Reply to: “Comment on: ’Geometric phase of neutrinos: Differences between Dirac and Majorana neutrinos’ [Phys. Lett. B 780 (2018) 216]” [Phys. Lett. B 818, 136376 (2021)]

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(Dated: July 20, 2021)

In this paper we reply to the comment presented in [1]. In that work the author raises several points about the geometric phase for neutrinos discussed in [2]. He affirms that the calculation is flawed due to incorrect application of the definition of noncyclic geometric phase and the omission of one term in Wolfenstein effective Hamiltonian. He claims that the results are neither gauge invariant nor lepton field rephasing invariant and presents an alternative calculation, solely in order to demonstrate that the Majorana CP-violating phase enters the geometric phase essentially by lepton field rephasing transformation. Finally he claims that the nontrivial dependence of geometric phase on Majorana CP-violating phase presented in [2] is unphysical and thus unmeasurable. We discuss each of the points raised in [1] and show that they are incorrect. In particular, we introduce geometric invariants which are gauge and reparametrization invariants and show that the omitted term in the Wolfenstein effective Hamiltonian has no effect on them. We prove that the appearance of the Majorana phase cannot be ascribed to a lepton field rephasing transformation and thus the incorrectness of the claim of unphysicality of geometric phase. In the end we show that the calculation presented in [1] is inconsistent and based on the erroneous assumption and implementation of the wavefunction collapse. We remark that the geometric invariants defined in the present paper show a difference between Dirac and Majorana neutrinos, since they depend on the CP-violating Majorana phase.

The paper is organized in three sections, each devoted to one of the specific points raised in [1].

1. The role of Majorana phase - We start by analyzing the possibility of the existence of observable physical quantities depending on the geometric phase of neutrinos. In Ref. [1] it is argued that the appearance of the Majorana phase in the expression of the geometric phase is a consequence of an unphysical rephasing transformation. In other words, the author affirmed that by appropriately choosing a rephasing transformation, it is possible to get rid of the Majorana phase. This argument of rephasing freedom is not new and it is essentially borrowed from [3]. We here discuss why both the original argument and the consequent assertions about the geometric phase are flawed.

The neutrino mass Hamiltonian for the propagation in the vacuum, but similar considerations apply to the propagation in matter, can be written, when expressed on the flavor basis, in either of the two ways

\[ H^{(1)} = \begin{pmatrix} m_{ee} & m_{e\mu} \\ m_{e\mu} & m_{\mu\mu} \end{pmatrix}, \quad H^{(2)} = \begin{pmatrix} m_{ee} & m_{e\mu}e^{i\alpha} \\ m_{e\mu}e^{-i\alpha} & m_{\mu\mu} \end{pmatrix}. \]  

(1)

It is fundamental to realize that the two Hamiltonians are equivalent if and only if the neutrinos are of the Dirac type. Indeed, if the neutrinos are Dirac fermions, the phase \( \alpha \) in \( H^{(2)} \) can be reabsorbed using a simple rephasing of the neutrino fields, under which the Dirac Lagrangian is invariant. On the contrary, the Majorana Lagrangian is not invariant under global \( U(1) \) transformations, and this fact implies that the phase freedom for the neutrino fields is lost. That the two Hamiltonians \( H^{(1)}, H^{(2)} \) are not equivalent in the Majorana case is also evident in their diagonalization. As the reference [1] itself mentions, the two Hamiltonians can be diagonalized by two distinct unitary matrices

\[ U^{(1)} = \begin{pmatrix} \cos \theta & \sin \theta e^{i\alpha} \\ -\sin \theta & \cos \theta e^{i\alpha} \end{pmatrix}, \quad U^{(2)} = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\alpha} \\ -\sin \theta & \cos \theta \end{pmatrix}, \]  

(2)

respectively for \( H^{(1)} \) and \( H^{(2)} \). It is an easy check to see that \( H^{(1)} \) cannot be diagonalized by \( U^{(2)} \) and \( H^{(2)} \) cannot be diagonalized by \( U^{(1)} \). Both mixing matrices bring the respective Hamiltonians to the same diagonal matrix according to

\[ (U^{(1)})^\dagger H^{(1)} U^{(1)} = H^{(1)} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} = H^{(2)} = (U^{(2)})^\dagger H^{(2)} U^{(2)}. \]  

