ON STRUCTURE THEOREMS AND NON-SATURATED EXAMPLES

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ABSTRACT. For any minimal system \((X, T)\) and \(d \geq 1\) there is an associated minimal system \((N_d(X), \mathcal{G}_d(T))\), where \(\mathcal{G}_d(T)\) is the group generated by \(T \times \cdots \times T\) and \(T \times T^2 \times \cdots \times T^d\) and \(N_d(X)\) is the orbit closure of the diagonal under \(\mathcal{G}_d(T)\). It is known that the maximal \(d\)-step pro-nilfactor of \(N_d(X)\) is \(N_d(X_d)\), where \(X_d\) is the maximal \(d\)-step pro-nilfactor of \(X\).

In this paper, we further study the structure of \(N_d(X)\). We show that the maximal distal factor of \(N_d(X)\) is \(N_d(X_{dis})\) with \(X_{dis}\) being the maximal distal factor of \(X\), and prove that as minimal systems \((N_d(X), \mathcal{G}_d(T))\) has the same structure theorem as \((X, T)\). In addition, a non-saturated metric example \((X, T)\) is constructed, which is not \(T \times T^2\)-saturated and is a Toeplitz minimal system.

1. INTRODUCTION

In the 1970’s, Furstenberg gave a dynamical proof of the Szemerédi’s theorem via establishing a version of multiple ergodic recurrence theorem (MERT for short), which states that in any measure preserving system \((X, \mathcal{X}, \mu, T)\), for any \(d \in \mathbb{N}\) and \(A \in \mathcal{X}\) with \(\mu(A) > 0\) there always exists \(n \in \mathbb{N}\) such that \(\mu(A \cap T^{-n} A \cap \cdots \cap T^{-dn} A) > 0\). A counterpart in topological dynamics is the topological multiple recurrence theorem (TMRT for short), which states that for any nonempty open set \(A\) in a minimal topological dynamical system \((X, T)\) and \(d \in \mathbb{N}\), there always exists \(n \in \mathbb{N}\) such that \(A \cap T^{-n} A \cap \cdots \cap T^{-dn} A \neq \emptyset\). An easy implication of TMRT is the classical van der Waerden’s theorem on arithmetic progressions.

The multiple recurrence theorems both in ergodic theory and topological dynamics are focusing on studying the orbit of \((x, x, \cdots, x)\) under \(\tau_d(T) = T \times \cdots \times T^d\) in the product space \(X \times \cdots \times X\). In ergodic theory, this can be captured by the limits of the following multiple ergodic averages

\[
\frac{1}{N} \sum_{k=0}^{N-1} f_1(T^k x) f_2(T^{2k} x) \cdots f_d(T^{dk} x),
\]

the convergence in \(L^2(\mu)\) of which has been finally solved in [14, 26] after nearly 30 years’ efforts by many mathematicians. In order to study the averages in (1.1), the characteristic factors were introduced to capture the limits of (1.1), which were finally shown to be an efficient way.

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In [11], Glasner established counterparts in topological dynamics which are called topological characteristic factors. Given a factor map \( \pi : (X, T) \to (Y, T) \) between minimal topological dynamical systems and \( d \geq 2 \), \( (Y, T) \) is said to be a \( d \)-step topological characteristic factor of \( (X, T) \) if there is a dense \( G_δ \) subset \( \Omega \) of \( X \) such that for any \( x \in \Omega \), the orbit closure \( L_x = \overline{\{ (x, \cdots, x), \tau_d(T) \}} = \pi^{(d)} := \pi \times \cdots \times \pi \)-saturated, which means that \( (\pi^{(d)})^{-1}(\pi^{(d)} L_x) = L_x \). Furthermore, a closely related system \( N_d(X) \) under the action of the group \( \mathcal{G}_d(T) := \langle \sigma_d(T), \tau_d(T) \rangle \) generated by \( \sigma_d(T) := T \times \cdots \times T \) (\( d \)-times) and \( \tau_d(T) \) was also studied in [11], where \( N_d(X) \) is the orbit closure of \( (x, \cdots, x) \) under \( \mathcal{G}_d(T) \). It was shown in [11] that \( (N_d(X), \mathcal{G}_d(T)) \) is also minimal for a minimal topological dynamical system \( (X, T) \).

In [12], the authors further studied the topological characteristic factors and the dynamics of \( (N_d(X), \mathcal{G}_d(T)) \). Up to an almost 1-1 modification, they showed that the topological characteristic factors are the pro-nilfactors (see [12, Theorem A]), which are the analogies in the ergodic situation. For other study on topological characteristic factors, see [4, 5].

Another interesting and useful result they showed on the dynamics of \( (N_d(X), \mathcal{G}_d(T)) \) is the following theorem [12, Theorem C].

**Theorem A:** Let \( (X, T) \) be minimal and \( n \in \mathbb{N} \). Then \( (X, T^n) \) is minimal if and only if \( N_d(T^n) = N_d(T) \) for any \( d \in \mathbb{N} \).

By this theorem, they showed that if \( (X, T^2) \) is minimal, then TMRT holds along odd numbers (actually they showed more, see [12, Theorem D]) and for a totally minimal system \( (X, T) \) and an integral quadratic polynomial \( p(n) = an^2 + bn + c \) with \( a \neq 0 \), there is a dense \( G_δ \) subset \( \Omega \) of \( X \) such that for any \( x \in \Omega \), \( \{ T^{p(n)} x : n \in \mathbb{Z} \} \) is dense in \( X \) (see [12, Theorem E]).

According to their powerful structure theorem of topological characteristic factors, they determined the maximal equicontinuous factors of \( N_d(X) \) ([12, Theorem B]), which says that the maximal equicontinuous factor of \( N_d(X) \) is the very \( N_d(X_{eq}) \), where \( X_{eq} \) is the maximal equicontinuous factor of \( X \). This is also crucial in the proof of Theorem A. Note that recently, Lian and Qiu [17] showed that for any \( k \in \mathbb{N} \), the maximal \( k \)-step nilfactor of \( N_d(X) \) is \( N_d(X_k) \), where \( X_k \) is the maximal \( k \)-step pro-nilfactor of \( X \).

In this paper, we will give another approach to Theorem A. That is, starting from the fact that the maximal equicontinuous factor of \( N_d(X) \) is \( N_d(X_{eq}) \), we use an equivalence condition for \( N_d(T) = N_d(T^n) \) (Theorem 3.2) to get the conclusion, instead of converting to the equicontinuous cases as did in [12].

Based on the results obtained in [12, 17] one naturally expect that the maximal distal factor of \( N_d(X) \) should be \( N_d(X_{dis}) \), where \( X_{dis} \) is the maximal distal factor of \( X \). We show that indeed it is the case.

**Theorem B:** Let \( (X, T) \) be a minimal topological dynamical system with \( X_{dis} \) being its maximal distal factor. Then for any \( d \geq 2 \), the maximal distal factor of \( N_d(X) \) is \( N_d(X_{dis}) \).

A powerful tool in studies of minimal systems is the structure theorem. In 1963, Furstenberg in [10] established the structure theorem for minimal distal systems. Then the structure theorem for pointed distal minimal systems was established by Veech in [22]. Finally, Ellis, Glasner and Shapiro in [8], McMahon [18], Veech [23] gave the structure
theorem for general minimal systems. In this paper, we will show the following result for $N_d(X)$ as a minimal system associated to $(X, T)$.

**Theorem C:** Let $(X, T)$ be a minimal topological dynamical system and $d \in \mathbb{N}$. Then $(N_d(X), \mathcal{G}_d(T))$ has the same structure theorem as $(X, T)$. In particular, if $(X, T)$ is distal (resp. PI, HPI), then so is $(N_d(X), \mathcal{G}_d(T))$.

The key ingredient to show the minimality of $N_d$ under the $\mathcal{G}_d$ action is that if $(x_1, \ldots, x_d)$ is minimal for $\sigma_d$ then it is minimal for $\mathcal{G}_d$. To show Theorem C, we need a generalization of this fact. Namely, let $n, d \in \mathbb{N}$. For $(x_1^1, \ldots, x_n^i) \in N_d(X)$, if $((x_1^1)^d_1, \ldots, (x_n^i)^d_1)$ is a minimal point in $(X^n, T)$ under the diagonal action, then $((x_1^1)^d_1, \ldots, (x_n^i)^d_1)$ is a minimal point in $((N_d(X))^n, \mathcal{G}_d(T))$ under the diagonal action, see Lemma 5.8.

As pointed in [12], the almost 1-1 modification in their structure theorem for topological characteristic factors is necessary due to an example given in [11], which describes an almost automorphic system whose maximal equicontinuous factor is not its characteristic factor. Unfortunately, the example given in [11] is not a metrizable system. Here we will construct a metric minimal system which is an almost 1-1 extension of its maximal equicontinuous factor such that the factor map is not saturated. We state it as follows.

**Theorem D:** There exists a minimal topological dynamical system $(X, T)$ such that

1. $\pi : (X, T) \to (X_{eq}, T)$ is an almost 1-1 extension, where $X_{eq}$ is its maximal equicontinuous factor;
2. $(X_{eq}, T)$ is not a topological characteristic factor of $(X, T)$, i.e. there does not exist a dense $G_δ$ subset $Ω$ of $X$ such that for any $x \in Ω$, $L_x = \overline{O}((x, x), T × T²)$ is $π × π$-saturated.

Actually, the system we construct is an irregular Toeplitz system.

**Organization of the paper.** In Section 2, we give some preliminaries. In Section 3, we give a proof of Theorem A. We prove Theorems B and C in Sections 4 and 5 respectively. In Section 6 we present some saturated examples and finally we prove Theorem D in Section 7 via constructing a desired Toeplitz sequence.

2. **Preliminaries**

In this section we provide notions and lemmas needed for our proofs.

2.1. **General topological dynamical systems.** By a topological dynamical system (t.d.s. or system for short) we mean a pair $(X, T)$ with $X$ being a compact Hausdorff space and $T$ being a homeomorphism of $X$. More generally, by a topological dynamical system we mean a triple $(X, G, φ)$ with a compact metric space $X$, a topological group $G$ and a homeomorphism $φ : G \to \text{Homeo}(X)$, where $\text{Homeo}(X)$ denotes the group of homeomorphisms of $X$. For brevity, we usually use $(X, G)$ to denote $(X, G, φ)$ and use $gx$ or $g(x)$ instead of $(φ(g))(x)$ for $g \in G$ and $x \in X$. Note that the system $(X, T)$ corresponds to the case of $G$ being $\mathbb{Z}$.

In this paper, we always assume that $X$ is a compact metric space with a metric $ρ$. Let $(X, G)$ be a t.d.s. For $x \in X$, the orbit of $x$ under the action of $G$ is defined to be $\{gx : g \in G\}$, which is denoted by $Gx$ or $φ(x, G)$ (or $φ(x, T)$ in case of $G = \mathbb{Z}$). Correspondingly,
\overline{G\mathcal{X}} \text{ or } \overline{O}(x, G) \text{ will denote the orbit closure of } x. \text{ Similarly, for a subset } A \subset X, \text{ the orbit of } A \text{ is } \{gx : x \in A, g \in G\}, \text{ which is denoted by } G(A) \text{ or } \mathcal{O}(A, G).

A subset \( A \subset X \) is called invariant under the action of \( G \) or \( G \)-invariant if \( G(A) = A \). A t.d.s. \((X, G)\) is said to be minimal if there is no nonempty proper \( G \)-invariant closed subset. Note that \((X, G)\) is minimal if and only if the orbit closure of every point is dense in \( X \). If there is a point of \( X \) whose orbit is dense in \( X \) then we say that the system is topologically transitive. Furthermore, if the product system \((X \times X, T)\) under the diagonal action (i.e. \( T(x, y) = (Tx, Ty) \)) is topologically transitive, then we say that \((X, T)\) is weakly mixing.

A factor map between the t.d.s. \((X, G)\) and \((Y, G)\) is a continuous surjective map \( \pi : X \to Y \) which intertwines the actions, i.e., \( \pi(gx) = g\pi(x) \) for any \( g \in G \) and \( x \in X \). In this case, we say that \((Y, G)\) is a factor of \((X, G)\) or \((X, G)\) is an extension of \((Y, G)\). Let \( \pi : (X, T) \to (Y, T) \) be a factor map. Then \( R_\pi = \{(x, y) \in X \times X : \pi(x) = \pi(y)\} \) is a closed \( G \)-invariant equivalence relation and \( Y \cong X/R_\pi \).

Let \( X \) be a set and \( d \in \mathbb{N} \) with \( d \geq 2 \). The diagonal of \( X^d \) is defined to be \( \Delta_d(X) = \{(x, x, \cdots, x) : x \in X\} \).

We use \( \Delta(X) \) to denote \( \Delta_2(X) \) for short. For \( x \in X \), let \( x^{(d)} \) denote \( (x, x, \cdots, x) \) \((d\text{-times})\) for short.

### 2.2. Distal, proximal and regionally proximal relations.

Let \((X, G)\) be a t.d.s. Given \((x, y) \in X^2\), it is a proximal pair if \( \liminf_{g \in G} \rho(gx, gy) = 0 \); it is a distal pair if it is not proximal. Denote by \( P(X, G) \) the set of proximal pairs of \((X, G)\). It is also called the proximal relation. Note that the proximal relation is \( G \)-invariant but not an equivalence relation in general. However, when \( P(X, G) \) is closed in \( X \times X \), it is an equivalence relation (see [2, Chapter 6, Corollary 11]). \((x, y) \in X^2\) is a regionally proximal pair if there exist sequences \((x_n), (y_n)\) of \( X \) with \( x_n \to x, y_n \to y \) and a sequence \((g_n)\) of \( G \) such that \( \rho(g_nx_n, g_ny_n) \to 0 \). The set of regionally proximal pair of \((X, G)\) is denoted by \( \text{RP}(X, G) \), which is a closed invariant relation but not an equivalence relation in general. However, when \( G \) preserves an invariant probability measure \( \text{RP}(X, G) \) is an equivalence relation (see [2, Chapter 9]).

A t.d.s. \((X, G)\) is distal if \( P(X, G) = \Delta(X) \), and is equicontinuous if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \rho(x, y) < \delta \) implies \( \rho(gx, gy) < \varepsilon \) for every \( g \in G \). Any equicontinuous system is distal. It is well known that every t.d.s. has a maximal equicontinuous factor, and a maximal distal factor respectively. Let \( S_{\text{eq}} \) (resp. \( S_{\text{dis}} \)) be the smallest closed invariant equivalence relation on \( X \) for which the factor \( X/S_{\text{eq}} \) (resp. \( X/S_{\text{dis}} \)) is an equicontinuous (resp. distal) system. It was shown in [7] that \( S_{\text{eq}} \) (resp. \( S_{\text{dis}} \)) is the smallest closed invariant equivalence relation containing the regionally proximal relation (resp. proximal relation).

In [3], Auslander and Glasner defined the capturing operation, a kind of reverse orbit closure, to characterize the equicontinuous and distal relations in minimal systems. If \( A \subset X \), the capturing set of \( A \) is defined by 
\[
C(A) = \{x \in X : \overline{O}(x, G) \cap A \neq \emptyset\}.
\]
Lemma 2.4. An equicontinuous extension.

Suppose that each product system is also proximal (resp. equicontinuous, distal).

Theorem 2.1. Let $(X, G)$ be a minimal system. Then $S_{dis}(X) = C(\partial(P))$.

Lemma 2.2. [3, Theorem 2.1] Let $\pi : (X, T) \to (Y, T)$ be an extension between minimal systems. Then $\pi(S_{dis}(X)) = S_{dis}(Y)$.

