LOCAL $A$-PACKETS AND DERIVATIVES

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Abstract. In this paper, we reformulate Mœglin’s explicit construction of local $A$-packets of split odd special orthogonal groups and symplectic groups. In the reformulation, we need only derivatives so that we can compute them explicitly by results of the previous paper [2]. Also, we give a non-vanishing criterion of our parametrization, and an algorithm to compute certain derivatives. Finally, we propose a conjectural formula for the Aubert duals of irreducible representations of Arthur type.

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1. Introduction

Let \( G_n \) be a split special odd orthogonal group \( \text{SO}_{2n+1}(F) \) or a symplectic group \( \text{Sp}_{2n}(F) \) of rank \( n \) over a \( p \)-adic field \( F \). An A-parameter for \( G_n \) is a homomorphism

\[
\psi : W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \rightarrow \hat{G}_n,
\]

where \( W_F \) is the Weil group of \( F \) and \( \hat{G}_n \) is the complex dual group of \( G_n \). In his magnificent work [1], Arthur constructed the (local) A-packet \( \Pi_\psi \) associated to \( \psi \), which is a multi-set over \( \text{Irr}_{\text{unit}}(G_n) \) together with a map

\[
\Pi_\psi \rightarrow \mathcal{S}_\psi, \quad \pi \mapsto \langle \cdot, \pi \rangle_\psi,
\]

where \( \mathcal{S}_\psi \) is the component group of \( \psi \) and \( \hat{\mathcal{S}}_\psi \) is its Pontryagin dual ([1, Theorem 1.5.1]). Here, \( \text{Irr}_{\text{unit}}(G_n) \) is the set of equivalence classes of irreducible (unitary) representations of \( G_n \). Arthur’s multiplicity formula [1, Theorem 1.5.2] says that the elements in the local A-packets are the local components of the square-integrable automorphic representations. For the theory of automorphic representations, it is an important problem to understand the local A-packets “explicitly”. This problem would also be central for the unitary dual problem (see e.g., [15, Conjecture 1.2]). However, the local A-packets are defined by endoscopic character identities so that the local meaning of A-packets is very unclear.

In her consecutive works [9, 10, 11, 12, 13], Mœglin constructed the local A-packets \( \Pi_\psi \) concretely. As a consequence, she proved that \( \Pi_\psi \) is multiplicity-free, i.e., a subset of \( \text{Irr}_{\text{unit}}(G_n) \). We explain this construction briefly. To construct A-packets, Mœglin considered the following filtration of A-parameters:

\[
(\text{elementary}) \subset (\text{discrete diagonal restriction}) \subset (\text{of good parity}) \subset (\text{general}).
\]

The methods in each step are as follows:

- Irreducible parabolic inductions from the good parity case to the general case (see Theorem 2.5 below);
- derivatives with respect to non-self-dual cuspidal representations from the case of the discrete diagonal restriction (DDR) to the good parity case;
- taking certain socles (maximal semisimple subrepresentations) from the elementary case to the case of DDR;
- partial Aubert involutions for the elementary case.

See [17] for more details. Here, we say that an A-parameter \( \psi \) is of good parity if \( \psi \) is a sum of irreducible self-dual representations of \( W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \) of the same type as \( \psi \). Since irreducible parabolic inductions are easy to understand, we will consider A-parameters of good parity only.

Derivatives are certain partial information of Jacquet modules. Fix an irreducible unitary supercuspidal representation \( \rho \) of \( \text{GL}_d(F) \). Let \( P_d \) be the standard parabolic subgroup of \( G_n \) with Levi subgroup \( \text{GL}_d(F) \times G_{n-d} \). For a smooth representation \( \pi \) of \( G_n \) of finite length, if the semisimplification of Jacquet module \( \text{Jac}_{P_d}(\pi) \) is

\[
[\text{Jac}_{P_d}(\pi)] = \bigoplus_i \tau_i \boxtimes \pi_i
\]

where \( \tau_i \) are irreducible representations of \( \text{GL}_d(F) \times G_{n-d} \).
with irreducible representations $\tau_i \boxtimes \pi_i$ of $\text{GL}_d(F) \times G_{n-d}$, we define the $\rho \cdot |^x$-derivative $D_{\rho|.|^x}(\pi)$ by

$$D_{\rho|.|^x}(\pi) = \bigoplus_{\tau_i \boxtimes \pi_i} \pi_i$$

for $x \in \mathbb{R}$. Also, set $D_{\rho|.|^x}^{(k)}(\pi) = \frac{1}{k!} D_{\rho|.|^x} \circ \cdots \circ D_{\rho|.|^x}(\pi)$ ($k$-times composition). If $D_{\rho|.|^x}^{(k)}(\pi) \neq 0$ but $D_{\rho|.|^x}^{(k+1)}(\pi) = 0$, we say that $D_{\rho|.|^x}^{(k)}(\pi)$ is the highest $\rho \cdot |^x$-derivative of $\pi$. It is known that if $\pi$ is irreducible and if $\rho \cdot |^x$ is not self-dual, then the highest $\rho \cdot |^x$-derivative $D_{\rho|.|^x}^{(k)}(\pi)$ is also irreducible. Moreover, by results of the previous paper ([2, Proposition 6.1, Theorem 7.1]), one can compute the correspondence $\pi \mapsto D_{\rho|.|^x}^{(k)}(\pi)$ and its converse explicitly. In particular, one would compute the $A$-packets for the good parity case if one were to know the $A$-packets in the case of DDR explicitly.

The elementary case is a main cause of the complexity of Mœglin’s construction. This case contains not only all discrete series representations and their (usual) Aubert duals, but also their “intermediates”. Partial Aubert involutions were introduced to construct these “intermediates”, but they are very artificial so that they lose the compatibility of the Jacquet functors. Because of the elementary case, in the case of DDR, one has to consider the socles of quite non-tempered parabolically induced representations. Therefore, it is very hard to compute $\Pi_\psi$ by Mœglin’s construction in the elementary case and in the case of DDR.

The reason why Mœglin used the above filtration would be that derivatives with respect to self-dual cuspidal representations are complicated (see cf., [2, §3.3]). The same difficulty appears when one tries to give an algorithm for the Aubert duality (see [6, §7]). To overcome this difficulty, in the previous paper [2], the author and Mínguez introduced new derivatives. These derivatives can be applied to construction of $A$-packets, i.e., they will make Mœglin’s construction of $A$-packets more simply and more computable. As a refinement of her construction, we will use the following filtration:

(non-negative discrete diagonal restriction) $\subset$ (of good parity) $\subset$ (general).

Basically, the methods are the same as above. In the case of the non-negative DDR, the socles of standard modules are taken. Hence one can understand these representations by definition. Although we have to consider $\rho \cdot |^x$-derivatives for any $x \in \mathbb{R}$ in the good parity case, thanks to the results in [2], one can compute them explicitly (see cf., the proof of Theorem 2.7 below).

To state our results more precisely, we will introduce a key notion. Recall in [1, Theorem 1.5.1] that if an $A$-parameter $\psi = \phi$ is tempered, i.e., if the restriction to the second $\text{SL}_2(\mathbb{C})$ is trivial, then $\Pi_\phi$ consists of irreducible tempered representations, and the map $\Pi_\phi \to \hat{S}_\phi$ is bijective. When $\pi \in \Pi_\phi$ corresponds to $\eta \in \hat{S}_\phi$, we write $\pi = \pi(\phi, \eta)$.

**Definition 1.1** (Definition 3.1). (1) An extended segment is a triple $([A, B]_\rho, l, \eta)$, where

- $[A, B]_\rho = \{\rho \cdot [A, \rho] \cdot [A^{-1}, \cdots, \rho \cdot [B]\}$ is a segment, with an irreducible unitary cuspidal representation $\rho$ of some $\text{GL}_d(F)$;
- $l \in \mathbb{Z}$ with $0 \leq l \leq \frac{b}{2}$, where $b = \# [A, B]_\rho = A - B + 1$;
- $\eta \in \{\pm 1\}$.
(2) An extended multi-segment for $G_n$ is an equivalence class of multi-sets of extended segments

$$\mathcal{E} = \sqcup_{\rho} \{([A_i, B_i], l, \eta)\}_{i \in (I_{\rho}, >)}$$

such that

- $I_{\rho}$ is a totally ordered finite set with a fixed totally order $>$ satisfying that $B_i > B_j \implies i > j$.
- $A_i + B_i \geq 0$ for all $\rho$ and $i \in I_{\rho}$;
- as a representation of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$,

$$\psi = \bigoplus_{\rho} \bigoplus_{i \in I_{\rho}} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}$$

where $(a_i, b_i) = (A_i + B_i + 1, A_i - B_i + 1)$, is an $A$-parameter for $G_n$ of good parity;
- a sign condition

$$\prod_{\rho} \prod_{i \in I_{\rho}} (-1)^{\frac{b_i}{2} + l_i} \eta_i^{-1} = 1$$

holds.

(3) Two extended segments $([A, B], l, \eta)$ and $([A', B'], l', \eta')$ are equivalent if

- $[A, B]_{\rho} = [A', B']_{\rho'}$;
- $l = l'$; and
- $\eta = \eta'$ whenever $l = l' < \frac{b_i}{2}$.

Similarly, $\mathcal{E} = \sqcup_{\rho} \{([A_i, B_i], l_i, \eta_i)\}_{i \in (I_{\rho}, >)}$ and $\mathcal{E}' = \sqcup_{\rho} \{([A'_i, B'_i], l'_i, \eta'_i)\}_{i \in (I_{\rho}, >)}$ are equivalent if $([A_i, B_i], l_i, \eta_i)$ and $([A'_i, B'_i], l'_i, \eta'_i)$ are equivalent for all $\rho$ and $i \in I_{\rho}$.

(4) The support of $\mathcal{E}$ is the multi-segment

$$\text{supp}(\mathcal{E}) = \sum_{\rho} \sum_{i \in (I_{\rho}, >)} [A_i, B_i]_{\rho}.$$ 

Here, $\rho$ is identified with an irreducible bounded representation of $W_F$ by the local Langlands correspondence for $GL_d(F)$, and $S_a$ is the unique irreducible algebraic representation of $\text{SL}_2(\mathbb{C})$ of dimension $a$.

As in §3.2, one can define a representation $\pi(\mathcal{E})$ for each extended multi-segment $\mathcal{E}$. It is defined by taking derivatives from an irreducible representation with explicit Langlands data. By results in [2], one can show that $\pi(\mathcal{E})$ is irreducible or zero. The first main theorem is a reformulation of Mœglin’s construction of $A$-packets.

**Theorem 1.2 (Theorems 3.3, 3.5).** Let $\psi = \oplus_{\rho} (\oplus_{i \in I_{\rho}} \rho \boxtimes S_{a_i} \boxtimes S_{b_i})$ be an $A$-parameter for $G_n$ of good parity. Set $A_i = (a_i + b_i)/2 - 1$ and $B_i = (a_i - b_i)/2$. Choose a totally order $>_{\psi}$ on $I_{\rho}$ satisfying that $B_i > B_j \implies i > j$. Then

$$\Pi_{\psi} = \left\{ \pi(\mathcal{E}) \left| \text{supp}(\mathcal{E}) = \sum_{\rho} \sum_{i \in (I_{\rho}, >_{\psi})} [A_i, B_i]_{\rho} \right. \right\} \setminus \{0\}.$$ 

Moreover, for $\mathcal{E}$ with $\text{supp}(\mathcal{E}) = \sum_{\rho} \sum_{i \in (I_{\rho}, >_{\psi})} [A_i, B_i]_{\rho}$, one can define $\eta_\mathcal{E} \in \hat{\mathcal{S}}_{\psi}$ explicitly (Definition 3.4) such that if $\pi(\mathcal{E}) \neq 0$, then

$$\langle \cdot, \pi(\mathcal{E}) \rangle_{\psi} = \eta_\mathcal{E}.$$
When $B_i \geq 0$ for all $\rho$ and $i \in I_\rho$, this construction is exactly the same as Mœglin's original construction. By Theorem 1.2, the remaining problems we have to consider are:

**Problem A:** Determine precisely when $\pi(\mathcal{E})$ is nonzero.

**Problem B:** Specify $\pi(\mathcal{E})$ precisely if it is nonzero.

The second main theorem reduces these problems to the non-negative case, i.e., the case where $\mathcal{E} = \bigcup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, \succ)}$ satisfies that $B_i \geq 0$ for all $\rho$ and $i \in I_\rho$.

**Theorem 1.3** (Theorem 3.6). Let $\mathcal{E} = \bigcup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, \succ)}$ be an extended multi-segment for $G_n$. Take a positive integer $t$ such that $B_i + t \geq 0$ for all $\rho$ and $i \in I_\rho$. Define $\mathcal{E}_t$ from $\mathcal{E}$ by replacing each $([A_i, B_i]_\rho, l_i, \eta_i)$ with $([A_i + t, B_i + t|_\rho, l_i, \eta_i])$. Then the representation $\pi(\mathcal{E})$ is nonzero if and only if $\pi(\mathcal{E}_t) \neq 0$ and the following condition holds for all $\rho$ and $i \in I_\rho$:

$$B_i + l_i \geq \begin{cases} 0 & \text{if } B_i \in \mathbb{Z}, \\ \frac{1}{2} & \text{if } B_i \notin \mathbb{Z}, \eta_i \neq (-1)^{\beta_i}, \\ -\frac{1}{2} & \text{if } B_i \notin \mathbb{Z}, \eta_i = (-1)^{\beta_i}, \end{cases}$$

where we set

$$\beta_i = \sum_{j \in I_\rho, j < i} (A_j - B_j).$$

Moreover, in this case, the Langlands data for $\pi(\mathcal{E})$ are the “shift” of those of $\pi(\mathcal{E}_t)$ by $-t$.

Therefore, we will consider Problems A and B for non-negative extended multi-segments $\mathcal{E}$.

First, we consider Problem A for non-negative extended multi-segments $\mathcal{E}$ for $G_n$. Since our parametrization $\pi(\mathcal{E})$ is the same as Mœglin’s one in this case, we can use several Xu’s results. In [13], Xu gave an algorithm to determine when Mœglin’s original parameterizations are nonzero. We will apply this algorithm to $\pi(\mathcal{E})$. When $\mathcal{E} = \bigcup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, \succ)}$ is non-negative, the totally order $> \in I_\rho$ is imposed only a weaker condition:

$$A_i > A_j, B_i > B_j \implies i > j.$$  

We call such an order $>$ on $I_\rho$ admissible. To give an algorithm, Xu prepared three necessary conditions for $\pi(\mathcal{E}) \neq 0$ on $([A_i, B_i]_\rho, l_i, \eta_i)$ and $([A_j, B_j]_\rho, l_j, \eta_j)$ for two adjacent elements $i > j$ in $I_\rho$ (Proposition 4.1). Roughly speaking, Xu’s algorithm to determine whether $\pi(\mathcal{E}) \neq 0$ is reformulated as the third main theorem as follows:

**Theorem 1.4** (Theorem 4.3). Let $\mathcal{E}$ be a non-negative extended multi-segment for $G_n$. Then $\pi(\mathcal{E}) \neq 0$ if and only if the three necessary conditions in Proposition 4.1 are satisfied for every pair of two elements $i, j \in I_\rho$ which are adjacent with respect to some admissible order $>'$ on $I_\rho$ for all $\rho$.

Next, we consider Problem B for non-negative extended multi-segments $\mathcal{E}$ for $G_n$. When $\mathcal{E} = \bigcup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I_\rho, \succ)}$ (with $B_i \geq 0$) satisfies that

- $A_i - B_i = A_j - B_j$ for any $i, j \in I_\rho$; or
- if $i > j$, then $A_i > A_j$ and $B_i > B_j$,
the Langlands data for $\pi(\mathcal{E})$ were obtained by Xu ([19] Theorems 1.1, 1.2, 1.3]). However, they are already quite complicated. To attack Problem B, it might not be good to try to give the Langlands data for $\pi(\mathcal{E})$. Instead of this, in this paper, we will give an algorithm to compute certain derivatives of $\pi(\mathcal{E})$, which determine $\pi(\mathcal{E})$ uniquely.

The starting point is the following proposition of Xu, which says that $A$-parameters and derivatives have some compatibility.

**Proposition 1.5** ([17] Proposition 8.3 (ii)). Let $\psi = \oplus_{i=1, \rho \in \mathbb{R}} \bigoplus_{A_i} \mathfrak{S}_{a_i} \otimes \mathfrak{S}_{b_i}$ be an $A$-parameter of good parity for $G_n$. If $i \in \Pi_\psi$ satisfies that $D_{\rho_1}^{(k)}(\pi) \neq 0$, then

$$k \leq \# \{ i \in I_\rho \mid 2x = a_i - b_i \}.$$

However, one has to notice that the inequality of this proposition is not always an equality. To overcome this problem, we turn the fact that $\Pi_\psi \cap \Pi_{\psi'} \neq \emptyset \Rightarrow \psi \cong \psi'$ into an advantage. Namely, for given $\psi$ and $\pi \in \Pi_\psi$, we will construct $\psi'$ with $\pi \in \Pi_{\psi'}$ such that $\psi'$ gives a better estimate in Proposition 1.5. More precisely, the fourth main theorem is stated as follows:

**Theorem 1.6** (Theorem 5.3). Let $\mathcal{E} = \cup_{\rho \in \mathbb{R}} \{ ([A_i, B_i]_{\rho}, l_i, \eta_i) \}_{i \in \{ I_{\rho}, > \}}$ be a non-negative extended multi-segment for $G_n$ such that $\pi(\mathcal{E}) \neq 0$. Set $B_{\text{max}} = \max \{ B_i \mid i \in I_\rho \}$. Then we can construct another extended multi-segment $\mathcal{E}' = \cup_{\rho \in \mathbb{R}} \{ ([A_i^*, B_i^*]_{\rho}, l_i, \eta_i) \}_{i \in \{ I_{\rho}, > \}}$ by Algorithm 5.3 such that $\pi(\mathcal{E}') \cong \pi(\mathcal{E})$. Moreover, if $B_{\text{max}} \geq 1$, with $I'_\rho(B_{\text{max}}) = \{ i \in I'_\rho \mid B_i^* = B_{\text{max}} \}$, write

$$\bigcup_{i \in I'_\rho(B_{\text{max}})} [A_i^*, B_i^*]_{\rho} = \{ \rho \cdot |x_1|^2, \ldots, \rho \cdot |x_l|^2 \}$$

as multi-sets such that $x_1 \leq \cdots \leq x_l$. Define $\mathcal{E}'$ from $\mathcal{E}'$ by replacing $[A_i^*, B_i^*]_{\rho}$ with $[A_i^* - 1, B_i^* - 1]_{\rho}$ for every $i \in I'_\rho(B_{\text{max}})$. Then $\pi(\mathcal{E}') \neq 0$ and

$$D_{\rho_1}^{(|x_1|^2)} \circ \cdots \circ D_{\rho_{l}}^{(|x_l|^2)} (\pi(\mathcal{E}')) \cong \pi(\mathcal{E}')$$

up to a multiplicity.

By Theorem 1.6 together with [19] Theorem 1.3 and [2] Theorem 7.1, one can compute the Langlands data for $\pi(\mathcal{E})$.

