Quantum marginal problem and N-representability

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Abstract. A variant of the quantum marginal problem was known from early sixties as N-representability problem. In 1995 it was designated by National Research Council of USA as one of ten most prominent research challenges in quantum chemistry. In spite of this recognition the progress was very slow, until a couple of years ago the problem came into focus again, now in framework of quantum information theory. In the paper I give an account of the recent development.

PACS numbers: 03.65.Ud, 03.67.-a
1. Introduction

The quantum marginal problem is about relations between spectrum of mixed state $\rho_{AB}$ of two (or multi) component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ and that of reduced states $\rho_A$ and $\rho_B$ of the components. This problem emerges a couple of years ago in framework of quantum information theory.

The problem can be stated in plain language as follows. Let $M = [m_{ijk}]$ be complex cubic matrix and $M_1, M_2, M_3$ be Gram matrices formed by Hermitian dot products of parallel slices of $M$. We are seeking for relations between spectra of these matrices.

In section 3 we survey some recent results that laid the ground of the quantum marginal problem and discuss a solution of this problem based on geometric invariant theory.

A variant of QMP dealing with system of $N$ fermions, like electrons in an atom or a molecule, was known from early sixties as $N$-representability problem (Coulson, 1960). Its solution allows to calculate nearly all properties of matter which are of interest for chemists and physicists (Coleman and Youkalow 2000). By this reason it was designated as one of ten most prominent research challenges in theoretical chemistry (Stillinger et al 1995). In section 3 we outline a general solution of the $N$-representability problem for one particle reduced density matrix. For systems of rank $\leq 8$ explicit sets of linear constraints are given for pure state $N$ representability. A representation theoretical interpretation of $N$-representability plays crucial role in the calculations.

The reduction $\rho_{AB} \mapsto \rho_A$ is known in mathematics as contraction. For example, Ricci curvature of a Riemann manifold is a contraction of its Riemann curvature. The results of this paper imply some inequalities between spectra of Riemann and Ricci curvatures, see Example 4.2.3. Recall that in general relativity Ricci curvature is governed by the energy-momentum tensor, i.e. by physical content of the space, while Riemann curvature is responsible for its geometry and topology. The above constraints impose some bounds on influence of matter on geometry.

2. Classical marginal problem

2.1. Marginal distributions

Let’s start with classical marginal problem which asks for existence of a “body” in $\mathbb{R}^n$ with given projections onto some coordinate subspaces $\mathbb{R}^I \subset \mathbb{R}^n, I \subset \{1, 2, \ldots, n\}$, i.e. existence of probability density $p(x) = p(x_1, x_2, \ldots, x_n)$ with given marginal distributions

$$p_I(x_I) = \int_{\mathbb{R}^J} p(x) dx_J, J = \{1, 2, \ldots, n\} \setminus I.$$

In discrete version the classical MP amounts to calculation of an image of a multidimensional symplex, say $\Delta = \{p_{ijk} \geq 0 | \sum p_{ijk} = 1\}$, under a linear map like

$$\pi : \mathbb{R}^{\ell mm} \to \mathbb{R}^{\ell m} \oplus \mathbb{R}^{mn} \oplus \mathbb{R}^{nl},$$

$$p_{ijk} \mapsto (p_{ij}, p_{jk}, p_{ki}).$$
Quantum marginal problem

\[ p_{ij} = \sum_k p_{ijk}, \quad p_{jk} = \sum_i p_{ijk}, \quad p_{ki} = \sum_j p_{ijk}. \]

The image is a convex hull of the projections of vertices of \( \Delta \). So the classical MP amounts to calculation of facets of a convex hull. In high dimensions it might be a computational nightmare (Pitowsky 1989).

2.2. Classical realism

Let \( X_i : \mathcal{H}_A \rightarrow \mathcal{H}_A \) be observables of quantum system \( A \). Actual measurement of \( X_i \) produces random quantity \( x_i \) with values in \( \text{Spec}(X_i) \) and density \( p_i(x_i) \) implicitly determined by expectations

\[ \langle f(x_i) \rangle = \langle \psi | f(X_i) | \psi \rangle \]

for all functions \( f \) on spectrum \( \text{Spec}(X_i) \). For commuting observables \( X_i, i \in I \) the random variables \( x_i, i \in I \) have joint distribution \( p_I(x_I) \) defined by similar equation

\[ \langle f(x_I) \rangle = \langle \psi | f(X_I) | \psi \rangle, \quad \forall f. \quad (2.1) \]

Classical realism postulates existence of a hidden joint distribution of all variables \( x_i \). This amounts to compatibility of the marginal distributions (2.1) for commuting sets of observables \( X_I \). Bell inequalities, designed to test classical realism, stem from the classical marginal problem.

2.2.1 Example. Observations of disjoint components of composite system \( \mathcal{H}_A \otimes \mathcal{H}_B \) always commute. For two qubits with two measurements per site their compatibility is given by 16 inequalities obtained from Clauser-Horne-Shimony-Holt inequality (Calauser et al. 1969)

\[ \langle a_1 b_1 \rangle + \langle a_2 b_1 \rangle + \langle a_2 b_2 \rangle - \langle a_1 b_2 \rangle + 2 \geq 0 \]

by spin flips \( a_i \mapsto \pm a_j \) and permutation of the components \( A \leftrightarrow B \). Here \( \langle a_i b_j \rangle \) is expectation of product of spin projections onto directions \( i, j \) in sites \( A, B \).

2.2.2 Example. For three qubits with two measurements per site the marginal constraints amounts to 53856 independent inequalities (Pitowsky et al. 2001). This example may help to disabuse us from overoptimistic expectations for the quantum marginal problem to be discussed below.

