GENERALIZED SPHERICAL MEAN VALUE OPERATORS ON EUCLIDEAN SPACE

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Abstract. We consider the Neumann version of the spherical mean value operator and its variants in the space of smooth functions, distributions and compactly supported ones. Surjectivity and range characterization issues are addressed from the viewpoint of convolution equations. AMS subject classification: 45E10, 47G10

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1. Introduction

Convolution equations are natural extensions of linear partial differential equations with constant coefficients. Malgrange [15] in 1955 proved that the convolution equation \( \mu * g = f \), where \( \mu \neq 0 \) is an arbitrary analytic functional, has an entire solution \( g \) for any entire function \( f \). Ehrenpreis [5] in 1956 studied equations with distribution kernels on the spaces differentiable functions, distributions, and real analytic functions. Such equations have been studied by many authors on various situations, for example, on spaces of holomorphic functions on convex domains, those with growth conditions near the boundary, hyperfunctions, Fourier hyperfunctions, etc. See Hörmander [7], Korobeinik [11], Kawai [12], Ishimura-Okada [10], Abanin-Ishimura-Khoi [1], Langenbruch [13] and the references therein. Necessary or sufficient conditions for solvability are often written in terms of the Fourier (or Laplace) transform of the kernel.

Lim [14] proved that the spherical mean value operators on Euclidean and hyperbolic spaces are surjective convolution operators. He employed techniques devised by Ehrenpreis and Hörmander. Christensen-Gonzalez-Kakehi [4] extended this result to the general case of noncompact symmetric spaces. In [4] and [14], the

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operators involve Dirichlet boundary values on spheres. So it is natural to consider the Neumann case. In the present paper, we study the surjectivity of the Neumann mean value operator on Euclidean space and higher order variants. We show that they are surjective on the space of smooth functions and that of distributions. Moreover, we characterize the ranges in the case of compact supports.

The proofs basically follow those in [14], but we have simplified some parts of the arguments.

Mean value operators are studied in various settings. See, for example, Agranovsky et al. [2] and Antipov et al. [3]. It may be possible to prove the Neumann versions of these results.

2. Distributions, convolution and the Fourier transform

Let $S(x, r)$ be the $n-1$ dimensional sphere centered at $x \in \mathbb{R}^n$ with radius $r > 0$. The reciprocal of its surface area is denoted by $c_r$. We have

$$c_r = \begin{cases} 
\frac{1}{2} (n = 1), \\
\frac{\Gamma(n/2)}{2\pi^{n/2}} (n \geq 2),
\end{cases}$$

where $\Gamma$ denotes the gamma function. The spherical mean value operator $M_r$ is defined by

$$M_r u(x) = \begin{cases} 
\frac{1}{2} \{ u(x - r) + u(x + r) \} & (n = 1), \\
c_r \int_{S(x, r)} u(y) \, dS_{x,r}(y) & (n \geq 2),
\end{cases}$$

where $dS_{x,r}$ is the surface area measure of $S(x, r)$. Let $\delta_{S(0, r)}$ be the distribution defined by

$$\delta_{S(0, r)} : C^\infty(\mathbb{R}^n) \to \mathbb{C}, \quad u(x) \mapsto M_r u(0).$$

Then we have

$$M_r u(x) = \delta_{S(0, r)} \ast u(x),$$

where $\ast$ denotes convolution.

Let $n = n_y$ be the outer unit normal of $S(x, r)$ at $y \in S(x, r)$. We introduce the Neumann version of the spherical mean value operator and its generalization by

$$M_r^{(\ell)} u(x) = c_r \int_{S(x, r)} \left( \frac{\partial}{\partial n_y} \right)^\ell u(y) \, dS_{x,r}(y)$$

for a smooth function $u(x)$ on $\mathbb{R}^n$ and a non-negative integer $\ell$. It is trivial that $M_r^{(0)} = M_r$.

The Fourier transform of $u$ is defined by $\hat{u}(\xi) = u_x(e^{-i\langle x, \xi \rangle}) = \langle u(x), e^{-i\langle x, \xi \rangle} \rangle$. In some cases, it is denoted by $\langle u \rangle$.

**Theorem 1.** We have

$$M_r^{(\ell)} u(x) = \frac{\partial^\ell}{\partial r^\ell} \delta_{S(0, r)} \ast u(x)$$

for a smooth function $u(x)$ on $\mathbb{R}^n$ and a non-negative integer $\ell$. 

