Small non-Leighton two-complexes†

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Abstract

How many 2-cells must two finite CW-complexes have to admit a common, but not finite common, covering? Leighton’s theorem says that both complexes must have 2-cells. We construct an almost (?) minimal example with two 2-cells in each complex.

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0. Introduction

Leighton Theorem [10] If two finite graphs have a common covering, then they have a common finite covering.

Alternative proofs and various generalisations of this result can be found, e.g., in [2], [4], [14], [15], [19], and references therein.

Does a similar result hold for any CW-complexes, i.e.

is it true that, if, for finite CW-complexes $K_1$ and $K_2$, there exist a CW-complex $K$ and cellular coverings $K_1 \leftarrow K \rightarrow K_2$, then there exists a finite CW-complex $K$ with this property?

This natural question was posed (in other terms) in [1] and [16]. Notice the cellularity requirement. Surely, we would obtain an equivalent question if we replace this condition with a formally stronger combinatorialness one: the image of each cell is a cell. However, without the cellularity condition, the answer would be negative: indeed, the torus and the genus-two surface have no finite common coverings (as the fundamental group the genus-two orientable surface $\langle x, y, z, t | [x, y][z, t] = 1 \rangle$ contains no abelian subgroups of finite index), while the universal coverings of these surfaces are homeomorphic, because they are the plane. The cellularity condition rules out such examples: if we take, e.g., the standard one-vertex cell structures on the torus and genus-two surface, then, on the covering plane, we obtain:

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(i) the usual square lattice on the (Euclidean) plane (in the torus-case);
(ii) and an octagonal lattice on the (Lobachevskii) plane (in the genus-two case);

(i.e. though the universal coverings are homeomorphic, the cell structure on them are principally different). This example cannot be saved by a complication of the cell structures on the torus and genus-two surface (as was noted in [1] and [16]; the authors of [1] even conjectured that the answer to the (cellular version of) the question is positive).

Nevertheless, the answer turned out to be negative as was shown in [18] (and actually, much earlier in [17]); the complexes $K_1, K_2$ forming such a non-Leighton pair from [18] contain as few as six 2-cells each. In [8], this number was reduced to four:

there exist two two-complexes containing four 2-cells each that have a common covering but have not finite common coverings.

(Henceforth, we omit the prefix “CW-” and word “cellular”: a complex means a CW-complex, and all mapping between complexes are assumed to be cellular in this paper.) The non-Leighton complexes $K_1$ and $K_2$ from [8] are the standard complexes of the following group presentations $\Gamma_i$, i.e. one-vertex complexes with edges corresponding to the generators and 2-cells attached by the relators:

$$\Gamma_1 = F_2 \times F_2 = \langle a, b, x, y \mid [a, x], [a, y], [b, x], [b, y] \rangle$$
and

$$\Gamma_2 = \langle a, b, x, y \mid axay, ax^{-1}by^{-1}, ay^{-1}bx^{-1}, bx^{-1}y^{-1} \rangle.$$

Both of these complex are covered by the Cartesian product of two trees (Cayley graphs of the free group $F_2$); and no finite common cover exists, because the fundamental group of such hypothetical covering complex would embed in both groups $\Gamma_i$ as finite-index subgroups, but, in $\Gamma_1$, any finite-index subgroup contains a finite-index subgroup which is the direct product of free groups, while $\Gamma_2$ has no such finite-index subgroups [8] (though, in this special case, it was recently obtained [6]). Henceforth,

$$x^k y \overset{\text{def}}{=} y^{-1} x^k y,$$
where $x$ and $y$ are elements of a group and $k \in \mathbb{Z}$.

Although the authors of [8] did not pursue this purpose; it was a byproduct of their results.
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In conclusion, note that results on coverings of two-complexes can imply nontrivial facts about graphs, because one can “model” 2-cells in graphs by means of additional vertices and edges, see [4]. Higher dimensional complexes are of little sense here: if complexes \( K_1 \) and \( K_2 \) form a non-Leighton pair, then their two-skeleta also form such a pair, as is easy to verify. A detailed exposition of the general theory of coverings and CW-complexes can be found, e.g., in [7].

