AUTOMORPHISMS OF THE ZERO-DIVISOR GRAPH OVER
2 × 2 MATRICES

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Abstract. The zero-divisor graph of a noncommutative ring $R$, denoted by $\Gamma(R)$, is a graph whose vertices are nonzero zero-divisors of $R$, and there is a directed edge from a vertex $x$ to a distinct vertex $y$ if and only if $xy = 0$. Let $R = M_2(F_q)$ be the $2 \times 2$ matrix ring over a finite field $F_q$. In this article, we investigate the automorphism group of $\Gamma(R)$.

1. Introduction

Zero-divisor graphs have received a lot of attention (see [1, 2, 4, 6, 13]), because they are helpful for revealing the ring-theoretic properties via their graph-theoretic properties. In 1988, I. Beck first introduced the concept of zero-divisor graphs of commutative rings in [7], where all elements of a commutative ring $R$ are defined to be vertices and distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. In such a graph, the vertex 0 is adjacent to every other vertex, and non-zero-divisors are adjacent only to 0. In order to better illustrate the zero-divisor structure of a ring, D. F. Anderson and P. S. Livingston [5] redefined the notion of a zero-divisor graph by cutting off 0 and non-zero-divisors from the graph. Let $R$ be a commutative ring (with 1) and let $Z(R)$ be the set of zero-divisors of $R$. The zero-divisor graph $\Gamma(R)$ of $R$ (defined by D. F. Anderson and P. S. Livingston [5]) is a graph with vertices $Z(R)^* = Z(R) - \{0\}$, the set of nonzero zero-divisors of $R$, and distinct vertices $x$ and $y$ are adjacent if and only if $xy = 0$. S. P. Redmond [16] further extended the concept of a zero-divisor graph to a noncommutative ring $R$, also written as $\Gamma(R)$, by taking the vertex set to be $Z(R)^*$, and there is a directed edge from a vertex...
x to a distinct vertex y if and only if \( xy = 0 \). In [9] and [10] F. DeMeyer et al. introduced the definition of zero-divisor graphs over arbitrary semigroups.

Automorphisms of graphs play an important role both in graph theory and in algebra, and characterization of the full automorphisms of a graph is often a difficult work. Until now, little is known (as far as we know) about the automorphisms of zero-divisor graphs. Now we list some known results related to ours. In 1999, D. F. Anderson and P. S. Livingston [5] proved that if \( n \geq 4 \) is a non-prime integer, then \( \text{Aut}(\Gamma(\mathbb{Z}_n)) \) is a direct product of some symmetric groups. In 2002, S. B. Mulay [14] established some group-theoretic properties of the group of graph-automorphisms of zero-divisor graphs over commutative rings. In 2002, F. DeMeyer and K. Schneider [11] studied the relationship between \( \text{Aut}(R) \) and \( \text{Aut}(\Gamma(R)) \) when \( R \) is a commutative ring. In 2008, J. Han (see [12, Theorem 3.9]) showed that \( \text{Aut}(\Gamma(M_2(\mathbb{Z}_p))) \) is isomorphic to the symmetric group \( S_{p+1} \) of degree \( p+1 \) when \( p \) is a prime. In 2011, S. Park and J. Han [15] generalized [12, Theorem 3.9] to an arbitrary finite field \( \mathbb{F}_q \) with \( q \) elements, and proved that \( \text{Aut}(\Gamma(M_2(\mathbb{F}_q))) \cong S_{q+1} \). When reading the proof of [15, Theorem 3.8], we find some major mistakes there (the automorphism \( \sigma \) they constructed in the proof fails to be a bijection), which inspires us to determine the full automorphisms of \( \Gamma(M_2(\mathbb{F}_q)) \) again. As an application of our main theorem, we show that [12, Theorem 3.9] and [15, Theorem 3.8] are both wrong.

This article is organized as follows. In Section 2, we give some preliminary results and introduce the compressed zero-divisor graph \( \Gamma_E(M_2(\mathbb{F}_q)) \). In Section 3, we determine the automorphisms of \( \Gamma_E(M_2(\mathbb{F}_q)) \). Section 4 is devoted to investigate the automorphism group of \( \Gamma(M_2(\mathbb{F}_q)) \) via what has been obtained in Section 3.

2. Preliminaries and notations

Let \( \Gamma \) be a directed graph with vertex set \( V(\Gamma) \). We write \( x \to y \) to mean that there is a directed edge from a vertex \( x \) to a distinct vertex \( y \), and write \( x \leftrightarrow y \) to mean \( x \to y \) and \( y \to x \). A graph \( H \) is called a subgraph of \( \Gamma \) if \( V(H) \subseteq V(\Gamma) \) and for any \( x,y \in V(H) \), \( x \to y \) in \( \Gamma \) whenever \( x \to y \) in \( H \). Further, \( H \) is called an induced subgraph of \( \Gamma \) if for any \( x,y \in V(H) \), \( x \to y \) in \( \Gamma \) if and only if \( x \to y \) in \( H \). An induced subgraph \( K \) of \( \Gamma \) is called a clique if \( x \leftrightarrow y \) for any distinct vertices \( x \) and \( y \) in \( K \). A set of vertices that induces a subgraph with no edges is called an independent set. For any vertex \( x \) of \( \Gamma \), \( N_l(x) = \{ y \in V(\Gamma) \mid x \to y \} \) and \( N_r(x) = \{ y \in V(\Gamma) \mid x \to y \} \) are respectively called the left neighborhood and the right neighborhood of \( x \). For any set \( X \), we denote by \( |X| \) the cardinality of \( X \). \( |N_l(x)| \) is called the in-degree of \( x \), and \( |N_r(x)| \) is called the out-degree of \( x \). A bijection \( \sigma \) on \( V(\Gamma) \) is said to be an automorphism of \( \Gamma \) if for any two vertices \( x,y \in V(\Gamma) \), \( x \to y \) if and only if \( \sigma(x) \to \sigma(y) \). Denote by \( \text{Aut}(\Gamma) \) the automorphism group of \( \Gamma \).
Hereafter, \( R \) will always denote \( M_2(F_q) \), the \( 2 \times 2 \) matrix ring over a finite field \( F_q \) with \( q \geq 2 \) elements. By \( M_{1 \times 2}(F_q) \) we mean the set of all \( 1 \times 2 \) matrices over \( F_q \), and by \( \alpha^t \) we mean the transpose of \( \alpha \in M_{1 \times 2}(F_q) \). Let \( M_{1 \times 2}^t(F_q) \) be a subset of \( M_{1 \times 2}(F_q) \) consisting of the vectors whose first nonzero component is 1, i.e., \( M_{1 \times 2}^t(F_q) = \{(0 1), (1 a) \mid a \in F_q \} \). By \( Z(R) \) we denote the set of all zero-divisors of \( R \), and by \( U(R) \) we denote the set of all units of \( R \). Then \( R = Z(R) \cup U(R) \). For every matrix \( A \in R \), let \( \det(A) \) be the determinant of \( A \), and let \( r(A) \) be the rank of \( A \). In \( R \), the matrix unit who has 1 in the \( (i,j) \) position and 0 elsewhere is denoted by \( E_{ij} \), the zero matrix is denoted by \( 0 \), and the identity matrix is denoted by \( I \). For any subset \( X \) of \( F_q \) (resp., \( R; M_{1 \times 2}(F_q) \)), let \( X^* = X - \{0\} \). If \( X \) is either an element or a subset of \( R \), then the left annihilator of \( X \) is \( \text{ann}_L X = \{A \in R \mid AX = 0\} \) and the right annihilator of \( X \), denoted by \( \text{ann}_R X \), is similarly defined. As usual, \( \Gamma(R) \) denotes the zero-divisor graph over \( R \), i.e., \( \Gamma(R) \) is a graph with vertex set \( Z(R)^* \), and \( A \rightarrow B \) if and only if \( A \neq B \) and \( AB = 0 \). By \( S_{F_q} \) we mean the symmetric group over \( F_q \), and \( S_{F_q}^* \) is similarly defined.

