ON ENTROPY FOR AUTOEQUIVALENCES OF THE DERIVED CATEGORY OF CURVES

KOHEI KIKUTA

ABSTRACT. To an exact endofunctor of a triangulated category with a split-generator, the notion of entropy is given by Dimitrov–Haiden–Katzarkov–Kontsevich, which is a (possibly negative infinite) real-valued function of a real variable. It is important to evaluate the value of the entropy at zero in relation to the topological entropy. In this paper, we study the entropy at zero of an autoequivalence of the derived category of a complex smooth projective curve, and prove that it coincides with the natural logarithm of the spectral radius of the induced automorphism on its numerical Grothendieck group.

1. Introduction

To an exact endofunctor of a triangulated category with a split-generator, the notion of entropy is given by Dimitrov–Haiden–Katzarkov–Kontsevich in [DHKK]. It is a (possibly negative infinite) real-valued function of a real variable motivated by an analogy with the topological entropy. It is known that the topological entropy of a surjective holomorphic endomorphism of a compact Kähler manifold coincides with the natural logarithm of the spectral radius of the induced action on the cohomology, which is the fundamental theorem of Gromov-Yomdin [Gro1, Gro2, Yom] (see also Theorem 3.6 in [Ogu]). Concerning the entropy at zero of an exact endofunctor of a complex smooth proper variety, there is a lower bound, under a certain technical assumption, by the natural logarithm of the spectral radius of the induced action on the Hochschild homology (Theorem 2.9 in [DHKK]). If the variety is projective, then the categorical and the topological entropies of surjective endomorphisms satisfying the assumption coincide (Theorem 2.12 in [DHKK]). It is indeed possible to show this without the technical assumption used in [DHKK], which is given in [KT].

In this paper, we study the entropy of autoequivalences of the derived category of a smooth projective curve, and prove the following theorem, which is a natural generalization of the fundamental theorem of Gromov-Yomdin.
Theorem 1.1 (Theorem 3.1). Let $C$ be a complex smooth projective curve and $F$ an autoequivalence of the bounded derived category $D^b(C)$ of coherent sheaves on $C$. The entropy $h(F)$ coincides with the natural logarithm of the spectral radius $\rho([F])$ of the induced automorphism $[F]$ on the numerical Grothendieck group $N(C)$ of $D^b(C)$. In particular, $\rho([F])$ is an algebraic number.

The contents of this paper is as follows. In Section 2, we recall the definition and some basic properties of the entropy of exact endofunctors by [DHKK]. We shall study the entropy for an autoequivalence of the bounded derived category of coherent sheaves on a smooth projective curve in Section 3. The non-trivial case is when a curve is an elliptic curve. We shall introduce the notion of the autoequivalence of type-$m$ (Definition 3.7), which behaves very well in two points: it gives a representative of a conjugacy class of the automorphism group of the numerical Grothendieck group preserving the Euler form (Proposition 3.8), and it enables us to compute explicitly the entropy (Proposition 3.9). In Section 4, we shall recall the notion of LLS-period and give a proof of Proposition 3.8, the key proposition for our main theorem.

Acknowledgements. I am grateful to my supervisor, Professor Atsushi Takahashi for guidances, supports and encouragements. He suggested to us Conjecture 2.14 motivated by the theorem of Gromov–Yomdin and Theorems 2.9, 2.12 in [DHKK]. I would like to thank Yuuki Shiraishi for giving many comments on this paper.

2. Preliminaries

2.1. Notations and terminologies. Throughout this paper, we work over the base field $\mathbb{C}$ and all triangulated categories are $\mathbb{C}$-linear and not equivalent to the zero category. The translation functor on a triangulated category is denoted by $[1]$.

A triangulated category $\mathcal{T}$ is called split-closed if every idempotent in $\mathcal{T}$ splits, namely, if it contains all direct summands of its objects, and it is called thick if it is split-closed and closed under isomorphisms. For an object $M \in \mathcal{T}$, we denote $\langle M \rangle$ by the smallest thick triangulated subcategory containing $M$. An object $G \in \mathcal{T}$ is called a split-generator if $\langle G \rangle = \mathcal{T}$. A triangulated category $\mathcal{T}$ is said to be of finite type if for all $M, N \in \mathcal{T}$ we have $\sum_{n \in \mathbb{Z}} \dim_{\mathbb{C}} \Hom_{\mathcal{T}}(M, N[n]) < \infty$. 
2.2. **Complexity.** From now on, $\mathcal{T}$ denotes a triangulated category of finite type.

**Definition 2.1** (Definition 2.1 in [DHKK]). For each $M, N \in \mathcal{T}$, define the function $\delta_{\mathcal{T},t}(M, N) : \mathbb{R} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ in $t$ by

$$
\delta_{\mathcal{T},t}(M, N) := \begin{cases} 
0 & \text{if } N \cong 0 \\
\inf \left\{ \sum_{i=1}^{p} \exp(n_it) \middle| 0 \to M[n_1] \to \cdots \to M[n_p] \to N \oplus N' \to 0 \right\} & \text{if } N \in \langle M \rangle \\
\infty & \text{if } N \not\in \langle M \rangle.
\end{cases}
$$

The function $\delta_{\mathcal{T},t}(M, N)$ is called the complexity of $N$ with respect to $M$.

**Remark 2.2.** If $\mathcal{T}$ has a split-generator $G$ and $M \in \mathcal{T}$ is not isomorphic to a zero object, then an inequality $1 \leq \delta_{\mathcal{T},0}(G, M) < \infty$ holds.

