Ambiguity of the one-loop calculations in a nonrenormalizable quantum gravity

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Abstract

The ambiguity in the calculations of one-loop counterterms by the background field method in nonrenormalizable theories of gravity is discussed. Some examples of such ambiguous calculations are given. The non-equivalence of the first and second order formalism in the quantum gravity is shown.

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1 Introduction

The construction of a quantum theory of gravity is an unresolved problem of modern theoretical physics. The Einstein gravity satisfies to the four classical experimental verifications [1]; however, this theory is incomplete. Yet at the classical level there are problems connected with the presence of the singularity [2] - [4], the definition of the energy-momentum tensor of gravitational field, etc. The main problems are connected with the quantum treatment of the gravitational interaction. Einstein's gravity is the finite theory at the one-loop level in the absence of both matter fields and a cosmological constant [5], but it is nonrenormalizable theory at the two-loop order [6], [7]. The interaction of the gravity with the matter fields gives rise to nonrenormalizable theories yet at the one-loop level [8]- [10]. Therefore, one needs to modify the theory or to show that difficulties presently encountered by the theory are only artifacts of the perturbation theory. For example, one can reject the perturbation renormalization of the theory as a main criterion of the true quantum gravity theory. We can consider the finite criterion: the Green functions can be divergent but all elements of the S-matrix must be finite on the mass-shell in each order of perturbation theory. This criterion is satisfactory in the supergravity theories.

Now there are several ways to modify the Einstein's gravity. The most interesting directions are following:

1. One can introduce terms quadratic in the curvature tensor in the action of the theory. This theory is renormalizable but it is not unitary because the ghosts and tachyons are present in the spectrum of the theory [11] - [15]. It is impossible to restore the unitarity of the theory by loop corrections or adding an interaction with matter fields (see also [16]).

2. One can consider the non-Riemannian geometry. This way is connected with the possibility for the quantum (and possible classical) treatment of space-time to involve more than the Riemannian space-time [17] - [21]. The most interesting non-Riemannian space-times are the space-time with torsion and affine-metric space-time. In these geometries, there are geometrical objects additional to the metric tensor such as torsion and nonmetricity tensors defined as independent variables. In these theories, there are additional symmetries connected with the local transformation of the connection fields [22], [23]. The presence of additional symmetries in the theory may improve the renormalization properties of the theory. However, all attempts to construct the perturbative renormalizable and unitary quantum gravity based on the non-Riemannian space-time failed.

3. One can consider the theories with an additional gauge symmetries. The most promising symmetry is supersymmetry. In supergravity we must use the finite criterion for obtaining the sensible quantum gravity. The simplest supergravity with \( N = 1 \) is the first example of the gravity theory interacting with the matter field which has the finite elements of the S-matrix on the mass-shell at the one-loop level. But at the three-loop level, there are nonvanishing counterterms violating the finiteness of the theory. The extended supergravities with \( N > 1 \)
may be finite up to $N$ loop. But at the present time, there is not a satisfactory 
supergravity model finite at all loop levels \cite{24}, \cite{25}.

In recent years, hopes of constructing a renormalizable theory of the quantum 
gravity have centered on the superstring \cite{26}. The question about the existence of a 
perturbative renormalizable quantum gravity in any string model is open.

The new promising non-perturbative treatment of the Einstein gravity is discussed in \cite{27}.

Modern quantum field theory is based on the principles like unitarity, renormaliz-
ability, the existence of the S-matrix and perturbation approach. All suggested models 
of the quantum gravity based on the Riemannian or non-Riemannian geometries can 
be divided into three classes:

1. the renormalizable, but non-unitary models

2. unitary, but nonrenormalizable models

3. nonrenormalizable and non-unitary models

Hence, all existing theories of the quantum gravity are unsatisfactory from the point 
of view of quantum field theory. In the gravity, quantum corrections give rise to 
very interesting results like the modification of the Newton law, disappearance of the 
classical singularity, corrections to the entropy of the black hole. All these results were 
obtained by means of the quantum field theory methods. Since all existing theories of 
gravity are unsatisfactory from the point of view of quantum field theory, the question 
arises about the validity of the results of loop calculations. In other words, one needs 
to investigate the consistency of the modern powerful tool of quantum field theory and 
existing theories of gravity.

In this paper, we will discuss only the validity of the results of one-loop calculations 
in the framework of the background field method in nonrenormalizable theories of the 
quantum gravity. We will concentrate our attention on the DeWitt-Kallosh and the 
equivalence theorems, which play the essential role in the modern methods of the loop 
calculations in quantum gravity.

The equivalence theorem states, that the $S$-matrix of the renormalizable theory is 
independent of the following change of variables:

$$\varphi^i \rightarrow \varphi^i = \varphi^i + (\varphi^2)^j + (\varphi^3)^j + \ldots$$ (1)

In the case of the quantum gravity, this statement is divided into two parts:

1. It is well know that there is considerable freedom in what one considers to be 
gravitational fields. For example, in the Einstein gravity we can consider an 
arbitrary tensor density $\tilde{g}_{\mu\nu} = g_{\mu\nu}(-g)^r$ or $\tilde{g}^{\mu\nu} = g^{\mu\nu}(-g)^s$ as gravitational vari-
bles. In accordance with the equivalence theorem the loop counterterms on the 
mass-shell must be independent of the choice of gravitational variables.
2. The loop counterterms on the mass-shell are independent of the redefinition of quantum fields of the form

\[ h_{\mu\nu} \rightarrow' h_{\mu\nu} = h_{\mu\nu} + k \left( h^2 \right)_{\mu\nu} + k^2 \left( h^3 \right)_{\mu\nu} + \ldots \]

This redefinition must influence only the higher loop results off the mass-shell.

By means of the corresponding choice of gravitational variables or the corresponding quantum field redefinition, one can considerably reduce the number and the type of interaction vertices. For example, if we consider \( g_{\mu\nu} \) as a gravitational variable, the number of three-point interactions in the Einstein gravity is equal to 13 [6]; if the tensor density \( g_{\nu\mu} \sqrt{-g} \) is selected as a dynamical variable, the number of a three-point interaction is equal to six [28]; combining both the method reduces the number of three-point interactions to two [7].

The main aim of our investigation is to show that the results of loop calculations within the background field method in nonrenormalizable theories of quantum gravity are ambiguous. As a consequence, we assert that in the nonrenormalizable theories of the quantum gravity the usual (background) effective action on and off shell does not give physical information.

