Spatial Correlations and Angular Schmidt Modes in Spontaneous Parametric Down-Conversion

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(Dated: December 19, 2011)

We report an experimental approach to entanglement characterization in high-dimensional quantum systems using Schmidt decomposition techniques. A particular experiment uses spatial degrees of freedom of biphotons. We present a technique to realize projective measurements in Schmidt basis, allowing us to measure the coefficients in Schmidt decomposition and to estimate the degree of entanglement directly. Issues of modeling the spatial part of biphoton amplitude with simple double-gaussian functions are discussed and shown to be in good agreement with experimental results.

PACS numbers: 03.67.Bg, 03.67.Mn, 42.65.Lm

I. INTRODUCTION

Spatial entanglement in spontaneous parametric down-conversion (SPDC) was a subject of intense research during the last decade. Besides fundamental issues, spatial states of biphoton pairs offer a platform for high-dimensional quantum states engineering motivating this interest. One can distinguish two complementary approaches to spatial qudit engineering with biphotons: one using “pixel entanglement” and similar schemes [1–4], and another one based on using high-order coherent (usually Laguerre-Gaussian) modes [5–16]. In both approaches achievable dimensionality and collection efficiency are figures of merit. Dimensionality of effective Hilbert space is limited by degree of spatial entanglement. In pixel entanglement schemes, for example, the pixel size should be made smaller than the coherence radius of the pump in the far zone, and since the pump is always divergent, even a plain wave, selected by point-like aperture would be correlated to a whole set of plain-wave modes in the conjugate beam. The same holds in general for other possible choices of modes. A natural question to ask is whether there exist a “preferred” basis among the multitude of possible coherent spatial modes, which consists only of pairwise correlated modes and allows us to fully exploit the spatial correlations in SPDC?

The answer is, of course, positive and well known – the decomposition into a set of Schmidt modes has all the required properties. Since it was used for the first time by Law and Eberly [17], it has become a common tool for entanglement analysis of infinite dimensional systems in general, and of spatial states of photons in particular. Nevertheless, direct experimental attempt to address spatial entanglement of SPDC biphotons in Schmidt basis was not reported until the recent work of authors [18]. Here we report a detailed description of experimental techniques used for Schmidt modes filtering and detection, models used to describe spatial entanglement in SPDC, and the structure of Schmidt decomposition revealed in our experiments.

The paper is organized as follows: in Section II we briefly review the main features of SPDC angular spectrum with emphasis on spatial entanglement, Section III reminds the main ideas behind Schmidt decomposition analysis of spatial entanglement, explicit expressions for Schmidt modes in both Hermite-Gaussian and Laguerre-Gaussian basis are discussed in Section IV with emphasis on Hermite-Gaussian case, Section V gives a brief review of experimental approaches to mode transformations and some recent results on estimating the degree of spatial entanglement and, finally, the detailed description of our experiments with spatial Schmidt modes is given in Section VI.

II. SPDC ANGULAR SPECTRUM AND SPATIAL CORRELATIONS OF PHOTONS

Biphotons generated in the SPDC process have continuous frequency and angular spectrum. Let us consider its structure in some details. SPDC can be phenomenologically described using the following effective interaction Hamiltonian [19]:

\[ H = \int_V d^3\vec{r} \langle \vec{r} | (\vec{r}) E_p^{(-)}(\vec{r}) E^{(+)}(\vec{r}) | \vec{r} \rangle \exp \left[ i \Delta \vec{r} \right] + \text{H.c.} \]  

(1)

Here \( E_p \) is the classical amplitude of the pump field and \( E \) is the scattered field operator. Considering pump to be monochromatic, the first order of perturbation theory gives the following expression for the state of the scattered field:

\[ |\Psi\rangle = |\text{vac}\rangle + \int d\vec{k}_1 d\vec{k}_2 \Psi(\vec{k}_1, \vec{k}_2) |1\rangle_{k_1} |1\rangle_{k_2}, \]

\[ \Psi(\vec{k}_1, \vec{k}_2) = \int_V d^3\vec{r} \langle \vec{r} | (\vec{r}) E_p^{(-)}(\vec{r}) E^{(+)}(\vec{r}) \exp \left[ i \Delta \vec{r} \right] \],

(2)

where \( \Delta = \vec{k}_1 + \vec{k}_2 - \vec{k}_p \), \( \omega_1 + \omega_2 = \omega_p \). In the case of collinear phase-matching and under wide crystal ap-
proximation one can obtain the following biphoton field amplitude \[ \Psi(k_1, k_2) = \mathcal{E}(k_{1\perp} + k_{2\perp}) \mathcal{F}(k_{1\perp} - k_{2\perp}), \] (3)

where \( \mathcal{E}(k_{1\perp} + k_{2\perp}) \) stands for angular spectrum of the pump, and \( \mathcal{F}(k_{1\perp} - k_{2\perp}) \) is a geometrical factor determined by phase-matching conditions.

