On the global convergence of a randomly perturbed dissipative nonlinear oscillator.

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Abstract

We consider in this work small random perturbations of a nonlinear oscillator with friction–type dissipation. We rigorously prove that under non–degenerate perturbations of multiplicative noise type, the perturbed system that describes the dynamics of the dissipative oscillator converges to local minimizers of the potential function in $\mathcal{O}(\ln(\varepsilon^{-1}))$ time on average, where $\varepsilon > 0$ is the scale of the random perturbation. Under a change of time scale, this indicates that for the diffusion process that approximates the stochastic heavy–ball method, it takes (up to logarithmic factor) only a linear time of the square root of inverse stepsize to evade from all saddle points and hence it implies a fast convergence of its discrete–time counterpart.

Keywords: Dissipative nonlinear oscillator, saddle point, Hamiltonian system, small random perturbations of dynamical system, heavy–ball method.

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1 Introduction.

1.1 Randomly perturbed dissipative nonlinear oscillator.

Consider the motion of a nonlinear oscillator system in \( \mathbb{R}^d \). The governing differential equation takes the form

\[
\ddot{X}(t) + \nabla_X f(X(t)) = 0 \quad X(0) \in \mathbb{R}^d.
\]  

(1)

Here \( X(t) = (X_1(t), \ldots, X_d(t)) \) describes the positions of the \( d \)-oscillators in the system, and \( f(X(t)) = f(X_1(t), \ldots, X_d(t)) \) is the potential function. One can add a friction to (1), so that we have a dissipative oscillator system

\[
\ddot{X}(t) + \alpha \dot{X}(t) + \nabla_X f(X(t)) = 0 \quad X(0) \in \mathbb{R}^d.
\]  

(2)

Here \( \alpha > 0 \) is the friction constant. System (2) can be written in Hamiltonian form

\[
\begin{align*}
\frac{dX}{dt} & = V(t) , \quad X(0) \in \mathbb{R}^d ; \\
\frac{dV}{dt} & = (-\alpha V(t) - \nabla_X f(X(t))) dt , \quad V(0) \in \mathbb{R}^d .
\end{align*}
\]  

(3)

System (3) is a Hamiltonian system with friction, and the Hamiltonian is given by

\[
H(X,V) = \frac{1}{2} V^2 + f(X) .
\]  

(4)

One can show that critical points of system (3) all lie on the \( X \)-axis. These critical points have \( (X,V) \) coordinates \( (X,0) \) in which \( X \) is a critical point of \( f(X) \). If \( X \) is a local minimum point of the function \( f(X) \), then \( (X,0) \) is a local minimum point of the Hamiltonian \( H \); If \( X \) is a local maximum or saddle point of the function \( f(X) \), then \( (X,0) \) is a saddle point of the Hamiltonian \( H \). When \( d = 1 \), Figure 1(a) shows the graph of the potential function \( f \); Figure 1(b) shows the phase flow of system (3) without the friction term \(-\alpha V(t)\), and the friction term is added as vector field \( b(X,V) = (0, -\alpha V) \) pointing towards the \( X \)-axis; Figure 1(c) shows the graph of the corresponding Hamiltonian function \( H(X,V) \).

Let us consider a small random perturbation of (3):

\[
\begin{align*}
\frac{dX^\varepsilon(t)}{dt} & = V^\varepsilon(t) dt + \varepsilon \sigma_1(X^\varepsilon(t), V^\varepsilon(t)) dW^1_t , \quad X^\varepsilon(0) \in \mathbb{R}^d ; \\
\frac{dV^\varepsilon(t)}{dt} & = (-\alpha V^\varepsilon(t) - \nabla_X f(X^\varepsilon(t))) dt + \varepsilon \sigma_2(X^\varepsilon(t), V^\varepsilon(t)) dW^2_t , \quad V^\varepsilon(0) \in \mathbb{R}^d .
\end{align*}
\]  

(5)

1 Formally (see [2]), we should actually introduce the momentum variable \( P = V \) here and write the Hamiltonian as \( H(X,P) = \frac{1}{2} \langle P, V \rangle + f(X) \) (\( \langle \cdot, \cdot \rangle \) is the pairing between momentum and velocity), but for simplicity of notation we shall just write everything in terms of \( V \).

2 Or \( X \)-plane, and we refer this to the \( X \)-axis throughout the text that follows. It just means that \( V = 0 \).
Figure 1: Phase portrait when $X$ has dimension 1.
Here $\varepsilon > 0$ is the intensity of the noise; $W_t^1$ and $W_t^2$ are two independent standard Brownian motions in $\mathbb{R}^d$; $\sigma_1(X,V)$ and $\sigma_2(X,V)$ are two matrices in $\mathbb{R}^d$ such that $a_i(X,V) \equiv \sigma_i(X,V)\sigma_i^T(X,V)$ are uniformly positive– definite for $i = 1, 2$.

When the intensity of the perturbation $\varepsilon > 0$ is small, the trajectory $(X^\varepsilon(t), V^\varepsilon(t))$ of system (5) will be close to $(X(t), V(t))$ of system (2) with the same initial conditions on any finite–time interval $[0, T]$, so that we have

$$\lim_{\varepsilon \to 0} \mathbb{P}_{(X(0), V(0))} \left( \max_{0 \leq t \leq T} (|X^\varepsilon(t) - X(t)|_{\mathbb{R}^d} + |V^\varepsilon(t) - V(t)|_{\mathbb{R}^d}) > \delta \right) = 0$$  \hspace{1cm} (6)

for any $\delta > 0$ and $T > 0$.

For any choice of $\alpha > 0$, the system (3) is a dissipative system in the sense that the Hamiltonian $H(X,V)$ decays along the flow, so that

$$H(X(t), V(t)) - H(X(0), V(0)) = -\alpha \int_0^t (V(s))^2 ds.$$  \hspace{1cm} (7)

From (7), we see that when $(V(t))^2 > 0$, the Hamiltonian function $H(X(t), V(t))$ is strictly decaying. However, when $V(t) = 0$, it is not a–priori guaranteed that the Hamiltonian $H(X(t), V(t))$ keeps decaying. If $V(t) = 0$ happens outside a neighborhood of a critical point $O$ of the flow of $(X(t), V(t))$, then the friction vanishes at the crossing point, but the flow is not zero there. Thus it is the Hamiltonian flow that brings the process immediately to a region where $(V(t))^2$ is still strictly positive. Combining this with (7) we see that even in this case the Hamiltonian $H(X(t), V(t))$ keeps strictly decaying along the Hamiltonian flow $(X(t), V(t))$. Nevertheless, when the Hamiltonian flow approaches a critical point $O$, the trajectory $(X(t), V(t))$ may be trapped there. In this case, the function $H(X(t), V(t))$ as a function of $t$ will not be strictly decaying.

From the above reasoning, we see that we expect the dissipative system (2) to converge to a neighborhood of a critical point of $H(X,V)$ after sufficiently long time. Notice that there are two types of critical points of the Hamiltonian function $H(X,V)$: saddle point or local minimum point (see Figure 1). Due to instability of the flow near the saddle point, one can expect that random perturbations in (5) help the process leave the saddle point after wandering in a neighborhood of the saddle point for sufficiently long time. Our goal in this paper is to characterize the asymptotic, as $\varepsilon \to 0$, of this convergence time. Combining this with (5), we see that the perturbed system $(X^\varepsilon(t), V^\varepsilon(t))$ converges only to local minimum points of $H(X,V)$ after sufficiently long time. In a rough way, our main result in this work can be summarized as the following

**Major Result.** Starting from a certain initial condition $(X(0), V(0)) \in \mathbb{R}^d \times \mathbb{R}^d$, as $\varepsilon \to 0$, the perturbed process $(X^\varepsilon(t), V^\varepsilon(t))$ in (5) converges to local minimizers of the Hamiltonian function $H(X,V)$ in (4) after time $\lesssim C \ln(\varepsilon^{-1})$. 

4
It is worth pointing out that, the constant $C > 0$ that appears in the convergence time asymptotic depends on the potential function $f(X)$ and the friction parameter $\alpha > 0$ explicitly.

From the point of view of the theme on random perturbations of Hamiltonian systems, many classical literature were studying the case when the friction parameter $\alpha > 0$ is diminishing as $\varepsilon \to 0$ (see [16, Chapter 8], [17], [12], [8], [14], [13], [3], [11]). In this aspect, techniques such as averaging principle have been introduced, and the asymptotic analysis of the perturbed process is studied effectively using diffusion processes on graphs (see [15]). On the other hand, there are also works addressing the over–damped asymptotic regime, that is, when $\alpha \to \infty$ as $\varepsilon \to 0$ (see [9]). The situation considered in our work addresses the intermediate case, as $\alpha > 0$ is assumed to be a moderate constant.

1.2 Background in optimization and statistical machine learning.

Our motivation comes from the smooth unconstrained optimization problem

$$\min_{x \in \mathbb{R}^d} f(x).$$

This problem can be solved by optimization methods that lead to second–order differential equations. These methods have been demonstrating acceleration towards convergence. As an example, Nesterov’s accelerated gradient method is a classical scheme of such type that has been used a lot in optimization. The original method can be “surrogated” by a limiting ODE of the form (see [32])

$$\ddot{X}(t) + \frac{3}{t} \dot{X}(t) + \nabla_X f(X(t)) = 0, \; X(0) \in \mathbb{R}^d.\quad (9)$$

However, in many practices of statistical machine learning and optimization (see, e.g., [34], [41], [39], [40], [6]), including the momentum variable does not require a time–decaying factor. Such methods date back to the “heavy ball method” in [31]. This leads to the consideration of the following gradient version of the heavy ball method algorithm

$$\begin{cases} x_t = x_{t-1} + \sqrt{s} v_t, \; x_0 \in \mathbb{R}^d; \\ v_t = (1 - \alpha \sqrt{s}) v_{t-1} - \sqrt{s} \nabla f(x_{t-1}), \; v_0 \in \mathbb{R}^d. \end{cases} \quad (10)$$

Here $s > 0$ is a small parameter and $\nabla_X f(x)$ is the gradient of the function $f$ with respect to the $X$–variable. As in [26, Theorem 1], it can be proved that after time–rescaling $t \to t/\sqrt{s}$, the family of processes $(x_{t/\sqrt{s}}, v_{t/\sqrt{s}})$ converges as $s \to 0$ to the solution $(X(t), V(t))$ of the system of differential equations (3). In other words, instead of the differential equation (3), we are using the equation (2) as the dynamical equation to approach local minimizers of the objective function $f(x)$ in (3).
The approximation of \((x_t, v_t)\) to \((X(t), V(t))\) can be proved in the strong sense, just as Theorem 2 of [32]. To be precise, we have, for all \(T > 0\), that

\[
\lim_{\varepsilon \to 0} \max_{0 \leq t \leq T} \left( |x_t|_{\sqrt{\varepsilon}} - X(t)|_{\mathbb{R}^d} + |v_t|_{\sqrt{\varepsilon}} - V(t)|_{\mathbb{R}^d} \right) = 0. \tag{11}
\]

In optimization practice, a major concern arises when the objective function \(f(x)\) from [33] is non–convex. In this case, process \(X(t)\) in (2) can be trapped at local maximum or saddle points of \(f\), and therefore the question arises whether or not one can add a noise to help the process \(X(t)\) escape from these critical points. Recall that critical points of system (3) all lie on the \(X\)–axis. These critical points have \((X, V)\) coordinates \((X, 0)\) in which \(X\) is a critical point of \(f(X)\). If \(X\) is a local minimum point of the function \(f(X)\), then \((X, 0)\) is a local minimum point of the Hamiltonian \(H\); If \(X\) is a local maximum or saddle point of the function \(f(X)\), then \((X, 0)\) is a saddle point of the Hamiltonian \(H\). Escaping from local maximum or saddle points of \(f(x)\) for the process \(X(t)\) equipped with noise is thus equivalent to escaping from saddle points of the Hamiltonian \(H(X, V)\) for the process \((X(t), V(t))\) equipped with noise, just as we saw in [34]. Obtaining the convergence rates of the perturbed process \((X^\varepsilon(t), V^\varepsilon(t))\), such as our major result stated in the previous subsection, is thus of interest from the point of view in optimization practice.

