Dynamical classical superfluid-insulator transition in a Bose-Einstein condensate on an optical lattice

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Abstract.
We predict a dynamical classical superfluid-insulator transition in a Bose-Einstein condensate (BEC) trapped in a combined optical and axially-symmetric harmonic potentials initiated by a periodic modulation of the radial trapping potential. The transition is marked by a loss of phase coherence in the BEC and a subsequent destruction of the interference pattern upon free expansion. For a weak modulation of the radial potential the phase coherence is maintained. For a stronger modulation and a longer time of holding in the modulated trap, the phase coherence is destroyed signaling a classical superfluid-insulator transition. The results are illustrated by a complete numerical solution of the axially-symmetric mean-field Gross-Pitaevskii equation for a repulsive BEC. Suggestion for future experiment is made.

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The experimental loading of a cigar-shaped Bose-Einstein condensate (BEC) in both one-\cite{1,2} and three-dimensional\cite{3} optical lattice potentials has allowed the study of quantum phase effects on a macroscopic scale such as interference of matter waves\cite{4}. There have been several theoretical studies on different aspects of a BEC in one-\cite{5} and three-dimensional\cite{6} optical lattice potentials. The phase coherence between different sites of a trapped BEC on an optical lattice has been established in recent experiments\cite{1,2,7,3,8} through the formation of distinct interference pattern when the traps are removed. In a one-dimensional optical lattice potential the expanding pattern consists of a large central piece and two smaller ones moving in opposite directions on a straight line\cite{7}. In a two-dimensional optical lattice potential the pattern consists of a large central piece and eight others on the sides of an expanding square\cite{6}. In a three-dimensional optical lattice potential the pattern consists of a large central piece and twenty-six others on the surface of an expanding cube\cite{3,6}.

The interference pattern is a consequence of phase coherence in the BEC generated due to free quantum tunneling of atoms from one optical lattice site to another originating in the superfluid state of the system\cite{3,9}. Equal phase at all points or a slowly (and orderly) varying phase are the ideal examples of coherent phase. On the other hand, a rapidly (or arbitrarily) varying phase in space is usually incoherent. It has been demonstrated for a three-dimensional optical trap potential by Greiner et al.\cite{3} that, as the strength of the optical potential traps is increased, the quantum tunneling of condensed atoms from one optical site to another is stopped resulting in a loss of superfluidity and phase coherence in the BEC. Consequently, no interference pattern is formed upon free expansion of such a BEC which is termed a Mott insulator state. This phenomenon represents a superfluid to Mott insulator quantum phase transition. The phase on an optical lattice site and the number of atoms in that site play the roles of conjugate variables obeying the Heisenberg uncertainty principle of quantum mechanics\cite{9}. In the superfluid state the coherent phase is considered to be known and consequently the number of atoms on each site is unknown thus allowing a free movement of atoms from one site to another\cite{3}. In the Mott insulator state the phase is entirely arbitrary across the optical lattice sites and the number of atoms at each site is fixed and their free passage from one site to another is stopped. As the strength of the optical potential traps in the Mott insulator state is reduced the superfluidity is restored in a short time via a Mott insulator to superfluid quantum phase transition. This reversible quantum phase transition may occur at absolute zero (0 K) and is driven by Heisenberg’s uncertainty principle\cite{9} and not by thermal fluctuations involving energy as in a classical phase transition. As the temperature approaches absolute zero all thermal fluctuations die out and at 0 K classical phase transitions are necessarily excluded.

