Sync-Maximal Permutation Groups Equal Primitive Permutation Groups

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Abstract. The set of synchronizing words of a given $n$-state automaton forms a regular language recognizable by an automaton with $2^n - n$ states. The size of a recognizing automaton for the set of synchronizing words is linked to computational problems related to synchronization and to the length of synchronizing words. Hence, it is natural to investigate synchronizing automata extremal with this property, i.e., such that the minimal deterministic automaton for the set of synchronizing words has $2^n - n$ states. The sync-maximal permutation groups have been introduced in [S. Hoffmann, Completely Reachable Automata, Primitive Groups and the State Complexity of the Set of Synchronizing Words, LATA 2021] by stipulating that an associated automaton to the group and a non-permutation has this extremal property. The definition is in analogy with the synchronizing groups and analog to a characterization of primitivity obtained in the mentioned work. The precise relation to other classes of groups was mentioned as an open problem. Here, we solve this open problem by showing that the sync-maximal groups are precisely the primitive groups. Our result gives a new characterization of the primitive groups. Lastly, we explore an alternative and stronger definition than sync-maximality.

Keywords: finite automata · synchronization · set of synchronizing words · primitive permutation groups · sync-maximal groups

1 Introduction

An automaton is synchronizing if it admits a word that drives every state into a single definite state. Synchronizing automata have a range of applications in software testing, circuit synthesis, communication engineering and the like, see [29,43,49]. The Černý conjecture states that the length of a shortest synchronizing word for a deterministic complete automaton with $n$ states has length at most $(n - 1)^2$ [13,14]. The best bound up to now is cubic [44]. This conjecture is one of the most famous open problems in combinatorial automata theory [49]. More specifically, the following bounds have been established:
Furthermore, the Černý conjecture [13] has been confirmed for a variety of classes of automata, just to name a few (without further explanation): circular automata [16,17,36], oriented or (generalized) monotonic automata [2,3,18], automata with a sink state [39], solvable and commutative automata [20,39,40], Eulerian automata [27], automata preserving a chain of partial orders [50], automata whose transition monoid contains a QI-group [7,8], certain one-cluster automata [46], automata that cannot recognize \( \{a, b\}^*ab\{a, b\}^* \) [1], aperiodic automata [48], certain aperiodically 1-contracting automata [15] and automata having letters of a certain rank [9].

Černý [13] gave an infinite family of synchronizing \( n \)-state automata with shortest synchronizing words of length \( p^2 - 1 \). Families of synchronizing automata with shortest synchronizing words close to \( p^n \) are called slowly synchronizing. There are only a few families of slowly synchronizing automata known, see [49].

The set of synchronizing words of an \( n \)-state automaton is a regular language and can be recognized by an automaton of size \( 2^n \). A property shared by most families of slowly synchronizing automata is that for them, every automaton for the set of synchronizing words needs exponentially many states [25,32,33]. Note that, of course, by taking an automaton and adjoining a letter mapping every state to a single state, as the extremal property of the set of synchronizing words is preserved by adding letters, automata whose sets of synchronizing words have exponential state complexity in the number of states are not necessarily slowly synchronizing. However, the evidence supports the conjecture that slowly synchronizing automata have this extremal property.

Testing if an automaton is synchronizing is doable in polynomial time [13,49]. However, computing a shortest synchronizing word is hard, more precisely, the decision variant of this problem is \( \text{NP} \)-complete [18,42], even for automata over a fixed binary alphabet. Moreover, variants of the synchronization problem for partial automata, or when restricting the set of allowed reset words, could even be \( \text{PSPACE} \)-complete [19,31].

The size of a smallest automaton for the set of synchronizing words seems to be also related to the difficulty to compute a shortest synchronizing word, or a synchronizing word subject to certain constraints. A first result in this direction was the realization that for commutative automata, i.e., where each permutation of an input word leads to the same state, and a fixed alphabet, we do not have
such an exponential blowup for the size of the minimal automaton for set of synchronizing words \cite{23,24}. As a consequence, the constrained synchronization problem for commutative input automata and a fixed constraint is always solvable in polynomial time \cite{23,24}. Note that for commutative input automata and a fixed alphabet, computing a shortest synchronizing can be done in polynomial time \cite{30}.

So, it is natural to focus on synchronizing automata such that the smallest automaton for the set of synchronizing words has maximal possible size.

After realizing that for certain special cases for which the Černý conjecture was established \cite{8,16,17,36,41}, this was due to the reason that certain permutation groups were contained in the transformation monoid of the automaton, the notion of synchronizing permutation groups was introduced \cite{7,8}. These are permutation groups with the property that if we adjoin a non-permutation to it, the generated transformation monoid contains a constant map. It was shown that these groups are contained strictly between the 2-transitive and the primitive groups \cite{7,34}. Meanwhile, a lot of related permutation groups have been introduced or linked to the synchronizing groups, for example: spreading, separating, \(QI\)-groups. See \cite{7} for a good survey and definitions. Furthermore, permutation groups in general have been investigated with respect to the properties of resulting transformation monoid if non-permutations were added \cite{4,5,6,7}.

The completely reachable automata have been introduced by Volkov & Bondar \cite{10,11}. This is a stronger notion than being synchronizing by stipulating that, starting from the whole state set, not only some singleton set is reachable, but every non-empty subset of states is reachable by some word. In fact, this property was also previously observed for many classes of synchronizing automata \cite{15,25,32,33}.

