Minimality of the well-rounded retract

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We prove that the well-rounded retract of $SO_n \backslash SL_n \mathbb{R}$ is a minimal $SL_n \mathbb{Z}$–invariant spine.

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1 Introduction

In this note we are interested in a certain $SL_n \mathbb{Z}$–invariant deformation retract of the symmetric space $S_n = SO_n \backslash SL_n \mathbb{R}$. To every element $A \in SL_n \mathbb{R}$ one can associate the lattice $AZ^n$ in $\mathbb{R}^n$. The element $A$ is well-rounded if the set of shortest nonzero vectors of the lattice $AZ^n$ generate $\mathbb{R}^n$ as a real vector space. This property is invariant under the left action of $SO_n$ and hence there is no ambiguity in saying that an element in $S_n$ is well-rounded. The subset $\mathcal{X}$ of $S_n$ consisting of well-rounded elements is homeomorphic to an $n(n-1)/2$–dimensional CW-complex and the right action of $SL_n \mathbb{Z}$ on $S_n$ induces a cocompact action on $\mathcal{X}$. Observe that if $n = 2$ then $\mathcal{X}$ is the dual to the Farey tesselation of $S_2 = \mathbb{H}^2$ and hence homeomorphic to the Bass–Serre tree of $SL_2 \mathbb{Z}$. For larger $n$, the set $\mathcal{X}$ does not have such a simple description, but Lannes and Soulé proved that $\mathcal{X}$ is a deformation retract of $S_n$ and hence contractible (see Soulé [8] for the case of $n = 3$, and Ash [3] for all $n$, treated in a more general setting). This is why the subset $\mathcal{X}$ is known as the well-rounded retract of $S_n$. Our goal is to show that $\mathcal{X}$ is a minimal $SL_n \mathbb{Z}$–invariant spine of $S_n$.

Definition 1.1 Let $\Gamma$ be a group acting discretely on a contractible space $S$. We say that a closed subset $\mathcal{X}$ of $S$ is a minimal $\Gamma$–invariant spine if it is $\Gamma$–invariant, contractible and does not properly contain any closed set with these properties.

We prove:

Theorem 1.2 The well-rounded retract $\mathcal{X}$ is a minimal $SL_n \mathbb{Z}$–invariant spine of the symmetric space $S_n = SO_n \backslash SL_n \mathbb{R}$.
It has long been known that the well-rounded retract does not contain any smaller dimensional $\text{SL}_n\mathbb{Z}$–invariant spines. This follows namely from the fact due to Borel–Serre [5] that the group $\text{SL}_n\mathbb{Z}$ has virtual cohomological dimension

$$\text{vcdim}(\text{SL}_n\mathbb{Z}) = \frac{n(n-1)}{2} = \dim \mathcal{X}.$$ 

In order to appreciate the difference between this statement and the claim of Theorem 1.2 it should be observed that the well-rounded retract contains interesting $\text{SL}_n\mathbb{Z}$–invariant subsets of dimension $n(n-1)/2$. For instance, recall that an element $A \in \text{SL}_n\mathbb{R}$ is well-rounded if the set of shortest nonzero vectors of the lattice $AZ^n$ generate $\mathbb{R}^n$ as a vector space; equivalently, they generate, as a group, a finite index lattice of $AZ^n$. We will say that $A \in \text{SL}_n\mathbb{R}$ is extremely well-rounded if the shortest nonzero vectors of $AZ^n$ generate the whole lattice $AZ^n$. The subset $\mathcal{X}'$ of $S_n$ consisting of extremely well-rounded elements is $\text{SL}_n\mathbb{Z}$–invariant and has dimension $n(n-1)/2$. While $\mathcal{X}' = \mathcal{X}$ for $n = 2, 3$ and 4, the set $\mathcal{X}'$ is a proper subset of the well-rounded retract for $n \geq 5$. In [7] we proved that $\mathcal{X}'$ is not contractible for $n \geq 5$. This result follows now directly from Theorem 1.2:

**Corollary 1.3** [7] The subset $\mathcal{X}' \subset S_n$ of extremely well-rounded elements is not contractible.

In order to prove Theorem 1.2 it suffices to show that whenever $\mathcal{Y}$ is a closed proper $\text{SL}_n\mathbb{Z}$–invariant subset of $\mathcal{X}$, there is a torsion-free, finite index subgroup $\Gamma \subset \text{SL}_n\mathbb{Z}$ such that the inclusion $\mathcal{Y}/\Gamma \hookrightarrow \mathcal{X}/\Gamma$ is not a homotopy equivalence. We proceed as follows: First we show that there is $A \in \mathcal{X} \setminus \mathcal{Y}$ with the property that there is a torsion-free, finite index subgroup $\Gamma$ of $\text{SL}_n\mathbb{Z}$ and a nontrivial homology class $[\alpha] \in H_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ represented by a cycle $\alpha$ which intersects the well-rounded retract exactly at $A$. Here $\bar{M}_\Gamma$ is the Borel–Serre compactification of the locally symmetric space $M_\Gamma = S_n/\Gamma$ and the homology is with coefficients in the ring $\mathbb{Z}/2\mathbb{Z}$. The class $[\alpha]$ is dual to some class $[\beta] \in H_{n(n-1)/2}(M_\Gamma)$. The fact that the cycle $\alpha$ does not intersect $\mathcal{Y}$ implies that $[\beta]$ is not in the image of $H_*(\mathcal{Y}/\Gamma)$ in $H_*(\mathcal{X}/\Gamma)$. This shows that the inclusion $\mathcal{Y}/\Gamma \hookrightarrow \mathcal{X}/\Gamma$ is not a homotopy equivalence.

