BOUND ED WEYL PSEUDODIFFERENTIAL OPERATORS IN FOCK SPACE

(A Calderón-Vaillancourt theorem in an infinite dimensional setting)

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ABSTRACT We aim at constructing an analog of the Weyl calculus in an infinite dimensional setting, in which the usual configuration and phase spaces are ultimately replaced by infinite dimensional measure spaces, the so-called abstract Wiener spaces. The Hilbert space on which the operators act can be seen as a Fock space or, equivalently, as a space of square integrable functions on the configuration space. The construction is not straightforward and needs to split the configuration space into two factors, of which the first one is finite dimensional. Then one defines, for a convenient symbol $F$, a hybrid calculus, acting on the finite dimensional factor as a Weyl operator and on the other one as an anti-Wick operator, defined thanks to an infinite dimensional Segal-Bargmann transformation. One can establish bounds on the hybrid operators. These bounds enable us to prove the convergence of any sequence of hybrid operators associated with an increasing sequence of finite dimensional factors. Their common limit is the Weyl operator $O_{h}^{\text{weyl}}(F)$, the analog of Calderón-Vaillancourt Theorem being a consequence of the upper mentioned bounds as well.

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1. Introduction.

For every infinitely differentiable $F$ on $\mathbb{R}^n \times \mathbb{R}^n$ with bounded derivatives and every $h > 0$, one denotes by $O_{h}^{\text{weyl}}(F)$ the bounded operator acting on $L^2(\mathbb{R}^n)$ which is formally defined by

$$
(1.1) \quad \left( O_{h}^{\text{weyl}}(F)(\varphi) \right)(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{1}{2}(x-y)\xi} F \left( \frac{x+y}{2}, \xi \right) \varphi(y) dyd\xi \quad \varphi \in \mathcal{S}(\mathbb{R}^n).
$$

According to Calderón-Vaillancourt Theorem [C-V], this operator is well defined and bounded in $L^2(\mathbb{R}^n)$. For further developments on pseudo-differential operators, see for example Hörmander [HO] or Lerner [LER].

Our aim is to establish an analog of this theorem in an infinite dimensional setting, replacing the set $\{1, ..., n\}$ by a countable set $\Gamma$, a typical example of which being a lattice in $\mathbb{R}^d$.

The result we shall prove in this article does not exactly take the shape of the initial theorem of [C-V] when it is restricted to a finite dimensional space. It is rather an analog of the results stated in Cordes [C], Hwang [HW] and Coifman Meyer [C-F].

Let $I_{m}(\{1, ..., n\})$ $(m \geq 1, n \geq 1)$ denote the set of multi-indices $(\alpha, \beta)$ such that $0 \leq \alpha_{j} \leq m$ and $0 \leq \beta_{j} \leq m$ for all $j \leq n$. The results of [C], [HW] and [C-M] were not exactly concerned with the Weyl formula (1.1), but with another one, used in those times to define pseudo-differential operators. These authors prove that, if $\partial_{\alpha}^{\beta} F$ is bounded for all $(\alpha, \beta)$ in $I_{1}(\{1, ..., n\})$, then the pseudo-differential operator associated with $F$ by their formula is bounded in $L^2(\mathbb{R}^n)$.

In the case of the Weyl calculus defined by (1.1), studying the proof given by A. Unterberger [U2] shows that, to ensure that $O_{h}^{\text{weyl}}(F)$ is bounded, it is sufficient to suppose that $\partial_{\alpha}^{\beta} F$ is bounded for all $(\alpha, \beta)$ in $I_{2}(\{1, ..., n\})$. Our first aim is to prove, under hypotheses similar to those of [C], [HW] and [C-M], an upper bound on the norm of $O_{h}^{\text{weyl}}(F)$ which can easily extend to the infinite dimensional case.

In the following theorem, one of the alternative statements requires relatively few derivatives (at most 2 in each variable $x_j$ or $\xi_j$). The other one uses derivatives up to order 4 in each variable but yields more precise bounds for certain examples.

For every non zero multi-index $(\alpha, \beta)$, let $S(\alpha, \beta)$ denote the support of $(\alpha, \beta)$, which is the largest subset $S$ of $\{1, ..., n\}$ such that $\alpha_{j} + \beta_{j} \geq 1$ for all $j \in S$.
Theorem 1.1. Let \( F \) be a continuous function defined on \( \mathbb{R}^{2n} \). Suppose there exist \( M > 0 \) and a family \( (\varepsilon_j)_{1 \leq j \leq n} \) of real nonnegative numbers such that the following properties are satisfied:

a) One has

\[
|F(x, \xi)| \leq M \quad (x, \xi) \in \mathbb{R}^{2n}
\]

b) For every multi-index \((\alpha, \beta)\) in \( I_2(\{1, \ldots, n\})\), the derivative \( \partial_\xi^\alpha \partial_x^\beta F \) is well defined, continuous, bounded on \( \mathbb{R}^{2n} \) and satisfies, if \((\alpha, \beta)\) is not zero:

\[
|\partial_\xi^\alpha \partial_x^\beta F(x, \xi)| \leq M \prod_{j \in S(\alpha, \beta)} \varepsilon_j^{\alpha_j + \beta_j} \quad (x, \xi) \in \mathbb{R}^{2n}.
\]

In this case the operator \( \text{Op}_h^{\text{w}^{\text{w}^{\text{e}}}}(F) \) is bounded in \( L^2(\mathbb{R}^n) \) and if \( 0 < h \leq 1 \):

\[
\|\text{Op}_h^{\text{w}^{\text{w}^{\text{e}}}}(F)\|_{L^2(\mathbb{R}^n)} \leq M \prod_{j=1}^n (1 + 225\pi K_2 \sqrt{\varepsilon_j})
\]

where \( K_2 = \sup_{j \leq n} \max(1, \varepsilon_j^3) \). If condition b) holds for every multi-index \((\alpha, \beta)\) in \( I_4(\{1, \ldots, n\})\), one has:

\[
\|\text{Op}_h^{\text{w}^{\text{w}^{\text{e}}}}(F)\|_{L^2(\mathbb{R}^n)} \leq M \prod_{j=1}^n (1 + 225\pi K_4 h \varepsilon_j^2)
\]

where \( K_4 = \sup_{j \leq n} \max(1, \varepsilon_j^5) \).

In order to discriminate between the roles of the \( x \) and \( \xi \) variables, one can alternatively assume that there exist two sequences of nonnegative real numbers \((\rho_j)\) and \((\delta_j)\) \((1 \leq j \leq n)\) such that, for every non zero multi-index \((\alpha, \beta)\) in \( I_4(\{1, \ldots, n\})\)

\[
|\partial_\xi^\alpha \partial_x^\beta F(x, \xi)| \leq M \prod_{j \in S(\alpha, \beta)} \rho_j^{\alpha_j} \delta_j^{\beta_j} \quad (x, \xi) \in \mathbb{R}^{2n}
\]

In this case, one derives from the second version of Theorem 1.1 that

\[
\|\text{Op}_h^{\text{w}^{\text{w}^{\text{e}}}}(F)\|_{L^2(\mathbb{R}^n)} \leq M \prod_{j=1}^n (1 + 225\pi K_4' h \rho_j \delta_j)
\]

where \( K_4' = \sup_{j \leq n} \max(1, (\rho_j \delta_j)^3) \).

The definition of the Weyl operator and the precise statement of the analog of Theorem 1.1 in an infinite dimensional setting cannot be given before Section 5. (Definition 5.1 and Theorem 5.4), the necessary notions being presented in Sections 2, 3, 4. The proof of Theorem 1.1 is mainly contained in the proof of Theorem 5.4 and will not be detailed for its own sake.

In the transition to an infinite dimensional situation, the set \( \{1, \ldots, n\} \) is replaced by a countable set \( \Gamma \) (for example a lattice in \( \mathbb{R}^d \) \((d \geq 1)\)). The space \( L^2(\mathbb{R}^n) \) on which the operators act is replaced by the symmetrized Fock space \( \mathcal{F}_\epsilon(\ell^2(\Gamma, \mathbb{C})) \) associated with the Hilbert space \( Z = \ell^2(\Gamma, \mathbb{C}) \). This space \( \mathcal{F}_\epsilon(\ell^2(\Gamma, \mathbb{C})) \) will be denoted by \( \mathcal{H}(\Gamma) \). The definition of Fock spaces in its abstract form is recalled in Section 3. One knows (cf. \([\text{RE-SI}], [\text{SI1}], [\text{SI2}], [J], [\text{LEV}]\)) that there exists an isomorphism between \( \mathcal{H}(\Gamma) \) and \( L^2(B(\Gamma), \mu_{\Gamma}^K) \), where \( B(\Gamma) \) is a convenient Banach space, playing the role of the configuration space and replacing \( \mathbb{R}^n \), and \( \mu_{\Gamma}^K \) is a Gaussian measure on \( B(\Gamma) \). This isomorphism, called the Segal isomorphism, will be recalled in Section 3. It will be denoted by \( J_{\Gamma}^K \).
If the set $\Gamma$ were finite, $B(\Gamma)$ would be the space $\mathbb{R}^\Gamma$ and the Gaussian measure would be defined by

\begin{equation}
\mu^K_{F,h}(u) = (\pi h)^{-|\Gamma|/2} e^{-\frac{1}{2h}|u|^2} d\lambda_F(u),
\end{equation}

where $\lambda_F$ is the Lebesgue measure on $\mathbb{R}^\Gamma$. If $\Gamma$ is infinite, there is no notion of Lebesgue measure, but the classical theory of Wiener spaces shows that the analog $\mu^K_{F,h}$ of the measure (1.5) can be constructed on an appropriate Banach space $B(\Gamma)$. This configuration space $B(\Gamma)$ is not unique. One can choose any Banach space satisfying the conditions required by the classical theorems related to Wiener spaces, which can be found in Kuo [KU] and will be recalled in Section 2 (Theorem 2.1). Here is an example of such a space.

**Definition 1.2.** Let $\Gamma$ be a countable set. Choose a family $b = (b_j)_{j \in E}$ of real positive numbers satisfying the following property. For every $\varepsilon > 0$, the family of positive real numbers

\begin{equation}
R_j(b_j, \varepsilon) = \int_{\varepsilon b_j}^{+\infty} e^{-\frac{x^2}{2}} dx \quad j \in \Gamma
\end{equation}

is summable. Let $B_b(\Gamma)$ denote the space of all families $(x_j)_{j \in E}$ such that $\left(\frac{|x_j|}{b_j}\right)_{j \in \Gamma}$ converges to zero when $j$ goes to infinity. We shall choose once for all such a family $(b_j)$ and the space $B_b(\Gamma)$ will be denoted by $B(\Gamma)$. If $E$ is a (finite or infinite) subset of $\Gamma$, let $B(E)$ denote the analogous space, corresponding to the restriction to $E$ of the same family $(b_j)$. This space has the norm

\begin{equation}
\|x\|_{B(E)} = \sup_{j \in E} \frac{|x_j|}{b_j}.
\end{equation}

One shows in Section 2 (Theorem 2.3) that, for every sequence $(b_j)_{j \in \Gamma}$ satisfying (1.6), the space $B_b(\Gamma)$ satisfies the hypotheses of L. Gross’s Theorem 2.1 about Wiener spaces. As a consequence, the infinite dimensional analog $\mu^K_{F,h}$ of the measure (1.5) is well defined as a measure on the Borel $\sigma$-algebra of $B_b(\Gamma)$. Remark that the choice of our configuration space does not rely on Hilbert-Schmidt operators (as is often the case in this kind of construction), which allows us to weaken the assumptions on the symbols. Let us give an example.

If $\Gamma$ is a lattice on $\mathbb{R}^d$ with $d \geq 1$ and if $|\cdot|$ is a norm on $\mathbb{R}^d$, then, for every $\gamma > 0$, the family $b = (b_j)_{j \in \Gamma}$ defined by

\begin{equation}
b_j = (1 + |j|)^\gamma
\end{equation}

satisfies the condition of Definition 1.2 (the corresponding family $R_j(b_j, \varepsilon)$ is summable for every $\varepsilon > 0$).

In the infinite dimensional version of Theorem 1.1, the functions $F$ (the symbols of the operators) are bounded and continuous on the phase space corresponding to the set $B(\Gamma)$ of Definition 1.2, that is $B(\Gamma) \times B(\Gamma)$. We can now list the hypothesis on the partial derivatives of the symbols. A multi-index is an application $\alpha$ from $\mathbb{N}$ to such that $\alpha_j = 0$ except for a finite number of indices $j \in \Gamma$. We denote by $I_m(\Gamma)$ ($m \geq 1$) the set of multi-indices $(\alpha, \beta)$ such that $0 \leq \alpha_j \leq m$ and $0 \leq \beta_j \leq m$ for all $j$ in $\Gamma$. For every non zero multi-index $(\alpha, \beta)$, $S(\alpha, \beta)$ denotes the largest set $S$ such that $\alpha_j + \beta_j \geq 1$ for all $j$ in $S$ and is called the support of $(\alpha, \beta)$. It is therefore finite.

**Definition 1.3.** Let $\Gamma$ be an infinite, countable set. Let $B(\Gamma)$ be the space of Definition 1.2. Let $\varepsilon = (\varepsilon_j)_{j \in \Gamma}$ be a family of nonnegative real numbers, indexed by the elements of $\Gamma$ and let $M$ be positive. A function $F$, bounded and continuous on $B(\Gamma) \times B(\Gamma)$, is said to satisfy the hypothesis $H_m(M, \varepsilon)$ (with $m \geq 1$) if

\begin{equation}
|F(x, \xi)| \leq M;
\end{equation}

a) For all $(x, \xi)$ in $B(\Gamma) \times B(\Gamma)$:
b) For every multi-index \((\alpha, \beta)\) in \(I_m(\Gamma)\), the partial derivative \(\partial_x^\alpha \partial_\xi^\beta F\) is well defined, continuous and bounded on \(B(\Gamma) \times B(\Gamma)\) and satisfies, for all \((x, \xi) \in B(\Gamma) \times B(\Gamma)\)

\[
|\partial_x^\alpha \partial_\xi^\beta F(x, \xi)| \leq M \prod_{j \in S(\alpha, \beta)} \varepsilon_j^{\alpha_j + \beta_j}.
\]

In addition to the Weyl calculus we shall need the anti-Wick calculus and consequently the Segal Bargmann transform in an infinite dimensional setting. One basically defines this transform (Definition 4.1) as a partial isometry from any Fock space \(F_s(Z)\) associated with a separable Hilbert space \(Z\), into the Fock space \(F_s(Z \times Z)\). In the case when \(Z = \ell^2(\Gamma, C)\), the Segal Bargmann transform, denoted by \(W_T\), is a partial isometry from the “configuration” Fock space \(F_s(\ell^2(\Gamma, C)) = \mathcal{H}(\Gamma)\) (on which the operators are defined) into the “phase” Fock space \(F_s(\ell^2(\Gamma, C) \times \ell^2(\Gamma, C))\), which will be denoted by \(\mathcal{H}_F(\Gamma)\). This space is isomorphic to \(L^2(B(\Gamma) \times B(\Gamma), \mathcal{F}^h)\), where \(\mathcal{F}^h\) is a Gaussian measure on \(B(E) \times B(E)\). This Segal isomorphism is denoted by \(J^F\). Composing the isometry \(W_T\) with both Segal isomorphisms yields a partial isometry from \(L^2(B(\Gamma), \mathcal{F}^h)\) into \(L^2(B(\Gamma) \times B(\Gamma), \mathcal{F}^h)\). We shall see in Section 4 the relation with the Bargmann transform as it is defined in Kree Raczka [K-R]. Moreover we shall give some equivalent characterizations of the subspace \(SB(E, h)\) of \(L^2(B(\Gamma) \times B(\Gamma), \mathcal{F}^h)\), which is the range of \(L^2(B(\Gamma), \mathcal{F}^h)\) by this application. One of these properties is due to Driver-Hall [D-H].

The composition \(J^F \circ W_T\) is a partial isometry from \(\mathcal{H}(\Gamma)\) into \(L^2(B(\Gamma) \times B(\Gamma), \mathcal{F}^h)\). Let \(F\) be bounded on \(B(\Gamma) \times B(\Gamma)\) and measurable with respect to the Borel \(\sigma\)-algebra. For all \(h > 0\), one can associate with \(F\) an Anti Wick operator \(Op^AW_h(F)\), which is bounded in \(\mathcal{H}(\Gamma)\) and satisfies, for all \(f\) and \(g\) in \(\mathcal{H}(\Gamma)\):

\[
< Op^AW_h(F) f, g >_{\mathcal{H}(\Gamma)} = \int_{B(\Gamma) \times B(\Gamma)} F(X) \left(J^F_{\Gamma,h} W_T f\right)(X) \left(J^F_{\Gamma,h} W_T g\right)(X) d\mathcal{F}^h(X).
\]

In this article we shall associate an operator depending on \(h > 0\) with every function \(F\) continuous and bounded on \(B(\Gamma) \times B(\Gamma)\) and satisfying

- either the hypothesis \(H_2(M, \varepsilon)\), where \(M > 0\) and \(\varepsilon = (\varepsilon_j)_{j \in \Gamma}\) is a summable family of real nonnegative numbers;
- or the hypothesis \(H_4(M, \varepsilon)\), where the family \((\varepsilon_j^2)_{j \in \Gamma}\) is summable.

In either case the associated operator \(Op^{weyl}_h(F)\) will be bounded in the Fock space \(\mathcal{H}(\Gamma) = F_s(\ell^2(\Gamma, C))\).

The precise definition of this operator cannot be given yet; let us just say that, for every finite subset \(E\) of \(\Gamma\), one defines a hybrid operator \(Op^{hyb,E}_h(F)\), acting as a Weyl operator with respect to the variables \(x_j\) \((j \in E)\) and as an Anti Wick operator with respect to the variables \(x_k\) \((k \notin E)\). In the next step one replaces the finite subset \(E\) by an increasing sequence \((\Lambda_n)\) of finite subsets of \(\Gamma\), whose union is \(\Gamma\) and one proves that the sequence of operators \(Op^{hyb,\Lambda_n}_h(F)\) is a Cauchy sequence in \(\mathcal{L}(\mathcal{H}(\Gamma))\) (Theorem 5.4). To establish an upper bound on the norms of the operators \(Op^{hyb,\Lambda_n}_h(F)\) or \(Op^{hyb,\Lambda_n}_h(F) - Op^{hyb,\Lambda_m}_h(F)\) one adapts, up to some details, the integration by parts method on which one proof of the Calderón-Vaillancourt Theorem is based. More precisely, one adapts the proof due to A. Unterberger [U2], which relies on coherent states.

The limit of the sequence of operators will be denoted by \(Op^{weyl}_h(F)\) and can be considered as the Weyl operator associated with \(F\). Under the hypothesis \(H_2(M, \varepsilon)\), with \(h > 0\) and a summable family \((\varepsilon_j)_{j \in \Gamma}\), its norm will satisfy

\[
||Op^{weyl}_h(F)||_{\mathcal{L}(\mathcal{H}(\Gamma))} \leq M \prod_{j \in \Gamma} (1 + 225\pi K_2 \sqrt{\varepsilon_j}),
\]
where $K_2 = \sup_{j \in \Gamma} \max(1, \varepsilon_j^3)$. Remark that, if the family $(\varepsilon_j)_{j \in \Gamma}$ is summable, then the infinite product converges. Under the hypothesis $H_4(M, \varepsilon)$ of Definition 1.3, it is sufficient that $(\varepsilon_j^2)_{j \in \Gamma}$ be summable. If $0 < h \leq 1$, one has

\[(1.9') \quad \|O_{p_h^{weyl}}(F)\|_{\mathcal{L}(\mathcal{H}(\Gamma))} \leq M \prod_{j \in \Gamma} (1 + 225 \pi K_4 h \varepsilon_j^2) ,\]

where $K_4 = \sup_{j \in \Gamma} \max(1, \varepsilon_j^6)$. Theorem 5.4 is the analog of Theorem 1.1 in an infinite dimensional setting and its proof (see Sections 6, 7, 8) contains that of Theorem 1.1.

One can wonder as to the relationship between this definition and former definitions of the Weyl calculus, used for example by B. Lascar [LA1] and Kree-Rączka [K-R] or Albeverio-Daletskii [A-D]. Those definitions suppose in general that the symbol $F$ is the Fourier transform of a finite measure on the Hilbert space $\ell^2(\Gamma, \mathbb{R}) \times \ell^2(\Gamma, \mathbb{R})$. On the other hand, there is no condition as to the existence and boundedness of partial derivatives. We shall see in Section 9 that, if a symbol $F$ is the Fourier transform of a finite measure on $\ell^2(\Gamma, \mathbb{R}) \times \ell^2(\Gamma, \mathbb{R})$ as well as verifies $H_2(M, \varepsilon)$ (with $(\varepsilon_j)_{j \in E}$ a summable family), then both definitions coincide (ours and, for example, B. Lascar’s). It would be interesting also to compare these definitions to Khrennikov’s [KH].

