HOW DO ELEMENTS REALLY FACTOR IN \( \mathbb{Z}[\sqrt{-5}] \)?

SCOTT T. CHAPMAN, FELIX GOTTI, AND MARLY GOTTI

Abstract. Most undergraduate level abstract algebra texts use \( \mathbb{Z}[\sqrt{-5}] \) as an example of an integral domain which is not a unique factorization domain (or UFD) by exhibiting two distinct irreducible factorizations of a nonzero element. But such a brief example, which requires merely an understanding of basic norms, only scratches the surface of how elements actually factor in this ring of algebraic integers. We offer here an interactive framework which shows that while \( \mathbb{Z}[\sqrt{-5}] \) is not a UFD, it does satisfy a slightly weaker factorization condition, known as half-factoriality. The arguments involved revolve around the Fundamental Theorem of Ideal Theory.

1. Introduction

Consider the integral domain

\[
\mathbb{Z}[\sqrt{-5}] = \{ a + b\sqrt{-5} \mid a, b \in \mathbb{Z} \}.
\]

Where have you seen this before? Your undergraduate abstract algebra text probably used it as the base example of an integral domain that is not a unique factorization domain (or UFD). The Fundamental Theorem of Arithmetic fails in \( \mathbb{Z}[\sqrt{-5}] \) as this domain contains elements with multiple factorizations into irreducibles; for example,

\((1)\)

\[6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5})\]

even though 2, 3, 1 - \(\sqrt{-5}\), and 1 + \(\sqrt{-5}\) are pairwise non-associate irreducible elements in \( \mathbb{Z}[\sqrt{-5}] \). To argue this, the norm on \( \mathbb{Z}[\sqrt{-5}] \), i.e.,

\((2)\)

\[N(a + b\sqrt{-5}) = a^2 + 5b^2,\]

plays an important role, as it is a multiplicative function satisfying that

- \(\alpha\) is a unit if and only if \(N(\alpha) = 1\) (i.e., \(\pm 1\) are the only units of \( \mathbb{Z}[\sqrt{-5}] \)),
- if \(N(\alpha)\) is prime, then \(\alpha\) is irreducible.

However, introductory abstract algebra books seldom dig deeper than what Equation \((1)\) does. The goal of this paper is to use ideal theory to describe exactly how elements in \( \mathbb{Z}[\sqrt{-5}] \) factor into products of irreducibles. In doing so, we will show that \( \mathbb{Z}[\sqrt{-5}] \) satisfies a nice factorization property, which is known as half-factoriality. Thus, we say that \( \mathbb{Z}[\sqrt{-5}] \) is a half-factorial domain (or HFD). Our journey will require nothing more than elementary algebra, but will give the reader a glimpse of how The Fundamental Theorem of Ideal Theory resolves the

The first author gratefully acknowledges support under an Academic Leave funded by Sam Houston State University. He would also like to thank his past REU students, who made him think very hard about how to explain extremely difficult mathematics clearly and quickly.
non-unique factorizations of $\mathbb{Z}[\sqrt{-5}]$. The notion that unique factorization in algebraic number rings could be recovered via ideals was important in the late 1800’s in attempts to prove Fermat’s Last Theorem (see [7, Chapter 11]).

Our presentation is somewhat interactive, as many steps that follow from standard techniques of basic algebra are left to the reader as exercises. The only background we expect from the reader are introductory courses in linear algebra and abstract algebra. Assuming such prerequisites, we have tried to present here a self-contained and friendly approach to the phenomenon of the non-unique factorization of $\mathbb{Z}[\sqrt{-5}]$. More advanced and general arguments (which apply to any ring of algebraic integers) can be found in [7] and [6].

2. Integral Bases and Discriminants

We start with a brief look at the structure of $\mathbb{Z}[\sqrt{-5}]$ with linear algebra in mind. Consider

$$\mathbb{Q}(\sqrt{-5}) = \{r_1 + r_2\sqrt{-5} \mid r_1, r_2 \in \mathbb{Q}\}.$$ 

Note that $\mathbb{Q}(\sqrt{-5})$ is a field containing $\mathbb{Z}[\sqrt{-5}]$. Moreover, $\mathbb{Q}(\sqrt{-5})$ is a two-dimensional vector space over $\mathbb{Q}$. We make the following definition.

**Definition 2.1.** The elements $\alpha_1, \ldots, \alpha_k$ in $\mathbb{Z}[\sqrt{-5}]$ form an **integral basis** for $\mathbb{Z}[\sqrt{-5}]$ if for each $\beta \in \mathbb{Z}[\sqrt{-5}]$ there are unique $z_1, \ldots, z_k \in \mathbb{Z}$ satisfying that $\beta = z_1\alpha_1 + \cdots + z_k\alpha_k$.

It is fairly obvious that $\{1, \sqrt{-5}\}$ is an integral basis for $\mathbb{Z}[\sqrt{-5}]$, but it is also clear that integral bases are not unique. Using what we know about $\mathbb{Q}(\sqrt{-5})$ can shed some light on the structure of an integral basis.

**Notation:** If $S$ is a subset of the complex numbers, then we let $S^\ast$ denote $S \setminus \{0\}$.

**Lemma 2.2.** An integral basis for $\mathbb{Z}[\sqrt{-5}]$ is a basis for $\mathbb{Q}(\sqrt{-5})$ as a vector space over $\mathbb{Q}$. Hence, every integral basis for $\mathbb{Z}[\sqrt{-5}]$ contains exactly two nonzero elements.

**Proof.** Suppose that $\{\alpha_1, \ldots, \alpha_k\}$ is an integral basis for $\mathbb{Z}[\sqrt{-5}]$, and take rational coefficients $q_1, \ldots, q_k$ such that

$$q_1\alpha_1 + \cdots + q_k\alpha_k = 0.$$ 

Multiplying the above equality by the common denominator of the nonzero $q_i$’s and using the fact that $\{\alpha_1, \ldots, \alpha_k\}$ is an integral basis for $\mathbb{Z}[\sqrt{-5}]$, we obtain that $q_1 = \cdots = q_k = 0$. Hence, $\{\alpha_1, \ldots, \alpha_k\}$ is a linearly independent set inside the two-dimensional vector space $\mathbb{Q}(\sqrt{-5})$. In particular, $k \leq 2$. Moreover, for $r_1 + r_2\sqrt{-5} \in \mathbb{Q}(\sqrt{-5})$, one can take $n_1, n_2 \in \mathbb{Z}^\ast$ such that $n_1r_1, n_2r_2 \in \mathbb{Z}$. As $\{\alpha_1, \alpha_2\}$ is an integral basis for $\mathbb{Z}[\sqrt{-5}]$, there exist $z_1, z_1', z_2, z_2' \in \mathbb{Z}$ satisfying $n_1r_1 = z_1\alpha_1 + z_2\alpha_2$ and $n_2r_2\sqrt{-5} = z_1'\alpha_1 + z_2'\alpha_2$. As a result,

$$r_1 + r_2\sqrt{-5} = \frac{z_1\alpha_1 + z_2\alpha_2}{n_1} + \frac{z_1'\alpha_1 + z_2'\alpha_2}{n_2} = \left(\frac{z_1}{n_1} + \frac{z_1'}{n_2}\right)\alpha_1 + \left(\frac{z_2}{n_1} + \frac{z_2'}{n_2}\right)\alpha_2.$$ 

Therefore the set $\{\alpha_1, \alpha_2\}$ spans $\mathbb{Q}(\sqrt{-5})$ over $\mathbb{Q}$. This implies that the integral basis $\{\alpha_1, \alpha_2\}$ is, indeed, a basis for the vector space $\mathbb{Q}(\sqrt{-5})$, which completes the proof. The second statement of the lemma follows immediately. □
Distinct integral bases for $\mathbb{Z}[\sqrt{-5}]$ do have something in common, which we will demonstrate in our next theorem.

**Definition 2.3.** If $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]$, then the **discriminant** of $\alpha, \beta$, denoted $\Delta[\alpha, \beta]$, is defined by

$$\Delta[\alpha, \beta] = \left( \det \begin{bmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{bmatrix} \right)^2 = (\alpha \overline{\beta} - \overline{\alpha} \beta)^2,$$

where $\overline{x}$ denotes the conjugate of $x$ as a complex number.

**Example 2.4.** For the integral basis $\{1, \sqrt{-5}\}$ it follows that

$$\Delta[1, \sqrt{-5}] = (1(-\sqrt{-5}) - 1\sqrt{-5})^2 = (-2\sqrt{-5})^2 = -20.$$ 

Moreover, if $a+b\sqrt{-5}$ and $c+d\sqrt{-5}$ are any two elements of $\mathbb{Z}[\sqrt{-5}]$, then a simple computation yields

$$\Delta[a + b\sqrt{-5}, c + d\sqrt{-5}] = -20(bc - ad)^2.$$ 

Thus, the discriminant of any two elements in $\mathbb{Z}[\sqrt{-5}]$ is an element of $\mathbb{Z}$.

By the end of this section, we will see that if the pair $\{a + b\sqrt{-5}, c + d\sqrt{-5}\}$ is an integral basis for $\mathbb{Z}[\sqrt{-5}]$, then $bc - ad$ in Example 2.4 is always 1.

**Proposition 2.5.** If $\{\alpha_1, \alpha_2\}$ is a vector space basis for $\mathbb{Q}(\sqrt{-5})$ contained in $\mathbb{Z}[\sqrt{-5}]$, then $\Delta[\alpha_1, \alpha_2] \in \mathbb{Z}$.

