BUBBLING SOLUTIONS FOR A PLANAR EXPONENTIAL NONLINEAR ELLIPTIC EQUATION WITH A SINGULAR SOURCE

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Abstract. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary, we study the following elliptic Dirichlet problem

\[
\begin{align*}
-\Delta v &= e^v - s\phi_1 - 4\pi a \delta_p - h(x) \quad \text{in} \quad \Omega, \\
v &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \( s > 0 \) is a large parameter, \( h \in C^0(\Gamma, \mathbb{R}), \) \( p \in \Omega, \alpha \in (-1, +\infty) \setminus \mathbb{N}, \delta_p \) denotes the Dirac measure supported at point \( p \) and \( \phi_1 \) is a positive first eigenfunction of the problem \( -\Delta \phi = \lambda \phi \) under Dirichlet boundary condition in \( \Omega. \) If \( p \) is a strict local maximum point of \( \phi_1, \) we show that such a problem has a family of solutions \( v_s \) with arbitrary \( m \) bubbles accumulating to \( p, \) and the quantity \( \int_{\Omega} e^{v_s} \to 8\pi(m + 1 + \alpha)\phi_1(p) \) as \( s \to +\infty. \)

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1. Introduction

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with smooth boundary. This paper deals with the analysis of solutions in the distributional sense for the following problem involving a singular source

\[
\begin{cases}
- \Delta v = e^v - s \phi_1 - 4\pi \alpha \delta_p - h(x) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( s > 0 \) is a large parameter, \( p \in \Omega, \alpha \in (-1, +\infty) \setminus \mathbb{N} \), \( \delta_p \) denotes the Dirac measure supported at point \( p \), \( h \in C^{0,\gamma}(\Omega) \) is given, \( \phi_1 > 0 \) is an eigenfunction of \(-\Delta\) with Dirichlet boundary condition corresponding to the first eigenvalue \( \lambda_1 \). Clearly, if we set \( \rho(x) = (-\Delta)^{-1} h \) in \( H^1_{0}(\Omega) \) and let \( G(x,y) \) be the Green’s function associated to \(-\Delta\) with Dirichlet boundary condition, namely

\[
\begin{cases}
- \Delta_x G(x,y) = 8\pi \delta_y(x), & x \in \Omega, \\
G(x,y) = 0, & x \in \partial \Omega,
\end{cases}
\]

and \( H(x,y) \) be its regular part defined as

\[
H(x,y) = G(x,y) - 4 \log \frac{1}{|x-y|},
\]

then equation (1.1) is equivalent to solving for \( u = v + \frac{1}{N} \phi_1 + \frac{2}{\pi} G(., p) + \rho \), the problem

\[
\begin{cases}
- \Delta u = |x-p|^{2\alpha} k(x)e^{-t \phi_1} e^u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( k(x) = e^{-\rho(x)-\frac{\alpha}{2} H(x,p)} \) and \( t = s/\lambda_1 \). We are interested in the existence of solutions of problem (1.4) (or (1.1)) which exhibit the concentration phenomenon when the parameter \( t \to +\infty \).

This work is directly motivated by the study of the regular case \( \alpha = 0 \) in equation (1.1), namely the following elliptic equation of Ambrosetti-Prodi type [1]:

\[
\begin{cases}
- \Delta v = e^v - s \phi_1 - h(x) & \text{in } \Omega, \\
v = 0 & \text{on } \partial \Omega,
\end{cases}
\]

or its equivalent form

\[
\begin{cases}
- \Delta u = k(x)e^{-t \phi_1} e^u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^N (N \geq 2) \). In the early 1980s, Lazer and McKenna conjectured that (1.5) has an unbounded number of solutions as \( s \to +\infty \) (see [2]). When \( N = 2 \), del Pino and Muñoz [3] proved the Lazer-McKenna conjecture for problem (1.5) by constructing non-simple bubbling solutions of (1.6) with the following properties

\[
k(x)e^{-t \phi_1} e^{u_t} \to 8\pi \sum_{i=1}^{m} m_i \delta_{\xi_i} \quad \text{and} \quad u_t = \sum_{i=1}^{m} m_i G(x, \xi_i) + o(1),
\]

where \( m_i > 1 \) and \( \xi_i \)'s are distinct maxima of \( \phi_1 \). Surprising enough, this multiple bubbling phenomenon is in strong opposition to a slightly modified but widely studied version of equation (1.6), namely the Liouville-type equation or sometimes referred to as the Gelfand equation

\[
\begin{cases}
- \Delta u = \varepsilon^2 k(x)e^u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
with \( \varepsilon \to 0 \), where \( \Omega \subset \mathbb{R}^2 \) is a bounded smooth domain and \( k(x) \in C^2(\overline{\Omega}) \) is a non-negative, not identically zero function. Indeed, the asymptotic analysis in [4, 5, 6, 7] shows that if \( u_\varepsilon \) is an unbounded family of solutions of equation (1.8) for which \( \varepsilon^2 \int_{\Omega} k(x)e^{u_\varepsilon} \) is uniformly bounded, then, up to a subsequence, there exists an integer \( m \geq 1 \) such that \( u_\varepsilon \) makes \( m \) distinct points simple blow-up on \( \mathcal{S} = \{\xi_1, \ldots, \xi_m\} \subset \Omega \) with \( \Omega = \{ x \in \Omega | k(x) > 0 \} \), more precisely

\[
\varepsilon^2 k(x) e^{u_\varepsilon} \to 8\pi \sum_{i=1}^{m} \delta_{\xi_i} \quad \text{and} \quad u_\varepsilon = \sum_{i=1}^{m} G(x, \xi_i) + o(1). \quad (1.9)
\]

Also the location of \( m \)-tuple of these bubbling points can be viewed as a critical point of a functional in terms of the Green’s function and its regular part. Conversely, the existence of solutions for equation (1.8) with these bubbling behaviors has been founded in [8, 9, 10, 11, 12]. In particular, the construction of solutions with arbitrary \( m \) distinct bubbling points is achieved in some special cases: for any \( m \geq 1 \) if \( \Omega \) is not simply connected ([10]), and for any \( m \in \{1, \ldots, h\} \) provided that \( \Omega \) is an \( h \)-dumbell with thin handles ([11]). Finally, we mention that in the recent paper [13], it has been proven that the Lazer-McKenna conjecture also holds true for problem (1.5) in dimension \( N \geq 3 \) with some symmetries, by constructing non-simple bubbling solutions to a two-dimensional anisotropic version of (1.6).

Our motivation also directly comes from the study of a slightly modified version of equation (1.1) (or (1.4)), namely the singular Liouville equation

\[
\begin{aligned}
-\Delta v &= \varepsilon^2 e^v - 4\pi \alpha \delta_p - h(x) \quad \text{in} \quad \Omega, \\
v &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\quad (1.10)
\]

or its equivalent form

\[
\begin{aligned}
-\Delta u &= \varepsilon^2 k(x)|x-p|^{2\alpha} e^{u} \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\quad (1.11)
\]

with \( \Omega \subset \mathbb{R}^2, -1 < \alpha \neq 0 \) and \( \varepsilon \to 0 \). This type of singular equation arises in the Tur-Yanovsky vortex pattern of planar stationary Euler equations for an incompressible and homogeneous fluid [14, 15], the construction of planar conformal metrics with conical singularity of order \( \alpha \) [16], and the several superconductivity theories of the self-dual regime, such as the Abelian Maxwell-Higgs and Chern-Simons-Higgs theories [17, 18].

For equation (1.10) involving \( \alpha > 0 \), solutions with \( m \) distinct bubbling points away from the singular source \( p \) have been founded first in [10] provided that \( m < 1 + \alpha \). Later in [19], this result has been extended to the case of multiple singular sources, and more specifically, it is shown that, under suitable restrictions on the weights, if several sources exist, then the more involved topology should generate a larger number of bubbling solutions than the singleton case considered in [10]. However, the problem of finding solutions of (1.10)\( |\alpha > 0 \) with additional bubbles around the singular source \( p \) is of different nature. Indeed, the asymptotic analysis in [20, 21] shows that if these solutions exist, then the bubble at the singularity provides an additional contribution of \( 8\pi(1 + \alpha)\delta_p \) in the limit of (1.9). More precisely, if \( u_\varepsilon \) is an unbounded family of solutions of (1.11)\( |\alpha > 0 \) for which \( \varepsilon^2 \int_{\Omega} k(x)|x-p|^{2\alpha} e^{u_\varepsilon} \) is uniformly bounded and \( u_\varepsilon \) is unbounded in any neighborhood of \( p \), then, up to a subsequence, there exists an integer \( m \geq 0 \) such that \( u_\varepsilon \) makes \( m + 1 \) distinct points simple blow-up on \( \mathcal{S} = \{ p, \xi_1, \ldots, \xi_m \} \subset \Omega \), namely

\[
\varepsilon^2 k(x)|x-p|^{2\alpha} e^{u_\varepsilon} \to 8\pi(1 + \alpha)\delta_p + 8\pi \sum_{i=1}^{m} \delta_{\xi_i} \quad \text{and} \quad u_\varepsilon = (1 + \alpha)G(x, p) + \sum_{i=1}^{m} G(x, \xi_i) + o(1). \quad (1.12)
\]

Moreover, the location of the \( m \) distinct points \( \xi_1, \ldots, \xi_m \in \Omega \setminus \{ p \} \) can be characterized as a critical point of some certain functional in terms of the Green’s function and its regular part. Reciprocally, the construction of solutions of equation (1.10)\( |\alpha > 0 \) with bubbles around \( p \) has been carried out first in [20] for the case \( \alpha \in (0, +\infty) \setminus \mathbb{N} \), later in [15] for the case \( \alpha \in \mathbb{N} \) where given any positive integer \( \alpha \) and any sufficiently small complex number \( a \), it is proven that there exists a solution of equation (1.10) with \( h(x) \equiv 0 \) and \( \delta_p \) replaced by \( \delta_{p_{n, \varepsilon}} \) for a suitable
\[ p_{a,\varepsilon} \in \Omega \text{ with } \alpha + 1 \text{ bubbling points at the vertices of a sufficiently tiny regular polygon centered in point } p_{a,\varepsilon}; \]

moreover \( p_{a,\varepsilon} \) lies close to a zero point of a vector field explicitly built upon derivatives of order \( \alpha + 1 \) of the regular part of Green’s function of the domain. Recently, for equation (1.11) with \( \alpha \in \mathbb{N} \) and the potential \( k(x) \) replaced by \( a(x)e^{-\frac{1}{2}(\alpha-1)H(x,p)} \), it has been proven in [22] that if the local potential \( a(x) \) and the geometry of the domain satisfy some conditions at the singular source \( p \), then there exists a solution \( u_{\varepsilon} \) bubbling only at \( p \) and satisfying \( \varepsilon^2 \int_\Omega a(x)e^{-\frac{1}{2}(\alpha-1)H(x,p)}|x - p|^{2\alpha}e^{\varepsilon x} \to 8\pi(1 + \alpha) \) as \( \varepsilon \to 0 \).

In the present paper, we consider the singular case of problem (1.1) (or (1.4)) involving \( \alpha \in (-1, +\infty) \setminus \mathbb{N} \) and try to prove the existence of its non-simple bubbling solutions in a constructive way. We find that if the singular source \( p \) is a strict local maximum point of \( \phi_1 \) in the domain, then problem (1.4) (or (1.1)) has a family of solutions with the accumulation of arbitrarily many bubbles at source \( p \). This can be stated as following:

**Theorem 1.1.** Let \( \alpha \in (-1, +\infty) \setminus \mathbb{N} \) and assume that \( p \) is a strict local maximum point of \( \phi_1(x) \) in \( \Omega \). Then for any integer \( m \geq 1 \), there exists \( t_m > 0 \) such that for any \( t > t_m \), problem (1.4) has a family of solutions \( u_t \) satisfying

\[
u_t(x) = \left[ \log \left( \frac{2}{\varepsilon_0^2/\mu_0^2 + |x - p|^{2(1 + \alpha)}^2} + (1 + \alpha)H(x, p) \right) + \sum_{i=1}^{m} \left[ \log \left( \frac{2}{\varepsilon_i^2/\mu_i^2 + |x - \xi_{i,t}|^{2}^2} + H(x, \xi_{i,t}) \right) \right] + o(1),\]

where \( o(1) \to 0 \), as \( t \to +\infty \), uniformly on each compact subset of \( \Omega \setminus \{p, \xi_1, t, \ldots, \xi_m, t\} \), the parameters \( \varepsilon_{0,t}, \varepsilon_{i,t}, \mu_{0,t} \) and \( \mu_{i,t} \) satisfy

\[
\varepsilon_{0,t} = e^{-\frac{1}{2}\phi_1(p)}, \quad \varepsilon_{i,t} = e^{-\frac{1}{2}\phi_1(\xi_{i,t})}, \quad \frac{1}{C} \leq \mu_{0,t} \leq C\varepsilon^{2m\beta}, \quad \frac{1}{C} \leq \mu_{i,t} \leq C\varepsilon^{2m+\alpha},
\]

for some \( C > 0 \), and \( (\xi_{1,t}, \ldots, \xi_{m,t}) \in \Omega^m \) satisfies

\[
\xi_{i,t} \to p \text{ for all } i, \quad \text{and} \quad |\xi_{i,t} - \xi_{j,t}| > t^{-\beta} \quad \forall \ i \neq j,
\]

with \( \beta = \frac{1}{2}(m + 1)(m + 1 + \alpha) \).

The equivalent result for problem (1.1) can be stated in the following form.

**Theorem 1.2.** Let \( \alpha \in (-1, +\infty) \setminus \mathbb{N} \) and assume that \( p \) is a strict local maximum point of \( \phi_1(x) \) in \( \Omega \). Then for any integer \( m \geq 1 \) and any \( s \) large enough, there exists a family of solutions \( v_s \) of problem (1.1) with \( m \) distinct bubbles accumulating to \( p \). Moreover,

\[
\lim_{s \to +\infty} \int_\Omega e^{v_s} = 8\pi(1 + \alpha)\phi_1(p).
\]

Moreover, for the case \( m = 0 \), we have the corresponding results for problems (1.1) and (1.4), respectively.

**Theorem 1.3.** Let \( \alpha \in (-1, +\infty) \setminus \mathbb{N} \). Then there exists \( t_0 > 0 \) such that for any \( t > t_0 \), problem (1.4) has a family of solutions \( u_t \) such that as \( t \) tends to +\infty,

\[
u_t(x) = \left[ \log \left( \frac{2}{\mu_0^2e^{-\phi_1(p)} + |x - p|^{2(1 + \alpha)}^2} + (1 + \alpha)H(x, p) \right) + o(1),\right.
\]

uniformly on each compact subset of \( \Omega \setminus \{p\} \), where the parameter \( \mu_0 \) satisfies \( 1/C \leq \mu_0 \leq C \) for some \( C > 0 \).

**Theorem 1.4.** Let \( \alpha \in (-1, +\infty) \setminus \mathbb{N} \). Then for any \( s \) large enough, there always exists a family of solutions \( v_s \) of problem (1.1) such that

\[
\lim_{s \to +\infty} \int_\Omega e^{v_s} = 8\pi(1 + \alpha)\phi_1(p).
\]
According to Theorems 1.1 and 1.2, it follows that if the singular source \( p \) is an isolated local maximum point of \( \phi_1 \), then for any integer \( m \geq 1 \) there exists a family of solutions of problem (1.4) which exhibits the phenomenon of \( m + 1 \)-bubbling at \( p \), namely, 
\[
|x - p|^{2\alpha} k(x) e^{-t\phi_1} e^{u_t} \rightarrow 8\pi (m + 1 + \alpha) \delta_p \quad \text{and} \quad u_t = (m + 1 + \alpha) G(x, p) + o(1).
\]
While for the case \( m = 0 \), by arguing exactly along the sketch of the proof of Theorem 1.1 we can prove the corresponding results in Theorems 1.3 and 1.4, and further find that problem (1.4) should always admit a family of solutions blowing up at the singular source \( p \) whether \( p \) is an isolated local maximum point of \( \phi_1 \) or not.

The strategy for proving our main results relies on a very well-known Lyapunov-Schmidt reduction procedure. In Section 2 we exactly describe the ansatz for the solution of problem (1.4) and rewrite problem (1.4) in terms of a linearized operator for which a solvability theory, subject to suitable orthogonality conditions, is performed through solving a linearized problem in Section 3. In Section 4 we solve a nonlinear projected problem. In Section 5 we set up a maximization problem. In the last section we show that the solution to the maximization problem indeed yields a solution of problem (1.4) with the qualitative properties as predicted in Theorem 1.1.

Throughout this paper, the symbol \( C \) will always denote a generic positive constant independent of \( t \), it could be changed from one line to another.