**Lemma 2.1.** The following three conditions are equivalent:

(i) \( A \in Z(R)^* \);

(ii) \( r(A) = 1 \);

(iii) \( A = \alpha \beta^t \) for \( \alpha, \beta \in M_{1 \times 2}(F_q)^* \).

**Proof.** Note that \( A \in Z(R) \) if and only if \( \det(A) = 0 \). Thus \( A \in Z(R)^* \) if and only if \( r(A) = 1 \), since \( A \) is a \( 2 \times 2 \) matrix. This yields (i)\(\Leftrightarrow\) (ii). Clearly, we have (ii)\(\Leftrightarrow\) (iii). \(\square\)

From the lemma above, each vertex \( A \) in \( \Gamma(R) \) can be written as \( A = \alpha \beta^t \) for some \( \alpha, \beta \in M_{1 \times 2}(F_q)^* \). The set of nonzero scalar multiples of \( A \in R \) is denoted by \( [A] \), i.e., \( [A] = \{aA \mid a \in F_q^*\} \). Then the multiplication \([A][B] = [AB]\) is well-defined, and \( [A] = [B] \) if and only if \( B = aA \) for some \( a \in F_q^* \). If \( A \in V(\Gamma(R)) \), then there exist unique \( \alpha, \beta \in M_{1 \times 2}(F_q) \) such that \([A] = [\alpha \beta^t]\). We call \( \alpha \beta^t \) the standard representation of \([A]\).

**Lemma 2.2.** If \( \alpha \in M_{1 \times 2}(F_q) \), then there exists a unique \( \beta \in M_{1 \times 2}(F_q) \) such that \( \alpha \beta^t = 0 \). Meanwhile \( \beta \alpha^t = 0 \), and

(i) if \( \alpha = (0 1) \), then \( \beta = (1 0) \),

(ii) if \( \alpha = (1 0) \), then \( \beta = (0 1) \),

(iii) if \( \alpha = (1 a) \) for some \( a \neq 0 \), then \( \beta = (1 - a^{-1}) \).

**Proof.** It is easy to get these results by direct calculations. \(\square\)

**Lemma 2.3.** If \( A \in Z(R)^* \), then \( A^2 = 0 \) if and only if \( A \in [E_{12}] \cup [E_{21}] \cup a \in F_q^* \left\{ \left[ \begin{array}{cc} 1 & a \\ -a^{-1} & 1 \end{array} \right] \right\} \).

**Proof.** Obviously, we get the ‘if’ part. For the ‘only if’ part, assume that \( \alpha \beta \) is the standard representation of \([A]\), where \( \alpha, \beta \in M_{1 \times 2}(F_q) \). Then it follows from \( A^2 = 0 \) if and only if \([A]^2 = 0\) that \( \beta \alpha^t = 0 \). By Lemma 2.2, we have
whose vertices are the equivalence classes induced by $\sim$, and only if $R^{\alpha}(1)D$.

Proof. Lemma 2.4. For $E \in \Gamma$, Definition 2.5. The compressed zero-divisor graph $\Gamma$ was later studied more extensively in [3, 8, 17]. Now, we extend this definition to noncommutative rings.

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$[A] = [\alpha^t \beta] = [(1 \ 0)^t(0 \ 1)] = [E_{12}]$, or $[A] = [\alpha^t \beta] = [(0 \ 1)^t(1 \ 0)] = [E_{21}]$.

or $[A] = [\alpha^t \beta] = [(1 - a^{-1})^t(1 \ a)] = \left(\frac{1}{-a^{-1} - 1}\right)$ for some $a \neq 0$. Thus $A \in [A] \subseteq [E_{12}] \cup [E_{21}] \cup_{a \in F_q} \left(\frac{1}{-a^{-1} - 1}\right).

Lemma 2.4. Let $A, B \in Z(R)^+$. Then $ann_{r}A \cap ann_{l}B \neq \{0\}$ and $ann_{r}A \cap ann_{l}B \neq \{0\}$ if and only if $[A] = [B]$. In particular, $ann_{r}A = ann_{l}B$ if and only if $[A] = [B]$. $\Box$.

Proof. Clearly, $\Leftarrow$ holds. In order to prove $\Rightarrow$, assume that $C \neq 0$ and $C \cap ann_{r}A \cap ann_{l}B$. Then $[C] \in ann_{r}A \cap ann_{l}B$. That means $[C][A] = 0$ and $[C][B] = 0$. Suppose that $\alpha_1^{\gamma_1}, \beta_1^{\gamma_2}, \beta_2, \gamma_1^{\gamma_2}$ are the standard representations of $[A], [B], [C]$, respectively. It follows from $[C][A] = 0$ and $[C][B] = 0$ that

$$\gamma_2 \alpha_1^{\gamma_1} = \gamma_2 \beta_1^{\gamma_2} = 0.$$  

Applying Lemma 2.2 on Eq. (1), we get $\alpha_1 = \beta_1$. Similarly, if $D \neq 0$ and $D \in ann_{r}A \cap ann_{r}B$, then $\alpha_2 = \beta_2$. Thus $[A] = [\alpha_1^t \alpha_2] = [\beta_1^t \beta_2] = [B]$. $\Box$.