Finally, we will give a conjecture on the Aubert dual of $\pi(\mathcal{E})$. In [3], Aubert defined an involution $\pi \mapsto \hat{\pi}$ on $\text{Irr}(G_n)$. We call $\hat{\pi}$ the Aubert dual of $\pi$. By Xu’s result [17] Appendix A] together with [3] §4.1, we know that

$$\{ \hat{\pi} \mid \pi \in \Pi_\psi \} = \Pi_{\hat{\psi}}$$

for any $A$-parameter (of good parity), where $\hat{\psi}$ is defined from $\psi$ by exchanging two $\text{SL}_2(\mathbb{C})$-factors. (It would also follow from Mœglin’s original construction.) In Definition 6.1 for any extended multi-segment $\mathcal{E} = \cup_{\rho \in \mathbb{R}} \{ ([A_i, B_i]_{\rho}, l_i, \eta_i) \}_{i \in \{ I_{\rho}, > \}}$, we will define another extended multi-segment $\hat{\mathcal{E}} = \cup_{\rho \in \mathbb{R}} \{ ([A_i, -B_i]_{\rho}, \hat{l}_i, \hat{\eta}_i) \}_{i \in \{ I_{\rho}, > \}}$ explicitly, and propose the following.

**Conjecture 1.7** (Conjecture 6.2). If $\pi(\mathcal{E}) \neq 0$, then its Aubert dual would be given by

$$\hat{\pi}(\mathcal{E}) \cong \pi(\hat{\mathcal{E}}).$$
Although we have an explicit algorithm to compute the Aubert duality ([2, Algorithm 4.1]), it is difficult to apply this to \( \pi(E) \) in general. It is because Algorithm [5,3] requires a changing of admissible orders.

This paper is organized as follows. In [2] we review representation theory of classical groups, including the theory of derivatives and known results on local A-packets. In [3] we introduce the notion of extended multi-segments and prove Theorems 1.2 and 1.3. We discuss about Problems A and B for non-negative case in [3] and [5] respectively. Finally, we give a conjectural formula for the Aubert duals in §6.

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**Notation.** Let \( F \) be a non-archimedean local field of characteristic zero. The normalized absolute value is denoted by \( |\cdot| \), which is also regarded as a character of \( GL_q(F) \) via composing the determinant map.

Let \( G = GL_n \) be a split special odd orthogonal group \( SO_{2n+1}(F) \) or a symplectic group \( Sp_{2n}(F) \) of rank \( n \) over \( F \). For a smooth representation \( \Pi \) of \( G_n \) or \( GL_n(F) \) of finite length, we write \([\Pi]\) for the semisimplification of \( \Pi \). Similarly, we denote by \( \soc(\Pi) \) the socle of \( \Pi \), i.e., the maximal semisimple subrepresentation of \( \Pi \). The set of equivalence classes of irreducible smooth representations of a group \( G \) is denoted by \( \Irr(G) \).

2. Preliminary

In this section, we review the representation theory of classical groups.

2.1. Langlands classification. First, we recall some notation for representations of \( GL_n(F) \). Let \( P \) be a standard parabolic subgroup of \( GL_n(F) \) with Levi subgroup \( M \cong GL_m(F) \times \cdots \times GL_{n_r}(F) \). For representations \( \tau_1, \ldots, \tau_r \) of \( GL_m(F), \ldots, GL_{n_r}(F) \), we denote by

\[
\tau_1 \times \cdots \times \tau_r := \text{Ind}_{P}^{GL_n(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r)
\]

the normalized parabolically induced representation.

Let \( \mathcal{C}_{\text{unit}}(GL_d(F)) \) be the set of equivalence classes of irreducible unitary supercuspidal representations of \( GL_d(F) \), and \( \mathcal{C}^\perp(GL_d(F)) \) be the subset consisting of self-dual elements. Set \( \mathcal{C}_{\text{unit}} = \cup_{d \geq 1} \mathcal{C}_{\text{unit}}(GL_d(F)) \) and \( \mathcal{C}^\perp = \cup_{d \geq 1} \mathcal{C}^\perp(GL_d(F)) \).

A segment \( [x, y]_\rho \) is a set of supercuspidal representations of the form

\[
[x, y]_\rho = \{ \rho \cdot |x|, \rho \cdot |x-1|, \ldots, \rho \cdot |y| \},
\]

where \( \rho \in \mathcal{C}_{\text{unit}}(GL_d(F)) \) and \( x, y \in \mathbb{R} \) such that \( x - y \in \mathbb{Z} \) and \( x \geq y \). For a segment \( [x, y]_\rho \), define a *Steinberg representation* \( \Delta_\rho[x, y] \) as a unique irreducible subrepresentation of

\[
\rho \cdot |x| \times \cdots \times \rho \cdot |y|.
\]

This is an essentially discrete series representation of \( GL_{d(x-y+1)}(F) \). Similarly, we define \( Z_\rho[y, x] \) as a unique irreducible quotient of the same induced representation.

The Langlands classification for \( GL_n(F) \) says that every \( \tau \in \Irr(GL_n(F)) \) is a unique irreducible subrepresentation of \( \Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \), where \( \rho_i \in \mathcal{C}_{\text{unit}} \) for \( i = 1, \ldots, r \).
such that \( x_1 + y_1 \leq \cdots \leq x_r + y_r \). In this case, we write

\[
\tau = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]).
\]

When \((x_{i,j})_{1 \leq i \leq t, 1 \leq j \leq d}\) satisfies that \(x_{i,j} = x_{1,1} - i + j\), the irreducible representation \(L(\Delta_{\rho}[x_{1,1}, x_{t,1}], \ldots, \Delta_{\rho}[x_{1,d}, x_{t,d}])\) is called a (shifted) Speh representation and is denoted by

\[
\left(\begin{array}{ccc}
x_{1,1} & \cdots & x_{1,d} \\
\vdots & \ddots & \vdots \\
x_{t,1} & \cdots & x_{t,d}
\end{array}\right)_\rho := L(\Delta_{\rho}[x_{1,1}, x_{t,1}], \ldots, \Delta_{\rho}[x_{1,d}, x_{t,d}]).
\]

Note that it is isomorphic to the unique irreducible subrepresentation of \(Z_{\rho}[x_{1,1}, x_{1,d}] \times \cdots \times Z_{\rho}[x_{t,1}, x_{t,d}].\)

An irreducibility criterion for the product of two Speh representations was obtained by Tadić [14]. For a more general situation, see Lapid–Mínguez [8]. The following is a part of [14] Theorem 1.1 or [8] Corollary 6.10.

**Theorem 2.1.** Let \((x_{i,j})_\rho\) and \((y_{i',j'})_{\rho'}\) be two Speh representations, where \(1 \leq i \leq t, 1 \leq j \leq d\) and \(1 \leq i' \leq t', 1 \leq j' \leq d'\). If the parabolically induced representation

\[
\left(\begin{array}{ccc}
x_{1,1} & \cdots & x_{1,d} \\
\vdots & \ddots & \vdots \\
x_{t,1} & \cdots & x_{t,d}
\end{array}\right)_\rho \times \left(\begin{array}{ccc}
y_{1,1} & \cdots & y_{1,d'} \\
\vdots & \ddots & \vdots \\
y_{t',1} & \cdots & y_{t',d'}
\end{array}\right)_{\rho'}
\]

is reducible, then the segments \([x_{1,d}, x_{t,1}]_\rho\) and \([y_{1,d'}, y_{t',1}]_{\rho'}\) are linked, i.e., \(\rho \cong \rho'\) and \([x_{1,d}, x_{t,1}]_\rho \nsubseteq [y_{1,d'}, y_{t',1}]_{\rho'}\), \([x_{1,d}, x_{t,1}]_\rho \nsubseteq [y_{1,d'}, y_{t',1}]_{\rho'}\) but \([x_{1,d}, x_{t,1}]_\rho \cup [y_{1,d'}, y_{t',1}]_{\rho'}\) is also a segment.

Next we recall some notation for representations of classical group \(G_n\). Fix an \(F\)-rational Borel subgroup of \(G_n\). Let \(P\) be a standard parabolic subgroup of \(G_n\) with Levi subgroup \(M \cong GL_{n_1}(F) \times \cdots \times GL_{n_r}(F) \times G_{n_0}\). For representations \(\tau_1, \ldots, \tau_r\) and \(\pi_0\) of \(GL_{n_1}(F), \ldots, GL_{n_r}(F)\) and of \(G_{n_0}\), respectively, denote by

\[
\tau_1 \times \cdots \times \tau_r \rtimes \pi_0 := \text{Ind}_P^{G_n}(\tau_1 \boxtimes \cdots \boxtimes \tau_r \boxtimes \pi_0)
\]

the normalized parabolically induced representation.

The Langlands classification for \(G_n\) says that every \(\pi \in \text{Irr}(G_n)\) is a unique irreducible subrepresentation of \(\Delta_{\rho_1}[x_1, y_1] \times \cdots \times \Delta_{\rho_r}[x_r, y_r] \rtimes \pi_0\), where

- \(\rho_1, \ldots, \rho_r \in \mathcal{C}_{\text{unit}}\);
- \(x_1 + y_1 \leq \cdots \leq x_r + y_r < 0\);
- \(\pi_0\) is an irreducible tempered representation of \(G_{n_0}\).

In this case, we write

\[
\pi = L(\Delta_{\rho_1}[x_1, y_1], \ldots, \Delta_{\rho_r}[x_r, y_r]; \pi_0).
\]

See [7] for more details.
2.2. Derivatives and socles. For a smooth representation $\pi$ of $G_n$ of finite length, denote by $\text{Jac}_{\rho_\pi}(\pi)$ its Jacquet module along the standard parabolic subgroup $P_d$ with Levi subgroup isomorphic to $\text{GL}_d(F) \times G_{n-d}$. Fix $\rho \in \text{Ind}^G_\rho(\text{GL}_d(F))$. For $x \in \mathbb{R}$, the $\rho \cdot |^x$-derivative $D_{\rho|\cdot|^x}(\pi)$ is a semisimple representation of $G_{n-d}$ satisfying that

$$[\text{Jac}_{\rho|\cdot|^x}(\pi)] = \rho \cdot |^x \otimes D_{\rho|\cdot|^x}(\pi) + \sum_i \tau_i \otimes \pi_i,$$

where $\tau_i \otimes \pi_i$ is an irreducible representation of $G_{n-d}$ such that $\tau_i \not\cong \rho \cdot |^x$. We also set $D^{(0)}_{\rho|\cdot|^x}(\pi) = \pi$ and

$$D^{(k)}_{\rho|\cdot|^x}(\pi) = \frac{1}{k} D_{\rho|\cdot|^x} \circ D^{(k-1)}_{\rho|\cdot|^x}(\pi) = \frac{1}{k!} D_{\rho|\cdot|^x} \circ \cdots \circ D_{\rho|\cdot|^x}(\pi).$$

When $D^{(k)}_{\rho|\cdot|^x}(\pi) \neq 0$ but $D^{(k+1)}_{\rho|\cdot|^x}(\pi) = 0$, we call $D^{(k)}_{\rho|\cdot|^x}(\pi)$ the highest $\rho \cdot |^x$-derivative of $\pi$. We say that $\pi$ is $\rho \cdot |^x$-reduced if $D^{(k)}_{\rho|\cdot|^x}(\pi) = 0$.

On the other hand, for a representation $\Pi$ of $\pi$, we denote by $\text{soc}(\Pi)$ the socle of $\Pi$, i.e., the maximal semisimple subrepresentation of $\Pi$. When $\Pi = (\rho \cdot |^x)^r \rtimes \pi$, we shortly write

$$S^{(r)}_{\rho|\cdot|^x}(\pi) = \text{soc}((\rho \cdot |^x)^r \rtimes \pi).$$

**Theorem 2.2** ([3] Lemma 3.1.3, [2] Propositions 3.3, 6.1, Theorem 7.1). Suppose that $x \neq 0$ so that $\rho \cdot |^x$ is not self-dual. Let $\pi$ be an irreducible representation of $G_n$.

1. The highest $\rho \cdot |^x$-derivative $D^{(k)}_{\rho|\cdot|^x}(\pi)$ is irreducible.
2. The socle $S^{(r)}_{\rho|\cdot|^x}(\pi)$ is irreducible for any $r \geq 0$.
3. They are related as

$$\pi = S^{(k)}_{\rho|\cdot|^x}(D^{(k)}_{\rho|\cdot|^x}(\pi))$$

and

$$D^{(k+r)}_{\rho|\cdot|^x}(S^{(r)}_{\rho|\cdot|^x}(\pi)) = D^{(k)}_{\rho|\cdot|^x}(\pi).$$

4. The Langlands data of $D^{(k)}_{\rho|\cdot|^x}(\pi)$ and $S^{(k)}_{\rho|\cdot|^x}(\pi)$ can be described from those of $\pi$ explicitly.

When $x = 0$, the $\rho$-derivative is difficult in general. As alternatives of $\rho$-derivative, we define the $\Delta_{\rho}[0, -1]$-derivative $D^{(k)}_{\Delta_{\rho}[0, -1]}(\pi)$ and the $Z_{\rho}[0, 1]$-derivative $D^{(k)}_{Z_{\rho}[0, 1]}(\pi)$ as semisimple representations of $G_{n-2dk}$ satisfying

$$[\text{Jac}_{\rho_{2dk}}(\pi)] = \Delta_{\rho}[0, -1]^k \otimes D^{(k)}_{\Delta_{\rho}[0, -1]}(\pi) + Z_{\rho}[0, 1]^k \otimes D^{(k)}_{Z_{\rho}[0, 1]}(\pi) + \sum_i \tau_i \otimes \pi_i,$$

where $\tau_i \otimes \pi_i$ is an irreducible representation of $\text{GL}_{2dk}(F) \times G_{n-2dk}$ such that $\tau_i \not\cong \Delta_{\rho}[0, -1]^k, Z_{\rho}[0, 1]^k$.

On the other hand, we set

$$S^{(r)}_{\Delta_{\rho}[0, -1]}(\pi) = \text{soc}(\Delta_{\rho}[0, -1]^r \rtimes \pi), \quad S^{(r)}_{Z_{\rho}[0, 1]}(\pi) = \text{soc}(Z_{\rho}[0, 1]^r \rtimes \pi).$$

**Theorem 2.3** ([2] Propositions 3.7, 3.8, §8). Let $\pi$ be an irreducible representation of $G_n$. Suppose that $\pi$ is $\rho \cdot |^{-1}$-reduced (resp. $\rho \cdot |^x$-reduced). Then the same assertions in Theorem 2.2 hold when $\rho \cdot |^x$ is replaced with $\Delta_{\rho}[0, -1]$ (resp. $Z_{\rho}[0, 1]$).
The following is a special case of [2, Lemma 3.5].

**Lemma 2.4.** Let \( \pi \) be an irreducible representation of \( G_n \). Assume that
- \( \pi \) is \( \rho \cdot \cdot (-1) \)-reduced (resp. \( \rho \cdot \cdot |1| \)-reduced);
- \( D_\phi \rho (\pi) \) is the highest \( \rho \)-derivative of \( \pi \);
- \( D_\phi \rho (\rho \cdot \cdot (-1) (D_\phi \rho (\pi)) \) (resp. \( D_\phi \rho (\rho \cdot \cdot |1| (D_\phi \rho (\pi)) \) ) the highest \( \rho \cdot \cdot (-1) \)-derivative (resp. \( \rho \cdot \cdot |1| \)-derivative) of \( D_\phi \rho (\pi) \).

Then \( D_{\Delta \rho (0, -1)}(\pi) = D_{\rho \cdot \cdot (-1)} \circ D_{\rho (\pi)} \) (resp. \( D_{\Delta \rho (0, 1)}(\pi) = D_{\rho \cdot \cdot |1|} \circ D_{\rho (\pi)} \)).

Finally, for a segment \([x, y]_{\rho} \), we set
\[
D_{\rho \cdot \cdot (-1)} \circ \cdots \circ D_{\rho \cdot \cdot |1|} \circ D_{\rho \cdot \cdot \rho (\pi)}(x) = D_{\rho \cdot \cdot (-1)} \circ \cdots \circ D_{\rho \cdot \cdot |1|} \circ D_{\rho \cdot \cdot \rho (\pi)}(y)
\]
and
\[
S_{\rho \cdot \cdot (-1)} \circ \cdots \circ S_{\rho \cdot \cdot |1|} \circ S_{\rho \cdot \cdot \rho (\pi)}(x) = S_{\rho \cdot \cdot (-1)} \circ \cdots \circ S_{\rho \cdot \cdot |1|} \circ S_{\rho \cdot \cdot \rho (\pi)}(y).
\]

For example, if \( \sigma \) is supercuspidal and \( y < x < 0 \), then
\[
D_{\rho \cdot \cdot (-1)} \circ \cdots \circ D_{\rho \cdot \cdot |1|} \circ D_{\rho \cdot \cdot \rho (\pi)}(x, y) = \sigma, \quad S_{\rho \cdot \cdot (-1)} \circ \cdots \circ S_{\rho \cdot \cdot |1|} \circ S_{\rho \cdot \cdot \rho (\pi)}(x, y) = L(\rho \cdot \cdot |1|, \ldots, \rho \cdot \cdot |1|, \sigma).
\]

### 2.3. Terminologies of A-parameters

Denote by \( \hat{G}_n \) the complex dual group of \( G_n \). Namely, \( \hat{G}_n = \text{Sp}_{2n}(\mathbb{C}) \) if \( G_n = \text{SO}_{2n+1}(F) \), and \( \hat{G}_n = \text{SO}_{2n+1}(\mathbb{C}) \) if \( G_n = \text{Sp}_{2n}(F) \). Recall that an A-parameter for \( G_n \) is a \( \hat{G}_n \)-conjugacy class of admissible homomorphisms
\[
\psi: W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \to \hat{G}_n
\]
such that the image of the Weil group \( W_F \) is bounded. By composing with the standard representation of \( \hat{G}_n \), we can regard \( \psi \) as a representation of \( W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \). It decomposes as
\[
\psi = \bigoplus_{\rho} \left( \bigoplus_{i \in I_{\rho}} \rho \otimes S_{a_i} \otimes S_{b_i} \right),
\]
where
- \( \rho \) runs over \( \mathcal{C}_{\text{unit}} \), which is identified with an irreducible bounded representation of \( W_F \) by the local Langlands correspondence for \( \text{GL}_d(F) \);
- \( S_a \) is the unique irreducible algebraic representation of \( \text{SL}_2(\mathbb{C}) \) of dimension \( a \).

Notice that \( a_i \) and \( b_i \) depend on \( \rho \), but we do not write it. We write \( \rho \otimes S_a \otimes S_b \) and \( \rho = \rho \otimes S_1 \otimes S_1 \) for short.

Let \( \psi \) be as above. We say that \( \psi \) is of good parity if \( \rho \otimes S_{a_i} \otimes S_{b_i} \) is self-dual of the same type as \( \psi \) for any \( \rho \) and \( i \in I_{\rho} \), i.e.,
- \( \rho \in \mathcal{C}^+ \) is orthogonal and \( a_i + b_i \equiv 0 \mod 2 \) if \( G_n = \text{Sp}_{2n}(F) \) (resp. \( a_i + b_i \equiv 1 \mod 2 \) if \( G_n = \text{SO}_{2n+1}(F) \)); or
- \( \rho \in \mathcal{C}^{-} \) is symplectic and \( a_i + b_i \equiv 1 \mod 2 \) if \( G_n = \text{Sp}_{2n}(F) \) (resp. \( a_i + b_i \equiv 0 \mod 2 \) if \( G_n = \text{SO}_{2n+1}(F) \)).