(3)

It is the above equations that define the mixing matrix: if \( H_f \) is the mass Hamiltonian in the flavor basis, the acceptable mixing matrices \( U \) are all and only those which satisfy \( U^\dagger H_f U = H_m \). Even if the matrices in the mass basis coincide (\( H^{(1)}_m = H^{(2)}_m \)), the different mixing matrices define two distinct and non-equivalent sets of linear relations between the flavor states and the mass states.
Nevertheless, in the ref. 3, the authors argued that one can freely move from the mixing matrix $U^{(1)}$ to the mixing matrix $U^{(2)}$ by rephasing the charged lepton fields in the weak charged current interaction:

$$\mathcal{L}_{\text{CC}} = \frac{g}{\sqrt{2}} \sum_{a=e,\mu} \sum_{k=1,2} \tau_\ell(x) \gamma^\rho U_{ak} \nu_k L(x) W_\rho(x) + h.c. \quad (4)$$

where $U_{ak}$ denotes the neutrino mixing matrix, $W_\rho$ is the charged weak gauge field and the subscript $L$ is used to denote the left-handed fermion fields.

In the case in which the neutrino fields are of Majorana type, one is not free to rephase $\nu_k L$, yet it is always possible to rephase the charged lepton fields $a_L$ which are of Dirac type. However, in the ref. 3, it is taken one step forward, i.e. the charged lepton field rephasing $a_L \rightarrow a_L e^{-i\phi}$ for $a = e, \mu$, is interpreted as defining the new mixing matrix

$$\tilde{U}_{ak} = e^{i\phi} U_{ak} \quad (5)$$

and clearly one can always choose the phases $\phi_{\alpha}$ in order to bring $\tilde{U}$ either in the form $U^{(1)}$ or $U^{(2)}$. Such an interpretation is not justified if the neutrino mixing matrix is originally to be defined from Eq. 3.

In other words, this interpretation would be legitimate if the neutrinos only participated in the weak interaction, evidently bringing no actual modification to the neutrino physics. However, neutrinos also oscillate according to the flavor Hamiltonian $H_f$ and it is not acceptable that the (unphysical) rephasing of the charged lepton fields brings along a change in the neutrino flavor Hamiltonian. Obviously, according to whether the mixing matrix is chosen as $U^{(1)}$ or $U^{(2)}$, one obtains respectively $H_f = H^{(1)}$ and $H_f = H^{(2)}$ which are in principle distinct.

We conclude that if the mixing matrix is to be defined from the flavor Hamiltonian via Eq. (3), then the identification in Eq. (5) is not legitimate, unless the new mixing matrix diagonalizes the same flavor Hamiltonian (which is not the case, of course, for $U^{(1)}$ and $U^{(2)}$). This argument is independent of whether the two Hamiltonians $H^{(1)}$ and $H^{(2)}$ produce distinct observable effects or not. We remark, however, that while the standard oscillation formulae are not affected by the Majorana phase $\alpha$ 3, the latter may well show up in other circumstances, for instance in the propagation in a dissipative medium 4 5 for which the oscillation formulas depend on the Majorana phase. Finally, it is not at all clear why the charged lepton field rephasing should have any effect on neutrinos, which should not be affected by the latter, because then it is obvious that any quantity depending only on the neutrino oscillatory evolution suffers no modification from the charged lepton field rephasing.

2. Geometric invariants and Majorana phase - The second point raised by the author of Ref. 1 is that in the paper 2, the calculations are flawed due to incorrect application of the definition of non-cyclic geometric phase and the omission of one term in the Wolfenstein effective Hamiltonian. Let us analyze these points in detail. The geometric phases at a time $t \sim z > 0$ for an initial electron neutrino state $|\nu_e\rangle$ and that evolves under the action of the mass Hamiltonian, is given by

$$\Phi_{\nu_e}^g(z) = \Phi_{\nu_e}^{\text{tot}}(z) - \Phi_{\nu_e}^{\text{dyn}}(z)$$

$$= \arg \left[ \langle \nu_e(0)|\nu_e(z)\rangle \right] - \Im \int_0^z \langle \nu_e(z')|\tilde{\nu}_e(z')\rangle \, dz' \quad (6)$$

$$= \arg \left[ \cos \left( \frac{\Delta m^2_{\nu e} z}{4E} \right) + i \cos 2\theta_m \sin \left( \frac{\Delta m^2_{\nu e} z}{4E} \right) \right] - \frac{\Delta m^2_{\nu e}}{4E} \cos 2\theta_m \ .$$

