Accurate computations with Said-Ball-Vandermonde matrices

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Abstract

A generalization of the Vandermonde matrices which arise when the power basis is replaced by the Said-Ball basis is considered. When the nodes are inside the interval (0, 1), then those matrices are strictly totally positive. An algorithm for computing the bidiagonal decomposition of those Said-Ball-Vandermonde matrices is presented, which allows to use known algorithms for totally positive matrices represented by their bidiagonal decomposition. The algorithm is shown to be fast and to guarantee high relative accuracy. Some numerical experiments which illustrate the good behaviour of the algorithm are included.

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Key words: Vandermonde matrix, Said-Ball basis, Totally positive matrix, Bidiagonal decomposition, High relative accuracy.

1 Introduction

Numerical computing with structured totally nonnegative matrices is a classical subject in the field of numerical linear algebra which has recently received a renewed attention, as can be seen in the recent survey paper [8], where several different classes of structured matrices are considered, among them totally positive matrices.

Classically, a matrix is said to be totally positive if all its minors are nonnegative [14]. Consequently, the matrices with that property are also called totally positive matrices.

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nonnegative matrices [11], and this term is becoming more used in recent literature.

The fact that a nonsingular totally nonnegative (TN) matrix can be decomposed as a product of nonnegative bidiagonal factors was used by Koev [21][22] to develop several accurate algorithms for the general class of TN matrices. A detailed survey of several results related to TN matrices, including the bidiagonal factorization, has been presented in [11].

Nevertheless, it must be stressed that the algorithms of Koev [21][22] start from the bidiagonal decomposition of a TN matrix $A$, which is stored in a matrix which is denoted there as $BD(A)$, and that such a decomposition needs to be computed for each particular class of TN matrices being considered. Using the words of the section devoted to conclusions and open problems in [22]:

*The caveat in our algorithms is that every TN matrix must be represented by its bidiagonal decomposition. While every TN matrix intrinsically possesses such a decomposition, and for many classes of structured matrices this decomposition is very easy to obtain accurately, there are important TN matrices for which we know of no accurate and efficient way to compute their bidiagonal decompositions.*

Examples of totally nonnegative matrices for which there are accurate and efficient algorithms for computing $BD(A)$ are Vandermonde [4][6][17][18], Cauchy [5], Cauchy-Vandermonde [24][25], generalized Vandermonde [10] and Bernstein-Vandermonde matrices [23].

On the other hand, it is not always recognized that while Neville elimination [11][12][13][14] is a key theoretical tool for the analysis of that bidiagonal decomposition, it generally fails to provide an accurate algorithm for computing $BD(A)$. This fact is explicitly noted in [20], where the author indicates that the function TNBD is the only function in the package TNTool that does not guarantee high relative accuracy.

Consequently, the accurate (and, if possible, fast) computation of $BD(A)$ is a previous task to be performed before applying Koev’s algorithms to a given class of TN matrices. The importance of those algorithms was very acknowledged in [27], while relevant previous results were presented in [9].

In this work we are extending to the class of Said-Ball-Vandermonde matrices the work we have recently carried out for the class of Bernstein-Vandermonde matrices [23]. A crucial fact for obtaining high relative accuracy in our algorithm is that it satisfies what is called in [8] the NIC (no inaccurate cancellation) condition:

**NIC:** The algorithm only multiplies, divides, adds (resp., substracts) real
numbers with like (resp., differing) signs, and otherwise only adds or substracts input data.

The Said-Ball basis is a generalization of the Ball basis \[123\], a well-known basis for cubic polynomials on a finite interval which is useful in the field of Computer-Aided Design. The Said-Ball basis was introduced for odd degree polynomials by Said in \[26\], and then its definition for polynomials of even degree was suggested in \[19\]. Its properties in connection with total positivity and shape preservation were studied by Goodman and Said for odd degree polynomials \[15\], and recently by Delgado an Peña in \[7\], where it was established that the Said-Ball basis is a normalized totally positive (NTP) basis for every value of the polynomial degree.

The rest of the paper is organized as follows. Some basic results on Neville elimination and total positivity are recalled in Section 2. In Section 3 the bidiagonal decomposition of a Said-Ball-Vandermonde matrix and of its inverse are presented. The algorithm for computing these bidiagonal factorizations is introduced in Section 4. In Section 5 the problems of linear system solving and eigenvalue computation for a Said-Ball-Vandermonde matrix are considered. Finally, Section 6 is devoted to illustrate the accuracy of our algorithms by means of some numerical experiments.

2 Basic results on Neville elimination and total positivity

In this section we will briefly recall some basic results on Neville elimination and total positivity which we will apply in Section 3. Our notation follows the notation used in \[1213\]. Given \( k, n \in \mathbb{N} \) \((1 \leq k \leq n)\), \( Q_{k,n} \) will denote the set of all increasing sequences of \( k \) positive integers less than or equal to \( n \).

Let \( A \) be a real square matrix of order \( n \). For \( k \leq n, m \leq n \), and for any \( \alpha \in Q_{k,n} \) and \( \beta \in Q_{m,n} \), we will denote by \( A[\alpha|\beta] \) the submatrix \( k \times m \) of \( A \) containing the rows numbered by \( \alpha \) and the columns numbered by \( \beta \).

