ORBITAL STABILITY AND SPECTRAL PROPERTIES OF
SOLITARY WAVES OF KLEIN–GORDON EQUATION WITH
CONCENTRATED NONLINEARITY

ANDREW COMECH
Texas A&M University, College Station, TX
Institute for Information Transmission Problems, Moscow, Russia

ELENA KOPYLOVA*
Vienna University, Vienna, Austria
Institute for Information Transmission Problems, Moscow, Russia

(Communicated by Hongjie Dong)

Abstract. We obtain explicit characterization of orbital and spectral stability
of solitary wave solutions to the U(1)-invariant 1D Klein–Gordon equation
coupled to an anharmonic oscillator. We also give the complete analysis of the
spectrum of the linearization at a solitary wave.

1. Introduction. We study U(1)-invariant Klein–Gordon equation on a line with
a concentrated nonlinearity:

\[ \ddot{\psi}(x, t) = \partial_x^2 \psi(x, t) - m^2 \psi(x, t) + \delta(x) a(|\psi(0, t)|^2) \psi(0, t), \quad \psi(x, t) \in \mathbb{C}, \ x \in \mathbb{R}, \tag{1.1} \]

where \( m > 0 \). Above, \( a(\cdot) \) is a real-valued differentiable function, so that the
model is U(1)-invariant. The equation is understood in the sense of distributions.
Physically, equation (1.1) describes the Klein–Gordon field coupled to a nonlinear
oscillator located at \( x = 0 \), with \( a(|\psi|^2) \psi \) being the oscillator force. Equation (1.1)
admits finite energy solutions of the form \( \varphi_\omega(x)e^{-i\omega t}, \ \omega \in \mathbb{R} \), called solitary waves.
The solitary waves form a two-dimensional solitary manifold in the Hilbert space
of finite energy states of the system.

Equation (1.1) was proposed to model electron’s transitions between Bohr’s
quantum orbits, which is one of the fundamental problems of Quantum Mechanics.
More precisely, (1.1) models the interaction of the electron (represented by
the anharmonic oscillator) with the electromagnetic radiation (represented by the
Klein–Gordon field). The global attraction of any finite energy solution to the set
of all solitary waves in local energy norms, which was established in [8], can be
interpreted as the relaxation of a perturbed electron to the Bohr orbit, where it no
longer loses the energy via the radiation, in the agreement with Bohr’s postulate
on quantum jumps.

2020 Mathematics Subject Classification. Primary: 35L70; Secondary: 47F05.
Key words and phrases. Klein-Gordon equation, concentrated nonlinearity, orbital stability,
spectral stability, solitary waves.

The second author is supported by the Austrian Science Fund (FWF), Grant P 34177-N.
* Corresponding author.
We pursue further properties of equation (1.1) since it is a convenient playground for establishing asymptotic stability results similar to [3, 10] (in the context of the Schrödinger equation with concentrated nonlinearity), allowing one to explicitly check all the spectral properties of the linearized equation. In the present article, we obtain the spectral and orbital stability results. Moreover, we obtain the complete description of the spectrum of the linearized equation. This detailed information is needed for the subsequent proof of asymptotic stability in this and similar models.

We point out that many pieces of the spectral analysis which we develop in this article can be carried over essentially verbatim to models similar to (1.1), such as the models where the concentrated nonlinearity is substituted by its regularized versions, such as the self-interaction based on the mean field [9].

We mention that the local and global well-posedness of (1.1) has already been proved in [8], and that the asymptotic stability of solitary wave solutions to the discrete Schrödinger and Klein–Gordon equations was obtained in [12, 13]. Let us also mention that related results on local well-posedness, orbital stability, and linear instability of solitary waves in the nonlinear Klein–Gordon equation in the external $\delta$-function potential were obtained in [4].

In the first part of the article we study the orbital stability of solitary wave solutions $e^{-it} \phi_\omega = e^{-it} [\varphi_\omega(x)]$ to the vector version of (1.1):

$$
\dot{\Psi} = \begin{bmatrix}
0 & 0 \\
\partial_x^2 - m^2 & 1
\end{bmatrix} \Psi + \delta(x) \begin{bmatrix}
0 \\
\alpha(|\psi|^2) \psi
\end{bmatrix}, \quad \Psi(x, t) = \begin{bmatrix}
\psi(x, t) \\
\pi(x, t)
\end{bmatrix} \in \mathbb{C}^2.
$$

(1.2)

We recall that $\phi_\omega$-orbit is the set \{ $e^{-it} \phi_\omega$; $t \in \mathbb{R}$ \} (see [6]). Denote $X := H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$.

**Definition 1.1.** The $\phi_\omega$-orbit is stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\| \Psi(0) - \phi_\omega \|_X < \delta$ then

$$
\sup_{0 < t < \infty} \inf_{s \in \mathbb{R}} \| \Psi(t) - e^{is} \phi_\omega \|_X < \varepsilon.
$$

Otherwise the $\phi_\omega$-orbit is called unstable.

In the second part of the paper we study the spectral stability of solitary waves:

**Definition 1.2.** The solitary wave $e^{-it} \phi_\omega$ is called spectrally stable if the corresponding linearization operator (the operator $A$ in (3.6) below) has purely imaginary spectrum:

$$
\sigma(A) \subset i\mathbb{R}.
$$

If this condition is not satisfied, the solitary wave is called linearly unstable.

Finally, we give a complete description of the spectrum of the linearization at a solitary wave. In particular, we give the condition for the absence of virtual levels. We plan to use these spectral properties in the study of asymptotic stability of solitary waves of the system (1.1).

The paper is organized as follows. The model and its solitary wave solutions are described in Section 2. The linearization at a solitary wave is carried out in Section 3, where the standard properties of the linearized operator are obtained. The orbital stability result (Theorems 4.4) is proved in Section 4. The spectral stability (Theorem 5.1) is proved in Section 5. In Section 6, we give a detailed structure of the spectrum of the linearized operator. For completeness, in Appendix A we present the explicit solution to the cubic equation satisfied by the eigenvalues of the linearized operator.
2. The model. We consider the oscillator force \( a(|\psi|^2)\psi, a \in C^1(\mathbb{R}) \), admitting a real-valued potential,
\[
a(|\psi|^2)\psi = -\nabla \text{Re} \psi, \text{Im} \psi U(\psi), \quad \psi \in \mathbb{C}, \quad U \in C^2(\mathbb{C}),
\]
where \( U(\psi) = u(|\psi|^2) \), with \( u(\tau) = -\frac{1}{2} \int_0^\tau a(s) \, ds \). The local and global existence result for the Cauchy problem for equation (1.2) was proved in [8, Theorem 2.1]:

**Theorem 2.1.** Assume that the potential is represented by \( U(\psi) = u(|\psi|^2) \) with \( u \in C^2(\mathbb{R}) \) and that \( U \) satisfies the inequality
\[
U(\psi) \geq A - B|\psi|^2 \quad \text{for} \quad \psi \in \mathbb{C}, \quad \text{where} \quad A \in \mathbb{R}, \quad 0 \leq B < m.
\]

Then for every \( \Psi_0 = [\psi_0] \in \mathcal{X} \) there is a unique solution \( \Psi \in C_b(\mathbb{R}, \mathcal{X}) \) of equation (1.2) satisfying \( \Psi|_{t=0} = \Psi_0 \). Moreover, there exists \( A(\Psi_0) > 0 \) such that
\[
\sup_{t \in \mathbb{R}} \|\Psi(t)\|_{\mathcal{X}} \leq A(\Psi_0) < \infty.
\]

Bound states of equation (1.2), known as solitary waves [6], are finite energy solutions of the form
\[
\Psi(x, t) = e^{-i\omega t} \varphi_\omega(x), \quad \varphi_\omega(x) = \begin{bmatrix} \varphi_\omega(x) \\ -i\omega \varphi_\omega(x) \end{bmatrix}, \quad \varphi_\omega \in H^1(\mathbb{R}), \quad \omega \in \mathbb{R}. \quad (2.2)
\]

By (1.2), the frequency \( \omega \in \mathbb{R} \) and the amplitude \( \varphi_\omega(x) \) solve the following nonlinear eigenvalue problem:
\[
\omega^2 \varphi_\omega(x) = -\partial_x^2 \varphi_\omega(x) + m^2 \varphi_\omega(x) - \delta(x)a(|\varphi_\omega(0)|^2)\varphi_\omega(0), \quad x \in \mathbb{R}. \quad (2.3)
\]

**Lemma 2.2.** Assume that the nonlinearity in (1.2) is represented by the function \( a \in C^1(\mathbb{R}_+) \). Then the set of all nonzero solitary waves of (1.2) is given by
\[
\left\{ \Psi(x, t) = e^{-i\omega t} \varphi_\omega(x); \quad \varphi_\omega(x) = e^{i\theta} \begin{bmatrix} \varphi_\omega(x) \\ -i\omega \varphi_\omega(x) \end{bmatrix}, \quad \varphi_\omega(x) = Ce^{-\alpha |x|} \right\}, \quad (2.4)
\]
where \( |\omega| < m, \alpha = \sqrt{m^2 - \omega^2} > 0, \quad \theta \in [0, 2\pi) \) and \( C > 0 \) satisfies the relation
\[
a(C^2) = 2\alpha. \quad (2.5)
\]

If \( a'(C^2) \neq 0 \), then \( C \) is locally a \( C^1 \)-function of \( \omega \).

**Proof.** Equation (2.3) implies that
\[
\partial_x^2 \varphi_\omega(x) = (m^2 - \omega^2) \varphi_\omega(x), \quad x \neq 0,
\]
and hence \( \varphi_\omega(x) = C_\pm e^{-\alpha_\pm |x|} \) for \( \pm x > 0 \), where \( \alpha_\pm \) satisfy \( \alpha_\pm^2 = m^2 - \omega^2 \). Since \( \varphi_\omega \in H^1(\mathbb{R}) \), we conclude that \( \alpha_\pm > 0, \ |\omega| < m \), and that \( \alpha_\pm = \sqrt{m^2 - \omega^2} > 0 \). Since the function \( \varphi_\omega(x) \) is continuous at \( x = 0 \), one has \( C_- = C_+ = C \). Thus, the solitary waves are solutions of the form
\[
\varphi_\omega(x) = Ce^{-\alpha |x|}, \quad \alpha = \alpha(\omega) := \sqrt{m^2 - \omega^2} > 0. \quad (2.6)
\]

The algebraic equation satisfied by the constant \( C > 0 \) is obtained by collecting coefficients at \( \delta(x) \) in (2.3):
\[
0 = \partial_x \varphi_\omega(0+) - \partial_x \varphi_\omega(0-) + a(|\varphi_\omega(0)|^2)\varphi_\omega(0).
\]

This implies that
\[
0 = -2\alpha C + a(C^2)C, \quad \text{hence} \quad a(C^2) = 2\alpha.
\]

We note that there is the following relation:
\[
-2\omega = \frac{d}{d\omega} \alpha^2 = \frac{d}{d\omega} \left( \frac{a(C^2)}{4} \right)^2 = a(C^2)a'(C^2)C \frac{dC}{d\omega},
\]
which shows that \( C \) is locally a \( C^1 \)-function of \( \omega \) if \( a'(C^2) \neq 0 \). \( \square \)
We consider a real version of (1.2). Namely, we set

$$
\Psi = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{bmatrix} = \begin{bmatrix} \text{Re } \psi \\ \text{Im } \psi \\ \text{Re } \pi \\ \text{Im } \pi \end{bmatrix} \in \mathbb{R}^4,
$$

and rewrite (1.2) in the following form:

$$
\dot{\Psi}(t) = \left[ \begin{array}{cc} 0 & I_2 \\ \partial_x^2 - m^2 & 0 \end{array} \right] \Psi(t) + \delta(x) \left[ \begin{array}{c} 0 \\ a(\Psi_1^2 + \Psi_2^2)\Psi_1 \\ a(\Psi_1^2 + \Psi_2^2)\Psi_2 \end{array} \right].
$$

