About Strongly Universal Cellular Automata

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In this paper, we construct a strongly universal cellular automaton on the line with 11 states and the standard neighbourhood. We embed this construction into several tilings of the hyperbolic plane and of the hyperbolic 3D space giving rise to strongly universal cellular automata with 10 states.

1 Introduction

Many papers about universality of cellular automata, especially those which try to minimize the number of states preserving this property, consider what is called weakly universal cellular automata. This term means that the initial configuration of the cellular automaton may be infinite, provided the following requirement is satisfied: the initial configuration must be periodic outside a bounded domain. In the case of the one dimensional line, it is accepted that the periodicity in the left-hand side infinite part may be different from the periodicity of the right-hand side one.

In this paper, we consider deterministic cellular automata in various settings. First, on the line with the standard neighbourhood which means, for each cell, the cell itself and its left- and right-hand side neighbours. Second, we shall implement the results found on the line into deterministic cellular automata in several tilings of the hyperbolic plane and one tiling of the hyperbolic 3D-space.

For these automata, we consider the construction of a strongly universal cellular automaton. There are two constraints in order to obey this definition which we take from Codd’s own definition, see [2]. The first restriction means that the initial configuration of the cellular automaton must be finite. This means that all cells except finitely many of them must be in a quiescent state at the initial state. We remind the reader that a state $s$ is quiescent if and only if a cell is in $s$ and its neighbours are also all in $s$ then it remains in $s$ at the next time. We shall also refer to this situation as starting from a blank background. The second restriction is about the halting: the computation of the cellular automaton halts if and only if there is the occurrence of two consecutive identical configurations. Now, it is in the definition of universality that when the simulated device halts, the simulating device has also to stop. This is the counterpart of the situation when the simulated device does not halt: then, by the very notion itself, the simulating device cannot halt. Now, few papers take into account this halting condition, although it is a natural one. Most papers follow different definitions of universality. Probably, the most studied case is what is called weak universality by this author and also a few ones, see [16] for further references. In weak universality, both above constraints are removed. For the halting, it is enough that the simulated machine halts, the simulating one may not halt. For the initial configuration, infinite ones are accepted provided they are not arbitrary. The usual constraint is, in the case of a 1D-cellular automaton that the configuration be periodic in both directions, starting from a certain point. It is not required that the periodic pattern should be the same on both sides. Accordingly much striking results were obtained. Among them, the most notable one is the cellular automaton with Rule 110, see [3, 15]. In this case we shall also say that the cellular automaton works on a non-blank periodic background.
The paper considers the cellular automaton with 7 states constructed in [9] which we later refer to as $\mathcal{L}N$. This cellular automaton is weakly universal. In [9], they claim that with two additional states, they obtain from $\mathcal{L}N$ a strongly universal cellular automaton. However, they do not give any hint on how these additional states are used and, moreover, they do not consider the halting of the their cellular automaton as mentioned above. Accordingly, such a result, if any, cannot be taken into consideration. Our construction starts from $\mathcal{L}N$ to which we append four states in order to be strongly universal as stated in the above definition. Then we transport the construction into two tessellations of the hyperbolic plane and one of the hyperbolic 3D space.

In Section 2, we remind the construction of [9] for self-containedness. Also in this section, we clarify the status of $\mathcal{L}N$ with respect to weak universality. Indeed, it was suspected to be not possible, due to a confusion about the Turing machine simulated by $\mathcal{L}N$. In Section 3 we show how the additional states allow us to transform $\mathcal{L}N$ into a strongly universal cellular automaton with 11 states.

In Section 4 we extend the result to cellular automata in hyperbolic spaces. There, it is possible to obtain strongly universal cellular automata with 10 states only in the tilings of the spaces which we shall consider.

2 Implementing a Turing machine in a one-dimensional cellular automaton

Weak universality results for cellular automata are rather easy to obtain from the simulation of a Turing machine. The idea is to embed the Turing tape into the cellular automaton by regularly putting the symbols of the Turing machine, two neighbouring symbols of the tape being separated by the same number of blanks of the cellular automaton.

In order to obtain a strongly universal cellular automaton, only a finite part of the Turing tape can be embedded in such a way. In order to perform the computation, especially if the computation turns out to be infinite, we have also to implement the continuation of the Turing tape. For the present moment, let us ignore this point to which we go back in Section 3. In Sub-section 2.1 we remind the general scheme of simulation, summarizing the presentation of [9]. In Sub-section 2.2 we explicitly indicate the Turing machine we consider, showing that it is strongly universal, and how it is implemented as in [9].

2.1 The guideline for the simulation

Consider a Turing machine $\mathcal{M}$ whose blank symbol is denoted by 0. Later, we shall explicitly indicate which Turing machine will play the role of $\mathcal{M}$. Now, for each such machine $\mathcal{M}$, we construct a deterministic cellular automaton which simulates $\mathcal{M}$ and we denote this automaton by $\mathcal{L}(\mathcal{M})$.

Let us first indicate how $\mathcal{L}(\mathcal{M})$ works, following the technique given in [9]. We say that a cell is blank if it is in the quiescent state. Accordingly, if a cell is blank as well as its left- and right-hand side neighbours, it remains blank at the next step of the computation.

The Turing tape is implemented by inserting the squares of the tape at regular places of the cellular automaton, leaving the same amount of blank cells in between two consecutive places. This means that outside a finite interval of the support of the cellular automaton, the pattern $0 \square \square \square \ldots \square 0$, where $k$ is the number of blank cells of $\mathcal{L}(\mathcal{M})$ in between two symbols of $\mathcal{M}$. In [9], $k$ is fixed as at least 2. Here, we also fix $k = 2$, as the value $k = 2$ will play an important role in Section 3. Accordingly, the initial configuration of $\mathcal{L}(\mathcal{M})$ looks like this: $(\square x \square)^\infty$.
However, in this representation, we do not know where is the head of the Turing machine and in which state it is. The Turing head is represented by T, following the notation of \[9\], assuming that T is not a letter of \(\mathcal{M}\). In order to indicate the state, T is accompanied by an auxiliary symbol which represents the state. But this representation is not direct. As in many papers, the basic idea of \[9\] is to mimic the space-time diagram of the Turing machine by that of \(\mathcal{L}(\mathcal{M})\). Under this consideration, T appears as a moving symbol which changes its neighbour: the left-hand side or the right-hand side one, depending on the direction of the move defined by the current instruction. The idea of \[9\] is to append a symbol to the left of T for the instructions moving to right and to append a symbol to the right of T for the instructions moving to left. Following \[9\], we call a pair \(\text{T a}\) or \(\text{a T}\) an impulsion, to left or to right respectively. As the impulsions travel on the support of \(\mathcal{L}(\mathcal{M})\), they meet with symbols of \(\mathcal{M}\). Such a meeting is called a collision in \[9\] and it is desirable to organize it in such a way that after the collision the support shows us the next configuration of the Turing machine: the result of the collision will be the application of the corresponding instruction.

Below, Tables 1 and 2 show us how the impulsions travel in the support of \(\mathcal{L}(\mathcal{M})\), and how instructions are performed. Note that in these tables the blank of \(\mathcal{L}(\mathcal{M})\) is denoted by the character ‘.’. The same convention will be used for tables based on executions by a computer program.

**Table 1** Standard situation for an impulsion whose direction is not changed by the collision. Left-hand side: impulsion to right; right-hand side: impulsion to left.

\[
\begin{align*}
\text{u a T v x y} & \quad \text{u v T a x y} \\
\text{u w b T x y} & \quad \text{u T b w x y}
\end{align*}
\]

Table 1 shows the execution of an instruction in the case when the application of the instruction does not change the direction of the motion of the Turing head. By contrast, Table 2 shows the situations when the execution of the instruction results in a change in the direction of the motion of the Turing head. This situation is usually called a half-turn of the Turing head.

**Table 2** Situations of a standard half-turn. Left-hand side: half-turn to left; right-hand side: half-turn to right.

\[
\begin{align*}
\text{u a T v x y} & \quad \text{u v T a x y} \\
\text{u T b w x y} & \quad \text{u w b T x y}
\end{align*}
\]

In Tables 1 and 2, we can say that the upper line represents the starting point of the collision and that the lower line, below the double bars, represents the final point. We can notice that the starting point represents a configuration of \(\mathcal{M}\) and that the final one represents the next configuration of \(\mathcal{M}\). However, the final point is never the next configuration of the starting point for \(\mathcal{L}(\mathcal{M})\). As can be noticed on the figures, the final point occurs at least three steps after the starting point. In \[9\], the number of steps of \(\mathcal{L}(\mathcal{M})\) between the starting point and the final one can be higher, and this will also be the case in our simulation.

Say that a configuration by the end of a collision is a standard one if it is of the form \(z y T\) or \(T y z\), where \(z\) is the new state after the collision which corresponds to the execution of the current instruction of the Turing machine on the currently scanned symbol. From Tables 1 and 2 we can see that if we consider three consecutive Turing symbols where the middle one is scanned by an impulsion then, after the collision is completed, the positions of the extremal Turing symbols are unchanged. This ensures that the working of the cellular automaton faithfully simulates the Turing machine.

Now, there are a few exceptional situations illustrated by Table 3, where the new state of the scanned
symbol appears one step later with respect to a standard configuration. However, if we look at the extension in space, we may have the situation illustrated by Table 3 but no more. We shall check them in Section 3. This indeed occurs in [9] with a few collisions.

**TABLE 3** In these cases, there is delay in the transformation of the new state by one step. Left-hand side: for an impulsion to right not changed by the collision; right-hand side: for a half-turn to right. There are no other cases involving such a delay.

\[
\begin{array}{cccccccc}
  u . a & T . v & \ldots & x & \ldots & y & \ldots \\
  u . w & . b & T . x & \ldots & y & \ldots \\
  u . z & \ldots & b & c & x & \ldots & y & . \\
  u . v & . T & a & x & \ldots & y & . \\
  u . z & . b & T & . x & \ldots & y & . \\
  u . w & \ldots & b & c & x & \ldots & y & . \\
\end{array}
\]

Let us now look at how many symbols are needed. The impulsions correspond to the indication of a state and of a move. If we analyze the table of \( M \) in this regard, there are \( s_\ell \) states in which all instructions go to the left, \( s_r \) of them, where all instructions go to the right and \( s_b \) states where we meet instructions of both kinds. Of course, \( s_\ell + s_r + s_b = n \), where \( n \) is the number of states of \( M \). Let us set \( n_\ell = s_\ell + s_b \) and \( n_r = s_r + s_b \). Then, the coding of the instructions to the left will require \( n_\ell \) symbols while the coding of those to the right will require \( n_r \) of them. In any case, we need at least \( \mu = \max(n_\ell, n_r) \) symbols to code the number of impulsions. The same symbol may be used to code one impulsion to the left and one impulsion to the right. This is why in Tables 1, 2 and 3 we cannot interpret the symbol which is appended to \( T \) as a state: it is simply an encoding of the impulsion. If \( \mu \leq m \), where \( m \) is the number of letters of \( M \), then we need just two extra symbols: \( T \) and \( \_ \) the blank of \( L(M) \). If \( m < \mu \) we still need \( \mu - m \) symbols. Note that the number of collisions is \( m(n_\ell + n_r) \).