2.2.3 Example. Univariant marginal distributions are always compatible, e.g. we can consider \( x_i \) as independent variables. However under additional constraints, say for a “body” of constant density, even univariant marginal problem becomes nontrivial. For its discrete version Gale-Ryser theorem (Gale 1957) tells that partitions \( \lambda, \mu \) are margins of a rectangular 0/1 matrix iff majorization inequality \( \lambda \prec \mu \) holds. Here marginal values arranged in decreasing order are treated as Young diagrams

\[ \lambda = (5, 4, 2, 1) = \begin{array}{cccc}
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\end{array} \]

\[ \lambda^t = (4, 3, 2, 2, 1) = \begin{array}{cccc}
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\hline \\
\end{array} \]
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\[ \mu^t \] stands for \textit{transposed} diagram, and the \textit{majorization order} \( \lambda \prec \nu \) is defined by inequalities

\[
\begin{align*}
\lambda_1 & \leq \nu_1 \\
\lambda_1 + \lambda_2 & \leq \nu_1 + \nu_2 \\
\lambda_1 + \lambda_2 + \lambda_3 & \leq \nu_1 + \nu_2 + \nu_3 \\
\vdots
\end{align*}
\]

3. Quantum marginal problem

3.1. Reduced states

Density matrix of composite system \( AB \) can be written as a linear combination of separable states

\[
\rho_{AB} = \sum_{\alpha} a_{\alpha} \rho_A^\alpha \otimes \rho_B^\alpha, \quad (3.1)
\]

where \( \rho_A^\alpha, \rho_B^\alpha \) are mixed states of the components \( A, B \) respectively, and the coefficients \( a_\alpha \) are \textit{not} necessarily positive. Its \textit{reduced matrices} or \textit{marginal states} may be defined by equations

\[
\begin{align*}
\rho_A &= \sum_{\alpha} a_{\alpha} \text{Tr}(\rho_B^\alpha) \rho_A^\alpha := \text{Tr}_B(\rho_{AB}), \\
\rho_B &= \sum_{\alpha} a_{\alpha} \text{Tr}(\rho_A^\alpha) \rho_B^\alpha := \text{Tr}_A(\rho_{AB}).
\end{align*}
\]

The reduced states \( \rho_A, \rho_B \) are independent of the decomposition (3.1) and can be characterized intrinsically by the following property

\[
\langle X_A \rangle_{\rho_{AB}} = \text{Tr}(\rho_{AB} X_A) = \text{Tr}(\rho_A X_A) = \langle X_A \rangle_{\rho_A}, \quad \forall \ X_A, \quad (3.2)
\]

which tells that \( \rho_A \) is a “visible” state of subsystem \( A \). This justifies the terminology.

3.1.1 Example. Let’s identify pure state of two component system

\[
\psi = \sum_{ij} \psi_{ij} \alpha_i \otimes \beta_j \in \mathcal{H}_A \otimes \mathcal{H}_B
\]

with its matrix \( [\psi_{ij}] \) in orthonormal bases \( \alpha_i, \beta_j \) of \( \mathcal{H}_A, \mathcal{H}_B \). Then the reduced states of \( \psi \) in respective bases are given by matrices

\[
\rho_A = \psi^\dagger \psi, \quad \rho_B = \psi \psi^\dagger, \quad (3.3)
\]

which have the same non negative spectra

\[
\text{Spec} \rho_A = \text{Spec} \rho_B = \lambda \quad (3.4)
\]

except extra zeros if \( \dim \mathcal{H}_A \neq \dim \mathcal{H}_B \). The isospectrality implies so called \textit{Schmidt decomposition}

\[
\psi = \sum_i \sqrt{\lambda_i} \psi_i^A \otimes \psi_i^B, \quad (3.5)
\]

where \( \psi_i^A, \psi_i^B \) are eigenvectors of \( \rho_A, \rho_B \) with the same eigenvalue \( \lambda_i \).
Quantum marginal problem

In striking contrast to classical case marginals of a pure state $\psi \neq \psi_A \otimes \psi_B$ are mixed ones, i.e. as Srödinger 1935 puts it “maximal knowledge of the whole does not necessarily includes the maximal knowledge of its parts.” He coined the term entanglement just to describe this phenomenon.

3.2. Statement of the problem

Quantum analogue of the classical marginal distribution is reduced state $\rho_A$ of composite system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. Accordingly, most general quantum marginal problem asks about existence of mixed state $\rho_I$ of composite system

$$\mathcal{H}_I = \bigotimes_{i \in I} \mathcal{H}_i$$

with given reduced states $\rho_J$ for some $J \subset I$ (cf. with classical settings section 2). Additional constraints on state $\rho_I$ may be relevant. Here we consider only two variations:

- **Pure quantum marginal problem**
  dealing with marginals of pure state $\rho_I = |\psi\rangle\langle\psi|$, and more general

- **Mixed quantum marginal problem**
  corresponding to a state with given spectrum $\lambda_I = \text{Spec} \rho_I$.

Both versions are nontrivial even for univariant margins (cf. Example 2.2.3). In this case reduced states $\rho_i$ can be diagonalized by local unitary transformations and their compatibility depends only on spectra $\lambda_i = \text{Spec} \rho_i$. Note that mixed QMP say for two component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ is formally equivalent to the pure one for three component system $\mathcal{H}_{AB} \otimes \mathcal{H}_A \otimes \mathcal{H}_B$.

Pure quantum marginal problem has no classical analogue, since projection of a point is a point. For two component system $\mathcal{H}_A \otimes \mathcal{H}_B$ marginal constraints amounts to isospectrality $\text{Spec} \rho_A = \text{Spec} \rho_B$, see Equation (3.4). For three component system the problem can be stated in plain language as follows.

**Problem 3.2.1.** Let $A = [A_{ijk}]$ be complex cubic matrix and $A_1, A_2, A_3$ be Gram matrices formed by Hermitian dot products of parallel slices of $A$. The question is what are relations between spectra of matrices $A_1, A_2, A_3$?

Unfortunately methods of this paper can’t be applied directly to overlapping marginals like $\rho_{AB}, \rho_{BC}, \rho_{CA}$.

3.3. Some known results

Here are some recent results that laid the ground of the quantum marginal problem. They all emerge from quantum information settings in two or three years.

**Theorem (Higuchi et al 2003).** For array of qubits $\bigotimes_{i=1}^n \mathcal{H}_i$, $\dim \mathcal{H}_i = 2$ all constraints on marginals $\rho_i$ of a pure state are given by polygonal inequalities

$$\lambda_i \leq \sum_{j \neq i} \lambda_j$$
for minimal eigenvalues $\lambda_i$ of $\rho_i$.

This result was proved independently by Sergey Bravyi who also managed to crack mixed two qubit problem.

**Theorem (Bravyi 2004).** For two qubits $\mathcal{H}_A \otimes \mathcal{H}_B$ solution of the mixed QMP is given by inequalities

$$\min(\lambda_A, \lambda_B) \geq \lambda_3^{AB} + \lambda_4^{AB},$$

$$\lambda_A + \lambda_B \geq \lambda_2^{AB} + \lambda_3^{AB} + 2\lambda_4^{AB},$$

$$|\lambda_A - \lambda_B| \leq \min(\lambda_1^{AB} - \lambda_3^{AB}, \lambda_2^{AB} - \lambda_4^{AB}),$$

where $\lambda_A, \lambda_B$ are minimal eigenvalues of $\rho_A, \rho_B$ and $\lambda_1^{AB} \geq \lambda_2^{AB} \geq \lambda_3^{AB} \geq \lambda_4^{AB}$ is spectrum of $\rho_{AB}$.