The fundamental tool for obtaining the theoretical results applied in this paper is the Neville elimination (see \[1213\]), a procedure that makes zeros in a matrix adding to a given row an appropriate multiple of the previous one. For a nonsingular matrix \( A = (a_{i,j})_{1 \leq i,j \leq n} \), it consists on \( n - 1 \) steps resulting in a sequence of matrices \( A := A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_n \), where \( A_t = (a_{i,j}^{(t)})_{1 \leq i,j \leq n} \) has zeros below its main diagonal in the \( t - 1 \) first columns. The matrix \( A_{t+1} \)
is obtained from $A_t (t = 1, \ldots, n)$ by using the following formula:

$$a_{i,j}^{(t+1)} := \begin{cases} a_{i,j}^{(t)}, & \text{if } i \leq t \\ a_{i,j}^{(t)} - (a_{i,t}^{(t)}/a_{i-1,j}^{(t)})a_{i-1,j}^{(t)}, & \text{if } i \geq t + 1 \text{ and } j \geq t + 1 \\ 0, & \text{otherwise.} \end{cases}$$

(2.1)

In this process the element

$$p_{i,j} := a_{i,j}^{(j)} \quad 1 \leq j \leq n; \quad j \leq i \leq n$$

is called pivot $(i, j)$ of the Neville elimination of $A$. The process would break down if any of the pivots $p_{i,j}$ ($j \leq i < n$) is zero. In that case we can move the corresponding rows to the bottom and proceed with the new matrix, as described in [12]. The Neville elimination can be done without row exchanges if all the pivots are nonzero, as it will happen in our situation. The pivots $p_{i,i}$ are called diagonal pivots. If all the pivots $p_{i,j}$ are nonzero, then $p_{i,1} = a_{i,1} \forall i$ and, by Lemma 2.6 of [12]

$$p_{i,j} = \frac{\det A[i-j+1, \ldots, i][1, \ldots, j]}{\det A[i-j+1, \ldots, i-1][1, \ldots, j-1]} \quad 1 < j \leq i \leq n. \quad (2.2)$$

The element

$$m_{i,j} = \frac{p_{i,j}}{p_{i-1,j}} \quad 1 \leq j \leq n; \quad j < i \leq n$$

(2.3)

is called multiplier of the Neville elimination of $A$. The matrix $U := A_n$ is upper triangular and has the diagonal pivots in its main diagonal.

The complete Neville elimination of a matrix $A$ consists on performing the Neville elimination of $A$ for obtaining $U$ and then continue with the Neville elimination of $U^T$. The pivot (respectively, multiplier) $(i, j)$ of the complete Neville elimination of $A$ is the pivot (respectively, multiplier) $(j, i)$ of the Neville elimination of $U^T$, if $j \geq i$. When no row exchanges are needed in the Neville elimination of $A$ and $U^T$, we say that the complete Neville elimination of $A$ can be done without row and column exchanges, and in this case the multipliers of the complete Neville elimination of $A$ are the multipliers of the Neville elimination of $A$ if $i \geq j$ and the multipliers of the Neville elimination of $A^T$ if $j \geq i$.

A matrix is called strictly totally positive) if all its minors are positive. The Neville elimination characterizes the strictly totally positive matrices as follows ([12]):

**Theorem 2.1.** A matrix is strictly totally positive if and only if its complete Neville elimination can be performed without row and column exchanges,
the multipliers of the Neville elimination of $A$ and $A^T$ are positive, and the diagonal pivots of the Neville elimination of $A$ are positive.

As it can be seen in [7], the Said-Ball-Vandermonde matrices are strictly totally positive when the real numbers satisfy $0 < t_1 < t_2 < \ldots < t_{n+1} < 1$, and this fact has inspired our search for a fast algorithm, but this result will also be shown to be a consequence of our Theorem 3.2.

## 3 Bidiagonal decomposition

The Said-Ball basis $S_n = \{ s^n_0(t), s^n_1(t), \ldots, s^n_n(t) \}$ of the space $\Pi_n(t)$ of the polynomials of degree less than or equal to $n$ on the interval $[0, 1]$ is defined by:

\[
\begin{align*}
  s^n_i(t) &= \binom{\lfloor n/2 \rfloor + i}{i} t^i (1-t)^{\lfloor n/2 \rfloor + 1}, \quad 0 \leq i \leq \lfloor (n-1)/2 \rfloor, \\
  s^n_{n-i}(t) &= \binom{\lfloor n/2 \rfloor + n-i}{n-i} t^{n-i} (1-t)^{\lfloor n/2 \rfloor + 1}, \quad [n/2] + 1 \leq i \leq n,
\end{align*}
\]

and, if $n$ is even

\[
s^n_{n/2}(t) = \binom{n}{n/2} t^{n/2} (1-t)^{n/2},
\]

where $\lfloor m \rfloor$ is the greatest integer less than or equal to $m$.

