Efficient recovery-based error estimation for the smoothed finite element method for smooth and singular linear elasticity

Octavio A. González-Estrada · Sundararajan Natarajan · Juan José Ródenas · Hung Nguyen-Xuan · Stéphane P. A. Bordas

Abstract An error control technique aimed to assess the quality of smoothed finite element approximations is presented in this paper. Finite element techniques based on strain smoothing appeared in 2007 were shown to provide significant advantages compared to conventional finite element approximations. In particular, a widely cited strength of such methods is improved accuracy for the same computational cost. Yet, few attempts have been made to directly assess the quality of the results obtained during the simulation by evaluating an estimate of the discretization error. Here we propose a recovery type error estimator based on an enhanced recovery technique. The salient features of the recovery are: enforcement of local equilibrium and, for singular problems a “smooth + singular” decomposition of the recovered stress. We evaluate the proposed estimator on a number of test cases from linear elastic structural mechanics and obtain efficient error estimations whose effectivities, both at local and global levels, are improved compared to recovery procedures not implementing these features.

Keywords Smoothed finite element method · Error estimation · Statical admissibility · SPR-CX · Singularity · Recovery

1 Introduction

The smoothed finite element for mechanics problems was introduced in 2006 by Liu et al. [1]. The main idea of the method is to relax the kinematic compatibility condition at the element level by replacing the standard compatible strain by its smoothed counterpart. The smoothing operation can be performed over domains of various shapes which can be obtained by dividing the computational domain into non-overlapping smoothing domains. These domains can be obtained by subdividing the elements (cell-based smoothing) as in [1–5], or using edge [6,7] or node-based geometrical information [8]. Each method has several advantages and drawbacks, summarized in, e.g. [9,10], but the strongest motivation for smoothed finite elements is certainly revealed in its enhancement of low order simplex elements (e.g. linear triangles and tetrahedral), alleviating overstiffness, locking and improving their accuracy significantly [6,11,12].

The applications of strain smoothing in finite elements are wide. Since the introduction of the smoothed finite element method (SFEM), the convergence, the stability, the accuracy and the computational complexity of this method were studied in [2,3] and the method was extended to treat various problems in solid mechanics such as plates [13], shells [14] and nearly incompressible elasticity [12]. Recently, Bordas et al. [15,16] combined strain smoothing with the XFEM to obtain the Smoothed eXtended Finite Element Method to solve problems with strong and weak discontinuities in 2D continuum.

In this paper, we focus on the cell-based smoothing, a review of which is provided in [4,12,17] along with
applications to plates, shells, three dimensional continuum and a coupling with the extended finite element method with applications to linear elastic fracture (continuum, plate).

The development of new numerical techniques based on the finite element method (e.g. GFEM, XFEM, . . .) aims at obtaining more accurate solutions for engineering problems. Despite the improvements introduced by the new techniques, numerical errors, especially the discretization error, are always present and have to be evaluated. Accuracy assessment techniques previously developed for the FE framework are commonly adapted to the framework of these new techniques [18–23]. As in any numerical method, the smoothed FEM approximation introduces an error that needs to be controlled to guarantee the quality of the numerical simulations. Although an adaptive node-based smoothed FEM has been developed in [24], a rather simple error estimator using a recovery procedure which initially is only valid for NS-FEM is used to guide the adaptive process. The technique evaluates a first-order recovered strain field interpolating the nodal values by means of the linear FEM shape functions.

The urge for quality assessment tools for smoothed FEM approximations, the promising results in [24] obtained with approximations, the Zienkiewicz and Zhu error estimator [26] commonly used in FEM, together with a recovery technique recently developed by the authors, specially tailored to the analysis of enriched approximations containing a smooth and a singular part and which locally enforces the fulfillment of equilibrium equations. The technique known as SPR-CX [21,27] was shown to lead to very good effectivity indices in FEM and XFEM.

The paper is organised as follows: In Sect. 2, the boundary value problem of linear elasticity is briefly introduced and the approximate solution using the SFEM is presented. In Sect. 3, we discuss basic concepts related to error estimation, especially recovery-based techniques. Section 4 is devoted to the proposed enhanced recovery technique and its application to SFEM approximations. Numerical examples are presented in Sect. 5 and the main concluding remarks in Sect. 6.