Finally for three qutrits the problem was solved by Matthias Franz using rather advanced mathematical technology and help of a computer. An elementary solution was found independently by Astashi Higuchi.

**Theorem (Franz 2002, Higuchi 2003).** All constraints on margins of a pure state of three qutrit system $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ are given by the following inequalities

$$\lambda_2^a + \lambda_1^a \leq \lambda_2^b + \lambda_1^b + \lambda_2^c + \lambda_1^c,$$

$$\lambda_3^a + \lambda_1^a \leq \lambda_2^b + \lambda_1^b + \lambda_3^c + \lambda_1^c,$$

$$\lambda_3^a + \lambda_2^a \leq \lambda_2^b + \lambda_1^b + \lambda_3^c + \lambda_2^c,$$

$$2\lambda_2^a + \lambda_2^c \leq 2\lambda_2^b + \lambda_1^b + 2\lambda_2^c + \lambda_1^c,$$

$$2\lambda_1^a + \lambda_2^a \leq 2\lambda_1^b + \lambda_1^b + 2\lambda_1^c + \lambda_2^c,$$

$$2\lambda_2^a + \lambda_3^a \leq 2\lambda_2^b + \lambda_1^b + 2\lambda_2^c + \lambda_3^c,$$

$$2\lambda_2^a + \lambda_3^a \leq 2\lambda_1^b + \lambda_2^b + 2\lambda_3^c + \lambda_2^c,$$

where $a, b, c$ is a permutation of $A, B, C$, and the marginal spectra are arranged in increasing order $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

Note that in contrast to the classical marginal problem, linearity of the quantum marginal constraints is a surprising nontrivial fact.

### 3.4. Main theorem

A general solution of the quantum marginal problem, based on geometric invariant theory, has been found recently (Klyachko 2004).

**Theorem 3.4.1.** For two component system $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ of format $m \times n$ all constraints on spectra $\lambda^{AB} = \text{Spec } \rho_{AB}$, $\lambda^A = \text{Spec } \rho_A$, $\lambda^B = \text{Spec } \rho_B$ arranged in decreasing order are given by linear inequalities

$$\sum_i a_i \lambda^A_{u(i)} + \sum_j b_j \lambda^B_{v(j)} \leq \sum_k (a + b)_k \lambda^{AB}_{w(k)}, \quad (3.6)$$

where $a : a_1 \geq a_2 \geq \cdots \geq a_m$, $b : b_1 \geq b_2 \geq \cdots \geq b_n$, $\sum a_i = \sum b_j = 0$ are “test spectra”, $a + b = \{a_i + b_j\} \downarrow$, and $u \in S_m$, $v \in S_n$, $w \in S_{mn}$ are permutations, subject to crucial condition $c^w_{uv}(a, b) \neq 0$ of topological nature to be explained below.
3.4.2 Remark. Here \((a + b)_k\) denotes \(k\)-th term of the sequence \(a_i + b_j\) arranged in decreasing order. The coefficient \(c^w_{uv}(a, b)\) depends only on the order in which quantities \(a_i + b_j\) appear in the spectrum \((a + b)_↓\). The order changes when the pair \((a, b)\) crosses hyperplane \(H_{ij|kl} : a_i + b_j = a_k + b_ℓ\).

The hyperplanes cut the set of all pairs \((a, b)\) into finite number of pieces called cubicles. For each cubicle one have to check inequality (3.6) only for its extremal edges. Hence the marginal constraints amounts to a finite system of inequalities, but the total number of extremal edges increases rapidly. Here are some sample data for arrays of qubits.

| # qubits | 2 | 3 | 4 | 5 | 6 |
|----------|---|---|---|---|---|
| # edges  | 2 | 4 | 12 | 125 | 11344 |

Unfortunately for most systems the marginal constraints are too numerous to be reproduced here. Therefore we give only summary table, which shows how complicated may be the answer.

| System | Rank | #Inequalities |
|--------|------|---------------|
| 2 × 2  | 2    | 7             |
| 2 × 2 × 2 | 3    | 40            |
| 2 × 3  | 3    | 41            |
| 2 × 4  | 4    | 234           |
| 3 × 3  | 4    | 387           |
| 2 × 2 × 3 | 4    | 442           |
| 2 × 2 × 2 × 2 | 4 | 805 |

3.5. Hidden geometry and topology

Here we explain the meaning of the coefficient \(c^w_{uv}(a, b)\) in statement of the theorem and show how they can be calculated. Consider set of Hermitian operators \(X_A : \mathcal{H}_A → \mathcal{H}_A\) with given spectrum \(\text{Spec}(X_A) = a\) and call it flag variety

\[
\mathcal{F}_a(\mathcal{H}_A) := \{X_A \mid \text{Spec}(X_A) = a\}.
\]

For two flag varieties \(\mathcal{F}_a(\mathcal{H}_A)\) and \(\mathcal{F}_b(\mathcal{H}_B)\) define map

\[
φ_{ab} : \mathcal{F}_a(\mathcal{H}_A) × \mathcal{F}_b(\mathcal{H}_B) → \mathcal{F}_{a+b}(\mathcal{H}_A ⊗ \mathcal{H}_B),
\]

\[
X_A × X_B \mapsto X_A ⊗ 1 + 1 ⊗ X_B.
\]

The coefficients \(c^w_{uv}(a, b)\) come from the induced morphism of cohomology

\[
φ^*_ab : H^*(\mathcal{F}_{a+b}(\mathcal{H}_{AB})) → H^*(\mathcal{F}_a(\mathcal{H}_A)) ⊗ H^*(\mathcal{F}_b(\mathcal{H}_B))
\]

written in the basis of Schubert cocycles \(σ^w\)

\[
φ^*_ab : σ^w → \sum_{u,v} c^w_{uv}(a, b)σ^u ⊗ σ^v.
\]
Quantum marginal problem