### 2. Ansatz for the solution

In this section we will provide an ansatz for solutions of problem (1.4). For the sake of convenience we always fix the point \( p \) as an isolated local maximum point of \( \phi_1 \) in \( \Omega \), and further assume
\[
\phi_1(p) = 1.
\]

The configuration space for \( m \) concentration points \( \xi = (\xi_1, \ldots, \xi_m) \) we try to seek is the following
\[
\mathcal{O}_t := \left\{ \xi = (\xi_1, \ldots, \xi_m) \in (B_d(p))^m \left| \xi_i - p \geq \frac{1}{2^t}, \ |\xi_i - \xi_j| \geq \frac{1}{2^t}, \ 1 - \phi_1(\xi_i) \leq \frac{1}{\sqrt{t}} \right. \right\},
\]
where \( d > 0 \) is a sufficiently small but fixed number, independent of \( t \), and \( \beta \) is given by
\[
\beta = \frac{(m + 1)(m + 1 + \alpha)}{2}.
\]

Let us fix \( \xi \in \mathcal{O}_t \). For numbers \( \mu_0 > 0 \) and \( \mu_i > 0 \), \( i = 1, \ldots, m \), yet to be determined, we define
\[
u_0(x) = \log \frac{8\mu_0^2(1 + \alpha)^2}{k(p)(\varepsilon_0^2 \mu_0^2 + |x - p|^{2(1 + \alpha)})^2}, \quad \nu_i(x) = \log \frac{8\mu_i^2}{k(\xi_i)|\xi_i - p|^{2\alpha}(\varepsilon_i^2 \mu_i^2 + |x - \xi_i|^{2(1 + \alpha)})^2},
\]
which satisfy in entire \( \mathbb{R}^2 \)
\[
-\Delta \nu_0 = \varepsilon_0^2 k(p)|x - p|^{2\alpha} e^{\nu_0}, \quad -\Delta \nu_i = \varepsilon_i^2 k(\xi_i)|\xi_i - p|^{2\alpha} e^{\nu_i},
\]
having the properties
\[
\int_{\mathbb{R}^2} \varepsilon_0^2 k(p)|x - p|^{2\alpha} e^{\nu_0} = 8\pi(1 + \alpha), \quad \int_{\mathbb{R}^2} \varepsilon_i^2 k(\xi_i)|\xi_i - p|^{2\alpha} e^{\nu_i} = 8\pi,
\]
where
\[
\varepsilon_0 = \varepsilon_0(t) \equiv e^{-\frac{1}{2^t}}, \quad \varepsilon_i = \varepsilon_i(t) \equiv e^{-\frac{1}{2^t} \phi_1(\xi_i)}.
\]

Our ansatz is then
\[
U(x) := \sum_{i=0}^m U_i(x) = \sum_{i=0}^m \left[ u_i(x) + H_i(x) \right],
\]
where \( H_i(x) \) is a correction term defined as the solution of
\[
\begin{align*}
\Delta H_i &= 0 \quad \text{in} \quad \Omega, \\
H_i &= -u_i \quad \text{on} \quad \partial \Omega.
\end{align*}
\]


Lemma 2.1. For any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \) and any \( t \) large enough, then we have

\[
H_0(x) = (1 + \alpha)H(x, p) - \log \frac{8\mu_0^2(1 + \alpha)^2}{k(p)} + O \left( \varepsilon_0^2 \mu_0^2 \right),
\]

and

\[
H_i(x) = H(x, \xi_i) - \log \frac{8\mu_i^2}{k(\xi_i)|\xi_i - p|^{2\alpha}} + O \left( \varepsilon_0^2 \mu_i^2 \right), \quad i = 1, \ldots, m,
\]

uniformly in \( \overline{\Omega} \), where \( H \) is the regular part of Green’s function defined in (1.3).

Proof. If we set \( z(x) = H_0(x) - (1 + \alpha)H(x, p) + \log \frac{8\mu_0^2(1 + \alpha)^2}{k(p)} \), then \( z(x) \) is a harmonic function. Hence by (1.3), (2.4), (2.9) and the maximum principle,

\[
\max_{x \in \overline{\Omega}} |z(x)| = \max_{x \in \partial \Omega} \left| -u_0(x) - 4(1 + \alpha) \log |x - p| + \log \frac{8\mu_0^2(1 + \alpha)^2}{k(p)} \right|
\]

\[
= \max_{x \in \partial \Omega} \left| \log \frac{1}{|x - p|^{4(1 + \alpha)}} - \log \frac{1}{(\varepsilon_0^2 \mu_0^2 + |x - p|^{2(\alpha + 1)})} \right| = O \left( \varepsilon_0^2 \mu_0^2 \right),
\]

uniformly in \( \overline{\Omega} \), as \( s \to +\infty \), which implies that expansion (2.10) holds. Furthermore, expansion (2.11) can be also obtained along these analogous arguments of (2.10). \( \square \)

Observe that \( u_0 \) and \( u_i, \) \( i = 1, \ldots, m \) are good approximations for a solution of problem (1.4) near points \( p \) and \( \xi_i \), \( i = 1, \ldots, m \), respectively. We expect that the ansatz in (2.8) is more accurate near \( p \) and each \( \xi_i \), namely the remainders \( U - u_0 = H_0 + \sum_{j \neq 0} (u_j + H_j) \) and \( U - u_i = H_i + \sum_{j \neq i} (u_j + H_j) \) vanish at main order near \( p \) or \( \xi_i \), respectively. This can be achieved by following the preceding precise choices of the concentration parameters \( \mu_0 \) and \( \mu_i \):

\[
\log \frac{8\mu_0^2(1 + \alpha)^2}{k(p)} = (1 + \alpha)H(p, p) + \sum_{j=1}^m G(p, \xi_j),
\]

(2.12)

\[
\log \frac{8\mu_i^2}{k(\xi_i)|\xi_i - p|^{2\alpha}} = H(\xi_i, \xi_i) + (1 + \alpha)G(\xi_i, p) + \sum_{j=1, j \neq i}^m G(\xi_i, \xi_j), \quad i = 1, \ldots, m.
\]

(2.13)

We thus fix \( \mu_0 \) and \( \mu_i \) a priori as functions of \( \xi \) in \( \mathcal{O}_t \) and write \( \mu_0 = \mu_0(\xi) \) and \( \mu_i = \mu_i(\xi) \) for all \( i = 1, \ldots, m \). Since \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \), there exists a constant \( C > 0 \) independent of \( t \) such that

\[
\frac{1}{C} \leq \mu_0 \leq Ct^{2m\beta} \quad \text{and} \quad |\partial_{\xi_k} \log \mu_0| \leq Ct^{\beta}, \quad \forall \ k = 1, \ldots, m, \ l = 1, 2,
\]

(2.14)

and

\[
\frac{1}{C} \leq \mu_i \leq Ct^{(2m+\alpha)\beta} \quad \text{and} \quad |\partial_{\xi_k} \log \mu_i| \leq Ct^{\beta}, \quad \forall \ i, k = 1, \ldots, m, \ l = 1, 2.
\]

(2.15)

Consider now the change of variables

\[
\omega(y) = u(\varepsilon_0y) - 2t \quad \forall \ y \in \overline{\Omega}_t,
\]

(2.16)

with

\[
\Omega_t = \varepsilon_0^{-1} \Omega = e^{-t} \Omega,
\]

then \( u(x) \) solves equation (1.4) if and only if \( \omega(y) \) satisfies

\[
\begin{cases}
-\Delta \omega = |\varepsilon_0y - p|^{2\alpha} q(y, t) e^\omega & \text{in } \Omega_t, \\
\omega = -2t & \text{on } \partial \Omega_t,
\end{cases}
\]

(2.17)

where

\[
q(y, t) \equiv k(\varepsilon_0y) \exp \left\{ -t \left[ \phi_1(\varepsilon_0y) - 1 \right] \right\}.
\]

(2.18)
Let us write \( p' = p/\varepsilon_0 \) and \( \xi'_i = \xi_i/\varepsilon_0, \ i = 1, \ldots, m \), and define the initial approximate solution of \((2.17)\) as

\[
V(y) = U(\varepsilon_0 y) - 2t,
\]

where \( U \) is defined in \((2.8)\). Moreover, set

\[
W(y) = |\varepsilon_0 y - p|^{2\alpha} q(y, t) e^{V(y)},
\]

and the “error term”

\[
E(y) = \Delta V(y) + |\varepsilon_0 y - p|^{2\alpha} q(y, t) e^{V(y)}.
\]

Let us see how well \(-\Delta V(y)\) match with \(W(y)\) through \(V(y)\) so that the “error term” \(E(y)\) is sufficiently small for any \(y \in \Omega_t\). A simple computation shows that

\[
-\Delta V(y) = -\varepsilon_0^2 \Delta_x U(x) = -\varepsilon_0^2 \sum_{i=0}^{m} \Delta_x U_i(x) = -\varepsilon_0^2 \sum_{i=0}^{m} \Delta_x [v_i(x) + H_i(x)]
\]

\[
= \varepsilon_0^3 k(p)|x - p|^{2\alpha} e^{u_0} + \varepsilon_0^2 \sum_{i=1}^{m} \varepsilon_0^2 k(\xi_i)|\xi_i - p|^{2\alpha} e^{\mu_i}
\]

\[
= \frac{8\varepsilon_0^4 \mu_0^2 (1 + \alpha)^2 |x - p|^{2\alpha}}{\varepsilon_0 \mu_0^2 + |x - p|^{2(1 + \alpha)}} + \sum_{i=1}^{m} \frac{8\varepsilon_0^2 \mu_i^2}{\varepsilon_0 \mu_i^2 + |x - \xi_i|^{2(1 + \alpha)}}
\]

\[
= \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{8(1 + \alpha)^2 |x - p|^{2\alpha}}{(1 + |x - p|^{2(1 + \alpha)})^2} + \sum_{i=1}^{m} \frac{1}{\gamma_i^2 (1 + \|x - \xi_i^2\|^2)^2},
\]

where

\[
\rho_0 := \varepsilon_0^{-1/\alpha} = \exp \left\{ -\frac{1}{2(1 + \alpha)} t \right\}, \quad v_0 := \mu_0^{1/\alpha}, \quad \gamma_i := \frac{1}{\varepsilon_0} \varepsilon_i \mu_i = \mu_i \exp \left\{ -\frac{1}{2} t [\phi_1(\xi_i) - 1] \right\}.
\]

Then if \(|y - \xi_i| \leq 1/(\varepsilon_0 t^{2\beta})\) for some index \(i \in \{1, \ldots, m\}\),

\[
-\Delta V(y) = \frac{1}{\gamma_i^2 (1 + |y - \xi_i|^2)} + O \left( \varepsilon_0^4 \mu_0^2 t^{4+2\alpha} \right) + \sum_{j=1, j \neq i} O \left( \varepsilon_0^4 \mu_i^2 t^{4+2\alpha} \right),
\]

and if \(|y - p'| \leq 1/(\varepsilon_0 t^{2\beta})\),

\[
-\Delta V(y) = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{8(1 + \alpha)^2 |x - p|^{2\alpha}}{(1 + |x - p|^{2(1 + \alpha)})^2} + \sum_{j=1}^{m} O \left( \varepsilon_0^2 \mu_j^2 t^{4+2\alpha} \right),
\]

while if \(|y - p'| > 1/(\varepsilon_0 t^{2\beta})\) and \(|y - \xi'_i| > 1/(\varepsilon_0 t^{2\beta})\) for all \(i = 1, \ldots, m\),

\[
-\Delta V(y) = O \left( \varepsilon_0^4 \mu_0^2 t^{8+4\alpha} \right) + \sum_{j=1}^{m} O \left( \varepsilon_0^2 \mu_j^2 t^{8+2\alpha} \right).
\]
On the other hand, let us first fix the index \(i \in \{1, \ldots, m\}\) and the region \(|y - \xi'_i| \leq 1/(\varepsilon_0 t^{2\beta})\). Then we have

\[
W(y) = \varepsilon_0^4 |\varepsilon_0 y - p|^{2\alpha} \exp \left\{ \sum_{j=0}^m \left[ u_j(\varepsilon_0 y) + H_j(\varepsilon_0 y) \right] \right\}
\]

\[
= \varepsilon_0^4 k(\varepsilon_0 y)|\varepsilon_0 y - p|^{2\alpha} \exp \left\{ -t \left[ \phi_1(\varepsilon_0 y) - 1 \right] + \log \frac{8\mu^2}{k(\xi_i)|\xi_i - p|^{2\alpha} (\varepsilon_0^4 \mu^2 + \varepsilon_0^4 |y - \xi'_i|)^2} \right\}
\]

\[
+ H_i(\varepsilon_0 y) + \sum_{j=0, j \neq i}^m \left[ u_j(\varepsilon_0 y) + H_j(\varepsilon_0 y) \right] \right\}
\]

\[
= \frac{1}{\gamma^2_i} \left( 1 + \frac{|y - \xi'_i|^2}{\gamma^2_i} \right)^2 \times \frac{k(\varepsilon_0 y)}{k(\xi_i)} \times \frac{|\varepsilon_0 y - p|^{2\alpha}}{|\xi_i - p|^{2\alpha}} \times \exp \left\{ -t \left[ \phi_1(\varepsilon_0 y) - \phi_1(\xi_i) \right] \right\}
\]

\[
\times \exp \left\{ H_i(\varepsilon_0 y) + \sum_{j=0, j \neq i}^m \left[ u_j(\varepsilon_0 y) + H_j(\varepsilon_0 y) \right] \right\}.
\]

(2.27)

From (2.4), (2.10), (2.11) and the fact that \(H(\cdot, x)\) is \(C^1(\Omega)\) for any \(x \in \Omega\), we have that for \(|y - \xi'_i| \leq 1/(\varepsilon_0 t^{2\beta})\),

\[
H_i(\varepsilon_0 y) + \sum_{j=0, j \neq i}^m \left[ u_j(\varepsilon_0 y) + H_j(\varepsilon_0 y) \right]
\]

\[
= H(\xi_i, \xi_i) \left( 1 + \alpha \right) \left[ \log \frac{1}{|\xi_i - p|^{2\alpha}} + H(\xi_i, p) + O \left( \frac{|\varepsilon_0 y - \xi'_i|}{|\xi_i - \xi'_i|} + \frac{\varepsilon_0^2 \mu^2}{|\xi_i - p|^{2(1+\alpha)}} \right) \right]
\]

\[
+ \sum_{j=1, j \neq i}^m \left[ \log \frac{1}{|\xi_i - \xi'_j|^{2\alpha}} + H(\xi_i, \xi_j) + O \left( \frac{|\varepsilon_0 y - \xi'_i|}{|\xi_i - \xi'_j|} + \frac{\varepsilon_0^2 \mu^2}{|\xi_i - \xi'_j|^{2\alpha}} \right) \right] + O(\varepsilon_0 |y - \xi'_i|) + \sum_{j=0}^m O(\varepsilon_0^2 \mu^2)
\]

\[
= H(\xi_i, \xi_i) \left( 1 + \alpha \right) \left[ \log \frac{1}{|\xi_i - p|^{2\alpha}} + H(\xi_i, p) + O \left( \frac{\varepsilon_0^2 \mu^2}{|\xi_i - p|^{2(1+\alpha)}} \right) \right]
\]

\[
+ \sum_{j=1, j \neq i}^m \left[ \log \frac{1}{|\xi_i - \xi'_j|^{2\alpha}} + H(\xi_i, \xi_j) + O \left( \frac{\varepsilon_0^2 \mu^2}{|\xi_i - \xi'_j|^{2\alpha}} \right) \right] + O(\varepsilon_0 |y - \xi'_i|) + \sum_{j=0}^m O(\varepsilon_0^2 \mu^2)
\]

(2.28)

where the last equality is due to the choice of \(\mu_i\) in (2.13). Thus if \(|y - \xi'_i| \leq 1/(\varepsilon_0 t^{2\beta})\) for some \(i \in \{1, \ldots, m\}\),

\[
W(y) = \frac{1}{\gamma_i^2} \left( 1 + \frac{|y - \xi'_i|^2}{\gamma_i^2} \right)^2 \left( 1 + O(\varepsilon_0 |y - \xi'_i|) + O(\varepsilon_0^2 \mu^2) + O(\varepsilon_0^4 \mu^{2(1+\alpha)}) + \sum_{j=1, j \neq i}^m O(\varepsilon_0^2 \mu_j^{2(1+\alpha)}) \right),
\]

(2.28)

and by (2.24),

\[
E(y) = \frac{1}{\gamma_i^2} \left( 1 + \frac{|y - \xi'_i|^2}{\gamma_i^2} \right)^2 \left( O(\varepsilon_0 |y - \xi'_i|) + O(\varepsilon_0^2 \mu_i^2) + O(\varepsilon_0^4 \mu_{0j}^{2(1+\alpha)}) + \sum_{j=1, j \neq i}^m O(\varepsilon_0^2 \mu_j^{2(1+\alpha)}) \right)
\]

\[
+ O(\varepsilon_0^4 \mu_0^{2(4+2\alpha)}) + \sum_{j=1, j \neq i}^m O(\varepsilon_0^2 \mu_j^{2(4+2\alpha)})
\]

(2.29)
Similarly, if $|y - p'| \leq 1/(\varepsilon_0 t^{2\beta})$, by (2.1), (2.4), (2.10), (2.11) and (2.12) we can compute

$$W(y) = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \left[ \frac{8(1 + \alpha)^2}{(1 + |\varepsilon_0 y - p'|^{2(1 + \alpha)})^2} \left[ 1 + O\left( \varepsilon_0 t^\beta |y - p'| \right) + O\left( \varepsilon_0^2 \mu_0^2 \right) + \sum_{j=1}^{m} O\left( \varepsilon_0^2 \mu_j^2 t^{2\beta} \right) \right] \right],$$

(2.30)

and by (2.25),

$$E(y) = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \left[ \frac{8(1 + \alpha)^2}{(1 + |\varepsilon_0 y - p'|^{2(1 + \alpha)})^2} \left[ O\left( \varepsilon_0 t^\beta |y - p'| \right) + O\left( \varepsilon_0^2 \mu_0^2 \right) + \sum_{j=1}^{m} O\left( \varepsilon_0^2 \mu_j^2 t^{2\beta} \right) \right] \right],$$