S. B. Mulay [14] introduced the compressed zero-divisor graph of a commutative ring while studying automorphisms. The relation on a commutative ring is an equivalence relation. The compressed zero-divisor graph $\Gamma_{E}(R)$ is the (undirected) graph whose vertices are the equivalence classes induced by $\sim$ other than $[0]$ and $[1]$, such that $[s]$ and $[t]$ are adjacent in $\Gamma_{E}(R)$ if and only if $st = 0$. This graph was later studied more extensively in [3, 8, 17]. Now, we extend this definition to noncommutative rings.

Definition 2.5. The relation on a noncommutative ring $R$ given by $s \sim t$ if and only if $ann_{r}s = ann_{l}t$ is an equivalence relation. The compressed zero-divisor graph of $R$, denoted by $\Gamma_{E}(R)$, is the directed graph whose vertices are the equivalence classes induced by $\sim$ other than $[0]$ and $[1]$, such that $[s] \to [t]$ in $\Gamma_{E}(R)$ if and only if $[s] \neq [t]$ and $st = 0$.

Let $\Gamma_{E}(R)$ be the compressed zero-divisor graph of $R = M_2(F_q)$, i.e., $\Gamma_{E}(R)$ is a graph with vertex set $\{[A] \mid A \in Z(R)^+\}$, and there is a directed edge from a vertex $[A]$ to a distinct $[B]$ if and only if $AB = 0$. For any vertex $[A]$ in $\Gamma_{E}(R)$, suppose that $A = \alpha^t \beta = (a_1 a_2)(b_1 b_2)$ and the first nonzero component of $\alpha$ (resp., $\beta$) is $a_1$ (resp., $b_1$), and call $[A]$ a vertex of type $(i, j)$. Then all vertices in $\Gamma_{E}(R)$ can be categorized into four types as follows.

- type $(1, 1)$: $[A] = [(1 \ 0)^t(1 \ b)] = [(\frac{1}{a} \ \frac{b}{a})]$, $a, b \in F$, 
- type $(1, 2)$: $[A] = [(1 \ a)^t(0 \ 1)] = [(\frac{a}{a} \ \frac{1}{a})]$, $a \in F$,
- type $(2, 1)$: $[A] = [(0 \ 1)^t(1 \ b)] = [(\frac{b}{0} \ \frac{1}{0})]$, $b \in F$,
- type $(2, 2)$: $[A] = [(0 \ 1)^t(0 \ 1)] = [E_{22}]$.

It is obvious to see that in $\Gamma_{E}(R)$, there are $q^2, q, q, 1$ vertices in types $(1, 1)$, $(1, 2), (2, 1), (2, 2)$ respectively. It follows that there are $(q + 1)^2$ vertices in $\Gamma_{E}(R)$, and $(q - 1)(q + 1)^2$ vertices in $\Gamma(R)$ since each vertex $[A] \in V(\Gamma_{E}(R))$ contains $q - 1$ different vertices in $\Gamma(R)$. 

Lemma 3.1. \( \sigma \) map over any given set, \( F \). Clearly, we get “\( \Rightarrow \)”. Conversely, suppose \( \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \) and \( P^{-1} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \). Since \( \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \), we get

\[
\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right).
\]

Thus \( \left( \begin{array}{cc} a & \frac{0}{0} \\ c & \frac{0}{0} \end{array} \right) \) and \( b_1 = c = 0 \). Hence

\[
P = \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) , \quad \sigma = \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right).
\]

From \( \sigma = \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \), we have

\[
\left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right).
\]

It follows that \( \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) \) and \( ac_1 = 0 \). Thus \( c_1 = 0 \), and therefore

\[
P = \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right), \quad \sigma = \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right).
\]

Since \( \sigma = \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \), it follows that

\[
\left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} a & 0 \\ 0 & d \end{array} \right) \left( \begin{array}{cc} a^{-1} & 0 \\ 0 & d^{-1} \end{array} \right) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right).
\]

Thus \( \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \). This gives \( ad^{-1} = 1 \). Thus \( a = d \), and \( P \) is a nonzero scalar matrix.

\[ \Box \]

Lemma 3.2. \( \Pi \cong U(R)/F_q^*I \). Proof. Set

\[
\psi : U(R) \to \Pi, \quad P \mapsto \sigma_P.
\]

Then \( \psi \) is well-defined and \( \psi \) is a surjection. It is easy to prove that \( \sigma_P \sigma_Q = \sigma_{PQ} \), and so \( \psi \) is a surjective homomorphism. By Lemma 3.1, we get \( \text{Ker} \psi = F_q^*I \). Thus \( \Pi \cong U(R)/F_q^*I \). \[ \Box \]
Next, let us introduce another automorphism $\tau_f$ of $\Gamma_E(R)$.

**Definition 3.3.** Let $f \in S_{F_q}$ that fixes 0, set $f^* \in S_{F_q}$ as follows, and call $f^*$ the companion permutation of $f$.

$$f^*(a) = \begin{cases} 
0, & a = 0, \\
-f(-a^{-1})^{-1}, & a \neq 0.
\end{cases}$$

**Lemma 3.4.** If $f, g \in S_{F_q}$ both fix 0, then $(f^*)^* = f$, $(f^{-1})^* = (f^*)^{-1}$, and $(fg)^* = f^*g^*$.

**Proof.** Apparently, $(f^*)^*(0) = f(0)$.

In the case $a \neq 0$, it follows from $f^*(-a^{-1})f(a) = -1$ and $(f^*)^*(a)f^*(-a^{-1}) = -1$ that $(f^*)^*(a) = f(a)$. Thus $(f^*)^* = f$.

Clearly, we have $(f^{-1})^*f^*(0) = 0$. For any $a \neq 0$, set $b = f(a)\tau$. Then $(f^{-1})^*f^*(a) = (f^{-1})^*(-f(-a^{-1})^{-1}) = (f^{-1})^*(-b^{-1}) = f^{-1}(b^{-1}) = (-a^{-1})^{-1} = a$. Thus $(f^{-1})^*f^* = 1$, and so $(f^{-1})^* = (f^*)^{-1}$.

Let $a \in F_q^*$. Set $b = g(-a^{-1})$. Then $(fg)^*(0) = f^*g^*(0)$ and $(fg)^*(a) = -(fg(-a^{-1}))^{-1} = -f(b^{-1}) = f^*(-b^{-1}) = f^*(-g(-a^{-1})^{-1}) = f^*g^*(a)$ yielding $(fg)^* = f^*g^*$.

