Diffeomorphism groups of convex polytopes

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Abstract
Let $M$ be a convex polytope in $\mathbb{R}^n$, with non-empty interior. We turn the group $\text{Diff}(M)$ of all $C^\infty$-diffeomorphisms of $M$ into a regular Lie group.

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1 Introduction and statement of results

Let $E$ be a finite-dimensional real vector space and $M \subseteq E$ be the convex hull of a finite subset of $E$, with non-empty interior $M^\circ$. For each $x \in M$, there exists a smallest face $M(x)$ of $M$ such that $x \in M(x)$; we set $E(x) := \text{span}_\mathbb{R}(M(x) - x) \subseteq E$.

For $k \in \mathbb{N}_0 \cup \{\infty\}$, endow the space $C^k(M,E)$ of all $C^k$-maps $f : M \to E$ with the compact-open $C^k$-topology (as in [15, Section 1.7]); then $C^k_\text{str}(M,E) := \{ f \in C^k(M,E) : (\forall x \in M) \ f(x) \in E(x) \}$ is a closed vector subspace (on which the induced topology will be used). We regard the functions $f \in C^k(M,E)$ as $C^k$-vector fields on $M$; the elements $f \in C^k_\text{str}(M,E)$ will be called stratified $C^k$-vector fields (cf. [9]). Let

$$\text{Diff}^k(M) := \{ \phi \in C^k(M,M) : (\exists \psi \in C^k(M,M)) : \phi \circ \psi = \psi \circ \phi = \text{id}_M \}$$

be the set of all $C^k$-diffeomorphisms $\phi : M \to M$, and abbreviate $\text{Diff}(M) := \text{Diff}^\infty(M)$. Then $\text{Diff}^k(M)$ is a group, using composition of diffeomorphisms as the group multiplication. The neutral element is the identity map $\text{id}_M$. Our main goal is the following result:

**Theorem 1.1** The group $\text{Diff}(M)$ admits a smooth manifold structure modeled on $C^\infty_\text{str}(M,E)$ making it a Lie group.

More details concerning the Lie group structure on $\text{Diff}(M)$ are now described. If $P \subseteq E$ is a convex polytope, we call a homeomorphism $\phi : P \to P$ face respecting if $\phi(F) = F$ for each face $F$ of $P$. Then also $\phi^{-1}$ is face respecting. For each $k \in \mathbb{N} \cup \{\infty\}$, the set

$$\text{Diff}^k_\text{fr}(M) := \{ \phi \in \text{Diff}^k(M) : \phi \text{ is face respecting} \}$$

is a normal subgroup of finite index in $\text{Diff}^k(M)$ (see Lemma 5.5) and

$$\Omega_k := \text{Diff}^k_\text{fr}(M) - \text{id}_M$$
is an open 0-neighbourhood in $C^k_{\text{str}}(M,E)$ (see Lemma 4.1). We give $\text{Diff}^k_{fr}(M)$ the smooth manifold structure modeled on $C^k_{\text{str}}(M,E)$ making the bijective map

$$\Phi_k: \text{Diff}^k_{fr}(M) \to \Omega_k, \quad \phi \mapsto \phi - \text{id}_M$$

a $C^\infty$-diffeomorphism. Then the following holds, using $C^{k,\ell}$-maps as in [2]:

**Proposition 1.2** For all $k \in \mathbb{N} \cup \{\infty\}$ and $\ell \in \mathbb{N}_0 \cup \{\infty\}$, the map

$$c_{k,\ell}: \text{Diff}^{k+\ell}_{fr}(M) \times \text{Diff}^k_{fr}(M) \to \text{Diff}^k_{fr}(M), \quad (\phi, \psi) \mapsto \phi \circ \psi$$

is $C^{\infty,\ell}$ (and thus $C^\ell$); moreover, the map

$$\iota_{k,\ell}: \text{Diff}^{k+\ell}_{fr}(M) \to \text{Diff}^k_{fr}(M), \quad \phi \mapsto \phi^{-1}$$

is $C^\ell$. Notably, $\text{Diff}^\infty_{fr}(M)$ is a smooth Lie group modeled on $C^\infty_{\text{str}}(M,E)$.

To establish Theorem 1.1 we shall give Diff$(M)$ a smooth Lie group structure modeled on $C^\infty_{\text{str}}(M,E)$ which turns $\text{Diff}^\infty_{fr}(M)$ into an open subgroup.

**Remark 1.3** The differentiability properties of the mappings $c_{k,\ell}$ and $\iota_{k,\ell}$ established in Proposition 1.2 show that $\text{Diff}^\infty_{fr}(M)$ fits into the framework of [17]. Thus $\text{Diff}^\infty_{fr}(M)$ (and hence also Diff$(M)$) is an $L^1$-regular Lie group in the sense of [13], by [16]. Notably, the Fréchet-Lie group Diff$(M)$ is $C^0$-regular (as in [12]) and hence regular in the sense of [20].

**Remark 1.4** The easiest – but most relevant – case of our construction is the case of a cube $M := [0,1]^n \subseteq \mathbb{R}^n$. The corresponding diffeomorphism group is of interest in numerical mathematics [6]. We mention that $[0,1]^n$ is a prime example of a smooth manifold with corners (as in [7, 8, 19]; cf. [18] for infinite-dimensional generalizations). Michor [19] discussed the diffeomorphism group of a paracompact, smooth manifold $M$ with corners, but his arguments contain a serious flaw: Contrary to claims in the book, local additions in Michor’s sense never exist if $\partial M \neq \emptyset$ (not even for $M = [0,\infty[ \text{ or } M = [0,1]$), as we show in an appendix. Yet, Michor’s conclusion is correct that Diff$(M)$ is a Lie group. More generally, Diff$(M)$ can be made a Lie group for each paracompact, locally polyhedral $C^\infty$-manifold $M$ (as in Remark 5.8); details will be given elsewhere.

As before, let $E$ be a finite-dimensional real vector space. If $M \subseteq E$ is any compact convex subset with non-empty interior, then a Lie group structure can be constructed on the group $\text{Diff}_{\partial M}(M)$ of all $C^\infty$-diffeomorphisms $\phi: M \to M$ such that $\phi(x) = x$ for all $x \in \partial M$ (see [14]). In the case that $M$ is a polytope, our discussion of the Lie group structure on $\text{Diff}^\infty_{fr}(M)$ proceeds along similar lines. Then $\text{Diff}_{\partial M}(M)$ is a closed normal subgroup and smooth submanifold of both $\text{Diff}^\infty_{fr}(M)$ and Diff$(M)$ (see Remark 5.9). For numerical applications, it is useful to know a lower bound for the 0-neighbourhoods $\Omega_k \subseteq C^k_{\text{str}}(M,E)$. Fixing any norm $\| \cdot \|$ on $E$ to calculate operator norms, we show:
Proposition 1.5 Let $M \subseteq E$ be a convex polytope with non-empty interior and $k \in \mathbb{N} \cup \{\infty\}$. Let $U_k$ be the set of all $f \in C^k_{\text{str}}(M,E)$ such that
\[ \|f\|_{\infty,\text{op}} := \sup_{x \in M} \|f'(x)\|_{\text{op}} < 1. \]
Then $U_k$ is an open $0$-neighbourhood in $C^k_{\text{str}}(M,E)$ and $U_k \subseteq \Omega_k$.

For $E = \mathbb{R}^n$ and $M = [0,1]^n$, see already [6]. An analogous result for $\text{Diff}_{\partial M}(M)$ was also established in [6], for compact convex subsets $M \subseteq \mathbb{R}^n$ with non-empty interior.

Remark 1.6 Note that $\|f\|_{\infty,\text{op}}$ is the smallest Lipschitz constant $\text{Lip}(f)$ for $f$ in the situation of Proposition 1.5 (see, e.g., [15, Lemma 1.5.3 (c)]).