We use the following notation:

\[ c = \hbar = 1; \quad k^2 = 16\pi G, \quad g = -\det(g_{\mu\nu}), \quad e = |\det(e^a_\mu)| \]

\[ \eta_{\mu\nu} = (++, --), \quad \varepsilon = \frac{4 - d}{2}, \quad \mu, \nu = 0, 1, 2, 3; \quad a, b = 0, 1, 2, 3; \]

\[ R^a_{\lambda\mu\nu} = \partial_\mu \Gamma^a_{\lambda\nu} - \partial_\nu \Gamma^a_{\mu\lambda} + \Gamma^a_{\alpha\mu} \Gamma^\alpha_{\lambda\nu} - \Gamma^a_{\alpha\nu} \Gamma^\alpha_{\mu\lambda}, \quad R_{\mu\nu} = R^a_{\mu\sigma\nu}, \quad R = R_{\mu\nu} g^{\mu\nu} \]

where \( \Gamma^a_{\mu\nu} \) is the Riemannian connection defined as

\[ \Gamma^a_{\mu\nu} = \frac{1}{2} g^{\sigma\lambda} (-\partial_\lambda g_{\mu\nu} + \partial_\nu g_{\mu\lambda} + \partial_\mu g_{\lambda\nu}) \quad (2) \]

Objects marked by the tilde \( ' \) are constructed by means of the Riemann-Cartan connection \( \tilde{\Gamma}^a_{\mu\nu} \). Other objects are the Riemannian objects.

## 2 Background field method

The background field method [29], [30] was suggested to obtain covariant results of the loop calculations. In the background field method, all dynamical fields \( \phi^j \) are expanded with respect to background values, according to

\[ \phi^j = \phi^j_b + \phi^j_q \]

and only the quantum fields \( \phi^j_q \) are integrated over in the path integral. The background fields \( \phi^j_b \) are effectively external sources. For the one-particle irreducible diagrams there is a difference between the normal field theory and the background field method insofar as the gauge-fixing term may introduce additional vertices. B.DeWitt has
proved that these additional vertices do not influence the $S$-matrix and the $S$-matrix in the formalism of the background field method is equivalent to the conventional $S$-matrix \[29\], \[30\]. This proof has later been extended in a lot of papers \[31\] - \[37\]. The physical quantities are gauge and parametrization independent elements of the $S$-matrix on the mass shell. The choice of external lines on the mass-shell in the background field formalism corresponds to using the classical equations of motion for the background fields. Hence, the counterterms on the mass-shell calculated by the background field method must be independent of the gauge-fixing parameters and the reparametrization of quantum fields. These statements are called the DeWitt-Kallosh theorem \[29\], \[30\], \[32\] and equivalence theorem \[38\] - \[41\], respectively. However, for nonrenormalizable theories the proofs of the DeWitt-Kallosh theorem and equivalence theorem are formal.

In the next chapter, we restrict ourselves only to one-loop calculations. Let us give the short notes about the one-loop calculations within the background field formalism.

In the gauge theories, the renormalization procedure may violate the gauge invariance at the quantum level, thus destroying the renormalizability of the theory. Therefore, one is bound to apply an invariant renormalization. This can be achieved by applying an invariant regularization and using the minimal subtraction scheme \[42\], \[43\]. It has been proved that the dimensional regularization \[44\] - \[47\] is an invariant regularization preserving all the symmetries of the classical action that do not depend explicitly on the space-time dimension \[43\], \[48\], \[49\]. It has been shown \[50\] that in general renormalizable and nonrenormalizable theories the background field formalism requires using an invariant renormalization procedure to obtain valid results. A noninvariant regularization or renormalization may break an implicit correlation between different diagrams, which is essential as one formally expands the action in the background and quantum fields. We will use the invariant regularization (dimensional renormalization and minimal subtraction scheme) in our calculations.

Let us consider the gauge theory with the classical action $S(\phi^j)$ where $\{\phi^j\}$ are the dynamical variables. In accordance with the background field method, all dynamical fields $\varphi^j$ are rewritten as a sum of the background and quantum fields:

$$\varphi^j = \phi^j_b + \phi^j_{qu}$$

and the fields $\phi^j_b$ satisfy the classical equations of motion

$$\frac{\delta S(\phi^j)}{\delta \phi^j_b} = 0$$

One expands the action $S(\varphi^j_b + \varphi^j_{qu})$ in powers of the quantum field and picks out the terms quadratic in the quantum fields we obtain

$$S_{eff} = \frac{1}{2} \phi^i_{qu} \delta \phi^i_{b} \delta \phi^j_{b} \phi^j_{qu}$$

This is an effective action for calculating the one-loop corrections. Due to the presence of the gauge invariance, one needs to introduce the gauge fixing term
\[ f^a = P_j^a(\phi_b)j_{\mu a} \]  \( 6 \)

\[ S_{gf} = \frac{1}{2\alpha} f^a f_a \]  \( 7 \)

where \( \alpha \) is an arbitrary constant and \( P_j^a(\phi_b) \) is the most general gauge in the background field method defined by the following conditions:

- Lorentz covariance
- linear in the quantum field
- the number of derivatives with respect to the quantum fields is smaller than or equal to one

When using the invariant renormalization the one-loop correction to the usual effective action is

\[ \Gamma^{(1)} = \frac{i}{2} \left( \ln \det \Delta_{ab} - 2 \ln \det \Delta_{FP} \right) \]  \( 8 \)

where \( \Delta_{FP} \) is the Faddeev-Popov ghost operator, defined in the standard way and

\[ \Delta_{ij} = \frac{\delta^2 S(\phi)}{\delta \phi^i \delta \phi^j} + P_i^a(\phi)P_{aj}^a(\phi) \]  \( 9 \)

The divergence part of the one-loop effective action obtained by means of the heat kernel method is

\[ \Gamma^{(1)}_{\infty} = -\frac{1}{32\pi^2 \varepsilon} \int d^4x \sqrt{-g} \left( B_4(\Delta_{ij}) - 2B_4(\Delta_{FP}) \right) \]  \( 10 \)

where \( B_4 \) is the second coefficient of the spectral expansion of the corresponding differential operator [51] - [53]. For the operator

\[ \Delta_{ij} = -\left( \nabla^2 1_{ij} + 2S^a_{ij} \nabla_\sigma + X_{ij} \right) \]  \( 11 \)

\( B_4 \) is equal to

\[ B_4(\Delta) = Tr \left( \frac{1}{180} \left( R^2_{\mu \nu \sigma \lambda} - R^2_{\mu \nu} \right) + \frac{1}{2} \left( Z + \frac{R}{6} \right) + \frac{1}{12} Y_{\mu \nu} Y^{\mu \nu} + \frac{1}{6} \left( \frac{R}{5} + Z \right) \right) \]  \( 12 \)

where

\[ Z = X - \nabla_\lambda S^\lambda - S_\lambda S^\lambda \]

\[ Y_{\mu \nu} = \nabla_\mu S_\nu - \nabla_\nu S_\mu + S_\mu S_\nu - S_\nu S_\mu + [\nabla_\mu, \nabla_\nu] 1 \]  \( 13 \)
3 Examples of ambiguity of the one-loop calculations

In this section, we remind some previous results on the ambiguity in one-loop calculations in nonrenormalizable theories of gravity (Examples 1 and 2).