In the case of [9], \( \mu = 5 \) and \( m = 4 \). This is why we need at least 7 symbols. It turns out that, using an infinite background as indicated above, with the Turing machine used in [9], it is enough.

### 2.2 Implementing Minsky’s Turing machine

Let us now turn to the Turing machine used in [9]. In Sub-subsection 2.2.1, we explain why Minsky’s Turing machine is strongly universal. In Sub-subsection 2.2.2 we remind its implementation in [9].

#### 2.2.1 A strongly universal Turing machine

This machine was devised by M. Minsky in [10]: it has seven states and four letters and, in [10], it was claimed to be strongly universal. We say **claimed** because the proof given in [10] cannot considered as completely satisfactory. This is why in [14, 13, 12, 7], several authors, including myself, considered that this machine is not universal: it was said that it damages its output before halting when the simulated machine halts. In [16], the authors claim that Minsky’s machine is indeed universal, but they do not give enough details to understand why it is the case. After discussing this point with Turlough Neary, I came to the conclusion that the claim of [16] is correct. After sometime, I also understood where the reason is exactly. I think it useful to clarify this point, which I try to do in some general form which seems to be interesting in itself.

Table 4 gives the table of Minsky’s machine which we later refer to as Minsky’s UTM. We follow the following notations, inspired by those of [10]. An instruction of a Turing machine is of the form \( pxMyq \). The head reads \( x \) in state \( p \). It then replaces \( x \) by \( y \), moves to the left-, right-hand side neighbour or remains on the same cell depending on whether \( M \) is \( L, R \) or \( S \) and the heads is now under the state \( q \). We read \( x \) in the heads of the columns and \( p \) on the heads of the lines in the Table. The corresponding
entry contains the triple $Myq$ of the instruction. In the table, we always define $M$ and we mention $y$ or $q$ only if it is different from $x$, $p$ respectively.

As proved in [11, 10], this machine is universal as it is able to simulate a tag-system with deletion number 2 which was proved to be universal in the same references. A tag-system itself is a particular kind of Post system. It consists of a finite alphabet $A$ together with a mapping $p$ from $A$ into $A^*$. The tag-system works as follows. First, it reads the first two letters of the current tagged word $w$, say $\ell_1\ell_2$. Then, it erases $\ell_1\ell_2$ from the beginning of $w$. This is why we say that this tag-system has deletion number 2. Then, the tag-system appends to what remains of $w$ the word $p(\ell_1)$: this is the new configuration of the tagged word after the application of one step of the tag-system. Among the letters of $A$, one of them at least is a halting letter: when the tag-system meets such a letter as the first letter of the current word, it performs the step and when this is completed, it halts the computation. When it halts, the computation of the tag system can be described as a sequence of $w_0, \ldots, w_n$, where $w_0$ is the initial tagged word and $w_n$ is the last one, subjected to the following condition: $w_{i+1}$ is the word obtained by the computation of the tag-system by the completion of the step which erased the last letter of $w_i$. We say that the transformation of $w_i$ into $w_{i+1}$ defines a pass of the tag-system over $w_i$. Note that a current tagged word is not necessary one of the $w_i$'s. Minsky imposes two assumptions on the tag-system as just described: there is a single halting letter and the tagged word must always be non-empty during the computation. It is an easy exercise, left to the reader, to check that these conditions are observed by the the tag system devised by Cocke and Minsky in their proof. The full solution is already in [10].

|   | 0  | 1  | y  | A  |
|---|----|----|----|----|
| 1 | L  | L2 | 0L | 1L |
| 2 | yR | AR | 0L1| yR6|
| 3 |   | AL | L  | 1A |
| 4 | yR5| L7 | L  | 1L |
| 5 | yL3| AR | R  | 1R |
| 6 | AL3| AR | R  | 1R |
| 7 | yR6| R  | 0R | 0R2|

It is interesting to understand why some people claimed that Minsky’s UTM damages its tape and why this is not necessary the case. Here I explain where the reason is.

Define an extended tag-system with deletion number 2 as given by two finite alphabets, $A$ and $T$, together with a mapping $p$ from $A$ into words on $(A+T)^*$. We call $T$ the terminal alphabet. The extended tag-system works exactly as a tag-system with deletion number 2 and we have exactly the same notion of pass. However, a halting computation is considered as valid only when a terminal letter is never the first letter of the current word during the computation, except possibly the final one, and the last current word is in $T^*$. A usual tag-system is of course an extended one whose terminal alphabet is the whole alphabet. We have the following property which, I think, is interesting per se:

**Lemma 1** — Let $E$ be an extended tag-system with deletion number 2. There is a tag-system $P$ also with deletion number 2, which simulates $E$ in the following meaning: if $E$ does not halt on $w$, $P$ also does not halt. If $w = w_1\ldots w_n$ is accepted by $E$, then $P$ halts on $\overline{w}$ if $n$ is odd, on $a\overline{w}$ if $n$ is even, with $\overline{w} = s_1s_1\ldots s_ks_k$ where $s_i$ are defined by $s_i = w_{2i+1}, i \in \{1,\ldots, \left\lfloor \frac{n-1}{2} \right\rfloor \}$ and $a \in T$. 


Proof. Let $A$ and $T$ be the alphabets of $E$, $T$ being the terminal one. Let $p$ be the mapping from $A$ into $(A + T)^*$ defining the computation of $E$. We construct $P$ as follows. Its alphabet is $(A + T + \{z\})^*$. Define the mapping $q$ of $P$ by $q(a) = p(a)$ for $a \in A \setminus T$, $aT$ not being the halting letter. For the halting letter $h$, $q(h) = p(h)zza$, $a$ being any letter of $T^*$, with the addition that $h$ is not a halting letter for $P$. For $b$ in $T$, $q(b) = bb$. The halting letter is $z$ and $q(z) = \varepsilon$ where $\varepsilon$ is the empty word.

From the definition of $P$, it is clear that when the computation of $E$ on $w$ does not halt, the computation of $P$ on the same word does not halt. If the computation of $E$ on $w$ is accepted, it halts and the final word is $\omega$ with $\omega \in T^*$. The computation with $P$ replaces $\omega$ by $\omega zza$ as far as, by definition, $\omega$ ends with $p(h)$ which is in $T^*$ by our assumption. Now, as all letters of $\omega$ are in $T$, $P$ transforms $\omega zza$ into $zaza\omega$ or $za\omega\omega$ depending of the parity of $|\omega|$ as mentioned in the statement of the lemma. Accordingly, when $P$ is applied to $za\omega$ or $za\omega\omega$, it produces $\omega$ or $\omega\omega$ and then, it halts.

Now, Minsky’s UTM does not simulate a tag system $P$ but, rather, some extension of $P$. Indeed, when Minsky’s UTM simulates the halting step of $P$, it copies the pattern $AAA$ at the end of the configuration and it does not stop. The halting of Minsky’s UTM requires that at some point, the head in state 3 meets a 0 on the tape. The tape of Minsky’s UTM contains an encoding of the productions on the left-hand side and an encoding of the tagged word on the right-hand side with a possible garbage of 0’s in between. The computation can be divided into cycles, each one of them being split into three steps: first, the localization of the production corresponding to the first letter; second, it copies the production. During these two steps, an appropriate marking of the configuration from the tagged word to the productions allows the machine to perform these tasks. The third stage consists in removing the markings and in detecting the first letter of the current word. The first step is accomplished by states 1 and 2. The copying is performed by states 3, 4, 5 and 6. The restoration and the transition to the next cycle are performed by states 6 and 7. During the copying, state 5 copies 0’s and state 6 copies 1’s, going from the productions to the end of the current word. When the symbol, 0 or 1 is copied, the return of the head to the productions is performed both by states 3 and 4: state 3 leads the head until it meets an $A$ at which moment state 4 takes the control, leading the head back to the production. It is important to note that this transition from state 3 to state 4 is made possible by the markings of the first steps which marks the configuration by $y$’s and $A$’s from the beginning of the current word to the last copied symbol in the productions. State 3 removes the markings in the current word and state 4 removes those of the production, the y’s being still present from the current word to the productions. State 7 changes back those y’s to 0. When the halting letter of the tag-system is detected, Minsky’s UTM copies the halting production at the end of the configuration. This consists in the pattern $AAA$. But the machine does not stop, so that it starts a new pass of the tag-system over the final word. As long as codes of words remain on the left-hand side of $AAA$, the computation of the tag system goes on, performing the last pass in Lemma[1]. Due to the erasing process, the machine eventually meets $AAA$ in state 7. The leftmost $A$ read under this state signals a new encoded letter. But the second $A$ read under state 2 shows that the end of the word is reached. And so there is no marking: the machine under state 6 appends an $A$ at the right-hand side end of the configuration. When it goes back in state 3, it will then meet a 0 in this state as there is no marking in between $AAA$ and the production. And so, the machine will halt.

This working reminds us that of an extended tag-system. From our lemma, it is plain that if we can present the computation of the execution of an extended tag-system and if we have $\omega = \overline{\omega}$ for the final word $\omega$, then we can recover the production of the initial tag-system. It is namely possible for the tag-system $\mathcal{T}$ devised by Cocke-Minsky to prove that tag-system with deletion number 2 can simulate any Turing machine on \{0, 1\}. This tag-system encodes the configuration of a Turing machine with four letters as follows:
$Aa(aa)^nBb(bb)^m$

where $n$ and $m$ are non-negative integers which encode the left- and the right-hand part, respectively, of the Turing tape, starting from the position of its head. We could change $Aa(aa)^nBb(bb)^m$ to:

$AA(aa)^nBB(bb)^m$

in order to be compatible with Lemma $[11]$. This does not change the proof of $[11]$ [10]. Now these four letters represent the states of the Turing machine: when an instruction of the Turing machine is performed, a new group of four letters appear, representing the new state of the head of the Turing machine. Accordingly, the final state is associated to a group of four letters:

$Hh(hh)^nKk(kk)^m$

where $H$ is the halting letter of $T$. Note that $h$, $K$ and $k$ never trigger a production of the tag-system. Note that here too, $Hh(hh)^nKk(kk)^m$ can be changed to $HH(hh)^nKK(kk)^m$ without altering the proof. Accordingly, the simulation of the Turing machine by $T$ can be presented as the computation in an extended tag-system. Moreover, this tag system has clearly the property that $w = \overline{w}$ for all current word $w$ during the computation. So that the lemma tells us how to transform $T$ into a standard tag-system $P$ which computes exactly the same thing as $T$. From the proof of the lemma and the behaviour of Minsky’s UTM as we presented above show us that Minsky’s UTM exactly executes $P$, which proves the strong universality of Minsky’s UTM.