Authors of \[21, 22\] give the following expression for \( \mathcal{F} \):

\[
\Psi(k_1, k_2) = N \mathcal{E}(k_{1\perp} + k_{2\perp}) \text{sinc} \left[ \frac{L(k_{1\perp} - k_{2\perp})^2}{4k_p} \right],
\] (4)

with \( L \) being the crystal length and \( N \) – a normalization constant. This expression is valid only in the case of small pump divergence. A detailed analysis shows that in the case of highly divergent pump one should take into account the dependence of refractive index for extraordinary wave on the propagation direction. This question was addressed in [21, 23], where it was shown that for highly divergent pump experimental results are better described by the following biphoton amplitude:

\[
\Psi(\vec{k}_{1\perp}, \vec{k}_{2\perp}) \propto E_p^*(\vec{k}_{1\perp} + \vec{k}_{2\perp}) \times \\
\text{sinc} \left[ \frac{L}{2} \left( \frac{\omega}{c} (\vec{k}_{1\perp} + \vec{k}_{2\perp}) \cdot \frac{\partial n_p}{\partial k_{p\perp}} + \frac{(\vec{k}_{1\perp} - \vec{k}_{2\perp})^2}{2k_p} \right) \right],
\] (5)

where \( k_p \) and \( \frac{\partial n_p}{\partial k_{p\perp}} \) are taken at \( \vec{k}_{p\perp} = 0 \).

Spatial correlations in SPDC angular spectrum manifest themselves in significant difference of single-particle and two-particle (conditional) distributions obtained in experimental scheme of Fig. 1.

The setup consists of two point-like detectors positioned in the far field and a coincidence circuit. One of the detectors is fixed, the other one may be scanned. The distribution of photocounts of the scanned detector corresponds to unconditional single-particle distribution \( du^{\text{uncond}} / dk_{1\perp x} \), while the coincidence distribution in such a scheme corresponds to a conditional distribution \( du^{\text{cond}} / dk_{1\perp x} \), where \( k_{1\perp x} \) is a transverse wavevector component of one of the photons along the direction of scanned detector’s motion. One can also study two-dimensional distributions by changing the scanning direction. These distributions are related to the biphoton amplitude in the following way:

\[
\frac{du^{\text{uncond}}}{dk_{1\perp x}} = \frac{1}{\Delta k_{1\perp}} \int dk_{2\perp x} |\Psi(k_{1\perp x}, k_{2\perp x})|^2
\]

\[
\frac{du^{\text{cond}}}{dx_1} = \frac{1}{\Delta k_{2\perp}} \int dk_{1\perp x} |\Psi(k_{1\perp x}, k_{2\perp x})|^2 |_{k_{2\perp x} = \text{const}}.
\] (6)

If the biphoton amplitude may be factorized in two terms depending on the wavevectors of each photon separately: \( \Psi(k_{1\perp x}, k_{2\perp x}) = \Psi_1(k_{1\perp}) \times \Psi_2(k_{2\perp}) \), i.e. if the biphoton state is separable, both distributions coincide. If this is not the case, and the state is entangled, the width of conditional distribution \( \Delta k_{1\perp} \) is less than the width of unconditional one \( \Delta k_{1\perp}^{(s)} \). It was proposed to use the ratio of these quantities \( R_k = \Delta k_{1\perp} / \Delta k_{1\perp}^{(s)} \) as a quantitative measure of spatial entanglement (which was later referred to as Fedorov ratio) [24]. Experiments have shown, that the value of \( R_k \) for SPDC biphoton states may reach values as large as \( R_k \sim 100 \), corresponding to high degree of entanglement [1, 4].

III. ANALYZING SPATIAL ENTANGLEMENT WITH SCHMIDT DECOMPOSITION

The most developed approach to quantitative analysis of spatial (and frequency) entanglement of SPDC biphoton states is based on using coherent modes decomposition. Biphoton spatial state space is “discretized” by switching from continuous distributions in plane-wave basis of the previous section to discrete distributions in a chosen basis of spatial mode functions \( \xi_i(\vec{k}_{1,2\perp}) \). For an arbitrary choice of mode functions in the decomposition of spatial state for each of the photons, the biphoton amplitude takes the following form:

\[
\Psi(\vec{k}_{1\perp}, \vec{k}_{2\perp}) = \sum_{i,j=0}^{\infty} C_{ij} \xi_i(\vec{k}_{1\perp}) \xi_j(\vec{k}_{2\perp}).
\] (7)

It turns out, that by appropriate choice of the basis mode functions one can transform the expression (7) to a single-sum form

\[
\Psi(\vec{k}_{1\perp}, \vec{k}_{2\perp}) = \sum_{i=0}^{\infty} \sqrt{\lambda_i} \psi_i(\vec{k}_{1\perp}) \psi_i(\vec{k}_{2\perp})
\] (8)

which is called Schmidt decomposition. In this case the basis functions \( \psi_i(\vec{k}_{1\perp}) \) should be eigenfunctions of single-photon density matrix \( \rho_{1,2}(\vec{k}_{1\perp}, \vec{k}_{1\perp}) \), and \( \lambda_i \) are the corresponding eigenvalues. It means that the appropriate mode functions may be found by solving the following integral equation:

\[
\int \rho_{1,2}(\vec{k}_{1\perp}, \vec{k}_{1\perp}) \psi_i(\vec{k}_{1\perp}) d\vec{k}_{1\perp} = \lambda_i \psi_i(\vec{k}_{1\perp}).
\] (9)
It is straightforward to note that for a factorized state only a single coefficient $\lambda_i$ in the Schmidt decomposition is nonzero. For a highly entangled state, to the contrary, $\lambda_i$ only slowly decrease with $i$. Since the biphoton state is pure, it is natural to use single-photon von Neumann entropy as a measure of entanglement \[ E(\rho) = -\sum_{i=1}^{\infty} \lambda_i \log_2 \lambda_i. \] (10)

