SPAN OF RESTRICTION OF HILBERT THETA FUNCTIONS

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ABSTRACT. In this paper, we study the diagonal restrictions of certain Hilbert theta series for a totally real field $F$, and prove that they span the corresponding space of elliptic modular forms when the $F$ is quadratic or cubic. Furthermore, we give evidence of this phenomenon when $F$ is quartic, quintic and sextic.

1. Introduction

Theta functions are classical examples of holomorphic modular forms. Given a positive definite, unimodular $\mathbb{Z}$-lattice $L$ of rank $8m$ with $m \in \mathbb{N}$, the associated theta function

$$\theta_L(\tau) := \sum_{\lambda \in L} q^{Q(\lambda)}, \quad q := e^{2\pi i \tau},$$

is in $M_{4m}$, the space of elliptic modular forms of weight $4m$ on $SL_2(\mathbb{Z})$. For example, the theta functions associated to the $E_8$ lattice and Leech lattice $\Lambda_{24}$ are explicitly given as

$$\theta_{E_8}(\tau) = E_4(\tau), \quad \theta_{\Lambda_{24}}(\tau) = E_4(\tau)^3 - 720\Delta(\tau),$$

where $E_{2k}(\tau)$ is the Eisenstein series of weight $2k$ and $\Delta(\tau)$ is the Ramanujan $\Delta$-function.

For $N \in \mathbb{N}$, we denote

$$M_Q^{(N)} := \bigoplus_{k \in \mathbb{N}} M_{Nk}$$

the finitely generated graded algebra of elliptic modular forms with weights divisible by $N$, and would like to consider the subalgebra $M_Q^\theta \subset M_Q^{(4)}$ generated by theta functions of unimodular lattices. Using the relation

$$\theta_{L_1 \oplus L_2}(\tau) = \theta_{L_1}(\tau)\theta_{L_2}(\tau),$$

for any two unimodular lattices $L_1, L_2$, we see that $M_Q^\theta$ is simply the span of such theta functions. Equation (1.2) and the fact $M_Q^{(4)} = \mathbb{C}[E_4, \Delta]$ imply that

$$M_Q^\theta = M_Q^{(4)}.$$

The construction of theta functions also extends to the case of Hilbert modular forms. Let $F$ be a totally real field of degree $d$ with ring of integers $\mathcal{O}_F$, and denote $\alpha_j \in \mathbb{R}$ the real
embeddings of $\alpha \in F$ for $1 \leq j \leq d$. For $N \in \mathbb{N}$, denote $M_F^{(N)}$ the algebra of holomorphic Hilbert modular forms of parallel weight $Nk$ for $k \in \mathbb{N}$. Given a totally positive definite, $\mathbb{Z}$-unimodular $O_F$-lattice $L$ of rank $8m$ (see Definition 1), the associated theta function

$$\theta_L(\tau) := \sum_{\lambda \in L} \prod_{j=1}^{d} q_j^{Q(\lambda)_j}, \quad \tau = (\tau_1, \ldots, \tau_d) \in \mathbb{H}^d, \quad q_j := e(\tau_j),$$

is a Hilbert modular form of parallel weight $4m$ on $SL_2(O_F)$. It is well-known that such lattice exists precisely when

$$m \in \frac{1}{d_2} \mathbb{N}, \quad d_2 := \gcd(2, d)$$

(see Prop. 2.1). However, their explicit constructions and classification have only been carried out when $d$ is small (see e.g. [Sch94, Wan14]). As a result, the relationship between $M_F^{(4/d_2)}$ and the subalgebra $M_F^\theta$ generated by such $\theta_L$ is not clear.

On the other hand, we have the following diagonal restriction map

$$M_F^{(N)} \to M_Q^{(Nd)}$$

$$f \mapsto f^\Delta(\tau) := f(\tau^\Delta),$$

where $\tau^\Delta = (\tau, \ldots, \tau) \in \mathbb{H}^d$. In this note, we will investigate the question about the image of $M_F^\theta$ under this map, which is denoted by $(M_F^\theta)^\Delta$ and contained in $M_Q^{(4d/d_2)}$. The main result is as follows.

**Theorem 1.1.** For a totally real field $F$ of degree $d = 2, 3$, we have

$$(M_F^\theta)^\Delta = M_Q^{(4d/d_2)}.$$

Based on this, it is then natural to make the following conjecture.

**Conjecture 1.** Equation (1.8) holds for any totally real field $F$ of degree $d$.

To prove Theorem 1.1 we apply an instance of the Siegel-Weil formula to see that the Hecke Eisenstein series $E_{F,k}$ defined in (2.5) is contained in $M_F^\theta$ for all $k \in (4d_2)\mathbb{N}$. Then we calculate the Petersson inner product between the diagonal restriction of $E_{F,k}$ and an elliptic cusp form. For $d = 2$, this inner product is related to Fourier coefficients of half-integral weight modular forms by a result of Kohnen-Zagier [KZ84]. For $d \geq 3$, we give an expression for this inner product in terms of a sum over the double coset $\Gamma_F/\Gamma_Q$ (see Prop. 3.1). When $d = 3$, we related this double coset to orders in a cubic field $F$ (see section 4). Using these results, we show that when $d = 2, 3$, $M_Q^{(4d/d_2)}$ can be generated by $E_{F,k}^\Delta$ and $\theta_L^\Delta$ for a $\mathbb{Z}$-unimodular $O_F$-lattice $L$.

The same approach can be used to check conjecture 1 numerically when $d \in \{4, 5, 6, 8, 10\}$. We list some results for $d = 4, 5, 6$ and $F$ has small discriminants in the last section (see Theorem 6.1).
Acknowledgement: We thank Jan Bruinier and Tonghai Yang for helpful discussion about Prop. 2.3. The authors are supported by the LOEWE research unit USAG, and by the Deutsche Forschungsgemeinschaft (DFG) through the Collaborative Research Centre TRR 326 “Geometry and Arithmetic of Uniformized Structures”, project number 444845124.

2. Preliminary

Let $F$ be a totally real field of degree $d$ with ring of integers $\mathcal{O}_F$ and different $d_F$. Denote $\mathrm{Cl}(F)$ the (wide) class group of $F$. Let $(V, Q)$ be an $F$-quadratic space of dimension $n$. We say that $V$ is \textit{totally positive} if $V \otimes_{\mathcal{O}_F} \mathbb{R}$ is totally positive for every real embedding $\iota: F \hookrightarrow \mathbb{R}$. In that case, $\text{SO}_V(\mathbb{R})$ is compact and the double quotient $\text{SO}_V(F) \backslash \text{SO}_V(\hat{F})/K$ is a finite set for any open compact subgroup $K \subset \text{SO}_V(\hat{F})$. Here $\mathbb{A}_F$ and $\hat{F}$ are the adele and finite adele of $F$.

A finitely generated $\mathcal{O}_F$-module $L \subset V$ is called a ($\mathcal{O}_F$-)lattice if $L \otimes \mathcal{O}_F = V$. We denote $\hat{L} := L \otimes \hat{\mathbb{Z}} \subset \hat{V} = V \otimes \hat{\mathbb{Q}}$. If $Q(L) \subset \mathfrak{d}_F^{-1}$, we say that $L$ is \textit{Z-even integral} and call the lattice

$$L' := \{ y \in V : (y, L) \subset \mathfrak{d}_F^{-1} \}$$

its \textit{Z-dual}. Viewed as a $\mathbb{Z}$-lattice with respect to $Q_Q(x) := \text{tr}_{F/Q}Q(x)$, such $L$ is even integral with dual $L'$.

Definition 1. An $\mathcal{O}_F$-lattice $L$ is said to be \textit{Z-unimodular} if $L' = L$.