(2.9)

Equation (2.9) can be written formally as the Hamiltonian system

$$
\dot{\Psi}(t) = \Sigma E'(\Psi(t)), \quad \text{with } \Sigma = \left[ \begin{array}{cc} 0 & I_2 \\ -I_2 & 0 \end{array} \right]
$$

(2.10)

and with the Hamiltonian functional

$$
E(\Psi) = \frac{1}{2} \int_\mathbb{R} \left( \sum_{j=1,2} (|\partial_x \Psi_j(x)|^2 + m^2|\Psi_j(x)|^2) + \sum_{j=3,4} |\Psi_j(x)|^2 \right) dx 
+ u(|\Psi_1(0)|^2 + |\Psi_2(0)|^2),
$$

(2.11)

which is conserved for finite energy solutions

$$
\Psi \in X := H^1(\mathbb{R}) \oplus H^1(\mathbb{R}) \oplus L^2(\mathbb{R}) \oplus L^2(\mathbb{R})
$$

by Theorem 2.1. Equation (2.9) is $U(1)$-invariant: if $\Psi(x,t)$ is a solution, then so is $e^{J\theta}\Psi(x,t)$ for any $\theta \in \mathbb{R}$, where

$$
J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}.
$$

(2.12)

The Nöther theorem implies the charge conservation: the value of the functional

$$
Q(\Psi) := \int_\mathbb{R} \left( \Psi_2(x)\Psi_3(x) - \Psi_1(x)\Psi_4(x) \right) dx
$$

(2.13)

is conserved for solutions to (2.9). In the terminology of [6],

$$
Q(\Psi) = \frac{1}{2} \langle B\Psi, \Psi \rangle,
$$

$$
B = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}.
$$

The bound states of equation (2.9) corresponding to (2.4) read as follows:

$$
\Psi(x,t) = e^{J\omega t}\Phi_\omega(x), \quad \Phi_\omega(x) = e^{J\theta} \begin{bmatrix} \varphi_\omega(x) \\ 0 \\ 0 \\ -\omega \varphi_\omega(x) \end{bmatrix}, \quad \theta \in [0, 2\pi).
$$

(2.14)

We note that, taking into account Lemma 2.2,

$$
Q(e^{J\omega t}\Phi_\omega) = Q(\Phi_\omega) = \omega \int_\mathbb{R} |\varphi_\omega(x)|^2 dx = \frac{\omega C^2}{\alpha},
$$

(2.15)

with $C$ satisfying (2.5).
Remark 1. One can readily check that $\Phi_\omega$ is a solution to the equation
\[ E'(\Phi_\omega) = \omega Q'(\Phi_\omega). \] (2.16)
In the terminology of [6] (see Definition on p.166), the solution $T(\omega t)\Phi_\omega$, with $T(\omega t) = e^{J\omega t}$, is the bound state solution to (1.2). We also note that $\Sigma$, $B$, and $T'(0)$ satisfy the relation
\[ \Sigma B = T'(0), \quad \text{with} \quad T'(0) = J \]
(cf. [6, Equation (2.8)]).

3. Linearization at a solitary wave. Without loss of generality, we assume that $\Phi_\omega$ is given by (2.14) with $\theta = 0$. Substituting
\[ \Psi(x,t) = e^{J\omega t}(\Phi_\omega(x) + Z(x,t)), \quad Z(x,t) = \begin{bmatrix} Z_1(x,t) \\ Z_2(x,t) \\ Z_3(x,t) \\ Z_4(x,t) \end{bmatrix} \in \mathbb{R}^4, \] (3.1)
into (2.9), we obtain:
\[ \mathbf{J}\omega(\Phi_\omega + Z) + \dot{Z} = \begin{bmatrix} 0 & 0 \\ 0 & (\partial_x^2 - m^2)I_2 \\ I_2 & 0 \\ 0 & 0 \end{bmatrix} (\Phi_\omega + Z) + \delta(x) \begin{bmatrix} 0 \\ a((C + Z_1)^2 + Z_2^2)(C + Z_1) \\ 0 \\ a((C + Z_1)^2 + Z_2^2)Z_2 \end{bmatrix}. \]
Further, equation (2.3) leads to
\[ \dot{Z} = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & \partial_x^2 - m^2 & \partial_x^2 - m^2 & \omega \\ \end{bmatrix} Z \]
\[ + \delta(x) \begin{bmatrix} 0 \\ a((C + Z_1)^2 + Z_2^2)(C + Z_1) - a(|C|^2)C \\ 0 \\ a((C + Z_1)^2 + Z_2^2)Z_2 \end{bmatrix}. \] (3.2)
Keeping the linear terms from (3.2), we arrive at the equation which describes the linearization at the solitary wave:
\[ \dot{Y}(x,t) = \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ 0 & 0 & 0 & -\omega \\ 0 & \partial_x^2 - m^2 & \partial_x^2 - m^2 & \omega \end{bmatrix} Y(x,t) + 2\delta(x) Y(x,t) + 2\delta(x) \begin{bmatrix} 0 \\ 0 \\ Y_1(0,t) \\ \alpha Y_2(0,t) \end{bmatrix}, \] (3.3)
where (cf. (2.5))
\[ a(C^2) = 2\alpha > 0, \quad \kappa := \frac{C^2 a'(C^2)}{a(C^2)}, \] (3.4)
with $C > 0$ the amplitude of the solitary wave at $x = 0$ (see Lemma 2.2).

Remark 2. We note that the above definition of $\kappa$ is compatible with the pure power case $a(\tau) = \tau^\kappa$, $\kappa > 0$, $\tau \geq 0$, when $C^2 a'(C^2) = \kappa a(C^2)$.

Let us denote
\[ L_\kappa(\omega) = -\partial_x^2 + m^2 - \omega^2 - 2(1 + 2\kappa)\alpha \delta(x), \quad \alpha = \sqrt{m^2 - \omega^2}. \] (3.5)
Remark 3. The operator $L_\kappa(\omega)$ is defined on the domain 
\[ \mathcal{D}(L_\kappa) = \left\{ \psi \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \cup \mathbb{R}_+) \mid \partial_x \psi(0^+) - \partial_x \psi(0^-) = -2(1+2\kappa)\alpha \psi(0) \right\}, \]
on which $L_\kappa(\omega)$ is a lower semi-bounded symmetric operator which admits the selfadjoint Friedrichs extension (see [1, §I.3.1]).

In terms of $L_\kappa(\omega)$ from (3.5), the system (3.3) reads as follows:
\[ \dot{Y}(x, t) = A(\omega, \kappa)Y(x, t), \quad A(\omega, \kappa) := \begin{bmatrix} 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \\ -L_\kappa(\omega) - \omega^2 & 0 & 0 & -\omega \\ 0 & -L_0(\omega) - \omega^2 & \omega & 0 \end{bmatrix}. \] (3.6)

Theorem 2.1 generalizes to equation (3.6): for every initial function $Y(x, 0) = Y_0 \in X$, the equation admits a unique solution $Y(x, t) \in C_b(\mathbb{R}, X)$. Note that $A(\omega, \kappa)$ is factored into
\[ A(\omega, \kappa) = \Sigma H(\omega, \kappa), \] (3.7)
where $\Sigma$ is defined in (2.10) and
\[ H(\omega, \kappa) = \begin{bmatrix} L_\kappa + \omega^2 & 0 & 0 & \omega \\ 0 & L_0 + \omega^2 & -\omega & 0 \\ 0 & -\omega & 1 & 0 \\ \omega & 0 & 0 & 1 \end{bmatrix}. \] (3.8)

We note that $H(\omega, \kappa) = E''(\Phi_\omega) - \omega Q''(\Phi_\omega)$ (cf. [6, Equation (2.17)]).

4. Orbital stability. Here we study the orbital stability of solitary waves following the approach of Grillakis-Shatah-Strauss [6]. First, we investigate the spectral properties of the operator $H(\omega, \kappa)$ from (3.8).

Theorem 4.1. Let $\omega \in (-m, m)$.
1. The essential spectrum of $H(\omega, \kappa)$ is given by
\[ \sigma_{\text{ess}}(H(\omega, \kappa)) = [c^-(\omega), 1] \cup [c^+(\omega), +\infty), \quad \kappa \in \mathbb{R}, \]
where
\[ c^\pm(\omega) = \frac{m^2 + 1 \pm \sqrt{(m^2 - 1)^2 + 4\omega^2}}{2} > 0. \] (4.1)
2. For $\kappa \in \mathbb{R} \setminus \{0\}$, the kernel of $H(\omega, \kappa)$ is spanned by $T'(0)\Phi_\omega = J\Phi_\omega$.
3. For $\kappa = 0$, the kernel of $H(\omega, \kappa)$ is spanned by $J\Phi_\omega$ and $\Phi_\omega$.
4. The operator $H(\omega, \kappa)$ has no negative eigenvalues for $\kappa \leq 0$, and it has exactly one simple negative eigenvalue for $\kappa > 0$.

Above, $J$ is from (2.12) and $\Phi_\omega(x)$ is from (2.14) (with $\theta = 0$).

Proof. We start with the essential and point spectra of operators $L_\kappa(\omega)$ defined in (3.5).

Lemma 4.2. Let $\omega \in (-m, m), \kappa \in \mathbb{R}$. The operator $L_\kappa(\omega)$ is selfadjoint and satisfies
\[ \sigma_{\text{ess}}(L_\kappa(\omega)) = [\alpha^2, +\infty), \]
\[ \sigma_p(L_\kappa(\omega)) = \begin{cases} \emptyset, & \kappa \leq -1/2, \\
\Lambda_\kappa(\omega) := -4\alpha^2(\kappa + \kappa^2), & \kappa > -1/2, \end{cases} \]
where $\alpha = \alpha(\omega) = \sqrt{m^2 - \omega^2}$. The eigenvalue $\Lambda_\kappa(\omega)$ is simple.
Proof. Solving the equation
\[- \partial_x^2 - 2(1 + 2\kappa)\alpha \delta(x) + \alpha^2)\psi = \Lambda \psi, \quad \psi \in L^2(\mathbb{R}),\]
we find that \(\Lambda\) has to satisfy \(\Lambda < \alpha^2\) and that the corresponding eigenfunction is given by
\[
\psi_\Lambda(x) = Ce^{-|x|\sqrt{\alpha^2 - \Lambda}}, \quad x \in \mathbb{R}, \quad C \neq 0,
\]
while \(\Lambda\) is obtained from the jump condition at \(x = 0:\)
\[
2(1 + 2\kappa)\alpha = 2\sqrt{\alpha^2 - \Lambda}.
\]
This shows that there could only be an eigenvalue if \(\kappa > -\frac{1}{2},\) and that \(\Lambda = -4\alpha^2(\kappa + \kappa^2).\) In particular, \(\sigma_p(L_0(\omega)) = \{0\};\) by (2.3), the corresponding eigenvector is \(\varphi_\omega.\)

Further, denote
\[
G_1 = G_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad G_2 = G_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{bmatrix}
\]
and consider the operator which is similar to \(H(\omega, \kappa)\):
\[
\tilde{H}(\omega, \kappa) = G_2 G_1 H(\omega, \kappa) G_1^{-1} G_2^{-1} = \begin{bmatrix} L_\kappa(\omega) + \omega^2 & \omega & 0 & 0 \\ \omega & 1 & 0 & 0 \\ 0 & 0 & L_0 + \omega^2 & \omega \\ 0 & 0 & \omega & 1 \end{bmatrix}
\]
Thus, the spectral problem for \(H(\omega, \kappa)\) is reduced to studying the spectrum of
\[
H_\kappa(\omega) = \begin{bmatrix} L_\kappa(\omega) + \omega^2 & \omega \\ \omega & 1 \end{bmatrix}.
\]

