Off-equatorial stable circular orbits for spinning particles

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Abstract

In this article, we investigate the motion of a spinning particle at a constant inclination, different from the equatorial plane, around a Kerr black hole. We mainly explore the possibilities of stable circular orbits for different spin supplementary conditions. The Mathisson-Papapetrau’s equations are extensively applied and solved within the framework of linear spin approximation. We explicitly show that for a given spin vector of the form $S = (0, S^r, S^\theta, 0)$, there exists an unique circular orbit at $\left(r_c, \theta_c\right)$ defined by the simultaneous minima of energy, angular momentum and Carter constant. This corresponds to the Innermost Stable Circular Orbit (ISCO) which is located on a non-equatorial plane. We further establish that the location $\left(r_c, \theta_c\right)$ of the ISCO for a given spinning particle depends on the radial component of the spin vector ($S^r$) as well as the angular momentum of the black hole ($J$). The implications of using different spin supplementary conditions are investigated.

1 Introduction

In this era of gravitational wave astronomy, modeling the relativistic two body problems are of utmost interests. Due to the non-linearity of the system, the relativistic two body problem can not be solved exactly within the framework of Einstein’s equations and one has to rely on approximations or numerical solutions. Two of well known approximations are : Post Newtonian or PN approximation [1] and Effective one body or EOB approach [2,3]. Even if these methods are good enough to explain the system in the linear region where the relativistic effects are small, in the non-linear domain these approximations breakdown. This is when the numerical tools become inevitable. In recent years, Numerical relativity has become an essential part of gravitational wave astronomy [4,5]. With the advance of computational techniques, the numerical methods are improving in a rapid rate and new physics emerges every moment beyond analytic understandings [6]. Besides these successes, Numerical relativity has its fair share of limitations. In particular, Numerical relativity is not vary efficient when the mass ratios become extremely large or small in a binary system [7]. In these scenarios, the approximate techniques such as the effective one body formalism are useful.

With these motivations, we study the motion of spinning objects within the pole-dipole approximation in curved spacetime. In astrophysical situations, this corresponds to the orbiting of a compact object,
representing a black hole or neutron star, of mass $M_1$ ($\sim$ few solar mass) around a massive black hole of mass $M_2$ ($\sim 10^4 M_\odot$ to $10^6 M_\odot$) such that $M_2 >> M_1$. This is usually referred as extreme mass ratio inspiral (EMRI) which is promising source of gravitational waves for proposed space based detectors such as LISA [8]. In these scenarios, the internal structures of the orbiting body is approximated to dipole and other higher moments are ignored. Even in this lowest order approximation, pole-dipole particles can have striking deviations from a geodesic trajectory. In curved spacetime, the orbits of these particles are described by the Mathission-Papapetrau equations [9, 10]. These equations has a long and substantial history spanning over few decades, an extensive literature survey can be found in [11]. We also refer our readers [12–16] for further insights. Though the exact solutions of these equations are extremely complicated, there are several approach for solving them with suitable approximations. In the present article, we have used the linear spin approximations in which we write the Mathission-Papapetrau equations upto the linear order in spin and neglect higher order terms. With this approximation, one can solve the orbit equations for an arbitrary inclination angle ($\theta$) and investigate the possible existence of circular orbits on $\theta = \text{constant}$ planes. In passing, we note that for a single monopole particle, there is no circular orbit either stable or unstable can exist in the Kerr spacetime at constant inclination angle except for $\theta = \pi/2$ [17].

In general, for an unequal mass binaries, especially with extreme mass ratios, the system undergoes a precession and orbits start to wobble along the off-equatorial planes whenever the angular momentums are not aligned along the orbital axis of the black hole. Here, we have shown that under specific conditions it is possible to have orbits without any wobbling. Apart from the wobbling in the off-equatorial directions, relativistic orbits can precess while confined to a particular orbital plane. This is usually known as the Periastron precession and in our solar system, it is called Perihelion precession [18, 19]. As in the present context we are only concentrating on the circular orbits, Periastron precession would identically vanish. What we consider, are families of stable circular orbits with an ISCO. We investigate the properties associated with these orbits for different spin supplementary conditions. The occurrence of such conditions are natural as the motion of a spinning object depends on the choice of a reference point and each choice would lead to a distinct spin supplementary condition. In this article, we mainly concentrate on Mathisson-Pirani [20] or Tulczyjew-Dixon spin supplementary condition [21] and Newton-Wigner spin supplementary condition [22]. Even if, Tulczyjew-Dixon and Mathisson-Pirani conditions are distinct from each other for the exact Mathisson-Papapetrau equations, they both merge in the limit of linear spin approximation.

The rest of the manuscript is organized as follows. In Section 2, we elaborately describe the motion of a spinning particle for different spin supplementary conditions while exclusively using the linear spin approximation. We then introduce the conserved quantities for a spinning particle such as, energy, angular momentum and Carter constant. Section 3 is devoted to study the motion of spinning particles numerically and discuss the existence of circular orbits at constant altitude. In Section 4, we discuss the stability of these circular orbits present in the non-equatorial plane. Finally, we close the article with a brief remark in Section 5.

