Nonholonomic Jet Deformations and Exact Solutions for Modified Ricci Soliton and Einstein Equations

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Abstract

Let $g$ be a pseudo–Riemannian metric of arbitrary signature on a manifold $V$ with conventional $n + n$ dimensional splitting, $n \geq 2$, determined by a nonholonomic (non–integrable) distribution $\mathcal{N}$ defining a generalized (nonlinear) connection and associated nonholonomic frame structures. We shall work with a correspondingly adapted linear metric compatible connection $\hat{D}$ and its nonzero torsion $\hat{T}$, both completely determined by $g$. Our first goal is to prove that there are certain generalized frame and/or jet transforms and prolongations with $(g, V) \rightarrow (\hat{g}, \hat{V})$ into explicit classes of solutions of some generalized Einstein equations $\hat{R}_{ic} = \Lambda \hat{g}$, $\Lambda = \text{const}$, encoding various types of (nonholonomic) Ricci soliton configurations and/or jet variables and symmetries. The second goal is to solve additional constraint equations for zero torsion, $\hat{T} = 0$, on generalized solutions constructed in explicit forms with jet variables and extract Levi–Civita configurations. This allows us to find generic off–diagonal exact solutions depending on all space time coordinates on $V$ via generating and integration functions and various classes of constant jet parameters and associated symmetries. We shall study (third goal) how such generalized metrics and connections can be related by so–called "half-conformal" and/or jet deformations of certain sub–classes of solutions with one, or two, Killing symmetries. Finally, there are considered some examples when exact solutions are constructed as nonholonomic jet prolongations of the Kerr metrics, with possible Ricci soliton deformations, and characterized by nonholonomic jet structures and generalized connections.

Keywords: Nonholonomic manifolds and jets, generalized connections, geometric methods and PDE, Ricci solitons, Einstein manifolds, modified gravity, exact solutions and mathematical relativity.

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1 Introduction

Various results and methods of the theory of nonholonomic manifolds, jets and connections can be combined and applied in order study symmetries of systems of nonlinear partial differential equations, PDEs, and constructing exact and approximate solutions. In modern physics, such fundamental field and evolution equations are related to the Ricci soliton geometry, mathematical relativity, particle physics and geometric mechanics [36, 38, 39, 40]. For instance, a jet space techniques was elaborated to analyze special features of the vacuum Einstein equations in general relativity, GR, which allows to define certain generalized symmetries and conservation laws. In a more general context, a Lagrangian formalism was elaborated on jet–gauge and jet–diffeomorphism
groups with the aim to unify gravity with internal gauge symmetries [1]. Another direction related to Finsler–Lagrange geometry and nonholonomic mechanics was considered by authors of papers [31, 32, 33, 34], when certain generalizations of Einstein equations were formulated on jet spaces endowed with nonlinear connection structure.

In recent years, a series of works has been devoted to elaborating geometric methods which allow to decouple (modified) Einstein equations for certain "auxiliary" connections and with respect to correspondingly adapted nonholonomic frames, and construct generic off–diagonal solutions depending on all spacetime coordinates, see reviews of results in [40, 43, 45]. Following the so–called anholonomic frame deformation method, AFDM, the solutions are generated in explicit forms via formulas determined by generating and integration functions and various commutative and noncommutative parameters. Such solutions can be with Killing, non–Killing solitonic and/or other type symmetries, which for respective boundary / initial / source conditions can be with nontrivial spacetime topology, may describe evolution and/or dynamical processes, or result in stochastic behaviour etc.

We can extract Levi–Civita configurations with zero torsion if we impose additional nonholonomic constraints on certain classes of generalized solutions. It should be noted that because such systems are nonlinear it is important to consider such restrictions via integration / generation functions and constants, symmetry / boundary / initial conditions "at the end", on some defined integral varieties. Prescribing from the very beginning only some special ansatz for metrics and connections which may simplify a system of equations (for instance, to transform it into a nonlinear system of ordinary differential equations), we can not decouple the PDEs in general forms and this cuts the bulk of nonlinear off–diagonal multi-variables.

The present work is aimed at analysing the basic properties of nonholonomic Ricci soliton and (modified) Einstein equations when certain class of metrics and (generalized) connections are generated by jet prolongations into certain classes of exact solutions of equations prolonged on jet spaces. We shall study also the constraints when certain general classes with generalized jet variables and symmetries are transformed into standard Einstein metrics with jet parametric dependence of generic off–diagonal metrics. Readers are referred to monographs [36, 23] on main results on jets and jet bundle geometry. The literature on nonholonomic jet manifolds and bundles is less popular and more sophisticated than that on holonomic jets. Experts on mathematical relativity and PDEs are less familiar with the geometry of nonholonomic manifolds elaborated in Vrˇ anceanu–Horak approach [17, 48, 49, 22, see recent results and applications in [6, 44]. We cite here some important works on generalized connections developed by different schools of differential geometry on nonholonomic jets, quasi-gets and theory of higher order connections, see [12, 34, 37, 10, 46, 38] and we shall sketch a few essential notions and necessary results using recent approaches formulated in Refs. [20, 27].

In this work, we follow three explicit goals motivated and stated in section 2.4.5. The first one is to develop the anholonomic frame deformation methods, (AFDM, see reviews of results in [10, 33, 15]) in such a form which allows us to decouple the nonholonomic $r$–jet deformations of the Ricci soliton, and Einstein equations, and integrate such equations for general classes of generic off–diagonal metric and nonlinear connection structures. The second goal is to extract from such extra dimensional jet configurations the Levi–Civita ones (in particular, physically important solutions in GR with jet parameters) by solving nonholonomic constraints for zero torsion conditions. Finally (third goal), we shall analyse explicit examples of exact solutions depending on jet parameters defining nonholonomic deformations of black hole solutions and gravitational solitonic waves. We study how nonholonomic and/or $r$–jet deformations of the Kerr metric may model physical effects of Ricci solitons, in massive gravity and other modified gravity models [3, 31, 32, 34, 35, 19, 20, 24, 5, 28, 15].

The article has the following structure. In section 2 we recall basic facts and definitions concerning nonholonomic manifolds and jets and elaborate on the concept of generalized connection structures. We provide an introduction into the geometry of nonholonomic manifolds and bundles endowed with nonlinear connection structure. There are outlined main results and stated respective denotations on nonholonomic maps and jets of (non) holonomic manifolds. The formalism of nonholonomic $r$–jet prolongation of Ricci soliton and (generalized/modified) equations is elaborated in details.

In section 3 we formulate and prove the main theorems on decoupling and integration of (modified) Ricci soliton and Einstein equations. The approach consists a generalization of results for nonholonomic jet prolongations of fundamental geometric and physical objects in generalized/ modified gravity theories and further developments of AFDM. The key idea is to consider nonholonomic $2+2+2+\ldots$ splitting with two dimensional

\footnote{which can not be diagonalized by coordinate transforms in a finite spacetime region, for instance, in GR}
gravitational polarizations. The main results are related to Ricci soliton modifications and modified gravity contributions and mimicking massive gravity terms with effective cosmological constant and symmetries and with Killing and non-Killing symmetries, deformations by Ricci soliton configurations, jet distributions. (2-d) shells of jet coordinates and adapting the geometric constructions to such nonholonomic spacetime and jet distributions.

Section 4 is devoted to explicit examples of exact solutions depending on jet coordinates, jet parameters and symmetries and with Killing and non-Killing symmetries, deformations by Ricci soliton configurations, modified gravity contributions and mimicking massive gravity terms with effective cosmological constant and gravitational polarizations. The main results are related to Ricci soliton modifications and \( r \)-jet prolongations of the Kerr metric which play a substantial role in the physics of black holes. Such black hole metrics can be extended in generic off–diagonal forms for various classes of modified gravity theories and extra dimensions \[10, 43, 15, 13\]. Even the AFDM allows us to construct very general integral varieties for such gravitational and geometric evolution like nonlinear systems of PDEs. For \( r \)-jet configurations, it is clear that new classes of gravitational and matter field equations at least possess certain jet type local symmetries and possible associate nonlinear gauge interior degrees of freedom. We show that such solutions can be constructed both with zero or non–zero canonical torsions, with possible rotoid symmetries for Kerr – de Sitter configurations and other classes of vacuum and non-vacuum jet prolongations.

In Appendix, we summarize some most important and necessary \( N \)-adapted coefficient formulas and provide technical details for some theorems.

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2 Nonholonomic Manifolds and Jet Bundles and Generalized Connections

We start by recalling a few basic definitions on the geometry of nonholonomic manifolds and bundles, related jet spaces and theory of generalized (nonlinear) connections \[36, 23, 26, 24\]. The geometric approach is generalized in a form to unify both the concepts of nonholonomic manifolds \[47, 48, 49, 22, 6, 44\] and that of classes of vacuum and non-vacuum jet prolongations. We shall work in the category of \( n + m \) dimensional nonholonomic manifolds \( \mathcal{V} \), with \( n, m \geq 2 \), of necessary smooth class (for instance, of class \( C^\infty \)), Hausdorff, finite dimensional and without boundaries. The solutions of certain systems of nonlinear partial differential equations (PDE) can be topologically nontrivial, with singularities and various type Killing and non–Killing symmetries. Such PDEs, nonholonomic constraint\[4\] and their solutions are for geometric models of (modified) gravity theories and Ricci soliton equations defined as certain stationary configurations in a nonholonomic geometric evolution system, with possible Wick rotations (for small deformations) and frame transforms between Lorentzian and Euclidean signatures of metrics.

2.1 Holonomic jets

Jets are certain equivalence classes of smooth maps between two manifolds \( M, \dim M = n \), and \( Q, \dim Q = m \), when maps are represented by Taylor polynomials. One writes this as \( f, g : M \to Q \) and say that a \( r \)-jet is determined at a point \( u \in M \) if there is a \( r \)-th order contact at \( u \). The idea is formalized mathematically using the concept the \( r \)-th order contact of two curves on a manifold.

**Definition 2.1 -Lemma:** Two curves \( \gamma, \delta : \mathbb{R} \to V \) have the \( r \)-th contact at zero if for every smooth function \( \varphi \) on \( M \) the difference \( \varphi \circ \gamma - \varphi \circ \delta \) vanishes to \( r \)-th order at \( 0 \in \mathbb{R} \). In this case, we have an equivalence relation \( \gamma \sim_r \delta \) when \( r = 0 \) means \( \gamma(0) = \delta(0) \). If \( \gamma \sim_r \delta \), then \( f \circ \gamma \sim_r f \circ \delta \) for every map \( f : b \to Q \).

Two maps \( f, g : V \to Q \) are said to determine the same \( r \)-jet at \( x \in M \), if for every curve \( \gamma : \mathbb{R} \to V \) with \( \gamma(0) = x \) the curves \( f \circ \gamma \) and \( g \circ \gamma \) have the \( r \)-th order contact at zero. In such a case, we write \( j^r_x f = j^r_x g \), or \( j^r f(x) = j^r g(x) \). An equivalence class of this relation is called an \( r \)-jets of \( M \) into \( Q \).

**Definition 2.2** The set of all \( r \)-jets of \( M \) into \( Q \) is denoted by \( J^r(M, Q) \); for an element \( X = j^r_x f \in J^r(M, Q) \), the point \( x := \alpha X \) is the source of \( X \) and the point \( f(x) := \beta X \) is the target of \( X \).

\( ^2 \)equivalently, anholonomic, i.e. non–integrable
One denotes by $\pi^r_s, 0 \leq s \leq r$ the projection $j^s r f \to j^s f$ of $r$-jets into $s$-jets. All $r$-jets form a category, the units of which are the $r$-jets of the identity maps of manifolds.

By $J^r_x(M,Q)$, or $J^r_x(M,Q)_y$ we mean the set of all $r$-jets of $x$ onto $Q$ with source $x \in M$, or tangent $y \in Q$, respectively, and we write

$$J^r_x(M,Q)_y = J^r_x(M,Q) \cap J^r_x(M,Q)_y \quad \text{and} \quad L^r_{n,m} = J^r_0(\mathbb{R}^n, \mathbb{R}^m)$$

In local coordinates $x^i$, we write

$$\partial_i f := \frac{\partial^{[i} f}{(\partial x^1)^{i_1} \cdots (\partial x^n)^{i_n}}$$

is the partial derivative of a function $f : U \subset \mathbb{R}^n \to \mathbb{R}$, with a multi-index $i$ of range $n$, which is a $m$-tuple $i = (i_1, \ldots, i_n)$ of non-negative integers. We write $|i| = i_1 + \ldots + i_n$, with $i_1! = i_1 i_2 \ldots i_n !$, $0! = 1$, and $x^i = (x^1)^{i_1} \cdots (x^n)^{i_n}$ for $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$.

Consider a local coordinate system $x^i$ on $M$ and a local coordinate system $y^a$ on $Q$. Two maps $f, g : M \to Q$ satisfy $j^r_x f = j^r_x g$ if and only if all the partial derivatives up to order $r$ of the components $f^a$ and $g^a$ of their coordinate expressions coincided at $x$. In this case the chain rule implies $f \circ \gamma \sim_r g \circ \gamma$. For the curves $x^i = \xi^i t$ with arbitrary $\xi^i$, these conditions read

$$\sum_{|i|=k} (\partial_i f^a(x)) \xi^i = \sum_{|i|=k} (\partial_i g^a(x)) \xi^i,$$

for $k = 0, 1, \ldots, r$.

The elements of $L^r_{n,m}$ can be identified with the $r$-th order Taylor expansions of the generating maps, i.e. with the $m$-tuples of polynomials of degree $r$ in $m$ variables without absolute term. Such an expression

$$\sum_{1 \leq |i| \leq r} \xi^a_i x^i$$

is the polynomial representative of an $r$-jet. Hence $L^r_{n,m}$ is a numerical space of the variables $\xi^a_i$.

Standard combinatorics yields $\dim L^r_{n,m} = m \left( \binom{n+r}{n} - 1 \right)$. The coordinates on $L^r_{n,m}$ are sometimes denoted more explicitly by $\xi^a_i, \xi^a_{ij}, \ldots, \xi^a_{i_1 \ldots i_r}$, symmetric in all subscripts. The projection $\pi^r_s : L^r_{m,n} \to L^s_{m,n}$ consists in suppressing all terms of degree $> s$.

The set of all invertible elements of $L^r_{n,m}$ with the jet composition is a Lie group $G^r_m$ called the $r$-th differential group of the $r$-th jet group in dimension $m$. For $r = 1$, the group $G^1_m$ is identified with $GL(m, \mathbb{R})$.

Let $p : Q \to M$ be a fibered manifold.

**Definition 2.3** A map $j^r f : M \to J^r(M,Q)$ is called a $r$-th jet prolongation of $f : M \to Q$. The set $J^r Q$ of all $r$-jets of the local sections of $Y$ is called the $r$-th jet prolongation of $Q$ and $J^r Q \subset J^r(M,Q)$ is a closed submanifold.

We note that if $Q \to M$ is a vector bundle, then $J^r Q$ is a also a vector bundle.

### 2.2 Nonholonomic manifolds and nonlinear connections

The concept of nonholonomic jet is elaborated in Refs. [12, 34, 27], when multi–indices are not symmetric and the jet spaces are subjected to certain non–integrable conditions. Nonholonomic structures with non–integrable constraints can be defined on the space of jets but also on the ‘prime’, $M$, and ‘target’, $Q$, manifolds. In our approach, we shall elaborate a geometric formalism encoding nonholonomic geometric structures both on manifolds and maps, i.e. on $M, Q$ and $J^r(M,Q)$.

A nonholonomic manifold $V$ is a usual one but endowed with a nonholonomic distribution $N$ in a sense of G. Vrânciu [47, 48, 49] and Z. Horák [22], see reviews [6, 14]. For our purposes, it is enough to consider such a $N$ which defines a nonlinear connection (N–connection) structure $N = \{N^a_i(x,y)\}$ as a Whitney sum

$$N : TV := hV \oplus vV,$$

---

Our definition of multi-index derivative $\partial_i f$ is similar to $D_i f$ used in [23]. We have to modify the system of notations in order to elaborate a geometric method of constructing exact solutions of PDEs with jet variables.
where $TV$ is the tangent bundle of $V$ and $hV$ and $vV$ are, respectively the horizontal (h) and vertical (v) subspaces for a nonholonomic fibration.\footnote{Local coordinates are with a conventional $2+2$ splitting, $u^α = (x^i, y^a)$, with $i,j,... = 1,2$ and $a,b,... = 3,4,...$; in brief, $u = (x,y) \in V$ for any point and its coordinates. We shall use boldface symbols in order to emphasize that certain spaces and/or geometric objects are provided/ adapted to a N-connection structure.} Such a geometric object was used in coordinate form by E. Cartan in his model of Finsler geometry\footnote{We refer readers to this monograph for a modern approach to differential geometry and main results on jets, Weil bundles and generalized connections.} considering $V = TM$ as a tangent bundle to a manifold $M$. In a similar form, we can work with a vector bundle, $V = E$, on $M$, $\dim E = n + m$, $\dim M = n$ (for $n,m \geq 2$) instead of $TM$. The global definition of $N$-connection is due to C. Ehresmann\footnote{This mean that certain geometric constructions are adapted to a h- and v-splitting stated by a N-connection distribution\footnote{We refer readers to this monograph for a modern approach to differential geometry and main results on jets, Weil bundles and generalized connections.}.} In\cite{23}, such connections are studied for fiber bundles and called generalized (Ehresmann) connections. We shall follow a different system of notations which was elaborated and used in the theory of nonholonomic (non) commutative Ricci flows, nonholonomic Dirac operators and Clifford bundles and deformation quantization of generalized geometries and gravity theories\cite{11,42,43}.

Any $N$ defines a $N$-adapted frame structure $e_α = (e_i, e_a)$, on $TV$, and co-frame structure $e^β = (e^j, e^b)$, on dual tangent bundle $T^*V$,

$$e_α = (e_i = ∂_i - N^b_i ∂_a, e_a = ∂_a) \text{ and } e^β = (e^j = dx^j, e^b = dy^b + N^b_i dx^i),$$

(2)

where the Einstein summation rule is applied on repeating indices and $∂_i = ∂/∂x^i$ and $∂_a = ∂/∂y^b$. In general, such local bases are nonholonomic, i.e. $e_α δ_β - e_β δ_α = W^γ_{αβ} e_γ$, with nontrivial anholonomy coefficients $W^γ_{αβ}$. We call certain geometric objects to be distinguished objects (d–objects), for instance d–tensors, d–vectors if they are determined by coefficients in N-adapted form, i.e. with respect N–elongated (co) bases\footnote{We refer readers to this monograph for a modern approach to differential geometry and main results on jets, Weil bundles and generalized connections.} and their tensor products. For instance, a vector $X ∈ TV$ can be written in a "not-adapted" coordinate form, $X = X^α ∂_α$, or as a d–vector, $\hat{X} = hX + vX = X^α e_α = X^i e_i + X^a e_a$.

Another two important characteristics of a $N$–connection are 1) the almost complex structure $J$, when $J(e_i) = -e_{2+i}$ and $J(e_{2+i}) = e_i$, with $J ∘ J = -I$, for unity matrix $I$ and 2) the Neijenhuis tensor (called also the curvature of $N$–connection)

$$N^α J[X,Y] := -[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY], \forall X, Y ∈ TV.$$

Linear connections on $(V,N)$ can be defined in $N$–adapted form as distinguished connections, $d$–connections, in order to preserve under parallel transports the distribution\footnote{We refer readers to this monograph for a modern approach to differential geometry and main results on jets, Weil bundles and generalized connections.}. Such a covariant differential operator splits as $D = (hD, vD)$. We can associate to $D$ a 1–form $Γ^γ_α = Γ^γ_{αβ} e^β$ and elaborate a $N$–adapted differential form calculus. The torsion and curvature are defined, respectively, by standard formulas:

$$T(X,Y) := D_X Y - D_Y X + [X,Y] \quad \text{and} \quad R(X,Y) := D_X D_Y - D_Y D_X - D[X,Y].$$

(3)

The Ricci d–tensor $Ric$ is constructed by contracting of indices in the curvature tensor $R = \{R^α_{βγμ}\}$, $Ric := \{R^α_{βγ} = R^α_{γβ}\}$,\footnote{We refer readers to this monograph for a modern approach to differential geometry and main results on jets, Weil bundles and generalized connections.} 

Let $g$ be a metric of arbitrary signature on a noholonomic manifold/ bundle $(V,N)$ which in $N$–adapted form\footnote{We refer readers to this monograph for a modern approach to differential geometry and main results on jets, Weil bundles and generalized connections.} is represented as symmetric d–tensor,

$$g = h + vh = g_{αβ}(u)e^α \otimes e^β = g_{ij}(x,y)dx^i \otimes dx^j + g_{ab}(x,y)e^a \otimes e^b.$$

For any metric structure $g$ on a nonholonomic manifold $(V,N)$, there are two "preferred" linear connections completely and uniquely defined by

$$g \rightarrow \left\{ \begin{array}{l}
\nabla : \nabla g = 0; \nabla T = 0, \\
\hat{D} : \hat{D}g = 0; h\hat{T} = 0, \nu\hat{T} = 0, 
\end{array} \right. \text{ the Levi–Civita connection;}$$

$$\text{the canonical d–connection.}$$

(4)

It should be noted that $\nabla$ is not a d–connection because it does not preserve under parallel transports the $h$-and $v$-splitting\footnote{We refer readers to this monograph for a modern approach to differential geometry and main results on jets, Weil bundles and generalized connections.}. Nevertheless, there is a unique $N$–adapted distortion relation

$$\hat{D} = \nabla + \hat{Z}$$

(5)
when both linear connections $\hat{\mathbf{D}}$ and $\nabla$ and the distorting d–tensor $\check{Z}$ are completely determined by the metric structure $g$ for a prescribed $N$–connection structure $N$. The Ricci and Riemannian tensors are different for $\hat{\mathbf{D}}$ and $\nabla$ because, in general, $\hat{T} \neq 0$ but $\nabla T = 0$. All geometric constructions with $(g, \nabla; V)$ can be transformed equivalently into similar ones with $(g, N; \mathbf{D}; V)$, and inversely, if distortion relations \([\mathbf{5}]\) are used.