We recall some basic properties of the complexity.

**Lemma 2.3** (Proposition 2.3 in [DHKK]). Let $M_1, M_2, M_3 \in \mathcal{T}$.

(i) If $M_1 \cong M_3$, then $\delta_{\mathcal{T},t}(M_1, M_2) = \delta_{\mathcal{T},t}(M_3, M_2)$.

(ii) If $M_2 \cong M_3$, then $\delta_{\mathcal{T},t}(M_1, M_2) = \delta_{\mathcal{T},t}(M_1, M_3)$.

(iii) If $M_2 \not\cong 0$, then $\delta_{\mathcal{T},t}(M_1, M_3) \leq \delta_{\mathcal{T},t}(M_1, M_2) \delta_{\mathcal{T},t}(M_2, M_3)$.

(iv) Let $\mathcal{T}'$ be a triangulated category of finite type. We have $\delta_{\mathcal{T}',t}(F(M_1), F(M_2)) \leq \delta_{\mathcal{T},t}(M_1, M_2)$ for any exact functor $F : \mathcal{T} \to \mathcal{T}'$.

**Lemma 2.4** (Lemma 2.4 in [DHKK]). Let $\mathcal{D}^b(\mathbb{C})$ be the bounded derived category of finite dimensional $\mathbb{C}$-vector spaces. For $M \in \mathcal{D}^b(\mathbb{C})$, we have the following inequality

$$
\delta_{\mathcal{D}^b(\mathbb{C}),t}(\mathbb{C}, M) = \sum_{l \in \mathbb{Z}} \left( \dim_{\mathbb{C}} H^l(M) \right) \cdot e^{-lt}.
$$

2.3. **Entropy of endofunctors.**

**Definition 2.5** (Definition 2.5 in [DHKK]). Let $G$ be a split-generator of $\mathcal{T}$ and $F$ an exact endofunctor of $\mathcal{T}$ such that $F^nG \not\cong 0$ for $n \geq 0$. The entropy of $F$ is the function $h_t(F) : \mathbb{R} \to \{-\infty\} \cup \mathbb{R}$ given by

$$
h_t(F) := \lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{T},t}(G, F^nG).
$$

It follows from Lemma 2.6 in [DHKK] that the entropy is well-defined.

**Lemma 2.6.** Let $G, G'$ be split-generators of $\mathcal{T}$ and $F$ an exact endofunctor of $\mathcal{T}$ such that $F^nG, F^nG' \not\cong 0$ for $n \geq 0$. Then

$$
h_t(F) = \lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{T},t}(G, F^nG').
$$
Proof. It follows from Lemma 2.3 (iii) and (iv) that
\[ \delta_{\mathcal{T},t}(G, F^m G') \leq \delta_{\mathcal{T},t}(G, F^n G) \delta_{\mathcal{T},t}(F^n G, F^m G') \leq \delta_{\mathcal{T},t}(G, G') \delta_{\mathcal{T},t}(G, F^n G). \]
Similarly, we have \( \delta_{\mathcal{T},t}(G, F^n G) \leq \delta_{\mathcal{T},t}(G', G) \delta_{\mathcal{T},t}(G, F^n G') \), which yields the statement. \( \square \)

Lemma 2.7 (Section 2 in [DHKK]). Let \( G \) be a split-generator of \( \mathcal{T} \) and \( F_1, F_2 \) exact endofunctors of \( \mathcal{T} \) such that \( F_i^n G \neq 0 \) for \( i = 1, 2 \) and \( n \geq 0 \).

1. If \( F_1 \cong F_2 \), then \( h_t(F_1) = h_t(F_2) \).
2. We have \( h_t(F^n_m) = mh_t(F_1) \) for \( m \geq 1 \).
3. We have \( h_t([m]) = mt \) for \( m \in \mathbb{Z} \).
4. If \( F_1 F_2 \cong F_2 F_1 \), then \( h_t(F_1 F_2) \leq h_t(F_1) + h_t(F_2) \).
5. If \( F_1 = F_2 [m] \) for \( m \in \mathbb{Z} \), then \( h_t(F_1) = h_t(F_2) + mt \).

Lemma 2.8. Let \( G \) be a split-generator of \( \mathcal{T} \) and \( F_1, F_2 \) exact endofunctors of \( \mathcal{T} \) such that \( F_i^n G \neq 0 \) for \( i = 1, 2 \) and \( n \geq 0 \). Then we have \( h_t(F_1 F_2) = h_t(F_2 F_1) \).

Proof. It follows from Lemma 2.3 (iii) and (iv) that
\[ \delta_{\mathcal{T},t}(G, (F_1 F_2)^n G) = \delta_{\mathcal{T},t}(G, F_1 (F_2 F_1)^{n-1} F_2 G) \]
\[ \leq \delta_{\mathcal{T},t}(G, F_1 G) \delta_{\mathcal{T},t}(F_1 G, F_1 (F_2 F_1)^{n-1} F_2 G) \]
\[ \leq \delta_{\mathcal{T},t}(G, F_1 G) \delta_{\mathcal{T},t}(G, (F_2 F_1)^{n-1} G) \]
\[ \cdot \delta_{\mathcal{T},t}((F_2 F_1)^{n-1} G, (F_2 F_1)^{n-1} F_2 G) \]
\[ \leq \delta_{\mathcal{T},t}(G, F_1 G) \delta_{\mathcal{T},t}(G, F_2 G) \delta_{\mathcal{T},t}(G, (F_2 F_1)^{n-1} G), \]
which gives \( h_t(F_1 F_2) \leq h_t(F_2 F_1) \). Similarly, we can show \( h_t(F_2 F_1) \leq h_t(F_1 F_2) \), which yields the statement. \( \square \)

Lemma 2.9. Let \( F \) be an exact endofunctor of \( \mathcal{T} = \langle G \rangle \) such that \( F^n G \neq 0 \) for \( n \geq 0 \), \( F' \) an autoequivalence of \( \mathcal{T} \), and \( F'^{-1} \) the quasi-inverse of \( F' \). The entropy is a class function, namely, \( h_t(F' F F'^{-1}) = h_t(F) \).