It has been proven in \cite{22} that a permutation group of degree \(n\) is primitive if and only if in the transformation monoid generated by the group and an arbitrary non-permutation with an image of size \(n - 1\), there exists, for every non-empty subset of the permutation domain, an element mapping the whole permutation domain to this subset, or said differently that an associated automaton is completely reachable. In the same paper \cite{22} the sync-maximal permutation groups were introduced by stipulating that, for an associated \(n\)-state automaton, the smallest automaton for the set of synchronizing words has size \(2^n - n\). It was shown that the sync-maximal permutation groups are contained between the 2-homogeneous and the primitive permutation groups, and it was posed as an open problem if they are properly contained between them, and if so, what the precise relation to other permutation groups is.

Here, we solve this open problem by showing that the sync-maximal permutation groups are precisely the primitive permutation groups, which also yields new characterizations of the primitive permutation groups.

## 2 Preliminaries

Let \(\Sigma\) be a finite set of symbols, called an alphabet. By \(\Sigma^*\), we denote the set of all finite sequences, i.e., of all words or strings. The empty word, i.e., the finite sequence of length zero, is denoted by \(\varepsilon\). The subsets of \(\Sigma^*\) are called
languages. For $n > 0$, we set $[n] = \{0, 1, \ldots, n-1\}$ and $[0] = \emptyset$. For a set $X$, we denote the power set of $X$ by $\mathcal{P}(X)$, i.e., the set of all subsets of $X$. Every function $f : X \to Y$ induces a function $\hat{f} : \mathcal{P}(X) \to \mathcal{P}(Y)$ by setting $\hat{f}(Z) := \{f(z) \mid z \in Z\}$. Here, we denote this extension also by $f$. Let $k \geq 1$. A $k$-subset of $X$ is a finite set of cardinality $k$. A 1-set is also called a singleton. We have, if Syn$(\mathcal{A}) \neq \emptyset$. We call $\mathcal{A}$ completely reachable, if for each non-empty $S \subseteq Q$ there exists a word $w \in \Sigma^*$ such that $\delta(Q, w) = S$. Note that every completely reachable automaton is synchronizing.

A (finite) automaton is a quintuple $\mathcal{A} = (\Sigma, Q, \delta, q_0, F)$ where $(\Sigma, Q, \delta)$ is a semi-automaton, $q_0 \in Q$ is the start state and $F \subseteq Q$ the set of final states. The languages recognized (by $\mathcal{A}$) is $L(\mathcal{A}) = \{w \in \Sigma^* \mid \delta(q_0, w) \in F\}$. An automaton with a start state and a set of final states is used for the description of languages, whereas, when we consider a semi-automaton, we are only concerned with the transition structure of the automaton itself. When the context is clear, we also call semi-automata simply automata and concepts and notions that do not use the start state or the final state carry over from semi-automata to automata and vice versa.

A language recognized by a finite automaton is called regular. An automaton $\mathcal{A}$ has the least number of states to recognize a language [26], i.e., is a minimal automaton, if and only if every state is reachable from the start state and every two distinct states $p, q \in Q$ are distinguishable, i.e., there exists $w \in \Sigma^*$ such that precisely one of the states $\delta(p, w)$ and $\delta(q, w)$ is a final state, but not the other. A minimal automaton is unique up to isomorphism [26], where two automata are isomorphic if one can be obtained from the other by renaming of states. Hence, we can speak about the minimal automaton.

If $\mathcal{A} = (\Sigma, Q, \delta)$ is a semi-automaton with a non-empty state set, then define $\mathcal{P}(\mathcal{A}) = (\Sigma, \mathcal{P}(Q) \setminus \{\emptyset\}, \delta, Q, F)$ where $\delta : \mathcal{P}(Q) \times \Sigma \to \mathcal{P}(Q)$ is the extension $\delta(S, w) = \{\delta(s, w) \mid s \in S\}$, for $S \subseteq Q$ and $w \in \Sigma^*$, of $\delta$ to subsets of states and $F = \{\{q\} \mid q \in Q\}$. As for functions $f : X \to Y$ introduced above, we drop the distinction between $\delta$ and $\hat{\delta}$ and denote both functions by $\delta$. We have, $\text{Syn}((\mathcal{A})) = \{w \in \Sigma^* \mid \delta(Q, w) \in F\}$. The states in $F$ can be merged to a single

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1 In group theory, usually the other convention is adopted, but we stick to the convention most often seen in formal language theory.
state to get a recognizing automaton for \( \text{Syn}(A) \). So, \( \text{Syn}(A) \) is recognizable by an automaton with \( 2^{|Q|} - |Q| \) states.

