A generalized “surfaceless” Stokes’ theorem

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We derive a generalized Stokes’ theorem, valid in any dimension and for arbitrary loops, even if self-intersecting or knotted. The generalized theorem does not involve an auxiliary surface, but inherits a higher rank gauge symmetry from the invariance under deformations of the surface used in the conventional formulation.
1. Introduction

Stokes’ theorem can be stated in general terms as

$$\Gamma[^\partial S] = \oint_{\partial S} A = \int_S dA,$$

(1.1)

where $S$ is a compact, orientable, $(n+1)$-dimensional manifold with boundary $\partial S$, $A$ is an $n$-form field on $S$ and $dA$ is its exterior derivative. We shall soon scale down this high-browed language of forms to that of everyday tensor analysis. But before we do that let us recall some (very) well known facts.

First, it is important to note that $\Gamma[^\partial S]$ in eq. (1.1) depends on the boundary $\partial S$ of $S$, but not on $S$ itself. If $S$ and $S'$ have the same boundary $\partial S$, they define a new closed orientable manifold $S - S'$, so we can use Stokes’ theorem again to show that the right hand sides of eq. (1.1), computed with $S$ and $S'$, differ by

$$\oint_{S - S'} dA = \int_{\partial S - \partial S'} dA = 0.$$

(1.2)

Secondly, $\Gamma[^\partial S]$ is invariant under the (generalized) Abelian gauge transformations given by

$$A \rightarrow A' = A + d\Lambda,$$

(1.3)

where $\Lambda$ is an $(n-1)$-form field. Under this transformations $dA$ itself is invariant

$$dA' = dA,$$

so we can think of the invariance of $\Gamma[^\partial S]$ as following from the right hand side of eq. (1.1). But we can also think of the invariance of $\Gamma[^\partial S]$ as a consequence of $\partial S$ being closed: using Stokes’ theorem once more, the change of $\Gamma[^\partial S]$ under the transformation in eq. (1.3) would be

$$\Gamma' - \Gamma = \oint_{\partial S} d\Lambda = 0.$$

The most important physical applications are with $n = 1$, in which case Stokes’ theorem relates the line integral of a covariant vector field $A$ along a closed curve $q$, to the flux of the curl $F$ of $A$ through a (compact, orientable) surface $S$ with boundary $\partial S = q$. As is often the case, we shall think of $q$ and $S$ as embedded in a $D$-dimensional manifold $M$. Then, using old-fashioned indices, Stokes’ theorem is

$$\Gamma[q] = \oint_q dx^\mu A_\mu(x) = \int_S d^2\sigma^{\mu\nu} F_{\mu\nu}(x),$$

(1.4)

where $d^2\sigma^{\mu\nu}$ is the usual surface element,

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),$$

(1.5)
that is, \( F = dA \), and the greek indices \( \mu, \nu, \ldots \) take values from 1 to \( D \). If \( D = 3 \), as in elementary vector calculus, the right hand side of eq. (1.4) can be written as

\[
\int_S d^2\sigma F_{\mu\nu}(x) = \int_S d^2\tilde{\sigma} \tilde{F}^\mu,
\]

which leads to the familiar relation between the circulation of the covariant vector field \( A \) (eg. the vector potential or the magnetic field) and the flux of the contravariant vector field \( \tilde{F} \) (correspondingly, the magnetic field or the current density).

We can restate our earlier remarks: \( \Gamma[q] \) in eq. (1.4) is invariant under the Abelian gauge transformations

\[
A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \Lambda,
\]

and depends on the curve \( q \) but not on the surface \( S \). As shown in eq. (1.2), this is a consequence of \( dF = dA = 0 \) or, in terms of the dual field

\[
\tilde{F}_{\mu_1\cdots\mu_{D-2}} = \frac{1}{2!} \epsilon_{\mu_1\cdots\mu_{D-2}\nu_1\nu_2} F^{\nu_1\nu_2},
\]

is a consequence of Bianchi’s identity

\[
\partial_{\mu_1} \tilde{F}^{\mu_1\cdots\mu_{D-2}} = 0. \tag{1.7}
\]