Proof. The inequality \( h_t(F' F F'^{-1}) \leq h_t(F) \) follows from
\[ \delta_{\mathcal{T},t}(G, (F' F F'^{-1})^n G) = \delta_{\mathcal{T},t}(G, F' F^n F'^{-1} G) \]
\[ \leq \delta_{\mathcal{T},t}(G, F' F^n G) \delta_{\mathcal{T},t}(F' F^n G, F' F^n F'^{-1} G) \]
\[ \leq \delta_{\mathcal{T},t}(G, F' G) \delta_{\mathcal{T},t}(F' G, F' F^n G) \delta_{\mathcal{T},t}(G, F'^{-1} G) \]
\[ \leq \{ \delta_{\mathcal{T},t}(G, F' G) \delta_{\mathcal{T},t}(G, F'^{-1} G) \} \delta_{\mathcal{T},t}(G, F^n G) \]
Considering \( F'' := F'^{-1} F' \), we have the reverse inequality. \( \square \)
2.4. The cases of smooth projective varieties. Let $X$ be a smooth projective variety. The bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves over $X$ is of finite type and it has a split-generator. In particular, $G := \bigoplus_{i=1}^{\dim X+1} \mathcal{O}_X(i)$ and its dual $G^* := \bigoplus_{i=1}^{\dim X+1} \mathcal{O}_X(-i)$ are known to be split-generators (cf. Theorem 4 in [Orl]). The following proposition enables us to compute entropies.

Proposition 2.10 (cf. Theorem 2.7 in [DHKK]). Let $G, G'$ be split-generators of $\mathcal{D}^b(X)$ and $F$ an autoequivalence of $\mathcal{D}^b(X)$. The entropy $h_t(F)$ is given by

$$h_t(F) = \lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(X)},t(G, F^n G'),$$

where

$$\delta_{\mathcal{D}^b(X)},t(M, N) := \sum_{m \in \mathbb{Z}} \left( \dim \text{Hom}_{\mathcal{D}^b(X)}(M, N[m]) \right) \cdot e^{-mt}, \quad M, N \in \mathcal{D}^b(X).$$

Proof. The following is proven in the proof of Theorem 2.7 in [DHKK].

Lemma 2.11. There exist $C_1(t), C_2(t)$ for $t \in \mathbb{R}$ such that

$$C_1(t) \delta_{\mathcal{D}^b(X)},t(G, M) \leq \delta_{\mathcal{D}^b(X)},t(G, M) \leq C_2(t) \delta_{\mathcal{D}^b(X)},t(G, M), \quad M \in \mathcal{D}^b(X).$$

In particular, for each $M \in \mathcal{D}^b(X)$ we have

$$\lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(X)},t(G, M) = \lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(X)},t(G, M). \quad (2.2)$$

Together with Lemma 2.6 we have

$$h_t(F) = \lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(X)},t(G, F^n G') = \lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(X)},t(G, F^n G').$$

We finished the proof of the proposition. \qed

In order to study the structure of the entropy we prepare some terminologies. For $M, N \in \mathcal{D}^b(X)$, set

$$\chi(M, N) := \sum_{n \in \mathbb{Z}} (-1)^n \dim \text{Hom}_{\mathcal{D}^b(X)}(M, N[n]). \quad (2.3)$$

It naturally induces a bilinear form on the Grothendieck group $K_0(X)$ of $\mathcal{D}^b(X)$, called the Euler form, which is denoted by the same letter $\chi$. Then numerical Grothendieck group $\mathcal{N}(X)$ is defined as the quotient of $K_0(X)$ by the radical of $\chi$ (which is well-defined by the Serre duality). It is known that $\mathcal{N}(X)$ is a free abelian group of finite rank by Hirzebruch-Riemann-Roch theorem.
The group of isomorphism classes of autoequivalences of \( D^b(X) \) is denoted by \( \text{Auteq}(D^b(X)) \). We denote the group of isomorphisms of \( N(X) \) preserving \( \chi \) by \( \text{Aut}_Z(N(X), \chi) \). Then we have the group homomorphism

\[
\text{Auteq}(D^b(X)) \to \text{Aut}_Z(N(X), \chi), \quad F \mapsto [F].
\]

**Definition 2.12.** For an \( F \in \text{Auteq}(D^b(X)) \), the spectral radius \( \rho([F]) \) is the maximum of absolute values of eigenvalues of the induced endomorphism \( [F] \in \text{Aut}_Z(N(X), \chi) \).

From now on, we shall only consider entropies at \( t = 0 \). For simplicity, set \( \delta_{D^b(X)} := \delta_{D^b(X),0} \), \( h(F) := h_0(F) \) and so on.