Let \( n \geq 0 \). Denote by \( S_n \) the symmetric group on \([n]\), i.e., the group of all permutations of \([n]\). A permutation group (of degree \( n \)) is a subgroup of \( S_n \). A permutation group \( G \) over \([n]\) is primitive, if it preserves no non-trivial equivalence relation on \([n] \), i.e., for no non-trivial equivalence relation \( \sim \subseteq [n] \times [n] \) we have \( p \sim q \) if and only if \( g(p) \sim g(q) \) for all \( g \in G \) and \( p, q \in [n] \) (recall that the elements of \( G \) are functions from \([n]\) to \([n]\)). Equivalently, a permutation group is primitive if there does not exist a non-empty proper subset \( \Delta \subseteq [n] \) with \( |\Delta| > 1 \) such that, for every \( g \in G \), we have \( g(\Delta) = \Delta \) or \( g(\Delta) \cap \Delta = \emptyset \). A permutation group \( G \) over \([n]\) is called \( k \)-homogeneous for some \( k \geq 1 \), if for every two \( k \)-subsets \( S, T \) of \([n]\), there exists \( g \in G \) such that \( g(S) = T \). A transitive permutation group is the same as a 1-homogeneous permutation group. Note that here, all permutation groups with \( n \leq 2 \) are primitive, and for \( n > 2 \) every primitive group is transitive. Because of this, some authors exclude the trivial group for \( n = 2 \) from being primitive. A permutation group \( G \) over \([n]\) is called \( k \)-transitive for some \( k \geq 1 \), if for two \( k \)-tuples \((p_1, \ldots, p_k), (q_1, \ldots, q_k) \in [n]^k \), there exists \( g \in G \) such that \( (g(p_1), \ldots, g(p_k)) = (q_1, \ldots, q_k) \).

By \( T_n \), we denote the set of all maps on \([n]\). The elements of \([n]\) are also called points in this context. A submonoid of \( T_n \) for some \( n \) is called a transformation monoid. If the set \( U \) is a submonoid (or a subgroup) of \( T_n \) (or \( S_n \)), we denote this by \( U \subseteq T_n \) (or \( U \subseteq S_n \)). For a set \( A \subseteq T_n \) (or \( A \subseteq S_n \)), we denote by \( \langle A \rangle \) the submonoid (or the subgroup) generated by \( A \). If \( A = \{a_1, \ldots, a_m\} \) we also write \( \langle a_1, \ldots, a_m \rangle = \langle A \rangle \). Let \( A = (\Sigma, Q, \delta) \) be a semi-automaton and for \( w \in \Sigma^* \) define \( \delta_w : Q \to Q \) by \( \delta_w(q) = \delta(q, w) \) for all \( q \in Q \). Then, we can associate with \( A \) the transformation monoid of the automaton \( \mathcal{T}_A = \{ \delta_w \mid w \in \Sigma^* \} \), where we can identify \( Q \) with \([n]\) for \( n = |Q| \). We have \( \mathcal{T}_A = \langle \{ \delta_x \mid x \in \Sigma \} \rangle \). The rank of a word \( w \in \Sigma^* \) is the cardinality of its image. For a given semi-automaton \( A = (\Sigma, Q, \delta) \), the rank of a word \( w \in \Sigma^* \) is the rank of \( \delta_w \). We call two sets \( S, T \subseteq [n] \) distinguishable in a transformation monoid \( M \subseteq T_n \) if there exists an element in \( M \) mapping precisely one of both sets to a singleton set and the other to a non-singleton set.

The following implies that we can check if a given semi-automaton is synchronizing by only looking at pairs of states \([13, 49]\). The proof basically works by repeatedly collapsing pairs of states to construct a synchronizing word \([49]\). It implies a polynomial time procedure to check synchronizability \([49]\).

**Theorem 1** (Černý \([13, 49]\)). Let \( A = (\Sigma, Q, \delta) \). Then, \( A \) is synchronizing if and only if for each \( p, q \in Q \) there exists \( w \in \Sigma^* \) such that \( \delta(p, w) = \delta(q, w) \). Hence, a transformation monoid \( M \subseteq T_n \) contains a constant map if and only if every two points can be mapped to a single point by elements in \( M \).

The next result appears in \([6, 7, 41]\) and despite it was never clearly spelled out by Rystsov himself, it is implicitly present in arguments used in \([11]\).

**Theorem 2** (Rystsov \([6, 7, 41]\)). A permutation group \( G \) on \([n]\) is primitive if and only if, for every map \( f : [n] \to [n] \) of rank \( n - 1 \), the transformation monoid \( \langle G \cup \{ f \} \rangle \) contains a constant map.

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\(^2\) The trivial equivalence relations on \([n]\) are \([n] \times [n] \) and \( \{(x, x) \mid x \in [n]\} \).
In [22], the following characterization of the primitive permutation groups, connecting them to completely reachable automata, was shown.

**Theorem 3 (Hoffmann [22]).** Let \( G = \langle g_1, \ldots, g_k \rangle \leq S_n \). Then the following are equivalent:

1. \( G \) is primitive;
2. for every transformation \( f : [n] \to [n] \) of rank \( n - 1 \) and non-empty \( S \subseteq [n] \), there exists \( g \in \langle G \cup \{f\} \rangle \) such that \( g([n]) = S \);
3. for every transformation \( f : [n] \to [n] \) of rank \( n - 1 \), the semi-automaton \( A = (\Sigma, Q, \delta) \), with \( \Sigma = \{g_1, \ldots, g_k, f\} \), \( Q = [n] \) and \( \delta(m, g) = g(m) \) for all \( m \in [n] \) and all \( g \in \Sigma \), is completely reachable.

In the same work [22], in analogy with Theorem 2 and 3, the sync-maximal permutation groups were introduced.

