n!},$$

When $x = 0$, $d_n^{(\alpha)} = d_n^{(\alpha)}(0)$ are called the type 2 Daehee numbers of order $\alpha$. 
For $k \in \mathbb{N} \cup \{0\}$, let us take $\alpha = -k$. Then, by (2.26) with $x = 0$ and $t$ replaced by $e^{\frac{t}{k}} - 1$, we get
\[
(\frac{2}{t})^k(e^{\frac{t}{k}} - e^{-\frac{t}{k}})^k = \sum_{l=0}^{\infty} d_l^{(-k)} \frac{l!}{n!}(e^{\frac{t}{k}} - 1)^l = \sum_{n=0}^{\infty} \left( \frac{1}{2^n} \sum_{l=0}^{n} d_l^{(-k)} S_2(n,l) \right) \frac{t^n}{n!}.
\] (2.27)

On the other hand, by Proposition 1, we get
\[
(\frac{2}{t})^k(e^{\frac{t}{k}} - e^{-\frac{t}{k}})^k = 2^k \frac{k!}{k} \sum_{n=0}^{\infty} T(n,k) \frac{t^n}{n!} = 2^k \sum_{n=0}^{\infty} T(n+k,k) \frac{1}{(n+k)!} \frac{t^n}{n!}.
\] (2.28)

Therefore, by (2.27) and (2.28), we get the following theorem.

**Theorem 2.6.** For $n, k \geq 0$, we have
\[
2^{n+k}T(n+k,k) = \binom{n+k}{k} \sum_{l=0}^{n} d_l^{(-k)} S_2(n,l).
\]

Now, we define the type 2 Bernoulli polynomials of order $\alpha$ by
\[
\left( \frac{t}{e^t - e^{-t}} \right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} b_n^{(\alpha)}(x) \frac{t^n}{n!}.
\] (2.29)

In particular, $\alpha = k \in \mathbb{N}$, we have
\[
t^k e^{xt} \underbrace{\text{csch} t \times \cdots \times \text{csch} t}_{k \text{ times}} = 2^k \sum_{n=0}^{\infty} b_n^{(\alpha)}(x) \frac{t^n}{n!}.
\]

When $x = 0$, $b_n^{(\alpha)} = b_n^{(\alpha)}(0)$ are called the type 2 Bernoulli numbers of order $\alpha$.

From by (2.26) and (2.29), we have the following Corollary.

**Corollary 2.7.** For $n \geq 0$, we have
\[
b_n^{(\alpha)}(x) = \sum_{l=0}^{n} S_2(n,l) d_l^{(\alpha)}(x),
\]
and
\[ d^{(\alpha)}_n(x) = \sum_{l=0}^{n} S_1(n, l) h^{(\alpha)}_l(x). \]

From (2.29), we observe that
\[ \frac{1}{t^k} (e^{\frac{x}{t}} - e^{-\frac{x}{t}})^k = \sum_{n=0}^{\infty} \frac{b_n^{(-k)}}{2^n n!} t^n. \]

(2.30)

On the other hand,
\[ \frac{1}{t^k} (e^{\frac{x}{t}} - e^{-\frac{x}{t}})^k = \frac{k!}{t^k} \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{T(n+k, k)}{\binom{n+k}{k}} \frac{t^n}{n!}. \]

(2.31)

Therefore, by (2.30) and (2.31), we get the following theorem.

**Theorem 2.8.** For \( n, k \geq 0 \), we have
\[ 2^{n+k} T(n+k, k) = \binom{n+k}{k} b^{(-k)}_n. \]

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**Department of Mathematics, Sogang University, Seoul 121-742, Republic of Korea**

*E-mail address: dskim@sogang.ac.kr*

**Department of Mathematics, Kwangwoon University, Seoul 139-701, Republic of Korea**

*E-mail address: tkkim@kw.ac.kr*