In [7], we used this strategy to prove Corollary 1.3. In that particular case we faced much simpler technical problems since it was possible to explicitly find a rational maximal flat intersecting $\mathcal{X}$ exactly once, at a point outside of $\mathcal{X}'$. Even in the case $n = 2$, it is easy to see that for a generic point $A \in \mathcal{X}$, every maximal flat through $A$ intersects $\mathcal{X}$ many times. To bypass this problem we give an elementary, though somewhat involved, construction of the cycle $\alpha$. 

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The paper is organized as follows: In Section 2 we review some facts about the symmetric space $S_n = \text{SO}_n \setminus \text{SL}_n \mathbb{R}$ and its quotients. In Section 3 we discuss some properties of the well-rounded retract, proving that a generic well-rounded element in $S_n$ has exactly $2n$ shortest vectors. In Section 4 we show that certain homology classes are nontrivial; all the results in this section are surely well known. In Section 5 we derive Theorem 1.2 from a result, Proposition 5.1, proved in Section 6. Proposition 5.1, the key point of this paper, yields nontrivial cycles in $C_{n-1}(\bar{M}_\Gamma, \partial \bar{M}_\Gamma)$ which intersect the well-rounded retract at a single point.

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**Notation** We denote by $\{e_1, \ldots, e_n\}$ and $|\cdot|$ the standard basis and Euclidean norm of $\mathbb{R}^n$. Sometimes we will write elements in $\mathbb{R}^n$ as columns and sometimes as rows; we hope that this does not cause any confusion. If $U$ is a linear subspace of $\mathbb{R}^n$, denote by $U^\perp$ its orthogonal complement with respect to the standard Euclidean product. We will use the same symbol to denote both an equivalence class and a representative of the equivalence class. For example, we use the same notation for an element in $\text{SL}_n \mathbb{R}$ and for the corresponding element in the symmetric space $S_n = \text{SO}_n \setminus \text{SL}_n \mathbb{R}$, or in even smaller quotients such as $S_n / \text{SL}_n \mathbb{Z}$. We will however consistently denote the homology class corresponding to a cycle $\alpha$ by $[\alpha]$. All the homology groups considered below have coefficients in the field $\mathbb{Z}/2\mathbb{Z}$ of two elements, although everything remains true with respect to any other commutative ring with unit.

## 2 The symmetric space $S_n = \text{SO}_n \setminus \text{SL}_n \mathbb{R}$

Up to scaling, the manifold $S_n = \text{SO}_n \setminus \text{SL}_n \mathbb{R}$ admits a unique symmetric metric invariant under the right action of $\text{SL}_n \mathbb{R}$; we shall always assume $S_n$ to be endowed with such a metric. The restriction of the right action of $\text{SL}_n \mathbb{R}$ on $S_n$ to $\text{SL}_n \mathbb{Z}$ is discrete. Moreover, any torsion-free subgroup $\Gamma$ of $\text{SL}_n \mathbb{Z}$ acts freely and hence the quotient $\bar{M}_\Gamma = S_n / \Gamma$ is a smooth locally symmetric manifold. It is well known that
SL\(_n\)\(\mathbb{Z}\) contains torsion-free finite index subgroups. If \(\Gamma \subset \operatorname{SL}_n\mathbb{Z}\) is any such subgroup, then the manifold \(M_\Gamma\) is not compact, but is homeomorphic to the interior of a compact manifold \(\tilde{M}_\Gamma\), the so-called Borel–Serre compactification of \(M_\Gamma\) [5].

For every \(v \in \mathbb{R}^n\), the length function
\[
l_v: S_n \rightarrow \mathbb{R}, \quad l_v(A) = |Av|
\]
is well-defined, analytic and convex. In particular we have
\[(2-1) \quad l_v(A'') \leq \max\{l_v(A), l_v(A')\}\]
for all \(A, A' \in S_n\) and every \(A''\) in the unique geodesic segment \([A, A']\) joining \(A\) and \(A'\) in \(S_n\). It should be observed that for every \(B \in \operatorname{SL}_n\mathbb{R}\) we have \(l_v(AB) = l_Bv(A)\). Since \(\operatorname{SL}_n\mathbb{Z}\) acts on the set \(\mathbb{Z}^n \setminus \{0\}\), this implies that the function
\[(2-2) \quad \text{syst}_1: S_n \rightarrow (0, \infty), \quad \text{syst}_1(A) = \min_{v \in \mathbb{Z}^n, v \neq 0} l_v(A)\]
is \(\operatorname{SL}_n\mathbb{Z}\)–invariant. The quantity \(\text{syst}_1(A)\) is said to be the systole, or first minimum, of \(A \in S_n\). The elements of the set
\[(2-3) \quad S_1(A) = \{v \in \mathbb{Z}^n \mid l_v(A) = \text{syst}_1(A)\}\]
are said to be the systoles or shortest vectors of \(A\).

