STABILITY AND DYNAMIC TRANSITION OF A TOXIN-PRODUCING PHYTOPLANKTON-ZOOPLANKTON MODEL WITH ADDITIONAL FOOD

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ABSTRACT. The article aims to investigate the dynamic transitions of a toxin-producing phytoplankton zooplankton model with additional food in a 2D-rectangular domain. The investigation is based on the dynamic transition theory for dissipative dynamical systems. Firstly, we verify the principle of exchange of stability by analysing the corresponding linear eigenvalue problem. Secondly, by using the technique of center manifold reduction, we determine the types of transitions. Our results imply that the model may bifurcate two new steady state solutions, which are either attractors or saddle points. In addition, the model may also bifurcate a new periodic solution as the control parameter passes critical value. Finally, some numerical results are given to illustrate our conclusions.

1. Introduction. It is well known that most aquatic life depends on plankton, which consists of phytoplankton and zooplankton. Phytoplankton is made of microscopic plants, living near the surfaces of all aquatic environments. On the one hand, phytoplankton provides food for zooplankton. On the other hand, phytoplankton produces a great deal of oxygen for other animals and absorbs carbon dioxide from environment through photosynthesis. Zooplankton is the animal in the plankton community, of which both herbivore and predator occur, with herbivore grazing on phytoplankton and then being eaten by zooplankton predator [20].

Due to the essential role in the ecology of the ocean, the dynamic analysis of marine plankton through mathematical models is an important subject. Some models have been utilized to investigate the dynamics of plankton. For example, Zheng and Sugie [29] analyzed a three-species model which comprises phytoplankton, zooplankton and fish and obtained sufficient conditions for global stability and equiasymptotic stability of equilibrium points. In recent years, toxin-producing phytoplankton...
has been attracted to some researchers’ attentions. Saha and Bandyopadhyay [23] investigated the Hopf-bifurcation of a toxin producing phytoplankton-zooplankton model. They did not consider the effect of diffusion in their models. That is, the mathematical models they studied are ordinary differential equations.

From a biological viewpoint, the diffusion phenomenon of population is common, which influences the distribution of population. Physical environment and other organisms in their spatial neighborhood can impact the diffusion. Therefore, some researchers have considered mathematical models with dissipative terms to investigate the dynamics of plankton. Rao [22] studied Turing bifurcation of a reaction-diffusion toxic-phytoplankton-zooplankton model with Holling type-II functional response. Jang et al. [9] proposed a mathematical model for phytoplankton-zooplankton interactions with toxin producing phytoplankton and explored the asymptotical stability of equilibrium point. Chakraborty et al. [1] investigated the linear stability and Turing instability of a nutrient-phytoplankton system with toxic effect on phytoplankton. For other related investigations, we refer to readers to [4,10,24,26,27].

In the real world, if zooplankton is strongly damaged by toxins from phytoplankton, it will alter its food for survival [25]. As a result, the supply of additional food is significant for the persistence of zooplankton, and it further enhances the consumption of phytoplankton [25]. For this reason, we utilize the toxin-producing phytoplankton-zooplankton model originally obtained by Wang et al. [25] as follows

$$\begin{align*}
\frac{\partial u_1}{\partial t} &= d_1 \Delta u_1 + u_1(1 - \frac{u_1}{\gamma}) - \frac{u_1 u_2}{1 + \alpha x + u_1}, \quad (x, t) \in \Omega \times (0, +\infty), \\
\frac{\partial u_2}{\partial t} &= d \Delta u_1 + d_2 \Delta u_2 + \frac{\delta(u_1 + \xi)u_2}{1 + \alpha x + u_1} - mu_2 - n \frac{u_1 u_2}{h + u_1},
\end{align*}$$

(1.1)

This system (1.1) subjects to the following initial-boundary conditions

$$\begin{align*}
\frac{\partial u_i}{\partial \nu} \big|_{\partial \Omega} &= 0, \quad (i = 1, 2), \\
u_i(0, x) &= u_{i0}(x),
\end{align*}$$

(1.2)

(1.3)

where \(\Omega = (0, l_1) \times (0, l_2) \subset \mathbb{R}^2\) and \(\nu\) is an outward normal vector on \(\partial \Omega\). For the descriptions of the unknown functions \(u, v\) and the parameters, we refer to readers to [25]. The boundary condition (1.2) implies that the phytoplankton and zooplankton have no external input from the boundary \(\partial \Omega\). Wang et al. [25] analyzed the combined effects of toxin liberation and additional food on stability and obtained the conditions for the occurrence of Turing instability.

As far as we know, there is little literature on the dynamic transitions of the model (1.1)-(1.3). Consequently, our object is to investigate the dynamic transitions of the model (1.1)-(1.3). We follow the way of the dynamic transition theory for nonlinear dissipative systems [17], which was developed recently by Ma and Wang. The main philosophy of the dynamic transition theory is to search for the full set of transition states, giving a complete characterization of stability and transition. The phase transitions in nonlinear science include three categories. One group is continuous transition, which indicates that the basic state bifurcates a local attractor. The other two groups are jump transition and mixed transition, respectively. The former indicates that a system jumps abruptly to another state and the later indicates that both continuous transition and jump transition are possible depending on the initial data. With advantages of the dynamic transition theory that can offer complete description of the transitions of different states, the dynamic transition theory has
been used extensively to solve biological problems \cite{15,18,19,28} and mathematical physics problems \cite{2,5–8,11–14,16,21}.

In this work, for the dynamic transitions of the model \((1.1)-(1.3)\) in a rectangular domain, we have two primary purposes. Firstly, we apply the dynamic transition theory \cite{17} to analyze a sufficient condition leading to dynamic transitions in the model. The sufficient condition is commonly derived by conducting the linear stability analysis, whose essence is to determine the spectrum of the corresponding linear differential operator of the model \((1.1)-(1.3)\) and verify the PES (principle of exchange of stability). Secondly, we aim to determine the possible types of the dynamic transitions and give the corresponding full set of transition states. The types of the dynamic transitions are described by the corresponding reduced equation derived from the model \((1.1)-(1.3)\) by using the method of center manifold reduction.

In order to obtain the PES, we establish a sufficient condition. We also give some numerical results to illustrate this sufficient condition can be satisfied. Moreover, both the transitions from a real eigenvalue and a pair of conjugate simple complex eigenvalues are considered. The results imply that the model \((1.1)-(1.3)\) may bifurcate two new equilibrium solutions, whose stability is determined by the sign of a non-dimensional coefficient. In addition, it may bifurcate a new periodic solution. Finally, we give some numerical examples to illustrate these results. Precisely, utilizing these results established in this work, we give the region-\((d, n)\) from which one can apparently know the types of transitions. Besides, using the finite difference method \cite{3,20}, we obtain the numerical solutions of the system \((1.1)-(1.3)\) near the critical value, which implies that the numerical results are consistent with the theoretical results.

The rest of this paper is organized as follows. In section 2, we investigate the constant steady state solution of the system \((1.1)-(1.3)\) and rewrite the system \((1.1)-(1.3)\) as an abstract equation. In section 3, we investigate the PES. In section 4, based on the PES and reduction on center manifold, the main results about dynamic transitions are given. In section 5, some numerical results are given to illustrate the theoretical results. In section 6, the conclusions are summarized.