A similar result is obtained for an initial muon neutrino state $|\nu_{\mu}\rangle$, indeed one has $\Phi_{\nu_{\mu}}^g(z) = -\Phi_{\nu_{\mu}}^g(z)$. Both $\Phi_{\nu_e}^g(z)$ and $\Phi_{\nu_{\mu}}^g(z)$ are gauge invariant and re-parametrization invariant.

In deriving the explicit expression of $\Phi_{\nu_e}^g(z)$ in eq. (6), we have neglected a term proportional to the identity in the neutrino Hamiltonian $H_0 = \left( \frac{m_e^2 + m_{\mu}^2}{4E} + \frac{G_F m_e}{\sqrt{2}} \right) I$. It is straightforward to show that such a term has no effect whatsoever on the previously defined invariants. The inclusion of this term, (a term proportional to the identity matrix), amounts to an additional time-dependent global phase factor $|\nu_{\nu_{e,\mu}}(z)\rangle \rightarrow e^{i\lambda(z)} |\nu_{\nu_{e,\mu}}(z)\rangle$, with $\lambda(z) = -\int_0^z dz' \left( \frac{m_e^2 + m_{\mu}^2}{4E} + \frac{G_F m_e}{\sqrt{2}} \right)$.

Computing the invariants taking into account the additional phase factor, one finds

$$\Phi_{\nu_e}^g(z) = \arg \left[ \langle \nu_e(0)|\nu_e(z)\rangle e^{i(\lambda(z) - \lambda(0))} \right] - \Im \int_0^z \langle \nu_e(z')|\tilde{\nu}_e(z')\rangle \, dz' - \Im i \int_0^z \lambda'(z') \, dz'$$

$$= \arg \left[ \langle \nu_e(0)|\nu_e(z)\rangle \right] + (\lambda(z) - \lambda(0)) - \Im \int_0^z \langle \nu_e(z')|\tilde{\nu}_e(z')\rangle \, dz' - (\lambda(z) - \lambda(0))$$

$$= \arg \left[ \langle \nu_e(0)|\nu_e(z)\rangle \right] - \Im \int_0^z \langle \nu_e(z')|\tilde{\nu}_e(z')\rangle \, dz' \ . \quad (7)$$
In addition to $\Phi^g_{\nu_e \rightarrow \nu_{\mu}} (z)$ and $\Phi^g_{\nu_\mu \rightarrow \nu_e} (z)$, one can also define two geometric invariants associated to the neutrino oscillation between different flavors, i.e.

$$\Phi^g_{\nu_e \rightarrow \nu_{\mu}} (z) = -3 \int_0^z \langle \nu_e (z') | \nu_{\mu} (z') \rangle dz', \quad (8)$$

$$\Phi^g_{\nu_\mu \rightarrow \nu_e} (z) = -3 \int_0^z \langle \nu_{\mu} (z') | \nu_e (z') \rangle dz'. \quad (9)$$

Explicitly, one has

$$\Phi^g_{\nu_e \rightarrow \nu_{\mu}} (z) = \Phi^g_{\nu_\mu \rightarrow \nu_e} (z) = \left( \frac{\Delta m^2_{21}}{4E} \sin 2\theta_m \cos \phi \right) z, \quad (10)$$