(2.31)

$$+ \sum_{j=1}^{m} O\left( \varepsilon_0^2 \mu_j^2 t^{2\beta} \right).$$

While if $|y - p'| > 1/(\varepsilon_0 t^{2\beta})$ and $|y - \xi_i| > 1/(\varepsilon_0 t^{2\beta})$ for all $i = 1, \ldots, m$, by (2.4), (2.10) and (2.11) we obtain

$$W(y) = O\left( \frac{\varepsilon_0^2 e^{-\phi y(y_0)}}{|\varepsilon_0 y - p'|^{1+2\alpha}} \prod_{i=1}^{m} \frac{1}{|\varepsilon_0 y - \xi_i|^4} \right).$$

(2.32)

Then by (2.26),

$$E(y) = O\left( \frac{\varepsilon_0^2 e^{-\phi y(y_0)}}{|\varepsilon_0 y - p'|^{1+2\alpha}} \prod_{i=1}^{m} \frac{1}{|\varepsilon_0 y - \xi_i|^4} \right) + O\left( \varepsilon_0^2 \mu_0^2 t^{8+4\alpha} \right) + \sum_{j=1}^{m} O\left( \varepsilon_0^2 \mu_j^2 t^{8\beta} \right).$$

(2.33)

In what remains of this paper we will seek solutions of problem (2.17) in the form $\omega = V + \phi$, where $\phi$ will represent a lower order correction. In terms of $\phi$, problem (2.17) becomes

$$\begin{cases}
\mathcal{L}(\phi) := -\Delta \phi - W \phi = E + N(\phi) \quad \text{in} \quad \Omega_t, \\
\phi = 0 \quad \text{on} \quad \partial \Omega_t,
\end{cases}$$

(2.34)

where the “nonlinear term” is given by

$$N(\phi) = W(e^\phi - 1 - \phi).$$

(2.35)

3. Solvability of a Linear Problem

In this section we consider the solvability of the following linear problems: given $h \in L^\infty(\Omega_t)$ and points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t$, we find a function $\phi$ such that for certain scalars $c_{ij}, i = 1, \ldots, m$, $j = 1, 2$, one has

$$\begin{cases}
\mathcal{L}(\phi) = -\Delta \phi - W \phi = h + \sum_{i=1}^{m} \sum_{j=1}^{2} c_{ij} \chi_i Z_{ij} \quad \text{in} \quad \Omega_t, \\
\phi = 0 \quad \text{on} \quad \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{ij} \phi = 0 \quad \forall \ i = 1, \ldots, m, \ j = 1, 2,
\end{cases}$$

(3.1)

where $W = |\varepsilon_0 y - p'|^{2\alpha} q(y, t)e^V$ satisfies (2.28), (2.30) and (2.32), and $Z_{ij}, \chi_i$ are defined as follows. Let $Z_p, Z_0, Z_1$ and $Z_2$ be

$$Z_p(z) = \frac{|z|^{2(1 + \alpha)} - 1}{|z|^{2(1 + \alpha)} + 1}, \quad Z_0(z) = \frac{|z|^2 - 1}{|z|^2 + 1}, \quad Z_j(z) = \frac{4z_j}{|z|^2 + 1}, \quad j = 1, 2.$$

(3.2)

It is well known that
any bounded solution to
\[ \Delta \phi + \frac{8(1 + \alpha)^2 |z|^2 \alpha}{(1 + |z|^{2(1+\alpha)})^2} \phi = 0 \quad \text{in } \mathbb{R}^2, \]  
(3.3)
where \(-1 < \alpha \in \mathbb{N}\), is proportional to \(Z_p\) (see [20, 23, 24]);

- any bounded solution to
  \[ \Delta \phi + \frac{8}{(1 + |z|^{2})^2} \phi = 0 \quad \text{in } \mathbb{R}^2, \]  
(3.4)
is a linear combination of \(Z_j\), \(j = 0, 1, 2\) (see [8, 9]).

Then we define
\[
Z_p(y) = \frac{\varepsilon_0}{\rho_0 v_0} Z_p \left( \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right) \quad \text{and} \quad Z_{ij}(y) = \frac{1}{\gamma_i} Z_j \left( \frac{y - \xi_i^j}{\gamma_i} \right), \quad i, j = 1, \ldots, m, \quad j = 0, 1, 2. \]  
(3.5)

Next, we consider a large but fixed positive number \(R_0\) and set a radial, smooth non-increasing cut-off function \(\chi(r)\) with \(0 \leq \chi(r) \leq 1\), \(\chi(r) = 1\) for \(r \leq R_0\) and \(\chi(r) = 0\) for \(r \geq R_0 + 1\). Let
\[
\chi_p(y) = \chi \left( \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right) \quad \text{and} \quad \chi_i(y) = \chi \left( \frac{y - \xi_i}{\gamma_i} \right), \quad i = 1, \ldots, m. \]  
(3.6)

Equation (3.1) will be solved for \(h \in L^\infty(\Omega_t)\), but we need to estimate the size of the solution in terms of the following \(L^\infty\)-weighted norm:
\[ \|h\|_{\ast} := \left\| \varepsilon_0 + \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \left( \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right)^2 + \sum_{i=1}^m \frac{1}{\gamma_i} \left( \frac{1}{1 + |y - \xi_i^j|^{1+\alpha}} \right) \right\|_{L^\infty(\Omega_t)}, \]  
(3.7)
where \(\alpha+1\) is a sufficiently small but fixed positive number, independent of \(t\), such that \(-1 < \alpha < \min \{\alpha, -2/3\}\).

**Proposition 3.1.** Let \(m\) be a positive integer. Then there exist constants \(t_m > 1\) and \(C > 0\) such that for any \(t > t_m\), any points \(\xi = (\xi_1, \ldots, \xi_m) \in \Omega_t\) and any \(h \in L^\infty(\Omega_t)\), there is a unique solution \(\phi \in L^\infty(\Omega_t)\), \(c_{ij} \in \mathbb{R}\), \(i = 1, \ldots, m, j = 1, 2\) to problem (3.1). Moreover
\[
\|\phi\|_{L^\infty(\Omega_t)} \leq C t \|h\|_{\ast}. \]  
(3.8)

**Proof.** The proof of this result consists of five steps which we state and prove next.

**Step 1:** The operator \(\mathcal{L}\) satisfies the maximum principle in \(\Omega_t := \Omega_t \setminus \left[ \bigcup_{i=1}^m B_{R_t, \gamma_i}^* (\xi_i^j) \cup B_{R_t, \rho_0 v_0 / \varepsilon_0}^* (p') \cup B_{2d, \varepsilon_0}^* (p') \right]\) for \(R_t\) large but \(d\) small independent of \(p\), namely if \(\psi\) satisfies \(\mathcal{L}(\psi) = -\Delta \psi - W \psi \geq 0\) in \(\Omega_t\), \(\psi \geq 0\) on \(\partial \Omega_t\), then \(\psi \geq 0\) in \(\Omega_t\). In order to prove it, we shall first find a function \(\mathcal{Z}\) such that \(\mathcal{L}(\mathcal{Z}) > 0\) and \(\mathcal{Z} > 0\) in \(\Omega_t\). Indeed, let
\[
\mathcal{Z}(y) = \Psi_0 (\varepsilon_0 y) + \tilde{\Psi}_p \left( \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right) + \sum_{i=1}^m \mathcal{Z}_0 \left( \frac{y - \xi_i^j}{\gamma_i} \right), \]  
(3.9)
where \(a > 0\), \(\Psi_0\) satisfies \(-\Delta \Psi_0 = 1\) in \(\Omega_t\), \(\Psi_0 = 2\) on \(\partial \Omega_t\), \(\mathcal{Z}_0\) is defined in (3.2) and
\[
\tilde{\Psi}_p(z) = \frac{\varepsilon_0^2 (1 + \hat{\alpha}) - 1}{\varepsilon_0^2 (1 + \hat{\alpha}) + 1}. \]

Observe that
\[
-\Delta \mathcal{Z} = \varepsilon_0^2 + \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \Delta \left( \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right) + \sum_{i=1}^m \frac{1}{\gamma_i} \left( \frac{8 a^2 \varepsilon_0^2 (1 + \hat{\alpha})}{(1 + |y - \xi_i^j|^{1+\alpha})^2} \right) \mathcal{Z}_0 \left( \frac{y - \xi_i^j}{\gamma_i} \right). \]
Then if $3^{1/(2+2\delta)}\rho_0 v_0/(\alpha e_0) \leq |y - p'| \leq 1/(\epsilon_0 t^{2\beta})$, by (2.30),
\[
\mathcal{L}(Z) \geq \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 a^2 \frac{4(1 + \hat{\alpha})^2}{(1 + |a\xi y - p|/|\rho_0 v_0|^{2(1+\alpha)})^2} \frac{25}{2} - W \|Z\|_{\infty}
\]
\[
\geq \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{(1 + \hat{\alpha})^2}{a^{2(1+\alpha)}} \frac{1}{|a\xi y - p|/|\rho_0 v_0|^{4+2\alpha}} - \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{8C(1 + \alpha)^2}{(1 + |a\xi y - p|/|\rho_0 v_0|^{4+2\alpha})^2}.
\]
\[
\geq \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{1}{|a\xi y - p|/|\rho_0 v_0|^{4+2\alpha}} \left[ \frac{(1 + \hat{\alpha})^2}{a^{2(1+\alpha)}} - \frac{8C(1 + \alpha)^2}{a^{2(1+\alpha)}} \right].
\]
Similarly, if $3^{1/2}\gamma_i/\alpha \leq |y - \xi_i| \leq 1/(\epsilon_0 t^{2\beta})$ for some index $i \in \{1, \ldots, m\}$, by (2.28),
\[
\mathcal{L}(Z) \geq \frac{1}{\gamma_i^2} \frac{1}{|y - \xi_i|^{4+2\alpha}} \left( \frac{1}{a^2} - 8C \right).
\]
While if $1/(\epsilon_0 t^{2\beta}) < |y - p'| < 2d/\epsilon_0$ and $|y - \xi_i| > 1/(\epsilon_0 t^{2\beta})$ for all $i = 1, \ldots, m$, by (2.32),
\[
\mathcal{L}(Z) \geq \epsilon_0^2 + \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{(1 + \hat{\alpha})^2}{a^{2(1+\alpha)}} \frac{1}{|a\xi y - p|/|\rho_0 v_0|^{4+2\alpha}} + \sum_{i=1}^{m} \frac{1}{\gamma_i^2} \frac{1}{a^2} |y - \xi_i|^{4+2\alpha} - C \frac{\varepsilon_0^2 e^{-\rho_1(\epsilon_0 y)}}{|\rho_0 y - p|^{4+2\alpha}} \prod_{i=1}^{m} \frac{1}{|\rho_0 y - \xi_i|}.
\]
Hence if $a$ is taken sufficiently small but fixed, and $R_1$ is chosen sufficiently large depending on this $a$, then by (2.23) we can easily conclude that $\mathcal{L}(Z) > 0$ in $\Omega_t$.

Next, we suppose that the operator $\mathcal{L}$ does not satisfy the maximum principle in $\Omega_t$. Since $Z > 0$ in $\Omega_t$, it follows that the function $\psi/Z$ has a negative minimum point $y_0$ in $\Omega_t$. A direct computation gives
\[
-\Delta \left( \frac{\psi}{Z} \right) = \frac{1}{\Delta t} [\mathcal{L}(\psi) - \psi \mathcal{L}(Z)] + \frac{1}{2} \nabla Z \nabla \left( \frac{\psi}{Z} \right).
\]
Then $-\Delta (\psi/Z)(y_0) > 0$, which contradicts to the fact that $y_0$ is a minimum point of $\psi/Z$ in $\Omega_t$.

**Step 2:** Let $R_1$ be as before. Since $\xi = (\xi_1, \ldots, \xi_m) \in \Omega_t$, $\rho_0 v_0 = o(1/t^{2\beta})$ and $\epsilon_0 \gamma_i = o(1/t^{2\beta})$ for $t$ large enough, we find $B_{R_1 \rho_0 v_0/\epsilon_0}(p')$ and $B_{R_1 t^{2\beta}}(\xi_i)$, $i = 1, \ldots, m$, disjointed and included in $\Omega_t$. Let us consider the following norm
\[
\|\phi\|_{**} = \sup_{y \in \bigcup_{i=1}^{m} B_{R_1 t^{2\beta}}(\xi_i) \cup B_{R_1 \rho_0 v_0/(\alpha e_0)}(p') \cup (\Omega_t \setminus B_{2d/\epsilon_0}(p'))} |\phi(y)|.
\]
We claim that there is a constant $C > 0$ independent of $t$ such that, if $\phi$ is the solution of the linear equation
\[
\begin{cases}
\mathcal{L}(\phi) = -\Delta \phi - W \phi = h & \text{in } \Omega_t, \\
\phi = 0 & \text{on } \partial \Omega_t,
\end{cases}
\]
then
\[
\|\phi\|_{L^\infty(\Omega_t)} \leq C (\|\phi\|_{**} + \|h\|_*)
\]
for any $h \in L^\infty(\Omega_t)$ and any points $\xi = (\xi_1, \ldots, \xi_m) \in \Omega_t$. We will establish this estimate with the use of suitable barriers. Let $M$ be a large number such that $\Omega \subset B(p, M)$ and $\Omega \subset B(\xi, M)$ for all $i = 1, \ldots, m$. Consider $\psi_0$ and $\psi_i$, $i = 1, \ldots, m$, respectively, as the solutions of the problems
\[
\begin{cases}
-\Delta \psi_0 = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{4}{|a\xi y - p|/|\rho_0 v_0|^{4+2\alpha}} + 4\varepsilon_0^2, \\
\psi_0(y) = 0, \quad \text{for } |a\xi y - p|/|\rho_0 v_0| = R_1, \\
\end{cases}
\]
where $M > \rho_0 v_0$. For $\psi_0$ and $\psi_i$, $i = 1, \ldots, m$, respectively, as the solutions of the problems
\[
\begin{cases}
-\Delta \psi_i = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{4}{|a\xi y - p|/|\rho_0 v_0|^{4+2\alpha}} + 4\varepsilon_0^2, \\
\psi_i(y) = 0, \quad \text{for } |a\xi y - p|/|\rho_0 v_0| = R_1, \\
\end{cases}
\]
for $M > \rho_0 v_0$.
Choosing $R_y$ and for $\psi_1(\gamma_i) = R_1$, we obtain that $\Delta \psi_1 = 1 \frac{4}{\gamma_i} < M_{\psi_1(\gamma_i)}$.

Then the solutions $\psi_0$ and $\psi_i, i = 1, \ldots, m$, are the positive functions, respectively given by

$$\psi_0(y) = -\left[ \frac{1}{(1 + \alpha)^2} \frac{\varepsilon_0 y - p_0}{\rho_0 v_0} \left( 1 + \frac{\alpha y - \alpha_0}{\gamma_i} \right)^2 \right] + \left[ \frac{1}{(1 + \alpha)^2} \frac{1}{R_1} + R_1 \right],$$

$$\psi_i(y) = -\left[ \frac{1}{(1 + \alpha)^2} \frac{\varepsilon_0 y - p_0}{\rho_0 v_0} \left( 1 + \frac{\alpha y - \alpha_0}{\gamma_i} \right)^2 \right] + \left[ \frac{1}{(1 + \alpha)^2} \frac{1}{R_i} + R_i \right], \quad i = 1, \ldots, m.$$

Clearly, the functions $\psi_0$ and $\psi_i, i = 1, \ldots, m$, are uniformly bounded from above by a constant independent of $t$. Let us consider the function $Z(y)$ defined in the previous step. We take the barrier

$$\tilde{\phi}(y) = 2\|\phi\|_\infty Z(y) + \|h\|_\infty \sum_{i=0}^m \psi_i(y).$$

Choosing $R_1$ larger if necessary, we have that for $y \in \bigcup_{i=1}^m \partial B_{R_1}(\xi_i^\alpha) \cup \partial B_{R_1 \rho_0 v_0 / \varepsilon_0}(p') \cup B_{2d/\varepsilon_0}(p')$, by (3.9),

$$\tilde{\phi}(y) \geq 2\|\phi\|_\infty Z(y) + \phi(y) \geq \|\phi\|_\infty \geq \|\phi(y)\| \geq \|\phi(y)\| \geq 0,$$

and for $y \in \Omega_t \setminus \bigcup_{i=1}^m B_{R_1}(\xi_i^\alpha) \cup B_{R_1 \rho_0 v_0 / \varepsilon_0}(p') \cup B_{2d/\varepsilon_0}(p')$, by (2.28), (2.30), (2.32) and (3.7),

$$\mathcal{L}(\tilde{\phi} \pm \tilde{\phi})(y) \geq \|h\| \sum_{i=0}^m \mathcal{L}(\psi_i)(y) \pm h(y) \geq \|h\| \left\{ - \sum_{i=0}^m \Delta \psi_i(y) - W \sum_{i=0}^m \psi_i(y) \right\} \pm h(y)$$

$$\geq \|h\| \left\{ \frac{\varepsilon_0}{\rho_0 v_0} \right\}^2 \frac{1}{4} + \sum_{i=1}^m \frac{1}{\gamma_i^2} \frac{4}{\gamma_i} \left( 1 + \frac{\varepsilon_0 y - p_0}{\rho_0 v_0} \right)^2 + \frac{1}{\gamma_i^2} \left( 1 + \frac{\varepsilon_0 y - p_0}{\rho_0 v_0} \right)^2 + C(\varepsilon_0^2)^{\beta (2m - 2 + \alpha)} e^{-t\phi(\varepsilon_0 y)} \right\} \pm h(y)$$

$$\geq \|h\| \pm h(y) \geq 0.$$
than those of (3.1) in the following way

\[ \int_{\Omega_t} \chi_t Z_{p} \phi = 0 \quad \text{and} \quad \int_{\Omega_t} \chi_t Z_{ij} \phi = 0, \quad i = 1, \ldots, m, \quad j = 0, 1, 2. \]  