2 Preliminaries and basic facts

We write $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The term “locally convex space” means a locally convex, Hausdorff topological vector space over the ground field $\mathbb{R}$. If $(E,\|\cdot\|)$ is a normed space, let $B_E^r(x) := \{y \in E : \|y - x\| < r\}$ and $\overline{B}_E^r(x) := \{y \in E : \|y - x\| \leq r\}$ for $x \in E$ and $r > 0$. For background concerning convex polytopes and their faces, the reader is referred to [5] and [22]. If $E$ is a real vector space and $S \subseteq E$ a subset, we write $\text{aff}(S)$ for the affine subspace of $E$ generated by $S$. In this article, we are working in the setting of infinite-dimensional calculus known as Keller’s $C^k_{\text{c}}$-theory (going back to [3]). Accordingly, the smooth manifolds and Lie groups we consider are modeled on locally convex spaces which need not have finite dimension; see [10], [15], and [21] for further information.

Let $E$ and $F$ be locally convex spaces. We recall from [15, Chapter 1]:

2.1 A map $f : U \to E$ on an open subset $U \subseteq E$ is called $C^1$ if $f$ is continuous, the directional derivative
\[ df(x,y) := (D_yf)(x) := \lim_{t \to 0} \frac{1}{t} (f(x + ty) - f(x)) \]
extists in $F$ for all $(x,y) \in U \times E$, and the map $df : U \times E \to F$ is continuous. Then $f'(x) := df(x,\cdot) : E \to F$ is linear for each $x \in U$.

2.2 Let $U \subseteq E$ be a subset which is locally convex (in the sense that each $x \in U$ has a convex neighbourhood in $U$) and whose interior $U^0$ is dense in $U$. A map $f : U \to F$ is called $C^0$ if $f$ is continuous. If $f$ is continuous, $f|_{U^0}$ is $C^1$ and $d(f|_{U^0}) : U^0 \times E \to F$ has a continuous extension
\[ df : U \times E \to F, \]
then $f$ is called $C^1$. Given $k \in \mathbb{N}$, we say that $f$ is $C^k$ if $f$ is $C^1$ and $df$ is $C^{k-1}$. If $f$ is $C^k$ for all $k \in \mathbb{N}_0$, then $f$ is called a $C^\infty$-map or also smooth.
If $E$ and $F$ are locally convex spaces, we write $L(E,F)_b$ for the locally convex space of all continuous linear maps $\alpha : E \to F$, endowed with the topology of uniform convergence on bounded subsets of $E$. For finite-dimensional domains, the $C^k$-property can be reformulated in the expected way:

**Lemma 2.3** Let $E$ be a finite-dimensional real vector space, $U \subseteq E$ be a locally convex subset with dense interior, $F$ be a locally convex space, and $k \in \mathbb{N}$. Then the following conditions are equivalent for a continuous map $f : U \to F$:

(a) $f$ is $C^k$;

(b) $f$ is $C^1$ and $f' : U \to L(E,F)_b$, $x \mapsto f'(x) = df(x,\cdot)$ is $C^{k-1}$.

**Proof.** To see that (a) implies (b), we may assume that $E \neq \{0\}$, excluding a trivial case. Let $b_1,\ldots,b_n$ be a basis for $E$. Then

$$
\psi : L(E,F)_b \to F^n, \quad \alpha \mapsto (\alpha(b_1),\ldots,\alpha(b_n))
$$

is an isomorphism of topological vector spaces. If $f$ is $C^k$, then $f$ is $C^1$ and $\psi \circ f' = df(\cdot,b_1),\ldots,df(\cdot,b_n)$ is $C^{k-1}$ (by [15, Lemma 1.4.6 and Proposition 1.4.10]), entailing that $f' = \psi^{-1} \circ (\psi \circ f')$ is $C^{k-1}$.

Conversely, assume that (b) holds. The projections $pr_1 : U \times E \to U$, $(x,y) \mapsto x$ and $pr_2 : U \times E \to E$, $(x,y) \mapsto y$ are smooth, being restrictions of continuous linear maps $E \times E \to E$. Since $E$ is finite dimensional and thus normable, the evaluation map $\varepsilon : L(E,F)_b \times E \to F$, $(\alpha,y) \mapsto \alpha(y)$ is continuous and thus smooth, being bilinear. Hence $df = \varepsilon \circ (f' \circ pr_1,pr_2)$ is $C^{k-1}$, using [15, Lemma 1.4.6 and Proposition 1.4.10].

If $f : X \to Y$ is a function between metric spaces $(X,d_X)$ and $(Y,d_Y)$, we define

$$
\text{Lip}(f) := \sup \left\{ \frac{d_Y(f(x),f(y))}{d_X(x,y)} : x,y \in X \text{ with } x \neq y \right\} \in [0,\infty].
$$

Thus $f$ is Lipschitz continuous if and only if $\text{Lip}(f) < \infty$, and $\text{Lip}(f)$ is the smallest Lipschitz constant for $f$ in this case. We shall use a version of the Lipschitz Inverse Function Theorem:

**Lemma 2.4** Let $(E,\|\cdot\|)$ be a normed space, $M \subseteq E$ a subset and $f : M \to E$ a Lipschitz continuous mapping such that $\text{Lip}(f) < 1$. Then $\phi := \text{id}_M+f : M \to E$ is injective and $\phi^{-1} : \phi(M) \to M$ is Lipschitz continuous with

$$
\text{Lip}(\phi^{-1}) \leq \frac{1}{1-\text{Lip}(f)}.
$$

If, moreover, $(E,\|\cdot\|)$ is a Banach space and $M$ is open in $E$, then $\phi(M)$ is open in $E$.
Proof. For all \( x, y \in M \),
\[
\| \phi(x) - \phi(y) \| \geq \| x - y \| - \| f(x) - f(y) \| \geq (1 - \text{Lip}(f)) \| x - y \|.
\]
Thus \( \phi(x) = \phi(y) \) entails \( x = y \), and also \(^{11}\) follows. If \((E, \| \cdot \|)\) is a Banach space and \( M \) is open in \( E \), then \( \phi \) is an open map, as a consequence of \(^{11}\) Theorem 5.3]; notably, \( \phi(M) \) is open. □

2.5 Let \( E_1, E_2, \) and \( F \) be locally convex spaces, \( U_1 \subseteq E_1 \) as well as \( U_2 \subseteq E_2 \) be locally convex subsets with dense interior, and \( r, s \in \mathbb{N}_0 \cup \{\infty\} \). Following \(^{2}\), we say that a map \( f : U_1 \times U_2 \to F \) is \( C^{r,s} \) if \( f \) is continuous and, for all \( i, j \in \mathbb{N}_0 \) with \( i \leq r \) and \( j \leq s \), there exists a continuous map
\[
d^{(i,j)}f : U_1 \times U_2 \times (E_1)^{i} \times (E_2)^{j} \to F
\]
such that the iterated directional derivatives
\[
(D_{(v_1,0)} \cdots D_{(v_i,0)} D_{(0,w_j)} \cdots D_{(0,w_1)})f)(x,y)
\]
exist for all \( x \in U_1^0, y \in U_2^0, v_1, \ldots, v_i \in E_1, w_1, \ldots, w_j \in E_2 \), and coincide with \( d^{(i,j)}f(x,y,v_1,\ldots,v_i,w_1,\ldots,w_j) \), using the interiors \( U_1^0 \) and \( U_2^0 \) (see also \(^{1}\)).

We mention that \( C^{r,s,t} \)-maps \( f : U_1 \times U_2 \times U_3 \to F \) can be defined analogously \(^{1}\).

