New integrals in few-body problems.

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Abstract

This work is concerned with multi-dimensional integrals, which are making their appearance in few-body atomic and nuclear physics. It is shown that the relevant two- and three-dimensional integrals can be reduced to one-dimensional form. This implies that the internal one- and two-dimensional integrals can be evaluated in explicit analytic form in term of the familiar generalized hypergeometric functions. Some of the integrals are presented here for the first time.

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I. INTRODUCTION

One of the effective tools for solving, e.g., $N$-body problem both in atomic and nuclear physics is introducing the Jacobi vectors (coordinates) $\xi_i$ which presents a linear combination of a standard vectors $\mathbf{r}_i$ ($i = 1, 2, \ldots N$) of the particles under consideration. Omitting the center of mass motion for a given choice of the Jacobi vectors, the hyperspherical coordinates are given by the so-called hyperradius $\rho$ and by a set $\Omega_{N-1}$ of angular variables. The latter can be expressed through the $2(N-1)$ polar angles $\omega_i \equiv (\theta_i, \phi_i)$ of the Jacobi vectors $\xi_i$ and $(N-2)$ hyperspherical angles $\varphi_i$ ($i = 2, \ldots, N-1$).

In order to estimate the physical properties of the considered system, one needs to calculate matrix elements of the proper Hamiltonian in the appropriate basis. For simplicity let us consider three-particle system (see, e.g., [1, 2]) in the most abundant basis set presenting the product

$$R_n(\rho)Y_{l_1,l_2,\mu}\cdot LM(\Omega_2),$$

where

$$Y_{l_1,l_2,\mu}\cdot LM(\Omega_2) = \mathcal{P}_{\mu}^{l_1,l_2}(\varphi_2) \sum_{l_1, l_2, m_1, m_2} \langle l_1 m_1 l_2 m_2 | LM \rangle Y_{l_1 m_1}(\theta_1, \phi_1) Y_{l_2 m_2}(\theta_2, \phi_2)$$

is the so called hyperspherical harmonic function with definite angular momentum associated to the quantum numbers $LM$. The RHS of Eq.(2) contains the function

$$\mathcal{P}_{\mu}^{l_1,l_2}(\varphi_2) = \mathcal{N}_{\mu}^{l_1,l_2}(\cos \varphi_2)^{l_2}(\sin \varphi_2)^{l_1} P_{\mu}^{l_1+1/2,l_2+1/2}(\cos 2\varphi_2),$$

where $P_{\mu}^{a,b}(z)$ are the Jacobi polynomials, $\mathcal{N}_{\mu}^{l_1,l_2}$ is normalization constant; the Clebsch-Gordan coefficients $\langle l_1 m_1 l_2 m_2 | LM \rangle$, and usual spherical harmonics $Y_{lm}(\theta, \phi)$. Here $\mu$ is non-negative integer, and $(l_i, m_i)$ are the quantum numbers corresponding to the angular momentum operator associated with the $i$th Jacobi vector.

The most commonly encountered form of hyperradial function [1, 2] is the following:

$$R_n(\rho) = C_{\alpha,n} L_n^{(5)}(\alpha \rho) e^{-\frac{1}{2} \alpha \rho},$$

where $L_n^{(k)}(z)$ are the generalized Laguerre polynomials, and $\alpha$ is a scale factor. The explicit forms of the normalization constants $C_{\alpha,n}$ and $\mathcal{N}_{\mu}^{l_1,l_2}$ are not important for the given treatment, in contrast to the form of Jacobi vectors

$$\xi_1 = \frac{1}{\sqrt{6}} (2 \mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_2), \quad \xi_2 = \frac{1}{\sqrt{2}} (\mathbf{r}_2 - \mathbf{r}_1)$$

(5)
for three particles of equal mass (see, e.g., [1, 3, 4]). Taking the inverse transformation for Jacobi vectors (5) plus the centre-of-mass vector $\xi_3 = \frac{1}{3} (r_1 + r + r_3)$ and keeping in mind that by definition $\xi_2 = \rho \cos(\varphi_2), \xi_1 = \rho \sin(\varphi_2)$ one obtains:

$$r^2_{12} = \rho^2(1 + \tau),$$

$$r^2_{13} = \frac{\rho^2}{2} \left[ 2 - \tau + \lambda \sqrt{3(1 - \tau^2)} \right],$$

$$r^2_{23} = \frac{\rho^2}{2} \left[ 2 - \tau - \lambda \sqrt{3(1 - \tau^2)} \right],$$

where $\tau = \cos 2\varphi_2$, and $\lambda$ presents cosine of the angle between vectors $\xi_1$ and $\xi_2$.