Also in this last evaluation we have neglected the constant term proportional to the identity operator in the neutrino Hamiltonian $H_0 = \left( \frac{\Delta m^2_{21}}{4E} + \frac{G_F m_e}{\sqrt{2}} \right)$ and we have taken into account that for any time $z$, we have $\langle \nu_e (z) | \nu_{\mu} (z) \rangle = 0$. It is straightforward to prove that also the geometric invariant of Eq. (8) is not affected by this additional term in the Hamiltonian:

$$\Phi^g_{\nu_e \rightarrow \nu_{\mu}} (z) = -3 \int_0^z e^{-i \lambda (z')} \langle \nu_e (z') | \frac{d}{dz} \left( e^{i \lambda (z')} \nu_{\mu} (z') \right) \rangle dz'$$

$$= -3 \int_0^z i \lambda' (z') \langle \nu_e (z') | \nu_{\mu} (z') \rangle dz' - 3 \int_0^z \langle \nu_e (z') | \nu_{\mu} (z') \rangle dz'$$

$$= -3 \int_0^z \langle \nu_e (z') | \nu_{\mu} (z') \rangle dz'. \quad (11)$$

The trivial calculation above also proves that $\Phi^g_{\nu_e \rightarrow \nu_{\mu}} (z)$ as defined in eq. (8) is indeed gauge invariant, since eq. (11) actually holds for any smooth function $\lambda (z)$. It is also easy to see that the Eqs. (8) are reparametrization invariant. Notice that the original full Mukunda-Simon definition (e.g. Eq. (6)) fails to be gauge invariant when the distinct neutrino states $\nu_e (z), | \nu_{\mu} (z) \rangle$ are used as in (8). This could be expected, because an invariant like (6) is defined for two distinct paths in Hilbert space, and strictly speaking it cannot be compared to a geometric phase, which is instead defined on a single trajectory in the Hilbert space.

However, differently from $\Phi^g_{\nu_e \rightarrow \nu_{\mu}} (z)$ and $\Phi^g_{\nu_\mu \rightarrow \nu_e} (z)$ the two geometric invariants introduced in eqs. (8) and (9) depend on the Majorana phase $\phi$. In the case of a Dirac neutrino, the phase $\phi$ can be removed by means of a rephasing of the neutrino fields and the invariants of Eqs. (10) reduce to

$$\Phi^g_{\nu_e \rightarrow \nu_{\mu}} (z) = \Phi^g_{\nu_\mu \rightarrow \nu_e} (z) = \left( \frac{\Delta m^2_{21}}{4E} \sin 2\theta_m \right) z.$$

We point out that for the discussion in the previous section, in the case of Majorana neutrinos, such dependence cannot be removed by a lepton field rephasing transformation, as mistakenly affirmed in Ref. [1]. Then the two geometric invariants defined in Eqs. (8) and (9) depend on the Majorana CP-violating phase, and as well-defined geometric quantities, they are in principle measurable.

3. Geometric phase and wavefunction collapse - The last point raised by the author of ref. [1] is that the calculation of the geometric invariants made in the article in [2] is wrong as it does not take into account the contribution to the geometric invariant due to the collapse of the wave function induced by the projective measurement operation. This claim is followed by an estimate in which the projective measurement process is carried out through a process in which, in a time $\tau$, the state at time $z = \tau$ is led into one of the two flavor eigenstates. In the end, the time $\tau$ is then made to tend to zero to recover the unitarity of the measurement process, a property considered central by the author of the ref. [1].

This criticism is flawed in at least two respects: from the conceptual point of view, the introduction of the wavefunction collapse is not justified and leads to wrong conclusions; from the technical point of view, the description of the collapse adopted in the ref. [1] is arbitrary and incorrect. The latter, in particular, gives an indication of the approximation with which the comment was written.

Let us focus on the first point. In order to define the geometric invariants associated to the mixing, namely Eqs. (8) and (9), it is not necessary that the transition $| \nu_e (t) \rangle \rightarrow | \nu_{\mu} (t) \rangle$ takes places, but it is sufficient to consider the evolved state $| \nu_{\mu} (t) \rangle$ which is by itself a combination of the initial flavor eigenstates. Asserting that the neutrino state at time $t$ is $| \nu_{\mu} (t) \rangle$ is
of course different from asserting that the neutrino state at time $t$ is $|\nu_\alpha(0)\rangle$. The last assertion, in particular, implies a non-unitary evolution of the neutrinos at some point. Moreover, even though it is not clear how to detect the geometric invariants of Eqs. (3) and (9), this fact does not imply that they are not observable in principle.

To justify the introduction of the wavefunction collapse at least one of the following hypotheses should be true:

(a) The detection relies on a projective measure.