An extension $\pi : (X, G) \to (Y, G)$ is called proximal if $R_\pi \subset P(X)$; $\pi$ is distal if $P(X) \cap R_\pi = \Delta(X)$; $\pi$ is equicontinuous if for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $(x_1, x_2) \in R_\pi$ with $\rho(x_1, x_2) < \delta$, we have $\sup_{y \in P} \rho(gx_1, gx_2) < \varepsilon$. An extension $\pi : (X, G) \to (Y, G)$ between minimal systems is called almost $1 - 1$ if the set $\{y \in Y : |\pi^{-1}(y)| = 1\}$ is a dense $G_\delta$ subset of $Y$, which is equivalent to the existence of a point in $Y$ whose fibre is a singleton.

Lemma 2.3. If $\pi : (X, G) \to (Y, G)$ is a proximal (resp. equicontinuous, distal) extension and $H$ is a subgroup of $G$ of finite index, then the induced extension $\pi_H : (X, H) \to (Y, H)$ is also proximal (resp. equicontinuous, distal).

Proof. Suppose $x_1, x_2 \in \pi^{-1}(y)$ for some $y \in Y$. Let $G = g_1 H \cup \cdots \cup g_n H$ for some $g_1, \cdots, g_n \in G$.

If $\inf_{g \in G} \rho(g x_1, g y) > 0$, then $\inf_{h \in H} \rho(h x_1, h y) > 0$. Thus if $\pi : (X, G) \to (Y, G)$ is a distal extension then so is $\pi_H : (X, H) \to (Y, H)$.

If $\inf_{h \in H} \rho(h x_1, h y) > 0$, then $\inf_{h \in H} \rho(g_i h x_1, g_i h y) > 0$ for any $i \in \{1, \cdots, n\}$. Thus $\inf_{g \in G} \rho(g x_1, g y) > 0$. Therefore, if $\pi : (X, G) \to (Y, G)$ is a proximal extension then so is $\pi_H : (X, H) \to (Y, H)$.

If $\pi : (X, G) \to (Y, G)$ is an equicontinuous extension, then for any $\varepsilon > 0$, there is some $\delta > 0$ such that for any $(x_1, x_2) \in R_\pi$ with $\rho(x_1, x_2) \leq \delta$, we have $\sup_{g \in G} \rho(g x_1, g x_2) < \varepsilon$. Then it follows trivially that $\sup_{h \in H} \rho(h x_1, h x_2) < \varepsilon$. Thus $\pi : (X, H) \to (Y, H)$ is also an equicontinuous extension.

Lemma 2.4. If $\pi_i : (X_i, G) \to (Y_i, G)$ is a proximal (resp. equicontinuous, distal) extension for each $i \in \{1, \cdots, n\}$, then the extension $\prod_{i=1}^n \pi_i : (\prod_{i=1}^n X_i, G) \to (\prod_{i=1}^n Y_i, G)$ of the product system is also proximal (resp. equicontinuous, distal).

Proof. Suppose that each $\pi_i$ is a proximal extension. We show that $\prod_{i=1}^n \pi_i$ is also a proximal extension just for $n = 2$. The general case follows by induction on $n$.

Let $(x_1, x_2), (x'_1, x'_2) \in R_{\pi_1 \times \pi_2}$. Then there is a sequence $(g_n)$ in $G$ such that $g_n x_1 \to z_1$ and $g_n x'_1 \to z'_1$ for some $z_1, z'_1 \in X_1$. By taking some subsequence, we may assume that $g_n x_2 \to z_2$ and $g_n x'_2 \to z'_2$ for some $z_2, z'_2 \in X_2$. Then $(z_2, z'_2) \in R_{\pi_2}$. So there is some sequence $(h_n)$ in $G$ such that $h_n z_2 \to z_3$ and $h_n z'_2 \to z'_3$ for some $z_3, z'_3 \in X_2$. We may further assume that $h_n z_1 \to z_0$ for some $z_0 \in X_1$. For each $k \geq 1$, there is some $n_k \geq 1$ such that for any $n \geq n_k$,

$$
\rho(h_k g_n x_1, h_k z_1), \rho(h_k g_n x'_1, h_k z'_1), \rho(h_k g_n x_2, h_k z_2), \rho(h_k g_n x'_2, h_k z'_2) < \frac{1}{k}.
$$
Then \( h_k g_n (x_1, x_2) \to (z_0, z_3) \) and \( h_k g_n (x'_1, x'_2) \to (z_0, z_3) \). Hence \( ((x_1, x_2), (x'_1, x'_2)) \in \mathcal{P}(X_1 \times X_2) \) and it follows that \( \pi_1 \times \pi_2 \) is also proximal.

It is trivial by definitions for equicontinuous and distal extensions. \( \square \)

### 2.3. Regionally proximal relation of higher order and Nil-factors.

For a t.d.s. \((X, T)\), Host, Kra and Maass [15] introduced the following definition. If \( n = (n_1, \ldots, n_d) \in \mathbb{Z}^d \) and \( \varepsilon \in \{0, 1\}^d \), we define

\[
\varepsilon \cdot n = \sum_{i=1}^{d} n_i \varepsilon_i.
\]

**Definition 2.5.** Let \((X, T)\) be a t.d.s. and let \( d \in \mathbb{N} \). The points \( x, y \in X \) are said to be **regionally proximal of order** \( d \) if for any \( \delta > 0 \), there exist \( x', y' \in X \) and a vector \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_d) \in \{0, 1\}^d \) such that \( \rho(x, x') < \delta, \rho(y, y') < \delta \), and

\[
\rho(T^n \varepsilon x', T^n \varepsilon y') < \delta
\]

for any \( \varepsilon \in \{0, 1\}^d \setminus \{0\} \), where \( 0 = (0, 0, \ldots, 0) \in \{0, 1\}^d \). The set of regionally proximal pairs of order \( d \) is denoted by \( \text{RP}^d \) (or by \( \text{RP}^d(X, T) \) in case of ambiguity), and is called the **regionally proximal relation of order** \( d \).

Similarly we can define \( \text{RP}^d(X, G) \) for a system \((X, G)\) with \( G \) being abelian. We note that \( \text{RP}^1 = \text{RP} \). It is easy to see that \( \text{RP}^d \) is a closed and invariant relation. Observe that

\[
\mathcal{P}(X) \subset \ldots \subset \mathcal{R}^d \subset \mathcal{R}^{d+1} \subset \ldots \subset \mathcal{R}^2 \subset \mathcal{R}^1 = \text{RP}.
\]

Host, Kra and Maass [15] showed that if a system is minimal and distal then \( \text{RP}^d \) is an equivalence relation, and a very deep result stating that \((X/\text{RP}^d, T)\) is the maximal \( d \)-step pro-nilfactor of the system. Shao and Ye [20] showed that all these results in fact hold for arbitrarily minimal systems of \( \mathbb{Z} \)-actions, and for abelian group actions, \( \text{RP}^d \) is an equivalence relation. See Glasner, Gutman and Ye [13] for similar results regarding general group actions.

It follows that for any minimal system \((X, T)\), \( \text{RP}^{\infty} := \cap_{d=1}^{\infty} \text{RP}^d \) is a closed invariant equivalence relation. Let \( X_\infty \) denote the quotient system of \((X, T)\) by \( \text{RP}^{\infty} \). In [6] \( , \) the notion of systems of order \( \infty \) was introduced. A minimal system \((X, T)\) is an \( \infty \)-step pro-nilsystem or a system of order \( \infty \), if \( \text{RP}^{\infty} = \Delta(X) \). The authors in [6, Theorem 3.6] further showed that a minimal system is an \( \infty \)-step pro-nilsystem if and only if it is an inverse limit of minimal nilsystems. One can show that \( X_\infty \) is the maximal factor of \((X, T)\) being order of \( \infty \).

#### 2.4. Saturatedness and topological characteristic factors.

**Definition 2.6.** Let \( \phi : X \to Y \) be a map between two sets. A subset \( L \) of \( X \) is \( \phi \)-**saturated** if \( \phi^{-1}(\phi(L)) = L \).

The following lemma is direct by the definition. For the completeness we afford a proof here.
**Lemma 2.7.** Let $X, Y, Z$ be sets, and let $\pi : X \to Y, \phi : X \to Z, \psi : Z \to Y$ be surjective maps such that $\pi = \psi \circ \phi$

\[ X \xrightarrow{\phi} Z \xleftarrow{\psi} Y \]

1. If $A \subset X$ is $\pi$ saturated, then $A$ is $\phi$ saturated and $\phi(A)$ is $\psi$ saturated.

2. If $A \subset X$ is $\phi$ saturated and $\phi(A)$ is $\psi$ saturated, then $A$ is $\pi$ saturated.

**Proof.** (1) Since $\pi^{-1} \circ \pi(A) = A$, we get $\phi^{-1} \circ \psi^{-1} \circ \psi \circ \phi(A) = A$. That is, $\psi^{-1} \circ \psi(\phi(A)) = \phi(A)$.

At the same time, $\phi^{-1} \circ \phi(A) \subset \phi^{-1} \circ \psi^{-1} \circ \psi(\phi(A)) = \pi^{-1} \circ \pi(A) = A$. That is, $\phi^{-1} \circ \phi(A) = A$.

(2) We know that $\psi^{-1} \circ \psi(\phi(A)) = \phi(A)$ and $\phi^{-1} \circ \phi(A) = A$. Thus,

\[ \pi^{-1} \circ \pi(A) = \phi^{-1} \circ \psi^{-1} \circ \psi(\phi(A)) = \phi^{-1} \circ \phi(A) = A. \]

Let $X, Y$ be compact metric spaces and $\phi : X \to Y$ a map between them. For $d \in \mathbb{N}$ with $d \geq 2$, we use $\phi^{(d)}$ to denote the map

\[ \phi \times \cdots \times \phi : X^d \to Y^d, (x_1, \ldots, x_d) \mapsto (\phi(x_1), \ldots, \phi(x_d)). \]

Let $(X, T)$ be a t.d.s. and $d \geq 2$. Define

\[ \sigma_d(T) = T^{(d)} = T \times \cdots \times T (d \text{-times}), \quad \tau_d(T) = T \times T^2 \times \cdots \times T^d, \]

and $\mathcal{G}_d(T) = \langle \sigma_d(T), \tau_d(T) \rangle$ be the group generated by $\sigma_d(T)$ and $\tau_d(T)$. Let

\[ N_d(X, T) = \overline{\mathcal{O}}(\Delta_d(X), \mathcal{G}_d(T)) = \{(T^{p+q}x, T^{p+2q}x, \ldots, T^{p+dq}x) : x \in \Delta_d(X), p, q \in \mathbb{Z}\}. \]

When there is no ambiguity we also use $N_d(T)$ or $N_d(X)$ to denote $N_d(X, T)$ for short. In [11], Glasner showed that $(N_d(X, T), \mathcal{G}_d(T))$ is also a minimal system provided that $(X, T)$ is minimal.

**Definition 2.8.** Let $\pi : (X, T) \to (Y, T)$ be a factor map between minimal systems and $d \in \mathbb{N}$ with $d \geq 2$. If there exists a dense $G_\delta$ subset $\Omega$ of $X$ such that for any $x \in \Omega$, the set

\[ L_x := \overline{\mathcal{O}}(\pi^{(d)}(x), \tau_d(T)) = \{(T^{n}x, T^{2n}x, \ldots, T^{dn}x) : n \in \mathbb{Z}\} \]

is $\pi^{(d)}$-saturated, i.e., $\pi^{(-d)} \big( \pi^{(d)}(L_x) \big) = L_x$, then we say that $(Y, T)$ is a $d$-step topological characteristic factor of $(X, T)$.

The following theorems characterizing topological characteristic factors were proved in [12].
Theorem 2.9. [12, Theorem A] Let \((X, T)\) be a minimal system, and \(\pi : X \to X_\infty\) be the factor map. Then there are minimal systems \(X^*\) and \(X^*_\infty\) which are almost one to one extensions of \(X\) and \(X_\infty\) respectively, and a commuting diagram below such that \(X^*_\infty\) is a \(d\)-step topological characteristic factor of \(X^*\) for all \(d \geq 2\),

\[
\begin{array}{ccc}
X & \xleftarrow{\sigma^*} & X^* \\
\downarrow{\pi} & & \downarrow{\pi^*} \\
X_\infty & \xleftarrow{\tau^*} & X^*_\infty
\end{array}
\]

Theorem 2.10. [12, Theorem 4.2] Let \(\pi : (X, T) \to (Y, T)\) be an extension of minimal systems. If \(\pi\) is open and \(X_\infty\) is a factor of \(Y\), then \(Y\) is a \(d\)-step topological characteristic factor of \(X\) for all \(d \in \mathbb{N}\).

\[
\begin{array}{c}
X \\
\phi \downarrow \\
X_\infty
\end{array}
\xleftarrow{\pi} 
\begin{array}{c}
\pi \\
\downarrow \\
Y
\end{array}
\]

Lemma 2.11. [12, Lemma 3.3] Let \(\pi : (X, T) \to (Y, T)\) be an open extension of minimal systems and \(d \in \mathbb{N}\). If \(Y\) is a \(d\)-step topological characteristic factor of \(X\), then \(N_{d+1}(X)\) is \(\pi^{(d+1)}\)-saturated, i.e. \((\pi^{(d+1)})^{-1}(N_{d+1}(Y)) = N_{d+1}(X)\).

We remark that the openness assumptions above are necessary.

3. ON THE MINIMALITY AND TOTAL MINIMALITY

In this section we will give a proof of Theorem A. First we give the following lemma.

Lemma 3.1. Let \((X, T)\) be a minimal system with \(x \in X\) and \(n, d \in \mathbb{N}\). For any \(i, j \in \{0, 1, \ldots, n-1\}\), let

\[
[N_d(T)]_i^j = (id \times T \times \cdots \times T^{d-1})^i(T \times T \times \cdots \times T)^j N_d(T^n).
\]

Then

1. \(N_d(T) = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{n-1} [N_d(T)]_i^j\); 
2. each member of \([N_d(T)]_i^j : i, j \in \{0, 1, \ldots, n-1\}\) is minimal under the action of \(G_d(T^n)\) and any two members are either identical or disjoint; 
3. if \((X, T^n)\) is minimal, then \(N_d(T) = \bigcup_{i=0}^{n-1} [N_d(T)]_i^i\) and each \([N_d(T)]_i^i\) is minimal under the action of \(G_d(T^n)\), where \([N_d(T)]_i^i = \bigcup_{j=0}^{n-1} [N_d(T)]_j^j\); 
4. if for any \(l \in \mathbb{Z}\) there exist \(y \in X\) and a sequence \((q_i)\) of integers such that

\[
T^{nq_i} x \to T^l y, T^{2nq_i} x \to T^{2l} y, \ldots, T^{(d+1)nq_i} x \to T^{(d+1)l} y,
\]

then \(N_{d+1}(T) = N_{d+1}(T^n)\).

Proof. It is clear that

\[
G_d(T) = \bigcup_{i=0}^{n-1} \bigcup_{j=0}^{n-1} (id \times T \times \cdots \times T^{d-1})^i(T \times T \times \cdots \times T)^j G_d(T^n).
\]
Thus (1) holds and the commutativity of $G_d(T)$ implies that (2) holds. If $(X, T^n)$ is minimal, then

$$N_d(T) = \bigcup_{i=0}^{n-1} (\text{id} \times T \times \cdots \times T^{d-1}) (\langle \sigma_d(T) \rangle x^{(d)}) = \bigcup_{i=0}^{n-1} (\text{id} \times T \times \cdots \times T^{d-1}) (\langle \sigma_d(T^n) \rangle x^{(d)})$$

Thus (3) holds.

Finally, we show (4). First we show that $(X, T^n)$ is minimal. To see this, let $Y_j$ to be the orbit closure of $T^jx$ under $T^n$ for $0 \leq j \leq n-1$. For $l = 1$, we have $y$ and $(q_i)$ such that (3.1) holds. We assume that $y \in Y_{j_0}$. Thus, we have $Y_0 = Y_{j_0+1} = Y_{j_0+2}$. This clearly implies that $Y_0 = Y_j$ for all $1 \leq j \leq n-1$ and hence $(X, T^n)$ is minimal.

