INTEGRABLE MODELS, SUSY GAUGE THEORIES
AND STRING THEORY*

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We consider the close relation between duality in $N = 2$ SUSY gauge theories and integrable models. Various integrable models ranging from Toda lattices, Calogero models, spinning tops, and spin chains are related to the quantum moduli space of vacua of $N = 2$ SUSY gauge theories. In particular, $SU(3)$ gauge theories with two flavors of massless quarks in the fundamental representation can be related to the spectral curve of the Goryachev-Chaplygin top, which is a Nahm’s equation in disguise. This can be generalized to the cases with massive quarks, and $N_f = 0, 1, 2$, where a system with seven dimensional phase space has the relevant hyperelliptic curve appear in the Painlevé test. To understand the stringy origin of the integrability of these theories we obtain exact nonperturbative point particle limit of type II string compactified on a Calabi-Yau manifold, which gives the hyperelliptic curve of $SU(2)$ QCD with $N_f = 1$ hypermultiplet.

1. Introduction

One of the challenging problems in theoretical physics is to understand nonperturbative behavior of field theories and string theories. Last several years we have witnessed a very important progress in understanding duality of $N = 2$ SUSY gauge theories. For the first time, we have tools to deal with exact nonperturbative calculations. The low energy description of these theories can be encoded by Riemann surfaces and the integrals of meromorphic one differentials over the periods of them. Exact effective actions of these theories can be described by holomorphic functions, so-called prepotentials. With these we can probe the strong coupling limits of the theories. It would desirable to have a better understanding of these structures, through other physical systems where similar structures appear. In fact in the study of the integrable models in two dimensions, which were long studied in the hopes of giving insights into higher dimensional systems, this structure on Riemann surfaces plays a crucial role. Among the methods of solving integrable

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models, in inverse scattering method we obtain the solitons solutions as potentials of a quantum mechanics problem, given the scattering data. The spectral parameter plays the role of the energy. If we consider the periodic soliton solutions, then the spectral parameter develops forbidden zones, just as we are familiar in solid state physics. Analytic continuation of the spectral parameter with the forbidden zones gives us the Riemann surface with genus $g > 0$. By now there are many works which connect these low-energy effective theories with known integrable systems. To relate effective quantum field theories with integrable systems, one needs averaging over fast oscillations, i.e. Whitham averaging. It was analyzed that the periods of the modulated Whitham solution of periodic Toda lattice give rise to the mass spectrum in the BPS saturated states. For the case of $SU(N_c)$ gauge theory with a single hypermultiplet in the adjoint representation, the corresponding integrable system was found and recognized to be the elliptic spin Calogero model, where short range interaction of Toda lattice is generalized to a long ranged integrable interaction. This connection was developed by identifying the coupling constant of Calogero system with the mass of a hypermultiplet in the adjoint representation, starting from the Lax operator for the Calogero model and calculating the spectral curve explicitly. The integrable system related to gauge theories with to massive hypermultiplets in the fundamental representation was also discussed. Here the relevant integrable models are spinning tops and/or spin chain models! These seemingly different systems share a common mathematical structure. More recently, Seiberg and Witten further studied the $N = 4$ gauge theories in 3 dimensions and structures such as Nahm’s equations, which appears in the study of moduli space of multimonopoles. Hyperkähler structures such as Atiyah-Hitchin space, its double covering, and Taub-NUT spaces with dihedral quotients appear. We can easily be puzzled by the plethora of models all claiming a relation to the nonperturbative SUSY dynamics. It is likely that there is a underlying unified point of view. In fact there is a mapping of monopoles spectral curves from the Nahm’s equations and that of Toda lattices. Furthermore we can map the Nahm’s equation to the generalized Kowalevski top, extending the bridges among the models. We can actually extend this and relate the Nahm’s equation to the GC top. Underlying scheme might as well be the self-dual Yang Mills equations: a wide class of low dimensional integrable models can be obtained from the self-dual Yang Mills equation. The relation between the spin chain models and the spinning tops is less clear, even though both appear in SW models with massive hypermultiplets. This clearly invokes a further study.