There are two canonical scalars determined by a d–metric $g$ via $\hat{\mathbf{D}}$, with $\hat{R} := g^{\beta\gamma} \hat{R}_{\beta\gamma}$ and the standard (pseudo) Riemannian scalar determined by $\nabla, R := g^{\beta\gamma} R_{\beta\gamma}$. Both values are related by a distortion relation which can be found by contracting with $g^{\beta\gamma}$ nonholonomic deformations of the Ricci tensor, $\hat{\text{Ric}} = \text{Ric} + \check{Z}_{\text{ic}}$, which are computed by introducing \([\mathbf{5}]\) into formulas \([\mathbf{3}]\).

\begin{definition}
An $\mathcal{X} \in \mathcal{J}(M, Q)$ is said to be a nonholonomic $r$–jet with the source $x \in M$ and the target $y \in Q$ if there is a local section $\sigma : M \rightarrow \mathcal{J}(M, Q)$ such that $\mathcal{X} = j_r^1 \sigma$ and $\beta(\sigma(x)) = y$.

We write $\mathcal{X} = j_r^1 \sigma$ (with calligraphic $\mathcal{X}$) instead of $X = j_r^1 \sigma$ from Definition \([\mathbf{2.2}]\) in order to emphasize that the jet map is defined, in genera, in nonholonomic form. There is a natural embedding $\mathcal{J}(M, Q) \subset \mathcal{J}(M, Q)$. In general, any $\mathcal{X}$ induces a nonholonomic map $\mu : (TT\ldots T M)_x \rightarrow (TT\ldots T Q)_y$, \([\mathbf{27}]\).
\end{definition}

### 2.3 Nonholonomic jets and N–adapted manifolds and maps

Nonholonomic jet structures can be introduced even the prime and target manifolds are considered only with holonomic distributions. In a more general context, all maps and manifolds can be nonholonomic.

#### 2.3.1 Nonholonomic maps of holonomic manifolds

Let us consider two holonomic manifolds $M$ and $Q$ and introduce the set of nonholonomic 1–jets $\mathcal{J}(M, Q) := J^1(M, Q)$ for $r = 1$.\(^7\) By induction, we can consider the source projection $\alpha : \mathcal{J}^{-1}(M, Q) \rightarrow M$ and the target projection $\beta : \mathcal{J}^{-1}(M, Q) \rightarrow Q$ the target projection of $(r - 1)$–th nonholonomic jets.

\begin{definition}
An $\mathcal{X} \in \mathcal{J}(M, Q)$ is said to be a complete nonholonomic $r$–jet with the source $u \in V$ and the target $u' \in V'$ if there is a local section $\sigma : V \rightarrow \mathcal{J}(V, V')$ such that $\mathcal{X} = j_r^1 \sigma$ and $\beta(\sigma(u)) = u'$.

For simplicity, we use the same nonholonomic jet symbol $\mathcal{X} = j_r^1 \sigma$ with boldface point $u \in V$. There are also defined natural embedding $\mathcal{J}(V, V') \subset \mathcal{J}(V, V') \subset \mathcal{J}(V, V')$, which can be parameterized by corresponding local coordinate and/or $N$–adapted frame systems and integrable or non-integrable maps. In general, any $\mathcal{X}$ induces a nonholonomic map $\mu : (TT\ldots T V)_u \rightarrow (TT\ldots T V')_{u'}$ splitting in corresponding horizontal and vertical components with $h, v, \ldots$ to $h', v', \ldots$.

We can generalize the concept of jet prolongation of fibered manifold, see Definition \([\mathbf{2.3}]\) to cases with nonholonomic maps and prime and target nonholonomic manifolds. Let $p : Q \rightarrow V$ be a fibered manifold when, in general, both $Q$ and $V$ are with nontrivial $N$–connection structures.

\(\)\(^7\)In \([\mathbf{27}]\), it is written $\mathcal{J}(M, Q)$ instead of boldface $J^1(M, Q)$. As we mentioned above, we use boldface letters in order to emphasize a $h$- and $v$-splitting via a $N$–connection structure of a class of geometric objects/ maps / spaces. We can consider such decompositions form the maps defining a jet structure (and write $\mathcal{J}$) even the respective prime and target manifold are holonomic ones, when $M$ and $Q$ are not boldface.
Definition 2.6 A nonholonomic map \( j^r f : V \to J^r(V, Q) \) is called a \( r \)-th jet prolongation of \( f : V \to Q \). The set \( J^r Q \) of all \( r \)-jets of the local sections of \( Q \) is called the \( r \)-th jet prolongation of \( Q \) and \( J^r Q \subset J^r(V, Q) \) is a closed submanifold.

We note that if \( Q \to V \) is a distinguished vector bundle with nonholonomic base and nonholonomic total spaces, then \( J^r Q \) is also a distinguished vector bundle.

2.3.3 Local expressions and \( h \)-\( v \)-coordinates

In order to construct exact solutions in explicit forms using geometric methods it is important use certain local, coordinate and \( N \)-adapted construction even geometric models are formulated in rigorous mathematical form in coordinate free and global forms. Let us establish necessary conventions: We use \( x = \{ x^i \} \) as local coordinates on a prime manifold \( M \), when \( i, j, \ldots = 1, 2, \ldots n \). We can use \( y = \{ y^a \} \) as local coordinates on a target manifold \( Q \), when \( a = n + 1, \ldots n+m \). On \( J^r(M, Q) \), we have local coordinates \( x^i, y^a \) and the induced coordinates \( v^n_{\alpha \beta \gamma} \) are symmetric on low indices \( i_1, i_2, \ldots i_p = 1, \ldots n \), for \( p = 1, \ldots r \). Working with nonholonomic jet spaces \( J^r(M, Q) \) for the same prime and target manifolds we use boldface induced coordinates \( v^n_{\alpha \beta \gamma} \) which are not symmetric on \( i_1, i_2, \ldots i_p \). We can consider corresponding coordinate systems with the same coordinate description of any \( J^r Y, J^r Y \) or \( J^r Y \).

Let us introduce parameterizations for indices and coordinates of \( N \)-adapted maps \( V \to V' \), when \( u = (x, y) = \{ u^\alpha = (x^i, y^a) \} \) are local coordinates on \( V \) and \( u' = (x', y') = \{ u'^\alpha = (x'^i, y'^a) \} \) are local coordinates on \( V' \). We denote

\[
\frac{\partial f}{\partial u^\alpha} := \frac{\partial^{[\alpha]} f}{(\partial u^1)^{\alpha_1} \cdots (\partial u^{n+m})^{\alpha_n+m}}
\]

is the partial derivative of a function \( f : U \subset \mathbb{R}^{n+m} \to \mathbb{R} \), with a multi-index \( \alpha \) of range \( n + m \), which is a \((n + m)\)-tuple \( \alpha = (\alpha_1, \ldots, \alpha_{n+m}) \) of non-negative integers. For such nonholonomic spaces, we write \( \alpha ! = \alpha_1! \alpha_2! \cdots \alpha_{n+m}! \), \( 0! = 1 \), and \( u^\alpha = (u^1)^{\alpha_1} \cdots (u^{n+m})^{\alpha_n+m} \) for \( u = (x_1, x_2, \ldots, x_n; y^{n+1}, \ldots, y^{n+m}) \in \mathbb{R}^{n+m} \).

The local coordinate system is conventionally split on both nonholonomic manifolds. Two \( N \)-adapted maps \( 1^r f : V \to V' \) and \( 2^r f : V \to V' \) satisfy \( j^r_1 f = j^r_2 f \) if and only if for the curves \( u^\alpha = \zeta^\alpha t \) with arbitrary \( \zeta^\alpha \) such

\[
\sum_{|\alpha|=k} (\partial_{\zeta^\alpha} 1^r f^\alpha(u)) \zeta^\alpha = \sum_{|\alpha|=k} (\partial_{\zeta^\alpha} 2^r f^\alpha(u)) \zeta^\alpha, \quad \text{for } k = 0, 1, \ldots, r.
\]

We can define jet distinguished groups, \( d \)-groups with elements \( L^r_{n+m,n'+m'} \) identified with the \( r \)-th order Taylor expansions of the generating maps. These are \( n+m \)-tuples of polynomials of degree \( r \) in \( n'+m' \) variables without absolute term, with a polynomial representative of an \( r \)-jet which can written in the form \( \sum_{1 \leq |\alpha| \leq r} \zeta^\alpha u^\alpha \).

\( L^r_{n+m,n'+m'} \) is a numerical space of the variables \( \zeta^\alpha \). A standard combinatoric calculus results in

\[
\dim L^r_{n+m,n'+m'} = (n'+m') \left( \begin{array}{c} n+m+r \\ n+m \end{array} \right) - 1.
\]

In explicit form, the coordinates on \( L^r_{n+m,n'+m'} \) are denoted by \( \zeta_0^\alpha, \zeta_{\alpha_1}^\alpha, \ldots, \zeta_{\alpha_r}^\alpha \) which are symmetric in all subscripts if such values are taken in natural coordinate frames. The projection \( \pi^r_{\alpha} : L^r_{n,m} \to L^r_{m,n} \) consists in suppressing all terms of degree \( > s \).

The set of all invertible elements of \( L^r_{n,m,n'+m'} \) with the jet composition is a Lie \( d \)-group \( G^r_{n+m} \) called the \( r \)-th differential \( d \)-group of the \( r \)-th jet \( d \)-group in dimension \( n + m \). For \( r = 1 \), the group \( G^1_{n+m} \) is identified with \( GL(n + m, \mathbb{R}) = GL(n, \mathbb{R}) \oplus GL(m, \mathbb{R}) \).

In this work, we study nonholonomic jet prolongations of the geometric objects from section 2.2 in \( J^r(V, V') \) framework with local coordinates

\[
u^{\alpha s} = (x^i, y^a, \zeta^{\alpha_1 \ldots \alpha_r}) = (x^i, y^a, \zeta^{\alpha s}).
\]
We use the label $s$ in order to perform a conventional splitting of dimensions, $\dim sV = 4 + 2s = 2 + 2 + \ldots + 2 \geq 4; s \geq 0$ for conventional finite dimensional (pseudo) Riemannian space $sV$. The jet coordinates $v^{a_1 \ldots a_s}_{\alpha_1 \ldots \alpha_s}$ are re-grouped in oriented two shells which allow us to apply the AFDM and to construct exact solutions for generalized Einstein equations and metrics $^s g$ with arbitrary signatures $(\pm 1, \pm 1, \pm 1, \ldots \pm 1)$. Such shells are determined by nonholonomic data which transforms into $c^a_{\alpha_1 \ldots \alpha_r}$ with symmetric low indices if the constructions are performed in coordinate bases. Let us establish conventions on (abstract) indices and coordinates $u^{a_s} = (x^i, y^a)$, for $s = 0, 1, 2, 3, \ldots$ labelling the oriented number of two dimensional, 2-d, "shells" added to a 4-d spacetime. For $s = 0$ (in a conventional form), we write $u^a = (x^i, y^a)$ and consider such local systems of coordinates:

\[
\begin{align*}
  s &= 1 : u^{a_1} = (x^a = u^a, v^{a_1}) = (x^i, y^a, \xi^1), \\
  s &= 2 : u^{a_2} = (x^{a_1} = u^{a_1}, v^{a_2}) = (x^i, y^a, \xi^1, \xi^2), \\
  s &= 3 : u^{a_3} = (x^{a_2} = u^{a_2}, v^{a_3}) = (x^i, y^a, \xi^1, \xi^2, \xi^3), \ldots
\end{align*}
\]

for $i, j, \ldots = 1, 2; a, b, \ldots = 3, 4; a_1, b_1, \ldots = 5, 6; a_2, b_2, \ldots = 7, 8; a_3, b_3, \ldots = 9, 10; \ldots$ and $i_1, j_1, \ldots = 1, 2, 3, 4; i_2, j_2, \ldots = 1, 2, 3, 4, 5, 6; i_3, j_3, \ldots = 1, 2, 3, 4, 5, 6, 7, 8; \ldots$ In brief, we shall write $u = (x, y); 1u = (u, 1 \xi) = (x, y, 1 \zeta), 2u = (1, 2 \zeta) = (x, y, 1 \zeta, 2 \zeta), \ldots$

We shall underline the indices in order to emphasize that certain values are in respect to local coordinate bases. The transformations between local frames, $e_{a_s}$, and coordinate frames, $\partial_{a_s} = \partial/\partial u^{a_s}$ on $sV$ are written $e_{a_s} = e_{a_s}^\alpha (s u) \partial/\partial u^\alpha$. General parameterizations of coefficients $e_{a_s}^\alpha$ result in nonholonomic anholonomy relations $e_{a_s} \beta \beta - e_{\beta \beta} e_{a_s} = W_{\alpha \beta}^a e_{a_s}$. The anholonomy coefficients $W_{\alpha \beta}^a = W_{\beta \alpha}^a (u)$ vanish for holonomic configurations. Using the condition $e_{a_s}^\alpha \partial/\partial u^\alpha = \delta_{a_s}^\alpha$, where the 'hook' operator $\hook$ corresponds to the inner derivative and $\delta_{a_s}^\alpha$ is the Kronecker symbol, we can construct dual frames, $e_{a_s}^\alpha = e_{a_s}^\alpha (s u) \partial/\partial u^{a_s}$.

It is important to distinguish the partial derivatives on spacetime coordinates (for instance, $\partial_i = \partial/\partial x^i, \partial_s = \partial/\partial y^a$ and $\partial_s = \partial/\partial u^{a_s}$) and on $r$-jet variables, when $\partial_s = \partial/\partial u^{a_s}$ is used for a $2 + 2 + \ldots$-conventional splitting of partial derivatives $\partial/\partial \xi^a_{\alpha_1 \ldots \alpha_r}$. In some sense, $\xi^a_{\alpha_1 \ldots \alpha_r}$ can be considered as extra dimension coordinates but with certain additional Lie group properties of $G_{n+m}$ considered above.

### 2.4 Jet prolonged Ricci soliton and Einstein equations

We can define canonical $N$-connection, frame, metric and distinguished metric structures on $J^r(V, V')$ determined by prolongations of respective prime objects on $V$, see Definition 2.3.

#### 2.4.1 Shell parameterized $N$-connection and associated frame structures

A map $j^r f : M \rightarrow J^r(M, Q)$ is called a $r$-th jet prolongation of $f : M \rightarrow Q$. The set $J^r Q$ of all $r$-jets of the local sections of $Y$ is called the $r$-th jet prolongation of $Q$ and $J^r Q \subset J^r(M, Q)$ is a closed submanifold.

**Theorem 2.1** Any $N$-connection structure $N$ on $V$ determines a $r$-th jet prolonged $N$-connection $^s N$ on $J^r(V, V')$ as Whitney sum

\[
^s N : T \ sV = hV \oplus vV \oplus vV \oplus 2vV \oplus \ldots \oplus sV,
\]

for a conventional horizontal (h) and vertical (v) "shell by shell" splitting.

**Proof.** It is a natural construction when the coefficients of $N$-connection are defined by jet prolongations and parameterized $^s N = N_{is}^a (u) dx^i \otimes \partial/\partial u^a$ on every chart on $J^r(V, V')$, i.e. for $^s V$.

\[ \square \] (end proof).

Using the coefficients of $N$-connection, we prove

**Corollary 2.1** $r$-th jet prolongations induce on $J^r(V, V')$ a system of $N$-elongated bases/ partial derivatives, $e_{a_s} = (e_i, e_{a_s})$, and cobases, $N$-adapted differentials, $e_{a_s} = (e^i, e^a)$.

\[ ^s \text{In a similar form, we can split odd dimensions, for instance, } \dim V = 3 + 2 + \ldots + 2. \]
**Proof.** Taking (2) for \( V \), we prolongate on \( s \geq 1 \) shells,

\[
\begin{align*}
e_{i_s} &= \frac{\partial}{\partial x^{i_s}} - N_i^{a_s} \partial_{a_s}, \quad e_{a_s} = \partial_{a_s} = \frac{\partial}{\partial \zeta^{a_s}}, \\
e^{i_s} &= dx^{i_s}, e_{a_s} = d\zeta^{a_s} + N_i^{a_s} dx^{i_s}.
\end{align*}
\]  

(9) (10)

\( \Box \)

The \( N \)-adapted operators (9) satisfy such anholonomy relations:

\[ [e_{a_s}, e_{\beta_s}] = e_{a_s} e_{\beta_s} - e_{\beta_s} e_{a_s} = W_{a_s}^{\gamma_s} e_{\gamma_s}, \]

(11)

when \( W_{i_{a_s}}^{b_s} = \partial_{a_s} N_i^{b_s} \) and \( W_{j_{i_s}}^{a_s} = J N_i^{a_s} \), where the Neijenhuis tensor, i.e. the curvature of the \( r \)-th jet prolongated \( N \)-connection, is \( J N_i^{a_s} = e_j (N_i^{a_s}) - e_i \left( N_j^{a_s} \right) \).

### 2.4.2 \( N \)-adapted shell prolongated \( d \)-connections

On \( J'(V, V') \) with prolongation of geometric objects from \( V \), we define linear connection structures in \( N \)-adapted form.

**Theorem 2.2 – Definition:** There are distinguished connection, \( d \)-connection, structures, \( {^s}D = \{ D_{a_s} \} \), with

\[ D = (hD; vD), \quad 1D = (hD; 1vD), \ldots, \quad s^{-1}D = (s^{-2}hD; s^{-1}vD), \quad sD = (s^{-1}hD; svD), \]

preserving under parallelism the \( N \)-connection splitting (8).

**Proof.** We construct in explicit form such a \( d \)-connection by considering \( N \)-adapted covariant derivatives

\[
D_{a_s} = (D_i; D_a), \quad D_{a_1} = (1D_{a_1}; D_{a_1}), \quad D_{a_2} = (2D_{a_1}; D_{a_2}), \ldots, D_{a_s} = (sD_{a_{s-1}}; D_{a_s}),
\]

for \( hD = (L_j^i; L_b^a) \), \( vD = (C_{j_c}^i; C_{bc}^a) \),

\[
1hD = (L_{\beta_s\gamma_s}^i; L_{b_1}^a), \quad 1vD = (C_{\beta_{c_1}}^i; C_{b_1}^a), \quad 2hD = (L_{\beta_1\gamma_1}^i; L_{b_2}^a), \quad 2vD = (C_{\beta_1 c_2}^i; C_{b_2}^a),
\]

\[
\ldots
\]

\[
shD = (L_{\beta_{s-1}\gamma_{s-1}}^i; L_{b_{s-1}^a}), \quad svD = (C_{\beta_{s-1} c_s}^i; C_{b_s}^a),
\]

when the coefficients

\[
\begin{align*}
\Gamma_{\beta_{s}}^\gamma &= (L_j^i; L_b^a; C_{j_c}^i; C_{bc}^a), \\
\Gamma_{\beta_1}^\gamma &= (L_{\beta_s\gamma_s}^i; L_{b_1}^a; C_{\beta_1 c_1}^i; C_{b_1}^a), \\
\Gamma_{\beta_2}^\gamma &= (L_{\beta_1\gamma_1}^i; L_{b_2}^a; C_{\beta_1 c_2}^i; C_{b_2}^a), \ldots
\end{align*}
\]

(12)

of such a \( d \)-connection \( {^s}D = \{ D_{a_s} \} \) can be computed in \( N \)-adapted form with respect to frames (9)–(10) following equations \( D_{a_s} e_{\beta_s} = \Gamma_{\beta_s}^{\gamma_s} e_{\gamma_s} \).

