On the duality of variable Triebel-Lizorkin spaces

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Abstract

The aim of this paper is to prove duality of Triebel-Lizorkin spaces $F^{\alpha(\cdot)}_{1,q(\cdot)}$. First, we prove the duality of associated sequence spaces. Then from the so-called $\varphi$-transform characterization in the sense of Frazier and Jawerth, we deduce the main result of this paper.

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1 Introduction

In recent years, there has been growing interest in generalizing classical spaces such as Lebesgue, Sobolev spaces, Besov spaces, Triebel-Lizorkin spaces to the case with either variable integrability or variable smoothness. The motivation for the increasing interest in such spaces comes not only from theoretical purposes, but also from applications to fluid dynamics [25], image restoration [6] and PDE with non-standard growth conditions. Leopold [17, 18, 19, 20] and Leopold & Schrohe [21] studied pseudo-differential operators, they introduced related Besov spaces with variable smoothness $B^{\alpha(\cdot)}_{p,p}$. Function spaces of variable smoothness have recently been studied by Besov [2, 3, 4]. Along a different line of study, J.-S. Xu [35], [36] has studied Besov spaces with variable $p$, but fixed $q$ and $\alpha$.

Besov spaces of variable smoothness and integrability, $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, initially appeared in the paper of A. Almeida and P. Hästö [1]. Several basic properties were established, such as the Fourier analytical characterisation and Sobolev embeddings. When $p, q, \alpha$ are constants they coincide with the usual function spaces $B^{\alpha}_{p,q}$.

Variable Besov-type spaces have been introduced in [11] and [12], where their basic properties are given, such as the Sobolev type embeddings and that under some conditions these spaces are just the variable Besov spaces. For constant exponents, these spaces unify and generalize many classical function spaces including Besov spaces, Besov-Morrey spaces (see, for example, [34, Corollary 3.3]). Independently, D. Yang, C. Zhuo and W. Yuan, [33] studied these function spaces where several properties are

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obtained such as atomic decomposition and the boundedness of trace operator, see \cite{29}, \cite{30} for further properties of variable Triebel-Lizorkin-type spaces. Also, A. I. Tyubin \cite{29}, \cite{30} has studied some new function spaces of variable smoothness. Triebel-Lizorkin spaces with variable exponents $F_{p,q}^{\alpha(\cdot)}$ were introduced by \cite{7}. They proved a discretization by the so called $\phi$-transform. Also atomic and molecular decomposition of these function spaces are obtained and used it to derive trace results. The Sobolev embedding of these function spaces was proved by J. Vybiral, \cite{31}. Some properties of these function spaces such as local means characterizations and characterizations by ball means of differences can be found in \cite{15} and \cite{16}. When $\alpha, p, q$ are constants they coincide with the usual function spaces $F_{p,q}^{\alpha}$. It is well-known that duality is an important concept when we study function spaces. It applied to real interpolation and embeddings. See \cite{26}, for the duality of the usual Besov spaces $B_{p,q}^{\alpha}$ and Triebel-Lizorkin spaces $F_{p,q}^{\alpha}$, M. Izuki and T. Noi \cite{23} have obtained the duality of $B_{p,q}^{\alpha(\cdot)}$ and $F_{p,q}^{\alpha(\cdot)}$, for $1 < p^- \leq p^+ < \infty$ and $1 < q^- \leq q^+ < \infty$, see \cite{22} for the duality of $B_{p,q}^{\alpha(\cdot)}$ and $F_{p,q}^{\alpha(\cdot)}$ spaces with variable $p$, but fixed $q$ and $\alpha$.

In the present paper we obtain the duality of variable Triebel-Lizorkin spaces $F_{1,q(\cdot)}^{\alpha(\cdot)}$.

2 Preliminaries

As usual, we denote by $\mathbb{R}^n$ the $n$-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq cg$ for some independent constant $c$ (and non-negative functions $f$ and $g$), and $f \approx g$ means $f \lesssim g \lesssim f$.

By $\text{supp} f$ we denote the support of the function $f$, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of $E$ and $\chi_E$ denotes its characteristic function.

The symbol $\mathcal{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathcal{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. We define the Fourier transform of a function $f \in \mathcal{S}(\mathbb{R}^n)$ by $\mathcal{F}(f)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx$. Its inverse is denoted by $\mathcal{F}^{-1}f$. Both $\mathcal{F}$ and $\mathcal{F}^{-1}$ are extended to the dual Schwartz space $\mathcal{S}'(\mathbb{R}^n)$ in the usual way.

For $v \in \mathbb{Z}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, let $Q_{v,m}$ be the dyadic cube in $\mathbb{R}^n$, $Q_{v,m} = \{(x_1, ..., x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}$. For the collection of all such cubes we use $Q := \{Q_{v,m} : v \in \mathbb{Z}, m \in \mathbb{Z}^n\}$. For each cube $Q$, we denote by $x_{Q,v,m}$ the lower left-corner $2^{-v}m$ of $Q = Q_{v,m}$, its side length by $l(Q)$ and for $r > 0$, we denote by $rQ$ the cube concentric with $Q$ having the side length $rl(Q)$. Furthermore, we put $v_Q = -\log_2 l(Q)$, $v_Q^+ = \max(v_Q, 0)$ and $\chi_{Q_{v,m}} = \chi_{Q_{v,m}}$

For $v \in \mathbb{Z}$, $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we set \( \tilde{\varphi}(x) := \varphi(-x) \), $\varphi_v(x) := 2^v \varphi(2^v x)$, and

$$
\varphi_{v,m}(x) := 2^{v/2} \varphi(2^v x - m) = |Q_{v,m}|^{1/2} \varphi_v(x - x_{Q_{v,m}}) \quad \text{if} \quad Q = Q_{v,m}.
$$

By $c$ we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g.
c(p) means that c depends on p, etc.). Further notation will be properly introduced whenever needed.

The variable exponents that we consider are always measurable functions p on \( \mathbb{R}^n \) with range in \([c, \infty)\) for some \( c > 0 \). We denote the set of such functions by \( \mathcal{P}_0 \). The subset of variable exponents with range \([1, \infty]\) is denoted by \( \mathcal{P} \). We use the standard notation \( p^- := \text{ess-inf} \ p(x), \ p^+ := \text{ess-sup} \ p(x) \).

The variable exponent modular is defined by \( \varrho_p(f) := \int_{\mathbb{R}^n} \varrho_p(f(x)) \, dx \), where \( \varrho_p(t) = t^p \). The variable exponent Lebesgue space \( L^{p(\cdot)} \) consists of measurable functions \( f \) on \( \mathbb{R}^n \) such that \( \varrho_p(\lambda f) < \infty \) for some \( \lambda > 0 \). We define the Luxemburg (quasi)-norm on this space by the formula \( \| f \|_{p(\cdot)} := \inf \{ \lambda > 0 : \varrho_p(\lambda f) \leq 1 \} \). A useful property is that \( \| f \|_{p(\cdot)} \leq 1 \) if and only if \( \varrho_p(f(x)) \leq 1 \), see \([3, \text{Lemma 3.2.4}]\). For variable exponents, Hölder’s inequality takes the form \( \| f g \|_{s(\cdot)} \lesssim \| f \|_{p(\cdot)} \| g \|_{q(\cdot)} \) where \( s \) is defined pointwise by \( \frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)} \). Often we use the particular case \( s(x) := 1 \) corresponding to the situation when \( q = p^* \) is the conjugate exponent of \( p \).

