No signaling and strong subadditivity condition for tomographic q-entropy of single qudit states

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1. Introduction

Quantum correlations such as the entanglement phenomenon associated with properties of a multipartite system states provide also specific entropic and information inequalities for such systems. In our study we consider entropic properties of systems without subsystems. The aim of the work is to review and obtain some new q-entropic inequalities for a single-qudit state tomogram, including a strong subadditivity condition for a noncomposite quantum system and to demonstrate the no-signaling property for such a system state. The subadditivity and strong subadditivity conditions for Shannon [1] and von Neumann [2] entropies are known for bipartite and three-partite system states, respectively, both for classical and quantum ones. Recently, the approach was suggested [3–8] to extend for composite systems the known entropic and information relations like equalities and inequalities to systems without subsystems. Among these relations there are inequalities for Tsallis entropy of the bipartite systems and other inequalities for von Neumann entropy proved and studied in [5, 9–31]. The idea of this approach is to use a map of integers $N$ on pairs of integers $(jk)$ or triples of integers $(jkl)$, etc. This map of integers provides the possibility to consider vectors $\beta$ with components $p_\beta$ as the matrix $P$ with matrix elements $P_{jk}$; i.e., we map the index $s$ onto the combined index $s \leftrightarrow (jk)$ or interpret the index $s$ as the function $s \equiv s(jk)$. Analogously, the matrices $\rho_{ss'}$ can be considered as matrices $P_{jk,j'k'}$ since, due to the described map, the integers $s$ and $s'$ are functions of the two variables $s(jk)$ and $s'(j'k')$. Also the vector with components $p_s$ can be considered as the table of numbers $P_{jk}$ due to the invertable map $s \leftrightarrow (jk)$. In this case the matrix $\rho_{ss'}$ can be considered as matrix $P_{jk,j'k'}$. After such a map, one can apply the analog of the partial tracing procedure used for probability vectors or joint probability distributions and density matrices of composite systems to get the reduced matrices. For specific maps such a tool was called the portrait method [32, 33] to be applied for studying entanglement of multiqudit states. The joint probability $P(ab|xy)$ of two random variables $a, b$ depending on two extra parameters $x$ and $y$ has the no-signaling property, which means that $\sum_x P(ab|xy)$ does not depend on parameter $y$, and $\sum_y P(ab|xy)$ does not depend on parameter $x$. The tomogram [34, 35] of two-qudit states has this property [3]. We will discuss this no-signaling phenomenon also for single-qudit state tomograms. The strong subadditivity condition was proved for the von Neumann (and Shannon) entropy of a three-partite system [13]. It is an inequality for entropies of the tomography system without subsystems. This inequality is not valid [11] for the Tsallis quantum entropy [36]. The new result which we present in the work is to show that the $q$-entropies associated with tomograms of quantum states both for a three-qudit
system and for a single-qudit system obey the strong subadditivity condition.

The paper is organized as follows.

In the second section, we study the no-signaling property of two-qudit state tomograms. In the third section, we discuss the no-signaling properties of single-qudit state tomograms. In the fourth section we consider an example of a qudit with $j = 5/2$ and study its no-signaling properties. In the fifth section, we discuss the q-deformed strong subadditivity condition in the case of a single-qudit state. In conclusion, we list our main results. In the appendix we present new formulas for arbitrary nonnegative Hermitian matrices.

2. No-signaling property of the two-qudit state tomograms

The tomogram $w(m_1, m_2 | u)$ of the two-qudit state with spins $j_1$ and $j_2$ and with density matrix $\rho(1, 2)_{m_1 m_2, n_1 n_2}$ where spin projections $m_1(m')_1, m_2(m')_2$ take semionteger values $-j_1, -j_1 + 1, \ldots j_1 - 1, j_1$ and $-j_2, -j_2 + 1, \ldots j_2 - 1, j_2$, respectively, is the fair probability distribution defined as a set of diagonal matrix elements of the matrix $u \rho(1, 2) u^\dagger$, i.e.,

$$w(m_1, m_2 | u) = \left( u \rho(1, 2) u^\dagger \right)_{m_1 m_2, m_2 m_1}.$$ (1)