2 Basic equations for a Spinning Particle

The trajectory of a single pole test particle in a gravitational field is given by the geodesic equation which is obtained by setting the acceleration to zero. Unlike Newtonian gravity, general relativity does not treat gravity as a force, instead, depicts it as an inbuilt manifestation of the spacetime itself. This is, in fact, one of the very basic postulate of Einstein’s gravity [23]. Motion of a particle can deviate from geodesic trajectories in presence of a force. This force can be external or internal, if the particle has higher order
mass multipoles. In a realistic situation, the astrophysical objects are expected to have complex internal structure. The first order correction to the single pole test particle would be to consider a dipolar mass moment along with the monopole moment to incorporate the internal angular momentum. By dipolar mass moments, we mean the center of mass of the spinning body in its rest frame does not coincide with the observed center of mass in the observer’s frame. This is because, for a spinning particle in a curved spacetime, in general, the center of mass is observer dependent [24]. The motion of these particles are described by the Mathission-Papapetrau equations and for a four momentum, $P^a$ and spin tensor, $S^{ab}$, these can be written as:

$$\frac{DP^a}{d\tau} = -\frac{1}{2} R^a_{\ bcd} U^b S^{cd}, \quad \frac{DS^{ab}}{d\tau} = P^a U^b - P^b U^a. \quad (1)$$

Here, $U^a$ is the four velocity of the particle and $R^a_{\ bcd}$ is the Riemann curvature tensor. For a limiting case of $S^{ab} \rightarrow 0$, one get back the geodesic equations, i.e., acceleration, $a^i = U^b \nabla_b U^i = 0$. The coupling of the spin tensor with the background geometry contribute to an acceleration and hence, the particle deviates from the usual geodesic trajectory.

In the case of spinning particles, the four momentum and four velocity are not proportional to each other. This will lead to total 14 unknown variables (four for each velocity and momentum and six for antisymmetric spin tensor), while we have only 10 equations in hand. In order to solve these set of equations consistently, we require additional four constraints. These are called spin supplementary condition and they are widely studied in the literature. Here we briefly introduce some of these conditions and describe their important features:

- **The Papapetrou and Corinaldesi condition**, $S^{0i} = 0$ [25]. This would simply imply there is no dipolar mass moment, i.e. the center of mass in the particle’s frame coincide with the observed center of mass in the chosen frame.

- **The Mathisson-Pirani supplementary condition**, $S^{ab} U_b = 0$ [20]. This condition is well studied up to some extend, and the predicted orbits are with helical structure. Initially it was believed to be unphysical, while recently it is has been shown that they have a physical interpretation [26, 27]. The rest mass with respect to $U^a$ is given as, $m = -P^a U_a = \text{constant}$.

- **The Tulczyjew-Dixon supplementary condition**, $S^{ab} P_b = 0$ [21], is extensively studied in several works [28–31]. This gives an exact physical solution of Mathission-Papapetrau equations. In this case the dynamical mass, $\mu = \sqrt{-P^a P_a}$ is conserved.

- **The Newton-Wigner condition**, $S^{ab} \omega_b = 0$ [22], gives a Hamiltonian approach to the motion of spinning particles [32]. The $\omega^a$ is given as, $\omega^a = P^a/\mu + \phi^b$, where $\phi^b$ is a timelike vector. This would help to improve the phenomenological approach to understand gravitational waves dynamics. At the same time, it neither conserves total spin nor mass of the test particle.

A detailed discussion on various spin supplementary condition and their connections to internal properties of the spinning particles can be found in [33]. However in the present context, we start with the Tulczyjew-Dixon or Mathisson-Pirani constraint and investigate various possibilities of circular orbits at constant altitudes. Following this, we shall study the similar situations with Newton-Wigner spin supplementary condition and compare the respective results.
2.1 Tulczyjew-Dixon or Mathisson-Pirani spin supplementary condition

In this section, we shall briefly discuss the evolution equations for a spinning particle within the framework of linear spin approximation. To do so, we have explicitly used the Tulczyjew-Dixon or Mathisson-Pirani condition which are similar in this particular limit. The difference of the four momentum and velocity can only be observed at $O(S^2)$. In addition, this assumption would also simplify the Mathission-Papapetrau equations significantly and can be written as [34]:

$$\frac{DU^k}{dt} = -\frac{1}{2m} R^k_{\;bcd} U^b S^{cd} + O(S^2), \quad \frac{DS^{ab}}{dt} = 0 + O(S^2), \quad \text{and} \quad m = \mu + O(S^2). \quad (2)$$

Now for computational convenience we use the standard spin four vector $S^a$ instead of the spin tensor, $S_{ab}$.

$$S^a = \frac{\epsilon^{abcd}}{2\sqrt{-g}} U_b S_{cd}, \quad S_{ab} = \frac{1}{\sqrt{-g}} \epsilon^{abcd} U_c S_d. \quad (3)$$

Where ‘$g$’ is the determinant of the metric and is always negative. Considering the circular orbits ($\dot{r} = \ddot{r} = 0$) at constant altitude ($\dot{\theta} = \ddot{\theta} = 0$), only non-vanishing components of the spin vector are $S^r$ and $S^\theta$, i.e., $S \equiv (0, S^r, S^\theta, 0)$.

