ON THE INTERSECTION OF SECTIONAL-HYPERBOLIC SETS

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Abstract. We analyse the intersection of positively and negatively sectional-
hyperbolic sets for flows on compact manifolds. First we prove that such
an intersection is hyperbolic if the intersecting sets are both transitive (this
is false without such a hypothesis). Next we prove that, in general, such an
intersection consists of a nonsingular hyperbolic set, finitely many singularities
and regular orbits joining them. Afterward we exhibit a three-dimensional star
flow with two homoclinic classes, one being positively (but not negatively)
sectional-hyperbolic and the other negatively (but not positively) sectional-
hyperbolic, whose intersection reduces to a single periodic orbit. This provides
a counterexample to a conjecture by Shy, Zhu, Gan and Wen ([24], [25]).

1. Introduction

Dynamical systems is concerned with the study of the asymptotic behavior of the
orbits of a given system. Certain hypothesis like Smale’s hyperbolicity guarantee
the knowledge of this behavior. Indeed, the celebrated Smale spectral decomposi-
tion theorem asserts that every hyperbolic system on a compact manifold comes
equipped with finite many pairwise disjoint compact invariant sets (homoclinic
classes or singularities) to which every trajectory converge. Although present in a
number of interesting examples, such a hypothesis is far from being abundant in
the dynamical forest. This triggered several attempts to extend it including the
sectional-hyperbolicity [17], committed to merge the hyperbolic theory to the so-
called geometric and multidimensional Lorenz attractors [1], [8], [13]. A number of
results from the hyperbolic theory have been carried out to the sectional-hyperbolic
context. This is nowadays matter of study in a number of works, see [2] and refer-
ences therein. One of these results was motivated by the well-known fact that two
different homoclinic classes contained in a common hyperbolic set are disjoint. It
was quite natural to ask if this statement is also true in the sectional-hyperbolic
context too. In other words, are two different homoclinic classes contained in a com-
mon sectional-hyperbolic set disjoint? But recent results dealing with this question
say that the answer is negative [20], [21]. Moreover, [21] studied the dynamics
of nontransitive sectional-Anosov flows with dense periodic orbits nowadays called
venice masks. It was proved that three-dimensional venice masks with a unique singularity exists [7] and that their maximal invariant set consists of two different
homoclinic classes with nonempty intersection [21]. Venice mask with \( n \) singulari-
ties can be constructed for \( n \geq 3 \) whereas ones with just two singularities have not
been constructed yet.

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These fruitful results motivate a related problem which is the analysis of the intersection of a sectional-hyperbolic set for the flow and a sectional-hyperbolic set for the reversed flow. For simplicity we keep the terms positively and negatively sectional-hyperbolic for these sets (respectively) which was coined by Shy, Gan and Wen in their recent paper [24]. After observing that every hyperbolic set can be realized as such an intersection, we show an example where such an intersection is not hyperbolic. Next we show that such an intersection is hyperbolic if the intersecting sets are both transitive. In general the intersection consists of a nonsingular hyperbolic set (possibly empty), finitely many singularities and regular orbits joining them. Finally, we construct a three-dimensional star flow exhibiting two homoclinic classes, one being positively (but not negatively) sectional-hyperbolic and the other being negatively (but not positively) sectional-hyperbolic, whose intersection reduces to a single periodic orbit. This will provide a counterexample to a conjecture by Zhu, Gan and Wen [25] (as amended by Shy, Gan and Wen [24]).

2. Statement of the results

Let $M$ be a differentiable manifold endowed with a Riemannian metric $\langle \cdot , \cdot \rangle$ an induced norm $\| \cdot \|$. We call flow any $C^1$ vector field $X$ with induced flow $X_t$ of $M$. If $\dim(M) = 3$, then we say that $X$ is a three-dimensional flow. We denote by $\text{Sing}(X)$ the set of singularities (i.e. zeroes) of $X$. By a periodic point we mean a point $x \in M$ for which there is a minimal $t > 0$ satisfying $X_t(x) = x$. By an orbit we mean $O(x) = \{X_t(x) : t \in \mathbb{R}\}$ and by a periodic orbit we mean the orbit of a periodic point. We say that $\Lambda \subset M$ is invariant if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$. In such a case we write $\Lambda^* = \Lambda \setminus \text{Sing}(X)$. We say that $\Lambda \subset M$ is transitive if there is $x \in \Lambda$ such that $\omega(x) = \Lambda$, where $\omega(x)$ is the $\omega$-limit set,

$$\omega(x) = \left\{ y \in M : y = \lim_{n \to \infty} X_{t_n}(x) \text{ for some sequence } t_n \to \infty \right\}.$$ 

The $\alpha$-limit set $\alpha(x)$ is the $\omega$-limit set for the reversed flow $-X$. If the set of periodic points of $X$ in $\Lambda$ is dense in $\Lambda$, we say that $\Lambda$ has dense periodic points.

