Ergodic cocycles of IDPFT systems and non-singular Gaussian actions

ALEXANDRE I. DANILENKO† and MARIUSZ LEMAŃCZYK‡
† B. I. Verkin Institute for Low Temperature Physics & Engineering of National Academy of Sciences of Ukraine, 47 Nauky Ave., Kharkiv 61103, Ukraine
(e-mail: alexandre.danilenko@gmail.com)
‡ Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12/18, 87-100 Toruń, Poland
(e-mail: mlem@mat.umk.pl)

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To the memory of Sergiy Sinel’shchikov, our colleague and friend

Abstract. It is proved that each Gaussian cocycle over a mildly mixing Gaussian transformation is either a Gaussian coboundary or sharply weak mixing. The class of non-singular infinite direct products $T$ of transformations $T_n$, $n \in \mathbb{N}$, of finite type is studied. It is shown that if $T_n$ is mildly mixing, $n \in \mathbb{N}$, the sequence of Radon–Nikodym derivatives of $T_n$ is asymptotically translation quasi-invariant and $T$ is conservative then the Maharam extension of $T$ is sharply weak mixing. This technique provides a new approach to the non-singular Gaussian transformations studied recently by Arano, Isono and Marrakchi.

Key words: ergodic transformation, Gaussian cocycle, Maharam extension

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1. Introduction

The original motivation of this paper was to tackle a problem (stated in [LeLeSk]) that is related to the theory of Gaussian dynamical systems. Let $T$ be an ergodic (equivalently, weakly mixing) Gaussian transformation on a standard probability space $(X, \mathcal{B}, \mu)$ and let $H$ be the corresponding invariant Gaussian subspace of the real Hilbert space $L^2_0(X, \mu)$.

Conjecture. For each function $f \in H$, either $f$ is a $T$-coboundary (equivalently, a Gaussian coboundary) or the skew product transformation $T_f$ acting on $X \times \mathbb{R}$ is ergodic.

In this paper we obtain an affirmative answer under an assumption which is slightly stronger than weak mixing. We say that an ergodic conservative non-singular transformation $R$ is sharply weak mixing if the direct product of $R$ with each ergodic conservative transformation is either totally dissipative or ergodic. We also recall that $T_f$ is conservative for each $f \in H$ [At].
**Theorem 1.1.** If $T$ is mildly mixing and $f$ is not a coboundary for $T$ then $T_f$ is sharply weak mixing.

To prove Theorem 1.1 we note that there exists a decomposition of $T$ into direct product of mildly mixing transformations $T_n$ in such a way that $f$ splits into a sum of coboundaries $f_n := a_n - a_n \circ T_n$ for $T_n$, $n \in \mathbb{N}$. Moreover, the sequence of distributions of the transfer functions $(a_n)_{n=1}^\infty$ satisfies a certain property that we call **asymptotic translation invariance** (ATI) in Definition 2.2. Then Theorem 1.1 follows from the following result.

**Theorem 1.2.** Given a locally compact second countable abelian group $G$, a sequence of mildly mixing dynamical systems $(X_n, \nu_n, T_n)$ and a sequence of functions $f_n : X_n \to G$, $n \in \mathbb{N}$, consider the infinite direct product $(X, \nu, T) := \bigotimes_{n=1}^\infty (X_n, \nu_n, T_n)$. Suppose that a function $f(x) := \sum_{n=1}^\infty (f_n(T_n x_n) - f_n(x_n)) \in G$ is well defined for $\nu$-almost every (a.e.) $x = (x_n)_{n=1}^\infty \in X$. If the sequence of distributions $(\nu_n \circ f_n^{-1})_{n \in \mathbb{N}}$ on $G$ is asymptotically translation invariant and the $f$-skew product extension $T_f : X \times G \to X \times G$ of $T$ is conservative then $T_f$ is sharply weak mixing.

The proof of Theorem 1.2 is based on the two ideas.

- The mild mixing and the product structure of $T_f$ yield that each $T_f$-invariant subset is also invariant under a large group of ‘finitary’ transformations, that is, transformations that ‘move’ finitely many coordinates only.
- The ATI property implies that this finitary group is ergodic via techniques related to computation of the essential values of cocycles.

The former idea was inspired by the proof [ArIsMa, Theorem D] of ergodic properties of some non-singular Gaussian group actions.

We then turn to classical problems of non-singular ergodic theory. We mention recent progress in providing natural examples for non-singular ergodic theory: non-singular Bernoulli and Markov shiftwise actions (see [DaLe, BjKoVa, KoSo, Av, MaVa] and references therein), non-singular Gaussian systems [ArIsMa] and non-singular Poisson systems [DaKoRo1, DaKoRo2]. In the present work we introduce one more natural family of non-singular transformations. We say that a non-singular transformation $T$ on a standard probability space $(X, \mu)$ is an **infinite direct product of finite types** (IDPFT) if there is a sequence of ergodic probability-preserving dynamical systems $(X_n, \nu_n, T_n)$ and a sequence of probability measures $\mu_n$ on $X_n$, $n \in \mathbb{N}$, such that $\mu_n \sim \nu_n$ for each $n$ and $(X, \mu, T) = \bigotimes_{n=1}^\infty (X_n, \mu_n, T_n)$. Kakutani’s theorem [Ka] provides a criterion where $\mu$ is quasi-invariant under $T$. We are interested in the case where $\mu \perp \nu$ and $\mu$ does not admit an equivalent $T$-invariant probability. It is possible that $(X, \mu, T)$ is totally dissipative.

**Theorem 1.3.** Let $(X_n, \nu_n, T_n)$ be mildly mixing for each $n > 0$. If $T$ is $\mu$-conservative and the sequence of distributions of the random variables $\log(d\mu_n/d\nu_n)$, $n \in \mathbb{N}$, is asymptotically translation quasi-invariant then $T$ is ergodic of stable type Krieger’s type $\text{III}_1$. Moreover, the Maharam extension of $T$ is sharply weak mixing.

The asymptotic translation quasi-invariance (ATQI) property (see Definition 3.8) in the statement of Theorem 1.3 is an analogue of ATI though neither ATI implies ATQI nor
vice versa. The scheme of the proof of Theorem 1.3 is similar to that of Theorem 1.2 and we again use the aforementioned two ideas. However, there is a ‘non-singular’ nuance. Namely, a formal repetition of the proof of Theorem 1.2 yields that the group of finitary transformations is ergodic with respect to the ‘wrong’ measure. Hence, it does not work. We recall that there are two different (mutually singular) natural measures associated with an IDPFT system: \( \nu \) (invariant) and \( \mu \) (quasi-invariant). Therefore a certain additional argument and the ATQI property rather than ATI are needed to prove ergodicity for the ‘right’ measure. We also provide examples of rigid IDPFT systems \( T \) of Krieger’s type \( \mathrm{III}_\lambda \) for an arbitrary \( \lambda \in (0, 1) \).

We have already mentioned that non-singular Gaussian systems were recently studied in [ArIsMa]. However, the exposition there is based heavily on affine geometry and often uses a non-standard (from the dynamical viewpoint) terminology. Therefore, we decided to provide here an alternative exposition of this important topic. We define the non-singular Gaussian systems as transformations on Hilbert spaces \( \mathcal{H} \) furnished with Gaussian measures stressing the fact that the systems are compositions of classical Gaussian automorphisms and totally dissipative transformations (given by non-singular rotations). Connections with the underlying Fock space, the first chaos and the exponential map are explicitly made. We also explain interrelation between non-singular Gaussian systems and non-singular Poisson systems. Our main observation is that Gaussian transformations (except for a ‘small’ family of degenerate ones) are a subclass of IDPFT systems. Hence we deduce [ArIsMa, Theorem D] (we consider only the case of \( \mathbb{Z} \)-actions) from Theorem 1.2. \( \mathcal{H}_0 \) below is a linear subspace of \( \mathcal{H} \) endowed with a new inner product; see § 4.

**Theorem 1.4.** Let an orthogonal operator \( V \) of a real Hilbert space \( \mathcal{H}_0 \) be mildly mixing. Let \( f \in \mathcal{H}_0 \) not be a \( V \)-coboundary (that is, \( f \neq Va - a \) for any \( a \in \mathcal{H}_0 \)). If the non-singular Gaussian transformation \( T(f, V) \) associated with the pair \( (f, V) \) is conservative then the Maharam extension of \( T(f, V) \) is sharply weak mixing. In particular, \( T(f, V) \) is of type \( \mathrm{III}_1 \).

The outline of the paper is as follows. In § 2 we introduce important definitions: Hellinger distance, weak mixing properties for non-singular actions, the ATI property, skew product extension, essential value of a cocycle, etc. Then we prove Theorem 1.2 (see Theorem 2.5) and deduce Theorem 1.1 from it (see Theorem 2.6). We also provide a generalization of Theorem 1.1 (see Conjecture II and discussion preceding it). In § 3 we consider non-singular versions of the problems studied in § 2. IDPFT systems are introduced in Definition 3.2. Radon–Nikodym cocycles, Maharam extensions and Krieger’s types \( \mathrm{III}_\lambda \), \( 0 \leq \lambda \leq 1 \), are discussed there. We show that each IDPFT system is either conservative or totally dissipative (Corollary 3.7), introduce the ATQI property (Definition 3.8) and prove Theorem 1.3 (Theorem 3.10). Type \( \mathrm{III}_\lambda \) rigid IDPFT systems are also constructed there for each \( \lambda \in (0, 1) \) (Proposition 3.12). The final § 4 is devoted to non-singular Gaussian systems. We first recall the definition of Gaussian measure in a separable Hilbert space. Then we discuss the main properties of the related Fock space and exponential map. Given an orthogonal operator \( V \) in a Hilbert space \( \mathcal{H}_0 \) and a vector \( f \in \mathcal{H}_0 \), we associate a non-singular transformation \( T(f, V) \) acting on the corresponding Hilbert space \( \mathcal{H} \supset \mathcal{H}_0 \) equipped with a Gaussian measure \( \mu \). We show that \( T(f, V) \) is the
composition of the classic Gaussian $\mu$-preserving transformation associated to $V$ with the (totally dissipative) rotation by $f$. It is well known that the non-singular transformation group $\{T_{(f,0)} \mid f \in \mathcal{H}_0\}$ generated by the rotations is ergodic (see, for example, [Gu]) but Krieger’s type has not been specified so far. We prove that it is $\text{III}_1$ (Theorem 4.7). We show that the Koopman operator generated by $T_{(f,V)}$ is the Weyl operator associated to the pair $(f/2, V)$. A criterion for the existence of an invariant equivalent probability measure for $T_{(f,V)}$ is established in Theorem 4.9 (cf. [DaKoRo1, Proposition 6.4] and [ArIsMa, Theorem 6.3(i)]). Theorem 1.4 in proved in this section (Theorem 4.12).

After completion of this paper we learnt of a work [MaVa] devoted to non-singular Gaussian actions of arbitrary groups. It was written independently of but at the same time as our work†. Some of our results overlap with theirs. For example, Theorem 4.7 is [MaVa, Theorem 3.1] and Theorem 1.1, though stated in a more general form, is, in fact, equivalent to [MaVa, Theorem 9.1(3)] in the case of $\mathbb{Z}$-actions. Our proofs are different. They are based solely on elementary techniques of the non-singular ergodic (measurable orbit) theory. We do not use affine geometry, representation theory or harmonic analysis.

2. Weak mixing cocycles of product type

2.1. Hellinger distance and Kakutani’s theorem. Let $\gamma$ and $\delta$ be two equivalent probability measures on a standard Borel space $(Y, \mathcal{F})$. The square of the Hellinger distance between $\gamma$ and $\delta$ is

$$H^2(\gamma, \delta) := \frac{1}{2} \int_Y \left(1 - \sqrt{\frac{d\gamma}{d\delta}}\right)^2 d\delta = 1 - \int_Y \sqrt{\frac{d\gamma}{d\delta}} d\delta.$$

By the Cauchy–Schwarz inequality, $0 \leq H(\gamma, \delta) < 1$. We also recall [Ni] the following inequalities between the Hellinger distance and the total variation:

$$H^2(\gamma, \delta) \leq \|\gamma - \delta\|_1 := \sup_{C \in \mathcal{F}} |\gamma(C) - \delta(C)| \leq \sqrt{2} H(\gamma, \delta). \quad (2.1)$$

We now state the Kakutani theorem on equivalence of infinite products of probability measures [Ka].

**THEOREM A.** Let $\mu_n$ and $\nu_n$ be two equivalent probability measures on a standard Borel space $(X_n, \mathcal{B}_n)$ for each $n \in \mathbb{N}$. Let $\mu$ and $\nu$ denote the infinite product measures $\otimes_{n \in \mathbb{N}} \mu_n$ and $\otimes_{n \in \mathbb{N}} \nu_n$ respectively on the standard Borel space $(X, \mathcal{B}) := \otimes_{n \in \mathbb{N}} (X_n, \mathcal{B}_n)$. If

$$\prod_{n=1}^{\infty} (1 - H^2(\mu_n, \nu_n)) > 0 \quad \text{or, equivalently,} \quad \sum_{n=1}^{\infty} H^2(\mu_n, \nu_n) < \infty \quad (2.2)$$

then $\mu \sim \nu$, $\prod_{n=1}^{\infty} (1 - H^2(\mu_n, \nu_n)) = 1 - H^2(\mu, \nu)$ and $(d\mu/d\nu)(x) = \prod_{n \in \mathbb{N}} (d\mu_n/d\nu_n)(x_n)$ at a.e. $x = (x_n)_{n \in \mathbb{N}} \in X$. If (2.2) does not hold then $\mu \perp \nu$.

2.2. Weak mixing properties of non-singular actions. We recall that, given a non-singular transformation $R$ of a standard Borel probability space $(Y, \mathcal{F}, \nu)$, there is a unique decomposition $Y = D(R) \sqcup C(R)$ (called Hopf’s decomposition) of $Y$ into two

† The two papers appeared on arXiv on two successive days. We thank S. Vaes for informing us about [MaVa].
Borel sets such that $D(R)$ is the disjoint union of the orbit of a wandering set $W$, that is, $D(R) = \bigsqcup_{n \in \mathbb{Z}} R^n W$ and $C(R) = Y \setminus D(R)$ contains no non-trivial wandering set. If $C(R) = Y$ then $R$ is called conservative and if $D(R) = Y$ then $R$ is called totally dissipative. As both parts $C(R)$ and $D(R)$ are $R$-invariant, each ergodic $R$ is either conservative or totally dissipative. An ergodic conservative non-singular transformation $R$ is called weakly mixing if, for each ergodic probability-preserving transformation $S$, the Cartesian product $R \times S$ is ergodic. We now introduce a stronger concept of weak mixing.

**Definition 2.1.** An ergodic conservative non-singular transformation $R$ is called sharply weak mixing if, for each ergodic conservative non-singular transformation $S$, the direct product $R \times S$ is either totally dissipative or ergodic.

If $S$ in the above definition admits an equivalent invariant probability measure (that is, $S$ is of type II$_1$) then $T \times S$ is conservative (see [Aa, Proposition 1.1.6, part 2]). Hence $R \times S$ is ergodic according to Definition 2.1. Thus, every sharply weak mixing transformation is weakly mixing. It follows from [SiTh] that every conservative non-singular transformation with property $K$ is sharply weak mixing (see also [AaLiWe, Theorem 6.7] for other examples). In [AdFrSi, Da] examples of weakly mixing infinite measure-preserving rank-one transformations $R$ were constructed such that $R \times R$ is conservative but not ergodic. Hence $R$ is not sharply weak mixing. We recall that an ergodic probability-preserving transformation $R$ defined on a space $(Y, \mathcal{C}, \nu)$ is called mildly mixing ([FuWe], see also [AaLiWe] and [ScWa]) if every function $f \in L^\infty(\nu)$ such that $\| f \circ T^{n_i} - f \|_1 \to 0$ for some sequence $n_i \to \infty$ is constant.

We will utilize the following result from [ScWa].