Ash proved in [2] that the systole function is a topological Morse function (see also Bavard [4] and Akrout [1]). Moreover, the induced function on \(S_n/\operatorname{SL}_n\mathbb{Z}\) is proper by the following theorem:

**Mahler’s compactness theorem** A closed subset \(K \subset S_n/\operatorname{SL}_n\mathbb{Z}\) is compact if and only if there is \(\epsilon > 0\) with \(\text{syst}_1(A) \geq \epsilon\) for all \(A \in K\).

We deduce from (2–1) and Mahler’s compactness theorem the following important observation:

**Lemma 2.1** Let \(\Gamma\) be a torsion-free subgroup of \(\operatorname{SL}_n\mathbb{Z}\), \(N\) a manifold, and \(f, g: N \rightarrow S_n\) two continuous maps such that for all \(\epsilon > 0\) there is a compact set \(K_\epsilon \subset N\) with the following property:
\[(*) \quad \text{For all } x \notin K_\epsilon \text{ there is } v \in \mathbb{Z}^n \setminus \{0\} \text{ with } l_v(f(x)), l_v(g(x)) < \epsilon.\]

Then the compositions of \(f\) and \(g\) with the projection \(\pi: S_n \rightarrow M_\Gamma\) are properly homotopic.

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Proof Let $H: N \times [0, 1] \to S_n$ be the geodesic homotopy from $f$ to $g$, i.e. $t \to H_t(x)$ traverses with constant velocity the geodesic segment $[f(x), g(x)]$. We claim that $h = \pi \circ H$ is proper. Let $C$ be a compact subset of $M_\Gamma = S_n / \Gamma$. By Mahler’s compactness theorem there is some $\epsilon$ positive with $\text{syst}_1(A) \geq \epsilon$ for all $A \in C$. For such an $\epsilon$, let $K_\epsilon \subset N$ be the compact subset provided by $(\ast)$. Then for $x \notin K_\epsilon$ there is some $v_x \in \mathbb{Z}$, $v_x \neq 0$, with $l_{v_x}(f(x)), l_{v_x}(g(x)) < \epsilon$. By (2–1) we have then $l_{v_x}(H_t(x)) < \epsilon$ for all $t \in [0, 1]$. This implies that $h^{-1}(C) \subset K_\epsilon \times [0, 1]$, proving that it is proper.

We will use Lemma 2.1 several times in the following situation.

**Corollary 2.2** Assume that $\Gamma$ is a finite index subgroup of $\text{SL}_n \mathbb{Z}$, and that $N \subset \text{SL}_n \mathbb{R}$ projects properly to $M_\Gamma = \text{SO}_n \setminus \text{SL}_n \mathbb{R} / \Gamma$. Then for every $B \in \text{SL}_n \mathbb{R}$ the projections of $N$ and of $BN = \{Bx, x \in N\}$ to $M_\Gamma$ are properly homotopic.

### 3 The well-rounded retract

In this section we discuss briefly some of the properties of the well-rounded retract. Recall the definition of the systole (2–2) and of the set of systoles (2–3) of a point $A \in S_n$. Let also

$$(3–1) \quad \Lambda_1(A) = \text{Span}_\mathbb{R}(S_1(A))$$

be the linear subspace of $\mathbb{R}^n$ generated by the set of systoles of $A$.

**Definition 3.1** An element $A \in S_n$ is well-rounded if $\Lambda_1(A) = \mathbb{R}^n$. The subset $\mathcal{X}$ of $S_n$ consisting of all well-rounded elements is called the well-rounded retract.

As mentioned in the introduction, Soulé [8] and Ash [3] proved that $\mathcal{X}$ is an $\text{SL}_n \mathbb{Z}$–invariant deformation retract. The idea behind this result is simple and beautiful, and so we explain it briefly here:

**Theorem 3.2** (Soulé, Ash) The well-rounded retract $\mathcal{X}$ is a deformation retract of $S_n$.

For $k = 1, \ldots, n$ let $\mathcal{X}_k$ be the set of those $A \in S_n$ for which we have $\dim \Lambda_1(A) \geq k$. We have the following chain of nested $\text{SL}_n \mathbb{Z}$–invariant subspaces:

$$\mathcal{X} = \mathcal{X}_n \subset \mathcal{X}_{n-1} \subset \cdots \subset \mathcal{X}_1 = S_n$$

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In order to prove Theorem 3.2 it suffices to show that for \( k = 1, \ldots, n - 1 \) the space \( X_{k+1} \) is an \( \text{SL}_n \mathbb{Z} \)-equivariant spine of \( X_k \); we construct a retraction. Given \( A \in X_k \) and \( \lambda \in \mathbb{R} \), consider the one-parameter family of linear maps:

\[
T^\lambda_A \in \text{SL}_n \mathbb{R}, \quad T^\lambda_A(v) = \begin{cases} e^{(n-k)\lambda}v & \text{for } v \in A\Lambda_1(A) \\ e^{-k\lambda}v & \text{for } v \in (A\Lambda_1(A))^\perp \end{cases}
\]

In other words, for positive \( \lambda \) the map \( T^\lambda_A \) expands the subspace generated by the image of the shortest vectors of \( A \), while contracting the orthogonal complement. Observe that for \( U \in \text{SO}_n \) we have \( T^\lambda_{UA}UA = UT^\lambda_AA \); hence the point \( T^\lambda_AA \in S_n \) depends only on \( A \) and not on the choice of representative.