2 Problem statement and SFEM solution

2.1 Finite element formulation of linear elastic BVPs and singular solutions at notches and corners

Let us consider the 2D linear elasticity problem. The unknown displacement field $u$, taking values in $\Omega \subset \mathbb{R}^2$, is the solution of the boundary value problem given by

$$
- \nabla \cdot \sigma(u) = b \quad \text{in} \quad \Omega \tag{1}
$$

$$
\sigma(u) \cdot n = t \quad \text{on} \quad \Gamma_N \tag{2}
$$

$$
u = 0 \quad \text{on} \quad \Gamma_D \tag{3}
$$

$$\epsilon(u) := \nabla u \tag{4}
$$

$$\sigma(u) = C : \epsilon(u) \tag{5}
$$

where $\Gamma_N$ and $\Gamma_D$ denote the Neumann and Dirichlet boundaries with $\partial \Omega = \Gamma_N \cup \Gamma_D$ and $\Gamma_N \cap \Gamma_D = \emptyset$. $\nabla u$ is the symmetric gradient of the displacements, $\sigma(u)$ and $\epsilon(u)$ are the stress and strain tensors of the Hooke’s law given by the four order tensor $C$. The Dirichlet boundary condition in (3) is assumed to be homogeneous for the sake of simplicity.

The weak form of the problem reads: Find $u \in V$ such that

$$
\forall v \in V = \{ v \mid v \in [H^1(\Omega)]^2, v|_{\Gamma_D}(x) = 0 \} \quad a(u, v) = l(v), \tag{6}
$$

where $V$ is the standard test space for the elasticity problem and

$$a(u, v) := \int_\Omega \epsilon(u)^T D \epsilon(v) \, d\Omega = \int_\Omega \sigma(u)^T D^{-1} \sigma(v) \, d\Omega \tag{7}
$$

$$l(v) := \int_\Omega b^T v \, d\Omega + \int_{\Gamma_N} t^T v \, d\Gamma, \tag{8}
$$

where $D$ is the elasticity matrix of the constitutive relation $\sigma = D \epsilon$, $\sigma$ and $\epsilon$ denote the stress and strain operators.

2.1.1 Singular problem

Figure 1 shows a portion of an elastic body with a reentrant corner (or V-notch), subjected to tractions on remote boundaries. No body loads are applied. For this kind of problem, the stress field exhibits a singular behaviour at the notch vertex.
The analytical solution corresponding to the stress distribution in the vicinity of the singular point is a linear combination of the singular and the non-singular terms. It is often claimed that the term with a highest order of singularity dominates over the other terms in a zone surrounding the singular point sufficiently closely. The analytical solution to this singular elastic problem in the vicinity of the singular point was first given by [28] and is described, for example, in [29,30]. Here, we reproduce those expressions for completeness. In accordance with the polar coordinate system of Fig. 1, the displacement and the stress fields at points sufficiently close to the corner can be described as:

\[
\begin{align*}
\mathbf{u}(r, \phi) &= \mathcal{K}_I r^{\lambda_1} \Phi_I(\lambda_1, \phi) + \mathcal{K}_II r^{\lambda_II} \Phi_I(\lambda_II, \phi) \\
\mathbf{\sigma}(r, \phi) &= \mathcal{K}_I r^{\lambda_1 - 1} \Phi_I(\lambda_1, \phi) + \mathcal{K}_II r^{\lambda_II - 1} \Phi_I(\lambda_II, \phi)
\end{align*}
\]

(9)

where \( r \) is the radial distance to the corner, \( \lambda_m \) (with \( m = I, II \)) are the eigenvalues that determine the order of the singularity, \( \Psi_m \) and \( \Phi_m \) are a set of trigonometric functions that depend on the angular position \( \phi \), and \( \mathcal{K}_m \) are the so-called generalised stress intensity factors (GSIFs). The GSIF is a multiplicative constant that depends on the loading of the problem and linearly determines the intensity of the displacement and the stress fields in the vicinity of the singular point. Hence, the eigenvalues \( \lambda \) and the GSIFs \( \mathcal{K} \) define the singular field.

