Circular Hessenberg Pairs

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Abstract

A square matrix is called Hessenberg whenever each entry below the subdiagonal is zero and each entry on the subdiagonal is nonzero. Let $M$ denote a Hessenberg matrix. Then $M$ is called circular whenever the upper-right corner entry of $M$ is nonzero and every other entry above the superdiagonal is zero. A circular Hessenberg pair consists of two diagonalizable linear maps on a nonzero finite-dimensional vector space, that each act on an eigenbasis of the other one in a circular Hessenberg fashion. Let $A, A^*$ denote a circular Hessenberg pair. We investigate six bases for the underlying vector space that we find attractive. We display the transition matrices between certain pairs of bases among the six. We also display the matrices that represent $A$ and $A^*$ with respect to the six bases. We introduce a special type of circular Hessenberg pair, said to be recurrent. We show that a circular Hessenberg pair $A, A^*$ is recurrent if and only if $A, A^*$ satisfy the tridiagonal relations. For a circular Hessenberg pair, there is a related object called a circular Hessenberg system. We classify up to isomorphism the recurrent circular Hessenberg systems. To this end, we construct four families of recurrent circular Hessenberg systems. We show that every recurrent circular Hessenberg system is isomorphic to a member of one of the four families.

Keywords: Leonard pair; tridiagonal pair; Hessenberg pair; circular Hessenberg pair.

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1 Introduction

This paper is about a linear algebraic object called a circular Hessenberg pair. Before we describe this object, we first give some motivation. The concept of a Leonard pair was introduced by Terwilliger [17]. A Leonard pair consists of two diagonalizable linear maps on a nonzero finite-dimensional vector space, that each act on an eigenbasis of the other one in an irreducible tridiagonal fashion; see Definition 2.1. A Leonard pair satisfies two relations called the tridiagonal relations [17]; see Lemma 2.4 below. Notable papers about Leonard pairs are [17–21]. There is a generalization of a Leonard pair called a tridiagonal pair. The concept of a tridiagonal pair was introduced by Ito, Tanabe, and Terwilliger [8]. A tridiagonal pair satisfies the tridiagonal relations [8]. Notable papers about tridiagonal pairs are [7–12]. For both Leonard pairs and tridiagonal pairs, there are connections to combinatorics [1, 16], representation theory [6, 8–11], special functions [2, 3, 21], and statistical mechanics [4, 5]. In [14], Godjali introduced the concept of a Hessenberg pair as a generalization of a tridiagonal pair. In [15], he considered a type of Hessenberg pair said to be thin, and he classified the thin Hessenberg pairs up to isomorphism. In [3], Baseilhac, Gainutdinov, and

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Vu introduced the concept of a cyclic tridiagonal pair as another generalization of a tridiagonal pair. They used this concept to study a higher-order generalization of the Onsager algebra.

In the present paper, we introduce the concept of a circular Hessenberg pair. This concept is a special case of both a thin Hessenberg pair and a cyclic tridiagonal pair; see Note 2.12. Roughly speaking, a circular Hessenberg pair consists of two diagonalizable linear maps on a nonzero finite-dimensional vector space, that each act on an eigenbasis of the other one in a circular Hessenberg fashion; see Definition 2.9. We now summarize our results. Let $A, A^*$ denote a circular Hessenberg pair. We discuss six bases for the underlying vector space that we find attractive. We display the transition matrices between certain pairs of bases among the six. We also display the matrices that represent $A$ and $A^*$ with respect to the six bases. We introduce a special type of circular Hessenberg pair, said to be recurrent; see Definition 2.20. We show that a circular Hessenberg pair $A, A^*$ is recurrent if and only if $A, A^*$ satisfy the tridiagonal relations; see Proposition 3.3 and the sentence beneath it. For a circular Hessenberg pair, there is a related object called a circular Hessenberg system; see Definition 2.11. We classify up to isomorphism the recurrent circular Hessenberg systems. To do this, we give a method for constructing a recurrent circular Hessenberg system; see Theorem 2.17. Using this method, we obtain four families of recurrent circular Hessenberg systems; see Examples 5.1–5.4. We prove that every recurrent circular Hessenberg system is isomorphic to a member of one of the four families; see Theorem 5.6. We conjecture that every circular Hessenberg system is recurrent.

This paper is organized as follows. In Section 2, we introduce the concept of a circular Hessenberg pair and a circular Hessenberg system. We also describe our main results. In Section 3, we consider a circular Hessenberg system. We discuss the corresponding eigenvalue sequence, dual eigenvalue sequence, and split sequence. In Section 4, we prove Theorem 2.17. In Section 5, we display four families of recurrent circular Hessenberg systems. We show that any recurrent circular Hessenberg system is isomorphic to a member of one of the four families. In Section 6, we consider any circular Hessenberg pair $A, A^*$. We identify six bases for the underlying vector space that we find attractive. We find the transition matrices between certain pairs of bases among the six. We also display the matrices that represent $A$ and $A^*$ with respect to each basis. In Section 7, we have some general comments about recurrent circular Hessenberg systems. We end the paper with an appendix that contains some formulas involving recurrent sequences.

## 2 Circular Hessenberg pairs

In this section, we introduce circular Hessenberg pairs and circular Hessenberg systems along with a detailed motivation. We use the following terms and notation. Throughout this paper, $\mathbb{F}$ denotes a field. All algebras and vector spaces discussed in this paper are over $\mathbb{F}$. We fix an integer $d \geq 3$. Let $V$ denote a vector space with dimension $d + 1$. Let $\text{End}(V)$ denote the algebra consisting of the $\mathbb{F}$-linear maps from $V$ to $V$. Let $\text{Mat}_{d+1}(\mathbb{F})$ denote the algebra consisting of the $d + 1$ by $d + 1$ matrices that have entries in $\mathbb{F}$. We index the rows and columns by $0, 1, \ldots, d$. Let $\mathbb{F}^{d+1}$ denote the vector space consisting of the column vectors of length $d + 1$ that have entries in $\mathbb{F}$. We view $\mathbb{F}^{d+1}$ as a left module for $\text{Mat}_{d+1}(\mathbb{F})$. Let $\{v_i\}_{i=0}^d$ denote a basis for $V$. For $B \in \text{End}(V)$ and $M \in \text{Mat}_{d+1}(\mathbb{F})$ we say $M$ represents $B$ with respect to $\{v_i\}_{i=0}^d$ whenever $Bv_j = \sum_{i=0}^d M_{ij}v_i$ for $0 \leq j \leq d$. There is an algebra isomorphism $\text{End}(V) \rightarrow \text{Mat}_{d+1}(\mathbb{F})$ that sends each $B \in \text{End}(V)$ to the unique matrix in $\text{Mat}_{d+1}(\mathbb{F})$ that represents $B$ with respect to $\{v_i\}_{i=0}^d$. A matrix in $\text{Mat}_{d+1}(\mathbb{F})$ is called tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the
superdiagonal. This tridiagonal matrix is called \textit{irreducible} whenever each entry on the subdiagonal is nonzero, and each entry on the superdiagonal is nonzero. The following matrices are tridiagonal:

\[
\begin{pmatrix}
2 & 1 & 0 & 0 \\
3 & 5 & 7 & 0 \\
0 & 1 & 3 & 6 \\
0 & 0 & 9 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 & 0 \\
3 & 5 & 7 & 0 \\
0 & 1 & 0 & 6 \\
0 & 0 & 9 & 2
\end{pmatrix}, \quad
\begin{pmatrix}
2 & 0 & 0 & 0 \\
3 & 5 & 0 & 0 \\
0 & 0 & 0 & 6 \\
0 & 0 & 9 & 2
\end{pmatrix}.
\]

Observe that the tridiagonal matrices on the left and middle are irreducible.

\textbf{Definition 2.1} (cf. [17, Definition 1.1]). By a \textit{Leonard pair} on \(V\) we mean an ordered pair \(A,A^*\) of elements in \(\text{End}(V)\) such that:

(i) There exists a basis for \(V\) with respect to which the matrix representing \(A\) is diagonal and the matrix representing \(A^*\) is irreducible tridiagonal.

(ii) There exists a basis for \(V\) with respect to which the matrix representing \(A^*\) is diagonal and the matrix representing \(A\) is irreducible tridiagonal.

\textbf{Note 2.2.} It is a common notational convention to use \(A^*\) to represent the conjugate-transpose of \(A\). We are not using this convention. In a Leonard pair \(A, A^*\) the linear maps \(A\) and \(A^*\) are arbitrary subject to (i) and (ii) of Definition 2.1.

We now recall the notion of a Leonard system. An element \(A \in \text{End}(V)\) is said to be \textit{diagonalizable} whenever \(V\) is spanned by the eigenspaces of \(A\). The element \(A\) is called \textit{multiplicity-free} whenever \(A\) is diagonalizable and each eigenspace of \(A\) has dimension one. Note that \(A\) is multiplicity-free if and only if \(A\) has \(d + 1\) mutually distinct eigenvalues in \(\mathbb{F}\). Assume that \(A\) is multiplicity-free. Let \(\{\theta_i\}_{i=0}^{d}\) denote an ordering of eigenvalues of \(A\). For \(0 \leq i \leq d\), let \(V_i\) denote the eigenspace of \(A\) corresponding to \(\theta_i\). Further, define \(E_i \in \text{End}(V)\) such that \((E_i - I)V_i = 0\) and \(E_i V_j = 0\) if \(j \neq i\) \((0 \leq j \leq d)\). Here \(I\) denotes the identity of \(\text{End}(V)\). We observe that (i) \(E_i E_j = \delta_{ij} E_i\) \((0 \leq i, j \leq d)\); (ii) \(A E_i = E_i A = \theta_i E_i\) \((0 \leq i \leq d)\); (iii) \(\sum_{i=0}^{d} E_i = I\); (iv) \(V_i = E_i V\) \((0 \leq i \leq d)\). Moreover,

\[E_i = \prod_{0 \leq j \leq d, j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq d).\]  

We call \(E_i\) the \textit{primitive idempotent} of \(A\) for \(V_i\) (or \(\theta_i\)).

\textbf{Definition 2.3} (cf. [17, Definition 1.4]). By a \textit{Leonard system} on \(V\) we mean a sequence

\[\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})\]

of elements in \(\text{End}(V)\) that satisfy the conditions (i)–(v) below.

(i) Each of \(A, A^*\) is multiplicity-free;

(ii) \(\{E_i\}_{i=0}^{d}\) is an ordering of the primitive idempotents of \(A\);

(iii) \(\{E_i^*\}_{i=0}^{d}\) is an ordering of the primitive idempotents of \(A^*\);

(iv) \(E_i A^* E_j = \begin{cases} 0 & \text{if } |i - j| > 1; \\
\neq 0 & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d);\)

(v) \(E_i A E_j = \begin{cases} 0 & \text{if } |i - j| > 1; \\
\neq 0 & \text{if } |i - j| = 1 \end{cases} \quad (0 \leq i, j \leq d);\)

(vi) \(\sum_{i=0}^{d} E_i = I\).
\begin{align*}
E_i^* A E_j^* &= \begin{cases}
0 & \text{if } |i-j| > 1; \\
\neq 0 & \text{if } |i-j| = 1
\end{cases} \quad (0 \leq i, j \leq d).
\end{align*}

We refer to $d$ as the \textit{diameter} of $\Phi$. We say that $\Phi$ is \textit{over} $F$.

The notion of isomorphism for Leonard systems was introduced in [17, Definition 1.5]. Leonard systems are classified up to isomorphism [17]. This classification amounts to a linear algebraic version of Leonard’s theorem [13].

**Lemma 2.4** (cf. [17, Theorem 1.12]). \textit{Let $A, A^*$ denote a Leonard pair on $V$. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $F$ such that both}

\begin{align*}
0 &= [A, A^2 A^* - \beta AA^* A + A^* A^2 - \gamma (AA^* + A^* A) - \varrho A^*], \\
0 &= [A^*, A^* 2 A^* - \beta A^* AA^* + AA^* 2 - \gamma^* (A^* A + AA^*) - \varrho^* A],
\end{align*}

\textit{where } $[r, s]$ \textit{means } $rs - sr$. \textit{The relations (2), (3) are called the tridiagonal relations.}

We now recall the notion of a Hessenberg pair. This notion generalizes the notion of a Leonard pair. A matrix in $\text{Mat}_{d+1}(F)$ is called \textit{Hessenberg} whenever each entry below the subdiagonal is zero and each entry on the subdiagonal is nonzero. The following matrices are Hessenberg:

\[
\begin{pmatrix}
2 & 1 & 4 & 9 \\
3 & 7 & 8 & \quad \\
0 & 1 & 3 & 6 \\
0 & 0 & 9 & 2
\end{pmatrix}, \quad \begin{pmatrix}
2 & 1 & 0 & 0 \\
3 & 5 & 7 & 0 \\
0 & 1 & 0 & 6 \\
0 & 0 & 9 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 9 & 0
\end{pmatrix}.
\]

**Definition 2.5** (cf. [15, Definition 1.1]). \textit{By a Hessenberg pair on $V$ we mean an ordered pair $A, A^*$ of elements in $\text{End}(V)$ such that:}

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is diagonal and the matrix representing $A^*$ is Hessenberg.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is diagonal and the matrix representing $A$ is Hessenberg.

**Note 2.6.** Our concept of a Hessenberg pair is slightly different from the one in [14,15]. What we call a Hessenberg pair is called a \textit{thin} Hessenberg pair in [15].

For a Hessenberg pair $A, A^*$ on $V$, each of $A, A^*$ is multiplicity-free [15, Lemma 2.1]. We now recall a Hessenberg system.

