Combining the Two-State System with a Matter-Wave Approach for Teaching Quantum Mechanics in High-School

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Abstract. In our teaching approach to quantum mechanics for high schools, we combined a two-state system approach with a matter-wave approach. We developed a simulation of a double well, which has been gradually improved with experience to include all the functionality needed to address the most important concepts of quantum physics. A complementary simulation of a double slit has also been developed to address the introduction of the wave function. A course, based on these simulations has been developed and is currently being tested for the fourth year, still undergoing improvements. Each year a series of formative questions and a pre- and post-test has been administered. We report here how the simulation is used to address the concepts of quantum physics and some preliminary results from the ongoing study of the course.

1. Introduction
The teaching of quantum mechanics in high-schools has become an important topic [1 – 3]. Due to the limited mathematical knowledge, approaches in high-school have to be tailored towards a more conceptual understanding. The approach with two-state systems is proposed due to its simplicity [1]. It is well suited to introduce states and the polarization first and spin first approaches are well suited to introduce incompatibility (the uncertainty relation). Since the position is not one of the observed properties, they can also be used to introduce entanglement and the fact that the concept of trajectory is not useful in quantum mechanics. A second group of approaches found in literature are those with various versions of amplitude summation (path integral [3], wave-function). These build on the experience that students have with waves and might be therefore closer to students in a constructivist sense. These necessarily include the time dependence and are therefore suitable to introduce the time evolution of the states. However, they are not well suited to introduce states, since there are an infinite number of states available. We combined the two-states approach and the wave-function approach, using also an exploratory simulation for the active engagement of students [4]. We developed a simulation of a double well with many tuneable settings, and we improved it over the course of the experimentation in class. We also developed a complementary simulation of the double slit experiment. The PhET simulation collection [5] includes both simulations, but they are designed more for visualisation after the wave-function has already been introduced, while we required a simulation that would allow students to build the concepts through exploration.

For the active engagement framework we chose ISLE (Investigative Science Learning Environment) [6], because it is designed to let students build their own knowledge in an epistemologically authentic
way. We hypothesize that such an approach, by building their own description, might help students overcome the usual difficulties in accepting the quantum description of reality.

2. The setting
The course is being experimented in a high-school class. The class is elective and is part of a natural-sciences oriented package of courses. The students that choose it are typically interested in either physics, chemistry or biology, or in general more interested in natural sciences than languages or humanist subjects. The usual number of students per year is 27. The number of hours at our disposal varies from year to year, because it depends on when the instructor is able to start the course. In the first year it has been 18, in the second 12 and the third year is still running. The class is taught by one of the authors (SF). In the first year, there was an exam in the middle of the course, but not at the end. In the second year there was no exam, but we did a pre- and post-test.

The sequence was also tested on three groups of motivated students, one group that attended a summer school for high-school students in Ljubljana (17 students, 3 hours, only the double well part), another group that attended the same summer school a year later (17 students, 3 hours, only the wave-function part) and a group of students who attended the summer school for high-school students in Udine (30 students, 3 hours, both parts). The data gathered from these three groups is mostly observational.

3. The simulation and the simulated experiments in the double well
The simulation’s graphic user interface (GUI) is shown in figure 1. The simulation allows the changing of the initial state of the particle (the preparation), the choice of quantity to be measured (position, $x$, or energy, $E$), the time of measurement and whether the particle is to be reset before subsequent measurements (this means whether the subsequent experiments are independent or subsequently made on the same particle). It is, of course, possible to automatically repeat the experiment a selected number of times. The results are displayed in form of histograms for each result, with the number of counts also displayed. In the second year a log of the results has also been added to allow subsequent analysis. The simulation is designed to allow the changing of the parameters of the wells, but most of this functionality is disabled to avoid being unnecessarily distracting. The capability of measuring the energy has been added only the second year to introduce the possibility of addressing the incompatibility without the introduction of the wave-function.