Let us consider the Gaussian-type central potentials of the form

$$V(\{r_{ij}\}) = \sum_{i>j} \sum_{k} A_k \exp(-\zeta_k r^2_{ij}).$$

The examples of such potentials can be presented by the two-nucleon potential models [5–7].

It can be shown that calculations of the matrix elements for the potentials of the form reduce to evaluating two-dimensional integral of the form

$$I_2 = \int_{-1}^1 (1 - \tau)^{l_1+1/2}(1 + \tau)^{l_2+1/2} P_{\mu_1}^{l_1+1/2,l_2+1/2}(\tau) P_{\mu_2}^{l_1+1/2,l_2+1/2}(\tau) d\tau \times$$

$$\int_0^\infty L_{n_1}^{(5)}(x) L_{n_2}^{(5)}(x) e^{-\gamma x^2} x^5 dx,$$

and three-dimensional integral of the form

$$I_3 = \int_{-1}^1 (1 - \tau^2) P_{\mu_1}^{l_1+1/2,l_1+1/2}(\tau) P_{\mu_2}^{l_2+1/2,l_2+1/2}(\tau) d\tau \times$$

$$\int_0^\infty L_{n_1}^{(5)}(x) L_{n_2}^{(5)}(x) e^{-\beta x^2} x^5 dx \int_{-1}^1 e^{-\kappa x^2} P_3(\lambda) P_2(\lambda) d\lambda,$$

where

$$\gamma \equiv \gamma(\tau) = \frac{\zeta_k}{\alpha^2} (1 + \tau),$$

$$\beta \equiv \beta(\tau) = \frac{\zeta_k}{\alpha^2} \left( 1 - \frac{\tau}{2} \right), \quad \kappa \equiv \kappa(\tau) = \frac{\zeta_k}{2\alpha^2} \sqrt{3(1 - \tau^2)},$$

and $P_l(\lambda)$ are the Legendre polynomials. One should emphasize that the 3-dimensional integral (11) corresponds only to the case of zero angular momentum ($L = 0$). However, for the potentials considered, the total orbital angular momentum is a good quantum number [1, 2], therefore only the wavefunctions with $L = 0$ were included.
It will be shown in the subsequent chapters that both 2-dimensional integral \( I_2 \) and 3-dimensional integral \( I_3 \) can be reduced to one-dimensional integrals. The latter implies that integrals over \( \lambda \) and \( x \) variables can be evaluated in the explicit (closed) form. At that, the final expressions for the integrals associated to 3-dimensional integral \( I_3 \) presents a new results, at least, from mathematical point of view.

II. TWO-DIMENSIONAL INTEGRAL

First, let us consider two-dimensional integral \( I_{10} \). Evaluation of the internal integral over \( x \) can be performed by making use of the known representation for the associated Laguerre polynomials

\[
L^{(k)}_{n_1,n_2}(\gamma) \equiv \int_0^\infty L^{(k)}_{n_1}(x) L^{(k)}_{n_2}(x) e^{-\gamma x^2-x^k} dx = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} C^{n_1}_{i,k} C^{n_2}_{j,k} J(k+i+j, \gamma)
\]  

followed by applying the integral presented on p.343(2.3.15(3)) \[8\]:

\[
J(\nu, \gamma) \equiv \int_0^\infty e^{-\gamma x^2-x^\nu} dx = \Gamma(\nu + 1)(2\gamma) e^{-\nu+1} e^{1/8} \gamma D_{-\nu-1} \left( \frac{1}{\sqrt{2}\gamma} \right).
\]

The coefficients introduced in Eq.(14) have a form:

\[
C_{i,k}^{n_1} = \frac{(n-i+1)_k (-1)^i}{(k+i)!i!}.
\]

Here, \( (a)_n \) denotes the Pochhammer symbol, and \( D_\sigma(z) \) is the parabolic-cylinder function. Alternatively, the latter function can be substituted for the Tricomi confluent hypergeometric function or Hermite function using relations (13.6.36) or (13.6.38), respectively \[9\]:

\[
e^{\frac{1}{8\gamma}} D_{-\nu-1} \left( \frac{1}{\sqrt{2}\gamma} \right) = 2^{-\nu+1/2} U \left( \nu + 1, \frac{1}{2}, \frac{1}{4\gamma} \right) = 2^{\nu+1/2} H_{-\nu-1} \left( \frac{1}{2\sqrt{\gamma}} \right).
\]

In order to derive the latter connection with Hermite function, we additionally made use the following relationship (see, e.g., (13.1.29) \[9\]):

\[
U(a, b, z) = z^{1-b} U(a - b + 1, 2 - b, z),
\]

that will be applied in the next Section too.