A compact invariant set $\Lambda$ is hyperbolic if there is a continuous invariant splitting $T \Lambda M = E^s \oplus E^X \oplus E^u$ and positive numbers $K, \lambda$ such that

1. $E^s$ is contracting, i.e., $\| DX_t(x)v_x^s \| \leq Ke^{-\lambda t}\|v_x^s\|$ for every $x \in \Lambda$, $v_x^s \in E_x^s$ and $t \geq 0$.
2. $E_x^X$ is the subspace generated by $X(x)$ in $T_x M$, for every $x \in \Lambda$.
3. $E^u$ is expanding, i.e., $\| DX_t(x)v_x^u \| \geq K^{-1}e^{\lambda t}\|v_x^u\|$ for every $x \in \Lambda$, $v_x^u \in E_x^u$ and $t \geq 0$.

A singularity or periodic orbit is hyperbolic if it does as a compact invariant set of $X$. The elements of a (resp. hyperbolic) periodic orbit will be called (resp. hyperbolic) periodic points. A singularity or periodic orbit is a sink (resp. source) if its unstable subbundle $E^u$ (resp. stable subbundle $E^s$) vanishes. Otherwise we call it saddle type.

The invariant manifold theory [14] asserts that through any point $x$ of a hyperbolic set it passes a pair of invariant manifolds, the so-called stable and unstable manifolds $W^s(x)$ and $W^u(x)$, tangent at $x$ to the subbundles $E_x^s$ and $E_x^u$ respectively. Saturating them with the flow we obtain the weak stable and unstable manifolds $W^{ws}(x)$ and $W^{wu}(x)$ respectively.

On the other hand, a compact invariant set $\Lambda$ has a dominated splitting with respect to the tangent flow if there are an invariant splitting $T \Lambda M = E \oplus F$ and
positive numbers $K, \lambda$ such that

$$
\|DX_t(x)e_x\|\|f_x\| \leq Ke^{-\lambda t}\|DX_t(x)f_x\|\|e_x\|, \quad \forall x \in \Lambda, t \geq 0, (e_x, f_x) \in E_x \times F_x.
$$

Notice that this definition allows every compact invariant set $\Lambda$ to have a dominated splitting with respect to the tangent flow: Just take $E_x = T_x M$ and $F_x = 0$ for every $x \in \Lambda$ (or $E_x = 0$ and $F_x = T_x M$ for every $x \in \Lambda$). However, such splittings need not to exist under certain constraints. For instance, not every compact invariant set has a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow which is nontrivial, i.e., satisfying $E_x \neq 0 \neq F_x$ for every $x \in \Lambda$.

A compact invariant set $\Lambda$ is partially hyperbolic if it has a partially hyperbolic splitting, i.e., a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow whose dominated subbundle $E$ is contracting in the sense of (1) above.

The Riemannian metric $\langle \cdot, \cdot \rangle$ of $M$ induces a 2-Riemannian metric [22],

$$
\langle u, v/w \rangle_p = \langle u, v \rangle_p \cdot \langle w, w \rangle_p - \langle u, w \rangle_p \cdot \langle v, w \rangle_p, \quad \forall p \in M, \forall u, v, w \in T_p M.
$$

This in turns induces a 2-norm [12] (or areal metric [15]) defined by

$$
\|u, v\| = \sqrt{\langle u, u/v \rangle_p}, \quad \forall p \in M, \forall u, v \in T_p M.
$$

Geometrically, $\|u, v\|$ represents the area of the parallelogram generated by $u$ and $v$ in $T_p M$.

If a compact invariant set $\Lambda$ has a dominated splitting $T_\Lambda M = E \oplus F$ with respect to the tangent flow, then we say that its central subbundle $F$ is sectionally expanding (resp. sectionally contracting) if

$$
\|DX_t(x)u, DX_t(x)v\| \geq K^{-1}e^{\lambda t}\|u, v\|, \quad \forall x \in \Lambda, u, v \in F_x, t \geq 0.
$$

(resp.

$$
\|DX_t(x)u, DX_t(x)v\| \leq Ke^{-\lambda t}\|u, v\|, \quad \forall x \in \Lambda, u, v \in F_x, t \geq 0.
$$

By a sectional-hyperbolic splitting for $X$ over $\Lambda$ we mean a partially hyperbolic splitting $T_\Lambda M = \hat{E} \oplus F$ whose central subbundle $F$ is sectionally expanding.

Now we define sectional-hyperbolic set.

**Definition 2.1.** A compact invariant set $\Lambda$ is sectional-hyperbolic for $X$ if its singularities are hyperbolic and if there is a sectional-hyperbolic splitting for $X$ over $\Lambda$. Following [24] we use the term positively (resp. negatively) sectional-hyperbolic to indicate a sectional-hyperbolic set for $X$ (resp. $-X$). The corresponding sectional-hyperbolic splitting will be termed positively (resp. negatively) sectional-hyperbolic splitting.