Calculating entropy is a rather demanding task, requiring the knowledge of all eigenvalues in the Schmidt decomposition. It is much more convenient to use “average number of Schmidt modes” determined by Schmidt number $K$ as an operational measure of entanglement\[ K = \frac{1}{\sum_{i=0}^{\infty} \lambda_i^2}. \] (11)

Degree of spatial entanglement of SPDC biphotons described by wavefunction $\Psi(\vec{k}_\perp)$ was analyzed by Law and Eberly in [17]. The pump was assumed to be Gaussian $\mathcal{E}_p = \exp\left[-\frac{1}{2} \frac{\lambda_1^2 + \lambda_2^2}{b^2} \right]$. Authors derived an analytical expression for Schmidt number by approximating $\mathcal{F}(k_1^\perp - k_2^\perp)$ function of (11) by a gaussian function:

\[ K_g = \frac{1}{4} \left( b\sigma + \frac{1}{b\sigma} \right)^2, \] (12)

where $b = \sqrt{\frac{L}{4\lambda_g}}$ is the waist of the gaussian function modeling $\mathcal{F}(k_1^\perp - k_2^\perp)$. In the following we will call this procedure “a double gaussian approximation”. The value of $K_g$ is determined by a single parameter $b\sigma$ and may reach very high values for $b\sigma \gg 1$ and $b\sigma \ll 1$. Numerical calculations performed for

\[ \mathcal{F}(k_1^\perp - k_2^\perp) = \sin \left[ b(k_1^\perp - k_2^\perp)^2 \right] \] (13)

showed that real value of $K$ is larger than $K_g$ for all values of $b\sigma$. Authors also proposed a method to increase the degree of entanglement by spatial filtering. It turned out that the number of significantly nonzero terms in the Schmidt decomposition increases if one cuts off part of angular spectrum with small values of $k_\perp$. Influence of spatial filtering on Schmidt number was further investigated in [20], where authors study the effect of irises and finite detection apertures.

IV. SPATIAL SCHMIDT MODES STRUCTURE

The choice of mode functions set in (7) is quite arbitrary and may be determined by convenience in analyzing a particular physical situation. If the situation corresponds to a beam propagating in free space, it is natural to chose the solutions of paraxial wave equation, i.e. Hermite-Gaussian or Laguerre-Gaussian modes, as a set of mode functions. Moreover it turns out, that these functions form the Schmidt decomposition for SPDC with the gaussian pump of moderate divergence.

A. Laguerre-Gaussian modes and orbital angular momentum entanglement

Let us introduce polar coordinates $\{k_\perp, \phi\}$ on the plane of transverse wave-vector components $k_\perp$. Laguerre-Gaussian modes are described by functions of the following form:

\[ LG_{pl}(k_\perp) \propto L_p^{|l|} \left( \frac{k_\perp^2}{(\Delta k_\perp)^2} \right) \exp \left( -\frac{k_\perp^2}{2(\Delta k_\perp)^2} \right) \times \exp \left( il\phi + i \left( p - \frac{|l|}{2} \right) \pi \right). \] (14)

Here $L_p^{|l|}(x)$ are associated Laguerre polynomials, $\Delta k_\perp$ is angular width of the mode. Radial index $p$ corresponds to number of zeros in radial direction and azimuthal index $l$ – to the phase shift on the closed loop around $k_\perp = 0$ point (topological charge of the beam). One can show, that Laguerre-Gaussian beams with nonzero $l$ possess orbital angular momentum (OAM) $|lh|$. This is also valid for single-photon beams: a photon in Laguerre-Gaussian mode has OAM of $lh$.

In Laguerre-Gaussian basis decomposition (7) takes the following form:

\[ \Psi(k_1^\perp, k_2^\perp) = \sum_{l_1, p_1, l_2, p_2} c_{l_1l_2}^{(l_1l_2)} LG_{p_1l_1}(k_1^\perp) LG_{p_2l_2}(k_2^\perp). \] (15)

It was shown in [29, 30] that OAM is conserved in the SPDC process, i.e. in the subspace of fixed $p_1$ and $p_2$ the identity $l_0 = l_1 + l_2$ holds, where $l_0$ is azimuthal index of the pump beam. This conservation law gives rise to OAM entanglement of photons in the same sense as transverse momentum conservation leads to spatial correlations considered in Sect. III. In the subspace of fixed radial indexes decomposition in terms of Laguerre-Gaussian modes takes the form of Schmidt decomposition:

\[ \Psi(k_1^\perp, k_2^\perp)|_{p_1p_2} = \sum_{l} c_{l_1l_2}^{(l_1l_2)} LG_{p_1l_1}(k_1^\perp) LG_{p_2l_2}(k_2^\perp). \] (16)

Properties of such decomposition were studied in [33] where authors propose to use its width (which they called “quantum spiral bandwidth”) to quantify the degree of OAM entanglement. Quantum spiral bandwidth estimates the effective dimensionality of OAM Hilbert space, but it is defined only for subspaces of fixed radial indexes. To get rid of double sum and transform (10) to a Schmidt decomposition form one should use double-gaussian approximation.