We now give an example of symbol satisfying our hypotheses.

**Example 1.4.** Let $\Gamma = \mathbb{Z}^d$. For all $j \in \Gamma$, set $b_j = (1 + |j|)^\gamma$, where $\gamma > 0$. Let $B(\Gamma)$ be the space of Definition 1.2, associated with this family. For all $X = (x, \xi)$ in $B(\Gamma) \times B(\Gamma)$, set

\[(1.10) \quad H(x, \xi) = \sum_{j \in \Gamma} g_j^2(x_j^2 + \xi_j^2) + \lambda \sum_{|j-k|=1} g_j g_k x_j x_k .\]

Here the norm $| \cdot |$ is an arbitrary norm on $\mathbb{R}^d$. If it is the supremum norm $\ell^\infty$, the function $H$ recalls a lattice of harmonic oscillators with a coupling between nearest neighbors. The constant $\lambda$ is such that the quadratic form $H$ is positive definite and $(g_j)_{j \in \Gamma}$ is a family of positive numbers. Set

\[(1.11) \quad F(x, \xi) = e^{-H(x, \xi)} \quad (x, \xi) \in B(\Gamma) \times B(\Gamma) .\]

If the family $(g_j)_{j \in \Gamma}$ satisfies

\[\sum_{j \in \Gamma} g_j^2 (1 + |j|)^{2\gamma} < +\infty , \]

then the function $F$ is continuous and bounded on $B(\Gamma) \times B(\Gamma)$. Moreover, for every integer $m$, the function $F$ satisfies hypothesis $H_m(1, \varepsilon)$, with $M = 1$ and $\varepsilon_j = C_m g_j$, where $C_m$ is a constant depending on $m$. Under these hypotheses, the condition $H_4(1, \varepsilon)$ holds and the family $(\varepsilon_j^2)_{j \in \Gamma}$ is summable. By the second version of Theorem 5.4, the Weyl operator associated with the function (1.11) will be bounded in $\mathcal{H}(\Gamma) = \mathcal{F}_s(\ell^2(\Gamma, \mathbb{C}))$.

Now let us give an application of Theorem 1.1.

**Example 1.5.** With the notations of Example 1.4, let $(E_N)$ be an increasing sequence of finite subsets of the lattice $\Gamma$, whose union is $\Gamma$. Let $V$ be a real valued bounded function in $C^\infty(\mathbb{R})$, whose derivatives are all bounded. For every integer $N$, set

\[H_N(x, \xi) = \sum_{j \in E_N} \xi_j^2 + \sum_{(j,k) \in E_N \times E_N} V(x_j - x_k) , \]

\[P_N(x, \xi) = e^{-\frac{1}{|E_N|} H_N(x, \xi)} .\]


The function $P_N$ satisfies condition a) of Theorem 1.1, with $n = |E_N|$ and $M = 1$. Condition b) is satisfied for the multi-indices $(\alpha, \beta)$ in $I_1(E_N)$, with $\varepsilon_J = C_1|E_N|^{-1/2}$, where $C_1$ is a real constant. By the second version of Theorem 1.1, the norm of $Op_{\text{wegl}}^1(P_N)$ satisfies:

$$\|Op_{\text{wegl}}^1(P_N)\|_{L^2(\mathbb{R}^{E_N})} \leq \left(1 + \frac{C_2}{|E_N|}\right)^{|E_N|},$$

where $C_2 > 0$ is a constant. This norm has therefore an upper bound which is independent of $N$.

Sections 2, 3 and 4 present the more or less classical notions, which will be needed to state the main theorem: abstract Wiener measures in Section 2, Fock spaces and Segal isomorphisms in Section 3, Segal Bargmann transform in an infinite dimensional situation in Section 4. The account about the Segal Bargmann transform goes further than needed in the rest of the article, but we thought it was useful to clarify the connections between the different definitions of this notion that can be found in the literature. In Section 5, we define the hybrid (Weyl- Anti Wick) operators associated with the finite subsets of $\Gamma$ (Definition 5.1) and we state Theorem 5.4, which can be considered as the main result. In Section 6, we establish the formula linking the hybrid (Weyl- Anti Wick) operators associated with the finite subsets of $\Gamma$ (Definition 5.1) and we state Theorem 5.4, which can be considered as the main result. In Section 6, we establish the formula linking two hybrid quantizations associated with two finite subsets, one containing the other. Sections 7 and 8 are devoted to the proof of the main result. In Section 9, we compare our definition of the Weyl pseudodifferential operator with the definition used in the articles of Kree Raszka and B. Lascar.

2. Measure spaces associated with the subsets of $\Gamma$.

The Weyl operator associated with a symbol will be the limit of a sequence of operators defined by a hybrid quantization, for which the set $\Gamma$ has to be split into complementary subsets $E_1$ and $E_2$ playing different roles. This method compels us to define, for each subset $E$ of $\Gamma$, a configuration space denoted by $B(E)$ and a phase space, which is naturally $B(E) \times B(E)$. For a finite $E$, $B(E) = \mathbb{R}^E$. Configuration and phase spaces are equipped with measures depending on a strictly positive parameter $h$, respectively denoted by $\mu^K_{E,h}$ and $\mu^\Phi_{E,h}$. For a finite $E$, the measure on $B(E) = \mathbb{R}^E$ is

$$d\mu^K_{E,h}(u) = (\pi h)^{-|E|/2} \prod_{j \in E} \left(e^{-\frac{1}{2}u_j^2} du_j\right).$$

The measure on the phase space $\mathbb{R}^E \times \mathbb{R}^E$ is analogous to $\mu^K_{E,h}$ but, for technical reasons linked with the coherent spaces, the variance is $2h$ instead of $h$. We have that

$$d\mu^\Phi_{E,h}(x, \xi) = (2\pi h)^{-|E|} \prod_{j \in E} \left(e^{-\frac{1}{2h}(x_j^2 + \xi_j^2)} dx_j d\xi_j\right).$$

If $E$ is infinite, the definition of both measures will use the theory of abstract Wiener spaces, [G2], [G3], Kuo [KU], Th. Lévy [LEV]). In order to apply the classical theorems allowing to define measures analogous to (2.1) and (2.2) on infinite dimensional spaces, we need the configuration space $B(E)$ and its Banach space norm to satisfy certain conditions. In the case when $E = \Gamma$, we shall find the space $B(\Gamma)$ from Definition 1.2 again.

Before defining the convenient phase spaces $B(E)$, we shall recall the classical results needed. Let $Z_\mathbb{R}$ be a separable Hilbert space with norm $|\cdot|$. Let $B$ be a separable Banach space with norm $\|\cdot\|_B$, into which $Z_\mathbb{R}$ is continuously embedded as a dense subset. In this case, $B' \subset Z_\mathbb{R}$. A cylinder set (or tame set) of $B$ is a subset $X \subset B$ of the form

$$(2.3) \quad X = \{x \in B, \quad (f_1(x), \ldots, f_n(x)) \in \Omega\},$$
where \( n \geq 1, f_1, \ldots, f_n \) belong to \( B' \) and \( \Omega \) is a Borel set of \( \mathbb{R}^n \). A cylinder (or tame) function on \( B \) is a function \( f : B \to \mathbb{R} \) such that there exist elements \( f_1, \ldots, f_n \) of \( B' \) (\( n \geq 1 \)) and a function \( g : \mathbb{R}^n \to \mathbb{R} \) measurable for the Borel-\( \sigma \)-algebra of \( \mathbb{R}^n \) with which we can express \( f(x) \) as \( f(x) = g(f_1(x), \ldots, f_n(x)) \) for all \( x \in B \). Since the \( f_j \) can be considered as elements of \( Z_{\mathbb{R}} \), one can always assume that in (2.3), they form an orthonormal system of \( Z_{\mathbb{R}} \). In this case, if \( X \) is a cylinder set defined by (2.3), one sets

\[
\mu_{t,B}(X) = (2\pi t)^{-n/2} \int_{\Omega} e^{-\frac{|x|^2}{2t}} dx
\]

for all \( t > 0 \).

This defines an additive mapping \( \mu_{t,B} \) on the set of all cylinder sets of \( B \).

One can define the notion of a cylinder set in \( Z_{\mathbb{R}} \), using the same formula (2.3), but this time the \( f_j \) belong to \( Z_{\mathbb{R}} \) and \( f_j(x) \) is the scalar product of \( f_j \) and \( x \) in \( Z_{\mathbb{R}} \). Similarly, the formula (2.4) gives the measure of \( X \) (provided the \( f_j \) are orthonormal). This defines an additive mapping on the set of all cylinder sets of \( Z_{\mathbb{R}} \). One shows (cf. Kuo [KU]) that, if \( Z_{\mathbb{R}} \) is infinite dimensional, the mapping \( \mu_{t,B} \) does not extend to a \( \sigma \)-additive measure on the \( \sigma \)-algebra generated by the cylinder sets of \( Z_{\mathbb{R}} \). On the other hand, the additive mapping \( \mu_{t,B} \) extends to a \( \sigma \)-additive measure on the \( \sigma \)-algebra generated by the cylinder sets of \( B \), under hypotheses that we shall state now.

A semi-norm \( N \) on \( Z_{\mathbb{R}} \) is \( \mu_{t,Z_{\mathbb{R}}} \)-measurable if, for every number \( \varepsilon > 0 \), there exists a finite dimensional subspace \( H_\varepsilon \) of \( Z_{\mathbb{R}} \) such that, for every finite dimensional subspace \( V \) of \( Z_{\mathbb{R}} \) orthogonal to \( H_\varepsilon \), the following inequality holds

\[
\mu_{t,Z_{\mathbb{R}}} \left( \{ x \in Z_{\mathbb{R}}, N(P_V(x)) > \varepsilon \} \right) < \varepsilon,
\]

where \( P_V \) is the orthogonal projection onto \( V \) (so the set appearing in (2.5) is a cylinder set).

**Theorem 2.1. (L. Gross)** If \( Z_{\mathbb{R}} \) is a real, separable Hilbert space and \( B \) a real separable Banach space into which \( Z_{\mathbb{R}} \) is continuously embedded as a dense subset, then the \( \sigma \)-algebra generated by the cylinder sets is the Borel \( \sigma \)-algebra of \( B \). Moreover, if the norm of \( B \), restricted to \( Z_{\mathbb{R}} \), is \( \mu_{t,Z_{\mathbb{R}}} \)-measurable, then the mapping \( \mu_{t,B} \) defined by (2.4) on the cylinder sets extends to a uniquely determined measure on the Borel \( \sigma \)-algebra of \( B \).

Both assertions are proved in Kuo [KU]. The first one ([KU] Theorem 4.2) does not require the \( \mu_{t,Z_{\mathbb{R}}} \)-measurability of the norm of \( B \). The second one is due to L. Gross [G2], (see also [KU], Theorem 4.1 and L. Gross [G3], Section 2, Theorem 1). The following Proposition shows that in the infinite dimensional case, \( Z_{\mathbb{R}} \) is \( \mu_{t,B} \)-negligible whereas \( \mu_{t,B}(B) = 1 \).

**Proposition 2.2.** With the notations and under the hypotheses of Theorem 2.1., if \( Z_{\mathbb{R}} \) is infinite dimensional, it is contained in a \( \mu_{t,B} \)-null Borel set.

**Proof.** Let \((e_n)_{n \geq 0}\) be an orthonormal basis of \( Z_{\mathbb{R}} \) whose vectors belong to \( B' \subset Z_{\mathbb{R}} \). Let

\[
A = \{ x \in B, \quad \sum_{n \geq 0} e_n(x)^2 < +\infty \}.
\]

Clearly, \( A \) is a Borel set of \( B \) containing \( Z_{\mathbb{R}} \). One defines a sequence \((\varphi_N)\) of cylinder functions by

\[
\varphi_N(x) = e^{-\sum_{k=0}^{N} e_k(x)^2}, \quad x \in B
\]

and denote by \( \varphi \) its pointwise limit. By the dominated convergence theorem,

\[
\int_B \varphi(x) d\mu_{t,B}(x) = \lim_{N \to \infty} \int_B \varphi_N(x) d\mu_{t,B}(x) = \lim_{N \to \infty} (1 + 2t)^{- (N+1)/2} = 0.
\]
Since $\varphi \geq 0$ and is strictly positive on $A$, it follows that $\mu_{t,B}(A) = 0$.

After recalling these results, we can apply them to prove the following theorem.

**Theorem 2.3.** Let $(b_j)_{j \in \Gamma}$ be a family of strictly positive real numbers, such that the family $R_j(b_j, \varepsilon)$ ($j \in \Gamma$) of (1.6) is summable for every $\varepsilon > 0$. For every subset $E \subseteq \Gamma$, let $B(E)$ be the space of the Definition 1.2. In this case, the space $Z_{\mathbb{R}} = \ell^2(E, \mathbb{R})$ is densely embedded in $B(E)$. Moreover, for all $t > 0$, the restriction to $Z_{\mathbb{R}}$ of the norm of $B(E)$ is $\mu_{t,Z_{\mathbb{R}}}$-measurable.

The proof of the last point uses the following result (cf. L. Gross [G3], Theorem 1, page 95).

**Theorem 2.4.** Let $\| \cdot \|_n$ be an increasing sequence of tame semi-norms on $Z_{\mathbb{R}}$. Let $t > 0$. If, for all $\varepsilon > 0$,

$$
(2.6) \quad \lim_{n \to \infty} \mu_{t,Z_{\mathbb{R}}}(\{x \in H, \|x\|_n \leq \varepsilon\}) > 0,
$$

then $\lim_{n \to \infty} \|x\|_n$ exists for all $x \in Z_{\mathbb{R}}$ and the limit defines a $\mu_{t,Z_{\mathbb{R}}}$-measurable semi-norm.

**Proof of Theorem 2.3 (last point).** One chooses an increasing sequence of finite subsets of $E$, whose union is $E$. One defines an increasing sequence $(\| \cdot \|_n)$ of tame semi-norms on $H = \ell^2(E, \mathbb{R})$ by setting, for all $x = (x_\lambda)_{\lambda \in E}$

$$
(2.7) \quad \|x\|_n = \sup_{j \in E_N} \left| x_j \right|_{b_j}.
$$

For all $\varepsilon > 0$, the set

$$
C_N = \{x \in Z_{\mathbb{R}} = \ell^2(E, \mathbb{R}), \|x\|_n \leq \varepsilon\}
$$

is a cylinder set of $H$. It can be written as in (2.3) with the Borel set $\Omega = \prod_{j \in E_N} [-\varepsilon b_j, \varepsilon b_j]$. Its $\mu_{t,H}$-measure is therefore

$$
\mu_{t,H}(C_N) = \prod_{j \in E_N} \left(2\pi t\right)^{1/2} \int_{-\varepsilon b_j}^{\varepsilon b_j} e^{-\frac{x^2}{2t}} dx.
$$

The sequence $\mu_{t,H}(C_N)$ decreases to a nonnegative limit. One has

$$
\mu_{t,H}(C_N) = \prod_{j \in E_N} \left[1 - 2(2\pi t)^{-1/2} \int_{\varepsilon b_j}^{+\infty} e^{-\frac{x^2}{2t}} dx\right] = \prod_{j \in E_N} \left[1 - 2(2\pi)^{-1/2} R_j(b_j, \frac{\varepsilon}{\sqrt{t}})\right],
$$

where $R_j(\varepsilon, \lambda)$ is defined by (1.6). Since the factors of this product are $> 0$, the limit is $> 0$ provided the family $R_j(b_j, \frac{\varepsilon}{\sqrt{t}})$ is summable. As a consequence, the sequence $(\| \cdot \|_N)$ satisfies the condition of Theorem 2.4 and the limit above, which is the restriction of the norm of $B$ to $Z_{\mathbb{R}}$, is $\mu_{t,Z_{\mathbb{R}}}$-measurable.

We shall now give an example of a family $(b_j)_{j \in \Gamma}$ satisfying the condition of Definition 1.2, in the case when $\Gamma$ is a lattice of $\mathbb{R}^d$.

**Proposition 2.5.** Let $\Gamma$ be a lattice of $\mathbb{R}^d$ (with a norm $| \cdot |$) and let $\gamma > 0$ be given. For all $j \in \Gamma$, set $b_j = (1 + |j|)^\gamma$. Then for all $\varepsilon > 0$, the family $R_j(b_j, \varepsilon)$ ($j \in \Gamma$) defined by (1.6) is summable.

**Proof.** For all $j \in \Gamma$ one has:

$$
2(2\pi)^{-1/2} \int_{\varepsilon(1 + |j|)^\gamma}^{+\infty} e^{-\frac{x^2}{2}} dx \leq e^{-\frac{\varepsilon^2(1 + |j|)^{2\gamma}}{4}} (2\pi t)^{-1/2} \int_{\mathbb{R}} e^{-\frac{x^2}{2t}} dx = \sqrt{2} e^{-\frac{\varepsilon^2(1 + |j|)^{2\gamma}}{4}}.
$$
Since \( \Gamma \) is a lattice of \( \mathbb{R}^d \), we know that the family

\[
\sum_{j \in I} e^{-\frac{\varepsilon^2 (t+|j|)^2}{4}}
\]

is summable, for every \( \varepsilon > 0 \) and every dimension \( d \).

According to Theorems 2.1 and 2.3, for every subset \( E \) of \( \Gamma \), if \( B(E) \) is the space of Definition 1.2, then the mapping \( \mu_{t,B(E)} \) defined by (2.4) on the cylinder sets of \( B(E) \) extends to a uniquely determined measure on the Borel \( \sigma \)-algebra, still denoted by \( \mu_{t,B(E)} \). In this paper, the measure on the configuration space is the measure \( \mu_{t,B(E)} \) defined as above with \( t = h/2 \) and an arbitrary \( h > 0 \). From now on, it will be denoted by \( \mu_{E,h}^K \). The measure on the phase space is \( \mu_{E,B(E)} \otimes \mu_{h,B(E)} \), this time with \( t = h \). It will be denoted by \( \mu_{E,h}^\Phi \) in the rest of the paper and will be used to define every integral on the phase space. If \( E \) is finite, then \( B(E) = \mathbb{R}^E \) and both measures coincide with those defined by (2.1) and (2.2).

One can now give an explicit expression of the integral of a function \( f \) in \( L^1(B(E), \mu_{E,h}^K) \), at least when \( f \) is a cylinder or tame function. Suppose that there exist a family \( (z_1, \ldots, z_n) \) in \( B(E)^n \), orthonormal with respect to the scalar product of \( Z_{\mathbb{R}} = \ell^2(E, \mathbb{R}) \) and a Borel measurable function \( F : \mathbb{R}^n \to \mathbb{C} \), such that

\[
f(x) = F(z_1(x), \ldots, z_n(x)) \quad x \in B(E).
\]

One has then

\[
\int_{B(E)} f(x) d\mu_{E,h}^K(x) = (\pi h)^{-n/2} \int_{\mathbb{R}^n} F(u_1, \ldots, u_n) e^{-\frac{|u|^2}{2h}} d\lambda(u),
\]

where \( \lambda \) is the Lebesgue measure on \( \mathbb{R}^n \). In Section 3, density arguments will allow us to extend this definition to the case when \( f \) is not a tame function but belongs to \( L^1(B(E), \mu_{E,h}^K) \).

Let us remark that, if \( E = E_1 \cup E_2 \) with disjoint \( E_1 \) and \( E_2 \), one has \( B(E) = B(E_1) \times B(E_2) \) and, for example for the phase space

\[
\mu_{E,h}^\Phi = \mu_{E_1,h}^\Phi \otimes \mu_{E_2,h}^\Phi.
\]

In particular, if \( E_1 \) is finite, then \( B(E) = \mathbb{R}^{E_1} \times B(E_2) \) and

\[
d\mu_{E,h}^\Phi(X_{E_1}, X_{E_2}) = (2\pi h)^{-|E_1|} e^{-\frac{|X_{E_1}|^2}{2h}} d\lambda_{E_1}(X_{E_1}) d\mu_{E_2,h}^\Phi(X_{E_2}),
\]

where \( \lambda_{E_1} \) is the Lebesgue measure on \( \mathbb{R}^{E_1} \times \mathbb{R}^{E_1} \).