It is worth noting that for integer \( \nu = n \) the parabolic-cylinder function can be presented in the form \[10\]:

\[
n! D_{-n-1}(z) = i^n 2^{1-n} \int_0^\infty e^{-\tau^2} \sum_{s=1}^{n} (-i)^s \left( \begin{array}{c} n \\ s \end{array} \right) H_{s-1} \left( \frac{z}{\sqrt{2}} \right) H_{n-s} \left( \frac{iz}{\sqrt{2}} \right) + \\
\sqrt{\pi} i^n 2^{1-n} (-1)^n e^{z^2} \text{erfc} \left( \frac{z}{\sqrt{2}} \right) H_n \left( \frac{iz}{\sqrt{2}} \right),
\]
where \( \text{erfc}(y) \) is the complementary error function, \( \binom{n}{k} \) is the binomial coefficient, and \( i \) represents the imaginary unit. Eq. (19) enables us to derive the additional representation for the integral (14):

\[
\mathcal{L}_{n_1,n_2}^{(k)}(\gamma) = \frac{(-1)^k}{n_1!n_2!(2\gamma)^{n_1+n_2+k}} \left[ \frac{\sqrt{\pi}}{\gamma} e^{\frac{\pi}{4\gamma}} \text{erfc} \left( \frac{1}{2\sqrt{\gamma}} \right) T_{n_1,n_2}^{(k)}(\gamma) - S_{n_1,n_2}^{(k)}(\gamma) \right]
\]

where

\[
T_{n_1,n_2}^{(k)}(\gamma) = n_1!n_2!(2\gamma)^{n_1+n_2+k} \times \sum_{i=0}^{n_1} \frac{(n_1 - i + 1)_{k+i}}{i!(k+i)!} \sum_{j=0}^{n_2} \frac{(n_2 - j + 1)_{k+j}}{j!(k+j)!} \left( -\frac{i}{2\sqrt{\gamma}} \right)^{i+j+k} H_{i+j+k} \left( \frac{i}{2\sqrt{\gamma}} \right)
\]

\[
S_{n_1,n_2}^{(k)}(\gamma) = -\frac{n_1!n_2!(2\gamma)^{n_1+n_2+k}}{\sqrt{\gamma}} \sum_{i=0}^{n_1} \frac{(n_1 - i + 1)_{k+i}}{i!(k+i)!} \sum_{j=0}^{n_2} \frac{(n_2 - j + 1)_{k+j}}{j!(k+j)!} \times \left( -\frac{i}{2\sqrt{\gamma}} \right)^{i+j+k} \sum_{s=1}^{i+j+k} \binom{i+j+k}{s} (-i)^s H_{i+j+k-s} \left( \frac{i}{2\sqrt{\gamma}} \right) H_{s-1} \left( \frac{1}{2\sqrt{\gamma}} \right)
\]

It is worth noting that both \( T_{n_1,n_2}^{(k)} \) and \( S_{n_1,n_2}^{(k)} \) present polynomials (in \( \gamma \)) with the integer coefficients, and possess properties:

\[
T_{n_1,n_2}^{(k)}(0) = S_{n_1,n_2}^{(k)}(0) = 1.
\]

### III. THREE-DIMENSIONAL INTEGRAL

The aim of this section is to derive the explicit (closed) expression for the internal two-dimensional integral

\[
\mathcal{B}_{n_1,n_2}^{(k)}(\beta, \kappa) \equiv \int_{0}^{\infty} L_{n_1}^{(k)}(x) L_{n_2}^{(k)}(x) e^{-\beta x^2 - \kappa x^2} dx \int_{-1}^{1} e^{-\lambda \kappa x^2} P_{l_1}(\lambda) P_{l_2}(\lambda) d\lambda
\]

associated to three-dimensional integral ([11]). To this end, one should first of all to apply the well-known Neumann-Adams formula which expresses the product of two Legendre polynomials as a sum of such polynomials, and then to make use of integral presented by formula (2.17.5(2)), p. 428 [11]. This yields:

\[
\int_{-1}^{1} e^{-\lambda \kappa x^2} P_{l_1}(\lambda) P_{l_2}(\lambda) d\lambda = \sum_{r=0}^{l_1} A_{l_1,l_2}^r \int_{-1}^{1} e^{-\lambda \kappa x^2} P_{l_1+l_2-2r}(\lambda) d\lambda =
\]

\[
x^{-1} \sqrt{\frac{2\pi}{\kappa}} \sum_{r=0}^{l_1} A_{l_1,l_2}^r (-1)^{l_1+l_2-2r} I_{l_1+l_2-2r+1/2}(\kappa x^2), \quad (l_1 \leq l_2)
\]

\[
(25)
\]
where
\[ A_{l_1,l_2}^r = \frac{(2l_1 - 2r - 1)!!(2r - 1)!!(2l_2 - 2r - 1)!!(l_1 + l_2 - r)!(2l_1 + 2l_2 - 4r + 1)}{(l_1 - r)!r!(l_2 - r)!(2l_1 + 2l_2 - 2r - 1)!!}, \] (26)
and \( I_{m+1/2}(z) \) are spherical modified Bessel functions of the first kind \((m \text{ is integer})\).