\( \Box \)

It is possible always to consider such frame transforms when all shell frames are \( N \)-adapted and

\[
1D_{a_s} = D_{a_s}, \quad 2D_{a_1} = D_{a_1}, \ldots, sD_{a_{s-1}} = D_{a_{s-1}}.
\]

**Corollary 2.2 – Definition:** There are natural \( r \)-th jet prolongations of the torsion and curvature \( d \)-tensors (3) defined on a prime \( V \) and elongated in \( N \)-adapted form on \( J'(V, V') \) with prescribed shell splitting on \( {^s}V \),

\[
\begin{align*}
{^s}T(X, Y) &:= {^s}D_X Y - {^s}D_Y X + [X, Y] \quad \text{and} \\
{^s}R(X, Y) &:= {^s}D_X {^s}D_Y - {^s}D_Y {^s}D_X - {^s}D_{[X,Y]},
\end{align*}
\]

(13) (14)

for any \( d \)-vectors \( X, Y \subset T {^s}V \).
Proof. To perform computations in N–adapted–shell form we can consider a differential connection 1–form $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} e^\gamma s$ and elaborate a differential form calculus with respect to skew symmetric tensor products of N–adapted frames \([9]–[10]\). Respectively, the torsion $T^\alpha s = \{T^\alpha_{\beta\gamma}s\}$ and curvature $R^\alpha_{\beta\gamma}s = \{R^\alpha_{\beta\gamma}s\}$ d–tensors of $^sD$ are computed

\[
\begin{align*}
T^\alpha s & := ^sDe^\alpha s = de^\alpha s + \Gamma^\alpha_{\beta\gamma} e^\beta s, \\
R^\alpha_{\beta\gamma}s & := ^sDr^\alpha_{\beta\gamma}s = dr^\alpha_{\beta\gamma}s - \Gamma^\alpha_{\beta\gamma} e^\gamma s = R^\alpha_{\beta\gamma}s e^\gamma s & \wedge & e^\delta s,
\end{align*}
\]

see Refs. \([33, 45]\) for explicit calculi of coefficients $R^\alpha_{\beta\gamma}s$ in higher dimensions. The formulas in jet shell adapted coordinates \([7]\) are very similar to those in N–adapted bases for extra dimensional (pseudo) Riemannian spaces. In standard r–jet coordinates \([9]\) for $J^r(V, V')$, $u^\alpha = (x^i, y^a, \zeta^{a_1}...a_r)$, additional contraction of up-low indices and symmetrization result in very cumbersome coefficient formulas.

\[\square\]

In Appendix \([A.1]\) we present two Theorems on computing N–adapted coefficients formulas for $T^\alpha_{\beta\gamma}s$ and $R^\alpha_{\beta\gamma}s$.

2.4.3 Jet prolonged d–metrics

On $^sV$, a metric tensor can be written in the form

\[
^s g = g_{\alpha\beta} e^\alpha s \otimes e^\beta s = g_{\alpha\beta} du^\alpha \otimes du^\beta , \quad s = 0, 1, 2, \ldots ,
\]

\[
= g_{\underline{\alpha}\underline{\beta}} du^{\underline{\alpha}} \otimes du^{\underline{\beta}} + g_{\underline{\alpha}+1\underline{\beta}+1} d\zeta^{\underline{\alpha}+1} \otimes d\zeta^{\underline{\beta}+1}
\]

where $du^\alpha \in T^*V$ and the indices are underlined in order to emphasize that we consider coordinate dual bases. The coefficients of such a metric are subjected to frame transform rules, $g_{\alpha\beta} = e^\alpha e_\beta g_{\underline{\alpha}\underline{\beta}}$, which can be respectively generalized for any tensor object. We can not preserve a $2 + 2 + 2 + \ldots$ splitting of dimensions under general frame/coordinate transforms.

Lemma 2.1 Any metric structure $^s g = \{g_{\alpha\beta}s\}$ on $^sV$ can be written as a distinguished metric (d–metric)

\[
^s g = g_{ij}(^su) e^i s \otimes e^j s + g_{ab}(^su) e^a s \otimes e^b s
\]

\[
= g_{ij}(x) e^i \otimes e^j + g_{ab}(u) e^a \otimes e^b + g_{aib1}(^1u) e^{a1} \otimes e^{b1} + \ldots + g_{aib}(^su) e^a \otimes e^b s.
\]

Proof. Using frame/coordinate transforms, we can parameterize any metric \([17]\) in such forms:

\[
^s g_{\alpha\beta}(u) = \begin{bmatrix}
  g_{ij} + h_{ab}N^a_i N^b_j & h_{ae}N^e_j \\
  h_{be}N^e_i & h_{ab}
\end{bmatrix}
\]

on the prime manifold and, for r–prolongations,

\[
^s g_{\alpha1\beta1} (^1\zeta) = \left[
\begin{array}{cc}
  g_{\alpha\beta} & h_{a1\beta} N^e_{\alpha1} \\
  h_{b1\beta} N^e_{\alpha1} & h_{ab1}
\end{array}
\right], \quad ^s g_{\alpha2\beta2} (^2\zeta) = \left[
\begin{array}{cc}
  g_{\alpha\beta} & h_{a2\beta} N^e_{\alpha2} \\
  h_{b2\beta} N^e_{\alpha2} & h_{ab2}
\end{array}
\right], \ldots
\]

\[
^s g_{\alpha\beta}s (^s\zeta) = \left[
\begin{array}{ccc}
  g_{ij} + h_{ab}N^a_i N^b_j & h_{ae}N^e_j & h_{ab} \\
  h_{be}N^e_i & h_{ab}
\end{array}
\right].
\]

Re–grouping terms shell by shell with respect to bases \([10]\), we obtain \([15]\).

\[\square\]

For extra dimensions, such parameterizations are similar to those introduced in the Kaluza–Klein theory when $\zeta^{a_1}, s \geq 1$, are considered as extra dimension coordinates with cylindrical compactification and $N^e_{\alpha1}(^s u) \sim A_{\alpha\alpha}(u) y^a$ are for certain (non) Abelian gauge fields $A_{\alpha\alpha}(u)$. Jet generalized gauge theories are with different symmetries than those with potentials taken values in Lie group algebras, see Ref. \([1]\) on unification of gravity with internal gauge interactions.
2.4.4 Canonical jet distortions and linear connections on jet bundles

For any \( r \)-jet prolonged (pseudo) Riemannian metric \( ^s g \), we can construct in standard form the Levi–Civita connection (LC–connection), \( ^s \nabla = \{ \Gamma_{\beta\gamma}^{\alpha} \} \). By definition such a metric is metric compatible, \( ^s \nabla( ^s g) = 0 \), and with zero torsion, \( \mathcal{T}^\alpha = 0 \) (we can used formulas (15) for \( ^s \mathcal{D} \rightarrow ^s \nabla \)). It should be emphasized that such a linear connection is not a d–connection because it does not preserve under general coordinate transforms a N–connection splitting (8). Nevertheless, one holds this

\[
\text{Theorem 2.3} \quad \text{There is a canonical distortion relation}
\]

\[
^s \mathcal{D} = ^s \nabla + ^s \mathcal{Z},
\]

for a canonical d–connection \( ^s \mathcal{D} \) which is completely and uniquely defined by a (pseudo) Riemannian metric \( ^s g \) for a chosen nonholonomic distribution \( ^s \mathcal{N} = \{ N^\alpha_i \} \) when \( ^s \mathcal{D}( ^s g) = 0 \) and the horizontal and vertical torsions are zero, i.e. \( ^h \mathcal{T} = \{ \mathcal{T}^i_{jk} \} = 0 \), \( ^v \mathcal{T} = \{ \mathcal{T}^a_{bc} \} = 0 \), \( ^1 \mathcal{T} = \{ \mathcal{T}^a_{b_1c_1} \} = 0, \ldots \), \( ^s \mathcal{T} = \{ \mathcal{T}^a_{b_1c_1} \} = 0 \); the distorting tensor \( ^s \mathcal{Z} = \{ \mathcal{Z}^\alpha_{\beta\gamma} \} \) is uniquely defined by the same date \( ^s \mathcal{N} \).

\( \Box \)

The N–adapted coefficients of the distortion d–tensor \( \mathcal{Z}^\alpha_{\beta\gamma} \) are algebraic combinations of \( \mathcal{T}^\alpha_{\beta\gamma} \) and vanish for zero torsion. The nonholonomic variables \( \{ ^s g, ^s \mathcal{N}, ^s \mathcal{D} \} \) are equivalent to the standard (pseudo) Riemannian ones \( \{ ^s \mathcal{N}, ^s \mathcal{D}, ^s \mathcal{Z} \} \). For instance, the GR theory in 4-d can be formulated equivalently using the connection \( ^s \nabla \) and/or \( ^s \mathcal{D} \) if the distorting relation (15) is used, see details in [10, 13, 15]. \( r \)-jet prolongations result in distortions (19). We can consider nonholonomic jet deformations of a 4–d (pseudo) Riemannian space to a \( J'(V, V') \) with a canonical nonzero d–torsion. In such cases, we are able to decouple modified Einstein equations and construct integral varieties with jet variables. At the end, we can impose additional nonholonomic constraints and fix the jet coordinates in order to generate exact solutions of Ricci soliton/ Einstein equations in 4-d, or higher dimensions, with \( r \)- jet symmetries.

Here we note that \( ^s \nabla \) and \( ^s \mathcal{D} \) are not tensor objects. \( ^s \mathcal{D} \) is a d–connection and such linear connections are subjected to different rules of coordinate transforms. It is possible to consider frame transforms with certain \( ^s \mathcal{N} = \{ N^\alpha_i \} \) when the conditions \( \Gamma^\gamma_{\alpha\beta} = \bar{\Gamma}^\gamma_{\alpha\beta} \) are satisfied with respect to some N–adapted frames (9)–(10). In general, \( ^s \nabla \neq ^s \mathcal{D} \) and the corresponding curvature tensors \( \mathcal{R}^\alpha_{\beta\gamma\delta} \neq \bar{\mathcal{R}}^\alpha_{\beta\gamma\delta} \) are different, but the Ricci tensor components may coincide for certain classes of nonholonomic constraints.

2.4.5 Prolongation of Ricci soliton and Einstein equations on nonholonomic jet configurations

In this section, we introduce important geometric and physical equations in nonholonomic variables on \( V \) and consider generalizations on \( J'(V, V') \).

\( \text{Definition 2.7} \quad \text{The geometric data} \quad \{ g, N, D; V \} \quad \text{define a gradient nonholonomic Ricci soliton if there exists a smooth potential function} \quad \kappa(x, y) \quad \text{such that}
\]

\[
\bar{\mathcal{R}}_{\beta\gamma} + \bar{\mathcal{D}}_{\beta} \bar{\mathcal{D}}_{\gamma} \kappa = \lambda g_{\beta\gamma}.
\]

There are three types of such Ricci solitons determined by a constant \( \lambda \): steady ones, for \( \lambda = 0 \); shrinking, for \( \lambda > 0 \); and expanding, for \( \lambda < 0 \).

The classification from above definition is related to Levi–Civita, LC, limits when shrinking solutions help us to understand the asymptotic behaviour of ancient solutions of the Ricci flow theory [17, 18, 33]. Generalizing and adapting the constructions to N–connection structures, we can describe geometric flows with nonholonomic constraints (11, 12). We omit a study of geometric analysis issues and generalized Ricci flow models in this work and restrict our research to nonholonomic \( r \)-jet prolongations of equations and important classes of solutions.
The $N$–adapted coefficients of the Ricci $d$–tensor $Ric = \{R_{\alpha \beta \gamma} := R_{\alpha \beta \gamma}^{\delta} \}$ of a $d$–connection $^{*}D$ in $J^{r}(V, \nabla')$ are computed by contracting of coefficients of the curvature tensor (10),

$$R_{\alpha \beta \gamma} = \{R_{i j s} := R_{i j k s}, \ R_{i 1 a 1} := -R_{i 1 k 1 a 1},...; \ R_{a 1 i s} := R_{b i a 1 s} \}. \tag{21}$$

Using the inverse matrix of $^{*}g$ (18), we can compute the scalar curvature of $^{*}D$,

$$^{*}R := \ g^{\alpha \beta}R_{\alpha \beta \gamma} = g^{i j}R_{i j s} + h^{a b}R_{a b s} = \ R + S + \ 1 S + ... + \ ^{*}S, \tag{22}$$

with respective h– and v–components of scalar curvature, $R = g^{i j}R_{i j}, \ S = h^{a b}R_{a b}, \ 1 S = h^{a b}R_{a 1 b},...; \ ^{*}S = h^{a b}R_{a b s}$. The sources $\Upsilon$ when $\Upsilon_{\delta}$.

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Theorem 2.4 In nonholonomic N–adapted r–jet variables, the gradient canonical Ricci jet–solitons are defined by equations

$$\tilde{R}_{\beta \gamma} + \tilde{D}_{\beta \gamma} \tilde{D}_{\alpha \beta} \kappa = \lambda g_{\beta \gamma \gamma}.$$  

(27)

Our first goal is elaborate a geometric method for decoupling the equations (20) and (27), which for certain classes of nonholonomic constraints transforms into systems of nonlinear PDE (25) with possible zero torsion (extract certain subvarieties of solutions when \(\gamma\) is parameterized in such forms which allow to integrate the differential equations in explicit forms. We shall prove that general classes of solutions \(g_{\alpha \beta}(u)\) depend generically on all spacetime coordinates via corresponding generating and integration functions and constants. This will be an explicit application of the geometry of nonholonomic distributions and generalized connections in mathematical relativity, modified gravity theories and the theory of physically important nonlinear systems of PDEs. The second goal is to find explicit solutions for the Levi-Civita (LC-configurations) (26), i.e. for additional generating functions and sources, \(\alpha \gamma\) depend generically on all spacetime coordinates via corresponding generating and integration functions and constants. This will be an explicit application of the geometry of nonholonomic distributions and generalized connections in mathematical relativity, modified gravity theories and the theory of physically important nonlinear systems of PDEs.

The second goal is to find explicit solutions for the Levi-Civita (LC-configurations) (26), i.e. for additional constraints when

$$\tilde{g}(x) = \delta = \text{diag}[1, 2, 3, 4, 5, 6],$$  

(28)

$$\tilde{g} = \text{diag}[1, 2, 3, 4, 5, 6],$$  

(29)

$$\tilde{g} = \text{diag}[1, 2, 3, 4, 5, 6],$$  

(30)

$$\tilde{g} = \text{diag}[1, 2, 3, 4, 5, 6],$$  

(31)

$$\tilde{g} = \text{diag}[1, 2, 3, 4, 5, 6],$$  

(32)

$$\tilde{g} = \text{diag}[1, 2, 3, 4, 5, 6],$$  

(33)

$$\tilde{g} = \text{diag}[1, 2, 3, 4, 5, 6],$$  

(34)

with respect to N–adapted frames (9) and (10). The effective source (anisotropically polarized cosmological constant)

$$\gamma^\alpha_\beta = \text{diag}[1, 2, 3, 4, 5, 6],$$  

(35)

is parameterized in such forms which allow to integrate the differential equations in explicit forms. We shall prove that general classes of solutions \(g_{\alpha \beta}(u)\) depend generically on all spacetime coordinates via corresponding generating and integration functions and constants. This will be an explicit application of the geometry of nonholonomic distributions and generalized connections in mathematical relativity, modified gravity theories and the theory of physically important nonlinear systems of PDEs.

The second goal is to find explicit solutions for the Levi-Civita (LC-configurations) (26), i.e. for additional constraints when

$$s^\alpha \tilde{T} = 0,$$  

(36)

(this formula follows from (13)), when considering some classes of solutions \((s^g, s^\gamma, s^\beta)\) of (28)–(34) we can extract certain subvarieties of solutions \((s^g, s^\gamma, s^\beta)\) of (28)–(34) for zero torsion, after re-scaling the generating functions and sources, \(s^g \rightarrow s^g, s^\gamma \rightarrow s^\gamma\) and \(s^\beta \tilde{T} \rightarrow s^\beta \tilde{T} \rightarrow const\). This way we shall formulate a geometric method of constructing exact solutions of the Einstein equations (24) with re–defined source and N–adapted frame structures when

$$R_{\alpha \beta \gamma} [s^\gamma] = \lambda g_{\alpha \beta \gamma},$$  

(37)

on nonholonomic manifolds/bundles and r–jet prolongations when the metrics are generic off–diagonal (i.e. can not be diagonalized via coordinate transforms), may depend on all spacetime coordinates and can be prescribed (via generating/integrating functions and constants) to satisfy various necessary type symmetry, boundary, Cauchy, topological conditions, with possible singularities and horizons etc. We note that the system (28)–(34) together with LC–conditions (26) is equivalent to (37). Both such systems of PDEs are nonlinear and parametric ones. So, it is important when certain nonholonomic constraints and ansatz conditions for frames, metrics and connections are imposed (at the end, when some solutions for \(s^\beta \tilde{T}\) have been found, or at the beginning, when \(s^\beta \tilde{T} \rightarrow s^\beta\)). We can not decouple such systems of equations in a general form if we work from the very beginning with the Levi-Civita connection \(s^\gamma\).
The third goal of this article is to state certain geometric conditions when a general (pseudo) Riemannian manifold \((g, V)\) (the metric \(g\) may be, or not, a solution of any (modified) Einstein or Ricci soliton equations) can be nonholonomically deformed via corresponding nonholonomic jet maps with generalized connection structures into certain geometrically/physically important classes of solutions of systems of type (38)–(40), or (37). In such cases

\[
\text{nonholonomic Jet} \quad (g, V) \rightarrow (\hat{g}, \hat{N}, \hat{D}, \hat{V}),
\]

when the target space \(\hat{V}\) and fundamental geometric structures \((\hat{g}, \hat{N}, \hat{D})\) constructed by nonholonomic jet transforms (Jet) are solutions of certain (modified/generalized) Einstein, or Ricci soliton, equations depending on generalized jet parameters with corresponding jet symmetries. We note, that for such geometric and physical models the jet variables are fixed to certain constant values (we do not put the left label \(s\)).

3 Decoupling and Integration of Jet Prolongated Einstein Equations

In this section, we prove that the system of nonlinear PDEs (38)–(41) with possible constraints (40) resulting in (37), can be decoupled in very general forms with respect to N-adapted frames with two dimensional shell parameterizations of jet variables. We show how such decoupled systems can be integrated in general forms for vacuum and non-vacuum cases in (modified) gravity and Ricci soliton theories.

3.1 Off–diagonal metrics for \(r\)-jet configurations with one Killing symmetry

We study nonholonomic jet deformations of "primary" geometric/physical data into "target" data,

\[
|\text{primary}| \rightarrow |\text{target}| \quad (s \hat{g}, s \hat{N}, s \hat{D}) \rightarrow (\hat{g}, \hat{N}, \hat{D}).
\]

The values labeled by "\(s\)" may define exact solutions in a Ricci soliton or gravity theory. The metrics with left label "\(\eta\)" will define a solution of modified gravitational field equations (38)–(41). The prime ansatz is taken in the form

\[
\hat{g} = \hat{g}_i(x^k)dx^i \otimes dx^j + \hat{h}_a(x^k, y^4)\hat{e}^a \otimes \hat{e}^b + \epsilon_{a_1}dy^{a_1} \otimes dy^{a_1} + \ldots + \epsilon_{a_s}dy^{a_s} \otimes dy^{a_s},
\]

\[
\hat{e}^a = dy^a + \hat{N}^3_i(x^k, y^4)dx^i, \quad \hat{N}^3_i = \hat{n}_i, \hat{N}^4_i = \hat{w}_i,
\]

for \(\epsilon_{a_s} = \pm 1\) depending on signature of extra dimensions and \((\hat{g}_i, \hat{h}_a; \hat{N}^3_i)\) defining, for instance, the Kerr black hole solution trivially imbedded into a \(4 + 2s\) jet prolonged spacetime. We express the N–adapted coefficients of a target ansatz (18) as

\[
g_{a_\alpha} = \eta_{a_\alpha}(u^{\beta_\alpha})\hat{g}_{a_\alpha}; \hat{N}^{a_\alpha}_i = \eta_{a_\alpha}N^{a_\alpha}_i(u^{\beta_\alpha-1}, y^{4+2s})
\]

\[
n_i = \eta_{i_\alpha}n_i, w_i = \eta_{i_\alpha}w_i, \quad \text{not summation on} \; i;
\]

with so–called gravitational "polarization" functions and extra dimensional N-coefficients, \(\eta_{a_\alpha}, \eta_{i_\alpha}\) and \(\eta_{a_\alpha}N_{i_\alpha}\). To be able to study certain limits \((s \hat{g}, s \hat{N}, s \hat{D}) \rightarrow (s \hat{g}, s \hat{N}, s \hat{D})\), for \(\varepsilon \rightarrow 0\), depending on a small parameter \(\varepsilon, 0 \leq \varepsilon \ll 1\), we shall introduce "small" polarizations of type \(\eta = 1 + \varepsilon \chi(u\ldots)\) and \(\varepsilon N_{a_\alpha} = \varepsilon n_{a_\alpha}(u\ldots)\).

The decoupling property of modified Einstein equations can be proven in the simplest form for certain ansatz with at least one Killing symmetry on a spacetime coordinate and certain parameterizations of nonholonomic \(r\)-jet prolongations. We consider target metrics of type (18) parameterized in the form

\[
\hat{g}_{\omega a} = g_{\omega}(x^k)dx^i \otimes dx^j + \hat{h}_{\omega a}(x^k, y^4)e^a \otimes e^b +
\]

\[
h_{a_1}(u^a, \xi^6) e^{a_1} \otimes e^{a_1} + h_{a_2}(u^{a_1}, \xi^8) e^{a_2} \otimes e^{b_2} + \ldots + h_{a_s}(u^{a_{s-1}}, \xi^a) e^{a_s} \otimes e^{a_s},
\]
Theorem 3.1 (see below) for non–vacuum configurations.

The N–adapted equations (A.5)–(A.12) decouple in this form:

$$\mathbf{e}^a = dy^a + N^a_i dx^i,$$
$$\mathbf{e}^{a_1} = d\zeta^{a_1} + N^{a_1}_a da^a,$$
$$\mathbf{e}^{a_2} = d\zeta^{a_2} + N^{a_2}_a da^{a_1},$$
$$\mathbf{e}^{a_s} = d\zeta^{a_s} + N^{a_s}_{a_1} da^{a_{s-1}},$$

for $N^{4+2s-1} = s_n a_1 (u^{\beta_1}, \zeta^{4+2s}), N^{4+2s} = s_w a_1 (u^{\beta_1}, \zeta^{4+2s}).$

Such ansatz contains also a jet Killing vector $\partial / \partial \zeta^{s-1}$ because the jet coordinate $\zeta^{s-1}$ is not contained in the coefficients of such metrics.

