Critical point theory for sparse recovery

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Abstract

We study the problem of sparse recovery in the context of compressed sensing. This is to minimize the sensing error of linear measurements by sparse vectors with at most \( s \) non-zero entries. We develop the so-called critical point theory for sparse recovery. This is done by introducing nondegenerate M-stationary points which adequately describe the global structure of this nonconvex optimization problem. We show that all M-stationary points are generically nondegenerate. In particular, the sparsity constraint is active at all local minimizers of a generic sparse recovery problem. Additionally, the equivalence of strong stability and nondegeneracy for M-stationary points is shown. We claim that the appearance of saddle points - these are M-stationary points with exactly \( s - 1 \) non-zero entries - cannot be neglected. For this purpose we derive a so-called Morse relation, which gives a lower bound on the number of saddle points in terms of the number of local minimizers. The relatively involved structure of saddle points can be seen as a source of well-known difficulty by solving the problem of sparse recovery to global optimality.

Keywords: sparse recovery, compressed sensing, critical point theory, nondegenerate M-stationarity, strong stability, genericity, saddle points, Morse relation

1 Introduction

Compressed sensing is concerned with the recovery of a sparse vector \( x \) from linear measurements \( Ax = b \), where \( A \in \mathbb{R}^{m \times n} \) is a sensing matrix and \( b \in \mathbb{R}^m \) is a measurement vector. For this purpose, it is usual to consider the following optimization problem, see e.g. Davenport et al. (2012):

\[
\min_{x \in \mathbb{R}^n} \|x\|_0 \quad \text{s. t.} \quad Ax = b,
\]

where the so-called \( \ell_0 \) "norm" counts non-zero entries of \( x \), i.e.

\[
\|x\|_0 = |\{i \in \{1, \ldots, n\} \mid x_i \neq 0\}|.
\]

If the linear measurements are prone to Gaussian noise, the optimization problem (1) can be modified as follows:

\[
\min_{x \in \mathbb{R}^n} \|x\|_0 \quad \text{s. t.} \quad \|Ax - b\|_2 \leq \varepsilon,
\]

where \( \varepsilon \) is a constant.

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where \( \varepsilon > 0 \) is the bound on the error magnitude with respect to the Euclidean norm. In this paper we consider an analogue formulation of Beck and Eldar (2013):

\[
\text{SR} : \min_{x \in \mathbb{R}^n} \frac{1}{2} \|Ax - b\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq s,
\]

where \( s \in \{0, 1, \ldots, n - 1\} \) is the bound on the number of non-zero entries of \( x \). Note that the sparse recovery problem consists of minimizing the sensing error by sparse vectors with at most \( s \) non-zero entries. Sometimes we write \( \text{SR}(A, b) \) for the problem of sparse recovery, in order to highlight the dependence on the data \((A, b)\). Throughout the paper we make the following not very restrictive assumption, cf. Beck and Eldar (2013).

**Assumption 1** The bound on the number of non-zero entries does not exceed the number of measurements in \( \text{SR} \), i.e. \( s \leq m \).

The difficulty of solving \( \text{SR} \) comes from the combinatorial nature of the sparsity constraint \( \|x\|_0 \leq s \). Although the objective function \( f \) of \( \text{SR} \) is convex, its feasible set is non-convex as a union of linear subspaces. Nevertheless, several attempts to tackle \( \text{SR} \) have been undertaken in recent years.

In the seminal paper [Beck and Eldar (2013)] a generalization of \( \text{SR} \) with an arbitrary smooth objective function is considered. The latter is referred to by the authors as *sparsity constrained nonlinear optimization*:

\[
\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \|x\|_0 \leq s.
\]

The notions of basic feasibility, \( L \)-stationarity, and CW-minimality have been introduced and shown to be necessary optimality conditions for (3). The formulation of \( L \)-stationarity mimics the techniques from convex optimization by using the orthogonal projection on the feasible set. The notion of CW-minimum incorporates the coordinate-wise optimality along the axes. Based on both stationarity concepts, algorithms that find points satisfying these conditions have been developed. These are the iterative hard thresholding method, as well as the greedy and partial sparse-simplex methods. In a series of subsequent papers [Beck and Hallak (2016, 2018)] elaborated the algorithmic approach based on \( L \)-stationarity and CW-minimality.

Another line of research started with [Burdakov et al. (2016)]. Here, in addition to an arbitrary smooth objective function also smooth equality and inequality constraints have been incorporated into the feasible set. For that, the authors coin the new term of *mathematical programs with cardinality constraints*:

\[
\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad \|x\|_0 \leq s, \quad h(x) = 0, \quad g(x) \geq 0.
\]

The key idea in [Burdakov et al. (2016)] is to provide a mixed-integer formulation of (4) whose standard relaxation still has the same solutions. For the relaxation the notion of S-stationary points is proposed. S-stationarity corresponds to the standard Karush-Kuhn-Tucker condition for the relaxed program. The techniques applied follow mainly those for mathematical programs with complementarity constraints. In particular, an appropriate regularization method for solving (4) is suggested. The latter is proved to converge towards so-called M-stationary points. M-stationarity
corresponds to the standard Karush-Kuhn-Tucker condition of the tightened program, where zero entries of a feasible point remain locally vanishing. Further research in this direction is presented in a series of subsequent papers Červinka et al. (2016), Bucher and Schwartz (2018).

The aim of this paper is to develop a critical point theory for the problem of sparse recovery. The main idea of critical point theory is to identify stationary points which roughly speaking induce the global structure of the underlying optimization problem. They have not only to include minimizers, but also all kinds of saddle points – just in analogy to the unconstrained case. Critical point theory for other non-convex optimization problems, such as e.g. mathematical programs with complementarity constraints, general semi-infinite programming, mathematical problems with vanishing constraints, is elaborated in Jongen et al. (2009), Jongen and Shikhman (2011), Dorsch et al. (2012), respectively.

Let us overview our main results on the the critical point theory for SR:

(i) It turns out that the concept of M-stationarity from Burdakov et al. (2016) is the adequate stationarity concept for our purposes. We introduce the notion of nondegeneracy for M-stationary points of SR. It is proved that all M-stationary points are generically nondegenerate, see Theorem 2. As an important consequence, the sparsity constraint must be active at all local minimizers of a generic SR, see Corollary 1.

(ii) Further, we introduce the notion of strongly stability of M-stationary points in the sense of Kojima (1980). The equivalence of strong stability and nondegeneracy for M-stationary points of SR is shown, see Theorem 3. In case of degeneracy a local minimizer of SR may bifurcate into multiple minimizers and a saddle point, see Example 1.

(iii) The role of saddle points play M-stationary points with exactly \( s - 1 \) non-zero entries. We derive a so-called Morse relation, which gives a lower bound on the number of saddle points in terms of the number of local minimizers, see Theorem 6. Hence, the appearance of saddle points cannot be neglected at least from the perspective of global optimization. As further novelty, a saddle point may lead to more than two different local minimizers. The relatively involved structure of saddle points can be seen as a source of well-known difficulty if solving mathematical programs with sparsity constraint to global optimality.

We would like to mention that in the recent preprint Lämmel and Shikhman (2019) the critical point theory for sparsity constrained nonlinear optimization (3) has been established. Note that although SR constitutes a subclass of (3), the adjustment of results from Lämmel and Shikhman (2019) for SR is by far not straight-forward. In fact, we cannot use either the corresponding results or their proof technique, in order to show the genericity of nondegenerate M-stationary points of SR. This is due to the fact that the data space of \((A, b)\) generates just a subset of \(C^2\)-functions via

\[
f(x) = \frac{1}{2} \| Ax - b \|^2_2.
\]

The issue of strong stability is new and has not been studied in Lämmel and Shikhman (2019). So is its equivalence to nondegeneracy for M-stationary points of SR. Finally, the derivation of Morse relation needs an SR specific notion of s-regularity of the sensing matrix \(A\) being introduced by Beck and Eldar (2013), see Lemma 4. For the readers’ convenience, we decided to make the exposition of the critical point theory for SR self-contained. This allows a potential reader, which
is just interested in the topic of sparse recovery, not to consult the previous paper at all.

The paper is organized as follows. In Section 2 we discuss the notion of a nondegenerate M-stationary point. In Section 3 we show that nondegeneracy is a generic property of M-stationary points. Section 4 is devoted to the strong stability of M-stationary points and its equivalence to their nondegeneracy. The global structure of SR is described in Section 5.