Here $u$ is a unitary matrix describing the global unitary transform in the Hilbert space $H = H_1 \times H_2$, and the Hilbert spaces $H_1$ and $H_2$ are the spaces of states for first and second qudits, respectively. The dimension of the space $H$ equals $N = nm$, where dimensions of spaces $H_1$ and $H_2$ are equal to integers $n$ and $m$. One has relations $n = 2j_1 + 1, m = 2j_2 + 1$. If the matrix $u$ equals the tensor product of the unitary $n \times n$ -matrix $u_1$ and the unitary $m \times m$-matrix $u_2$, i.e., $u = u_1 \otimes u_2$, the tomogram of the quantum state reads

$$w(m_1, m_2 | u_1, u_2) = \left( u_1 \otimes u_2 \rho(1, 2) u_1^\dagger \otimes u_2^\dagger \right)_{m_1 m_2, m_2 m_1}.$$ (2)

If the matrices $u_1$ and $u_2$ are matrices of irreducible representations of the group SU(2) corresponding to spins $j_1$ and $j_2$, respectively, the tomogram of the quantum state of the qudits is spin-tomogram

$$w(m_1, m_2 | n_1, n_2) = \left( n_1 \otimes n_2 \rho(1, 2) n_1^\dagger \otimes n_2^\dagger \right)_{m_1 m_2, m_2 m_1}.$$ (3)

Here $n_1$ and $n_2$ are unit vectors depending on angles $\phi_1, \theta_1$ and $\phi_2, \theta_2$ perpendicular to the Poincare sphere. The physical meaning of the tomogram (3) is the following. It equals the joint probability distribution to get the spin projections $m_1$ and $m_2$ on the quantization axes determined by vectors $n_1$ and $n_2$. From the physical meaning of the spin tomogram follows the relation for the marginal probability distribution $w(m_1 | u) = \sum_{m_2} w(m_1, m_2 | u)$, and the relation is

$$w_1(m_1 | u_1 \otimes u_2) \equiv w_1(m_1 | u_1).$$ (4)

For spin tomogram (3) one has

$$w_1(m_1 | n_1, n_2) = \sum_{m_2=-j_2}^{j_2} w_1(m_1, m_2 | n_1, n_2) \equiv w_1(m_1 | n_1).$$ (5)

It means that such parameters of a second subsystem (second qudit) like $u_2$ or $n_2$ do not determine the tomographic probability distribution of the first spin (first qudit). This property is called the no-signaling property. On the other hand, obvious from the point of view of physical properties of two random observables, this feature is a numerical relation for matrix elements of the matrix $u \rho(1, 2) u^\dagger$. This observation provides the possibility to find the analog of the no-signaling property for other systems, e.g., for the single-qudit state with spin $j$. We follow the approach [5, 9–17, 21, 29–31] to use the bijective maps of integers or semiontegers to get the no-signaling property for a system without subsystems.

3. No signaling for single-qudit state tomogram

Given the density $N \times N$-matrix $\rho$ of a single-qudit state where $N = nm = 2j + 1$, let us use the map $| - j \leftrightarrow 1, - j + 1 \leftrightarrow 2, \ldots, j - 1 \leftrightarrow N - 1, j \leftrightarrow N$. The matrix elements of the density matrix of the single-qudit state in this case are $\rho_{j\alpha}$, where $\alpha, \beta = 1, 2, \ldots, N$. The matrix $\rho$ can be presented in block form

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn} \end{pmatrix}.$$ (6)

