Separate universe approach to evaluate nonlinear matter power spectrum for non-flat ΛCDM model

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The spatial curvature (Ω_K) of the Universe is one of the most fundamental quantities that could give a link to the early universe physics. In this paper we develop an approximate method to compute the nonlinear matter power spectrum, P(k), for “non-flat” ΛCDM models using the separate universe (SU) ansatz which states that the effect of the curvature on structure formation is equivalent to that of long-wavelength density fluctuation (δ_b) in a local volume in the “flat” ΛCDM model, via the specific mapping between the background cosmological parameters and redshifts in the non-flat and flat models. By utilizing the fact that the normalized response of P(k) to δ_b (equivalently Ω_K), which describes how the non-zero Ω_K alters P(k) as a function of k, is well approximated by the response to the Hubble parameter h within the flat model, our method allows one to generalize the prediction of P(k) for flat cosmologies via fitting formulae or emulators to that for non-flat cosmologies. We use N-body simulations for the non-flat ΛCDM models with |Ω_K| ≤ 0.1 to show that our method can predict P(k) for non-flat models up to k ≃ 6 hMpc^{-1} in the redshift range z ≃ 0,1,5, to the fractional accuracy within ~ 1% that roughly corresponds to requirements for weak lensing cosmology with upcoming surveys. We find that the emulators, those built for flat cosmologies such as EuclidEmulator, can predict the non-flat P(k) with least degradation.

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I. INTRODUCTION

The spatial curvature of the Universe is one of the most fundamental quantities that could give a link to the early universe physics. In this paper we develop an approximate method to compute the nonlinear matter power spectrum, P(k), for “non-flat” ΛCDM models using the separate universe (SU) ansatz which states that the effect of the curvature on structure formation is equivalent to that of long-wavelength density fluctuation (δ_b) in a local volume in the “flat” ΛCDM model, via the specific mapping between the background cosmological parameters and redshifts in the non-flat and flat models. By utilizing the fact that the normalized response of P(k) to δ_b (equivalently Ω_K), which describes how the non-zero Ω_K alters P(k) as a function of k, is well approximated by the response to the Hubble parameter h within the flat model, our method allows one to generalize the prediction of P(k) for flat cosmologies via fitting formulae or emulators to that for non-flat cosmologies. We use N-body simulations for the non-flat ΛCDM models with |Ω_K| ≤ 0.1 to show that our method can predict P(k) for non-flat models up to k ≃ 6 hMpc^{-1} in the redshift range z ≃ 0,1,5, to the fractional accuracy within ~ 1% that roughly corresponds to requirements for weak lensing cosmology with upcoming surveys. We find that the emulators, those built for flat cosmologies such as EuclidEmulator, can predict the non-flat P(k) with least degradation.

1 If the curvature is as small as |Ω_K| ∼ 10^{-5}, we cannot distinguish between the global curvature and the primordial curvature “perturbations”. Hence, the target goal for a hunt of the non-zero curvature is in the range |Ω_K| ≥ a few × 10^{-5}.
etry: \(\Omega_K = -0.0001 \pm 0.0018\) \cite{16}. Ideally we want to use only galaxy BAO measurements at multiple redshifts to constrain the curvature, without employing the CMB prior on the BAO scale, to address whether the CMB and galaxy datasets have consistency within ΛCDM cosmologies, as motivated by the possible tensions between the CMB (early-time) and late-time universe datasets [e.g. see \cite{17} for the recent review].

On the other hand, the growth constraint on the curvature is still in the early stage. Weak lensing and galaxy clustering, observed from wide-area galaxy surveys, are powerful methods to constrain cosmological parameters. However, most of the previous cosmological analyses assume a flat geometry and focus on the parameters to characterize the clustering amplitudes such as \(S_8\) and \(\Omega_m\) [e.g. see \cite{18} for the attempt to constrain \(\Omega_K\) from the weak lensing data]. Although the curvature effect on the linear growth factor is accurately known, the linear-regime information is weaker than the BAO constraint. To obtain a tighter constraint on the curvature, we need an accurate model of the clustering observables that are applicable to the nonlinear regime. N-body and hydrodynamical simulations of cosmic structure formation are among the most powerful, accurate methods for such a purpose. However, simulations are still expensive to construct the theoretical templates, especially in a multidimensional parameter space such as the vanilla ΛCDM model plus the curvature parameter. A more practical method at this stage is using the fitting formula or “emulation” based method [e.g. \cite{19–30}]. However, such efforts developing the emulation method are usually done assuming flat-geometry cosmologies due to the computational expense.

Hence the purpose of this paper is to develop, as the first step, an approximate method for computing the nonlinear matter power spectrum, \(P(k)\), for non-flat cosmologies, which is the fundamental quantity for weak lensing cosmology \cite{31,32}. In fact the existing weak lensing measurements have been used to obtain tight constraints on the cosmological parameters \cite{33–38}. The current and upcoming weak lensing surveys require a 1%-level or even better accuracy in the theoretical template of \(P(k)\) up to \(k \sim 1\,\text{hMpc}^{-1}\) in order not to have a significant bias in cosmological parameters such as dark energy parameters \cite{39}. In this paper we employ the separate universe (SU) approach to study \(P(k)\) for non-flat ΛCDM cosmologies. The SU ansatz states that the effect of the curvature on structure formation in a given non-flat ΛCDM model is equivalent to the effect of the long-wavelength (super-box) density fluctuation on the evolution of short-wavelength (sub-box) fluctuations in the counterpart flat-geometry ΛCDM model \cite{30,34,40,41}, where the cosmological parameters and redshifts in between the non-flat and flat models have to be mapped in the specific way. To study structure formation in the two mapped models, it is useful to use the “response” function of \(P(k)\) which quantifies how \(P(k)\) responds to the long-wavelength density fluctuation or equivalently the non-zero curvature, as a function of \(k\). To develop our method, we further utilize the approximate identity that the response of \(P(k)\) to the curvature, normalized relative to the response in the linear regime, is approximated by the normalized response of \(P(k)\) to the Hubble parameter \(h\) \cite{13}. By using the response to \(h\), we can express \(P(k)\) for a target non-flat ΛCDM model in terms of quantities for the corresponding flat ΛCDM model. That is, our method allows us to extend fitting formula or emulator, developed for flat-geometry cosmologies, to predicting \(P(k)\) for non-flat model, which eases the computational cost for constructing the theoretical templates. We will validate our method using a set of N-body simulations for flat and non-flat ΛCDM models with \(|\Omega_K| \leq 0.1\). We will also assess the performance of the publicly available emulator for computing \(P(k)\) for non-flat models.

This paper is organized as follows. In Section II we first review the SU approach and then describe our approximate method for computing the nonlinear matter power spectrum for non-flat ΛCDM models. In Section III we describe details of N-body simulations for flat and non-flat ΛCDM models. In Section IV we present the main results of this paper. We first validate the approximation for the normalized growth response as we described above, and then show the accuracy of our method for predicting the nonlinear matter power spectrum for non-flat ΛCDM model. Section V is devoted to discussion and conclusion. In Appendix A we give justification of our method based on the halo model. Throughout this paper we use notations \(\Omega_m\) and \(\Omega_f\) to denote the density parameters for non-relativistic matter and the cosmological constant, respectively.

II. SU ESTIMATOR OF \(P(k)\) FOR NON-FLAT ΛCDM MODEL

In this section we develop a method to compute \(P(k)\) for non-flat ΛCDM model, from quantities for the corresponding flat ΛCDM model based on the SU approach \cite{40,44}.

A. Preliminary

Before going to our method we would like to introduce a motivation to use the SU approach. One might naively think that we can use a Taylor expansion of \(P(k, z; \Omega_K)\) treating \(\Omega_K\) as an expansion parameter; \(P(k, z; \Omega_K) \approx P(k, z)|_{\Omega_K = 0} + (\partial P/\partial \Omega_K)|_{\Omega_K = 0} \Omega_K\), where \(P(k, z)|_{\Omega_K = 0}\) is the power spectrum for a flat model. However, there is no unique way to define this partial derivative operation. In particular, we have to satisfy the identity \(\Omega_K = 1 - (\Omega_m + \Omega_f)\) and vary cosmological parameters other than \(\Omega_K\) simultaneously. In addition, there is ambiguity in how the time variable is matched between the flat and curved models. The simplest examples are to match the redshift, or the physical time, while these
might not be optimal. As a working example, we utilize
the SU approach for connecting the power spectra for
non-flat and flat ΛCDM models.

