A GENUINELY STABLE LAGRANGE–GALERKIN SCHEME FOR CONVECTION-DIFFUSION PROBLEMS

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Abstract. We present a Lagrange–Galerkin scheme free from numerical quadrature for convection-diffusion problems. Since the scheme can be implemented exactly as it is, theoretical stability result is assured. While conventional Lagrange–Galerkin schemes may encounter the instability caused by numerical quadrature error, the present scheme is genuinely stable. For the $P_k$-element we prove error estimates of $O(\Delta t + h^{2 + k + 1})$ in $\ell^\infty(L^2)$-norm and of $O(\Delta t + h^{2 + k})$ in $\ell^\infty(H^1)$-norm. Numerical results reflect these estimates.

1. Introduction

The Lagrange–Galerkin method, which is also called characteristics finite element method or Galerkin-characteristics method, is a powerful numerical method for flow problems such as the convection-diffusion equations and the Navier–Stokes equations. In this method the material derivative is discretized along the characteristic curve, which originates the robustness for convection-dominated problems. Although, as a result of the discretization along the characteristic curve, a composite function term at the previous time appears, it is converted to the right-hand side in the system of the linear equations. Thus, the coefficient matrix in the left-hand side is symmetric, which allows us to use efficient linear solvers for symmetric matrices such as the conjugate gradient method and the minimal residual method.

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Stability and error analysis of LG schemes has been done in [1, 3, 4, 6, 9, 10, 11, 12, 13, 14, 16]; see also the bibliography therein. Pironneau [11] analyzed convection-diffusion problems and the Navier–Stokes equations to obtain suboptimal convergence results. Optimal convergence results were obtained by Douglas–Russell [6] for convection-diffusion problems and by Süli [16] for the Navier–Stokes equations. Optimal convergence results of second order in time were obtained by Boukir et al. [4] for the Navier–Stokes equations in multi-step method and by Rui–Tabata [14] for convection-diffusion problems in single-step method. All these theoretical results are derived under the condition that the integration of the composite function term is computed exactly. Since, in real problems, it is difficult to get the exact integration value, numerical quadrature is usually employed. It is, however, reported that instability may occur caused by numerical quadrature error in [9, 17, 18]. That is, the theoretical stability results may collapse by the introduction of numerical quadrature.

Several methods have been studied to avoid the instability. The map of a particle from a time to the previous time along the trajectory, which is nothing but to solve a system of ordinary differential equations (ODEs), is simplified in [3, 13]. Morton–Priestley–Suli [9] solved the ODEs only at the centroids of the elements, and Priestley [13] did only at the vertices of the elements. The map of the other points is approximated by linear interpolation of those values. It becomes possible to perform the exact integration of the composite function term with the simplified map. Bermejo–Saavedra [3] used the same simplified map as [13] to employ a numerical quadrature of high accuracy to the composite function term. Tanaka–Suzuki–Tabata [19] approximated the map by a locally linearized velocity and the backward Euler approximation for the solution of the ODEs in $P_1$-element. The approximate map makes possible the exact integration of the composite function term with the map. Pironneau–Tabata [12] used mass lumping in $P_1$-element to develop a scheme free from quadrature for convection-diffusion problems.

In this paper we prove the stability and convergence for the scheme with the same approximate map as [19] in $P_k$-element for convection-diffusion problems. Since we neither solve the ODEs nor use numerical quadrature, our scheme can be precisely implemented to realize the theoretical results. It is, therefore, a genuinely stable Lagrange–Galerkin scheme. Our convergence results are of $O(\Delta t + h^2 + h^k)$ in $\ell^\infty(L^2)$-norm and of $O(\Delta t + h^2 + h^k)$ in $\ell^\infty(H^1)$-norm. They are best possible in both norms for $P_1$-element and in $\ell^\infty(H^1)$-norm for $P_2$-element.

The contents of this paper are as follows. In the next section we describe the convection-diffusion problem and some preparation. In section 3 after recalling the conventional Lagrange–Galerkin scheme, we present our genuinely stable Lagrange–Galerkin scheme. In section 4 we show stability and convergence results, which are proved in section 5. In section 6 we show some numerical results, which reflect the theoretical convergence order. In section 7 we give conclusions.

2. Preliminaries

We state the problem and prepare notation used throughout this paper.

Let $\Omega$ be a polygonal or polyhedral domain of $\mathbb{R}^d$ ($d = 2, 3$) and $T > 0$ be a time. We use the Sobolev spaces $L^p(\Omega)$ with the norm $\|\cdot\|_{0,p}$, $W^{s,p}(\Omega)$ and $W^{0,p}_0(\Omega)$ with the norm $\|\cdot\|_{s,p}$ and the semi-norm $|\cdot|_{s,p}$ for $1 \leq p \leq \infty$ and a positive integer $s$. We will write $H^s(\Omega) = W^{s,2}(\Omega)$ and drop the subscript $p = 2$ in the corresponding
norms. The $L^2$-norm $\|\cdot\|_0$ is simply denoted by $\|\cdot\|$. The dual space of $H^1_0(\Omega)$ is denoted by $H^{-1}(\Omega)$. For the vector-valued function $w \in W^{1,\infty}(\Omega)^d$ we define the semi-norm $|w|_{1,\infty}$ by

$$\left\| \left\{ \sum_{i,j=1}^d \left( \frac{\partial u_i}{\partial x_j} \right)^2 \right\}^{1/2} \right\|_{0,\infty}.$$ 

The parenthesis $(\cdot, \cdot)$ shows the $L^2$-inner product $(f,g) \equiv \int_{\Omega} fg \, dx$. For a Sobolev space $X(\Omega)$ we use abbreviations $H^m(\Omega) = H^m(0,T; X(\Omega))$ and $C(\Omega) = C([0,T]; X(\Omega))$.

We define a function space $X$ where $\| \cdot \|_X$ is the boundary of $\Omega$ and $\nu > 0$ be a time increment, $\| \cdot \|$ defined in $\Omega$ are denoted by $\| \cdot \|_H(\Omega) \equiv \| \cdot \|_{H^1(\Omega)}$. For the vector-valued function $u \in L^1_{\text{loc}}(\Omega)$, we define the semi-norm $\|u\|_{1,\text{loc}}(\Omega)$ by

$$\|u\|_{1,\text{loc}}(\Omega) \equiv \sum_{m=0}^n \|u_m\|_{L^1(\Omega)}.$$ 

and denote $Z^m(0,T)$ by $Z^m$.

We consider the convection-diffusion problem: find $\phi : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi - \nu \Delta \phi = f, \quad (x,t) \in \Omega \times (0,T),$$

$$\phi = 0, \quad (x,t) \in \partial \Omega \times (0,T),$$

$$\phi = \phi^0, \quad x \in \Omega, \ t = 0,$$  

where $\partial \Omega$ is the boundary of $\Omega$ and $\nu > 0$ is a diffusion constant which is less than or equal to a given $\nu_0$. Functions $u : \Omega \times (0,T) \to \mathbb{R}^d$, $f \in C(\Omega)$ and $\phi^0 \in C(\Omega)$ are given.

**Remark 1.** As usual, in place of (1b), we can deal with the inhomogeneous boundary condition $\phi = g$ by replacing the unknown function $\phi$ by $\tilde{\phi} \equiv \phi - \bar{g}$ if the function $g$ defined on $\partial \Omega \times (0,T)$ can be extended to a function $\tilde{g}$ in $\Omega \times (0,T)$ appropriately.