Let \( p, q \in \mathcal{P}_0 \). The mixed Lebesgue-sequence space \( \ell^{q(\cdot)}(L^{p(\cdot)}) \) is defined on sequences of \( L^{p(\cdot)} \)-functions by the modular

\[
\mathcal{E}(L^{p(\cdot)})(\{f_v\}_v) := \inf \left\{ \sum_v \varrho_p \left( \frac{f_v}{\lambda_v^{1/q(\cdot)}} \right) \leq 1 \right\}.
\]

The (quasi)-norm is defined from this as usual:

\[
\| \{f_v\}_v \|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf \left\{ \mu > 0 : \mathcal{E}(L^{p(\cdot)})(\{f_v\}_v) \leq \mu \right\} \equiv \mathcal{E}(L^{p(\cdot)})(\{f_v\}_v).
\]

If \( q^+ < \infty \), then we can replace (1) by the simpler expression \( \mathcal{E}(L^{p(\cdot)})(\{f_v\}_v) := \sum_v \| f_v \|_{q(\cdot)} \). Furthermore, if \( p \) and \( q \) are constants, then \( \ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p) \). The case \( p := \infty \) can be included by replacing the last modular by \( \mathcal{E}(L^{p(\cdot)})(\{f_v\}_v) := \sum_v \| f_v \|_{\infty} \). Let \( p, q \in \mathcal{P}_0 \). Then \( \mathcal{E}(L^{p(\cdot)}(L^{q(\cdot)}) \) is continuous if \( p^+ < \infty \) and \( q^+ < \infty \), see \([1]\).

It is known, cf. \([1] \text{ and } [3]\), that \( \ell^{q(\cdot)}(L^{p(\cdot)}) \) is a norm if \( q(\cdot) \geq 1 \) is constant almost everywhere (a.e.) on \( \mathbb{R}^n \) and \( p(\cdot) \geq 1 \), or if \( \frac{1}{p(x)} + \frac{1}{q(x)} \leq 1 \) a.e. on \( \mathbb{R}^n \), or if \( 1 \leq q(x) \leq p(x) < \infty \) a.e. on \( \mathbb{R}^n \).

We state also the definition of the space \( L^{p(\cdot)}(\ell^{q(\cdot)}) \) which is much more intuitive then the definition of \( \ell^{q(\cdot)}(L^{p(\cdot)}) \). One just takes the \( \ell^q \) norm of \( (f_v(x))_v \) for every \( x \in \mathbb{R}^n \) and then the \( L^{p(\cdot)} \)-norm with respect to \( x \in \mathbb{R}^n \), i.e.

\[
\| (f_v)_v \|_{L^{p(\cdot)}(\ell^{q(\cdot)})} := \| (f_v(x))_v \|_{\ell^{q(\cdot)}}(x) \|_{p(\cdot)}.
\]

It is easy to show that \( L^{p(\cdot)}(\ell^{q(\cdot)}) \) is always a quasi-normed space and it is a normed space, if \( \min(p(x), q(x)) \geq 1 \) holds point-wise.

We say that \( g : \mathbb{R}^n \to \mathbb{R} \) is locally log-Hölder continuous, abbreviated \( g \in C^{\log}_{\text{loc}} \), if there exists \( c_{\log}(g) > 0 \) such that

\[
|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + 1/|x - y|)}
\]
We define the following class of variable exponents decay constant for all constants if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The and we use the convention 1

It is known that for Hölder continuous

Recall that \( \eta \) were introduced in [13, Section 2]. We define \( 1/p_\infty := \lim_{|x| \to \infty} 1/p(x) \) and we use the convention \( \frac{1}{\infty} = 0 \). Note that although \( \frac{1}{p} \) is bounded, the variable exponent \( p \) itself can be unbounded. If \( p \in P^0 \), then the convolution with a radially decreasing \( L^1 \)-function is bounded on \( L^{p(\cdot)} \): 

It is known that for \( p \in P^0 \) we have

Also,

for small balls \( B \subset \mathbb{R}^n \) (\( |B| \leq 2^n \)), with constants only depending on the log-Hölder constant of \( p \) (see, for example, [8] Section 4.5)). These properties are hold if \( p \in P^0 \), since \( \| \chi_B \|_{p(\cdot)} = \| \chi_B \|_{1/a} \) and \( \xi \in P^0 \) if \( p^- \geq a \).

Recall that \( \eta_{v,N}(x) := 2^{nv}(1 + 2^n |x|)^{-N} \), for any \( x \in \mathbb{R}^n \), \( v \in \mathbb{N}_0 \) and \( N > 0 \). Note that \( \eta_{v,N} \in L^1 \) when \( N > n \) and that \( \| \eta_{v,N} \|_1 = c_N \) is independent of \( v \). We introduce the abbreviations

The following lemma is the \( \ell^{q(\cdot)}(L^{p(\cdot)}_v) \)-version of Lemma 4.7 from A. Almeida and P. Hästö [1] (we use it, since the maximal operator is in general not bounded on \( \ell^{q(\cdot)}(L^{p(\cdot)}) \), see [1] Example 4.1]).

**Lemma 1** Let \( p \in P^0 \), \( q \in P^0 \) with \( 0 < q^- \leq q^+ < \infty \) and \( p^- > 1 \). For \( N > 2n + c_\log (1/p) + c_\log (1/q) \), there exists \( c > 0 \) such that

The arguments in [11], Lemma 2.12], are true to prove this property, in view of the fact that \( \| \chi_P \|_{p(\cdot)} \approx |P|^{1/p(\cdot)} \), since \( |P| \leq 1 \). The proof of the following lemma is given in [7, Theorem 3.2].

**Lemma 2** Let \( p,q \in P^0 \) with \( 1 < p^- \leq p^+ < \infty \) and \( 1 < q^- \leq q^+ < \infty \). For \( N > n \), there exists \( c > 0 \) such that

for all \( x,y \in \mathbb{R}^n \). We say that \( g \) satisfies the log-Hölder decay condition, if there exists \( g_\infty \in \mathbb{R} \) and a constant \( c_\log > 0 \) such that

\[
|g(x) - g_\infty| \leq \frac{c_\log}{\log(e + |x|)}
\]

for all \( x \in \mathbb{R}^n \). We say that \( g \) is globally-log-Hölder continuous, abbreviated \( g \in C^\log \), if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants \( c_\log (g) \) and \( c_\log \) are called the locally log-Hölder constant and the log-Hölder decay constant, respectively. We note that all functions \( g \in C^\log \) always belong to \( L^{\infty} \).
3 Spaces of variable smoothness and integrability

In this section we recall the definition of the spaces $\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)}$ and $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ as given in [12] and [7]. Let $\Psi$ be a function in $S(\mathbb{R}^n)$ satisfying $0 \leq \Psi(x) \leq 1$ for all $x$, $\Psi(x) = 1$ for $|x| \leq 1$ and $\Psi(x) = 0$ for $|x| \geq 2$. We put $\mathcal{F}\varphi_0(x) = \Psi(x)$, $\mathcal{F}\varphi(x) = \Psi\left(\frac{x}{2}\right) - \Psi(x)$ and $\mathcal{F}\varphi_v(x) = \mathcal{F}\varphi(2^{-v+1}x)$ for $v = 1, 2, 3, \ldots$. Then $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ is a resolution of unity, $\sum_{v=0}^{\infty} \mathcal{F}\varphi_v(x) = 1$ for all $x \in \mathbb{R}^n$. Thus we obtain the Littlewood-Paley decomposition

$$f = \sum_{v=0}^{\infty} \varphi_v * f$$

of all $f \in \mathcal{S}'(\mathbb{R}^n)$ (convergence in $\mathcal{S}'(\mathbb{R}^n)$).