B. SU approach for \( P(k) \)

The effect of curvature (\( \Omega_K \)) on structure formation
appears only in the late universe. In other words, the
curvature does not affect structure formation in the early
universe such as CMB physics (as long as the curvature
parameter is small as indicated by current observa-
tions). Hence throughout this paper we employ a model
where structure formation in the early universe is identi-
cal. This is equivalent to keeping the parameters,
\[
\{ \omega_c, \omega_b, A_s, n_s \},
\]
fixed, where \( \omega_c (= \Omega_c h^2) \) and \( \omega_b (= \Omega_b h^2) \) are the phy-
sical density parameters of CDM and baryon, respectively,
and \( A_s \) and \( n_s \) are the amplitude (at the pivot scale
\( k_{\text{pivot}} = 0.05 \text{Mpc}^{-1} \)) and the spectral tilt of the power
spectrum of primordial curvature perturbations. The
linear matter power spectrum is given as
\[
P_L(k, z) = \left( \frac{D(z)}{D(z_i)} \right)^2 P_L(k, z_i),
\]
where \( z_i \) is the initial redshift in the linear regime satis-
fying \( z_i \gg 1 \) yet well after the matter-radiation equality
such that residual perturbations in radiation do not play
a role and \( D(z) \) is the linear growth factor. The
superscript “L” stands for the linear-theory quantities. In
our method, we keep the linear power spectrum at \( z_i \),
\( P_L(k, z_i) \), fixed.

Let us consider a non-flat ΛCDM model, denoted as
\( \Omega_K - \Lambda \text{CDM} \), as a target model for which we want to es-
timate the nonlinear matter power spectrum at \( z \) in
the late universe. The background cosmology for this target
\( \Omega_K - \Lambda \text{CDM} \) model is specified by
\[
\Omega_K - \Lambda \text{CDM} : \{ \Omega_K, \Omega_m, h \},
\]
where \( \Omega_m \) is the density parameter of total matter (CDM
plus baryon: \( \Omega_m = \Omega_c + \Omega_b \)). The density parameter of
the cosmological constant is given by the identity, \( \Omega_K = 1 - (\Omega_m + \Omega_\Lambda) \). The following discussion can be applied
only to ΛCDM model, so we do not consider a model
with dynamical dark energy [e.g., see \( \text{[19]} \) for discussion
on the separate universe approach for dynamical dark
energy model].

The SU approach gives a mapping between non-flat
ΛCDM and flat ΛCDM models by assigning the degree
of \( \Omega_K \) in the former cosmology to the “long-wavelength”
density fluctuation, denoted as \( \delta_b(t) \), in the latter flat
ΛCDM model. We call the “fake” flat-ΛCDM model as
\( f \Lambda \text{CDM} \) model. Following Li et al. \([42]\), in the SU ap-
proach the physical matter densities in the two models
are related as
\[
\rho_m(t) = \rho_{mf}(t) [1 + \delta_b(t)].
\]
Here and throughout this paper we assume \( \delta_b(t) \) evolves
according to the linear growth factor \( D_f(t) \) as \( \delta_b(t) \propto D_f(t) \),
and we denote quantities in the \( f \Lambda \text{CDM} \) model by
subscript “\( f \)”. The above equation gives
\[
\Omega_m h^2 = \Omega_{mf} h_{f}^2 [1 + \delta_b(t)],
\]
where we defined cosmological parameters of \( f \Lambda \text{CDM} \) model as
\( \rho_{mf}(a_f = 1) = 3H_f^2 \Omega_{mf}/8\pi G \) and \( H_f = 100 h_f \text{ km s}^{-1} \text{ Mpc}^{-1} \).
At very high redshift in the early universe, where \( |\delta_b(t)| \ll 1 \),
we can find \( \Omega_{mf} h_{f}^2 = \Omega_m h^2 \).
This condition gives a mapping between the scale factors:
\[
a_f(t) [1 + \delta_b(t)]^{-1/3} = a(t).
\]
Here we stress that the mapping between quantities in
\( \Omega_K - \Lambda \text{CDM} \) model and the fake flat universe should be
found at the same cosmic time (\( t \)). The above equation
allows us to find the scale factor \( a_f(t) \) in \( f \Lambda \text{CDM} \) model,
corresponding to \( a(t) \) in \( \Omega_K - \Lambda \text{CDM} \) model, at the same
cosmic time. Equivalently we can find the mapping for
redshift as \( (1 + z_f)[1 + \delta_b(z_f)]^{1/3} = 1 + z \). The redshift \( z_f \)
in the fake universe corresponding to the target redshift \( z \)
can be found by solving numerically the above equation.

Once the transfer function in the early universe is fixed
(Eqs. \([1] \) and \([2] \)), we can take \( \Omega_K \) and \( h \) as the free par-
eters to specify the background cosmology in \( \Omega_K - \Lambda \text{CDM} \)
model. For a given set of \( \Omega_K \) and \( h \), we can find that the
 corresponding \( f \Lambda \text{CDM} \) model is specified by

\[
\delta_b(t) = \frac{3 \Omega_K}{D_f(t)} = \frac{3 \Omega_K}{5 \Omega_m},
\]
\[
h_f = h(1 + \delta_h),
\]
with

\[
\delta_h \equiv (1 - \Omega_K)^{1/2} - 1.
\]
Since \( \delta_b(t) \propto D_f(t) \), \( \delta_b(t)/D_f(t) \) is a constant quantity;
for example, its value at the present is specified by the

\footnote{Note that, even if we include massless and massive neutrinos,
throughout this paper we fix those neutrino parameters so that
the early universe physics remains unchanged [e.g., see \([20]\) \([19]\)
for the method].}

\footnote{In reality, at the typical starting redshifts of simulations, the
residual radiation perturbations are not completely negligible,
leaving scale-dependent corrections to the linear growth factor.
Here we assume a situation that, as usually done when setting
up the initial conditions of an N-body simulation, one can first
evolve the baryon and CDM perturbations until today (until the
two components catch up with each other), and then trace back
the “single-fluid” perturbation to the initial redshift by using
the linear growth factor for a given cosmological model.}
The condition $\Omega_{mf} h_f^2 = \Omega_m h^2$ yields a mapping for $\Omega_{mf}$ as

$$\Omega_{mf} = \Omega_m (1 + \delta_h)^{-2}.$$  \hfill(9)

The flat-geometry condition for f-ΛCDM model, i.e. $\Omega_{Kf} = 0$ (or $\Omega_{mf} + \Omega_{Af} = 1$), gives the following identity:

$$\Omega_{Af} = \Omega_A (1 + \delta_h)^{-2}.$$  \hfill(10)

In the SU approach, the effect of $\Omega_K$ on structure formation is realized by the effect of $\delta_h$ on structure formation in a local volume in thefake flat universe. According to the SU approach \[42, 43\], the power spectrum at $z$ in $\Omega_K$-ΛCDM model can be approximated by the power spectrum at $z_f$ in f-ΛCDM model as

$$\hat{P}(k, z; \Omega_K) \simeq P_f(k, z_f; \delta_h)$$

$$\simeq P_f(k, z_f)|_{\delta_h=0} + \frac{\partial P_f(k, z_f; \delta_h)}{\partial \delta_h}|_{\delta_h=0} \delta_h$$

$$= P_f(k, z_f)|_{\delta_h=0} \left[ 1 + \frac{\partial \ln P_f(k, z_f; \delta_h)}{\partial \delta_h}|_{\delta_h=0} \right],$$  \hfill(11)

where $\delta_h \equiv \delta_h(z_f)$. The relation between $z$ and $z_f$ is given by Eq. \[6\]. We often call $\partial P_f(k)/\partial \delta_h$ the growth “response” which describes how the power spectrum at $k$ responds to the long-wavelength mode $\delta_h$ in f-ΛCDM model. We have put the tilde symbol ~ in $P(k, z; \Omega_K)$ to explicitly denote that $\hat{P}$ is an “estimator” of the nonlinear matter power spectrum for $\Omega_K$-ΛCDM model. Note that we need to compute these quantities at $k$ in the comoving wavenumbers of the target $\Omega_K$-ΛCDM model, so we need not include the dilation effect, i.e. the mapping between comoving wavenumbers in between the non-flat and flat models, differently from the method in Ref. \[42\].