From now on, we will call Said-Ball-Vandermonde matrices (SB-Vandermonde matrices in the sequel) the generalization of the Vandermonde matrices obtained when considering the Said-Ball basis instead of the power basis. The SB-Vandermonde matrices are therefore

\[
A = \begin{pmatrix}
\left( \frac{n-1}{0} \right) t_1 \frac{n+1}{2} & \left( \frac{n-1}{0} \right) t_2 \frac{n+1}{2} & \cdots & \left( \frac{n-1}{0} \right) t_{n+1} \frac{n+1}{2} \\
\left( \frac{n-1}{1} \right) t_1 \frac{n+1}{2} & \left( \frac{n-1}{1} \right) t_2 \frac{n+1}{2} & \cdots & \left( \frac{n-1}{1} \right) t_{n+1} \frac{n+1}{2} \\
\vdots & \vdots & \ddots & \vdots \\
\left( \frac{n-1}{n-1} \right) t_1 \frac{n+1}{2} & \left( \frac{n-1}{n-1} \right) t_2 \frac{n+1}{2} & \cdots & \left( \frac{n-1}{n-1} \right) t_{n+1} \frac{n+1}{2} \\
\left( \frac{n+1}{0} \right) t_1 \frac{n+1}{2} & \left( \frac{n+1}{0} \right) t_2 \frac{n+1}{2} & \cdots & \left( \frac{n+1}{0} \right) t_{n+1} \frac{n+1}{2} \\
\end{pmatrix}^T
\]
in the case of odd \( n \), and

\[
A = \begin{pmatrix}
\left( \frac{a}{0} \right) (1 - t_1)^{n+2} & \left( \frac{a}{0} \right) (1 - t_2)^{n+2} & \cdots & \left( \frac{a}{0} \right) (1 - t_n)^{n+2} \\
\left( \frac{n+1}{2} \right) t_1 (1 - t_1)^{n+2} & \left( \frac{n+1}{2} \right) t_2 (1 - t_2)^{n+2} & \cdots & \left( \frac{n+1}{2} \right) t_{n+1} (1 - t_{n+1})^{n+2} \\
\vdots & \vdots & \ddots & \vdots \\
\left( \frac{n-1}{2} \right) t_1^{n+2} (1 - t_1)^{n+2} & \left( \frac{n-1}{2} \right) t_2^{n+2} (1 - t_2)^{n+2} & \cdots & \left( \frac{n-1}{2} \right) t_{n+1}^{n+2} (1 - t_{n+1})^{n+2} \\
\left( \frac{n}{2} \right) t_1^{n+2} (1 - t_1)^{n+2} & \left( \frac{n}{2} \right) t_2^{n+2} (1 - t_2)^{n+2} & \cdots & \left( \frac{n}{2} \right) t_{n+1}^{n+2} (1 - t_{n+1})^{n+2} \\
\left( \frac{n+1}{2} \right) t_1^{n+2} (1 - t_1) & \left( \frac{n+1}{2} \right) t_2^{n+2} (1 - t_2) & \cdots & \left( \frac{n+1}{2} \right) t_{n+1}^{n+2} (1 - t_{n+1}) \\
\left( \frac{n}{2} \right) t_1^{n+2} & \left( \frac{n}{2} \right) t_2^{n+2} & \cdots & \left( \frac{n}{2} \right) t_{n+1}^{n+2}
\end{pmatrix}^T
\]

in the case of even \( n \).

It must be observed that the SB–Vandermonde matrix \( A \) is the coefficient matrix associated with the following interpolation problem in the Said-Ball basis \( S_n \): given the interpolation nodes \( \{ t_i : i = 1, \ldots, n + 1 \} \) and the interpolation data \( \{ b_i : i = 1, \ldots, n + 1 \} \) find the polynomial

\[
p(t) = \sum_{k=0}^{n} a_k s_k^n(t)
\]

such that \( p(t_i) = b_i \) for \( i = 1, \ldots, n + 1 \).

From now on, we will assume \( 0 < t_1 < t_2 < \ldots < t_{n+1} < 1 \).

**Proposition 3.1.** The determinant of the SB–Vandermonde matrix \( A \) defined above is

\[
\det A = \left[ \left( \frac{n-1}{2} \right) \left( \frac{n+1}{2} \right) \left( \frac{n+3}{2} \right) \cdots \left( \frac{n-2}{2} \right) \left( \frac{n-1}{2} \right) \right]^2 \prod_{1 \leq i < j \leq n+1} (t_j - t_i),
\]

if \( n \) is odd, and

\[
\det A = \left[ \left( \frac{n}{2} \right) \left( \frac{n+2}{2} \right) \left( \frac{n+4}{2} \right) \cdots \left( \frac{n-1}{2} \right) \right]^2 \prod_{1 \leq i < j \leq n+1} (t_j - t_i),
\]

if \( n \) is even.

**Proof.** Here we include the proof for the case in which \( n \) is odd. The proof in the even case is completely analogous.
Looking at [19], it can be observed that the matrix of change of basis from the Bernstein basis to the Said-Ball basis is a block-diagonal matrix $M$ with triangular diagonal blocks, and whose determinant is

$$\det M = \left[ \frac{n+1}{2} \frac{n+1}{2} \cdots \frac{n-2}{2} \frac{n-1}{2} \right]^2 \frac{n}{n-1} \cdots \frac{n}{n}.$$  

As it can be seen, for example, in [23], the matrix of change of basis from the Bernstein basis $B_n = \{b_i^{(n)}(t) = \binom{n}{i}(1-t)^{n-i}t^i, \quad i = 0, \ldots, n\}$

to the power basis $\{1, t, t^2, \ldots, t^n\}$ is a lower triangular matrix $N$ of order $n+1$ whose determinant is

$$\det N = \binom{n}{0} \binom{n}{1} \cdots \binom{n}{n}.$$  

(3.2)

Taking this into account, the matrix of change of basis from the power basis to the Said-Ball basis is $MN^{-1}$, and consequently,

$$\det A = \frac{\det M \det N}{\det V},$$

where $V$ is the Vandermonde matrix

$$V = \begin{pmatrix} 1 & t_1 & t_1^2 & \cdots & t_1^n \\ 1 & t_2 & t_2^2 & \cdots & t_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_{n+1} & t_{n+1}^2 & \cdots & t_{n+1}^n \end{pmatrix}.$$  

Using the well-known formula for the determinant of a Vandermonde matrix

$$\det V = \prod_{1 \leq i < j \leq n+1} (t_j - t_i)$$

and the equations (3.1) and (3.2), the proof is concluded. $\square$

The following two theorems will be essential in the construction of our algorithm for computing $BD(A)$ of a SB–Vandermonde matrix.