Let $\Delta t > 0$ be a time increment, $N_T \equiv [T/\Delta t]$, $t^n \equiv n\Delta t$ and $\psi^n \equiv \psi(\cdot, t^n)$ for a function $\psi$ defined in $\Omega \times (0,T)$. For a set of functions $\Psi = \{ \psi^n \}_{n=0}^{N_T}$, two norms $\| \cdot \|_{L^\infty(L^2)}$ and $\| \cdot \|_{L^2(\Omega)}$ are defined by

$$\|\psi\|_{L^\infty(L^2)} \equiv \max\{\|\psi^n\| : n = 0, \ldots, N_T\},$$

$$\|\psi\|_{L^2(\Omega)} \equiv \left( \sum_{n=1}^n \|\psi^n\|^2 \right)^{1/2}$$

and denote $\|\psi\|_{L^2(\Omega)}$ by $\|\psi\|_{L^2(\Omega)}$.

Let $u$ be smooth. The characteristic curve $X(t;x,s)$ is defined by the solution of the system of the ordinary differential equations,

$$\frac{dX}{dt}(t;x,s) = u(X(t;x,s), t), \quad t < s,$$

$$X(s;x,s) = x.$$  

Then, we can write the material derivative term $\frac{\partial \phi}{\partial t} + u \cdot \nabla \phi$ as

$$\left( \frac{\partial \phi}{\partial t} + u \cdot \nabla \phi \right)(X(t), t) = \frac{d}{dt} \phi(X(t), t).$$
For $w : \Omega \to \mathbb{R}^d$ we define the mapping $X_1(w) : \Omega \to \mathbb{R}^d$ by
\[
(X_1(w))(x) \equiv x - w(x)\Delta t.
\]

\textbf{Remark 2.} The image of $x$ by $X_1(u(\cdot,t))$ is nothing but the backward Euler approximation of $X(t - \Delta t; x, t)$.

The symbol \( \circ \) stands for the composition of functions, e.g., \((g \circ f)(x) \equiv g(f(x))\).

Let $T_h = \{ K \}$ be a triangulation of $\Omega$ and $h \equiv \max_{K \in T_h} \text{diam}(K)$ be the maximum element size. Throughout this paper we consider a regular family of triangulations $\{ T_h \}_{h \downarrow 0}$. Let $k$ be a fixed positive integer and $V_h \subset H^1_0(\Omega)$ be the $P_k$-finite element space,
\[
V_h \equiv \{ v_h \in C(\Omega) \cap H^1_0(\Omega); \ v_h|_K \in P_k(K), \ \forall K \in T_h \},
\]
where $P_k(K)$ is the set of polynomials on $K$ whose degrees are less than or equal to $k$. Let $\phi_h \in V_h$ be the Poisson projection of $\phi \in H^1_0(\Omega)$ defined by
\[
(\nabla(\phi_h - \phi), \nabla \psi_h) = 0, \ \forall \psi_h \in V_h.
\]

We use $c$ to represent a generic positive constant independent of $h$, $\Delta t$, $\nu$, $f$ and $\phi$ which may take different values at different places. The notation $c(A)$ means that $c$ depends on a positive parameter $A$ and that $c$ increases monotonically when $A$ increases. The constants $c_0$, $c_1$ and $c_2$ stand for $c_0 = c(\|u\|_{C(L^\infty)})$, $c_1 = c(\|u\|_{C(W^{1,\infty})})$ and $c_2 = c(\|u\|_{C(W^{2,\infty})})$. We also use fixed positive constants $\alpha_c$ and $\delta_\epsilon$ defined in Lemma 4 in the next section and in Lemma 5 in Section 5, respectively.

3. A Genuinely Stable Lagrange–Galerkin Scheme

The conventional Lagrange–Galerkin scheme, which we call Scheme LG, is described as follows.

\textbf{Scheme LG.} Let $\phi_h^n \equiv \widehat{\phi}_h^n$. Find $\{ \phi_h^n \}_{n=1}^{N_T} \subset V_h$ such that for $n = 1, \ldots, N_T$
\[
\left( \frac{\phi_h^n - \phi_h^{n-1} \circ X^n_1}{\Delta t}, \psi_h \right) + \nu(\nabla \phi_h^n, \nabla \psi_h) = (f^n, \psi_h), \ \forall \psi_h \in V_h,
\]
where $X^n_1 = X_1(u^n)$.

For this scheme error estimates
\[
\|\phi_h^n - \phi\|_{L^\infty(L^2)} \leq c(h^k + \Delta t), \ c(1/\nu)(h^{k+1} + \Delta t),
\]
\[
\|\phi_h^n - \phi\|_{L^\infty(H^1)} \leq c(1/\nu)(h^k + \Delta t)
\]
are proved in [9], where the composite function term $(\phi_h^{n-1} \circ X^n_1, \psi_h)$ is assumed to be exactly integrated.

Although the function $\phi_h^{n-1}$ is a polynomial on each element $K$, the composite function $\phi_h^{n-1} \circ X^n_1$ is not a polynomial on $K$ in general since the image $X^n_1(K)$ of an element $K$ may spread over plural elements. Hence, it is hard to calculate the composite function term $(\phi_h^{n-1} \circ X^n_1, \psi_h)$ exactly. In practice, the following numerical quadrature has been used. Let $g : K \to \mathbb{R}$ be a continuous function. A numerical quadrature $I_h[g; K]$ of $\int_K g \, dx$ is defined by
\[
I_h[g; K] \equiv \text{meas}(K) \sum_{i=1}^{N_S} w_i \ g(a_i),
\]
where $\text{meas}(K)$ is the measure of $K$.
where \( N_q \) is the number of quadrature points and \((w_i, a_i) \in \mathbb{R} \times K\) is a pair of weight and point for \(i = 1, \ldots, N_q\). We call the practical scheme using numerical quadrature Scheme LG'.

**Scheme LG'.** Let \( \phi_h^0 = \tilde{\phi}_h^0 \). Find \( \{\phi_h^n\}_{n=1}^{N_T} \subset V_h \) such that for \(n = 1, \ldots, N_T\)

\[
\frac{1}{\Delta t} (\phi_h^n, \psi_h) - \frac{1}{\Delta t} \sum_{K \in T_h} I_h[(\phi_h^{n-1} \circ X_h^n)\psi_h; K] + \nu (\nabla \phi_h^n, \nabla \psi_h) = (f^n, \psi_h), \quad \forall \psi_h \in V_h, \tag{8}
\]

where \( X_h^n = X_1(u^n) \).

It is reported that the results (6) do not hold for Scheme LG' \([9, 17, 18, 19]\).

We denote by \( \Pi_h^{(1)} \) the Lagrange interpolation operator to the \( P_1 \)-finite element space. The following lemma is well-known \([5]\).