Let $f \in S_{F_q}$ that fixes 0, and let $\Sigma = \{\tau_f | f \in S_{F_q}, f(0) = 0\}$, where the map $\tau_f : V(\Gamma_E(R)) \rightarrow V(\Gamma_E(R))$ is defined as the following.

$$\tau_f : [A] = [(1 a)^t(1 b)] \mapsto [(1 f^*(a))^t(1 f(b))], a, b \in F_q,$$

$$[A] = [(1 a)^t(0 1)] \mapsto [(1 f^*(a))^t(0 1)], a \in F_q,$$

$$[A] = [(0 1)^t(1 b)] \mapsto [(0 1)^t(1 f(b))], b \in F_q,$$

$$[A] = [(0 1)^t(0 1)] \mapsto [(0 1)^t(0 1)].$$

**Lemma 3.5.** (i) $\tau_f \tau_g \tau_f^{-1} = \tau_f^{-1}$ for every $\tau_f, \tau_g \in \Sigma$.

(ii) $\Sigma$ is a group, and $\Sigma \cong S_{F_q}$.

**Proof.** (i) For any $a, b \in F_q$, we have

$$\tau_f g([(1 a)^t(1 b)]) = [(1 (fg)^*(a))^t(1 fg(b))]$$

$$= [(1 f^*g^*(a))^t(1 fg(b))]$$

$$= \tau_f([(1 g^*(a))^t(1 g(b))])$$

$$= \tau_f \tau_g([(1 a)^t(1 b))].$$

Thus

$$\tau_f g([A]) = \tau_f \tau_g([A])$$

for each $[A]$ of type $(1,1)$. Similarly, Eq. (2) holds for any $[A]$ of types $(1,2), (2,1), (2,2)$. Hence $\tau_f g = \tau_f \tau_g$, and so $\tau_f \tau_1 \tau_f = \tau_f \tau_1 = 1$. 

(ii) By (i) we know $\Sigma$ is a group. Apparently, $S_{F_q}$ is a group and $S_{F_q} \cong \{f \in S_{F_q} | f(0) = 0\}$. Each $f \in S_{F_q}$ that fixes 0 induces a $\tau_f$ in $\Sigma$. Let

$$\varphi : \{f \in S_{F_q} | f(0) = 0\} \to \Sigma, \ f \mapsto \tau_f.$$ 

We show that $\varphi$ is an isomorphism, thus $\Sigma \cong \{\tau_f | f \in S_{F_q}, f(0) = 0\} \cong S_{F_q}$. Indeed, by (i) we get that the surjection $\varphi$ satisfies $\varphi(fg) = \varphi(f)\varphi(g)$. In what follows, we prove that $\varphi$ is injective, and so $\varphi$ is an isomorphism. In fact, if there exist $f, g \in S_{F_q}$ fix 0 such that $\tau_f = \tau_g$, then for every $b \in F_q$ we have $\tau_f([[0 1]^t(1 b)]) = \tau_g([[0 1]^t(1 b)])$. Thus $f(b) = g(b)$, and so $f = g$. Hence $\varphi$ is injective.

Now we show that $\tau_f$ is an automorphism of $\Gamma_E(R)$.

**Lemma 3.6.** $\tau_f$ is an automorphism of $\Gamma_E(R)$, and $\Sigma$ is a subgroup of $\text{Aut}(\Gamma_E(R))$.

**Proof.** Clearly, $\tau_f$ preserves the type of each vertex in $\Gamma_E(R)$. Since $f$ and $f^*$ are permutations over $F_q$, it follows that $\tau_f$ is bijective. If we can prove that $[A] \to [B]$ if and only if $\tau_f([A]) \to \tau_f([B])$ for each $[A], [B] \in V(\Gamma_E(R))$, then $\tau_f$ is an automorphism of $\Gamma_E(R)$. In fact, it follows from $\tau_f([A]) \to \tau_f([B])$ that $[A] \to [B]$ since $\tau_f^{-1}\tau_f = 1$. Conversely, if $[A] \to [B]$, then $[A] \neq [B]$ and $[\alpha_f \alpha_2][\beta_f \beta_2] = 0$, where $\alpha_f \alpha_2$ and $\beta_f \beta_2$ are respectively the standard representations of $[A]$ and $[B]$. Thus $\alpha_f \beta_1 = 0$. Lemma 2.2 yields $\alpha_2, \beta_1$ satisfy one of the three cases as follows.

(i) If $\alpha_2 = (0 1)$, then $\beta_1 = (1 0)$, and so

$$\tau_f([A])\tau_f([B]) = \left(\begin{array}{cc} \ast & \ast \\ 0 & 1 \end{array}\right)^t[(0 1)^t(\ast \ast)] = 0.$$ 

(ii) If $\alpha_2 = (1 0)$, then $\beta_1 = (0 1)$. Now

$$\tau_f([A])\tau_f([B]) = \left(\begin{array}{cc} \ast & \ast \\ 1 & 0 \end{array}\right)^t[(1 0)^t(\ast \ast)] = 0.$$ 

(iii) If $\alpha_2 = (1 a)$ for some $a \neq 0$, then $\beta_1 = (1 - a^{-1})$. Thus

$$\tau_f([A])\tau_f([B]) = \left(\begin{array}{cc} \ast & \ast \\ 1 & 0 \end{array}\right)^t[(1 f(a)^t(\ast \ast)] = 0.$$ 

By (i)-(iii), we always have $\tau_f([A])\tau_f([B]) = 0$. Note that, $[A] \neq [B]$ and $\tau_f$ is bijective, so we get $\tau_f([A]) \neq \tau_f([B])$. Hence $\tau_f([A]) \to \tau_f([B])$. \hfill \Box

**Lemma 3.7.** Let $P, Q \in U(R)$, and let $f, g \in S_{F_q}$ that fix 0. Then

(i) $\tau_f = 1$ if and only if $f = 1$,

(ii) if $f$ fixes 1 and $\sigma_f = \tau_f$, then $\sigma_f = \tau_f = 1$,

(iii) if $f, g \in S_{F_q}$ fix 1 and $\sigma_f \tau_f = \sigma_g \tau_g$, then $f = g$ and $P = aQ$ for some $a \in F^*$. 

**Proof.** (i) Clearly, if $f = 1$, then $f^* = 1$, and so $\tau_f = 1$. Conversely, by the definition of $\tau_f$, we conclude $f = 1$ if $\tau_f = 1$.

(ii) Note that $\sigma_f([E_{11}]) = \tau_f([E_{11}]) = [E_{11}], \sigma_f([E_{12}]) = \tau_f([E_{12}]) = [E_{12}], \sigma_f([0 1]) = \tau_f([0 1]) = [0 1]$, it follows from Lemma 3.1 that $\tau_f = \sigma_f = 1$. 