Here the blocks $\rho_{jk}$ with $j, k = 1, 2, \ldots, n$ are $m \times m$-matrices. One can construct two matrices $\rho_1$ and $\rho_2$ using the blocks $\rho_{jk}$. The $n \times n$ matrix $\rho_1$ has the matrix elements $(\rho_1)_{jk} = \text{Tr} \rho_{jk}$. The $m \times m$-matrix $\rho_2 = \sum_{j,k} \rho_{jk}$. In the case of a two-qudit state with density matrix $\rho \equiv \rho(1, 2)$, considered in the previous section, the matrix $\rho_1$ coincides with the density matrix of the first qudit state, and the matrix $\rho_2$ coincides with the density matrix of the second qudit state. In this case the tomogram $w_1(m_1 | u_1)$ is determined by the diagonal elements of the matrix $(u_1 \rho_1 u_1^\dagger)_{m_1 m_1}$, and the diagonal tomogram $w_2(m_2 | u_2) = (u_2 \rho_2 u_2^\dagger)_{m_2 m_2}$. On the other hand, the tomographic probability distribution which determines the density matrix $\rho$ of a single-qudit state equals the diagonal matrix element of the $N \times N$-matrix $u \rho u^\dagger$. Since available numerical relations for the matrix elements of the matrices do not depend on any interpretation of the matrices we can write these relations for the Hermitian nonnegative $N \times N$-matrix $\rho$ with $\text{Tr} \rho = 1$ and unitary matrices $u, u_1$ and $u_2$. The relations can be presented in the form of relations for probability vector $\bar{w}_\rho(u)$ with $N$ components equal to the diagonal matrix elements of the matrix $u \rho u^\dagger$. The probability vector reads [37]

$$\bar{w}_\rho(u) = |u u_0|^2 \bar{p}.$$ (7)
Here $\hat{\rho}$ is the $N$-vector with components equal to eigenvalues of the matrix $\rho$. The unitary $N \times N$-matrix $u_0$ has the columns which are corresponding eigenvectors of the matrix $\rho$. The notation for the $N \times N$-matrix $|a|^2$ means that the matrix elements of the matrix $|a|_a^j = |a_a^j|^2$. Let us introduce stochastic $N \times N$-matrix $M^{(1)}$ given in the block form analogous to (6), i.e.,

$$M^{(1)} = \begin{pmatrix} M_{11}^{(1)} & M_{12}^{(1)} & \cdots & M_{1n}^{(1)} \\ M_{21}^{(1)} & M_{22}^{(1)} & \cdots & M_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n1}^{(1)} & M_{n2}^{(1)} & \cdots & M_{nn}^{(1)} \end{pmatrix}. \quad (8)$$

The blocks $M_{jk}^{(1)}$ for $j \geq 2$ are zero $m \times m$-matrices. The blocks $M_{nn}^{(1)}$ have all nonzero matrix elements equal to one in $k$th row. The probability $N$-vector

$$\vec{w}_1(u) = M^{(1)} |u u_0^j \hat{\rho}.$$ \quad (9)

has the property

$$\vec{w}_1(u_1 \otimes u_2) = M^{(1)} |u_1 \otimes 1 |u_2^j \hat{\rho}. \quad (10)$$

In (10) the unitary matrix $u_2 = 1$ is the $m \times m$-matrix. This probability vector (10) has only $n$ first nonzero components. The $n$-vector with these components is the tomographic probability distribution, which determines the $n \times n$-matrix $\rho_1$. The components of the vector are equal to diagonal elements of the $n \times n$-matrix $u_1 \rho_1 u_1^\dagger$.

Analogously, we introduce the stochastic $N \times N$-matrix $M^{(2)}$ of the block form with blocks $M_{jk}^{(2)}$, where the only nonzero blocks are blocks $M_{11}^{(2)}$, which are equal to unity matrices. The probability $N$-vector determined by the matrix $M^{(2)}$ reads

$$\vec{w}_2(u) = M^{(2)} |u u_0^j \hat{\rho}. \quad (11)$$

It has first $m$ nonzero components. The property of this vector (11) analogous to (10) is

$$\vec{w}_2(u_1 \otimes u_2) = M^{(2)} |1 \otimes u_2^j |u_1 \otimes u_2 |u_2^j \hat{\rho}. \quad (12)$$

In (12) the unitary matrix $u_1 = 1$ is the $n \times n$-matrix. The independence of tomographic probability vectors $\vec{w}_1(u_1 \otimes u_2)$ and $\vec{w}_2(u_1 \otimes u_2)$ on the unitary transforms $u_2$ and $u_1$, respectively, reflects the no-signaling property. For the bipartite system the vectors are just tomographic probability distributions of the subsystem states.

Thus we conclude that for any single-qudit quantum state with density $N \times N$-matrix $\rho$, if $N = mn$, there exist properties of probability distribution associated with probability vector (7), which are analogous to the no-signaling properties of the tomograms of bipartite system states. We illustrate the presented results on examples of a qudit with $j = 5/2$ and an analogous bipartite qubit-qutrit composite system.