**Lemma 1.**

(i) There exists a positive constant \( c_{\Pi} \) such that for \( w \in W^{2, \infty}(\Omega)^d \)

\[
\| \Pi_h^{(1)} w - w \|_{0, \infty, \Omega} \leq c_{\Pi} h^2 |w|_{2, \infty}. \tag{10}
\]

(ii) There exists a positive constant \( \alpha_* \geq 1 \) such that for \( w \in W^{1, \infty}(\Omega)^d \)

\[
| \Pi_h^{(1)} w |_{1, \infty} \leq \alpha_* |w|_{1, \infty}. \tag{10}
\]

We now present our genuinely stable scheme GSLG, which is free from quadrature and exactly computable. We define a locally linearized velocity \( u_h \) and a mapping \( X_{1h} \) by

\[
u_h = \Pi_h^{(1)} u, \quad X_{1h} = X_1(u^n_h).
\]

**Scheme GSLG.** Let \( \phi_h^0 = \tilde{\phi}_h^0 \). Find \( \{\phi_h^n\}_{n=1}^{N_T} \subset V_h \) such that for \(n = 1, \ldots, N_T\)

\[
\left( \frac{\phi_h^n - \phi_h^{n-1} \circ X_{1h}^n}{\Delta t} \psi_h \right) + \nu (\nabla \phi_h^n, \nabla \psi_h) = (f^n, \psi_h), \quad \forall \psi_h \in V_h. \tag{9}
\]

We show that the integration \( (\phi_h^{n-1} \circ X_{1h}^n, \psi_h) \) can be calculated exactly. At

**Figure 1.** Elements \( K_0, K_1 \) and a polygon \( E_1 \)

first we prepare two lemmas. The next lemma on the mapping \( \tilde{\phi}_h^0 \) is proved in \([14]\).

**Lemma 2** (\([14\text{, Proposition 1]}\)). Suppose

\[
w \in W_0^{1, \infty}(\Omega)^d \text{ and } \Delta t |w|_{1, \infty} < 1. \tag{10}
\]

Let \( F \equiv X_1(w) \) be the mapping defined in \([3]\). Then, \( F : \Omega \rightarrow \Omega \) is bijective.

**Lemma 3.** Let \( K_0, K_1 \in T_h \) and \( F : K_0 \rightarrow \mathbb{R}^d \) be linear and one-to-one. Let \( E_1 \equiv K_0 \cap F^{-1}(K_1) \) and \( \text{meas}(E_1) > 0 \). Then, the following hold.
(i) \( E_1 \) is a polygon \((d = 2)\) or a polyhedron \((d = 3)\).

(ii) \( \phi_h \circ F |_{E_1} \in P_k(E_1), \quad \forall \phi_h \in P_k(K_1). \)

**Proof.** (i) Since both \( K_0 \) and \( F^{-1}(K_1) \) are triangles \((d = 2)\) or tetrahedra \((d = 3)\), the intersection is a polygon or a polyhedron. See Fig. 1.

(ii) \( F \in P_1(K_0)^d \) implies that \( F \in P_1(E_1)^d \) and it holds that \( F(E_1) \subset K_1. \) Hence, \( \phi_h \circ F |_{E_1} \) is well defined and \( \phi_h \circ F |_{E_1} \in P_k(E_1). \)

**Proposition 1.** Let \( \phi_h, \psi_h \in V_h, w \in W^{1,\infty}_0(\Omega) \) and \( X_{1h} \equiv X_1(\Pi_h(1)w) \), where \( X_1 \) is the operator defined in \( [3] \). Suppose \( \alpha \Delta t |w|_{1,\infty} < 1. \) Then, \( \int_{\Omega}(\phi_h \circ X_{1h})\psi_h dx \) is exactly computable.

**Proof.** It is sufficient to show that \( \int_{K_0}(\phi_h \circ X_{1h})\psi_h dx \) can be computed exactly for any \( K_0 \in T_h \). The mapping \( X_{1h} : \Omega \rightarrow \Omega \) is bijective since we can apply Lemma 2 thanks to

\[
\Delta t |\Pi_h(1)w|_{1,\infty} \leq \alpha \Delta t |w|_{1,\infty} < 1. \tag{11}
\]

Let \( \Lambda(K_0) \equiv \{ l; K_0 \cap X_{1h}^{-1}(K_l) \neq \emptyset \} \) and \( E_l \equiv K_0 \cap X_{1h}^{-1}(K_l) \) for \( l \in \Lambda(K_0) \). Noting that

\[
\bigcup_{l \in \Lambda(K_0)} E_l = K_0 \cap \bigcup_{l \in \Lambda(K_0)} X_{1h}^{-1}(K_l) = K_0
\]

and that \( \text{meas}(E_l \cap E_m) = 0 \) for \( l \neq m, l, m \in \Lambda(K_0) \), we can divide the integration on \( K_0 \) into the sum of those on \( E_l \) for \( l \in \Lambda(K_0) \),

\[
\int_{K_0}(\phi_h \circ X_{1h})\psi_h dx = \sum_{l \in \Lambda(K_0)} \int_{E_l}(\phi_h \circ X_{1h})\psi_h dx.
\]

Since Lemma 3 with \( F = X_{1h} \) implies that both \( \phi_h \circ X_{1h} \) and \( \psi_h \) are polynomials on \( E_l \), we can execute the exact integration. \( \square \)

**Remark 3.** In the case of \( d = 2 \), Priestley \( [3] \) approximated \( X(t^{n-1};x,t^n) \) by

\[
\tilde{X}_{1h}(x) = B_1\lambda_1(x) + B_2\lambda_2(x) + B_3\lambda_3(x), \quad x \in K_0
\]
on each \( K_0 \in T_h \), where \( B_i = X(t^{n-1};x,t^n) \), \( \{A_i\}_{i=1}^3 \) are vertices of \( K_0 \) and \( \{\lambda_i\}_{i=1}^3 \) are the barycentric coordinates of \( K_0 \) with respect to \( \{A_i\}_{i=1}^3 \). Since \( \tilde{X}_{1h}(x) \) is linear in \( K_0 \), the decomposition

\[
\int_{K_0}(\phi_h \circ \tilde{X}_{1h})\psi_h dx = \sum_{l \in \Lambda(K_0)} \int_{E_l}(\phi_h \circ \tilde{X}_{1h})\psi_h dx,
\]

\[
\Lambda(K_0) \equiv \{ l; K_0 \cap \tilde{X}_{1h}^{-1}(K_l) \neq \emptyset \}, \quad E_l \equiv K_0 \cap \tilde{X}_{1h}^{-1}(K_l)
\]

makes the exact integration possible. However, \( B_i = X(t^{n-1};x,t^n) \) are the solutions of a system of ordinary differential equations and they cannot be solved exactly in general. In practice, some numerical method, e.g., Runge–Kutta method, is required, which introduces another error.

**4. MAIN RESULTS**

We show the main results, the stability and convergence of Scheme GSLG.

**Hypothesis 1.** (i) \( u \in C((W^{1,\infty}_0)^d) \), (ii) \( u \in C((W^{1,\infty}_0 \cap W^{2,\infty})^d) \).

**Hypothesis 2.** \( \phi \in H^1(H^{k+1}) \cap Z^2 \).
Hypothesis 3. The time increment $\Delta t$ satisfies $0 < \Delta t \leq \Delta t_0$, where

$$\Delta t_0 \equiv \frac{\delta_s}{\alpha_s |u|_{C(W^{1,\infty})}},$$

and $\alpha_s$ and $\delta_s$ are the constants stated in Lemma 1 (Section 3) and Lemma 5 (Section 5), respectively.

Hypothesis 4. There exists a positive constant $c_P$ such that, for $\psi \in H^{k+1}(\Omega) \cap H^1_0(\Omega)$,

$$\|\hat{\psi}_h - \psi\|_0 \leq c_P h^{k+1} \|\psi\|_{k+1},$$

where $\hat{\psi}_h$ is the Poisson projection defined in 4.

