STABILITY RESULTS FOR SOLUTIONS OF
OBSTACLE PROBLEMS WITH MEASURE DATA

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Abstract

In this paper we study the continuous dependence with respect to obstacles for obstacle problems with measure data. This is deeply investigated introducing a suitable type of convergence, which gives stability under very general hypotheses. Moreover stability with respect to $H^1$ and uniform convergent obstacles is proved.

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1. Introduction

Given a regular bounded open set $\Omega$ of $\mathbb{R}^N$, $N \geq 1$, and a linear elliptic operator $A$ of the form

$$Au = -\sum_{j,j=1}^{N} D_i(a_{ij}D_ju),$$

with $a_{ij} \in L^\infty(\Omega)$, we study obstacle problems for the operator $A$ in $\Omega$ with homogeneous Dirichlet boundary conditions on $\partial\Omega$, when the datum $\mu$ is a bounded Radon measure on $\Omega$ and the obstacle $\psi$ is an arbitrary function on $\Omega$.

According to [12], a function $u$ is a solution of this problem, which will be denoted by $OP(\mu,\psi)$, if $u$ is the smallest function with the following properties: $u \geq \psi$ in $\Omega$ and $u$ is a solution in the sense of Stampacchia [22] of a problem of the form

$$\begin{cases}
Au = \mu + \lambda & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}$$

for some bounded Radon measure $\lambda \geq 0$. The measure $\lambda$ which corresponds to the solution of the obstacle problem is called the obstacle reaction.

Existence and uniqueness of the solution of $OP(\mu,\psi)$ have been proved in [12], provided that there exists a measure $\lambda$ such that the solution of (1.1) is greater than or equal to $\psi$. These results have been extended to the non-linear case in [17], when $\mu$ vanishes on all sets with capacity zero. For a different approach to obstacle problems for non-linear operators with measure data see [6], [4], [5], [19] and [20].

If the measure $\mu$ belongs to the dual $H^{-1}(\Omega)$ of the Sobolev space $H^1_0(\Omega)$, and if there exists a function $w \in H^1_0(\Omega)$ above the obstacle $\psi$, then the solution of the obstacle problem $OP(\mu,\psi)$ according to the previous definition coincides with the solution $u$ of the variational inequality

$$\begin{cases}
u \in H^1_0(\Omega), & u \geq \psi, \\
\langle Au, v - u \rangle \geq \langle \mu, v - u \rangle, & \forall v \in H^1_0(\Omega) \text{ s.t. } v \geq \psi
\end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1(\Omega)$ and $H^1_0(\Omega)$. In this case the obstacle reaction $\lambda$ belongs to $H^{-1}(\Omega)$. We also know it is concentrated on the contact set $\{u = \psi\}$ if $\psi$ is continuous, or, more in general, quasi upper semicontinuous.

An important role in this problem is played by the space $\mathcal{M}_b^0(\Omega)$ of all bounded Radon measures on $\Omega$ which are absolutely continuous with respect to the harmonic capacity. If the datum $\mu$ belongs to $\mathcal{M}_b^0(\Omega)$ (it is actually enough that its negative part $\mu^{-}$ is such), so does the obstacle reaction, provided that there exists a measure $\lambda \in \mathcal{M}_b^0(\Omega)$ such that the solution of (1.1) is greater than or equal to $\psi$ (see [12], theorem 7.5). In this case the obstacle
reaction is concentrated on the contact set \( \{ u = \psi \} \), whenever the obstacle \( \psi \) is quasi upper semicontinuous (see [17], theorem 2.9).

It can be seen that in general this does not occur when \( \mu^- \not\in \mathcal{M}_b^0(\Omega) \). This case was studied in [11]. Remembering that \( \mu^- \) can be decomposed as \( \mu^- = \mu_a^- + \mu_s^- \), where \( \mu_a^- \in \mathcal{M}_b^0(\Omega) \) and \( \mu_s^- \) is concentrated on a set of capacity zero, it was proved that under some natural assumptions on the obstacle, the singular part \( \mu_s^- \) can be neglected and the obstacle problems \( OP(\mu, \psi) \) and \( OP(\mu^+ - \mu_a^-, \psi) \) have the same solutions.

The topic of continuous dependence with respect to data was already treated in [12].

As for stability with respect to the right hand side, it was proved that, if \( \mu_n, \mu \in \mathcal{M}_b(\Omega) \) are such that \( \mu_n \rightharpoonup \mu \) strongly in \( \mathcal{M}_b(\Omega) \), then \( u_n \rightharpoonup u \) strongly in \( W^{1,q}(\Omega) \), where \( u_n \) and \( u \) are the solutions of \( OP(\mu_n, \psi) \) and \( OP(\mu, \psi) \) respectively. Trying to use weak-* convergence, it was seen that in general \( \mu_n \rightharpoonup \mu \) weakly-* does not imply that \( u_n \rightarrow u \), even with the obstacle \( \psi \equiv 0 \), but we know only that for any measure \( \mu \in \mathcal{M}_b(\Omega) \), there exists a special sequence \( \mu_k \rightharpoonup \mu \) weakly-* in \( \mathcal{M}_b(\Omega) \), with \( \mu_k \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega) \), such that \( u_k \rightarrow u \) strongly in \( W^{1,q}(\Omega) \).

In this paper we consider stability with respect to obstacles. To study this question we introduce a kind of convergence of functions, the level set convergence, which yields the convergence of solutions under very mild assumptions.

The convergence of \( \psi_n \) to \( \psi \) in the sense of level sets, defined precisely in definition 3.1, is verified in particular when

\[
\text{cap}(\{ \psi > t \} \cap B) = \lim_{n \to +\infty} \text{cap}(\{ \psi_n > t \} \cap B)
\]

for all \( t \in \mathbb{R} \) and for all \( B \subset \subset \Omega \) (see also remark 3.2).

We will see that without further hypothesis it can only be proved that, calling \( u_n \) and \( u \) the solutions of \( OP(\mu, \psi_n) \) and of \( OP(\mu, \psi) \) respectively, if \( \psi_n \rightharpoonup^{\text{lev}} \psi \) then, up to a subsequence, \( u_n \) converges to some function \( u^* \) which is always greater than or equal to \( u \) (proposition 3.9).

Then we will obtain, from the level set convergence of the obstacles, that \( u_n \) converges to \( u \), under some conditions: in particular, by means of the Mosco convergence of convex sets, we obtain that

(i) if \( \mu^- \in H^{-1}(\Omega) \) then \( u_n \rightarrow u \) strongly in \( H^1(\Omega) \);
(ii) if \( \mu^- \in \mathcal{M}_b^0(\Omega) \) then \( u_n \rightarrow u \) strongly in \( W^{1,q}(\Omega) \);
(iii) if \( \psi \) is suitably controlled below, then \( u_n \rightarrow u \) strongly in \( W^{1,q}(\Omega) \).

In section 4 we consider the case \( \psi_n \leq \psi \) and show that we have the convergence of solutions for any datum \( \mu \in \mathcal{M}_b(\Omega) \).

We conclude this study considering two cases in which the assumptions that the obstacles converge in a stronger way allows to obtain a stronger convergence also for the solutions.
When the difference $\psi_n - \psi$ belongs to $H^1_0(\Omega)$ and tends to zero strongly in this space, then we obtain the same type of convergence for the solutions, for any $\mu \in M_b(\Omega)$.

In section 6, we extend the theory so far developed to the case of nonzero boundary values. For any function $g \in H^1(\Omega)$, we can define the function $u$ to be the solution of $OP(\mu, g, \psi)$ if and only if $u - u^g_0$ is the solution of $OP(\mu, \psi)$, where $u^g_0$ is the solution of

$$\begin{cases}
Au^g_0 = 0 & \text{in } H^{-1}(\Omega) \\
u^g_0 - g \in H^1_0(\Omega).
\end{cases}$$

All the results developed in the case of homogeneous boundary conditions can be extended, thanks to the linearity of $A$.

Using this extension we prove a new characterization: the solution of $OP(\mu, g, \psi)$ is the minimum element among all the supersolutions of $A - \mu$ which are above the obstacle and greater than or equal to $g$ on the boundary $\partial \Omega$. From this we easily prove that if the obstacles converge uniformly then so do the solutions of the corresponding obstacle problems for any $\mu \in M_b(\Omega)$.

2. Notations and basic results.

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$, $N \geq 1$, with Lipschitz boundary.

Let $A(u) = -\text{div}(A(x)\nabla u)$ be a linear elliptic operator with coefficients in $L^\infty(\Omega)$, that is $A(x) = (a_{ij}(x))$ is an $N \times N$ matrix such that

$$a_{ij} \in L^\infty(\Omega) \quad \text{and} \quad \sum a_{ij}(x)\xi_i\xi_j \geq \gamma |\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \ a.e. \ in \ \Omega,$$

with $\gamma > 0$.