4. Example of no signaling for qudit $j = 5/2$

Let $6 \times 6$-matrix $\rho$ be the density matrix of a qudit a state for $j = 5/2$. It is presented in block form (6), i.e., for $n = 2$, $m = 3$ one has

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \quad (13)$$

where the blocks $\rho_{jk}$, $(j, k = 1, 2)$, are $3 \times 3$-matrices. In this case the stohastic $6 \times 6$-matrices $M^{(1)}$ and $M^{(2)}$ read

$$M^{(1)} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$M^{(2)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (14)$$

The $2 \times 2$-matrix $\rho_1$ and $3 \times 3$-matrix $\rho_2$ are

$$\rho_1 = \begin{pmatrix} \text{Tr} \rho_{11} & \text{Tr} \rho_{12} \\ \text{Tr} \rho_{21} & \text{Tr} \rho_{22} \end{pmatrix}, \quad \rho_2 = \rho_{11} + \rho_{22}. \quad (15)$$

The 6-vector $\vec{w}(u)$ has the components

$$(w(+5/2|u), w(+3/2|u), w(+1/2|u), w(-1/2|u), w(-3/2|u), w(-5/2|u))$$

which are diagonal elements of the matrix $u_\mu^\dagger u$. We used natural notation $w(\bar{m}|u)$, where $\bar{m}$ is the spin projection value in the reference frame rotated by means of a unitary $6 \times 6$-matrix in the Hilbert space $H$ of the qudit states. It means that we use the map of indices $1 \leftrightarrow +5/2$, $2 \leftrightarrow +3/2$, $3 \leftrightarrow +1/2$, $4 \leftrightarrow -1/2$, $5 \leftrightarrow -3/2$, $6 \leftrightarrow -5/2$ to label the matrix elements of the density matrix $\rho_{\bar{m}\bar{n}}$, ($\alpha, \beta = 1, 2, \ldots, 6$). The tomogram of the matrix $\rho$, which is a 6-vector $\vec{w}(u)$, gives two 6-vectors $\vec{w}_1(u) = M^{(1)} \vec{w}(u)$ and $\vec{w}_2(u) = M^{(2)} \vec{w}(u)$. The 6-vector $\vec{w}_1(u)$ has two nonzero first components, and the 6-vector $\vec{w}_2(u)$ has three nonzero first components. The nonzero components provide 2-vector and 3-vector, which are analogs of vectors given by tomographic probability distributions associated with matrices $\rho_1$ and $\rho_2$, respectively. By construction these probability vectors correspond to marginal probability distributions associated with the artificial joint probability distribution given by 6-vector $\vec{w}(u)$. The 6-vectors $\vec{w}_1(u)$ and $\vec{w}_2(u)$ have the no-signaling properties; i.e., if $u = u_1 \otimes u_2$ one has

$$\vec{w}_1(u_1 \otimes u_2) \equiv \vec{w}_1(u_1 \otimes 1), \quad \vec{w}_2(u_1 \otimes u_2) \equiv \vec{w}_2(1 \otimes u_2). \quad (16)$$

These properties can be proved by direct checking. For a qubit-qutrit bipartite system with density matrix $\rho$ (13), the matrix $\rho_1$ is the density matrix of the qubit state, and the matrix $\rho_2$ is the density matrix of the qutrit state. These matrices are connected with the matrix $\rho$ by the partial tracing procedure. Nonzero components of the 6-vectors $\vec{w}_1(u_1 \otimes 1)$ and $\vec{w}_2(1 \otimes u_2)$ provide in this case the tomographic
probability vectors with components obtained as diagonal elements of the matrix \( u_i \rho u_i^\dagger \) and \( u_j \rho u_j^\dagger \), respectively.