Remark 4. (i) It is well-known that the $H^1$-estimate

$$\|\hat{\psi}_h - \psi\|_1 \leq c_P h^k \|\psi\|_{k+1}$$

holds without any specific condition. On the other hand, Hypothesis 4 holds, for example, if $\Omega$ is convex, by Aubin–Nitsche lemma 5.

(ii) Hypothesis 2 implies $\phi \in C(H^{k+1})$ and $\phi^0 \in H^{k+1}(\Omega)$.

Theorem 1. Suppose Hypotheses 1 (i) and 2. Then, there exists a positive constant $c_*$ independent of $h, \Delta t, \nu, \phi$ and $f$ such that

$$\|\phi_h\|_{L^\infty(L^2)} + \sqrt{\nu} \|\nabla \phi_h\|_{L^2(L^2)} \leq c_* \left( \|\phi^0_h\| + \|f\|_{L^2(L^2)} \right).$$

Theorem 2. Suppose Hypotheses 1 (ii), 2 and 3.

(i) There exists a positive constant $c_*$ independent of $h, \Delta t, \nu$ and $\phi$ such that

$$\|\phi - \phi_h\|_{L^\infty(L^2)} + \sqrt{\nu} \|\nabla (\phi - \phi_h)\|_{L^2(L^2)} \leq c_* \left( \frac{\Delta t \|\phi\|_{Z^2} + h^k (\|\frac{\partial \phi}{\partial t}\|_{L^2(H^{k+1})} + \|\phi|_{L^\infty(H^{k+1})} + \|\phi\|_{L^2(0,N_T;H^{k+1})}) + h^2 \|\nabla \phi\|_{L^2(0,N_T-1;L^2)} \right).$$

(ii) There exists a positive constant $c_*$ independent of $h, \Delta t, \phi$ (but dependent on $1/\nu$) such that

$$\|\phi - \phi_h\|_{L^\infty(H^1)} \leq c_* \left( \frac{\Delta t \|\phi\|_{Z^2} + h^k (\|\frac{\partial \phi}{\partial t}\|_{L^2(H^{k+1})} + \|\phi|_{L^\infty(H^{k+1})} + \|\phi\|_{L^2(0,N_T;H^{k+1})}) + h^2 \|\nabla \phi\|_{L^2(0,N_T-1;L^2)} \right).$$

(iii) Moreover, suppose Hypothesis 4. Then, there exists a positive constant $c_{**}$ independent of $h, \Delta t, \nu$ and $\phi$ such that

$$\|\phi - \phi_h\|_{L^\infty(L^2)} \leq c_{**} \left( \frac{\Delta t \|\phi\|_{Z^2} + h^{k+1} (\|\frac{\partial \phi}{\partial t}\|_{L^2(H^{k+1})} + \|\phi|_{L^\infty(H^{k+1})} + \|\phi\|_{L^2(0,N_T-1;L^2)}) + h^2 \|\nabla \phi\|_{L^2(0,N_T-1;L^2)} \right).$$
Remark 5. From Theorem [2] we have
\[ \| \phi - \phi_h \|_{\ell^\infty(L_2)} \leq c(\Delta t + h^2 + h^k), \]
\[ \| \phi - \phi_h \|_{\ell^\infty(H_1)} \leq c \left( \frac{1}{\nu} \right)(\Delta t + h^2 + h^k). \]

In the case of \( P_k \)-element, \( k = 1, 2 \), the estimate \([15]\) shows the optimal \( L^2 \)-convergence rate \( O(\Delta t + h^k) \) independent of \( \nu \). The dependency on \( \nu \) in \([16],[17]\) is also inevitable in Scheme LG.

5. PROOFS OF MAIN THEOREMS

We recall some results used in proving main theorems. For their proofs we only show outlines or refer to the bibliography.

**Lemma 4** ([14, Lemma 1]). Suppose \( w \in W^{1,\infty}_0(\Omega)^d \) and
\[ \Delta t |w|_{1,\infty} < 1. \] (18)

Let \( F \equiv X_1(w) \) be the mapping defined in \([4]\). Then, there exists a positive constant \( c(|w|_{1,\infty}) \) such that for \( \psi \in L^2(\Omega) \)
\[ \| \psi \circ F \| \leq (1 + c\Delta t) \| \psi \|. \]

The proof is given in \([14]\).

**Lemma 5.** There exists a constant \( \delta_* \in (0,1) \) such that, for \( w \in W^{1,\infty}_0(\Omega)^d \) and \( \Delta t \) satisfying \( \Delta t |w|_{1,\infty} \leq \delta_* \),
\[ \frac{1}{2} \leq \left| \frac{\partial X_1(w)}{\partial x} \right| \leq \frac{3}{2}, \]
where \( |\partial X_1(w)/\partial x| \) is the Jacobian of the mapping \( X_1(w) \) defined in \([4]\).

**Lemma 5** is easily proved by the fact,
\[ \left( \frac{\partial X_1(w)}{\partial x} \right)_{ij} = \delta_{ij} - \Delta t \frac{\partial w_i}{\partial x_j}. \]

**Lemma 6.** Let \( w_i \in W^{1,\infty}_0(\Omega)^d \) and \( F_i \equiv X_1(w_i) \) be the mapping defined in \([3]\) for \( i = 1, 2 \). Under the condition \( \Delta t |w_i|_{1,\infty} \leq \delta_* \), \( i = 1, 2 \), we have for \( \psi \in H^1(\Omega) \)
\[ \| \psi \circ F_1 - \psi \circ F_2 \| \leq \sqrt{2}\Delta t \| w_1 - w_2 \|_{0,\infty} \| \nabla \psi \|. \]

**Lemma 6** is a direct consequence of \([1]\, Lemma 4.5\) and Lemma 5.

**Lemma 7.** Let \( w \in W^{1,\infty}_0(\Omega)^d \) and \( F \equiv X_1(w) \) be the mapping defined in \([3]\). Under the condition \( \Delta t |w|_{1,\infty} \leq \delta_* \), there exists a positive constant \( c(|w|_{1,\infty}) \) such that for \( \psi \in L^2(\Omega) \)
\[ \| \psi - \psi \circ F \|_{H^{-1}(\Omega)} \leq c\Delta t \| \psi \|. \]

**Lemma 7** is obtained from \([6]\, Lemma 1\) and Lemma 5.
Lemma 8 (discrete Gronwall inequality). Let $a_0$ and $a_1$ be non-negative numbers, $\Delta t \in (0, \frac{1}{2a_0}]$ be a real number, and $\{x^n\}_{n \geq 0}, \{y^n\}_{n \geq 1}$ and $\{b^n\}_{n \geq 1}$ be non-negative sequences. Suppose

$$\frac{x^n - x^{n-1}}{\Delta t} + y^n \leq a_0 x^n + a_1 x^{n-1} + b^n, \forall n \geq 1.$$ 

Then, it holds that

$$x^n + \Delta t \sum_{i=1}^{n} y^i \leq \exp \{ (2a_0 + a_1) n \Delta t \} \left( x^0 + \Delta t \sum_{i=1}^{n} b^i \right), \forall n \geq 1.$$ 

Lemma 8 is shown by using the inequalities

$$\frac{1}{1 - a_0 \Delta t} \leq 1 + 2a_0 \Delta t \leq \exp(2a_0 \Delta t).$$

Outline of the proof of Theorem 2. We substitute $\phi^n_h$ into $\psi_h$ in (19). We can apply Lemma 4 with $w = u^n_h$ and $\psi = \phi^{n-1}_h$ by virtue of $\Delta t |u_h|_{C([-1, 1]} < 1$. The rest of the proof is similar to [14, Theorem 1]. We, therefore, omit it.