Let us consider the following noisy scheme of the heavy–ball method, which we call the **stochastic heavy–ball method**:

\[
\begin{align*}
x_t &= x_{t-1} + \sqrt{s} \tilde{v}_t, \ x_0 \in \mathbb{R}^d ; \\
v_t &= (1 - \alpha \sqrt{s}) v_{t-1} - \sqrt{s} \nabla^2_X f(x_{t-1}), \ v_0 \in \mathbb{R}^d. \tag{12}
\end{align*}
\]

Here \(\tilde{v}_t = v_t + \text{unbiased noise}\), and the noisy gradient \(\nabla^2_X f = \nabla^2_X f + \text{unbiased noise}\). Let us pick the noise in the \(x\)–variable iteration to be independent of the noise chosen in the \(v\)–variable iteration. Following the general scheme in [20], [27], [7], [5], [24], [21] this leads to a counterpart of the system (4) that has non–degenerate random noise as follows

\[
\begin{align*}
dx(t) &= \sqrt{s} v(t) dt + \sqrt{s} \sigma_1(x(t), v(t)) dW_t^1, \ x(0) \in \mathbb{R}^d ; \\
dv(t) &= (-\alpha \sqrt{s} v(t) - \sqrt{s} \nabla^2_X f(x(t))) dt + \sqrt{s} \sigma_2(x(t), v(t)) dW_t^2, \ v(0) \in \mathbb{R}^d. \tag{13}
\end{align*}
\]

Here \(W_t^1\) and \(W_t^2\) are two independent \(d\)–dimensional Brownian motions, and \(\sigma_1(x(t), v(t)), \sigma_2(x(t), v(t))\) come from the correlation in the noises within \(\tilde{v}_t\) and \(\nabla^2_X f(x)\). Let us set \(\varepsilon = \sqrt{s}\). Then we have \(x(t/\sqrt{s}, v(t/\sqrt{s})) = (X^\varepsilon(t), V^\varepsilon(t))\), in which \((X^\varepsilon(t), V^\varepsilon(t))\) is the process [33]. Therefore, in terms of optimization, our major result can be formulated roughly as the following:
Convergence of the diffusion limit of the stochastic heavy–ball method. Starting from a certain initial condition \( X(0) \in \mathbb{R}^d \), as \( s \to 0 \), the process \( x(t) \) in (13) converges to local minimizers of the objective function \( f(x) \) in (8) after time \( \lesssim C \sqrt{s^{-1}} \ln(s^{-1}) \). Here the constant \( C > 0 \) depends on the function \( f(x) \) as well as the friction constant \( \alpha > 0 \).

The paper is organized as follows. In Section 2 we establish the basic Hamiltonian set–up of the problem. In Section 3 we consider exit behavior of the randomly perturbed process near one specific saddle point. In Section 4 we consider global convergence and analyze the case when there is a chain of saddle points, and henceforce obtain our major result there. In Section 5 we address, as a side remark, the oscillation behavior of the perturbed process near a local minimum point. Finally, Section 6 is dedicated to some further discussions.

### 2 Setup: Hamiltonian formulation of the nonlinear dissipative oscillator.

Throughout the text, \( \nabla_X \) or \( \nabla_V \) will denote gradient with respect to \( X \) or \( V \) variable, and \( \nabla_X^2, \nabla_V^2 \), etc. are denoted similarly; \( \nabla \) or \( D \) will denote gradient with respect to \( (X, V) \) variable, and \( \nabla^2, D^2 \), etc. are denoted similarly; if we use \( \frac{\partial}{\partial X} \) or \( \frac{\partial}{\partial V} \), then it means the corresponding gradients with respect to \( X \) or \( V \) variable, and \( \frac{\partial^2}{\partial X^2} \) or \( \frac{\partial^2}{\partial V^2} \), etc. are defined similarly. Standard Euclidean norms in \( \mathbb{R}^d \) are defined either by \( \| \cdot \|_{\mathbb{R}^d} \) or \( | \cdot | \).

We consider a Hamiltonian system

\[
\begin{aligned}
    dX(t) &= V(t) dt, \quad X(0) \in \mathbb{R}^d; \\
    dV(t) &= -\nabla_X f(X(t)) dt, \quad V(0) \in \mathbb{R}^d.
\end{aligned}
\]  

(14)

We add friction to (14), and we get

\[
\begin{aligned}
    dX(t) &= V(t) dt, \quad X(0) \in \mathbb{R}^d; \\
    dV(t) &= (-\alpha V(t) - \nabla_X f(X(t))) dt, \quad V(0) \in \mathbb{R}^d.
\end{aligned}
\]  

(15)

We will denote the flow map of (15) to be \( S^t \), so that \( (X(t), V(t)) = S^t(X(0), V(0)) \).

Let us define the Hamiltonian

\[
H(X, V) = \frac{1}{2} V^2 + f(X),
\]

so that

\[
\frac{\partial H}{\partial V} = V \quad \text{and} \quad \frac{\partial H}{\partial X} = \nabla_X f(X).
\]
Before we continue with the analysis, we will make some additional structural assumptions on the function $f(x)$.

**Definition 2.1.** A smooth function $f : \mathbb{R}^d \to \mathbb{R}$ is a Morse function if it has all critical points being non-degenerate, i.e., the Hessian $\nabla^2 f(x_0)$ of $f(\bullet)$ at any critical point $x_0$ is non-degenerate. This implies that all eigenvalues $\lambda_1 \leq \ldots \leq \lambda_d$ of the Hessian matrix $\nabla^2 f(x_0)$ are real and nonzero.

Morse functions admit local quadratic re-parametrization at each critical point, which is the content of the so-called Morse Lemma [29, Lemma 2.2]. To ensure that the perturbation helps the process $X(t)$ escape from saddle points, we introduce the following “strict saddle property” (compare with [18], [33]) as follows.

**Definition 2.2 (strict saddle property).** Given fixed $\gamma_1 > 0$ and $\gamma_2 > 0$, we say a Morse function $f$ defined on $\mathbb{R}^d$ satisfies the “strict saddle property” if each point $x \in \mathbb{R}^d$ belongs to one of the following: (i) $|\nabla f(x)| \geq \gamma_2 > 0$; (ii) $|\nabla f(x)| \leq \gamma_2$ and $\lambda_{\min}(\nabla^2 f(x)) \leq -\gamma_1 < 0$; (iii) $|\nabla f(x)| \leq \gamma_2$ and $\lambda_{\min}(\nabla^2 f(x)) \geq \gamma_1 > 0$.

Here $\lambda_{\min}(\nabla^2 f(x))$ is the smallest eigenvalue of the Hessian matrix $\nabla^2 f(x)$ at point $x$.

We will call a saddle point $x \in \mathbb{R}^d$ of the function $f$ a “strict saddle” if Definition 2.2 (ii) holds at $x$. Thus a Morse function $f$ that satisfies the strict saddle property has all its saddle points that are strict saddle points.

For the sake of proof, it is natural to assume that all eigenvalues of the Hessian $\nabla^2 f$ at critical points are uniformly bounded away from $0$. This leads to our new notion of “strong saddle property” as follows.

**Definition 2.3 (strong saddle property).** Let the Morse function $f(\bullet)$ satisfy the strict saddle property with parameters $\gamma_1 > 0$ and $\gamma_2 > 0$. We say the Morse function $f(\bullet)$ satisfy the “strong saddle property” if for some $\gamma_3 > 0$ and any $x \in \mathbb{R}^d$ such that $\nabla f(x) = 0$, all eigenvalues $\lambda_i, i = 1, 2, \ldots, d$ of the Hessian $\nabla^2 f(x)$ at $x$ satisfying (ii) but not (i) in Definition 2.2 are bounded away from $\gamma_3 > 0$ in absolute value, i.e., $|\lambda_i| \geq \gamma_3 > 0$ for any $1 \leq i \leq d$.

We will call a saddle point $x \in \mathbb{R}^d$ of the function $f$ a “strong saddle” if Definition 2.3 holds at $x$. Thus a Morse function $f$ that satisfies the strong saddle property has all its saddle points that are strong saddle points. Throughout this paper we will work under Definition 2.3 for the potential function $f$.

Let $(X_0, V_0)$ be a critical point of $H$. The above implies that $V_0 = 0$ and $X_0$ is a critical point of $f(X)$. From (16) we have a formal expansion

\[ H(X, V) = H(X_0, 0) + \frac{1}{2}V^2 + (X - X_0)^T \nabla^2_X f(X)(X - X_0) + \text{higher order terms}. \]
If $X_0$ is a local minimum point of $f(X)$, then (17) tells us that $(X_0, 0)$ is a local minimum point of $H(X, V)$. If $X_0$ is a local maximum or saddle point of $f(X)$, then (17) tells us that $(X_0, 0)$ is a saddle point of $f(X)$. In particular, by our strong saddle property assumption for the potential function $f$, we know that all saddle points of the Hamiltonian function $H(X, V)$ are strong saddle points.

To summarize, we have the following.

**Lemma 2.1.** Critical points of system (15) all lie on the $X$-axis. These critical points have $(X, V)$ coordinates $(X, 0)$ in which $X$ is a critical point of $f(X)$. If $X$ is a local minimum point of the function $f(X)$, then $(X, 0)$ is a local minimum point of the Hamiltonian $H$; If $X$ is a local maximum or saddle point of the function $f(X)$, then $(X, 0)$ is a saddle point of the Hamiltonian $H$. All saddle points of the Hamiltonian function $H(X, V)$ are strong saddle points under Definition 2.3.

Define the column vector $Y(t) = \left( X(t), V(t) \right) \in \mathbb{R}^d \times \mathbb{R}^d$ and the skew gradient

$$\nabla \perp H(X, V) = \begin{pmatrix} \frac{\partial H}{\partial V} \\ -\frac{\partial H}{\partial X} \end{pmatrix}. \tag{18}$$

Then the system (15) can be written in a standard Hamiltonian form with friction

$$\frac{dY}{dt} = \nabla \perp H(Y(t)) + b(Y(t)), Y(0) = Y_0 \in \mathbb{R}^d \times \mathbb{R}^d. \tag{19}$$

Here $b(X, V) = (0, -\alpha V)^T$ is the friction vector field. Notice that $\text{div}b = -\alpha < 0$ corresponds to the classical friction case. Furthermore, $b(X, V)$ is a gradient field in the sense that we can write $b(X, V) = -\nabla B(X, V)$, in which $B(X, V) = \frac{1}{2} \alpha V^2$.

Let us introduce the standard symplectic matrix

$$J = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}. \tag{20}$$

The standard symplectic matrix $J$ has the property that $J^2 = -I_{2d}, J^T = -J = J^{-1}$.

The Hamiltonian vector field $\nabla \perp H(Y)$ can be written as $\nabla \perp H(Y) = J \nabla H(Y)$. The local behavior of this vector field near its critical point is characterized by the “skew Hessian matrix”

$$\nabla J \nabla H(X, V) = \begin{pmatrix} \frac{\partial^2 H}{\partial X \partial V} & \frac{\partial^2 H}{\partial V \partial X} \\ -\frac{\partial^2 H}{\partial X^2} & -\frac{\partial^2 H}{\partial X \partial V} \end{pmatrix} = \begin{pmatrix} 0 & I_d \\ -\nabla^2_X f(X) & 0 \end{pmatrix}, \tag{21}$$
in the sense that we have the expansion around a critical point \((X_0, V_0)\) of \(\nabla_{\perp} H\):

\[
\nabla_{\perp} H(X, V) = \nabla J \nabla H(X_0, V_0) \begin{pmatrix} X - X_0 \\ V - V_0 \end{pmatrix} + \psi(X - X_0, V - V_0)(|X - X_0|^2_{\mathbb{R}^d} + |V - V_0|^2_{\mathbb{R}^d}) ,
\]

where \(\psi(X, V)\) is some bounded smooth vector–function in the variables \((X, V)\).