**Gaussian Vectors.**

One knows (cf. Th. Lévy [LEV]) that, if a real Hilbert space \( Z_{\mathbb{R}} \), a Banach space \( B \) and a real number \( t > 0 \) satisfy the hypothesis of Theorem 2.2, then the complexified \( Z_{\mathbb{C}} \) of \( Z_{\mathbb{R}} \) is isomorphic to a subspace of \( L^2(B, \mu_t,B) \). The elements of \( Z_{\mathbb{R}} \) are sent on elements of \( L^2(B, \mu_t,B) \) generally called **Gaussian random variables**. We shall now give the precise form of this isomorphism in the case when \( Z_{\mathbb{R}} = \ell^2(E, \mathbb{R}) \), for a subset \( E \) of \( \Gamma \).

**Theorem 2.6.** Let \( E \) be an infinite subspace of \( \Gamma \). Let \( a \) belong to \( \ell^2(E, \mathbb{C}) \). For every finite subset \( F \) of \( E \), set

\[
\ell_{a,F}(x) = \sum_{j \in F} a_j x_j , \quad E_{a,F}(x) = e^{\ell_{a,F}(x)}.
\]

Let \( (F_n)_{n \geq 0} \) be an increasing sequence of finite subspaces of \( E \), whose union is \( E \). The sequences of functions \( (\ell_{a,F_n}) \) and \( (E_{a,F_n}) \) are Cauchy sequences in \( L^2(B(E), \mu_{E,h}^K) \). Their limits, respectively denoted by \( \ell_a \) and
$E_a$, are independent of the sequence $(F_n)$. The function $\ell_a$ belongs to $L^p(B(E), \mu^K_{E,h})$ $(1 \le p < +\infty)$ as well and the sequence $(\ell_{a,F_n})$ converges to $\ell_a$ in $L^p(B(E), \mu^K_{E,h})$. One has

(2.9) \[ \|\ell_a\|_{L^2(B(E))}^2 = \frac{h}{2} |a|_{L^2(E)}^2 \quad \text{and} \quad \|E_a\|_{L^2(B(E))}^2 = e^{h \|\text{Re}(a)\|_{L^2(E)}}. \]

One can write $E_a(x) = e^{i\ell(x)}$ too. The system of all functions $\ell_a(x)^p$ $(p \text{ integer } \ge 0, a \text{ in } \ell^2(E, \mathbb{R}))$ is total in $L^2(B(E), \mu^K_{E,h})$.

For the Hilbert space $\ell^2(E, \mathbb{R}) \times \ell^2(E, \mathbb{R})$ and the phase space $B(E) \times B(E)$ one proceeds similarly, setting

$\ell_{a,b}(x, \xi) = \ell_a(x) + \ell_b(\xi)$.

Since the measure on $B(E)$ is $\mu^K_{E,h}$, we need to alter the computations (2.9). Writing $Z_C$ instead of $\ell^2(E, C)$, one gets, for all $a$ in $Z_C \times Z_C$:

\[ \|\ell_a\|_{L^2(B(E) \times B(E), \mu^K_{E,h})}^2 = h |a|_{Z_C \times Z_C}^2, \quad \|E_a\|_{L^2(B(E) \times B(E), \mu^K_{E,h})}^2 = e^{2h \|\text{Re}(a)\|_{L^2(\mathbb{R} \times \mathbb{R})}}. \]

**Proof.** If $n < m$, an explicit computation shows that

\[ \|\ell_{a,F_n}\|_{L^2(B(E))}^2 = \frac{h}{2} \sum_{j \in F_n} |a_j|^2, \quad \|\ell_{a,F_n} - \ell_{a,F_m}\|_{L^2(B(E))}^2 = \frac{h}{2} \sum_{j \in F_m \setminus F_n} |a_j|^2. \]

The assertions concerning the function $\ell_a$ easily follow from this. One can see that

\[ \|E_{a,F_n}\|_{L^2(B(E))}^2 = e^{h \sum_{j \in F_n} |\text{Re}(a_j)|^2} \]

and that, if $n < m$:

\[ \|E_{a,F_n} - E_{a,F_m}\|_{L^2(B(E))}^2 = \ldots \]

\[ = e^{h \sum_{j \in F_n} |\text{Re}(a_j)|^2} \left[ e^{h \sum_{k \in F_m \setminus F_n} |\text{Re}(a_k)|^2} - \frac{1}{e^2} \sum_{k \in F_m \setminus F_n} a_k^2 - e^{h \sum_{k \in F_m \setminus F_n}} 1 \right]. \]

Remarking that, for all $z \in C$, one has $|e^z - 1| \le |z| e^{\text{sup}(\text{Re}(z), 0)}$, one gets

\[ \|E_{a,F_n} - E_{a,F_m}\|_{L^2(B(E))}^2 \le 3he^{h \|\text{Re}(a)\|_{L^2(\mathbb{R})}} \sum_{j \in F_m \setminus F_n} |a_j|^2, \]

from which one can easily deduce the assertions concerning $E_a$. For the spaces $L^p(B(E))$ it amounts to showing it for a real number $a$ and, via Hölder, for an even integer $p \ge 2$. In this case one uses the inequality $\frac{|a|^p}{p^2} \le \frac{1}{2} (e^a + e^{-a}) - 1$. One can pass from this to

\[ \frac{1}{p^2} \int \int_{B(E)} |\ell_{a,F_n}(x) - \ell_{a,F_m}(x)|^p d\mu^K_{E,h}(x) \le \frac{h}{4} e^{h \|\text{Re}(a)\|_{L^2(E)}} \sum_{j \in F_m \setminus F_n} |a_j| |a_j|^2. \]

The results about the convergence in $L^p(B(E))$ are a consequence of this. The result about the total system is classical (cf. Janson [J]). It is based on the fact that, if a function $f$ belonging to $L^2(B(E), \mu^K_{E,h})$ were orthogonal to all $\ell^a(x)^p$ $(p \text{ integer } \ge 0, a \text{ in } \ell^2(E, \mathbb{R}))$, it would be orthogonal to all functions $x \to e^{if(x)}$. Accordingly, its Fourier transform with respect to the Gaussian measure would be identically zero.

3. Fock spaces and Segal isomorphisms.
The definition of the hybrid Weyl-anti-Wick quantization stated in Section 5 involves the Hilbert space $L^2(B(E), \mu^E_{\Gamma})$ (the configuration space) together with the Hilbert space $L^2(B(E) \times B(E), \mu_E^k)$ (the phase space), for every subset $E$ of $\Gamma$, where $B(E)$ is given by Definition 1.2 and where the two measures are defined in Section 2.

Each of these two Hilbert spaces (the configuration and the phase space) is isomorphic to some more abstract Hilbert space, namely, the Fock space. The two corresponding Fock spaces will be used in Section 4 in order to simplify its content.

A. Symmetric Fock space over an Hilbert space.

Even if symmetric Fock spaces over Hilbert spaces are standard (c.f. [RE-SI], [SI1], [SI2], [J]), it may be useful to recall their definitions here. Let $Z_C$ be a separable complex Hilbert space. For all $n \geq 2$, we denote by $Z_C^n$ the $n$-fold tensor product $Z_C \otimes \cdots \otimes Z_C$. The scalar product of $X = u_1 \otimes \cdots \otimes u_n$ and $Y = v_1 \otimes \cdots \otimes v_n$ belonging to $Z_C^n$ ($u_j \in Z$, $v_j \in Z$) is defined by:

$$< X, Y > = < u_1, v_1 > \cdots < u_n, v_n > .$$

The set $Z_C^0$ is the subspace of $Z_C^n$ containing only symmetric elements. This space is also often designated by $S_n Z_C$. The subspace $Z_C^{0,n}$ of $Z_C^n$ is associated with the above scalar product when restricted to $Z_C^n$.

We agree that $Z_C^0 = C$, and that $Z_C^1 = Z$. The real number 1 is considered as an element of $Z_C^0$, it is denoted by $\Omega$ and called the vacuum state. The algebraic direct sum of the $Z_C^n$ ($n \geq 0$) is associated with the scalar product satisfying that the spaces $Z_C^n$ are $Z_C^n$ are orthogonal when $n \neq p$. The completion of this direct sum under the above scalar product is denoted by $F_s(Z_C)$ and is called the symmetric Fock space over $Z_C$.

The symmetrized tensor product of $n$ elements $u_1, \ldots, u_n$ in $Z_C$ is defined by the following element of $Z_C^n$:

$$u_1 \circ \cdots \circ u_n = \frac{1}{\sqrt{n!}} \sum_{\sigma \in S_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} .$$

This notation is borrowed to Th. Levy [LEV] and is different of those in Reed-Simon [RE-SI]. The space $Z_C^n$ is generated by these symmetrized tensor products. The scalar product of two elements in $Z_C^n$ and $Z_C^m$ vanishes if $n \neq m$. When $m = n$,

$$\left< u_1 \circ \cdots \circ u_n, v_1 \circ \cdots \circ v_n \right> = \sum_{\sigma \in S_n} \prod_{j=1}^n < u_{\sigma(j)}, v_{\sigma(j)} > .$$

For each $X \in Z_C$, the creation operator $a^*(X)$ acts in the algebraic direct sum of the $Z_C^n$ and is defined by:

$$a^*(X)(u_1 \circ \cdots \circ u_n) = X \otimes u_1 \circ u_2 \cdots \circ u_n$$

for any $u_1, \ldots, u_n$ in $Z_C$ and for each $n \geq 1$. It maps the vacuum state to $a^*(X)(\Omega) = X$. The annihilation operator $a(X)$ associated with $X \in Z_C$ is given by:

$$a(X)(u_1 \circ \cdots \circ u_n) = \sum_{j=1}^n < u_j, X > u_1 \circ \cdots \circ \hat{u}_j \circ \cdots \circ u_n$$

for each $n \geq 2$. The term $u_j$ is omitted in the symmetrized product in the above right hand-side. For $n = 1$, we have $a(X)(u_1) = < u_1, X > \Omega$, and if $n = 0$ then $a(X)(\Omega) = 0$. Note that the mapping $X \rightarrow a^*(X)$ is $C-$ linear, whereas the mapping $X \rightarrow a(X)$ is antilinear.
For all $X$ in $Z_C$, the Segal field is the unbounded operator $\Phi_S(X)$ defined by:

$$\Phi_S(X)(A) = \frac{1}{\sqrt{2}}(a(X) + a^*(X))(A)$$

for all $A$ in the algebraic sum of the $Z_C^{\otimes n}$. It is isomorphic to a subspace of $B$. For $(3.6)$ now recall the construction of this isomorphism when $Z$. It is known that the Fock space $J$ following two identities:

$$e^{i\Phi_S(X)}\Omega = \sum_{n \geq 0} \frac{i^n e^{-|X|^2/2n}}{2^n/2n!} X \otimes \cdots \otimes X.$$  

(The $e^{i\Phi_S(X)}$ are often called coherent states of $F_S(Z_C)$.)

**Proposition 3.1.** The set of $e^{i\Phi_S(X)}\Omega$, $X \in Z_C$, is complete in $F_S(Z_C)$.

This result is probably very well-known. To prove it, we note that the set of subspaces $Z_C^{\otimes n}$ ($n \geq 0$) is complete in $F_S(Z_C)$ and by polarization (c.f. Janson [J], Theorem D.1), the set of elements $X \otimes \cdots \otimes X$ ($X \in Z_C$) is complete in $Z_C^{\otimes n}$. Next, we deduce from (3.7) that (c.f. Rodnianski Schlein [RO-SC]):

$$\frac{1}{2\pi} \int_0^{2\pi} e^{-i n \theta} e^{i \Phi_S(e^{i\theta}X)}\Omega d\theta = \frac{i^n e^{-|X|^2/2n}}{2^n/2n!} X \otimes \cdots \otimes X$$

for each $X$ in $Z_C$ and each $n \geq 0$. Proposition 3.1 thus easily follows.

**Application.** Let $E$ be a subset of $\Gamma$. The symmetric Fock spaces over the Hilbert spaces $Z_C = \ell^2(E, C)$ and $Z_C = (\ell^2(E, C))^2$ are denoted by $\mathcal{H}(E)$ and $\mathcal{H}_\Phi(E)$ respectively. These two spaces shall be our configuration and phase spaces in the following sections. The corresponding vacuum states are denoted by $\Omega_K(E)$ and $\Omega_\Phi(E)$ respectively. The spaces $\mathcal{H}_K^{\text{fin}}(E)$ and $\mathcal{H}_\Phi^{\text{fin}}(E)$ stands for the algebraic direct sum of the subspaces $Z_C^{\otimes n}$. The (Hilbertian) completion of the tensor product of two complex Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is denoted by $\mathcal{H}_1 \otimes \mathcal{H}_2$. Let us recall that

$$F_S(Z_1 \oplus Z_2) \simeq F_S(Z_1) \otimes F_S(Z_2),$$

where $Z_1$ and $Z_2$ are two complex Hilbert spaces. In particular, when $E_1$ and $E_2$ are two disjoint subsets of $\Gamma$, the previous identity applied with $Z_j = \ell^2(E_j, C)$ and with $Z_j = \ell^2(E_j, C)^2$, ($1 \leq j \leq 2$) gives the following two identities:

$$\mathcal{H}(E_1 \cup E_2) \simeq \mathcal{H}(E_1) \otimes \mathcal{H}(E_2),$$

$$\mathcal{H}_\Phi(E_1 \cup E_2) \simeq \mathcal{H}_\Phi(E_1) \otimes \mathcal{H}_\Phi(E_2).$$

**B. The Segal isomorphism.**

It is known that the Fock space $F_S(Z_C)$ is isomorphic to $L^2(B, \mu_B)$ if the real Hilbert space $Z_\mathbb{R}$ and the Banach space $B$ satisfy the hypotheses of Theorem 2.1 and if $Z_C$ is the complexification of $Z_\mathbb{R}$. A proof of this result appears in Janson [J] (Theorem 4.1). The starting point in [J] relies on the fact that $Z_C$ is isomorphic to a subspace of $L^2(B, \mu_B)$. This point is also proved in our case in Theorem 2.6. Let us now recall the construction of this isomorphism when $Z_\mathbb{R} = \ell^2(E, \mathbb{R})$ with $E$ being any subset of $\Gamma$. This isomorphism shall be denoted by $J^K_{Eh}$.

It is sufficient to define $J^K_{Eh}$ restricted to the subspaces $Z_C^{\otimes n}$ ($n \geq 0$).

For $n = 0$, we set

$$J^K_{Eh}(\mathcal{M} \Omega_K(E)) = \lambda$$

$$\lambda \in C,$$
where $\Omega_K(E)$ is the vacuum state of $\mathcal{H}(E)$. The right hand-side is the constant function defined on $B(E)$ equal to $\lambda$.

For $n = 1$, for any $u$ in $Z_{\mathcal{C}}^1 = Z_{\mathcal{C}} = \ell^2(E, \mathcal{C})$, we set

$$J_{E,h}^K(u)(x) = \sqrt{\frac{2}{h}}\ell_u(x) \tag{3.10}$$

for a.e. $x$ in $B(E)$, where $\ell_u$ is the function belonging to $L^2(B(E), \mu_{E,h}^K)$ defined in Theorem 2.6.

For $n \geq 2$, it suffices to define $J_{E,h}^K(u_1 \odot \cdots \odot u_n)$ for $u_1, \ldots, u_n$ in $Z_{\mathcal{C}} = \ell^2(E, \mathcal{C})$. Let $P_n$ be the subspace of $L^2(B(E), \mu_{E,h}^K)$ spanned by $1$ and by the functions $\ell_{v_1}, \ldots, \ell_{v_k}$ ($v_1, \ldots, v_k$ in $Z_{\mathcal{C}}$, $k \leq n$). Here, the product is the multiplication product for functions on $B(E)$ and it belongs to $L^2(B(E), \mu_{E,h}^K)$. Let $\Pi_n$ be the orthogonal projection in $P_n$ on the orthogonal to $P_{n-1}$, with the scalar product of $L^2(B(E), \mu_{E,h}^K)$. Then set

$$J_{E,h}^K(u_1 \odot \cdots \odot u_n) = \Pi_n(J_{E,h}^K(u_1) \ldots J_{E,h}^K(u_n)) \tag{3.11}$$

for all $u_1, \ldots, u_n$ in $Z_{\mathcal{C}}$. The function $J_{E,h}^K(u_1 \odot \cdots \odot u_n)$ is denoted by $J_{E,h}^K(u_1 \cdots u_n)$ by Th. Levy [LEV] (when $h = 1$).

The above construction is identical to the one of Janson [J]. It is also noticed that, the mapping $J_{E,h}^K$ is extended by density as an isometric isomorphism between $\mathcal{H}(E)$ and $L^2(B(E), \mu_{E,h}^K)$, for any $h > 0$, and each (finite or infinite) subset $E$ of $\Gamma$.

Now setting $Z_{\mathcal{C}} = \ell^2(E, \mathcal{C})^2$, we naturally proceed similarly for the construction of the mapping $J_{E,h}^B$ associated with the phase space. The only difference with the construction associated with the configuration space being that, (3.10) becomes: for all $(u, v)$ in $\ell^2(E, \mathcal{C})^2$,

$$J_{E,h}^B(u, v)(x, y) = \sqrt{\frac{1}{h}}[\ell_u(x) + \ell_v(y)] \tag{3.12}$$

for a.e. $(x, y)$ in $B(E) \times B(E)$. The function $\ell_u$ is defined in Theorem 2.6. Note that $2h$ appears for the phase space case whereas it is $h$ in the configuration space case. The remaining part of the construction is identical to the one concerning the configuration space.

C. Segal isomorphism and Hilbertian bases.

Let $E$ be a subset of $\Gamma$. We shall define the two Hilbertian bases of $\mathcal{H}(E)$ and $L^2(B(E), \mu_{E,h}^K)$ (resp. of $\mathcal{H}_B(E)$ and $L^2(B(E) \times B(E), \mu_{E,h}^B)$). The Segal isomorphism realizes a bijection between the two bases associated with the configuration space (resp. with the phase space). We first fix some notations concerning multi-indices.

We call a multi-index any map $\alpha$ from $E \subseteq \Gamma$ into $\mathbb{N}$ such that $\alpha_j = 0$ except for a finite number of indices $j$. The sum of $\alpha_j$ ($j \in E$) is denoted by $|\alpha|$. Let $S(\alpha)$ be the largest subset $S$ of $E$ such that $\alpha_j \geq 1$ for all $j \in S$. It is necessarily finite. We set:

$$|\alpha|! = \prod_{j \in S(\alpha)} \alpha_j! .$$

Let $(e_j)_{j \in E}$ be the canonical basis of $Z_{\mathcal{C}} = \ell^2(E, \mathcal{C})$. For every multi-index $\alpha$, $e^\alpha \in \mathcal{H}(E)$ stands for the symmetrized product of $|\alpha|$ factors, where each factor in the symmetrized product is an $e_j$ ($j \in S(\alpha)$), and where each $e_j$ appears exactly $\alpha_j$ times in the product.
Hermite polynomials is the sequence of polynomials \( H_n \) on \( \mathbb{R} \), being an orthogonal basis of \( L^2(\mathbb{R}, \nu) \), where \( \nu \) is the measure \( (2\pi)^{-1/2}e^{-x^2/2}dx \), \( H_n \) is of degree \( n \), and the coefficient of \( x^n \) in \( H_n \) equals to 1. The \( L^2(\mathbb{R}, \nu) \) norm of \( H_n \) is \( \sqrt{n!} \). For every multi-index \( \alpha \), we set:

\[
(3.13) \quad P_{\alpha, h}(x) = \prod_{j \in S(\alpha)} H_{\alpha_j} \left( x_j \sqrt{\frac{2}{h}} \right).
\]

**Proposition 3.2.** Let \( c_\alpha = (\alpha!)^{-1/2} \), for every multi-index \( \alpha \). The set of \( c_\alpha e^\alpha \) is an Hilbertian basis of \( \mathcal{H}(E) \). The set of \( c_\alpha P_{\alpha, h} \) is an Hilbertian basis of \( L^2(B(E), \mu_{E,h}^K) \). The isomorphism \( J_{E,h}^K \) satisfies:

\[
(3.14) \quad J_{E,h}^K(e^\alpha) = P_{\alpha h}
\]

for every multi-index \( \alpha \).