This definition is slightly different from the original one given in Definition 2.3 of [17] (which requires, for instance, that the central subbundle be two-dimensional at least). Such a difference permits every hyperbolic set $\Lambda$ to be both positively and negatively sectional-hyperbolic. Indeed, if $T_\Lambda M = E^s \oplus E^X \oplus E^u$ is the respective hyperbolic splitting, then $T_\Lambda M = E^s \oplus E^{se}$ with $E^{se} = E^X \oplus E^u$ and $T_\Lambda M = \hat{E}^s \oplus \hat{E}^{se}$ with $\hat{E}^s = E^u$ and $\hat{E}^{se} = E^s \oplus E^X$ define positively and negatively sectional-hyperbolic splittings respectively over $\Lambda$. In particular, every hyperbolic set is the intersection of a positively and a negatively sectional-hyperbolic set.

One can ask if the hyperbolic sets are the sole possible intersection between a positively and a negatively sectional-hyperbolic set, but they aren’t. In fact, there are nonhyperbolic compact invariant sets which, nevertheless, are both positively
and negatively sectional-hyperbolic. This is the case of the example described in Figure 1. In such a figure $O(x)$ represents the orbit of $x \in W^s(\sigma_1) \cap W^u(\sigma_2)$ whereas a singularity of a three-dimensional flow is Lorenz-like for $X$ if it has three real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ satisfying $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$.

\[ \Lambda = \{\sigma_1, \sigma_2\} \cup O(x) \]

Lorenz-like for $X$: $\lambda^u > -\lambda^s$  
Lorenz-like for $-X$: $\gamma^u < -\gamma^s$

\[ W^s(\sigma_1) \]
\[ W^u(\sigma_1) \]
\[ W^s(\sigma_2) \]
\[ W^u(\sigma_2) \]

**Figure 1.** Nonhyperbolic but positively and negatively sectional-hyperbolic.

This counterexample motivates the search of sufficient conditions under which the intersection of a positively and a negatively sectional-hyperbolic set be hyperbolic. Our first result is about this problem.

**Theorem 2.2.** The intersection of a transitive positively sectional-hyperbolic set and a transitive negatively sectional-hyperbolic set is hyperbolic.

Consequently,

**Corollary 2.3.** Every transitive set which is both positively and negatively sectional-hyperbolic is hyperbolic.

The similar results replacing transitivity by denseness of periodic orbits hold.

By looking at Figure 1 we observe that this example consists of two singularities and a regular point $x$ whose $\omega$-limit and $\alpha$-limit set is a singularity. This observation is the motivation for the result below.

**Theorem 2.4.** Every compact invariant set which is both positively and negatively sectional-hyperbolic is the disjoint union of a (possibly empty) nonsingular hyperbolic set $H$, a (possibly empty) finite set of singularities $S$ and a (possibly empty) set of regular points $R$ such that $\alpha(x) \subset H \cup S$ and $\omega(x) \subset H \cup S$ for every $x \in R$.

Since the intersection of a positively and a negatively sectional-hyperbolic set is both positively and negatively sectional-hyperbolic, we obtain the following corollary.

**Corollary 2.5.** The intersection of a positively and a negatively sectional-hyperbolic set is a disjoint union of a (possibly empty) nonsingular hyperbolic set $H$, a (possibly empty) finite set of singularities $S$ and a (possibly empty) set of regular points $R$ such that $\alpha(x) \subset H \cup S$ and $\omega(x) \subset H \cup S$ for every $x \in R$.

Our next result is an example of nontrivial transitive sets which are positively and negatively sectional-hyperbolic (resp.) whose intersection is the simplest possible, i.e., a single periodic orbit.
Denote by $\text{Cl}(\cdot)$ the closure operation. We say that $H \subset M$ is a \textit{homoclinic class} if there is a hyperbolic periodic point $x$ of saddle type such that
\[ H = \text{Cl}\{q \in W^{wu}(x) \cap W^{wu}(x) : \dim(T_qW^{wu}(x) \cap T_qW^{wu}(x)) = 1\} \].

It follows from the Birkhoff-Smale Theorem that every homoclinic class is a transitive set with dense periodic orbits.

Given points $x, y \in M$, if for every $\epsilon > 0$ there are sequences of points $\{x_i\}_{i=0}^n$ and times $\{t_i\}_{i=0}^{n-1}$ such that $x_0 = x, x_n = y, t_i \geq 1$ and $d(X_{t_i}(x_i), x_{i+1}) < \epsilon$ for every $0 \leq i \leq n-1$, then we say that $x$ is in the chain stable set of $y$. If $x$ is in the chain stable set of $y$ and viceversa, then one says that $x$ and $y$ are \textit{chain related}. If $x$ is chain related to itself, one says that $x$ is a \textit{chain recurrent point}. The set of chain recurrent points is the \textit{chain recurrent set} denoted by $\text{CR}(X)$. It is clear that the chain related relation is in equivalence on $\text{CR}(X)$. By using this equivalence, one splits $\text{CR}(X)$ into equivalence classes denominated \textit{chain recurrent classes}.