We want to consider the obstacle problem also in the case of thin obstacles, so we will need the techniques of capacity theory. For this theory we refer, for instance, to [15].

We recall very briefly that, given a set $E \subseteq \Omega$ its capacity with respect to $\Omega$ is given by

$$\text{cap}(E) = \inf \{ \|z\|^2_{H^1(\Omega)} : z \in H^1_0(\Omega), z \geq 1 \ a.e. \ in \ a \ neighbourhood \ of \ E \}.$$ 

A property holds quasi everywhere (abbreviated as q.e.) when it holds up to sets of capacity zero.

A set $A$ is said to be quasi open (resp. quasi closed) if for any $\varepsilon > 0$ there exists an open (resp. closed) set $V$ such that $\text{cap}(A \Delta V) < \varepsilon$.

A function $v : \Omega \to \overline{\mathbb{R}}$ is quasi continuous (resp. quasi upper semicontinuous) if, for every $\varepsilon > 0$ there exists a set $E$ such that $\text{cap}(E) < \varepsilon$ and $v|_{\Omega \setminus E}$ is continuous (resp. upper semicontinuous) in $\Omega \setminus E$. 
We recall also that if \( u \) and \( v \) are quasi continuous functions and \( u \leq v \) a.e. then also \( u \leq v \) q.e. in \( \Omega \).

A function \( u \in H^1_0(\Omega) \) always has a quasi continuous representative, that is there exists a quasi continuous function \( \tilde{u} \) which equals \( u \) a.e.

Consider the function \( \psi : \Omega \to \mathbb{R} \), and let the convex set be

\[
K_\psi(\Omega) := \{ z \text{ quasi continuous} : z \geq \psi \text{ q.e. in } \Omega \}.
\]

Without loss of generality we may suppose that \( \psi \) is quasi upper semicontinuous thanks to the following proposition (it is a consequence of proposition 1.5 in [10]).

**Proposition 2.1.** Let \( \psi : \Omega \to \mathbb{R} \). Then there exists a quasi upper semicontinuous function \( \hat{\psi} : \Omega \to \mathbb{R} \) such that:

1. \( \hat{\psi} \geq \psi \text{ q.e. in } \Omega \);
2. if \( \varphi : \Omega \to \mathbb{R} \) is quasi upper semicontinuous and \( \varphi \geq \psi \text{ q.e. in } \Omega \) then \( \varphi \geq \hat{\psi} \text{ q.e. in } \Omega \).

Thus, in particular, \( K_\psi(\Omega) = K_{\hat{\psi}}(\Omega) \).

In their natural setting, obstacle problems are part of the theory of variational inequalities (for which we refer to well known books such as [16] and [23]).

For any datum \( F \in H^{-1}(\Omega) \) the variational inequality with obstacle \( \psi \)

\[
\begin{align*}
\begin{cases}
\langle Au, v - u \rangle \geq \langle F, v - u \rangle & \forall v \in K_\psi(\Omega) \cap H^1_0(\Omega) \\
um \in K_{\hat{\psi}}(\Omega) \cap H^1_0(\Omega)
\end{cases}
\end{align*}
\]

(2.1)

(has a unique solution, whenever the set \( K_\psi(\Omega) \cap H^1_0(\Omega) \) is nonempty, i.e.

\[
\exists z \in H^1_0(\Omega) : z \geq \psi \text{ q.e. in } \Omega.
\]

(2.2)

In this case we will say that the obstacle is \( VI \)-admissible.

Let now \( \mathcal{M}_b(\Omega) \) be the space of bounded Radon measures, with the norm given by \( \| \mu \|_{\mathcal{M}_b(\Omega)} = |\mu|(\Omega) \). \( \mathcal{M}_b^0(\Omega) \) is the subspace of measures of \( \mathcal{M}_b(\Omega) \) vanishing on sets of zero capacity. \( \mathcal{M}_b^+(\Omega) \) and \( \mathcal{M}_b^{0+}(\Omega) \) are the corresponding cones of non negative measures. Recall that \( H^{-1}(\Omega) \not\subseteq \mathcal{M}_b(\Omega) \) but \( H^{-1}(\Omega) \cap \mathcal{M}_b(\Omega) \subseteq \mathcal{M}_b^0(\Omega) \).

Any measure \( \mu \in \mathcal{M}_b(\Omega) \) can be decomposed as \( \mu = \mu_a + \mu_s \) (see [14]), where \( \mu_a \in \mathcal{M}_b^0(\Omega) \) and \( \mu_s \) is concentrated on a set of capacity zero.

If \( x \in \Omega \), we denote by \( \delta_x \) the Dirac’s delta centered at \( x \).

When the datum is a measure, equations and inequalities can not be studied in the variational framework.

In [22] G. Stampacchia gave the following definition
Definition 2.2. A function $u_\mu \in L^1(\Omega)$ is a solution in the sense of Stampacchia (also called solution by duality) of the equation
\[
\begin{array}{ll}
Au_\mu = \mu & \text{in } \Omega \\
u_\mu = 0 & \text{on } \partial \Omega
\end{array}
\]  
(2.3)
if
\[
\int_{\Omega} u_\mu g \, dx = \int_{\Omega} u_\mu^* d\mu, \quad \forall g \in L^\infty(\Omega),
\]
where $u_\mu^*$ is the solution of
\[
\begin{array}{ll}
A^* u_\mu^* = g & \text{in } H^{-1}(\Omega) \\
u_\mu^* \in H^1_0(\Omega)
\end{array}
\]
and $A^*$ is the adjoint of $A$.

Throughout the paper $q$ will be any exponent satisfying $1 < q < \frac{N}{N-1}$; Stampacchia proved that a solution $u_\mu$ exists, is unique, and belongs to $W^{1,q}_0(\Omega)$; moreover if the datum $\mu$ is more regular, namely belongs to $M_b(\Omega) \cap H^{-1}(\Omega)$, then the solution coincides with the variational one. It is possible to prove that, when the data converge weakly-$\ast$ in $M_b(\Omega)$, the solutions converge strongly in $W^{1,q}_0(\Omega)$.

If $u$ is such a solution then $T_k(u) \in H^1_0(\Omega)$, for any $k \in \mathbb{R}^+$, where $T_k(s) := (-k) \vee (s \wedge k)$ is the usual truncation function. Moreover
\[
\int_{\Omega} |DT_k(u)|^2 \, dx \leq k |\mu|(\Omega).
\]
(2.4)
These facts imply that $u$ has a quasi continuous representative which is finite q.e. in $\Omega$. In the rest of the paper we shall always identify $u$ with its quasi continuous representative.

We will use the following notation: $u_\mu$ denotes the solution of the equation (2.3).

The following definition of solution for obstacle problems with measure data was given in [12].

Definition 2.3. We say that the function $u$ is a solution of the obstacle problem with datum $\mu$ and obstacle $\psi$ (shortly $OP(\mu, \psi)$) if

1. $u \in K_\psi(\Omega)$ and there exists a positive bounded measure $\lambda \in M^+_b(\Omega)$ such that
\[
u = u_\mu + u_\lambda;
\]
2. for any $\nu \in M^+_b(\Omega)$, such that $v = u_\mu + u_\nu$ belongs to $K_\psi(\Omega)$, we have
\[
u \leq v \text{ a.e. in } \Omega.
\]

For the problem to make sense let us assume that the obstacle $\psi$ satisfies a minimal hypothesis, instead of (2.2), namely
\[
\exists \rho \in M_b(\Omega) : u_\rho \geq \psi \text{ q.e. in } \Omega.
\]
(2.5)
In this case we will say that $\psi$ is $OP$-admissible.
Theorem 2.4. Given $\mu \in M_b(\Omega)$ and $\psi$ OP-admissible (namely $\psi \leq u_\rho$ q.e. in $\Omega$), the obstacle problem $OP(\mu, \psi)$ has a unique solution $u = u_\mu + u_\lambda$. The measure $\lambda \in M_b^+(\Omega)$ satisfies

$$\|\lambda\|_{M_b(\Omega)} \leq \| (\mu - \rho)^- \|_{M_b(\Omega)}$$

(2.6).

Moreover, if $\rho \in M_b(\Omega) \cap H^{-1}(\Omega)$, there exists a sequence $\mu_k = AT_k(u_\mu - u_\rho) + \rho \in M_b(\Omega) \cap H^{-1}(\Omega)$ such that the solutions $u_k$ of $OP(\mu_k, \psi)$ converge strongly in $W^{1, q}(\Omega)$. This sequence does not depend on the obstacle but only on the measure $\rho$.

Also here the positive measure $\lambda$ associated with the solution is called the obstacle reaction.

We mention here a very simple and very useful result whose proof is immediate, but it is worth stating it on its own.

Lemma 2.5. Let $\mu, \nu \in M_b(\Omega)$ and $\psi$ is OP-admissible. Then $u$ is the solution of $OP(\mu + \nu, \psi)$ if and only if $u - u_\nu$ is the solution of $OP(\mu, \psi - u_\nu)$.