As a convention, the trivial lattice in the trivial $F$-vector space is totally positive and $\mathbb{Z}$-unimodular. Consider the monoid

$$\mathcal{U}^+_F := \{(L, Q) : L \text{ is an even Z-unimodular } \mathcal{O}_F\text{-lattice and totally positive definite} \}$$

with respect to $\oplus$, and denote $\mathcal{U}^{+\cdot n}_F \subset \mathcal{U}^+_F$ the subset of lattices of rank $n$. We first have the following result.

Proposition 2.1. The set $\mathcal{U}^{+\cdot n}_F$ is non-empty precisely when $(8/d_2) \mid n$.

Proof. Satz 1 in [Cha70] implies that there exists definite, unimodular $\mathcal{O}_F$-lattices in the sense loc. cit. if and only if $(8/d_2) \mid n$. Furthermore since $n$ is even, all of the $2^d$ possible definite signatures will appear in the set of definite, unimodular $\mathcal{O}_F$-lattices of rank $n$. One can then use the fact that the class $\mathfrak{d}_F$ in the class group is a square to translate this result to the existence $\mathbb{Z}$-unimodular lattices. (see the proof of Prop. 2.5 in [Li21] for details). □

Remark 2.2. For $L \in \mathcal{U}^{+\cdot n}_F$ and $h \in \text{SO}_V(\hat{Q})$ with $V = L \otimes \mathcal{O}_F F$, the lattice

$$h \cdot L := (h \cdot \hat{L}) \cap V \subset V$$

is also in $\mathcal{U}^{+\cdot n}_F$. 
For each \( L \in U_{F, n} \), let \( \theta_L(\tau) \) be the associated theta function defined in (1.6). It is a Hilbert modular form of parallel weight \( n/2 \) for \( SL_2(O_F) \). Now, the Siegel-Weil formula [Sie66, Wei65] gives us the following result.

**Proposition 2.3.** Let \( F \) be a totally real field of degree \( d \). Then

\[
\int_{SO_V(F) \backslash SO_V(A_F)/SO_V(\mathbb{R})} \theta_{h, L}(\tau) \, dh = \kappa E_{F,n/2}(\tau),
\]

for some positive constant \( \kappa \), where \( E_{F,k} \) is the Hecke Eisenstein series of parallel weight \( k \) defined by

\[
E_{F,k}(\tau) := 1 + \zeta_F(k)^{-1} \sum_{A=[a] \in Cl(F)} \text{Nm}(a)^k \sum_{(c,d) \in \mathcal{O}_F^2 \cap a^2 \mathcal{O}_F^2, \, c \neq 0} \prod_{j=1}^{d} (c_j \tau_j + d_j)^{-k}
\]

In particular, \( E_{F,k} \in \mathcal{M}_{F}^{0} \) for all \( k \in (4/d_2) \mathbb{N} \).

**Remark 2.4.** The Hecke Eisenstein series have the following well-known Fourier expansion (see [Sie69, Zag76])

\[
E_{F,k}(\tau) = 1 + \frac{2^d}{\zeta_F(1-k)} \sum_{t \in \mathcal{O}_F^1, \, t \geq 0} \sigma_{k-1}(t \mathcal{O}_F) \prod_{j=1}^{d} q_j^{(t \tau_j)}
\]

with \( \sigma_r(a) := \sum_{b \mid a, \, b \in \mathcal{O}_F} \text{Nm}(b)^r \) for any integral ideal \( a \) and \( r \in \mathbb{N} \).

**Proof.** By the Siegel-Weil formula, the left hand side of (2.4) equals to the Eisenstein series

\[
E_L(\tau) = v^{-n/4} \sum_{\gamma \in B(F) \backslash SL_2(F)} \Phi_L(\gamma g_F, n/2 - 1),
\]

where \( B \subset SL_2 \) is the standard Borel subgroup, and \( \Phi_L \) is the Siegel-Weil section associated to the lattice \( L \) (see e.g. [Kud08, section I.3]). For \( t \in F^\times \), the \( t \)-th Fourier coefficient of \( E_L \) is given by

\[
\prod_{p < \infty} W_{t,p}(1, n/2 - 1, \Phi_{L,p})
\]

up to constant independent of \( t \). Here \( W_{t,p}(g, s, \phi) \) is the local Whittaker function (see e.g. [Yan05]). Since \( L \) is \( \mathbb{Z} \)-unimodular, the local lattice \( L \otimes \mathcal{O}_{F,p} \) in \( V \otimes F_p \) is self-dual for every finite place \( p \). Standard calculations (see e.g. [KY10]) then gives us

\[
W_{t,p}(1, s, \Phi_{L,p}) = \sum_{m=0}^{\text{ord}_p(t \mathcal{O}_{F,p})} \text{Nm}(p)^s \text{Nm}(p)^{s}
\]

when \( t \in \mathcal{O}_{F,p}^{-1} \), and zero otherwise. So up to a constant, the Eisenstein series \( E_L \) and \( E_{F,n/2} \) have the same non-constant term Fourier coefficients, hence agree. Now the left hand side of
(2.4) is just a sum of $\theta_{L_j}$ over certain $L_j \in \mathcal{U}_F^m$ by Remark 2.2. Combining this with Prop. 2.1 finishes the proof. □

We can rewrite the Hecke-Eisenstein series $E_{F,k}$ as

$$E_{F,k}(\tau) := 1 + \sum_{\mathcal{A} = [a] \in \text{Cl}(F)} \left( \frac{\text{Nm}(a)}{\text{Nm}(c)} \right)^k \prod_{j=1}^{d} (\tau_j + d_j/c_j)^{-k}$$

For any $\beta \in F$, there is unique $\mathcal{A} = [a]$ and $(c,d) \in a^2/O_F^\times$ with $c \neq 0$ such that $a = O_Fc + OFd$ and $\beta = d/c$. Therefore, we denote

$$A_\beta := \frac{\text{Nm}(c)}{\text{Nm}(a)} \in \mathbb{Z} - \{0\}.$$ (2.7)

It is easy to check this definition does not depend on the choice of the representative $a$, and

$$A_{\beta + a,k} = A_{\beta,k}$$ (2.8)

for all $a \in \mathbb{Z}$. Then we have

$$E_{F,k}(\tau) = 1 + \sum_{\beta \in F} A_{\beta,k}^k \prod_{j=1}^{d} (\tau_j + \beta_j)^{-k}. \tag{2.9}$$

3. Petersson Inner Product Calculations

In this section, let $F/\mathbb{Q}$ be totally real with degree $d \geq 3$. We will give an expression for the Petersson inner product between the diagonal restriction of the Hecke Eisenstein series $E_{F,k}$ and an elliptic cusp form $f$ of weight $dk$.

For $\alpha \in M_{m,n}(F)$ and $1 \leq j \leq d$, we write $\alpha_j \in M_{m,n}(\mathbb{R})$ with $1 \leq j \leq d$ for the real embeddings of $\alpha$. We identify $\mathbb{P}^1(F) \cong B(F)\backslash \text{SL}_2(F)$ via

$$\beta \mapsto \begin{cases} (1 \hspace{1cm} \beta) & \beta \in F, \\ (0 \hspace{1cm} 1) & \beta = \infty. \end{cases} \tag{3.1}$$

Let $S_0 \cup \{\infty\} \subset \mathbb{P}^1(F)$ be a set of representatives of the double coset $B(F)\backslash \text{SL}_2(F)/\text{SL}_2(\mathbb{Z})$. Then $S_0 \subset F - \mathbb{Q}$ and we can use (2.9) to express the diagonal restriction of $E_{F,k}$ as

$$E_{F,k}^\Delta(\tau) = E_{dk} + \sum_{\beta \in S_0} E_{F,k,\beta}(\tau), \quad E_{F,k,\beta}(\tau) := \sum_{\gamma \in \text{SL}_2(\mathbb{Z})} A^{-k}_{-\gamma^{-1} \cdot (-\beta)} \prod_{j=1}^{d} (\tau - \gamma^{-1} \cdot (-\beta_j))^{-k} \tag{3.2}$$

with $\tau \in \mathbb{H}$. Note that $E_{dk}$ is just the elliptic Eisenstein series of weight $dk$. 