We use the tetrad formalism with the components of the tetrads are given by [35]:

$$e^{(0)}_{\mu} = \left( \sqrt{\frac{\Delta}{\Sigma}}, 0, 0, -a \sin^2 \theta \sqrt{\frac{\Delta}{\Sigma}} \right),$$

$$e^{(1)}_{\mu} = \left( 0, \sqrt{\frac{\Sigma}{\Delta}}, 0, 0 \right),$$

$$e^{(2)}_{\mu} = \left( 0, 0, \sqrt{\Sigma}, 0, 0 \right),$$

$$e^{(3)}_{\mu} = \left( -a \sin \theta \sqrt{\frac{\Sigma}{\Delta}}, 0, 0, \frac{r^2 + a^2}{\sqrt{\Sigma}} \sin \theta \right). \quad (4)$$

Here, ‘$\Delta$’ and ‘$\Sigma$’ has usual meanings, $\Delta = r^2 - 2Mr + a^2$ and $\Sigma = r^2 + a^2 \cos^2 \theta$. The inverse of the tetrad given in Eq. (4) can be easily computed with the relation:

$$e^a_{(\mu)} = \eta_{(\mu)(\nu)} g^{ab} e^b_{(\nu)}.$$

Where, $\eta_{(\mu)(\nu)}$ is given as,

$$\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}$$

From Eq. (2), for circular orbits at constant inclination angle, both $\frac{DU^0}{dt}$ and $\frac{DU^3}{dt}$ will identically vanish. The nonzero components correspond to the radial and angular equations can be reduced to:

$$\Lambda_1 + \Lambda_2 \Omega^2 + 2\Lambda_3 \Omega = -e^{(1)}_{(1)} \left[ 3R_{(1)(3)(1)(3)} \Omega S^{(2)} + R_{(1)(3)(0)(2)} S^{(1)} \left( 1 + \Omega^2 \right) \right], \quad (6)$$

$$\tilde{\Lambda}_1 + \tilde{\Lambda}_2 \Omega^2 + 2\tilde{\Lambda}_3 \Omega = -e^{(2)}_{(2)} \left[ 3R_{(1)(3)(1)(3)} \Omega S^{(1)} - R_{(1)(3)(0)(2)} S^{(2)} \left( 1 + \Omega^2 \right) \right], \quad (7)$$
with,

\[ \Lambda_1 = \Gamma_{33}^1 (e_3(0))^2 + \Gamma_{00}^1 (e_0(0))^2 + 2\Gamma_{03}^1 (e_0(0)e_3(0)), \]
\[ \Lambda_2 = \Gamma_{33}^1 (e_3(3))^2 + \Gamma_{00}^1 (e_0(3))^2 + 2\Gamma_{03}^1 (e_0(3)e_3(3)), \]
\[ \Lambda_3 = \Gamma_{33}^1 (e_0(0)e_3(3)) + \Gamma_{00}^1 (e_0(3)e_0(0)) + \Gamma_{03}^1 (e_0(3) + e_3(0)), \]
\[ \tilde{\Lambda}_1 = \Gamma_{33}^2 (e_3(0))^2 + \Gamma_{00}^2 (e_0(0))^2 + 2\Gamma_{03}^2 (e_0(0)e_3(0)), \]
\[ \tilde{\Lambda}_2 = \Gamma_{33}^2 (e_3(3))^2 + \Gamma_{00}^2 (e_0(3))^2 + 2\Gamma_{03}^2 (e_0(3)e_3(3)), \]
\[ \tilde{\Lambda}_3 = \Gamma_{33}^2 (e_0(0)e_3(3)) + \Gamma_{00}^2 (e_0(3)e_0(0)) + \Gamma_{03}^2 (e_0(3) + e_3(0)) \] \quad (8)

Where, \( \Gamma \)'s are the Christoffel symbols and we define \( \Omega = \frac{U(3)}{U(0)} \) to be the angular velocity of the particle in the tetrad frame. These equations describe the circular motion in only one parameter, \( \Omega \). The extremal values of \( \Omega \) are bounded by the angular velocity of photons, \( \Omega_{ph} = \pm 1 \). Now in principle one can solve Eq. (6) for \( \Omega \) and substitute in Eq. (7), and get relation between \( r \) and \( \theta \) for different spin values,

\[ \Omega = \Omega(r, \theta, S^{(1)}, S^{(2)}, a), \]
\[ \theta = \theta(r, S^{(1)}, S^{(2)}, a). \] \quad (9)

We use the numerical approach to solve these equations in the next section.