**THEOREM B.** Let $R$ be a mildly mixing transformation of a standard probability space $(Y, \mathcal{C}, \nu)$ and let $C$ be a conservative non-singular transformation of a standard probability space $(Z, \mathcal{F}, \tau)$. If a function $F \in L^\infty(Y \times Z, \nu \otimes \tau)$ is invariant under $R \times C$ then there is $f \in L^\infty(Z, \tau)$ such that $F(y, z) = f(z)$ almost everywhere.

We note that Theorem B was proved in [ScWa] for the ergodic conservative $C$ only, but the proof remains valid for an arbitrary conservative $C$ as well. Direct products of finitely (and countably) many mildly mixing transformations are mildly mixing.

It follows from Theorem B that an ergodic finite measure-preserving transformation is sharply weak mixing if and only if it is mildly mixing. In Theorems 2.5 and 2.6 below we provide examples of mildly mixing transformations (including the zero-entropy case) which have locally compact group extensions that are sharply weak mixing infinite measure-preserving (and hence not mildly mixing).

2.3. **ATI property.** Fix a locally compact second countable abelian group $G$. Denote by $\lambda_G$ a Haar measure on $G$.

**Definition 2.2.** A sequence $(\xi_n)_{n=1}^\infty$ of probability Borel measures on $G$ is called asymptotically translation invariant if
\[
\lim_{m \to \infty} \|\xi_n * \xi_{n+1} \cdots * \xi_{n+m} \|_1 = 0
\]
for each \( n \in \mathbb{N} \) and \( a \in G \).

**Example 2.3.** Let \( N_{a, \sigma^2} \) denote the normal distribution on \( \mathbb{R} \) with parameters \( a \) and \( \sigma^2 \), that is, \( N_{a, \sigma^2}(t) = e^{iat-(1/2)\sigma^2 t^2} \) for all \( t \in \mathbb{R} \). We leave verification of the formula
\[
H^2(N_{a, \sigma^2}, N_{b, \sigma^2}) = 1 - \frac{2\sigma}{\sigma^2 + \tau^2} e^{-(1/4)((a-b)^2/(\sigma^2 + \tau^2))} \quad \text{for all } a, b, \sigma, \tau \in \mathbb{R},
\]
as an exercise for the reader. Given two sequences \((a_n)_{n \in \mathbb{N}}\) and \((\sigma_n)_{n=1}^{\infty}\) of reals such that \( \sum_{n=1}^{\infty} \sigma_n^2 = +\infty \), the sequence of probabilities \((N_{a_n, \sigma_n^2})_{n=1}^{\infty}\) is asymptotically translation invariant. Indeed,
\[
N_{a_n, \sigma_n^2} * \cdots * N_{a_{n+m}, \sigma_{n+m}^2} = N_{\sum_{k=n}^{n+m} a_k, \sum_{k=n}^{n+m} \sigma_k^2},
\]
and
\[
H^2(N_{a, \sum_{k=n}^{n+m} a_k}, N_{b, \sum_{k=n}^{n+m} \sigma_k^2}) = 1 - e^{-(1/8)2^2/(\sum_{k=n}^{n+m} \sigma_k^2)} \to 0
\]
as \( m \to \infty \). Hence \((\xi_n)_{n=1}^{\infty}\) is asymptotically translation invariant in view of (2.1).

2.4. **Ergodic cocycles of ergodic transformation groups.** Given a standard Borel \( \sigma \)-finite measure space \((Y, \mathcal{C}, \nu)\), we denote by Aut\((Y, \nu)\) the group of all \( \nu \)-non-singular invertible Borel transformations on \( Y \). Let Aut\(_0(Y, \nu)\) denote the subgroup of \( \nu \)-preserving transformations from Aut\((Y, \nu)\). Let \( \Gamma \) be an ergodic countable subgroup in Aut\((Y, \nu)\). The full group \([\Gamma]\) of \( \Gamma \) is defined by
\[
[\Gamma] := \{ \theta \in \text{Aut}(Y, \nu) \mid \theta y \in \{ y y \mid y \in \Gamma \} \text{ at.e. } y \in Y \}.
\]
Let \( G \) be a locally compact second countable abelian group and let \( \lambda_G \) be a Haar measure on \( G \). A measurable map \( \alpha : \Gamma \times Y \to G \) is called a cocycle of \( \Gamma \) if
\[
\alpha(\gamma_1, \gamma_2, y) = \alpha(\gamma_1, \gamma_2 y) + \alpha(\gamma_2, y) \quad \text{at.e. } y \in Y
\]
for all \( \gamma_1, \gamma_2 \in \Gamma \). From now on we assume that \( \Gamma \) is free, that is, if \( \gamma \in \Gamma \setminus \{I\} \) then \( \gamma y \neq y \) for a.e. \( y \). Then \( \alpha \) can be ‘extended’ to \([\Gamma]\) if we set
\[
\alpha(\theta, y) := \alpha(\gamma, y) \quad \text{where } \gamma \text{ is defined uniquely by } \theta y = \gamma y.
\]
It is straightforward to verify that (2.3) holds if we replace \( \gamma_1 \) and \( \gamma_2 \) with arbitrary elements from \([\Gamma]\). A cocycle \( \alpha \) is a coboundary if there is a measurable map \( a : Y \to G \) such that
\[
\alpha(\gamma, y) = a(\gamma y) - a(y) \quad \text{at.e. } y \in Y
\]
for all \( \gamma \in \Gamma \). Given a pair \((\Gamma, \alpha)\), we can construct a transformation group \( \Gamma_\alpha := \{ \gamma_\alpha \mid \gamma \in \Gamma \} \subset \text{Aut}(Y \times G, \nu \times \lambda_G) \), where
\[
\gamma_\alpha(y, g) := (\gamma y, \alpha(\gamma, y) + g) \quad \text{for all } y \in Y, g \in G.
\]
The group \( \Gamma_\alpha \) is called the \( \alpha \)-skew product extension of \( \Gamma \). If \( \Gamma \) preserves \( \nu \) then \( \Gamma_\alpha \) preserves the product measure \( \nu \otimes \lambda_G \). If \( \Gamma_\alpha \) is ergodic then \( \alpha \) is called ergodic.
coboundary is never ergodic (unless G is a singleton). It is easy to verify that if \( \Gamma = \{ R^n \mid n \in \mathbb{Z} \} \) for a transformation \( R \in \text{Aut}(Y, \nu) \) then each measurable function \( f : Y \to G \) uniquely defines a cocycle \( \alpha_f \) of \( \Gamma \) via the condition

\[
\alpha_f(R, y) := f(y) \quad \text{for each } y \in Y.
\]

For brevity we will write \( R_f \) for the \( \alpha_f \)-skew product extension \( R_{\alpha_f} \) of \( R \).

We now recall an important concept of essential value for a cocycle.

**Definition 2.4.** Suppose that \( \Gamma \) preserves \( \nu \). An element \( g \in G \) is called an **essential value** of \( \alpha \) if, for each subset \( A \subset Y \) of positive measure and a neighborhood \( U \) of \( g \), there are a Borel subset \( B \subset A \) and an element \( \gamma \in \Gamma \) such that \( \nu(B) > 0 \), \( \gamma B \subset A \) and \( \alpha(\gamma, y) \in U \) for all \( y \in B \).

It appears that the set \( r(\alpha) \) of all essential values of a cocycle is a closed subgroup of \( G \). Our interest in the essential values of \( \alpha \) is explained by the fact that \( \alpha \) is ergodic if and only if \( r(\alpha) = G \) [Sc]. It is often easier to check the aforementioned condition on essential values not for each subset \( A \in \mathcal{C} \) of positive measure but only for a dense subfamily of subsets in \( \mathcal{C} \). However, in this case we have to strengthen this condition. More precisely, we will use the following folklore lemma.

**Lemma C.** Let \( (Y, \mathcal{C}, \nu) \) be a standard probability space, \( \mathfrak{A} \) a dense subset in \( \mathcal{C} \), \( \Gamma \) an ergodic countable subgroup of \( \text{Auto}_0(Y, \nu) \) and \( \alpha : \Gamma \times Y \to G \) a Borel cocycle of \( \Gamma \). If, for some \( a \in G \) and each subset \( B \in \mathfrak{A} \) and each neighborhood \( U \) of \( 0 \) in \( G \), there are a measurable subset \( D \subset B \) and an element \( \theta \in \Gamma \) such that \( \theta D \subset B \), \( \nu(D) > 0.5 \nu(B) \) and \( \alpha(\theta, x) \in a + U \) for all \( x \in D \) then \( a \) is an essential value of \( \alpha \).

### 2.5. Sharp weak mixing of skew products for cocycles of product type

In this subsection we prove the following theorem.

**Theorem 2.5.** Let \( T_n \) be a mildly mixing transformation of a standard probability space \( (X_n, \mathfrak{B}_n, \nu_n) \) for each \( n \in \mathbb{N} \). Let

\[
(X, \mathfrak{B}, \nu, T) := \bigotimes_{n \in \mathbb{Z}}(X_n, \mathfrak{B}_n, \nu_n, T).
\]

Suppose that, for a measurable function \( f : X \to G \), there are functions \( f_n : X_n \to G \) such that

\[
f(x) = \sum_{n \in \mathbb{N}}(f_n(T_n x_n) - f_n(x_n))
\]

at \( \nu \)-a.e. \( x = (x_n)_{n=1}^\infty \in X \) and the sequence of measures \( (\nu_n \circ f_n^{-1})_{n \in \mathbb{N}} \) is asymptotically translation invariant. If the skew product extension \( T_f : X \times G \to X \times G \) of \( T \) is conservative then \( T_f \) is sharply weak mixing.

**Proof.** Let \( C \) be an ergodic conservative transformation of a standard probability space \( (Z, \mathcal{Z}, \kappa) \). Suppose that \( T_f \times C \) is not totally dissipative. Then it follows from [Aa, Proposition 1.2.4] that \( T_f \times C \) is conservative. All that remains is to show that \( T_f \times C \) is ergodic.

Let a function \( F \in L^\infty(X \times G \times Z, \mu \otimes \lambda_G \otimes \kappa) \) be invariant under \( T_f \times C \). We first show that \( F \) is also invariant under a huge group of transformations. Fix \( n > 0 \). For each \( x \in X \), we write \( x^n := (x_1, \ldots, x_n) \in X_1 \times \cdots \times X_n \) and \( x_{n+1}^\infty := (x_{n+1}, x_{n+2}, \ldots) \in \)
$X_{n+1} \times X_{n+2} \times \cdots$. Then $x = (x_1^n, x_{n+1}^\infty)$. We define a measure-preserving automorphism $E_n$ of $(X \times G \times Z, \mu \otimes \lambda_G \otimes \kappa)$ and a non-singular automorphism $V_n$ of $(\bigotimes_{k=n+1}^\infty (X_k, \nu_k)) \otimes (G, \lambda_G) \otimes (Z, \kappa)$ respectively by setting

$$E_n(x, g, z) := \left(x, g + \sum_{k=1}^n f_k(x_k), z\right)$$

and

$$V_n((x_k)_{k=n+1}^\infty, g, z) := \left((T_k x_k)_{k=n+1}^\infty, g + \sum_{k>n} (f_k(T_k x_k) - f_k(x_k)), Cz\right).$$

A straightforward verification shows that

$$E_n^{-1}(T_f \times C)E_n = (T_1 \times \cdots \times T_n) \times V_n.$$

Since $V_n$ is a factor of the transformation $E_n^{-1}(T_f \times C)E_n$ and the latter transformation is conservative, it follows that $V_n$ is conservative. On the other hand, the function $F \circ E_n$ is invariant under $E_n^{-1}(T_f \times C)E_n$. Utilizing these two facts, we deduce from Theorem B that $F \circ E_n$ does not depend on the coordinates $x_1, \ldots, x_n$. Hence, for each transformation $S \in \text{Aut}_0(X_1 \times \cdots \times X_n, \bigotimes_{k=1}^n \nu_k)$, we have that $F \circ E_n \circ (S \times I) = F \circ E_n$. Therefore $F$ is invariant under the transformation $E_n(S \times I)E_n^{-1} \in \text{Aut}_0(X \times G \times Z, \mu \otimes \lambda_G \otimes \kappa)$ and

$$E_n(S \times I)E_n^{-1}(x, g, z) = (Sx_1^n, x_{n+1}^\infty, g - A_n(x_1^n) + A_n(Sx_1^n), z),$$

(2.4)

where $A_n$ stands for the mapping

$$X_1 \times \cdots \times X_n \ni (x_1, \ldots, x_n) \mapsto A_n(x_1, \ldots, x_n) := \sum_{k=1}^n f_k(x_k).$$

Thus, we have shown that $F$ is invariant under each transformation from the set

$$G := \bigcup_{n>0} E_n\left(\text{Aut}_0\left(X_1 \times \cdots \times X_n, \bigotimes_{k=1}^n \nu_k\right) \times \{I\}\right)E_n^{-1}.$$

We now consider a new dynamical system. The space of this system is the product $(X, \mathfrak{B}, \mu)$. Denote by $\Gamma$ the group of transformations of this space generated by mutually commuting measure-preserving transformations $\widehat{T}_1, \widehat{T}_2, \ldots$, where

$$\widehat{T}_n x = (x_1^{n-1}, T_n x_n, x_{n+1}^\infty), \quad n \in \mathbb{N}.$$

Then $\Gamma$ is countable, abelian† and ergodic. For each $n > 0$, we consider a coboundary

$$\alpha_n : X \ni x \mapsto \alpha_n(x) := f_n(T_n x_n) - f_n(x_n) \in G$$

of $\widehat{T}_n$. It is straightforward to verify‡ that the $\alpha_n$-skew product extensions $\left(\widehat{T}_n\right)_{\alpha_n}$ of $\widehat{T}_n$, $n \in \mathbb{N}$, mutually commute. It follows that a cocycle $\alpha : \Gamma \times X \to G$ of $\Gamma$ with values in $G$ is well defined by the formula

$$\alpha(\widehat{T}_n, x) := \alpha_n(x), \quad n \in \mathbb{N}.$$

† It is isomorphic to $\bigotimes_{n=1}^\infty \mathbb{Z}$.
‡ This follows from the fact that each function $\alpha_n$ depends only on a single coordinate $x_n$, $n = 1, 2, \ldots$. 
we can find $m > n$ such that $E_n(I \times T_n \times I) E_n^{-1} = (\hat{T}_n)_{\alpha_n} \times I_Z$.

Hence $(\hat{T}_n)_{\alpha_n} \times I_Z \in \mathcal{G}$. Although, each $\alpha_n$ is a coboundary for the $\mathbb{Z}$-action given by $T_n$, the cocycle $\alpha$ is not a coboundary for $\Gamma$. In fact, we will now show the following claim.