**Definition 2.13.** For an \( f \in \text{Aut}(X) \), the categorical entropy \( h_{\text{cat}}(f) \) is the entropy \( h(F^*) \) of its derived functor at \( t = 0 \).

It is known that the topological entropy \( h_{\text{top}}(f) \) of a surjective holomorphic endomorphism \( f \in \text{End}(M) \) of a compact Kähler manifold \( M \) coincides with the natural logarithm of the spectral radius \( \rho(f) \) of the induced action on the cohomology, which is the fundamental theorem of Gromov-Yomdin (see also Theorem 3.6 in [Ogu]). Concerning the entropy \( h(F) \) of an exact endofunctor \( F \) of \( D^b(X) \) of a complex smooth proper variety \( X \), there is a lower bound, under a certain technical assumption, by the natural logarithm of the spectral radius of the induced action on the Hochschild homology (Theorem 2.9 in [DHKK]). If \( X \) is projective, \( h_{\text{cat}}(f) = \log \rho(f) \) for a surjective endomorphism \( f \in \text{End}(X) \) satisfying the assumption (Theorem 2.12 in [DHKK]). It is indeed possible to show that \( h_{\text{cat}}(f) = \log \rho([f^*]) = \log \rho'(f) = h_{\text{top}}(f) \) without the technical assumption used in [DHKK], which is given in [KT].

From the above, it is natural to expect the following conjecture in general.

**Conjecture 2.14.** Let \( X \) be a smooth projective variety. For each \( F \in \text{Auteq}(D^b(X)) \), the entropy should coincide with the natural logarithm of the spectral radius:

\[
h(F) = \log \rho([F]). \tag{2.4}
\]

The conjecture is true in the case of the ample canonical or anti-canonical sheaf ([KT]).

### 3. Entropy for curves

The main result of this paper is the following.

**Theorem 3.1.** Let \( C \) be a smooth projective curve. For each \( F \in \text{Auteq}(D^b(C)) \), the entropy coincides with the natural logarithm of the spectral radius:

\[
h(F) = \log \rho([F]). \tag{3.1}
\]
3.1. Standard autoequivalences.

Definition 3.2. The standard autoequivalence group is the subgroup of \( \text{Auteq}(\mathcal{D}^b(X)) \) given by

\[
\text{Auteq}^{st}(\mathcal{D}^b(X)) := \text{Aut}(X) \ltimes (\text{Pic}(X) \times \mathbb{Z}[1]).
\]

Proposition 3.3. Let \( C \) be a smooth projective curve. For each standard autoequivalence \( F \in \text{Auteq}^{st}(\mathcal{D}^b(C)) \), we have \( h(F) = 0 \).

Proof. Each standard autoequivalence \( F \) is represented as \( F = f^*(- \otimes \mathcal{L})[m] \). Since Lemma 2.7 (v) gives \( h(F) = h(f^*(- \otimes \mathcal{L})) \), we can assume \( F = f^*(- \otimes \mathcal{L}) \). Let \( \mathcal{O}_C(1) \) be a very ample invertible sheaf and set \( G := \mathcal{O}_C(1) \oplus \mathcal{O}_C(2) \). Recall that \( G \) and \( G^* = \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-2) \) are split-generators of \( \mathcal{D}^b(C) \). By Lemma 2.11, we have

\[
h(F) = \lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(C)}(G, F^n G^*) = \lim_{n \to \infty} \frac{1}{n} \log \delta'_{\mathcal{D}^b(C)}(G, F^n G^*).
\]

If \( \deg \mathcal{L} > 0 \), then \( G^* \otimes F^n G^* \) becomes a sum of invertible sheaves with positive degrees for large \( n \), and hence we have \( \text{Hom}_{\mathcal{D}^b(X)}(G, F^n G^*[i]) = 0 \) for \( i \neq 0 \) and large \( n \). If \( \deg \mathcal{L} \leq 0 \), then \( G^* \otimes F^n G^* \) becomes a sum of invertible sheaves with negative degrees for \( n \geq 0 \), and hence we have \( \text{Hom}_{\mathcal{D}^b(X)}(G, F^n G^*[i]) = 0 \) for \( i \neq 0 \) and \( n \geq 0 \). Since \( f^* \) acts trivially on \( \mathcal{N}(C) \), we obtain

\[
\lim_{n \to \infty} \frac{1}{n} \log \delta'_{\mathcal{D}^b(C)}(G, F^n G^*) = \lim_{n \to \infty} \frac{1}{n} \log |\chi(G, F^n G^*)| = \lim_{n \to \infty} \frac{1}{n} \log |\chi(G, G^* \otimes \mathcal{L}^n)|.
\]

It follows from the polynomial-growth of \( \chi(G, G^* \otimes \mathcal{L}^n) \) (cf. Lemma 2.14 in \([\text{DHKK}]\)) that

\[
h(F) = \lim_{n \to \infty} \frac{1}{n} \log |\chi(G, G^* \otimes \mathcal{L}^n)| = 0.
\]

We finished the proof of the proposition. \( \square \)

Let \( C \) be a smooth projective curve which is not an elliptic curve. Then it follows from \([\text{BO}]\) that \( \text{Auteq}(\mathcal{D}^b(C)) = \text{Auteq}^{st}(\mathcal{D}^b(C)) \). Therefore, for the proof of our main theorem we only need to consider the case of an elliptic curve.