The particular case in which the datum $\mu$ belongs to $M_b^0(\Omega)$ has been investigated in detail. Then the obstacle reaction itself belongs to $M_b^0(\Omega)$, provided the obstacle satisfies the following condition

$$\exists \sigma \in M_b^0(\Omega) : u_\sigma \geq \psi \text{ q.e. in } \Omega;$$

(2.7)

This will be shortened by saying that $\psi$ is $OP^0$-admissible.

Notice that if the datum $\mu$ is in $M_b^0(\Omega)$, but the obstacle is only OP-admissible, then the reaction $\lambda$ in general does not belong to $M_b^0(\Omega)$. This is shown in the following simple example.

Example 2.6. Let $\mu = 0$ and $\psi = u_{\delta_{x_0}}$, where $\delta_{x_0}$ is the Dirac’s delta centered at $x_0 \in \Omega$. Then the solution of $OP(0, \psi)$ is $u_{\delta_{x_0}}$ itself and hence $\lambda = \delta_{x_0} \notin M_b^0(\Omega)$.

The interaction between obstacles and solutions is studied deeply in [11], where the following result was proved.

Theorem 2.7. Let $\mu \in M_b(\Omega)$ and let $\psi : \Omega \to \overline{\mathbb{R}}$ be such that

$$-u_\tau - u_\sigma - \varphi \leq \psi \leq u_\sigma \text{ q.e. in } \Omega$$

(2.8)

where $\varphi \in H^1(\Omega)$, $\sigma \in M_b^0(\Omega)$ and $\tau \in M_b(\Omega)$ such that $\tau \perp \mu_\sigma^-$. Then the solutions

$$u = u_\mu + u_\lambda \text{ of } OP(\mu, \psi) \quad \text{and} \quad u_\mu^+ - u_{\mu^-} + u_\lambda \text{ of } OP(\mu^+ - \mu^-; \psi)$$

are the same. Moreover $\lambda = \lambda_1 + \mu_\sigma^-$ with $\lambda_1 \in M_b^{0,+}(\Omega)$.

Here condition (2.8) is given in its full generality, for instance it is satisfied by obstacles in $H^1(\Omega)$ that are OP-admissible.
Remark 2.8. If the obstacle \( \psi \) satisfies (2.8) with \( \tau = 0 \) then the conclusion holds for every \( \mu \in M_b(\Omega) \).

The presence of \( \tau \), which depends on \( \mu \), in (2.8) allows anyway to treat situations like the following one. If \( \mathcal{A} = -\Delta \), \( \Omega = B_1(0) \), the obstacle is \( -u_{\delta_0} \) and the datum is \( -\delta x_0 \) for any \( x_0 \neq 0 \), then the solution of the obstacle problem is zero, because the theorem applies, and because the solution must be less than or equal to zero.

In this paper we will be concerned with the continuous dependence of the solutions with respect to various types of convergence of the obstacles. To our knowledge the only result that was proved on this problem for an arbitrary measure \( \mu \) is the following (proved in [12]) which deals with a very special case.

Proposition 2.9. Let \( \psi_n : \Omega \to \overline{\mathbb{R}} \) be obstacles such that
\[
\psi_n \leq \psi \quad \text{and} \quad \psi_n \to \psi \quad \text{q.e. in } \Omega,
\]
\( \psi \) \( \text{OP-admissible} \), and let \( u_n \) and \( u \) be the solutions of \( \text{OP}(\mu, \psi_n) \) and \( \text{OP}(\mu, \psi) \), respectively. Then
\[
u_n \to u \quad \text{strongly in } W^{1,q}(\Omega).
\]

This result will be generalized in section 4.

In order to study the problem in the most general way we introduce in section 3 a notion of convergence that was used in [10]. It will be called “level set convergence” and in most cases it is equivalent to the convergence of convex sets introduced by U. Mosco in [18], see also [1].

Mosco proved that this type of convergence is the right one for the stability of variational inequalities with respect to obstacles. This is the main theorem of his theory.

Theorem 2.10. Let \( \psi_n \) and \( \psi \) be \( \text{VI-admissible} \). Then
\[
K_{\psi_n}(\Omega) \cap H^1_0(\Omega) \xrightarrow{M} K_{\psi}(\Omega) \cap H^1_0(\Omega),
\]
if and only if, for any \( f \in H^{-1}(\Omega) \),
\[
u_n \to u \quad \text{strongly in } H^1(\Omega)
\]
where \( u_n \) and \( u \) are the solutions of \( \text{VI}(f, \psi_n) \) and \( \text{VI}(f, \psi) \), respectively.

3. The level set convergence, and the related stability properties.

In this section we will define a kind of convergence of functions which will prove to be a good one for the obstacles in obstacle problems with measure data: it is rather general and allows to obtain the convergence of the solutions under very mild assumptions.
Definition 3.1. Let $\psi_n$ and $\psi$ be quasi upper semicontinuous function from $\Omega$ to $\overline{\mathbb{R}}$. We say that $\psi_n$ tends to $\psi$ in the sense of level sets and write

$$\psi_n \xrightarrow{\text{lev}} \psi$$

if

$$\text{cap}(\{\psi > t\} \cap B) \leq \liminf_{n \to +\infty} \text{cap}(\{\psi_n > s\} \cap B')$$

(3.1)

$$\limsup_{n \to +\infty} \text{cap}(\{\psi_n > t\} \cap B) \leq \text{cap}(\{\psi > s\} \cap B')$$

(3.2)

for all $s, t \in \mathbb{R}$, $s < t$, and for all $B \subset\subset B' \subset\subset \Omega$.

Remark 3.2. From the definition it is clear that the level set convergence is implied by

$$\text{cap}(\{\psi > t\} \cap B) = \lim_{n \to +\infty} \text{cap}(\{\psi_n > t\} \cap B).$$

(3.3)

for all $t \in \mathbb{R}$ and for all $B \subset\subset \Omega$

Remark 3.3. From the definition it follows that, if $\psi_n$ converge to $\psi$ locally in capacity, i.e.

$$\text{cap}(\{|\psi_n - \psi| > t\} \cap A) \to 0, \quad \forall t \in \mathbb{R}^+, \forall A \subset\subset \Omega,$$

then $\psi_n \xrightarrow{\text{lev}} \psi$.

To prove that the limit in the sense of level set is unique we give the following lemma from capacity theory which can be found in [13].

Lemma 3.4. Let $E$ and $F$ be quasi closed subsets of $\Omega$ such that

$$\text{cap}(E \cap A) \leq \text{cap}(F \cap A), \quad \forall A \subset\subset \Omega \text{ open},$$

(3.4)

then $\text{cap}(E \setminus F) = 0$ (we say also that $E$ is quasi contained in $F$).

Proposition 3.5. Let $\psi_n$, $\psi$ and $\varphi$ be quasi upper semicontinuous functions. If

$$\psi_n \xrightarrow{\text{lev}} \psi \quad \text{and} \quad \psi_n \xrightarrow{\text{lev}} \varphi,$$

then $\psi = \varphi$.

Proof. Let us fix an open set $A \subset\subset \Omega$ and two real numbers $s < t$. Take now two subsets $A'$ and $A''$ such that $A'' \subset\subset A' \subset\subset A$ and real numbers $t'$ and $t''$ such that $s < t' < t'' < t$. Then

$$\text{cap}(\{\psi > t''\} \cap A'') \leq \liminf_{n \to +\infty} \text{cap}(\{\psi_n > t'\} \cap A')$$

$$\leq \limsup_{n \to +\infty} \text{cap}(\{\psi_n > t'\} \cap A') \leq \text{cap}(\{\varphi \geq s\} \cap A).$$
Hence, since \( \{\psi \geq t\} \subseteq \{\psi > t'\} \), we have
\[
\text{cap}(\{\psi \geq t\} \cap A''') \leq \text{cap}(\{\varphi \geq s\} \cap A),
\]
from which, invading \( A \) by means of \( A'' \subset \subset A \),
\[
\text{cap}(\{\psi \geq t\} \cap A) \leq \text{cap}(\{\varphi \geq s\} \cap A).
\]
Using the fact that \( \psi \) and \( \varphi \) are quasi upper semicontinuous and thanks to lemma 3.4 we deduce that \( \{\psi \geq t\} \) is quasi contained \( \{\varphi \geq s\} \). Now, given \( t \), consider two sequences \( t_k \searrow t \) and \( s_k \searrow t \), with \( t_k > s_k \), so that
\[
\{\psi \geq t_k\} \rightharpoonup \{\psi > t\} \quad \text{and} \quad \{\varphi \geq s_k\} \rightharpoonup \{\varphi > t\},
\]
and we get that
\[
\{\psi > t\} \text{ is quasi contained in } \{\varphi > t\},
\]
for all \( t \in \mathbb{R} \).

Exchanging the roles of \( \psi \) and \( \varphi \) we get the reverse inclusion so that \( \{\psi > s\} \) and \( \{\varphi > s\} \) coincide up to sets of capacity zero.