B. Hermite-Gaussian modes as approximate Schmidt modes for SPDC biphoton field

Another convenient set of modes which is somewhat less used in literature is the set of Hermite-Gaussian...
modes:
\[
\text{HG}_{nm}(k_x,k_y) \propto H_n \left( \frac{k_x^2}{(\Delta k_x)^2} \right) H_m \left( \frac{k_y^2}{(\Delta k_y)^2} \right) \exp \left( -\frac{k_x^2 + k_y^2}{2(\Delta k)^2} \right),
\]
where \( H_n(x) \) are Hermite polynomials, and \( \{k_x, k_y\} \) are transverse wave-vector components. This decomposition was studied in [34]. Authors investigate dependence of coefficients of the decomposition on modal structure of the pump (which is also expressed in terms of Hermite-Gaussian modes) and on the relative width of the gaussian function in (17) and the pump mode. In this case the spatial parity of the beam is the conserved quantity: parity of product \( \text{HG}_{n_1,n_2}(k_{1x},k_{1y}) \times \text{HG}_{n,m_2}(k_{2x},k_{2y}) \) equals to the parity of pump mode.

As well as for Laguerre-Gaussian modes, one can get rid of two indexes in decomposition (17) and transform it to a form of Schmidt decomposition using double-gaussian approximation [17, 32]. Let us consider the case of moderately divergent pump, when one may not consider the linear term in (19) [49]. For small \( \sigma \) we can make a substitution \( \text{sinc} \left( \frac{x}{2\sigma_a} \right) \rightarrow \exp (-\gamma \frac{x^2}{2\sigma_a^2}) \), where \( \gamma \) is a coefficient chosen to make both functions "close" to each other. A good approximation is provided by choosing a value of \( \gamma = 0.86 \) [33]. The biphoton wavefunction now takes the following form:
\[
\Psi(k_1,k_2) \propto \exp \left( -\frac{(k_{1\perp} + k_{2\perp})^2}{2a^2} \right) \exp \left( -\frac{(k_{1\perp} - k_{2\perp})^2}{2b^2} \right),
\]
where \( a \) determines the angular bandwidth of the pump, and \( b = \sqrt{4k_p/\gamma L} \) - the phase-matching bandwidth. Since the wavefunction is a product of functions depending only on \( k_{1x} \) and only on \( k_{2x} \), it is sufficient to consider the problem in one dimension:
\[
\Psi(k_{1x},k_{2x}) = \sqrt{\frac{2}{\pi a b}} \exp \left( -\frac{(k_{1x} + k_{2x})^2}{2a^2} \right) \exp \left( -\frac{(k_{1x} - k_{2x})^2}{2b^2} \right).
\]

One can show, that solutions of (9) for such a wavefunction have the form [35]:
\[
\psi_n(k_{1x,2x}) = \left( \frac{2}{ab} \right)^{1/4} \phi_n \left( \sqrt{\frac{2}{ab}} k_{1x,2x} \right),
\]
where \( \phi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x) \). For corresponding eigenvalues and Schmidt number we obtain:
\[
\lambda_n = 4ab \left( \frac{a - b}{a + b} \right)^{2n} \frac{1}{(n+1)}, \quad K_x = \frac{a^2 + b^2}{2ab}.
\]
So we have the following form of Schmidt decomposition for SPDC biphoton state under the double-gaussian approximation:
\[
\Psi(k_1,k_2) = \sum_{mn} \sqrt{\lambda_n \lambda_m} \psi_n(k_{1x}) \psi_m(k_{1y}) \times \psi_n(k_{2x}) \psi_m(k_{2y}).
\]

Degree of entanglement for this two-dimensional wavepacket is given by Schmidt number:
\[
K = K_x \times K_y = (a^2 + b^2)^2/4a^2b^2.
\]

Let us note once again, that (22) is not the only possible form of Schmidt decomposition. It may as well be described in terms of Laguerre-Gaussian modes as in (17) [36]. The value of Schmidt number is, of course, basis independent, as was explicitly shown in a recent preprint by Miatto et al. [37]. In fact, the difference between these two representations corresponds to the choice of polar or cartesian coordinates on the plane of transverse momentum components.

V. EXPERIMENTAL METHODS OF SPATIAL MODES TRANSFORMATION

There are several well developed methods to transform a gaussian laser beam to higher Laguerre-Gaussian and Hermite-Gaussian modes. The general idea is based on introducing a spatially varying phase shift corresponding to the desired mode to a beam [38–40]. This is achieved when a beam is diffracted on a phase mask or a hologram, acquiring a desired phase structure. Examples of phase holograms corresponding to Hermite-Gaussian and Laguerre-Gaussian modes of lower orders are shown in Fig. 2. Usually these masks are combined with blazed diffraction grating, “highlighting” the first diffraction order, which allows to get rid of nondiffracted components increasing the quality of mode transformation. Using this type of holograms allows effective transformation to relatively high order modes [31].