Our notation is standard. The cardinality of a finite set $S$ is denoted by $|S|$. The $n$-dimensional Euclidean space is denoted by $\mathbb{R}^n$ with the coordinate vectors $e_i$, $i = 1, \ldots, n$. For $J \subset \{1, \ldots, n\}$ we denote by $\text{conv}(e_j, j \in J)$ the convex hull of the coordinate vectors $e_j, j \in J$. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted by $\|x\|_2$, and by $x \geq 0$ we refer to the componentwise comparison $x_i \geq 0$ for all $i = 1, \ldots, n$. The entries of the subvector $x_I$ correspond to those of $x \in \mathbb{R}^n$ with respect to a given index set $I \subset \{1, \ldots, n\}$. The space of real $(m, n)$-matrices is denoted by $\mathbb{R}^{m \times n}$. For $A \in \mathbb{R}^{m \times n}$ the transposed matrix is denoted by $A^T \in \mathbb{R}^{n \times m}$. If $A \in \mathbb{R}^{m \times n}$ is of full rank $n \leq m$, then $A^+ = (A^T A)^{-1} A^T$ denotes the Moore-Penrose inverse of $A$. For an index set $I \subset \{1, \ldots, n\}$ we denote by $A_I$ the submatrix of $A \in \mathbb{R}^{m \times n}$ with the columns corresponding to the set $I$. Additionally, we denote by $A^T_I$ the transposition of $A_I$.

## 2 Nondegeneracy

For $0 \leq k \leq n$ we use the notation

$$\mathbb{R}^{n,k} = \{x \in \mathbb{R}^n \mid \|x\|_0 \leq k\}.$$ 

Using the latter, the feasible set of SR can be written as

$$\mathbb{R}^{n,s} = \{x \in \mathbb{R}^n \mid \|x\|_0 \leq s\}.$$ 

For a feasible point $x \in \mathbb{R}^{n,s}$ we define the following complementary index sets:

$$I_0(x) = \{i \in \{1, \ldots, n\} \mid x_i = 0\}, \quad I_1(x) = \{j \in \{1, \ldots, n\} \mid x_j \neq 0\}.$$ 

Without loss of generality, we assume throughout the whole paper that at the particular point of interest $\bar{x} \in \mathbb{R}^{n,s}$ it holds:

$$I_0(\bar{x}) = \{1, \ldots, n - \|\bar{x}\|_0\}, \quad I_1(\bar{x}) = \{n - \|\bar{x}\|_0 + 1, \ldots, n\}.$$ 

Using this convention, the following local description of SR feasible set can be deduced. Let $\bar{x} \in \mathbb{R}^{n,s}$ be a feasible point of SR. Then, there exist neighborhoods $U_{\bar{x}}$ and $V_0$ of $\bar{x}$ and 0, respectively, such that under the linear coordinate transformation $\Phi(x) = x - \bar{x}$ we have:

$$\Phi(\mathbb{R}^{n,s} \cap U_{\bar{x}}) = \left(\mathbb{R}^{n-k-\|\bar{x}\|_0, s-\|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0}\right) \cap V_0, \quad \Phi(\bar{x}) = 0. \quad (5)$$

For a feasible point $\bar{x}$ of SR we formulate necessary optimality conditions with respect to the free variables from $I_1(\bar{x})$:

$$\frac{\partial f}{\partial x_j}(\bar{x}) = 0 \quad \text{for all } j \in I_1(\bar{x}).$$
Recalling \( f(x) = \frac{1}{2} \|Ax - b\|_2^2 \), we get:

\[
(A^T A\bar{x} - A^T b)_j = 0 \quad \text{for all } j \in I_1(\bar{x}).
\]

Due to \( \bar{x}_{I_0(\bar{x})} = 0 \), it holds equivalently:

\[
A^T_{I_1(\bar{x})} A_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - A^T_{I_1(\bar{x})} b = 0.
\]

Note that \( A_{I_1(\bar{x})} \) stands for the submatrix of \( A \) with the columns corresponding to the set \( I_1(\bar{x}) \). Analogously, \( x_{I_1(\bar{x})} \) stands for the subvector of \( x \) with the entries corresponding to the set \( I_1(\bar{x}) \).

The previous derivation gives rise to the following definition.

**Definition 1 (M-stationarity, Burdakov et al. (2016))** A feasible point \( \bar{x} \in \mathbb{R}^{n,s} \) is called M-stationary for SR if

\[
A^T_{I_1(\bar{x})} A_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - A^T_{I_1(\bar{x})} b = 0.
\]

Obviously, a local minimizer of SR is an M-stationary point.

Let us check the second-order sufficient optimality condition with respect to the free variables from \( I_1(\bar{x}) \). We have:

\[
\frac{\partial^2 f}{\partial x_j \partial x_k}(\bar{x})_{j,k \in I_1(\bar{x})} = A^T_{I_1(\bar{x})} A_{I_1(\bar{x})}.
\]

For the latter matrix to be positive definite, it is enough to assume that \( A_{I_1(\bar{x})} \) has full rank.

Further, we examine the first-order behavior of \( f \) on the sparse variables from \( I_0(\bar{x}) \):

\[
\frac{\partial f}{\partial x_i}(\bar{x}) = (A^T A\bar{x} - A^T b)_i = \left( A^T_{I_0(\bar{x})} A_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - A^T_{I_0(\bar{x})} b \right)_i \quad \text{for all } i \in I_0(\bar{x}).
\]

The following definition of nondegeneracy additionally requires the derivatives of the SR objective function with respect to the sparse variables be non-vanishing.

**Definition 2 (Nondegeneracy)** An M-stationary point \( \bar{x} \in \mathbb{R}^{n,s} \) of SR is called nondegenerate if the following conditions hold:

\( ND1: \) if \( \|\bar{x}\|_0 < s \) then all entries of the vector \( A^T_{I_0(\bar{x})} A_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - A^T_{I_0(\bar{x})} b \) are non-vanishing,

\( ND2: \) the matrix \( A_{I_1(\bar{x})} \) is of full rank, i.e. \( \text{rank}(A_{I_1(\bar{x})}) = \|\bar{x}\|_0 \).

Otherwise, we call \( \bar{x} \) degenerate.

We point out that nondegeneracy is closely related to the property of \( s \)-regularity of the matrix \( A \) introduced in Beck and Eldar (2013).

**Definition 3 (s-regularity, Beck and Eldar (2013))** A matrix \( A \in \mathbb{R}^{m \times n} \) is called \( s \)-regular if for every index set \( I \subset \{1, \ldots, n\} \) with \( |I| = s \) it holds: \( \text{rank}(A_I) = s \).

**Lemma 1 (s-regularity and ND2)** If \( A \) is \( s \)-regular, then ND2 is satisfied at all M-stationary points of SR.
Proof:
Let $\bar{x}$ be an M-stationary point of SR. Since we have $|I_1(\bar{x})| \leq s$, there exists an index set $I \subset \{1, \ldots, n\}$ with $|I| = s$ and $I_1(\bar{x}) \subset I$. The $s$-regularity of $A$ implies that $\text{rank}(A_I) = s$. In particular, it follows that $\text{rank}(A_{I_1(\bar{x})}) = \|\bar{x}\|_0$. \hfill $\square$

Lemma 2 (s-regularity and finiteness, Beck and Eldar (2013)) If $A$ is $s$-regular, then there are finitely many M-stationary points of $\text{SR}$. 

Proof:
If $\bar{x}$ is an M-stationary point of $\text{SR}$, then by using (6) we have:

$$\bar{x}_{I_1(\bar{x})} = \left( A_{I_1(\bar{x})}^T A_{I_1(\bar{x})} \right)^{-1} A_{I_1(\bar{x})}^T b \quad \text{and} \quad \bar{x}_{I_0(\bar{x})} = 0,$$

where the matrix $A_{I_1(\bar{x})}^T A_{I_1(\bar{x})}$ is nonsingular due to the $s$-regularity of $A$. Since $|I_1(\bar{x})| \leq s$, and the number of subsets of $\{1, 2, \ldots, n\}$ with at most $s$ elements is finite, the result follows. \hfill $\square$

Conditions ND1 and ND2 from Definition 2 allow to derive a relatively simple local representation of $\text{SR}$ around a nondegenerate M-stationary point. In comparison to the corresponding result by L"ammel and Shikhman (2019) for the sparsity constrained nonlinear optimization (3), the so-called quadratic index is vanishing here. This leads to the absence of negative squares in the representation (7).