Proof. 

\[ M_r^{(t)} u(x) = \frac{\partial^t}{\partial r^t} c_1 \int_{S(0,1)} u(x + r\omega) dS_1(\omega) \]

\[ = \frac{\partial^t}{\partial r^t} (M_r u)(x) = \frac{\partial^t}{\partial r^t} \left[ \delta_{S(0,r)} * u(x) \right] = \frac{\partial^t}{\partial r^t} \delta_{S(0,r)} * u(x) \]  

\[ \square \]

**Remark 2.** The convolution operator \((\partial^t \delta_{S(0,r)}/\partial r^t)^*\) is well-defined on \(D'(\mathbb{R}^n)\). We will study \((\partial^t \delta_{S(0,r)}/\partial r^t)^*\) as endomorphisms on \(C^\infty(\mathbb{R}^n)\), \(D'(\mathbb{R}^n)\), \(\mathcal{E}'(\mathbb{R}^n)\) and \(C^\infty_0(\mathbb{R}^n)\).

### 3. Invertibility

**Theorem 3** ([\(\text{[9]}\) Theorem 16.3.9, 16.3.10]). For \(u \in \mathcal{E}'(\mathbb{R}^n)\), the following statements are equivalent.

(i) There is a constant \(A > 0\) such that we have

\[ \sup \{ |\hat{u}(\xi)| : \xi \in \mathbb{C}^n, |\xi - \xi| < A \log(2 + |\xi|) \} > (A + |\xi|)^{-A} \]

for any \(\xi \in \mathbb{R}^n\).

(ii) If \(w \in \mathcal{E}'(\mathbb{R}^n)\) and \(\hat{w}/\hat{u}\) is an holomorphic function, then \(\hat{w}/\hat{u}\) is the Fourier transform of a distribution in \(\mathcal{E}'(\mathbb{R}^n)\).

(iii) If \(v \in \mathcal{E}'(\mathbb{R}^n)\) satisfies \(u * v \in C^\infty(\mathbb{R}^n)\), then \(v \in C^\infty(\mathbb{R}^n)\).

**Definition 4** ([\(\text{[9]}\) Definition 16.3.12]). We say that an element \(u\) of \(\mathcal{E}'(\mathbb{R}^n)\) is invertible if it satisfies the conditions in Theorem 3.

**Remark 5.** The statements (a) and (b) below are equivalent. If they are true, then so is (i) in Theorem 3. In other words, (a) and (b) are sufficient conditions for \(u\) to be invertible.

(a) There is a constant \(A > 0\) such that we have

\[ \sup \{ |\hat{u}(\eta)| : \eta \in \mathbb{R}^n, |\eta - \xi| < A \log(2 + |\xi|) \} > (A + |\xi|)^{-A} \]

for any \(\xi \in \mathbb{R}^n\).

(b) There are constants \(A, B > 0\) such that we have

\[ \sup \{ |\hat{u}(\eta)| : \eta \in \mathbb{R}^n, |\eta - \xi| < A \log(2 + |\xi|) \} > (A + |\xi|)^{-A} \]

for any \(\xi \in \mathbb{R}^n\) satisfying \(|\xi| > B\).

We employ the normalized Bessel function \(j_\nu(z)\) defined by

\[ j_\nu(z) = \Gamma(\nu + 1) \left( \frac{2}{z} \right)^\nu J_\nu(z), \quad \nu > -1, \]

where \(J_\nu(z)\) is the usual Bessel function of the first kind of order \(\nu\). The advantage of \(j_\nu(z)\) over \(J_\nu(z)\) is the fact that the former is an even entire function. Indeed, we have

\[ j_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\nu + 1)}{k! \Gamma(k + \nu + 1)} \left( \frac{z}{2} \right)^{2k}, \quad z \in \mathbb{C}. \]

Recall that all the zeros of \(J_\nu(z)\) are real and simple except possibly at \(z = 0\). Therefore all the zeros of \(j_\nu(z)\) are real and simple and they appear in pairs of the form \(\pm x_j \in \mathbb{R} \setminus \{0\} \).
Theorem 6. We have

\begin{equation}
\tag{3.1}
J'_\nu(z) = -\frac{z J_{\nu+1}(z)}{2(\nu + 1)}.
\end{equation}

For \(\nu > -1\) and \(\ell \geq 0\) fixed, there exist constants \(C_{\ell,k} = C_{\ell,k}^{(\nu)}\) such that

\begin{equation}
\tag{3.2}
J_{\nu}^{(\ell)}(z) = \begin{cases} 
\sum_{k=\ell/2}^{\ell} C_{\ell,k} z^{2k-\ell} j_{\nu+k}(z) & (\ell \text{ : even}), \\
\sum_{k=(\ell+1)/2}^{\ell} C_{\ell,k} z^{2k-\ell} j_{\nu+k}(z) & (\ell \text{ : odd}).
\end{cases}
\end{equation}

We have

\[C_{\ell,k} = C_{\ell,k}^{(\nu)} = (-1)^{\ell} \frac{(2k - \ell + 1)_{2(\ell-k)}}{2^\ell (\ell-k)! (\nu+1)_k}\]

where \(\ell/2 \leq k \leq \ell\) if \(\ell\) is even and \((\ell + 1)/2 \leq k \leq \ell\) if \(\ell\) is odd. Here \((a)_p\) is the rising factorial: \((a)_0 = 1\), \((a)_p = a(a+1) \cdots (a+p-1)\).