Let $f(\tau) = \sum_{n \geq 1} c_n q^n \in S_d$ be a cusp form. We are interested in estimating its inner product with $E_{F,k}^\Delta$. By the usual unfolding process, we obtain

$$
\langle E_{F,k}^\Delta, f \rangle = \sum_{\beta \in S_0} \int_{\Gamma_\infty \backslash \mathbb{H}} E_{F,k,\beta}(\tau) \overline{f(\tau)} v^{d-1} \frac{dv}{v^2}
$$

$$
= \sum_{\beta \in S_0} \int_0^\infty \sum_{n \geq 1} e(a_{F,k,\beta}(n,v)) e(-2\pi n \nu v^{d-1}) \frac{dv}{v},
$$

where $\Gamma_\infty := B(\mathbb{Q}) \cap \text{SL}_2(\mathbb{Z})$ and

$$
E_{F,k,\beta}(\tau) := \sum_{\gamma \in \Gamma_\infty} A_{\gamma^{-1}(-\beta)}^k \prod_{j=1}^d (\tau - \gamma^{-1} \cdot (-\beta_j))^{-k}
$$

(3.3)

$$
= 2A_{\beta}^k \prod_{j=1}^d (\tau + \beta_j + b)^{-k} = \sum_{n \in \mathbb{Z}} a_{F,k,\beta}(n,\nu) e(nv).
$$

for $\beta = d/c \in S_0$. Here we have $r_{-\gamma(-\beta)} = r_\beta$ for all $\gamma \in \Gamma_\infty$ by (2.8). It is easy to see that

$$
a_{F,k,\beta}(n,\nu) = 2A_{\beta}^k \int_{\mathbb{R}} \prod_{j=1}^d (u + iv + \beta_j)^{-k} e(-nu) du
$$

(3.4)

$$
= 4\pi i (-A_{\beta})^{-k} \sum_{z \in Z(\beta)} \text{Res}_{x=z} \left( e(nx) \prod_{j=1}^d (x - (\beta_j + iv))^{-k} \right),
$$

where $Z(\beta) := \{ \beta_j + iv : 1 \leq j \leq d \} \subset \mathbb{H}$ since

$$
\sum_{z \in Z(\beta)} \text{Res}_{x=z} \left( e(nx) \prod_{j=1}^d (x - z_j)^{-k} \right) = \frac{1}{2\pi i} \int_{\mathbb{R}} e(nx) \prod_{j=1}^d (x - z_j)^{-k} dx.
$$

(3.5)

Suppose $\beta_j$'s are all distinct. Then

$$
\sum_{z \in Z(\beta)} \text{Res}_{x=z} \left( e(nx) \prod_{j=1}^d (x - (\beta_j + iv))^{-k} \right) = \frac{1}{\Gamma(k)} \sum_{j=1}^d \left( \frac{d}{dx} \frac{e(nx)}{\prod_{j'=1, j' \neq j}^d (x - (\beta_{j'} + iv))} \right) \bigg|_{x=\beta_j+iv}
$$

$$
= \frac{e(niv)}{\Gamma(k)} \sum_{j=1}^d \sum_{\ell=0}^{k-1} (2i\nu)^{k-1-\ell} e(n\beta_j) e^{-2\pi \nu v} \binom{k-1}{\ell} \frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}} (\beta_j - \beta_1, \ldots, \beta_j - \beta_d),
$$

where $P_{m,k,\ell}, Q_{m,r} \in \mathbb{Q}[x_1, \ldots, x_m]$ are symmetric polynomials of degrees $(m-1)\ell$ and $mr$ defined by

$$
P_{m,k,\ell}(x_1, \ldots, x_m) := (x_1 \ldots x_m)^{k+\ell}(\partial_{x_1} + \cdots + \partial_{x_m})^\ell (x_1 \ldots x_m)^{-k},
$$

$$
Q_{m,r}(x_1, \ldots, x_m) := (x_1 \ldots x_m)^r.
$$

(3.6)
Note that

\[
\frac{P_{m,k,\ell}}{Q_{m,k+\ell}}(x_1, \ldots, x_m) = (-1)^\ell \sum_{r=(r_j)\in\mathbb{N}^m, \sum_j r_j = \ell} \left( \begin{array}{c} k \\ r \end{array} \right) \prod_{j=1}^m x_j^{-k-r_j},
\]

where \(\left( \begin{array}{c} k \\ r \end{array} \right) := \frac{k(r_1) \cdots k(r_m)}{r_1! \cdots r_m!}\) for \(r = (r_1, \ldots, r_m) \in \mathbb{N}^m\) with \(k(n) := k(k+1) \cdots (k+n-1)\). Substituting this into the unfolding gives us the following result.

**Proposition 3.1.** Suppose \(F\) is a totally real field of degree \(d \geq 3\) and there is no intermediate field between \(F\) and \(\mathbb{Q}\). For any \(k \in 2\mathbb{N}\) and \(f(\tau) = \sum_{n \geq 1} c(n)q^n \in S_{dk}\), we have

\[
\langle E_{\Delta, F,k}, f \rangle = \frac{i \Gamma(dk-1)}{(4\pi)^{dk-2} \Gamma(k)} \sum_{\ell=0}^{k-1} (2\pi i)^{-1-\ell} \sum_{\beta \in S_0} A_{\beta}^{-k} \times \sum_{j=1}^d \left( \frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}} \right) (\beta_j - \beta_1, \ldots, \beta_j - \beta_{j-1}, \beta_{j+1}, \ldots, \beta_j - \beta_d) \sum_{n \geq 1} \frac{e(n\beta_j)}{n^{(d-1)k+\ell}},
\]

where the polynomials \(P_{m,k,\ell}\) and \(Q_{m,r}\) are defined in (3.6).

**Remark 3.2.** The condition that there is no intermediate field between \(F\) and \(\mathbb{Q}\) implies that \(\beta_i = \beta_j\) if and only if \(i = j\) for all \(\beta \in F - \mathbb{Q}\). A similar but more complicated formula for the inner product can be derived without this condition.

**Example 3.3.** Let \(d = 3\) and \(k = 2\). Then

\[
\frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}}(x, y) = \begin{cases} 1/(xy)^2, & \ell = 0, \\ -2(x + y)/(xy)^2, & \ell = 1. \end{cases}
\]

Set \(\gamma_1 := \beta_2 - \beta_3, \gamma_2 := \beta_3 - \beta_1, \gamma_3 := \beta_1 - \beta_2\), we have

\[
\sum_{\ell=0}^{k-1} (2\pi in)^{k-1-\ell} \sum_{j=1}^d \frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}} (\beta_j - \beta_1, \ldots, \beta_j - \beta_d) e(n\beta_j) = \left( \frac{2\pi in}{(\gamma_2\gamma_3)^2} + \frac{2(\gamma_3 - \gamma_2)}{(\gamma_2\gamma_3)^3} \right) e(n\beta_1) + \left( \frac{2\pi in}{(\gamma_1\gamma_3)^2} + \frac{2(\gamma_1 - \gamma_3)}{(\gamma_1\gamma_3)^3} \right) e(n\beta_2) + \left( \frac{2\pi in}{(\gamma_1\gamma_2)^2} + \frac{2(\gamma_2 - \gamma_1)}{(\gamma_1\gamma_2)^3} \right) e(n\beta_3).
\]