1. Algebraic lemmata

The following fact is well known [13], we give a short proof for the reader’s convenience.

**Commutator Lemma.** In the group \( H = BS(3, 5) = \langle c, d \mid c^{3d} = c^5 \rangle \), the commutator \( h = [c^d, c] \) belongs to any finite-index subgroup.

**Proof.** Each finite-index subgroup contains a normal finite-index subgroup (see, e.g., [9]). Therefore, it suffices to show that \( h \) lies in the kernel of any homomorphism \( \varphi : H \to K \) to any finite group \( K \).

The elements \( \varphi(c^3) \) and \( \varphi(c^5) \) have the same order (because they are conjugate); hence, the order of \( \varphi(c) \) is not divisible by three. Therefore, \( \varphi(c) \in \langle \varphi(c^3) \rangle \). Thus, \( \varphi(c)^{\varphi(d)} \in \langle \varphi(c) \rangle \) and \( h = [c^d, c] \) belongs to the kernel of \( \varphi \). This completes the proof.

**Bottle Lemma.** If a group \( G \) has a subgroup \( \langle a, b \rangle = \langle a, b \mid a^p = a^{-1} \rangle \simeq BS(1, -1) \), and the element \( b \) lies in all finite-index subgroups of \( G \), then any finite-index subgroup of \( G \) contains a subgroup isomorphic to the Klein-bottle group \( BS(1, -1) \).

**Proof.** Any finite-index subgroup contains all elements conjugate to \( b \), because the intersection \( R \) of all finite-index subgroups is normal. Therefore, \( a^2 = b^{-1}b^a \in R \) and \( \langle a^2, b \rangle \subseteq R \). It remains to note that \( a^{2b} = a^{-2} \), and the groups \( \langle a^2 \rangle \) and \( \langle b \rangle \) are infinite; hence, the subgroup \( \langle a^2, b \rangle \) is isomorphic to \( BS(1, -1) \), because,

\[
\text{in any group, infinite-order elements } x \text{ and } y \text{ such that } x^y = x^{-1} \text{ generate a subgroup isomorphic to the Klein-bottle group. (1)}
\]

Indeed, there is obvious epimorphism

\[
\varphi : BS(1, -1) = \langle a, b \rangle \longrightarrow \langle x, y \rangle.
\]

Any element \( g \in BS(1, -1) \) can be written in the form \( g = a^{k}b^{l} \). If \( g = a^{k}b^{l} \in \ker \varphi \), then \( \ker \varphi \ni [b, g] = b^{-1}b^{-1}a^{-k}ba^{k}b^{l} = a^{\pm 2k} \). Therefore, \( k = 0 \) (because \( |\langle x \rangle| = \infty \)). But then \( l = 0 \) too, because \( 1 = \varphi(g) = \varphi(b^{l}) = y^{l} \), and \( |\langle y \rangle| = \infty \). Thus, \( \ker \varphi = \{ 1 \} \) and \( \varphi \) is an isomorphism. This completes the proof.

**No-Bottle Lemma.** The amalgamated free product

\[
G = \left\{ a, c, d \mid [a, [c^d, c]] = 1, \ c^{3d} = c^5 \right\} = \langle a, b \mid [a, b] = 1 \rangle \ast_{b=[c^d, c]} \left\{ c, d \mid c^{3d} = c^5 \right\}
\]

of the free abelian group and the Baumslag–Solitar group \( BS(3, 5) \) contains no subgroups isomorphic to the Klein-bottle group \( K = BS(1, -1) \).