Let \( \Psi(G_n) \supset \Psi_{\text{gp}}(G_n) \) be the sets of equivalence classes of \( A \)-parameters and \( A \)-parameters of good parity, respectively. Also, we let \( \Phi_{\text{temp}}(G_n) \) be the subset of \( \Psi(G_n) \) consisting of \textit{tempered} \( A \)-parameters, i.e., \( A \)-parameters \( \phi \) which are trivial on the second \( \text{SL}_2(\mathbb{C}) \). Finally, we set \( \Phi_{\text{gp}}(G_n) = \Psi_{\text{gp}}(G_n) \cap \Phi_{\text{temp}}(G_n) \).
For $\psi = \oplus_\rho(\oplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}) \in \Psi_{\text{gp}}(G_n)$, define the enhanced component group by

$$\mathcal{A}_\psi = \bigoplus_{\rho \in I_\rho} (\mathbb{Z}/2\mathbb{Z}) \alpha_{\rho,i},$$

i.e., $\mathcal{A}_\psi$ is a $(\mathbb{Z}/2\mathbb{Z})$-vector space with a canonical basis $\alpha_{\rho,i}$ corresponding to $\rho \boxtimes S_{a_i} \boxtimes S_{b_i}$. The component group $\mathcal{S}_\psi$ is the quotient of $\mathcal{A}_\psi$ by the subgroup generated by

- $\alpha_{\rho,i} + \alpha_{\rho,j}$ such that $\rho \boxtimes S_{a_i} \boxtimes S_{b_i} = \rho \boxtimes S_{a_j} \boxtimes S_{b_j}$; and
- $z_\psi = \sum_{\rho} \sum_{i \in I_\rho} \alpha_{\rho,i}$, which is called the central element of $\mathcal{A}_\psi$.

Let $\hat{\mathcal{S}}_\psi \subset \hat{\mathcal{A}}_\psi$ be the Pontryagin duals of $\mathcal{S}_\psi$ and $\mathcal{A}_\psi$, respectively. When $\varepsilon \in \hat{\mathcal{A}}_\psi$, we write $\varepsilon(\rho \boxtimes S_{a_i} \boxtimes S_{b_i}) = \varepsilon(\alpha_{\rho,i}) \in \{\pm 1\}$.

2.4. $A$-packets. Let $\text{Irr}_{\text{unit}}(G_n)$ (resp. $\text{Irr}_{\text{temp}}(G_n)$) be the set of equivalence classes of irreducible unitary (resp. tempered) representations of $G_n$. To an $A$-parameter $\psi \in \Psi(G_n)$, Arthur [1] Theorem 2.2.1 associated an $A$-packet $\Pi_\psi$, which is a finite multi-set over $\text{Irr}_{\text{unit}}(G_n)$. It is characterized by twisted endoscopic character identities. Moreover, if $\phi \in \Phi_{\text{temp}}(G_n)$ is a tempered $A$-parameter, then $\Pi_\phi$ is a subset of $\text{Irr}_{\text{temp}}(G_n)$ and

$$\text{Irr}_{\text{temp}}(G_n) = \bigcup_{\phi \in \Phi_{\text{temp}}(G_n)} \Pi_\phi \text{ (disjoint union)}.$$  

Moeglin [13] showed that $\Pi_\psi$ is multiplicity-free, i.e., a subset of $\text{Irr}_{\text{unit}}(G_n)$. To prove it, she constructed $\Pi_\psi$ concretely. The purpose of this paper is to describe $\Pi_\psi$ more explicit. In general, $\Pi_\psi \cap \Pi_\psi' \neq \emptyset$ even if $\psi_1 \neq \psi_2$. We will use this fact to give an algorithm to compute elements of $\Pi_\psi$.

If $\psi = \oplus_\rho(\oplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i})$, set

$$\tau_\psi = \bigotimes_{i \in I_\rho} \left( \begin{array}{ccc} \frac{a_i-b_i}{2} & \ldots & \frac{a_i+b_i}{2} -1 \\ \vdots & \ddots & \vdots \\ \frac{a_i+b_i}{2} +1 & \ldots & \frac{a_i-b_i}{2} \end{array} \right)$$

to be the product of (unitary) Speh representations, which is an irreducible unitary representation of $\text{GL}_d(F)$ with $d = \dim(\psi)$.

Any $\psi \in \Psi(G_n)$ can be decomposed as

$$\psi = \psi_1 \oplus \psi_0 \oplus \psi_1^\vee$$

for some representation $\psi_1$ of $W_F \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C})$ and some $\psi_0 \in \Psi_{\text{gp}}(G_{n_0})$.

**Theorem 2.5** ([1] Theorem 6, [17] Proposition 8.11). For $\pi_0 \in \Pi_\psi$, the parabolically induced representation $\tau_\psi \times \pi_0$ is irreducible and independent of a choice of $\psi_1$. Moreover,

$$\Pi_\psi = \{ \tau_\psi \times \pi_0 \mid \pi_0 \in \Pi_\psi^0 \}.$$  

Hence we may only consider $\psi \in \Psi_{\text{gp}}(G_n)$.

Through this paper, we implicitly fix a Whittaker datum for $G_n$. Let $\psi \in \Psi_{\text{gp}}(G_n)$ so that we have defined the component group $\mathcal{S}_\psi$. Arthur [1] Theorem 2.2.1 gave a map

$$\Pi_\psi \to \hat{\mathcal{S}}_\psi, \ \pi \mapsto \langle \cdot, \pi \rangle_\psi.$$  

If $\psi = \phi \in \Phi_{\text{gp}}(G_n)$ is tempered, this map is bijective. When $\pi \in \Pi_\phi$ corresponds to $\varepsilon \in \hat{\mathcal{S}}_\psi$, we write $\pi = \pi(\phi, \varepsilon)$. 
Now we recall some relation between $A$-parameters and derivatives. Let $\psi = \oplus_{\rho}(\oplus_{i \in I_{\rho}} \rho \boxtimes S_{a_{i}} \boxtimes S_{b_{i}}) \in \Psi_{\text{sp}}(G_{n})$. Define

$$A_{i} = \frac{a_{i} + b_{i}}{2} - 1, \quad B_{i} = \frac{a_{i} - b_{i}}{2}.$$  

Note that the definition of $B_{i}$ is not the same as the one of Mœglin and Xu. By the compatibility of twisted endoscopic character identities and Jacquet modules ([16, §6], see also [17, §8]), Xu gave the following proposition.

**Proposition 2.6 ([17, Proposition 8.3 (ii)])**. Let $\psi = \oplus_{\rho}(\oplus_{i \in I_{\rho}} \rho \boxtimes S_{a_{i}} \boxtimes S_{b_{i}}) \in \Psi_{\text{sp}}(G_{n})$. If $\pi \in \Pi_{\psi}$ satisfies that $D^{(k)}_{\rho|\psi}(\pi) \neq 0$, then

$k \leq \#\{i \in I_{\rho} \mid x = B_{i}\}$.

We call a total order $>_{\psi}$ on $I_{\rho}$ admissible if it satisfies the following condition:

(\(P\)) \quad For \(i, j \in I_{\rho}\), if $A_{i} > A_{j}$ and $B_{i} > B_{j}$, then $i > \psi j$.

More strongly, we also consider the following condition:

(\(P'\)) \quad For \(i, j \in I_{\rho}\), if $B_{i} > B_{j}$, then $i > \psi j$.

Using these orders, we have the following theorem.

**Theorem 2.7.** Let $\psi = \oplus_{\rho}(\oplus_{i \in I_{\rho}} \rho \boxtimes S_{a_{i}} \boxtimes S_{b_{i}}) \in \Psi_{\text{sp}}(G_{n})$. Fix an admissible order $>_{\psi}$ on $I_{\rho}$ for each $\rho$, and write $I_{\rho} = \{1, \ldots, m\}$ with $1 <_{\psi} \cdots <_{\psi} m = m_{\rho}$. Assume that $>_{\psi}$ on $I_{\rho}$ satisfies (\(P'\)) if $B_{i} < 0$ for some $i \in I_{\rho}$. Take $\psi' = \oplus_{\rho}(\oplus_{i \in I_{\rho}} \rho \boxtimes S_{a_{i}+2t_{i}} \boxtimes S_{b_{i}})$ with non-negative integers $t_{1}, \ldots, t_{m}$ such that

$$0 \leq B_{1} + t_{1} \leq A_{1} + t_{1} < B_{2} + t_{2} \leq A_{2} + t_{2} < \cdots < B_{m} + t_{m} \leq A_{m} + t_{m}.$$  

Then

$$\Pi_{\psi} = \left\{ o_{\rho} o_{i \in I_{\rho}} (D_{\rho|\psi_{i+1}} \cdots D_{\rho|\psi_{1}}) (\pi') \mid \pi' \in \Pi_{\psi'} \right\} \setminus \{0\}.$$  

Here, we write $o_{i \in I_{\rho}} D_{i} = D_{m} \cdots \circ D_{1}$.

**Proof.** For simplicity, write $D = o_{\rho} o_{i \in I_{\rho}} (D_{\rho|\psi_{i+1}} \cdots D_{\rho|\psi_{1}}) (\pi')$. When $B_{i} \geq 0$ for all $\rho$ and $i \in I_{\rho}$, the assertion is already known (see [17, §8]). In general, by the compatibility of twisted endoscopic character identities and Jacquet modules [16, §6], as virtual representations, we have

$$\sum_{\pi \in \Pi_{\psi}} \langle s_{\psi}, \pi \rangle_{\psi} \pi = \sum_{\pi' \in \Pi_{\psi'}} \langle s_{\psi'}, \pi' \rangle_{\psi'} D(\pi') \tag{*}$$

for certain elements $s_{\psi} \in S_{\psi}$ and $s_{\psi'} \in S_{\psi'}$. When $B_{i} \geq 0$ or $B_{i} \in (1/2)\mathbb{Z} \setminus \mathbb{Z}$ for any $\rho$ and $i \in I_{\rho}$, by Theorem 2.2 and Proposition 2.6, we see that

1. $D(\pi')$ is irreducible or zero for any $\pi' \in \Pi_{\psi'}$;
2. for $\pi_{1}', \pi_{2}' \in \Pi_{\psi'}$,

$$D(\pi_{1}') \cong D(\pi_{2}') \neq 0 \implies \pi_{1}' \cong \pi_{2}'.$$
On the other hand, since the \( \rho \)-derivative \( D_\rho \) is not injective, there might be a cancelation in the right hand side of \((*)\). However, since \( A_i \geq 0 \) for all \( \rho \) and \( i \in I_\rho \), if \( D_\rho \) appears in the definition of \( D \), it appears as \( D_{\rho,|1|} = D_{\rho,|1|} \circ D_\rho \). By Lemma 2.4 together with the condition \( (P') \) and Proposition 2.6, we may replace \( D_{\rho,|1|} \) with \( D_{Z_\rho[0,1]} \). By Theorem 2.3, we have the same conclusions as (1) and (2), and hence there is no cancelation in \((*)\). \( \square \)

3. Extended multi-segments and their associated representations

To describe \( A \)-packets, we introduce the notion of extended multi-segments, and define associated representations.

3.1. Extended segments. In this subsection, we define extended (multi-)segments.

**Definition 3.1.** (1) An extended segment is a triple \( ([A,B]_\rho,l,\eta) \), where

- \( [A,B]_\rho = \{ \rho : [A,\ldots,A,\rho,B] \} \) is a segment;
- \( l \in \mathbb{Z} \) with \( 0 \leq l \leq \frac{b}{2} \), where \( b = \# [A,B]_\rho = A - B + 1 \);
- \( \eta \in \{ \pm 1 \} \).

(2) An extended multi-segment for \( G_n \) is an equivalence class of multi-sets of extended segments

\[
\mathcal{E} = \bigcup_\rho \{ ([A_i,B_i]_\rho,l_i,\eta_i) \}_{i \in (I_\rho,>} \]

such that

- \( I_\rho \) is a totally ordered finite set with a fixed admissible order \( > \);
- \( A_i + B_i \geq 0 \) for all \( \rho \) and \( i \in I_\rho \);
- as a representation of \( \text{WF} \times \text{SL}_2(\mathbb{C}) \times \text{SL}_2(\mathbb{C}) \),

\[
\psi = \bigoplus_\rho \bigoplus_{i \in I_\rho} \rho \otimes S_{a_i} \otimes S_{b_i} \in \Psi_{\text{gp}}(G_n),
\]

where \( a_i = A_i + B_i + 1 \) and \( b_i = A_i - B_i + 1 \);
- a sign condition

\[
\prod_\rho \prod_{i \in I_\rho} (-1)^{b_i} l_i^{a_i} b_i^{b_i} = 1
\]

holds.

(3) Two extended segments \( ([A,B]_\rho,l,\eta) \) and \( ([A',B']_\rho,l',\eta') \) are equivalent if

- \( [A,B]_\rho = [A',B']_\rho' \);
- \( l = l' \); and
- \( \eta = \eta' \) whenever \( l = l' < \frac{b}{2} \).

Similarly, \( \mathcal{E} = \bigcup_\rho \{ ([A_i,B_i]_\rho,l_i,\eta_i) \}_{i \in (I_\rho,>} \) and \( \mathcal{E}' = \bigcup_\rho \{ ([A_i',B_i']_\rho,l_i',\eta_i') \}_{i \in (I_\rho,>} \) are equivalent if \( ([A_i,B_i]_\rho,l_i,\eta_i) \) and \( ([A_i',B_i']_\rho,l_i',\eta_i') \) are equivalent for all \( \rho \) and \( i \in I_\rho \).

(4) The support of \( \mathcal{E} \) is the multi-segment

\[
\text{supp}(\mathcal{E}) = \sum_\rho \sum_{i \in (I_\rho,>)} [A_i,B_i]_\rho.
\]

Let \( \mathcal{E} = \bigcup_\rho \mathcal{E}_\rho \) be an extended multi-segments with \( \mathcal{E}_\rho = \{ ([A_i,B_i]_\rho,l_i,\eta_i) \}_{i \in (I_\rho,>} \). We regard \( \mathcal{E}_\rho \) as the following symbol: When \( \mathcal{E}_\rho = \{ ([A,B]_\rho,l,\eta) \} \) is a singleton, we write

\[
\mathcal{E}_\rho = \begin{pmatrix}
\begin{array}{c}
B \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\downarrow & \cdots & \downarrow & \cdots & \downarrow & \cdots & \uparrow & \cdots & \uparrow \\
\end{array}
\end{pmatrix}_{\rho}.
\]
where $\odot$ is replaced with $\oplus$ and $\ominus$ alternately, starting with $\oplus$ if $\eta = +1$ (resp. $\ominus$ if $\eta = -1$). In general, we put each symbols vertically.

**Example 3.2.** When $\mathcal{E}_{\rho} = \{([A_i, B_i]_{\rho}, l_i, \eta_i)\}_{1 \leq i \leq 4}$ with

- $[A_1, B_1] = [3, 1], \ [A_2, B_2] = [5, 2], \ [A_3, B_3] = [6, 3]$ and $[A_4, B_4] = [5, 4]$;
- $(l_1, l_2, l_3, l_4) = (0, 1, 2, 0)$ and $(\eta_1, \eta_2, \eta_4) = (-1, -1, -1),

the symbol is

$$\mathcal{E}_{\rho} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \ominus & \ominus & \oplus & \ominus & \ominus & \ominus \end{pmatrix}.$$

This symbol does not depend on $\eta_3$.

As in this example, the symbol is determined by the equivalence class of extended multisegments. The number of $\ominus$ appearing in $\mathcal{E}_{\rho}$ is odd if and only if $\prod_{i \in I_{\rho}} (-1)^{\left(\frac{l_i}{2}\right) + l_i} \eta_i = -1$. Hence the sign condition in Definition 3.1 (2) means that $\ominus$ appears even times among all symbols.

### 3.2. Associated representations.

Let $\mathcal{E} = \cup_{\rho}(\{([A_i, B_i]_{\rho}, l_i, \eta_i)\}_{i \in (I_{\rho}, >)}$ be an extended multisegment for $G_{\eta}$. Assume that the admissible order $>$ on $I_{\rho}$ satisfies $(P')$ if $B_i < 0$ for some $i \in I_{\rho}$. We will define a representation $\pi(\mathcal{E})$ as follows. First, we assume that

- for $i, j \in I_{\rho}$, if $i > j$, then $B_i > A_j$;
- $B_i \geq 0$ for any $i \in I_{\rho}$.

In this case, we define

$$\pi(\mathcal{E}) = \text{soc} \left( \bigotimes_{i \in I_{\rho}} \left( \bigotimes_{\rho} D_{\rho^{|i = l_i + 1}} \right) \right) \times \pi(\phi, \varepsilon)$$

with

$$\phi = \bigoplus_{\rho} \bigoplus_{i \in I_{\rho}} \rho \bigotimes \left( S_{2(B_i + l_i + 1)} \oplus \cdots \oplus S_{2(A_i - l_i + 1)} \right)$$

and $\varepsilon(\rho \bigotimes S_{2(B_i + l_i + k) + 1}) = (-1)^{k} \eta_i$ for $0 \leq k \leq b_i - 2l_i - 1$. Note that

- the shifted Speh representations appearing in the above induced representation are commutative to each other by Theorem 2.11;
- the parabolically induced representation is isomorphic to a standard module so that $\pi(\mathcal{E})$ is an irreducible representation.

In general, take non-negative integers $t_1, \ldots, t_m$ such that $\mathcal{E}' = \cup_{\rho}(\{([A_i + t_i, B_i + t_i]_{\rho}, l_i, \eta_i)\}_{i \in (I_{\rho}, >)}$ satisfies the above conditions, and define

$$\pi(\mathcal{E}) = c_{\rho} \bigotimes_{i \in I_{\rho}} \left( D_{\rho^{|i = l_i + t_i + 1}} \bigotimes_{\rho} D_{\rho^{|i = l_i + t_i + 1}} \right) (\pi(\mathcal{E}')).$$

This definition does not depend on the choice of $t_i$. By Theorems 2.22 and 2.23 we see that $\pi(\mathcal{E})$ is irreducible or zero (see the proof of Theorem 2.7). For examples, see [3.5] below.

The following is a reformulation of Mœglin’s construction of $A$-packets.
Theorem 3.3. Let $\psi = \bigoplus_{\rho} (\oplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}) \in \Psi_{\text{gp}}(G_n)$. Set $A_i = (a_i + b_i)/2 - 1$ and $B_i = (a_i - b_i)/2$. Choose an admissible order $>_{\psi}$ on $I_\rho$ satisfying $(P')$ for each $\rho$. Then 
\[
\bigoplus_{\pi \in \Pi_\psi} \pi \equiv \bigoplus_{\mathcal{E}} \pi(\mathcal{E}),
\]
where $\mathcal{E}$ runs over all extended multi-segments with $\text{supp}(\mathcal{E}) = \sum_{i} \sum_{i \in (I_\rho, >_\psi)} [A_i, B_i]_\rho$.