Proof of Theorem 2. We first show the estimate (15). Let

$$e_h \equiv \phi_h - \hat{\phi}_h, \quad \eta \equiv \phi - \hat{\phi}_h,$$

where $\hat{\phi}_h$ is the Poisson projection defined in (4). From (18) and (9), we have

$$\left( \frac{e^n_h - e^{n-1}_h \circ X^n_{1h}}{\Delta t}, \psi_h \right) + \nu(\nabla e^n_h, \nabla \psi_h) = \sum_{i=1}^{4} (R^n_i, \psi_h)$$

for $\psi_h \in V_h$, where

$$R^n_1 = \frac{\partial \phi^n_h + u^n \cdot \nabla \phi^n - \phi^n - \phi^{n-1} \circ X^n_{1h}}{\Delta t},$$

$$R^n_2 = \frac{\phi^{n-1} \circ X^n_{1h} - \phi^{n-1} \circ X^n_{1h}}{\Delta t},$$

$$R^n_3 = \frac{\eta^n - \eta^{n-1}}{\Delta t}, \quad R^n_4 = \frac{\eta^{n-1} - \eta^{n-1} \circ X^n_{1h}}{\Delta t}.$$ 

Substituting $e^n_h$ into $\psi_h$, applying Lemma 4 with $F = X^n_{1h}$ and $\psi = e^n_{h-1}$, and evaluating the first term of the left-hand side as

$$\left( \frac{e^n_h - e^{n-1}_h \circ X^n_{1h}}{\Delta t}, e^n_h \right) \geq \frac{1}{2 \Delta t} \left( \| e^n_h \|^2 - \| e^{n-1}_h \circ X^n_{1h} \|^2 \right)$$

$$\geq \frac{1}{2 \Delta t} \left( \| e^n_h \|^2 - (1 + c_1 \Delta t)^2 \| e^{n-1}_h \|^2 \right)$$

$$= \frac{1}{2 \Delta t} \left( \| e^n_h \|^2 - \| e^{n-1}_h \|^2 \right) - \frac{c_1}{2} (2 + c_1 \Delta t) \| e^{n-1}_h \|^2,$$

we have

$$\frac{1}{2 \Delta t} \left( \| e^n_h \|^2 - \| e^{n-1}_h \|^2 \right) + \nu \| \nabla e^n_h \|^2 \leq c_1 \| e^{n-1}_h \|^2 + \sum_{i=1}^{4} \frac{1}{4c_1} \| R^n_i \|^2 + \left( \sum_{i=1}^{4} \| \xi_i \|^2 \right) \| e^n_h \|^2,$$

where $\{\xi_i\}_{i=1}^{4}$ are positive constants satisfying $\Delta t_0 \leq \frac{1}{4c_1}, \xi_0 \equiv \sum_{i=1}^{4} \xi_i$. 

\[ \]
We evaluate $R_i$, $i = 1, \ldots, 4$. Setting
\[ y(x, s) = x + (s - 1)\Delta t \ u^n(x), \quad t(s) = t^{n-1} + s\Delta t, \]
we have
\[ \frac{\phi^n - \phi^{n-1} \circ X^1_1}{\Delta t} = \frac{1}{\Delta t} \left[ \phi(y(\cdot, s), t(s)) \right]_{s=0}^1, \]
which implies
\[ R_1^n = \frac{\partial \phi^n}{\partial t} + u^n \cdot \nabla \phi^n - \int_0^1 \left\{ u^n(\cdot) \cdot \nabla \phi + \frac{\partial \phi}{\partial t} \right\} (y(\cdot, s), t(s)) ds \]
\[ = \Delta t \int_0^1 ds \int_s^1 \left\{ \left( u^n(\cdot) \cdot \nabla + \frac{\partial}{\partial t} \right)^2 \phi \right\} (y(\cdot, s_1), t(s_1)) ds_1 \]
\[ = \Delta t \int_0^1 s_1 \left\{ \left( u^n(\cdot) \cdot \nabla + \frac{\partial}{\partial t} \right)^2 \phi \right\} (y(\cdot, s_1), t(s_1)) ds_1. \]
Hence, we have
\[ \| R_1^n \| \leq \Delta t \int_0^1 s_1 \left\{ \left( u^n(\cdot) \cdot \nabla + \frac{\partial}{\partial t} \right)^2 \phi \right\} (y(\cdot, s_1), t(s_1)) ds_1 \]
\[ \leq c_0 \sqrt{\Delta t} \| \phi \|_{L^2(t^{n-1}, t^n)}, \quad (23) \]
where we have used the transformation of independent variables from $x$ to $y$ and $s_1$ to $t$, and the estimate $|\partial x/\partial y| \leq 2$ by virtue of Lemma 5.

From $\Delta t |u|_{C(W^{1, \infty})}, \Delta t |u_h|_{C(W^{1, \infty})} \leq \delta_*$, and Lemmas 6 and 11 it holds that
\[ \| R_2^n \| \leq \sqrt{2} \| \nabla \phi^{n-1} \| \| \Pi_h^{(1)} u^n - u^n \|_{0, \infty} \leq c_2 h^2 \| \nabla \phi^{n-1} \|. \quad (24) \]
$R_3^n$ is evaluated as
\[ \| R_3^n \| = \left\| \int_0^1 \frac{\partial \eta}{\partial t}(\cdot, t(s)) ds \right\| \leq \frac{c_p h k}{\sqrt{\Delta t}} \| \frac{\partial \phi}{\partial t} \|_{L^2(t^{n-1}, t^n, H^{k+1})}, \quad (25) \]
where we have used (11).

From $\Delta t |u_h|_{C(W^{1, \infty})} \leq \delta_*$ and Lemma 6 it holds that
\[ \| R_4^n \| \leq \sqrt{2} \| \nabla \eta^{n-1} \| \| \Pi_h^{(1)} u^n \|_{0, \infty} \leq c_0 h^k \| \phi^{n-1} \|_{k+1}. \quad (26) \]
Combining (22) and (24), we have
\[ \frac{1}{2\Delta t} \left( \| e_h^n \|^2 - \| e_h^{n-1} \|^2 \right) + \nu \| \nabla e_h^n \|^2 \leq \varepsilon_0 \| e_h^n \|^2 + c_1 \| e_h^{n-1} \|^2 \]
\[ + c_2 \left\{ \Delta t \| \phi \|_{L^2(t^{n-1}, t^n)} + h^4 \| \nabla \phi^{n-1} \|^2 \]
\[ + \frac{h^{2k}}{\Delta t} \| \frac{\partial \phi}{\partial t} \|_{L^2(t^{n-1}, t^n, H^{k+1})} + h^{2k} \| \phi^{n-1} \|_{k+1} \right\}. \]
From Lemma 8 we obtain for \( n = 1, \ldots, N_T \)
\[
\|e_h^n\|^2 + 2\nu \Delta t \sum_{j=1}^{N_T} \left\| \nabla e_h^j \right\|^2 \leq c_2 \left( \|e_h^0\|^2 + \Delta t^2 \|\phi\|_{L^2}^2 \right) + h^{2k} \left\| \frac{\partial \phi}{\partial t} \right\|_{L^2(H^{k+1})}^2 + h^{2k} \Delta t \sum_{j=0}^{N_T-1} \|\phi^j\|_{k+1}^2 + h^4 \Delta t \sum_{j=0}^{N_T-1} \left\| \nabla \phi^j \right\|^2 ,
\]
which implies (15) by virtue of (16), and the triangle inequalities,
\[
\|\phi - \phi_h\|_{L^\infty(L^2)} \leq \|e_h\|_{L^\infty(L^2)} + \|\eta\|_{L^\infty(L^2)} \leq \|e_h\|_{L^\infty(L^2)} + c_ph^k \|\phi\|_{L^\infty(H^{k+1})} , \tag{27}
\]
\[
\|\nabla (\phi - \phi_h)\|_{L^2(L^2)} \leq \|\nabla e_h\|_{L^2(L^2)} + \|\nabla \eta\|_{L^2(L^2)} \leq \|\nabla e_h\|_{L^2(L^2)} + c_ph^k \|\phi\|_{L^2(H^{k+1})} .
\]