**Definition 4.** A permutation group \( G = \langle g_1, \ldots, g_k \rangle \leq S_n \) is called sync-maximal, if for every map \( f : [n] \to [n] \) of rank \( n - 1 \), for the automaton \( A = (\Sigma, Q, \delta) \), with \( \Sigma = \{g_1, \ldots, g_k, f\} \) and \( \delta(m, g) = g(m) \) for all \( m \in [n] \) and \( g \in \Sigma \), the minimal automaton of \( \text{Syn}(A) \) has \( 2^n - n \) states.

This definition is independent of the choice of generators for \( G \). In purely combinatorial language, using the characterization of the minimal automaton, applied to \( \mathcal{P}_A \), this means the following.

**Theorem 5.** A permutation group \( G \leq S_n \) is sync-maximal if and only if for every transformation \( f : [n] \to [n] \) of rank \( n - 1 \), we have that (1) for every non-empty subset \( S \subseteq [n] \) of size at least two, there exists \( h \in \langle G \cup \{f\} \rangle \) such that \( S = h([n]) \), and (2) for two distinct non-empty and non-singleton subsets of \( [n] \), there exists a transformation in \( \langle G \cup \{f\} \rangle \) mapping precisely one to a singleton but not the other.

In [22] it was shown that every sync-maximal permutation groups is primitive. The main result of the present work is that the converse implication holds true.

**Proposition 6 ([22]).** Every sync-maximal permutation group is primitive.

In [22] Lemma 3.1 it was shown that distinguishability of all sets reduces to distinguishability of the 2-subsets. Formulated without reference to automata, this gives the next result.

**Theorem 7 ([22]).** Let \( M \leq T_n \) be a transformation monoid. Then, for every two distinct non-empty and non-singleton \( S, T \subseteq [n] \) there exists a transformation in \( M \) mapping precisely one to a singleton but not the other if and only if this condition holds true for every two distinct 2-subsets of \( [n] \).

The next lemma is obvious, as it basically states the definition of injectivity for the restriction of a function to a subset, and stated for reference.

**Lemma 8.** Let \( f : [n] \to [n] \) and \( S \subseteq [n] \). Then, \( |S \cap f^{-1}(x)| \leq 1 \) for each \( x \in [n] \) if and only if \( f \) acts injective on \( S \), i.e., \( |f(S)| = |S| \).

\(^3\) As \( f \) has rank \( n - 1 \), this also implies that at least one singleton set is reachable. In fact, even more holds true, with [22] Lemma 4.1 we can deduce that \( G \) is transitive and so for every non-empty \( S \subseteq [n] \) there exists \( h \in \langle G \cup \{f\} \rangle \) with \( S = h([n]) \).
sync-maximal permutation groups equal primitive permutation groups – the proof

here, we will prove the following theorem.

theorem 9. let $g = \langle g_1, \ldots, g_k \rangle \leq s_n$. then the following are equivalent:

1. $g$ is primitive;
2. for every transformation $f : [n] \rightarrow [n]$ of rank $n - 1$ and $\{a, b\}, \{c, d\} \subseteq [n]$ with $\{a, b\} \neq \{c, d\}$, there exists $g \in \langle g \cup \{f\} \rangle$ such that precisely one of the subsets $g(\{a, b\})$ and $g(\{c, d\})$ is a singleton but not the other;
3. $g$ is sync-maximal.

at the heart of our result is the following statement.

proposition 10. let $g \leq s_n$ be a permutation group and $f : [n] \rightarrow [n]$ be an idempotent map of rank $n - 1$. suppose $\langle g \cup \{f\} \rangle$ contains a constant map. then, for all $\{a, b\}, \{c, d\} \subseteq [n]$ with $\{a, b\} \neq \{c, d\}$ there exists a transformation in $\langle g \cup \{f\} \rangle$ mapping precisely one set to a singleton but not the other.

proof. suppose $g = \langle g_1, \ldots, g_k \rangle$. let $\{a, b\}, \{c, d\} \subseteq [n]$ be two distinct 2-sets and $f : [n] \rightarrow [n]$ being idempotent and of rank $n - 1$. without loss of generality, we can suppose $f(0) = f(1) = 1$ and $f(i) = i$ for $i \in \{2, \ldots, n - 1\}$. as $\langle g \cup \{f\} \rangle$ contains a constant map, we can map $\{a, b\}$ to a singleton set. choose a transformation $h \in \langle g \cup \{f\} \rangle$ represented by a shortest possible word in the generators of $g$ and $f$ such that $h(\{a, b\}) = 1$. then, we can write $h = f_{u_m}f_{u_{m-1}}f \cdots f_{u_2}f_{u_1}$ with $u_i \in g$, $m \geq 1$. note that, by the minimal choice, the transformation $f$ is applied at the end\(^4\). if $h(\{c, d\})$ is not a singleton set, we are done. so, suppose $h(\{c, d\})$ is also a singleton set. for $i \in \{1, \ldots, m\}$, set $h_i = f_{u_i}f_{u_{i-1}}f \cdots f_{u_2}f_{u_1}$. then, $h = h_m$. by minimality of the representation in the generators of $g$ and $f$, for all $i \in \{1, \ldots, m - 1\}$ we have $|h_i(\{a, b\})| = 2$. hence, if there exists $i \in \{1, \ldots, m - 1\}$ such that $h_i(\{c, d\})$ is a singleton set, we are also done. so, suppose this is not the case.