### 2.2 Newton-Wigner spin supplementary condition

In this section, we discuss the Newton-Wigner formalism in detail and explicitly use it in the framework of linear spin approximation. As already described earlier, we define a timelike vector \( \omega^a \) such that,

\[ \omega^a = P^a / \mu + \phi^a. \] \quad (10)

where, \( \phi^a \) is an unit timelike vector. In this case, the Newton-Wigner constraint can be written in terms of \( \omega^a \) as:

\[ S^{ab} \omega_b = S^{ab}(\nu_b + \phi_b) = 0. \] \quad (11)

With \( \nu^b \) defines as the normalized momenta, \( P^b = \mu \nu^b \). It should noted that neither mass (\( \mu \) or \( m \)) nor the total spin is conserved in this formalism, while their difference appear only in the \( \mathcal{O}(S^2) \). In the present context, neglecting terms containing \( \mathcal{O}(S^2) \), we may consider them as conserved quantities. In addition, we constrain \( \phi^a \) to satisfy,

\[ \nu^a \nu_a = -1, \quad \phi^a \phi_a = -1, \quad \text{and} \quad \nu^a \phi_a = -k. \] \quad (12)

Where, \( k \) is a constant. We may now express the vectors in the chosen tetrad given in Eq. (4),

\[ \nu^a = a_0 e^a_{(0)} + a_3 e^a_{(3)}, \]
\[ \phi^a = c_0 e^a_{(0)} + c_3 e^a_{(3)}. \] \quad (13)
Using Eq. (12) along with Eq. (13), we obtain,
\[ c_0 = a_0 k + a_3 \sqrt{k^2 - 1}, \]
\[ c_3 = a_3 k + a_0 \sqrt{k^2 - 1}. \] (14)
As in the previous case, here also we introduce a spin four vector to simplify calculations. However, in this case, the vector need to be defined with respect to \( \omega^a \) instead of four momentum \((P^a)\) or velocity \((U^a)\):
\[ S^{ab} = \epsilon^{abcd} \omega_c \sqrt{-g}(-\omega^m \omega_m) = \epsilon^{abcd} \omega_c S_d \sqrt{-g(1+k)}. \] (15)
The denominator of the above equation would contribute a factor of 2 which we have incorporated in the spin vector. Note, with \( k = 1 \) would give the trivial results with \( c_0 = a_0 \) and \( c_3 = a_3 \). From Eq. (14) and Eq. (15), we can write the orbit equations as:
\[ \Lambda_1 + \Lambda_2 \Lambda_3 = -\epsilon^{(1)} R_{(1)(3)(1)(3)} (3\Lambda_1 + \alpha_1) S^{(2)} + R_{(1)(3)(0)(2)} S^{(1)} \left( 1 + \Lambda_3^2 + \beta \right), \]
\[ \Lambda_1 + \Lambda_2 \Lambda_3 = -\epsilon^{(2)} R_{(1)(3)(1)(3)} (3\Lambda_1 + \alpha_2) S^{(1)} - R_{(1)(3)(0)(2)} S^{(2)} \left( 1 + \Lambda_3^2 + \beta \right). \] (16)
The \( \alpha \)'s and \( \beta \) are given as:
\[ \alpha_1 = \sqrt{k - 1 \over k + 1} \left( 2 + \Lambda_3^2 \right), \quad \alpha_2 = \sqrt{k - 1 \over k + 1} \left( 1 + 2\Lambda_3^2 \right), \quad \text{and} \quad \beta = 2\Lambda_3 \sqrt{k - 1 \over k + 1}. \] (17)
As one can see, the general dependence of these equations on \( k \) is weak as the prefactor goes as, \( \sqrt{k - 1 \over k + 1} \) and for a large value of \( k \) it is close to unity. Hence, the orbit equations in both these formalism would differ by a small amount. The equations correspond to Tulczyjew-Dixon or Mathisson-Pirani spin supplementary condition can be easily obtained by setting \( \alpha_1, \alpha_2 \) and \( \beta \) to zero. Now we may rewrite the Eq. (16) similar to the previous case as in Eq. (9) and numerically solve them to compute the non-equatorial circular orbits:
\[ \bar{\Omega}_{NW} = \bar{\Omega}_{NW}(r, \theta, S^{(1)}, S^{(2)}, a), \]
\[ \theta = \theta (r, S^{(1)}, S^{(2)}, a). \] (18)
Similar to the previous case, we shall solve these equations in the next section.