Now we show $N_{d+1}(T) = N_{d+1}(T^n)$. For any $l \in \mathbb{Z}$, we have $y$ and $(q_i)$ such that (3.1) holds. For any $k \in \mathbb{Z}$ there is a sequence $(p_i)$ of integers such that

$$T^{p_i}x \rightarrow T^{k}y,$$

since $(X, T^n)$ is minimal. Thus, combining (3.1) and (3.2) we get that there are subsequences $(p'_i)$ and $(q'_i)$ of $(p_i)$ and $(q_i)$ such that

$$T^{p_i'}y \rightarrow T^{k+l}y, T^{p_i'+2d}y \rightarrow T^{k+2l}y, \ldots, T^{p_i' + (d+1)d}y \rightarrow T^{k+(d+1)l}y.$$

This implies that $\sigma^k T^{l}y^{(d+1)} \subset N_{d+1}(T^n)$ and in turn implies $\sigma^k T^{l}y^{(d+1)} \subset N_{d+1}(T^n)$ by the fact that $(X, T^n)$ is minimal. Since $l$ and $k$ are arbitrary, we get that $N_{d+1}(T) \subset N_{d+1}(T^n)$. \hfill \square

Next we give two equivalent characterizations for the equality between $N_d(T)$ and $N_d(T^n)$. The equivalence between (1) and (3) in the following theorem was given in [12, Theorem 6.2]. However, the equivalence with (2) is new, which is the main ingredient for our proof of Theorem A.

**Theorem 3.2.** Let $(X, T)$ be a minimal system, and $d, n \in \mathbb{N}$. Then the following statements are equivalent.

1. $N_{d+1}(T) = N_{d+1}(T^n)$.
2. There is a dense $G_\delta$ subset $X_0$ of $X$ such that for any $x \in X_0$, there exists $y \in X$ such that for any $l \in \mathbb{Z}$ there is a sequence $(q_i)$ of integers satisfying

$$T^{mq_i}x \rightarrow T^iy, T^{2mq_i}x \rightarrow T^{2i}y, \ldots, T^{(d+1)mq_i}x \rightarrow T^{(d+1)i}y.$$

3. There is a dense $G_\delta$ subset $X_0$ of $X$ such that for any $l \in \mathbb{Z}$ and $x \in X_0$, there is a sequence $(q_i)$ of integers such that

$$T^{mq_i}x \rightarrow T^iy, T^{2mq_i}x \rightarrow T^{2i}y, \ldots, T^{dq_i}x \rightarrow T^{di}y.$$

**Proof.** It remains to show the equivalence between (1) and (2). It is clear that (2) implies (1), by Lemma 3.1 (4). Next we are going to show that (1) implies (2).
First we show the case \( d = 1 \). By the assumption \( N_2(T) = N_2(T^n) \) we know that for a given \( x \in X \) and \( l \in \mathbb{Z} \), there are sequences \( (p_i) \) and \( (q_i) \) of integers such that

\[
T^{np_i+nq_i}x \rightarrow T^lx, T^{np_i+2nq_i}x \rightarrow T^{2l}x.
\]

(3.3)

Given \( \varepsilon > 0 \) and \( l \in \mathbb{Z} \), there are \( x' \in X \) and \( q \in \mathbb{Z} \) such that

\[
\rho(T^{nq}(x'), T^lx) < \varepsilon, \quad \rho(T^{2nq}(x'), T^{2l}x) < \varepsilon,
\]

with \( x' = T^{np}x \) for some \( p \in \mathbb{Z} \) by (3.3).

Let \( k \in \mathbb{N} \) and

\[
A_k = \left\{ z \in X : \exists z' \in X \text{ s.t. } \forall l \in \mathbb{Z} \ \exists q \in \mathbb{Z} \text{ with } \rho(T^{nq}(z), T^lx), \rho(T^{2nq}(z), T^{2l}x) < \frac{1}{k} \right\}.
\]

It is clear that \( A_k \) is non-empty and open. To show it is dense we need to use the minimality of \((X,T)\). To see this let \( U \) be a non-empty open subset of \( X \) and assume that \( X = \bigcup_{N=1}^{\infty} T^NU \) for some \( N \in \mathbb{N} \). Choose \( j > 0 \) such that \( \rho(x_1, x_2) < \frac{1}{j} \) implies that \( \rho(T^{i_1}x_1, T^{i_2}x_2) < \frac{1}{k} \) for \( 1 \leq i \leq N \). Given \( l \in \mathbb{Z} \), choose \( q \in \mathbb{Z} \) satisfying \( \rho(T^{nq}(x'), T^lx) < \frac{1}{j} \) and assume that \( T^ix' \in U \) for some \( 1 \leq i \leq N \). We have

\[
\rho(T^{nq}(T^lx'), T^l(T^ix')) < \frac{1}{k}, \quad \rho(T^{2nq}(T^lx'), T^{2l}(T^ix')) < \frac{1}{k}
\]

and hence \( T^lx' \in U \cap A_k \).

Let \( X_0 = \bigcap_{k=1}^{\infty} A_k \) and \( w \in X_0 \), then there is \( w_k \in X \) and sequences \( (q^{(l)}_k) \) such that for any \( l \in \mathbb{Z} \), we have

\[
\rho(T^{nq^{(l)}_k}(w), T^lw_k) < \frac{1}{k}, \rho(T^{2nq^{(l)}_k}(w), T^{2l}w_k) < \frac{1}{k}.
\]

We may assume that \( \lim w_k = w' \) we have \( T^{nq^{(l)}_k}(w) \rightarrow T^lw \) and \( T^{2nq^{(l)}_k}(w) \rightarrow T^{2l}w' \), for any \( l \in \mathbb{Z} \).

Next we turn to the general case. By the assumption \( N_{d+1}(T) = N_{d+1}(T^n) \) we know that for a given \( x \in X \) and \( l \in \mathbb{Z} \), there are sequences \( (p_i) \) and \( (q_i) \) such that

\[
T^{np_i+nq_i}x \rightarrow T^lx, T^{np_i+2nq_i}x \rightarrow T^{2l}x, \ldots, T^{np_i+(d+1)nq_i}x \rightarrow T^{(d+1)l}x.
\]

(3.4)

Given \( \varepsilon > 0 \) and \( l \in \mathbb{Z} \), there is \( x' \in X \) and \( q \in \mathbb{Z} \) such that

\[
\rho(T^{nq}(x'), T^lx) < \varepsilon, \quad \rho(T^{2nq}(x'), T^{2l}x) < \varepsilon, \ldots, \rho(T^{(d+1)nq}(x'), T^{(d+1)l}x) < \varepsilon
\]

with \( x' = T^{np}x \) for some \( p \in \mathbb{Z} \) by (3.4).

For \( k \in \mathbb{N} \) let

\[
A_k = \left\{ z \in X : \exists z' \in X \text{ s.t. } \forall l \in \mathbb{Z} \ \exists q \in \mathbb{Z} \text{ with } \rho(T^{nq}(z), T^lx) < \frac{1}{k} \text{ for } i = 1, \ldots, d+1 \right\}.
\]

Let \( U \) be a non-empty open subset of \( X \) and assume that \( X = \bigcup_{i=1}^{N} T^lU \) for some \( N \in \mathbb{N} \). Choose \( j > 0 \) such that \( \rho(x_1, x_2) < \frac{1}{j} \) implies that \( \rho(T^ix_1, T^ix_2) < \frac{1}{k} \) for \( 1 \leq i \leq N \). Given
Theorem 3.4. [12, Theorem B] Let \( (X,T) \) be a minimal system and \( d \in \mathbb{N} \). Then the maximal equicontinuous factor of \( (N_d(X,T),\langle \sigma_d, \tau_d \rangle) \) is \( (N_d(X_1,T),\langle \sigma_d, \tau_d \rangle) \), where as above \( X_1 \) is the maximal equicontinuous factor of \((X,T)\).

We now give another approach of Theorem A based on Theorem 3.4 and Theorem 3.2.

Proof of Theorem A. It suffices to show that the minimality of \((X,T^n)\) implies the equality \(N_d(T^n) = N_d(T)\) for any \(d \in \mathbb{N}\).

By Lemma 3.1, we have \(N_d(T) = [N_d(T)]^0 \cup [N_d(T)]^1 \cup \cdots \cup [N_d(T)]^{n-1}\) and each \([N_d(T)]^i\) is \(G(T^n)\)-minimal. Thus any pair of \([N_d(T)]^i\) and \([N_d(T)]^j\) are either identical or disjoint.

Next we are going to show that \([N_d(T)]^i = [N_d(T)]^1\) for each \(i \in \{0,1,\ldots,n-1\}\) by induction on \(d\), which will implies that \(N_d(T) = N_d(T^n)\), by noting that \(N_d(T^n) = [N_d(T)]^0\).
For $d = 1, 2$, it is clear that $N_1(T) = N_1(T^n) = X$ and $N_2(T) = N_2(T^n) = X \times X$, by the minimality of $(X, T^n)$. Now we assume that $N_d(T) = N_d(T^n)$ holds for some $d \geq 2$. We are going to show $N_{d+1}(T) = N_{d+1}(T^n)$.

For $j \in \{0, 1, \ldots, n-1\}$, by taking $l = -j$ in Theorem 3.2 (2), there is a dense $G_\delta$ subset $X_0$ of $X$ such that for each $x \in X_0$, there is some $u = u(x) \in X$ and a sequence $(q_i)$ of integers with $q_i \equiv j \mod n$ satisfying

$$T^{q_i}x \rightarrow u, T^{2q_i}x \rightarrow u, \ldots, T^{dq_i}x \rightarrow u,$$

which implies that $(x, u) \in \mathcal{RP}(X, T)$ and $(x, u, \ldots, u) \in [N_{d+1}(T)]^n$. Thus

$$(x^{(d+1)}, (x, u, \ldots, u)) \in \mathcal{RP}(N_{d+1}(T))$$

according to Theorem 3.4. By noting that $x^{(d+1)} \in [N_{d+1}(T)]^0$ and Lemma 3.3, we have $[N_{d+1}(T)]^0 = [N_{d+1}(T)]^n$ whence $N_{d+1}(T) = N_{d+1}(T^n)$. This completes the proof. □

For a minimal distal system $(X, T)$, we have an approach of Theorem A (without using Theorem 3.4) by using the following theorem which is a direct consequence of Theorem 2.9.

**Theorem 3.5.** [12, Theorem 4.3] Let $(X, T)$ be a minimal system which is an open extension of its maximal distal factor, and $d \in \mathbb{N}$. Then $X_d$ is a $(d+1)$-step topological characteristic factor of $X$.

**Proof of Theorem A for minimal distal systems.** We use induction on $d$. It is trivial for $d = 1, 2$. Now assume that $N_d(T) = N_d(T^n)$ for some $d \geq 2$. Then, by applying Theorem 3.2 (2), there is a dense $G_\delta$ subset $X_0$ such that for any $x \in X_0$ and each $l \in \mathbb{Z}$, there is some $u = u(x) \in X$ and a sequence $(q_i)$ of integers with $q_i \equiv -l \mod n$ satisfying

$$T^{q_i}x \rightarrow u, T^{2q_i}x \rightarrow u, \ldots, T^{dq_i}x \rightarrow u,$$

which implies that $(x, u) \in \mathcal{RP}^{(d-1)}(X, T) = \mathcal{RP}^{(d-1)}(X, T^n)$. Since $(X, T^n)$ is minimal, the maximal $d-1$-step pronilfactor of $(X, T^n)$ is the same as the one of $(X, T)$. WLOG we assume that $X_0$ is also the set of saturation points both for $T$ and $T^k$, which is also a dense $G_\delta$ set.

Since $(X, T)$ is distal, $L_x$ is minimal and $(X, T^k)$ is minimal, we know that $\pi_{d-1}^{-1}\pi_{d-1}(x) \subset X_0$, where $\pi_{d-1} : X \rightarrow X_{d-1}$ is the factor map. By Theorem 3.5 (applying to $(X, T^n)$) there is a sequence $(p_i)$ of integers with $p_i \equiv 0 \mod n$ for each $i$ such that

$$T^{p_i}u \rightarrow x, T^{2p_i}u \rightarrow x, \ldots, T^{dp_i}u \rightarrow x,$$

Combining (3.6) and (3.7), there is a sequence $(r_i)$ of integers satisfying

$$T^{nr_i}x \rightarrow T^lx, T^{2nr_i}x \rightarrow T^{2l}x, \ldots, T^{dnr_i}x \rightarrow T^{dl}x,$$

which implies that $N_{d+1}(T) = N_{d+1}(T^n)$ according to Theorem 3.2. Thus we complete the proof. □

### 4. Maximal Distal Factors

In this section, we will show that the maximal distal factor of $N_d(X)$ is $N_d(X_{dis})$. 
4.1. **Closed proximal relations.** In this subsection, we characterize the maximal distal factor of \( N_d(X) \) under the condition that the proximal relation of \( X \) is closed.

**Lemma 4.1.** Let \( (X, T) \) be a t.d.s. and \( d \in \mathbb{N} \). Then

1. for any \( n \in \mathbb{N} \), \( P(X, T) = P(X, T^n) \);
2. if \( P(X) \) is closed, then

\[
(4.1) \quad P(N_d(X)) = \{( (x_i)^d_1, (y_i)^d_1 ) \in N_d(X) \times N_d(X) : (x_i, y_i) \in P(X, T), i = 1, \ldots, d \}.
\]

**Proof.** (1) It is clear that \( P(X, T^n) \subset P(X, T) \). By the proof of Lemma 2.3, we have \( P(X, T) \subset P(X, T^n) \). Thus (1) holds.

(2) It is also clear that \( P(N_d(X)) \subset \text{RHS} \) of (4.1), where \( \text{RHS} \) stands for the right hand side. Next we show \( P(N_d(X)) \supset \text{RHS} \) of (4.1) by induction on \( d \). It is obvious for \( d = 1 \). Now assume it holds for \( d \).

Suppose that \( ((x_i)^{d+1}_1, (y_i)^{d+1}_1) \in N_{d+1}(X) \times N_{d+1}(X) \) satisfying for any \( i = 1, \ldots, d + 1, (x_i, y_i) \in P(X, T) \). By induction assumption, we have \( ((x_i)^d_1, (y_i)^d_1) \in P(N_d(X)) \). Then there exist sequences \((p_i, g_i)\) of integers such that

\[
T^{p_i} \times T^{p_i+g_i} \times T^{p_i+2g_i} \times \ldots \times T^{p_i+(d-1)g_i} ( (x_i)^d_1, (y_i)^d_1 ) \rightarrow \text{a point in } \Delta(N_d(X)).
\]

By passing to some subsequences, we may assume that \( T^{p_i+dq_i} x_{d+1} \rightarrow z_1 \) and \( T^{p_i+dq_i} y_{d+1} \rightarrow z_2 \) for some \( z_1, z_2 \in X \). Since \( P(X) \) is closed, we have \( \{z_1, z_2\} \in P(X) \). Thus there is a sequence \((r_i)\) of integers such that \( T^{r_i} (z_1, z_2) \) tends to a point in \( \Delta(X) \). For any \( k \in \mathbb{N} \), by uniform continuity, there exists some positive integers \( m_k \) and \( n_k \) such that for any \( n \geq n_k \),

\[
\rho \left( T^{r_{m_k}+p_n} \times \ldots \times T^{r_{m_k}+p_n+dn_k} (x_i)^{d+1}_1, T^{r_{m_k}+p_n} \times \ldots \times T^{r_{m_k}+p_n+dn_k} (y_i)^{d+1}_1 \right) < \frac{1}{k}.
\]

We may assume that \( (m_k) \) and \( (n_k) \) are increasing. Then

\[
\rho \left( T^{r_{m_k}+p_n} \times \ldots \times T^{r_{m_k}+p_n+dn_k} (x_i)^{d+1}_1, T^{r_{m_k}+p_n} \times \ldots \times T^{r_{m_k}+p_n+dn_k} (y_i)^{d+1}_1 \right) \rightarrow 0.
\]

So \( (x_i)^{d+1}_1, (y_i)^{d+1}_1 \) \( \in \) \( P(N_{d+1}(X)) \). Thus (2) holds.