**Definition 2.7** (cf. [15, Definition 2.2]). \textit{By a Hessenberg system on $V$ we mean a sequence}

\[
\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)
\]

\textit{of elements in $\text{End}(V)$ that satisfy the conditions (i)–(v) below.}

(i) Each of $A, A^*$ is multiplicity-free;
(ii) \(\{E_i\}_{i=0}^{d} \) is an ordering of the primitive idempotents of \(A\);

(iii) \(\{E^*_i\}_{i=0}^{d} \) is an ordering of the primitive idempotents of \(A^*\);

(iv) \(E_i A^* E_j = \begin{cases} 0 & \text{if } i - j > 1; \\ \neq 0 & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d); \)

(v) \(E^*_i A E^*_j = \begin{cases} 0 & \text{if } i - j > 1; \\ \neq 0 & \text{if } i - j = 1 \end{cases} \quad (0 \leq i, j \leq d). \)

We refer to \(d\) as the diameter of \(\Phi\). We say that \(\Phi\) is over \(\mathbb{F}\).

**Note 2.8.** What we call a Hessenberg system is called a thin Hessenberg system in [15].

We remark that Hessenberg pairs do not satisfy the tridiagonal relations (2), (3) in general.

We now define a circular Hessenberg pair. This is a special case of a Hessenberg pair. Assume that \(M \in \text{Mat}_{d+1}(\mathbb{F})\) is Hessenberg. Then \(M\) is called *circular* whenever the \((0, d)\)-entry of \(M\) is nonzero and every other entry above the superdiagonal is zero. The following matrices are circular Hessenberg:

\[
\begin{pmatrix}
2 & 1 & 0 & 9 \\
3 & 5 & 7 & 0 \\
0 & 1 & 3 & 6 \\
0 & 0 & 9 & 2
\end{pmatrix},
\begin{pmatrix}
2 & 0 & 0 & 9 \\
3 & 5 & 7 & 0 \\
0 & 1 & 0 & 6 \\
0 & 0 & 9 & 2
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 9 \\
3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 9 & 0
\end{pmatrix}.
\]

**Definition 2.9.** By a *circular Hessenberg pair* (or *CH pair*) on \(V\) we mean an ordered pair \(A, A^*\) of elements in \(\text{End}(V)\) such that:

(i) There exists a basis for \(V\) with respect to which the matrix representing \(A\) is diagonal and the matrix representing \(A^*\) is circular Hessenberg.

(ii) There exists a basis for \(V\) with respect to which the matrix representing \(A^*\) is diagonal and the matrix representing \(A\) is circular Hessenberg.

**Lemma 2.10.** Let \(A, A^*\) denote a CH pair on \(V\). Then the pair \(A, A^*\) is Hessenberg. Moreover, each of \(A, A^*\) is multiplicity-free.

**Proof.** The first assertion directly follows from the definition of a circular Hessenberg matrix. The second assertion follows from the first assertion and the comment below Note 2.6. \(\blacksquare\)

We now introduce a *circular Hessenberg system*.

**Definition 2.11.** By a *circular Hessenberg system* (or *CH system*) on \(V\) we mean a sequence

\[
\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E^*_i\}_{i=0}^{d})
\]

of elements in \(\text{End}(V)\) that satisfy the conditions (i)–(v) below.

(i) Each of \(A, A^*\) is multiplicity-free;

(ii) \(\{E_i\}_{i=0}^{d}\) is an ordering of the primitive idempotents of \(A\);
(iii) \( \{E_i^*\}_{i=0}^{d} \) is an ordering of the primitive idempotents of \( A^* \);

(iv) \( E_iA^*E_j = \begin{cases} 0 & \text{if } 1 < i - j \text{ or } 1 < j - i < d; \\ \neq 0 & \text{if } 1 = i - j \text{ or } j - i = d \end{cases} \quad (0 \leq i, j \leq d); \)

(v) \( E_i^*AE_j^* = \begin{cases} 0 & \text{if } 1 < i - j \text{ or } 1 < j - i < d; \\ \neq 0 & \text{if } 1 = i - j \text{ or } j - i = d \end{cases} \quad (0 \leq i, j \leq d). \)

We refer to \( d \) as the diameter of \( \Phi \). We say that \( \Phi \) is over \( \mathbb{F} \).

CH pairs and CH systems are related as follows. Let \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) denote a CH system on \( V \). Then \( A, A^* \) is a CH pair on \( V \). Conversely, let \( A, A^* \) denote a CH pair on \( V \). Then each of \( A, A^* \) is multiplicity-free by Lemma 2.10. Moreover, there exists an ordering \( \{E_i\}_{i=0}^d \) of the primitive idempotents of \( A \), and there exists an ordering \( \{E_i^*\}_{i=0}^d \) of the primitive idempotents of \( A^* \), such that \((A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)\) is a CH system on \( V \).

**Note 2.12.** Let \( A, A^* \) denote a CH pair on \( V \). By Lemma 2.10, \( A, A^* \) is a Hessenberg pair on \( V \). We next explain why \( A, A^* \) is a cyclic tridiagonal pair in the sense of [3, Definition 1.1]. By the comment above the note, there is a CH system \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) on \( V \). Define \( N = d+1 \) and consider the cyclic group \( \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z} \). The group elements are denoted by 0, 1, \ldots, \( d \). By Definition 2.11(iv),(v) we have

\[
A^*E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V,
AE_i^*V \subseteq E_{i-1}^*V + E_i^*V + E_{i+1}^*V
\]

for \( i \in \mathbb{Z}_N \). Let \( W \) denote a nonzero subspace of \( V \) such that \( AW \subseteq W \) and \( A^*W \subseteq W \). We show that \( W = V \). Let \( i \in \mathbb{Z}_N \). By (1) and since \( AW \subseteq W \), we have \( E_iW \subseteq W \). Moreover, \( E_iW \neq 0 \) if and only if \( E_iW = E_iV \). A similar comment applies to \( E_i^* \). Since \( W \neq 0 \), there exists \( j \in \mathbb{Z}_N \) such that \( E_jW \neq 0 \). Therefore, \( E_jV \subseteq W \). By Definition 2.11(iv) we have \( E_{i+1}^*AE_i \neq 0 \) for \( i \in \mathbb{Z}_N \). Taking \( i = j \) we find that \( E_{j+1}V \subseteq W \). Repeating this argument, we have \( E_rV \subseteq W \) for all \( r \in \mathbb{Z}_N \). Therefore, \( W = V \). By these comments, \( A, A^* \) is a cyclic tridiagonal pair on \( V \).

We will be discussing the notion of isomorphism for CH pairs and CH systems (cf. [14, Definition 2.5]). We now clarify what it means. Let \( A, A^* \) denote a CH pair on \( V \) and let \( B, B^* \) denote a CH pair on \( V' \). By an isomorphism of CH pairs from \( A, A^* \) to \( B, B^* \), we mean a vector space isomorphism \( \sigma : V \rightarrow V' \) such that \( \sigma A = B \sigma \) and \( \sigma A^* = B^* \sigma \). We say the CH pairs \( A, A^* \) and \( B, B^* \) are isomorphic whenever there exists an isomorphism of CH pairs from \( A, A^* \) to \( B, B^* \). Let \( \sigma : V \rightarrow V' \) be an isomorphism of vector spaces. For \( X \in \text{End}(V) \) abbreviate \( X^\sigma = \sigma X \sigma^{-1} \). Observe that the map \( \text{End}(V) \rightarrow \text{End}(V') \), \( X \mapsto X^\sigma \) is an isomorphism of algebras. For a CH system \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) on \( V \), write \( \Phi^\sigma = (A^\sigma; \{E_i^\sigma\}_{i=0}^d; A^*; \{E_i^{\sigma*}\}_{i=0}^d) \). Observe that \( \Phi^\sigma \) is a CH system on \( V' \). Let \( \Phi' \) denote a CH system on \( V' \). By an isomorphism of CH systems from \( \Phi \) to \( \Phi' \), we mean a vector space isomorphism \( \sigma : V \rightarrow V' \) such that \( \Phi^\sigma = \Phi' \). We say that the CH systems \( \Phi \) and \( \Phi' \) are isomorphic whenever there exists an isomorphism of CH systems from \( \Phi \) to \( \Phi' \).

Regarding Lemma 2.4, it is natural to ask if a CH pair satisfies the tridiagonal relations (2), (3).
Conjecture 2.13. Let $A, A^*$ denote a CH pair on $V$. Then there exists a sequence of scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ taken from $F$ such that both (2) and (3) hold.

Shortly we will give another version of the above conjecture. We have some comments about CH pairs and CH systems. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a CH system on $V$. For $0 \leq i \leq d$, let $\theta_i$ (resp. $\theta_i^*$) denote the eigenvalue of $A$ (resp. $A^*$) corresponding to $E_i$ (resp. $E_i^*$).

We refer to $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) as the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$. By [15, Proposition 5.9] there exists a unique sequence $\{\phi_i\}_{i=1}^d$ of nonzero scalars in $F$ with the following property: there exists a basis for $V$ with respect to which the matrices representing $A$ and $A^*$ are

$$A : \begin{pmatrix} \theta_d & \theta_{d-1} & 0 \\ 1 & \theta_{d-2} & \ddots \\ 0 & \ddots & \ddots & \theta_0 \end{pmatrix}, \quad A^* : \begin{pmatrix} \theta_0^* & \phi_1 & \phi_2 & \cdots \\ \phi_1 & \theta_1^* & \theta_2^* & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \phi_d & \theta_d^* \end{pmatrix}. \quad (5)$$

The sequence $\{\phi_i\}_{i=1}^d$ is called the $\Phi$-split sequence. The above basis is called a $\Phi$-split basis. In Section 3, we shall discuss the $\Phi$-split sequence and the $\Phi$-split basis in detail. For the rest of this paper we use the following notation.

Notation 2.14. Let $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\phi_i\}_{i=1}^d$ denote scalars in $F$ such that

(i) $\theta_i \neq \theta_j, \; \theta_i^* \neq \theta_j^*$ if $i \neq j \; (0 \leq i, j \leq d)$,

(ii) $\phi_i \neq 0 \; (1 \leq i \leq d)$.

Definition 2.15. Referring to Notation 2.14, define the matrices $A, A^* \in \text{Mat}_{d+1}(F)$ by

$$A = \begin{pmatrix} \theta_d & \theta_{d-1} & 0 \\ 1 & \theta_{d-2} & \ddots \\ 0 & \ddots & \ddots & \theta_0 \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & \phi_1 & \phi_2 & \cdots \\ \phi_1 & \theta_1^* & \theta_2^* & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \phi_d & \theta_d^* \end{pmatrix}. $$

Observe that $A, A^*$ are multiplicity-free. For $0 \leq i \leq d$, let $E_i$ (resp. $E_i^*$) denote the primitive idempotent of $A$ (resp. $A^*$) with respect to $\theta_i$ (resp. $\theta_i^*$).

Recall the scalars $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\phi_i\}_{i=1}^d$ from Notation 2.14. In [17, Section 11], the following scalars $\{\vartheta_i\}_{i=0}^{d+1}$ are introduced:

$$\vartheta_i = \phi_i - (\theta_i^* - \theta_i^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d), \quad \vartheta_0 = 0, \quad \vartheta_{d+1} = 0. \quad (6)$$

Next we discuss how the scalars $\{\vartheta_i\}_{i=0}^{d+1}$ are related to the tridiagonal relations (2), (3). In what follows, we will use the concept of a $\beta$-recurrent sequence; see Definition 8.1.
**Lemma 2.16** (cf. [17, Theorem 12.5]). Let the scalars \( \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d \) be as in Notation 2.14. Let the matrices \( A, A^* \) be as in Definition 2.15. Let \( \beta \) denote any scalar in \( F \). Then there exist scalars \( \gamma, \gamma^*, \varrho, \varrho^* \) in \( F \) such that

\[
0 = [A, A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \varrho A^*],
\]

\[
0 = [A^*, A^{2*}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \varrho^*A]
\]

if and only if (i)–(iii) hold below.

(i) The sequence \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent.

(ii) The sequence \( \{\theta_i^*\}_{i=0}^d \) is \( \beta \)-recurrent.

(iii) The sequence \( \{\vartheta_i\}_{i=0}^{d+1} \) from (6) is \( \beta \)-recurrent.

We now state our first main theorem.

**Theorem 2.17.** Referring to Notation 2.14 and Definition 2.15, assume that there exists \( \beta \in F \) such that (i) the sequence \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent; (ii) the sequence \( \{\theta_i^*\}_{i=0}^d \) is \( \beta \)-recurrent; (iii) the sequence \( \{\vartheta_i\}_{i=0}^{d+1} \) from (6) is \( \beta \)-recurrent. Further assume that \( \vartheta_1 \neq \vartheta_d \). Then the sequence

\[
\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)
\]

(7)

is a CH system on \( \mathbb{F}^{d+1} \).

The proof of Theorem 2.17 appears in Section 4.

Motivated by Theorem 2.17, we make a definition.

**Definition 2.18.** Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a CH system on \( V \), with eigenvalue sequence \( \{\theta_i\}_{i=0}^d \) and dual eigenvalue sequence \( \{\theta_i^*\}_{i=0}^d \). Let \( \{\phi_i\}_{i=1}^d \) denote the \( \Phi \)-split sequence. Then for \( \beta \in \mathbb{F} \), we say that \( \Phi \) is \( \beta \)-recurrent whenever

(i) the sequence \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent,

(ii) the sequence \( \{\theta_i^*\}_{i=0}^d \) is \( \beta \)-recurrent,

(iii) the sequence \( \{\vartheta_i\}_{i=0}^{d+1} \) from (6) is \( \beta \)-recurrent.