2.3 Conserved quantities: energy, momentum and Carter constant

In the case of spinning particles, the conserved quantities get modified depending on the spin of the particle. For a killing vector field \( K^a \), the corresponding conserved quantity is written as [36,37]:
\[ C = K^a P_a - {1 \over 2} S^{ab} K_{ab}. \] (19)
Where the semicolon (\( ; \)) is defined as the covariant derivative. As the Kerr spacetime has two Killing vectors, a timelike (\( \xi^a \)) and a spacelike (\( \eta^a \)), the corresponding conserved quantities are given by:
\[ E = -C_t = -\xi^a P_a + {1 \over 2} S^{ab} \xi_{a;b}, \quad \text{and} \quad J_z = C_\phi = \eta^a P_a - {1 \over 2} S^{ab} \eta_{a;b}. \] (20)
Unlike the geodesics, neither energy \((-\xi^a P_a)\) nor the angular momentum \((\eta^a P_a)\) is conserved in case of a spinning particle. Instead, we have the conserved quantities are, \(-C_t\) and \(C_\phi\) and they become energy and angular momentum only for specific case \(S = 0\).

In addition to the above conserved quantities, there is another constant of motion related to total angular momentum of a particle. This is called Carter constant [38-40]. A general prescription to define total angular momentum is more involved in general relativity. In fact, it did not receive much attention until Carter came up with this non-trivial constant to describe the geodesic motion in a Kerr black hole. It turns out that this constant is closely related to the total angular momentum of a particle and for a static spacetime it is exactly same as the square of total angular momentum [41-43]. Presence of this constant make the trajectories completely integrable in the Kerr spacetime. Unfortunately, so far there is no general notion of Carter like constant in case of spinning particles, but a super-symmetrical approach is available in the literature [44]. This is a continuation of pseudo-classical spinning particles in general relativity, which is a limiting case of the Dirac particles in Quantum Mechanics. It is explicitly used by Tanaka et al. to demonstrate that the adiabatic approximation can be applicable in case of spinning particle upto linear order on the equatorial plane [45]. This is given by:

\[
\frac{Q}{m^2} = \left\{ \left( \Sigma(U^{(0)})^2 - r^2 \right) \right\} - \frac{2a \sin \theta}{\sqrt{\Sigma}} \left\{ r \left( U^{(0)} S^{(1)(3)} - 2U^{(3)} S^{(1)(0)} \right) + a \cos \theta U^{(3)} S^{(2)(3)} \right\} - \frac{2\sqrt{\Sigma}}{\Sigma} \left\{ a \cos \theta \left( 2U^{(0)} S^{(2)(3)} - U^{(3)} S^{(2)(0)} \right) - rU^{(0)} S^{(1)(0)} \right\}. \tag{21}
\]

Before dealing with the more general case of a rotating black hole, we first investigate the properties of the above constant in a Schwarzschild black hole. By setting \(a = 0\), Eq. (21) become,

\[
\frac{Q}{m^2} = r^2 \left\{ \left( U^{(2)} \right)^2 + \left( U^{(3)} \right)^2 \right\} + 2\sqrt{\Sigma} U^{(0)} S^{(1)(0)}. \tag{22}
\]

The first term in the above expression can be easily recognized as the square of total angular momentum \((L)\), while the second term can be written in terms of the spin vector.

\[
\frac{L^2}{m^2} = (U_\theta)^2 + \frac{(U_\phi)^2}{\sin^2 \theta}. \tag{23}
\]

So we may conclude,

\[
Q = L^2 + 2m^2 \Delta r \sin \theta U^\phi U^\phi S^\theta. \tag{24}
\]

It is interesting to see that the extra term is proportional to the spin vector \((S^\theta)\) and for a limit \(S \to 0\), \(Q \to L^2\). Now we compute the total angular momentum (orbital+spin) of a spinning test particle and explicitly show this matches with Eq. (24).

We have already discussed how a killing vector is useful to exploit various symmetries in a geometry. Unlike a Kerr black hole, Schwarzschild spacetime is endowed with spherical symmetry and contain three spacelike killing vectors:

\[
\eta_1^a = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}, \quad \eta_2^a = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}, \quad \text{and} \quad \eta_3^a = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}. \tag{25}
\]
Now from Eq. (19), we write down the each conserved quantities explicitly:

\[
\frac{J_x}{m} = \{ - \sin \phi U_\phi - \cot \theta \cos \phi U_\phi \} + (S^r U^t - S^t U^r) \left( \cos \phi \cos \frac{2\theta}{2\sin \theta} - \frac{\cos \phi}{2\sin \theta} \right) - r \sin \theta \sin \phi (1 - 2M/r) (S^t U^\phi - S^\phi U^t) - r \cos \theta \cos \phi (1 - 2M/r) (S^\phi U^t - S^t U^\phi),
\]

\[
\frac{J_y}{m} = \{ \cos \phi U_\phi - \cot \theta \sin \phi U_\phi \} + (S^r U^t - S^t U^r) \left( \sin \phi \cos \frac{2\theta}{2\sin \theta} - \frac{\sin \phi}{2\sin \theta} \right) + r \sin \theta \cos \phi (1 - 2M/r) (S^t U^\phi - S^\phi U^t) - r \cos \theta \sin \phi (1 - 2M/r) (S^\phi U^t - S^t U^\phi),
\]

\[
\frac{J_z}{m} = U_\phi + (r - 2M) \sin \theta \left( U^t S^\theta - U^\theta S^t \right) + \cos \theta \left( S^t U^r - S^r U^t \right).
\]

With the above equations, it is easy to show that

\[
J^2 = J_x^2 + J_y^2 + J_z^2 = L^2 + 2m^2 \Delta r \sin \theta U^t U^\phi S^\theta + O(S^2) \approx Q.
\]  

So as we claimed earlier, the Carter constant for spinning particle with the linear spin approximation is similar to the total angular momentum of the particle.

