Abstract. We introduce the $\varphi A$-differentiability, the corresponding generalized Cauchy-Riemann equations ($\varphi A$-CREs), the Cauchy-integral theorem, and consider the problem of when a given system of two linear first order partial differential equations results the $\varphi A$-CREs for some function $\varphi$ and a two dimensional algebra $A$. We show that the four dimensional vector fields associated with triangular billiards are $\varphi A$-differentiable.

Keyword: Vector fields, Lorch differentiability, Generalized Cauchy-Riemann equations

MSC[2010]: 37C10, 58C20, 53C22.

Introduction

Consider a linear systems of partial differential equations (PDEs) of the form

$$
\begin{align*}
    a_{111}u_x + a_{121}v_x + a_{131}w_x + a_{112}u_y + a_{122}v_y + a_{132}w_y &= 0 \\
    a_{211}u_x + a_{221}v_x + a_{231}w_x + a_{212}u_y + a_{222}v_y + a_{232}w_y &= 0 \\
    a_{311}u_x + a_{321}v_x + a_{331}w_x + a_{312}u_y + a_{322}v_y + a_{332}w_y &= 0
\end{align*}
$$

(1)

where $a_{ijk}$ are functions of $(x, y, z)$, $u_x = \frac{\partial u}{\partial x}$ and so on. In this paper we define a type of differentiability for which in particular cases the “Generalized Cauchy-Riemann equations” are linear systems of partial differential equations having the form (1). It will be called $\varphi A$-differentiability.

The algebrizability of vector fields and differential equations has been studied in [1], [2], [3], [6], [7], and [8]. The case of algebrizability of systems of two first order partial differential equations is being developed in [5]. In this paper we define the $\varphi A$-algebrizability of vector fields and autonomous ordinary differential equations. The corresponding Cauchy-Riemann equations ($\varphi A$-CREs) are introduced and a version of the Cauchy-integral Theorem is showed. We give examples of generalized systems of Cauchy-Riemann equations for the $\varphi A$-differentiability. The following inverse problem is considered: when a given homogeneous linear system of two first order partial differential equations results the CREs for some function $\varphi$ and a two dimensional
algebra $\mathbb{A}$. We show that two-dimensional quadratic complex vector fields associated with triangular billiards are $\varphi\mathbb{A}$-algebrizable. A triangular billiard has a periodic orbit if and only if the real system of four differential equations associated with a complex system of two differential equations has a periodic orbit, see [1].

The organization of this paper is the following. In Section 1 we recall the definition of an algebra which we denote by $\mathbb{A}$, introduce the $\varphi\mathbb{A}$-differentiability and give some results related to this like the $\varphi\mathbb{A}$-CREs. In Section 2 the problem of when a given first order linear system of two partial differential equations results the $\varphi\mathbb{A}$-CREs for some differentiable function $\varphi$ and an algebra $\mathbb{A}$, is considered. In Section 3 the $\varphi\mathbb{A}$-line integral is defined, a corresponding Cauchy-integral theorem is given, a generalization of the fundamental theorem of calculus is a consequence obtained, and examples are given. Moreover, the $\varphi\mathbb{A}$-differential equations are introduced, an existence and uniqueness of solutions theorem is given, and it is showed that systems associated to triangular billiards are $\varphi\mathbb{A}$-differentiable.

1 $\varphi\mathbb{A}$-differentiability

1.1 Algebras

We call the $\mathbb{R}$-linear space $\mathbb{R}^n$ an algebra $\mathbb{A}$ if it is endowed with a bilinear product $\mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A}$ denoted by $(u, v) \mapsto uv$, which is associative and commutative $u(vw) = (uv)w$ and $uv = vu$ for all $u, v, w \in \mathbb{A}$; furthermore, there exists a unit $e = e_\mathbb{A} \in \mathbb{A}$, which satisfies $eu = ue = u$ for all $u \in \mathbb{A}$, see [10].

In this paper $\mathbb{A}$ denotes an algebra. An element $u \in \mathbb{A}$ is called regular if there exists $u^{-1} \in \mathbb{A}$ called the inverse of $u$ such that $u^{-1}u = uu^{-1} = e$. We also use the notation $e/u$ for $u^{-1}$, where $e$ is the unit of $\mathbb{A}$. If $u \in \mathbb{A}$ is not regular, then $u$ is called singular. $\mathbb{A}^*$ denotes the set of all the regular elements of $\mathbb{A}$. If $u, v \in \mathbb{A}$ and $v$ is regular, the quotient $u/v$ means $uv^{-1}$.

The $\mathbb{A}$ product between the elements of the canonical basis $\{e_1, \cdots, e_n\}$ of $\mathbb{R}^n$ is given by $e_i e_j = \sum_{k=1}^{n} c_{ijk} e_k$ where $c_{ijk} \in \mathbb{R}$ for $i, j, k \in \{1, \cdots, n\}$ are called structure constants of $\mathbb{A}$. The first fundamental representation of $\mathbb{A}$ is the injective linear homomorphism $R : \mathbb{A} \rightarrow M(n, \mathbb{R})$ defined by $R : e_i \mapsto R_i$, where $R_i$ is the matrix with $[R_i]_{jk} = c_{ikj}$, for $i = 1, \cdots, n$.

1.2 Definition of $\varphi\mathbb{A}$-differentiability

We use notation $u = (u_1, \cdots, u_k)$. The usual differential of a function $f$ will be denoted by $df$.

Let $\mathbb{A}$ be the linear space $\mathbb{R}^n$ endowed with an algebra product. Consider two differentiable function in the usual sense $f, \varphi : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ defined in an an open set $U$. We say $f$ is $\varphi\mathbb{A}$-differentiable on $U$ if there exists a function $f'_\varphi : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ that we call $\varphi\mathbb{A}$-derivative such that for all $u \in U$

$$
\lim_{\xi \rightarrow 0, \xi \in \mathbb{R}^k} \frac{f(u + \xi) - f(u) - f'_\varphi(u) d\varphi(u) \xi}{||\xi||} = 0,
$$

2
where \( f'_\varphi(u) d\varphi_u(\xi) \) denotes the \( \mathbb{A} \)-product of \( f'_\varphi(u) \) and \( d\varphi_u(\xi) \). That is, \( f \) is \( \varphi \mathbb{A} \)-differentiable if \( df_u(\xi) = f'_\varphi(u) d\varphi_u(\xi) \) for all \( \xi \in \mathbb{R}^k \).

If \( k = n \) and \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) is the identity transformation \( \varphi(x) = x \), the \( \varphi \mathbb{A} \)-differentiability will be called \( \mathbb{A} \)-differentiability and the \( \mathbb{A} \)-derivative of \( f \) will be denoted by \( f' \). This last differentiability is known as Lorch differenciability, see [9]. Differentiability related to commutative and noncommutative algebras is considered in [4].

### 1.3 Algebrizability of planar vector fields

The algebrizability (\( \mathbb{A} \)-differentiability for some algebra) of planar vector fields \( F = (u, v) \) is characterized in [1] and [6]. A vector field \( F \) is algebrizable on an open set \( \Omega \subset \mathbb{R}^2 \) if and only if \( F \) satisfies at least one of the following homogeneous first order systems of PDEs

a) \( u_x + \beta v_x - v_y = 0, \ u_y - \alpha v_x = 0, \)

b) \( u_x + \gamma u_y - v_y = 0, \ v_x - \delta u_y = 0, \) and

c) \( u_y = 0, \ v_x = 0. \)

For case a) we take \( \mathbb{A} = \mathbb{A}^2_1(\alpha, \beta) \), for b) \( \mathbb{A} = \mathbb{A}^2_2(\gamma, \delta) \), and for c) \( \mathbb{A} = \mathbb{A}^2_{1,2} \). These systems are called Cauchy–Riemann equations associated with \( \mathbb{A} \) (\( \mathbb{A} \)-CREs), where \( \alpha, \beta, \gamma, \delta \in \mathbb{R} \) are parameters, see [12]. For \( \mathbb{A}^2_1(\alpha, \beta) \) the product is

\[
\begin{array}{c|cc}
\cdot & e_1 & e_2 \\

e_1 & e_1 & e_2 \\

e_2 & \alpha e_1 + \beta e_2 & e_2 \\
\end{array}
\]

(2)

hence the unit is \( e = e_1 \). The structure constants are

\[
\begin{align*}
c_{111} &= 1, & c_{112} &= 0, & c_{121} &= 0, & c_{122} &= 1, \\
c_{211} &= 0, & c_{212} &= 1, & c_{221} &= \alpha, & c_{222} &= \beta,
\end{align*}
\]

(3)

or equivalently, its first fundamental representation is

\[
R(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R(e_2) = \begin{pmatrix} 0 & \alpha \\ 1 & \beta \end{pmatrix}.
\]

(4)