**Proof.** We have already seen that $\Delta[\alpha_1, \alpha_2] \in \mathbb{Z}$. So suppose, by way of contradiction, that $\Delta[\alpha_1, \alpha_2] = 0$. Taking $\{\beta_1, \beta_2\}$ to be an integral basis for $\mathbb{Z}[\sqrt{-5}]$, one obtains

$$\alpha_1 = z_{1,1}\beta_1 + z_{1,2}\beta_2,$$

$$\alpha_2 = z_{2,1}\beta_1 + z_{2,2}\beta_2,$$

with the $z_{i,j}$ in $\mathbb{Z}$. Now

$$\Delta[\alpha_1, \alpha_2] = \left( \det \begin{bmatrix} \alpha_1 & \overline{\alpha_2} \\ \alpha_2 & \overline{\alpha_1} \end{bmatrix} \right)^2 = \left( \det \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix} \det \begin{bmatrix} \beta_1 & \overline{\beta}_1 \\ \beta_2 & \overline{\beta}_2 \end{bmatrix} \right)^2 \Delta[\beta_1, \beta_2].$$

If $\beta_1 = 1$ and $\beta_2 = \sqrt{-5}$, then

$$\det \begin{bmatrix} z_{1,1} & z_{2,1} \\ z_{1,2} & z_{2,2} \end{bmatrix} = \det \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix} = 0,$$

and so there are elements $q_1, q_2 \in \mathbb{Q}$ not both zero with

$$\begin{bmatrix} z_{1,1} & z_{2,1} \\ z_{1,2} & z_{2,2} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence,

$$0 = \beta_1(q_1 z_{1,1} + q_2 z_{2,1}) + \beta_2(q_1 z_{1,2} + q_2 z_{2,2}) = q_1(z_{1,1}\beta_1 + z_{1,2}\beta_2) + q_2(z_{2,1}\beta_1 + z_{2,2}\beta_2) = q_1\alpha_1 + q_2\alpha_2,$$

which is a contradiction because $\{\alpha_1, \alpha_2\}$ is linearly independent in $\mathbb{Q}(\sqrt{-5})$. Thus, $\Delta[\alpha_1, \alpha_2] \neq 0$. \qed
Using Lemma 2.2, we obtain the following important result.

**Corollary 2.6.** The discriminant of each integral basis for \( \mathbb{Z}[\sqrt{-5}] \) is in \( \mathbb{Z}^* \).

**Notation:** Let \( \mathbb{N} \) denote the set of positive integers, and set \( \mathbb{N}_0 = \{0\} \cup \mathbb{N} \).

**Theorem 2.7.** Any two integral bases for \( \mathbb{Z}[\sqrt{-5}] \) have the same discriminant.

**Proof.** Let \( \{\alpha_1, \alpha_2\} \) and \( \{\beta_1, \beta_2\} \) be integral bases, and let \( z_{i,j} \) be defined as in the proof of Proposition 2.5. As \( \Delta[\alpha_1, \alpha_2] \) and \( \Delta[\beta_1, \beta_2] \) are both integers, Equation (3) in the proof of Proposition 2.5 along with the fact that \( \text{det} \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix} \in \mathbb{N} \), implies that \( \Delta[\beta_1, \beta_2] \) divides \( \Delta[\alpha_1, \alpha_2] \) in \( \mathbb{Z} \). Using a similar argument, we can show that \( \Delta[\alpha_1, \alpha_2] \) divides \( \Delta[\beta_1, \beta_2] \). Since both discriminants have the same sign, \( \Delta[\alpha_1, \alpha_2] = \Delta[\beta_1, \beta_2] \). \( \square \)

Using Example 2.4, we obtain the following corollary.

**Corollary 2.8.** Every integral basis for \( \mathbb{Z}[\sqrt{-5}] \) has discriminant \(-20\).

3. **General Properties of Ideals**

Let \( R \) be a commutative ring with identity. We know from our first class in algebra that the units of \( R \) are precisely the invertible elements, while a nonunit \( x \in R \setminus \{0\} \) is irreducible if whenever \( x = uv \) in \( R \), then one has that either \( u \) or \( v \) is a unit.

To truly understand factorizations in \( \mathbb{Z}[\sqrt{-5}] \), we will need to know first how ideals of \( \mathbb{Z}[\sqrt{-5}] \) are generated. Recall that a subset \( I \) of a commutative ring \( R \) with identity is called an ideal of \( R \) provided that \( I \) is a subring with the property that \( rI \subseteq I \) for all \( r \in R \). It follows immediately that if \( x_1, \ldots, x_k \in R \), then the set

\[
I = \langle x_1, \ldots, x_k \rangle = \{r_1x_1 + \cdots + r_kx_k \mid \text{ each } r_i \in R \}
\]

is an ideal of \( R \), that is, the ideal generated by \( x_1, \ldots, x_k \). Recall that \( I \) is called principal if \( k = 1 \), and \( R \) is said to be a principal ideal domain (or PID) if each ideal of \( R \) is principal. The zero ideal \( \langle 0 \rangle \) and the entire ring \( R = \langle 1 \rangle \) are principal ideals. May it be that all the ideals of \( \mathbb{Z}[\sqrt{-5}] \) are principal?

**Example 3.1.** The ring of algebraic integers \( \mathbb{Z}[\sqrt{-5}] \) is not a PID. We argue that the ideal

\[
I = \langle 2, 1 + \sqrt{-5} \rangle
\]

is not principal. If \( I = \langle \alpha \rangle \), then \( \alpha \) divides both \( 2 \) and \( 1 + \sqrt{-5} \). The reader will verify in Exercise 3.2 below that both of these elements are irreducible and nonassociate. Hence, \( \alpha = \pm 1 \) and \( I = \langle \pm 1 \rangle = \mathbb{Z}[\sqrt{-5}] \). We show that \( 3 \notin I \). Suppose there exist \( a, b, c, d \in \mathbb{Z} \) so that

\[
(a + b\sqrt{-5})2 + (c + d\sqrt{-5})(1 + \sqrt{-5}) = 3.
\]

Expanding the previous equality, we obtain

\[
2a + c - 5d = 3
\]
\[
2b + c + d = 0.
\]

After subtracting, we are left with \( 2(a - b) - 6d = 3 \) which implies that \( 2 \mid 3 \) in \( \mathbb{Z} \), a contradiction.
Exercise 3.2. Show that the elements 2 and $1 + \sqrt{-5}$ are irreducible and non-associate in $\mathbb{Z}[\sqrt{-5}]$.

Let us recall that a proper ideal $I$ of a commutative ring $R$ with identity is said to be prime if whenever $xy \in I$ for $x, y \in R$, then either $x \in I$ or $y \in I$. In addition, we know that an element $p \in R \setminus \{0\}$ is said to be prime if the principal ideal $(p)$ is prime. It follows immediately that each prime in $\mathbb{Z}[\sqrt{-5}]$ is irreducible.

Exercise 3.3. Let $P$ be an ideal of a commutative ring $R$ with identity. Show that $P$ is prime if and only if the fact that $IJ \subseteq P$ for some ideals $I$ and $J$ of $R$ implies that either $I \subseteq P$ or $J \subseteq P$.

Example 3.4. We argue that the ideal $I = (2)$ is not prime in $\mathbb{Z}[\sqrt{-5}]$ and will in fact use Equation (1). Since $(1 - \sqrt{-5})(1 + \sqrt{-5}) = 2 \cdot 3$, it follows that $$(1 - \sqrt{-5})(1 + \sqrt{-5}) \in (2).$$

Now if $1 - \sqrt{-5} \in (2)$, then there is an $\alpha \in \mathbb{Z}[\sqrt{-5}]$ with $1 - \sqrt{-5} = 2\alpha$. But then $\alpha = \frac{1}{2} - \frac{\sqrt{-5}}{2} \notin \mathbb{Z}[\sqrt{-5}]$, a contradiction. A similar argument works with $1 + \sqrt{-5}$. Hence, $(2)$ is not a prime ideal.

We remind the reader that a proper ideal $I$ of a commutative ring $R$ with identity is called maximal if for each ideal $J$ the fact that $I \subseteq J \subseteq R$ implies that either $J = I$ or $J = R$. What we ask the reader to verify in the next exercise is a well-known result from basic abstract algebra.

Exercise 3.5. Let $I$ be an ideal of a commutative ring $R$ with identity, and let $R/I = \{r + I \mid r \in R\}$ be the quotient of $R$ by $I$.

(1) Show that $I$ is prime if and only if $R/I$ is an integral domain.
(2) Show that $I$ is maximal if and only if $R/I$ is a field.

Example 3.6. We expand our analysis of $I = (2, 1 + \sqrt{-5})$ in Example 3.1 by showing that $I$ is a prime ideal of $\mathbb{Z}[\sqrt{-5}]$. To do this, we first argue that an element $\alpha = z_1 + z_2\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ is contained in $I$ if and only if $z_1$ and $z_2$ have the same parity. If $\alpha \in I$, then there are integers $a, b, c,$ and $d$ so that $z_1 + z_2\sqrt{-5} = (a + b\sqrt{-5})2 + (c + d\sqrt{-5})(1 + \sqrt{-5})$.

Adjusting the equations from (1) yields

\[
\begin{align*}
2a + c - 5d &= z_1 \\
2b + c + d &= z_2.
\end{align*}
\]

Notice that if $c \equiv d \pmod{2}$, then both $z_1$ and $z_2$ are even, while $c \not\equiv d \pmod{2}$ implies that both $z_1$ and $z_2$ are odd. Hence, $z_1$ and $z_2$ must have the same parity. Conversely, suppose that $z_1$ and $z_2$ have the same parity. As, clearly, every element of the form $2k_1 + 2k_2\sqrt{-5} = 2(k_1 + k_2\sqrt{-5})$ is in $I$, let us assume that $z_1$ and $z_2$ are both odd. The equations in (1) form a linear system that obviously has solutions over $\mathbb{Q}$ for any choice of $z_1$ and $z_2$ in $\mathbb{Z}$. By solving this system, we find that $a$ and $b$ are dependent variables and

\[
a = \frac{z_1 - c + 5d}{2} \quad \text{and} \quad b = \frac{z_2 - c - d}{2}.
\]
Letting \( c \) be any even integer and \( d \) any odd integer now yields a solution with both \( a \) and \( b \) integers. Thus, \( z_1 + z_2\sqrt{-5} \in I \).