3.1 Example 1

Let us consider the matter field in the external gravitational background \[55\]. Consider the interaction of the gravity based on the Riemannian space-time with a real scalar field \(\phi\) described by the action

\[
S_1(g, \phi) = \int d^4x \, g^{\mu\nu} \sqrt{g} \left( \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{k^2} R_{\mu\nu}(g) + \frac{1}{12} \phi^2 R_{\mu\nu}(g) \right)
\]  

(14)

Now we make the change of the variables

\[
g_{\mu\nu} = G_{\mu\nu} \cosh^2 (k \psi)
\]

(15)

\[
\phi = k^{-1} \tanh (k \psi)
\]

(16)

The new action is given by

\[
S_2(G, \psi) = \int d^4x \, G^{\mu\nu} \sqrt{G} \left( \frac{1}{2} \partial_\mu \psi \partial_\nu \psi - \frac{1}{k^2} R_{\mu\nu}(G) \right)
\]

(17)

In accordance with the equivalence theorem, the S-matrix corresponding to the action \(S_1\) coincides with the S-matrix corresponding to the action \(S_2\). The DeWitt-Kallosh theorem asserts that the one-loop counterterms on the mass-shell calculated by the background field method must be gauge and parametrization invariant. As consequence, the one-loop counterterms of the theory described by the action \(S_1\) must coincide with the one-loop counterterms of the theory described by the action \(S_2\). As has been shown by M. J. Duff \[55\] when only the scalar field is quantized, which corresponds to quantum field theory in the external curved space-time, the one-loop counterterms are

\[
\Delta_1(\infty) = \frac{1}{1920 \pi^2 \xi} \int d^4x \, \sqrt{g} \, C_{\alpha\beta\mu\nu}(g) C^{\alpha\beta\mu\nu}(g)
\]

(18)

and

\[
\Delta_2(\infty) = \frac{1}{1920 \pi^2 \xi} \int d^4x \, \sqrt{G} \left( C_{\alpha\beta\mu\nu}(G) C^{\alpha\beta\mu\nu}(G) + \frac{5}{2} R^2(G) \right)
\]

(19)

for the actions \(S_1\) and \(S_2\), respectively. Here \(C_{\alpha\beta\mu\nu}\) is the Weyl tensor.

Since

\[
C_{\alpha\beta\mu\nu}(g) C^{\alpha\beta\mu\nu}(g) \sqrt{g} = C_{\alpha\beta\mu\nu}(G) C^{\alpha\beta\mu\nu}(G) \sqrt{G}
\]

(20)

we see that
\[ \triangle_{1\infty} \neq \triangle_{2\infty} \quad (21) \]

Hence, the equivalence theorem is violated.

### 3.2 Example 2

Consider the interaction of the Einstein gravity with the scalar field \( \phi \) described by the action

\[ S(g, \phi) = \int d^4x \ g^{\mu\nu} \sqrt{g} \left( \frac{1}{2} \partial_\mu \phi \partial_\nu \phi - \frac{1}{k^2} R_{\mu\nu}(g) \right) \quad (22) \]

For the calculation of the one-loop counterterms within the background field method, we use the following gauge \([56]\):

\[ L_{gf} = \frac{1}{2\beta_1} C^\mu C_\mu \quad (23) \]

\[ C_\mu = \nabla_\nu h^\nu - \frac{1}{2} \alpha_1 \nabla_\mu h - \alpha_2 k \nabla_\mu \phi + \alpha_3 \phi \partial_\mu \nabla_\nu h \quad (24) \]

where \( h_{\mu\nu} \) and \( \varphi \) are the quantum metric and scalar fields, respectively, and

\[ \alpha_1 = 1 - 2\alpha, \quad \beta_1 = 1 - 2\beta, \quad \alpha_2 = 1 + \xi \]

\[ |\alpha| \ll 1, \quad |\beta| \ll 1, \quad |\alpha_3| \ll 1 \]

and \( \xi \) is arbitrary.

The one-loop counterterms on the mass-shell in the topological trivial space-time are

\[ \triangle = \frac{1}{8\pi^2\varepsilon} \frac{1}{5760} \int d^4x \ \sqrt{g} (\partial_\mu \phi \partial_\nu \phi)^2 \left( -3654 + 60\alpha(-3\xi^4 + 24\xi^3 + 35\xi^2 + 11\xi) 
+ 720\alpha_3(-\xi + 2) + 5\beta(90\xi^4 - 273\xi^2 + 324\xi + 116) + O(\alpha, \alpha_3, \beta)^2 \right) \quad (25) \]

Hence the DeWitt-Kallosh theorem is violated.

### 4 One-loop counterterms in the first-order gravity with the Gilbert-Einstein action

Let us consider the Riemann-Cartan space-time. Let us briefly describe the mathematical tool of the space-time with the torsion. In the Riemann-Cartan space-time there are two equivalent approaches for describing the geometry of the space-time with torsion.

The first approach, so-called the Poincarè gauge approach, considers the vierbein field \( e^a_\mu \) and local Lorentz connection \( \tilde{w}^a_{\beta\mu} \) as independent dynamical variables. In this approach, there are two gauge field strengths. One is the translational gauge field strength defined by
\[ Q_{\mu\nu}^a(e, \bar{w}) \equiv -\frac{1}{2} \left( \partial_\mu e^a_\nu - \partial_\nu e^a_\mu + \bar{w}^a_{b\mu} e^b_\nu - \bar{w}^a_{b\nu} e^b_\mu \right) \] (26)

This tensor is the strength tensor of the vierbein \( e^a_\mu \). The other is the Lorentz gauge field strength defined by the following relation:

\[ \tilde{R}_{b\mu\nu}(\bar{w}) \equiv \partial_\mu \bar{w}^a_{b\nu} - \partial_\nu \bar{w}^a_{b\mu} + \bar{w}^a_{c\mu} \bar{w}^c_{b\nu} - \bar{w}^a_{c\nu} \bar{w}^c_{b\mu} \] (27)