Proof. When $\psi$ satisfies that
\begin{itemize}
  \item for $i, j \in I_\rho$, if $i >_{\psi} j$, then $B_i > A_j$;
  \item $B_i \geq 0$ for any $i \in I_\rho$,
\end{itemize}
this is just Mœglin's construction (see [17, §7]). In general, the assertion follows from Theorem 2.7 and the definition of $\pi(\mathcal{E})$. □

3.3. Characters of component groups. Let $\psi \in \Psi_{\text{gp}}(G_n)$. Recall that Arthur [1] Theorem 2.2.1] established $\Pi_\psi$, together with a map 
\[
\Pi_\psi \to \widehat{\mathcal{S}}_\psi, \quad \pi \mapsto \langle \cdot, \pi \rangle_\psi.
\]
This map plays an important role for global applications. In this subsection, we describe this map.

The following is a reformulation of [17, Definition 5.2].

Definition 3.4. Let $\psi = \bigoplus_{\rho} (\oplus_{i \in I_\rho} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}) \in \Psi_{\text{gp}}(G_n)$. With $A_i = (a_i + b_i)/2 - 1$ and $B_i = (a_i - b_i)/2$, we choose an admissible order $>_{\psi}$ on $I_\rho$ satisfying $(P')$.

1. For $i \in I_\rho$, define $Z(\psi)_{\rho, i}$ by the set of $j \in I_\rho$ such that $\# [A_i, B_j]_\rho \neq \# [A_j, B_i]_\rho \mod 2$ and 
\[
\begin{cases}
  j < i & \implies A_j + B_j > \frac{A_i + B_i}{2}, \quad \# [A_j, B_i]_\rho > \# [A_i, B_j]_\rho, \\
  j > i & \implies A_j + B_j < \frac{A_i + B_i}{2}, \quad \# [A_j, B_i]_\rho < \# [A_i, B_j]_\rho.
\end{cases}
\]

2. For an extended multi-segment $\mathcal{E} = \bigcup_{\rho} \{([A_i, B_j]_\rho, \eta_i)\}_{i \in (I_\rho, >)}$ for $G_n$, define $\eta_\mathcal{E} \in \widehat{\mathcal{S}}_\psi$ by
\[
\eta_\mathcal{E}(\rho \boxtimes S_{a_i} \boxtimes S_{b_i}) = (-1)^{\# Z(\psi)_{\rho, i} + [b_i^0] + \# \eta_i^0}.
\]

Theorem 3.5. If $\pi(\mathcal{E}) \neq 0$ so that $\pi(\mathcal{E}) \in \Pi_\psi$, we have 
\[
\langle \cdot, \pi(\mathcal{E}) \rangle_\psi = \eta_\mathcal{E}.
\]

Proof. When $B_i \geq 0$ for all $\rho$ and $i \in I_\rho$, the assertion is [17, Propositions 5.7, 8.2]. Remark that the character of [17, Definition 8.1] is trivial in this case. The general case follows from the compatibility of standard endoscopic character identities and Jacquet modules ([16, §6], see also the proof of [17, Theorem 7.5]). □

3.4. Reduction to the non-negative case. In this subsection, we reduce problems for $\pi(\mathcal{E})$ to the non-negative case. Let $\mathcal{E} = \bigcup_{\rho} \{([A_i, B_j]_\rho, \eta_i)\}_{i \in (I_\rho, >)}$ be an extended multi-segment for $G_n$. We assume that the fixed admissible order $>$ satisfied that:

$(P')$
\[\text{For } i, j \in I_\rho, \text{ if } B_i > B_j, \text{ then } i > j.\]

Take a non-negative integer $t$ such that $t + B_i \geq 0$ for any $\rho$ and $i \in I_\rho$. Define $\mathcal{E}_t$ from $\mathcal{E}$ by replacing $[A_i, B_i]_\rho$ with $[A_i + t, B_i + t]_\rho$ for any $\rho$ and $i \in I_\rho$. 
Theorem 3.6. The representation $\pi(\mathcal{E})$ is nonzero if and only if $\pi(\mathcal{E}_i) \neq 0$ and the following condition holds for any $\rho$ and $i \in I_\rho$:

\[
(*) \quad B_i + l_i \geq \begin{cases} 
0 & \text{if } B_i \in \mathbb{Z}, \\
1/2 & \text{if } B_i \notin \mathbb{Z}, \eta_i \neq (-1)^{\beta_i}, \\
-1/2 & \text{if } B_i \notin \mathbb{Z}, \eta_i = (-1)^{\beta_i}, 
\end{cases}
\]

where we set

\[
\beta_i = \sum_{j \in I_\rho, j < i} (A_j - B_j).
\]

Moreover, in this case, if $\pi(\mathcal{E}_i) \cong L(\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_r}[x_r, -y_r]; \pi(\phi, \varepsilon))$ with $\phi = \bigoplus_{j=1}^s \rho_j \boxtimes S_{a_j}$, then

- $x_i + y_i + 1 \geq 2t$ for any $1 \leq i \leq r$;
- $a_j \geq 2t$ for any $1 \leq j \leq s$,

and

\[
\pi(\mathcal{E}) \cong L(\Delta_{\rho_1}[x_1 - t, -(y_1 - t)], \ldots, \Delta_{\rho_r}[x_r - t, -(y_r - t)]; \pi(\phi_{-t}, \varepsilon_{-t}))
\]

where $\phi_{-t} = \bigoplus_{j=1}^s \rho_j^{\prime} \boxtimes S_{a_j-2t}$ and $\varepsilon_{-t}(\rho_j^{\prime} \boxtimes S_{a_j-2t}) = \varepsilon(\rho_j \boxtimes S_{a_j})$.

Proof. We prove the assertion by induction on $\sum \rho_i(\# I_\rho - 1)$. Write $I_\rho = \{1, \ldots, m\}$ with $1 < \cdots < m$. First of all, if $0 \leq B_1 < A_1 < B_2 < A_2 < \cdots < B_m < A_m$ for any $\rho$, then the assertion is obvious from the definition of $\pi(\mathcal{E})$.

Now we consider the general case. Take a positive integer $t'$, and define $\mathcal{E}'$ (resp. $\mathcal{E}_{i}'$) from $\mathcal{E}$ (resp. $\mathcal{E}_i$) by replacing $[A_m, B_m]_\rho$ with $[A_m + t', B_m + t']_\rho$ (resp. $[A_m + t, B_m + t]_\rho$) with $[A_m + t + t', B_m + t + t']_\rho$. When $t' \gg 0$, the calculation of the definition of $\pi(\mathcal{E}')$ is the same if we replace $[A_m + t', B_m + t']_\rho$ with $[A_m + t', B_m + t']_\rho$ for some $\rho'$, i.e., we can replace $I_\rho$ with the union of $I_\rho = \{1, \ldots, m-1\}$ and $I_\rho = \{m\}$. Therefore, we may apply the induction hypothesis to $\mathcal{E}'$ and $\mathcal{E}_{i}'$. Moreover, we have

\[
(3.1) \quad \pi(\mathcal{E}) = D_{\rho_1,1}[B_{m+1}, \ldots, B_m] \cdot \cdots \cdot D_{\rho_s,1}[B_{m+t'} = \ldots = B_{m+t'}(\pi(\mathcal{E}')),
\]

\[
(3.2) \quad \pi(\mathcal{E}_i) = D_{\rho_1,1}[B_{m+1}, \ldots, B_m] \cdot \cdots \cdot D_{\rho_s,1}[B_{m+t'} = \ldots = B_{m+t'}(\pi(\mathcal{E}_i))).
\]

We show that if $\pi(\mathcal{E}) \neq 0$, then the condition $(*)$ for $i = m$. To do this, we may assume that $B_m < 0$. We note that $B_1 \leq \cdots \leq B_m$ by the condition (P'). First, we prove that $\pi(\mathcal{E}) \neq 0 \implies B_m + l_m \geq -1/2$. Suppose that $\pi(\mathcal{E}) \neq 0$ but $B_m + l_m < -1/2$. When $t'$ is big enough, by definition, we have

\[
\pi(\mathcal{E}') \supseteq \begin{pmatrix} B_m + t' & \ldots & B_m + t' + l_m - 1 \\
\vdots & \ddots & \vdots \\
-(A_m + t') & \ldots & -(A_m + t' - l_m + 1) \end{pmatrix}_\rho \ni \pi(\mathcal{E}''),
\]

where $\mathcal{E}''$ is defined from $\mathcal{E}'$ by replacing $([A_m + t', B_m + t']_\rho, l_m, \eta_m)$ by $([A_m + t - l_m, B_m + t' + l_m]_\rho, 0, \eta_m)$. Hence if $\pi(\mathcal{E}) \neq 0$, then (using [16, Lemma 5.6]), we see that

\[
(3.3) \quad D_{\rho,1}[B_{m+1} \ldots m] \cdot D_{\rho,1}[B_{m+2} \ldots + t' + l_m] \cdot \cdots \cdot D_{\rho,1}[B_{m+t'} + l_m] \cdot \pi(\mathcal{E}'')) \neq 0.
\]
Note that \( \# [A_m + t' - l_m, B_m + t' + l_m]_\rho > 0 \) since \( A_m + B_m \geq 0 \). We may redefine \( \mathcal{E}'' \) by splitting \(( [A_m + t' - l_m, B_m + t' + l_m]_\rho, 0, \eta_m) \) into
\[
\{(B_m + t' + l_m, B_m + t' + l_m)_\rho, 0, \eta_m), ([A_m + t' - l_m, B_m + t' + l_m + 1]_\rho, 0, -\eta_m)\}.
\]
Using the twisted endoscopy, we transfer (3.3) to a general linear group. By the compatibility of twisted endoscopic character identities and Jacquet modules [16, §6] (cf., see Proposition 2.6 and Theorem 2.7), the Steinberg representation \( \Delta_\rho[B_m + t' + l_m, -(B_m + t' + l_m)] \) should be embedded into
\[
\rho \cdot |B_m + t' + l_m| \times \cdots \times \rho \cdot |B_m + t' + l_m + 1| \times \rho \cdot -(B_m + t' + l_m)
\]
for some nonzero representation \( \rho \). However, in this case, \( \rho \) is a representation of \( GL_d(2(B_m + l_m + 1)) \) (where \( \rho \in \mathcal{E}^{-1}(GL_d(F)) \). Therefore, we must have \( 2(B_m + l_m + 1) \geq 0 \), which is a contradiction.

When \( B_m + l_m = -1/2 \) and \( \eta_m \neq (-1)^{\beta_i} \), one can see that
\[
D_{\rho | l/2} \circ D_{\rho | 3/2} \circ \cdots \circ D_{\rho | b_m + t' + l_m} (\pi(\mathcal{E}'')) = 0.
\]
This comes from the special (formal) understanding of [2 Theorem 5.3] for \( x = 1/2 \) (see also [6, Theorem 3.3]). Or, it is also understood by Theorem 4.3 below. In particular, if \( \pi(\mathcal{E}) \neq 0 \), then the condition (\(*\)) must holds for \( i = m \).

Under the condition (\(*\)) on \( B_m + l_m \), we will show that the computations of the derivatives in the equations (3.1) and (3.2) are exactly the same (up to the shift by \( t \)). Let \( D_{\rho | l/\pi \circ \cdots \circ D_{\rho | l/\pi} \circ \cdots \circ D_{\rho | l/\pi} \circ \cdots \circ D_{\rho | l/\pi} \) denote the compositions of derivatives appearing in (3.1) and (3.2), respectively. Fix \( 0 \leq s' \leq s', \) and set
\[
\pi = D_{\rho | l/\pi} \circ \cdots \circ D_{\rho | l/\pi} (\pi(\mathcal{E}')), \quad \pi_t = D_{\rho | l/\pi} \circ \cdots \circ D_{\rho | l/\pi} (\pi(\mathcal{E}')).
\]
By induction on \( \pi_t \), we claim that if \( \pi_t = L(\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_r}[x_r, -y_r]; \pi(\phi, \epsilon)) \), then \( \pi = L(\Delta_{\rho_1}[x_1 - t, -(y_1 - t)], \ldots, \Delta_{\rho_r}[x_r - t, -(y_r - t)]; \pi(\phi_t, \epsilon)) \). Namely, we compare \( D_{\rho | l/\pi} (\pi) \) and \( D_{\rho | l/\pi} (\pi_t) \).

When \( x \leq 0 \), by the condition (\(*\)) on \( B_m + l_m \), we may assume that \( \rho | l/\pi \equiv \rho | l/\pi \) so that \( D_{\rho | l/\pi} (\pi) \) and \( D_{\rho | l/\pi} (\pi_t) \) are both nonzero. However, by Proposition 2.6, they are both irreducible (up to multiplicities). Moreover, they are given by replacing \( \Delta_{\rho_1}[x_1, -y_1] \) and \( \Delta_{\rho_1}[x_1 - t, -(y_1 - t)] \) with \( \Delta_{\rho_1}[x_1 - t, -(y_1 - t)] \) and \( \Delta_{\rho_1}[x_1 - t - 1, -(y_1 - t)] \), respectively. It shows our claim.

Hence we may assume that \( x > 0 \). In this case, to compute \( D_{\rho | l/\pi} (\pi_t) \), according to [2 Theorem 7.1], we consider
\[
A_{\rho | l/\pi} = \{ i \in \{1, \ldots, r \} \mid \rho_i \equiv \rho, x_i = x + t \}.
\]
and relevant matching functions \( f : A^0_{\rho | l/\pi + t - 1} \to A^0_{\rho | l/\pi + t} \) and \( g : B^0_{\rho | l/\pi + t - 1} \to B^0_{\rho | l/\pi + t} \). See [2 §6.1] for the definition of matching functions. When we compute \( D_{\rho | l/\pi} (\pi) \), we need to consider similar (totally ordered) sets \( A_{\rho | l/\pi}, A_{\rho | l/\pi} \), \( B_{\rho | l/\pi}, B_{\rho | l/\pi} \) and matching functions. However, since \( \Delta_{\rho}[-x - 1, -x] = \Delta_{\rho}[-x, -(x + 1)] = 1_{GL_0(F)} \), we have to remove them in the definitions of \( B_{\rho | l/\pi} \) and \( B_{\rho | l/\pi} \). It follows from the definition of \( \pi(\mathcal{E}_t) \) that the multiplicity of \( \Delta_{\rho}[-x +
$t - 1, -(x + t)$ in the multi-set \(\{\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_r}[x_r, -y_r]\}\) is greater than or equal to that of \(\Delta_{\rho}[x - t, -(x + t - 1)]\). If \(\Delta_{\rho}[-x + t - 1, -(x + t)]\) is in \(\{\Delta_{\rho_1}[x_1, -y_1], \ldots, \Delta_{\rho_r}[x_r, -y_r]\}\), the corresponding index is the maximal element in \(B_{\rho_1}^{(t)}\). Moreover, by the definition of the best matching function (see [2] §6.1, 7.1), it is the image of an index \(B_{\rho_1}^{(t)}\) corresponding to \(\Delta_{\rho}[-x + t, -(x + t - 1)]\) via \(f\). Therefore, the complement \(B_{\rho_1}^{(t)} \setminus B_{\rho_1}^{(0)}\) is equal to \(B_{\rho_1}^{(t - 1)} \setminus B_{\rho_1}^{(0)}\). Finally, if \(x = 1/2\), then by the condition (\(\ast\)), the multiplicity of \(\rho \otimes S_{2(x + t) - 1}\) in \(\phi\) is at most one. Therefore, by [2] Theorem 7.1, the computations of \(D_{\rho_1}^{(t)}(\pi)\) and \(D_{\rho_1}^{(t - 1)}(\pi_t)\) are exactly the same (up to the shift by \(t\)).

Finally, by the proof of the claim, we see that \((x_i - t) + (y_i - t) + 1 \geq 0\) and \(a_j - 2t \geq 0\). This completes the proof of Theorem 3.6

In §4 and §5 below, we consider \(\pi(\mathcal{E})\) for \(\mathcal{E} = \cup_{\rho}\{([A_i, B_i], l_i, \eta_i)\}_{i \in (I_{\rho}, >)}\) with \(B_i \geq 0\) for all \(\rho\) and \(i \in I_{\rho}\).

### 3.5. Examples of A-packets

In this subsection, we set \(\rho = 1_{\text{GL}_3(F)}\) and we drop \(\rho\) from the notation. When \(\phi = \rho \otimes (S_{2x_1 + 1} \oplus \cdots \oplus S_{2x_r + 1})\) with \(\epsilon(\rho \otimes S_{2x_i + 1}) = \epsilon_i\), we write \(\pi(\phi, \epsilon) = \pi(x_1^{\epsilon_1}, \ldots, x_r^{\epsilon_r})\).

**Example 3.7.** Let us compute the A-packet \(\Pi_{\psi}\) for

\[
\psi = 1 \otimes S_6 + 1 \otimes S_2 + S_4 \otimes 1 \in \Psi_{\text{gp}}(\text{SO}_{13}(F)).
\]

It is an elementary A-parameter (see [17] §6). By Theorem 3.6 together with the sign condition in Definition 3.4 (2), we see that \(\Pi_{\psi}\) has at most 4 irreducible representations and they are associated to

\[
\begin{align*}
\mathcal{E}_1 &= \begin{pmatrix}
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\end{pmatrix}, \\
\mathcal{E}_2 &= \begin{pmatrix}
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\end{pmatrix}, \\
\mathcal{E}_3 &= \begin{pmatrix}
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\end{pmatrix}, \\
\mathcal{E}_4 &= \begin{pmatrix}
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\theta & \theta & \theta & \theta & \theta & \theta \\
\end{pmatrix}.
\end{align*}
\]

We compute \(\pi(\mathcal{E}_i)\) for \(i = 1, 2, 3, 4\). Taking \((t_1, t_2, t_3) = (0, 4, 4)\), we consider \(\mathcal{E}_i\) as in §4. Then \(\pi(\mathcal{E}_i) \cong D_3 \circ D_2(\pi(\mathcal{E}_i))\) with

\[
\begin{align*}
D_2 &= D_{\epsilon, 1/2, 1/2} \circ D_{\epsilon, 1/2, 3/2} \circ D_{\epsilon, 1/2, 3/2} \circ D_{\epsilon, 1/2, 7/2} \circ D_{\epsilon, 1/2, 7/2} \circ D_{\epsilon, 1/2, 9/2}, \\
D_3 &= D_{\epsilon, 1/2} \circ D_{\epsilon, 7/2} \circ D_{\epsilon, 9/2} \circ D_{\epsilon, 11/2}.
\end{align*}
\]

(1) For \(i = 1\), by [2] Theorem 7.1, we have

\[
\pi(\mathcal{E}_1) \cong D_3 \circ D_2 \left( L(\{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot; \Delta(7/2, -9/2); \pi((11/2)^+)\}) \right)
\cong L(\{\cdot, \cdot, \cdot, \cdot, \cdot, \cdot; \pi((3/2)^+)\}).
\]
(2) For $i = 2$, by [4] Theorem 7.1, we have

$$\pi(\mathcal{E}_2) \cong D_3 \circ D_2 \left( L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}, \cdot | -\frac{1}{2}; \pi((7/2)^-, (9/2)^+, (11/2)^-)) \right)$$

$$\cong D_3 \circ D_1 \circ D_1 \left( L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}, \cdot | -\frac{1}{2}; \pi((1/2)^-, (3/2)^+, (11/2)^-)) \right)$$

$$\cong D_3 \left( L(\cdot | -\frac{5}{2}, \pi((1/2)^-, (3/2)^+, (11/2)^-)) \right)$$

$$\cong D_1 \left( L(\cdot | -\frac{5}{2}, \pi((1/2)^-, (3/2)^+, (5/2)^-)) \right).$$

In particular, $\pi(\mathcal{E}_2)$ is a supercuspidal representation by [16] Theorem 3.3.