**Theorem 4.2.** Let \( (X, T) \) be a minimal t.d.s. If the proximal relation of \( (X, T) \) is closed and \( d \in \mathbb{N} \), then the maximal distal factor of \( (N_d(X, T), \mathcal{G}_d) \) is \( (N_d(X_{dis}, T), \mathcal{G}_d) \), where \( X_{dis} \) is the maximal distal factor of \( (X, T) \).

**Proof.** By Lemma 4.1, we have

\[
(4.2) \quad P(N_d(X)) = \{( (x_i)^d_1, (y_i)^d_1 ) \in N_d(X) \times N_d(X) : (x_i, y_i) \in P(X, T), i = 1, \ldots, d \}.
\]

Thus \( P(N_d(X)) \) is closed. Recall that if a proximal relation is closed, then it is an equivalence relation (see [2, Chapter 6, Corollary 11]). It follows that \( X_{dis} = X/P(X) \) and \( (N_d(X))_{dis} = N_d(X)/P(N_d(X)) \). By (4.2), we conclude that \( (N_d(X))_{dis} = N_d(X_{dis}) \).

We end the subsection with the following remark.

**Remark 4.3.** We remark that even in the assumption \( (X, T) \) is minimal and \( \pi : X \rightarrow X_{eq} \) is almost 1-1, it is not true that \( (x_1, x_2, x_3) \in N_3 \) if \( \pi(x_1) = \pi(x_2) = \pi(x_3) \), as \( N_3 \) can be seen as a subset of \( Q^{[2]} \), see [21].
4.2. **Open extensions.** We start with some properties of topological characteristic factors.

Let $Z$ be a compact metric space. Let $2^Z$ be the space of nonempty closed subsets of $Z$, which is also a compact metric space endowed with Hausdorff metric. For a sequence $(A_n)$ in $2^Z$, define $\lim\inf A_n = \{ z \in Z : \exists z_n \in A_n \text{ s.t. } z_n \to z \}$ and $\lim\sup A_n = \{ z \in Z : \exists \text{ subsequence } (n_i) \text{ and } z_{n_i} \in A_{n_i} \text{ s.t. } z_{n_i} \to z \}$. Let $Y$ be a metric space. A map $f : Y \to 2^Z$ is **lower semi-continuous** (resp. **upper semi-continuous**) at $y$ if for any sequence $(y_n)$ with $y_n \to y$, $\lim\inf f(y_n) \supset f(y)$ (resp. $\lim\sup f(y_n) \subset f(y)$). Note that $f$ is lower semi-continuous at $y$ if for any open set $V$ with $f(y) \cap V \neq \emptyset$, $\{ y' \in Y : f(y') \cap V \neq \emptyset \}$ is a neighborhood of $y$, and $f$ is upper semi-continuous at $y$ if for any open set $V$ of $Z$ containing $f(y)$, $\{ y' \in Y : f(y') \subset V \}$ is a neighborhood of $y$.

**Lemma 4.4.** Let $(X, T)$ be a minimal system and $d \geq 2$, then for a dense $G_\delta$ subset $X_0 \subset X$ such that for $x \in X_0$ one has $N_d[x] := \{ (x_1, \ldots, x_d) \in N_d(X) : x_1 = x \} = \{ x \} \times L_x$, where $L_x$ is the orbit closure of $x^{(d-1)}$ under $T \times T^2 \times \ldots \times T^{d-1}$ action.

**Proof.** Consider $\Phi : X \to 2^X$ given by $x \mapsto \{ x \} \times L_x$. It is easy to check that this map is lower-semi-continuous (see [12, Lemma 6.1, Claim 1]). It is well known that the set of continuity points of $\Phi$ is a dense $G_\delta$ subset $X_0 \subset X$. We now show $\bigcup_{x \in X} \{ x \} \times L_x$ is dense in $N_d(X)$. To see this let $(y_1, \ldots, y_d) \in N_d$. Then there are subsequences $\{p_i\}$ and $\{i\}$ of $\mathbb{Z}$ and $x \in X$ such that $T^{p_1,i} \to y_1, T^{p_2,i} \to y_2, \ldots, T^{p_d,i} \to y_d$. Let $z_i = T^{p_i,i} \to y_i, T^{q_i,z_i} \to y_2, \ldots, T^{(d-1)q_i,z_i} \to y_d$. It is clear that $(z_i, T^{q_i,z_i}, \ldots, T^{(d-1)q_i,z_i}) \in \{ z_i \} \times L_{z_i}$.

Thus, it follows that at each point $x \in X_0$ we must have $N_d[x] = \{ x \} \times L_x$. Indeed let $x_0 \in X_0$ and assume $N_d[x] \neq \{ x \} \times L_x$, i.e. $\{ x \} \times L_x \not\subset N_d[x]$. Let $U$ be an open set in $N_d(X)$ so that $\{ x_0 \} \times L_{x_0} \subset N_d(X) \cap U \neq N_d[x]$. As $\Phi$ is continuous at $x_0$ the set $\{ x \in X \} \times L_x \subset U$ is a neighborhood of $x_0$ and it follows $\bigcup_{x \in X} \{ x \} \times L_x$ is not dense in $N_d(X)$. \qed

To sum up we have

**Theorem 4.5.** Let $\pi : (X, T) \to (Y, T)$ be an extension of minimal systems and $d \in \mathbb{N}$.

1. If $Y$ is a $d$-step topological characteristic factor of $X$ and $\pi$ is open, then $N_{d+1}(X)$ is $\pi^{(d+1)}$-saturated, i.e. $(\pi^{(d+1)})^{-1}(N_{d+1}(Y)) = N_{d+1}(X)$.

2. If $N_{d+1}(X)$ is $\pi^{(d+1)}$-saturated, then $Y$ is a $d$-step topological characteristic factor of $X$.

**Proof.** (1) follows from Lemma 2.11.

To show (2) we use Lemma 4.4. Indeed, by Lemma 4.4 we know that there is a dense $G_\delta$ subset $X_0 \subset X$, such that for $x \in X_0$ one has $N_d[x] = \{ x \} \times L_x$. Now let $(x_1, \ldots, x_{d-1}) \in L_x$, then we have $(x, x_1, \ldots, x_{d-1}) \in N_d(X)$. For any $(y_1, \ldots, y_{d-1}) \in X^{(d-1)}$ with $\pi(x_i) = y_i$, $1 \leq i \leq d-1$, we have $(x, y_1, \ldots, y_{d-1}) \in N_d(X)$. Thus, $(x, y_1, \ldots, y_{d-1}) \in N_d[x] = \{ x \} \times L_x$, i.e. $(y_1, \ldots, y_{d-1}) \in L_x$. \qed
4.3. **Proof of Theorem B.** First we need a simple lemma.

**Lemma 4.6.** Let \( \pi : X \rightarrow Y \) be an extension between minimal systems. If \( R_\pi \subset S_{\text{dis}}(X) \), then \( X_{\text{dis}} = Y_{\text{dis}} \). Particularly, this holds for proximal extensions.

**Proof.** It is clear that \( Y_{\text{dis}} \) is a distal factor of \( X \). By the maximality of \( X_{\text{dis}} \), \( Y_{\text{dis}} \) is a factor of \( X_{\text{dis}} \). If \( R_\pi \subset S_{\text{dis}}(X) \), then \( X_{\text{dis}} = X / S_{\text{dis}}(X) \) is a factor of \( Y = X / R_\pi \). Since \( X_{\text{dis}} \) is distal, we have that \( X_{\text{dis}} \) is also a factor of \( Y_{\text{dis}} \). Thus \( X_{\text{dis}} = Y_{\text{dis}} \).

When \( \pi \) is proximal we have \( R_\pi \subset \mathcal{P}(X) \subset S_{\text{dis}}(X) \). Thus the conclusion holds for proximal extensions. \( \square \)

The following lemma is direct from the definition.

**Lemma 4.7.** Let \( \pi : X \rightarrow Y \), \( Z \) is a factor of \( Y \) and distal. If the maximal distal factor of \( X \) is \( Z \), then \( Z \) is also the maximal distal factor of \( Y \).

**Proof of Theorem B.** Let \( \pi : X \rightarrow X_{\text{dis}} \) be the factor map. Then there are minimal systems \( X^* \) and \( Y \) such that \( \phi : X^* \rightarrow X \) and \( \psi : Y \rightarrow X_{\text{dis}} \) are almost 1-1 extension, \( \pi^* : X^* \rightarrow Y \) is open and the following commutative diagram holds:

\[
\begin{array}{ccc}
X & \xleftarrow{\phi} & X^* \\
\downarrow{\pi} & & \downarrow{\pi^*} \\
X_{\text{dis}} & \xleftarrow{\psi} & Y
\end{array}
\]

(4.3)

By Lemma 4.6, we have \( X_{\text{dis}} = (X^*)_{\text{dis}} \). Let \( \tau = \pi \circ \phi : X^* \rightarrow X_{\text{dis}} \). Then (4.3) naturally induces the following diagram:

\[
\begin{array}{ccc}
N_d(X) & \xleftarrow{\phi^{(d)}} & N_d(X^*) \\
\pi^{(d)} & & \pi^{(d)} \\
N_d(X_{\text{dis}}) & \xleftarrow{\psi^{(d)}} & N_d(Y)
\end{array}
\]

**Claim.** \( R_{\tau^{(d)}} \subset S_{\text{dis}}(N_d(X^*)) \).

If the Claim holds, then by Lemma 4.6 we have \( (N_d(X^*))_{\text{dis}} = N_d(X_{\text{dis}}) \). It follows from Lemma 4.7 that the maximal distal factor of \( N_d(X) \) is \( N_d(X_{\text{dis}}) \).

Now we are going to prove the claim. Fix \((x_i^d, x_i'_{d}) \in R_{\tau^{(d)}}\) for each \( 1 \leq i \leq d \). Let \( y_i = \pi^*(x_i) \) and \( y'_i = \pi^*(x'_i) \) for each \( 1 \leq i \leq d \). By Lemma 2.2, we have \( (y_i, y'_i) \in S_{\text{dis}}(Y) \) for each \( 1 \leq i \leq d \). Since \( \psi \) is almost 1-1 and \( X_{\text{dis}} \) is distal, we have \( R_\psi = \mathcal{P}(Y) = S_{\text{dis}}(Y) \). By Lemma 4.1, we have \( R_{\psi^{(d)}} \subset S_{\text{dis}}(N_d(Y)) = \mathcal{P}(N_d(Y)) \). Thus there is a sequence \((g_n) \in \mathcal{G}_d \) such that \( g_n \left( ((y_i^d, y'_i^d)) \right) \rightarrow \Delta(N_d(Y)) \). Then \( \overline{\mathcal{G}} \left( ((x_i^d, x_i'_{d})) \right) \cap R_{(\pi^*)^{(d)}} \neq \emptyset \).

Since \( X^*_\infty \) is a factor of \( X_{\text{dis}} \), it follows from Theorems 2.10 and 4.5 that \( N_d(X^*) \) is \((\pi^*)^{(d)}\)-saturated. Let \((z_i^d, z_i'_{d}) \in \overline{\mathcal{G}} \left( ((x_i^d, x_i'_{d})) \right) \cap R_{(\pi^*)^{(d)}} \). It follows from the
saturatedness of \((\pi^*)^{(d)}\) that
\[
(z_1, \cdots, z_i, z_{i+1}', \cdots, z_d') \in N_d(X^*),
\]
for each \(0 \leq i \leq d\).

First consider the points \(((z_1, \ldots, z_d), (z_1', z_2, \ldots, z_d)) \in (N_d(X^*))^2\). Since \(\pi^*(z_1) = \pi^*(z_1')\), we have \(\tau(z_1) = \tau(z_1')\) and hence \((z_1, z_1') \in S_{\text{dis}}(X^*)\). It follows from Theorem 2.1 that
\[
\overline{O}((z_1, z_1'), T) \cap \overline{P(N_d(X^*))} \neq \emptyset
\]
which implies
\[
\overline{O}(((z_1, \ldots, z_d), (z_1', \ldots, z_d)), \mathcal{G}_d) \cap \overline{P(N_d(X^*))} \neq \emptyset.
\]
Thus
\[
((z_1, \ldots, z_d), (z_1', z_2, \ldots, z_d)) \in S_{\text{dis}}(N_d(X^*)).
\]
Next consider \(((z_1', z_2, \ldots, z_d), (z_1', z_2', z_3, \ldots, z_d)) \in (N_d(X^*))^2\) in the same way. Inductively, for any \(i = 2, 3, \ldots, d\), we have
\[
((z_1', \ldots, z_{i-1}', z_i', \ldots, z_d), (z_1', \ldots, z_{i-1}', z_i, z_{i+1}, \ldots, z_d)) \in S_{\text{dis}}(N_d(X^*)).
\]
Since \(S_{\text{dis}}(N_d(X^*))\) is an invariant closed equivalence relation, it follows that
\[
((z_1, \ldots, z_d), (z_1', \ldots, z_d)) \in S_{\text{dis}}(N_d(X^*)).
\]
Thus
\[
\overline{O}(((z_1, \ldots, z_d), (z_1', \ldots, z_d)) \cap \overline{P(N_d(X^*))} \neq \emptyset.
\]
Recall that \(((x_i)^d_1, (x_i')^d_1) \in \overline{O}(((x_i)^d_1, (x_i')^d_1))\). So we have
\[
\overline{O}(((x_i)^d_1, (x_i')^d_1)) \cap \overline{P(N_d(X^*))} \neq \emptyset.
\]
Hence \(((x_i)^d_1, (x_i')^d_1) \in S_{\text{dis}}(N_d(X^*)).\) This proves the claim.

\section{5. Extensions and the structure theorems}

In this section, we will show that minimal t.d.s. \((X, T)\) and \((N_d(X), \mathcal{G}_d)\) have the same structure theorems.

\subsection{5.1. Structure theorem for minimal systems.}
We say that a minimal system \((X, G)\) is a \textit{strictly PI system} if there is a countable ordinal \(\eta\) and a family of minimal systems \((X_\lambda, G)\) \((0 \leq \lambda \leq \eta)\) such that
(1) \(X_0\) is a singleton;
(2) for every successor ordinal \(\lambda < \eta\), there exists an extension \(\phi_\lambda : (X_{\lambda+1}, G) \to (X_\lambda, G)\) which is either proximal or equicontinuous;
(3) for a limit ordinal \(\lambda < \eta\) the system \((X_\lambda, G)\) is the inverse limit of the systems \((X_\mu, G)_{\mu < \lambda}\);
(4) \(X_\eta = X\).
A minimal system \((X, G)\) is a **PI system** if there exists a strictly PI system \((\tilde{X}, \tilde{G})\) and a proximal extension \(\pi : (\tilde{X}, \tilde{G}) \to (X, G)\). If we replace the proximal extensions by almost \(1-1\) extensions in the definition of strictly PI system, the resulting system is called an **HPI system**. If we replace the proximal extensions by trivial extensions, the resulting system is called an **I system**.

**Theorem 5.1** (Structure Theorem of Minimal Distal Systems). *Every minimal distal system is an I system.*

An extension \(\phi : (X, G) \to (Y, G)\) is called a **weakly mixing extension** if \(R_\phi\) under the diagonal action of \(G\) is topologically transitive.

An extension \(\phi : (X, G) \to (Y, G)\) is called a **RIC** (relatively incontractible) extension if \(\pi\) is an open map and for any \(n \geq 1\) the minimal points are dense in \(R^n_\phi = \{ (x_1, \ldots, x_n) \in X^n : \phi(x_1) = \cdots = \phi(x_n) \}\). The notion of RIC extension was introduced in [8]. Here we use an equivalent definition avoiding extra notions (see [1, Theorem A.2]).