**Definition 2.19.** Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a CH system on \( V \). We say that \( \Phi \) is recurrent whenever there exists \( \beta \in \mathbb{F} \) such that \( \Phi \) is \( \beta \)-recurrent.

**Definition 2.20.** Let \( A, A^* \) denote a CH pair on \( V \). We say that \( A, A^* \) is recurrent whenever there exists a CH system \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) on \( V \) that is recurrent.

By Definitions 2.18 and 2.19, we observe that the CH system from Theorem 2.17 is recurrent. We now give another version of Conjecture 2.13.

**Conjecture 2.21.** Every CH system on \( V \) is recurrent.

We remark that Conjecture 2.13 is equivalent to Conjecture 2.21. Indeed, if Conjecture 2.21 is true, then Conjecture 2.13 is also true by Lemma 2.16. Conversely, suppose that Conjecture 2.13 is true. Let \( \Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d) \) denote a CH system on \( V \). Then by Lemma 2.16, \( \Phi \) is \( \beta \)-recurrent, where \( \beta \) is from Conjecture 2.13. By Definition 2.19, \( \Phi \) is recurrent. Therefore, Conjecture 2.21 is true.
3 The $\Phi$-split sequence and the $\Phi$-split basis

We continue to discuss the CH system $\Phi = (A; \{E_i^d\}_{i=0}^d; A^*; \{E_i^d\}_{i=0}^d)$ from Definition 2.11. In the previous section, we discussed the $\Phi$-split sequence and the $\Phi$-split basis. In this section, we discuss these topics in more detail. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^d\}_{i=0}^d$) denote the eigenvalue sequence (resp. dual eigenvalue sequence) of $\Phi$. By a decomposition of $V$, we mean a sequence $\{U_i\}_{i=0}^d$ of one-dimensional subspaces of $V$ such that

$$V = \sum_{i=0}^d U_i$$

(direct sum).

For example, each of the sequences $\{E_i V\}_{i=0}^d$ and $\{E_i^d V\}_{i=0}^d$ is a decomposition of $V$. We now recall the $\Phi$-split decomposition [15, Section 4]. For $0 \leq i \leq d$ define

$$U_i = (E_0^i V + E_1^i V + \cdots + E_i^i V) \cap (E_0 V + E_1 V + \cdots + E_{d-i} V).$$

Then the sequence $\{U_i\}_{i=0}^d$ is a decomposition of $V$. Observe that

$$U_0 = E_0^* V, \quad U_d = E_0 V. \quad (8)$$

Moreover,

$$(A - \theta_{d-i} I) U_i = U_{i+1} \quad (0 \leq i \leq d - 1), \quad (A - \theta_0 I) U_d = 0, \quad (9)$$

$$(A^* - \theta_i^d I) U_i = U_{i-1} \quad (1 \leq i \leq d), \quad (A^* - \theta_0^d I) U_0 = 0. \quad (10)$$

The sequence $\{U_i\}_{i=0}^d$ is called the $\Phi$-split decomposition of $V$. By (8) and (9),

$$U_i = (A - \theta_{d-i+1} I) \cdots (A - \theta_{d-1} I)(A - \theta_d I) E_0^i V \quad (0 \leq i \leq d). \quad (11)$$

Combining (9) and (10) we find that for $1 \leq i \leq d$,

$$(A - \theta_{d-i+1} I)(A^* - \theta_i^d I) U_i = U_i. \quad (12)$$

Observe that $U_i$ is invariant under $(A - \theta_{d-i+1} I)(A^* - \theta_i^d I)$ and the corresponding eigenvalue is a nonzero element of $\mathbb{F}$. We denote this eigenvalue by $\phi_i$.

**Lemma 3.1.** With the above notation, for $1 \leq i \leq d$ the subspace $U_{i-1}$ is invariant under $(A^* - \theta_i^d I)(A - \theta_{d-i+1} I)$ and the corresponding eigenvalue is $\phi_i$.

**Proof.** Pick $0 \neq u \in U_{i-1}$ and $0 \neq v \in U_i$. By (9), there exists $0 \neq \lambda \in \mathbb{F}$ such that $(A - \theta_{d-i+1} I)u = \lambda v$. By (10), there exists $0 \neq \mu \in \mathbb{F}$ such that $(A^* - \theta_i^d I)v = \mu u$. By these comments, we have $(A - \theta_{d-i+1} I)(A^* - \theta_i^d I)v = \lambda \mu v$. By the comment below (12), we find $\lambda \mu = \phi_i$. Observe that $(A^* - \theta_i^d I)(A - \theta_{d-i+1} I)u = \lambda \mu u = \phi_i u$. The result follows.

Fix a nonzero vector $u^* \in E_0^i V$. For $0 \leq i \leq d$ define

$$v_i = (A - \theta_{d-i+1} I) \cdots (A - \theta_{d-1} I)(A - \theta_d I) u^*. \quad (13)$$

Comparing (11) and (13) we see that $v_i$ is a nonzero element in $U_i$. Therefore, the vectors $\{v_i\}_{i=0}^d$ form a basis for $V$. 

9
Proposition 3.2. Consider the basis \( \{ v_i \}_{i=0}^d \) for \( V \) from (13). With respect to this basis, the matrices representing \( A \) and \( A^* \) are

\[
A = \begin{pmatrix} \theta_d & 0 & \cdots & 0 \\ 1 & \theta_{d-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \theta_0 \end{pmatrix}, \quad A^* = \begin{pmatrix} \theta_0^* & \phi_1 & \cdots & 0 \\ \phi^*_1 & \theta_1^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \phi_d \end{pmatrix}.
\]

Proof. Consider the action of \( A \) on \( \{ v_i \}_{i=0}^d \). By (13), for \( 0 \leq i \leq d-1 \) we have \( (A - \theta_{d-i} I)v_i = v_{i+1} \), and therefore \( Av_i = \theta_{d-i}v_i + v_{i+1} \). By the equation on the right in (9), \( v_d \) is an eigenvector of \( A \) with eigenvalue \( \theta_0 \). By these comments, the matrix on the left in (14) represents \( A \) with respect to \( \{ v_i \}_{i=0}^d \). Next, we consider the action of \( A^* \) on \( \{ v_i \}_{i=0}^d \). By Lemma 3.1, for \( 1 \leq i \leq d \) we have \( (A^* - \theta_i^* I)v_i = (A^* - \theta_i^* I)(A - \theta_{d-i+1} I)v_{i-1} = \phi_i v_{i-1} \), and therefore \( A^* v_i = \theta_i^* v_i + \phi_i v_{i-1} \). By the equation on the right in (10), \( v_0 \) is an eigenvector of \( A^* \) with eigenvalue \( \theta_0^* \). By these comments, the matrix on the right in (14) represents \( A^* \) with respect to \( \{ v_i \}_{i=0}^d \).

We comment on Proposition 3.2. Comparing (5) and (14), we see that \( \{ \phi_i \}_{i=1}^d \) is the \( \Phi \)-split sequence and \( \{ v_i \}_{i=0}^d \) is a \( \Phi \)-split basis for \( V \).

Proposition 3.3. Let \( \beta \in \mathbb{F} \) and let \( \Phi = (A; \{ E_i \}_{i=0}^d; A^*; \{ E_i^* \}_{i=0}^d) \) denote a CH system on \( V \). Then the following (i), (ii) are equivalent:

(i) \( \Phi \) is \( \beta \)-recurrent.

(ii) There exists a sequence of scalars \( \gamma, \gamma^*, \varphi, \varphi^* \) taken from \( \mathbb{F} \) such that both (2) and (3) hold.

Proof. By Lemma 2.16, Definition 2.18 and Proposition 3.2.

From Proposition 3.3, we find that a CH pair \( A, A^* \) is recurrent if and only if \( A, A^* \) satisfy the tridiagonal relations (2) and (3).

We now define the dual of a CH system.

Definition 3.4. Let \( \Phi = (A; \{ E_i \}_{i=0}^d; A^*; \{ E_i^* \}_{i=0}^d) \) denote a CH system on \( V \). Observe that \( \Phi^* = (A^*; \{ E_i^* \}_{i=0}^d; A; \{ E_i \}_{i=0}^d) \) is a CH system on \( V \). We call \( \Phi^* \) the dual of \( \Phi \).

Lemma 3.5. Let \( \Phi = (A; \{ E_i \}_{i=0}^d; A^*; \{ E_i^* \}_{i=0}^d) \) denote a CH system on \( V \). Let \( \{ U_i \}_{i=0}^d \) denote the \( \Phi \)-split decomposition of \( V \). Then \( \{ U_{d-i} \}_{i=0}^d \) is the \( \Phi^* \)-split decomposition of \( V \). Moreover, the eigenvalue sequences, the dual eigenvalue sequences, and the split sequences of \( \Phi \) and \( \Phi^* \) are related as follows.

| CH system | eigenvalue sequence | dual eigenvalue sequence | split sequence |
|-----------|---------------------|-------------------------|---------------|
| \( \Phi \) | \( \{ \theta_i \}_{i=0}^d \) | \( \{ \theta_i^* \}_{i=0}^d \) | \( \{ \phi_i \}_{i=0}^d \) |
| \( \Phi^* \) | \( \{ \theta_i^* \}_{i=0}^d \) | \( \{ \theta_i \}_{i=0}^d \) | \( \{ \phi_{d-i+1} \}_{i=0}^d \) |

Proof. Routine.
Earlier we defined the Φ-split basis of $V$. We now describe the $Φ^*$-split basis of $V$. Fix a nonzero vector $u ∈ E_0 V$. For $0 ≤ i ≤ d$ define

$$v_i^* = (A^* - θ_{d-i+1}^* I) \cdots (A^* - θ_{d-1}^* I)(A^* - θ_d^* I)u.$$  

(15)

Then $\{v_i^*\}_{i=0}^d$ is the $Φ^*$-split basis for $V$. Applying Proposition 3.2 to $Φ^*$, we find that the matrices representing $A$ and $A^*$ with respect to $\{v_i^*\}_{i=0}^d$ are

$$A : \begin{pmatrix} θ_0 & φ_d & 0 & \cdots & 0 \\ θ_1 & φ_{d-1} & \theta_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \phi_1 & \theta_d \\ \end{pmatrix}, \quad A^* : \begin{pmatrix} θ_d^* & 0 & \cdots & 0 \\ 1 & \theta_{d-1}^* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & θ_0^* \\ \end{pmatrix}. \quad (16)$$

Next, we discuss how the $Φ^*$-split bases and the $Φ^*$-split bases are related.

**Definition 3.6.** Let $Φ = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a CH system on $V$. Recall the nonzero vector $u^* ∈ E_0^* V$ from above line (13) and the nonzero vector $u ∈ E_0 V$ from above line (15). Observe that $E_0 u^* ∈ E_0 V$ and $E_0^* u ∈ E_0^* V$. Therefore, there exist scalars $ε, ε∗ ∈ F$ such that

$$E_0 u^* = εu, \quad E_0^* u = ε^* u^*.$$  

**Lemma 3.7.** With reference to Definition 3.6, $\text{tr}(E_0 E_0^*) = ε ε^*.$

**Proof.** Recall the equation $E_0 u^* = εu$. Applying $E_0^*$ to both sides and using $E_0^* u = ε^* u^*$, we have $E_0^* E_0 u^* = ε ε^* u^*$. Since $u^* = E_0^* u^*$, it follows that $E_0^* E_0 u^* = ε ε^* E_0^* u^*$. Therefore, we have $E_0^* E_0 E_0^* = ε ε^* E_0^*$. Take the trace of both sides to get $\text{tr}(E_0^* E_0 E_0^*) = ε ε^* \text{tr}(E_0^*)$. Since $\text{tr}(E_0^* E_0 E_0^*) = \text{tr}(E_0^* E_0^* E_0^*) = \text{tr}(E_0^* E_0^*)$ and $\text{tr}(E_0^*) = 1$, the result follows. \[\Box\]

**Lemma 3.8.** With reference to Definition 3.6,

$$ε ε^* = \frac{φ_1 φ_2 \cdots φ_d}{(θ_0 - θ_1)(θ_0 - θ_2) \cdots (θ_0 - θ_d)(θ_0^* - θ_1^*)(θ_0^* - θ_2^*) \cdots (θ_0^* - θ_d^*)}.$$  

**Proof.** By Lemma 3.7 and [15, Lemma 7.5, Lemma 7.6]. \[\Box\]

The scalars $ε, ε^*$ are nonzero by Lemma 3.8.