For convenience of our discussion, we introduce the normalized response, from Eq. \[11\], as

$$\hat{P}(k, z; \Omega_K) \simeq P_f(k, z_f) \left[ 1 + \frac{26}{21} T_{\delta_h}(k, z_f) \delta_h(z_f) \right],$$  \hfill(12)

with

$$T_{\delta_h}(k, z_f) \equiv \left[ 2 \frac{\partial \ln D_f(z_f)}{\partial \delta_h} \right]^{-1} \left[ 1 + \frac{\partial \ln P_f(k, z_f; \delta_h)}{\partial \delta_h} \right]|_{\delta_h=0}.$$  \hfill(13)

The normalized response has an asymptotic behavior of $T_{\delta_h} \to 1$ at the linear limit $k \to 0$, because $P_f(k) \propto (D_f)^2 P_L(k, z_i)$ in such linear regime (see Eq \[2\]). The coefficient, $26/21$, in the second term in the square bracket on the r.h.s. comes from the linear limit of $k \to 0$ \[40, 41\]. As we will show below or discussed in Ref. \[43\] (around Fig. 6 in the paper), we propose that the power spectrum for the target $\Omega_K$-ΛCDM model is well approximated by replacing the normalized response to $\delta_h$ with the normalized response with respect to $h$ within the flat model:

$$\hat{P}(k, z; \Omega_K) \simeq P_f(k, z_f) \left[ 1 + \frac{26}{21} T_h(k, z_f) \delta_h(z_f) \right],$$  \hfill(14)

with

$$T_h(k, z_f) \equiv \left[ 2 \frac{\partial \ln D_f(z_f)}{\partial h_f} \right]^{-1} \left[ 1 + \frac{\partial \ln P_f(k, z_f)}{\partial h_f} \right],$$  \hfill(15)

where the partial derivative $\partial/\partial h_f$ is the derivative with respect to $h_f$, while keeping the other cosmological parameters (Eq. \[1\]) fixed to their fiducial values; more explicitly we vary $h$ with keeping $\Omega_m h^2$ fixed, and accordingly we have to change $\Omega_m$ (and $\Omega_\Lambda$ from the identity $\Omega_\Lambda = 1 - \Omega_m$ for flat models). Here we defined the normalized response satisfying $T_h(k) \to 1$ at $k \to 0$. If $T_{\delta_h}(k, z_f) \simeq T_h(k, z_f)$ for an input set of $k$ and $z_f$ as we will show below, we can use Eq. \[14\] to approximate the nonlinear matter power spectrum at $z$ for the target $\Omega_K$-ΛCDM model.

**C. Linear limit**

The SU picture has an analogy with the spherical collapse model \[50, 52\], where a spherical top-hat over- or under-density fluctuation is embedded into the FRW homogeneous background and then the time-evolution of the top-hat interior density can be fully tracked up to the fully nonlinear regime. As described in Wagner et al. \[44\], we can find a mapping between the full growth factor of such a spherical top hat density and the linearly-extrapolated density fluctuation $\delta_h$ up to the full order of $\delta_h$. In the SU setup this is equivalent to expressing the growth factor of density fluctuations in a local volume with $\delta_h$ denoted as $D_f(z_f; \delta_h)$, in terms of the growth factor in the background of the fake universe [from Eq. 22 in Ref. \[44\]]:

$$\hat{D}_f(z_f; \delta_h) \simeq D_f(z_f) \left[ 1 + \frac{13}{21} \delta_h + \frac{71}{189} \delta_h^2 + \frac{29609}{130877} \delta_h^3 \right],$$  \hfill(16)

where $\delta_h = \delta_h(z_f)$. Comparison of Eqs. \[12\] and \[16\] clarifies that the expression of Eq. \[12\] corresponds to the linear-order expansion of the linear growth factor in terms of $\delta_h$, because $P_f(k, z_f; \delta_h) \sim D_f(z_f, \delta_h)^2$ at the linear limit. The coefficient $26/21$ on the r.h.s. in Eq. \[14\] comes from the first-order expansion of the linear growth factor in the above equation: $(D_f/D_f)^2 \simeq 1 + (26/21) \delta_h$. Because we know the exact mapping between the linear growth factors in the $\Omega_K$-ΛCDM and f-ΛCDM models, we can fully account for the mapping at the linear limit. We will later include this linear-limit correction.

Fig. \[7\] shows the accuracy of the approximation of the growth factor, $[\hat{D}_f(z_f; \delta_h)/D(z; \Omega_K)]^2$ when truncated at
FIG. 1: An accuracy of the approximation that gives the growth factor for \( \Omega_K\text{-}\Lambda\text{CDM} \) model in terms of the growth factor for the corresponding flat \( \Lambda\text{CDM} \) model and the Taylor expansion of \( \delta_b \) in the SU approach (Eq. [16]). Here \( \delta_b \) is related to the curvature parameter \( \Omega_K \), in the \( x \)-axis, of each \( \Omega_K\text{-}\Lambda\text{CDM} \) model, via Eq. [17]. Note that we consider \( z_f = 0 \) and fixed the other cosmological parameters \( \Omega_{m, f} \) and \( h_f \) as \( \Omega_{m, f} = 0.3156 \) and \( h_f = 0.6727 \) in the flat \( \Lambda\text{CDM} \) model. Here we assess \( [\tilde{D}\delta_b(z, f)]^2 / D(z; \Omega_K)^2 \), where \( D(z; \Omega_K) \) is the true growth factor for each \( \Omega_K\text{-}\Lambda\text{CDM} \) model, because the ratio corresponds to the linear limit of the approximation of matter power spectrum we study in this paper. The dashed, solid and dotted curves denote the results for the approximations obtained when including the zeroth-, first- or second-order expansion of \( \delta_b \) in Eq. [16].

some finite order in \( \delta_b \), as a function of the input \( \Omega_K \) in the \( x \)-axis, where \( D(z; \Omega_K) \) is the true growth factor for \( \Omega_K\text{-}\Lambda\text{CDM} \) model. The value of \( \delta_b \) is specified by the input \( \Omega_K \) in the \( x \)-axis, from Eq. [17]. The dashed, solid and dotted curves show the ratio when including only the zeroth term, or up to the 1st or 2nd term, respectively, in the square bracket of the r.h.s. of Eq. [16]. The 1st-order expansion (solid curve) corresponds to the approximation of the power spectrum, \( \tilde{P}(k, z; \Omega_K) / P(k, z; \Omega_K) \) (Eq. [12]) at linear limit \( (k \to 0) \). Encouragingly, the 1st-order approximation (solid curve) is accurate to within about 2% in the fractional amplitude for the range of \( |\Omega_K| < 0.1 \), which is very broad compared to the current constraint, \( |\Omega_K| < 0.1 \) (2\( \sigma \) level) from the Planck CMB data alone [19]. Note that, if we do not take into account the mapping of redshift \( (z_f \leftrightarrow z) \) or we forcibly use the power spectrum at \( z \) in the fake universe, the accuracy of the 1st-order approximation is significantly degraded. We also note that the results are similar for other redshifts, but have better accuracy with the increase of redshift.