Lemma 4.3. Let \(\omega \in (-m, m)\) and \(\kappa \in \mathbb{R}.
1. The essential spectrum of \(H_\kappa(\omega)\) is given by
\[
\sigma_{\text{ess}}(H_\kappa(\omega)) = \begin{cases} [m^2, +\infty) \cup \{1\}, & \omega = 0, \\ [c^-(\omega), 1] \cup [c^+(\omega), +\infty), & \omega \in (-m, m) \setminus \{0\}, \end{cases}
\]
where \(c^\pm(\omega)\) are defined in (4.1). We note that
\[
0 < c^-(\omega) \leq \min\{1, m^2\}, \quad c^+(\omega) \geq \max\{1, m^2\}.
\]
2. The point spectrum of \(H_\kappa(\omega)\) is given by
\[
\sigma_p(H_\kappa(\omega)) = \begin{cases} \{0\}, & \omega \in (-m, m) \setminus \{0\}, \quad \kappa \leq -1/2, \\ \{1\}, & \omega = 0, \quad \kappa \leq -1/2, \\ \{\lambda^\pm_\kappa(\omega)\}, & \omega \in (-m, m), \quad \kappa > -1/2, \end{cases}
\]
with the eigenvalues \(\lambda^\pm_\kappa(\omega)\) given by
\[
\lambda^\pm_\kappa(\omega) = \frac{1}{2} \left( \Lambda_\kappa(\omega) + \omega^2 + 1 \pm \sqrt{\left(\Lambda_\kappa(\omega) + \omega^2 + 1\right)^2 - 4\Lambda_\kappa(\omega)} \right),
\]
where \(\Lambda_\kappa(\omega)\) (defined for \(\kappa > -1/2\)) is from Lemma 4.2.
3. The multiplicity of eigenvalue \( \lambda = 1 \) is infinite (it is present in the spectrum if and only if \( \omega = 0; \kappa \in \mathbb{R} \) is arbitrary). The corresponding eigenspace contains \[
\begin{bmatrix}
0 \\
u_2
\end{bmatrix}
\] with arbitrary \( u_2 \in L^2(\mathbb{R}) \).

4. All eigenvalues of \( H_\kappa(\omega) \) which are different from \( \lambda = 1 \) are simple.

5. \( 0 \in \sigma_p(H_\kappa(\omega)) \) if and only if \( \kappa = 0 \), and
\[
\sigma_p(H_0(\omega)) = \{0\} \cup \{\omega^2 + 1\},
\]
with the corresponding eigenvectors given by
\[
\psi_0(x) = \begin{bmatrix} \varphi_\omega(x) \\ -\omega \varphi_\omega(x) \end{bmatrix}, \quad \psi_{\omega^2+1}(x) = \begin{bmatrix} \omega \varphi_\omega(x) \\ \varphi_\omega(x) \end{bmatrix}.
\]

\textbf{Proof.} Let us compute the essential spectrum of \( H_\kappa(\omega) \). Since \( \delta(x) \) is a relatively compact perturbation of the Laplacian in one dimension, by the Weyl theorem, when computing the essential spectrum of \( H_\kappa(\omega) \), we can replace \( L_\kappa(\omega) + \omega^2 \) in (4.5) by \(-\partial_x^2 + m^2\). Then the values of the essential spectrum are those \( \lambda \in \mathbb{C} \) that satisfy
\[
\det \begin{bmatrix} \xi^2 + m^2 - \lambda & \omega \\ \omega & 1 - \lambda \end{bmatrix} = (1 - \lambda) \left( \xi^2 + m^2 - \lambda - \frac{\omega^2}{1 - \lambda} \right) = 0
\]
for some \( \xi \in \mathbb{R} \). Thus, either \( \lambda = 1 \), or, expressing \( \xi^2 \) in terms of \( \lambda \), there is the inequality
\[
\xi^2 = -m^2 + \lambda + \frac{\omega^2}{1 - \lambda} = \frac{\lambda^2 - (m^2 + 1)\lambda + m^2 - \omega^2}{\lambda - 1} \geq 0.
\]
The two roots \( \lambda = c^\pm(\omega) \) of the numerator satisfy
\[
0 < c^-(\omega) < 1 < c^+(\omega), \quad \omega \in (-m, m) \setminus \{0\}.
\]
(We note that when \( \omega = 0 \), one has \( c^-(0) = 1 \) if \( m^2 \geq 1 \) and \( c^+(0) = 1 \) if \( m^2 \leq 1 \).)

We conclude that
\[
\sigma_{\text{ess}}(H_\kappa(\omega)) = \left[ c^-(\omega), 1 \right] \cup \left[ c^+(\omega), +\infty \right).
\]

Let us now study the point spectrum of \( H_\kappa(\omega) \). First let us find out when \( \lambda = 1 \) is an eigenvalue of \( H_\kappa(\omega) \). One has to solve:
\[
\begin{bmatrix} L_\kappa(\omega) + \omega^2 & \omega \\ \omega & \omega \end{bmatrix} \begin{bmatrix} u_1 \\
u_2 \end{bmatrix} = \begin{bmatrix} 1 \\
u_2 \end{bmatrix}, \quad \begin{bmatrix} u_1 \\
u_2 \end{bmatrix} \in L^2(\mathbb{R}, \mathbb{C}^2),
\]
with \( u_1 \) from the domain of \( L_\kappa \) (see Remark 3). If \( \omega \neq 0 \), then from the second equation \( u_1 = 0 \), and then from the first equation also \( u_2 = 0 \); thus, \( \omega = 0 \). Then \( u_2 \in L^2(\mathbb{R}) \) is arbitrary, while \( u_1 \) is an \( L^2 \)-solution to \( L_\kappa(0)u_1 = u_1 \) (\( u_1 \) may be nontrivial: by Lemma 4.2, \( A_\kappa(0) \) equals 1 for some \( \kappa \in (-1/2, 0) \) if \( m^2 > 1 \)). It follows that eigenvalue \( \lambda = 1 \) is of infinite multiplicity.

Now we consider the case \( \lambda \neq 1 \). The operator
\[
H_\kappa(\omega) - \lambda I_2 = \begin{bmatrix} L_\kappa(\omega) + \omega^2 - \lambda & \omega \\ \omega & 1 - \lambda \end{bmatrix}
\]
has zero eigenvalue if and only if so does the Schur complement of \( 1 - \lambda \), which is given by
\[
L_\kappa(\omega) + \omega^2 - \lambda - \frac{\omega^2}{1 - \lambda} = L_\kappa(\omega) - \lambda - \frac{\lambda \omega^2}{1 - \lambda}.
\]
Thus, \( \lambda \neq 1 \) is an eigenvalue of \( H_\kappa(\omega) \) if and only if
\[
\Lambda = \lambda + \frac{\lambda \omega^2}{1 - \lambda}
\] (4.8)
is an eigenvalue of \( L_\kappa(\omega) \). Using (4.8) together with Lemma 4.2, one obtains the expressions (4.6) for \( \lambda_{\pm}^\kappa(\omega) \) (see also Figure 1).

\[
\begin{align*}
\sigma(L_\kappa(\omega)) & \quad \sigma(H_\kappa(\omega)) \\
\Lambda - \lambda_{-}^\kappa(\omega) & \quad \Lambda - \lambda_{+}^\kappa(\omega) \\
0 & \quad 0
\end{align*}
\]

**Figure 1.** The relation between \( \lambda \in \sigma(H_\kappa(\omega)) \) and \( \Lambda \in \sigma(L_\kappa(\omega)) \) for \( \omega \in (-m, m) \setminus \{0\}, \kappa > 0 \).

Let us point out that the eigenfunctions corresponding to the eigenvalues \( \lambda_{\pm}^\kappa(\omega) \neq 1 \) are given by (cf. (4.2))
\[
\begin{align*}
\psi_{\pm}(x) &= \begin{bmatrix} 1 - \lambda_{\pm}^\kappa(\omega) \\ -\omega \end{bmatrix} e^{-|x|\sqrt{m^2 - \omega^2} - \Lambda(\omega)}.
\end{align*}
\] (4.9)

Denote
\[
\begin{align*}
\Xi_1 &= \begin{bmatrix} 0 & 0 \\
\phi_{\omega}(x) & -\omega \phi_{\omega}(x) 
\end{bmatrix}, \\
\Xi_2 &= \begin{bmatrix} \phi_{\omega}(x) \\ -\omega \phi_{\omega}(x) \\
0 & 0
\end{bmatrix}.
\end{align*}
\] (4.10)

Due to (4.4) and (4.7),
\[
\ker(\mathbf{H}(\omega, \kappa)) = \begin{cases} \text{Span}\{\Xi_1\}, & \kappa \neq 0, \\
\text{Span}\{\Xi_1, \Xi_2\}, & \kappa = 0.
\end{cases}
\] (4.11)

To complete the proof of Theorem 4.1, it remains to note that \( \mathbf{G}_1 \mathbf{G}_2 \Xi_1 = -i \mathbf{J} \Phi_{\omega} \), \( \mathbf{G}_1 \mathbf{G}_2 \Xi_2 = \Phi_{\omega} \) with \( \Phi_{\omega}(x) \) from (2.14) (with \( \theta = 0 \)).

**Corollary 1.** The geometric multiplicity of eigenvalue \( \lambda = 0 \in \sigma_p(\mathbf{H}(\omega, \kappa)) \) equals 1 if \( \kappa \neq 0 \); it equals 2 if \( \kappa = 0 \).

The following theorem gives the orbital stability result (cf. Definition 1.1).

**Theorem 4.4.** Let \( a \in C^1(\mathbb{R}) \). Assume that there is an interval \( \mathcal{I} \subset (-m, m) \) such that for \( \omega \in \mathcal{I} \) there are bound states \( e^{-i\omega t} \phi_{\omega}(x) \) of equation (1.2), with
\[
\phi_{\omega} = \begin{bmatrix} \phi_{\omega}(x) \\ -i\omega \phi_{\omega}(x) \end{bmatrix}, \quad \phi_{\omega}(x) = Ce^{-|x|\sqrt{m^2 - \omega^2}}, \quad C = C(\omega) > 0, \quad \omega \in \mathcal{I},
\] (4.12)
with $C(\omega)$ a $C^1$-function of $\omega$ satisfying $a'(C^2) = 2\sqrt{m^2 - \omega^2} > 0$ (cf. Lemma 2.2). The $\phi_\omega$-orbit is stable if $\kappa < \frac{\omega^2}{m^2}$ and unstable if $\kappa > \frac{\omega^2}{m^2}$, where $\kappa = C^2a'(C^2)/a(C^2)$.

**Remark 4.** If there is a solitary wave (4.12) such that $\kappa \neq 0$, then $a'(C^2) \neq 0$, and, by Lemma 2.2, $C(\omega)$ is locally $C^1$, defined for $\omega$ from some interval $I \subset (-m, m)$.

**Remark 5.** The stability of solitary waves corresponding to the critical case $\kappa = \omega^2/m^2$ requires further analysis. In a particular case when $\omega$ is on the border of stability and instability regions, the solitary waves can be shown to be unstable following the approach of [5] (let us also mention the approach [15] developed for the proof of instability of the critical solitary waves of the Klein–Gordon equation in $\mathbb{R}^n$, $n \geq 2$, with pure power nonlinearity).

**Proof.** We follow the Grillakis–Shatah–Strauss theory [6]. Let us first consider the case $\kappa < 0$. In this case, by Theorem 4.1, the operator $\tilde{H}(\omega, \kappa)$ has simple eigenvalue $\lambda = 0$ with the corresponding eigenvector $\tilde{\Phi}_\omega$ and the rest of its spectrum is positive and bounded away from zero. Then all assumptions of [6, Theorem 1] are satisfied, and that theorem shows that in this case ($\kappa < 0$) the bound state $e^{i\omega t}\tilde{\Phi}_\omega$ is orbitally stable. Obviously, the stability of the $\Phi_\omega$-orbit of (2.9) implies the stability of the $\phi_\omega$-orbit of (1.2).