**CLAIM I.** The cocycle $\alpha$ of $\Gamma$ is ergodic.

**Proof.** For this purpose we show that each element $a \in G$ is an essential value of $\alpha$. Given $n > 0$ and a subset $B \subset X_1 \times \cdots \times X_n$, denote by $[B]^n_1 \subset X$ the corresponding cylinder with ‘head’ $B$, that is, $[B]^n_1 := \{ x \in X \mid x^n_1 \in B \}$. Let $U$ be a symmetric neighborhood of $0$ in $G$. Choose a countable partition $\mathcal{P}$ of $G$ into Borel subsets $\Delta$ such that $g - h \in U$ for all $g, h \in \Delta$ and each $\Delta \in \mathcal{P}$. Let $\psi_k := \nu_k \circ f_k^{-1}$ for each $k > 0$. Using the ATI assumption, we can find $m > n$ such that

$$\| \psi_{n+1} \cdot \cdot \cdot \psi_m \cdot \delta_a - \psi_{n+1} \cdot \cdot \cdot \psi_m \|_1 < \epsilon. \quad (2.5)$$

For each $\Delta \in \mathcal{P}$, we let

$$A_\Delta := \left\{ y = (y_k)^m_{k=n+1} \in X_{n+1} \times \cdots \times X_m \mid \sum_{k=n+1}^m f_k(y_k) \in \Delta \right\} \quad \text{and}$$

$$B_\Delta := \left\{ y = (y_k)^m_{k=n+1} \in X_{n+1} \times \cdots \times X_m \mid a + \sum_{k=n+1}^m f_k(y_k) \in \Delta \right\}. \quad (2.6)$$

Then $\{A_\Delta\}_{\Delta \in \mathcal{P}}$ and $\{B_\Delta\}_{\Delta \in \mathcal{P}}$ are two measurable partitions of $X_{n+1} \times \cdots \times X_m$. It follows from $(2.5)$ that

$$\sum_{\Delta \in \mathcal{P}} |v_{n+1}^m(A_\Delta) - v_{n+1}^m(B_\Delta)| = \sum_{\Delta \in \mathcal{P}} |\psi_{n+1} \cdot \cdot \cdot \psi_m \cdot \delta_a(\Delta) - \psi_{n+1} \cdot \cdot \cdot \psi_m(\Delta)|$$

$$\leq \| \psi_{n+1} \cdot \cdot \cdot \psi_m \cdot \delta_a - \psi_{n+1} \cdot \cdot \cdot \psi_m \|_1 < \epsilon,$$

where $v_{n+1}^m$ denotes the direct product $\bigotimes_{k=n+1}^m v_k$. We can find subsets $A'_\Delta \subset A_\Delta$ and $B'_\Delta \subset B_\Delta$ such that

$$v_{n+1}^m(A'_\Delta) = v_{n+1}^m(B'_\Delta) = \min(v_{n+1}^m(A_\Delta), v_{n+1}^m(B_\Delta)). \quad (2.7)$$

Note that the group $\Gamma_{n+1,m}$ generated by $m - n$ mutually commuting transformations $T_{n+1} \times I \times \cdots \times I, \ I \times T_{n+2} \times I \times \cdots \times I, \ldots, I \times \cdots \times I \times T_m \in \text{Aut}_0(X_{n+1} \times \cdots \times X_m, v_{n+1}^m)$ is ergodic. Hence, in view of $(2.7)$, Hopf’s lemma [HaOs, Lemma 10] yields that there is a transformation $S_0 \in [\Gamma_{n+1,m}]$ such that $S_0 A'_\Delta = B'_\Delta$ for each $\Delta \in \mathcal{P}$. (We recall that Hopf’s equivalence lemma claims that, given an ergodic conservative measure-preserving countable transformation group $\Sigma$ of a standard $\sigma$-finite measure space $(Y, \mathcal{M}, \omega)$ and two subsets $A, B \in \mathcal{M}$ with $\omega(A) = \omega(B)$, there is a transformation $\gamma \in [\Sigma]$ such that $\gamma A = B \ mod \ \omega$. The lemma is proved via the standard exhaustion

Since $\alpha_n(x) = A_n((I \times T_n)x^n_1) - A_n(x^n_1)$, it follows from $(2.4)$ that

$$E_n(I \times T_n \times I) E_n^{-1} = (\hat{T}_n)_{\alpha_n} \times I_Z.$$
argument.) We note that
\[
\sum_{\Delta \in \mathcal{P}} v_{n+1}^m (A_\Delta \setminus A'_\Delta) \leq \sum_{\Delta \in \mathcal{P}} |v_{n+1}^m (A_\Delta) - v_{n+1}^m (B_\Delta)| < \epsilon.
\]
It follows that \(v_{n+1}^m (\bigcup_{\Delta \in \mathcal{P}} A'_\Delta) > 1 - \epsilon\). On the other hand, in view of (2.6), for each \(y \in A^+ := \bigcup_{\Delta \in \mathcal{P}} A'_\Delta\),
\[
\left( \sum_{k=n+1}^m f_k \right)(y) - \left( \sum_{k=n+1}^m f_k \right)(S_0 y) \in a + U.
\]
We now 'extend' \(S_0\) to a transformation \(S \in \text{Aut}_0(X, \mu)\) by setting
\[
Sx := (x_1^n, S_0 x_{n+1}^m, x_{m+1}^\infty) \in X \quad \text{for all} \ x \in X.
\]
Then \(S \in \Gamma\) and
\[
\alpha(S, x) \in a + U \quad \text{whenever} \ x_{n+1}^m \in A^+.
\]
(2.8)
Then we have that \([B \times A^+]_1^m \subset [B]_1^m\), \(S[B \times A^+]_1^m \subset [B]_1^m\), \(\mu([B \times A^+]_1^m) > \frac{1}{2} \mu([B]_1^m)\) and (2.8) holds for all \(x \in [B \times A^+]_1^m\). Since the set of all cylinders is dense in \(\mathfrak{B}\), it follows from Lemma C that \(a\) is an essential value of \(\alpha\). Thus, Claim 1 is proved. 

To complete the proof of the theorem, we have already noticed that \((\hat{T}_n)_{\alpha_n} \times I_Z \in \mathcal{G}\) for each \(n \in \mathbb{N}\). Hence \(F(\gamma_\alpha(x, g), z) = F(x, g, z)\) at a.e. \((x, g, z) \in X \times G \times Z\) for each \(\gamma \in \Gamma\). Claim 1 yields that there is a function \(M : Z \to \mathbb{R}\) such that \(F(x, g, z) = M(z)\) at a.e. \((x, g, z) \in X \times G \times Z\). Since \(F\) is invariant under \(T_f \times C\), we obtain that \(M\) is invariant under \(C\). Since \(C\) is ergodic, \(M\) is constant almost everywhere and hence \(F\) is constant almost everywhere, that is, \(T_f \times C\) is ergodic. 

We call the cocycle \(f\) in the statement of Theorem 2.5 a cocycle of product type.

2.6. Application to Gaussian cocycles. Let \((X, \mathfrak{B}, \mu, T)\) be an ergodic Gaussian dynamical system. It is completely determined by a restriction of the corresponding Koopman unitary operator \(U_T\) to a closed (real) Gaussian subspace \(H \subset L^2_0(X, \mu)\), called the first chaos (see, for example, [LePaTh] for the definitions). Let \(\kappa\) denote the maximal spectral type of \(U_T | H\). It is known that \(T\) is ergodic if and only if \(T\) is weakly mixing if and only if \(\kappa\) is non-atomic [Mar]. Take \(f \in H\). Then the measurable map \(f : X \to \mathbb{R}\) considered as a cocycle of \(T\) is called a Gaussian cocycle. It was shown in [LeLeSk] that if \(f\) is a \(T\)-coboundary, that is, \(f = h \circ T - h\) for a measurable function \(h : X \to \mathbb{R}\), then \(h \in H\). We now recall a conjecture from [LeLeSk].

Conjecture I. If a Gaussian cocycle \(f\) is not a coboundary then \(f\) is ergodic.

We now prove this conjecture (in fact, we prove a stronger result) under an additional assumption that \(T\) is mildly mixing.

THEOREM 2.6. If \(T\) is a mildly mixing Gaussian transformation and \(f\) is a Gaussian cocycle of \(T\) which is not a coboundary then \(T_f\) is sharply weak mixing.
Proof. Since \( f \in H \), it follows that \( \int_X f \, d\mu = 0 \). Hence, by Atkinson’s theorem \([At]\), \( T_f \) is conservative. Consider now the spectral decomposition for the pair \((H, U_T)\):

\[
H = \int_T^{\oplus} \mathcal{H}_z \, d\kappa(z) \quad \text{and} \quad U_T = \int_T^{\oplus} zI_z \, d\kappa(z),
\]

where \( T \ni z \mapsto \mathcal{H}_z \) is the corresponding measurable field of Hilbert spaces and \( I_z \) is the identity operator in \( \mathcal{H}_z \). In other words, we can consider an element \( h \) of \( H \) as a measurable map \( T \ni z \mapsto h(z) \in \mathcal{H}_z \) such that \( \|h\|^2 = \int_T \|h(z)\|^2 \, d\kappa(z) < \infty \). We now let \( \Delta_n := \{z \in T \mid (1/(n+1) < |z - 1| \leq (1/n)\} \). Then we obtain a countable partition \( \bigcup_{n=1}^{\infty} \Delta_n \) of \( T \setminus \{1\} \). Since \( \kappa([1]) = 0 \), this countable partition generates a decomposition of \( H \) into a direct sum \( \bigoplus_{n \in \mathbb{N}} H_n \) of closed \( U_T \)-invariant subspaces \( H_n \) consisting of the measurable maps \( h : T \ni z \mapsto h(z) \in \mathcal{H}_z \) such that \( \|h\| < \infty \) and \( h(z) = 0 \) whenever \( z \notin \Delta_n \). This decomposition induces a decomposition of \((X, \mu, T)\) into the infinite direct product \((X, \mu, T) = \bigotimes_{n=1}^{\infty} (X_n, \mu_n, T_n)\), where \((X_n, \mu_n, T_n)\) is the Gaussian dynamical system associated with the pair \((H_n, U_T \upharpoonright H_n)\) for each \( n \in \mathbb{N} \). Now we can expand \( f \) into an orthogonal sum \( f = \bigoplus_{n=1}^{\infty} f_n \) with \( f_n \in H_n \) for each \( n \in \mathbb{N} \). We claim that, for each \( n > 0 \), there is \( a_n \in H_n \) such that \( f_n = U_T a_n - a_n \). Indeed, it follows from this equation and (2.9) that if we represent \( f_n \) as a measurable map \( \Delta_n \ni z \mapsto f_n(z) \in \mathcal{H}_z \) then \( f_n(z) = z a_n(z) - a_n(z) \) for a.e. \( z \in \Delta_n \). Solving this equation, we obtain that \( a_n(z) = (z - 1)^{-1} f_n(z) \) for a.e. \( z \in \Delta_n \). Since \( |z - 1|^{-1} < n + 1 \) for all \( z \in \Delta_n \), we obtain that \( a_n \in H_n \). This yields an expansion

\[
f = \bigoplus_{n=1}^{\infty} (U_T a_n - a_n) = \bigoplus_{n=1}^{\infty} (a_n \circ T_n^{-1} - a_n)
\]

of \( f \) into an infinite sum of \( T_n \)-coboundaries. Of course, \( \sum_{n \in \mathbb{N}} \|a_n\|^2 = +\infty \). Otherwise, the series \( \sum_{n \in \mathbb{N}} a_n \) converges in \( H \) and hence \( f \) would be a coboundary, which contradicts the assumption of the theorem. We have that \( \mu_n \circ a_n^{-1} = \mathcal{N}_{\|a_n\|^2} \) for each \( n \in \mathbb{N} \). Passing, if necessary, to a subsequence, we may assume without loss of generality that the convergence in (2.10) is almost everywhere. Example 2.3 yields that the sequence \( (\mu_n \circ a_n^{-1})_{n=1}^{\infty} \) is asymptotically translation invariant. It now follows from Theorem 2.5 that \( T_f \) is sharply weak mixing. \( \square \)

Consider now the general case. Then there is a maximal (with respect to \( \kappa \)) subset \( A \) of \( T \) such that \( U_T \) restricted to the closed subspace \( \int_A^{\oplus} \mathcal{H}_z \, d\kappa(z) \) of \( H \) is mildly mixing. We note that \( A \) is symmetric. Then \( \kappa \) decomposes into a sum of two orthogonal measures: \( \kappa_{mm} := \kappa \upharpoonright A \) (the mildly mixing part of \( \kappa \)) and \( \kappa_r := \kappa \upharpoonright (T \setminus A) \) (the rigid part of \( \kappa \)). This decomposition defines a decomposition of \((X, \mu, T)\) into a direct product \((X_1, \mu_{mm}, M) \times (X_2, \mu_r, R)\), where \((X_1, \mu_{mm}, M)\) is the Gaussian dynamical system corresponding to the pair \((\int_A^{\oplus} \mathcal{H}_z \, d\kappa_{mm}(z), U_T)\) and \((X_2, \mu_r, R)\) is the Gaussian dynamical system corresponding to the pair \((\int_{T \setminus A}^{\oplus} \mathcal{H}_z \, d\kappa_r(z), U_T)\). Also, we obtain a decomposition of \( f \) into a sum \( f_{mm} + f_r \), where \( f_{mm} := f 1_A \) and \( f_r = f 1_{T \setminus A} \). There are two possible cases: either \( f_{mm} \) is a coboundary or \( f_{mm} \) is not a coboundary. In the first case \( T_f \) is isomorphic to \( Q \times R_f \). Moreover, \( f_r \) is not a coboundary because otherwise \( f \) would be a coboundary. Since \( Q \) is mildly mixing and \( T_f \) is conservative, \( T_f \) is ergodic if
and only if $R_f$ is ergodic. In the second case, $T_f$ is isomorphic to $Q_{f_{mm}} \times R_f$, and $Q_{f_{mm}}$ is sharply weak mixing by Theorem 2.5. Since $T_f$ is conservative, it follows that $T_f$ is ergodic if and only if $R_f$ is ergodic. Thus, we have reduced the conjecture from [LeLeSk] as follows.

Conjecture II. If a Gaussian cocycle $f$ is not a coboundary and $\kappa$ has only rigid part then $f$ is ergodic.

In [LeLeSk], there were constructed some concrete rigid Gaussian transformations admitting ergodic Gaussian cocycles. In [MaVa] this result was extended to arbitrary rigid Gaussian transformations which have at least one Gaussian non-coboundary. However, it is unknown whether ergodicity holds for each Gaussian non-coboundary in those examples.

3. Krieger’s type of infinite direct products of dynamical systems of finite type

3.1. IDPFT systems. Let $T_n$ be a non-singular invertible transformation of a standard probability space $(X_n, \mathfrak{B}_n, \mu_n)$ for each $n \in \mathbb{N}$. Denote by $T$ the infinite direct product of $T_n$, $n \in \mathbb{N}$, acting on the infinite product space $(X, \mathfrak{B}, \mu) := \bigotimes_{n \in \mathbb{Z}} (X_n, \mathfrak{B}_n, \mu_n)$. By Theorem A, $T$ is $\mu$-non-singular if and only if

$$\prod_{n=1}^{\infty} (1 - H^2(\mu_n \circ T_n^{-1}, \mu_n)) > 0 \quad \text{or} \quad \sum_{n=1}^{\infty} H^2(\mu_n \circ T_n^{-1}, \mu_n) < \infty. \quad (3.1)$$

If (3.1) does not hold then $\mu \circ T^{-1} \perp \mu$. If $T$ is $\mu$-non-singular then

$$\frac{d\mu \circ T^{-1}}{d\mu}(x) = \prod_{n=1}^{\infty} \frac{d\mu_n \circ T_n^{-1}}{d\mu_n}(x_n) \quad \text{at a.e. } x \in X.$$

Suppose now that $T_n$ is of finite type, that is, there exists a $\mu_n$-equivalent probability measure $\nu_n$ which is invariant under $T_n$ for each $n \in \mathbb{N}$. We then put $\phi_n := d\mu_n/d\nu_n$. Since $1 - H^2(\mu_n \circ T_n^{-1}, \mu_n) = \int_{X_n} \sqrt{(\phi_n \circ T_n^{-1})/\phi_n} \phi_n d\nu_n$, formula (3.1) and Theorem A yield the following corollary.

**Corollary 3.1.** $T$ is $\mu$-non-singular if and only if

$$\prod_{n=1}^{\infty} \int_{X_n} \sqrt{\phi_n \cdot \phi_n \circ T_n^{-1}} d\nu_n > 0. \quad (3.2)$$

$\mu \perp \nu$ if and only if

$$\prod_{n=1}^{\infty} \int_{X_n} \sqrt{\phi_n} d\nu_n = 0. \quad (3.3)$$

**Definition 3.2.** If $T$ is $\mu$-non-singular and $T_n$ is of finite type for all $n > 0$ then we say that the dynamical system $(X, \mathfrak{B}, \mu, T)$ is an infinite direct product of finite types.

Our purpose in this section is to investigate dynamical properties of IDPFT systems. The first result is about ergodicity of conservative IDPFT systems under the mild mixing assumption on the factors.
Proposition 3.3. Let \((X_n, \nu_n, T_n)\) be mildly mixing for each \(n > 0\) and (3.2) and (3.3) hold. Suppose that \(T\) is \(\mu\)-conservative. Then \(T\) is \(\mu\)-sharply weak mixing and \(\mu \perp \nu\).