We’ll give below an algorithm for their calculation. For this we need another description of the cohomology of flag varieties due to (Bernstein et al 1973). Specifically, for simple spectrum $a$ eigenspaces of operator $X_A \in F_a(H_A)$ of given eigenvalue $a_i$ form a line bundle $L^A_i$ on $F_a(H_A)$. Their Chern classes $x_i^A = c_1(L^A_i)$ generate cohomology ring $H^*(F_a(H_A))$ and in this setting morphism $\varphi_{ab}^*$ admits a simple description

$$\varphi_{ab}^*(L^A_k) = L^A_i \boxtimes L^B_j, \quad \varphi_{ab}^*: x^{AB}_k \mapsto x^A_i + x^B_j$$

where $(a + b)_k = a_i + b_j$. In terms of the canonical generators $x_i = c_1(L_i)$ Schubert cocycle $\sigma^w$ is given by Schubert polynomial (Macdonald 1991)

$$S_w(x_1, x_2, \ldots) = \partial_{w-1}w_0(x_1^{n-1}x_2^{n-2}\cdots x_{n-1})$$

where $w \in S_n$ is a permutation of $1, 2, \ldots, n$, $w_0 = (n, n-1, \ldots, 2, 1)$ reversion of the order, operator $\partial_w = \partial_{i_1}\partial_{i_2} \cdots \partial_{i_{\ell}}$, is defined via minimal decomposition $w = s_{i_1}s_{i_2} \cdots s_{i_{\ell}}$ into product of transpositions $s_i = (i, i + 1)$, $\ell = \ell(w)$ is the number of inversion of $w$ called its length, and finally

$$\partial_if = \frac{f(\ldots, x_i, x_{i+1}, \ldots) - f(\ldots, x_{i+1}, x_i, \ldots)}{x_i - x_{i+1}}$$

is finite difference operator. This leads to the computational formula

$$c^w_{uv}(a, b) = \partial^A_u \partial^B_v S_w(x^{AB}) \bigg|_{x^{AB} = x^A_i + x^B_j}$$

(3.7)

where $(a + b)_k = a_i + b_j$ and $\ell(w) = \ell(u) + \ell(v)$. It can be easily implemented into a computer program. Recall that in order to get a finite system of inequalities one have also to find all the extremal edges and use them as the test spectra $(a, b)$.

3.5.1 Example. Note that for identical permutations $u, v, w$ the coefficient $c^w_{uv}(a, b)$ is equal to 1. Hence inequality

$$\sum_i a_i\lambda^A_i + \sum_j b_j\lambda^B_j \leq \sum_k (a + b)_k\lambda^{AB}_k$$

holds for all test spectra $(a, b)$. This amounts to finite system of basic inequalities (Han et al 2004)

$$\lambda^A_1 + \lambda^A_2 + \cdots + \lambda^A_k \leq \lambda^{AB}_1 + \lambda^{AB}_2 + \cdots + \lambda^{AB}_{kn}, \quad k \leq m, \quad \lambda^B_1 + \lambda^B_2 + \cdots + \lambda^B_\ell \leq \lambda^{AB}_1 + \lambda^{AB}_2 + \cdots + \lambda^{AB}_{m\ell}, \quad \ell \leq n.$$  

3.6. Array of qubits

The calculations can be essentially reduced using the following result.

**Theorem 3.6.1.** In the setting of Theorem 3.4.1 all marginal constraints are given by inequalities (3.6) with $c^w_{uv}(a, b) = 1$.

3.6.2 Example. One can check that for array of qubits $\otimes^n_{i=1} H_i$ all inequalities (3.6) with odd coefficient $c^w_{uv12\ldots n}(a_1, a_2, \ldots, a_n)$ can be obtained from the basic inequality

$$\sum_i a_i(\lambda^{(i)}_1 - \lambda^{(i)}_2) \leq \sum_{\pm}(\pm a_1 \pm a_2 \pm \cdots \pm a_n)k\lambda_k$$
by transposition $\lambda_k \leftrightarrow \lambda_{k+1}$, $k = \text{odd}$, combined with sign change of a summand in LHS. Here $a_i \geq -a_i$ are the test spectra, $\lambda$ and $\lambda^{(i)}$ are spectra of state $\rho$ of the system and its one qubit reduced state $\rho^{(i)}$ respectively. To get a finite system one has only to find the extremal edges. For large $N$ this may be a challenge, see Remark 3.4.2.

3.6.3 Example. For 3-qubit the theorem returns the following list of marginal inequalities grouped by their extremal edges. The first inequality in each group is the basic one. The transposed eigenvalues in modified inequalities typeset in bold face. We expect $\Delta_i = \lambda_1^{(i)} - \lambda_2^{(i)}$ to be arranged in increasing order $\Delta_1 \leq \Delta_2 \leq \Delta_3$.

\begin{align*}
\Delta_3 & \leq \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8, \\
\Delta_2 + \Delta_3 & \leq 2\lambda_1 + 2\lambda_2 - 2\lambda_7 - 2\lambda_8, \\
\Delta_1 + \Delta_2 + \Delta_3 & \leq 3\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - 3\lambda_8, \\
-\Delta_1 + \Delta_2 + \Delta_3 & \leq 3\lambda_2 + \lambda_1 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - 3\lambda_8, \\
-\Delta_1 + \Delta_2 + \Delta_3 & \leq 3\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_8 - 3\lambda_7, \\
\Delta_1 + \Delta_2 + 2\Delta_3 & \leq 4\lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_6 - 2\lambda_7 - 4\lambda_8, \\
-\Delta_1 + \Delta_2 + 2\Delta_3 & \leq 4\lambda_2 + 2\lambda_1 + 2\lambda_3 - 2\lambda_6 - 2\lambda_7 - 4\lambda_8, \\
-\Delta_1 + \Delta_2 + 2\Delta_3 & \leq 4\lambda_1 + 2\lambda_2 + 2\lambda_4 - 2\lambda_6 - 2\lambda_7 - 4\lambda_8, \\
-\Delta_1 + \Delta_2 + 2\Delta_3 & \leq 4\lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_5 - 2\lambda_7 - 4\lambda_8, \\
-\Delta_1 + \Delta_2 + 2\Delta_3 & \leq 4\lambda_1 + 2\lambda_2 + 2\lambda_3 - 2\lambda_6 - 2\lambda_8 - 4\lambda_7.
\end{align*}