### 2.2.2 The Lindgren-Nordahl cellular automaton with 7 states

The implementation of Minsky’s UTM in a cellular automaton was performed in [9] as mentioned in Subsection [2.1]

For the sake of the reader, we mention the exact implementation.

From Table 4 we can see that we have the following impulsons to right: states 2, 5, 6 and 7 and that we have five impulsons to left: states 1, 2, 3, 4 and 7. They are accordingly encoded as $YT$, $0T$, $1T$, $AT$ and $Ty$, $T0$, $T1$, $TA$, $TB$ respectively.

**Table 5** Table of the collisions induced by the table of Minsky’s Turing machine, see Table 4 impulsons to right.

| $yT0$ | $yT1$ | $yT\_y$ | $yT\_A$ |
|-------|-------|---------|---------|
| $y\_yT$ | $A\_yT$ | $Ty\_0$ | $y\_1T$ |
| $0T0$ | $0T1$ | $0T\_y$ | $0T\_A$ |
| $T1\_y$ | $A\_0T$ | $y\_0T$ | $1\_0T$ |
| $1T0$ | $1T1$ | $1T\_y$ | $1T\_A$ |
| $T1\_A$ | $A\_1T$ | $y\_1T$ | $1\_1T$ |
| $AT0$ | $AT1$ | $AT\_y$ | $AT\_A$ |
| $y\_1T$ | $1\_AT$ | $0\_AT$ | $0\_yT$ |
### Table 6
Table of the collisions induced by the table of Minsky's Turing machine, see Table 4 impulses to left.

| 0 - Ty | 1 - Ty | y - Ty | A - Ty |
|--------|--------|--------|--------|
| Ty 0   | T 0 1  | Ty 0   | Ty 1   |
| 0 _ T 0| 1 _ T 0| y _ T 0| A _ T 0|
| y _ y  T| A _ y  T| Ty 0   | y _ 1 T|
| 0 _ T 1| 1 _ T 1| y _ T 1| A _ T 1|
| _ 0 _ | T 1 _ A| T 1 _ y| T A _ 1|
| 0 _ T A| 1 _ T A| y _ T A| A _ T A|
| y _ 0 T| T B _ 1| T A _ y| T A _ 1|
| 0 _ T B| 1 _ T B| y _ T B| A _ T B|
| y _ 1 T| 1 _ A T| 0 _ A T| 0 _ y T|

Tables 5 and 6 represent the collisions of the various impulsions at the meeting with a symbol of Minsky’s UTM in a cell as we have seen in Sub-section 2. In the tables, note that there may be states with impulsions of both kinds. This is the case for states 2 and 7 exactly. Accordingly, there are two times four collisions devoted to each state, which means that we have a total of 36 collisions.

### Table 7
Table of the cellular automaton with seven states as stated in [9].

| . 0 y A B T | 0 0 y A B T | 1 0 y A B T |
|-------------|-------------|-------------|
| . . . . . . T | . . . . . . T | . . . . . . T |
| 0 . . . . . . T | 0 0 . . . . . . T | 0 0 . . . . . . T |
| 1 . . . . . . 1 | 1 0 . . . . . . 1 | 1 0 . . . . . . 1 |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
| . . . . . . . A | . . . . . . . A | . . . . . . . A |
| B . . . . . . . | B . . . . . . . | B . . . . . . . |
| . . . . . . . T | . . . . . . . T | . . . . . . . T |
| . . . . . . . y | . . . . . . . y | . . . . . . . y |
Table 8 Execution of the rules for the collisions of a right-hand side impulsion with a Turing letter. Note that the blank is represented by .

| 1 T . 0 | 1 T . 1 | 1 T . y | 1 T . A |
|-------|-------|-------|-------|
| 0 B 0 | 0 0 1 | 0 B y | 0 A A |
| 1 y y | y 0 T | 1 A 0 | B T T |
| T I y | A . 0 T | 1 0 T | 1 . 0 T |
| T I . y | B . 0 T | . y . 0 T |
| . 1 T . 0 | . 1 T . 1 | . 1 T . y | . 1 T . A |
| . 1 B 0 | . 1 0 1 | . 1 B y | . 1 A A |
| . A 0 y | . B 1 T | . A A 0 | . 1 T |
| . 1 y A | . A . 1 T | . 0 1 T | . 1 . 1 T |
| . T I 0 | . y . 1 T |
| . T I B B |
| . T I . A |
| . y T . 0 | . y T . 1 | . y T . y | . y T . A |
| . y B 0 | . y 0 1 | . y B y | . y A A |
| . B I y | . A y T | . B y O | . 0 1 T |
| . A A y | . A . y T | . T y A | . y . 1 T |
| . 0 0 A | . T y . 0 |
| . y y T |
| . y . y T |
| . A T . 0 | . A T . 1 | . A T . y | . A T . A |
| . A B 0 | . A 0 1 | . A B y | . A A A |
| . B A y | . 1 A T | . B A 0 | . 0 y T |
| . y A A | . 1 . A T | . y A T | . 0 . y T |
| . 0 1 T | . 1 . A T | . 0 . A T |
| . y . 1 T | . i . A |

Table 7 gives the rules of \( \mathcal{L} \mathcal{N} \) as described in [9]. Our presentation is slightly different. The table in \( \mathcal{L} \mathcal{N} \) is split into seven small tables labelled by a state of \( \mathcal{L} \mathcal{N} \). The sub-table \( x \) gives all rules of the form \( l x r \rightarrow y \), where \( x \) is the state of the cell, \( l \) the state of its left-, right-hand side neighbour respectively, and \( y \) is the new state of the cell after the application of the rule. The entry \( y \) is at the intersection of the row \( 1x \) and the column \( r \).
Table 9: Execution of the rules for the collisions of a left-hand side impulsion with a Turing letter. Here too, the blank is represented by .

. 0 . T 0 . . 1 . T 0 . . y . T 0 . . A . T 0 . .
. 0 y 0 . . 1 1 0 . . . y y 0 . . A 1 0 . .
. 0 0 A . . 1 0 B . . . y 1 A . . O A B . .
. y y T . . B 1 B . . . T y A . . B 0 y . .
. y . y T . . A y T . . . T y . 0 . . y 0 A . .
. A . y T . . A 1 T . .
. . . . 0 1 i T . .
. . . . y . . 1 T . .
. . 0 . T 1 . . 1 . T 1 . . y . T 1 . . A . T 1 . .
. 0 y 1 . . 1 1 1 . . . y y 1 . . A 1 1 . .
. 0 T y . . 1 y 0 . . . y 1 y . . 0 0 0 . .
. 0 . . . T 1 A . . . T 1 y . . . y A 1 . .
. T 1 A . . . T 1 . y . . . O B A . .
. . . . . y 1 y . .
. . . . . T A y . .
. . . . . T A B A . .
. . . . . T A . . 1 . .
. . 0 . T y . . 1 . T y . . . y . T y . . A . T y . .
. 0 . y y . . 1 1 y . . . y y y . . A 1 y . .
. 0 . 0 y . . 1 0 y . . . y B y . . O A y . .
. . y 1 A . . B B A . . . B y 0 . . B y A . .
. . T y A . . y 1 1 . . . T y A . . T T y 0 . .
. . T y . 0 . . T O 0 . . . T y 0 . . T O y B A . .
. . T O . . 1 . .
. . . . . y 1 y . .
. . . . . B y A . .
. . . . . T A B A . .
. . . . . T A . . 1 . .
. . . . . y . . . . T A B . . . T A . . 1 . .
. . . . . T B 0 0 . . . T A . . y . . T A . . 1 . .
. . . . . y . . . T B . . 1 . . . A . . y . .
. . . . . T B . . 1 . . . y . . T B . . A . . T B . .
. . . . . y B . . 1 1 B . . . y y B . . A 1 B . .
. . . . . 0 1 T . . 0 1 A T . . . y A T . . 0 y T . .
. . . . . y . 1 T . . 1 . A T . . 0 . A T . . 0 . y T . .

Tables 8 and 9 reproduce those of [9]. They were produced by a computer program mimicking \(L_N\), applying the rules of Table 7. This allows us to check that the collisions depicted by Tables 5 and 6 can be performed according to the guidelines of Subsection 2.1.

3 Strongly universal cellular automata on a one-dimensional line

In this section, we complete the description of \(L_N\) so that it now satisfies the requirements of strong universality as defined in Section 1.

As mentioned in Section 2.2, \(L_N\) is not yet strongly universal: it works in an infinite non-blank background. The idea is to start with a finite initial configuration, but we modify it in such a way that it constructs the background so that each time the Turing head goes out from the current configuration to the right, it meets the required pattern of the background.

We can improve \(L_N\) by providing a construction of the background during the computation of the automaton. There is a very simple way to do that by using three additional states, say \#, \$ and \&.

From the explanations of Sub-subsection 2.2.1 as Minsky’s UTM simulates a tag-system, there is a square \(c\) of the Turing tape such that the head of the machine never goes to the left of \(c\). And so, we
can consider that all cells to the left of \( c \) are blank. As the initial configuration of finite, there is a cell \( d \) such that all cells on the right-hand side of \( d \) are blank. And so we can choose \( d \) so that the interval \([c, d]\) contains the encoding of the initial configuration of Minsky’s UTM. From Sub-subsection 2.2.1 again, we remark that when Minsky’s UTM halts, the head is in between the final tagged word and the productions, looking at a 0.

Consider the leftmost 0 which is on the right-hand side of \( d \). Replace 0 by #. Table 10 provides us with a scheme of execution which shows how it is possible to construct the background needed for \( \mathcal{L} \mathcal{N} \) during its computation:

**TABLE 10** Using three states to construct the background for \( \mathcal{L} \mathcal{N} \).

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
|   | # |   |   |   |   |   |   |
|   | 0 | $ |   |   |   |   |   |
|   | 0 | & |   |   |   |   |   |
|   | 0 |   | # |   |   |   |   |

However, this scheme is not suitable for our purpose: when the Turing machine halts, a signal should be sent in order to reach $, & or # in order to stop the construction of the background for \( \mathcal{L} \mathcal{N} \). But this cannot be achieved: The maximal speed of a signal is 1 and that of the above symbols in the scheme of Table 10 is also 1, so that it can never be reached. We use another pattern given by Table 11 which makes use of two additional states only.