Let us also calculate the Hessian matrix of the Hamiltonian function \(H\) as

\[
\nabla^2 H(X, V) = \begin{pmatrix}
\frac{\partial^2 H}{\partial X^2} & \frac{\partial^2 H}{\partial X \partial V} \\
\frac{\partial^2 H}{\partial X \partial V} & \frac{\partial^2 H}{\partial V^2}
\end{pmatrix} = \begin{pmatrix}
\nabla^2_X f(X) & 0 \\
0 & I_d
\end{pmatrix} .
\]

From (21) and (23), taking into account that \(\nabla^2_X f\) is a symmetric matrix, we see that we have the relation

\[
J \nabla^2 H(X, V) = \nabla J \nabla H(X, V) .
\]

Since \(b(X, V) = (0, -\alpha V)^T\), we have \(b(X, V) = 0\) when \(V = 0\), i.e., the following

**Lemma 2.2.** The friction term \(b(X, V)\) vanishes on the \(X\)-axis.

Let us denote the matrix

\[
I^0 = \begin{pmatrix} 0 & 0 \\ 0 & I_d \end{pmatrix} .
\]

Then we can write the friction term \(b(X, V)\) as

\[
b(Y) = -\alpha I^0 Y
\]

where \(Y = \begin{pmatrix} X \\ V \end{pmatrix} .\)

Let us consider a small random perturbation of (15):

\[
\begin{cases}
dX^\varepsilon(t) = V^\varepsilon(t) dt + \varepsilon \sigma_1(X^\varepsilon(t), V^\varepsilon(t)) dW_1^\varepsilon , \quad X^\varepsilon(0) \in \mathbb{R}^d \\
dV^\varepsilon(t) = (-\alpha V^\varepsilon(t) - \nabla_X f(X^\varepsilon(t))) dt + \varepsilon \sigma_2(X^\varepsilon(t), V^\varepsilon(t)) dW_2^\varepsilon , \quad V^\varepsilon(0) \in \mathbb{R}^d
\end{cases}
\]

Here \(\varepsilon > 0\) is the intensity of the noise; \(W_1^\varepsilon\) and \(W_2^\varepsilon\) are two independent standard Brownian motions in \(\mathbb{R}^d\); \(\sigma_1(X, V)\) and \(\sigma_2(X, V)\) are two matrices in \(\mathbb{R}^d\) such that \(a_i(X, V) \equiv \sigma_i(X, V) \sigma_i^T(X, V)\) are uniformly positive–definite for \(i = 1, 2\).

Set \(Y^\varepsilon(t) = \begin{pmatrix} X^\varepsilon(t) \\ V^\varepsilon(t) \end{pmatrix} .\) In standard Hamiltonian formulation (19), one can introduce the non–degenerate matrix \(\Sigma(Y) = \begin{pmatrix} \sigma_1(Y) & 0 \\ 0 & \sigma_2(Y) \end{pmatrix} ,\) so that (27) can be written as
\[ dY^\varepsilon(t) = [\nabla^\perp H(Y^\varepsilon(t)) + b(Y^\varepsilon(t))]dt + \varepsilon \Sigma(Y^\varepsilon(t))dW_t, \quad Y^\varepsilon(0) \in \mathbb{R}^{2d}. \]  

System (27) is standard formulation of a randomly perturbed Hamiltonian system with dissipation (see [16, Chapter 8], [17]).

3 Exit behavior near one specific saddle point.

The central problem discussed in this work is about how the dynamics of (28) escapes from a strict saddle point of the objective function \( f(x) \). In this section, we would like to study the exit behavior of system (27) near one specific saddle point. Some preliminary analysis and arguments are already found in [26].

Mathematical analysis of noise–induced exit near a saddle point has been long investigated in the stochastic analysis community. Starting from early works in [23], some most notable works along this thread are [28], [10], [4], among many other literature dedicated to this topic. Our approach here follows mainly the method developed in [4], and works in the framework of our previous work [19]. The major difference here is that we are working under the particular features of Hamiltonian dynamics.

In the case when the friction term \( b(X,V) = (0, -\alpha V) \) is small in the parameter \( \varepsilon > 0 \), the general program of dealing with a randomly perturbed Hamiltonian system such as (28) is to consider the evolution of the Hamiltonian \( Z^\varepsilon(t) = H(Y^\varepsilon(t)) \). In general, for fixed \( \varepsilon > 0 \), the process \( Z^\varepsilon(t) \) will not be a Markov process. However, if we set \( \varepsilon \to 0 \) and consider the weak limit of \( Z^\varepsilon(t) \) to a process \( Z^0(t) \) on the metric graph \( \Gamma \) of the Hamiltonian \( H \), we get limiting characterization of the dynamics of \( H(Y^\varepsilon(t)) \) in small \( \varepsilon > 0 \) regime (see [15]). In the case of dimension \( d = 1 \), this problem is completely solved in [8], [12], [15], [16, Chapter 8]. In higher dimensions, more interesting phenomena occur as a result of the different possibilities of the Morse index at the saddle (see [14], [17, Chapter 9]).

In our case, the friction term \( b(X,V) = (0, -\alpha V) \) has a magnitude that is moderate when compared to the Hamiltonian flow \( \nabla^\perp H \), and it does not go to zero together with the small parameter \( \varepsilon > 0 \). In this case, averaging principle does not apply. Rather, one has to carefully analyze the behavior of the process (28) near the saddle point.

According to (22), the linear part of the Hamiltonian vector field \( \nabla^\perp H = J\nabla H \) in (28) should presumably be given by \( \nabla J \nabla H(O) \) as in (21). By (24), we know that \( \nabla J \nabla H(O) = J\nabla^2 H(O) \), which is not necessarily a symmetric matrix, and in fact it will even produce complex eigenvalues that may have either or both nonzero real and imaginary parts (think of the symplectic matrix \( J \) itself, the eigenvalues are \( \pm i \)). In this case, using (22), we know that the linear part of the Hamiltonian vector field \( \nabla^\perp H \) with
dissipation near the saddle point \( O \) is given by the matrix \( A = \nabla^2 H(O) \). Here \( Q \) is the symmetric Hessian matrix \( Q = \nabla^2 H(O) \).

Let us also note that in [4] the author has to assume that all the eigenvalues of \( JQ \) are real and simple, and in our case it is natural to expect complex eigenvalues. This will be a difference between our case and the case considered in [4], where all eigenvalues of the matrix \( A \) are real and simple. Due to this reason, the linearization idea used in equation (8.1) of [4] needs some further investigation, which we postpone the discussion to the next section. However, even in the case of complex eigenvalues (with all eigenvalues having non–zero real parts), the argument of [23] still works, but is only able to help us analyze the exit behavior near one specific saddle point. To apply this argument, in our case for the system (28), we have to consider (technically) the eigenvalues and eigenvectors of the matrix \( A = JQ - \alpha I^0 \).

Let us first imagine that \( I^0 \) is replaced by a full rank identity matrix \( I = I_{2d} \). Then all complex eigenvalues of \( A = JQ - \alpha I \) differ from the ones of \( JQ \) by a shift \(-\alpha\) in their real parts. As a consequence, if \( \alpha > 0 \) is very large, it is possible that none of the complex eigenvalues of \( A \) have positive real part. In this case, the exit to the boundary of a neighborhood of the saddle point \( O \) takes exponentially long time due to large deviation effects (see [16, Chapter 4]). Figure 2 shows the phase portrait when \( d = 1 \). Geometrically, by looking at Figure 2 we see that the projection of the unstable manifold of \( \nabla^2 H \) to the \( X \) direction will always create an unstable component pointing from the saddle \( O \). Given the half–rank matrix \( I^0 \), we see that the friction term \( b(Y) = -\alpha I^0 Y \) has zero projection to the \( X \)–direction. Such intuitive reasoning can be improved into the following technical result in linear algebra.

Let the saddle point \( O \) be \( O = (X_0, 0) \). Since \( \nabla^2_X f(X_0) \) is a symmetric matrix, we can find an orthonormal basis \( \xi_1, ..., \xi_d \) (viewed as column vectors) in \( \mathbb{R}^d \) such that \( \nabla^2_X f(O) \xi_i = \lambda_i \xi_i \). By Lemma 2.1 \( X_0 \) is a local maximum or saddle point of \( f(x) \), and thus we see that without loss of generality we can assume that

\[
\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k < 0 < \lambda_{k+1} \leq ... \leq \lambda_d
\]

for some \( 1 \leq k \leq d \).

Set

\[
\mu_0 = \frac{-\alpha + \sqrt{\alpha^2 - 4\lambda_1}}{2} > 0 .
\]

Proposition 3.1. There exist an invertible \( 2d \times 2d \) matrix \( P \) with real or complex terms such that for the matrix \( A = JQ - \alpha I^0 \) we have

\[
P^{-1}AP = \text{diag}(A_1, ..., A_d)
\]

with each \( A_i \) being a \( 2 \times 2 \) block matrix. Moreover, the invertible matrix \( P \) can be taken of the form

\[
P = (u_1^+, u_1^-, ..., u_d^+, u_d^-)
\]
Figure 2: Hamiltonian flow when $d = 1$.

in which $u_i^\pm$ are $2d$-dimensional real or complex vectors for $i = 1, 2, ..., d$.

Here for some integers $l, m$ such that $1 \leq k \leq l \leq m \leq d$ we have

1. For $i = 1, ..., k$, we have $\lambda_i < 0$, $A_i = \begin{pmatrix} \mu_i^+ & 0 \\ 0 & \mu_i^- \end{pmatrix}$, $\mu_i^\pm = -\alpha \pm \sqrt{\alpha^2 - 4\lambda_i}$, $\mu_i^\pm$ are real, $\mu_i^+ > 0 > \mu_i^-$, and $u_i^\pm = \begin{pmatrix} \xi_i \\ \mu_i^\pm \xi_i \end{pmatrix}$ are real $2d$-dimensional vectors;

2. For $i = k+1, ..., l$, we have $0 < \lambda_i < \alpha^2 / 4$, $A_i = \begin{pmatrix} \mu_i^+ & 0 \\ 0 & \mu_i^- \end{pmatrix}$, $\mu_i^\pm$ are real, $0 > \mu_i^+ > \mu_i^-$, $\mu_i^\pm = -\alpha \pm \sqrt{\alpha^2 - 4\lambda_i}$, and $u_i^\pm = \begin{pmatrix} \xi_i \\ \mu_i^\pm \xi_i \end{pmatrix}$ are real $2d$-dimensional vectors;

3. For $i = l+1, ..., m$, we have $\lambda_i = \alpha^2 / 4$, $A_i = \begin{pmatrix} -\alpha & 0 \\ 0 & -\alpha \end{pmatrix}$, $\mu_i^+ = \mu_i^- = -\alpha / 2$, and

   $u_i^+ = \begin{pmatrix} \xi_i \\ -\alpha \xi_i \end{pmatrix}$, $u_i^- = a_i$ are two linearly independent real $2d$-dimensional vectors,

   in which $a_i$ satisfies $(A - \lambda_i I)a_i = \begin{pmatrix} \xi_i \\ -\alpha \xi_i \end{pmatrix}$;

4. For $i = m+1, ..., d$, we have $0 < \alpha^2 / 4 < \lambda_i$, $A_i = \begin{pmatrix} \mu_i^+ & 0 \\ 0 & \mu_i^- \end{pmatrix}$, $\mu_i^\pm$ are complex
and are of the form $\mu_i^\pm = -\alpha/2 \pm \sqrt{\lambda_i - \alpha^2}/2$, and $u_i^\pm = \begin{pmatrix} \xi_i \\ \mu_i^\pm \xi_i \end{pmatrix}$ are complex 2d–dimensional vectors.

**Proof.** Consider the matrix

$$A = JQ - \alpha I^0 = \begin{pmatrix} 0 & I_d \\ -\nabla^2_X f(X_0) & -\alpha I_d \end{pmatrix}.$$ 

Suppose an eigenvector of the matrix $A$ has the form $\begin{pmatrix} \xi \\ v \end{pmatrix}$ with eigenvalue $\mu$ ($\mu$ may be complex). Here $\xi$ and $v$ are two column vectors in $\mathbb{R}^d$. Then we have

$$A \begin{pmatrix} \xi \\ v \end{pmatrix} = \begin{pmatrix} 0 & I_d \\ -\nabla^2_X f(X_0) & -\alpha I_d \end{pmatrix} \begin{pmatrix} \xi \\ v \end{pmatrix} = \begin{pmatrix} \xi \\ -\nabla^2_X f(X_0)\xi - \alpha v \end{pmatrix} = \mu \begin{pmatrix} \xi \\ v \end{pmatrix}.$$ 

This implies that $v = \mu \xi$ and $-\nabla^2_X f(X_0)\xi = (\mu + \alpha)v = (\mu + \alpha)\mu \xi$. Therefore $\xi$ must be an eigenvector of $\nabla^2_X f(X_0)$ with eigenvalue $\lambda = (\mu + \alpha)\mu$.