We show the estimate (16). The equation (20) can be rewritten as
\[
\frac{1}{\Delta t} (e_h^n - e_h^{n-1}, \psi_h) + \nu (\nabla e_h^n, \nabla \psi_h) = \sum_{i=1}^{5} (R^n_i, \psi_h),
\]
where
\[
R^0_h \equiv \frac{1}{\Delta t} (e_h^{n-1} \circ X_{\psi_h} - e_h^{n-1}).
\]

From Lemma 6 it holds that
\[
\|R^0_h\| \leq \sqrt{2} \|\nabla e_h^{n-1}\| \|\Pi^0_h u^n\|_{0, \infty} \leq c_0 \|\nabla e_h^{n-1}\|.
\]
Substituting \( \overline{D} \Delta t e_h^n \equiv \frac{1}{\Delta t} (e_h^n - e_h^{n-1}) \) into \( \psi_h \), and using (23)–(26) for \( R_1, \ldots, R_4 \), we have
\[
\left\| \overline{D} \Delta t e_h^n \right\|^2 + \frac{1}{\Delta t} \left( \frac{\nu}{2} \|\nabla e_h^n\|^2 - \frac{\nu}{2} \|\nabla e_h^{n-1}\|^2 \right) + \frac{\nu}{2\Delta t} \left\| \nabla (e_h^n - e_h^{n-1}) \right\|^2 \leq c_2 \left\{ \Delta t \left( \frac{\nu}{2} \|\nabla \phi\|^2_{L^2(L^2)} + h^4 \|\nabla \phi^{n-1}\|^2 + \frac{h^{2k}}{\Delta t} \left\| \frac{\partial \phi}{\partial t} \right\|_{L^2(H^{k+1})} \right. 
+ \left. h^{2k} \left\| \phi^{n-1}\right\|^2_{k+1} + \frac{c_0}{\nu} \left( \frac{\nu}{2} \|\nabla e_h^{n-1}\|^2 \right) + \frac{1}{\Delta t} \left\| \overline{D} \Delta t e_h^n \right\|^2 \right\}.
\]

From Lemma 8 we have for \( n = 1, \ldots, N_T \)
\[
\frac{\Delta t}{2} \sum_{j=1}^{N_T} \left\| \overline{D} \Delta t e_h^j \right\|^2 + \frac{\nu}{2} \|\nabla e_h^j\|^2 \leq c_2 \exp \left( \frac{c_0 T}{\nu} \right) \left( \|\nabla e_h^0\|^2 + \Delta t^2 \|\phi\|_{L^2}^2 \right) + h^{2k} \left\| \frac{\partial \phi}{\partial t} \right\|_{L^2(H^{k+1})}^2 + h^{2k} \Delta t \sum_{j=0}^{N_T-1} \|\phi^j\|^2_{k+1} + h^4 \Delta t \sum_{j=0}^{N_T-1} \left\| \nabla \phi^j \right\|^2 ,
\]
which implies (16) by virtue of (15), the triangle inequality,
\[
\|\nabla (\phi - \phi_h)\|_{L^\infty(L^2)} \leq \|\nabla e_h\|_{L^\infty(L^2)} + \|\nabla \eta\|_{L^\infty(L^2)} \leq \|\nabla e_h\|_{L^\infty(L^2)} + c_ph^k \|\phi\|_{L^\infty(H^{k+1})}
\]
and the Poincaré inequality,
\[
\|v\|_1 \leq c \|\nabla v\| , \quad \forall v \in H^1_0(\Omega). \tag{28}
\]
Hence, it holds that $u_{\Omega}$ is not a polygon and

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From Lemma 7 we have

\[ R^2_{\Omega} \leq c_1 \| \eta^{n-1} \| \leq c_1 h^{k+1} \| \phi^{n-1} \|_{k+1}. \]

Hence, it holds that

\[ (R^n_\Omega, e^n_\Omega) \leq \| R^n_\Omega \|_{H^{-1}(\Omega)} \| e^n_\Omega \|_1 \leq \frac{c_1}{\nu} h^{2(k+1)} \| \phi^{n-1} \|_{k+1}^2 + \frac{\nu}{2} \| \nabla e^n_\Omega \|_2^2, \]

where we have used the Poincaré inequality \(28\). Using this inequality instead of \( \frac{1}{4\alpha} \| R^n_\Omega \|_2^2 + \frac{1}{4\nu} \| e^n_\Omega \|_2^2 \) in \(22\) and replacing the last term of \(27\) by \( c_1 h^{k+1} \| \phi \|_{L^\infty(H^{k+1})} \), we obtain \(17\).

\[ \Box \]

6. Numerical results

We show numerical results in $d = 2$. We compare the conventional scheme (Scheme LG) with the present one (Scheme GSLG). We use FreeFem++ \(8\) for the triangulation of the domain. Both $P_1$- and $P_2$-elements are used. For Scheme LG we use the seven points quadrature formula of degree five \(7\). A relative error $E_X$ is defined by

\[ E_X \equiv \frac{\| \Pi_h^{(k)} \phi - \phi_h \|_{L^\infty(X)}}{\| \Pi_h^{(k)} \phi \|_{L^\infty(X)}}, \]

where $\Pi_h^{(k)}$ is the Lagrange interpolation operator to the $P_k$-finite element space and $X = L^2(\Omega)$ or $H^1_0(\Omega)$.

Example 1 (The rotating Gaussian hill \(14\)). In \(11\), $\Omega$ is an unit disk, and we set $T = 2\pi, \nu = 10^{-5}$,

\[ u(x, t) \equiv (-x_2, x_1), \ f \equiv 0, \ \phi^0 \equiv \phi_e(\cdot, 0), \]

where

\[ \phi_e(x, t) \equiv \frac{\sigma}{\sigma + 4\nu t} \exp \left\{ -\frac{(\bar{x}_1(t) - x_1, e)^2 + (\bar{x}_2(t) - x_2, e)^2}{\sigma + 4\nu t} \right\}, \]

\[ (\bar{x}_1, x_2)(t) \equiv (x_1 \cos t + x_2 \sin t, -x_1 \sin t + x_2 \cos t), \]

\[ (x_1, x_2, e) \equiv (0.25, 0), \ \sigma \equiv 0.01. \]

In this problem the identity $\Pi_h^{(1)} u = u$ holds. This problem does not satisfy our setting because $\Omega$ is not a polygon and $u \neq 0$ on $\partial \Omega$. The function $\phi_e$ in Fig. 2 (left) satisfies \(13\) and \(14\) but does not satisfy the boundary condition \(11\). However, we may apply the schemes and treat $\phi_e$ as the solution since the value of $\phi_e$ on $\partial \Omega$ is almost equal to zero, less than $10^{-15}$, and we may neglect the effect of the boundary value and the term $\int_K (\phi_h^{n-1} \circ X^n) \psi_h dx$ and $\int_K (\phi_h^{n-1} \circ X^n_h) \psi_h dx$ on the element $K$ touching the boundary.