**Proof.** The fact that the family of \( c_\alpha e^\alpha \) is orthonormal follows from (3.3). The fact that the set of functions \( c_\alpha P_{\alpha, h} \) is orthonormal in \( L^2(B(E), \mu_{E,h}^K) \) comes from the above properties on Hermite polynomials. Let us now show that this system is complete. We know that the set of functions on \( B(E) \)

\[
(3.15) \quad f(X) = \ell_a(X)^p
\]

is complete in \( L^2(B(E), \mu_{E,h}^K) \), where \( p \in \mathbb{N} \), \( a \) belongs to \( Z_C = \ell^2(E, C) \), and \( \ell_a \) is defined in Theorem 2.6 (c.f. Janson [J] or see Theorem 2.6). Let \( f \) be a function written as in (3.15), with \( a \) in \( Z_C \). There is a sequence \( (a^{(\nu)})_{\nu \geq 0} \) of elements in \( Z_C \) such that \( a^{(\nu)} \) is the vanishing sequence except for a finite number of indices, and such that the sequence \( (a^{(\nu)})_{\nu \geq 0} \) converges to \( a \) in \( Z_C \) as \( \nu \) tends to \( +\infty \). The following function:

\[
f^{(\nu)}(X) = (\ell_{a^{(\nu)}}(X))^p
\]

is a polynomial depending on a finite number of variables. In Theorem 2.6, it is seen that:

\[
\lim_{\nu \to +\infty} \| \ell_a - \ell^{(\nu)}_a \|_{L^p(B(E), \mu_{E,h}^K)} = 0.
\]

For functions \( f \) as in (3.15), we deduce that:

\[
\lim_{\nu \to +\infty} \| f - f^{(\nu)} \|_{L^2(B(E), \mu_{E,h}^K)} = 0.
\]

Consequently, the set of polynomial functions depending on a finite number of variables is dense in the space \( L^2(B(E), \mu_{E,h}^K) \). Since each polynomial functions depending on a finite number of variables is a linear combination of the \( P_{\alpha, h} \), then the set of \( c_\alpha P_{\alpha, h} \) is an Hilbertian basis of \( L^2(B(E), \mu_{E,h}^K) \). We can similarly show that the set of \( c_\alpha e^\alpha \) is an Hilbertian basis of \( \mathcal{H}(E) \). Equality (3.14) follows from (3.11), (3.10) and the following identity:

\[
\left( \frac{2}{h} \right)^{|\alpha|/2} \left( \Pi_{\alpha}(f_{\alpha}) \right)(x) = \prod_{j \in S(\alpha)} H_{\alpha_j} \left( x_j \sqrt{\frac{2}{h}} \right),
\]

\[
f_{\alpha}(x) = \prod_{j \in S(\alpha)} x_j^{\alpha_j}
\]

for every multi-index \( \alpha \), with \( n = |\alpha| \).

\[\square\]

In order to consider the phase space, we set \( u_j = (e_j, 0) \) and \( v_j = (0, e_j) \), for all \( j \in \Gamma \). For every multi-index \( (\alpha, \beta) \), we define \( u^\alpha v^\beta \) in the phase Fock space \( \mathcal{H}_\phi(E) \) as the symmetrized product of \( |\alpha| + |\beta| \) factors, each
of them being either one of the \( u_j \) \((j \in S(\alpha, \beta))\) or one of the \( v_j \) \((j \in S(\alpha, \beta))\), where each \( u_j \) appears \( \alpha_j \) times and each \( v_j \) appears \( \beta_j \) times in the symmetrized product. Set:

\[
(3.16) \quad P_{\alpha, \beta, h}(x, \xi) = \prod_{j \in S(\alpha, \beta)} H_{\alpha_j} \left( \frac{x_j}{\sqrt{h}} \right) H_{\beta_j} \left( \frac{\xi_j}{\sqrt{h}} \right)
\]

for every multi-index \((\alpha, \beta)\).

**Proposition 3.3.** The set of \( c_{\alpha} c_{\beta} u^{\alpha} v^{\beta} \) (with \( c_{\alpha} = (\alpha !)^{-1/2} \)) is an Hilbertian basis of the Fock space \( \mathcal{H}_G(E) \). The set of functions \( c_{\alpha} c_{\beta} P_{\alpha, \beta, h} \) is an Hilbertian basis of \( L^2(B(E) \times B(E), \mu_{E, h}^{\phi}) \). The Segal isomorphism \( J_{E, h}^{\phi} \) verifies:

\[
(3.17) \quad J_{E, h}^{\phi}(u^{\alpha} v^{\beta}) = P_{\alpha, \beta, h}
\]

for every multi-index \((\alpha, \beta)\).

The proof is the same as the one of proposition 3.2.

**D. Covariance formulas.**

In view of identities (3.8) we shall specify the two isomorphisms \( J_{E, h}^{K} \) and \( J_{E, h}^{\phi} \) when \( E = E_1 \cup E_2 \) with disjoint \( E_1 \) and \( E_2 \). Let \( X_E = (X_{E_1}, X_{E_2}) \) be the running variable in \( \mathbb{R}^E \times \mathbb{R}^{E_2} \). We have:

\[
(3.18) \quad \left( J_{E, h}^{\phi}(f_1 \otimes f_2) \right)(X_{E_1}, X_{E_2}) = \left( J_{E_1, h}^{\phi}(f_1) \right)(X_{E_1}) \left( J_{E_2, h}^{\phi}(f_2) \right)(X_{E_2})
\]

for every \( f_1 \) in \( \mathcal{H}_G(E_1) \) and every \( f_2 \) in \( \mathcal{H}_G(E_2) \).

**Theorem 3.4.** For any (finite or infinite) subset \( E \) of \( \Gamma \), for every \( f \) in \( \mathcal{H}(E) \), for each \( h > 0 \) and for all \( a + ib \) in \( Z_C = \ell^2(E, \mathbb{C}) \), \((a \text{ and } b \text{ being real numbers})\), we have:

\[
(3.19) \quad \left( J_{E, h}^{K} e^{i \Phi_S(a+ib)} f \right)(u) = e^{-\frac{i}{2}|b|^2 + \frac{i}{\sqrt{h}} \Omega(a, b) + \frac{i}{\sqrt{h}} \ell_a(x, \xi)} \left( J_{E, h}^{K} f \right)(u + \sqrt{h}b)
\]

for a.e. \( u \) in \( B(E) \). Similarly, for every \( F \) in \( \mathcal{H}(E) \), for each \( h > 0 \) and for all \((a + ib, a' + ib')\) in \( Z_C \times Z_C \), for a.e. \((x, \xi)\) in \( B(E) \times B(E)\), we have:

\[
(3.20) \quad \left( J_{E, h}^{K} e^{i \Phi_S(a+ib, a'+ib')} F \right)(x, \xi) = e^{\psi \left( J_{E, h}^{K} F \right)}(x + \sqrt{2hb}, \xi - \sqrt{2hb})
\]

\[
(3.21) \quad \psi = -\frac{1}{2}(|b|^2 + |b'|^2) + \frac{i}{2}(a.b + a'.b') + \frac{i}{\sqrt{2h}} \left( \ell_{a+ib}(x) + \ell_{a'+ib'}(\xi) \right)
\]

**Proof. First step:** we prove here (3.19) for \( f = \Omega_K(E) \). Let \( X = a + ib \) be in \( Z_C \). The isomorphism \( J_{E, h}^{K} \) is applied to both sides of equality (3.7). We use the definition (3.10)-(3.11) of this isomorphism together with the following notations (c.f. Janson [J]): the element \( \ell_{u_1} \cdots \ell_{u_n} : (\ell_{u_n} : \text{when } u_1 = \cdots = u_n = u) \) stands for the function \( \Pi_n(\ell_{u_1} \cdots \ell_{u_n}) \). It is called Wick product. We obtain, with obvious notations:

\[
(3.22) \quad J_{E, h}^{K} \left( e^{i \Phi_S(X)} \Omega_K(E) \right) = e^{-\frac{i}{2}X^2} \sum_{n \geq 0} \frac{i^n}{h^{n/2} n!} \sqrt{n} : \ell_{X}^n := e^{-\frac{i}{2n} X} : e^{\frac{i}{\sqrt{h}} \ell_{X}} :
\]

According to Janson [J] (Theorem 3.33):

\[
: e^{\frac{i}{\sqrt{h}} \ell_{X}} := e^{\frac{i}{\sqrt{h}} \ell_{X} + \frac{1}{2n} E(\ell_{X})}
\]
where, with standard notations:

\[ E(\ell_X^2) = \int_{B(E)} (\ell_a(u) + i \ell_b(u))^2 d\mu_{E,h}(u) = \frac{\hbar}{2} |a|^2 - |b|^2 + 2i a \cdot b \] .

We then deduce (3.19) for \( f = \Omega_K(E) \) since \( J_{Eh}^K(\Omega_K(E)) = 1 \).

**Second step:** we now prove equality (3.19) when \( f = e^{i \Phi_s(Y)} \Omega_K(E) \), with \( Y \) in \( Z_C = \ell^2(E, C) \). From a standard formula on the product of Weyl operators (c.f. Reed-Simon [RE-SI], Theorem X.41, (X.65)), which is also a particular case of the Campbell Hausdorff formula, we have:

\[ e^{i \Phi_s(X)} e^{i \Phi_s(Y)} = e^{2 \Im(X \overline{Y})} e^{i \Phi_s(X+Y)} . \]

Consequently:

\[ J_{Eh}^K e^{i \Phi_s(X)} e^{i \Phi_s(Y)} \Omega_K(E) = e^{2 \Im(X \overline{Y})} J_{Eh}^K e^{i \Phi_s(X+Y)} \Omega_K(E) . \]

We apply the first step with \( X \) replaced by \( X + Y \) and with \( X \) replaced by \( Y \). Combining these two formulas with (3.23) leads by direct computations to equality (3.19), for \( f = e^{i \Phi_s(Y)} \Omega_K(E) \), with \( Y \) in \( Z_C = \ell^2(E, C) \).

These elements of \( F_\psi(Z_C) \) are \( \mathcal{H}(E) \) (coherent states) form a total family in \( \mathcal{H}(E) \) (proposition 3.1). Therefore, equality (3.19) is valid for all \( f \) in \( \mathcal{H}(E) \). The proof of (3.20)(3.21) is a straightforward modification. Also note that the parameter \( h \) becomes \( 2h \) when considering \( J_{Eh}^\Phi \) instead of \( J_{Eh}^K \).

4. **Segal Bargmann spaces and transforms.**

We shall use the coherent states, a standard family of functions of \( L^2(\mathbb{R}^E, \lambda_E) \), where \( \lambda_E \) is the Lebesgue measure on \( \mathbb{R}^E \), for every finite subset \( E \) of \( \Gamma \). These functions may depend on the parameters \( X = (x, \xi) \) in \( \mathbb{R}^E \times \mathbb{R}^E \) and \( h > 0 \). These functions are here denoted by \( \Psi_{X,h} \) and are defined by:

\[ \Psi_{X,h}(u) = (\pi h)^{-|E|/4} e^{-\frac{|u-x|^2}{2h}} e^{i u \cdot x - \frac{i u \cdot \xi}{h}} , \quad u \in \mathbb{R}^E , \]

where the norm and the scalar product are those of \( \mathbb{R}^E \). The exponent \( E \) may be omitted from the notation. It is known that:

\[ < f, g > = (2\pi h)^{|E|/2} \int_{\mathbb{R}^E} < f, \Psi_{X,h} > < \Psi_{X,h}, g > d\lambda_E(X) \]

for all \( f \) and \( g \) in \( L^2(\mathbb{R}^E) \). In other word, one defines the mapping \( \tilde{T}_{Eh} \) from \( L^2(\mathbb{R}^E) \) into \( L^2(\mathbb{R}^E \times \mathbb{R}^E) \) (where \( \mathbb{R}^E \) is associated with the Lebesgue measure) by:

\[ (\tilde{T}_{Eh} f)(x, \xi) = (2\pi h)^{|E|/2} \int_{\mathbb{R}^E} f(u) \overline{\Psi_{(x,\xi),h}(u)} d\lambda_E(u) . \]

Since our aim will be to work in infinite dimension, we rather use the gaussian measures \( \mu^K_{E,h} \) and \( \mu^\Phi_{E,h} \) for the configuration and phase spaces, defined in Section 2. This leads us to define the following transform:

\[ (T_{Eh} f)(x, \xi) = \int_{\mathbb{R}^E} f(u) e^{\frac{i u \cdot (x-\xi)}{h} - \frac{u \cdot (x-\xi)^2}{2h}} d\mu^K_{E,h}(u) . \]

This mapping is indeed a partial isometry from \( L^2(\mathbb{R}^E, \mu^K_{E,h}) \) into \( L^2(\mathbb{R}^E \times \mathbb{R}^E, \mu^\Phi_{E,h}) \), and in finite dimension, it is called Segal Bargmann transform. One may see [FA][FO] for its properties in finite dimension. The range of this transform is the closed subspace of functions in \( L^2(\mathbb{R}^E \times \mathbb{R}^E, \mu^\Phi_{E,h}) \) which are antiholomorphic once \( \mathbb{R}^E \times \mathbb{R}^E \) is identified to \( \mathbb{C}^E \). This subspace is called the Segal Bargmann space. We note that the integral transform in (4.4) has been extended by J. Sjöstrand [SJ] when the exponent in the right hand-side is a quadratic form or a more general function.
When the set $E$ is infinite, we shall also define a mapping, also called the Segal Bargmann transform. It will be seen either as a mapping from $L^2(B(E), \mu^K_{E,h})$ into $L^2(B(E) \times B(E), \mu^\Phi_{E,h})$, which extend the one in (4.4) or, as a mapping $W_E$ from the configuration Fock space $\mathcal{H}(E)$ into the phase Fock space $\mathcal{H}_\Phi(E)$. The two points of view are equivalent according to the two Segal isomorphisms that are recalled in Section 3. Some difficulties arise when defining an analog to the integral in (4.4) in infinite dimension. Therefore, we found easier to define the Segal Bargmann transform in the abstract Fock spaces $\mathcal{H}(E)$ and $\mathcal{H}_\Phi(E)$.

A. Definition of the Segal Bargmann transform.

We shall first define this transform as a partial isometry $W_E$ from the configuration Fock space $\mathcal{H}(E)$ to the phase Fock space $\mathcal{H}_\Phi(E)$ associated with $E$, for every (finite or infinite) subset $E$.

We start by defining a partial isometry $T$ from $Z_C = \ell^2(E, C)$ and taking values in $Z_C \times Z_C$ by setting:

$$T(u) = \frac{1}{\sqrt{2}}(u, -iu)$$

for all $u \in Z_C$

There is a canonical functor, usually denoted by $\Gamma$, which associates to any continuous linear map $T$ from a complex Hilbert space $Z_1$ into a complex Hilbert space $Z_2$, with a norm smaller than 1, a map $\Gamma(T)$ from the Fock space $\mathcal{F}_s(Z_1)$ into $\mathcal{F}_s(Z_2)$, (see Derezinski Gérard [D-G] (lemma 2.6) or Reed-Simon [RE-SI], Section X.7, when $Z_1 = Z_2$). If $T$ is a partial isometry from $Z_1$ into $Z_2$, $\Gamma(T)$ is a partial isometry from $\mathcal{F}_s(Z_1)$ into $\mathcal{F}_s(Z_2)$. We shall recall the definition of $\Gamma(T)$ assuming (using the notations in Section 3), that $Z_1 = Z_C = \ell^2(E, C)$ and $Z_2 = Z_C \times Z_C$.

It is sufficient to define $\Gamma(T)$ restricted to any subspace $Z_C^\otimes n$ ($n \geq 0$). For $n = 0$, we set $\Gamma(T)(\Omega_K(E)) = \Omega_\Phi(E)$. For $n = 1$, we set $\Gamma(T)(u) = T(u)$, for every $u \in Z_C$. When $n \geq 2$, let:

$$\Gamma(T)(u_1 \otimes \cdots \otimes u_n) = T(u_1) \otimes \cdots \otimes T(u_n)$$

for all $u_1, \ldots, u_n$ in $Z_C$, ($n \geq 2$). If $T$ has a norm smaller or equal than 1 then the mapping $\Gamma(T)$ defined above is extended as a continuous linear mapping from $\mathcal{F}_s(Z_C) = \mathcal{H}(E)$ into $\mathcal{F}_s(Z_C \times Z_C) = \mathcal{H}_\Phi(E)$.

Definition 4.1. For every subset $E$ of $\Gamma$, we call Segal Bargmann transform associated with $E$, the mapping $W_E$ from $\mathcal{H}(E) = \mathcal{F}_s(Z_C)$ ($Z_C = \ell^2(E, C)$) into $\mathcal{H}_\Phi(E) = \mathcal{F}_s(Z_C \times Z_C)$, defined by $W_E = \Gamma(T)$ where $T$ is the mapping from $Z_C$ into $Z_C \times Z_C$ defined in (4.5).

We note that:

$$W_{E_1 \cup E_2} = W_{E_1} \otimes W_{E_2}$$

when $E = E_1 \cup E_2$ with disjoint $E_1$ and $E_2$.

One obtains a partial isometry $\theta_{E,h}$ from $L^2(B(E), \mu^K_{E,h})$ in $L^2(B(E) \times B(E), \mu^\Phi_{E,h})$ defined by:

$$\theta_{E,h} = J^\Phi_{E,h} \circ W_E \circ \left(J^K_{E,h}\right)^{-1}$$

when composing $W_E$ defined above with the two Segal isomorphisms.

The Lebesgue measure $\lambda_E$ becomes available again when $E$ is finite and one may define the isomorphisms $J^K_{E,h}$ and $J^\Phi_{E,h}$ between the configuration Fock space $\mathcal{H}(E)$ (resp. phase Fock space $\mathcal{H}_\Phi(E)$) and $L^2(\mathbb{R}^E, \lambda_E)$ (resp. $L^2(\mathbb{R}^E \times \mathbb{R}^E, \lambda_E \times \lambda_E)$). These isomorphisms are defined by:

$$J^K_{E,h}(f)(u) = (\pi \hbar)^{-|E|/4} J^K_{E,h}(f)(u) e^{-\frac{|u|^2}{\hbar}}$$

$$u \in \mathbb{R}^E$$
\[ \mathcal{J}_{Eh}^\Phi(f)(x, \xi) = (2\pi h)^{-|E|/2} J_{Eh}^\Phi(f)(x, \xi) e^{-\frac{|x|^2+|\xi|^2}{4h}} \quad (x, \xi) \in \mathbb{R}^E \times \mathbb{R}^E. \]

Then, we can also define a mapping \( \tilde{\theta}_{Eh} \) from \( L^2(\mathbb{R}^E, \lambda_E) \) into \( L^2(\mathbb{R}^E \times \mathbb{R}^E, \lambda_E \times \lambda_E) \) by:

\[ \tilde{\theta}_{Eh} = \mathcal{J}_{Eh}^\Phi \circ W_E \circ \left( \mathcal{J}_{Eh}^K \right)^{-1}. \]

We shall verify (Theorem 4.3) that, if \( E \) is finite then the two mappings \( T_{Eh} \) and \( \theta_{Eh} \) respectively defined in (4.4) and (4.8) are equal. This holds true with the mappings \( \tilde{T}_{Eh} \) and \( \tilde{\theta}_{Eh} \) defined (4.3) and (4.11).

B. Segal Bargmann transform and Hilbertian bases.

Let \((e_j)_{j \in E}\) be the canonical basis of \( Z = \ell^2(E, \mathbb{C}) \). Set

\[ w_j = \frac{1}{\sqrt{2}} (e_j, -ie_j). \]

For each multi-index \( \alpha \), we define an element \( w^\alpha \) of the Fock space \( \mathcal{H}_\Phi(E) \) as the symmetrized product of \(|\alpha|\) factors where each of the factor is a \( w_j \) (\( j \in S(\alpha) \)) and where each factor \( w_j \) (\( j \in S(\alpha) \)) appears exactly \( \alpha_j \) times in the symmetrized product. We also set

\[ Q_{\alpha h}(x, \xi) = (2h)^{-|\alpha|/2} \prod_{j \in S(\alpha)} (x_j - i\xi_j)^{\alpha_j}. \]

**Proposition 4.2.** Let \( c_\alpha = (\alpha!)^{-1/2} \). The set of elements \( c_\alpha w^\alpha \) (where \( S(\alpha) \subseteq E \)) is an orthonormal system of \( \mathcal{H}_\Phi(E) \). The set of functions \( c_\alpha Q_{\alpha h} \) is an orthonormal system in \( L^2(B(E) \times B(E), \mu_{E, h}) \). The \( e^\alpha \) in Section 3 satisfy:

\[ W_E e^\alpha = w^\alpha. \]

We have:

\[ J_{Eh}^\Phi w^\alpha = Q_{\alpha h}. \]

**Proof.** Only (4.15) needs to be proved. We may write:

\[ w^\alpha = \sum_{\beta + \gamma = \alpha} 2^{-|\alpha|/2} \alpha! \gamma! \frac{u^\beta v^\gamma}{\beta! \gamma!}, \]

where the \( u^\beta v^\gamma \) are defined in Section 3. From proposition 3.3, one has: \( J_{Eh}^\Phi(u^\beta v^\gamma) = P_{\alpha \beta h} \) where \( P_{\alpha \beta h} \) is defined in (3.16). A basic formula on Hermite polynomials shows that:

\[ Q_{\alpha h} = \sum_{\beta + \gamma = \alpha} 2^{-|\alpha|/2} \alpha! \gamma! \frac{P_{\alpha \beta h}}{\beta! \gamma!}. \]

The equality used, once written in one dimension, is the following one and is probably standard:

\[ (x - i\xi)^m = \sum_{p=0}^{m} \binom{m}{p} (-i)^{m-p} H_p(x) H_{m-p}(\xi). \]

Equality (4.15) then follows from (4.16), proposition 3.3, and (4.17).
C. Integral form (cylindrical case).

