Projectively equivariant quantization and symbol calculus: noncommutative hypergeometric functions

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Abstract

We extend projectively equivariant quantization and symbol calculus to symbols of pseudo-differential operators. An explicit expression in terms of hypergeometric functions with noncommutative arguments is given. Some examples are worked out, one of them yielding a quantum length element on $S^3$.

Keywords: Quantization, projective structures, hypergeometric functions.
1 Introduction

Let $M$ be a smooth manifold and $\mathcal{S}(M)$ the space of smooth functions on $T^*M$, polynomial on the fibers; the latter is usually called the space of symbols of differential operators. Let us furthermore assume that $M$ is endowed with an action of a Lie group $G$. The aim of equivariant quantization [12, 7, 8] (see also [4], and [2, 3]) is to associate to each symbol a differential operator on $M$ in such a way that this quantization map intertwines the $G$-action.

The existence and uniqueness of equivariant quantization in the case where $M$ has a flat projective (resp. conformal) structure, i.e., when $G = \text{SL}(n+1, \mathbb{R})$ with $n = \dim(M)$ (resp. $G = \text{SO}(p+1,q+1)$ with $p + q = \dim(M)$) has recently been proved in the above references.

More precisely, let $\mathcal{F}_\lambda(M)$ stand for the space of (complex-valued) tensor densities of degree $\lambda$ on $M$ and $\mathcal{D}_{\lambda,\mu}(M)$ for the space of linear differential operators from $\mathcal{F}_\lambda(M)$ to $\mathcal{F}_\mu(M)$. These spaces are naturally modules over the group of all diffeomorphisms of $M$. The space of symbols corresponding to $\mathcal{D}_{\lambda,\mu}(M)$ is therefore $\mathcal{S}_\delta(M) = \mathcal{S}(M) \otimes \mathcal{F}_\delta(M)$ where $\delta = \mu - \lambda$. There is a filtration

$$\mathcal{D}^0_{\lambda,\mu} \subset \mathcal{D}^1_{\lambda,\mu} \subset \cdots \subset \mathcal{D}^k_{\lambda,\mu} \subset \cdots$$

and the associated module $\mathcal{S}_\delta(M) = \text{gr} (\mathcal{D}_{\lambda,\mu})$ is graded by the degree of polynomials:

$$\mathcal{S}_\delta = \mathcal{S}_{0,\delta} \oplus \mathcal{S}_{1,\delta} \oplus \cdots \oplus \mathcal{S}_{k,\delta} \oplus \cdots$$

The problem of equivariant quantization is the quest for a quantization map:

$$Q_{\lambda,\mu} : \mathcal{S}_\delta(M) \rightarrow \mathcal{D}_{\lambda,\mu}(M) \quad (1.1)$$

that commutes with the $G$-action. In other words, it amounts to an identification of these two spaces which is canonical with respect to the geometric structure on $M$.

The inverse of the quantization map:

$$\sigma_{\lambda,\mu} = (Q_{\lambda,\mu})^{-1} \quad (1.2)$$

is called the symbol map.
In this Letter, we will restrict considerations to the projectively equivariant case. Without loss of generality, we will assume $M = S^n$ endowed with its standard $\text{SL}(n+1, \mathbb{R})$-action. The explicit formulae for the maps (1.1) and (1.2) can be found in [4] for $n = 1$ and in [12] for $\lambda = \mu$ in any dimension. Our purpose is to rewrite the expressions for $Q_{\lambda,\mu}$ and $\sigma_{\lambda,\mu}$ in a more general way which, in particular, extends the quantization to a bigger class of symbols of pseudo-differential operators.

2 Projectively equivariant quantization map

In terms of affine coordinates on $S^n$, the vector fields spanning the canonical action of the Lie algebra $\text{sl}(n+1, \mathbb{R})$ are as follows

$$\frac{\partial}{\partial x^i}, \quad x^i \frac{\partial}{\partial x^j}, \quad x^i x^j \frac{\partial}{\partial x^j},$$

with $i, j = 1, \ldots, n$ (the Einstein summation convention is understood).