Conversely, if $\xi$ is an eigenvector of $\nabla^2_X f(X_0)$ with eigenvalue $\lambda$, say $\nabla^2_X f(X_0)\xi = \lambda \xi$, then $\begin{pmatrix} \xi \\ \mu \xi \end{pmatrix}$ is an eigenvector of $A = JQ - \alpha I^0$ with $A \begin{pmatrix} \xi \\ \mu \xi \end{pmatrix} = \mu \begin{pmatrix} \xi \\ \mu \xi \end{pmatrix}$ and the eigenvalue $\mu$ satisfies $\mu(\mu + \alpha) = \lambda$.

From the above we see that there is a correspondence between eigenvectors/eigenvalues of $\nabla^2_X f(X_0)$ and eigenvectors/eigenvalues of $A = JQ - \alpha I^0$. In fact, each eigenvector $\xi_i$ ($i = 1, 2, ..., d$) of $\nabla^2_X f(X_0)$ with eigenvalue $\lambda_i$ corresponds to two eigenvectors $\begin{pmatrix} \xi \\ \mu_i^\pm \xi \end{pmatrix}$ and $\begin{pmatrix} \xi \\ \mu_i^- \xi \end{pmatrix}$ of $A = JQ - \alpha I^0$ with eigenvalues $\mu_i^+$ and $\mu_i^-$.3

The two eigenvalues $\mu_i^\pm$ are the two roots of the equation

$$\mu_i^2 + \alpha \mu_i + \lambda_i = 0,$$ 

so that

$$\mu_i^\pm = -\frac{\alpha \pm \sqrt{\alpha^2 - 4 \lambda_i}}{2}.$$ 

(31)

Let us analyze the eigenvalues $\mu_i^\pm$ from (31). Recall that we have $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_k < 0 < \lambda_{k+1} \leq ... \leq \lambda_d$. We discuss the following cases:

1. For $i = 1, ..., k$, we have $\lambda_i < 0$. The two eigenvalues $\mu_i^\pm$ are real and $\mu_i^+ > 0 > \mu_i^-$.  

3It can happen that $\mu_i^+ = \mu_i^-$, and in that case the two eigenvectors may alternatively be replaced by a two–dimensional invariant subspace. We will discuss this case later in this proof.
2. For \( i = k + 1, \ldots, l \), we have \( 0 < \lambda_i < \frac{\alpha^2}{4} \) and \( \mu_i^\pm \) are real and \( 0 > \mu_i^+ > \mu_i^- \).

3. For \( i = l + 1, \ldots, m \), we have \( \lambda_i = \frac{\alpha^2}{4} \) and \( \mu_i^+ = \mu_i^- = -\frac{\alpha}{2} \).

4. For \( i = m + 1, \ldots, d \), we have \( 0 < \frac{\alpha^2}{4} < \lambda_i \), and \( \mu_i^\pm \) are complex and are of the form

\[
\mu_i^\pm = -\frac{\alpha}{2} \pm \frac{\sqrt{4\lambda_i - \alpha^2}}{2} i;
\]

In summary, the only eigenvalues of \( A = JQ - \alpha I^0 \) that have positive real parts are \( \mu_{1+}^+, \ldots, \mu_k^+ \), and all the other eigenvalues of \( A = JQ - \alpha I^0 \) have negative real parts.

Recall that \( \lambda_1 < 0 \) is the negative eigenvalue with largest absolute value among all eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k < 0 \). If we set \( \mu_0 = \frac{-\alpha + \sqrt{\alpha^2 - 4\lambda_1}}{2} \), then we have

\[
\max_{i=1,2,\ldots,d} \text{Re} \mu_i^\pm = \mu_0 > 0.
\]

Notice that when \( \alpha^2 \neq 4\lambda_i \), the two eigenvalues \( \mu_i^+ \neq \mu_i^- \), and thus the eigenvectors

\[
\begin{pmatrix}
\xi_i \\
\mu_i^+ \xi_i
\end{pmatrix}, \quad
\begin{pmatrix}
\xi_i \\
\mu_i^- \xi_i
\end{pmatrix}
\]

are linearly independent. In fact, if we have

\[
c_i^+ \begin{pmatrix}
\xi_i \\
\mu_i^+ \xi_i
\end{pmatrix} + c_i^- \begin{pmatrix}
\xi_i \\
\mu_i^- \xi_i
\end{pmatrix} = 0,
\]

then we have

\[
\begin{cases}
c_i^+ + c_i^- = 0, \\
c_i^+ \mu_i^+ + c_i^- \mu_i^- = 0,
\end{cases}
\]

from which we derive \( c_i^+ = c_i^- = 0 \) under \( \mu_i^+ \neq \mu_i^- \). In this case the two dimensional linear invariant subspace \( V_i = \text{span} \begin{pmatrix}
\xi_i \\
\mu_i^+ \xi_i
\end{pmatrix}, \begin{pmatrix}
\xi_i \\
\mu_i^- \xi_i
\end{pmatrix}\) splits further into two independent 1–dimensional subspaces \( V_i = V_i^+ \oplus V_i^- \), with \( V_i^+ = \text{span} \begin{pmatrix}
\xi_i \\
\mu_i^+ \xi_i
\end{pmatrix}\). The corresponding Jordan block for the invariant subspace \( V_i \) is diagonal

\[
\begin{pmatrix}
\mu_i^+ & 0 \\
0 & \mu_i^-
\end{pmatrix}.
\]

If for some \( i = k + 1, \ldots, d \) we have \( \alpha^2 = 4\lambda_i \), then from the eigenvector

\[
\begin{pmatrix}
\xi_i \\
-\frac{\alpha}{2} \xi_i
\end{pmatrix}
\]

and the real eigenvalue \( \mu_i^\pm = -\frac{\alpha}{2} \) of \( A = JQ - \alpha I^0 \) we solve the generalized eigenvalue/eigenvector equation

\[
(A - \lambda_i I)\mathbf{a}_i = \begin{pmatrix}
\xi_i \\
-\frac{\alpha}{2} \xi_i
\end{pmatrix},
\]

15
and we obtain another vector $a_i \in \mathbb{R}^{2d}$ independent of $\left(\begin{array}{c} \xi_i \\ -\frac{\alpha}{2} \xi_i \end{array}\right)$. The two-dimensional linear subspace $V_i = \text{span}\left\{\left(\begin{array}{c} \xi_i \\ -\frac{\alpha}{2} \xi_i \end{array}\right), a_i\right\}$ forms an invariant subspace of the matrix $A = JQ - \alpha I^0$ for the eigenvalue $\mu_i^\pm = -\frac{\alpha}{2}$, which corresponds to the Jordan block $\left(\begin{array}{cc} -\frac{\alpha}{2} & 1 \\ 0 & -\frac{\alpha}{2} \end{array}\right)$.

Notice that if for some $i \neq j$ and $i,j \in \{k + 1, \ldots, d\}$ we have $\alpha^2 = 4\lambda_i = 4\lambda_j$, then the eigenvectors $\left(\begin{array}{c} \xi_i \\ -\frac{\alpha}{2} \xi_i \end{array}\right)$ and $\left(\begin{array}{c} \xi_j \\ -\frac{\alpha}{2} \xi_j \end{array}\right)$ are linearly independent since $\xi_i$ and $\xi_j$ are linearly independent. This implies that the invariant subspace for the matrix $A = JQ - \alpha I^0$ corresponding to the eigenvalue $-\frac{\alpha}{2}$ with a possible multiplicity splits into two-dimensional invariant subspaces as described in the previous paragraph.

Summarizing the above discussion, we see that we can find an invertible matrix of the form

$$P = (u_1^+, u_1^-, \ldots, u_k^+, u_k^-, u_{k+1}^+, u_{k+1}^-, \ldots, u_i^+, u_i^-, \ldots, u_{m+1}^+, u_{m+1}^-, \ldots, u_d^+, u_d^-)$$

such that $u_i^\pm = \left(\begin{array}{c} \xi_i \\ \mu_i^\pm \xi_i \end{array}\right)$ for $i = 1, 2, \ldots, k$ and $i = k + 1, \ldots, l$. Here for $i = 1, 2, \ldots, k$ we have $\lambda_i < 0$, and $\mu_i^\pm$ are chosen according to case 1 discussed above; for $i = k + 1, \ldots, l$ we have $0 < \lambda_i < \alpha^2/4$, and $\mu_i^\pm$ are chosen according to case 2 discussed above. When $i = l + 1, \ldots, m$ we have $0 < \lambda_i = \alpha^2/4$, and in this case $\mu_i^\pm = -\alpha/2$, so that $u_i^\pm = \left(\begin{array}{c} \xi_i \\ -\frac{\alpha}{2} \xi_i \end{array}\right)$, $u_i^- = a_i$ are chosen according to case 3 discussed above. When $i = m + 1, \ldots, d$ we have $0 < \alpha^2/4 < \lambda_i$ and in this case $u_i^\pm = \left(\begin{array}{c} \xi_i \\ \mu_i^\pm \xi_i \end{array}\right)$ and $\mu_i^\pm$ are chosen according to case 4 discussed above.

Finally by picking the matrix $P$ as above we have

$$P^{-1}AP = \text{diag}(A_1, \ldots, A_d)$$

with each $A_i$ being a $2 \times 2$ block matrix, and $A_i = \left(\begin{array}{cc} \mu_i^+ & 0 \\ 0 & \mu_i^- \end{array}\right)$ for $i = 1, 2, \ldots, k, k + 1, \ldots, l, m + 1, \ldots, d$; $A_i = \left(\begin{array}{cc} -\alpha/2 & 1 \\ 0 & -\alpha/2 \end{array}\right)$ for $i = l + 1, \ldots, m$.

\[\square\]

Remarks.

1. No matter how large $\alpha$ is, the eigenvalues $\mu_1^+, \ldots, \mu_k^+$ of $A$ are always real and positive. This means that the number of unstable directions for the Hamiltonian
field $\nabla^\perp H$ is always equal to the number of unstable directions of the Hessian matrix $\nabla^2_X f(X_0)$. In particular, it means that the “partially damped” friction $b(Y) = -\alpha I^0 Y$ cannot kill all unstable directions of $\nabla^\perp H$, no matter how large $\alpha > 0$ is. The latter fact can be very intuitively seen from the following physical argument. In fact, the matrix $JQ$ can be thought of as the infinitesimal local generator of the Hamiltonian field $\nabla^\perp H$ near the saddle $O = (X_0, 0)$. The friction $b(Y) = -\alpha I^0 Y$ is in the downward $V$–direction and it has 0 projection onto the $X$–direction (Figure 2). Therefore when $\alpha > 0$ is large, if the friction $b(Y) = -\alpha I^0 Y$ kills all unstable directions of $\nabla^\perp H$, it must happen that the latter direction is pointing only to the $V$–direction. Keeping in mind that the Hamiltonian $H$ is preserved under the flow $\nabla^\perp H$, this implies that locally the function $H(X, V) = \frac{1}{2} V^2 + f(X)$ is a constant as $V$ increases or decreases. If it were true that the unstable direction of $\nabla^\perp H$ at the saddle point $O$ is pointing only to the $V$–direction, then $X$ remains locally to be a constant. This contradicts with the conservation of $H(X, V) = \frac{1}{2} V^2 + f(X)$ along the Hamiltonian flow.

2. Imagine $\alpha \to \infty$, then $\mu_0$ as in [30] has limit $\mu_0 \to 0^+$. This can also be seen from the above physical argument, because as the friction becomes very large, it kills the $V$–component of all possible unstable directions of $\nabla^\perp H$, but it ignores the $X$–direction. In the limit as $\alpha \to \infty$, the $X$–components of the stable and unstable directions of $\nabla^\perp H$ at the saddle point $O = (X_0, 0)$ cancel with each other, since they are both given by unstable directions of $\nabla_X f$ at $X_0$ (Figure 2).

3. It is also very interesting to look at the case of the differential equation for the original Nesterov’s method [9]. We see that in this case, the friction coefficient $\alpha$ becomes time–dependent $\alpha(t) = \frac{3}{t}$. Imagine that we add non–denegerate noise to [9]. Let us start the noisy Nesterov’s method at an initial condition that lies in a neighborhood of a saddle point. At the beginning, when $t$ is small, we are in the asymptotic regime in which $\alpha > 0$ is large, so that the process starts to move according to the unstable direction pointed by the eigenvector that corresponds to $\mu^+_1 > 0$ in Proposition 3.1. However, as will be shown in Theorem 3.2, this exit takes logarithmically long time, so that during this evolution of time $\alpha(t) = \frac{3}{t}$ starts to decay. This means that the process is always “altering” its exit direction that is pointed by the eigenvector corresponding to the eigenvalue $\mu^+_1$. However, since $\alpha$ is decaying, it is expected that we are guaranteed with logarithmic exit behavior (see Theorem 3.2).