3 Proof of Proposition \(^{15}\)

By definition of the compact-open \( C^k \)-topology, the map
\[
C^k_{\text{str}}(M,E) \to C(M \times E,E), \quad f \mapsto df
\]
is continuous if we endow \( C(M \times E,E) \) with the compact-open topology. Hence
\[
U_k = \{ f \in C^k_{\text{str}}(M,E) : df(M \times B^k_E(0)) \subseteq B^k_E(0) \}
\]
is open in \( C^k_{\text{str}}(M,E) \). Moreover, \( 0 \in U_k \). We prove the remaining assertion of Proposition \(^{15}\) by induction on the dimension of \( E \). The case \( \dim(E) = 0 \) is trivial, since \( f = 0 \) for all \( f \in C^k_{\text{str}}(M,E) \) in this case and thus \( \text{id}_M + f = \text{id}_M \in \text{Diff}^k(M) \). Let \( \dim(E) > 0 \) now and assume that the assertion has been established for all finite-dimensional vector spaces of dimension \( < \dim(E) \), in place of \( E \). Given \( f \in U_k \), our goal is to show that \( \phi := \text{id}_M + f \) is a face-respecting \( C^k \)-diffeomorphism of \( M \). As a first step, we show that \( \phi(F) = F \) for each face \( F \neq M \) of \( M \). Pick an element \( x \) in the algebraic interior \( \text{algint}(F) \) of \( F \) (called the “relative interior” in \(^{51} 22\)). Then \( F = M(x) \); we endow
\[
E(x) = \text{span}_R(F - x)
\]
with the norm induced by \( \| \cdot \| \). For each \( z \in \text{algint}(F) \), we have \( f(z) \in E(z) = \text{aff}(F) - z = \text{aff}(F) - x = E(x) \). Then \( f(z) \in E(x) \) for all \( z \in F \), as \( f \) is
compact and hence closed in $\varphi$. By Lemma 2.4, $\varphi$ is connected and $\varphi(M)$ is closed in $x$.

Since $x$ are continuous, after replacing $a \lambda$ by a linear functional $g$ holds, showing that $\lambda(M) \subseteq M$. If this was false, we could find $x \in M$ such that $\lambda(x) \notin M$. Now $M$ being an intersection of half-spaces, we would find a linear functional $\lambda: E \to \mathbb{R}$ and $a \in \mathbb{R}$ such that

$$\lambda(M) \subseteq [-\infty, a] \quad \text{and} \quad \lambda(\phi(x)) > a.$$ 

Then $\lambda \neq 0$, entailing that $\lambda$ is an open map. Since $M$ is compact and $\lambda \circ \phi$ is continuous, after replacing $x$ if necessary we may assume that

$$\lambda(\phi(x)) = \max \lambda(\phi(M)).$$

Since $\phi(F) = F \subseteq M$ for each proper face $F$ of $M$ by the first step, we must have $x \in M^0$. Since $\phi(M^0)$ is open in $E$ by Lemma 2.4, $\lambda(\phi(M^0))$ is an open neighbourhood of $\lambda(\phi(x))$ in $\mathbb{R}$, contradicting the maximality of $\lambda(\phi(x))$. Hence $x$ cannot exist and $\phi(M) \subseteq M$ must hold.

As the third step, we show that $\phi(M) = M$. Since $\partial M$ is the union of all proper faces $F$ of $M$ and $\phi(F) = F$ by Step 1, we have $\phi(\partial M) = \partial M$. Now $M$ being closed in $E$, we have $M = M^0 \cup \partial M$ with $M^0 \cap \partial M = \emptyset$. We already observed that $\phi(M^0)$ is open in $E$, by Lemma 2.4, thus $\phi(M^0) \subseteq M^0$. Since $\phi(M)$ is compact and hence closed in $M$, the intersection

$$\phi(M) \cap M^0 = (\phi(M^0) \cup \phi(\partial M)) \cap M^0 = \phi(M^0)$$

is closed in $M^0$. But $\phi(M^0)$ is also open in $E$, and hence open in $M^0$. Since $M^0$ is connected and $\phi(M^0) \neq \emptyset$, we deduce that $M^0 = \phi(M^0)$. As a consequence, $\phi(M) = \phi(M^0 \cup \partial M) = M^0 \cup \partial M = M$.

By Lemma 2.4, $\phi: M \to M$ is a homeomorphism. By Steps 1 and 3, $\phi(F) = F$. 

6
for each face $F$ of $M$, whence $\phi$ is face respecting.

The inversion map $j: \text{GL}(E) \to \text{GL}(E)$, $\alpha \mapsto \alpha^{-1}$ is smooth on the general linear group $\text{GL}(E) := L(E, E)^\times$. For each $x \in M^0$, we have $\phi'(x) - \text{id}_E = f'(x)$ with $\|f'(x)\|_{\text{op}} < 1$, whence $\phi'(x): E \to E$ is invertible (by means of Neumann’s series). Thus $\phi|_{M^0}$ is a local $C^k$-diffeomorphism at $x$, by the Inverse Function Theorem. As a consequence, the bijection $\phi|_{M^0}: M^0 \to M^0$ is a $C^k$-diffeomorphism. Now $\phi^{-1}: M \to E$ is a continuous map and $\phi^{-1}|_{M^0} = (\phi|_{M^0})^{-1}$ is $C^1$ with

$$(\phi^{-1}|_{M^0})' = ((\phi|_{M^0})^{-1})' = j \circ (\phi|_{M^0})' \circ (\phi|_{M^0})^{-1}.$$  

By the preceding, the map

$$g := j \circ \phi \circ \phi^{-1}: M \to L(E, E)_b$$

is a continuous extension of $(\phi^{-1}|_{M^0})': M^0 \to L(E, E)_b$. Then $g^\wedge: U \times E \to E$, $(x, y) \mapsto g(x)(y) = \varepsilon(g(x), y)$ is a continuous extension of $d(\phi^{-1}|_{M^0})$, using that the evaluation map $\varepsilon: L(E, E)_b \times E \to E$ is continuous. Hence $\phi^{-1}$ is $C^1$ with $d(\phi^{-1}) = g$ and thus

$$(\phi^{-1})' = j \circ \phi' \circ \phi^{-1}. \quad (2)$$

By induction on $\ell \in \mathbb{N}$ with $\ell \leq k$, we may assume that $\phi^{-1}$ is $C^{\ell-1}$. Then $(\phi^{-1})'$ is $C^{\ell-1}$, by (2), whence $\phi^{-1}$ is $C^{\ell}$, by Lemma 2.3 Thus $\phi^{-1}$ is $C^k$, entailing that $\phi \in \text{Diff}^k_{fr}(M)$ and hence $f = \phi - \text{id}_M \in \Omega_k$. Thus $U_k \subseteq \Omega_k$.

4 Proof of Proposition 1.2

We first show that $\Omega_k$ is open.

**Lemma 4.1** For each $k \in \mathbb{N} \cup \{\infty\}$, the set $\Omega_k := \{\phi - \text{id}_M : \phi \in \text{Diff}^k_{str}(M, E)\}$ is an open $0$-neighbourhood in $C^k_{str}(M, E)$.

**Proof.** If $\phi \in \text{Diff}^k_{fr}(M)$, then $\phi - \text{id}_M: M \to E$ is a $C^k$-map. For each $x \in M$, we have $\phi(x) - x \in \phi(M(x)) - x = M(x) - x \subseteq E(x)$, showing that $\phi - \text{id}_M$ is a stratified vector field and thus $\phi - \text{id}_M \in C^k_{str}(M, E)$. By Proposition [Theorem 1.5] $\Omega_k$ is a $0$-neighbourhood in $C^k_{str}(M, E)$. It remains to show that $\Omega_k$ is open. Let $V := C^k(M, E)$. For each $g \in \Omega_k$, the map

$$R_g: C^k(M, E) \to C^k(M, E), \quad f \mapsto f \circ (\text{id}_M + g)$$

is continuous linear, by [Theorem 1.7.11]. Now $h := (\text{id}_M + g)^{-1} - \text{id}_M \in \Omega_k$. Since $\text{id}_M + h = (\text{id}_M + g)^{-1}$, we see that $R_h \circ R_g = R_g \circ R_h = \text{id}_V$. Thus $R_g$ is an isomorphism of topological vector spaces, with $(R_g)^{-1} = R_h$. Also

$$\tau: C^k(M, E) \to C^k(M, E), \quad f \mapsto \text{id}_M + f$$

is a homeomorphism. We deduce that

$$r_g := \tau^{-1} \circ R_g \circ \tau: C^k(M, E) \to C^k(M, E), \quad f \mapsto g + f \circ (\text{id}_M + g)$$

7
is a homeomorphism with \( r_g^{-1} = \tau^{-1} \circ R_h \circ \tau = r_h \). Using that \( \phi := \text{id}_M + g \in \text{Diff}^r_\mathcal{P}(M) \) is face respecting, we now show that

\[
\hat{r}_g(f) \in C^k_{\text{str}}(M, E) \quad \text{for each} \quad f \in C^k(M, E). \tag{3}
\]

