Quasi-exactly solvable models based on special functions

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We suggest a systematic method of extension of quasi-exactly solvable (QES) systems. We construct finite-dimensional subspaces on the basis of special functions (hypergeometric, Airy, Bessel ones) invariant with respect to the action of differential operators of the second order with polynomial coefficients. As an example of physical applications, we show that the known two-photon Rabi Hamiltonian becomes quasi-exactly solvable at certain values of parameters when it can be expressed in terms of corresponding QES operators related to the hypergeometric function.

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I. INTRODUCTION

Usually, the models of the quantum mechanics are considered to be exactly-solvable (when the eigenvalues and eigenfunctions are known) or non-solvable at all (when the eigenvalues and eigenfunctions are unknown, and they can be found numerically only). It became a great surprise that an intermediate case is also possible, when inside the Hilbert space there is an invariant subspace for which the eigenvalues and eigenfunctions could be found from algebraic equations. This type of systems is called quasi-exactly solvable (QES). For the wide range of the one-dimensional QES the Hamiltonian possesses hidden Lie algebra $sl(2)$ and represents a bilinear form of first-order differential operators which satisfy the same commutator relations as the spin ones [1], [2], [3], [4]. The QES Schrödinger operators based on the $sl(2)$ representation were studied in [1], [5], [6], [7], [8]. Later it became possible to go beyond the Lie algebraic context. In particular, starting from the second-order differential operators

$$T = q(x) \frac{d^2}{dx^2} + p(x) \frac{d}{dx} + r(x)$$

(1)

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such that they possess the finite-dimensional representations both in the subspace $\mathcal{P}_n = \text{span}\{1, x, \ldots, x^{n-1}, x^n\}$ of all monomials of the degree $\leq n$ and in the monomial subspace $\text{span}\{1, x^2, \ldots, x^{n-1}, x^n\}$ [9], [10], [11]. In [12], [13], the extension of the corresponding subspace was suggested

$$\nu = \mathcal{P}_n \cup g(x)\mathcal{P}_n = \text{span}\{1, x^2, \ldots, x^{n-1}, x^n, g(x), g(x) \cdot x, \ldots, g(x) \cdot x^n\}$$ (2)

for certain functions $g(x)$ [12], [13].

In the present paper we consider the problem of constructing QES operators which preserve subspaces of a more general form:

$$\mathcal{M}_n = \text{span}\{f_1(x), f_2(x), \ldots, f_{n-1}(x), f_n(x)\}. \tag{3}$$

The general features of differential operators with invariant subspace $\mathcal{M}_n$ (3) were discussed in [15]. The main result obtained is the following: any linear operator $P_n \in \mathcal{P}(\mathcal{M}_n)$, that preserve $\mathcal{M}_n$ (3), may be represented as follows (a formula 5.8 in [15]):

$$P_n = \sum_{i,j} a_{ij} \cdot f_i(x) \cdot L_j + \sum R_m \cdot K_m, \tag{4}$$

where $a_{ij}$ are arbitrary constants, $R_v \in \mathcal{D}$ are arbitrary linear differential operators, $\mathcal{D} = \mathcal{D}(\mathcal{F})$ is a space of the differential operators with coefficients belonging to the functional space $\mathcal{F}$ (hereafter denoted $\mathcal{P}_n$), $K_m$ are operators belonging to the annihilator $\mathcal{A}(\mathcal{M}_n)$:

$$\mathcal{A}(\mathcal{M}_n) = \{K \in \mathcal{D} \mid K(f(x)) = 0, \text{ for all } f(x) \in \mathcal{M}_n\}, \tag{5}$$

$L_j$ are operators belonging to the affine annihilator $\mathcal{K}(\mathcal{M}_n)$:

$$\mathcal{K}(\mathcal{M}_n) = \{L \in \mathcal{D} \mid L(f(x)) = c \in \mathbb{R}, \text{ for all } f(x) \in \mathcal{M}_n\} \tag{6}$$

c is a constant. All the definitions and symbols are in agreement with the ones used in [15].

In spite of the fact that the theorem of existence specifies a general form of the required differential operators (4), their finding is, generally speaking, not trivial task. The purpose of the given work consists in explicit construction of such QES operators, for which the set of functions $f_n(x)$ does not reduce to polynomials and represents special functions. With this purpose, we offer a method of QES extension. It consists in the following. To construct the QES Hamiltonians on the basis of the subspace (3), we select in (6) the subspace of differential operators $\mathcal{K}_2(\mathcal{M}_n)$ of the degree two or less:

$$\mathcal{K}_2(\mathcal{M}_n) = \{L \in \mathcal{K}(\mathcal{M}_n) \mid \text{order}(L) \leq 2\}. \tag{7}$$
The general approach suggested in [15] does not supply us with information, whether or not the space $K_2(M_n)$ (7) is empty, without the constructing $K(M_n)$ for the chosen $M_n$ (3).

To simplify matter, we start from the two-dimensional subspace $M_2 = \text{span}\{f_1(x), f_2(x)\}$, for which operators $L_i \in K_2(M_2)$ are known to exist and can be explicitly constructed [15]. Let us try to extend the subspace $M_2$ by adding function $f_3(x)$ to it in such a way that the new subspace $M_3 = \text{span}\{f_1(x), f_2(x), f_3(x)\}$ be the three-dimensional and the space of corresponding operators $K_2(M_3)$ (7) be non-empty. Thus, transformation $M_2 \rightarrow M_3$ has to lead to the change of the coefficients at $\frac{d}{dx}$ in the operators $L_i \in K_2(M_2)$, with the order of the operators $L_i \in K_2(M_2)$ preserved: $\text{order}(L_i) \leq 2$. Repeating this operation $n$ times, we obtain the QES-extension of $M_2$. In other words, the initial subspace $M_2 = \text{span}\{f_1(x), f_2(x)\}$ permits the QES-extension if $M_2$ may be extended to $M_n$ under the condition that the affine annihilator $K_2(M_n)$ (7) is not empty. (It is worth noting another method of constructing invariant subspaces (3) which is based on the conditional symmetries [14], but it will not be discussed in the present paper.)