Presenting the product of the Laguerre polynomials in its explicit polynomial form, as it was done in the previous section, and making use of Eq.(26), one obtains:
\[ B(k)_{n_1,n_2}(\beta, \kappa) = \sqrt{\frac{2\pi}{\kappa}} \sum_{r=0}^{l_1} A_{l_1,l_2}^r (-1)^{l_1+l_2-2r} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} C_{i,k}^{n_1} C_{j,k}^{n_2} \times \]
\[ \int_0^\infty x^{i+j+k-1} e^{-\beta x^2-x} I_{l_1+l_2-2r+1/2}(\kappa x^2) \, dx, \] (27)
where coefficients \( C_{i,k}^{n_1} \) are defined by Eq.(16).

Much of what follows are devoted to deriving the explicit analytic expression for evaluating the integral
\[ K_p^\mu(\beta, \kappa) = \int_0^\infty e^{-\beta x^2-x} I_\mu(\kappa x^2) x^p \, dx. \] (28)
We have not found a solution of this problem both in mathematical and physical literature even for integer values of \( p \) corresponding to Eq.(27).

Final results that will be obtained here are valid for any real \( p \) allows the integral (28) to be convergent at the given half-integer \( \mu \). On the other hand, it is clear that this integral converges only for \( \beta \geq \kappa \). We shall consider real \( \beta > 0 \) and \( \kappa > 0 \). Note, that the case of \( \beta = \kappa \) is presented in [11] (see,2.15.6(1), p.306). However, it is easy to make sure that according to definition (13)
\[ 0 \leq \frac{\kappa}{\beta} \equiv \frac{\sqrt{3(1-\tau^2)}}{2-\tau} \leq 1 \] (29)
for values of \(-1 \leq \tau \leq 1\).

First, let us present series expansion for the modified Bessel function of the first kind [9]:
\[ I_\mu(\kappa x^2) = \left(\frac{\kappa x^2}{2}\right)^\mu \sum_{k=0}^\infty \frac{\left(\frac{x^2}{2}\right)^{2k}}{\Gamma(k+\mu+1)k!}. \] (30)
Inserting this representation into Eq.(28), one obtains:
\[ K_p^\mu(\beta, \kappa) = \left(\frac{\kappa}{2}\right)^\mu \sum_{k=0}^\infty \frac{\left(\frac{x^2}{2}\right)^{2k}}{k!\Gamma(k+\mu+1)} \int_0^\infty e^{-\beta x^2-x} x^{4k+2\mu+p} \, dx = \]
\[ \frac{\left(\frac{\kappa}{8\beta}\right)^\mu}{(4\beta)^{p+1}} \sum_{k=0}^\infty \frac{\Gamma(4k+2\mu+p+1)}{\Gamma(k+\mu+1)k!} \left(\frac{\kappa}{8\beta}\right)^{2k} U \left(2k+\mu+p+\frac{1}{2};\frac{1}{2},\frac{1}{4\beta}\right). \] (31)
Instead of Tricomi confluent hypergeometric function, one can use any of equivalent functions as provided by Eqs.(15)-(19). It is worth noting that the cutoff expansion (31) can be successfully applied for computing integral (28) with any values of \((\kappa/\beta) < 1\). However, the more value of \((\kappa/\beta)\) requires the more length of expansion (31). Therefore, the latter is especially effective for very small values of \((\kappa/\beta)\).

