On the coercivity of continuously differentiable vector fields

Răzvan M. Tudoran

Abstract

Given an arbitrary fixed continuously differentiable vector field on $\mathbb{R}^n$, we prove that this vector field is coercive if and only if its conservative part is coercive. We apply this result in order to provide sufficient conditions to guarantee the coexistence of equilibrium states of a continuously differentiable vector field and its conservative part.

MSC 2010: 37C10; 47H05.
Keywords: coercive vector field; conservative vector field; equilibrium states.

1 Introduction

The aim of this short note is to provide an alternative point of view regarding the classical notion of coercivity in the class of continuously differentiable vector fields on $\mathbb{R}^n$.

More precisely, in the next section, using Presnov’s decomposition of continuously differentiable vector fields, we prove that an arbitrary given $C^1$ vector field in $\mathbb{R}^n$ is coercive if and only if its conservative part is a coercive vector field. Consequently, we obtain that in the class of $C^1$ vector fields in $\mathbb{R}^n$, the coercivity is determined actually only by the conservative part of the vector field. In the third section we present two consequences of the main result, each of them providing sufficient conditions to guarantee the co-existence of equilibrium states of a continuously differentiable vector field and its conservative part.

2 Continuously differentiable coercive vector fields

In this section we present the main result of this note which gives a characterization of the coercivity of a continuously differentiable vector field on $\mathbb{R}^n$ in terms of the coercivity of its conservative part. We start by recalling the definition of a coercive vector field on $\mathbb{R}^n$.

Definition 2.1 A vector field $X \in \mathfrak{X}(\mathbb{R}^n)$ is called coercive if

$$\lim_{\|x\| \to \infty} \frac{\langle X(x), x \rangle}{\|x\|} = \infty,$$
where $\langle \cdot, \cdot \rangle$, $\| \cdot \|$, stand for the canonical inner product on $\mathbb{R}^n$, and the associated norm respectively.

Let $X \in \mathfrak{X}(\mathbb{R}^n)$ be a continuously differentiable vector field. Recall from [2] that $X$ can be decomposed uniquely in a conservative part and a sphere invariant part, such that

$$X(x) = \nabla H_X(x) + u(x), \forall x \in \mathbb{R}^n,$$

(2.1)

where $H_X(0) = 0$ and $\langle u(x), x \rangle = 0$, $\forall x \in \mathbb{R}^n$. The potential function which generates the conservative part of $X$, that is $\nabla H_X$, is given by

$$H_X(x) = \int_0^1 \langle X(tx), x \rangle dt, \forall x \in \mathbb{R}^n.$$

(2.2)

Note that this decomposition, known as Presnov’s decomposition, is tight related to the canonical Euclidean structure of the ambient space. For a geometric extension of the Presnov decomposition (2.1), see [3].

A direct consequence of the decomposition (2.1) is the following characterization of coercivity of continuously differentiable vector fields on $\mathbb{R}^n$, which shows that the coercivity is determined actually only by the conservative part of the vector field.

**Theorem 2.2** Let $X \in \mathfrak{X}(\mathbb{R}^n)$ be a continuously differentiable vector field on $\mathbb{R}^n$. Then $X$ is coercive if and only if its conservative part $\nabla H_X$ is coercive.

**Proof.** Using the decomposition (2.1) it follows that for any $x \in \mathbb{R}^n$

$$\langle X(x), x \rangle = \langle \nabla H_X(x) + u(x), x \rangle = \langle \nabla H_X(x), x \rangle + \langle u(x), x \rangle = \langle \nabla H_X(x), x \rangle,$$

and hence

$$\langle X(x), x \rangle = \langle \nabla H_X(x), x \rangle, \forall x \in \mathbb{R}^n.$$ 

(2.3)

Consequently, we obtain that

$$\lim_{\|x\| \to \infty} \frac{\langle X(x), x \rangle}{\|x\|} = \infty \quad \text{if and only if} \quad \lim_{\|x\| \to \infty} \frac{\langle \nabla H_X(x), x \rangle}{\|x\|} = \infty.$$

Let us recall from [2] that the uniqueness of the decomposition (2.1) is guaranteed by the fact that $X$ is continuously differentiable on the whole $\mathbb{R}^n$. Nevertheless the equality (2.3) holds true for an arbitrary continuously differentiable vector field defined on an open subset of $\mathbb{R}^n$, star–shaped with respect to the origin.

