Nonlinearity analysis for cosmological inflation with minimal and non-minimal coupling of scalar field from horndeski theory for special cases: de sitter expansion and decaying scalar field

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Abstract. In this work, we consider De Sitter expansion and decaying scalar field in cosmological model with combination of minimal and non-minimal coupling of scalar field from Horndeski theory. As a continuity of the previous work, after the background solution and condition obtained, we continue the analysis to the nonlinearity aspect of perturbations via the spectral index of the perturbations. Spectral index of the perturbations and tensor to-scalar ratio of these special cases give sign of inflation model.

1. Introduction

Horndeski theory [1], with Lagrangian,

\[ L = \sum_{i=2}^{5} L_i, \]

where,

\[ L_2 = K(\phi, X), \]
\[ L_3 = -G_3(\phi, X) \Box \phi, \]
\[ L_4 = G_4(\phi, X) R - 2G_{4X}(\phi, X)[(\Box \phi)^2 - \phi_{\mu\nu} \phi^{\mu\nu}], \]
\[ L_5 = G_5(\phi, X) G_{\mu\nu} \phi^{\mu\nu} + \frac{1}{3} G_{5X}(\phi, X) [(\Box \phi)^3 - 3(\Box \phi)(\phi_{\mu\nu} \phi^{\mu\nu}) + 2(\phi_{\mu\nu} \phi^{\mu\nu} \phi^{\sigma\tau})], \]

with the coefficient functions, \( X = \partial_{\mu} \phi \partial^{\mu} \phi \) and \((K(\phi, X), G_i(\phi, X))\), can be assigned specifically, known to be the most general scalar-tensor theory. Some cosmological model can be derived from this theory [2][3][4][5][6][7].
In line with previous work, we assign our cosmological model as,
\[ K = \omega(\phi)X; \quad G_3 = 0; \quad G_4 = \frac{M_{pl}^2}{2} - \frac{1}{2} \zeta \phi^2; \quad G_5 = \xi \phi, \tag{6} \]
where \( M_{pl}^2 = \frac{1}{8\pi G} \approx 1 \times 10^{-5} \). The background solution analysis give specific range of coupling constant, \( 0 < \zeta \leq 0.021 \sim 10^{-2} \) for balanced coupling approximation \( \xi = \zeta \). In this work, we analyze de Sitter expansion and vanishing scalar field approximation for the spectral index of the perturbations, and show the deviation from the invariance scale, as the evidence of inflation at early times.

This paper is organized as follows, section 2 is about the framework of perturbations we worked as general theory. In section 3, we insert de Sitter expansion and vanishing scalar field approximation to the expression of the spectral index, and gain the picture of the evolution of it to give insight about inflation phenomena. The last section is for conclusion.

2. Dynamical Equation of Cosmological Perturbations
The background framework for this work is flat Friedmann-Robertson-Walker metric, with scalar (\( \Theta \)) and tensor perturbation (\( \gamma_{ij} \)),
\[ ds^2 = -dt^2 + h_{ij}dx^idx^j, \tag{7} \]
where,
\[ h_{ij} = a(t)^2 e^{2\Theta} (\delta_{ij} + \gamma_{ij} + \frac{1}{2} \gamma_{il} \gamma_{lj}), \tag{8} \]
with \( a(t) \) is a scale factor. Tensor perturbation (\( \gamma_{ij} \)) is traceless and divergence-free, so \( \gamma_{ii} = \partial_i \gamma_{ij} = 0 \), as known as Maldacena Gauge \[9\] which is defined only to second order because any higher order of \( h_{ij} \) will not significantly contribute to the action. Because of the property that scalar and tensor modes decouple from each other, we can analyze each perturbation modes by oneself. \( \gamma_{ij} \) can be described in two polarization modes,
\[ \gamma_{ij} = h + \gamma_{ij}^+ + h \times \gamma_{ij}^\times \tag{9} \]
where ini Fourier space, \( \gamma_{ij}^+ \) and \( \gamma_{ij}^\times \) comply with the normalization condition \( \gamma_{ij}(k) \gamma_{ij}(-k)^* = 2 \) and \( \gamma_{ij}^+(k) \gamma_{ij}^\times(-k)^* = 0 \[10\].

