Eight-Dimensional Hermitian Lie Groups Conformally Foliated by Minimal $\text{SU}(2) \times \text{SU}(2)$ Leaves

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Abstract

We investigate the 8-dimensional Riemannian Lie groups $G^8$, carrying a left-invariant, conformal and minimal foliation $\mathcal{F}$, with leaves diffeomorphic to the subgroup $\text{SU}(2) \times \text{SU}(2)$ of $G^8$. Such groups have been classified by E. Ghandour, S. Gudmundsson and T. Turner in their recent work [2]. They show that these 8-dimensional Lie groups form a real 13-dimensional family.

For each left-invariant Hermitian structure $J_V$ on $\text{SU}(2) \times \text{SU}(2)$, we extend this to an almost Hermitian structures $J$ on $G$ adapted to the foliation $\mathcal{F}$ i.e. respecting the leaf structure on $G$ induced by $\mathcal{F}$. We then classify those 8-dimensional Lie groups $G$ for which the almost Hermitian structures $J$ are integrable ($\mathcal{W}_3 \oplus \mathcal{W}_4$), semi-Kähler ($\mathcal{W}_4$), locally conformal Kähler ($\mathcal{W}_3$) or even Kähler ($\mathcal{K}$).

It turns out that for each $J_V$ we obtain a 9-dimensional family of Lie groups $G$ for which $J$ is integrable. In the case of $J$ being semi-Kähler, we yield a 3-dimensional family of such groups. We then show that in the cases of $J$ being Kähler or locally conformal Kähler there are no solutions i.e. such 8-dimensional Lie groups do not exist.

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are either assumed to be too well known or the fruits of my own efforts.
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Chapter 1

Riemannian Manifolds

We assume that the readers of this thesis have some basic knowledge of differential geometry. They can refer to the textbook [4] by Sigmundur Gudmundsson for a good introduction. In this chapter, we introduce the main object that we investigate and cover some notions and results that will be used later in this thesis. All submanifolds are assumed to be smoothly embedded.

1.1 Lie Groups

A finite-dimensional Lie group is a smooth manifold $G$ with a group structure such that the multiplication and the inversion are smooth. Let $G$ and $H$ be Lie groups, a map $\phi : G \to H$ is called a Lie group isomorphism if it is both a group homomorphism and a diffeomorphism. If such a map exists, $G$ and $H$ are said to be isomorphic. A subgroup $H$ of $G$ is called a Lie subgroup if $H$ is also a submanifold of $G$.

The Lie algebra $\mathfrak{g}$ of a Lie group $G$ is the vector space of left-invariant vector fields on $G$. This can be identified with the tangent space at the identity $e \in G$. An inner product structure on $\mathfrak{g}$ induces a left-invariant Riemannian metric $g$ on $G$ by the left translations. A Riemannian Lie group is a Lie group $G$ equipped with a left-invariant Riemannian metric.

A Lie algebra over a field $\mathbb{F}$ is a vector space $V$ over $\mathbb{F}$ endowed with a bilinear map $[,] : V \times V \to V$ satisfying

(i) antisymmetry: $[X,Y] = -[Y,X]$ for any $X,Y \in V$,

(ii) Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$ for any $X,Y,Z \in V$.

Let $\mathfrak{g}, \mathfrak{h}$ be two Lie algebras over a field $\mathbb{F}$, a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ is said to be a Lie algebra homomorphism if it preserves the Lie brackets i.e. $\phi([X,Y]) = [\phi(X), \phi(Y)]$. It is said to be a Lie algebra isomorphism if it is also bijective. The Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are said to be isomorphic if there exists a Lie algebra isomorphism between them.

The Lie algebra of a finite-dimensional Lie group is a finite-dimensional Lie algebra. Moreover, simply connected Lie groups can be classified by their Lie algebras. Specifically, we have the following result.

Theorem 1.1 ([6, page 531]). For any finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{R}$, there exists a simply connected Lie group $G$ whose Lie algebra is isomorphic to
Furthermore, if $G$ and $H$ are simply connected Lie groups with isomorphic Lie algebras, then $G$ and $H$ are isomorphic.

Another result about the correspondence between Lie subgroups and Lie subalgebras will be useful when we classify Lie groups containing $K$ as a Lie subgroup.

Theorem 1.2 ([6, page 506]). Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. If $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, then there exists a unique connected Lie subgroup $K$ of $G$ with Lie algebra $\mathfrak{k}$.

1.2 Foliations

Let $M$ be a smooth $m$-dimensional manifold with tangent bundle $TM$. A smooth distribution $V$ of rank $k$ is a $k$-dimensional subbundle of $TM$. In other words, it is the collection of some $k$-dimensional subspaces of the tangent spaces of $M$ that vary smoothly over $M$.

A smooth distribution $V$ on $M$ is said to be involutive if for any vector fields $X, Y$ in $V$, the vector field $[X, Y]$ also belongs to $V$. Given a smooth distribution $V$ on $M$, a submanifold $N$ of $M$ is said to be an integral manifold of $V$ if $T_pN = V_p$ for any $p \in M$. A connected integral manifold of $V$ is said to be maximal if it is not contained in any other connected integral manifold of $V$.

Next, we define the notion of a foliation on a smooth manifold $M$.

Definition 1.3. Let $\mathcal{F} = \{L_{\alpha} | \alpha \in I\}$ be a collection of disjoint, connected, nonempty $k$-dimensional submanifolds of $M$. Then $\mathcal{F}$ is called a foliation of dimension $k$ on $M$ if

(i) $\bigcup_{\alpha \in I} L_{\alpha} = M$,

(ii) for any point $p \in M$, there exists a chart $(U, \phi)$ about $p$ such that $\phi(U)$ is a cube in $\mathbb{R}^m$ and the intersection $L_{\alpha} \cap U$ for each $L_{\alpha} \in \mathcal{F}$ is either empty or a countable union of $k$-dimensional slices of the form $x^{k+1} = c^{k+1}, \ldots, x^n = c^n$.

An element $L_{\alpha}$ of a foliation is called a leaf. And (ii) is equivalent to the following condition: for each point $p \in M$, there is an open neighborhood $W$ of $p$ in $M$, a $(m-k)$-dimensional smooth manifold $N$ and a smooth submersion $\phi : W \to N$ such that for each $L_{\alpha} \in \mathcal{F}$, the connected components of $W \cap L_{\alpha}$ are the connected components of the fibres of $\phi$.

Given a smooth submersion $\phi : M \to N$, by Definition 1.3 and the implicit function theorem, we see that the collection $\mathcal{F}$ of the connected components of the fibres of $\phi$ is a smooth foliation on $M$. We will call $\mathcal{F}$ the foliation associated to $\phi$. And a foliation is said to be simple if it is associated to some global smooth submersion with connected fibres. The neighborhood $W$ in Definition 1.3 is called a $\mathcal{F}$-simple open set.

Foliations are related to involutive distributions by the following important global Frobenius theorem.

Theorem 1.4 ([6, page 502]). Let $V$ be an involutive distribution on a smooth manifold $M$. Then the collection of all maximal connected integral manifolds of $V$
forms a foliation of \( M \). Conversely, given a foliation \( \mathcal{F} \) on \( M \), the collection of tangent spaces of the leaves of \( \mathcal{F} \) forms a distribution on \( M \), which is called the associated distribution of \( \mathcal{F} \). Furthermore, the associated distribution of a foliation \( \mathcal{F} \) on \( M \) is involutive.

A subgroup of a Lie group gives us examples of distributions and foliations that are of particular interest to us.

**Definition 1.5.** A distribution \( \mathcal{V} \) on a Lie group \( G \) is said to be left-invariant if it is closed under left translations.

**Definition 1.6.** Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). Then the distribution on \( G \) generated by left translations of a subalgebra \( \mathfrak{k} \) of \( \mathfrak{g} \) is involutive. It is called the left-invariant distribution generated by \( \mathfrak{k} \).

By Theorem (1.4), the left-invariant distribution generated by a Lie subalgebra \( \mathfrak{k} \) produces a foliation on \( M \). We call it the left-invariant foliation on \( M \) generated by \( \mathfrak{k} \).

### 1.3 Minimal Submanifolds

In this section we introduce the mean curvature vector field and minimal submanifolds in preparation for the next section. Readers can find more details about this part in the textbook [5] by Kobayashi and Nomizu.

Let \((N,h)\) be an \((m + r)\)-dimensional Riemannian manifold and \( M \) be an \( m \)-dimensional submanifold of \( N \) equipped with the metric \( g \) induced by \( h \). The Levi-Civita connection on \( N \) is denoted by \( \nabla \). Let \( X \) be a vector field on \( M \) and \( Z \in \mathcal{C}^\infty (NM) \) be a vector field in the normal bundle \( NM \) of \( M \). We define a vector field by

\[
S_Z(X) = - (\nabla_X Z)^\top
\]

where \( \top \) means the orthogonal projection onto \( TM \).

The operator \( S \) generalizes the shape operator of a surface in \( \mathbb{R}^3 \) by the following Weingarten equation.

**Proposition 1.7** ([5, page 14]). The map \( S: C^\infty (NM) \times C^\infty (TM) \to C^\infty (TM) \) is a tensor field on \( M \). Therefore, for any point \( p \in M \), the value of \( (S_Z(X))_p \) only depends on the values of \( Z \) and \( X \) at \( p \). Therefore, \( S \) induces a well-defined bilinear map \( S: T_p^\perp M \times T_p M \to T_p M \). Furthermore, we define the second fundamental form \( B \) of \( M \) by

\[
B(X,Y) = (\nabla_X Y)^\perp
\]

where \( X,Y \in T_p M \) and \( \perp \) is the orthogonal projection onto \( NM \). Then we have

\[
\langle S_Z(X), Y \rangle = \langle B(X,Y), Z \rangle
\]

for any \( X,Y \in T_p M \) and \( Z \in T_p M^\perp \).

For a fixed \( Z \in T_p M^\perp \), the map \( X \mapsto S_Z(X) \) is a symmetric linear endomorphism of \( T_p M \) by the symmetry of the second fundamental form and ([1.4]). So we have a linear functional on \( T_p M^\perp \) defined by

\[
Z \mapsto \frac{1}{m} \text{trace} S_Z.
\]
From linear algebra we know that there exists a unique normal vector \( \eta \in T_pM^\perp \) such that
\[
g(\eta, Z) = \frac{1}{m} \text{trace } S_Z
\]
for any \( Z \in T_pM^\perp \). We call \( \eta \) the mean curvature vector of \( M \) at \( p \). We shall derive a formula for the mean curvature vector in an orthonormal basis. Let \( \xi_1, \ldots, \xi_r \) be an orthonormal basis for \( T_pM^\perp \) and \( \beta_1, \ldots, \beta_m \) be an orthonormal basis for \( T_pM \). The operator \( S_{\xi_j} \) is abbreviated as \( S_j \). Since the operator \( S_Z \) is symmetric with respect to \( g \), by (1.1) and (1.3) we have
\[
\eta = \sum_{j=1}^r \langle \eta, \xi_j \rangle \xi_j
= \sum_{j=1}^r \left( \frac{1}{m} \text{trace } S_j \right) \xi_j
= \frac{1}{m} \sum_{j=1}^r \sum_{k=1}^m \langle S_j(\beta_k), \beta_k \rangle \xi_j
= \frac{1}{m} \sum_{j=1}^r \sum_{k=1}^m \langle B(\beta_k, \beta_k), \xi_j \rangle \xi_j
= \frac{1}{m} \sum_{j=1}^r \left( \sum_{k=1}^m B(\beta_k, \beta_k) \right) \xi_j
= \frac{1}{m} \sum_{k=1}^m B(\beta_k, \beta_k)
\] (1.2)

Applying (1.2) to a local orthonormal frame of \( TM \), we know that there is a well-defined smooth vector field \( \eta^M \in C^\infty(NM) \) that takes the value of mean curvature vector of \( M \) at every point. We call \( \eta^M \) the mean curvature vector field of \( M \) in \( (N, h) \).

**Definition 1.8.** Let \((M, g)\) be a Riemannian manifold isometrically embedded into \((N, h)\). Then \( M \) is said to be a minimal submanifold if \( \eta^M \) vanishes identically. By (1.2) we see that \( M \) is minimal if and only if
\[
\eta^M = \text{trace } B = \sum_{k=1}^m B(X_k, X_k) = 0,
\]
for any local orthonormal frame \( \{X_1, \ldots, X_m\} \) for \( TM \).