In modern experiments phase holograms are usually realized using active liquid crystal spatial light modulators (SLM). Devices based on parallel-aligned nematic liquid crystals (VAN) matrices on silicon substrate (LCOS) are most commonly used for mode transformation tasks [42]. Every pixel of such matrix works as a birefringent phase plate introducing a phase delay for a polarization component parallel to the crystals alignment plane. The delay
is determined by the orientation of the crystals and is controlled by voltage applied to the pixel. Commercially available devices have pixels of 10 µm size.

Adapting the technique of mode transformations to the single photon level allows to realize both transformations and measurements in spatial modes basis. A principal sketch of experimental setup for projective measurements is shown in Fig. 3. The crucial element here is a single mode fiber guiding only the fundamental gaussian $HG_{00} \equiv LG_{00}$ mode. This mode is coupled with a lens to a gaussian part of incident light, thus realizing a projection on gaussian mode. To realize higher order projections the hologram transforming a desired mode to a gaussian one should be used. In such a setup only the mode corresponding to a particular hologram passes through the fiber and gives a click in the single photon counter, thus realizing a projective measurement.

As far as we know, this scheme was realized for the first time at the single photon level in the work of Zeilinger’s group \[5\]. Authors have experimentally demonstrated OAM entanglement of SPDC photons. Following works of the same group showed the possibility to engineer qutrit \[8, 9\] and higher dimensional qudit \[43\] states and to realize quantum key distribution protocols with spatial encoding.

Experimental preparation of photons with OAM and study of OAM-entanglement of photon pairs is a rather developed field with numerous experimental contributions \[6, 7, 10–16\] (see review \[44\] for a much more comprehensive list of references). In context of our work it is important to mention recent works of Boyd’s group \[10, 11\], where the quantum spiral bandwidth was directly measured. The same quantity was estimated with excellent precision using indirect measurements in \[13\]. Authors obtained the coefficients of decomposition in Laguerre-Gaussian basis by Fourier transforming the decomposition of Hong-Ou-Mandel-type interference visibility $\alpha$ and $\gamma$ of the same group showed the possibility to engineer OAM entanglement of SPDC photons. Following works of the same group \[5\]. Authors have experimentally demonstrated the possibility of realizing quantum key distribution protocols with spatial encoding.

Up to our knowledge, no experimental attempts to analyze spatial entanglement in SPDC in Hermite-Gaussian basis were reported. The question of properly adjusting the experimental conditions to make the detected set of modes as close to Schmidt decomposition as possible was not addressed as well. These were the main concerns of our work.

VI. EXPERIMENTAL REALIZATION OF MEASUREMENTS IN SCHMIDT BASIS

Our main goal was to demonstrate the possibility of performing direct measurements in spatial Schmidt basis and to experimentally demonstrate all features specific to Schmidt decomposition. We have chosen to work in Hermite-Gaussian basis. In this basis Schmidt decomposition is symmetric in indexes $m$ and $n$, making it convenient for characterizing complete two-dimensional entanglement structure.

First of all let us address the issue of applicability of double-gaussian approximation in our experimental conditions. We used a 2 mm BBO crystal pumped by a CW He-Cd laser with $\lambda_p = 325$ nm wavelength. The crystal was cut for collinear frequency-degenerate Type-I phase-matching. Angular bandwidth of phase-matching in such crystal (neglecting the pump divergence) – $b$ parameter in \[18\] is $b = 0.037$. It was convenient for our purposes to select the value of pump divergence corresponding to a moderate Schmidt number. We focused the pump inside the crystal with a 150 mm quartz lens and measured the divergence – $a$ parameter in \[18\] to be $a = (5.8 \pm 0.1) \times 10^{-3}$. We calculated eigenvalues and eigenfunctions for the reduced single-photon density matrix, corresponding to the precise SPDC wavefunction \[10\] numerically. The calculation was performed as follows: Hermite-Gaussian modes corresponding to the approximate function \[10\] were chosen as a basis, we have restricted ourselves to 10 lower order modes (giving the Schmidt number with 3 decimal digits precision) and calculated the matrix elements of the precise density matrix in this basis. Diagonalizing the calculated matrix, we obtained eigenvalues and eigenfunctions. The results for lower order modes are shown in Fig. 4. One can see reasonable correspondence, at least, we should expect that phase holograms for Schmidt modes should be close to those of Hermite-Gaussian modes of appropriate divergence. We should note that measured waist size \[52\] of the pump beam in the focal plane of the lens was $w_p = (25 \pm 1) \mu \text{m}$, corresponding to $M^2 = 1.4$. That means, the pump beam is aberrated and is not really gaussian, and that may cause some deviations from Hermite-Gaussian shape of Schmidt modes as well.

We have used an LCoS SLM with VAN matrix produced by Cambridge Correlators. The matrix has 1027 \times 768 pixels of 10 µm size each. It is an 8 bit device, capable of introducing a phase shift of up to 0.8π. Since larger phase shifts are required for our holograms we used a double reflection scheme. We used two polymer film polarizers in front and after the SLM to reduce the un-
respectively), and the distance between phase steps for transition at the SLM (in horizontal and vertical directions, modes in a unique way. If we define “visibility” for mode parameters define the shape of holograms for other modes in a unique way. We have actually adjusted three parameters: the position of phase step for wanted polarization rotations by an additional dielectric mirror necessary in such scheme (see insets in Fig. 5).