Following a suggestion by Smerzi et al.\cite{10}, Cataliotti et al.\cite{11} have demonstrated in a novel experiment the loss of phase coherence and superfluidity in a BEC trapped in a one-dimensional optical-lattice and harmonic potentials when the center of the harmonic potential is suddenly displaced along the optical lattice through a distance larger than a critical value. Then a modulational instability takes place in the BEC and
it cannot reorganize itself quickly enough and the phase coherence and superfluidity of the BEC are lost. The loss of superfluidity is manifested in the destruction of the interference pattern upon free expansion. However, for displacements smaller than the critical distance the BEC can reorganize itself and the superfluidity is maintained [7,11]. Distinct from the quantum phase transition observed by Greiner et al. [3], this modulational instability responsible for the superfluid-insulator transition is classical in nature [10,11]. This process is also different from the Landau dissipation mechanism [10,12], occurring when the fluid velocity is greater than local speed of sound. When Landau instability occurs, the system lowers energy by emitting phonons [12]. The present classical dynamical transition can be well described [13,10,12] by the mean-field Gross-Pitaevskii (GP) equation [14].

The above modulational instability is not the unique dynamical classical process leading to a superfluid-insulator transition. Many other classical processes leading to a rapid movement in the condensate can lead to such a transition [15]. The movement should be rapid enough so that the BEC cannot reorganize itself to evolve through phase coherent states. In [11] a rapid translation of the BEC through the optical lattice sites leads to the destruction of phase coherence. Here we suggest that a rapid oscillation of the BEC may also lead to a superfluid-insulator transition. The oscillation is initiated by a periodic modulation of the magnetic trapping potential $\sim \omega^2$ in the radial direction via $\omega^2 \rightarrow \omega^2(1 + A\sin(\Omega\tau))$ where $\tau$ is time, $A$ an amplitude, $\omega$ is the radial trapping frequency, and $\Omega$ is the frequency of modulation. Such modulation of the trapping potential is known to generate resonant (collective) excitations in the BEC which have been studied both theoretically [16] and experimentally [17] in the absence of an optical lattice potential. The study of such excitations in the presence of an optical lattice potential has just began [18]. Similar collective excitation generated by a periodic modulation of the atomic scattering length [19] has also been shown to lead to a classical superfluid-insulator transition [20].

In the quantum phase transition [3] the Mott insulator state has a perfectly smooth probability distribution (modulus of the wave function) across the optical lattice sites whereas the phase of the wave function across the optical lattice sites remains entirely arbitrary. In the dynamical classical transition considered in this work, because of classical oscillation of the BEC, the insulator state in the joint traps is marked by a partially disturbed (nonsmooth) probability distribution across the optical lattice sites in addition to the loss of phase coherence. However, the information about the destruction of superfluidity in both quantum and classical cases is not solely contained in the initial probability distribution. Consequently, the BEC needs to be released from the joint traps and the formation of the interference pattern studied for a definite conclusion about the destruction of superfluidity.
As the present transition is classical or mean-field-type in nature, we base the present study on the numerical solution of the time-dependent mean-field axially-symmetric GP equation \[ 14 \] in the presence of a combined harmonic and optical potential traps. The time-dependent BEC wave function \( \Psi(r; t) \) at position \( r \) and time \( t \) is described by the following mean-field nonlinear GP equation \[ 14 \]
\[
\left[-i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2 \nabla^2}{2m} + V(r) + gN|\Psi(r; t)|^2\right] \Psi(r; t) = 0, 
\]
where \( m \) is the mass and \( N \) the number of atoms in the condensate, \( g = 4\pi\hbar^2a/m \) the strength of interatomic interaction, with \( a \) the atomic scattering length. In the presence of the combined axially-symmetric and optical lattice traps \( V(r) = \frac{1}{2}m\omega^2(\rho^2 + \nu^2z^2) + V_{\text{opt}} \) where \( \omega \) is the angular frequency of the harmonic trap in the radial direction \( \rho \), \( \nu \omega \) that in the axial direction \( z \), with \( \nu \) the aspect ratio, and \( V_{\text{opt}} \) is the optical lattice trap introduced later. The normalization condition is \( \int d\mathbf{r} |\Psi(r; t)|^2 = 1 \).