To this end, let \( f \in C^k_{\text{str}}(M, E) \). For \( x \in M \), the image \( \phi(M(x)) = M(x) \) is a face containing the element \( \phi(x) \), whence \( M(\phi(x)) \subseteq M(x) \). Replacing \( x \) with \( \phi(x) \) and \( \phi \) with its inverse, the same argument shows that \( M(x) = M(\phi^{-1}(\phi(x))) \subseteq M(\phi(x)) \). Thus \( \phi(M(x)) = M(\phi(x)) \). As a consequence, \( \text{aff} M(x) = \text{aff} M(\phi(x)) \) and hence

\[
E(x) = (\text{aff } M(x)) - x = (\text{aff } M(\phi(x))) - \phi(x) = E(\phi(x)).
\]

Now \( r_g(f)(x) = g(x) + f(x + g(x)) = g(x) + f(\phi(x)) \in E(x) + E(\phi(x)) = E(x) \), establishing \( \text{Proposition 1.2} \). By \( \text{Proposition 1.2} \), the map \( r_g \) restricts to a self-map

\[
\rho_g : C^k_{\text{str}}(M, E) \to C^k_{\text{str}}(M, E)
\]

which is continuous as we endowed \( W := C^k_{\text{str}}(M, E) \) with the topology induced by \( C^k(M, E) \). As \( \rho_g \circ \rho_h = \rho_h \circ \rho_g = \text{id}_W \), we deduce that \( \rho_g \) is a homeomorphism with \( (\rho_g)^{-1} = \rho_h \). We claim that

\[
\rho_g(\Omega_k) \subseteq \Omega_k.
\]

If this is true, then \( \Omega_k \) is a \( g \)-neighbourhood in \( C^k_{\text{str}}(M, E) \) (which completes the proof), as \( \Omega_k \) is a 0-neighbourhood, \( \rho_g \) a homeomorphism, and \( \rho_g(0) = g \).

To establish the claim, let \( f \in \Omega_k \). Then \( \text{id}_M + f \in \text{Diff}^r_\mathcal{P}(M) \) and hence \( (\text{id}_M + f) \circ (\text{id}_M + g) \in \text{Diff}^r_\mathcal{P}(M) \), the latter being closed under composition. Thus \( \rho_g(f) = g + f \circ (\text{id}_M + g) = (\text{id}_M + f) \circ (\text{id}_M + g) - \text{id}_M \in \Omega_k \). □

We write \( \text{Diff}^r(K) := \{ \phi \in C^r(K, K) : (\exists \psi \in C^r(K, K)) \phi \circ \psi = \psi \circ \phi = \text{id}_K \} \) in the next lemma.

**Lemma 4.2** Let \( F \) be a locally convex space, \( U \subseteq F \) be a locally convex subset with dense interior, \( E \) be a finite-dimensional real vector space, \( K \subseteq E \) be a compact convex subset with non-empty interior and \( r \in \mathbb{N}_0 \cup \{ \infty \} \). If a map \( f : U \times K \to K \subseteq E \) is \( C^r \) and \( f_z := f(x, \cdot) \in \text{Diff}^r(K) \) for all \( x \in U \), then also the following map is \( C^r \):

\[
g : U \times K \to K, \quad (x, z) \mapsto (f_z)^{-1}(z).
\]

**Proof.** If \( r = 0 \), then the graph of \( f \) is closed in \( U \times K \times K \). As a consequence, the graph of \( g \) is closed in \( U \times K \times K \) (being obtained from the former by flipping the second and third component). Since \( K \) is compact, continuity of \( g \) follows (see, e.g., [13] Lemma 2.1). For \( r \in \mathbb{N} \cup \{ \infty \} \), we can repeat the proof of [13] Theorem C] without changes. □

**Proof of Proposition 1.2** The evaluation map

\[
\varepsilon_k : C^k(M, E) \times M \to E, \quad (f, x) \mapsto f(x)
\]
Lemma 3.2.7]. By [1, Theorem 3.25], the latter will hold if
\[ \text{Diff}_k \]
Lemma 1.3.19]. Then also
\[ k \]
For the discussion of the inversion maps, note that, for each
\[ k \]
P \subset C \text{C} \text{tor subspace}
\[ C \]
C \text{C} \text{tor}
\[ \text{Diff}_k \]
Hence
\[ \epsilon \]
is \( C^{\infty,k} \) and thus \( C^{\ell,k} \), by [2 Proposition 3.20]. Likewise, the evaluation map
\[ \epsilon_{k+\ell}: C^{k+\ell}(M,E) \times M \to E \]
is \( C^{\infty,k+\ell} \). Let us show that the map
\[ H_{k,\ell}: C^{k+\ell}(M,E) \times \Omega_k \to C^k(M,E), \quad (f,g) \mapsto g + f \circ (\text{id}_M + g) \]
is \( C^{\ell} \), for all \( k \in \mathbb{N} \cup \{\infty\} \) and \( \ell \in \mathbb{N}_0 \cup \{\infty\} \). It suffices to show that
\[ \Gamma_{k,\ell}: C^{k+\ell}(M,E) \times \Omega_k \to C^k(M,E), \quad (f,g) \mapsto f \circ (\text{id}_M + g) \]
is \( C^{\ell} \), since \( H_{k,\ell}(f,g) = g + \Gamma_{k,\ell}(f,g) \). This will hold if we can show that \( \Gamma_{k,\ell} \) is \( C^{\infty,\ell} \) (see [2, Lemma 3.15]). By [1 Theorem 3.20], it suffices to show that
\[ \Gamma^\ast_{k,\ell}: C^{k+\ell}(M,E) \times \Omega_k \times M \to E, \quad (f,g,x) \mapsto f(x+g(x)) = \epsilon_{k+\ell}(f,x+\epsilon_k(g,x)) \]
is \( C^{\infty,\ell,k} \). Now \( \Omega_k \times M \to M, (g,x) \mapsto x \) is a smooth map and hence \( C^{\ell,k} \). As also \( \epsilon_k \) is \( C^{\ell,k} \), we find that
\[ h_2: \Omega_k \times M \to M, \quad (g,x) \mapsto x + g(x) = x + \epsilon_k(g,x) \]
is \( C^{\ell,k} \). The identity map \( h_1: C^{k+\ell}(M,E) \to C^{k+\ell}(M,E), f \mapsto f \) is \( C^{\infty} \). Since \( \epsilon_{k+\ell} \) is \( C^{\infty,k+\ell} \), the Chain Rule in the form [11 Lemma 3.16] shows that
\[ \Gamma^\ast_{k,\ell} = \epsilon_{k+\ell} \circ (h_1 \times h_2) \]
is \( C^{\infty,\ell,k} \). Thus \( H_{k,\ell} \) is \( C^{\ell} \), whence also \( H_{k,\ell}|_{\Omega_{k+\ell} \times \Omega_k} \) is \( C^{\ell} \). Now
\[ H_{k,\ell}(f,g) = (\text{id}_M + f) \circ (\text{id}_M + g) - \text{id}_M \in \Omega_k \subseteq C^{k}_{\text{str}}(M,E) \]
for all \( f \in \Omega_{k+\ell} \subseteq \Omega_k \) and \( g \in \Omega_k \), since \( \text{Diff}^k(M) \) is closed under composition. Hence \( H_{k,\ell}|_{\Omega_{k+\ell} \times \Omega_k} \) co-restricts to a map
\[ h_{k,\ell}: \Omega_{k+\ell} \times \Omega_k \to \Omega_k \subseteq C^{k}_{\text{str}}(M,E) \]
which is \( C^{\ell} \) as the vector subspace \( C^{k}_{\text{str}}(M,E) \) of \( C^{k}(M,E) \) is closed (see [15, Lemma 1.3.19]). Then also \( \epsilon_{k,\ell} = \Phi^{-1}_{k+\ell} \circ h_{k,\ell} \circ (\Phi_{k+\ell} \times \Phi_k) \) is \( C^{\ell} \).