The second approach to description of the Riemann-Cartan space-time, so-called geometrical approach, considers the metric tensor \( g_{\mu\nu} \) and linear affine connection \( \tilde{\Gamma}^\sigma_{\mu\nu} \) as independent dynamical variables. These variables satisfy the metric condition

\[ \tilde{\nabla}_\sigma g_{\mu\nu} \equiv \partial_\sigma g_{\mu\nu} - \tilde{\Gamma}^\alpha_{\mu\sigma} g_{\alpha\nu} - \tilde{\Gamma}^\alpha_{\nu\sigma} g_{\alpha\mu} = 0 \] (28)

By means of these variables we can define two geometrical objects, the curvature and the torsion tensors, characterizing the Riemann-Cartan space-time. The torsion and curvature tensors are defined by the following expressions:

\[ Q^\sigma_{\mu\nu}(\tilde{\Gamma}) \equiv \frac{1}{2} \left( \tilde{\Gamma}^\sigma_{\mu\nu} - \tilde{\Gamma}^\sigma_{\nu\mu} \right) \] (29)

\[ \tilde{R}^a_{\lambda\mu\nu}(\tilde{\Gamma}) \equiv \partial_\mu \tilde{\Gamma}^\sigma_{\lambda\nu} - \partial_\nu \tilde{\Gamma}^\sigma_{\lambda\mu} + \tilde{\Gamma}^\sigma_{\alpha\mu} \tilde{\Gamma}^\alpha_{\lambda\nu} - \tilde{\Gamma}^\sigma_{\alpha\nu} \tilde{\Gamma}^\alpha_{\lambda\mu} \] (30)

The Poincaré gauge approach and geometrical approach are related with each other by the following constraint equations:

\[ g_{\mu\nu} = e^a_\sigma e^b_\nu \eta_{ab} \] (31)

\[ \tilde{\nabla}_\sigma e^a_\mu \equiv \partial_\sigma e^a_\mu + \bar{w}^a_{\sigma\mu} e^b_\nu - \tilde{\Gamma}^\lambda_{\mu\sigma} e^a_\lambda = 0 \] (32)

Due to equations (31) and (32) the connection between the Poincaré gauge approach and the geometrical approach becomes very clear. The translational gauge field strength \( Q_{\mu\nu}^a(e, \bar{w}) \) is the torsion field

\[ Q_{\mu\nu}^a(e, \bar{w}) = e^a_\sigma Q^\sigma_{\mu\nu}(\tilde{\Gamma}) \] (33)

and the Lorentz gauge field strength \( \tilde{R}^a_{\mu\nu}(\bar{w}) \) is the curvature tensor

\[ \tilde{R}^a_{\mu\nu}(\bar{w}) = e^a_\sigma e^b_\nu \tilde{R}^\sigma_{\lambda\mu\nu}(\tilde{\Gamma}) \] (34)

Greek and Latin indices is converted into Latin or Greek indices with the help of the vierbein field, for example

\[ K^a_{b\nu} = e^a_\sigma e^\mu_\sigma K^\sigma_{\mu\nu} \] (35)

Having solved equation (28) and using the constrain equations (31) and (32) we obtain the following decompositions of the linear affine connection \( \tilde{\Gamma}^\sigma_{\mu\nu} \) and local Lorentz connection \( \bar{w}^a_{b\mu} \):
\[ \tilde{\Gamma}^\sigma_{\mu\nu} = \Gamma^\sigma_{\mu\nu} + K^\sigma_{\mu\nu} \]  
(36)

\[ \tilde{w}^a_{\mu b} = w^a_{\mu b} + K^a_{\mu b} \]  
(37)

where

\[ \Gamma^\sigma_{\mu\nu} \]  
is the Riemannian connection defined in \((2)\),

\[ K^\sigma_{\mu\nu} \equiv Q^\sigma_{\mu\nu} + Q_{\mu\sigma} + Q_{\nu\sigma} \]  
(38)

\[ w^a_{\mu b} \]  
is the Ricci rotation coefficients given by first derivatives of the vierbein fields

\[ w_{abm} = C_{abm} + C_{bma} + C_{mba} \]  
(39)

where

\[ C_{abm} = \frac{1}{2} e^\mu_b e^\nu_m (\partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}) \]  
(40)

The tensor \( K^a_{\mu b} \) is a contorsion tensor defined in \((35)\).

Let us consider the following action:

\[ S_1 = -\frac{1}{k^2} \int d^4x \ e \left( \tilde{R}(e, \tilde{w}) - 2\Lambda \right) \]  
(41)

Using the decomposition of the Lorentz connection \( \tilde{w}^a_{\mu b} \) into its irreducible parts \( (37) \), it is possible to rewrite the action \((41)\) in the following form:

\[ S_2 = -\frac{1}{k^2} \int d^4x \ e \left( R(e) - 2\Lambda - 4\nabla_\sigma Q^\sigma - 4Q^\nu Q_\sigma + Q^{\mu\nu} Q_{\sigma\nu} + 2Q^{\mu\nu} Q_{\nu\mu} \right) \]  
(42)

The third term in expression \((42)\) is the full derivatives

\[ \int d^4x \sqrt{g} \nabla_\mu Q^\mu = \int d^4x \partial_\mu (\sqrt{g} Q^\mu) \]  
(43)

and in space-time without boundaries we can neglect this term.

In the Poincarè gauge approach describing the Riemann-Cartan space-time there are two sets of dynamical variables: \((e^a_{\mu}, \tilde{w}^a_{\mu b})\) and \((e^a_{\mu}, Q^a_{\mu\nu})\) (we can consider the contorsion tensor \( K^a_{\mu b} \) instead of the torsion tensor \( Q^a_{\mu\nu} \)). The equations of motion of the theory described by the action \((41)\) or classical equivalent action \((42)\) are independent of the choice of the dynamical variables and have the following form:

\[ R_{\mu\nu}(e) = \Lambda g_{\mu\nu} \]  
(44)

\[ Q^a_{\mu\nu} = 0 \]  
(45)

From equation \((45)\) we see that the torsion field \( Q^a_{\mu\nu} \) is a nonpropagating auxiliary field that can be excluded from the Lagrangian by means of the equation of motion. In other words, the equation of motion of the torsion tensor \( Q^a_{\mu\nu} \) is the second-class
constraint. So the theories described by the actions (41) and (42) are equivalent to Einstein’s theory with the cosmological constant at the tree level in the absence of the matter fields. From general consideration of the theories with constrains [57] one knows that in the renormalizable theory with the second-class constraints the loop calculations can be done by the two equivalent methods:

1. One excludes auxiliary fields from the Lagrangian by means of the equations of motion (these equations are in general the second-class constraints) at the classical level and quantizes the obtained theory.

2. One considers auxiliary fields as independent dynamical variables and quantizes the theory with the existing sets of fields.