**TABLE 11** Using an appropriate pattern to construct the background needed by \( \mathcal{L} \mathcal{N} \).

|   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|
|   | U | U |   |   |   |   |   |
|   | U | U | B |   |   |   |   |
|   | U | U | U |   |   |   |   |
|   | U | U | U | B |   |   |   |
|   | U | U | U | U |   |   |   |
|   | U | U | U | U | B |   |   |
|   | U | U | U | U | U |   |   |

The advantage of this new pattern is that its speed is \( \frac{1}{2} \), so that it will be reached by a signal sent later at speed 1. But the pattern has also to be not too slow: it must provide the background in advance and so, it must go faster than the advance of the computation itself. We can see in Tables 9 and 8 that the speed of the computation of \( \mathcal{L} \mathcal{N} \) oscillates between \( \frac{1}{7} \) and \( \frac{1}{3} \).

This pattern leaves a whole interval of U’s growing to the right-hand side and which starts from the right-hand side neighbour of the cell \( d \).

We also have to implement the signal which will reach the pattern constructing the background. An additional state 3 is used to erase the U’s until reaching the last occurrence of B which is erased too. We call it the stopping signal. Note that 3 is raised by the neighbouring \( 0Ty \) of \( \mathcal{L} \mathcal{N} \). We simply replace the rule \( 0Ty \rightarrow 0 \) by the rule \( 0Ty \rightarrow 3 \). But there is an additional constraint: signal 3 starts from the left-hand end of the final tagged word and it has to reach the right-hand side end of the configuration. Accordingly, it must cross the tagged word without damaging it. For this purpose, we introduce two more states: 4 and V. States 3 and 4 replace and then restore 0, 1 respectively, while U and V perform the same operations on y and A respectively.

Now, when the Turing head starts to examine the cell \( d \), the rules we have seen in Sub-subsection 2.2.2 do not apply. So that we have to append new rules for the impulsions going to the right and meeting a 0: now, U has the meaning of a 0. Another state is needed, 4, also interpreted as 0. The role of state 4 is to create the two blanks which separate the Turing symbols on the tape. Table 12 shows how this works.
Table 12 also contains the corresponding rules and Table 15 was obtained by a computer applying the rules of Table 12.

**Table 12** Table of the rules to complete the cellular automaton $\mathcal{L}$ into the rules for $\mathcal{L}_{11}$.

| .  | 0 1 y A B T 3 4 U V | .  | 0 1 y A B T 3 4 U V | 1  | 0 1 y A B T 3 4 U V |
|----------------|---------------------|----------------|---------------------|----------------|---------------------|
| . . | . . . . . T . . . | . 0 | 0 y y 0 B 1 . 0 | . 1 | 1 B 1 T 1 A . 1 |
| 0. | . . . . y . . | 0 0 | 1 y 0 1 y | 0 1 | T . |
| 1. | . . . . 1 . | 1 1 | 0 0 y 0 B A . |
| y. | . . . . y . | y0 | 1 A y 1 . |
| A. | . . . . . . . . | A0 | T A y |
| B. | . . . . y . | B0 | T y 0 B |
| T. | T B O B A . B | T0 | . . |
| 3. | 3 3 3 3 | 30 | 3 |
| 4. | 3 | 3 1 |
| U. | B y | U0 | |
| V. | 3 | V0 | |

| y. | 0 1 y A B T 3 4 U V | A. | A 0 1 A B y 1 |
|----------------|---------------------|----------------|---------------------|
| . 0 | 0 y y 0 B 1 . 0 | 0 0 | 1 y A B T 3 4 U V |
| . 1 | 1 B 1 T 1 A . 1 | . 1 | V y |
| 0 0 | 0 y B A . |
| 1 1 | 0 0 y B A . |
| y0 | 1 A y 1 . |
| A0 | T A y |
| B0 | T y 0 B |
| T0 | . . |
| 30 | 3 |
| 3 1 |
| U0 | |
| V0 | |

| y | . 0 1 y A B T 3 4 U V | A | A 0 1 A B y 1 |
|----------------|---------------------|----------------|---------------------|
| . 0 | 0 y y 0 B 1 . 0 | 0 0 | 1 y A B T 3 4 U V |
| . 1 | 1 B 1 T 1 A . 1 | . 1 | V y |
| 0 0 | 0 y B A . |
| 1 1 | 0 0 y B A . |
| y0 | 1 A y 1 . |
| A0 | T A y |
| B0 | T y 0 B |
| T0 | . . |
| 30 | 3 |
| 3 1 |
| U0 | |
| V0 | |

| T | . 0 1 y A B T 3 4 U V | 3 | . 0 1 y A B T 3 4 U V | 4 | . 0 1 y A B T 3 4 U V |
|----------------|---------------------|----------------|---------------------|----------------|---------------------|
| . . | . . . . . T . . . | . 0 | 0 y y 0 B 1 . 0 | . 1 | 1 B 1 T 1 A . 1 |
| 0. | . . . . y . . | 0 0 | 1 y 0 1 y | 0 1 | T . |
| 1. | . . . . 1 . | 1 1 | 0 0 y 0 B A . |
| y. | . . . . y . | y0 | 1 A y 1 . |
| A. | . . . . . . . . | A0 | T A y |
| B. | . . . . y . | B0 | T y 0 B |
| T. | T B O B A . B | T0 | . . |
| 3. | 3 3 3 3 | 30 | 3 |
| 4. | 3 | 3 1 |
| U. | B y | U0 | |
| V. | 3 | V0 | |

Table 13 shows how the states are used for this crossing. For a technical reason, the crossing of a y is slightly more complex: as we need the rule U . . $\rightarrow$ B, we cannot have a rule . U . $\rightarrow$ y as the other patterns of the Table would require. Instead, we use the simultaneous occurrence of V and B to avoid any conflict with the rules defining the motion indicated by Table 11.
The rules given in Table 12 perform the computations given in Tables 11 and 13. They also allow to perform the computations given in Table 14 which show how the stopping signal halts the progression of the background.

**TABLE 13** The progression of the stopping signal across the final tagged word. The first line of the right-hand side table reproduce the last line of the left-hand side one for clarity reason.

| 3 . y . y . A . 0 . 1 | . y . y . 3 A . 0 . 1 |
| . 3 y . y . A . 0 . 1 | . y . y . V . 0 . 1 |
| . U . y . A . 0 . 1 | . y . y . A 3 . 0 . 1 |
| . V B . y . A . 0 . 1 | . y . y . A . 3 . 1 |
| . y 3 . y . A . 0 . 1 | . y . y . A . 0 . 3 . 1 |
| . y . 3 y . A . 0 . 1 | . y . y . A . 0 . 3 . 1 |
| . y . U . A . 0 . 1 | . y . y . A . 0 . 3 . 1 |
| . y . V B . A . 0 . 1 | . y . y . A . 0 . 4 . 1 |
| . y . y 3 . A . 0 . 1 | . y . y . A . 0 . 1 . 3 |
| . y . y . y 3 . A . 0 . 1 | . y . y . A . 0 . 1 . 3 |

Let us now call $\mathcal{L}N_{11}$ the cellular automaton with 11 states whose rules are given in Table 12. As Table 7 the table is organized in two-dimensional sub-tables, according to the same pattern.

It can be noticed that there are a lot of empty entries. This corresponds to the fact that the table of $\mathcal{L}N_{11}$ is constructed for specific configurations attached to our purpose. The configurations we consider are assumed to encode the current configuration of a Turing machine so that not any triple of the states occurs during these configurations. This particularity will be useful for the adaptation of $\mathcal{L}N_{11}$ to a strongly universal cellular automaton in the hyperbolic tilings we shall consider in Section 4.

**TABLE 14** When the stopping signal arrives to the end of the zone of U’s.

| 3 U U U U | . . . . . |
| . 3 U U U B | . . . . . |
| . . 3 U U U | . . . . . |
| . . . 3 U B | . . . . . |
| . . . . 3 U | . . . . . |
| . . . . . 4 U | . . . . . |
| . . . . . . 4 B | . . . . . |

**TABLE 15** Execution of the rules of Table 12 for collisions with the zone of U’s.

| y T . U U U | . . . 0 T . U U U | . . . 1 T . U U U | . . . A T . U U U | . . . B I y U U U | . . . 1 y y U U U | . . . A 0 y U U U | . . . B A y U U U | . . . A A y 4 U U B | . . . T 1 y 4 U U B | . . . 1 y A 4 U U B | . . . y A A 4 U U B | . . . 0 0 A . U U U | . . . T 1 . y . U U | . . . T 1 0 . U U | . . . O 1 T . U U | . . . y y T . U U B | . . . T 1 B B . U U B | . . . y . 1 T . U U B | . . . y . y T . U U U | . . . T 1 . A . U U U | . . . y . 1 B U U U | . . . y . y B U U B | . . . T 1 . U U U | . . . T 1 . A . U U U | . . . y . 1 B U U U | . . . y . y B U U B | . . . T 1 . U U U | . . . T 1 . A . U U U | . . . y . 1 B U U U | . . . y . y B U U B |

Accordingly, $\mathcal{L}N_{11}$ allows us to prove:

**THEOREM 1** There is a deterministic cellular automaton on the line, with the standard neighbourhood and 11 states which is strongly universal.

We have an interesting corollary:

**COROLLARY 1** There is a deterministic cellular automaton on the line, with the standard neighbourhood and 9 states whose halting problem when starting from a finite configuration is undecidable.
Proof. If we look only at the halting problem, when the simulated machine halts, it is no more necessary to keep the final result of the tag-system. This allows us to spare the additional states $3$ and $V$. However, we need a stopping signal. This can be performed by state $4$. We replace the rule $0T_y \rightarrow 3$ by $0T_y \rightarrow 4$. Later, we can decide that state $4$ destroys any symbol. The rules involving $3$ and $V$ are cancelled. We have the following new rules:

$$\begin{align*}
_4\alpha \rightarrow & \quad 4\alpha \rightarrow 4 & _4\_ \rightarrow & \quad 4\_ \rightarrow 4 & _4U \rightarrow & \quad 4B \rightarrow & \quad 4B \_ \rightarrow \_ ,
\end{align*}$$

where $\alpha$ stands for $0, 1, y,$ or $A$, as $B$ and $T$ disappeared and as far as the simulated machine halted. We keep the rules $\_4U \rightarrow \_ $ and $4UU \rightarrow \_ $.

\[\square\]

4 Strongly universal cellular automata in hyperbolic spaces

We shall start from the automaton $L^N$ to which we appended the states $U$ and $4$ with the rules producing Tables 11 and 15. This automaton, call it $L^N_9$ has 9 states but it does not halt when the simulated machine halts.