Theorem 4.3. For every finite subset $E$ of $\Gamma$, the two mappings $T_{Eh}$ and $\theta_{Eh}$, defined in (4.4) and (4.8), are equal. This is also true with $\tilde{T}_{Eh}$ and $\tilde{\theta}_{Eh}$ defined in (4.3) and (4.11). If $E$ is any (finite or infinite) subset of $\Gamma$ and if $f \in L^2(B(E), \mu^K_{E,h})$ depends only on the variables $u_j$ ($j \in S$), where $S$ is a finite subset of $E$, then:

\begin{equation}
(\theta_{Eh}f)(x, \xi) = \int_{\mathbb{R}^S} f(u) e^{\frac{1}{2} \varphi(x, \xi, u)} d\mu^K_{E,h}(u),
\end{equation}

where:

\begin{equation}
\varphi(x, \xi, u) = \sum_{j \in S} u_j (x_j - i \xi_j) - \frac{1}{4} (x_j - i \xi_j)^2
\end{equation}

For any finite $E$, for every $X = (x, \xi)$ in $\mathbb{R}^E \times \mathbb{R}^E$, let $\varphi_{Xh}$ be the element $\mathcal{H}(E)$ defined by:

\begin{equation}
\varphi_{Xh} = e^{-\phi_S(\xi - ix)} \Omega_K(E)
\end{equation}

with $\Phi_S$ given in (3.6). Let $\Psi^E_{Xh}$ be the coherent state defined in (4.1). Then, we have:

\begin{equation}
\tilde{T}^K_{Eh} \varphi_{Xh} = \Psi^E_{2\xi h}.
\end{equation}

Proof. If $S$ is finite then the mapping $T_{Sh}$ defined in (4.4) is the Segal Bargmann transform in finite dimension and its properties are well-known. It is a partial isometry from $L^2(\mathbb{R}^S, \mu_K^{S,h})$ into $L^2(\mathbb{R}^S \times \mathbb{R}^S, \mu^\Phi_{S,h})$ and verifies:

\begin{equation}
T_{Sh}(P_{ah}) = Q_{ah} \quad S(\alpha) \subset S
\end{equation}

where $P_{ah}$ and $Q_{ah}$ are defined in (3.13) and (4.13). From propositions 3.2 and 4.2, the mapping $\theta_{Sh}$ defined in (4.8) also satisfies:

\begin{equation}
\theta_{Sh}(P_{ah}) = Q_{ah} \quad S(\alpha) \subset S
\end{equation}

and it is also a partial isometry from $L^2(\mathbb{R}^S, \mu_K^{S,h})$ into $L^2(\mathbb{R}^S \times \mathbb{R}^S, \mu^\Phi_{S,h})$. Thus, this two mappings $T_{Eh}$ and $\theta_{Eh}$, defined in (4.4) and (4.8) are equal. Consequently, the two mappings $\tilde{T}_{Eh}$ and $\tilde{\theta}_{Eh}$ defined in (4.3) and (4.11) are also equal. When $E$ is an arbitrary subset of $\Gamma$ and if $f$ belonging to $L^2(B(E), \mu^K_{E,h})$ depends only on the variables $u_j$ ($j \in S$), where $S$ is a finite subset of $E$, then we see that $\theta_{Eh} f$ depends only on the variables $(x_j, \xi_j)$ ($j \in S$) and may be identified to the function $\theta_{Sh} f$, and then to the function $T_{Sh} f$, which is the right hand-side of (4.18). In order to prove (4.20), we apply Theorem 3.4 with $f$ replaced by $\Omega_K(E)$, $a$ by $\frac{i}{\sqrt{h}}$ and $b$ by $-\frac{i}{\sqrt{h}}$, while taking into account that $J^K_{Eh} \Omega_K(E) = 1$. We then immediately deduce (4.20) using the definition (4.9) of the isomorphism $\tilde{T}^K_{Eh}$ and the definition (4.1) of coherent states.

Theorem 4.3 provides another way to determine the mapping $\theta_{Eh}$ of (4.8) when $E$ is infinite. For each $f$ in $L^2(\mathbb{R}^E, \mu^K_{E,h})$, there is a sequence of functions $f_n$ in the same space and a sequence of finite subspaces $S_n$ of $\Gamma$, such that $f_n$ depends only on the variables $u_j$ ($j \in S_n$) and such that the sequence $(f_n)$ tends to $f$ in $L^2(\mathbb{R}^E, \mu^K_{E,h})$. This fact follows from proposition 3.2. The transforms $\theta_{Eh}(f_n)$ are determined by (4.18) with $S$ replaced by $S_n$. The sequence $(\theta_{Eh}(f_n))$ is a Cauchy sequence in $L^2(\mathbb{R}^E, \mu^K_{E,h})$ and its limit is $\theta_{Eh}(f)$.

D. The Segal Bargmann space.

We have to give a characterization of the range of the Fock space $\mathcal{H}(E)$ by $J^\Phi_{Eh} \circ W_E$, that is to say, the range of the space $L^2(B(E), \mu^K_{E,h})$ by the operator $\theta_{Eh}$. It is a closed subspace of $L^2(B(E) \times B(E), \mu^\Phi_{E,h})$. This point is well-known when $E$ is finite: the range is the subspace of $L^2(B(E) \times B(E), \mu^\Phi_{E,h})$ constituted of antiholomorphic functions. The article of B. Hall [HA] remarks that the set of (anti)holomorphic is not a closed subspace of $L^2(B(E) \times B(E), \mu^\Phi_{E,h})$. The Segal Bargmann space defined here has been introduced by
Driver-Hall [D-H]. It is defined in [D-H] as the $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$ closure of the set of antiholomorphic functions which are cylindrical, that it is to say, depending on a finite number of variables. In other words, their definition is the property i) below.

**Theorem 4.4.** Let $E$ be an infinite subset of $\Gamma$ and $h > 0$. Then, the following properties define equivalently the same closed subspace $SB(E, h)$ of $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$:

i) $SB(E, h)$ is the closure in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$ of the subspace of functions depending only on a finite number of variables, and which, additionally, are antiholomorphic when identifying $B(E) \times B(E)$ with the complexification of $B(E)$.

ii) $SB(E, h)$ is the closure in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$ of the subspace spanned by the functions $Q_{a,h}$ $(S(\alpha) \subseteq E)$.

iii) $SB(E, h)$ is the range of the configuration Fock space $H(E)$ by the mapping $J_{Eh}$.

iv) $SB(E, h)$ is the range of the space $L^2(B(E), \mu_{E,h}^\Phi)$ by the mapping $\theta_{Eh}$ (4.8).

**Proof.** In order to derive that the space defined in i) is included into the one defined in ii), we consider a function $f$, depending on a finite number of variables and being antiholomorphic. Taking Proposition 3.3 into account, we may write:

$$f = \sum_{S(\alpha, \beta) \subseteq E} f_{\alpha\beta} c_{\alpha} c_{\beta} P_{\alpha\beta h} , \quad \sum_{S(\alpha, \beta) \subseteq E} |f_{\alpha\beta}|^2 = \|f\|^2 .$$

Let $\Pi$ be the orthogonal projection on the subspace $SB(E, h)$ defined by i). Since $f = \Pi f$, we have:

$$f = \lim_{N \to +\infty} f_N , \quad f_N = \sum_{S(\alpha, \beta) \subseteq E} f_{\alpha\beta} c_{\alpha} c_{\beta} \Pi P_{\alpha\beta h} .$$

We see that $\Pi P_{\alpha\beta h}$ is a linear combination of the $Q_{\gamma,h}$ such that $|\gamma| \leq |\alpha| + |\beta|$ and $S(\gamma) \subseteq S(\alpha, \beta)$. Consequently, $f$ is in the space defined in ii). Thus, the space defined in i) is included in the space defined in ii). These two spaces are then equal. We remark that the spaces defined in ii) and iii) are equal from proposition 4.2 using that the set of $c_\alpha e^\alpha$ is an Hilbertian basis of $H(E)$ and since $J_{Eh}^\Phi \circ W_E$ is a partial isometry. The equality between the spaces defined in iii) and iv) comes from the fact that $J_{Eh}^\Phi$ is an isomorphism from $H(E)$ to $L^2(B(E), \mu_{E,h}^\Phi)$.

**Proposition 4.5.** Let $F$ and $G$ be in $SB(E, h)$. Then, for every $a$ and $b$ in $Z_R = \ell^2(E, R)$, we have:

$$\int_{B(E) \times B(E)} e^{-\frac{i}{2} \ell_{a\alpha}(x + i\xi)} F(x + a, \xi + b) G(x, \xi) d\mu_{E,h}^\Phi(x, \xi) = \int_{B(E) \times B(E)} F(X) G(X) d\mu_{E,h}^\Phi(X) .$$

**Proof.** We define the operator $T$ in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$ by

$$(T_{ab} \varphi)(x, \xi) = e^{-\frac{i}{2} \ell_{a\alpha}(x + i\xi)} \varphi(x + a, \xi + b) \quad \varphi \in L^2(B(E) \times B(E), \mu_{E,h}^\Phi) .$$

This operator is bounded in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$ with a norm smaller than $e^{W_{ab}}$. Since the functions $F$ and $G$ are in $SB(E, h)$, there are two sequences $(F_n)$ and $(G_n)$ of cylindrical (depending on a finite number of variables) antiholomorphic functions converging to $F$ and $G$ in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$. For every $n$, we have:

$$\langle (T_{ab} - I) F_n, G_n \rangle = 0 .$$

Indeed, the proof of the proposition is elementary in finite dimension. According to the continuity of $T_{ab}$, we then deduce that $\langle (T_{ab} - I) F, G \rangle = 0$, which is Proposition 4.5.
The next issue is now to give an explicit integral expression of the Bargmann transform in infinite dimension, which possibly extend (4.4). We may, according to Section 4.C, approximate any arbitrary function in $L^2(B(E), \mu^E)$ by a sequence $(f_n)$ of cylindrical functions, on which we may apply the transform (4.18), which is the cylindrical analog of (4.4). There is also a transform analog to (4.4) in [K-R], but leaving the cylindrical framework. This transform is used in Lascar [LA1]. We shall see in Theorem 4.10 that in some sense, this integral transform is the transform $\theta_{E,h}$ defined in (4.8), once restricted to $Z_\mathbb{R} \times Z_\mathbb{R}$. If $E$ is infinite then $Z_\mathbb{R} \times Z_\mathbb{R}$ is of measure zero in $B(E) \times B(E)$ (Proposition 2.2), and this notion of restriction should be first clarified.

**Theorem 4.6.** For every subset $E$ of $\Gamma$ and for each $X = (x, \xi)$ in $Z_\mathbb{R} \times Z_\mathbb{R}$, the mapping $F \to F(X)$, defined on the space spanned by the $Q_{\alpha \beta}$ $(S(\alpha) \subseteq E)$, can be extended in an unique way to a continuous linear form on the space $SB(E \times h)$. This extension is denoted by $\rho_X$. For every $F$ in $SB(E \times h)$ and for any $X = (x, \xi)$ in $Z_\mathbb{R} \times Z_\mathbb{R}$, we have:

$$\rho_X(F) = \int_{B(E) \times B(E)} e^{\frac{1}{h} \ell_{x-i}(y+i\eta)} F(Y) d\mu^E_{X,h}(Y),$$

where the above exponential is the function defined in Theorem 2.6. One also has:

$$\rho_X(F) = \int_{B(E) \times B(E)} e^{\frac{1}{h} \ell_{x+i}(y-i\eta)} F(X + Y) d\mu^E_{X,h}(Y).$$

For each $F$ in $SB(E \times h)$, the function $X \mapsto \rho_X F$ is continuous on $Z_\mathbb{R} \times Z_\mathbb{R}$ and Gâteaux antiholomorphic when identifying $Z_\mathbb{R} \times Z_\mathbb{R}$ with $Z_C$.

The integral operator $\rho_X$ is often called reproducing kernel. However, this terminology seems appropriate only when $E$ is finite, since in that case $Z_\mathbb{R} = B(E) = \mathbb{R}^E$.

**Proof of Theorem 4.6.** For each $X$ in $Z_\mathbb{R} \times Z_\mathbb{R}$, set:

$$E_X(y, \eta) = e^{\frac{1}{h} \ell_{x-i}(y+i\xi)}.$$

Then, the mapping $X \to E_X$ is continuous from $Z_\mathbb{R} \times Z_\mathbb{R}$ into $L^2(B(E) \times B(E), \mu^E_{X,h})$ and we have:

$$\|E_X\| = e^{\frac{1}{h} |\text{Re} X|^2_{\ell_2(E)}}.$$

Consequently, the integral in (4.22) properly defines a continuous linear form $\rho_X$ on $L^2(B(E) \times B(E), \mu^E_{X,h})$. The preceding remarks imply that the mapping $X \to \rho_X(F)$ is continuous on $Z_\mathbb{R} \times Z_\mathbb{R}$ and Gâteaux antiholomorphic, for each $F$ in $L^2(B(E) \times B(E), \mu^E_{X,h})$. Moreover, $\rho_X(F)$ can be expressed as in (4.23). If $F$ is a linear combination of the $Q_{\alpha \beta}$ then there is a finite subset $S$ of $E$ such that $F$ depends only on the variables $x_j$ and $\xi_j \ (j \in S)$. In this situation, this also holds in the integral (4.22) which may be written as:

$$\rho_X(F) = \rho_X(S) = (2\pi h)^{-|S|} \int_{R^S \times R^S} e^{\frac{1}{h} \ell_{x-S-i}(y-S+i\eta)} F(Y_S) e^{-\frac{1}{h} |Y_S|^2} d\lambda_S(Y_S)$$

for all $X$ in $Z_\mathbb{R} \times Z_\mathbb{R}$ and where $\lambda_S$ is the Lebesgue measure on $R^S \times R^S$. This integral makes sense since $S$ is finite. Since $F$ is identified to an antiholomorphic function on $C^F$, then the reproducing kernels theory in finite dimension shows that $\rho_X(F) = F(X)$, for every function $F$ written as a finite linear combination of the $Q_{\alpha \beta}$ and for each $X$ in $Z_\mathbb{R} \times Z_\mathbb{R}$. Therefore, $\rho_X$ defined by (4.22) or by (4.23) is the unique extension by continuity of the mapping $F \to F(X)$ defined by finite linear combinations of the $Q_{\alpha \beta}$.

In view of (4.22), we may give a fifth characterization of the space $SB(E, h)$ equivalent to those in Theorem 4.4. For each finite subset $S \subseteq E$, one may define a cylindrical reproducing kernel $\rho^S$ in the following way.
Set $T = E \setminus S$ and denote by $(X_S, X_T)$ the variable in $B(E) \times B(E)$ with $X_S$ in $\mathbb{R}^S \times \mathbb{R}^S$ and $X_T$ in $B(T) \times B(T)$. The operator $\rho^S$ is defined by:

$$
\rho^S F(X_S, X_T) = (2\pi h)^{-|S|} \int_{\mathbb{R}^S \times \mathbb{R}^S} e^{\frac{1}{h}|x_S - i\xi_S|} (y_S + i\eta_S) \ F(Y_S, X_T) e^{-\frac{1}{2h}|Y_S|^2} \ d\lambda_S(Y_S)
$$

for all $F$ in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$. This operator is bounded in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$.

**Proposition 4.7.** Let $E$ be a finite subset of $\Gamma$ and let $h > 0$. The space $SB(E, h)$ in Theorem 4.4 is also characterized by the following property:

v) $SB(E, h)$ is the set of all $F$ in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$ satisfying $\rho^S F = F$ for every finite subset $S$ of $E$.

**Proof.** Let $F$ be in the space defined by v). For any integer $N$ and every finite subset $S$ of $E$, let $\Pi_{NS}$ be the orthogonal projection operator in $L^2(B(E) \times B(E), \mu_{E,h}^\Phi)$, on the subspace spanned by the $P_{\alpha\beta h}$ with $|\alpha| + |\beta| \leq N$ and $S(\alpha, \beta) \subseteq S$. For every $\varepsilon > 0$, there are $N$ and a (finite) $S$ satisfying $\|F - \Pi_{NS} F\| < \varepsilon$. We remark that $\rho^S$ and $\Pi_{NS}$ commute. Consequently, $\Pi_{NS} F$ is a stable space by $\rho^S$. Thus, since $S$ is finite then $\Pi_{NS}$ is a Bergman anti-projection and its range consists of antiholomorphic functions. Thus, every function $F$ in the space defined by v) is a sequence of antiholomorphic functions depending only on a finite number of variables, and is therefore in the space defined by i) in Theorem 4.4.

E. Covariance formulas.

**Proposition 4.8.** For any $f$ in $\mathcal{H}(E)$, for any $Y = y + i\eta$ in $Z_C = \ell^2(E, C)$ and for a.e. $(x, \xi)$ in $B(E) \times B(E)$, we have:

$$
(4.25) \quad \left( J_{Eh}^\Phi W_E e^{\sum_{\Phi_S} (iy)} f \right) (x, \xi) = e^{-\frac{1}{h}|Y|^2 - \frac{1}{h} \sum_{y = \eta}(x - \xi)} \left( J_{Eh}^\Phi W_E f \right) (x + y, \xi + \eta).
$$

**Proof.** From the definitions of the mapping $W_E$ and of the Segal field $\Phi_S$:

$$
(4.26) \quad W_E \Phi_S(Y) f = \frac{1}{\sqrt{2}} \Phi_S(Y, -iY) W_E f
$$

for every $Y$ in $Z_C$ and every $f$ in $\mathcal{H}^{fin}(E)$. In the above right hand-side (resp. left hand-side), the Segal field acts in $\mathcal{H}_{\Phi}(E)$, (resp. in $\mathcal{H}(E)$). Taking the exponential, we deduce that:

$$
(4.27) \quad W_E e^{i\Phi_S(Y)} f = e^{i\Phi_S(\frac{y}{\sqrt{2}} e^{-\frac{1}{2h}})} W_E f
$$

for all $f$ in $\mathcal{H}(E)$ and any $Y$ in $Z_C$. Consequently,

$$
\left( J_{Eh}^\Phi W_E e^{\sum_{\Phi_S} (iy)} f \right) (x, \xi) = \left( J_{Eh}^\Phi e^{\sum_{\Phi_S} (iy)} W_E f \right) (x, \xi).
$$

Applying Theorem 3.4 (point 2) with $F = W_E f$ and $b = a' = \frac{y}{\sqrt{2h}}$, $b' = -a = \frac{y}{\sqrt{2h}}$, we obtain (4.25).

**Theorem 4.9.** For every subset $E$ of $\Gamma$, for every $f$ and $g$ in $\mathcal{H}(E)$, for each $Y = y + i\eta$ in $Z_C = \ell^2(E, C)$:

$$
(4.28) \quad < e^{-\frac{i}{h\sqrt{2}} \Phi_S(y + i\eta)} f, g >_{\mathcal{H}(E)} = \ldots
$$

$$
\ldots = e^{-\frac{i}{h\sqrt{2}}} \int_{B(E) \times B(E)} e^{-\frac{1}{2h}(\ell_y(x) + \ell_{\eta}(\xi))} \left( J_{Eh}^\Phi W_E f \right) (x, \xi) \left( J_{Eh}^\Phi W_E g \right) (x, \xi) d\mu_{E,h}(x, \xi).
$$

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Proof. Let $A$ be the left hand-side of (4.28). Applying the isometries to the two functions in the scalar product, we see that:

$$A = \int_{B(E) \times B(E)} \left( J_{E,h}^\Phi W_{E} e^{-\frac{1}{\pi} \Phi \pi (y + i\eta)} f \right)(x, \xi) \overline{\left( J_{E,h}^\Phi W_{E} g \right)(x, \xi)} d\mu_{E,h}(x, \xi).$$

From Proposition 4.8, $A$ satisfies:

$$A = \int_{B(E) \times B(E)} \Phi_Y(x, \xi) \overline{\left( J_{E,h}^\Phi W_{E} g \right)(x, \xi)} d\mu_{E,h}(x, \xi),$$

$$\Phi_Y(x, \xi) = e^{-\frac{1}{\pi} \ell \pi |\xi|^2 - \frac{1}{\pi} \ell \pi (x + i\xi)(x - i\xi)} \left( J_{E,h}^\Phi W_{E} f \right)(x - \eta, \xi + y).$$

The function $\Phi$ is in $SB(E, h)$. The function $G = J_{E,h}^\Phi W_{E} g$ is also in $SB(E, h)$. Applying Proposition 4.5 to the functions $\Phi$ and $G$ with $a = \eta$ and $b = -\eta$, we learn:

$$A = \int_{B(E) \times B(E)} e^{-\frac{1}{\pi} \ell \pi (x + i\xi)} \Phi_Y(x + \eta, \xi - y) \overline{\left( J_{E,h}^\Phi W_{E} g \right)(x, \xi)} d\mu_{E,h}(x, \xi).$$

A direct computation then gives (4.28).