3.2. Elliptic curves. Let \( X = (E, x_0) \) be an elliptic curve with a closed point \( x_0 \in E \). The numerical Grothendieck group \( \mathcal{N}(E) \) has a canonical basis given by \( [\mathcal{O}_E] \) and \( [\mathcal{O}_{x_0}] \) and the Euler form \( \chi \) with respect to this basis is given by

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\]

which yields the isomorphism \( \text{Aut}_{\mathbb{Z}}(\mathcal{N}(E), \chi) \cong \text{SL}(2, \mathbb{Z}) \). Consider the Fourier-Mukai functor \( \Phi_{\mathcal{P}} := \mathbb{R}p_{2*}(\mathbb{L}p_1^*(-) \otimes^{\mathbb{L}} \mathcal{P}) \), an autoequivalence of \( \mathcal{D}^b(E) \) first introduced by Mukai \([\text{Muk}]\), where \( \mathcal{P} \in \text{coh}(E \times E) \) is the Poincaré bundle and \( p_1 \) (resp. \( p_2 \)) is the projection
\( E \times E \rightarrow E \) to the first (resp. second) component. Set \( S := \Phi_p, T := - \otimes \mathcal{O}_E(x_0) \). These satisfy the following relations [Muk, ST]:

\[
S^2 \cong (-1)^*[1], \ (TS)^3 \cong S^2.
\] (3.3)

With a canonical basis of \( \mathcal{N}(E) \) given by \([\mathcal{O}_E]\) and \([\mathcal{O}_{x_0}]\), we have

\[
[S] = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ [T] = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\] (3.4)

It is a well-known fact that the natural map \( \phi : \text{Auteq}(\mathcal{D}^b(E)) \rightarrow \text{Aut}(\mathcal{N}(E), \chi) \) is surjective and gives the following exact sequence

\[
\{1\} \rightarrow \text{Aut}(E) \times (\text{Pic}^0(E) \times \mathbb{Z}[2]) \rightarrow \text{Auteq}(\mathcal{D}^b(E)) \rightarrow \text{SL}(2, \mathbb{Z}) \rightarrow \{1\}. \]

(3.5)

**Lemma 3.4.** The map \( h : \text{Auteq}(\mathcal{D}^b(E)) \rightarrow \mathbb{R}_{\geq 0}, F \mapsto h(F) \) factors through \( \text{SL}(2, \mathbb{Z}) \).

**Proof.** Set \( \mathcal{O}_E(1) := \mathcal{O}_E(3x_0) \) and \( G := \mathcal{O}_E(1) \oplus \mathcal{O}_E(2) \). Suppose that an element \( F \in \text{Auteq}(\mathcal{D}^b(E)) \) is of the form \( F = F'F_1 \) with \( F' \in \text{Auteq}(\mathcal{D}^b(E)) \) and \( F_1 \in \text{Aut}(E) \times (\text{Pic}^0(E) \times \mathbb{Z}[2]) \). Then there exist \( F_2, \ldots, F_n \in \text{Aut}(E) \times (\text{Pic}^0(E) \times \mathbb{Z}[2]) \) such that \( F^n = (F'F_1)^n = F^nF_n \cdots F_1 \). We have

\[
\delta_{\mathcal{D}^b(E)}(G, F^nG^*) \leq \delta_{\mathcal{D}^b(E)}(G, F^nG)\delta_{\mathcal{D}^b(E)}(G, F_1 \cdots F_1G^*),
\]

and hence,

\[
h(F) \leq h(F') + \lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(E)}(G, F_n \cdots F_1G^*).
\]

Since \( F_i \) is of the form \( f_i(- \otimes \mathcal{L}_i)[2\ell] \) for some \( f_i \in \text{Aut}(E) \), \( \mathcal{L}_i \in \text{Pic}^0(E) \) and \( \ell_i \in \mathbb{Z}, \ G^* \otimes F_n \cdots F_1G^*[-2\ell_n - \cdots - 2\ell_1] \) is a sum of anti-ample invertible sheaves. Therefore, we have \( \delta'_{\mathcal{D}^b(E)}(G, F_n \cdots F_1G^*) = |\chi(G, F_n \cdots F_1G^*)| \). Since \( F_i \) acts trivially on \( \mathcal{N}(E) \), we have \( |\chi(G, F_n \cdots F_1G^*)| = |\chi(G, G^*)| \), which implies

\[
\lim_{n \to \infty} \frac{1}{n} \log \delta_{\mathcal{D}^b(E)}(G, F_n \cdots F_1G^*) = \lim_{n \to \infty} \frac{1}{n} \log \delta'_{\mathcal{D}^b(E)}(G, F_n \cdots F_1G^*) \quad \text{(by Lemma 2.11)}
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \log |\chi(G, F_n \cdots F_1G^*)| = \lim_{n \to \infty} \frac{1}{n} \log |\chi(G, G^*)| = 0.
\]

Therefore \( h(F) \leq h(F') \). We also have \( h(F') \leq h(F) \) since \( F' = FF_1^{-1} \) and \( F_1^{-1} \) belongs to \( \text{Aut}(E) \times (\text{Pic}^0(E) \times \mathbb{Z}[2]) \). \( \square \)

**Lemma 3.5.** The entropies of \( S, T, TS \) are all equal to zero.

**Proof.** From the relations (3.3) and Lemma 2.7 (ii) \( h(S) = 0 \) and \( h(TS) = 0 \) follow, \( h(T) = 0 \) is given by Proposition 3.3. \( \square \)
The characteristic polynomial of a matrix $A \in \text{SL}(2, \mathbb{Z})$ is given by

$$x^2 - (\text{tr}A)x + 1 = 0. \quad (3.6)$$

If $|\text{tr}[F]| \leq 2$, then $\rho(A) = 1$. If $\text{tr}[F] > 2$, then eigenvalues of $A$ are $\rho, \rho^{-1}$, where $\rho$ is a real number greater than 1. In particular, the spectral radius $\rho(A)$ of $A$ is $\rho$, which is an algebraic number. Since $h(F) = h(S^2F)$ and $[S^2F] = -[F]$ for any $F \in \text{Auteq}(\mathcal{D}^b(E))$, we do not need to consider the case of $\text{tr}[F] < -2$.