The eigenvalue \( \lambda \) is easily known because it depends only on the corner angle \( \alpha \), and can be obtained as the smallest positive root of the following characteristic equations [28]:

\[
\begin{align*}
\sin \lambda_1 \alpha + \lambda_1 \sin \alpha &= 0 \\
\sin \lambda_II \alpha - \lambda_II \sin \alpha &= 0
\end{align*}
\]

(11)

The smallest positive root yields the highest order of singularity and determines the term that dominates the elastic fields given by (9) in the vicinity of the notch vertex. Strictly speaking, (11) corresponds to the symmetric part of the elastic fields with respect to \( \phi = 0 \) (i.e. the bisector line BB in Fig. 1) and (12) to the antisymmetric solution. These solutions are also called mode I and mode II solutions, respectively. The trigonometric functions for the mode I displacement and stress fields in (9, 10) are given by [29]:

\[
\begin{align*}
\Psi_I(\lambda_1, \phi) &= \begin{bmatrix} \Psi_{I,xx}(\lambda_1, \phi) \\ \Psi_{I,yy}(\lambda_1, \phi) \end{bmatrix} \\
&= \frac{1}{2\mu} \begin{bmatrix} (\kappa - Q(\lambda_1 + 1)) \sin \lambda_1 \phi - \lambda_1 \cos(\lambda_1 - 2) \phi \\ (\kappa + Q(\lambda_1 + 1)) \sin \lambda_1 \phi + \lambda_1 \sin(\lambda_1 - 2) \phi \end{bmatrix} \\
\Phi_I(\lambda_1, \phi) &= \begin{bmatrix} \Phi_{I,xx}(\lambda_1, \phi) \\ \Phi_{I,yyyy}(\lambda_1, \phi) \\ \Phi_{I,yyyy}(\lambda_1, \phi) \end{bmatrix} \\
&= \begin{bmatrix} (2 - Q(\lambda_1 + 1)) \cos(\lambda_1 - 1) \phi - (\lambda_1 - 1) \cos(\lambda_1 - 3) \phi \\ (2 + Q(\lambda_1 + 1)) \cos(\lambda_1 - 1) \phi + (\lambda_1 - 1) \cos(\lambda_1 - 3) \phi \\ Q(\lambda_1 + 1) \sin(\lambda_1 - 1) \phi + (\lambda_1 - 1) \sin(\lambda_1 - 3) \phi \end{bmatrix}
\end{align*}
\]

(13)

where \( \mu \) is the shear modulus and \( \kappa \) is the Kolosov constant, defined as functions of \( E \) (Young's modulus) and \( v \) (Poisson's coefficient) according to the following expressions:

\[
\mu = \frac{E}{2(1+v)}, \quad \kappa = \frac{3 - 4v}{3 - v} \quad \text{plane strain}
\]

\[
\mu = \frac{E}{1+v}, \quad \kappa = \frac{3 - v}{3 - v} \quad \text{plane stress}
\]

In the same way, for mode II we have:

\[
\begin{align*}
\Psi_{II}(\lambda_II, \phi) &= \begin{bmatrix} \Psi_{II,xx}(\lambda_II, \phi) \\ \Psi_{II,yy}(\lambda_II, \phi) \end{bmatrix} \\
&= \frac{1}{2\mu} \begin{bmatrix} (\kappa - Q(\lambda_II + 1)) \sin \lambda_II \phi - \lambda_II \sin(\lambda_II - 2) \phi \\ -(\kappa + Q(\lambda_II + 1)) \cos \lambda_II \phi - \lambda_II \cos(\lambda_II - 2) \phi \end{bmatrix} \\
\Phi_{II}(\lambda_II, \phi) &= \begin{bmatrix} \Phi_{II,xx}(\lambda_II, \phi) \\ \Phi_{II,yy}(\lambda_II, \phi) \\ \Phi_{II,yyyy}(\lambda_II, \phi) \end{bmatrix} \\
&= \begin{bmatrix} (2 - Q(\lambda_II + 1)) \sin(\lambda_II - 1) \phi - (\lambda_II - 1) \sin(\lambda_II - 3) \phi \\ (2 + Q(\lambda_II + 1)) \sin(\lambda_II - 1) \phi + (\lambda_II - 1) \sin(\lambda_II - 3) \phi \\ Q(\lambda_II + 1) \cos(\lambda_II - 1) \phi + (\lambda_II - 1) \cos(\lambda_II - 3) \phi \end{bmatrix}
\end{align*}
\]

(15)

Note that the components of the displacement and the stress fields are expressed in Cartesian coordinates. In addition, \( Q \) is a constant for a given notch angle:

\[
\begin{align*}
Q_I &= -\frac{\cos((\lambda_I - 1) \alpha)}{\cos((\lambda_I + 1) \alpha)}, \quad Q_{II} = -\frac{\sin((\lambda_{II} - 1) \alpha)}{\sin((\lambda_{II} + 1) \alpha)}
\end{align*}
\]