To check the quality of mode transformation with this device we used the setup sketched in Fig. 4. We used an attenuated 650 nm diode laser as a source. The beam was mode filtered with single mode fiber and focused with a 20x microscope objective to obtain the divergence similar to that of an HG00 Schmidt mode in Fig. 4 and the waist at the position of the crystal. So we obtained a single mode gaussian beam modeling the zero order Schmidt mode of SPDC beam. The beam was collimated with 145 mm lens and after reflection from SLM was focused to a 2 mm BBO crystal with a 150 mm quartz lens L1, a second lens L2 with F = 145 mm was set confocal with L1 to collimate the beam. Pump radi-

transformations as the ratio of counting rates with holograms for HGnm modes to that for untransformed gaussian mode: \( V = (R_{00} - R_{mn})/(R_{00} + R_{mn}) \), then for almost all of the modes with \( 0 \leq m, n \leq 4 \) it exceeds 97%, corresponding to high quality of mode transformations. The visibility is slightly lower for HG01 and HG11 modes (still exceeding 90%), which we explain by poorer adjustment in vertical direction due to the design of optomechanical components used. Histogram of counting rates for various modes is shown in Fig. 6.

To check whether the spatial structure of transformed modes is really close to Hermite-Gaussian, we scanned the fiber tip in the focal plane of O3 objective. The counting rate dependence on fiber position is determined by the convolution of a corresponding Hermite-Gaussian function and a fundamental gaussian mode of the fiber:

\[
R(x) \propto \int_{-\infty}^{\infty} H_{nm}(\sqrt{2}x/w) \exp\left(-\frac{x^2}{w^2}\right) \exp\left(-\frac{(x-\tilde{x})^2}{w^2}\right) dx^2
\]

where \( w \) is the gaussian mode waist. Experimental dependencies are shown in Fig. 4 and have a characteristic shape of double-peak curves. Distance between maxima depends on the mode number, and is shown in Fig. 5 for “horizontal” HG_{n0} and “vertical” HG_{0m} modes, together with theoretical predictions for Hermite-Gaussian modes. To plot the theoretical predictions correctly we estimated the waist size by fitting the convolution for HG_{00} mode with a gaussian curve, obtaining \( w = (3.87 \pm 0.07) \) \( \mu m \).

When the attenuated laser beam is substituted with SPDC radiation, the described scheme realizes projective measurements in Hermite-Gaussian basis. Full scheme of experimental setup is shown in Fig. 6. Pump was focused to a 2 mm BBO crystal with a 150 mm quartz lens L1, a second lens L2 with \( F = 145 \) mm was set confocal with L1 to collimate the beam. Pump radi-
Figure 7: Counting rate dependence on the position of fiber tip for an attenuated laser beam transformed to various modes. Fiber is scanned in horizontal direction (1) and in vertical one (2).

Figure 8: Dependence of maxima positions for fiber tip scan on mode number for laser beam transformations. "Horizontal" $H_{G_{m0}}$ (blue bars), "vertical" $H_{G_{0m}}$ (red bars) modes and theoretical predictions (solid black line and dots). Theoretical predictions are calculated for gaussian waist $a = 3.9 \mu m$ (see text for details).

Figure 9: Experimental setup. L1 – 150 mm quartz lens; L2 – 145 mm lens; BBO – 2 mm BBO crystal placed in the joint focus of L1 and L2; UVM – UV mirror cutting off the pump; IF – interference filter for 650 nm with 40 nm bandwidth; BS – non-polarizing 50/50 beam-splitter; 0.1, 2 – 8× microscope objectives; PM – spatial light modulator (is shown as transmitting mask for simplicity, real alignment is shown on the inset); PM2 – phase mask made of thin glass plates; SMF - single mode fiber; SMF/MMF – single or multi-mode fiber depending on the experiment (see text for details); D1,2 – single photon counters (Perkin Elmer). A 200 \mu m vertical slit S was used in "ghost" imaging experiments.

Schmidt modes are really close to Hermite-Gaussian ones, we expect coincidences to appear only when similar modes are selected in both channels of the setup. If different modes are selected in the transmitted and the reflected channels, no coincidences should appear, since only terms with equal indexes are present in (8). This fact may be used as an experimental criteria of how well Hermite-Gaussian modes detected in our setup approximate precise spatial Schmidt modes. Fig. 10 (1) shows a histogram of coincidence counting rate for the case when $H_{G_{00}}$ with $0 \leq m, n \leq 4$ modes were consequently selected in the transmitted channel and $H_{G_{00}}$ mode was selected in the reflected channel. We obtained values of visibility $V = (R_{00} - R_{mn})/(R_{00} + R_{mn})$, where $R_{mn}$ is a counting rate for $H_{G_{mn}}$ mode, over 94% for all modes except $H_{G_{11}}$ and $H_{G_{11}}$ modes. Comparing the results to those of Fig. 8, we conclude that non-100% visibility for these modes is rather a result of technical imperfections, common for schemes with attenuated laser beam and SPDC, than some physical discrepancy between detected modes and Schmidt modes. So we can state, that our scheme indeed realizes projective measurements in Schmidt basis.