**Theorem 3.2.** Let $A = (a_{i,j})_{1 \leq i,j \leq n+1}$ be a SB–Vandermonde matrix whose nodes satisfy $0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < 1$. Then $A^{-1}$ admits a
factorization in the form

\[ A^{-1} = G_1 G_2 \cdots G_n D^{-1} F_n F_{n-1} \cdots F_1, \quad (3.3) \]

where \( G_i \) are upper triangular bidiagonal matrices, \( F_i \) are lower triangular bidiagonal matrices \( (i = 1, \ldots, n) \), and \( D \) is a diagonal matrix.

**Proof.** The matrix \( A \) is a strictly totally positive matrix (see [6,7]) and therefore, by Theorem 2.1, the complete Neville elimination of \( A \) can be performed without row and column exchanges providing the following factorization of \( A^{-1} \) (see [12,13]):

\[ A^{-1} = G_1 G_2 \cdots G_n D^{-1} F_n F_{n-1} \cdots F_1, \]

where \( F_i \) \( (1 \leq i \leq n) \) are bidiagonal matrices of the form

\[
F_i = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots \\
0 & -m_{i+1,i} & 1 & \ddots & \ddots & \ddots \\
-1 & -m_{i+2,i} & 1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & 1 & 1 \\
& & & & & & 1 \\
& & & & & & & 1 \\
& & & & & & & & 1 \\
& & & & & & & & & 1
\end{pmatrix}, \quad (3.4)
\]

\( G^T_i \) \( (1 \leq i \leq n) \) are bidiagonal matrices of the form

\[
G^T_i = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 1 \\
0 & 1 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \ddots \\
0 & -\tilde{m}_{i+1,i} & 1 & \ddots & \ddots & \ddots \\
-1 & -\tilde{m}_{i+2,i} & 1 & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & 1 & 1 \\
& & & & & & 1 \\
& & & & & & & 1 \\
& & & & & & & & 1 \\
& & & & & & & & & 1 \\
& & & & & & & & & & 1
\end{pmatrix}, \quad (3.5)
\]

and \( D \) is the diagonal matrix whose \( i \)th \( (1 \leq i \leq n + 1) \) diagonal entry is the
diagonal pivot $p_{i,j} = a_{i,j}^{(i)}$ of the Neville elimination of $A$:

$$D = \text{diag}\{p_{1,1}, p_{2,2}, \ldots, p_{n+1,n+1}\}. \quad (3.6)$$

First we obtain the expressions for the multipliers $m_{i,j}$ and $\bar{m}_{i,j}$, and for the diagonal pivots $p_{i,i}$ in the case of odd $n$.

Taking into account that the minors of $A$ with $j$ initial consecutive columns and $j$ consecutive rows starting with row $i$ are

$$\det A[i, \ldots, i+j-1] = (1-t_i)^{n-j} \prod_{l \leq k \leq i+j-1} (t_l - t_k),$$

if $j \leq \frac{n+1}{2}$, and

$$\det A[i, \ldots, i+j-1] = (1-t_i)^{n-j} \prod_{l \leq k \leq i+j-1} (t_l - t_k),$$

if $j > \frac{n+1}{2}$, a result that follows from the properties of the determinants and Proposition 3.1, and that $m_{i,j}$ are the multipliers of the Neville elimination of $A$, we obtain that

$$m_{i,j} = \begin{cases} 
\frac{(1-t_i)^{n-j}}{(1-t_{i-1})^{n-j}} \frac{t_{i-1}-t_{i-k}}{t_{i-1}-t_{i-k}} \prod_{l=2}^{i-1} (t_{i-1} - t_{i-l-k}), & j = 1, \ldots, \frac{n+1}{2}; \ i = j + 1, \ldots, n + 1, \\
\frac{(1-t_i)^{n-j}}{(1-t_{i-1})^{n-j}} \frac{t_{i-1}-t_{i-k}}{t_{i-1}-t_{i-k}} \prod_{l=2}^{i-1} (t_{i-1} - t_{i-l-k}), & j = \frac{n+3}{2}, \ldots, n; \ i = j + 1, \ldots, n + 1. 
\end{cases} \quad (3.7)$$

As for the minors of $A^T$ with $j$ initial consecutive columns and $j$ consecutive rows starting with row $i$, they are:

$$\det A^T[i, \ldots, i+j-1] = (1-t_1)^{n-j} \prod_{l \geq k \geq i+j-1} (t_l - t_k),$$

if $i \leq \frac{n+1}{2}$ and $i+j-1 \leq \frac{n+1}{2}$,

$$\det A^T[i, \ldots, i+j-1] = (1-t_1)^{n-j} \prod_{l \geq k \geq i+j-1} (t_l - t_k),$$

if $i \geq \frac{n+1}{2}$ and $i+j-1 \geq \frac{n+1}{2}$.
Finally, the diagonal entries of $D$ are:

$$p_{i,i} = \left\{ \begin{array}{ll}
\frac{n+1+i-1}{n-i-1} (1 - t_i) \frac{n+1}{n-i} \prod_{k<i} (t_i - t_k), & i = 1, \ldots, \frac{n+1}{2}, \\
(\frac{n+1}{n+i+1}) (1-t_i)^{n+i+1} \prod_{k<i} (t_i - t_k), & i = \frac{n+3}{2}, n+1.
\end{array} \right. \tag{3.9}$$

The formulas for $p_{i,i}$ are obtained by using the expressions for the minors of $A$ with initial consecutive columns and initial consecutive rows.