II. CONSTRUCTION OF THE INvariant SUBSPACES

Below we suggest the method that enables one to construct invariant subspaces $M_n$. Let us start from the simplest case $n = 2$. Let us choose a linear independent basis $\{f_0^+(x), f_0^-(x)\}$ for the subspace $M_2 = \text{span}\{f_0^+(x), f_0^-(x)\}$ in the following way. Let us consider the function $f(x)$, which satisfies the second order homogeneous differential equation with polynomial coefficients: $q(x) \frac{d^2}{dx^2} f(x) + p(x) \frac{d}{dx} f(x) + r(x) f(x) = 0$. The case when $f(x) = \text{constant} \cdot \frac{d}{dx} f(x)$ is eliminated. We select the function $f_0^+(x) = f(x)$ and its derivative $f_0^-(x) = \frac{d}{dx} f(x) = f'(x)$ as the basis components of $M_2$:

$$M_2 = \text{span}\{f_0^+(x), f_0^-(x)\}. \quad (8)$$

For simplicity, we shall work directly with the operator (4) instead of the operators $L_i$ representing affine annihilator $K(M_2)$ (6).

Our strategy can be described as follows:

I. We find a general form of the operator of the second order $P_2$ for which subspace $M_2$ (9) is preserved. The existence of such an operator is supported of Lemma 4.10 [15].
II We make extension of the subspace $\mathcal{M}_2 \rightarrow \mathcal{M}_4 = \text{span}\{f_0^+, f_1^+, f_0^-, f_1^-\}$.

III We find a general form of the operator of the second order $P_4$ for which subspace $\mathcal{M}_4$ is preserved. If a non-trivial solution $P_4 \neq \text{const}$ exists for the chosen way of the extension, we pass to item IV, otherwise we change a way of extension.

IV We make comparison of two operators $P_2$ and $P_4$ and try to guess a general form of coefficients which enable us to repeat extension in the chain ($\mathcal{M}_2 \rightarrow \mathcal{M}_4 \rightarrow \ldots \rightarrow \mathcal{M}_{2(N+1)}$) ($N = 0, 1, 2, \ldots$) that leaves these subspaces invariants. As a result, we obtain the explicit form of operator $P_{2(N+1)}$ that acts on the elements of the subspace $\mathcal{M}_{2(N+1)} = \{f_0^+, f_1^+, f_2^+, \ldots, f_N^+, f_0^-, f_1^-, f_2^-, \ldots, f_N^-\}$, $N = 0, 1, 2, \ldots$. This is not the end of story since the general form does not fix by itself the concrete expression for the coefficients.

V With the guessed general form of the operator $P_{2(N+1)}$ at hand, we require that it leave the corresponding subspace invariant, whence we find the concrete values of its coefficients.

For explicit demonstration of the aforementioned algorithm, we shall consider a concrete choice of the function $f_0^+(x) : f_0^+(x) = {}_0F_1\left[\begin{smallmatrix}-s \\ s \end{smallmatrix}; x\right]$ [16] where ${}_0F_1\left[\begin{smallmatrix}-s \\ s \end{smallmatrix}; x\right]$ is a hypergeometric function, $f_0^-(x) = {}_0F_1\left[\begin{smallmatrix}s+1 \\ s \end{smallmatrix}; x\right]$ is its derivative up to a constant factor.

The differential equation which the hypergeometric function obeys $(x \cdot \frac{d^2}{dx^2} + s \cdot \frac{d}{dx} - 1) f_0^+(x) = 0$ as well as its other properties can be found, e.g., in [16]. Then, we have:

$$\mathcal{M}_2 = \text{span}\left\{ {}_0F_1\left[\begin{smallmatrix}-s \\ s \end{smallmatrix}; x\right], {}_0F_1\left[\begin{smallmatrix}s+1 \\ s \end{smallmatrix}; x\right] \right\},$$

(9)

Let us choose the operator of the second order $P_2 = a_3(x) \frac{d^2}{dx^2} + a_2(x) \frac{d}{dx} + a_0(x)$ and write down the condition of invariance of the subspace $\mathcal{M}_2$ (9) for the given operator:

$$P_2\left(f_0^+\right) = c_1 \cdot f_0^+ + c_2 \cdot f_0^-$$

(10)

$$P_2\left(f_0^-\right) = c_3 \cdot f_0^+ + c_4 \cdot f_0^-$$

(11)

Using rules of differentiation $\frac{d}{dx} f_0^+ = \frac{f_0^+}{x} - \frac{f_0^-}{x} \cdot \frac{d}{dx} f_0^- = -s (s+1) \frac{f_0^+}{x^2} + (s + s^2 + x) \frac{f_0^-}{x^2}$ and
equating coefficients at functions $f_0^+, f_0^-$ (10, 11) we obtain the solution:

$$a_3(x) = -c_3 \cdot \frac{x^2}{s} + c_2 \cdot s \cdot x,$$

$$a_2(x) = -c_3 \cdot x + c_2 \cdot s (s + 1),$$

$$a_1(x) = c_3 \cdot \frac{x}{s} - c_2 \cdot s + c_4 + c_3$$

$c_1 = c_4 + c_3$. Substituting obtained coefficients $a_k(x), (k = 1, 2, 3)$ (12) into the expression for the operator $P_2$ we have:

$$P_2 = c_2 \cdot s \cdot P_2^1 - \frac{c_3}{s} \cdot P_2^2 + c_4 + c_3 + s \cdot c_2$$