Now we recover the values of \( \psi \) and \( \varphi \) at quasi every point \( x \in \Omega \) thanks to the well known formula
\[
\varphi(x) = \sup_{s \in Q} s \chi_{\{\varphi > s\}}(x).
\]
Since the level sets are the same, the two functions coincide quasi everywhere.

The main result on level sets convergence is the following theorem, which shows the connection with the Mosco convergence introduced in [18] (for the proof see theorem 5.9 in [10]).

**Theorem 3.6.** Let \( \psi_n \) and \( \psi \) be functions \( \Omega \to \overline{\mathbb{R}} \). If
\[
K_{\psi_n}(\Omega) \cap H^1_0(\Omega) \xrightarrow{M} K_{\psi}(\Omega) \cap H^1_0(\Omega).
\]
then
\[
\psi_n \xrightarrow{\text{lev}} \psi.
\]
If moreover the obstacles are equicontrolled from above, namely
\[
\psi_n, \psi \leq u_\rho, \quad \text{with } \rho \in M_b(\Omega) \cap H^{-1}(\Omega),
\]
then also the reverse implication holds.

Notice that, though very similar to Mosco convergence, the level set convergence concerns also the case of obstacles that are not \( VI \)-admissible.

Another simple observation, which requires no proof, but which is useful to state separately is the following.
Lemma 3.7. Let $\psi_n \overset{\text{lev}}{\to} \psi$ and let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuous non decreasing function. Then
\[ \Phi(\psi_n) \overset{\text{lev}}{\to} \Phi(\psi). \]

In the next lemmas we will denote the solution of $OP(\mu, \psi_n)$ and of $OP(\mu, \psi)$ by $u_n$ and $u$, respectively.

Let us show that in general the Mosco convergence (and so also the level set convergence) of the obstacles does not imply the convergence of the solutions for an arbitrary measure.

Example 3.8. Let $\Omega = B_1(0) \subseteq \mathbb{R}^N$, with $N > 2$, $A = -\Delta$ and $\mu = -\delta_0$, the Dirac delta in the origin.

Let the obstacles $\psi_n = -n$, so that clearly
\[ K_{\psi_n}(\Omega) \cap H^1_0(\Omega) \overset{\text{M}}{\to} H^1_0(\Omega) \]
and, by theorem 3.6, also $\psi_n \overset{\text{lev}}{\to} -\infty$.

It is immediate to see that the solutions $u_n = u_{-\delta_0} + u_{\lambda_n}$ of $OP(-\delta_0, -n)$ are less than or equal to zero since the latter satisfies condition 1 of definition 2.3. So $u_n = T_n(u_n)$ and hence is in $H^1_0(\Omega)$. But then $-\delta_0 + \lambda_n \in H^{-1}(\Omega) \cap M_b(\Omega) \subset M_b^0(\Omega)$, and it must be a positive measure and hence $u_n = 0$ for each $n$. On the other hand $u = u_{-\delta_0}$ and cannot be the limit of the $u_n$.

What can be proved without further assumptions is the following result.

Proposition 3.9. Let $\psi, \psi \leq u_\rho$ q.e. in $\Omega$ with $\rho \in M_b(\Omega)$. Assume that
\[ \psi_n \overset{\text{lev}}{\to} \psi \]
Then there exists a subsequence $u_{n'}$ and a quasi continuous function $u^* \in W^{1,q}_0(\Omega)$, such that
\[ u_{n'} \to u^* \text{ strongly in } W^{1,q}(\Omega), \]
and
\[ u^* \geq u \text{ q.e. in } \Omega. \]

Proof. By theorem 2.4
\[ \|\lambda_n\|_{M_b(\Omega)} \leq \|(\mu - \rho)^-\|_{M_b(\Omega)} \tag{3.5} \]
so that there exists a subsequence $\{\lambda_{n'}\}$ and a measure $\lambda^* \in M^+_b(\Omega)$ such that $\lambda_n \rightharpoonup \lambda^*$, weakly-* in $M_b(\Omega)$ and hence $u_{n'} = u_{\mu} + u_{\lambda_{n'}} \to u^* = u_{\mu} + u_{\lambda^*}$ strongly in $W^{1,q}(\Omega)$.

If we show that $u^* \geq \psi$ q.e. in $\Omega$ we will have that $u^*$ satisfies condition 1 of definition 2.3 and get the thesis, by definition 2.3.
Given \( k > 0 \), observe that, thanks to (2.4) and to (3.5)
\[
\int_{\Omega} |DT_k(u_n')|^2 dx \leq kc
\]
and hence \( T_k(u_n') \rightharpoonup T_k(u^*) \) weakly in \( H^1_0(\Omega) \).

From lemma 3.7 it follows that
\[
T_k(\psi_n) \rightharpoonup T_k(\psi), \quad \forall k \in \mathbb{R}^+;
\]
Since \( T_k(u_n) \geq T_k(\psi_n) \) q.e. in \( \Omega \) for each \( n \) and \( k \), and using theorem 3.6 and the definition of Mosco convergence, we get \( T_k(u^*) \geq T_k(\psi) \) q.e. in \( \Omega \).

Now we can pass to the limit as \( k \to +\infty \) and obtain \( u^* \geq \psi \) q.e. in \( \Omega \).

We prove now the central lemma of this section.

**Lemma 3.10.** Let \( \psi_n \) and \( \psi \) be quasi upper semicontinuous functions controlled above by \( u_\rho \) with \( \rho \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega) \), and let \( w \) be a quasi continuous function. If

\[
\psi_n \rightharpoonup \psi\quad \forall k \in \mathbb{R}^+;
\]

then

\[
\psi_n - w \rightharpoonup \psi - w.
\]

To prove this lemma we need the following result.

**Lemma 3.11.** Given a quasi continuous function \( w \), for each \( A \subset \Omega \) and for each \( \varepsilon > 0 \), there exists \( u \in C_0^\infty(\Omega) \) such that

\[
\text{cap}(\{|u - w| > \varepsilon\} \cap A) < \varepsilon. \tag{3.6}
\]

**Proof.** By definition of quasi continuity there exists a relatively closed subset \( C \) such that \( \text{cap}(\Omega \setminus C) < \varepsilon \) and \( w|_C \) is continuous. By Tietze’s theorem there exists a continuous function \( g \) which extends \( w|_{\Omega \setminus C} \) to \( \mathbb{R}^N \).

Obviously, for any \( A \subset \Omega \), we have \( \{|w - g| > 0\} \cap A \subset \Omega \setminus C \) so that

\[
\text{cap}(\{|w - g| > 0\} \cap A) < \varepsilon.
\]

On its turn \( g \) can be approximated in \( A \) with a function \( u \in C_0^\infty(\Omega) \) so that \( \sup_A |u - g| < \varepsilon \), and again from \( \{|w - u| > \varepsilon\} \cap A \subset \{|w - g| > 0\} \cap A \) we get \( \text{cap}(\{|w - u| > \varepsilon\} \cap A) < \varepsilon \).
Proof of 3.10. It is immediate to observe that the thesis is true when, instead of \( w \), we have a function \( u \) in \( H^1_0(\Omega) \). This is because in our hypotheses level set convergence is equivalent to Mosco convergence (see theorem 3.6) and translating \( K_{\psi_n}(\Omega) \) and \( K_{\psi}(\Omega) \) by \( u \in H^1_0(\Omega) \) we get

\[
K_{(\psi_n-u)} \cap H^1_0(\Omega) \xrightarrow{M} K_{(\psi-u)} \cap H^1_0(\Omega).
\]

or equivalently

\[
\psi_n - u \xrightarrow{\text{lev}} \psi - u
\]

We want to show the inequalities (3.1) and (3.2) for \( \psi_n - w \) and \( \psi - w \).

Let us fix \( B \subset \subset \Omega \), \( \varepsilon > 0 \) and a function \( u \in C^\infty_0(\Omega) \) such that (3.6) holds with respect to \( B \).