Proof. The well-known formula \((z^{-\nu} J_{\nu}(z))' = -z^{-\nu} J_{\nu+1}(z)\) ([16, 10.6.6]) gives

\[\left( z^{-\nu} \frac{z^{\nu} J_{\nu}(z)}{2\nu \Gamma(\nu+1)} \right)' = -z^{-\nu} \frac{z^{\nu+1} J_{\nu+1}(z)}{2\nu+1 \Gamma(\nu+2)} \]

and (3.1) follows. The general formula (3.2) can be proved by induction. We have

\[\{z^{2k-\ell} J_{\nu+k}(z)\}' = (2k - \ell) z^{2k-(\ell+1)} J_{\nu+k}(z) - \frac{z^{2(k+1)-(\ell+1)}}{2(k+1)} J_{\nu+(k+1)}(z).\]

Here notice that \(2k - \ell\) vanishes for \(k = \ell/2\). If \(\ell\) is even, we have

\[J_{\nu}^{(\ell+1)}(z) = \sum_{k=\ell/2+1}^{\ell} C_{\ell,k} (2k - \ell) z^{2k-(\ell+1)} J_{\nu+k}(z) \]

\[= \sum_{k=(\ell+2)/2}^{\ell} C_{\ell,k'} (2k' - \ell) z^{2k'-(\ell+1)} J_{\nu+k'}(z) \]

\[= \sum_{k'=(\ell+2)/2}^{\ell+1} C_{\ell,k'-1} \frac{z^{2k'-(\ell+1)}}{2(\nu + k')} J_{\nu+k'}(z) \]

\[= \sum_{k'=[(\ell+1)+1]/2}^{\ell+1} C_{\ell+1,k'} z^{2k'-(\ell+1)} J_{\nu+k'}(z), \]

where

\[\begin{cases} 
C_{\ell+1,\ell+1} = -\frac{C_{\ell,\ell}}{2(\nu + \ell + 1)}, \\
C_{\ell+1,k} = (2k - \ell) C_{\ell,k} - \frac{C_{\ell,k-1}}{2(\nu + k)} \left( \frac{(\ell + 1) + 1}{2} \leq k \leq \ell \right). 
\end{cases} \]
If \( \ell \) is odd, we have

\[
\begin{align*}
j_{\nu}^{(\ell+1)}(z) &= \sum_{k=(\ell+1)/2}^{\ell} C_{\ell,k} (2k - \ell) z^{2k - (\ell+1)} j_{\nu+k}(z) \\
&\quad - \sum_{k=(\ell+1)/2}^{\ell} C_{\ell,k} \frac{2(k+1) - (\ell+1)}{2(\nu + k)} j_{\nu+(k+1)}(z) \\
&= \sum_{k'=(\ell+1)/2}^{\ell} C_{\ell,k'} (2k' - \ell) z^{2k' - (\ell+1)} j_{\nu+k'}(z) \\
&\quad - \sum_{k'=(\ell+1)/2+1}^{\ell+1} C_{\ell,k'-1} \frac{2(k'+1) - (\ell+1)}{2(\nu + k')} z^{2k' - (\ell+1)} j_{\nu+k'}(z) \\
&= \sum_{k'=(\ell+1)/2}^{\ell+1} C_{\ell+1,k'} z^{2k' - (\ell+1)} j_{\nu+k'}(z),
\end{align*}
\]

where

\[
\begin{align*}
C_{\ell+1,\ell+1} &= -\frac{C_{\ell,\ell}}{2(\nu + \ell + 1)}, \\
C_{\ell+1,k} &= (2k - \ell) C_{\ell,k} - \frac{C_{\ell,k-1}}{2(\nu + k)} \left( \frac{\ell + 1}{2} + 1 \leq k \leq \ell \right), \\
C_{\ell+1,(\ell+1)/2} &= C_{\ell,(\ell+1)/2}.
\end{align*}
\]

Combining \( C_{0,0} = 1 \) with the recurrence relation, we get the expression of \( C_{\ell,k} \).

\[\square\]

**Lemma 7.** Fix \( r > 0 \) and \( n \geq 1 \). Let \( \nu = n/2 - 1 \).

(i) There exist constants \( A, B > 0 \) such that

\[
(3.3) \quad \sup \left\{ \left| j_{\nu}^{(\ell)}(r|\eta|) \right| : \eta \in \mathbb{R}^n, |\eta - \xi| < A \log(2 + |\xi|) \right\} > (A + |\xi|)^{-A}
\]

for any \( \xi \in \mathbb{R}^n \) with \( |\xi| > B \).