**Theorem 5.2** (Structure Theorem of Minimal Systems). *Let \((X, T)\) be a minimal system. Then there exist a countable ordinal \(\eta\) and a canonically defined commutative diagram of minimal systems (a PI tower):

\[
\begin{align*}
X &= X_0 \leftarrow \phi_1 X_1 \leftarrow \cdots \leftarrow X_v \leftarrow \phi_{v+1} X_{v+1} \leftarrow \cdots \leftarrow X_\eta \\
\pi_0 \downarrow \sigma_1 & \uparrow \pi_1 \quad \cdots \quad \pi_v \downarrow \sigma_{v+1} & \uparrow \pi_{v+1} \quad \cdots \quad \pi_\eta \\
\{pt\} &= Y_0 \rho_1 Z_1 \psi_1 Y_1 \cdots Y_v \rho_{v+1} Z_{v+1} \psi_{v+1} Y_{v+1} \cdots Y_\eta
\end{align*}
\]

where for \(v \leq \eta\), \(\pi_v\) is RIC, \(\phi_v\) and \(\psi_v\) are proximal, \(\rho_v\) are equicontinuous. \(\pi_\eta\) is RIC and weakly mixing. For a limit ordinal \(\lambda\), \(X_\lambda, Y_\lambda, \pi_\lambda\), etc. are the inverse limits of \(X_\lambda, Y_\lambda, \pi_\lambda\), etc. for \(\lambda < \eta\). \{pt\} denotes the trivial system.

Then it naturally induces the following commutative diagram associated to \(N_d(X)\):

\[
\begin{align*}
N_d(X) & \leftarrow \cdots \leftarrow N_d(X_v) \leftarrow N_d(X_{v+1}) \cdots \leftarrow N_d(X_\eta) \\
\pi_0^{(d)} \downarrow \phi_1^{(d)} & \uparrow \sigma_1^{(d)} \quad \cdots \quad \pi_v^{(d)} \downarrow \phi_{v+1}^{(d)} & \uparrow \sigma_{v+1}^{(d)} \quad \cdots \quad \pi_{v+1}^{(d)} \downarrow \phi_{v+1}^{(d)} & \uparrow \sigma_v^{(d)} \quad \cdots \quad \pi_\eta^{(d)} \\
N_d(Y_0) & \leftarrow \cdots \leftarrow N_d(Y_v) \leftarrow N_d(Y_{v+1}) \cdots \leftarrow N_d(Y_\eta)
\end{align*}
\]

Our main aim in this section is to show the following theorem which will imply Theorem C.

**Theorem 5.3.** *Let \((X, T)\) be a minimal system. Then \((N_d(X), \mathcal{G}_d)\) has the same structure theorem as \((X, T)\). Precisely, in the commutative diagram 5.1, \(v \leq \eta\), \(\pi_v^{(d)}\) is RIC, \(\phi_v^{(d)}\) and \(\psi_v^{(d)}\) are proximal, \(\rho_v^{(d)}\) are equicontinuous. \(\pi_\eta^{(d)}\) is RIC and weakly mixing.*

### 5.2. PI extension.

**Lemma 5.4.** *If \(\pi : X \to Y\) is a non-trivial extension, then so is \(\pi^{(d)} : N_d(X) \to N_d(Y)\) for any \(d \in \mathbb{N}\).*
Proof. Since $\pi$ is non-trivial, there is some $y \in Y$ with $|\pi^{-1}(y)| \geq 2$. For each $x \in \pi^{-1}(y)$, we have $x(d) \in N_d(X)$ and $\pi(d)(x(d)) = y(d)$. Thus, $|(\pi(d))^{-1}(y(d))| \geq |\pi^{-1}(y)| \geq 2$. Hence $\pi(d)$ is also non-trivial. 

Lemma 5.5. Suppose that $\pi : X \to Y$ is an almost 1-1 extension (resp. proximal, equicontinuous, distal) between minimal systems. Then $\pi(d) : N_d(X) \to N_d(Y)$ is also almost 1-1 (resp. proximal, equicontinuous, distal) for any $d \in \mathbb{N}$.

Proof. Suppose that $\pi$ is an almost 1-1 extension. Then there is $y \in Y$ such that $\pi^{-1}(y) = \{x\}$. Thus $(\pi(d))^{-1}(y(d)) = \{x(d)\}$. Hence $\pi(d) : N_d(X) \to N_d(Y)$ is also an almost 1-1 extension.

Now suppose that $\pi$ is equicontinuous or distal, then so is $\pi(d)$ just by the definitions. Suppose that $\pi$ is proximal. It follows from the proof of Lemma 4.1 by replacing $P(X)$ by $R_\pi$.

Lemma 5.6. Let $\lambda$ be a limit ordinal and $(X_\mu, T)_\mu \leq \lambda$ be a collection of minimal systems. If $(X_\lambda, T) = \lim (X_\mu, T)_\mu < \lambda$, then for any $d \in N$,

$$(N_d(X_\lambda), G_d(T)) = \lim (N_d(X_\mu), G_d(T))_{\mu < \lambda}.$$ 

Proof. Let $\phi_{\mu, v} : X_v \to X_\mu$ be the homomorphisms associated in the inverse limit for any ordinals $\mu \leq v < \lambda$ and let $\phi_{\mu, v}^{(d)} : N_d(X_v) \to N_d(X_\mu)$ be the naturally induced homomorphisms. Then

$$X_\lambda = \left\{ \left(x_\alpha \right) \in \prod_{\alpha < \lambda} X_\alpha : \forall \mu \leq v \leq \xi < \lambda, \phi_{\mu, \xi}(x_\xi) = \phi_{\mu, v} \circ \phi_{v, \xi}(x_\xi) \right\}.$$ 

By identifying $(\prod_{\mu < \lambda} X_\mu)^d$ with $\prod_{\mu < \lambda} X_\mu^d$, we write the point of $(\prod_{\mu < \lambda} X_\mu)^d$ in the form of $(x_1^{(d)}, \ldots, x_\mu^{(d)})_{\mu < \lambda}$. If $(x_1^{(d)}, \ldots, x_\mu^{(d)})_{\mu < \lambda} \in N_d(X_\lambda)$, then there is $(x_\mu)_{\mu < \lambda} \in X_\lambda$ and sequences $(k_i)$ and $(l_i)$ of integers such that

$$(x_1^{(d)}, \ldots, x_\mu^{(d)})_{\mu < \lambda} = \lim_{i \to \infty} (T^{l_i+k_i}x_\mu, T^{l_i+2k_i}x_\mu, \ldots, T^{l_i+d_k}x_\mu)_{\mu < \lambda}.$$ 

For any $\mu \leq v \leq \xi < \lambda$,

$$\phi_{\mu, \xi}^{(d)} \left((x_1^{(d)}, \ldots, x_\xi^{(d)})\right) = \lim_{i \to \infty} \left(T^{l_i+k_i} \phi_{\mu, \xi}(x_\xi), \ldots, T^{l_i+d_k} \phi_{\mu, \xi}(x_\xi)\right) = \lim_{i \to \infty} \left(T^{l_i+k_i} \phi_{\mu, v} \circ \phi_{v, \xi}(x_\xi), \ldots, T^{l_i+d_k} \phi_{\mu, v} \circ \phi_{v, \xi}(x_\xi)\right) = \phi_{\mu, v} \circ \phi_{v, \xi}^{(d)} \left((x_1^{(d)}, \ldots, x_\xi^{(d)})\right),$$

and $(x_1^{(d)}, \ldots, x_\mu^{(d)}) \in N_d(X_\mu)$. Let

$$E = \left\{ (x_1^{(d)}, \ldots, x_\mu^{(d)})_{\mu < \lambda} \in \prod_{\mu < \lambda} N_d(X_\mu) : \forall \mu \leq v \leq \xi < \lambda, \phi_{\mu, \xi}^{(d)} \left((x_1^{(d)}, \ldots, x_\xi^{(d)})\right) = \phi_{\mu, v} \circ \phi_{v, \xi}^{(d)} \left((x_1^{(d)}, \ldots, x_\xi^{(d)})\right) \right\}.$$

Then $N_d(X_\lambda) \subset E$. Note that $N_d(X_\lambda)$ is minimal under the action of $\mathcal{G}_d(\prod_{\mu < \lambda} T)$ and $E$ is minimal under the action of $\prod_{\mu < \lambda} \mathcal{G}_d(T) \cong \mathcal{G}_d(\prod_{\mu < \lambda} T)$. It follows that $N_d(X_\lambda) = E$. Hence $(N_d(X_\lambda), \mathcal{G}_d(T))$ is the inverse limit of $(N_d(X_\mu), \mathcal{G}_d(T))_{\mu < \lambda}$. □

**Theorem 5.7.** If a minimal system $(X, T)$ is distal (resp. PI, HPI), then so is $N_d(T)$.

**Proof.** It follows directly by the definitions and Lemma 5.4, 5.5, 5.6. □

5.3. **RIC extension.** In this section, we show that if $\phi : (X, T) \to (Y, T)$ is a RIC extension (resp. RIC weakly mixing) between minimal systems, then so is $\phi^{(d)}$ for any $d \in N$. We need the following lemma.

**Lemma 5.8.** Let $(X, T)$ be a minimal system and $d \in \mathbb{N}$. For $(x_1, \ldots, x_d), (y_1, \ldots, y_d) \in N_d(T)$, $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ is minimal in $(X^{2d}, T)$ under the diagonal action, then $((x_1, \ldots, x_d), (y_1, \ldots, y_d))$ is a minimal point in $(N_d(T) \times N_d(T), \mathcal{G}_d(T))$ under the diagonal action.

Generally, let $n, d \in \mathbb{N}$. For $(x_1^1)^d, \ldots, (x_n^d)^d \in N_d(T)$, if $((x_1^1)^d, \ldots, (x_n^d)^d)$ is a minimal point in $(X^n, T)$ under the diagonal action, then $((x_1^1)^d, \ldots, (x_n^d)^d)$ is a minimal point in $(N_d(T)^n, \mathcal{G}_d(T))$ under the diagonal action.

**Proof.** It is clear for $d = 1$. For $d = 2$, suppose that $x = (x_1, x_2, y_1, y_2)$ is $S = T \times T \times T \times T$ minimal. Let $X_4$ be the orbit closure of $x$ under the action of $S$. Then $(X_4, S)$ is minimal, and hence $(x, x)$ is a minimal point under the action of $\langle S \times S, S \times S \rangle$. That is, $((x_1, x_2, y_1, y_2), (x_1, x_2, y_1, y_2))$ is minimal under the action of

$$\{T^{n+m} \times T^{n+m} \times T^{n+m} \times T^{n+2m} \times T^{n+2m} \times T^{n+2m} \times T^{n+2m} : n, m \in \mathbb{Z}\}.$$  

By projecting to the first, third, sixth and eighth coordinates, it follows that $(x_1, x_2, y_1, y_2)$ is minimal under the action $\{T^{n+m} \times T^{n+2m} \times T^{n+m} \times T^{n+2m} : n, m \in \mathbb{Z}\}$.

For $d > 2$, assume that $x = (x_1, \ldots, x_d, y_1, \ldots, y_d)$ is $S = \sigma_2(T)$-minimal. Let $X_2d$ be the orbit closure of $x$ under $S$. Then $(X_2d, S)$ is minimal, and thus $x^{(d)}$ is a minimal point for $\langle S \times S \times \ldots \times S, S \times S \times \ldots \times S \rangle$. That is, $(x_1, \ldots, x_d, y_1, \ldots, y_d)^{(d)}$ is minimal under the action of $\{S^{n+m} 	imes S^{n+2m} \times \ldots \times S^{n+2m} : n, m \in \mathbb{Z}\}$. By projecting to the $(k + (k - 1)d)^{th}$ and $(k + kd)^{th}$ coordinates for $k \in \{1, \ldots, d\}$, it follows that $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ is minimal under the action of

$$\{T^{n+m} \times T^{n+2m} \times \ldots \times T^{n+2m} \times T^{n+m} \times T^{n+2m} \times \ldots \times T^{n+2m} : n, m \in \mathbb{Z}\}.$$  

In general, let $n, d \in \mathbb{N}$ and $(x_1^1)^{d}, \ldots, (x_n^d)^d \in N_d(X)$. If $x = ((x_1^1)^d, \ldots, (x_n^d)^d)$ is $S = \sigma_{nd}(T)$-minimal, then $(X_{nd}, S)$ is minimal, where $X_{nd}$ denotes the orbit closure of $x$ under $S$. Thus $x^{(d)}$ is a minimal point under the action of $\langle S \times S \times \ldots \times S, S \times S \times \ldots \times S \rangle$. That is, $(x_1^1)^d, \ldots, (x_n^d)^d)^{(d)}$ is minimal under the action of

$$\{S^{n+m} \times S^{n+2m} \times \ldots \times S^{n+2m} : n, m \in \mathbb{Z}\}.$$  

By projecting to the $(k + (k - 1)d)^{th}$ coordinates for $k, j \in \{1, \ldots, d\}$, it follows that $(x_1, \ldots, x_d, y_1, \ldots, y_d)$ is minimal under the action of

$$\{(T^{n+m} \times T^{n+2m} \times \ldots \times T^{n+2m}):(n) : n, m \in \mathbb{Z}\}.$$  

Then $((x_1^1)^d, \ldots, (x_n^d)^d)$ is minimal in $(N_d(T)^n, \mathcal{G}_d(T))$ under the diagonal action. □
Lemma 5.9. [24, A.8] Let $f : X \rightarrow Y$ be a continuous surjective map between compact metric spaces. Then

1. the map $F : Y \rightarrow 2^X, y \mapsto f^{-1}\{y\}$ is upper semi-continuous;
2. $F$ is continuous if and only if $f$ is open.

Lemma 5.10. [23] Let $\pi : (X, T) \rightarrow (Y, T)$ be a RIC weakly mixing extension. Then $(\mathcal{R}^\pi_d, T^{(n)})$ is topologically transitive for any $n \in \mathbb{N}$.

Theorem 5.11. Let $\pi : (X, T) \rightarrow (Y, T)$ be a RIC (resp. RIC weakly mixing) extension between minimal systems with $X_\infty$ being a factor of $Y$. Then for each $d \in \mathbb{N}$, $\pi^{(d)} : \mathcal{N}_d(X) \rightarrow \mathcal{N}_d(Y)$ is also RIC (resp. RIC and weakly mixing).

Proof. Claim 1. For any $d \in \mathbb{N}$, we have

$$\mathcal{N}_d(T) \supseteq \mathcal{R}_d^\pi = \{(x_1, \ldots, x_d) \in X^d : \pi(x_1) = \cdots = \pi(x_d)\}.$$

Proof of Claim. It is equivalent to show that for any $y \in Y$, $\pi^{-1}(y) \times \cdots \times \pi^{-1}(y) \subset \mathcal{N}_d(X)$. By Lemma 5.9, the map

$$\Pi^{(d)} : Y^d \rightarrow 2^{X^d}, (y_1, \ldots, y_d) \mapsto \pi^{-1}(y_1) \times \cdots \times \pi^{-1}(y_d)$$

is continuous since $\pi$ is open. By Theorem 2.10, $Y$ is a $d$-step characteristic factor of $X$, which means there is a dense $G_\delta$ subset $\Omega$ of $X$ such that for any $x \in \Omega$, $L_x = \overline{\mathcal{O}(x, T)}$ is $\pi^{(d)}$-saturated. Thus for any $y' \in \pi(\Omega)$, $\pi^{-1}(y') \times \cdots \times \pi^{-1}(y') \subset \mathcal{N}_d(X)$. Note that $\pi(\Omega)$ is dense in $Y$. Then for any $y \in Y$, there exists a sequence $(y_n)$ in $\pi(\Omega)$ with $y_n \rightarrow y$. By the continuity of $\Pi^{(d)}$, $\pi^{-1}(y_n) \times \cdots \times \pi^{-1}(y_n)$ converges to $\pi^{-1}(y) \times \cdots \times \pi^{-1}(y)$ in the hyperspace. From the definition of the convergence in hyperspaces, we conclude that $\pi^{-1}(y) \times \cdots \times \pi^{-1}(y) \subset \mathcal{N}_d(X)$. This proves the claim.