For \( \mathbb{A}^2_2(\gamma, \delta) \) the product is

\[
\begin{array}{c|cc}
\cdot & e_1 & e_2 \\

e_1 & \gamma e_1 + \delta e_2 & e_1 \\

e_1 & e_1 & e_2 \\
\end{array}
\]

(5)

hence the unit is \( e = e_2 \). The structure constants are

\[
\begin{align*}
c_{111} &= \gamma, & c_{112} &= \delta, & c_{121} &= 1, & c_{122} &= 0, \\
c_{211} &= 1, & c_{212} &= 0, & c_{221} &= 0, & c_{222} &= 1,
\end{align*}
\]

(6)
or equivalently, its first fundamental representation is
\[ R(e_1) = \begin{pmatrix} \gamma & 1 \\ \delta & 0 \end{pmatrix}, \quad R(e_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \] (7)

For \( A_{1,2} \) the product is
\[ e_1 e_2 e_1 = e_1 e_2 e_2. \] (8)

hence the unit is \( e = e_1 + e_2 \). The structure constants are
\[ c_{111} = 1, \quad c_{112} = 0, \quad c_{121} = 0, \quad c_{122} = 0, \]
\[ c_{211} = 0, \quad c_{212} = 0, \quad c_{221} = 0, \quad c_{222} = 1, \] (9)
or equivalently, its first fundamental representation is
\[ R(e_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R(e_2) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \] (10)

1.4 On \( \varphi A \)-differentiability

The \( \varphi A \)-derivative \( f'(u) \) is unique unless \( d\varphi_u(\mathbb{R}^k) \) is contained in the singular set of \( A \). The function \( \varphi : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n \) is \( \varphi A \)-differentiable and \( \varphi'(u) = e \) for all \( u \in U \) where \( e \in A \) is the unit. Also, the \( A \)-combinations (linear) and \( A \)-products of \( \varphi A \)-differentiable functions are \( \varphi A \)-differentiable functions and they satisfy the usual rules of differentiation. In the same way if \( f \) is \( \varphi A \)-differentiable and has image in the regular set of \( A \), then the function \( \frac{f}{n} \) is \( \varphi A \)-differentiable for \( n \in \{1, 2, \cdots \} \), and
\[ \left( \frac{e}{f^n} \right)'_{\varphi} = -n \frac{f'_n}{f^{n+1}}. \] (11)

A \( \varphi A \)-polynomial is an expression having the form
\[ c_0 + c_1 \varphi(u) + \cdots + c_m(\varphi(u))^m \]
where \( c_0, c_1, \cdots, c_m \in A \) are constants and variable \( u \) represent the variable in \( \mathbb{R}^k \). We call \( \varphi A \)-polynomial function to a function \( p : \mathbb{R}^k \rightarrow \mathbb{R}^n \) defined by a \( \varphi A \)-polynomial. A \( \varphi A \)-rational function is defined by a quotient of two \( \varphi A \)-polynomials. Then, \( \varphi A \)-polynomial functions and \( \varphi A \)-rational functions are \( \varphi A \)-differentiable and the usual rules of differentiation are satisfied for the \( \varphi A \)-derivative.

In general, the rule of chain does not have sense since \( \varphi \) is \( \varphi A \)-differentiable however the composition \( \varphi \circ \varphi \) only is defined when \( k = n \). Even in this case the rule of the chain can not be verified. Suppose that \( \varphi \) is a linear isomorphism and that the rule of chain is satisfied. Thus the Jacobian matrix of \( \varphi \circ \varphi \) satisfies
\[ J(\varphi \circ \varphi) = MM = R((\varphi \circ \varphi)'_\varphi)M, \]
where $M$ is the matrix associated with $\varphi$ respect to the canonical basis of $\mathbb{R}^n$ and $R$ is the first fundamental representation of $A$. Then

$$J(\varphi \circ \varphi)M^{-1} = M \in R(A).$$

(12)

Therefore, if $M$ has determinant $\det(M) \neq 0$ and $M \notin R(A)$, the rule of chain is not valid for the $\varphi A$-differentiability. By (12) $M \in R(A)$.

We have the following first version of the rule of chain.

**Lemma 1.1** If $g : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is $A$-differentiable with $A$-derivative $g'$, $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is $\varphi A$-differentiable, and $f(U) \subset \Omega$, then $g \circ f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is a $\varphi A$-differentiable function with $\varphi A$-derivative

$$(g \circ f)'_\varphi = (g' \circ f)f'_\varphi.$$

**Proof.** The function $g \circ f$ is differentiable in the usual sense and

$$d(g \circ f)u(\xi) = dg_{f(u)}df_u(\xi) = g'(f(u))f'_\varphi(u)d\varphi_u(\xi). \quad \square$$

Lemma 1.1 has the following converse: each $\varphi A$-differentiable function $f$ can be expressed as $g \circ \varphi$ where $g$ is an $A$-algebrizable vector field, as we see in the following lemma.

**Lemma 1.2** If $\varphi : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism defined on an open set $U$ and $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ a is $\varphi A$-differentiable on $U$, then there exists an $A$-differentiable vector field $g$ such that $f(u) = g \circ \varphi(u)$ for all $u \in U$, and $g'(\varphi(u)) = f'_\varphi(u)$.

**Proof.** Define $g = f \circ \varphi^{-1}$, thus

$$dg_{\varphi(u)} = df_{\varphi^{-1}}d\varphi^{-1}_u = f'(u)d\varphi_u d\varphi^{-1}_u = f'_\varphi(u).$$

This means that $g$ is $A$-differentiable at $\varphi(u)$ and its $A$-derivative is $g'(\varphi(u)) = f'_\varphi(u). \quad \square$

We have the following proposition.

**Proposition 1.1** Let $\varphi : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism defined on an open set $U$. The following three statements are equivalent

a) $f : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is $\varphi A$-differentiable on $U$.

b) $g = f \circ \varphi^{-1}$ is $A$-differentiable.

c) $f$ is differentiable in the usual sense on $U$ and $Jf_u(J\varphi_u)^{-1} \in R(A)$ for all $u \in U$.

**Proof.** Suppose a), by Lemma 1.2 we have b).

Suppose b), then $g = f \circ \varphi^{-1}$ is $A$-differentiable. Since $\varphi$ is a diffeomorphism $f = g \circ \varphi$ is differentiable in the usual sense, the rule of the chain gives $dg_{\varphi(u)} = df_u \circ d\varphi^{-1}_u$, and $Jg_{\varphi(u)} = Jf_u J\varphi_u^{-1} \in R(A)$. That is, b) implies c).

Suppose c). Since $f$ is differentiable in the usual sense $Jg_{\varphi(u)} = Jf_u J\varphi_u^{-1} \in R(A)$ implies $Jf_u = Jg_{\varphi(u)}J\varphi_u$. That is, $df_u = dg_{\varphi(u)}d\varphi_u$. Thus, $f$ is $\varphi A$-differentiable. \(\square\)
Corollary 1.1  Let $f(x, y) = (u(x, y), v(x, y))$ be a vector field for which there exists a diffeomorphism $\phi(s, t) = (x(s, t), y(s, t))$ such that some of the following conditions is satisfied:

a) There exist constants $\alpha$ and $\beta$ such that
\[
uxx + uyy + \beta(vxx + vyy) - (vx + vy) = 0,
\]
\[
ux + uyt - \alpha(vx + vy) = 0.
\]

b) There exist constants $\gamma$ and $\delta$ such that
\[
uxx + uyy + \gamma(ux + uyt) - (vx + vy) = 0,
\]
\[
vxx + vyy - \delta(ux + uyt) = 0.
\]

c) $ux + uyt = 0$ and $uxx + vyy = 0$.

In case a) we take $A = A^2_1(\alpha, \beta)$, in b) $A = A^2_2(\gamma, \delta)$, and in c) $A = A^2_{1,2}$. If $\phi = \phi^{-1}$, then $f$ is $\phi A$-differentiable.

Proof. In the three cases the systems of partial differential equations are the generalized Cauchy-Riemann equations given in Section 1.3 for $g = f \circ \phi^{-1}$, then $g$ is $A$-differentiable. Thus, by Proposition 1.1 $f$ is $\phi A$-differentiable. □

We also have the following second version of the rule of chain.