(16)

(17)

2.2 FEM solution with strain smoothing

2.2.1 Finite element formulation

Let \( \mathbf{u}^h \) be a finite element approximation to \( \mathbf{u} \). The solution lies in a functional space \( V^h \subset V \) associated with a mesh of isoparametric finite elements of characteristic size \( h \), and it is such that

\[
\forall \mathbf{v}^h \in V^h \quad a(\mathbf{u}^h, \mathbf{v}^h) = l(\mathbf{v}^h)
\]

(18)

Using a variational formulation of the BVP problem in Sect. 2.1 and (18), and a finite element approximation \( \mathbf{u}^h = \mathbf{N} \mathbf{u}^e \), where \( \mathbf{N} \) denotes the shape functions of order \( p \), we obtain a system of linear equations to solve the displacements at nodes \( \mathbf{u}^e \):

\[
\mathbf{K} \mathbf{u}^e = \mathbf{f}
\]

(19)

where \( \mathbf{K} \) is the stiffness matrix, \( \mathbf{U} \) is the vector of nodal displacements and \( \mathbf{f} \) is the load vector.

2.2.2 Strain smoothing in FEM

Inspired by the work of Chen et al. [31] on stabilized conforming nodal integration (SCNI), Liu et al. [1] introduced...
where all shape functions \( I \in \{1, \ldots, 4\} \), the \( 3 \times 2 \) submatrix \( \tilde{B}_{CI} \) represents the contribution to the strain displacement matrix associated with shape function \( I \) and cell \( C \) and writes for 2D problems

\[
\forall I \in \{1, 2, \ldots, 4\}, \forall C \in \{1, 2, \ldots, nc\} \tilde{B}_{CI} = \int_{S_C} \begin{bmatrix} n_x & 0 \\ 0 & n_y \end{bmatrix} (x) N_I(x) dS
\]

or, since (26) is computed on the boundary of \( \Omega_C \) and one Gauß point is sufficient for an exact integration (in the case of a bilinear approximation):

\[
\tilde{B}_{CI}(x_C) = \frac{1}{A_C} \sum_{b=1}^{n_b} \begin{bmatrix} n_x & 0 \\ 0 & n_y \end{bmatrix} N_I(x_b^G) n_y N_I(x_b^G) n_x
\]

where \( n_b \) is number of edges of the subcell, \((n_x, n_y)\) is the outward normal to the smoothing cell, \( \Omega_C \), \( x_b^G \) and \( l_b^C \) are the center point (Gauß point) and the length of \( l_b^C \), respectively.

### 3 Error estimation by gradient smoothing in the complementary energy norm

The discretization error in the standard finite element approximation is defined as the difference between the exact solution \( u \) and the finite element solution \( u^h \): \( e = u - u^h \). Since the exact solution is in practice unknown, in general, the exact error can only be estimated. To obtain an estimation of \( e \), norms that allow a better global interpretation of the error are normally used. Considering the complementary energy norm of the error \( e \) written as

\[
|||e||| = |||u - u^h||| = \left( \int_{\Omega} (\sigma - \sigma^h)^T D^{-1} (\sigma - \sigma^h) d\Omega \right)^{1/2}
\]

Zienkiewicz and Zhu [26] proposed to evaluate an approximation of \( |||e||| \) using the following expression (known as the ZZ error estimator):

\[
|||e_{esl}||| = \left( \int_{\Omega} (\sigma^* - \sigma^h)^T D^{-1} (\sigma^* - \sigma^h) d\Omega \right)^{1/2}
\]

where \( \sigma^* \) is an enhanced or recovered stress field, which is supposed to be more accurate than the FE solution \( \sigma^h \). The domain \( \Omega \) could refer to the full domain of the problem or a local subdomain (element).