Stokes’ theorem is used occasionally in situations involving a flux through a physical surface. However, very often the surface \( S \) is an auxiliary object, while the importance of the theorem stems from the interest in exchanging the “gauge potential” \( A_\mu \) in favor of its gauge invariant “field strength” \( F_{\mu\nu} \). From this point of view, Stokes’ theorem as formulated in eq. (1.4) is very limited. It requires the use of a fiducial surface \( S \), with no immediate physical significance, and more important, it’s application is limited to those loops which are boundaries of orientable surfaces. The gauge invariant left hand side of eq. (1.4) is well defined for a very wide class of loops, including self intersecting and knotted loops (with the appropriate limitations arising from the singularities of \( A_\mu \)). But Stokes’ theorem, as formulated in eq. (1.4), cannot be used to obtain an expression in terms of the manifestly gauge invariant field strength \( F_{\mu\nu} \), except for the simplest loops. This limitation is of increasing practical importance in view of the growing role of the so-called Loop Space variables in the analysis of gauge theories in general [1], of quantum gravity [2], of topological field theories [3], and more recently in such classical areas as the theory of turbulence [4]. In many applications the Abelian invariance, as expressed in eqs. (1.3) and (1.6), is also a limitation. A non-Abelian extension has been discussed elsewhere [5,6], but we shall not consider it here.

For Abelian gauge fields one can always fix the gauge completely and then invert eq. (1.5) to obtain \( A_\mu \) as a functional of \( F_{\mu\nu} \). Thus, it is clear that in principle one can always write the line integral in the left hand side of eq. (1.4) in terms of the gauge invariant field strength \( F_{\mu\nu} \). The question is then whether one can derive a simple geometrical
expression providing a direct generalization of Stokes’ theorem in eq. (1.4) for general loops in arbitrary dimension $D$. From the preceding discussion it should be clear that such a generalization must not involve any auxiliary surface, so that whatever limitations apply follow solely from the loop $q$ and the gauge potential $A_\mu$. In the following sections we derive and discuss various aspects of that generalization of Stokes’ theorem.

2. Vector and Tensor Currents

Consider a closed loop $q$ embedded in a $D$-dimensional manifold $\mathcal{M}$ which we shall assume to be flat and, for simplicity, we take to be Euclidean. The loop $q$ will be parametrized by a parameter $s \in [0,1]$, so it is given by a trajectory $q^\mu(s)$ in $\mathcal{M}$, with $q^\mu(0) = q^\mu(1)$. Associated to the loop there is a vector current given by

$$ j^\mu[q; x] = \int_0^1 ds \; \dot{q}^\mu(s) \; \delta(x - q(s)) \tag{2.1} $$

where $\delta(x)$ is the $D$-dimensional delta in $\mathcal{M}$, and $\dot{q}^\mu(s) = dq^\mu(s)/ds$. Then, the loop integral in the left hand side of eq. (1.4) is

$$ \Gamma[q] = \int_0^1 ds \; \dot{q}^\mu(s) \; A_\mu(q(s)) = \int d^Dx \; j^\mu[q; x] A_\mu(x), \tag{2.2} $$

and its invariance under the gauge transformation in eq. (1.6) is a consequence of $j^\mu$ being conserved

$$ \partial_\mu j^\mu[q; x] = 0, \tag{2.3} $$

which in turn follows from the definition of the current in eq. (2.1) and the fact that $q$ is a closed loop.

A key ingredient in the generalization of Stokes’ theorem is the fact that the current $j^\mu$ is itself the divergence of an antisymmetric tensor current,

$$ j^\mu[q; x] = \partial_\nu \tilde{\theta}^{\mu\nu}[q; x] \tag{2.4} $$

with

$$ \tilde{\theta}^{\mu\nu} = -\tilde{\theta}^{\nu\mu}. $$

Indeed, consider

$$ \tilde{\theta}_{\mu\nu}[q; x] = -\int_0^1 ds \; [\dot{q}_\mu(s) \partial_\nu - \dot{q}_\nu(s) \partial_\mu] G_D(x - q(s)), \tag{2.5} $$

were $G_D(x)$ is the inverse of the laplacian in $D$-dimensions

$$ -\partial^2 G_D(x) = \delta(x). \tag{2.6} $$
The divergence of $\tilde{\theta}{}^{\mu\nu}$ is
\[
\partial_\nu \tilde{\theta}{}^{\mu\nu}[q; x] = \int_0^1 ds \left[ \dot{q}^\nu(s) \partial_\nu \partial^\mu - q^\mu(s) \partial^2 \right] G_D(x - q(s))
\]

The first term in the right hand side vanishes for closed loops
\[
\int_0^1 ds \dot{q}^\nu(s) \partial_\nu \partial^\mu G_D(x - q(s)) = - \int_0^1 ds \frac{d}{ds} \partial_\mu G_D(x - q(s)) = 0,
\]
so eq. (2.4) follows after using eq. (2.6). Then, substituting eq. (2.4) into eq. (2.2) we have
\[
\int d^Dx \, j^\mu[q; x] \, A_\mu(x) = \int d^Dx \, \partial_\nu \tilde{\theta}{}^{\mu\nu}[q; x] \, A_\mu(x)
\]
\[
= - \int d^Dx \, \tilde{\theta}{}^{\mu\nu}[q; x] \, \partial_\nu A_\mu(x).
\]
Thus,
\[
\int_0^1 ds \, \dot{q}^\mu(s) \, A_\mu(q(s)) = \int d^Dx \, j^\mu[q; x] \, A_\mu(x)
\]
\[
= \frac{1}{2} \int d^Dx \, \tilde{\theta}{}^{\mu\nu}[q; x] \, F_{\mu\nu}(x),
\]
which is the generalized version of Stokes’ theorem we were after.