For \(d = 3\) and \(k - 1 \geq \ell \geq 0\), we can write explicitly

\[
\sum_{j=1}^d \frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}} (\beta_j - \beta_1, \ldots, \beta_j - \beta_d) e(n\beta_j) = \frac{P_{2,k,\ell}}{Q_{2,k+\ell}} (\gamma_3, -\gamma_2) e(n\beta_1) + \frac{P_{2,k,\ell}}{Q_{2,k+\ell}} (-\gamma_3, \gamma_1) e(n\beta_2) + \frac{P_{2,k,\ell}}{Q_{2,k+\ell}} (-\gamma_2, -\gamma_1) e(n\beta_3).
\]
Using the inequalities $k^{(a)} k^{(b)} \leq k^{(a+b)}$, $(x_1 + x_2 + x_3)^2 \leq 3(x_1^2 + x_2^2 + x_3^2),$

\begin{equation}
(3.9) \quad \sum_{\sigma \in S_3} x^{a \sigma(1)} x^{b \sigma(2)} x^{c \sigma(3)} \leq \frac{abbc!}{(a + b + c)!} (x_1 + x_2 + x_3)^{a+b+c}, \quad x_i, a, b, c \geq 0
\end{equation}

and Equation (3.7), we obtain the bound

\begin{align*}
&\left| d \sum_{j=1}^{d} \frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}} (\beta_j - \beta_1, \ldots, \beta_j - \beta_d) \mathbf{e}(n \beta_j) \right| \\
&\leq \left| \frac{P_{2,k,\ell}}{Q_{2,k+\ell}} (\gamma_3, -\gamma_2) \right| + \left| \frac{P_{2,k,\ell}}{Q_{2,k+\ell}} (-\gamma_3, \gamma_1) \right| + \left| \frac{P_{2,k,\ell}}{Q_{2,k+\ell}} (-\gamma_2, -\gamma_1) \right| \\
&\leq \frac{\ell!}{\gamma_1 \gamma_2 \gamma_3} \sum_{a+b=\ell} \frac{k^{(a)} k^{(b)}}{ab!} \left( |\gamma_1 b_{1a2} a_{k+\ell} | + |\gamma_2 b_{2a3} a_{k+\ell} | + |\gamma_3 b_{3a1} a_{k+\ell} | \right) \\
&\leq \ell! \frac{(\gamma_1 + \gamma_2 + \gamma_3) \ell}{\gamma_1 \gamma_2 \gamma_3} \leq \frac{(k+1) \ell!}{(k+2\ell)!} \frac{2}{2} \leq \frac{\ell!}{(k+2\ell)!} \left( \frac{k+1}{2} \right)^{3/2} \left( \gamma_1^2 + \gamma_2^2 + \gamma_3^2 \right)^{3/2}.
\end{align*}

4. Double Coset and Binary Cubic Forms

When $d = 3$, we can identify the double coset $B(F) \backslash SL_2(F)/SL_2(\mathbb{Z}) - \{ \infty \}$ with orders in $\mathcal{O}_F$ in the following way. Let

$\mathcal{Q}_F := \{ f(X, Y) = AX^3 + BX^2Y + CXY^2 + D Y^3 \in \mathbb{Z}[X, Y] : f(\beta, 1) = 0 \text{ for some } \beta \in F \setminus \mathbb{Q} \}$

be the set of integral binary cubic forms with a root in $F - \mathbb{Q}$. A form is primitive if its coefficients have no common factor. There is a natural action of $SL_2(\mathbb{Z})$ on $\mathcal{Q}_F$ that preserves the discriminant

\begin{equation}
(4.1) \quad \Delta(f) := A^6 ((\beta_1 - \beta_2) (\beta_1 - \beta_3) (\beta_2 - \beta_3))^2
\end{equation}

\begin{align*}
&= 18ABC D + B^2C^2 - 4AC^3 - 4B^3D - 27A^2D^2,
\end{align*}

and the subset of primitive forms. The quantity

\begin{equation}
(4.2) \quad P(f) := B^2 - 3AC > 0
\end{equation}

is the leading coefficient of the Hessian of $f$, which is a positive definite quadratic form and a coinvariant of $f$. For every $f \in \mathcal{Q}_F$, Prop. 2 in [Cre99] gives us $f' \sim_{SL_2(\mathbb{Z})} f$ satisfying

\begin{equation}
(4.3) \quad P(f') \leq \sqrt{\Delta(f')} = \sqrt{\Delta(f)}.
\end{equation}

Given $\beta \in F - \mathbb{Q}$, we can associate to it a primitive element $f_\beta \in \mathcal{Q}_F$ defined by

\begin{equation}
(4.4) \quad f_\beta(X, Y) := A_\beta \prod_{j=1}^{3} (X - \beta_j Y) = A_\beta X^3 + B_\beta X^2Y + C_\beta XY^2 + D_\beta Y^3 \in \mathcal{Q}_F.
\end{equation}
Note that \( f_\beta(\beta, 1) = 0 \) and the right action of \( \text{SL}_2(\mathbb{Z}) \) on \( B(F) \setminus \text{SL}_2(F) \) corresponds to its natural action on \( \mathbb{Q}_F \).

To any binary cubic form \( f \) with non-zero discriminant and \( f(\beta, 1) = 0 \) we can associate the free \( \mathbb{Z} \)-module of rank 3
\[
\mathcal{O}_f := \mathbb{Z} + \mathbb{Z}A\beta + \mathbb{Z}(A\beta^2 + B\beta + C) \subset \mathbb{Q}(\beta),
\]
which is also a commutative ring. A classical result of Delone and Faddeev tells us that this gives a bijection between \( \text{GL}_2(\mathbb{Z}) \)-classes of binary cubic forms with non-zero discriminants and isomorphism classes of commutative rings that are free \( \mathbb{Z} \)-modules of rank 3 \([DF64]\). If we restrict \( \beta \) to be in a fixed field \( F \), then \( \mathcal{O}_f \) is an order in \( \mathcal{O}_F \), and \( \mathcal{O}_{f_1}, \mathcal{O}_{f_2} \subset \mathcal{O}_F \) are the same if and only if \( f_1, f_2 \in \mathcal{Q}_F \) are \( \text{GL}_2(\mathbb{Z}) \)-equivalent (see e.g. \([Nak98, \text{Lemma 3.1}]\)). Furthermore, we have
\[
\Delta(f) = \Delta(\mathcal{O}_f) = D_F[\mathcal{O}_F : \mathcal{O}_f]^2
\]
with \( \Delta(\cdot) \) the discriminant. For \( s = [\beta] \in \mathbb{P}^1(F)/\text{SL}_2(\mathbb{Z}) \setminus \{\infty\} \), we then denote
\[
\mathcal{O}_s := \mathcal{O}_{f_\beta}, \Delta(s) := \Delta(\mathcal{O}_s).
\]
The discussions above lead to the following result.

**Proposition 4.1.** The map 
\[
\mathbb{P}^1(F)/\text{SL}_2(\mathbb{Z}) \setminus \{\infty\} \to \{\mathcal{O} : \mathcal{O} \subset \mathcal{O}_F \text{ is an order}\}/ \cong \quad s \mapsto \mathcal{O}_s
\]
is well-defined and \( (2|\text{Aut}(\mathcal{O}_F)|) \)-to-1.

**Remark 4.2.** The quantity \( |\text{Aut}(\mathcal{O}_F)| \) is either 3 or 1 depending on \( F/\mathbb{Q} \) is Galois or not.