4. In [30], the authors obtain in their Theorem 4 and Corollary 5 some linear algebra results that are similar to our Proposition 3.1. Yet there are several differences. First, the set–up of [30] does not make use of Hamiltonian structure, and we have
revealed the Hamiltonian structure behind the heavy-ball scheme. Second, we are taking advantage of the continuous approximation, so that we can make use of very delicate results of Kifer about exit behavior (see [23] and Theorem 3.2). Third, the analysis of [30] focuses more on showing that the heavy ball method does not converge to saddle point, which is originated from [23]. In our case, we follow our previous papers (see [20], [19]) and reduce the problem to the analysis of the dynamics given by stochastic differential equations with small diffusion term. Our method in a sense is an attempt to investigate the question about exit from saddle points raised at the end of [30].

Let \( O = (X_0, 0) \) be a saddle point of the Hamiltonian vector field \( \nabla H \), in which \( X_0 \) is a local maximum or saddle point of \( f(x) \). Let \( G \) be an open neighborhood in \( \mathbb{R}^{2d} \) of the saddle point \( O \), such that \( O \) is the only critical point of the Hamiltonian flow \( \nabla H \) inside \( G \) (see Figure 3). Let the process \( Y^\varepsilon_t = (X^\varepsilon(t), V^\varepsilon(t))^T \) defined in (27) start from initial condition \( Y^\varepsilon_0 = (x, v) \in G \). Let

\[
\tau^\varepsilon_{(x,v)} = \inf \{ t > 0 : Y_0 = (x, v), Y^\varepsilon_t \in \partial G \}. \tag{33}
\]

Let us also consider the corresponding deterministic Hamiltonian flow with friction \([15]\). Let the flow map of \([15]\) be defined as \( S^t \). The author of [23] introduced a decomposition

\[
G \cup \partial G = 0 \cup A_1 \cup A_2 \cup A_3,
\]

where \( A_1 \) is a set of points \((x,v) \in G \cup \partial G\) such that if \((x,v) \in A_1\) then \( S^u(x,v) \in G \) for \( u > s \) and \( S^u(x,v) \notin G \cup \partial G \) if \( u \leq s \) for some \( s = s(x,v) \leq 0 \) and \( S^t(x,v) \to 0 \) as \( t \to \infty \); \( A_2 \) is a set of points \((x,v) \in G \cup \partial G\) such that if \((x,v) \in A_2\) then \( S^u(x,v) \in G \) for \( u < s \) and \( S^u(x,v) \notin G \cup \partial G \) if \( u > s \) for some \( s = s(x,v) \geq 0 \) and \( S^t(x,v) \to 0 \) as \( t \to -\infty \); \( A_3 \) is a set of points \( x \in G \cup \partial G\) such that if \((x,v) \in A_3\) then \( S^u(x,v) \in G \) provided \( s_1 < u < s_2 \) and \( S^u(x,v) \notin G \cup \partial G \) if either \( u > s_2 \) or \( u < s_1 \) for some \( s_1 = s_1(x,v) \leq 0 \) and \( s_2 = s_2(x,v) \geq 0 \). It is noted in [23] that \( A_1 \) and \( A_3 \) may be simultaneously empty.

If \((x,v) \in A_2 \cup A_3\), then \( S^t(x,v) \) leaves \( G \) after some time, so that there is a finite

\[
t(x,v) = \inf \{ t > 0 : S^t(x,v) \in \partial G \}. \tag{34}
\]

Consider the eigenvalue distribution of \( \nabla^2 f(O) \) in [29]. Let us suppose that there exist some \( 1 \leq k^o \leq k \) such that \( \lambda_1 = \lambda_2 = \ldots = \lambda_{k^o} \). As in [23], let us denote by \( \Gamma_{max} \) the eigenspace of \( A \) which corresponds to the eigenvalues \( \mu^+_1, \ldots, \mu^+_k \). Then as in [23], there exists a \( k^o \)-dimensional sub–manifold \( W_{max}^{\infty} \) tangent to \( \Gamma_{max} \) at 0 and is invariant with respect to \( S^t \). We see that \( Q_{max} = W_{max}^{\infty} \cap \partial G \) is not empty. If \( k^o > 1 \) then \( Q_{max} \) is a sub–manifold of \((k^o - 1)\)-dimensions on the boundary \( \partial G \). If \( k^o = 1 \) then \( Q_{max} \) consists of two points.
Figure 3: Analysis in a neighborhood of one specific saddle point.

Combining Proposition 3.1 and Theorems 2.1–2.3 of [23] (see also Theorem 3.3 in [20], or Theorem 2.1 in [19]), we then get the following.

**Theorem 3.2.** (a) (Exit time) For each fixed \((x,v) \in G\) we have
\[
\limsup_{\varepsilon \to 0} \frac{\mathbb{E}_{\varepsilon} \tau_{(x,v)}}{\ln(\varepsilon^{-1})} \leq \frac{1}{\mu_0},
\]
where \(\mu_0 > 0\) is defined in (30).

(b) (Exit distribution) If \((x,v) \in (0 \cup A_1) \setminus \partial G\) then for any open set \(Q\) of \(\partial G\) such that \(Q \supset Q_{\text{max}}\) we have
\[
\lim_{\varepsilon \downarrow 0} \mathbb{P}(Y_{\varepsilon \tau_{(x,v)}}^{(x,v)} \in Q) = 1.
\]
If \((x,v) \in A_2 \cup A_3\) then for any Borel measurable set \(Q\) of \(\partial G\) we have
\[
\lim_{\varepsilon \downarrow 0} \mathbb{P}(Y_{\varepsilon \tau_{(x,v)}}^{(x,v)} \in Q) = 1 \left( S^{d(x,v)}(x,v) \in Q \right).
\]

Theorem 3.2 characterizes the asymptotic exit time for the exit from one specific saddle point of the Hamiltonian flow \(\nabla H\).

To deal with the case of a chain of saddle points, as in the next section, similarly as in [19], we will make a linearization assumption. By the classical Hartman–Grobman
Theorem (see [1 §13]), for any strong saddle point $O$ that we consider, there exists an open neighborhood $G$ of $O$, and a $C^0$ homeomorphism mapping $h : G \to \mathbb{R}^n$, such that the Hamiltonian flow given by (14) is mapped by $h$ into a linear flow. The homeomorphism $h$ is called a (linear) conjugacy mapping. To make our argument work, we will have to put an additional

**Linerization Assumption.** The homeomorphism $h$ provided by the Hartman–Grobman Theorem can be taken to be $C^2$.

It is known that a sufficient condition for the validity of the $C^2$ Linerization Assumption is the so called non–resonance condition (see, for example, the Sternberg linerization Theorem [22, Theorem 6.6.6]).

Let us also notice here that the first equation in (27) is already linearized, so that presumably we can first linearize the gradient field $-\nabla_X f$ (see [26]). However, since (28) comes from a Hamiltonian system, a linearization of the system (28) requires simultaneously making the $-\nabla_X f$ term linear and rotate the whole system to match with the eigen–directions. The linear part of the Hamiltonian vector field $\nabla_\perp H = J\nabla H$ in (28) should presumably be given by $\nabla J\nabla H(0)$ as in (21). By (21), we know that $\nabla J\nabla H(0) = J\nabla^2 H(0)$. As in the previous section, we can set $A = JQ - \alpha I^0$, where $Q = \nabla^2 H(0)$. The matrix $A$ characterizes the linear part of the Hamiltonian vector field $\nabla_\perp H$ with dissipation near the saddle point $O$.

Let the open neighborhood $U$ of the saddle point $O$ be chosen so small that $\text{dist}(U \cup \partial U, \partial G) > 0$. Let $t(x,v)$ be defined as in (34). As in [19, Section 2], set

$$\partial G_{U \cup \partial U \to \text{out}} = \{S^{t(x,v)}(x,v) \text{ for some } (x,v) \in U \cup \partial U \text{ with finite } t(x,v) \} \cup Q_{\text{max}}.$$  

For small $\kappa > 0$ we let

$$Q^\kappa = \{(x,v) \in \partial G \, , \, \text{dist}((x,v), \partial G_{U \cup \partial U \to \text{out}}) < \kappa\}$$

(see Figure 3).

Then we have as in [19 Theorem 2.2] the following Theorem, which is an uniform version of Theorem 3.2. We would like to point out here, that improving Theorem 3.2 to Theorem 3.3 requires essential exploitation of the Linerization Assumption, and it is only this uniform version Theorem 3.3 that can enable our later proof in the case of a chain of saddle points.

**Theorem 3.3.** For any $r > 0$, there exist some $\varepsilon_0 > 0$ so that for all $(x,v) \in U \cup \partial U$ and all $0 < \varepsilon < \varepsilon_0$ we have

$$\frac{E^\varepsilon_{(x,v)}}{\ln(\varepsilon^{-1})} \leq \frac{1}{\mu_0} + r.$$  

(38)
For any small $\kappa > 0$ and any $\rho > 0$, there exist some $\varepsilon_0 > 0$ so that for all $x \in U \cup \partial U$ and all $0 < \varepsilon < \varepsilon_0$ we have
\[
P(Y_{\tau_{(x,v)}}^\varepsilon \in Q^\kappa) \geq 1 - \rho .
\] (39)

Here the stopping time $\tau_{(x,v)}^\varepsilon$ is defined as in (33), and $\mu_0$ is defined as in (30).

Proof. Let $h$ be the conjugacy mapping in the Linerization Assumption. In this case, by considering $Y_{\varepsilon}(t) = h(Y^\varepsilon(t))$ in which $h$ is the linearization homeomorphism that we discussed above, we can make (28) into the following equation
\[
dY_{\varepsilon}(t) = \left[ (JQ - \alpha I^0)Y_{\varepsilon}(t) + \varepsilon^2 \Psi(Y_{\varepsilon}(t)) \right] dt + \varepsilon \Sigma(Y_{\varepsilon}(t))dW_t , \quad Y_{\varepsilon}(0) \in \mathbb{R}^{2d}.
\] (40)

Here the additional term $\Psi$ is smooth in $Y$. As before, we set $A = JQ - \alpha I^0$ and we have the mild solution of (40) written as
\[
Y_{\varepsilon}(t) = e^{At}Y_{\varepsilon}(0) + \varepsilon \int_0^t e^{A(t-s)}\Sigma(Y_{\varepsilon}(s))dW_s + \varepsilon^2 \int_0^t e^{A(t-s)}\Sigma(Y_{\varepsilon}(s))ds .
\] (41)

Let us pick an orthonormal basis $\zeta_1, \ldots, \zeta_{2d}$ in $\mathbb{R}^{2d}$ such that $\zeta_i = \frac{1}{\sqrt{1 + (\mu_i^+)^2 \xi_i}} \begin{pmatrix} \xi_i \\ \mu_i^+ \xi_i \end{pmatrix}$ for $i = 1, 2, \ldots, k$. Set the orthogonal matrix
\[
M = \begin{pmatrix} \zeta_1 & \zeta_2 & \ldots & \zeta_k & \zeta_{k+1} & \ldots & \zeta_{2d} \end{pmatrix}.
\] (42)

Then we have
\[
AM = M \begin{pmatrix} \text{diag}(\mu_1^+, \ldots, \mu_k^+) & 0 \\ 0 & \tilde{A} \end{pmatrix} = M \tilde{A} ,
\] (43)
in which $\tilde{A}$ is a matrix of size $(2d - k) \times (2d - k)$, and $\tilde{A} = \begin{pmatrix} \text{diag}(\mu_1^+, \ldots, \mu_k^+) & 0 \\ 0 & \tilde{A} \end{pmatrix}$. In this way we have
\[
e^{At} = M^T \begin{pmatrix} \text{diag}(e^{\mu_1^+ t}, \ldots, e^{\mu_k^+ t}) & 0 \\ 0 & e^{\tilde{A} t} \end{pmatrix} M = M^T e^{\tilde{A} t} M ,
\] (44)
so that
\[
e^{\tilde{A} t} = \begin{pmatrix} \text{diag}(e^{\mu_1^+ t}, \ldots, e^{\mu_k^+ t}) & 0 \\ 0 & e^{\tilde{A} t} \end{pmatrix} .
\]