(iii) Since $\sigma_P \tau_f = \sigma_Q \tau_g$, it follows that $\sigma_Q^{-1} \sigma_P = \tau_g \tau_f^{-1}$. Then

$$\sigma_Q^{-1} \sigma_P = \tau_g \tau_f^{-1} = \tau_{gf^{-1}}.$$  \hspace{1cm} \text{(3)}$$

By Eq. (3) and (ii), we have

$$\sigma_Q^{-1} \sigma_P = \tau_{gf^{-1}} = 1.$$  \hspace{1cm} \text{(4)}

By (i), Eq. (4), and Lemma 3.1, we get $Q^{-1} P$ is a nonzero scalar matrix and $gf^{-1} = 1$. Thus $f = g$, and $P = aQ$ for some $a \in F^*$.

In the following of this section, we develop some lemmas to show that any automorphism of $\Gamma_E(R)$ can be expressed as $\sigma_P \tau_f$, where $\sigma_P$ and $\tau_f$ are the automorphisms we constructed as above.

**Lemma 3.8.** The in-degree and out-degree of each vertex $[A]$ in $\Gamma_E(R)$ are both equal to $q + 1$ if $A^2 \neq 0$, and are both equal to $q$ if $A^2 = 0$.

**Proof.** For any $[B], B \in Z(R)^*$ satisfies $[B][A] = 0$, assume that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are respectively the standard representations of classes $[A], [B]$. Thus $\alpha_1, \alpha_2, \beta_1, \beta_2 \in M_{1 \times 2}(F_q)$ and $[B][A] = 0$ implies $\beta_2 \alpha_1 = 0$. By Lemma 2.2, $\beta_2$ is uniquely determined by $\alpha_1$. Thus the cardinality of $\{|[B]|[B][A] = 0, B \in Z(R)^*\}$ is depending on the choice of $\beta_1 \in M_{1 \times 2}(F_q)$. Hence there are $q + 1$ nonzero classes in $\{|[B]|[B][A] = 0, B \in Z(R)^*\}$ since $|M_{1 \times 2}(F_q)| = q + 1$ and different $\beta_1$ implies different $[\beta_1^*]$. Then $N_{\Gamma}(|[A]|) = \{|[B]|[B][A] = 0, B \in Z(R)^*\} - \{|[A]|\}$, and so the in-degree of $[A]$ is $|N_{\Gamma}(|[A]|)| = q$ if $A^2 = 0$, the in-degree of $[A]$ is $|N_{\Gamma}(|[A]|)| = q + 1$ if $A^2 \neq 0$. By a similar argument, we can get the out-degree of each vertex $[A]$.

**Lemma 3.9.** If $\sigma$ is an automorphism of $\Gamma_E(R)$ and $\sigma([A]) = [B]$, then $A^2 = 0$ if and only if $B^2 = 0$.

**Proof.** It immediately follows from Lemma 3.8 since $\sigma$ preserves the in-degree and the out-degree of each vertex.

**Lemma 3.10.** If $\sigma$ is an automorphism of $\Gamma_E(R)$ that fixes $[E_{ii}], i = 1, 2$, then it preserves the type of each vertex.

**Proof.** Note that, there is exactly one vertex $[E_{22}]$ in type $(2,2)$, and $\sigma$ already fixes $[E_{22}]$.

From $[E_{11}] \rightarrow ([\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}])$ we know $\sigma([E_{11}]) \rightarrow \sigma(([\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]))$. Assume that $\sigma(([\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}])) = ([\begin{smallmatrix} b_1 & b_2 \\ b_3 & b_4 \end{smallmatrix}])$. Then $\sigma([E_{11}])\sigma(([\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}])) = 0$ implies $E_{11} (b_1, b_2, b_3, b_4) = 0$. Thus $b_1 = b_2 = 0$, and hence $\sigma(([\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}])) = ([\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}])$. Since $\sigma$ is a bijection and $\sigma([E_{22}]) = [E_{22}]$, we get $b_3 \neq 0$. That means

$$\sigma(([\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}])) = ([\begin{smallmatrix} 0 & 0 \\ b_3 & b_4 \end{smallmatrix}]) = ([\begin{smallmatrix} 0 & 0 \\ 1 & b_3^{-1} b_4 \end{smallmatrix}]).$$

This yields $\sigma$ preserves vertices of type $(2,1)$.

After a similar argument on $([\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}]) \rightarrow [E_{11}]$, we get that $\sigma$ preserves vertices of type $(1,2)$. 

\[ \text{□} \]
Since $\sigma$ is bijective and it preserves vertices of types $(1,2)$, $(2,1)$, $(2,2)$, it follows that $\sigma$ preserves vertices of type $(1,1)$. \hfill \square

**Lemma 3.11.** If $\sigma$ is an automorphism of $\Gamma_E(R)$ that fixes $[E_{ii}]$, $i = 1, 2$, then there exists an $f \in S_{F_2}$ fixes $0$ such that $\sigma = \tau_f$.

**Proof.** Since $\sigma$ is an automorphism of $\Gamma_E(R)$ and fixes $[E_{ii}]$, $i = 1, 2$, it follows from Lemma 3.10 that $\sigma$ preserves the type of $(2,1)$. For every $a \in F_q$, $\sigma([\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}]) = [\begin{smallmatrix} 0 & 0 \\ 1 & b \end{smallmatrix}]$ for some $b \in F_q$. Set $f(a) = b$. Then $f \in S_{F_2}$. By Lemma 3.9, we have $\sigma([E_{21}]) = [E_{21}]$, and so $f(0) = 0$. Thus $\tau_f$ is well defined. Now, we show that $\sigma = \tau_f$.

(i) Clearly, $\sigma([A]) = \tau_f([A])$ holds for $[A]$ of type $(2,2)$, since $\sigma([E_{22}]) = [E_{22}] = \tau_f([E_{22}])$.

(ii) $\sigma([A]) = \tau_f([A])$ holds for $[A]$ of type $(2,1)$.

For any $a \in F_q$, we get

$$
\sigma([\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}]) = [\begin{smallmatrix} 0 & 0 \\ 1 & f(a) \end{smallmatrix}] = \tau_f([\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}]).
$$

(iii) $\sigma([A]) = \tau_f([A])$ holds for $[A]$ of type $(1,2)$.