3.2 Decoupling in nonholonomic $r$–jet shell variables

Let us consider an ansatz (40) with $g_i(x^k) = \epsilon_i e^{\psi(x^k)}$, where $\epsilon_i = \pm 1$, and the $\gamma, \alpha, \beta$–coefficients are defined by respective generating functions $\phi, \star \phi$ following formulas

$$\gamma := \partial_t (\ln |h_3|^{3/2} / |h_4|), \quad \alpha_i = (\partial_i \phi)(\partial_t h_3) / 2 h_3, \quad \beta = (\partial_\gamma \phi)(\partial_t h_3) / 2 h_3,$$

for generating function $\phi = \ln |\partial_t h_3 / \sqrt{|h_3 h_4|}|$.

$$1^\gamma := \partial_t (\ln |h_5|^{3/2} / |h_6|), \quad 1^\alpha_r = (\partial_r 1^\gamma \phi)(\partial_t h_5) / 2 h_5, \quad 1^\beta = (\partial_r 1^\gamma \phi)(\partial_t h_5) / 2 h_5,$$

for r–jet generating function $1^\gamma \phi = \ln |(\partial_t h_5) / \sqrt{|h_5 h_6|}|$.

$$2^\gamma := \partial_t (\ln |h_7|^{3/2} / |h_8|), \quad 2^\alpha_r = (\partial_r 2^\gamma \phi)(\partial_t h_7) / 2 h_7, \quad 2^\beta = (\partial_r 2^\gamma \phi)(\partial_t h_7) / 2 h_7,$$

for r–jet generating function $2^\gamma \phi = \ln |(\partial_t h_7) / \sqrt{|h_6 h_8|}|$.

with nonzero $\partial_\lambda \phi, \partial_\lambda h_5, \partial_\alpha 1^\gamma \phi, \partial_\alpha 2^\gamma \phi, \partial_\beta 2^\gamma \phi$.

We assume that via N–adapted frame transforms the sources $\Upsilon_{\beta, \delta a}$ for equations (28), (29), (31) and (33) can be parameterized in the form

$$\Psi_1 \equiv \Psi_2 \equiv \psi(x^k, y^l) + \frac{1}{2} \lambda(u^b, \zeta^8) + \frac{1}{2} \lambda(u^b, \zeta^8) + \frac{1}{2} \lambda(u^b, \zeta^8),$$

Such parameterizations are very general for (effective) $\Upsilon_{\beta, \delta a}$ with arbitrary contributions of Ricci soliton or (modified) gravity and matter fields and further r–jet generalizations when the N–adapted coefficients are modeled in certain systems of references by "polarized" cosmological constants $\lambda(x^k), \psi(x^k, y^l), \psi(u^b, \zeta^8), \psi(u^b, \zeta^8)$ etc. For certain models of extra dimension gravity, we can consider, for simplicity, $\lambda = \psi = \frac{1}{2} \lambda = \frac{1}{2} \lambda = const.$ We can always introduce such effective sources by re–defining the generating functions (see below) for non–vacuum configurations.

Theorem 3.1 For a general off–diagonal ansatz (40) with Killing symmetry on $\partial_4$ and N–adapted parameterizations for generating functions (41) and sources (47), the system of modified Einstein equations (see N–adapted equations (A.3)–(A.12)) decouple in this form:

For a nonholonomic 2+2 spacetime splitting,

$$\epsilon_1 \partial_1 \psi + \epsilon_2 \partial_2 \psi = 2 \Lambda(x^k),$$

$$\partial_4 \psi = 2 h_3 h_4 \psi(x^k, y^l),$$

$$\partial_4 n_i + \gamma \partial_4 n_i = 0,$$

$$\beta w_i - \alpha_i = 0;$$
and, on nonholonomic $r$–jet variables,

$$\begin{align}
(\partial_6^{-1} \phi)(\partial_6 h_5) &= 2 h_5 h_6 v \Lambda(u^2, \zeta^6), \\
\partial_6^2 n_r + 1 \gamma \partial_6^{-1} n_r &= 0, \\
1 \beta_1 w_r - 1 \alpha_r &= 0, \\
\text{...}
\end{align}$$

(50)

(51)

(52)

$$\begin{align}
(\partial_2^s \phi)(\partial_2 h_{2s-1}) &= 2 h_{2s-1} h_{2s} v \Lambda(u^{\beta_s-1}, \zeta^{2s}), \\
\partial_2^s 2 n_r + 2 \gamma \partial_2 2 n_r &= 0, \\
2 \beta \partial_2 2 w_r - 2 \alpha_r &= 0, \\
\text{...}
\end{align}$$

(53)

**Proof.** It is a tedious technical proof following an explicit calculation of nontrivial components of the Ricci d–tensor for the mentioned ansatz and parameterizations of (effective) sources and parametrization functions (see Appendix A.3). Such equations are straightforward consequences of symmetries of the Einstein and Ricci d–tensors A.13 for the canonical d–connection $^s\hat{D}$.

Let us explain in brief the decoupling property for nonholonomic 4–d and $r$–jet equations (46)–(53):

1. The equation (46) is just a 2-d Laplace, or d’Alambert (depending on prescribed signature for $\varepsilon_i = \pm 1$) with solutions determined by any source $\Lambda(x^k)$.

2. The equation (47) contains only the partial derivative $\partial_4$ and consists together with the algebraic formula for the coefficient $h_2$ a system of two equations four functions: $h_3(x^i, y^4)$, $h_4(x^i, y^4)$ and $\phi(x^i, y^4)$ and source $v \Lambda(x^k, y^4)$. Prescribing two any such functions, we can define (integrating on $y^4$, or differentiating on this coordinate) other two such functions. Such functions can be re–defined in order to transform $v \Lambda(x^k, y^4)$ into an effective constant. The function $\phi(x^i, y^4)$ and/or any its functional can be considered as a generating function which can be prescribed following certain geometric or physical arguments on symmetries, boundary conditions, any explicit singular or non–singular behaviour, smooth class conditions, stochastic conditions, topological configurations etc. This allows us to compute $h_n(x^i, y^4)$ in explicit form. If necessary, we can consider, for instance, the coefficient $h_3(x^i, y^4)$ (or $h_4(x^i, y^4)$) to be a generating function and compute $h_4$ (or $h_3$) and $\phi$ for a given source $v \Lambda(x^k, y^4)$. We note that if we consider vacuum solutions for $^s\hat{D}$ with $v \Lambda = 0$ in (47), we are constrained to study only configurations with N–adapted coefficients $\partial_4 h_3 = 0$ and/or $\partial_4 \phi = 0$. Such solutions are also important to study geometric and physical properties of vacuum off–diagonal configurations and their possible diagonal limits. The decoupling property is explicit for such vacuum and non–vacuum equations because there are contained only two coefficients of the metric, $h_4$; coupling with other diagonal and/or off–diagonal coefficients (like $g_4$, $w_4$, $n_4$, and/or jet prolongations) are not involved.

3. Having computed the coefficient $\gamma$ (41), the N–connection coefficients $n_i$ can be defined after two integrations on $y^4$ in (48). This defines a part of N–connection coefficients. A value $n_i$ does not depend on other coefficients of a d–metric excepting the coefficient $\gamma$ determined by $h_n$.

4. Using $h_3$ and $\phi$ from previous point, we compute the coefficients $\alpha_i$ and $\beta_i$, see (41), which allows us to define $w_i$ from the algebraic equations (49). Such $w_i$ define another part of the N–connection coefficients can determine off–diagonal coefficients of the metric but also may contribute in the diagonal coefficients if such metrics are written in coordinate bases. Nevertheless, the functions $w_i$ are independent from other coefficients of the d–metric and N–connection with respect to N–adapted frames.

5. The procedure 2–4 can be repeated step by step for any shell with $r$–jet variables. The equations (50)–(52) are completely similar to (47)–(49) but contain additional dependencies on jet coordinates and derivatives.
\( \partial \Phi \) on respective jet coordinates. For instance, the equation (50) and formula (44) with partial derivative \( \partial_6 \) is for functions \( h_5(x^i, y^\alpha, \zeta^6), h_6(x^i, y^\beta, \zeta^6) \) and \( 1 \phi(x^i, y^\rho, \zeta^6) \) and source \( \Lambda(u^\beta, \zeta^6) \). We can compute any two such functions integrating on \( \zeta^6 \) if two other ones are prescribed. In a similar form, we follow points 3 and 4 with \( 1 \alpha, 1 \beta, 1 \gamma \), see (43), and compute the higher order N–connection coefficients \( 1 n_r \) and \( 1 w_r \).

6. The splitting property holds on any 2-d shell as it is stated by (53). Here we note that it is not clear if any splitting of equations could be proven in general form for 3–d shells. It is obvious that the topological properties of 2-d and 3–d shells are very different. The equations of type (17), (50) degenerate for 1–d shells. That why, our AFDM is based on 2+2+2+... splitting which allows to decouple and solve such nonlinear systems of PDEs in general form. We can consider in similar form splitting of type 3+2+2+... for 3–d bases when the point 1 refers to 3-d Laplace, or d’Alambert equations. For certain configurations, we can generate extra–dimension and jet configurations by imbedding 1,2 and 3 dimensional metrics into some d–metric configurations with splitting 2+2+2+... In arbitrary systems of reference, such effective vacuum and non–vacuum nonholonomic dynamical systems depend on spacetime coordinates. They may be of nonlinear evolution type, or Ricci soliton fixed configurations, for respectively prescribed signatures. Nevertheless, the assumption on 2–d shall is a condition imposing certain (2-d) topological restrictions on jet variables extensions.

Finally, we emphasize that the splitting property of nonholonomic and holonomic Einstein equations for higher dimensions was proven in [43]. Geometric and physical models with jet variables and extra dimension coordinates are similar in certain sense but with different jet symmetry conditions. Formal 2+2+2+... splitting are possible only in correspondingly adapted nonholonomic systems of reference and the derived nonholonomic dynamical/evolution equations encode a different interior gauge like dynamics.

### 3.3 Jet integral varieties for off–diagonal metrics and generalized connections

The system of nonlinear PDEs (46) – (53) in spacetime and jet variables can be integrated in general forms for any 2–d shell \( \text{dim} \ \mathbf{V} \geq 4 \). We note that the coefficients \( g_1 = e^\psi(x^k) \) are defined by solutions of corresponding Laplace/ d’Alambert equation (46) which does not contain jet coordinates in N–adapted frames. General solutions will be considered for "vertical" spacetime and jet variables.

#### 3.3.1 4–d non–vacuum spacetime nonholonomic configurations

We can solve (17) and (12) for any \( \partial_4 \Phi \neq 0, h_\alpha \neq 0 \) and \( \Lambda^\alpha \neq 0 \). Let us re-write respectively those equations,

\[
 h_3 h_4 = (\partial_4 \Phi)(\partial_4 h_3)/2 \Lambda^\alpha \quad \text{and} \quad |h_3 h_4| = (\partial_4 h_3)^2 e^{-2\phi}.
\]  

(54)

Considering a new generating function \( \Phi := e^\phi \) and introducing the first equation into the second one,

\[
|\partial_4 h_3| = \frac{\partial_4 (e^{2\phi})}{4 |\Lambda^\alpha|} - \frac{\partial_4 [\Phi^2]}{4 |\Lambda^\alpha|}.
\]  

(55)

Integrating on \( y^4 \), we find

\[
h_3 [\Phi, \Lambda^\alpha] = 0 h_3(x^k) + \frac{\epsilon_3 \epsilon_4}{4} \int dy^4 \frac{\partial_4 [\Phi^2]}{\Lambda^\alpha},
\]  

(56)

where \( 0 h_3 = 0 h_3(x^k) \) is an integration function and \( \epsilon_3, \epsilon_4 = \pm 1 \). To compute \( h_4 \) we can use the first equation in (54) when

\[
h_4 [\Phi, \Lambda^\alpha] = \frac{(\partial_4 \phi)}{\Lambda^\alpha} \partial_4 (\ln |h_3|) = \frac{1}{2} \frac{\partial_4 [\Phi^2]}{\Lambda^\alpha} \frac{\partial_4 h_3}{h_3}.
\]  

(57)

The formulas (56) and (57) for \( h_\alpha [\Phi, \Lambda^\alpha] \) can be re–parameterized in a more convenient form with effective cosmological constant \( \Lambda_0 = \text{const} \neq 0 \). Let us re–define the generating function \( \Phi \to \tilde{\Phi} \), when \( \frac{\partial_4 [\Phi^2]}{\Lambda^\alpha} = \frac{\partial_4 [\tilde{\Phi}^2]}{\Lambda_0} \), i.e.

\[
\Phi^2 = \Lambda_0^{-1} \int dy^4 (\Lambda^\alpha)\partial_4(\tilde{\Phi}^2) \quad \text{and} \quad \tilde{\Phi}^2 = \Lambda_0 \int dy^4 (\Lambda^\alpha)^{-1} \partial_4(\tilde{\Phi}^2).
\]  

(58)
Introducing the integration function $0 h_3(x^k)$ and $\epsilon_3$ and $\epsilon_4$ in $\Phi$ and, respectively, in $\nu \Lambda$, we express the solutions for $h_{a}$ as functionals on $[\Phi, \Lambda_0, \Xi]$,

$$h_3[\Phi, \Lambda_0] = \frac{\tilde{\Phi}^2}{4\Lambda_0}$$

and

$$h_4[\Phi, \Lambda_0, \Xi] = \frac{(\partial_4 \tilde{\Phi})^2}{\Xi}. \quad (59)$$

The functional

$$\Xi[\nu \Lambda, \tilde{\Phi}] = \int dy^4 (\nu \Lambda) \partial_4 (\tilde{\Phi}^2)$$

in the last formula can be considered as a re–defined source for a prescribed generating function $\tilde{\Phi}$, $\nu \Lambda \rightarrow \Xi$, when $\nu \Lambda = \partial \Lambda / \partial (\tilde{\Phi}^2)$ (it contains information on Ricci soliton contribution, effective and/or energy-momentum tensor of matter in modified gravity theories). We can work with two couples of generating data, $(\Phi, \nu \Lambda)$ and $(\tilde{\Phi}, \Xi)$, related by formulas (58) for a prescribed effective cosmological constant $\Lambda_0$.

Using the values $h_{a}$ (59), we compute the coefficients $\alpha_i$, $\beta$ and $\gamma$ from (41). The resulting solutions for N–coefficients, i.e of respective equations (48) and (49), can be expressed recurrently,

$$n_k = 1n_k + 2n_k \int dy^4 h_4/(\sqrt{|h_3|})^3 = 1n_k + 2\tilde{n}_k \int dy^4 (\partial_4 \tilde{\Phi})^2 / (\tilde{\Phi}^2 \Xi), \text{ and}$$

$$w_i = \partial_i \phi / \partial_4 \phi = \partial_4 \phi / \partial_4 \Phi = \partial_4 \Phi^2 / \partial_4 \Phi^2 = \int dy^4 \partial_4 [(\nu \Lambda) \partial_4 (\tilde{\Phi}^2)] / (\nu \Lambda) \partial_4 (\tilde{\Phi}^2) = \partial_4 \Xi / \partial_4 \Xi, \quad (60)$$

where $1n_k(x^i)$ and $2n_k(x^i)$, or $2\tilde{n}_k(x^i) = 8 2n_k(x^i)|\tilde{\Lambda}|^3/2$, are integration functions. Putting together the formulas for coefficients (59)–(60), we prove:

**Theorem 3.2** The system of nonlinear PDEs (46)–(49) for non–vacuum 4–d configurations with Killing symmetry on $\partial_3$ is integrated in general form by such quadratic elements

$$ds^2_{4[dK]} = g_{\alpha \beta}(x^k, y^4)du^\alpha du^\beta = \epsilon_i e^{\psi(x^k)}(dx^i)^2 + \frac{\tilde{\Phi}^2}{4\Lambda_0} \left[ dy^3 + \left(1n_k + 2\tilde{n}_k \int dy^4 (\partial_4 \tilde{\Phi})^2 / (\tilde{\Phi}^2 \Xi) \right) dx^k \right]^2 + \frac{(\partial_4 \tilde{\Phi})^2}{\Xi} \left[ dy^4 + \partial_4 \Xi / \partial_4 \Xi dx^4 \right]^2. \quad (61)$$

This element defines a family of generic off–diagonal solutions with Killing symmetry $\partial / \partial y^3$ of the 4–d nonholonomic Einstein equations (28)–(30) with source parametrization of type (15) and for the canonical d–connection $\mathbf{D}$ (the label 4dK is for "nonholonomic 4d Killing solutions). We can verify by straightforward computations that the anholonomy coefficients $W_{\alpha \beta \gamma}$ in (11) are not zero if arbitrary generating function $\phi$ and integration funtions $(0 \nu a, 1n_k$ and $2n_k)$ are considered. This means that such metrics can not be diagonalized by coordinate transforms in a finite spacetime region. The class of solutions (61) is with nontrivial canonical d–torsion (15) which can be proven by using explicit N–adapted coefficient formulas (A.4) for the canonical d–connection (A.3). In section 3.3.3 we shall state additional conditions when such solutions define LC–configurations. Vacuum nonholonomic spacetime quadratic elements are considered in section A.4.1.

### 3.3.2 Nonholonomic r–jet prolongations of non–vacuum solutions

The solutions with jet variables can be constructed in certain forms which are similar to the 4–d ones but using new classes of generating and integration functions with dependencies on r–jet shell coordinates. We can generate solutions of the system (50)–(52) with coefficients (11) and (13) following a formal analogy when the generating functions and (effective) sources from the previous paragraph are generalized in the form:

$$\partial_4 \rightarrow \partial_6, \phi(x^k, y^4) \rightarrow 1\phi(u^\tau, \zeta^6), \quad \nu \Lambda(x^k, y^4) \rightarrow \nu_1 \Lambda(u^\tau, \zeta^6) \ldots$$

and re–defined values $\tilde{\Phi}(x^k, y^4) \rightarrow 1\tilde{\Phi}(u^\tau, \zeta^6)$ and $\tilde{\Lambda}_0 \rightarrow 1\tilde{\Lambda}_0 = const.$

The first set of r–jet coefficients of the d–metric are computed

$$h_5[1\tilde{\Phi}, \tilde{\Lambda}] = \frac{1\tilde{\Phi}^2}{4\nu_1\Lambda} \quad \text{and} \quad h_6[1\tilde{\Phi}] = \frac{(\partial_6 1\tilde{\Phi})^2}{\nu_1\Lambda}. \quad (59)$$
for \( 1\Xi = \int d\zeta^6 (\frac{v}{\Lambda}) \partial_6 (\frac{1}{\Phi^2}) \) and, for \( N \)-coefficients,

\[
1n_r = \frac{1}{4} n_r + \frac{1}{2} r \int d\zeta^6 h_6/(\sqrt{|h_5|})^3 = \frac{1}{4} n_r + \frac{1}{2} n_k \int d\zeta^6 (\partial_6 \frac{1}{\Phi})^2/(\frac{1}{\Phi})^3 \Xi,
\]

\[
1w_r = \partial_r \frac{1}{\Phi} / \partial_6 \frac{1}{\Phi} = \partial_r \frac{1}{\Phi} / \partial_6 \frac{1}{\Phi} = \partial_r \frac{1}{\Xi} / \partial_6 \frac{1}{\Xi},
\]

where \( h_{a_r} = 0 h_{a_r}(u^\tau) \), \( n_k(u^\tau) \) and \( n_k(u^\tau) \) are integration functions.

A general class of quadratic line elements with one shell jet variables defining generic off–diagonals solutions of the nonholonomic canonical deformations of the Einstein equations can be parameterized in the form

\[
ds^2_{4+2[dK]} = ds^2_{4+dK} + \frac{2\Phi^2}{4 A} \left[ d\zeta^7 + \left( \frac{1}{2} n_{r_s} + \frac{1}{2} n_k \int d\zeta^8 (\partial_8 \frac{2}{\Phi})^2 \right) du^r \right] + \frac{(\partial_{3+2s} \frac{2}{\Phi})^2}{2 \Xi} \left[ d\zeta^8 + \frac{1}{4} \Xi du^r \right]^2,
\]

where \( ds^2_{4+dK} \) is given by formula (61) and \( \tau = 1, 2, 3, 4 \). This quadratic line element is with Killing jet symmetry on \( \phi_3 \) (in \( N \)-adapted frames, the metric does not depend on \( \zeta^5 \)).

Extending the constructions to the jet shell \( s = 2 \) with \( \partial_6 \to \partial_8 \), \( 1\phi(u^\tau, \zeta^6) \to 2\phi(u^\tau, \zeta^8), v \Lambda(u^\tau, \zeta^6) \to 2 \Lambda(u^\tau, \zeta^8) \), ..., with \( \Phi(u^\tau, \zeta^8) \), \( 2 \Lambda_0 \), where \( \tau_1 = 1, 2, ..., 5, 6 \), we generate off–diagonal solutions in 8–d jet modified gravity model,

\[
ds^2_{4+2[dK]} = ds^2_{4+2[dK]} + \frac{2\Phi^2}{4 A} \left[ d\zeta^7 + \left( \frac{1}{2} n_{r_s} + \frac{1}{2} n_k \int d\zeta^8 (\partial_8 \frac{2}{\Phi})^2 \right) du^r \right] + \frac{(\partial_{3+2s} \frac{2}{\Phi})^2}{2 \Xi} \left[ d\zeta^8 + \frac{1}{4} \Xi du^r \right]^2,
\]

where \( ds^2_{4+2[dK]} \) is given by formula (62), \( 2\Xi = \int d\zeta^8 (\frac{v}{\Lambda}) \partial_8 (\frac{2}{\Phi})^2 \), and corresponding integration/generating functions \( h_{a_r}(u^7); a_r = 7, 8; n_{r_s}(u^7) \) and \( n_{r_s}(u^7) \) are integration functions.

