The q\bar{q} relativistic interaction in the Wilson loop approach

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We study the q\bar{q} relativistic interaction starting from the Feynman–Schwinger representation of the gauge-invariant quark-antiquark Green function. We focus on the one-body limit and discuss the obtained non-perturbative interaction kernel of the Dirac equation.

1. INTRODUCTION

The dynamics of a system composed by two heavy quarks is well understood in terms of a potential interaction (static plus relativistic corrections) obtained from the semirelativistic reduction of the QCD dynamics \cite{1-5}.

If at least one of the quarks is light the system behaves relativistically (the light quark can no more be considered static) and a pure relativistic treatment becomes necessary (via Dirac or Bethe–Salpeter equations). A lot of phenomenologically justified relativistic equations can be found in the literature but up to now we miss a relativistic treatment which follows directly from QCD. Our work goes in this direction. In order to simplify the problem we focus on the heavy-light mesons in the non-recoil limit (i.e. infinitely heavy antiquark). Only at the end we will briefly discuss the two-body case. Our starting point is the quark-antiquark gauge-invariant Green function taken in the infinite mass limit of one particle. The only dynamical assumption is on the behaviour of the Wilson loop (i.e. on the nature of the non-perturbative vacuum). The gauge invariance of the formalism guarantees that the relevant physical information are preserved at any step of our derivation. In this way an unified description. We discuss our result with respect to the old-standing problem of the Lorentz structure of the Dirac kernel for a confining interaction \cite{6-8}. The main results presented here can be found in \cite{9}.

2. THE RELATIVISTIC INTERACTION IN THE ONE-BODY LIMIT

The quark-antiquark Green function is given in the quenched approximation by

$$G_{\text{inv}}(x, u, y, v) = \left\langle \text{Tr} i S^{(1)}(x, y; A) U(y, v) \right.$$ 

$$\times i S^{(2)}(v, u; A) U(u, x) \right\rangle,$$ \hspace{1cm} (1)

where the points x, y, u, v are defined as in Fig. 1, \langle \rangle means the normalized average over the gauge field A\textsubscript{\mu}, S\textsuperscript{(i)} is the fermion propagator in the external field A\textsubscript{\mu} associated with the particle i and the strings

$$U(y, x) \equiv \text{P} \exp \left\{ ig \int_0^1 ds (y - x)^\mu A_\mu(x + s(y - x)) \right\},$$

are needed in order to have gauge invariant initial and final bound states. A very convenient way to represent it is the so-called Feynman–Schwinger representation (see \cite{10,11} and refs. therein), where the fermion propagators are expressed in terms of quantomechanical path integrals over the quark trajectories (z\textsubscript{1}(t\textsubscript{1}) and z\textsubscript{2}(t\textsubscript{2}))

$$G_{\text{inv}}(x, u, y, v) = \frac{1}{4} \left\langle \text{Tr} \text{P} (i \partial_y^{(1)} + m_1) \right.$$ 

...
Given by the quark paths connecting non-vanishing contribution to the Wilson loop is

\[ W(\Gamma; A) \equiv \text{Tr} \mathcal{P} \exp \left\{ ig \int_{\Gamma} dz^\mu A_\mu(z) \right\}, \tag{3} \]

where \( \Gamma \) is the closed curve defined by the quark trajectories and the endpoint strings \( U(y, v) \) and \( U(u, x) \).

In order to treat a simpler case, let us assume, now, that the antiquark moving on the second fermion line becomes infinitely heavy. The only trajectory surviving in the path integral of Eq. (3) associated with the second particle is the static straight line propagating from \( v \) to \( u \). The corresponding Wilson loop of the system is represented in Fig. 1. As already noted in [12] in this case it turns out to be convenient to choose the following gauge condition:

\[ A_\mu(x_0, 0) = 0, \quad x^I A_I(x_0, \mathbf{x}) = 0. \tag{4} \]

Notice that this gauge choice is possible since the formalism is completely gauge invariant. Within this gauge it is possible to express the gauge field in terms of field strength tensor,

\[ A_\mu(x) = \int_0^1 d\alpha \, \alpha^{\alpha(x_0, \mathbf{x})} x^\mu \mathcal{E}_{\mu}(x_0, \mathbf{x}), \]

where \( n(0) = 0 \) and \( n(i) = 1 \). Moreover the only non-vanishing contribution to the Wilson loop is given by the quark paths connecting \( x \) with \( y \), and we have

\[ W(\Gamma; A) = \text{Tr} \mathcal{P} \exp \left\{ ig \int_{\Gamma} dz^\mu A_\mu(z) \right\}. \tag{5} \]