Now, we define the spaces under consideration.

**Definition 1** Let $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ be a resolution of unity, $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $p, q \in \mathcal{P}_0$.

(i) The Besov-type space $\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P \in \mathcal{Q}} \left( \sum_{v \geq v_P} \|2^{vq} \varphi_v * f\|_p \chi_P \right) < \infty,$$

(ii) Let $0 < p^- \leq p^+ < \infty$ and $0 < q^- \leq q^+ < \infty$. The Triebel-Lizorkin space $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}} := \left( \sum_{v \geq 0} \|2^{vq} \varphi_v * f\|_{L^p(\mathbb{R}^n)} \right) < \infty.$$

The definition of the spaces $\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)}$ and $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is independent of the chosen resolution of unity ([5]) if $\alpha \in C^{\log}_{\text{loc}}$, $p, q \in \mathcal{P}_0^{\log}$ and $0 < q^- \leq q^+ < \infty$, $(0 < p^- \leq p^+ < \infty$ in the $F$ case) and that different choices yield equivalent quasi-norms. Using the system $\{\varphi_v\}_{v\in\mathbb{N}_0}$ we can define the norm

$$\|f\|_{B^{\alpha,\gamma}_{p,q}} := \sup_{P \in \mathcal{Q}} \frac{1}{|P|^\gamma} \left( \sum_{v \geq v_P} 2^{vq} \|2^{vq} \varphi_v * f\|_p \chi_P \right)^{1/q}$$

for constants $\alpha$ and $p, q \in (0, \infty]$. The Besov-type space $B^{\alpha,\gamma}_{p,q}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ for which $\|f\|_{B^{\alpha,\gamma}_{p,q}} < \infty$. It is well-known that these spaces do not depend on the choice of the system $\{\mathcal{F}\varphi_v\}_{v\in\mathbb{N}_0}$ (up to equivalence of quasinorms). Further details on the classical theory of these spaces can be found in [9] and [34]; see also [10] for recent developments. One recognizes immediately that if $\alpha$, $p$ and $q$ are constants, then $\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)} = B^{\alpha,1/p}_{p,q}$ and $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)} = F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ is the classical Triebel-Lizorkin spaces, see [27], [28] and [24] for the history of these spaces. When, $q := \infty$ the Besov-type space $B^{\alpha(\cdot),p(\cdot)}_{p(\cdot),\infty}$ consist of all distributions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\sup_{P \in \mathcal{Q}, v \geq v_P} \frac{2^{vq} \varphi_v * f}{|P|^{1/p} \chi_P} < \infty.$$

If we replace dyadic cubes $P$ in Definition [11] by arbitrary cubes $P$, we then obtain equivalent quasi-norms. It is clear that if $\alpha$ and $p$ are constants, then $\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)} = F^{\alpha(\cdot)}_{\infty,p}$, see [13] for the properties of $F^{\alpha(\cdot)}_{\infty,p}$. We refer to the papers [11], [12] and [33], where various results on variable Besov-type spaces were obtained.
Lemma 3 A tempered distribution $f$ belongs to $\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{\mu,\nu(\cdot)}$ if and only if,
\[ \|f\|_{\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{\mu,\nu(\cdot)}} := \sup_{\{P \in \mathcal{Q} : |P| \leq 1\}} \left\| \left( \frac{2^{\nu(\alpha(\cdot))} \varphi_{\nu(\cdot)} * f}{|P|^{1/p(\cdot)}} \chi_P \right)_{v \geq |P|} \right\|_{\ell^\nu(\cdot)(\ell^p(\cdot))} < \infty. \]

Furthermore, the quasi-norms $\|f\|_{\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{\mu,\nu(\cdot)}}$ and $\|f\|_{\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{\mu,\nu(\cdot)}}$ are equivalent.

The proof of this lemma is given in [12]. One of the key tools to prove the duality of Triebel-Lizorkin spaces $F^{\alpha(\cdot)}_{\mu,\nu(\cdot)}$ is to transfer the problem from function spaces to their corresponding sequence spaces.

Definition 2 Let $p, q \in \mathcal{P}_\infty$ and let $\alpha : \mathbb{R}^n \to \mathbb{R}$. Then for all complex valued sequences $\lambda = \{\lambda_v, m \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$ we define
\[ \widetilde{b}^{\alpha(\cdot),p(\cdot)}_{\mu(\cdot),q(\cdot)} := \left\{ \lambda : \|\lambda\|_{\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{\mu(\cdot),q(\cdot)}} < \infty \right\} \]
where
\[ \|\lambda\|_{\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{\mu(\cdot),q(\cdot)}} := \sup_{P \in \mathcal{Q}} \left\| \left( \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot) + n/2)} \lambda_v, m \chi_{v, m} \frac{|P|^{1/p(\cdot)}}{|P|^{1/1/p(\cdot)}} \chi_P \right)_{v \geq |P|} \right\|_{\ell^\mu(\cdot)(\ell^q(\cdot))}. \]

If we replace dyadic cubes $P$ by arbitrary balls $B_J$ of $\mathbb{R}^n$ with $J \in \mathbb{Z}^n$, we then obtain equivalent quasi-norms, where the supremum is taken over all $J \in \mathbb{Z}^n$ and all balls $B_J$ of $\mathbb{R}^n$.

Remark 1 Let $\alpha \in C^0_{\text{log}}$, $p, q \in \mathcal{P}_0^\text{log}$ and $0 < q^+ < \infty$. Then
\[ \|f\|_{\widetilde{B}^{\alpha(\cdot),p(\cdot)}_{q, q(\cdot)}} \approx \sup_{\{P \in \mathcal{Q} : |P| \leq 1\}} \left\| \left( \frac{2^{\nu(\alpha(\cdot))} \varphi_{\nu(\cdot)} * f}{|P|^{1/p(\cdot)}} \chi_P \right)_{v \geq |P|} \right\|_{L^{q(\cdot)}(q(\cdot))} \]
for any $f \in \widetilde{B}^{\alpha(\cdot),p(\cdot)}_{q, q(\cdot)}$. For all complex valued sequences $\lambda = \{\lambda_v, m \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n\} \in \widetilde{b}^{\alpha(\cdot),p(\cdot)}_{q, q(\cdot)}$ we have
\[ \|\lambda\|_{\widetilde{b}^{\alpha(\cdot),p(\cdot)}_{q, q(\cdot)}} \approx \sup_{\{P \in \mathcal{Q} : |P| \leq 1\}} \left\| \left( \sum_{m \in \mathbb{Z}^n} 2^{\nu(\alpha(\cdot) + n/2)} \lambda_v, m \chi_{v, m} \frac{|P|^{1/p(\cdot)}}{|P|^{1/q(\cdot)}} \chi_P \right)_{v \geq |P|} \right\|_{L^{\nu(\cdot)}(\nu(\cdot))} \]
for any $\lambda \in \widetilde{b}^{\alpha(\cdot),p(\cdot)}_{q, q(\cdot)}$, see [12].