Theorem 1 (Morse-Lemma for $\text{SR}$) Suppose that $\bar{x}$ is a nondegenerate M-stationary point of $\text{SR}$. Then, there exist neighborhoods $U_{\bar{x}}$ and $V_0$ of $\bar{x}$ and $0$, respectively, and a local coordinate system $\Psi : U_{\bar{x}} \to V_0$ of $\mathbb{R}^n$ around $\bar{x}$ such that:

$$f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{i \in I_0(\bar{x})} y_i + \sum_{j \in I_1(\bar{x})} y_j^2; \tag{7}$$

where $y \in \mathbb{R}^{n-\|\bar{x}\|_0, s-\|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0}$. 

Proof:
Let $\bar{x}$ be a nondegenerate M-stationary point of $\text{SR}$. By using the linear coordinate transformation $\Phi$ from (5), we put $\bar{f} := f \circ \Phi^{-1}$ on the set $(\mathbb{R}^{n-\|\bar{x}\|_0, s-\|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0}) \cap V_0$. As new coordinates we put $y = (y_{I_0(\bar{x})}, y_{I_1(\bar{x})})$. Then, it holds:

$$\frac{\partial \bar{f}}{\partial y_i(0)} = \left( A_{I_0(\bar{x})}^T A_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - A_{I_0(\bar{x})}^T b \right)_i \quad \text{for all } i \in I_0(\bar{x}),$$

$$\frac{\partial \bar{f}}{\partial y_j(0)} = \left( A_{I_1(\bar{x})}^T A_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - A_{I_1(\bar{x})}^T b \right)_j \quad \text{for all } j \in I_1(\bar{x}),$$

$$\left( \frac{\partial^2 \bar{f}}{\partial y_j \partial y_k (0)} \right)_{j,k \in I_1(\bar{x})} = A_{I_1(\bar{x})}^T A_{I_1(\bar{x})}.$$

Due to ND1, M-stationarity of $\bar{x}$, and ND2, respectively, we have:
(i) if \( \|\bar{x}\|_0 < s \) then \( \frac{\partial \bar{f}}{\partial y_i}(0) \neq 0 \) for all \( i \in I_0(\bar{x}) \),

(ii) \( \frac{\partial \bar{f}}{\partial y_j}(0) = 0 \) for all \( j \in I_1(\bar{x}) \),

(iii) the matrix \( \left( \frac{\partial^2 \bar{f}}{\partial y_j \partial y_k}(0) \right)_{j,k \in I_1(\bar{x})} \) is positive definite.

In what follows, we denote \( \bar{f} \) by \( f \) again. Under the following coordinate transformations the set \( \mathbb{R}^{n-\|\bar{x}\|_0, s-\|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0} \) will be equivariantly transformed in itself. It holds:

\[
f(y) = \int_0^1 \frac{d}{dt} f(ty_{I_0(\bar{x})}, y_{I_1(\bar{x})}) \, dt + f(0, y_{I_1(\bar{x})}) = \sum_{i \in I_0(\bar{x})} y_id_i(y) + f(0, y_{I_1(\bar{x})}),
\]

with linear functions \( d_i, i \in I_0(\bar{x}) \).

Due to (ii)-(iii), we may apply the standard Morse lemma on the quadratic function \( f(0, y_{I_1(\bar{x})}) \) without affecting the coordinates \( y_{I_0(\bar{x})} \), see e.g. Jongen et al. (2000). The corresponding coordinate transformation is linear. Denoting the transformed functions again by \( f \) and \( d_i \), we obtain

\[
f(y) = f(\bar{x}) + \sum_{i \in I_0(\bar{x})} y_id_i(y) + \sum_{j \in I_1(\bar{x})} y_j^2.
\]

In case \( \|\bar{x}\|_0 = s \), we need to consider \( f \) locally around the origin on the set

\[
\mathbb{R}^{n-\|\bar{x}\|_0, s-\|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0} = \mathbb{R}^{n-\|\bar{x}\|_0,0} \times \mathbb{R}^{\|\bar{x}\|_0} = \{0\}^{n-\|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0}.
\]

Hence, \( y_i = 0 \) for \( i \in I_0(\bar{x}) \), and we immediately obtain the representation (7).

In case \( \|\bar{x}\|_0 < s \), (i) provides that \( d_i(0) = \frac{\partial f}{\partial y_i}(0) \neq 0 \) for \( i \in I_0(\bar{x}) \). Hence, we may take

\[
y_id_i(y), i \in I_0(\bar{x}), \quad y_j, j \in I_1(\bar{x})
\]

as new local coordinates by a straightforward application of the inverse function theorem. Denoting the transformed function again by \( f \), we obtain (7). Here, the coordinate transformation \( \Psi \) is understood as the composition of all previous ones.

By means of Theorem 1 the following important result follows.

**Proposition 1 (Nondegenerate minimizers)** Let \( \bar{x} \) be a nondegenerate M-stationary point for \( SR \). Then, \( \bar{x} \) is a local minimizer for \( SR \) if and only if the sparsity constraint is active, i.e. \( \|\bar{x}\|_0 = s \).

**Proof:**

Let \( \bar{x} \) be a nondegenerate M-stationary point of \( SR \). The application of Morse Lemma from Theorem 1 says that there exist neighborhoods \( U_{\bar{x}} \) and \( V_0 \) of \( \bar{x} \) and 0, respectively, and a local \( C^\infty \)-coordinate system \( \Psi : U_{\bar{x}} \to V_0 \) of \( \mathbb{R}^n \) around \( \bar{x} \) such that:

\[
f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{i \in I_0(\bar{x})} y_i + \sum_{j \in I_1(\bar{x})} y_j^2, \quad (8)
\]
where \( y \in \mathbb{R}^{n - \|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0} \). Therefore, \( \bar{x} \) is a local minimizer for SR if and only if 0 is a local minimizer of \( f \circ \Psi^{-1} \) on the set \( (\mathbb{R}^{n - \|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0}) \cap V_0 \). If we have \( \|\bar{x}\|_0 = s \), the formula in (8) reads as

\[
f \circ \Psi^{-1}(y) = f(\bar{x}) + \sum_{j \in I_1(\bar{x})} y_j^2,
\]

(9)

where \( y \in \{0\}^{n-s} \times \mathbb{R}^s \). Thus, 0 is a local minimizer for (9). Vice versa, if 0 is a local minimizer for (8), then obviously \( \|\bar{x}\|_0 = s \).

3 Genericity

Let us show that \( s \)-regularity is likely to be satisfied in the context of compressed sensing. This issue has been already mentioned in Beck and Eldar (2013).

**Lemma 3 (Genericity of \( s \)-regularity)** Let \( A \) denote the subset of \( s \)-regular matrices \( A \). Then, \( A \) is open and dense in \( \mathbb{R}^{m \times n} \).

**Proof:**
We consider the sets

\[
\Gamma_{I,r} = \{ A \in \mathbb{R}^{m \times n} \mid \text{rank}(A_I) = r \},
\]

where \( I \subset \{1, \ldots, n\} \) with \( |I| = s \), and \( r = 0, 1, \ldots, s \). According to Example 7.3.23 from Jongen et al. (2000), \( \Gamma_{I,r} \) is a submanifold of \( \mathbb{R}^{m \times n} \) with codimension \((m-r)(s-r)\) – recall that we have \( r \leq s \leq m \) by Assumption [1]. In other words, \( \Gamma_{I,r} \) is generically empty for \( r = 0, 1, \ldots, s-1 \), and \( \Gamma_{I,s} \) is dense in \( \mathbb{R}^{m \times n} \). Thus, \( \text{rank}(A_I) = s \) holds for all \( I \) in generic sense, which provides the assertion.

Next, we show that ND1 and ND2 are fulfilled at all M-stationary points of SR for almost all data \((A, b) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m\) with respect to Lebesgue measure, i.e. they are generically nondegenerate.

**Theorem 2 (Genericity of nondegeneracy)** Let \( \mathcal{D} \) denote the subset of data \((A, b)\) for which each M-stationary point of SR is nondegenerate. Then, \( \mathcal{D} \) is open and dense in \( \mathbb{R}^{m \times n} \times \mathbb{R}^m \).