4. N-representability problem

4.1. Physical background

The quantum marginal problem may be complicated by additional constraints on state $\psi$. For example, Pauli principle implies that state space of $N$ identical particles shrinks to symmetric tensors $S^N H \subset H^\otimes N$ for bosons and to skew symmetric tensors $\wedge^N H$ for fermions. For such systems reduced density matrices appear in the second quantization formalism in the form

\begin{align*}
\rho^{(1)} &= \langle \psi | a_i^\dagger a_j | \psi \rangle = 1 \text{ particle RDM}, \\
\rho^{(2)} &= \langle \psi | a_i^\dagger a_j a_k a_l | \psi \rangle = 2 \text{ particle RDM, etc.}
\end{align*}

Their physical importance stems from the observation that, say for fermionic system, like multi electron atom or molecule, with pairwise interaction

\begin{equation}
H = \sum_i H_i + \sum_{i<j} H_{ij}
\end{equation}

the energy of state $\psi$ depends only on 2-point RDM

\begin{equation}
E = \binom{N}{2} \text{Tr} (H^{(2)} \rho^{(2)}),
\end{equation}
where \( H^{(2)} = \frac{1}{N-1} [H_1 + H_2] + H_{12} \) is reduced two particle Hamiltonian. This allows, for example, to express the energy of ground state \( E_0 \) via 2-point RDM

\[
E_0 = \left( \begin{array}{c} N \end{array} \right)_2 \min_{\rho^{(2)} = \text{RDM}} \text{Tr}(H^{(2)} \rho^{(2)}).
\]

The problem however is that it is not obvious what conditions the RDM itself should satisfy. This is what the quantum marginal problem is about. In this settings it is known from early sixties as \( N \)-representability problem (Coulson 1960, Coleman 1963). Later in mid 90-th the problem was regarded as one of ten most prominent research challenges in quantum chemistry (Stillinger et al 1995). Its solution allows to calculate nearly all properties of matter which are of interest to chemists and physicists. For current state of affairs and more history see (Coleman and Yukalov 2000, Coleman 2001).

4.2. One point \( N \)-representability

Here we outline a solution of the problem for one point reduced states. Following chemists we treat them as electron density and accordingly use normalization \( \text{Tr} \rho^{(1)} = N \) while keeping \( \text{Tr} \rho = 1 \). There are few cases where complete solution of one point \( N \)-representability was known prior 2005:

- Pauli principle: \( 0 \leq \lambda_i \leq 1, \text{ } \lambda = \text{Spec } \rho^{(1)}. \) This condition provides a criterion for mixed \( N \)-representability (Coleman 1963).
- Criterion for pure \( N \)-representability for two particles \( \wedge^2 \mathcal{H}_r \) or two holes \( \wedge^{r-2} \mathcal{H}_r \) is given by even degeneration of all eigenvalues of \( \rho^{(1)} \), except 0 (resp. 1) for odd \( r = \dim \mathcal{H}_r \) (Coleman 1963).
- For system of three fermions of dimension six \( \wedge^3 \mathcal{H}_6 \) all constraints on one point reduced matrix of a pure state are given by the following (in)equalities

\[
\lambda_1 + \lambda_6 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = 1, \text{ } \lambda_4 \leq \lambda_5 + \lambda_6,
\]

where \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5 \geq \lambda_6 \) is spectrum of \( \rho^{(1)}. \)

The last result belongs to Borland and Dennis 1972 who commented it as follows:

“\text{We have no apology for consideration of such a special case. The general } N \text{-representability problem is so difficult and yet so fundamental for many branches of science that each concrete result is useful in shedding light on the nature of general solution.}”

For more then 30 years passed after this theorem no other solution of \( N \)-representability problem has been found. Borland and Dennis derived their criterion from an extensive computer experiment, and later proved it with help provided by M.B. Ruskai and R.L. Kingsly. They also conjectured solutions for systems \( \wedge^3 \mathcal{H}_7, \wedge^4 \mathcal{H}_7, \wedge^4 \mathcal{H}_8 \), e.g. for \( \wedge^3 \mathcal{H}_7 \) one point pure representability is given by 4 inequalities

\[
\lambda_1 + \lambda_6 + \lambda_7 \geq 1, \text{ } \lambda_2 + \lambda_5 + \lambda_7 \geq 1,
\]

\[
\lambda_3 + \lambda_4 + \lambda_7 \geq 1, \text{ } \lambda_3 + \lambda_5 + \lambda_6 \geq 1,
\]

(4.2)
Theorem 4.2.1. For mixed state $\rho$ of $n$-fermion system $\wedge^n \mathcal{H}$ all constraints on spectra $\nu = \text{Spec} \rho$ and $\lambda = \text{Spec} \rho^{(1)}$ are given by inequalities

$$\sum_i a_i \lambda_{(i)} \leq \sum_j (\wedge^n a)_j \nu_{(j)}$$

(4.3)

for all “test spectra” $a : a_1 \geq a_2 \geq \cdots \geq a_r$, $\sum a_i = 0$. Here $\wedge^n a = \{a_{i_1} + a_{i_2} + \cdots + a_{i_n}\}$ consists of all sums $a_{i_1} + a_{i_2} + \cdots + a_{i_n}$, $i_1 < i_2 < \cdots < i_n$ arranged in decreasing order, and $v \in S_r, w \in S_{(r)}$ are permutations subject to topological condition $c_w^v(a) \neq 0$ to be explained below.

4.2.2 Remark. Recall that the spectra $\lambda$ and $\nu$ are arranged in decreasing order and normalized to trace $n$ and $1$ respectively. Similarly to Theorem 3.4.1 the coefficients $c_w^v(a)$ are defined via flag variety $\mathcal{F}_a(\mathcal{H}) := \{A : \mathcal{H} \to \mathcal{H} | \text{Spec}(A) = a\}$ and morphism

$$\varphi_a : \mathcal{F}_a(\mathcal{H}) \to \mathcal{F}_{\wedge^n a}(\wedge^n \mathcal{H})$$

$$A \mapsto A^{(a)}$$

where operator $A^{(a)} : \wedge^n \mathcal{H} \to \wedge^n \mathcal{H}$ acts as differential

$$A^{(a)} : x_1 \wedge x_2 \wedge \ldots \wedge x_n \mapsto \sum_i x_1 \wedge x_2 \wedge \ldots \wedge Ax_i \wedge \ldots \wedge x_n.$$

The coefficients $c_w^v(a)$ come from the induced morphism of cohomology

$$\varphi_a^* : H^*(\mathcal{F}_{\wedge^n a}(\wedge^n \mathcal{H})) \to H^*(\mathcal{F}_a(\mathcal{H}))$$

written in the basis of Schubert cocycles $\sigma_w$

$$\varphi_a^* : \sigma_w \mapsto \sum_v c_w^v(a) \sigma_v.$$

They can be calculated by equation

$$c_w^v(a) = \partial_v S_w(z) \big|_{z_k = x_{i_1} + x_{i_2} + \cdots + x_{i_n}},$$

where $a_{i_1} + a_{i_2} + \cdots + a_{i_n}$ is $k$-th term of the sequence $\wedge^n a$, cf. section 3.3.

