Holographic Interpolation between $a$ and $F$

Teruhiko Kawano,¹ Yuki Nakaguchi,¹,² and Tatsuma Nishioka¹

¹Department of Physics, Faculty of Science, The University of Tokyo, Bunkyo-ku, Tokyo 113-0033, Japan
²Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo, 5-1-5 Kashiwa-no-Ha, Kashiwa City, Chiba 277-8568, Japan

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An interpolating function $\tilde{F}$ between the $a$-anomaly coefficient in even dimensions and the free energy on an odd-dimensional sphere has been proposed recently and is conjectured to monotonically decrease along any renormalization group flow in continuous dimension $d$. We examine $\tilde{F}$ in the large-$N$ CFT’s in $d$ dimensions holographically described by the Einstein-Hilbert gravity in the AdS$_{d+1}$ space. We show that $\tilde{F}$ is smooth function of $d$ and correctly interpolates the $a$ coefficients and the free energies. The monotonicity of $\tilde{F}$ along an RG flow follows from the analytic continuation of the holographic $c$-theorem to continuous $d$, which completes the proof of the conjecture.

I. INTRODUCTION

A measure of degrees of freedom in a quantum field theory (QFT) remains to be elucidated in arbitrary $d$ dimensions. Physically, it decreases monotonically as the energy scale is lowered because of the decoupling of massive particles. Implementation of such a measure in any QFT in diverse dimensions is intriguing and desirable to characterize the behavior under a renormalization group (RG) flow.

For even $d$, the conformal anomaly in the stress-energy tensor [1]

$$\langle T_\mu^\nu \rangle = \frac{(-1)^{d+1}}{2} a E_d + \sum_i b_i I_i , \quad (1)$$

defines the unique $a$ coefficient for the Euler density $E_d$ and several $b_i$ coefficients for the Weyl invariants $I_i$ labeled by an integer $i$. The $a$ coefficients are believed to be monotonically decreasing along any RG flow, namely the value $a_{UV}$ at the ultra-violet (UV) fixed point is equal or greater than that $a_{IR}$ at the infra-red (IR) fixed point, $a_{UV} \geq a_{IR}$. This statement was established in two dimensions by the Zamolodchikov’s $c$-theorem [2] and in four dimensions by the $a$-theorem [3–5]. On the other hand, the $F$-theorem asserts that the free energy, $F \equiv (-1)^{\frac{d+1}{2}} \log Z_{S^d}$, defined by the conformal invariant partition function $Z_{S^d}$ on $S^d$ of radius $R$, decreases under any RG flow in odd dimensions [6, 7]. A proof for $d = 3$ was presented by [8] through the relation of the free energy to the entanglement entropy $S$ across an entangling surface $S^{d-2}$ of radius $R$ in $\mathbb{R}^{1,d-1}$ [9]

$$F = (-1)^{\frac{d+1}{2}} S , \quad (2)$$

that holds for odd $d$ up to UV divergences.

These two proposals look quite different at first sight, but share the fact that both the $a$ coefficient and the free energy can be read off on $S^d$; the former arises from the integration of the trace of the stress-energy tensor (1) and the latter from the partition function. To interpolate between the $a$ coefficient and the free energy, Giombi and Klebanov define a new function [10]

$$\tilde{F} = \sin \left( \frac{\pi d}{2} \right) \log Z_{S^d} , \quad (3)$$

which correctly reduces to the free energies for odd $d$. They show as $d$ approaches to even integers [11] (see also [12] as a related work)

$$\tilde{F} = \frac{\pi}{2} a . \quad (4)$$

Note that the partition function $Z_{S^d}$ used in (3) is conformal invariant and UV divergent for even $d$. The relation (4) follows from the fact that the conformal invariant partition function in $d = 2n + \epsilon$ dimensions behaves as $\log Z_{S^d} = (-1)^{\frac{d}{2}} \frac{\epsilon}{d} + O(1)$ for small $\epsilon$. This is because one has to add a local counter term

$$I_{\text{c.t.}} = (-1)^{\frac{d+1}{2}} \frac{a}{2\epsilon} \int_{S^d} d^d x \sqrt{g} E_{2n} , \quad (5)$$

to the partition function to obtain the renormalized partition function $\log Z_{S^d}^{\text{ren}} = \log Z_{S^d} + I_{\text{c.t.}}$, reproducing the conformal anomaly $\log Z_{S^d}^{\text{ren}} = (-1)^{n+1} a \log R$ on $S^{2n}$ of radius $R$ in $\epsilon \to 0$ limit.

The function $\tilde{F}$ is also defined for non-integer $d$ and therefore smoothly interpolates between the $a$ coefficients in even dimensions and the free energies in odd dimensions. They conjecture that $\tilde{F}$ is positive and decreases along any RG flow in arbitrary $d$ dimensions, based on several examples including a double-trace deformation of the large-$N$ conformal field theory (CFT). We will call their proposal the $\tilde{F}$-theorem.