(ii) There exists a sequence of real numbers \( \{a_m^{(\nu,\ell)}\}_m \) such that \( j_{\nu}^{(\ell)}(\pm a_m^{(\nu,\ell)}) = 0 \) and \( a_m^{(\nu,\ell)} \to \infty \) (\( m \to \infty \)).

**Proof.** The function \( j_{\nu}(x) (\mu > -1) \) has the asymptotic behavior ([10] 10.17.3)]

\[
\begin{align*}
j_{\nu}(x) &= \frac{\Gamma(\mu + 1)}{\sqrt{\pi}} \left( \frac{2}{x} \right)^{\nu + 1/2} \left\{ \cos \left( x - \frac{\mu \pi}{2} - \frac{\pi}{4} \right) + O(x^{-1}) \right\} \quad (\mathbb{R} \ni x \to \infty).
\end{align*}
\]

In the right-hand side of (3.2), the dominant term is the one corresponding to \( k = \ell \).

We have

\[
(3.4) \quad j_{\nu}^{(\ell)}(x) \sim \frac{\Gamma(\nu + 1) C_{\ell,\ell} 2^\ell}{\sqrt{\pi}} \left( \frac{2}{x} \right)^{\nu + 1/2} \left\{ \cos \left( x - \frac{\nu \pi + \ell}{2} - \frac{\pi}{4} \right) + O(x^{-1}) \right\}
\]

and the limit supremum of \( |j_{\nu}^{(\ell)}(x)| \) on \( x > \text{const.} \) decays in the order of \( x^{-(\nu+1/2)} \). Therefore the estimate (3.3) holds true if \( A \) and \( B \) are sufficiently large. Moreover, (3.3) implies that the real valued function \( j_{\nu}^{(\ell)}(x) \) for \( x > 0 \) is oscillatory, that is, it changes sign infinitely many times as \( x \to \infty \).

\[\square\]
**Theorem 8.** For any \( \xi \in \mathbb{R}^n \), we have
\[
\hat{\delta}_{S(0,r)}(\xi) = j_{n/2-1}(r|\xi|).
\]
For \( \ell \geq 0 \), the Fourier transform of \( \partial^\ell \delta_{S(0,r)}/\partial r^\ell \) is
\[
\left( \frac{\partial^\ell \delta_{S(0,r)}}{\partial r^\ell} \right)^\wedge(\xi) = |\xi|^{\ell} j_{n/2-1}(r|\xi|)
\]
\[
= \begin{cases} 
\sum_{k=\ell/2}^{\ell} C_{\ell,k} r^{2k-\ell} |\xi|^{2k} j_{n/2-1+k}(r|\xi|) & (\ell: \text{even}), \\
\sum_{k=(\ell+1)/2}^{\ell} C_{\ell,k} r^{2k-\ell} |\xi|^{2k} j_{n/2-1+k}(r|\xi|) & (\ell: \text{odd}).
\end{cases}
\]

**Proof.** By [6] Introduction, Lemma 3.6,.
\[
\hat{\delta}_{S(0,1)}(\xi) = c_0 (2\pi)^{n/2} J_{n/2-1}(|\xi|) |\xi|^{n/2-1} = \Gamma(n/2) (2\pi)^n/2 J_{n/2-1}(|\xi|) |\xi|^{n/2-1}
\]
\[
= 2^{n/2-1} \Gamma(n/2) J_{n/2-1}(|\xi|) = j_{n/2-1}(|\xi|).
\]
For general \( r > 0 \),
\[
\hat{\delta}_{S(0,r)}(\xi) = \int_{\mathbb{R}^n} e^{-i(x,\xi)} \delta_{S(0,r)}(x) \, dx = \int_{\mathbb{R}^n} e^{-i(x,\xi)} r^{-n} \delta_{S(0,1)}(x/r) \, dx
\]
\[
= \int_{\mathbb{R}^n} e^{-ir(\nu,\xi)} \delta_{S(0,1)}(r\nu) \, dy = \hat{\delta}_{S(0,1)}(r\xi) = j_{n/2-1}(r|\xi|).
\]
The formula (3.5) has been proved and (3.6) follows. \( \square \)