These two methods of calculations give rise to the identical results of loop calculations in the renormalizable theories. In particular, the equivalence of quantum theory in the first and second-order formalism is based on these two equivalent quantization methods.

We consider two sets of the independent dynamical variables \((e^a_{\mu}, Q^a_{\mu\nu})\) and \((e^a_{\mu}, \tilde{w}^a_{\mu})\). The corresponding actions called \(S_2\) and \(S_1\), respectively, are given in (42) and (41).

Let us consider the action \(S_2\). Using the first method of quantization we obtain that after excluding the torsion fields \(Q^a_{\mu\nu}\) by means of the equation of motion (45) the action \(S_2\) reduces to the ordinary action of the Einstein gravity with the cosmological constant in the vierbein formalism

\[
S_2 \to S_2^{\text{mod}} = -\frac{1}{k^2} \int d^4x \, e (R(e) - 2\Lambda) \tag{46}
\]

The vierbein fields have sixteen components: in addition to their ten (metric) symmetrical components they have six antisymmetrical components, expressing the freedom of homogeneous transformations of the local Lorentz frames, which introduce additional dynamical content, especially at the quantum level. The theory (46) has two kinds of gauge invariance: the usual coordinate freedom and the local Lorentz rotations. Both gauges must be fixed in the covariant quantization scheme by adding gauge-breaking terms. In the special gauge fixing the local Lorentz invariance, the contribution of the antisymmetrical vierbein components and their ghosts disappear from the quantized theory. In this gauge, the vierbein and metric formulations are equivalent at the quantum domain in the absence of the spinor fields [9]. Hence, the results of the one-loop calculations in the metric and vierbein formulations coincide. Using the results obtained in paper [54] it is possible to write the one-loop counterterms of the action (46) in the vierbein formalism.

\[
\triangle_\infty = -\frac{1}{32\pi^2\varepsilon} \int d^4x \, e \left( \frac{53}{45} R_{\mu\nu\sigma\lambda}(e)R^{\mu\nu\sigma\lambda}(e) - \frac{58}{5} \Lambda^2 \right) \tag{47}
\]

Now we consider the second method of quantization. Rewriting all the dynamical variables as a sum of classical and quantum fields

\[
e^a_{\mu} = e^a_{\mu} + k\lambda^a_{\mu} \tag{48}
\]
and expanding the action $S_2$ in powers of the quantum field up to terms quadratic in the quantum fields, we obtain the effective Lagrangian for the calculation of the one-loop counterterms

$$L_{\text{eff}} = L^{GR}_{\text{eff}} - \frac{1}{2}q^{\alpha \beta}_{\mu \nu} H_{a b}^{\mu \nu} q^{\alpha \beta}_{a b} - \frac{1}{2} \lambda^a_\mu X^\nu_{a b} \lambda^b_\nu - \lambda^b_\sigma Z^\sigma_{a b \mu \nu} q^{\alpha \beta}_{a b}$$

(50)

where $L^{GR}_{\text{eff}}$ is the effective Lagrangian of the Einstein gravity with the cosmological constant quadratic in the quantum field $\lambda^a_\mu$, $X^\nu_{a b}$ is proportional to $Q^2$, $Z^\sigma_{a b \mu \nu}$ is proportional to $Q$ and $H_{a b}^{\mu \nu} \lambda^b_\tau$ consists of the metric tensors and Kronecker’s symbols

$$H^{\mu \nu \alpha \beta}_{\sigma \rho} = \left( \delta^\beta_\alpha \delta^\nu_\beta g^{\alpha \mu} - \delta^\alpha_\sigma \delta^\alpha_\beta g^{\beta \mu} - \delta^\beta_\sigma \delta^\nu_\beta g^{\alpha \nu} + \delta^\alpha_\sigma \delta^\nu_\beta g^{\beta \nu} \right) + g_{\sigma \rho} \left( g^{\alpha \mu} g^{\beta \nu} - g^{\alpha \nu} g^{\beta \mu} \right)$$

(51)

To get the diagonal form of the effective Lagrangian, we are to replace the dynamical variables in the following way:

$$q^{\sigma \mu \nu} \rightarrow \tilde{q}^{\sigma \mu \nu} = q^{\sigma \mu \nu} - H^{-1 \sigma \rho}_{\mu \nu \alpha \beta} Z^\tau_{a b} \lambda^b_\tau$$

(52)

where $H^{-1 \sigma \rho}_{\mu \nu \alpha \beta}$ satisfies three conditions

$$H^{-1 \sigma \rho}_{\mu \nu \alpha \beta} = H^{-1 \rho \sigma}_{\alpha \beta \mu \nu}$$

(53)

$$H^{-1 \sigma \rho}_{\mu \nu \alpha \beta} = - H^{-1 \sigma \rho}_{\nu \mu \alpha \beta} = - H^{-1 \sigma \rho}_{\mu \nu \beta \alpha}$$

(54)

$$H^{-1 \sigma \rho}_{\mu \nu \alpha \beta} H^\alpha_\rho H^\beta_\omega H^\kappa_\omega = \frac{1}{2} \delta^\sigma_\omega \left( \delta^\kappa_\nu \delta^\kappa_\mu - \delta^\kappa_\mu \delta^\kappa_\nu \right)$$

(55)

It is known that $H^{-1 \sigma \rho}_{\mu \nu \alpha \beta}$ satisfying the conditions (53)-(54) exist [58]. In the extended theory of gravity additional symmetries connected with the local transformation of the connection field may be present in the theory. In this sort of a theory the expression like $H^{-1 \sigma \rho}_{\mu \nu \alpha \beta}$ does not exist [22], [23].

The replacement (52) does not change the functional measure

$$\det \left| \frac{\partial (\lambda^a_\kappa, q^{\sigma \mu \nu}_a)}{\partial (\lambda^a_\tau, q^{\sigma \mu \nu}_{a \beta})} \right| = 1$$

(56)

Since we are interested only in the results on the mass-shell, it is possible to consider the effective Lagrangian (50) and the replacement of the variables (52) only on the mass-shell (45). Taking into account the on-shell identities

$$Z^\tau_{a b} \lambda^b_\tau = X^\mu_{a b}$$

(57)

we obtain, on the mass-shell, the diagonal effective Lagrangian. The one-loop counterterms are the sum of the contributions of the quantum fields $\lambda^\alpha_\mu$ and $q^{\sigma \mu \nu}_a$. On
the mass-shell, the one-loop contribution of the torsion fields to the effective action is proportional to $\delta^4(0) \det \left( H_{\mu \nu \rho}^{\alpha \beta} \right)$. In the dimensional regularization, $[\delta^4(0)]_R = 0$ and the one-loop contribution of the torsion fields to the one-loop counterterms is equal to zero. Hence, the one-loop counterterms depend only on the contribution of the quantum fields $\lambda^a_{\mu}$ and coincide with the standard result (47). This result is the consequence of the equivalence of the two above-mentioned methods of calculation.