Consider $Y_{\varepsilon}(t) = \sum_{i=1}^{2d} y_i^\varepsilon(t)\zeta_i$ for some $y^\varepsilon(t) = (y_1^\varepsilon(t), \ldots, y_{2d}^\varepsilon(t))^T$ in $\mathbb{R}^{2d}$ with fixed $\varepsilon > 0$ and $t \geq 0$. Then the equation for $y^\varepsilon(t)$ in terms of mild solution takes the form

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\[ y^\varepsilon(t) = e^{\tilde{A}t}y^\varepsilon(0) + \varepsilon \int_0^t e^{\tilde{A}(t-s)}\Sigma(y^\varepsilon(s))dW_s + \varepsilon^2 \int_0^t e^{\tilde{A}(t-s)}\Psi(y^\varepsilon(s))ds \]  \hspace{1cm} (45)

Write \( \Sigma(y) = \begin{pmatrix} \Sigma_1(y) \\ \vdots \\ \Sigma_k(y) \\ \tilde{\Sigma}(y) \end{pmatrix} \) in which \( \tilde{\Sigma}(y) \) is a matrix of size \((2d - k) \times 2d\), \( \Psi(y) = \begin{pmatrix} \Psi_1(y) \\ \vdots \\ \Psi_k(y) \\ \tilde{\Psi}(y) \end{pmatrix} \) in which \( \tilde{\Psi}(y) \) is a column vector of size \(2d - k\), and

\[ y^\varepsilon_i(t) = e^{\mu_it}y^\varepsilon_i(0) + \varepsilon \int_0^t e^{\mu_i(t-s)}\Sigma_i(y^\varepsilon(s))dW_s + \varepsilon^2 \int_0^t e^{\mu_i(t-s)}\Psi_i(y^\varepsilon(s))ds \] \hspace{1cm} (46)

and

\[ \tilde{y}^\varepsilon(t) = e^{\tilde{A}t}\tilde{y}^\varepsilon(0) + \varepsilon \int_0^t e^{\tilde{A}(t-s)}\tilde{\Sigma}(y^\varepsilon(s))dW_s + \varepsilon^2 \int_0^t e^{\tilde{A}(t-s)}\tilde{\Psi}(y^\varepsilon(s))ds \] \hspace{1cm} (47)

By the spectral radius theorem (see [42]) we know that

\[ \lim_{s \to \infty} \| e^{\tilde{A}s} \|^{1/s} \leq e^{-\mu^- t} \] \hspace{1cm} (48)

for some \( \mu^- > 0 \). This implies that there is some positive constant \( C^{(1)} > 0 \) such that

\[ \| e^{\tilde{A}s} \| \leq C^{(1)}e^{-\mu^- s} \] \hspace{1cm} (49)

for all \( s \geq 0 \). From here, we can argue with the same reasoning as in [4, Section 8] and [19, Appendix A]. We see that, to analyze the exit time and exit distribution of the process \( y^\varepsilon(t) \), it suffices to look at the vector consisting of the first \( k \)-components \( y^\varepsilon(t) = (y^\varepsilon_1(t), \ldots, y^\varepsilon_k(t))^T \). The latter process \( y^\varepsilon(t) \) is governed by the equations (46) where \( i = 1, 2, \ldots, k \). We can then make full use of the arguments in [4, Section 8] and [19, Appendix A], and we conclude the statement of this Theorem.

4 Chain of saddle points and the global convergence.

Upon exit from a neighborhood of a saddle point, we will analyze the process \( Y^\varepsilon_t \) in (28) in the case when there is a chain of saddle points by using the method we
developed in [19]. Since many of the arguments we use here will be the same as in our previous work [19], we will make our arguments here more brief, and we will emphasize the differences.

Figure 2 shows the Hamiltonian flow when $d = 1$ and the saddle points $O_1$ and $O_2$ form a chain of saddle points. Figure 4 shows the case when friction is added to the Hamiltonian flow. In general, we can assume that there are $k$–consecutive strong saddle points $O_1, ..., O_k$ of the Hamiltonian $H(X, V)$ such that $H(O_1) > H(O_2) > ... > H(O_k)$. Let $O^*$ be a local minimum point of $H(X, V)$ such that $H(O_k) > H(O^*)$ and there are no other critical points $o$ such that $H(O_k) > H(o) > H(O^*)$.

Notice that any critical point of the Hamiltonian flow $\nabla^* H$ has $V$–component equal to zero. Furthermore, the saddle points $O_1, ..., O_k$ are of the form $O_j = (x^{O_j}, 0)$, $j = 1, 2, ..., k$ where $x^{O_j}$ is a local maximum point or saddle point (under Definition 2.3) of the potential function $f(x)$. In the same way we have $O^* = (x^*, 0)$ where $x^*$ is a local minimum point of the potential function $f(x)$.

Let us consider the perturbed process $Y_t^\varepsilon$ in (23). Set small $\varepsilon > 0$ and let

$$T_{(X, V)}^{H, \varepsilon} = \inf\{t \geq 0 : Y_0^\varepsilon = (X, V) \land H(Y_t^\varepsilon) < H((X^*, 0)) + \varepsilon\}. \quad (50)$$

Recall that $Y_t^\varepsilon = (X_t^\varepsilon, V_t^\varepsilon)^T$ as in (27). In an equivalent way, we can set

$$T_{(X, V)}^{f, \varepsilon} = \inf\{t \geq 0 : X_0 = X \land f(X_t^\varepsilon) < f(x^*) + \varepsilon\}, \quad (51)$$

so that $T_{(X, V)}^{H, \varepsilon} = T_{(X, V)}^{f, \varepsilon}$.

In this section we aim at characterizing the asymptotic upper bound as $\varepsilon \to 0$ of the times $T_{(X, V)}^{H, \varepsilon}$ (or $T_{(X, V)}^{f, \varepsilon}$).

For each saddle point $O_i$, we consider a nested pair of open neighborhoods $U_i \subseteq G_i$, containing $O_i$ (see Figure 3). Let us first pick all $G_i$, $i = 1, 2, ..., k$ in such a way that $O_i \in G_i$ is the only stationary point inside $G_i$, and for any $i \neq j$ we have $G_i \cap G_j = \emptyset$. Let us then pick $U_i \subset G_j$ such that $\text{dist}(\partial U_i, \partial G_j) > 0$ for all $i = 1, 2, ..., k$. For each saddle point $O_i$, let us denote by $\Gamma_{i, \max}$ the eigenspace of $J\nabla^2 H(O_i) - \alpha I^0$ which corresponds to the eigenvalues of $J\nabla^2 H(O_i) - \alpha I^0$ with largest positive real parts (see the paragraph right above Theorem 3.2). For each open neighborhood $G_i$, as in [23], we can construct a submanifold $W_{i, \max}$ that is tangent to $\Gamma_{i, \max}$ at $O_i$ and is invariant with respect to $S^t$.

Let $Q_{i, \max} = W_{i, \max} \cap \partial G_i$. By the classification of points in $G_i \cup \partial G_i$ as $O_i$, $A_1$, $A_2$, $A_3$ as in [23] presented right above Theorem 3.2 we see that for any point $(x, v) \in U_i \cup \partial U_i$, either there exist some finite $t(x, v)$ such that $S^{t(x,v)}(x, v) \in \partial G_i$, or $S^t(x, v) \to O_i$ as $t \to \infty$. Let

$$\partial G_i, U_i \cup \partial U_i \rightarrow \text{out} = \{S^{t(x,v)}(x, v) \mid (x, v) \in U_i \cup \partial U_i \text{ with finite } t(x, v)\} \cup Q_{i, \max}.$$
For a very small \( e > 0 \), let us set
\[
U = \{(x, v) \in \mathbb{R}^d : H(x, v) < H(O^*) + e \text{ for some local minimum point } O^*\}.
\]
The set \( U = \bigcup_j U_j \) where each \( U_j \) is an open neighborhood of one of the local minimum points of the Hamiltonian \( H(X, V) \). We can set \( e > 0 \) to be so small that each of \( U_i \) and \( U_j \) are mutually disjoint, and they are also disjoint with any of the \( G_i 's. \)

Let us pick the neighborhoods \( G_i \) to be so small that starting from any point \( (x, v) \) on \( G_i \cup \partial G_i \), the deterministic flow \( S^t(x, v) \) of (15) will never return to \( G_i \cup \partial G_i \). This can be achieved by the friction term \( b(X, V) \) in (19) (see Figure 4). Let \( U = \bigcup_{i=1}^{k} U_i \) and \( G = \bigcup_{i=1}^{k} G_i \). Let
\[
K = \inf \left\{ |\nabla H(x, v)| : (x, v) \notin U \cup U \right\}.
\]
(52)
From our construction above we see that \( K > 0 \) and it is independent of \( \varepsilon \). Set \( Y_{0}^{\varepsilon} = (X_{0}, V_{0})^{T} \) and the stopping times
\[
0 = \sigma_0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \sigma_2 \leq ... \quad (53)
\]
such that
\[
\tau_j = \inf \{ t > \sigma_{j-1} : Y_{t}^{\varepsilon} \in \partial U \cup \partial U \},
\]
and
\[
\sigma_j = \inf \{ t > \tau_j : Y_{t}^{\varepsilon} \in \partial G \cup \partial U \}.
\]
(55)
Starting from the initial condition \( Y_0 = (X_0, V_0) \) that is outside of \( G \cup U \), such that for some \( a_1 < b_1 \) we have \( a_1 < H(X_0, V_0) < b_1 \), the process \( Y_t^{\varepsilon} \) travels for time \( \tau_1 = \tau_1(\varepsilon) \) before it enters \( U \cup \mathcal{U} \). The following lemma is parallel to Lemma 3.3 in [19].

**Proposition 4.1.** There exists some \( \varepsilon_0 > 0 \) uniformly for all \( Y_0 = (X_0, V_0) \) with \( a_1 < H(X_0, V_0) < b_1 \), such that for all \( 0 < \varepsilon < \varepsilon_0 \), there exist some finite \( C > 0 \) independent of \( \varepsilon \) such that
\[
E_{x} \tau_1 \leq C.
\]
(56)

**Proof.** Let us take the Lyapunov function as the Hamiltonian
\[
H(X(t), V(t)) = \frac{1}{2}(V(t))^2 + f(X(t)).
\]
Then along the flow of (15) we have
\[
H(X(t), V(t)) - H(X(0), V(0)) = -\alpha \int_{0}^{t} (V(s))^2 ds.
\]
(57)
From the equation (57) we see that when the process \( Y(t) \) is a distance away from the \( X \)-axis, the Hamiltonian function \( H(X(t), V(t)) \) is strictly decaying. However, if the process \( Y(t) \) crosses the \( X \)-axis, then \( V(s) \) takes 0 value along the trajectory and it is not guaranteed that the Hamiltonian \( H(X(t), V(t)) \) keeps decaying. It is in this aspect that we see the effect of interacting the friction with the Hamiltonian flow.

A crossing through \( X \)-axis happens in the following two cases: (a) It is outside a neighborhood of a critical point \( O \) of the Hamiltonian flow \( \nabla^\perp H \), that is either a saddle point or a local minimum. Then the friction vanishes at the crossing point, but the Hamiltonian flow is not zero there, since \( |\nabla^\perp H| \geq K > 0 \), and thus it is the Hamiltonian flow that brings the process immediately to a region where \( (V(s))^2 \) is still strictly positive. Combining this with (57) we see that even in this case the Hamiltonian \( H(X(t), V(t)) \) keeps strictly decaying along the flow of \( Y(t) \); (b) The flow of \( Y(t) \) approaches a critical point \( O \) of the Hamiltonian flow \( \nabla^\perp H \), that is either a saddle point or a local minimum. In this case, we are entering a neighborhood of the critical point.

Suppose we start the process \( Y(t) \) in (15) from an initial point \( Y_0 = (X_0, V_0) \) that stays away from a neighborhood of a saddle point \( O \) of the Hamiltonian flow \( \nabla^\perp H \). From the above reasoning we see that, in finite time \( T_0 > 0 \) the process \( Y(t) \) reaches a neighborhood of another critical point of the Hamiltonian flow \( \nabla^\perp H \), that is either a local minimum point or a saddle. Combining this with the arguments that lead to Lemma 3.3 in [19], we conclude the validity of this proposition.