Finally, the following Dirichlet series
\[
\eta_F(s) := \sum_{\mathcal{O} \subset \mathcal{O}_F \text{ order}} [\mathcal{O}_F : \mathcal{O}]^{-s} = \sum_{\mathcal{O} \subset \mathcal{O}_F \text{ order}} \frac{D_F^{s/2}}{\Delta(\mathcal{O})^{s/2}}.
\]
can be factorized in the following way by a result of Datskovsky and Wright \([DW86]\) (see \([Nak98, \text{Lemma 3.2}]\))
\[
\eta_F(s) = \frac{\zeta_F(s)}{\zeta(2s)} \zeta(3s) \zeta(3s - 1).
\]

### 5. Proof of Theorem \[\text{I.1}^{\text{I.1}}\]

We are now ready to prove Theorem \[\text{I.1}^{\text{I.1}}\]. The cases of \( d = 2, 3 \) are proved separately.
Proof of Theorem 1.1 for $d = 2$. For $k = 2, 4$, the space $M_{2k}$ is 1-dimensional and spanned by the Eisenstein series $E_{2k}$. Since $\theta_L^\Delta$ is non-trivial for any $L \in \mathcal{U}^F$, the claim follows in these two base cases as $M_{F,k}^\theta$ is non-trivial by Prop. 2.1 (see also [Sch94] for an explicit construction). More generally, we know that $M_{Q}^{\Delta} = \mathbb{Q}[E_4, \Delta]$. Therefore, it suffices to show that $\Delta \in S_{12}$ is in $(M_{F,6}^\theta)^\Delta$. As $M_{12}$ is 2-dimensional and
\begin{equation}
E_4^3 = E_{12} + \frac{432000}{691} \Delta \in (M_{F,6}^\theta)^\Delta,
\end{equation}
we just need to produce a form $f \in (M_{F,6}^\theta)^\Delta$ linearly independent from $E_4^3$. For this purpose, we apply Prop. 2.3 with $k = 6$ to get
\begin{equation}
f(\tau) := (E_{F,6}^\Delta(\tau) = 1 + \frac{4}{\zeta_F(-5)} \sum_{m \geq 1} q^m \sum_{\nu \in \mathfrak{D}_F, \nu \gg 0, \text{tr} (\nu) = m} \sigma_5((\nu) \mathfrak{D}_F).\end{equation}
By Theorem 6 in [KZ84], we know that
\begin{equation}
f = E_{12} - \frac{12}{691 \zeta_F(-5)} \Delta,
\end{equation}
where $c(D)$ is the $D$-th Fourier coefficient of the half-integral weight form
\begin{equation}
g(\tau) = \sum_{D \in \mathbb{N}} c(D) q^D := \frac{1}{8\pi i} (2E_4(4\tau)\theta'(\tau) - E_4'(4\tau)\theta(\tau))
\end{equation}
spanning the Kohnen plus space $S_{13/2}^+$. Now using the easy estimate $L(k, \chi_D) > 2 - \zeta(k)$ for $k \geq 2$ (see e.g. Equation (3) in [CK13]) we know that $\zeta_F(1-k) = D^{k-1/2} \frac{4 \Gamma(k)^2}{(-4\pi)^k} \zeta_F(k)$ satisfies 
\begin{equation}
|\zeta_F(-5)| > 0.01 \cdot D^{11/2}.
\end{equation}
On the other hand, the Hecke bound for $c(D)$ yields
\begin{equation}
|c(D)| \leq c \cdot D^{13/4}, \quad c := e^{2\pi} \max_{\tau \in \mathbb{H}} |g(\tau)| v^{13/4} < 10
\end{equation}
Comparing with (5.1), it is clear that $f$ and $E_4^3$ are linearly independent for all fundamental discriminant $D > 0$. This finishes the proof of Theorem 1.1 for $d = 2$. 

Using the calculation in section 3 and the correspondence in section 4, we can prove the following lemma.

**Lemma 5.1.** For $d = 3, k \geq 3$ and $f(\tau) = \sum_{n \geq 1} c_f(n) q^n \in S_{3k}$, let $c_f > 0$ be a constant such that 
\begin{equation}
|c_f(n)| \leq c_f \cdot n^{3k/2}
\end{equation}
for all $n \geq 1$. Then we have the bound
\begin{equation}
|\langle E_{F,k}^\Delta, f \rangle| \leq C_k c_f D_F^{-k/4}
\end{equation}
for all cubic field \( F \), with \( C_k := 6c_k \frac{(k/2)^3}{\zeta(3k/2 - 1)} \) and the constant \( c_k \) given in \((5.4)\).

**Proof.** Let \( a_k := \frac{\Gamma(3k-1)}{\Gamma(k)} (4\pi)^{2-3k} \). For \( \beta \in S_0 \subset F \), recall that \( f_\beta \) is the binary cubic form associated to it in \((4.4)\), which has coefficients \( A_\beta, B_\beta, C_\beta, D_\beta \). Using \((3.8)\), the estimate in Example 3.3 and \((4.3)\), we obtain the bound

\[
|\langle E_{F,k}, f \rangle| \leq a_k \sum_{\ell=0}^{k-1} (2\pi)^{k-1-\ell} \sum_{n \geq 1} \left| \frac{c_f(n)}{n^{2k+\ell}} \right| \sum_{\beta \in S_0} A_{\beta}^{-k} \sum_{j'=1}^{d} \frac{P_{d-1,k,\ell}}{Q_{d-1,k+\ell}} (\beta_{j'} - \beta_1, \ldots, \beta_{j'} - \beta_d) e(n\beta_{j'})
\]

\[
\leq c_f \cdot a_k \sum_{\ell=0}^{k-1} (2\pi)^{k-1-\ell} \zeta(k/2 + \ell) \frac{(k-1+\ell)}{(k+2\ell)} (\ell + 1)! \frac{3^{k/2+\ell}}{2}
\]

\[
\times \sum_{\beta \in S_0} A_{\beta}^{-k} \frac{((\beta_1 - \beta_2)^2 + (\beta_2 - \beta_3)^2 + (\beta_3 - \beta_1)^2)^{k/2+\ell}}{((\beta_1 - \beta_2)^2(\beta_2 - \beta_3)^2(\beta_3 - \beta_1)^2)^{(k+\ell)/2}}
\]

\[
\leq 2^{-1} c_f \cdot a_k \sum_{\ell=0}^{k-1} (2\pi)^{k-1-\ell} \zeta(k/2 + \ell) \frac{(k-1+\ell)}{(k+2\ell)} (\ell + 1)! 6^{k/2+\ell} \sum_{\beta \in S_0} \frac{P(f_{\beta})^{k/2+\ell}}{\Delta(f_{\beta})^{(k+\ell)/2}}
\]

\[
\leq c_f \cdot c_k \sum_{\beta \in S_0} \Delta(f_{\beta})^{-k/4} \leq c_f \cdot c_k \cdot 2 |\text{Aut}(\mathcal{O}_F)| \cdot D_F^{-k/4} \eta_F \left( \frac{k}{2} \right)
\]

Here the constant \( c_k \) is defined by

\[
(5.4) \quad c_k := \frac{\Gamma(3k-1)}{2\Gamma(k)} (4\pi)^{2-3k} \sum_{\ell=0}^{k-1} (2\pi)^{k-1-\ell} \zeta(k/2 + \ell) \frac{(k-1+\ell)}{(k+2\ell)} (\ell + 1)! 6^{k/2+\ell}.
\]

For the last steps, we used Prop. 4.1. Combining this with \((1.9)\) and applying \( \zeta_F(s) \leq \zeta(s)^3 \) for \( s > 1 \), we have

\[
|\langle E_{F,k}, f \rangle| \leq c_f c_k 2 |\text{Aut}(\mathcal{O}_F)| \frac{\zeta_F(k)}{\zeta_F(k)} \zeta(k) \zeta(\frac{3k}{2} - 1) D_F^{-k/4} \leq 6c_f c_k \frac{\zeta(k/2)^3}{\zeta(k)^2} \zeta(\frac{3k}{2} - 1) D_F^{-k/4}
\]

for \( k \geq 3 \). This finishes the proof. \( \square \)

**Remark 5.2.** For \( k = 4 \), the bound above gives \( C_4 < 5.79 \). We can obtain a better bound by estimating the second to the last line in Example 3.3 case by case for each \( \ell = 0, 1, 2, 3 \), instead of using \((3.9)\). The improved bound is

\[
|\langle E_{F,4}, f \rangle| \leq 0.067 c_f D_F^{-1}
\]

for all totally real cubic field \( F \).