(3) For $i = 3$, by [4] Theorem 7.1, we have

$$\pi(\mathcal{E}_3) \cong D_3 \circ D_2 \left( L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}, \Delta[7/2, -9/2]; \pi((1/2)^-, (11/2)^-)) \right)$$

$$\cong D_3 \circ D_1 \circ D_1 \left( L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}, \Delta[-1/2, -3/2]; \pi((1/2)^-, (11/2)^-)) \right)$$

$$\cong D_3 \left( L(\cdot | -\frac{5}{2}, \Delta[-1/2, -3/2]; \pi((1/2)^-, (11/2)^-)) \right)$$

$$\cong L(\cdot | -\frac{5}{2}, \Delta[-1/2, -3/2]; \pi((1/2)^-, (3/2)^-)).$$

(4) For $i = 4$, by [4] Theorem 7.1, we have

$$\pi(\mathcal{E}_4) \cong D_3 \circ D_2 \left( L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}; \pi((1/2)^-, (7/2)^-, (9/2)^+, (11/2)^+)) \right)$$

$$\cong D_3 \circ D_1 \circ D_1 \left( L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}; \pi((1/2)^-, (1/2)^-, (3/2)^+, (11/2)^+)) \right)$$

$$\cong D_3 \left( L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}; \pi((3/2)^+, (11/2)^+)) \right)$$

$$\cong D_1 \left( L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}; \pi((1/2)^+, (3/2)^+)) \right)$$

$$\cong L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}, \cdot | -\frac{1}{2}; \pi((1/2)^+, (3/2)^+)).$$

Therefore, $\Pi_\psi$ consists of 4 irreducible representations

$$\pi(\mathcal{E}_1) = L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}, \cdot | -\frac{1}{2}, \cdot | -\frac{1}{2}; \pi((3/2)^+)),$$

$$\pi(\mathcal{E}_2) = \pi((1/2)^-, (3/2)^+, (5/2)^-),$$

$$\pi(\mathcal{E}_3) = L(\cdot | -\frac{5}{2}, \Delta[-1/2, -3/2]; \pi((1/2)^-, (3/2)^-)),$$

$$\pi(\mathcal{E}_4) = L(\cdot | -\frac{5}{2}, \cdot | -\frac{3}{2}, \cdot | -\frac{1}{2}; \pi((1/2)^+, (3/2)^+)).$$

One can also compute $\Pi_\psi$ by using Mœglin’s original construction (see [17] §6), but it would be much harder.

**Example 3.8.** Let us consider $\psi = S_4 \boxtimes S_6 + S_3 \boxtimes S_3 \in \Psi_{sp}(\text{Sp}_{32}(F))$. By Theorems [3.4] and [4.4] below, we can see that $\#\Pi_\psi = 7$. The extended multi-segments $\mathcal{E}$ with $\pi(\mathcal{E}) \neq 0$ and the
characters ηE are listed as follows. Here, η ∈ $\hat{S}_\psi$ is identified with $(\eta(S_4 \boxtimes S_6), \eta(S_3 \boxtimes S_3)) \in \{\pm 1\}^2$.

\[ \eta_1 = (-, -), \eta_2 = (-, -), \eta_3 = (+, +), \eta_4 = (+, +), \eta_5 = (+, +), \eta_6 = (-, -), \eta_7 = (-, -). \]

The associated representations are listed as follows.

\[ \pi(\mathcal{E}_1) \cong L(\Delta[-1, -4], \Delta[0, -3], \Delta[0, -2], \pi(1^-, 1^-, 2^+)), \]
\[ \pi(\mathcal{E}_2) \cong L(\Delta[-1, -2], \Delta[0, -1], \pi(0^+, 1^+, 2^-, 3^+, 4^-)), \]
\[ \pi(\mathcal{E}_3) \cong L(\Delta[-1, -4], \Delta[0, -3], \Delta[0, -2], \Delta[1, -2], \pi(1^+)), \]
\[ \pi(\mathcal{E}_4) \cong L(\Delta[-1, -4], \Delta[0, -2], \pi(0^-, 1^+, 1^-, 2^-, 3^+)), \]
\[ \pi(\mathcal{E}_5) \cong L(\Delta[-1, -2], \Delta[0, -4], \Delta[0, -1], \pi(1^-, 2^+, 3^-)), \]
\[ \pi(\mathcal{E}_6) \cong L(\Delta[-1, -4], \Delta[0, -3], \Delta[1, -2], \pi(0^+, 1^-, 2^-)), \]
\[ \pi(\mathcal{E}_7) \cong L(\Delta[-1, -2], \Delta[0, -4], \Delta[1, -3], \pi(0^-, 1^+, 2^-)). \]

4. A NON-VANISHING CRITERION

We fix an extended multi-segment $\mathcal{E} = \cup_{\rho} \{(A_i, B_i, l_i, \eta_i)\}_{i \in (I_\rho, >)}$ for $G_n$ such that $B_i \geq 0$ for all $\rho$ and $i \in I_\rho$. In this section, we discuss about the non-vanishing of $\pi(\mathcal{E})$. 
4.1. Necessary conditions. In [18], Xu established an algorithm to determine whether \( \pi(E) \neq 0 \). To do this, he gave three necessary conditions for \( \pi(E) \neq 0 \). Recall that \( I_\rho \) is a totally ordered set with a fixed admissible order \( > \). For \( [A_i, B_i]_\rho \), we set \( b_i = \# [A_i, B_i] = A_i - B_i + 1 \).

**Proposition 4.1** ([18] Lemmas 5.5, 5.6, 5.7). Let \( E = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i) \}_{i \in (I_\rho, \rho)} \) be an extended multi-segment for \( G_n \) such that \( B_i \geq 0 \) for all \( \rho \) and \( i \in I_\rho \). Let \( k > k - 1 \) be two adjacent elements in \( I_\rho \). Suppose that \( \pi(E) \neq 0 \).

1. If \( A_k \geq A_{k-1} \) and \( B_k \geq B_{k-1} \), then
   \[
   \begin{cases}
   \eta_k = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \iff A_k - l_k \geq A_{k-1} - l_{k-1}, & B_k + l_k \geq B_{k-1} + l_{k-1}, \\
   \eta_k \neq (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \iff B_k + l_k > A_{k-1} - l_{k-1}.
   \end{cases}
   \]
   In particular, if \( [A_k, B_k]_\rho = [A_{k-1}, B_{k-1}]_\rho \), then \( \eta_k = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \) and \( l_k = l_{k-1} \).

2. If \( [A_{k-1}, B_{k-1}]_\rho \subset [A_k, B_k]_\rho \), then
   \[
   \begin{cases}
   \eta_k = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \iff 0 \leq l_k - l_{k-1} \leq b_k - b_{k-1}, \\
   \eta_k \neq (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \iff l_k + l_{k-1} \geq b_{k-1}.
   \end{cases}
   \]

3. If \( [A_{k-1}, B_{k-1}]_\rho \supset [A_k, B_k]_\rho \), then
   \[
   \begin{cases}
   \eta_k = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \iff 0 \leq l_{k-1} - l_k \leq b_{k-1} - b_k, \\
   \eta_k \neq (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \iff l_k + l_{k-1} \geq b_k.
   \end{cases}
   \]

In a particular case, the condition in Proposition 4.1 (1) is sufficient.

**Proposition 4.2** ([19] Theorem A.3]). Let \( E = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i) \}_{i \in (I_\rho, \rho)} \) be an extended multi-segment for \( G_n \) such that \( B_i \geq 0 \) for all \( \rho \) and \( i \in I_\rho \). Suppose that for any \( \rho \) and any two adjacent elements \( k > k - 1 \) of \( I_\rho \), we have \( A_k \geq A_{k-1} \) and \( B_k \geq B_{k-1} \). Then \( \pi(E) \neq 0 \) if and only if the condition in Proposition 4.1 (1) holds for all adjacent elements \( k > k - 1 \).

4.2. Changing of admissible orders. Now suppose that \( E = \cup_\rho \{([A_i, B_i]_\rho, l_i, \eta_i) \}_{i \in (I_\rho, \rho)} \) satisfies the three necessary conditions in Proposition 4.1. In general, there are many choices of admissible orders on \( I_\rho \), and there is no canonical choice. We recall the behavior of \( (l_i, \eta_i)_{i \in I_\rho} \) under changing admissible orders.

For a positive integer \( b \), let \( (\mathbb{Z}/b\mathbb{Z})/\{\pm 1\} \) be the quotient of \( \mathbb{Z}/b\mathbb{Z} \) by the multiplication by \( \pm 1 \). Then we can identify \( \{l \in \mathbb{Z} \mid 0 \leq l \leq \frac{b}{2}\} \) with \( (\mathbb{Z}/b\mathbb{Z})/\{\pm 1\} \). In particular, we regard \( l_i \) as an element in \( (\mathbb{Z}/b\mathbb{Z})/\{\pm 1\} \), where \( b_i = \# [A_i, B_i] = A_i - B_i + 1 \).

Let \( >' \) be the order on \( I_\rho \) which is given from \( > \) by changing \( k - 1 >' k \). Suppose that \( >' \) is also admissible, which is equivalent that \( [A_{k-1}, B_{k-1}]_\rho \subset [A_k, B_k]_\rho \) or \( [A_{k-1}, B_{k-1}]_\rho \supset [A_k, B_k]_\rho \).

When \( [A_{k-1}, B_{k-1}]_\rho \subset [A_k, B_k]_\rho \), we define \( l'_i \) and \( \eta'_i \) for \( i \in I_\rho \) as follows:

- If \( i \neq k, k - 1 \), then \( l'_i = l_i \) and \( \eta'_i = \eta_i \);
- \( l'_{k-1} = l_{k-1} \) and \( \eta'_{k-1} = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \);
- \( l'_k = l_k + \epsilon(b_{k-1} - 2l_{k-1}) \) in \( (\mathbb{Z}/b\mathbb{Z})/\{\pm 1\} \), where \( \epsilon = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \eta_k \in \{\pm 1\} \);
- if \( \eta_k = (-1)^{A_{k-1}-B_{k-1}} \eta_{k-1} \) and \( b_k - 2l_k < 2(b_{k-1} - 2l_{k-1}) \), then \( \eta'_k = (-1)^{A_{k-1}-B_{k-1}} \eta_k \); and otherwise \( \eta'_k = (-1)^{A_{k-1}-B_{k-1}} \eta_k \).

Similarly, when \( [A_{k-1}, B_{k-1}]_\rho \supset [A_k, B_k]_\rho \), we define \( l'_i \) and \( \eta'_i \) for \( i \in I_\rho \) as follows:
• If \( i \neq k, k-1 \), then \( t'_i = t_i \) and \( \eta'_i = \eta_i \);
• \( t'_k = t_k \) and \( \eta'_k = (-1)^{A_{k-1} - B_{k-1}} \eta_k \);
• \( t'_{k-1} = t_{k-1} + \epsilon(b_k - 2l_k) \) in \( \mathbb{Z}/b_k \mathbb{Z} \) \( \{\pm 1\} \), where \( \epsilon = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} \eta_k \in \{\pm 1\} \);
• if \( \eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} \) and \( b_{k-1} - 2l_{k-1} < 2(b_k - 2l_k) \), then \( \eta'_{k-1} = (-1)^{A_k - B_k} \eta_{k-1} \); and otherwise \( \eta'_{k-1} = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} \).

Set \( \mathcal{E}' = \bigcup_{\rho} \{(A_i, B_i)_{\rho}, t'_i, \eta'_i\} \in \mathcal{T}_{(\rho, l^\prime)} \). One can check that the necessary conditions in Proposition 4.1 hold for \( \mathcal{E}' \) with respect to \( k - 1 >^l k \).

**Theorem 4.3 (\cite{Theorem 6.1}).** Suppose that \( [A_{k-1}, B_{k-1}]_{\rho} \subset [A_k, B_k]_{\rho} \) or \( [A_{k-1}, B_{k-1}]_{\rho} \supset [A_k, B_k]_{\rho} \). With the above notation, we have \( \pi(A) \equiv \pi(A') \).

We describe the relation \( \mathcal{E}_\rho \leftrightarrow \mathcal{E}'_\rho \) in the case where \( [A_{k-1}, B_{k-1}]_{\rho} \supset [A_k, B_k]_{\rho} \).

1. If \( \eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} \) and \( b_{k-1} - 2l_{k-1} \geq 2(b_k - 2l_k) \), with \( \alpha = b_k - 2l_k \), we have

\[
\begin{pmatrix}
B_{k-1} & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_{k-1} \\
\alpha & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_k \\
\end{pmatrix}_\rho
\leftrightarrow
\begin{pmatrix}
B_k & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_k \\
\alpha & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_{k-1} \\
\end{pmatrix}_\rho
\]

2a. If \( \eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} \) and \( b_{k-1} - 2l_{k-1} < 2(b_k - 2l_k) \), and if \( \alpha = (b_k - 2l_k - 1) - (b_k - 2l_k) \geq 0 \), then noting that \( b_{k-1} - 2l_{k-1} > 2\alpha \), we have

\[
\begin{pmatrix}
B_{k-1} & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_{k-1} \\
\alpha & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_k \\
\end{pmatrix}_\rho
\leftrightarrow
\begin{pmatrix}
B_k & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_k \\
\alpha & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_{k-1} \\
\end{pmatrix}_\rho
\]

2b. If \( \eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} \) and \( b_{k-1} - 2l_{k-1} < 2(b_k - 2l_k) \), and if \( \alpha = (b_k - 2l_k - 1) - (b_k - 2l_k - 1) \geq 0 \), then noting that \( \alpha \leq l_{k-1} \) by Proposition 4.1, we have

\[
\begin{pmatrix}
B_{k-1} & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_{k-1} \\
\alpha & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_k \\
\end{pmatrix}_\rho
\leftrightarrow
\begin{pmatrix}
B_k & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_k \\
\alpha & \cdots & \alpha & \cdot \cdot \cdot & \alpha & \cdot \cdot \cdot & A_{k-1} \\
\end{pmatrix}_\rho
\]
(3) If $\eta_k \neq (-1)^{A_{k-1}-B_{k-1}}\eta_{k-1}$, then noting that $\alpha = b_k - 2l_k \leq l_{k-1}$ by Proposition 4.1, we have

$$
\begin{pmatrix}
B_{k-1} & \cdots & \alpha & \cdots & \alpha & \cdots & A_{k-1} \\
\odot & \cdots & \odot & \cdots & \odot & \cdots & \odot \\
B_k & \cdots & \alpha & \cdots & \alpha & \cdots & \alpha
\end{pmatrix}_\rho
$$

\leftrightarrow

$$
\begin{pmatrix}
B_{k-1} & \cdots & \alpha & \cdots & \alpha & \cdots & A_{k} \\
\odot & \cdots & \odot & \cdots & \odot & \cdots & \odot \\
B_k & \cdots & \alpha & \cdots & \alpha & \cdots & A_{k}
\end{pmatrix}_\rho.
$$

4.3. Non-vanishing criterion. Let $\mathcal{E} = \cup_\rho \{([A_i, B_i]_{\rho, l_i}, \eta_i)\}_{i \in (I_\rho, >)}$ be an extended multi-segment for $G_n$ such that $B_i \geq 0$ for all $\rho$ and $i \in I_\rho$. Define $I_\rho^{2, \text{adj}}$ to be the set of triples $(i, j, >)$, where $>$ is an admissible order on $I_\rho$, and $i > j$ are two adjacent elements in $I_\rho$ with respect to $>$. Now we can reformulate Xu’s algorithm [18, §8] for $\pi(\mathcal{E}) \neq 0$ as follows.

**Theorem 4.4.** Let $\mathcal{E} = \cup_\rho \{([A_i, B_i]_{\rho, l_i}, \eta_i)\}_{i \in (I_\rho, >)}$ be an extended multi-segment for $G_n$ such that $B_i \geq 0$ for all $\rho$ and $i \in I_\rho$. Then the representation $\pi(\mathcal{E})$ is nonzero if and only if for every $(i, j, >)$ $\in I_\rho^{2, \text{adj}}$, the three necessary conditions in Proposition 4.1 are satisfied for $\mathcal{E}$ with respect to $i > j$, where $\mathcal{E}' = \cup_\rho \{([A_i, B_i]_{\rho, l_i}, \eta_i)\}_{i \in (I_\rho, >)}$ is such that $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$.

**Proof.** The only if part is Proposition 4.1. We will prove the if part by induction on $\sum_\rho (\# I_\rho - 1)$. Suppose that the three necessary conditions in Proposition 4.1 are satisfied with respect to any $(i, j, >) \in I_\rho^{2, \text{adj}}$. The case where $|I_\rho| \leq 2$ for any $\rho$ follows from Propositions 4.1 and 4.2 using “Pull” ([18, Proposition 7.1]) if necessary. Hence we may assume that $|I_\rho| \geq 3$ for a fixed $\rho$. To prove $\pi(\mathcal{E}) \neq 0$, we apply Xu’s algorithm [18, §8].

First, we assume that an element $m \in I_\rho$ is maximal for every admissible order $>$. We may assume that the original admissible order $>$ satisfies that for $i, j \in I_\rho \setminus \{m\}$,

$$
i > j \iff B_i > B_j, \text{ or } B_i = B_j \text{ and } A_i \geq A_j.$$

Write $I_\rho = \{1, \ldots, m\}$ such that $1 < \cdots < m$. In this situation, we can use “Expand” ([18, Proposition 7.4]). Let $t = B_m - B_{m-1}$. (Note that we assume that $m \geq 3$.) Define $\mathcal{E}'$ from $\mathcal{E}$ by replacing $([A_m, B_m]_{\rho, l_m}, \eta_m)$ with $([A_m + t, B_m - t, l_m + t], \eta_m)$. Then [18, Proposition 7.4] says that

$$
\pi(\mathcal{E}) \neq 0 \iff \pi(\mathcal{E}') \neq 0.
$$

We claim that $\mathcal{E}'$ satisfies the three necessary conditions in Proposition 4.1 with respect to any $(i, j, >) \in I_\rho^{2, \text{adj}}$. It is non-trivial only when $i = m$ or $j = m$. We may assume that $i = m$ and hence $j \neq m$. In this case, by changing $>$ suitably, we may assume that

- $B_k = B_{m-1}$ for any $j < k < m$;
- $B_{k-1} \leq B_k$ and $A_{k-1} \leq A_k$ for any $j < k < m$, where $k > k - 1$ are adjacent elements with respect to $>$. 