Now we consider the fields $e^a_{\mu}$ and $\tilde{w}^{a}_{b\mu}$ as independent dynamical variables. It has been shown [59] - [62] that after solving the second-class constraints that exist in the theory described by the action (41), it is possible to express all $\tilde{\omega}^a_{b\mu}$ as functions of $e^a_{\mu}$ and exclude $\tilde{\omega}^a_{b\mu}$ as physical degrees of freedom from the theory. Then, the Lagrangian can be written as

$$S_1 \rightarrow S_1^{\text{mod}} = -\frac{1}{k^2} \int d^4 x \ e (R(e) - 2\Lambda)$$

(58)

The results of the one-loop calculations on the mass-shell are given by the expression (47).

Now we consider the second method of calculation. In accordance with the background field method, all dynamical variables are rewritten in the following form:

$$e^a_{\mu} = e^a_{\mu} + k\lambda^a_{\mu}$$
$$\tilde{w}^{a}_{b\mu} = \tilde{w}^{a}_{b\mu} + k\gamma^a_{b\mu}$$

(59)

where $\lambda^a_{\mu}$ and $\gamma^a_{b\mu}$ are the quantum fields and $e^a_{\mu}$ and $\tilde{w}^{a}_{b\mu}$ are the classical fields satisfying the equations of motion (44) and (45).

The effective Lagrangian for the calculation of one-loop counterterms is

$$L_{\text{eff}} = -\left( \frac{1}{2\gamma} b_{\mu} F_{a \mu} b_{\nu} g_{a \nu} + \frac{1}{2\lambda} D_{b \mu} F_{a \mu} \lambda^a_{\nu} + \lambda^m_n \left( G_{m \ j} b_{c} \nabla_{j} + T_{m \ a} b_{c} \gamma_{a} \right) \right) e$$

(60)

where

$$G_{m \ j} b_{c} = \delta^m_{j} b_{a} g_{b c} - \delta^m_{j} b_{a} g_{b c} + \delta^j_{m} b_{a} g_{b c} - \delta^j_{m} b_{a} g_{b c} + \delta^c_{m} b_{a} g_{b c} - \delta^c_{m} b_{a} g_{b c}$$

(61)

$$D_{m \ c} = \left( R - 2\Lambda \right) \left( \delta^m_{c} \delta^f_{m} - \delta^f_{m} \delta^m_{c} \right) + 2R_{m \ c}^{f} + \frac{1}{2} \left( R_{m \ c}^{m} \delta^f_{m} - R_{m \ c}^{m} \delta^f_{m} + R_{m \ c}^{m} \delta^f_{m} - R_{m \ c}^{m} \delta^f_{m} \right)$$

(62)

$$T_{m \ a} = \left( 2\delta^m_{b} Q_{a} g_{b c} - 2\delta^m_{b} Q_{a} g_{b c} + \delta^m_{b} K_{a} c - g_{b c} K_{a} c + \delta^m_{b} K_{a} c + \delta^m_{b} K_{a} c - \delta^m_{b} K_{a} c \right)$$

(63)
\[ F_{\alpha \beta \mu \nu} = \frac{1}{4} \left( g^{\beta \lambda} \delta_{\mu}^\sigma - g^{\beta \lambda} \delta_{\mu}^\tau - g^{\beta \lambda} \delta_{\mu}^\sigma + g^{\beta \lambda} \delta_{\mu}^\sigma \right) \]

The corresponding variables in the following way:

\[ \gamma^a_{bc} \rightarrow \tilde{\gamma}^a_{bc} = \gamma^a_{bc} + F^{-1a}_{bc \ k mn} \left( G_{\beta \mu \ k} \nabla_s - T_{\beta \mu \ k} \right) \lambda^p \]

where \( F^{-1a}_{bc \ k mn} \) satisfies three conditions

\[ F^{-1\sigma}_{\mu \nu \ alpha \ beta} = F^{-1\rho \sigma}_{\alpha \beta \ mu \ nu} \]

\[ F^{-1\sigma}_{\mu \nu \ alpha \ beta} = -F^{-1\sigma}_{\nu \mu \ alpha \ beta} = -F^{-1\sigma}_{\mu \ beta \ alpha \ nu} \]

\[ F^{-1\sigma}_{\mu \nu \ alpha \ beta} F^\sigma_{\beta \nu \ \rho \ \omega} = \frac{1}{2} \delta^\sigma \left( \delta^\gamma_{\mu \nu} - \delta^\mu_{\beta \nu} \right) \]

The corresponding \( F^{-1\sigma}_{\mu \nu \ alpha \ beta} \), defined in paper [58], have the following form:

\[ F^{-1\alpha \beta \ \mu \ \nu \ \lambda} = \frac{1}{4} \left( g^{\alpha \mu} g_{\beta \sigma} g_{\lambda \nu} - g^{\alpha \mu} g_{\beta \nu} g_{\lambda \sigma} + g^{\alpha \mu} g_{\sigma \nu} g_{\lambda \beta} - g^{\alpha \mu} g_{\sigma \nu} g_{\lambda \beta} + g_{\sigma \lambda} \delta^\mu_{\beta \nu} - g_{\sigma \lambda} \delta^\mu_{\beta \nu} \right) \]

The replacement (65) does not influence the functional integral measure

\[ \det \frac{\partial \left( \lambda_{\tau \sigma}^{\alpha \beta \ mu \ nu} \right)}{\partial \left( \lambda^{\nu \sigma}_{\beta \nu \ \mu \ \alpha} \right)} = 1 \]

The effective Lagrangian (60) is invariant under the general coordinate transformation

\[ x^\mu \rightarrow \ 'x^\mu = x^\mu + k \xi^\mu (x) \]

\[ e^a_{\mu}(x) \rightarrow \ 'e^a_{\mu}(x) = -k \partial_\mu \xi^\nu e^a_{\nu}(x) - k \xi^\nu \partial_\nu e^a_{\mu}(x) + O(k^2) \]

\[ \tilde{w}^a_{b\mu}(x) \rightarrow \ 'w^a_{b\mu}(x) = -k \partial_\mu \xi^\nu \tilde{w}^a_{b\nu} - k \xi^\nu \partial_\nu \tilde{w}^a_{b\mu}(x) + O(k^2) \]

and under the local Lorentz rotations
\( x^\mu \rightarrow 'x^\mu = x^\mu + k\Theta^\mu_\nu(x)x^\nu \)
\( e^a_\mu(x) \rightarrow 'e^a_\mu(x) = k\Theta^a_\nu e^b_\mu(x) + O(k^2) \)
\( \tilde{w}^a_{\beta\mu}(x) \rightarrow 'w^a_{\beta\mu}(x) = k\Theta^a_\nu \tilde{w}^c_{\beta\nu}(x) - k\Theta^a_\nu \tilde{w}^c_{\nu\mu}(x) - k\partial_\mu \Theta^a_\nu + O(k^2) \) (72)