**Corollary 9.** The Fourier transform of \( \partial^\ell \delta_{S(0,r)}/\partial r^\ell \) is an even entire function and we have
\[
\hat{\delta}_{S(0,r)}(\xi) = j_{n/2-1}(r \sqrt{\zeta^2})
\]
\[
= \Gamma \left( \frac{n}{2} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+n/2)} \left( \frac{r}{2} \right)^{2k} \zeta^{2k}
\]
\[
\left( \frac{\partial \delta_{S(0,r)}}{\partial r} \right)^\wedge(\zeta) = - \frac{\zeta^2}{n} j_{n/2}(r \sqrt{\zeta^2})
\]
\[
= - \frac{\zeta^2}{n} \Gamma \left( \frac{n}{2} \right) \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \Gamma(k+n/2+1)} \left( \frac{r}{2} \right)^{2k} \zeta^{2k}
\]
\[
\left( \frac{\partial^\ell \delta_{S(0,r)}}{\partial r^\ell} \right)^\wedge(\zeta) = \begin{cases} 
\sum_{k=\ell/2}^{\ell} C_{\ell,k} r^{2k-\ell} (\zeta^2)^k j_{n/2-1+k}(r \sqrt{\zeta^2}) & (\ell: \text{even}), \\
\sum_{k=(\ell+1)/2}^{\ell} C_{\ell,k} r^{2k-\ell} (\zeta^2)^k j_{n/2-1+k}(r \sqrt{\zeta^2}) & (\ell: \text{odd}).
\end{cases}
\]

Here \( \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n, \zeta^2 = \zeta_1^2 + \cdots + \zeta_n^2. \) Notice that \( j_\nu(\cdot) \) is an even function and \( j_\nu(r \sqrt{\zeta^2}) \) is well-defined.

**Proposition 10.** The distribution \( \partial^\ell \delta_{S(0,r)}/\partial r^\ell (\ell \geq 0) \) is invertible.
Proof. The proposition follows from Remark 5 (b), Lemma 7 and Theorem 8. □

4. Surjectivity on the space of smooth functions

We introduce the notion of \( \mu \)-convexity for supports of a pair of open sets \((X_1, X_2)\). When \( \mu \in E'(\mathbb{R}^n) \), we set

\[
\hat{\mu}(\phi) = \mu(\tilde{\phi}), \quad \tilde{\phi}(x) = \phi(-x),
\]

where \( \phi \in C_0^\infty(\mathbb{R}^n) \) is a test function. In some cases, \( \hat{\mu} \) is denoted by \((\mu)^\vee\).

Definition 11 ([9, Definition 16.5.4]). Assume \( \mu \in E'(\mathbb{R}^n) \). Let \( X_1 \) and \( X_2 \) be non-empty open subsets of \( \mathbb{R}^n \) satisfying \( X_2 - \text{supp} \mu \subset X_1 \). We say that \((X_1, X_2)\) is \( \mu \)-convex for supports if for every compact set \( K_1 \subset X_1 \) one can find a compact set \( K_2 \subset X_2 \) such that \( \text{supp} v \subset K_2 \) if \( v \in C_0^\infty(X_2) \) and \( \hat{\mu} \ast v \subset K_1 \).

We will need the case of \( X_1 = X_2 = \mathbb{R}^n \) only. The condition \( X_2 - \text{supp} \mu \subset X_1 \) is trivial in that case.

Theorem 12 ([9, Theorem 16.5.7]). Assume \( \mu \in E'(\mathbb{R}^n) \). Let \( X_1 \) and \( X_2 \) be non-empty open subsets of \( \mathbb{R}^n \) satisfying \( X_2 - \text{supp} \mu \subset X_1 \). Then the following two statements are equivalent.

(i) The convolution operator \( \mu \ast : \mathcal{C}^\infty(X_1) \rightarrow \mathcal{C}^\infty(X_2) \) is surjective.

(ii) The distribution \( \mu \) is invertible and the pair \((X_1, X_2)\) is \( \mu \)-convex for supports.

Proposition 13. The pair \((\mathbb{R}^n, \mathbb{R}^n)\) is \( \partial^\ell \delta_{(0, r)} / \partial r^\ell \)-convex for supports for \( \ell \geq 0 \).

Proof. Recall that

\[
\text{ch supp } u_1 \ast u_2 = \text{ch supp } u_1 + \text{ch supp } u_2
\]

holds for any \( u_1, u_2 \in E'(\mathbb{R}^n) \) ([3, Theorem 4.3.3]), where \( \text{ch} \) denotes the convex hull. It implies \( \text{ch supp } u_1 \subset \text{ch supp } u_1 \ast u_2 - \text{ch supp } u_2 \).