In case of a rotating geometry, we shall describe a new quantity as effective Carter constant, \( K_s = Q - (J_z - aE)^2 \). Note, this quantity would vanish in case of a geodesic trajectory on the equatorial plane. But in the present context we consider orbits lie close to the equatorial plane, \( \theta = \pi/2 + \eta \),

\[
K_s \approx 2aS^z - \frac{2S^{(1)}}{r^2 \sqrt{\Delta}} \left\{ E J_z r^3 - 2aM (J_z - aE)^2 \right\} \eta + O(\eta^2).
\]

The first term is a coupling between the spin component of the particle with the angular momentum of the black hole, while the second term is related to the square of momenta. It is interesting to note that the second term is proportional to both \( S^r \) and \( \eta \).

3 Non equatorial orbits: constraining r and theta (\( \theta \))

Before delving into the spinning particle, we first investigate the possibilities of circular geodesics on the non-equatorial plane of a Kerr black hole. We start with the effective potential in radial \( V_r \) and angular \( V_\phi \) direction [46, 47]:

\[
V(\theta) = Q - L_z^2 \cot^2 \theta + a^2 (E^2 - m^2) \cos^2 \theta,
\]

\[
V(r) = Er^4 - (L_z^2 - a^2 E^2) r^2 + 2(L_z - aE)^2 r - (m^2 r^2 + Q) \Delta.
\]

The necessary and sufficient condition for a circular orbit is given by, \( V(r) = 0 \) and \( \frac{dV(r)}{dr} = 0 \). In addition to this, a circular orbit at constant altitude also has to satisfy \( V(\theta) = 0 \) and \( \frac{dV(\theta)}{d\theta} = 0 \),

\[
\frac{dV(\theta)}{d\theta} = 2 \cos \theta \left\{ L_z^2 \csc^2 \theta - a^2 (E^2 - m^2) \sin \theta \right\} = 0.
\]

This immediately suggests, either \( \theta = \pi/2 \) or \( L_z^2 = a^2 (E^2 - m^2) \sin^4 \theta \). In the first case with, \( \theta = \pi/2 \), it is easy to show that \( V(\theta) \) vanishes only when \( Q = 0 \). Now one can employ the radial potential \( V(r) \)
to show that \( Q = 0 \) indeed describes a circular orbit on the equatorial plane. On the other hand, for \( L_z^2 = a^2(E^2 - m^2)\sin^2\theta \), bound circular orbits are unlikely to appear as they consist with \( E^2 - m^2 < 0 \) and this is inconsistent with \( L_z^2 > 0 \). Hence, one can conclude that circular orbits only exist on the equatorial plane of a Kerr black hole.

The situation is quite different in the case of a spinning particle and non equatorial circular orbits can be obtained from Mathission-Papapetrau equations. Next we shall discuss these orbits for both the spin supplementary conditions.

### 3.1 Tulczyjew-Dixon or Mathisson-Pirani spin supplementary condition:

Here we numerically solve Eq. (9) and obtain \((r, \theta)\) for a given value of the spin parameters. For each polar angle \( \theta \), there is only one possible radial coordinate \( r \) that satisfies the equation of motion. The plot for \( \theta \) as a function of \( r \) is shown in Fig. (1) for a given spin vector, \( S^{(i)} = (0, -0.01, -0.01, 0) \) and angular momentum \( a = \{0, 0.5M, M\} \) for both co-rotating and counter-rotating orbits. These orbits behave as a small perturbation from the geodesic trajectories as they appear very close to the equatorial plane. It is shown that the deviation from the equatorial plane not only depends on the sign of the spin vectors, but also on their direction of rotations. The counter rotating orbits cease to exist beyond \( r = 4M \) for a maximally Kerr black hole \((a = M)\), while co-rotating orbits continue appear even close to the horizon as shown Fig. (1a).

(a) The co-rotating circular orbits are shown in the non-equatorial planes for different angular momentum of the black hole. For a large angular momentum of the black hole, the orbits are dragged close to the horizon. The spin components are fixed at \( S^{(i)} = (0, -0.01, -0.01, 0) \) while extrema of the angular momentum ‘\( a \)’ is shown in the inset.

(b) The counter-rotating circular orbits are shown for \( S^{(i)} = (0, -0.01, -0.01, 0) \). They move away from the horizon as one increase the value of the black hole’s angular momentum. As the spinning particle move close to the horizon, it gets more deviated from the equatorial plane. The extrema of the black hole momenta is shown in the inset.

Figure 1: Non-equatorial circular orbits are shown for spinning particles.