(3.14)

Namely, we prove that there exists a constant \( C > 0 \) independent of \( t \) such that for any \( h \in L^\infty(\Omega_t) \) and any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \),

\[ \|\phi\|_{L^\infty(\Omega_t)} \leq C\|h\|_*, \]  

(3.15)

for \( t \) large enough. By contradiction, assume that there are sequences of parameters \( t_n \to +\infty \), functions \( h_n, W_n \) and associated solutions \( \phi_n \) of equation (3.11) with orthogonality conditions (3.14) such that

\[ \|\phi_n\|_{L^\infty(\Omega_{t_n})} = 1 \quad \text{and} \quad \|h_n\|_* \to 0, \quad \text{as} \quad n \to +\infty. \]  

(3.16)

Let us set

\[ \hat{\phi}^n_{p,x}(x) = \phi_n(x/\varepsilon^n_0), \quad \hat{h}^n_{p,x}(x) = h_n(x/\varepsilon^n_0) \quad \text{for all} \quad x \in \Omega \setminus B_{2d}(p), \]

and

\[ \hat{\phi}^n_i(z) = \phi_n((\rho^n_0\varepsilon^n_0 z + p)/\varepsilon^n_0), \quad \hat{h}^n_i(z) = h_n((\rho^n_0\varepsilon^n_0 z + p)/\varepsilon^n_0), \]

and for all \( i = 1, \ldots, m \),

\[ \hat{\phi}^n_i(z) = \phi_n(\gamma^n_i z + (\xi^n_i)'), \quad \hat{h}^n_i(z) = h_n(\gamma^n_i z + (\xi^n_i)'), \]

where \( \mu^n = (\rho^n_0, \mu^n_1, \ldots, \mu^n_m), \) \( \varepsilon^n_0 = \exp \{-2t_n\}, \) \( v^n_0 = (\mu^n_0/\varepsilon^n_0) \) and \( \gamma^n_i = \frac{1}{\varepsilon^n_0} e^{\gamma^n_i} \mu^n_i = \mu^n_i \exp \{-2t_n[\phi_1(\xi^n_i) - 1]\} \). First, using the expansion of \( W_n \) in (3.22), we have that \( \hat{\phi}^n_{p,x}(x) \) satisfies

\[ -\Delta \hat{\phi}^n_{p,x}(x) + O \left( \frac{e^{-t_n \phi_1(x)}}{|x - p|^{4+2\alpha}} \prod_{i=1}^{m} \frac{1}{|x - \xi^n_i|^4} \right) \hat{\phi}^n_{p,x}(x) = \left( \frac{1}{\varepsilon^n_0} \right)^{2} \hat{h}^n_{p,x}(x) \quad \text{in} \quad \Omega \setminus B_{2d}(p), \]

on \( \partial \Omega. \)

By the definition of the \( \| \cdot \|_* \) norm in (3.7) we find that \( (\frac{1}{\varepsilon^n_0})^2 |\hat{h}^n_{p,x}(x)| \leq C\|h_n\|_* \to 0 \) uniformly in \( \Omega \setminus B_{2d}(p) \). Obviously, elliptic regularity theory implies that \( \hat{\phi}^n_{p,x} \) converges uniformly in \( \Omega \setminus B_{2d}(p) \) to a trivial solution \( \hat{\phi}^\infty_{p,x} \), namely \( \hat{\phi}^\infty_{p,x} \equiv 0 \) in \( \Omega \setminus B_{2d}(p) \).

Next, using the expansion of \( W_n \) in (2.30), we find that \( \hat{\phi}^n_i(z) \) satisfies

\[ -\Delta \hat{\phi}^n_i(z) - \frac{8(1 + \alpha)^2|z|^{2\alpha}}{1 + |z|^{2(1+\alpha)}} \left[ 1 + O \left( \frac{\rho^n_0\varepsilon^n_0 t^n_{i,n}|z|}{1 + |z|^{2(1+\alpha)}} \right) + o(1) \right] \hat{\phi}^n_i(z) = \left( \frac{\rho^n_0\varepsilon^n_0}{\varepsilon^n_0} \right)^2 \hat{h}^n_i(z) \]

for any \( z \in B_{R_0+2}(0) \). Thanks to the definition of the \( \| \cdot \|_* \) norm in (3.7), we have that for any \( q \in (1, -1/\alpha), \)

\[ (\frac{\rho^n_0\varepsilon^n_0}{\varepsilon^n_0})^2 \hat{h}^n_i \to 0 \text{ in } L^q(B_{R_0+2}(0)). \]

Since \( \frac{8(1+\alpha)^2|z|^{2\alpha}}{1 + |z|^{2(1+\alpha)}} \) is bounded in \( L^q(B_{R_0+2}(0)) \), elliptic regularity theory readily implies that \( \hat{\phi}^n_i \) converges uniformly over compact subsets near the origin to a bounded solution \( \hat{\phi}^\infty_i \) of equation (3.3), which satisfies

\[ \int_{\mathbb{R}^2} \chiacobian Z_{p} \hat{\phi}^\infty_{p} = 0. \]  

(3.17)

Then \( \hat{\phi}^\infty_{p} \) is proportional to \( Z_p \). Since \( \int_{\mathbb{R}^2} \chiacobian Z_{p}^2 > 0 \), by (3.17) we deduce that \( \hat{\phi}^\infty_{p} \equiv 0 \) in \( B_{R_0}(0) \).

Finally, using the expansion of \( W_n \) in (2.28) and elliptic regularity, we can derive that for each \( i = 1, \ldots, m \), \( \hat{\phi}^n_i \) converges uniformly over compact subsets near the origin to a bounded solution \( \hat{\phi}^\infty_i \) of equation (3.4), which satisfies

\[ \int_{\mathbb{R}^2} \chiacobian Z_{j} \hat{\phi}^\infty_{j} = 0 \quad \text{for} \quad j = 0, 1, 2. \]  

(3.18)
Then $\hat{\phi}_i^\infty$ is a linear combination of $Z_j$, $j = 0, 1, 2$. Notice that $\int_{\mathbb{R}^2} \chi Z_i Z_i = 0$ for $j \neq l$ and $\int_{\mathbb{R}^2} \chi Z_j^2 > 0$. Hence (3.18) implies $\hat{\phi}_i^\infty \equiv 0$ in $B_{R_i}(0)$. As a consequence, by definition (3.10) we find $\lim_{n \to +\infty} \|\phi_n\|_{**} = 0$. But (3.12) and (3.16) tell us $\lim\inf_{n \to +\infty} \|\phi_n\|_{**} > 0$, which is a contradiction.

**Step 4:** We establish uniform an a priori estimate for solutions $\phi$ to equation (3.11), when $h \in L^\infty(\Omega_t)$ and $\phi$ only satisfies the orthogonality conditions in (3.1)

$$\int_{\Omega_t} \chi_i Z_i \phi = 0, \quad i = 1, \ldots, m, \ j = 1, 2.$$  \hspace{1cm} (3.19)

More precisely, we prove that there exists a constant $C > 0$ independent of $t$ such that for any $h \in L^\infty(\Omega_t)$ and any points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t$,

$$\|\phi\|_{L^\infty(\Omega_t)} \leq Ct\|h\|_\ast,$$  \hspace{1cm} (3.20)

for $t$ large enough.

Let $R > R_0 + 1$ be a large but fixed number. Set

$$\hat{Z}_p(y) = Z_p(y) - \frac{\epsilon_0}{\rho_0 v_0} + a_p G(\xi_0, y, p), \quad \hat{Z}_{i0}(y) = Z_{i0}(y) - \frac{1}{\gamma_i} + a_{i0} G(\epsilon_0 y, \xi_i),$$  \hspace{1cm} (3.21)

where

$$a_p = \frac{\epsilon_0}{\rho_0 v_0 [H(p, p) - 4 \log(\rho_0 v_0 R)]}, \quad a_{i0} = \frac{1}{\gamma_i [H(\xi_i, \xi_i) - 4 \log(\epsilon_0 \gamma_i R)].}$$  \hspace{1cm} (3.22)

Note that by estimates (2.14)-(2.15), and definitions (2.2), (2.7) and (2.23),

$$C_1 |\log \epsilon_0| \leq - \log(\rho_0 v_0 R) \leq C_2 |\log \epsilon_0|, \quad C_1 |\log \epsilon_i| \leq - \log(\epsilon_0 \gamma_i R) \leq C_2 |\log \epsilon_i|,$$  \hspace{1cm} (3.23)

and

$$\hat{Z}_p(y) = O\left(\frac{\epsilon_0 G(\xi_0 y, p)}{\rho_0 v_0 |\log \epsilon_0|}\right), \quad \hat{Z}_{i0}(y) = O\left(\frac{G(\epsilon_0 y, \xi_i)}{\gamma_i |\log \epsilon_i|}\right).$$  \hspace{1cm} (3.24)

Let $\eta_1$ and $\eta_2$ be radial smooth cut-off functions in $\mathbb{R}^2$ such that

$$0 \leq \eta_1 \leq 1; \quad |\nabla \eta_1| \leq C \text{ in } \mathbb{R}^2; \quad \eta_1 \equiv 1 \text{ in } B_R(0); \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{R+1}(0);$$

$$0 \leq \eta_2 \leq 1; \quad |\nabla \eta_2| \leq C \text{ in } \mathbb{R}^2; \quad \eta_2 \equiv 1 \text{ in } B_{3d}(0); \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{5d}(0),$$

where $d > 0$ can be chosen as a sufficiently small but fixed number independent of $t$ such that $B_{3d}(p) \subset \Omega$. Set

$$\eta_{p1}(y) = \eta_1 \left(\frac{|\epsilon_0 y - p|}{\rho_0 v_0}\right), \quad \eta_{i1}(y) = \eta_1 \left(\frac{|y - \xi_i|}{\gamma_i}\right),$$  \hspace{1cm} (3.25)

and

$$\eta_{p2}(y) = \eta_2 \left(\frac{|\epsilon_0 |y - p|}{\rho_0 v_0}\right), \quad \eta_{i2}(y) = \eta_2 \left(\frac{|y - \xi_i|}{\gamma_i}\right).$$  \hspace{1cm} (3.26)

We define the two test functions

$$\tilde{Z}_p = \eta_{p1} Z_p + (1 - \eta_{p1}) \eta_{p2} \hat{Z}_p, \quad \tilde{Z}_{i0} = \eta_{i1} Z_{i0} + (1 - \eta_{i1}) \eta_{i2} \hat{Z}_{i0}. \hspace{1cm} (3.27)$$

Given $\phi$ satisfying (3.11) and (3.19), let

$$\tilde{\phi} = \phi + d_p \tilde{Z}_p + \sum_{i=1}^m d_i \tilde{Z}_{i0} + \sum_{i=1}^m \sum_{j=1}^2 e_{ij} \chi_i Z_{ij}.$$  \hspace{1cm} (3.28)

We will first prove the existence of $d_p$, $d_i$ and $e_{ij}$ such that $\tilde{\phi}$ satisfies the orthogonality conditions in (3.14). Remark that $\tilde{Z}_{i0}$ coincides with $Z_{i0}$ in $B_{R\gamma_i}(\xi_i)$ and hence $\tilde{Z}_{i0}$ is still orthogonal to $\chi_i Z_{ij}$ for $j = 1, 2$. Testing
(3.28) against \( \chi_i Z_{ij} \) and using the orthogonality conditions in (3.14) and (3.19) for \( j = 1, 2 \) and the fact that \( \chi_i \chi_k = 0 \) if \( i \neq k \), we can write

\[
eq = \frac{1}{d_p} \int_{\Omega_t} \chi_i Z_{ij} \tilde{Z}_p - \frac{1}{d_k} \int_{\Omega_t} \chi_i Z_{ij} \tilde{Z}_{k0} \right) \left/ \int_{\Omega_t} \chi_i^2 Z_{ij}^2, \quad i = 1, \ldots, m, \quad j = 1, 2. \tag{3.29}\]

Notice that \( \int_{\Omega_t} \chi_i^2 Z_{ij}^2 = c > 0 \) for all \( i, j \), and by (3.24) and (3.27),

\[
\int_{\Omega_t} \chi_i Z_{ij} \tilde{Z}_p = O \left( \frac{\varepsilon_0 \gamma_i \log t}{\rho_0 v_0 |\log \varepsilon_0|} \right), \quad \int_{\Omega_t} \chi_i Z_{ij} \tilde{Z}_{k0} = O \left( \frac{\gamma_i \log t}{\gamma_k |\log \varepsilon_k|} \right), \quad k \neq i.
\]

Then

\[
|e_{ij}| \leq C \left( |d_p| \frac{\varepsilon_0 \gamma_i \log t}{\rho_0 v_0 |\log \varepsilon_0|} + \sum_{k \neq i} |d_k| \frac{\gamma_i \log t}{\gamma_k |\log \varepsilon_k|} \right). \tag{3.30}
\]

We need just to consider \( d_p \) and \( d_i \). Testing (3.28) against \( \chi_p Z_p \) and \( \chi_k Z_{k0} \), respectively, and using the orthogonality conditions in (3.14) for \( p \) and \( j = 0 \), we get a system of \( (d_p, d_1, \ldots, d_m) \),

\[
d_p \int_{\Omega_t} \chi_p Z_p \tilde{Z}_p + \sum_{i=1}^m d_i \int_{\Omega_t} \chi_p Z_p \tilde{Z}_{i0} = - \int_{\Omega_t} \chi_p Z_p \phi, \tag{3.31}
\]

and

\[
d_p \int_{\Omega_t} \chi_k Z_{k0} \tilde{Z}_p + \sum_{i=1}^m d_i \int_{\Omega_t} \chi_k Z_{k0} \tilde{Z}_{i0} = - \int_{\Omega_t} \chi_k Z_{k0} \phi, \quad k = 1, \ldots, m.
\]

But

\[
\int_{\Omega_t} \chi_p Z_p \tilde{Z}_p = \int_{\Omega_t} \chi_p Z_p^2 = C_1 > 0, \quad \int_{\Omega_t} \chi_p Z_p \tilde{Z}_{i0} = O \left( \frac{\rho_0 v_0 \log t}{\varepsilon_0 \gamma_i |\log \varepsilon_0|} \right),
\]

and

\[
\int_{\Omega_t} \chi_k Z_{k0} \tilde{Z}_p = O \left( \frac{\rho_0 v_0 \log t}{\varepsilon_0 \gamma_k |\log \varepsilon_0|} \right), \quad \int_{\Omega_t} \chi_k Z_{k0} \tilde{Z}_{i0} = C_2 > 0, \quad \int_{\Omega_t} \chi_k Z_{k0} \tilde{Z}_{i0} = O \left( \frac{\gamma_k \log t}{\gamma_i |\log \varepsilon_i|} \right), \quad i \neq k.
\]

Let us denote \( \mathcal{M} \) the coefficient matrix of system (3.31). From the above estimates it follows that \( P^{-1} \mathcal{M} P \) is diagonally dominant and then invertible, where \( P = \text{diag} \left( \frac{\rho_0 v_0}{\varepsilon_0}, \gamma_1, \ldots, \gamma_m \right) \). Hence \( \mathcal{M} \) is also invertible and \( (d_p, d_1, \ldots, d_m) \) is well defined.

Estimate (3.20) is a direct consequence of the following two claims.

Claim 1.

\[
\| \mathcal{L}(\tilde{Z}_p) \|_* \leq \frac{C \varepsilon_0 \log t}{\rho_0 v_0 |\log \varepsilon_0|}, \tag{3.32}
\]

and

\[
\| \mathcal{L}(\chi_i Z_{ij}) \|_* \leq \frac{C}{\gamma_i}, \quad \| \mathcal{L}(\tilde{Z}_{i0}) \|_* \leq \frac{C \log t}{\gamma_i |\log \varepsilon_i|}. \tag{3.33}
\]

Claim 2.

\[
|d_p| \leq C \frac{\rho_0 v_0 |\log \varepsilon_0|}{\varepsilon_0} \| h \|_*, \quad |d_i| \leq C \gamma_i |\log \varepsilon_i| \| h \|_*, \quad |e_{ij}| \leq C \gamma_i \log t \| h \|_* \tag{3.34}
\]

In fact, by the definition of \( \tilde{\phi} \) in (3.28) we get

\[
\mathcal{L}(\tilde{\phi}) = h + d_p \mathcal{L}(\tilde{Z}_p) + \sum_{i=1}^m d_i \mathcal{L}(\tilde{Z}_{i0}) + \sum_{i=1}^m \sum_{j=1}^m e_{ij} \mathcal{L}(\chi_i Z_{ij}) \quad \text{in} \quad \Omega_t. \tag{3.35}
\]
Since (3.14) hold, by estimate (3.15) we conclude
\[
\|\vec{\phi}\|_{L^\infty(\Omega_t)} \leq C \left[ \|h\|_* + |d_p|\|L(\bar{Z}_p)\|_* + \sum_{i=1}^m |d_i|\|L(\bar{Z}_i)\|_* + \sum_{i=1}^m \sum_{j=1}^2 |e_{ij}|\|L(\chi_i Z_{ij})\|_* \right].
\] (3.36)

Using the definition of \(\vec{\phi}\) again and the fact that
\[
\|\bar{Z}\|_{L^\infty(\Omega_t)} \leq \frac{C \rho_0}{\rho_0 v_0}, \quad \|\bar{Z}_{i\theta}\|_{L^\infty(\Omega_t)} \leq \frac{C}{\gamma_i}, \quad \|\chi_i Z_{ij}\|_{L^\infty(\Omega_t)} \leq \frac{C}{\gamma_i}.
\] (3.37)
estimate (3.20) then follows from (2.7), (3.36), Claims 1 and 2.