**Proposition 2.3** Let $X \in \mathfrak{X}(\Omega)$ be a continuously differentiable vector field defined on an open set $\Omega \subseteq \mathbb{R}^n$, star–shaped with respect to the origin. Then the following relation holds true

$$\langle X(x), x \rangle = \langle \nabla H_X(x), x \rangle, \forall x \in \Omega,$$

where

$$H_X(x) = \int_0^1 \langle X(tx), x \rangle dt, \forall x \in \Omega.$$
Proof. Let \( h_X \in C^1(\Omega, \mathbb{R}) \) be given by \( h_X(x) := \langle X(x), x \rangle, \forall x \in \Omega \). Using this notation, the following relation holds true

\[
H_X(x) = \int_0^1 \frac{1}{t} h_X(tx) \, dt, \forall x \in \Omega.
\]

Thus, for any \( x = (x_1, \ldots, x_n) \in \Omega \) we obtain successively that

\[
\langle \nabla H_X(x), x \rangle = \sum_{i=1}^n x_i \frac{\partial H_X}{\partial x_i}(x) = \sum_{i=1}^n x_i \int_0^1 \frac{1}{t} \frac{\partial h_X}{\partial x_i}(tx) \, dt = \int_0^1 \left[ \sum_{i=1}^n x_i \frac{\partial h_X}{\partial x_i}(tx) \right] \, dt = \int_0^1 \left[ \frac{d}{dt} h_X(tx) \right] \, dt = h_X(x) = \langle X(x), x \rangle,
\]

and hence we get the conclusion. \( \blacksquare \)

3 Two results concerning the existence of equilibria of vector fields

In this section we present two consequences of the results from the previous section, regarding the co-existence of equilibrium states of a continuously differentiable vector field and its conservative part. In order to do that we recall a classical result which provides a boundary condition that guarantees the existence of solutions to the equation \( f(x) = 0 \), where \( f : \overline{B}_r(0) \to \mathbb{R}^n \) is a continuous function defined on the \( n \)-dimensional closed ball of radius \( r > 0 \) centered at the origin, \( \overline{B}_r(0) := \{ x \in \mathbb{R}^n : \|x\| \leq r \} \). For details regarding the proof of this result, see e.g. [1].

Theorem 3.1 ([1]) Let \( f : \overline{B}_r(0) \to \mathbb{R}^n \) be a continuous mapping satisfying the boundary condition

\[
\langle f(x), x \rangle > 0, \forall x \in \partial \overline{B}_r(0).
\]

Then the equation \( f(x) = 0 \) admits at least one solution \( x \in B_r(0) \).

Next, we prove that in the case of a continuously differentiable vector field \( X \), the hypothesis of Theorem 3.1 forces the conclusion to holds true not only for the vector field \( X \), but also for its conservative part \( \nabla H_X \), where the formula of the potential function \( H_X \) is given by the equality (2.2). More precisely, we obtain the following result.

Proposition 3.2 Let \( X \in X(\overline{B}_r(0)) \) be a \( C^1 \) vector field defined on the closed ball \( \overline{B}_r(0) \subset \mathbb{R}^n \), satisfying the boundary condition

\[
\langle X(x), x \rangle > 0, \forall x \in \partial \overline{B}_r(0).
\]

Then both the vector field \( X \) and its conservative part \( \nabla H_X \), admits at least one equilibrium state in \( B_r(0) \).
Proof. Using the Proposition 2.3 we get that $\langle \nabla H_X(x), x \rangle = \langle X(x), x \rangle$, $\forall x \in \overline{B}_r(0)$, and hence the boundary condition (3.1) becomes

$$\langle \nabla H_X(x), x \rangle = \langle X(x), x \rangle > 0, \forall x \in \partial \overline{B}_r(0).$$

Now the conclusion follows by applying Theorem 3.1 to the vector field $X$ and also to its conservative part $\nabla H_X$. ■

Next result presents a sufficient condition that guarantees the existence of equilibrium states of a continuously differentiable vector filed and its conservative part, when both are perturbed by an arbitrary constant vector field.

**Proposition 3.3** Let $X \in X(\mathbb{R}^n)$ be a $C^1$ coercive vector field. Then for any fixed $b \in \mathbb{R}^n$, each of the vector fields $X + b$ and $\nabla H_X + b$, admits at least one equilibrium state.

**Proof.** Let us start by fixing an arbitrary vector $b \in \mathbb{R}^n$. Using the Theorem 2.2 we obtain that the conservative part of $X$ is also a coercive vector field, and hence using the definition of coercivity, there exists $\rho = \rho(b) > 0$ such that

$$\langle \nabla H_X(x) + b, x \rangle = \langle X(x) + b, x \rangle > 0, \forall x \in \partial \overline{B}_\rho(0).$$

Now the conclusion follows by Theorem 3.1 applied to each of the vector fields $X + b$ and $\nabla H_X + b$. ■

**References**

[1] J. Francu, Monotone operators. A survey directed to applications to differential equations, *Aplikace Matematiky*, 35(4)(1990), 257–301.

[2] E. Presnov, Non-local decomposition of vector fields, *Chaos, Solitons and Fractals*, 12(2002), 759-764.

[3] R.M. Tudoran, A global geometric decomposition of vector fields and applications to topological conjugacy, *Acta Appl Math*, (2019), https://doi.org/10.1007/s10440-019-00258-0

R.M. Tudoran  
West University of Timișoara  
Faculty of Mathematics and Computer Science  
Department of Mathematics  
Blvd. Vasile Pârvan, No. 4  
300223 - Timișoara, România.  
E-mail: razvan.tudoran@e-uvt.ro