The recovered stress field \( \sigma^* \) is usually interpolated in each element using the shape functions \( N \) of the underlying

\[
\begin{bmatrix} \tilde{B}_{C1} & \tilde{B}_{C2} & \tilde{B}_{C3} & \tilde{B}_{C4} \end{bmatrix}
\]

\( ^1 \) The subcells \( \Omega_C \) form a partition of the element \( \Omega^h \).
FE approximation and the values of the recovered stress field calculated at the nodes \( \sigma^* \), given by:

\[
\sigma^*(x) = \sum_{I=1}^{n_v} N_I(x) \sigma^*_I(x_I), \tag{30}
\]

where \( n_v \) is the number of nodes in the element under consideration and \( \sigma^*_I(x_I) \) are the stresses provided by a recovery technique at node \( I \). The superconvergent patch recovery technique (SPR) proposed by Zienkiewicz and Zhu [33] is commonly used to evaluate the components \( (j = xx, yy, xy) \) of \( \sigma^*_I \) using a polynomial expansion, \( \sigma^*_I = p a_j \). This expansion is defined over a set of contiguous elements connected to node \( I \) called patch, where \( p \) is the polynomial basis and \( a_j \) are the unknown coefficients obtained using a least squares fitting to the values of the FE stresses evaluated at integration points in the patch, being \( p \), normally, of the same order as the interpolation of displacements. The ZZ error estimator is asymptotically exact (i.e. the approximate error converges to the exact error as the mesh size goes to zero) if the recovered solution used in the error estimation converges at a higher rate than the finite element solution [33,34].

As it can be seen in (29), the accuracy of the error estimate is closely related to the quality of the recovered field. For this reason, several techniques have been developed aiming to improve the quality of \( \sigma^* \). Since the first publications by Zienkiewicz and Zhu many enhancements of the SPR technique have been proposed to improve the quality of the solution, e.g. considering equilibrium conditions, either by (moving) least squares methods of Lagrangian extensions [35–37]. The authors have proposed different techniques mostly for the FEM/XFEM context as the extended moving least squares recovery (XMLS) and the extended global recovery techniques proposed by Duflot and Bordas in a series of papers [19,38,39], the SPR-C and the SPR-CX by Ródenas et al. [21,40], which were used later as the basis for the development of recovery-based error bounding techniques [27,41]. The next section presents the SPR-CX technique which improves the recovered field by enforcing equilibrium and effectively dealing with singular fields.

Remark In mathematics is common to consider that one can only speak about an error estimator if sharp or at least approximated upper - and desired - also lower error bounds can be proven, reserving the word indicator when the technique does not necessarily bound the error. However, this terminology is not general and many other authors, usually from the engineering community, use the term error estimator even when the technique is not able to provide error bounds. This is the case for example in [26,35,36] and also our case.

4 SPR-CX recovery technique

The SPR-CX recovery technique first introduced by Ródenas et al. [21] is an enhancement of the SPR in [33], which incorporates the ideas of the SPR-C technique proposed in [40] to improve the quality of the recovered stress field \( \sigma^* \) by introducing information of the exact solution known a priori. In [21,25,27] a set of key ideas are proposed to modify the standard SPR allowing its use with singular problems.

The recovered stresses \( \sigma^* \) are directly evaluated at a sampling point (e.g an integration point) \( x \) through the use of a partition of unity procedure, properly weighting the stress interpolation polynomials obtained from the different patches formed at the vertex nodes of the element containing \( x \):

\[
\sigma^*(x) = \sum_{I=1}^{n_v} N_I(x) \sigma^*_I(x), \tag{31}
\]

where \( N_I \) are the shape functions associated to the vertex nodes \( n_v \). To obtain the nodal values \( \sigma_I \), we solve a least squares approximation of the stresses evaluated at a set of sampling points distributed within the domain of the patch of node \( I \) (elements connected to \( I \)). In FEM, such points usually correspond to the integrations points used in the finite element approximation. In SFEM, we map the constant strains at each subcell to a \( 2 \times 2 \) Gauß quadrature distribution in the subcell used as sampling points. This way we have a sufficient number of points at each patch to solve the linear system of the least squares approximation, see Fig. 2. Note that as in the other versions of the SFEM (NS-FEM, ES-FEM) the elements are also subdivided into subcells, a similar approach can be used to perform the mapping of the stresses to sampling points. Therefore, the proposed error estimation technique can be used with all SFEM implementations.