(13)

where

$$P_2^1 = x \frac{d^2}{dx^2} + (s + 1) \cdot \frac{d}{dx}, \quad P_2^2 = x^2 \frac{d^2}{dx^2} + s \cdot x \frac{d}{dx} - x.$$

(14)

One can check that operators $P_2^1$, $P_2^2$ act on the basis functions according to the formulas $P_2^1 (f_0^+) = f_0^+ + \frac{1}{s} \cdot f_0^-$, $P_2^1 (f_0^-) = f_0^-$, $P_2^2 (f_0^+) = 0$, $P_2^2 (f_0^-) = s \cdot f_0^- - s \cdot f_0^+$ and, thus, preserve the subspace $\mathcal{M}_2$ (9). This is agreement with the fact that the affine annihilator $\mathcal{K}_2 (\mathcal{M})$ has the property of the vector space and possesses the operator basis (14) (see in [15]).

The extension of the subspace $\mathcal{M}_2$ (9) can be realized in many ways. The basic requirement here consists in that after adding new elements of the basis $\mathcal{M}_2 \cup \text{span} \{f_1^+, f_1^\prime\} = \text{span} \{f_0^+, f_1^+, f_0^-, f_1^\prime\} = \mathcal{M}_4$ the operator basis of $P_2$ (13, 14) should remain two-dimensional for the new subspace $\mathcal{M}_4$. We consider the simplest way of extension - multiplication of the basis functions by the power function $x^n$: $\mathcal{M}_2 \rightarrow \mathcal{M}_4 \rightarrow ... \rightarrow \mathcal{M}_{2(N+1)}$ ($N = 0, 1, 2, ...$),

$$\{f_0^+, f_0^-\} \rightarrow \{f_0^+, f_1^+, f_0^-, f_1^\prime\} \rightarrow ... \rightarrow \{f_0^+, f_1^+, ... f_{N-1}^+, f_0^-, f_1^\prime, ..., f_N\}$$

(15)

where

$$f_n^+ = x \cdot f_{n-1}^+ = x^n \cdot f_0^+, f_n^- = x \cdot f_{n-1}^- = x^n \cdot f_0^-,$$

(16)

$n = 0, 1, ..., N - 1, N$. To demonstrate that suggested extension is suitable, we shall consider subspace $\mathcal{M}_4 = \{f_0^+, f_1^+, f_0^-, f_1^\prime\}$ and shall write down the condition of its invariance with respect to the operator $P_4 = b_3(x) \frac{d^2}{dx^2} + b_2(x) \frac{d}{dx} + b_0(x)$:

$$P_4 (\vec{f}) = \hat{d} \cdot \vec{f}$$

(17)

where $\vec{f} = \{f_0^+, f_1^+, f_0^-, f_1^\prime\}^T$ - is vector, $\hat{d} = [d_{i,j}], (i = 1, ..., 4; j = 1, ..., 4) -$ is the matrix of coefficients. The condition (17) represents the system of the linear equations on functions
$b_k(x), k = 1, 2, 3$ and quantities $d_{i,j}$, ($i = 1, ..., 4; j = 1, ..., 4$). Solving this linear system of the equations we have:

\[ b_3(x) = d_{13} \cdot s \cdot x - d_{14} \cdot s \frac{x^2}{2}, \]  
\[ b_2(x) = d_{13} \cdot s \cdot (s + 1) + d_{14} \cdot x \cdot s \left(1 - \frac{s}{2}\right), \]  
\[ b_1(x) = d_{13} \cdot (-s) + d_{14} \cdot \left(\frac{sx}{2} + s^2 - s + 1\right) \]

(18)

\[ d_{11} = d_{14} \cdot (s^2 - s) + d_{44}, d_{21} = d_{13} \cdot (s^2 + s), d_{22} = d_{14} \cdot \frac{s^2}{2} + d_{44}, d_{24} = 3d_{13}, \]  
\[ d_{31} = d_{14} \cdot \frac{3s^2}{2}, d_{33} = -d_{14} \cdot \left(\frac{s^2}{2} + s\right) + d_{44}, d_{41} = 2d_{13} \cdot s^2, \]  
\[ d_{42} = d_{14} \cdot \frac{s^2}{2}, d_{43} = d_{13} \cdot (s - s^2), d_{12} = d_{23} = d_{32} = d_{34} = 0. \]  

(21)

The existence of the non-trivial solution (18-20) means that the way by which extension (15) was made is suitable.

Let us analyze the general form of the operator $P_4$ for the obtained solution:

\[ P_4 = d_{13} \cdot s \cdot P_4^1 - d_{14} \cdot \frac{s}{2} P_4^2 + d_{44} - d_{13} \cdot s + d_{14} \cdot (s^2 - s) \]  

(24)

where

\[ P_4^1 = x \frac{d^2}{dx^2} + (s + 1) \cdot \frac{d}{dx}, P_4^2 = x^2 \frac{d^2}{dx^2} + (s - 2) \cdot x \frac{d}{dx} - x. \]  

(25)

The operator $P_4$ (24) as well as operator $P_2$ (13), up to an additive constant, depends on two free parameters $d_{13}, d_{14}$ (24). From the general solution (24) we select two operators $P_4^1, P_4^2$ (25) which do not depend on parameters $d_{13}, d_{14}, d_{44}$. We see that the coefficients of the operator $P_4^1$ coincide with those of the operator $P_2^1$ (14). This suggests the idea to implement the expression (25) for the operator $P_{2(N+1)}^1$ for any $N$ and subspace (15). The situation with the operator $P_{2m}^2$ is somewhat more complicated. The natural general guessed form for this operator reads

\[ P_{2(N+1)}^2 = x^2 \frac{d^2}{dx^2} + (s - \alpha_N) \cdot x \frac{d}{dx} - x \]  