![Figure 1. The graphic user interface of the simulation: (A) the representation of the potential, (B) the representation of a 1D double well, (C) The preparation settings, (D) the measurement settings, (E) resetting or not resetting the particle after measurement, (F) graphical representation of the results of the measurements, (G) log of the results and (H) parameters of the wells.](image-url)
Technically, the simulation is written in HTML 5 to allow native portability to different platforms. It is available on-line [6] and can be run from portable devices, which is how students perform the simulated experiments. So far we have observed some issues with some mobile devices. It appears that the time between automatic repetitions is too short for some devices and some particles “get lost”, they are not recorded. The reason for the slowness is mainly in the fact that there is a graphical blink of the sensor each time it detects a particle, which has to last for a minimum amount of time to be noticeable by the human eye. We might improve this in later modifications to the software. The energy states are calculated from the settings with a numerical algorithm. However, to obtain the position states as localized in one well as possible, the energy states are very close together and the numerical calculation has sometimes failed to identify the first two roots. That is part of the reason why some user adjustments of the well have been disabled. In the future we plan to address this issue and find a more reliable numerical algorithm.

The simulated experiments that can be performed with the simulation are listed below, but first let us introduce the terminology that we use.

3.1. Terminology
When we talk about quantum, we talk about states, quantities and properties. The property is the value of a quantity, and we observed for now that it is better to use the term “value” with students. Thus when we measure the quantity “position” in a double well, we can get two properties: either left well, denoted x:L or right well, denoted x:R. Likewise the energy properties are denoted E:E1 and E:E2. The state is something that best describes the system. In classical physics, it can be often imagined as a collection of properties, while in quantum mechanics our suggestion is to imagine it as a collection of properties of which some cannot be exactly known, but instead have a probability distribution over various possible values. This is a representation that we plan to further explore.

3.2. Stochasticity and probability
Preparing the particle in a known energy state, students measure the particle’s position at time 0. The results are randomly distributed between the left and the right well. Thus the properties x:L and x:R are equally likely to occur. The questions for students are: “Can you predict the outcome of the next experiment?” “What can you predict about the outcome?” “Do both sensors ever blink at the same time?” The findings that arise from these questions are that (i) the single result is stochastic, unpredictable, (ii) that the probability (or the distribution of results) is predictable, and (iii) that the particle in undividable, it is never found in both wells at the same time.

3.3. The meaning of state
Students prepare the particle in a position state and immediately afterwards measure its position. They always get the property that is consistent with the prepared state. Thus we distinguish pure states, those with a selected property known, and non-pure states, later relabelled superposition states, those with a statistical distribution of the selected property.

3.4. Time evolution
Students prepare the particle in a known position state and then measure its position at different times. We suggest that they try values of the order of tens of femtoseconds. The students’ task is to identify patterns and describe them mathematically, if possible. Students are asked to reiterate what the columns in the histogram represent, that is probability, and thus clarify that it is the probability of the result of a position measurement that changes with time. Next, students prepare the particle in an energy state and repeat the experiment. With the same task they notice that in this case the probability distribution does not change with time. They further identify that there is a period for the transition of the probability from one well to the other and back again. It is important to focus students attention each time on the fact that it is the probability distribution that we are talking about.
3.5. Formal description
At this point we ask students to suggest a formal description of a state which has one probability for the result of a position measurement to be \( x:L \) and a different probability for the result to be \( x:R \). Students most often suggest either \( a|L \rangle + b|R \rangle \) or \( a|L \rangle \text{ OR } b|R \rangle \). More than 50\% of students in our experience suggest the first option. This means that the formal notation \( a|L \rangle + b|R \rangle \) is quite intuitive, although its exact meaning still needs to be explained.

3.6. Trajectory
With just the position measurement students can explore some key concepts of quantum mechanics, starting with the trajectory. In the case of a double well, the only meaningful definition of trajectory is the sequence of position properties at different times. An example of a trajectory would be \( x:L; t:0 \), \( x:L; t:20\text{fs} \), \( x:R; t:40\text{fs} \), \( x:R; t:60\text{fs} \) ... Students have thus far determined the period of transition of the probability between wells to be 120 fs. The number is not necessarily realistic, we were aiming for the right order of magnitude only. Thus at 60 fs the probability will be entirely in the right well. Students are asked to suggest experiments to determine how a particle moved from one well to the other. For example where was it at 20 fs or at some other time in-between. In a class discussion the idea to simply measure at a time in-between always comes from the students. However, we observed that some additional discussion is needed to decide to not reset the particle after the first measurement. The log of the measurement shows that the “trajectory” is not unique. Some particles might have a trajectory \( x:L; t:0 \), \( x:L; t:20\text{fs} \), \( x:R; t:60\text{fs} \), while others \( x:L; t:0 \), \( x:R; t:20\text{fs} \), \( x:R; t:60\text{fs} \). The finding from this is that the trajectory is at best probabilistic.