We have

\[
\text{supp } (\partial^\ell \delta_{(0, r)} / \partial r^\ell)^\vee = \text{supp } (\partial^\ell \delta_{(0, r)}) / \partial r^\ell = \{ x \in \mathbb{R}^n ; |x| = r \}.
\]

Let \( K_1 \subset \mathbb{R}^n \) be an arbitrary compact set and assume that \( v \in C_0^\infty(\mathbb{R}^n) \) satisfies \( \text{supp } (\partial^\ell \delta_{(0, r)} / \partial r^\ell)^\vee \ast v \subset K_2 \). We have

\[
\text{supp } v \subset \text{ch supp } (\partial^\ell \delta_{(0, r)} / \partial r^\ell)^\vee \ast v - \text{ch supp } (\partial^\ell \delta_{(0, r)} / \partial r^\ell)^\vee
\subset \text{ch } K_2 - \{ x \in \mathbb{R}^n ; |x| \leq r \}.
\]

Recall that the convex hull of a compact subset in \( \mathbb{R}^n \) is compact. Therefore \( K_2 = \text{ch } K_1 - \{ x \in \mathbb{R}^n ; |x| \leq r \} \) is a compact set independent of \( v \). □

Theorem 14. Let \( r > 0 \). The convolution operator \( \partial^\ell \delta_{(0, r)} / \partial r^\ell \ast : \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n) \) is surjective.

Proof. Apply Theorem 12 and Propositions 10 and 13. □

We conclude this section by showing that \( \partial^\ell \delta_{(0, r)} / \partial r^\ell \ast : \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n) \) is not injective for any \( \ell \geq 0 \).

Theorem 15. The kernel of the operator \( \partial^\ell \delta_{(0, r)} / \partial r^\ell \ast : \mathcal{C}^\infty(\mathbb{R}^n) \rightarrow \mathcal{C}^\infty(\mathbb{R}^n) \) is infinite-dimensional for any \( \ell \geq 0 \).
Proof. Set
\[ d_t(\zeta) = \left( \frac{\partial^t \delta_{S(0,r)}}{\partial r^t} \right)^\wedge (\zeta_1, 0, 0, \ldots, 0). \]
Then by Lemma 7 (ii), \( d_t(\zeta_1) \) admits infinitely many (real) zeros and so does \( (\partial^t \delta_{S(0,r)}/\partial r^t)^\wedge(\zeta) \). Since \( (\partial^t \delta_{S(0,r)}/\partial r^t)^\wedge(\zeta) \) is a function of \( \zeta^2 \), there exists a countably infinite subset \( S \) of \( \mathbb{C} \) such that \( (\partial^t \delta_{S(0,r)}/\partial r^t)^\wedge(\zeta) \) vanishes on \( T = \{ \zeta; \zeta^2 \in S \} \).

On the other hand, we have
\[
\exp(\pm i(\zeta, x)) \ast \frac{\partial^t \delta_{S(0,r)}}{\partial r^t}(x) = \int_{\mathbb{R}^n} \frac{\partial^t \delta_{S(0,r)}}{\partial r^t}(y) \exp(\pm i(\zeta, x - y)) \, dy
= \left( \frac{\partial^t \delta_{S(0,r)}}{\partial r^t} \right)^\wedge(\zeta) \exp(\pm i(\zeta, x))
\]
and \( \exp(\pm i(\zeta, x)) \) belong to the kernel if \( \zeta \in T \). It implies the infinite-dimensionality of the kernel. By integrating over \( T \) with respect to measures, one can construct a large variety of homogeneous solutions to \( \partial^t \delta_{S(0,r)}/\partial r^t * u = 0 \).

Notice that in the particular cases \( t = 0, 1 \), we can get a concrete description of the location of those zeros. Indeed, they correspond to the zeros of the Bessel function. If \( \pm r \sqrt{\zeta^2} \) are zeros of \( j_{n/2-1}(\cdot) \), then \( \exp(\pm i(\zeta, x)) \) belong to the kernel of \( \delta_{S(0,r)} * \). If \( \pm r \sqrt{\zeta^2} \) are zeros of \( j_{n/2}(\cdot) \), then \( \exp(\pm i(\zeta, x)) \) belong to the kernel of \( \partial \delta_{S(0,r)}/\partial r^t \). \( \square \)

Remark 16. There are elements of the kernel that have very simple expressions. If \( a_m \in \mathbb{R} (m \geq 0) \) satisfy \( j_{n/2-1}^{(t)}(\pm ra_m) = 0 \), we have \( d_t(\pm a_m) = 0 \) and
\[
(\partial^t \delta_{S(0,r)}/\partial r^t)^\wedge(\pm a_m, 0, 0, \ldots, 0) = (\partial^t \delta_{S(0,r)}/\partial r^t)^\wedge(0, \pm a_m, 0, \ldots, 0)
= \cdots = (\partial^t \delta_{S(0,r)}/\partial r^t)^\wedge(0, \ldots, 0, \pm a_m) = 0.
\]
Then \( \exp(\pm ia_mx_j) \) belong to the kernel for any \( j = 1, \ldots, n \).