(b) The collapse of the wave function is intrinsically associated with the physical phenomenon being analyzed.

The second assumption is certainly wrong. Indeed the oscillation between two levels is a well-known phenomenon in several fields of quantum mechanics such as, to provide an example in a very different field, the vacuum Rabi oscillation [6], and never involve a collapse of the wave function process. On the other hand, we make no claim about how a possible experimental detection of these geometrical invariants is to be performed.

In addition, the author of Ref. [1] seems to be unaware of the fact that not all the possible measures that can be done on a quantum system must include demolition of the quantum state [7]. The design of a non-demolition measure for neutrinos would be a difficult task to achieve, but if one wants to prove that a quantity cannot be observed, one must consider all the possible ways to realize a measurement.

Even assuming the necessity of including the wavefunction collapse, for whatever reason, the definition of the geometric invariant presented in [1] is not correct. Indeed, the wavefunction collapse is taken into account considering an arbitrary non-unitary evolution, for which unitarity is restored in the limit of a vanishing collapse time $\tau$ (see eq. (14) of the ref. [1]). In any case, while it can be assumed that the wavefunction collapse is very rapid, one cannot expect it to be a unitary process. In fact, it is well known that the measurement of an observable on a pure state which is not an eigenstate of the corresponding operator produces a mixed state with reduced purity [8]. This general fact implies that the wavefunction collapse cannot be represented by a unitary operator preserving the purity of the state.

A coherent analysis of the wavefunction collapse process, should certainly start from considering, during the measurement process, the neutrino as an open system which interacts with the experimental device. The dynamics of the neutrino during the measurement process must be described by a Gorini-Kossakowski-Sudarshan-Lidbland master equation [9, 10]

$$\frac{\partial \rho(t)}{\partial t} = -\frac{i}{\hbar} [H, \rho(t)] + D[\rho(t)].$$

(12)

Here $\rho(t)$ is the time-dependent density matrix of the neutrino while the non unitary part of the evolution is described by the dissipator $D[\rho(t)]$ defined as

$$D[\rho(t)] = \frac{1}{2} \sum_{i=1}^{N-1} a_{ij} \left( [F_i \rho(t), F_j^\dagger] + [F_i, \rho(t) F_j^\dagger] \right).$$

(13)

The coefficients $a_{ij}$ of the Kossakowski matrix, can be derived by the properties of the measurement device [11,12] while $F_i$ are a set of traceless operators such that $Tr(F_i^\dagger F_j) = \delta_{ij}$. In the two flavor neutrino mixing $F_i$ can be represented by the Pauli matrices, and in the three flavor generalization the $F_i$ can be represented by the Gell-Mann matrices $\lambda_i$. Therefore the eq. (14) of ref. [1] and the following eqs. are flawed and not physically consistent.

4. Conclusions - In conclusion, in this reply to the comment presented in [1], we discussed each of the raised points and shown their incorrectness. In particular, we defined geometric invariants which are gauge and reparametrization invariant, showing that any term proportional to the identity in the Hamiltonian does not affect them. We have demonstrated that the geometric invariants related to neutrino mixing depend on the Majorana phase and we have proven that the appearance of this dependence cannot be explained in terms of a simple lepton rephasing transformation. We thus discussed why the claim of unphysicality and unmeasurability of the geometric invariants is baseless. Finally we discussed the geometric phase as defined in the reference [1] and we have shown its physical inconsistency.

ACKNOWLEDGEMENTS

A.C. and A.Q. acknowledge partial financial support from MIUR and INFN, A.C. also acknowledges the COST Action CA1511 Cosmology and Astrophysics Network for Theoretical Advances and Training Actions (CANTATA), S.M.G acknowledges support from the European Regional Development Fund for the Competitiveness and Cohesion Operational Programme (KK.01.1.06–RBI TWIN SIN) and the Croatian Science Funds Project No. IP-2016–6–3347 and No. IP-2019–4–3321. S.M.G also acknowledges the QuantiXLie Center of Excellence, a project co–financed by the Croatian Government and European
Union through the European Regional Development Fund—the Competitiveness and Cohesion Operational Programme (Grant KK.01.1.1.01.0004). B.C.H. acknowledges the support of the Austrian Science Fund (FWFP26783).

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