Finally by using the arguments of [19, Section 3], we see that we have the following analogue of Theorem 3.4 in [19].

**Theorem 4.2.** Consider the process \( Y^\varepsilon_t \) defined as in (28). Assume the diffusion matrix \( \Sigma(Y)\Sigma^T(Y) \) is uniformly positive definite for all choices of \( Y \). Then we have

(i) For any small \( \rho > 0 \), with probability at least \( 1 - \rho \), the process \( Y^\varepsilon_t \) in (28) converges to the local minimum point \( O^* \) for sufficiently small \( \varepsilon \) after passing through all \( k \) saddle points \( O_1, ..., O_k \);

(ii) As \( \varepsilon \downarrow 0 \), conditioned on the above convergence of \( Y^\varepsilon_t \) to \( O^* \), we have

\[
\limsup_{\varepsilon \to 0} \frac{E T^H_{(X,V)}(X,V)}{\ln(\varepsilon^{-1})} \leq \frac{k}{2\gamma_1} \left( \sqrt{\alpha^2 + 4\gamma_1} + \alpha \right). \tag{58}
\]

Here the stopping time \( T^H_{(X,V)} \) is defined as in (50), and \( \gamma_1 \) is as in Definition 2.2.

**Proof.** The proof of this Theorem follows the same lines of arguments as those in the proof of Theorem 3.4 in [19]. As \( Y^\varepsilon_t \) is a strong Markov process, we see that each of \( \sigma_j - \tau_j \) in distributed in the same way as \( \tau^\varepsilon_{(x,v)} \) in (53). However, we note that in this case, the initial condition \((x,v)\) in \( \tau^\varepsilon_{(x,v)} \) will in general be random. In this case, based
on Theorem 3.3, which is a uniform version of Theorem 3.2, one can show that each of \( \sigma_j - \tau_j \) is distributed in such a way that for any \( r > 0 \) there exist some \( \varepsilon_0^{(1)} > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0^{(1)} \) we have

\[
\frac{\mathbb{E}(\sigma_j - \tau_j)}{\ln(\varepsilon^{-1})} \leq \frac{1}{\mu_0^{(i)}} + r.
\]

(59)

Here \( \mu_0^{(i)} = -\alpha + \sqrt{\alpha^2 - 4\lambda_1^{(i)}} \) corresponds to formula (30) but for the saddle point \( O_i \), in which \( \lambda_1^{(i)} \) is from (29) but for the saddle point \( O_i \). Notice that by Definition 2.2, we have

\[
-\lambda_1^{(i)} = \frac{-\alpha + \sqrt{\alpha^2 - 4\gamma_1}}{2} \geq \gamma_1,
\]

so that

\[
\mu_0^{(i)} \geq -\alpha + \sqrt{\alpha^2 + 4\gamma_1}.
\]

(60)

Moreover, by Lemma 5.1, we can show that there exist some \( \varepsilon_0^{(2)} > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0^{(2)} \) we have

\[
\mathbb{E}(\tau_j - \sigma_{j-1}) \leq C
\]

(61)

for some constant \( C > 0 \). Now we decompose

\[
T_{(X,V)}^H,\varepsilon = \sigma_0 + (\tau_1 - \sigma_0) + (\sigma_1 - \tau_1) + \ldots.
\]

(62)

We notice the fact that when the deterministic process (15) leaves each of the \( G_i \), it never returns to the same \( G_i \). Therefore, the expansion (62) will terminate after passing through at most \( k \)-saddle points, i.e,

\[
T_{(X,V)}^H,\varepsilon = \sigma_0 + (\tau_1 - \sigma_0) + (\sigma_1 - \tau_1) + \ldots + (\tau_k - \sigma_{k-1}) + (\sigma_k - \tau_k) + (\sigma_{k+1} - \sigma_k).
\]

(63)

We can then see the validity of this Theorem from (63), (59), (61), (60).

The above Theorem is formulated for the Hamiltonian function \( H(X,V) = \frac{1}{2}V^2 + f(X) \). Taking into account that the original optimization problem (8) is for the objective function \( f(x) \), and each saddle point \( O \) of the Hamiltonian \( H \) is given by \( O = (X,0) \), in which \( X \) is a local maxima or saddle point of \( f \), we can reformulate the above Theorem as follows.

**Corollary 4.3.** Consider the process \( X_t^\varepsilon \) defined as in (27). Let \( x^* \) be the unique local minimum of \( f \) within an open neighborhood \( U(x^*) \) such that \( f(x^*) < f(x^{O_k}) \). Then we have

- (i) For any small \( \rho > 0 \), with probability at least \( 1 - \rho \), the process \( X_t^\varepsilon \) in (27) converges to the minimizer \( x^* \) for sufficiently small \( \varepsilon \) after passing through all \( k \) saddle points \( x^{O_1}, \ldots, x^{O_k} \).
Figure 4: Chain of saddle points.

(ii) As $\varepsilon \downarrow 0$, conditioned on the above convergence of $X_t^\varepsilon$ to $x^*$, we have

$$\limsup_{\varepsilon \to 0} \frac{E T_{(X,V)}^{f,\varepsilon}}{\ln(\varepsilon^{-1})} \leq \frac{k}{2\gamma_1} \left( \sqrt{\alpha^2 + 4\gamma_1 + \alpha} \right). \quad (64)$$

Here $T_{(X,V)}^{f,\varepsilon}$ is defined as in (51), and $\gamma_1$ is as in Definition 2.2.

Finally, we shall formulate a convergence result for the diffusion approximation of the stochastic heavy ball method (13). Taking into account that the process $(x(t), v(t))$ in (13) is related to the process $(X(t), V(t))$ in (27) via a time change $(X(t), V(t)) = (x(t/\sqrt{s}), v(t/\sqrt{s}))$ and $\varepsilon = \sqrt{s}$, we see that we have the following Theorem.

**Theorem 4.4.** Consider the process $x(t)$ defined as in (13). Let the potential function $f(x)$ satisfy Definition 2.3. Let $x^*$ be the unique local minimum of $f$ within an open neighborhood $U(x^*)$ such that $f(x^*) < f(x^{O_k})$. Then we have

(i) For any small $\rho > 0$, with probability at least $1 - \rho$, the process $x(t)$ in (13) converges to the minimizer $x^*$ for sufficiently small $s > 0$ after passing through all $k$ saddle points $x^{O_1}, ..., x^{O_k}$;

(ii) Set $\varepsilon > 0$ small and let

$$T_x = \inf\{t \geq 0 : x(0) = x, f(x(t)) < f(x^*) + \varepsilon\}.$$  

(65)
Then as $s \downarrow 0$, conditioned on the above convergence of $x(t)$ to $x^*$, we have
\[
\limsup_{\epsilon \to 0} \frac{E T_x}{\sqrt{s^{-1} \ln(s^{-1})}} \leq \frac{k}{8\gamma_1} \left( \sqrt{\alpha^2 + 4\gamma_1} + \alpha \right).
\]
(66)

Here $\gamma_1$ is as in Definition 2.2.

5 Asymptotic behavior of the process when approaching a local minimum point.

In Theorems 4.2–4.4, the results are formulated about convergence time of the processes under consideration to a neighborhood of the local minimum point $O^*$ of $H(X,V)$ (or $x^*$ of $f(x)$). A natural question to ask is about how much fluctuations, due to the injection of noise in (28), can the process have when entering a neighborhood of a local minimum point. The following results are formulated very generally and they address that the fluctuations are about the order $O(\epsilon)$ near $O^*$ if we run the process at sufficiently long time scale (between logarithmic and exponential). Although the results can be applied to the framework of this work, the current section itself has independent interest, and it uses notations and symbols that are independent of the other sections.

Let us consider a smooth deterministic dynamical system
\[
\dot{x}_t = b(x_t) , \ x_0 = x \in \mathbb{R}^d .
\]
(67)

Suppose $0 \in \mathbb{R}^d$ is the only isolated asymptotic stable equilibrium of (67). Let the dynamical system defined via (67) be denoted by $x_t = S^t x$. Assume that there exist some $\kappa > 0$ such that for any $x \in \mathbb{R}^d$ and some constant $C_1 = C_1(x) > 0$ we have
\[
|S^t x - 0| \leq C_1 e^{-\kappa t} .
\]
(68)

In particular, although the constant $C_1$ depends on $x$, it can be chosen independent of the initial point $x$ for all $x \in B(0, R)$. Here $B(0, R)$ is a ball centered at $0$ with radius $R > 0$.

Let us consider a small random perturbation of (67), so that we have
\[
dX_t^\epsilon = b(X_t^\epsilon) dt + \epsilon \sigma(X_t^\epsilon) dW_t , \ X_0^\epsilon = x \in \mathbb{R}^d .
\]
(69)

Here we will assume that the $d \times d$ diffusion matrix $a(x) = \sigma(x)\sigma^T(x)$ is smooth and is uniformly positive definite.

As $b(x)$ is a smooth vector field in $\mathbb{R}^d$, when $x \in U \subset \mathbb{R}^d$ is in some open neighborhood of the origin $0$, we have the expansion
\[
b(x) = \Lambda x + \psi(x)|x|^2
\]
(70)
for some bounded smooth function \( \psi(x) \). Here \( \Lambda \) is a matrix having eigenvalues with nonzero real parts. Let \( \lambda_1, \ldots, \lambda_d \) be the eigenvalues of the matrix \( \Lambda \) (counting multiplicity), and assume that \(-c = \text{Re}\lambda_1 \leq \cdots \leq \text{Re}\lambda_d < 0\). From here we see that there exist some \( \alpha_0 > 0 \) such that

\[
 b(x) \cdot x = \langle \Lambda x, x \rangle + \langle \psi(x), 1 \rangle |x|^3 \leq -\alpha_0 |x|^2
\]

for \( x \in V \subset U \subset \mathbb{R}^d \), in which \( V \) is an open neighborhood of the origin 0.

Pick some small \( r > 0 \) that is independent of \( \varepsilon > 0 \), and let \( B(0, r) \) be an open ball centered at point 0 with radius \( r > 0 \), such that \( B(0, 2r) \subset V \subset U \). From (68) we have the following

**Lemma 5.1.** There exist some finite time \( t_0 = t_0(\kappa, r) \) with \( 0 \leq t_0 \leq C_2\kappa^{-1} \ln(r^{-1}) \) for some constant \( C_2 = C_2(x) > 0 \), such that \( |S^t x| \leq r \) for all \( t \geq t_0 \). In particular, for all \( x \in B(0, R) \) inside a ball centered at 0 with radius \( R > 0 \), we can pick the constant \( C_2 > 0 \) to be independent of the choice of \( x \).

From Theorem 1.2 in [16, Chapter 2], together with its proof, we conclude the following statement.

**Lemma 5.2.** For any initial point \( x \in \mathbb{R}^d \), any \( \rho > 0 \), for all \( t > 0 \) and \( \delta > 0 \), there exist some \( \varepsilon_0 = \varepsilon_0(\rho, \delta, t, x) > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \) we have

\[
 P \left\{ \max_{0 \leq s \leq t} |X^\varepsilon_s - x_s| > \delta \right\} < \rho .
\]

In particular, for any \( x \) chosen inside a ball \( B(0, R) \) of radius \( R > 0 \) centered at 0, the parameter \( \varepsilon_0 \) can be chosen independent of \( x \).

Let us pick \( \delta > 0 \) so small that we have set inclusion

\[
 \{ y \in \mathbb{R}^d : |y - x| \leq \delta \text{ for some } x \in B(0, r) \} \subset B(0, 2r) .
\]

In particular, we see that this choice of \( \delta \) depends on \( r \).