(26)

where the quantity $\alpha_N$ depends on $N$ and, besides, satisfies the conditions $\alpha_0 = 0, \alpha_1 = 2$ as it follows from (14, 25). To find dependence $\alpha_N$ from $N$, we shall consider the result of the action of the operator $P_{2(N+1)}^2$ on the element $f_n^+$ (15):

\[ P_{2(N+1)}^2 (f_n^+) = n (n + s - 1 - \alpha_N) \cdot f_n^+ + \frac{(2n - \alpha_N)}{s} \cdot f_{n+1}. \]  

(27)
If we put \( n = N \) in (27) and require that the subspace be finite-dimensional, the condition of cut off at \( n = N \) gives us \( a_N = 2 \cdot N \), whence the operator \( P_{2(N+1)}^2 \) looks like \( P_{2(N+1)}^2 = x^2 \frac{d^2}{dx^2} + (s - 2 \cdot N) \cdot x \frac{d}{dx} - x \). As a result, we obtained a pair of operators that indeed leave the subspace \( \mathcal{M}_{2(N+1)} \) invariant. Correspondingly, any linear combination \( P_{2(N+1)} = \kappa_1 \cdot P_{1}^1 + \kappa_2 \cdot P_{2(N+1)}^2 \) where \( \kappa_1, \kappa_2 \) arbitrary constants has the same property.

Following the way we described above, we found a series of concrete examples 1, 2, 4, 5, 6. The list of obtained finite-dimensional subspaces together with the basic operators for which corresponding subspaces are invariant is given below. Meanwhile, the suggested way of the QES - extension (16) of the operators (4) is not unique. As the alternative approach, in an example 3 we used another ”natural” procedure of QES - extension in that an integer \( n \) labeling the basis was added to the parameter of the function: \( \alpha \rightarrow \alpha + n \).

1) Finite dimensional function subspace \( \mathcal{R}_N^1 = \text{span}\{f_0^+, \ldots, f_N^+, f_0^-, \ldots, f_N^-\} \) \( (N = 0, 1, 2, \ldots) \), \( \dim(\mathcal{R}_N^1) = 2 \cdot (N + 1) \), formed by functions \( f_n^+ = x^n \cdot {}_0F_1 \left[ \frac{-}{s}; x \right] \), \( f_n^- = x^n \cdot {}_0F_1 \left[ \frac{-}{s+1}; x \right] \) \( (n = 0, 1, \ldots, N - 1, N) \) [16], is invariant for the operators \( J_1^1 \equiv P_{1}^1, J_1^- \equiv P_{2(N+1)}^2 \):

\[
\begin{align*}
J_1^+ &= x^2 \frac{d^2}{dx^2} + (s + 1) \frac{d}{dx} \\
J_1^- &= x^2 \frac{d^2}{dx^2} + (s - 2N) \cdot x \frac{d}{dx} - x
\end{align*}
\]  \( \text{(28)} \)

The operators \( J_1^+ \), \( J_1^- \) act on the functions \( f_n^+(x) \), \( f_n^-(x) \in \mathcal{R}_N^1 \) as follows:

\[
\begin{align*}
J_1^+(f_n^+|f_n^-) &= \left( \frac{n (A_n - 1 + s) \cdot f_n^+ + 2B_n \cdot f_{n+1}^-}{(n - s) (A_n - 1) \cdot f_n^- + s (2B_n - 1) \cdot f_n^+} \right) \\
J_1^-(f_n^+|f_n^-) &= \left( \frac{f_n^+ + n (n + s) \cdot f_{n-1}^+ + 1 + 2n}{f_n^- + n (n - s) \cdot f_{n-1}^- + 2ns \cdot f_{n+1}^-} \right)
\end{align*}
\]  \( \text{(29)} \)

, where \( A_n = n - 2N, B_n = n - N \). The function \( f_0^+(x) \) satisfies the differential equation \( (x^2 \frac{d^2}{dx^2} + s \frac{d}{dx} - 1) f_0^+ = 0 \).

2) Finite dimensional function subspace \( \mathcal{R}_N^2 = \text{span}\{f_0^+, \ldots, f_N^+, f_0^-, \ldots, f_N^-\} \) \( (N = 0, 1, 2, \ldots) \), \( \dim(\mathcal{R}_N^2) = 2 \cdot (N + 1) \), formed by functions \( f_n^+ = x^n \cdot {}_1F_1 \left[ \frac{\alpha}{s}; x \right] \), \( f_n^- = x^n \cdot {}_1F_1 \left[ \frac{\alpha+1}{s+1}; x \right] \) \( (n = 0, 1, \ldots, N - 1, N) \) [16], is invariant for the operators \( J_2^+, J_2^- \):

\[
\begin{align*}
J_2^+ &= x^2 \frac{d^2}{dx^2} + (1 + s - x) \frac{d}{dx} \\
J_2^- &= x^2 \frac{d^2}{dx^2} + (s - 2N - x) \cdot x \frac{d}{dx} + x (N - \alpha)
\end{align*}
\]  \( \text{(30)} \)
The operators $J_2^+$, $J_2^-$ act on the functions $f_n^+(x)$, $f_n^-(x) \in \mathcal{R}_N^2$ as follows:

$$J_2^+ \left( \frac{f_n^+}{f_n^-} \right) = \left( \frac{n (A_n - 1 + s) \cdot f_n^+ - B_n \cdot f_{n+1}^+ + 2\alpha B_n / s \cdot f_{n+1}^-}{(s - n) (1 - A_n) \cdot f_n^- + B_n \cdot f_{n+1}^- + s (2B_n - 1) \cdot f_n^+} \right)$$

$$J_2^- \left( \frac{f_n^+}{f_n^-} \right) = \left( \frac{(\alpha - n) f_n^+ + n (n + s) \cdot f_{n+1}^+ + \alpha (1 + 2n) / s \cdot f_n^-}{(\alpha + n + 1) f_n^- + n (n - s) \cdot f_{n-1}^- + 2ns \cdot f_{n+1}^-} \right)$$