One major modification of the original SPR technique for the evaluation of the recovered stresses on patches \( \sigma^*_I(x) \) to be used in (31) is the introduction of a splitting procedure to perform the recovery. As shown in [29], in linear elasticity, the solution around a singular point can be expressed as an asymptotic expansion where the first term is singular and dominates the stress field near the singularity. Therefore, the
The constraint equations are introduced via Lagrange multipliers into the linear system used to solve for the coefficients of the polynomial expansion of the recovered stresses on each patch. These include the satisfaction of the:

- Internal equilibrium equations (see (1)).
- Boundary equilibrium equation: A point collocation approach is used to impose the satisfaction of a second order approximation to the tractions along the Neumann boundary (see (2)).
- Compatibility equation: This additional constraint is also imposed to further increase the sharpness of the recovered stress field. The compatibility condition in 2D is given by:

\[
\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2}
\] (35)

To evaluate the recovered field, quadratic polynomials are used in the patches along the boundary and linear polynomials for the remaining patches. As more information about the solution is available along the boundary, polynomials one degree higher are useful to improve the quality of the recovered stress field.

Once we have the recovered smooth part of the stress field on each patch we add the singular part \(\sigma_{\text{sing}}\) which could be calculated at any point in \(\Omega\). The field at each patch \(\sigma^*_l\) is evaluated as \(\sigma^*_l = \sigma^*_{l,\text{smo}} + \sigma^*_{l,\text{sing}}\). Notice that, as indicated in Fig. 3, the stress splitting is not applied in patches of nodes outside the splitting area. This will avoid discontinuities of the recovered field locally at patches in elements neighbouring those with contributions from the singular part. Finally, we can use the partition of unity procedure in (31) to obtain the field \(\sigma^*\).

The enforcement of equilibrium equations provides an equilibrated recovered stress field locally on patches. However, the process used to obtain a continuous field \(\sigma^*\) shown in (31) introduces a small lack of equilibrium as explained in [27,41]. The reader is referred to [27,40,41,43,44] for more details regarding the implementation and characteristics of the recovery method.

In order to estimate the error in SFEM approximations we can follow a similar procedure. To build the patches we use the topological information of the SFEM discretization. The recovered stress field is evaluated at the centre of the subcells and then projected to the sampling points as explained before.

**Remark** The recovery method proposed in this paper is general, and could also be applied, although this is beyond the scope of this paper, to problems with corner singularities at

![Fig. 3](image-url)
5 Numerical results

In this section, numerical tests considering 2D benchmark problems with exact solution have been used to investigate the quality of the proposed technique. The performance of the technique has been evaluated using the effectiveness index of the error in energy norm, both at global and local levels. Globally, we have considered the value of the effectiveness index $\theta$ given by

$$\theta = \frac{\|\|e_{ex}\|\|}{\|\|e\|\|}$$  \hspace{1cm} (36)$$

where $\|\|e\|\|$ denotes the exact error in the energy norm, and $\|\|e_{ex}\|\|$ represents the evaluated error estimate. At element level, the distribution of the local effectiveness index $D$, its mean value $m(D)$ and standard deviation $\sigma(D)$ have been analysed, as described in [40]:

$$D = \theta^e - 1 \text{ if } \theta^e \geq 1$$

$$D = 1 - \frac{1}{\theta^e} \text{ if } \theta^e < 1 \text{ with } \theta^e = \frac{\|\|e_{ex}\|\|}{\|\|e\|\|}$$  \hspace{1cm} (37)$$

Note that $\theta^e \in (0, 1)$ when the error is underestimated and $\theta^e \in (1, +\infty)$ when it is overestimated. The definition of $D$ fairly compares the underestimation of the error ($D < 0$) and the overestimation ($D > 0$). The good local behaviour of the estimates results in values of $D$ close to zero. The global effectiveness index $\theta$ is used to evaluate global results. The mean value $m(|D|)$ and the standard deviation $\sigma(D)$ of the local effectiveness are also used to evaluate the global quality of the error estimator as these parameters are useful to take into account error compensations in the evaluation of $\theta$.

5.1 Thick-wall cylinder subjected to an internal pressure.

The geometrical model for this problem is shown in Fig. 4. Due to symmetry conditions, only one part of the section is modelled. Plane strain conditions are assumed.

The exact solution to this problem is given by the following expressions, where for a point $(x, y)$, $c = b/a, r = \frac{b}{a}$.
The recovered solution also has an associated error in energy norm defined as \(||e^*||| = |||u - u^*|||\), which would be evaluated using the exact and recovered stresses, \(\sigma\) and \(\sigma^*\). In Fig. 8 we represent the evolution of the exact errors for the SFEM solution and the recovered field. The error in the recovered field has a higher convergence rate, which is in agreement with the expected quality for the field \(\sigma^*\). According to [34], this also serves to verify the asymptotic exactness of the proposed error estimator.
Fig. 8 Cylinder under internal pressure. Evolution of the exact error $|||e|||$ and convergence rate $s$ for the SFEM solution using four subcells ($|||e|||_{s_{avg}} = 0.49$) and for the error in the recovered solution ($|||e^*|||_{s_{avg}} = 0.65$).