**Note 3.9.** In [15, Section 7], a nonzero scalar $ν$ was introduced to study Hessenberg systems. We have $ν^{-1} = ε ε^*.$

**Proposition 3.10.** Recall the $Φ$-split basis $\{v_i\}_{i=0}^d$ from (13) and the $Φ^*$-split basis $\{v_i^*\}_{i=0}^d$ from (15). Then the following (i), (ii) hold:

(i) $v_i = \frac{ε (θ_0 - θ_1)(θ_0 - θ_2) \cdots (θ_0 - θ_d)}{φ_1 φ_2 \cdots φ_d} v_{d-i}^*$ \quad (0 ≤ i ≤ d).$

(ii) $v_i^* = \frac{ε (θ_0^* - θ_1^*)(θ_0^* - θ_2^*) \cdots (θ_0^* - θ_d^*)}{φ_1 φ_2 \cdots φ_{d-i}} v_{d-i}$ \quad (0 ≤ i ≤ d).
Proof. (i): By the matrix on the left in (14), we have \( A^*v_i = \theta_i^*v_i + \phi_iv_{i-1} \) for \( 1 \leq i \leq d \). This implies that \( v_{i-1} = (A^* - \theta_i^*I)v_i/\phi_i \). By induction on \( i \), we have
\[
v_i = \frac{(A^* - \theta_{i+1}^*I)(A^* - \theta_{i+2}^*I)\cdots(A^* - \theta_d^*I)v_d}{\phi_{i+1}\phi_{i+2}\cdots\phi_d} \quad (0 \leq i \leq d-1).
\] (17)

Evaluating (13) at \( i = d \) and using (1), we have
\[
v_d = (A - \theta_1I)(A - \theta_2I)\cdots(A - \theta_dI)u^* = (\theta_0 - \theta_1)(\theta_0 - \theta_2)\cdots(\theta_0 - \theta_d)E_0u^*.
\] (18)

Eliminate \( v_d \) in (17) using (18) along with \( E_0u^* = \varepsilon u \), and simplify the result to get
\[
v_i = \varepsilon \frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2)\cdots(\theta_0 - \theta_d)}{\phi_{i+1}\phi_{i+2}\cdots\phi_d} (A^* - \theta_{i+1}^*I)(A^* - \theta_{i+2}^*I)\cdots(A^* - \theta_d^*I)u
\] (19)
for \( 0 \leq i \leq d-1 \). By (15) we note that \( v_{d-i}^* = (A^* - \theta_{i+1}^*I)(A^* - \theta_{i+2}^*I)\cdots(A^* - \theta_d^*I)u \). Hence, (i) follows.

(ii): Similar to (i).

We finish this section with a comment. Let \( \Phi \) denote a CH system on \( V \). By the parameter array of \( \Phi \), we mean the sequence \( \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d \) where \( \{\theta_i\}_{i=0}^d \) is the eigenvalue sequence of \( \Phi \), \( \{\theta_i^*\}_{i=0}^d \) is the dual eigenvalue sequence of \( \Phi \), and \( \{\phi_i\}_{i=1}^d \) is the \( \Phi \)-split sequence.

Lemma 3.11. Let \( \Phi \) and \( \Phi' \) denote CH systems over \( \mathbb{F} \). Then \( \Phi \) and \( \Phi' \) are isomorphic if and only if they have the same parameter array.

Proof. By [15, Theorem 6.3].

4 The proof of Theorem 2.17

In this section we prove Theorem 2.17. Throughout this section, we refer to Notation 2.14 and Definition 2.15. Shortly we will be referring to the results of [22]. We note that the element \( E_i \) in [22] corresponds to the \( E_{d-i} \) in the present paper.

Lemma 4.1. The following (i), (ii) hold. For \( 0 \leq i, j \leq d \),

\[
(i) \quad E_iA^*E_j = \begin{cases} 0 & \text{if } i - j > 1, \\ \neq 0 & \text{if } i - j = 1. \end{cases}
\]

\[
(ii) \quad E_i^*AE_j^* = \begin{cases} 0 & \text{if } i - j > 1, \\ \neq 0 & \text{if } i - j = 1. \end{cases}
\]

Proof. (i): Use [22, Proposition 7.6], [22, Lemma 7.7].

(ii): By [22, Proposition 7.6].

Lemma 4.2. For \( 0 \leq i \leq d \), both \( E_iE_0^* \neq 0 \) and \( E_i^*E_0 \neq 0 \).

Proof. By Lemma 4.1(ii) and [22, Lemma 7.5].
Lemma 4.3. Both

\[ \sum_{i=2}^{d} E_0 A^i E_0^* (\theta_i - \theta_1) = E_0 E_0^* (\vartheta_1 - \vartheta_d), \]  
(20)

\[ \sum_{i=2}^{d} E_0^* A E_0 (\theta_i^* - \theta_1^*) = E_0 E_0^* (\vartheta_1 - \vartheta_d). \]  
(21)

Proof. By [22, Proposition 8.4] and [22, Lemma 8.5].

Lemma 4.4. Assume that there exists \( \beta \in \mathbb{F} \) such that each of \( \{ \theta_i \}_{i=0}^{d} \), \( \{ \theta_i^* \}_{i=0}^{d} \), \( \{ \vartheta_i \}_{i=0}^{d+1} \) is \( \beta \)-recurrent. Then the following (i)–(iv) hold.

(i) \( E_i A^* E_j = 0 \) if \( 1 < j - i < d \) \((0 \leq i, j \leq d)\).

(ii) \( E_i^* A E_j^* = 0 \) if \( 1 < j - i < d \) \((0 \leq i, j \leq d)\).

(iii) \( E_0 A^* E_d \neq 0 \) if and only if \( \vartheta_1 \neq \vartheta_d \).

(iv) \( E_0^* A E_d^* \neq 0 \) if and only if \( \vartheta_1 \neq \vartheta_d \).

Proof. (i): By a slight modification of the proof of line (60) in [22, Section 17]. (ii): Similar to the proof of (i). (iii): Recall the equation (20). By (i), we have \( E_0 A^* E_j = 0 \) for \( 1 < j < d \). Applying this to the equation (20), we have the equation \( E_0 A^* E_d E_0^* (\theta_d - \theta_1) = E_0 E_0^* (\vartheta_1 - \vartheta_d) \). Note that \( \theta_d \neq \theta_1 \). Also, \( E_0 E_0^* \neq 0 \) by Lemma 4.2. Therefore, \( \vartheta_1 \neq \vartheta_d \) if and only if \( E_0 A^* E_d E_0^* \neq 0 \). By Lemma 4.2 and [22, Proposition 6.4] with \( X = E_0 A^* \) and \( i = d, \) \( E_0 A^* E_d E_0^* \neq 0 \) if and only if \( E_0 A^* E_d \neq 0 \). (iv): Similar to the proof of (iii).

We are now ready to prove Theorem 2.17.

Proof of Theorem 2.17. To prove that the sequence \( \Phi = (A; \{ E_i \}_{i=0}^{d}; A^*; \{ E_i^* \}_{i=0}^{d}) \) is a CH system, we show that \( \Phi \) satisfies the conditions (i)–(v) of Definition 2.11. By construction, the conditions (i)–(iii) hold. The condition (iv) follows from Lemma 4.1(i) and Lemma 4.4(i),(iii). The condition (v) follows from Lemma 4.1(ii) and Lemma 4.4(ii),(iv). The result follows.

5 Four families of CH systems

In this section, we classify up to isomorphism the recurrent CH systems. To do this, we first display four families of recurrent CH systems. We then show that every recurrent CH system is isomorphic to a member of one of the four families.

For the rest of this section, let \( \{ \theta_i \}_{i=0}^{d} \), \( \{ \theta_i^* \}_{i=0}^{d} \), \( \{ \vartheta_i \}_{i=1}^{d} \) denote scalars in \( \mathbb{F} \). Let \( \overline{\mathbb{F}} \) denote the algebraic closure of \( \mathbb{F} \).

Example 5.1. Fix \( 0 \neq q \in \mathbb{F} \) such that \( q + q^{-1} \in \mathbb{F} \). Assume that \( q^i \neq 1 \) for \( 1 \leq i \leq d \) and \( q^{d+1} = 1. \) Notice that \( q \neq \pm 1 \) since \( d \geq 3 \). Set \( \beta = q + q^{-1} \). Observe that \( \beta \neq \pm 2 \). Let \( a, b, c, a^*, b^*, c^* \) be scalars taken from \( \overline{\mathbb{F}} \). Assume

\[ \theta_i = a + bq^i + cq^{-i}, \]  
(22)

\[ \theta_i^* = a^* + b^* q^i + c^* q^{-i}. \]  
(23)
for $0 \leq i \leq d$. Assume that $c \neq bq^i$ and $c^* \neq b^*q^i$ for $1 \leq i \leq 2d - 1$. By these assumptions, the scalars $\{\theta^i\}_{i=0}^d$ are mutually distinct and $\beta$-recurrent. Similarly, the scalars $\{\theta^*_i\}_{i=0}^d$ are mutually distinct and $\beta$-recurrent. Pick distinct $y, z \in \mathbb{F}$. Assume

$$\phi_i = (q^i - 1)(y - zq^{-i}) + (q^i - 1)(q^{-i} - 1)(b - cq^i)(b^* - c^*q^{-i})$$

(24)

for $1 \leq i \leq d$. Assume that $y, z$ are chosen so that $\phi_i \neq 0$ for $1 \leq i \leq d$. From (22)–(24), the scalars $\{\vartheta^i\}_{i=0}^{d+1}$ in (6) satisfy

$$\vartheta_i = (q^i - 1)(y - zq^{-i})$$

(25)

for $1 \leq i \leq d$ along with $\vartheta_0 = 0$ and $\vartheta_{d+1} = 0$. Observe that the sequence $\{\vartheta^i\}_{i=0}^{d+1}$ is $\beta$-recurrent. Also, observe that $\vartheta_1 - \vartheta_d = q(1 - d^{-1})(y - z)$, so $\vartheta_1 \neq \vartheta_d$. We have shown that the scalars $\{\theta^i\}_{i=0}^d$, $\{\theta^*_i\}_{i=0}^d$, $\{\vartheta^i\}_{i=0}^{d+1}$ satisfy the conditions of Theorem 2.17. By this theorem, the sequence

$$\Phi = (A; \{E^i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$$

(26)

from Definition 2.15 forms a CH system on $\mathbb{F}^{d+1}$. By construction, $\Phi$ is $\beta$-recurrent.

**Example 5.2.** Assume that $\text{Char}(\mathbb{F}) = d + 1$. Set $\beta = 2$. Let $a, b, c, a^*, b^*, c^*$ be scalars taken from $\mathbb{F}$. Assume

$$\theta_i = a + bi + ci(i - 1)/2,$$

(27)

$$\theta^*_i = a^* + b^*i + c^*i(i - 1)/2$$

(28)

for $0 \leq i \leq d$. Assume that $2b \neq c(1 - i)$ and $2b^* \neq c^*(1 - i)$ for $1 \leq i \leq 2d - 1$. By these assumptions, the scalars $\{\theta^i\}_{i=0}^d$ are mutually distinct and $\beta$-recurrent. Similarly, the scalars $\{\theta^*_i\}_{i=0}^d$ are mutually distinct and $\beta$-recurrent. Pick $y, z \in \mathbb{F}$ such that $2y \neq z$. Assume

$$\phi_i = i(y + z(i - 1)/2) - i^2(b + c(d - i)/2)(b^* + c^*(i - 1)/2)$$

(29)

for $1 \leq i \leq d$. Assume that $y, z$ are chosen so that $\phi_i \neq 0$ for $1 \leq i \leq d$. From (27)–(29), the scalars $\{\vartheta^i\}_{i=0}^{d+1}$ in (6) satisfy

$$\vartheta_i = yi + zi(i - 1)/2$$

(30)

for $1 \leq i \leq d$ along with $\vartheta_0 = 0$ and $\vartheta_{d+1} = 0$. Observe that the sequence $\{\vartheta^i\}_{i=0}^{d+1}$ is $\beta$-recurrent. Also, observe that $\vartheta_1 - \vartheta_d = 2y - z$, so $\vartheta_1 \neq \vartheta_d$. We have shown that the scalars $\{\theta^i\}_{i=0}^d$, $\{\theta^*_i\}_{i=0}^d$, $\{\vartheta^i\}_{i=0}^{d+1}$ satisfy the conditions of Theorem 2.17. By this theorem, the sequence

$$\Phi = (A; \{E^i\}_{i=0}^d; A^*; \{E^*_i\}_{i=0}^d)$$

(31)

from Definition 2.15 forms a CH system on $\mathbb{F}^{d+1}$. By construction, $\Phi$ is $\beta$-recurrent.

**Example 5.3.** Assume that $d$ is odd and $d \geq 5$ and $\text{Char}(\mathbb{F}) = (d + 1)/2$. Set $\beta = -2$. Let $a, b, c, a^*, b^*, c^*$ be scalars taken from $\mathbb{F}$. Assume

$$\theta_i = a + b(-1)^i + ci(-1)^i,$$

(32)

$$\theta^*_i = a^* + b^*(-1)^i + c^*i(-1)^i$$

(33)

for $0 \leq i \leq d$. Assume that none of $b, b^*, c, c^*$ is zero. Also assume that $2b \neq -ic$ and $2b^* \neq -ic^*$ for $1 \leq i \leq 2d - 1$ with $i$ odd. By these assumptions, the scalars $\{\theta^i\}_{i=0}^d$ are mutually distinct.
and $\beta$-recurrent. Similarly, the scalars $\{\theta_i^*\}_{i=0}^d$ are mutually distinct and $\beta$-recurrent. Pick $y, z \in F$ such that $z \neq 0$. Assume

$$
\phi_i = y \left( (-1)^i - 1 \right) + zi(-1)^i + \left( b((-1)^i - 1) - ci(-1)^i \right) \left( b^*((-1)^i - 1) + c^*i(-1)^i \right)
$$

(34)

for $1 \leq i \leq d$. Assume that $y, z$ are chosen so that $\phi_i \neq 0$ for $1 \leq i \leq d$. From (32)–(34), the scalars $\{\vartheta_i\}_{i=0}^{d+1}$ in (6) satisfy

$$
\vartheta_i = y((-1)^i - 1) + zi(-1)^i
$$

(35)

for $1 \leq i \leq d$ along with $\vartheta_0 = 0$ and $\vartheta_{d+1} = 0$. Observe that the sequence $\{\vartheta_i\}_{i=0}^{d+1}$ is $\beta$-recurrent. Also, observe that $\vartheta_1 - \vartheta_d = z(d-1)$, so $\vartheta_1 \neq \vartheta_d$. We have shown that the scalars $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\vartheta_i\}_{i=0}^{d+1}$ satisfy the conditions of Theorem 2.17. By this theorem, the sequence

$$
\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)
$$

(36)

from Definition 2.15 forms a CH system on $F^{d+1}$. By construction, $\Phi$ is $\beta$-recurrent.