Lemma 1.3  If $\phi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is differentiable on an open set $U$, $g : V \subset \mathbb{R}^l \to \mathbb{R}^k$ is differentiable on an open set $V$ with $g(V) \subset U$, and $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is $\phi A$-differentiable on $U$, then $h = f \circ g$ is $\phi A$-differentiable on $V$ for $\phi = \phi \circ g$, and $h'_{\phi}(v) = f'_{\phi}(g(v))$

Proof. We have
\[
dh_v = d(f \circ g)_v = df_{g(v)} dg_v = f'_{\phi}(g(v)) d\phi_{g(v)} dg_v = f'_{\phi}(g(v)) d\phi_v.
\]
Thus, $h'_{\phi}(v) = f'_{\phi}(g(v))$. □

1.5  Cauchy-Riemann equations for the $\phi A$-differentiability

The canonical basis of $\mathbb{R}^k$ and $\mathbb{R}^n$ will be denoted by $\{e_1, \cdots, e_k\}$ and $\{e_1, \cdots, e_n\}$, respectively, according to the context of the uses it will be determined if $e_i$ belongs to $\mathbb{R}^k$ or to $\mathbb{R}^n$. The directional derivatives of a function $f$ with respect to a direction $u \in \mathbb{R}^k$ (a direction $u$ is a vector with $\|u\| = 1$) is denoted by $f_u = (f_{1u}, \cdots, f_{nu})$, and the directional derivative of $f$ with respect to $e_i$ by
\[
f_{ei} = f_{1i} + \cdots + f_{ni} e_i.
\]

The Cauchy-Riemann equations for $(\phi, A)$ ($\phi A$-CREs) is the linear system of $n(k - 1)!$ PDEs obtained from
\[
d\phi(e_j)f_{ui} = d\phi(e_i)f_{uj},
\]
(13)
for $i, j \in \{1, \ldots, k\}$. For $i = 1, \ldots, k$ suppose $\varphi = (\varphi_1, \ldots, \varphi_n)$, then

$$d\varphi(e_i) = \varphi_{u_i} = \sum_{l=1}^{n} \varphi_{lu_i} e_l. \tag{14}$$

In the following theorem the $\varphi_A$-CREs are given.

**Theorem 1.1** Let $f = (f_1, \ldots, f_n)$ be an $\varphi_A$-differentiable function. Thus, the $\varphi_A$-CREs are given by

$$\sum_{m=1}^{n} \sum_{l=1}^{n} (f_{mu_i} \varphi_{lu_j} - f_{mu_j} \varphi_{lu_i}) C_{lmq} = 0 \tag{15}$$

for $1 \leq i < j \leq k$ and $q = 1, \ldots, n$, which is a system of $n(k - 1)!$ partial differential equations.

**Proof.** The equalities (13) and (14) give

$$\sum_{q=1}^{n} \left( \sum_{m=1}^{n} \sum_{l=1}^{n} (f_{mu_i} \varphi_{lu_j} - f_{mu_j} \varphi_{lu_i}) C_{lmq} \right) e_q = 0. \quad \Box$$

The directional derivatives of $\varphi_A$-differentiable functions are given in the following lemma.

**Lemma 1.4** If $f$ is $\varphi_A$-differentiable, for each direction $u \in \mathbb{R}^k$ we have

$$f_u = f'_\varphi d\varphi(u). \tag{16}$$

**Proof.** The proof is obtained directly from the $\varphi_A$-differentiability of $f$. \Box

The $\varphi_A$-differentiability implies the $\varphi_A$-CREs, as we see in the following proposition.

**Proposition 1.2** Let $f : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a differentiable function in the usual sense on an open set $U$, and $k \in \{2, \ldots, n\}$. Thus, if $f$ is $\varphi_A$-differentiable, then $d\varphi(e_j) f_{u_i} = d\varphi(e_i) f_{u_j}$. That is, the components of $f$ satisfy the $\varphi_A$-CREs.

**Proof.** By using (16) we have $f_{u_i} = f'_\varphi d\varphi(e_i)$ and $f_{u_j} = f'_\varphi d\varphi(e_j)$. Then

$$d\varphi(e_j) f_{u_i} = d\varphi(e_j) f'_\varphi d\varphi(e_i) = d\varphi(e_i) f'_\varphi d\varphi(e_j) = d\varphi(e_i) f_{u_j}. \quad \Box$$

We say $\varphi$ has an $A$-regular direction $\xi$ if $\xi : U \rightarrow S^1$ is a function $u \mapsto \xi_u$ such that $d\varphi_u(\xi_u)$ is a regular element of $A$ for all $u \in U$, where $S^1 \subset \mathbb{R}^k$ denotes the unit sphere centered at the origin. If $\varphi$ has an $A$-regular direction, Proposition 1.2 has a converse, as we see in the following theorem.

**Theorem 1.2** Let $f : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a differentiable function in the usual sense on an open set $U$, and $k \in \{2, \ldots, n\}$. Suppose that $\varphi$ has regular directions on $U$. Thus, if the components of $f$ satisfies the $\varphi_A$-CREs, then $f$ is $\varphi_A$-differentiable.
Proof. Let $\xi$ be a regular direction of $\varphi$. Since the components of $f$ satisfy the $\varphi\mathbb{A}$-CREs we have that $d\varphi(e_j)f_{u_i} = d\varphi(e_i)f_{u_j}$ for $1 \leq i, j \leq k$. Thus,

$$
d\varphi(u)f_{u_i} = f_{u_i}\sum_{j=1}^{n} u_j d\varphi(e_j) = \sum_{j=1}^{n} u_j d\varphi(e_j)f_{u_i}
$$

$$
= \sum_{j=1}^{n} u_j d\varphi(e_i)f_{u_j} = \sum_{j=1}^{n} u_j f_{u_j} d\varphi(e_i)
$$

$$
= f_{u_i}d\varphi(e_i).
$$

Then, $f_{u_i} = \frac{f_{e_i}}{d\varphi(\xi)} d\varphi(e_i)$. We take $g_{\varphi} = \frac{f_{e_i}}{d\varphi(\xi)}$. By proof of Theorem 1.2 we have that $df(x) = \sum_{i=1}^{k} x_i f_{u_i}$ and $g_{\varphi}d\varphi(x) = \sum_{i=1}^{k} x_i g_{\varphi}d\varphi(e_i)$. Under these conditions we have that $df(x) = g_{\varphi}d\varphi(x)$ for all $x \in \mathbb{R}^k$. That is, $f$ is $\varphi\mathbb{A}$-differentiable and $f_{\varphi} = g_{\varphi}$. □

### 1.6 Examples for the complex field

In the following examples $\mathbb{A}$ is the complex filed $\mathbb{C}$.

We first consider $\varphi(x, y) = (y, x)$.

**Example 1.1** Let $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\varphi(x, y) = (y, x)$. The CREs are given by

$$
\varphi(e_2)(u, v)_x = \varphi(e_1)(u, v)_y.
$$

Then

$$
e_1(u_x, v_x) = e_2(u_y, v_y) = (-v_y, u_y),
$$

from which we obtain the $\varphi\mathbb{A}$-CREs for the $\varphi\mathbb{A}$-differentiability

$$
u_x = -v_y, \quad v_x = u_y.
$$

The function $f(x, y) = (y^2 - x^2, 2xy)$ satisfies $f(x, y) = (\varphi(x, y))^2$. In this case we have $u(x, y) = y^2 - x^2$ and $v(x, y) = 2xy$, and they satisfy the $\varphi\mathbb{A}$-CREs.

Now, we consider $\varphi(x, y) = (y, 0)$.

**Example 1.2** Let $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ be given by $\varphi(x, y) = (y, 0)$. The CREs are given by

$$
\varphi(e_2)(u, v)_x = \varphi(e_1)(u, v)_y.
$$

Then $e_1(u_x, v_x) = (0, 0)$, from which we obtain the $\varphi\mathbb{A}$-CREs for the $\varphi\mathbb{A}$-differentiability

$$
u_x = 0, \quad v_x = 0.
$$

Thus, the $\varphi\mathbb{A}$-differentiable functions $f(x, y)$ are differentiable functions depending only on $y$. 
Now, we consider \( \varphi(x, y) = (y, x + y) \).

**Example 1.3** Let \( \varphi : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by \( \varphi(x, y) = (y, x + y) \). The CREs are given by

\[
\varphi(e_2)(u, v)_x = \varphi(e_1)(u, v)_y.
\]

Then

\[
(e_1 + e_2)(u_x, v_x) = e_2(u_y, v_y),
\]

from which we obtain the \( \varphi_\mathbb{A} \)-CREs for the \( \varphi_\mathbb{A} \)-differentiability

\[
v_y = v_x - u_x, \quad u_y = u_x + v_x.
\]

The function \( f(x, y) = (-x^2 - 2xy, 2xy + 2y^2) \) satisfies \( f(x, y) = (\varphi(x, y))^2 \). In this case we have \( u(x, y) = -x^2 - 2xy \), \( v(x, y) = 2xy + 2y^2 \) and they satisfy the \( \varphi_\mathbb{A} \)-CREs.