Let the initial point \( X_0^\varepsilon = x_0 = x \in B(0, R) \) be inside a ball centered at 0 with radius \( R > 0 \). Set

\[
 \tau_1 = \inf\{ t \geq 0 : X_t^\varepsilon \in B(0, 2r) \} .
\]

Combining Lemma 5.1, Lemma 5.2 and (73), we see that we have the following

**Proposition 5.3.** For any \( \rho_1 > 0 \), any initial point \( x \in B(0, R) \), there exist some \( \varepsilon_0 = \varepsilon_0(\rho_1, r) > 0 \) such that for any \( 0 < \varepsilon < \varepsilon_0 \) we have

\[
 P \left\{ \tau_1 \leq t_0 \right\} \geq 1 - \rho_1 .
\]

Here \( t_0 \) is as chosen in Lemma 5.1.
Let $Y^\varepsilon_s = X^\varepsilon_{s+\tau_1}$. By strong Markov property of the process $X^\varepsilon_t$, we see that $Y^\varepsilon_s$ is the same process $X^\varepsilon_t$ starting at some $y \in B(0, 2r) \subset V$. Let

$$
\tau_2 = \inf \{ s \geq 0 : Y^\varepsilon_s \in \partial V \} .
$$

(76)

We can then apply standard results in large deviation theory. From Theorem 4.2 in [16, Chapter 4], we have

**Proposition 5.4.** For any $\rho_2 > 0$, there exist some $\varepsilon_0 = \varepsilon_0(\rho_2, r, V)$, such that for any $0 < \varepsilon < \varepsilon_0$ we have

$$
P \left\{ \tau_2 > e^{C_3 \varepsilon^{-2}} \right\} \geq 1 - \rho_2
$$

(77)

for some constant $C_3 > 0$.

**Proposition 5.5.** For any $\rho > 0$, there exist some $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ we have, with probability at least $1 - \rho$, that

$$
E |Y^\varepsilon_s|^2 \lesssim O(\varepsilon^2)
$$

(78)

for $O \left( \ln \left( \frac{1}{\varepsilon^2} \right) \right) \ll s \ll O(e^{C_3 \varepsilon^{-2}})$, in which $C_3 > 0$ is as in Proposition 5.4.

**Proof.** We can apply Itô’s formula to the process $|Y^\varepsilon_s|^2$, so that for any $0 \leq s_1 < s_2 < \infty$ we have

$$
|Y^\varepsilon_{s_2}|^2 - |Y^\varepsilon_{s_1}|^2 = 2 \int_{s_1}^{s_2} Y^\varepsilon_q \cdot b(Y^\varepsilon_q)dq + 2 \int_{s_1}^{s_2} Y^\varepsilon_q \cdot \sigma(Y^\varepsilon_q)dW_q + \varepsilon^2 \int_{s_1}^{s_2} \text{Tr} [\sigma(Y^\varepsilon_q)\sigma^T(Y^\varepsilon_q)]dq .
$$

(79)

We can assume that $|\text{Tr} [\sigma(Y^\varepsilon_q)\sigma^T(Y^\varepsilon_q)]| \leq \Phi < \infty$ for any $q \geq 0$ and some positive constant $0 < \Phi < \infty$. Therefore for $0 \leq s_1 < s_2 \leq \tau_2$ we have, by (71), that

$$
E |Y^\varepsilon_{s_2}|^2 - E |Y^\varepsilon_{s_1}|^2 \leq -2\alpha_0 \int_{s_1}^{s_2} E |Y^\varepsilon_q|^2 dq + \varepsilon^2 (s_2 - s_1) \Phi .
$$

This implies that for any $0 \leq s \leq \tau_2$ we have

$$
\frac{d}{ds}E |Y^\varepsilon_s|^2 \leq -2\alpha_0 E |Y^\varepsilon_s|^2 + \varepsilon^2 \Phi .
$$

(80)

That is to say, for any $0 \leq s \leq \tau_2$ we have

$$
E |Y^\varepsilon_s|^2 \leq \frac{\varepsilon^2 \Phi}{2\alpha_0} + e^{-2\alpha_0 s} \left( |y|^2 - \frac{\varepsilon^2 \Phi}{2\alpha_0} \right) .
$$

(81)

Let us pick $\varepsilon^{(1)}_0 > 0$ very small, so that for any $0 < \varepsilon < \varepsilon^{(1)}_0$ we have

$$
\ln \left( \frac{1}{\varepsilon^2} \right) \ll e^{C_3 \varepsilon^{-2}} .
$$
Given any $\rho > 0$, set some $\varepsilon(2) > 0$ as in Proposition 5.4, so that for any $0 < \varepsilon < \varepsilon(2)$ we have $P\left\{ \tau_2 > e^{C_3\varepsilon^{-2}} \right\} < 1 - \rho$. In this way, by noticing that when $s \geq \frac{1}{2\alpha_0} \ln \left( \frac{1}{\varepsilon^2} \right)$, we have $e^{-2\alpha_0 s} \lesssim O(\varepsilon^2)$, we see that we have, for $0 < \varepsilon < \varepsilon(1) \wedge \varepsilon(2)$, that

$$P \left\{ E|Y_s^\varepsilon|^2 \lesssim O(\varepsilon^2) \text{ when } O(\ln \left( \frac{1}{\varepsilon^2} \right)) \ll s \ll O(e^{C_3\varepsilon^{-2}}) \right\} > 1 - \rho. \quad (82)$$

In particular, it is easy to see that the result of Proposition 5.5 can be applied to our system (28), and we conclude that the local fluctuation near a local minimum point of the process $Y_t^\varepsilon$ in (28) is of order $O(\varepsilon)$ at sufficiently long time scale (between logarithmic and exponential).

6 Some further discussions.

1. The original heavy–ball scheme, hypo–elliptic perturbations.

A standard way of introducing small noise to system (10) is to consider replacing $\nabla Xf$ in (10) by a noisy gradient $\tilde{\nabla} Xf = \nabla Xf + \text{unbiased noise}$. This leads to the following noisy gradient version of the “heavy ball method” scheme

$$\begin{align*}
x_t &= x_{t-1} + \sqrt{s}v_t , \ x_0 \in \mathbb{R}^d ; \\
v_t &= (1 - \alpha\sqrt{s})v_{t-1} - \sqrt{s}\nabla Xf (x_{t-1} + \sqrt{s}(1 - \alpha\sqrt{s})v_{t-1}) , \ v_0 \in \mathbb{R}^d .
\end{align*} \quad (83)$$

Following the general program in [20], [27], as well as monographs [7], [5], [24], we can introduce the following system of stochastic differential equations (SDE)

$$\begin{align*}
dx(t) &= \sqrt{s}v(t)dt , \ x(0) \in \mathbb{R}^d ; \\
dv(t) &= (-\alpha\sqrt{s}v(t) - \sqrt{s}\nabla Xf(x(t)))dt + \sqrt{s}\sigma(x(t), v(t))dW^2_t , \ v(0) \in \mathbb{R}^d .
\end{align*} \quad (84)$$

Here $\sigma(x(t), v(t))$ comes from the correlation in the noise within $\tilde{\nabla} Xf(x)$, and $W^2_t$ is a $d$–dimensional Brownian motion. The solution $(x(t), v(t))$ can be considered as a (weak) approximation to the original discrete process $(x_t, v_t)$ (see [20], [27], [7], [5], [24]). Conversely, the original discrete process $(x_t, v_t)$ can be viewed as a numerical scheme of the continuous–time process $(x(t), v(t))$ in (84). To summarize, we use the continuous process as a “surrogate” to the discrete process. We can do a time rescaling $t \rightarrow t/\sqrt{s}$, so that we end up with the process $Y^s(t) = \begin{pmatrix} X^s(t) \\ V^s(t) \end{pmatrix}$, which is a random perturbation of (31), i.e.,
\[
\begin{align*}
\left\{ \begin{array}{l}
\, dX^s(t) = V^s(t)dt , \, X^s(0) \in \mathbb{R}^d ; \\
\, dV^s(t) = (\, -\alpha V^s(t) - \nabla_X f(X^s(t)))dt + \sqrt{s} \sigma(X^s(t), V^s(t))dW_t^2 , \, V^s(0) \in \mathbb{R}^d . \\
\end{array} \right.
\end{align*}
\] (85)

Set the small parameter \( \varepsilon = \sqrt{s} > 0 \). We let \( Y^\varepsilon(t) = \left( X^\varepsilon(t), V^\varepsilon(t) \right) \). In this way, (85) can be written as

\[
\begin{align*}
\left\{ \begin{array}{l}
\, dX^\varepsilon(t) = V^\varepsilon(t)dt , \, X^\varepsilon(0) \in \mathbb{R}^d ; \\
\, dV^\varepsilon(t) = (\, -\alpha V^\varepsilon(t) - \nabla_X f(X^\varepsilon(t)))dt + \varepsilon \sigma(X^\varepsilon(t), V^\varepsilon(t))dW_t^2 , \, V^\varepsilon(0) \in \mathbb{R}^d . \\
\end{array} \right.
\end{align*}
\] (86)

Our discussion above is about the case of system (27), in which we have non–degenerate random perturbation. In the case of system (86), in which we have degenerate random perturbation, we need further improvements of the result of Kifer [23]. Let us note that in the case of system (86), the random perturbation is in the \( V \)–direction, and it can be decomposed into the eigen–directions pointed by the Hamiltonian flow at the saddle point \( O \) (Figure 1). Non–degeneracy here brings in dependence of the noise at different eigen–directions, which may further cause “less degree of freedom” in picking up the unstable direction to exit. Improving Kifer’s result in [23] to degenerate perturbations (for example, hypo–elliptic perturbations) is in itself an interesting and challenging problem, which can lead to an independent paper.

2. Connection with the Bregman Lagrangian flow framework.

In [30, 37, 38, 35], the authors proposed a general variational perspective on accelerated methods in optimization. Their continuous–time dynamical model can be used to interpret many different kinds of accelerated methods in optimization in a uniform way, and it takes into account both the non–Euclidean setting as well as higher–order case. The method goes as follows: let the Bregman Lagrangian be defined by

\[
\mathcal{L}(X, V, t) = e^{\alpha t + \gamma t} \left( D_h(X + e^{-\alpha t} V, X) - e^{\beta t} f(X) \right) .
\] (87)

Here position \( X \in \mathbb{R}^d \), velocity \( V \in \mathbb{R}^d \), and time parameter \( t \in \mathbb{T} \subset \mathbb{R} \). The convex function \( h \) is used to define an alternative measure of distance in \( \mathbb{R}^d \) for \( x, y \in \mathbb{R}^d \) via its Bregman divergence

\[
D_h(y, x) = h(y) - h(x) - \langle \nabla h(x), y - x \rangle .
\] (88)

The functions \( \alpha, \beta, \gamma : \mathbb{T} \to \mathbb{R} \) are arbitrary smooth (continuously differentiable) functions of time that determine the weighting of the velocity, the potential function, and the overall damping of the Lagrangian. The authors in these works put the following ideal scaling conditions:
\begin{align}
\dot{\beta}_t &\leq e^{\alpha_t}, \quad \dot{\gamma}_t = e^{\alpha_t}. 
\end{align}

In [37], the authors pick \( \alpha_t = \ln p - \ln t \), \( \beta_t = p \ln t + \ln C \), \( \gamma_t = p \ln t \). Under this parametrization, the Euler–Lagrangian equation with Lagrangian \( \mathcal{L}(X, V, t) \) in (88) given by
\begin{align}
\frac{d}{dt} \left\{ \frac{\partial \mathcal{L}}{\partial V}(X_t, \dot{X}_t, t) \right\} &= \frac{\partial \mathcal{L}}{\partial X}(X_t, \dot{X}_t, t), 
\end{align}
reduces to the second-order differential equation
\begin{align}
\ddot{X}_t + \frac{p+1}{t} \dot{X}_t + Cp^{2p-2} \left[ \nabla^2 h \left( X_t + \frac{t}{p} \dot{X}_t \right) \right]^{-1} \nabla f(X_t) &= 0. 
\end{align}

In particular, if \( p = 2, \ C = \frac{1}{4} \) and \( h(X) = \frac{1}{2} \| X \|^2 \) (the Euclidian case), we come back to (9).

Let \( h^* \) be the convex dual of \( h \), and let \( P \) be the momentum that is the dual variable to velocity \( V \). Then the Euler–Lagrangian equation (90) corresponds to the Hamiltonian equation for the following Bregman Hamiltonian
\begin{align}
\mathcal{H}(X, P, t) = e^{\alpha_t + \gamma_t} \left( D_{h^*}(\nabla h(X) + e^{-\gamma_t} P, \nabla h(X)) + e^{\beta_t} f(X) \right). 
\end{align}

In [36], [37], [38] the authors consider various discretizations of the system (91) that form numerical schemes. An interesting question would be to mimic our approach in this work and inject random inputs into these algorithms, and consider their weak limit processes. This might lead to better understandings of how the resulting processes exit from saddle points by using analysis that are similar to our work.
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