**Example 5.4.** Assume that $d = 3$ and $\text{Char}(F) = 2$. Set $\beta = 0$. Let $a, b, c, a^*, b^*, c^*$ be scalars taken from $F$. Assume

$$
\theta_i = a + bi + c \binom{i}{2},
$$

(37)

$$
\theta_i^* = a^* + b^*i + c^* \binom{i}{2}
$$

(38)

for $0 \leq i \leq 3$ and where we interpret the binomial coefficient as follows:

$$
\binom{n}{2} = \begin{cases} 
0 & \text{if } n = 0 \text{ or } n = 1 \pmod{4}, \\
1 & \text{if } n = 2 \text{ or } n = 3 \pmod{4}.
\end{cases}
$$

(39)

Assume that none of $b, b^*, c, c^*$ is zero. Also assume that $b \neq c$ and $b^* \neq c^*$. By these assumptions, the scalars $\{\theta_i\}_{i=0}^3$ are mutually distinct and $\beta$-recurrent. Similarly, the scalars $\{\theta_i^*\}_{i=0}^3$ are mutually distinct and $\beta$-recurrent. Pick $y, z \in F$ such that $z \neq 0$. Assume

$$
\phi_i = yi + z \binom{i}{2} + (bi + c \binom{i+1}{2}) \left( b^*i + c^* \binom{i}{2} \right)
$$

(40)

for $1 \leq i \leq 3$. Assume that $y, z$ are chosen so that $\phi_i \neq 0$ for $1 \leq i \leq 3$. The scalars $\{\vartheta_i\}_{i=0}^4$ from (6) satisfy

$$
\vartheta_i = yi + z \binom{i}{2}
$$

(41)

for $1 \leq i \leq 3$ along with $\vartheta_0 = 0$ and $\vartheta_4 = 0$. Observe that the sequence $\{\vartheta_i\}_{i=0}^4$ is $\beta$-recurrent. Also, observe that $\vartheta_1 - \vartheta_3 = z$, so $\vartheta_1 \neq \vartheta_3$. We have shown that the scalars $\{\theta_i\}_{i=0}^3$, $\{\theta_i^*\}_{i=0}^3$, $\{\vartheta_i\}_{i=0}^4$ satisfy the conditions of Theorem 2.17. By this theorem, the sequence

$$
\Phi = (A; \{E_i\}_{i=0}^3; A^*; \{E_i^*\}_{i=0}^3)
$$

(42)

from Definition 2.15 forms a CH system on $F^4$. By construction, $\Phi$ is $\beta$-recurrent.
We have displayed four families of recurrent CH systems. Our next goal is to show that every recurrent CH system over $\mathbb{F}$ is isomorphic to a member of one of the four families. In order to obtain this result, we need a lemma.

**Lemma 5.5.** Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a recurrent CH system over $\mathbb{F}$. Let the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ denote the parameter array of $\Phi$. Recall the scalars $\{\vartheta_i\}_{i=0}^{d+1}$ from (6). Then $\vartheta_1 \neq \vartheta_d$.

**Proof.** Since $\Phi$ is recurrent, there exists $\beta \in \mathbb{F}$ such that each of $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\vartheta_i\}_{i=0}^{d+1}$ is $\beta$-recurrent. We have $E_0A^*E_d \neq 0$ by Definition 2.11(iv). The result follows from these comments and Lemma 4.4(iii). □

**Theorem 5.6.** Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a recurrent CH system over $\mathbb{F}$. Then $\Phi$ is isomorphic to a member of one of the four families in Examples 5.1–5.4.

**Proof.** Let the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ denote the parameter array of $\Phi$. Recall the scalars $\{\vartheta_i\}_{i=0}^{d+1}$ from (6). Since $\Phi$ is recurrent, there exists $\beta \in \mathbb{F}$ such that each of $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$, $\{\vartheta_i\}_{i=0}^{d+1}$ is $\beta$-recurrent. We have $\vartheta_1 \neq \vartheta_d$ by Lemma 5.5. Consider the $\Phi$-split sequence $\{\phi_i\}_{i=1}^d$. By (6),

$$\phi_i = \vartheta_i + (\theta_i^* - \theta_0^*)(\theta_{d-i+1} - \theta_0) \quad (1 \leq i \leq d).$$

(43)

We show that $\Phi$ is isomorphic to a member of one of the Examples 5.1–5.4. We divide the argument into the following four cases: (i) $\beta \neq \pm 2$; (ii) $\beta = 2$ and $\text{Char}(\mathbb{F}) \neq 2$; (iii) $\beta = -2$ and $\text{Char}(\mathbb{F}) \neq 2$; (iv) $\beta = 0$ and $\text{Char}(\mathbb{F}) = 2$.

**Case (i): $\beta \neq \pm 2$.**

Pick $0 \neq q \in \mathbb{F}$ such that $q + q^{-1} = \beta$. By Lemma 8.2(i) there exist scalars $a, b, c, a^*, b^*, c^* \in \mathbb{F}$ such that

$$\theta_i = a + bq_i + cq_i^{-1},$$

$$\theta_i^* = a^* + b^*q_i + c^*q_i^{-1}$$

(44)

(45)

for $0 \leq i \leq d$. We claim that $q^{d+1} = 1$ and $q^i \neq 1$ for $1 \leq i \leq d$. Since the sequence $\{\vartheta_i\}_{i=0}^{d+1}$ is $\beta$-recurrent, by Lemma 8.2(i) there exist scalars $x, y, z \in \mathbb{F}$ such that

$$\vartheta_i = x + yq_i + zq_i^{-1}$$

(46)

for $0 \leq i \leq d + 1$. Using $\vartheta_0 = 0$ and $\vartheta_{d+1} = 0$, we have $x + y + z = 0$ and

$$x + yq^{d+1} + zq^{-d-1} = 0.$$  

(47)

Eliminate $x$ in (47) using $x = -y - z$ and evaluate the result to get

$$(1 - q^{d+1})(y - zq^{-d-1}) = 0.$$  

(48)

Suppose that $q^{d+1} \neq 1$. Then $y = zq^{-d-1}$. Using this equation, eliminate $y$ in (46) to obtain

$$\vartheta_i = x + zq^{i-d-1} + zq^{-i} = \vartheta_{d-i+1}$$

(49)

for $0 \leq i \leq d + 1$. In particular, for $i = 1$ we have $\vartheta_1 = \vartheta_d$, a contradiction. Therefore, $q^{d+1} = 1$.

Next, we show that $q^i \neq 1$ for $1 \leq i \leq d$. Recall the eigenvalue sequence $\{\theta_i\}_{i=0}^d$ of $\Phi$. If $q^i = 1$ for
some $1 \leq i \leq d$, then $\theta_i = \theta_0$ by (44); this is a contradiction since the scalars $\{\theta_i\}_{i=0}^d$ are mutually distinct. Thus, $q^i \neq 1$ for $1 \leq i \leq d$. We have proved the claim.

Next, we discuss the scalars $\{\phi_i\}_{i=1}^d$. Evaluate the right-hand side of (43) using (44)–(46) to obtain
\[
\phi_i = (q^{i-1} - 1)(y - zq^{-i}) + (q^i - 1)(q^{i-1} - 1)(b - cq^i)(b^* - c^* q^{-i})
\] (50)
for $1 \leq i \leq d$. Comparing (22), (23), (24) with (44), (45), (50), respectively, and using Lemma 3.11, we find that $\Phi$ is isomorphic to the CH system shown in (26) of Example 5.1.

**Case (ii): $\beta = 2$ and Char($\mathbb{F}$) $\neq 2$.**

By Lemma 8.2(ii) there exist scalars $a, b, c, a^*, b^*, c^* \in \mathbb{F}$ such that
\[
\theta_i = a + bi + ci(i - 1)/2,
\] (51)
\[
\theta_i^* = a^* + b^*i + c^*i(i - 1)/2
\] (52)
for $0 \leq i \leq d$. We claim that Char($\mathbb{F}$) $= d + 1$. Since the sequence $\{\vartheta_i\}_{i=0}^{d+1}$ is $\beta$-recurrent, by Lemma 8.2(ii) there exist scalars $x, y, z \in \mathbb{F}$ such that
\[
\vartheta_i = x + yi + zi(i - 1)/2
\] (53)
for $0 \leq i \leq d + 1$. Using $\vartheta_0 = 0$ and $\vartheta_{d+1} = 0$, we have $x = 0$ and
\[
(d + 1)(y + zd/2) = 0.
\] (54)
Suppose that $d + 1 \neq 0$ in $\mathbb{F}$. Then $y = -zd/2$. Using this equation, eliminate $y$ in (53) to obtain
\[
\vartheta_i = \frac{(i - d - 1)i}{2} z = \vartheta_{d-i+1}
\] (55)
for $0 \leq i \leq d + 1$. In particular, for $i = 1$ we have $\vartheta_1 = \vartheta_d$, a contradiction. Therefore, $d + 1 = 0$ in $\mathbb{F}$. Next, we show that $i \neq 0$ in $\mathbb{F}$ for $1 \leq i \leq d$. Let $i$ be given. If $i = 0$ in $\mathbb{F}$, then $\theta_i = \theta_0$ by (51); this is a contradiction since the scalars $\{\theta_i\}_{i=0}^d$ are mutually distinct. Thus, $i \neq 0$ in $\mathbb{F}$. We have proved the claim.

Next, we discuss the scalars $\{\phi_i\}_{i=1}^d$. Evaluate the right-hand side of (43) using (51)–(53) to obtain
\[
\phi_i = i(y + z(i - 1)/2) - i^2(b + c(d - i)/2)(b^* + c^*(i - 1)/2)
\] (56)
for $1 \leq i \leq d$. Comparing (27), (28), (29) with (51), (52), (56), respectively, and using Lemma 3.11, we find that $\Phi$ is isomorphic to the CH system shown in (31) of Example 5.2.

**Case (iii): $\beta = -2$ and Char($\mathbb{F}$) $\neq 2$.**

By Lemma 8.2(iii) there exist scalars $a, b, c, a^*, b^*, c^* \in \mathbb{F}$ such that
\[
\theta_i = a + b(-1)^i + ci(-1)^i,
\] (57)
\[
\theta_i^* = a^* + b^*(-1)^i + c^*i(-1)^i
\] (58)
for $0 \leq i \leq d$. We claim that $d$ is odd, $d \geq 5$, and Char($\mathbb{F}$) $= (d + 1)/2$. We first show that $d$ is odd. Since the sequence $\{\vartheta_i\}_{i=0}^{d+1}$ is $\beta$-recurrent, by Lemma 8.2(iii) there exist scalars $x, y, z \in \mathbb{F}$ such that
\[
\vartheta_i = x + y(-1)^i + zi(-1)^i
\] (59)
for $0 \leq i \leq d+1$. Since $\vartheta_0 = 0$ and $\vartheta_{d+1} = 0$, we have $x+y = 0$ and
\[ x + y(-1)^{d+1} + z(d+1)(-1)^{d+1} = 0. \tag{60} \]
Eliminate $x$ in (60) using $x = -y$ and evaluate the result to get
\[ y((-1)^{d+1} - 1) + z(d+1)(-1)^{d+1} = 0. \tag{61} \]
Suppose that $d$ is even. Simplify the equation in (61) to get $2y = -(d+1)z$. Using this equation along with $x+y = 0$, eliminate $x$ and $y$ in (59) and simplify the result to obtain
\[ \vartheta_i = z \left( \frac{d+1}{2} - \frac{d+1}{2}(-1)^i \right) = \vartheta_{d-i+1}. \tag{62} \]
for $0 \leq i \leq d+1$. In particular, for $i = 1$ we have $\vartheta_1 = \vartheta_d$, a contradiction. Therefore, $d$ is odd.

Next, we show that $\text{Char}(\mathbb{F}) = (d+1)/2$. Abbreviate $p = \text{Char}(\mathbb{F})$. Recall that $p$ is zero or a prime number. Since $d$ is odd, we simplify the equation (61) to get $z(d+1) = 0$. If $d+1 \neq 0$ in $\mathbb{F}$, then $z = 0$. So, by (59) it follows that $\vartheta_i = -y + y(-1)^i$ for $0 \leq i \leq d+1$. However, this is a contradiction as $\vartheta_1 = -2y = \vartheta_d$. Thus, $d+1 = 0$ in $\mathbb{F}$. This implies that $p$ is a prime factor of $d+1$.

Next, we show that $(d+1)/2 \leq p$. Suppose not. Then $2p \leq d-1$. Setting $i = 2p$ in (57), we have $\theta_i = \theta_0$, a contradiction. We have shown that $(d+1)/2 \leq p$ and $p$ divides $d+1$. Therefore, either $p = (d+1)/2$ or $p = d+1$. We have $p$ is prime and $d$ is odd and $d \geq 3$, so $p \neq d+1$. Therefore, $p = (d+1)/2$. Lastly, we show that $d \geq 5$. If $d = 3$, then $p = 2$. But, $p \neq 2$ by assumption. Therefore, $d \geq 5$. We have proved the claim.