Let $\Phi$ and $\varphi$ satisfy
\[ \text{supp}\mathcal{F}\Phi \subset \overline{B(0, 2)} \text{ and } |\mathcal{F}\Phi(\xi)| \geq c \text{ if } |\xi| \leq \frac{5}{3} \]
and
\[ \text{supp}\mathcal{F}\varphi \subset \overline{B(0, 2) \setminus B(0, 1/2)} \text{ and } |\mathcal{F}\varphi(\xi)| \geq c \text{ if } \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \]
where $c > 0$. It easy to see that $\int_{\mathbb{R}^n} x^\gamma \varphi(x) dx = 0$ for all multi-indices $\gamma \in \mathbb{N}_0^n$. By [14 pp. 130–131], there exist functions $\Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (9) and $\psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (10) such that for all $\xi \in \mathbb{R}^n$

$$\mathcal{F} \tilde{\Phi}(\xi) \mathcal{F} \Psi(\xi) + \sum_{j=1}^{\infty} \mathcal{F} \tilde{\varphi}(2^{-j} \xi) \mathcal{F} \psi(2^{-j} \xi) = 1, \quad \xi \in \mathbb{R}^n. \quad (11)$$

Furthermore, we have the following identity for all $f \in \mathcal{S}'(\mathbb{R}^n)$; see [14 (12.4)]

$$f = \Psi \ast \tilde{\Phi} \ast f + \sum_{v=1}^{\infty} \psi_v \ast \tilde{\varphi}_v \ast f$$

$$= \sum_{m \in \mathbb{Z}^n} \tilde{\Phi} \ast f(m) \Psi(\cdot - m) + \sum_{v=1}^{\infty} 2^{-vn} \sum_{m \in \mathbb{Z}^n} \tilde{\varphi}_v \ast f(2^{-v} m) \psi_v(\cdot - 2^{-v} m).$$

Recall that the $\varphi$-transform $S_{\varphi}$ is defined by setting $(S_{\varphi})_{0,m} = \langle f, \Phi_m \rangle$ where $\Phi_m(x) = \Phi(x - m)$ and $(S_{\varphi})_{v,m} = \langle f, \varphi_{v,m} \rangle$ where $\varphi_{v,m}(x) = 2^{vm/2} \varphi(2^v x - m)$ and $v \in \mathbb{N}$. The inverse $\varphi$-transform $T_\psi$ is defined by

$$T_\psi f = \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \Phi_m + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \varphi_{v,m},$$

where $\lambda = \{\lambda_{v,m} \in \mathbb{C}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$, see [14].

To prove of the main result of this paper we need the following $\varphi$-transform characterization of $\tilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)}$, see [12].

**Theorem 1** Let $\alpha \in C^1_{log}$ and $p, q \in P^1_{log}$ with $0 < q^+ < \infty$. Suppose that $\Phi, \Psi \in \mathcal{S}(\mathbb{R}^n)$ satisfying (9) and $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ satisfy (10) such that (11) holds. The operators

$$S_{\varphi} : \tilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)} \rightarrow \tilde{b}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)} \text{ and } T_{\psi} : \tilde{b}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)} \rightarrow \tilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)}$$

are bounded. Furthermore, $T_{\psi} \circ S_{\varphi}$ is the identity on $\tilde{B}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)}$.

### 4 Duality

This section is devoted to the duality of Triebel-Lizorkin spaces $F^{\alpha(\cdot)}_{1,q(\cdot)}$. In the case of constant indices $q$ and $\alpha$, this is a classical part of the theory of function spaces. Before proving the duality of these function spaces we present some results, which appeared in the paper of Frazier and Jawerth [14] for constant exponents.

**Proposition 1** Let $\alpha \in C^1_{log}$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$ and $p, q \in P^1_{log}$ with $0 < q^- \leq q^+ < \infty$. Suppose that for each dyadic cube $Q_{v,m}$ there is a set $E_{Q,v,m} \subseteq Q_{v,m}$ with $|E_{Q,v,m}| > \varepsilon|Q_{v,m}|$, $\varepsilon > 0$. Then

$$||\lambda||_{\tilde{b}^{\alpha(\cdot),p(\cdot)}_{p(\cdot),q(\cdot)}} \approx \sup_{\{P \in \mathcal{Q} \mid |P| \leq 1\}} \left( \frac{\sum_{m \in \mathbb{Z}^n} 2^{\nu(\alpha(\cdot) + n/2)} \lambda_{v,m} \chi_{E_{Q,v,m}}}{|P|^{1/p(\cdot)}} \chi_P \right)_{v \geq v_P} \|\ell^{\nu(\cdot)}_{p(\cdot)}(L^{p(\cdot)}) \|. $$
Proof. Obviously, the problem can be reduced to the case when \( \ell^q(\ell^p) \) is a normed space. Since \( \chi_{E_Q} \leq \chi_Q \) for all \( Q \in \mathcal{Q} \), one the direction is trivial. For the other, we use the estimate \( \chi_{Q_{v,m}} \leq c \eta_{v,N} * \chi_{E_{Q_{v,m}}} \) for all \( Q_{v,m} \in \mathcal{Q} \) and all \( N > 2n + c_{\log}(1/p) + c_{\log}(1/q) \). Now Lemma 1 implies that \( \|\lambda\|_{\ell^q(\ell^p)} \) is bounded by

\[
\sup_{\{P \in \mathcal{Q}, |P| \leq 1\}} \left\| \left( \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)) + n/2} \lambda_{v,m} \chi_{E_{Q_{v,m}}} \right)_{v \geq v_P} \right\|_{\ell^q(\ell^p)}.
\]

Proposition 2 Let \( \alpha \in C_{\log}^{\log}, v \in \mathbb{N}_0, m \in \mathbb{Z}^n \) and \( q \in \mathcal{P}_{\log}^0 \) with \( 0 < q^- \leq q^+ < \infty \). Suppose that for each dyadic cube \( Q_{v,m} \) there is a set \( E_{Q_{v,m}} \subseteq Q_{v,m} \) with \( |E_{Q_{v,m}}| > \varepsilon |Q_{v,m}|, \varepsilon > 0 \). Then

\[
\|\lambda\|_{\ell^q(\ell^p)} \leq c \left( \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)) + n/2} \|\lambda_{v,m}\|_{\ell^q(\ell^p)} \chi_{E_{Q_{v,m}}} \right)^{1/q}(\cdot) \|_{\ell^q(\ell^p)}.
\]