Now consider \( \mathbb{Z}[\sqrt{-5}]/I \). As \( I \) is not principal (Example 3.1), \( 1 \notin I \). Therefore \( 1 + I \neq 0 + I \). If \( c_1 + c_2\sqrt{-5} \notin I \), then \( c_1 \) and \( c_2 \) have opposite parity. If \( c_1 \) is odd and \( c_2 \) even, then \( ((c_1 - 1) + c_2\sqrt{-5}) + I = 0 + I \) implies that \( (c_1 + c_2\sqrt{-5}) + I = 1 + I \). If \( c_1 \) is even and \( c_2 \) odd, then \( ((c_1 - 1) + c_2\sqrt{-5}) + I = 0 + I \) again implies that \( (c_1 + c_2\sqrt{-5}) + I = 1 + I \). Hence, \( \mathbb{Z}[\sqrt{-5}]/I \cong \{0 + I, 1 + I\} \cong \mathbb{Z}_2 \). Since \( \mathbb{Z}[\sqrt{-5}]/I \) is an integral domain, \( I \) is a prime ideal (by Exercise 3.4).

**Exercise 3.7.** Show that \( \langle 3, 1 - 2\sqrt{-5} \rangle \) and \( \langle 3, 1 + 2\sqrt{-5} \rangle \) are prime ideals of \( \mathbb{Z}[\sqrt{-5}] \).

Let \( R \) be a commutative ring with identity. If every ideal of \( R \) is finitely generated, then \( R \) is called Noetherian ring. In addition, \( R \) is said to satisfy the Ascending Chain Condition (ACC) if every increasing sequence of ideals of \( R \) eventually stabilizes.

**Exercise 3.8.** Let \( R \) be a commutative ring with identity. Show that \( R \) is Noetherian if and only if it satisfies the ACC.

By Theorem 4.3 the ring \( \mathbb{Z}[\sqrt{-5}] \) is Noetherian and, therefore, satisfies the ACC.

## 4. Ideals in \( \mathbb{Z}[\sqrt{-5}] \)

In this section we explore the algebraic structure of all ideals of \( \mathbb{Z}[\sqrt{-5}] \) under ideal multiplication, encapsulating the basic properties of multiplication of ideals. Let us begin by generalizing the notion of an integral basis, which also plays an important role in ideal theory.

**Definition 4.1.** Let \( I \) be a proper ideal of \( \mathbb{Z}[\sqrt{-5}] \). A set of elements \( \{\alpha_1, \alpha_2\} \) of \( I \) is called an integral basis for \( I \) if every element of \( I \) can be written uniquely in the form \( z_1\alpha_1 + z_2\alpha_2 \) with \( z_1, z_2 \in \mathbb{Z} \).

Notice that if \( \{\alpha_1, \alpha_2\} \) is an integral basis for \( I \), then \( I = \langle \alpha_1, \alpha_2 \rangle \). Care is needed here as the converse is not true. For instance, if \( I = \langle 3 \rangle \), then \( \{3\} \) is not an integral basis for \( I \) (note that \( 3\sqrt{-5} \notin I \)).

**Exercise 4.2.** Argue that \( \{3, 3\sqrt{-5}\} \) is an integral basis for \( I = \langle 3 \rangle \).

We now show that every proper ideal of \( \mathbb{Z}[\sqrt{-5}] \) has an integral basis.

**Theorem 4.3.** Every proper ideal of \( \mathbb{Z}[\sqrt{-5}] \) has an integral basis consisting of two elements. Hence, every ideal of \( \mathbb{Z}[\sqrt{-5}] \) is finitely generated.

**Proof.** Suppose \( I \) is a proper ideal of \( \mathbb{Z}[\sqrt{-5}] \) with an integral basis \( S \). By the definition of an integral basis and an argument similar to that given in the proof of Lemma 2.2 the set \( S \) must be linearly independent in \( \mathbb{Q}(\sqrt{-5}) \) as a vector space over \( \mathbb{Q} \). Hence, \( |S| \leq 2 \). To argue that \( I \) cannot have an integral basis consisting of only one element, assume, by contradiction, that \( \{\gamma\} \subset \mathbb{Z}[\sqrt{-5}]^\bullet \) is an integral basis for \( I \). Now, take any integral basis \( \{\beta_1, \beta_2\} \) for \( \mathbb{Z}[\sqrt{-5}] \) and any \( \alpha \in I^\bullet \). As \( \alpha\beta_1, \alpha\beta_2 \in I \), they are both integral multiples of \( \gamma \) and, therefore, \( \{\alpha\beta_1, \alpha\beta_2\} \)
is a linearly dependent set in $\mathbb{Q}(\sqrt{-5})$. This implies that $\{\beta_1, \beta_2\}$ is also linearly dependent in $\mathbb{Z}[(\sqrt{-5})]$. But in this case, Lemma 2.2 would yield that $\{\beta_1, \beta_2\}$ fails to be an integral basis of $\mathbb{Z}[(\sqrt{-5})]$, a contradiction.

We must now show that each proper ideal of $\mathbb{Z}[\sqrt{-5}]$ has an integral basis. Consider all subsets $\{\delta_1, \delta_2\}$ of $I$ which form a vector space basis for $\mathbb{Q}(\sqrt{-5})$ (note that the linearly independent set $\{\alpha \beta_1, \alpha \beta_2\}$ in the previous paragraph one is one of such sets). Proposition 2.5 ensures that $\Delta[\delta_1, \delta_2] \in \mathbb{Z}^*$. Assume that $\{\delta_1, \delta_2\}$ has the minimum possible $|\Delta[\delta_1, \delta_2]|$. We show that $\{\delta_1, \delta_2\}$ is an integral basis for $I$.

Assume, by way of contradiction, that $\{\delta_1, \delta_2\}$ is not an integral basis for $I$. Then there is an element $y \in I$ with $y = q_1 \delta_1 + q_2 \delta_2$ with not both $q_1$ and $q_2$ in $\mathbb{Z}$. Without loss of generality, we can assume that $q_1 \in \mathbb{Q} \setminus \mathbb{Z}$. Thus, $\{\delta_1, \delta_2\}$ is also a vector space basis for $\mathbb{Q}(\sqrt{-5})$. Write $q_1 = z + r$, where $z \in \mathbb{Z}$ and $0 < r < 1$. Let

$$
\delta_1^* = y - z \delta_1 = (q_1 - z) \delta_1 + q_2 \delta_2
$$

$$
\delta_2^* = \delta_2.
$$

It is easy to verify that $\{\delta_1^*, \delta_2^*\}$ is linearly independent and thus is another vector space basis of $\mathbb{Q}(\sqrt{-5})$ which consists of elements of $I$. Now

$$
\Delta[\delta_1^*, \delta_2^*] = r^2 \Delta[\delta_1, \delta_2]
$$

since

$$
\left(\det \begin{bmatrix} q_1 - z & q_2 \\ 0 & 1 \end{bmatrix}\right)^2 = r^2.
$$

As $0 < r < 1$, we have $|\Delta[\delta_1^*, \delta_2^*]| < |\Delta[\delta_1, \delta_2]|$, contradicting the minimality of $|\Delta[\delta_1, \delta_2]|$. Hence, $\{\delta_1, \delta_2\}$ is an integral basis for $I$.

Theorem 4.3 yields this important corollary.

**Corollary 4.4.** If $I$ is a proper ideal of $\mathbb{Z}[\sqrt{-5}]$, then there exist elements $\alpha, \beta \in I$ such that $I = (\alpha, \beta)$.

**Remark 4.5.** One can actually say much more. The following stronger statement is true: if $I$ is a proper ideal of $\mathbb{Z}[\sqrt{-5}]$ and $\alpha \in I^*$, then there exists $\beta \in I$ such that $I = (\alpha, \beta)$. This condition is known as the $1\frac{1}{2}$-generator property. The interested reader can find a proof of this in [7, Theorem 9.3].

A set $M$ on which a binary operation $*$ has been defined is called a monoid if $*$ is associative and there exists $e \in M$ satisfying that $e * x = x * e = x$ for all $x \in M$. The element $e$ is called the identity element. The monoid $M$ is called commutative if $*$ is commutative.

Let $R$ be a commutative ring with identity. Recall that we have a natural multiplication on the set consisting of all the ideals of $R$, that is, for any two ideals $I$ and $J$ of $R$, the product

$$
IJ = \left\{ \sum_{i=1}^{k} a_i b_j \mid \text{each } a_i \in I \text{ and each } b_j \in J \right\}
$$

is again an ideal. It is not hard to check that ideal multiplication is associative and commutative and also satisfies that $RI = I$ for each ideal $I$ of $R$. This amounts to arguing the following exercise.

**Exercise 4.6.** Let $R$ be a commutative ring with identity. Show that the set of all ideals of $R$ is a commutative monoid under ideal multiplication.
Example 4.7. To give the reader a notion of how ideal multiplication works, we show that

\[(2, 1 + \sqrt{-5})^2 = (2).\]

It follows by (6) that ideal multiplication can be achieved by merely multiplying generators. For instance,

\[(2, 1 + \sqrt{-5})^2 = (2, 1 + \sqrt{-5})(2, 1 + \sqrt{-5})
= \langle 4, 2(1 + \sqrt{-5}), 2(1 + \sqrt{-5}), -2(2 - \sqrt{-5}) \rangle\]

Since 2 divides each of the generators of \((2, 1 + \sqrt{-5})^2\) in \(\mathbb{Z}[\sqrt{-5}]\), we clearly have \((2, 1 + \sqrt{-5})^2 \subseteq (2).\) To verify the reverse inclusion, let us first observe that \(2\sqrt{-5} = 4 - 2(2 - \sqrt{-5}) \in (2, 1 + \sqrt{-5})^2.\) As \(2\sqrt{-5} \in (2, 1 + \sqrt{-5})^2,\) one immediately sees that \(2 = 2(1 + \sqrt{-5}) - 2\sqrt{-5} \in (2, 1 + \sqrt{-5})^2.\) Hence, \((2) \subseteq (2, 1 + \sqrt{-5})^2,\) and equality follows.

Exercise 4.8. Verify that the following equalities hold:

\[\langle 3 \rangle = (3, 1 - 2\sqrt{-5})(3, 1 + 2\sqrt{-5}),\]
\[\langle 1 - \sqrt{-5} \rangle = (2, 1 + \sqrt{-5})(3, 1 + 2\sqrt{-5}),\]
\[\langle 1 + \sqrt{-5} \rangle = (2, 1 + \sqrt{-5})(3, 1 - 2\sqrt{-5}).\]

The example above is no accident. Every nonprincipal ideal of \(\mathbb{Z}[\sqrt{-5}]\) can be sent via multiplication to a principal ideal.