In the case $\kappa > 0$, Theorem 4.1 implies that $\tilde{H}(\omega, \kappa)$ has exactly one simple negative eigenvalue $\lambda^-_\omega(\omega)$ and has its kernel spanned by $T'(0)\Phi_\omega$, while the rest of its spectrum is positive and separated away from zero. Then all the assumptions of [6, Theorems 2,3] are satisfied, and, according to these theorems, the $\Phi_\omega$-orbit of (2.9) is stable (and consequently so is the $\phi_\omega$-orbit of (1.2)) if the function $d(\omega) = \mathcal{E}(\Phi_\omega) - \omega Q(\Phi_\omega)$ satisfies the condition $d''(\omega) > 0$, and is unstable if $d''(\omega) < 0$ (the case $d''(\omega) = 0$ requires additional analysis). Differentiating $d(\omega)$ and using (2.13) and (2.16), we derive:

$$d'(\omega) = -\mathcal{Q}(\Phi_\omega) = -\omega\|\phi_\omega\|^2, \quad d''(\omega) = -\partial_\omega(\omega\|\phi_\omega\|^2).$$

By (2.15), $\omega\|\phi_\omega\|^2 = \frac{\omega(C(\omega))^2}{\alpha}$, where $\alpha = \sqrt{m^2 - \omega^2}$. Taking into account (2.8), we derive:

$$\partial_\omega(\omega\|\phi_\omega\|^2) = \frac{C^2}{\alpha} + \omega\left(\frac{2C}{\alpha} \frac{dC}{d\omega} + \omega \frac{C^2}{\alpha^3}\right) = \frac{1}{\alpha^3} \left(m^2C^2 - \frac{\omega^2a(C^2)}{a'(C^2)}\right) = \frac{C^2}{\alpha^3} \left(m^2 - \frac{\omega^2}{\kappa}\right) < 0 \iff \kappa < \frac{\omega^2}{m^2},$$

where we used the definition (3.4) of $\kappa$ and the assumption $0 < \kappa$.

To treat the case $\kappa = 0$, we consider the linearization of equation (1.2) at the solitary wave (2.2) directly, substituting $\Psi(x, t) = e^{-i\omega t}([\phi(x) - \omega \phi(x)] + \zeta(x, t))$ with $\zeta(x, t) \in \mathbb{C}^2$ into (1.2). The linearized equation on $\zeta$,

$$\dot{\zeta}(x, t) = A(\omega)\zeta(x, t), \quad \zeta(x, t) \in \mathbb{C}^2,$$

is $\mathbb{C}$-linear, with the linearization operator $A(\omega)$ given by

$$A(\omega) = J\tilde{H}_0(\omega), \quad \text{where} \quad \tilde{H}_0(\omega) = \begin{bmatrix} L_0(\omega) + \omega^2 & -i\omega \\ i\omega & 1 \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Due to (4.5), the operator $\tilde{H}_0(\omega) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is similar to $H_0(\omega)$ and hence has the same spectrum. Namely, by Lemma 4.3,

$$\sigma_{ess}(\tilde{H}_0(\omega)) = \{c^-(\omega), 1\} \cup \{c^+(\omega), +\infty\}, \quad \sigma_p(\tilde{H}_0(\omega)) = \{0\} \cup \{\omega^2 + 1\},$$

where $c^-(\omega)$ and $c^+(\omega)$ are the lower and upper limits of $\{\omega^2 + 1, \omega^2 - 1\}$.
with eigenvalue $\lambda = 0$ being simple. Hence, [6, Theorem 1] applies, showing that the solitary wave solution $e^{-i\omega t}\phi_\omega(x)$ to equation (1.2) is orbitally stable.

The above considerations yield the orbital stability for all values $\kappa < \omega^2/m^2$. The instability of the $\phi_\omega$-orbit in the case $\kappa \in (\omega^2/m^2, +\infty)$ follows from [6, Theorem 3].

5. Spectral stability. Now we study the spectral stability (cf. Definition 1.2).

**Theorem 5.1.** Let $e^{-i\omega t}\phi_\omega(x)$ be a solitary wave with $\phi_\omega$ as in (4.12). The bound state $e^{-i\omega t}\phi_\omega(x)$ is spectrally stable if $\kappa \leq \omega^2/m^2$ and linearly unstable if $\kappa > \omega^2/m^2$.

**Proof.** We follow the arguments of Kolokolov [11] and Grillakis–Shatah–Strauss [6].

**Step 1.** Fixing $\lambda \in i\mathbb{R}$ with $|\lambda| \geq m - |\omega|$ and considering the Weyl sequences supported away from $x = 0$, we have:

$$\sigma_{\text{ess}}(A(\omega, \kappa)) = \mathbb{R} \setminus (-m + |\omega|, m - |\omega|))$$

We note that if $\lambda$ belongs to $\sigma_p(A(\omega, \kappa))$, then so do $\lambda, -\lambda$, and $-\lambda$, since the spectrum of $A(\omega, \kappa)$ is symmetric with respect to both $\mathbb{R}$ and $i\mathbb{R}$; indeed, $A(\omega, \kappa)$ has real coefficients, while, by (3.7),

$$A(\omega, \kappa)^* = (\Sigma H(\omega, \kappa))^* = -H(\omega, \kappa)\Sigma = -\Sigma^{-1}(\Sigma H(\omega, \kappa))\Sigma$$

is similar to $-\Sigma H(\omega, \kappa)$.

**Step 2.** Let us prove that

$$\sigma_p(A(\omega, \kappa)) \subset \mathbb{R} \cup i\mathbb{R}.$$ (5.1)

To achieve this, we consider the eigenvalue problem for

$$\hat{A}(\omega, \kappa) = G_2 G_1 A(\omega, \kappa) G_1^{-1} G_2^{-1}$$

(cf. (4.4)):

$$\hat{A}(\omega, \kappa) = \hat{\Sigma} H(\omega, \kappa) = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} H_\kappa(\omega) & 0 \\ 0 & H_0(\omega) \end{bmatrix},$$ (5.2)

where $\sigma_1$ is the first Pauli matrix and $\hat{\Sigma}$ and $\hat{H}(\omega, \kappa)$ are similar to $\Sigma$ and $H(\omega, \kappa)$:

$$\hat{\Sigma} = G_2 G_1 \Sigma G_1^{-1} G_2^{-1} = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \quad \hat{H}(\omega, \kappa) = G_2 G_1 H(\omega, \kappa) G_1^{-1} G_2^{-1}.$$ (5.3)

If $\kappa \leq 0$, then $\hat{H}(\omega, \kappa)$ is nonnegative and selfadjoint, hence one can extract the square root, which is also nonnegative and selfadjoint; therefore, since $\hat{H}^{1/2} \hat{\Sigma} \hat{H}^{1/2}$ is antiselfadjoint,

$$\sigma_d(\hat{\Sigma} \hat{H}) \setminus \{0\} = \sigma_d(\hat{H}^{1/2} \hat{\Sigma} \hat{H}^{1/2}) \setminus \{0\} \subset \mathbb{R}^+.$$ (5.4)

Now we concentrate on the case $\kappa > 0$. The equation $\hat{A}(\omega, \kappa)\psi = \lambda \psi$ reads:

$$\begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} H_\kappa(\omega) & 0 \\ 0 & H_0(\omega) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} u \\ v \end{bmatrix},$$ (5.5)

Eliminating $v$, we derive:

$$-\sigma_1 H_0(\omega) \sigma_1 H_\kappa(\omega) u = \lambda^2 u, \quad u \in L^2(\mathbb{R}, \mathbb{C}^2).$$ (5.6)

Let us first consider (5.4) in the case $\omega = 0$:

$$\begin{bmatrix} 0 & 1 \\ L_0(0) & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ L_\kappa(0) & 0 \end{bmatrix} u = - \begin{bmatrix} L_\kappa(0) & 0 \\ 0 & L_0(0) \end{bmatrix} u = \lambda^2 u.$$ (5.7)
By Lemma 4.2, one has:

$$\lambda^2 \in \sigma_p(L_0(0)) \cup \sigma_p(L_0(0)) = \begin{cases} \{0\}, & \kappa \leq -1/2, \\ \{0, -\Lambda_\kappa(0) = 4(\kappa + \kappa^2)m^2\}, & \kappa > -1/2. \end{cases} \tag{5.5}$$

This leads to the existence of eigenvalues $\lambda = \pm 2m\sqrt{\kappa + \kappa^2}$ for $\kappa > -1/2$, showing that in the case $\omega = 0$ there is linear instability ($\lambda > 0$) if and only if $\kappa > 0$.

Let us now consider the relation (5.4) in the case when $\omega \neq 0$. For $\lambda \neq 0$, one can see that $u$ is orthogonal to

$$\ker (\sigma_1 H_0(\omega) \sigma_1) = \ker \left( \begin{bmatrix} 1 & \omega \\ \omega & L_0 + \omega^2 \end{bmatrix} \right) = \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix} \tag{5.6}$$

(cf. (2.3)), hence we can write

$$H_\kappa(\omega) u = -\lambda^2 (\sigma_1 H_0(\omega) \sigma_1)^{-1} u, \tag{5.7}$$

where $(\sigma_1 H_0(\omega) \sigma_1)^{-1}$ is considered as an operator on $(\ker(\sigma_1 H_0(\omega) \sigma_1))^\perp$. Coupling (5.7) with $u$ and taking into account that $(u, (\sigma_1 H_0(\omega) \sigma_1)^{-1} u) > 0$, we conclude that $\lambda^2 \in \mathbb{R}$.

**Step 3.** To find out whether $-\lambda^2$ can be negative in the case $\kappa > 0$ and $\omega \neq 0$ (thus corresponding to linear instability), one considers the minimization problem

$$\mu = \inf \left\{ (u, H_\kappa(\omega) u), \quad (u, u) = 1, \quad u \in H^2(\mathbb{R}, \mathbb{C}), \right\}$$

$$u \in \left( \ker (\sigma_1 H_0(\omega) \sigma_1) \right) \perp \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}. \tag{5.8}$$

The solution $u$ to this minimization problem satisfies

$$H_\kappa(\omega) u = \mu u + \nu \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}, \tag{5.9}$$

with $\mu, \nu \in \mathbb{R}$ the Lagrange multipliers. We claim that if $\mu \leq 0$, then $\nu \neq 0$, or else one would have $\mu = \lambda_\kappa^-$ (the only negative eigenvalue of $H_\kappa(\omega)$), which is not possible since $u^-(x)$ from (4.9) corresponding to eigenvalue $\lambda_\kappa^-$ is not orthogonal to $\ker (\sigma_1 H_0(\omega) \sigma_1)$: one has

$$(u^-)^* \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix} = \begin{bmatrix} 1 - \lambda_\kappa^- \end{bmatrix} \left[ \begin{bmatrix} -\omega \\ 1 \end{bmatrix} e^{-|x|\sqrt{m^2 - \omega^2 - \lambda}} e^{-|x|} \right]$$

$$= - (2 - \lambda_\kappa^-) x e^{-|x|\sqrt{m^2 - \omega^2 - \lambda}} e^{-|x|}, \tag{5.10}$$

where $\omega \neq 0$ by assumption and $2 - \lambda_\kappa^-$ is strictly positive (since in the case $\kappa > 0$ and $\omega \neq 0$ the smaller root of (4.6) satisfies $\lambda_\kappa^- < 0$).