3. The two-dimensional case

To get a feeling for the contents of this result let us consider first the $D = 2$ case, when eq. (2.7) can be written as
\[
\int_0^1 ds \, \dot{q}^\mu(s) \, A_\mu(q(s)) = \int d^2x \, \theta[q; x] \, \tilde{F}(x),
\]
where
\[
\tilde{F} = \frac{1}{2!} \epsilon^{\mu\nu} F_{\mu\nu}
\]
and
\[
\theta[q; x] = \frac{1}{2!} \epsilon_{\mu\nu} \tilde{\theta}{}^{\mu\nu}[q; x].
\]
Substituting here eq. (2.5), we get
\[
\theta[q; x] = \int_0^1 ds \, \dot{q}^\mu(s) \, B_\mu(x - q(s)),
\]
where $B_\mu$ can be chosen to be

$$B_\mu(x) = -\epsilon_{\mu\nu} \partial^\nu G_2(x). \quad (3.4)$$

For our discussion in higher dimensions it will be relevant to note here that this choice for $B_\mu$ is not unique. Indeed, eq. (3.3) defines $B_\mu$ only up to a gauge transformation

$$B_\mu(x) \rightarrow B_\mu(x) + \partial_\mu \Phi(x), \quad (3.5)$$

under which $\theta$ and $\tilde{\theta}^{\mu\nu}$, and consequently $j^\mu$, remain invariant.

To proceed, recall that in $D = 2$ the inverse laplacian can be chosen to be

$$G_2(x) = -\frac{1}{4\pi} \ln m^2 x^2, \quad (3.6)$$

so $B_\mu$ in eq. (3.4) becomes

$$B_\mu(x) = \frac{1}{2\pi} \epsilon_{\mu\nu} \frac{x^\nu}{x^2}. \quad (3.7)$$

Thus, $B_\mu(x)$ can be thought of as the “gauge potential” at the point $x$ due to a (anti-) monopole at the origin, so $\theta[q; x]$ in eq. (3.3), and therefore $\tilde{\theta}^{\mu\nu}[q; x]$, is the winding number of the loop $q$ around the point $x$. Then eq. (3.1) is the usual Stokes’ theorem in $D = 2$, except that the factor $\theta[q; x]$ in the integrand provides the correct weight to each area element, rendering the theorem valid for arbitrary loops, even if they are self intersecting or self overlapping [5].

### 4. Hidden gauge invariance of higher rank

In higher dimensions the situation is more involved. We can think of the right hand side of eq. (2.7) as a sum of projections onto the individual $\mu - \nu$ planes, with $\tilde{\theta}^{\mu\nu}[q; x]$ providing the correct weight to the surface element in each integral. But for $D > 2$ there is more structure than that in this generalized Stokes’ theorem. Although we have disposed of the surface used in the usual formulation of the theorem, we can expect that the symmetry under deformations of that surface may remain hidden somewhere.

Analogous to $\theta[q; x]$ in eq. (3.2), for $D > 2$ we have

$$\theta_{\mu_1 \ldots \mu_{D-2}}[q; x] = \frac{1}{2!} \epsilon_{\mu_1 \ldots \mu_{D-2} \nu_1 \nu_2} \tilde{\theta}^{\nu_1 \nu_2}[q; x] \quad (4.1)$$

and conversely,

$$\tilde{\theta}^{\mu\nu}[q; x] = \frac{1}{(D - 2)!} \epsilon_{\mu \nu \lambda_1 \ldots \lambda_{D-2}} \theta^{\lambda_1 \ldots \lambda_{D-2}}[q; x]. \quad (4.2)$$

Then, as in eq. (3.3), we now have

$$\theta_{\mu_1 \ldots \mu_{D-2}}[q; x] = \int_0^1 ds \dot{q}^\nu(s) B_{\mu_1 \ldots \mu_{D-2} \nu}(x - q(s)), \quad (4.3)$$
where we can choose
\[ \mathcal{B}_{\mu_1 \cdots \mu_{D-1}}(x) = -\epsilon_{\mu_1 \cdots \mu_{D-1} \lambda} \partial^\lambda G_D(x). \] (4.4)