\textbf{Proof.} Let \(S\) be an ergodic conservative transformation of a standard probability space \((Y, \mathfrak{C}, \nu)\). By [Aa, Proposition 1.2.4], \(T \times S\) is either totally dissipative or conservative. Suppose that \(T \times S\) is conservative. We have to prove that it is ergodic. Let a subset \(A \in \mathfrak{B} \otimes \mathfrak{C}\) be invariant under \(T \times S\). It follows from Theorem B that, for each \(n > 0\), \(A\) belongs to the \(\sigma\)-algebra \(\{\emptyset, X_1 \times \cdots \times X_n\} \otimes \mathfrak{B}_{n+1} \otimes \mathfrak{B}_{n+2} \otimes \cdots \otimes \mathfrak{C}\) (When applying Theorem B, we consider the measure \((\otimes_{k=1}^{n} \nu_k) \otimes (\otimes_{k>n} \mu_k) \otimes \nu\) on \(X \times Y\). This measure is equivalent to \(\mu \otimes \nu\)). By the Kolmogorov 0–1 law, the intersection of these \(\sigma\)-algebras is \(\mathfrak{F} \otimes \mathfrak{C}\), where \(\mathfrak{F}\) is the trivial \(\sigma\)-algebra on \(X\). Thus \(A = X \times D\) for some subset \(D \in \mathfrak{C}\). Since \(A\) is invariant under \(T \times S\), it follows that \(D\) is invariant under \(S\). Since \(S\) is ergodic, we obtain that either \(\mu \otimes \nu(A) = 0\) or \(\mu \otimes \nu(A) = 1\). \(\square\)

Remark 3.4. In §4 we give examples of \((X, \nu, T)\) and \(\mu\) such that \((X, \mu, T)\) is of type III1. In particular, there is no \(\mu\)-equivalent invariant probability measure. On the other hand, we do not know of examples where \((X, \mu, T)\) is of type II1, that is, \(T\) is mildly mixing with respect to a \(\mu\)-equivalent invariant probability measure.

3.2. Radon–Nikodym cocycle and type III1. Let \(\Gamma\) be an ergodic countable subgroup of \(\text{Aut}(Y, \nu)\). Denote by \(\rho_\nu : \Gamma \times Y \to \mathbb{R}\) the logarithm of the Radon–Nikodym cocycle of \(\Gamma\), that is,

\[\rho_\nu(\gamma, y) := \log \frac{dv \circ \gamma}{d\nu}(y).\]

The \(\rho_\nu\)-skew product extension \(\Gamma_{\rho_\nu}\) of \(\Gamma\) is called the Maharam extension of \(R\). We note that \(\Gamma_{\rho_\nu}\) preserves an equivalent \(\sigma\)-finite measure \(\nu \otimes \kappa\), where \(\kappa\) is a Lebesgue absolutely continuous \(\sigma\)-finite measure on \(\mathbb{R}\) such that \(d\kappa(t) = e^{-t}dt\) for all \(t \in \mathbb{R}\). Similar to the finite measure-preserving case, \(\rho_\nu\) ‘extends’ to the full group \([\Gamma]\) in such a way that the cocycle identity holds. Moreover, we do not need the freeness condition for \(\Gamma\) to define this extension.

We note that \(\rho_\nu\) is a coboundary if and only if there is a \(\Gamma\)-invariant \(\nu\)-equivalent \(\sigma\)-finite measure on \((Y, \mathfrak{C})\).

By the Maharam theorem (see [Sc]), \(\Gamma_{\rho_\nu}\) is conservative if and only if \(\Gamma\) is conservative. However, if \(\Gamma\) is ergodic then \(\Gamma_{\rho_\nu}\) is not necessarily ergodic. If the Maharam extension of \(\Gamma\) is ergodic then \(\Gamma\) is called of Krieger’s type III1. If, for each homomorphism \(\vartheta : \Gamma \to \text{Aut}_0(Y, \nu)\) such that the image \(\{\vartheta(\gamma) \mid \gamma \in \Gamma\}\) is ergodic, the direct product \(\{\gamma \times \vartheta(\gamma) \mid \gamma \in \Gamma\}\) is ergodic and of type III1 then \(\Gamma\) is said to be of stable Krieger’s type III1.

It is possible to define essential values of \(\rho_\nu\) in the same way as in the finite measure-preserving case.

Definition 3.5. An element \(g \in \mathbb{R}\) is called an essential value of \(\rho_\nu\) if, for each subset \(A \subset Y\) of positive measure and a neighborhood \(U\) of \(g\), there are a Borel subset \(B \subset A\) and an element \(\gamma \in \Gamma\) such that \(\nu(B) > 0, \gamma B \subset A\) and \(\rho_\nu(\gamma, y) \in U\) for all \(y \in B\).
We refer to [Sc, HaOs] for the proof of the following results:

- The set \( r(\rho_\nu) \) is a closed subgroup in \( \mathbb{R} \).
- \( \Gamma \) is of type III if and only if \( r(\rho_\nu) = \mathbb{R} \).

If there is \( \lambda \in (0, 1) \) such that \( r(\rho_\nu) = \{ n \log \lambda \mid n \in \mathbb{Z} \} \) then \( \Gamma \) is said to be of Krieger’s type III\( _\lambda \).

We will need the following analog of Lemma C. We do not provide a proof of the lemma because it is routine.

**Lemma D.** An element \( a \in \mathbb{R} \) is an essential value of \( \rho_\nu \) if there exists \( \delta > 0 \) such that for each \( \epsilon > 0 \) and each subset \( B \) from a dense collection \( \mathcal{C}_0 \) of subsets in \( \mathcal{C} \), there are a subset \( B_0 \subset B \) and a transformation \( \theta \in \Gamma \) such that for all \( y \in B_0 \) or \( |\rho_\nu(\theta, y) + a| \leq \epsilon \) for all \( y \in B_0 \).

3.3. **On conservativeness of IDPFT systems.** In this subsection we first establish a general result on conservativeness of infinite direct product systems. We note that the argument used below to prove conservativeness of IDPFT systems is similar to the argument used in [Ko, Da] to prove ergodicity of non-singular Bernoulli and Markov systems.

**Proposition 3.6.** Let \( (X_n, \mathcal{B}_n, \mu_n, T_n) \) be an ergodic non-singular dynamical system on a standard probability space for each \( n \in \mathbb{N} \) and let (3.1) hold. Let \( (X, \mathcal{B}, \mu, T) = \bigotimes_{n=1}^{\infty} (X_n, \mathcal{B}_n, \mu_n, T_n) \). If, for each \( n \in \mathbb{N} \), there is a function \( \alpha_n : X_n \to [1, +\infty) \) such that, for each \( k \in \mathbb{N} \),

\[
\alpha_n(x) - 1 \leq \frac{d\mu_n \circ T^k}{d\mu_n}(x) \leq \alpha_n(x)
\]

at a.e. \( \mu_n \)-a.e. \( x \in X_n \) then the dynamical system \( (X, \mathcal{B}, \mu, T) \) is either conservative or totally dissipative. Moreover, if \( (Y, \mathcal{C}, \nu, S) \) is an ergodic conservative non-singular dynamical system then the direct product \( T \times S \) is either conservative or totally dissipative.

**Proof.** We will prove the second claim only. By the Hopf criterion [Aa, Proposition 1.3.1],

\[
\mathcal{D}(T \times S) = \left\{ (x, y) \in X \times Y \left| \sum_{k=1}^{\infty} \frac{d(\mu \otimes \nu) \circ (T \times S)^k}{d(\mu \otimes \nu)}(x, y) < \infty \right. \right\}.
\]

For each \( r > 0 \), we consider a transformation \( \gamma_r \) of \( X \) by setting \( \gamma_r(x_1, x_2, \ldots) := (x_1, \ldots, x_{r-1}, T_r x_r, x_{r+1}, \ldots) \). Of course, \( \gamma_r \in \text{Aut}(X, \mu) \). Denote by \( \Gamma \) the transformation group generated by \( \gamma_r, r \in \mathbb{N} \). It follows from the Kolmogorov 0–1 law that \( \Gamma \) is ergodic. We claim that \( \mathcal{D}(T \times S) \) is invariant under \( \gamma_r \times 1 \) for each \( r \). Let \( (x, y) \in \mathcal{D}(T \times S) \). Since, for each \( k > 0 \),

\[
\frac{d\mu \circ T^k}{d\mu}(\gamma_r x) = \frac{d\mu_r \circ T_r^k}{d\mu_r}(T_r x_r) \left( \frac{d\mu_r \circ T_r^k}{d\mu_r}(x_r) \right)^{-1} \prod_{n=1}^{\infty} \frac{d\mu_n \circ T_n^k}{d\mu_n}(x_n) \leq \alpha_r(T_r x_r) \alpha_r(x_r)^{-1} \frac{d\mu \circ T^k}{d\mu}(x),
\]

where \( \alpha_r(T_r x_r) \alpha_r(x_r)^{-1} \leq \max \left\{ \alpha_r(T_r x_r), \alpha_r(x_r)^{-1} \right\} \).
it follows that

\[
\sum_{k=1}^{\infty} \frac{d(\mu \otimes \nu) \circ (T \times S)^k(y, x)}{d(\mu \otimes \nu)}(y) = \sum_{k=1}^{\infty} \frac{d\mu \circ T^k}{d\mu}(y) \frac{d\nu \circ S^k}{d\nu}(y) \leq \alpha_r(T, x_r) \sum_{k=1}^{\infty} \frac{d\mu \circ T^k}{d\mu}(x) \frac{d\nu \circ S^k}{d\nu}(y) = \alpha_r(T, x_r) \sum_{k=1}^{\infty} \frac{d(\mu \otimes \nu) \circ (T \times S)^k(x, y)}{d(\mu \otimes \nu)}(x, y) < \infty.
\]

Thus, \((y, x) \in D(T \times S)\). Since \(D(T \times S)\) is invariant under \(I \times S\), we obtain that \(D(T \times S)\) is invariant under an ergodic transformation group on \(X \times Y\) generated by \(I \times S\), \(\gamma \times I\), \(\gamma \in \Gamma\). Hence, either \((\mu \otimes \nu)(D(T \times S)) = 0\) or \((\mu \otimes \nu)(D(T \times S)) = 1\), as desired.

We now apply Proposition 3.6 to IDPFT systems.

**Corollary 3.7.** Let \((X_n, B_n, \mu_n, T_n)\) be an ergodic non-singular dynamical system on a standard probability space for each \(n \in \mathbb{N}\) and let (3.1) hold. Suppose that, for each \(n \in \mathbb{N}\), there is a \(\mu_n\)-equivalent \(T_n\)-invariant probability measure on \(X_n\). Let \((X, B, \mu, T) := \bigotimes_{n=1}^{\infty} (X_n, B_n, \mu_n, T_n)\). Then the dynamical system \((X, B, \mu, T)\) is either conservative or totally dissipative. Moreover, if \((Y, C, \nu, S)\) is an ergodic conservative non-singular dynamical system then the direct product \(T \times S\) is either conservative or totally dissipative.

**Proof.** Let \(\phi_n := \frac{d\mu_n}{d\nu_n}\) for each \(n > 0\). If, for each \(n > 0\), there is a real \(\alpha_n \geq 1\) such that \(\alpha_n^{-1} \leq \phi_n \leq \alpha_n\) almost everywhere then the claim of the corollary follows directly from Proposition 3.6. We now show that the general case can be reduced to the ‘bounded’ case. Indeed, for each \(n > 0\), we can find a probability measure \(\tilde{\mu}_n \sim \mu_n\) such that \(H^2(\tilde{\mu}_n, \mu_n) \leq 2^{-n}\) and the Radon–Nikodym derivative \(d\tilde{\mu}_n/d\nu_n\) is bounded from above and separated from 0 from below (For that purpose, take the Radon–Nikodym derivative \(d\mu_n/d\nu_n\) and change it on a subset of very small measure to get the boundedness. The ‘modified’ function will be the Radon–Nikodym derivative \(d\tilde{\mu}_n/d\mu_n\)). Since \(\sum_{n=1}^{\infty} H^2(\tilde{\mu}_n, \mu_n) < \infty\), it follows from Theorem A that \(\mu \sim \tilde{\mu} := \bigotimes_{n=1}^{\infty} \tilde{\mu}_n\). All that remains is to note that the conservativeness of a dynamical system does not depend on the choice of quasi-invariant measure within its equivalence class. \(\square\)

### 3.4. Sharp weak mixing for Maharam extensions of IDPFT systems

We first introduce a ‘non-singular analogue’ of the ATI property (cf. Definition 2.2).

**Definition 3.8.** A sequence \((\xi_n)_{n=1}^{\infty}\) of probability non-atomic Borel measures on \(G\) is called asymptotically translation quasi-invariant if, for each \(a \in G\), there exists \(\zeta_a > 0\) such that, for every \(n \in \mathbb{N}\), there are \(m > n\) and a Borel subset \(W_{n, m} \subset G\)
such that
\[
\zeta_a \leq (\xi_n * \xi_{n+1} * \cdots * \xi_{n+m})(W_{n,m}),
\]
\[
\xi_n * \xi_{n+1} * \cdots * \xi_{n+m} * \delta_a < \xi_n * \xi_{n+1} * \cdots * \xi_{n+m} \quad \text{and}
\]
\[
\zeta_a \leq \frac{d(\xi_n * \xi_{n+1} * \cdots * \xi_{n+m} * \delta_a)}{d(\xi_n * \xi_{n+1} * \cdots * \xi_{n+m})}(t) \quad \text{for each } t \in W_{n,m}.
\]

We will need the following lemma on continuous measures.

**Lemma 3.9.** Given a standard non-atomic probability space \((Y, \mathcal{C}, \nu)\), a non-negative function \(\phi \in L^1(Y, \nu)\) and \(\delta \in (0, 1)\), then

\[
\max \left\{ \int_A \phi \, d\nu \mid \nu(A) = \delta \right\} \geq \frac{\delta}{2} \int_Y \phi \, d\nu.
\]

**Proof.** Let \(\alpha := \max \{\int_A \phi \, d\nu \mid \nu(A) = \delta\}\). Find \(n \geq 1\) such that \(\delta \leq (1/n)\). Then there is a partition \(Y = Y_1 \sqcup \cdots \sqcup Y_{n+1}\) of \(Y\) into subsets \(Y_k\) such that \(\nu(Y_k) = \delta\) for each \(k = 1, \ldots, n\) and \(\mu(Y_{n+1}) \leq \delta\). We now have

\[
\int_Y \phi \, d\nu = (n + 1)\alpha \leq (\delta^{-1} + 1)\alpha < \frac{2}{\delta}\alpha.
\]

The next theorem is a non-singular analogue of Theorem 2.5. The skeleton of the proof is similar to that of Theorem 2.5.

**Theorem 3.10.** Let a dynamical system \((X_n, \mathfrak{B}_n, \nu_n, T_n)\) be mildly mixing for each \(n > 0\). Let \(\mu_n\) be a probability on \(X_n\) such that \(\mu_n \sim \nu_n\) for each \(n \in \mathbb{N}\). Let \(\phi_n := d\mu_n/d\nu_n\) and (3.2) hold. We set

\[
(X, \mathfrak{B}, \nu, T) := \bigotimes_{n \in \mathbb{N}} (X_n, \mathfrak{B}_n, \nu_n, T_n)
\]

and \(\mu := \bigotimes_{n=1}^{\infty} \mu_n\). If \(T \) is \(\mu\)-conservative and the sequence of probability measures \((\nu_n \circ (\log \phi_n)^{-1})_{n=1}^{\infty}\) is asymptotically translation quasi-invariant then \(T \in \text{Aut}(X, \mu)\) is ergodic of stable type III₁. Moreover, the Maharam extension of \(T\) is sharply weak mixing.

**Proof.** By the Maharam theorem, the Maharam extension \(T_{\mu_n}\) is conservative. Let \(C\) be an ergodic conservative transformation of a standard probability space \((Z, \mathcal{Z}, \eta)\). As in the proof of Theorem 2.5, we see that \(T_{\mu_n} \times C\) is either totally dissipative or conservative. Suppose that it is conservative and prove that it is ergodic.