Let us proceed with derivation of the closed analytic expression for the integral (28). The primary definition for the Tricomi confluent hypergeometric function yields [12] (see, 7.2.2.2.(2), p.434):

\[
U \left( 2k + \mu + \frac{p+1}{2}, \frac{1}{2}, \frac{1}{4\beta}; \frac{1}{2}, \frac{1}{4} \right) = \frac{\sqrt{\pi} \, \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( 2k + \frac{p+1}{2} + \mu + 1 \right)} - \frac{\sqrt{\pi} \, \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( 2k + \frac{p+1}{2} + \mu \right)} \sqrt{\beta} \, \Gamma \left( 2k + \frac{p+1}{2} + \mu \right).
\]  

(32)

Inserting the first term of the RHS of Eq.(32) into Eq.(31) and then changing the order of summation, one obtains:

\[
K_1 = \sqrt{\frac{\pi}{(4\beta)^{1/2}}} \sum_{k=0}^{\infty} \frac{\Gamma(4k + 2\mu + p + 1)}{\Gamma(k + \mu + 1)} \left( \frac{\kappa}{8\beta} \right)^{2k} \frac{1F1 \left( 2k + \frac{p+1}{2} + \mu; \frac{1}{2}; \frac{1}{4} \right)}{\Gamma \left( 2k + \frac{p+1}{2} + \mu + 1 \right)} = 2F1 \left( \frac{n + \mu + p}{2}, \frac{n + \mu + p + 3}{4}; 1 + \mu; z^2 \right),
\]

(33)

We used for the latter derivation and we will use in what follows, the well-known duplication formula (6.1.18) [9] for the gamma functions.

Next step is making use of the relationship between the Gauss hypergeometric functions of the form \(2F1(a, a + 1/2; c; z^2)\) and the associated Legendre functions of the first kind [12] (7.3.1(101), p.460), which yields for the case of interest:

\[
2F1 \left( \frac{n + \mu}{2}, \frac{n + \mu}{2}; 1 + \mu; z^2 \right) = \Gamma(\mu + 1) \left( \frac{2}{1z} \right)^\mu (1 - z^2) \frac{\Gamma(n + \mu + p + 1)}{\Gamma(n + \mu + p + 1 + 1)} \frac{1}{\sqrt{1 - z^2}},
\]

(34)

with \(z = \kappa/\beta\).

In this stage, one needs to express the Legendre functions presented in Eq.(34) through finite sums with the limits (of summation), which are not dependent on \(n\). Note, that
relationship 3.2(9) \cite{13} enables us to express the associated Legendre functions of the first kind through two Gauss hypergeometric functions by different manners, which are presented by formulas (3.2.14)-(3.2.31) \cite{13}. From the latter representations it is seen that only the case of a half-integer \( \mu \) could give the mentioned above Gauss hypergeometric functions with one (of the first two) negative integer parameter, which is not dependent on \( n \).

Thus, introducing denotation

\[
\mu = m + \frac{1}{2},
\]

where \( m \) is a non-negative integer, and making use representation 3.2(30) \cite{13}, one obtains:

\[
\left[ \sqrt{1-z} \left( \frac{1-z}{1+z} \right)^{\frac{2n+1}{4}} \frac{(-1)^n \Gamma(n+p/2)}{\Gamma(n+p/2 + m + 1)} 2F_1 \left( m + 1, -m; 1 - n - \frac{p}{2}; \frac{1+z}{2z} \right) - \sqrt{1+z} \left( \frac{1+z}{1-z} \right)^{\frac{2n+1}{4}} \frac{\Gamma(n+p/2 - m)}{\Gamma(n+p/2 + 1)} 2F_1 \left( m + 1, -m; 1 + n + \frac{p}{2}; \frac{1+z}{2z} \right) \right].
\]

(36)

Note, that factor \( e^{i\pi \left( \frac{\mu - \frac{3}{2}}{2} \right)} \) is correct, but it is different from the corresponding factor presented in Ref.\cite{13}.

One should emphasize, that due to the factor \( \Gamma(n+p/2 - m) \) Eq.(36) is not valid only for the case of even \( p \leq 2m \) \((n \geq 0)\). This case will be considered later.

Now, one needs to insert Eq.(36) with the hypergeometric functions presented in explicit (polynomial) form into Eq.(34). Then, inserting the result into Eq.(33) and changing the order of summation, one obtains for the real values of \( p > -2m - 2 \), excluding the case of even \( p \leq 2m \):

\[
\sqrt{8\pi\kappa} K_1 (-\text{even } p \leq 2m) = \left[ \left( \frac{1}{\beta - \kappa} \right)^{p/2} \sum_{n=0}^{\infty} \Gamma \left( m + n + 1 + \frac{p}{2} \right) \Gamma \left( n + \frac{p}{2} - m \right) \frac{1}{(2n)!\Gamma(1 + n + \frac{p}{2})} 2F_1 \left( m + 1, -m; 1 + n + \frac{p}{2}; \frac{\beta + \kappa}{2\kappa} \right) - \left( \frac{1}{\beta + \kappa} \right)^{p/2} (-1)^n \sum_{n=0}^{\infty} \Gamma \left( n + \frac{p}{2} \right) \frac{1}{(2n)!\Gamma(\beta + \kappa)^n} 2F_1 \left( m + 1, -m; 1 - n - \frac{p}{2}; \frac{\beta + \kappa}{2\kappa} \right) \right] =
\]