As for the case in which $n$ is even, proceeding analogously as in the odd case we obtain the following expressions for the multipliers $m_{i,j}$ and $\tilde{m}_{i,j}$, and the diagonal pivots $p_{i,i}$:

$$m_{i,j} = \left\{ \begin{array}{ll}
\frac{(1-t_i)^{n+j}}{(1-t_{i-1})^{n+j}} \prod_{k=2}^{j} (t_{i-k} - t_{i-k-1})^{n+1}, & j = 1, \ldots, \frac{n}{2}; \ i = j + 1, \ldots, n + 1, \\
(1-t_i)^{n+j+1} (1-t_{i-j}) \prod_{k=1}^{j} (t_i - t_{i-k}) \prod_{k=2}^{j} (t_{i-1} - t_{i-k}), & j = \frac{n+2}{2}, \ldots, n; \ i = j + 1, \ldots, n + 1.
\end{array} \right. \tag{3.10}$$
\[
\tilde{m}_{i,j} = \begin{cases} 
\frac{n+i-1}{i-1} t_j, & i = 2, \ldots, \frac{n}{2}, \ j = 1, \ldots, i - 1, \\
\frac{2t_i}{\prod_{k=1}^{j} (1-t_k)}, & i = \frac{n+2}{2}, \ j = 1, \ldots, \frac{n}{2}, \\
\frac{n-i+2}{n-i+2} \frac{1}{1-t_j}, & i = \frac{n+6}{2}, \ n+1; j = 1, \ldots, i - \frac{n+4}{2}, \\
\frac{n-i+2}{n-i+2} t_i, & i = \frac{n+4}{2}, \ n+1; j = i - \frac{n+2}{2}, \ldots, i - 1, 
\end{cases} 
\]  
(3.11) 

and 

\[
p_{i,i} = \begin{cases} 
\binom{\frac{n+i-1}{i-1}}{i-1} (1-t_i)^{\frac{n+2}{2}} \prod_{k<i} (t_i - t_k), & i = 1, \ldots, \frac{n}{2}, \\
\binom{\frac{n-i+1}{i-1}}{\frac{n-i+1}{n-i+1}} (1-t_i)^{\frac{n-i+1}{n-i+1}} \prod_{k<i} (t_i - t_k) / \prod_{k=i}^{n} (1-t_k), & i = \frac{n+2}{2}, \ldots, n + 1. 
\end{cases} 
\]  
(3.12) 

Moreover, by using the same arguments of [24], it can be seen that this factorization is unique among factorizations of this type, that is to say, factorizations in which the matrices involved have the properties shown by formulae (3.4)-(3.6).

Let us observe that the formulae obtained in the proof of Theorem 3.2 for the minors of \(A\) with \(j\) initial consecutive columns and \(j\) consecutive rows, and for the minors of \(A^T\) with \(j\) initial consecutive columns and \(j\) consecutive rows show that they are not zero, and so the complete Neville elimination of \(A\) can be performed without row and column exchanges. Looking at equations (3.7)-(3.12) it is easily seen that \(m_{i,j}, \tilde{m}_{i,j}\) and \(p_{i,i}\) are positive. Therefore, taking into account Theorem 2.1, this confirms that the matrix \(A\) is strictly totally positive.

**Theorem 3.3.** Let \(A = (a_{i,j})_{1 \leq i,j \leq n+1}\) be a SB–Vandermonde matrix whose nodes satisfy \(0 < t_1 < t_2 < \ldots < t_n < t_{n+1} < 1\). Then \(A\) admits a factorization in the form 

\[A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n\]

where \(F_i\) are lower triangular bidiagonal matrices, \(G_i\) are upper triangular \((i = 1, \ldots, n)\), and \(D\) is a diagonal matrix.

**Proof.** The matrix \(A\) is a strictly totally matrix [7] and therefore, by Theorem 2.1, the complete Neville elimination of \(A\) can be performed without row and column exchanges providing the following factorization of \(A\) (see [14]):

\[A = F_n F_{n-1} \cdots F_1 D G_1 \cdots G_{n-1} G_n,\]
where \( F_i (1 \leq i \leq n) \) are bidiagonal matrices of the form

\[
F_i = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & m_{i+1,1} & 1 \\
0 & 1 & 0 & \cdots & 0 & m_{i+2,1} & 1 \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & 0 & 1 & m_{n,n-i} & 1
\end{pmatrix},
\]

(3.9)