### 1.4 Foliations on Riemannian Manifolds

Now we can define some particular types of foliations on Riemannian manifolds. Readers are referred to [1] for more details about this part.

Let \((M, g)\) be a Riemannian manifold of dimension \( m \) and denote the Levi-Civita connection of \( M \) by \( \nabla \). Let \( \mathcal{V} \) be a \( q \)-dimensional distribution on \( M \) and denote the orthogonal distribution \( \mathcal{V}^\perp \) by \( \mathcal{H} \). We call \( \mathcal{V} \) the vertical distribution and \( \mathcal{H} \)
the horizontal distribution. The orthogonal projections from $TM$ onto $\mathcal{V}, \mathcal{H}$ are also denoted as $\mathcal{V}, \mathcal{H}$, respectively.

Then we introduce second fundamental forms on distributions. More specifically we have the following definition.

**Definition 1.9.** The second fundamental form of the vertical distribution $\mathcal{V}$ is the tensor field $B^\mathcal{V}$ of type $(1, 2)$ on $M$ defined by

$$B^\mathcal{V}(X, Y) = \frac{1}{2}(\mathcal{H}(\nabla_\mathcal{V}X \mathcal{V}Y) + \mathcal{H}(\nabla_\mathcal{V}Y \mathcal{V}X)) \quad \text{for any } X, Y \in C^\infty(TM).$$  \hfill (1.3)

Similarly, the second fundamental form of the horizontal distribution $\mathcal{H}$ is defined by

$$B^\mathcal{H}(X, Y) = \frac{1}{2}(\mathcal{V}(\nabla_\mathcal{H}X \mathcal{H}Y) + \mathcal{V}(\nabla_\mathcal{H}Y \mathcal{H}X)) \quad \text{for any } X, Y \in C^\infty(TM).$$  \hfill (1.4)

Notice that if the distribution $\mathcal{V}$ is involutive, then for any point $p \in M$ and vectors $X, Y \in \mathcal{V}_p$, by Definition 1.9 and the fact the $\nabla$ is torsion-free, we have

$$B^\mathcal{V}(X, Y) = \frac{1}{2}(\mathcal{H}(\nabla_\mathcal{V}X \mathcal{V}Y) + \mathcal{H}(\nabla_\mathcal{V}Y \mathcal{V}X)) = \frac{1}{2}(\mathcal{H}(\nabla X Y) + \mathcal{H}(\nabla Y X)) = \mathcal{H}(\nabla X Y).$$

So Definition 1.9 agrees with the usual notion of the second fundamental form on the integral manifolds of $\mathcal{V}$ when $\mathcal{V}$ is involutive.

**Definition 1.10.** An involutive distribution $\mathcal{V}$ on a Riemannian manifold $(M, g)$ is said to be

(i) **minimal**, if the integral manifolds of $\mathcal{V}$ are all minimal i.e.

$$\text{trace } B^\mathcal{V} = \sum_{j=1}^{m} B^\mathcal{V}(E_j, E_j) = 0,$$

for some and hence for all orthonormal basis $\{E_1, \ldots, E_m\}$ of $\mathcal{V}_p$ at any point $p$,

(ii) **totally geodesic**, if the integral manifolds of $\mathcal{V}$ are all totally geodesic i.e.

$$B^\mathcal{V} \equiv 0,$$

(iii) **conformal**, if for vector fields $X, Y$ in the horizontal distribution $\mathcal{H}$, there exists a vector field $V$ in $\mathcal{V}$ such that

$$B^\mathcal{H}(X, Y) = g(X, Y) \otimes V,$$

(iv) **Riemannian**, if $V \equiv 0$ in (iii) i.e. $B^\mathcal{H} \equiv 0$.

A smooth foliation $\mathcal{F}$ on a Riemannian manifold $(M, g)$ is said to be **minimal**, **totally geodesic**, **conformal**, or **Riemannian** if the same applies to the associated vertical distribution $\mathcal{V}$.
1.5 Harmonic Morphisms

In this section, we give a brief introduction to harmonic morphisms and explain how foliations and harmonic morphisms are closely related. Finally we introduce the main object we investigate in this thesis.

First we define the musical isomorphisms between the tangent and cotangent bundles of a Riemannian manifold.

**Definition 1.11.** Let \((M, g)\) be a Riemannian manifold. For any vector field \(X \in C^\infty(TM)\), there exists a unique 1-form \(X^\flat\) called the flat of \(X\) such that
\[ X^\flat(Y) = g(X, Y), \]
for any vector field \(Y \in C^\infty(TM)\). Similarly, for any 1-form \(\omega\) on \(M\), there exists a unique vector field \(\omega^\sharp\) called the sharp of \(\omega\) such that
\[ g(\omega^\sharp, Y) = \omega(Y), \]
for any vector field \(Y \in C^\infty(TM)\). The operators \(\flat\) and \(\sharp\) are mutually inverse vector bundle isomorphisms between \(TM\) and \(T^*M\).

Let \(X\) be a smooth vector field on a Riemannian manifold \((M, g)\). Then there is a \(C^\infty(M)\)-linear map from \(C^\infty(TM) \times C^\infty(TM)\) to \(C^\infty(M)\) defined by
\[ (Y, Z) \mapsto g(\nabla_Y X, Z). \] (1.5)
The map (1.5) gives rise to a bilinear form on \(T_pM\) for each point \(p \in M\). We can take the trace of this map with respect to the metric tensor \(g\). In particular, let \(\{E_1, \ldots, E_m\}\) be a local orthonormal frame, then we have
\[ \text{trace} \nabla X = \sum_{j=1}^m g(\nabla_{E_j} X, E_j). \]

**Definition 1.12.** Let \(f : M \to \mathbb{R}\) be a smooth function on a Riemannian manifold \((M, g)\). Then the gradient of \(f\) is the vector field given by
\[ \text{grad} f = (df)^\sharp. \] (1.6)
Let \(X\) be a smooth vector field on \(M\). Then the divergence of \(X\) is defined by
\[ \text{div} X = \text{trace} \nabla X. \] (1.7)

By expanding (1.6) and (1.7) by an orthonormal frame, we see that Definition 1.12 agrees with the usual definition of the gradient and divergence in \(\mathbb{R}^n\). A smooth function \(f : M \to \mathbb{R}\) on a Riemannian manifold \((M, g)\) is said to be harmonic if
\[ \Delta f = \text{div} \text{grad} f \equiv 0 \]
and the operator \(\Delta = \text{div} \text{grad}\) is called the Laplace-Beltrami operator.

Now we can define a harmonic morphism between Riemannian manifolds.

**Definition 1.13.** A smooth map \(\phi : M \to N\) between Riemannian manifolds \((M, g)\) and \((N, h)\) is called a harmonic morphism if for every harmonic function \(f : V \to \mathbb{R}\) defined on an open subset \(V\) of \(N\) such that \(f^{-1}(V) \neq \emptyset\), the function \(f \circ \phi\) is harmonic on \(\phi^{-1}(V)\).
In this thesis we are particularly interested in conformal foliations of codimension 2 because of the following result, see page 128 in [1].

**Theorem 1.14.** Let \( \mathcal{F} \) be a conformal foliation of codimension 2 on a Riemannian manifold \( M \). Then \( \mathcal{F} \) is minimal if and only if for any \( \mathcal{F} \)-simple open set \( U \) of \( \mathcal{F} \), there is a submersive harmonic morphism from \( U \), to a 2-dimensional Riemannian manifold \( N \), whose associated foliation is \( \mathcal{F}|_U \).

So finding conformal and minimal foliations of codimension 2 will give us many interesting examples of harmonic morphisms. In the paper [2], the authors prove that for every 8-dimensional Lie group \( G^8 \) containing \( K = \text{SU}(2) \times \text{SU}(2) \) as a Lie subgroup, the foliation \( \mathcal{F} \) generated by \( K \) is Riemannian and totally geodesic. Thus they obtain a 13-dimensional family of 8-dimensional Lie groups carrying a Riemannian foliation of codimension 2 with totally geodesic leaves. And this thesis is about investigating some structures from complex geometry on this family of Lie groups.
Chapter 2

Almost Hermitian Manifolds

2.1 Almost Hermitian Structures

In this section, we introduce some interesting notions and results from complex geometry.

By an n-dimensional complex manifold we mean a second-countable Hausdorff space $M$ that admits an open cover $\{U_\alpha\}$ where each $U_\alpha$ is homeomorphic to some open subset of $\mathbb{C}^n$, which is called a complex chart, and the transition maps are holomorphic.

Let $M$ be a smooth manifold of real dimension $2n$. Then by an almost complex structure on $M$ we mean a tensor field $J$ of type $(1, 1)$ satisfying $J^2 = -id.$

If an almost complex structure $J$ is defined on $M$, then $(M, J)$ is called an almost complex manifold. A complex manifold of complex dimension $n$ has a canonical almost complex structure induced by its complex local charts. Let $z_k = x_k + iy_k, \ k = 1,\ldots,n,$ be the complex coordinates for some complex chart. Then $x_k, y_k, \ k = 1,\ldots,n,$ are local coordinates for the real manifold structure if we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. The map defined by

$$\frac{\partial}{\partial x_k} \mapsto \frac{\partial}{\partial y_k}, \quad \frac{\partial}{\partial y_k} \mapsto -\frac{\partial}{\partial x_k}, \quad k = 1,\ldots,n,$$

is independent of the choice of the chart thus gives a canonical almost complex structure on $M$ induced by the complex one.

Conversely, an almost complex structure $J$ on a smooth manifold $M$ is said to be integrable if it is induced by some complex structure on $M$. The integrability problem can be converted into the calculation of some tensor fields by the following important Newlander-Nirenberg theorem.

**Theorem 2.1** ([9]). Let $J$ be an almost complex structure on a smooth manifold $M$. The Nijenhuis tensor $N_J$ of $J$ is the tensor field of type $(1, 2)$ defined by

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY],$$

(2.1)
for all vector fields \(X, Y\) on \(M\). Then \(J\) is integrable if and only if \(N_J \equiv 0\).

Now let \((M, g)\) be a \(2n\)-dimensional Riemannian manifold. An almost complex structure \(J\) is said to be compatible with \(g\) if
\[
g(X, Y) = g(JX, JY),
\]
for any point \(p \in M\) and \(X, Y \in T_p M\). A Riemannian manifold \((M, g)\) with a compatible almost complex structure \(J\) is called an almost Hermitian manifold. If this almost complex structure \(J\) is integrable, then \(g\) induces a Hermitian metric on the complex structure induced by \(J\) and we call \((M, g, J)\) a Hermitian manifold.

In this thesis we are particularly interested in the left-invariant almost Hermitian structures on Riemannian Lie groups.

**Definition 2.2.** Let \(G\) be a \(2n\)-dimensional Lie group. Let \(J_e\) be a complex structure on its Lie algebra \(g\). Then the left-invariant almost complex structure \(J\) on \(G\) induced by \(J_e\) is defined by
\[
J(X) = dL_p \circ J_e \circ dL_p^{-1}(X),
\]
for any \(p \in G\), \(X \in T_p G\), where \(L_p\) denotes the left translation on \(G\) by \(p\).

**Definition 2.3.** A Hermitian Lie group is a Riemannian Lie group \((G, g)\) equipped with a left-invariant complex structure \(J\) that is compatible with \(g\). So in this case \(G\) is both a Lie group and a Hermitian manifold.

Given a left-invariant almost Hermitian structure \(J\) on a Riemannian Lie group \((G, g)\), the integrability problem of \(J\) can be simplified by the following well-known result.

**Proposition 2.4.** Let \(J\) be a left-invariant almost complex structure on a \(2n\)-dimensional Lie group \(G\). Then the Nijenhuis tensor \(N_J\) vanishes if and only if \(N_J\) is zero at the tangent space \(T_e G \cong g\).