Motivated by the works in SUSY gauge theories, the duality really blossomed in the context of string theories. Even though the current string duality is very useful in understanding strong coupling regime utilizing the weak coupling expansion in the dual model, it is at the moment not clear how the intermediate coupling regime will be described. Certain self duality might be useful there. Among these, the $N = 2$ type II/heterotic duality in four dimensions has been proposed and further studied in many subsequent papers. In fact, it was extended to the $F$-theory/heterotic
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duality in eight dimensions where the heterotic strings compactified on $T^2$ is dual to $F$-theory compactified on $K3$ which admits an elliptic fibration. Further compactification in six dimensions leads to the duality between $F$-theories compactified on Calabi Yau (CY) manifolds and heterotic strings on $K3$. Among the many ways to check the consistency on this duality one can consider the point like limit of four dimensional $N = 2$ SUSY compactifications of heterotic strings, and see the resulting gauge theory which reproduces the exact field theory results. Additional question would be whether one can get also matter from the point like limit of the string theory compactification. We would like to see how the $N = 2$ SUSY QCD is embedded in this compactification of string theory.

2. SUSY Gauge Theories

We first consider the $N = 2$ SUSY $SU(N_c)$ gauge theories with $N_c$ colors and $N_f$ flavors. The field content of the theories consists, in terms of $N = 1$ superfields, a vector multiplet $W_\alpha$, a chiral multiplet $\Phi$, and two chiral superfields $Q_i^a$ and $\tilde{Q}_{ia}$ where $i = 1, \ldots, N_f$ and $a = 1, \ldots, N_c$. The curve representing the moduli space with $N_f < N_c$ case is as follows:

$$y^2 = (x^{N_c} - \sum_{i=2}^{N_c} u_i x^{N_c-i})^2 - \Lambda_{N_f}^{2N_c-N_f} \prod_{i=1}^{N_f} (x + m_i), \quad (1)$$

where the moduli $u_i$’s are the vacuum expectation values of a scalar field of the $N = 2$ chiral multiplet, and $m_i$’s are the bare quark masses. It turns out that from the point of view of integrable theory, $u_i$’s correspond to the integrals of motion. The second term in Eq.(1) is due to the instanton corrections. For the $N_c \leq N_f < 2N_c$ case, the correction due to matter is different. By inspection we see that the case of $N_f = 0$ corresponds to the periodic Toda lattice with $N_c$-particles, after an appropriate rescaling of the variables. In general the following type of hyperelliptic curve appears

$$y^2 = P_{N_c}(x)^2 - Q_m(x), \quad (2)$$

where $P_n(x)$ and $Q_m(x)$ are polynomials of order $n$ and $m$. It is natural to ask which integrable theories have such spectral curves. The form is indicative of Riemann surfaces with punctures as well as genus. We will start with the known case of $y^2 = P_3(x)^2 - ax^2$ ($a$ is a constant) which corresponds to the so called Goryachev-Chaplygin (GC) top.

3. Integrable Models

Let us now review the classical mechanics of rotation of a heavy rigid body around a fixed point, which is described by the following Hamiltonian:

$$H(M, p) = \frac{M_1^2}{2I_1} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_3} + \gamma_1 p_1 + \gamma_2 p_2 + \gamma_3 p_3. \quad (3)$$

It was noted in that there exists such a connection.
The phase space of this system is six dimensional: $M_i$’s are the components of the angular momentum and $p_i$’s are the linear momenta. $I_i$’s are the principal moments of inertia of the body and $\gamma_i$’s are the coordinates of the center of mass. There are four known integrable cases for the Hamiltonian in Eq. (3). In all these cases there is always one obvious integral of motion, the energy. It is necessary to get one extra integral independent of the energy for complete integrability according to Liouville’s theorem.

Apart from the better known cases of Euler’s and Lagrange’s tops, we have the following two other cases: i) Kowalewski’s case: $(I_1 = I_2 = 2I_3, \gamma_3 = 0)$. The extra integral can be found by the Painlevé test or the Kowalewski’s asymptotic method. Here the symmetry group is $SO(3, 2)$. ii) Goryachev-Chaplygin’s case: $(I_1 = I_2 = 4I_3, \gamma_3 = 0)$. We need $M_1p_1 + M_2p_2 + M_3p_3 = 0$, which leads to a new integral of motion together with $H$ the Hamiltonian and $G$ the GC integral. The Lax operator for the GC top is given as follows, where we have written it down in a form useful when comparing to Nahm’s equation:

$$zL(z) = \begin{pmatrix} 0 & -ip_3 & 0 \\ ip_3 & 0 & p_2 - ip_1 \\ 0 & p_2 + ip_1 & 0 \end{pmatrix} - 2iz \begin{pmatrix} 0 & 0 & \frac{iM_2 + M_1}{2} \\ 0 & -M_3 & 0 \\ -\frac{iM_2 + M_1}{2} & 0 & M_3 \end{pmatrix} + z^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix}. \quad (4)$$