As shown in [10][11] in order to evaluate Eq. (3) we need to know the Wilson loop average over the gauge field. We evaluate it via the cumulant expansion described in [3]. Keeping only bilocal cumulants we obtain:

\[ \langle W(\Gamma, A) \rangle = \exp \left\{ -\frac{g^2}{2} \int_x^x u^\mu \int_{x}^{y^\mu} d\gamma^\nu D_{\mu\nu}(x', y) \right\}, \]

\[ D_{\mu\nu}(x, y) \equiv x^k y^l \int_0^1 d\alpha \alpha^{\alpha(x)} x^\mu y^\nu \mathcal{F}_{\mu}(x_0, \alpha x) \mathcal{F}_{\nu}(y_0, \beta y), \tag{6} \]

Assumption (3) corresponds to the so-called stochastic vacuum model. Inserting expression (3) in Eq. (3) and expanding the exponential we obtain the following expression for the propagator \( S_D \) of the quark (which is \( G_{\text{inv}} \) “projected” on the first fermion line):

\[ S_D = S_0 + S_0 K S_0 + S_0 K S_0 K S_0 + \cdots. \tag{7} \]

\( S_0 \) is the free fermion propagator. Taking into account only the first planar graph (since we are interested only in contributions proportional to the gluon condensate), we have \( K(y', x') = \gamma^\nu S_0(y', x') \gamma^\mu D_{\mu\nu}(x', y') \). A graphical representation of \( K \) is given in Fig. 2. Eq. (6) can be written in closed form as \( S_D = S_0 + S_0 KS_D \) (or in terms of the wave-function, \( (\bar{\psi} - m - i K)^{-1}\psi = 0 \); \( m \equiv m_1 \)). Therefore, \( K \) can be interpreted as the interaction kernel of the Dirac equation associated with the motion of a quark in the field generated by an infinitely heavy antiquark.

Assuming that the correlator \( \langle F_{\mu I}(x) F_{\nu I}(y) \rangle \) depends only on the difference between the coor-
Figure 2. The interaction kernel $K$.

dinates, we define:

$$
\langle F_{\mu\nu}(x^0, \alpha x) F_{\rho\sigma}(y^0, \beta y) \rangle \\
\equiv f_{\mu\nu\rho\sigma}(x^0 - y^0, \alpha x - \beta y).
$$

With this assumption $K$ can be written in momentum space as:

$$
K(q, p) \\
= -g^2(2\pi)\delta(p^0 - q^0) \int_{-\infty}^{+\infty} d\tau \int_{0}^{1} d\alpha \alpha^\nu \xi^\mu \\
\times \int_{0}^{1} d\beta \beta^\nu \xi^\rho \partial^\rho \partial^\mu \int d^3r e^{i(p - q) \cdot r} \\
\times \gamma^\nu \{ \theta(-\tau) \Lambda_+(t) \gamma^0 c^{(p^0 - E_t)\tau} \\
- \theta(\tau) \Lambda_-(t) \gamma^0 c^{-(p^0 + E_t)\tau} \} \gamma^\mu \\
\times f_{\mu\nu\rho\sigma}(\tau, (\alpha - \beta)\tau),
$$

where $t \equiv (\beta p - \alpha q)/(\beta - \alpha)$, $E_t = \sqrt{t^2 + m^2}$ and $\Lambda_\pm(t) = E_t \pm (m - t \cdot \gamma)\gamma^0/2E_t$.

Equation (8) is our basic expression. It contains the perturbative interaction up to order $g^2$ and the non-perturbative one carried by a single insertion of a second order cumulant. From now on we want to focus our attention only on the purely non-perturbative interaction. The Lorentz structure of the non-perturbative relevant part of $f_{\mu\nu\rho\sigma}$ is

$$
f_{\mu\nu\rho\sigma}^{n.p.}(x) = \frac{\langle F^2(0) \rangle}{24N_c} (g_{\mu\nu} g_{\lambda\rho} - g_{\mu\rho} g_{\nu\lambda}) D(x^2)\tag{9}
$$

where $\langle F^2(0) \rangle$ is the gluon condensate, $\frac{1}{24N_c}$ the identity matrix of SU(3) and $D$ is a non-perturbative form factor normalized to unit at the origin. Lattice simulations have shown that $D$ falls off exponentially (in Euclidean space-time) at long distances with a correlation length $a^{-1} \sim (1 \text{ GeV})^{-1}$. As shown in [3] this behaviour of $D$ is sufficient to give confinement at least in some kinematic regions.