set $g_i = u_if_{u_{i-1}}f \cdots f_{u_2}f_{u_1}$. then, $h_i = f_{g_i}$. we must have $0 \notin g_i(\{a, b\})$ for all $i$, for otherwise, as $f$ acts as the identity on $\{1, \ldots, n - 1\}$, we can leave $f$ out, i.e., $g_i(\{a, b\}) = h_i(\{a, b\})$, in the expression for $h$ and get a shorter representing word, contradicting the minimal choice of the expression representing $h$. similarly, $0 \notin g_i(\{c, d\})$ for all $i$, as otherwise we can leave a single instance of $f$ out again and have a word that maps $\{c, d\}$ to a singleton, but not $\{a, b\}$ by the minimal choice of $h$ in the length of a representing word.

note that, as $|g_m(\{a, b\})| = |g_m(\{c, d\})| = 2$, $|h_m(\{a, b\})| = |h_m(\{c, d\})| = 1$ and $h_m = fg_m$, we must have $g_m(\{a, b\}) = g_m(\{c, d\}) = \{0, 1\}$.

next, we will show by induction on $j \in \{1, \ldots, m\}$ that $g_j(\{a, b\}) = g_j(\{c, d\})$, where the base case is $j = m$. then, $g_1(\{a, b\}) = g_1(\{c, d\})$ implies $\{a, b\} = \{c, d\}$ as $g_1 = u_1 \in g$ is a permutation. however, this is a contradiction as both 2-sets are assumed to be distinct. hence, the case that $h_m$ maps both to a singleton and $h_i$ for all $i \neq m$ maps both to 2-sets is not possible. as noted, for the base

\(^4\) recall that by our convention, the leftmost function is applied last.
case we have \( g_m(\{a, b\}) = g_m(\{c, d\}) = \{0, 1\} \). Now suppose \( j \in \{1, \ldots, m - 1\} \) and \( g_{j+1}(\{a, b\}) = g_{j+1}(\{c, d\}) \). Then, as \( g_{j+1} = u_{j+1}f_{g_{j}} = u_{j+1}h_{j} \), we can deduce \( h_{j}(\{a, b\}) = h_{j}(\{c, d\}) \) as they only differ by the application of the permutation \( u_{j+1} \in G \). As written above \( 0 \in g_{j}(\{a, b\}) \cap g_{j}(\{c, d\}) \). This implies, as \( |h_{j}(\{a, b\})| = |h_{j}(\{c, d\})| = 2 \), that \( 1 \neq g_{j}(\{a, b\}) \cup g_{j}(\{c, d\}) \). So, \( |f^{-1}(x) \cap g_{j}(\{a, b\})| \leq 1 \) and \( |f^{-1}(x) \cap g_{j}(\{c, d\})| \leq 1 \) for every \( x \in [n] \). As \( h_{j} = f g_{j} \), we can write \( f(g_{j}(\{a, b\})) = f(g_{j}(\{c, d\})) \). Applying Lemma 8 then yields \( g_{j}(\{a, b\}) = g_{j}(\{c, d\}) \). 

The following lemma allows us to assume the transformation of rank \( n - 1 \) in Theorem 9 is idempotent.

**Lemma 11.** Let \( G \leq S_n \) be a transitive permutation group and \( f : [n] \rightarrow [n] \) be a transformation of rank \( n - 1 \). Then, there exists an idempotent transformation \( h : [n] \rightarrow [n] \) of rank \( n - 1 \) such that \( \langle G \cup \{h\} \rangle \leq \langle G \cup \{f\} \rangle \).

So, now we have everything together to prove Theorem 9.

**Proof (Proof of Theorem 9).** We can assume \( n > 2 \), as we have not included the assumption of transitivity in our definition of primitivity (which is implied for \( n > 2 \), see [12]) and so, for \( n \leq 2 \) every subgroup is primitive and also fulfills the second condition vacuously, as then we cannot find two distinct 2-sets. Also, for \( n \leq 2 \), every group is sync-maximal, as is easily seen by case analysis.

So first, let \( G = \langle g_1, \ldots, g_r \rangle \leq S_n \) be primitive and suppose \( f : [n] \rightarrow [n] \) is a transformation of rank \( n - 1 \). By Lemma 11 there exists an idempotent transformation \( f' \in \langle G \cup \{f\} \rangle \). By Theorem 2 in \( \langle G \cup \{f'\} \rangle \) we find a constant map. Then, by Proposition 10 for distinct 2-sets there exists an element in \( \langle G \cup \{f'\} \rangle \leq \langle G \cup \{f\} \rangle \) mapping precisely one of both 2-sets to a singleton set.

Now, suppose the second condition holds true. First, note that the second condition implies for \( n > 2 \) and \( \{a, b\} \subseteq [n] \) that there must exist \( g \in G \) such that \( g(\{a, b\}) \neq \{a, b\} \). Assume this is not the case. Then, for \( c \neq \{a, b\} \), we have

\[
\{g(\{a, c\}), g(\{b, c\})\} \cap \{\{a, b\}\} = \emptyset
\]

for every \( g \in G \) and, more generally, we have \( \{g(\{d, e\}), g(\{d', e'\})\} \cap \{\{a, b\}\} = \emptyset \) for every \( g \in G \) and \( \{d, e\}, \{d', e'\} \) not equal to \( \{a, b\} \). Choose \( c \in [n] \backslash \{a, b\} \) and \( f' : [n] \rightarrow [n] \) idempotent of rank \( n - 1 \) with \( f'(a) = f'(b) = \emptyset \). Then, with Equation 11 we can deduce \( \{h(\{a, c\}), h(\{b, c\})\} \cap \{\{a, b\}\} = \emptyset \) for \( h \in \langle G \cup \{f'\} \rangle \).