In the axially-symmetric configuration, the wave function can be written as \( \Psi(r; t) = \psi(\rho, z, t) \). Now transforming to dimensionless variables \( \hat{\rho} = \sqrt{2}\rho/l, \hat{z} = \sqrt{2}z/l, \tau = t\omega, l \equiv \sqrt{\hbar/(m\omega)} \), and \( \varphi(\hat{\rho}, \hat{z}; \tau) \equiv \hat{\rho}\sqrt{8}\psi(\rho, z; t) \), \[ 11 \] becomes \[ 21 \]
\[
\left[-i\frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial \hat{\rho}^2} + \frac{1}{\hat{\rho}} \frac{\partial}{\partial \hat{\rho}} - \frac{\partial^2}{\partial \hat{z}^2} + \frac{1}{4} \left( \hat{\rho}^2 + \nu^2\hat{z}^2 \right)\right] \varphi(\hat{\rho}, \hat{z}; \tau) = 0,
\]
where nonlinearity \( n = Na/l \). In terms of the one-dimensional probability \( P(z, t) \equiv 2\pi \int_0^\infty d\hat{\rho} |\varphi(\hat{\rho}, \hat{z}; \tau)|^2/\hat{\rho} \), the normalization of the wave function is given by \( \int_{-\infty}^\infty d\hat{z} P(z, t) = 1 \). The probability \( P(z, t) \) is useful in the study of the present problem under the action of the optical lattice potential, specially in the investigation of the formation and evolution of the interference pattern after the removal of the trapping potentials.

In the experiment of Cataliotti et al. \[ 7 \] with repulsive \(^{87}\)Rb atoms in the hyperfine state \( F = 1, m_F = -1 \), the radial trap frequency was \( \omega = 2\pi \times 92 \) Hz. The optical potential created with the standing wave laser field of wavelength \( \lambda = 795 \) nm is given by \( V_{\text{opt}} = V_0E_R\cos^2(k_Lz) \), with \( E_R = \hbar^2k_L^2/(2m) \), \( k_L = 2\pi/\lambda \), and \( V_0 \) (< 12) the strength. For the mass \( m = 1.441 \times 10^{-25} \) kg of \(^{87}\)Rb the harmonic oscillator length \( l = \sqrt{\hbar/(m\omega)} = 1.126 \) \( \mu \)m and the present dimensionless time unit \( \omega^{-1} = 1/(2\pi \times 92) \) \( s \) = 1.73 ms. In terms of the dimensionless laser wave length \( \lambda_0 = \sqrt{2}\lambda/l \approx 1 \), the dimensionless standing-wave energy parameter \( E_R/(\hbar\omega) = 4\pi^2/\lambda_0^2 \). Hence in dimensionless unit \( V_{\text{opt}} \) of \[ 2 \] is
\[
\frac{V_{\text{opt}}}{\hbar\omega} = V_0\frac{4\pi^2}{\lambda_0^2} \left[ \cos^2 \left( \frac{2\pi}{\lambda_0} \hat{z} \right) \right] .
\]
Although we employ the dimensionless space units \( \hat{\rho} \) and \( \hat{z} \) and time unit \( \tau \) in numerical calculation, the results are reported in actual units \( r \) \( \mu \)m, \( z \) \( \mu \)m and \( t \) ms. In the conversion we used the parameters of the experiment of Cataliotti et al. \[ 7 \], e.g., \( \rho = 0.8\hat{\rho} \) \( \mu \)m, \( z = 0.8\hat{z} \) \( \mu \)m, and \( t = 1.73\tau \) ms.
We solve (2) numerically using a split-step time-iteration method with the Crank-Nicholson discretization scheme described recently [22]. The time iteration is started with the harmonic oscillator solution of (2) with \( n = 0 \):

\[
\varphi(\hat{\rho}, \hat{z}) = [\nu/(8\pi^3)]^{1/4} \hat{\rho} e^{-(\hat{\rho}^2 + \nu\hat{z}^2)/4} [21].
\]

The nonlinearity \( n \) and the optical lattice potential parameter \( V_0 \) are slowly increased by equal amounts in 10000 steps of time iteration until the desired value of \( n \) and \( V_0 \) are attained. Then, without changing any parameter, the solution so obtained is iterated 50 000 times so that a stable solution is obtained independent of the initial input and time and space steps.