Let $N$ be the division number of the circle. We set $h \equiv 2\pi/N$, $N = 32, 64, 128$ and 256. Figure 2 (right) shows the triangulation of $\Omega$ for $N = 64$. The time increment $\Delta t$ is set to be $c_1 h$ and $c_2 h^2$ for $P_1$-element ($c_1 = \frac{1}{3\pi} \equiv 0.255$, $c_2 = \frac{4}{3\pi^2} \equiv 1.30$),
The function $\phi_e(\cdot, 0)$ (left) and the triangulation of $\bar{\Omega}$ for $N = 64$ (right) in Example 1.

c_3 h^2$ and $c_4 h^3$ for $P_2$-element ($c_3 = \frac{128}{5\pi} \approx 2.59$, $c_4 = \frac{2048}{5\pi} \approx 13.21$) so that we can observe the convergence behavior of $O(h^k)$ for $E_{H_0^1}$, and $O(h^k)$ and $O(h^{k+1})$ for $E_{L^2}$ when $P_k$-element is employed.

In the following figures we use the symbols shown in Table 1. Figure 3 shows the log-log graphs of $E_{L^2}$ and $E_{H_0^1}$ versus $h$. The left graph shows the results of $P_1$-element and Tables 2 shows the values of them. The convergence order of $E_{L^2}$ with $\Delta t = O(h)$ is less than 1 in Scheme LG' (□) and more than 1 in Scheme GSLG (●). The orders of $E_{L^2}$ with $\Delta t = O(h^2)$ are almost 2 for small $h$ in both schemes (□, ●). The convergence of $E_{H_0^1}$ is not observed in Scheme LG' (■) while the order is almost 1 in Scheme GSLG (●). The right graph of Fig. 3 shows the results of $P_2$-element and Tables 3 shows the values of them. The errors $E_{L^2}$ with $\Delta t = O(h^2)$ are too large at $N = 128$ and 256 to be plotted in the graph in Scheme LG' (□) while the convergence order is almost 2 in Scheme GSLG (●). The error $E_{L^2}$ with $\Delta t = O(h^3)$ is large at $N = 128$, but it becomes small again at $N = 256$ in Scheme LG' (□). We will discuss the reason why such a behavior occurs in a forthcoming paper. The order is greater than 2.5 in Scheme GSLG (●). The errors $E_{H_0^1}$ are too large at $N = 128$ and 256 to be plotted in the graph in Scheme LG' (■) while we can observe the convergence of $E_{H_0^1}$ but the order is less than 2 in Scheme GSLG (●). The errors of Scheme GSLG are smaller than those of Scheme LG' in both cases of $P_1$- and $P_2$-element.

| Scheme       | $X$ | $\ell^\infty(L^2)$ | $\ell^\infty(L^2)$ | $\ell^\infty(H_0^1)$ | $O(h^k)$ | $O(h^{k+1})$ |
|--------------|-----|---------------------|---------------------|-----------------------|--------|---------------|
| LG'          | □   | □                   | □                   | □                     |        |               |
| GSLG         | ●   | ●                   | ●                   | ●                     |        |               |

In the case of $P_1$-element, the solution of Scheme LG' is oscillatory while that of Scheme GSLG is much better though a small ruggedness is observed. In the case of $P_2$-element, the solution of Scheme LG' is quite oscillatory while that of Scheme GSLG is stable.
Figure 3. Graphs of $E_{L^2}$ and $E_{H^1_0}$ versus $h$ in Example 1 by $P_k$-element. $k = 1$ (left) and $k = 2$ (right).

Figure 4. Solutions $\phi^n_h (n \Delta t \approx 2\pi)$ in Example 1 by Scheme LG’ (top left) and Scheme GSLG (top right) for $P_1$-element, and by Scheme LG’ (bottom left) and Scheme GSLG (bottom right) for $P_2$-element.
Table 2. The values of errors and orders of the graph in Fig. 3 by $P_1$-element

| $N$ | order | order | order |
|-----|-------|-------|-------|
| 32  | 7.58E-01 | 7.58E-01 | 9.52E-01 |
| 64  | 5.65E-01 | 0.42   | 5.53E-01 | 0.45 |
| 128 | 3.93E-01 | 0.52   | 1.87E-01 | 1.56 |
| 256 | 2.04E-01 | 0.95   | 4.15E-02 | 2.17 |

Table 3. The values of errors and orders of the graph in Fig. 3 by $P_2$-element

| $N$ | order | order | order |
|-----|-------|-------|-------|
| 32  | 6.86E-01 | 6.86E-01 | 9.22E-01 |
| 64  | 4.06E-01 | 0.76   | 3.97E-01 | 0.79 |
| 128 | 1.67E+02 | -8.68  | 8.30E-01 | -1.06 |
| 256 | 1.42E+27 | -82.81 | 3.05E-03 | 8.09 |

Example 2. In (1), $\Omega$ is the square $(0,1) \times (0,1)$, and we set $T = 1$, $\nu = 10^{-2}$ and $10^{-5}$, $u(x,t) \equiv (\sin \pi x \sin \pi x_2, \sin \pi x \sin \pi x_2)$, $f \equiv \frac{\partial \phi_e}{\partial t} + u \cdot \nabla \phi_e - \nu \Delta \phi_e$, $\phi^0 \equiv \phi_e(\cdot,0)$, $\phi_e(x,t) \equiv \cos(2\pi t) \sin^2(\pi x_1) \sin(2\pi x_2)$.

In this problem, $\Pi_h^{(1)} u \neq u$. Let $N$ be the division number of each side of $\Omega$. We set $h = 1/N$. $N = 8, 16, 32$ and 64. Figure 4 shows the triangulation of $\Omega$ for $N = 16$. The time increment $\Delta t$ is set to be $c_1 h$ and $c_2 h^2$ for $P_1$-element ($c_1 = 0.125, c_2 = 1$), $c_3 h^3$ and $c_4 h^3$ for $P_2$-element ($c_3 = 1, c_4 = 5.12$) so that we can observe the convergence behavior of $O(h^k)$ for $E_{L^2}$, and $O(h^k)$ and $O(h^{k+1})$ for $E_{H^1}$ when $P_k$-element is employed.