**Proposition 3.6.** For every autoequivalence $F \in \text{Auteq}(\mathcal{D}^b(E))$ with $|\text{tr}[F]| \leq 2$,

$$h(F) = 0 = \log \rho([F]).$$

**Proof.** First we consider the case of $\text{tr}[F] = -1, 0, 1$. From the form of characteristic polynomial of $[F]$, we have

$$[F]^2 = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad [F]^3 = \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From Lemma 2.7(ii), clearly their entropies are equal to 0. Then in the case of $|\text{tr}[F]| = 2$, representatives of conjugacy classes are given as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = [(TST)^n] \quad (n > 0).$$

We have

$$h((TST)^n) = h((S^{-1}T^{-1}S^{-1})^n) \quad \text{(by (3.3))}$$

$$= h(S^{-1}T^{-1}S^{-2}T^{-1} \cdots T^{-1}S^{-2}T^{-1}S^{-1})$$

$$= h(S^{-2}T^{-1}S^{-2}T^{-1} \cdots T^{-1}S^{-2}T^{-1})$$

$$= h(T^{-n}) = nh(T^{-1}) = 0.$$

Clearly $\rho([F]) = 1$ in both cases. We finished the proof of the proposition. \qed

We shall introduce the notion of the autoequivalence of type-$\mathbf{m}$, which behaves very well in two points: it gives a representative of a conjugacy class of $\text{SL}(2, \mathbb{Z})$, and it enables us to compute explicitly the entropy.

**Definition 3.7.** Let $\mathbf{m} = (m_2, \ldots, m_1)$, $n \geq 1$ be an ordered sequence of positive integers. An autoequivalence $F \in \text{Auteq}(\mathcal{D}^b(E))$ is of type-$\mathbf{m}$ if $F$ is isomorphic to following types:

$$F_{\mathbf{m}} = \begin{cases} S^{2T^{m_2n}}ST^{-m_2n-1}S \cdots T^{m_2ST^{-m_1}}S & \text{if } n: \text{odd} \\ T^{m_2n}ST^{-m_2n-1}S \cdots T^{m_2ST^{-m_1}}S & \text{if } n: \text{even}. \end{cases}$$
Proposition 3.8. For any \( F \in \text{Auteq}(\mathcal{D}^b(E)) \) with \( \text{tr}[F] > 2 \), there exists an autoequivalence \( F_m \) of type-\( \mathbf{m} \) such that \( F \) is conjugate to \( F_m \).

Since the proof is elementary but technical, we shall give it in Appendix.

Proposition 3.9. For each \( F \in \text{Auteq}(\mathcal{D}^b(E)) \) of type-\( \mathbf{m} \), we have \( h(F) = \log \rho([F]) \).

**Proof.** First, we show the following lemma.

Lemma 3.10. Let \( F \) be an autoequivalence of type-\( \mathbf{m} \). For each indecomposable locally free sheaf \( \mathcal{E} \) on \( E \) with positive degree, \( F(\mathcal{E}) \) is an indecomposable locally free sheaf with positive degree up to even translations.

**Proof.** Note that each autoequivalence sends an indecomposable object to an indecomposable one. It follows from equation (3.4) that if \( \mathcal{E} \) is an indecomposable locally free sheaf with non-zero degree then \( S(\mathcal{E}) \) is a locally free sheaf up to translations since the abelian category \( \text{coh}(E) \) is hereditary. Obviously, \( T \) sends a locally free sheaf to a locally free sheaf. In addition, for positive numbers \( r_1, d_1 \), we have

\[
\begin{align*}
(r_2) & := [T^{-m_2i-1}S] (r_1) = \begin{pmatrix} 0 & 1 \\ -1 & -m_2i-1 \end{pmatrix} r_1 = \begin{pmatrix} d_1 \\ -r_1 - m_2i-1 d_1 \end{pmatrix} \in \left( \mathbb{Z}_{>0}, \mathbb{Z}_{<0} \right), \\
(d_2) & := [T^{m_2i}S] (d_2) = \begin{pmatrix} 0 & 1 \\ -1 & m_2i \end{pmatrix} d_2 = \begin{pmatrix} d_2 \\ -r_2 + m_2d_2 \end{pmatrix} \in \left( \mathbb{Z}_{<0}, \mathbb{Z}_{<0} \right), \\
(r_3) & := [T^{-m_2i+1}S] (r_3) = \begin{pmatrix} 0 & 1 \\ -1 & -m_2i+1 \end{pmatrix} r_3 = \begin{pmatrix} d_3 \\ -r_3 - m_2i+1 d_3 \end{pmatrix} \in \left( \mathbb{Z}_{<0}, \mathbb{Z}_{>0} \right), \\
(d_3) & := [T^{m_2i+2}S] (d_3) = \begin{pmatrix} 0 & 1 \\ -1 & m_2i+2 \end{pmatrix} d_3 = \begin{pmatrix} d_4 \\ -r_4 + m_2i+2d_4 \end{pmatrix} \in \left( \mathbb{Z}_{>0}, \mathbb{Z}_{>0} \right),
\end{align*}
\]

which implies that \( F(\mathcal{E}) \) is a locally free sheaf with positive degree up to even translations. \( \Box \)