**Proof of Claim 1.** Let us begin with inequality (3.32). Consider four regions
\[
\Omega_1 = \left\{ \frac{\varepsilon_0 y - p}{\rho_0 v_0} \leq R \right\}, \quad \Omega_2 = \left\{ R < \frac{\varepsilon_0 y - p}{\rho_0 v_0} \leq R + 1 \right\},
\]
\[
\Omega_3 = \left\{ R + 1 < \frac{\varepsilon_0 y - p}{\rho_0 v_0} \leq \frac{3d}{\rho_0 v_0} \right\}, \quad \Omega_4 = \left\{ \frac{3d}{\rho_0 v_0} < \frac{\varepsilon_0 y - p}{\rho_0 v_0} \leq \frac{6d}{\rho_0 v_0} \right\}.
\]
Observe first that, by (3.2), (3.3) and (3.5),
\[
L(Z_p) = -\Delta Z_p - WZ_p = \left[ \frac{\varepsilon_0}{\rho_0 v_0} \right]^2 \frac{8(1 + \alpha)^2 |\varphi - p|^2 \rho_0}{(1 + |\varphi - p|^{2(1+\alpha)})^2} - W \right] Z_p.
\] (3.38)
In \(\Omega_1\), by (2.30), (2.27) and (3.38),
\[
L(\bar{Z}_p) = L(Z_p) = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^3 \frac{8(1 + \alpha)^2 |\varphi - p|^2 \rho_0}{(1 + |\varphi - p|^{2(1+\alpha)})^2} \left[ O \left( \varepsilon_0 \rho_0 \left| y - p' \right| \right) + O \left( \varepsilon_0 \rho_0^2 \right) + \sum_{j=1}^m O \left( \varepsilon_0^2 \rho_0^{2j+2} \right) \right].
\] (3.39)
In \(\Omega_2\), by (1.2), (3.21) and (3.27),
\[
L(\bar{Z}_p) = L(Z_p) - (1 - \eta_{p1})L(Z_p - \bar{Z}_p) - 2\nabla \eta_{p1} \nabla(Z_p - \bar{Z}_p) - (Z_p - \bar{Z}_p) \Delta \eta_{p1}
\]
\[
= L(Z_p) + (1 - \eta_{p1})W(Z_p - \bar{Z}_p) - 2\nabla \eta_{p1} \nabla(Z_p - \bar{Z}_p) - (Z_p - \bar{Z}_p) \Delta \eta_{p1}.
\] (3.40)
Notice that, by (3.21)-(3.22),
\[
Z_p - \bar{Z}_p = \frac{\varepsilon_0}{\rho_0 v_0} - a_p G(\varepsilon_0 y, p) = \frac{\varepsilon_0}{\rho_0 v_0} \left[ \frac{H(p, p) - 4 \log(\rho_0 v_0 R)}{R \rho_0 v_0} + O \left( \varepsilon_0 \left| y - p' \right| \right) \right],
\] (3.41)
and then in \(\Omega_2\), by (3.23),
\[
|Z_p - \bar{Z}_p| = O \left( \frac{\varepsilon_0}{R \rho_0 v_0 \log \varepsilon_0} \right), \quad |\nabla(Z_p - \bar{Z}_p)| = O \left( \frac{\varepsilon_0^2}{R \rho_0^2 v_0^2 \log \varepsilon_0} \right).
\] (3.42)
Moreover, \(|\nabla \eta_{p1}| = O(\varepsilon_0/(\rho_0 v_0))\) and \(|\Delta \eta_{p1}| = O(\varepsilon_0^2/(\rho_0 v_0^2))\). By (2.30), (3.38), (3.40) and (3.42) we have that in \(\Omega_2\),
\[
L(\bar{Z}_p) = O \left( \frac{\varepsilon_0^3}{R \rho_0^2 v_0 \log \varepsilon_0} \right).
\] (3.43)
In \(\Omega_3\), by (1.2), (3.21), (3.27) and (3.38),
\[
L(\bar{Z}_p) = L(\bar{Z}_p) - L(Z_p - \bar{Z}_p)
\]
\[
= \left[ \frac{\varepsilon_0}{\rho_0 v_0} \right]^2 \frac{8(1 + \alpha)^2 |\varphi - p|^2 \rho_0}{(1 + |\varphi - p|^{2(1+\alpha)})^2} - W \right] Z_p + W \left[ \frac{\varepsilon_0}{\rho_0 v_0} - a_p G(\varepsilon_0 y, p) \right] = A_1 + A_2.
\]
For the estimates of these two terms, we decompose $\Omega_3$ to some subregions:

$$\Omega_p = \left\{ R + 1 < \frac{\varepsilon_0 y - p}{\rho_0 v_0} \leq \frac{1}{\rho_0 v_0^{2\beta}} \right\},$$

$$\Omega_{3,k} = \Omega_3 \cap \left\{ |y - \xi_k| \leq 1/\varepsilon_0 t^{2\beta} \right\} \quad \text{and} \quad \tilde{\Omega}_3 = \Omega_3 \setminus \bigcup_{k=1}^{m} \Omega_{3,k} \cup \Omega_p.$$ 

By (2.30), (2.32) and (3.5),

$$A_1 = \left\{ \varepsilon_0^3 \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^3 8(1 + \alpha)^2 \left| \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right|^{2\alpha} \left( \frac{\rho_0 v_0}{\rho_0 v_0} \right)^{2(1+\alpha)} \right\} \left[ \frac{O \left( \varepsilon_0 t^{2\beta} \right)}{O \left( \varepsilon_0 \rho_0 \right)} + O \left( \frac{\varepsilon_0^2 \rho_0}{\rho_0 v_0} + \sum_{j=1}^{m} O \left( \frac{\varepsilon_0^2 \rho_0^{2\beta}}{\rho_0 v_0} \right) \right) \right] \text{ in } \Omega_p,$$

$$O \left( \frac{\varepsilon_0^3 \rho_0^2 \alpha (2+\alpha)}{\rho_0 v_0} \right) + O \left( \frac{\varepsilon_0^3 \rho_0^{2\beta}}{\rho_0 v_0} e^{-\varepsilon_0 (\varepsilon_0 y) t^{2\beta}} \right) \text{ in } \tilde{\Omega}_3.$$ 

Moreover, by (3.23) and (3.41),

$$A_2 = \left\{ \varepsilon_0^3 \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^3 8(1 + \alpha)^2 \left| \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right|^{2\alpha} \left( \frac{\rho_0 v_0}{\rho_0 v_0} \right)^{2(1+\alpha)} \right\} \left[ \frac{O \left( \varepsilon_0 t^{2\beta} \right)}{O \left( \varepsilon_0 \rho_0 \right)} + O \left( \frac{\varepsilon_0^2 \rho_0}{\rho_0 v_0} + \sum_{j=1}^{m} O \left( \frac{\varepsilon_0^2 \rho_0^{2\beta}}{\rho_0 v_0} \right) \right) \right] \text{ in } \Omega_p,$$

$$O \left( \frac{\varepsilon_0^3 \rho_0^2 \alpha (2+\alpha)}{\rho_0 v_0} \right) + O \left( \frac{\varepsilon_0^3 \rho_0^{2\beta}}{\rho_0 v_0} e^{-\varepsilon_0 (\varepsilon_0 y) t^{2\beta}} \right) \text{ in } \tilde{\Omega}_3.$$ 

Then in $\Omega_p \cup \tilde{\Omega}_3$,

$$\mathcal{L}(\tilde{Z}_p) = \mathcal{L}(\tilde{Z}_p) = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^3 8(1 + \alpha)^2 \left| \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right|^{2\alpha} \left( \frac{\rho_0 v_0}{\rho_0 v_0} \right)^{2(1+\alpha)} \frac{O \left( \varepsilon_0 t^{2\beta} \right)}{O \left( \varepsilon_0 \rho_0 \right)} + \sum_{j=1}^{m} O \left( \frac{\varepsilon_0^2 \rho_0^{2\beta}}{\rho_0 v_0} \right) \frac{\varepsilon_0 G(\varepsilon_0 y, p)}{\rho_0 v_0} \frac{\varepsilon_0 \rho_0 \log \xi_k}{\rho_0 v_0 \log \varepsilon_0} \bigg( \frac{1}{\varepsilon_0^2} \bigg).$$ 

Finally in $\Omega_4$, by (3.21) and (3.27),

$$\mathcal{L}(\tilde{Z}_p) = -\eta_{p_2} \Delta Z_p - \eta_{p_2} W \tilde{Z}_p - 2\nabla \eta_{p_2} \nabla \tilde{Z}_p - \tilde{Z}_p \Delta \eta_{p_2}.$$ 

Note that from the previous choice of the number $d$ we get that for any $y \in \Omega_4$ and any $k = 1, \ldots, m$,

$$|y - \xi_k| \geq |y - p'| - |p' - \xi_k| \geq \frac{3d}{\varepsilon_0} - \frac{d}{\varepsilon_0} = \frac{2d}{\varepsilon_0} > \frac{1}{\varepsilon_0 t^{2\beta}}.$$ 

This combined with (2.32) gives

$$W = O \left( \frac{\varepsilon_0^2 \rho_0}{\rho_0 v_0} \right) \frac{\varepsilon_0 \log t}{\rho_0 v_0} \log \varepsilon_0 \bigg( \frac{1}{\varepsilon_0^2} \bigg).$$ 

In addition, $|\nabla \eta_{p_2}| = O(\varepsilon_0)$, $|\Delta \eta_{p_2}| = O(\varepsilon_0)$ and

$$|\tilde{Z}_p| = O \left( \frac{\varepsilon_0}{\rho_0 v_0} \log \varepsilon_0 \right) \text{ in } \Omega_4.$$ 

(3.47)

$$|\nabla \tilde{Z}_p| = O \left( \frac{\varepsilon_0^2 \rho_0}{\rho_0 v_0} \log \varepsilon_0 \right) \text{ in } \Omega_4.$$ 

(3.48)
Hence by (3.38), (3.46), (3.47) and (3.48), we find that in $\Omega_4$,

$$L(\tilde{Z}_p) = O\left(\frac{\varepsilon_0^3}{\rho_0 v_0 |\log \varepsilon_0|}\right). \tag{3.49}$$

Combining (3.7), (3.39), (3.43), (3.44), (3.45) and (3.49), we readily conclude

$$\|L(\tilde{Z}_p)\|_* = O\left(\frac{\varepsilon_0 |\log t|}{\rho_0 v_0 |\log \varepsilon_0|}\right).$$

The inequalities in (3.33) are easy to establish as they are very similar to the consideration of inequality (3.32), so we leave the detailed proof for readers.

**Proof of Claim 2.** Let us prove the first two inequalities in (3.34). Testing (3.35) against $\tilde{Z}_p$ and using estimates (3.36) and (3.37), we find

$$d_p \int_{\Omega_t} \tilde{Z}_p L(\tilde{Z}_p) + \sum_{k=1}^m d_k \int_{\Omega_t} \tilde{Z}_p L(\tilde{Z}_{k0})$$

$$= -\int_{\Omega_t} h \tilde{Z}_p + \int_{\Omega_t} \tilde{\phi} L(\tilde{Z}_p) - \sum_{k=1}^m \sum_{l=1}^2 c_{kl} \int_{\Omega_t} \chi_k Z_{kl} L(\tilde{Z}_p)$$

$$\leq \frac{C \varepsilon_0}{\rho_0 v_0} \|h\|_* + C \|L(\tilde{Z}_p)\|_* \left(\|\tilde{\phi}\|_{L^\infty(\Omega_t)} + \sum_{k=1}^m \sum_{l=1}^2 \frac{1}{\gamma_k} |c_{kl}|\right)$$

$$\leq \frac{C \varepsilon_0}{\rho_0 v_0} \|h\|_* + C \|L(\tilde{Z}_p)\|_* \left[\|h\|_* + |d_p| \|L(\tilde{Z}_p)\|_* + \sum_{k=1}^m |d_k| \|L(\tilde{Z}_{k0})\|_* + \sum_{k=1}^m \sum_{l=1}^2 \frac{1}{\gamma_k} \left(\frac{1}{\gamma_k} + \|L(\chi_k Z_{kl})\|_*\right)\right],$$

where we have applied the following two inequalities:

$$\left(\frac{\varepsilon_0}{\rho_0 v_0}\right)^2 \int_{\Omega_t} \frac{|e_{0,0} - e_0|^2}{(1 + \frac{\rho_0 v_0}{\rho_0 v_0})^{4+2\alpha + 2\alpha}} \, dy \leq C$$

and

$$\int_{\Omega_t} \frac{1}{\gamma_i^2} \left(1 + \frac{|e_x|}{\gamma_i}\right)^{4+2\alpha} \, dy \leq C, \quad i = 1, \ldots, m.$$
By (3.5) and (3.39), we get

\[ I_1 = \int_{\Omega} Z_p \mathcal{L}(Z_p) = \int_{\Omega} \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \frac{(1 + \alpha)^2}{(1 + \frac{\varepsilon_0 v - p}{\rho_0 v_0})^{2(1 + \alpha)}} \left[ O \left( \varepsilon_0 t^3 |y - p'| \right) + O \left( \varepsilon_0^2 \mu_0^2 \right) + \sum_{j=1}^{m} O \left( \varepsilon_0^2 \mu_j^2 t^{2j} \right) \right] \]

\[ = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \left[ O \left( \varepsilon_0 v_0 t^3 \right) + O \left( \varepsilon_0^2 \mu_0^2 \right) + \sum_{j=1}^{m} O \left( \varepsilon_0^2 \mu_j^2 t^{2j} \right) \right]. \tag{3.53} \]

By (3.24), (3.44) and (3.45), we have

\[ I_3 = \int_{\Omega_{\lambda,k}} \mathcal{Z}_p \mathcal{L}(\mathcal{Z}_p) + \sum_{k=1}^{m} \int_{\Omega_{\lambda,k}} \mathcal{Z}_p \mathcal{L}(\mathcal{Z}_p) \]

\[ = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 O \left( \int_{\Omega_{\lambda,k}} \frac{1}{r^{1+2\alpha} \log(\varepsilon_0)|\log \varepsilon_0|^2} \right) \int_{\Omega_{\lambda,k}} \int_{0}^{1/(\varepsilon_0 \gamma_0 t^{2\beta})} \frac{r \log^2 t}{(1 + r^2)^2} \log \varepsilon_0 | \log \varepsilon_0 |^2 \right) \]

\[ = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \left[ O \left( \frac{1}{R^{2(1+\alpha)}|\log \varepsilon_0|^2} \right) + O \left( \frac{\varepsilon_0^2 \mu_0^2}{|\log \varepsilon_0|^2} \right) \right]. \tag{3.54} \]

By (3.48) and (3.49), we derive that

\[ I_4 = \int_{\Omega_2} \mathcal{Z}_p \mathcal{L}(\mathcal{Z}_p) = \int_{\Omega_2} \mathcal{Z}_p \mathcal{L}(\mathcal{Z}_p) + \sum_{k=1}^{m} \int_{\Omega_{\lambda,k}} \mathcal{Z}_p \mathcal{L}(\mathcal{Z}_p) \]

\[ \int_{\Omega_{\lambda,k}} \int_{0}^{1/(\varepsilon_0 \gamma_0 t^{2\beta})} \frac{r \log^2 t}{(1 + r^2)^2} \log \varepsilon_0 | \log \varepsilon_0 |^2 \right) \]

\[ \int_{\Omega_{\lambda,k}} \int_{0}^{1/(\varepsilon_0 \gamma_0 t^{2\beta})} \frac{r \log^2 t}{(1 + r^2)^2} \log \varepsilon_0 | \log \varepsilon_0 |^2 \right) \]

By (3.48) and (3.49), we derive that

\[ I_4 = \int_{\Omega_2} \mathcal{Z}_p \mathcal{L}(\mathcal{Z}_p) = \int_{\Omega_2} \mathcal{Z}_p \mathcal{L}(\mathcal{Z}_p) + \sum_{k=1}^{m} \int_{\Omega_{\lambda,k}} \mathcal{Z}_p \mathcal{L}(\mathcal{Z}_p) \]

\[ = \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 O \left( \frac{1}{R^{2(1+\alpha)}|\log \varepsilon_0|^2} \right) \int_{\Omega_2} \int_{0}^{1/(\varepsilon_0 \gamma_0 t^{2\beta})} \frac{r \log^2 t}{(1 + r^2)^2} \log \varepsilon_0 | \log \varepsilon_0 |^2 \right) \]

\[ \int_{\Omega_{\lambda,k}} \int_{0}^{1/(\varepsilon_0 \gamma_0 t^{2\beta})} \frac{r \log^2 t}{(1 + r^2)^2} \log \varepsilon_0 | \log \varepsilon_0 |^2 \right) \]

By (2.14), (2.23), (3.2), (3.5), (3.22), (3.25) and (3.41), we conclude

\[ I_2 = -\int_{\Omega_2} \mathcal{Z}_p (Z_p - \mathcal{Z}_p) \Delta \eta_0 = 2 \int_{\Omega_2} \mathcal{Z}_p \nabla \eta_0 \nabla (Z_p - \mathcal{Z}_p) \]

\[ + \int_{\Omega_2} \mathcal{Z}_p \mathcal{L}(Z_p) \]

\[ = I_{21} + I_{22} + I_{23} + I_{24} + \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 O \left( \frac{1}{R^{3+2\alpha}|\log \varepsilon_0|^2} \right). \tag{3.56} \]