**Example 1.4** Let \( \varphi : \mathbb{R}^3 \to \mathbb{R}^2 \) be defined by \( \varphi(x, y, z) = (x + z, y) \). Thus, the \( \varphi_\mathbb{A} \)-CREs are given by

\[
\begin{align*}
  u_x &= v_y, & v_x &= -u_y, \\
  u_y &= u_z, & v_y &= v_z.
\end{align*}
\]  

(17)

Since the components of the function \( f(x, y, z) = \varphi(x, y, z) \) satisfy equations (18) and \( \varphi(e_1) = e \), by Theorem 1.2 we have that \( f \) is \( \varphi_\mathbb{A} \)-differentiable. Thus, polynomial functions \( p \) of the form

\[
p(x, y, z) \mapsto c_0 + c_1 \varphi(x, y, z) + \cdots + c_m(\varphi(x, y, z))^m
\]

where \( c_0, c_1, \ldots, c_m \in \mathbb{A} \) are \( \varphi_\mathbb{A} \)-differentiable. Moreover, the rational functions obtained by quotients of these polynomials are \( \varphi_\mathbb{A} \)-differentiable on their domain of definition.

The \( f_1(x, y, z) = x^2 + z^2 + 2xz - y^2 \) and \( f_2(x, y, z) = 2xy + 2yz \) satisfy the \( \varphi_\mathbb{A} \)-CREs, thus \( f(x, y, z) = (x^2 - 2xy, 2xy + 2yz) \) is \( \varphi_\mathbb{A} \)-differentiable. Since \( f(x, 0, 0) = x^2(1, 0) \), we have \( f(x, y, z) = (\varphi(x, y, z))^2 \).

We consider an example in which \( \varphi \) in non linear.

**Example 1.5** Let \( \varphi : \mathbb{R}^3 \to \mathbb{R}^2 \) be defined by \( \varphi(x, y, z) = (x^2 + z, 1/y) \). Thus, the \( \varphi_\mathbb{A} \)-CREs are given by

\[
\begin{align*}
  -\frac{1}{y^2}u_x &= 2xv_y, & \frac{1}{y^2}v_x &= 2xu_y, \\
  u_x &= 2xu_z, & v_x &= 2xv_z.
\end{align*}
\]  

(18)

Since the components of the function \( f(x, y, z) = \varphi(x, y, z) \) satisfy equations (18) and \( d\varphi(e_3) = e \), by Theorem 1.2 we have that \( f \) is \( \varphi_\mathbb{A} \)-differentiable. Thus, polynomial functions \( p \) of the form

\[
p(x, y, z) \mapsto c_0 + c_1 \varphi(x, y, z) + \cdots + c_m\varphi^m(x, y, z)
\]

where \( c_0, c_1, \ldots, c_m \in \mathbb{A} \) are \( \varphi_\mathbb{A} \)-differentiable. Moreover, the rational functions obtained by quotients of these polynomials are \( \varphi_\mathbb{A} \)-differentiable on their domain of definition.

The systems of \( \varphi_\mathbb{A} \)-CREs presented in this section do not satisfy the conditions given in [12], then the \( \varphi_\mathbb{A} \)-differentiable functions related to these examples are not \( \mathbb{A} \)-differentiable for all the algebras \( \mathbb{A} \).
1.7 Examples for $\mathbb{A} = \mathbb{A}^3_3(p_1, \cdots, p_6)$

Consider the algebra $\mathbb{A} = \mathbb{A}^3_3(p_1, \cdots, p_6)$ with unit $e = (1, 0, 0)$ given in Theorem 1.3 of [7]. That is, $\mathbb{A}$ is the linear space $\mathbb{R}^3$ endowed with the product

\[
\begin{array}{c|ccc}
   \cdot & e_1 & e_2 & e_3 \\
\hline
   e_1 & e_1 & e_2 & e_3 \\
   e_2 & e_2 & p_7e_1 + p_1e_2 + p_2e_3 & p_8e_1 + p_3e_2 + p_4e_3 \\
   e_3 & e_3 & p_9e_1 + p_3e_2 + p_4e_3 & p_9e_1 + p_5e_2 + p_6e_3 \\
\end{array}
\] (19)

where $e = e_1$ and $p_1, p_2, \cdots, p_9$ are parameters satisfying the equalities

\[
\begin{align*}
p_7 & = p_1p_4 + p_2p_6 - p_2p_3 - p_4^2, \\
p_8 & = p_3p_4 - p_2p_5, \\
p_9 & = p_1p_5 + p_3p_6 - p_4p_5 - p_5^2.
\end{align*}
\] (20)

In the following examples $\mathbb{A}$ is the algebra $\mathbb{A}^3_3(p_1, \cdots, p_6)$.

**Example 1.6** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $\varphi(x, y) = (x, y, 0)$. Thus, the $\varphi\mathbb{A}$-CREs are given by

\[
\begin{align*}
u_y & = p_7v_x + p_8w_x \\
v_y & = u_x + p_1v_x + p_3w_x \\
w_y & = p_2v_x + p_4w_x
\end{align*}
\] (21)

Since the components of the function $f(x, y) = \varphi(x, y)$ satisfy equations (22) and $\varphi(1, 0) = e$, by Theorem 1.2 we have that $f$ is $\varphi\mathbb{A}$-differentiable. Thus, polynomial functions $p$ of the form

\[
p(x, y) \mapsto c_0 + c_1\varphi(x, y) + \cdots + c_m(\varphi(x, y))^m
\]

where $c_0, c_1, \cdots, c_m \in \mathbb{A}$ are $\varphi\mathbb{A}$-differentiable. Moreover, the rational functions obtained by quotients of these polynomials are $\varphi\mathbb{A}$-differentiable on their domain of definition.

**Example 1.7** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $\varphi(x, y) = (x, 0, y)$. Thus, the $\varphi\mathbb{A}$-CREs are given by

\[
\begin{align*}
u_y & = p_8v_x + p_9w_x \\
v_y & = p_3v_x + p_5w_x \\
w_y & = u_x + p_4v_x + p_6w_x
\end{align*}
\] (22)

Since the components of the function $f(x, y) = \varphi(x, y)$ satisfy equations (22) and $\varphi(1, 0) = e$, by Theorem 1.2 we have that $f$ is $\varphi\mathbb{A}$-differentiable. Thus, polynomial functions $p$ of the form

\[
p(x, y) \mapsto c_0 + c_1\varphi(x, y) + \cdots + c_m(\varphi(x, y))^m
\]

where $c_0, c_1, \cdots, c_m \in \mathbb{A}$ are $\varphi\mathbb{A}$-differentiable. Moreover, the rational functions obtained by quotients of these polynomials are $\varphi\mathbb{A}$-differentiable on their domain of definition.
Example 1.8 Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $\varphi(x, y) = (0, x, y)$. Thus, the $\varphi_\mathbb{A}$-CREs are given by

\[
\begin{align*}
p_8 v_x + p_9 w_x - p_7 v_y - p_8 w_y &= 0, \\
p_3 v_x + p_5 w_x - u_y - p_1 v_y - p_3 w_y &= 0, \\
u_x + p_4 v_x + p_6 w_x - p_2 v_y - p_4 w_y &= 0.
\end{align*}
\] (23)

Consider the function $f(x, y) = \varphi(x, y)$, then $df_{(a,b)}(x, y) = (0, x, y)$. That is, $f$ is $\varphi_\mathbb{A}$-differentiable and $f'(x, y) = e$ for all $(x, y) \in \mathbb{R}^2$. Thus, polynomial functions $p$ of the form

\[ p(x, y) \mapsto c_0 + c_1 \varphi(x, y) + \cdots + c_m (\varphi(x, y))^m \]

where $c_0, c_1, \cdots, c_m \in \mathbb{A}$ are $\varphi_\mathbb{A}$-differentiable. Moreover, the rational functions obtained by quotients of these polynomials are $\varphi_\mathbb{A}$-differentiable on their domain of definition.

Example 1.9 Any function $f(x, y) = (0, f_2(x, y), f_3(x, y))$ satisfies $\varphi_\mathbb{A}$-CREs (23) for $\mathbb{A} = \mathbb{A}_3^1(0, \cdots, 0)$ and $\varphi(x, y) = (0, x, y)$, but not all the differentiable (in the usual sense) functions $f$ having this form are $\varphi_\mathbb{A}$-differentiable. This can happen because $\varphi(\mathbb{R}^2)$ is contained in the singular set of $\mathbb{A}$.

Suppose that $f(x, y) = (0, x + y, x - y)$ is $\varphi_\mathbb{A}$-differentiable. We have that $df_{(0,0)}(x, y) = (0, x + y, x - y)$. Thus, if $(a, b, c) = f'_\varphi(0,0)$, then

\[ (0, x + y, x - y) = (a, b, c)(0, x, y) = (0, ax, ay) \]

which is a contradiction. Therefore, $f$ satisfies equations (23) but $f$ is not $\varphi_\mathbb{A}$-differentiable.