The one-dimensional pattern of BEC on the optical lattice for a specific nonlinearity and the interference pattern upon free expansion of such a BEC have been recently studied using the numerical solution of (2) [13]. Here we study the destruction of this interference pattern after the application of a periodic modulation of the radial trapping potential in (2) via

\[
\hat{\rho}^2/4 \rightarrow (\hat{\rho}^2/4)[1 + A \sin(\Omega\tau)],
\]

while the axial trapping potential is left unchanged [16,17]. In the present model study we employ nonlinearity \( n = 5 \), the axial trap parameter \( \nu = 0.5 \), and the optical lattice strength \( V_0 = 6 \) throughout. First we calculate the ground-state wave function in the combined harmonic and optical lattice potentials.

Modulation (4) of the trapping potential may lead to resonant oscillation of the BEC and such resonances have been studied in the case of harmonic trap alone [16]. For a very small \( A \), prominent resonances appear in the BEC oscillation when the modulation frequency \( \Omega \) is an integral multiple \( N \) of the harmonic oscillator frequency \( \omega \) \((\Omega = N\omega)\) [16]. This is quite expected from our wisdom in linear classical physics where resonances appear when the driving frequency is a multiple of the characteristic frequency of oscillation. Actually, for a finite \( A \), resonances appear for a band of modulation frequency [16] \((\Omega = N\omega \pm \Delta_N)\) where \( \Delta_N \) defines the spread of frequency values. The resonance becomes more prominent with the increase of the parameter \( A \) or \( N \). At resonance the BEC executes rapid oscillation which is responsible for the destruction of superfluidity in the BEC via a classical dynamical transition provided that the time of stay of the BEC in the modulated magnetic trap, called hold time, is larger than a critical value. A large value of \( A \) and/or \( \Omega \) facilitates the dynamical classical superfluid-insulator transition. We illustrate this fact in the following for \( N = 1 \) and 2. A careful study of the present phenomenon will aid in the understanding of resonance in nonlinear physics. Although the generation of resonance in different linear problems is well understood, the same in nonlinear physics is just starting.

As the present calculation is performed with the full wave function without approximation, phase coherence among different wells of the optical lattice is automatically guaranteed in the initial state. As a result when the condensate is released from the combined trap, a matter-wave interference pattern is formed in a few milliseconds as described in [13]. The atom cloud released from one lattice site expand, and overlap and interfere with atom clouds from neighboring sites to form the robust
Figure 1. One-dimensional probability $P(z, t)$ vs. $z$ and $t$ for the BEC on optical lattice under the action of modulation (4) with $\Omega = \omega$ and $A = 0.5$ in the radial magnetic trap and upon the removal of the combined traps after hold times (a) 35 ms, (b) 52 ms, and (c) 69 ms, in the modulated radial trap.

interference pattern. The pattern consists of a central peak and two symmetrically spaced peaks, each containing about 10% of total number of atoms moving in opposite directions [7].

Next we consider an oscillating BEC in the combined harmonic and optical traps. If we introduce the modulation of the radial trapping potential (4) after the formation of the BEC in the combined trap, the condensate will be out of equilibrium and start
to oscillate. As the height of the potential-well barriers on the optical lattice is much larger than the energy of the system, the atoms in the condensate will move by tunneling through the potential barriers. This fluctuating transfer of Rb atoms across the potential barriers is due to Josephson effect in a neutral quantum liquid [7]. We demonstrate that the phase coherence between different wells of the condensate can be destroyed during this process for rapid oscillations with large amplitude and/or frequency and no matter-wave interference pattern will be formed after the removal of the joint traps.