So, if $\mu \leq 0$, then $\nu \neq 0$. Then one concludes from (5.8) that $\mu > \lambda_\kappa^-$. We rewrite (5.9) as

$$(H_\kappa(\omega) - \mu) u = \nu \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}, \quad u = \nu (H_\kappa(\omega) - \mu)^{-1} \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}. \tag{5.11}$$

The sign of $\mu$ could be obtained from the condition that $u$ is orthogonal to $\ker (\sigma_1 H_0(\omega) \sigma_1)$:

$$\left( \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix}, (H_\kappa(\omega) - z)^{-1} \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \end{bmatrix} \right) = 0.$$
We consider
\[ h(z) = \left\langle \begin{bmatrix} -\omega \varphi_w \\ \varphi_w \end{bmatrix}, (H_\kappa(\omega) - z)^{-1} \begin{bmatrix} -\omega \varphi_w \\ \varphi_w \end{bmatrix} \right\rangle, \quad z \in (\lambda^-, c^-) \subset \rho(H_\kappa(\omega)). \]

Since \( h(z) \) is monotonically increasing on \( \rho(H_\kappa(\omega)) \cap \mathbb{R} \), the sign of \( \mu \) is opposite to the sign of \( h(0) \), which is given by
\[
 h(0) = \left\langle \begin{bmatrix} -\omega \varphi_w \\ \varphi_w \end{bmatrix}, H_\kappa(\omega)^{-1} \begin{bmatrix} -\omega \varphi_w \\ \varphi_w \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} -\omega \varphi_w \\ \varphi_w \end{bmatrix}, \begin{bmatrix} -\partial_\omega \varphi_w \\ \omega \partial_\omega \varphi_w + \varphi_w \end{bmatrix} \right\rangle = \partial_\omega (\omega \|\varphi_w\|^2). \tag{5.11} \]

Above, we used the relation
\[
 H_\kappa(\omega) \begin{bmatrix} -\partial_\omega \varphi_w \\ \omega \partial_\omega \varphi_w + \varphi_w \end{bmatrix} = \begin{bmatrix} L_\kappa + \omega^2 \\ \omega \\ 1 \end{bmatrix} \begin{bmatrix} -\partial_\omega \varphi_w \\ \omega \partial_\omega \varphi_w + \varphi_w \end{bmatrix} = \begin{bmatrix} -\omega \varphi_w \\ \varphi_w \end{bmatrix} \tag{5.12} \]

which in turn follows from taking the \( \omega \)-derivative of (2.3) (after we substitute \( \varphi_w(x) \)), which yields
\[
 L_\kappa \partial_\omega \varphi_w = 2\omega \varphi_w. \tag{5.13} \]

We conclude from (5.11) that there is \( \lambda^2 > 0 \) (hence, there is a pair of a positive and a negative eigenvalues) if and only if \( h(0) = \partial_\omega (\omega \|\varphi_w\|^2) = -\partial_\omega Q(\Phi_\omega) > 0 \), with \( Q(\Phi_\omega) \) given by (2.15). Hence, the solitary wave \( e^{-i\omega t} \partial_\omega \) from Lemma 2.2 is spectrally stable if and only if \( \partial_\omega Q(\Phi_\omega) \leq 0 \). The above is in agreement with the Kolokolov stability condition \( \partial_\omega Q(\Phi_\omega) < 0 \) derived in [11] in the context of the nonlinear Schrödinger equation.

In conclusion, we investigate the multiplicity of eigenvalue \( \lambda = 0 \) of the operator \( \mathbf{A}(\omega, \kappa) \).

**Lemma 5.2.** Let \( \omega \in (-m, m) \), \( \kappa \in \mathbb{R} \). The geometric multiplicity of eigenvalue \( \lambda = 0 \in \sigma_p(\mathbf{A}(\omega, \kappa)) \) equals 1 if \( \kappa \neq 0 \); it equals 2 if \( \kappa = 0 \).

The algebraic multiplicity of eigenvalue \( \lambda = 0 \) equals 2 if \( \kappa \neq \omega^2/m^2 \); it equals 4 if \( \kappa = \omega^2/m^2 \).

**Proof.** The statement of the geometric multiplicity follows from \( \dim \ker(\mathbf{A}(\omega, \kappa)) = \dim \ker(\mathbf{H}(\omega, \kappa)) \) and Corollary 1. Hence, it suffices to compute the algebraic multiplicity of zero eigenvalue \( \lambda = 0 \).

We first consider the case \( \omega \in (-m, m), \kappa \neq 0 \). By (4.11), \( \ker(\tilde{\mathbf{A}}(\omega, \kappa)) = \text{Span}\{\Xi_1\} \). Further, (5.12) and (4.10) imply
\[
 \tilde{\mathbf{A}}(\omega, \kappa) \begin{bmatrix} -\partial_\omega \varphi_w \\ \omega \partial_\omega \varphi_w + \varphi_w \\ 0 \\ 0 \end{bmatrix} = -i \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix} \begin{bmatrix} H_\kappa(\omega) \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -\partial_\omega \varphi_w \\ \omega \partial_\omega \varphi_w + \varphi_w \\ 0 \\ 0 \end{bmatrix} = -i \Xi_1;
\]

thus eigenvalue \( \lambda = 0 \) is of algebraic multiplicity at least two. To be able to extend this Jordan chain, solving
\[
 \tilde{\mathbf{A}}(\omega, \kappa) \Xi = \begin{bmatrix} -\partial_\omega \varphi_w \\ \omega \partial_\omega \varphi_w + \varphi_w \\ 0 \\ 0 \end{bmatrix}, \quad \Xi \in \mathbb{C}^4, \tag{5.14} \]
we need to make sure that the right-hand side is orthogonal to the kernel of the
adjoint of the operator in the left-hand side, which is given by
\[
\ker(\tilde{A}(\omega, \kappa)^*) = \ker(\tilde{H}(\omega, \kappa)\hat{\Sigma}) = \text{Span}\{\Sigma \Xi_1\} = \text{Span}\left\{\begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \\ 0 \\ 0 \end{bmatrix}\right\}.
\]
Thus, the condition to have a Jordan block of a larger size is
\[
0 = \begin{bmatrix} -\omega \varphi_\omega \\ \varphi_\omega \\ 0 \\ 0 \end{bmatrix}, \quad \partial_\omega (\omega\|\varphi_\omega\|^2) = 0,
\]
(5.15)
Since \(\kappa \neq 0\), (4.13) implies that \(\partial_\omega (\omega\|\varphi_\omega\|^2) = 0\) if and only if \(\kappa = \omega^2/m^2\). If \(\kappa \neq \omega^2/m^2\), the condition (5.15) is not satisfied and equation (5.14) has no \(L^2\)-
solutions. Hence the algebraic multiplicity of \(\lambda = 0\) is exactly two. If \(\kappa = \omega^2/m^2\),
then equation (5.14) has an \(L^2\)-solution; the algebraic multiplicity jumps by at least
two due to the symmetry of \(\sigma(A(\omega, \kappa))\) with respect to \(\mathbb{R}\) and \(i\mathbb{R}\).
In the case \(\kappa = 0\), one has \(\ker(\tilde{A}(\omega, \kappa)^*) = \text{Span}\{\Sigma \Xi_1, \Sigma \Xi_2\}\). Therefore, the
condition to have a Jordan block corresponding to \(\Xi_j\) for some \(j = 1, 2\) reads as follows:
\[
\langle \Xi_j, \tilde{\Sigma} \Xi_k \rangle = 0, \quad 1 \leq k \leq 2.
\]
Evidently, \(\langle \Xi_j, \tilde{\Sigma} \Xi_k \rangle = 0\) for all \(j \neq k\) and for any \(\omega \in (-m, m)\), while \(\langle \Xi_j, \tilde{\Sigma} \Xi_j \rangle = 2i\|\varphi_\omega\|^2 = 0\) if and only if \(\omega = 0\). Thus, if \(\kappa = 0\), the
eigenvalue \(\lambda = 0\) is of algebraic multiplicity larger than two if and only if \(\omega = 0\).
It remains to note that standard considerations (see e.g. [5]) show that for \(\kappa = \omega^2/m^2\), the algebraic multiplicity of zero eigenvalue cannot be larger than four.
We will not do this since this result follows from Lemma 6.5 below.

6. The spectrum of the linearization operator. Now we are going to perform
the detailed analysis of the spectrum of the linearization operator \(A(\omega, \kappa)\). We will
focus on the case \(\omega \neq 0\). For \(x \neq 0\), substituting \(\Psi \in \mathbb{C}\),
\(\text{Re} \nu \geq 0\), into the equation \((A(\omega, \kappa) - \lambda I)\Psi = 0\), we get
\[
\begin{bmatrix}
-\lambda & -\omega & 1 & 0 \\
\omega & -\lambda & 0 & 1 \\
\nu^2 - m^2 & 0 & -\lambda & -\omega \\
0 & \nu^2 - m^2 & \omega & -\lambda
\end{bmatrix}w = 0, \quad \nu \in \mathbb{C}, \quad w \in \mathbb{C}^4, \quad w \neq 0,
\]
(6.1)
which, via the Schur complement idea, is equivalent to
\[
\begin{bmatrix}
-\lambda & -\omega & 1 & 0 \\
\omega & -\lambda & 0 & 1 \\
S(\omega, \lambda, \nu) & 0 & 0 & 0
\end{bmatrix}w = 0, \quad w \in \mathbb{C}^4, \quad w \neq 0,
\]
(6.2)
with \(S(\omega, \lambda, \nu) \in \text{End}(\mathbb{C}^2)\) the Schur complement of the top right block \(I_2\):
\[
S(\omega, \lambda, \nu) = \begin{bmatrix}
\nu^2 - m^2 & 0 \\
0 & \nu^2 - m^2
\end{bmatrix} - \begin{bmatrix}
-\lambda & -\omega \\
\omega & -\lambda
\end{bmatrix}^2
= \begin{bmatrix}
\nu^2 - m^2 + \omega^2 - \lambda^2 & -2\lambda \omega \\
2\lambda \omega & \nu^2 - m^2 + \omega^2 - \lambda^2
\end{bmatrix}.
\]
(6.3)
The condition to have a nonzero solution \( w \in \mathbb{C}^4 \) to (6.2) is equivalent to

\[
\det S(\omega, \lambda, \nu) = (m^2 - \omega^2 + \lambda^2 - \nu^2)^2 + 4\lambda^2\omega^2 = 0.
\]

This gives \( \nu^2 = m^2 - (\omega + i\lambda)^2 \). We choose the cuts in the complex plane \( \lambda \) from the branching points to infinity:

\[
C_+ := (-i\infty, -i(m - \omega)] \cup [i(m + \omega), +i\infty), \quad C_- := (-i\infty, -i(m + \omega)] \cup [i(m - \omega), +i\infty),
\]

defining

\[
\nu_\pm(\omega, \lambda) = \sqrt{m^2 - (\omega \pm i\lambda)^2}, \quad (6.4)
\]

with

\[
\Re \nu_\pm(\omega, \lambda) > 0, \quad \lambda \in \mathbb{C} \setminus C_\pm. \quad (6.5)
\]

Eigenvalue zero of the Schur complement \( S \) from (6.3) corresponds to two eigenvectors \( u_\pm \in \mathbb{C}^2 \), depending on the choice \( \nu = \nu_\pm \); these eigenvectors are given by

\[
\left[ 2\lambda \omega 
\begin{array}{c}
-m^2 + \omega^2 + \nu^2 
\end{array}
\begin{array}{c}
-2\lambda \omega 
\end{array}
\right] = \left[ 2\lambda \omega 
\begin{array}{c}
-m^2 + \omega^2 + (m^2 - \omega^2 + \lambda^2 \mp 2i\lambda\omega) - \lambda^2 
\end{array}
\begin{array}{c}
2\lambda \omega 
\end{array}
\right] = 2\lambda \omega \left[ \frac{1}{1 \mp i} \right],
\]

so we can use \( u_\pm = [1 \pm i] \). By (6.2), the corresponding vector from the null space of \( A(\omega, \kappa) - \lambda I \) is then \( w_\pm = [u_\pm] \in \mathbb{C}^4 \), with

\[
u_\pm = 
\begin{bmatrix}
-\lambda \\
\omega \\
-\omega \\
-\lambda
\end{bmatrix} u_\pm = 
\begin{bmatrix}
\lambda + i\omega \\
-\omega + i\lambda
\end{bmatrix} \in \mathbb{C}^2.
\]