4.2.3 Example. For system $\wedge^2 \mathcal{H}_4$ the marginal constraints on $\nu = \text{Spec} \rho$ and $\lambda = \text{Spec} \rho^{(1)}$ are given by inequalities

$$2\lambda_1 \leq \nu_1 + \nu_2 + \nu_3$$

$$2\lambda_4 \geq \nu_4 + \nu_5 + \nu_6$$

$$2(\lambda_1 - \lambda_4) \leq \nu_1 + \nu_2 - \nu_5 - \nu_6$$

$$\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 \leq \nu_1 - \nu_6$$

$$\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 \leq \min(\nu_1 - \nu_5, \nu_2 - \nu_6)$$

$$|\lambda_1 - \lambda_2 - \lambda_3 + \lambda_4| \leq \min(\nu_1 - \nu_4, \nu_2 - \nu_5, \nu_3 - \nu_6)$$

$$2\max(\lambda_1 - \lambda_3, \lambda_2 - \lambda_4) \leq \min(\nu_1 + \nu_3 - \nu_5 - \nu_6, \nu_1 + \nu_2 - \nu_4 - \nu_6)$$

$$2\max(\lambda_1 - \lambda_2, \lambda_3 - \lambda_4) \leq \min(\nu_1 + \nu_3 - \nu_4 - \nu_6, \nu_2 + \nu_3 - \nu_5 - \nu_6, \nu_1 + \nu_2 - \nu_4 - \nu_5).$$
In these inequalities we use standard normalization $\text{Tr}\rho = \text{Tr}\rho^{(1)}$. Recall that all spectra are written in decreasing order.

4.2.4 Example. Similar compatibility conditions for system $\wedge^2\mathcal{H}_5$ contain 460 independent inequalities which can’t be reproduced here.

4.2.5 Remark. In tensor algebra the reduction $\rho_{AB} \mapsto \rho_A$ is known as contraction. Most mathematicians are familiar with this procedure from differential geometry, where say Ricci curvature tensor $\text{Ric}: \mathcal{T} \to \mathcal{T}$ is defined as contraction of Riemann curvature $R: \wedge^2\mathcal{T} \to \wedge^2\mathcal{T}$ (here $\mathcal{T}$ stands for tangent bundle). This allows to interpret inequalities (4.4) as relations between spectra of Riemann and Ricci curvatures of four-manifold. Recall that in general relativity Ricci curvature is governed by energy-momentum tensor, i.e. by physical content of the space, while Riemann curvature is responsible for its geometry and topology. The above constraints impose some bounds on influence of matter on geometry.

4.3. Pure $N$-representability in dimension $\leq 8$

Here I’ll give an account of our joint work with Murat Altunbudak. The details will be published elsewhere.

Formally solution of pure marginal problem can be deduced from inequalities (4.3) of Theorem 4.2.1 by putting $\nu_i = 0$ for $i \neq 1$. However for a system like $\wedge^4\mathcal{H}_8$ we confront with an immense symmetric group of degree $\left(\begin{array}{c}8 \\
4\end{array}\right) = 70$. A representation theoretical interpretation of $N$-representability discussed below allows to circumvent this difficulty.

Let’s start with decomposition of a symmetric power of $\wedge^n\mathcal{H}$, called plethysm, into irreducible components

$$S^m(\wedge^n\mathcal{H}) = \sum_{\lambda} m_{\lambda}\mathcal{H}_{\lambda}$$

of the unitary group $U(\mathcal{H})$. The components $\mathcal{H}_{\lambda}$, entering into the decomposition with some multiplicities $m_{\lambda} \geq 0$, can be parameterized by Young diagrams

$$\lambda: \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0$$

of size $|\lambda| = \sum_{i} \lambda_i = n \cdot m$ that fit into $r \times m$ rectangular, $r = \dim \mathcal{H}$. It is instructive to treat the diagrams as spectra. We are interested in asymptotic of these spectra as $m \to \infty$ and therefor normalize them to a fixed size $\bar{\lambda} = \lambda/m$, $\text{Tr}\bar{\lambda} = n$.

**Theorem 4.3.1.** Every $\bar{\lambda}$ obtained from irreducible component $\mathcal{H}_{\lambda} \subset S^d(\wedge^n\mathcal{H})$ is spectrum one point reduced matrix $\rho^{(1)}$ of a pure state $\psi \in \wedge^n\mathcal{H}$. Moreover every one point reduced spectrum is a convex combination of the spectra $\lambda$ with bounded $m \leq M$.

A similar result holds in standard settings of the quantum marginal problem (Franz 2002, Christandl and Mitchison 2004, Klyachko 2004, Christandl et al 2005).

Note that representation $S^m(\wedge^r-n\mathcal{H})$ is dual to $S^m(\wedge^n\mathcal{H})$ and hence

$$S^m(\wedge^r-n\mathcal{H}) = \sum_{\lambda} m_{\lambda}\mathcal{H}_{\lambda^*},$$

(4.6)
where $\lambda^*$ is complement of the diagram $\lambda$ to the rectangle $r \times m$ and the multiplicity $m_\lambda$ is the same as in (4.3). Thus we arrived at the following particle-hole duality.

**Corollary 4.3.2.** Marginal constraints on spectrum of reduced matrix of a pure state for system $\wedge^{r-n} H_r$ can be obtained from that of the system $\wedge^n H_r$ by substitution $\lambda_i \mapsto 1 - \lambda_{r+1-i}$.

4.3.3 Example. There are few cases where decomposition (4.5) is explicitly known, for example

$$S^m(\wedge^2 H_r) = \sum_{|\lambda|=2m, \lambda=\text{even}} H_\lambda,$$

where the sum is extended over diagrams $\lambda \subset r \times m$ with even multiplicity of every nonzero row (Macdonald 1995). Together with theorem 4.3.1 and the particle-hole duality this implies Coleman’s criteria of pure N-representability for systems of two particles $\wedge^2 H_r$ and two holes $\wedge^{r-2} H_r$ mentioned at the beginning of section 4.2.