For the discussion of the inversion maps, note that, for each \( k \in \mathbb{N} \cup \{\infty\} \), \( \text{Diff}^k(M) \) is a smooth submanifold of \( C^{k}(M,E) \) modeled on the closed vector subspace \( C^{k}_{\text{str}}(M,E) \) of \( C^{k}(M,E) \). In fact, since \( \Omega_k \) is an open subset of \( C^{k}_{\text{str}}(M,E) \) whose topology is induced by the compact-open \( C^k \)-topology on \( C^{k}(M,E) \), we find an open subset \( Q \subseteq C^{k}(M,E) \) with \( Q \cap C^{k}_{\text{str}}(M,E) = \Omega_k \). Then \( P := \text{id}_M + Q = \{\text{id}_M + f: f \in Q\} \) is an open subset of \( C^{k}_{\text{str}}(M,E) \) and
\[ \kappa: P \to Q, \quad \phi \mapsto \phi - \text{id}_M \]
is a chart for \( C^{k}(M,E) \) such that \( \kappa(P \cap \text{Diff}^k_{\text{str}}(M)) = \Omega_k = Q \cap C^{k}_{\text{str}}(M,E) \). As a consequence, for \( k \in \mathbb{N} \cup \{\infty\} \) and \( \ell \in \mathbb{N}_0 \cup \{\infty\} \), the map \( t_{k,\ell} \) will be \( C^{\ell} \) to \( \text{Diff}^k(M) \) if we can show that \( t_{k,\ell} \) is \( C^{\ell} \) as a map to \( C^{k}(M,E) \) (see [15, Lemma 3.2.7]). By [1, Theorem 3.25], the latter will hold if
\[ t^\ast_{k,\ell}: \text{Diff}^{k+\ell}_{\text{str}}(M) \times M \to E, \quad (\phi,z) \mapsto t_{k,\ell}(\phi)(z) = \phi^{-1}(z) \]
is \( C^{\ell,k} \). But \( \iota_{k,\ell}' \) even is \( C^{k+\ell} \); in fact,

\[
f: \text{Diff}^{k+\ell}_r(M) \times M \to E, \quad (\phi, x) \mapsto \phi(x)
\]
is \( C^{\infty,k+\ell} \) (and hence \( C^{k+\ell} \)) as the restriction to a submanifold of the evaluation mapping \( \varepsilon_{k+\ell}: C^{k+\ell}(M, E) \times M \to E \), which is \( C^{\infty,k+\ell} \). Since \( \text{Diff}^{k+\ell}_r(M) \) is \( C^{\infty} \)-diffeomorphic to the open set \( \Omega_{k+\ell} \subseteq C^{k+\ell}_r(M, E) \), the \( C^{k+\ell} \)-property of \( \iota_{k,\ell}'\): \( (\phi, z) \mapsto f(\phi, \cdot)^{-1}(z) \) follows from Lemma 4.2.

Note that \( c_{\infty,\infty} \) is the group multiplication of \( \text{Diff}^{\infty}_r(M) \) and \( c_{\infty,\infty} \) the group inversion. As both of these mappings are smooth, \( \text{Diff}^{\infty}_r(M) \) is a Lie group. □

### 5 Proof of Theorem 1.1

Two lemmas will be useful for the proof of Theorem 1.1.

**Lemma 5.1** Let \( E \) be a finite-dimensional real vector space and \( M \subseteq E \) be a convex polytope with non-empty interior. If \( r > 0 \) and \( \gamma: [0, r] \to M \) is a \( C^1 \)-map, then there exists \( \varepsilon > 0 \) such that

\[
\gamma(0) + t\gamma'(0) \in M \quad \text{for all} \quad t \in [0, \varepsilon].
\]

**Proof.** We may assume \( E \neq \{0\} \), omitting a trivial case. There are a finite set \( \Lambda \neq \emptyset \) of non-zero linear functionals \( \lambda: E \to \mathbb{R} \) and numbers \( a_{\lambda} \in \mathbb{R} \) such that

\[
M = \{ x \in E : (\forall \lambda \in \Lambda) \lambda(x) \leq a_{\lambda} \}.
\]

Let \( \Lambda_0 := \{ \lambda \in \Lambda : \lambda(\gamma(0)) = a_{\lambda} \} \). Then

\[
\lambda(\gamma'(0)) = (\lambda \circ \gamma)'(0) = \lim_{t \to 0^+} \frac{\lambda(\gamma(t)) - \lambda(\gamma(0))}{t} \leq 0
\]

for each \( \lambda \in \Lambda_0 \), as \( \lambda(\gamma(t)) \leq a_{\lambda} \) and \( \lambda(\gamma(0)) = a_{\lambda} \). As a consequence,

\[
\lambda(\gamma(0) + t\gamma'(0)) = a_{\lambda} + t\lambda(\gamma'(0)) \leq a_{\lambda}
\]

for all \( \lambda \in \Lambda_0 \) and \( t \geq 0 \). For all \( \lambda \in \Lambda \setminus \Lambda_0 \), we have \( \lambda(\gamma(0)) < a_{\lambda} \). As \( \Lambda \setminus \Lambda_0 \) is a finite set, we find \( \varepsilon > 0 \) such that \( \lambda(\gamma(0)) + t\lambda(\gamma'(0)) \leq a_{\lambda} \) for all \( t \in [0, \varepsilon] \) and all \( \lambda \in \Lambda \setminus \Lambda_0 \). For all \( t \in [0, \varepsilon] \), we then have \( \lambda(\gamma(0) + t\gamma'(0)) \leq a_{\lambda} \) for all \( \lambda \in \Lambda \). Thus \( \gamma(0) + t\gamma'(0) \in M \), by \( \square \).

**Lemma 5.2** Let \( E \) and \( F \) be finite-dimensional real vector spaces, \( M \subseteq E \) and \( N \subseteq F \) be convex polytopes with non-empty interior and \( \phi: U \to V \) be a \( C^1 \)-map between open subsets \( U \subseteq M \) and \( V \subseteq N \). For \( x \in M \), let \( M(x) \) be the face of \( M \) generated by \( x \), and \( E(x) := \text{span}_\mathbb{R}(M(x) - x) \). For \( y \in N \), let \( N(y) \) be the face of \( N \) generated by \( y \), and \( F(y) := \text{span}_\mathbb{R}(N(y) - y) \). For \( x \in U \), we have:

- (a) \( \phi'(x)(E(x)) \subseteq F(\phi(x)) \);
Proof. (a) Let \( w \in E(x) \). Since \( x \in \text{algint } M(x) \), there exists \( r > 0 \) such that \( x + tw \in M(x) \subseteq M \) for all \( t \in [0, r] \). After shrinking \( r \), we may assume that \( x + tw \in U \) for all \( t \in [0, r] \) and thus \( \phi(x + tw) \in N \). Since \( \pm \phi'(x)(w) = \left. \frac{d}{dt} \right|_{t=0} \phi(x + tw) \), Lemma 5.1 provides \( \varepsilon > 0 \) such that

\[
\phi(x) + t\phi'(x)(w) \in N \quad \text{for all } t \in [0, \varepsilon].
\]

Notably, \( v_+ := \phi(x) + \varepsilon\phi'(x)(w) \in N \) and \( v_- := \phi(x) - \varepsilon\phi'(x)(w) \in N \). Since \( (1/2)v_+ + (1/2)v_- = \phi(x) \in N(\phi(x)) \) and \( N(\phi(x)) \) is a face of \( N \), we deduce that \( v_+, v_- \in N(\phi(x)) \). Thus \( \varepsilon\phi'(x)(w) = v_+ - \phi(x) \in N(\phi(x)) - \phi(x) \subseteq F(\phi(x)) \), whence also \( \phi'(x)(w) \in F(\phi(x)) \).

(b) is immediate from (a).

(c) Set \( y := \phi(x) \). By (a), we have \( \phi'(x)(E(x)) \subseteq F(y) \) and \( (\phi^{-1})'(y)(F(y)) \subseteq E(\phi^{-1}(y)) = E(x) \). Since \( (\phi^{-1})'(y) = (\phi'(x))^{-1} \), applying \( \phi'(x) \) to the latter inclusion we get \( F(y) \subseteq \phi'(x)(E(x)) \), whence \( F(y) = \phi'(x)(E(x)) \). \( \Box \)

Lemma 5.2(c) is analogous to the well-known boundary invariance for manifolds with corners (as in [18, Theorem 1.2.12 (a)]).