Theorem 4.9. Let \(I\) be an ideal in \(\mathbb{Z}[\sqrt{-5}]\). Then there exists a nonzero ideal \(J\) of \(\mathbb{Z}[\sqrt{-5}]\) such that \(IJ\) is principal.

Proof. If \(I\) is a principal ideal, then the result follows by letting \(J = (1).\) So suppose \(I = \langle \alpha, \beta \rangle\) is not a principal ideal of \(\mathbb{Z}[\sqrt{-5}],\) where \(\alpha = a + b\sqrt{-5}\) and \(\beta = c + d\sqrt{-5}.\) Notice that it is enough to verify the existence of such an ideal \(J\) when \(\gcd(a, b, c, d) = 1,\) and we make this assumption. It is easy to check that \(a\beta + \alpha\beta = 2ac + 10bd \in \mathbb{Z}.\) Hence, \(a\alpha, \alpha\beta + \alpha\beta,\) and \(\beta\beta\) are all integer numbers. Let

\[f = \gcd(a\alpha, \alpha\beta + \alpha\beta, \beta\beta)\]
\[= \gcd(a^2 + 5b^2, 2ac + 10bd, c^2 + 5d^2).\]

Take \(J = \langle \alpha, \beta \rangle.\) We claim that \(IJ = (f).\) Since \(f = \gcd(a\alpha, \alpha\beta + \alpha\beta, \beta\beta),\) there are integer numbers \(z_1, z_2,\) and \(z_3\) so that

\[f = z_1a\alpha + z_2\beta\beta + z_3(\alpha\beta + \alpha\beta).\]

Because \(IJ = \langle a\alpha, \alpha\beta, \beta\beta, \beta\beta \rangle,\) we have that \(f\) is a linear combination of the generating elements. Thus, \(f \in IJ\) and, therefore, \((f) \subseteq IJ.\)

To prove the reverse containment, we first show that \(f\) divides \(bc - ad.\) Suppose, by way of contradiction, that this is not the case. Notice that \(25 \nmid f;\) otherwise \(25 \mid a^2 + 5b^2\) and \(25 \mid c^2 + 5d^2\) would imply that \(5 \mid \gcd(a, b, c, d).\) On the other hand, \(4 \mid f\) would imply \(4 \mid a^2 + 5b^2\) and \(4 \mid c^2 + 5d^2,\) forcing \(a, b, c,\) and \(d\) to be even, which is not possible as \(\gcd(a, b, c, d) = 1.\) Hence, \(4 \nmid f\) and \(25 \nmid f.\) Because

\[2c(a^2 + 5b^2) - a(2ac + 10bd) = 10b(bc - ad)\]
\[2a(c^2 + 5d^2) - c(2ac + 10bd) = 10d(ad - bc),\]
f must divide both $10b(bc - ad)$ and $10d(bc - ad)$. As, by assumption, $f \nmid bc - ad$, there must be a prime $p$ and a natural $n$ such that $p^n \mid f$ but $p^n \nmid bc - ad$. If $p = 2$, then $4 \nmid f$ forces $n = 1$. In this case, both $a^2 + 5b^2$ and $c^2 + 5d^2$ would be even, and so $2 \mid a - b$ and $2 \mid c - d$, which implies that $2 \mid bc - ad$, a contradiction. Thus, $p \neq 2$. On the other hand, if $p = 5$, then again $n = 1$. In this case, $5 \mid a^2 + 5b^2$ and $5 \mid c^2 + 5d^2$ and so $5$ would divide both $a$ and $c$, contradicting that $5 \nmid bc - ad$.

Then, we can assume that $p \notin \{2, 5\}$. As $p^n \mid 10b(bc - ad)$ but $p^n \nmid bc - ad$, we have that $p \mid 10b$. Similarly, $p \mid 10d$. Since $p \notin \{2, 5\}$, it follows that $p \mid b$ and $p \mid d$. Now the fact that $p$ divides both $a^2 + 5b^2$ and $c^2 + 5d^2$ yields that $p \mid a$ and $p \mid c$, contradicting that $\gcd(a, b, c, d) = 1$. Hence, $f \mid bc - ad$.

Let us verify now that $f \mid ac + 5bd$. If $f$ is odd, then $f \mid ac + 5bd$. Assume, therefore, that $f = 2f_1$, where $f_1 \in \mathbb{Z}$. As $4 \nmid f$, the integer $f_1$ is odd. Now, $f \mid a^2 + 5b^2$ implies that $a$ and $b$ have the same parity. Similarly, one sees that $c$ and $d$ have the same parity. As a consequence, $ac + 5bd$ is even. Since $f_1$ is odd, it must divide $(ac + 5bd)/2$, which means that $f$ divides $ac + 5bd$, as desired.

Because $f$ divides both $\overline{\alpha \beta}$ and $\overline{\beta \beta}$ in $\mathbb{Z}$, proving that $IJ \subseteq \langle f \rangle$ amounts to verifying that $f$ divides both $\overline{\alpha \beta}$ and $\overline{\beta \beta}$ in $\mathbb{Z}^{\{\sqrt{5}\}}$. Since $f$ divides both $ac + 5bd$ and $bc - ad$ in $\mathbb{Z}$, one has that

$$x = \frac{ac + 5bd}{f} \in \mathbb{Z} \quad \text{and} \quad y = \frac{bc - ad}{f} \in \mathbb{Z}.$$  

Therefore

$$\overline{\alpha \beta} = ac + 5bd + \langle bc - ad \rangle \sqrt{5} = (x + y\sqrt{5})f \in \langle f \rangle.$$  

Also, $\overline{\beta \beta} = (x - y\sqrt{5})f \in \langle f \rangle$. Hence, the reverse inclusion $IJ \subseteq \langle f \rangle$ also holds, which completes the proof. \hfill \Box

A commutative monoid $(M, \ast)$ is said to be cancellative if for all $a, b, c \in M$, the equality $a \ast b = a \ast c$ implies that $b = c$. By Exercise 4.6, the set

$$\mathcal{I} := \{I \mid I \text{ is an ideal of } \mathbb{Z}^{\{\sqrt{5}\}}\}$$

is a commutative monoid. As the next corollary states, the set $\mathcal{I}^\ast := \mathcal{I} \setminus \{(0)\}$ is indeed a commutative cancellative monoid.

**Corollary 4.10.** The set $\mathcal{I}^\ast$ under ideal multiplication is a commutative cancellative monoid.

**Proof.** Because $\mathcal{I}$ is a commutative monoid under ideal multiplication, it immediately follows that $\mathcal{I}^\ast$ is also a commutative monoid. To prove that $\mathcal{I}^\ast$ is cancellative, take $I, J, K \in \mathcal{I}^\ast$ such that $IJ = IK$. By Theorem 4.9, there exists an ideal $I'$ of $\mathbb{Z}^{\{\sqrt{5}\}}$ and $x \in \mathbb{Z}^{\{\sqrt{5}\}}$ with $I'I = \langle x \rangle$. Then

$$\langle x \rangle J = I'J = I'K = \langle x \rangle K.$$  

As $x \neq 0$ and the product in $\mathbb{Z}^{\{\sqrt{5}\}}$ is cancellative, $J = K$. \hfill \Box
5. The Fundamental Theorem of Ideal Theory

We devote this section to prove a version of the Fundamental Theorem of Ideal Theory for the ring of integers $\mathbb{Z}[\sqrt{-5}]$. To do this, we need to develop a few tools. In particular, we introduce the concept of a fractional ideal of $\mathbb{Z}[\sqrt{-5}]$ and show that the set of such fractional ideals is an abelian group.

Let us begin by exploring the relation between the concepts of prime and maximal ideals. We recall that every proper ideal of a commutative ring $R$ with identity is contained in a maximal ideal, which implies, in particular, that maximal ideals always exist.

**Exercise 5.1.** Show that every maximal ideal of a commutative ring with identity is prime.

Prime ideals, however, are not necessarily maximal. The following example sheds some light upon this observation.

**Example 5.2.** Let $\mathbb{Z}[X]$ denote the ring of polynomials with integer coefficients. Clearly, $\mathbb{Z}[X]$ is an integral domain. It is not hard to verify that the ideal $(X)$ of $\mathbb{Z}[X]$ is prime. Because $2 \notin (X)$, one obtains that $(X) \subsetneq (2, X)$. It is left to the reader to argue that $(2, X)$ is a proper ideal of $\mathbb{Z}[X]$. Since $(X) \subseteq (2, X) \subsetneq \mathbb{Z}[X]$, it follows that $(X)$ is not a maximal ideal of $\mathbb{Z}[X]$.

In the ring of integers $\mathbb{Z}[\sqrt{-5}]$ the notions of prime and maximal ideals are somehow equivalent.

**Proposition 5.3.** Every nonzero prime ideal of $\mathbb{Z}[\sqrt{-5}]$ is maximal.

**Proof.** Let $P$ be a nonzero prime ideal of $\mathbb{Z}[\sqrt{-5}]$, and let $\{\alpha_1, \alpha_2\}$ be an integral basis for $\mathbb{Z}[\sqrt{-5}]$. Fix $\beta \in P^\circ$. Note that $n := N(\beta) = \beta\overline{\beta} \in P \cap \mathbb{N}$. Consider the finite subset

$$S = \{r_1\alpha_1 + r_2\alpha_2 + P \mid 0 \leq r_1, r_2 < n\}$$

of $\mathbb{Z}[\sqrt{-5}]/P$. Take $x \in \mathbb{Z}[\sqrt{-5}]$. As $\{\alpha_1, \alpha_2\}$ is an integral basis, there exist $n_1, n_2 \in \mathbb{Z}$ such that $x = n_1\alpha_1 + n_2\alpha_2$ and, therefore, $x + P = r_1\alpha_1 + r_2\alpha_2 + P \in S$, where $r_i \in \{0, \ldots, n - 1\}$ and $r_i \equiv n_i \pmod{n}$. Hence, $\mathbb{Z}[\sqrt{-5}]/P = S$, which implies that $\mathbb{Z}[\sqrt{-5}]/P$ is finite. It follows by Exercise 3.5(1) that $\mathbb{Z}[\sqrt{-5}]/P$ is an integral domain. As a result, $\mathbb{Z}[\sqrt{-5}]/P$ is a field (see Exercise 5.3 below). Thus, Exercise 5.3(2) guarantees that $P$ is a maximal ideal. □

**Exercise 5.4.** Let $R$ be a finite integral domain. Show that $R$ is a field.

Although the concepts of (nonzero) prime and maximal ideals coincide in $\mathbb{Z}[\sqrt{-5}]$, we will keep on using both terms depending on the ideal property we are willing to apply.