The idea is to implement $L^N_9$ on a line which is continued in such a way that, taking advantage of the hyperbolic structure, we can simplify the process which halts the computation. We shall take the approach given in [7]. There, the construction of a line supporting the computation of $1D$-cellular automaton is performed starting from a finite initial segment of the cells which contains the implementation of the finite initial configuration of the Turing machine. Such a construction can be viewed as an independent cellular automaton $P$. The simple juxtaposition of both automata, $P$ and the implementation of $L^N_9$, leads to a new cellular automaton. With some minor tuning, this will allow us to prove the results claimed in the introduction, see Theorem 3.

In Sub-section 4.1, a minimal presentation of the tilings in which we implement cellular automata is given. In Sub-section 4.2, we remind the reader the main lines of the construction of $P$. In Subsection 4.3, the combination of $P$ with $L^N_9$ is performed, proving the results stated by Theorem 2 and 3.

4.1 Tilings of the hyperbolic plane and cellular automata

It is not possible to remind here all the properties from hyperbolic geometry needed for the paper. I very sketchily remind here what is needed to understand the arguments developed in this section. The reader is referred to [5, 8] for a more informative introduction to the field, especially suited for the paper.

In this paper, we again use Poincaré’s disc model of the hyperbolic plane as well as its generalization to the 3D ball as a model of the hyperbolic 3D-space.

Poincaré’s disc model is illustrated by Figure 1. The points of the hyperbolic plane are the points of a fixed open disc $D$ whose border $\partial D$ is drawn in the figure. The points of $\partial D$ are called the points at infinity and they do not belong to the hyperbolic plane. The lines are represented by the traces in $D$ of diameters or of circles which are orthogonal to $\partial D$. In the figure, a line $\ell$ is represented together with a point $A$ out of $\ell$. Three kinds of lines are represented in the figure. The first kind of lines is represented by $s$: they all cut $\ell$ and are thus called secant with $\ell$. Two lines constitute the second kind: they share with $\ell$ a point at infinity. They are illustrated by $p$ and $q$ in the figure. The lines $p$ and $q$ pass through $A$ and meet $\ell$ exactly at the points at infinity $P$ and $Q$ respectively. Accordingly, they do not meet $\ell$ in the hyperbolic plane and are therefore called parallel to $\ell$. And so, in the model, through each point out of a line, we can draw two distinct parallels to the line. Now we have a third kind of lines which has no counterpart in Euclidean geometry. It consists of the lines passing through $A$ which do not meet $\ell$ at all:
neither in the hyperbolic plane, nor at infinity, nor outside the closure of $D$ in the model. These lines are illustrated in the figure by $m$ and are called **non-secant** with $\ell$. Non-secant lines are characterized by the fact that they have a unique common perpendicular.

![Figure 1](image_url)

**Figure 1** Poincaré’s disc. The three kinds of lines passing through a point out of a given line.

A tessellation is defined by the following process. We start from a convex regular polygon $P$, the **basis** and we replicate $P$ by reflection in its sides and, recursively, the images in their sides. We call $P$ and these images of $P$ **copies** of $P$. We say that this process defines a **tessellation** if and only if the copies do not overlap and they cover all points of the plane. In the Euclidean plane, up to similarity, there are three tessellations exactly. The bases are the square, the regular hexagon and the equilateral triangle.

The important news is that in the hyperbolic plane, there are infinitely many tessellations. A basis $P$ of such a tessellation is defined by two numbers: the number $p$ of sides of $P$ and the number $q$ of copies of $P$ which can be put around a point $A$ in order to cover a neighbourhood of $A$ with no overlapping. The tessellation itself is denoted by $\{p,q\}$. A fundamental theorem proved by Poincaré says that $P$ generates a tessellation of the hyperbolic plane if and only if

$$\frac{1}{p} + \frac{1}{q} < \frac{1}{2}. \quad (\star)$$

Note that we recover the Euclidean tessellations by replacing $< \text{ by } = \text{ in } (\star)$.

In the hyperbolic plane, the smallest values for $q$ are 3 and 4 and the smallest corresponding values of $p$ such that $(\star)$ holds are 7 and 5 respectively. The corresponding tilings are called **heptagrid** and **pentagrid** respectively, see Figure 2.

As mentioned in the introduction of the sub-section, Poincaré’s disc model generalizes to the hyperbolic 3D space and, in fact, to any higher dimension. Planes are the trace in the unit open ball $B$ of diametral planes or spheres which are orthogonal to the border $\partial B$ of $B$. Lines are intersections of secant planes. There are four tessellations in the hyperbolic 3D space and we shall consider one of them, only, the **dodecaagrid**. It is based on what is called Poincaré’s dodecahedron which is constructed upon the convex rectangular regular pentagon in the same way as a Euclidean dodecahedron is constructed upon a convex regular pentagon. It turns out that this dodecahedron tiles the hyperbolic 3D space.
Figure 2 Hyperbolic tessellations. To left: the pentagrid; to right: the heptagrid.

Figure 3 illustrates how we represent dodecahedra and the dodecagrid in this setting. For our purpose, we need to fix two perpendicular planes $H$ and $V$ which support faces of dodecahedra of the tiling.

In all these figures, we use the projection of a dodecahedron on the plane of its face 0. It is a central projection from a point which we can imagine on the vertical line which passes through the centre of the face, in the direction of face 11 which is opposite to face 0. This projection was first devised by Schlegel, a geometer of the 19th century: accordingly, the projection is called a Schlegel diagram. The leftmost picture of the figure indicates how the faces of a dodecahedron are numbered. The other two pictures illustrate the representation of the part of the dodecagrid which is used by the construction of the cellular automaton. The dodecahedra are aligned along the line which is the intersection of $H$ and $V$. The middle picture represents the dodecahedra which are above $H$, the right-hand side picture represents those which are below $H$. In both cases, each dodecahedron is projected on its face 0 which, by definition is the face in contact with $H$. 
In this representation, the faces which are along an edge belonging to two neighbouring faces 0 are represented twice as the face they represent belongs to both dodecahedra which share it. This property is used to indicate the state of a cell in the pictures of this section. In the pictures, states are represented by colours. As a face perpendicular to \( \mathcal{H} \) appears twice, it takes the colour of the cell to which it does not belong. This is why the state of a cell \( c \) appears in the picture as a ring of faces of the same colour which surround the face 0 of \( c \).

The correspondence between the colours and the states will be given later.

Now, it is clear that on a tessellation we can construct cellular automata. In our constructions, we shall deal with rotation invariant cellular automata. This means that if we perform a rotation around a cell which leaves the tiling globally invariant, this does not change the state of the cell. In the case of the plane, such a rotation is simply a circular permutation on the numbers of the neighbours of the cell. Checking that a set of rules is rotation invariant amounts to check that any circular permutation on the neighbours does not change the new state: the corresponding new rule also belongs to the set. In the hyperbolic 3D space, the characterization is more complex as the group of the rotations leaving the tiling globally invariant is in fact the group \( A_5 \). However, checking the rotation invariance of a set of rules can be performed by a computer program according to a rather simple algorithm, see for instance [8].

For implementation purposes, it is important to locate the cells as more conveniently as possible. We refer the reader to [5, 6, 8] where such a convenient system is proposed, based on a tree structure. It is there illustrated by many applications, in particular in [8] by the construction of many weakly universal cellular automata. However, for the presentation of this work, we do not need to introduce this localisation system as the concerned cells are mainly placed along a line.

4.2 Continuing a segment

In this sub-section, we remind the construction of [7] for continuing a segment along a given line \( \delta \) we call the guideline.

The constraint of the construction lies in the fact that we wish to construct a rotation invariant cellular automaton. From this constraint, the cells do not know \textit{a priori} their orientation, so that we have to provide them such information. In all the tilings we consider, the cells are constructed along a line which does not cross the cells but which defines their common border. Sub-subsections 4.2.1 4.2.2 and 4.2.3 deal with the implementation in the pentagrid, the heptagrid and the dodecagrid respectively.

4.2.1 In the pentagrid

The guideline \( \delta \) is defined by the common side shared by a yellow cell and a red one and it is also shared by the two green cells when they appear. The guideline defines two half-planes \( \pi_1 \) and \( \pi_2 \) whose intersection is the guideline itself. The key of the process is that the guideline is continued by the construction of two sequences of cells in parallel. One sequence lies in \( \pi_1 \) while the other lies in \( \pi_2 \).

This is illustrated by figures 4. In this figure, we assume that the initial configuration consists of six cells: see the leftmost picture in the first row of pictures in Figure 4.

The process is the following. We have five states denoted by \( \overline{\text{w}}, \overline{\text{y}}, \overline{\text{r}}, \overline{\text{py}} \) and \( \overline{\text{g}} \), which we call \textbf{blank}, \textbf{yellow}, \textbf{red}, \textbf{pale yellow} and \textbf{green} respectively. The blank is the quiescent state of our automaton. In the figure, the blank is represented by a blue colour and we use several hues of blue in order to remember the tree structure of the tiling.

The rules are indicated in Table 16 after the following format: \( w_0 w_1 w_2 w_3 w_4 w_5 w_6 \) where \( w_0 \) is the current state of the cell, \( w_i \) is the state of its neighbour \( i, i \in \{1..5\} \) and \( w_6 \) is the new state of the cell.
after the application of the rule. The neighbours are increasingly numbered by counter-clockwise turning around the cell. As the rules are rotation invariant, it is not important which is neighbour 1.

![Figure 4 The continuation process in the pentagrid.](image)

We can formulate the rules as follows. The initial configuration is given by the first picture. It consists of three red cells, a yellow one and two green ones in the configuration indicated by the picture. A blank cell with a single green neighbour becomes pale yellow. A green cell with a red, yellow neighbour becomes red, yellow respectively. A pale yellow cell with a single non blank neighbour becomes blank. A pale yellow cell with three blank neighbours and a pale yellow one becomes green. Yellow and red cells keep their colour.

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | W | W | W | W | W | W | W |
| 0 | W | R | W | W | W | W | W |
| 0 | W | Y | W | W | W | W | W |
| 0 | Y | Y | W | R | W | W | R |
| 0 | Y | R | R | W | W | W | R |
| 0 | R | R | R | W | W | W | R |
| 0 | R | R | Y | G | W | W | R |
| 0 | Y | Y | R | Y | W | W | Y |

|   | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| 0 | Y | Y | R | R | W | W | Y |
| 0 | Y | R | Y | pY | pY | pY | Y |
| 0 | Y | Y | Y | pY | pY | pY | R |
| 0 | Y | R | R | W | W | W | R |
| 0 | Y | Y | Y | W | W | W | W |

### Table 16 Rules for the continuation of the line in the pentagrid. The rules are rotationally independent.

In the heptagrid

Due to the angle $\frac{2\pi}{3}$, the guideline is not defined by the side of a tile. It is defined by the line which joins the midpoints of consecutive sides belonging alternately to the same and the consecutive heptagons, see [5, 8]. Consequently it crosses the tiles of both sequences whose growth consists in the continuation of the process. We have again five states and we use the same colours as in the case of the pentagrid to represent the states.