The general coordinate invariance is violated by the following gauge

\[
F_\mu = \frac{1}{2} \left( \nabla_\nu h^\nu_\mu + \nabla_\nu h^\mu_\nu - \nabla_\mu h \right)
\]
\[
L_{gh} = \frac{1}{2} F_\mu F^\mu \] (73)

The action of the coordinate ghost is

\[
L^{(\text{coor})}_{gh} = \bar{c}^\mu (g_{\mu\nu} \nabla^\alpha \nabla_\alpha + R_{\mu\nu}) c^\nu \] (74)

We fix the Lorentz invariance by means of the Landau gauge

\[
f_{ab} = h_{ab} - h_{ba} \] (75)

\[
L^{\text{Lorentz}}_{gh} = \lim_{\alpha \to 0} \frac{1}{2\alpha} f_{ab} f^{ab} = \delta(f_{ab}) \] (76)

where \( \delta(f_{ab}) \) is the delta-function.

The corresponding Lagrangian of the Lorentz ghost is

\[
L^{(\text{Lor})}_{gh} = -\bar{\omega}_{ab} \frac{\delta f^{ab}}{\delta \omega_{mn}} \omega^{mn}
= -\bar{\omega}_{ab} \left( \left( \delta^a_m \delta^b_n - \delta^a_n \delta^b_m - \delta^a_m \delta^b_n + \delta^b_m \delta^a_n \right) \omega^{mn} + \nabla_a c_b - \nabla_b c_a \right) c
\] (77)

After some irrelevant redefinitions of the ghosts fields, we may drop the term \( \bar{\omega} c \) in equation (77) as it alone is insufficient for a closed-loop diagram containing \( \omega \) and \( c \) fields.

Hence, the contribution of the Lorentz ghost to the one-loop effective action is proportional to \( \delta^4(0) \) and in the dimensional regularization is equal to zero.

Summarizing all contributions we obtain that the one-loop counterterms on the mass-shell including the contributions of the quantum and the ghost fields are

\[
\Delta_\infty = -\frac{1}{32\pi^2} \int d^4x \ e \left( \frac{19}{360} R_{\mu\nu\sigma\lambda}(e) R^{\mu\nu\sigma\lambda}(e) - \frac{89}{15} \Lambda^2 \right)
\] (78)

This result does not coincide with the previous one [47].
5 Discussion

Now we discuss the results of the previous chapters. Let us consider the example 1. At first, one needs to verify that the changes of the variables (15) and (16) satisfy the condition (1) of the equivalence theorem.

Expressions (15) and (16) can be written in the following form:

\[
\begin{align*}
\phi &= \psi + O(\psi^2) \\
g_{\mu\nu} &= G_{\mu\nu}(1 + k^2\psi^2 + O(\psi^4))
\end{align*}
\]

These changes of the variables satisfy the condition (1). Hence, the equivalence theorem must be fulfilled. Then, the question arises: what does the result (21) mean? We can suggest that such situation when one field is quantized and not others (the theory in the external field) is not consistent with the equivalence theorem in non-renormalizable theories. Only when both fields quantizing the equivalence theorem must fulfil. Then, in accordance with M.J.Duff [55], the result (21) can be considered as a consequence of the inconsistency of quantum field theory in an external gravitational field. Indeed, the actions (14) and (15) describe the same classical theory written in a different way. Starting from the same classical theory written in a different way, we obtain inequivalent quantum results on the mass-shell (21). There is no obvious principle which singles out one particular choice of the classical action. An arbitrary classical theory can be written in many different ways. In the semi-classical approach we cannot select one choice of variables as ”correct” and reject all the others. Then the semi-classical approach is inconsistent because it yields ambiguous results and one has no criterion for deciding which is correct. In this way, inconsistency of quantum field theory in the external gravitational field is the consequence of the affirmation that the results of the loop calculations on the mass-shell have some physical significance. However, if we suggest that the results of the loop calculations within the background field method on the mass-shell in the nonrenormalizable theories are physically meaningless, then the result (21) has a simple explanation. Both the actions (14) and (15) are equivalent at the classical and quantum domains. But since the one-loop counterterms on the mass-shell do not have physical significance and do not give information about the S-matrix of the theory, the demanding that \(\Delta_{1\infty}\) must be equal to \(\Delta_{2\infty}\) is an additional, nonphysical request. Both the results (18) and (19) are true and both results are physically meaningless.

Now we discuss example 2. Since the results of the loop calculations on the mass-shell depend on the gauge it is possible to choose such a gauge that the theory described by the action (22) will be finite at the one-loop level. Hence, the affirmation that the Einstein gravity interacting with the matter fields (in particular, scalar field) is a nonrenormalizable (no-finite) theory at the one-loop level is wrong. This result can be explained by the assumption that the results of the loop calculations on the mass-shell do not have physical significance and do not give information about the S-matrix of the theory. The problems connected with the use of the gauge (23) and (24) have been discussed also in paper [63]. There is the other explanation of the result of example 2
The gauge (24) mixes the one-loop and the two-loop order of perturbation theory. In order to obtain the gauge independent result on the mass-shell in the gauge (24) one needs to take into account the two-loop counterterms.

Recently the one-loop counterterms for Einstein gravity within the class of gauge suggested in [56] have been calculated in paper [65]. The resultant form for divergent part of one-loop counterterms on the mass-shell does not coincide with the result of paper [56] and does not depend on gauge.

Let me discuss the results of the chapter 4. Unlike example 1 we quantize both the fields existing in the theory, and the conditions of the equivalence theorem are fulfilled. On the mass-shell the term \( \int d^4x \, e R_{\mu\nu\sigma\lambda}^2 \) can be rewritten as \( \int d^4x \, e \left( R_{\mu\nu\sigma\lambda}^2 - 4 R_{\mu\nu}^2 + R^2 \right) \). This expression is topologically invariant, so-called Euler number, defined by

\[
\chi = \frac{1}{32\pi^2} \int d^4x \, e \left( R_{\mu\nu\sigma\lambda}^2 - 4 R_{\mu\nu}^2 + R^2 \right)
\]

Then the results (47) and (78) can be written in the following form

\[
\Delta_1 = -\frac{1}{\varepsilon} \left( \frac{53}{45} \chi + \frac{29k^2\Lambda S}{160\pi^2} \right)
\]

where \( S \) is the classical action on the mass-shell.