\( G^T_i (1 \leq i \leq n) \) are bidiagonal matrices of the form

\[
G^T_i = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & \tilde{m}_{i+1,1} & 1 \\
0 & 1 & 0 & \cdots & 0 & \tilde{m}_{i+2,1} & 1 \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & 0 & 1 & \tilde{m}_{n,n-i} & 1
\end{pmatrix},
\]

(3.10)

and \( D \) is the diagonal matrix

\[
D = \text{diag}\{p_{1,1}, p_{2,2}, \ldots, p_{n+1,n+1}\}.
\]

The expressions of the multipliers \( m_{i,j} (1 \leq i, j \leq n + 1) \) of the Neville elimination of \( A \), the multipliers \( \tilde{m}_{i,j} (1 \leq i, j \leq n + 1) \) of the Neville elimination of \( A^T \), and the diagonal pivots \( p_{i,i} (1 \leq i, \leq n + 1) \) of the Neville elimination of \( A \) are also in this case the ones given by Eq. (3.7) and Eq. (3.10), Eq. (3.8) and Eq. (3.11), and Eq. (3.9) and Eq. (3.12), respectively. \( \square \)

It must be observed that the matrices \( F_i \) and \( G_i \) \((i = 1, \ldots, n)\) that appear in the bidiagonal factorization of \( A \) are not the same bidiagonal matrices that appear in the bidiagonal factorization of \( A^{-1} \), nor their inverses (see Theorem 3.2 and Theorem 3.3). The multipliers of the Neville elimination of \( A \) and \( A^T \) give us the bidiagonal factorization of \( A \) and \( A^{-1} \), but obtaining the bidiagonal factorization of \( A \) from the bidiagonal factorization of \( A^{-1} \) (or vice versa) is
not straightforward. The structure of the bidiagonal matrices that appear in both factorizations is not preserved by the inversion, that is, in general, \( F_i^{-1} \) and \( G_i^{-1} \) \((1 \leq i, j \leq n)\) are not bidiagonal matrices. See [14] for a more detailed explanation.

4 The algorithm

In this section we present a fast and accurate algorithm for computing \( \mathcal{BD}(A) \) for a totally positive SB–Vandermonde matrix \( A \). Let us point out here that given \( A \) the matrix \( \mathcal{BD}(A) \) represents both the bidiagonal decomposition of \( A \), and that of its inverse \( A^{-1} \) (see Theorem 3.2 and Theorem 3.3).

The algorithm will compute the multipliers \( m_{ij} \) of the Neville elimination of \( A \), the multipliers \( \tilde{m}_{ij} \) of the Neville elimination of \( A^T \) and the diagonal pivots \( p_{ii} \) of the Neville elimination of \( A \), which are the entries of the matrix \( \mathcal{BD}(A) \).

We include here the algorithm for the case in which \( n \) is an odd number, the algorithm for the even case being analogous.

The algorithm for computing the \( m_{i,j} \) given by Eq. (3.7) is:

\[
\text{for } i = 2 : n + 1 \\
\quad m_{i,1} = \frac{(1-t_i)^{n+1}}{(1-t_{i-1})^{n+2}} \\
\quad \text{for } j = 1 : \min(i - 2, \frac{n-1}{2}) \\
\quad \quad m_{i,j+1} = \frac{t_i-t_{i+j}}{t_{i-1}-t_{i+j-1}} \cdot m_{i,j} \\
\quad \text{end} \\
\text{end} \\
\text{for } i = \frac{n+5}{2} : n + 1 \\
\quad m_{i,,\frac{n+3}{2}} = \frac{(1-t_i)^{n+3}(t_i-t_{i-1})^{n+1}}{(1-t_i)(t_{i-1}-t_{i-\frac{n+3}{2}})} \cdot m_{i,\frac{n+1}{2}} \\
\quad \text{for } j = \frac{n+2}{2} : i - 2 \\
\quad \quad m_{i,j+1} = \frac{(1-t_{i-1})(1-t_{i-j-1})(t_i-t_{i-1})}{(1-t_i)(1-t_{i-j})(t_{i-1}-t_{i-j-1})} \cdot m_{i,j} \\
\quad \text{end}
\]
The algorithm for the computation of the $\tilde{m}_{i,j}$ given by Eq. (3.8) is:

for $i = 2 : \frac{n+1}{2}$

\[
\text{aux} = \frac{\frac{n+i-1}{i-1}}
\]

for $j = 1 : i - 1$

\[
\tilde{m}_{i,j} = \text{aux} \cdot t_j
\]

end

end

$\tilde{m}_{n+3,1} = \frac{t_1}{1-t_1}$

for $j = 1 : \frac{n-1}{2}$

\[
\tilde{m}_{n+3,j+1} = \frac{t_{j+1}}{t_j(1-t_{j+1})} \cdot \tilde{m}_{n+3,j}
\]

end

for $i = \frac{n+5}{2} : n + 1$

\[
\text{aux} = \frac{\frac{n-i+2}{\frac{i}{2}+n-i+2}}
\]

for $j = 1 : i - \frac{n+3}{2}$

\[
\text{int} = \frac{1}{1-t_j}
\]

\[
\tilde{m}_{i,j} = \text{aux} \cdot \text{int}
\]

end

for $j = i - \frac{n+1}{2} : i - 1$

\[
\text{int} = \frac{t_j}{1-t_j}
\]

\[
\tilde{m}_{i,j} = \text{aux} \cdot \text{int}
\]

end

end

The algorithm for computing the diagonal pivots $p_{i,i}$ given by Eq. (3.9) is:
\[ q = 1 \]
\[ p_{1,1} = (1 - t_1)^{\frac{n+1}{2}} \]