Now Fix $d \in \mathbb{N}$ and let $\phi = \pi^{(d)} : \mathcal{N}_d(X) \rightarrow \mathcal{N}_d(Y)$.

By the definition of RIC extensions, it follows that $\pi$ is open and hence $\phi$ is open. It remains to show that for any $n \in \mathbb{N}$, the minimal points in $\mathcal{R}^\phi_n$ are dense, where

$$\mathcal{R}^\phi_n = \left\{ \left( (x_1^1)^d, \ldots, (x_n^d)^d \right) \in \mathcal{N}_d(X)^n : \phi \left( (x_1^1)^d \right) = \cdots = \phi \left( (x_n^d)^d \right) \right\}.$$

Let $M$ denote the set of minimal points in $\mathcal{R}_n^\phi$ under the diagonal action of $\mathcal{G}_d(T)$ and $N$ denote the set of minimal points in $\mathcal{R}_n^\pi$ under $T^{(nd)}$. According to Lemma 5.8 and Claim 1, we have

Claim 2. $N \subset M$.

Thus for any $y \in Y$ and $x \in \phi^{(-1)}(y^{(nd)})$, $x \in \overline{M}$. Given $((x_1^1)^d, \ldots, (x_n^d)^d) \in \mathcal{R}^\phi_n$, it follows that

$$\pi(x_1^i) = \pi(x_2^i) = \cdots = \pi(x_n^i) = y_i, \quad i = 1, 2, \ldots, d,$$

for some $(y_1, y_2, \ldots, y_d) \in \mathcal{N}_d(Y)$. By the definition of $\mathcal{N}_d$, there exist a sequence $(g_n)$ in $\mathcal{G}_d(T)$ and $u_n \in Y$ with $g_n(u_n, \ldots, u_n) \rightarrow (y_1, \ldots, y_d)$. Since $\Pi^{(d)}$ is continuous, there exists $(w_1^1)^d, \ldots, (w_n^d)^d \in \phi^{(-1)}(u_n^{(d)})$ satisfying

$$g_n((w_1^1)^d) \rightarrow (x_1^1)^d, \ldots, g_n((w_n^1)^d) \rightarrow (x_n^1)^d.$$
Thus $((x_1^1)^d,\ldots,(x_n^d)^d)\in\mathcal{M}$ by noting that $(g_n(w_1^1)^d,\ldots,g_n(w_n^d)^d)\in\mathcal{M}$. Hence $M$ is dense in $R^n_{\phi}$.

Now we discuss RIC weakly mixing extensions. It remains to show that $\phi$ is weakly mixing. Since $\pi$ is RIC weakly mixing, $(R^n_{\pi},T^{(2d)})$ is transitive by Lemma 5.10. Then there exists $y\in Y$ and $z_1,\ldots,z_{2d}\in\pi^{-1}(y)$ such that

$$R^n_{\pi} = \overline{\{(z_1,\ldots,z_{2d}),T^{(2d)}\}}.$$ 

Suppose that $((x_1,\ldots,x_d),(x_1',\ldots,x_d'))\in R_{\phi}$. Then we have $\pi(x_i) = \pi(x_i') = y_i$ for some $y_i\in Y$ and $(y_1,\ldots,y_d)\in N_d(Y)$. There exist a sequence $(g_n)$ in $\mathcal{G}_d(T)$ and a sequence $(v_n)$ in $Y$ satisfying $g_n(v_n^{(d)})\to (y_1,\ldots,y_d)$. By the continuity of $\Pi^{(d)}$, there are sequences $(w_1^n,\ldots,w_d^n),(w_1'^n,\ldots,w_d'^n)\in\pi^{-1}(u_n)\times\cdots\times\pi^{-1}(u_n)$ with

$$g_n(w_1^n,\ldots,w_d^n)\to (x_1,\ldots,x_d), \quad g_n(w_1'^n,\ldots,w_d'^n)\to (x_1',\ldots,x_d').$$ 

Note that $((w_1^n,\ldots,w_d^n),(w_1'^n,\ldots,w_d'^n))\in\overline{\mathcal{O}((z_1,\ldots,z_{2d}),T^{(2d)})}$. Thus

$$((x_1,\ldots,x_d),(x_1',\ldots,x_d'))\in\overline{\mathcal{O}((z_1,\ldots,z_{2d}),\mathcal{G}_d(T))}.$$ 

Therefore $R_{\phi}$ is topologically transitive under the diagonal action of $\mathcal{G}_d(T)$. □

Finally we have the following remark.

Remark 5.12. In Theorem 5.11, we have the assumption that $X_{\mu}$ is a factor of $Y$. It is clear that $Y_{\eta}$ in Theorem 5.2 satisfies this assumption.

6. Saturation examples

In this section we give saturation examples. First we need some simple lemmas.

Lemma 6.1. Let $(X,T)$ be minimal, $\pi:X\to X_{eq}$ be the factor map, and $d\geq 2$. For a given $x\in X$, let $(x_1,x_2,\ldots,x_d)\in L_x$. Then for any subsequence $\{n_i\}\subset\mathbb{Z}$ with $T^{n_i}x\to x_j$ for some $1\leq j\leq d$, we have $T^{kn_i}x\to x'_k$ with $\pi(x_k) = \pi(x'_k)$ for any $1\leq k\leq d$.

Proof. Let $y = \pi(x)$ and $y_i = \pi(x_i), 1\leq i\leq d$. Then we have $(y_1,y_2,\ldots,y_d)\in L_y$. So, there is $g\in X_{eq}$ such that $y_i = y + ig, 1\leq i\leq d$. Since $T^{n_i}x\to x_j$, we have $T^{n_i}y\to y_j = y + jg$. Assume that $T^{n_i}y,T^{2n_i}y,\ldots,T^{dn_i}y\to (y+g',y+y+2g',\ldots,y+dg')$ for some $g'\in X_{eq}$. This implies that $g' = g$ and hence

$$(T^{n_i}y,T^{2n_i}y,\ldots,T^{dn_i}y)\to (y+g,y+2g,\ldots,y+dg).$$ 

As $T^{kn_i}x\to x'_k$, we get that $T^{kn_i}y\to \pi(x'_k)$ for each $1\leq k\leq d$. Thus, $\pi(x'_k) = y+kg = y_k = \pi(x_k)$ for each $1\leq k\leq d$. □

For any abelian group $G, A\subset G$ and a rational number $r = \frac{p}{q}$ we define

$$rA = \{g\in G : qg = pa \text{ for some } a\in A\}.$$ 

With above lemmas we can show

Theorem 6.2. Let $(X,T)$ be minimal, $d\geq 2$ and $\pi:X\to Y = X_{eq}$ be almost 1-1. Let $A = \{y\in X_{eq} : |\pi^{-1}y| \geq 2\}$. We have
(1) If $2A - A$ is also first category, then $\pi$ is saturated for $T \times T^2$.

(2) If $B_d =: \bigcup_{1 \leq i \leq d} \left( \frac{j_i}{j_i} A - \frac{1}{j_i} A \right)$ is also first category, then $\pi$ is saturated for $T \times T^2 \times \ldots \times T^d$ and $d \geq 3$.

**Proof.** First we consider the case when $d = 2$.

Let $x \in X$ and $L_x = \overline{O}((x,x), T \times T^2)$. Let $y = \pi(x)$. Then

\[ L_y = \{(y + g, y + 2g) : g \in Y\}. \]

If $y + g = y_1 \in A$ and $y + 2g = y_2 \in A$, then $y = 2y_1 - y_2 \in 2A - A$.

Let $\Omega = \pi^{-1}(2A - A)^c$. Then $\Omega$ is a dense $G_\delta$ set. We are going to show that for each $x \in \Omega$, $L_x$ is saturated. Let $y = \pi(x)$. For $(x_1, x_2) \in L_x$ we have the following 3 cases.

**Case (1):** $\pi(x_1), \pi(x_2) \in (2A - A)^c$.

In this case, $|\pi^{-1}(x_1)| = 1$ and $|\pi^{-1}(x_2)| = 1$. It is clear that $\pi^{-1}(x_1) \times \pi^{-1}(x_2) \in L_x$.

**Case (2):** $\pi(x_1) \in 2A - 2A, \pi(x_2) \in (2A - A)^c$.

In this case we may assume that $|\pi^{-1}(x_1)| \geq 2$ and $|\pi^{-1}(x_2)| = 1$. For any $z \in \pi^{-1}(x_1)$, there is a subsequence $\{n_i\}$ of $\mathbb{Z}$ with $T^{n_i} x \rightarrow z$. It is clear that $T^{2n_i} x \rightarrow x_2$ by Lemma 6.1. This implies that $\pi^{-1}(x_1) \times \pi^{-1}(x_2) \in L_x$.

**Case (3):** $\pi(x_1) \in (2A - A)^c, \pi(x_2) \in (2A - A)$.

In this case we may assume that $|\pi^{-1}(x_1)| = 1$ and $|\pi^{-1}(x_2)| \geq 2$. First we note that there is $z \in \pi^{-1}(x_2)$ such that $(x_1, z) \in L_x$. This implies that there is a subsequence $\{n_i\}$ of $\mathbb{Z}$ with $T^{n_i} x \rightarrow x_1$ and $T^{2n_i} x \rightarrow z$. This implies that $x, z \in X_1$, where $X_1$ is minimal under $T^2$. Since $(X, T)$ is almost 1-1, and hence $\pi$ is proximal. By Lemma 3.3, $\pi^{-1}(x_2) \subseteq X_1$.

Thus, for each $z' \in \pi^{-1}(x_2)$ there is a subsequence $\{m_i\}$ of $\mathbb{Z}$ such that $T^{2m_i} x \rightarrow z'$. Then we have $T^{m_i} x \rightarrow x_1$ by Lemma 6.1. This implies that $\pi^{-1}(x_1) \times \pi^{-1}(x_2) \in L_x$.

To sum up we have proved that for each $x \in \Omega$, $L_x$ is saturated for $T \times T^2$.

For the general case when $d \geq 3$ and $x \in X$, let $L_x = \overline{O}((x, \ldots, x), T \times \ldots, \times T^d)$. Let $y = \pi(x)$. Then

\[ L_y = \{(y + g, y + 2g, \ldots, y + dg) : g \in Y\}. \]

If $y + ig = y_1 \in A$ and $y + jg = y_2 \in A$ with $1 \leq i < j \leq d$, then $(j - i)g = y_2 - y_1$, and so, $y = \frac{i}{j-i} y_1 - \frac{j}{j-i} y_2 \in \frac{i}{j-i} A - \frac{j}{j-i} A$.

Since $B_d$ is also of first category, $\Omega = \pi^{-1} B^c$ is a dense $G_\delta$ subset of $X$ and the same proof is applied, since for any $x \in \Omega$ and $g \in Y$, there exists at most one $1 \leq j \leq d$ such that $\pi(x) + jg \in A$.

To sum up we have proved that $L_x$ is saturated for each $x \in \Omega$ under $T \times \ldots \times T^d$. □

**Remark 6.3.** By what we have proved we know that Denjoy, Froyad examples are saturated since for such systems, since $B_d$ is countable.
7. THE CONSTRUCTION OF A NON-SATURATED EXAMPLE

Our aim in this section is to construct a non-saturated minimal system which is a proximal extension of its maximal equicontinuous factor. The following theorem tells us that it suffices to find counterexamples in almost automorphic systems.

**Theorem 7.1.** Assume that \((X, T)\) is minimal and \(\pi : X \rightarrow X_{eq}\) is a proximal extension. If it is not \(\pi\)-saturated, then there is \(\tau^* : X_{eq}^* \rightarrow X_{eq}\) which is almost 1-1 and is not \(\tau^*\)-saturated, where \(X_{eq}^*\) is minimal.

**Proof.** If \(\pi\) is almost one to one, we are done. Thus, we may assume that \(\pi\) is not almost one to one. It is clear that \(\pi\) is not open by [12].

Now there are minimal systems \(X^*\) and \(X_{eq}^*\), and factor maps \(\pi^* : X^* \rightarrow X_{eq}^*\), \(\sigma^* : X^* \rightarrow X\) and \(\tau^* : X_{eq}^* \rightarrow X_{eq}\) such that \(\pi^*\) is open and \(\sigma^*, \tau^*\) are almost 1-1 with \(\tau^* \circ \pi^* = \pi \circ \sigma^*\).

\[
\begin{array}{ccc}
X & \xleftarrow{\sigma^*} & X^* \\
\downarrow{\pi} & & \downarrow{\pi^*} \\
X_{eq} & \xleftarrow{\tau^*} & X_{eq}^*
\end{array}
\]

By [12], \(\pi^*\) is saturated. We are going to show that \(\tau^*\) is not saturated.

Assume the contrary that \(\tau^*\) is saturated. Thus, there is a dense \(G_\delta\) set \(\Omega_1\) of \(X^*\) such that for each \(x^* \in \Omega_1\), \(L_{x^*}\) is \(\pi^*\)-saturated, and there is a dense \(G_\delta\) set \(\Omega_2\) of \(X_{eq}^*\) such that for each \(x_{eq}^* \in \Omega_2\), \(L_{x_{eq}^*}\) is \(\pi^*\)-saturated. Let \(\Omega_3 = \Omega_1 \cap (\pi^*)^{-1}\Omega_2\) and \(\Omega = \sigma^*\Omega_3\). It is clear that \(\Omega\) is a dense \(G_\delta\) set of \(X\).

For \(x \in \Omega\) pick \(x^* \in \Omega_3\) with \(\sigma^*(x^*) = x\). It is clear that \(L_{x^*}\) is \(\pi^*\)-saturated and \(\pi^*(L_{x^*}) = L_{\pi^*x^*}\) is \(\tau^*\)-saturated. This implies that \(L_{x^*}\) is \(\tau^* \circ \pi^*\)-saturated by Lemma 2.7. Thus, \(L_x = \sigma^* L_{x^*}\) is \(\pi^*\)-saturated again by Lemma 2.7, a contradiction.

\[\Box\]

In the following we will construct a minimal system \((X, T)\) which is an almost one to one extension of an equicontinuous system but not saturated for \(T \times T^2\). First we need some lemmas.

7.1. Some lemmas.

**Lemma 7.2.** [9, Proposition 2.41] Let \(G\) be a locally compact abelian group. If \(f \in L^p(G), g \in L^q(G)\), where \(1 < p < \infty\) and \(\frac{1}{p} + \frac{1}{q} = 1\), then the convolution \(f \ast g \in C_c(G)\).

**Lemma 7.3.** Let \(G\) be a locally compact abelian group and let \(\mu\) be the Haar measure of \(G\). Suppose that \(A, B\) are compact subsets of \(G\). If \(\mu(A) > 0\) and \(\mu(B) > 0\), then \(A + B\) contains a nonempty open subset.
Proof. For every $x \in G$, 
\[ \chi_A \ast \chi_B(x) = \int \chi_A(x-y)\chi_B(y)dy \]
\[ = \int \chi_{x-A}(y)\chi_B(y)dy \]
\[ = \int \chi_{(x-A)\cap B}(y)dy \]
\[ = \mu((x-A)\cap B). \]
Thus 
\[ \int \chi_A \ast \chi_B(x)dx = \int \int \chi_A(x-y)\chi_B(y)dydx = \mu(A)\mu(B) > 0. \]

Let $E = \{x \in G : \chi_A \ast \chi_B(x) > 0\} = \{x \in G : \mu((x-A)\cap B) > 0\}$. Note that for every $x \in E$, $(x-A)\cap B \neq \emptyset$ which is equivalent to $x \in A+B$. Thus $E \subset A+B$. By Lemma 7.2, $\chi_A \ast \chi_B$ is continuous. Therefore, $E$ is a nonempty open subsets contained in $A+B$. \qed

Lemma 7.4. Let $G$ be a compact metrizable monothetic group and let $\mu$ be the Haar measure of $G$. If $A \subset G$ with positive measure, then $\mu(nA) > 0$ for any $n \in \mathbb{Z} \setminus \{0\}$.