In the topological trivial space-time (\( \chi = 0 \)), the considered theory described by the action (11) or (12) is renormalizable on the mass-shell. Two different sets of dynamical variables give rise to different renormalization group functions

\[
\mu^2 \frac{\partial \lambda}{\partial \mu^2} = -\frac{29}{160} \lambda^2
\]

\[
\mu^2 \frac{\partial \lambda}{\partial \mu^2} = -\frac{89}{960} \lambda^2
\]

where \( \lambda = k^2\Lambda \) is the dimensionless constant and equations (83) and (84) are connected with the results (81) and (82), respectively.

It has been argued in paper [54] that in the topological non-trivial space-time (\( \chi \neq 0 \)) to obtain the one-loop renormalizable theory the term \( \int d^4x \, e \left( R_{\mu\nu\sigma\lambda}^2 - 4 R_{\mu\nu}^2 + R^2 \right) \) must be added to the classical action with the coefficient \( \alpha \). Since \( \chi \) is topologically invariant, this can be done without damage to the field equation and one-loop counterterms in the space-time without boundaries. Then, the one-loop counterterms can be absorbed into a renormalization of the new topological coupling constant \( \alpha \) and cosmological constant \( \Lambda \). The renormalization group equations describing the behaviour of the topological constant are the following:

\[
\mu^2 \frac{\partial \alpha}{\partial \mu^2} = -\frac{106}{45}
\]
\[ \mu^2 \frac{\partial \alpha}{\partial \mu^2} = -\frac{19}{180} \]  
\hspace{1cm} (86)

where equations (85) and (86) are connected with the results (81) and (82), respectively.

Two sets of the dynamical variables \((e^a_{\mu}, Q^a_{\mu\nu})\) and \((e^a_{\mu}, \tilde{\omega}^a_{b\mu})\) must be equivalent at the classical and quantum level because the transformation from one to another set of the variables does not change the functional integral measure

\[
\det \left| \frac{\partial (e^a_{\mu}, \tilde{\omega}^a_{b\mu})}{\partial (e^m_{\sigma}, Q^\lambda_{\alpha\beta})} \right| = 1 \]  
\hspace{1cm} (87)

Instead of the dynamical variables \((e^a_{\mu}, Q^a_{\mu\nu})\) or \((e^a_{\mu}, \tilde{\omega}^a_{b\mu})\) we can consider the other two sets \((g_{\mu\nu}, Q^a_{\mu\nu})\) or \((g_{\mu\nu}, \tilde{\Gamma}^a_{\mu\nu})\) where the connection \(\tilde{\Gamma}^a_{\mu\nu}\) satisfies the metric condition (28). The results of the one-loop calculation within these variables will coincide with (47) and (78) respectively.

The one-loop counterterms in the first-order gravity with the Gilbert-Einstein action without the cosmological constant have been calculated in the paper [66]. However in our opinion, in this paper there is inconsistency between the equations of motion and the choice of the dynamical variables. In the affine-metric theory with the metric and connection fields as independent dynamical variables the tensor of connection defect \((A^a_{\mu\nu})\) in the notations of paper [66] is not equal to zero (see [22], [23]).

The classical Lagrangian (46) written in the two different classically equivalent ways give rise to different quantum results. We can introduce in the theory, described by the action (41) or (42), the matter fields interacting only with the vierbein (or metric) field. In this theory, the torsion fields will be auxiliary, nonpropagating fields. However, the results of the loop calculations will also depend on the choice of dynamical variables.

### 6 Conclusion

The main aim of this paper was to show that the loop calculations in the nonrenormalizable quantum gravity are ambiguous ones. It has been investigated in the previous chapters that the classical theory written in a different way leads to the inequivalent quantum results depending on the choice of dynamical variables, gauge fixing term and the choice of parametrization. The modern point of view is that the physical observation quantities must be independent of the choice of such non-physical parameters as gauge and parametrization. As a consequence, we obtain that the results of the loop calculations depending on the nonphysical parameters must be physically meaningless. Due to violation of the equivalence theorem and DeWitt-Kallosh theorem in the nonrenormalizable theories of the quantum gravity, the question arises about the criteria of a sensible theory of the quantum gravity. It is possible that an arbitrary theory nonrenormalizable by power counting like the Einstein gravity can be finite order by order in perturbation theory by the choice of the non-standard gauge fixing term.

The other question is what will be with the renormalizable theory of the quantum gravity with higher derivatives? The calculations like example 2 with the non-standard gauge do not take the place. All one-loop counterterms in this theory have been
calculated only in the Landau-DeWitt gauge [16]. The validity of the DeWitt-Kallosh theorem in the theory of the quantum gravity with higher derivatives must be verified by the calculations of the one-loop counterterms in the arbitrary parametrization and by means of the gauge distinct from the Landau-DeWitt gauge [63]. However, the example 1 and chapter 4 take the place also in the renormalizable theory of the gravity. It has been shown [17], [68] that the necessity to introduce the term $\xi \phi^2 R$ is demanded by the renormalizability of the gravity interacting with the scalar fields. Hence, in the renormalizable theory of gravity the ambiguity of the loop calculations will also be present.

What is the reason for these strange results? All methods and theorems of quantum field theory are based on some principles like renormalizability and unitary. All existing theories of gravity (based on the Riemannian and non-Riemannian space-time structure with and without supersymmetry) do not satisfy these principles. There are unitary, but nonrenormalizable theories (like the Einstein gravity) or renormalizable, but non-unitary theories (like the theory with higher derivatives) or nonrenormalizable and non-unitary theories. Hence, all existing theories of quantum gravity are unsatisfactory from the point of view of quantum field theory. We suggest that in an arbitrary existing theory of the quantum gravity all results of the loop calculations do not have physical significance. In our opinion, to construct a sensible theory of the quantum gravity, one needs to use non-standard methods of calculation, for example, non-perturbative methods of calculation of the quantum corrections.

The only way to avoid this ambiguity is to suggest that loop counterterms on the mass-shell in the nonrenormalizable theories are physically meaningless. Then, the results (17) and (78) have the same physical ground.

To summarize, the only way to explain the results (21), (25) and (78) is to take that in the nonrenormalizable theories of the gravity the results of the loop calculations on and off mass-shell do not have physical significance. As a consequence, all physical predictions and calculations performed on the basis of the loop calculations in the nonrenormalizable theories of the quantum gravity are meaningless.

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