Next, we discuss the scalars $\{\phi_i\}_{i=1}^d$. Evaluate the right-hand side of (43) using (57)--(59) to obtain
\[ \phi_i = y((-1)^i - 1) + zi(-1)^i + \left(b((-1)^i - 1) - ci(-1)^i\right)\left(b^*((-1)^i - 1) + c^*i(-1)^i\right) \tag{63} \]
for $1 \leq i \leq d$. Comparing (32), (33), (34) with (57), (58), (63), respectively, and using Lemma 3.11, we find that $\Phi$ is isomorphic to the CH system shown in (36) of Example 5.3.

Case (iv): $\beta = 0$ and $\text{Char}(\mathbb{F}) = 2$.

By Lemma 8.2(iv) there exist scalars $a, b, c, a^*, b^*, c^* \in \mathbb{F}$ such that
\[ \theta_i = a + bi + c \begin{pmatrix} i \\ 2 \end{pmatrix}, \tag{64} \]
\[ \theta_i^* = a^* + b^*i + c^* \begin{pmatrix} i \\ 2 \end{pmatrix} \tag{65} \]
for $0 \leq i \leq d$. We claim that $d = 3$. If $d \geq 4$, then $\theta_0 = \theta_4$ by (64); this is a contradiction since the scalars $\{\theta_i\}_{i=0}^4$ are mutually distinct. Since $d \geq 3$, the claim follows.

Next, we discuss the scalars $\{\phi_i\}_{i=1}^3$. Since the sequence $\{\vartheta_i\}_{i=0}^3$ is $\beta$-recurrent, by Lemma 8.2(iv) there exist scalars $x, y, z \in \mathbb{F}$ such that
\[ \vartheta_i = x + yi + z \begin{pmatrix} i \\ 2 \end{pmatrix} \tag{66} \]
for $0 \leq i \leq 4$. Since $\vartheta_0 = 0$, we have $x = 0$. Evaluate the right-hand side of (43) using (64)–(66) to obtain

$$\phi_i = yi + z\binom{i}{2} + \left( bi + c \binom{i + 1}{2} \right) \left( b^*i + c^* \binom{i}{2} \right)$$

(67)

for $1 \leq i \leq 3$. Comparing (37), (38), (40) with (64), (65), (67), respectively, and using Lemma 3.11, we find that $\Phi$ is isomorphic to the CH system shown in (42) of Example 5.4. The proof is complete. 

We finish this section with a comment. Pick $\beta \in \mathbb{F}$. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a $\beta$-recurrent CH system over $\mathbb{F}$. Let $\{\vartheta_i\}_{i=0}^d$, $\{\vartheta_i^*\}_{i=0}^d$, $\{\phi_i\}_{i=0}^d$ be the parameter array of $\Phi$. Let the scalars $\{\vartheta_i\}_{i=0}^{d+1}$ be from (6). Recall from Lemma 5.5 that $\vartheta_1 \neq \vartheta_d$. In the following lemma, we express $\vartheta_i (1 \leq i \leq d)$ as a linear combination of $\vartheta_1$ and $\vartheta_d$.

**Lemma 5.7.** With the above notation, for $1 \leq i \leq d$ the following (i)–(iv) hold.

(i) Assume $\beta \neq \pm 2$. Then

$$\vartheta_i = \frac{(q^i - 1)(q^{d-i} - 1)}{(q - 1)(q^{d-1} - 1)} \vartheta_1 + \frac{(q^i - 1)(q^{d-i+1} - 1)}{(q - 1)(q^{d-1} - 1)} \vartheta_d,$$

(68)

where $q + q^{-1} = \beta$.

(ii) Assume $\beta = 2$ and $\text{Char}(\mathbb{F}) \neq 2$. Then

$$\vartheta_i = \frac{i(d - i)}{d - 1} \vartheta_1 + \frac{(i - 1)(d - i + 1)}{d - 1} \vartheta_d.$$

(69)

(iii) Assume $\beta = -2$, $\text{Char}(\mathbb{F}) \neq 2$, and $d$ odd. Then

$$\vartheta_i = \begin{cases} \frac{i}{d - 1} \vartheta_1 + \frac{d - i + 1}{d - 1} \vartheta_d & \text{if } i \text{ is even;} \\ \frac{d - i}{d - 1} \vartheta_1 + \frac{i - 1}{d - 1} \vartheta_d & \text{if } i \text{ is odd.} \end{cases}$$

(70)

(iv) Assume $\beta = 0$, $\text{Char}(\mathbb{F}) = 2$, and $d = 3$. Then

$$\vartheta_2 = \vartheta_1 + \vartheta_3.$$

(71)

**Proof.** (i): Since the sequence $\{\vartheta_i\}_{i=0}^d$ is $\beta$-recurrent, by Lemma 8.1(i) there exist scalars $x, y, z \in \mathbb{F}$ such that

$$\vartheta_i = x + yq^i + zq^{-i} \quad (0 \leq i \leq d+1).$$

(72)

Since $\vartheta_0 = 0$, we have $x + y + z = 0$. Using this equation, eliminate $x$ in (72) to get

$$\vartheta_i = (q^i - 1)y + (q^{-i} - 1)z \quad (0 \leq i \leq d+1).$$

(73)

Note that $q^{d+1} = 1$ as we saw below (49). Evaluate (73) at $i = 1$ and $i = d$, and solve these two equations for $y$ and $z$. Simplify the result using $q^{d+1} = 1$ to get

$$y = -\frac{\vartheta_1 + q^d \vartheta_d}{(q - 1)(q^{d-1} - 1)},$$

(74)

$$z = -\frac{q^d \vartheta_1 + \vartheta_d}{(q - 1)(q^{d-1} - 1)}.$$

(75)
Eliminate $y$ and $z$ in (73) using (74) and (75), and simplify the result to get (68).

(ii)–(iv): Similar. ■

6 Six bases for $V$

Throughout this section, let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a CH system on $V$. Let the sequence $\{(\theta_i)_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d\}$ denote the parameter array of $\Phi$. In this section, we obtain the following results. We display six bases for $V$ that we find attractive. We display the transition matrices between certain pairs of bases among the six. We display the matrices that represent $A$ and $A^*$ with respect to the six bases.

We now describe the first of the six bases. Consider the decomposition $\{E_i V\}_{i=0}^d$ of $V$. Recall the nonzero vector $u^* \in E_0^* V$ from above line (13). By [15, Lemma 8.1], the sequence $\{E_i u^*\}_{i=0}^d$ is a basis for $V$. We say this basis is $\Phi$-standard. In the next two results, we recall some characterizations of the $\Phi$-standard basis.

Lemma 6.1 (cf. [15, Proposition 8.9]). Let $\{u_i\}_{i=0}^d$ denote a sequence of vectors in $V$, not all zero. Then the sequence $\{u_i\}_{i=0}^d$ is a $\Phi$-standard basis for $V$ if and only if both

(i) $u_i \in E_i V$ for $0 \leq i \leq d$,

(ii) $\sum_{i=0}^d u_i \in E_0^* V$.

Proof. Suppose that $\{u_i\}_{i=0}^d$ is a $\Phi$-standard basis for $V$. Then there exists $0 \neq u^* \in E_0^* V$ such that $u_i = E_i u^*$ for $0 \leq i \leq d$. This implies (i). Moreover, $\sum_{i=0}^d u_i = \sum_{i=0}^d E_i u^* = I u^* = u^* \in E_0^* V$, and thus (ii) holds. Conversely, suppose that the sequence $\{u_i\}_{i=0}^d$ satisfies (i) and (ii). Set $u^* = \sum_{i=0}^d u_i$. Then $u^*$ is nonzero and contained in $E_0^* V$. From (i) we have $E_i u^* = E_i (\sum_{j=0}^d u_j) = u_i$ for $0 \leq i \leq d$. Therefore, $\{u_i\}_{i=0}^d$ is a $\Phi$-standard basis for $V$. ■

Let $B$ denote a matrix in $\text{Mat}_{d+1}(\mathbb{F})$ and let $\alpha$ denote a scalar in $\mathbb{F}$. Then $B$ is said to have constant row sum $\alpha$ whenever $B_{i0} + B_{i1} + \cdots + B_{id} = \alpha$ for $0 \leq i \leq d$.

Lemma 6.2 (cf. [15, Proposition 8.10]). Let $\{u_i\}_{i=0}^d$ denote a basis for $V$. Let $B$ (resp. $B^*$) denote the matrix representing $A$ (resp. $A^*$) with respect to $\{u_i\}_{i=0}^d$. Then $\{u_i\}_{i=0}^d$ is a $\Phi$-standard basis for $V$ if and only if both

(i) $B = \text{diag}(\theta_0, \theta_1, \ldots, \theta_d),$

(ii) $B^*$ has constant row sum $\theta_0^*$.

Proof. Recall that for $0 \leq i \leq d$, $E_i V$ is the eigenspace of $A$ associated with $\theta_i$. Therefore $u_i \in E_i V$ for $0 \leq i \leq d$ if and only if $B = \text{diag}(\theta_0, \theta_1, \ldots, \theta_d)$. Next, we observe that

$$A^* \sum_{j=0}^d u_j = \sum_{j=0}^d A^* u_j = \sum_{j=0}^d \sum_{i=0}^d B_{ij} u_i = \sum_{i=0}^d \sum_{j=0}^d B_{ij} u_i = \sum_{i=0}^d (B_{i0} + B_{i1} + \cdots + B_{id}) u_i.$$

From this, it follows that $\sum_{j=0}^d u_j \in E_0^* V$ if and only if $B^*$ has constant row sum $\theta_0^*$. The result follows from these comments and Lemma 6.1. ■
We have discussed the \( \Phi \)-standard basis for \( V \). We now discuss five more bases for \( V \). Recall the \( \Phi \)-split basis \( \{ v_i \}_{i=0}^d \) for \( V \) from (13). The sequence \( \{ v_{d-i} \}_{i=0}^d \) is a basis for \( V \). We call this basis the \textit{inverted } \( \Phi \)-split basis. Recall the dual CH system \( \Phi^* = (A^*, \{ E_i^* \}_{i=0}^d, A, \{ E_i \}_{i=0}^d) \). Recall the nonzero vector \( u \in E_0 \) from above line (15). By definition, the sequence \( \{ E_i^* u \}_{i=0}^d \) is a \( \Phi^* \)-standard basis for \( V \). Recall the \( \Phi^* \)-split basis \( \{ v_{i} \}_{i=0}^d \) for \( V \) from (15). We consider the inverted \( \Phi^* \)-split basis \( \{ v_{d-i} \}_{i=0}^d \) for \( V \). We have now described six bases for \( V \). The six bases are shown in the table below.

| name                        | basis                                      |
|-----------------------------|--------------------------------------------|
| \( \Phi \)-standard basis   | \( \{ E_i^* u \}_{i=0}^d \)               |
| \( \Phi \)-split basis      | \( \{ v_i \}_{i=0}^d \)                   |
| inverted \( \Phi \)-split basis | \( \{ v_{d-i} \}_{i=0}^d \)          |
| \( \Phi^* \)-standard basis | \( \{ E_i^* u \}_{i=0}^d \)               |
| \( \Phi^* \)-split basis    | \( \{ v_i \}_{i=0}^d \)                   |
| inverted \( \Phi^* \)-split basis | \( \{ v_{d-i} \}_{i=0}^d \)          |

Our next goal is to describe the transition matrices between certain pairs of bases among the six. First, we recall the notion of a transition matrix. Suppose we are given two bases for \( V \), denoted \( \{ x_i \}_{i=0}^d \) and \( \{ y_i \}_{i=0}^d \). By the \textit{transition matrix} from \( \{ x_i \}_{i=0}^d \) to \( \{ y_i \}_{i=0}^d \), we mean the matrix \( T \in \text{Mat}_{d+1}(\mathbb{F}) \) such that

\[
y_j = \sum_{i=0}^d T_{ij} x_i \quad \text{for } 0 \leq j \leq d.
\]

Consider the following diagram:

\[
\begin{array}{ccc}
\{ E_i^* u \}_{i=0}^d & \cdots & \{ E_i^* u \}_{i=0}^d \\
\{ v_{d-i} \}_{i=0}^d & \cdots & \{ v_{d-i} \}_{i=0}^d \\
\{ v_i \}_{i=0}^d & \cdots & \{ v_i \}_{i=0}^d \\
\{ v_{d-i} \}_{i=0}^d & \cdots & \{ E_i^* u \}_{i=0}^d
\end{array}
\]

We will display the transition matrices between each pair of bases that are adjacent in the above diagram.

**Lemma 6.3.** The transition matrix from \( \{ E_i^* u \}_{i=0}^d \) to \( \{ v_{d-i} \}_{i=0}^d \) is upper triangular with \( (i, j) \)-entry

\[
(\theta_i - \theta_{j+1})(\theta_i - \theta_{j+2})\cdots(\theta_i - \theta_d)
\]

for \( 0 \leq i \leq j \leq d \). Moreover, the transition matrix from \( \{ v_{d-i} \}_{i=0}^d \) to \( \{ E_i^* u \}_{i=0}^d \) is upper triangular with \( (i, j) \)-entry

\[
\frac{1}{(\theta_j - \theta_1)\cdots(\theta_j - \theta_{j-1})(\theta_j - \theta_{j+1})\cdots(\theta_j - \theta_d)}
\]

for \( 0 \leq i \leq j \leq d \).