Suppose that $[A] = [(1 \ a)^t]$, $a \in F_q$. By Lemma 3.10 we can assume $\sigma([(1 \ a)^t]) = [(1 \ b)^t]$. Apparently, by Lemma 3.9 we get $\sigma([E_{12}]) = [E_{12}] = [(1 \ f(t))^t(0 \ 1)] = \tau_f([E_{12}])$. In the case $a \neq 0$, we have $[(1 \ a^{-1})] \rightarrow [(1 \ b^{-1})]$, and so $\sigma([(1 \ a^{-1})]) \rightarrow \sigma([(1 \ b^{-1})])$. By (ii) we know

$$
\sigma([\begin{smallmatrix} 0 & 0 \\ 1 & a^{-1} \end{smallmatrix}]) = \tau_f([\begin{smallmatrix} 0 & 0 \\ 1 & f(-a) \end{smallmatrix}]),
$$

and so $\sigma([(0 \ 1 \ 0 \ 1)]) = \sigma([(0 \ 0 \ 1 \ 0)]) = 0$. Thus

$$
\sigma([\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}]) = [\begin{smallmatrix} 0 & 0 \\ 1 & f(a) \end{smallmatrix}] = \tau_f([\begin{smallmatrix} 0 & 0 \\ 1 & a \end{smallmatrix}]).
$$

(iv) $\sigma([A]) = \tau_f([A])$ holds for $[A]$ of type $(1,1)$.

For each $[A] = [(1 \ a)^t(1 \ b)]$, $a, b \in F_q$, assume $\sigma([A]) = [(1 \ b)^t(1 \ a)]$. Then

$$
\sigma([(1 \ a)^t(1 \ b)]) = [(1 \ f(a))^t(1 \ f(b))] = \tau_f([(1 \ a)^t(1 \ b)]), a, b \in F_q.
$$

Indeed, if $a = 0$, then applying $\sigma$ on $[E_{22}] \rightarrow [\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}]$ we know

$$
\sigma([\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}]) = [\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}].
$$

In the case $a \neq 0$, from $[(0 \ 1)^t(1 \ a^{-1})] \rightarrow [(1 \ a)^t(1 \ b)]$ we have $\sigma([(0 \ 1)^t(1 \ a^{-1})]) \rightarrow \sigma([(1 \ a)^t(1 \ b)])$. Thus $\sigma([(0 \ 1)^t(1 \ a^{-1})]) = \sigma([(1 \ a)^t(1 \ b)]) = 0$. Note that, by (ii) we get $\sigma([(0 \ 1)^t(1 \ a^{-1})]) = \tau_f([(0 \ 1)^t(1 \ a^{-1})]) = [(0 \ 1)^t(1 \ f(-a^{-1}))]$, thus

$$
\sigma([(1 \ a)^t(1 \ b)]) = [(1 \ f(a))^t(1 \ a)].
$$
If $b = 0$, then applying $\sigma$ on $[(1 \ a)^t (1 \ 0)] \to [(0 \ 1)^t (0 \ 1)]$ and by Eqs. (5)-(6), we have

\[
\sigma([(1 \ a)^t (1 \ 0)]) = [(1 \ f^*(a))^t (1 \ 0)] = \tau_f([(1 \ a)^t (1 \ 0)]).
\]

In the case $b \neq 0$, applying $\sigma$ on $[(1 \ a)^t (1 \ b)] \to [(1 - b^{-1})^t (0 \ 1)]$ and by Eqs. (5)-(6), we get that

\[
\sigma([(1 \ a)^t (1 \ b)]) = [(1 \ f^*(a))^t (1 \ f(b))] = \tau_f([(1 \ a)^t (1 \ b)]).
\]

From (i)-(iv), we conclude that $\sigma([A]) = \tau_f([A])$ for any $[A]$ of all types. Thus $\sigma = \tau_f$.

\[\text{Theorem 3.12. If } \sigma \text{ is an automorphism of } \Gamma_E(R), \text{ then } \sigma = \sigma_P \tau_f, \text{ where } \sigma_P \in \Pi \text{ and } \tau_f \in \Sigma.\]

\[\text{Proof. The proof is divided into two steps as follows.}\]

\[\text{Step 1. There exists } P \in U(R) \text{ such that } \sigma_P \sigma \text{ fixes } [E_{ii}], \text{ i = 1, 2.}\]

Since the rank of every matrix in $Z(R)^*$ is 1, there exist $P_1, Q_1 \in U(R)$ such that $\sigma([E_{11}]) = [P_1E_{11}Q_1]$. Write $\sigma_1 = \sigma_{P_1^{-1}} \sigma$. Then

\[
\sigma_1([E_{11}]) = [P_1^{-1}P_1E_{11}Q_1P_1] = [E_{11}Q_1P_1] = [aE_{11} + bE_{12}]
\]

for some $a, b \in F_q$. Since $E_{11} \neq 0$, by Lemma 3.9 we know $(aE_{11} + bE_{12})^2 \neq 0$. Thus $a \neq 0$. Set $P_2 = (\begin{smallmatrix} 1 & -b \\ 0 & 1 \end{smallmatrix})$ and set $\sigma_2 = \sigma_{P_2} \sigma_1$. We have

\[
\sigma_2([E_{11}]) = [P_2 \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix} P_2^{-1}] = [E_{11}].
\]

Since $[E_{22}] \in N_t([E_{11}]) \cap N_r([E_{11}])$, we get

\[
\sigma_2([E_{22}]) \in N_t(\sigma_2([E_{11}])) \cap N_r(\sigma_2([E_{11}]))
= N_t([E_{11}]) \cap N_r([E_{11}])
= \begin{bmatrix} 1 & \ast \\ 0 & \ast \end{bmatrix} \cap \begin{bmatrix} 0 & \ast \\ \ast & \ast \end{bmatrix}
= \{[E_{22}]\}.
\]

From what is above, if we set $P = P_2P_1^{-1}$, then $P \in U(R)$ and $\sigma_2 = \sigma_P \sigma$ preserves $[E_{ii}], i = 1, 2$.

\[\text{Step 2. There exists } f \in S_{F_q} \text{ that fixes 0 such that } \sigma_P \sigma = \tau_f.\]

Thanks to Lemma 3.11, we immediately get this step.

By Steps 1-2, if $\sigma$ is an automorphism of $\Gamma_E(R)$, then there exist $P \in U(R)$ and $f \in S_{F_q}$ that fixes 0 such that $\sigma_P \sigma = \tau_f$. Rewrite $P^{-1}$ as $P$. Then $\sigma = \sigma_P \tau_f$.

\[\text{In order to calculate the number of automorphisms of } \Gamma_E(R) \text{ more easily, we develop the following theorem.}\]

\[\text{Theorem 3.13. If } \sigma \text{ is an automorphism of } \Gamma_E(R), \text{ then } \sigma = \sigma_P \tau_f, \text{ where } P \in U(R) \text{ and } f \in S_{F_q} \text{ that fixes 0 and 1.}\]
Similarly, we get

Lemma 4.1. $\sigma_1 = \sigma_3\sigma_2$ preserves $[E_{ii}] = E_{ii}(i = 1, 2)$.