For any positive integer \( l \), \( F^l \mathcal{O}_E(1), F^l \mathcal{O}_E(2) \) are indecomposable locally free sheaves with positive degrees up to even translations by Lemma 3.10. Hence, it easily follows from Atiyah’s results (cf. Lemma 1.1 in [Har]) that \( \text{Hom}_{\mathcal{D}^b(E)}(G^*, F^lG[1]) = 0 \) for \( l \geq 0 \) and any odd integer \( n \), which gives

\[
\begin{align*}
h(F) &= \lim_{l \to \infty} \frac{1}{l} \log \delta_{\mathcal{D}^b(E)}(G^*, F^lG) = \lim_{l \to \infty} \frac{1}{l} \log \delta'_{\mathcal{D}^b(E)}(G^*, F^lG) \quad \text{(by Lemma 2.11)} \\
&= \lim_{l \to \infty} \frac{1}{l} \log \chi(G \otimes F^lG).
\end{align*}
\]

We obtain \( \lim_{l \to \infty} \sqrt[l]{\chi(G \otimes F^lG)} = \rho([F]) \), in this case by some elementary computations of linear algebra, which finished the proof of Proposition 3.9. \( \Box \)
By Propositions 3.6, 3.8 and 3.9 we obtain the following

**Theorem 3.11.** For each \( F \in \text{Auteq}(\mathcal{D}^b(E)) \), we have \( h(F) = \log \rho([F]) \).

To summarize, we finished the proof of our main theorem, Theorem 3.1.

4. APPENDIX

4.1. LLS-period. This subsection is based on Chapter 7 in [Kar].

**Definition 4.1** (Definition 7.12 in [Kar]). Let \( A \) be a matrix in \( \text{SL}(2, \mathbb{Z}) \) represented as

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

If \( 0 < a \leq c < d \), we call \( A \) a reduced matrix.

**Remark 4.2.** The definition of a reduced matrix is slightly different from the one in [Kar], where \( A \) is a reduced matrix if and only if \( 0 \leq a \leq c < d \). A reduced matrix in the sense of [Kar] with \( a = 0 \) is represented as

\[
\begin{pmatrix} 0 & -1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & d - 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad d > 2,
\]

namely, it is conjugate to a reduce matrix in the sense of this paper. Thus, for the proof of Proposition 3.8 it is sufficient to consider only the case of \( a > 0 \).

**Proposition 4.3** (Theorem 7.13 in [Kar]). For any matrix \( A \in \text{SL}(2, \mathbb{Z}) \) with \( \text{tr} A > 2 \), there exists a reduced matrix \( A' \) such that \( A \) is conjugate to \( A' \).

For each \( A \in \text{SL}(2, \mathbb{Z}) \) with \( \text{tr} A > 2 \), one can associate a cyclically ordered sequence \( \text{LLS}(A) \) of an even length with positive integers, called the LLS-period. For the details, see Definition 7.10 in [Kar].

**Proposition 4.4** (Theorem 7.14 in [Kar]). Let \( A \in \text{SL}(2, \mathbb{Z}) \) be a reduced matrix. Suppose that there exists positive integers \( a_1, \ldots, a_{2n-1}, a_{2n}, n \geq 1 \) such that

\[
\frac{c}{a} = [a_1; a_2, \ldots, a_{2n-1}] := a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_{2n-2} + \frac{1}{a_{2n-1}}}}}, \quad \lfloor \frac{d - 1}{c} \rfloor = a_{2n},
\]

where \( [a_1; a_2, \ldots, a_{2n-1}] \) is the canonical continued fraction representation of \( c/a \) and \( \lfloor \frac{d - 1}{c} \rfloor \) is the integer part of \( (d - 1)/c \). Then the LLS-period \( \text{LLS}(A) \) of \( A \) is given by

\[
\text{LLS}(A) = (a_1, a_2, \ldots, a_{2n-1}, a_{2n}). \tag{4.1}
\]

**Proposition 4.5** (Proposition 7.11 in [Kar]). For reduced matrices \( A, A' \in \text{SL}(2, \mathbb{Z}) \) with \( \text{tr} A, \text{tr} A' > 2 \), \( A \) is conjugate to \( A' \) if and only if \( \text{LLS}(A) \) coincides with \( \text{LLS}(A') \).
4.2. The proof of Proposition 3.8. Recall that \( m = (m_{2n}, \ldots, m_1) \), \( n > 1 \), where \( m_i \) are all positive integers.

**Proposition 4.6** (Proposition 3.8). For each \( F \in \text{Auteq}(\mathcal{D}_b(E)) \) with \( \text{tr}[F] > 2 \), there exists \( F_m \in \text{Auteq}(\mathcal{D}_b(E)) \) of type-\( m \) such that \( [F] \) is conjugate to \( [F_m] \).

**Proof.** The following lemma is essential.

**Lemma 4.7.** For each \( F_m \in \text{Auteq}(\mathcal{D}_b(E)) \) of type-\( m \), \( [F_m] \) is a reduced matrix such that \( \text{LLS}([F_m]) = m \).