We will denote by $\text{aff}(n, \mathbb{R})$ the affine subalgebra spanned by the first-order vector fields. We will find it convenient to identify locally, in each affine chart, the spaces $S_\delta$ and $D_{\lambda,\mu}$ via the “normal ordering” isomorphism

$$\mathcal{I}: P(x)^{i_1 \ldots i_k} \xi_i \ldots \xi_{i_k} \mapsto (-i\hbar)^k P(x)^{i_1 \ldots i_k} \frac{\partial}{\partial x^{i_1}} \ldots \frac{\partial}{\partial x^{i_k}}$$

which is already equivariant with respect to $\text{aff}(n, \mathbb{R})$. An equivalent means of identification is provided by the Fourier transform

$$(\mathcal{I}(P)\phi)(x) = \frac{1}{(2\pi \hbar)^{n/2}} \int_{\mathbb{R}^{2n}} e^{(i/\hbar)\langle \xi, x-y \rangle} P(y, \xi)\phi(y) \, dy \, d\xi$$

where $P(y, \xi) = \sum_{k=0}^{\infty} P(y)^{i_1 \ldots i_k} \xi_{i_1} \ldots \xi_{i_k}$ and where $\phi$ is a compactly supported function (representing a $\lambda$-density in the coordinate patch). This mapping extends to the space of pseudo-differential symbols (defined in the chosen affine coordinate system).

The purpose of projectively equivariant quantization is to modify the map $\mathcal{I}$ in (2.1) in order to obtain an identification of $S_\delta$ and $D_{\lambda,\mu}$ that does not depend upon a chosen affine coordinate system, and is, therefore, globally defined on $S^n$.

Recall [13] that the (locally defined) operators on $S_\delta$, namely

$$\mathcal{E} = \xi_i \frac{\partial}{\partial \xi_i}, \quad \mathcal{D} = \frac{\partial}{\partial x^i} \frac{\partial}{\partial \xi_i},$$

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(where the $\xi_i$ are the coordinates dual to the $x^i$) commute with the $\text{aff}(n, \mathbb{R})$-action on $T^*S^n$. The Euler operator, $\mathcal{E}$, is the degree operator on $\mathcal{S}_\delta = \bigoplus_{k=0}^{\infty} \mathcal{S}_{k,\delta}$ while the divergence operator $D$ lowers this degree by one.

Let us now recall (in a slightly more general context) the results obtained in [12, 4]. The $\text{SL}(n + 1, \mathbb{R})$-equivariant quantization map (1.1) is given on every homogeneous component by

$$Q_{\lambda,\mu}|_{\mathcal{S}_{k,\delta}} = \sum_{m=0}^{k} C_{m}^k (i\hbar D)^m|_{\mathcal{S}_{k,\delta}}$$

(2.4)

where the constant coefficients $C_m^k$ are determined by the following relation

$$C_{m+1}^k = \frac{k - m - 1 + (n + 1)\lambda}{(m+1)(2k-m-2 + (n + 1)(1-\delta))} C_m^k$$

(2.5)

and the normalization condition: $C_0^k = 1$.

As to the projectively equivariant symbol map (1.1), it retains the form

$$\sigma_{\lambda,\mu}|_{\mathcal{S}_{k,\delta}} = \sum_{m=0}^{k} \widetilde{C}_m^k \left( \frac{D}{i\hbar} \right)^m|_{\mathcal{S}_{k,\delta}}$$

(2.6)

where the coefficients $\widetilde{C}_m^k$ are such that

$$\widetilde{C}_{m+1}^k = -\frac{k + (n + 1)\lambda}{(m+1)(2k-m + (n + 1)(1-\delta))} \widetilde{C}_m^k$$

(2.7)

and, again, $\widetilde{C}_0^k = 1$ for all $k$.

**Remark 2.1.** The expressions (2.4) and (2.6) make sense if

$$\delta \neq 1 + \frac{\ell}{n+1}$$

with $\ell = 0, 1, 2, \ldots$ For these values of $\delta$, the quantization and symbol maps do not exist for generic $\lambda$ and $\mu$; see [11] for a detailed classification.