Figure 9 shows the evolution of global indicators $\theta$, $m(|D|)$ and $\sigma(D)$ for the SFEM using equilibrated recovery (SPR-CX curve), without equilibrium constraints (SPR curve) and the standard FEM with equilibrium (SPR-CX (FEM) curve). The equilibrated SFEM and the FEM recoveries exhibit similar results, with good effectivity of the error estimator and decrease of $m(|D|)$ and $\sigma(D)$ for finer meshes. The non–equilibrated SFEM recovery (SPR curve).
The use of different numbers of subcells for the SFEM approximation is also considered for comparison. Figure 10 shows the convergence of the estimated error in energy norm for two, four and eight subcells. All the curves exhibit the same convergence rate \( s = 0.49 \), close to the theoretical value \( s = 0.5 \).

Figure 11 shows the evolution of global indicators \( \theta, m(|D|) \) and \( \sigma(D) \) for two, four and eight subcells. The effectivity indices for all the subcells types shown converge asymptotically to the theoretical value and are very sharp \((1.08 > \theta > 1)\). The local effectivity index goes to zero at the same rate as shown in the curves \( m(|D|) \) and \( \sigma(D) \).

In Fig. 12 we show the influence of the order of the polynomial expansion used for the local recovery on patches. We compare the evolution of the global parameters for first order polynomials, previously represented in Fig. 11, with the corresponding curves considering second order polynomials. We can see that the increase of the polynomial order does not produce and improvement of the effectivity. Local behaviour in \( m(|D|) \) and \( \sigma(D) \) indicates even worse results as we increase the number of degrees of freedom. This is in correspondence with previous results observed in the FEM context [47], where an increase of the polynomial order not necessarily derived in better effectivities.
5.2 L-shape domain under mode I load

This singular problem consists of a portion of an infinite L-shaped domain. The model is loaded on the boundary with the tractions corresponding to the first symmetric term of the asymptotic expansion that describes the exact solution under mode I loading conditions around the singular vertex, see Fig. 13. The exact values of boundary tractions on the emphasized boundaries have been imposed in the FE analyses.

The exact displacement and stress fields for this problem are given by \((9, 10)\). For \(\alpha = 3\pi/2\) one obtains \(\lambda_I = 0.544483736782464, \lambda_{II} = 0.908529189846099\) and \(Q = 0.543075578836737\). Exact values of the GSIFs have been taken as \(K_I = 1\) and \(K_{II} = 0\). The material parameters are Young’s modulus \(E = 1000\), and Poisson’s ratio \(\nu = 0.3\). The splitting radius is \(\rho = 0.5\). To evaluate the SIF we use the equivalent interaction integral defined with a plateau function with radius 0.9 as indicated in [21]. As the analytical solution of this problem is singular at the reentrant corner of the plate, we apply the singular+smooth decomposition of the stress field as explained in Sect. 4. We use the expression in \((10)\) and the estimated GSIF evaluated using the interaction integral to obtain an approximation of the singular part \(\sigma_{sing}\).

Figure 14 shows the distribution of the local effectivity index for the sequence of graded meshes in Fig. 15. In the

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Fig. 12 Cylinder under internal pressure. Global indicators \(\theta, m(|D|)\) and \(\sigma(D)\) for elements with two (2SC), four (4SC) and eight (8SC) subcells using 1st and 2nd order polynomials for the recovery on patches

Fig. 13 L-shaped domain under mode I load

[Diagram of L-shaped domain]

The tractions corresponding to the first symmetric term of the asymptotic expansion that describes the exact solution under mode I loading conditions around the singular vertex, see Fig. 13. The exact values of boundary tractions on the emphasized boundaries have been imposed in the FE analyses.