Now \( T^0_AA = A \), and if \( A \in X_k \setminus X_{k+1} \), there is some \( \lambda \) positive with \( T^\lambda_AA \in X_{k+1} \). For \( A \in X_k \), let \( \tau(A) \geq 0 \) be maximal such that

\[
T^\lambda_A \in X_k \setminus X_{k+1} \text{ for all } \lambda \in [0, \tau(A)).
\]

By definition \( \tau(A) = 0 \) for \( A \in X_{k+1} \). The function \( A \mapsto \tau(A) \) is continuous on \( X_k \), which implies that

\[
[0,1] \times X_k \to X_k, \quad (t, A) \mapsto T^{t\tau(A)}_AA
\]

is continuous as well. By definition, this homotopy is \( \text{SL}_n \mathbb{Z} \)-equivariant, starts with the identity, and ends with a projection of \( X_k \) to \( X_{k+1} \). This proves that \( X_{k+1} \) is an \( \text{SL}_n \mathbb{Z} \)-equivariant spine of \( X_k \) for \( k = 1, \ldots, n - 1 \), concluding the sketch of the proof of Theorem 3.2.

It is not difficult to prove that \( X_k \) is a codimension \( k - 1 \) semi-algebraic set, i.e., that it is given by a locally finite collection of inequalities and (quadratic) algebraic equations. Hence \( X \) is homeomorphic to a CW-complex of dimension \( \dim(X) = \dim S_n - (n - 1) = n(n - 1)/2 \). It is also easy to see that \( X/\Gamma \) is compact. We prove now that a generic point in \( X \) has exactly \( 2n \) shortest vectors:

**Proposition 3.3** The set of those \( A \in X \) for which there are \( v_1, \ldots, v_n \in \mathbb{Z}^n \) linearly independent with \( S_1(A) = \{\pm v_1, \ldots, \pm v_n\} \) is dense in \( X \).

In order to prove Proposition 3.3 we will use the following not very surprising but also not completely obvious geometric lemma.

**Lemma 3.4** Assume that \( S \) is a finite subset of the sphere \( S^{n-1} \) in \( \mathbb{R}^n \) with the property that \( \mathbb{R}^n = \text{Span}_\mathbb{R} S \) and assume that if \( v \in S \) then \(-v \in S\) as well. Then there is basis \( B \) of \( \mathbb{R}^n \) contained in \( S \) and a linear map \( F: \mathbb{R}^n \to \mathbb{R}^n \) close to the identity such that for \( v \in S \) we have \( |Fv| = |v| \) if \( \pm v \in B \) and \( |Fv| > |v| \) otherwise.
Assuming Lemma 3.4, we prove Proposition 3.3. Given \( A \in \mathcal{A} \) choose a representative in \( SL_n \mathbb{R} \), again denoted by \( A \). By definition, the image \( AS_1(A) \) of the set of systoles of \( A \) generates \( \mathbb{R}^n \) and is contained in the round sphere \( S_{syst_1(A)}^{n-1} \) of radius syst_1(A). Let \( \mathcal{B} \subset AS_1(A) \) and \( F: \mathbb{R}^n \to \mathbb{R}^n \) be the basis and the linear map provided by Lemma 3.4. We set \( A^{-1} \mathcal{B} = \{ v_1, \ldots, v_n \} \) and \( A' = (1/\sqrt{n/\det(F)}) FA \). Since we may assume that \( F \) is very close to the identity, we have that \( A' \) is very close to \( A \), and hence \( S_1(A') \subset S_1(A) \). It follows now from Lemma 3.4 that \( S_1(A') = \{ \pm v_1, \ldots, \pm v_n \} \). This concludes the proof of Proposition 3.3.

We prove now Lemma 3.4:

**Proof of Lemma 3.4** We use induction on the number of elements in \( S \). There is nothing to show if \( S \) has 2n elements, so assume that we have proved the lemma for all sets with at most \( 2k \geq 2n \) elements, and that \( S \) has \( 2(k+1) \) elements. Observe that there is a codimension one linear subspace \( U \subset \mathbb{R}^n \) generated by \( U \cap S \) such that there are at least four elements in \( S \) which don’t belong to \( U \) (recalling that if \( v \in S \), then \(-v \in S \) as well). We first describe a map \( F_1: \mathbb{R}^n \to \mathbb{R}^n \) which will allow us to apply our inductive hypothesis.

We choose \( v \in S \), \( v \notin U \) with minimal angle \( \angle(U,v) = \theta \in (0, \pi/2) \). Let \( V \) be the codimension one linear subspace containing \( v \) and the intersection \( (\mathbb{R}v)^\perp \cap U \) of the orthogonal complement of \( \mathbb{R}v \) and \( U \). The planes \( U \) and \( V \) have angle \( \theta \) and divide \( \mathbb{R}^n \) into two open sectors, \( C_1 \) and \( C_2 \) with angle \( \theta \), and two also open sectors, \( C_3 \) and \( C_4 \) with angle \( \pi - \theta \). By the minimality of \( \theta \), any vector in \( S \) which is not in \( U \cup \{ \pm v \} \) has angle at least \( \theta \) with \( U \) and so is not contained in \( V \). Moreover, for the same reason, we have \( S \cap (C_1 \cup C_2) = \emptyset \), but \( S \cap (C_3 \cup C_4) \neq \emptyset \).