5. Surjectivity on the space of distributions

Definition 17 (Definition 16.5.13). Assume \( \mu \in \mathcal{E}'(\mathbb{R}^n) \). Let \( X_1, X_2 \) be non-empty open subsets of \( \mathbb{R}^n \) satisfying \( X_2 - \text{sing supp } \mu \subset X_1 \). We say that \( (X_1, X_2) \) is \( \mu \)-convex for singular supports if for every compact set \( K_1 \subset X_1 \) one can find a compact set \( K_2 \subset X_2 \) such that \( \text{sing supp } v \subset K_2 \) if \( v \in \mathcal{E}'(X_2) \) and \( \text{sing supp } \mu \ast v \subset K_1 \).

We recall two important facts.

Theorem 18 (Corollary 16.5.19). Assume \( \mu \in \mathcal{E}'(\mathbb{R}^n) \). Let \( X_1 \) and \( X_2 \) be non-empty open subsets of \( \mathbb{R}^n \). Assume \( X_2 - \text{supp } \mu \subset X_1 \). Then \( \mu \ast \mathcal{D}'(X_1) = \mathcal{D}'(X_2) \) if and only if \( \mu \) is invertible and \( (X_1, X_2) \) is \( \mu \)-convex for supports and singular supports.

Theorem 19 (Corollary 16.3.15). Assume that \( u \in \mathcal{E}'(\mathbb{R}^n) \) is invertible. Then we have
\[
\text{ch sing supp } v \subset \text{ch sing supp } (u \ast v) - \text{ch sing supp } u, \quad v \in \mathcal{E}'(\mathbb{R}^n).
\]
The following is an analogue of Proposition 13.
Proposition 20. The pair \((\mathbb{R}^n, \mathbb{R}^n)\) is \(\partial^\ell \delta_S(0, r)/\partial r^\ell\)-convex for singular supports for \(\ell \geq 0\).

Proof. The proof is almost the same as that of Proposition \(\ref{corollary9}\) and Proposition \(\ref{proposition23}\). We can use Proposition \(\ref{proposition10}\) and Theorem \(\ref{theorem3}\).

Our main result in this section is the following.

Theorem 21. The convolution operator \(\partial^\ell \delta_S(0, r)/\partial r^\ell \ast : \mathcal{D}'(\mathbb{R}^n) \rightarrow \mathcal{D}'(\mathbb{R}^n)\) is surjective.

Proof. Apply Theorem \(\ref{theoremm23}\) and Propositions \(\ref{proposition10} \ref{corollary9} \ref{proposition20}\).

6. COMPACTLY SUPPORTED DISTRIBUTIONS

In this and the following sections, we restrict our consideration to the cases of \(\ell = 0, 1\) and \(n \geq 2\).

Lemma 22. Let \(a(\zeta)\) and \(b(\zeta)\) be holomorphic functions on an open subset \(U\) of \(\mathbb{C}^n\). Set \(Z = \{\zeta \in U; a(\zeta) = 0\}\) and assume \(da(\zeta) \neq 0\) on \(Z\). If \(b(\zeta)\) vanishes on \(Z\), then \(b(\zeta)/a(\zeta)\) is holomorphic in \(U\).

Proof. We have only to prove the claim in a neighborhood of each point of \(Z\). Let \(p \in Z\). In a neighborhood \(V\) of \(p\), there is a new system of coordinates \((z_1, \ldots, z_n)\) such that \(a(\zeta) = z_1\), \(V \cap Z = \{z_1 = 0\}\). Then the function \(b\) can be divided by \(z_1\).

Proposition 23. Assume \(n \geq 2\). Set \(f_1(\zeta) = j_{n/2}(r\sqrt{\zeta^2})\), \(f_2(\zeta) = \zeta^2\). Let \(Z = \{\zeta \in \mathbb{C}^n; f_1(\zeta)f_2(\zeta) = 0\}\). If \(g(\zeta)\) is holomorphic in a neighborhood of \(Z\) and \(g(\zeta) = 0\) on \(Z\), then \(g(\zeta)/[f_1(\zeta)f_2(\zeta)]\) is holomorphic.

Proof. Set \(Z_j = \{\zeta \in \mathbb{C}^n; f_j(\zeta) = 0\} (j = 1, 2)\). Then \(Z = Z_1 \cup Z_2\), \(Z_1 \cap Z_2 = \emptyset\), \(0 \notin Z_1, 0 \in Z_2\).