This Lax operator depends on the phase space variables, $M_i$, $p_i$ and on the spectral parameter $z$. Now it is easy to calculate the spectral curve from the equation $\text{Det}(L(z) - xI) = 0$, which gives the spectral curve as follows:

$$x^3 + 2xH - 2iG - x(4z^2 + \frac{\lambda^2}{z^2}) = 0, \quad (5)$$

where $H = \frac{1}{2}(M_1^2 + M_2^2 + 4M_3^2) - 2p_1$ is the Hamiltonian, and $G = M_3(M_1^2 + M_2^2) + 2M_1p_3$ is the GC integral. We also have the following constraints:

$$p_1^2 + p_2^2 + p_3^2 = \lambda^2, \quad \text{and} \quad M_1p_1 + M_2p_2 + M_3p_3 = 0. \quad (6)$$

Now we see that the spectral curve depends on special combinations of $M_i$, $p_i$’s, which are nothing but the integrals of motion. By introducing variable $y = x(4z^2 - \frac{\lambda^2}{z^2})$, we thus get

$$y^2 = (x^3 + 2Hx - 2iG)^2 - 16\lambda^2x^2, \quad (7)$$

which are the same as the curve for GC top with some rescalings. To relate this to the curve of SUSY gauge theory we make the following simple substitutions:

$$H \to -\frac{1}{2}u_2, \quad G \to -\frac{i}{2}u_3, \quad \lambda^2 \to \frac{1}{16}\Lambda_2^4. \quad (8)$$
It is easy to see that Eq.(7) exactly coincides with Eq.(1) for the particular case of $N_c = 3, N_f = 2$ and $m_1 = m_2 = 0$!

As mentioned in the introduction, the GC top can be viewed as the Nahm’s equation. To see this clearly, let us cast the Lax operator of GC top in a form useful to compare with the Nahm’s equation. We see that we can actually utilize Hitchin’s parametrization and put the Lax operator as follows:

$$
\begin{align*}
    zL(z) &= (T_1 + iT_2) - 2izT_3 + z^2(T_1 - iT_2), \\
    A(z) &= T_0 - iT_1 + z(T_1 - iT_2),
\end{align*}
$$

(9)

where $T_0$ and $T_i$’s ($i = 1, 2, 3$) are certain $3 \times 3$ matrices. Then clearly the Lax equation $dL/dt = [L, A]$ can be recast as the Nahm’s equation:

$$
\frac{dT_i}{dt} = [T_i, T_0] + \frac{1}{2} \epsilon_{ijk}[T_j, T_k].
$$

(10)

The components of $T_i$’s can easily be written down. Ward observed that the Nahm’s equation can be written as a Lax equation, and that it also can be regarded as a Yang-Baxter equation. We can gauge away $T_0$. The Nahm’s equation and can be mapped into Toda lattice when $T_i$’s are of a special form and also to the generalized Kowalewski’s top. The spectral curve of the Nahm’s equation obtained from

$$
\text{Det}
\begin{pmatrix}
    x + i \sum_{j=1}^{3} \eta_j T_j
\end{pmatrix}
= 0,
$$

(11)

where $\eta_1 = -i(1 + z^2), \eta_2 = 1 - z^2, \eta_3 = -2z$, describes the moduli space of multimonopole configuration, and arises in the twister formulation for monopoles. We note that in a recent work of Seiberg and Witten of SUSY gauge theories on compactified three dimensional spacetime, Dancer’s spectral curve appeared in the context of $SU(2) N_f = 1$ case.

4. The Massive Case

Since we have seen the intimate relation between the GC top and the SUSY $SU(3)$ gauge theory with two flavor massless hypermultiplets, it is natural for us to extend this to the massive case. For this purpose we need an integrable system which has both the GC top and the three body Toda lattice as particular limits, because the latter corresponds to pure gauge theory with no matter. The Hamiltonian system which realizes this is hard to imagine, but there exists a system of coupled seven nonlinear differential equations in mathematical literature. This system has the following nonlinear “equations of motion”:

$$
\begin{align*}
    \dot{z}_1 &= -8z_7, & \dot{z}_2 &= 4z_5, & \dot{z}_3 &= 2(z_4z_7 - z_5z_6), & \dot{z}_4 &= 4z_2z_5 - z_7, \\
    \dot{z}_5 &= z_6 - 4z_2z_4, & \dot{z}_6 &= -z_1z_5 + 2z_2z_7, & \dot{z}_7 &= z_1z_4 - 2z_2z_6 - 4z_3.
\end{align*}
$$