Proof. From the fact that \( \ell^q(\ell^p) = L^q(\ell^p) \), we have for any dyadic cube \( P \), with \( |P| \leq 1 \)

\[
\left\| \left( \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)) + n/2} \lambda_{v,m} \chi_{E_{Q_{v,m}}} \right)_{v \geq v_P} \right\|_{L^q(\ell^p)} = \left\| \left( \sum_{v=v_P}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)) + n/2} \|\lambda_{v,m}\|_{\ell^q(\ell^p)} \chi_{E_{Q_{v,m}}} \right)^{1/q}(\cdot) \chi_{E_{Q_{v,m}}} \right\|_{q(\cdot)}.
\]

Obviously this term is bounded by

\[
\left\| \left( \sum_{v=v_P}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot)) + n/2} \|\lambda_{v,m}\|_{\ell^q(\ell^p)} \chi_{E_{Q_{v,m}}} \right)^{1/q}(\cdot) \right\|_{\infty} \left\| \frac{1}{|P|^{1/q(\cdot)}} \chi_{E_{Q_{v,m}}} \right\|_{q(\cdot)}.
\]

Therefore, we obtain the desired inequality. 

For any dyadic cube \( P \), with \( |P| \leq 1 \), we set

\[
G_P^{\alpha(\cdot),q(\cdot)}(\lambda)(x) := \left( \sum_{v=v_P}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(x)+n/2)q(x)} \|\lambda_{v,m}\|_{\ell^q(\cdot)} \chi_{E_{Q_{v,m}}}(x) \right)^{1/q(x)}(x).
\]

We put

\[
m_P^{\alpha(\cdot),q(\cdot)}(\lambda) := \inf \left\{ \varepsilon : \{x \in P : G_P^{\alpha(\cdot),q(\cdot)}(\lambda)(x) > \varepsilon\} < \frac{|P|}{4} \right\}. \tag{12}
\]

We also set

\[
m^{\alpha(\cdot),q(\cdot)}(\lambda)(x) = \sup_P m_P^{\alpha(\cdot),q(\cdot)}(\lambda) \chi_P(x).
\]

Then we obtain.
Proposition 3 Let $\alpha \in C_{\text{loc}}^q$, $q \in P^q$ with $1 < q^{-} \leq q^{+} < \infty$ and $\lambda = \{\lambda_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \tilde{b}_{d, q(\cdot), q(\cdot)}^\alpha$. Then
\[
\|\lambda\|_{\tilde{b}_{d, q(\cdot), q(\cdot)}^\alpha} \approx \|m_{d, q(\cdot), q(\cdot)}(\lambda)\|_{\infty}.
\]

Proof. We use the arguments of [14, Proposition 5.5]. Let $P$ be any dyadic cube, with $|P| \leq 1$. We use the Chebyshev inequality,
\[
|x \in P: G_P^{\alpha(\cdot), q(\cdot)}(\lambda)(x) > \varepsilon| \\
\leq \frac{1}{\varepsilon} \int_P G_P^{\alpha(\cdot), q(\cdot)}(\lambda)(x) \, dx \\
\leq c \frac{|P|}{\varepsilon} \left\| \chi_P \right\|_{q(\cdot)} \left\| G_P^{\alpha(\cdot), q(\cdot)}(\lambda) \chi_P \right\|_{q(\cdot)},
\]
by Hölder’s inequality. Using the properties (3), (4) and (8), to estimate the last expression by
\[
c \frac{|P|}{\varepsilon} \left\| G_P^{\alpha(\cdot), q(\cdot)}(\lambda) \chi_P \right\|_{q(\cdot)} = c \frac{|P|}{\varepsilon} \left\| G_P^{\alpha(\cdot), q(\cdot)}(\lambda) \right\|_{q(\cdot)} \\
\leq c \frac{|P|}{\varepsilon} \left\| \chi_P \right\|_{q(\cdot)} \\
\leq c \frac{|P|}{\varepsilon} \left\| \lambda \right\|_{\tilde{b}_{d, q(\cdot), q(\cdot)}^\alpha}.
\]
This term is less than to $\frac{|P|}{4} \varepsilon$ if $\varepsilon > c 4 \left\| \lambda \right\|_{\tilde{b}_{d, q(\cdot), q(\cdot)}^\alpha}$. Hence,
\[
\left\| m_{d, q(\cdot), q(\cdot)}(\lambda) \right\|_{\infty} \leq c \left\| \lambda \right\|_{\tilde{b}_{d, q(\cdot), q(\cdot)}^\alpha}.
\]

Now let
\[
\tilde{j}(x) \\
= \inf \left\{ j \in \mathbb{N}_0 : \left( \sum_{v=j}^{\infty} \sum_{h \in \mathbb{Z}^n} 2^{v(\alpha(x)+n/2)q(x)} |\lambda_{v,h}|^{q(x)} \chi_{v,h}(x) \right)^{1/q(x)} \leq m_{d, q(\cdot), q(\cdot)}(\lambda)(x) \right\},
\]
and
\[
E_{v,h} = \{ x \in Q_{v,h} : 2^{-j(x)} \geq l(Q_{v,h}) \} \\
= \left\{ x \in Q_{v,h} : G_{Q_{v,h}}^{\alpha(\cdot), q(\cdot)}(\lambda)(x) \leq m_{d, q(\cdot), q(\cdot)}(\lambda)(x) \right\}
\]
for any dyadic cube $Q_{v,h}, v \in \mathbb{N}_0$ and $h \in \mathbb{Z}^n$. By (12), $|E_{Q_{v,h}}| \geq \frac{3|Q_{v,h}|}{4}$, and
\[
\left( \sum_{v=0}^{\infty} \sum_{h \in \mathbb{Z}^n} 2^{v(\alpha(x)+n/2)q(x)} |\lambda_{v,h}|^{q(x)} \chi_{E_{v,h}}(x) \right)^{1/q(x)} \leq c \left\| m_{d, q(\cdot), q(\cdot)}(\lambda) \right\|_{\infty}(x).
\]
for each $x \in \mathbb{R}^n$. Multiplying by $|P|^{-1/q'(x)}$, $\chi_P(x)$ ($P$ is a dyadic cube such that $Q_{v,h} \subset P$ and $|P| \leq 1$) and then taking the $L^{q'}(\cdot)$-norm, we obtain
\[
\left\| \left( \sum_{v=v(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot) + n/2)q'(\cdot)} |\lambda_{v,h}|^{q'(\cdot)} \chi_{E_{v,h}} \right)^{1/q'(\cdot)} \chi_P \right\|_{q'(\cdot)} \lesssim \|m^{\alpha(\cdot)q'(\cdot)}(\cdot)\|_{\infty}.
\]
From the last estimate and Proposition 1, we deduce that
\[
\|\lambda\|_{b^\alpha(\cdot)q'(\cdot)} \lesssim \|m^{\alpha(\cdot)q'(\cdot)}(\cdot)\|_{\infty}.
\]
By this proposition and Proposition 1 we obtain another equivalent norm of $b^\alpha(\cdot)q'(\cdot)$.