2. Mathematical setting. It is clear that the system \((1.1)\) has the following steady state solutions:

\[
E_0 = (0, 0), \quad E_1 = (\gamma, 0), \quad E_3 = (u_1^*, u_2^*),
\]

where

\[
\begin{align*}
  u_1^* &= \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0a_2}}{2a_0}, \\
  u_2^* &= (1 + \alpha \xi + u_1^*)(1 - \frac{u_1^*}{\gamma}),
\end{align*}
\]

(2.1)

with

\[
  a_0 = \delta - m - n; \quad a_2 = \delta \xi h - mh - m\alpha \xi h, \\
  a_1 = \delta \xi + \delta h - m\alpha \xi - mh - n\alpha \xi - n - m.
\]

There exists only one positive equilibrium point \(E_3\) in the system \((1.1)\), provided \(a_0a_2 < 0\) and \(0 < u_1^* < \gamma\). The unique positive solution has biological significance. Consequently, we shall consider the dynamic transitions from the unique positive steady state solution.
Let $E_3 = (u_1^*, u_2^*)$ be the positive equilibrium point of the system (1.1). We denote

$$u_i' = u_i - u_i^*, \quad (i = 1, 2). \tag{2.2}$$

Then, substituting (2.2) into (1.1) and dropping the primes, we obtain

$$\begin{cases}
\frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + u_1 - \frac{2u_1^*}{\gamma} u_1 + f_1(u_1, u_2), \\
\frac{\partial u_2}{\partial t} = d \Delta u_1 + d_2 \Delta u_2 - mu_2 + f_2(u_1, u_2),
\end{cases}$$

where

$$f_1(u_1, u_2) = u_1^* - \frac{u_1^2}{\gamma} - \frac{(u_1^*)^2}{\gamma} - \frac{(u_1 + u_1^*)(u_2 + u_2^*)}{1 + \alpha \xi + u_1 + u_1^*},$$

$$f_2(u_1, u_2) = \frac{\delta(u_1 + u_1^* + \xi)(u_2 + u_2^*)}{1 + \alpha \xi + u_1 + u_1^*} - mu_2 - \frac{n(u_1 + u_1^*)(u_2 + u_2^*)}{h + u_1 + u_1^*}.$$

The Taylor expansions of $f_1, f_2$ at $u = (0, 0)$ are respectively expressed by

$$f_1(u_1, u_2) = -\frac{(1 + \alpha \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^2} u_1 - \frac{u_1^*}{1 + \alpha \xi + u_1^*} u_2 + \left[ \frac{u_2^* (1 + \alpha \xi)}{(1 + \alpha \xi + u_1^*)^3} - \frac{1}{\gamma} \right] u_1^2,$$

and

$$f_2(u_1, u_2) = \left[ \frac{\delta(1 + \alpha \xi - \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^2} - \frac{nu_2^*}{(1 + \alpha \xi + u_1^*)^3} u_1 + \left[ \frac{\delta(u_1^* + \xi)}{1 + \alpha \xi + u_1^*} - \frac{nu_1^*}{h + u_1^*} \right] u_2^2 \right] u_1,$$

Then, the system (1.1)-(1.3) can be rewritten as

$$\begin{cases}
\frac{\partial u_1}{\partial t} = d_1 \Delta u_1 + \left[ 1 - \frac{2u_1^*}{\gamma} - \frac{(1 + \alpha \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^2} \right] u_1 - \frac{u_1^*}{1 + \alpha \xi + u_1^*} u_2 + g_1(u_1, u_2), \\
\frac{\partial u_2}{\partial t} = d \Delta u_1 + d_2 \Delta u_2 + \frac{\delta u_2^*(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nu_2^*}{(h + u_1^*)^3} u_1 + g_2(u_1, u_2),
\end{cases} \tag{2.3}$$

and

$$\frac{\partial u_i}{\partial \nu} |_{\partial \Omega} = 0, \quad (i = 1, 2), \tag{2.4}$$

$$u_i(0, x) = u_{i0} - u_i^*, \tag{2.5}$$

where

$$g_1(u_1, u_2) = \left[ \frac{u_2^* (1 + \alpha \xi)}{(1 + \alpha \xi + u_1^*)^3} - \frac{1}{\gamma} \right] u_1^2 - \frac{1 + \alpha \xi}{(1 + \alpha \xi + u_1^*)^2} u_1 u_2,$$

and

$$g_2(u_1, u_2) = \frac{(1 + \alpha \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^4} u_1^2 + \frac{(1 + \alpha \xi)}{(1 + \alpha \xi + u_1^*)^3} u_1^2 u_2 + o(|u|^3), \tag{2.6}$$
as well as
\[
g_2(u_1, u_2) = \begin{bmatrix} -\delta u_2^2(1 + \alpha \xi - \xi) + \frac{nh u_2^2}{(h + u_1^i)^3} & u_1^2 + \left[ \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^i)^2} \right] \\
- \frac{nh}{(h + u_1^i)^2} & u_1 u_2 + \left[ \frac{\delta u_2^2(1 + \alpha \xi - \xi)}{(h + u_1^i)^2} - \frac{nh u_2^2}{(h + u_1^i)^3} \right] u_1^3 
\end{bmatrix}
\]

where \( \lambda \) is a constant on \( \Omega \). Moreover, we define the nonlinear operator \( \mathcal{H} : H_1 \rightarrow H \) as follows
\[
\mathcal{H}(u, \lambda) = \begin{bmatrix} g_1(u_1, u_2) \\
g_2(u_1, u_2) \end{bmatrix},
\]
where \( g_1(u_1, u_2), g_2(u_1, u_2) \) are given by (2.6) and (2.7). Combining (2.8) and (2.9), the system (2.3)-(2.5) is referred to the abstract equation as follows
\[
\begin{cases}
\frac{du}{dt} = L_\lambda u + \mathcal{H}(u, \lambda), \\
\end{cases}
\]

where \( u_0 = (u_{10} - u_1^*'), u_{20} - u_2^*') \).

3. Eigenvalues and principle of exchange of stabilities. In this section, we shall investigate the eigenvalues of the linear equations of (2.3)-(2.5). The corresponding eigenvalue problem reads
\[
\begin{cases}
d_1 \Delta u_1 + \left[ 1 - 2u_1^i \lambda - \frac{(1 + \alpha \xi)u_1^i}{(1 + \alpha \xi + u_1^i)^2} \right] u_1 - \frac{u_1^i}{1 + \alpha \xi + u_1^i} u_2 = \beta_1 u_1, \\
d \Delta u_1 + d_2 \Delta u_2 + \left[ \frac{\delta u_2^2(1 + \alpha \xi - \xi)}{(h + u_1^i)^2} - \frac{nh u_2^2}{(h + u_1^i)^3} \right] u_1 = \beta_2 u_2,
\end{cases}
\]

where \( \beta_1, \beta_2 \).
By the separation of variables, the solutions of the following eigenvalue equations
\[
\begin{align*}
-\Delta \varphi_k &= \rho_k \varphi_k, \quad x \in (0, l_1) \times (0, l_2), \\
\frac{\partial \varphi_k}{\partial x_1} &= 0, \quad x_1 = 0, l_1, \\
\frac{\partial \varphi_k}{\partial x_2} &= 0, \quad x_2 = 0, l_2, \\
\int_{\Omega} |\varphi_k|^2 dx &= 1,
\end{align*}
\]
are given by
\[
\rho_k = \left(\frac{k_1 \pi}{l_1}\right)^2 + \left(\frac{k_2 \pi}{l_2}\right)^2, \quad \varphi_k = \frac{2}{\sqrt{l_1 l_2}} \cos\left(\frac{k_1 \pi}{l_1} x_1\right) \cos\left(\frac{k_2 \pi}{l_2} x_2\right), \quad k = (k_1, k_2) \in \mathbb{N}^2,
\]
\[
\varphi_k = \sqrt{\frac{2}{l_1 l_2}} \cos\left(\frac{k_1 \pi}{l_1} x_1\right), k = (k_1, 0), \quad \varphi_k = \sqrt{\frac{2}{l_1 l_2}} \cos\left(\frac{k_2 \pi}{l_2} x_2\right), k = (0, k_2).
\]
In particular, \(\varphi_0 = \frac{1}{\sqrt{l_1 l_2}}\).