(12)
There are five constants of motion of the system:

$$6a = z_1 + 4z_2^2 - 8z_4, \quad 2b = z_1z_2 + 4z_6, \quad c = z_4^2 + z_6^2 + z_3,$$

$$d = z_4z_6 + z_5z_7 + z_2z_3, \quad e = z_6^2 + z_7^2 - z_1z_3. \quad (13)$$

Although the Lax operator for this system is not readily available, we can still apply the asymptotic method due to Kowalewski to this system. The integrable system generally possesses the Painlevé property, i.e., solutions have only movable poles in the complex plane. We thus take:

$$z_i = t - n_i \sum_{j=0}^{\infty} A_{ij} t^j$$

where $n_i$'s are positive integers. Substituting these Laurent expansions into the system of Eqs. (12) and (13), one finds $n_i = 1$ for $i = 1, 2, 3$, $n_i = 2$ for $i = 4, 5, 6, 7$ and a relation between the coefficients of $A_{ij}$'s. Then we obtain the Laurent solutions for this system with seven parameters, five of which are from the constants of motion, $a, b, c, d, e$ and two additional ones $x$ and $y$ where they satisfy the equation for an hyperelliptic curve:

$$y^2 = (2x^3 - 3ax + b)^2 - 4(4cx^2 + 4dx + e). \quad (14)$$

We clearly see that with the following substitution this gives the algebraic curves given in Eq. (1) of $N = 2$ SUSY $SU(3)$ gauge theories with massive quarks of two flavors of masses $m_1$ and $m_2$:

$$y \rightarrow 2y, \quad a \rightarrow \frac{2}{3}w_2, \quad b \rightarrow -2w_3, \quad c \rightarrow \frac{1}{4} \Lambda_2^4,$$

$$d \rightarrow \frac{\Lambda_2^4}{4}(m_1 + m_2), \quad e \rightarrow \Lambda_2^4 m_1 m_2. \quad (15)$$

When we consider the case of $c = 0$, then this leads to gauge theory coupled to one massive quark of mass $m_1$ or massless one ($N_f = 1$) after similar substitution. For the case of $c = d = 0$, the usual periodic Toda lattice is recovered, and for $d = e = 0$ we get back GC top. So clearly we have a unifying model of two seemingly different systems.

5. String Theory

Now let us consider the point like limit of string theories, where the $N = 2$ SUSY QCD is embedded in a compactification of string theory. We obtain exact nonperturbative point particle limit of a four dimensional $N = 2$ SUSY compactification of heterotic strings. Using Heterotic/type II duality, we show how $N = 2$ SUSY QCD with one flavor of massless quark arises in type II string compactification on Calabi-Yau manifolds.

Such analyses were performed for the following two cases. First is the case where the $E_8 \times E_8$ heterotic string compactified on $K3 \times T^2$ is dual to the type IIB (or type IIA) theory compactified on a CY manifold (or its mirror), which is the weighted projective space of weights 1,1,2,2,6. The point like limit of this model was shown to yield the exact results of Seiberg and Witten with pure $N = 2$ Yang-Mills...
theory with gauge group $SU(2)$. Second case is where the weighted projective space has weights 1,1,2,8,12, the point like limit is known to be the that of pure $N = 2$ $SU(3)$ Yang-Mills theory. By going to the conifold locus of the CY manifold and blowing it up, one can indeed obtain the algebraic curves for all the cases of $SU(n)$ gauge groups.

In order to relate these gauge theories with matter with string compactification scheme on a CY manifold, we look for the known cases where the explicit forms of the discriminant and the Picard-Fuchs operators of the CY manifolds have been worked out. One of the strong candidate is that of the weighted projective space with weights 1,1,1,6,9. This is because if we look at the discriminant locus in terms of the coordinates describing the large moduli parameters, the singularity structure of this is identical to that of $N = 2$ $SU(2)$ gauge theory coupled to single ($N_f = 1$) flavor in the fundamental representation.

Consider the moduli space of the mirror of the weighted projective space with weights 1,1,1,6,9 CY manifold with Hodge numbers $h_{1,1} = 2, h_{2,1} = 272$ whose defining polynomial given as follows:

$$p = x_1^{18} + x_2^{18} + x_3^{18} + x_4^3 + x_5^2 - 18\psi x_1 x_2 x_3 x_4 x_5 - 3\phi x_1^6 x_2^6 x_3^6 = 0. \quad (16)$$

This CY manifold has 2 vector multiplets whose scalar expectation values correspond to $\psi$ and $\phi$ and 273 hypermultiplets including a dilaton field. It is convenient to introduce the following variables that were used for the complex moduli of the mirror:

$$x = \frac{3\phi}{(18\psi)^{1/3}}, \quad y = \frac{1}{(3\phi)^{1/3}}. \quad (17)$$

The discriminant can be written as

$$\Delta = (1 - \bar{x})^3 - \bar{x}^3 \bar{y}, \quad \text{where} \quad \bar{x} = 243 x, \quad \bar{y} = 3^3 y. \quad (18)$$

For weak coupling, $\bar{y} \rightarrow 0$, there exists a triple singularity at $\bar{x} = 1$. The locus on which the CY manifold acquires a conifold point is where $\Delta = 0$.