**Proposition 4** Let $\alpha \in C^\infty_{\text{loc}}$ and $q \in \mathcal{P}^\infty$ with $1 < q^- \leq q^+ < \infty$. Then $\lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n \} \in b^\alpha(\cdot)q'(\cdot)$ if and only if for each dyadic cube $Q_{v,m}$ there is a subset $E_{Q_{v,m}} \subset Q_{v,m}$ with $|E_{Q_{v,m}}| > |Q_{v,m}/2$ (or any other, fixed, number $0 < \varepsilon < 1$) such that
\[
\left\| \left( \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot) + n/2)q'(\cdot)} |\lambda_{v,m}|^{q'(\cdot)} \chi_{E_{Q_{v,m}}(\cdot)} \right)^{1/q'(\cdot)} \right\|_{\infty} < \infty.
\]
Moreover, the infimum of this expression over all such collections $\{E_{Q_{v,m}}\}_{v,m}$ is equivalent to $\|\lambda\|_{b^\alpha(\cdot)q'(\cdot)}$.

Suppose that $1 \leq p \leq \infty$, and $1/p + 1/q = 1$. In the classical Lebesgue space,
\[
\|f\|_p = \sup \left| \int f(x) g(x) dx \right|,
\] where the supremum is taken over all $g \in L^q$ with $\|g\|_q \leq 1$. Our aim is to extend this result to $b^\alpha(\cdot)q'(\cdot)$; see [3] for variable Lebesgue spaces. Let $q \in \mathcal{P}$, $\alpha : \mathbb{R}^n \to \mathbb{R}$. We define the conjugate norm to $b^\alpha(\cdot)q'(\cdot)$. This is the functional $\|\lambda\|^{p'}_{b^{-\alpha(\cdot)q'(\cdot)}}$ given by
\[
\|\lambda\|^{p'}_{b^{-\alpha(\cdot)q'(\cdot)}} = \sup \left| \int_P \frac{1}{|P|} \sum_{v=v(P)}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} s_{v,m}(x) \chi_{v,m}(x) dx \right|, \quad \lambda = \{\lambda_{v,m} \in \mathbb{C} : v \in \mathbb{N}_0, m \in \mathbb{Z}^n \}
\] where the supremum is taking all dyadic cube $P$, with $|P| \leq 1$ and over all sequence of functions $s = \{s_{v,m} \}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$ such that
\[
\|s\|^{p'}_{b^{-\alpha(\cdot)q'(\cdot)}} = \sup_{\{P \in \mathcal{Q} : |P| \leq 1\}} \left\| \left( \sum_{m \in \mathbb{Z}^n} 2^{v(\alpha(\cdot) - n/2)q'(\cdot)} s_{v,m}(\cdot) \chi_{v,m}(\cdot) \right)^{1/q'(\cdot)} \chi_P \right\|_{L^{p'}(\cdot)} \leq 1.
\] Let us start with the following lemma.
Lemma 4 Let \( q \in \mathcal{P}, 1 < q^- \leq q^+ < \infty, \alpha : \mathbb{R}^n \to \mathbb{R}, \lambda = \{\lambda_{v,m} \in \mathbb{C}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \) and \( \mathcal{Q}_0 = \{Q \in \mathcal{Q} : |Q| \leq 1\}. \) If \( \|\lambda\|_{b_{q'(\cdot), q'(\cdot)}}^\alpha \leq 1 \) and

\[
\rho_{L^\alpha(\mathcal{L}^{q'(\cdot)})}(\left( \frac{1}{|P|^{1/q'(\cdot)}} \sum_{m \in \mathbb{Z}^n} 2^{q(-\alpha() + n/2)} |\lambda_{v,m}| \chi_{v,m} \right)_{v \geq v_P}) < \infty
\]

for any \( P \in \mathcal{Q}_0, \) then

\[
\rho_{L^\alpha(\mathcal{L}^{q'(\cdot)})}(\left( \frac{1}{|P|^{1/q'(\cdot)}} \sum_{m \in \mathbb{Z}^n} 2^{q(-\alpha() + n/2)} |\lambda_{v,m}| \chi_{v,m} \right)_{v \geq v_P}) \leq 1
\]

for any \( P \in \mathcal{Q}_0. \)

**Proof.** Assume, for the sake of contradiction, that

\[
\rho_{L^\alpha(\mathcal{L}^{q'(\cdot)})}(\left( \frac{1}{|P|^{1/q'(\cdot)}} \sum_{m \in \mathbb{Z}^n} 2^{q(-\alpha() + n/2)} |\lambda_{v,m}| \chi_{v,m} \right)_{v \geq v_P}) > 1
\]

for \( P \in \mathcal{Q}_1 \subset \mathcal{Q}_0. \) Then by the continuity of the modular there exists \( d > 1 \) such that

\[
\rho_{L^\alpha(\mathcal{L}^{q'(\cdot)})}(\left( \frac{1}{|P|^{1/q'(\cdot)}} \sum_{m \in \mathbb{Z}^n} 2^{q(-\alpha() + n/2)} \left| \frac{\lambda_{v,m}}{d} \right| \chi_{v,m} \right)_{v \geq v_P}) = 1.
\]

for \( P \in \mathcal{Q}_1. \) Let \( s = \{s_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \) be a sequence of functions defined by

\[
s_{v,m}(x) = \begin{cases} 
2^{q(-\alpha() + n/2)q'(x)} |\lambda_{v,m}| q'(x)^{-1} \chi_{v,m}(x) \sgn \lambda_{v,m} & \text{if } Q_{v,m} \subset P \in \mathcal{Q}_1 \\
2^{q(-\alpha() + n/2)q'(x)} |\lambda_{v,m}| q'(x)^{-1} \chi_{v,m}(x) \sgn \lambda_{v,m} & \text{if } Q_{v,m} \subset P \in \mathcal{Q}_0 \setminus \mathcal{Q}_1.
\end{cases}
\]

Then

\[
\rho_{L^\alpha(\mathcal{L}^{q'(\cdot)})}(\left( \frac{1}{|P|^{1/q'(\cdot)}} \sum_{m \in \mathbb{Z}^n} 2^{q(-\alpha() - n/2)} |s_{v,m}| \right)_{v \geq v_P}) = \rho_{L^\alpha(\mathcal{L}^{q'(\cdot)})}(\left( \frac{1}{|P|^{1/q'(\cdot)}} \sum_{m \in \mathbb{Z}^n} 2^{q(-\alpha() + n/2)} \left| \frac{\lambda_{v,m}}{d} \right| \chi_{v,m} \right)_{v \geq v_P}) = 1,
\]

if \( P \in \mathcal{Q}_1 \) and

\[
\rho_{L^\alpha(\mathcal{L}^{q'(\cdot)})}(\left( \frac{1}{|P|^{1/q'(\cdot)}} \sum_{m \in \mathbb{Z}^n} 2^{q(-\alpha() - n/2)} |s_{v,m}| \right)_{v \geq v_P}) \leq 1,
\]

if \( P \in \mathcal{Q}_0 \setminus \mathcal{Q}_1, \) so \( \|s\|_{b_{q'(\cdot), q'(\cdot)}}^\alpha \leq 1. \) Therefore, by the definition of \( \|\lambda\|_{b_{q'(\cdot), q'(\cdot)}}^\alpha \),

\[
\|\lambda\|_{b_{q'(\cdot), q'(\cdot)}}^\alpha \geq \int_P \frac{1}{|P|} \sum_{v = v_P}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} s_{v,m}(x) \chi_{v,m}(x) dx.
\]
for any $P \in Q_1$. But the last expression is

$$\frac{d}{|P|} \sum_{v=1}^{\infty} \left( \sum_{m \in \mathbb{Z}^n} 2^{\nu(-\alpha(x)+n/2)q'(x)} \left| \frac{\lambda_{v,m}}{d} \right|^{q'(x)} \chi_{\nu,m}(x, v_{\nu,m}) \right) \leq d.$$ 

This contradicts our hypothesis on $\lambda = \{\lambda_{v,m} \in \mathbb{C} \}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n}$, so the desired inequality holds.