Therefore, one can denote
\[
M_k(\lambda) = \begin{pmatrix}
-d_1 \rho_k + 1 - 2u_1^* \lambda - \frac{(1 + \alpha \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^2} & -u_1^* \\
-d_2 \rho_k + \delta(1 + \alpha \xi - \xi) u_2^* & -d_2 \rho_k - \frac{n h u_2^*}{(1 + \alpha \xi + u_1^*)^2} - \frac{n h u_2^*}{(h + u_1^*)^2}
\end{pmatrix}
\]
Thus, all eigenvalues \(\beta_k\) and the eigenvectors \(e_k\) corresponding to \(\beta_k\) are completely determined by the eigenvalue problem
\[
M_k(\lambda) \eta_k = \beta_k \eta_k, \quad e_k = \eta_k \varphi_k.
\]
Consequently, we know that the eigenvalues \(\beta_k\) satisfy the following quadratic equation
\[
\beta_k^2 - B_k(\lambda) \beta_k + C_k(\lambda) = 0,
\]
where
\[
B_k(\lambda) = \text{trace}(M_k) = -d_1 \rho_k - d_2 \rho_k + 1 - 2u_1^* \lambda - \frac{(1 + \alpha \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^2},
\]
\[
C_k(\lambda) = \text{det}(M_k) = \left[ -d_1 \rho_k + 1 - 2u_1^* \lambda - \frac{(1 + \alpha \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^2} \right](-d_2 \rho_k)
\]
\[
+ \frac{u_1^*}{1 + \alpha \xi + u_1^*} \left[ -d_2 \rho_k + \delta(1 + \alpha \xi - \xi) u_2^* - \frac{n h u_2^*}{(1 + \alpha \xi + u_1^*)^2} - \frac{n h u_2^*}{(h + u_1^*)^2} \right] .
\]
After some straightforward computations, one can derive that
\[
\beta_{k_1} = \frac{B_k + \sqrt{B_k^2 - 4C_k}}{2}, \quad \beta_{k_2} = \frac{B_k - \sqrt{B_k^2 - 4C_k}}{2}.
\]
From (3.2), if the eigenvalues \(\beta_{k_i}(i = 1, 2)\) are real, then we derive that the eigenvectors \(e_{k_i}\) corresponding to \(\beta_{k_i}\) are given by
\[
e_{k_i} = \begin{pmatrix} \varphi_k h_{k_i} \\ \varphi_k \end{pmatrix}, \quad h_{k_i} = \frac{-\beta_{k_i} + d_2 \rho_k}{d_2 \rho_k - \left[ \delta(1 + \alpha \xi - \xi) u_2^* - \frac{n h u_2^*}{(1 + \alpha \xi + u_1^*)^2} - \frac{n h u_2^*}{(h + u_1^*)^2} \right]}. \tag{3.3}
\]
In addition, let us denote the dual operator of \(L_\lambda\) as \(L_\lambda^*\). Consequently, we know that the eigenvalues \(\beta_{k_i}^*\) of \(L_\lambda^*\) satisfy \(\beta_{k_i}^* = \beta_{k_i}(i = 1, 2)\). The eigenvectors \(e_{k_i}^*\) corresponding to \(\beta_{k_i}^*\) are
\[
e_{k_i}^* = \begin{pmatrix} \varphi_k h_{k_i}^* \\ \varphi_k \end{pmatrix}, \quad h_{k_i}^* = \frac{-\beta_{k_i} + d_2 \rho_k(1 + \alpha \xi + u_1^*)}{u_1^*}. \tag{3.4}
\]
Theorem 3.1. Let $\rho$, $\beta$, and $\Lambda$ be given by (3.5) and (3.6), respectively, and assume $d, d_1, d_2, \alpha, \xi, \delta, m, n, h$ and $u_1^*$ satisfy the following conditions

$$\frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(u_1^* + h)^2} > 0, \quad (3.10)$$

and

$$(2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*})u_1^* d_1 \rho_1 - (u_1^*)^2 \left[\frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(u_1^* + h)^2}\right] > 0, \quad (3.11)$$

where $\rho_1 = (\frac{\xi}{2})^2, (l_1 < l_2)$. Then, for the linearized equations (3.1), the following assertions hold true.
(1) If \( \lambda_c < \Lambda_c \), then, for \( \lambda \) in the vicinity of \( \Lambda_c \), the eigenvalue \( \beta_{k1} \) is real, and all eigenvalues of (3.1) satisfy

\[
\beta_{k1}(\lambda) = \begin{cases} 
< 0, & \lambda > \Lambda_c, \\
= 0, & \lambda = \Lambda_c, \\
> 0, & \lambda < \Lambda_c,
\end{cases}
\]

(3.12)

\[\text{Re}\beta_{k1}(\Lambda_c) < 0, \quad \text{for any} \quad k \neq K = (K_1, K_2), \quad \text{for any} \quad k \in Z^2.\]

(3.13)

(3.14)

(2) If \( \lambda_c > \Lambda_c \), then for \( \lambda \) in the vicinity of \( \lambda_c \), the eigenvalues \( \beta_{01}, \beta_{02} \) are a pair of conjugate complex numbers, and all eigenvalues of (3.1) satisfy

\[\text{Re}\beta_{01}(\lambda) = \text{Re}\beta_{2}(\lambda) \begin{cases} 
< 0, & \lambda > \lambda_c, \\
= 0, & \lambda = \lambda_c, \\
> 0, & \lambda < \lambda_c,
\end{cases}\]

(3.15)

\[\text{Im}\beta_{01}(\lambda_c) = -\text{Im}\beta_{02}(\lambda_c) \neq 0, \quad \text{Re}\beta_{k1}(\lambda_c) = \text{Re}\beta_{k2}(\lambda_c) < 0, \quad \text{for any} \quad k \neq (0, 0).\]

(3.16)

(3.17)

Proof. This proof is divided into two steps. Step 1, we shall show the assertion (1).

Using the definition of \( \lambda_c \) and \( \Lambda_c \), we derive that

\[
C_K(\Lambda_c) = \left[ -d_1 \rho_K + 1 - 2u_1^* \Lambda_c - \frac{(1 + \alpha \xi)u_2^*}{(1 + \alpha \xi + u_1^*)^2} \right] (-d_2 \rho_K) \\
+ \frac{u_1^*}{1 + \alpha \xi + u_1^*} \left[ -d_\rho_K + \frac{\delta(1 + \alpha \xi - \xi)u_2^*}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh u_2^*}{(h + u_1^*)^2} \right] \\
= \left[ (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*})u_1^* d_\rho_K - (u_2^*)^2 \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(h + u_1^*)^2} \right] \Lambda_c \\
+ d_2 \rho_K (d_1 \rho_K - 1) + \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*} d_2 \rho_K \\
+ u_1^* \left[ \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(h + u_1^*)^2} \right] - \frac{u_1^*}{1 + \alpha \xi + u_1^*} d_\rho_K = 0.
\]

Moreover, we have

\[
B_K(\Lambda_c) = -d_1 \rho_K - d_2 \rho_K + 1 - 2u_1^* \Lambda_c - \frac{(1 + \alpha \xi)u_2^*}{(1 + \alpha \xi + u_1^*)^2} = (1 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) - (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) \Lambda_c u_1^* - d_1 \rho_K - d_2 \rho_K \\
< (1 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) - (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) \Lambda_c u_1^* - d_1 \rho_K - d_2 \rho_K \\
= -d_1 \rho_K - d_2 \rho_K \leq 0.
\]

Thus, we obtain that \( \beta_{K1}(\Lambda_c) = 0 \) and \( \beta_{K2}(\Lambda_c) < 0 \).

As \( \lambda < \Lambda_c \), we derive that

\[
C_K(\lambda) = \left[ (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*})u_1^* d_\rho_K - (u_2^*)^2 \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(h + u_1^*)^2} \right] \lambda \\
+ d_2 \rho_K (d_1 \rho_K - 1) + \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*} d_2 \rho_K
\]
and \( \beta \) can show \( \beta \) where we have used the condition (3.11). Then \( \Lambda > \Lambda_c \). Consequently, the conclusion (1) is valid.

In addition, for case \( k \neq (0, 0) \), we have

\[
B_k(\Lambda_c) = (1 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) - (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) \Lambda_c u_1^* - d_1 \rho_k - d_2 \rho_k
\]

\[
< (1 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) - (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) \lambda_c u_1^* - d_1 \rho_k - d_2 \rho_k
\]

\[
= -d_1 \rho_k - d_2 \rho_k \leq 0,
\]

and

\[
C_k(\Lambda_c) = \left[ (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*} u_1^* d_2 \rho_k - (u_1^*)^2 \right] \left[ \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(h + u_1^*)^2} \right] \Lambda_c
\]

\[
+ d_2 \rho_k (d_1 \rho_k - 1) + \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*} d_2 \rho_k - \frac{u_1^*}{1 + \alpha \xi + u_1^*} d \rho_k
\]

\[
< \frac{u_1^* d \rho_k}{1 + \alpha \xi + u_1^*} - \frac{u_1^*}{1 + \alpha \xi + u_1^*} \left[ \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(h + u_1^*)^2} \right] d_2 \rho_k (d_1 \rho_k - 1)
\]

\[
- \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*} d_2 \rho_k + d_2 \rho_k (d_1 \rho_k - 1) + \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*} d_2 \rho_k
\]

\[
+ \frac{u_1^*}{1 + \alpha \xi + u_1^*} \left[ \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(h + u_1^*)^2} \right] - \frac{u_1^*}{1 + \alpha \xi + u_1^*} d \rho_k = 0.
\]

For case \( k = (0, 0) \), it is clear that we have

\[
B_0(\Lambda_c) = (1 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) - (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) \Lambda_c u_1^*
\]

\[
< (1 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) - (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) \lambda_c u_1^* = 0,
\]

as well as

\[
C_0(\Lambda_c) = \frac{u_1^*}{1 + \alpha \xi + u_1^*} \left[ \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(n + h)^2} \right] u_2 > 0.
\]

where we have used the condition (3.10). Consequently, the conclusion (1) is valid.