In contradistinction with the operators $\mathcal{E}$ and $D$ defined in (2.3), the quantization map $Q_{\lambda,\mu}$ and the symbol map $\sigma_{\lambda,\mu}$ are globally defined on $T^*S^n$, i.e., they are independent of the choice of an affine coordinate system.
3 Noncommutative hypergeometric function

Our main purpose is to obtain an expression for $Q_{\lambda, \mu}$ and $\sigma_{\lambda, \mu}$ valid for a larger class of symbols, namely for symbols of pseudo-differential operators. We will rewrite the formulæ (2.4), (2.5) and (2.6), (2.7) in terms of the $\text{aff}(n, \mathbb{R})$-invariant operators $E$ and $D$ in a form independent of the degree, $k$, of polynomials.

It turns out that our quantization map (1.1) involves a certain hypergeometric function; let us now recall this classical notion. A hypergeometric function with $p + q$ parameters is defined (see, e.g., [9]) as the power series in $z$ given by

$$F \left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_q} \bigg| z \right) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{z^m}{m!}$$

with $(a)_m = a(a+1) \cdots (a+m-1)$. This hypergeometric function is called confluent if $p = q = 1$.

**Theorem 3.1.** The projectively equivariant quantization map is of the form

$$Q_{\lambda, \mu} = F \left( \frac{A_1, A_2}{B_1, B_2} \bigg| Z \right)$$

(3.2)

where the parameters

$$A_1 = E + (n + 1)\lambda, \quad A_2 = 2E + (n + 1)(1 - \delta) - 1,$$

$$B_1 = E + \frac{1}{2}(n + 1)(1 - \delta) - \frac{1}{2}, \quad B_2 = E + \frac{1}{2}(n + 1)(1 - \delta),$$

are operator-valued, as well as the variable

$$Z = \frac{i\hbar D}{4}.$$  

(3.3)

**Proof.** Recall that for a hypergeometric function (3.1), one has

$$F \left( \frac{a_1, \ldots, a_p}{b_1, \ldots, b_q} \bigg| z \right) = \sum_{m=0}^{\infty} c_m z^m$$

with

$$\frac{c_{m+1}}{c_m} = \frac{1}{m+1} \left[ \frac{(a_1 + m) \cdots (a_p + m)}{(b_1 + m) \cdots (b_q + m)} \right].$$
Let us replace $k - m$ by the degree operator $E$ in the coefficients $C^k_m$; the expression (2.4) can be therefore rewritten as $Q_{\lambda,\mu} = \sum_{m=0}^{k} C_m(E)(i\hbar D)^m$. From the recursion relation (2.5), one readily obtains
\[
\frac{C_{m+1}(E)}{C_m(E)} = \frac{1}{4(m+1)} \left[ \frac{(\mathcal{E} + (n+1)\lambda + m)(2\mathcal{E} + (n+1)(1 - \delta) - 1 + m)}{(\mathcal{E} + \frac{1}{2}(n+1)(1 - \delta) - \frac{1}{2} + m)(\mathcal{E} + \frac{1}{2}(n+1)(1 - \delta) + m)} \right]
\]
completing the proof.

**Corollary 3.2.** The quantization map is given by the series
\[
Q_{\lambda,\mu} = \sum_{m=0}^{k} C_m(E)(i\hbar D)^m
\tag{3.5}
\]
where
\[
C_m(E) = \frac{1}{m!} \frac{(\mathcal{E} + (n+1)\lambda)_m}{(2\mathcal{E} + (n+1)(1 - \delta) + m - 1)_m}.
\tag{3.6}
\]

**Remark 3.3.** Let us stress that the operator-valued parameters (3.3) and the variable (3.4) entering the expression (3.2) do not commute. We have therefore chosen an ordering that assigns the divergence operator $D$ to the right.

In the particular and most interesting case of half-densities (cf. [7, 8]), the expression (3.2) takes a simpler form.

**Corollary 3.4.** If $\lambda = \mu = \frac{1}{2}$, the quantization map (3.3) reduces to the confluent hypergeometric function
\[
Q_{\frac{1}{2},\frac{1}{2}} = F\left(\frac{2\mathcal{E}}{E} \bigg| \frac{i\hbar D}{4}\right)
\tag{3.7}
\]
with the notation: $E = \mathcal{E} + \frac{1}{2}n$.