**Proof:**
Due to Lemma 3 the set \( A \) of \( s \)-regular matrices is open and dense in \( \mathbb{R}^{m \times n} \). Then, Lemma 1 implies that ND2 generically holds. Now, we prove that ND1 is a generic condition for all M-stationary points \( \bar{x} \) with \( \|\bar{x}\|_0 < s \). By setting \( S = I_1(\bar{x}) \), we write (6) as

\[
\bar{x}_S = (A_S^T A_S)^{-1} A_S^T b = A_S^+ b,
\]

where \( A_S^+ \) denotes the Moore-Penrose inverse of \( A_S \). Hence, the vector under consideration in ND1 becomes:

\[
A_S^T A_S \bar{x}_S - A_S^T b = -A_S^T (I - A_S A_S^+ ) b = -((I - A_S A_S^+ ) A_S^+)^T b,
\]

(10)
where we use the identity matrix $I \in \mathbb{R}^{m \times m}$, and the fact that $A SA_S^+$ is symmetric:

$$(A SA_S^+)^T = (A S^+)^T A S^+ = (A S^+)^+ A S^+ = A S \left( A S^+ A S \right)^{-1} A S^+ = A S A_S^+.$$  

Note that the entries of the vector in (10) have to be shown generically non-vanishing, i.e.

$$(\left( I - A S A_S^+ \right) A (i))^T b \neq 0 \text{ for all } i \in S^c.$$  

For that, we define the sets

$$\Omega_{S,i} = \left\{ A \in \mathcal{A} \mid (I - A S A_S^+) A (i) = 0 \right\},$$

where $S \subset \{1, \ldots, n\}$ with $|S| < s$, and $i \in S^c$. Let us show that $\Omega_{S,i}$ is a submanifold. The condition $(I - A S A_S^+) A (i) = 0$ means that the vector $A (i)$ lies in the nullspace of $I - A S A_S^+$, i.e. $A (i) \in N(I - A S A_S^+)$. Let us determine the dimension of $N(I - A S A_S^+)$. We start with the matrix $A S A_S^+$. It holds for the latter:

$$(A S A_S^+) A S A_S^+ = A S \left( A S^+ A S \right)^{-1} A S^+ = A S \left( A S^+ A S \right)^{-1} A S^+ = A S A_S^+.$$  

Furthermore, the Sylvester’s rank inequality provides

$$\text{rank} \left( A S A_S^+ \right) \geq \text{rank} \left( A S \right) + \text{rank} \left( A_S^+ \right) - |S| = |S|,$$

due to $\text{rank} \left( A_S^+ \right) = \text{rank} \left( A S \right) = |S|$ and the $s$-regularity of $A$. Additionally, we have:

$$\text{rank} \left( A S A_S^+ \right) \leq \min \left\{ \text{rank} \left( A S \right), \text{rank} \left( A_S^+ \right) \right\} = |S|.$$  

Altogether, $A S A_S^+$ is an orthogonal projection with $\text{rank} \left( A S A_S^+ \right) = |S|$. Hence, $I - A S A_S^+$ is also an orthogonal projection, and for the dimension of its nullspace we have:

$$\text{dim} \left( N \left( I - A S A_S^+ \right) \right) = \text{rank} \left( A S A_S^+ \right) = |S|.$$  

Since the dimension of $N \left( I - A S A_S^+ \right)$ remains constant under sufficiently small perturbations of $A S$, the condition $A (i) \in N \left( I - A S A_S^+ \right)$ provides exactly $m - |S|$ stable equations. Hence, $\Omega_{S,i}$ is a submanifold of codimension $m - |S|$. Due to $m - |S| > m - s \geq 0$, we conclude that all $\Omega_{S,i}$ are generically empty. In particular, the submanifold

$$\Omega_S = \left\{ A \in \mathcal{A} \mid (I - A S A_S^+) A (i) \neq 0 \text{ for all } i \in S^c \right\}$$

is of dimension $mn$ and, hence, dense in $\mathbb{R}^{m \times n}$.

Finally, we define the sets

$$\Upsilon_{S,i} = \left\{ (A, b) \in \Omega_S \times \mathbb{R}^m \mid \left( \left( I - A S A_S^+ \right) A (i) \right)^T b = 0 \right\},$$

where $S \subset \{1, \ldots, n\}$ with $|S| < s$, and $i \in S^c$ as above. Since $A \in \Omega_S$, the vector $(I - A S A_S^+) A (i)$ does not vanish. Hence, the equation $\left( (I - A S A_S^+) A (i) \right)^T b = 0$ is nondegenerate. We conclude
that $\Upsilon_{S,1}$ is a submanifold of codimension 1, and can be therefore generically avoided. Overall, we have shown that the condition ND1 holds in generic sense.

The openness part follows due to the continuity of ND1 and ND2 with respect to sufficiently small perturbations of $A$.

We deduce the following important corollary on the structure of minimizers for SR.

**Corollary 1 (Sparsity constraint at minimizers)** Generically, each minimizer $\bar{x} \in \mathbb{R}^{n,s}$ of SR is nondegenerate with the active sparsity constraint, i.e. $\|\bar{x}\|_0 = s$.

**Proof:** Note that every local minimizer of SR has to be M-stationary. Nondegenerate M-stationary points are generic by Theorem 2. The rest follows by means of Proposition 1. □

### 4 Stability

Let us fix an arbitrary norm $\|(A,b)\|$ on the data space $(A,b) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m$. For M-stationary points we define the notion of strong stability in the sense of Kojima (1980).

**Definition 4 (Strong stability)** An M-stationary point $\bar{x}$ of $\text{SR}(A,b)$ is called strongly stable if for some $r > 0$ and each $\varepsilon \in (0, r]$ there exists $\delta > 0$ such that whenever

$$\left(\bar{A}, \bar{b}\right) \in \mathbb{R}^{m \times n} \times \mathbb{R}^m \quad \text{and} \quad \left\|\left(\bar{A}, \bar{b}\right) - (A, b)\right\| \leq \delta,$$

the ball $B(\bar{x}, \varepsilon)$ contains an M-stationary point $\tilde{x}$ of $\text{SR}\left(\bar{A}, \bar{b}\right)$ that is unique within the ball $B(\bar{x}, r)$.

Let us illustrate a possible failure of strong stability of M-stationary points caused by their degeneracy.

**Example 1 (Instability)** Let the following sensing matrix and measurement vector be given:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$  

We consider the corresponding sparse recovery problem with $s = 1$:

$$\text{SR}(A,b) : \min_{x_1, x_2} \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \quad \text{s. t.} \quad \|(x_1, x_2)\|_0 \leq 1.$$  

Obviously, $\bar{x} = (0, 0)$ is the unique minimizer of $\text{SR}(A,b)$. Further, let us perturb the data by means of an arbitrarily small $\varepsilon > 0$ as follows:

$$\bar{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \bar{b} = \begin{pmatrix} \varepsilon \\ \varepsilon \end{pmatrix}.$$
We obtain as perturbed sparse recovery problem:

\[
SR(\tilde{A}, \tilde{b}) : \min_{x_1, x_2} \frac{1}{2} (x_1 - \varepsilon)^2 + \frac{1}{2} (x_2 - \varepsilon)^2 \quad \text{s.t.} \quad \|(x_1, x_2)\|_0 \leq 1.
\]

It is easy to see that \(SR(\tilde{A}, \tilde{b})\) has now two solutions \(\tilde{x}^a_1 = (\varepsilon, 0)\) and \(\tilde{x}^b_1 = (0, \varepsilon)\). Here, we observe a bifurcation of the minimum \(\bar{x}\) of the original problem \(SR(A, b)\) into two minima \(\tilde{x}^a_1\) and \(\tilde{x}^b_1\) of the perturbed problem \(SR(\tilde{A}, \tilde{b})\). Let us explain this bifurcation in terms of M-stationarity. The bifurcation is caused by the degeneracy of \(\bar{x}\) viewed as an M-stationary point of \(SR(A, b)\). Note that ND1 is violated at the M-stationary point \(\bar{x}\) of \(SR(A, b)\). More interestingly, there is another M-stationary point \(\tilde{x}^2 = (0, 0)\) of the perturbed problem. In fact, due to \(\|\tilde{x}^2\|_0 = 0\) and the validity of ND1, \(\tilde{x}^2\) is a nondegenerate M-stationary point of \(SR(\tilde{A}, \tilde{b})\). For the latter we have

\[
\|\tilde{x}^2\|_0 = s - 1,
\]

meaning that \(\tilde{x}^2\) is a saddle point which connects two nondegenerate minimizers \(\tilde{x}^a_1\) and \(\tilde{x}^b_1\) of \(SR(\tilde{A}, \tilde{b})\). Overall, we conclude that the degenerate minimum \(\bar{x}\) of the original problem \(SR(A, b)\) is not strongly stable. Moreover, it bifurcates into two nondegenerate minima \(\tilde{x}^a_1\) and \(\tilde{x}^b_1\), as well as leads to one nondegenerate saddle point \(\tilde{x}^2\) of the perturbed problem \(SR(\tilde{A}, \tilde{b})\). It is not hard to see that every of the M-stationary points \(\tilde{x}^a_1, \tilde{x}^b_1,\) and \(\tilde{x}^2\) are strongly stable for \(SR(\tilde{A}, \tilde{b})\). \(\square\)

It turns out that Example[4] is typical in the context of sparse recovery. Namely, strong stability and nondegeneracy are equivalent properties of M-stationary points.