By (2.14), (2.23), (3.2), (3.5), (3.22), (3.25) and (3.41), we conclude

\[ I_{21} = -a_p \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \left[ \int_{R^3} \frac{1}{|\varepsilon_0 \gamma_0 t^{2\beta} - p|^2} \right] \frac{1}{|y - p'|} \eta_0' \left( \frac{\varepsilon_0 y - p}{\rho_0 v_0} \right) \left( 4 + o(1) \right) \]

\[ = -8\pi a_p \varepsilon_0 \int_{R^3} \eta_0' (r) \left[ 1 + O \left( \frac{1}{r^{2(1+\alpha)}} \right) \right] \]

\[ = \frac{2\pi (1 + \alpha)}{|\log \varepsilon_0|^2} \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \left[ 1 + O \left( \frac{1}{R^{2(1+\alpha)}} \right) \right]. \tag{3.57} \]

By (3.2), (3.5), (3.25) and (3.42) we find $|\nabla \eta_0| = O \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)$ and $|\nabla \mathcal{Z}_p| = O \left( \frac{\varepsilon_0^2}{R^{3+2\alpha} \rho_0 v_0} \right)$ in $\Omega_2$. Furthermore,

\[ I_{22} = O \left( \frac{\varepsilon_0^2}{R^{3+2\alpha} \rho_0 v_0 |\log \varepsilon_0|^2} \right); \quad I_{23} = O \left( \frac{\varepsilon_0^2}{R^{3+2\alpha} \rho_0 v_0 |\log \varepsilon_0|^2} \right); \quad I_{24} = O \left( \frac{\varepsilon_0^2}{R^{3+2\alpha} \rho_0 v_0 |\log \varepsilon_0|^2} \right). \tag{3.58} \]
Substituting estimates (3.53)-(3.58) into (3.52), we conclude that for \( R \) and \( t \) large enough,

\[
\int_{\Omega_1} \bar{Z}_p \mathcal{L}(\bar{Z}_p) = \frac{2\pi(1+\alpha)}{[\log \varepsilon_0]} \left( \frac{\varepsilon_0}{\rho_0 v_0} \right)^2 \left[ 1 + O \left( \frac{1}{R^{2(1+\alpha)}} \right) \right].
\]

(3.59)

According to (3.50), we need just to calculate \( \int_{\Omega_2} \bar{Z}_{k0} \mathcal{L}(\bar{Z}_p) \) for all \( k \). By the above estimates of \( \mathcal{L}(\bar{Z}_p) \) and \( \bar{Z}_{k0} \), we can easily prove that

\[
\int_{\Omega_1} \bar{Z}_{k0} \mathcal{L}(\bar{Z}_p) = O \left( \frac{\varepsilon_0 (\rho_0 v_0 t^\beta + \sum_{j=0}^{m} \varepsilon_j^2 \mu_j^2 t^{2\beta}) \log t}{\rho_0 v_0 \gamma_k |\log \varepsilon_k|} \right),
\]

\[
\int_{\Omega_2} \bar{Z}_{k0} \mathcal{L}(\bar{Z}_p) = O \left( \frac{\varepsilon_0 (\rho_0 v_0 \gamma_k |\log \varepsilon_0| |\log \varepsilon_k|)}{\rho_0 v_0 \gamma_k |\log \varepsilon_k|} \right),
\]

and

\[
\int_{\Omega_3} \bar{Z}_{k0} \mathcal{L}(\bar{Z}_p) = O \left( \frac{\varepsilon_0 \log^2 t}{\rho_0 v_0 \gamma_k |\log \varepsilon_0| |\log \varepsilon_k|} \right) \quad \text{for all} \quad l \neq k.
\]

It remains to calculate the integral over \( \Omega_3,k \). From (3.27) and an integration by parts we have

\[
\int_{\Omega_3,k} \bar{Z}_{k0} \mathcal{L}(\bar{Z}_p) = \int_{\Omega_3,k} \bar{Z}_p \mathcal{L}(\bar{Z}_{k0}) - \int_{\partial \Omega_3,k} \bar{Z}_{k0} \frac{\partial \bar{Z}_p}{\partial \nu} + \int_{\partial \Omega_3,k} \bar{Z}_p \frac{\partial \bar{Z}_{k0}}{\partial \nu}.
\]

Observe that

\[
\int_{\Omega_3,k} \bar{Z}_p \mathcal{L}(\bar{Z}_{k0}) = \left( \int_{\{1-\varepsilon_0 k \leq 1\}} \frac{1}{\rho_0 v_0 \gamma_k |\log \varepsilon_k|} \right) \mathcal{L}(\bar{Z}_{k0}) = O \left( \frac{\varepsilon_0 t^\beta}{\gamma_k} \right),
\]

for \( \gamma_k R < |y-x_k| \leq \gamma_k(R+1) \),

\[
\mathcal{L}(\bar{Z}_{k0}) = \mathcal{L}(\bar{Z}_k) = O \left( \frac{\varepsilon_0 t^\beta}{\gamma_k^2} \right),
\]

and for \( \gamma_k(R+1) < |y-x_k| \leq 1/\varepsilon_0 t^{2\beta} \),

\[
\mathcal{L}(\bar{Z}_{k0}) = \mathcal{L}(\bar{Z}_k) = O \left( \frac{\varepsilon_0 t^\beta}{\gamma_k^2 |\log \varepsilon_k|} \right).
\]

These, together with the estimate of \( \bar{Z}_p \) in (3.24), give

\[
\int_{\Omega_3,k} \bar{Z}_p \mathcal{L}(\bar{Z}_{k0}) = O \left( \frac{\varepsilon_0 \log t}{\rho_0 v_0 \gamma_k |\log \varepsilon_0| |\log \varepsilon_k|} \right).
\]

As on \( \partial \Omega_3,k \), by (2.2) and (3.24),

\[
\bar{Z}_p = O \left( \frac{\varepsilon_0 \log t}{\rho_0 v_0 |\log \varepsilon_0|} \right), \quad |\nabla \bar{Z}_p| = O \left( \frac{\varepsilon_0 t^{\beta}}{\rho_0 v_0 |\log \varepsilon_0|} \right),
\]

and

\[
\bar{Z}_{k0} = O \left( \frac{\log t}{\gamma_k |\log \varepsilon_k|} \right), \quad |\nabla \bar{Z}_{k0}| = O \left( \frac{\varepsilon_0 t^{2\beta}}{\gamma_k |\log \varepsilon_k|} \right).
\]

Then

\[
\int_{\Omega_3,k} \bar{Z}_{k0} \mathcal{L}(\bar{Z}_p) = O \left( \frac{\varepsilon_0 \log t}{\rho_0 v_0 \gamma_k |\log \varepsilon_0| |\log \varepsilon_k|} \right).
\]
By the above estimates, we readily have
\[
\int_{\Omega_t} \tilde{Z}_{k0} L(\tilde{Z}_p) = O\left( \frac{\varepsilon_0 \log^2 t}{\rho_0 v_0 |\log \varepsilon_0|} \right), \quad k = 1, \ldots, m.
\] (3.60)

Inserting estimates (3.59) and (3.60) into (3.50), we get
\[
\frac{\varepsilon_0 |d_p|}{\rho_0 v_0 |\log \varepsilon_0|} \leq C\|h\|_a + C \log^2 t \left( \frac{\varepsilon_0 |d_p|}{\rho_0 v_0 |\log \varepsilon_0|} + \sum_{k=1}^m \frac{|d_k|}{\gamma_k |\log \varepsilon_k|} \right).
\] (3.61)

On the other hand, similar to the above arguments in (3.59)-(3.60), we can show that for \( R \) and \( t \) large enough,
\[
\int_{\Omega_t} \tilde{Z}_{i0} L(\tilde{Z}_{i0}) = \frac{2\pi}{\gamma_i^2 |\log \varepsilon_i|} \left[ 1 + O \left( \frac{1}{R^2} \right) \right],
\] (3.62)
and
\[
\int_{\Omega_t} \tilde{Z}_{k0} L(\tilde{Z}_{i0}) = O \left( \frac{\log^2 t}{\gamma_i \gamma_k |\log \varepsilon_i| |\log \varepsilon_k|} \right) \quad \text{for all } k \neq i.
\] (3.63)

These, together with (3.51) and (3.60), imply
\[
\frac{|d_i|}{\gamma_i |\log \varepsilon_i|} \leq C\|h\|_a + C \log^2 t \left( \frac{\varepsilon_0 |d_p|}{\rho_0 v_0 |\log \varepsilon_0|} + \sum_{k=1}^m \frac{|d_k|}{\gamma_k |\log \varepsilon_k|} \right).
\] (3.64)

As a result, using linear algebra arguments, by (2.7), (3.61) and (3.64) we can prove Claim 2 for \( d_p \) and \( d_i \), and then complete the proof by (3.30).

**Step 5:** Proof of Proposition 3.1. We begin by establishing the validity of the a priori estimate (3.8). Using estimate (3.20) and the fact that \( \|\chi_i Z_{ij}\|_a = O(\gamma_i) \), we deduce
\[
\|\phi\|_{L^\infty(\Omega_i)} \leq C t \left( \|h\|_a + \sum_{i=1}^m \sum_{j=1}^2 \gamma_i |c_{ij}| \right).
\] (3.65)
So it suffices to estimate the values of the constants \( c_{ij} \). Let us consider the cut-off function \( \eta_{2i} \) defined in (3.26). Multiplying (3.1) by \( \eta_{2i} Z_{ij} \) and integrating by parts, we find
\[
\int_{\Omega_t} \phi L(\eta_{2i} Z_{ij}) = \int_{\Omega_t} h \eta_{2i} Z_{ij} + \sum_{k=1}^m \sum_{l=1}^2 c_{kl} \int_{\Omega_t} \chi_k Z_{kl} \eta_{2i} Z_{ij}.
\] (3.66)
Notice that
\[
L(\eta_{2i} Z_{ij}) = \eta_{2i} L(Z_{ij}) - \Delta \eta_{2i} - 2 \nabla \eta_{2i} \nabla Z_{ij} = \left[ \frac{1}{\gamma_i^2} \left( 1 + \frac{y - \xi_i}{\gamma_i} \right)^2 - W \right] \eta_{2i} Z_{ij} + O \left( \varepsilon_0^3 \right).
\]
For the estimate of the first term, we decompose \( \text{supp}(\eta_{2i}) \) to some subregions:
\[
\hat{\Omega}_p = \text{supp}(\eta_{2i}) \cap \{ |y - p'| \leq 1/(\varepsilon_0 t^{2\beta}) \}, \quad \hat{\Omega}_k = \text{supp}(\eta_{2i}) \cap \{ |y - \xi_k'| \leq 1/(\varepsilon_0 t^{2\beta}) \}, \quad k = 1, \ldots, m,
\]
\[
\hat{\Omega}_2 = \text{supp}(\eta_{2i}) \setminus \left( \bigcup_{k=1}^m \hat{\Omega}_k \cup \hat{\Omega}_p \right),
\]
where \( \text{supp}(\eta_{2i}) = \{ |y - \xi_i'| \leq 6d/\varepsilon_0 \} \). Notice that, by (2.2),
\[
|y - \xi_i'| \geq |\xi_i' - p'| - |y - p'| \geq |\xi_i' - p'| - \frac{1}{\varepsilon_0 t^{2\beta}} \geq \frac{1}{\varepsilon_0 t^{2\beta}} \left( 1 - \frac{1}{t^{2\beta}} \right)
\] (3.67)
uniformly in $\bar{\Omega}_p$, and
\[ |y - \xi_i| \geq |\xi_i' - \xi_k'| - |y - \xi_k'| \geq |\xi_i' - \xi_k'| - \frac{1}{\varepsilon_0 t^{2\beta}} \left( 1 - \frac{1}{t^3} \right) \] (3.68)
uniformly in $\bar{\Omega}_{k1}$ with $k \neq i$. By (2.23), (2.28), (2.30), (2.32) and (3.5) we have that in $\bar{\Omega}_{i1}$,
\[
\left[ \frac{1}{\gamma_i^2 (1 + |y - \xi_i'|^2)^2} - W \right] \eta_{2i} Z_{ij} = \frac{1}{\gamma_i^2} \left( 1 + \frac{8}{\gamma_i^2 |y - \xi_i'|^2} \right)^{3/2} \left[ O \left( \varepsilon_0 t^{2\beta} \left| y - \xi_i' \right| \right) + O \left( \frac{\varepsilon_i^2 \mu_i^2}{\gamma_i} \right) + O \left( \frac{m_0 \mu_i^{2\beta} (1 + \alpha)}{\gamma_i} \right) \right] + \sum_{j=1, j \neq i}^m O \left( \frac{1}{\gamma_i^2} \mu_j^{2\beta} \right),
\]
and in $\bar{\Omega}_p$, by (3.67),
\[
\left[ \frac{1}{\gamma_i^2 (1 + |y - \xi_i'|^2)^2} - W \right] \eta_{2i} Z_{ij} = \left[ O \left( \frac{\gamma_i^2}{|y - \xi_i'|^4} \right) + O \left( \frac{\varepsilon_0}{\rho_0 \varepsilon_0} \right)^2 \left[ O \left( \frac{8(1 + \alpha)^2 \varepsilon_0 \varepsilon_0}{\rho_0 \varepsilon_0} \right)^2 \left( \frac{1 + |y - \xi_i'|^2}{1 + |y - \xi_i'|^2} \right)^2 \right] O \left( \frac{1}{|y - \xi_i'|} \right),
\]
and in $\bar{\Omega}_{k1}$, $k \neq i$, by (3.68),
\[
\left[ \frac{1}{\gamma_i^2 (1 + |y - \xi_i'|^2)^2} - W \right] \eta_{2i} Z_{ij} = \left[ O \left( \frac{\gamma_i^2}{|y - \xi_i'|^4} \right) + O \left( \frac{1}{\gamma_k^2} \left( 1 + \frac{8}{\gamma_k^2 |y - \xi_k'|^2} \right)^2 \right) O \left( \frac{1}{|y - \xi_i'|} \right),
\]
and in $\bar{\Omega}_2$,
\[
\left[ \frac{1}{\gamma_i^2 (1 + |y - \xi_i'|^2)^2} - W \right] \eta_{2i} Z_{ij} = O \left( \varepsilon_0^2 \varepsilon_i^2 \mu_i^{2\beta} t^{10\beta} \right) + O \left( \varepsilon_0^2 \varepsilon_i^2 \mu_i^{2\beta} (4m+5+2\alpha) e^{-t\phi_1(\varepsilon_0 y)} \right).
\]
Then
\[
\left| \int_{\Omega} \phi L(\eta_{2i} Z_{ij}) \right| \leq C \varepsilon_0 t^{2\beta} \| \phi \|_{L^\infty(\Omega_i)}. \] (3.69)
On the other hand, since $\| \eta_{2i} Z_{ij} \|_{L^\infty(\Omega_i)} \leq C \gamma_i^{-1}$, we know that
\[
\int_{\Omega} h \eta_{2i} Z_{ij} = O \left( \frac{\|h\|_*}{\gamma_i} \right). \] (3.70)
Moreover, if $k = i$, by (3.2), (3.5) and (3.6),
\[
\int_{\Omega} \chi_k Z_k(\eta_{2i} Z_{kj}) = \int_{\mathbb{R}^2} \chi(|z|) Z_i(z) Z_j(z) dz = C \delta_{ij}, \] (3.71)
while if $k \neq i$, by (3.68),
\[
\int_{\Omega} \chi_k Z_k(\eta_{2i} Z_{ij}) = O \left( \gamma_k \varepsilon_0 t^{2\beta} \right). \] (3.72)
As a consequence, substituting estimates (3.69)-(3.72) into (3.66), we find
\[
|c_{ij}| \leq C \left( \varepsilon_0 t^{2\beta} \| \phi \|_{L^\infty(\Omega_i)} + \frac{1}{\gamma_i} \| h \|_* + \sum_{k \neq i} \sum_{l=1}^m \gamma_k \varepsilon_0 t^{2\beta} |c_{kl}| \right),
\]
and then, by (2.23),
\[
|c_{ij}| \leq C \left( \varepsilon_0 t^{2\beta} \| \phi \|_{L^\infty(\Omega_i)} + \frac{1}{\gamma_i} \| h \|_* \right).
Combing this estimate with (3.65), we conclude

$$|c_{ij}| \leq C \frac{1}{\gamma_i} \|h\|_s,$$

which proves (3.8).

Now, we consider the Hilbert space

$$H_\xi = \left\{ \phi \in H^1_0(\Omega_t) \mid \int_{\Omega_t} \chi_i Z_{ij} \phi = 0 \quad \forall \ i = 1, \ldots, m, \ j = 1, 2 \right\}$$

with the norm $\|\phi\|_{H_\xi} = \|\nabla \phi\|_{L^\infty(\Omega_t)}$. Equation (3.1) is equivalent to find $\phi \in H_\xi$, such that

$$\int_{\Omega_t} \nabla \phi \nabla \psi - \int_{\Omega_t} W \phi \psi = \int_{\Omega_t} h \psi, \quad \forall \ \psi \in H_\xi.$$ 

By Fredholm’s alternative this is equivalent to the uniqueness of solutions to this problem, which in turn follows from estimate (3.8).

The result of Proposition 3.1 implies that the unique solution $\phi = T(h)$ of (3.1) defines a bounded linear map from the Banach space $C_*$ of all functions $h$ in $L^\infty$ for which $\|h\|_* < \infty$, into $L^\infty$.