The “trajectory” experiment above can now be turned into a testing experiment for the following hypothesis: for every particle moving from the left well to the right well, a probabilistic trajectory can be determined. The prediction for the outcome of the experiment, if the hypothesis is correct, is that all trajectories will start with the property \( x:L \) and end with the property \( x:R \), while the properties at some intermediate time will be stochastically distributed with a repeatable probability distribution between \( x:L \) and \( x:R \). The results of this experiment show that the trajectory can be determined for each particle, but that the final state is not always \( x:R \) anymore (see figure 2). So far we have not asked the students to make a conclusion about the hypothesis based on this outcome. Instead we point out that the concept of a “trajectory” in a classical sense seems to be of no practical use in quantum mechanics. We do not want to cognitively burden the students with the existence or non-existence of a trajectory, it suffices that the concept is not useful.

![Figure 2. A measurement during the time evolution changes the final state of the system.](image)

3.7. Collapse
From the experiment with the trajectory, we see that the measurement affects the system. Students are now asked to propose experiments to explore in which way it affects the system. In a class discussion,
the idea to perform two measurements in close succession emerges from the students as a predominant suggestion. In our simulation, for a particle prepared in a position state, the times of 23.0 fs and 23.1 fs are a good choice for the two times. The outcome of the experiment, the log of the measurement results, shows that the distribution of probabilities remains exactly the same and that there is a perfect correlation between the results of the first and the second measurement. As a comparison, a measurement at the same times, but resetting the particle, shows the same distribution, but no correlation (except the statistical one, due to the distribution).

This is an opportunity for students to propose explanations for the observed correlation. The details of this part of the activity are still being developed, but the reasoning should go like this: If the result of the first measurement is x:L, then the result of the second will always be x:L. The only state that always produces the result x:L is the state |L>. Likewise for x:R. In general the state after the measurement is the pure state consistent with the result of the measurement. There are several testing experiments that can be proposed to test this hypothesis. One is to use the hypothesis to explain the change of the result in the “trajectory” experiment (section 3.6).

3.8. Incompatibility
So far we have not discussed the energy states and the energy measurements. Introducing them allows us to discuss the incompatibility of properties (the uncertainty relation). Students prepare a particle in a pure position state and measure position, reset the particle to the pure position state and measure energy. The position property is the same 100% of the time, while the energy properties are evenly distributed. The opposite is true for a pure energy state. Question for students: “can you find a state where both energy and position are 100% certain?” They cannot find such a state. In fact the two pure position states are obviously not pure energy states. And there are no others pure position states to try. Another question for students: “Does that mean that when we measure energy, we cannot measure position at all?” No. It just means that the result is unpredictable.

3.9. Statistical mixture vs. superposition
It is known that students interpret the superposition as a statistical mixture of states. We can address this with a testing experiment suggesting the following two hypotheses: 100 particles in a superposition state c|L> + c|R> is approximately equal to 50 particles in state |L> and 50 particles in state |R>. Performing only position measurements we cannot differentiate between the two. However, performing energy measurements we can. We choose 50 particles prepared in state |L> and 50 particles prepared in state |R>. Then we measure their energy. Both groups will have equally distributed energy properties. We prepare 100 particles in a state c|L> + c|R>, which is a pure energy state, and measure their energy. We will always get the same energy property. Clearly then the results of the two groups are not necessarily the same, and therefore the states are not the same.

3.10. The state in the middle of the time evolution
This question arose from a group of talented high-school students taking a shortened version of the course at a summer school. The state in the time evolution at 1/4 of the period of transition between the wells, is a state of the form [50%]|L> + [50%]|R> (this notation allows us to express probability while ignoring the phase of the coefficients). The students asked whether this is a pure energy state. A simple measurement reveals it is not, since the energy properties are equally distributed.

3.11. Complex coefficients
The question of the state in the middle (section 3.10.) allows us to address complex coefficients. An important concept for students to know before reasoning in this activity is the concept of linear independence (we do not put any effort in a visual representation of states as vectors, specifically because the visualization becomes difficult once complex coefficients are introduced). But the concept of linear independence is still important, so we define it as: two states |U> and |V> are independent, if there is no (complex) number C, which can transform |V> into |U>; |U> ≠ C|V> for all C. All
different states must be linearly independent. The latter rule is axiomatic, but we are developing ways to make it emerge in a more organic way.