**Theorem 3 (Characterization of strong stability)** An M-stationary point \(\bar{x}\) of \(SR(A, b)\) is strongly stable if and only if it is nondegenerate.

**Proof:**

We start with the necessity part. Let \(\bar{x} \in \mathbb{R}^{n,s}\) be a nondegenerate M-stationary point of \(SR(A, b)\). For any \((\tilde{A}, \tilde{b})\) chosen sufficiently close to \((A, b)\), we show that there exists a unique M-stationary point \(\bar{x} \in \mathbb{R}^{n,s}\) of \(SR(\tilde{A}, \tilde{b})\) in a neighborhood of \(\bar{x}\). First, for all \(\tilde{x} \in \mathbb{R}^{n,s}\) being sufficiently close to \(\bar{x}\) we have by continuity arguments that

\[
I_1(\tilde{x}) \supset I_1(\bar{x}).
\]

We claim that if \(\tilde{x}\) is additionally an M-stationary point of \(SR(\tilde{A}, \tilde{b})\) then actually the equality holds above, i.e.

\[
I_1(\tilde{x}) = I_1(\bar{x}).
\]

To see this we consider the following cases:

(i) The sparsity constraint is active, i.e. \(\|\tilde{x}\|_0 = s\). Then, due to \(\|\tilde{x}\|_0 \leq s\), we have trivially \(I_1(\tilde{x}) = I_1(\bar{x})\).
(ii) The sparsity constraint is not active, i.e. \( \|\bar{x}\|_0 < s \). We assume in contrary that there exists \( \bar{i} \in I_1(\bar{x}) \setminus I_1(x) \). By having \( \bar{i} \in I_0(x) \) and recalling ND1 for \( x \), we obtain

\[
A_1^T A_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - A_1^T b = 0.
\]

By continuity, it follows that

\[
\bar{A}_1^T \bar{A}_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - \bar{A}_1^T \bar{b} \neq 0
\]

for any \((\bar{A}, \bar{b})\) from a sufficiently small neighborhood of \((A, b)\). Due to \( \bar{i} \in I_1(\bar{x}) \), the latter contradicts the fact that \( \bar{x} \) is M-stationary for \( \text{SR}(\bar{A}, \bar{b}) \).

Further, for \( \bar{x} \in \mathbb{R}^{n,s} \) to be M-stationary for \( \text{SR}(\bar{A}, \bar{b}) \) the following holds, see Definition II

\[
\bar{A}_1^T \bar{A}_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - \bar{A}_1^T \bar{b} = 0.
\]

Since \( I_1(\bar{x}) = I_1(\bar{x}) \), we get locally:

\[
\bar{A}_1^T \bar{A}_{I_1(\bar{x})} \bar{x}_{I_1(\bar{x})} - \bar{A}_1^T \bar{b} = 0.
\]

Moreover, by continuity and ND2 for \( \bar{x} \), the matrix \( \bar{A}_{I_1(\bar{x})} \) is of full rank, i.e. \( \text{rank}(\bar{A}_{I_1(\bar{x})}) = \|\bar{x}\|_0 \).

Consequently, the unique M-stationary point of \( \text{SR}(\bar{A}, \bar{b}) \) in a neighborhood of \( \bar{x} \) is

\[
\bar{x}_{I_1(\bar{x})} = (\bar{A}_1^T \bar{A}_{I_1(\bar{x})})^{-1} \bar{A}_1^T \bar{b} \quad \text{and} \quad \bar{x}_{I_0(\bar{x})} = 0,
\]

which depends continuously on \((\bar{A}, \bar{b})\).

We proceed with the sufficiency part. Let \( \bar{x} \in \mathbb{R}^{n,s} \) be a strongly stable M-stationary point of \( \text{SR}(A, b) \). We show by contradiction that \( \bar{x} \) is also nondegenerate.

First, let us assume that ND2 is violated for \( \bar{x} \), hence, the matrix \( A_{I_1(\bar{x})} \) is not of full rank, i.e. \( \text{rank}(A_{I_1(\bar{x})}) < \|\bar{x}\|_0 \). We consider the following system of linear equations with respect to \( x \):

\[
A_{I_1(\bar{x})}^T A_{I_1(\bar{x})} x_{I_1(\bar{x})} - A_{I_1(\bar{x})}^T b = 0 \quad \text{and} \quad x_{I_0(\bar{x})} = 0.
\]

(11)

Note that \( \bar{x} \) solves (11) as an M-stationary point of \( \text{SR}(A, b) \). Since the matrix \( A_{I_1(\bar{x})}^T A_{I_1(\bar{x})} \) is singular, the solution set of (11) is a linear subspace of dimension \( \|\bar{x}\|_0 - \text{rank}(A_{I_1(\bar{x})}) > 0 \). Any solution \( x \) of (11) is feasible for \( \text{SR}(A, b) \), since \( \|x\|_0 \leq \|\bar{x}\|_0 \leq s \). Moreover, within a sufficiently small neighborhood of \( \bar{x} \) it holds \( x_i \neq 0 \) for all \( i \in I_1(\bar{x}) \), hence, \( I_1(x) = I_1(\bar{x}) \). Altogether, the solutions of (11) around \( \bar{x} \) are M-stationary for \( \text{SR}(A, b) \). Thus, \( \bar{x} \) is not isolated as an M-stationary point of \( \text{SR}(A, b) \) and, therefore, cannot be strongly stable, a contradiction.

Second, we assume that ND1 is violated for \( \bar{x} \), but ND2 is fulfilled. Then, we have \( \|\bar{x}\|_0 < s \) and there exists \( i \in I_0(\bar{x}) \) with

\[
A_{I_1(\bar{x})}^T A_{I_1(\bar{x})} x_{I_1(\bar{x})} - A_{I_1(\bar{x})}^T b = 0.
\]

(12)
As an auxiliary claim, we show that the matrix \( A_{I_1(x)} \cup \{ i \} \) is of full rank, i.e.

\[
\text{rank} \left( A_{I_1(x)} \cup \{ i \} \right) = \| x \|_0 + 1. \tag{13}
\]

Let us assume for a moment that (13) is not fulfilled. We come to a contradiction by considering the following system of linear equations with respect to \( x \):

\[
A_{I_1(x)}^T \cdot A_{I_1(x)} \cdot x_{I_1(x)} - A_{I_1(x)}^T \cdot A_{I_1(x)} \cdot b = 0 \quad \text{and} \quad x_{I_0(x) \setminus \{ i \}} = 0. \tag{14}
\]

Note that \( x \) solves (14) as an M-stationary point of \( \text{SR}(A, b) \) and due to (12). Since we suppose that \( \text{rank} \left( A_{I_1(x)} \cup \{ i \} \right) < \| x \|_0 + 1 \), the matrix \( A_{I_1(x)}^T \cdot A_{I_1(x)} \) is singular, and the solution set of (14) is a linear subspace of dimension

\[
\| x \|_0 + 1 - \text{rank} \left( A_{I_1(x)} \cup \{ i \} \right) > 0.
\]

Any solution \( x \) of (14) is feasible for \( \text{SR}(A, b) \), since \( \| x \|_0 \leq \| x \|_0 + 1 \leq s \). Moreover, within a sufficiently small neighborhood of \( x \) it holds \( x_i \neq 0 \) for all \( i \in I_1(x) \). Let us assume for a moment that for all solutions \( x \) of (14) within an arbitrarily small neighborhood of \( x \), it holds \( x_i = 0 \). Then, \( x \) solves (14). Due to ND2, the matrix \( A_{I_1(x)}^T \cdot A_{I_1(x)} \) is nonsingular, which implies that \( x = x \). This would mean that (14) is uniquely solvable in a neighborhood of \( x \), a contradiction to the singularity of \( A_{I_1(x)}^T \cdot A_{I_1(x)} \). Altogether, in any sufficiently small neighborhood of \( x \) there exist solutions \( x \) of (14) such that \( I_1(x) = I_1(x) \cup \{ i \} \). Hence, those points \( x \) are M-stationary for \( \text{SR}(A, b) \). Thus, \( x \) is not isolated as an M-stationary point of \( \text{SR}(A, b) \) and, therefore, cannot be strongly stable. By contradiction, we just conclude that (13) is fulfilled.