Close to the equatorial plane, one can write the slope as:

\[
\frac{d\theta}{dr} = - \frac{S^{(1)} \Omega}{r + 2r(r - 1)\Omega + (r + r^3)\Omega^2 + 6S^{(2)}(1 + \Omega^2)}.
\] (31)
For a co-rotating orbit with $S^{(1)} < 0$, the orbits getting close to the $\theta = 0$ axis while the behavior is completely opposite for a counter-rotating orbit. In addition, the polar angle depends only on the spin component $S^{(1)}$ and independent of $S^{(2)}$.

### 3.2 Newton-Wigner spin supplementary condition

In the Newton-Wigner condition one has to solve Eq. (18) along with Eq. (16) to compute the non-equatorial orbits, these are shown in Fig. (2). It should be noted that the dependence of these results on $\mathbf{a} = 0.0\mathbf{a} = 0.5M\mathbf{a} = \mathbf{M}$

\[ \begin{align*}
\text{Eq. Plane} & \quad 1 & 2 & 3 & 4 & 5 & 6 \\
1.560 & 1.562 & 1.564 & 1.566 & 1.568 & 1.570 \\
\end{align*} \]

Radial Distance $(r/M)$

Theta in Radian

(a) The above figure shows the co-rotating circular orbits in non-equatorial planes with the spin vector remain similar to the previous case, $S^{(i)} = (0, -0.01, -0.01, 0)$.

(b) The counter-rotating circular orbits are shown with the Newton-Wigner condition.

Figure 2: Non-equatorial circular orbits, both co-rotating and counter-rotating, are shown for spinning particles with Newton-Wigner constraint. The value of $k$ is fixed at 2.

$k$ is weak and almost negligible. This is related to the prefactor $\sqrt{k - 1 \over k + 1}$ which has a maximum value of one as discussed earlier. The overall behavior is similar to the previous case, while the numerical values differ by a small amount, as shown in Fig. (3). The difference between the various spin supplementary conditions become significant only for larger values of spin.

It is evident from Fig. (1) and Fig. (2) that, the radial coordinate $(r)$ for each non-equatorial orbit is related to a specific value of angular coordinate $(\theta)$. For a given spin value, if one choose to have a circular orbit at $\theta = \theta_s$, the corresponding radial coordinate takes a particular value of $r = r_s$. We can estimate the radius of such circular orbits at constant altitude as, $R_s = r_s \sin \theta_s$. As one get closer to the horizon, it deviates furthermore from the equatorial plane, and also the radius ($R_s$) starts to decrease. This can be better explain in a graphical representation as shown in Fig. (4).

### 4 Circular orbits and stability analysis

In this section, we shall discuss the stability of the non-equatorial circular orbits for a spinning particle. But before investigating the spinning particle, let us revisit the stability properties of the geodesic trajectories
(a) The above figure shows the co-rotating circular orbits for $S^{(s)} = (0, -0.01, -0.01, 0)$ with two different spin supplementary condition. Though the nature of the plots remain same as seen from Fig. (1) and Fig. (2), they differ in a small scale.

(b) The dependence of Mathission-Papapetrau equations for different spin supplementary condition certainly increases with an increase of the spin parameters. A considerable amount of difference is achieved with $S^{(s)} = (0, -0.05, -0.05, 0)$.

Figure 3: Figure shows a comparative study of non-equatorial orbits for different spin supplementary condition with $a = 0.8M$.

Figure 4: The circular orbits are shown explicitly in the non-equatorial planes. The origin is located at $(0, 0, r_s \cos \theta_s)$ while the radius is $R_s = r_s \sin \theta_s$. Scale of $r_s \cos \theta_s$ is raised by the square of logarithmic to realize the difference properly.
around a Schwarzschild black hole. In this case, the energy and angular momentum are easily derivable from the radial potential:

\[ E_{\text{sbh}}^2 = \frac{1}{r} \frac{(r - 2M)^2}{(r - 3M)}, \quad \text{and} \quad L_{\text{sbh}}^2 = \frac{Mr^2}{(r - 3M)}. \]  

(32)

Where \( M \) is the mass of the black hole. Both energy and momentum reaches a simultaneous minima at \( r = 6M \), which is precisely the innermost stable circular orbit (ISCO) for a timelike particle. Beyond this limit, no stable circular orbit is possible in a Schwarzschild spacetime. A similar situation appears around a Kerr black hole with energy \( E_{\text{kbh}} \) and momentum \( L_{\text{kbh}} \). But in that case, ISCO depends on the angular momentum of the black hole. For example, at \( a = 0.4M \), the ISCO for a co-rotating geodesic appears at \( r = 4.614M \). A schematic diagram to demonstrate the ISCO is given in Fig. (5).

![Figure 5](image)

Figure 5: Energy and angular momentum of a timelike geodesic is shown in a rotating as well as in a non-rotating gravitational field. The ISCO’s always appear at a point where energy and angular momentum both simultaneously become minimum.