, where $A_n = n - 2N$, $B_n = n - N$. The function $f_0^+(x)$ satisfies the differential equation

$$\left( x \frac{d^2}{dx^2} + (s - x) \frac{d}{dx} - \alpha \right) f_0^+(x) = 0.$$

3) Finite dimensional function subspace $\mathcal{R}_N^3 = \text{span}\{f_0, f_1, ..., f_N\}$ $(N = 0, 1, 2,...)$, dim $(\mathcal{R}_N^3) = N + 1$, formed by functions $f_n = \mathcal{F}_1 \left[ \frac{\alpha + n}{s} ; x \right]$ $(n = 0, 1, ..., N - 1, N)$ [16], is invariant for the operators $J_3^+$, $J_3^-$:

$$J_3^- = x \frac{d^2}{dx^2} + (s - x) \frac{d}{dx}$$

$$J_3^+ = x^2 \frac{d^2}{dx^2} + (s - N - x) \cdot x \frac{d}{dx} - \alpha x$$

The operators $J_3^+$, $J_3^-$ act on the functions $f_n(x) \in \mathcal{R}_N^3$ as follows:

$$J_3^- (f_n) = (n + \alpha) \cdot f_n$$

$$J_3^+ (f_n) = (sn + (\alpha + n) C_n) \cdot f_n + (\alpha + n) B_n \cdot f_{n+1} + n (\alpha + n - s) \cdot f_{n+1}$$

, where $C_n = N - 2n$, $B_n = n - N$ . It is worthwhile to note that replacement

$(s \rightarrow s - 1, \alpha \rightarrow \alpha - \frac{1}{2} + \frac{N}{2}, N \rightarrow \frac{N}{2} - \frac{1}{2})$ transforms the operators $J_2^+$, $J_2^-$ into $J_3^+$, $J_3^-$ , if $N$ is odd.

4) Finite-dimensional function subspace $\mathcal{R}_N^4 = \text{span}\{f_0^+, ..., f_N^+, f_0^-, ..., f_N^-\}$ $(N = 0, 1, 2,...)$, dim $(\mathcal{R}_N^4) = 2(N + 1)$, formed by functions $f_n^+ = x^n \cdot \text{Airy}(x)$, $f_n^- = x^n \cdot \frac{d}{dx} \text{Airy}(x)$ $(n = 0, 1, ..., N - 1, N)$ [16], is invariant for the operators $J_4^+, J_4^-:

$$J_4^- = x \frac{d^2}{dx^2} - (1 + 2N) \frac{d}{dx} - x^2$$

$$J_4^+ = \frac{d^2}{dx^2} - x$$

The operators $J_4^+$, $J_4^-$ act on the functions $f_n^+(x)$, $f_n^-(x) \in \mathcal{R}_N^4$ as follows:

$$J_4^+ \left( \frac{f_n^+}{f_n^-} \right) = \left( \frac{n (n - 1) \cdot f_{n-2}^+ + 2n \cdot f_{n-1}^-}{(1 + 2n) \cdot f_n^+ + n (n - 1) \cdot f_{n-2}^+} \right)$$

$$J_4^- \left( \frac{f_n^+}{f_n^-} \right) = \left( \frac{(A_n - 2) n \cdot f_{n-1}^+ + (2B_n - 1) \cdot f_n^-}{2B_n \cdot f_{n+1}^+ + (A_n - 2) n \cdot f_{n-1}^-} \right)$$

where $A_n = n - 2N$, $B_n = n - N$. The function $f_0^+(x)$ to satisfy the differential equation

$$\left( \frac{d^2}{dx^2} - x \right) f_0^+(x) = 0.$$
5) Finite dimensional function subspace $\mathcal{R}_N^5 = \text{span}\{f_0^+, \ldots, f_N^+, f_0^-, \ldots, f_N^-\} \ (N = 0, 1, 2, \ldots)$, $\dim (\mathcal{R}_N^5) = 2 (N + 1)$, formed by functions $f_n^+ = x^n \cdot \text{BesselK} (\nu, x)$, $f_n^- = x^n \cdot \text{BesselK} (\nu + 1, x) \ (n = 0, 1, \ldots, N - 1, N)$ [16], is invariant for the operators $J_5^+, J_5^-:$

$$J_5^- = x \frac{d^2}{dx^2} + 2 \frac{d}{dx} - \frac{\nu^2 + \nu + x^2}{x},$$

$$J_5^+ = x^2 \frac{d^2}{dx^2} + x (1 - 2N) \frac{d}{dx} - x^2$$

(38)

The operators $J_5^+, J_5^-$ act on the functions $f_n^+(x), f_n^-(x) \in \mathcal{R}_N^5$ as follows:

$$J_5^+ \left( \frac{f_n^+}{f_n^-} \right) = \left( \frac{(\nu + n) (\nu + A_n)}{(n - 1 - \nu) (A_n - 1 - \nu)} \cdot f_n^+ - 2B_n \cdot f_{n+1}^- \right)$$

(39)

$$J_5^- \left( \frac{f_n^+}{f_n^-} \right) = \left( - \frac{n (1 + n + 2\nu)}{1 + 2n} \cdot f_{n-1}^+ - (1 + 2n) \cdot f_n^- \right)$$