2.1. Scalar Perturbations
The dynamical equation of motion for each mode perturbations is obtained from the second order Lagrangian equation,
\[ \mathcal{L}^{(s)}_2 = a^3 Q_s [\dot{\Theta}^2 - \frac{c_s^2}{a^2} (\partial \Theta)^2]; \tag{10} \]
\[ Q_s \equiv \frac{2L_S(9\Lambda^2 + 8L_sw)}{W^2}; \tag{11} \]
\[ c_s^2 \equiv \frac{2}{Q_s}(\dot{\mathcal{M}} + H\mathcal{M} - \mathcal{E}), \tag{12} \]
where,
\begin{align}
L_S &= \frac{1}{2}[1 - \zeta \phi^2 - \xi \dot{\phi}^2] \quad (13) \\
\mathcal{W} &= 2[H - \zeta (H \phi^2 + \dot{\phi}) - \frac{3 \xi H \dot{\phi}^2}{2}] \quad (14) \\
w &= 3[-9H^2 + 9\zeta (H^2 \phi^2 + 2H \dot{\phi}) + 27\xi H^2 \dot{\phi}^2] \quad (15) \\
\mathcal{M} &= \frac{4\xi^2 \dot{\phi}^4 + 4\zeta \xi \phi^2 \dot{\phi}^2 + \xi^2 \dot{\phi}^4 + 16L_S - 4}{8[H(2L_S - \zeta \phi^2)]} \quad (16) \\
\mathcal{E} &= \frac{1}{2}[1 - \zeta \phi^2 - \xi \dot{\phi}^2] \quad (17)
\end{align}

with \( H = \frac{\dot{a}}{a} \) and \( \dot{a} = \frac{\partial a}{\partial t} \). In Fourier space, the equation of motion for scalar modes (\( \Theta \)) can be written as
\begin{equation}
\ddot{\Theta} + (3H + \frac{\dot{Q}_s}{Q_s}) \dot{\Theta} + c_s^2 k^2 a^2 \Theta = 0. \quad (18)
\end{equation}

To find the solution of (18), we take Bunch-Davies vacuum function, then the solution for each modes respectively,
\begin{equation}
\Theta(\tau, k) = iHe^{-ic_s k \tau} \frac{2\pi^2}{k_1^3} P_\Theta (k_1)(2\pi)^3 \delta(3)(k_1 + k_2). \quad (19)
\end{equation}

In super horizon scales, \( c_s k \ll aH \), the power spectrum is defined at \( \tau \approx 0 \) by,
\begin{equation}
\langle \Theta(0, k_1)\Theta(0, k_2) \rangle = \frac{2\pi^2}{k_1^3} P_\Theta (k_1)(2\pi)^3 \delta(3)(k_1 + k_2). \quad (20)
\end{equation}

So, if we consider solution (19) for \( \tau \approx 0 \) limit,
\begin{equation}
\Theta(0, k) = iH \frac{e^{-ic_s k \tau}}{2(c_s k)^{3/2} \sqrt{Q_s}}. \quad (21)
\end{equation}

the power spectrum can be obtained,
\begin{equation}
P_\Theta = \frac{H^2}{8\pi^2 c_s^3 Q_s}. \quad (22)
\end{equation}

The characterization of scalar fluctuation spectrum is described by a spectral index \( n_s \) defined as,
\begin{equation}
n_s - 1 \equiv \frac{d \ln P_\Theta}{d \ln k} |_{cs = aH} = \frac{2H}{H} - \frac{\dot{Q}_s}{HQ_s} - 3 \frac{c_s}{Hc_s}, \quad (23)
\end{equation}

where because of the perturbations evolve towards constant in this scales, it is good estimation to evaluate the power spectrum at \( c_s k = aH \) during inflation. From Planck Collaboration data [11], the scalar spectral index is bounded as \( n_s = 0.9665 \pm 0.0038 \) at 68\% confidence level.