Let a function \(F \in L^\infty(X \times \mathbb{R} \times Z, \mu \otimes \kappa \otimes \eta)\) be invariant under \(T_{\mu_n} \times C\). We first show that \(F\) is also invariant under a huge group of transformations. Fix \(n > 0\). We define a non-singular automorphism \(T^{(n)}\) of \((\bigotimes_{k=n+1}^{\infty} X_k, \mu^{(n)}), \) where \(\mu^{(n)} := \bigotimes_{k=n+1}^{\infty} \mu_k\), and a measure-preserving isomorphism \(E_n\) of \((X \times \mathbb{R} \times Z, \mu \otimes \kappa \otimes \eta)\) onto the product space \((X \times \mathbb{R} \times Z, \bigotimes_{k=1}^{\infty} \nu_k) \otimes \mu^{(n)} \otimes \kappa \otimes \eta)\) by setting

\[
T^{(n)}(x_k)_{k=n+1}^{\infty} := (T_k x_k)_{k=n+1}^{\infty}
\]

and

\[
E_n(x, t, z) := \left( x, t + \sum_{k=1}^{n} \log \phi_k(x_k), z \right).
\]
Since
\[ T_{\rho_{\mu}}(x, t) = \left( T_1 x_1, T_2 x_2, \ldots, t + \sum_{k=1}^{\infty} \log \frac{d\mu_k \circ T_k}{d\mu_k}(x_k) \right) \]
\[ = \left( T_1 x_1, T_2 x_2, \ldots, t + \sum_{k=1}^{\infty} (\log \phi_k(T_k x_k) - \log \phi_k(x_k)) \right), \]
it follows that
\[ E_n^{-1}(T_{\rho_{\mu}} \times C)E_n = (T_1 \times \cdots \times T_n) \times (T^{(n)})_{\rho_{\mu(n)}} \times C. \tag{3.4} \]
Since \( T_{\rho_{\mu}} \times C \) is conservative, it follows from (3.4) that the product \((T^{(n)})_{\rho_{\mu(n)}} \times C\) is also conservative. On the other hand, the function \( F \circ E_n \) is invariant under \( E_n^{-1}(T_f \times C)E_n \).

Utilizing these two facts plus the mild mixing of the transformation \( T_1 \times \cdots \times T_n \), we deduce from Theorem B that \( F \circ E_n \) does not depend on the coordinates \( x_1, \ldots, x_n \).

Hence, for each transformation \( S \in \text{Aut}_0(X_1 \times \cdots \times X_n, \otimes_{k=1}^n \nu_k) \), we have that \( F \circ E_n \circ (S \times I \times I_Z) = F \circ E_n \). Therefore \( F \) is invariant under the transformation \( E_n(S \times I \times I_Z)E_n^{-1} \in \text{Aut}(X \times \mathbb{R} \times Z, \mu \otimes \kappa \otimes \eta) \) and
\[ E_n(S \times I \times I_Z)E_n^{-1} = (S \times I)_{\rho_{\mu}} \times I_Z. \tag{3.5} \]
Denote by \( \Gamma \) the group of non-singular transformations of \((X, \mathcal{B}, \mu)\) generated by \( I \times T_n \times I, n \in \mathbb{N} \). Then \( \Gamma \) is an ergodic abelian countable subgroup of \( \text{Aut}(X, \mu) \) and \( F \) is invariant under \( \{ \gamma_{\rho_{\mu}} \times I_Z \ | \ \gamma \in \Gamma \} \) by (3.5).

**CLAIM II.** We claim that \( \Gamma \) is of type III1.

**Proof.** Equivalently, we will show that each \( a \in \mathbb{R} \) is an essential value for the cocycle \( \rho_{\mu} \) of \( \Gamma \). For that purpose, fix \( n > 0, \epsilon > 0 \) and a Borel subset \( B \subset X_1 \times \cdots \times X_n \). Denote by \( \psi_k \) the pushforward of \( \nu_k \) under \( \log \phi_k \) for each \( k > 0 \). By ATQI, there is \( \zeta_a > 0 \) (which does not depend on \( n \)), \( m > n \) and a subset \( W_{n+1,m} \subset \mathbb{R} \) such that
\[ \zeta_a \leq (\psi_{n+1} \cdots \psi_m)(W_{n+1,m}), \]
\[ \psi_{n+1} \cdots \psi_m \delta_a < \psi_{n+1} \cdots \psi_m \quad \text{and} \quad \zeta_a \leq \frac{d(\psi_{n+1} \cdots \psi_{n+m} \delta_a)}{d(\psi_{n+1} \cdots \psi_{n+m})}(t) \quad \text{for each} \ t \in W_{n+1,m}. \tag{3.6} \]
Choose a countable partition \( \mathcal{P} \) of \( W_{n+1,m} \) into subsets of diameter no more than \( \epsilon \). For each \( \Delta \in \mathcal{P} \), we let
\[ A_\Delta := \left\{ y = (y_k)_{k=n+1}^m \in X_{n+1} \times \cdots \times X_m \left| \sum_{k=n+1}^m \log \phi_k(y_k) \in \Delta \right. \right\} \]
\[ B_\Delta := \left\{ y = (y_k)_{k=n+1}^m \in X_{n+1} \times \cdots \times X_m \left| a + \sum_{k=n+1}^m \log \phi_k(y_k) \in \Delta \right. \right\}. \]
Let \( \mu_{n+1}^m := \otimes_{k=n+1}^m \mu_k \), \( \nu_{n+1}^m := \otimes_{k=n+1}^m \nu_k \) and \( \phi_{n+1}^m := d\mu_{n+1}^m / d\nu_{n+1}^m \). Dropping off some atoms of \( \mathcal{P} \) if necessary, we may assume without loss of generality that
Lemma D that $\Gamma_{n+1,m}$ generated by $m-n$ mutually commuting transformations $T_{n+1} \times I \times \cdots \times I$, $I \times T_{n+2} \times I \times \cdots \times I$, $\ldots$, $I \times \cdots \times I \times T_m \in \text{Aut}_0(X_{n+1} \times \cdots \times X_m, \nu_{n+1})$ is ergodic. Suppose that $v_{n+1}^m(A_\Delta) > v_{n+1}^m(B_\Delta)$ for some $\Delta \in \mathcal{P}$. We now apply Lemma 3.9 to the space $A_\Delta$ equipped with the conditional measure $v_{n+1}^m(\cdot)/v_{n+1}^m(A_\Delta)$, the function $v_{n+1}^m(A_\Delta)\phi_{n+1}^m$ and $\delta := v_{n+1}^m(B_\Delta)/v_{n+1}^m(A_\Delta)$. Then there is a Borel subset $A'_\Delta \subset A_\Delta$ such that $v_{n+1}^m(A'_\Delta)/v_{n+1}^m(A_\Delta) = \delta$ and

$$\int_{A'_\Delta} \phi_{n+1}^m \, d\nu_{n+1}^m \geq \frac{\delta}{2} \int_{A_\Delta} \phi_{n+1}^m \, d\nu_{n+1}^m.$$ 

In other words, $\mu_{n+1}^m(A'_\Delta) \geq (\delta/2)\mu_{n+1}^m(A_\Delta)$. It follows from (3.6) that

$$\delta = \frac{v_{n+1}^m(B_\Delta)}{v_{n+1}^m(A_\Delta)} = \frac{\psi_n * \psi_{n+1} * \cdots * \psi_{n+m} * \delta_\Delta(\Delta)}{\psi_n * \psi_{n+1} * \cdots * \psi_{n+m}(\Delta)} \geq \zeta_a.$$ 

Therefore $\mu_{n+1}^m(A'_\Delta) \geq (\zeta_a/2)\mu_{n+1}^m(A_\Delta)$. By Hopf’s lemma, there is a transformation $S_0 \in [\Gamma_{n+1,m}]$ such that

- $S_0A_\Delta \subset B_\Delta$ if $v_{n+1}^m(A_\Delta) \leq v_{n+1}^m(B_\Delta)$ and
- $S_0A'_\Delta = B_\Delta$ if $v_{n+1}^m(A_\Delta) > v_{n+1}^m(B_\Delta)$, because in this case we have that $v_{n+1}^m(A'_\Delta) = \delta v_{n+1}^m(A_\Delta) = v_{n+1}^m(B_\Delta)$.

Let

$$A^+ := \bigsqcup_{v_{n+1}^m(A_\Delta) \leq v_{n+1}^m(B_\Delta)} A_\Delta \sqcup \bigsqcup_{v_{n+1}^m(A_\Delta) > v_{n+1}^m(B_\Delta)} A'_\Delta.$$ 

Then $\mu_{n+1}^m(A^+) \geq (\zeta_a/2)\mu_{n+1}^m(\bigsqcup_{\Delta \in \mathcal{P}} A_\Delta) = (\zeta_a/2)(\psi_{n+1} * \cdots * \psi_m)(W_{n+1,m}) \geq (\zeta_a^2/2)$. Of course, for each $y \in A^+$,

$$\left( \sum_{k=n+1}^m \log \phi_k \right)(y) - \left( \sum_{k=n+1}^m \log \phi_k \right)(S_0 y) = a \pm \epsilon. \quad (3.7)$$

We now ‘extend’ $S_0$ to a transformation $S \in \text{Aut}(X, \mu)$ by setting

$$S := I \times S_0 \times I.$$ 

Then $S \in [\Gamma]$ and, in view of (3.7),

$$\rho_\mu(S, x) = -a \pm \epsilon \quad \text{whenever} \ x_{n+1}^m \in A^+.$$ 

We now have that

$$[B \times A^+]^m_1 \subset [B]^m_1, \quad S[B \times A^+]^m_1 \subset [B]^m_1, \quad \mu([B \times A^+]^m_1) \geq \frac{\xi^2}{2} \mu([B]^m_1)$$

and (3.8) holds for all $x \in [B \times A^+]^m_1$. Since the set of all cylinders is dense in $\mathfrak{B}$, it follows from Lemma D that $a$ is an essential value of $\alpha$. Thus, Claim II is proved.

The assertion of the theorem follows from Claim II in the very same way as the assertion of Theorem 2.5 follows from Claim I (in the proof of Theorem 2.5).
Remark 3.11. In this remark we clarify some subtle points in the proof of Theorem 3.10. Let \( \mathfrak{B}_0 \) stand for the collection of all cylinders in \( X \). Then \( \mathfrak{B}_0 \) is dense in \( \mathfrak{B} \) both with respect to \( \mu \) and with respect to \( \nu \). Though \( \mu \perp \nu \), the two measures are equivalent on \( \mathfrak{B}_0 \), that is, \( \mu(B) = 0 \) if and only if \( \nu(B) = 0 \) whenever \( B \in \mathfrak{B}_0 \). Given a transformation \( \theta \in \text{Aut}(X, \mu) \), the Radon–Nikodym derivative \( (d\mu \circ \theta)/d\mu \) is defined up to a subset of zero \( \mu \)-measure. Hence it makes no sense as a function on \( (X, \nu) \). However, if we consider transformations of a specific product structure, say \( (pn) \) is the only \( \mathbb{Z} \)-invariant Borel probability measure on \( X \), then \( (d\mu \circ \gamma)/d\mu \) is defined, in fact, up a subset of zero \( \mu \)-measure from \( \text{Aut}(X, \mu) \) as well as from \( \text{Aut}_0(X, \nu) \). Thus, although an element of the full group \( [\Gamma] \) is defined up to subset of zero measure, the relation \( S \in [\Gamma] \) is well defined with respect to \( \mu \) as well as with respect to \( \nu \).

3.5. On type III\( \lambda \) for rigid IDPFT systems. We would like to emphasize that the conclusion of Theorem 3.10 does not hold if we drop the mild mixing condition on \( T_n \) and the ATQI property. We illustrate this on a family of IDPFT systems consisting of infinite products of periodic transformations. Let \( (p_n)_{n=1}^\infty \) be a sequence of mutually coprime positive integers such that \( p_n > 2p_1 \cdots p_{n-1} \) for each \( n \in \mathbb{N} \). Below we will specify more conditions on the growth of \( (p_n)_{n=1}^\infty \). For \( n \in \mathbb{N} \), we set \( X_n := \{0, 1, \ldots, p_n - 1\} \) and identify \( X_n \) with the cyclic group \( \mathbb{Z}/p_n\mathbb{Z} \). Then \( T_n : X_n \to X_n \), given by \( T_n x = x + 1 \) (mod \( p_n \)), is a bijection of \( X_n \). The infinite product \( T = \bigotimes_{n=1}^\infty T_n \) is a minimal rotation on the compact totally disconnected abelian group \( X := \bigotimes_{n=1}^\infty X_n \). Of course, the Haar measure \( \nu \) on \( X \) is the only \( T \)-invariant Borel probability measure on \( X \). This measure is the infinite direct product of the equidistributions on \( X_n, n \in \mathbb{N} \).

Fix \( \lambda \in (0, 1) \). For each \( n > 0 \), let \( \mu_n \) denote the unique probability measure on \( X_n \) such that:

- \( \mu_n(j) = \mu_n(0) \) for each \( j \leq p_n/2 \);
- \( \mu_n(j) = \mu_n(p_n - 1) \) for each \( j > p_n/2 \);
- \( \mu_n(p_n - 1)/\mu_n(0) = \lambda \).

It is straightforward to verify that \( H^2(\mu_n, \mu_n \circ T_n^{-1}) \to 0 \) as \( n \to \infty \). Hence, passing to a subsequence in \( (p_n)_{n=1}^\infty \), we may assume without loss of generality that

\[
\sum_{n=1}^\infty H^2(\mu_n, \mu_n \circ T_n^{-1}) < \infty.
\]

We now let

\[
Y_n := \left\{ x_n \in X_n \mid x_n < \frac{p_n}{2} - \prod_{k=1}^{n-1} p_k \right\} \cup \left\{ x_n \in X_n \mid \frac{p_n}{2} < x_n < p_n - \prod_{k=1}^{n-1} p_k \right\},
\]

\[
Z_n := \{ x_n \in X_n \mid p_1 \cdots p_{n-1} < x_n < p_n/2 \}.
\]

Passing to a further subsequence in \( (p_n)_{n=1}^\infty \), we will assume that the following two conditions are satisfied:

- (•) \( \mu_n(Y_n) > 1 - 2^{-n-1} \) for each \( n > 0 \);
- (⋆) \( \mu_n(Z_n) > (1/(2(\lambda + 1))) \) for each \( n > 0 \).
Of course, (6) is satisfied for this subsequence as well. We now let \( \mu = \bigotimes_{n=1}^\infty \mu_n \). By Theorem A, in view of (6), \( T \) is \( \mu \)-non-singular (Although the topological system \((X, T)\) is a topological odometer, that is, a minimal rotation on a monothetic compact totally disconnected abelian group, the non-singular system \((X, \mu, T)\) should not be confused with the non-singular product odometers which have been well studied in the literature (see, for example, [DaSi, HaOs, Sc]) because \( \mu \) does not split into an infinite product when \( X \) is written in the product form suitable for the odometer ‘addition with carry’). It follows from the Kolmogorov 0–1 law that \( T \) is \( \mu \)-ergodic.

**Proposition 3.12.** \((X, \mu, T)\) is of Krieger’s type III\(_1\).

**Proof.** Since \( \log(d\mu \circ T^{-1}/d\mu) \in \{n \log \lambda \mid n \in \mathbb{Z}\} \) at a.e. \( x \in X \), it suffices to show that \( \log \lambda \) is an essential value of the Radon–Nikodym cocycle \( \rho_\mu \) of \( T \). For each \( n > 0 \), denote by \( l_n \) the positive integer such that \( l_n p_1 \cdots p_{n-1} \leq p_n < (l_n + 1) p_1 \cdots p_{n-1} \).