We notice here that the most general form of $f_{\mu\nu\rho\sigma}^{n.p.}(x)$ could also contain a term of the type

$$
\frac{1}{24N_c} \left\{ \partial_\mu (x_\nu g_{\lambda\rho} - x_\rho g_{\nu\lambda}) \\
+ \partial_\lambda (x_\nu g_{\mu\rho} - x_\rho g_{\mu\nu}) \right\} D_1(x^2), \tag{10}
$$

where $D_1$ is another unknown non-perturbative form factor. Since this is the Lorentz structure of the perturbative part of $f$, it is not surprising to discover that, substituting Eq. (10) inside the Wilson loop in the large time separation limit, the terms depending on $D_1$ give rise to the kernel

$$
g^2 \langle F^2(0) \rangle \frac{2}{24N_c} \int_{-\infty}^{+\infty} d\tau \int_{0}^{r} d\lambda D_1(\tau^2 - \lambda^2) \gamma^0,
$$

which is of the Coulomb type. The treatment of this kernel is trivial and gives contribution of the perturbative type. Therefore in this work we will not consider terms containing $D_1$.

In what follows we study expression (8) for different choices of the parameters which are the correlation length $a$, the mass $m$, the binding energy ($p^0 - m$) and the momentum transfer $(p - q)$.

A. Heavy quark potential case ($m_a > p^0 - m$)

If we assume $a$ to be bigger than the binding energy ($p^0 - m$) and smaller than the mass $m$ of the quark, since $a \sim 1 \text{ GeV}$, the quark turns out to be sufficiently heavy to be considered non-relativistic. In order to obtain the $1/m^2$ potential we can neglect the “negative energy states” contributions to (8) by writing

$$
K(q, p) \simeq -g^2(2\pi)\delta(p^0 - q^0) \\
\times \int_{0}^{1} d\alpha \alpha^\nu \xi^\mu \xi^\rho \partial^\rho \partial^\mu \\
\times \int d^3r e^{i(p - q) \cdot r} \\
\times \gamma^\nu \Lambda_+(t) \gamma^0 c^{(p^0 - E_t)\tau} \gamma^\mu f_{\mu\nu\rho\sigma}^{n.p.}(\tau, (\alpha - \beta)\tau). \tag{11}
$$

Now, inserting Eq. (8) and by means of the usual reduction techniques, we obtain up to order $1/m^2$ the static and spin dependent potential

$$
V(r) = g^2 \langle F^2(0) \rangle \frac{2}{24N_c} \int_{-\infty}^{+\infty} d\tau \int_{0}^{r} d\lambda
$$
\[ \times (r - \lambda) D(\tau^2 - \lambda^2) \]
\[ + \frac{\sigma \cdot \mathbf{L}}{4m^2r} g^2 \frac{\langle F^2(0) \rangle}{24N_c} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} d\tau d\lambda \times \left( \frac{2\lambda}{r} - 1 \right) D(\tau^2 - \lambda^2). \] (12)

This result agrees with the one body limit of the potential given in [3,5]. In particular for \( r \to \infty \) identifying the string tension \( \sigma = g^2 \frac{\langle F^2(0) \rangle}{24N_c} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} d\tau d\lambda D(\tau^2 - \lambda^2) \) we obtain the well-known Eichten and Feinberg result [4],

\[ V(r) = \sigma r - C = \frac{\sigma \cdot \mathbf{L}}{4m^2r}, \] (13)

where \( C = g^2 \frac{\langle F^2(0) \rangle}{24N_c} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} d\tau d\lambda D(\tau^2 - \lambda^2) \). We observe that the Lorentz structure which gives origin to the negative sign in front of the spin-orbit potential in (13) is in our case not simply a scalar (\( K \simeq \sigma r \)).

**B. Sum rules case** (\( a < p^0 - m, a < m \)) Let us consider now the case in which the binding energy of the quark is bigger than the correlation length, which can be considered zero respect to all the scales of the problem.

\[ K(q, p) \simeq -g^2(2\pi)\delta(p^0 - q^0) \frac{\langle F^2(0) \rangle}{24N_c} \times (g_{\mu\nu}g_{st} - g_{\mu t}g_{\nu k}) \]
\[ \times \int_{0}^{1} d\alpha \alpha^{\alpha(\mu)} \int_{0}^{1} d\beta \beta^{\beta(\nu)} \frac{\partial}{\partial p^0} \frac{\partial}{\partial q^0} \times (\gamma^\nu S_0(p)\gamma^\mu(2\pi)^3\delta^3(p - q)). \] (14)