### 2.2. Tensor Perturbations

For tensor perturbations, the second order Lagrangian equation,
\begin{align}
\mathcal{L}_2^{(h)} &= \frac{a^3}{4} Q_t [\gamma_{ij}^2 - \frac{\xi}{a^2} (\partial_k \gamma_{ij})^2]; \quad (24) \\
Q_t &= \frac{1}{2} [1 - \zeta \phi^2 - \xi \dot{\phi}^2]; \quad (25) \\
c_t^2 &= \frac{1}{2} [1 - \zeta \phi^2 + \xi \dot{\phi}^2]. \quad (26)
\end{align}
Analogous with scalar modes, each of the tensor polarization modes $h_\lambda(\lambda = +, \times)$ satisfies the equation of motions,

$$\ddot{h}_\lambda + (3H + \frac{\dot{Q}_t}{Q_s}) \dot{h}_\lambda + \epsilon_t^2 \frac{k^2}{a^2} h_\lambda = 0, \quad (27)$$

therefore, the solution for each modes is given by,

$$h_\lambda(\tau, k) = \frac{i He^{-ic_t k\tau}}{2(c_t k)^{3/2} \sqrt{Q_t}} (1 + ic_t k\tau). \quad (28)$$

In the $\tau \approx 0$ limit, the solution becomes,

$$h_\lambda(0, k) = \frac{i H}{2(c_t k)^{3/2} \sqrt{Q_t}}, \quad (29)$$

so the power spectrum of tensor pertubations can be obtained,

$$P_h = \frac{H^2}{2\pi^2 Q_t c_t^3}, \quad (30)$$

because of the polarization of the tensor modes. The tensor spectral index at $c_t k = aH$, as we consider for scalar perturbations shows,

$$n_t \equiv \frac{d\ln P_h}{d\ln k} |_{c_s = aH} = 2 \frac{\dot{H}}{H} \frac{\dot{Q}_t}{HQ_t} - 3 \frac{\dot{c}_t}{Hc_t}. \quad (31)$$

The tensor spectral index is close to scale-invariant, $n_t \approx 0$, where before the inflation ends, the tensor-to-scalar ratio can be approximated as,

$$r = \frac{P_h}{P_\Theta} = 4 \frac{Q_s c_s^3}{Q_t c_t^3}. \quad (32)$$

Planck Collaboration data [11] gives $r < 0.10$ in 95% confidence level.

3. De Sitter Expansion and Decaying Scalar Field Case

The cases considered in this work are de Sitter Expansion approximation,

$$a(t) \sim e^{(H_0 t)} \rightarrow H(t) = \frac{\dot{a}}{a} = H_0, \quad (33)$$

and decaying scalar field,

$$\phi \sim e^{(\phi_0 t)}, \quad (34)$$

$$\dot{\phi} \sim \phi_0 e^{(\phi_0 t)}, \quad (35)$$

$$\ddot{\phi} \sim \phi_0^2 e^{(\phi_0 t)}. \quad (36)$$

To ensure the expansion and the decaying, $H_0 > 0$ and $\phi_0 < 0$ condition is considered. After we substitute (33-36) to each perturbation modes, we plot contour plot of the spectral index with some variation of $\phi_0$. Because of the fact that inflation happens if universe expands exponentially for more than 60 e-folds [12], we take $H_0 = 60$. Furthermore, based on the background works [8], we take $\zeta = 10^{-2}, 10^{-3}, 10^{-4}$, and $10^{-5}$.
3.1. Scalar Perturbation

Considering (33-36) in scalar perturbations, the spectral index (31) of scalar modes can be written as,

\[ n_s - 1 = - \frac{\dot{Q}_{s0}}{H_0 Q_{s0}} - 3 \frac{c_{s0}}{H_0 c_{s0}}, \]

where,

\[
Q_{s0} = -9 \left[ \begin{array}{c}
\zeta e^{(6\phi_0 t)} [H_0 \phi_0^6 - H_0 \phi_0^5 + \frac{1}{2} \phi_0^4 - 2H_0 \phi_0^3 + \phi_0^2 - H_0 2]
+ \zeta e^{(4\phi_0 t)} [2H_0 \phi_0^3 - \phi_0^2 + H_0^2] + \zeta e^{(2\phi_0 t)} H_0^2 \phi_0^2 - H_0^2 \\
- \zeta e^{(\phi_0 t)} [3H_0^2 \phi_0^2 + 2H_0 \phi_0 + H_0^2]
\end{array} \right],
\]