Step 2. We shall prove the conclusion (2). Similarly, from the definition of \( \lambda_c \) and \( \Lambda_c \), we obtain \( B_0(\lambda_c) = 0 \) and

\[
B_0(\lambda) = (1 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) - (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*}) \lambda u_1^*
\]
Proof. Denote the linear space spanned by types of (2.3)-(2.5) in a rectangular domain \( \Omega = (0, \lambda_c) \times (0, l_2) \). Hence, as \( \lambda < \lambda_c \), near \( \lambda_c \), we can show that \( \Re \beta_{01} = \Re \beta_{02} > 0 \). As \( \lambda > \lambda_c \), near \( \lambda_c \) we can show that \( \Re \beta_{01} = \Re \beta_{02} < 0 \) in the same way. Similarly, a simple calculation shows that \( B_k(\lambda_c) < 0 \) and \( C_k(\lambda_c) > 0 \). Consequently, the conclusion (2) holds true. \( \square \)

4. Nonlinear dynamic transitions. In this section, we focus on the transition types of (2.3)-(2.5) in a rectangular domain \( \Omega = (0, l_1) \times (0, l_2) \). We firstly consider the dynamic transition from a real eigenvalue. To this end, we introduce the following parameter

\[
b(\lambda) = \left( 2R_1h_{K1} - R_2 \right) h_{K1}^* + (2R_3h_{K1} + R_4) \left[ \frac{1}{\sqrt{l_1l_2}} (\Phi_{01}h_{01} + \Phi_{02}h_{02}) \right.
\]

\[
+ \frac{1}{2\sqrt{l_1l_2}} (\Phi_{2K1}h_{2K1} + \Phi_{2K2}h_{2K2}) + \frac{1}{\sqrt{l_1l_2}} (\Phi_{2K1}h_{2K1} + \Phi_{2K2}h_{2K2})
\]

\[
\left. + \frac{1}{\sqrt{l_1l_2}} (\Phi_{2K1}h_{2K1} + \Phi_{2K2}h_{2K2}) \right] + \left[ - R_2h_{K1}h_{K1}^* + R_4h_{K1} \right]
\]

\[
\cdot \left[ \frac{1}{\sqrt{l_1l_2}} (\Phi_{01} + \Phi_{02}) + \frac{1}{\sqrt{l_1l_2}} (\Phi_{2K1} + \Phi_{2K2}) + \frac{1}{\sqrt{l_1l_2}} (\Phi_{2K1} + \Phi_{2K2}) \right]
\]

\[
+ \frac{9}{4l_1l_2} \left( -S_1h_{K1}^3 + S_2h_{K1}^2 \right) h_{K1}^*
\]

\[
+ (S_3h_{K1}^3 + S_4h_{K1}^2)
\]

(4.1)

where \( h_{K1}, h_{K1}^*, \Phi_{k,i}, R_j \) and \( S_j \) are given respectively by (3.3), (3.4), (4.9), (4.10) and (4.14).

Theorem 4.1. Under the conditions of Theorem 3.1, if \( \lambda_c < \Lambda_c \), then the following assertions are valid

1. If the parameter \( b(\Lambda_c) < 0 \), then the system (2.3)-(2.5) undergoes a continuous transition from \( (0, \Lambda_c) \) and bifurcates two singular points \( u_{1,2} \) on \( \Lambda < \Lambda_c \), which are asymptotically stable. The topological structure of phase portraits are shown in Figure 1 and Figure 2.

2. If the parameter \( b(\Lambda_c) > 0 \), then the system (2.3)-(2.5) undergoes a jump transition from \( (0, \Lambda_c) \) and bifurcates two saddle points \( u_{1,2} \) with the Morse index one on \( \Lambda > \Lambda_c \). The topological of phase portraits are shown in Figure 3 and Figure 4.

3. The bifurcated points \( u_{1,2} \) are expressed by

\[
u_{1,2} = \pm \sqrt{- \frac{P_{K1} \beta_{K1}}{b(\Lambda_c)} \epsilon_{K1} + \Phi \left( - \frac{P_{K1} \beta_{K1}}{b(\Lambda_c)} \right) + o(3)},
\]

where \( \Phi \) is given by (4.13).

Proof. Denote the linear space spanned by \( \{ \epsilon_{K1} \} \) as \( E_1 \). Let \( E_2 \) be spanned by the rest of eigenvectors. By the spectral theory of linear completely continuous field, the spaces \( H \) and \( H_1 \) can be decomposed into

\[
H = E_1 \oplus E_2, \quad H_1 = \bar{E}_1 \oplus \bar{E}_2.
\]
Then, in the vicinity of \( \Lambda_c \), the solution of (2.3)-(2.5) can be written as

\[
\begin{align*}
    u & = x_K e_{K1} + y, \\
    y & = \sum_{k \neq K} x_k e_k + \sum_{k=0}^{\infty} x_k e_{k2},
\end{align*}
\]

where \( x_K e_K \in E_1 \) and \( y \in E_2 \). Thus, in the space \( E_1 \), the system (2.3)-(2.5) can be reduced to

\[
\frac{dx_K}{dt} = \beta_K x_K + \frac{1}{\langle e_{K1}, e_{K1}^* \rangle} \langle G(x_K e_K + \Phi, \lambda), e_{K1}^* \rangle, \tag{4.2}
\]

where \( \Phi : E_1 \to E_2 \) is the corresponding center manifold function associated with PES (3.12).
Next, we calculate the center manifold function $\Phi$. Utilizing the approximation formula of center manifold function [17], we have

$$- L_\lambda \Phi = P_2 G_2(x_K e_{K1}, \lambda) + o(|x_K|^2) + O(|\beta_{K1}||x_K|^2),$$

(4.3)

where $P_2 : H \to E_2$ is the canonical projection.

By (2.9), we obtain

$$G(u, \lambda) = G_2(u, \lambda) + G_3(u, \lambda),$$

(4.4)

where

$$G_2(u, \lambda) = \left( \frac{u_3^2(1+\alpha+\xi)}{(1+\alpha+\xi+u_3^2)} - \lambda u_1^2 \right) - \frac{1+\alpha}{(1+\alpha+\xi+u_1^2)} u_1 u_2 - \frac{1+\alpha}{(1+\alpha+\xi+u_1^2)} u_1 u_2,$$

(4.5)

and

$$G_3(u, \lambda) = \left( \frac{u_3^2(1+\alpha+\xi)}{(1+\alpha+\xi+u_3^2)} - \lambda u_1^2 \right) - \frac{1+\alpha}{(1+\alpha+\xi+u_1^2)} u_1 u_2 - \frac{1+\alpha}{(1+\alpha+\xi+u_1^2)} u_1 u_2.$$  

(4.6)

Then, we get

$$G_2(x_K e_{K1}, \lambda) = x_2^2 \varphi_K^2 \left( \frac{u_3^2(1+\alpha+\xi)}{(1+\alpha+\xi+u_3^2)} - \lambda \right) h_{K1}^2 - \frac{1+\alpha}{(1+\alpha+\xi+u_1^2)} h_{K1}^2 - \frac{1+\alpha}{(1+\alpha+\xi+u_1^2)} h_{K1}^2.$$

(4.7)

We notice that

$$\varphi_K^2 = \frac{4}{l_1 l_2} \cos^2 \left( \frac{K_1 \pi}{l_1} x_1 \right) \cos^2 \left( \frac{K_2 \pi}{l_2} x_2 \right)$$

$$= \frac{1}{l_1 l_2} \left[ \cos \left( \frac{2 K_1 \pi}{l_1} x_1 \right) \cos \left( \frac{2 K_2 \pi}{l_2} x_2 \right) + \cos \left( \frac{2 K_1 \pi}{l_1} x_1 \right) + \cos \left( \frac{2 K_2 \pi}{l_2} x_2 \right) + 1 \right],$$

which implies that the center manifold function should be taken the following form

$$\Phi = \left[ \Phi_{01} e_{01} + \Phi_{02} e_{02} + \Phi_{2K_1 e_{2K1}} + \Phi_{2K_2 e_{2K2}} + \Phi_{2K_1 e_{2K1}} + \Phi_{2K_2 e_{2K2}} \right] x_2^2 + o(2).$$

(4.8)