**Proof.** Let \(p\) be an element of \(G\) and \(X_p, Y_p\) be arbitrary vectors in \(T_p G\). Then there exist left-invariant vector fields \(X, Y\) with the values \(X_p, Y_p\) at \(p\), respectively. Therefore
\[
N_J(X_p, Y_p) = N_J(X, Y)(p)
= [X_p, Y_p] + J[JX_p, Y_p] + J[X_p, JY_p] - [JX_p, JY_p].
\]
Since the vector fields \(X, Y\) are left-invariant, we have
\[
J[JX_p, Y_p] = dL_p \circ J \circ dL_p^{-1}([dL_p \circ J \circ dL_p^{-1}(X_p), Y_p])
= dL_p \circ J([J \circ dL_p^{-1}(X_p), dL_p^{-1}(Y_p)])
= dL_p(J[JX_e, Y_e])
\]
and similarly for the other terms that
\[
[X_p, Y_p] = dL_p([X_e, Y_e]),
J[X_p, JY_p] = dL_p(J[X_e, JY_e]),
[JX_p, JY_p] = dL_p([JX_e, JY_e]).
\]
Therefore we have
\[
N_J(X_p, Y_p) = dL_p(N_J(X_e, Y_e))
\]
which accomplishes the proof. \(\square\)
2.2 A Classification of Almost Hermitian Structures

In this section, we present the famous classification of almost Hermitian structures from the paper [3], where the authors A. Gray and L. M. Hervella classify them into 16 different types.

For an almost Hermitian manifold \((M^{2n}, g, J)\), the Kähler form \(\omega\) is the differential 2-form defined by

\[
\omega = g(JY, Z), \quad Y, Z \in C^\infty(TM).
\]  

(2.3)

Denote the Levi-Civita connection of \(M\) by \(\nabla\). Then there is a well-defined 3-covariant tensor field \(\nabla \omega\) on \(M\) defined by

\[
(\nabla_X \omega)(Y, Z) = X(\omega(Y, Z)) - \omega(\nabla_X Y, Z) - \omega(Y, \nabla_X Z),
\]  

(2.4)

where \(X, Y, Z \in C^\infty(TM)\).

Proposition 2.5. Let \((M^{2n}, g, J)\) be an almost Hermitian manifold, then the Kähler form \(\omega\) satisfies

(i) \((\nabla_X \omega)(Y, Z) = - (\nabla_X \omega)(Z, Y)\),

(ii) \((\nabla_X \omega)(Y, Z) = - (\nabla_X \omega)(JY, JZ)\),

for any vector fields \(X, Y, Z\) on \(M\).

Proof. The first equality follows directly from (2.3) and (2.4). Now we consider (ii). Since the Levi-Civita connection is compatible with the metric \(g\), we have

\[
(\nabla_X \omega)(Y, JY) = X(g(JY, JY)) - \omega(\nabla_X Y, JY) - \omega(Y, \nabla_X JY) \\
= X(g(JY, JY)) - g(J\nabla_X Y, JY) - g(JY, \nabla_X JY) \\
= X(g(JY, JY)) - g(\nabla_X JY) - g(JY, \nabla_X JY) \\
= X(g(JY, JY)) - \frac{1}{2}X(g(JY, JY)) \\
= 0.
\]

Then the equality (ii) follows by expanding \((\nabla_X \omega)(Y + Z, Y + Z) = 0\).

Let \(V\) be a 2\(n\)-dimensional real vector space with an inner product \(\langle , \rangle\) and an almost complex structure \(J\) such that \(\langle X, Y \rangle = \langle JX, JY \rangle\), for any \(X, Y \in V\). Let \(V^*\) be the dual space of \(V\). By a direct calculation we see that \(W\) defined by

\[
W = \{ \alpha \in \otimes^3 V^* | \alpha(X, Y, Z) = -\alpha(X, Z, Y), \alpha(X, Y, Z) = -\alpha(X, JY, JZ) \text{ for all } X, Y, Z \in V \}.
\]  

(2.5)

is a linear subspace of \(\otimes^3 V^*\). Then there is a natural inner product on \(W\) defined by

\[
\langle\langle \alpha, \beta \rangle\rangle = \sum_{j,k,l=1}^{2n} \alpha(E_j, E_k, E_l)\beta(E_j, E_k, E_l),
\]  

where \(E_j, E_k, E_l \in V\).
for any orthonormal basis \( \{E_1, ..., E_{2n}\} \) of \( V \). Following the notations in [3], for \( \alpha \in W \), we define \( \tilde{\alpha} \in V^* \) by

\[
\tilde{\alpha}(X) = -\sum_{j=1}^{2n} \alpha(E_i, E_i, X),
\]

(2.6)

where \( \{E_1, ..., E_{2n}\} \) is an arbitrary orthonormal basis for \( V \). We also define the following four subspaces of \( W \) by

\[
\begin{align*}
W_1 &= \{\alpha \in W | \alpha(X, X, Z) = 0 \text{ for all } X, Z \in V\}, \\
W_2 &= \{\alpha \in W | \alpha(X, Y, Z) + \alpha(Z, X, Y) + \alpha(Y, Z, X) = 0 \text{ for all } X, Y, Z \in V\}, \\
W_3 &= \{\alpha \in W | \alpha(X, Y, Z) - \alpha(JX, JY, Z) = 0, \tilde{\alpha}(Z) = 0 \text{ for all } X, Y, Z \in V\}, \\
W_4 &= \{\alpha \in W | \alpha(X, Y, Z) = -\frac{1}{2(n-1)} (\langle X, Y \rangle \tilde{\alpha}(Z) - \langle X, Z \rangle \tilde{\alpha}(Y) \\
&\quad - \langle X, JY \rangle \tilde{\alpha}(JZ) + \langle X, JZ \rangle \tilde{\alpha}(JY)) \text{ for all } X, Y, Z \in V\}.
\end{align*}
\]

Since \( J \) is compatible with the inner product of \( V \), we can equip it with a natural structure of a Hermitian vector space of complex dimension \( n \) by setting

\[
(a + bi)X = aX + bJX,
\]

where \( a, b \in \mathbb{R} \) and \( X \in V \). Hence, if we fix an orthonormal basis \( \{E_1, ..., E_n\} \) for \( V \), there is a natural representation \( \rho \) of the unitary group \( U(n) \) on \( \otimes^3 V^* \) defined by

\[
\rho(A)(\phi)(X, Y, Z) = \phi(AX, AY, AZ),
\]

where \( A \in U(n), \phi \in \otimes^3 V^* \) and \( X, Y, Z \in V \).

By Definition 2.5 we see that \( W \) is invariant under the action of the unitary group \( U(n) \). So \( \rho \) induces a representation of \( U(n) \) on the vector space \( W \). The following result is vital to the classification.

**Theorem 2.6 ([3]).** The vector space \( W \) has the following orthogonal decomposition

\[
W = W_1 \oplus W_2 \oplus W_3 \oplus W_4,
\]

where each component \( W_j \) is invariant under the action of \( U(n) \). Furthermore, the induced representation of \( U(n) \) on each \( W_j \) is irreducible.

Let \( (M^{2n}, g, J) \) be an almost Hermitian manifold and \( p \) be an element of \( M \). By comparing Proposition 2.5 with (2.5), we can put \( V = T_p M \), then \( (\nabla\omega)|_p \in W \). We denote the four irreducible components of this \( W \) by \( W_{p,j}, j = 1, 2, 3, 4 \).

Now we can classify all of the almost Hermitian manifolds according to \( \nabla\omega \).

**Definition 2.7.** The class of almost Hermitian manifolds \( (M^{2n}, g, J) \) whose Kähler form \( \omega \) satisfies

\[
(\nabla\omega)|_p \in W_{p,j},
\]

for each point \( p \in M \) is denoted by \( \mathcal{W}_j \), where \( j = 1, 2, 3, 4 \). And \( W_{p,j} \) can be replaced by \( W_{p,j} \oplus W_{p,k} \) when the class is denoted as \( \mathcal{W}_j \oplus \mathcal{W}_k \), similarly for the other direct sums. In particular, we denote \( \mathcal{K} = \bigcap_{j=1}^{4} \mathcal{W}_j \).
Hence, we have 16 classes of different almost Hermitian structures. In the paper [3], Gray and Hervella prove that each class defined in this way coincides with some well-known class of almost Hermitian manifolds. We give a brief account of their proofs for the case $W_3 \oplus W_4$, $W_3$, $W_4$ and $K$.

**Definition 2.8.** An almost Hermitian manifold $(M, g, J)$ is said to be

(i) **semi-Kähler** if $\delta \omega = 0$,
(ii) **locally conformal Kähler** if the tensor field $\mu = 0$,
(iii) **Kähler** if $J$ is integrable and $d\omega = 0$.

Here, the operator $\delta$ in (i) is the codifferential operator given by

$$\delta : \Omega^{k+1}(M) \to \Omega^k(M), \quad \delta = -\sum_{j=1}^{2n} E_j \lrcorner \nabla E_j, \quad (2.7)$$

see page 101 of [8]. The $(1,2)$ tensor field $\mu$ in (ii) is defined by

$$\langle \mu(X,Y), Z \rangle = (\nabla_X \omega)(Y,Z) + \frac{1}{2(n-1)} \{ \langle X,Y \rangle \cdot \delta \omega(Z) - \langle X,Z \rangle \cdot \delta \omega(Y) - \langle X,JY \rangle \cdot \delta \omega(JZ) + \langle X,JZ \rangle \cdot \delta \omega(JY) \}, \quad (2.8)$$

for any $X,Y,Z \in C^\infty(TM)$. It is a locally conformal invariant of almost Hermitian manifolds and vanishes when $M$ is Kähler (see Lemma 4.1 in [3]). Besides, it can be verified that the condition $d\omega = 0$ in (iii) is equivalent to $\nabla \omega = 0$.

**Theorem 2.9 ([3]).** An almost Hermitian manifold $(M, g, J)$ belongs to $W_3 \oplus W_4$, $W_3$, $W_4$ or $K$ if and only if $M$ is Hermitian, Hermitian semi-Kähler, locally conformal Kähler or Kähler, respectively.

**Proof.** First we show that

$$W_3 \oplus W_4 = \{ \alpha \in W \mid \alpha(X,Y,Z) - \alpha(JX,JY,Z) = 0 \text{ for all } X,Y,Z \in V \}. \quad (2.9)$$

If $\alpha \in W$ satisfies $\alpha(X,Y,Z) - \alpha(JX,JY,Z) = 0$, we define $\alpha_1, \alpha_2$ by

$$\alpha_2(X,Y,Z) = -\frac{1}{2(n-1)} \{ \langle X,Y \rangle \hat{\alpha}(Z) - \langle X,Z \rangle \hat{\alpha}(Y) - \langle X,JY \rangle \hat{\alpha}(JZ) + \langle X,JZ \rangle \hat{\alpha}(JY) \},$$

$$\alpha_1(X,Y,Z) = \alpha(X,Y,Z) - \alpha_2(X,Y,Z). \quad (2.10)$$

By a direct calculation, we have

$$\hat{\alpha}_2(Z) = \hat{\alpha}(Z).$$

Hence, $\alpha = \alpha_1 + \alpha_2$ is a well-defined decomposition of $\alpha$ in $W_3 \oplus W_4$. The inclusion in the opposite direction can be verified in straightforward way.

Gray and Hervella prove the following equalities between $N_J$ and $\omega$.

$$\langle N_J(X,Y), JZ \rangle = \langle (\nabla_X J)(Y) - (\nabla_{JX} J)(JY) - (\nabla_Y J)(X) + (\nabla_{JY} J)(JX), Z \rangle,$$
\[2\langle(\nabla_X J)(Y) - (\nabla_{JX} J)(JY), Z \rangle = \langle N_J(X, Y), JZ \rangle - \langle N_J(Y, Z), JX \rangle + \langle N_J(Z, X), JY \rangle.\]

Since
\[\langle(\nabla_X J)(Y), Z \rangle = \langle \nabla_X JY - J\nabla_X Y, Z \rangle = (\nabla_X \omega)(Y, Z),\]
we have
\[\langle(\nabla_X J)(Y) - (\nabla_{JX} J)(JY), Z \rangle = (\nabla_X \omega)(Y, Z) - (\nabla_{JX} \omega)(JY, Z).\]
Combining these results together, we see that \(M\) is Hermitian if and only if
\[(\nabla_X \omega)(Y, Z) - (\nabla_{JX} \omega)(JY, Z) = 0.\]
So \(\mathcal{W}_3 \oplus \mathcal{W}_4\) is the class of Hermitian manifolds.

By comparing (2.7) with the definition of \(\tilde{\alpha}\), we see that a manifold belongs to \(\mathcal{W}_3\) if and only if \(\delta \omega = 0\) i.e. \(\mathcal{W}_3\) is the class of Hermitian semi-Kähler manifolds. Similarly, a manifold belongs to \(\mathcal{W}_4\) if and only if \(\mu = 0\) i.e. \(\mathcal{W}_4\) is the class of locally conformal Kähler manifolds. Finally, \(\mathcal{K}\) is the class of Kähler manifolds since it contains exactly the manifolds satisfying \(\nabla \omega = 0\).
Chapter 3

Hermitian Foliations

In this chapter, we give a detailed account of the 13-dimensional family of Lie groups $G^8$ introduced in Chapter 1. We define a left-invariant Hermitian structure $J$ on $G^8$ and investigate the conditions for $J$ to be integrable.

