Massless Scalar Field Theory in a Quantised Space-Time

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ABSTRACT

A method is developed to construct a non-local massless scalar field theory in a flat quantised space-time generated by an operator algebra. Implicit in the operator algebra is a fundamental length scale of the space-time. The fundamental two-point function of free fields is constructed by assuming that the causal Green functions still have support on the light cone in the operator algebra quantised space-time. In contrast to previous stochastic approaches, the method introduced here requires no explicit averaging over spacetime coordinates. The two- and four-point functions of $g\varphi^4$ theory are calculated to the one-loop level, and no ultraviolet divergences are encountered. It is also demonstrated that there are no IR divergences in the processes considered.

Int. Class. for Physics: 0370, 1110, 04, 0460
1. Introduction

The existence of a minimum observable length in string theories [1] and the evidence suggesting the existence of fundamental length scales in quantum gravity [2] suggests that the construction of a quantum field theory (QFT) in space-times which possess a fundamental scale becomes an important question. Manifestly non-local actions for QFTs contain derivatives of infinite order and necessarily contain fundamental length scales while retaining the continuous nature of spacetime [3].

An alternative approach of the formulation of QFT in a (flat) quantised space-time was first attempted by Snyder [4]. This quantised space-time naturally introduces a fundamental length scale, which as well as describing the space-time itself, can also regulate UV divergences of the QFT. This latter aspect was the motivation for Snyder’s work.

The Snyder space-time is constructed essentially by adding non-commuting operators for the space-time coordinates to the usual Poincaré group. The spatial coordinate operators have spectra that are integer multiples of a fundamental length $a$, while the time coordinate operator admits a continuous spectrum. Momentum space remains continuous, with commuting momentum operators. These features result in a space-time structure violating the Born reciprocity principle, or the symmetry between configuration and momentum space representations of field theory.

This lack of reciprocity makes the construction of field theories difficult, due to reliance on this feature of most of the presently established formalisms. The common operations of analysis are either extremely cumbersome, having to be carried out in terms of summations (see Lee [5]), or cannot be carried out at all. Snyder [6] was able to formulate a version of the classical electromagnetic field in his quantised space-time but there is no evidence that this approach was effective for the case of a quantum field theory.

Gol’fand [7,8], advocated the construction of a quantum field theory in a momentum space of constant curvature. This is in some respects similar to the Snyder construction since the Snyder spacetime is generated by an operator algebra formulated in an abstract space of constant curvature. However, in Gol’fand’s theory, the
addition of momenta are non-commutative, resulting in changes to the laws of conservation of momentum and energy.

Attempts to restore symmetry between coordinate and momentum space, thus maintaining access to established methods of QFT are the basis of the work of Namsrai [8–13], Namsrai and Dineykhan [14], Dineykhan and Namsrai [15–18], and Dineykhan, Efimov and Namsrai [19]. These researchers use the concept of a stochastic space-time, in which coordinates are assumed to undergo quantum fluctuations. Explicit averaging over these stochastic coordinates removes the asymmetry between configuration and momentum space. This allows Namsrai [10] to make use of the existing formalism of field theory, with certain changes introduced by the averaging.

The combination of the space-time stochasticity and the averaging results in the theory having a manifestly non-local nature, as a result of the explicit dependence of the Green’s functions, or propagators, on derivatives of infinite order [3,20]. This approach allows the familiar construction of field theories in terms of an action principle with appropriately modified propagators, which have the important property of being sufficiently convergent that the resulting theory is ultraviolet finite. The averaging over the space-time, however, has the disadvantage of being rather ad hoc, and not contained within the dynamics of the theory.

In this paper a new method will be used to construct a massless scalar field in a quantised space-time generated by an operator algebra. Section 2 develops the algebra of the space-time, and it is shown that the Poincaré algebra remains an invariant subgroup.

The massless free scalar fields are considered in Section 3 leading to a modified free propagator, which depends on the fundamental scale of the space-time. The one-loop two-, and four-point functions are calculated in Section 4 for massless scalar $g\varphi^4$ in the quantised space-time. The free propagator is compared to the propagator of the Namsrai theory and found to have remarkably similar behaviour in the energy regime $p^2 < 1/a^2$. The large $p^2$ asymptotic behaviour, however, is quite different. This difference is perhaps not surprising since failures of the stochastic averaging at small scales where the algebra of the quantized spacetime becomes crucial will lead to distinct large $p^2$ behaviour.
2. Operator Algebra for a Quantised Space-Time

2.1. The Snyder Algebra

Here we present the algebra for a quantised space-time as it was developed by Snyder [4]. The algebra is formulated in an abstract five-dimensional space, which allows the construction of the Poincaré group, and position operators $x_\mu$.