\[
(\beta - \kappa)^{-\frac{p}{2}} \Gamma \left( \frac{p}{2} + m + 1 \right) \Gamma \left( \frac{p}{2} - m \right) \times \sum_{k=0}^{m} \frac{(m+k)!}{k!(m-k)!\Gamma \left( 1 + k + \frac{p}{2} \right)} 2F_2 \left[ \frac{p}{2} - m; \frac{p}{2} + m + 1; \frac{1}{2}, \frac{p}{2} + k + 1; \frac{1}{4(\beta - \kappa)} \right] - \]

\[
(\beta + \kappa)^{-\frac{p}{2}} (-1)^m \sum_{k=0}^{m} \frac{(m+k)!\Gamma \left( \frac{p}{2} - k \right) \left( \frac{\beta + \kappa}{2\kappa} \right)^k}{k!(m-k)!} 1F_1 \left[ \frac{p}{2} - k; \frac{1}{2}, \frac{1}{4(\beta + \kappa)} \right].
\]

(37)
For the even values of $p \leq 2m$, one needs to divide a summation over $n$ by two ranges: $[0 : m - p/2]$ and $[m - p/2 + 1 : \infty]$, and then - to perform the same procedure, as in the previous case. This yields:

$$
\sqrt{8\pi\kappa} K_1 \text{ (even } p \leq 2m) = \frac{2\sqrt{\pi} \left( \frac{\kappa}{2\beta} \right)^{m+1/2}}{\Gamma \left( m + \frac{3}{2} \right) \beta^{3/2}} \sum_{n=0}^{m} \frac{\Gamma \left( \frac{p}{2} + m + n + 1 \right)}{(2n)! \beta^{n}} 2F_1 \left( \frac{n + m + 1}{2}, \frac{n + m + 2}{2} ; \frac{p}{4} ; \frac{m + \frac{3}{2} \kappa^2}{\beta^2} \right) + \frac{1}{(2m + 2 - p)} \left\{ \left( -\frac{1}{\beta + \kappa} \right)^{m+1} \times \sum_{k=0}^{m} \frac{(m + k)! \left( \frac{\beta + \kappa}{2\kappa} \right)^{k}}{k!} 2F_2 \left[ 1, 1 - k + m; m + \frac{3 - p}{2}, m + 2 - \frac{p}{2}; \frac{1}{4(\beta + \kappa)} \right] + \left( \frac{1}{\beta - \kappa} \right)^{m+1} \left( 2m + 1 \right)! \times \sum_{k=0}^{m} \frac{(-\frac{\beta + \kappa}{2\kappa})^{k}}{k!(m - k)!(m + k + 1)} 3F_3 \left[ 1, 1, 2 + 2m; 2 + k + m, m + \frac{3 - p}{2}, 2 + m - \frac{p}{2}; \frac{1}{4(\beta - \kappa)} \right] \right\}. \tag{38}
$$

One can proceed to consideration of the residual RHS of Eq.(31). Inserting the second term of the RHS of Eq.(32) into Eq.(31) and then changing the order of summation, one obtains:

$$
K_2 = -\frac{\sqrt{\pi} \left( \frac{\kappa}{2\beta} \right)^{m+1/2}}{(4\beta)^{m+1/2}} \sum_{k=0}^{\infty} \frac{\Gamma \left( 4k + 2m + p + 2 \right)}{k! \Gamma \left( k + m + 3/2 \right)} \left( \frac{\kappa}{8\beta} \right)^{2k} \frac{1}{\sqrt{\beta}} \frac{1}{\Gamma \left( 2k + \frac{p+3}{2} + m + 1 \right)} = -\frac{\left( \frac{\kappa}{2\beta} \right)^{m+1/2}}{2 \Gamma \left( m + 3/2 \right) \beta^{3/2}} \sum_{n=0}^{\infty} \frac{\Gamma \left( n + m + \frac{p+3}{2} \right)}{(2n + 1)! \beta^{n}} 2F_1 \left[ \frac{n + m}{2}, \frac{n + m + 1}{4}, \frac{n + m + 5}{4}, \frac{n + m + 3}{2}; \frac{3}{2}, \frac{\kappa^2}{2\beta^2} \right]. \tag{39}
$$