5. Strong subadditivity condition for q-entropy of single qudit state

For a composite three-partite system with the diagonal density matrix \( \rho(1, 2, 3) \), it is known [9, 10] that the Tsallis q-entropy satisfies the strong subadditivity condition. It means that for \( q \geq 1 \)

\[
S_q(\rho(1, 2, 3)) + S_q(\rho(2)) \leq S_q(\rho(1, 2)) + S_q(\rho(2, 3)).
\]

(17)

Here \( \rho(1, 2) \), \( \rho(2, 3) \), and \( \rho(2) \) are diagonal density matrices determined by the partial tracing procedure

\[
\rho(1, 2) = \text{Tr}_3 \rho(1, 2, 3), \quad \rho(2, 3) = \text{Tr}_1 \rho(1, 2, 3), \quad \rho(2) = \text{Tr}_3 \rho(2, 3).
\]

(18)

In fact, the diagonal matrix elements of the matrix \( \rho(1, 2, 3) \) provide classical joint probability distribution of three random variables. For any density matrix the q-entropy is defined as

\[
S_q(\rho) = -\text{Tr} \rho^{1-q} \rho^{1-q} - 1 - q,
\]

where \( q \geq 1 \). We extend the inequality (17), which is valid for a composite system to the case of a noncomposite system. We will do this for tomographic probability distribution (qudit tomogram) \( w(m|u) \) or probability vector \( \vec{w}(u) \), given by (7).

Let \( 2j + 1 = N = n_1 n_2 n_3 \), where \( n_k \) are integers. Let us use the bijective map of integers onto spin projections \( 1, 2, \ldots, N \leftrightarrow s \), \( j \), \( j + 1 \), \ldots, \( j = 1, \) \( j \). Then we introduce the map of the integers onto triplets of integers, i.e., \( s \leftrightarrow s(ijk) \) where \( s = 1, 2, \ldots, N \), \( i = 1, 2, \ldots, n_1 \), \( k = 1, 2, \ldots, n_2 \), \( l = 1, 2, \ldots, n_3 \). It means that we introduce the function of three variables \( s(ijk) \).

Thus the probability vector \( \vec{w}(u) \) with \( N \) components \( w_j(u) \) can be considered as the probability vector with components \( \bar{w}_j(u)_{s(ijk)} \). Then one can construct three probability N-vectors \( \bar{w}_{12}(u), \bar{w}_{23}(u), \bar{w}_{12}(u) \) with nonzero components

\[
\left( \bar{w}_{12}(u) \right)_{s(ijk)} = \sum_{i=1}^{n_1} \left( \bar{w}(u) \right)_{s(ijk)},
\]

\[
\left( \bar{w}_{23}(u) \right)_{s(ijk)} = \sum_{i=1}^{n_1} \left( \bar{w}(u) \right)_{s(ijk)},
\]

\[
\left( \bar{w}_{12}(u) \right)_{s(ijk)} = \sum_{i=1}^{n_1} \sum_{i=1}^{n_2} \left( \bar{w}(u) \right)_{s(ijk)}.
\]

(20)

Other components of the probability vectors equal zero. The strong subadditivity condition for the quantum tomogram \( w(m|u) \) written in terms of the probability vector components reads

\[
-\frac{1}{1-q} \sum_{k=1}^{m} \sum_{k=1}^{n} \left( \bar{w}(u) \right)_{s(ijk)} \left( \bar{w}(u) \right)_{s(ijk)} - 1 - q - q
\]

\[
-\frac{1}{1-q} \sum_{k=1}^{m} \sum_{k=1}^{n} \left( \bar{w}_{12}(u) \right)_{s(ijk)} \left( \bar{w}_{12}(u) \right)_{s(ijk)} - 1
\]

\[
-\frac{1}{1-q} \sum_{k=1}^{m} \sum_{k=1}^{n} \left( \bar{w}_{23}(u) \right)_{s(ijk)} \left( \bar{w}_{23}(u) \right)_{s(ijk)} - 1
\]

\[
-\frac{1}{1-q} \sum_{k=1}^{m} \sum_{k=1}^{n} \left( \bar{w}_{12}(u) \right)_{s(ijk)} \left( \bar{w}_{12}(u) \right)_{s(ijk)} - 1.
\]

(21)