Observe now that for any \( t \in \mathbb{R} \) we have

\[
\{\psi_n - w > t\} \cap B \subseteq \{(\psi_n - u > t - \varepsilon) \cap B\} \cup \{(u - w > \varepsilon) \cap B\},
\]

hence, by subadditivity,

\[
\text{cap} (\{\psi_n - w > t\} \cap B) \leq \text{cap} (\{\psi_n - u > t - \varepsilon\} \cap B) + \text{cap} (\{|u - w| > \varepsilon\} \cap B).
\]

Passing to the limsup and using (3.6), we obtain

\[
\limsup_{n \to +\infty} \text{cap} (\{\psi_n - w > t\} \cap B) \leq \limsup_{n \to +\infty} \text{cap} (\{\psi_n - u > t - \varepsilon\} \cap B) + \varepsilon.
\]

We know that for \( \psi - u \), (3.1) holds true, so we can use it with \( t - \varepsilon \) and \( t - 2\varepsilon \) instead of \( t \) and \( s \), so that, for \( B \subset \subset B' \subset \subset \Omega \), we get

\[
\limsup_{n \to \infty} \text{cap} (\{\psi_n - w > t\} \cap B) \leq \text{cap} (\{\psi - u > t - 2\varepsilon\} \cap B') + \varepsilon. \tag{3.7}
\]

With an argument similar to before, from

\[
\{\psi - u > t - 2\varepsilon\} \cap B' \subseteq (\{\psi - u > t - 3\varepsilon\} \cap B') \cup (\{w - u > \varepsilon\} \cap B'),
\]

we obtain,

\[
\text{cap} (\{\psi - u > t - 2\varepsilon\} \cap B') \leq \text{cap} (\{\psi - u > t - 3\varepsilon\} \cap B') + \varepsilon,
\]

and substituting in (3.7) we get

\[
\limsup_{n \to \infty} \text{cap} (\{\psi_n - w > t\} \cap B) \leq \text{cap} (\{\psi - w > t - 3\varepsilon\} \cap B') + 2\varepsilon.
\]

For any choice of \( s \) and \( t \), \( \varepsilon \) can be taken sufficiently small so that \( s < t - 3\varepsilon \). Then we can let \( \varepsilon \to 0 \) and conclude

\[
\limsup_{n \to \infty} \text{cap} (\{\psi_n - w > t\} \cap B) \leq \text{cap} (\{\psi - w > s\} \cap B''').
\]

Here nothing depends on \( u \) so this holds for all \( s, t \in \mathbb{R}, \ s < t \) and for all \( B \subset \subset B' \subset \subset \Omega \).

Inequality (3.1) is proved in a similar way and this concludes the proof.\qed
Theorem 3.12. Let \( \psi_n, \psi \leq u_\rho \) with \( \rho \in \mathcal{M}_b(\Omega) \cap H^{1}(\Omega) \). Let \( \mu \in \mathcal{M}_b(\Omega) \) with \( \mu^- \in H^{-1}(\Omega) \) and let \( u_n \) and \( u \) be the solutions of \( OP(\mu, \psi_n) \) and of \( OP(\mu, \psi) \) respectively. If

\[
\psi_n \xrightarrow{\text{lev}} \psi
\]

then \( u_n - u \) belongs to \( H^1_0(\Omega) \) and tends to zero strongly in \( H^1_0(\Omega) \).

Proof. Thanks to the previous lemma we have that

\[
\psi_n - u_\mu \xrightarrow{\text{lev}} \psi - u_\mu,
\]

and then, since \( \psi_n - u_\mu, \psi - u_\mu \leq u_\rho + u_\mu^- \) q.e. in \( \Omega \), by theorem 3.6,

\[
K_{(\psi_n - u_\mu)} \cap H^1_0(\Omega) \xrightarrow{\mathcal{M}} K_{(\psi - u_\mu)} \cap H^1_0(\Omega).
\]

Hence all solutions of variational inequalities converge. In particular if \( v_n \) and \( v \) are the solutions of \( VI(0, \psi_n - u_\mu) \) and \( VI(0, \psi - u_\mu) \), respectively, then \( v_n \to v \) strongly \( H^1_0(\Omega) \).

By lemma 2.5 we have \( u_n = v_n + u_\mu \) and \( u = v + u_\mu \). This implies that \( u_n - u = v_n - v \) and the conclusion follows. \( \square \)

The minimal hypothesis on the obstacles \( \psi_n \) and \( \psi \) in order to have the solutions of \( OP(\mu, \psi_n) \) and \( OP(\mu, \psi) \) is that they are \( OP \)-admissible. Nevertheless if in this theorem we drop the request that they are controlled by a function which is also in \( H^1_0(\Omega) \), the conclusion fails. Indeed there is the following example which derives from example 3.8.

Example 3.13. Let us consider the operator \( A = -\Delta \), the domain \( \Omega = B_1(0) \subseteq \mathbb{R}^N \), with \( N > 2 \), and the datum \( \mu = 0 \).

Consider now as obstacles \( \psi_n = u_{\delta_0} - n \), where \( \delta_0 \) is the Dirac delta centred at zero. They are clearly \( OP \)-admissible, and also bounded by the same function \( u_\rho \), but in this case \( \rho = \delta_0 \not\in H^{-1}(\Omega) \).

Now for each \( n \) the solution \( u_n \) of \( OP(0, \psi_n) \) is \( u_{\delta_0} \) itself. Indeed, according to lemma 2.5, \( u_n - u_{\delta_0} \) is the solution of \( OP(-\delta_0, -n) \), that, as seen in example 3.8, is zero.

But then we have that \( \psi_n \xrightarrow{\text{lev}} -\infty \) and \( u_n \to u_{\delta_0} \), while the solution of \( OP(0, -\infty) \) is \( u = 0 \).

When the negative part of the measure \( \mu \) is only in \( \mathcal{M}_0^b(\Omega) \), we can not use the same trick because the the sets \( K_{\psi_n - u_\mu} \) might be empty, but anyway we do not fall into the pathology of the example 3.8, and in fact we can prove the following theorem which gives the convergence of the solutions as well, though in a weaker sense.
Theorem 3.14. Let $\psi_n, \psi \leq u_\rho$ with $\rho \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, $\mu \in \mathcal{M}_b(\Omega)$ such that $\mu^- \in \mathcal{M}^0_b(\Omega)$, and let $u_n$ and $u$ be the solutions of $OP(\mu, \psi_n)$ and of $OP(\mu, \psi)$ respectively. If $\psi_n \xrightarrow{lev} \psi$, then $u_n \to u$ strongly in $W^{1,q}(\Omega)$.

Proof. From [9] we know that $\mu^- - \mu_k^-$ can be written as $g \nu$ where $\nu \in \mathcal{M}^+_b(\Omega) \cap H^{-1}(\Omega)$ and $g \in L^1(\Omega, \nu)$, $g \geq 0$. Hence the measure $\mu_k^- := (g \wedge k) \nu$ is in $H^{-1}(\Omega)$, so that $\mu_k := \mu^+ - \mu_k^-$ satisfies the hypothesis of the previous theorem.

Call $u_n^k$ and $u^k$ the solutions of $OP(\mu^k, \psi_n)$ and $OP(\mu^k, \psi)$, respectively. By theorem 3.12,

$$u_n^k \to u^k, \text{ strongly in } H^1(\Omega), \quad \forall k > 0.$$ 

Now, observing that $\mu^- - \mu_k^-$ is a positive measure, we easily obtain by comparison that $u_n^k \geq u_n$ and $u^k \geq u$, q.e. in $\Omega$.

On the other hand

$$u_n + u_{(\mu^- - \mu_k^-)} = u_\mu + u_{(\lambda_n + \mu^+ - \mu_k^-)},$$  \hspace{1cm} (3.8)

where $\lambda_n \geq 0$ is the obstacle reaction of $OP(\mu, \psi_n)$. Since also $u_n + u_{(\mu^- - \mu_k^-)} \geq u_n \geq \psi_n$, by definition 2.3 and eq. (3.8) we have $u_n + u_{(\mu^- - \mu_k^-)} \geq u_n^k$ q.e. in $\Omega$. In the same way we prove that $u + u_{(\mu^- - \mu_k^-)} \geq u^k$ q.e. in $\Omega$.

Since $\mu^- - \mu_k^- \to 0$ strongly in $\mathcal{M}_b(\Omega)$, we have, by proposition 4.2 in [12] $u_{(\mu^- - \mu_k^-)} \to 0$ strongly in $W^{1,q}(\Omega)$, so, from

$$u + u_{(\mu^- - \mu_k^-)} \geq u^k \geq u \quad \text{q.e. in } \Omega$$

letting $k \to \infty$ we get that $u^k \to u$ a.e. in $\Omega$.

Recalling proposition 3.9, let us fix a subsequence $\{u_{n'}\}$ which converges to a function $u^*$ strongly in $W^{1,q}(\Omega)$, so that from

$$u_{n'} + u_{(\mu^- - \mu_k^-)} \geq u_{n'}^k \geq u_{n'} \quad \text{q.e. in } \Omega$$

letting first $n' \to \infty$ and then $k \to \infty$ we obtain $u^* \geq u \geq u^*$ q.e. in $\Omega$. Therefore $u_n^k \to u$, since the limit does not depend on the subsequence.

As seen in example 3.13 the request that the obstacles be well controlled can not be dropped, even if the datum is regular. On the other hand example 3.8 showed that the control from above can be not enough to have convergence for all data $\mu \in \mathcal{M}_b(\Omega)$.

In the following theorem we show how, provided we strengthen the assumptions on the obstacles in the way given by theorem 2.7, we can give up any assumption on the data $\mu$.