A flow is \textit{star} if it exhibits a neighborhood $U$ (in the space of $C^1$ flows) such that every periodic orbit or singularity of every flow in $U$ is hyperbolic.

With these definitions we obtain the following result.

\textbf{Theorem 2.6.} There is a star flow $X$ in the sphere $S^3$ whose chain recurrent set is the disjoint union of two periodic orbits $O_1$ (a sink), $O_2$ (a source); two singularities $s_-$ (a source), $s_+$ (a saddle); and two homoclinic classes $H_-, H_+$ with the following properties:
\begin{itemize}
    \item $H_-$ is negatively (but not positively) sectional-hyperbolic;
    \item $H_+$ is positively (but not negatively) sectional-hyperbolic;
    \item $H_- \cap H_+$ is a periodic orbit.
\end{itemize}

Recall that the \textit{nonwandering set} of a flow $X$ is defined as the set of points $x \in M$ such that for every neighborhood $U$ of $x$ and $T > 0$ there is $t \geq T$ satisfying $X_t(U) \cap U \neq \emptyset$. Given a certain subset $O$ of the space of $C^1$ flows, we say that a $C^1$ generic flow in $O$ satisfies another property (Q) if there is a residual subset of flows $R$ of $O$ such that every flow in $R$ satisfying (P) also satisfies (Q).

There are two current conjectures relating star flows and sectional-hyperbolicity. These are based on previous results in the literature e.g. [11], [19].

\textbf{Conjecture 2.7 (Zhu-Shy-Gan-Wen [24, 25]).} The chain recurrent set of every star flow is the disjoint union of a positively sectional-hyperbolic set and a negatively sectional-hyperbolic set.

\textbf{Conjecture 2.8 (Arbieto [11]).} The nonwandering set of a $C^1$ generic star flow is the disjoint union of finitely many transitive sets which are positively or negatively sectional-hyperbolic.

However, the union $H_- \cup H_+$ of the homoclinic classes $H_-$ and $H_+$ in Theorem 2.6 is a chain recurrent class of the corresponding flow $X$ (because $H_- \cap H_+ \neq \emptyset$). Therefore, Theorem 2.6 gives a counterexample for Conjecture 2.7 in dimension 3. Similar counterexamples can be obtained in dimension $\geq 3$.

\textbf{Corollary 2.9.} There is a star flow in $S^3$ whose chain recurrent set is not the disjoint union of a positively sectional-hyperbolic set and a negatively sectional-hyperbolic set.
Another interesting feature regarding this counterexample is the existence of a chain recurrent class without any nontrivial dominated splitting with respect to the tangent flow. Moreover, every ergodic measure supported on this class is hyperbolic saddle. These features are related to [9] or [18]. Notice also that the star flow in Corollary 2.9 can be $C^1$ approximated by ones exhibiting the heteroclinic cycle obtained by joining the unstable manifold $W^u(\sigma_1)$ of $\sigma_1$ to the stable manifold $W^s(\sigma_2)$ of $\sigma_2$ in Figure 1. Such a cycle was emphasized in the figure after the statement of Lemma 3.3 in p.951 of [25]. This put in evidence the role of robust transitivity in the proof of such a lemma.

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3. Proof of Theorems 2.2 and 2.4

First we prove Theorem 2.4. For this we use the following technical definition.

**Definition 3.1.** A compact invariant set $\Lambda$ of a flow $X$ is almost hyperbolic if:

1. Every singularity in $\Lambda$ is hyperbolic.
2. There are continuous invariant subbundles $E^s, E^u$ of $T\Lambda M$ such that $E^s$ is contracting, $E^u$ is expanding and
   $$T\Lambda \cdot M = E^s \oplus E^X \oplus E^u.$$  

Notice that this definition is symmetric with respect to the reversing-flow operation. Moreover, hyperbolic sets are almost hyperbolic but not conversely by the example in Figure 1. Likewise sectional-hyperbolic sets, the almost hyperbolic sets satisfy

**Lemma 3.2** (Hyperbolic Lemma). Every compact invariant subset without singularities of an almost periodic set is hyperbolic.

More properties will be obtained from the lemma below. We denote by $B(x, \delta)$ the open $\delta$-ball operation, $\delta > 0$. If $\sigma \in \text{Sing}(X)$ is hyperbolic, then we denote by $W^s_\delta(\sigma)$ (resp. $W^u_\delta(\sigma)$) the connected component of $B(\sigma, \delta) \cap W^s(\sigma)$ (resp. $B(\sigma, \delta) \cap W^u(\sigma)$) containing $\sigma$.

**Lemma 3.3.** For every almost hyperbolic set $\Lambda$ of a flow $X$ there is $\delta > 0$ such that $\Lambda \cap B(\sigma, \delta) \subset W^s_\delta(\sigma) \cup W^u_\delta(\sigma)$ for every $\sigma \in \text{Sing}(X) \cap \Lambda$.