D. Summary: Estimator of \( P(k) \) for non-flat \( \Lambda\text{CDM} \)

By combining Eqs. [14] and [16], we propose the following approximation to compute the nonlinear matter power spectrum at \( z \) for \( \Omega_K\text{-}\Lambda\text{CDM} \) model that is specified by the parameters (\( \Omega_K, \Omega_m, h \)):

\[
\tilde{P}(k, z) \simeq P_f(k, z_f) \left( \frac{D(z; \Omega_K)}{D_f(z_f)} \right)^2 \times \left[ 1 + \frac{26}{21} \{ T_h(k, z_f) - 1 \} \delta_h(z_f) \right],
\]

(17)

with

\[
\delta_h(z_f) = -D_f(z_f) \frac{3\Omega_K}{5\Omega_m},
\]

\[
(1 + z_f) \left[ 1 + \delta_h(z_f) \right]^{1/3} = 1 + z,
\]

\[
T_h(k, z_f) = \left[ 2 \frac{\partial \ln D_f(z_f)}{\partial h_f} \right]^{-1} \frac{\partial \ln P_f(k, z_f)}{\partial h_f}.
\]

(18)

Note that the parameters \( (h, \Omega_{m, f}, \Omega_{\Lambda, f}) \) for \( f\Lambda\text{CDM} \) model are given by Eqs. [7], [9] and [10], and the other cosmological parameters to specify the transfer function and the primordial perturbations \( (\Omega_c, \omega_b, A_s, n_s) \) are kept fixed in the \( \Omega_K\text{-}\Lambda\text{CDM} \) and \( f\Lambda\text{CDM} \) models.

We employed the modification of Eq. [17] from Eq. [14] to fully take into account the modification in the linear growth factor up to the full order of \( \delta_b \); Eq. [17], by design, reproduces the underlying true power spectrum for \( \Omega_K\text{-}\Lambda\text{CDM} \) model at the linear limit, i.e. \( P(k, z) = P(k, z) \) at \( k \to 0 \) (also from Eq. [2]).

All the terms on the r.h.s. of Eq. [17], except for \( D(z) \), are given by quantities for the flat-geometry \( f\Lambda\text{CDM} \) model, which are specified by the cosmological parameters of \( \Omega_K\text{-}\Lambda\text{CDM} \) model. That is, if Eq. [17] is a good approximation, we can evaluate the nonlinear matter power spectrum for an arbitrary \( \Omega_K\text{-}\Lambda\text{CDM} \) model from the quantities for the counterpart flat model in the SU approach. For example, if we use the fitting formulae or emulators of nonlinear matter power spectrum calibrated for flat \( \Lambda\text{CDM} \) cosmologies, we can compute the nonlinear matter power spectrum for the target \( \Omega_K\text{-}\Lambda\text{CDM} \) model. This would be a useful approximation, and we will below give validation of our method, and quantify the accuracy of our method.

III. SIMULATION DATA

A. \( N \)-body simulations

To validate our method (Eq. [17], we use cosmological \( N \)-body simulations. Our simulations follow the method in Nishimichi et al. [26], and here we give a brief summary of the simulations used in this paper.

We use Gadget-2 [53] to carry out \( N \)-body simulation for a given cosmological model. The initial conditions are set up at redshift \( z_i = 59 \) using the second-order Lagrangian perturbation theory [55, 56] implemented by Nishimichi et al. [57] and then parallelized in Valageas
and Nishimichi [58]. We use the public code CAMB [59] to compute the transfer function for a given model, which is used to compute the input linear power spectrum. For all simulations in this paper, we use the same simulation box size in Gpc (i.e., without $h$ in the units) and the same number of particles: $L = 1\ h^{-1}\text{Gpc} \simeq 1.49\ \text{Gpc}$ (without $h$ in units) and $N_p = 2048^3$, which correspond to the particle Nyquist wavenumber, $k = 6.4 \ h\ \text{Mpc}^{-1}$. In the following we will show the results at wavenumbers smaller than this Nyquist wavenumber.

In this paper we use simulations for 5 different cosmological models, denoted as “fiducial” flat $\Lambda$CDM, “$\Omega_K$-CDM1”, “$\Omega_K$-CDM2”, “$\Omega_K$-CDM3”, and “$\delta h$-CDM” models, respectively, as given in Table I. Here the cosmological parameters for the “fiducial” model are chosen to be consistent with those for the Planck 2015 best-fit cosmology [60]. The cosmological parameters for each of the non-flat cosmological models are chosen so that it has the fiducial $\Lambda$CDM model as the “fake” flat $\Lambda$CDM model in the SU approach. We use paired simulations for “$\Omega_K$-CDM1” model to compute the power spectrum response with respect to $\delta h$ ($T_{\delta h}$), where the curvature parameters are specified by $\delta h = \pm 0.01$ at $z_f = 0$. The “$\delta h$-CDM” model is for computing the response with respect to $h$ ($T_h$): here, we chose a step size of $\delta h = \pm 0.02$ for the numerical derivative. Note that the setup of these simulations is designed to compute the “growth” response by taking the numerical derivative at fixed comoving wavenumbers $k$ [see Section III B in Ref. [72]]. We also use the simulations for non-flat $\Lambda$CDM models with $\Omega_K = \pm 0.05$ or $\pm 0.1$, named as “$\Omega_K$-CDM2” and “$\Omega_K$-CDM3”, to assess how our method can approximate the matter power spectrum for non-flat models.

Table I gives the values of $\Omega_K$ and $h$, and we use the fixed values of other cosmological parameters, given as $(\omega_c, \omega_b, A_s, n_s) = (0.1198, 0.02215, 2.2065 \times 10^{-9}, 0.9645)$, which specify the transfer function and the primordial power spectrum, or equivalently the linear matter power spectrum. Note that $\Omega_m$ and $\Omega_A$ are specified by a given set of the parameters for each model: $\Omega_m = \Omega_m h^2 / h^2$ and $\Omega_K = 1 - (\Omega_m + \Omega_A)$. For each model, we use the outputs at 4 redshifts, $z_f \simeq 0.55, 1.03, 1.48$. Since the “fiducial” flat $\Lambda$CDM model is the fake flat model in the SU method, each redshift for the fiducial flat model corresponds to a slightly different redshift in each non-flat model, which is computed from Eq. (6).

Note that all the $N$-body simulations for different cosmological models are designed to have the fixed mass resolution, $m_p \simeq 1.52 \times 10^{10} M_\odot$ (in units without $h$). Hence the comoving mass density in the $N$-body box is kept fixed: $\rho_{m0} = N_p m_p / V_{\text{com}} \simeq 3.96 \times 10^{11} M_\odot \text{Mpc}^{-3}$. We utilize this fact to define a sample of halos in the same mass bins, in units of $M_\odot$, for all the cosmological

| Name         | $\Omega_K$ | $h$  | $N_{\text{real}}$ | Angulo-Pontzen redshift ($z$) |
|--------------|------------|------|-------------------|-------------------------------|
| flat (fiducial) | 0          | 0.6727 | 2                  | Yes.                          |
| $\Omega_K$-CDM1 | 0.00663   | 0.6749 | 10                 | No.                           |
|              | -0.00672  | 0.6705 | 10                 | No.                           |
| $\Omega_K$-CDM2 | 0.05       | 0.6902 | 2                  | Yes.                          |
|              | -0.05     | 0.6565 | 2                  | Yes.                          |
| $\Omega_K$-CDM3 | 0.1        | 0.7091 | 2                  | Yes.                          |
|              | -0.1      | 0.6144 | 2                  | Yes.                          |
| $\delta h$-CDM | 0          | 0.6927 | 10                 | No.                           |
|              | 0         | 0.6527 | 10                 | No.                           |

Note that we also include the effect of massive neutrinos on the linear matter power spectrum, assuming $\Omega_\nu h^2 = 0.00064$ corresponding to $m_\nu = 0.06\text{eV}$, the lower limit inferred from the terrestrial experiments [see Ref. [20] for details]. Hence the physical density parameter of total matter is $\Omega_m h^2 = \omega_c + \omega_b + \omega_\nu$. 

\footnote{Note that we also include the effect of massive neutrinos on the linear matter power spectrum, assuming $\Omega_\nu h^2 = 0.00064$ corresponding to $m_\nu = 0.06\text{eV}$, the lower limit inferred from the terrestrial experiments [see Ref. [20] for details]. Hence the physical density parameter of total matter is $\Omega_m h^2 = \omega_c + \omega_b + \omega_\nu$.}
models. This makes it easier to compute the response of halo mass function with respect to \( \delta_b \) or \( h \), which is used to study the power spectrum responses based on the halo model (see Appendix A).