The proof is divided into three steps as follows.

**Step 1.** There exists $P \in U(R)$ such that $\sigma_1 = \sigma_3\sigma_2$ fixes $[E_{ii}]$, $i = 1, 2$.

See the proof of Step 1 in Theorem 3.12.

**Step 2.** There exists $Q \in U(R)$ such that $\sigma_2 = \sigma_3\sigma_1$ preserves $[E_{ii}] = E_{ii}(i = 1, 2)$ and $\{(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})\}$.

By Lemmas 3.9-3.10, we know $\sigma_1$ preserves the type of each vertex and sends a square-zero class to a square-zero one. Thus $\sigma_1([E_{12}]) = [E_{12}]$ and $\sigma_1([E_{21}]) = [E_{21}]$. By Lemma 2.3, we know $A^2 = 0$ if and only if $A \in [E_{12}] \cup [E_{21}] \cup_{a \in F_q} \{(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})\}$ for some unique $a \in F_q$. Set $Q = (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ and set $\sigma_2 = \sigma_3\sigma_1$, then $\sigma_2([E_{ii}]) = [E_{ii}] (i = 1, 2)$ and $\sigma_3([\{(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})\}]) = [\{(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})\}]$.

**Step 3.** There exists $f \in S_{F_q}$ that fixes 0 and 1 such that $\sigma_2 = \tau_f$.

By Lemma 3.11, and note that $\sigma_2([\{(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})\}]) = [\{(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix})\}]$, we have $f(1) = 1$.

By Steps 1-3, if $\sigma$ is an automorphism of $\Gamma(E_2)$, then there exist $P, Q \in U(R)$ and $f \in S_{F_q}$ that fixes 0, 1 such that $\sigma Q \sigma P \sigma = \tau_f$. Rewrite $P^{-1}Q^{-1}$ as $P$. Then $\sigma = \sigma P \tau_f$.

**Corollary 3.14.** $|\text{Aut}(\Gamma(E_2))| = (q + 1)!$.

Proof. Since $||I| = |U(R)/F_q^*| = \frac{(q^2 - 1)(q^2 - q)}{q - 1} = q(q^2 - 1)$, $\{|f \in S_{F_q} | f(0) = 0, f(1) = 1\| = (q - 2)!$, and by Lemma 3.7(iii) and Theorem 3.13 we have $|\text{Aut}(\Gamma(E_2))| = q(q^2 - 1) \cdot (q - 2)! = (q + 1)!$.

4. **Automorphisms of $\Gamma(R)$**

In this section, we discuss the automorphism group of $\Gamma(R)$. Let $\sigma$ be an automorphism of $\Gamma(R)$, and let $\sigma[A] = \{\sigma(B) | B \in [A]\}$.

**Lemma 4.1.** Let $\sigma$ be an automorphism of $\Gamma(R)$. Then $\sigma[A] = [\sigma(A)]$, $A \in Z(R)^*$.

Proof. For every $A \in Z(R)^*$, by Lemma 3.8 we have $|N_1([A])| \geq q$ in $\Gamma(E_2)$, and so $|N_1([A])| \geq q(q - 1) \geq 2$ in $\Gamma(R)$. Thus $N_1(A) \cap N_1(aA) \neq \emptyset$, $a \in F_q^*$. That means $N_1(\sigma(A)) \cap N_1(\sigma(aA)) \neq \emptyset$, and so $\text{ann}_1\sigma(A) \cap \text{ann}_1\sigma(aA) \neq \emptyset$.

Similarly, we get $\text{ann}_n\sigma(A) \cap \text{ann}_n\sigma(aA) \neq \emptyset$.

By Lemma 2.14, we have $[\sigma(A)] = [\sigma(aA)]$. Thus $\sigma(aA) \in [\sigma(A)]$ for any $a \in F_q^*$, and so $\sigma[A] \subseteq [\sigma(A)]$. Note that, $\sigma$ is a bijection and $|[A]| = |[\sigma(A)]| = q - 1$, so we get $\sigma[A] = [\sigma(A)]$.

We can assume that $V(\Gamma(R)) = \bigcup_{i=1}^{(q+1)^2} [A_i]$, since there are $(q + 1)^2$ vertices in $\Gamma(E_2)$. Let $\sigma_i$ be a bijection on $V(\Gamma(R))$ satisfying $\sigma_i[A_i] = [A_i]$ and $\sigma_i(\Gamma) = A$ if $A \notin [A_i]$, and let $S_{[A_i]}$ be the set consisting of all such bijections. Clearly,
\(S_{[A_i]}\) is isomorphic to the symmetric group over set \([A_i]\). Set \(K(\Gamma(R)) = \prod_{i=1}^{(q+1)^2} S_{[A_i]}\). Then \(|K(\Gamma(R))| = ((q-1)!)^{(q+1)^2}\).

**Lemma 4.2.** \(K(\Gamma(R))\) is a subgroup of \(\text{Aut}(\Gamma(R))\).

**Proof.** Let \(\sigma_i \in S_{[A_i]}\). Then \(\sigma_i[A_i] = [A_i]\) and \(\sigma_i\) fixes any other vertex in \(\Gamma(R)\). Note that, if \(A^2 \neq 0\), then \([A_i]\) induces an independent set, and each of the vertices in \([A_i]\) have the same left neighborhood and the same right neighborhood. This gives \(\sigma_i\) an automorphism of \(\Gamma(R)\), and \(S_{[A_i]}\) is a subgroup of \(\text{Aut}(\Gamma(R))\). If \(A^2 = 0\), then \([A_i]\) induces a clique, and \(N_i(A) - [A_i] = N_i(B) - [A_i]\), \(N_i(A) - [A_i] = N_i(B) - [A_i]\) for every \(A, B \in [A_i]\). Again \(\sigma_i\) is an automorphism of \(\Gamma(R)\) and \(S_{[A_i]}\) is a subgroup of \(\text{Aut}(\Gamma(R))\). Thus \(K(\Gamma(R)) = \prod_{i=1}^{(q+1)^2} S_{[A_i]}\) is a subgroup of \(\text{Aut}(\Gamma(R))\) since \(K(\Gamma(R))\) is generated by \(\cup_{i=1}^{(q+1)^2} S_{[A_i]}\).