After this preliminary considerations, we construct a perturbation \( (\tilde{A}, \tilde{b}) \) arbitrarily close to \( (A, b) \) such that \( \text{SR}(\tilde{A}, \tilde{b}) \) has at least two M-stationary points \( \tilde{x}^1 \neq \tilde{x}^2 \) in a proximity to \( x \). We may assume that for all \( \tilde{b} \) close to \( b \) it holds:

\[
\tilde{b} \neq A_{I_1(\tilde{x})} \tilde{x}_{I_1(\tilde{x})}.
\]

In fact, if \( b = A_{I_1(\tilde{x})} \tilde{x}_{I_1(\tilde{x})} \) then any \( \tilde{b} \neq b \) suffices. Hence, there exists a normalized vector \( c \in \mathbb{R}^m \) such that for all \( \tilde{b} \) close, but not necessarily equal to \( b \) it holds:

\[
c^T \left( A_{I_1(\tilde{x})} \tilde{x}_{I_1(\tilde{x})} - \tilde{b} \right) \neq 0.
\]

We define the following family of perturbations \( \tilde{A} \in \mathbb{R}^{m,n} \) depending on the parameter \( t \in \mathbb{R}^2 \):

\[
\tilde{A}(i) = A(i) + tc, \quad \tilde{A}(i) = A(i) + c.
\]

Note that \( \tilde{A} \) differs from \( A \) only with respect to the \( i \)-th column. Moreover, for \( t = 0 \) both matrices \( \tilde{A} \) and \( A \) coincide.

1) Construction of \( \tilde{x}^1 \). We consider the following system of linear equations with respect to \( x \):

\[
\tilde{A}_{I_1(\tilde{x})}^T \tilde{A}_{I_1(\tilde{x})} \cdot x_{I_1(\tilde{x})} - \tilde{A}_{I_1(\tilde{x})}^T \tilde{A}_{I_1(\tilde{x})} \cdot \tilde{b} = 0 \quad \text{and} \quad x_{I_0(\tilde{x})} = 0. \tag{15}
\]
Since $\bar{t} \notin I_1 (\bar{x})$, the system (15) is equivalent to
\[
A_{I_1(\bar{x})}^T A_{I_1(\bar{x})} x_{I_1(\bar{x})} - A_{I_1(\bar{x})}^T \bar{b} = 0 \quad \text{and} \quad x_{I_0(\bar{x})} = 0.
\]
Due to ND2, the unique solution of (15) is then
\[
\bar{x}_{I_1(\bar{x})}^1 = (A_{I_1(\bar{x})}^T A_{I_1(\bar{x})})^{-1} A_{I_1(\bar{x})}^T \bar{b} \quad \text{and} \quad \bar{x}_{I_0(\bar{x})}^1 = 0.
\]
Note that $\bar{x}^1$ is independent of $t$. We see that $\bar{x}^1$ is feasible for SR $(A, \bar{b})$, since $\|\bar{x}^1\|_0 \leq \|\bar{x}\|_0 < s$.

Since $\bar{x}^1$ depends continuously on $\bar{b}$, the point $\bar{x}^1$ falls into a sufficiently small neighborhood of $\bar{x}$. Hence, it holds $\bar{x}^1_t \neq 0$ for all $i \in I_1 (\bar{x})$ or, equivalently, $I_1 (\bar{x}^1) = I_1 (\bar{x})$. Altogether, $\bar{x}^1$ is M-stationary for SR $(A, \bar{b})$.

2) Construction of $\bar{x}^2$. We consider the following system of linear equations with respect to $x$:
\[
\bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{A}_{I_1(\bar{x})\cup\{i\}} x_{I_1(\bar{x})\cup\{i\}} - \bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{b} = 0 \quad \text{and} \quad x_{I_0(\bar{x})\setminus\{i\}} = 0.
\]
As we showed in (13), the matrix $A_{I_1(\bar{x})\cup\{i\}}$ is of full rank. Due to continuity reasons, the matrix $\bar{A}_{I_1(\bar{x})\cup\{i\}}$ is also of full rank at least for $t$ sufficiently close to zero. The unique solution of (16) is then
\[
\bar{x}_{I_1(\bar{x})\cup\{i\}}^2 = (\bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{A}_{I_1(\bar{x})\cup\{i\}})^{-1} \bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{b} \quad \text{and} \quad \bar{x}_{I_0(\bar{x})\setminus\{i\}}^2 = 0.
\]
We see that $\bar{x}^2$ is feasible for SR $(A, \bar{b})$, since $\|\bar{x}^2\|_0 \leq \|\bar{x}\|_0 + 1 \leq s$. Now, we use the fact that $\bar{x}$ is the unique solution of (14), i.e.
\[
\bar{x}_{I_1(\bar{x})\cup\{i\}}^1 = (\bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{A}_{I_1(\bar{x})\cup\{i\}})^{-1} \bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{b} \quad \text{and} \quad \bar{x}_{I_0(\bar{x})\setminus\{i\}}^1 = 0.
\]

As consequence, $\bar{x}^2$ falls into an arbitrarily small neighborhood of $\bar{x}$ as soon as $(\bar{A}, \bar{b})$ is sufficiently close to $(A, b)$. This implies that $\bar{x}^2_t^i \neq 0$ for all $i \in I_1 (\bar{x})$. In order to show that $\bar{x}^2_t^i \neq 0$, we compute the derivative of $\bar{x}^2_{I_1(\bar{x})\cup\{i\}}$ at $t = 0$ by using the implicit function theorem for the system of equations (10). For its left-hand side we set
\[
G (x_{I_1(\bar{x})\cup\{i\}}, t) = \bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{A}_{I_1(\bar{x})\cup\{i\}} x_{I_1(\bar{x})\cup\{i\}} - \bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{b}.
\]
It is straightforward to see that
\[
D_{x_{I_1(\bar{x})\cup\{i\}}} G (x_{I_1(\bar{x})\cup\{i\}}, 0) = \bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{A}_{I_1(\bar{x})\cup\{i\}}
\]
where \( c^T (A_{I_1(\bar{x})}x_{I_1(\bar{x})} - \bar{b}) \) is the \( i \)-th component of \( D_t G(\bar{x}_{I_1(\bar{x})}, t) \). It follows that

\[
\frac{d\bar{x}_2}{dt} = -\left(D_{\bar{x}_{I_1(\bar{x})}\cup\{i\}} G(\bar{x}_{I_1(\bar{x})}\cup\{i\}, 0)\right)^{-1} D_t G(\bar{x}_{I_1(\bar{x})}\cup\{i\}, t)
\]

\[
= -\left(\bar{A}_{I_1(\bar{x})\cup\{i\}}^T \bar{A}_{I_1(\bar{x})\cup\{i\}}\right)^{-1} c^T (A_{I_1(\bar{x})}x_{I_1(\bar{x})} - \bar{b})
\]

Let us assume for a moment that \( \bar{x}_2 = 0 \) for all \( t \) within a neighborhood of zero. Then, every \( \bar{x}_2 \) solves also (15), hence, \( \bar{x}_2 = \bar{x}_1 \). In this case \( \bar{x}_2 = x_{I_1(\bar{x})\cup\{i\}} \) is constant and its derivative with respect to \( t \) vanishes around zero. Substituting into (17), we obtain

\[
c^T (A_{I_1(\bar{x})}x_{I_1(\bar{x})} - \bar{b}) = 0,
\]

a contradiction to the choice of \( \bar{b} \). We have just shown that in any sufficiently small neighborhood of \( \bar{x} \) there exist solutions \( \bar{x}_2 \) of (15) such that \( I_1(\bar{x}_2) = I_1(\bar{x})\cup\{i\} \). Altogether, \( \bar{x}_2 \) is M-stationary for \( \text{SR}(\bar{A}, \bar{b}) \). We have also shown that \( \bar{x}_1 \neq \bar{x}_2 \), since \( \bar{x}_1 = 0 \) and \( \bar{x}_2 \neq 0 \). By this, the M-stationary point \( \bar{x} \) of \( \text{SR}(A, b) \) is not strongly stable. \( \square \)

We point out that the equivalence of strong stability and nondegeneracy of stationary point is by far not usual in nonsmooth optimization. Exemplarily, let us compare the relation of both notions in the context of mathematical programs with complementarity constraints.