This is illustrated by Figure[5]. In this figure, we assume that the initial configuration consists of five cells: see the leftmost picture of the first row in Figure[5]. Although each rule is longer than those for the pentagrid, we shall see that the rules for the heptagrid are simpler than those of Sub-subsection 4.2.1.
The format of the rules is alike that for the pentagrid: here, each cell has simply seven neighbours instead of five of them. Hence the new state taken by the cell after the rule is applied is \( w_8 \) in Table 17.

However the rules are a bit simpler as can be seen on the following informal presentation:

A white cell with a red and a green neighbour around the same vertex, the red cell first while counter-clockwise turning around the cell, becomes pale yellow. The green cell remains green if it has no pale yellow neighbour. If it has a pale yellow neighbour, it becomes yellow. A pale yellow cell becomes red at the next time. A blank cell with a pale yellow and a green neighbour, in this order, sharing a common vertex becomes green. Yellow and red cells keep their colour.

**TABLE 17** Rules for the continuation of the line in the heptagrid. The rules are rotationally independent.

| -- | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| -- | 0 | W | W | w | w | w | w | w | w |
| 0 | 0 | W | w | W | W | w | w | W | W |
| 0 | 0 | W | R | w | W | w | w | W | W |
| 0 | 0 | W | G | W | W | w | w | W | W |
| 0 | 0 | W | Y | R | W | W | W | W | W |
| 0 | 0 | W | R | R | W | W | W | W | W |
| 0 | 0 | W | G | Y | W | W | W | W | W |
| 0 | 0 | R | Y | R | W | W | W | W | R |
| 0 | 0 | R | R | Y | R | W | W | W | R |
| 0 | 0 | Y | Y | R | Y | R | W | W | Y |
| 0 | 0 | G | R | R | R | W | W | W | G |
| 0 | 0 | G | Y | W | w | w | W | G |
| 1 | 0 | W | R | G | W | W | W | W | pY |

**FIGURE 5** The continuation process in the heptagrid.

4.2.3 **In the dodecagrid**

Here, the guideline is again a line which does not cross the tiles: it is the intersection of the planes \( \mathcal{H} \) and \( \mathcal{V} \).

Basically, the process looks like that of the pentagrid. However, we have a small adaptation for the 3D situation. The guideline is shared by four tiles while, in the case of the pentagrid it is shared by two of them only. This means that here, the continuation requires the construction in parallel of four sequences of tiles. The planes \( \mathcal{H} \) and \( \mathcal{V} \) define four regions \( \eta_1, \eta_2, \eta_3 \) and \( \eta_4 \) of the hyperbolic 3D space. Each
of them is the half of a half-space. Each sequence of tiles lies in its own region $\eta_i$, different from the regions of the others. Moreover, we distinguish a upper part and a lower with respect to $\mathcal{H}$: two regions are above $\mathcal{H}$ while the two others are below it.

We use again the colours defined for the pentagrid. One sequence of the upper ones will be the yellow one, the other will be mauve, a new colour, denote by $m$. Both sequences below $\mathcal{H}$ will be red. Accordingly, each sequence knows whether it is above or below $\mathcal{H}$. But we shall see that this may sometimes be disturbed.

![Figure 6](image)

FIGURE 6 The continuation process in the dodecagrid. In each pair, the left-, right hand side picture is the projection of the cells above, below $\mathcal{H}$ respectively.

Figure 6 illustrates the process, while Table 18 gives the rules which control the process.

Initially, there are twelve cells: six of them above $\mathcal{H}$ and six of them below $\mathcal{H}$. The yellow cell, three mauve ones and two green ones which are above $\mathcal{H}$ reproduce the initial configuration for the pentagrid, see Figure 4. Below $\mathcal{H}$, the four red cells are below the mauve and yellow ones. The two green cells are below those which are above $\mathcal{H}$.

The format of the rules is a bit different from what we have seen in the planar tilings. Here, a rule is written as $w_0w_1w_2w_3w_4w_5w_6w_7w_8w_9w_{10}w_{11}w_n$, where $w_0$ is the current state of the cell, $w_i$ with $i$ in $\{0..11\}$ the state of the neighbour $i$, i.e. the one which seen through the face $i$ of the cell. The new state of the cell after the rule is applied is $w_n$.

The rules can be formulated like this. A blank cell which sees a green one, state $dG$, becomes pink, state $dY$. A pink cell becomes blank unless it has two pink neighbours through faces sharing an edge: one pink neighbour is above $\mathcal{H}$ and the other is below. In that case, the cell becomes green. At last, the green cell takes the colour of its neighbour on the same line: in particular, if the cell is on the yellow line, it takes the blank of $L \mathcal{N}_0$. Mauve, yellow and red cells keep their colour. Figure 18 illustrates this
process giving, at each time two projections onto $H$: on the left-hand side, the cells above the plane, on the right-hand side, the cells below it.

**TABLE 18** Rules for the propagation of the 1D-structure in the dodecagrid.
The rules are rotationally independent.

| old | new |
|-----|-----|
| 0   | w w w w w w w w w w w w |
| 0   | w w w w w w w w w w w w |
| 0   | w w w w w w w w w w w w |
| 0   | M M Y Y w w w w w w w w |
| 0   | M M M M w w w w w w w w |
| 0   | Y R M M G w w w w w w |
| 0   | R M R R W w w w w w w |
| 0   | R M R R G w w w w w w |
| 0   | R Y G R w w w w w w |
| 0   | W G w w w w w w w w w w |
| 0   | W G Y Y w w w w w w |
| 0   | G G G G w w w w w w |
| 0   | G G G G w w w w w w |
| 0   | M R Y M Y w w w w w w |
| 0   | Y R Y M Y w w w w w w |
| 0   | Y R Y M Y w w w w w w |
| 0   | R M R R W w w w w w w |
| 0   | R Y R R W w w w w w w |
| 0   | W pY w w w w w w w w w w |
| 0   | pY pY pY pY pY pY pY pY pY pY |
| 0   | Y R M M G w w w w w w |
| 0   | G G G G w w w w w w |
| 0   | G G G G w w w w w w |
| 0   | M R Y M Y w w w w w w |
| 0   | Y R Y M Y w w w w w w |
| 0   | Y R Y M Y w w w w w w |
| 0   | R M R R W w w w w w w |
| 0   | R Y R R W w w w w w w |
| 0   | W pY w w w w w w w w w w |
| 0   | pY pY pY pY pY pY pY pY pY pY |
| 0   | Y R M M G w w w w w w |
| 0   | G G G G w w w w w w |
| 0   | G G G G w w w w w w |
| 0   | M R Y M Y w w w w w w |
| 0   | Y R Y M Y w w w w w w |
| 0   | Y R Y M Y w w w w w w |
| 0   | R M R R W w w w w w w |
| 0   | R Y R R W w w w w w w |
| 0   | W pY w w w w w w w w w w |
| 0   | pY pY pY pY pY pY pY pY pY pY |
| 0   | Y R M M G w w w w w w |
| 0   | G G G G w w w w w w |
| 0   | G G G G w w w w w w |
| 0   | M R Y M Y w w w w w w |
| In the next subsection, the automaton defined by these rules will be denoted by $\mathcal{P}$. Note that $\mathcal{P}$ has five states in the case of the planar tilings and it has six states in the case of the dodecagrid.

### 4.3 The results and their proofs

As already mentioned, we take into consideration $\mathcal{L}\mathcal{N}_9$ only. We are now with a cellular automaton with 9 states which is actually universal up to the halting of its own computation. The automaton has 5 states or 6 states depending on whether we consider the planar tilings or the dodecagrid respectively. This means that the simple superposition of $\mathcal{L}\mathcal{N}_9$ with $\mathcal{P}$ would give us 14 or 15 states respectively.

Following the idea given in [7], we can improve such a result by identifying states of $\mathcal{P}$ with those of $\mathcal{L}\mathcal{N}_9$. In [7], the identifications where performed according to the scheme $(S_1)$. In this scheme, the first line indicates the states of $\mathcal{L}\mathcal{N}_9$, the second and the third ones those of $\mathcal{P}$ for the planar tilings and
the dodecagrid respectively. The identified states are placed one upon another. Of course, \( Y \) is identified with any state of \( \mathcal{L} \mathcal{N} \). This means that, a priori, several states have a double interpretation. As an example, \( A \) is a state of \( \mathcal{L} \mathcal{N} \) while, by the identification, it is also the pink state in \( \mathcal{P} \). The distinction is obtained by the neighbouring of the considered cell. To take this example, a pink cell has a lot of \( W \)-neighbours, more than a yellow cell. This entails that a cell containing \( A \) knows which rules it has to apply: those of \( \mathcal{L} \mathcal{N} \) or those of \( \mathcal{P} \).

\[
\begin{array}{cccccccc}
0 & 1 & y & A & B & T & U & 4 \\
y & Y & Y & Y & pY & Y & R & W & G \\
y & Y & Y & Y & pY & M & R & W & G \\
\end{array}
\]  

\((S_1)\)

The situation may be more intricate with other states, but it is not very difficult, we refer the reader to \([7]\) for corresponding explanations. There is just a difference with \([7]\) where there was no process of continuation in the 1D-cellular automaton. Here we have two continuation processes. The first one concerns the construction of the hyperbolic structure. We can notice that the speed of the growth of this structure is \( \frac{1}{2} \), as a new cell is appended each second time. Now, in the yellow cells, we also have the continuation process of \( \mathcal{L} \mathcal{N} \) which also advances at the speed \( \frac{1}{2} \): the distance between both processes remains constant as long as the stopping signal did not reach the continuation process of \( \mathcal{L} \mathcal{N} \). But this raises no difficulty and can easily be absorbed by the scenario depicted in \([7]\).

This allows us to state:

**Theorem 2** In each of the following tilings: the pentagrid and the heptagrid of the hyperbolic plane, there is a deterministic, rotation invariant cellular automaton with radius 1 which has 11 states and which is strongly universal.

The goal of this section is to prove the following result:

**Theorem 3** In each of the following tilings: the pentagrid and the heptagrid of the hyperbolic plane and also the dodecagrid of the hyperbolic 3D-space, there is a deterministic, rotation invariant cellular automaton with radius 1 which has 10 states and which is strongly universal.

Of course, Theorem 2 is also a corollary of Theorem 3 and as such, it also holds for the dodecagrid. However, following \([7]\), it can be proved for the pentagrid and the heptagrid, without reference to Theorem 3.