In case of a spinning particle, neither energy nor momentum is a conserved quantity. In stead, some spin dependent functions such as, \( C_t \), \( C_\phi \) and \( K_s \) become constant of motion. Hence, the ISCO is located at \((r_c, \theta_c)\), such that:

\[ \left( \frac{\partial C_t}{\partial r} \right)_{r=r_c} = 0, \quad \left( \frac{\partial C_\phi}{\partial r} \right)_{r=r_c} = 0, \quad \text{and} \quad \left( \frac{\partial K_s}{\partial r} \right)_{r=r_c} = 0, \]  

(33)

\[ \left( \frac{\partial C_t}{\partial \theta} \right)_{\theta=\theta_c} = 0, \quad \left( \frac{\partial C_\phi}{\partial \theta} \right)_{\theta=\theta_c} = 0, \quad \text{and} \quad \left( \frac{\partial K_s}{\partial \theta} \right)_{\theta=\theta_c} = 0. \]  

(34)

We first compute \( U^{(0)} \) and \( U^{(3)} \) as:

\[ U^{(0)} = \frac{1}{\sqrt{1 - \Omega^2}}, \quad \text{and} \quad U^{(3)} = \frac{\Omega}{\sqrt{1 - \Omega^2}}. \]  

(35)

Substituting the numerical values of \( \Omega, r \) and \( \theta \), we plot the variation of \( C_t, C_\phi \) and \( K_s \), as shown in Fig. (6) and Fig. (7). It can be noted that all of them has a simultaneous minima in a non-equatorial orbit. This
corresponds to ISCO for the spinning particles and unlike geodesics, it appear in a off-equatorial plane. The ISCO’s for Newton-Wigner spin supplementary condition is shown in Fig. (8) and the comparison between two spin supplementary condition is depicted in Fig. (9). It is easy to notice that the difference for different spin supplementary conditions is small in the linear spin approximation.

Figure 6: The conserved energy and total angular momentum is shown for co-rotating orbits with $S^{(1)} = S^{(2)} = -0.01$ in the Tulczyjew-Dixon spin supplementary condition. The ISCO is shown as the minimum of energy and angular momentum, which is placed in a non equatorial plane. The variation of ISCO is shown for different angular momentum of the black hole. Black hole’s angular momentum is zero for the upper branch and $a = M$ for the lower branch. In between $a$ is increasing from top to bottom. It is shown that for large values of ‘a’, the ISCO exist in a more deviated non-equatorial plane while for $a \to 0$, the ISCO exist very close to the equatorial plane.
Figure 7: The Carter constant is shown for $S^{(1)} = S^{(2)} = -0.01$ with the angular momentum of the geometry varies as shown in the figure. Similar to the other constants such as energy and angular momentum, Carter constant also reaches a minima in $r$ and $\theta$. It clearly suggests that the particle will eventually settle down in a non-equatorial orbit.

Figure 8: The conserved quantities are shown in the Newton-Wigner spin supplementary condition. Similar to the previous case of Tulczyjew-Dixon spin supplementary condition, the minima appear in a non-equatorial plane.
Newton-Wigner
Tulczyjew-Dixon
0.0 0.2 0.4 0.6 0.8 1.0
1
2
3
4
5
6
Angular momentum of the black hole (a / M)
Raidal Distance of ISCO (r / M)

(a) The radial coordinate for the ISCO’s are shown in the above figure for different angular momentum of the black hole while the spin parameters are fixed at $S^{(1)} = S^{(2)} = -0.01$.

(b) The difference of ISCO’s for different spin supplementary condition increase with the increase of spin parameter. In this case, we set $S^{(1)} = S^{(2)} = -0.05$.

Figure 9: The ISCO’s are shown for different spin supplementary condition.

5 Discussion

The circular motion of spinning particles are discussed on the $\theta = \text{constant}$ plane in a Kerr background. We numerically solve the Mathission-Papapetrau equations and explicitly shown the existence of such orbits in a rotating geometry for different spin supplementary conditions. The deviation from the equatorial plane is proportional to the radial spin component ($S^r$) of the particle as well as the angular momentum of the black hole. But the direction of the deviation is related to the sign of $S^{(1)}\Omega$. More precisely, as shown in Fig. (1), for $S^{(1)} < 0$ the counter-rotating orbits deviate in the direction of $\theta = \pi$, while an opposite phenomena appear for co-rotating orbits.

The study of different spin supplementary conditions are carried out in the linear spin framework. The nature of the plots remain similar as shown in Fig. (1) and Fig. (2), and the difference is very small for different spin supplementary conditions.

We provided a better understanding of the stabilities of these circular orbits in terms of conserved quantities such as energy, angular momentum and Carter constant. Similar to the geodesics, there exist a point $(r_c, \theta_c)$ where all the conserved quantities reaches their respective minima simultaneously and this corresponds to the ISCO. Spinning particles not only can move in the non-equatorial circular orbits, it may settle down in such planes.

Though in this study, we considered that the equations of motion are linear order in spin, in principle, this analysis can be extended for large spin.

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