For \( \eta > \theta \) with \( \eta - \theta \) small we can consider the linear map \( F_1: \mathbb{R}^n \to \mathbb{R}^n \) which is the identity on \( U \), an isometry when restricted to \( V \), and which opens \( C_1 \) and \( C_2 \) to angle \( \eta \). The map \( F_1 \) preserves the length of vectors in \( U \cup V \), reduces the length of vectors in \( C_1 \cup C_2 \) and increases the length of vectors in \( C_3 \cup C_4 \). In particular, \( F_1 \) maps \( (S \cap U) \cup \{ \pm v \} \) to the subset \( (S \cap U) \cup \{ \pm F_1(v) \} \) of \( \mathbb{R}^n \) which still generates \( \mathbb{R}^n \), and increases the length of the (at least two) remaining vectors in \( S \).

The induction hypothesis now applies to the set \( (S \cap U) \cup \{ \pm F_1(v) \} \) of cardinality at most \( 2k \) : there is a basis \( \mathcal{B}_1 \) of \( \mathbb{R}^n \) contained in \( (S \cap U) \cup \{ \pm F_1(v) \} \), and a map \( F_2: \mathbb{R}^n \to \mathbb{R}^n \) which preserves the lengths of the elements of \( \mathcal{B}_1 \) (and their negatives) and increases the lengths of all other vectors in \( (S \cap U) \cup \{ \pm F_1(v) \} \). We require that \( F_2 \) be close enough to the identity that the vectors in \( F_1(S) \) of length greater than one remain so after applying \( F_2 \). Now the basis \( B = F_1^{-1}(\mathcal{B}_1) \) and the map \( F = F_2 \circ F_1: \mathbb{R}^n \to \mathbb{R}^n \) satisfy the requirements of the lemma for the set \( S \).
4 A bit of homology

In this section we give elementary proofs of some homological results which are probably well known to experts and nonexperts alike.

As mentioned above, $\text{SL}_n\mathbb{Z}$ contains torsion-free subgroups of finite index, and any such subgroup acts freely and discretely on $S_n$; as always, we denote the quotient manifold by $M_\Gamma = S_n/\Gamma$ and its Borel–Serre compactification by $\overline{M}_\Gamma$. If $U \subset \overline{M}_\Gamma$ is a regular neighborhood of $\partial \overline{M}_\Gamma$, we have $H_*(\overline{M}_\Gamma, U) \cong H_*(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$. In particular, we can consider every properly immersed submanifold of $M_\Gamma$ as a cycle in $C_*(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$. Recall that we always consider homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$.

Before stating the main result of this section, we recall that by Lefschetz duality there is a nondegenerate pairing

$$\iota: H_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma) \times H_{n(n-1)/2}(M_\Gamma) \to \mathbb{Z}/2\mathbb{Z}$$

which can be computed as follows. Given homology classes $[\alpha] \in H_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$ and $[\beta] \in H_{n(n-1)/2}(M_\Gamma)$, represent them by cycles $\alpha$ and $\beta$ in general position. Then $\iota([\alpha], [\beta])$ is just the parity of the cardinality of the set $\alpha \cap \beta$. Observe that in order to prove that a cycle $\beta \in C_{n(n-1)/2}(M_\Gamma)$ represents a nontrivial homology class, it suffices to find a cycle $\alpha \in C_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$ which intersects $\beta$ transversally at a single point; if this is the case we will say that the two classes $[\alpha]$ and $[\beta]$ are dual to each other. This is the argument used in [7] to prove:

**Proposition 4.1** Let $\Gamma$ be a finite index torsion-free subgroup of $\text{SL}_n\mathbb{Z}$, $\Delta$ the connected component of the identity in the diagonal subgroup of $\text{SL}_n\mathbb{R}$ and $\text{Nil}$ the subgroup of $\text{SL}_n\mathbb{R}$ consisting of upper triangular matrices with units in the diagonal. Then the projection of $\Delta$ and $\text{Nil}$ to $M_\Gamma$ represent dual, and hence nontrivial, homology classes in $H_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$ and $H_{n(n-1)/2}(M_\Gamma)$, respectively. \hfill $\Box$

Proposition 4.1 is surely well known, as is the following slightly more general version.

**Corollary 4.2** Given $B \in \text{GL}_n\mathbb{Q}$, assume that $\Gamma \subset \text{SL}_n\mathbb{Z}$ is a finite index torsion-free subgroup with $B^{-1}\Gamma B \subset \text{SL}_n\mathbb{Z}$, and that $\Delta$ and $\text{Nil}$ are as in Proposition 4.1. Then the projections of $B\Delta B^{-1}$ and $B\text{Nil} B^{-1}$ to $M_\Gamma$ represent dual, and hence nontrivial, homology classes in $H_{n-1}(\overline{M}_\Gamma, \partial \overline{M}_\Gamma)$ and $H_{n(n-1)/2}(M_\Gamma)$, respectively.