Proof. Since $G$ is monothetic, it is abelian. Note that $\mu(-nA) = \mu(nA)$ by the uniqueness of the Haar measure. So, it remains to show the theorem for $n \geq 2$. We show the case when $n = 2$, and the proof for the rest $n > 2$ is similar.

Assume that $g_0 \in G$ with $\{ng_0 : n \in \mathbb{Z}\}$ is dense in $G$. Let $G_1 = \{2ng_0 : n \in \mathbb{Z}\}$, then $G = G_1 \cup (G_1 + g_0)$. There are two cases: $G_1 = G$ or $G_1 \neq G$.

In the first case we let $\phi : G \longrightarrow G$ with $g \mapsto 2g$ and $\nu = \phi_*\mu$. And in the second case let $\phi : G \longrightarrow G_1$ with $g \mapsto 2g$ and $\nu = \phi_*\mu$. It is clear that $\phi$ is surjective.

In the first case for any $B \subset G$ and $g \in G$ we claim that 
\[ \phi^{-1}(B+g) = \phi^{-1}B + g' \text{ with } 2g' = g. \]
(Note that the existence of $g'$ follows from the subjectivity of $\phi$.) To see this, let $y \in \phi^{-1}(B+g)$. Then $2y \in B+g$ which implies that $2(y-g') \in B$. Thus, $y-g' \in \phi^{-1}B$, i.e. $y \in \phi^{-1}B + g'$. The converse is true by the same reason.

This shows that $\nu = \mu$. Thus, $\mu(2A) = \mu(\phi^{-1}(2A)) = \mu(A + \ker \phi) \geq \mu(A) > 0$.

In the second case for any $B \subset G_1$ we claim that $\phi^{-1}(B+g) = \phi^{-1}B + g'$ with $2g' = g$. This shows that $\nu$ is the Haar measure $\mu_1$ on $G_1$. Without loss of generality we assume that $A \subset G_1$, otherwise we replace $A$ by $A + g_0$ which has the same measure as $A$.

Thus, we have that $\mu_1(2A) = \mu(\phi^{-1}(2A)) \geq \mu(A) > 0$. It deduces that $\mu(2A) \geq \frac{1}{2} \mu_1(2A) > 0$, as $\mu = \frac{1}{2} \mu_1 + \frac{1}{2} \mu_2$, where $\mu_2 = \phi_* \mu_1$ with $\phi : G_1 \longrightarrow G_1 + g_0$, $g \mapsto g + g_0$. \qed

Remark 7.5. We remark that

1. The monothetic assumption is essential. $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots$ is a counterexample.
2. For the adding machine $X$ and $y \in X$, it is not true that there $x \in X$ with $2x = y$. For example, let $a = (1,0,\ldots)$. Let $b = (x_1,x_2,\ldots) \in X$ then $2b = (0,y_2,\ldots)$. So, there is no $b \in X$ with $2b = a$. This indicates the second case can happen.
(3) Lemma 7.4 can be proved by using the structure theorem of a compact metrizable abelian monothetic group. Such a group is an inverse limit of groups $G_i = \mathbb{T}^{\eta_i} \times \mathbb{Z}_{\rho_i}$. So, the problem can be deduced to the same one on $G_i$. This can be done by a direct computation.

**Corollary 7.6.** Let $G$ be a compact metrizable abelian monothetic group and let $\mu$ be the Haar measure of $G$. If $A \subset G$ with $\mu(A) > 0$, then $2A - A$ contains a non-empty open set $U$.

7.2. Toeplitz flows. Let $\Sigma$ be a finite set and $X = \Sigma^\mathbb{Z}$. Then $X$ is a compact metric space with a metric $d$ defined by

$$
\rho(x, y) = \sum_{n=-\infty}^{\infty} \frac{|x(n) - y(n)|}{2^n}.
$$

Let $T$ denote the left shift on $X$, i.e. $Tx(n) = x(n + 1)$ for all $x \in X$ and $n \in \mathbb{Z}$.

For and $x \in X$, $\sigma \in \Sigma$ and a positive integer $p \in \mathbb{N}^*$, set

$$
\text{Per}_p(x, \sigma) = \{n \in \mathbb{Z} : x(n') = x(n) = \sigma \text{ for all } n' \equiv n \mod p\},
$$

$$
\text{Per}_p(x) = \bigcup_{\sigma \in \Sigma} \text{Per}_p(x, \sigma),
$$

$$
\text{Aper}(x) = \mathbb{Z} \setminus (\bigcup_{p \in \mathbb{N}^*} \text{Per}_p(x)).
$$

A sequence $x \in X$ is called a Toeplitz sequence if $\text{Aper}(x) = \emptyset$.

By the $p$-skeleton of $x \in X$ we mean the part of $x$ which is periodic with period $p$. We call that $p$ is an essential period of $x$ if the $p$-skeleton of $x$ is not periodic with any smaller period.

**Definition 7.7.** A periodic structure for a non-periodic Toeplitz sequence $\eta$ is an increasing sequence $(p_i)_{i \in \mathbb{N}^*}$ of positive integers satisfying

1. $p_i$ is an essential period of $\eta$ for all $i \in \mathbb{N}$,
2. $p_i | p_{i+1}$ for all $i$,
3. $\bigcup_{i=1}^{\infty} \text{Per}_{p_i}(\eta) = \mathbb{Z}$.

Note that every non-periodic Toeplitz sequence has a periodic structure. In the following, we will assume that $\eta$ is a non-periodic Toeplitz sequence and fix a periodic structure $(p_i)$ of $\eta$. Let $G$ be the inverse limit group $\mathbb{Z}/p_i\mathbb{Z}$, i.e.

$$
G = \{(n_i) : n_i \in \mathbb{Z}/p_i\mathbb{Z} \text{ and } n_j \equiv n_i \mod p_i \text{ for all } i < j\}.
$$

Let $\hat{1}$ denote the element (1) in $G$ and $\hat{n} = n \cdot \hat{1}$ for $n \in \mathbb{Z}$. Then $G$ a compact monothetic group generated by $\hat{1}$.

For each $i \in \mathbb{N}^*$, $n \in \mathbb{Z}/p_i\mathbb{Z}$, set

$$
A_n^i = \{T^m \eta : m \equiv n \mod p_i\}.
$$

**Lemma 7.8.** [25, Lemma 2.3] We have

1. $A_n^i$ is exactly the set of all $\omega \in \overline{\sigma}(\eta)$ with the same $p_i$-skeleton as $T^n \eta$;
2. $\{A_n^i : n \in \mathbb{Z}/p_i\mathbb{Z}\}$ is a partition of $\overline{\sigma}(\eta)$ into relatively open sets;
3. $A_n^i \supset A_m^i$ for $i < j$ and $m \equiv n \mod p_i$;
4. $TA_n^i = A_{n+1}^i$. 

For $g = (n_i) \in G$, set $A_g = \bigcap_{i=1}^{\infty} A_{n_i}$. It is clear that $A_g$ is nonempty. Define a map $\pi: \mathcal{O} \to G$ by $\pi^{-1}(g) = A_g$.

**Theorem 7.9.** [25] We have

1. $(G, \hat{1})$ is the maximal equicontinuous factor of $(\mathcal{O}, T)$ with the factor map $\pi$.
2. $\pi(\omega) = \pi(\omega')$ if and only if $\omega$ and $\omega'$ have the same $p_i$-skeleton, for all $i \in \mathbb{N}^*$.

In particular, $\pi$ is one-to-one on the set of Toeplitz sequences in $\mathcal{O}$.

Since $\text{Per}_{p_i}(\eta)$ is periodic, it has a density in $\mathbb{Z}$ given by

$$d_i = \frac{1}{p_i} \cdot \# \{ n \in \mathbb{Z}/p_i\mathbb{Z} : n \in \text{Per}_{p_i}(\eta) \}.$$ 

Then $(d_i)$ is increasing. Set $d = \lim_{i \to \infty} d_i$. The Toeplitz sequence $\eta$ is called regular if $d = 1$.

Let $R = \{ g \in G : |\pi^{-1}(g)| = 1 \}$. Then $\pi^{-1}(R)$ consists of Toeplitz sequences in $\mathcal{O}$.

**Lemma 7.10.** [25] $R$ is measurable and

$$m(R) = \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{if } d < 1, \end{cases}$$

where $m$ denotes the Haar measure on $G$.

**7.3. The construction.** Let $\Sigma = \{0, 1, 2, 3, 4\}$ and let $(q_i)$ be an increasing sequence of even integers satisfying $q_1 \geq 6$ and $\sum_{i=1}^{\infty} \frac{1}{q_i}$ converges. Let $p_0 = 1$ and $p_i = p_{i-1} q_i$, for every $i \in \mathbb{N}^*$. Now we construct a Toeplitz sequence $\eta$ inductively.

**Step 1.** Set

$$\eta(n) = \begin{cases} 0, & n \equiv 0 \mod q_1, \\
1, & n \equiv 1 \mod q_1, \\
2, & n \equiv 2 \mod q_1, \\
3, & n \equiv \frac{q_1}{2} + 1 \mod q_1, \\
4, & n \equiv -1 \mod q_1. \end{cases}$$

For each $k \in \mathbb{Z}$, let

$$J(1, k) = [kp_1 + 3, \frac{q_1}{2}] \cup [\frac{q_1}{2} + 2, (k + 1)p_1 - 2].$$

**Step 2.** For all $n \in J(1, k)$, set

$$\eta(n) = \begin{cases} 0, & k \equiv 0 \mod q_2, \\
1, & k \equiv 1 \mod q_2, \\
2, & k \equiv 2 \mod q_2, \\
3, & k \equiv \frac{q_2}{2} + 1 \mod q_2, \\
4, & k \equiv -1 \mod q_2. \end{cases}$$

For each $k \in \mathbb{Z}$, let $J(2, k)$ denote the set of $n \in [kp_2 + 1, (k + 1)p_2 - 1]$ that has not been defined at the end of the second step.
Assume that we have completed the $i$th step. For each $k \in \mathbb{Z}$, let $J(i, k)$ denote the set of $n \in [kp_i + 1, (k + 1)p_i - 1]$ that has not been defined at the end of $i$th step.

**Step** $i + 1$. For all $n \in J(i, k)$, set

$$
\eta(n) = \begin{cases} 
0, & k \equiv 0 \mod q_{i+1}, \\
1, & k \equiv 1 \mod q_{i+1}, \\
2, & k \equiv 2 \mod q_{i+1}, \\
3, & k \equiv \frac{q_{i+1}}{2} + 1 \mod q_{i+1}, \\
4, & k \equiv -1 \mod q_{i+1}.
\end{cases}
$$

Note that $\eta$ has been defined on $[-p_i, 3p_i]$ at the end of $(i + 1)$th step and the construction is periodic at each step, so $\eta$ is a Toeplitz sequence. Furthermore, it is clear that $\eta$ is aperiodic.

**Lemma 7.11.** The sequence $\eta$ is regular if and only if $\sum_{i=1}^{\infty} \frac{1}{q_i}$ diverges.

**Proof.** Recall that $d_i = \frac{1}{p_i} \cdot \# \{ n \in \mathbb{Z} / p_i \mathbb{Z} : n \in \text{Per}_{p_i}(\eta) \}$. Then $d_1 = \frac{4}{p_1}$. For $i \geq 1$,

$$d_{i+1} = d_i + (1 - d_i) \frac{4}{q_{i+1}}.
$$

Thus

$$1 - d_{i+1} = (1 - d_1) \prod_{j=1}^{i} \left( 1 - \frac{4}{q_{j+1}} \right).$$

Then $d_i \to 1$ if and only if $\prod_{j=1}^{i} \left( 1 - \frac{4}{q_{j+1}} \right) \to 0$, if and only if $\sum_{i=1}^{\infty} \frac{1}{q_i}$ diverges..

**Lemma 7.12.** [25, Lemma 3.3]

1. For all $\omega \in \overline{\mathcal{O}}(\eta)$, $\omega(n)$ is constant on $\text{Aper}(\omega)$.
2. For each $g \in G$ and $\sigma \in \Sigma$, there is an $\omega \in \pi^{-1}(g)$ with $\omega(n) = \sigma$ for all $n \in \text{Aper}(g)$.

**Proof.** (1) Let $\pi(\omega) = g = (n_1, n_2, \cdots)$. For each $i$, $\omega$ has the same $p_i$-skeleton as $T^{n_i} \eta$ and so

$$[-n_i, p_i - n_i) \cap \text{Aper}(\omega) \subset [-n_i, p_i - n_i) \setminus \text{Per}_{p_i}(T^{n_i} \eta) = J(i, 0) - n_i.$$ 

If $T^m \eta \in \mathcal{A}_{n_i}$ then $m = n_i + kp_i$ for some $k \in \mathbb{Z}$, then $T^m \eta$ is constant on $J(i, 0) - n_i$ since $\eta$ is constant on $J(i, k)$. Since $\omega \in \mathcal{A}_{n_i}$, $\omega$ must also be constant on $J(i, 0) - n_i$. Hence $\omega$ is constant on $[-n_i, p_i - n_i) \cap \text{Aper}(\omega)$ for all $i$. If $\text{Aper}(\omega) \neq \emptyset$, then $-n_i \to -\infty$ and $p_i - n_i \to \infty$. Therefore, $\omega$ is constant on $\text{Aper}(\omega)$.

(2) Let $g = (n_i) \in G$ and $\sigma \in \Sigma$. The sequences $T^{n_i} \eta$ all have the same $p_i$-skeleton for $i > j$, so $T^{n_j} \eta(n)$ is eventually constant for each $n \in \text{Aper}(g)$. For any positive integer $i$, $T^{n_i} \eta(n) = 0$ for $n \in J(i, 0) - n_i$, which contains $[-n_i, p_i - n_i) \cap \text{Aper}(\omega)$; $T^{n_i + p_i} \eta(n) = 1$ for $n \in J(i, 1) - n_i - p_i$, which contains $[-n_i, p_i - n_i) \cap \text{Aper}(\omega)$; $T^{n_i + 2p_i} \eta(n) = 0$ for $n \in J(i, 2) - n_i - 2p_i$, which contains $[-n_i, p_i - n_i) \cap \text{Aper}(\omega)$; $T^{n_i + \left(\frac{q_{i+1}}{2} + 1\right)p_i} \eta(n) = 3$ for $n \in J(i, -1) - n_i - \left(\frac{q_{i+1}}{2} + 1\right)p_i$, which contains $[-n_i, p_i - n_i) \cap \text{Aper}(\omega)$; $T^{n_i - p_i} \eta(n) = 4$.
Lemma 7.13. For a nonnegative integer \( n \), if we write \( n = n_0 + n_1p_1 + n_2p_2 + n_3p_3 + \cdots \), where \( n_i \in \{0, 1, \cdots, q_i+1\} \), then \( j = \min \{ i : n_i \in \{0, 1, 2, q_i+1/2 + 1, q_i+1 - 1\} \} \) is the smallest integer such that \( n \in \Per_{p_{j+1}}(\eta) \).

Proof. From the construction of \( \eta \), \( n \) is defined for \( \eta \) at the \( j \)th step and the lemma is followed.