As with \( \mathcal{B}_\mu \) in eq. (3.4), this choice for \( \mathcal{B} \) is not unique. Let us show that the invariance under the gauge transformation of \( \mathcal{B} \), given in \( D = 2 \) by eq. (3.5), now generalizes to
\[ \mathcal{B}_{\mu_1 \cdots \mu_{D-1}}(x) \rightarrow \mathcal{B}_{\mu_1 \cdots \mu_{D-1}}(x) + (D - 1) \partial_{[\mu_1} \Phi_{\mu_2 \cdots \mu_{D-1}]}(x), \] (4.5)
where \( \Phi_{\mu_1 \cdots \mu_{D-2}} = \Phi_{[\mu_1 \cdots \mu_{D-2}]^2} \) is a completely antisymmetric tensor field. Substituting into eq. (4.3) one finds that under this transformation \( \theta \) transforms as
\[ \theta_{\mu_1 \cdots \mu_{D-2}}[q, x] \rightarrow \theta_{\mu_1 \cdots \mu_{D-2}}[q, x] + (D - 2) \partial_{[\mu_1} \Psi_{\mu_2 \cdots \mu_{D-2}]}[q, x] \] (4.6)
where
\[ \Psi_{\mu_1 \cdots \mu_{D-3}}[q, x] = \int_0^1 ds \dot{q}^\nu(s) \Phi_{\mu_1 \cdots \mu_{D-3} \nu}(x - q(s)). \] (4.7)
Correspondingly, for \( \tilde{\theta} \) we get, with eq. (4.2),
\[ \tilde{\theta}^{\mu \nu}[q, x] \rightarrow \tilde{\theta}^{\mu \nu}[q, x] + \frac{1}{(D - 3)!} \epsilon^{\mu \nu \lambda_1 \cdots \lambda_{D-2}} \partial_{\lambda_1} \Psi_{\lambda_2 \cdots \lambda_{D-2}}[q, x]. \] (4.8)
Hence, in \( D > 2 \) neither \( \theta \) nor \( \tilde{\theta} \) are invariant, but the divergence of \( \tilde{\theta} \) is invariant
\[ \partial_\nu \tilde{\theta}^{\mu \nu}[q; x] \rightarrow \partial_\nu \tilde{\theta}^{\mu \nu}[q; x]. \]

Then, in order to have \( j^\mu = \partial_\nu \tilde{\theta}^{\mu \nu} \) invariant, as needed by Stokes’ theorem in eq. (2.7), \( \theta[q, x] \) is the line integral of a gauge potential \( \mathcal{B}(x) \) of rank \( (D - 1) \) (in the sense of Kalb-Ramond-Nambu [7]). Moreover, \( \theta[q, x] \) itself transforms as a gauge potential of rank \( (D - 2) \), but the corresponding gauge transformation depends on the loop \( q \). The sources for these potentials are different: eq. (4.4) shows that \( \mathcal{B}(x) \) has as source an ordinary point singularity, while the source for \( \theta[q, x] \) has a singularity along the loop \( q \) as seen from eq. (4.3). In \( D = 2 \) the potential \( \mathcal{B} \) becomes of rank 1, while \( \theta \) is invariant.

Finally, let us clarify the relation between this higher rank gauge invariance in \( D > 2 \), and the invariance under deformations of the auxiliary surface in the usual formulation of Stokes’ theorem, also in \( D > 2 \). The generalized Stokes’ theorem in eq. (2.7) is
\[ \Gamma[q] = \frac{1}{2} \int dx \tilde{\theta}^{\mu \nu}[q; x] F_{\mu \nu}(x) \]
\[ = \frac{1}{(D - 2)!} \int dx \theta_{\mu_1 \cdots \mu_{D-2}}[q; x] \tilde{F}^{\mu_1 \cdots \mu_{D-2}}(x). \]
Then, under the transformation of \( \theta \) in eq. (4.6), \( \Gamma \) would change by
\[ \delta \Gamma = \frac{1}{(D - 3)!} \int dx \partial_{\mu_1} \Psi_{\mu_2 \cdots \mu_{D-2}}[q, x] \tilde{F}^{\mu_1 \cdots \mu_{D-2}}(x) \]
\[ = -\frac{1}{(D - 3)!} \int dx \tilde{\theta}^{\mu_1 \cdots \mu_{D-2}}[q, x] \partial_{\mu_1} \tilde{F}^{\mu_1 \cdots \mu_{D-2}}(x), \]
which vanishes by virtue of Bianchi’s identity in eq. (1.7). Thus, the higher rank gauge invariance exhibited in $D > 2$ has the same geometrical contents as the invariance under deformations of the auxiliary surface in ordinary Stokes’ theorem. It may be noticed that in $D = 2$ there is neither Bianchi identity, nor room where to deform the surface.

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