F. Connection to the definition of Kree Rączka.

In Kree Rączka [K-R], the Segal Bargmann transform is defined as the mapping from $L^2(B(E), \mu_{E,h}^K)$ taking values in the space of continuous functions being Gâteaux antiholomorphic on $Z_C = \ell^2(E, C)$. This integral transform is defined by the right hand-side below in equality (4.29). Identifying $Z_C$ and $Z_{\mathbb{R}} \times Z_{\mathbb{R}}$, we shall see that the transform of $f$ defined in [K-R] is actually also the function $X \rightarrow \rho_X(\theta_{E,h}(f))$, where $\theta_{E,h}$ is defined in (4.8) and $\rho_X$ is defined in Theorem 4.6. We shall also remark that any element $F$ in the Segal Bargmann space $SB(E, h)$ is uniquely determined by its "restriction" to $Z_{\mathbb{R}} \times Z_{\mathbb{R}}$, that is to say, by the function $X \rightarrow \rho_X(F)$ defined on $Z_{\mathbb{R}} \times Z_{\mathbb{R}}$. In other words, the transform of $f$ defined in Kree Rączka [K-R] uniquely determines $\theta_{E,h}(f)$. The integral in (4.29) is the finite dimensional analog of (4.4), but it makes sense, in general, only if $X = (x, \xi)$ is in $Z_{\mathbb{R}} \times Z_{\mathbb{R}}$ instead of $B(E) \times B(E)$.

Theorem 4.10. For every $f$ in $L^2(B(E), \mu_{E,h}^K)$ and for any $X = (x, \xi)$ in $Z_{\mathbb{R}} \times Z_{\mathbb{R}}$, $(Z_{\mathbb{R}} = \ell^2(E, \mathbb{R}))$, we have:

$$(4.29) \quad \rho_X(\theta_{E,h} f) = \int_{B(E)} f(u) e^{\frac{1}{2} i \ell \pi (x - i\xi)(u - i\xi) - \frac{1}{8} \pi (x - i\xi)^2} d\mu_{E,h}^K(u).$$

The proof uses the proposition below.

Proposition 4.11. For any subset $E$, for every $f$ in $\mathcal{H}(E)$, for each $Y = (y, \eta)$ in $\ell^2(E, \mathbb{R})^2$ identified to $Y = y + i\eta$ in $\ell^2(E, C)$, we have:

$$(4.30) \quad e^{-\frac{1}{8} \pi |Y|^2} \rho_Y(J_{E,h}^\Phi W_{E} f) \leq e^{\frac{1}{2} \pi \Phi \pi (Y, Y)} f, \Omega_K(E) > .$$

Proof. Fix $f$ in $\mathcal{H}(E)$. The function $F = J_{E,h}^\Phi W_{E} f$ belongs to $SB(E, h)$. Take $Y$ in $\ell^2(E, \mathbb{R})^2$. According to the second expression (4.23) of $\rho_Y$, we note:

$$(4.31) \quad \rho_Y(J_{E,h}^\Phi W_{E} f) = \int_{B(E) \times B(E)} e^{-\frac{1}{2} \pi (\ell \pi + i\ell \pi)(x - i\xi)} \left( J_{E,h}^\Phi W_{E} f \right)(X + Y) d\mu_{E,h}^\Phi(X).$$
In view of Proposition 4.8, we obtain:

\[ e^{-\frac{1}{\hbar}((\xi_\alpha + i\xi_\beta)(x - \xi))} \left( J_{Eh}^\Phi W_E f \right)(X + Y) = e^{-\frac{1}{\hbar}|Y|^2} \left( J_{Eh}^\Phi W_E e^{i\hbar \Phi_S(\eta, y)} f \right)(X) \]

for a.e. \( X = (x, \xi) \) in \( B(E) \times B(E) \). Consequently:

\[ e^{-\frac{1}{\hbar}|Y|^2} \rho_Y \left( J_{Eh}^\Phi W_E f \right) = \langle J_{Eh}^\Phi W_E e^{i\hbar \Phi_S(\eta, y)} f, 1 \rangle. \]

We deduce (4.30) since \( 1 = J_{Eh}^\Phi W_E \Omega_K(E) \) and since \( J_{Eh}^\Phi W_E \) is an isometry.

**End of the proof of Theorem 4.10.** From Proposition 4.11, we have:

\[ \rho_X \left( \theta_{Eh} f \right) = e^{\frac{1}{\hbar}|X|^2} < \left( J_{Eh}^\Phi \right)^{-1} f, e^{-\frac{1}{\hbar} \Phi_S(\xi X)} \Omega_K(E) >_H(E) \]

\[ = e^{\frac{1}{\hbar}|X|^2} < f, J_{Eh}^\Psi e^{-\frac{1}{\hbar} \Phi_S(\xi X)} \Omega_K(E) >_{L^2(B(E), \mu_{Eh})}. \]

We then apply Theorem 3.4 with the element \( \Omega_K(E) \). Thus, we derive (4.29) by direct computations.

Let us now prove that any function \( F \) in \( SB(E, h) \) is uniquely determined by its "restriction" to \( Z_\mathbb{R} \times Z_\mathbb{R} \).

**Theorem 4.12.** Suppose that \( F \) in \( SB(E, h) \) satisfies \( \rho_X(f) = 0 \) for all \( X \) in \( Z_\mathbb{R} \times Z_\mathbb{R} \). Then \( F = 0 \).

**Proof.** Since \( F \) is in \( SB(E, h) \), there is \( f \) in \( H(E) \) such that \( F = J_{Eh}^\Psi W_E f \). From Proposition 4.11, we have for each \( X = (x, \xi) \) in \( Z_\mathbb{R} \times Z_\mathbb{R} = Z_\mathbb{C} \):

\[ \rho_X(F) = e^{\frac{1}{\hbar}|X|^2} < f, e^{-\frac{1}{\hbar} \Phi_S(\xi X)} \Omega_K(E) >_H(E). \]

Under our hypothesis, the element \( f \) is then orthogonal to all coherent states of \( H(E) \), namely, the elements \( e^{-\frac{1}{\hbar} \Phi_S(\xi X)} \Omega_K(E) (X \text{ in } Z_\mathbb{C}) \). According to proposition 3.1, the set of all these elements is complete in \( H(E) \). Therefore \( f = 0 \) and then \( F = 0 \).

**5. Hybrid Weyl-anti-Wick quantization.**

Let \( E \) be a finite subset of \( \Gamma \) and \( G \) be a bounded continuous function on \( \mathbb{R}^E \times \mathbb{R}^E \), such that \( \partial_\alpha^\phi \partial_\beta G \) is well-defined, bounded and continuous on \( \mathbb{R}^E \times \mathbb{R}^E \), for every multi-index \( (\alpha, \beta) \) in \( I_2(\mathbb{E}) \). These hypotheses imply that the Weyl operator \( Op_{weyl}^h(G) \) is bounded in \( L^2(\mathbb{R}^E) \). This is a result proved by H.O. Cordes for a formula similar to (1.1). It is also valid for the Weyl formula (1.1) itself in view of the proof by A. Unterberger [U2]. For all functions \( \varphi \) and \( \psi \) in \( S(\mathbb{R}^E) \), we have:

\[ \langle Op_{weyl}^h(G) \varphi, \psi \rangle = (2\pi \hbar)^{-|E|} \int_{(\mathbb{R}^E)^3} e^{\frac{i}{\hbar}(u-v) \cdot X} G \left( \frac{u+v}{2}, t \right) \varphi(v) \psi(u) d\lambda_E(u) d\lambda_E(v) d\lambda_E(t). \]

In order to be self-contained, let us mention that (5.1) makes sense even without proving an \( L^2 \) estimate for this operator. To see this, it suffices to integrate by parts and to redefine the integral by

\[ \langle Op_{weyl}^h(G) \varphi, \psi \rangle = \ldots \]

\[ = (2\pi \hbar)^{-|E|} \int_{(\mathbb{R}^E)^3} e^{\frac{i}{\hbar}(u-v) \cdot X} G \left( \frac{u+v}{2}, t \right) \left( 1 + \frac{4|t|^2}{\hbar} \right)^N \left( 1 - (\partial_\alpha - \partial_\beta)^2 \right)^N (\varphi(v) \psi(u)) d\mu \]

\[ d\mu = d\lambda_E(u) d\lambda_E(v) d\lambda_E(t). \]

This integral is convergent when choosing \( N \) large enough, which gives a meaning to (5.1).
To a subset $E$ and a function $G$ we may also associate an anti-Wick operator. In that case, the set $E$ may be infinite and the function $G$ is measurable and bounded on $B(E) \times B(E)$. The operator $\text{Op}_{\hbar}^{AW,E}(G)$ is the bounded operator in $H(E)$ satisfying,

$$<\text{Op}_{\hbar}^{AW,E}(G)f, g>_{H(E)} = \int_{B(E) \times B(E)} G(X)(J_{Eh}^{\Phi}W_{E}f)(X)\overline{(J_{Eh}^{\Phi}W_{E}g)(X)}d\mu_{E,A}(X)$$

for every $f$ and $g$ in $H(E)$, where the Segal isomorphism $J_{Eh}^{\Phi}$ has been defined in Section 3.B.

We shall now define a hybrid operator $\text{Op}_{\hbar}^{hyb,E}(F)$ bounded in $H(\Gamma)$, for every finite subset $E$ of $\Gamma$ and every function $F$ on $B(\Gamma) \times B(\Gamma)$. It suffices to define the scalar product $<\text{Op}_{\hbar}^{hyb,E}(F)f, g>$ for all $f$ and $g$ in $H(\Gamma)$. For the sake of clarity, let us first begin with the case when:

$$f = f_{E} \otimes f_{E^{c}}, \quad g = g_{E} \otimes g_{E^{c}},$$

with $f_{E}$ and $g_{E}$ in $H(E)$, $f_{E^{c}}$ and $g_{E^{c}}$ in $H(E^{c})$. The running variable in $B(\Gamma) \times B(\Gamma)$ may be written as $(X_{E}, X_{E^{c}})$. For every $X_{E^{c}}$ in $B(E^{c}) \times B(E^{c})$, $F_{X_{E^{c}}}$ stands for the function defined on $\mathbb{R}^{E} \times \mathbb{R}^{E}$ by:

$$F_{X_{E^{c}}}(X_{E}) = F(X_{E}, X_{E^{c}}) \quad X_{E} \in \mathbb{R}^{E} \times \mathbb{R}^{E}.$$ 

When $f$ and $g$ are written as in (5.3), the hybrid operator shall be defined as, at least formally:

$$<\text{Op}_{\hbar}^{hyb,E}(F)f, g>_{H(\Gamma)} = <\text{Op}_{\hbar}^{AW,E}(\Phi)f_{E^{c}}, g_{E^{c}} >_{H(E^{c})},$$

where $\Phi$ is the function defined on $B(E^{c}) \times B(E^{c})$ by:

$$\Phi(X_{E^{c}}) = <\text{Op}_{\hbar}^{weyl,E}(F_{X_{E^{c}}})f_{E}, g_{E} >_{H(E)},$$

with $F_{X_{E^{c}}}$ given by (5.4).

Next, we consider $f$ and $g$ in $H(\Gamma)$, not necessarily written as in (5.3). We shall use the coherent states $\Psi_{X_{E,h}}$ defined in (4.1) with finite $E$. For every $X_{E}$ in $\mathbb{R}^{E} \times \mathbb{R}^{E}$ ($E$ finite), $i_{X_{E}}^{*}$ denotes the mapping from $H(\Gamma)$ into $H(E^{c})$ satisfying:

$$<i_{X_{E}}^{*}f, g>_{H(E^{c})} = <f, \phi_{X_{E,h}} \otimes g >_{H(\Gamma)},$$

with $\phi_{X_{E,h}}$ defined in (4.19). Following (4.2), we may write, at least formally

$$f = (2\pi \hbar)^{-|E|} \int_{\mathbb{R}^{E} \times \mathbb{R}^{E}} \phi_{X_{E,h}} \otimes i_{X_{E}}^{*}f \ d\lambda_{E}(X_{E})$$

for any $f$ in $H(\Gamma)$, with a finite $E$ and where $\lambda_{E}$ is the Lebesgue measure.

The expansion (5.8) may be used to verify that the definition of the hybrid operator in (5.5),(5.6) is a particular case of Definition 5.1 below. In view of Definitions (5.1) and (5.2) of the Weyl and anti-Wick operators, we are led in the general case to the following definition.

**Definition 5.1.** For every bounded continuous function $F$ on $B(\Gamma) \times B(\Gamma)$ and for any finite subset $E$ of $\Gamma$, we denote by $\text{Op}_{\hbar}^{hyb,E}(F)$ the operator in $H(\Gamma)$ such that, for all $f$ and $g$ in $H(\Gamma)$:

$$<\text{Op}_{\hbar}^{hyb,E}(F)f, g> = ...$$

$$= (2\pi \hbar)^{-2|E|} \int_{\Delta(E)} <\text{Op}_{\hbar}^{weyl,E}(F_{Z_{E^{c}}})\Psi_{X_{E,h}}^{E}, \Psi_{Y_{E,h}}^{E} > (J_{Eh}^{\Phi}W_{E^{c}}i_{X_{E,h}}^{*}f)(Z_{E^{c}}) \overline{(J_{Eh}^{\Phi}W_{E^{c}}i_{Y_{E,h}}^{*}g)(Z_{E^{c}})}...$$
Cauchy sequence in \( L^2 \) and satisfies, if \( 0 \leq \varepsilon \leq 1 \). Let \( \Gamma \) be a finite subset of \( \Gamma \). Then, the operator \( \mathcal{O}_{\mathbf{op}}^{\mathbf{hyb}, \Lambda}(\mathbf{F}) \) in Definition 5.1 is bounded in \( \mathcal{H}(\Gamma) \) and we have:

\[
\| \mathcal{O}_{\mathbf{op}}^{\mathbf{hyb}, \Lambda}(\mathbf{F}) \|_{\mathcal{L}(\mathcal{H})} \leq M \prod_{j \in \Lambda} (1 + 225 \pi K_2 \sqrt{\hbar} \varepsilon_j),
\]

where \( K_2 = \sup_{\varepsilon \in \mathbb{R}} \max(1, \varepsilon^3) \). Under hypothesis \( H_4(M, \varepsilon) \), we have:

\[
\| \mathcal{O}_{\mathbf{op}}^{\mathbf{hyb}, \Lambda}(\mathbf{F}) \|_{\mathcal{L}(\mathcal{H})} \leq M \pi \prod_{j \in \Lambda} (1 + 225 \pi K_4 \hbar \varepsilon_j^2),
\]

where \( K_4 = \sup_{\varepsilon \in \mathbb{R}} \max(1, \varepsilon^6) \).

**Theorem 5.3.** Let \( 0 < \hbar \leq 1 \). Suppose that \( \mathbf{F} \) is a function as in Theorem 5.2 and assume that \( \Lambda \) and \( \Lambda' \) are two finite subsets of \( \Gamma \) such that \( \Lambda \subseteq \Lambda' \). Then, we have:

\[
\| \mathcal{O}_{\mathbf{op}}^{\mathbf{hyb}, \Lambda}(\mathbf{F}) - \mathcal{O}_{\mathbf{op}}^{\mathbf{hyb}, \Lambda'}(\mathbf{F}) \|_{\mathcal{L}(\mathcal{H})} \leq M 225 \pi K_2 \sqrt{\hbar} \sum_{j \in \Lambda' \setminus \Lambda} \varepsilon_j \prod_{k \in \Lambda'} (1 + 225 \pi K_4 \hbar \varepsilon_k^2). \]

If \( H_4(M, \varepsilon) \) is satisfied, then

\[
\| \mathcal{O}_{\mathbf{op}}^{\mathbf{hyb}, \Lambda}(\mathbf{F}) - \mathcal{O}_{\mathbf{op}}^{\mathbf{hyb}, \Lambda'}(\mathbf{F}) \|_{\mathcal{L}(\mathcal{H})} \leq M 225 \pi K_4 \hbar \sum_{j \in \Lambda' \setminus \Lambda} \varepsilon_j^2 \prod_{k \in \Lambda'} (1 + 225 \pi K_4 \hbar \varepsilon_k^2). \]

**Theorem 5.4.** Suppose that \( \mathbf{F} \) is a function as in Theorem 5.2, and assume that \( \{ \Lambda_n \} \) is an increasing sequence of finite subsets of \( \Gamma \) with union \( \Gamma \). Then, the sequence of operators \( (\mathcal{O}_{\mathbf{op}}^{\mathbf{weyl}, \Lambda_n}(\mathbf{F}))_{n \in \mathbb{N}} \) is a Cauchy sequence in \( \mathcal{L}(\mathcal{H}(\Gamma)) \). Its limit is denoted by \( \mathcal{O}_{\mathbf{op}}^{\mathbf{weyl}}(\mathbf{F}) \). It is independent of the chosen sequence and satisfies, if \( 0 < \hbar \leq 1 \):

\[
\| \mathcal{O}_{\mathbf{op}}^{\mathbf{weyl}}(\mathbf{F}) \|_{\mathcal{L}(\mathcal{H})} \leq M \prod_{j \in \Gamma} (1 + 225 \pi K_2 \sqrt{\hbar} \varepsilon_j). \]

When the hypothesis \( H_4(M, \varepsilon) \) is verified, we have:

\[
\| \mathcal{O}_{\mathbf{op}}^{\mathbf{weyl}}(\mathbf{F}) \|_{\mathcal{L}(\mathcal{H})} \leq M \prod_{j \in \Gamma} (1 + 225 \pi K_4 \hbar \varepsilon_j^2). \]

Theorem 5.4 is a consequence of Theorem 5.2 and Theorem 5.3. If \( (\varepsilon_j)_{j \in \Gamma} \) is a summable family then

\[
\sum_{n \to \infty} \sum_{j \not\in \Lambda_n} \varepsilon_j = 0.
\]
where \((A_n)\) is a sequence as in Theorem 5.4 and the infinite product in (5.14) is convergent. The same holds true if \((\varepsilon_j^2)_{j \in \Gamma}\) is summable.

6. Changing the subset \(E\).

We have to study the relation between \(O_{\Phi}^{hyb,E_1}(F)\) and \(O_{\Phi}^{hyb,E_2}(F)\) when the two subsets \(E_1\) and \(E_2\) of \(\Gamma\) are finite and verify \(E_1 \subset E_2\) and when the function \(F\) satisfies our hypotheses.

**Proposition 6.1.** For every finite subsets \(E_1\) and \(E_2\) of \(\Gamma\) such that \(E_1 \subset E_2\) and for all functions \(F\) satisfying the hypothesis in Theorem 5.2, we have:

\[
O_{\Phi}^{hyb,E_1}(F) = O_{\Phi}^{hyb,E_2}
\left(e^{\frac{i}{\hbar} \Delta_{E_2 \setminus E_1}} F\right),
\]

where:

\[
\Delta_{E_2 \setminus E_1} = \sum_{j \in E_2 \setminus E_1} \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2}.
\]

We make use of the following notations throughout the proof. We set \(Z = E_2 \setminus E_1\) and \(\Delta = (\Lambda_n)^\circ\). Thus, the variable in \(\mathbb{R}^{E_2}\) may be written as \((x_{E_1}, x_D)\) and the one in \(B(E_1^c)\) is denoted by \((x_D, x_S)\). The proof is obtained by combining Definition 5.1 with \(E_1\) and \(E_2\), together with the following two lemmas.