Notice that in the examples is always the limit obstacle the one that gives troubles. Indeed we see here that it is enough to require the control from below only for the limit.
Theorem 3.15. Let $\psi, \psi \leq u_{\rho}$ with $\rho \in M_b(\Omega) \cap H^2(\Omega)$, let $\mu \in M_b(\Omega)$ and let $\psi$ be an obstacle. Let $u_n$ and $u$ be the solutions of $OP(\mu, \psi_n)$ and of $OP(\mu, \psi)$, respectively. If $\psi$ and $\mu$ satisfy condition (2.8) and $\psi_n \xrightarrow{\text{lev}} \psi$ then $u_n \to u$ strongly in $W^{1,q}(\Omega)$.

Proof. From proposition 3.9 we know that, up to a subsequence, $u_n \to u^*$ strongly in $W^{1,q}(\Omega)$, and $u^* \geq u$ q.e. in $\Omega$.

Now consider the $v_n$'s solutions of $OP(\mu^+ - \mu^-; \psi_n)$. These, by theorem 3.14, converge to $v$, the solution of $OP(\mu^+ - \mu^-; \psi)$, but, according to theorem 2.7, $v = u$.

On the other side $v_n = u^* + u\lambda_n + u\mu_n$, with $\lambda_n \in M^+_b(\Omega)$, and $v_n \geq \psi$ q.e. in $\Omega$ and so, by definition 2.3, we have

$$v_n \geq u_n \text{ q.e. in } \Omega.$$ 

Letting $n$ go to $+\infty$ we obtain $u \geq u^*$ q.e. in $\Omega$. Therefore $u_n \to u$ strongly in $W^{1,q}(\Omega)$.

The limit doesn't depend on the subsequence, so the whole sequence $u_n$ converges to $u$. $\square$

Let us show now a further example, which clarifies more deeply in which cases there is not convergence of the solutions.

In particular we see that theorems 2.7 and 3.15 do not hold for some obstacles $\psi$ which are too singular only at one point.

Example 3.16. Let us choose $\mathcal{A} = -\Delta$, $\Omega = B_1(0) \subseteq \mathbb{R}^N$, with $N > 2$, and $\mu = -\delta_0$.

Let us consider the obstacles $\psi = -u\delta_0$ and

$$\psi_n(x) = \begin{cases}
  -\frac{1}{2}u\delta_0(x) & \text{if } |x| < a_n \\
  -n & \text{if } a_n < |x| < b_n \\
  -u\delta_0 & \text{if } b_n < |x|
\end{cases} \quad (3.9)$$

where $a_n$ and $b_n$ are appropriate constants, which tend to zero as $n \to +\infty$ (see picture).

It is easy to verify that $\psi_n \xrightarrow{\text{lev}} \psi$ and that the solution of the limit problem $OP(-\delta_0, \psi)$ is clearly $-u\delta_0$ itself.

Let us prove that the solution of $OP(-\delta_0, \psi_n)$ is $-\frac{1}{2}u\delta_0$.

This function satisfies condition 1 of definition 2.3 because it is of the form $u\mu + u\frac{1}{2}\delta_0$ and it is above the obstacle for each $n$.

Fix $n$ and suppose $\nu_n \in M^+_b(\Omega)$ such that $u\mu + u\nu_n$ is the solution. Then it is smaller than or equal to $u\mu + u\frac{1}{2}\delta_0$, or also

$$u\nu \leq u\frac{1}{2}\delta_0 \text{ q.e. in } \Omega.$$
In the small circle \( B_{\alpha_n}(0) \) they must be equal. In \( B_1(0) \setminus B_{\alpha_n}(0) \), \( u_\nu \) is superharmonic and \( u_{\frac{1}{2}\delta_0} \) is harmonic and they have the same boundary data. So \( u_\nu \geq u_{\frac{1}{2}\delta_0} \) and they must coincide.

This proves that the solution \( u_n \) of \( OP(-\delta_0, \psi_n) \) is \( -\frac{1}{2}u_{\delta_0} \) independently of \( n \), and that \( u_n \) does not converge to the solution \( u = u_{\delta_0} \) of \( OP(-\delta_0, \psi) \).

As of remark 2.8 we point out that in the example it is crucial that the deltas involved are centered in the same point. If for instance, with the same obstacles, we had as datum \( \mu = -\delta_{x_0} \) for any \( x_0 \neq 0 \), we would obtain, thanks to theorem 2.7 that the solutions of \( OP(-\delta_{x_0}, \psi_n) \) and of \( OP(-\delta_{x_0}, \psi) \) are all identically zero.

The last consideration of this section concerns the fact that passing from theorem 3.12 to theorem 3.14 we loose something on the convergence of the solutions. To see that this loss is not due to the technique of the proof we can consider the following example.

**Example 3.17.** Let \( \Omega = B_{\frac{1}{2}}(0) \subseteq \mathbb{R}^N \), with \( N > 2 \), and let \( f \in L^1(\Omega) \) be the function defined by

\[
f(x) = \begin{cases} 1 & \text{if } x \in \Omega, x \neq 0 \\ \frac{1}{|x|^N(-\log |x|)^\vartheta} & \text{if } x = 0 \\
\end{cases}
\]

with \( \vartheta > 1 \). L. Orsina in [21] noticed that the solution \( u_f \) of the equation

\[
\begin{cases} -\Delta u_f = f & \text{in } \Omega \\ u_f = 0 & \text{on } \partial \Omega
\end{cases}
\]

belongs to \( W^{1,q}(\Omega) \) for any \( q < \frac{N}{N-1} \) but does not belong to \( W^{1,N}(\Omega) \).

With this choice of \( A \) and \( \Omega \), take as datum \( \mu = -f \), which clearly belongs to \( \mathcal{M}_b^0(\Omega) \), and as limit obstacle \( \psi = -u_f \), which satisfies condition (2.8) with \( \sigma = f \), so that the solution of \( OP(-f,-u_f) \) is \( -u_f \) itself.
If we set $\psi_n = -(u_f \land n)$ then the solution $u_n$ of $OP(-f, -(u_f \land n))$ is between 0 (because $f$ is positive and $u_{-f} + u_f$ is a supersolution) and $-n$. Hence $u_n = T_n(u_n)$ and this implies that $u_n \in H^1_0(\Omega)$.

Now it is easy to see (use for instance remark 3.2) that $\psi_n \xrightarrow{\text{lev}} \psi$ so that, by proposition 3.15, $u_n \to u$ strongly in $W^{1,q}(\Omega)$. Nevertheless it is not possible to have this convergence in the norm of $W^{1,p}(\Omega)$ with $p \geq \frac{N}{N-1}$, because the fact that $u_n \in H^1(\Omega)$ would imply also that $u \in W^{1,p}(\Omega)$, which is false since $u = -u_f$.

4. The case $\psi_n \leq \psi$.

In the previous section we have seen that the level sets convergence of the obstacles in general it is not enough to give the convergence of the solutions for any $\mu$.

In this section we want to generalize proposition 2.9. Pointwise convergence is replaced by level sets convergence, the condition $\psi_n \leq \psi$ q.e. in $\Omega$ is strong enough to give the result for any measure $\mu \in M_b(\Omega)$ with no control from below on the limit obstacle.

**Proposition 4.1.** Let $\psi_n$ and $\psi$ be $OP$-admissible and such that

$$\psi_n \xrightarrow{\text{lev}} \psi.$$ 

If in addition $\psi_n \leq \psi$ q.e. in $\Omega$ then

$$u_n \to u \quad \text{q.e. in } \Omega.$$

**Proof.** First apply proposition 3.9 from which we know that, up to a subsequence, $u_n \to u^*$ strongly in $W^{1,q}(\Omega)$ and that $u^* \geq u$. The reverse inequality is guaranteed by the fact that $u \geq u_n$, for all $n$ thanks to condition 1 of definition 2.3.

Let us remark that this is a generalized version of proposition 2.9. Indeed under the assumption that $\psi_n \leq \psi$, we have that quasi everywhere convergence implies level sets convergence.

If $\psi_n$ is monotone increasing, this is an easy consequence of the continuity of capacities on increasing sequences of sets.

The general case, $\psi_n \leq \psi$ but $\psi_n$ not necessarily increasing, is proved by considering the sequence $\varphi_n := \inf_{k \geq n} \psi_k$, which is increasing, converge pointwise to $\psi$ and satisfies

$$\text{cap}(\{\varphi_n > t\} \cap B) \leq \text{cap}(\{\psi_n > t\} \cap B) \leq \text{cap}(\{\psi > t\} \cap B)$$

for all $t \in \mathbb{R}$, and $B \subset \subset \Omega$. Passing to the limit, thanks to the previous step, we conclude the proof.
5. Obstacles converging in the energy space.