As is expected, when a phase mask corresponding to $H_{G_{10}}$ mode is inserted in the second channel, maximal coincidence rate appears for the same mode in the transmitted channel, which is clearly seen as a peak shift on the histogram of Fig. 10 (2). We believe lower quality of phase mask used in the reflected channel to be the origin of somewhat lower visibility in this case.

Further evidence of similarity of Schmidt modes to Hermite-Gaussian ones may be obtained by analyzing the dependencies of single counts and coincidences on the fiber tip position in the focal plane of the focusing microscope objective. We expect the dependence for coinci-
Figure 10: Coincidence counting rate for SPDC radiation for various masks in the channel with SLM. (1) – HG$_{00}$ mode is selected in the reflected channel, (2) – HG$_{01}$ is selected in the reflected channel.

Figure 11: Coincidence (1) and single counts (2) rate dependence on the fiber tip position in the focal plane of the microscope objective in the channel with SLM for different modes. Fiber is scanned in horizontal direction.

Figure 12: Coincidence counting rate dependence on the fiber tip position in the reflected channel (without SLM) for different modes. Fiber tip is scanned in horizontal direction.

Figure 13: Dependence of maxima positions for coincidence distributions of Fig. 11 on the mode number (red bars). Calculated dependence for Hermite-Gaussian modes with $w = (3.0 \pm 0.1) \mu m$ corresponding to HG$_{00}$ waist size (grey bars) is provided for comparison.

Figure 14: Coincidence counting rate dependence on the slit position with multi-mode fiber in the reflected channel. "Ghost images" of the modes selected in the transmitted channel are observed. Results for first three modes HG$_{00}$, HG$_{10}$, HG$_{20}$ are shown in Fig. 14. Solid curves are fit with Hermite-Gaussian functions.

Differences to be described by (24). Experimental curves for the case when fiber in the transmitted channel is scanned are shown in Fig. 11. Note that the double-peak structure characteristic for Hermite-Gaussian modes appears only in coincidences dependence, while single counts behave monotonously, as is expected for spatially multi-mode radiation. The maximal value of single counting rate, however, decreases with increasing value of mode indexes. We will pay special attention to this dependence later. Distance between maxima behaves analogously to the case of attenuated laser beam, as shown in Fig. 13.

We obtained same dependencies of coincidence counting rate when the fiber tip was scanned in the reflected channel (see Fig. 12). In this case single counts, obviously, do not depend on the mode selected in the conjugate channel at all. This effect is a straightforward consequence of intermodal correlations in SPDC and may be thought of as a sort of "ghost interference" 17. We should note that almost zero coincidence counting rate in the central position of the fiber is an interference effect demonstrating spatial coherence of detected modes. So this result may be considered as an experimental demonstration of one of the main features of Schmidt modes – their spatial coherence.

One can also observe the shape of Schmidt modes directly. For this purpose we substituted the fiber in the reflected channel with a multi-mode one, which serves as a "bucket" detector, collecting all spatial modes. A 200 $\mu m$ vertical slit was inserted in front of the objective in the reflected channel. Dependence of coincidence counting rate on the slit position forms a "ghost image" of the mode selected in the conjugate (transmitted) channel. Experimental results for the first three modes HG$_{00}$, HG$_{10}$, HG$_{20}$ are shown in Fig. 14. Solid curves are fit with Hermite-Gaussian functions.
As we noted above, single counting rate in the channel with SLM for the central position of the single mode fiber decreases for higher order detected modes. Indeed, a state of the single photon of a pair is described by a reduced density matrix, which has the following form in the basis of Schmidt modes $\psi_{nm}(k_i)$:

$$\rho(k_i, k'_i) = \sum_{n,m} \lambda_{nm} \psi_{nm}(k_i) \psi_{mn}(k'_i), \quad (25)$$

where index $i = 1, 2$ numbers photons. Counting rate of detector D1: $R_{1mn} \sim \langle \psi_{nm}|\rho|\psi_{mn} \rangle \sim \lambda_{nm}$ is proportional to the eigenvalue – "weight" of the corresponding mode in the Schmidt decomposition. Analyzing single counting rate in the channel with SLM we can determine the eigenvalues in Schmidt decomposition. Histogram of single counting rate for detector D1 is shown in Fig. 15 (1). It should be compared to the calculated eigenvalues of single-photon density matrix in Hermite-Gaussian basis shown on Fig. 15 (2). We can use fidelity:

$$F = \text{Tr} \sqrt{\sqrt{\rho} (\rho^{(exp)}) \sqrt{\rho}} = \sum_{m,n} \sqrt{\lambda_{mn}} \sqrt{\lambda_{mn}^{(exp)}}, \quad (26)$$

as a quantitative measure of correspondence between the measured and calculated distributions. Experimental estimate for eigenvalues $\lambda_{mn}^{(exp)} = (R_{1mn} - R_0) / \sum_{n,m} R_{1mn}$ takes into account normalization, implying unit trace for the density matrix, and substitution of a constant noise level $R_0$, originated both from stray light and non-diffracted SPDC radiation. Experimental data give value of fidelity $F = 0.92 \pm 0.03$.