**Lemma 5.5.** If $I \in \mathcal{I}^\circ$, then there exist nonzero prime ideals $P_1, \ldots, P_n$ which satisfy that $P_1 \cdots P_n \subseteq I$.

**Proof.** Assume, by way of contradiction, that the statement of the lemma does not hold. Because $\mathbb{Z}[\sqrt{-5}]$ satisfies the ACC, there exists $I \in \mathcal{I}$ that is maximal among all the ideals failing to satisfy the statement. Clearly, $I$ cannot be prime. By Exercise 3.3 there exist $J, K \in \mathcal{I}$ such that $JK \subseteq I$ but neither $J \subseteq I$ nor $K \subseteq I$. 

10 SCOTT T. CHAPMAN, FELIX GOTTI, AND MARLY GOTTI
Now notice that the ideals $J' = I + J$ and $K' = I + K$ both strictly contain $I$ and. The maximality of $I$ implies that both $J'$ and $K'$ contain products of nonzero prime ideals. Now the fact that $J'K' \subseteq I$ would also imply that $I$ contains a product of nonzero prime ideals, a contradiction. □

**Definition 5.6.** For $I \in \mathcal{I}$ and $\alpha \in \mathbb{Z}[\sqrt{-5}]^*$, the subset $\alpha^{-1}I$ of $\mathbb{Q}(\sqrt{-5})$ is called a fractional ideal of $\mathbb{Z}[\sqrt{-5}]$. Let $\mathcal{F}$ denote the set of all fractional ideals of $\mathbb{Z}[\sqrt{-5}]$.

It is clear that every ideal is a fractional ideal. However, fractional ideals are not necessarily ideals. The product of fractional ideal is defined similarly to the product of standard ideals. Therefore it is easily seen that the product of two fractional ideals is again a fractional ideal. Indeed, for $\alpha, \beta \in \mathbb{Z}[\sqrt{-5}]^*$ and $I, J \in \mathcal{I}$, it follows that $(\alpha^{-1}I)(\beta^{-1}J) = (\alpha\beta)^{-1}IJ$. Let $\mathcal{F}^*$ denote $\mathcal{F} \setminus \{0\}$.

**Definition 5.7.** For an ideal $I$ of $\mathbb{Z}[\sqrt{-5}]$, the set

$$ I^{-1} := \{ \alpha \in \mathbb{Q}(\sqrt{-5}) \mid \alpha I \subseteq \mathbb{Z}[\sqrt{-5}] \} $$

is called the inverse of $I$.

**Exercise 5.8.** For each $I \in \mathcal{I}$, show that the set $I^{-1}$ is a fractional ideal.

**Lemma 5.9.** If $I$ is a proper ideal of $\mathbb{Z}[\sqrt{-5}]$, then $\mathbb{Z}[\sqrt{-5}]$ is strictly contained in the fractional ideal $I^{-1}$.

**Proof.** Let $I$ be a proper ideal of $\mathbb{Z}[\sqrt{-5}]$, and let $M$ be a maximal ideal containing $I$. Take $\alpha \in M^*$. By the definition of the inverse of an ideal, $\mathbb{Z}[\sqrt{-5}] \subseteq M^{-1}$. Lemma 5.5 ensures the existence of $m \in \mathbb{N}$ and prime ideals $P_1, \ldots, P_m$ of $\mathbb{Z}[\sqrt{-5}]$ such that $P_1 \cdots P_m \subseteq (\alpha) \subseteq M$. Assume that $m$ is the minimum natural number satisfying this property. Since $M$ is a prime ideal (Exercise 5.1), by Exercise 5.3 there exists $P \in \{P_1, \ldots, P_m\}$ such that $P \subseteq M$. By Proposition 5.3, the ideal $P$ is maximal, which implies that $M = P$. By the minimality of $m$, there exists $\alpha' \in P_2 \cdots P_m \setminus \langle \alpha \rangle$. This implies that $\alpha^{-1}\alpha' \notin \mathbb{Z}[\sqrt{-5}]$ and $\alpha'M \subseteq \langle \alpha \rangle$, that is $\alpha^{-1}\alpha'M \subseteq \langle 1 \rangle = \mathbb{Z}[\sqrt{-5}]$. As a result, $\alpha^{-1}\alpha' \in M^{-1} \setminus \mathbb{Z}[\sqrt{-5}]$. Hence, we find that $\mathbb{Z}[\sqrt{-5}] \subseteq M^{-1} \setminus I^{-1}$, and the proof follows. □

**Lemma 5.10.** If $I \in \mathcal{I}^*$ and $\alpha \in \mathbb{Q}(\sqrt{-5})$, then $\alpha I \subseteq I$ implies $\alpha \in \mathbb{Z}[\sqrt{-5}]$.

**Proof.** Let $I$ and $\alpha$ be as in the statement of the lemma. By Theorem 4.9 there exists a nonzero ideal $J$ of $\mathbb{Z}[\sqrt{-5}]$ such that $IJ = \langle \beta \rangle$ for some $\beta \in \mathbb{Z}[\sqrt{-5}]$. Then $\alpha(\beta) = \alpha I J \subseteq I J = \langle \beta \rangle$, which means that $\alpha\beta = \sigma\beta$ for some $\sigma \in \mathbb{Z}[\sqrt{-5}]$. As $\beta \neq 0$, it follows that $\alpha = \sigma \in \mathbb{Z}[\sqrt{-5}]$. □

**Theorem 5.11.** The set $\mathcal{F}^*$ is an abelian group under multiplication of fractional ideals.
Corollary 5.12. Proper ideal of \( \mathbb{Z} \).

Proof. Clearly, multiplication of fractional ideals is associative. In addition, it immediately follows that the fractional ideal \( \mathbb{Z}[\sqrt{-5}] = 1^{-1}(1) \) is the identity. The most involved part of the proof consists in arguing that each fractional ideal is invertible.

Let \( M \in \mathfrak{I}^* \) be a maximal ideal of \( \mathbb{Z}[\sqrt{-5}] \). By definition of \( M^{-1} \), we have that \( MM^{-1} \subseteq \mathbb{Z}[\sqrt{-5}] \), which implies that \( MM^{-1} \in \mathfrak{I}^* \). As \( M = M\mathbb{Z}[\sqrt{-5}] \subseteq MM^{-1} \) and \( M \) is maximal, \( MM^{-1} = M \) or \( MM^{-1} = \mathbb{Z}[\sqrt{-5}] \). As \( M \) is proper, Lemma 5.39 ensures that \( M^{-1} \) strictly contains \( \mathbb{Z}[\sqrt{-5}] \), which implies, by Lemma 5.10 that \( MM^{-1} \neq \mathbb{Z}[\sqrt{-5}] \). As a result, each maximal ideal of \( \mathbb{Z}[\sqrt{-5}] \) is invertible.

Now suppose, by way of contradiction, that not every ideal in \( \mathfrak{I}^* \) is invertible. Among all the nonzero non-invertible ideals take one, say \( J \), maximal under inclusion (this is possible because \( \mathbb{Z}[\sqrt{-5}] \) satisfies the ACC). Because \( \mathbb{Z}[\sqrt{-5}] \) is an invertible fractional ideal, \( J \subseteq \mathbb{Z}[\sqrt{-5}] \). Let \( M \) be a maximal ideal containing \( J \).

By Lemma 5.39 one has that \( \mathbb{Z}[\sqrt{-5}] \subseteq M^{-1} \subseteq J^{-1} \). This, along with Lemma 5.10 yields \( J \subseteq JM^{-1} \subseteq JJ^{-1} \subseteq \mathbb{Z}[\sqrt{-5}] \). Thus, \( JM^{-1} \) is an ideal of \( \mathbb{Z}[\sqrt{-5}] \) strictly containing \( J \). The maximality of \( J \) now implies that \( JM^{-1}(JM^{-1})^{-1} = \mathbb{Z}[\sqrt{-5}] \) and, therefore, \( M^{-1}(JM^{-1})^{-1} \subseteq J^{-1} \). Then

\[
\mathbb{Z}[\sqrt{-5}] = JM^{-1}(JM^{-1})^{-1} \subseteq JJ^{-1} \subseteq \mathbb{Z}[\sqrt{-5}],
\]

which forces \( JJ^{-1} = \mathbb{Z}[\sqrt{-5}] \), a contradiction.

Finally, take \( F \in \mathfrak{F}^* \). Then there exists an ideal \( I \in \mathfrak{I}^* \) and \( \alpha \in \mathbb{Z}[\sqrt{-5}]^* \) such that \( F = \alpha^{-1}I \). Then one obtains

\[
(\alpha I^{-1})F = (\alpha I^{-1})(\alpha^{-1}I) = I^{-1}I = \mathbb{Z}[\sqrt{-5}].
\]

As a consequence, the fractional ideal \( \alpha I^{-1} \) is the inverse of \( F \) in \( \mathfrak{F}^* \). Because each nonzero fractional ideal of \( \mathbb{Z}[\sqrt{-5}] \) is invertible, \( \mathfrak{F}^* \) is a group. Since the multiplication of fractional ideals is commutative, \( \mathfrak{F}^* \) is abelian. \( \square \)

Corollary 5.12. If \( I \in \mathfrak{I}^* \) and \( \alpha \in \mathfrak{I}^* \), then \( II = (\alpha) \) for some \( J \in \mathfrak{I}^* \).