It is well known that in the case of variational inequalities the convergence of obstacles in the norm of $H^1(\Omega)$ implies the convergence of the corresponding solutions. In particular we have the following result

**Theorem 5.1.** Let $\psi_1, \psi_2 : \Omega \to \mathbb{R}$ be VI-admissible. Suppose that $\psi_1 - \psi_2 \in H^1_0(\Omega)$. Let $f \in H^{-1}(\Omega)$ and let $u_1$ and $u_2$ be the solutions of $VI(f, \psi_1)$ and $VI(f, \psi_2)$, respectively. Then

$$\|u_n - u\|_{H^1_0(\Omega)} \leq \frac{C}{\gamma} \|\psi_n - \psi\|_{H^1_0(\Omega)},$$

where $\gamma$ is the ellipticity constant and $C$ is such that $|\langle Au, v \rangle| \leq C\|u\|_{H^1_0(\Omega)}\|v\|_{H^1_0(\Omega)}$, so they depend only on the operator $A$.

Also for the solutions of obstacle problems, we want to investigate the dependence on $H^1(\Omega)$ convergence. In this frame, as we have seen with example 3.17, this can not follow directly from Mosco convergence, as it was in the variational case. The next theorem concerns the case in which the obstacles “have the same boundary value”.

**Theorem 5.2.** Let $\psi_n, \psi : \Omega \to \mathbb{R}$ be such that $\psi_n, \psi \leq u_{\rho}$ q.e. in $\Omega$, with $\rho \in \mathcal{M}_b(\Omega)$. Assume $\psi_n - \psi \in H^1_0(\Omega)$, and let $\psi_n - \psi \to 0$ strongly in $H^1(\Omega)$. Let $\mu \in \mathcal{M}_b(\Omega)$ and let $u_n$ and $u$ be the solutions of $OP(\mu, \psi_n)$ and $OP(\mu, \psi)$, respectively. Then

$$u_n - u \in H^1_0(\Omega) \text{ and } u_n - u \to 0 \text{ strongly in } H^1(\Omega),$$

**Proof.** As a first step assume that $\rho$ belongs also to $H^{-1}(\Omega)$, and consider the special sequence $\mu_k$ of measures in $\mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$, $\mu_k \rightharpoonup^* \mu$, weakly- in $\mathcal{M}_b(\Omega)$, such that the solutions of the corresponding obstacle problems converge (see theorem 2.4). In particular $u_n^k \to u_n$, strongly in $W^{1,q}(\Omega)$ for all $n$ and $u^k \to u$, strongly in $W^{1,q}(\Omega)$.

Thanks to (5.1), for all $k$ we also have

$$\|u_n^k - u^k\|_{H^1_0(\Omega)} \leq c\|\psi_n - \psi\|_{H^1_0(\Omega)},$$

so that the sequence $\{u_n^k - u^k\}_k$ is bounded in $H^1_0(\Omega)$, for each $n$ fixed. Thus, up to a subsequence, there is a limit function $z$. But we already know that the sequence converges, strongly in $W^{1,q}(\Omega)$, to $u_n - u$, so this must be also the weak limit in $H^1(\Omega)$.

By lower semicontinuity of the norm we have

$$\|u_n - u\|_{H^1_0(\Omega)} \leq \liminf_{k \to \infty} \|u_n^k - u^k\|_{H^1_0(\Omega)} \leq c\|\psi_n - \psi\|_{H^1_0(\Omega)}.$$
This says that $u_n - u$ belongs to $H_0^1(\Omega)$ (while $u_n$ and $u$, in general, do not) and gives the thesis in the first case.

Let now $\rho$ be only in $\mathcal{M}_b(\Omega)$. Set $\psi^h := \psi \wedge h$ and $\psi^h_n := \psi_n - \psi + \psi^h$. These obstacles are equi OP-admissible, because $\psi^h_n \leq \psi_n$ and $\psi^h \leq \psi$ q.e. in $\Omega$. They are also equi VI-admissible since, if $\psi \leq u_\rho$, then $\psi^h \leq T_h(u_\rho) \in H_0^1(\Omega)$ and $\psi^h_n \leq \psi_n - \psi + T_h(u_\rho) \in H_0^1(\Omega)$, and we can find a function $\varphi \in H_0^1(\Omega)$ such that $\varphi \geq \psi_n - \psi$ for all $n$.

It is easy to see that in this case there exists $\rho^h \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega)$ such that $\psi^h_n \leq u_{\rho^h}$ q.e. in $\Omega$. Hence we are in the hypothesis of the first step. Moreover $\|\psi^h_n - \psi^h\|_{H_0^1(\Omega)} = \|\psi_n - \psi\|_{H_0^1(\Omega)}$.

So by the first step, for each $h$

$$\|u^h_n - u^h\|_{H_0^1(\Omega)} \leq \gamma \|\psi_n - \psi\|_{H_0^1(\Omega)}.$$ 

On the other side we know that, since $\psi^h_n \nearrow \psi_n$ and $\psi^h \nearrow \psi$, by proposition 2.9 $u^h_n - u^h \rightharpoonup u_n - u$ strongly in $W^{1,q}(\Omega)$, so that we can conclude as in the first step. 

We want to remark that if, more generally, the obstacles are such that $\psi_n - \psi \to 0$ in $H^1(\Omega)$ then they also converge in the sense of level sets, so we can deduce the convergence of the solutions in all the situations given by theorems 3.12, 3.14 and 3.15, but here we obtain a stronger convergence with no further assumptions on the obstacles and on the data.

We may now wonder what happens when the obstacles converge in the space $W^{1,q}(\Omega)$, with $1 < q < \frac{N}{N-1}$. In general this is not enough to obtain the convergence of the solutions. Indeed reconsider example 3.16. Let us prove that $\psi_n \to \psi$ strongly in $W^{1,q}_0(\Omega)$. We have

$$\psi_n - \psi = \begin{cases} \frac{1}{2} u_{\delta_0} & \text{in } |x| < a_n \\ u_{\delta_0} - n & \text{in } a_n < |x| < b_n \\ 0 & \text{otherwise} \end{cases}$$

so that

$$\|\psi_n - \psi\|_{W^{1,q}(\Omega)}^q = \frac{1}{2} \int_{B_{a_n}(0)} |Du_{\delta_0}|^q dx + \int_{B_{b_n}(0) \setminus B_{a_n}(0)} |Du_{\delta_0}|^q dx,$$

which tends to zero, since $a_n$ and $b_n$ tend to zero and by the absolute continuity of the integral. But, as already seen in example 3.16, the solutions of the obstacle problems do not converge.

Anyway it is possible to prove the following result.

**Proposition 5.3.** Let $\mu$ be in $\mathcal{M}_b(\Omega)$ and let $\psi_n$ and $\psi$ be OP-admissible and such that $\psi_n - \psi = u_{\rho_n}$ with $\rho_n \in \mathcal{M}_b(\Omega)$, $\|\rho_n\|_{\mathcal{M}_b(\Omega)} \to 0$. Then

$$u_n \to u \text{ strongly in } W^{1,q}(\Omega),$$
where \( u_n \) and \( u \) are the solutions of \( \text{OP}(\mu, \psi_n) \) and \( \text{OP}(\mu, \psi) \), respectively.

**Proof.** Since \( \psi_n = \psi - u_{\rho_n} \), we have (using lemma 2.5) that \( u_n - u_{\rho_n} \) is the solution of \( \text{OP}(\mu + \rho_n, \psi) \). So from theorem 4.2 in [12] we get that

\[
u_n - u_{\rho_n} \rightarrow u \quad \text{strongly in } W^{1,q}(\Omega). \tag{5.2}\]

Then

\[
u_n - u \leq \|\nu_n - u\|_{W^{1,q}(\Omega)} + \|u_{\rho_n}\|_{W^{1,q}(\Omega)}; \]

the first term goes to zero because of (5.2), the second one by hypothesis, and we get the thesis. \( \square \)

### 6. Problems with nonzero boundary data and uniform convergence

In this section we extend the theory of obstacle problems with measure data developed in [12] to problems with nonzero boundary data. This is standard for variational inequalities and also in this case this generalization is very simple; we will only point out what has to be settled.

Let \( g \in H^1(\Omega) \) we will denote by \( u_0^g \) the solution of

\[
\begin{cases} 
Au_0^g = 0 & \text{in } H^{-1}(\Omega) \\
u_0^g - g \in H_0^1(\Omega).
\end{cases}
\]

We will look for solutions of obstacle problems which take the value \( g \) on the boundary \( \partial \Omega \). So we have to change accordingly the notion of admissibility for the obstacles.

An obstacle \( \psi : \Omega \rightarrow \mathbb{R} \) is said to be \( \text{OP}_g \)-admissible if

\[
\exists \rho \in M^+_b(\Omega) \text{ s.t. } \psi \leq u_\rho + u_0^g \quad \text{q.e. in } \Omega.
\]

Given a measure \( \mu \in M_b(\Omega) \), a boundary datum \( g \in H^1(\Omega) \) and an \( \text{OP}_g \)-admissible obstacle \( \psi \), the solution of the obstacle problem \( \text{OP}(\mu, g, \psi) \), if it exists, is the minimum element of the set

\[
\mathcal{F}_\psi^g(\mu) := \{ v \in W^{1,q}(\Omega) : \exists \nu \in M_b^+(\Omega), v = u_\mu + u_0^g + u_\nu; v \geq \psi \text{ q.e. in } \Omega \}.
\]

It is immediate to prove the following

**Theorem 6.1.** Let \( \mu \in M_b(\Omega) \) and let \( \psi \) be \( \text{OP}_g \)-admissible. Then there exists a unique solution of \( \text{OP}(\mu, g, \psi) \).