Using 2+2+... symmetries of off–diagonal parameterizations (62) and (63), we can construct exact solutions for arbitrary finite sets of \( r \)-jet shells on \( ^s V \) for a nonholonomic \( ^J \) on \( V, V' \). The corresponding quadratic line elements are

\[
ds^2_{4+2s[dK]} = ds^2_{2+2s[dK]} + \frac{s\Phi^2}{4 A} \left[ d\zeta^{3+2s} + \left( \frac{1}{2} n_{r_{s-1}} + \frac{1}{2} n_k \int d\zeta^{4+2s} (\partial_{4+2s} \frac{2}{\Phi})^2 \right) du^{r_{s-1}} \right] + \frac{(\partial_{4+2s} \frac{2}{\Phi})^2}{2 \Xi} \left[ d\zeta^{4+2s} + \frac{1}{4} \Xi du^{r_{s-1}} \right]^2,
\]

where \( 2\Xi = \int d\zeta^{4+2s} (\frac{v}{\Lambda}) \partial_{4+2s} (\frac{2}{\Phi})^2 \), and corresponding integration/generating functions; \( n_{r_{s-1}}(u^{r_{s-1}}) \) and \( n_{r_{s-1}}(u^{r_{s-1}}) \) are integration functions.

### 3.3.3 The Levi–Civita conditions

All classes of general solutions constructed in this section and (for vacuum configurations) in Appendix define certain subclasses of generic off–diagonal metrics (10) for the canonical d–connections \( ^D \) and nontrivial nonholonomically induced d–torsion coefficients \( T_{\alpha_1 \beta_2}^{\gamma_3 \alpha_4} \) (11). It is natural to have such torsion fields in theories with gauge like jet symmetries of gravitational and matter fields. Nevertheless, we can perform r–jet prolongations in such forms that the nonholonomic induced torsion vanishes for a subclass of nonholonomic distributions with necessary types of parameterizations of the generating and integration functions and sources.

In explicit form, we construct LC–configurations by imposing additional constraints, including certain "shell by shell on jet variables", on the d–metric and N–connection coefficients. By straightforward computations (see details in Refs. [44], and Appendix A.5 for \( r \)-jet variables), we can verify that if in \( N \)-adapted frames

\[
\partial_4 w_i = e_i \ln \sqrt{\left| h_{14} \right|}, e_i \ln \sqrt{\left| h_{35} \right|} = 0, \partial_i w_j = \partial_j w_i \text{ and } \partial_k n_i = 0;
\]

\[
s = 1 : \partial_6 1w_\alpha = e_\alpha \ln \sqrt{\left| h_{61} \right|}, e_\alpha \ln \sqrt{\left| h_{36} \right|} = 0, \partial_\alpha 1w_\beta = \partial_\beta 1w_\alpha \text{ and } \partial_6 1w_\gamma = 0;
\]

\[
 s = 2 : \partial_8 2w_\alpha = 2e_\alpha \ln \sqrt{\left| h_{82} \right|}, 2e_\alpha \ln \sqrt{\left| h_{38} \right|} = 0, \partial_\alpha 2w_\beta = \partial_\beta 2w_\alpha \text{ and } \partial_8 2n_\gamma = 0;

\]

\[
... \]
the torsion coefficients become zero. For $n$–coefficients, such conditions are satisfied if $2n_k(x^i) = 0$ and
\[ \partial_i \n_j(x^k) = \partial_j \n_i(x^k); \quad \n_\alpha (u^\beta) = 0 \quad \text{and} \quad \n_\tau (u^\beta) = \partial_\tau \n_i(u^\beta) \quad \text{etc.} \] The explicit form of solutions of constraints on $w_k$ derived from (65) depend on the class of vacuum or non–vacuum metrics and their jet prolongations.

Let us find explicit solutions for the LC–conditions (65) in spacetime coordinates. Such nonholonomic constraints can not be solved in explicit form for arbitrary data $(\Phi, \nu \Lambda)$, $(\Bar{\Phi}, \Xi, \Lambda_0)$, and all types of nonzero integration functions $\n_j(x^k)$ and $2n_k(x^i) = 0$. One of the conditions which allow us to write solutions in explicit form if via coordinate and frame transforms we can fix $2n_k(x^i) = 0$ and $\n_j(x^k) = \partial_j n(x^k)$ for a function $n(x^k)$. We use the property that
\[
\mathbf{e}_i \Phi = (\partial_i - w_i \partial_4) \Phi \equiv 0
\]
for any $\Phi$ if $w_i = \partial_i \Phi / \partial_4 \Phi$, see (60). The equality
\[
\mathbf{e}_i H = (\partial_i - w_i \partial_4) H = \frac{\partial H}{\partial \Phi} (\partial_i - w_i \partial_4) \Phi \equiv 0
\]
holds for any functional $H[\Phi]$. We can restrict our construction to a subclass of generating data $(\Phi, \nu \Lambda)$ and $(\Bar{\Phi}, \Xi, \Lambda_0)$ related via formulas (58) when $H = \Bar{\Phi}[\Phi]$ is a functional which allows us to generate LC–configurations in explicit form. Using $h_3[\Bar{\Phi}] = \Bar{\Phi}^2 / 4 \Lambda$ (59) for $H = \Bar{\Phi} = \ln \sqrt{h_3}$, we satisfy the second condition, $\mathbf{e}_i \ln \sqrt{h_3} = 0$, in (65).

Now, we solve the first condition in (65) for spacetime coordinates. Taking the derivative $\partial_4$ of $w_i = \partial_i \Phi / \partial_4 \Phi$ (60), we obtain
\[
\partial_4 w_i = \frac{\partial_4 \partial_4 \Phi (\partial_4 \Phi) - (\partial_4 \Phi) \partial_4 \partial_4 \Phi}{(\partial_4 \Phi)^2} = \frac{\partial_4 \partial_4 \Phi}{\partial_4 \Phi} - \frac{\partial_4 \Phi}{\partial_4 \Phi} \partial_4 \Phi.
\]
Choosing a generating function $\Phi = \Bar{\Phi}$ for which
\[
\partial_4 \partial_4 \Phi = \partial_4 \partial_4 \Bar{\Phi}
\]
and using (66), we compute $\partial_4 w_i = \mathbf{e}_i \ln |\partial_4 \Phi|$. Taking $h_4[\Phi, \nu \Lambda]$ (57), we have
\[
\mathbf{e}_i \ln \sqrt{|h_4|} = \mathbf{e}_i [\ln |\partial_4 \Phi| - \ln \sqrt{|\nu \Lambda|}],
\]
(see also the conditions (67) and that $\mathbf{e}_i \Bar{\Phi} = 0$). Using the last two formulas, we can obtain
\[
\partial_4 w_i = \mathbf{e}_i \ln \sqrt{|h_4|}
\]
if
\[
\mathbf{e}_i \ln \sqrt{|\nu \Lambda|} = 0.
\]
This is possible for $\nu \Lambda = \text{const}$, or if $\nu \Lambda$ can be expressed as a functional $\nu \Lambda(x^i, y^4) = \nu \Lambda[\Phi]$.

Finally, we note that the third condition, $\partial_i w_j = \partial_j w_i$, in (65), can be solved for any $A = \Bar{A}(x^k, y^4)$ for which $w_i = \Bar{w}_i = \partial_i \Bar{A} / \partial_4 \Phi = \partial_i \Bar{A}$.

In shell jet variables, we can extend above constructions for "shell" generating functions:

\[
s = 1: \quad \begin{align*}
\Phi &= \Phi(u^\tau, \zeta, \xi) = \partial_\theta \partial_\tau \Phi &= \partial_\theta \partial_\theta \Phi; \\
\Phi &= \Phi(u^\tau, \zeta) = \partial_\theta \partial_\tau \Phi = \partial_\theta \partial_\theta \Phi; \\
\Phi &= \Phi(u^\tau, \zeta, \xi) = \partial_\theta \partial_\tau \Phi = \partial_\theta \partial_\theta \Phi;
\end{align*}
\]
\[
s = 2: \quad \begin{align*}
\Phi &= \Phi(u^\tau, \zeta, \xi, \zeta) = \partial_\theta \partial_\tau \Phi = \partial_\theta \partial_\theta \Phi; \\
\Phi &= \Phi(u^\tau, \zeta, \xi, \zeta) = \partial_\theta \partial_\tau \Phi = \partial_\theta \partial_\theta \Phi;
\end{align*}
\]
\[
\Phi &= \Phi(u^\tau, \zeta, \xi, \zeta) = \partial_\theta \partial_\tau \Phi = \partial_\theta \partial_\theta \Phi;
\]
\[
\ldots
\]

We can re–define the generating functions are functionals of "inverse hat" values, when
\[
\begin{align*}
\hat{\Phi}^2 &= (\Bar{\Lambda}_0)^{-1} \int dy^4 (\nu \Lambda) \partial_4 (\hat{\Phi}^2) \quad \text{and} \quad \hat{\Phi}^2 = \Bar{\Lambda} \int dy^4 (\nu \Lambda)^{-1} \partial_4 (\hat{\Phi}^2); \\
\hat{\Phi}^2 &= (\nu \Lambda)^{-1} \int d\zeta^6 (\nu \Lambda) \partial_6 (\hat{\Phi}^2) \quad \text{and} \quad \hat{\Phi}^2 = \nu \Lambda \int d\zeta^6 (\nu \Lambda)^{-1} \partial_6 (\hat{\Phi}^2); \\
\hat{\Phi}^2 &= (2 \nu \Lambda)^{-1} \int d\zeta^8 (2 \nu \Lambda) \partial_8 (\hat{\Phi}^2) \quad \text{and} \quad \hat{\Phi}^2 = 2 \nu \Lambda \int d\zeta^8 (2 \nu \Lambda)^{-1} \partial_8 (\hat{\Phi}^2),
\end{align*}
\]
and compute the values $\Xi(\tilde{\Phi}[\tilde{\Phi}])$, $^1\Xi(1\Phi[1\Phi])$ and $^2\Xi(2\Phi[2\Phi])$ as in (63). This way, we can construct quadratic line elements for LC–configurations.

$$\begin{align*}
\int_{\Sigma} \frac{1}{4}s_0^2 = e_i e^\phi (x^i) (dx^i)^2 + \frac{(\tilde{\Phi}_0[\tilde{\Phi}])^2}{4 \Lambda_0} \left[ dy^3 + (\partial r_n) dx^i \right]^2 + \frac{(\tilde{\Phi}_1[\tilde{\Phi}])^2}{4 \Xi(1\Phi[1\Phi])} \left[ dy^4 + (\partial x^\Lambda) dx^i \right]^2 \\
+ \frac{(\tilde{\Phi}_2[\tilde{\Phi}])^2}{4 \Lambda_0} \left[ d\zeta^5 + (\partial r_n) du^r \right]^2 + \frac{(\tilde{\Phi}_3[\tilde{\Phi}])^2}{2 \Xi(2\Phi[2\Phi])} \left[ \Xi(2\Phi[2\Phi]) \right]^2 \\
+ \frac{(\tilde{\Phi}_4[\tilde{\Phi}])^2}{4 \Lambda_0} \left[ d\zeta^6 + (\partial r_n) du^s \right]^2 + \frac{(\tilde{\Phi}_5[\tilde{\Phi}])^2}{2 \Xi(3\Phi[3\Phi])} \left[ \Xi(3\Phi[3\Phi]) \right]^2 \\
+ \cdots \\
+ \frac{(\tilde{\Phi}[\tilde{\Phi}])^2}{4 \Lambda_0} \left[ d\zeta^{3+2s} + (\partial r_{s-1} n) du^{r_{s-1}} \right]^2 + \frac{(\tilde{\Phi}[\tilde{\Phi}])^2}{2 \Xi(4\Phi[4\Phi])} \left[ \Xi(4\Phi[4\Phi]) \right]^2 \\
+ \frac{(\tilde{\Phi}[\tilde{\Phi}])^2}{4} \left[ d\zeta^{4+2s} + (\partial r_{s-1} A) du^{r_{s-1}} \right].
\end{align*}$$

The torsions of such non–vacuum exact solutions (69) generated by respective data ($^*\mathbf{g}, ^*\mathbf{N}, ^*\mathbf{V}$) are zero, which is different from the class of exact solutions (64) with nontrivial canonical d–torsions (A.4) completely determined by arbitrary data ($^*\mathbf{g}, ^*\mathbf{N}, ^*\mathbf{D}$) with Killing symmetry on $\partial_\phi$. For an arbitrary shell s, we have a Killing symmetry on $\partial_s$.

### 3.4 Violation of Killing symmetries and jet prolongations

Considering prolongations of 4–d nonholonomic Ricci soliton and Einstein equations on jet variables we can generate new classes of solutions with non–Killing symmetries both on spacetime coordinates and on jet shells. On $\mathcal{J}^r(\mathbf{V}, V')$, there are two general possibilities to generate "non–Killing" configurations mentioned in Refs. [43, 15] which in this work are generalized for nonholonomic jet variables: 1) to perform a formal embedding into, for instance, higher dimension jet prolonged vacuum spacetimes and/or by 2) "vertical" conformal nonholonomic deformations, in general, with jet variables.

#### 3.4.1 Imbedding into a jet prolonged vacuum solution

Let us analyze an example when a subclass of off–diagonal metrics for 6–d space with jet variables via nonholonomic constraints and re–parameterizations transform into 4–d non–Killing vacuum solutions. We consider geometric data: $\Lambda = ^*\Lambda = ^1\Lambda = 0$; $h_3 = \epsilon_3, h_5 = \epsilon_5, n_k = 0$ and $^1n_\alpha = 0$ with a 2-d h–metric $e_i e^\phi (x^\Lambda, \Lambda = 0) (dx^i)^2$. The coefficients of the Ricci d–tensor are zero, see formulas (A.9)–(A.11). For such conditions, we can not use the equations (46)–(52) derived for $\partial_4 h_3 \neq 0, \partial_6 h_9 \neq 0$ etc. because such conditions do not allow, for instance, values $h_3 = \epsilon_3, h_5 = \epsilon_5$, for any nontrivial data $h_4(x^i, y^4), w_k(x^i, y^4), h_6(x^i, y^4, \zeta^6), w_k(x^i, y^4, \zeta^6)$. Such functions depending, in general, on spacetime and jet variables can be considered as generating functions for vacuum quadratic line elements

$$\begin{align*}
\int_{\Sigma} \frac{1}{4}s_0^2 = e_i e^\phi (x^\Lambda, \Lambda = 0) (dx^i)^2 + \epsilon_3 (dy^3)^2 + h_4 (dy^4 + w_k dx^k)^2 \\
+ \epsilon_5 (d\zeta^5)^2 + h_6 (d\zeta^6 + w_k dx^k + w_4 dy^4)^2
\end{align*}$$

on first 2–d jet shell on $\mathcal{J}^r(\mathbf{V}, V')$. This class of vacuum 6-d metrics with two jet variables are with nonzero nonholonomically induced d–torsion (A.4). Such solutions can not be considered as a subclass of vacuum solutions (A.18) when $h_3 \rightarrow \epsilon_3$ and $h_5 \rightarrow \epsilon_5$ because the conditions $\partial_4 h_3 \neq 0$ and $\partial_6 h_9 \neq 0$ impose additional constraints on the class of possible generating functions $h_4$ and $h_6$. Fixing from the very beginning certain configurations with $\partial_4 h_3 = 0$ and $\partial_6 h_9 = 0$, we can consider the values $h_4, h_6$ and $w_k, w_4$ as independent generating functions.

We generate LC–configurations if the coefficients of the d–metric (70) are subjected additionally to the constraints (65) up to $s = 1$. We can follow a formal procedure which is similar to that outlined in section 3.3.3. For any constant $h_3 = \epsilon_3$ and $h_5 = \epsilon_5$, the conditions $\mathbf{e}_i \ln \sqrt{|h_3|} = 0$ and $\mathbf{e}_a \ln \sqrt{|h_5|} = 0$ are satisfied. The
class of generating functions can be restricted to solve the conditions
\[ \partial_i w_i(x^i, y^4) = \epsilon_i \ln \sqrt{1 + h_4 (x^i, y^4) \partial_i w_j = \partial_j w_i}, \quad (71) \]
\[ \partial_6 w_\alpha (x^i, y^4, \zeta^6) = \epsilon_\alpha \ln \sqrt{1 + h_6 (x^i, y^4, \zeta^6)}, \quad \partial_\alpha \partial_\beta w_\beta = \partial_\beta \partial_\alpha w_\alpha, \]

Such equations do not depend on spacetime coordinate \( y^3 \) and on jet variable \( \zeta^5 \). Prescribing any values of \( h_4 \) and \( h_6 \) we can find LC-admissible \( \omega \)-coefficients solving systems of first order partial derivative equations in \( (71) \). In general, such solutions are defined by certain nonholonomic constraints, i.e. in "non-explicit" form. If the respective \( d \)-metric and N-connection coefficients \( h_4 [\Phi], h_6 [1 \Phi] \) and \( w_k [\Phi], w_k [1 \Phi], w_4 [1 \Phi] \) are determined by \( \Phi (x^i, y^4) \) and \( 1 \Phi (x^i, y^4, \zeta^6) \) satisfying conditions \( (57) \) and \( (58) \) (for such configurations, \( h_3 \) and \( h_5 \) may be not functionals of type \( (57) \)), we can solve the equations \( (71) \) in explicit form.

Choosing any generating functions \( \Phi \) and \( 1 \Phi \) and functionals \( h_4 [\Phi], h_6 [1 \Phi] \) we compute
\[ w_i = \partial_i \Phi / \partial \Phi = \partial_i \tilde{A} \quad \text{and} \quad 1w_i = \partial_i 1\Phi / \partial_6 \Phi = \partial_i 1\tilde{A}, \quad 1w_4 = \partial_4 1\Phi / \partial_6 \Phi = \partial_4 1\tilde{A}, \]
\[ w_i = \partial_i \Phi / \partial \Phi = \partial_i \tilde{A} \quad \text{and} \quad 1w_i = \partial_i 1\Phi / \partial_6 \Phi = \partial_i 1\tilde{A}, \quad 1w_4 = \partial_4 1\Phi / \partial_6 \Phi = \partial_4 1\tilde{A}, \]

for some \( \tilde{A} (x^i, y^4) \) and \( 1\tilde{A} (x^i, y^4, \zeta^6) \) which are necessary to satisfy the equalities \( \partial_i w_j = \partial_j w_i \) and \( \partial_\alpha \partial_\beta w_\beta = \partial_\beta \partial_\alpha w_\alpha \). Applying the functional derivatives of type \( (56) \) and N-coefficients of type \( (52) \) when \( H[\Phi] = \ln \sqrt{h_4} \) and \( 1H[1 \Phi] = \ln \sqrt{h_6} \), we can satisfy the LC-conditions \( (71) \). The constructions from last two paragraphs allow to define a subclass of metrics \( (70) \) determined by generic off–diagonal metrics as solutions of 6–d vacuum Einstein equations with two jet variables from the first shell,
\[ ds_6^{2} = e_i^e \psi (x^k) \Lambda = 0 (dx^i)^2 + e_3 (dy^3)^2 + h_4 [\Phi] (dy^4 + \partial_k \tilde{A} dx^k)^2 + e_5 (d\zeta^5)^2 + h_6 [1 \Phi] (d\zeta^6 + \partial_k \tilde{A} dx^k + \partial_4 \tilde{A} dy^4)^2. \]

The terms \( e_3 (dy^3)^2 \) and \( e_5 (d\zeta^5)^2 \) are for trivial extensions from 4-d to 6-d configurations but imbedded in a nontrivial form in a jet extra dimensional vacuum background. Re-defining the coordinate \( \zeta^6 \to y^3 \), we generate vacuum solutions in 4-d gravity with metrics \( (73) \) depending on all four coordinates \( x^i, y^3 \) and \( y^4 \). This way we mimic certain 4-d gravitational interactions on a jet prolongated 3–d spacetime manifold. Finally, we note that the anholonomy coefficients \( (11) \) are not zero and that such metrics can not be diagonalized by coordinate or jet coordinates transforms. This class of 4–d vacuum spacetimes do not possess, in general, Killing symmetries.