**Proof.** The group \( BS(3, 5) \) does not contain subgroups isomorphic to \( K \) [11] and is torsion-free. Therefore, applying once again (1), we obtain that the quotient
has no nonidentity elements conjugate their inverse. Therefore, any element of $G$ by the normal closure of $(a, G)$ implies that $x$ or $x\mapsto n$:

(i) either $\hat{x}_1^2 \in \langle b \rangle$ for some $\hat{x}_1 \in (\langle a \rangle_\infty \times \langle b \rangle_\infty) \setminus \{b\}$;

(ii) or $\hat{x}_1^2 \in([[c^d, c]] \setminus \langle [c^d, c] \rangle$ for some $\hat{x}_1 \in \{c, d \mid c^{3d} = c^5\} \setminus \langle [c^d, c] \rangle$.

The first is impossible of course. The impossibility of the second case can be verified, e.g., as follows:

(i) the quotient group $Q = \langle c, d \mid c^{3d} = c^5 \rangle / \langle [[c^d, c]] \rangle$ is torsion-free; indeed, $Q$ is the HNN-extension $Q = \langle c, e, d \mid [e, c] = 1, e^3 = c^5, c^d = e \rangle$ of the abelian group $A = \langle c, e \mid [e, c] = 1, e^3 = c^5 \rangle$, which is torsion-free (moreover, it is easy to verify that $A \simeq \mathbb{Z}$ and $Q \simeq \text{BS}(3, 5)$);

(ii) therefore, $\hat{x}_1$ lies in the normal closure $F = \langle [[c^d, c]] \rangle$, which is a free group, because, by the Karrass–Solitar theorem (see, e.g., [12]), any subgroup of an HNN-extension is free if it intersects conjugates of the base trivially. It remains to show that $[c^d, c]$ is not a square in $F$ (because in a free group an inclusion $a^2 \in \langle b \rangle$ implies that $\langle a, b \rangle$ is cyclic by the Nielsen–Schreier theorem and, hence, $\alpha \in \langle b \rangle$ if $\beta$ is not a square). The commutator $[c^d, c]$ is not a square in $F$, because, assuming the contrary and noting that automorphic images of squares are squares too, we obtain $F = \langle [[c^d, c]] \rangle = \langle \hat{x}_1^2 \rangle \subseteq \langle \{f^2 \mid f \in F\} \rangle$, which cannot hold in a nontrivial free group $F$. This completes the proof.

2. Proof of the main theorem

Take the fundamental groups of the torus and the Klein bottle:

$$G_1 = \text{BS}(1, 1) = \langle a, b \mid [a, b] = 1 \rangle \quad \text{and} \quad G_{-1} = \text{BS}(1, -1) = \left\langle a, b \mid a^b = a^{-1} \right\rangle$$
and consider the amalgamated free products $H_\varepsilon = G_\varepsilon \ast_{b = h} H$ of $G_\varepsilon$ and a group

$$H = \langle X \mid R \rangle \supseteq \langle h \rangle_\infty$$

(henceforth $\varepsilon = \pm 1$). Let $K_\varepsilon$ be the standard complex of the (standard) presentation of $H_\varepsilon$:

$$H_\varepsilon = \left\langle \{a\} \sqcup X \mid \{\hat{h}a^{-\varepsilon}\} \sqcup R \right\rangle,$$

where $\hat{h}$ is a word in the alphabet $X^{\pm 1}$ representing the element $h \in H$.

The Cayley graphs of $G_\varepsilon$ are isomorphic surely (as abstract undirected graphs), the same is true for the universal coverings of the standard complexes of presentations of the groups $G_\varepsilon$ (these covering complexes are planes partitioned on squares, Figure 1).

A slightly less trivial observation is that, for groups $H_\varepsilon$, the universal coverings are isomorphic too:

for any infinite-order element $h$ of any group $H$, the universal coverings of complexes $K_\varepsilon$ are isomorphic.

($\ast$)

In what follows, we explain this simple fact in details; the readers who regard this fact as obvious, can skip to Observation ($\ast\ast$).