**Proof.** Consider the obstacle \( \psi - u_0^g \). It is \( \text{OP} \)-admissible. So there exists a unique solution \( v \) of \( \text{OP}(\mu, \psi - u_0^g) \). Then \( v + u_0^g \) is our solution: indeed it belongs to \( \mathcal{F}_\psi^g(\mu) \), and it is less than or equal to any \( z \in \mathcal{F}_\psi^g(\mu) \). \( \square \)
Remark 6.2. From theorem 6.1 and (2.6) it follows also that, if \( u_\mu + u_0^g + u_\lambda \) is the solution of \( OP(\mu, g, \psi) \), then
\[
\|\lambda\|_{\mathcal{M}_b(\Omega)} \leq \| (\mu - \rho)^- \|_{\mathcal{M}_b(\Omega)},
\]
indpendently of \( g \).

Remark 6.3. Since \( \psi \) is \( OP \)-admissible if and only if \( \psi - u_0^g \) is \( OP_g \)-admissible and \( u_n \to u \) strongly in \( W^{1,q}(\Omega) \) if and only if \( u_n - u_0^g \to u - u_0^g \) strongly in \( W^{1,q}(\Omega) \), all the theorems on continuous dependence on the data hold without modifications, in particular propositions 2.9 which will be useful in the following.

Remark 6.4. When \( \mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega) \) and \( \psi \leq u_\rho + u_0^g \) q.e. in \( \Omega \) with \( \rho \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega) \) then the solution of \( OP(\mu, g, \psi) \) coincides with the solution of \( VI(\mu, g, \psi)^* \).

We come now to discuss the continuous dependence of the solutions on the obstacles when these converge uniformly.

To do this we will use a characterization via supersolutions similar to the one that holds in the variational case (see [16]).

To this aim, let us introduce the set \( G^g_\psi(\mu) \) of all the functions \( v \in W^{1,q}(\Omega) \) with \( v \geq \psi \) q.e. in \( \Omega \), such that \( v = u_\mu + u_0^h + u_\nu \), where \( \nu \in \mathcal{M}_b^+(\Omega) \) and \( h \in H^1(\Omega) \) such that \( h \geq g \) on \( \partial \Omega \), i.e. \( (h - g)^- \in H_0^1(\Omega) \).

We see now that the solution of \( OP(\mu, g, \psi) \) can be compared not only with the functions of \( F^g_\psi(\mu) \), but also with all those that have boundary datum greater than or equal to \( g \).

Proposition 6.5. Let \( \mu \in \mathcal{M}_b(\Omega) \) and \( \psi \) be \( OP_g \)-admissible. If \( u \) is the solution of \( OP(\mu, g, \psi) \) then it is the minimum element of \( G^g_\psi(\mu) \).

Proof. Step 1. First consider \( \mu \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega) \) and \( \psi \) both \( VI \)- and \( OP \)-admissible.

Let \( v = u_\mu + u_0^h + u_\nu \in G^g_\psi(\mu) \). We approximate \( \nu \) by means of the sequence \( \nu_k := \lambda T_k(u_\nu) \). We have that \( \nu_k \in \mathcal{M}_b(\Omega) \cap H^{-1}(\Omega) \) and that \( \nu_k \to \nu \) weakly-* in \( \mathcal{M}_b(\Omega) \). Moreover observe that \( u_{\nu_k} = T_k(u_\nu) \) tends to \( u_\nu \) q.e. in \( \Omega \) and, since \( u_\nu \) is nonnegative it is an increasing sequence.

Hence if we define \( v_k := u_\mu + u_0^h + u_{\nu_k} \), then \( v_k \nearrow v \) q.e. in \( \Omega \), and setting \( \psi_k := \psi \land v_k \) also \( \psi_k \nearrow \psi \) q.e. in \( \Omega \).

Let now \( u_k \) be the solutions of \( VI(\mu, g, \psi_k) \). So, by theorem II.6.4 in [16], \( v_k \geq u_k \).

Using proposition 2.9 and remark 6.3 we know that \( u_k \to u \) a.e. in \( \Omega \). Then \( v \geq u \) a.e. in \( \Omega \) and then also q.e. in \( \Omega \).

Step 2. Consider now \( \mu \in \mathcal{M}_b(\Omega) \) and \( \psi \) still both \( VI \)- and \( OP \)-admissible. Take again \( v \in G^g_\psi(\mu) \).

* There is no need to define explicitly what is \( VI(\mu, g, \psi) \).
Let $\mu_k = AT_k(u_\mu - u_\rho) + \rho$ be the sequence of measures given in theorem 2.4, so that we know that if $u_k$ are the solutions of $VI(\mu_k, g, \psi)$ then $u_k \to u$ strongly in $W^{1,q}(\Omega)$.

Taking now $v_k = u_{\mu_k} + u_0^h + u_\nu$ it is easy to verify that $v_k \geq \psi$ q.e. in $\Omega$ for all $k > 0$, and then, by definition 2.3, $v_k \geq u_k$ q.e. in $\Omega$. Also $v_k \to v$ strongly in $W^{1,q}(\Omega)$ so, passing to the limit, we obtain $v \geq u$ a.e. in $\Omega$ and then also q.e. in $\Omega$.

**Step 3.** Finally consider the general case $\mu \in M_b(\Omega)$ and $\psi$ OP-admissible. The obstacles $\psi_k := \psi \land k$ are also VI-admissible and such that $\psi_k \nearrow \psi$ q.e. in $\Omega$. So, if $u_k$ is the solution of $OP(\mu, g, \psi_k)$, by proposition 2.9 and remark 6.3, we have that $u_k \to u$ strongly in $W^{1,q}(\Omega)$.

Taken any $v \in G^g_\psi(\mu)$, then $v \geq \psi_k$, for all $k$. Hence, by definition 2.3, $v \geq u_k$ q.e. in $\Omega$. Passing to the limit, we get $v \geq u$ a.e. in $\Omega$ and also q.e. in $\Omega$.

From this we point out that the sets $F^g_\psi(\mu)$ and $G^g_\psi(\mu)$ have the following lattice property.

**Proposition 6.6.** Let $\mu \in M_b(\Omega)$, $g \in H^1(\Omega)$ and $\psi$ OP$_g$-admissible. Then

(i) If $u, v \in F^g_\psi(\mu)$ then $u \land v \in F^g_\psi(\mu)$;

(ii) If $u, v \in G^g_\psi(\mu)$ then $u \land v \in G^g_\psi(\mu)$.

**Proof.** Let us prove only the first statement, the proof of the second being alike.

Set $w := u \land v$ and let $z$ be the solution of $OP(\mu, g, w)$. Then $u, v \in F^g_\psi(\mu)$ and hence also $w \geq z$.

On the other hand $z \geq w$ and hence they are equal. So $u \land v$ is of the form $u_\mu + u_0^h + u_\nu$ and is above $\psi$, and hence belongs to $F^g_\psi(\mu)$. □

We can prove now the following continuity result

**Theorem 6.7.** Let $\mu \in M_b(\Omega)$, $g \in H^1(\Omega)$, and $\psi_n$ and $\psi$ be OP-admissible and let $u_n$ and $u$ be the solutions of $OP(\mu, \psi_n)$ and $OP(\mu, \psi)$, respectively. Assume that $\psi_n - \psi \in L^\infty(\Omega)$ and $\psi_n - \psi \to 0$ in $L^\infty(\Omega)$. Then

$$u_n - u \in L^\infty(\Omega) \quad \text{and} \quad u_n - u \to 0 \text{ in } L^\infty(\Omega).$$

**Proof.** Set $c_n := \|\psi_n - \psi\|_{L^\infty(\Omega)}$. Obviously $c_n = u_0^{c_n}$, so that

$$u + c_n = u_\mu + u_0^{c_n} + u_\lambda \quad \text{and} \quad u + c_n \geq \psi_n \quad \text{q.e. in } \Omega$$

hence $u + c_n \in G^g_\psi(\mu)$ and hence $u + c_n \geq u_n$.

The same can be done the other way round to obtain that $u_n + c_n \geq u$. In the end we get $|u_n - u| \leq c_n$, and, taking the sup over $x \in \Omega$, the thesis. □
Remark 6.8. Also in this case we have to remark that the uniform convergence of the obstacles implies their level set convergence (via remark 3.4). But the result we have obtained in this section does not require that the obstacles be equicontrrolled, and the convergence of the solutions is in a different norm.

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