Furthermore, we use simulations that are run using the “paired-and-fixed” method in Angulo and Pontzen [53], where the initial density field in each Fourier mode is generated from the fixed amplitude of the power spectrum \( P(k) \) and the paired simulations with reverse phases, i.e. \( \delta_k \) and \( -\delta_k \), are run. The mean power spectrum of the paired runs fairly well reproduces the ensemble average of many realizations even in the nonlinear regime [53, 11]. The paired-and-fixed simulations allow us to significantly reduce the sample variance in the power spectrum estimation. The column “Angulo-Pontzen” in Table I denotes whether we use the paired-and-fixed simulations. For the paired-and-fixed simulations, “2” on the column \( N_{\text{real}} \) denotes one pair of the pared-and-fixed simulations.

B. Measurements of power spectrum and growth response

To calculate the power spectrum from each simulation output, we assign the particles on 2048 \(^ 3 \) grids using the cloud-in-cells (CIC) method [62] to obtain the density field. After performing the Fourier transform, we correct for the window function of CIC following the method described in Takahashi et al. [24]. In addition, to evaluate the power spectrum at small scales accurately, we fold the particle positions into a smaller box by replacing \( x \rightarrow x^\prime/(L/10^n) \), where the operation \( a^\%b \) stands for the remainder of the division of \( a \) by \( b \). This procedure leads to effectively \( 10^n \) times higher resolution. Here we adopt \( n = 0, 1 \). We use the density fluctuation \( \delta_k \) up to half the Nyquist frequency determined by the box size \( L/10^n \) with the grid number, and we will show the results at wavenumbers smaller than \( k = 6.4 \, h_f \, \text{Mpc}^{-1} \).

Since we use the fixed box size and the same particle number, we use the same binning to estimate the average of \( |\delta_k|^2 \) in each \( k \) bin to estimate the band power. We then use the two-side numerical derivative method to compute the power spectrum responses. To reduce statistical stochasticity (or sample variance), we employ the same initial seeds as those for the “fiducial” model. The column “\( N_{\text{real}} \)” in Table I denotes the number of realizations for paired simulations, where each pair uses the same initial seeds. For \( \Omega_{k}\text{-}\Lambda \text{CDM}1 \) and \( \delta h\text{-}\Lambda \text{CDM} \) models, we further run 9 paired simulations to estimate the statistical scatters; hence we use 10 paired simulations in total to estimate the power spectrum responses at each redshift, \( \delta_h(k, z_f) \) and \( T_h(k, z_f) \).

IV. RESULTS

A. Power spectrum responses

In Fig. 2 we study the normalized growth responses of matter power spectrum, \( T_h(k) \) and \( T_h(k) \), at the four redshifts, which are computed from the \( N \)-body simulations for \( \Omega_k\text{-}\Lambda \text{CDM}1 \) and \( \delta h\text{-}\Lambda \text{CDM} \) models, respectively, in Table I. It is clear that the approximate identity of \( T_{\delta_b} \approx T_h \) holds over the range of scales and for all the redshifts. To be more precise, the two responses agree with each other to within 2 (16)% in the fractional amplitudes for \( k \lesssim 1 \) (6.4) \( h_f \) \, \text{Mpc} \(^{-1} \). Our results confirm the result of Li et al. [53] (see Fig. 6 in the paper). However, a closer look of Fig. 2 reveals a slight discrepancy at \( k \gtrsim 1 \) \( h_f \) \, \text{Mpc} \(^{-1} \). As we showed in Appendix A, the responses at these small scales are mainly from modifications in the mass density profiles of halos. Hence we conclude that the identity of \( T_{\delta_b} \approx T_h \) is not exact, but approximately valid for models around the fiducial \( \Lambda \text{CDM} \) models we consider.

Since \( \delta_b \) and \( h \) are varied at fixed initial power spectrum, their impact on the power spectrum in the linear regime at a fixed comoving scale comes solely from changing the linear growth factor \( D \). Since we normalize the response to account for this linear dependence on \( D \), the data points converge to unity at the low-\( k \) limit by design.

For the quasi nonlinear regime, the perturbation theory of structure formation predicts that the higher-order loop corrections to the power spectrum are well approximated by a separable form in terms of time and scale: an exact result for the Einstein-de Sitter (EdS) cosmology, which is usually generalized to \( \Lambda \text{CDM} \) cosmology by replacing the scale factor \( a \) with the linear growth factor \( D \). Possible corrections to this arising from the non-separability is known to have a weak dependence on \( \Omega_m(z) \), and this can be usually ignored in the modeling of mildly nonlinear regime [61–64]. Under this approximation, the nonlinear power spectrum is fully specified by its linear counterpart evaluated at the same redshift. Therefore the perturbation theory also predicts that \( T_{\delta_b} \approx T_h \) should be valid due to the matched shape of the linear power spectrum.

The agreement \( T_{\delta_b} \approx T_h \) in the nonlinear regime suggests that nonlinear matter power spectrum is approximately given by a functional form of the input linear power spectrum, \( \Delta_{\text{NL}}^2(k) = F_{\text{NL}}[\Delta^2_k] \) \( (\Delta^2 \equiv k^3 P(k)/(2\pi^2)) \), as implied by the stable clustering ansatz for a CDM model [19, 68–71]. If the ansatz holds, the identity \( T_{\delta_b} = T_h \) holds exactly. In Appendix A, we also show that the approximation \( T_{\delta_b} \approx T_h \) can be found from the halo model picture, which is derived using the growth responses of the halo mass function and the halo mass density profile to \( \delta_b \) and \( h \) that are estimated from \( N \)-body simulations. Thus the results of Fig. 2 suggest that the stable clustering ansatz is approximately valid.

Before proceeding, we comment on the normalized growth response to the primordial
FIG. 2: Normalized growth response of matter power spectrum with respect to \( \delta_h \) and \( h \), \( T_{\delta_h}(k) \) and \( T_h(k) \), at the four redshifts as denoted by the legend in each panel. The horizontal line denotes the linear limit: \( T_{\delta_h}, T_h = 1 \). We use the \( 10 \) paired simulations for \( \Omega_k-\Lambda \text{CDM} \) and \( \delta_h-\Lambda \text{CDM} \) models in Table I to compute these responses. The circle or triangle symbols denote the mean of \( T_{\delta_h} \) or \( T_h \) in each \( k \) bin, and the error bars (although not visible in some \( k \) bins) denote the statistical errors for simulation box with side length \( L = 1 h^{-1} \text{Gpc} \), which are estimated from the standard deviations among the \( 10 \) paired simulations. Note that the range of \( y \)-axis is different in different panels.

The normalized growth response is given by the ratio of the linear matter power spectrum \( P_L(k) \) to the nonlinear matter power spectrum \( P_N(k) \): 

\[
\frac{P_N(k)}{P_L(k)} = 1 + \frac{\partial \ln P_N(k)}{\partial \ln P_L(k)} - 1 = 1 + \frac{\partial \ln P_N(k)}{\partial A_s} - 1.
\]

A change in \( A_s \) does not alter the shape of the linear matter power spectrum. If the stable clustering is exact, one would expect \( T_{A_s} = T_{\delta_h} \). However, as shown in Fig. 6 of Li et al. [13] also see [22], \( T_{A_s} \) shows a sizable discrepancy from \( T_{\delta_h} \) (or \( T_h \)) at \( k \gtrsim 0.1 h^{-1} \text{Mpc} \). This means that a change in \( A_s \) leads to a larger change in the transition scale \( k_{\text{NL}} \) between the linear and nonlinear regimes, or a larger change in the halo profile (e.g., the halo concentration). Hence, we again stress that \( T_{\delta_h}(k) \approx T_h(k) \) is an approximate identity around the \( \Lambda \text{CDM} \) model.