Proof. Let \( I \) and \( \alpha \) be as in the statement of the corollary. As \( \alpha^{-1}I \) is a nonzero fractional ideal, there exists a nonzero fractional ideal \( J \) such that \( \alpha^{-1}I \subseteq J \subseteq \mathbb{Z}[\sqrt{-5}] \), that is \( IJ = (\alpha) \). Since \( \beta J \subseteq JJ = (\alpha) \subseteq I \) for all \( \beta \in J \), Lemma 5.10 guarantees that \( J \subseteq \mathbb{Z}[\sqrt{-5}] \). Hence, \( J \) is a nonzero ideal of \( \mathbb{Z}[\sqrt{-5}] \). \( \square \)

Theorem 5.13. [The Fundamental Theorem of Ideal Theory] Let \( I \) be a nonzero proper ideal of \( \mathbb{Z}[\sqrt{-5}] \). There exist a unique (up to order) list of prime ideals \( P_1, \ldots, P_k \) of \( \mathbb{Z}[\sqrt{-5}] \) such that \( I = P_1 \cdots P_k \).

Proof. Suppose, by way of contradiction, that not every ideal in \( \mathfrak{I}^* \) can be written as the product of prime ideals. From the set of ideals of \( \mathbb{Z}[\sqrt{-5}] \) which are not the product of primes ideals, take one, say \( I \), maximal under inclusion. Clearly, \( I \) is not prime. Therefore \( I \) is contained in a maximal ideal \( P_1 \), and such containment must be strict by Exercise 5.21. By Lemma 5.39 one has that \( \mathbb{Z}[\sqrt{-5}] \not\subseteq P_1^{-1} \) and so \( I \subseteq IP_1^{-1} \). Now Lemma 5.10 ensures that the latter inclusion is strict. The maximality of \( I \) now implies that \( IP_1^{-1} = P_2 \cdots P_k \) for some prime ideals \( P_2, \ldots, P_k \). This, along with Theorem 5.11 ensures that \( I = P_1 \cdots P_k \), a contradiction.
To argue uniqueness, let us assume, by contradiction, that there exists an ideal having two distinct prime factorizations. Let \( m \) be the minimum natural number such that there exists \( I \in I \) with two distinct factorizations into prime ideals, one of them containing \( m \) factors. Suppose that
\[
I = P_1 \cdots P_m = Q_1 \cdots Q_n.
\]
Because \( Q_1, \ldots, Q_n \subseteq P_m \), there exists \( Q \in \{Q_1, \ldots, Q_n\} \) such that \( Q \subseteq P_m \) (Exercise 3.3). By Proposition 5.3, both \( P_m \) and \( Q \) are maximal ideals, which implies that \( P_m = Q \). As \( IQ^{-1} \subseteq I^{-1} \subseteq \mathbb{Z}[-5] \), it follows that \( IQ^{-1} \in I \). Multiplying the equality \((7)\) by the fractional ideal \( Q^{-1} \), we obtain that \( IQ^{-1} \) is an ideal of \( \mathbb{Z}[-5] \) with two distinct factorizations into prime ideals such that one of them, namely \( P_1 \cdots P_{m-1} \), contains less than \( m \) factors. As this contradicts the minimality of \( m \), uniqueness follows. \( \square \)

An element \( a \) of a commutative monoid \( M \) is said to be an atom if for all \( x, y \in M \) such that \( a = xy \), either \( x \) is a unit or \( y \) is a unit (i.e., has an inverse). A commutative cancellative monoid is called atomic if every nonzero nonunit element can be factored into atoms.

**Corollary 5.14.** The monoid \( I^* \) is atomic.

### 6. The Class Group

To understand the phenomenon of non-unique factorization in \( \mathbb{Z}[-5] \), we first need to understand certain classes of ideals of \( \mathbb{Z}[-5] \). Let
\[
\mathcal{P} := \{ I \in I \mid I \text{ is a principal ideal of } \mathbb{Z}[-5] \}.
\]
Two ideals \( I, J \in I \) are equivalent if \( \langle \alpha \rangle \cdot I = \langle \beta \rangle \cdot J \) for some \( \alpha, \beta \in \mathbb{Z}[-5]^* \). In this case, we write \( I \sim J \). It is clear that \( \sim \) defines an equivalence relation on \( \mathbb{Z}[-5] \). The equivalence classes of \( \sim \) are called ideal classes. Let \( IP \) denote the ideal class of \( I \), and we also let \( \mathcal{C}(\mathbb{Z}[-5]) \) denote the set of all nonzero ideal classes. Now define a binary operation \( * \) on \( \mathcal{C}(\mathbb{Z}[-5]) \) by
\[
IP * JP = (IJ)P.
\]
It turns out that \( \mathcal{C}(\mathbb{Z}[-5]) \) is, indeed, a group under the \( * \) operation.

**Theorem 6.1.** The set of ideal classes \( \mathcal{C}(\mathbb{Z}[-5]) \) is an abelian group under \( * \).

**Proof.** Because the product of ideals is associative and commutative, so is \( * \). Also, it follows immediately that \( \langle 1 \rangle P * IP = \langle (1)I \rangle P = IP \) for each \( I \in I^* \), which means that \( \mathcal{P} = \langle 1 \rangle \mathcal{P} \) is the identity element of \( \mathcal{C}(\mathbb{Z}[-5]) \). In addition, as any two nonzero principal ideals are in the same ideal class, Theorem 4.9 ensures that, for any \( IP \in \mathcal{C}(\mathbb{Z}[-5]) \), there exists \( J \in I^* \) such that \( IP * JP = IJ \in \mathcal{P} = \langle 1 \rangle \mathcal{P} \). So \( JIP \) is the inverse of \( IP \) in \( \mathcal{C}(\mathbb{Z}[-5]) \). Hence, \( \mathcal{C}(\mathbb{Z}[-5]) \) is an abelian group. \( \square \)

**Definition 6.2.** The group \( \mathcal{C}(\mathbb{Z}[-5]) \) is called the class group of \( \mathbb{Z}[-5] \), and the order of \( \mathcal{C}(\mathbb{Z}[-5]) \) is called the class number of \( \mathbb{Z}[-5] \).

Recall that if \( \theta : R \to S \) is a ring homomorphism, then \( \ker \theta = \{ r \in R \mid \theta(r) = 0 \} \) is an ideal of \( R \). Moreover, the First Isomorphism Theorem for rings states that \( R/\ker \theta \cong \theta(R) \).
**Definition 6.3.** If $I \in \mathcal{I}$, then the norm of $I$, denoted by $N(I)$, is the size of the quotient ring $\mathbb{Z}[\sqrt{-5}]/I$.

**Proposition 6.4.** $N(I)$ is finite for all $I \in \mathcal{I}^\ast$.

**Proof.** Take $n = \alpha \bar{\alpha}$ for any $\alpha \in \mathcal{I}^\ast$. Then $n \in I \cap \mathbb{N}$. As $(n) \subseteq I$, it follows that $|\mathbb{Z}[\sqrt{-5}]/I| \leq |\mathbb{Z}[\sqrt{-5}]/(n)|$. In addition, each element of $\mathbb{Z}[\sqrt{-5}]/(n)$ has a representative $a + b\sqrt{-5}$ with $0 \leq a, b < n$. Hence, $\mathbb{Z}[\sqrt{-5}]/(n)$ is finite and, therefore, $N(I) = |\mathbb{Z}[\sqrt{-5}]/I| < \infty$.

As ideal norms generalize the notion of standard norms given in (2), we expect they satisfy some similar properties. Indeed, this is the case.

**Exercise 6.5.** Let $I$ and $P$ be a nonzero ideal and a nonzero prime ideal of $\mathbb{Z}[\sqrt{-5}]$, respectively. Show that $|\mathbb{Z}[\sqrt{-5}]/P| = |I/IP|$.

**Proposition 6.6.** $N(IJ) = N(I)N(J)$ for all $I, J \in \mathcal{I}^\ast$.

**Proof.** By factoring $J$ as the product of prime ideals (Theorem 5.13) and applying induction on the number of factors, we can assume that $J$ is a prime ideal. Consider the ring homomorphism $\theta : \mathbb{Z}[\sqrt{-5}]/IJ \to \mathbb{Z}[\sqrt{-5}]/I$ defined by $\theta(a + IJ) = a + I$. It follows immediately that $\theta$ is surjective and $\ker \theta = \{a + IJ \mid a \in I\}$. Therefore
\[
\frac{\mathbb{Z}[\sqrt{-5}]/IJ}{IJ} \cong \mathbb{Z}[\sqrt{-5}]/I
\]
by the First Isomorphism Theorem. As $IJ$ is nonzero, $|\mathbb{Z}[\sqrt{-5}]/IJ| = N(IJ)$ is finite and so $|\mathbb{Z}[\sqrt{-5}]/IJ| = |\mathbb{Z}[\sqrt{-5}]/I| \cdot |I/IJ|$. Since $J$ is prime, we can use Exercise 6.5 to conclude that
\[
N(IJ) = |\mathbb{Z}[\sqrt{-5}]/IJ| = |\mathbb{Z}[\sqrt{-5}]/I| \cdot |I/IJ|
= |\mathbb{Z}[\sqrt{-5}]/I| \cdot |\mathbb{Z}[\sqrt{-5}]/J| = N(I)N(J).
\]

**Corollary 6.7.** If $N(I)$ is prime for some $I \in \mathcal{I}^\ast$, then $I$ is a prime ideal.

Let us verify now that the ideal norm is consistent with the standard norm on principal ideals.