Once again one should make use of the relationship between the Gauss hypergeometric functions of the form $2F_1(a, a + 1/2; c; z^2)$ and the associated Legendre functions of the first kind \[12\] (7.3.1(101), p.460). However, in this case one sets $a = \left( \frac{n + m}{2} \right) + (p + 3)/4$, $c = m + 3/2$, $z = \kappa/\beta$. Then, one needs to apply representation 3.2(9) \[13\] with the parameters presented by 3.2(30) \[13\]. Thus, one obtains:

$$
2F_1 \left[ \frac{n + m}{2} + \frac{p + 3}{4}, \frac{n + m + 3}{2}; \frac{1}{2} - \frac{z}{2}, z^2 \right] = \frac{\Gamma \left( \frac{m + 3}{2} \right)}{2 \Gamma \left( m + 3/2 \right)} \left( \frac{2}{z} \right)^{m+1} \left( 1 - z^2 \right)^{-\frac{2n+p}{4}} \left[ \sqrt{1 + \frac{(1 + z)^{2n+p}}{1 - z}} \Gamma \left( \frac{n + \frac{p+1}{2} - m}{\Gamma \left( n + \frac{p+3}{2} + m \right)} 2F_1 \left[ m + 1, -m; \frac{n + p + 3}{2}, \frac{z + 1}{2z} \right] - \sqrt{1 - \frac{(1 + z)^{2n+p}}{1 - z}} \right] \right. \tag{40}
$$
One should notice that due to the factor $\Gamma \left( \frac{n + p + 1}{2} - m \right)$ Eq. (40) is not valid for odd values of $p \leq 2m - 1$.

Substituting representation (40) with the explicit (polynomial) expressions for the hypergeometric functions into Eq. (39), and then changing the order of summation, one obtains:

$$\sqrt{8\pi\kappa} K_2\left(\text{odd } p \leq 2m - 1 \right) =$$

$$(-1)^{m}(\beta + \kappa)^{-\frac{p+1}{4}} \sum_{k=0}^{m} \frac{(m+k)!}{(m-k)!k!} \frac{(\frac{p+1}{2} - k)}{\left(\frac{\beta + \kappa}{2}\right)^k} _1F_1\left[p+1; \frac{3}{2}; 4(\beta + \kappa) - \frac{1}{2}\right] -$$

$$\left(\beta - \kappa\right)^{-\frac{p+1}{4}} \Gamma\left(\frac{p+1}{2} - m\right) \Gamma\left(\frac{p+3}{2} + m\right) \times$$

$$\sum_{k=0}^{m} \frac{(m+k)!}{(m-k)!k!} \frac{(-1)^k}{\left(\frac{\beta + \kappa}{2\kappa}\right)^k} 2F_2\left[p+1, m, \frac{p+3}{2}; m + \frac{3}{2}, \frac{p+3}{2} + k; 1\right].$$

(41)

For the odd values of $p \leq 2m - 1$, one needs to divide a summation over $n$ in Eq. (39) by two ranges: $[0 : m - (p+1)/2]$ and $[m - (p-1)/2 : \infty]$. And then, it is necessary to perform the same procedure, as in the previous case. This yields:

$$\sqrt{8\pi\kappa} K_2\left(\text{odd } p \leq 2m - 1 \right) =$$

$$-\frac{2\sqrt{\pi}}{\Gamma\left(m + \frac{3}{2}\right)} \frac{\left(\frac{\kappa}{2\beta}\right)^{m+1}}{\beta^{\frac{p+1}{2}}} \sum_{n=0}^{m-\frac{p+1}{2}} \frac{\Gamma\left(m + n + \frac{p+2}{2}\right)}{(2n+1)!\beta^n} _2F_1\left(\frac{n+m}{2}, \frac{p+3}{4}; \frac{n+m}{2} + \frac{p+5}{4}; m + \frac{3}{2}; \frac{\kappa^2}{\beta^2}\right) -$$

$$\frac{1}{(2m+2-p)!} \left\{ \left(-\frac{1}{\beta + \kappa}\right)^{m+1} \timesight.$$

$$\sum_{k=0}^{m} \frac{(m+k)!}{k!} \frac{\left(\beta + \kappa\right)^k}{2^k} 2F_2\left[1, 1 - k + m; m + \frac{3-p}{2}, m + 2 - \frac{p}{2}; 1\right] +$$

$$\left(\frac{1}{\beta - \kappa}\right)^{m+1} \left(2m + 1\right)! \times$$

$$\sum_{k=0}^{m} \frac{\left(-\frac{\beta + \kappa}{2\kappa}\right)^k}{k!(m-k)!(m+k+1)} 3F_3\left[1, 1, 2 + 2m; 2 + k + m, m + \frac{3-p}{2}, 2 + m - \frac{p}{2}; 1\right].$$

(42)