In Fig. 3 we assess the accuracy of the publicly-available fitting formula of \( P(k) \) for predicting the normalized growth response. Here we employ the two versions of Halofit in Smith et al. [20] (Smith+03) and Takahashi et al. [23] (Takahashi+12), respectively, and the \( \text{HMcode} \) in Mead et al. [28] (Mead+20). All the fitting formulae are primarily functional of the linear power spectrum at the target redshift (although each formula includes terms that have an extra dependence on cosmological parameters). Among these fitting formulae, only Smith+03 Halofit was calibrated against \( N \)-body simulations for models including non-flat CDM model. Note that we here compute \( T_{\delta_h}(k) \) from Eq. (12) based on the SU method; we compute \( \tilde{P}(k, z) \) for varied \( \Omega_k-\Lambda \text{CDM} \) models assuming that the fitting formula is valid for non-flat cosmologies, respectively, and then compute \( T_{\delta_h}(k) \) from numerical derivative. Here we adopt \( \delta_h = \pm 0.01 \) at \( z_f = 0 \). On the other hand, for \( T_h(k) \), we vary only \( h \) with keeping the linear matter power spectrum fixed (keeping \( \Omega_m h^2 \) fixed as discussed around Table I) in flat models, and then compute \( T_h(k) \) from numerical derivative of the fitting formula predictions, where we adopt variations of \( \delta h = \pm 0.02 \). The figure shows that none of the fitting formulae reproduces the approximate identity of \( T_{\delta_h} \approx T_h \) in nonlinear regime at the level that we

\[
\frac{P_N(k)}{P_L(k)} = 1 + \frac{\partial \ln P_N(k)}{\partial A_s} - 1.
\]
see in the responses measured from N-body simulations [also see Ref. [27] for the similar discussion]. This implies that the fitting formulae have a degraded accuracy for non-flat cosmologies in the nonlinear regime, because the response to $\delta_h$ is equivalent to the dependence of $P(k)$ on $\Omega_K$. Nevertheless it is intriguing to find that all the fitting formulae give a closer prediction to the simulation result for $T_h$ and $T_{bh}$, respectively. The triangle symbols are the same as in Fig. 2. We omit the simulation result for $T_{bh}$, as it is very close to the triangle symbols, to avoid crowdedness in the figure.

In Fig. 3, we also show the response computed using RESPRESSO [63]. It reconstructs the nonlinear power spectrum from an input linear power spectrum at a target redshift for a target cosmological model based on the perturbation theory motivated method starting from a nonlinear matter power spectrum at a fiducial cosmology measured from N-body simulations. The difference in the nonlinear power spectra in $k$ bin between the two cosmologies is computed by summing up the contributions from the band power of the linear power spectrum in each $q$ bin. To do this, the diagrams in the perturbative expansion relevant to the response of the nonlinear power spectrum to the linear counterpart are precomputed at different values of $\Omega_m$, and this lookup table is inferred along a path between the fiducial and the target cosmology. Note that RESPRESSO outputs the predictions up to $k \sim 1 \ h \text{Mpc}^{-1}$. Since RESPRESSO needs only the linear power spectrum of the target cosmology as input, its prediction is unique for models with the same linear power spectrum (e.g., two models with different values of $A_s$, but evaluated at different redshifts to match the normalization of the linear power spectrum), even when the growth history is different. Hence $T_h$ and $T_{bh}$ computed by RESPRESSO are indistinguishable except for the slight difference due to numerical accuracy. However, since RESPRESSO is motivated by the perturbation theory, these predictions begin to differ from the responses measured from N-body simulations around $k \sim 0.2 \ h \text{Mpc}^{-1}$, which corresponds to the scale where the perturbation theory fails. This is similar to the results from Ref. [72] (see Fig. 2 in the paper), where the perturbation theory (1-loop) predictions were used to compute the response up to quasi nonlinear regime. These results indicate that the perturbation theory motivated model fails to predict the responses in the nonlinear regime. That is, as we discussed above, the nonlinear power spectrum has other dependencies besides the linear power spectrum (also see Appendix A for the similar discussion).

In Fig. 4 we use the publicly-available emulators of $P(k)$, built for flat cosmologies, to assess accuracy for predicting the response to $h$. Here we used CosmicEmu [24], NGenHalofit [29], and EuclidEmulator [30]. The later two emulators fairly well reproduce the simulation results, although a jagged feature in the numerical derivative is seen, probably due to a $k$-binning issue (or interpolation issue) in the output.
B. Accuracy of the approximation of \( P(k) \) for non-flat ΛCDM

In this section we assess an accuracy of the approximation (Eq. 17) to evaluate \( P(k) \) for Ω\(_K\)-ΛCDM model, by comparing the predictions based on the method with power spectra directly measured from N-body simulations for Ω\(_K\)-ΛCDM model.

The data points in Fig. 5 show \( P(k) \) estimated from the simulations, in Table I for Ω\(_K\)-ΛCDM models with Ω\(_K\) = ±(0.05, 0.1), at each of the four redshifts. The curves in each panel show the predictions computed based on Eq. 17, where we used the normalized response \( T_h \), computed from the simulations (the results in Fig. 2), and the power spectrum \( P_f(k, z_f) \) computed from the fΛCDM simulation (the triangle symbols). The figure shows that the estimator reproduces the simulation result at an accuracy better than ~1% in the amplitude over the wide range of wavenumbers, except for ~2% accuracy for Ω\(_K\) = 0.1 at \( k \sim 1 \text{h} \text{Mpc}^{-1} \) that corresponds to the largest \( \delta_v \). The 1% accuracy at \( k \sim 1 \text{h} \text{Mpc}^{-1} \) roughly meets requirements on \( P(k) \) for upcoming weak lensing surveys 39.

In Fig. 6, we compare the performance of the estimator predictions (Eq. 17) with public codes for each cosmological model. Here we consider two fitting formulae, Halofit in Takahashi et al. 23 and HMcode 28. In addition, we consider RESPRESSO 68. Although these models are calibrated under Ω\(_K\) = 0, we use their direct predictions for non-flat cosmologies using an extrapolation to Ω\(_K\) ≠ 0. While the accuracy of Halofit, HMcode and RESPRESSO vary with the redshift, the estimator prediction displays a better performance than the public codes especially in the nonlinear regime.

Now we study the accuracy of the emulators of \( P(k) \), calibrated only for flat cosmologies, to predict \( P(k) \) for...
FIG. 6: Similar to the previous figure, but this figure compares the nonlinear matter power spectra that are computed from our method and the public codes, for the Ω_K-ΛCDM3 model with Ω_K = ±0.1. Here we consider Halofit [23], HMcode [28] and RESPRESSO [33]. The solid line in each panel is the same as in the previous figure, while the other lines denote the results computed from the codes, where we used the direct predictions for non-flat models. The horizontal dashed and solid lines denote ±1% and ±2% fractional accuracy, respectively.

V. DISCUSSION AND CONCLUSION

In this paper we have developed an approximate method to compute the nonlinear matter power spectrum, \( P(k) \), for a “non-flat” ΛCDM model, from quantities computed for the counterpart flat ΛCDM model, based on the separate universe (SU) method. To do this, we need to employ a specific mapping of the cosmological parameters and redshifts between the non-flat and flat models, with keeping the initial power spectrum fixed. In addition we utilized the fact that the normalized response of \( P(k) \) to the long-wavelength fluctuation mode \( \delta_0 \) in the flat model is well approximated by the normalized response to \( h \) for the flat model, which was validated by using the N-body simulations. We showed that our method (Eq. 17) enables to compute \( P(k) \) for non-flat models with |Ω_K| ≤ 0.1, to the fractional accuracy of ~ 1% compared to the N-body simulation results, over the range of scales up \( k \lesssim 6 \, h\text{Mpc}^{-1} \) and in the range \( 0 \leq z \leq 1.5 \), if the accurate response function is avail-
able. Encouragingly, even if we use the publicly available emulator of $P(k)$, which is calibrated for flat cosmologies (e.g. EuclidEmulator [30]), our method allows one to compute $P(k)$ for non-flat model, to within the accuracy of $\sim 2\%$.