**Example 2 (Complementarity constraints)** Let the following sensing matrix and measurement vector be given:

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.
\]
We consider the corresponding sparse recovery problem with $s = 1$:

$$SR(A, b) : \min_{x_1, x_2} \frac{1}{2}(x_1 + 1)^2 + \frac{1}{2}x_2^2 \quad \text{s.t.} \quad \|x_1, x_2\|_0 \leq 1.$$  

Obviously, $(-1, 0)$ is the nondegenerate minimizer of $SR(A, b)$. However, there exists another $M$-stationary point of $SR(A, b)$, namely, $\bar{x} = (0, 0)$. Due to the violation of $ND1$, $\bar{x}$ is degenerate and, hence, cannot be strongly stable for $SR(A, b)$ in view of Theorem 3. Now, we consider the following mathematical program with complementarity constraints:

$$MPCC(A, b) : \min_{x_1, x_2} f(x_1, x_2) = \frac{1}{2}(x_1 + 1)^2 + \frac{1}{2}x_2^2 \quad \text{s.t.} \quad x_1 \cdot x_2 = 0, \quad x_1, x_2 \geq 0. \quad (18)$$

The objective function $f$ of $MPCC(A, b)$ is the same as of $SR(A, b)$, but the sparsity constraint $\|x_1, x_2\|_0 \leq 1$ is substituted by the complementarity constraint $x_1 \cdot x_2 = 0, x_1, x_2 \geq 0$. We see that $\bar{x} = (0, 0)$ is the unique minimizer of $MPCC(A, b)$. In particular, $\bar{x}$ is a so-called C-stationary point of $MPCC(A, b)$, see e.g. Jongen et al. (2012) for details. In fact, at $\bar{x}$ the derivatives of the objective function with respect to biactive variables are of the same sign:

$$\frac{\partial f}{\partial x_1} (\bar{x}) = 1, \quad \frac{\partial f}{\partial x_2} (\bar{x}) = 0.$$  

However, $\bar{x}$ is degenerate, since one of the above derivatives vanishes. Nevertheless, $\bar{x}$ is a strongly stable C-stationary point of $MPCC(A, b)$. This is due to Corollary 3.1 from Jongen et al. (2012), where the following sufficient condition for the strong stability of C-stationary points is given:

$$\frac{\partial f}{\partial x_2} (\bar{x}) = 0, \quad \frac{\partial^2 f}{\partial x_2^2} (\bar{x}) \cdot \frac{\partial f}{\partial x_1} (\bar{x}) > 0.$$  

Here, the latter is fulfilled due to

$$\frac{\partial^2 f}{\partial x_2^2} (\bar{x}) = 1, \quad \frac{\partial f}{\partial x_1} (\bar{x}) = 1.$$  

We conclude that for mathematical programs with complementarity constraints the equivalence of strong stability and nondegeneracy is not valid as it is the case of sparse recovery. $\square$

5 Global aspects

Let us study the topological properties of $SR$ lower level sets

$$M^a = \{ x \in \mathbb{R}^{n,s} \mid f(x) \leq a \},$$

where $a \in \mathbb{R}$ is varying. For that, we define intermediate sets for $a < b$:

$$M^b_a = \{ x \in \mathbb{R}^{n,s} \mid a \leq f(x) \leq b \}.$$
For the topological concepts used below we refer to Spanier (1966).

In what follows, we mention several consequences of the $s$-regularity of $A$ for the topological properties of the lower level sets.

**Lemma 4 (Lower level sets)** Let $A$ be an $s$-regular matrix. Then, all lower level sets $M^a$ are bounded. Moreover, for all sufficiently large $a \in \mathbb{R}$ they are also connected.

**Proof:**
First, we show that the lower level sets

$$M^a = \left\{ x \in \mathbb{R}^{n,s} \mid \|Ax - b\|_2^2 \leq a \right\}$$

are bounded for any $a \in \mathbb{R}$. We write for the SR feasible set:

$$\mathbb{R}^{n,s} = \bigcup_{S \subset \{1, \ldots, n\}, \ |S| \leq s} X_S,$$

where

$$X_S = \{ x \in \mathbb{R}^n \mid x_S^c = 0 \}.$$

Hence,

$$M^a = \bigcup_{S \subset \{1, \ldots, n\}, \ |S| \leq s} M^a_S,$$

(19)

where

$$M^a_S = \left\{ x \in X_S \mid \|Ax - b\|_2^2 \leq a \right\}.$$

It holds for $x \in X_S$:

$$\|Ax - b\|_2^2 = \|A_S x_S - b\|_2^2.$$

Since $A$ is $s$-regular and $|S| \leq s$, we have rank $(A_S) = |S|$. Hence, the sets

$$M^a_S = \left\{ x \in X_S \mid \|A_S x_S - b\|_2^2 \leq a \right\}$$

are $|S|$-dimensional ellipsoids and, thus, bounded. Therefore, the set $M^a$ is bounded as well, namely, as a finite union of bounded sets $M^a_S$, cf. the representation (19).

Further, it is possible to increase $a \in \mathbb{R}$ in order to guarantee that $M^a$ is also connected. To have this, let us assume that $a \geq \|b\|_2^2$. Hence, we have $0 \in M^a_S$ for all $S \subset \{1, \ldots, n\}$ with $|S| \leq s$. Moreover, the sets $M^a_S$ from the representation (19) are connected as $|S|$-dimensional ellipsoids. As a consequence, the set $M^a$ is connected as a finite union of connected sets $M^a_S$, all of them having nonempty intersection, i.e.

$$0 \in \bigcap_{S \subset \{1, \ldots, n\}, \ |S| \leq s} M^a_S.$$

This concludes the proof. \qed
We show that the lower level sets do not undergo topological changes when passing a non-M-stationary level.

**Theorem 4 (Deformation for SR)** Let $A$ be an $s$-regular matrix and $M_{a}^b$ contain no M-stationary points for SR. Then, $M^a$ is homeomorphic to $M^b$.

**Proof:**
We apply Proposition 3.2 from Part I in Goresky and MacPherson (1988). The latter provides the deformation for general Whitney stratified sets with respect to critical points of proper maps. Note that the SR feasible set admits a Whitney stratification:

$$
\mathbb{R}^{n,s} = \bigcup_{I \subset \{1, \ldots, n\}} \bigcup_{|J| \leq s} Z_{I,J},
$$

where

$$
Z_{I,J} = \{ x \in \mathbb{R}^n \mid x_I = 0, x_J > 0, x_{I \setminus J} < 0 \}.
$$

The notion of criticality used in Goresky and MacPherson (1988) can be stated for SR as follows. A point $\bar{x} \in \mathbb{R}^{n,s}$ is called critical for $f$ on $\mathbb{R}^{n,s}$ if it holds:

$$
Df(\bar{x})_{|T_Z} = 0,
$$

where $Z$ is the stratum of $\mathbb{R}^{n,s}$ which contains $\bar{x}$, and $T_Z$ is the tangent space of $Z$ at $\bar{x}$. By identifying $I = I_1(\bar{x})$ and, hence, $I^c = I_0(\bar{x})$, we see that the concepts of criticality and M-stationarity coincide. It remains to note that, due to Lemma 4, the restriction of $f$ on $\mathbb{R}^{n,s}$ is proper, i.e. $f^{-1}(K) \cap \mathbb{R}^{n,s}$ is compact for any compact set $K \subset \mathbb{R}$.

Now, we turn our attention to the topological changes of lower level sets when passing an M-stationary level. Traditionally, they are described by means of the so-called cell-attachment. We first consider a special case of cell-attachment dealt with already in Lämmel and Shikhman (2019). For that, let $N^\epsilon$ denote the lower level set of a special linear function on $\mathbb{R}^{p,q}$, i.e.

$$
N^\epsilon = \left\{ x \in \mathbb{R}^{p,q} \mid \sum_{i=1}^{p} x_i \leq \epsilon \right\},
$$

where $\epsilon \in \mathbb{R}$, and the integers $q < p$ are nonnegative.