It is a remarkable fact that the expression for inverse symbol map (1.2) is much simpler. It is given by a confluent hypergeometric function for any $\lambda$ and $\mu$.

**Theorem 3.5.** The projectively equivariant symbol map (1.2) is given by
\[
\sigma_{\lambda,\mu} = F\left(\frac{\mathcal{E} + (n+1)\lambda}{2\mathcal{E} + (n+1)(1 - \delta)} \bigg| -\frac{D}{i\hbar}\right).
\tag{3.8}
\]

The proof is analogous to that of Theorem 3.1.

It would be interesting to obtain expressions of the projectively equivariant quantization and symbol maps as integral operators similar to (2.2).
4 Some examples

We wish to present, here, a few applications of the projectively equivariant quantization to some special Hamiltonians on $T^*S^n$.

The first example deals with the geodesic flow. Denote by $g$ the standard round metric on the unit $n$-sphere and by $H = g^{ij} \xi_i \xi_j$ the corresponding quadratic Hamiltonian. In an affine coordinate system, it takes the following form

$$H = (1 + \|x\|^2) \left( \delta^{ij} + x^i x^j \right) \xi_i \xi_j$$ (4.1)

where $\|x\|^2 = \delta_{ij} x^i x^j$ with $i, j = 1, \ldots, n$. Moreover, we will consider a family of such Hamiltonians belonging to $S_\delta$, namely $H_\delta = H \sqrt{g^\delta}$ where $g = \det(g_{ij})$.

In order to provide explicit formulæ, we need to recall the expression of the covariant derivative of $\lambda$-densities, namely $\nabla_i = \partial_i - \lambda \Gamma^j_{ij}$.

**Proposition 4.1.** The projectively equivariant quantization map $[\mathcal{L}]$ associates to $H_\delta$ the following differential operator

$$Q_{\lambda, \mu}(H_\delta) = -\hbar^2 (\Delta + C_{\lambda, \mu} R)$$ (4.2)

where $\Delta = g^{ij} \nabla_i \nabla_j$ is the Laplace operator; the constant coefficient is

$$C_{\lambda, \mu} = \frac{(n+1)^2 \lambda (\mu - 1)}{(n-1)(1-\delta)(n+1)+1}$$ (4.3)

and $R = n(n-1)$ is the scalar curvature of $S^n$.

**Proof.** The quantum operator (4.2) is obtained, using (3.2)–(3.4), by a direct computation. However, the formula (4.2) turns out to be a particular case of (5.4) in [1] since the Levi-Civita connection is projectively flat. \qed

Another example is provided by the $\alpha$-th power $H^\alpha$ of the Hamiltonian $H$, where $\alpha \in \mathbb{R}$. We will only consider the case $\lambda = \mu$ in the sequel.

**Proposition 4.2.** For

$$\alpha = \frac{1-n}{4}$$ (4.4)

one has $Q_{\lambda, \lambda}(H^\alpha) = H^\alpha$.

**Proof.** Straightforward computation leads to

$$D(H^\alpha) = 2\alpha(4\alpha + n - 1)H^{\alpha-1}(1 + \|x\|^2)\langle \xi, x \rangle$$

and (2.4) therefore yields the result. \qed
We have just shown that the Fourier transform \((2.3)\) of \(H^{(1-n)/4}\) is well-defined on \(S^n\) and actually corresponds to the projectively equivariant quantization of this pseudo-differential symbol.

If we want to deal with operators acting on a Hilbert space, we have to restrict now considerations to the case \(\lambda = \mu = \frac{1}{2}\).

For the 3-sphere only, the above quantum Hamiltonian on \(T^*S^n \setminus S^n\) is as follows

\[
Q_{\frac{1}{2}, \frac{1}{2}}(H^{-\frac{1}{2}}) = \frac{1}{\hbar} \frac{1}{\sqrt{-\Delta}}
\]  

and can be understood as a quantized “length element” in the sense of \([5]\).

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