Now we are ready to prove Theorem 1.1 in the cubic case.
Proof of Theorem 1.1 for $d = 3$. Since $M^{(12)}_Q = C[E_{12}, \Delta]$, we only have to check that $(M^6_F)_{\Delta} \cap M^{(12)}_Q$ is 2 dimensional. For any $L \in U^+_F$, the diagonal restriction $\theta^\Delta_L$ is the theta function for a unimodular lattice $P$ over $\mathbb{Z}$. So we know that $\theta_p \in (M^6_F)_{\Delta}$ for some Niemeier lattice $P$. To see that it is linearly independent from $E^\Delta_{F,4} = 1 + c(1)q + O(q^2)$, it suffices to show that $c(1)$ is not integral. We have checked this numerically for any cubic $F$ with $D_F < 70000$.

More generally, we have

$$\theta_P = E_{12} + (N_2(P) - 65520/691)\Delta,$$

with $N_2(P)$ is the number of norm 2 vectors in $P$. From Table V in [CS82], we obtain a list of $N_2(P)$ and 

$$|\langle \theta_P, \Delta \rangle| = |N_2(P) - 65520/691|\langle \Delta, \Delta \rangle > 1.22 \times 10^{-6}$$

for any Niemeier lattice $P$. On the other hand by taking $c_\Delta = 1$, the upper bound found in Lemma 5.1 and improved in Remark 5.2 gives us

$$|\langle E^\Delta_{F,4}, \Delta \rangle| < \frac{0.067}{D_F}.$$ 

So $E^\Delta_{F,4}$ and $\theta_P$ are linearly independent for $D_F \geq 60000$. This finishes the proof.

6. Numerical Evidence for Conjecture 1

In this section, we approach numerically Conjecture 1 in the case $F$ is a totally real field of degree $d \in \{4, 5, 6\}$. For these choices of $d$ the space $M^{(4d/d_2)}_Q$ can be in principle generated by the restriction of Eisenstein series and of (at most) one theta function $\theta_L$ of rank $8/d_2$. Conjecture 1 reduces then to the verification of the linear independence of $\theta^\Delta_L$ and $E^\Delta_{F,4/d_2}$ for $d = 5, 6$, and of monomials in $\theta^\Delta_L$, $E^\Delta_{F,4/d_2}$, and $E^\Delta_{F,k}$ in general for suitable weights $k$. This approach gives data supporting Conjecture 1 in the case $d = 4, 5$, and in the case $d = 6$ except for two fields $F$. Our result, for which evidence is given in this final section, is the following.

Theorem 6.1. Conjecture 1 holds for

1. $d = 4$ and $D_F \leq 10^5$;
2. $d = 5$ and $D_F \leq 2 \times 10^6$;
3. $d = 6$ and $D_F \leq 5 \times 10^6$ except for the fields of discriminant 453789 and 1397493.

6.1. A note on the computations. For $l, k \in \mathbb{Z}_{\geq 0}$, let $\sigma_{k-1}$ be as in Remark 2.4 and define

$$s^F_l(k) := \sum_{\nu \in P^{-1} \atop \nu \geq 0 \atop \text{tr}(\nu) = l} \sigma_{k-1}(\nu)\theta_F.$$
Then the diagonal restriction of $E_{F,k}$ has the following $q$-expansion at $\infty$ by (2.6)

\begin{equation}
E_{F,k}^\Delta(\tau) = 1 + \frac{2^d}{\zeta_K(1-k)} \sum_{l=0}^{\infty} s_l^F(k).
\end{equation}

We computed the first few coefficients of the above expansion with PARI/GP [The21]. As (6.1) shows, this reduces to the determination of the functions $s_l^F(k)$ for small values of $l$ (up to $l = 5$ in the case $d = 5$) and different values of $k$. The main difficulty is to find the totally positive $\nu \in \mathcal{D}_F^{-1}$ of fixed trace $l$. Let $(\nu_1, \ldots, \nu_d)$ be an integral basis for $\mathcal{D}_F^{-1}$. Then any $\nu \in \mathcal{D}_F^{-1}$ is of the form $\nu = v_1\nu_1 + \cdots + v_d\nu_d$ for $(v_1, \ldots, v_d) \in \mathbb{Z}^d$ and conversely every vector in $\mathbb{Z}^d$ gives an element $\nu \in \mathcal{D}_F^{-1}$. If $Q(x_1, \ldots, x_d)$ denotes the quadratic form $x_1^2 + \cdots + x_d^2$, we have, for a totally positive $\nu \in \mathcal{D}_F^{-1}$, that $Q(\sigma_1(\nu), \ldots, \sigma_d(\nu)) < \text{tr}(\nu)^2$. This implies that if $A = (\sigma_i(\nu_j))_{i,j}$ denotes the matrix of the real embeddings of the basis of $\mathcal{D}_F^{-1}$, we can search the totally positive $\nu \in \mathcal{D}_F^{-1}$ of fixed trace $l$ among of vectors $v = (v_1, \ldots, v_d) \in \mathbb{Z}^d$ satisfying

\[v^T(ATA)v = Q(\nu) < l^2.\]

This gives a finite (but large as $l$ and $D_F$ grow) set of vectors on which we can perform the final search. Once the suitable $\nu \in \mathcal{D}_F^{-1}$ have been determined, it is straightforward to compute $\sigma_k((\nu)\mathcal{D}_F)$ for every value of $k$ by using the basic PARI functions.

**Remark 6.2.** It is possible to investigate also the cases $d = 8, 10$ with the method outlined at the beginning of this section. For the case $d = 8$ we need to compute five coefficients of the $q$-expansion (6.1), while for $d = 10$ we need to compute six coefficients. This, together with the size of the discriminants of these fields ($D_F \geq 282300416$ for $d = 8$ and $D_F \geq 443952558373$ for $d = 10$), makes it hard to collect significant data in these cases.

6.2. Tables.

$d=4$. Let $F$ be a totally real field with $[F : \mathbb{Q}] = 4$. In this case, the proof of Conjecture 1 reduces to the statement that $\mathcal{M}_Q^{(8)}(\Gamma_F)$ is spanned by restrictions of Hilbert Eisenstein series on $\Gamma_F$. It is easy to see that $\{E_2^F, \Delta E_4, \Delta^2\}$ is a generating set for $\mathcal{M}_Q^{(8)}(\Gamma_F)$. By a dimension argument, $E_{F,2}^\Delta = E_4^\Delta$. It follows that $\Delta E_4$ and $\Delta^2$ can be obtained by restriction of Eisenstein series on $\Gamma_F$ respectively if the sets $\{E_{F,4}^\Delta, (E_{F,2}^\Delta)^2\}$ and $\{(E_{F,2}^\Delta)^3, E_{F,2}^\Delta E_{F,4}^\Delta, E_{F,6}^\Delta\}$ are both linearly independent.