For a Borel subset \( B \subset X_1 \times \cdots \times X_n \), we set \( A := B \times Z_{n+1} \times Y_{n+2} \times Y_{n+3} \times \cdots \subset X \). Then \( A \) is a Borel subset of the cylinder \([B]_1^n\). Of course, \( T^{p_1 \cdots p_n} [B]_1^n = [B]_1^n \) and hence \( T^{p_1 \cdots p_n l_{n+1}} A \subset [B]_1^n \). Since

- \( T^{p_1 \cdots p_n} = I \) for each \( m = 1, \ldots, n \),
- \((d\mu_{n+1} \circ T^{p_1 \cdots p_n l_{n+1}})/(d\mu_{n+1})(x_{n+1}) = \lambda \) if \( x_{n+1} \in Z_{n+1} \) \( \lambda \) and
- \((d\mu_m \circ T^k)/(d\mu_m)(x_m) = 1 \) if \( x_m \in Y_m \) and \( 0 \leq k \leq p_1 \cdots p_{m-1} \) and every \( m > n + 1 \),

it follows that, for each \( x = (x_m)_{m=1}^\infty \in A \),

\[
\frac{d\mu \circ T^{p_1 \cdots p_n l_{n+1}}}{d\mu}(x) = \prod_{m=1}^\infty \frac{d\mu_m \circ T^{p_1 \cdots p_n l_{n+1}}}{d\mu_m}(x_m) = \lambda.
\]

We also note that \( \mu(A) > (\mu([B]_1^n)/2(\lambda + 1)) \prod_{m=1}^\infty (1 - 2^{-m-1}) \) in view of (7) and (8).

Since the family of Borel cylinders \([ [B]_1^n \mid B \subset X_1 \times \cdots \times X_n, n \in \mathbb{N} \] \) generates a dense subring of the entire Borel \( \sigma \)-algebra on \( X \), it follows from Lemma D that \( \log \lambda \) is an essential value of \( \rho_\mu \).

\( \square \)

4. Gaussian dynamical systems

4.1. Integration in Hilbert spaces. Let \( \mathcal{H} \) denote a separable infinite-dimensional real Hilbert space. Given a Borel probability measure \( \mu \) on \( \mathcal{H} \), we denote by \( \widehat{\mu} \) the characteristic functional of \( \mu \), that is,

\[
\widehat{\mu}(y) := \int_{\mathcal{H}} e^{i \langle x, y \rangle} d\mu(x), \quad y \in \mathcal{H}.
\]

We note that each Borel probability measure on \( \mathcal{H} \) is defined completely by its characteristic functional. If there is a vector \( h \in \mathcal{H} \) and a bounded linear operator \( B > 0 \) in \( \mathcal{H} \) such that \( \widehat{\mu}(y) = e^{i \langle h, y \rangle -(1/2)\langle By, y \rangle} \) for all \( y \in \mathcal{H} \) then \( \mu \) is called the (non-degenerate) Gaussian measure with covariance operator \( B \) and mean \( h \). Then, for each \( t \in \mathbb{R} \) and \( y \in \mathcal{H} \),

\[
\int_{\mathbb{R}} e^{its} d(\mu \circ (\cdot, y)^{-1})(s) = \int_{\mathcal{H}} e^{i \langle x, y \rangle} d\mu(x) = e^{it \langle h, y \rangle -(1/2)t^2 \langle By, y \rangle}.
\]
Therefore the continuous linear functional $\mathcal{H} \ni x \mapsto \langle x, y \rangle$ has normal distribution $\mathcal{N}_{\langle h, y \rangle, \langle By, y \rangle}$. In particular, each continuous linear functional belongs to $L^2(\mathcal{H}, \mu_B)$. By the Minlos–Sazonov theorem, $B$ is a nuclear operator, that is, $\text{tr}(B) < \infty$ [Sk, Theorem 1 and Example from §4]. Conversely, each strictly positive nuclear operator $B$ in $\mathcal{H}$ determines a unique Gaussian measure on $\mathcal{H}$ with zero mean and covariance operator $B$. We denote this measure by $\mu_B$. Thus $\widehat{\mu}_B(y) = e^{-(1/2)(By, y)}$ for all $y \in \mathcal{H}$. We note that $\int_{\mathcal{H}} \langle x, y \rangle d\mu_B(y) = 0$ for each $h \in \mathcal{H}$. It is well known (see, for instance, [DalFo, Ch. II, §1.2]) that

$$
\int_{\mathcal{H}} \langle x, y \rangle \langle z, y \rangle d\mu_B(y) = (Bx, z)
$$

and hence $\text{tr}(B) = \int_{\mathcal{H}} \|y\|^2 d\mu_B(y)$. We now let $\mathcal{H}_0 := B^{1/2} \mathcal{H} \subset \mathcal{H}$ and define an inner product and the corresponding norm on $\mathcal{H}_0$ by setting

$$
\langle x, y \rangle_0 := \langle B^{-1/2} x, B^{-1/2} y \rangle \text{ and } \|x\|_0^2 := \langle x, x \rangle_0 \quad \text{for } x, y \in \mathcal{H}_0.
$$

Then $(\mathcal{H}_0, \langle .., .. \rangle_0)$ is a Hilbert space. We now show that there is a canonical isometric embedding of $\mathcal{H}_0$ into $L^2(\mathcal{H}, \mu_B)$. To do so, we first take $\theta \in B\mathcal{H} \subset \mathcal{H}_0$. Then the mapping

$$
l_\theta : \mathcal{H} \ni y \mapsto \langle B^{-1} \theta, y \rangle
$$

is a continuous linear functional on $\mathcal{H}$. Moreover, for all $\theta, \eta \in B\mathcal{H}$,

$$
\langle l_\theta, l_\eta \rangle_{L^2(\mathcal{H}, \mu_B)} = \int_{\mathcal{H}} \langle B^{-1} \theta, y \rangle \langle B^{-1} \eta, y \rangle d\mu_B(y) = \langle \theta, B^{-1} \eta \rangle = \langle \theta, \eta \rangle_0.
$$

In particular, the linear mapping

$$
l : B\mathcal{H} \ni \theta \mapsto l_\theta \in L^2(\mathcal{H}, \mu_B)
$$

is isometric.† We note that $B\mathcal{H}$ is dense in $\mathcal{H}_0$. Indeed, since the linear span $\mathcal{L}$ of the orthonormal basis in $\mathcal{H}$ consisting of eigenvectors for $B$ is dense in $\mathcal{H}$, it follows that $B^{1/2} \mathcal{L}$ is dense in $\mathcal{H}_0$ because $B^{1/2}$ is an isometric isomorphism of $(\mathcal{H}, \langle .., .. \rangle)$ onto $(\mathcal{H}_0, \langle .., .. \rangle_0)$. All that remains is to observe that $B^{1/2} \mathcal{L} = B\mathcal{L} = \mathcal{L}$. Since $B\mathcal{H}$ is dense in $\mathcal{H}_0$, the isometry $l$ extends by continuity to an isometry from $\mathcal{H}_0$ to $L^2(\mathcal{H}, \mu_B)$. Thus, for each $y \in \mathcal{H}_0$, there is a sequence $(\theta_n)_{n=1}^\infty$ of elements from $B\mathcal{H}$ such that $\|y - \theta_n\| \to 0$ and the sequence $(l_{\theta_n})_{n=1}^\infty$ converges to some element $l_y \in L^2(\mathcal{H}, \mu_B)$. Hence a subsequence $(l_{\theta_{n_k}})_{k=1}^\infty$ converges to $l_y$ almost everywhere. Let $D_y$ denote the set of all $x \in \mathcal{H}$ such that the sequence $(l_{\theta_{n_k}}(x))_{k=1}^\infty$ converges. It is easy to verify that $D_y$ is a (Borel) linear subspace of $\mathcal{H}$, $\mu_B(D_y) = 1$ and $l_y$ is linear on $D_y$. That is why $l_y$ is often called a measurable linear functional on $\mathcal{H}$. Moreover, $D_y \supset \mathcal{H}_0$ and $l_y$ is defined uniquely by the restriction to $\mathcal{H}_0$ even though $\mu_B(\mathcal{H}_0) = 0$. It is often convenient to write $\langle B^{-1} y, x \rangle$ instead of $l_y(x)$ for $\mu_B$-a.e. $x \in \mathcal{H}$. We note that the distribution of $l_\theta$ is $\mathcal{N}_{\langle \theta, By - \theta \rangle} = \mathcal{N}_{\theta, \|y\|_0^2}$ for each $\theta \in B\mathcal{H}$. Passing to a limit, we obtain that the distribution of $l_y$ is $\mathcal{N}_{\theta, \|y\|_0^2}$ for each $y \in \mathcal{H}_0$.

† If $B\mathcal{H}$ is furnished with $\|..\|_0$. 

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For each \(y \in \mathcal{H}\), we denote by \(L_y\) the rotation by \(y\), that is, \(L_yx = x + y\) for all \(x \in \mathcal{H}\).

By the Cameron–Martin theorem (see [Gu, Corollary 7.4], [Sk]),

\[
\mathcal{H}_0 = \{ y \in \mathcal{H} \mid \mu_B \sim \mu_B \circ L_y^{-1} \} \quad \text{and, for each } y \in \mathcal{H}_0, \\
\frac{d \mu_B \circ L_y^{-1}}{d \mu_B}(x) = e^{(b^{-1}y,x) - (1/2)\|y\|^2_0} \quad \text{at a.e. } x \in \mathcal{H}.
\]  

(4.1)

4.2. Fock space and exponential map. Given a separable Hilbert space \(\mathcal{K}\), the \((bosonic)\) Fock space \(\mathcal{F}(\mathcal{K})\) built over \(\mathcal{K}\) is the Hilbert space \(\bigoplus_{n=0}^{\infty} \mathcal{K}^\otimes n\). The subspace \(\mathcal{K}^\otimes n\) of \(\mathcal{F}(\mathcal{K})\) is called the \(n\)-chaos in \(\mathcal{F}(\mathcal{K})\), \(n \in \mathbb{Z}_+\). Given \(h \in \mathcal{K}\), we let \(\exp_h := \bigoplus_{n=0}^{\infty}(h^\otimes n/\sqrt{n!}) \in \mathcal{F}(\mathcal{K})\). In particular, \(\exp_0 = (1, 0, 0, \ldots, )\) is called the \(vacuum vector\) in \(\mathcal{F}(\mathcal{H})\). The map \(\exp : \mathcal{K} \ni h \mapsto \exp_h \in \mathcal{F}(\mathcal{K})\) is called the \(exponential map\). It satisfies the following properties [Gu]:

(i) \(\exp\) is continuous;

(ii) \(\langle \exp_h, \exp_k \rangle_{\mathcal{F}(\mathcal{K})} = e^{(h,k)_\mathcal{K}}\) for all \(h, k \in \mathcal{K}\);

(iii) the set \(\{\exp_h \mid h \in \mathcal{K}\}\) is linearly independent and total in \(\mathcal{F}(\mathcal{K})\).

Given an orthogonal operator \(V\) in \(\mathcal{K}\), we can define a linear operator \(\exp V\) of \(\mathcal{F}(\mathcal{K})\), called the \(second quantization\) of \(V\), by setting

\[
(\exp V)h^\otimes n := (Vh)^\otimes n \quad \text{for all } n \geq 0 \text{ and } h \in \mathcal{K}.
\]

Then \(\exp V\) preserves each chaos in \(\mathcal{F}(\mathcal{K})\) and the restriction of \(\exp V\) to the first chaos is \(V\). Of course, \((\exp V) \exp_h = \exp_{Vh}\) for each \(h \in \mathcal{K}\). The most important property of the Fock spaces is the following: given a decomposition \(\mathcal{K} = \bigoplus_{j=1}^{\infty} \mathcal{K}_j\) of \(\mathcal{K}\) into an orthogonal sum of subspaces \(\mathcal{K}_j\), there is a unique unitary isomorphism \(\Phi\) of \((\mathcal{F}(\mathcal{K}), \exp_0)\) onto \(\bigotimes_{j=1}^{\infty}(\mathcal{F}(\mathcal{K}_j), \exp_0)\) such that

\[
\Phi(\exp \bigoplus_{j=1}^{\infty} h_j) = \bigotimes_{j=1}^{\infty} \Phi(\exp h_j)
\]

for each vector \(\bigoplus_{j=1}^{\infty} h_j \in \mathcal{K}\) such that \(h_j = 0\) for all but finitely many \(j\) [Gu, Proposition 2.3] (We consider the infinite tensor product in the category of Hilbert spaces furnished with unit vectors (see [Gu, Appendix A]). It is assumed that the unitary isomorphism in this category intertwines the corresponding unit vectors.).

Denote the orthogonal group of \(\mathcal{K}\) by \(\mathcal{O}(\mathcal{K})\). Let \(\text{Aff}(\mathcal{K}) := \mathcal{K} \rtimes \mathcal{O}(\mathcal{K})\) stand for the group of affine operators in \(\mathcal{K}\). We recall that an operator \(A = (f, V) \in \text{Aff}(\mathcal{K})\) acts on \(\mathcal{K}\) by the formula \(Ah := f + Vh\). One can verify that the multiplication law in \(\text{Aff}(\mathcal{K})\) is given by

\[
(f, V)(f', V') := (f + Vf', VV').
\]

We note that \(\text{Aff}(\mathcal{K})\) is a Polish group if endowed with the product of the norm topology on \(\mathcal{K}\) and the weak operator topology on \(\mathcal{O}(\mathcal{K})\). We recall the well-known \(Weyl\) \(unitary\) \(representation\) \(W = (W_{(f,V)})_{(f,V) \in \text{Aff}(\mathcal{K})}\) of \(\text{Aff}(\mathcal{K})\) in \(\mathcal{F}(\mathcal{K})\) [Gu, §2.2]:

\[
W_{(f,V)} \exp_h := e^{-(f,Vh)_\mathcal{K} - (1/2)\|f\|^2_\mathcal{K}} \exp_{f + Vh}, \quad h \in \mathcal{K}.
\]

(4.2)

It is well defined due to (ii) and (iii). Of course, \(W_{(0,V)} = \exp V\) for each \(V \in \mathcal{O}(\mathcal{K})\).
By [Gu, Theorem 7.1], there is a unique (canonical) unitary isomorphism of $L^2(\mathcal{H}, \mu_B)$ with $\mathcal{F}(\mathcal{H}_0)$ such that (For simplicity, we write $L^2(\mathcal{H}, \mu_B) = \mathcal{F}(\mathcal{H}_0)$ and hence identify $\exp_h$ with an $L^2$-function on $(\mathcal{H}, \mu_B)$, $h \in \mathcal{H}_0$.)

$$\exp_h(x) := e^{(B^{-1}h, x) - (1/2)\|h\|_0^2}, \quad \text{for a.e. } x \in \mathcal{H}. \quad (4.3)$$

Moreover, the map $\mathcal{H}_0 \ni h \mapsto l_h \in L^2(\mathcal{H}, \mu_B)$ identifies (isometrically) $\mathcal{H}_0$ with the first chaos in $L^2(\mathcal{H}, \mu_B)$. It follows from (4.1) and (4.3) that

$$\exp_h = \frac{d\mu_B \circ L^{-1}_h}{d\mu_B} \quad \text{for each } h \in \mathcal{H}_0. \quad (4.4)$$

It is straightforward to verify that the following additional properties for $\exp$ hold:

(iv) $\exp_h > 0$ for each $h \in \mathcal{H}_0$;
(v) $\exp_h \in \bigcap_{p=1}^{\infty} L^p(\mathcal{H}, \mu_B)$ because the map $\mathcal{H} \ni x \mapsto \langle B^{-1}h, x \rangle - (1/2)\|h\|_0^2$ has normal distribution $N_{-(1/2)\|h\|_0^2, \|h\|_0^2}$ and (4.3) holds;
(vi) $\|\exp_h\|_1 = 1$ for each $h \in \mathcal{H}_0$;
(vii) the cone $\{\sum_{k=1}^n a_k \exp_h \mid a_1, \ldots, a_n > 0, h_1, \ldots, h_n \in \mathcal{H}_0, n \in \mathbb{N}\}$ is dense in the cone $L^2_+(\mathcal{H}, \mu_B)$ of non-negative functions from $L^2(\mathcal{H}, \mu_B)$;
(viii) $\exp_h \cdot \exp_k = e^{(h,k)0} \exp_{h+k}$ for all $h, k \in \mathcal{H}_0$, and hence
(ix) $\sqrt{\exp_h} = e^{-(1/8)\|h\|^2} \exp_h/2$ for each $h \in \mathcal{H}_0$;
(x) $\exp_h \circ L_{f}^{-1} = e^{-(h,f)0} \exp_h = e^{-(h,f)0} \exp_h$ for all $h, f \in \mathcal{H}_0$.