So, it is not possible to map one of \( \{a, c\} \) and \( \{b, c\} \) to a singleton set. But this is excluded by assumption, so there must exist an element in \( G \) mapping \( \{a, b\} \) to a different 2-subset.

So, now let \( f : [n] \rightarrow [n] \) be an arbitrary transformation of rank \( n - 1 \) and \( \mathcal{A} = (\Sigma, Q, \delta) \) be the automaton with \( \Sigma = \{g_1, \ldots, g_k, f\} \), \( Q = [n] \) and \( \delta(i, x) = x(i) \) for \( i \in [n] \) and \( x \in \Sigma \). Then, the second condition precisely says that all non-empty 2-sets are distinguishable in \( \mathcal{P}_\mathcal{A} \). With Theorem 7, then all non-empty subsets with at least two elements are distinguishable in \( \mathcal{P}_\mathcal{A} \). So, we only need to show that all non-empty subsets with at least two elements are reachable and at least one singleton subset is reachable in \( \mathcal{A} \). In fact, we will establish the stronger statement that \( \mathcal{A} \) is completely reachable. Let \( \{a, b\} \subseteq [n] \).
be a 2-subset. As shown above, we can choose $g \in G$ with $g\{a, b\} \neq \{a, b\}$. Then, by assumption, there exists $h \in \langle G \cup \{f\} \rangle$ such that precisely one of $h\{a, b\}$ or $(hg)\{a, b\}$ is a singleton set. By Theorem 1 as $\{a, b\}$ was arbitrary, $\langle G \cup \{f\} \rangle$ contains a constant map. As $f : [n] \to [n]$ was arbitrary of rank $n-1$, by Theorem 2 the group $G$ is primitive and so, by Theorem 3 the automaton $A$ is completely reachable.

Finally, suppose the last condition is fulfilled, i.e, $G$ is sync-maximal. Then, by Proposition 6, $G$ is primitive.

Note that, by Lemma 11, in the statements of Theorem 3 and 9 it is enough if the mentioned conditions hold for idempotent transformations of rank $n-1$ only.

With a little more work, we can show the following statement. By Lemma 11 we can always restrict to idempotent transformations for the mentioned characterizations, hence every statement entails two statements: one for all transformations of rank $n-1$ and one for idempotent transformations of rank $n-1$ only. Both formulations are put into a single statement by putting the word “idempotent” in square brackets in Theorem 12.

**Theorem 12.** Let $G \leq S_n$ be a permutation group and $n \geq 5$. Then the following are equivalent:

1. $G$ is primitive;
2. for every [idempotent] transformation $f : [n] \to [n]$ of rank $n-1$, in the transformation semigroup $\langle G \cup \{f\} \rangle$ we find, for each non-empty $S \subseteq [n]$, an element $g \in \langle G \cup \{f\} \rangle$ such that $g([n]) = S$;
3. for every [idempotent] transformation $f : [n] \to [n]$ of rank $n-1$ and 2-sets $\{a, b\}, \{c, d\} \subseteq [n]$ with $\{a, b\} \neq \{c, d\}$, there exists a transformation in $\langle G \cup \{f\} \rangle$ mapping precisely one to a singleton, but not the other;
4. for every [idempotent] transformation $f : [n] \to [n]$ of rank $n-1$ and two distinct non-empty and non-singleton subsets $S, T \subseteq [n]$, there exists a transformation in $\langle G \cup \{f\} \rangle$ mapping one to a singleton but not the other;
5. for every [idempotent] transformation $f : [n] \to [n]$ of rank $n-1$ and two distinct non-empty and non-singleton subsets $S, T \subseteq [n]$, there exists a transformation in $\langle G \cup \{f\} \rangle$ mapping both to subsets of different cardinality;
6. for every [idempotent] transformation $f : [n] \to [n]$ of rank $n-1$ and two disjoint non-empty 2-sets $\{a, b\}, \{c, d\} \subseteq [n]$, there exists a transformation in $\langle G \cup \{f\} \rangle$ mapping precisely one to a singleton, but not the other.

### 4 Strongly Sync-Maximal Permutation Groups

As the sync-maximal groups turned out to be precisely the primitive permutation groups, can we alter the definition to give us a new class of permutation groups related to the size of the minimal automata for the set of synchronizing words? One first approach might be to demand that, for each transformation of rank

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5 In case of Theorem 3 which was proven in the conference version [22], this was communicated to me, for which I am thankful, by an anonymous referee of [22].
if we add this to the group, in the resulting transformation monoid every non-empty set of size at most \(n - k\) is reachable and two distinct non-empty and non-singleton subsets \(S, T\) of states with \(|S|, |T| \in \{m \mid m \leq n - k\} \cup \{n\}\) can be mapped to sets of different cardinality. However, it is easy to show that every such group is \(k\)-reachable as introduced in [22]. So, also with the results from [22], for \(6 \leq k \leq n - 6\) this condition is only fulfilled by the symmetric or the alternating groups.