The tomogram of the single-qudit state satisfies also the subadditivity condition for q-entropy. If we use notation \( n_1 = n, n_2 n_3 = m, N = nm \), the condition reads

\[
-\frac{1}{1-q} \sum_{k=1}^{m} \sum_{k=1}^{n} \left( \bar{w}(u) \right)_{s(ijk)} \left( \bar{w}(u) \right)_{s(ijk)} - 1 - q - q
\]

\[
-\frac{1}{1-q} \sum_{k=1}^{m} \sum_{k=1}^{n} \left( \bar{w}_{12}(u) \right)_{s(ijk)} \left( \bar{w}_{12}(u) \right)_{s(ijk)} - 1
\]

\[
-\frac{1}{1-q} \sum_{k=1}^{m} \sum_{k=1}^{n} \left( \bar{w}_{23}(u) \right)_{s(ijk)} \left( \bar{w}_{23}(u) \right)_{s(ijk)} - 1
\]

\[
-\frac{1}{1-q} \sum_{k=1}^{m} \sum_{k=1}^{n} \left( \bar{w}_{12}(u) \right)_{s(ijk)} \left( \bar{w}_{12}(u) \right)_{s(ijk)} - 1.
\]

(22)

Here we introduce map of integers \( s = 1, 2, \ldots, N \) onto pairs of integers \( i = 1, 2, \ldots, m \), \( k = 1, 2, \ldots, n \). It means that we introduce the function \( s(ijk) \). The nonzero components of probability N-vectors \( \bar{w}_1(u) \) and \( \bar{w}_2(u) \) are defined as

\[
\left( \bar{w}_1(u) \right)_{s(ijk)} = \sum_{k=1}^{m} \left( \bar{w}(u) \right)_{s(ijk)},
\]

\[
\left( \bar{w}_2(u) \right)_{s(ijk)} = \sum_{i=1}^{n} \left( \bar{w}(u) \right)_{s(ijk)}. \]

(23)

The subadditivity condition (22) corresponds to the known for bipartite systems q-entropy subadditivity condition for matrix \( \rho \) and matrices \( \rho_1 \) and \( \rho_2 \)

\[
-\text{Tr} \rho^{1-q} - 1 - q - q
\]

\[
-\text{Tr} \rho_1^{1-q} - 1 - q - q
\]

\[
-\text{Tr} \rho_2^{1-q} - 1 - q - q
\]

(24)

The matrix \( \rho \) is the density matrix of the single-qudit state. The matrices \( \rho_1 \) and \( \rho_2 \) are obtained by means of the analog of the partial tracing procedure used for composite quantum systems. Also one can get the analogous inequality using a change of notation, \( n_1 n_2 = n, n_3 = m \). If \( N \neq n_1 n_2 n_3 \), one can introduce the matrix

\[
\bar{\rho} = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}.
\]

where \( N = k = n_1 n_2 n_3 \), and choosing the corresponding integer \( k \). The tomogram \( \vec{w}(m|u) \) will satisfy in this case inequalities (22) and (24) with obvious substitutions \( w \rightarrow \vec{w} \) in these formulas.
6. Example of strong subadditivity condition for qudit with $j = 7/2$

Let us consider the density matrix $\rho_{mm'}$, where

$$m, m' = -7/2, -5/2, -3/2, -1/2, +1/2, +3/2, +5/2, +7/2.$$  

The tomogram $w(m|u)$ of the quantum state with this matrix, where $u$ is the unitary 8 $\times$ 8-matrix, is described by the probability 8-vector (7) $\vec{w}(u)$ with components $w(m|u)$. Let us introduce three stochastic matrices $M^{(12)}$, $M^{(23)}$, and matrix $M^{(2)}$ of the form

$$M^{(12)} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$M^{(23)} = \begin{pmatrix} 1_4 & 1_4 \\ 0_4 & 0_4 \end{pmatrix}$$

Here $0_4$ is a zero $4 \times 4$-matrix, and $1_4$ is an identity $4 \times 4$-matrix. The new inequality, which is the strong subadditivity condition for the tomographic probability vector $\vec{w}(u)$ of the single-qudit state, reads

$$-\frac{1}{1 - q} \sum_{m=-7/2}^{7/2} \left[ (M^{(23)}w(u))_m \right]^q \times \left[ \left( (M^{(23)}w(u))_m \right)^{1-q} - 1 \right] \leq -\frac{1}{1 - q} \sum_{m=-7/2}^{7/2} \left[ (M^{(23)}w(u))_m \right]^q \times \left[ \left( (M^{(23)}w(u))_m \right)^{1-q} - 1 \right].$$  