![Figure 3](image-url) Figure 3. Three different states with equal probability for x:L and x:R (blue). On the right of each diagram there is a diagram of the energy measurement distribution for each state (red). These are not equal for all states, which makes the states different.

In our experiments, we have encountered three independent position states with equal probability for |L⟩ and |R⟩ (see figure 3). We write them in a generic way $a|L⟩ + b|R⟩$, where $a$ and $b$ can be different for each of the states, but $|a| = |b| = c$. The latter we can assume for now, because $a$ and $b$ represent probability in some way, and probability is always positive (and real). Question for students: “What are possible mathematical values of $a$ and $b$, so that all three states will be linearly independent?”

So far we have helped students by suggesting sets $\{a, b\} = \{c, c\}$ and $\{a, b\} = \{c, -c\}$ for the two pure energy states and showing that they are linearly independent. In two groups so far (motivated students of the summer school, to whom we have not previously mentioned complex numbers, and a regular class, to whom we have mentioned complex numbers more than three weeks before) the suggestion of complex numbers came from the class in a class discussion. We thus propose the third set to be $\{a, b\} = \{c^2 + ic^2, c^2 - ic^2\}$. There is no need to be so specific, though. The mere introduction of complex numbers suffices.

Once we introduce the possibility of negative and complex coefficients, it becomes clear that coefficients cannot directly represent probability. However, their modulus square can.

With this the activities related to the double well are concluded. We have introduced stochasticity, probability, time evolution, incompatibility, collapse, the difference between statistical mixture and superposition and complex coefficients. Next we pass to the wave-function.

4. The activities related to the wave-function
Most of the wave-function related concepts are introduced in a traditional way, so we will only briefly describe them, but there are some passages that relate uniquely to the material with the double well, and those we will describe in a little more detail.

4.1. Passing to an infinite number of position states
Instead of putting only two sensors on the double well, we can put more (see figure 4a). Thus we introduce more than just two position states and ask ourselves what could be the coefficients in front of those states. This is an introductory question that sets the scene for introducing the wave-function.
4.2. The double slit experiment and the wave-function

We developed a simulation for the double slit experiment which intentionally resembles the one with the double well. The building of the pattern is particle-by-particle. Students are asked to compare the pattern to their previous experience and all classes so far identified the pattern as the interference pattern of the experiment with light. From there we pass to the introduction of the wave-function – a wave-like description of particles, which is necessary to explain the observed pattern.

We relate this description to the question asked before: what coefficients to put in front of the position states? Figure 4b represents the reasoning: we have unknown coefficients in front of the various position states and we have a wave description, which includes the position parameter. Could we just match the position state with the corresponding value of the wave-function?

An efficient way to do this is still being investigated. At the moment we address it by just paying much attention to the visualization, such as that in figure 4b, but other ways are being investigated. We see the potential of this approach in the fact that we can gradually move from two position states to a finite number and then to an infinite number of states, but the best way to do so is still being investigated.

4.3. Some other experiments

The experiment with the diffraction of electrons on a crystal is used to introduce the relation between momentum (velocity) and wavelength. Students already know which parameters affect the diffraction pattern and by changing the voltage across the electron gun, the relation between kinetic energy/momentum/velocity and wavelength becomes apparent.

The PhET simulation Tunnelling [8], of a wave-function in a potential is used to familiarize students with the wave-function description of electrons. Various activities are performed where students are asked to predict the shape of the wave-function in a given potential. All with the goal to familiarize students with this new description.

The PhET simulation Fourier – making waves [9] is used to reiterate the uncertainty principle. Students are asked to produce a spatially localized wave and they discover that the only way to do that is to use plane waves with different wavelengths. With the relation between wavelength and momentum (velocity), we can reiterate the uncertainty relation between position and momentum (velocity) and even relate it to diffraction.
4.4. The atom
Students are asked to predict the behaviour of the wave-function in a confined space. The standing waves always emerge in a class discussion. These are then related to the shape of the orbitals, using animations of a 2D membrane [10] and the fact that students already know the shape of the orbitals from chemistry. Now they have an explanation for these shapes.

4.5. Back to the double well
In the end we go back to the double well and explain all the observations. With the PhET simulation Bound states [11], which can simulate a double well, we discuss the energy states and their superposition to produce what we called position states, $|L>$ and $|R>$. We observe the time evolution of the “position states” and the coefficients. Here we can discuss also the complex nature of the actual wave-function, although it can be discussed at any point in the wave-function part of the course. But here, or in the atom activity (section 4.4.), it becomes relevant, because it can be related to the fact that particles do not simply periodically disappear (as it would happen, if there was only a real part).