So, in the following definition, we only demand the distinguishability conditions, but not the reachability condition. Note that in the characterizations of primitive groups given above, both conditions – either distinguishability or reachability – are equivalent if we add a transformation of rank \(n - 1\).

**Definition 13.** A permutation group \(G \leq S_n\) is called strongly sync-maximal if for each transformation \(f : [n] \to [n]\) of rank \(r\) with \(2 \leq r \leq n - 1\) in \(<G \cup \{f\}>\) all 2-subsets are distinguishable.

**Proposition 14.** Every 4-transitive group is strongly sync-maximal and every strongly sync-maximal group is primitive.

Whether the strongly sync-maximal groups are properly contained between the above mentioned groups is an open problem. If so, the precise relation to the synchronizing groups and other classes of groups is an open problem and remains for future work.

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A Proofs for Section 2 (Preliminaries)

**Theorem 5.** A permutation group $G \leq S_n$ is sync-maximal if and only if for every transformation $f : [n] \rightarrow [n]$ of rank $n - 1$, we have that (1) for every non-empty subset $S \subseteq [n]$ of size at least two there exists $h \in \langle G \cup \{f\} \rangle$ such that $S = h([n])$, and (2) for two distinct non-empty and non-singleton subsets of $[n]$, there exists a transformation in $\langle G \cup \{f\} \rangle$ mapping precisely one to a singleton but not the other.

*Proof.* As $G$ is finite, there exists a generating set $g_1, \ldots, g_k$. If we associate the semi-automaton $A = (\Sigma, [n], \delta)$ with $\Sigma = \{g_1, \ldots, g_k, f\}$ and $\delta(m, g) = g(m)$ for $g \in \Sigma$ and $m \in [n]$ to $G$, then, as $G$ is sync-maximal, the minimal automaton for $\text{Syn}(A)$ has $2^n - n$ states. Hence, in $P_A$ all non-empty and non-singleton sets are reachable and at least one singleton set, which gives the first condition, and every two distinct non-empty and non-singleton sets are distinguishable in $P_A$, which gives the second claim.

Conversely, suppose conditions (1) and (2) hold true and let $f : [n] \rightarrow [n]$ be of rank $n - 1$. If we associate an automaton $A$ with $G$ and $f$ as before, then condition (2) says that that all states in $P_A$ are distinguishable. Condition (1) implies that every subset of size at least two is reachable in $P_A$. Let $a, b \in [n]$ be such that $f(a) = f(b)$ and $h \in \langle G \cup \{f\} \rangle$ with $h([n]) = \{a, b\}$. Then, $|f(h([n]))| = 1$ and so at least one singleton set is reachable. In fact, by Lemma [Hof21, Lemma 4.1], $G$ is transitive and so every subset is reachable in $P_A$.

**Theorem 7 ([Hof21]).** Let $M \leq T_n$ be a transformation monoid. Then, for every two distinct non-empty and non-singleton $S, T \subseteq [n]$ there exists a transformation in $M$ mapping precisely one to a singleton but not the other if and only if this condition holds true for every two distinct 2-subsets of $[n]$.

*Proof.* The method of proof used in [Hof21, Lemma 3.1] works in the general case of transformation monoids. However, we give a simple self-contained (and shorter than the original) proof here. Assume the claim holds true for the 2-subsets of $[n]$. We do induction on $|S| + |T|$. As by assumption $\min\{|S|, |T|\} \geq 2$, we have $|S| + |T| \geq 4$. If $|S| + |T| = 4$, then both are 2-subsets and this is precisely the assumption. If $|S| + |T| > 4$, then, as $S \neq T$, there exist $\{a, b\} \subseteq S$ and $\{c, d\} \subseteq T$ with $\{a, b\} \neq \{c, d\}$. By assumption there exists $g \in M$ mapping precisely one to a singleton. Without loss of generality, suppose this is $\{a, b\}$. Then $|g(S)| < |S|$ and $|g(S)| + |g(T)| < |S| + |T|$. Then, either $g(S)$ is a singleton set and $|g(T)| \geq 2$ or, by the induction hypothesis, we find $h \in M$ mapping precisely one of $g(S)$ and $g(T)$ to a singleton set. In this case, $hg$ maps precisely one of $S$ and $T$ to a singleton set.

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6 As $f$ has rank $n - 1$, this also implies that at least one singleton set is reachable. In fact, even more holds true, with [Hof21, Lemma 4.1] we can deduce that $G$ is transitive and so for every non-empty $S \subseteq [n]$ there exists $h \in \langle G \cup \{f\} \rangle$ with $S = h([n])$.

7 This argument, for which I am thankful, was communicated to me by an anonymous reviewer of the present work.
B Proofs for Section 3 (Sync-Maximal Permutation Groups Equal Primitive Permutation Groups – The Proof)

Lemma 11. Let \( G \leq S_n \) be a transitive permutation group and \( f : [n] \to [n] \) be a transformation of rank \( n - 1 \). Then, there exists an idempotent transformation \( h : [n] \to [n] \) of rank \( n - 1 \) such that \( \langle G \cup \{ h \} \rangle \leq \langle G \cup \{ f \} \rangle \).