**Proof** The map $\phi: S_n \to S_n$ given by $\phi(X) = XB^{-1}$ induces a diffeomorphism $\Phi: M_{B^{-1}\Gamma B} \to M_\Gamma$. By Proposition 4.1 the projections of $\Delta$ and $\text{Nil}$ represent dual homology classes in $M_{B^{-1}\Gamma B}$. Pushing forward with $\Phi$, we obtain dual cycles $\Delta B^{-1}$ and $\text{Nil} B^{-1}$. By Corollary 2.2, these cycles are properly homotopic, and hence homologous, to the cycles $B\Delta B^{-1}$ and $B\text{Nil} B^{-1}$. The claim follows. \hfill $\Box$
5 Proof of Theorem 1.2

In the next section we will show:

Proposition 5.1 Assume that $A \in X$ is such that there are $v_1, \ldots, v_n \in \mathbb{Z}^n$ linearly independent with $S_1(A) = \{ \pm v_1, \ldots, \pm v_n \}$. Let $B \in \text{GL}_n \mathbb{Q}$ be the matrix with columns $v_1, \ldots, v_n$, and let $\Gamma$ be a finite index torsion-free subgroup of $\text{SL}_n \mathbb{Z} \cap B \text{SL}_n \mathbb{Z} B^{-1}$. Then the nontrivial homology class $[B\Delta B^{-1}]$ is represented by a cycle $\alpha \in C_{n-1}(\tilde{M}_\Gamma, \partial \tilde{M}_\Gamma)$ whose support intersects the well-rounded retract $X$ only in $A$.

Assuming Proposition 5.1, we prove the main theorem:

Theorem 1.2 The well-rounded retract $X$ is a minimal $\text{SL}_n \mathbb{Z}$–invariant spine of the symmetric space $S_n = \text{SO}_n \setminus \text{SL}_n \mathbb{R}$.

Proof Assume that $Y \subset X$ is a proper, closed, $\text{SL}_n \mathbb{Z}$–invariant subset of $X$. As mentioned in the introduction, in order to show that $Y$ is not contractible, it suffices to prove that for some $\tilde{\alpha} = \text{SL}_n \mathbb{Z}$ the induced map $Y = \tilde{\alpha}/\tilde{\alpha}$ is not a homotopy equivalence.

By Proposition 3.3 there is $A \in X \setminus Y$ and a linearly independent subset $\{v_1, \ldots, v_n\} \subset \mathbb{Z}$ with $S_1(A) = \{ \pm v_1, \ldots, \pm v_n \}$. Let $B \in \text{GL}_n \mathbb{Q}$ be the matrix with columns $v_1, \ldots, v_n$. The subgroups $\text{SL}_n \mathbb{Z}$ and $B \text{SL}_n \mathbb{Z} B^{-1}$ are commensurable and hence there is a torsion-free finite index subgroup $\Gamma \subset \text{SL}_n \mathbb{Z} \cap B \text{SL}_n \mathbb{Z} B^{-1}$. By Proposition 5.1, the homology class $[B\Delta B^{-1}] \in H_{n-1}(\tilde{M}_\Gamma, \partial \tilde{M}_\Gamma)$ is represented by a cycle $\alpha$ with $\alpha \cap X = \{A\}$. On the other hand, the class $[B\Delta B^{-1}]$ is dual to some class $[\beta] \in H_{n(n-1)/2}(\tilde{M}_\Gamma)$ by Corollary 4.2. Since $\alpha$ represents $[B\Delta B^{-1}]$ and intersects $X$ only at $A$, we deduce that every cycle contained in $X/\Gamma$ and representing $[\beta]$ has to contain $A$ in its support. In particular, the map

$$H_{n(n-1)/2}(Y/\Gamma) \to H_{n(n-1)/2}(X/\Gamma)$$

is not surjective. This implies that the map $Y/\Gamma \to X/\Gamma$ is not a homotopy equivalence.

$\square$

6 Flags of systoles

In this section we prove Proposition 5.1. The first step is to construct a certain continuous map

$$\Phi: S_n \times [0, \infty) \to S_n$$
which essentially pushes points in $S_n \setminus \mathcal{X}$ away from $\mathcal{X}$.

To begin with, recall the definition of the systole $\text{syst}_i(A)$ of $A \in S_n$. We can extend this definition as follows: for $i = 1, \ldots, n$, the $i$–th systole of $A$ is given by

\begin{equation}
\text{syst}_i(A) = \inf \{ r \mid \dim_{\mathbb{R}}(\text{Span}_{\mathbb{R}} \{ v \in \mathbb{Z} \mid |Av| < r \}) \geq i \}.
\end{equation}

In other words, $\text{syst}_i(A)$ is the infimum of those $r$ for which the set of vectors $v$ in $\mathbb{Z}^n$ whose image $Av$ has length less than $r$ generates a subspace of $\mathbb{R}^n$ with dimension at least $i$. Equivalently,

\begin{equation}
\text{syst}_i(A) = \sup \{ r \mid \dim_{\mathbb{R}}(\text{Span}_{\mathbb{R}} \{ v \in \mathbb{Z} \mid |Av| < r \}) < i \}.
\end{equation}

The $i$–th systole coincides with Minkowski’s $i$–th successive minimum of the lattice $A\mathbb{Z}^n$ with respect to the ball $B_1$ of radius 1 in $\mathbb{R}^n$. See Martinet [6] for more about successive minima.