4.3.4 Example. Borland-Dennis equations (4.1) imply that every component $H_\lambda \subset S^m(\wedge^3 H_6)$ is selfdual $\lambda = \lambda^*$. It seems mathematicians missed this fact, which holds only for this specific system. Note that wedge product ensure selfduality of $\wedge^3 H_6$ and hence of the plethysm $S^m(\wedge^3 H_6)$. However apparently there is no simple way to extend this to every component $H_\lambda \subset S^m(\wedge^3 H_6)$.

Theorem 4.3.1 for any fixed $M$ gives an inner approximation to the set of all possible reduced spectra, while any set of inequalities (4.3) of theorem 4.2.1 amounts to its outer approximation. This suggests the following approach to pure N-representability problem, which combines both theorems.

- Find all irreducible components $H_\lambda \subset S^m(\wedge^n H)$ for $m \leq M$ starting with $M = 1$.
- Calculate convex hull of the corresponding reduced spectra $\tilde{\lambda}$.
- Check whether or not all inequality defining facets of the convex hull fit into the form (4.3) of Theorem 4.2.1
- If they do then all inequalities are found. Otherwise increase $M \mapsto M + 1$.

4.3.5 Remark. The success of this approach depends on the degrees of generators of the module of covariants of the system $\wedge^n H_r$. Generically the degrees are expected to be huge as well as the whole number of the resulting inequalities. However for systems of rank $r \leq 8$ and for $r = 9$, $n \neq 4, 5$ the module of covariants is free (Vinberg and Popov 1992) and the degrees of the generators should be reasonably small.

Indeed an inexpensive PC, assisted with some dirty tricks, managed to resolve N-representability problem for rank $r \leq 8$. Recall that for two fermions or two holes the answer is known, see section 4.2. Together with particle-hole duality this bound us to the range $3 \leq n \leq r/2$. The corresponding constraints are listed below. They are grouped by the extremal edges and use chemical normalization $\sum_i \lambda_i = n$.

- $\wedge^3 H_6$.

$$\lambda_1 + \lambda_6 = \lambda_2 + \lambda_5 = \lambda_3 + \lambda_4 = 1, \quad \lambda_4 \leq \lambda_5 + \lambda_6$$
Quantum marginal problem

- $\Lambda^3 \mathcal{H}_7$.

- $-4\lambda_1 + 3\lambda_2 + 3\lambda_3 + 3\lambda_4 + 3\lambda_5 - 4\lambda_6 - 4\lambda_7 \leq 2$
- $3\lambda_1 - 4\lambda_2 + 3\lambda_3 + 3\lambda_4 - 4\lambda_5 + 3\lambda_6 - 4\lambda_7 \leq 2$
- $3\lambda_1 + 3\lambda_2 - 4\lambda_3 - 4\lambda_4 - 3\lambda_5 + 3\lambda_6 - 4\lambda_7 \leq 2$
- $3\lambda_1 + 3\lambda_2 - 4\lambda_3 + 3\lambda_4 - 4\lambda_5 - 4\lambda_6 + 3\lambda_7 \leq 2$

- $\Lambda^3 \mathcal{H}_8$.

- $3\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 + 3\lambda_8 \leq 1$
- $-\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 \leq 1$
- $\lambda_1 + \lambda_2 - \lambda_3 - \lambda_4 + \lambda_5 + \lambda_6 - \lambda_7 - \lambda_8 \leq 1$
- $\lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 + \lambda_7 - \lambda_8 \leq 1$
- $\lambda_1 - \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 + \lambda_6 - \lambda_7 - \lambda_8 \leq 1$
- $2\lambda_1 + \lambda_2 - 2\lambda_3 - \lambda_4 - \lambda_6 + \lambda_8 \leq 1$
- $2\lambda_1 - \lambda_2 - \lambda_4 + \lambda_6 - 2\lambda_7 + \lambda_8 \leq 1$
- $\lambda_3 + 2\lambda_4 - 2\lambda_5 - \lambda_6 - \lambda_7 + \lambda_8 \leq 1$
- $\lambda_1 + 2\lambda_2 - 2\lambda_3 - \lambda_5 - \lambda_6 + \lambda_8 \leq 1$
- $2\lambda_1 - \lambda_2 + \lambda_4 - 2\lambda_5 - \lambda_6 + \lambda_8 \leq 1$

- $5\lambda_1 + 5\lambda_2 - 7\lambda_3 - 3\lambda_4 - 3\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8 \leq 3$
- $5\lambda_1 - 3\lambda_2 - 3\lambda_3 + \lambda_4 + \lambda_5 + 5\lambda_6 - 7\lambda_7 + \lambda_8 \leq 3$
- $5\lambda_1 + \lambda_2 - 3\lambda_3 + \lambda_4 - 3\lambda_5 + \lambda_6 - 3\lambda_7 + \lambda_8 \leq 3$
- $\lambda_1 + \lambda_2 + \lambda_3 + 5\lambda_4 - 3\lambda_5 - 3\lambda_6 - 3\lambda_7 + \lambda_8 \leq 3$
- $\lambda_1 + 5\lambda_2 - 3\lambda_3 + \lambda_4 + \lambda_5 - 3\lambda_6 - 3\lambda_7 + \lambda_8 \leq 3$
- $9\lambda_1 + \lambda_2 - 7\lambda_3 - 7\lambda_4 - 7\lambda_5 + \lambda_6 + \lambda_7 + 9\lambda_8 \leq 3$
- $9\lambda_1 - 7\lambda_2 - 7\lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - 7\lambda_7 + 9\lambda_8 \leq 3$