The identification which we shall now prove is given by \((S_2)\).

\[
\begin{array}{cccccccc}
0 & 1 & y & A & B & T & U & 4 \\
W & Y & Y & Y & pY & Y & R & G \\
W & Y & Y & Y & pY & M & R & G \\
\end{array}
\]  

\((S_2)\)

In Sub-subsection 4.3.1 we explain why the identification is possible, and in Sub-subsection 4.3.2 we give the rules of the automaton proving Theorem 3 after the identification.

Before turning to this problem, let us see why we cannot go further than \((S_2)\) in the identifying process.

In \([7]\) and in the construction we perform to prove Theorem 3 the stopping signal is a state of \( \mathcal{P} \) which is captured by the red line, the mauve line in the case of the dodecagrid, and which travels to the right-hand side of the configuration. During this motion, the signal destroys all states which do not belong to the Turing machine, including the constructing pattern of the lines when it reaches the pattern. In \([7]\) and here, the stopping signal is identified to \( G \). Due to its nature, the stopping signal must not be seen on the Turing part of the line before the end of the Turing computation and it also must not be seen
until this end by an ordinary red or mauve cell, otherwise, the end of the computation would occur too soon, ruining the simulation.

As shall soon be seen, \( W \) is identified with the blank of \( L \mathcal{N}_9 \). This identification is not trivial and, except the case of the heptagrid, it requires to modify \( P \) for the pentagrid and for the dodecagrid. Now, it would be necessary to identify \( G \) with a state of \( L \mathcal{N}_9 \) too. And this is not possible: besides \( \top \) and \( 4 \), all states of \( L \mathcal{N}_9 \) occur many times during the computation and are repeated in many places seen by ordinary red or mauve cells. For this reason, as it is seen from the beginning by ordinary red or mauve cells, \( 4 \) cannot be the stopping signal. For what is \( \top \), it would occur in between two blanks of \( L \mathcal{N}_9 \). This means that two heads would be raised, starting new computations which are not at all connected with the simulation. And so, \( G \) remains the single candidate for the stopping signal and from what we have seen, it cannot be identified with a state of \( L \mathcal{N}_9 \). We denote by \( D \) the new automaton resulting from this identification and applying the rules of \( L \mathcal{N}_9 \) to the cells of the yellow line and the rules of \( P \) for the cells of another type.

4.3.1 Proof of that the identification \( (S_2) \) holds

We successively examine the situation in the pentagrid, in the heptagrid and in the dodecagrid. The initial configuration is similar in all cases and it will be precisely described in the case of the pentagrid only.

Pentagrid

In this case, the identification of \( \_ \) with \( W \) raises a contradiction between the rules of \( L \mathcal{N}_9 \) and those of \( P \). During the computation, many Turing symbols remain some time in between two blanks. This is the case for \( \_ \) for instance. Now, such a state must not be changed, as long as it is not involved in a collision with an impulsion. Now, according to the identification, the neighbourhood of such \( \_ \) in the scenario depicted in Sub-subsection 4.2.1 is \( \top \mathcal{T} w w w w \), as it can see only a red cell as a non-blank one. The rule then should be \( \top \mathcal{T} w w w w w \). This is in contradiction with the rule \( \top \mathcal{T} w w w w w w \) telling that a pale yellow cell seeing a single non-blank cell becomes blank.

Accordingly, the process of Sub-subsection 4.2.1 must be tuned.

Necessarily, we have to keep the rule \( \top \mathcal{T} w w w w w \). The solution is to reinforce the way which allows a cell to know for sure that it is on the yellow line. For that purpose, we introduce a new pattern for the yellow cell. We decide that such a cell have a neighbourhood of the form \( \top \mathcal{T} 1 \mathcal{B} \mathcal{B} \mathcal{B} \mathcal{r} \), where \( \mathcal{r}, \mathcal{l} \) is the state of the left-, right-hand side neighbour respectively of the considered yellow cell \( c \) whose state we denote by \( c \).

Figure 7 illustrates the new implementation of the whole computation based on this new setting for the yellow cells. The three rows of the figure illustrate the three steps of the computation.

During the first step, \( D \) performs the computation of the simulated Turing machine and \( P \) constructs the line needed for the computation of \( L \mathcal{N}_9 \). At this point, we can take advantage of \( P \) in order to provide the needed background to the yellow line. For this purpose, we no more need the action between the states \( U \) and \( B \) as depicted in Section 2. It is enough to decide that the yellow cell produced by the transformation of the green cell of the yellow line is the state \( U \). Then, the rules of \( L \mathcal{N}_9 \) with the states \( U \) and \( 4 \) allow to create the needed two blanks between two occurrences of \( 0 \) each time the head of the Turing machine goes out outside the current configuration on its right-hand side end. Accordingly, this will slightly simplify the action of the stopping signal. Also, in the initial configuration, the segment \( [c, d] \) contains the initial configuration of \( L \mathcal{N}_9 \). Giving 1 as coordinate for the first yellow cell at the left-hand side of the configuration, \( c \) can be given the coordinate 5, cell 1 being in \( B \), the cells 2 up to 4 being blank. We decide that \( d \) contains the last Turing symbol. Then, starting from \( d+2 \) until \( d+4 \) the
cells contain $U$ and the cell $d+5$ is green.

The last two rows of Figure 7 are devoted to the propagation of the stopping signal and its action.

The second row of the figure shows us the occurrence of the stopping signal on the yellow line, its immediate transfer to the red line and there, its progression towards the right-hand side end of the configuration. The last row shows us what happens when the stopping signal arrives very close to the end of the configuration and how it stops its progression.

Provided that rules can be defined to perform these various actions, we shall look at this point in Sub-subsection 4.3.2, we can now look at the different neighbourhood of these cells in order to check that the pattern we devised for the yellow cells also avoids any confusion in the computation of $P$. As we identified the states of $P$ with those of $\mathcal{L}N_9$, $G$ being excepted, we shall formulate the new rules for $P$ in terms of states of $\mathcal{L}N_9$ and $G$ when this latter state is involved.

Let us first look at the yellow line.

We decide that the neighbourhood of a yellow cell is of the form $TlBr$, where $l$, $r$ is the state of the left-, right-hand side neighbour of the cell on the yellow line. We have to check that this allows a yellow cell to know that it is such a cell and to correctly identify its left- and right-hand side neighbours on the yellow line.

Indeed, if $l$ or $r$ is neither $B$ nor $T$, denote by $a$ the neighbour which is different from both $B$ and $T$. Then, necessarily, $a$ is in between $B$ and the $T$ which belongs to the red line. This allows to identify correctly whether $a$ is $l$ or $r$. Now, consider the case when both $l$ and $r$ are equal to $B$ or $T$. They cannot be both equal to $T$ as there is a single $T$ on the yellow line. The can be both equal to $B$, in which case $c = 1$. But this situation gives rise to a single rule. And so, we consider the case when $\{l, r\} = \{B, T\}$. Depending on which $T$ is considered as that of the red line we have to consider the neighbourhood $BeT$ or $TeB$. Now, Table 12 indicates that $c$ cannot be in $\_\_$. $B$, $U$, $4$. When $c$ is $T$ or $0$, only the neighbourhood $BeT$ exists on the yellow line which fixes the correct orientation and a single rule can be applied. When $c$ is $y$ or $A$, the only neighbourhood is $TeB$, also giving rise to a single rule. The remaining case is $c = 1$ but the two possible rules give the same state. And so, whatever the choice of the red $T$, the correct new state will be defined.
We remain with the other situations regarding the red and yellow line when the computation of the simulated machine has been completed and what happens at the right-hand side end of the configuration. We have two other states: the detection of the stopping signal with its progression and the halting of the computation, when the stopping signal reached the end of the configuration. Tables 19 and 20 show the rules in a way where these contexts are clearly separated both for the yellow and the for the red cells. A close examination of the tables show that the neighbourhoods involved in these different kinds of cells can never be confused.

And so, we can conclude that the new automaton satisfies the requirement of the statement of Theorem 3 for what is the pentagrid. Accordingly, the theorem is proved in this case.

Heptagrid

In the case of the heptagrid, the same problem as in the case of the pentagrid happens with the state A when the blank of $L\text{N}_9$ is identified with that of $P$. Indeed, the rule $A T G W W W W W T$ may be raised in the yellow line, far from the end. This would be the case for a yellow state under the state A when the stopping signal running over the red line arrives in front of cell in $U$ on the yellow line. For that latter cell, we need the rule $A T G W W W W W A$.

As in the case of the pentagrid, the solution is to reinforce the identification of the yellow cells. We decide that they have a neighbourhood of the form T Y B W B Y T. These two B’s are present in the initial configuration and the construction of the line appends the new ones as indicated in the first two rows of Figure 8. In the situation of a cell in the state A, we now have two different rules: $A T G W W W W W T$ and $A G W B W B W T A$. The new structure of the rules on the yellow line can also be seen in the rules of the cellular automaton for the heptagrid as displayed by Table 21. We can check on the rules that the cells know their position without ambiguity. The pattern BWB allows a more clear identification that the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure8.png}
\caption{The new continuation in the heptagrid. First line: before the stopping signal. Second line: the stopping signal was just triggered. Third line: the stopping signal reaches the pattern which constructs the line.}
\end{figure}
pattern BB in the case of the pentagrid. This can clearly be seen from the neighbourhood $T Y B W B Y T$ which we can rewrite as $T x B W B z T$. It is plain that whatever the values of $x$ and $z$ among the states of $\mathcal{L}N_9$, as B never occurs on the red line, the pattern BWB cannot be placed at another site in the word.

**Dodecagrid**

In the dodecagrid, the neighbourhood of a yellow cell is $T B z W W x W W W W W W$ after the identification $(S_2)$. If $x = B$ and $z = W$, the cell knows that its back is the face in $T$, but it may hesitate on which $U$ is its face 1. If it takes one choice, it considers that it has to apply the instruction raised by $B y W$ and if it takes the other $U$, the instruction is raised by $W y B$. However, Table 12 shows the rules $B0 \rightarrow y$ and $0B \rightarrow 1$ and a lot of other examples which are kept by $L N_9$.

This forces us to define a new pattern which, as previously, reinforce the identification of the yellow cells. But this time, the new scenario will reinforce the identification of each line. The new pattern is illustrated by Figure 9 and it consists in putting a non-blank cell on the faces 6 and 7 of the cells, see Figure 6 in Subsection 4.2.