*Proof.* It suffices to prove that if $x_n \in \Lambda^*$ is a sequence converging to some singularity $\sigma \in \Lambda$, then $x_n \in W^s(\sigma) \cup W^u(\sigma)$ for $n$ large enough.

Let $T_\sigma M = E^s_\sigma \oplus E^X_\sigma \oplus E^u_\sigma$ be the hyperbolic splitting of $\sigma$. By definition $T_{x_n} M = E^s_{x_n} \oplus E^X_{x_n} \oplus E^u_{x_n}$ so

$$\dim(E^s_{x_n}) + \dim(E^u_{x_n}) = \dim(M) - 1, \quad \forall n.$$
Passing to the limit we obtain

\[ \dim(E^s_\sigma) + \dim(E^u_\sigma) = \dim(M) - 1. \]

Since \( E^s_\sigma \) and \( E^u_\sigma \) are contracting and expanding respectively, we obtain \( E^s_\sigma \subset F^s_\sigma \) and \( E^u_\sigma \subset F^u_\sigma \).

If \( \dim(F^s_\sigma) > \dim(E^s_\sigma) + 1 \) we would have

\[ \dim(E^s) = \dim(M) - 1 - \dim(E^s_\sigma) > \dim(M) - \dim(F^s_\sigma) = \dim(F^u_\sigma), \]

which is impossible. Then \( \dim(F^s_\sigma) \leq \dim(E^s_\sigma) + 1 \). Analogously, \( \dim(F^u_\sigma) \leq \dim(E^u_\sigma) + 1 \). Therefore, \( \dim(F^s_\sigma) = \dim(E^s_\sigma) \) or \( \dim(F^u_\sigma) = \dim(E^u_\sigma) + 1 \).

But we cannot have \( \dim(F^s_\sigma) = \dim(E^s_\sigma) + 1 \) and \( \dim(F^u_\sigma) = \dim(E^u_\sigma) + 1 \) simultaneously because

\[ \dim(M) = \dim(F^s_\sigma) + \dim(F^u_\sigma) = \dim(E^s_\sigma) + \dim(E^u_\sigma) + 2 = \dim(M) + 1 \]

which is absurd. All together imply

\[ \dim(E^s_\sigma) = \dim(F^s_\sigma) \quad \text{or} \quad \dim(E^u_\sigma) = \dim(F^u_\sigma). \]

Suppose \( \dim(E^s_\sigma) = \dim(F^s_\sigma) \). If \( y \in \Lambda \cap (W^s(\sigma) \setminus \{\sigma\}) \) is sufficiently close to \( \sigma \), then \( \dim(E^s_\sigma) = \dim(E^s_\sigma) = \dim(T_y W^s(\sigma)) \).

On the other hand, \( E^s \) is contracting thus \( E^s_y \subset T_y W^s(\sigma) \). From these remarks we obtain that if \( \dim(E^s_\sigma) = \dim(F^s_\sigma) \), then \( E^s_y = T_y W^s(\sigma) \) for all \( y \in \Lambda \cap (W^s(\sigma) \setminus \{\sigma\}) \) close to \( \sigma \). Analogously if \( \dim(E^u_\sigma) = \dim(F^u_\sigma) \), then \( E^u_y = T_y W^u(\sigma) \) for all \( y \in \Lambda \cap (W^u(\sigma) \setminus \{\sigma\}) \) close to \( \sigma \).

Now suppose by contradiction that \( x_n \not\in W^s(\sigma) \cup W^u(\sigma) \) for all \( n \) (say). Then, by flowing the orbit of \( x_n \) nearby \( \sigma \), as described in Figure 2, we obtain two sequences \( x^s_n, x^u_n \) in the orbit of \( x_n \) such that \( x^s_n \to y^s \) and \( x^u_n \to y^u \) for some \( y^s \in W^s(\sigma) \setminus \{\sigma\} \) and \( y^u \in W^u(\sigma) \setminus \{\sigma\} \) close to \( \sigma \).

\[ \text{Figure 2. Proof of Lemma 3.3} \]

If \( \dim(E^s_\sigma) = \dim(F^s_\sigma) \) then \( E^s_{y^s} = T_{y^s} W^s(\sigma) \) but also \( E^X_{y^s} \subset T_{y^s} W^s(\sigma) \) since \( W^s(\sigma) \) is an invariant manifold. Therefore, \( E^X_{y^s} \subset E^s_{y^s} \) and then \( E^X_{y^s} = 0 \) since the sum \( T_{y^s} M = E^s_{y^s} + E^X_{y^s} + E^u_{y^s} \) is direct. This is a contradiction. Analogously we obtain a contradiction if \( \dim(E^u_\sigma) = \dim(F^u_\sigma) \) and the proof follows. \( \square \)

Now we relate sectional and almost hyperbolicity.