3.1 Left-invariant Complex Structures on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$

The left-invariant complex structures on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ have been studied by L. Magnin in his paper [7]. Specifically, he proves the following result.

**Proposition 3.1.** The set of integrable left-invariant almost complex structures on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ consists of matrices of the form

\[
\begin{pmatrix}
\lambda_1^2 \xi & -\lambda_1 \mu_1 \xi + \nu_1 & \lambda_1 \mu_1 \xi + \mu_1 & \eta \lambda_1 \lambda_2 & -\eta \lambda_1 \mu_2 & \eta \lambda_1 \nu_2 \\
-\lambda_1 \mu_1 \xi + \nu_1 & \mu_1^2 \xi & \lambda_1 - \mu_1 \nu_1 \xi & -\eta \mu_1 \lambda_2 & \eta \mu_1 \mu_2 & -\eta \mu_1 \nu_2 \\
-\frac{\xi^2 + \lambda_1}{\eta} \lambda_1 \lambda_2 & -\frac{\xi^2 + \lambda_1}{\eta} \mu_1 \lambda_2 & \frac{\xi^2 + \lambda_1}{\eta} \nu_1 \lambda_2 & -\frac{\lambda_2^2}{2} \xi & \lambda_2 \mu_2 \xi + \nu_2 & -\lambda_2 \nu_2 \xi + \mu_2 \\
-\frac{\xi^2 + \lambda_1}{\eta} \mu_1 \mu_2 & -\frac{\xi^2 + \lambda_1}{\eta} \mu_1 \nu_2 & -\frac{\xi^2 + \lambda_1}{\eta} \nu_1 \nu_2 & -\lambda_2 \mu_2 \xi - \nu_2 & -\mu_2^2 \xi & \lambda_2 + \mu_2 \nu_2 \xi \\
-\frac{\xi^2 + \lambda_1}{\eta} \lambda_1 \nu_2 & -\frac{\xi^2 + \lambda_1}{\eta} \mu_1 \nu_2 & -\frac{\xi^2 + \lambda_1}{\eta} \nu_1 \nu_2 & -\lambda_2 \nu_2 \xi - \mu_2 & -\lambda_2 + \mu_2 \nu_2 \xi & -\nu_2^2 \xi
\end{pmatrix},
\]

where

\[(\xi, \eta) \in \mathbb{R} \times \mathbb{R}^*, \quad \begin{pmatrix} \lambda_j \\ \mu_j \\ \nu_j \end{pmatrix} \in \mathbb{S}^2, \quad j = 1, 2.\]

Furthermore, these matrices form a closed 6-dimensional smooth submanifold of $\mathbb{R}^{36}$ diffeomorphic to $\mathbb{R} \times \mathbb{R}^* \times \mathbb{S}^2 \times \mathbb{S}^2$.

Magnin notices that the matrix $J_y$ in Proposition 3.1 can be written in a different way. Recall the Hopf fibration $p : \mathbb{S}^3 \to \mathbb{S}^2$ defined by

\[p(x, y, z, w) = (2(wy - xz), 2(wx + yz), 2x^2 + 2y^2 - 1).\]

Hence, for $(\lambda_j, \mu_j, \nu_j) \in \mathbb{S}^2$, $j = 1, 2$, there exists $(x_j, y_j, z_j, w_j) \in \mathbb{S}^3$, $j = 1, 2$, such that

\[
\lambda_j = 2(w_j y_j - x_j z_j), \\
\mu_j = 2(w_j x_j + y_j z_j), \\
\nu_j = 2x_j^2 + 2y_j^2 - 1.
\]
There is a double cover of $SO(3)$ by $SU(2) \cong S^3$ that maps $q = (x_j, y_j, z_j, w_j) \in S^3$ to

$$M_j = \begin{pmatrix} x_j^2 - y_j^2 - z_j^2 + w_j^2 & -2(x_jy_j + z_jw_j) & 2(-x_jz_j + w_jy_j) \\ 2(-w_jz_j + x_jy_j) & x_j^2 - y_j^2 + z_j^2 - w_j^2 & -2(w_jx_j + y_jz_j) \\ 2(w_jy_j + x_jz_j) & 2(w_jx_j - y_jz_j) & x_j^2 + y_j^2 - z_j^2 - w_j^2 \end{pmatrix} \in SO(3),$$

$j = 1, 2$. By a direct calculation, we see that the matrix $J_V$ in Proposition 3.1 can be written as

$$J_V = \Phi \Psi(\xi, \eta) \Phi^{-1}$$

(3.2)

where

$$\Phi = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}$$

and

$$\Psi(\xi, \eta) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi & 0 & 0 & \eta \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -\frac{1 + \xi^2}{\eta} & 0 & 0 & -\xi \end{pmatrix}.$$  

Conversely, we can verify that any matrix of the form (3.2) can be written as (3.1) by reversing the process above.

### 3.2 Hermitian Lie Groups

Let $G^8$ be a simply connected 8-dimensional Lie group containing $K = SU(2) \times SU(2)$ as a Lie subgroup. Denote the Lie algebras of $G^8$ and $K$ by $\mathfrak{g}$ and $\mathfrak{k}$, respectively. By Theorem 1.1 and 1.2 the Lie group structure of $G^8$ is totally determined by the Lie algebra $\mathfrak{g}$ containing $\mathfrak{k} = su(2) \oplus su(2)$ as a Lie subalgebra up to isomorphism. The standard structure equations of $\mathfrak{k}$ are given by

$$[A, B] = 2C, \quad [B, C] = 2A, \quad [C, A] = 2B,$$

$$[R, S] = 2T, \quad [S, T] = 2R, \quad [T, R] = 2S,$$

where $\mathfrak{k}$ is equipped with the standard metric induced by its Killing form and $\{A, B, C, R, S, T\}$ is the standard orthonormal basis for $\mathfrak{k}$ i.e. each copy of $su(2)$ has the orthonormal basis $\{A, B, C\}, \{X, Y, Z\}$, respectively. We can extend $\{A, B, C, R, S, T\}$ to a basis for $\mathfrak{g}$ by adding vectors $X$ and $Y$. By declaring

$$\mathcal{B} = \{A, B, C, R, S, T, X, Y\}$$

to be an orthonormal basis for $\mathfrak{g}$, we can define a left-invariant metric on $G^8$. The orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, which is denoted by $\mathfrak{m}$, has the orthonormal basis $\{X, Y\}$. The Lie algebra $\mathfrak{g}$ is well-defined if and only if the Lie brackets relations among basis vectors is antisymmetric and satisfy the Jacobi identity introduced in Chapter 1. The problem of classifying such Lie algebras has been completely solved by the following result, see Proposition 4.1 in [2].
Proposition 3.2. Let \( G \) be an 8-dimensional Riemannian Lie group containing \( K = \text{SU}(2) \times \text{SU}(2) \) as a Lie subgroup, where \( K \) is endowed with the standard Riemannian metric induced by its Killing form. Then the Lie bracket relations for \( \mathfrak{g} \) take the following form

\[
[A, X] = -b_{11}B - c_{11}C, \quad [A, Y] = -b_{21}B - c_{21}C, \\
[B, X] = b_{11}A - c_{12}C, \quad [B, Y] = b_{21}A - c_{22}C, \\
[C, X] = c_{11}A + c_{12}B, \quad [C, Y] = c_{21}A + c_{22}B, \\
[R, X] = -s_{14}S - t_{14}T, \quad [R, Y] = -s_{24}S - t_{24}T, \\
[S, X] = s_{14}R - t_{15}T, \quad [S, Y] = s_{24}R - t_{25}T, \\
[T, X] = t_{14}R + t_{15}S, \quad [T, Y] = t_{24}R + t_{25}S, \\
[X, Y] = \rho X + \theta_1 A + \theta_2 B + \theta_3 C + \theta_4 R + \theta_5 S + \theta_6 T,
\]

where

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6 \\
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
-\rho c_{12} + b_{11}c_{21} - b_{21}c_{11} \\
\rho c_{11} + b_{11}c_{22} - b_{21}c_{12} \\
-\rho b_{11} + c_{11}c_{22} - c_{12}c_{21} \\
-\rho t_{15} + s_{14}t_{24} - s_{24}t_{14} \\
\rho t_{14} + s_{14}t_{25} - s_{24}t_{15} \\
-\rho s_{14} + t_{14}t_{25} - t_{15}t_{24} \\
\end{pmatrix}
\]

and the 13 coefficients \( b_{jk}, c_{jk}, s_{jk}, t_{jk}, \rho \) are arbitrary real numbers.

Let \( \mathcal{V}, \mathcal{H} \) be the left-invariant distributions generated by \( \mathfrak{k}, \mathfrak{m} \), respectively. Then \( TM \) has an orthogonal decomposition \( TM = \mathcal{V} \oplus \mathcal{H} \). Let \( J \) be a left-invariant almost Hermitian structure on \( G^8 \) such that \( J(V) = V \) and \( J(H) = H \). Then we can assume that \( J(X) = Y \) and \( J(Y) = -X \), otherwise we interchange \( X, Y \) and the resulting equations only differ by a sign. Therefore, we can write \( J \) as a 8 \( \times \) 8 block matrix

\[
J = \begin{pmatrix} J_Y & 0 \\ 0 & J_H \end{pmatrix}, \quad \text{where} \quad J_H = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathbb{R}^{2 \times 2}
\]

and \( J_Y \in \mathbb{R}^{6 \times 6} \) satisfies \( J_Y^2 = -I_6 \). So \( J_Y \) is a complex structure on \( \mathfrak{k} \). Since \( J(V) \subseteq V \) and \( \mathcal{V}, \mathcal{V} \subseteq V \), we can restrict \( N_J \) to the Nijenhuis tensor \( N_J \) on \( V \). Hence, \( N_J = 0 \) implies that \( N_J = 0 \) i.e. \( J_Y \) must be an integrable left-invariant almost complex structure on \( \mathfrak{su}(2) \oplus \mathfrak{su}(2) \), which is given by (3.3).

The structure \( J \) preserves the metric i.e. \( J \in \text{SO}(8) \) if and only if \( J_Y \in \text{SO}(6) \), which is equivalent to \( \Psi(\xi, \eta) \in \text{SO}(6) \) by (3.2). So it suffices to solve the following system of equations:

\[
\begin{align*}
\xi^2 + \eta^2 &= 1, \\
\left(\frac{1 + \xi^2}{\eta}\right)^2 + (-\xi)^2 &= 1, \\
-\frac{1 + \xi^2}{\eta} - \xi \eta &= 0.
\end{align*}
\]

The only solution to the system is \( \xi = 0, \eta = \pm 1 \).
When $\eta = 1$ we have

$$\Psi(\xi, \eta) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$  

And when $\eta = -1$ we have

$$\Psi(\xi, \eta) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$  

When $\eta = 1$, consider the action of the almost complex structure $J$ on $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ geometrically. Firstly, $J$ rotates the two copies of $\mathfrak{su}(2)$ by the corresponding matrix $M^1, M^2 \in \text{SO}(3)$, respectively. Then it gives the three mutually orthogonal subspaces spanned by $\{A, B\}, \{C, T\}, \{R, S\}$ a structure of complex plane respectively, i.e. the two copies of $\mathfrak{su}(2)$ are interconnected by the basis vector $C$ and $T$. Finally, the two copies are rotated back according to $M_1, M_2 \in \text{SO}(3)$ respectively.