Consider another extended multi-segment

$$
\mathcal{E}' = \{([A_i, B_i]_{\rho, l_i}, \eta_i)\}_{i \in I_\rho, i > j} \cup \mathcal{E}_0
$$

for $G_{n'}$, where $\mathcal{E}_0$ is an “easy” auxiliary data. Then we can apply Proposition 4.2 and hence the conditions for $(m, j, >)$ follow from Proposition 4.1.
Therefore, we may replace $\pi(\mathcal{E})$ with $\pi(\mathcal{E}^*)$. In other words, we may assume that $I_\rho = \{1, \ldots, m\}$ with $1 < \cdots < m$ such that the order $>'$ on $I_\rho$ given by

$$1 <' \cdots <' m - 2 <' m <' m - 1$$

is also admissible. Moreover, we may assume that $A_m = \max\{A_i \mid i \in I_\rho\}$ and

$$A_{m-1} = \max\{A_i \mid i \in I_\rho \setminus \{m\}, [A_i, B_i]_\rho \subset [A_m, B_m]_\rho\}.$$  

In this situation, we can use “Pull” ([8] Propositions 7.1, 7.3). For $t \gg 0$, define

1. $\mathcal{E}^2$ from $\mathcal{E}$ by replacing $([A_m, B_m]_\rho, l_m, \eta_m)$ (resp. $([A_{m-1}, B_{m-1}]_\rho, l_{m-1}, \eta_{m-1})$) with $([A_m + t, B_m + t]_\rho, l_m, \eta_m)$ (resp. $([A_{m-1} + t, B_{m-1} + t]_\rho, l_{m-1}, \eta_{m-1})$);
2. $\mathcal{E}^3$ from $\mathcal{E}$ by replacing $([A_m, B_m]_\rho, l_m, \eta_m)$ with $([A_m + t, B_m + t]_\rho, l_m, \eta_m)$;
3. $\mathcal{E}^2$ from $\mathcal{E}'$ by replacing $([A_{m-1}, B_{m-1}]_\rho, l'_{m-1}, \eta'_{m-1})$ with $([A_{m-1} + t, B_{m-1} + t]_\rho, l'_{m-1}, \eta'_{m-1})$,

where $\mathcal{E}' = \bigcup_{\rho'}\{([A_i, B_i]_\rho, l'_i, \eta'_i)\}_{i \in (I_\rho >')}$ is such that $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$.

Then [8] Propositions 7.1, 7.3] say that $\pi(\mathcal{E}) \neq 0$ if and only if the three representations

$$\pi(\mathcal{E}^2), \quad \pi(\mathcal{E}^3), \quad \pi(\mathcal{E}^2)$$

are all nonzero. However, when $t \gg 0$, the calculation of the definition of $\pi(\mathcal{E}^2)$ is the same if we replace $\{([A_{m-1} + t, B_{m-1} + t]_\rho, l_{m-1}, \eta_{m-1})\}, ([A_m + t, B_m + t]_\rho, l_m, \eta_m)$ with $\{([A_{m-1} + t, B_{m-1} + t]_\rho, l_{m-1}, \eta_{m-1})\}, ([A_m + t, B_m + t]_\rho, l_m, \eta_m)$ for some $\rho'$, i.e., we can replace $I_\rho$ with the union of $I_\rho = \{1, \ldots, m - 2\}$ and $I_{\rho'} = \{m - 1, m\}$. Hence by the induction hypothesis, we have $\pi(\mathcal{E}^2) \neq 0$. Similarly, using the partition $I_\rho = \{1, \ldots, m - 2, m - 1\} \sqcup \{m - 1\}$ (resp. $I_\rho = \{1, \ldots, m - 2, m - 1\} \sqcup \{m - 1\}$), by the induction hypothesis, we have $\pi(\mathcal{E}^3) \neq 0$ (resp. $\pi(\mathcal{E}^2) \neq 0$). Therefore, we conclude that $\pi(\mathcal{E}) \neq 0$, as desired.  

4.4. Xu’s example. As in [8] Example B.1], let us determine the cardinality $\#\Pi_\psi$ for

$$\psi = \rho \boxtimes (S_{51} \boxtimes S_{31} \oplus S_{31} \boxtimes S_{45} \oplus S_{13} \boxtimes S_{5}).$$

Set $(a_i, b_i) = (13, 5)$, $(a_j, b_j) = (31, 45)$ and $(a_k, b_k) = (51, 31)$ so that $[A_i, B_i]_\rho = [8, 4]_\rho$, $[A_j, B_j]_\rho = [37, -7]_\rho$ and $[A_k, B_k]_\rho = [40, 10]_\rho$. There are exactly two admissible orders $j < i < k$ and $i < j' < k'$, and the first one satisfies the condition $(P')$. Let

$$\mathcal{E} = \{([A_i, B_i]_\rho, l_i, \eta_i), ([A_j, B_j]_\rho, l_j, \eta_j), ([A_k, B_k]_\rho, l_k, \eta_k)\},$$

$$\mathcal{E}' = \{([A_i, B_i]_\rho, l'_i, \eta'_i), ([A_j, B_j]_\rho, l'_j, \eta'_j), ([A_k, B_k]_\rho, l'_k, \eta'_k)\},$$

where the admissible order $>$ (resp. $>'$) is used for $\mathcal{E}$ (resp. $\mathcal{E}'$), be such that $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$.

Note that

$$\begin{pmatrix}
\begin{array}{cccccccccc}
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\end{array}
\end{pmatrix}_{\rho}$$

$$\quad = \begin{pmatrix}
\begin{array}{cccccccccc}
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots & \circ & \cdots \\
\end{array}
\end{pmatrix}_{\rho}$$

By Theorems [8] and [4], $\pi(\mathcal{E})$ is nonzero if and only if

- $l_j \geq 7$;
• the condition in Proposition 4.1 (3) holds for \( i > j \);
• the condition in Proposition 4.1 (1) holds for \( k > i \);
• the condition in Proposition 4.1 (1) holds for \( k > j \).

However, when \( l_j \geq 7 > \# [A_i, B_i] = 5 \), the condition in Proposition 4.1 (3) holds for \( i > j \). Similarly, since \( B_k > A_k \), the condition in Proposition 4.1 (1) for \( k > i \) is trivial.

Now we determine when the condition in Proposition 4.1 (1) holds for \( k > j \) (under assuming \( l_j \geq 7 \)). Note that \( (\eta_i, \eta_j, \eta_k) \) is determined by the equations between \( \eta_1, \eta_2, \eta_3 \) since the sign condition \( \eta_i \eta_j \eta_k = (-1)^{l_i+ l_j + l_k} \) is required.

To exchange the first and second lines, we separate several cases.

1. Suppose that \( \eta_i = \eta_j \) and \( 45 - 2l_i \geq 2(5 - 2l_i) \), i.e., \( 7 \leq l_j \leq 17 + 2l_i \). By Theorem 4.3 we have \((l'_j, \eta'_j) = (l_j + 5 - 2l_i, -\eta_j)\), i.e.,

\[
E' = \begin{pmatrix}
0 & 1 & \ldots & l_i - 2 - 2l_i & \ldots & l_j - 5 - 2l_i & \ldots & l_k - 23 + 2l_i \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

where the last \( \odot \) in the second line is \( -\eta_j \). Therefore, in this case, the condition in Proposition 4.1 (1) says that

• if \( \eta_k \neq \eta_j \), then \( l_j - 2 - 2l_i \leq 10 + l_k \) and \( 32 - l_j + 2l_i \leq 40 - l_k \), i.e., \( \max\{0, l_j - 2l_i - 12\} \leq l_k \leq \min\{15, 8 + l_j - 2l_i\} \);

• if \( \eta_k = \eta_j \), then \( 33 - l_j + 2l_i \leq 10 + l_k \), i.e., \( \max\{0, 23 - l_j + 2l_i\} \leq l_k \leq 15 \).

In particular, for fixed \( \eta_i = \eta_j \) and \((l_i, l_j)\) with \( 7 \leq l_j \leq 17 + 2l_i \),

\[
\#\{(l_k, \eta_k) \mid \pi(E') \neq 0\} = \begin{cases}
9 + l_j - 2l_i & \text{if } 7 \leq l_j \leq 12 + 2l_i \\
21 & \text{if } 12 + 2l_i \leq l_j \leq 17 + 2l_i.
\end{cases}
\]

2. Suppose that \( \eta_i = \eta_j \) and \( 45 - 2l_j < 2(5 - 2l_i) \), i.e., \( 17 + 2l_1 < l_j \leq 15 \). By Theorem 4.3 we have \((l'_2, \eta'_2) = (40 + 2l_1 - l_2, \eta_2)\), i.e.,

\[
E' = \begin{pmatrix}
0 & 1 & \ldots & l_1 - 2 - 2l_2 & \ldots & l_j - 5 - 2l_2 & \ldots & l_k - 23 + 2l_2 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

where the last \( \odot \) in the second line is \( \eta_2 \). Therefore, in this case, the condition in Proposition 4.1 (1) says that

• if \( \eta_k = \eta_j \), then \( 33 + 2l_1 - l_j \leq 10 + l_k \) and \( -3 - 2l_1 + l_j \leq 40 - l_k \), i.e., \( \max\{0, 23 + 2l_1 - l_j\} \leq l_k \leq \min\{15, 43 + 2l_1 - l_j\} \);

• if \( \eta_k \neq \eta_j \), then \( -2 - 2l_1 + l_j \leq 10 + l_k \), i.e., \( \max\{0, -12 - 2l_1 + l_j\} \leq l_k \leq 15 \).
Since we assume that $17 < l_j - 2l_i \leq 15$, it is equivalent that

- if $\eta_k = \eta_j$, then $23 + 2l_i - l_j \leq l_k \leq 15$;
- if $\eta_k \neq \eta_j$, then $-12 - 2l_i + l_j \leq l_k \leq 15$.

In particular, for fixed $\eta_i = \eta_j$ and $(l_i, l_j)$ with $17 + 2l_i < l_j \leq 22$, we have

$$\#\{ (l_k, \eta_k) \mid \pi(\mathcal{E}) \neq 0 \} = 21.$$  

(3) Suppose that $\eta_i \neq \eta_j$. By Theorem 4.3, we have $(l'_i, \eta'_i) = (l_j + 2l_i - 5, \eta_i)$, i.e.,

$$\mathcal{E'} = \left( \begin{array}{cccc}
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5.1. **Deformation.** The key idea to describe derivatives of \(\pi(\mathcal{E})\) is to study the condition in Proposition [4.1] (1) more deeply. Now we assume that

\[
\mathcal{E} = \{([A_{k-1}, B_{k-1}], \rho_{k-1}, \eta_{k-1}), ([A_k, B_k], \rho_k, \eta_k)\}
\]

consists of two extended segments with order \(k > k - 1\). Recall that \(b_{k-1} = \#(A_{k-1}, B_{k-1}) = A_{k-1} - B_{k-1} + 1\). Suppose that \(A_k > A_{k-1}, B_k > B_{k-1}\) and one of the following:

1. \(\eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} - 1\) and \(A_k - l_k = A_{k-1} - l_{k-1}\);
2. \(\eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} - 1\) and \(B_k + l_k = B_{k-1} + l_{k-1}\);
3. \(\eta_k \neq (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} - 1\) and \(B_k + l_k = A_{k-1} - l_{k-1} + 1\).

Note that (1) implies \(l_k > l_{k-1}\), whereas (2) implies \(l_k < l_{k-1}\). In particular, (1) and (2) cannot occur simultaneously. We will define \(\mathcal{E}' = \{([A'_{k-1}, B'_{k-1}], \rho_{k-1}, \eta'_{k-1}), ([A'_k, B'_k], \rho_k, \eta'_k)\}\) so that \([A'_{k-1}, B'_{k-1}], \rho = [A_{k-1}, B_{k-1}], \rho \cup [A_k, B_k], \rho\) and \([A'_k, B'_k], \rho = [A_{k-1}, B_{k-1}], \rho \cap [A_k, B_k], \rho\) and so that \(\pi(\mathcal{E}) \cong \pi(\mathcal{E}')\).

**Theorem 5.1.** Set \([A'_{k-1}, B'_{k-1}], \rho = [A_{k-1}, B_{k-1}], \rho\) and \([A'_k, B'_k], \rho = [A_{k-1}, B_k], \rho\). Define \(l'_1\) and \(\eta'_i\) for \(i \in \{k-1, k\}\) as follows.

1. When \(\eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} - 1\) and \(A_k - l_k = A_{k-1} - l_{k-1}\), set

\[
(l'_{k-1}, l'_k, \eta'_{k-1}, \eta_k) = (l_{k-1}, l_k - (A_k - A_{k-1}), \eta_{k-1}, (-1)^{A_{k-1} - A_{k-1}} \eta_{k-1}).
\]

Note that \(l_k = (A_k - A_{k-1}) = l_{k-1}\).

2. When \(\eta_k = (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} - 1\) and \(B_k + l_k = B_{k-1} + l_{k-1}\), set

\[
(l'_{k-1}, l'_k, \eta'_{k-1}, \eta_k) = \left\{ \begin{array}{l}
(l_{k-1} + (A_k - A_{k-1}), l_k, \eta_{k-1}, (-1)^{A_{k-1} - A_{k-1}} \eta_{k-1}) \\
(b_{k-1} - l_{k-1}, l_k, \eta_{k-1}, (-1)^{A_{k-1} - A_{k-1}} \eta_{k-1})
\end{array} \right. \]

according to \(b_{k-1} - 2l_{k-1} \geq A_k - A_{k-1}\) or not. Note that \(b_{k-1} + (A_k - A_{k-1}) = A_k - B_{k-1} + 1\) so that if \(b_{k-1} - 2l_{k-1} = A_k - A_{k-1}\), then \(\mathcal{E}'\) does not depend on \(\eta'_k\).

3. When \(\eta_k \neq (-1)^{A_{k-1} - B_{k-1}} \eta_{k-1} - 1\) and \(B_k + l_k = A_{k-1} - l_{k-1} + 1\), set

\[
(l'_{k-1}, l'_k, \eta'_{k-1}, \eta_k) = \left\{ \begin{array}{l}
(l_{k-1}, l_k, \eta_{k-1}, (-1)^{A_{k-1} - A_{k-1}} \eta_{k-1}) \\
(l_{k-1}, l_k, \eta_{k-1}, (-1)^{A_{k-1} - A_{k-1}} \eta_{k-1})
\end{array} \right. \]

if \(l_k \leq l_{k-1}\),

\[
(l_{k-1}, l_k, \eta_{k-1}, (-1)^{A_{k-1} - A_{k-1}} \eta_{k-1})
\]

if \(l_k > l_{k-1}\).

Note that if \(l_k = l_{k-1}\), then \(\mathcal{E}'\) does not depend on \(\eta'_k\).

Then we have \(\pi(\mathcal{E}) \cong \pi(\mathcal{E}')\).

**Proof.** For simplicity, we write \(k = 2\). We will prove the assertion by using the symbol of \(\mathcal{E} = \mathcal{E}_\rho\). It might be a formal proof, but it is justified by the explicit formula for \(\pi(\mathcal{E})\) (Theorem 1.3).

In the case (1), \(\mathcal{E} = \mathcal{E}_\rho\) is of the form

\[
\begin{pmatrix}
B_1 & \cdots & \langle \otimes \rangle & \cdots & \cdots & A_1 - t_1 \\
\langle \cdots & \langle \otimes \rangle & \cdots & \cdots & \cdots & \langle A_1 \\
B_2 & \cdots & \langle \otimes \rangle & \cdots & \cdots & A_2 - t_2 \\
\end{pmatrix}_\rho
\]

such that the sign of the first \(\otimes\) in the first line (resp. the second line) is \(\eta_1\) (resp. \(\eta_2\)). Taking the socle

\[
S_\rho |A_2, \ldots, \rho|A_2 - t_1 + 1 \cdots \rho |A_1 - t_1 + 1, \ldots, \rho |A_1 - t_1 + 2,
\]
it becomes
\[
\begin{pmatrix}
B_1 & \cdots & \odot & \cdots & \cdots & A_1-l_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_2 & \cdots & \odot & \cdots & \cdots & A_2-l_2 \\
\end{pmatrix},
\]
whose representation is isomorphic to the one of
\[
\begin{pmatrix}
B_1 & \cdots & \odot & \cdots & \cdots & A_1-l_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_2 & \cdots & \odot & \cdots & \cdots & A_2-l_2 \\
\end{pmatrix},
\]
where the first $\odot$ in the second line is equal to the last one in the first line which is $(-1)^{A_2-B_1}\eta_1 = (-1)^{A_2-A_1}\eta_2$. By Taking the derivative
\[
D_{\rho|\cdot|A_1-l_2+1,\ldots,\rho|\cdot|A_1+1} \circ D_{\rho|\cdot|A_2-l_1+1,\ldots,\rho|\cdot|A_2},
\]
with $(l'_1, l'_2) = (l_1, l_1)$ and $(\eta'_1, \eta'_2) = (\eta_1, (-1)^{A_2-A_1}\eta_2)$, we see that $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$. This proves (1).

In the case (2), $\mathcal{E} = \mathcal{E}_\rho$ is of the form
\[
\begin{pmatrix}
B_1 & \cdots & B_1+l_1 & \cdots & B_2 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_2 & \cdots & B_2+l_2 & \cdots & B_2 & \cdots \\
\end{pmatrix},
\]
such that the sign of the first $\odot$ in the first line (resp. the second line) is $\eta_1$ (resp. $\eta_2$). We will compute the socle
\[
S_{\rho|\cdot|A_1+1,\ldots,\rho|\cdot|B_1+l_1+1}
\]
by separating the several cases when $b_1 - 2l_1 > 0$. We write $b_1 - 2l_1 = 2\alpha + \delta$ for $\alpha \in \mathbb{Z}$ and $\delta \in \{0,1\}$.

(a) When $\delta = 0$ and $\alpha > 0$, we write
\[
\begin{pmatrix}
B_1 & \cdots & \odot & \cdots & \cdots & A_1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_2 & \cdots & \odot & \cdots & \cdots & A_2 \\
\end{pmatrix},
\]
Using [2, Theorem 7.1], by taking the socle, it becomes
\[
\begin{pmatrix}
B_1 & \cdots & \odot & \cdots & \cdots & A_1+1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_2 & \cdots & \odot & \cdots & \cdots & A_2 \\
\end{pmatrix},
\]
whose representation is isomorphic to the one of
\[
\begin{pmatrix}
B_1 & \cdots & \odot & \cdots & \cdots & A_1+1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
B_2 & \cdots & \odot & \cdots & \cdots & A_2 \\
\end{pmatrix},
\]
(b) When \( \delta = 1 \) and \( \alpha > 0 \), we write
\[
\begin{pmatrix}
B_1 & \alpha & \alpha & \ldots & \alpha & A_1 & \triangleright \\
\triangleleft & \cdot & \cdot & \ldots & \cdot & \triangleright \\
B_2 & \alpha & \alpha & \ldots & \alpha & \triangleright \\
\end{pmatrix}_\rho.
\]

Using [2] Theorem 7.1, by taking the socle, it becomes
\[
\begin{pmatrix}
B_1 & \alpha & \alpha & \ldots & \alpha & A_{1+1} & \triangleright \\
\triangleleft & \cdot & \cdot & \ldots & \cdot & \triangleright \\
B_2 & \alpha & \alpha & \ldots & \alpha & \triangleright \\
\end{pmatrix}_\rho.
\]

Here, we used the following fact (or its sign change). The socle map \( S_{\rho|1^x} \) gives
\[
\pi\left( \begin{pmatrix} \ominus & \ominus & \oplus \end{pmatrix}_\rho \right) \mapsto \pi\left( \begin{pmatrix} \ominus & \ominus & \oplus \end{pmatrix}_\rho \right) \cong \pi\left( \begin{pmatrix} \triangleleft & \cdot & \cdot \end{pmatrix}_\rho \right).
\]

The representation associated to the last symbol is isomorphic to the one of
\[
\begin{pmatrix}
B_1 & \alpha & \alpha & \ldots & \alpha & A_{1+1} & \triangleright \\
\triangleleft & \cdot & \cdot & \ldots & \cdot & \triangleright \\
B_2 & \alpha & \alpha & \ldots & \alpha & \triangleright \\
\end{pmatrix}_\rho.
\]

(c) When \( \delta = 1 \) and \( \alpha = 0 \), we write
\[
\begin{pmatrix}
B_1 & \alpha & \alpha & \ldots & \alpha & A_1 & \triangleright \\
\triangleleft & \cdot & \cdot & \ldots & \cdot & \triangleright \\
B_2 & \alpha & \alpha & \ldots & \alpha & \triangleright \\
\end{pmatrix}_\rho.
\]

Using [2] Theorem 7.1, by taking the socle, it becomes
\[
\begin{pmatrix}
B_1 & \alpha & \alpha & \ldots & \alpha & A_{1+1} & \triangleright \\
\triangleleft & \cdot & \cdot & \ldots & \cdot & \triangleright \\
B_2 & \alpha & \alpha & \ldots & \alpha & \triangleright \\
\end{pmatrix}_\rho.
\]

Here, we used the same fact as in (b) above.