### 3.4.2 "Vertical" nonholonomic conformal and jet deformations

We mention in brief another possibility to generate off–diagonal solutions depending on all spacetime coordinates and, in general, with nontrivial sources of type \((A, 13)\) \((B, 13)\). To work with jet variable is neccessary a formal re-definition of extra dimension coordinates into nonholonomic shell jet coordinates. By straightforward but tedious computations, we can prove

**Corollary 3.1** Any metric
\[ g = g_l (x^k) dx^i \otimes dx^i + \omega^2 (u^a) h_a (x^k, y^4) e^a \otimes e^a \]
\[ + \omega^2 (u^a) h_a (x^k, \zeta^6) e^a \otimes e^a + \cdots + \omega^2 (u^a s) h_a (u^a s, \zeta^4 + 2 s) e^a \otimes e^a, \]

with the conformal \( v \)-factors subjected to the conditions
\[ e_k \omega = \partial_k \omega + n_k \partial_\omega + w_k \partial_4 \omega = 0, \]
\[ e_\beta \omega = \partial_\beta \omega + n_\beta \partial_5 \omega + w_\beta \partial_6 \omega = 0, \]
\[ 2e_\beta \omega = \partial_\beta \omega + 2 n_\beta \partial_7 \omega + 2 w_\beta \partial_8 \omega = 0, \]

... does not change the Ricci d–tensor \((A, \tilde{A})\) \((A, 13)\).

In result of this Corollary, any class of solutions considered in this section can be generalized to non–Killing configurations using "vertical" nonholonomic conformal and jet transformations and deformations.
4 Nonholonomic Jet Prolongations of the Kerr Metric and Ricci Solitons

In this section, we study nonholonomic off–diagonal and/or jet deformations the Kerr black hole solution. The approach develops the results from section 4 of Ref. [15] for jet variables and Ricci soliton configurations when the constructions for massive gravity are re–considered for jet modified gravity theories. A series of new classes of exact solutions when metrics are nonholonomically deformed into general or ellipsoidal stationary configurations in four dimensional gravity with Ricci soliton correction and/or extra dimensions treated jet variables. We cite the monographs [21, 25, 29] for the standard methods and bibliography on stationary black holes.

4.1 N–adapted parameterizations of the Kerr vacuum solution

A 4-d ansatz

\[ ds^2_{[0]} = Y^{-1}e^{2h}(d\rho^2 + dz^2) - \rho^2Y^{-1}dt^2 + Y(d\varphi + Adt)^2 \]

parameterized in terms of three functions \((h, Y, A)\) on coordinates \((\rho, z)\) defines the Kerr solution of the vacuum Einstein equations (for rotating black holes) if we chose

\[
Y = \frac{1 - (p\tilde{x}_1)^2 - (q\tilde{x}_2)^2}{(1 + p\tilde{x}_1)^2 + (q\tilde{x}_2)^2}, \quad A = \frac{2Mq}{p(1 - (p\tilde{x}_1) - (q\tilde{x}_2))}, \quad e^{2h} = \frac{1 - (p\tilde{x}_1)^2 - (q\tilde{x}_2)^2}{p^2[(\tilde{x}_1)^2 + (\tilde{x}_2)^2]}, \quad \rho^2 = M^2(\tilde{x}_1^2 - 1)(1 - \tilde{x}_2^2), \quad z = M\tilde{x}_1\tilde{x}_2,
\]

where \(M = \text{const}\) and \(\rho = 0\) states the horizon \(\tilde{x}_1 = 0\) with the "north / south" segments of the rotation axis, \(\tilde{x}_2 = +1/-1\). For our purposes, such a metric is written in the form

\[ ds^2_{[0]} = (dx^1)^2 + (dx^2)^2 - \rho^2Y^{-1}(e^3)^2 + Y(e^4)^2, \]

(76)

with some coordinates \(x^1(\tilde{x}_1, \tilde{x}_2)\) and \(x^2(\tilde{x}_1, \tilde{x}_2)\) whom

\[(dx^1)^2 + (dx^2)^2 = M^2e^{2h}(\tilde{x}_1^2 - \tilde{x}_2^2)Y^{-1}\left(\frac{dx_1^2}{\tilde{x}_1^2 - 1} + \frac{dx_2^2}{1 - \tilde{x}_2^2}\right)\]

and \(y^3 = t + \tilde{y}^3(x^1, x^2), y^4 = \varphi + \tilde{y}^4(x^1, x^2, t)\). We write

\[ e^3 = dt + (\partial_i\tilde{y}^3)dx^i, e^4 = dy^4 + (\partial_i\tilde{y}^4)dx^i, \]

for some functions \(\tilde{y}^a, a = 3, 4, \) with \(\partial_i\tilde{y}^a = -A(x^k)\).

The Boyer–Linquist coordinates for the Kerr metric were introduced as \((r, \vartheta, \varphi, t)\), where \(r = m_0(1 + p\tilde{x}_1), \tilde{x}_2 = \cos \vartheta\). The parameters \(p, q\) are related to the total black hole mass, \(m_0\) and the total angular momentum, \(am_0\), for the asymptotically flat, stationary and axisymmetric Kerr spacetime. The formulas \(m_0 = Mp^{-1}\) and \(a = Mq^{-1}\) when \(p^2 + q^2 = 1\) implies \(m_0^2 - a^2 = M^2\). In such variables, the metric (76) is written

\[ ds^2_{[0]} = (dx^1)^2 + (dx^2)^2 + \frac{1}{\mathcal{A}(\mathcal{E}^3)^2 + (\mathcal{C} - \mathcal{B}^2/\mathcal{A})(\mathcal{E}^4)^2}, \]

\[ e^3 = dt + d\varphi\mathcal{B}/\mathcal{A} = dy^3 - \partial_i(\tilde{y}^3 + \varphi\mathcal{B}/\mathcal{A})dx^i, e^4 = dy^4 = d\varphi, \]

(77)

In such quadratic expressions, we consider coordinate functions

\[ x^1(r, \vartheta), \ x^2(r, \vartheta), \ y^3 = t + \tilde{y}^3(r, \vartheta, \varphi) + \varphi\mathcal{B}/\mathcal{A}, y^4 = \varphi, \ \partial_i\tilde{y}^3 = -\mathcal{B}/\mathcal{A}, \]

for which \((dx^1)^2 + (dx^2)^2 = \Xi(\Delta^{-1}d\vartheta^2 + d\varphi^2)\), when the coefficients are

\[
\mathcal{A} = -\Xi^{-1}(\Delta - a^2\sin^2\vartheta), \quad \mathcal{B} = \Xi^{-1}a\sin^2\vartheta[\Delta - (r^2 + a^2)],
\]

\[
\mathcal{C} = -\Xi^{-1}\sin^2\vartheta[r^2 + a^2 - \Delta a^2\sin^2\vartheta], \quad \text{and}
\]

\[
\Delta = r^2 - 2m_0 + a^2, \quad \Xi = r^2 + a^2\cos^2\vartheta.
\]

(78)
We consider prime data

\[
\hat{g}_1 = 1, \hat{g}_2 = 1, \hat{h}_3 = -\rho^2 Y^{-1}, \hat{h}_4 = Y, \tilde{N}_i = \partial_i \tilde{y}^i, \tag{79}
\]

or \[\hat{g}_V = 1, \hat{g}_{2V} = 1, \hat{h}_{3V} = \tilde{A}, \hat{h}_{4V} = \tilde{C} - \tilde{B}^2/\tilde{A}, \]

\[\tilde{N}^i_{3V} = -\partial_i (\tilde{y}^{3V} + \varphi \tilde{B}/\tilde{A}), \tilde{N}^i_{4V} = \tilde{w}_V = 0 \]

for the quadratic linear elements \((76), \) or \((77), \) define exact solutions with rotating spherical symmetry of the vacuum Einstein equations parameterized in the form \((25) \) and \((26)\) with zero sources. The Kerr vacuum solution in 4-d GR consists a "degenerate" case of 4-d off–diagonal vacuum solutions determined by primary metrics with data \((77)\) when the diagonal coefficients depend only on two "horizontal" \(N–\)adapted coordinates and the off–diagonal terms are induced by rotating frames.

### 4.2 Deformations of Kerr metrics by an effective Ricci soliton source

Let us consider the coefficients \((79)\) for the Kerr metric as the data for a prime metric \(\hat{g}.\) Our goal is to study nonholonomic off–diagonal deformations of the Kerr solution into a Ricci soliton configuration, i.e. when the vacuum Einstein equations are modified by a Ricci soliton, with

\[\hat{g}, \tilde{N}, \tilde{Y} = 0, \tilde{Y} = 0 \rightarrow (\tilde{g}, \tilde{N}, \tilde{Y} = \Lambda(x^k), \tilde{Y} = \Lambda(x^k, y^4))\]

where the target source \((45)\) is parameterized \((\tilde{Y})^{1}_1 = \tilde{Y}^{2}_3 = \tilde{h}_4 = \tilde{Y}^{3}_4 = \tilde{Y} = \Lambda(x^k)\) and encode contributions of gradient function \(\kappa\) and constant \(\lambda\) from the Ricci soliton equations \((20)\) into solutions of equations \((28)–(30)\). The target metric \(\tilde{g}\) is constrained to define a generic off–diagonal solution of field equations with effective \(h–\) and \(v–\)polarized gravitational constants. In some sense, the Ricci soliton contributions may induce a mass term of type \(\Lambda_0 = \mu^2/\lambda,\) like considered in \([15]\), for respective parameterizations. The \(N–\)adapted deformations of coefficients of metrics and frames are written

\[\{\tilde{g}_i, \tilde{h}_a, \tilde{w}_i, \tilde{n}_i, \} \rightarrow \{\tilde{g}_i = \tilde{n}_a \tilde{g}_i, \tilde{h}_3 = \tilde{n}_4 \tilde{h}_3, \tilde{h}_4 = \tilde{n}_4 \tilde{h}_4, \tilde{w}_i = \tilde{w}_i + \tilde{n}_w \tilde{w}_i, \tilde{n}_i = \tilde{n}_i + \tilde{n}_n \tilde{n}_i\}
\]

where the values \(a, \tilde{w}_i, \tilde{n}_i, \) and \(v\) are functions on three coordinates \((x^k, y^4 = \varphi)\) and \(\tilde{n}_i(x^k)\) depend only on \(h–\)coordinates \(x^k.\) The prime data \(\tilde{g}_i, \tilde{h}_a, \tilde{w}_i, \tilde{n}_i, \) for a Kerr metric are given by coefficients depending only on \((x^k).\) The quadratic elements determined by target solutions of type \((61)\) are parameterized in the form

\[
ds^2_{[dK]} = e^{\psi(x^k)}[(dx'^V)^2 + (dx'^{2V})^2] - \frac{\Phi^2}{4\Lambda_0} \left[ dy^3 + \left( 1n_k + 2 \tilde{n}_2 \int d\varphi \frac{(\partial_\varphi \tilde{\Phi})^2}{\Phi^3/\Xi} \right) dx \right]^2 + \left( \frac{\partial_\varphi \tilde{\Phi}}{\Xi} \right)^2 \left[ d\varphi + \frac{\partial_\varphi \Xi}{\Xi} dx \right]^2,
\]

where \(\Xi = [\varphi \Lambda, \tilde{\Phi}] = \int d\varphi \left( \varphi \Lambda \right) \partial_\varphi (\tilde{\Phi}^2).\)

In terms of \(\eta–\)functions \((39)\) resulting in \(h_4^2 \neq 0, \) \(\tilde{g}_i = c_i e^{\psi(x^k)}\) \(\) and LC–configurations, the solutions of type \((61)\) with effective cosmological constant \(\Lambda_0\) induced by off–diagonal Ricci soliton configurations and \(2n_{k'} = 0\) can be re–written in the form

\[
ds^2 = e^{\psi(x^k)}[(dx'^V)^2 + (dx'^{2V})^2] - \tilde{n}_2 \tilde{A} [dy^3 + \left( \partial_{k'} \eta_n(x^k) - \partial_{k'} (\tilde{y}^{3V} + \varphi \tilde{B}/\tilde{A}) \right) dx^k] + 2 \tilde{n}_2 \left( \tilde{C} - \tilde{B}^2/\tilde{A} \right) [d\varphi + \left( \partial_\varphi \tilde{\eta}_\tilde{A} \right) dx^V], \tag{80}
\]

where there are used "primed" coordinates and prime Kerr data \((77)\) and \((79)\). The gravitational polarizations \((\eta, \eta_0)\) and \(N–\)coefficients \((n_i, w_i)\) are computed

\[
e^{\psi(x^k)} = \tilde{n}_2 = \tilde{n}_2 = \tilde{\Phi}^2/4\Lambda_0 \tilde{A}, \tilde{\eta}_k = \frac{(\partial_\varphi \tilde{\Phi})^2}{\Xi (\tilde{C} - \tilde{B}^2/\tilde{A})}, \tag{81}
\]

\[\tilde{w}_i = \tilde{w}_i + \eta \tilde{w}_i = \partial_i (\tilde{\eta} \tilde{A} \tilde{\Phi}), \tilde{n}_k = \tilde{n}_k + \eta \tilde{n}_k = \partial_k (\tilde{\eta} \tilde{y}^{3V} + \varphi \tilde{B}/\tilde{A} + \eta n),
\]

where \(\eta \tilde{A}(x^k, \varphi)\) is introduced via formulas and assumptions similar to \((88),\) for \(s = 1, \) and \(\psi(x^k)\) is a solution of 2–d Poisson equation,

\[
\partial_1^2 \psi + \partial_2^2 \psi = 2 \Lambda(x^k).
\]
To extract LC–configurations there are used the parameterizations \( \{60\} \) when \( \hat{h}_{ij} \hat{h}_{ij} = \overline{AC} - \overline{B}^2 \) and the N–coefficients are computed as
\[
w_\nu = \dot{w}_\nu + \eta w_\nu = \partial_\nu (\dot{\Phi} \sqrt{|\overline{AC} - \overline{B}^2|}) / \partial_\nu \dot{\Phi} \sqrt{|\overline{AC} - \overline{B}^2|} = \partial_\nu \eta \overline{A}
\]
for \( 1n_\nu = \partial_\nu \eta n(x^k) \) computed for an arbitrary function \( \eta n(x^k) \).

**Theorem 4.1** Quadratic elements \( \{81\} \) define nonholonomic deformations of a prime Kerr solution \( \{\tilde{g}_i, \tilde{h}_a, \tilde{w}_i, \tilde{\eta}_i\} \) \( \{79\} \) into target Ricci soliton LC–configurations with Killing symmetry on \( \partial / \partial \tilde{y}^3 \) determined by polarization functions \( \{81\} \) generated by data \( \{\psi(x^k), \tilde{\eta}_\nu(x^k, \phi), \eta \overline{A}(\tilde{\eta}_\nu), \eta n(x^k), v \Lambda(x^k, \phi), \overline{\Lambda}_0\} \).

**Proof.** Let us show that \( \tilde{\eta}_\nu \) can be defined by \( \tilde{\eta}_\nu \) which can be considered as a generating function instead of \( \dot{\Phi} \). Considering the second formula in \( \{81\} \), we express
\[
\dot{\Phi}^2 = 4\overline{\Lambda}_0 \overline{A} \tilde{\eta}_\nu
\]
and compute \( \Xi = 4\overline{\Lambda}_0 \overline{A} \int d\phi (\dot{\phi} \Lambda) \partial_\phi (\tilde{\eta}_\nu) \). We introduce these formulas into the third formula in \( \{81\} \) and find
\[
\tilde{\eta}_\nu = \overline{A} \left( \partial_\phi \sqrt{|\tilde{y}_\nu|} \right)^2 / (\overline{AC} - \overline{B}^2) \int d\phi (\dot{\phi} \Lambda) \partial_\phi (\tilde{\eta}_\nu).
\]
So, prescribing any polarization function \( \tilde{\eta}_\nu(x^k, \phi) \) and \( v \)–source \( \dot{\phi} \Lambda(x^k, \phi) \), we can compute \( \tilde{\eta}_\nu \). The polarizations \( \tilde{\eta}_\nu = \eta_\nu \) are determined by function \( \psi(x^k) \), i.e. by source \( \Lambda(x^k) \). Finally, prescribing any functional \( \eta \overline{A}(\tilde{\eta}_\nu) \) and function \( \eta n(x^k) \) we can compute the N–connection coefficients for any fixed effective cosmological constant \( \overline{\Lambda}_0 \).

The solutions \( \{80\} \) are for stationary LC–configurations generated canonically as off–diagonal Ricci solitons from Kerr black holes when the new class of spacetimes are with Killing symmetry on \( \partial / \partial y^3 \) and generic dependence on three (from maximally four) coordinates, \( (x^i, r, \phi) \). Off–diagonal modifications are possible even for very small values of the effective cosmological constant which can mimic gravitational effects determined by a gravitational mass parameter \( \mu_g \).

### 4.2.1 Nonholonomically induced torsion and Ricci soliton modified gravity

If we do not impose the LC–conditions \( \{26\} \), a nontrivial source \( v \Lambda(x^k, \phi) \) induces stationary configuration with nontrivial d–torsion \( \{4.1\} \). For simplicity, we can study nonholonomic torsion effects for a \( v \)–source not depending on coordinate \( \phi \), i.e. for \( v \Lambda(x^k) \) The torsion coefficients are determined by metrics of type \( \{61\} \) with nontrivial \( \overline{\Lambda}_0 \) and certain parameterizations of coefficients of an associated N–connection, canonical d–torsion and coordinates distinguishing the prime data for a Kerr metric \( \{79\} \). The corresponding quadratic elements can be written in the form
\[
\begin{align*}
\frac{ds^2}{\overline{A}} &= e^{\psi(x^k)}[(dx^i)^2 + (dx^k)^2] - \frac{\Phi^2}{4|\overline{\Lambda}_0|} A[d\tilde{y}^3] + \left( 1n_k(x^i) + 2n_k(x^i) \partial_{\nu} A \right) dx^k)^2 \\
&= \left( \partial_{\nu} \Phi \right)^2 \left( \frac{\Phi}{\overline{A}} \right)^2 \left( C - B^2 / \overline{A} \right) d\phi + \partial_{\nu} \Phi \partial_{\nu} \Phi d\phi^2,
\end{align*}
\]

where nonzero values of \( 2n_k(x^i) \) are considered. We can see that Ricci soliton effects may result in nontrivial stationary off–diagonal torsion effects if the integration function \( 2n_k \neq 0 \). Considering two different classes of off–diagonal solutions \( \{82\} \) and \( \{80\} \), we can study the issue if a Ricci modified gravity theory is with induced torsion or characterized by additional nonholonomic constraints as in GR (resulting in zero torsion).

It should be noted that configurations of type \( \{52\} \) can be constructed in various theories with noncommutative, brane, extra–dimension, warped and trapped brane type variables in string, or Finsler like and/or Hořava–Lifshits theories \( \{10, 13, 15\} \) when nonholonomically induced torsion effects play a substantial role.
4.2.2 Small Ricci soliton modifications of Kerr metrics and modelling modified and massive gravity

We can construct off–diagonal solutions for superposition of Ricci soliton effects and f–modified and massive gravity interactions, see original contributions and reviews of results in Refs. [8, 31, 32, 14, 35, 19, 20, 24, 5, 28, 15]. Small nonlinear effects and modifications can be distinguished in explicit form if we consider for additional f–deformations, for instance, a "prime" solution for massive gravity/ effective modelled in GR with source $\mu \Lambda = \mu^2 g \lambda(x^k)$, or re–defined to $\mu \Lambda = \mu^2 g \lambda = \text{const.}$ Adding a "small" value $\Lambda$ determined by f–modifications, we work in N–adapted frames with an effective source $\tilde{\Upsilon} = \tilde{\Lambda} + \lambda$. We construct a class of off–diagonal solutions in modified f–gravity generated from the Kerr black hole solution as a result of two nonholonomic deformations

$$(\tilde{g}, \tilde{N}, \tilde{\Upsilon} = 0, \tilde{\Upsilon} = 0) \rightarrow (\tilde{g}, \tilde{N}, \tilde{\Upsilon} = \lambda, \tilde{\Upsilon} = \lambda) \rightarrow (\varepsilon \tilde{g}, \varepsilon \tilde{N}, \varepsilon \tilde{\Upsilon} = \varepsilon \tilde{\Lambda} + \mu \tilde{\Lambda}, \varepsilon \tilde{\Upsilon} = \varepsilon \tilde{\Lambda} + \mu \tilde{\Lambda}),$$

when the target data $g = \varepsilon \tilde{g}$ and $N = \varepsilon \tilde{N}$ depend on a small parameter $\varepsilon$, $0 < \varepsilon \ll 1$. For simplicity, we construct generic off–diagonal solutions with $|\varepsilon \tilde{\Lambda}| \ll |\mu \tilde{\Lambda}|$, when f–modifications in N–adapted frames are much smaller than massive gravity effects. A similar analysis for nonlinear interactions with $|\varepsilon \tilde{\Lambda}| \gg |\mu \tilde{\Lambda}|$ is omitted. The corresponding N–adapted transforms are parameterized

$$(\hat{g}_i, \hat{h}_{ia}, \hat{w}_i, \hat{n}_i) \rightarrow (\hat{g}_i, \hat{h}_{ia}, \hat{w}_i, \hat{n}_i) = (1 + \varepsilon \chi_1 \tilde{g}_i, \hat{h}_{ia}, \varepsilon w_i = w_i + \varepsilon w_i, \varepsilon n_i = n_i + \varepsilon n_i),$$

$$\tilde{\Upsilon} = \mu \tilde{\Lambda}(1 + \varepsilon \tilde{\Lambda}/\mu \tilde{\Lambda}); \quad \varepsilon \tilde{\phi} = \tilde{\phi}(x^k)[1 + \varepsilon \tilde{\phi}(x^k, \varphi)\tilde{\phi}(x^k, \varphi)] = \exp[\varepsilon \varepsilon \tilde{\phi}(x^k, \varphi)],$$

$$d s^2 = \varepsilon^2(1 + \varepsilon \chi_1)e^{\psi(x^k)}(d x^i)^2 + \frac{\varepsilon^2}{4} \tilde{\Upsilon} \left[ d y^2 \right] + \frac{\varepsilon^2}{\tilde{\Upsilon}^2} \left[ d y^2 \right] + \frac{\left( \partial_x \tilde{\phi} \right)^2}{\mu \tilde{\Lambda}} \left[ d x^2 \right].$$

which for LC–configurations, $\partial_x \varepsilon \tilde{\Lambda} = \partial_x \varepsilon \tilde{\Lambda} = \varepsilon \tilde{\Lambda}$ is determined by $\varepsilon \tilde{\phi} = \tilde{\phi} + \varepsilon \tilde{\phi}$ following conditions (72). The values labeled by "e" and "n" are taken as in previous sections but, for simplicity, we omit priming of indices and consider $\varepsilon \pi_i = 0$. The $\chi$– and $w$–values are computed for $\varepsilon$–deformed LC–configurations, see formulas (85) for spacetime components, as solutions of the system (13) in the form (14) for a source $\Upsilon = \mu \Lambda + \varepsilon \Lambda$.