In this thesis, we solve the integrability conditions of $J$ for the case $\eta = 1$. The other cases when $\eta = -1$ can be solved in a similar way. When $\xi = 0, \eta = 1$, the matrix $J$ becomes

$$J \begin{pmatrix} 0 & \nu_1 & \mu_1 & \lambda_1\lambda_2 & -\lambda_1\mu_2 & \lambda_1\nu_2 & 0 & 0 \\ -\nu_1 & 0 & \lambda_1 & -\mu_1\lambda_2 & \mu_1\mu_2 & -\mu_1\nu_2 & 0 & 0 \\ -\mu_1 & -\lambda_1 & 0 & \nu_1\lambda_2 & -\nu_1\mu_2 & \nu_1\nu_2 & 0 & 0 \\ -\lambda_1\lambda_2 & \mu_1\lambda_2 & -\nu_1\lambda_2 & 0 & \nu_2 & \mu_2 & 0 & 0 \\ \lambda_1\mu_2 & -\mu_1\mu_2 & \nu_1\mu_2 & -\nu_2 & 0 & \lambda_2 & 0 & 0 \\ -\lambda_1\nu_2 & \mu_1\nu_2 & -\nu_1\nu_2 & -\nu_2 & -\lambda_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$  

(3.4)

For convenience, the basis $\{A, B, C, R, S, T, X, Y\}$ is denoted by $\{e_1, \ldots, e_8\}$. We use the two notations interchangeably.

Now we can state the main result of this thesis.

**Theorem 3.3.** Let $G^8$ be a 8-dimensional Riemannian Lie group containing $K = \text{SU}(2) \times \text{SU}(2)$ as a Lie subgroup, where $K$ is endowed with the standard Riemannian metric induced by its Killing form. The Lie bracket relations are given by Proposition 3.2. Let $J$ be the left-invariant almost Hermitian structure on $G^8$ defined by (3.4), where $\lambda_j^2 + \mu_j^2 + \nu_j^2 = 1, j = 1, 2$. Without loss of generality, we assume that $\lambda_1, \lambda_2 \neq 0$. Then $J$ is integrable if and only if the coefficients of $\mathfrak{g}$ satisfy the following relations

$$c_{11} = \frac{\lambda_1^2 + \mu_1^2}{\lambda_1} b_{21} + \frac{\mu_1}{\lambda_1} c_{12} - \frac{\mu_1 \nu_1}{\lambda_1} c_{21} - \nu_1 c_{22},$$
We denote $J$ by (3.1), there exists exactly a 9-dimensional family of coefficients $f$ or the Lie bracket relations of $g$ such that $(G^8, g, J)$ is a Hermitian Lie group. The space of solutions to the problem therefore forms a vector bundle of rank 9 over $S^2 \times S^2$.

Remark 3.4. For each case, the dimension of the solutions to the system is 8. Notice that in the Lie bracket relations of $g$, the coefficient $\rho$ is an arbitrary real number. Therefore, for any left-invariant almost Hermitian structure $J$ on $G^8$ given by (3.1), there exists exactly a 9-dimensional family of coefficients for the Lie brackets relations of $g$ such that $(G^8, g, J)$ is a Hermitian Lie group. The space of solutions to the problem therefore forms a vector bundle of rank 9 over $S^2 \times S^2$.

Proof. We denote $\langle N_J(e_j, e_\alpha), e_k \rangle$ by $N_J(e_j, e_\alpha)_k$ and the dimension of $G^8$ by $2n$. Since $J_V$ is integrable and $N_J$ restricts to $N_{J_V}$, we have

$$N_J(e_j, e_k) = 0,$$

where $j, k \in \{1, \ldots, 6\}$. By the formula of $J$, we have

$$N_J(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$$

$$= [X, Y] + J[Y, Y] + J[X, -X] - [Y, -X]$$

$$= 0.$$

Since the Nijenhuis tensor $N_J$ is antisymmetric, for $\alpha, \beta \in \{7, 8\}$, we have

$$N_J(e_\alpha, e_\beta) = 0.$$

Now we consider the case when $j \in \{1, \ldots, 6\}$ and $\alpha \in \{7, 8\}$. Since $J(V) \subseteq V$ and $\mathcal{H}([E, F]) = 0$ for $E \in V, F \in \mathcal{H}$, we have

$$N_J(e_j, e_\alpha)_\beta = \langle [e_j, e_\alpha] + J[e_j, e_\alpha] + J[e_j, J e_\alpha] - [J e_j, J e_\alpha], e_\beta \rangle$$

$$= 0,$$

where $\beta \in \{7, 8\}$. So it suffices to check

$$N_J(e_j, e_\alpha)_k = 0,$$

where $j, k \in \{1, \ldots, 6\}$ and $\alpha \in \{7, 8\}$.

There are $6 \times 6 \times 2 = 72$ equations to solve. To simplify the system of equations, first of all we prove the following equality

$$N_J(e_j, e_\alpha)_k + N_J(e_k, e_\alpha)_j = 0 \quad (3.5)$$

for any $j, k \in \{1, 2, 3, 4, 5, 6\}$ and $\alpha \in \{7, 8\}$. By definition, we have

$$N_J(e_j, e_\alpha)_k = \langle [e_j, e_\alpha] + J[e_j, e_\alpha] + J[e_j, J e_\alpha] - [J e_j, J e_\alpha], e_k \rangle,$$

$$N_J(e_k, e_\alpha)_j = \langle [e_k, e_\alpha] + J[e_k, e_\alpha] + J[e_k, J e_\alpha] - [J e_k, J e_\alpha], e_j \rangle.$$
From the Lie bracket relations \((3.3)\) for \(G^8\) we can directly see that
\[
\langle [e_j, e_\alpha], e_k \rangle = -\langle [e_k, e_\alpha], e_j \rangle.
\] (3.6)

Next we show that the sum of the third item of \(N_J(e_j, e_\alpha)_k\) and the fourth item of \(N_J(e_k, e_\alpha)_j\) is zero. By definition, we have
\[
\langle J[e_j, Je_\alpha], e_k \rangle - \langle J[e_k, Je_\alpha], e_j \rangle
= -\left\{ \langle [e_j, Je_\alpha], Je_k \rangle + \langle [Je_k, Je_\alpha], e_j \rangle \right\}
= -\left\{ \sum_{l=1}^{2n} \langle [e_j, Je_\alpha], e_l \rangle e_l + \sum_{l=1}^{2n} \langle Je_k, e_l \rangle e_l \right\}
= -\left\{ \sum_{l=1}^{2n} \langle [e_j, Je_\alpha], e_l \rangle \langle Je_k, e_l \rangle + \sum_{l=1}^{2n} \langle Je_k, e_l \rangle \langle [e_j, Je_\alpha], e_l \rangle \right\}
= -\sum_{l=1}^{2n} \langle [e_j, Je_\alpha], e_l \rangle \langle Je_k, e_l \rangle - \sum_{l=2n-1}^{2n} \langle [e_j, Je_\alpha], e_l \rangle \langle Je_k, e_l \rangle
= -\sum_{l=1}^{2n-2} \langle [e_j, Je_\alpha], e_l \rangle \langle Je_k, e_l \rangle,
\] (3.7)
where the second last equality holds since \(\mathcal{H}(Je_k) = 0\) by \((3.3)\), and the last equality follows from \((3.6)\). By the symmetry in \(j, k\), we see that the sum of the fourth item of \(N_J(e_j, e_\alpha)_k\) and the third item of \(N_J(e_k, e_\alpha)_j\) is also zero.

For the second item in \(N_J(e_j, e_\alpha)_k\), we have
\[
\langle J[e_j, e_\alpha], Je_k \rangle = -\langle [Je_j, e_\alpha], Je_k \rangle
= -\left\{ \sum_{l=1}^{2n} \langle [Je_j, e_l], e_\alpha \rangle e_l \langle Je_k, e_l \rangle \right\}
= -\left\{ \sum_{r=1}^{2n} \langle [\sum_{l=1}^{2n} \langle [Je_j, e_l], e_\alpha \rangle e_l, e_r \rangle \langle Je_k, e_r \rangle \right\}
= -\sum_{r=1}^{2n} \langle [\sum_{l=1}^{2n} \langle Je_j, e_l \rangle e_l, e_\alpha \rangle \langle Je_k, e_r \rangle \right\}
= -\sum_{r=1}^{2n} \sum_{l=1}^{2n} \langle Je_j, e_l \rangle \langle e_l, e_\alpha \rangle \langle Je_k, e_r \rangle = 0.
\]
Switching $k$ and $j$, by the symmetry in $j, k$, we see that
\[
\langle J[e_k, e_\alpha], e_j \rangle = - \sum_{r=1}^{2n} \sum_{l=1}^{2n} \langle J[e_k, e_l] \rangle \langle [e_l, e_\alpha], e_r \rangle \langle J[e_j, e_r] \rangle
\]
\[
= - \sum_{r=1}^{2n} \sum_{l=1}^{2n} \langle J[e_j, e_l] \rangle \langle [e_l, e_\alpha], e_r \rangle \langle J[e_k, e_r] \rangle,
\]
where the last equality follows by renaming the indices $l, r$. Adding the two equalities together, by (3.6), we have
\[
\langle J[e_j, e_\alpha], e_k \rangle + \langle J[e_k, e_\alpha], e_j \rangle = - \sum_{r=1}^{2n} \sum_{l=1}^{2n} \langle J[e_j, e_l] \rangle \langle [e_l, e_\alpha], e_r \rangle \langle J[e_k, e_r] \rangle
\]
\[
= - \sum_{r=1}^{2n} \sum_{l=1}^{2n} \langle J[e_j, e_l] \rangle \langle [e_l, e_\alpha], e_r \rangle \langle J[e_k, e_r] \rangle
\]
\[
= 0,
\]
where the second equality holds since $\mathcal{H}(J[e_j]) = \mathcal{H}(J[e_k]) = 0$ by (3.3).

The proof of (3.5) is accomplished by adding (3.6), (3.7) and (3.8). Therefore, it suffices to solve the system of $2 \times \binom{6}{2} = 30$ equations given by
\[
N_J(e_j, e_\alpha)_{k} = 0,
\]
where $j, k \in \{1, 2, 3, 4, 5, 6\}$, $j < k$, and $\alpha \in \{7, 8\}$. These 30 equations can be divided into 2 cases

(i) $j \in \{1, 2, 3\}$, $k \in \{4, 5, 6\}$, $\alpha \in \{7, 8\};$

(ii) $j, k \in \{1, 2, 3\}$ or $j, k \in \{4, 5, 6\}$, $\alpha \in \{7, 8\}.

Case (i) consists of $3 \times 3 \times 2 = 18$ equations, and case (ii) has $2 \times 2 \times \binom{3}{2} = 12$ equations. By a standard calculation, we see that the two cases satisfy the following linear relations
\[
N_J(e_1, e_\alpha)_{4} = \lambda_2 N_J(e_2, e_\alpha)_{3} + \lambda_1 N_J(e_5, e_\alpha)_{6},
\]
\[
N_J(e_1, e_\alpha)_{5} = -\mu_2 N_J(e_2, e_\alpha)_{3} - \lambda_1 N_J(e_4, e_\alpha)_{6},
\]
\[
N_J(e_1, e_\alpha)_{6} = \nu_2 N_J(e_2, e_\alpha)_{3} + \lambda_1 N_J(e_4, e_\alpha)_{5},
\]
\[
N_J(e_2, e_\alpha)_{4} = -\lambda_2 N_J(e_1, e_\alpha)_{3} + \mu_1 N_J(e_5, e_\alpha)_{6},
\]
\[
N_J(e_2, e_\alpha)_{5} = \mu_2 N_J(e_1, e_\alpha)_{3} + \mu_1 N_J(e_4, e_\alpha)_{6},
\]
\[
N_J(e_2, e_\alpha)_{6} = -\nu_2 N_J(e_1, e_\alpha)_{3} - \mu_1 N_J(e_4, e_\alpha)_{5},
\]
\[
N_J(e_3, e_\alpha)_{4} = \lambda_2 N_J(e_1, e_\alpha)_{2} + \nu_1 N_J(e_5, e_\alpha)_{6},
\]
\[
N_J(e_3, e_\alpha)_{5} = -\mu_2 N_J(e_1, e_\alpha)_{2} - \nu_1 N_J(e_4, e_\alpha)_{6},
\]
\[
N_J(e_3, e_\alpha)_{6} = \nu_2 N_J(e_1, e_\alpha)_{2} + \nu_1 N_J(e_4, e_\alpha)_{5},
\]
where $\alpha \in \{7, 8\}$. We check the first equality in (3.9) where $\alpha = 7$. By definition, we have
\[
N_J(e_1, e_7)_{4} = \langle [e_1, e_7] + J[e_1, e_7] + J[e_1, J[e_7]] - [J[e_1, J[e_7]], e_4] \rangle.
\]
By a direct calculation, we see that
\[
\begin{align*}
\langle [e_1, e_7], e_4 \rangle &= 0, \\
\langle J[Je_1, e_7], e_4 \rangle &= \lambda_2 (-c_{12} + \lambda_1^2 c_{12} + \lambda_1 \mu_1 c_{11} + \lambda_1 \nu_1 b_{11}) \\
&\quad + \lambda_1 (-t_{15} + \lambda_2^2 t_{15} + \lambda_2 \mu_2 t_{14} + \lambda_2 \nu_2 s_{14}) \\
&= \lambda_2 \left( \langle e_2, e_7 \rangle + J[J(e_2, e_7), e_3] \right) + \lambda_1 \left( \langle e_5, e_7 \rangle + J[J(e_5, e_7), e_6] \right), \\
\langle J[e_1, Je_7], e_4 \rangle &= \lambda_2 (-\mu_1 b_{21} + \nu_1 c_{21}) \\
&= \lambda_2 \left( \langle J[e_2, Je_7] - [Je_2, Je_7], e_3 \rangle \right), \\
-\langle [Je_1, Je_7], e_4 \rangle &= \lambda_1 (-\mu_2 s_{24} + \nu_2 t_{24}) \\
&= \lambda_1 \left( \langle J[e_5, Je_7] - [Je_5, Je_7], e_6 \rangle \right).
\end{align*}
\]

Adding these equalities together gives us
\[
N_J(e_1, e_7)_4 = \lambda_2 N_J(e_2, e_7)_3 + \lambda_1 N_J(e_5, e_7)_6.
\]

The other equalities follow by a similar calculation. From (3.9) we see that the solutions to case (ii) also solve case (i). Hence, it suffices to consider case (ii).