A key ingredient in our approach is that how the derivative operation with respect to $\Omega_K$ should be performed exactly in a multi-dimensional input parameter space under a constraint, $\Omega_K = 1 - (\Omega_m + \Omega_\Lambda)$. In our case, the SU approach guides us to use $\delta_h$ to fully specify the direction along which the derivative is taken. Then, we numerically find that this derivative coincides well with the derivative with respect to $h$ within flat cosmologies. This turns out to be practically useful to model non-flat cosmologies using only the knowledge within flat cosmologies. We can consider different ways to match flat-$\Lambda$CDM and $\Omega_K$-CDM models, or more general models such as $w$CDM models with the equation-of-state parameter $w$ for dark energy. It might be of interest to study more about the similarities and differences of the responses with respect to different combinations of parameters, as well as the time variable, at the nonlinear level beyond the applicable range of the one-to-one correspondence between $P(k)$ and $P^L(k)$, which is valid within the EdS approximation and is explicitly used in methods such as RESPRESSO. We will postpone to address a more comprehensive study in this direction as a future work.

An obvious application is to apply the method to actual data for constraining the curvature parameter $\Omega_K$. We will explore this direction in our future work. It is really interesting to explore a constraint of $\Omega_K$ from galaxy surveys, independently from the CMB constraint. As discussed in Ref. [9], if we have precise BAO measurements at multiple redshifts (more than 3 redshifts), we can constrain $\Omega_K$ without employing any prior on the sound horizon (BAO scale) from CMB, because such multi-redshift BAO measurements can give sufficient information on the sound horizon scale and the cosmological distances that depend on $\Omega_m$ and $\Omega_K$ ($\Omega_\Lambda$ is set by

FIG. 7: Comparison of nonlinear power spectra for flat cosmology at $z_f = 1.476$ (top) $z_f = 0$ (bottom) computed from our simulations and the public emulators. Here we consider CosmicEmu [25], NGenHalofit [29], and EuclidEmulator [30]. The horizontal dashed and solid lines denote $\pm 1\%$ and $\pm 2\%$ fractional accuracy, respectively.

FIG. 8: Performance of our method (Eq. 17) for predicting the nonlinear matter power spectrum for non-flat $\Lambda$CDM model with $\Omega_K = \pm 0.1$, at $z_f = 1.476$ (top) and $z_f = 0$ (bottom). Here we use EuclidEmulator [30] predictions for $P_f(k, z_f)$ and $T_h(k, z_f)$ in Eq. (17). The horizontal dashed and solid lines denote $\pm 1\%$ and $\pm 2\%$ fractional accuracy, respectively.
the identity $\Omega_K = 1 - (\Omega_m + \Omega_\Lambda)$ for non-flat ΛCDM models. However, note that the galaxy BAO geometrical constraints need to assume the existence of the standard ruler (i.e. BAO scale) over the multiple redshifts, as supported by the adiabatic initial condition, and the constraints would be degraded if employing further extended models such as time-varying dark energy models. In addition, we should try to explore the curvature information from the growth history of cosmic structures, in addition to the geometrical constraints. Once such high-precision constraint on $\Omega_K$ is obtained from galaxy surveys, we can address whether the Planck constraint and galaxy surveys, or more generally the late-time universe, are consistent with each other within the adiabatic ΛCDM framework. Any deviation or inconsistency in these tests would be a smoking gun evidence of new physics beyond the standard ΛCDM model, and this will be definitely an interesting and important direction to explore with actual datasets.

The response $T_{\delta h}$ is also a key quantity for calibrating the super sample covariance (SSC), which is a dominant source of the non-Gaussian errors in the correlation functions of cosmic shear [41] [73] [75]. For future weak lensing surveys, it is important to obtain an accurate calibration of the non-Gaussian covariance, e.g. to have a proper assessment of the best-fit model compared with the statistical errors and not to have any significant bias in estimated parameters in the parameter inference [76]. Estimating the SSC term for an arbitrary cosmological model is computationally expensive, because it requires to run a sufficient number of SU simulations (including the simulations in non-flat cosmologies) to have an accurate estimation of the SSC terms. For this, the approximation of $T_{\delta h} \approx T_h$ is also useful because we can use the public code of $P(k)$ to compute the SSC for a given cosmological model. However, note that, to model the total power of the SSC term, we further need to take into account the dilation effect [42], which is straightforward to compute from the numerical derivative of the nonlinear power spectra with respect to $k$.

The SSC term is also significant or not negligible for galaxy-galaxy or galaxy clustering, respectively [77]. The SSC terms for these correlation functions arise from the responses of the matter-galaxy or galaxy-galaxy power spectra, $P_{gm}$ or $P_{gg}$, to the super survey mode, $\delta_h$. Since the galaxy-halo connection is modeled by the halo occupation distribution (HOD) model, it is interesting to study whether the normalized growth response of the matter-galaxy (matter-halo) or galaxy-galaxy (halo-halo) power spectra to $\delta_h$ is approximated by the normalized response to $h$ similarly to the case for the matter power spectrum. This is our future work and will be presented elsewhere.

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**Appendix A: Validation of the power spectrum responses with halo model**

In this section we study whether the approximate identity of $T_{\delta h}(k) \approx T_h(k)$ is reproduced by the halo model [78].

1. halo model approach

The halo model gives a useful (semi-)analytical description of the nonlinear clustering statistics, and allows us to study the power spectrum responses [see41] [42] [79] for the similar study]. In the halo model, the power spectrum is given by sum of the 1- and 2-halo terms as

$$P(k) = P^{1h}(k) + P^{2h}(k), \quad (A1)$$

where

$$P^{1h}(k) \equiv \int dM \; n(M) \left( \frac{M}{\rho_m} \right)^2 \tilde{u}_M(k)^2 \quad (A2)$$

and

$$P^{2h}(k) \equiv \left[ I^1_l(k) \right]^2 P^L(k). \quad (A3)$$

with the function defined as

$$I^1_l(k_1, \cdots, k_\mu) \equiv \int dM \; n(M) \left( \frac{M}{\rho_m} \right)^\mu \tilde{b}_\mu(M) \prod_{i=1}^\mu \tilde{u}_M(k_i), \quad (A4)$$

where $n(M) dM$ is the number density of halos in the mass range $[M, M + dM]$ (i.e. the halo mass function), $\tilde{b}_\mu(M)$ is the bias parameter for halos with mass $M$, defined in that $b_0 = 1$ and $b_1(M)$ is the linear bias parameter, and $\tilde{u}_M(k)$ is the Fourier transform of the mass
density profile of halos with mass $M$. Note that the halo profile is normalized so as to satisfy $\delta_M(k) \to 1$ at $k \to 0$. With this normalization, $I_1^1(k)$ should be normalized at very small $k$ so as to satisfy the linear limit $I_1^1(k) \to 1$ at $k \to 0$ in that the 2-halo term reproduces the linear matter power spectrum, $P_{2h}(k) \approx P_L(k)$. For the halo mass density profile, we assume the Navarro-Frenk-White (NFW) halo profile [80] in the following, where we estimate the halo concentration for halos in each mass bin from simulations.

We can formally express the power spectrum response with respect to a parameter $p$ ($p = \delta_b$ or $h$) as

$$\frac{\partial P(k)}{\partial p} = \frac{\partial P_{1h}(k)}{\partial p} + \frac{\partial P_{2h}(k)}{\partial p} \tag{A5}$$

where the 1-halo term response is given as

$$\frac{\partial P_{1h}(k)}{\partial p} = \int dM \frac{M}{\rho_{\text{m0}}} \left( \frac{M}{\rho_{\text{m0}}} \right) \delta_M(k)^2 \times \left[ \frac{\partial \ln n(M)}{\partial p} + 2 \frac{\partial \ln \delta_M(k)}{\partial p} \right]. \tag{A6}$$

The 2-halo term response is given as

$$\frac{\partial P_{2h}(k)}{\partial p} = 2I_1^1(k) \frac{\partial I_1^1(k)}{\partial p} P_L(k) + [I_1^1(k)]^2 \frac{\partial P_L(k)}{\partial p}. \tag{A7}$$

Here the 2nd term, i.e. the linear power spectrum response $\partial P_L/\partial p$, is equivalent to the response of the linear growth factor as discussed in Section 11C. Hence it is straightforward to compute the 2nd term using the linear growth factor. Since the 2-halo term gives a dominant contribution to the total power in the linear regime, where $I_1^1 \approx 1$ as discussed above, we ignore the 1st term for the following results, for simplicity.