**Proof.** Set \( [F_m] = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \), \( \alpha_n := b_n - m_1a_n \), \( \beta_n := d_n - m_1c_n \).

We shall show the following five properties by induction on \( n \): (i) \( 0 < a_n \), (ii) \( a_n \leq c_n \), (iii) \( \frac{c_n}{a_n} = [m_{2n}; m_{2n-1}, \ldots, m_2] \), (iv) \( 0 \leq \alpha_n < a_n \), (v) \( 0 < \beta_n \leq c_n \).

If \( n = 1 \), then all the properties follows from the computation

\[
[S^2T^{m_2}ST^{-m_1}S] \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} 1 & m_1 \\ m_2 & m_1m_2 + 1 \end{pmatrix}.
\]

Suppose that the properties holds for \( n \) and we shall show them for \( n + 1 \). The property (i) is clear by

\[
\begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & m_{2n+1} \\ m_{2n+2} & m_{2n+1}m_{2n+2} + 1 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} = \begin{pmatrix} a_n + c_nm_{2n+1} & b_n + d_nm_{2n+1} \\ a_nm_{2n+2} + c_n(m_{2n+1}m_{2n+2} + 1) & b_nm_{2n+2} + d_n(m_{2n+1}m_{2n+2} + 1) \end{pmatrix}.
\]

The properties (ii) and (iii) follow from the computation

\[
\frac{c_{n+1}}{a_{n+1}} = \frac{a_nm_{2n+2} + c_n(m_{2n+1}m_{2n+2} + 1)}{a_n + c_nm_{2n+1}} = m_{2n+2} + \frac{c_n}{a_n + c_nm_{2n+1}} = m_{2n+2} + \frac{1}{m_{2n+1} + \frac{1}{a_n}}.
\]
The property (iv) holds since we have

\[
b_{n+1} = b_n + d_n m_{2n+1}
\]

\[
= (m_1a_n + \alpha_n) + d_n m_{2n+1}
\]

\[
= (m_1a_n + m_1m_{2n+1}c_n) - m_1m_{2n+1}c_n + \alpha_n + d_n m_{2n+1}
\]

\[
= m_1a_{n+1} + (\alpha_n + m_{2n+1}(d_n - m_1c_n))
\]

\[
= m_1a_{n+1} + (\alpha_n + m_{2n+1}\beta_n)
\]

\[
< m_1a_{n+1} + (a_n + m_{2n+1}c_n)
\]

\[
= m_1a_{n+1} + a_{n+1}.
\]

Finally, we see the property (v) by

\[
d_{n+1} = b_n m_{2n+2} + d_n(m_{2n+1}m_{2n+2} + 1)
\]

\[
= (m_1a_n + \alpha_n)m_{2n+2} + (m_1c_n + \beta_n)(m_{2n+1}m_{2n+2} + 1)
\]

\[
= m_1c_{n+1} + \alpha_n m_{2n+2} + \beta_n(m_{2n+1}m_{2n+2} + 1)
\]

\[
\leq m_1c_{n+1} + \{a_n m_{2n+2} + c_n(m_{2n+1}m_{2n+2} + 1)\}
\]

\[
= m_1c_{n+1} + c_{n+1}.
\]

Consequently, all the properties hold for \(n+1\) and hence they do for all positive integers.

From the properties (i),(ii),(v) and the definition of \(\beta_n\), we have \(0 < a_n \leq c_n < d_n\), which means that \([F_m]\) is a reduced matrix. By the equality

\[
\frac{d_n - 1}{c_n} = m_1 + \frac{\beta_n - 1}{c_n}
\]

and the property (v), we have

\[
LLS([F_m]) = (m_{2n}, \ldots, m_1) = m.
\]

We finished the proof of the lemma. \(\square\)

Lemma 4.7 and Proposition 4.5 yield the statement of the proposition. \(\square\)

References

[BO] A. Bondal, D. Orlov, Reconstruction of a variety from the derived category and groups of autoequivalences, Compositio Math. 125 (2001), 327–344.

[DHKK] G. Dimitrov, F. Haiden, L. Katzarkov, M. Kontsevich, Dynamical systems and categories, Contemporary Mathematics, 621 (2014), 133–170, doi:10.1090/conm/621.

[Gro1] M. Gromov, On the entropy of holomorphic maps, Enseign. Math. 49 (2003), 217–235.

[Gro2] M. Gromov, Entropy, homology and semialgebraic geometry, Astérisque 145–146 (1987), 225–240.

[Har] R. Hartshorne, Ample vector bundles on curves, Nagoya Math. J. 43 (1971), 73–89.
[Kar] O. Karpenkov, *Geometry of Continued Fractions*, Springer (2013).

[KT] K. Kikuta, A. Takahashi, *On the categorical entropy and the topological entropy*, arXiv:1602.03463

[Muk] S. Mukai, *Duality between D(X) and D(\hat{X}) with its applications to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.

[Ogu] K. Oguiso, *Some aspects of explicit birational geometry inspired by complex dynamics*, Proceedings of the ICM Seoul 2014, Vol.II, 695–721.

[Orl] D. Orlov, *Remarks on generators and dimensions of triangulated categories*, Moscow Math. J. **9** (2009), no. 1, 153–159.

[ST] P. Seidel, R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37–108.

[Yom] Y. Yomdin, *Volume growth and entropy*, Israel J. Math. **57** (1987), 285–300.

Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka Osaka, 560-0043, Japan

E-mail address: k-kikuta@cr.math.sci.osaka-u.ac.jp