Making use of (4.7), upon substituting the preceding expression (4.8) into (4.3) and comparing the corresponding coefficients on both sides, one can derive that

$$\Phi_{01} = \frac{1}{\beta_{01} \sqrt{l_1 l_2}} \left[ |R_{1} h_{K1}^2 - R_{2} h_{K1}^2| h_{K1}^2 + R_{3} h_{K1}^2 + R_{4} h_{K1} \right],$$

$$\Phi_{02} = \frac{1}{\beta_{02} \sqrt{l_1 l_2}} \left[ |R_{1} h_{K1}^2 - R_{2} h_{K1}^2| h_{K1}^2 + R_{3} h_{K1}^2 + R_{4} h_{K1} \right],$$

$$\Phi_{2K_1} = \frac{1}{\sqrt{2} \beta_{2K_1} \sqrt{l_1 l_2}} \left[ |R_{1} h_{K1}^2 - R_{2} h_{K1}^2| h_{2K_1}^2 + R_{3} h_{K1}^2 + R_{4} h_{K1} \right],$$

$$\Phi_{2K_2} = \frac{1}{\sqrt{2} \beta_{2K_2} \sqrt{l_1 l_2}} \left[ |R_{1} h_{K1}^2 - R_{2} h_{K1}^2| h_{2K_2}^2 + R_{3} h_{K1}^2 + R_{4} h_{K1} \right].$$
\( \Phi_{2K,1} = -\frac{1}{2\beta_{2K,1}\sqrt{h_2}} \left\{ [R_1 h_{K1}^2 - R_2 h_{K1} h_{2K,1}^* + R_3 h_{K1}^2 + R_4 h_{K1}] \right\}, \\
\Phi_{2K,2} = -\frac{1}{2\beta_{2K,2}\sqrt{h_2}} \left\{ [R_1 h_{K1}^2 - R_2 h_{K1} h_{2K,2}^* + R_3 h_{K1}^2 + R_4 h_{K1}] \right\}, \\
(4.9) \)

where

\[
R_1 = \frac{u_2^*(1 + \alpha \xi)}{(1 + \alpha \xi + u_1^*)^3} - \lambda, \quad R_2 = \frac{1 + \alpha \xi}{(1 + \alpha \xi + u_1^*)^2}, \\
R_3 = -\frac{\delta(1 + \alpha \xi - \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^3} + \frac{nh u_2^*}{(h + u_1^*)^3}, \quad R_4 = \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(h + u_1^*)^2}. \\
(4.10) \]

Let us denote

\[
\phi_1 = \Phi_{01} h_{01} \varphi_0 + \Phi_{02} h_{02} \varphi_0 + \Phi_{2K1} h_{2K1} \varphi_{2K1} + \Phi_{2K2} h_{2K2} \varphi_{2K2}, \\
\phi_2 = \Phi_{01} h_{01} \varphi_0 + \Phi_{02} h_{02} \varphi_0 + \Phi_{2K1} h_{2K1} \varphi_{2K1} + \Phi_{2K2} h_{2K2} \varphi_{2K2}, \\
(4.11) \]

and

\[
\phi_2 = \Phi_{01} h_{01} \varphi_0 + \Phi_{02} h_{02} \varphi_0 + \Phi_{2K1} h_{2K1} \varphi_{2K1} + \Phi_{2K2} h_{2K2} \varphi_{2K2} + \Phi_{2K1} h_{2K1} \varphi_{2K1} + \Phi_{2K2} h_{2K2} \varphi_{2K2}, \\
(4.12) \]

Then, the center manifold function \( \Phi \) can be written as the following form

\[
\Phi = (\phi_1, \phi_2)^T x_K^2 + o(2). \\
(4.13) \]

As a result, with the help of (4.4), (4.11), (4.12) and (4.13), one can derive that

\[
\begin{align*}
\langle G(x_K e_{K1} + \Phi, \lambda), e_{K1}^* \rangle &= \int_{\Omega} [R_1 (x_K h_{K1} \varphi_K + \phi_1 x_K^2) - R_2 (x_K h_{K1} \varphi_K + \phi_1 x_K^2)(x_K \varphi_K + \phi_2 x_K^2)] \\
&\quad \cdot h_{K1}^* \varphi_K dx + \int_{\Omega} [R_3 (x_K h_{K1} \varphi_K + \phi_1 x_K^2) + R_4 (x_K h_{K1} \varphi_K + \phi_1 x_K^2)] \\
&\quad \cdot (x_K \varphi_K + \phi_2 x_K^2) \varphi_K dx + \int_{\Omega} [-S_1 (x_K h_{K1} \varphi_K + \phi_1 x_K^2)] \\
&\quad + S_2 (x_K h_{K1} \varphi_K + \phi_1 x_K^2)^2 (x_K \varphi_K + \phi_2 x_K^2)] h_{K1}^* \varphi_K dx + \int_{\Omega} [S_3 (x_K h_{K1} \varphi_K + \phi_1 x_K^2)^2 (x_K \varphi_K + \phi_2 x_K^2)] \varphi_K dx \\
&= x_K^3 [(2R_1 h_{K1}^2 - R_2) h_{K1}^* + (2R_3 h_{K1} + R_4)] \int_{\Omega} \varphi_K^2 \phi_1 dx + x_K^3 \int_{\Omega} \varphi_K^2 \phi_2 dx \\
&\quad + R_4 h_{K1} \int_{\Omega} \varphi_K^2 \phi_2 dx + x_K^3 [(S_1 h_{K1}^* + S_2 h_{K1}) h_{K1}^* \\
&\quad + (S_3 h_{K1}^* + S_4 h_{K1})] \int_{\Omega} \varphi_K^2 dx + o(|x_K|^3) = b(\lambda) x_K^2 + o(|x_K|^3),
\end{align*} \\
\]

where

\[
S_1 = -\frac{(1 + \alpha \xi) u_2^2}{(1 + \alpha \xi + u_1^*)^3}, \quad S_2 = \frac{1 + \alpha \xi}{(1 + \alpha \xi + u_1^*)^3}, \\
S_3 = \frac{\delta(1 + \alpha \xi - \xi) u_2^2}{(h + u_1^*)^3}, \quad S_4 = \frac{-\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} + \frac{nh}{(h + u_1^*)^3}. \\
(4.14) \]
Therefore, (4.2) is reduced to
\[
\frac{dx_K}{dt} = \beta K x_K + \frac{1}{P_K} b(\lambda) x_K^3 + o(|x_K|^3).
\] (4.15)
Consequently, by the Theorem 2.3.1 in [17], we obtain the assertions.

Similarly, based on Theorem 3.1 and with the help of center manifold reduction, we obtain the following results about transition types. For convenience, let us introduce the following parameter
\[
\tilde{b}(\lambda_c) = \frac{3\pi}{4} (a_{01}^1 + a_{02}^2) + \frac{\pi}{2} \frac{a_2^5}{\beta_{01}} (a_{02}^1 a_{02}^2 - a_{02}^1 a_{02}^2)
\]
where \(a_{pi}^j(2 \leq p + q \leq 3, i = 1, 2)\) are given by (4.28)-(4.29).

**Theorem 4.2.** Under the conditions of Theorem 3.1, if \(\lambda > \lambda_c\), then the system (2.3)-(2.5) undergoes a Hopf bifurcation on \(\lambda = \lambda_c\) and the following assertions hold true

1. If \(\tilde{b}(\lambda_c) < 0\), then the system (2.3)-(2.5) undergoes a continuous transition and bifurcates a periodic orbit from \((0, \lambda_c)\), which is an attractor.
2. If \(\tilde{b}(\lambda_c) > 0\), then the system (2.3)-(2.5) undergoes a jump transition and bifurcates an unstable periodic orbit.
3. The bifurcated periodic solution is expressed as
   \[
u = x e_{01} + y e_{02} + o(\text{Re} \beta_{01}),
   \]
   where
   \[
x(t) = \sqrt{\frac{\text{Re} \beta_{01}(\lambda)}{|b|}} \cos(\text{Im} \beta_{01} t) + o(\text{Re} \beta_{01}),
   \]
   \[
y(t) = \sqrt{\frac{\text{Re} \beta_{01}(\lambda)}{|b|}} \sin(\text{Im} \beta_{01} t) + o(\text{Re} \beta_{01}).
   \]