In this letter, we provide a further evidence to the $\tilde{F}$-theorem from the holographic viewpoint. To this end, we take advantage of the relation (2) and calculate the holographic entanglement entropy [13, 14] across a sphere $S^{d-2}$ in the Einstein-Hilbert gravity on the AdS$_{d+1}$ space. We perform the dimensional regularization in the bulk and obtain the analytic result of $\tilde{F}$ that is a positive and smooth function of dimension $d$. We show...
that the equality (4) holds for even \( d \) and furthermore prove the \( \tilde{F} \)-theorem that follows from the holographic c-theorem [15–18] assuming the dimensional continuation of the null energy condition.

II. HOLOGRAPHIC PROOF OF THE \( \tilde{F} \)-THEOREM

We will evaluate \( \tilde{F} \) with the relation (2) between the free energy on \( S^d \) and the entanglement entropy across \( S^{d-2} \). The latter can be holographically calculated by the Ryu-Takayanagi formula in the Einstein-Hilbert gravity [13, 14]

\[
S = \frac{\text{Area}(\gamma)}{4G_N^{(d+1)}},
\]

where \( G_N^{(d+1)} \) is the Newton constant, and \( \gamma \) stands for the \( (d-1) \)-dimensional minimal surface in the AdS\(_{d+1} \) space, whose boundary is the entangling surface \( S^{d-2} \). Since the boundary of the AdS\(_{d+1} \) space is the flat space \( \mathbb{R}^{1,d-1} \), we will use the Poincaré coordinates

\[
ds^2 = L^2 \left( dz^2 - dt^2 + dr^2 + r^2 d\Omega_{d-2}^2 \right),
\]

where \( L \) is the AdS radius. The entangling surface is located at \( t = 0 \) and \( r = R \) at the boundary \( z = 0 \). In these coordinates, the minimal surface \( \gamma \) in the bulk is a semi-hypersphere satisfying \( r^2 + z^2 = R^2 \) [13, 14]. This solution leads the entanglement entropy across \( S^{d-2} \)

\[
S = \frac{1}{4G_N^{(d+1)}} L^{d-1} \text{Vol}(S^{d-2}) \int_{\epsilon/R}^1 dy \left(1 - y^2 \right)^{\frac{d-3}{2}} y^{d-1},
\]

where we introduced a small cutoff at \( z = \epsilon \) to regularize the UV divergence and \( \text{Vol}(S^{d-2}) \) is the volume of a unit \( (d-2) \)-dimensional round sphere. Expanding the integrand with respect to \( y \) and performing the integration, one obtains the UV divergent parts of the entanglement entropy. We, however, want to employ the dimensional regularization instead of putting the UV cutoff at \( z = \epsilon \) for our purpose. So we take \( \epsilon = 0 \) and carry out the integral in the range \( 1 < d < 2 \), that yields

\[
S = \frac{L^{d-1}}{4G_N^{(d+1)}} \pi^{\frac{d-2}{2}} - \Gamma \left( 1 - \frac{d}{2} \right).
\]

Then we analytically continue \( d \) to any real value. It is clear that there are poles at even \( d \) in the entanglement entropy (9) corresponding to the conformal anomalies. Finally, using the relations (2) and (3), and the formula \( \Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z) \), we obtain \( \tilde{F} \) in the holographic theories

\[
\tilde{F} = \frac{L^{d-1}}{4G_N^{(d+1)}} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} \right).
\]

This is manifestly a positive and smooth function of dimension \( d \) without poles at even \( d \).

Now let us extrapolate the holographic values of \( \tilde{F} \) to even dimensions and see if the relation (4) holds. The \( a \) coefficients holographically computed in the Einstein-Hilbert gravity are known to be [17–20]

\[
a = \frac{L^{d-1}}{2\pi G_N^{(d+1)}} \pi^{\frac{d}{2}} \Gamma \left( \frac{d}{2} \right).
\]

Combining it with (10), we confirm the relation (4) between \( \tilde{F} \) and \( a \). Moreover, imposing the null energy condition in the bulk, the holographic c-theorem states that the \( a \) coefficient given by (11) satisfies the monotonicity, \( a_{UV} \geq a_{IR} \), for positive integer \( d \) [15–18]. Assuming the analytic continuation of dimension \( d \) in the gravity, the holographic c-theorem holds for \( d \geq 1 \) [21] which assures the \( \tilde{F} \)-theorem due to the relation (4).

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[21] The null energy condition $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$ is crucial in the proof of the holographic c-theorem [16–18] where the $d$-dimensional null vector $\xi$ has only two non-zero components $\xi^z$ and $\xi^t$. Thus defining a formal null vector $\xi = (\xi^z, \xi^t, 0, \cdots, 0)$ in continuous $d$ dimensions, the proof can be carried over for $d \geq 1$. 