The type of $\varphi_\mathbb{A}$-CREs systems considered in this section are not considered in [12].

2 Linear systems of two first order PDEs

2.1 First order linear systems of two PDEs

Consider the linear first order PDEs

\[
\begin{align*}
a_{11} u_x + a_{12} u_y + a_{13} v_x + a_{14} v_y &= f, \\
a_{21} u_x + a_{22} u_y + a_{23} v_x + a_{24} v_y &= g,
\end{align*}
\] (24)

where $a_{ij}, f$ and $g$ are differentiable functions of $x, y$. The corresponding homogenous system is given by

\[
\begin{align*}
a_{11} u_x + a_{12} u_y + a_{13} v_x + a_{14} v_y &= 0, \\
a_{21} u_x + a_{22} u_y + a_{23} v_x + a_{24} v_y &= 0.
\end{align*}
\] (25)

We use notation

\[
\left\langle \left(\begin{array}{cccc}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24}
\end{array}\right) : \left(\begin{array}{cc}
u_x & u_y \\
v_x & v_y
\end{array}\right)\right\rangle = \left(\begin{array}{c}
f \\
g
\end{array}\right),
\] (26)
for system (24), where \( \langle \cdot, \cdot \rangle \) denotes the inner product. The corresponding homogenous system is expressed by
\[
\left( \begin{array}{cccc}
 a_{11} & a_{12} & a_{13} & a_{14} \\
 a_{21} & a_{22} & a_{23} & a_{24}
\end{array} \right) : \left( \begin{array}{cc}
 u_x & u_y \\
 v_x & v_y
\end{array} \right) = \left( \begin{array}{c}
 0 \\
 0
\end{array} \right),
\]
(27)
If \( A \) is the \( 2 \times 4 \) matrix function formed by the \( a_{ij} \), \( w = (u, v) \), \( F = (f, g) \) and \( dw \) denotes the usual derivative, then (26) is written by \( \langle A : dw \rangle = F \).

**Definition 2.1** If \( A_1 \), \( A_2 \) are two \( 2 \times 4 \) matrix valued functions, and \( F_1 \), \( F_2 \) are two \( 2 \times 1 \) matrix valued functions, we say that systems \( \langle A_1 : dw \rangle = F_1 \) and \( \langle A_2 : dw \rangle = F_2 \) are equivalent if there exists a non singular \( 2 \times 2 \) matrix valued function \( M = M(x, y) \) such that \( A_2 = MB_1 \) and \( F_2 = MF_1 \).

If \( \langle A_1 : dw \rangle = F_1 \) and \( \langle A_2 : dw \rangle = F_2 \) are equivalent, then they share the same solutions set.

**Proposition 2.1** Let \( \langle A_1 : dU \rangle = F_1 \) and \( \langle A_2 : dw \rangle = F_2 \) be systems that share the same solutions set. If there exists a \( 2 \times 2 \) matrix valued function \( M \) such that \( F_1 = MF_2 \) and there exists a solution \( w \) with \( dw \) not singular, then the given systems are equivalent.

**Proof.** Let \( w \) be a solution with \( dw \) being non singular, then
\[
\langle A_1 - MA_2 : dw \rangle = \langle A_1 : dw \rangle - \langle MA_2 : dw \rangle = F_1 - MF_2 = 0,
\]
that is, \( A_1 - MA_2 = 0 \). Thus, the given systems are equivalent. \( \square \)

If \( w_p \) is a particular solution of system (24), then each solution \( w \) of system (24) is expressed by \( w = w_h + w_p \), where \( w_h \) is a solution of the associated homogenous system (25). Therefore, if the homogenous system results equivalent to a set of CREs for the \( \varphi A \)-derivative and a particular solution \( w_p \) of (24) is known, then the solutions of (24) have the form \( w = w_h + w_p \), where \( w_h \) is a \( \varphi A \)-differentiable function.

### 2.2 Homogeneous linear systems of two first order PDEs

The problem considered in this section is when a given homogenous linear system of two PDEs is the CREs for the \( \varphi A \)-differentiability for some differentiable function \( \varphi \) and an algebra \( A \). The general system we consider has the form
\[
\begin{align*}
 a_{11}u_x + a_{12}u_y + a_{13}v_x + a_{14}v_y &= 0 \\
 a_{21}u_x + a_{22}u_y + a_{23}v_x + a_{24}v_y &= 0,
\end{align*}
\]
(28)
where \( a_{ij} \) are functions of \((x, y)\) for \( i = 1, 2 \) and \( j = 1, \ldots, 4 \). Suppose that
\[
(a_{11}, a_{12}, a_{13}, a_{14}) \neq \alpha(a_{21}, a_{22}, a_{23}, a_{24})
\]
for all differentiable scalar functions \( \alpha(x, y) \).

We are interested in the existence of an algebra \( A \) and a scalar function \( \varphi(x, y) \) such that system (28) is equivalent to a system which is the CREs for the \( \varphi A \)-derivative.

We have the following propositions and examples.
**Proposition 2.2** Suppose that \( F_1 = (-b_{12}, b_{11}) \) and \( F_2 = (-b_{22}, b_{21}) \) are conservative vector fields with potential functions \( \varphi_i \) for \( i = 1, 2 \), such that system \( \langle A : dw \rangle = 0 \) is equivalent to system
\[
\begin{align*}
 b_{11}u_x + b_{12}u_y + \alpha b_{21}v_x + \alpha b_{22}v_y & = 0 \\
 b_{21}u_x + b_{22}u_y + (b_{11} + \beta b_{21})v_x + (b_{12} + \beta b_{22})v_y & = 0
\end{align*}
\]  
for some parameters \( \alpha, \beta \), then the \( \varphi A \)-differentiable functions are solutions of \( \langle A : dw \rangle = 0 \), where \( \varphi = (\varphi_1, \varphi_2) \) and \( A = A_1^2(\alpha, \beta) \) is the algebra given in Section 1.

**Proof.** Under these conditions in the theorem the Jacobian matrix of \( \varphi \) is given by
\[
 J\varphi = \begin{pmatrix} -b_{12} & b_{11} \\ -b_{22} & b_{21} \end{pmatrix}.
\]
The \( \varphi A \)-CREs are given by \( d\varphi(e_2)(u_x, v_x) = d\varphi(e_1)(u_y, v_y) \), where \( d\varphi \) denotes the usual derivative. That is,
\[
 (b_{11}, b_{21})(u_x, v_x) = -(b_{12}, b_{22})(u_y, v_y).
\]
By using the \( A \)-product we obtain system \( (29) \).

**Example 2.1** Consider the system
\[
\begin{align*}
yu_x + xu_y - \alpha xv_x + \alpha yv_y & = 0 \\
xu_x - yu_y - (y - \beta x)v_x - (x + \beta y)v_y & = 0
\end{align*}
\]  
Since \( F_1 = (2x, -2y) \), \( F_2 = (2y, 2x) \) are conservative vector fields with potential functions \( \varphi_1(x, y) = x^2 - y^2 \), \( \varphi_2(x, y) = 2xy \), by Proposition 1.2 and Theorem 1.2 we have for \( \varphi = (\varphi_1, \varphi_2) \) and \( A = A_1^2(\alpha, \beta) \) that the set of \( \varphi A \)-differentiable functions is the set of the solutions of system \( (30) \).

By Lemma 1.2 we have that \( \varphi A \)-differentiable functions \( f \) have the form \( f = g \circ \varphi \), where \( g \) is an \( A \)-differentiable function. So that,
\[
f(x, y) = g(x^2 - y^2, 2xy),
\]
is solution of \( (30) \) if \( g \) is an \( A \)-differentiable function.

**Proposition 2.3** Suppose that \( F_1 = (-b_{14}, b_{13}) \) and \( F_2 = (-b_{24}, b_{23}) \) are conservative vector fields with potential functions \( \varphi_i \) for \( i = 1, 2 \), such that system \( \langle A : dw \rangle = 0 \) is equivalent to system
\[
\begin{align*}
 (\gamma b_{13} + b_{23})u_x + (\gamma b_{14} + b_{24})u_y + b_{13}v_x + b_{14}v_y & = 0 \\
 \delta b_{13}u_x + \delta b_{14}u_y + b_{23}v_x + b_{24}v_y & = 0
\end{align*}
\]  
for some parameters \( \gamma, \delta \), then the \( \varphi A \)-differentiable functions are solutions of system \( \langle A : dw \rangle = 0 \), where \( \varphi = (\varphi_1, \varphi_2) \) and \( A = A_2^2(\gamma, \delta) \) is the algebra given in Section 1.