Proof. By the spectral theory of linear completely continuous field, the spaces \(H\) and \(H_1\) are decomposed into
\[
H = E_1 \oplus E_2, \quad H_1 = E_1 \oplus E_2,
\]
where \(E_1 = \text{span}\{e_{01}, e_{02}\}\) and \(E_2 = E_1^+\). \(\{e_{ki}\}\) is the eigenvector corresponding to \(\beta_{ki}\). Then the solution of (2.3)-(2.5) is expressed by
\[
u = x e_{01} + y e_{02} + z,
\]
where \(z \in E_2\). Hence, (2.3)-(2.5) are reduced to
\[
\frac{dx}{dt} = \text{Re} \beta_{01} x - \text{Im} \beta_{01} y + \frac{1}{\langle e_{01}, e_{01}^* \rangle} (G(x e_{01} + y e_{02} + \Phi, \lambda), e_{01}^*),
\]
\[
\frac{dy}{dt} = \text{Re} \beta_{01} y + \text{Im} \beta_{01} x + \frac{1}{\langle e_{02}, e_{02}^* \rangle} (G(x e_{01} + y e_{02} + \Phi, \lambda), e_{02}^*),
\]
where \(\Phi : E_1 \rightarrow E_2\) is the center manifold function.
At the critical value \(\lambda = \lambda_c\), we have \(\text{Re} \beta_{01}(\lambda_c) = 0\). Therefore, (4.17)-(4.18) are written as
\[
\frac{dx}{dt} = -\text{Im} \beta_{01} y + \frac{1}{\langle e_{01}, e_{01}^* \rangle} (G(x e_{01} + y e_{02} + \Phi, \lambda_c), e_{01}^*),
\]
\[
\frac{dy}{dt} = \text{Re} \beta_{01} y + \text{Im} \beta_{01} x + \frac{1}{\langle e_{02}, e_{02}^* \rangle} (G(x e_{01} + y e_{02} + \Phi, \lambda_c), e_{02}^*).
\]
\[
\frac{dy}{dt} = Im\beta_{01}x + \frac{1}{(e_{02}, e_{02}^*)}(G(xe_{01} + ye_{02} + \Phi, \lambda_c), e_{02}^*).
\]

Next, we shall calculate the eigenvectors \(e_{01}(\lambda_c), e_{02}(\lambda_c)\) and their conjugates \(e_{01}(\lambda_c)^*, e_{02}(\lambda_c)^*\). Let us denote

\[
e_{01} = (\tilde{\eta}_1 \varphi_0, \eta_2 \varphi_0), \quad e_{02} = (\tilde{\eta}_1 \varphi_0, \tilde{\eta}_2 \varphi_0),
\]

\[
e_{01}^* = (\tilde{\eta}_1^* \varphi_0, \eta_2^* \varphi_0), \quad e_{02}^* = (\tilde{\eta}_1^* \varphi_0, \tilde{\eta}_2^* \varphi_0).
\]

(4.19)

Substituting (4.19) into (3.6) and (3.7) respectively, we derive that

\[
\begin{pmatrix}
1 - 2u_1^* \lambda_c - \frac{(1 + \alpha \xi)u_2^*}{(1 + \alpha \xi + u_1^*)^2} - \frac{u_1^*}{1 + \alpha \xi + u_1^*} - \frac{u_1^*}{(h + u_1^*)^2} \\
\delta u_2^*(1 + \alpha \xi - \xi) \frac{nu_2^*}{(1 + \alpha \xi + u_1^*)^2} - \frac{u_1^*}{(h + u_1^*)^2}
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_2
\end{pmatrix}
= Im\beta_{01}
\begin{pmatrix}
\tilde{\eta}_1 \\
\tilde{\eta}_2
\end{pmatrix},
\]

(4.20)

as well as

\[
\begin{pmatrix}
1 - 2u_1^* \lambda_c - \frac{(1 + \alpha \xi)u_2^*}{(1 + \alpha \xi + u_1^*)^2} \\
\delta u_2^*(1 + \alpha \xi - \xi) \frac{nu_2^*}{(1 + \alpha \xi + u_1^*)^2}
\end{pmatrix}
\begin{pmatrix}
\eta_1^* \\
\eta_2^*
\end{pmatrix}
= -Im\beta_{01}
\begin{pmatrix}
\tilde{\eta}_1^* \\
\tilde{\eta}_2^*
\end{pmatrix},
\]

(4.21)

We notice that

\[
1 - 2u_1^* \lambda_c - \frac{(1 + \alpha \xi)u_2^*}{(1 + \alpha \xi + u_1^*)^2} = 0,
\]

(4.24)

and

\[
(Im\beta_{01})^2 = \frac{u_1^*}{1 + \alpha \xi + u_1^*} \left[ \frac{\delta(1 + \alpha \xi - \xi) u_2^*}{(1 + \alpha \xi + u_1^*)^2} - \frac{nu_2^*}{(h + u_1^*)^2} \right].
\]

(4.25)

Using (4.20)-(4.25), we obtain

\[
\eta_1 = \frac{1}{Im\beta_{01}} \frac{u_1^*}{1 + \alpha \xi + u_1^*}, \quad \eta_2 = 1, \quad \tilde{\eta}_1 = -\frac{1}{Im\beta_{01}} \frac{u_1^*}{1 + \alpha \xi + u_1^*}, \quad \tilde{\eta}_2 = 1,
\]

(4.26)

and

\[
\eta_1^* = Im\beta_{01} \frac{1 + \alpha \xi + u_1^*}{u_1^*}, \quad \eta_2^* = 1, \quad \tilde{\eta}_1^* = -Im\beta_{01} \frac{1 + \alpha \xi + u_1^*}{u_1^*}, \quad \tilde{\eta}_2^* = 1.
\]

(4.27)

Then, one can derive that

\[
\langle e_{01}, e_{01}^* \rangle = \langle e_{02}, e_{02}^* \rangle = 2 \int_\Omega \varphi_0^2 dx = 2, \quad \langle e_{01}, e_{02}^* \rangle = \langle e_{02}, e_{01}^* \rangle = 0.
\]

Next, we shall calculate the center manifold function \(\Phi\). Using the approximation of center manifold function [17], we have

\[
\Phi = \Phi_1 + \Phi_2 + \Phi_3 + o(2),
\]

where \(\Phi_1, \Phi_2, \Phi_3\) satisfy

\[
- L_{\lambda_c} \Phi_1 = x^2G_{11} + y^2G_{22} + xy(G_{12} + G_{21}),
\]

\[
- \left((-L_{\lambda_c})^2 + 4(Im\beta_{01})^2\right) L_{\lambda_c} \Phi_2 = 2(Im\beta_{01})^2 \left[ (x^2 - y^2)(G_{22} - G_{11}) - 2xy(G_{12} + G_{21}) \right],
\]

\[
L_{\lambda_c} \Phi_3 = 0.
\]
\[
\left[-L_{\lambda}\right]^2 + 4\text{Im}\beta_0^2 \right] \Phi_3 = \text{Im}\beta_0 \left[ (x^2 - y^2)(G_{12} + G_{21}) + 2xy(G_{11} - G_{22}) \right],
\]
and \( G_{ij} = P_2G_2(e_{0i}, e_{0j}, \lambda_c), 1 \leq i, j \leq 2. \)

From (4.5), we obtain
\[
G_2(e_i, e_j, \lambda_c) = \left( R_1 e_{01}^1 e_{0j}^1 \right) - \left( R_2 e_{01}^1 e_{0j}^2 \right) = \left( R_3 e_{01}^1 e_{0j}^1 \right) + \left( R_4 e_{01}^1 e_{0j}^2 \right) \varphi_0^2,
\]
where \( e_{0i} = (e_{01}^1, e_{02}^2) = (\eta_{12} \varphi_0, \eta_{22} \varphi_0) \) and
\[
(\eta_{11}, \eta_{12}) = \left( \frac{u_1^*}{\text{Im}\beta_0 1 + \alpha \xi + u_1^*}, 1 \right), \quad (\eta_{21}, \eta_{22}) = \left( -\frac{1}{\text{Im}\beta_0 1 + \alpha \xi + u_1^*}, 1 \right).
\]

Then we derive that \( G_{ij} = P_2G_2(e_{0i}, e_{0j}) = 0. \) Thus, \( \Phi = o(2). \)