**Proposition 6.8.** $N((\alpha)) = N(\alpha)$ for all $\alpha \in \mathbb{Z}[\sqrt{-5}]^\ast$.

**Proof.** Set $S = \{a + b\sqrt{-5} \mid 0 \leq a, b < n\}$. Clearly, $|S| = n^2$. In addition,
\[
\mathbb{Z}[\sqrt{-5}]/(n) = \{s + (n) \mid s \in S\}.
\]
Note that if $s + (n) = s' + (n)$ for $s, s' \in S$, then we have $s = s'$. As a consequence, $N((n)) = n^2 = N(n)$ for each $n \in \mathbb{N}$. It is also easily seen that the map $\theta : \mathbb{Z}[\sqrt{-5}] \to \mathbb{Z}[\sqrt{-5}]/(\tilde{\alpha})$ defined by $\theta(x) = \tilde{x} + (\tilde{\alpha})$ is a surjective ring homomorphism with $\ker \theta = (\alpha)$. Therefore the rings $\mathbb{Z}[\sqrt{-5}]/(\alpha)$ and $\mathbb{Z}[\sqrt{-5}]/(\tilde{\alpha})$ are isomorphic by the First Isomorphism Theorem. This implies that $N((\alpha)) = N((\tilde{\alpha}))$. Because $\alpha \tilde{\alpha} \in \mathbb{N}$, using Proposition 6.6 one obtains
\[
N((\alpha)) = \sqrt{N((\alpha))N((\tilde{\alpha}))} = N((\alpha \tilde{\alpha})) = \alpha \tilde{\alpha} = N(\alpha).
\]
Example 4.7 and Exercise 4.8: first, we verify that every nonzero ideal $\langle a \rangle$ with $N(a) \leq 6N(I)$. For $I \in \mathcal{I}$, take $B = \lceil \sqrt{N(I)} \rceil$ and define $S_I := \{a + b\sqrt{-5} \mid 0 \leq a, b \leq B\} \subseteq \mathbb{Z}[\sqrt{-5}]$. Observe that $|S_I| = (B + 1)^2 > N(I)$. Thus, there exist $a_1 = a_1 + b_1\sqrt{-5} \in S_I$ and $\alpha_2 = a_2 + b_2\sqrt{-5} \in S_I$ such that $\alpha = a_1 - \alpha_2 \in I \setminus \{0\}$ and $N(a) = (a_1 - a_2)^2 + 5(b_1 - b_2)^2 \leq 6B^2 \leq 6N(I)$.

Now, let $IP$ be a nonzero ideal class of $\mathbb{Z}[\sqrt{-5}]$. Also, take $J \in \mathcal{I}$ satisfying $IJP = P$. By the argument given in the previous paragraph, there exists $\beta \in J^*$ such that $N(\beta) = 6N(J)$. By Corollary 5.12, there exists an ideal $K \in \mathcal{I}$ such that $JK = \langle \beta \rangle$. Using Proposition 6.6 and Proposition 6.8, one obtains $N(J)N(K) = N(\langle \beta \rangle) = N(\beta) \leq 6N(J)$, which implies that $N(K) \leq 6$. Because $KJ \sim IJ$ (they are both principal), it follows that $K \in IP$. Hence, every nonzero ideal class of $\mathbb{Z}[\sqrt{-5}]$ contains an ideal whose norm is at most 6.

To show that the class group of $\mathbb{Z}[\sqrt{-5}]$ is $\mathbb{Z}_2$, let us first determine the congruence relations among ideals of norm at most 6. Every ideal $P$ of norm $p \in \{2, 3, 5\}$ must be prime by Corollary 6.7. Moreover, by Lemma 6.9, Theorem 5.13, and Proposition 6.6, the ideal $P$ must show in the prime factorization

\[(8) \quad \langle p \rangle = P_1^{n_1} \cdots P_k^{n_k}
\]

of the ideal $\langle p \rangle$. The following ideal factorizations have been already verified in Example 4.7 and Exercise 4.8:

\[(2) = \langle 2, 1 + \sqrt{-5} \rangle^2,
\]

\[(3) = \langle 3, 1 - 2\sqrt{-5} \rangle \langle 3, 1 + 2\sqrt{-5} \rangle,
\]

\[(5) = \langle \sqrt{-5} \rangle^2.
\]

In addition, we have proved in Example 3.6 and Exercise 3.7 that the ideals on the right-hand side of the first two equalities in (8) are prime. Also, the fact that $N(\langle \sqrt{-5} \rangle) = N(\sqrt{-5}) = 5$ implies that the ideal $\langle \sqrt{-5} \rangle$ is prime. It follows now by the uniqueness of Theorem 5.13 that the ideals on the right-hand side of the equalities (8) are the only ideals of $\mathbb{Z}[\sqrt{-5}]$ having norm in the set $\{2, 3, 5\}$. Once again, combining Lemma 6.9, Theorem 5.13, and Proposition 6.6, we obtain that any ideal $I$ whose norm is 4 must be a product of prime ideals dividing (2), which...
forces $I = \langle 2 \rangle$. Similarly, any ideal $J$ with norm 6 must be the product of ideals dividing the ideals $\langle 2 \rangle$ and $\langle 3 \rangle$. As we ask to verify below,

\begin{align*}
1 - \sqrt{-5} &= (2, 1 + \sqrt{-5})(3, 1 + 2\sqrt{-5}) \\
1 + \sqrt{-5} &= (2, 1 + \sqrt{-5})(3, 1 - 2\sqrt{-5}).
\end{align*}

Therefore $1 - \sqrt{-5}$ and $1 + \sqrt{-5}$ are the only two ideals having norm 6. Now that we know all ideals of $\mathbb{Z}[\sqrt{-5}]$ with norm at most 6, it is not difficult to check that $|\mathcal{C}(\mathbb{Z}[\sqrt{-5}])| = 2$. Because each principal ideal of $\mathbb{Z}[\sqrt{-5}]$ represents the identity ideal class $\mathcal{P}$, we find that

\[ (1)\mathcal{P} = (2)\mathcal{P} = (\sqrt{-5})\mathcal{P} = (1 - \sqrt{-5})\mathcal{P}. \]

On the other hand, we have seen that the product of $\langle 2, 1 + \sqrt{-5} \rangle$ and each of the three nonprincipal ideals with norm at most 6 is a principal ideal. Thus,

\[ \langle 2, 1 + \sqrt{-5} \rangle\mathcal{P} = (3, 1 + 2\sqrt{-5})\mathcal{P} = (3, 1 - 2\sqrt{-5})\mathcal{P}. \]

Since there are only two ideal classes, $\mathcal{C}(\mathbb{Z}[\sqrt{-5}]) = \mathbb{Z}_2$. \(\square\)

**Exercise 6.11.** Verify the equalities \((10)\), and \((11)\).

From this observation, we deduce an important property of the ideals of $\mathbb{Z}[\sqrt{-5}]$.

**Corollary 6.12.** If $I, J \in \mathcal{I}^*$ are not principal, then $IJ$ is principal.

### 7. Half-factoriality

The class group, in tandem with The Fundamental Theorem of Ideal Theory, will allow us to determine exactly what elements of $\mathbb{Z}[\sqrt{-5}]$ are irreducible.

**Proposition 7.1.** Let $\alpha$ be a nonzero nonunit element in $\mathbb{Z}[\sqrt{-5}]$. Then $\alpha$ is irreducible in $\mathbb{Z}[\sqrt{-5}]$ if and only if

1. $\alpha$ is a prime ideal in $\mathbb{Z}[\sqrt{-5}]$ (and hence $\alpha$ is a prime element), or
2. $\alpha = P_1P_2$ where $P_1$ and $P_2$ are nonprincipal prime ideals of $\mathbb{Z}[\sqrt{-5}]$.

**Proof.** ($\Rightarrow$) Suppose $\alpha$ is irreducible in $\mathbb{Z}[\sqrt{-5}]$. If $\langle \alpha \rangle$ is a prime ideal, then we are done. Assume $\langle \alpha \rangle$ is not a prime ideal. Then by Theorem 6.13 there are prime ideals $P_1, \ldots, P_k$ of $\mathbb{Z}[\sqrt{-5}]$ with $\langle \alpha \rangle = P_1 \cdots P_k$ for some $k \geq 2$. Suppose that one of the $P_i$’s is a principal ideal. Without loss of generality, assume that $P_1 = \langle \beta \rangle$ for some prime $\beta$ in $\mathbb{Z}[\sqrt{-5}]$. Using the class group, $P_2 \cdots P_k = \langle \gamma \rangle$, where $\gamma$ is a nonzero nonunit of $\mathbb{Z}[\sqrt{-5}]$. Thus, $\langle \alpha \rangle = \langle \beta \rangle \langle \gamma \rangle$ implies that $\alpha = (u\beta)\gamma$ for some unit $u$ of $\mathbb{Z}[\sqrt{-5}]$. This contradicts the irreducibility of $\alpha$ in $\mathbb{Z}[\sqrt{-5}]$. Therefore all the $P_i$’s are nonprincipal. Since the class group of $\mathbb{Z}[\sqrt{-5}]$ is $\mathbb{Z}_2$, it follows that $k$ is even. Now suppose that $k > 2$. Using Corollary 6.12 and proceeding in a manner similar to the previous argument, $P_1P_2 = \langle \beta \rangle$ and $P_3 \cdots P_k = \langle \gamma \rangle$, and again $\alpha = u\beta\gamma$ for some unit $u$, which contradicts the irreducibility of $\alpha$. Hence, either $k = 1$ and $\alpha$ is a prime element, or $k = 2$.

($\Leftarrow$) If $\langle \alpha \rangle$ is a prime ideal, then $\alpha$ is prime and so irreducible. Then suppose that $\langle \alpha \rangle = P_1P_2$, where $P_1$ and $P_2$ are nonprincipal prime ideals of $\mathbb{Z}[\sqrt{-5}]$. Let $\alpha = \beta\gamma$ for some $\beta, \gamma \in \mathbb{Z}[\sqrt{-5}]$, and assume, without loss of generality, that $\beta$ is a nonzero nonunit of $\mathbb{Z}[\sqrt{-5}]$. Notice that $\langle \beta\gamma \rangle = \langle \beta \rangle \langle \gamma \rangle = P_1P_2$. Because $P_1$ and
$P_2$ are nonprincipal ideals, $\langle \beta \rangle \notin \{P_1, P_2\}$. As a consequence of Theorem 6.13 we have that $\langle \beta \rangle = P_1 P_2$. This forces $\langle \gamma \rangle = \langle 1 \rangle$, which implies that $\gamma \in \{\pm 1\}$. Thus, $\alpha$ is irreducible.