**Proof.** The proof is similar to that of 2.2 □
Example 2.2 Consider the system
\[
\begin{align*}
(\gamma y - x)u_x + (\gamma x + y)u_y + yv_x + xv_y &= 0, \\
\delta yu_x + \delta xu_y - xv_x + yv_y &= 0.
\end{align*}
\] (32)

Since \( F_1 = (2x, -2y) \), \( F_2 = (2y, 2x) \) are conservative vector fields with potential functions \( \varphi_1(x, y) = x^2 - y^2 \), \( \varphi_2(x, y) = 2xy \), by Proposition 1.2 and Theorem 1.2 we have for \( \varphi = (\varphi_1, \varphi_2) \) and \( \mathbb{A} = \mathbb{A}_2^2(\gamma, \delta) \) that the set of \( \varphi\mathbb{A} \)-differentiable functions is the set of the solutions of system (32).

By Lemma 1.2 we have that \( \varphi\mathbb{A} \)-differentiable functions \( f \) have the form \( f = g \circ \varphi \), where \( g \) is an \( \mathbb{A} \)-differentiable function. So that,

\[
\begin{align*}
f(x, y) &= g(x^2 - y^2, 2xy),
\end{align*}
\]

is solution of (32) if \( g \) is an \( \mathbb{A} \)-differentiable function.

Proposition 2.4 Suppose that \( F_1 = (-b_2, b_1) \) and \( F_2 = (-b_3, b_3) \) are conservative vector fields with potential functions \( \varphi_i \) for \( i = 1, 2 \), such that system \( \langle A: dw \rangle = 0 \) is equivalent to system
\[
\begin{align*}
b_1u_x + b_2u_y &= 0, \\
b_3v_x + b_4v_y &= 0,
\end{align*}
\] (33)

then the \( \varphi\mathbb{A} \)-differentiable functions are solutions of system \( \langle A: dw \rangle = 0 \), where \( \mathbb{A} = \mathbb{A}_{1,2}^2 \) is the algebra given in Section I.

Proof. The proof is similar to that of 2.2. □

Example 2.3 Consider the system
\[
\begin{align*}
yu_x + xu_y &= 0, \\
xv_x - yv_y &= 0.
\end{align*}
\] (34)

Since \( F_1 = (x, -y) \), \( F_2 = (y, x) \) are conservative vector fields with potential functions \( \varphi_1(x, y) = \frac{x^2}{2} - \frac{y^2}{2} \), \( \varphi_2(x, y) = xy \), by Proposition 1.2 and Theorem 1.2 we have for \( \varphi = (\varphi_1, \varphi_2) \) and \( \mathbb{A} = \mathbb{A}_{1,2}^2 \) that the set of \( \varphi\mathbb{A} \)-differentiable functions is the set of the solutions of system (34).

The \( \mathbb{A} \)-differentiable functions have the form \( g(x, y) = (g_1(x), g_2(y)) \) where \( g_1 \) and \( g_2 \) are one variable differentiable functions in the usual sense. By Lemma 1.2 we have that \( \varphi\mathbb{A} \)-differentiable functions \( f \) have the form \( f = g \circ \varphi \), where \( g \) is an \( \mathbb{A} \)-differentiable function. So that,

\[
\begin{align*}
f(x, y) &= \left( g_1\left(\frac{x^2}{2} - \frac{y^2}{2}\right), g_2(xy)\right),
\end{align*}
\]

is solution of (34) if \( g_1 \) and \( g_2 \) are differentiable functions of one variable.
3 \ \varphi A\text{-differential equations}

3.1 The Cauchy-integral theorem for the \varphi A\text{-differentiability}

If \( f : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n \) is a \varphi A\-differentiable function defined in an open set \( U \). The \varphi A\-line integral of \( f \) is defined by

\[
\int_\gamma f d\varphi = \int_\gamma f(v) d\varphi(v') := \int_0^{t_1} f(\gamma(s)) d\varphi(\gamma'(s)) ds
\]

(35)

where \( \gamma \) is a differentiable function of \( t \) with values in \( U \), \( \gamma(0) = u_0 \), \( \gamma(t_1) = u \), \( f(\gamma(s)) d\varphi(\gamma'(s)) \) represents the \( A \)-product of \( f(\gamma(s)) \) and \( d\varphi(\gamma'(s)) \), and the right hand of (35) represents the usual line integral in \( \mathbb{R}^n \).

A version of the Cauchy integral theorem for the \varphi A\-line integral is given in the following theorem, see [8] and Corollary 10.11 pg. 49 of [4] for another version of the Cauchy-integral theorem relative to algebras.

Theorem 3.1 Let \( \varphi : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n \) be a \( C^2 \)-function defined on a simply-connected open set \( U \) and \( f : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n \) a \varphi A\-differentiable function. If \( \gamma \) is a closed differentiable path contained in \( U \), then the \varphi A\-line integral (35) is equal to zero.

Proof. We will show that \( f d\varphi(\gamma') = \sum_{q=1}^n \langle G_q, \gamma' \rangle e_q \), where the \( G_q \) are \( n \)-dimensional conservative vector fields. Remember that \( d\varphi(e_j) = \sum_{l=1}^n \varphi_{lu_j} e_l \).

The \( A \)-product of \( f \) and \( \varphi(\gamma') \) is given by

\[
fd\varphi(\gamma') = \left( \sum_{m=1}^n f_m e_m \right) \left( \sum_{j=1}^k \gamma'_j d\varphi(e_j) \right) = \sum_{m=1}^n \sum_{j=1}^k \sum_{l=1}^n f_m \gamma'_j \varphi_{lu_j} e_l e_m
\]

\[
= \sum_{q=1}^n \left( \sum_{m=1}^n \sum_{j=1}^k \sum_{l=1}^n f_m \gamma'_j \varphi_{lu_j} C_{lmq} \right) e_q
\]

\[
= \sum_{q=1}^n \left( \sum_{j=1}^k \left( \sum_{m=1}^n \sum_{l=1}^n f_m \varphi_{lu_j} C_{lmq} \right) e_j, \sum_{j=1}^k \gamma'_j e_j \right) e_q,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product of the vector field \( G_q \) and \( \gamma' \), and

\[
G_q = \sum_{j=1}^k \left( \sum_{m=1}^n \sum_{l=1}^n f_m \varphi_{lu_j} C_{lmq} \right) e_j
\]

for \( q = 1, \ldots, n \). By taking the exterior derivative of the dual 1-form of \( F_q \), using the \varphi A\-CREs given by (15), and the commutativity of the second partial derivatives of the components of \( \varphi \), we show that this 1-form is exact. Therefore, \( G_q \) is a conservative vector field. See [8] for the proof of this theorem for the \( A \)-differentiable functions. \( \square \)
If $U$ is a simply connected open set containing $u$ and $u_0$, Theorem 3.1 permit us to define
\[ \int_{u_0}^{u} fd\varphi = \int_{\gamma} fd\varphi, \]
where $\gamma$ is a differentiable function of $t$ with values in $U$, $\gamma(0) = u_0$, and $\gamma(t_1) = u$, as in Definition 35.

**Corollary 3.1** Let $\varphi : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a $C^2$-function on an open set $U$ and $f : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$ be a $\varphi A$-differentiable function. The vector fields
\[ G_q = \sum_{j=1}^{k} \left( \sum_{m=1}^{n} \sum_{l=1}^{n} f_m \varphi_{lu_j} C_{lmq} \right) e_j \]
for $q = 1, \ldots, n$ are conservative, where $\varphi_{u_j} = \sum_{l=1}^{n} \varphi_{lu_j} e_l$.

**Example 3.1** Consider the algebra $A = A_3^{1}(-1, \ldots, -1)$ with unit $e = (1, 0, 0)$ given in Subsection 1.7 which is given by
\[ \begin{array}{c|ccc}
 & e_1 & e_2 & e_3 \\
e_1 & e_1 & e_2 & e_3 \\
e_2 & e_2 & e_2 + e_3 & e_2 + e_3 \\
e_3 & e_3 & e_2 + e_3 & e_2 + e_3 \\
\end{array} \] (36)

The structure constants of $A$ are given by
\[
\begin{align*}
C_{111} &= 1, & C_{112} &= 0, & C_{113} &= 0, \\
C_{121} &= 0, & C_{122} &= 1, & C_{123} &= 0, \\
C_{131} &= 0, & C_{132} &= 0, & C_{133} &= 1, \\
C_{211} &= 0, & C_{212} &= 1, & C_{213} &= 0, \\
C_{221} &= 0, & C_{222} &= 1, & C_{223} &= 1, \\
C_{231} &= 0, & C_{232} &= 1, & C_{233} &= 1, \\
C_{311} &= 0, & C_{312} &= 0, & C_{313} &= 1, \\
C_{321} &= 0, & C_{322} &= 1, & C_{323} &= 1, \\
C_{331} &= 0, & C_{332} &= 1, & C_{333} &= 1.
\end{align*}
\]

Let $\varphi(x, y) = (x, y, 0)$. The function $f(x, y, z) = \varphi(x, y)^{-1}$ is $\varphi A$-differentiable and
\[ f(x, y) = \left( \frac{1}{x^3 + 2x^2 y}, \frac{-xy - y^2}{x^3 + 2x^2 y}, \frac{y^2}{x^3 + 2x^2 y} \right). \]

Thus, the conservative vector fields $G_i$ for $i = 1, 2, 3$ are given by
\[
\begin{align*}
G_1 &= (f_1, 0) = \left( \frac{1}{x}, 0 \right), \\
G_2 &= (f_2, f_1 + f_2 + f_3) = \left( \frac{-xy - y^2}{x^3 + 2x^2 y}, \frac{x + y}{x^3 + 2x^2 y} \right), \\
G_3 &= (f_3, f_2 + f_3) = \left( \frac{y^2}{x^3 + 2x^2 y}, -\frac{xy}{x^3 + 2x^2 y} \right).
\end{align*}
\]
If $U$ is a simply-connected open set and $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is $\varphi\mathbb{A}$-differentiable on $U$, then the function
\[ F(u) = \int_{u_0}^u f(v) d\varphi(v'), \]
for $u_0, u \in U$ is well defined.