In order to go to the point like limit of strings ($\alpha' \rightarrow 0$) we would like to identify $\bar{x} - 1$ with the vacuum expectation value of 4D gauge theory $u$ upto leading order of $\alpha'$. In fact, to be dimensionally correct we need

$$\bar{x} = 1 + \alpha' u + O(\alpha'^2) = 1 + \frac{\epsilon}{\Lambda_1} u + O(\epsilon^2), \quad (19)$$

where $\Lambda_1$ is the renormalization scale parameter of the theory with $N_f = 1$. At the conifold locus, we have

$$\bar{y} = \frac{(1 - \bar{x})^3}{\bar{x}^3} \frac{1}{u^3}. \quad (20)$$

When we expand for $\psi$ and $\phi$ we get

$$\psi = \frac{1}{18} \epsilon^{-1/2} (1 + \epsilon \psi_1 + \cdots), \quad \phi = \frac{1}{3} \epsilon^{-1} (1 + \epsilon u + \cdots), \quad (21)$$

where $\psi_1$ is independent of $u$. With the expressions in the defining polynomial, we can now compare with the the curve of SUSY QCD. From the requirement that
the coefficient of the term linear in $u$ be order of $\epsilon$, we immediately see that the product of $x_1^6 x_2^6 x_3^6$ should be order of $\epsilon$. Taking the following expansion,

\begin{align}
x_1 &= \epsilon \hat{x}_1 a_1 + \cdots, \quad x_2 = \epsilon \hat{x}_2 a_2 + \cdots, \quad x_3 = a_3 (1 + \epsilon b_3 + \cdots), \\
x_4 &= a_4 (1 + \epsilon b_4 + \cdots), \quad x_5 = a_5 (1 + \epsilon b_5 + \cdots),
\end{align}

(21)

and by requiring that $p$ has the following form up to the first power of $\epsilon$, we recover the hyperelliptic curve for $SU(2) N_f = 1$ gauge theory:

\[ p = \epsilon \left( 2u - 2x^2 + \hat{z} + \frac{A_n^3(x + m)}{\hat{z}} + v^2 + w^2 \right) + O(\epsilon^2), \]

(22)

once we fix the functions in an appropriate form. The change of variable $y = \hat{z} - P_2(x)$ gives rise to the explicit form of the curve given in Eq. (1).

Now we consider the periods. As is the case of pure SUSY Yang-Mills theory, $p = 0$ differs from (1) by quadratic terms. On the other hand, the holomorphic 3-form $\Omega$ is

\[ \Omega = d \left( \ln \frac{\hat{z}}{\sqrt{Q(x)}} \right) \wedge \left[ \frac{dv \wedge dx}{dP_3} \right]. \]

(23)

In order to integrate $\Omega$ over $v$, we solve for $w$ from $p = 0$, and plug this value of $w$ into (23). Then the integral over $v$ becomes trivial, leading to the following result:

\[ \int v \Omega = dx d\ln \frac{\hat{z}}{\sqrt{Q(x)}} = d \left( x d\ln \frac{\hat{z}}{\sqrt{Q(x)}} \right). \]

(24)

Now we see that the integral of $\Omega$ on a 3-cycle of the CY manifold produces an integral of $dS$ over the cycle of Riemann surface.

### 6. Conclusion

If one wishes to obtain the prepotentials which are needed for exact effective action in SUSY gauge theory, we should consider quasiclassical $\tau$ functions in the context of integrable theory as in the case of pure gauge theory. Of course the Nahm’s equation can be subjected to averaging. There are algebraic curves for higher rank cases with generic $N_c$ and $N_f$. Does this mean that a ‘higher’ dimensional generalization of GC top exists? Although there exists multi-dimensional generalization of Kowalewski top, it is not available for GC top yet. The relation to the Nahm’s equation might be helpful here. String duality and integrability of SUSY gauge models are closely related, but still needs further systematic investigation, especially when we have matter.

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