In conclusion, if \( b_1 - 2l_1 > 0 \), i.e., if \( \ominus \) appears in the first line, after taking the socle, the number of \( \ominus \) in the first line becomes \( b_1 - 2l_1 - 1 \), and \((l_1, l_2, \eta_1, \eta_2)\) becomes \((l_1 + 1, l_2, \eta_1, -\eta_2)\).

In particular, if \( b_1 - 2l_1 \geq (A_2 - A_1) \), then by repeating these operations \((A_2 - A_1)\) times, with \((l'_1, l'_2) = (l_1 + A_2 - A_1, l_2)\) and \((\eta'_1, \eta'_2) = (\eta_1, (-1)^{A_2-A_1} \eta_2)\), we have \( \pi(\xi) \cong \pi(\xi') \).

From now, we assume that \( b_1 - 2l_1 < (A_2 - A_1) \). Then after the \((b_1 - 2l_1)\)-th step, we have the following symbol
\[
\begin{pmatrix}
B_1 & \alpha & \alpha & \ldots & \alpha & A_{b_1-2l_1} & \triangleright \\
\triangleleft & \cdot & \cdot & \ldots & \cdot & \triangleright \\
B_2 & \alpha & \alpha & \ldots & \alpha & \triangleright \\
\end{pmatrix}_\rho.
\]
where the first $\odot$ in the second line is $(-1)^{b_1} \eta_2 = -\eta_1$. By taking the appropriate socle, it becomes
\[
\begin{pmatrix}
\begin{array}{ccc}
B_1 & \cdots & b_{1-2l_1} \\
\odot & \cdots & \odot \\
B_2 & \cdots & A_2
\end{array}
\end{pmatrix},
\]
whose representation is isomorphic to the one of
\[
\begin{pmatrix}
\begin{array}{ccc}
B_1 & \cdots & b_{1-2l_1} \\
\odot & \cdots & \odot \\
B_2 & \cdots & A_2
\end{array}
\end{pmatrix},
\]
where the first $\odot$ in the first line (resp. the second line) is $-\eta_1$ (resp. $(-1)^{A_2-A_1} \eta_2$). Therefore, with $(l'_1, l'_2) = (b_1 - l_1, l_2)$ and $(\eta'_1, \eta'_2) = (- \eta_1, (-1)^{A_2-A_1} \eta_2)$, we have $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$. This proves (2).

In the case (3), $\mathcal{E} = \mathcal{E}_1$ is of the form
\[
\begin{pmatrix}
\begin{array}{ccc}
B_1 & \cdots & A_{1-l_1+1} \\
\odot & \cdots & A_1 \\
B_2 & \cdots & A_2
\end{array}
\end{pmatrix},
\]
such that the sign of the first $\odot$ in the first line (resp. the second line) is $\eta_1$ (resp. $\eta_2$). We take the socle
\[
S_{\rho_{|A_2, \ldots, \rho_{|A_2-l_1+1} \odot \cdots \odot S_{\rho_{|A_1+2, \ldots, \rho_{|A_1-l_1+3} \odot S_{\rho_{|A_1+1, \ldots, \rho_{|A_1-l_1+2}}}}.
\]
If $l_1 \leq l_2$, it becomes
\[
\begin{pmatrix}
\begin{array}{ccc}
B_1 & \cdots & A_2 \\
\odot & \cdots & \odot \\
B_2 & \cdots & A_2
\end{array}
\end{pmatrix},
\]
whose representation is isomorphic to the one of
\[
\begin{pmatrix}
\begin{array}{ccc}
B_1 & \cdots & A_2 \\
\odot & \cdots & \odot \\
B_2 & \cdots & A_2
\end{array}
\end{pmatrix},
\]
Hence, with $(l'_1, l'_2) = (l_1, l_2)$ and $(\eta'_1, \eta'_2) = (\eta_1, (-1)^{A_2-A_1} \eta_2)$, we have $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$. If $l_1 < l_2$, it becomes
\[
\begin{pmatrix}
\begin{array}{ccc}
B_1 & \cdots & A_2 \\
\odot & \cdots & \odot \\
B_2 & \cdots & A_2
\end{array}
\end{pmatrix},
\]
whose representation is isomorphic to the one of
\[
\begin{pmatrix}
\begin{array}{ccc}
B_1 & \cdots & A_2 \\
\odot & \cdots & \odot \\
B_2 & \cdots & A_2
\end{array}
\end{pmatrix},
\]
where the first $\odot$ in the second line is $(-1)^{A_2-B_1}\eta_1 = -(-1)^{A_2-A_1}\eta_2$. Hence, with $(l'_1, l'_2) = (l_1, l_2)$ and $(\eta'_1, \eta'_2) = (\eta_1, -(−1)^{A_2-A_1}\eta_2)$, we have $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$. This proves (3). $\square$

The same also holds in general.

**Corollary 5.2.** Let $\mathcal{E} = \cup_{\rho} \{([A_i, B_i], \rho, l_i, \eta_i)\}_{i \in (I_{\rho}, >)}$ be an extended multi-segment for $G_n$ such that $B_i \geq 0$ for all $\rho$ and $i \in I_{\rho}$. Assume that two adjacent elements $k > k - 1$ of $I_{\rho}$ satisfy one of the conditions (1), (2) or (3). Define $\mathcal{E}'$ from $\mathcal{E}$ by replacing $([A_k, B_k], \rho, l_k, \eta_k)$ (resp. $([A_{k-1}, B_{k-1}], \rho, l_{k-1}, \eta_{k-1})$) with $([A'_k, B'_k], \rho, l'_k, \eta'_k)$ (resp. $([A'_{k-1}, B'_{k-1}], \rho, l'_{k-1}, \eta'_{k-1})$) as in Theorem 5.1. Then $\pi(\mathcal{E}) \cong \pi(\mathcal{E}')$.

**Proof.** We may assume that $B_i$ is big enough for any $i \in I_{\rho}$ with $i > k$. For positive integers $t_k > t_{k-1} \gg 0$, we define $\mathcal{E}_1$ from $\mathcal{E}$ by replacing $([A_k, B_k], \rho, l_k, \eta_k)$ (resp. $([A_{k-1}, B_{k-1}], \rho, l_{k-1}, \eta_{k-1})$) with $([A_k + t_k, B_k + t_k], \rho, l_k, \eta_k)$ (resp. $([A_{k-1} + t_{k-1}, B_{k-1} + t_{k-1}], \rho, l_{k-1}, \eta_{k-1})$). Define $\mathcal{E}'_1$ from $\mathcal{E}'$ similarly. Then we may also assume that

$$
\pi(\mathcal{E}) = o_{i \in \{k-1, k\}} \left( D_{|\rho|} |\beta_i| \cdots |\beta_{i+1} | \circ \cdots \circ D_{|\rho|} |\beta_{i+1} + 1 | \circ \cdots \circ D_{|\rho|} |\beta_0| \circ \pi(\mathcal{E}_1),
\right)
$$

$$
\pi(\mathcal{E}') = o_{i \in \{k-1, k\}} \left( D_{|\rho|} |\beta'_i| \cdots |\beta'_{i+1} | \circ \cdots \circ D_{|\rho|} |\beta'_{i+1} + 1 | \circ \cdots \circ D_{|\rho|} |\beta_0| \circ \pi(\mathcal{E}'_1) \right).
$$

Consider

$$
\pi_0 = D_{|\rho|} |\beta_k| \circ \cdots \circ D_{|\rho|} |\beta_k + t_k| \circ \pi(\mathcal{E}_1),
$$

$$
\pi'_0 = D_{|\rho|} |\beta'_k| \circ \cdots \circ D_{|\rho|} |\beta'_k + t_k| \circ \pi(\mathcal{E}'_1).
$$

If $\pi(\mathcal{E})$ (resp. $\pi(\mathcal{E}')$) is nonzero, then $\pi_0$ (resp. $\pi'_0$) is also nonzero (and irreducible) by Theorem 4.4 (at least when $t_{k-1} \gg 0$). Moreover, by Theorem 5.1 we see that $\pi_0 \cong \pi'_0$ (so that both of them are nonzero).

Now assume that $\pi(\mathcal{E})$ is nonzero. Then by [17] Proposition 8.5, we see that

$$
\pi(\mathcal{E}_1) \hookrightarrow \begin{pmatrix}
B_{k-1} + t_{k-1} & \cdots & A_{k-1} + t_{k-1} \\
\vdots & \ddots & \vdots \\
B_{k-1} + 1 & \cdots & A_{k-1} + 1
\end{pmatrix}_{\rho} \times \begin{pmatrix}
B_k & \cdots & A_k + t_k \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_{\rho} \times \pi(\mathcal{E}).
$$

Since

$$
\begin{pmatrix}
B_k & \cdots & A_k + t_k \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_{\rho} \hookrightarrow \begin{pmatrix}
B_k & \cdots & A_k + t_k \\
\vdots & \ddots & \vdots \\
B_k + t_{k-1} + 1 & \cdots & A_k + t_{k-1} + 1
\end{pmatrix}_{\rho} \times \begin{pmatrix}
B_k & \cdots & A_k + t_k \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_{\rho},
$$

and since $\pi_0 \neq 0$, by [3] Proposition 6.15, the above embedding of $\pi(\mathcal{E}_1)$ factors through

$$
\pi(\mathcal{E}_1) \hookrightarrow \text{soc} \begin{pmatrix}
B_k & \cdots & A_k + t_k \\
\vdots & \ddots & \vdots \\
B_k + t_{k-1} + 1 & \cdots & A_k + t_{k-1} + 1
\end{pmatrix}_{\rho} \times \begin{pmatrix}
B_{k-1} & \cdots & A_k + t_{k-1} \\
\vdots & \ddots & \vdots \\
B_{k-1} + 1 & \cdots & A_{k-1} + 1
\end{pmatrix}_{\rho},
$$

then $\pi(\mathcal{E}_1) \cong \pi(\mathcal{E}'_1)$.
\[
\times \begin{pmatrix}
B_k + t_{k-1} & \cdots & A_k + t_{k-1} \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_\rho \times \pi(\mathcal{E}).
\]

This implies that
\[
\pi_0 \mapsto \begin{pmatrix}
B_{k-1} + t_{k-1} & \cdots & A_{k-1} + t_{k-1} \\
\vdots & \ddots & \vdots \\
B_{k-1} + 1 & \cdots & A_{k-1} + 1
\end{pmatrix}_\rho \times \begin{pmatrix}
B_k + t_{k-1} & \cdots & A_k + t_{k-1} \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_\rho \times \pi(\mathcal{E}).
\]

Hence
\[
\pi(\mathcal{E}') \mapsto \begin{pmatrix}
B_k + t_k & \cdots & A_{k-1} + t_k \\
\vdots & \ddots & \vdots \\
B_k + t_{k-1} + 1 & \cdots & A_{k-1} + t_{k-1} + 1
\end{pmatrix}_\rho \times \pi_0 \times \begin{pmatrix}
B_k + t_{k-1} & \cdots & A_{k-1} + t_{k-1} \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_\rho \times \pi(\mathcal{E}).
\]

Since \(\pi_0' \neq 0\), on the first two Speh representations in the last inclusion, \(\pi(\mathcal{E}')\) factors though
\[
\text{soc} \left( \begin{pmatrix}
B_k + t_k & \cdots & A_{k-1} + t_k \\
\vdots & \ddots & \vdots \\
B_k + t_{k-1} + 1 & \cdots & A_{k-1} + t_{k-1} + 1
\end{pmatrix}_\rho \times \begin{pmatrix}
B_k + t_k & \cdots & A_{k-1} + t_k \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_\rho \right). 
\]

This is also a subrepresentation of
\[
\begin{pmatrix}
B_{k-1} + t_{k-1} & \cdots & B_k + t_{k-1} - 1 \\
\vdots & \ddots & \vdots \\
B_{k-1} + 1 & \cdots & B_k
\end{pmatrix}_\rho \times \begin{pmatrix}
B_k + t_{k-1} & \cdots & A_{k-1} + t_k \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_\rho .
\]

Note that two shifted Speh representations
\[
\begin{pmatrix}
B_k + t_k & \cdots & A_{k-1} + t_k \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_\rho \quad \text{and} \quad \begin{pmatrix}
B_k + t_{k-1} & \cdots & A_k + t_{k-1} \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_\rho
\]

are commutative by Theorem 2.1 Therefore,
\[
\pi(\mathcal{E}') \mapsto \begin{pmatrix}
B_{k-1} + t_{k-1} & \cdots & B_k + t_{k-1} - 1 \\
\vdots & \ddots & \vdots \\
B_{k-1} + 1 & \cdots & B_k
\end{pmatrix}_\rho \times \begin{pmatrix}
B_k + t_{k-1} & \cdots & A_k + t_{k-1} \\
\vdots & \ddots & \vdots \\
B_k + 1 & \cdots & A_k + 1
\end{pmatrix}_\rho.
\]
Remark 5.4. Let $E$ be the extended multi-segment for $G_\rho$ such that $\pi$ satisfies one of the three conditions in §5.1 with respect to $\pi$. Then using Theorem 5.1, replace $\pi(E') = \pi(E)$. By the same argument, if $\pi(E') \neq 0$, then we can show that $\pi(E) \cong \pi(E')$. This completes the proof.

5.2. Algorithm for derivatives. Let $E = \bigcup_\pi \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I^\rho, >)}$ be an extended multi-segment for $G_\rho$ such that $B_i \geq 0$ for all $\rho$ and $i \in I^\rho$. Assume that $\pi(E) \neq 0$. Now we give an algorithm to compute certain derivatives (or socles) of $\pi(E)$.

Set $B^{\max} = \max\{B_i \mid i \in I^\rho\}$, and $I^\rho(B^{\max}) = \{i \in I^\rho \mid B_i = B^{\max}\}$. We may assume that if $i \in I^\rho(B^{\max})$ and $j \in I^\rho \setminus I^\rho(B^{\max})$, then $j < i$. Recall that $I_2^\rho$ is the set of triples $(i, j, >')$, where $>'$ is an admissible order on $I_2^\rho$, and $i >' j$ are two adjacent elements in $I_2^\rho$ with respect to $> '$.

Algorithm 5.3. With the notation as above, we proceed the following:

Step 1: Set $i$ to be the minimal element in $I_2^\rho(B^{\max})$, and go to Step 2.

Step 2: If there exist an element $j \in I^\rho$ and an admissible order $>'$ such that

- $(i, j, >') \in I_2^\rho$;
- $A_i > A_j$ and $B_i > B_j$; and
- $E'$ satisfies one of the three conditions in §5.1 with respect to $i >' j$, where $E' = \bigcup_\pi \{([A^\prime_i, B^\prime_i]_\rho, l^\prime_i, \eta^\prime_i)\}_{i \in (I^\rho, >')}$ such that $\pi(E) \cong \pi(E')$, which is obtained in

then using Theorem 5.4, replace $E = \bigcup_\pi \{([A_i, B_i]_\rho, l_i, \eta_i)\}_{i \in (I^\rho, >)}$ so that

$[A_i, B_i]_\rho \sim [A_i, B_i]_\rho \cap [A_j, B_j]_\rho$, $[A_j, B_j]_\rho \sim [A_i, B_i]_\rho \cup [A_j, B_j]_\rho$, $A_i > A_j$ and $B_i > B_j$;

and so that $\pi(E)$ does not change. If the new $[A_i, B_i]_\rho$ is empty, we understand that $I^\rho(B^{\max})$ is replaced with $I^\rho(B^{\max}) \setminus \{i\}$. Go to Step 3.

Step 3: When Theorem 5.4 was applied in Step 2,

- if $I^\rho(B^{\max}) = \emptyset$, then the procedure is ended;
- otherwise, go back to Step 1.

When Theorem 5.4 was not applied in Step 2,

- if $i$ is not maximal element, replace $i \in I^\rho(B^{\max})$ with the next element, and go back to Step 2;
- otherwise, the procedure is ended.

Let $E^*$ be the resulting extended multi-segment so that $\pi(E) \cong \pi(E^*)$.

Remark 5.4. Recall that we set $b_i = \#([A_i, B_i])$. By induction on

$$t = \sum_{i \in I^\rho(B^{\max})} b_i,$$

we see that Algorithm 5.3 stops for any $E$.

Using Algorithm 5.3 we can compute certain derivatives of $\pi(E)$.
Theorem 5.5. Let $\mathcal{E}$ and $B^{\max}$ be as above. Suppose that $\pi(\mathcal{E}) \neq 0$ and $B^{\max} \geq 1$. Let $\mathcal{E}^* = \bigcup_{\rho} \{ (A_i^*, B_i^*) \}_{i \in I_\rho(B^{\max})}$ be the extended multi-segment obtained by Algorithm 5.3, so that $\pi(\mathcal{E}^*) \cong \pi(\mathcal{E})$. With $I_\rho(B^{\max}) = \{ i \in I_\rho | B_i^* = B^{\max} \}$, write

$$\bigcup_{i \in I_\rho(B^{\max})} [A_i^*, B_i^*] \rho = \{ \rho \cdot \cdot \cdot \rho \cdot \cdot \cdot \rho \}$$

as multi-sets such that $x_1 \leq \cdots \leq x_t$. (Here, the left hand side is regarded as a disjoint union.) Define $\mathcal{E}'$ from $\mathcal{E}^*$ by replacing $[A_i^*, B_i^*] \rho$ with $[A_i^* - 1, B_i^* - 1] \rho$ for every $i \in I_\rho(B^{\max})$. Then $\pi(\mathcal{E}') \neq 0$ and

$$D_{\rho \cdot \cdot \cdot \rho \cdot \cdot \cdot \rho} (\pi(\mathcal{E}^*)) \cong \pi(\mathcal{E}')$$

up to a multiplicity. In particular,

$$\pi(\mathcal{E}) \cong S_{\rho \cdot \cdot \cdot \rho \cdot \cdot \cdot \rho} (\pi(\mathcal{E}')).$$

Proof. Since $\pi(\mathcal{E}^*) \neq 0$, by Theorem 4.4, we see that $\mathcal{E}^*$ satisfies the three necessary conditions in Proposition 4.1 with respect to all $(i, j, >') \in I_\rho^{2, \text{adj}}$. However, by the construction of $\mathcal{E}^*$ (Algorithm 5.3), we see that $\mathcal{E}'$ also satisfies the same conditions. Therefore $\pi(\mathcal{E}') \neq 0$ by Theorem 4.4.