for \( i = 1 : \frac{n-1}{2} \)

\[ q = \frac{n-1+i}{2} \cdot q \]

\[ aux = 1 \]

for \( k = 1 : i \)

\[ aux = (t_{i+1} - t_k) \cdot aux \]

end

\[ p_{i+1,i+1} = q \cdot (1 - t_{i+1})^{\frac{n+1}{2}} \cdot aux \]

end

\[ aux = 1 \]

for \( k = 1 : \frac{n+1}{2} \)

\[ aux = (1 - t_k) \cdot aux \]

end

\[ q = \frac{q}{aux} \]

\[ aux = 1 \]

for \( k = 1 : \frac{n+1}{2} \)

\[ aux = (t_{n+3} - t_k) \cdot aux \]

end

\[ p_{n+3,n+3} = q \cdot (1 - t_{n+3})^{n-\frac{n+1}{2}} \]

for \( i = \frac{n+3}{2} : n \)

\[ q = \frac{n-i+1}{n+1+n-i+1} \cdot \frac{1}{1-t_i} \cdot q \]

\[ aux = 1 \]

for \( k = 1 : i \)
\[ aux = (t_{i+1} - t_k) \cdot aux \]
\[ end \]

\[ p_{i+1,i+1} = q \cdot (1 - t_{i+1})^{n-i} \cdot aux \]
\[ end \]

Looking at this algorithm is enough to conclude that:

- The computational complexity of the computation of \( m_{ij}, \tilde{m}_{ij} \) and \( p_{ii} \), i.e. of the computation of \( BD(A) \) is \( O(n^2) \).
- The algorithm has high relative accuracy because it only involves arithmetic operations that avoid inaccurate cancellation.
- The algorithm does not construct the SB–Vandermonde matrix, it only works with the nodes \( \{t_i\}_{1 \leq i \leq n+1} \).

As for the even case, the properties of the algorithm are exactly the same.

5 Accurate computations with SB–Vandermonde matrices

In this section algorithms for solving linear systems and for eigenvalue computation are presented for the case of a totally positive SB–Vandermonde matrix \( A \). The algorithms are both accurate and efficient and are based on the algorithm presented in Section 4 for computing \( BD(A) \).

Let us observe here that, of course, one could try to solve these problems by using standard algorithms. However the solution provided by them will generally be less accurate since SB–Vandermonde matrices are ill conditioned (see the numerical experiments in Section 6) and these algorithms can suffer from inaccurate cancellation, since they do not take into account the structure of the matrix, which is crucial in our approach.

5.1 Linear system solving

Let \( Ax = b \) be a linear system whose coefficient matrix \( A \) is a SB–Vandermonde matrix of order \( n + 1 \) generated by the nodes \( \{t_i\}_{1 \leq i \leq n+1} \), where \( 0 < t_1 < \ldots < t_{n+1} < 1 \).

The following algorithm solves \( Ax = b \) in a fast way.

INPUT: The nodes \( \{t_i\}_{1 \leq i \leq n+1} \) and the data vector \( b \in \mathbb{R}^{n+1} \).
OUTPUT: The solution vector $x \in \mathbb{R}^{n+1}$.

- **Step 1**: Computation of $BD(A)$ by using the algorithm introduced in Section 4.
- **Step 2**: Computation of

$$x = A^{-1}b = G_1G_2 \cdots G_nD^{-1}F_nF_{n-1} \cdots F_1b.$$ 

Step 2 can be carried out by using the algorithm `TNSolve` of P. Koev [20]. Given the bidiagonal factorization of the matrix $A$, `TNSolve` solves $Ax = b$ by computing the above matrix product.

Although $BD(A)$ is computed with high relative accuracy, the accuracy of the solution vector will generally depend on the data vector $b$ [23].

Taking into account that, as we have shown in Section 4, the computational cost of Step 1 is of $O(n^2)$ arithmetic operations, and the cost of computing the whole product in Step 2 (from right to left) is also of $O(n^2)$ arithmetic operations, the computational complexity of the algorithm for solving $Ax = b$ is $O(n^3)$.

### 5.2 Eigenvalue computation

Let $A$ be a SB–Vandermonde matrix of order $n + 1$ generated by the nodes $\{t_i\}_{1 \leq i \leq n+1}$, where $0 < t_1 < \ldots < t_{n+1} < 1$. The following algorithm computes accurately the eigenvalues of $A$.

**INPUT**: The nodes $\{t_i\}_{1 \leq i \leq n+1}$.

**OUTPUT**: A vector $x \in \mathbb{R}^{n+1}$ containing the eigenvalues of $A$.

- **Step 1**: Computation of $BD(A)$ by using the algorithm introduced in Section 4.
- **Step 2**: Given the result of Step 1, computation of the eigenvalues of $A$ by using the algorithm `TNEigenvalues`.

`TNEigenvalues` is an algorithm of P. Koev [21] which computes accurate eigenvalues of a totally positive matrix starting from its bidiagonal factorization. The computational cost of `TNEigenvalues` is of $O(n^3)$ arithmetic operations (see [21]) and its implementation in MATLAB can be taken from [20]. In this way, as the computational cost of Step 1 is of $O(n^2)$ arithmetic operations, the cost of the whole algorithm is of $O(n^3)$ arithmetic operations.
6 Numerical experiments

In this section we present two numerical experiments illustrating the accuracy of the two algorithms we have introduced in the previous section.