The nonholonomic deformations (83) of the off–diagonal metrics (80) result in a new class of $\varepsilon$–deformed solutions with

$$\chi_1 = \chi_2 = \chi, \quad \text{for} \quad \partial_{x^1} \varepsilon \chi + \varepsilon \partial_{x_2} \chi = 2 \tilde{\Lambda};$$

$$\chi_3 = 2 \frac{\tilde{\phi}^2}{\mu \tilde{\Lambda}} \frac{\tilde{\Upsilon}}{\tilde{\Lambda}}(1 + \varepsilon[2 \tilde{\phi}^2 + 2 - \tilde{\Lambda}/\mu \tilde{\Lambda}][\tilde{\Upsilon}^2 + 2 \tilde{\phi}^2 + \tilde{\phi} \tilde{\phi} - 2 \tilde{\phi}^2 + \varphi B/\tilde{\Lambda}]) d x^k \right)^2 +$$

$$d s^2 = e^\psi(x^k)(1 + \varepsilon \chi(x^k))((dx^l)^2 + (dx^m)^2) - \frac{\tilde{\phi}^2}{\mu \tilde{\Lambda}} \frac{\tilde{\Upsilon}}{\tilde{\Lambda}}(1 + \varepsilon[2 \tilde{\phi}^2 + 2 - \tilde{\Lambda}/\mu \tilde{\Lambda}][\tilde{\Upsilon}^2 + 2 \tilde{\phi}^2 + \tilde{\phi} \tilde{\phi} - 2 \tilde{\phi}^2 + \varphi B/\tilde{\Lambda}]) d x^k \right)^2 +$$

$$ \left( \partial_x \tilde{\phi} \right)^2 \left( \tilde{\Upsilon}^2 - \tilde{\Lambda}^2 \right) \left[ d x^2 \right].$$

We can consider $\varepsilon$–deformations of type (83) for (82) and generate new classes of off–diagonal solutions with nonholonomically induced torsion determined both by Ricci soliton, massive and f–modifications of GR. Such geometric and physical models can not be considered as an effective ones with anisotropic polarizations in GR which also result in different r–jet symmetries and prolongations.
4.3 Nonholonomic $r$–jet off–diagonal Ricci soliton prolongations of the Kerr solution

In [15], generic off–diagonal deformations of the Kerr metric into solutions on higher dimensional spacetimes were studied. Prolongations on $r$–jet variables can performed following similar methods generalized to include nonholonomic variables.

4.3.1 Jet one shell deformations with nontrivial cosmological constant

Jet symmetries impose certain constraints on possible off–diagonal deformations of a Kerr metric generalized for a corresponding class of solutions with any nontrivial cosmological constant in 6–d. (In a similar form we can generalize the constructions for any finite number of shells). The corresponding class of Kerr– de Sitter jet prolongated configurations are generated by nonholonomic deformations $(\bar{g}, \bar{\mathbf{N}}, \bar{v} \bar{\Upsilon} = 0, \bar{\Upsilon} = 0) \rightarrow (\bar{g}, \bar{\mathbf{N}}, \bar{v} \bar{\Upsilon} = \Lambda, \bar{\Upsilon} = \Lambda, \bar{v}_1 \bar{\Upsilon} = \Lambda)$ when solutions are characterized by a jet Killing symmetry on $\partial/\partial \zeta^5$ and parameterized as

$$ds^2 = e^{\psi(x')}|(dx^{1'})^2 + (dx^{2'})^2| - \frac{\Phi^2}{4 \Lambda}[dy^{3'} + \left( \partial_{r'} n(x') - \partial_{y'} (\tilde{y}^{3'} + \varphi E/A) \right) dx^{k'}|^2 +$$

$$\frac{(\partial_{x'} \Phi)^2}{\Lambda \Phi^2} (AC - B^2)[d \varphi + (\partial_{r'} n) dx^{2'}]^2 + \frac{(\partial_{y'} \Phi)^2}{4 \Lambda} [d \zeta^5 + (\partial_{r'} 1) du^*]^2 + \frac{(\partial_{y} \Phi)^2}{\Lambda \Phi^2} [d \zeta^6 + (\partial_{r'} 1) du^*]^2.$$

The generating functions for such d–metrics are parameterized

$$\Phi = \Phi(x', \varphi), \quad 1 \Phi(u^k, \zeta^k) = \Phi(x', t, \varphi, \zeta^6); \quad n = n(x'),$$

$$1_n = 1_n(u^k, \zeta^6); \quad \eta \tilde{A} = \eta \tilde{A}(x', \varphi), \quad 1 \tilde{A} = 1 \tilde{A}(u^k, \zeta^6),$$

and subject to LC–conditions and conditions of integrability and the "primary" data $A, B, C$ are taken for the Kerr one solution in the form [(18)].

Imposing additional symmetries and constraints on the spacetime generating functions, we can "extract" ellipsoid configurations for a subclass of metrics with $\varepsilon$–deformations,

$$ds^2 = e^{\psi(x')}|(dx^{1'})^2 + (dx^{2'})^2| - \frac{\Phi^2}{4 \Lambda}[dy^{3'} + \left( \partial_{r'} n(x') - \partial_{y'} (\tilde{y}^{3'} + \varphi E/A) \right) dx^{k'}|^2 +$$

$$\frac{(\partial_{x'} \Phi)^2}{\Lambda \Phi^2} (AC - B^2)[d \varphi + (\partial_{r'} n) dx^{2'}]^2 + \frac{(\partial_{y'} \Phi)^2}{4 \Lambda} [d \zeta^5 + (\partial_{r'} 1) du^*]^2 + \frac{(\partial_{y} \Phi)^2}{\Lambda \Phi^2} [d \zeta^6 + (\partial_{r'} 1) du^*]^2,$$

where $\zeta, \omega$ and $\varphi_0$ are certain constants determining gravitational toroidal configurations with eccentricity $\varepsilon$. For small values of $\varepsilon$, such metrics describe "slightly" deformed Kerr black holes embedded self–consistently into a generic off–diagonal jet prolonged 6–d spacetime.

4.3.2 Two shell effective 8–d jet prolongations

Applying the AFDM, we can construct two shell nonholonomic jet prolongations of the Kerr metric which, in general, are with nontrivial nontrivial induced torsion for an effective 8-d spacetime with interior jet symmetries. The nonholonomic deformations are defined by data $(\bar{g}, \bar{\mathbf{N}}, \bar{v} \bar{\Upsilon} = 0, \bar{\Upsilon} = 0) \rightarrow (\bar{g}, \bar{\mathbf{N}}, \bar{v} \bar{\Upsilon} = \Lambda, \bar{\Upsilon} = \Lambda, \bar{v}_1 \bar{\Upsilon} = \Lambda, \bar{v}_2 \bar{\Upsilon} = \Lambda)$ extending on jet variables the 4–d quadratic element [(82)] but for a different source (we consider a cosmological constant $\Lambda$ for all dimensions). The corresponding class of solutions is determined by

$$ds^2 = e^{\psi(x')}|(dx^{1'})^2 + (dx^{2'})^2| - \frac{\Phi^2}{4 \Lambda}[dy^{3'} + \left( 1 \nu_k(x') + 2 \nu_k(x') \frac{(\partial_{x'} \Phi)^2}{\Phi^2} - \partial_{y'} (\tilde{y}^{3'} + \varphi E/A) \right) dx^{k'}|^2 +$$

$$+ \frac{(\partial_{x'} \Phi)^2}{\Lambda \Phi^2} (AC - B^2)[d \varphi + \frac{\partial_{x'} \Phi}{\partial_{y'} \Phi} dx^{2'}|^2 + \frac{(\partial_{y'} \Phi)^2}{4 \Lambda} [d \zeta^5 + (\partial_{r'} 1) du^*]^2 + \frac{(\partial_{y} \Phi)^2}{\Lambda \Phi^2} [d \zeta^6 + (\partial_{r'} 1) du^*]^2$$

$$+ \frac{2 \Phi^2}{4 \Lambda} [d \zeta^7 + (\partial_{r_1} 2 n) du^{r_1}]^2 + \frac{(\partial_{y} \Phi)^2}{\Lambda \Phi^2} [d \zeta^8 + (\partial_{r_1} 2 \tilde{A}) du^{r_1}]^2.$$

(87)
The generating functions depend on spacetime and jet variables,

\[
\begin{align*}
\Phi &= \Phi(x^k, \varphi), \quad 1\Phi(u^\beta, \zeta^6) = 1\Phi(x^k, t, \varphi, \zeta^6), \quad 2\Phi(u^\beta, \zeta^8) = 2\Phi(x^k, t, \varphi, \zeta^5, \zeta^6, \zeta^8); \\
1_n &= 1_n(u^\beta, \zeta^6), \quad 2_n = 2_n(u^\beta, \zeta^8), \quad \eta \bar{A} = \eta \bar{A}(x^k, \varphi), \quad 1\bar{A} = 1\bar{A}(x^k, t, \varphi, \zeta^6), \quad 2\bar{A} = 2\bar{A}(x^k, t, \varphi, \zeta^5, \zeta^6, \zeta^8).
\end{align*}
\]  

Such values are chosen in such forms when the nonholonomically induced torsion \( [\Lambda, \bar{A}] \) is effectively modeled on a 4–d pseudo–Riemannian spacetime but on jet shells \( s = 1 \) and \( s = 2 \) the torsion fields are zero. This mean that there are jet coordinate transforms to certain classes of holonomic variables. We can generate jet depending nontrivial torsion \( N \)-adapted coefficients if nontrivial integration functions of type \( 2n_k(x^k) \) are extended to contain jet variables.

### 4.3.3 Kerr Ricci soliton deformations and vacuum \( r \rightarrow \) jet prolongations

There are classes of solutions with jet variables describing vacuum ellipsoid spacetime configurations prolonged on two shell jet variables when the source is of type \( Y = \lambda + \varepsilon(\Lambda + \lambda) = 0 \), with effective mass gravity term \( \mu \Lambda = \mu_g^2 | \lambda | \), result in ellipsoidal off–diagonal configurations in GR. For such metrics, \( \varepsilon = -\mu \Lambda/(\Lambda + \lambda) \ll 1 \) can be considered as an eccentricity parameter. The corresponding models of off–diagonal jet interior gravitational interactions are with \( f \)-modifications when \( \Lambda \) compensates nonholonomic contributions via effective constant \( \Lambda \) and relates the constructions to massive gravity deformations of a Kerr solution. This subclass of solutions for \( \varepsilon \)-deformations into vacuum solutions is parameterized by target ansats

\[
d s^2 = e^{\psi(x^k)}(1 + \varepsilon \chi(x^k))[(dx^1)^2 + (dx^2)^2] - \frac{\Phi^2}{4 \mu \Lambda \Phi^2}[1 + 2\varepsilon \xi(\omega \varphi + \varphi_0)]dy^2 + \left( \partial_{x^k} \eta \n(x^k) - \partial_{x^k} \tilde{\eta}(\tilde{y}^3 + \varphi_0) \right) dx^k]^2 + \frac{1}{4(\Lambda + \lambda)}[d\zeta^5 + (\partial_{\lambda} \nu \zeta^1)du^1]^2 + \frac{1}{4(\Lambda + \lambda)}[d\zeta^7 + (\partial_{\lambda} \nu \zeta^2)du^2]^2 + \frac{1}{4(\Lambda + \lambda)}[d\zeta^8 + (\partial_{\lambda} \nu \zeta^3)du^3]^2.
\]  

The jet components are generated by functions \( 1\Phi, 2\Phi \) and \( N \)-coefficients similarly to solutions (87) but with modified effective jet prolonged sources, \( \Lambda \rightarrow \Lambda + \lambda \). This result shows that interior jet interactions can mimic \( \varepsilon \)-deformations in order to compensated contributions from \( f \)-modifications and even effective vacuum configurations for the 4–d horizontal part. In general, vacuum metrics (89) encode jet modifications/ polarizations of physical constants and coefficients of metrics under nonlinear polarizations of an effective 8-d vacuum distinguishing 4–d nonholonomic configurations and Ricci soliton or massive gravity contributions. Jet variables and \( f \)-modified contributions are described by terms proportional to eccentricity \( \varepsilon \).

### 4.3.4 Jet ellipsoid like Kerr – de Sitter configurations

Using the solutions (87), we construct a class of non–vacuum 8–d jet prolongation solutions with rotoid configurations if we chose for \( \varepsilon \)-deformations (see a similar formula (84) for 4-d) a small polarization

\[
\chi_3 = 2 \frac{\Phi}{\Phi} - (\Lambda + \lambda)/\mu \Lambda = 2\Phi \sin(\omega_0 \varphi + \varphi_0).
\]

Re–expressing \( 1\Phi = \Phi[ (\Lambda + \lambda)/2 \mu \Lambda + \mu \Lambda + \zeta \sin(\omega_0 \varphi + \varphi_0)] \) and (88), and generate a class of generic off–diagonal jet prolonged metrics for ellipsoid Kerr– de Sitter configurations

\[
d s^2 = e^{\psi(x^k)}(1 + \varepsilon \chi(x^k))[(dx^1)^2 + (dx^2)^2] - \frac{\Phi^2}{4 \mu \Lambda \Phi^2}[1 + 2\xi \xi(\omega \varphi + \varphi_0)]dy^2 + \left( \partial_{x^k} \eta \n(x^k) - \partial_{x^k} \tilde{\eta}(\tilde{y}^3 + \varphi_0) \right) dx^k]^2 + \frac{1}{4(\Lambda + \lambda)}[d\zeta^5 + (\partial_{\lambda} \nu \zeta^1)du^1]^2 + \frac{1}{4(\Lambda + \lambda)}[d\zeta^7 + (\partial_{\lambda} \nu \zeta^2)du^2]^2 + \frac{1}{4(\Lambda + \lambda)}[d\zeta^8 + (\partial_{\lambda} \nu \zeta^3)du^3]^2.
\]
These metrics possess a respective Killing symmetry on $\delta \gamma$ and define $\varepsilon$–deformations of Kerr – de Sitter black holes into ellipsoid configurations with effective cosmological constants determined, respectively, by constants in Ricci soliton models, massive gravity, $f$–modifications and jet prolongation contributions.

A N–adapted Coefficients and Proofs

We provide a set of necessary N-adapted coefficient formulas which are important for proofs and applications. A series of results in [10, 13, 15] are reformulated and generalized for nonholonomic $r$–jet variables with conventional $2 + 2 + \ldots$ splitting.

A.1 Torsions and Curvature of d–connections on $J^r(V, V')$ with 2-d shells

For any d–connection structure $^s\mathbf{D}$ and $r$–jet 2d shell prolongations with coefficients (12), there are two important theorems:

Theorem A.1 The N–adapted coefficients of d–torsion $^s\mathbf{T} = \{ \mathbf{T}^{\alpha \beta}_{\gamma \delta} \}$ from (15) are computed recurrently "shall by shell" following formulas

$$
T^{ij}_{jk} = L^i_{jk} - L^j_{kj}, \ T^{ia}_{ja} = C^{ia}_{jb}, \ T^{ai}_{ji} = -N^a_{ji}, \ (A.1)
$$

$$
T^c_{ai} = L^c_{ai} - \partial_a(N^c_j), T^a_{bc} = C^a_{bc} - C^a_{cb}, \ \text{spacetime components};
$$

$$
T^i_{ja,ks} = L^i_{ja,ks} - L^i_{ja,js}, \ T^i_{ja,sa} = C^i_{ja,bc}, \ T^a_{ja,sa} = -N^a_{ja,sa};
$$

$$
T^a_{sa,js} = L^a_{sa,js} - \partial_a(N^c_j), T^b_{bc,cs} = C^b_{bc,cs} - C^b_{cs,bc}, \ r$–jet variables.
$$

Proof. The coefficients (A.4) are computed for any $\tilde{\mathbf{D}} = \{ \Gamma^{\alpha \beta}_{\gamma \delta} \}$ and N–adapted frames (1) and (10) using standard differential form calculus with (15) (or, in operator form, using the formula (13)).

Theorem A.2 The N–adapted coefficients of d–curvature $^s\mathbf{R} = \{ \mathbf{R}^{\alpha \beta}_{\gamma \delta} \}$ from (16) are computed recurrently shell by shell following formulas

$$
R^i_{hjk} = \partial_k L^i_{bj} - \partial_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{ha} N^a_{jk},
$$

$$
R^{a}_{bjk} = \partial_k L^a_{bj} - \partial_j L^a_{bk} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{bc} N^c_{jk},
$$

$$
R^i_{jka} = \partial_a L^i_{jk} - \partial_k L^i_{ja} + C^i_{ja,bc}, \ R^a_{bka} = \partial_a L^a_{bk} - D_k C^a_{bc} + C^a_{bd} T^k_{ka},
$$

$$
R^i_{jbc} = \partial_c C^i_{jb} - \partial_b C^i_{jc} + C^h_{jb} C^i_{hc} - C^h_{jc} C^i_{hb}, \ R^a_{bcd} = \partial_d C^a_{bc} - \partial_c C^a_{bd} + C^a_{be} C^e_{cd} - C^e_{bd} C^a_{ce}, \ \text{spacetime components};
$$

Proof. The coefficients (A.2) are computed for any $\tilde{\mathbf{D}} = \{ \Gamma^{\alpha \beta}_{\gamma \delta} \}$ and N–adapted frames (1) and (10) using a standard differential form calculus with (16) (or, in operator form, using the formula (14)). For $2 + 2 + \ldots$ shell decompositions, such formulas are similar to the coefficients of curvature in higher dimensional spacetime but with re–parameterized (in our case) nonholonomic $r$–jet variables.

\[\square\]
A.2 Sketch of proof of theorem 2.3

We can check by straightforward computations that the conditions of metric compatibility and zero \( h \)- and \( v \)- torsions are satisfied by \( ^s\tilde{\mathbf{D}} = \{ \tilde{\Gamma}^{\gamma}_{\alpha \beta \delta} \} \) with coefficients computed recurrently

\[
\begin{align*}
\hat{L}^i_{jk} &= \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}) , \\
\hat{L}^i_{bk} &= e_b (N^a_k) + \frac{1}{2} h^{ac} (e_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d) , \\
\hat{C}^i_{jc} &= \frac{1}{2} g^{jk} e_c g_{jk}, \quad \hat{C}^i_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_d h_{cd} - e_d h_{bc}) , \\
\end{align*}
\]

(A.3)

The torsion d–tensor \((\text{15})\) of \( ^s\tilde{\mathbf{D}} \) is completely defined by \( ^s\tilde{\mathbf{g}} \) \((18)\) for any chosen \( ^s\mathbf{N} = \{ N^a_i \} \) if the above coefficients \((A.3)\) are introduced "shell by shell" into formulas

\[
\begin{align*}
\hat{T}^i_{jk} &= \hat{L}^i_{jk} - \hat{L}^i_{kj}, \quad \hat{T}^i_{ja} = \hat{C}^i_{jb} \hat{T}^b_{ja} = - N^i_{ja}, \quad \hat{T}^a_{bc} = \hat{C}^a_{bd} \hat{T}^d_{bc} = - \hat{C}^a_{bc} , \\
\end{align*}
\]

(A.4)

We can impose as additional nonholonomic conditions certain equations when all coefficients \((A.4)\) are zero. In this case we extract from \((A.3)\) various LC–configurations. The coefficients of the LC–connection \( \Gamma^{\gamma}_{\alpha \beta \gamma} \) can be computed in standard form in coordinate bases and/or with respect to \( N \)-adapted frames. Taking differences between \( \tilde{\Gamma}^{\gamma}_{\alpha \beta \delta} \) and \( \Gamma^{\gamma}_{\alpha \beta \gamma} \) we find the \( N \)-adapted coefficients of the distortion d–tensor \( \hat{\mathbf{Z}}^{\alpha}_{\beta \gamma} \) (similar formulas are given for Corollary 2.1 and (22) in Ref. \([43]\), in extra dimension coordinates but without jet configurations).