By Lemma \(\ref{lemma22}\) \(g(\zeta)/f_1(\zeta)\) is holomorphic in a neighborhood of \(Z\). By Lemma \(\ref{lemma22}\) again, \(g(\zeta)/[f_1(\zeta)f_2(\zeta)]\) is holomorphic in a neighborhood of \(Z \setminus \{0\}\). Therefore \(0\) is an isolated singularity. By Hartogs’s extension theorem, it is removable.

Theorem 24. Assume \(\ell = 0, 1\) and \(n \geq 2\). Then we have the following characterization of the range of the endomorphism \(\partial^\ell \delta_S(0, r)/\partial r^\ell \ast\) on \(\mathcal{E}'(\mathbb{R}^n)\).

\[
\begin{equation}
\text{range} \left( \partial^\ell \delta_S(0, r)/\partial r^\ell \ast : \mathcal{E}'(\mathbb{R}^n) \rightarrow \mathcal{E}'(\mathbb{R}^n) \right) = \left\{ w \in \mathcal{E}'(\mathbb{R}^n); \hat{w}(\zeta) = 0 \text{ if } (\partial^\ell \delta_S(0, r)/\partial r^\ell)^\wedge(\zeta) = 0 \right\}.
\end{equation}
\]

The case of \(\ell = 0, n \geq 1\) was proved in \(\ref{theoremm13}\) Theorem 2.17.

Proof. Here we show the case of \(\ell = 1\). The proof of the case of \(\ell = 0\) is easier. Let the left- and right-hand sides of \(\ref{equation6.1}\) be \(R\) and \(W\) respectively. The Fourier transform of \((\partial^\ell \delta_S(0, r)/\partial r^\ell)^\wedge(\zeta)v(\zeta)\) and the inclusion \(R \subset W\) follows immediately.

Next we show \(R \subset W\). If \(w\) belongs to \(W\), then \(\hat{w}(\zeta)/(\partial^\ell \delta_S(0, r)/\partial r^\ell)^\wedge(\zeta)\) is entire by Corollary \(\ref{corollary10}\) and Proposition \(\ref{proposition23}\). By Proposition \(\ref{proposition10}\) and Theorem \(\ref{theorem3}\)

\[
\hat{w}(\zeta)/(\partial^\ell \delta_S(0, r)/\partial r^\ell)^\wedge(\zeta) = \hat{v}(\zeta)\]

for some distribution \(v\) in \(\mathcal{E}'(\mathbb{R}^n)\). Then \(\hat{w}(\zeta) = (\partial^\ell \delta_S(0, r)/\partial r^\ell)^\wedge(\zeta)\hat{v}(\zeta)\) and \(w = \partial^\ell \delta_S(0, r)/\partial r^\ell v\).
7. COMPACTLY SUPPORTED SMOOTH FUNCTIONS

**Theorem 25.** Assume $\ell = 0, 1$ and $n \geq 2$. Then we have the following characterization of the range of the endomorphism $\partial^\ell \delta_{S(0, r)} / \partial r^\ell \ast$ on $C_0^\infty(\mathbb{R}^n)$.

\begin{equation}
\begin{aligned}
\text{range} \left( \partial^\ell \delta_{S(0, r)} / \partial r^\ell \ast : C_0^\infty(\mathbb{R}^n) \to C_0^\infty(\mathbb{R}^n) \right) \\
= \{ w \in C_0^\infty(\mathbb{R}^n); \hat{w}(\zeta) = 0 \text{ if } (\partial^\ell \delta_{S(0, r)} / \partial r^\ell \ast)^\wedge(\zeta) = 0 \}.
\end{aligned}
\end{equation}

The case of $\ell = 0, n \geq 1$ was proved in [14, Theorem 2.23].

**Proof.** Here we show the case of $\ell = 1$. The proof of the case of $\ell = 0$ is easier. Let the left- and right-hand sides be $R'$ and $W'$ respectively. The inclusion $R' \subset W'$ is trivial.

Next we show $R' \supset W'$. If $w$ belongs to $W'$, then $w = \partial \delta_{S(0, r)} / \partial r \ast v$ for some distribution $v$ in $\mathcal{E}'(\mathbb{R}^n)$ by Theorem 24. Since $w \in C_0^\infty(\mathbb{R}^n)$, the invertibility of $\partial \delta_{S(0, r)} / \partial r$ and Theorem 3 (iii) imply that $v \in C^\infty(\mathbb{R}^n)$. Hence $v \in \mathcal{E}'(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$ and $w \in R'$.

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