**Lemma 3.4.** Every compact invariant set which is both positively and negatively sectional-hyperbolic is almost hyperbolic.
Proof. Let $\Lambda$ be a compact invariant set which is both positively and negatively sectional-hyperbolic. Then, every singularity in $\Lambda$ is hyperbolic. Moreover, there are positively and negatively sectional-hyperbolic splittings

$$T_\Lambda M = E^s \oplus E^{sc}, \quad \text{and} \quad T_\Lambda M = \hat{E}^s \oplus \hat{E}^{sc},$$

Taking $E^u = \hat{E}^s$ and $E^{sc} = \hat{E}^{sc}$ we obtain an expanding and a sectional contracting subbundles of $T_\Lambda M$. Since $E^s$ is contracting, we have $E^X \subset E^{sc}$ by Lemma 3.2 in [3]. Similarly, $E^X \subset E^{sc} \cap E^{sc}$.

On the other hand, since $E^u$ is expanding and $E^{sc}$ expanding, the angle $\langle E^s, E^u \rangle$ is bounded away from zero. Then, the dominating condition implies

$$E^u \subset E^{sc} \quad \text{and} \quad E^s \subset E^{sc}.$$

From this we have $T_\Lambda M = E^{sc} + E^{sc}$ and so

$$\dim(M) = \dim(E^{sc}) + \dim(E^{sc}) - \dim(E^{sc} \cap E^{sc}).$$

At regular points we cannot have a vector outside $E^X$ contained in $E^{sc} \cap E^{sc}$. Then,

$$E^X = E^{sc} \cap E^{sc} \quad \text{and so} \quad \dim(E^{sc} \cap E^{sc}) = 1$$

in $\Lambda^*$. Replacing above we get

$$\dim(M) = \dim(E^{sc}) + \dim(E^{sc}) - 1.$$ But we also have $\dim(M) = \dim(E^u) + \dim(E^{sc})$ so

$$\dim(E^u) = \dim(E^{sc}) - 1.$$ Since $E^u$ is expanding, we have $E^u \cap E^X = \{0\}$ thus

$$T_\Lambda M = E^s \oplus E^X \oplus E^u$$

proving the result. \qed

Proof of Theorem 2.3. Let $\Lambda$ be the intersection of a positively and a negatively sectional-hyperbolic set of a flow $X$. Then, it is both positively and negatively sectional-hyperbolic and so almost hyperbolic by Lemma 3.4. From this we can select $\delta > 0$ as in Lemma 3.3. Clearly we can take $\delta$ such that the balls $B(\sigma, \delta)$ are pairwise disjoint for $\sigma \in S$, where $S = \text{Sing}(X) \cap \Lambda$.

Define

$$H = \bigcap_{(t, \sigma) \in \mathbb{R} \times S} X_t(\Lambda \setminus B(\sigma, \delta))$$

and $R = \Lambda \setminus (H \cup S)$.

Clearly $S$ consists of finitely many singularities. Moreover, $H$ is nonsingular hence hyperbolic by the Hyperbolic Lemma. Now take $x \in R$. Then, there is $(t, \sigma) \in \mathbb{R} \times S$ such that $X_t(x) \in B(\sigma, \delta)$. By Lemma 3.3 we obtain $X_t(x) \in W^s(\sigma) \cup W^u(\sigma)$ hence $x \in W^s(\sigma) \cup W^u(\sigma)$.

If $x \in W^s(\sigma)$ we obtain $\omega(x) \subset H \cup S$. If $X_r(x) \notin \cup_{\sigma \in S} B(\sigma, \delta)$ for all $r \leq 0$ then $\alpha(x) \subset H$. Otherwise, there is $(r, \rho) \in \mathbb{R} \times S$ such that $X_r(x) \in B(\rho, \delta)$ and so $x \in W^u(\rho)$. All together yields $\alpha(x) \subset H \cup S$. Similarly, we have $\alpha(x) \subset H \cup S$ and $\omega(x) \subset H \cup S$ if $x \in W^u(\sigma)$ and the result follows. \qed

To prove Theorem 2.2 we use the following lemma. Recall that an invariant set is nontrivial if it does not reduces to a single orbit.
Lemma 3.5. Let $\Lambda$ be a nontrivial transitive positively sectional-hyperbolic set of a flow $X$. If $\sigma \in \text{Sing}(X) \cap \Lambda$, then the hyperbolic and the respective hyperbolic and positively sectional-hyperbolic splittings $T_\sigma M = F^s_\sigma \oplus F^u_\sigma$ and $T_\sigma M = E^s_\sigma \oplus E^{sc}_\sigma$ of $\sigma$ satisfy $\dim(E^{sc}_\sigma \cap F^s_\sigma) = 1$.

Proof. Clearly $E^s_\sigma \subset F^s_\sigma$. Suppose for a while that $E^s_\sigma = F^s_\sigma$. Then, $\dim(E^s_\sigma) = \dim(T_y W^s(\sigma))$ for every $y \in \Lambda \cap W^s(\sigma)$ close to $\sigma$. As clearly $E^s_y \subset T_y W^s(\sigma)$ for all such points $y$, we obtain $E^s_y = T_y W^s(\sigma)$ for every $y \in \Lambda \cap W^s(\sigma)$ close to $\sigma$.