(40)

where $A_n = n - 2N, B_n = n - N$. The function $f_0^+(x)$ satisfies the differential equation

$$\left( x^2 \frac{d^2}{dx^2} + x \frac{d}{dx} - (x^2 + \nu^2) \right) f_0^+(x) = 0.$$

6) Finite dimensional function subspace $\mathcal{R}_N^6 = \text{span}\{f_0^+, \ldots, f_N^+, f_0^-, \ldots, f_N^-\} \ (N = 0, 1, 2, \ldots)$, $\dim (\mathcal{R}_N^6) = 2 (N + 1)$, formed by functions $f_n^+ = x^n \cdot _1F_1 \left[ \frac{\alpha + 1}{3/2}; x^2 \right], f_n^- = x^n \cdot _1F_1 \left[ \frac{\alpha + 1}{3/2}; x^2 \right] \ (n = 0, 1, \ldots, N - 1, N)$ [16], is invariant for the operators $J_6^+, J_6^-:$

$$J_6^- = x^2 \frac{d^2}{dx^2} - 2x \frac{d}{dx}$$

(41)

$$J_6^+ = x^2 \frac{d^2}{dx^2} - (2x^2 + 1 + 2N) \frac{d}{dx} + 2x (N - 2\alpha)$$

The operators $J_6^+, J_6^-$ act on the functions $f_n^+(x), f_n^-(x) \in \mathcal{R}_N^6$ as follows:

$$J_6^+ \left( \frac{f_n^+}{f_n^-} \right) = \left( - \frac{2B_n \cdot f_{n+1}^+ + n (A_n - 2) \cdot f_{n-1}^+ + 4\alpha (2B_n - 1) \cdot f_n^-}{n (A_n - 2) \cdot f_{n-1}^- + (2B_n - 1) \cdot f_n^+ + 2B_n \cdot f_{n+1}^-} \right)$$

(42)

$$J_6^- \left( \frac{f_n^+}{f_n^-} \right) = \left( \frac{2 (2\alpha - n) \cdot f_n^+ + (n^2 - n) \cdot f_{n-2}^+ + 8\alpha n \cdot f_{n-1}^-}{2n \cdot f_{n-1}^+ + 2 (2\alpha + 1 + n) \cdot f_n^- + (n^2 - n) \cdot f_{n-2}^-} \right)$$

(43)

where $A_n = n - 2N, B_n = n - N$.

For the proof of equalities (29, 31, 32, 34, 36, 37, 39, 40, 42, 43) it is enough to use rules of differentiation of special functions or their representations by the power series [16]. The commutation rules for the operators $J_k^+, J_k^- \ (k = 1 \ldots 6)$ are given in appendix A. Let us now consider an explicit example of application of the constructed subspaces to a physical system - two-photon Rabi Hamiltonian.
The two-photon Rabi Hamiltonian (TPRH) is an obvious extension of the original Rabi Hamiltonian [18], which takes into account the atomic transitions induced by the absorption and emission of two photons rather than one [17]. The corresponding system as a whole is not integrable. Let us prove that (33, 34), at some choice of parameters, is invariant subspace for TPRH:

$$H = \frac{\omega_0}{2} \sigma_z + \omega \cdot b^+ b + g \left( b^2 + (b^+)^2 \right) \cdot (\sigma_+ + \sigma_-)$$

(44)

where $\sigma_z, \sigma_\pm = \sigma_x \pm i \cdot \sigma_y$ are Pauli matrices, $b$ and $b^+$ are the annihilation and creation operators respectively ($[b, b^+] = 1$). Following the approach of the article [17], apply the Bogoliubov transformation to the operators $b$ and $b^+$ ($b(t) = \cos(t) \cdot b + \sin(t) \cdot b^+$, $b^+(t) = \cos(t) \cdot b^+ - \sin(t) \cdot b$), where $t \in [0, 2\pi]$ is a parameter. We will use the coherent state representation (Fock-Bargman representation): $b = \frac{d}{dx}$, $b^+ = z$. The eigevalue problem for the operator (44) takes the form:

$$\begin{bmatrix} \omega_0/2 & 0 & 1 \\ 0 & \hat{a} & 0 \\ 1 & 0 & \hat{c} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_1 \\ \psi_2 \end{bmatrix} = E \begin{bmatrix} \psi_1 \\ \psi_1 \\ \psi_2 \end{bmatrix},$$

(45)

where $\hat{a} = g \left( b(t)^2 + b^+(t)^2 \right) = 2g \left( \frac{d^2}{dx^2} + z^2 \right)$, $\hat{c} = \omega \cdot b^+(t) b(t) = \frac{\omega^2}{2} \cdot \left[ -\sin(2t) \frac{d^2}{dx^2} + 2 \cos(2t) z \frac{d}{dx} + \sin(2t) z^2 + \cos(2t) - 1 \right]$. Removing the function $\psi_1(z)$ from the system (45) we come to the fourth-order differential equation for the function $\psi_2(z)$:

$$L(t) \psi_2(z) \equiv \left[ \frac{1}{\omega} \cdot (\hat{a} + \hat{c}) \left( \hat{a} - \hat{c} \right) + \frac{2E}{\omega^2} \cdot \hat{c} + \frac{\omega^2}{4\omega^2} - \frac{E^2}{\omega^2} \right] \psi_2(z) = 0$$

(46)