![Figure 2](image_url)

**Figure 2.** One-dimensional probability $P(z,t)$ vs. $z$ and $t$ for the BEC on optical lattice under the action of modulation (4) with $\Omega = 2\omega$ and $A = 0.5$ in the radial magnetic trap and upon the removal of the combined traps after hold times (a) 17 ms, and (b) 35 ms in the modulated radial trap.

Now we explicitly study the destruction of superfluidity in the condensate upon the application of modulation (4) when the BEC is allowed to stay in this modulated trap for a certain interval of time more than a critical value (hold time). For small $A$ and $\Omega$ away from resonance, the BEC executes slow oscillation maintaining the phase coherence. For large $A$ and $\Omega$ and near resonance, the BEC executes rapid oscillation [17, 18] which results in a destruction of superfluidity. The destruction of superfluidity for a larger hold time in the modulated trap manifests in the disappearance of the interference pattern upon free expansion which can be studied experimentally.

For numerical simulation we allow the BEC to evolve on a lattice with $r \leq 15$
\( \mu \text{m} \) and \( 20 \mu \text{m} \geq z \geq -20 \mu \text{m} \) after the modulation (4) with \( \Omega = \omega \) and \( A = 0.5 \) is applied and study the the system after different hold times. The probability densities \( P(z, t) \) are plotted in figures 1 (a), (b) and (c), for hold times 35 ms, 52 ms, and 69 ms, respectively. For the hold time of 35 ms prominent interference pattern is formed upon free expansion as we can see in figure 1 (a). The interference pattern is slowly destroyed as hold time is increased as we can see in figures 1 (b) and (c). In figure 1(a) three separate pieces in the interference pattern corresponding to three distinct trails can be identified. However, as the hold time in the displaced trap increases the maxima of the interference pattern mixes up and finally for the hold time of 69 ms the interference pattern is completely destroyed as we find in figure 1 (c).

As the BEC is allowed to evolve for a substantial interval of time after the modulation (4) of the radial trapping potential is applied, a dynamical instability of classical nature sets in which destroys superfluidity [10, 11]. This has been explicitly demonstrated in the present simulation which results in the destruction of the interference pattern.

The destruction of superfluidity is facilitated as the amplitude \( A \) or frequency \( \Omega \) of the radial modulation (4) is increased. We demonstrate this for an increase in \( \Omega \) in the following. In figures 2 (a) and (b) we present the evolution of probability \( P(z, t) \) after the application of the modulation with \( A = 0.5 \) and \( \Omega = 2\omega \). With the increase of \( \Omega \) from \( \omega \) to \( 2\omega \), the destruction of superfluidity is facilitated as one can find from figures 2. The superfluidity is destroyed for a hold time of 35 ms for \( \Omega = 2\omega \) (figure 2), whereas it is maintained for the same hold time for \( \Omega = \omega \) (figure 1). An increase in the value of the parameter \( A \) also increases the resonant oscillation and we verified that the destruction of superfluidity is also facilitated in the process. However, we do not present that study here.

In conclusion, using the explicit numerical solution of the GP equation we have studied in detail the destruction of superfluidity in a cigar-shaped condensate loaded in a combined axially-symmetric harmonic and optical lattice traps upon the application of a modulation of the radial trapping potential near resonance. In the absence of modulation, the formation of the interference pattern upon the removal of the combined traps clearly demonstrates the phase coherence [6, 13]. The superfluidity is maintained for a slow modulation (4) of the radial trapping potential away from resonance. For rapid modulation of large amplitude and/or frequency near resonance, there is a superfluid-insulator classical dynamical transition, provided that the BEC is kept in the modulated trap for a certain hold time. Consequently, after release from the combined trap no interference pattern is formed. For smaller amplitude \( A \) and/or frequency \( \Omega \) of modulation (4) near resonance, the superfluid-insulator transition occurs for a larger hold time and vice versa. It is possible to study this novel phenomenon experimentally and compare with the present theoretical prediction.
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Reference