\[c_{s0}^2 = 2 \frac{Q_{s0}}{Q_{s0}} (\dot{M}_0 - H_0 \dot{M}_0 - \dot{E}_0),\]

\[\dot{Q}_{s0} = \frac{\partial Q_{s0}}{\partial t}; \quad \dot{c}_{s0} = \frac{1}{2c_{s0}} \frac{\partial (c_{s0})}{\partial t},\]

with,

\[\dot{M}_0 = \frac{\zeta^2 e^{(4\phi_0 t)} [\frac{1}{4} \phi_0^4 + \phi_0^2 + 1] - \zeta e^{(2\phi_0 t)} [\phi_0^2 + 2] + 1}{2H_0 - \zeta e^{(2\phi_0 t)} [2H_0 + 2\phi_0 + 3H_0 \phi_0^2]},\]

\[\dot{\dot{M}}_0 = \frac{\partial \dot{M}_0}{\partial t},\]

\[\dot{E}_0 = \frac{1}{2} (1 - \zeta e^{(\phi_0 t)} + \frac{1}{2} \phi_0^2 e^{(2\phi_0 t)}).\]

Therefore, the plot spectral index as the function of time and \( \phi_0 \) can be seen in figure 1.

**Figure 1.** Contour plot of spectral index of scalar pertubations as the function of time and \( \phi_0 \).
As we can see in figure 1, the difference of $\zeta$ does not give significantly different value of spectral index. However, in each $\zeta$ case, in some $\phi_0$ value, the spectral index still near the scale-invariance ($n_s \simeq 1$), with a little deviation for it.

3.2. Tensor Perturbation

Analogous with scalar perturbations, the spectral index of tensor perturbations (23) can be written as,

$$n_t = -\frac{\dot{Q}_{t0}}{H_0 Q_{t0}} - 3 \frac{\dot{c}_{t0}}{H_0 c_{t0}},$$  \hspace{1cm} (44)

where,

$$Q_{t0} = \frac{1}{2} [1 - \zeta e^{2\phi_0 t}(1 + \frac{\phi_0^2}{2})],$$  \hspace{1cm} (45)

$$c_{t0}^2 = \frac{1 - \zeta e^{2\phi_0 t}(1 + \frac{\phi_0}{2})}{1 - \zeta e^{2\phi_0 t}(1 - \frac{\phi_0}{2})},$$  \hspace{1cm} (46)

$$\dot{Q}_{t0} = \frac{\partial Q_{t0}}{\partial t}, \quad \dot{c}_{t0} = \frac{1}{2c_{t0}} \frac{\partial (c_{t0}^2)}{\partial t}. $$  \hspace{1cm} (47)

Therefore the plot of spectral index as function of time and $\phi_0$ can be seen in figure 2.

Figure 2. Contour plot of spectral index of tensor pertubations as the function of time and $\phi_0$.

As we can see in figure 2, same with scalar perturbation, the difference of $\zeta$ does not give significantly different value of spectral index. However, in each $\zeta$ case, in some $\phi_0$ value, the spectral index still near the scale-invariance ($n_t \simeq 0$), with a little deviation for it. Furthermore, we can obtain the tensor-to-scalar ratio,

$$r = \frac{4Q_{s0} c_{s0}^3}{Q_{t0} c_{t0}^3}. $$  \hspace{1cm} (48)
The plot of tensor-to-scalar ratio as function of time and $\phi_0$ can be seen in figure 3.

Based on Planck Collaboration data [11], $r < 0.10$ in 95% confidence level, so from figure 3 we can obtain the value that inside the range of the data.

4. Conclusion
The deviation of spectral index from the scale invariant, $n_s \approx 1$ and $n_t \approx 0$, is associated with the behaviour of scalar (curvature) and tensor perturbations when inflation occurs and also depends on the model considered. In this work, at special condition where universe expands exponentially and the scalar field “forced” to decay at late time, a little deviation from scale invariant is observed and still in the range of observational data.

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