An $\varphi\mathbb{A}$-antiderivative of a function $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ defined on an open set $U$ is a function $F : U \subset \mathbb{R}^k \to \mathbb{R}^n$ whose $\varphi\mathbb{A}$-derivative is given by $F'_{\varphi} = f$.

For $\varphi\mathbb{A}$-polynomial functions the $\varphi\mathbb{A}$-antiderivative can be calculated in the usual way. The $\varphi\mathbb{A}$-line integral of $\varphi\mathbb{A}$-differentiable functions gives $\varphi\mathbb{A}$-antiderivatives, as we have in the following corollary which is a generalization of the fundamental theorem of calculus.

**Corollary 3.2** Let $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a $C^2$-function defined on a simply-connected open set $U$ and $f : U \subset \mathbb{R}^k \to \mathbb{R}^n$ a $\varphi\mathbb{A}$-differentiable function. If $u_0, u \in U$ and
\[ F(u) = \int_{u_0}^u f(v) d\varphi(v'), \]
then $F'_{\varphi} = f$.

**Proof.** We take the curve $\gamma(t) = u + t\xi$ joining $u$ and $u + \xi$, thus $\gamma'(t) = \xi$. The rest of the proof is a consequence of Theorem 3.1. \[\Box\]

### 3.2 Existence and uniqueness of solutions

Let $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field defined on an open set $\Omega$. A $\varphi\mathbb{A}$-differential equation is
\[ w'_{\varphi} = F(w), \quad w(\tau_0) = w_0, \]
which is understand as the problem of finding a $\varphi\mathbb{A}$-differentiable function $w : V_{\tau_0} \subset \mathbb{R}^k \to \mathbb{R}^n$ defined in a neighborhood $V_{\tau_0}$ of $\tau_0$ such that $w'_{\varphi}(\tau) = F(w(\tau))$ for all $\tau \in V_{u_0}$, and satisfying the initial condition $w(\tau_0) = w_0$.

We find the following existence and uniqueness Theorem for $\mathbb{A}$-algebrizable vector fields and $\varphi\mathbb{A}$-differential equations.

**Theorem 3.2** Let $\varphi : U \subset \mathbb{R}^k \to \mathbb{R}^n$ be a $C^2$-function defined on an open set $U$ and $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ a $\mathbb{A}$-differentiable vector field defined on an open set $\Omega$ with $\varphi(U) \subset \Omega$. For every initial condition $w_0 \in \Omega$ there exists an unique $\varphi\mathbb{A}$-differentiable function $w : V_{\tau_0} \subset \mathbb{R}^k \to \mathbb{R}^n$ with $w(\tau_0) = w_0$ and satisfying (45), where $V_{\tau_0} \subset U$ is a neighborhood of $\tau_0$.

**Proof.** Define
\[ w_{n+1}(\tau) = \int_{\tau_0}^\tau F \circ w_n(v) d\varphi(v'), \quad w_0(v) = w_0. \]
The function \( w_0(v) \) is \( \varphi \mathbb{A} \)-differentiable, and by (3.2) we have that \( w_1(v) \) is \( \varphi \mathbb{A} \)-differentiable for all \( n \in \mathbb{N} \). Thus, we apply induction and show that \( w_n(v) \) is \( \varphi \mathbb{A} \)-differentiable. The remaining arguments are similar to the usual Existence and uniqueness theorem for ordinary differential equations. □

Let \( \mathbb{A} \) be an algebra which as linear space is \( \mathbb{R}^n \), and \( \varphi : V \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) a differentiable function defined on open set \( V \). Consider a function \( F : \Omega \subset \mathbb{R}^k \times \mathbb{R}^n \rightarrow \mathbb{A} \) defined on an open set \( \Omega \). We say \( F \) is \( (\varphi \mathbb{A}, \mathbb{A}) \)-differentiable if \( F(\tau, w) \) as a function of \( \tau \) (with \( w \) being fixed) is \( \varphi \mathbb{A} \)-differentiable and as a function of \( w \) (with \( \tau \) being fixed) is \( \mathbb{A} \)-differentiable. For the identity map \( \varphi : \mathbb{A} \rightarrow \mathbb{A} \), we also say \( F \) is \( (\mathbb{A}, \mathbb{A}) \)-differentiable if \( F(\tau, w) \) is \( (\varphi \mathbb{A}, \mathbb{A}) \)-differentiable. We say a function \( f : U \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}^n \) defined on an open set \( U \) has a lifting \( F : \Omega \subset \mathbb{A} \times \mathbb{A} \rightarrow \mathbb{A} \) if \( f(t, x) = F(te, x) \) for all \( (t, x) \in U \).

A non-autonomous \( \varphi \mathbb{A} \)-ordinary differential equation \( (\varphi \mathbb{A} \text{-ODE}) \) is written by

\[
    w'_{\varphi} = F(\tau, w), \quad w(\tau_0) = w_0, \tag{38}
\]

where it is understood as the problem of finding a \( \varphi \mathbb{A} \)-differentiable function \( w : V_{\tau_0} \subset \mathbb{R}^k \rightarrow \mathbb{R}^n \) defined in a neighborhood \( V_{\tau_0} \) of \( \tau_0 \) whose \( \varphi \mathbb{A} \)-derivative \( w'_{\varphi}(\tau) \) satisfies \( w'_{\varphi}(\tau) = F(\tau, w(\tau)) \). The corresponding existence and uniqueness of solutions can be stated for \( (\varphi \mathbb{A}, \mathbb{A}) \)-differentiable functions \( F = F(\tau, w) \).

The \( \mathbb{A} \)-line integral is defined by the \( \varphi \mathbb{A} \)-line integral when \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the identity map. Some \( \varphi \mathbb{A} \)-differential equations can be solved, as we see in the following proposition.

**Proposition 3.1** Suppose \( H(\tau, w) \) is \( (\varphi \mathbb{A}, \mathbb{A}) \)-differentiable and it can be expressed in the form \( H(\tau, w) = K(\tau)L(w) \) respect to the \( \mathbb{A} \)-product, where \( L \) has domain contained in \( \mathbb{A} \) and image contained in the regular set of \( \mathbb{A} \). Then, \( \mathbb{A} \) and \( \varphi \mathbb{A} \)-line integrals

\[
    \int^w \frac{dv}{L(v)} = \int^\tau K(s) d\varphi(s') + C, \tag{39}
\]

implicitly defines \( w \) as a \( \varphi \mathbb{A} \)-differentiable function of \( \tau \) which solves the \( \varphi \mathbb{A} \)-differential equation (38).

**Proof.** Suppose \( H(\tau, w) = K(\tau)L(w) \) is a \( (\varphi \mathbb{A}, \mathbb{A}) \)-differentiable function and \( L \) has image in the regular set of \( \mathbb{A} \). Furthermore, assume \( \varphi : \Omega \subset \mathbb{A} \rightarrow \mathbb{A} \) is a differentiable map in the usual sense. Supposing \( w \) is implicitly defined as a function of \( \tau \), applying Lemma □ to the left hand of (39) we calculate:

\[
    \left( \int^w \frac{dv}{L(v)} \right)'_{\varphi} = \left( \frac{d}{dw} \int^w \frac{dv}{L(v)} \right)w'_{\varphi} = \frac{w'_{\varphi}}{L(w)}.
\]

The \( \varphi \mathbb{A} \)-line integral of a function defines a \( \varphi \mathbb{A} \)-antiderivative. Thus, \( \varphi \mathbb{A} \)-derivative of the right hand of (39) is given by

\[
    \left( \int^\tau K(s) d\varphi(s') + C \right)'_{\varphi} = K(\tau).
\]

Therefore, (39) implicitly defines solutions \( w(\tau) \) of the \( \varphi \mathbb{A} \)-differential equation (38). □

We consider the following example.
Example 3.2  Consider an algebra $A$, a differentiable function in the usual sense $\varphi$, and the $\varphi A$-ODE $w'_\varphi = K(\tau)$, where $K(\tau)$ is a $\varphi A$ differentiable function. Then, by (39) $w = \int^\tau K d\varphi + C$, where $C$ is a constant in $A$. Thus, the solutions are given by

$$w(\tau) = \int^\tau K d\varphi + C.$$  

Another example can be given by

Example 3.3  Consider an algebra $A$, a differentiable function in the usual sense $\varphi$, and the $\varphi A$-ODE $w'_\varphi = \varphi(\tau)w^2$. Then, by (39) $-w^{-1} = (\varphi(\tau))^2/2 + C$, where $C$ is a constant in $A$. Thus, the solutions are given by

$$w(\tau) = \frac{-e}{(\varphi(\tau))^2 + C}.$$  

Another $\varphi A$-ODEs can be solved as in the following example.