A.3 Proof of theorem 3.1

Such a proof is possible by explicit computing of the \( N \)-adapted coefficients of the canonical Ricci d–tensor on \( \mathbf{J}^{(\mathbf{V}, \mathbf{V}')} \) with 2-d shells. Let us consider an ansatz \((\text{40})\) with \( \partial_1 h_a \neq 0, \partial_0 h_{a_1} \neq 0, \ldots, \partial_2 h_{a_2} \neq 0 \), when the partial derivatives are denoted in the forms \( \partial_i h = \partial h / \partial x^i, \partial_i h = \partial h / \partial y^i, \partial^2 h / \partial y^i \partial y^j \) and \( \partial^2 h / \partial x^i \partial x^j \), where indices taking values \( 5, 6, \ldots \) are for \( 2 + 2 + \cdots \) jet parameterized variables. We can construct more special classes of solutions when the mentioned conditions are not satisfied which requests analyses of more special classes of solutions. For simplicity, we suppose that via frame transforms it is always possible to introduce necessary type parameterizations for d–metrics when necessary types partial derivatives of some coefficients are not zero.

**Lemma A.1** With respect to \( N \)-adapted frames \((\text{4})\) and \((\text{10})\), the nonzero coefficients of the Ricci d–tensor \( \tilde{\mathbf{R}}^{\alpha}_{\beta \gamma} \) \((\text{27})\) for ansatz \((\text{40})\) with Killing symmetry on \( \partial_3 \) possess symmetries determined by formulas: for space
time components with partial derivative operator $\partial$,

$$\tilde{R}_1^i = \tilde{R}_2^i = -\frac{1}{2g_1g_2}[\partial^2_{11}g_2 - \frac{(\partial_1g_1)(\partial_2g_2)}{2g_1} - \frac{(\partial_2g_1)(\partial_1g_2)}{2g_2} + \partial_2^2g_1 - \frac{(\partial_2g_1)(\partial_2g_2)}{2g_2} - \frac{(\partial_2g_1)^2}{2g_1}], \quad (A.5)$$

$$\tilde{R}_3^\kappa = \tilde{R}_4^\kappa = -\frac{1}{2h_3h_4}[\partial^2_{14}h_3 - \frac{(\partial_4h_3)(\partial_1h_4)}{2h_3} - \frac{(\partial_1h_3)(\partial_4h_4)}{2h_4}], \quad (A.6)$$

$$\tilde{R}_{3k} = \frac{h_3}{2h_1}\partial^2_{14}h_k + \left(\frac{h_3}{h_4} - \frac{3}{2}\partial_4h_3\right)\partial_4n_k \frac{h_4}{2h_1}, \quad (A.7)$$

$$\tilde{R}_{4k} = \frac{w_k}{2h_3}[\partial^2_{14}h_3 - \frac{(\partial_4h_3)(\partial_1h_4)}{2h_3} + \frac{(\partial_1h_3)(\partial_4h_4)}{2h_4}] + \frac{\partial_4h_3}{4h_3}\frac{h_3 + h_4}{h_4} - \frac{\partial_k(\partial_4h_3)}{2h_3}, \quad (A.8)$$

and for $r$–jet components with partial derivative operator $\partial$ on jet variables, on shell $s = 1, \tau = 1, 2, 3, 4$;

$$\tilde{R}_5^\tau = \tilde{R}_6^\tau = -\frac{1}{2h_5h_6}[\partial_{66}h_5 - \frac{(\partial_6h_5)^2}{2h_5} - \frac{(\partial_6h_5)(\partial_6h_6)}{2h_5}], \quad (A.9)$$

$$\tilde{R}_{5r} = \frac{h_5}{2h_6}\partial^2_{66}1_{\tau} + \left(\frac{h_5}{h_6}\partial_{66}h_6 - \frac{3}{2}\partial_6h_5\right)\partial_61_{\tau} \frac{h_6}{2h_6}, \quad (A.10)$$

$$\tilde{R}_{6r} = \frac{1}{2h_5}[\partial^2_{66}h_5 - \frac{(\partial_6h_5)^2}{2h_5} - \frac{(\partial_6h_5)(\partial_6h_6)}{2h_6}] + \frac{\partial_6h_5}{4h_5}\frac{h_5 + \partial_5h_6}{h_6} - \frac{\partial_5(\partial_6h_5)}{2h_5}, \quad (A.11)$$

and, for extra shells with number $s$,

$$\tilde{R}_{2s-1}^{2s} = \tilde{R}_{2s}^{2s} = -\frac{1}{2h_3+2s_h_4+2s}[\partial^2_{1+2s+4+2s}h_3+2s - \frac{(\partial_{4+2s}h_3+2s)}{2h_3+2s} - \frac{(\partial_{4+2s}h_3+2s)(\partial_{4+2s}h_4+2s)}{2h_4+2s}], \quad (A.12)$$

when $\tau_1 = 1, 2, 3, 4, 5, 6$;

Proof. We introduce the coefficients of the canonical d–connection $\tilde{\Gamma}^{\gamma_\alpha_{\beta\kappa}}_{\alpha\beta\kappa}$ (A.3) for the d–metric ansatzs (40) and compute the N–adapted d–curvature coefficients (A.2) and (A.1). Then, contracting the indices (following formulas (21), (22) and (23)) we find the nontrivial values of the N–adapted coefficients for the Ricci d–tensor, scalar curvature and Einstein d–tensor of $\tilde{\mathcal{D}}$. Explicit proofs of formulas (A.3)–(A.8) for 4-d and extra dimensional indices are provided in a series of our works, for instance, in [45, 43, 40]. We do not repeat that calculus in this paper.

Introducing $r$–jet variables, we observe that on the first shell, with $s = 1$, the formulas (A.6)–(A.8) are generalized in a similar form but for the partial derivatives $\partial$ on jet variables, with respective indices 5 and 6 for a nonholonomic 2+2+2+... splitting. To avoid ambiguities, we put left labels $s = 1$ on necessary geometric objects and coefficients. On this shell, the first four coordinates $\alpha = 1, 2, 3, 4$ are treated as "base type" ones but $a_1, a_2, ..., = 5, 6$ as conventional "fiber/jet" ones. In symbolic form, the equations (A.9)–(A.11) are constructed via a formal increasing by 2 of respective values of 4-d spacetime indices and introducing dependences on all "base/ spacetime" coordinates.

For shells $s = 2, 3, ..., "fiber/jet"$ indices are labeled with values of type $3 + 2s$ and $4 + 2s$ and the previous (base type) take values $1, 2, ..., 2 + 2s$. The equations (A.12) present a "recurrent" generalizations for a finite number of shells, $s$, of the 1st jet shell when $s = 1$.

Let us analyze some important nonholonomic symmetries of of the canonical Ricci and Einstein d–tensors:
For $s = 1$ and using above formulas, we can compute the Ricci scalar (22) for $1 \tilde{D}$. $1 \tilde{R} = 2(\tilde{R}_1^1 + \tilde{R}_3^3 + \tilde{R}_5^5)$. There are certain $N$–adapted symmetries of the Einstein d–tensor (23) for the ansatz (40), $\tilde{E}_1^1 = \tilde{E}_2^2 = -(\tilde{R}_3^3 + \tilde{R}_5^5), \tilde{E}_4^4 = -(\tilde{R}_1^1 + \tilde{R}_3^3), \tilde{E}_5^5 = \tilde{E}_6^6 = -(\tilde{R}_1^1 + \tilde{R}_3^3)$.

In a similar form, we find symmetries for $s = 2$:

\[
\begin{align*}
\tilde{E}_1^1 &= \tilde{E}_2^2 = -(\tilde{R}_3^3 + \tilde{R}_5^5 + \tilde{R}_7^7), \tilde{E}_4^4 = -(\tilde{R}_1^1 + \tilde{R}_3^3 + \tilde{R}_7^7), \\
\tilde{E}_5^5 &= \tilde{E}_6^6 = -(\tilde{R}_1^1 + \tilde{R}_3^3 + \tilde{R}_5^5).
\end{align*}
\]

We conclude that the nonholonomically jet modified Einstein equations (A.5)–(A.12) for $s = 2$ jet shells with nontrivial $\Lambda$–sources can be written in $N$–adapted form as

\[
\begin{align*}
\hat{R}_1 &= \hat{R}_2 = -\Lambda(x^k), \hat{R}_3 = \hat{R}_4 = -v\Lambda(x^k, y^4), \\
\hat{R}_5 &= \hat{R}_6 = -v\Lambda(u^\beta, \zeta^6), \hat{R}_7 = \hat{R}_8 = -v\Lambda(u^\beta, \zeta^8), \\
&\ldots,
\end{align*}
\]

which can be extended for an arbitrary finite number of jets’ shells.

### A.4 Nonholonomic spacetime and $r$–jet vacuum solutions

#### A.4.1 4–d nonholonomic vacuum configurations

To consider vacuum solutions for $\tilde{D}$ with $v\Lambda = 0$ in (17) we study configurations with $N$–adapted coefficients when $\partial_4 h_3 = 0$ and/or $\partial_4 \phi = 0$. The limits to off–diagonal solutions with $\Lambda = v\Lambda = 0$ are not smooth because multiples ($v\Lambda)^{-1}$ are considered in various coefficients and re–defined generating functions for solutions (61).

Let us analyze the conditions when the nontrivial coefficients of the Ricci d–tensor (A.5)–(A.8) are zero for ansatz (40). The first equation is a typical example of 2–d wave, or Laplace, equation. We can express such solutions in a similar form $g_i = \epsilon_i e^{(x^k, \Lambda = 0)(dx^i)^2}$.

There are three classes of off–diagonal metrics resulting in zero coefficients (A.6)–(A.8).

1. We impose the condition $\partial_4 h_3 = 0$, $h_3 \neq 0$, which results only in one nontrivial equation, see (A.7), $\partial_4^2 n_k + \partial_4 n_k \partial_1 \ln |h_4| = 0$, where $h_4(x^i, y^4) \neq 0$ and $w_k(x^i, y^4)$ are arbitrary functions. If $\partial_4 h_4 = 0$, we must take $\partial_4^2 n_k = 0$. For $\partial_4 h_4 \neq 0$, we get

\[
n_k = n_k + 2n_k \int dy^4/h_4 \quad (A.14)
\]

with integration functions $1n_k(x^i)$ and $2n_k(x^i)$. The corresponding class of nonholonomic vacuum solutions is defined by quadratic line element

\[
d s_{v1}^2 = \epsilon_i e^{(x^k, \Lambda = 0)(dx^i)^2 + h_3(x^k, y^4)[dy^3 + (1n_k(x^i) + 2n_k(x^i) \int dy^4/h_4)dx^i]^2} + h_4(x^i, y^4)[dy^4 + w_i(x^k, y^4)dx^i].
\]

2. Let us assume $\partial_4 h_3 \neq 0$ and $\partial_4 h_4 \neq 0$. We can solve (A.6) and/or (17) for $v\Lambda = 0$ if $\partial_4 \phi = 0$ for coefficients (12) and (11). For $\phi = \phi_0 = const$, we can consider arbitrary functions $w_i(x^k, y^4)$ as generating functions because $\beta = \alpha_i = 0$ for such configurations. The condition (11) is satisfied by any

\[
h_4 = 0h_4(x^k)(\partial_4 \sqrt{|h_3|})^2, \quad (A.15)
\]

where $0h_3(x^k)$ is an integration function and $h_3(x^k, y^4)$ is any generating function. The coefficients $n_k$ are found from (A.7), see (A.14). The corresponding class of nonholonomic vacuum metrics is defined by the quadratic element

\[
d s_{v2}^2 = \epsilon_i e^{(x^k, \Lambda = 0)(dx^i)^2 + h_3(x^k, y^4)[dy^3 + (1n_k(x^i) + 2n_k(x^i) \int dy^4/h_4)dx^i]^2 + 0h_4(x^k)(\partial_4 \sqrt{|h_3|})^2[dy^4 + w_i(x^k, y^4)dx^i]^2}.
\]
3. Another type configurations are stated by \( \partial_3 h_3 \neq 0 \) but \( \partial_1 h_4 = 0 \). The equation \( (\ref{A.6}) \) is \( \partial_4^2 h_3 - \frac{(\partial_3 h_3)^2}{2h_3} = 0 \), with general solution is \( h_3(x^k, y^4) = [c_1(x^k) + c_2(x^k)y^4]^2 \), where \( c_1(x^k), c_2(x^k) \) are generating functions and \( h_4 = 0h_4(x^k) \). For \( \phi = \phi_0 = const \), we can take any values \( w_i(x^k, y^4) \) because \( \beta = \alpha_i = 0 \). The coefficients \( n_i \) are determined by equation \( (\ref{A.7}) \) and/or, equivalently, \( (\ref{A.8}) \) with \( \gamma = \frac{1}{2}\partial_4 h_3 \). We find

\[
n_i = n_i(x^k) + 2n_i(x^k) \int dy^4 h_3|^{-3/2} = n_i(x^k) + 2\tilde{n}_i(x^k)[c_1(x^k) + c_2(x^k)y^4]^{-2},
\]

with integration functions \( n_i(x^k) \) and \( 2n_i(x^k) \), or re-defined \( 2\tilde{n}_i = -2n_i/2c_2 \). The quadratic line elements for this class of vacuum nonholonomic solutions is given by

\[
d_{s_{c3}}^2 = \epsilon_1 e^{\psi(x^k, L=0)}(dx^i)^2 + [c_1(x^k) + c_2(x^k)y^4]^2 dy^3 + \left( n_k(x^k) + 2\tilde{n}_k(x^k)[c_1(x^k) + c_2(x^k)y^4]^{-2}\right) dx^i\]  
\[
+ 0h_4(x^k)[dy^4 + w_i(x^k, y^4)dx^i]^2.
\]

(A.17)

Finally, we note that such solutions are with nontrivial induced torsions \( (A.4) \) and that additional assumptions are necessary in order extract vacuum LC–configurations.

### A.4.2 Nonholonomic r–jet prolongations of vacuum solutions:

The quadratic elements \( (\ref{A.16}), (\ref{A.17}), (\ref{A.18}) \) for off–diagonal jet prolongations of generic off–diagonal solutions have been constructed for nontrivial sources \( ^v\Lambda(x^k, y^4), ^3\Lambda(u^r, \zeta^6), ^6\Lambda(u^r, \zeta^6), ... \) In a similar manner, we can generate jet prolongated vacuum configurations with effective zero cosmological constants extending with r–jet variables the 4-d vacuum metrics of type \( ds_{c1}^2, ds_{c3}^2, \) \( ds_{c3}^2, \) \( ds_{c3}^2 \) etc. It is possible to generate solutions when the sources are zero on some shells and nonzero on other shells.

Let us consider an example of quadratic line element for jet prolongated effective 6-d gravity derived as a \( s = 1 \) generalization of \( (\ref{A.16}) \). For such solutions, \( \partial_4 h_{a} \neq 0, \partial_3 h_{a1} \neq 0, ... \) and \( \phi = \phi_0 = const \), \( \psi = 1 \phi_0 = const, ... \)

\[
d_{s_{c23}}^2 = \epsilon_1 e^{\psi(x^k, L=0)}(dx^i)^2 + h_3(x^i, y^4)[dy^3 + \left( n_k(x^k) + 2n_k(x^k) \int dy^4/h_4 \right) dx^i]^2 + \]

\[
0h_4(x^k)(\partial_4 \sqrt{|h_3|})^2[dy^4 + w_i(x^k, y^4)dx^i]^2 + h_5(u^r, \zeta^6)[d\zeta^5 + \left( \frac{1}{2}n_\lambda(u^r) + \frac{1}{2}n_\lambda(u^r) \int d\zeta^5/h_6 \right) du^\lambda]^2
\]

\[
+ 0h_6(u^r)(\partial_6 \sqrt{|h_5|})^2[d\zeta^6 + w_\lambda(u^r, \zeta^6)du^\lambda]^2,
\]

where \( 0h_3(x^k), 0h_5(u^r), \frac{1}{2}n_\lambda(u^r), \frac{1}{2}n_\lambda(u^r) \) are integration functions. The values \( h_4(x^k, y^4) \) and \( h_6(u^r, \zeta^6) \) are any generating functions on spacetime and jet prolongated variables. We can consider arbitrary functions \( w_i(x^k, y^4) \) \( w_\lambda(u^r, \zeta^6) \) because, respectively, \( \beta = \alpha_i = 0 \) and \( \frac{1}{2}\beta = \frac{1}{2}\alpha_i = 0 \) for such configurations, see formulas \( (\ref{4}), (\ref{3}) \) and \( (\ref{4}), (\ref{3}) \).

### A.5 The LC–conditions

We can consider nonholonomic frame deformations of N–coefficients and ansatz \( (\ref{10}) \) when all coefficients of a nonholonomically induced torsion \( (\ref{4}) \) are zero and \( \Gamma^{\gamma s}_{\alpha s \beta s} = \tilde{\Gamma}^{\gamma s}_{\alpha s \beta s} \). For simplicity, we analyze such conditions for a 4–d spacetimes (generalizations to extra jet shell can be performed recurrently as we explained in section \( 3 \)).

The trivial coefficients of d–torsion \( (\ref{4}) \) are \( \tilde{T}^{i}_{jk} = \tilde{L}^{i}_{jk} - \tilde{L}^{i}_{kj} = 0, \tilde{T}^{i}_{ja} = \tilde{C}^{i}_{ja} = 0, \tilde{T}^{a}_{bc} = \tilde{C}^{a}_{bc} = \tilde{C}^{a}_{cb} = 0 \) for any ansatz \( (\ref{10}) \). Let us compute the nontrivial coefficients \( T^{c}_{aj} = \tilde{L}^{c}_{aj} - e_a(N^c_j) \) and \( T^{a}_{ji} = -N^a_{ji} \). For a 2+2 spacetime splitting, the values

\[
\tilde{L}^{a}_{bi} = \partial_b N^a_i + \frac{1}{2}h^{ac}(\partial_i h_{bc} - N^c_i \partial_b h_{ci} - h_{dc} \partial_b N^d_i - h_{db} \partial_c N^d_i),
\]

\[
\tilde{T}^{a}_{aj} = \frac{1}{2}h^{ac}(\partial_i h_{bc} - N^c_i \partial_b h_{ci} - h_{dc} \partial_b N^d_i - h_{db} \partial_c N^d_i).
\]
are computed for $N^3_i = n_i(x^k, y^k), N^4_i = w_i(x^k, y^k)$; $h_{bc} = \text{diag}[h_3(x^k, y^k), h_4(x^k, y^k)];\ h^{ac} = \text{diag}[(h_3)^{-1}, (h_4)^{-1}]$.

We write

$$\hat{T}_{bi}^3 = \frac{1}{2} h^{3c} (\partial_i h_{bc} - N^c_i \partial_c h_{bc} - h_{dc} \partial_b N^d_i - h_{db} \partial_c N^d_i) = \frac{1}{2 h_3} (\partial_i h_{bc} - w_i \partial_4 h_{bc} - h_3 \partial_b n_i),$$

e.i. $\hat{T}_{3i}^2 = \frac{1}{2 h_3} (\partial_i h_3 - w_i \partial_4 h_3), \hat{T}_{4i}^3 = \frac{1}{2} \partial_4 n_i$.

In a similar form, we compute

$$\hat{T}_{bi}^4 = \frac{1}{2} h^{4c} (\partial_i h_{bc} - N^c_i \partial_c h_{bc} - h_{dc} \partial_b N^d_i - h_{db} \partial_c N^d_i) = \frac{1}{2 h_4} (\partial_i h_{bc} - w_i \partial_4 h_{bc} - h_4 \partial_b w_i - h_3 \partial_4 n_i - h_{4b} \partial_4 w_i),$$

e.i. $\hat{T}_{3i}^4 = -\frac{h_3}{2 h_4} \partial_4 n_i, \hat{T}_{4i}^4 = \frac{1}{2 h_4} (\partial_i h_4 - w_i \partial_4 h_4) - \partial_4 w_i$.

The coefficients of the N-connection curvature $N J^a_{ij} = e_j(N^a_i) - e_i(N^a_j)$ are expressed

$$N J^a_{ij} = \partial_j (N^a_i) - \partial_i (N^a_j) = N^b_i \partial_b N^a_j + N^b_j \partial_b N^a_i = \partial_j (N^a_i) - \partial_i (N^a_j) - w_j \partial_4 N^a_i + w_i \partial_4 N^a_j$$

with such nontrivial values:

$$N J^3_{21} = - N J^3_{21} = \partial_2 n_1 - \partial_1 n_2 - w_2 \partial_4 n_1 + w_1 \partial_4 n_2,$$
$$N J^3_{12} = - N J^3_{12} = \partial_2 w_1 - \partial_1 w_2 - w_2 \partial_4 w_1 + w_1 \partial_4 w_2.$$  \hspace{1cm} (A.19)

Summarizing above formulas for $\partial_4 n_i = 0$ and $\partial_4 n_1 = \partial_4 n_2 = 0$, we get the condition of zero torsion for ansatz [40] with $n_k = \partial_k n(x^k)$,

$$\frac{1}{2 h_3} (\partial_i h_3 - w_i \partial_4 h_3) = 0, \quad \frac{1}{2 h_4} (\partial_i h_4 - w_i \partial_4 h_4) = \partial_4 w_i, \quad \partial_2 w_1 - \partial_1 w_2 - w_2 \partial_4 w_1 + w_1 \partial_4 w_2 = 0.$$  \hspace{1cm} (A.20)

form, we can define a LC-configuration. The final step is impose the condition that the coefficients $n_k$ do not depend on $y^4$. This can be fixed the form $n_k(x^i) = \partial_k n(x^i)$ and $2n_k = 0$, i.e. $n_k = \partial_k n(x^i)$.

Finally, we note that the LC-conditions can be formulated recurrently, in similar forms, for higher order shells of jet coordinates using the partial derivative operator $\partial$ both for zero and non-zero sources.

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