Definition 5.3 Let \( E \) be a finite-dimensional real vector space of dimension \( n \) and \( M \subseteq E \) be a convex polytope with non-empty interior. The index of \( x \in M \) is defined as

\[
\text{ind}_M(x) := \dim(E/E(x)) = n - \dim E(x) = n - \dim M(x),
\]

where \( M(x) \) is the face of \( M \) generated by \( x \) and \( E(x) := \text{span}_E(M(x) - x) \). For \( i \in \{0, 1, \ldots, n\} \), we define

\[
\partial_i(M) := \{ x \in M : \text{ind}_M(x) = i \}.
\]

In the situation of Definition 5.3 the following holds.

Lemma 5.4 For each \( i \in \{0, 1, \ldots, n\} \), the set \( \partial_i(M) \) is an \((n-i)\)-dimensional smooth submanifold of \( E \). If \( \mathcal{F}_{n-i}(M) \) is the finite set of all faces of \( M \) of dimension \( n-i \), then the topology induced by \( M \) on \( \partial_i(M) \) makes the latter the topological sum of the algebraic interiors \( \text{algint}(F) \) for \( F \in \mathcal{F}_{n-i}(M) \). The connected components of \( \partial_i(M) \) are the sets \( \text{algint}(F) \) for \( F \in \mathcal{F}_{n-i}(M) \); they are open and closed in \( \partial_i(M) \).

Proof. If \( F, G \in \mathcal{F}_{n-i}(M) \) such that \( F \neq G \) and \( F \cap G \neq \emptyset \), then \( F \cap G \) is a face of \( M \) of dimension \( < n - i \) and \( F \cap G \subseteq \text{algint}(F) \) as well as \( F \cap G \subseteq \text{algint}(G) \), entailing that \( \text{algint}(F) \cap \text{algint}(G) = \emptyset \). Thus

\[
\text{algint}(F) \cap \text{algint}(G) = \emptyset \quad \text{for all } F, G \in \mathcal{F}_{n-i}(M) \text{ such that } F \neq G. \quad (5)
\]
Let $F \in \mathcal{F}_{n-1}(M)$. For each $G \in \mathcal{F}_{n-1}(M)$, we have \( \text{algint}(F) \cap G \subseteq F \cap G \subseteq F \setminus \text{algint}(F) \) by the preceding and thus \( \text{algint}(F) \cap G = \emptyset \). Thus

$$K := \bigcup_{G \in \mathcal{F}_{n-1}(M) \setminus \{F\}} G$$

is a compact subset of $E$ such that \( \text{algint}(F) \cap K = \emptyset \). Note that $F = H \cap M$ for some hyperplane $H$ in $E$ (faces of polyhedra being exposed), whence

$$F = M \cap \text{aff}(F).$$

Now \( \text{algint}(F) = U \cap \text{aff}(F) \) for some open subset $U \subseteq E$. After replacing $U$ by its intersection with the open set $E \setminus K$ which contains \( \text{algint}(F) \), we may assume that $U \cap \text{algint}(G) = \emptyset$ for all $G \in \mathcal{F}_{n-1}(M) \setminus \{F\}$ and hence

$$\partial_i(M) \cap U = U \cap \text{algint}(F) = \text{algint}(F),$$

showing that \( \text{algint}(F) \) is open in $\partial_i(M)$. The topology induced by $E$ therefore makes $\partial_i(M)$ the topological sum of the \( \text{algint}(F) \) with $F \in \mathcal{F}_{n-1}(M)$. If we choose $x \in \text{algint}(F)$, then $E(x) = \text{aff}(F) - x$ and

$$\phi: U \to U - x, \quad y \mapsto y - x$$

is a $C^\infty$-diffeomorphism between open subsets of $E$ such that

$$\phi(U \cap \partial_i(M)) = \phi(\text{algint}(F)) = \phi(U \cap \text{aff}(F)) = (U - x) \cap E(x).$$

Thus $\partial_i(M)$ is a submanifold of $E$ of dimension $\dim(E(x)) = n - i$. \hfill \( \square \)

**Lemma 5.5** Let $E$ be a finite-dimensional real vector space, $M \subseteq E$ be a convex polytope with non-empty interior, and $k \in \mathbb{N} \cup \{\infty\}$. Then $\text{Diff}^k(M)$ is a normal subgroup of finite index in $\text{Diff}^k(M)$.

**Proof.** We write $\mathcal{F}(M)$ for the set of all faces of $M$. Let $n := \dim(E)$ and $\phi \in \text{Diff}^k(M)$. By Lemma 5.2(c), we have $\text{ind}_M(\phi(x)) = \text{ind}_M(x)$ for each $x \in M$, whence $\phi(\partial_i(M)) \subseteq \partial_i(M)$ for each $i \in \{0, 1, \ldots, n\}$ and thus $\phi(\partial_i(M)) = \partial_i(M)$, as $M$ is the disjoint union of $\partial_0(M), \ldots, \partial_n(M)$ and $\phi$ is surjective. Since $\partial_i(M)$ is a submanifold of $E$, the inclusion map $\partial_i(M) \to E$ is smooth, and thus also the inclusion map $j_i: \partial_i M \to M$. Thus $\phi|_{\partial_i(M)} = \phi \circ j_i$ is $C^k$. As this map has image in $\partial_i(M)$, which is a submanifold of $E$, we deduce that

$$\phi_i := \phi|_{\partial_i(M)}: \partial_i(M) \to \partial_i(M)$$

is $C^k$. Note that $(\phi^{-1})_i$ is inverse to $\phi_i$, whence $\phi_i$ is a $C^k$-diffeomorphism and hence a homeomorphism. For each $F \in \mathcal{F}_{n-1}(M)$, the algebraic interior $\text{algint}(F)$ is open and closed in the topological sum $\partial_i(M)$. As a consequence, $\phi(\text{algint}(F)) = \phi_i(\text{algint}(F))$ is both open and closed in $\partial_i(M)$. Being also non-empty and connected, $\phi(\text{algint}(F))$ is a connected component of $\partial_i(M)$ and thus $\phi(\text{algint}(F)) = \text{algint}(G)$ for some $G \in \mathcal{F}_{n-1}(M)$. As a consequence,

$$\phi(F) = \phi(\overline{\text{algint}(F)}) = \overline{\phi(\text{algint}(F))} = \overline{\text{algint}(G)} = G.$$
Thus $\phi_*(F) := \phi(F) \in \mathcal{F}(M)$ for each $F \in \mathcal{F}(M)$. Since $(\text{id}_M)_*$ is the identity map on $\mathcal{F}(M)$ and $(\phi \circ \psi)_* = \phi_* \circ \psi_*$ for all $\phi, \psi \in \text{Diff}^k(M)$, we get a group homomorphism

$$\pi: \text{Diff}^k(M) \to \text{Sym}(\mathcal{F}(M)), \quad \phi \mapsto \phi_*$$

to the group of all permutations of the finite set $\mathcal{F}(M)$. Thus $\text{Diff}^k_{fr}(M) = \ker(\pi)$ is a normal subgroup of $\text{Diff}^k(M)$ of finite index. 