Thus, one has

\[
w_\pm = 
\begin{bmatrix}
1 \\
\pm i \\
\lambda + i\omega \\
\omega + i\lambda
\end{bmatrix} \in \mathbb{C}^4.
\]

Therefore, the eigenfunction corresponding to an eigenvalue \( \lambda \in \mathbb{C} \) is of the form

\[
\Psi(x, \omega, \lambda) = A w_+ e^{-\nu_+ |x|} + B w_- e^{-\nu_- |x|},
\]

with \( A, B \in \mathbb{C} \) not simultaneously zeros. The values of \( A \) and \( B \) are obtained from the jump conditions at \( x = 0 \); substituting \( \Psi \) into \( (A(\omega, \kappa) - \lambda)\Psi = 0 \) and collecting the terms with \( \delta \)-function gives:

\[
\begin{cases}
(-\nu_+ + (1 + 2\kappa)\alpha)A + (-\nu_- + (1 + 2\kappa)\alpha)B = 0, \\
(-\nu_+ + \alpha)iA + (-\nu_- + \alpha)(-iB) = 0.
\end{cases} \quad (6.7)
\]

The condition to have \( A, B \in \mathbb{C} \) not simultaneously zeros,

\[
\det
\begin{bmatrix}
-\nu_+ + (1 + 2\kappa)\alpha & -\nu_- + (1 + 2\kappa)\alpha \\
-\nu_+ + \alpha & -\nu_- + \alpha
\end{bmatrix} = 0,
\]

takes the form

\[
D_{\omega, \kappa}(\lambda) = 0, \quad (6.8)
\]

with

\[
D_{\omega, \kappa}(\lambda) := (1 + \kappa)^2\alpha^2 - (\nu_+ + \nu_-)(1 + \kappa)\alpha + \nu_+\nu_- - \kappa^2\alpha^2. \quad (6.9)
\]
The function $D_{\omega, \kappa}(\lambda)$ is analytic in $\lambda$ in $\mathbb{C} \setminus (\mathbb{C}_{-} \cup \mathbb{C}_{+})$. Since there are two possible values for each of the square roots in the definition (6.4) of $\nu_{\pm}$, the function $D_{\omega, \kappa}(\lambda)$ can be continued analytically through the cuts $\mathbb{C}_{-}$ and $\mathbb{C}_{+}$ to an analytic function on the four-sheet cover of $\mathbb{C}$, which we also denote by $D_{\omega, \kappa}(\lambda)$. We call the sheet defined by conditions (6.5) the physical sheet of $D_{\omega, \kappa}(\lambda)$.

6.1. Embedded eigenvalues and virtual levels. We start by studying embedded eigenvalues and virtual levels (also known as a threshold resonance) of the operator $A(\omega, \kappa)$.

**Definition 6.1.** ([2, 7, 14]) There are the virtual level at a threshold point $\mu$ of the essential spectrum of an operator $K$ (i.e. the endpoint of the essential spectrum or the point where the continuous spectrum changes its multiplicity) if the equation $K\psi = \mu\psi$ has a nonzero solution $\psi \in L^{\infty}(\mathbb{R}) \setminus L^{2}(\mathbb{R})$.

**Lemma 6.2.** 1. There are embedded eigenvalues $\lambda = \pm 2i\omega$ if and only if $\kappa = 0$ and $|\omega| \geq m/3$.
2. For $\kappa \in \mathbb{R}$ and $\omega \in (-m, m) \setminus \{0\}$ there are no virtual levels at the embedded thresholds $\lambda = \pm i(m + |\omega|)$.
3. There are virtual levels at $\lambda = \pm i(m - |\omega|)$ if and only if $\kappa \in \left[-\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ and $\omega = \mp \mathcal{T}_{\kappa}$, where

$$\mathcal{T}_{\kappa} = m \frac{(1 + 2\kappa)^{2}}{3 + 4\kappa}. \quad (6.10)$$

**Proof.** Because of the symmetry with respect to the sign of $\omega$ and $\lambda$, it suffices to consider the case $0 < \omega < m$, $\lambda = +i\mu$.

1. Let $\lambda = i\Lambda$ with $\Lambda \geq m - \omega$. Then $\nu_{-} = \sqrt{m^{2} - (\omega + \Lambda)^{2}} \in i\mathbb{R}$, and in the expression for the corresponding eigenfunction (6.6) one would need to take $B = 0$. There are two cases to consider:

- If $\kappa \neq 0$, the system (6.7) shows that $A = 0$; hence, $\lambda$ cannot be an eigenvalue.
- If $\kappa = 0$, the system (6.7) is satisfied for $B = 0$ and any $A \in \mathbb{C}$ if and only if $\Lambda = 2\omega$ (and hence $\nu_{+} = \alpha$). For $\Lambda \geq m - \omega$ one need $\omega \geq m/3$.

2. To find virtual levels, one needs to find for which $\omega$ and $\kappa$ the function $D_{\omega, \kappa}$ from (6.9) vanishes at one of the thresholds $\pm i(m \pm \omega)$. In the case $\omega > 0$ and $\lambda = i(m + \omega)$, one has $\nu_{+} = 0$ and $\nu_{-} = \sqrt{m^{2} - (m + 2\omega)^{2}} \in i\mathbb{R} \setminus \{0\}$. Thus, we have:

$$D_{\omega, \kappa}(i(m + \omega)) = (1 + 2\kappa)\alpha^{2} - \nu_{+}(1 + \kappa)\alpha \neq 0.$$  

3. Finally, let $\lambda = i(m - \omega)$. Then $\nu_{-} = 0$ and $\nu_{+} = 2\sqrt{\omega(m - \omega)} > 0$ by (6.5).

We need to solve:

$$0 = D_{\omega, \kappa}(i(m - \omega)) = (1 + 2\kappa)\alpha^{2} - \nu_{+}(1 + \kappa)\alpha.$$  

The equation has no solution for $\kappa = -1$. In the case $\kappa \neq -1$, it is equivalent to

$$\sqrt{m^{2} - \omega^{2}} \frac{1 + 2\kappa}{1 + \kappa} = 2\sqrt{\omega(m - \omega)},$$

which leads to

$$\omega = \mathcal{T}_{\kappa} = m \frac{(1 + 2\kappa)^{2}}{3 + 4\kappa} = m \left(\kappa + \frac{1}{4} + \frac{1}{4(3 + 4\kappa)}\right). \quad (6.11)$$

We note that $0 < \mathcal{T}_{\kappa} < m$ as long as $\kappa \in (-1/2, 1/\sqrt{2})$.

**Remark 6.** It is easy to check that in the case $\omega = 0$, there are virtual levels at the edges $\lambda = \pm im$ if and only if $\kappa = -1/2$.  

6.2. **Point spectrum.** Now we study the point spectrum of $A(\omega, \kappa)$. We recall that for all $\omega \in (-m, m)$ and $\kappa \in \mathbb{R}$, the operator $A(\omega, \kappa)$ always has eigenvalue $\lambda = 0$; for its algebraic and geometric multiplicities, see Lemma 5.2. Recall that the case $\omega = 0$ is covered in Theorem 5.1. Namely, (5.5) implies

$$\sigma_p(A(0, \kappa)) = \begin{cases} 0, & \kappa \leq -1/2; \\ \{0, \pm 2m\sqrt{\kappa(1 + \kappa)}\}, & \kappa > -1/2, \end{cases} \quad (6.12)$$

where the eigenvalues $\lambda = \pm 2m\sqrt{\kappa(1 + \kappa)}$ are simple for $\kappa \neq 0$: real if $\kappa > 0$ and purely imaginary if $-1/2 < \kappa < 0$.

Let us consider the case $\kappa = 0$.

**Lemma 6.3.**

$$\sigma_p(A(\omega, 0)) = \{0, \pm 2\omega i\}, \quad 0 < |\omega| < m.$$  

The eigenvalues $\lambda = \pm 2\omega i$ are simple (they are embedded into the continuous spectrum if $|\omega| \geq m/3$; see Lemma 6.2 -1).

**Proof.** Considering for which values $\lambda$ the matrix $A(\omega, 0) - \lambda I_4$ has eigenvalue zero reduces to studying this question for the Schur complement of the top right block $I_2$, 

$$S = \begin{bmatrix} -L_0(\omega) - \omega^2 & 0 \\ 0 & -L_0(\omega) - \omega^2 \end{bmatrix} - \begin{bmatrix} -\lambda & -\omega \\ \omega & -\lambda \end{bmatrix}^2 = \begin{bmatrix} -L_0(\omega) - \lambda^2 & -2\lambda \omega \\ 2\lambda \omega & -L_0(\omega) - \lambda^2 \end{bmatrix}.$$  

By Lemma 4.2, $L_0(\omega)$ has the only eigenvalue 0; we conclude that $\lambda$ can be obtained from the equation

$$\det \begin{bmatrix} -\lambda^2 & -2\lambda \omega \\ 2\lambda \omega & -\lambda^2 \end{bmatrix} = 0,$$

which gives $\lambda = 0$ and $\lambda = \pm 2\omega i$. \qed

Denote

$$K_\omega = \frac{\sqrt{m^2 - (2|\omega| - m)^2} - \alpha}{2\alpha - \sqrt{m^2 - (2|\omega| - m)^2}} = \frac{2\sqrt{|\omega| - 1}}{2 - 2\sqrt{|\omega|}}, \quad \omega \in (-m, m).$$

We note that

$$K_\omega \leq 0 \quad \text{for} \quad \omega \leq m/3, \quad \text{and} \quad K_\omega \geq 0 \quad \text{for} \quad \omega \geq m/3. \quad (6.13)$$

We note also that for $\omega \in [0, m)$ the function $\omega \mapsto K_\omega$ is the inverse to the function $\kappa \mapsto T_\kappa$ defined in (6.10). The next lemma gives conditions on $\omega \neq 0$ and $\kappa \neq 0$ ensuring the absence of a nonzero point spectrum.