Figure 5 shows the log-log graphs of $E_{L^2}$ and $E_{H^1}$ versus $h$ with $\nu = 10^{-2}$. The left graph shows the results of $P_1$-element and Table 4 shows the values of them. The convergence orders of $E_{L^2}$ with $\Delta t = O(h)$ are almost 1 in both schemes ($\square$, $\bigcirc$). The orders of $E_{L^2}$ with $\Delta t = O(h^2)$ are almost 2 in both schemes ($\square$, $\bigcirc$). The orders of $E_{H^1}$ are almost 1 in both schemes ($\blacksquare$, $\bullet$). The right graph of Fig. 6 shows
Figure 5. The triangulation of $\bar{\Omega}$ for $N = 16$ in Example 2

Figure 6. Graphs of $E_{L^2}$ and $E_{H^1}$ versus $h$ in Example 2 with $\nu = 10^{-2}$ by $P_k$-element. $k = 1$ (left) and $k = 2$ (right)

Figure 7. Graphs of $E_{L^2}$ and $E_{H^1}$ versus $h$ in Example 2 with $\nu = 10^{-5}$ by $P_k$-element. $k = 1$ (left) and $k = 2$ (right)
Table 4. The values of errors and orders of the graph in Fig. 6 by $P_1$-element

| $N$ | order | $E_L$ | order | $E_H$ | order |
|-----|-------|-------|-------|-------|-------|
| 8   | 8.14E-02 | 8.14E-02 | 1.10E-01 | |
| 16  | 3.64E-02 | 1.10   | 2.10   | 4.36E-02 | 1.34 |
| 32  | 1.70E-02 | 1.10   | 2.05   | 1.87E-02 | 1.22 |
| 64  | 8.53E-03 | 0.99   | 1.94   | 9.18E-03 | 1.03 |
| $N$ | order | $E_L$ | order | $E_H$ | order |
| 8   | 8.97E-02 | 8.97E-02 | 1.09E-01 | |
| 16  | 3.68E-02 | 1.29   | 2.07   | 4.23E-02 | 1.37 |
| 32  | 1.78E-02 | 1.05   | 2.14   | 1.92E-02 | 1.14 |
| 64  | 8.90E-03 | 1.00   | 1.90   | 9.43E-03 | 1.03 |

Table 5. The values of errors and orders of the graph in Fig. 6 by $P_2$-element

| $N$ | order | $E_L$ | order | $E_H$ | order |
|-----|-------|-------|-------|-------|-------|
| 8   | 7.01E-02 | 4.50E-02 | 7.37E-02 | |
| 16  | 1.77E-02 | 1.99   | 2.98   | 1.85E-02 | 1.99 |
| 32  | 4.44E-03 | 2.00   | 3.01   | 4.63E-03 | 2.00 |
| 64  | 1.11E-03 | 2.00   | 3.00   | 1.15E-03 | 2.01 |
| $N$ | order | $E_L$ | order | $E_H$ | order |
| 8   | 6.31E-02 | 3.87E-02 | 6.60E-02 | |
| 16  | 1.58E-02 | 2.00   | 2.49   | 1.64E-02 | 2.01 |
| 32  | 3.98E-03 | 1.99   | 2.28   | 4.12E-03 | 1.99 |
| 64  | 9.90E-04 | 2.01   | 2.08   | 1.02E-03 | 2.01 |

Table 6. The values of errors and orders of the graph in Fig. 7 by $P_1$-element

| $N$ | order | $E_L$ | order | $E_H$ | order |
|-----|-------|-------|-------|-------|-------|
| 8   | 9.53E-02 | 9.53E-02 | 2.90E-01 | |
| 16  | 3.94E-02 | 1.27   | 1.19   | 2.00E-01 | 0.54 |
| 32  | 1.82E-02 | 1.11   | 1.65   | 1.09E-01 | 0.88 |
| 64  | 9.07E-03 | 1.00   | 1.95   | 5.12E-02 | 1.09 |
| $N$ | order | $E_L$ | order | $E_H$ | order |
| 8   | 9.70E-02 | 9.70E-02 | 2.23E-01 | |
| 16  | 3.93E-02 | 1.30   | 1.76   | 1.23E-01 | 0.86 |
| 32  | 1.89E-02 | 1.06   | 1.95   | 6.02E-02 | 1.03 |
| 64  | 9.45E-03 | 1.00   | 1.96   | 3.08E-02 | 0.97 |

the results of $P_2$-element and Table 5 shows the values of them. The convergence orders of $E_{L^2}$ with $\Delta t = O(h^2)$ are almost 2 in both schemes ($\cap$, $\circ$). The order of $E_{L^2}$ with $\Delta t = O(h^3)$ is almost 3 in Scheme LG' ($\square$) and almost 2 in Scheme GSLG ($\diamond$). The orders of $E_{H^1}$ are almost 2 in both schemes ($\blacksquare$, $\bullet$). These results
Table 7. The values of errors and orders of the graph in Fig. 7 by $P_2$-element

| $N$ | $E_{L^2}$ | $E_{H^1}$ | order | $E_{L^2}$ | $E_{H^1}$ | order |
|-----|-----------|-----------|-------|-----------|-----------|-------|
| 8   | 7.45E-02  | 4.74E-02  | 1.88E-01 | 1.25E-01  | 0.59      |
| 16  | 1.81E-02  | 2.04      | 6.77E-03 | 2.81      | 1.25E-01  | 0.59  |
| 32  | 3.94E+00  | -7.77     | 1.16E-03 | 2.81      | 1.25E-01  | 0.59  |
| 64  | 1.10E+00  | 1.84      | 1.17E-04 | 3.31      | 1.25E-01  | 0.59  |

are consistent with the theoretical ones of Scheme GSLG, $E_{L^2} = O(\Delta t + h^2 + h^k+1)$ and $E_{H^1} = O(\Delta t + h^2 + h^k)$.

Figure 7 shows the log-log graphs of $E_{L^2}$ and $E_{H^1}$ versus $h$ with $\nu = 10^{-5}$. The left graph shows the results of $P_1$-element and Table 6 shows the values of them. The convergence orders of $E_{L^2}$ with $\Delta t = O(h)$ are almost 1 in both schemes (□, ○). The orders of $E_{L^2}$ with $\Delta t = O(h^2)$ are almost 2 for small $h$ in both schemes (□, ○). The orders of $E_{H^1}$ are almost 1 in both schemes (□, ○). The right graph of Fig. 7 shows the results of $P_2$-element and Table 7 shows the values of them. The errors $E_{L^2}$ with $\Delta t = O(h^2)$ are too large at $N = 32$ and 64 to be plotted in the graph in Scheme LG’ (□) while the convergence order is almost 2 in Scheme GSLG (○). The order $E_{L^2}$ with $\Delta t = O(h^3)$ is almost 3 for small $h$ in Scheme LG’ (□) and almost 2 in Scheme GSLG (○). The errors $E_{H^1}$ are too large at $N = 32$ and 64 to be plotted in the graph in Scheme LG’ (□) while we can observe the convergence but the order is less than 2 in Scheme GSLG (○). In order to obtain the theoretical convergence order $O(h^2)$, it seems that finer mesh will be necessary.

7. Conclusions

We have presented a genuinely stable Lagrange–Galerkin scheme for convection-diffusion problems. In the scheme locally linearized velocities are used and the integration is executed exactly without numerical quadrature. For the $P_k$-element we have shown error estimates of $O(\Delta t + h^2 + h^{k+1})$ in $L^\infty(L^2)$-norm and of $O(\Delta t + h^2 + h^k)$ in $L^\infty(H^1)$-norm. We have also obtained error estimate, $c(\Delta t + h^2 + h^k)$ in $L^\infty(L^2)$-norm, where the coefficient $c$ is dependent on the exact solution $\phi$ but independent of the diffusion constant $\nu$. Numerical results have reflected these estimates. The extension to the Navier–Stokes equations will be discussed in a forthcoming paper.

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