### A. Final formula for the new integral

The results obtained above for the integral (28) with half-integer parameter $\mu$ can be processed and presented in the compact form. To this end, let us introduce three auxiliary
functions:

\[ F_1(s) \equiv F_1(s, p, m; \beta, \kappa) = \frac{2\sqrt{\pi} \left( \frac{\kappa}{2\rho} \right)^{m+1}}{\Gamma \left( m + \frac{3}{2} \right)^{\beta + \frac{1}{2}}} \times \left( \sum_{n=0}^{m-\frac{s}{2}} \frac{\Gamma \left( m + n + 1 + \frac{p+s}{2} \right)}{(2n+s)!\beta^n} \right)^{2F_1} \left[ \frac{n+m+1}{2} + \frac{p+s}{4}, \frac{n+m+2}{2} + \frac{p+s}{4}; m + \frac{3}{2} \kappa^2 \right], \quad (43) \]

\[ F_2 \equiv F_2(p, m; \beta, \kappa) = \frac{1}{(2m+2-p)!} \left\{ \left( -\frac{1}{\beta + \kappa} \right)^{m+1} \times \sum_{k=0}^{m} \frac{(m+k)! \left( \frac{\beta+\kappa}{2} \right)^k}{k!} \right\} \left( \frac{\beta+\kappa}{2k} \right)^{2F_2} \left[ 1, 1 - k + m; m + \frac{3-p}{2}, m + 2 - \frac{p}{2}; 1, \frac{1}{4(\beta + \kappa)} \right] + \frac{(2m+1)!}{(\beta - \kappa)^{m+1}} \times \sum_{k=0}^{m} \frac{(-\beta + \kappa)^k}{k!(m-k)!(m+k+1)} \left( \frac{\beta+\kappa}{2k} \right)^{3F_3} \left[ 1, 1, 2 + m; 2 + k + m, \frac{3-p}{2} + m, 2 + m - \frac{p}{2}; \frac{1}{4(\beta - \kappa)} \right], \quad (44) \]

\[ F_3(s) \equiv F_3(s, p, m; \beta, \kappa) = \frac{(-1)^m}{(\beta + \kappa)^{\frac{p+s}{2}}} \sum_{k=0}^{m} \frac{(m+k)! \Gamma \left( \frac{p+s}{2} - k \right) \left( \beta + \kappa \right)^k}{(m-k)!k!} \left( \frac{\beta+\kappa}{2k} \right)^{1F_1} \left[ p + s \right. \left. \frac{2}{2} - k; \frac{2s+1}{2}; \frac{1}{4(\beta + \kappa)} \right] + \frac{\Gamma \left( \frac{p+s}{2} - m \right) \Gamma \left( \frac{p+s}{2} + m + 1 \right)}{(\beta - \kappa)^{\frac{p+s}{2}}} \sum_{k=0}^{m} \frac{(m+k)!(-1)^k}{(m-k)!k!\Gamma \left( \frac{p+s}{2} + k + 1 \right)} \left( \frac{\beta+\kappa}{2k} \right)^k \times \left( \frac{p+s}{2} - m, \frac{p+s}{2} + m + 1; \frac{2s+1}{2}, \frac{p+s}{2} + k + 1; \frac{1}{4(\beta - k)} \right]. \quad (45) \]

The integral of interest can be expressed in term of these functions as follows:

\[ \int_{0}^{\infty} e^{-\beta x^2 - x} I_{m+\frac{1}{2}}(\kappa x^2) x^p dx = \frac{1}{\sqrt{8\pi \kappa}} \begin{cases} F_1(0) + F_2 - F_3(1) & \text{even } p \leq 2m \\ -F_1(1) - F_2 + F_3(0) & \text{odd } p < 2m \\ F_3(0) - F_3(1) & \text{otherwise} \end{cases} \quad (46) \]

It is seen that representation (46) cannot be used for very small values of parameter \( \kappa \). However, in this case, the cutoff representation (31) can be applied with advantage. Moreover, one should notice that the closed form (46) presents the difference of two large quantities, because \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) have different signs, but their absolute values are very close. At that, the relation \( |\mathcal{K}_1/\mathcal{K}_2| \) increases very quickly with parameter \( m \) of the spherical modified Bessel function of the first kind. Therefore, the quantities \( \mathcal{K}_1 \) and \( \mathcal{K}_2 \) (i.e., functions \( F_i \)) have to be calculated with high accuracy, especially for large \( m \).
It is worth noting that a few-body problem is related to the integral (46) with integer power \( p \) and even \( m \), whereas formulas (43)-(46) are valid for any integer \( m \geq 0 \) and any real \( p > -2m - 2 \).
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