Consequently, combining (4.4) and (4.19), we derive that
\[
\langle G(xe_{0i} + ye_{02} + \Phi, \lambda_c), e_{01}^* \rangle
\]
\[
= \int_{\Omega} \left[ (x\eta_1 + y\eta_1^*) R_1 - (x\eta_2 + y\eta_2) R_2 \right] \eta_1^* \varphi_0^3 dx
\]
\[
+ \int_{\Omega} \left[ (x\eta_1 + y\eta_1^*) R_3 + (x\eta_2 + y\eta_2) R_4 \right] \eta_2^* \varphi_0^3 dx
\]
\[
+ \int_{\Omega} \left[ -S_1(x\eta_1 + y\eta_1^*)^3 + S_2(x\eta_1 + y\eta_1^*)^2(x\eta_2 + y\eta_2) \right] \eta_1^* \varphi_0^3 dx
\]
\[
+ \int_{\Omega} \left[ S_3(x\eta_1 + y\eta_1^*)^3 + S_4(x\eta_1 + y\eta_1^*)^2(x\eta_2 + y\eta_2) \right] \eta_2^* \varphi_0^3 dx
\]
\[
= \sum_{2 \leq p + q \leq 3} a_{pq}^1 y^p + o(3),
\]
where
\[
a_{20}^1 = \frac{1}{\sqrt{l_1l_2}} \left[ \eta_1^* (\eta_1^2 R_1 - \eta_1^2 R_2) + \eta_2^* (\eta_2^2 R_3 + \eta_1 \eta_2 R_4) \right],
\]
\[
a_{11}^1 = \frac{1}{\sqrt{l_1l_2}} \left[ \eta_1^* (2 \eta_1 \eta_1 R_1 - (\eta_1 \eta_2 + \eta_2 \eta_1) R_2) \right.
\]
\[
\left. + \eta_2^* (2 \eta_1 \eta_2 R_3 + (\eta_1 \eta_2 + \eta_2 \eta_1) R_4) \right],
\]
\[
a_{02}^1 = \frac{1}{\sqrt{l_1l_2}} \left[ \eta_1^* (\eta_1^2 R_1 - \eta_1 \eta_2 R_2) + \eta_2^* (\eta_2^2 R_3 + \eta_1 \eta_2 R_4) \right],
\]
\[
a_{30}^1 = \frac{1}{l_1l_2} \left[ \eta_1^* (-S_1 \eta_1^3 + S_2 \eta_2^3 \eta_1) + \eta_2^* (S_3 \eta_1^3 + S_4 \eta_2^3 \eta_1) \right],
\]
\[
a_{21}^1 = \frac{1}{l_1l_2} \left[ \eta_1^* (-3 S_1 \eta_1^2 \eta_1 + (\eta_1^2 \eta_2 + 2 \eta_1 \eta_1 \eta_2) S_2) \right.
\]
\[
\left. + \eta_2^* (3 S_3 \eta_1^2 \eta_1 + (\eta_1^2 \eta_2 + 2 \eta_1 \eta_1 \eta_2) S_4) \right],
\]
\[
a_{12}^1 = \frac{1}{l_1l_2} \left[ \eta_1^* (-3 S_1 \eta_1^2 \eta_1 + (\eta_1^2 \eta_2 + 2 \eta_1 \eta_1 \eta_2) S_2) \right.
\]
\[
\left. + \eta_2^* (3 S_3 \eta_1^2 \eta_1 + (\eta_1^2 \eta_2 + 2 \eta_1 \eta_1 \eta_2) S_4) \right],
\]
\[
a_{03}^1 = \frac{1}{l_1l_2} \left[ \eta_1^* (-3 S_1 \eta_1^2 \eta_1^3 + \eta_1^2 \eta_2 S_2) + \eta_2^* (S_3 \eta_1^3 + \eta_1 \eta_2 S_4) \right],
\]
(4.28)
and \( \eta_1, \eta_2, \eta_3 (i = 1, 2), R_j, S_j (j = 1, 2, 3, 4) \) are given respectively by (4.26)-(4.27),
(4.10) and (4.14).
Similarly, some direct calculations indicate

\[ (G(xe_0 + ye_0 + \Phi, \lambda_c), e_{02}^p) = \sum_{2 \leq p + q \leq 3} a_{pq}^2 x^p y^q + o(3), \]

where

\[
\begin{align*}
    a_{20}^2 &= \frac{1}{\sqrt{l_1 l_2}} \left[ \eta_1^* (\eta_1^2 R_1 - \eta_1 \eta_2 R_2) + \eta_2^* (\eta_2^2 R_3 + \eta_1 \eta_2 R_4) \right], \\
    a_{11}^2 &= \frac{1}{\sqrt{l_1 l_2}} \left[ \eta_1^* (2 \eta_1 \eta_1 R_1 - (\eta_1 \eta_2 + \eta_2 \eta_1) R_2) \\
    &+ \eta_2^* (2 \eta_1 \eta_1 R_3 + (\eta_1 \eta_2 + \eta_2 \eta_1) R_4) \right], \\
    a_{21}^2 &= \frac{1}{\sqrt{l_1 l_2}} \left[ \eta_1^* (2 \eta_1 \eta_1 R_1 - \eta_1 \eta_2 R_2 + \eta_2^2 R_3 + \eta_1 \eta_2 R_4) \right], \\
    a_{30}^2 &= \frac{1}{\sqrt{l_1 l_2}} \left[ \eta_1^* (-S_1 \eta_1^3 + S_2 \eta_1^2 \eta_2) + \eta_2^* (S_3 \eta_1^3 + S_4 \eta_1^2 \eta_2) \right], \\
    a_{12}^2 &= \frac{1}{l_1 l_2} \left[ \eta_1^* (-3 S_1 \eta_1^2 \eta_1 + (\eta_1^2 \eta_2 + 2 \eta_1 \eta_1 \eta_2) S_2) \\
    &+ \eta_2^* (3 S_3 \eta_1^3 \eta_1 + (\eta_1^2 \eta_2 + 2 \eta_1 \eta_1 \eta_2) S_4) \right], \\
    a_{22}^2 &= \frac{1}{l_1 l_2} \left[ \eta_1^* (-3 S_1 \eta_1^2 \eta_1 + (\eta_1^2 \eta_2 + 2 \eta_1 \eta_1 \eta_2) S_2) \\
    &+ \eta_2^* (3 S_3 \eta_1^3 \eta_1 + (\eta_1^2 \eta_2 + 2 \eta_1 \eta_1 \eta_2) S_4) \right], \\
    a_{33}^2 &= \frac{1}{l_1 l_2} \left[ \eta_1^* (-S_1 \eta_1^3 + \eta_2^2 \eta_2 S_2) + \eta_2^* (S_3 \eta_1^3 + \eta_2^2 \eta_2 S_4) \right],
\end{align*}
\]

and \( \eta_i, \tilde{\eta}_i, \tilde{\eta}_i^* (i = 1, 2), R_j, S_j (j = 1, 2, 3, 4) \) are given respectively by (4.26), (4.10) and (4.14).

As a result, we have

\[
\begin{align*}
    \frac{dx}{dt} &= -Im \beta_{011} y + \frac{1}{2} \sum_{2 \leq p + q \leq 3} a_{pq}^1 x^p y^q + o(3), \\
    \frac{dy}{dt} &= Im \beta_{010} x + \frac{1}{2} \sum_{2 \leq p + q \leq 3} a_{pq}^2 x^p y^q + o(3).
\end{align*}
\]

Consequently, by the Theorem 2.3.7 in [17], the conclusions hold true. \( \square \)

5. Numerical results and discussions. In the preceding section, the nonlinear dynamic transition theorem for the system (1.1)-(1.3) is established. For the purpose of illustration, we give some numerical examples to illustrate the main results for some specified parameters. We consider the rectangular domain as \( \Omega = (0, 10) \times (0, 20) \). Choosing the parameter \( d_1 = 0.1, d_2 = 29.7, \alpha = 0.5, \delta = 4.2, \xi = 0.7, m = 2.6, h = 9.3,\) \( d \in [6, 13] \) and \( n \in [0.1, 1.5] \), respectively, we can obtain the graph of the critical parameters \( \Lambda_c \) and \( \lambda_c \), shown in Figure 5.

From Figure 5, the sufficient conditions (3.10)-(3.11) are satisfied for each \( d \in [6, 13] \) and \( n \in [0.1, 1.5] \). In addition, we know that the critical parameter value \( \lambda_c \) is less than the critical parameter value \( \Lambda_c \) for all \( d \in [11, 13] \) and \( n \in [0.1, 1] \), respectively, we can obtain the graph of the critical parameters \( \Lambda_c \) and \( \lambda_c \), shown in Figure 5.