**Remark 4.1.**

(i) We recall that $\mathcal{H}_0$ is determined by the pair $(\mathcal{H}, B)$ (see §4.1). Conversely, if $\mathcal{H}_0$ is given beforehand as an abstract Hilbert space, then it uniquely determines the probability space $(\mathcal{H}, \mu_B)$ for some pair $(\mathcal{H}, B)$ such that $\mathcal{H}_0 = B^{1/2} \mathcal{H}$. Indeed, if there is another Hilbert space $\mathcal{K}$ and a non-degenerated nuclear operator $C > 0$ on $\mathcal{K}$ such that the space $\mathcal{K}_0 := C^{1/2} \mathcal{K}$ furnished with the corresponding Hilbert norm is unitarily isomorphic to $\mathcal{H}_0$ via some unitary isomorphism $\Psi$ then, according to [Gu, Theorem 7.1] and (4.3), there is a unique unitary isomorphism $\Phi$ of $L^2(\mathcal{H}, \mu_B)$ with $L^2(\mathcal{K}, \mu_C)$ which maps $\exp_h$ onto $\exp_{\Phi^{-1}h}$ for each $h \in \mathcal{H}_0$. Hence in view of (vii), $\Phi$ maps $L^2_+(\mathcal{K}, \mu_C)$ onto $L^2_+(\mathcal{K}, \mu_C)$. Moreover, $\Phi 1 = 1$. Therefore $\Phi$ is *spatial*, that is, there is a measure-preserving isomorphism $\theta : (\mathcal{H}, \mu_B) \to (\mathcal{K}, \mu_C)$ such that $\Phi h = h \circ \theta^{-1}$ for each $h \in \mathcal{H}$.

(ii) Another useful observation is that, given a Hilbert space $\mathcal{K}_0$, there is another Hilbert space $\mathcal{K} \supseteq \mathcal{K}_0$ and a nuclear operator $C$ of $\mathcal{K}$ such that $C^{1/2}$ is a unitary isomorphism of $\mathcal{K}$ onto $\mathcal{K}_0$.

**Remark 4.2.** Given a decomposition $\mathcal{H}_0 = \bigoplus_{j=1}^{\infty} \mathcal{H}_0,j$ of $\mathcal{H}_0$ into an orthogonal sum of subspaces $\mathcal{H}_0,j$, consider the corresponding decomposition $\mathcal{H} = \bigoplus_{j=1}^{\infty} \mathcal{H}_j$ of $\mathcal{H}$ into an orthogonal sum of subspaces $\mathcal{H}_j := B^{-(1/2)} \mathcal{H}_0,j$, $j \in \mathbb{N}$. Let $P_j : \mathcal{H} \to \mathcal{H}_j$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_j$ and let $B_j := P_j B P_j^\ast$. Then $B_j : \mathcal{H}_j \to \mathcal{H}_j$ is a nuclear operator and $B_j^{1/2} \mathcal{H}_j = \mathcal{H}_0,j$ for each $j \in \mathbb{N}$. Moreover, $(\mathcal{H}, \mu_B)$ splits into the direct product $(\mathcal{H}, \mu_B) = \bigotimes_{j=1}^{\infty} (\mathcal{H}_j, \mu_B,j)$ of Gaussian probability spaces $(\mathcal{H}_j, \mu_B,j)$ in...
such a way that \{\exp_h \mid h \in \mathcal{H}_{0,j}\} is total in \(L^2(H_j, \mu_{B_j})\) and \(\mu_{B_j} = \mu_B \circ P_j^*\) for each \(j \in \mathbb{N}\).

4.3. Non-singular Gaussian action of \(\text{Aff} \, \mathcal{H}_0\). Let \((Y, \mathfrak{C}, \nu)\) be a standard non-atomic probability space. Denote by \(\mathcal{U}(L^2(Y, \nu))\) the group of unitary operators in \(L^2(Y, \nu)\) and by \(\mathcal{U}_\mathbb{R}(L^2(Y, \nu))\) the subgroup of unitaries that preserve the subspace \(L^2_\mathbb{R}(Y, \nu)\) of real-valued functions in \(L^2(Y, \nu)\). Let

\[
U : \text{Aut}(Y, \nu) \ni T \mapsto U_T \in \mathcal{U}_\mathbb{R}(L^2(Y, \nu))
\]

stand for the unitary Koopman representation of \(\text{Aut}(Y, \nu)\) in \(L^2(Y, \nu)\). We recall that

\[
U_T f := f \circ T^{-1}/((d\mu \circ T^{-1})/d\mu) \quad \text{for all } f \in L^2(Y, \nu).
\]

The following results are well known:

1. \(\{U_T \mid T \in \text{Aut}(Y, \nu)\} = \{V \in \mathcal{U}_\mathbb{R}(L^2(Y, \nu)) \mid VL^2_+(Y, \nu) = V_L^2(Y, \nu)\}\);
2. \(\{U_T \mid T \in \text{Aut}_0(Y, \nu)\} = \{V \in \mathcal{U}_\mathbb{R}(L^2(Y, \nu)) \mid VL^2_+(Y, \nu) = V_L^2(Y, \nu), V1 = 1\}\).

We also note that \(U\) is one-to-one and the image of \(U\) is closed in \(\mathcal{U}_\mathbb{R}(L^2(Y, \nu))\) in the weak (and the strong) operator topology.

Let \(\mathbb{R}^+\) denote the multiplicative group of reals. It is straightforward to verify that, for each \(t \in \mathbb{R}^+\), the map \(\alpha_t : \text{Aff}(\mathcal{H}_0) \to \text{Aff}(\mathcal{H}_0)\) given by

\[
(f, V) \mapsto \alpha_t(f, V) := (tf, V)
\]

is a continuous automorphism of \(\text{Aff}(\mathcal{H}_0)\). Moreover, \(\alpha_{t_1} \alpha_{t_2} = \alpha_{t_1 t_2}\) for all \(t_1, t_2 \in \mathbb{R}^+\).

It is straightforward to verify that, for each \(A \in \text{Aff}(\mathcal{H}_0)\), the corresponding Weyl unitary operator \(W_A\) (see (4.2)) preserves the cone

\[
\left\{ \sum_{k=1}^n a_k \exp_{h_k} \mid a_k > 0, h_k \in \mathcal{H}_0, \text{ for each } k = 1, \ldots, n \text{ and } n \in \mathbb{N} \right\}.
\]

Hence it preserves \(L^2_+(\mathcal{H}, \mu_B)\) in view of (vii) from §4.2. Therefore, by (1), there is a (unique) transformation \(T_A \in \text{Aut}(\mathcal{H}, \mu_B)\) such that \(U_{T_A} = W_{\alpha_{1/2}(A)}\).

**Definition 4.3.** \(T_A\) is called the non-singular Gaussian transformation generated by \(A \in \text{Aff}(\mathcal{H}_0)\).

Since the image of \(\text{Aff}(\mathcal{H}_0)\) under the unitary Weyl representation is closed in the unitary group of the space \(L^2(\mathcal{H}, \mu_B)\) [Gu, Theorem 2.1], it follows that the group \(\{T_A \mid A \in \text{Aff}(\mathcal{H}_0)\}\) of non-singular Gaussian transformations is closed in \(\text{Aut}(\mathcal{H}, \mu_B)\).

**Proposition 4.4.**

(i) If \(V \in \mathcal{O}(\mathcal{H}_0)\) then \(T_{(0,V)}\) is the usual (classic) measure-preserving Gaussian transformation generated by the orthogonal operator \(V\), that is, \(U_{T_{(0,V)}} = \exp V\) (see [LePaTh, Lemma 2]).

(ii) If \(f \in \mathcal{H}_0\) then \(T_{(f,1)} = L_f\).

**Proof.** (i) We note that

\[
U_{T_{(0,V)} \exp h} = W_{(0,V)} \exp h = \exp V \exp h = \exp V \exp h.
\]

Hence \(U_{T_{(0,V)}} = \exp V\).
(ii) Using (4.4) and (viii)–(x) from 4.2, we obtain that
\[
U_{L_f} \exp_h = \sqrt{\frac{d\mu_B \circ L_f^{-1}}{d\mu_B}} \exp_h \circ L_f^{-1} \\
= \sqrt{\exp_f e^{-(f,h)_{0}}} \exp_h \\
= e^{-(1/8)f h_{0}^2} \exp_{f/2} e^{-(h,f)_{0}} \exp_h \\
= e^{-(1/8)f h_{0}^2-(f,h)_{0}+(1/2)(f,h)_{0}} \exp_{f/2+h}.
\]
Hence \(U_{L_f} \exp_h = e^{-(1/8)f h_{0}^2-(1/2)(f,h)_{0}} \exp_{f/2+h} = W(f/2,I) \exp_h = U_{T(f,I)} \exp_h\). It follows that \(T(f,I) = L_f\).

**COROLLARY 4.5.** Every non-singular Gaussian transformation \(T(f,V)\) is the composition of the classic \(\mu_B\)-preserving Gaussian transformation \(T(0,V)\) and a \(\mu_B\)-non-singular translation \(L_f = T(f,I)\) which is totally dissipative (Let \(K\) stand for the orthogonal complement in \(H\) to the one-dimensional subspace generated by \(f\). Then the set \(\{sf + k \mid 0 \leq s < 1, k \in K\} \subset H\) is a Borel fundamental domain for \(L_f\)). These two transformations commute if and only if \(Vf = f\).

**Remark 4.6.** Let \((X, \mathcal{B}, \mu)\) be a standard \(\sigma\)-finite non-atomic measure space. Let a transformation \(S \in \text{Aut}(X, \mu)\) be such that \(\sqrt{(d\mu \circ S^{-1})/d\mu} - 1 \in L^2(X, \mu)\). Then a non-singular Poisson suspension \(S_\gamma\) of \(S\) is well defined on a standard probability space \((X^*, \mathcal{B}^*, \mu^*)\) [DaKoRo1]. Let \(A := (U_S, \sqrt{(d\mu \circ S^{-1})/d\mu} - 1) \in \text{Aff}(L^2(X, \mu))\). It was shown in [DaKoRo1] that \(U_{S_\gamma}\) is unitarily equivalent to \(W_A\). It follows that each non-singular Poisson transformation is unitarily equivalent to a non-singular Gaussian transformation: \(S_\gamma\) is unitarily equivalent to \(T_{\alpha_2(A)}\) (see (4.5)). The converse is not true even in the classic (finite measure preserving) case: there is no \(\Pi_\infty\) automorphism which is spectrally isomorphic to a Gaussian–Kronecker automorphism (see [Ro, Theorem 4.13]).

It is well known that the transformation group \(\{T(f,I) \mid f \in \mathcal{H}_0\} \subset \text{Aut}(\mathcal{H}, \mu_B)\) is ergodic (see [Gu, Sk]). However, its Krieger’s type has not been determined so far. We will show that it is type III\(_1\), that is, a dense countable subgroup of it is of type III\(_1\) (hence every dense countable subgroup is of type III\(_1\)).

**THEOREM 4.7.** \(\{T(f,I) \mid f \in \mathcal{H}_0\}\) is of type III\(_1\).

**Proof.** Let \(\{e_n \mid n \in \mathbb{N}\}\) be an orthonormal basis of \(\mathcal{H}\) consisting of the eigenvectors of \(B\). As \(B\) is positive and nuclear, \(Be_n = \lambda_n e_n, \lambda_n > 0\) for each \(n \in \mathbb{N}\) and \(\sum_{n=1}^{\infty} \lambda_n < \infty\). Denote by \(\Gamma\) the group generated by translations \(L_{\sqrt{\lambda}} e_k\) for all \(k \in \mathbb{N}\). Then \(\Gamma\) is an ergodic countable abelian subgroup of \(\text{Aut}(\mathcal{H}, \mu_B)\). We will show that \(\Gamma\) is of type III\(_1\).

Denote by \(\mathcal{B}_n\) the smallest Borel \(\sigma\)-algebra on \(\mathcal{H}\) such that the map \(\mathcal{H} \ni x \mapsto \langle x, e_k \rangle \in \mathbb{R}\) is \(\mathcal{B}_n\)-measurable for each \(k = 1, \ldots, n\). Then \(\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots\) and the union \(\bigcup_{n>0} \mathcal{B}_n\) is dense in \(\mathcal{B}\). We deduce from (4.3) and (4.4) that, for each \(n > 0\),
\[
\log \frac{d\mu_B \circ L_{\sqrt{\lambda_n}} e_n}{d\mu_B}(x) = \frac{\langle x, e_{n+1} \rangle}{\sqrt{\lambda_{n+1}}} - \frac{1}{2}.
\]
Take $a \in \mathbb{R}$ and $\epsilon > 0$. We now let

$$D_n := \left\{ x \in \mathcal{H} \ \bigg| \ a + \frac{1}{2} - \epsilon < \frac{\langle x, e_{n+1} \rangle}{\sqrt{\lambda_{n+1}}} < a + \frac{1}{2} + \epsilon \right\}.$$ 

Since $e_{n+1} \perp e_k$ for each $k = 1, \ldots, n$ and the random variable $\mathcal{H} \ni x \mapsto \langle x, e_k \rangle \in \mathbb{R}$ is Gaussian for all $k = 1, \ldots, n+1$ (and the joint distribution is also Gaussian), it follows that $D_n$ is independent of $\mathfrak{B}_n$. Moreover, the measure

$$\mu_B(D_n) = \frac{1}{\sqrt{2\pi}\lambda_{n+1}} \int_{(a+(1/2) - \epsilon, a+(1/2) + \epsilon)\cdot\sqrt{\lambda_{n+1}}} e^{-t^2/2\lambda_{n+1}} \, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{(a+(1/2) - \epsilon, a+(1/2) + \epsilon)} e^{-t^2/2} \, dt$$

of $D_n$ does not depend on $n$ (We use here the fact that the random variable $\langle \cdot, e_{n+1} \rangle$ has normal distribution $N(0,\lambda_{n+1})$). We denote it by $\delta > 0$. Since each subset $A \in \mathfrak{B}_n$ depends only on the ‘first $n$ coordinates’ $x_1, \ldots, x_n$ while the translation $L_{e_{n+1}}$ changes only the $(n+1)$th coordinate, we have that $L_{e_{n+1}} A = A$ and hence

- $(A \cap D_n) \cup L_{e_{n+1}}(A \cap D_n) \subset A$,
- $\mu_B(A \cap D_n) = \mu_B(A) \mu_B(D_n) = \delta \mu_B(A)$ and
- $\log((d\mu_B \circ L_{e_{n+1}})/d\mu_B)(x) = a \pm \epsilon$ for each $x \in A \cap D_n$.

It follows from Lemma D that $a$ is an essential value of the logarithm of the Radon–Nikodym cocycle of $\Gamma$. Since $a$ is an arbitrary element of $\mathbb{R}$, the Radon–Nikodym cocycle is ergodic, that is, $\Gamma$ is of type III$_1$. \hfill $\square$

4.4. When non-singular Gaussian systems are of type III$_1$. We recall a standard definition.

**Definition 4.8.** Given $V \in \mathcal{O}(\mathcal{H}_0)$, we say that a vector $f \in \mathcal{H}_0$ is a $V$-coboundary if there is $a \in \mathcal{H}_0$ such that $f = a - V a$.

In this subsection we prove the following statement (cf. [DaKoRo1, Proposition 6.4] and [ArIsMa]).

**Theorem 4.9.** Let $(f, V) \in \text{Aff}(\mathcal{H}_0)$. For $n \in \mathbb{Z}$, we define $f^{(n)} \in \mathcal{H}_0$ by setting $(f, V)^n = (f^{(n)}, V^n)$. The following statements are equivalent.

(i) $T_{(f, V)}$ admits an equivalent invariant probability measure.

(ii) $f$ is a $V$-coboundary.

(iii) The affine operator $(f, V)$ has a fixed point.

(iv) The sequence $(f^{(n)})_{n \in \mathbb{Z}}$ is bounded in $\mathcal{H}_0$.