Using directly measured eigenvalues we may estimate the Schmidt number. Let us analyze the one-dimensional section of eigenvalues distribution shown in Fig. 16 for HG$_{m0}$ modes. This distribution is excellently approximated with a gaussian one: $R_{1m0} = R_0 + C \exp(-\alpha n)$ (correlation coefficient $R^2 = 0.993$). Using expression (21) for the Schmidt number it follows that $\lambda_{n+1}/\lambda_n = (K_x - 1)/(K_x + 1)$, and it is straightforward to obtain the following estimate for the Schmidt number:

$$K_x = \frac{e^\alpha + 1}{e^\alpha - 1}. \quad (27)$$

Using experimental results for HG$_{0m}$ modes we estimate the "horizontal" Schmidt number to be $K_x = 3.1 \pm 0.9$. Analogous procedure for "vertical" modes HG$_{0n}$ gives $K_y = 2.7 \pm 0.5$. Both values are in agreement with $K = 2.97$ calculated from (21) for experimental values of $a = 3.5 \times 10^{-3}$ and $b = 0.020$ (values are taken inside the crystal).

Experimental dependencies of eigenvalues are in good qualitative correspondence both with exponential behavior for double-gaussian approximation and with the results of numerical calculations. Small qualitative discrepancies are still above the level of statistical fluctuations. They can be partially explained by systematic fluctuations of signal level on long time scales caused primarily by temperature fluctuations. Experiments have shown, that both coincidence and single counts rates are extremely sensitive to small beam displacements. So work of an air-conditioning system stabilizing the room temperature caused small but noticeable periodic change of temperature caused small but noticeable periodic change of signal with a period of 10 min. Error bars shown on Fig. 16 correspond to the amplitude of this fluctuations since the exposition time was comparable with their period. Complete thermal isolation of the setup should remove this source of errors. Another significant source of errors may be the difference between detected angular aperture and full angular bandwidth of SPDC. Sharp dependence of counting rates on the position of fiber tip relatively to the microscope objective shows maximum in the position slightly different from the focal plane. Moreover, the pump, assumed to be gaussian, was actually aberrated to a double-gaussian approximation and with the results to be clear evidence of possibility to realize projective measurements in spatial Schmidt modes basis for SPDC biphotons, and the implemented scheme is useful for quantitative measurements and state reconstruction.
VII. CONCLUSION

We have analyzed spatial entanglement in SPDC in terms of spatial Schmidt decomposition and shown that under reasonable assumptions, applicable to the particular experiment, these modes are close to Hermite-Gaussian modes. Using this fact we have experimentally realized a scheme of projective measurements in Schmidt basis using an active spatial light modulator. Experimental results prove high quality of gaussian beam transformation to higher order Hermite-Gaussian modes. For a spatial multi-mode SPDC radiation such spatial filtering allowed us to realize projective measurements in Schmidt basis. We have experimentally verified similarity of spatial Schmidt modes for SPDC to Hermite-Gaussian ones. We have performed direct measurement of eigenvalues in Schmidt decomposition and experimentally estimated the Schmidt number, obtaining good agreement with predictions of a simple double-gaussian model of SPDC spectrum.

Spatial modes are interesting objects for quantum state engineering. The Hilbert space is vast and its dimension may be chosen at will by simply focusing the pump properly. One can both obtain single-mode biphotons and highly entangled EPR-like states as extreme cases. However, intermediate case, considered in our work also offers interesting possibilities. We foresee at least two lines of research in this area: quantum tomography with high-dimensional spatial states and quantum state engineering of non-trivial states. Schmidt modes measurement techniques developed here should be a useful tool for both tasks.

Another interesting question is the behavior of Schmidt-like internodal correlations in the classical limit. What would stand for the Schmidt number, and whether the modal structure may be described in terms of some simple decompositions like in the case of SPDC, are interesting issues to address both theoretically and experimentally. Our results for quasi-thermal light sources will be reported elsewhere.

We are grateful to M. V. Fedorov for stimulating discussions. This work has been founded by RFBR grants 10-02-00204 and 12-02-00288. S. S. Straupe and I. B. Bobrov are grateful to the "Dynasty" foundation for financial support.
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[48] OAM conservation in the general case of SPDC is questioned by some authors (see for example [31, 32]). We will not go into details here, since all models predict OAM conservation in near-collinear case considered here.

[49] We may use this approximation since anisotropy of angular distributions is insignificant for the values of pump divergence and crystal length used in our experiment.

[50] We have tried several criteria for choosing $\gamma$, for example equality of FWHM for both functions, which is used in [35] and gives $\gamma = 0.249$. However, $\gamma = 0.86$ better describes experimental data, which is not completely understood, but may be caused by limited angular detection aperture.

[51] One should pay attention to the fact, that all formulas in previous sections do not take into account the refraction on the crystal surface, so to compare with experiment all angular variables for the pump should be divided and for SPDC multiplied on $n_0(\lambda_p) = n_0(\lambda_s) = 1.667$.

[52] To avoid misunderstanding, we define "waist size" to be $w$ in the intensity dependence on the distance to beam axis: $I(r) = I_0 e^{-2r^2/w^2}$.

[53] Multiplied by an additional gaussian term to take into account change in the integral intensity caused by use of the slit instead of circular aperture.