**Theorem 4.3.** \(\text{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \text{Aut}(\Gamma_E(R))\).

**Proof.** If \(\sigma\) is an automorphism of \(\Gamma(R)\), then by Lemma 4.1 we have \(\sigma[A] = [\sigma(A)]\). And for each \(A, B \in Z(R)^*\),

\[
A \rightarrow B \iff \sigma(A) \rightarrow \sigma(B).
\]

Thus

\[
A \neq B, AB = 0 \iff \sigma(A) \neq \sigma(B), \ \sigma(A)\sigma(B) = 0,
\]

and so

\[
(7) \quad A \neq B, AB = 0 \iff \sigma(A) \neq \sigma(B), \ \sigma(A)\sigma(B) = 0.
\]

Set \([A] \mapsto [\sigma(A)]\). Then \([\sigma]\) is an automorphism of \(\Gamma_E(R)\). Indeed, \([\sigma]\) is a bijection, and by Eq. (7) we know \([A] \mapsto [B]\) in \(\Gamma_E(R)\) if and only if \([\sigma([A]) \mapsto [\sigma([B])]\) in \(\Gamma_E(R)\).

Set

\[
\phi : \text{Aut}(\Gamma(R)) \mapsto \text{Aut}(\Gamma_E(R)), \sigma \mapsto [\sigma].
\]

Then we prove that \(\phi\) is a surjective homomorphism, and show that \(\text{Ker} \ \phi = K(\Gamma(R))\). Thus \(\text{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \text{Aut}(\Gamma_E(R))\).

(i) \(\phi\) is surjective.

For any \(\delta \in \text{Aut}(\Gamma_E(R))\), let \(\sigma\) be a bijection on \(V(\Gamma(R))\) such that \(\sigma[A] = \delta([A])\). Then \(\sigma\) is well defined since \(|\sigma(A)| = |\delta([A])| = q - 1\). Now we prove that \(\sigma \in \text{Aut}(\Gamma(R))\). (ia) If \(A \rightarrow B\) in \(\Gamma(R)\), then \(A, B \in Z(R)^*\) are distinct and satisfy \(AB = 0\). If \([A] = [B]\), then from \(AB = 0\) we get \(A^2 = 0\), and so \(\sigma[A] = \delta([A])\) is a square-zero class by Lemma 3.9. This gives \(\sigma(A)\sigma(B) \in (\sigma[A])^2 = 0\), and so \(\sigma(A) \rightarrow \sigma(B)\) in \(\Gamma(R)\). If \([A] \neq [B]\), then \(\delta([A]) \rightarrow \delta([B])\) in \(\Gamma_E(R)\) since \([A][B] = 0\). Thus \(\sigma(A)\sigma(B) \in \sigma[A]\sigma[B] = \delta([A])\delta([B]) = 0\), and so \(\sigma(A) \rightarrow \sigma(B)\) in \(\Gamma(R)\). (ib) Conversely, if \(\sigma(A) \rightarrow \sigma(B)\) in \(\Gamma(R)\), then \(A \neq B\) and

\[
(8) \quad \sigma(A)\sigma(B) = 0.
\]
Note that \( \sigma(A) \in \sigma[A] = \delta([A]) \), and so \( \sigma(A) \sigma[A] = \delta([A]) \). Similarly, we have \( [\sigma(B)] = \sigma[B] = \delta([B]) \). Thus, by Eq. (8) we get

\[
\delta([A])\delta([B]) = 0. \tag{9}
\]

If \( \delta([A]) \neq \delta([B]) \), then \( \delta([A]) \to \delta([B]) \) in \( \Gamma_E(R) \). It follows that \( [A] \to [B] \) in \( \Gamma_E(R) \). Hence \( AB = 0 \), and so \( A \to B \) in \( \Gamma(R) \). If \( \delta([A]) = \delta([B]) \), then \( [A] = [B] \). And by Eq. (9) we have \( (\delta([A]))^2 = 0 \). By Lemma 3.9 we get \( A^2 = 0 \). And note that \( [A] = [B] \), so we have \( AB = 0 \). Thus \( A \to B \) in \( \Gamma(R) \). Hence \( \sigma \in \text{Aut}(\Gamma(R)) \), and so \( \phi \) is a surjection.

(ii) \( \phi(\sigma_1 \sigma_2) = \phi(\sigma_1) \phi(\sigma_2) \).

\[ [\sigma_1 \sigma_2][A] = [\sigma_1][\sigma_2][A] = [\sigma_1][\sigma_2][A], \quad A \in Z(R)^* \quad \text{yields} \quad [\sigma_1 \sigma_2] = [\sigma_1][\sigma_2], \quad \text{and so} \quad \phi(\sigma_1 \sigma_2) = \phi(\sigma_1) \phi(\sigma_2). \]

(iii) \( \text{Ker} \phi = K(\Gamma(R)) \).

Write \( V(\Gamma(R)) = \bigcup_{i=1}^{q+1} [A_i] \). If \( [\sigma] \) is the identity automorphism of \( \Gamma_E(R) \), then \( \sigma[A_i] = [\sigma][A_i] = [A_i] \) for any \( A_i \in V(\Gamma(E(R))) \). Thus \( \sigma \in K(\Gamma(R)) \).

From (i)-(iii), we have \( \text{Aut}(\Gamma(R))/K(\Gamma(R)) \cong \text{Aut}(\Gamma(E(R))) \). \( \square \)

**Corollary 4.4.** \( |\text{Aut}(\Gamma(R))| = ((q - 1)!)(q+1)^q \cdot (q + 1)! \)

**Proof.** It immediately follows from \( |K(\Gamma(R))| = ((q - 1)!)(q+1)^q \cdot (q + 1)! \), Corollary 3.14 and Theorem 4.3. \( \square \)

**Remark 4.5.** [15, Theorem 3.8] (resp., [12, Theorem 3.9]) said that the automorphism group of \( \Gamma(R) \) (resp., \( \Gamma(M_2(\mathbb{Z}_q)) \)) is isomorphic to the symmetric group \( S_{q+1} \) where \( q \) is a prime and \( q > 1 \), which means \( |\text{Aut}(\Gamma(R))| = (q + 1)! \) (resp., \( |\text{Aut}(\Gamma(M_2(\mathbb{Z}_q)))| = (q + 1)! \)) is much greater than \( (q + 1)! \) in general. Hence [12, Theorem 3.9] and [15, Theorem 3.8] are both incorrect. In fact, the automorphism \( \sigma \) that constructed in [12, Theorem 3.9] (resp., [15, Theorem 3.8]) fails to be a bijection.

**Acknowledgements.** We are very grateful to the referee whose comments and suggestions greatly improved the presentation of this paper.

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