2. Evaluation with N-body simulation

In this section we use the $N$-body simulations in Table IV to calibrate each term of the 1-halo term response (Eq. A6), more exactly, the responses of the halo mass function and the halo mass density profile.

First we need to define halos from each output of $N$-body simulations. We follow the method in Nishimichi et al. [26], so please see the paper for further details. As we emphasize around Table IV all the $N$-body simulations employ the same box size ($L \simeq 1.49$ Gpc), the same $N$-body particle number (2048$^3$) and the same $N$-body mass scale ($m_p = 1.52 \times 10^{10} M_\odot$). In this setting, the mean comoving mass density $\bar{\rho}_{\text{m0}}$ is the same for all the simulations. To identify halos in each simulation output, we use the public software Rockstar [81] that identifies halos and subhalos based on the clustering of $N$-body particles in phase space. For each halo/subhalo, we compute the spherical overdensity, $\Delta = 200$, to define mass of each halo/subhalo in the comoving coordinates, $M = (4\pi/3)(R_{200\text{m}})^3 \bar{\rho}_{\text{m0}} \Delta$. Note that the halo mass definition is different from that used in the study of halo bias calibration using the SU simulations [e.g. 46], where the spherical overdensity is set to be $\Delta = 200/(1 + \delta_b)$ in the SU simulation so that halos are identified using the same physical overdensity as the corresponding global universe. By using this halo definition, we can estimate only the “growth” response for the 1-halo term, in the decomposition of “growth” and “dilation” responses [42]. Hence this response calibration is different from the method in [46].

After we identified halo candidates, we determine whether they are central or satellite halos. When the separation of two different halos (between their centers) is closer than $R_{200\text{m}}$ of the more massive one, we mark the less massive one as a satellite halo. In the following we use only central halos with mass containing more than 100 particles. With these definitions, each halo in all the simulations contains exactly the same number of member particles, which allows a cleaner calibration of the power spectrum responses in the halo model approach.

We first estimate the halo mass function from the halo catalog in each simulation realization. We use the following fitting function, which is a modified version of the earlier work in Press and Schechter [82] [also see 83], to fit the mass function estimated from the simulation:

$$n(M) \equiv \frac{dn}{dM} = f(\sigma_M) \frac{\bar{\rho}_{\text{m0}}}{M} \frac{d \ln \sigma_M^{-1}}{dM}, \tag{A8}$$

with

$$f(\sigma_M) = A \left( \frac{\sigma_M}{b} \right)^a + 1 \exp \left( -\frac{c}{\sigma_M^2} \right), \tag{A9}$$

where $A$, $a$, $b$ and $c$ are fitting parameters. The mass variance $\sigma_M^2$ is defined as

$$\sigma_M^2(z) \equiv \int \frac{k^2dk}{2\pi^2} P(k,z) W_R(k)^2, \tag{A10}$$

where $W_R(k)$ is the Fourier transform of a top-hat filter of radius $R$ that is specified by an input halo mass via $R = (3M/4\pi \bar{\rho}_{\text{m0}})^{1/3}$.

For each of the simulations for $\Omega_K$-CDM and $\delta$-CDM models in Table IV we estimate the best-fit values of $A$ and $a$ by fitting the above formula to the mass function measured from each simulation, assuming the Poisson noise in each halo mass bin. For the parameters $b$ and $c$, we fixed their values to those in Tinker et al. [84]. We use 10 realizations for each of the models with $\delta_b = \pm 0.01$ at $z_f = 0$ ($\Omega_K$-CDM1) and the plus or negative variations of $h$ from its fiducial value ($\delta h$-CDM). We then estimate the responses of the halo mass function with respect to $\delta_b$ or $h$ from the averaged mass function, using the two-side numerical derivatives: $\partial \ln n(M)/\partial \delta_b$, or $\partial \ln n(M)/\partial h$, which is the first term of the 1-halo term response (Eq. A6). Note that this corresponds to the growth response of halo mass function due to the reason we described above. Fig. 9 shows the results for the mass
function responses at $z_f = 0$. Here we normalized the responses in the same way as those in $T_{h_0}$ and $T_h$ using the responses of the linear growth factor (see around Eq. 12). The figure shows that the responses are in remarkably nice agreement with each other. This agreement supports that the halo mass function is approximately given by a “universal” form, i.e. $f(\nu)$, where $\nu = \delta_c/\sigma(M(z))$ ($\delta_c$ is a critical collapse threshold) or $\nu \propto 1/\sigma(M(z))$, for different cosmological models. In this case, the halo mass function response is given by $\partial \ln n(M)/\partial p \propto \partial \ln f/\partial \nu \times \partial \nu/\partial p = -\partial \ln f/\partial \ln \nu \times \partial \ln \sigma/\partial p$, since $\sigma(M(z)) \propto D(z)$. Note that we estimated the parameters $A$ and $a$ independently for different cosmological models, so the universality breaks down if the parameters $A$ and $a$ differ in the different models.

Next we employ the following method to estimate the responses of the halo mass density profile, which is the 2nd term of Eq. [A9], in each simulation. We divide halos into $20$ logarithmically-spaced mass bins in the range of $M = [10^{12.45}, 10^{15.45}] h^{-1} M_\odot$, and measure the “averaged” halo mass profile of halos in each bin. We fit each of the estimated mass profiles by an NFW profile to estimate the best-fit concentration parameter, assuming the Poisson errors according to the number of N-body particles contained in each of the radial bins. We then compare the best-fit NFW profiles to estimate the responses of the halo mass concentration with respect to the variations of $\delta_h = \pm 0.01$ at $z_f = 0$ and $\delta h = \pm 0.02$, from the simulations for $\Omega_K-\Lambda$CDM1 and $\delta h-\Lambda$CDM models. Fig. 10 shows the responses of $u_M(k)$ with respect to $\delta_h$ and $h$ for halos with $10^{13} h^{-1} M_\odot$ at $z_f = 0$, where we employ the same normalization as in Fig. 9. To estimate these responses, we plug in the variations of the halo concentration parameters into the Fourier transform of NFW profile. Note that the responses are by definition vanishing in small $k$ bins, where the normalized profile $u_M(k) = 1$. The figure shows that the halo profile responses show a sizable difference at scales, where $u_M(k) < 1$. The difference implies that the two responses do not exactly agree with each other at large $k$ in the nonlinear regime. This difference would be the origin of the slight discrepancy in $T_{h_0}$ and $T_h$ at $k \gtrsim 1 h^{-1} \text{Mpc}^{-1}$.

Fig. 11 shows the normalized growth responses of matter power spectrum with respect to $\delta_h$ and $h$, $T_{h_0}(k)$ and $T_h(k)$, which are computed using the halo model: the sum of the 1-halo and 2-halo terms (Eqs. [A6] and [A7]). To compute these results, we used the results of Figs. 9 and 10. First, the figure clearly shows that the halo model predictions for $T_{h_0}$ and $T_h$ agree well with each other. As expected, the scale-dependent responses arise from the 1-halo term, and the responses at $k \gtrsim 1 h^{-1} \text{Mpc}^{-1}$ arises from the responses of the halo mass density profile. Thus these results give another confirmation of the approximate consistency of $T_{h_0}$ and $T_h$ for $\Lambda$CDM model.

However, the halo model cannot well reproduce the simulation results for the power spectrum response, especially at transition scales between the 1- and 2-halo terms, reflecting the limitation of the halo model.
FIG. 11: Solid lines show the halo model predictions for the normalized responses of matter power spectrum to $\delta_b$ and $h$, i.e. $T_{\delta_b}$ and $T_h$ at $z_f = 0$. The dotted and dot-dashed lines show the 1- and 2-halo term predictions, respectively. The dashed lines denote the halo model predictions where we ignored the responses of the halo mass density profile or equivalently when we include only the halo mass function responses. For comparison, we show the simulation results for $T_{\delta_b}$ and $T_h$ which are the same as those in the lower-right panel of Fig. 2.

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