- $7\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 + 7\lambda_6 - 9\lambda_7 - \lambda_8 \leq 5$
- $7\lambda_1 - \lambda_2 - \lambda_3 + 7\lambda_4 - 9\lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 \leq 5$
- $7\lambda_1 + 7\lambda_2 - 9\lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 \leq 5$
- $-\lambda_1 - \lambda_2 + 7\lambda_3 + 7\lambda_4 - \lambda_5 - \lambda_6 - 9\lambda_7 - \lambda_8 \leq 5$
- $-\lambda_1 + 7\lambda_2 - \lambda_3 + 7\lambda_4 - \lambda_5 - 9\lambda_6 - \lambda_7 - \lambda_8 \leq 5$
- $-\lambda_1 + 7\lambda_2 - \lambda_3 - \lambda_4 + 7\lambda_5 - \lambda_6 - 9\lambda_7 - \lambda_8 \leq 5$
\[ -3\lambda_1 + 5\lambda_2 + 5\lambda_3 + 13\lambda_4 - 11\lambda_5 - 3\lambda_6 - 11\lambda_7 + 5\lambda_8 \leq 7 \]
\[ 5\lambda_1 + 13\lambda_2 - 11\lambda_3 + 5\lambda_4 - 11\lambda_5 - 3\lambda_6 - 3\lambda_7 + 5\lambda_8 \leq 7 \]
\[ 5\lambda_1 - 3\lambda_2 + 5\lambda_3 + 13\lambda_4 - 11\lambda_5 - 11\lambda_6 - 3\lambda_7 + 5\lambda_8 \leq 7 \]
\[ 5\lambda_1 + 13\lambda_2 - 11\lambda_3 - 3\lambda_4 + 5\lambda_5 - 11\lambda_6 - 3\lambda_7 + 5\lambda_8 \leq 7 \]
\[ 19\lambda_1 + 11\lambda_2 - 21\lambda_3 - 13\lambda_4 - 5\lambda_5 - 5\lambda_6 + 3\lambda_7 + 11\lambda_8 \leq 9 \]
\[ 19\lambda_1 - 13\lambda_2 - 5\lambda_3 - 5\lambda_4 + 3\lambda_5 + 11\lambda_6 - 21\lambda_7 + 11\lambda_8 \leq 9 \]
\[ 11\lambda_1 + 19\lambda_2 - 21\lambda_3 - 5\lambda_4 - 13\lambda_5 - 5\lambda_6 + 3\lambda_7 + 11\lambda_8 \leq 9 \]
\[ -5\lambda_1 + 3\lambda_2 + 11\lambda_3 + 19\lambda_4 - 21\lambda_5 - 13\lambda_6 - 5\lambda_7 + 11\lambda_8 \leq 9 \]

• $\wedge^4 \mathcal{H}_8$.

\[ 5\lambda_1 + \lambda_2 + \lambda_3 - 3\lambda_4 + \lambda_5 - 3\lambda_6 - 3\lambda_7 + \lambda_8 \leq 4 \]
\[ \lambda_1 + \lambda_2 + 5\lambda_3 - 3\lambda_4 + \lambda_5 + \lambda_6 - 3\lambda_7 - 3\lambda_8 \leq 4 \]
\[ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 5\lambda_5 - 3\lambda_6 - 3\lambda_7 - 3\lambda_8 \leq 4 \]
\[ \lambda_1 + 5\lambda_2 + \lambda_3 - 3\lambda_4 + \lambda_5 - 3\lambda_6 + \lambda_7 - 3\lambda_8 \leq 4 \]
\[ 5\lambda_1 - 3\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - 3\lambda_7 - 3\lambda_8 \leq 4 \]
\[ 5\lambda_1 + \lambda_2 + \lambda_3 - 3\lambda_4 - 3\lambda_5 + \lambda_6 + \lambda_7 - 3\lambda_8 \leq 4 \]
\[ 5\lambda_1 + \lambda_2 - 3\lambda_3 + \lambda_4 + \lambda_5 - 3\lambda_6 + \lambda_7 - 3\lambda_8 \leq 4 \]
\[ -\lambda_1 + 3\lambda_2 + 3\lambda_3 - \lambda_4 + 3\lambda_5 - \lambda_6 - \lambda_7 - 5\lambda_8 \leq 4 \]
\[ 3\lambda_1 + 3\lambda_2 - \lambda_3 - \lambda_4 + 3\lambda_5 - 5\lambda_6 - \lambda_7 - \lambda_8 \leq 4 \]
\[ 3\lambda_1 + 3\lambda_2 + 3\lambda_3 - 5\lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 \leq 4 \]
\[ 3\lambda_1 - \lambda_2 + 3\lambda_3 - \lambda_4 + 3\lambda_5 - \lambda_6 - 5\lambda_7 - \lambda_8 \leq 4 \]
\[ 3\lambda_1 + 3\lambda_2 - \lambda_3 - \lambda_4 - \lambda_5 - \lambda_6 + 3\lambda_7 - 5\lambda_8 \leq 4 \]
\[ 3\lambda_1 - \lambda_2 - \lambda_3 + 3\lambda_4 + 3\lambda_5 - \lambda_6 - \lambda_7 - 5\lambda_8 \leq 4 \]
\[ 3\lambda_1 - \lambda_2 + 3\lambda_3 - \lambda_4 - \lambda_5 + 3\lambda_6 - \lambda_7 - 5\lambda_8 \leq 4 \]

4.3.6 Remark. The marginal inequalities are independent and written in the form \[ (\text{1.6}) \] of theorem \[ (\text{1.2.1}) \] Using the normalization equation $\text{Tr} \rho = n$ they can be transformed in many different ways. For example, the above constraints for system $\wedge^3 \mathcal{H}_7$ are equivalent to inequalities \[ (\text{1.2}) \]. The inequalities for $\wedge^4 \mathcal{H}_8$ can be recast into a nice form found experimentally by Borland and Dennis

\[ |x_1| + |x_2| + |x_3| + |x_4| + |x_5| + |x_6| + |x_7| \leq 4, \]

where

\[ x_1 = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 - \lambda_5 - \lambda_6 - \lambda_7 - \lambda_8 \]
\[ x_2 = \lambda_1 + \lambda_2 + \lambda_5 + \lambda_6 - \lambda_3 - \lambda_4 - \lambda_7 - \lambda_8 \]
\[ x_3 = \lambda_1 + \lambda_3 + \lambda_5 + \lambda_7 - \lambda_2 - \lambda_4 - \lambda_6 - \lambda_8 \]
\[ x_4 = \lambda_1 + \lambda_4 + \lambda_6 + \lambda_7 - \lambda_2 - \lambda_3 - \lambda_5 - \lambda_8 \]
$x_5 = \lambda_2 + \lambda_3 + \lambda_6 + \lambda_7 - \lambda_1 - \lambda_4 - \lambda_5 - \lambda_8$

$x_6 = \lambda_2 + \lambda_4 + \lambda_5 + \lambda_7 - \lambda_1 - \lambda_3 - \lambda_6 - \lambda_8$

$x_7 = \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6 - \lambda_1 - \lambda_2 - \lambda_7 - \lambda_8$

Borland and Dennis numerical data were inconclusive for the system $\lambda^3 H_8$ described by 31 inequalities. One may wonder whether they can be written in a nice symmetric form.

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