We notice that the system of equations of case (ii) can be written as
\[
\begin{pmatrix}
P_{11} & 0 \\ 0 & P_{22}
\end{pmatrix}
\begin{pmatrix}
u_1 \\ \nu_2
\end{pmatrix} = 0,
\]
where \(P_{11}, P_{22} \in \mathbb{R}^{6 \times 6}\), \(u = (b_{11}, b_{21}, c_{11}, c_{12}, c_{21}, c_{22})^T, v = (s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25})^T\). We denote the system of equations \(P_{11}u = 0\) by \(S_1\) and the system \(P_{22}v = 0\) by \(S_2\).

The system \(S_1\) includes the cases when \(j, k \in \{1, 2, 3\}, j < k\) and \(\alpha \in \{7, 8\}\).

Since we have assumed that \(\lambda_1 \neq 0\), the first row and the third row are linearly independent. Denote the \(j\)th row of the matrix as \(R_j\), then we have
\[
\begin{align*}
-\frac{\mu_1 \nu_1}{\lambda_1} \cdot R_1 + \frac{-\nu_1^2 + 1}{\lambda_1} \cdot R_3 &= R_2, \\
-\frac{\nu_1}{\lambda_1} \cdot R_1 + \frac{-\mu_1}{\lambda_1} \cdot R_3 &= R_4, \\
\frac{1 - \mu_1^2}{\lambda_1} \cdot R_1 + \frac{\mu_1 \nu_1}{\lambda_1} \cdot R_3 &= R_5, \\
-\mu_1 \cdot R_1 + \nu_1 \cdot R_3 &= R_6.
\end{align*}
\]
So the solution of $S_1$ is 4-dimensional, which is

$$c_{11} = \frac{\lambda_1^2 + \mu_1^2}{\lambda_1} b_{21} + \frac{\mu_1}{\lambda_1} c_{12} - \frac{\mu_1 \nu_1}{\lambda_1} c_{21} - \nu_1 c_{22},$$

$$b_{11} = \frac{\mu_1 \nu_1}{\lambda_1} b_{21} + \frac{\nu_1}{\lambda_1} c_{12} - \frac{\lambda_1^2 + \nu_1^2}{\lambda_1} c_{21} + \mu_1 c_{22},$$

where $b_{21}, c_{12}, c_{21}, c_{22} \in \mathbb{R}$.

The system $S_2$ includes the cases when $j, k \in \{4, 5, 6\}, j < k$, and $\alpha \in \{7, 8\}$.

Similarly, since $\lambda_2 \neq 0$, $S_1$ has the following 4-dimensional solution

$$t_{14} = \frac{\lambda_2^2 + \mu_2^2}{\lambda_2} s_{24} + \frac{\mu_2 \nu_2}{\lambda_2} t_{15} - \frac{\nu_2}{\lambda_2} t_{24} - \nu_2 t_{25},$$

$$s_{14} = \frac{\mu_2 \nu_2}{\lambda_2} s_{24} + \frac{\nu_2}{\lambda_2} t_{15} - \frac{\lambda_2^2 + \nu_2^2}{\lambda_2} t_{24} + \mu_2 t_{25},$$

where $s_{24}, t_{15}, t_{24}, t_{25} \in \mathbb{R}$.

Since $S_1$ and $S_2$ are independent of each other, case (ii) thus the whole system of 72 equations has 8-dimensional solutions. Since in the structure equations of $g$, $\rho$ can change freely in $\mathbb{R}$, we have shown that for any fixed $J$ taking the form (3.4), there exists exactly a 9-dimensional family of coefficients for $g$ such that $(G, g, J)$ is a Hermitian manifold. The Lie bracket relations are given by (3.3) and the coefficients satisfy the following linear relations

$$c_{11} = \frac{\lambda_1^2 + \mu_1^2}{\lambda_1} b_{21} + \frac{\mu_1}{\lambda_1} c_{12} - \frac{\mu_1 \nu_1}{\lambda_1} c_{21} - \nu_1 c_{22},$$

$$b_{11} = \frac{\mu_1 \nu_1}{\lambda_1} b_{21} + \frac{\nu_1}{\lambda_1} c_{12} - \frac{\lambda_1^2 + \nu_1^2}{\lambda_1} c_{21} + \mu_1 c_{22},$$

$$t_{14} = \frac{\lambda_2^2 + \mu_2^2}{\lambda_2} s_{24} + \frac{\mu_2 \nu_2}{\lambda_2} t_{15} - \frac{\nu_2}{\lambda_2} t_{24} - \nu_2 t_{25},$$

$$s_{14} = \frac{\mu_2 \nu_2}{\lambda_2} s_{24} + \frac{\nu_2}{\lambda_2} t_{15} - \frac{\lambda_2^2 + \nu_2^2}{\lambda_2} t_{24} + \mu_2 t_{25},$$

where $\rho, b_{21}, c_{12}, c_{21}, c_{22}, s_{24}, t_{15}, t_{24}, t_{25} \in \mathbb{R}$. 

$\square$
Chapter 4

Hermitian Semi-Kähler Manifolds

In this chapter we look for the conditions when $G^8$ belongs to the class of Hermitian semi-Kähler manifolds i.e. $\mathcal{W}_3$. Since we have solved the integrability condition for $G^8$ in Chapter 3, it suffices to check when $\hat{\alpha}(Z) = 0$ for all tangent vectors $Z \in T_0 G^8 \cong g$. By the Koszul formula, for any $1 \leq j \leq 8$, we have

$$
(\nabla_{e_j} \omega)(e_j, Z) = e_j(\omega(e_j, Z)) - \omega(\nabla_{e_j} e_j, Z) - \omega(e_j, \nabla_{e_j} Z)
$$

$$
= e_j(g(Je_j, Z)) - g(J\nabla_{e_j} e_j, Z) - g(Je_j, \nabla_{e_j} Z)
$$

$$
= g(\nabla_{e_j} Je_j, Z) + g(Je_j, \nabla_{e_j} Z) - g(J \nabla_{e_j} e_j, Z) - g(Je_j, \nabla_{e_j} Z)
$$

$$
= g(\nabla_{e_j} Je_j - J \nabla_{e_j} e_j, Z)
$$

$$
= g(\nabla_{e_j} Je_j, Z) + g(\nabla_{e_j} e_j, JZ)
$$

$$
= \frac{1}{2} (g([e_j, Je_j], Z) - g([e_j, Z], Je_j) - g([Je_j, Z], e_j) - 2g([e_j, JZ], e_j)),
$$

where we use the same notation for a vector in $g$ and the left-invariant vector field generated by it since $\nabla \omega$ is tensorial. Hence for a basis vector $e_k$ of $g$, we have

$$
\hat{\alpha}(e_k) = \frac{1}{2} \sum_{j=1}^{8} \left( g([e_j, Je_j], e_k) - g([e_j, e_k], Je_j) - g([Je_j, e_k], e_j) - 2g([e_j, J e_k], e_j) \right),
$$

where $k \in \{1, \ldots, 8\}$. By a standard calculation, we obtain the following results

$$
\hat{\alpha}(e_1) = -\frac{1}{2}(-\rho c_{12} + b_{11} c_{21} - b_{21} c_{11} - 4\lambda_1),
$$

$$
\hat{\alpha}(e_2) = -\frac{1}{2}(-\rho c_{11} + b_{11} c_{22} - b_{21} c_{12} + 4\mu_1),
$$

$$
\hat{\alpha}(e_3) = -\frac{1}{2}(-\rho b_{11} + c_{11} c_{22} - c_{12} c_{21} - 4\nu_1),
$$

$$
\hat{\alpha}(e_4) = -\frac{1}{2}(-\rho t_{15} + s_{14} t_{24} - s_{24} t_{14} - 4\lambda_2),
$$

$$
\hat{\alpha}(e_5) = -\frac{1}{2}(\rho t_{14} + s_{14} t_{25} - s_{24} t_{15} + 4\mu_2),
$$

$$
\hat{\alpha}(e_6) = -\frac{1}{2}(\rho s_{14} + t_{14} t_{25} - t_{15} t_{24} - 4\nu_2),
$$

$$
\hat{\alpha}(e_7) = 0,
$$

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Recall that in (3.3), the coefficients $\theta_j = \langle [X, Y], e_j \rangle$ are given by

$$
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3 \\
\theta_4 \\
\theta_5 \\
\theta_6
\end{pmatrix} = \frac{1}{2}
\begin{pmatrix}
-\rho c_{12} + b_{11}c_{21} - b_{21}c_{11} \\
\rho c_{11} + b_{11}c_{22} - b_{21}c_{12} \\
-\rho b_{11} + c_{11}c_{22} - c_{12}c_{21} \\
-\rho t_{15} + s_{14}t_{24} - s_{24}t_{14} \\
\rho t_{14} + s_{14}t_{25} - s_{24}t_{15} \\
-\rho s_{14} + t_{14}t_{25} - t_{15}t_{24}
\end{pmatrix}.
$$

By comparing $\theta_j$ and (4.1), we see that $G^8$ belongs to $\mathcal{W}_3$ if and only if

$$
\begin{align*}
\theta_1 &= 2\lambda_2, \quad \theta_2 = -2\mu_1, \quad \theta_3 = 2\nu_1, \\
\theta_4 &= 2\lambda_2, \quad \theta_5 = -2\mu_2, \quad \theta_6 = 2\nu_2.
\end{align*}
$$

Since $\lambda_j^2 + \mu_j^2 + \nu_j^2 = 1$, where $j = 1, 2$, at least two of $\theta_k$ must be nonzero. Therefore, for any $G^8$ in the class $\mathcal{W}_3$, we have

$$
[X, Y] \notin \mathfrak{m},
$$

i.e. the horizontal distribution is not integrable.