Now, for simplicity, we rewrite $\mathcal{E}^* = \mathcal{E}$, and write $I_\rho(B^{\max}) = \{ 1, 2, \ldots, m \}$ with $1 < \cdots < m$. Take integers $0 < t_1 < \cdots < t_m$ and define $\mathcal{E}_1$ from $\mathcal{E}$ by replacing $[A_i, B_i] \rho$ with $[A_i + t_i, B_i + t_i] \rho$ for $i \in I_\rho(B^{\max})$. By choosing $0 < t_1 < \cdots < t_m$ appropriately, we may assume that

$$\pi(\mathcal{E}) = \circ_{i \in I_\rho(B^{\max})} \left( D_{\rho \cdot \cdot \cdot \rho \cdot \cdot \cdot \rho} (\pi(\mathcal{E}_1)) \right),$$

$$\pi(\mathcal{E}') = \circ_{i \in I_\rho(B^{\max})} \left( D_{\rho \cdot \cdot \cdot \rho \cdot \cdot \cdot \rho} (\pi(\mathcal{E}_1)) \right).$$

Since $\pi(\mathcal{E}') \neq 0$, by [17], Proposition 8.5, we have

$$\pi(\mathcal{E}_1) \mapsto \left( \begin{array}{cccccc} B_1 + t_1 & \ldots & A_1 + t_1 \\ \vdots & \ddots & \vdots \\ B_i & \ldots & A_i \\ \end{array} \right) \times \cdots \times \left( \begin{array}{cccccc} B_m + t_m & \ldots & A_m + t_m \\ \vdots & \ddots & \vdots \\ B_i & \ldots & A_i \\ \end{array} \right) \times \pi(\mathcal{E}').$$

Since $B_1 = \cdots = B_m = B^{\max}$, by Theorem 2.1, the two Speh representations $Z_{\rho}[B_j, A_j]$ and $Z_{\rho}[B_j + t_j, A_j]$ are commutative for any $i, j \in I_\rho(B^{\max})$. Therefore, we have

$$\pi(\mathcal{E}) \mapsto Z_{\rho}[B_1, A_1] \times \cdots \times Z_{\rho}[B_m, A_m] \times \pi(\mathcal{E}').$$

Since $\cup_{i=1}^m [A_i, B_i] \rho = \{ \rho \cdot \cdot \cdot \rho \cdot \cdot \cdot \rho \}$ as multi-sets, we have

$$D_{\rho \cdot \cdot \cdot \rho \cdot \cdot \cdot \rho} (\pi(\mathcal{E})) \cong \pi(\mathcal{E}')$$

up to a multiplicity. This completes the proof. \qed
5.3. Example for derivatives. Let us consider \( \mathcal{E} = \mathcal{E}_\rho \) with

\[
\mathcal{E} = \begin{pmatrix}
0 & 1 & 2 & 3 \\
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\end{pmatrix}_\rho.
\]

Note that by definition (or [19, Theorem 1.3]), we know that

\[
\pi(\mathcal{E}) = L(\Delta_\rho[1, -3], \Delta_\rho[1, -2]; \pi(0^+, 1^-, 2^+, 3^-)).
\]

We compute some derivatives using Algorithm 5.3. First, by Theorem 4.3,

\[
\pi(\mathcal{E}) = \pi \begin{pmatrix}
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\end{pmatrix}_\rho \cong \pi \begin{pmatrix}
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\end{pmatrix}_\rho.
\]

This is in the situation of Theorem 5.1 for the second and third lines. By this theorem and Theorem 4.3, we have

\[
\pi(\mathcal{E}) \cong \pi \begin{pmatrix}
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\end{pmatrix}_\rho \cong \pi \begin{pmatrix}
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\end{pmatrix}_\rho.
\]

By Theorem 5.5 together with Theorems 5.1 and 4.3 we have

\[
D_{\rho\mid -3} \circ D_{\rho\mid -2} \circ D_{\rho\mid -1} \circ (\pi(\mathcal{E})) \cong \pi \begin{pmatrix}
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\end{pmatrix}_\rho \cong \pi \begin{pmatrix}
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\end{pmatrix}_\rho \cong L(\Delta_\rho[0, -3], \Delta_\rho[1, -2]; \pi(0^+, 0^+, 0^+, 1^-, 2^-)).
\]

We denote this representation by \( \pi(\mathcal{E}') \). By [2, Proposition 3.8], we have

\[
D_{\rho\mid -3} \circ D_{\rho\mid -2} \circ D_{\Delta_\rho[0, -1]}^{(1)} (\pi(\mathcal{E}')) \cong L(\Delta_\rho[1, -2]; \pi(0^+, 0^+, 0^+, 1^-, 2^-))
\]

\[
\cong \pi \begin{pmatrix}
\oplus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\ominus & \ominus & \ominus & \ominus \\
\end{pmatrix}_\rho.
\]
Hence
\[D_{\rho|1^3} \circ D_{\rho|2} \circ D_{\rho|0}(\pi(E')) \cong \pi \left( \begin{array}{c} \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \end{array} \right) \rho \cong \pi \left( \begin{array}{c} \ominus \\ \ominus \\ \ominus \\ \ominus \\ \ominus \end{array} \right) \rho.\]

If we denote this representation by \(\pi(E'')\), we have
\[D_{\rho|1} \circ D_{\rho|2} \circ D_{\rho|0}(\pi(E'')) \cong \pi \left( \begin{array}{c} \ominus \\ \ominus \\ \ominus \\ \ominus \end{array} \right) \rho \cong \pi \left( \begin{array}{c} \ominus \\ \ominus \\ \ominus \\ \ominus \end{array} \right) \rho.\]

Hence
\[D_{\rho} \circ D_{\rho|0}(\pi(E'')) \cong \pi \left( \begin{array}{c} \ominus \\ \ominus \end{array} \right) \rho = \pi(0^+, 0^+, 0^+),\]
which is an irreducible induction \(\rho \rtimes \pi(0^+)^{\dagger}\) from a supercuspidal representation.

6. A CONJECTURAL FORMULA FOR AUBERT DUALITY

In [3], Aubert defined an involution on \(\text{Irr}(G_n)\), which is a generalization of the Zelevinsky involution given in [20]. In this section, we give a conjectural formula for the Aubert dual of \(\pi(E)\).

6.1. Definition and algorithm. Aubert [3] showed that for any irreducible representation \(\pi\) of \(G_n\), there exists a sign \(\epsilon \in \{\pm 1\}\) such that the virtual representation
\[\hat{\pi} = \epsilon \sum_P (-1)^{\dim A_M[\text{Ind}_{P}(\text{Jac}_P(\pi))]}\]
is again an irreducible representation, where \(P = MN\) runs over all standard parabolic subgroups of \(G_n\), and \(A_M\) is the maximal split torus of the center of \(M\). We call \(\hat{\pi}\) the Aubert dual of \(\pi\).

In the previous work with Mínguez, we gave an algorithm [2, Algorithm 4.1] to compute \(\hat{\pi}\) for any \(\pi \in \text{Irr}(G_n)\). It says that by using the compatibilities of the Aubert duality with derivatives
\[\hat{\pi} \cong S^{(k)}_{\rho|1}(D_{\rho|1}^{(k)}(\pi)) \quad \text{for } x \neq 0; \text{ and} \]
\[\hat{\pi} \cong S^{(k)}_{\rho|0}(D_{\rho|0}^{(k)}(\pi)) \quad \text{if } \pi \text{ is } \rho|x^{-1}\text{-reduced},\]
the computation of \(\hat{\pi}\) can be reduced to the one for an easier representation ([2, Proposition 5.4]).
6.2. **Conjecture.** It is expected that the Aubert duality would preserve the unitarity. For \( \psi = \oplus_{\rho}(\oplus_{i \in I_{\rho}} \rho \boxtimes S_{a_i} \boxtimes S_{b_i}) \in \Psi_{sp}(G_n) \), define \( \hat{\psi} = \oplus_{\rho}(\oplus_{i \in I_{\rho}} \rho \boxtimes S_{b_i} \boxtimes S_{a_i}) \). By the compatibility of the Aubert duality with parabolic inductions, and by the result on the Zelevinsky duals of Speh (ladder) representations ([4, §4.1]), we see that the Zelevinsky dual of \( \hat{\psi} \) is equal to \( \tau_{\hat{\psi}} \). In particular, by the compatibility of twisted endoscopic character identities and the Aubert duality ([17, §A]), we have

\[
\{ \hat{\pi} \mid \pi \in \Pi_{\hat{\psi}} \} = \Pi_{\hat{\psi}}.
\]

It would also follow from Moeglin’s original construction of \( \Pi_{\hat{\psi}} \).

**Definition 6.1.** Let \( \mathcal{E} = \bigcup_{\rho} \{ ([A_i, B_i], l_i, \eta_i) \}_{i \in (I_{\rho}, >)} \) be an extended multi-segment for \( G_n \).

We assume that the fixed admissible order \( > \) on \( I_{\rho} \) satisfies \( (\mathcal{P}') \), i.e., \( B_i > B_j \implies i > j \). We define

\[
\hat{\mathcal{E}} = \bigcup_{\rho} \{ ([A_i, -B_i], \hat{l}_i, \hat{\eta}_i) \}_{i \in (I_{\rho}, >)}
\]

as follows:

- \( > \) is the totally order on \( I_{\rho} \) given by \( i > j \iff i < j \);
- if \( B_i \in \mathbb{Z} \), we set
  \[
  \hat{l}_i = l_i + B_i, \quad \hat{\eta}_i = (-1)^{\beta_i} \eta_i,
  \]
  where \( b_i = \#([A_i, B_i]_{\rho}) \) and \( \beta = \sum_{j \in I_{\rho}} b_j \);
- if \( B_i \not\in \mathbb{Z} \), we set
  \[
  \hat{l}_i = \begin{cases} 
  l_i + B_i + 1/2 & \text{if } \eta_i = (-1)^{\beta_i}, \\
  l_i + B_i - 1/2 & \text{if } \eta_i \neq (-1)^{\beta_i},
  \end{cases}
  \]
  where
  \[
  \begin{align*}
  \alpha_i &= \sum_{j \in I_{\rho}, j > i} (A_j + B_j) = \sum_{j \in I_{\rho}, j > i} (a_j - 1), \\
  \beta_i &= \sum_{j \in I_{\rho}, j < i} (A_j - B_j) = \sum_{j \in I_{\rho}, j < i} (b_j - 1).
  \end{align*}
  \]

Here, when \( l_i = b_i/2 \), we regard \( \eta_i = (-1)^{\beta_i} \).

Note that \( > \) is an admissible order on \( I_{\rho} \) since we assume that \( > \) satisfies \( (\mathcal{P}') \). Also, it is easy to check that the correspondence \( \mathcal{E} \mapsto \hat{\mathcal{E}} \) is an involution.

**Conjecture 6.2.** If \( \pi(\mathcal{E}) \neq 0 \), then its Aubert dual would be given by

\[
\hat{\pi}(\mathcal{E}) \cong \pi(\hat{\mathcal{E}}).
\]

The simplest case can be proven.

**Proposition 6.3.** Suppose that \( \#I_{\rho} \leq 1 \) for any \( \rho \). If \( \pi(\mathcal{E}) \neq 0 \), then \( \hat{\pi}(\mathcal{E}) \cong \pi(\hat{\mathcal{E}}) \).

**Proof.** We prove the assertion by induction on the rank of \( G_n \). Write \( \mathcal{E} = \bigcup_{\rho} \{ ([A, B]_{\rho}, l, \eta) \} \) so that \( \hat{\mathcal{E}} = \bigcup_{\rho} \{ ([A, -B]_{\rho}, \hat{l}, \hat{\eta}) \} \), where

\[
\hat{l} = \begin{cases} 
  B + l & \text{if } B \in \mathbb{Z}, \\
  B + l + 1/2 & \text{if } B \not\in \mathbb{Z}, \; \eta = +1, \\
  B + l - 1/2 & \text{if } B \not\in \mathbb{Z}, \; \eta = -1,
  \end{cases}
\]

\[
\hat{\eta} = \begin{cases} 
  \eta & \text{if } B \in \mathbb{Z}, \\
  -\eta & \text{if } B \not\in \mathbb{Z}.
  \end{cases}
\]
By Theorem 4.4, $\pi(E) \neq 0$ if and only if $\hat{l} \geq 0$ for any $\rho$.

When $\hat{l} > 0$ (for some $\rho$), if we define $E'$ from $E$ by replacing $([A, B]_\rho, l, \eta)$ with $([A - 1, B - 1]_\rho, \hat{l} - 1, \eta)$, we have

$$D_{\rho|B, \ldots, \rho|A}(\pi(E)) = \pi(E').$$

On the other hand, since $\hat{E}'$ is given from $\hat{E}$ by replacing $(\rho|A - 1, B - 1, \hat{l} - 1, \eta)$ with $(\rho|A - 1, B - 1, \hat{l} - 1 + 1, \hat{l} - 1, \eta)$, we have

$$D_{\rho|A - 1 \circ \cdots \circ \rho|B}(\pi(\hat{E})) = \pi(\hat{E}').$$

Since we know that $\hat{\pi}(E') \cong \pi(\hat{E}')$ by the induction hypothesis, we conclude that $\hat{\pi}(E) \cong \pi(\hat{E})$.

Hence we may assume that $\hat{l} = 0$. By exchanging the roles of $E$ and $\hat{E}$, we can also assume that $l = 0$. However, if $l = \hat{l} = 0$ (for all $\rho$), then $\pi(E)$ is supercuspidal so that $\hat{\pi}(E) = \pi(E)$.

Moreover, by the definition of $\hat{E}$, we have $\pi(E) \cong \pi(\hat{E})$ in this case. This completes the proof. $\square$

Since we have an algorithm (Algorithm 5.3 and Theorem 5.5) to compute derivatives of $\pi(E)$, one might seem that Conjecture 6.2 could be proven by a similar argument. However, in this algorithm, an admissible order which does not satisfy $(P')$ is used in general. In this situation, the dual order $\hat{\rho}$ on $I_\rho$ may not admissible, and hence $\pi(\hat{E})$ cannot be considered. Nevertheless, Conjecture 6.2 seems to be true. See Example 6.4 below.

### 6.3. Examples of Aubert duals

We give some examples for Conjecture 6.2.

**Example 6.4.** Let us consider $E = E_\rho$ with

$$E = \begin{pmatrix} -1 & 0 & 1 & 2 & 3 \\ \triangle & \oplus & \triangledown & \triangledown \\ \triangle & \oplus & \oplus & \triangledown \\ \oplus & \oplus & \end{pmatrix}_\rho.$$

Then $\hat{E}$ is defined as

$$\hat{E} = \begin{pmatrix} -1 & 0 & 1 & 2 & 3 \\ \triangle & \oplus & \oplus & \triangledown \\ \triangle & \oplus & \oplus & \triangledown \\ \oplus & \oplus & \end{pmatrix}_\rho.$$

The associated representations are given by

$$\pi(E) \cong L(\rho|\cdot|^{-1}, \Delta_\rho[0, -3], \Delta_\rho[1, -2]; \pi(0^+, 1^-, 2^-)),$$

$$\pi(\hat{E}) \cong L(\rho|\cdot|^{-1}, \Delta_\rho[0, -2]; \pi(0^-, 1^+, 1^+, 2^-, 3^+)).$$

By applying [Algorithm 4.1], one can check that $\hat{\pi}(E) \cong \pi(\hat{E})$. On the other hand, if we apply Algorithm 5.3 to $E$, we have

$$\pi \begin{pmatrix} -1 & 0 & 1 & 2 & 3 \\ \triangle & \oplus & \triangledown & \triangledown \\ \triangle & \oplus & \oplus & \triangledown \\ \oplus & \oplus & \end{pmatrix}_\rho \cong \pi \begin{pmatrix} -1 & 0 & 1 & 2 & 3 \\ \triangle & \oplus & \triangledown & \triangledown \\ \triangle & \oplus & \oplus & \triangledown \\ \oplus & \oplus & \end{pmatrix}_\rho.$$
Hence \( D_{\rho|^{\frac{1}{5}/2}}(\pi(\mathcal{E})) \neq 0 \) but \( D_{\rho|^{\frac{1}{5}/2},\rho|^{\frac{1}{2}}}^{\frac{2}{5}}(\pi(\mathcal{E})) = 0 \). Moreover, on the second and the third extended multi-segments, the admissible orders do not satisfy \( \mathcal{P}' \) so that they are not in the situation of Conjecture 6.2.

For other examples, one can check that Tadić’s computations [15, Propositions 4.1, 5.2, 6.1, 7.2] are compatible with Conjecture 6.2.

Finally, we consider Jantzen’s example.

**Example 6.5.** As in [6, §4.1], let us consider \( \pi = L(\Delta_{\rho|^{1/2}, -5/2}, \rho|^{-1/2}; \pi((1/2)^-, (1/2)^-)) \).

It is not of Arthur type. Nevertheless, we would write

\[
\pi \cong \pi \left( \begin{array}{cccc}
-\frac{1}{2} & \frac{1}{2} & \frac{3}{2} & \frac{5}{2} \\
\uparrow & \downarrow & \uparrow & \downarrow \\
\uparrow & \uparrow & \downarrow & \uparrow \\
\uparrow & \uparrow & \uparrow & \uparrow \\
\end{array} \right). 
\]

Hence by the definition of \( \pi(\mathcal{E}) \), we have

\[
D_{\rho|^{\frac{1}{5}/2}}(\pi) \cong \pi \left( \begin{array}{cccc}
-\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\
\uparrow & \downarrow & \uparrow \\
\uparrow & \uparrow & \downarrow \\
\downarrow & \uparrow & \downarrow \\
\end{array} \right). 
\]

Here, we regard \( \eta_1 = -1 \) and \( \eta_4 = +1 \) by Definition 6.4. According to Conjecture 6.2, its Aubert dual would be given by

\[
\hat{\pi} \cong S_{\rho|^{-\frac{1}{5}/2}} \left( \begin{array}{cccc}
-\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\
\uparrow & \downarrow & \uparrow \\
\uparrow & \uparrow & \downarrow \\
\downarrow & \uparrow & \downarrow \\
\end{array} \right) \cong L \left( \Delta_{\rho|[-1/2, -3/2], (\rho|^{-\frac{1}{2}})^2; \pi((1/2)^+, (1/2)^+)} \right). 
\]

Therefore,

\[
\hat{\pi} \cong S_{\rho|^{-\frac{1}{5}/2}} \left( D_{\rho|^{\frac{1}{5}/2}}(\pi) \right) \cong L \left( \Delta_{\rho|[-1/2, -3/2], (\rho|^{-\frac{1}{2}})^2; \pi((1/2)^+, (1/2)^+)} \right). 
\]

This is consistent with the conclusion of [6, §4.1].

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