**Lemma 6.4.** For $0 < |\omega| < m$ and $\kappa \in (-\infty, 0) \cup (0, K_\omega)$, one has

$$\sigma_p(A(\omega, \kappa)) = \{0\}.$$  

**Proof.** Because of the symmetry with respect to the sign of $\omega$, it suffices to consider $\omega \in (0, m)$. Due to Lemma 5.2, it suffices to prove that

$$D_{\omega, \kappa}(\lambda) \neq 0, \quad \lambda \in i\mathbb{R} \setminus \{0\}, \quad \kappa \in (-\infty, 0) \cup (0, K_\omega), \quad \omega \in (0, m), \quad (6.14)$$

with $D_{\omega, \kappa}(\lambda)$ defined in (6.9). Note that $\lambda = \pm 2\omega i$ are not roots of $D_{\omega, \kappa}(\lambda)$ for $\omega > 0$ and $\kappa \neq 0$. Indeed, for $\lambda = 2\omega i$ one has $\alpha - \nu_+ = 0$, and then

$$D_{\omega, \kappa}(2\omega i) = \alpha \kappa (\alpha - \nu_+) = \alpha \kappa \left(\sqrt{m^2 - \omega^2} - \sqrt{m^2 - 9\omega^2}\right) \neq 0;$$

$$D_{\omega, \kappa}(2\omega i) = \alpha \kappa (\alpha - \nu_+) = \alpha \kappa \left(\sqrt{m^2 - \omega^2} - \sqrt{m^2 - 9\omega^2}\right) \neq 0;$$
the case \( \lambda = -2\omega i \) is treated similarly. For \( \lambda \neq \{0, \pm 2\omega i\} \), we may rewrite \( D_{\omega,\kappa}(\lambda) \) as
\[
D_{\omega,\kappa}(\lambda) = (\alpha - \nu_+)(\alpha - \nu_-) \left( 1 + \kappa\alpha \left( \frac{1}{\alpha - \nu_+} + \frac{1}{\alpha - \nu_-} \right) \right). \tag{6.15}
\]

It remains to prove that for \( \omega \in (0, m) \) and \( \kappa \in (-\infty, 0) \cup (0, K_\omega) \), the equation
\[
1 + \kappa\alpha \left( \frac{1}{\alpha - \nu_+} + \frac{1}{\alpha - \nu_-} \right) = 0
\]
has no solutions \( \lambda \in \mathbb{R} \setminus \{0, \pm 2\omega i\} \). Denoting \( \Lambda = i\lambda \), we rewrite the above equation as
\[
1 + \kappa\alpha Q(\Lambda) = 0, \tag{6.16}
\]
where
\[
Q(\Lambda) = \frac{1}{\sqrt{m^2 - \omega^2 - \omega^2} - (\omega + \Lambda)^2} + \frac{1}{\sqrt{m^2 - \omega^2 - \omega^2} - (\omega - \Lambda)^2}
= \frac{1}{\Lambda + 2\omega} \left[ \frac{\alpha + \sqrt{\alpha^2 - \lambda^2 - 2\omega\Lambda}}{\Lambda + 2\omega} + \frac{\alpha + \sqrt{\alpha^2 - \lambda^2 + 2\omega\Lambda}}{\Lambda - 2\omega} \right]. \tag{6.17}
\]

Due to (6.13), it suffices to prove that there are no solution to (6.16) in the following domains:

1. \( 0 < \omega < m/3 \), \( \kappa < K_\omega < 0 \), \( 2\omega < \Lambda < m - \omega \);
2. \( 0 < \omega < m/3 \), \( \kappa < K_\omega < 0 \), \( 0 < \Lambda < 2\omega \);
3. \( m/3 < \omega < m \), \( 0 < \kappa < K_\omega \), \( 0 < \Lambda < m - \omega \).

(1) In the first domain,
\[
Q(\Lambda) > 0, \quad Q'(\Lambda) = -\sum_{\pm} \frac{\Lambda \pm \omega}{\sqrt{m^2 - (\Lambda \pm \omega)^2}} < 0.
\]
Hence,
\[
\inf_{(2\omega, m-\omega)} Q(\Lambda) = Q(m - \omega) = \frac{1}{\alpha} + \frac{1}{\sqrt{m^2 - (2\omega - m)^2}}
= \frac{1}{\alpha} \frac{2\sqrt{m^2 - \omega^2} - 2\sqrt{\omega(m - \omega)}}{\sqrt{m^2 - \omega^2} - 2\sqrt{\omega(m - \omega)}} = \frac{1}{\alpha} \frac{2\sqrt{m + \omega} - 2\sqrt{\omega}}{\sqrt{m + \omega} - 2\sqrt{\omega}}
= \frac{1}{\alpha} \frac{2 - 2\sqrt{\frac{\omega}{m + \omega}}}{1 - 2\sqrt{\frac{\omega}{m + \omega}}} = \frac{1}{\alpha} |K_\omega|^{-1} > 0, \tag{6.18}
\]
and \(|\kappa|\alpha Q(\Lambda) > |K_\omega|\alpha Q(m - \omega) = 1\). Therefore, (6.16) has no solutions in the first domain.

(2) In the second and third domains one has \( 0 < \Lambda < \min\{2\omega, m - \omega\} \). Hence, \(|(\Lambda \pm 2\omega)\Lambda| < \alpha^2\), and the following expansion holds:
\[
\sqrt{m^2 - (\omega + \Lambda)^2} = \sqrt{\alpha^2 - (\Lambda \pm 2\omega)^2} = \alpha \left( 1 - \frac{(\Lambda \pm 2\omega)\Lambda}{2\alpha^2} - \sum_{n=2}^{\infty} \frac{(2n)!((\Lambda \pm 2\omega)^n\Lambda^n)}{(2n-1)(n!)^2\alpha^{2n}} \right).
\]
The above leads to
\[
\frac{\alpha + \sqrt{m^2 - (\omega + \Lambda)^2}}{(\Lambda \pm 2\omega)\Lambda} = \frac{2\alpha}{(\Lambda \pm 2\omega)\Lambda} - \frac{1}{2\alpha} - \sum_{n=1}^{\infty} c_n((\Lambda \pm 2\omega)^n\Lambda^n),
\]
where \( c_n := \frac{(2n + 2)!}{(2n + 1)((2n + 1)!)^2 4^{n+1} 2^n} \). Hence, (6.17) leads to

\[
Q(\Lambda) = -\frac{1}{\alpha} - \frac{4\alpha}{4\omega^2 - \Lambda^2} \sum_{n=1}^{\infty} c_n \Lambda^n ((\Lambda + 2\omega)^n + (\Lambda - 2\omega)^n) < 0,
\]

(6.19)
since each term under the summation sign is positive. This implies that (6.16) has no solution in the second domain.

(3) In the third domain one has \( m - \omega < 2\omega \). Let us show that \( Q(\Lambda) \) decreases monotonically in the interval \((0, m - \omega)\). Indeed, (6.19) implies that

\[
Q'(\Lambda) = -\frac{8\Lambda\alpha}{(4\omega^2 - \Lambda^2)^2}
\]

\[-\sum_{n=1}^{\infty} n c_n \Lambda^{n-1} ((\Lambda + 2\omega)^n + (\Lambda - 2\omega)^n + \Lambda((\Lambda + 2\omega)^{n-1} + (\Lambda - 2\omega)^{n-1})) < 0.
\]

Hence, similarly to (6.18),

\[
\sup_{(0, m - \omega)} |Q(\Lambda)| = |Q(m - \omega)| = \frac{1}{\sqrt{m^2 - (2\omega - m)^2}} - \frac{1}{\alpha} = \frac{1}{\alpha K_\omega},
\]

and \( \kappa \alpha |Q(\Lambda)| < K_\omega \alpha |Q(m - \omega)| = 1 \). Therefore, (6.16) has no solutions in this domain. \(\square\)

**Lemma 6.5.** If \( \kappa = \omega^2/m^2 \), \( \omega \in (-m, m) \), then \( \lambda = 0 \) is a root of \( D_{\omega, \kappa}(\Lambda) \) of multiplicity 4.

**Proof.** In the case \( \omega = 0 \), the statement of the lemma follows from (6.12). Now let us consider the case \( \omega \in (-m, m) \setminus \{0\} \). By (6.15),

\[
D_{\omega, \kappa}(\Lambda) = (\alpha - \nu_+)(\alpha - \nu_-)(1 + \kappa \alpha Q(i\lambda)).
\]

(6.20)
The product of the first two factors in (6.20) yields

\[
(\alpha - \nu_+)(\alpha - \nu_-) = \left(\alpha - \sqrt{\alpha^2 + \lambda^2 - 2i\lambda\omega}\right)\left(\alpha - \sqrt{\alpha^2 + \lambda^2 + 2i\lambda\omega}\right)
\]

\[
= \frac{\lambda^2 \omega^2}{\alpha^2} + O(\lambda^3), \quad \lambda \to 0.
\]

At the same time, for the last factor in (6.20), the relation (6.19) leads to

\[
1 + \kappa \alpha Q(i\lambda) = 1 + \kappa \alpha \left(\frac{1}{\alpha} - \frac{\alpha}{\omega^2 (1 + \frac{\lambda^2}{4\omega^2})} + \frac{\lambda^2 \omega^2}{4\alpha^2\omega^2} + \frac{\lambda^2 \omega^2}{2\alpha^2\omega^2} + O(\lambda^4)\right)
\]

\[
= 1 - \frac{\kappa m^2}{\omega^2} + \frac{\lambda^2 \omega^2}{4\alpha^2\omega^2} + O(\lambda^4)
\]

\[
= \begin{cases} O(1), & \kappa \neq \omega^2/m^2, \\ O(\lambda^2), & \kappa = \omega^2/m^2, \quad \lambda \to 0. \end{cases}
\]

We note that the coefficient at \( \lambda^2 \) in the expression above is strictly positive. \(\square\)

Let us recall at this point that, due to Theorem 5.1, there are nonzero real eigenvalues for \( \kappa > \omega^2/m^2 \), while for \( \kappa < \omega^2/m^2 \) only purely imaginary nonzero eigenvalues are possible. In Appendix A, we show that in the vicinity of the curve \( \kappa = \omega^2/m^2 \) there are exactly two simple nonzero eigenvalues.

Finally, we collect the detailed description of the point spectrum of the operator \( A(\omega, \kappa) \) obtained above (See Figure 2):
Lemma 6.6. 1. For $\kappa = \omega^2/m^2$, one has $\sigma_p(A(\omega, \kappa)) = \{0\}$;
2. For $\omega \in (-m, m) \setminus \{0\}$, $\kappa = 0$, one has $\sigma_p(A(\omega, 0)) = \{0, \pm 2\omega i\}$ (we note that $\pm 2\omega i \in \sigma_{res}(A(\omega, \kappa))$ if $|\omega| \geq m/3$);
3. For $\kappa > \omega^2/m^2$, one has $\sigma_p(A(\omega, \kappa)) = \{0, \pm \lambda\}$ with some $\lambda > 0$;
4. For $\kappa \in (K_\omega, \omega^2/m^2)$, one has $\sigma_p(A(\omega, \kappa)) = \{0, \pm \lambda, \Lambda\}$, $\Lambda > 0$;
5. For $\kappa \leq K_\omega$, $\kappa \neq 0$, one has $\sigma_p(A(\omega, \kappa)) = \{0\}$.
Moreover, eigenvalue $\lambda = 0$ is of algebraic multiplicity 4 if $\kappa = \omega^2/m^2$ and of algebraic multiplicity 2 otherwise. All nonzero eigenvalues of $A(\omega, \kappa)$, $\omega \in (-m, m)$, $\kappa \in \mathbb{R}$, are simple.

Remark 7. The parabola $\kappa = \omega^2/m^2$ on Figure 2 corresponds to the collision of two simple eigenvalues at $\lambda = 0$. For $\kappa \neq \omega^2/m^2$, the two nonzero eigenvalues move away from the origin, becoming purely imaginary for $\kappa < \omega^2/m^2$ and purely real for $\kappa > \omega^2/m^2$. These two purely imaginary eigenvalues hit the threshold points $\pm i(m - |\omega|)$, at $\omega = \pm T_\kappa$, becoming virtual levels. The condition for $T_\kappa$ to be smaller than $m$ (so that the root of $D_{\omega, \kappa}(\lambda)$ indeed arrives at $i(m - |\omega|)$ when $\omega = \pm T_\kappa$) is $(1 + 2\kappa)^2 < 3 + 4\kappa$, which gives the requirement $\kappa \in (-\frac{1}{2}, \frac{3}{4})$. When $\omega = \pm T_\kappa$ (equivalently, when $\kappa = K_\omega$, $\kappa \neq 0$), the nonzero roots of $D_{\omega, \kappa}(\lambda)$ touch the thresholds $\pm (m - |\omega|)$; when $|\omega| > T_\kappa$, these roots move onto the unphysical sheet of the Riemann surface of $D_{\omega, \kappa}(\lambda)$. We also recall that in the case $\kappa = 0$, $|\omega| \geq \frac{m}{\sqrt{2}}$ the point spectrum contains embedded eigenvalues $\pm 2\omega i$ by Lemma 6.2-1.