**Proof.** Let \( P \) denote the upper triangular matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) with \( (i, j) \)-entry (78) for \( 0 \leq i \leq d \).
By (13) we find that for $0 \leq j \leq d$,

$$v_{d-j} = (A - \theta_{j+1}I)(A - \theta_{j+2}I) \cdots (A - \theta_d I)u^*$$

$$= \sum_{i=0}^{d} E_i (A - \theta_{j+1}I)(A - \theta_{j+2}I) \cdots (A - \theta_d I)u^*$$

$$= \sum_{i=0}^{d} (\theta_i - \theta_{j+1})(\theta_i - \theta_{j+2}) \cdots (\theta_i - \theta_d) E_i u^*$$

$$= \sum_{i=0}^{d} P_{ij} E_i u^*.$$

By these comments, $P$ is the transition matrix from $\{E_i u^*\}_{i=0}^d$ to $\{v_{d-i}\}_{i=0}^d$.

To get the second assertion, we find the inverse of $P$. Since $P$ is upper triangular, the inverse of $P$ is also upper triangular. Let $H$ denote the upper triangular matrix in $\text{Mat}_{d+1}(F)$ with $(i,j)$-entry (79) for $0 \leq i \leq j \leq d$. We claim that $H$ is the inverse of $P$. To this end, it suffices to show that $PH = I$. Observe that $PH$ is upper triangular since $P$ and $H$ are upper triangular. Consider the $(i,j)$-entry of $PH$ for $0 \leq i \leq j \leq d$. If $i = j$, then $(PH)_{ii} = P_{ii}H_{ii} = 1$ by (78) and (79). Next, assume $i < j$. We show that $(PH)_{ij} = 0$. Since $\theta_0, \theta_1, \ldots, \theta_d$ are mutually distinct, it suffices to show that $(\theta_i - \theta_j)(PH)_{ij} = 0$. Since $P_{i\ell} = 0$ for $0 \leq \ell \leq i - 1$ by (78) and $H_{\ell j} = 0$ for $j + 1 \leq \ell \leq d$ by (79), we have

$$(PH)_{ij} = \sum_{\ell=0}^{i-1} P_{i\ell} H_{\ell j} + \sum_{\ell=i}^{j} P_{i\ell} H_{\ell j} + \sum_{\ell=j+1}^{d} P_{i\ell} H_{\ell j}$$

$$= \sum_{\ell=i}^{j} P_{i\ell} H_{\ell j}$$

Therefore,

$$(\theta_i - \theta_j)(PH)_{ij} = \sum_{\ell=i}^{j} P_{i\ell} H_{\ell j} (\theta_i - \theta_j)$$

$$= \sum_{\ell=i}^{j} P_{i\ell} H_{\ell j} (\theta_i - \theta_\ell + \theta_\ell - \theta_j)$$

$$= \sum_{\ell=i}^{j} P_{i\ell} (\theta_i - \theta_\ell) H_{\ell j} - \sum_{\ell=i}^{j} P_{i\ell} H_{\ell j} (\theta_j - \theta_\ell)$$

$$= \sum_{\ell=i+1}^{j} P_{i,\ell-1} H_{\ell j} - \sum_{\ell=i}^{j-1} P_{i\ell} H_{\ell+1, j}.$$

We notice that the last two sums are the same, so it follows that $(\theta_i - \theta_j)(PH)_{ij} = 0$. Combining all our above comments, we find that $H$ is the inverse of $P$. Therefore, $H$ is the transition matrix from $\{v_{d-i}\}_{i=0}^d$ to $\{E_i u^*\}_{i=0}^d$. ■

Recall the scalar $\varepsilon^*$ from Definition 3.6.
Lemma 6.4. The transition matrix from \( \{v_{d-i}\}_{i=0}^d \) to \( \{v^*_i\}_{i=0}^d \) is diagonal with \((i,i)\)-entry
\[
\frac{\varepsilon^*(\theta_0^* - \theta_1^*) (\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{\phi_i \phi_{i+2} \cdots \phi_{d-i}}
\]
for \( 0 \leq i \leq d \). Moreover, the transition matrix from \( \{v_i\}_{i=0}^d \) to \( \{v_{d-i}\}_{i=0}^d \) is diagonal with \((i,i)\)-entry
\[
\frac{\varepsilon^*(\theta_0^* - \theta_1^*) (\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{\phi_i \phi_{i+2} \cdots \phi_{d-i}}
\]
for \( 0 \leq i \leq d \).

Proof. By Proposition 3.10.

Let \( Z \) denote the matrix in \( \text{Mat}_{d+1}(\mathbb{F}) \) with \((i,j)\)-entry
\[
Z_{ij} = \begin{cases} 
1 & \text{if } i + j = d, \\
0 & \text{if } i + j \neq d
\end{cases} \quad (0 \leq i, j \leq d).
\] (80)

Observe that \( Z^2 = I \).

Lemma 6.5. The transition matrix from \( \{v_i\}_{i=0}^d \) to \( \{v_{d-i}\}_{i=0}^d \) is equal to \( Z \). Moreover, the transition matrix from \( \{v_{d-i}\}_{i=0}^d \) to \( \{v_i\}_{i=0}^d \) is equal to \( Z \).

Proof. Immediate from (80).

Lemma 6.6. The transition matrix from \( \{v^*_i\}_{i=0}^d \) to \( \{v^*_{d-i}\}_{i=0}^d \) is equal to \( Z \). Moreover, the transition matrix from \( \{v^*_{d-i}\}_{i=0}^d \) to \( \{v^*_i\}_{i=0}^d \) is equal to \( Z \).

Proof. Immediate from (80).

Recall the scalar \( \varepsilon \) from Definition 3.6.

Lemma 6.7. The transition matrix from \( \{v^*_{d-i}\}_{i=0}^d \) to \( \{v_i\}_{i=0}^d \) is diagonal with \((i,i)\)-entry
\[
\frac{\varepsilon (\theta_0 - \theta_1) (\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)}{\phi_{i+1} \phi_{i+2} \cdots \phi_d}
\]
for \( 0 \leq i \leq d \). Moreover, the transition matrix from \( \{v_i\}_{i=0}^d \) to \( \{v^*_{d-i}\}_{i=0}^d \) is diagonal with \((i,i)\)-entry
\[
\frac{\varepsilon (\theta_0 - \theta_1) (\theta_0 - \theta_2) \cdots (\theta_0 - \theta_d)}{\phi_{i+1} \phi_{i+2} \cdots \phi_d}
\]
for \( 0 \leq i \leq d \).

Proof. By Proposition 3.10.

Lemma 6.8. The transition matrix from \( \{E_i^* u\}_{i=0}^d \) to \( \{v^*_{d-i}\}_{i=0}^d \) is upper triangular with \((i,j)\)-entry
\[
(\theta_i^* - \theta_{j+1}^*) (\theta_i^* - \theta_{j+2}^*) \cdots (\theta_i^* - \theta_d^*)
\]
for \( 0 \leq i \leq j \leq d \). Moreover, the transition matrix from \( \{v^*_{d-i}\}_{i=0}^d \) to \( \{E_i^* u\}_{i=0}^d \) is upper triangular with \((i,j)\)-entry
\[
\frac{1}{(\theta_j^* - \theta_i^*) \cdots (\theta_j^* - \theta_{j-1}^*) (\theta_j^* - \theta_{j+1}^*) \cdots (\theta_j^* - \theta_d^*)}
\]
for \( 0 \leq i \leq j \leq d \).
Proof. Apply Lemma 6.3 to $\Phi^*$.  

We have a comment. In Lemmas 6.3–6.8, we found the transition matrices between any pair of bases that are adjacent in the diagram (77). Using these matrices and linear algebra, one can compute the transition matrix between any pair of bases among the six bases in (76).

We now describe the matrices representing $A$ and $A^*$ with respect to each basis in (76). Recall from Proposition 3.2 that the matrices representing $A$ and $A^*$ with respect to the $\Phi$-split basis $\{v_i\}_{i=0}^d$ are given in (14). Similarly, the matrices representing $A$ and $A^*$ with respect to the $\Phi^*$-split basis $\{v^*_i\}_{i=0}^d$ are given in (16).

**Lemma 6.9.** The matrices representing $A$ and $A^*$ with respect to the inverted $\Phi$-split basis $\{v_{d-i}\}_{i=0}^d$ are

\[
A : \begin{pmatrix}
\theta_0 & 1 & 0 \\
\theta_1 & 1 & \\
\theta_2 & & . \\
. & & \\
0 & 1 & \theta_d \\
\end{pmatrix}, \quad A^* : \begin{pmatrix}
\theta^*_d & \theta^*_{d-1} & 0 \\
\phi_d & \theta^*_{d-1} & \theta^*_{d-2} \\
. & & . \\
0 & & \phi_1 \\
\end{pmatrix}.
\]

*Proof. Conjugate each matrix in (14) by the matrix $Z$ from (80).*  

**Lemma 6.10.** The matrices representing $A$ and $A^*$ with respect to the inverted $\Phi^*$-split basis $\{v^*_d-i\}_{i=0}^d$ are

\[
A : \begin{pmatrix}
\theta_d & \theta_{d-1} & 0 \\
\phi_1 & \theta_{d-1} & \theta_{d-2} \\
. & & . \\
0 & \phi_d & \theta_0 \\
\end{pmatrix}, \quad A^* : \begin{pmatrix}
\theta^*_0 & 1 & \theta^*_d \\
\theta^*_1 & 1 & . \\
\theta^*_2 & & . \\
0 & & \theta^*_d \\
\end{pmatrix}.
\]

*Proof. Conjugate each matrix in (16) by the matrix $Z$ from (80).*  

**Proposition 6.11.** The matrix representing $A$ with respect to the $\Phi$-standard basis $\{E_iu^*_i\}_{i=0}^d$ is

\[
\text{diag}(\theta_0, \theta_1, \theta_2, \ldots, \theta_d).
\]

The matrix representing $A^*$ with respect to $\{E_iu^*_i\}_{i=0}^d$ is circular Hessenberg:

\[
\begin{pmatrix}
C_1^* & C_2^* & C_3^* & \cdots & C_d^* \\
C_2^* & C_3^* & C_4^* & \cdots & \vdots \\
C_3^* & C_4^* & C_5^* & \cdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_d^* & \cdots & \cdots & \cdots & C_1^* \\
\end{pmatrix},
\]

where

\[
\xi^* = \theta^*_0 - a^*_0 - b^*_0.
\]

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Proposition 6.12. The matrix representing $A^*$ with respect to the $\Phi^*$-standard basis $\{E_iu^*_i\}^d_{i=0}$ is

$$\text{diag}(\theta_0^*, \theta_1^*, \theta_2^*, \ldots, \theta_d^*).$$

Proof. The first assertion follows from Lemma 6.2(i). For the second assertion, let $B^*$ denote the matrix representing $A^*$ with respect to $\{E_iu^*_i\}^d_{i=0}$. By Definition 2.11(v), the matrix $B^*$ has the circular Hessenberg form (82). By Lemma 6.2(ii), we have (83). We now show (84)–(90). Recall the transition matrix from the circular Hessenberg form (82). By Lemma 6.2(ii), we have (83). We now show (84)–(90). Recall (84). Next, for 0 \leq i \leq d evaluate the (i, i)-entry of both sides of (91) to find

$$b^*_0 = (\theta_0 - \theta_d)(\theta_0 - \theta_d-1) \cdots (\theta_0 - \theta_1) \times \left( \frac{a^*_0}{\theta_0 - \theta_1} + \frac{\phi_2}{\theta_0 - \theta_1} \right) \times \left( \frac{b^*_0}{\theta_0 - \theta_2} \right),$$

(88)

by (85)–(87), we have (84)–(90). Recall (84). Next, for 0 \leq i \leq d evaluate the (i, i)-entry of both sides of (91) to find

$$a^*_i P_{i+1} = \phi^*_d P_{i+1},$$

(92)

\begin{align*}
    c^*_i &= \frac{(\theta_i - \theta_d)(\theta_i - \theta_{d-1}) \cdots (\theta_i - \theta_{i+1})}{(\theta_{i-1} - \theta_d)(\theta_{i-1} - \theta_{d-1}) \cdots (\theta_{i-1} - \theta_{i})} \phi^*_{d-i+1} (1 \leq i \leq d), \\
    a^*_0 &= \theta^*_0 d + \frac{\phi_d}{\theta_0 - \theta_1}, \\
    a^*_i &= \theta^*_{d-i} + \frac{\phi_{d-i}}{\theta_i - \theta_{i+1}} + \frac{\phi_{d-i+1}}{\theta_{i} - \theta_{i-1}} (1 \leq i \leq d - 1), \\
    a^*_d &= \theta^*_0 + \frac{\phi_1}{\theta_d - \theta_{d-1}}, \\
    b^*_0 &= (\theta_0 - \theta_d)(\theta_0 - \theta_d-1) \cdots (\theta_0 - \theta_2) \times \left( \frac{a^*_0}{\theta_0 - \theta_1} + \frac{\phi_2}{\theta_0 - \theta_1} \right) \times \left( \frac{b^*_0}{\theta_0 - \theta_2} \right). \\
    b^*_{i-1} &= (\theta_{i-1} - \theta_d)(\theta_{i-1} - \theta_{d-1}) \cdots (\theta_{i-1} - \theta_{i+2}) \\
    &\times \left( \frac{\phi_{d-i-1}}{\theta_{i-1} - \theta_{i+2}} + \frac{\phi_{d-i}}{\theta_{i-1} - \theta_i} + \frac{\phi_{d-i+1}}{\theta_{i-1} - \theta_{i+1}} \right) (1 \leq i \leq d - 2), \\
    b^*_{d-1} &= \theta^*_0 - \theta^*_1 + \frac{\phi_1}{\theta_d - \theta_{d-1}} + \frac{\phi_2}{\theta_d - \theta_{d-2}}.
\end{align*}