The following lemma is the $\tilde{b}^{-\alpha(\cdot),q'(\cdot)}$-version of (13), and then we obtain an equivalent norm on $\tilde{b}^{-\alpha(\cdot),q'(\cdot)}$.

**Lemma 5** Let $q \in \mathcal{P}$, $1 < q^- < q^+ < \infty$ and $\alpha : \mathbb{R}^n \to \mathbb{R}$. Then

$$\|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}} \leq \|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}} \leq 2 \|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}}.$$ 

**Proof.** Since $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$, by Hölder’s inequality

$$\|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}} \leq \|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}} \leq 2 \|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}}.$$ 

Now let us prove that $\|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}} \leq \|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}}$. By the scaling argument, it suffices to consider the case $\|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}} \leq 1$ and show that the modular of the sequence $\lambda$ on the left-hand side is bounded. By Lemma 4

$$\frac{1}{|P|} \int_P \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} 2^{\nu(-\alpha(x)+n/2)q'(x)} |\lambda_{v,m}|^{q'(x)} \chi_{\nu,m}(x, v_{\nu,m}) dx \leq 1$$ 

for any dyadic cube $P$, with $|P| \leq 1$, which is the desired inequality. To prove the main result of this paper, we need the following result.

**Theorem 2** Let $\alpha \in C_{loc}^{\log}$ and $q \in \mathcal{P}^{\log}$ with $1 < q^- \leq q^+ < \infty$. Then

$$(f_{1,q'(\cdot)}^{\alpha(\cdot)})^* = \tilde{b}^{-\alpha(\cdot),q'(\cdot)}.$$ 

In particular, if $\lambda = \{\lambda_{v,m} \}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \tilde{b}^{-\alpha(\cdot),q'(\cdot)}$, then the map

$$s = \{s_{v,m} \}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \to T_\lambda(s) = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} s_{v,m} \lambda_{v,m}$$

defined a continuous linear functional on $f_{1,q'(\cdot)}^{\alpha(\cdot)}$ with

$$\|T_\lambda\|_{(f_{1,q'(\cdot)}^{\alpha(\cdot)})^*} \approx \|\lambda\|_{\tilde{b}^{-\alpha(\cdot),q'(\cdot)}},$$

and every $T \in (f_{1,q'(\cdot)}^{\alpha(\cdot)})^*$ is of this form for some $\lambda \in \tilde{b}^{-\alpha(\cdot),q'(\cdot)}$. 

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Proof. We will use some idea from [14] Theorem 5.9. Let

\[ E_{v,h} = \left\{ x \in Q_{v,h} : G_{Q_{v,h}}^{-\alpha(\cdot),q'(\cdot)}(\lambda)(x) \leq m^{-\alpha(\cdot),q'(\cdot)}(\lambda)(x) \right\} \]

for any dyadic cube \( Q_{v,h} \), with \( v \in \mathbb{N}_0 \) and \( h \in \mathbb{Z}^n \). Then \( |E_{Q_{v,h}}| \geq \frac{3|Q_{v,h}|}{4} \) and

\[ |s_{v,m}|\|\lambda_{v,h}| = \frac{1}{|E_{v,h}|} \int_{E_{v,h}} |s_{v,h}|\|\lambda_{v,h}| dx \leq \frac{4}{3|Q_{v,h}|} \int_{E_{v,h}} |s_{v,h}|\|\lambda_{v,h}| dx. \]

Using the H"older inequality, we find

\[ |T_\lambda(s)| \leq \frac{4}{3} \int \sum_{v=0}^\infty \sum_{h \in \mathbb{Z}^n} 2^{v(\alpha(x)+n/2)} |s_{v,h}| 2^{v(-\alpha(x)+n/2)} |\lambda_{v,h}| \chi_{E_{v,h}}(x) dx \]

\[ \leq \frac{4}{3} \int \left( \sum_{v=0}^\infty \sum_{h \in \mathbb{Z}^n} 2^{v(\alpha(x)+n/2)q(x)} |s_{v,h}|^q(x) \right)^{1/q(x)} \sum_{v=0}^\infty \sum_{h \in \mathbb{Z}^n} 2^{v(-\alpha(x)+n/2)q'(x)} |\lambda_{v,h}|^q'(x) \chi_{E_{v,h}}(x)^{1/q'(x)} dx. \]

The last term is bounded by

\[ c \|s\|_{\ell_1^{\alpha(\cdot)}(\cdot)} \left\| \left( \sum_{v=0}^\infty \sum_{h \in \mathbb{Z}^n} 2^{v(-\alpha(x)+n/2)q'(x)} |\lambda_{v,h}| q'(x) \chi_{E_{v,h}}(x) \right)^{1/q'(x)} \right\|_{\infty} \]

\[ \lesssim \|s\|_{\ell_1^{\alpha(\cdot)}(\cdot)} \left\| m^{-\alpha(\cdot),q'(\cdot)}(\lambda) \right\|_{\infty} \lesssim \|s\|_{\ell_1^{\alpha(\cdot)}(\cdot)} \|\lambda\|_{\ell_1^{-\alpha(\cdot),q'(\cdot)}}, \]

by Proposition 3. Therefore,

\[ \|T_\lambda(f_{1,q(\cdot)}^{\alpha(\cdot)}) \| \lesssim \|\lambda\|_{\ell_1^{-\alpha(\cdot),q'(\cdot)}}, \]

Clearly every \( T \in (f_{1,q(\cdot)}^{\alpha(\cdot)})^* \) is of the form \( \sum_{v=0}^\infty \sum_{h \in \mathbb{Z}^n} t_{v,h} \lambda_{v,h} \) for some \( \lambda = \{\lambda_{v,h}\}_{v \in \mathbb{N}_0, h \in \mathbb{Z}^n} \).