Proof. Let \( f : [n] \to [n] \) be a transformation of rank \( n - 1 \). Let \( a \in [n] \) be such that \( f([n]) = [n] \setminus \{a\} \). As \( G \) is transitive, there exists \( g \in G \) such that \( a \neq g(f([n])) \) and \( gf \) permutes \( [n] \setminus \{a\} \). Then, some power of \( gf \) acts as the identity on \( [n] \setminus \{a\} \), i.e., is idempotent. \( \square \)

The following separation result that will be needed in the proof of Theorem 12.

Theorem 15. [ACS17, Neu75, BBMN09] Let \( G \leq S_n \) be transitive and \( A, B \subseteq [n] \) be such that \( |A| \cdot |B| < n \). Then, there exists \( g \in G \) such that \( g(A) \cap B = \emptyset \).

Theorem 12. Let \( G \leq S_n \) be a permutation group and \( n \geq 5 \). Then the following are equivalent:

(1) \( G \) is primitive;
(2) for every idempotent transformation \( f : [n] \to [n] \) of rank \( n - 1 \), in the transformation semigroup \( \langle G \cup \{ f \} \rangle \) we find, for each non-empty \( S \subseteq [n] \), an element \( g \in \langle G \cup \{ f \} \rangle \) such that \( g([n]) = S \);
(3) for every idempotent transformation \( f : [n] \to [n] \) of rank \( n - 1 \) and 2-sets \( \{a, b\}, \{c, d\} \subseteq [n] \) with \( \{a, b\} \neq \{c, d\} \), there exists a transformation in \( \langle G \cup \{ f \} \rangle \) mapping precisely one to a singleton, but not the other;
(4) for every idempotent transformation \( f : [n] \to [n] \) of rank \( n - 1 \) and two distinct non-empty and non-singleton subsets \( S, T \subseteq [n] \), there exists a transformation in \( \langle G \cup \{ f \} \rangle \) mapping both to subsets of different cardinality;
(5) for every idempotent transformation \( f : [n] \to [n] \) of rank \( n - 1 \) and two disjoint non-empty 2-sets \( \{a, b\}, \{c, d\} \subseteq [n] \), there exists a transformation in \( \langle G \cup \{ f \} \rangle \) mapping precisely one to a singleton, but not the other.

Proof. By Lemma 11, each of the properties holds true if and only if it holds true for idempotent transformations of rank \( n - 1 \) only.

1 implies 2: This is implied by Theorem 6
2 implies 3: This is implied by Theorem 9
3 implies 4: This is implied by Theorem 7
4 implies 5: This is implied by Theorem 7
5 implies 6: This is clear.
implies (1): Let \( f : [n] \to [n] \) be a transformation of rank \( n - 1 \). Let \( \{a, b\} \subseteq [n] \) be a 2-set. As \( n > 4 \), by Theorem 15 there exists \( g \in G \) such that \( g(\{a, b\}) \cap \{a, b\} = \emptyset \). By Property (6) there exists \( h \in \langle G \cup \{f\} \rangle \) such that precisely one of \( \{a, b\}, g(\{a, b\}) \) is mapped to a singleton set. Then, either \( h \) or \( gh \) collapse \( \{a, b\} \). So, with Theorem 1 the transformation monoid \( \langle G \cup \{f\} \rangle \) contains a constant map. Then, with Theorem 2 we find that \( G \) is primitive.

Remark 1. The assumption \( n \geq 5 \) is necessary for Theorem 12 to hold true. For example, let \( n = 4 \) and consider the non-primitive permutation group \( G = \langle g \rangle \) with \( g(0) = 1, g(1) = 2, g(2) = 0 \) and \( g(3) = 3 \). Let \( f : [4] \to [4] \) be a transformation of rank 3 and \( \{a, b\}, \{c, d\} \) be two disjoint 2-sets. The orbits on the 2-sets are

\[
\{0,3\}^h \mid h \in G = \{0,1\}, \{1,3\}, \{2,3\}.
\]

and every 2-set is in one of both sets. So, as \( \{a, b\} \) must be in a different orbit than \( \{c, d\} \) as they are disjoint, we can find \( fh \) such that precisely one of the two 2-sets \( \{a, b\} \) and \( \{c, d\} \) is mapped to a singleton set. However, note that Property (1) and Property (2) of Theorem 12 are also equivalent for \( n \in \{3, 4\} \), see [Hof21].

C Proofs for Section 4 (Strongly Sync-Maximal Permutation Groups)

Proposition 14. Every 4-transitive group is strongly sync-maximal and every strongly sync-maximal group is primitive.

Proof. Let \( G \leq S_n \) be 4-transitive and \( f : [n] \to [n] \) be a transformation of rank \( r \) with \( 2 \leq r \leq n - 1 \). Then, there exist \( \{a, b\}, \{c, d\} \subseteq [n] \) with \( |f(\{a, b\})| = 2 \) and \( |f(\{c, d\})| = 1 \). Now, suppose we have arbitrary distinct 2-subsets \( \{a_1, a_2\}, \{a_3, a_4\} \) of \( [n] \). By assumption, there exists \( g \in G \) such that \( g(\{a_1, a_2\}) = \{a, b\} \) and \( g(\{a_3, a_4\}) = \{c, d\} \). Then, \( fg \) maps precisely one of these two 2-subsets to a singleton but not the other.

Now, suppose \( G \leq S_n \) is strongly sync-maximal. By Theorem 5 by only considering transformations of rank \( n - 1 \), we find that \( G \) is primitive.

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