In order to study this problem, we compute the restriction of $E_{F,k}$ for $k = 4, 6$. As bases for $M_{16}$ and $M_{24}$, we choose $\{E_4^\Delta, E_4^\Delta \Delta\}$ and $\{E_6^\Delta, E_6^\Delta \Delta, \Delta^2\}$ respectively. We have

\begin{align*}
E_{F,4}^\Delta &= E_4^\Delta + bE_4^\Delta, \\
E_{F,6}^\Delta &= E_6^\Delta - c_1E_4^\Delta + c_2\Delta^2,
\end{align*}

for some coefficients $b, c_1, c_2 \in \mathbb{Q}$ that depend on $F$. To prove Conjecture 1, it suffices to check that $b$ and $c_2$ are both non-zero. We computed the coefficients $b, c_1, c_2$ for the first 30
totally real quartic fields $F$. The results are reported in Table \ref{table}. For these fields it is enough to specify the discriminant $D_F$ to uniquely identify the field $F$ (check the number field database \cite{LMF22}). This remark applies also for the fields we consider in the cases $d = 5, 6$.

It turns out that the numerical values of $b, c_1, c_2$ are very close to 955, 1439, and $-129930$ respectively. These numbers are related to the Eisenstein series of weight 16 and 24 since

$$E_{16} = E_4^4 + b(E_{10})E_4\Delta, \quad E_{24} = E_4^5 + c_1(E_{24})E_4^2\Delta + c_2(E_{24})\Delta^2,$$

with

$$b(E_{16}) = -\frac{3456000}{3617} \sim 955, \quad c_1(E_{24}) = \frac{340364160000}{236364991} \sim 1439, \quad c_2(E_{24}) = -\frac{30710845440000}{236364991} \sim 129930.$$

In other words, it seems that the diagonal restriction of $E_{F,4}$ and $E_{F,6}$ are close to $E_{16}$ and $E_{24}$ respectively. In analogy with the proof of Theorem \ref{thm1} in the case $d = 3$, Conjecture \ref{conj1} holds for $D_F \gg 0$ if the Petersson products of $E_{F,4}$ and $E_{F,6}$ with all cusp forms of weight 16 and 24 respectively can be bounded by small quantities as $D_F \to \infty$. If $F$ ranges over the totally real quartic fields with no non-trivial subfields, the decay of the Petersson products as $D_F \to \infty$ can be observed from the data. We expect similar strategy for the proof of Theorem \ref{thm2} when $d = 3$ to work in this case. When $F$ ranges instead over extensions of the form $\mathbb{Q} \subset K \subset F$, where $K$ is a fixed real quadratic field, the data suggest that

$$\langle E_{F,k}^\Delta, f \rangle \to \langle E_{K,2k}^\Delta, f \rangle \quad \text{as disc}(F) \to \infty.$$

The proof of Conjecture \ref{conj1} may be obtained then in two steps: first proving that $E_{F,k}$ restrict to the Hilbert Eisenstein series $E_{K,2k}$ on $\Gamma_K$ as $F \to \infty$, and then using Theorem \ref{thm1} for the real quadratic field $K$.

d=5. Let $F$ be a totally real field of degree 5. The space $M_{20}^{(20)}$ is generated by the set $\{E_{20}, E_8\Delta, E_4\Delta^3, \Delta^5\}$. In order to get this space by restriction of Hilbert theta series (Conjecture \ref{conj1}), we only need to consider a Hilbert theta function $\theta_L$ for $L \in U_{F,5}^+$ and the Eisenstein series $E_{F,4}, E_{F,8}$, and $E_{F,12}$. Fixing basis for $M_{20}, M_{40},$ and $M_{60}$, we find the expressions

$$E_{F,4}^\Delta = E_4^5 + bE_4^2\Delta,$$

$$E_{F,8}^\Delta = E_4^{10} + c_1E_4^7\Delta + c_2E_4^4\Delta^2 + c_3E_4\Delta^3,$$

$$E_{F,12}^\Delta = E_4^{15} + d_1E_4^{12}\Delta + d_2E_4^9\Delta^2 + d_3E_4^6\Delta^3 + d_4E_4^3\Delta^4 + d_5\Delta^5,$$

for $b, c, d \in \mathbb{Q}$ that depends on $F$. Since $\theta_L^\Delta = 1 + \sum_{n \geq 1} a_n q^n$ with $a_n \in \mathbb{Z}$, in order to prove linear independence of $\theta_L^\Delta$ and $E_{F,4}^\Delta$, it suffices to show that $b \notin \mathbb{Z}$. If this holds true, we only need that $c_3 \neq 0$ and $d_5 \neq 0$ to prove Conjecture \ref{conj1}. The results of the computation of $b, c_3$, and $d_5$, for the first 30 totally real quintic fields $F$ (ordered by discriminant) can be found in Table \ref{table2}. Similarly to the case $d = 4$, the numerical values of $b, c, d$ are close to
the coefficients appearing in the expression of the Eisenstein series $E_{20}$, $E_{40}$, and $E_{60}$ with respect to the bases specified above:

$$E_{20} = E_4^5 + b(E_{20})E_4^2\Delta, \quad E_{40} = E_4^{10} + \sum_{i=1}^{3} c_i(E_{40})E_4^{10-3i}\Delta^i, \quad E_{60} = E_4^{15} + \sum_{i=1}^{5} d_i(E_{60})E_4^{15-3i}\Delta^i,$$

the relevant values being

$$b(E_{20}) \approx \frac{209520000}{174611} \sim 1199, \quad c_3(E_{40}) = \frac{27014542428736906240000000000}{261082718496449122051} \sim 103471200,$$

$$d_5(E_{60}) = \frac{14231522539047393662157818174002937318490940798202464041491}{12152331404837555720403049940798202464041491} \sim 1171094011917.$$

In Table 2 we do not write the numerical values of $c_3$, $d_5$, but of their difference with the coefficients $c_3(E_{40})$ and $d_5(E_{60})$ respectively. Analogously to the case $d = 4$, it seems that the diagonal restriction of $E_{F,4}$, $E_{F,8}$, and $E_{F,12}$ are close to the Eisenstein series $E_{20}$, $E_{40}$, and $E_{60}$ respectively. In particular, since $\langle E_{20}, E_4^2\Delta \rangle = 0$, this implies that the Petersson product

$$\langle E_{F,4}^\Delta, E_4^2\Delta \rangle = |b - b(E_{20})| \langle E_4^2\Delta, E_4^2\Delta \rangle$$

is small for any field $F$ and may decay as $D_F \to \infty$. Similar considerations apply to the cases $E_{F,8}^\Delta$ and $E_{F,12}^\Delta$.

$d = 6$. We have that $\mathcal{M}_Q^{12} = \mathbb{C}[E_4^3, \Delta]$. We only have to check that

$$E_{F,2}^\Delta = E_4^3 + b \cdot \Delta$$

is not the restriction of a Hilbert theta function $\theta_L$. We know this is the case if $b$ is not an integer, as explained in the proof of Theorem 1.1 in the case $d = 3$. However, looking at the values of $b$ computed for the first 30 totally real sextic fields $F$ in Table 3, this is not always the case. Since $\theta_L^\Delta = 1 + N_2(L)q + \cdots$, we have to compare, for integral values of $b$, the number $720 + b$ with the possible values of $N_2(L)$ listed in table V of [CS82] to check whether they differ or not. This happens in all cases but two: the field of discriminant 453789 has $720 + b = 0 = N_2(\Lambda_{24})$, the field of discriminant 1397493 has $720 + b = 72 = N_2(\Lambda_{12}^2)$. For these fields our argument can not confirm the validity of Conjecture 1. We checked fields up to $D_F = 5 \times 10^6$ (144 fields) and found no other such instances.

As in the cases $d = 4, 5$, in Table 3 we also compare the value of $b$ with $b(E_{12}) = -\frac{432000}{691}$ (see (5.1)).