![Figure 9](image-url)

**Figure 9** Note that now, faces 6 and 7 are non-blank.

As far as faces 6 and 7 are no more blank, there is a single blank face, face 9, which is surrounded by blank faces of the dodecahedron. Face 1 is opposite to face 9, so that face 1 is clearly identified. Now, we decide that in the case of yellow cells, faces 6 and 7 are in $B$ which is identified with the mauve colour. Consider again a cell of the yellow line. There is at least one face in $T$ and two contiguous faces in $B$, not in contact with the face in $T$ and these three faces are around face 1 which is immediately identified. Let us check that the cell can now identify which face is face 0. If there is no neighbour of the cell in $T$ on the yellow line, then the cell has a single neighbour in $T$ and it is seen through face 0 necessarily. Assume that there is another cell in $T$ and there is necessarily at most a single one. There cannot be another one as there is at most one cell in $T$ on the yellow line. Necessarily, this $T$-face is in contact with face 0 and with the faces in $B$ around face 1. There could be a confusion if the fifth face around face 1 would also be in $B$. But the confusion cannot happen. Looking at the patterns $BxT$ and $TxB$ in the computation of $\mathcal{L}N_9$, see Table 12 there is no such patterns when $x \in \{ \_B, 4, U \}$, one of them exist but not the others when $x \in \{0, y, A, T \}$. The single case when both patterns exist is when $x = 1$ and, in both cases, the corresponding rule produces the same state. And so, in all cases, but one only, face 0 is clearly identified and in all cases, the applied rule always yields the right result.

Let us decide that faces 6 and 7 are red on the mauve line and on the red lines too. This allows the cells of these lines to easily know to which line they belong too and which is their face 0. Remember that a cell of the yellow line has at most two $T$-neighbours after the identification of $(S_2)$: the face which looks at the red line below $\mathcal{H}$ and possibly, either face 2 or face 5 which are shared by its neighbours on the yellow line. A cell of the mauve line has three or four $T$-neighbours exactly: face 0, face 6 and face 7
and, possibly, its face 1. As its faces 2 and 5 and the cell itself are mauve or at most one of them is green, this allows the cell to recognize that it is mauve. A cell of the red line has at least five T-neighbours: its faces 1, 2, 5, 6 and 7. It is not important for a cell of this line to identify its face 0: the cell is always red.

Note that this argument has to be tuned for the cells which are at the ends of the configuration. At the fixed end, the face 2 or 5, depending on the side with respect to \( V \), are \( \text{w} \). As these cells belong to the initial configuration, faces 6 and 7 are fitted with the appropriate colour. This does not change the distinction we have noted, so that each cell knows to which kind it belongs. Consider the other end where we have the constructing pattern. Initially, the four cells are green. They have nine \( \text{w} \)-neighbours, two green neighbours and the last one has the colour of the line to which they belong. Accordingly, at the next step, they take the appropriate colour. Note that as noted previously, the yellow cell has the state \( G \). The nine \( \text{w} \)-cells turn to \( A \). Now, as can be checked on Figures 10 and 12 as well as in Table 22, faces 6 and 7 take the appropriate colour: they have the colour of the cell they can see through their face 0 and through face 1 they can see another \( A \)-cell. The cell need not know which is its face 0. What is important is the number of cells and the other non-blank colours. If there are two of them, if the yellow colour is among them, remember that here the yellow colour is \( G \), the cell turns to mauve, otherwise it turns to red.

Accordingly, the process described in Subsection 4.2 is correctly implemented.

This allows us to transport the construction indicated by Figure 7 in the case of the pentagrid. This can be seen on Figures 10, 11 and 12. Each figure consists of sets of two pictures as mentioned in Figure 3: one picture is the projection onto \( H \) from above of the cells in contact with this plane and which lie upon it. The other picture, somehow smaller, is the projection onto \( H \) again, but from below, of the cells in contact with \( H \) and which are below the plane. In the pictures of Figures 10 and 11 showing the projection above \( H \), we can see that the computation is exactly the same as in Figure 7.

Table 22 also shows that the process illustrated by Figure 12 is correctly implemented. In particular, we note the rules which make the last cells of the yellow and mauve lines disappear, involving the successive vanishings of the green cells: when no more green cell is generated, the process stops as the \( A \)-cells with no support disappear.

**Figure 10** The simultaneous construction of the lines.
Figure 11  Propagation of the stopping signal along the mauve line.

Figure 12  The stopping signal reaches the end and halts the computation.
4.3.2 Completing the proof of Theorem 3: the rules

In this subsection, we summarize the previous analyzes by giving the rules of the cellular automaton which proves Theorem 3. We have an instance of the automaton for each tiling we consider. We shall successively consider the case of the pentagrid, the heptagrid and then the dodecagrid. In each case, we consider that the format of a rule is \( w_0 w_1 w_2 w_3 w_4 w_5 w_6 \) in the pentagrid, \( w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 \) in the heptagrid and \( w_c w_0 w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8 w_9 w_{10} w_{11} w_n \) for the dodecagrid where, in that latter case, \( w_c \) is the current state of the cell and \( w_n \) its new state after the application of the rule. In each case, the context of the rule \( \rho \) written as a word, is the subword \( \kappa \) such that \( \rho = \kappa w_n \) where \( w_n \) is the new state of the cell.

**Pentagrid**

The rules for the cellular automaton in this case are listed in Tables 19 and 20. The first table gives the rule for the cells belonging to the yellow line and their neighbours which do not belong to the red one. The rules are adapted to the description of the initial configuration given in Sub-subsection 4.3.1.

Each table is divided into three parts: the continuation of the line, the occurrence of the stopping signal and its progression on the red line to the end of the configuration and, when the end is reached, the halting of the cellular automaton.

**Table 19** New rules for the pentagrid: continuation of the line, transfer of the stopping signal and completion of the computation for the yellow cells and their non-red neighbours. In the table, \( Y \) replace any state of \( L.N \). Different occurrences of \( Y \) in the same rule may represent different states. Also, \( x \) represents the symbols of the Turing machine: 0, 1, \( y \) or \( A \). The rules are rotationally independent.
TABLE 20  New rules for the pentagrid: continuation of the line, transfer of the stopping signal and completion of the computation for the red cells and their non-yellow neighbours. The rules are rotationally independent.

|            | -- red line and its non-yellow neighbours | -- stopping signal: source and motion | -- halting of the computation |
|------------|------------------------------------------|---------------------------------------|-------------------------------|
| -- the line| T T G T W W G                             | T G U G W W G                         | T G U G W W G                 |
| -- 0 1 2 3 4 5 6 | T T W W G W W T                         | T G U A A A W                         | G G W W W W W                 |
| -- T B T W W W T | T T Y G A A A W | W W U A A A W | A A W W W W W |
| -- T T Y T W W T | G T Y T W W T | A A W W W W | A A W W W W |

The rules were established from a careful analysis of Figure 7, going cell after cell along the lines, first the yellow one and then the red one. It is not difficult to check that there is no contradiction in the rules and that they completely describe the working of the cellular automaton. This completes the proof of Theorem 3 for the pentagrid. They are also rotation invariant.

Heptagrid

The rules for the cellular automaton in the heptagrid are listed in Table 21. They are also established from Figure 8 exactly in the same way as the rules for the pentagrid were deduced from Figure 7. As can be seen from Figure 8 there are much less rules than in the case of the heptagrid. This is why we have a single table. The rules for the cells of the yellow line need two columns while the last one is devoted to the cells of the red line. Here too, we can check that the rules are coherent, rotation invariant and they completely describe the motion of the cellular automaton.
New rules for the heptagrid: continuation of the line, transfer of the stopping signal and completion of the computation. First two columns: the rules for the yellow cells and their non-red neighbours. Third column: the rules for the red cells and their non-yellow neighbours. In the table, $Y$ replace any state of $\mathcal{L}_9$. Different occurrences of $Y$ in the same rule may represent different states. Also, $x$ represents the symbols of the Turing machine: 0, 1, $y$ or $A$. The rules are rotationally independent.

| -- | -- | -- |
| --- | --- | --- |
| yellow line and its non-red neighbours | stopping signal: source and motion | red line and its non-yellow neighbours |
| the computation | 0 1 2 3 4 5 6 7 8 | the computation |
| and continuing | -- | -- |
| the line | T T O B W B y T G | the line |
| 0 1 2 3 4 5 6 7 8 | y T T B W B W T W | 0 1 2 3 4 5 6 7 8 |
| -- | -- | -- |
| W W W W W W W W | T T T B W B B W G Y | T W B W B T W |
| W T W W W W W W W | Y T Y B W B Y G Y | T B T W W W W W T |
| W W W W W W W W | Y G Y B W B Y T Y | T T B T W W W W T |
| W Y W W W W W W W | U T Y B W B U G W | T T U G W W W W T |
| B B W T T T W W B | W T Y B W B U G W | -- |
| W T B B W B W T W | -- | stopping signal: source and motion |
| Y T Y B W B Y T Y | -- | halting of the computation |
| B Y Y W W W W W B | -- | computation |
| B G U W W W W W B | -- | -- |
| W G B W W W W W W | U G U B W B G T W | T T W G T W W W G |
| W T G W W W W W A | W T W B W B U W W | T T Y Y G W W W T |
| W A G W W W W W G | W T W B W B U W W | T G Y Y T W W W G |
| W G B W W W W W W | U T W W B W W G W | G T Y Y T W W W T |
| W B Y B W W W W B | B U W W W W W B B | -- |
| G T U W W W W W G | B G W W W W W W B | -- |
| G A T U B W W W U | B W W W W W W B | -- |
| W G U W W W W W B | -- | computation |
| W T U B W B B W W | T T U B W B W W W | T G U G A W W W W |
| A G W W W W W W W | -- | -- |
| G A W W W W W W W | -- | -- |

This completes the proof of Theorem 3 in the case of the heptagrid.

Dodecagrid

Table 22 gives the rules for the computation of the cellular automaton in the case of the dodecagrid. The rules were established from an analysis of Figures 10, 11 and 12.

We remark that the scenario is very close to what was performed in the pentagrid. As in the previous tables, the states are those of $\mathcal{L}_9$ to which we append the state $G$. As noted in the study of the figures, most of the cells have a structure in which six faces are blank forming a ring around a face in a single way. This remarkable feature allows us to establish the rotation invariance of the cellular automaton as well as to check the coherence of the rules. Also, the rules completely describe the working of the automaton.

This completes the proof of Theorem 3 in the case of the dodecagrid.

5 Conclusion

Theorems 2 and 3 significantly improve the result of [7, 8]. As far as known to the author, Theorem 1 is the best result on a small strongly universal cellular automaton on the line.
Table 22. Rules for the propagation of the 1D-structure in the dodecagrid and for the stopping of the computation of $P$ once $L_N$ detected the halting of the simulated computation. Remember that the column $c$ represents the current state of a cell of the automaton and that the column $n$ represents its new state after the application of the rule. The rules are rotationally independent.

| $c$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | $n$ |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|
| $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $A$ |
| $B$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $W$ | $B$ | $B$ | $A$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ | $Y$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ | $B$ |
| $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ | $G$ |
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
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