Let us try to reduce eq. (46) to the algebraic problem using the finite-dimensional presentation of the operators $J^+_3, J^-_3$ (33). For this purpose, we shall present the operator $L(t)$ (46) as polynomial on the operators $J^+_3, J^-_3$. To this end, we use a gauge transformation $\phi(z)^{-1} L(t) \phi(z) = P(J^+_3, J^-_3) |_{x=\mu(z)}$, where $P(x_1, x_2)$ is a polynomial of the second degree in non-commutative variables $x_1$ and $x_2$. We restrict ourselves by a special case of $\phi(z)$ and $\mu(z)$: $\phi(z) = \phi_1(z) = \exp(\eta \cdot z^2)$ or $\phi(z) = \phi_2(z) = z \cdot \exp(\eta \cdot z^2)$, $\mu(z) = \xi \cdot z^2$. Thus, a search of a polynomial representation for $L(t)$ is reduced to a search of coefficients of polynomial $P(x_1, x_2)$ and constants $\eta, \xi, t$. The obtained solutions read:
Type I, \( s = \frac{1}{2}, \alpha = -\frac{1}{4} - \frac{N}{2} \):

\[
\mathbf{L}(t_0) e^{\eta z^2} = \frac{e^{\eta z^2}}{3} \left( 2 \left( J_3^+ \right)^2 + \left[ J_3^+, J_3^- \right] - 7J_3^+ + 4J_3^- + C_1 \right) \big|_{x=-\xi z^2} \tag{47}
\]

Type II, \( s = \frac{3}{2}, \alpha = \frac{5}{4} - \frac{N}{2} \):

\[
\mathbf{L}(-t_0) e^{-\eta z^2} = \frac{ze^{-\eta z^2}}{3} \left( 2 \left( J_3^- \right)^2 + \left[ J_3^+, J_3^- \right] - 7J_3^+ - 8J_3^- + C_2 \right) \big|_{x=\xi z^2} \tag{48}
\]

where \( t_0 = \frac{1}{4} \arctan \left( \frac{10\sqrt{2}}{23} \right), \frac{E}{\omega} = \frac{N+1}{\sqrt{2}}, \eta = \frac{\sqrt{2}}{8}, C_1 = 3\eta^2 - N^2 - \frac{N}{4} + \frac{1}{8}, C_2 = 3\frac{\omega^2}{4\omega^2} - N^2 + \frac{13N}{4} + \frac{40}{8}, \omega = \frac{1}{2\sqrt{6}}, \xi = \frac{3\sqrt{2}}{8} \). Two types of the solutions (47,48) have different symmetries with respect to the parity operator \( \Pi = \exp(i\pi b^+b) \) [17] ([\Pi, H] = 0).

Further, we will consider the solution (47) (type-I) in more details, the second solution (48) can be considered in a similar way. Taking into account the equality (47), the problem (46) can be reduced to a search of vectors \( \phi_k \) satisfying the matrix equation

\[
\left( 2 \left( j_3^- \right)^2 + \left[ j_3^+, j_3^- \right] - 7j_3^+ + 4j_3^- + C_1 \right) \phi_k = 0,
\]

where \( j_3^+, j_3^- \) are finite-dimensional representations of the operators \( J_3^+, J_3^- \) (33) in the subspace \( \mathcal{R}_N^3 \). For each of \( \phi_k \), taking into account (47), we have \( \mathbf{L}(t_0) e^{\eta z^2} \phi_k = 0 \) that is equivalent to (46) under the condition \( \psi_2 \equiv \psi_2^k = e^{\eta z^2} \phi_k \). The functions \( \psi_2^k \) obtained are entire ones and belong to the Fock space:

\[
|\psi^k_2(z)|^2 = O \left( |z|^p \exp \left( -\frac{\sqrt{7}}{4} (z^2 + \tilde{z}^2) \right) \right) \text{ (type I)},
\]

\[
|\psi^k_2(z)|^2 = O \left( |z|^p \exp \left( \frac{\sqrt{7}}{4} (z^2 + \tilde{z}^2) \right) \right) \text{ (type II)}.
\]

We list the corresponding functions explicitly for the special case \( N = 2 \).

**type I:**

\[
\psi_2 = e^{\eta z^2} \left( 1 F_1 \left[ \begin{array}{c} -5/4, -\xi z^2 \\ 1/2 \end{array} \right] \cdot \left( \frac{57}{49} + \frac{7\sqrt{7}}{15} \right) + 1 F_1 \left[ \begin{array}{c} -1/4, -\xi z^2 \\ 1/2 \end{array} \right] \cdot \left( \frac{5}{3} + \frac{\sqrt{7}}{10} \right) + 1 F_1 \left[ \begin{array}{c} 3/4, -\xi z^2 \\ 1/2 \end{array} \right] \right).
\]

\[
\omega_0 \over 2\omega = \sqrt{\frac{11}{12} + \frac{\sqrt{42}}{3}}.
\]

**type II:**

\[
\psi_2 = ze^{-\eta z^2} \left( 1 F_1 \left[ \begin{array}{c} 1/4, \xi z^2 \\ 3/2 \end{array} \right] \cdot \left( \frac{\sqrt{10}}{7} \right) + 1 F_1 \left[ \begin{array}{c} 5/4, \xi z^2 \\ 3/2 \end{array} \right] \cdot \left( \frac{\sqrt{10}}{42} \right) + 1 F_1 \left[ \begin{array}{c} 9/4, \xi z^2 \\ 3/2 \end{array} \right] \right).
\]

\[
\omega_0 \over 2\omega = \sqrt{\frac{\sqrt{10}}{3} - \frac{5}{12}}.
\]
where $\xi = \frac{3\sqrt{2}}{8}$, $\eta = \frac{\sqrt{2}}{8}$, $E = 3\sqrt{3} - \frac{1}{2}$. The second component $\psi_1$ of the spectral problem (45) can be obtained from the equation $\psi_1 = \frac{2}{\omega_0} \cdot (E + \hat{a} - \hat{c}) \psi_2$. In table 1 the values of relative frequencies $\frac{2\omega}{\omega_0}$ for both types I and II are given for different dimensions of the subspace $\mathcal{R}_N^3$. The obtained solutions (47, 48 and table 1) concern a non-resonant case $\frac{2\omega}{\omega_0} \neq 1$ and are related to the known isolated solutions (Juddian solution), found earlier in [17]. It is worth stressing that in [17] the eigenfunctions were constructed on the basis of elementary function, whereas the solutions (47, 48) are constructed on the basis of functions $\binom{\alpha+n}{s} x$. For the values of the parameters $s = 1/2$, $\alpha = -1/4 - N/2$ (type I), $s = 3/2$, $\alpha = 5/4 - N/2$ (type II) the hypergeometric function does not degenerate into polynomial ($\alpha \notin \mathbb{N}$).