**Lemma 3.2.** For any integer $m \geq 1$, the operator $T$ is differentiable with respect to the variables $\xi = (\xi_1, \ldots, \xi_m)$ in $\Omega_t$, precisely for any $k = 1, \ldots, m$ and $l = 1, 2$,

$$\|\partial_{\xi_k}^l T(h)\|_{L^\infty(\Omega_t)} \leq Ct^2 \|h\|_*.$$ (3.74)

**Proof.** Differentiating (3.1) with respect to $\xi_{kl}^i$, formally $Z = \partial_{\xi_{kl}^i} \phi$ should satisfy

$$\left\{
\begin{align*}
\mathcal{L}(Z) &= \frac{\partial \partial_{\xi_{kl}^i} W + \sum_{i=1}^m \sum_{j=1}^2 \left[c_{ij} \partial_{\xi_{kl}^i} (\chi_i Z_{ij}) + \bar{c}_{ij} \chi_i Z_{ij}\right]}{\Omega_t}, \\
Z &= 0 \quad \text{in} \ \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{ij} Z &= -\int_{\Omega_t} \phi \partial_{\xi_{kl}^i} (\chi_i Z_{ij}) \quad \forall \ i = 1, \ldots, m, \ j = 1, 2,
\end{align*}\right. $$

where (still formally) $\bar{c}_{ij} = \partial_{\xi_{kl}^i} (c_{ij})$. Furthermore, if we consider the constants $b_{ij}$ defined as

$$b_{ij} \int_{\Omega_t} \chi_i Z_{ij}^2 = \int_{\Omega_t} \phi \partial_{\xi_{kl}^i} (\chi_i Z_{ij}),$$

and set

$$\tilde{Z} = Z + \sum_{i=1}^m \sum_{j=1}^2 b_{ij} \chi_i Z_{ij},$$

then we have

$$\left\{
\begin{align*}
\mathcal{L}(\tilde{Z}) &= f + \sum_{i=1}^m \sum_{j=1}^2 \bar{c}_{ij} \chi_i Z_{ij} \quad \text{in} \ \Omega_t, \\
\tilde{Z} &= 0 \quad \text{on} \ \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{ij} \tilde{Z} &= 0 \quad \forall \ i = 1, \ldots, m, \ j = 1, 2,
\end{align*}\right.$$ 

where

$$f = \frac{\partial \partial_{\xi_{kl}^i} W + \sum_{i=1}^m \sum_{j=1}^2 b_{ij} \mathcal{L}(\chi_i Z_{ij}) + \sum_{i=1}^m \sum_{j=1}^2 c_{ij} \partial_{\xi_{kl}^i} (\chi_i Z_{ij})}. $$
From Proposition 3.1 it follows that this equation has a unique solution $\tilde{Z}$ and $\tilde{c}_{ij}$, and hence $\partial_{\xi_i} T(h) = T(f) - \sum_{i=1}^{m} \sum_{j=1}^{2} b_{ij} \chi_{i} Z_{ij}$ is well defined. Moreover, by (3.8) we get

$$\|\partial_{\xi_i} T(h)\|_{L^{\infty}(\Omega_t)} \leq \|T(f)\|_{L^{\infty}(\Omega_t)} + C \sum_{i=1}^{m} \sum_{j=1}^{2} |b_{ij}| \leq Ct\|f\|_{*} + C \sum_{i=1}^{m} \sum_{j=1}^{2} |\gamma_i| |b_{ij}|. \quad (3.75)$$

Now, to prove estimate (3.74), we first estimate $\partial_{\xi_i} W$. Notice that $\partial_{\xi_i} W = W \partial_{\xi_i} V$. Obviously, by (2.28), (2.30), (2.32) and (3.7) we find $\|W\|_{*} = O(1)$. On the other hand, similar to the proof of Lemma 2.1, by (2.14)-(2.15) we can compute that

$$\partial_{\xi_i} H_0(\varepsilon_0 y) = O(\varepsilon_0 t^3) \quad \text{and} \quad \partial_{\xi_i} H_i(\varepsilon_0 y) = O(\varepsilon_0 t^3), \quad i = 1, \ldots, m, \quad (3.76)$$

uniformly in $\overline{\Omega}_t$. Furthermore, by (2.4), (2.8), (2.14), (2.15), (2.23), (3.2) and (3.5) we can directly check that

$$\partial_{\xi_i} V(y) = Z_{kl}(y) + O(\varepsilon_0 t^3). \quad (3.77)$$

This, together with the fact that $\frac{1}{\gamma_k} \leq C$ uniformly on $t$, immediately implies

$$\|\partial_{\xi_i} V\|_{L^{\infty}(\Omega_t)} = O(1) \quad \text{and} \quad \|\partial_{\xi_i} W\|_{*} = O(1). \quad (3.78)$$

Next, by definitions (3.5)-(3.6), a straightforward computation gives

$$\|\partial_{\xi_i} (\chi_{i} Z_{ij})\|_{*} = \begin{cases} O(\varepsilon_0 \gamma_i t^3) & \text{if } i \neq k, \\ O(1) & \text{if } i = k. \end{cases} \quad (3.79)$$

Furthermore,

$$|b_{ij}| = \begin{cases} O(\varepsilon_0 \gamma_i t^3) \|\phi\|_{L^{\infty}(\Omega_t)} & \text{if } i \neq k, \\ O(1) \|\phi\|_{L^{\infty}(\Omega_t)} & \text{if } i = k. \end{cases} \quad (3.80)$$

Finally, by (3.8), (3.33), (3.73), (3.79) and (3.80), we obtain

$$\|f\|_{*} \leq Ct\|h\|_{*} \quad \text{and} \quad |b_{ij}| \leq Ct\|h\|_{*}. \quad (3.81)$$

Inserting these into (3.75), we then prove (3.74).

\[ \square \]

4. THE NONLINEAR PROJECTED PROBLEM

In this section we solve the nonlinear projected problem: for any integer $m \geq 1$ and any points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t$, we find a function $\phi$ and scalars $c_{ij}, i = 1, \ldots, m, j = 1, 2$, such that

$$\begin{cases} \mathcal{L}(\phi) = -\Delta \phi - W \phi = E + N(\phi) + \sum_{i=1}^{m} \sum_{j=1}^{2} c_{ij} \chi_{i} Z_{ij} & \text{in } \Omega_t, \\ \phi = 0 & \text{on } \partial \Omega_t, \\ \int_{\Omega_t} \chi_{i} Z_{ij} \phi = 0 & \forall \ 1, \ldots, m, \ j = 1, 2, \end{cases} \quad (4.1)$$

where $W$ is as in (2.28), (2.30) and (2.32), and $E, N(\phi)$ are given by (2.21) and (2.35), respectively.

**Proposition 4.1.** Let $m$ be a positive integer. Then there exist constants $t_m > 1$ and $C > 0$ such that for any $t > t_m$ and any points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t$, problem (4.1) admits a unique solution $\phi \in L^{\infty}(\Omega_t)$, and scalars $c_{ij} \in \mathbb{R}, \ i = 1, \ldots, m, \ j = 1, 2$, such that

$$\|\phi\|_{L^{\infty}(\Omega_t)} \leq Ct \max \left\{ (\rho_0 v_0)^{\min(1,2(\alpha-\delta))} t^\beta, \varepsilon_0 \gamma_1 t^\beta, \ldots, \varepsilon_0 \gamma_m t^\beta, \|e^{-\frac{4}{\lambda} t} \phi\|_{L^{\infty}(\Omega_t)} \right\}. \quad (4.2)$$
Furthermore, the map $\xi' \mapsto \phi(\xi') \in C(\overline{\Omega})$ is $C^1$, precisely for any $k = 1, \ldots, m$ and $l = 1, 2$, 

$$
\|\partial_{\xi'_k} \phi \|_{L^\infty(\Omega_t)} \leq C t^2 \max \left\{ (\rho_0 v_0)^{\min\{1,2(\alpha - \tilde{\alpha})\}} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta}, \| e^{-\frac{1}{2} t \phi_1} \|_{L^\infty(\Omega_t)} \right\},
$$

where $\xi' := (\xi'_1, \ldots, \xi'_m) = (\frac{1}{\sqrt{n}} \xi_1, \ldots, \frac{1}{\sqrt{n}} \xi_m)$.

\textbf{Proof.} Let $T$ be the operator as defined in Proposition 3.1. Then $\phi$ solves (4.1) if and only if 

$$
\phi = T(E + N(\phi)) \equiv A(\phi).
$$

For a given number $k > 0$, let us consider the region 

$$
F_k = \left\{ \phi \in C(\overline{\Omega}_t) \mid \|\phi\|_{L^\infty(\Omega_t)} \leq k \max \left\{ (\rho_0 v_0)^{\min\{1,2(\alpha - \tilde{\alpha})\}} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta}, \| e^{-\frac{1}{2} t \phi_1} \|_{L^\infty(\Omega_t)} \right\} \right\}.
$$

Observe that, by (2.29), (2.31), (2.33) and (3.7), 

$$
\|E\|_* \leq C \max \left\{ (\rho_0 v_0)^{\min\{1,2(\alpha - \tilde{\alpha})\}} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta}, \| e^{-\frac{1}{2} t \phi_1} \|_{L^\infty(\Omega_t)} \right\}.
$$

Moreover, by definition (2.35) of $N(\phi)$ and Lagrange’s theorem we have that for $\phi, \phi_1, \phi_2 \in F_k$, 

$$
\|N(\phi)\|_* \leq C \|W\|_* \|\phi\|_{L^\infty(\Omega_t)} \leq C \|\phi\|_{L^\infty(\Omega_t)}^2,
$$

$$
\|N(\phi_1) - N(\phi_2)\|_* \leq C (\max_{i = 1, 2} \|\phi_i\|_{L^\infty(\Omega_t)}) \|\phi_1 - \phi_2\|_{L^\infty(\Omega_t)}.
$$

where $C > 0$ is independent of $k$ and $t$. Hence by (2.7), (2.14), (2.23) and Proposition 3.1, 

$$
\|A(\phi)\|_{L^\infty(\Omega_t)} \leq C t (\|E\|_* + \|N(\phi)\|_*) \leq C t \max \left\{ (\rho_0 v_0)^{\min\{1,2(\alpha - \tilde{\alpha})\}} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta}, \| e^{-\frac{1}{2} t \phi_1} \|_{L^\infty(\Omega_t)} \right\},
$$

$$
\|A(\phi_1) - A(\phi_2)\|_{L^\infty(\Omega_t)} \leq C t \|N(\phi_1) - N(\phi_2)\|_* < \frac{1}{2} \|\phi_1 - \phi_2\|_{L^\infty(\Omega_t)}.
$$

This means that for all $t$ large enough, $A$ is a contraction on $F_k$ and thus a unique fixed point of $A$ exists in the region.

We now analyze the differentiability of the map $\xi' \mapsto \phi$. Assume for instance that the partial derivative $\partial_{\xi'_k} \phi$ exists. Then, formally 

$$
\partial_{\xi'_k} \phi = T \left( \partial_{\xi'_k} E + \partial_{\xi'_k} N(\phi) \right) + \left( \partial_{\xi'_k} T \right) (E + N(\phi)).
$$

By (3.74) and (4.5), we get 

$$
\left\| \left( \partial_{\xi'_k} T \right) (E + N(\phi)) \right\|_{L^\infty(\Omega_t)} \leq C t^2 (\|E\|_* + \|N(\phi)\|_*) \leq C t^2 \max \left\{ (\rho_0 v_0)^{\min\{1,2(\alpha - \tilde{\alpha})\}} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta}, \| e^{-\frac{1}{2} t \phi_1} \|_{L^\infty(\Omega_t)} \right\}.
$$

Observe that 

$$
\partial_{\xi'_k} N(\phi) = \partial_{\xi'_k} W \left( e^\phi - \phi - 1 \right) + W \left( e^\phi - 1 \right) \partial_{\xi'_k} \phi,
$$

so that, by (3.78), 

$$
\|\partial_{\xi'_k} N(\phi)\|_* \leq C \|\phi\|_{L^\infty(\Omega_t)} \left( \|\phi\|_{L^\infty(\Omega_t)} + \|\partial_{\xi'_k} \phi\|_{L^\infty(\Omega_t)} \right).
$$

Also, thanks to the expansion of $\partial_{\xi'_k} V$ in (3.77), by (2.22), (2.28), (2.30) and (2.32) we can directly check that 

$$
\|\partial_{\xi'_k} E\|_* \leq C \max \left\{ (\rho_0 v_0)^{\min\{1,2(\alpha - \tilde{\alpha})\}} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta}, \| e^{-\frac{1}{2} t \phi_1} \|_{L^\infty(\Omega_t)} \right\}.
$$

Hence by Proposition 3.1, we then prove 

$$
\|\partial_{\xi'_k} \phi\|_{L^\infty(\Omega_t)} \leq C t^2 \max \left\{ (\rho_0 v_0)^{\min\{1,2(\alpha - \tilde{\alpha})\}} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta}, \| e^{-\frac{1}{2} t \phi_1} \|_{L^\infty(\Omega_t)} \right\}.
$$

The above computation can be made rigorous by using the implicit function theorem and the fixed point representation (4.4) which guarantees $C^1$ regularity of $\xi'$.
5. THE REDUCED PROBLEM: A MAXIMIZATION PROCEDURE

In this section we study a maximization problem involving the variational reduction. Let us consider the energy function \( J_t \) associated to problem (1.4), namely

\[
J_t(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega |x - p|^{2\alpha} k(x) e^{-t\phi_t} e^u, \quad u \in H^1_0(\Omega). \tag{5.1}
\]

For any integer \( m \geq 1 \), we take its finite dimensional restriction

\[
F_t(\xi) = J_t(U(\xi) + \tilde{\phi}(\xi)) \quad \forall \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t,
\tag{5.2}
\]

where \( U(\xi) \) is our approximate solution defined in (2.8) and \( \tilde{\phi}(\xi)(x) = \phi(\frac{x}{\xi_1, \ldots, \xi_m}), x \in \Omega \), with \( \phi = \phi_\xi \) the unique solution to problem (4.1) given by Proposition 4.1. Define

\[
\mathcal{M}_m^t = \max_{(\xi_1, \ldots, \xi_m) \in \mathcal{O}_t} F_t(\xi_1, \ldots, \xi_m).
\tag{5.3}
\]

From the results obtained in Proposition 4.1 and the definition of function \( U(\xi) \) we have clearly that for any integer \( m \geq 1 \), the map \( F_t : \mathcal{O}_t \to \mathbb{R} \) is of class \( C^1 \) and then this maximization problem has a solution over \( \mathcal{O}_t \).

**Proposition 5.1.** For any integer \( m \geq 1 \) and any \( t \) large enough, the maximization problem

\[
\max_{(\xi_1, \ldots, \xi_m) \in \mathcal{O}_t} F_t(\xi_1, \ldots, \xi_m)
\tag{5.4}
\]

has a solution \( \xi_t = (\xi_{1,t}, \ldots, \xi_{m,t}) \in \mathcal{O}_t^l \), i.e., the interior of \( \mathcal{O}_t \).

**Proof.** The proof of this result consists of three steps which we state and prove next.

**Step 1:** With the choices for the parameters \( \mu_0 \) and \( \mu_i, i = 1, \ldots, m \), respectively given by (2.12) and (2.13), let us prove that the following expansion holds

\[
J_t(U(\xi)) = 8\pi(1 + \alpha)t + 8\pi t \sum_{i=1}^m \phi_1(\xi_i) + 16\pi(2 + \alpha) \sum_{i=1}^m \log |\xi_i - p| + 16\pi \sum_{i \neq j}^m \log |\xi_i - \xi_j| + O(1)
\tag{5.5}
\]

uniformly for all points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \) and for all \( t \) large enough.