Example 3.4  Consider an algebra $A$, a differentiable function in the usual sense $\varphi$, and the $\varphi A$-ODE $w'_\varphi = K(\tau)w^2$, where $K(\tau)$ is a $\varphi A$ differentiable function. Then, by (39) $-w^{-1} = \int^\tau K d\varphi + C$, where $C$ is a constant in $A$. Thus, the solutions are given by

$$w(\tau) = \frac{-e}{\int^\tau K d\varphi + C}.$$  

If $k = n$ and $F$ is a $\varphi A$-differentiable vector field, we also say $F$ is a $\varphi A$-algebrizable vector field. Consider an ordinary differential equation (ODE)

$$\frac{dx}{dt} = f(t, x),$$  

(40)

where $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ and $U$ is an open set. Suppose $A$ is an algebra such that $f$ has a lifting $F : \Omega \subset A \times A \to A$ for some set $\Omega$, then the $A$-ODE associated with (40) is

$$\frac{dw}{d\tau} = F(\tau, w),$$  

(41)

where $\frac{dw}{d\tau}$ denotes the $A$-derivative. When $F$ is ($A, A$)-differentiable, solutions $w(\tau)$ of (41) define solutions $x(t) = w(te)$ of (40), where $e$ is the unit of $A$. If $H$ is the function defined by $H(\tau, w) = (d\varphi_t(e))^{-1}F(\tau, w)$, the $\varphi A$-differential equation associated with (40) is given by

$$w'_\varphi = H(\tau, w).$$  

(42)

When $H$ is ($\varphi A, A$)-differentiable, solutions $w(\tau)$ of (42) define solutions $x(t) = w(te)$ of (40).

We say: a) the ODE (40) is $A$-algebrizable if $F$ is ($A, A$)-differentiable, and b) the ODE (40) is $\varphi A$-algebrizable if $H$ is ($\varphi A, A$)-differentiable.
3.3 A \( \varphi A \)-algebrizable vector field associated to triangular billiards

Consider the four-dimensional vector field defined on \( \mathbb{R}^4 \)

\[
f(x_1, y_1, x_2, y_2) = \begin{pmatrix}
  b(x_1^2 - y_1^2) - (b + c)(x_1x_2 - y_1y_2) \\
  2bx_1y_1 - (b + c)(x_1y_2 + x_2y_1) \\
  a(x_2^2 - y_2^2) - (a + c)(x_1x_2 - y_1y_2) \\
  2ax_2y_2 - (a + c)(x_1y_2 + x_2y_1)
\end{pmatrix}^T.
\]  

(43)

It can be written as the two dimensional complex vector field

\[
F(u, v) = (bu^2 - (b + c)uv, av^2 - (a + c)uv),
\]

for \( u = (x_1, y_1) \) and \( v = (x_2, y_2) \).

If \( \mathbb{A} \) is the linear space \( \mathbb{C}^2 \) endowed with an algebra structure over \( \mathbb{C} \). We still using \{\( e_1, e_2 \)\} for the standard basis of \( \mathbb{C}^2 \) as a linear space over \( \mathbb{C} \). For a \( \mathbb{C} \)-linear transformation \( \varphi : \mathbb{C}^2 \rightarrow \mathbb{C}^2 \), the \( \varphi A \)-differentiability, the \( \varphi A \)-differential equations, and the \( \varphi A \)-algebrizability of vector fields and autonomous ordinary differential equations are defined in the same way for algebras over \( \mathbb{C} \) as the definitions for algebras over the real field \( \mathbb{R} \), given in Section 3 and Subsection 3.2. Let \( G : \Omega \subset \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) be a complex vector field defined on an open set \( \Omega \). An \( \varphi A \)-differential equation is

\[
w'_\varphi = G(\tau, w), \quad w(\tau_0) = w_0,
\]

(45)

which is understand as the problem of finding a \( \varphi A \)-differentiable function \( w : V_{\tau_0} \subset \mathbb{C}^2 \rightarrow \mathbb{C}^2 \) defined in a neighborhood \( V_{\tau_0} \subset \Omega \) of \( \tau_0 \) such that \( w'_\varphi(\tau) = G(\tau, w(\tau)) \) for all \( \tau \in V_{\tau_0} \).

Consider the complex vector field \( F \) given in (44). By using Proposition 3.1, we want to show that \( F \) is a \( \varphi A \)-algebrizable vector field for some algebra \( A \) and a \( \mathbb{C} \)-linear transformation \( \varphi \). Its complex Jacobian matrix is

\[
JF(u, v) = \begin{pmatrix}
  2bu - (b + c)v & -(b + c)u \\
  -(a + c)v & 2av - (a + c)u
\end{pmatrix}
\]

\[
= u \begin{pmatrix}
  2b & -(b + c) \\
  0 & -(a + c)
\end{pmatrix} + v \begin{pmatrix}
  -(b + c) & 0 \\
  -(a + c) & 2a
\end{pmatrix}.
\]

By multiplying \( JF \) by the matrix

\[
M = \frac{-1}{(a + c)} \begin{pmatrix}
  -(a + c) & b + c \\
  0 & 2b
\end{pmatrix}
\]

(46)

we have

\[
JFM = u \begin{pmatrix}
  2b & 0 \\
  0 & 2b
\end{pmatrix} + v \begin{pmatrix}
  -(b + c) & \frac{(b + c)^2}{(a + c)} \\
  -(a + c) & (b + c) - \frac{4ab}{a + c}
\end{pmatrix}.
\]

Consider

\[
\alpha = -\frac{(b + c)^2}{(a + c)^2}, \quad \beta = -2\frac{b + c}{a + c} + \frac{4ab}{(a + c)^2}.
\]

(47)
and the algebra $\mathbb{A}$ defined by $\mathbb{C}^2$ endowed with the product given in (2). Therefore, the structure constants $c_{ijk}$ of $\mathbb{A}$ are given (3). The first fundamental representation of $\mathbb{A}$ is given by (4). Under these conditions $JF(u,v)M \in \mathbb{R}(\mathbb{A})$. By Proposition 1, $F$ is $\varphi\mathbb{A}$-differentiable, where $\varphi$ is the linear transformation with associated matrix

$$M^{-1} = \begin{pmatrix} 1 & \frac{b+c}{2b} \\ 0 & -\frac{(a+c)}{2b} \end{pmatrix}.$$  \hspace{1cm} (48)

Thus,

$$\varphi(u,v) = \left( u - \frac{b+c}{2b}v, -(a+c)2b v \right).$$  \hspace{1cm} (49)

Now, we will look for the expression of $F$ in terms of the variable $w$ of $\mathbb{A}$. By evaluating $F$ along the unit $e \in \mathbb{A}$ we have

$$F(u, 0) = b(u, 0)^2 = b(\varphi(u, 0))^2 = b(\varphi(u, 0))^2.$$  

Thus, the complex vector field $F$ in terms of the variable $w = (u, v)$ of $\mathbb{A}$ is given by

$$F(w) = b(\varphi(w))^2,$$  \hspace{1cm} (50)

where $(\varphi(w))^2$ is defined by the $\mathbb{A}$-product. Therefore, $F$ is $\varphi\mathbb{A}$-algebrizable. The corresponding complex ordinary differential equation can be written by

$$\frac{dw}{dz} = b(\varphi(w))^2,$$  \hspace{1cm} (51)

and its associated $\mathbb{A}$-ODE is given by

$$\frac{dw}{d\tau} = b(\varphi(w))^2.$$  \hspace{1cm} (52)

Since $d\varphi_w = \varphi$ and $\varphi(e) = e$, the corresponding $\varphi\mathbb{A}$-ODE is given by

$$w'_{\varphi} = b(\varphi(w))^2.$$  \hspace{1cm} (53)

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