Table 1. Values of parameters Hamiltonian (44,45), for different dim($\mathcal{R}_N^3$), $\frac{2}{\omega} = \frac{1}{2\sqrt{\omega_0}}$.

| dim($\mathcal{R}_N^3$) | 3     | 5     | 6     | 7     | 8     |
|------------------------|-------|-------|-------|-------|-------|
| I $\frac{2\omega}{\omega_0}$ | 0.44315 | 1.68889 | 3.03496 | 2.72766 | 2.10305 |
| |       |       |       | 3.60267 | 3.74421 | 3.90266 |
| II $\frac{2\omega}{\omega_0}$ | 0.79838 | 0.79838 | 2.23006 | 3.43545 | 2.66128 |
| |       |       |       | 2.75234 |        | 4.08801 |
| $\frac{E}{\omega}$ | 1.23205 | 2.38675 | 2.96410 | 3.54145 | 4.11880 |

IV. CONCLUSION

To the best of my knowledge, the only previously known example of QES related to special functions was found in [19], where these function appeared "by chance" for a particular problem connected with quartic Bose Hamiltonians. Now, we developed a systematic approach of QES-extension that enables us to generate new QES operators based on special functions. This extends considerably the family of QES systems and can find physical applications, one of which (two-photon Rabi Hamiltonian) was discussed in the present article. The main features of our approach include 1) the construction of the affine annihilator $\mathcal{K}(\mathcal{M}_2)$ [15]; 2) multiplication at the power function $x^n$. One can think that the approach suggested in the given work, will give rise to further essential expansion of classes of quasi exactly solvable models.
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VI. APPENDIX A.

The constructed operators $J_k^+, J_k^- (k = 1 \ldots 6)$, satisfy to the following commutation relations:

\[
[J_k^-, J_k^+] = S_k
\]  
(55)

\[
[J_k^+, S_k] = c_1^+ \cdot (J_k^-)^2 + c_3^+ \cdot J_k^+ J_k^- + c_4^+ \cdot S_k + c_5^+ \cdot J_k^+ + c_6^+ \cdot J_k^- + c_7^+
\]  
(56)

\[
[J_k^-, S_k] = c_1^- \cdot (J_k^-)^2 + c_2^- \cdot (J_k^+)^2 + c_5^- \cdot J_k^+ + c_6^- \cdot J_k^- + c_7^{-}
\]  
(57)

where $k = 1 \ldots 6$, $c_i^\pm (i = 1, \ldots, 7)$ are constants, theirs value are given in the table 2. The operators $S_k$ have the general structure $S_k = \beta_3 (x) \frac{d^3}{dx^3} + \beta_2 (x) \frac{d^2}{dx^2} + \beta_1 (x) \frac{d}{dx} + \beta_0 (x)$, $\beta_m (x) \in \mathcal{P}_n (m = 0, 1, 2, 3)$. However, we do not list them here explicitly since the corresponding expressions are rather cumbersome.

Table 2. Values of constants $c_i^\pm (i = 1, \ldots, 7)$ included in the commutation relations (55-57), $C_N = 2 + 2\alpha + s$, $\alpha_N = 2 + N + 2\alpha$, $\beta_N = -4 \cdot (1 + \nu^2 + 2N + \nu)$, $\gamma_N = (\alpha - N) \cdot (s + 1) \cdot C_N$, $\delta_N = -8 \cdot (2\alpha - N) \cdot \alpha_N$, $S_N = s \cdot \alpha \cdot (s - N - 2)$, $A_N = s + N + 2\alpha$, $B_N = (2N - s) \cdot (s - 2 - 2N)$, $D_N = (s + 1) \cdot (s - 2 - 2N)$, $G_N = (\alpha - N) \cdot (s + 1)$. 
\[\begin{array}{ccccccc}
  k = 1 & k = 2 & k = 3 & k = 4 & k = 5 & k = 6 \\
  c_1^k & 0 & 0 & 0 & 0 & 0 & -6 \\
  c_3^k & 2 & 2 & 2 & 0 & 2 & 0 \\
  c_2^k & 0 & 0 & 0 & 6 & 0 & 0 \\
  c_4^k & -4 & -4 & -4 & 0 & -4 & 0 \\
  c_5^k & -2 & -2 & -2 & 0 & -2 & 0 \\
  c_6^k & 2 & C_N & A_N & -2 & 0 & 0 \\
  c_7^k & 0 & 1 & 1 & 0 & 4 & 4 \\
  c_8^k & B_N & B_N & s - N & 0 & 1 - 4N^2 & 12 + 32\alpha \\
  c_9^k & -2 & -C_N & -A_N & 2 & 0 & 0 \\
  c_{18}^k & D_N & \gamma_N & S_N & 0 & 0 & \delta_N \\
  c_{19}^k & 0 & G_N & s \cdot \alpha & 0 & \beta_N & 0 \\
\end{array}\]

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