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### Table 1. $d = 4$

| $D_F$ | $E_{F,4}^\Delta$ | $|b - b(E_{16})|$ | $|c_1 - c_1(E_{24})|$ | $|c_2 - c_2(E_{24})|$ |
|-------|-------------------|-------------------|-------------------|-------------------|
| 725   | $\frac{518400}{541}$ | 2.7375349         | 0.00050313732     | 25.886498        |
| 1125  | $\frac{1299600}{1201}$ | 3.7507260         | 0.00054525118     | 81.739221        |
| 1600  | $\frac{16588800}{1747}$ | 0.80419080       | 0.00021600333     | 72.207992        |
| 1957  | $\frac{3379968}{3541}$ | 0.96439255        | 0.00038594892     | 17.453573        |
| 2000  | $\frac{3628800}{3703}$ | 1.2217550        | 0.00025214822     | 55.822134        |
| 2048  | $\frac{83358720}{87439}$ | 2.1522766         | 0.00086000436     | 17.157301        |
| 2225  | $\frac{4406400}{4601}$ | 2.2168733        | 0.00044417599     | 65.944997        |
| 2304  | $\frac{6996480}{7437}$ | 1.8993132        | 0.00078107824     | 34.635539        |
| 2525  | $\frac{40953600}{42787}$ | 1.6625629        | 0.00038679430     | 60.388956        |
| 2624  | $\frac{31242240}{32681}$ | 0.48766988       | 0.00016760431     | 11.280096        |
| 2777  | $\frac{30326400}{31739}$ | 0.0052682944      | 2.49791 $\times 10^{-5}$ | 3.191617         |
| 3600  | $\frac{3948080}{4117}$ | 1.7138725        | 0.00032163274     | 63.391164        |
| 3981  | $\frac{22988800}{23631}$ | 0.0065088042     | 1.13683 $\times 10^{-5}$ | 16.924484        |
| 4205  | $\frac{81112320}{84937}$ | 0.51758364        | 7.99169 $\times 10^{-5}$ | 20.235531        |
| 4225  | $\frac{31168000}{32567}$ | 1.5789962        | 0.00036468807     | 64.303710        |
| 4352  | $\frac{14613696}{15031}$ | 0.40686765        | 0.00042977636     | 2.3880196        |
| 4400  | $\frac{28713600}{30007}$ | 1.7697819        | 0.00034047188     | 63.576584        |
| 4525  | $\frac{315005600}{329717}$ | 2.0167958        | 0.00041971303     | 62.764065        |
| 4752  | $\frac{9472160}{99107}$ | 0.77303737        | 0.00019694052     | 7.4011277        |
| 4913  | $\frac{358572006}{375437}$ | 0.40870317       | 4.15983 $\times 10^{-5}$ | 2.8025931        |
| 5125  | $\frac{24364800}{25453}$ | 1.7587165        | 0.00037508012     | 63.196773        |
| 5225  | $\frac{262310400}{273971}$ | 1.9505876        | 0.00039428701     | 63.490510        |
| 5725  | $\frac{716947200}{748883}$ | 1.8674479        | 0.00042362838     | 63.447454        |
| 5744  | $\frac{727626240}{761737}$ | 0.26820601       | 7.42018 $\times 10^{-5}$ | 5.6160966        |
| 6125  | $\frac{454636800}{474913}$ | 1.8174699        | 0.00042240976     | 63.162128        |
| 6224  | $\frac{204894720}{214357}$ | 0.028287048      | 2.32205 $\times 10^{-5}$ | 5.7256495        |
| 6809  | $\frac{87570720}{91729}$ | 0.75775312      | 0.00019944285     | 7.0978686        |
| 7053  | $\frac{1504154880}{1573751}$ | 0.28894424      | 0.00013645348     | 2.0848135        |
| 7056  | $\frac{191034720}{200123}$ | 0.90144417       | 0.00037862134     | 11.709300       |
| 7168  | $\frac{670104576}{701855}$ | 0.72584168     | 0.00033000104     | 3.6107465        |
\[
\begin{array}{c|cccc}
D_F & -b & E_{F,4}^\Delta |b - b(E_{20})| & E_{F,8}^\Delta |c_3 - c_3(E_{40})| & E_{F,12}^\Delta |d_5 - d_5(E_{50})| \\
\hline
14641 & 1017360000 & 0.060027104 & 20.049846 & 602.44929 \\
24217 & 539081460 & 0.0056153314 & 3.0959986 & 626.69793 \\
36497 & 228980816 & 0.020731861 & 3.7691249 & 625.79357 \\
38569 & 1372671360 & 0.065169297 & 7.1399961 & 53.593811 \\
65657 & 17909631360 & 0.050318395 & 1.4956754 & 21.447253 \\
70601 & 2278694920 & 0.023943997 & 6.3509437 & 57.580627 \\
81509 & 1255163040 & 0.013653680 & 0.14871660 & 33.681371 \\
81589 & 157427145 & 0.0040921029 & 3.7633773 & 5.5844793 \\
89417 & 3299933520 & 0.010842438 & 0.88686411 & 7.1415794 \\
101833 & 2742237340 & 0.00095029875 & 2.8922290 & 147.56474 \\
106069 & 8416770640 & 0.020817098 & 1.2397693 & 6.2320866 \\
11768 & 726761960 & 0.029096490 & 0.024328539 & 11.486101 \\
122821 & 264596160 & 0.019992669 & 3.3965204 & 46.863235 \\
124817 & 169474446720 & 0.0057786083 & 1.2672930 & 6.7827247 \\
126032 & 1800970320 & 0.0082151643 & 0.36230593 & 11.688370 \\
130576 & 2936881640 & 0.014954319 & 0.35027520 & 21.034030 \\
138136 & 32809823 & 0.0084035031 & 0.15151378 & 21.202474 \\
138917 & 3092368752 & 0.010994082 & 3.3294478 & 176.27092 \\
144209 & 3510535200 & 0.013203831 & 1.2246131 & 5.0019384 \\
147109 & 7942261280 & 0.0041847312 & 1.4237043 & 8.3262747 \\
14916 & 3162452560 & 0.028634117 & 1.8194260 & 15.033097 \\
153424 & 24509153664 & 0.023174277 & 3.4016473 & 25.402792 \\
157457 & 76544972964 & 0.0031668548 & 1.0412335 & 7.2747406 \\
160801 & 63790577 & 0.0072814568 & 0.29819082 & 10.519558 \\
161121 & 663937120 & 0.0038819428 & 0.12885785 & 18.108623 \\
170701 & 12569281600 & 0.019242412 & 2.2046583 & 104.80927 \\
173513 & 53005990464 & 0.026193754 & 0.10356861 & 42.247614 \\
176281 & 18783713640 & 0.0054077065 & 1.9881737 & 26.798380 \\
176684 & 6024872936 & 0.011580917 & 1.0387400 & 12.861239 \\
179024 & 63851084520 & 0.014289468 & 0.34114599 & 1.2188617 \\
\end{array}
\]
Table 3. $d = 6$

| $D_F$ | $-b$ | $E_{F,2}^\Delta$ | $b - b(E_{12})$ |
|-------|------|-----------------|-----------------|
| 300125 | $\frac{21600}{37}$ | 41.397113 | |
| 371293 | $\frac{11808}{19}$ | 3.7072130 | |
| 434581 | $\frac{8352}{13}$ | 17.280641 | |
| 453789 | 720 | 94.819103 | |
| 485125 | $\frac{7200}{11}$ | 29.364557 | |
| 592661 | 672 | 46.819103 | |
| 703493 | $\frac{2048}{3}$ | 57.485769 | |
| 722000 | $\frac{4800}{7}$ | 60.533388 | |
| 810448 | $\frac{3456}{5}$ | 66.019103 | |
| 820125 | $\frac{43200}{73}$ | 33.400075 | |
| 905177 | $\frac{3348}{5}$ | 44.419103 | |
| 966125 | 675 | 49.819103 | |
| 980125 | 675 | 49.819103 | |
| 1075648 | $\frac{8352}{13}$ | 17.280641 | |
| 1081856 | $\frac{3072}{5}$ | 10.780897 | |