Namely, the assertion (1) of Theorem 3.1 is valid. Moreover, it is shown that the critical parameter value \( \lambda_c \) is greater than the critical parameter \( \Lambda_c \) for all \( d \in [6, 8] \) and \( n \in [1, 1.2] \). That is, the assertion (2) of Theorem 3.1 holds true.
Furthermore, with the help of (4.1) and (4.16), we obtain the region-\((d, n)\) shown in Figure 6 from which one can clearly know the transition types. More precisely, for each \((d, n)\) in the region A, we obtain \(\lambda_c < \Lambda_c\) and \(b(\Lambda_c) < 0\) which implies that the system undergoes dynamic transitions of continuous type. Consequently, there exists a fork bifurcation in the system as \(\Lambda < \Lambda_c\), and two asymptotically stable points are bifurcated from the equilibrium point \((u^*_1, u^*_2)\). Similarly, we know that \(\lambda_c > \Lambda_c\) and \(b(\Lambda_c) > 0\) if \((d, n)\) is chosen in the region B. Namely, the system undergoes dynamic transitions of jump type and an unstable periodic solution is bifurcated from the equilibrium point \((u^*_1, u^*_2)\).

More precisely, we firstly choose \(d_1 = 0.1, d_2 = 29.7, d = 12.6, \alpha = 0.5, \delta = 4.2, \xi = 0.7, m = 2.6, n = 0.8, h = 9.3\). These values of parameters are the same as
in [25]. Using (2.1) and (3.10)-(3.11), we obtain
\[ u_1^* = 0.3914, \quad \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(u_1^* + h)^2} = 0.8210, \]
as well as
\[ (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1^*})u_1^* d_2 \rho_1 - (u_1^*)^2) \cdot \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1^*)^2} - \frac{nh}{(u_1^* + h)^2} = 0.2255 > 0. \]

Therefore, the conditions (3.10) and (3.11) are valid. Furthermore, using the (3.8) and (1.1)-(1.3), we obtain a continuous transition and bifurcates a stable equilibrium point. Namely, the system (1.1)-(1.3) bifurcates a stable equilibrium point from (0.3914, 1.3696) in the vicinity of \( \lambda = 0.5455 \).

Furthermore, using the finite difference method [3,20], we can obtain the numerical solutions near \( \Lambda_c \). The difference scheme is given as follows
\[
\begin{align*}
\partial_{n1} u_{1, ij}^{n+1} &= d_1 \Delta h_i, u_{1, ij}^{n} + \tilde{f}(u_{1, ij}^{n}, u_{2, ij}^{n}), \\
\partial_{n2} u_{2, ij}^{n+1} &= d_2 \Delta h_i, u_{2, ij}^{n} + \tilde{g}(u_{1, ij}^{n}, u_{2, ij}^{n}),
\end{align*}
\]
where
\[
\begin{align*}
\tilde{f}(u_{1, ij}^{n}, u_{2, ij}^{n}) &= u_{1, ij}^{n} - \lambda u_{1, ij}^{n}|u_{1, ij}^{n-1}| - \frac{u_{1, ij}^{n-1} - u_{2, ij}^{n}}{1 + \alpha \xi + |u_{1, ij}^{n-1}|}, \\
\tilde{g}(u_{1, ij}^{n}, u_{2, ij}^{n}) &= d \Delta h_i, u_{1, ij}^{n-1} + \frac{\delta(u_{1, ij}^{n-1} + \xi)u_{2, ij}^{n}}{1 + \alpha \xi + |u_{1, ij}^{n-1}|} - m u_{1, ij}^{n} - \frac{n u_{1, ij}^{n-1} - u_{2, ij}^{n}}{h + |u_{1, ij}^{n-1}|},
\end{align*}
\]
and
\[
\begin{align*}
u_{k, ij}^{n+1} &= u_k(n_1 \Delta t, l_i + ih_1, 1 + jh_1), \\
\partial_{ni} u_{k, ij}^{n+1} &= \frac{u_{k, ij}^{n+1} - u_{k, ij}^{n-1}}{\Delta t}, \\
\Delta h_i u_{k, ij}^{n+1} &= \frac{u_{k, ij}^{n+1} + u_{k, (i+1), j} + u_{k, (i-1), j} + u_{k, (i+1), (j-1)} - 4u_{k, ij}^{n}}{h_1^2},
\end{align*}
\]
where \( h_1, \Delta t \) denote space step and time step, respectively.

Choosing space step \( h_1 = 1/10 \) on the \( x \) axis and \( h_2 = 1/5 \) on the \( y \) axis as well as time step \( \Delta t = 1/200 \) as well as initial data \( u_{10} = 0.3914 + 10^{-2}(x-5)(y-15), u_{20} = 1.3696 \), we obtain the numerical solutions of (1.1)-(1.3) at time \( T = 600 \) shown in Figure 7 and Figure 8. From Figure 7 and Figure 8, it is clear that the system (1.1)-(1.3) undergoes a continuous transition and bifurcates a stable equilibrium point from (0.3914, 1.3696) near the critical value \( \Lambda_c = 0.5455 \).

Second, we choose the parameters \( d_1 = 0.1, d_2 = 29.7, d = 6, \alpha = 0.5, \delta = 4.2, \xi = 0.7, m = 2.6, n = 1.3, h = 9.3 \). Some computations yield to that
\[ u_1 = 0.4180, \quad \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1)^2} - \frac{nh}{(n + h)^2} = 0.7453 > 0, \]
as well as
\[ (2 - \frac{1 + \alpha \xi}{1 + \alpha \xi + u_1})u_1^* d_2 \rho_1 - (u_1^*)^2) \cdot \frac{\delta(1 + \alpha \xi - \xi)}{(1 + \alpha \xi + u_1)^2} - \frac{nh}{(n + h)^2} = 0.2485 > 0. \]
Consequently, the conditions stated in Theorem 3.1 are also satisfied. In addition, we obtain $\lambda_c = 0.4574$ and $\Lambda_c = 0.4361$. The critical value $\Lambda_c$ is obtained as $\rho_k = 0.3208$ and $K = (1, 3)$. The conclusion (2) stated in Theorem 3.1 is valid. Furthermore, with the help of (4.16), we obtain $b(\lambda_c) = 0.0046 > 0$. As a result, by the assertion (2) of Theorem 4.2, the system (2.3)-(2.8) undergoes a jump transition and bifurcates an unstable periodic solution from $(\lambda_c, 0)$. That is, the system (1.1)-(1.3) bifurcates an unstable periodic solution from $(0.4180, 1.4299)$ near the critical value $\lambda_c = 0.4574$.

6. **Conclusions.** Toxin-producing phytoplankton-zooplankton model is a popular topic which has caused wide public attention. Wang et al. [25] discussed Turing instability. However, this expression of the critical parameter for Turing instability is not given. We discuss the bifurcation and instability problem from the perspective of dynamic transition which is different from the investigation in [25]. Firstly, Turing instability analysis is based on the principle that pattern formation in a system of partial differential equations will occur if a system is stable to spatially homogeneous perturbations but unstable if spatially heterogeneously perturbed. A dissipative dynamical system undergoes dynamic transition if it satisfies PES(principle of exchange of stability). Secondly, amplitude equations for
Turing patterns are obtained by using multiple-scale method [25]. Reduction equations which determine the type of dynamic transition are established by using the method of center manifold reduction. The critical value (3.8)-(3.9) and the PES (Theorem 3.1) for dynamic transitions are established by investigating the linear eigenvalue problem. With the help of the PES and reduction on center manifold, the mathematical model (1.1)-(1.3) can be reduced to an ordinary differential system (4.15) or (4.30) whose bifurcation is equivalent to the bifurcation of (1.1)-(1.3).

According to Theorem 4.1 and Theorem 4.2, there exist two critical parameters \( \lambda_c \) and \( \Lambda_c \) such that if \( \lambda_c < \Lambda_c \), the pattern formation formed by plankton is stationary, and if \( \lambda_c > \Lambda_c \), a time oscillation behavior will occur. The bifurcated solution is also established in Theorem 4.1 or Theorem 4.2. It is noted that the bifurcated steady state solution \( u_{1,2} \) stated in Theorem 4.1 describes the distribution of plankton when the dynamic transition is continuous.

Making use of the results (Theorem 4.1 and Theorem 4.2) addressed in the paper, we give some numerical examples. From Figure 5 and Figure 6, the model (1.1)-(1.3) bifurcates a stable equilibrium point for most of the parameters \( (d,n) \) and bifurcates an unstable periodic solution for a small number of parameters \( (d,n) \). Namely, when we choose the parameter \( (d,n) \) in the region \( A \), there exists a critical value \( \Lambda_c \) below which new pattern formation of plankton can be observed. When we choose the parameter \( (d,n) \) in the region \( B \), a time oscillation behavior will occur. However, in real life, this periodic phenomenon is invisible. From Figure 7-Figure 8, the numerical results are consistent with the theoretical results.

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