**Lemma 5 (Normal Morse data, Lämmel and Shikhman (2019))** For any $\epsilon > 0$ the set $N^\epsilon$ is homotopy-equivalent to $N^{-\epsilon}$ with $\left(\frac{p}{q}-1\right)$ cells of dimension $q$ attached. The latter cells are the $q$-dimensional simplices from the collection

$$
\{ \text{conv}(e_j, j \in J) \mid J \subset \{1, \ldots, p\}, 1 \in J, |J| = q + 1 \}.
$$

The general case of cell-attachment can be shown by using Lemma 5.
Theorem 5 (Cell-Attachment for SR) Let $A$ be an $s$-regular matrix and $M^b_a$ contain exactly one $M$-stationary point $\bar{x}$ for SR. If $a < f(\bar{x}) < b$, then $M^b_a$ is homotopy-equivalent to $M^a$ with $(n-\|\bar{x}\|_0-1)$ cells of dimension $s-\|\bar{x}\|_0$ attached, namely:

$$\bigcup_{J \subset \{1, \ldots, n-\|\bar{x}\|_0\}} \text{conv}(e_j, j \in J).$$

Proof:
Theorem 4 allows deformations up to an arbitrarily small neighborhood of the $M$-stationary point $\bar{x}$. In such a neighborhood, we may assume without loss of generality that $\bar{x} = 0$ and $f$ has the following form as from Theorem 1:

$$f(x) = f(\bar{x}) + \sum_{i \in I_0(\bar{x})} x_i + \sum_{j \in I_1(\bar{x})} x_j^2,$$

where $x \in \mathbb{R}^{n-\|\bar{x}\|_0, n-\|\bar{x}\|_0}$. In terms of Goresky and MacPherson (1988) the set $\mathbb{R}^{n-\|\bar{x}\|_0, n-\|\bar{x}\|_0} \times \mathbb{R}^{\|\bar{x}\|_0}$ can be interpreted as the product of the tangential part $\mathbb{R}^{\|\bar{x}\|_0}$ and the normal part $\mathbb{R}^{n-\|\bar{x}\|_0, n-\|\bar{x}\|_0}$. The cell-attachment along the tangential part is standard. Analogously to the unconstrained case, one cell of dimension zero has to be attached on $\mathbb{R}^{\|\bar{x}\|_0}$. The cell-attachment along the normal part is more involved. Due to Lemma 5, we need to attach $(n-\|\bar{x}\|_0-1)$ cells on $\mathbb{R}^{n-\|\bar{x}\|_0, n-\|\bar{x}\|_0}$, each of dimension $s-\|\bar{x}\|_0$. Finally, we apply Theorem 3.7 from Part I in Goresky and MacPherson (1988), which says that the local Morse data is the product of tangential and normal Morse data. Hence, the dimensions of the attached cells add together. This provides the assertion. 

Let us present a global interpretation of our results for SR. For that, we consider $M$-stationary points $\bar{x}$ with exactly $s-1$ non-zero entries, i.e.

$$\|\bar{x}\|_0 = s - 1.$$ 

We refer to them as saddle points.

Theorem 6 (Morse relation for SR) Let $A$ be an $s$-regular matrix, and all $M$-stationary points of SR be nondegenerate with pairwise different functional values of the objective function. Then, it holds:

$$(n-s)r_1 \geq r-1,$$

where $r$ is the number of local minimizers, and $r_1$ is the number of saddle points of SR.

Proof:
Let $q_a$ denote the number of connected components of the lower level set $M^a$. We focus on how $q_a$ changes as $a \in \mathbb{R}$ increases. Due to Theorem 4, $q_a$ can change only if passing through a value corresponding to an $M$-stationary point $\bar{x}$, i.e. $a = f(\bar{x})$. In fact, Theorem 4 allows homeomorphic deformations of lower level sets up to an arbitrarily small neighborhood of the $M$-stationary point.
\( \bar{x} \). Then, we have to estimate the difference between \( q_a \) and \( q_{a-\epsilon} \), where \( \epsilon > 0 \) is arbitrarily, but sufficiently small, and \( a = f(\bar{x}) \). This is done by a local argument. We use Theorem 5 which says that \( M^a \) is homotopy-equivalent to \( M^{a-\epsilon} \) with a cell-attachment of

\[
\bigcup_{J \subset \{1, \ldots, n - \|\bar{x}\|_0\}} \text{conv}(e_j, j \in J).
\]

(22)

Let us distinguish the following cases:

1) \( \bar{x} \) is a local minimizer, i.e. \( \|\bar{x}\|_0 = s \). Then, by (22) we attach the cell \( \text{conv}(e_1) \) of dimension zero to \( M^{a-\epsilon} \). Consequently, a new connected component is created, and it holds:

\[
q_a = q_{a-\epsilon} + 1.
\]

2) \( \bar{x} \) is a saddle point, i.e. \( \|\bar{x}\|_0 = s - 1 \). Then, by (22) we attach \( n - s + 1 \) cells of dimension one to \( M^{a-\epsilon} \), namely:

\[
\bigcup_{j = 2, \ldots, n - s + 1} \text{conv}(e_1, e_j)
\]

Consequently, at most \( n - s \) connected components disappear, and it holds:

\[
q_{a-\epsilon} - (n - s) \leq q_a \leq q_{a-\epsilon}.
\]

For illustration we refer to Figure 1.

3) \( \bar{x} \) is an M-stationary point with \( \|\bar{x}\|_0 < s - 1 \). The boundary of the cell-attachment in (22) is

\[
\bigcup_{J \subset \{1, \ldots, n - \|\bar{x}\|_0\}} \partial \text{conv}(e_j, j \in J).
\]

The latter set is connected if \( \|\bar{x}\|_0 < s - 1 \). Consequently, the number of connected components of \( M^a \) remains unchanged, and it holds:

\[
q_a = q_{a-\epsilon}.
\]

Now, we proceed with the global argument. Since the objective function is lower bounded by zero, there exists \( c < 0 \) such that \( M^c \) is empty, thus, \( q_c = 0 \). Due to Lemma 2 the number of M-stationary points is finite. We conclude that there exists a lower level set \( M^d \) which contains all, but finitely many, M-stationary points of SR. Due to Lemma 4 it is possible to increase \( d \in \mathbb{R} \) in order to additionally guarantee that \( M^d \) is connected, i.e. \( q_d = 1 \). Let us now vary the level \( a \) from \( c \) to \( d \) and describe how the number \( q_a \) of connected components of the lower level sets \( M^a \) changes. It follows from the local argument that \( r \) new connected components are created, where \( r \) is the number of local minimizers. Let \( q \) denote the actual number of disappearing connected components if passing the levels corresponding to saddle points, and let \( r_1 \) denote the number of saddle points. The local argument provides that at most \((n - s)r_1\) connected components might
disappear while doing so, i.e. \( q \leq (n-s)r_1 \).

Altogether, we have:
\[
r - (n-s)r_1 \leq r - q = q_d - q_c.
\]

By recalling that \( q_d = 1 \) and \( q_c = 0 \), we get Morse relation (21).

\[ \Box \]

We illustrate Theorem 6 by discussing the perturbed SR from Example 1.

**Example 3 (Saddle point)** Let the following sensing matrix and measurement vector be given:

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

We consider the corresponding sparse recovery problem with \( s = 1 \):

\[
\begin{aligned}
\text{SR:} \quad & \min_{x_1, x_2} \frac{1}{2} (x_1 - 1)^2 + \frac{1}{2} (x_2 - 1)^2 \\
& \text{s. t.} \quad \| (x_1, x_2) \|_0 \leq 1.
\end{aligned}
\]

As we have seen in Example 1, both M-stationary points \((1, 0)\) and \((0, 1)\) are nondegenerate minimizers. Thus, we have \( r = 2 \). Morse relation (21) from Theorem 6 provides:

\[
r_1 \geq 1.
\]

Hence, there should exist a saddle point. In fact, \((0, 0)\) is this nondegenerate saddle point, cf. Example 1. Note that, due to \( r_1 = 1 \), Morse relation (21) holds with equality here.

\[ \Box \]

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