The exact displacement and stress fields for this problem are given by \((9, 10)\). For \(\alpha = 3\pi/2\) one obtains \(\lambda_I = 0.544483736782464, \lambda_{II} = 0.908529189846099\) and \(Q = 0.543075578836737\). Exact values of the GSIFs have been taken as \(K_I = 1\) and \(K_{II} = 0\). The material parameters are Young’s modulus \(E = 1000\), and Poisson’s ratio \(\nu = 0.3\). The splitting radius is \(\rho = 0.5\). To evaluate the SIF we use the equivalent interaction integral defined with a plateau function with radius 0.9 as indicated in [21]. As the analytical solution of this problem is singular at the reentrant corner of the plate, we apply the singular+smooth decomposition of the stress field as explained in Sect. 4. We use the expression in \((10)\) and the estimated GSIF evaluated using the interaction integral to obtain an approximation of the singular part \(\sigma_{sing}\).

Figure 14 shows the distribution of the local effectivity index for the sequence of graded meshes in Fig. 15. In the
The local index $D$ decreases with the refinement of the meshes and the obtained values are within a narrow range. The first mesh of the sequence is an uniform mesh. This kind of meshes is known to produce pollution error for singular problems. The highly underestimated areas at the right of the plate are explained by this pollution error, which cannot be controlled with local techniques.

Figure 16 shows the convergence of the estimated error in energy norm for different configurations of the recovery procedure: SPR-CX that consider equilibrium and stress decomposition, SPR-X that considers stress decomposition only, SPR-C that consider equilibrium only, and a conventional SPR. The exact error ||$\varepsilon$|| is shown for comparison.

Figure 17 shows the evolution of global indicators $m(|D|)$ and $\sigma(D)$ for the different configurations: SPR-CX, SPR-X, SPR-C, SPR. The best results are for the SPR-CX. The SPR-C and the SPR cannot efficiently recover the field close to the singularity. The SPR seems to provide good global effectivity results, however, this is only due to compensation between underestimated and overestimated areas. The real behaviour is clear when analysing the evolution of $m(|D|)$ and $\sigma(D)$ and even more patent if we represent $D$ as seen in Fig. 18, where we represent the results for the different configurations of the recovery technique. The effect of equilibrium enforcement and singular decomposition is clearly shown. A more homogeneous distribution of $D$ is obtained when using SPR-CX. SPR and SPR-X are not equilibrated along the boundary and therefore they underestimate the error in the elements along it. SPR and SPR-C overestimates the error in the vicinity of the reentrant corner.
6 Conclusions

In this paper, an *a posteriori* recovery-based error estimator which makes use of a modified version of the SPR technique previously used in the FEM and XFEM contexts has been adapted to the SFEM. The recovery technique considers the local enforcement of equilibrium equations and the singular + smooth decomposition of the stress field for singular problems. The technique has been applied to the cell-based smoothed FEM but could also be used with the node-based and edge-based smoothed FEM implementations.

The method yielded sharp estimations of the error in the complementary energy norm, both locally and globally, for the numerical examples presented in this paper. It can be inferred that enforcing equilibrium constraints is required to obtain accurate results. The influence in the recovery of the number of subcells used to formulate the smoothed finite elements is also studied, showing that the technique performs adequately for the different configurations. The error estimator based on the use of the SPR-CX recovery technique efficiently captures the discretization error both in smooth and singular problems. Moreover, it could be used to guide *h*-adaptive refinements and the recovered field $\sigma^*$ can be used as an enhanced solution, more accurate than the stress field provided by the approximation.

Future work includes the comparison of the proposed estimators with other recently developed error estimators for the extended finite element method when dealing with singular problems [19,38,39] for three dimensional fracture problems, the focus of our current research. We will also analyse the behaviour of strain smoothing for real-life three dimensional fracture mechanics problems, which is the topic of the EPSRC project which funded this work. The extension of the proposed technique to 3D problems following the SPR approach should result rather simple, although a higher dimension will imply a higher computational effort and a different level of complexity. For this reason, examples considering adaptive meshes in three dimensions in order to evaluate the real efficiency of the different error estimators are expected, as well as comparisons with recently developed

![Figure 17](image-url)
Fig. 18 L-shaped domain under mode I load. Distribution of the effectivity index \( D \) for the different configurations of the recovery technique. SPR-CX: equilibrium and stress decomposition, SPR-X: stress decomposition, SPR-C: equilibrium, and a conventional SPR explicit residual error estimators. Further mathematical convergence and error analysis of the technique to investigate the properties (robustness, bounding, consistency,...\[48\]) of the error estimates are also an issue to consider in future publications.

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