**Lemma 6.2.** For every continuous function \(F\) on \(B(\Gamma) \times B(\Gamma)\) satisfying the hypotheses in Theorem 5.2, for any \(X_{E_2} = (X_{E_1}, X_D)\) and \(Y_{E_2} = (Y_{E_1}, Y_D)\) in \((\mathbb{R}^{E_2})^2\), we have:

\[
< O_{\Phi}^{(weyl,E_2)} \left(e^{\frac{i}{\hbar} \Delta_D} F_{Z_S} \right) \Psi_{X_{E_1}, X_D, h}, \Psi_{Y_{E_1}, Y_D, h} > = \ldots
\]

\[
= (2\pi\hbar)^{-|\Omega|} \int_{(\mathbb{R}^{E_2})^2} \langle O_{\Phi}^{(weyl,E_1)} (F_{Z_D, Z_S}) \Psi_{X_{E_1}, h}, \Psi_{Y_{E_1}, h} \rangle < \Psi_{X_{D}, h} < \Psi_{Z_D, h} > < \Psi_{Z_D, h}, \Psi_{Y_D, h} > d\lambda_D(Z_D).
\]

The scalar product in the above left hand-side is the scalar product of \(L^2(\mathbb{R}^{E_2})\) and in the right hand-side, it is the scalar product of \(L^2(\mathbb{R}^{E_1})\) (for the first one) and \(L^2(\mathbb{R}^D)\) (for the last two ones). Also, \(F_{Z_D, Z_S}\) stands for the function defined on \((\mathbb{R}^{E_1})^2\) by \(F_{Z_D, Z_S}(Z_{E_1}) = F(Z_{E_1}, Z_D, Z_S)\), for all \(Z_{E_1}\), in \((\mathbb{R}^{E_1})^2\).

**Proof.** The left hand-side of (6.3) is expressed using the Weyl calculus definition (5.1), replacing \(E\) by \(E_2 = E_1 \cup D\), \(G\) being replaced by \(e^{\frac{i}{\hbar} \Delta_D} F_{Z_S}\), \(u\) and \(v\) and \(t\) being replaced by \((u_{E_1}, u_D)\), \((v_{E_1}, v_D)\) and \((t_{E_1}, t_D)\), \(\varphi(v)\) by \(\Psi_{X_{E_1}, h}(v_{E_1})\Psi_{X_D, h}(v_D)\), and \(\psi(u)\) by \(\Psi_{Y_{E_1}, h}(u_{E_1})\Psi_{Y_D, h}(u_D)\). Next, we write:

\[
\left(\frac{u_{E_1} + v_{E_1}}{2}, \frac{u_D + v_D}{2}, t_{E_1}, t_D\right) = \ldots
\]

\[
= (\pi\hbar)^{-|\Omega|} \int_{(\mathbb{R}^D) \times (\mathbb{R}^D)} e^{-\frac{1}{\hbar} \int_{z_D \setminus \zeta_D} |z_D^2 - |t_D - \zeta_D|^2| F_{Z_S} \left(\frac{u_{E_1} + v_{E_1}}{2}, z_D, t_{E_1}, \zeta_D\right) dz_D d\zeta_D}. 
\]

Then, setting \(Z_D = (z_D, \zeta_D)\) we remark that:

\[
(\pi\hbar)^{-|\Omega|} \int_{\mathbb{R}^D} e^{-\frac{1}{\hbar} \int_{z_D \setminus \zeta_D} |z_D^2 - |t_D - \zeta_D|^2| e^{\frac{i}{\hbar} (u_D - v_D) t_D} dt_D = \Psi_{Z_D, h}(u_D)\Psi_{Z_D, h}(v_D). 
\]

According to Definition 5.1 with \(E\) now replaced by \(E_1\) and \(G\) replaced by \(F_{Z_D, Z_S}\), it leads to the right hand-side of (6.3).
Lemma 6.3. For any $f$ in $\mathcal{H}(\Gamma)$, for any $X_{E_1}$ in $(\mathbb{R}^{E_1})^2$, for any $Z_D$ in $(\mathbb{R}^D)^2$, for a.e. $Z_S$ in $B(S) \times B(S)$, we have:

\begin{equation}
(6.4) \quad e^{-\frac{|x|^2}{2\hbar^2}} \left( J_{DhS,h}^\Phi W_{DhS,i_{X_{E_1}}^*} f \right)(Z_D, Z_S) = ...
\end{equation}

\[ ...
= (2\pi \hbar)^{-|D|} \int_{(\mathbb{R}^D)^2} <\Psi_{X_D,h}, \Psi_{Z_D,h}>|L^2(\mathbb{R}^D)| \left( J_{DhS,h}^\Phi W_{S,h,i_{X_{E_1}}^*} f \right)(Z_S) d\lambda(X_D) .
\]

Proof. It is sufficient to prove (6.4) when $f = f_{E_1} \otimes f_D \otimes f_S$, with $f_{E_1}$ in $\mathcal{H}(E_1)$, $f_D$ in $\mathcal{H}(D)$, $f_S$ in $\mathcal{H}(S)$. Then, from (5.7):

\[ i_{X_{E_1}} f = \langle f_{E_1}, \varphi_{X_{E_1}, h} \rangle_{\mathcal{H}(E_1)} f_D \otimes f_S .
\]

Consequently, from (3.18) and (4.7):

\[ \left( J_{DhS,h}^\Phi W_{DhS,i_{X_{E_1}}^*} f \right)(Z_D, Z_S) = \langle f_{E_1}, \varphi_{X_{E_1}, h} \rangle_{\mathcal{H}(E_1)} \left( J_{DhS,h}^\Phi W_{S,h} f_D \right)(Z_D) \left( J_{S,h} f_S \right)(Z_S) .
\]

Since $D$ is finite, we have, using Proposition 4.11:

\[ e^{-\frac{|x|^2}{2\hbar^2}} \left( J_{DhS}^\Phi W_{Dh} f_D \right)(Z_D) = \langle f_D, \varphi_{Z_D, h} \rangle_{\mathcal{H}(D)}
\]

and in view of (4.2):

\[ ...
= (2\pi \hbar)^{-|D|} \int_{\mathbb{R}^D \times \mathbb{R}^D} <f_D, \varphi_{X_D, h} \rangle_{\mathcal{H}(D)} <\Psi_{X_D, h}, \Psi_{Z_D, h}>|L^2(\mathbb{R}^D)| d\lambda_D (X_D) .
\]

According to (5.7), we also have:

\[ \langle f_{E_1}, \varphi_{X_{E_1}, h} \rangle_{\mathcal{H}(E_1)} <f_D, \varphi_{X_D, h} \rangle_{\mathcal{H}(D)} f_S = i_{X_{E_1}, X_D} f .
\]

Equality (6.4) then follows when $f = f_{E_1} \otimes f_D \otimes f_S$. In view of (3.8), it also holds true in the general case using linearity and continuity of the two hand-sides.

End of the proof of Proposition 6.1. With the above notations, it suffices to show:

\begin{equation}
(6.5) \quad \langle Op_h^{(hyb, E_2)} (e^{\frac{\Phi}{2\hbar}} F) f, g \rangle_{\mathcal{H}(\Gamma)} = \langle Op_h^{(hyb, E_1)} (F) f, g \rangle
\end{equation}

for all $f$ and $g$ in $\mathcal{H}(\Gamma)$. The left hand-side is rewritten using Definition 5.1 with the set $E_2 = E_1 \cup D$ and the function $e^{\frac{\Phi}{2\hbar}} F$. The $L^2(\mathbb{R}^{E_2})$ scalar product in the new expression is rewritten using Lemma 6.2. Then, Lemma 6.3 is applied twice with $f, X_{E_1}$ and $Z_D$ and once with $g, Y_{E_1}$ and $Z_D$. Next we observe, from (2.8):

\[ d\mu_{DhS,h}^\Phi (Z_D, Z_S) = (2\pi \hbar)^{-|D|} e^{-\frac{|x|^2}{2h^2}} d\lambda_D (Z_D) d\mu_{S,h}^\Phi (Z_S) .
\]

In this new expression, it appears the right hand-side of (6.5) using Definition 5.1 with the set $E_1$ and with the function $F$. This proves (6.5) and the proof of the proposition is completed.

7. First reductions.

Let $\Lambda$ be a finite subset of $\Gamma$. Denoting by $I_\Lambda$ the identity operator in the space of bounded continuous functions on $\mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$, we have:

\begin{equation}
(7.1) \quad I_\Lambda = \sum_{E \subseteq \Lambda} \prod_{j \in E} (I - e^{\frac{\Phi}{2\hbar}}) \prod_{j \in \Lambda \setminus E} e^{\frac{\Phi}{2\hbar}} .
\end{equation}
In the above equality, the sum is running over all subsets of \( \Lambda \) including the empty subset and
\[
(7.2) \quad \Delta_j = \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2}.
\]

The operator \( T_h(E) \) stands for:
\[
(7.3) \quad T_h(E) = \prod_{j \in E} (I - e^{\Delta_j})
\]
for every finite subset \( E \) of \( \Gamma \) and every \( h > 0 \). If \( E \) is empty then \( T_h(E) = I \). From equality (7.1) and Proposition 6.1 we obtain the following identity.

**Proposition 7.1.** For every function \( F \) satisfying hypothesis \( H_2(M, \varepsilon) \) in Definition 1.3 and for any finite subset \( \Lambda \) in \( \Gamma \), we have:
\[
(7.4) \quad \mathcal{O}_h^{(\text{hyb}, \Lambda)}(F) = \sum_{E \subseteq \Lambda} \mathcal{O}_h^{(\text{hyb}, E)}(T_h(E)F).
\]
Again, the sum is running over all subsets of \( \Lambda \) including the empty subset. We have
\[
\mathcal{O}_h^{(\text{hyb}, \emptyset)}(F) = \mathcal{O}_h^{AW, \Gamma}(F).
\]

For every finite subset \( \Lambda \) and \( \Lambda' \) with \( \Lambda \subset \Lambda' \), we have:
\[
(7.5) \quad \mathcal{O}_h^{(\text{hyb}, \Lambda')}(F) - \mathcal{O}_h^{(\text{hyb}, \Lambda)}(F) = \sum_{E \in P(\Lambda, \Lambda')} \mathcal{O}_h^{(\text{hyb}, E)}(T_h(E)F),
\]
where \( P(\Lambda, \Lambda') \) is the set of all the \( E \subseteq \Lambda' \), which are not included in \( \Lambda \) and, in particular, not empty.

The most technical part of this work is the following proposition. It will be proved in Section 8.

**Proposition 7.2.** For every function \( F \) verifying hypothesis \( H_2(M, \varepsilon) \) in Definition 1.3 and for any finite subset \( E \) of \( \Gamma \), we have
\[
\| \mathcal{O}_h^{(\text{hyb}, E)}(T_h(E)F) \|_{L(H(\Gamma))} \leq M(225\pi K_2 \sqrt{h}) |E| \prod_{j \in E} \varepsilon_j,
\]
where \( K_2 = \sup_{j \in \Gamma} \max(1, \varepsilon^3_j) \). If \( E \) is empty then the norm is bounded by \( M \). If hypothesis \( H_4(M, \varepsilon) \) is satisfied then:
\[
\| \mathcal{O}_h^{(\text{hyb}, E)}(T_h(E)F) \|_{L(H(\Gamma))} \leq M(225\pi K_4 h) |E| \prod_{j \in E} \varepsilon_j^2,
\]
where \( K_4 = \sup_{j \in \Gamma} \max(1, \varepsilon^6_j) \).

Theorem 5.2 and Theorem 5.3 are easily deduced from the two above propositions.

**Proof of Theorem 5.2.** If \( H_2(M, \varepsilon) \) is verified, it is seen from Proposition 7.1 (point (7.4)) and Proposition 7.2 that:
\[
\| \mathcal{O}_h^{(\Lambda)}(F) \|_{L(H)} \leq M \sum_{E \subseteq \Lambda} (225\pi K_2 \sqrt{h}) |E| \prod_{j \in E} \varepsilon_j = M \prod_{j \in \Lambda} (1 + 225\pi K_2 \sqrt{h} \varepsilon_j),
\]
where \( K_2 = \sup_{j \in \Gamma} \max(1, \varepsilon^3_j) \). When \( H_4(M, \varepsilon) \) is verified, we have:
\[
\| \mathcal{O}_h^{(\Lambda)}(F) \|_{L(H)} \leq M \sum_{E \subseteq \Lambda} (225\pi K_4 h) |E| \prod_{j \in E} \varepsilon_j^2 = M \prod_{j \in \Lambda} (1 + 225\pi K_4 h \varepsilon_j^2),
\]
29
where $K_4 = \sup_{j \in \Gamma} \max(1, \varepsilon^5)$.

**Proof of Theorem 5.3.** According to Proposition 7.1 (point (7.5)) and Proposition 7.2, when $\Lambda$ and $\Lambda'$ are two finite subsets of $\Gamma$ with $\Lambda \subset \Lambda'$ and if $H_2(M, \varepsilon)$ is satisfied, we have:

$$\|O_{\Lambda}^{(\Lambda')} - O_{\Lambda'}^{(\Lambda)}\|_{\mathcal{L}(\mathcal{H})} \leq M \sum_{E \in P(\Lambda, \Lambda')} (225 \pi K_2 \sqrt{h})|E| \prod_{j \in E} \varepsilon_j,$$

where $P(\Lambda, \Lambda')$ is the set of $E$ in $\Lambda'$ not being included in $\Lambda$. Inequality (5.12) then follows.

The norm $N(f)$ defined below for every $f$ in $\mathcal{H}(\Gamma)$ and for any finite subset $E$ of $\Gamma$ shall be involved in the proof of Proposition 7.2 and then also in our main results. Set

$$N_E(f)^2 = (2\pi h)^{-|E|} \int_{(\mathbb{R}^2)^2} |(J_{E\times \Lambda}W_{E\times \Lambda}i_{X_{E\times \Lambda}}f)(Z_{E\times \Lambda})|^2 dX_E d\mu_E (Z_{E\times \Lambda}),$$

The next proposition will be useful.

**Proposition 7.3.** For every $f$ in $\mathcal{H}(\Gamma)$ and for any finite subset $E$ of $\Gamma$, we have $N_E(f) = \|f\|_{\mathcal{H}(\Gamma)}$.

**Proof.** Since $J_{E\times \Lambda}$ is an isometric isomorphism between $\mathcal{H}_E$ and $L^2(B(E)^2, \mu_{E\times \Lambda})$ and since $W_{E\times \Lambda}$ is a partial isometry from $\mathcal{H}(E)$ to $\mathcal{H}_E$, then we have:

$$(2\pi h)^{-|E|} \int_{(\mathbb{R}^2)^2} |(J_{E\times \Lambda}W_{E\times \Lambda}i_{X_{E\times \Lambda}}f)(Z_{E\times \Lambda})|^2 d\mu_{E\times \Lambda} (Z_{E\times \Lambda}) = \|W_{E\times \Lambda}i_{X_{E\times \Lambda}}f\|_{\mathcal{H}_E}^2 = \|i_{X_{E\times \Lambda}}f\|_{\mathcal{H}_E}^2$$

for all $X_{E\times \Lambda}$ in $(\mathbb{R}^2)^2$. We shall now show that:

$$(2\pi h)^{-|E|} \int_{(\mathbb{R}^2)^2} \|i_{X_{E\times \Lambda}}f\|_{\mathcal{H}(E\times \Lambda)}^2 = \|\langle f \rangle_{\mathcal{H}(\Gamma)} \|^2_{\mathcal{H}(\Gamma)}.$$

More generally, we shall prove that:

$$(2\pi h)^{-|E|} \int_{(\mathbb{R}^2)^2} \langle i_{X_{E\times \Lambda}}f, i_{X_{E\times \Lambda}}g \rangle_{\mathcal{H}(E\times \Lambda)} dX_E$$

for all $f$ and $g$ in $\mathcal{H} = \mathcal{H}(\Gamma)$. We first prove (7.10) when $f = f_E \otimes f_{E'}$ and $g = g_E \otimes g_{E'}$ with $f_E$ and $g_E$ in $\mathcal{H}(E)$, $f_{E'}$ and $g_{E'}$ in $\mathcal{H}(E')$. In that case, we have:

$$(2\pi h)^{-|E|} \int_{(\mathbb{R}^2)^2} \langle i_{X_{E\times \Lambda}}f, i_{X_{E\times \Lambda}}g \rangle_{\mathcal{H}(E\times \Lambda)} dX_E = ...$$

... $= (2\pi h)^{-|E|} \int_{(\mathbb{R}^2)^2} \langle \Phi_{X_{E\times \Lambda}} h \rangle_{\mathcal{H}(E\times \Lambda)} \langle \Phi_{X_{E\times \Lambda}} g \rangle_{\mathcal{H}(E\times \Lambda)} dX_E$

We have here used (4.2). Equality (7.10) then follows in the general case since both terms are continuous bilinear mappings on $\mathcal{H}(\Gamma)$ and since $\mathcal{H}(\Gamma)$ is the completion of the tensor product $\mathcal{H}(E) \otimes \mathcal{H}(E')$.

8. **Proof of Proposition 7.2.**

We shall first give a bound on the scalar product appearing in the integral (5.9) defining the hybrid quantization, when $F$ is replaced by $T_h(E)F$ with $F$ satisfying hypothesis $H_2(M, \varepsilon)$ and where $T_h(E)$ is defined in (7.3).
Proposition 8.1. If $F$ verifies hypothesis $H_2(M, \varepsilon)$ in Definition 1.3 with a constant $M > 0$ and a summable family $(\varepsilon_j)_{j \in \Gamma}$, if $E$ is a finite subset of $\Gamma$, if $0 < h \leq 1$ and if $Z_{E^c}$ is in $B(E^c) \times B(E^c)$, then:

\begin{equation}
\langle < O_{\phi}^{w, E}(T_h(E)F_{Z_{E^c}})\Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} \rangle \Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} > | \leq \ldots
\end{equation}

\begin{equation}
\ldots \leq M(450K_2\sqrt{\eta})^{(|E|)} \prod_{j \in E} \varepsilon_j \left( 1 + \frac{|x_j - y_j|^2}{h} \right)^{-1} \left( 1 + \frac{|\xi_j - \eta_j|^2}{h} \right)^{-1},
\end{equation}

where $K_2 = \sup_{j \in \Gamma} \max(1, \varepsilon_j^3)$. When the hypothesis $H_4(M, \varepsilon)$ is satisfied, we have:

\begin{equation}
\langle < O_{\phi}^{w, E}(T_h(E)F_{Z_{E^c}})\Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} \rangle \Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} > | \leq \ldots
\end{equation}

\begin{equation}
\ldots \leq M(450K_4\varepsilon)^{|E|} \prod_{j \in E} \varepsilon_j \left( 1 + \frac{|x_j - y_j|^2}{h} \right)^{-1} \left( 1 + \frac{|\xi_j - \eta_j|^2}{h} \right)^{-1},
\end{equation}

where $K_4 = \sup_{j \in \Gamma} \max(1, \varepsilon_j^9)$.

Proposition 8.1 will rely on propositions 8.2, 8.3 and 8.4 below. The first one is concerned with an integral expression of the left hand-side of (8.1).

Proposition 8.2. For any finite subset $E$ of $\Gamma$, for every bounded continuous function $G$ on $\mathbb{R}^E \times \mathbb{R}^E$, for all $X_E$ and $Y_E$ in $(\mathbb{R}^E)^2$, we have:

\begin{equation}
\langle < O_{\phi}^{w, E}(G)\Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} \rangle \Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} > = (\pi h)^{|E|} \int_{(\mathbb{R}^E)^2} G(Z_E)e^{-\frac{1}{\pi h} \frac{|x_j - y_j|^2}{h}} e^{\frac{i}{\pi} \varphi(X_E, Y_E, Z_E)}d\lambda(E)(Z_E)
\end{equation}

setting $X_E = (x_E, \xi_E)$, $Y_E = (y_E, \eta_E)$, $Z_E = (z_E, \zeta_E)$ and

\begin{equation}
\varphi(X_E, Y_E, Z_E) = z_E \cdot (\xi_E - \eta_E) - \zeta_E \cdot (x_E - y_E) + \frac{1}{2}(x_E \cdot \eta_E - y_E \cdot \xi_E).
\end{equation}

This proposition is derived by Unterberger [U2] (formula (1.3)). In [U2], the integral kernel in (8.2) is defined as the Wigner function associated with the two coherent states $\Psi_{X_{E^c}, \psi_{Y_{E^c}}}$ and $\Psi_{X_{E^c}, \psi_{Y_{E^c}}}$, taken at the point $Z_E$. It is expressed in (1.4) of [U2] and direct computations give (8.2) and (8.3).

Proposition 8.3. For every function $G$ satisfying hypothesis $H_2(M, \varepsilon)$ in Definition 1.3, for each $Z_{E^c}$ in $B(E^c) \times B(E^c)$ and for each $h > 0$:

\begin{equation}
\prod_{j \in E} \left( 1 + \frac{|x_j - y_j|^2}{h} \right)^{-1} \left( 1 + \frac{|\xi_j - \eta_j|^2}{h} \right)^{-1} | < O_{\phi}^{w, E}(G_{Z_{E^c}})\Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} \rangle \Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} > | \leq \ldots
\end{equation}

\begin{equation}
\ldots \leq C^{(|E|)} \sum_{(\alpha, \beta) \in I_m(E)} h^{(|\alpha| + |\beta|)/2} ||\partial_{\alpha}^{\beta} G_{Z_{E^c}}||_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)},
\end{equation}

where $C = 25$ and $I_m(E) = \{0, 1, \ldots, m\}^E \times \{0, 1, \ldots, m\}^E$ for all $m \geq 1$.