Example 6.1. Let $S_{15}$ be the Said–Ball basis of the space of polynomials with degree less than or equal to 15 in $[0, 1]$, and let $A$ be the SB–Vandermonde matrix of order 16 generated by the following nodes:

\[
1 < \frac{1}{16} < \frac{1}{13} < \frac{2}{11} < \frac{3}{10} < \frac{4}{9} < \frac{7}{10} < \frac{15}{13} < \frac{17}{15} < \frac{26}{20} < \frac{9}{6} < \frac{5}{3} < \frac{21}{15}.
\]

The condition number of $A$ is: $\kappa_2(A) = 3.2e + 08$. Let us consider the data vector

\[
b = (12, -3, 0, 1, 5, -7, 0, 2, 21, -4, 0, 9, -11, 6, -8, 0)^T.
\]

We compute the exact solution $x_e$ of the linear system $Ax = b$ by using the command `linsolve` of Maple 10 and we use it for comparing the accuracy of the results obtained in MATLAB by means of:

1. The algorithm presented in Section 5.1. We will call it MM.
2. The algorithm TNBD of Plamen Koev [20] that computes $\mathcal{BD}(A)$ without taking into account the structure of $A$.
3. The command $A\backslash b$ of MATLAB.

In (2), the second stage in the solution of the linear system is the computation of the fast product (from right to left) of the bidiagonal matrices and the vector $b$. It is done in MATLAB by using the same command as in (1): `TNSolve` of Koev [20].

We compute the relative error of a solution $x$ of the linear system $Ax = b$ by means of the formula:

\[
err = \frac{\|x - x_e\|_2}{\|x_e\|_2}.
\]

The relative errors of the solutions of $Ax = b$ computed by means of the approaches (1), (2) and (3) are reported in Table 1.

|        | MM      | TNBD    | $A\backslash b$ |
|--------|---------|---------|-----------------|
|        | 5.1e-16 | 2.2e-09 | 3.9e-10         |

Table 1
Relative errors in Example 6.1

Example 6.2. Let $A$ be the SB–Vandermonde matrix of order 16 considered in Example 6.1. In Table 2 we present the eigenvalues $\lambda_i$ of $A$ and the relative errors obtained when computing them by means of:

1. The algorithm presented in Section 5.2. We will call it MM.
(2) The algorithm \texttt{TNBD} \cite{20} that computes $BD(A)$ without taking into account the structure of $A$.

(3) The command \texttt{eig} from \textsc{Matlab}.

In (2), the second stage in the computation of the eigenvalues is done in \textsc{Matlab} by using the same command as in (1): \texttt{TNEigenvvalues} of P. Koev \cite{20}.

The relative error of each computed eigenvalue is obtained by using the eigenvalues computed in \textsc{Maple 10} with 50-digit arithmetic.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\lambda_i$ & MM & TNDB & \texttt{eig} \\
\hline
1.0e + 00 & 4.4e − 16 & 1.0e − 12 & 1.8e − 15 \\
9.4e − 01 & 1.3e − 15 & 2.1e − 11 & 1.2e − 15 \\
7.0e − 01 & 9.6e − 16 & 2.5e − 11 & 6.4e − 16 \\
5.2e − 01 & 6.3e − 16 & 1.3e − 11 & 2.1e − 16 \\
3.1e − 01 & 5.4e − 16 & 7.9e − 12 & 2.7e − 15 \\
1.4e − 01 & 1.3e − 15 & 1.5e − 11 & 1.3e − 15 \\
6.0e − 02 & 5.7e − 16 & 1.1e − 11 & 1.1e − 15 \\
3.0e − 02 & 4.6e − 16 & 6.2e − 12 & 4.6e − 16 \\
8.6e − 03 & 4.1e − 16 & 4.6e − 12 & 1.3e − 14 \\
2.6e − 03 & 9.9e − 16 & 1.0e − 11 & 3.7e − 14 \\
6.1e − 04 & 5.4e − 16 & 2.3e − 11 & 7.2e − 14 \\
6.2e − 05 & 0 & 1.0e − 11 & 3.1e − 13 \\
8.3e − 06 & 4.1e − 16 & 1.8e − 11 & 6.4e − 13 \\
9.1e − 07 & 1.2e − 16 & 4.6e − 11 & 3.2e − 12 \\
5.5e − 08 & 2.0e − 15 & 1.2e − 10 & 3.1e − 10 \\
5.0e − 09 & 3.0e − 15 & 2.3e − 09 & 2.0e − 09 \\
\hline
\end{tabular}
\caption{Table 2}
\end{table}

Relative errors in Example 6.2

The results appearing in Table 1 and Table 2 illustrate the good behaviour of our approach. In particular, the very different results obtained for the approaches (1) and (2) show the importance of computing $BD(A)$ with high relative accuracy, since in both approaches the second stage is exactly the same.
For this specific matrix $A$ the relative error obtained when computing the matrix $BD(A)$ by using the algorithm we have presented in Section 4 is $2.8e-15$, while the relative error obtained when computing it by means of the command TMBD is $6.8e-10$. These relative errors have been computed for each solution $B$ by using

$$err = \frac{\|B - B_e\|_2}{\|B_e\|_2},$$

where $B_e$ is the exact $BD(A)$ computed in Maple 10.

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