Observe first that by (2.8) and (2.9),

\[
\frac{1}{2} \int_\Omega |\nabla U|^2 = -\frac{1}{2} \int_\Omega U_0 \Delta U_0 - \sum_{j=1}^m \int_\Omega U_j \Delta U_0 - \frac{1}{2} \sum_{i,j=1}^m \int_\Omega U_j \Delta U_i
\]

\[
= -\frac{1}{2} \int_\Omega (u_0 + H_0) \Delta u_0 - \sum_{j=1}^m \int_\Omega (u_j + H_j) \Delta u_0 - \frac{1}{2} \sum_{i,j=1}^m \int_\Omega (u_j + H_j) \Delta u_i.
\tag{5.6}
\]

Let us analyze the behavior of the first term. By (2.4), (2.5) and (2.10) we get

\[
-\int_\Omega (u_0 + H_0) \Delta u_0 = \int_\Omega \frac{8\varepsilon_{\mu_0}^2(1 + \alpha)^2|x - p|^{2\alpha}}{(\varepsilon_{\mu_0}^2 + |x - p|^{2(1+\alpha)})^2} \left[ \log \frac{1}{(\varepsilon_{\mu_0}^2 + |x - p|^{2(1+\alpha)})^2} + (1 + \alpha)H(x, p) + O(\varepsilon_{\mu_0}^2) \right].
\]

Making the change of variables \( \rho_0 v_0 = z - p \), we can derive that

\[
-\int_\Omega (u_0 + H_0) \Delta u_0 = \int_{\Omega_{\rho_0 v_0}} \frac{8(1 + \alpha)^2|z|^{2\alpha}}{(1 + |z|^{2(1+\alpha)})^2} \left[ \log \frac{(\varepsilon_{\mu_0}^2)^{-4}}{(1 + |z|^{2(1+\alpha)})^2} + (1 + \alpha)H(p, p) + O(\rho_0 v_0 |z|) + O(\varepsilon_{\mu_0}^2) \right],
\]

where \( \Omega_{\rho_0 v_0} = \rho_0 v_0 (\Omega - \{p\}) \). Note that

\[
\int_{\Omega_{\rho_0 v_0}} \frac{8(1 + \alpha)^2|z|^{2\alpha}}{(1 + |z|^{2(1+\alpha)})^2} = 8\pi(1 + \alpha) + O(\varepsilon_{\mu_0}^2),
\]

where...
and
\[
\int_{\Omega_{\rho_0\mu_0}} 8(1 + \alpha)^2 |z|^{2\alpha} \log \frac{1}{1 + |z|^{2(1 + \alpha)}} = -16\pi (1 + \alpha) + O(\varepsilon_0^2 \mu_0^2).
\]

Then
\[
- \int_{\Omega} (u_0 + H_0) \Delta u_0 = 8\pi (1 + \alpha) [(1 + \alpha)H(p, p) - 2 - 4 \log(\varepsilon_0 \mu_0)] + O(\rho_0 \varepsilon_0) + O(\varepsilon_0^2 \mu_0^2 \log(\varepsilon_0 \mu_0)).
\] (5.7)

For the second term of (5.6), by (2.4), (2.5), (2.11) and the change of variables \(\rho_0 \varepsilon_0 z = x - p\) we have that for any \(j = 1, \ldots, m\),
\[
- \int_{\Omega} (u_j + H_j) \Delta u_0 = \int_{\Omega} \frac{8\varepsilon_0^2 \mu_0^2 (1 + \alpha)^2 |x - p|^{2\alpha}}{(\varepsilon_0^2 \mu_0^2 + |x - p|^{2(1 + \alpha)})^2} \left[ \log \frac{1}{(\varepsilon_j^2 \mu_j^2 + |x - \xi_j|^2)^2} + H(x, \xi_j) + O(\varepsilon_j^2 \mu_j^2) \right] dx
\]
\[
= \int_{\Omega_{\varepsilon_0 \mu_0}} \frac{8(1 + \alpha)^2 |z|^{2\alpha}}{(1 + |z|^{2(1 + \alpha)})^2} \left[ \log \frac{1}{|p - \xi_j|^4} + H(p, \xi_j) + O(\rho_0 \varepsilon_0 t^2 |z|) + O(\varepsilon_j^2 \mu_j^2 t^{2\beta}) \right] dz
\]
\[
= 8\pi (1 + \alpha) G(p, \xi_j) + O(\rho_0 \varepsilon_0 t^2) + O(\varepsilon_j^2 \mu_j^2 t^{2\beta}).
\] (5.8)

As for the last term of (5.6), by (2.4), (2.5), (2.11) and the change of variables \(\varepsilon_i \mu_i z = x - \xi_i\) we observe that for any \(i, j = 1, \ldots, m\),
\[
- \int_{\Omega} (u_j + H_j) \Delta u_i = \int_{\Omega_{\varepsilon_i \mu_i}} \frac{8\varepsilon_i^2 \mu_i^2}{|x - \xi_j|^2} \left[ \log \frac{1}{(\varepsilon_j^2 \mu_j^2 + |x - \xi_j|^2)^2} + H(x, \xi_j) + O(\varepsilon_j^2 \mu_j^2) \right] dx
\]
\[
= \int_{\Omega_{\varepsilon_i \mu_i}} \frac{8}{|x|^2} \log \frac{1}{(\varepsilon_j^2 \mu_j^2 + |\xi_j - \xi_j + \varepsilon_i \mu_i z|^2)^2} + H(\xi_i, \xi_j) + O(\varepsilon_i \mu_i |z|) + O(\varepsilon_j^2 \mu_j^2) \right] dz,
\]
where \(\Omega_{\varepsilon_i \mu_i} = \frac{1}{\varepsilon_i \mu_i}(\Omega - \{\xi_i\})\). Then for all \(i, j = 1, \ldots, m\),
\[
- \int_{\Omega} (u_j + H_j) \Delta u_i = \begin{cases} 8\pi [H(\xi_i, \xi_j) - 2 - 4 \log(\varepsilon_i \mu_i)] + O(\varepsilon_i \mu_i) & \forall i = j, \\
8\pi G(\xi_i, \xi_j) + O(\varepsilon_i \mu_i t^2) + O(\varepsilon_j^2 \mu_j^2 t^{2\beta}) & \forall i \neq j.
\end{cases}
\] (5.9)

On the other hand, by (2.18), (2.19), (2.20) and the change of variables \(x = \varepsilon_0 y = e^{-t/4} y\), we obtain
\[
\int_{\Omega} |x - p|^{2\alpha} k(x)e^{-tφ_i}e^U = \int_{\Omega_{\xi_i}} \varepsilon_0 y - p|^{2\alpha} k(\varepsilon_0 y)e^{-t[φ_i(\varepsilon_0 y) -1]} e^{U(\varepsilon_0 y) -2t} dy
\]
\[
= \int_{\Omega_{\xi_i}} \left[ \bigcup_{i=1}^m B_{\frac{1}{\varepsilon_0 \mu_0}}(\xi_i) \cup \bigcup_{i=1}^m B_{\frac{1}{\varepsilon_0 \mu_0}}(p') \right] W dy + \int_{\Omega_{\xi_i}} \left[ \bigcup_{i=1}^m B_{\frac{1}{\varepsilon_0 \mu_0}}(p') \right] W dy + \sum_{i=1}^m \int_{\Omega_{\xi_i}} \left[ \bigcup_{i=1}^m B_{\frac{1}{\varepsilon_0 \mu_0}}(p') \right] W dy.
\]

By (2.28), (2.30) and (2.32) we obtain
\[
\int_{\Omega_{\xi_i}} \left[ \bigcup_{i=1}^m B_{\frac{1}{\varepsilon_0 \mu_0}}(\xi_i) \cup \bigcup_{i=1}^m B_{\frac{1}{\varepsilon_0 \mu_0}}(p') \right] W dy = \int_{\Omega_{\xi_i}} \left[ \bigcup_{i=1}^m B_{\frac{1}{\varepsilon_0 \mu_0}}(\xi_i) \cup \bigcup_{i=1}^m B_{\frac{1}{\varepsilon_0 \mu_0}}(p') \right] O\left(\frac{\varepsilon_0^2 e^{-tφ(\varepsilon_0 y)}}{|\varepsilon_0 y - p|^{1+2\alpha} \prod_{i=1}^m |\varepsilon_0 y - \xi_i|^4} dy \right)
\]
\[
= O(1),
\]
and
\[
\int_{B_{\frac{1}{\varepsilon_0 \mu_0}}(p')} W dy = \int_{B_{\frac{1}{\varepsilon_0 \mu_0}}(p')} \frac{\varepsilon_0}{\rho_0 \varepsilon_0} \left(1 + \alpha\right)^2 \frac{|\varepsilon_0 y - p|^{2\alpha} \rho_0 \varepsilon_0}{(1 + |\varepsilon_0 y - p|^{2(1 + \alpha)})^2} \left[ 1 + O(\varepsilon_0 t^2 |y - p|) + o(1) \right] dy
\]
\[
= 8\pi (1 + \alpha) + o(1),
\]
Thanks to (5.1), (5.6)-(5.10) we conclude that

\[ J_t(U(\xi)) = -16\pi(1 + \alpha) \log(\varepsilon_0 \mu_0) - 16\pi \sum_{i=1}^{m} \log(\varepsilon_i \mu_i) + 8\pi \sum_{i=1}^{m} \left[ (1 + \alpha)G(p, \xi_i) + \sum_{j=i+1}^{m} G(\xi_j, \xi_i) \right] + O(1), \]

which, together with the definitions of \( \varepsilon_0, \varepsilon_i \) in (2.7) and the choices of \( \mu_0, \mu_i \) in (2.12)-(2.13), implies that expansion (5.5) holds.

**Step 2:** For any integer \( m \geq 1 \) and any \( t \) large enough, let us claim that the following expansion holds

\[ F_t(\xi) = J_t(U(\xi)) + o(1) \]  

uniformly on points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega_t \). Indeed, let

\[ I_t(\omega) = \frac{1}{2} \int_{\Omega_t} |\nabla \omega|^2 - \int_{\Omega_t} |\varepsilon_0 y - p|^{2}\gamma q(y, t)e^{\omega}, \quad \omega \in H^1_0(\Omega_t). \]  

By (2.7) and (2.19) we obtain

\[ F_t(\xi) - J_t(U(\xi)) = I_t(V(\xi') + \phi_\xi') - I_t(V(\xi')). \]

Using \( D I_t(V + \phi_\xi')[\phi_\xi'] = 0 \), a Taylor expansion and an integration by parts, we give

\[ F_t(\xi) - J_t(U(\xi)) = \int_0^1 D^2 I_t(V + \tau \phi_\xi')[\phi_\xi']^2(1 - \tau)d\tau \]

\[ = \int_0^1 \left\{ \int_{\Omega_t} [N(\phi_\xi') + E|\phi_\xi' - W[1 - e^{\tau \phi_\xi'}]\phi_\xi']^2 \right\}(1 - \tau)d\tau. \]

Thanks to \( \|\phi_\xi'\|_{L^\infty(\Omega_t)} \leq C_t \max \left\{ (\rho_0 \nu_0)^{\min(1,2(\alpha - \hat{\alpha}))} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta}, \|(1 - \hat{\xi})\phi_1\|_{L^\infty(\Omega_t)} \right\} \) and the estimates in Lemma 3.2 and Proposition 4.1, we have readily

\[ F_t(\xi) - J_t(U(\xi)) = O \left( t^{\min \left\{ (\rho_0 \nu_0)^{\min(2,4(\alpha - \hat{\alpha}))} t^{\beta}, \varepsilon_0 \gamma_1 t^{\beta}, \ldots, \varepsilon_0 \gamma_m t^{\beta} \right\}}, \varepsilon_0 \gamma_{m+1} t^{\beta}, \|e^{\hat{\xi} - \hat{\xi}}\|_{L^\infty(\Omega_t)} \right) = o(1). \]

The continuity in \( \xi \) of the above expression is inherited from that of \( \phi_\xi' \) in the \( L^\infty \) norm.

**Step 3:** Proof of Proposition 5.1. Let \( \xi_t = (\xi_{t1}, \ldots, \xi_{tm}, t) \) be the maximizer of \( F_t \) over \( \bar{\Omega}_t \). We need to prove that \( \xi_t \) belongs to the interior of \( \Omega_t \). First, we obtain a lower bound for \( F_t \) over \( \bar{\Omega}_t \). Let us fix the point \( p \) as a strict local maximum point of \( \phi_1 \) in \( \Omega \) and set

\[ \xi^0_t = p + \frac{1}{\sqrt{t}} \hat{\xi}_t, \]

where \( \hat{\xi}_t = (\hat{\xi}_1, \ldots, \hat{\xi}_m) \) is a \( m \)-regular polygon in \( \mathbb{R}^2 \). Clearly, \( \xi^0_t = (\xi^0_{t1}, \ldots, \xi^0_{tm}) \in \Omega_t \) because \( \beta > 1 \) and \( \phi_1(\xi^0_t) = 1 + O(t^{-1}) \). By (5.5) and (5.11) we find

\[ \max_{\xi \in \bar{\Omega}_t} F_t(\xi) \geq 8\pi(1 + \alpha)t + 8\pi \sum_{i=1}^{m} \phi_i(\xi^0_t) + 16\pi(2 + \alpha) \sum_{i=1}^{m} \log|m - p| + 16\pi \sum_{i \neq j} \log|m - \xi^0_t - \xi^0_j| + O(1) \]

\[ \geq 8\pi m(1 + \alpha)t - 8\pi m(m + 1 + \alpha) \log t + O(1). \]  

Next, we suppose \( \xi_t = (\xi_{t1}, \ldots, \xi_{tm}, t) \in \partial \Omega_t \). Then there exist three possibilities:

**C1.** There exists an \( i_0 \) such that \( \phi_1(\xi_{i_0}, t) = 1 - \frac{1}{\sqrt{t}} \).
C2. There exist indices \( i_0, j_0 \) such that \( |\xi_{i_0,t} - \xi_{j_0,t}| = t^{-\beta} \).

C3. There exists an \( i_0 \) such that \( |\xi_{i_0,t} - p| = t^{-\beta} \).

For the first case, we have

\[
\max_{\xi \in \mathcal{C}_t} F_t(\xi) = F_t(\xi_t) \leq 8\pi(1 + \alpha)t + 8\pi t \left[ (m - 1) + 1 - \frac{1}{\sqrt{t}} \right] + O(\log t),
\]

which contradicts (5.13). For the second case, we have

\[
\max_{\xi \in \mathcal{C}_t} F_t(\xi) = F_t(\xi_t) \leq 8\pi(m + 1 + \alpha)t - 16\pi \beta \log t + O(1).
\]

For the last case, we have

\[
\max_{\xi \in \mathcal{C}_t} F_t(\xi) = F_t(\xi_i) \leq 8\pi(m + 1 + \alpha)t - 16\pi(2 + \alpha)\beta \log t + O(1).
\]

Combining (5.15)-(5.16) with (5.13), we give

\[
16\pi(2 + \alpha)\beta \log t + O(1) \leq 8\pi m(m + 1 + \alpha) \log t + O(1),
\]

which is impossible by the choice of \( \beta \) in (2.3). \( \square \)

6. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. According to Proposition 4.1, we have that for any integer \( m \geq 1 \), any points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \) and any \( t \) large enough, there exists a function \( \phi' \) such that

\[
-\Delta (V'(\xi') + \phi') - |\varepsilon_0 y - p|^{2\alpha} q(y, t)e^{V'(\xi') + \phi'} = \sum_{i=1}^{m} \sum_{j=1}^{2} c_{ij}(\xi') \chi_{ij} \int_{\Omega_t} \chi_{ij} \phi' = 0
\]

for some coefficients \( c_{ij}(\xi') \), \( i = 1, \ldots, m, \ j = 1, 2 \). Therefore, in order to construct a solution to problem (2.17) and hence to the original problem (1.4), we need to adjust \( \xi \in \mathcal{O}_t \) such that the above coefficients \( c_{ij}(\xi') \) satisfy

\[
c_{ij}(\xi') = 0 \quad \text{for all } i = 1, \ldots, m, \ j = 1, 2. \tag{6.1}
\]

On the other hand, from Proposition 5.1, there is a \( \xi_t = (\xi_{t,1}, \ldots, \xi_{t,m,t}) \in \mathcal{O}_t^{\gamma} \) that achieves the maximum for the maximization problem in Proposition 5.1. Let \( \omega_t = V(\xi_t') + \phi_{\xi_t}' \). Then we have

\[
\partial_{\xi_{kl}} F_t(\xi_t) = 0 \quad \text{for all } k = 1, \ldots, m, \ l = 1, 2. \tag{6.2}
\]

Notice that by (5.1), (5.2) and (5.12),

\[
\partial_{\xi_{kl}} F_t(\xi_t) = \partial_{\xi_{kl}} J_t(U(\xi_t) + \phi(\xi_t)) = \frac{1}{\varepsilon_0} \partial_{\xi_{kl}} J_t(V(\xi_t') + \phi_{\xi_t}' - \sum_{i=1}^{m} \sum_{j=1}^{2} c_{ij}(\xi') \chi_{ij} \int_{\Omega_t} \chi_{ij} \phi' = 0.
\]

Then for all \( k = 1, \ldots, m \) and \( l = 1, 2 \),

\[
\sum_{i=1}^{m} \sum_{j=1}^{2} c_{ij}(\xi') \int_{\Omega_t} \chi_{ij} [\partial_{\xi_{kl}} V(\xi_t') + \partial_{\xi_{kl}} \phi_{\xi_t}'] = 0.
\]

Since \( \partial_{\xi_{kl}} V(\xi_t')(y) = Z_{kl}(y) + O(\varepsilon_0 t^\beta) \) and \( \| \partial_{\xi_{kl}} \phi_{\xi_t}' \|_{L^\infty(\Omega_t)} \leq C t^2 \max \{ (\rho_0 v_0)^{\min(1,2(\alpha-\delta))} t^\beta, \varepsilon_0^0 t^\beta, \ldots, \varepsilon_0^{m-1} t^\beta, \| e^{-\frac{1}{2} \phi_{\xi_t}'} \|_{L^\infty(\Omega_t)} \} \), by (2.7), (2.14), (2.15) and (2.23) we get the validity of a system of equations of the form

\[
\sum_{i=1}^{m} \sum_{j=1}^{2} c_{ij}(\xi') \int_{\Omega_t} \chi_{ij} [Z_{kl}(y) + o(1)] = 0, \quad k = 1, \ldots, m, \ l = 1, 2. \tag{6.3}
\]
Note that
\[ \int_{\Omega} \chi_i Z_{ij} Z_{kl}(y) = \begin{cases} \int_{\mathbb{R}^2} \chi(|z|) Z_j(z) Z_l(z) dz = C \delta_{jl} & \text{if } i = k, \\ O(\varepsilon \alpha_i \beta) & \text{if } i \neq k. \end{cases} \]

Hence the coefficient matrix of system (6.3) is strictly diagonal dominant and then \( c_{ij}(\xi) = 0 \) for all \( i = 1, \ldots, m, j = 1, 2 \). As a consequence, we obtain a solution \( u_t \) to problem (1.4) of the form \( U(\xi_t) + \phi(\xi_t) \) with the qualitative properties as predicted in Theorem 1.1. \( \square \)

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