**Proof of Theorem 1.1.** In view of the local description of Lie group structures (analogous to Proposition 18 in [3, Chapter III, §1, no. 9]), we need only show that the group homomorphism

$$I_\psi: \text{Diff}^\infty_{fr}(M) \to \text{Diff}^\infty_{fr}(M), \quad \phi \mapsto \psi \circ \phi \circ \psi^{-1}$$

is smooth for each $\psi \in \text{Diff}(M)$. Since $\text{Diff}^\infty_{fr}(M)$ is a smooth submanifold of $C^\infty(M,E)$, we need only show that $I_\psi$ is smooth as a map to $C^\infty(M,E)$, which will hold if we can show that the map

$$I_\psi^\wedge: \text{Diff}^\infty_{fr}(M) \times M \to E, \quad (\phi, x) \mapsto I_\psi(\phi)(x) = \psi(\phi(\psi^{-1}(x)))$$

is smooth (see [2, Theorem 3.25 and Lemma 3.22]). We know that the evaluation map $\varepsilon: C^\infty(M,E) \times M \to E, (f, x) \mapsto f(x)$ is smooth (see [2, Proposition 3.20 and Lemma 3.22]). Since $\text{Diff}^\infty_{fr}(M)$ is a submanifold of $C^\infty(M,E)$, also the restriction of $\varepsilon$ to a map

$$\text{Diff}^\infty_{fr}(M) \times M \to E$$

is smooth. As this map takes its values in $M$, also its corestriction

$$\theta: \text{Diff}^\infty_{fr}(M) \times M \to M, \quad (\phi, x) \mapsto \phi(x)$$

is smooth. The formula

$$I_\psi^\wedge(\phi, x) = \psi(\theta(\phi, \psi^{-1}(x)))$$

now shows that $I_\psi^\wedge$ is a smooth function of $(\phi, x)$, as required. 

**Remark 5.6** If $E = \mathbb{R}^n$ and $M \subseteq E$ is a polytope with non-empty interior, then

$$\text{Diff}_{\partial M}(M) := \{ \phi \in \text{Diff}(M) : \phi(x) = x \text{ for all } x \in \partial M \}$$

is a subgroup of $\text{Diff}^\infty_{fr}(M)$. Since

$$C^\infty_{\partial M}(M,E) := \{ f \in C^\infty(M,E) : f(x) = 0 \text{ for all } x \in \partial M \}$$

is a closed vector subspace of $C^\infty_{str}(M,E)$ and the chart

$$\Phi_{\infty}: \text{Diff}^\infty_{fr}(M) \to \Omega_{\infty} \subseteq C^\infty_{str}(M,E)$$

satisfies $\Phi_{\infty}(\text{Diff}_{\partial M}(M)) = \Omega_{\infty} \cap C^\infty_{\partial M}(M,E)$, we see that $\text{Diff}_{\partial M}(M)$ is a submanifold (and hence a Lie subgroup) of $\text{Diff}^\infty_{fr}(M)$ modeled on $C^\infty_{\partial M}(M,E)$, and thus also of $\text{Diff}(M)$ (in the sense of [15, Definition 3.1.10]).

13
Remark 5.7} Let \( E \) be a finite-dimensional real vector space, \( n := \dim(E) \), \( M \subseteq E \) be a convex polytope with non-empty interior, and \( k \in \mathbb{N}_0 \cup \{ \infty \} \). Using Lemma 5.4, we obtain the following more intuitive characterization: A function \( f \in C^k(M, E) \) is stratified (and thus in \( C^{\infty}_s(M, E) \)) if and only if \((\text{id}_M, f)(\partial_i(M)) \subseteq T(\partial_i(M))\) for each \( i \in \{0, \ldots, n\} \), identifying \( T(\partial_i(M)) \) with a subset of \( TM = M \times \mathbb{R}^n \) as usual. That is, \( f|_{\partial_i(M)} \) can be considered as a \( C^k \)-vector field of the \((n - i)\)-dimensional \( C^\infty \)-manifold \( \partial_i(M) \) for all \( i \in \{0, \ldots, n\} \).

Remark 5.8} Given \( n \in \mathbb{N}_0 \), we define a \emph{locally polyhedral} \( C^\infty \)-manifold of dimension \( n \) as a Hausdorff topological space \( M \), together with a set \( \mathcal{A} \) of homeomorphisms \( \phi: U_\phi \to V_\phi \) from open subsets \( U_\phi \subseteq M \) onto open subsets of a polytope \( P_\phi \subseteq \mathbb{R}^n \) with non-empty interior, such that \( \mathcal{A} \) is a \emph{polyhedral smooth atlas} in the sense that \( \bigcup_{\phi \in \mathcal{A}} U_\phi = M \) and \( \phi \circ \psi^{-1} \) is \( C^\infty \) for all \( \phi, \psi \in \mathcal{A} \), and \( \mathcal{A} \) is maximal among such atlases. Notably, \( \mathcal{A} \) is a rough \( C^\infty \)-atlas in the sense of [15], whence \( M \) can be considered as a smooth manifold with rough boundary in the sense of [15]. By Lemma 5.2 for all \( x \in M \) we have

\[
\text{ind}_{P_\phi}(\phi(x)) = \text{ind}_{P_\psi}(\psi(x))
\]

for all \( \phi, \psi \in \mathcal{A} \) such that \( x \in U_\phi \) and \( x \in U_\psi \), whence

\[
\text{ind}_M(x) := \text{ind}_{P_\phi}(\phi(x))
\]

is a well-defined integer in \( \{0, 1, \ldots, n\} \). If we set

\[
\partial_i(M) := \{x \in M : \text{ind}_M(x) = i\},
\]

then \( \phi(U_\phi \cap \partial_i(M)) = V_\phi \cap \partial_i(P_\phi) \) is an \((n - i)\)-dimensional submanifold of \( \mathbb{R}^n \) for each \( \phi \in \mathcal{A} \), entailing that \( \partial_i(M) \) is an \((n - i)\)-dimensional manifold (and a submanifold of \( M \) in the sense of [15] Definition 3.5.14); cf. also [15] Remark 3.5.16 (a)).

\section{Non-existence of Michor-type local additions}

If \( M \) is a finite-dimensional smooth manifold with corners of dimension \( n \), one can define its tangent bundle \( TM \) with fibre \( \mathbb{R}^n \) and obtains the subset \( ^1TM \) of all tangent vectors \( v \) of the form

\[
v = \dot{\gamma}(0) \in T_{\gamma(0)}M
\]

for some smooth curve \( \gamma: [0, 1[ \to M \), the so-called \emph{inner} tangent vectors (see [19] p. 20]). If \( M = [0, \infty[^k \times \mathbb{R}^{n-k} \) with \( k \in \{0, \ldots, n\} \), we identify \( TM \) with \( M \times \mathbb{R}^n \) as usual; then

\[
^1TM = \{(x, y) \in M \times \mathbb{R}^n : (\forall j \in \{1, \ldots, k\}) \ x_j = 0 \implies y_j \geq 0\},
\]

writing \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) in components. Hence \( ^1TM \) is a convex subset of \( \mathbb{R}^n \times \mathbb{R}^n \) and its interior \( M^0 \times \mathbb{R}^n \) is dense in \( ^1TM \).
Notably, $\mathcal{T}M$ is a locally convex subset of $\mathbb{R}^{2n}$ with dense interior. Since every $n$-dimensional smooth manifold $M$ with corners locally looks like $[0, \infty)^n$, we see that $\mathcal{T}M$ locally looks like $\mathcal{T}([0, \infty]^n)$, whence $\mathcal{T}M$ is a smooth manifold with rough boundary in the sense of Chapter 3. It therefore makes sense to speak about smooth functions on $\mathcal{T}M$. (A smaller class of modeling sets and corresponding generalized manifolds is proposed in [19, p. 20, Remark]). According to [19, §10.2, p. 90], local additions on a smooth manifold $M$ with corners are defined as follows:

A.1 A local addition $\tau$ on $M$ is a smooth mapping $\tau: \mathcal{TM} \to M$ satisfying

(A1) $(\tau, \pi_{\mathcal{TM}}): \mathcal{TM} \to M \times M$ is a diffeomorphism onto an open neighbourhood of the diagonal in $M \times M$;

(A2) $\tau(0_x) = x$ for all $x \in M$.