Appendix A. Reducing $D_{\omega, \kappa}(\lambda) = 0$ to a cubic with explicit solution. For the completeness, we prove that the function $D_{\omega, \kappa}(\lambda)$ (see (6.9)) with $\kappa \approx \omega^2/m^2$, $\omega \neq 0$, $\kappa \neq \omega^2/m^2$, has exactly three roots: root $\lambda = 0$ of multiplicity 2 and two simple roots $\lambda_{\pm} \in \mathbb{R} \cup i\mathbb{R}$; if $\omega \neq 0$, $\kappa = \omega^2/m^2$, then $\lambda = 0$ is a root of multiplicity 4.

Denote $\Sigma = \nu_+ + \nu_-$, with $\nu_{\pm} = \nu_{\pm}(\omega, \lambda)$ from (6.4); one has

$$\nu_+\nu_- = \frac{\Sigma^2 - \nu_+^2 - \nu_-^2}{2} = \frac{\Sigma^2 - 2m^2 + 2\omega^2 - 2\lambda^2}{2} = \frac{1}{2} \Sigma^2 - \lambda^2 - \alpha^2. \quad (A.1)$$
Then the equation \( D_{\omega,\kappa}(\lambda) = 0 \) (see (6.8)) takes the form
\[
(1 + \kappa)^2 \alpha^2 - (1 + \kappa)\alpha \Sigma + \frac{1}{2} \Sigma^2 - \lambda^2 - \alpha^2 - \kappa^2 \alpha^2 = 0,
\]
which we rewrite as
\[
\Sigma^2 - 2(1 + \kappa)\alpha \Sigma + 2(2\kappa\alpha^2 - \lambda^2) = 0. \tag{A.2}
\]
Equation (A.2) yields:
\[
\Sigma = (1 + \kappa)\alpha \pm \sqrt{(1 - \kappa)^2 \alpha^2 + 2\lambda^2}. \tag{A.3}
\]
Using (A.1), (A.2), and (A.3), we derive the relation
\[
\nu_+\nu_- = \frac{1}{2} \Sigma^2 - \lambda^2 - \alpha^2 = (1 + \kappa)\alpha \Sigma + \kappa^2 \alpha^2 - (1 + \kappa)^2 \alpha^2 = (1 + \kappa)\alpha((1 + \kappa)\alpha \pm \sqrt{(1 - \kappa)^2 \alpha^2 + 2\lambda^2}) + \kappa^2 \alpha^2 - (1 + \kappa)^2 \alpha^2 = \kappa^2 \alpha^2 \pm (1 + \kappa)\alpha \sqrt{(1 - \kappa)^2 \alpha^2 + 2\lambda^2},
\]
so
\[
\nu_+^2\nu_-^2 = \alpha^4 \kappa^4 + (1 + \kappa)^2 \alpha^2 ((1 - \kappa)^2 \alpha^2 + 2\lambda^2) \pm 2(1 + \kappa)\alpha^3 \kappa \sqrt{(1 - \kappa)^2 \alpha^2 + 2\lambda^2}. \tag{A.4}
\]
On the other side, the definition (6.4) of \( \nu_\pm \) implies that
\[
\nu_+^2\nu_-^2 = ((\omega + i\lambda)^2 - m^2)((\omega - i\lambda)^2 - m^2) = ((\omega - m)^2 + \lambda^2)((\omega + m)^2 + \lambda^2) = (\omega^2 - m^2)^2 + \lambda^2 + 2(\omega^2 + m^2)\lambda^2 = \alpha^4 + \lambda^4 + 4\lambda^2 m^2 - 2\lambda^2 \alpha^2. \tag{A.5}
\]
From (A.4) and (A.5), simplifying and denoting \( x = \lambda^2 \), we obtain:
\[
x^2 + 4xm^2 - 4x\alpha^2 - 2\alpha^4 \kappa^4 + 2\alpha^4 \kappa^2 - 2x\kappa^2 \alpha^2 - 4x\kappa\alpha^2 = \pm 2\alpha^3 (1 + \kappa)\kappa \sqrt{(1 - \kappa)^2 \alpha^2 + 2x}. \tag{A.6}
\]
Defining
\[
c(\omega, \kappa) = 4m^2 - 4\alpha^2 - 4\kappa \alpha^2 - 2\kappa^2 \alpha^2 \tag{A.7}
\]
and squaring (A.6) yields
\[
(x^2 + cx + 2(1 - \kappa^2)\kappa^2 \alpha^4)^2 = 4(1 - \kappa^2)^2 \kappa^4 \alpha^8 + 8(1 + \kappa)^2 \kappa^4 \alpha^6 x.
\]
There is a root \( x = 0 \). To find nonzero roots, we simplify the above and cancel \( x \), arriving at
\[
x^3 + 2cx^2 + x \left[c^2 + 4(1 - \kappa^2)\kappa^2 \alpha^4 \right] + 4c(1 - \kappa^2)\kappa^2 \alpha^4 - 8(1 + \kappa)^2 \kappa^4 \alpha^6 = 0. \tag{A.8}
\]
Writing
\[
x = y - \frac{2c}{3}, \tag{A.9}
\]
we reduce the cubic equation (A.8) to the form
\[
y^3 + py + q = 0 \tag{A.10}
\]
with
\[
p = -\frac{c^2}{3} + 4(1 - \kappa^2)\kappa^2 \alpha^4, \quad q = -\frac{2c^3}{27} + \frac{4c(1 - \kappa^2)\kappa^2 \alpha^4}{3} - 8(1 + \kappa)^2 \kappa^4 \alpha^6. \tag{A.11}
\]
If the discriminant
\[
\Delta(\omega, \kappa) = -4p^3 - 27q^2, \tag{A.12}
\]
with \( c = c(\omega, \kappa) \) from (A.7), is negative, then there is exactly one real root \( y \in \mathbb{R} \) of equation (A.10) (and two complex conjugate roots with nonzero imaginary parts).

The real root is given by

\[
y_1 = \left( -\frac{q}{2} + \frac{(-\Delta)^{1/2}}{108} \right)^{1/3} + \left( -\frac{q}{2} - \frac{(-\Delta)^{1/2}}{108} \right)^{1/3},
\]

and then (A.9) yields

\[
x_1 = \left( -\frac{q}{2} + \frac{(-\Delta)^{1/2}}{108} \right)^{1/3} + \left( -\frac{q}{2} - \frac{(-\Delta)^{1/2}}{108} \right)^{1/3} - \frac{2c}{3}.
\]

Let us mention that roots of (A.10) with nonzero imaginary part correspond to roots \( \lambda = \pm \sqrt[3]{y - 2c/3} \) of \( D_{\omega, \kappa}(\lambda) \) with nonzero real and imaginary parts. By (5.1), these roots do not correspond to eigenvalues of \( A(\omega, \kappa) \).

We are going to show that the discriminant \( \Delta(\omega, \kappa) \) is negative for \( \kappa \approx \omega^2/m^2 \), \( \omega \neq 0 \), so that there is exactly one real solution to (A.10) and therefore exactly one pair of roots to \( D_{\omega, \kappa}(\lambda) = 0 \) which are located on the Riemann sheet corresponding to eigenvalues of \( A(\omega, \kappa) \) (that is, when the real parts of \( \nu_+ \) and \( \nu_- \) from (6.4) are positive).

**Lemma A.1.** Let \( \omega \in (m, m) \setminus \{0\} \). If \( \kappa \in (0, 1) \) is sufficiently close to \( \omega^2/m^2 \), then \( \Delta(\omega, \kappa) < 0 \).

**Proof.** Due to the continuity of \( \Delta(\omega, \kappa) \) with respect to \( \kappa \), it suffices to prove that

\[
\Delta(\omega, \kappa) < 0 \quad \text{for} \quad \kappa \approx \omega^2/m^2 > 0.
\]

For convenience, we put \( m = 1 \) and denote \( u = 1 + \kappa > 0, \ v = 1 - \kappa > 0 \). Then, using (A.7), we have:

\[
c(\omega, \kappa) = 4 - 4(1 - \kappa) - 4(1 - \kappa)\kappa - 2(1 + \kappa)\kappa^2 = 2(1 + \kappa)\kappa^2 = 2\kappa^2u,
\]

\[
4(1 - \kappa^2)\kappa^2\alpha^4 = 4(1 - \kappa^2)(1 - \kappa^2)\kappa^2 = 4\kappa^2uv^3,
\]

hence (A.11) leads to

\[
p = \frac{4}{3} \kappa^2(-\kappa^2u^2 + 3uv^3), \quad q = -\frac{16}{27} \kappa^3(\kappa^3u^3 + 9\kappa u^2v^3),
\]

and (A.12) implies that for \( \kappa = \omega^2/m^2 \) one has

\[
\Delta(\omega, \kappa) = \frac{512}{27} \kappa^6\left( (\kappa^2u^2 - 3uv^3)^3 - (\kappa^3u^3 + 9\kappa u^2v^3)^2 \right)
\]

\[
= -27(\kappa^4u^5v^6 + u^3v^9 + 3\kappa^2u^4v^6) < 0.
\]

**REFERENCES**

[1] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*, 2nd edition, American Mathematical Society, Providence, RI, 2005.

[2] N. Boussaïd and A. Comech, Virtual levels and virtual states of linear operators in Banach spaces. Applications to Schrödinger operators, preprint, arXiv:2101.11979.

[3] V. Buslaev, A. Comech, E. Kopylova and D. Stuart, On asymptotic stability of solitary waves in Schrödinger equation coupled to nonlinear oscillator, *Commun. Partial Differ. Equa.*, 33 (2008), 669–705.

[4] E. Csoógyi, F. Genoud, M. Ohta, and J. Royer, Stability of standing waves for a nonlinear Klein–Gordon equation with delta potentials, *J. Differ. Equ.*, 268 (2019), 353–388.

[5] A. Comech and D. Pelinovsky, Purely nonlinear instability of standing waves with minimal energy, *Commun. Pure Appl. Math.*, 56 (2003), 1565–1607.

[6] M. Grillakis, J. Shata and W. Strauss, Stability theory of solitary waves in the presence of symmetry, *J. Funct. Anal.*, 74 (1987), 160–197.
[7] A. Jensen and T. Kato, Spectral properties of Schrödinger operators and time-decay of the wave functions, *Duke Math. J.*, 46 (1979), 583–611.

[8] A. Komech and A. Komech, Global attractor for a nonlinear oscillator coupled to the Klein–Gordon field, *Arch. Ration. Mech. Anal.*, 185 (2007), 105–142.

[9] A. Komech and A. Komech, Global attraction to solitary waves for Klein–Gordon equation with mean field interaction, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26 (2009), 855–868.

[10] A. Komech, E. Kopylova and D. Stuart, On asymptotic stability of solitons in a nonlinear Schrödinger equation, *Commun. Pure Appl. Anal.*, 11 (2012), 1063–1079.

[11] A. Kolokolov, Stability of the dominant mode of the nonlinear wave equation in a cubic medium, *J. Appl. Mech. Tech. Phys.*, 14 (1973), 426–428.

[12] E. Kopylova, On the asymptotic stability of solitary waves in the discrete Schrödinger equation coupled to a nonlinear oscillator, *Nonlinear Analysis: Theory, Methods & Applications*, 71 (2009), 3031–3046.

[13] E. Kopylova, On asymptotic stability of solitary waves in discrete Klein–Gordon equation coupled to a nonlinear oscillator, *Appl. Anal.*, 89 (2010), 1467–1492.

[14] M. Murata, Asymptotic expansions in time for solutions of Schrödinger-type equations, *J. Funct. Anal.*, 49 (1982), 10–56.

[15] M. Ohta and G. Todorova, Strong instability of standing waves for the nonlinear Klein–Gordon equation and the Klein–Gordon–Zakharov system, *SIAM J. Math. Anal.*, 38 (2007), 1912–1931.

Received August 2020; revised February 2021.

*E-mail address: comech@gmail.com*

*E-mail address: elena.kopylova@univie.ac.at*