For $i = 1, \ldots, n$, the $i$–th systole function

$$\text{syst}_i : S_n \to (0, \infty)$$

is well-defined and $\text{SL}_n \mathbb{Z}$–equivariant. We claim that it is continuous. In fact, if $(A_k)$ is a sequence in $S_n$ converging to some $A \in S_n$ then for all $r$ the finite sets \{ $v \in \mathbb{Z}^n, |A_k v| < r$ \} converge in the Gromov-Hausdorff topology to the (again finite) set \{ $v \in \mathbb{Z}^n, |Av| < r$ \}. Since $\mathbb{Z}^n$ is discrete, we have that for all sufficiently large $k$

$$\{ v \in \mathbb{Z}^n, |A_k v| < r \} = \{ v \in \mathbb{Z}^n, |Av| < r \}.$$

Together with (6–2), this implies that $\text{syst}_i$ is lower semi-continuous. Likewise (6–3) and the same argument yield upper semi-continuity.

**Lemma 6.1** The function $\text{syst}_i : S_n \to (0, \infty)$ is continuous and $\text{SL}_n \mathbb{Z}$–equivariant for $i = 1, \ldots, n$. □

Recall now the definition of $\Lambda_1(A)$ given in (3–1). We extend this definition, setting for $i = 1, \ldots, n$

$$\Lambda_i(A) = \text{Span}_{\mathbb{R}}(\{ v \in \mathbb{Z}^n, |Av| \leq \text{syst}_i(A) \}).$$

In order to avoid treating special cases we set $\Lambda_0(A) = 0$ for all $A \in S_n$. By definition

\begin{equation}
0 \subseteq \Lambda_1(A) \subseteq \cdots \subseteq \Lambda_n = \mathbb{R}^n
\end{equation}

and $\dim_{\mathbb{R}}(\Lambda_i(A)) \geq i$. Observe that for $i < n$ this last inequality is strict if $A$ is well-rounded. In particular, we cannot expect that the subspaces $\Lambda_i(A)$ depend continuously
on $A$. However we have the following weak continuity, which can be proved with essentially the same argument as Lemma 6.1:

**Lemma 6.2** Assume that $(A_k)$ is a sequence in $S_n$ converging to some $A \in S_n$. Then there is $k_0$ such that for all $k \geq k_0$ and $i \in \{1, \ldots, n\}$ there is a unique $\kappa(k, i) \in \{1, \ldots, n\}$ with

- $\Lambda_{\kappa(k, i)}(A_k) = \Lambda_i(A)$, and
- if $\kappa(k, i) \neq n$ then $\Lambda_{\kappa(k, i) + 1}(A_k) \neq \Lambda_i(A)$.

If moreover $i'$ is minimal with $\text{syst}_{i'}(A) = \text{syst}_i(A)$ then

$$\lim_{k \to \infty} \text{syst}_{j_k}(A_k) = \text{syst}_i(A)$$

for all choices of $j_k$ with $\kappa(k, i') - 1 < j_k \leq \kappa(k, i)$.

We use the flag $(6–4)$ to construct the continuous map $(6–1)$. To begin with we consider for $i = 1, \ldots, n$ the subspace

$$\Theta_i(A) = (A \Lambda_{i-1}(A))^\perp \cap (A \Lambda_i(A)).$$

In more plain language, $\Theta_i(A)$ is the orthogonal complement of the image of $\Lambda_{i-1}(A)$ under $A$ within the image of $\Lambda_i(A)$. We have thus the orthogonal decomposition

$$\mathbb{R}^n = \Theta_1(A) \oplus \cdots \oplus \Theta_n(A)$$

together with the associated orthogonal projections

$$\pi_{\Theta_i(A)} : \mathbb{R}^n \to \Theta_i(A).$$

We define now for $x \in \mathbb{R}^n$

$$(6–5) \quad \Phi_t(A)x = \frac{1}{\sqrt{n} \prod_{i=1}^n \text{syst}_i(A)^t \dim \Theta_i(A)} \sum_{i=1}^n \text{syst}_i(A)^t \pi_{\Theta_i(A)}(Ax).$$

The multiplicative factor in $(6–5)$ ensures that $\Phi_t(A) \in \text{SL}_n \mathbb{R}$ for all $A \in \text{SL}_n \mathbb{R}$. Moreover, for all $U \in \text{SO}_n$ we have $\Phi_t(UA) = U \Phi_t(A)$. In particular, we have a well-defined map

$$(6–6) \quad \Phi_t : S_n \times [1, \infty) \to S_n$$

It is easy to check that the map $(6–6)$ is $\text{SL}_n \mathbb{Z}$–equivariant, and its continuity follows.
from Lemma 6.2. Moreover, since $\text{syst}_1(A) \leq \text{syst}_i(A)$ for all $i$, we have for all $x \in \mathbb{R}^n$

$$|\Phi_t(A)x| \geq \left( \frac{\text{syst}_1(A)}{\sqrt{\prod_{i=1}^n \text{syst}_i(A)^{\dim \Theta_i(A)}}} \right)^t |Ax|$$

with equality if and only if $x \in \Lambda_1(A)$. In particular we see that $\Lambda_1(\Phi_t(A)) = \Lambda_1(A)$ for all $t \geq 0$. Moreover, if $\Lambda_1(A) \neq \mathbb{R}^n$ then the exponentiated quantity in (6–7) is less than 1 and hence

$$\lim_{t \to \infty} \text{syst}_1(\Phi_t(A)) = 0$$

On the other hand, if $\Lambda_1(A) = \mathbb{R}^n$ then $\Phi_t(A) = A$ for all $t$.