(89)

(90)

For 1 \leq i \leq d, evaluate the (i, i-1)-entry of both sides of (91) to find

$$c^*_i P_{i-1,i-1} = \phi^*_{d-i+1} P_{i,i},$$

(92)

Solve the equation (92) for $c^*_i$ and simplify the result to get (84). Next, for 0 \leq i \leq d evaluate the (i, i)-entry of both sides of (91) to find

$$a^*_i P_{i,i} + c^*_i P_{i-1,i+1} = \theta^*_d P_{i,i+1} + \phi^*_{d-i} P_{i,i+1}.$$  

(93)

Solve the equation (93) for $a^*_i$ and simplify the result to get (85)–(87). Next, for 0 \leq i \leq d - 1 evaluate the (i, i+1)-entry of both sides of (91) to find

$$c^*_i P_{i-1,i+1} + a^*_i P_{i+1,i+1} + b^*_i P_{i+1,i+1} = \theta^*_d P_{i,i+1} + \phi^*_{d-i} P_{i,i+2}.$$ 

(94)

Solve the equation (94) for $b^*_i$ and simplify the result to get (88)–(90). The result follows. 

\[\blacksquare\]
The matrix representing $A$ with respect to $\{E_i^*u\}_{i=0}^d$ is circular Hessenberg:

$$
\begin{pmatrix}
  a_0 & b_0 & \cdots & \xi \\
  c_1 & a_1 & b_1 & \cdots \\
  & c_2 & a_2 & \cdots \\
  & & \ddots & \ddots & \ddots \\
  & & & \cdots & \cdots & b_{d-1} \\
  & & & & 0 & c_d & a_d
\end{pmatrix},
$$

(96)

where

$$\xi = \theta_0 - a_0 - b_0,$$

(97)

and

$$c_i = \frac{(\theta_i^* - \theta_0^*)(\theta_i^* - \theta_{i-1}^*) \cdots (\theta_i^* - \theta_{i+d-1}^*)}{(\theta_{i-1}^* - \theta_d^*)(\theta_{i-d}^* - \theta_{i-d-1}^*) \cdots (\theta_{i-d}^* - \theta_d^*)^d} \phi_i \quad (1 \leq i \leq d),$$

(98)

$$a_0 = \theta_d + \frac{\phi_1}{\theta_0^* - \theta_1^*},$$

(99)

$$a_i = \theta_{d-i} + \frac{\phi_{i+1}}{\theta_i^* - \theta_{i+1}^*} + \frac{\phi_i}{\theta_i^* - \theta_{i-1}^*} \quad (1 \leq i \leq d-1),$$

(100)

$$a_d = \theta_0 + \frac{\phi_d}{\theta_d^* - \theta_d^*},$$

(101)

$$b_0 = \frac{(\theta_0^* - \theta_d^*)(\theta_0^* - \theta_{d-1}^*) \cdots (\theta_0^* - \theta_2^*)}{(\theta_1^* - \theta_d^*)(\theta_1^* - \theta_{d-1}^*) \cdots (\theta_1^* - \theta_2^*)} \times \left( \theta_{d-1} - \theta_d + \frac{\phi_2}{\theta_0^* - \theta_2^*} + \frac{\phi_1}{\theta_1^* - \theta_2^*} \right),$$

(102)

$$b_i = \frac{(\theta_i^* - \theta_d^*)(\theta_i^* - \theta_{i+1}^*) \cdots (\theta_i^* - \theta_{i+d-1}^*) (\theta_{i+1}^* - \theta_{i+2}^*)}{(\theta_{i-1}^* - \theta_d^*)(\theta_{i-1}^* - \theta_{i-1}^*) \cdots (\theta_{i-1}^* - \theta_{i+2}^*)} \times \left( \theta_{d-i-1} - \theta_{d-i} + \frac{\phi_{i+2}}{\theta_i^* - \theta_{i+2}^*} + \frac{\phi_{i+1}}{\theta_{i+1}^* - \theta_i^*} + \frac{\phi_i}{\theta_{i-1}^* - \theta_{i+1}^*} \right) \quad (1 \leq i \leq d-2),$$

(103)

$$b_{d-1} = \theta_0 - \theta_1 + \frac{\phi_d}{\theta_d^* - \theta_1^*} + \frac{\phi_{d-1}}{\theta_d^* - \theta_{d-1}^*}.$$  

(104)

Proof. Apply Proposition 6.11 to $\Phi^*$.  

7 The scalars $\xi$ and $\xi^*$

Throughout this section, let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a recurrent CH system over $\mathbb{F}$. Let $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d$ denote the parameter array of $\Phi$. In Propositions 6.11 and 6.12, we encountered the scalars $\xi$ and $\xi^*$. Our goal for this section is to obtain a formula for $\xi$ and $\xi^*$. We give two versions; these are Proposition 7.2 and Corollary 7.3.

Lemma 7.1. We have

$$
\prod_{i=2}^{d-1} \frac{\theta_0 - \theta_i}{\theta_1 - \theta_i} = 1, \quad \prod_{i=2}^{d-1} \frac{\theta_0^* - \theta_{i+1}^*}{\theta_1^* - \theta_i^*} = 1.
$$

(105)
Proof. We first prove the equation on the left in (105). In this equation, let \( \psi \) denote the product on the left. Since \( \Phi \) is recurrent, there exists \( \beta \in \mathbb{F} \) such that the sequence \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent. We divide the argument into four cases: (i) \( \beta \neq \pm 2 \), (ii) \( \beta = 2 \) and \( \text{Char}(\mathbb{F}) \neq 2 \), (iii) \( \beta = -2 \) and \( \text{Char}(\mathbb{F}) \neq 2 \), (iv) \( \beta = 0 \) and \( \text{Char}(\mathbb{F}) = 2 \). In case (i), choose \( 0 \neq q \in \mathbb{F} \) such that \( q + q^{-1} = \beta \). Apply Lemma 8.3(i) to \( \psi \) and evaluate the result to get

\[
\psi = \frac{(1 - q^d)(1 - q^{d-1})}{q^{d-2}(1 - q)(1 - q^2)}.
\]  

(106)

We saw below (49) that \( q^{d+1} = 1 \). Using this fact, simplify the right-hand side of (106) to get \( \psi = 1 \), as desired. We have now proved the argument for case (i). The remaining cases (ii)–(iv) are proved in a similar fashion. We have proved the equation on the left in (105). Apply this result to \( \Phi^* \) to get the equation on the right in (105).

Proposition 7.2. Recall the scalars \( \xi \) from (96) and \( \xi^* \) from (82). Then both

\[
\xi = \frac{\phi_1 - \phi_d}{\theta_1 - \theta_d} + \frac{(\theta_1 - \theta_0)(\theta_0^* - \theta_0^*) - (\theta_d - \theta_0)(\theta_1^* - \theta_0^*)}{\theta_1^* - \theta_d^*},
\]

(107)

\[
\xi^* = \frac{\phi_d - \phi_1}{\theta_1 - \theta_d} + \frac{(\theta_1^* - \theta_0^*)(\theta_d - \theta_0) - (\theta_2^* - \theta_0^*)(\theta_1 - \theta_0)}{\theta_1 - \theta_d}.
\]

(108)

Proof. We first prove (107). Evaluate \( b_0 \) in (102) using the equation on the right in (105) to get

\[
b_0 = \frac{\theta_0^* - \theta_2^*}{\theta_1^* - \theta_0^*} \left( \theta_{d-1} - \theta_d + \frac{\phi_2}{\theta_0^* - \theta_2^*} + \frac{\phi_1}{\theta_1^* - \theta_0^*} \right).
\]

(109)

Recall from (97) that \( \xi = \theta_0 - a_0 - b_0 \). Eliminate \( a_0 \) and \( b_0 \) in this equation using (99) and (109) to get

\[
\xi = \theta_0 - \theta_d - \frac{\phi_1}{\theta_0^* - \theta_1^*} - \frac{\theta_0^* - \theta_2^*}{\theta_2^* - \theta_0^*} \left( \theta_{d-1} - \theta_d + \frac{\phi_1}{\theta_1^* - \theta_0^*} + \frac{\phi_2}{\theta_0^* - \theta_2^*} \right).
\]

(110)

Consider the quantity that is the right-hand side of (107) minus the right-hand side of (110). Evaluate this quantity using the closed forms for the scalars \( \{\theta_i\}_{i=0}^d \), \( \{\theta_i^*\}_{i=0}^d \), \( \{\phi_i\}_{i=1}^d \) that are presented in Examples 5.1–5.4. For each example, the above quantity is zero. We have proved (107). To prove (108), apply (107) to \( \Phi^* \).

Corollary 7.3. Recall the scalars \( \{\vartheta_i\}_{i=0}^{d+1} \) from (6). We have

\[
\xi = \frac{\vartheta_1 - \vartheta_d}{\theta_1^* - \theta_d^*}, \quad \xi^* = \frac{\vartheta_d - \vartheta_1}{\theta_1 - \theta_d}.
\]

(111)

Proof. In (107) and (108), eliminate \( \phi_1 \) and \( \phi_d \) using (6) and simplify the result.

Note 7.4. Referring to (111), \( \xi \) and \( \xi^* \) are nonzero by Lemma 5.5.
8 Appendix

In this appendix, we recall some formulas involving recurrent sequences that are used in the main body of the paper. Throughout this appendix, let \( \beta \) denote any scalar in \( \mathbb{F} \) and let \( \{\theta_i\}_{i=0}^d \) denote an arbitrary sequence of scalars taken from \( \mathbb{F} \).

**Definition 8.1** ([17, Definition 8.2]). The sequence \( \{\theta_i\}_{i=0}^d \) is said to be \( \beta \)-recurrent whenever

\[
\theta_{i-2} - (\beta + 1)\theta_{i-1} + (\beta + 1)\theta_i - \theta_{i+1} = 0
\]

for \( 2 \leq i \leq d-1 \).

**Lemma 8.2** (cf. [17, Lemma 9.2]). Assume that the sequence \( \{\theta_i\}_{i=0}^d \) is \( \beta \)-recurrent. Then the following (i)–(iv) hold.

(i) Suppose \( \beta \neq \pm 2 \) and choose \( 0 \neq q \in \overline{\mathbb{F}} \) such that \( q + q^{-1} = \beta \). Then there exist scalars \( \alpha_1, \alpha_2, \alpha_3 \) in \( \mathbb{F} \) such that

\[
\theta_i = \alpha_1 + \alpha_2 q^i + \alpha_3 q^{-i}, \quad 0 \leq i \leq d.
\]

(ii) Suppose \( \beta = 2 \) and \( \text{Char}(\mathbb{F}) \neq 2 \). Then there exist scalars \( \alpha_1, \alpha_2, \alpha_3 \) in \( \mathbb{F} \) such that

\[
\theta_i = \alpha_1 + \alpha_2 i + \alpha_3 (i-1)/2, \quad 0 \leq i \leq d.
\]

(iii) Suppose \( \beta = -2 \) and \( \text{Char}(\mathbb{F}) \neq 2 \). Then there exist scalars \( \alpha_1, \alpha_2, \alpha_3 \) in \( \mathbb{F} \) such that

\[
\theta_i = \alpha_1 + \alpha_2 (-1)^i + \alpha_3 (-1)^i, \quad 0 \leq i \leq d.
\]

(iv) Suppose \( \beta = 0 \) and \( \text{Char}(\mathbb{F}) = 2 \). Then there exists \( \alpha_1, \alpha_2, \alpha_3 \) in \( \mathbb{F} \) such that

\[
\theta_i = \alpha_1 + \alpha_2 i + \alpha_3 \left(\begin{array}{c} i \\ 2 \end{array}\right), \quad 0 \leq i \leq d,
\]

where we interpret the binomial coefficient as follows:

\[
\left(\begin{array}{c} i \\ 2 \end{array}\right) = \begin{cases} 0 & \text{if } i = 0 \text{ or } i = 1 \pmod{4}, \\ 1 & \text{if } i = 2 \text{ or } i = 3 \pmod{4}. \end{cases}
\]

**Lemma 8.3** (cf. [17, Lemma 9.4]). Assume that the scalars \( \{\theta_i\}_{i=0}^d \) are mutually distinct and \( \beta \)-recurrent. Pick any integers \( i, j, r, s \) (\( 0 \leq i, j, r, s \leq d \)) such that \( i + j = r + s \), \( r \neq s \). Then (i)–(iv) hold below.

(i) Suppose \( \beta \neq \pm 2 \). Then

\[
\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \frac{q^j - q^i}{q^r - q^s},
\]

where \( q + q^{-1} = \beta \).

(ii) Suppose \( \beta = 2 \) and \( \text{Char}(\mathbb{F}) \neq 2 \). Then

\[
\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \frac{i - j}{r - s}.
\]
Suppose \( \beta = -2 \) and \( \text{Char}(\mathbb{F}) \neq 2 \). Then
\[
\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \begin{cases} 
(-1)^{i+j}(i-j)/(r-s) & \text{if } i+j \text{ is even;} \\
(-1)^{i+j} & \text{if } i+j \text{ is odd.}
\end{cases}
\]

Suppose \( \beta = 0 \) and \( \text{Char}(\mathbb{F}) = 2 \). Then
\[
\frac{\theta_i - \theta_j}{\theta_r - \theta_s} = \begin{cases} 
0 & \text{if } i = j; \\
1 & \text{if } i \neq j.
\end{cases}
\]

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