Proof. Integrating by parts (8.2) with $G$ replaced by $G_{Z_{E^c}}$ yields:

\begin{equation}
| x_j - y_j |^2 \langle < O_{\phi}^{w, E}(G_{Z_{E^c}})\Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} \rangle \Psi_{X_{E^c}, \psi_{Y_{E^c}}}^{E} > = \ldots
\end{equation}

\begin{equation}
= -h^2 (\pi h)^{-|E|} \int_{(\mathbb{R}^E)^2} e^{\frac{1}{\pi} \varphi(X_E, Y_E, Z_E)} \frac{\partial^2}{\partial X_j^2} \left[ G(Z_E, Z_{E^c})e^{-\frac{1}{\pi h} \frac{|x_j - y_j|^2}{h}} \right] d\lambda(Z_E)
\end{equation}

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for \( j \in E \), where \( \varphi \) is defined in (8.3). Iterating this process, we obtain:

\[
\prod_{j \in E} \left( 1 + \frac{|x_j - y_j|^2}{h} \right) \left( 1 + \frac{|\zeta_j - \eta_j|^2}{h} \right) \leq O p_h^{\text{wexp}.E}(G_{Z_{E^c}}) \Psi_{X_{E^c h}, \Psi_{Y_{E^c h}}} \leq \ldots
\]

\[
\leq (\pi h)^{-|E|} \int_{(\mathbb{R}^E)^2} |H(X_E, Y_E, Z_E, Z_{E^c})| \, d\lambda(Z_E),
\]

where:

\[
H(X_E, Y_E, Z_E, Z_{E^c}) = \prod_{j \in E} \left( 1 - h \frac{\partial^2}{\partial z_j^2} \right) \left( 1 - h \frac{\partial^2}{\partial \zeta_j^2} \right) \left[ G(Z_E, Z_{E^c}) e^{-\frac{1}{2} \left| Z_{E^c} - \frac{X_E + Y_E}{2} \right|^2} \right].
\]

Clearly:

\[
H(X_E, Y_E, Z_E, Z_{E^c}) = e^{-\frac{1}{2} \left| Z_{E^c} - \frac{X_E + Y_E}{2} \right|^2} \prod_{j \in E} L_{z_j} L_{\zeta_j} G(Z_E, Z_{E^c}),
\]

where we use the notation:

\[
L_{z_j} = \sum_{k=0}^{3} p_k \left( h^{-1/2} \left( z_j - \frac{x_j + y_j}{2} \right) \right) h^{k/2} \frac{\partial^k}{\partial z_j^k},
\]

where

\[
p_0(x) = 3 - 4x^2 \quad \quad p_1(x) = 4x \quad \quad p_2(x) = -1
\]

and with similar notations for \( L_{\zeta_j} \). We may write:

\[
\prod_{j \in E} L_{z_j} L_{\zeta_j} = \sum_{(\alpha, \beta) \in I_E(E)} h^{[\alpha + |\beta|]/2} A_{\alpha \beta} \left( h^{-1/2} \left( Z_E - \frac{X_E + Y_E}{2} \right) \right) \partial_{\alpha} \partial_{\zeta}^\beta
\]

with:

\[
A_{\alpha \beta}(z, \zeta) = \prod_{j \in E} p_{\alpha_j}(z_j) p_{\beta_j}(\zeta_j).
\]

Let \( C > 0 \) be such that:

\[
(8.5) \quad C \frac{\pi}{2} \geq \pi^{-1/2} \int_{\mathbb{R}} |p_k(x)| e^{-x^2} \, dx \quad k = 0, 1, 2.
\]

Inequality (8.4) then holds true. One may choose \( C = 25 \) in order to satisfy (8.5).

Now, we shall apply Proposition 8.3 setting \( G = T_h(E)F \) where \( T_h(E) \) is defined in (7.3) and \( F \) verifies hypothesis \( H_2(M, \varepsilon) \). Replacing \( G \) by \( T_h(E)F \), the constants in the right hand-side of (8.4) will be improved.

**Proposition 8.4.** Suppose \( 0 \leq h < 1 \) and assume that \( F \) satisfies hypothesis \( H_2(M, \varepsilon) \). For any \( Z_{E^c} \in B(E^c) \times B(E^c) \), we have:

\[
(8.6) \quad \sup_{(\alpha, \beta) \in I_E(E)} h^{[\alpha + |\beta|]/2} \| \partial_{\alpha} \partial_{\zeta}^\beta T_h(E)F_{Z_{E^c}} \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)} \leq 2^{|E|} \sup_{(\alpha, \beta) \in I_E(E)} h^{[\alpha + |\beta|]/2} \| \partial_{\alpha} \partial_{\zeta}^\beta F_{Z_{E^c}} \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)}
\]

where \( I_E(E) \) is the set of multi-indices \( (\alpha, \beta) \) in \( I_E(E) \), such that \( \alpha_j + \beta_j \geq 1 \) for all \( j \in E \). We also have, for any \( 0 \leq h < 1 \) and for any \( F \) verifying the assumption \( H_4(M, \varepsilon) \), the following estimates:

\[
(8.7) \quad \sup_{(\alpha, \beta) \in I_E(E)} h^{[\alpha + |\beta|]/2} \| \partial_{\alpha} \partial_{\zeta}^\beta T_h(E)F_{Z_{E^c}} \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)} \leq \sup_{(\alpha, \beta) \in I_E(E)} h^{[\alpha + |\beta|]/2} \| \partial_{\alpha} \partial_{\zeta}^\beta F_{Z_{E^c}} \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)}
\]

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where $\tilde{I}_4(E)$ is the set of every multi-indices $(\alpha, \beta)$ in $I_4(E)$ satisfying $\alpha_j + \beta_j \geq 2$ for each $j \in E$.

Proof. We first note that:

$$e^{\frac{h}{4} \Delta_j} - I = \frac{h}{4} \Delta_j V_{jh}$$

with the operator $V_{jh}$ given by:

$$(V_{jh} F)(x, \xi) = (\pi h)^{-1} \int_{\mathbb{R}^2 \times [0,1]} e^{-\frac{h}{4} (u^2 + v^2)} 2\theta F(x + \theta u e_j, \xi + \theta v e_j) du d\theta$$

where $(e_j)_{j \in E}$ is the canonical basis of $\mathbb{R}^E$. Next, we observe that the operator $V_{jh}$ is bounded in $L^\infty(\mathbb{R}^E \times \mathbb{R}^E)$ with a norm smaller than 1. Besides, the operator $e^{\frac{h}{4} \Delta_j} - I$ is also bounded in $L^\infty(\mathbb{R}^E \times \mathbb{R}^E)$ with a norm smaller than 2. Consequently:

$$\| \partial^\alpha \partial^\beta T_h(E) F \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)} \leq 2^{|E|} \| (\prod_{j \in E} M_j) F \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)}$$

for any multi-index $(\alpha, \beta)$ in $I_2(E)$, where

$$M_j = \partial^{\alpha_j} \partial^{\beta_j} \partial^{\gamma_j} \partial^{\delta_j} \quad \text{if} \quad \alpha_j + \beta_j + \gamma_j + \delta_j \geq 1 \quad \text{or} \quad \alpha_j = \beta_j = 0 .$$

We then deduce (8.6). When $F$ satisfies $H_4(M, \varepsilon)$, we also have:

$$\| \partial^2 \partial^2 T_h(E) F \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)} \leq \| (\prod_{j \in E} M'_j) F \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)}$$

for every multi-index $(\alpha, \beta)$ in $I_2(E)$, with

$$M'_j = \frac{h}{4} \Delta_j \partial^{\alpha_j} \partial^{\beta_j} \partial^{\gamma_j} \partial^{\delta_j}$$

for all $j \in E$. Then we obtain (8.7) .

Proof of Proposition 8.1. We apply Proposition 8.3 with $G = T_h(E) F$ and Proposition 8.4 with $F$ where $F$ satisfies hypothesis $H_2(M, \varepsilon)$. We notice that the number of multi-indices $(\alpha, \beta)$ in $I_2(E)$ is $9^{|E|}$. We also remark that, $h^{(|\alpha| + |\beta|)/2} \leq h^{|E|/2}$, for $0 < h \leq 1$, for all $(\alpha, \beta)$ in $\tilde{I}_2(E)$ and, $h^{(|\alpha| + |\beta|)/2} \leq h^{|E|}$, for all $(\alpha, \beta)$ in $\tilde{I}_4(E)$. We then observe that, under the assumption $H_2(M, \varepsilon)$, we have:

$$\| \partial^2 \partial^2 F_{Z,E} \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)} \leq MK^{|E|}_2 \prod_{j \in E} \varepsilon_j \quad (\alpha, \beta) \in \tilde{I}_2(E)$$

and when $H_4(M, \varepsilon)$ is verified:

$$\| \partial^2 \partial^2 F_{Z,E} \|_{L^\infty(\mathbb{R}^E \times \mathbb{R}^E)} \leq MK^{|E|}_4 \prod_{j \in E} \varepsilon_j^2 \quad (\alpha, \beta) \in \tilde{I}_4(E) .$$

Proposition 8.1 is then deduced.

End of the proof of Proposition 7.2. In view of Definition 5.1 concerning the hybrid quantization and proposition 8.1, together with the Schur lemma, we deduce that:

$$| < O_{\text{ph}_h}^{h} \psi \mathcal{E}(T_h(E) F) \rangle f, g > | \leq M \left( \frac{CK^2 \sqrt{M} \pi}{2} \right)^{|E|} \prod_{j \in E} \varepsilon_j N_E(f) N_E(g)$$
for every $f$ and $g$ in $\mathcal{H}(\Gamma)$ and for all $F$ satisfying $H_{2}(M, \varepsilon)$, where $C = 450$ is the constant appearing in Proposition 8.1, $K_{2} = \sup_{j \in \Gamma} \max(1, \varepsilon_{j}^{3})$, and $N_{E}(f)$ is defined by (7.8). In the case when $F$ verifies $H_{4}(M, \varepsilon)$, we have:

\begin{equation}
\langle O_{\bar{h}}^{ab,E}(T_{\bar{h}}(E)F)\rangle \leq M \left( CK_{4}h_{\pi} \right)^{\frac{|E|}{2}} \left( \prod_{j \in E} \varepsilon_{j}^{2} \right) N_{E}(f)N_{E}(g) ,
\end{equation}

where $K_{4} = \sup_{j \in \Gamma} \max(1, \varepsilon_{j}^{6})$. When applying the Schur lemma, we have used the fact that:

\begin{equation*}
(2\pi h)^{-1/2} \int_{\mathbb{R}} \left( 1 + \frac{x^{2}}{h} \right)^{-1} dx = \sqrt{\frac{\pi}{2}} .
\end{equation*}

From Proposition 7.3, we have $N_{E}(f) = \|f\|_{\mathcal{H}(\Gamma)}$ and Proposition 7.2 is proved.

9. Comparison with previous definitions of the Weyl calculus.

Some standard works consider the case when the symbol $F$ is a continuous function on $B(\Gamma) \times B(\Gamma)$ and the Fourier transform of a bounded measure. We then assume that there exists a bounded measure $\rho$ on $Z_{\mathbb{R}} \times Z_{\mathbb{R}}$, where $Z_{\mathbb{R}} = \ell^{2}(\Gamma, \mathbb{R})$, such that:

\begin{equation}
F(x, \xi) = \int_{Z_{\mathbb{R}} \times Z_{\mathbb{R}}} e^{-i(\ell_{y}(x)+\ell_{y}(\xi))} d\rho(y, \eta) ,
\end{equation}

where $\ell_{y}$ is the function defined on $B(\Gamma)$ in Theorem 2.6, for all $y$ in $Z_{\mathbb{R}}$. Thus, $F$ is a bounded function on $B(\Gamma) \times B(\Gamma)$.

When $E$ is a finite subset of $\Gamma$, combining Theorem 3.4 (point (3.19)) and the definition (4.9) of the isomorphism $J_{\bar{h}}^{K}$ concerning the Lebesgue measure, we have:

\begin{equation*}
\left( J_{\bar{h}}^{K}e^{-\sqrt{\pi} \Phi_{S}(a+ib)} \left( J_{\bar{h}}^{K}f \right)^{-1} \right)(u) = f(u + b)e^{\frac{h}{\pi}(a+u+\frac{1}{2}a-b)}
\end{equation*}

for any $a$ and $b$ in $\mathbb{R}$ and for every $f$ in $L^{2}(\mathbb{R}, \lambda_{E})$. Besides, it is well-known that the Weyl calculus has the following property (in finite dimension):

\begin{equation*}
E(x, \xi) = e^{\frac{h}{\pi}(a+ib)} \Rightarrow \left( O_{\bar{h}}^{w}(E)f \right)(u) = f(u + b)e^{\frac{h}{\pi}(a+u+\frac{1}{2}a-b)} .
\end{equation*}

Therefore, it is natural to define the Weyl operator associated with a symbol $F$ verifying (9.1) by:

\begin{equation}
O_{\bar{h}}^{ab,w}(F) = \int_{Z_{\mathbb{R}} \times Z_{\mathbb{R}}} e^{-i\sqrt{\pi} \Phi_{S}(y+in)} d\rho(y, \eta) ,
\end{equation}

where the unbounded operator $\Phi_{S}(y + in)$, formally self-adjoint, is associated with the element $y + in$ of $Z_{\mathbb{C}}$ as in (3.6), for all $(y, \eta)$ in $Z_{\mathbb{R}} \times Z_{\mathbb{R}}$. The operator in (9.2) is indeed bounded in the symmetric Fock space $\mathcal{H}(\Gamma)$ and we have:

\begin{equation}
\|O_{\bar{h}}^{ab,w}(F)\|_{\mathcal{L}(\mathcal{H}(\Gamma))} \leq \int_{Z_{\mathbb{R}} \times Z_{\mathbb{R}}} d|\rho|(y, \eta) ,
\end{equation}

where $|\rho|$ is the absolute value measure of the bounded measure $\rho$. Definition (9.2) is considered by Kree Rakzka [K-R], Lascar [L1] and more recently by Albeverio Daletskii [A-D].
Theorem 9.1. Assume that $F$ is a continuous function on $B(\Gamma) \times B(\Gamma)$ written as in (9.1) (where $\rho$ is a bounded measure on $\mathbb{Z}_R \times \mathbb{Z}_R$ with $\mathbb{Z}_R = \ell^2(\Gamma, \mathbb{R})$) and also verifying hypothesis $H_3(M, \varepsilon)$ in Definition 1.3, where $(\varepsilon_j)$ is a summable family. Then, the two operators $O_{p_h}^{\text{weyl}}(F)$ and $O_{p_h}^{\text{old-weyl}}(F)$, respectively defined by (5.4) and (9.2), are equal.

Let $(\Lambda_n)$ be an increasing sequence of finite subsets of $\Gamma$ with union $\Gamma$. For every $y$ in $\ell^2(\Gamma, \mathbb{R})$ and any $n \geq 0$, let:

$$(p_n(y))_j = \begin{cases} y_j & \text{if } j \in \Lambda_n \\ 0 & \text{if } j \in \Lambda_n^c \end{cases}$$

and set $q_n = I - p_n$.

Lemma 9.2. Under the assumptions of Theorem 9.1, the operator $O_{p_h}^{\text{hyb}, \Lambda_n}(F)$ given by Definition 5.1 is satisfying:

$$(9.4) \quad O_{p_h}^{\text{hyb}, \Lambda_n}(F) = \int_{\mathbb{Z}_R \times \mathbb{Z}_R} e^{-i\sqrt{4}(y + iq_n)\varphi(y, \eta)} e^{-\frac{4}{\varphi}(|q_n(y)|^2 + |q_n(\eta)|^2)} d\rho(y, \eta).$$

Proof of the lemma. We need to prove that:

$$(9.5) \quad \langle O_{p_h}^{\text{hyb}, \Lambda_n}(F)f, g\rangle > \int_{\mathbb{Z}_R \times \mathbb{Z}_R} \langle e^{-i\sqrt{4}(y + iq_n)\varphi(y, \eta)} f, g \rangle > e^{-\frac{4}{\varphi}(|q_n(y)|^2 + |q_n(\eta)|^2)} d\rho(y, \eta)$$

for all $f$ and all $g$ in $\mathcal{H}$. It suffices to prove this equality when:

$$(9.6) \quad f = f_{\Lambda_n} \otimes f_{\Lambda_n^c} \quad \quad g = g_{\Lambda_n} \otimes g_{\Lambda_n^c}$$

with $f_{\Lambda_n}$ and $g_{\Lambda_n}$ in $\mathcal{H}(\Lambda_n)$, $f_{\Lambda_n^c}$ and $g_{\Lambda_n^c}$ in $\mathcal{H}(\Lambda_n^c)$. In this situation, we use Definition 5.1 of the hybrid operator $O_{p_h}^{\text{hyb}, \Lambda_n}(F)$. In this definition appears the operator $O_{p_h}^{\text{weyl}, \Lambda_n}(F_{\Lambda_n^c})$. For this operator, we may replace the definition in (5.1) by the one in (9.2). These two definitions are indeed equivalent since $\Lambda_n$ is finite. We obtain:

$$\langle O_{p_h}^{\text{hyb}, \Lambda_n}(F)f, g\rangle = \int_{(\Lambda_n^c) \times (\Lambda_n^c) \times (\Lambda_n) \times (\Lambda_n^c) \times (\Lambda_n) \times (\Lambda_n^c)} \left( \int_{\mathbb{Z}_R \times \mathbb{Z}_R} e^{-i\sqrt{4}(y + iq_n)\varphi(y, \eta)} f_{\Lambda_n} \cdot g_{\Lambda_n} \cdot e^{-\frac{4}{\varphi}(|q_n(y)|^2 + |q_n(\eta)|^2)} d\rho(y, \eta) \right).$$

From Theorem 4.8 applied with $E = \Lambda_n^c$ and $Y$ replaced by $h_{q_n}(Y)$, we see that:

$$e^{-\frac{4}{\varphi}(|q_n(y)|^2 + |q_n(\eta)|^2)} < e^{-i\sqrt{4}(y + iq_n)\varphi(y, \eta)} f_{\Lambda_n^c}, g_{\Lambda_n^c} > = \int_{(\Lambda_n^c) \times (\Lambda_n^c)} e^{-i\varepsilon_n(s_{\Lambda_n^c}) + i\varepsilon_{q_n}(s_{\Lambda_n^c})} \left( \int_{\mathbb{Z}_R \times \mathbb{Z}_R} \left( \int_{\mathbb{Z}_R \times \mathbb{Z}_R} e^{-\frac{4}{\varphi}(|q_n(y)|^2 + |q_n(\eta)|^2)} d\rho(y, \eta) \right) \right) d\mu_{\Lambda_n^c, \Lambda_n^c}.$$

Thus, we obtain (9.5) in the case (9.6). We then deduces (9.5) in the general case applying linearity, density and continuity arguments to both sides.

End of the proof of Theorem 9.1. Suppose that $O_{p_h}^{\text{old-weyl}}(F)$ denotes the operator defined by the standard relation (9.2) then, from (9.2), (9.4) and (9.3),

$$\|O_{p_h}^{\text{hyb}, \Lambda_n}(F) - O_{p_h}^{\text{old-weyl}}(F)\|_{\mathcal{L}(\mathcal{H})} \leq \int_{\mathbb{Z}_R \times \mathbb{Z}_R} \left| 1 - e^{-\frac{4}{\varphi}(|q_n(y)|^2 + |q_n(\eta)|^2)} \right| d\rho(y, \eta),$$

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Lebesgue Theorem implies that

$$\lim_{n \to +\infty} \|O_p^{(h_{\gamma}, \Lambda_n)}(F) - O_p^{\text{old-weyl}}(F)\|_{L(H(\Gamma))} = 0.$$ 

In view of Theorem 5.4, we also have, since hypothesis $H_2(M, \varepsilon)$ is satisfied and since $(\varepsilon_j)_{j \in \Gamma}$ is a summable family,

$$\lim_{n \to +\infty} \|O_p^{(h_{\gamma}, \Lambda_n)}(F) - O_p^{\text{weyl}}(F)\|_{L(H(\Gamma))} = 0.$$ 

Therefore, if $F$ is as in (9.1) and verifies hypothesis $H(M, \varepsilon)$, we deduce that the operator $O_p^{\text{old-weyl}}(F)$ given by the standard definition (which uses (9.1)) and $O_p^{\text{weyl}}(F)$ constructed in this work (which uses hypothesis $H(M, \varepsilon)$), are equal.

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