Now we have the following system of equations for $G^8$ to be Hermitian semi-Kähler when $\lambda_1, \lambda_2 \neq 0$ and $\eta = 1$.

$$
\begin{align*}
-\rho c_{12} + b_{11}c_{21} - b_{21}c_{11} &= 4\lambda_1, \\
\rho c_{11} + b_{11}c_{22} - b_{21}c_{12} &= -4\mu_1, \\
-\rho b_{11} + c_{11}c_{22} - c_{12}c_{21} &= 4\nu_1, \\
(\lambda_1^2 + \mu_1^2)b_{21} - \lambda_1c_{11} + \mu_1c_{12} - \mu_1\nu_1c_{21} - \lambda_1\nu_1c_{22} &= 0, \\
-\lambda_1b_{11} + \mu_1\nu_1b_{21} + \nu_1c_{12} - (\lambda_1^2 + \nu_1^2)c_{21} + \lambda_1\nu_1c_{22} &= 0, \\
-\rho t_{15} + s_{14}t_{24} - s_{24}t_{14} &= 4\lambda_2, \\
\rho t_{14} + s_{14}t_{25} - s_{24}t_{15} &= -4\mu_2, \\
-\rho s_{14} + t_{14}t_{25} - t_{15}t_{24} &= 4\nu_2, \\
(\lambda_2^2 + \mu_2^2)s_{24} - \lambda_2t_{14} + \mu_2t_{15} - \mu_2\nu_2t_{24} - \lambda_2\nu_2t_{25} &= 0, \\
-\lambda_2s_{14} + \mu_2\nu_2s_{24} + \nu_2t_{15} - (\lambda_2^2 + \nu_2^2)t_{24} + \lambda_2\mu_2t_{25} &= 0.
\end{align*}
$$

**Theorem 4.1.** Let $G^8$ be a 8-dimensional Riemannian Lie group containing $K = \text{SU}(2) \times \text{SU}(2)$ as a Lie subgroup, where $K$ is endowed with the standard Riemannian metric induced by its Killing form. The Lie bracket relations are given by Proposition 3.2. Let $J$ be the left-invariant almost Hermitian structure on $G^8$ defined by (3.4), where $\lambda_j^2 + \mu_j^2 + \nu_j^2 = 1, j = 1,2$. Without loss of generality, we assume that $\lambda_1, \lambda_2 \neq 0$. Then $(G^8, g, J)$ belongs to the class $\mathcal{W}_3$ if and only if $\rho \neq 0$ and there exist $r_1, r_2 \in \mathbb{R}$ such that

$$
\begin{align*}
c_{12} &= -\frac{4\lambda_1}{\rho}, \quad c_{11} = -\frac{4\mu_1}{\rho}, \quad b_{11} = -\frac{4\nu_1}{\rho}, \quad c_{22} = r_1\lambda_1, \quad c_{21} = r_1\mu_1, \quad b_{21} = r_1\nu_1, \\
t_{15} &= -\frac{4\lambda_2}{\rho}, \quad t_{14} = -\frac{4\mu_2}{\rho}, \quad s_{14} = -\frac{4\nu_2}{\rho}, \quad t_{25} = r_2\lambda_2, \quad t_{24} = r_2\mu_2, \quad s_{24} = r_2\nu_2.
\end{align*}
$$
Remark 4.2. By Theorem 4.1, for a fixed matrix representation of the almost Hermitian structure $J$, there exists exactly a 3-dimensional family of the coefficients for $g$ such that $G^8$ is Hermitian semi-Kähler. The space of solutions to the problem forms a fibre bundle over $S^2 \times S^2$ with fibre $\mathbb{R}^* \times \mathbb{R}^2$.

Proof. Notice that for a fixed value of $\rho$, the equations in (4.2) that involves $b_{jk}, c_{jk}$ are independent of those involving $s_{jk}, t_{jk}$. Furthermore, these two subsystems have the same structure if we replace $b_{11}, b_{21}, c_{11}, c_{21}, c_{22}, \lambda_1, \mu_1, \nu_1$ by $s_{14}, s_{24}, t_{14}, t_{15}, t_{24}, t_{25}, \lambda_2, \mu_2, \nu_2$, respectively. So it suffices to solve the system of the equations about $b_{jk}, c_{jk}$. The subsystem of equations is given by

$$
- \rho c_{12} + b_{11} c_{21} - b_{21} c_{11} = 4 \lambda_1,
\rho c_{11} + b_{11} c_{22} - b_{21} c_{12} = -4 \mu_1,
- \rho b_{11} + c_{11} c_{22} - c_{12} c_{21} = 4 \nu_1,
$$

$$(\lambda_1^2 + \mu_1^2) b_{21} - \lambda_1 c_{11} + \mu_1 c_{12} - \mu_1 \nu_1 c_{21} - \lambda_1 \nu_1 c_{22} = 0,
- \lambda_1 b_{11} + \mu_1 \nu_1 b_{21} + \nu_1 c_{12} - (\lambda_1^2 + \nu_1^2) c_{21} + \lambda_1 \mu_1 c_{22} = 0.
$$

Denote $(c_{12}, -c_{11}, b_{11})^T$, $(c_{22}, -c_{21}, b_{21})^T$, $(\lambda_1, -\mu_1, \nu_1)^T$ by $u$, $v$, $w$, respectively. Then the system of equations can be rewritten as

$$
- \rho u + u \times v = 4w, \quad (4.3)
$$

$$(\mu_1) \cdot u + \left(\begin{array}{c}
-\lambda_1 \nu_1 \\
\mu_1 \nu_1 \\
\lambda_1^2 + \mu_1^2
\end{array}\right) \cdot v = 0,
(u) \cdot u + \left(\begin{array}{c}
\lambda_1 \mu_1 \\
\lambda_1^2 + \nu_1^2 \\
\mu_1 \nu_1
\end{array}\right) \cdot v = 0.
$$

Substitute $u, v$ by $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, respectively. Then the system of the last two equations is equivalent to

$$
\mu_1 x_1 + \lambda_1 y_1 = C_1,
\nu_1 x_1 - \lambda_1 z_1 = C_2,
- \lambda_1 \nu_1 x_2 + \mu_1 \nu_1 y_2 + (\lambda_1^2 + \mu_1^2) z_2 = -C_1,
\lambda_1 \mu_1 x_2 + (\lambda_1^2 + \nu_1^2) y_2 + \mu_1 \nu_1 z_2 = -C_2,
$$

where $C_1, C_2 \in \mathbb{R}$. The general solution to the first two equations is

$$
\begin{pmatrix}
x_1 \\
y_1 \\
z_1
\end{pmatrix} = l_1 \begin{pmatrix}
\lambda_1 \\
-\mu_1 \\
\nu_1
\end{pmatrix} = l_1 w, \quad (4.4)
$$

where $l_1 \in \mathbb{R}$. By substitution, we see that this is also the general solution to the last two equations, i.e.

$$
\begin{pmatrix}
x_2 \\
y_2 \\
z_2
\end{pmatrix} = r_1 \begin{pmatrix}
\lambda_1 \\
-\mu_1 \\
\nu_1
\end{pmatrix} = r_1 w, \quad (4.5)
$$

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where \( r_1 \in \mathbb{R} \). Therefore, we have
\[
\begin{align*}
u &= u_0 + l_1 w, \\
v &= v_0 + r_1 w,
\end{align*}
\]
where \( l_1, r_1 \in \mathbb{R} \) and \( u_0, v_0 \) are special solutions to the system for fixed \( C_1, C_2 \).
Especially, we can choose \( u_0, v_0 \) such that they are orthogonal to \( w \), i.e. \( u_0 \) is the solution to
\[
\begin{align*}
\mu_1 x + \lambda_1 y &= C_1, \\
\nu_1 x - \lambda_1 z &= C_2, \\
\lambda_1 x - \mu_1 y + \nu_1 z &= 0,
\end{align*}
\]
and \( v_0 \) is the solution to
\[
\begin{align*}
-\lambda_1 \nu_1 x + \mu_1 \nu_1 y + (\lambda_1^2 + \mu_1^2) z &= -C_1, \\
\lambda_1 \mu_1 x + (\lambda_1^2 + \nu_1^2) y + \mu_1 \nu_1 z &= -C_2, \\
\lambda_1 x - \mu_1 y + \nu_1 z &= 0.
\end{align*}
\]
Solving the two systems gives us
\[
u_0 = \begin{pmatrix}
\nu C_1 + \mu C_2 \\
-\frac{1}{\lambda_1} (\mu \nu C_1 + \mu \nu C_2 - C_2) \\
\frac{1}{\lambda_1} (\nu^2 C_1 - C_1 + \mu \nu C_2)
\end{pmatrix}
\]
(4.6)
and
\[
v_0 = \begin{pmatrix}
\frac{1}{\lambda_1} (-\mu C_1 + \nu C_2) \\
-C_1 \\
-C_2
\end{pmatrix}
\]
(4.7)
By a direct calculation, we see that
\[
u_0 \cdot v_0 = 0.
\]
Hence, \( u_0, v_0, w \) are pairwise orthogonal to each other. Furthermore, we have
\[
w \times v_0 = \begin{vmatrix}
e_1 & e_2 & e_3 \\
\lambda & \nul & \nu_1 \\
\frac{1}{\lambda_1} (-\mu \nu C_1 + \nu \nu C_2) & -C_1 & -C_2
\end{vmatrix} = u_0.
\]
Since \( w \) and \( v_0 \) are orthogonal and \( \|w\| = 1 \), we have
\[
\|u_0\| = \|v_0\|.
\]
Therefore, the basis \( [w, v_0, u_0] \) is positively oriented and
\[
u_0 \times w = v_0.
\]
Moreover, since \( \|u_0\| = \|v_0\| \) and \( \|w\| = 1 \), we have
\[
u_0 \times v_0 = -\|v_0\|^2 w.
\]
Using these results, for (4.3), we have
\[- \rho u + u \times v = 4w \]
\[\iff - \rho (u_0 + l_1w) + (u_0 + l_1w) \times (v_0 + r_1w) = 4w \]
\[\iff - \rho (u_0 + l_1w) + u_0 \times v_0 + r_1u_0 \times w + l_1w \times v_0 = 4w \]
\[\iff - \rho (u_0 + l_1w) - \|v_0\|^2 w + r_1v_0 + l_1u_0 = 4w \]
\[\iff (4 + \rho l_1 + \|v_0\|^2)w - r_1v_0 + (\rho - l_1)u_0 = 0. \quad (4.8) \]

When \(\|v_0\| \neq 0\), the vectors \(w, v_0, u_0\) form a basis for \(\mathbb{R}^3\). Hence, the equation is equivalent to
\[4 + \rho l_1 + \|v_0\|^2 = 0, \quad -r_1 = 0, \quad \rho - l_1 = 0.\]

Substituting the third equation \(l_1 = \rho\) into the first one, we see that
\[4 + \rho^2 + \|v_0\|^2 > 0.\]

Hence, the system has no solution when \(\|v_0\| \neq 0\). It suffices to consider the case when \(\|v_0\| = 0\). By (4.7), this is equivalent to
\[C_1 = C_2 = 0.\]

In this situation, the system is solved by the general solutions (4.4) and (4.5). By (4.8), we have
\[- \rho u + u \times v = 4w \iff (4 + \rho l_1)w = 0 \iff 4 + \rho l_1 = 0.\]

Therefore, the system of equations has solutions if and only if \(\rho \neq 0\), and the solutions are given by setting \(l_1 = -\frac{2}{\rho}\) in the general solution (4.4), i.e.
\[x_1 = -\frac{4\lambda_1}{\rho}, \quad y_1 = \frac{4\mu_1}{\rho}, \quad z_1 = -\frac{4\nu_1}{\rho}, \quad x_2 = r_1\lambda_1, \quad y_2 = -r_1\mu_1, \quad z_2 = r_1\nu_1,\]
where \(\rho \in \mathbb{R}^*\) and \(r_1 \in \mathbb{R}\).

As we have addressed before, the same argument can be applied to the other part of the system which involves \(s_{jk}, t_{lm}\). Therefore, by the above discussion, we have proven that \(G^8\) belongs to \(W_3\) if and only if
\[c_{12} = -\frac{4\lambda_1}{\rho}, \quad c_{11} = -\frac{4\mu_1}{\rho}, \quad b_{11} = -\frac{4\nu_1}{\rho}, \quad c_{22} = r_1\lambda_1, \quad c_{21} = r_1\mu_1, \quad b_{21} = r_1\nu_1,\]
\[t_{15} = -\frac{4\lambda_2}{\rho}, \quad t_{14} = -\frac{4\mu_2}{\rho}, \quad s_{14} = -\frac{4\nu_2}{\rho}, \quad t_{25} = r_2\lambda_2, \quad t_{24} = r_2\mu_2, \quad s_{24} = r_2\nu_2,\]
where \(\rho \in \mathbb{R}^*\), \(r_1, r_2 \in \mathbb{R}\). Thus the solution to this problem is 3-dimensional. \(\square\)
Chapter 5

Locally Conformal Kähler Manifolds

In this chapter, we investigate the conditions when $G^8$ belongs to the class $W_4$. By the tensoriality of $\mu$ and the left translations on $G^8$, it suffices to check the condition for any basis vectors $e_j, e_k, e_l$ in the orthonormal basis of $\mathfrak{g}$.