**Proof.** (ii) $\iff$ (iv) is classic; see [BeKaVal, Proposition 2.2.9], for a proof.

(iii) is obvious because the equality $(f, V)a = a$ for some $a \in \mathcal{H}_0$ means $f + V a = a$, that is, $f$ is a $V$-coboundary.

(ii) $\implies$ (i). In view of Proposition 4.4 and Corollary 4.5,

$$\frac{d\mu_B \circ T_{(f, V)}^{-1}}{d\mu_B} = \frac{d(\mu_B \circ T_{(0, V)}^{-1}) \circ T_{(f, I)}^{-1}}{d\mu_B} = \frac{d\mu_B \circ T_{(f, I)}^{-1}}{d\mu_B} = \frac{d\mu_B \circ L_{f}^{-1}}{d\mu_B}.$$
Therefore, by (4.4), we obtain that
\[
\frac{d\mu_B \circ T_{(f,V)}^{-1}}{d\mu_B} = \exp_f.
\]

Let \( f = a - Va \) for some \( a \in \mathcal{H_0} \). We claim that
\[
\exp_f = \frac{\exp_a}{\exp_a \circ T_{(f,V)}^{-1}}.
\]  
(4.6)

Indeed, applying Proposition 4.4 and (viii) and (x) from 4.2, we obtain that
\[
\exp_f \exp_a \circ T_{(f,V)}^{-1} = \exp_f ((\exp V) \exp_a) \circ L_f^{-1} = \exp_f \exp V_a e^{-(Vaf)} = \exp_{f+Va} = \exp a.
\]

Since \( \exp a \in L^1(\mathcal{H}, \mu_B) \), (i) follows from (4.6).

(i) \( \implies \) (iv). We first note that, for each \( h \in \mathcal{H_0} \),
\[
\| \exp h \|_1 = e^{-\langle h, h \rangle / 8}.
\]

The rest of the argument is almost a repetition of the proof that (4) implies (1) in [DaKoRo1, Proposition 6.4]. Suppose that the sequence \( (f^{(n)})_{n=1}^\infty \) is unbounded. Then there is an increasing sequence \( n_1 < n_2 < \cdots \) such that \( \| f^{(n_k)} \|_0^2 \to +\infty \) as \( k \to \infty \). Hence \( \langle (U_{T(f,V)})^{n_k} 1, 1 \rangle \to 0 \) as \( k \to \infty \). Since the operator \( U_{T(f,V)} \) is positive with respect to the cone \( L^2_+ (\mathcal{H}, \mu_B) \), it follows that \( U_{T(f,V)}^{n_k} \to 0 \) weakly as \( k \to \infty \). Since \( T_{(f,V)} \) admits an equivalent invariant probability measure, \( U_{T(f,V)} \) is unitarily equivalent to the Koopman operator of a probability-preserving transformation. The latter does not have subsequences weakly converging to zero because 1 is a fixed point of this operator.

Remark 4.10. In fact, we showed more: if \( f = a - Va \) and \( \nu \) is a \( \mu_B \)-equivalent \( T_{(f,V)} \)-invariant measure then \( d\nu / d\mu_B = \exp a \).
4.5. Gaussian transformations as IDPFT systems. Suppose that we are given an affine operator \((f, V) \in \text{Aff}(\mathcal{H}_0)\). Suppose also that \(V\) has no non-trivial invariant vectors\(^\dagger\). Using the spectral decomposition of \(V\), as in the proof of Theorem 2.6, we can choose an orthogonal decomposition \(\mathcal{H}_0 = \mathop{\bigoplus}_{r=1}^{\infty} \mathcal{H}_{0,r}\) of \(\mathcal{H}_0\) in such a way that \(V \mathcal{H}_{0,r} = \mathcal{H}_{0,r}\) and the orthogonal projection \(f_r\) of \(f\) onto \(\mathcal{H}_{0,r}\) is a \(V\)-coboundary for each \(r \in \mathbb{N}\). Let \(V_r := V \upharpoonright \mathcal{H}_r\). Then \((f_r, V_r) \in \text{Aff}(\mathcal{H}_{0,r})\) for each \(r \in \mathbb{N}\) and \((f, V) = \mathop{\bigoplus}_{r=1}^{\infty} (f_r, V_r)\). Let \(\mathcal{H}_r\) and \(\mu_r\) stand for the Hilbert space and a Gaussian measure on \(\mathcal{H}_r\) respectively such that \(\mathcal{F}(\mathcal{H}_{0,r})\) is canonically isomorphic to \(L^2(\mathcal{H}_r, \mu_r)\) (see Remark 4.1(ii)). Then the standard probability space \((\mathcal{H}, \mu_B, \mathcal{H}(f_r, V_r))\) according to Remark 4.2. It follows that

\[
(\mathcal{H}, \mu_B, T(f, V)) = \mathop{\bigotimes}_{r=1}^{\infty} (\mathcal{H}_r, \mu_r, T(f_r, V_r)).
\] (4.7)

Since \(f_r\) is a \(V_r\)-coboundary, there is \(a_r \in \mathcal{H}_{0,r}\) such that \(f_r = a_r - V_r a_r\) for each \(r \in \mathbb{N}\). By Theorem 4.9, the system \((\mathcal{H}_r, \mu_r, T(f_r, V_r))\) admits an equivalent invariant probability measure \(v_r\). Moreover, \(d\mu_r / dv_r = \exp_{-a_r}\) for each \(r \in \mathbb{N}\) in view of Remark 4.10. Thus, we have shown that each non-singular Gaussian dynamical system \((\mathcal{H}, \mu_B, T(f, V))\) such that \(V\) has no non-trivial invariant vectors is IDPFT (see (4.7)). Therefore, Corollary 3.7 yields the following result.

**Corollary 4.11.** If \(V\) has no non-trivial invariant vectors then the non-singular Gaussian dynamical system \((\mathcal{H}, \mu_B, T(f, V))\) is either conservative or totally dissipative. In fact, if \((Y, \mathcal{E}, \nu, S)\) is an ergodic conservative non-singular dynamical system then the direct product \(T(f, V) \times S\) is either conservative or totally dissipative.

The following theorem was first proved in [ArIsMa] in the case of mixing \(V\). We extend it to the mildly mixing case with a different proof.

**Theorem 4.12.** Let \(T(0, V)\) be mildly mixing and let \(f\) not be a \(V\)-coboundary. If \(T(f, V)\) is conservative then the Maharam extension of \(T(f, V)\) is sharply weak mixing. In particular, \(T(f, V)\) is of type III\(_1\).

**Proof.** Since \(T(f, V)\) is conservative, it follows from (4.7) and Proposition 3.3 that \(T(f, V)\) is sharply weak mixing. Let \(a_r, \mu_r\) and \(v_r\) be as above in this subsection. Since \(d\mu_r / dv_r = \exp_{-a_r}\) for each \(r \in \mathbb{N}\), it follows from (4.1) and (4.4) that the distribution \(\psi_r\) of \(\log(d\mu_r / dv_r)\) defined on \((\mathcal{H}_{0,r}, v_r)\) is \(N_{-\|a_r\|_0^2/2,\|a_r\|_0^2}\). Hence, for all \(m > n\), we have

\[
\psi_{n+1} \ast \cdots \ast \psi_m = N_{-0.5 \sum_{r=n+1}^{m} \|a_r\|_0^2, \sum_{r=n+1}^{m} \|a_r\|_0^2}
\]

and

\[
\psi_{n+1} \ast \cdots \ast \psi_m \ast \delta_{a} = N_{-0.5 \sum_{r=n+1}^{m} \|a_r\|_0^2, \sum_{r=n+1}^{m} \|a_r\|_0^2}
\]

\(^\dagger\) Equivalently, the measure of maximal spectral type of \(V\) has no atom at 1.
for each \( a \in \mathbb{R} \). We will show that the sequence \( (\psi_t)_{t=1}^{\infty} \) is asymptotically translation quasi-invariant. First, it is straightforward to verify that, for each \( \sigma \in \mathbb{R} \) and \( b > 0 \),

\[
\log \left( \frac{dN_{b-\sigma^2/2, \sigma^2}}{dN_{-\sigma^2/2, \sigma^2}}(t) \right) = \frac{b(2t + \sigma^2 - b)}{2\sigma^2} = \frac{bt}{\sigma^2} + \frac{b(\sigma^2 - b)}{2\sigma^2}, \quad t \in \mathbb{R}.
\]

Hence, if \( t \geq -\sigma^2 \) and \( \sigma^2 \geq 2b \) then

\[
\frac{dN_{b-\sigma^2/2, \sigma^2}}{dN_{-\sigma^2/2, \sigma^2}}(t) \geq e^{-b + (b(\sigma^2 - b)/2\sigma^2)} > e^{-(3b/4)}.
\]  

(4.8)

Moreover,

\[
\int_{-\sigma^2}^{+\infty} dN_{-\sigma^2/2, \sigma^2}(t) = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\sigma^2}^{+\infty} e^{-(1/2)((t+\sigma^2)/\sigma)^2} dt = \frac{1}{\sigma \sqrt{2\pi}} \int_{-\sigma^2/2}^{+\infty} e^{-(t^2)/2\sigma^2} dt = \frac{1}{\sqrt{2\pi}} \int_{-\sigma/2}^{+\infty} e^{-(t^2/2)} dt,
\]

that is, \( N_{-\sigma^2/2, \sigma^2}((-\sigma^2, +\infty)) = N_{0,1}((-\sigma/2, +\infty)) \). Obviously, we have

\[
\psi_{n+1} \ast \cdots \ast \psi_m \ast \delta_a \sim \psi_{n+1} \ast \cdots \ast \psi_m \quad \text{for all } n < m.
\]

We now set \( \xi_a := e^{-(3a/4)} \). Next, we note that \( f \) is not a \( V \)-coboundary if and only if \( \sum_{r=1}^{\infty} \|a_r\|_0^2 = \infty \). Hence for each \( n > 0 \), there is \( m > n \) such that \( \sum_{r=n+1}^{m} \|a_r\|_0^2 > 2a \).

Let \( W_{n+1,m} := \{ - \sum_{r=n+1}^{m} \|a_r\|_0^2, +\infty \} \subset \mathbb{R} \). Then (4.8) yields that

\[
\frac{d(\psi_{n+1} \ast \cdots \ast \psi_m \ast \delta_a)}{d(\psi_{n+1} \ast \cdots \ast \psi_m)}(t) \geq \xi_a \quad \text{for all } t \in W_{n+1,m}.
\]

Moreover, \( (\psi_{n+1} \ast \cdots \ast \psi_m)(W_{n+1,m}) = N_{0,1}((-0.5 \sqrt{\sum_{r=n+1}^{m} \|a_r\|_0^2}, +\infty)) \approx 1 \) if \( m \) is large. Hence \( (\psi_t)_{t=1}^{\infty} \) is asymptotically translation quasi-invariant. It follows now from Theorem 3.10 that the Maharam extension of \( T(f,V) \) is sharply weak mixing.

\[ \square \]

4.6. One-parametric family of non-singular Gaussian systems. We note that (4.5) determines a one-to-one homomorphism \( \mathbb{R}^+ \ni t \mapsto \alpha_t \) from the multiplicative group \( \mathbb{R}^+ \) to the group of continuous automorphisms of \( \text{Aff}(\mathcal{H}_0) \). Therefore, for each \( A \in \text{Aff}(\mathcal{H}_0) \), one can consider a one-parametric family of non-singular Gaussian transformations \( T_{\alpha_t}(A) \in \text{Aut}(\mathcal{H}, \mu_B), t \in \mathbb{R}^+ \). Our purpose is this section is to investigate how the dynamical properties of \( T_{\alpha_t}(A) \) depend on \( t \). It is straightforward to verify that the linear operator \(-I \) of \( \mathcal{H} \) preserves \( \mu_B \) and conjugates \( T_{\alpha_t}(A) \) with \( T_{\alpha_{-t}}(A) \). Therefore it suffices to consider only the transformations \( T_{\alpha_{t}}(A) \) with \( t \in \mathbb{R}^+ \).

**Proposition 4.13.** [ArIsMa] Given \( A = (f, V) \in \text{Aff}(\mathcal{H}_0) \) such that \( V \) has no non-zero invariant vectors, there is \( t_{\text{diss}}(A) \in [0, +\infty) \) such that the transformation \( T_{\alpha_t}(A) \) is conservative if \( 0 < t < t_{\text{diss}}(A) \) and totally dissipative if \( t > t_{\text{diss}}(A) \).

\[ \text{T} \]

We note that the map \( \mathbb{R}^+ \ni t \mapsto T_{\alpha_t}(A) \) is not a group homomorphism.
Therefore, there is PROPOSITION 4.14. 

\[ n > N \implies x \] for all 

\[ \sum \] \( \delta A \) at a.e.

Hence, there is \( N_x > 0 \) such that \( \langle B^{-1} f^{(n)}, x \rangle - \frac{1}{2} \| f^{(n)} \|_{0}^2 < 0 \) for all \( n > N_x \). It follows that 

\[ \langle t B^{-1} f^{(n)}, x \rangle - \frac{t^2 \| f^{(n)} \|^2_{0}}{2} < t \left( \langle B^{-1} f^{(n)}, x \rangle - \frac{\| f^{(n)} \|^2_{0}}{2} \right) < \langle B^{-1} f^{(n)}, x \rangle - \frac{\| f^{(n)} \|^2_{0}}{2} \]

for all \( n > N_x \). Hence \( \sum_{n=0}^{\infty} e^{t(B^{-1} f^{(n)}, x) - (t^2/2) \| f^{(n)} \|^2_{0}} < \infty \) for \( \mu_B \)-a.e. \( x \in \mathcal{H} \). Since \( tf^{(n)} = (tf)^{(n)} \), we deduce from the Hopf criterion that \( T_{\alpha t}(A) \) is dissipative, as desired.

We recall that the Poincaré exponent of \( A = (f, V) \in \text{Aff}(\mathcal{H}_0) \) \([\text{ArIsMa}]\) is 

\[ \delta_A := \inf \left\{ \alpha > 0 \mid \sum_{n=1}^{\infty} e^{-\alpha \| f^{(n)} \|^2_{0}} < +\infty \right\} \in [0, +\infty] \].

For completeness of our argument we give a proof of the following proposition.

**PROPOSITION 4.14.** \([\text{ArIsMa}]\) \( \sqrt{2\delta_A} \leq t_{\text{diss}}(A) \leq 2\sqrt{\delta_A} \).

**Proof** \([\text{ArIsMa}]\). Let \( t > t_{\text{diss}}(A) \). Since \( T_{\alpha t}(A) \) is isomorphic to \( T_{\alpha t'}(A) \), the two transformations are dissipative. Therefore, by the Hopf criterion, 

\[ \sum_{n=0}^{\infty} e^{t(B^{-1} f^{(n)}, x) - (t^2/2) \| f^{(n)} \|^2_{0}} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} e^{-t(B^{-1} f^{(n)}, x) - (t^2/2) \| f^{(n)} \|^2_{0}} < \infty \]

at a.e. \( x \). Since \( e^{t(B^{-1} f^{(n)}, x)} + e^{-t(B^{-1} f^{(n)}, x)} \geq 2 \) for each \( x \in X \), it follows that \( \sum_{n=0}^{\infty} e^{-t^2/2 \| f^{(n)} \|^2_{0}} < \infty \), that is, \( \delta_A \leq t^2/2 \) and hence \( \delta_A \leq t_{\text{diss}}(A)^2/2 \).

On the other hand, if \( t < t_{\text{diss}}(A) \) then \( T_{\alpha t}(A) \) is conservative and hence 

\[ \sum_{n=0}^{\infty} e^{t(B^{-1} f^{(n)}, x) - (t^2/2) \| f^{(n)} \|^2_{0}} = +\infty. \]

Therefore, 

\[ +\infty = \sum_{n=0}^{\infty} \int_{\mathcal{H}} e^{(1/2)t(B^{-1} f^{(n)}, x) - (t^2/4) \| f^{(n)} \|^2_{0}} d\mu_B(x) = \sum_{n=0}^{\infty} e^{-(t^2/8) \| f^{(n)} \|^2_{0}}. \]

Hence \( \delta_A \geq (t^2/8) \) and therefore \( \delta_A \geq (t_{\text{diss}}(A)^2/8) \).

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