Thus we come to a full circle of activities which brought us from a simple two-state system to the wave-function and the explanations for the phenomena observed in the first part.

5. Observations from the classes and some preliminary data
Here we report some preliminary data from the first year of implementation and some observations from the following years.

Each year we used a pre- and post-survey to assess the ideas of the students. In the first year, the survey had three questions. The results are shown in figure 5. The first question was to depict a single slit experiment, which they observed during the course with light, and with electrons as part of the activity with the double slit (section 4.2.). In the pre-survey the most used representation was with trajectories (“traj.” in the figure), not identifying the diffraction at all, a classical ball-like idea. In the post-survey many students avoided representing any path between the slit and the screen (“avoid”) and represented the result with hits on the screen (“hits”). Very few used a wave-function (“w.f.”). The representation with trajectories was still present, but to a lesser extent.

The second question asked students to represent an electron in a triple potential well. The answers suggesting a quantum picture (labelled “Q” in figure 5) include representations using the wave-function, probability distribution, measurement results and formal description using Dirac notation. As classical (“C”) are labelled all the representations, which rely on a particle picture, without any hint to any quantum idea. The number of unanswered forms (“N. a.”) dropped to zero in the post-survey.

The third question asked students to identify the key elements of the quantum world. The mention of the idea that the measurement affects the system (labels “coll.” and “meas.” in figure 5) is dominating with more than 76% of students identifying this as a key feature. We observed that students started reasoning using the collapse early in the course, and often used it as the first resource to explain a new observation. A typical example was when asked how they could explain the change in the interference pattern when placing sensors on the slits. The answer of many students was that collapse happens at the slits. While this does not indicate that their understanding of collapse is adequate, it does indicate the importance of collapse in their reasoning. The second most mentioned distinguishing feature of quantum mechanics is its probabilistic nature (“prob.”). The wave nature (“wave”) is mentioned by a third of the students. Duality (“dual.”) is almost not mentioned. Such a result is expected since we avoid discussing duality in the course.
As a second tool we used a questionnaire designed to probe the students ideas about some concrete topics covered in the course. The questionnaire was comprised of statements that students rated on a scale from strongly agree to strongly disagree. This scale has been converted to a scale between −1 and 1 representing the fraction of students who correctly assessed a statement (+1 if disagreed with a false statement or if agreed with a true statement, etc.). We administered the questionnaire also to a comparison group of third-year physics education students at the Faculty of Education (pre-service teachers), who in that year attended two one-semester courses on special relativity and quantum mechanics.

The results of the questionnaire are in Fig. 6. The questions are grouped into different categories. Both the high-school students and the pre-service teachers scored highest in the categories of atomic structure, probability and wave-function and lowest on collapse. It appears that although the collapse was identified by the students as a key feature, more training is needed in its application to problems. The high-school students outperformed the pre-service teachers in every category. The uncertainty relation was given little emphasis, so the poor result is not surprising.

**Figure 5.** The figure shows results to a pre- and post-survey. The survey had three questions. The first group of columns represents the distribution of answer types to the first question, the second group to the second question and the third group to the third question. The meaning of the labels is described in the text.

**Figure 6.** Comparison between the answers of high-school students and pre-service teachers on various categories. The positive scores represent correct assessments of the statements and the negative scores the incorrect ones.
We observed that students also displayed a surprisingly good ability of argumentation using quantum ideas. The students were asked questions such as: “How would you respond to someone who suggests that the electron in a double slit experiment passes through one slit, we just cannot know which.” Many responses went along the lines that we can measure the trajectory, but then the result changes so the phenomenon is not the same anymore. Coupled to the number of students who avoided depicting a trajectory in the survey (figure 5) we believe that the fact that trajectory is not a useful concept in quantum mechanics was well understood.

6. Conclusions
We presented a course based on exploration of simulated experiments with a two-state system. We showed how the simulation can be used to build quantum concepts and how using two position states can facilitate the transition to an infinite number of position states. The preliminary findings presented here show that the course successfully built many quantum ideas. We also found that students overwhelmingly identified the measurement disturbing the system (collapse) as a key feature of quantum mechanics. However, the correct application of the collapse appears to require more exercise than we provided. The course is still being developed, and we recognize that some steps need further thought. We are continuing to research and develop the activities and investigate their efficiency.

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