Summing up, we have:

**Proposition 6.3** There is a continuous map $\Phi : S_n \times [0, \infty) \to S_n$, $\Phi(A, t) = \Phi_t(A)$, with the following properties:

- $\Phi_0(\cdot) = \text{Id}$,
- $\Phi_t(A) \in \mathcal{X}$ if and only if $A \in \mathcal{X}$, and
- if $A \notin \mathcal{X}$ then $\lim_{t \to \infty} |\Phi_t(A)v| = 0$ for all $v \in \Lambda_1(A)$.

We are now ready to prove Proposition 5.1:

**Proposition 5.1** Assume that $A \in \mathcal{X}$ is such that there are $v_1, \ldots, v_n \in \mathbb{Z}^n$ linearly independent with $S_1(A) = \{ \pm v_1, \ldots, \pm v_n \}$, let $B \in \text{GL}_n \mathbb{Q}$ be the matrix with columns $v_1, \ldots, v_n$ and $\Gamma$ a finite index torsion-free subgroup in $\text{SL}_n \mathbb{Z} \cap B \text{SL}_n \mathbb{Z} B^{-1}$. Then the nontrivial homology class $[B\Delta B^{-1}]$ is represented by a cycle $\alpha \in C_{n-1}(\tilde{M}_\Gamma, \partial \tilde{M}_\Gamma)$ whose support intersects the well-rounded retract $\mathcal{X}$ only at $A$.

Recall that $\Delta$ is the connected component of the identity in the diagonal subgroup of $\text{SL}_n \mathbb{R}$.

**Proof** In order to construct the cycle $\alpha$ we start with the map

$$g_1 : \Delta \to M_\Gamma, \quad g_1(X) = BXB^{-1}$$

The cycle $g_1(\Delta)$ represents a nontrivial homology class in $H_{n-1}(\tilde{M}_\Gamma, \partial \tilde{M}_\Gamma)$ by Corollary 4.2. The point $A$ may not belong to the image of $g_1(\Delta)$, but this can be easily corrected by considering the map

$$g_2 : \Delta \to M_\Gamma, \quad g_2(X) = AXB^{-1}$$
Corollary 2.2 implies that \( g_1(\Delta) \) and \( g_2(\Delta) \) are properly homotopic and hence homologous.

Now we have \( g_2(\text{Id}) = A \), but it is not clear at all how many other times \( g_2(\Delta) \) may intersect \( \mathcal{X} \). We correct this problem by constructing a third map \( g_3 \) properly homotopic to \( g_2 \). Before going further we identify \( \Delta \) with \( \mathbb{R}^{n-1} \) via the following map

\[
(a_1, \ldots, a_{n-1}) \mapsto \begin{pmatrix}
e^{a_1} & 0 & \cdots & 0 & 0 \\
0 & e^{a_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & e^{a_{n-1}} & 0 \\
0 & 0 & \cdots & 0 & e^{-a_1-a_2-\cdots-a_{n-1}}
\end{pmatrix}
\]

A simple computation shows:

**Lemma 6.4** There is some \( \epsilon > 0 \) such that for all \( x \in B_\epsilon \subset \mathbb{R}^{n-1} = \Delta \), \( g_2(x) \in \mathcal{X} \) if and only if \( x = 0 \). If moreover \( x \in B_\epsilon \), \( x \neq 0 \) and \( v \in S_1(g_2(x)) \) then we have

\[
\lim_{t \to \infty} l_v(g_2(tx)) = 0.
\]

Here \( B_\epsilon \) is the ball of radius \( \epsilon \) centered at 0 in \( \mathbb{R}^{n-1} \simeq \Delta \).

We can now define the map \( g_3 : \mathbb{R}^{n-1} \to M_- \). With \( \epsilon \) as in Lemma 6.4 and \( \Phi \) the map provided by Proposition 6.3, we set

\[
g_3(x) = \begin{cases} 
g_2(x) & |x| \leq \epsilon \\
\Phi_{|x| - \epsilon}(g_2(\epsilon \frac{x}{|x|})) & |x| > \epsilon.
\end{cases}
\]

In other words we extend radially, using the map \( \Phi \) and the restriction of \( g_2 \) to \( B_\epsilon \). Since \( g_2(x) \notin \mathcal{X} \) for \( x \) with \( |x| = \epsilon \), we deduce from Proposition 6.3 that \( g_3(x) \notin \mathcal{X} \) for all \( x \) with \( |x| \geq \epsilon \). On the other hand, for \( |x| \leq \epsilon \) we have \( g_3(x) = g_2(x) \). Hence

\[
g_3(\mathbb{R}^{n-1}) \cap \mathcal{X} = \{A\}.
\]

If \( v \in \mathbb{Z}^n \) is a systole for \( g_2(x) \) with \( |x| = \epsilon \), then we have by (6–8)

\[
\lim_{t \to \infty} l_v(g_2(tx)) = 0
\]

and by Proposition 6.3

\[
\lim_{t \to \infty} l_v(g_3(tx)) = \lim_{t \to \infty} l_v(\Phi_{t-1}(g_2(x))) = 0.
\]

Lemma 2.1 implies now that the maps \( g_2 \) and \( g_3 \) are properly homotopic to each other. Hence the cycle \( \alpha = g_3(\Delta) \) represents the nontrivial homology class \([B \Delta B^{-1}] \in H_{n-1}(M_-, \partial M_-)\) and \( \alpha \cap \mathcal{X} = \{A\} \).
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