On the other hand, we also have that $E^s_y \subset T_y W^s(\sigma)$ for all such points $y$. From this we conclude that $E^s_y \subset E^s_y$ for every point $y \in \Lambda \cap W^s(\sigma)$ close to $\sigma$. Now we observe that since $\Lambda$ is transitive we obtain $E^s_X \subset E^{sc}$. Using again that $\Lambda$ is nontrivial transitive (see Figure 2) we obtain $y = y^s \in \Lambda \cap W^s(\sigma)$ close to $\sigma$. For such a point we obtain $0 \neq E^s_y \subset E^s_y \cap E^{sc}_y$ which is absurd. Therefore, $E^s_\sigma \neq F^s_\sigma$.

Next we observe that $\dim(E^{sc}_\sigma \cap F^s_\sigma) \leq 1$ by sectional expansivity. Suppose for a while that $\dim(E^{sc}_\sigma \cap F^s_\sigma) = 0$. Clearly $E^s \cap F^u_\sigma = 0$ and so $E^u_\sigma \subset E^{sc}_\sigma$ by domination. From this we obtain $T_\Lambda M = F^s_\sigma \oplus F^u_\sigma$ thus $\dim(E^u_\sigma) = \dim(F^u_\sigma) = \dim(M) = \dim(F^s_\sigma) + \dim(F^u_\sigma)$ yielding $\dim(E^{sc}_\sigma) = \dim(F^u_\sigma)$ so $E^{sc}_\sigma = F^u_\sigma$ thus $E^s_\sigma = F^s_\sigma$ which is absurd. Therefore, $\dim(E^{sc}_\sigma \cap F^s_\sigma) = 1$ and we are done. \[\square\]

Proof of Theorem 2.2. Let $\Lambda_+ \text{ and } \Lambda_-$ be transitive sets of a flow $X$ such that $\Lambda_+$ is positively sectional-hyperbolic and $\Lambda_-$ is negatively sectional-hyperbolic. If one of these sets reduces to a single orbit, then the intersection $\Lambda_- \cap \Lambda$ reduces to that orbit and the result follows.

So, we can assume both $\Lambda_+$ and $\Lambda_-$ are nontrivial. Let $T_{\Lambda_+} M = E^s \oplus E^{sc}$ and $T_{\Lambda_-} M = \tilde{E}^s \oplus \tilde{E}^{sc}$ be the positively and negatively sectional-hyperbolic splittings of $\Lambda_+$ and $\Lambda_-$ respectively. Denoting $E^u = \tilde{E}^s$ and $E^{sc} = \tilde{E}^{sc}$ we obtain an expanding subbundle and a sectionally contracting subbundle of $T_\Lambda M$.

Suppose for a while that there is $\sigma \in \Lambda_- \cap \Lambda_+ \cap \text{Sing}(X)$. By Lemma 3.5 applied to $X$, we have that $\sigma$ has a real negative eigenvalues $\lambda^s$ corresponding to the one-dimensional eigendirection $E^{sc}_\sigma \cap F^s_\sigma$. Similarly, applying the lemma to $-X$, we obtain a real positive eigenvalue $\lambda^u$ corresponding to the one-dimensional eigendirection $E^{sc}_\sigma \cap F^u_\sigma$.

Take unitary vectors $v^s \in E^{sc}_\sigma \cap F^s_\sigma$ and $v^u \in E^{sc}_\sigma \cap F^u_\sigma$. Since

$$\left( E^{sc}_\sigma \cap F^s_\sigma \right) \cap \left( E^{sc}_\sigma \cap F^u_\sigma \right) \subset F^s_\sigma \cap F^u_\sigma = 0,$$

we have that $v^s$ and $v^u$ are linearly independent. Then, $\|v^s, v^u\| \neq 0$. Since $F^u_\sigma \subset E^{sc}_\sigma$, we have $v^s, v^u \in E^{sc}_\sigma$ so

$$e^{\lambda^s t} \cdot e^{\lambda^u t} \|v^s, v^u\| = \|DX(\sigma) v^s, DX(\sigma) v^u\| \to \infty \quad \text{as} \quad t \to \infty$$

by sectionally expansiveness. Then

$$\lambda^s + \lambda^u > 0.$$ 

Similarly, since $F^s_\sigma \subset E^{sc}_\sigma$, we have $v^s, v^u \in E^{sc}_\sigma$ so

$$e^{-\lambda^u t} \cdot e^{-\lambda^s t} \|v^s, v^u\| = \|DX(-\sigma) v^s, DX(-\sigma) v^u\| \to \infty \quad \text{as} \quad t \to \infty$$

by sectionally expansiveness with respect to $-X$. Then,

$$\lambda^s + \lambda^u < 0.$$ 


which is absurd. We conclude that $\Lambda_- \cap \Lambda_+ \cap \text{Sing}(X) = \emptyset$. Now we can apply the hyperbolic lemma for sectional-hyperbolic sets to obtain that $\Lambda_- \cap \Lambda_+$ is hyperpolic. This finishes the proof.