It suffices to show that some coverings $\hat{K}_\varepsilon \rightarrow K_\varepsilon$ have isomorphic $\hat{K}_\varepsilon$; we prefer to take the coverings corresponding to the normal closure $\langle\langle a \rangle\rangle$ of $a \in H_\varepsilon$. In explicit form, these complexes $\hat{K}_\varepsilon$ are the following ones:

(i) the vertices are elements of $H$;
(ii) the edges with labels from $X$ are drawn as in the Cayley graph of the group $H$: an edge with label $x \in X$ go from each vertex $h' \in H$ to the vertex $h'x \in H$;
(iii) in addition, to each vertex $h' \in H$, a directed loop (edge) $a_{h'}$, labelled by $a$ is attached;
(iv) to each cycle whose label is a relator from $R$, an oriented 2-cell is attached;
to each cycle with label $a^\hat{h}a^{-\varepsilon}$, an oriented 2-cell (a special cell) is attached; thus, going along the boundary of a special cell in the positive direction, we meet two edges labelled by $a$, namely, $a_{h'}$ and $a_{h'}^{-\varepsilon}$, where, as usual, $a_{h'}^{-1}$ means that the edge $a_{h'}$ is traversed against its direction.

The isomorphism $\Phi_1: \hat{K}_1 \to \hat{K}_{-1}$ is the following:

(i) the vertices, edges with labels from $X$ and nonspecial 2-cells (corresponding to relators from $R$) are mapped identically;

(ii) to define the mapping $\Phi_1$ on edges labelled by $a$ and special 2-cells, we choose a set $T$ of left-coset representatives of $\langle h \rangle$ in $H$, and put $\Phi_1(a_{th}k) = a_{-1}(k)$ for all $t \in T$ and $k \in \mathbb{Z}$ (i.e., in each coset, each second loop labelled by $a$ is inverted); then the mapping of singular cells are defined naturally: a cell of $\hat{K}_1$ with edges $a_{h'}$ and $a_{h'}^{-1}$ on its boundary is mapped to the cell of $\hat{K}_{-1}$ containing $a_{h'}$ and $a_{h'}^{-1}$ on its boundary.

The next simple observation is that:

if $h \in H$ belongs to all finite-index subgroups of $H$, and the complexes $K_\varepsilon$ have a finite common covering, then the group $H_1$ contains a subgroup isomorphic to the Klein-bottle group $BS(1, -1)$.

Indeed, in $H_{-1}$, the element $b = h$ is contained in all subgroups of finite index (because the intersection of each such subgroup with $H$ is of finite index in $H$ and, therefore, contains $h$). By the bottle lemma (applied to $G = H_{-1}$), we obtain that each finite-index subgroup contains a subgroup isomorphic to the Klein-bottle group. It remains to note that, if a finite complex $\hat{K}$ covers $K_1$ and $K_{-1}$, then its fundamental group $\pi_1(\hat{K})$ embeds into $\pi_1(K_\varepsilon) = H_\varepsilon$ as a finite-index subgroup.

Now, we take a particular group $H$, namely, let $H$ be the Baumslag–Solitar group: $H = BS(3, 5) = \langle c, d \mid c^3d = c^5 \rangle$, and let $h \in H$ be the commutator: $h = [c^d, c]$. This element $h$ is contained in any finite-index subgroup of $H$ by the commutator lemma. According to (**), this means that, if complexes $K_\varepsilon$ would have a common finite covering, then $H_1 = \langle a, c, d \mid [a, [c^d, c]] = 1, c^3d = c^5 \rangle$ would contain the Klein-bottle group as a subgroup, which contradicts the no-bottle lemma. Therefore, there are no finite common coverings for complexes $K_\varepsilon$; while an infinite common covering exists according to (*). Thus, the following fact is proven.

**Main Theorem.** The standard complexes of presentations

$$H_\varepsilon = \left\{ a, c, d \mid a^{[c^d, c]} = d^\varepsilon, c^3d = c^5 \right\},$$

where $\varepsilon = \pm 1$, containing two 2-cells and one vertex, and three edges have a common covering, but have no finite common coverings.

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