Here $0_x \in T_xM$ is the 0-vector in the tangent space $T_xM$ for $x \in M$. Michor claims that every smooth manifold with corners admits a local addition (see [19, p. 90, Lemma]). However:

**Proposition A.2** $M := [0, \infty[$ does not admit a local addition in the sense defined by Michor, as in A.1.

**Proof.** We identify $TM$ with $M \times \mathbb{R}$. Thus

$$\mathcal{T}M = (\{0\} \times [0, \infty[) \cup ([0, \infty[ \times \mathbb{R}).$$

Suppose that $\tau: \mathcal{T}M \to M$ was a local addition. We shall derive a contradiction. Since $\phi: \mathcal{T}M \to M \times M, (x, y) \mapsto (\tau(x, y), x)$ is a diffeomorphism onto on open neighbourhood of the diagonal, the Jacobi matrix $J_{\phi}(0, 0)$ must be invertible. By (A2), we have $\tau(0, 0) = x$, whence $\frac{\partial \tau}{\partial y}(0, 0) = 1$. Thus

$$J_{\phi}(0, 0) = \begin{pmatrix} 1 & \frac{\partial \tau}{\partial y}(0, 0) \\ 1 & 0 \end{pmatrix},$$

and invertibility implies that $\frac{\partial \tau}{\partial y}(0, 0) \neq 0$. Since $\tau(0, y) \in [0, \infty[$ for all $y \in [0, \infty[$, we have $\tau(0, y) \geq 0$ and thus $\frac{\partial \tau}{\partial y}(0, 0) \geq 0$, using that $\tau(0, 0) = 0$. Thus $\frac{\partial \tau}{\partial y}(0, 0) > 0$. Choose $\theta > 0$ such that

$$\theta \frac{\partial \tau}{\partial y}(0, 0) > 1;$$

thus $\varepsilon := \theta \frac{\partial \tau}{\partial y}(0, 0) - 1 > 0$. Consider the smooth map

$$h: [0, \infty[ \to M \subseteq \mathbb{R}, \quad t \mapsto \tau(t, -\theta t).$$

Then

$$h'(0) = \frac{\partial \tau}{\partial x}(0, 0) - \theta \frac{\partial \tau}{\partial y}(0, 0) = -\varepsilon.$$
Since $h(0) = 0$, Taylor’s Theorem yields

$$h(t) = -\varepsilon t + R(t)$$

with $R(t)/t \to 0$ for $t \to 0$. There exists $\delta > 0$ such that $|R(t)|/t < \varepsilon$ for all $t \in (0, \delta]$, whence $R(t) < \varepsilon t$ and thus

$$h(t) < -\varepsilon t + \varepsilon t = 0.$$  

This contradicts $h(t) \geq 0$, which holds as $h(t) \in M$. $\square$

It would not help to assume that, instead, the local addition $\tau$ is only defined on an open neighbourhood $\Omega$ of the 0-section in $\mathfrak{T}M$. In fact, such a neighbourhood $\Omega$ of $[0, \infty[^k \times \{0\}$ in $\mathfrak{T}([0, \infty[) \times \{0\}$ contains $[0, \rho[^k \times [-\rho, \rho[$ for some $\rho > 0$. There exists $\delta \in [0, \rho]$ such that $\theta \delta < \rho$ and thus $(t, -\theta t) \in \Omega$ for all $t \in [0, \delta]$. Hence $h(t) := \tau(t, -\theta t) \in M$. After shrinking $\delta$ if necessary, a contradiction is obtained as in the preceding proof.

An analogous remark applies in the following more general situation. We consider smooth manifolds with corners as in [19] here, which are assumed finite-dimensional and paracompact.

**Proposition A.3** If $M$ is a smooth manifold with corners such that $\partial M \neq \emptyset$, then $M$ does not admit a smooth local addition in Michor’s sense.

**Proof.** Suppose we could find a smooth local addition $\tau: \mathfrak{T}M \to M$ in Michor’s sense. We derive a contradiction. Let $n \in \mathbb{N}$ be the dimension of $M$. For $p \in \partial M$, there exist $k \in \{1, \ldots, n\}$, an open $p$-neighbourhood $U \subseteq M$ and a $C^\infty$-diffeomorphism $\phi: U \to [0, \infty[^k \times \mathbb{R}^{n-k} =: V$ such that $\phi(p) = 0$. Let

$$F := \mathfrak{T}V \subseteq TV = V \times \mathbb{R}^n$$

be the set of all $(x, y) \in V \times \mathbb{R}^n$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ such that $y_j \geq 0$ for all $j \in \{1, \ldots, k\}$ such that $x_j = 0$. Then $F$ is a convex subset of $\mathbb{R}^n \times \mathbb{R}^n$ with dense interior. Moreover, $\tau^{-1}(U)$ is an open $0_p$-neighbourhood in $\mathfrak{T}M$ and

$$Q := T\phi((TU) \cap \tau^{-1}(U))$$

is an open $(0, 0)$-neighbourhood in $F \subseteq V \times \mathbb{R}^n$, whence $Q$ is a locally convex subset of $\mathbb{R}^n \times \mathbb{R}^n$ with dense interior. We obtain a smooth map

$$\sigma := \phi \circ \tau \circ T\phi^{-1}|_Q: Q \to V \subseteq \mathbb{R}^n$$

such that

$$\psi := (\sigma, \text{pr}_1): Q \to V \times V, \quad (x, y) \mapsto (\sigma(x, y), x)$$

is a $C^\infty$-diffeomorphism onto an open neighbourhood of the diagonal in $V \times V$. For some $\rho > 0$, we have $P := [-\rho, \rho[^2 \cap F \subseteq Q$. Write $\sigma|_P = (\sigma_1, \ldots, \sigma_n)$ in terms of the components $\sigma_1, \ldots, \sigma_n: P \to \mathbb{R}$. Using the partial maps $\sigma_0 :=
\( \sigma(0, \cdot) : [0, \infty]^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n \) and \( \sigma^0 := \sigma(\cdot, 0) : [0, \rho]^k \times [0, \rho^{n-k}] \to \mathbb{R}^n \) of \( \sigma|_P \), the Jacobian of \( \psi \) at \((0, 0) \in P \subseteq \mathbb{R}^n \times \mathbb{R}^n \) can be regarded as the block matrix

\[
J_\psi(0, 0) = \begin{pmatrix}
J_{\sigma^0}(0) & J_{\sigma_0}(0) \\
1 & 0
\end{pmatrix}.
\]

The invertibility of the Jacobian implies that the final \( n \) columns must be linearly independent. Notably, we must have \( \partial \sigma_1 \partial y_j(0, 0) \neq 0 \) for some \( j \in \{1, \ldots, n\} \). Let \( e_1, \ldots, e_n \subseteq \mathbb{R}^n \) be the standard basis vectors. For small \( t \geq 0 \), we have \((0, te_j) \in Q\), whence \((0, te_j) \in V\), entailing that \( \sigma_1(0, te_j) \geq 0 \) and thus

\[
\frac{\partial \sigma_1}{\partial y_j}(0, 0) = \lim_{t \to 0^+} \frac{\sigma_1(0, te_j)}{t} \geq 0,
\]

exploiting that \( \sigma_1(0, 0) = 0 \). There exists \( \theta > 0 \) such that

\[
\varepsilon := \theta \frac{\partial \sigma_1}{\partial y_j}(0, 0) - 1 > 0.
\]

There exists \( \delta > 0 \) such that

\[
t(1, \ldots, 1, -\theta e_j) \in Q
\]

for all \( t \in [0, \delta] \) (with 1 in the first \( n \) slots). Then

\[
h : [0, \delta] \to \mathbb{R}, \quad t \mapsto \sigma_1(t(1, \ldots, 1, -\theta e_j))
\]

is a smooth function such that

\[
h'(0) = \sum_{i=1}^{n} \frac{\partial \sigma_1}{\partial x_i}(0, 0) - \theta \frac{\partial \sigma_1}{\partial y_j}(0, 0) = 1 - \theta \frac{\partial \sigma_1}{\partial y_j}(0, 0) = -\varepsilon < 0,
\]

where \( \delta_{1,i} \) denotes Kronecker’s delta. Since \( h(0) = 0 \), after shrinking \( \delta \), we can achieve that \( h(t) < 0 \) for all \( t \in [0, \delta] \). Thus \( \sigma_1(\delta(1, \ldots, 1, -\theta e_j)) = h(\delta) < 0 \), contradicting the fact \( \sigma(Q) \subseteq V \) and thus \( \sigma_1(x, y) \geq 0 \) for all \((x, y) \in Q\). \( \Box \)

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