In particular from Eq. (14) we obtain the well-known leading contribution to the heavy quark condensate [15]:

\[ \langle \bar{Q}Q \rangle \]
\[ = - \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} \text{Tr} \{ S_0(q)K(q, p)S_0(p) \} \]
\[ = - \frac{1}{12} \frac{\langle \alpha F^2(0) \rangle}{\pi m}. \] (15)

**C. Light quark case** (\( a > m \)) Since we have reproduced the known results concerning heavy quarks, Eq. (15) should maintain some physical meaning also when considering heavy-light mesons with a strange quark (like Ds and Bs). In this case the light quark mass is smaller than \( a \): \( m_s \sim 200 \text{ MeV} < 1 \text{ GeV} \). Actually the case \( a > m \) has to be considered as the only realistic one concerning heavy-light mesons. Under this condition either the exponent \( (p^0 - E_t) \) as well as \( (p^0 + E_t) \) can be neglected with respect to \( a \).

Therefore we have:

\[ K(q, p) \simeq -g^2(2\pi)\delta(p^0 - q^0) \]
\[ \times \int_{0}^{+\infty} d\tau \int_{0}^{1} d\alpha \alpha^{n(\mu)} \int_{0}^{1} d\beta \beta^{n(\nu)} \frac{\partial}{\partial p^0} \frac{\partial}{\partial q^0} \times \int d^3r e^{i(p - q)\tau} \gamma^\mu (\Lambda_+(t) - \Lambda_-(t)) \gamma^0 \gamma^\mu \]
\[ \times f_{\pi p}^0(\tau, (\alpha - \beta)\mathbf{r}). \] (16)

We observe that in the zero mass limit this expression gives a chirally symmetric interaction (while a purely scalar interaction breaks chiral symmetry at any mass scale). This means on one side that our interaction keeps the main feature of QCD i.e. in the zero mass limit chiral symmetry is broken only spontaneously. On the other side this seems to suggest that for very light quarks the projectors \( \Lambda_+ \) and \( \Lambda_- \) which appear in (13) should be taken from the chiral broken solution of the corresponding Dyson–Schwinger equation.

**3. CONCLUSIONS**

In the literature, also recently, a Dirac equation with scalar confining kernel (i.e. \( K \simeq \sigma r \)) has been used in order to evaluate non-recoil contributions to the heavy-light meson spectrum [4,6,7]. The main argument in favor of this type of kernel is the nature of the spin-orbit potential for heavy quarks. This turns out to have a long-range vanishing magnetic contribution (according to the Buchmüller picture of confinement) and is completely described by the Thomas precession term. This situation is compatible with a scalar confining kernel. However, assuming more sophisticated confinement models with a bigger sensitivity to the intermediate distance region, the spin-orbit interaction has no more such a simple behaviour. In particular non zero corrections to the magnetic spin-orbit potential show up. Moreover, the velocity dependent sector of the potential seems not to be compatible with a scalar
kernel (we refer the reader to [5] for an exhaustive discussion). Therefore also from the point of view of the potential theory there are strong indications that the Lorentz structure of the confining kernel should be more complicated than a simple scalar one. This emerges also in our approach. The kernel (8) follows simply from the assumption on the gauge fields dynamics given by Eq. (6) and by taking only one non-perturbative gluon insertion on the quark fermion line. When performing the potential reduction of this kernel in the heavy quark case (\(A\)) we obtain exactly the expected static and spin-dependent potentials. Therefore our conclusion is that there exists at least one non scalar kernel which reproduces for heavy quark not only the Eichten and Feinberg potentials in the long distances limit, but also the entire stochastic vacuum model spin-dependent potential. Moreover when considering \(a\), the inverse of the correlation length, small with respect to all the energy scales (case \(B\)), the kernel (8) gives back the leading heavy quark sum rules results. It is possible to try to extend the range of applicability of Eq. (8) to more realistic cases, like \(D_s\) and \(B_s\) mesons where the light quark mass is smaller than the characteristic correlation length of the two point cumulant (case \(C\)). The relevant part of the kernel is also in this case not a simply scalar one. We mention that a similar picture of the Lorentz structure of the confining Dirac kernel emerges also in the different approach of [6].

An attempt to extend the present approach to the two-body case is given in [11]. The equivalent graphs of Fig. 2 seem to play a crucial role (in the two-body case such kind of graph exists for any fermion line and as exchange graph). Nevertheless these graphs are not sufficient in order to provide a complete relativistic description of the two-body system. The main difficulty is that in this case it does not exist a gauge like (6) which automatically cancels the contributions coming from the end-point strings. These contributions are necessary in order to restore gauge invariance. From this point of view the situation seems to be quite more complicated than in the heavy-light system which remains the most natural context to test the formalism.

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