We define the function $\psi$ by

$$\psi(j, k, l) = \alpha(e_j, e_k, e_l) + \frac{1}{2(n-1)}(g(e_j, e_k)\tilde{\alpha}(e_l) - g(e_j, e_l)\tilde{\alpha}(e_k)) - g(e_j, Je_k)\tilde{\alpha}(Je_l) + g(e_j, Je_l)\tilde{\alpha}(Je_k)).$$

The Hermitian manifold $G^8$ belongs to $W_4$ if and only if $\psi(j, k, l)$ vanishes for any $1 \leq j, k, l \leq 8$. By a direct calculation, we have:

$$\psi(8, 8, 1) = \frac{1}{3}\theta_1 + \frac{\lambda_1}{3}, \psi(8, 8, 2) = \frac{1}{3}\theta_2 - \frac{\mu_1}{3}, \psi(8, 8, 3) = \frac{1}{3}\theta_3 + \frac{\nu_1}{3},$$

$$\psi(8, 8, 4) = \frac{1}{3}\theta_4 + \frac{\lambda_2}{3}, \psi(8, 8, 5) = \frac{1}{3}\theta_5 - \frac{\mu_2}{3}, \psi(8, 8, 6) = \frac{1}{3}\theta_6 + \frac{\nu_2}{3}.$$

Therefore, if $G^8$ belongs to $W_4$, we must have

$$\theta_1 = -\lambda_1, \theta_2 = \mu_1, \theta_3 = -\nu_1, \theta_4 = -\lambda_2, \theta_5 = \mu_2, \theta_6 = -\nu_2. \quad (5.1)$$

Substituting $\theta_j$ by \([5.1]\), we have

$$\psi(1, 1, 4) = \frac{1}{2}\lambda_2(1 - \lambda_1^2), \quad \psi(1, 1, 5) = \frac{1}{2}\mu_2(\lambda_1^2 - 1), \quad \psi(1, 1, 6) = \frac{1}{2}\nu_2(1 - \lambda_1^2),$$

$$\psi(4, 4, 1) = \frac{1}{2}\lambda_1(1 - \lambda_2^2), \quad \psi(4, 4, 2) = \frac{1}{2}\mu_1(\lambda_2^2 - 1), \quad \psi(4, 4, 3) = \frac{1}{2}\nu_1(1 - \lambda_2^2).$$

Since $\lambda_j^2 + \mu_j^2 + \nu_j^2 = 1$, we must have

$$\lambda_1^2 = 1, \quad \lambda_2^2 = 1, \quad \mu_1 = \nu_1 = 0, \quad \mu_2 = \nu_2 = 0.$$ 

However, substituting $\mu_j, \nu_j$ by 0, we have

$$\psi(2, 3, 4) = \frac{\lambda_1\lambda_2}{2} \neq 0.$$

Hence, $G^8$ never belongs to $W_4$. 

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Appendix A

The Maple Programme for $G^8$

In this appendix, we give a brief account of the Maple programme that we use to check our computations. First, we set up the environment.

```maple
restart;
with(DifferentialGeometry);
with(LieAlgebras);
with(LinearAlgebra);
with(ArrayTools);
```

Then we import the structure equations of $G^8$ from the paper [2] to the program.

```maple
BasisVectors := [A, B, C, R, S, T, X, Y]:
StructureEquations := [
    [A, B] = 2*C, [C, A] = 2*B, [B, C] = 2*A,
    [R, S] = 2*T, [T, R] = 2*S, [S, T] = 2*R,
    [A, X] = -B*b[11] - C*c[11],
    [A, Y] = -B*b[21] - C*c[21],
    [B, X] = A*b[11] - C*c[12],
    [B, Y] = A*b[21] - C*c[22],
    [C, X] = A*c[11] + B*c[12],
    [C, Y] = A*c[21] + B*c[22],
    [R, X] = -S*s[14] - T*t[14],
    [R, Y] = -S*s[24] - T*t[24],
    [S, X] = R*s[14] - T*t[15],
    [S, Y] = R*s[24] - T*t[25],
    [T, X] = R*t[14] + S*t[15],
    [T, Y] = R*t[24] + S*t[25],
    [X, Y] = rho*X
+ (1/2)*(-rho*c[12] + b[11]*c[21] - b[21]*c[11])*A
+ (1/2)*(rho*c[11] + b[11]*c[22] - b[21]*c[12])*B
+ (1/2)*(-rho*b[11] + c[11]*c[22] - c[12]*c[21])*C
+ (1/2)*(-rho*t[15] + s[14]*t[24] - s[24]*t[14])*R
+ (1/2)*(rho*t[14] + s[14]*t[25] - s[24]*t[15])*S
+ (1/2)*(-rho*s[14] + t[14]*t[25] - t[15]*t[24])*T];
```
Initialize the equations for checking the integrability. The parameters
\[ \lambda_1, \mu_1, \nu_1, \lambda_2, \mu_2, \nu_2 \]
are replaced by
\[ x_1, y_1, z_1, x_2, y_2, z_2 \]
for convenience.

\[ \text{LAD := LieAlgebraData(StructureEquations, BasisVectors, alg):} \]
\[ \text{DGsetup(LAD):} \]
\[ \text{N := numelems(BasisVectors);} \]
\[ \text{ProcBasis := [cat(e, 1 .. N)];} \]
\[ \text{NJ := array(1 .. 8, 1 .. 8, 1 .. 8);} \]
\[ \text{Equs := \{x1^2 + y1^2 + z1^2 = 1, x2^2 + y2^2 + z2^2 = 1\};} \]

The function \text{con()} converts a data from the form of a vector (in LinearAlgebra) to a vector (in LieAlgebra) in \text{g}.

\[ \text{con := proc(V)} \]
\[ \text{local a, i;} \]
\[ \text{a := 0;} \]
\[ \text{for i to N do} \]
\[ \text{a := a + V[i]*ProcBasis[i];} \]
\[ \text{end do;} \]
\[ \text{end;} \]

The following procedure computes the Nijenhuis tensor for given vectors \text{u}, \text{v} and a given almost complex structure \text{J}.

\[ \text{Nijenhuis := proc(u, v, index_u, index_v)} \]
\[ \text{local i, result, Ju, Jv, U, V, Juv, uJv, JJuv, JuJv;} \]
\[ \text{global Equs, NJ;} \]
\[ \text{result := 0;} \]
\[ \text{U := Vector(GetComponents(u, ProcBasis));} \]
\[ \text{V := Vector(GetComponents(v, ProcBasis));} \]
\[ \text{result := result + LieBracket(u, v);} \]
\[ \text{Ju := con(MatrixVectorMultiply(J, U));} \]
\[ \text{Jv := con(MatrixVectorMultiply(J, V));} \]
\[ \text{result := result - LieBracket(Ju, Jv);} \]
\[ \text{Juv := Vector(GetComponents(LieBracket(Ju, v), ProcBasis));} \]
\[ \text{JJuv := con(MatrixVectorMultiply(J, Juv));} \]
\[ \text{uJv := Vector(GetComponents(LieBracket(u, Jv), ProcBasis));} \]
\[ \text{JuJv := con(MatrixVectorMultiply(J, uJv));} \]
\[ \text{result := result + JJuv + JuJv;} \]
\[ \text{if (result != 0) then} \]
\[ \text{for i to N do} \]
\[ \text{if (GetComponents(result, ProcBasis)[i] != 0) then} \]
\[ \text{print(u, v, i);} \]
\[ \text{print(GetComponents(result, ProcBasis)[i]);} \]

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The procedure CheckIntegrability() lists the result of Nijenhuis() and we use the output to analyze the symmetry in the Nijenhuis tensor.

CheckIntegrability := proc()
local i, j;
for i to N - 1 do
  for j from i + 1 to N do
    Nijenhuis(ProcBasis[i], ProcBasis[j], i, j);
  end do;
end do;
end proc:

We import the result by Magnin to our program and run the procedure CheckIntegrability().

Now we define a function Nabla() corresponding to \( \alpha(e_j, e_k, e_l) \) in the thesis.

Nabla := proc(index_u, index_v, index_w)
  local result, u, v, w, U, V, W, JU, JV, Ju, Jv, Jw, UJV, UW, JwV, UV, UJW, VJW;
global Equs, NJ;
  result := 0;
  u := ProcBasis[index_u];
  v := ProcBasis[index_v];
  w := ProcBasis[index_w];
  U := Vector(GetComponents(u, ProcBasis));
  V := Vector(GetComponents(v, ProcBasis));
  W := Vector(GetComponents(w, ProcBasis));
  JU := MatrixVectorMultiply(J, U);
  JV := MatrixVectorMultiply(J, V);
end proc:
JW := MatrixVectorMultiply(J, W);
Ju := con(JU);
Jv := con(JV);
Jw := con(JW);
UJV := Vector(GetComponents(LieBracket(u, Jv), ProcBasis));
UW := Vector(GetComponents(LieBracket(u, w), ProcBasis));
JW := Vector(GetComponents(LieBracket(Jv, w), ProcBasis));
UV := Vector(GetComponents(LieBracket(u, v), ProcBasis));
UJW := Vector(GetComponents(LieBracket(u, Jw), ProcBasis));
VJW := Vector(GetComponents(LieBracket(v, Jw), ProcBasis));
result := 1/2*result
+ 1/2*DotProduct(UJV, W, conjugate = false)
- 1/2*DotProduct(UW, Jv, conjugate = false)
- 1/2*DotProduct(JVW, U, conjugate = false)
+ 1/2*DotProduct(UV, JW, conjugate = false)
- 1/2*DotProduct(UJW, V, conjugate = false)
- 1/2*DotProduct(VJW, U, conjugate = false);
return result;
end:

The function NablaOmegajjk() calculates $\nabla e_j \omega(e_j, e_k)$.

NablaOmegajjk := proc(u, v)
local result, U, V, JU, JV, Ju, Jv, UJV, UJU, JUV, UV;
result := 0;
U := Vector(GetComponents(u, ProcBasis));
V := Vector(GetComponents(v, ProcBasis));
JU := MatrixVectorMultiply(J, U);
JV := MatrixVectorMultiply(J, V);
Ju := con(JU); Jv := con(JV);
UJV := Vector(GetComponents(LieBracket(u, Jv), ProcBasis));
UJU := Vector(GetComponents(LieBracket(u, Ju), ProcBasis));
JUV := Vector(GetComponents(LieBracket(Ju, v), ProcBasis));
UV := Vector(GetComponents(LieBracket(u, v), ProcBasis));
result := (1/2)*DotProduct(UJU, V, conjugate = false)
- (1/2)*DotProduct(UV, Ju, conjugate = false)
- (1/2)*DotProduct(JUV, U, conjugate = false)
- DotProduct(UJV, U, conjugate = false);
return result;
end:

Then we can define $\bar{\alpha}(e_k)$ for basis vectors $e_k$ in the thesis.

BarAlphaek := proc(v)
local j, result;
result := 0;
for j to N do
    result := result + NablaOmegajjk(ProcBasis[j], v);
end do;
return result;
end:
And we can calculate $\bar{\alpha}(v)$ for any $v \in \mathfrak{g}$.

BarAlpha := proc(v)
    local j, V, result;
    global N;
    result := 0;
    V := Vector(GetComponents(v, ProcBasis));
    for j to N do
        result := result + V[j]*BarAlphaek(ProcBasis[j]);
    end do;
    return result;
end:

The function $\psi(j, k, l)$ in Chapter 5 in the thesis is calculated by the following programme.

W4 := proc(index_u, index_v, index_w)
    local result, u, v, w, U, V, W, JV, JW, Jv, Jw;
    global Equs, NJ, N;
    result := 0;
    u := ProcBasis[index_u];
    v := ProcBasis[index_v];
    w := ProcBasis[index_w];
    U := Vector(GetComponents(u, ProcBasis));
    V := Vector(GetComponents(v, ProcBasis));
    W := Vector(GetComponents(w, ProcBasis));
    JV := MatrixVectorMultiply(J, V);
    JW := MatrixVectorMultiply(J, W);
    Jv := con(JV);
    Jw := con(JW);
    result := Nabla(index_u, index_v, index_w)
               + (DotProduct(U, V, conjugate = false)*BarAlpha(w)
               - DotProduct(U, W, conjugate = false)*BarAlpha(v)
               - DotProduct(U, JW, conjugate = false)*BarAlpha(Jw)
               + DotProduct(U, JW, conjugate = false)*BarAlpha(Jv))/(N - 2);
    return result;
end:

Finally, the following procedure gives the condition for $G^8 \in \mathcal{W}_4$.

CheckW4 := proc()
    local j, k, l;
    for j to N do
        for k to N do
            for l to N do
                print(j, k, l);
                print(W4(j, k, l));
            end do;
        end do;
    end do;
end:
end do;
end do;
end:

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