Let us use Proposition 7.1 to analyze the factorizations presented in (11) at the beginning of the exposition. As the product of any two nonprincipal ideals of $\mathbb{Z}[\sqrt{-5}]$ is a principal ideal, the decompositions

$$\begin{align*}
(6) &= \langle 2 \rangle \langle 3 \rangle = \langle 2, 1 + \sqrt{-5} \rangle^2 \langle 3, 1 - \sqrt{-5} \rangle \langle 3, 1 + \sqrt{-5} \rangle \\
&= \langle 2, 1 + \sqrt{-5} \rangle \langle 3, 1 + \sqrt{-5} \rangle \langle 2, 1 + \sqrt{-5} \rangle \langle 3, 1 - \sqrt{-5} \rangle \\
&= \langle 1 + \sqrt{-5} \rangle \langle 1 - \sqrt{-5} \rangle
\end{align*}$$

yield that $2 \cdot 3$ and $(1 + \sqrt{-5})(1 - \sqrt{-5})$ are the only two irreducible factorizations of 6 in $\mathbb{Z}[\sqrt{-5}]$. Thus, any two irreducible factorizations of 6 in $\mathbb{Z}[\sqrt{-5}]$ have the same factorization length. We can take this observation a step further.

**Theorem 7.2.** If $\alpha$ is a nonzero nonunit of $\mathbb{Z}[\sqrt{-5}]$ and $\beta_1, \ldots, \beta_s, \gamma_1, \ldots, \gamma_t$ are irreducible elements of $\mathbb{Z}[\sqrt{-5}]$ with $\alpha = \beta_1 \cdots \beta_s = \gamma_1 \cdots \gamma_t$, then $s = t$.

**Proof.** Let $\alpha = \omega_1 \cdots \omega_m$ be a factorization into irreducibles of $\alpha$ in $\mathbb{Z}[\sqrt{-5}]$. By Theorem 5.13 there are unique prime ideals $P_1, \ldots, P_k$ of $\mathbb{Z}[\sqrt{-5}]$ satisfying that $\langle \alpha \rangle = P_1 \cdots P_k$. Suppose that exactly $d$ of these prime ideals are principal and assume, without loss, that $P_i = \langle \alpha_i \rangle$ for all $i \in \{1, \ldots, d\}$, where each $\alpha_i$ is prime in $\mathbb{Z}[\sqrt{-5}]$. Since the class group of $\mathbb{Z}[\sqrt{-5}]$ is $\mathbb{Z}_2$, there exists $n \in \mathbb{N}$ such that $k - d = 2n$. Hence,

$$\langle \alpha \rangle = \langle P_1 \cdots P_d \rangle \langle P_{d+1} \cdots P_k \rangle = \langle \alpha_1 \cdots \alpha_d \rangle \langle P_{d+1} \cdots P_k \rangle,$$

and any factorization into irreducibles of $\alpha$ will be of the form $u \alpha_1 \cdots \alpha_d \beta_1 \cdots \beta_u$, where each ideal $\langle \beta_i \rangle$ is the product of two ideals chosen from $P_{d+1}, \ldots, P_k$. As a result, $m = d + n$ and, clearly, $s = t = m$, completing the proof.

Thus, while some elements of $\mathbb{Z}[\sqrt{-5}]$ admit many factorizations into irreducibles, the number of irreducible factors in any two factorizations of a given element is the same. As we mentioned in the introduction, this phenomenon is called half-factoriality. Since the concept of half-factoriality does not involve the addition of $\mathbb{Z}[\sqrt{-5}]$, it can also be defined for commutative monoids.

**Definition 7.3.** An atomic monoid $M$ is called half-factorial if any two factorizations of each nonzero nonunit element of $M$ have the same number of irreducible factors.

Half-factorial domains and monoids have been systematically studied since the 1950’s, when Carlitz gave a characterization theorem of half-factorial rings of integers, which generalizes the case of $\mathbb{Z}[\sqrt{-5}]$ considered in this exposition.

**Theorem 7.4** (Carlitz [1]). Let $R$ be the ring of integers in a finite extension field of $\mathbb{Q}$. Then $R$ is half-factorial if and only if $R$ has class number less than or equal to two.

A few families of half-factorial domains are presented in [5]. We will conclude this paper by exhibiting two simple examples of half-factorial monoids, using the
Therefore every element of $H$ is congruent to 3 modulo 4. Hence, any factorization of an element of $H$ pairing the prime factors of $p$ into irreducibles, each of them contains five factors: $p \equiv 1 \pmod{4}$ and $p \equiv 3 \pmod{4}$.

Example 7.5 (Hilbert monoid). It is easily seen that $H = \{1 + 4k \mid k \in \mathbb{N}_0\}$ is a multiplicative submonoid of $\mathbb{N}$. The monoid $H$ is called Hilbert monoid. It is not hard to verify (Exercise 7.6) that the irreducible elements of $H$ are

1. the prime numbers $p$ satisfying $p \equiv 1 \pmod{4}$ and
2. $p_1p_2$, where $p_1$ and $p_2$ are prime numbers satisfying $p_i \equiv 3 \pmod{4}$.

Therefore every element of $H$ is a product of irreducibles. Also, in the factorization of any element of $H$ into primes, there must be an even number of prime factors congruent to 3 modulo 4. Hence, any factorization of an element $x \in H$ comes from pairing the prime factors of $x$ that are congruent to 3 modulo 4. This implies that $H$ is half-factorial. For instance, $x = 5^2 \cdot 3^2 \cdot 11 \cdot 13 \cdot 19$ has exactly two factorizations into irreducibles, each of them contains five factors:

$$x = 5^2 \cdot 13 \cdot (3^2) \cdot (11 \cdot 19) = 5^2 \cdot 13 \cdot (3 \cdot 11) \cdot (3 \cdot 19).$$

Exercise 7.6. Argue that the irreducible elements of the Hilbert monoid are precisely those described in Example 7.5.

Definition 7.7. Let $p$ be a prime number.

1. We say that $p$ is **inert** if $\langle p \rangle$ is a prime ideal in $\mathbb{Z}[\sqrt{-5}]$.
2. We say that $p$ is **ramified** if $\langle p \rangle = P^2$ for some prime ideal $P$ of $\mathbb{Z}[\sqrt{-5}]$.
3. We say that $p$ **splits** if $\langle p \rangle = PP'$ for two distinct prime ideals of $\mathbb{Z}[\sqrt{-5}]$.

Prime numbers $p$ can be classified according to the above definition. Indeed, we have seen that $p$ is ramified when $p \in \{2, 5\}$. It is also known that $p$ splits if $p \equiv 1, 3, 7, 9 \pmod{20}$ and is inert if $p \not\equiv 1, 3, 7, 9 \pmod{20}$ (except 2 and 5). A proof of this result is given in [3].

Example 7.8. When $n \geq 2$, the submonoid $\mathbb{X}_n$ of the additive monoid $\mathbb{N}_0^{n+1}$ given by

$$\mathbb{X}_n = \{(x_1, \ldots, x_{n+1}) \mid x_i \in \mathbb{N}_0 \text{ and } x_1 + \cdots + x_n = x_{n+1}\}$$

is a half-factorial Krull monoid with divisor class group $\mathbb{Z}_2$ (see [4, Section 2] for more details). Following [3], we will use $\mathbb{X}_n$ to count the number of distinct factorizations into irreducibles of a given nonzero nonunit $\alpha \in \mathbb{Z}[\sqrt{-5}]$. Let

$$\langle \alpha \rangle = P_1^{n_1} \cdots P_k^{n_k} Q_1^{m_1} \cdots Q_t^{m_t},$$

where the $P_i$'s are distinct prime ideals in the trivial class ideal of $\mathbb{Z}[\sqrt{-5}]$, the $Q_j$'s are distinct prime ideals in the nontrivial class ideal of $\mathbb{Z}[\sqrt{-5}]$, and $m_1 \leq \cdots \leq m_t$.

Then the desired number of factorizations $\eta(\alpha)$ of $\alpha$ in $\mathbb{Z}[\sqrt{-5}]$ is given by

$$\eta(\alpha) = \eta_{\mathbb{X}_t}(m_1, \ldots, m_t, m_1 + \cdots + m_t),$$

which, when $t = 3$, can be computed by the formula

$$\eta_{\mathbb{X}_3}(x_1, x_2, x_3, x_4) = \sum_{j=0}^{\lfloor x_1/2 \rfloor} \sum_{k=0}^{x_1-2j} \left( \left\lfloor \frac{\min\{x_2 - k, x_3 - x_1 + 2j + k\}}{2} \right\rfloor + 1 \right).$$
For instance, let us find how many factorizations $1980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$ has in $\mathbb{Z}[\sqrt{-5}]$. We have seen that 5 ramifies as $\langle 5 \rangle = P_1^2$, where $P_1$ is principal. As 11 is inert, $P_2 = \langle 11 \rangle$ is prime. In addition, 3 splits as $\langle 3 \rangle = Q_1 Q_2$, where $Q_1$ and $Q_2$ are nonprincipal. Finally, 2 ramifies as $\langle 2 \rangle = Q_3^2$, where $Q_3$ is nonprincipal. Therefore one has that $\langle 1980 \rangle = P_1^2 P_2 Q_1^2 Q_2 Q_3^4$, and so

$$\eta(1980) = \eta_{\mathbb{Z}_5}(2, 2, 4, 4) = \sum_{j=0}^{1} \sum_{k=0}^{2-2j} \left( \left\lfloor \frac{\min\{2-k, 2+2j+k\}}{2} \right\rfloor + 1 \right) = 6.$$ 

References

[1] L. Carlitz, A characterization of algebraic number fields with class number two, Proc. Amer. Math. Soc. 11 (1960) 391–392.
[2] S. T. Chapman, A tale of two monoids: A friendly introduction to nonunique factorizations, Math. Mag. 87 (2014) 163–173.
[3] S. T. Chapman, J. Herr, and N. Rooney, A factorization formula for class number two, J. Number Theory 79 (1999) 58–66.
[4] S. T. Chapman, U. Krause, and E. Oeljeklaus, Monoids determined by a homogeneous linear diophantine equation and the half-factorial property, J. Pure Appl. Algebra 151 (2000) 107–133.
[5] H. Kim, Examples of half-factorial domains, Canad. Math. Bull. 43 (2000) 362–367.
[6] D. Marcus, Number Fields, Vol. 1995, Springer, New York, 1977.
[7] H. Pollard and H. G. Diamond, The Theory of Algebraic Numbers, Courier Corporation, New York, 1998.