Now, the norm \( \|\lambda\|_{\ell_1^{-\alpha(\cdot),q'(\cdot)}} \) is bounded by

\[ \sup \left| \int_P \frac{1}{|P|} \sum_{v \in \mathbb{N}_0} \sum_{h \in \mathbb{Z}^n} \lambda_{v,h} s_{v,h}(x) \chi_{v,h}(x) dx \right|, \]

where the supremum is taking all dyadic cube \( P \), with \( |P| \leq 1 \) and over all sequence of functions \( s = \{s_{v,h}\}_{v \in \mathbb{N}_0, h \in \mathbb{Z}^n} \) such that \( \|s\|_{\ell_1^{\alpha(\cdot)-n,q(\cdot)}} \leq 1 \), see Lemma 5. The integral, \( \int_P \cdots dx \) can be estimated by

\[ \sum_{v=v_P}^\infty \sum_{h \in \mathbb{Z}^n} |\lambda_{v,h}| \int_P \frac{1}{|P|} |s_{v,h}(x)| \chi_{v,h}(x) dx = \sum_{v=0}^\infty \sum_{h \in \mathbb{Z}^n} |\lambda_{v,h}| D_{v,h,P} \leq \|T\| (f_{1,q(\cdot)}^{\alpha(\cdot)})^* \|D_P\| f_{1,q(\cdot)}^{\alpha(\cdot)}, \]

where \( D_P = \{D_{v,h,P}\}_{v \in \mathbb{N}_0, h \in \mathbb{Z}^n} \) and

\[ D_{v,h,P} = \begin{cases} 0 & \text{if } v < v_P \\ 0 & \text{if } v \geq v_P \text{ and } Q_{v,h} \cap P = \emptyset \\ \int_P \frac{1}{|P|} \chi_{v,h}(x) dx & \text{if } v \geq v_P \text{ and } Q_{v,h} \subset P. \end{cases} \]
Let us prove that \( \|D_P\|_{F_{1,q}^{\alpha}} \leq 1 \). The claim can be reformulated as showing that

\[
\int_{P} \left( \sum_{v \in P} \sum_{h \in \mathbb{Z}^n} 2^{
u(y)+n/2} D_{v,h,P}^{\nu(y)}(y) \right)^{1/q(y)} dy \lesssim 1. \tag{14}
\]

By the fact \( 2^{
u(x)} \approx 2^\nu(x) \) for any \( x, y \in Q_{v,h} \),

\[
\frac{2^{
u(y)} - 2^{
u(x)}}{|Q_{v,h}|} \lesssim \frac{1}{|Q_{v,h}|} \int_{Q_{v,h}} \frac{2^{
u(x) - n/2}}{|P|} |s_{v,h}(\cdot)|_{x_{v,h}}(x) dx, \quad y \in Q_{v,h} \subset P
\]

for any \( y \in \mathbb{R}^n \) and any \( N > n \). Therefore, the left-hand side of (14) is bounded by

\[
c \int_{P} \left( \sum_{v \in P} \left( \sum_{h \in \mathbb{Z}^n} 2^{
u(y)} |s_{v,h}(\cdot)|_{x_{v,h}}(y) \right)^{1/q(y)} \right) dy
\]

by the Hölder inequality and Lemma \( \text{[14]} \). We can move \( \|\chi_P\|_{q'} \) inside the norm and using the properties \( \text{[3]} \) and \( \text{[1]} \) to estimate the last expression by

\[
c \left( \sum_{v \in P} \sum_{h \in \mathbb{Z}^n} 2^{
u(y)} |s_{v,h}(\cdot)|_{x_{v,h}}(y) \right)^{1/q(y)} \chi_P \lesssim \|s\|_{F_{1,q}^{\alpha}} \lesssim 1.
\]

Therefore, \( \|\lambda\|_{\widetilde{B}^{-\alpha}_{q'}^{\mu}, q'} \leq \|T\|_{F_{1,q}^{\alpha}} \) and hence completes the proof of this theorem. 

Using the notation introduced above, we may now state the main result of this paper.

**Theorem 3** Let \( \alpha \in C_{\log}^\infty \) and \( q \in \mathcal{P}^\log \) with \( 1 < q^- \leq q^+ < \infty \). Then

\[
\left( F_{1,q}^{\alpha} \right)^* = \widetilde{B}^{-\alpha}_{q'}^{\mu}.
\]

In particular, if \( g \in \widetilde{B}^{-\alpha}_{q'}^{\mu} \), then the map, given by \( l_g(f) = \langle f, g \rangle \), defined initially for \( f \in \mathcal{S}(\mathbb{R}^n) \) extends to a continuous linear functional on \( F_{1,q}^{\alpha} \) with \( \|g\|_{B^{-\alpha}_{q'}^{\mu}} \approx \|l_g\|_{F_{1,q}^{\alpha}} \) and every \( l \in (F_{1,q}^{\alpha})^* \) satisfies \( l = l_g \) for some \( g \in \widetilde{B}^{-\alpha}_{q'}^{\mu} \).

**Proof.** Here we use the same arguments of \( \text{[14]} \) Theorem 5.13. In \( \text{[1]} \) we may choose \( \Phi = \Psi \) and \( \varphi = \psi \). If \( g \in \widetilde{B}^{-\alpha}_{q'}^{\mu} \) and \( f \in \mathcal{S}(\mathbb{R}^n) \), then \( \langle f, g \rangle = \langle S_{\varphi}f, S_{\varphi}g \rangle \) and applying Theorems \( \text{[2]} \) and \( \text{[1]} \) to obtain

\[
\|\langle f, g \rangle \| \leq \|S_{\varphi}f\|_{F_{1,q}^{\alpha}} \|S_{\varphi}g\|_{\widetilde{B}^{-\alpha}_{q'}^{\mu}} \lesssim \|f\|_{F_{1,q}^{\alpha}} \|g\|_{\widetilde{B}^{-\alpha}_{q'}^{\mu}}.
\]
Hence \( \|l\|_{(F_\alpha^{(\cdot)})^*} \lesssim \|g\|_{B^{-\alpha(\cdot),q'}_{q'(\cdot)}} \). Suppose \( l \in (F_\alpha^{(\cdot)})^* \), then \( l_1 = l \circ T_\psi \in (F_\alpha^{(\cdot)})^* \), so by Theorem 2 there exists \( \lambda = \{\lambda_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \tilde{B}^{-\alpha(\cdot),q'}_{q'(\cdot)} \) such that

\[
l_1(\lambda) = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} s_{v,m} \lambda_{v,m}
\]

for \( s = \{s_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in f_{1,q'(\cdot)}^{(\cdot)} \) and \( \|\lambda\|_{\tilde{B}^{-\alpha(\cdot),q'}_{q'(\cdot)}} \approx \|l_1\|_{(F_\alpha^{(\cdot)})^*} \lesssim \|l\|_{(F_\alpha^{(\cdot)})^*} \), since \( T_\psi \) is bounded. By Theorem 1

\[
l_1 \circ S_\varphi = l \circ T_\psi \circ S_\varphi = l.
\]

Hence putting

\[
g = T_\psi \lambda = \sum_{m \in \mathbb{Z}^n} \lambda_{0,m} \psi_m + \sum_{v=1}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \psi_{v,m},
\]

we obtain

\[
l(f) = l_1(S_\varphi f) = \langle S_\varphi f, \lambda \rangle = \langle f, g \rangle.
\]

Then \( l = l_g \) and again by Theorem 2

\[
\|g\|_{B^{-\alpha(\cdot),q'}_{q'(\cdot)}} \lesssim \|\lambda\|_{\tilde{B}^{-\alpha(\cdot),q'}_{q'(\cdot)}} \lesssim \|l\|_{(F_\alpha^{(\cdot)})^*}.
\]

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