From (27) follows the new entropic inequality

$$-\frac{1}{1 - q} \text{Tr} \rho^q(q_{1-q} - 1) - \frac{1}{1 - q} \text{Tr} \rho_{2}^{1-q} - 1 \leq -\frac{1}{1 - q} \sum_{m=-7/2}^{7/2} \left[ (M^{(23)}w(u))_m \right]^q \times \left[ \left( (M^{(23)}w(u))_m \right)^{1-q} - 1 \right].$$

In (28) the $2 \times 2$-matrix $\rho_3$ has matrix elements

$$\rho_{211} = \rho_{7/2,7/2} + \rho_{5/2,5/2} + \rho_{-1/2,-1/2} + \rho_{-3/2,-3/2},$$

$$\rho_{212} = 1 - (\rho_{211})^*,$$

and

$$\rho_{212} = (\rho_{212})^* = \rho_{-7/2,-7/2} + \rho_{-5/2,-5/2} + \rho_{1/2,1/2} + \rho_{3/2,3/2}.$$  

In the right-hand side of the equation (28) the unitary matrix $u$ is an arbitrary unitary matrix of the product form $u = u_1 \times u_2 \times u_3$, where the local transform matrices correspond to integers $n_1, n_2, n_3$. For $q \rightarrow 1$, one has the entropic inequality for the von Neumann entropy associated with the tomogram of the single-qudit state.

7. Conclusion

To summarize, we list the main results of our work. We show that the tomographic probability distribution determining the single-qudit state has the no-signaling property, which was known for joint tomographic probability distribution. We considered the no-signaling property for the tomogram of the qudit (spin) state with $j = 5/2$. We found a new entropic inequality for the single-qudit-state, and this inequality provides the relation for the Tsallis $q$-entropy associated with the qudit-state tomogram and probability vectors obtained from this tomographic probability vector by the action of stochastic matrices. In the case of a three-partite system, the inequality coincides with the strong subadditivity condition for $q$-entropies associated with tomographic probability vectors of the system and its three subsystems. As a partial case of the obtained inequality, we derived the inequality for the sum of the Tsallis quantum entropy of the single-qudit-state and the analog of one of the subsystems of the system. This new inequality in the limit $q \rightarrow 1$ is compatible with the analog of the strong subadditivity condition available for Shannon.
entropies of a three-partite system. The new inequalities obtained in this work are given by (27)–(29).

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Appendix

We present the new inequality, which is valid for arbitrary the $N \times N$-matrix $\rho$, where $N = nm$ and $\rho^\dagger = \rho$, $\text{Tr} \rho = 1$. The matrix $\rho$ is nonnegative, and we consider also the arbitrary unitary $N \times N$-matrix $u$. Then for $q \geq 1$, one has the inequality

$$1 + \sum_{a=1}^{N} \left\{ \left( u^{a} \right)_{aa} \right\}^{q} \leq \sum_{j=1}^{N} \left\{ \sum_{a=1}^{N} M_{j}^{(1)}(u)_{aa}^{q} \right\} + \sum_{a=1}^{N} M_{j}^{(2)}(u)_{aa}, \tag{29}$$

Here the stochastic $N \times N$-matrices $M_{j}^{(1)}$ and $M_{j}^{(2)}$ are given in block form (8), with the $m \times m$-blocks $M_{j}^{(1)}(u)$ and $M_{j}^{(2)}(u)$, $j, k = 1, 2, \ldots, n$. All the $m \times m$-blocks $M_{j}^{(1)}(u)$ and $M_{j}^{(2)}(u)$ for $j = 1$ have nonzero matrix elements equal to 1 only in the $a$th row for $a = 1$. All the blocks $M_{j}^{(2)}(u)$ are either zero or unity $m \times m$-matrices. For example, the $6 \times 6$-matrices $\rho$ and $u$ satisfy the inequality (29) for $q \geq 1$, where the matrices $M_{j}^{(1)}(u)$ and $M_{j}^{(2)}(u)$ are given by equation (14). The written matrix inequality gives the subadditivity condition for the system state tomogram in a case where the matrix $\rho$ is a density matrix of a bipartite system, e.g., of a two-qudit system with $2j_{1} + 1 = n, 2j_{2} + 1 = m$.

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