\section{Proof of Theorem 2.6}

Roughly speaking, the proof consists of glueing the so-called singular horseshoe \cite{10} with its time reversed counterpart.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{horseshoe.png}
\caption{Smale Horseshoe}
\end{figure}

We start with the standard Smale horseshoe which is the map in the 2-disk on the left of Figure 3. It turns out that its nonwandering set consists of a sink and a hyperbolic homoclinic class containing the saddle. Its suspension is the flow described in the right-hand picture of the figure. It is a flow in the solid torus whose nonwandering set is also a periodic sink $O_1$ together with a hyperbolic homoclinic class.

The next Figure 4 describes a procedure of inserting singularities in the suspended Smale horseshoe. We select an horizontal interval $I$ and a point $x$ in the square forming the horseshoe.

The selection is done in order to place $I$ in the stable manifold of a Lorenz-like equilibrium $\sigma_+$, and $x$ in the stable manifold of a Lorenz-like equilibrium for the reversed flow $\sigma_-$. This construction requires to add two additional singularities, a source $s_-$ to which the unstable branch of $\sigma_-$ not containing $x$ goes; and a saddle $s_+$ close to $\sigma_+$. See Figure 5.

An accurate description of the aforementioned procedure is done in \cite{8} and \cite{23}.

Next we observe that the resulting flow’s return map presents a cut along $I$ and a blowup circle derived from $x$.

We now proceed to deform the flow in order to obtain a deformation of the return map by pushing up one branch of the circle, and pushing down the cusped region derived from the cutting as indicated in Figure 6.
We keep doing this deformation (see Figure 7) up to arrive to the final flow whose return map is described in Figure 8.

This flow is defined in a solid torus, transversal to the boundary and pointing inward there.

The final return map (denoted by $R$) is described with some detail in Figure 9. We are in position to describe the homoclinic classes $H_-$ and $H_+$ in Theorem 2.6.

They are precisely the maximal invariant set of $R$ in the upper and lower rectangles $Q_+$ and $Q_-$ forming the rectangle $Q$ in Figure 9. These maximal sets are located in the intersections $A \cap B \cap A' \cap B'$ (for $H_-$) and $C \cap D \cap E \cap C' \cap E' \cap D'$.
Figure 6. Deforming.

Figure 7. Still deforming.
(for $H_+$). A rough description of $H_-$ and $H_+$ is that $H_+$ is the singular horseshoe in [16] and $H_-$ its time reversal.

The proof that $H_-$ and $H_+$ are nontrivial homoclinic classes is done as in [6], [7]. The analysis in [5] or [16] shows that $H_+$ is a sectional-hyperbolic set for the (final) flow and that $H_+$ is a sectional-hyperbolic set for the reversed flow. We assume that the horizontal cone field $x \in Q \mapsto C^s_1(x)$ and the vertical cone field $x \in Q \mapsto C^u_1(x)$,
where
\[ C_\alpha(x) = \left\{ (a, b) \in \mathbb{R}^2 : \frac{|b|}{|a|} \leq \alpha \right\} \quad \text{and} \quad C_\alpha^*(x) = \left\{ (a, b) \in \mathbb{R}^2 : \frac{|a|}{|b|} \leq \alpha \right\}, \quad \forall \alpha > 0, \]
are contracting and expanding (respectively) for the return map \( R \) in the sense that there is \( \rho > 1 \) with the following properties:

1. If \( x \in R^{-1}(Q) \cap Q \) then
   \[ DR(x)C_\alpha^1(x) \subset C_\alpha^1(R(x)) \quad \text{and} \quad \| DR(x)v^n \| \geq \rho \| v^n \|, \quad \forall v^n \in C_\alpha^1(x). \]

2. If \( x \in R(Q) \cap Q \) then
   \[ DR^{-1}(x)C_\alpha^1(x) \subset C_\alpha^1(R^{-1}(x)) \quad \text{and} \quad \| DR^{-1}(x)v^s \| \geq \rho \| v^s \|, \quad \forall v^s \in C_\alpha^1(x). \]

(See Figure 9.) Since such conefields do not allow the existence of nonhyperbolic periodic points, and are preserved by small perturbations, we obtain that the final flow is star in its solid torus domain.

Next we observe that \( H_- \) is not hyperbolic, since it contains the singularity \( \sigma_- \) and, analogously, \( H_+ \) is not hyperbolic for it contains \( \sigma_+ \). Since every homoclinic class is transitive, we conclude from Theorem 2.3 that \( H_- \) is not positively sectional-hyperbolic and \( H_+ \) is not negatively sectional-hyperbolic.

To complete the proof we extend the final flow from its solid torus domain to the whole \( S^3 \). This is done by gluing it with another solid torus whose core is a periodic source \( O_2 \). This completes the proof. \( \square \)

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