Lemma 7.14. For any \( g = (g_1, g_2, \cdots) \in G \), the fiber of \( g \) is a singleton if and only if there exists an increasing subsequence \((n_i)\) such that for each \( i \),

\[
\frac{g_{n_i+1} - g_{n_i}}{p_{n_i}} \in \left\{ 0, 1, 2, \frac{q_{n_i+1}}{2} + 1, q_{n_i+1} - 1 \right\}.
\]

Proof. Suppose that \( |\pi^{-1}(g)| = 1 \) and let \( x = \pi^{-1}(g) \). Then \( x \) is a Toeplitz sequence. Thus for any \( i \), there exists \( j \) such that \( [-g_i, p_i - g_i] \subset \Per_{p_j}(x) \). It is clear that \( j \geq i + 1 \), otherwise \( x \) is periodic. Since \( x \) has the same \( p_j \)-skeleton as \( T^{g_j/\eta} \), we have \( [-g_i, p_i - g_i] \subset \Per_{p_j}(T^{g_j/\eta}) \). Hence \( [-g_i, p_i - g_i] + g_j \subset \Per_{p_j}(\eta) \). For any \( n \in [0, p_i) \),

\[
g_j - g_i + n = n + (g_{i+1} - g_i) + (g_{i+2} - g_{i+1}) + \cdots + (g_j - g_{j-1})
\]

\[
= n + \frac{g_{i+1} - g_i}{p_i} p_i + \frac{g_{i+2} - g_{i+1}}{p_{i+1}} p_{i+1} + \cdots + \frac{g_j - g_{j-1}}{p_j} p_j.
\]

Taking \( n = 3 + 3p_1 + \cdots + 3p_{i-1} \), by Lemma 7.13, there exists \( k \) with \( i + 1 \leq k \leq j \) such that \( \frac{g_{k} - g_{k-1}}{p_k} \in \{ 0, 1, 2, \frac{q_{k+1}}{2} + 1, q_{k+1} - 1 \} \). Since \( i \) is arbitrary, we can find an increasing subsequence \((n_i)\) such that for each \( i \),

\[
\frac{g_{n_i+1} - g_{n_i}}{p_{n_i}} \in \left\{ 0, 1, 2, \frac{q_{n_i+1}}{2} + 1, q_{n_i+1} - 1 \right\}.
\]

Conversely, suppose that there exists an increasing subsequence \((n_i)\) such that for each \( i \),

\[
\frac{g_{n_i+1} - g_{n_i}}{p_{n_i}} \in \left\{ 0, 1, 2, \frac{q_{n_i+1}}{2} + 1, q_{n_i+1} - 1 \right\}.
\]

We may assume that \( -g_i \to -\infty \) and \( p_i - g_i \to \infty \), otherwise there \( g \) is \( \hat{m} \) for some integer \( m \) whence \( |\pi^{-1}(g)| = 1 \). For any \( k \), there exists \( i \) such that \( k < n_i \). Then for any \( n \in [-g_k, p_k - g_k) \),

\[
g_{n_i} - g_k + n = n + (g_{k+1} - g_k) + (g_{k+2} - g_{k+1}) + \cdots + (g_{n_i} - g_{n_i-1})
\]

\[
= n + \frac{g_{k+1} - g_k}{p_k} p_k + \frac{g_{k+2} - g_{k+1}}{p_{k+1}} p_{k+1} + \cdots + \frac{g_{n_i} - g_{n_i-1}}{p_{n_i-1}} p_{n_i-1}.
\]

Since \( \frac{g_{n_i} - g_{n_i-1}}{p_{n_i-1}} \in \{ 0, 1, 2, \frac{q_{n_i}}{2} + 1, q_{n_i} - 1 \} \), we have \( g_{n_i} - g_k + n \in \Per_{p_{n_i}}(\eta) \). For \( x \in \pi^{-1}(g) \),

\[
-g_k + n \in \Per_{p_{n_i}}(T^{g_{n_i}} \eta) = \Per_{p_{n_i}}(x).
\]

Since \( k \) and \( n \) are arbitrary, \( x \) is a Toeplitz sequence. Hence \( |\pi^{-1}(g)| = 1 \).
7.4. The verification.

**Theorem 7.15.** $\pi : (\hat{O}(\eta), T) \to (G, \hat{1})$ is almost 1-1 and not saturated.

**Proof.** Let $A = \{ g \in G : |\pi^{-1}(g)| > 1 \}$. By Lemma 7.12, $A = \{ g \in G : |\pi^{-1}(g)| = 5 \}$. By Lemma 7.10, $m(A) = 1$. Let $G_0 = \{(g_1,g_2,\cdots) \in G : 0 \leq g_i < \frac{g_0}{2} \}$ and let $A_0 = A \cap G_0$. Then $A_0$ has positive Haar measure. Hence $2A_0 - A_0$ contains a nonempty set $U$. Next we show that for any $x \in \pi^{-1}(U \setminus A)$, $L_x = \hat{O}((x,x), T \times T^2)$ is not saturated.

Let $g = \pi(x)$. Recall that $\pi(x) \in 2A_0 - A_0$. Thus $g = 2a - b$ for some $a, b \in A_0$. Suppose that $T^k\eta \to x_1 \in \pi^{-1}(a)$ and $T^{2k}\eta \to x_2 \in \pi^{-1}(b)$.

Firstly, assume $x = \eta$. Then $2a = b$ and $L_\eta$ is not saturated by the following claim.

**Claim.** If $x_1(n) = 0$ for $n \in \text{Aper}(a)$, then $x_2(n) \neq 2$ for $n \in \text{Aper}(b)$.

**Proof of Claim.** Let $r(i)$ be a positive integer such that $\hat{k}_i$ agrees with $a$ on $[1, r(i)]$. Then we can write $k_i$ in the form of

$$k_i = a_{r(i)} + s_0p_{r(i)} + s_1p_{r(i)+1} + s_2p_{r(i)+2} + \cdots,$$

where $s_j \in \{0, 1, \cdots, q_{r(i)+j+1} - 1\}$. It follows that $T^{k_i}\eta$ has the same $p_{r(i)}$-skeleton as $T^{a_{r(i)}}\eta$. Thus $T^{k_i}\eta$ has the same $p_{r(i)}$-skeleton as $x_1$. Set

$$t_i = \min \left\{ j \geq 0 : s_j \in \{0, 1, 2, \frac{q_{r(i)+j+1}}{2} + 1, q_{r(i)+j+1} - 1\} \right\}.$$

Then for any $n \in \text{Aper}(x_1) \cap [-a_{r(i)}, p_{r(i)} - a_{r(i)}], x_1(n) = 0$ and

$$T^{k_i}\eta(n) = \eta(n + k_i) = \begin{cases} 0, & \text{if } s_{t_i} = 0, \\ 1, & \text{if } s_{t_i} = 1, \\ 2, & \text{if } s_{t_i} = 2, \\ 3, & \text{if } s_{t_i} = \frac{q_{r(i)+t_i+1}}{2} + 1, \\ 4, & \text{if } s_{t_i} = q_{r(i)+t_i+1} - 1. \end{cases}$$

Since $\hat{k}_i \to a$, $r(i) \to \infty$. By passing to some subsequence, we may assume that $r(i)$ is increasing with respect to $i$. Note that $-a_{r(i)} \to -\infty$ and $p_{r(i)} - a_{r(i)} \to \infty$, otherwise $a = \hat{m}$ for some integer $m$ and then $a \notin A$. Thus for all sufficiently large $i$, $\text{Aper}(x_1) \cap [-a_{r(i)}, p_{r(i)} - a_{r(i)}] \neq \emptyset$. Note that $[-a_{r(i)}, p_{r(i)} - a_{r(i)}]$ is nested with respect to $i$. Therefore, there exists $i_0$ such that for any $i \geq i_0$, $s_{t_i} = 0$.

Note that

$$2k_i = 2a_{r(i)} + 2s_0p_{r(i)} + 2s_1p_{r(i)+1} + 2s_2p_{r(i)+2} + \cdots,$$

$$= b_{r(i)} + s'_0p_{r(i)} + s'_1p_{r(i)+1} + s'_2p_{r(i)+2} + \cdots,$$
where $s_j' \in \{0, 1, \cdots, q_{r(i)+j+1} - 1\}$. Then for each $j$,

\[
{s_j'} = \begin{cases} 
2s_j + 2s_j + 1, & \text{if } s_j < \frac{q_{r(i)+j+1}}{2}, \\
2s_j - q_{r(i)+j+1} \text{ or } 2s_j - q_{r(i)+j+1} + 1, & \text{if } s_j \geq \frac{q_{r(i)+j+1}}{2}.
\end{cases}
\]

Set

\[
t_i' = \min\left\{ j \geq 0 : s_j' \in \{0, 1, 2, \frac{q_{r(i)+j+1}}{2} + 1, q_{r(i)+j+1} - 1\} \right\}.
\]

Then for any $n \in \text{Aper}(x_1) \cap [-b_{r(i)}, p_{r(i)} - b_{r(i)}]$,

\[
T^{2k_i} \eta(n) = \eta(n + 2k_i) = \begin{cases} 
0, & \text{if } s_j' = 0, \\
1, & \text{if } s_j' = 1, \\
2, & \text{if } s_j' = 2, \\
3, & \text{if } s_j' = \frac{q_{r(i)+j+1}}{2} + 1, \\
4, & \text{if } s_j' = q_{r(i)+j+1} - 1.
\end{cases}
\]

For any $i \geq i_0$, since $s_{i_0} = 0$, we have $s_{i} = 0$ or $1$. Thus $t_i' \leq t_i$. In the following, we assume that $i \geq i_0$. If $s_j' = 2$, then $t_i' < t_i$ and $s_j' = 1$ or $\frac{q_{r(i)+j+1}}{2} + 1$. This contradicts the choice of $t_i$ which is the smallest integer $j$ such that

\[
s_j \in \left\{0, 1, 2, \frac{q_{r(i)+j+1}}{2} + 1, q_{r(i)+j+1} - 1\right\}.
\]

Therefore, for any $i \geq i_0$ and $n \in \text{Aper}(x_2) \cap [-b_{r(i)}, p_{r(i)} - b_{r(i)}]$, $T^{2k_i}(n) \neq 2$. Hence $x_2(n) \neq 2$ for $n \in \text{Aper}(b)$. This proves the claim.

Now we are going to deal with the general case. We write $a = g + h$ and $b = g + 2h$ for some $h \in G$. In this case we also claim that if $x_1(n) = 0$ for $n \in \text{Aper}(a)$, then $x_2(n) \neq 2$ for $n \in \text{Aper}(b)$.

Let $r(i)$ be a positive integer such that $\hat{k}_i$ agrees with $h$ on $[1, r(i)]$. Furthermore, we may assume that $r(i)$ is increasing with respect to $i$. Then we can write $k_i$ in the form of

\[
k_i = h_{r(i)} + s_0 p_{r(i)} + s_1 p_{r(i)+1} + s_2 p_{r(i)+2} + \cdots,
\]

where $s_j \in \{0, 1, \cdots, q_{r(i)+j+1} - 1\}$. It follows that $T^{k_i} x$ has the same $p_{r(i)}$-skeleton as $T^{h_{r(i)} x}$. Thus $T^{k_i} x$ has the same $p_{r(i)}$-skeleton as $x_1$. Then for any $n \in \text{Aper}(x_1) \cap [-a_{r(i)}, p_{r(i)} - a_{r(i)}]$, $x_1(n) = 0$ and

\[
n + k_i \in [-g_{r(i)}, p_{r(i)} - g_{r(i)}] + l_i,
\]

where

\[
l_i = \begin{cases} 
k_i - h_{r(i)}, & \text{if } a_{r(i)} = g_{r(i)} + h_{r(i)}, \\
k_i - h_{r(i)} + p_{r(i)}, & \text{if } a_{r(i)} = g_{r(i)} + h_{r(i)} - p_{r(i)}.
\end{cases}
\]

Since $a_{r(i)} < b_{r(i)} < \frac{p_{r(i)}}{2}$, if $a_{r(i)} = g_{r(i)} + h_{r(i)}$ then $b_{r(i)} = g_{r(i)} + 2h_{r(i)}$, and if $a_{r(i)} = g_{r(i)} + h_{r(i)} - p_{r(i)}$ then $b_{r(i)} = g_{r(i)} + 2h_{r(i)} - p_{r(i)}$. Thus for any $m \in \text{Aper}(x_2) \cap
where

\[ l'_i = \begin{cases} 
2(k_i - h_{r(i)}), & \text{if } a_{r(i)} = g_{r(i)} + h_{r(i)}; \\
2(k_i - h_{r(i)}) + p_{r(i)}, & \text{if } a_{r(i)} = g_{r(i)} + h_{r(i)} - p_{r(i)}. 
\end{cases} \]

Write \( l_i = s_0 p_{r(i)} + s_1 p_{r(i)+1} + s_2 p_{r(i)+2} + \cdots \), where \( s_j \in \{0, 1, \ldots, q_{r(i)+j+1} - 1\} \). Set

\[ t_i = \min \left\{ j \geq 0 : s_j \in \{0, 1, 2, \frac{q_{r(i)+j+1}}{2} + 1, q_{r(i)+j+1} - 1\} \right\}. \]

Then for any \( n \in \text{Aper}(x_1) \cap [-a_{r(i)}, p_{r(i)} - a_{r(i)}), x_1(n) = 0 \) and

\[ T^{k_i}x(n) = x(n + k_i) = \begin{cases} 
0, & \text{if } s_{t_i} = 0, \\
1, & \text{if } s_{t_i} = 1, \\
2, & \text{if } s_{t_i} = 2, \\
3, & \text{if } s_{t_i} = \frac{q_{r(i)+j+1}}{2} + 1, \\
4, & \text{if } s_{t_i} = q_{r(i)+t_i+1} - 1.
\end{cases} \]

Case 1. \( a_{r(i)} = g_{r(i)} + h_{r(i)} \).

Then we have \( l'_i = 2l_i \). Similar to the case that \( x = \eta \), we have \( T^{2k_i}x(n) \neq 2 \) for any \( n \in \text{Aper}(x_2) \cap [-b_{r(i)}, p_{r(i)} - b_{r(i)}). \)

Case 2. \( a_{r(i)} = g_{r(i)} + h_{r(i)} - p_{r(i)} \).

In this case, we have \( l'_i = 2l_i - p_i \). We write \( 2l_i = s'_0 p_{r(i)} + s'_1 p_{r(i)+1} + s'_2 p_{r(i)+2} + \cdots \), where \( s'_j \in \{0, 1, \ldots, q_{r(i)+j+1} - 1\} \).

Case 2a. \( s'_0 = 0 \).

Then \( l'_i = (q_{r(i)+1} - 1)p_{r(i)} + sp_{r(i)+1} \) for some nonnegative integer \( s \). For any \( n \in \text{Aper}(x_2) \cap [-b_{r(i)}, p_{r(i)} - b_{r(i)}), \) we have \( T^{2k_i}x(n) = 0 \).

Case 2b. \( s'_0 > 0 \).

Then \( l'_i = (s'_0 - 1)p_{r(i)} + s'_1 p_{r(i)+1} + s'_2 p_{r(i)+2} + \cdots \). Set

\[ t'_i = \min \left\{ j \geq 0 : s'_j \in \{0, 1, 2, \frac{q_{r(i)+j+1}}{2} + 1, q_{r(i)+j+1} - 1\} \right\}. \]

Note that \( s'_0 = 2s_0 \) or \( 2s_0 - p_{r(i)} \) and for each \( j > 0, \)

\[ s'_j = \begin{cases} 
2s_j \text{ or } 2s_j + 1, & \text{if } s_j < \frac{q_{r(i)+j+1}}{2}, \\
2s_j - q_{r(i)+j+1} \text{ or } 2s_j - q_{r(i)+j+1} + 1, & \text{if } s_j \geq \frac{q_{r(i)+j+1}}{2}.
\end{cases} \]

If \( t'_i = 0 \) and \( s'_0 - 1 = 2 \), then \( s'_0 = 3 \). This is impossible since \( p_{r(i)} \) is an even number. If \( t'_i > 0 \), then \( s'_0 \neq 2 \) by the same reason as the discussion in case of \( x = \eta \). In any case, we have \( T^{2k_i}x(n) \neq 2 \) for any \( n \in \text{Aper}(x_2) \cap [-b_{r(i)}, p_{r(i)} - b_{r(i)}). \)

By the above discussion, we have \( x_2(n) \neq 2 \) for \( n \in \text{Aper}(b) \). Hence \( L_\eta \) is not saturated.

\[ \square \]

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