[1] Anderson B P and Kasevich M A 1998 Science 282 1686
[2] Orzel C, Tuchman A K, Fenselau M L, Yasuda M and Kasevich M 2001 Science 291 2386
[3] Greiner M, Mandel O, Esslinger T, Hänsch T W and Bloch I 2002 Nature (London) 415 39
Greiner M, Mandel O, Hänsch T W and Bloch I 2002 Nature (London) 419 51
[4] Andrews M R, Townsend C G, Miesner H J, Durfee D S, Kurn D M and Ketterle W 1997 Science 275 637
[5] Band Y B and Trippenbach M 2002 Phys. Rev. A 65 053602
Liu W M, Fan W B, Zheng W M, Liang J Q and Chui S T 2002 Phys. Rev. Lett. 88 170408
Kramer M, Pitaevskii L and Stringari S 2002 Phys. Rev. Lett. 88 180404
Goral K, Santos L and Lewenstein M 2002 Phys. Rev. Lett. 88 170406
Sukhorukov A A and Kivshar Y S 2002 Phys. Rev. E 65 036609
Massignan P and Modugno M 2002 e-print cond-mat/0205516
[6] Adhikari S K and Muruganandam P 2003 Phys. Lett. A 310 229
[7] Cataliotti F S, Burger S, Fort C, Middaloni P, Minardi F, Trombettoni A, Smerzi A and Inguscio M 2001 Science 293 843
[8] Morsch O, Cristiani M, Müller J H, Ciampini D and Arimondo E 2002 Phys. Rev. A 66 021601
Müller J H, Morsch O, Cristiani M, Ciampini D and Arimondo E 2002 e-print cond-mat/0211079
[9] Stoof H T C 2002 Nature (London) 415 25
[10] Smerzi A, Trombettoni A, Kevrekidis P G and Bishop A R 2002 Phys. Rev. Lett. 89 170402
[11] Cataliotti F S, Fallani L, Ferlaino F, Fort C, Middaloni P, Inguscio M, Smerzi A, Trombettoni A, Kevrekidis P G and Bishop A R 2002 e-print cond-mat/0207139
[12] Wu B and Niu Q 2002 Phys. Rev. Lett. 89 088901
[13] Adhikari S K 2003 Eur. Phys. J. D in press (e-print cond-mat/0208275)
[14] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 Rev. Mod. Phys. 71 463
[15] Adhikari S K 2003 Phys. Lett. A 308 302
[16] García-Ripoll J J, Perez-García V M and Torres P 1999 Phys. Rev. Lett. 83 1715
[17] Jin D S, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1996 Phys. Rev. Lett. 77 420
Mewes M O, Andrews M R, van Druten N J, Kurn D M, Durfee D S, Townsend C G and Ketterle W 1996 Phys. Rev. Lett. 77 988
Maragó O, Hechenblaikner Hodby G E and Foot C 2001 Phys. Rev. Lett. 86 3938
[18] Fort C, Cataliotti F S, Fallani L, Ferlaino F, Middaloni P and Inguscio M 2002 e-print cond-mat/0210240
[19] Adhikari S K 2003 J. Phys. B: At. Mol. Opt. Phys. 36 1109
Abdullaev F K, Bronski J C and Galimzyanov R M 2002 e-print cond-mat/0205464
Abdullaev F K, Caputo J G, Kraenkel R A and Malomed B A 2003 Phys. Rev. A 67 013605
Adhikari S K 2003 Phys. Lett. A in press
[20] Adhikari S K 2003 Phys. Lett. A 308 302
[21] Adhikari S K 2002 Phys. Rev. E 65 016703
[22] Adhikari S K and Muruganandam P 2002 J. Phys. B: At. Mol. Opt. 35 2831
Adhikari S K and Muruganandam P 2003 J. Phys. B: At. Mol. Opt. Phys. 36