The Snyder algebra is based on the invariance of the homogeneous quadratic form

$$-\eta^2 = \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 - \eta_4^2$$

(2.1)

in an abstract $\eta$-space which is known as a de Sitter space. It is known that the symmetry group of a de Sitter space is isomorphic to the Poincaré group in the limit of vanishing curvature [21]. The coordinate operators are defined over this space as

$$
\begin{align*}
  x &= ia \left( \eta_4 \frac{\partial}{\partial \eta_1} - \eta_1 \frac{\partial}{\partial \eta_4} \right) \\
  y &= ia \left( \eta_4 \frac{\partial}{\partial \eta_2} - \eta_2 \frac{\partial}{\partial \eta_4} \right) \\
  z &= ia \left( \eta_4 \frac{\partial}{\partial \eta_3} - \eta_3 \frac{\partial}{\partial \eta_4} \right) \\
  t &= i \frac{a}{c} \left( \eta_4 \frac{\partial}{\partial \eta_0} + \eta_0 \frac{\partial}{\partial \eta_4} \right)
\end{align*}
$$

(2.2)

where $a$ is the fundamental length introduced by this theory. Rotation operators, ie., angular momentum ($L$) and boosts ($M$), are defined analogously to the conventional ones:

$$
\begin{align*}
  L_x &= i\hbar \left( \eta_3 \frac{\partial}{\partial \eta_2} - \eta_2 \frac{\partial}{\partial \eta_3} \right) \\
  L_y &= i\hbar \left( \eta_1 \frac{\partial}{\partial \eta_3} - \eta_3 \frac{\partial}{\partial \eta_1} \right) \\
  L_z &= i\hbar \left( \eta_2 \frac{\partial}{\partial \eta_1} - \eta_1 \frac{\partial}{\partial \eta_2} \right) \\
  M_x &= i\hbar \left( \eta_0 \frac{\partial}{\partial \eta_1} + \eta_1 \frac{\partial}{\partial \eta_0} \right) \\
  M_y &= i\hbar \left( \eta_0 \frac{\partial}{\partial \eta_2} + \eta_2 \frac{\partial}{\partial \eta_0} \right) \\
  M_z &= i\hbar \left( \eta_0 \frac{\partial}{\partial \eta_3} + \eta_3 \frac{\partial}{\partial \eta_0} \right)
\end{align*}
$$

(2.3)
Each of these operators is a symmetry operation on the de Sitter space, which leaves the quadratic form (2.1) invariant. It can easily be seen that the coordinate operators do not commute. Their commutators are 
\[ [x, y] = (i a^2 / \hbar) L_z \]
and cyclic permutations, and those involving time obey 
\[ [t, x] = (i a^2 / \hbar c) M_x, \]
etc. (Hereafter we will put \( c = \hbar = 1 \).) It is also seen that in the limit \( a \to 0 \) all commutators return to those associated with conventional, non-quantised spacetime. This correspondence is regarded as essential in all following work.

In a covariant form, the algebra becomes

\[ [x^\mu, x^\nu] = i a^2 J^{\mu\nu} \]  
\[ [x^\mu, J^{\rho\sigma}] = i (g^{\mu\rho} x^\sigma - g^{\mu\sigma} x^\rho) \]  
\[ [J_{\mu\nu}, J_{\rho\sigma}] = i (g_{\nu\rho} J_{\mu\sigma} - g_{\mu\rho} J_{\nu\sigma} + g_{\mu\sigma} J_{\nu\rho} - g_{\nu\sigma} J_{\mu\rho}) \]

where \( J_{\mu\nu} = -J_{\nu\mu} \) \( (L_x = J_{23}, \ L_y = J_{31}, \ M_x = J_{10} \ etc) \) satisfy the usual algebra of Lorentz transformations. The algebra defined by (2.4) is actually the de Sitter group which has the following two casimir invariant operators

\[ C_1 = x^2 - \frac{1}{2} a^2 J_{\mu\nu} J^{\mu\nu} \]  
\[ C_2 = \sum_{i=0}^{4} W_i W_i \ ; \ W_i = \epsilon_{ijklm} J^{jk} J^{lm} \]

where \( J_{kl} = -J_{lk} \) is defined by

\[ J_{kl} \equiv a J_{\mu\nu} \ ; \ \mu = k, \ \nu = l, \ k, l = 0, 1, 2, 3 \]  
\[ J_{4i} \equiv x_\mu \ ; \ \mu = i = 0, 1, 2, 3 \]

In addition to the ten operators defined above, to take the place of momentum operators we define

\[ p_\mu = \frac{1}{a} \left( \begin{array}{c} \eta_\mu \\ \eta_4 \end{array} \right) \]

which have the required property of being translation operators. It is worth noting that with these definitions we will have \([p_\mu, p_\nu] = 0\). Also, it can easily be verified that we can write \( L_x = yp_z - zp_y \), etc., in the standard way. This constitutes the complete set of objects in the Snyder algebra.
We note from equation (2.2) that the Snyder coordinate operators are very similar in structure to the usual angular momentum operators of quantum mechanics. This demonstrates the quantised nature of the spectra of these operators. The time operator, however, can be shown to possess a continuous spectrum although since the time and space operators do not commute simultaneous space and time eigenstates cannot exist. This latter point will be discussed further in the following sections.

2.2. Covariant Form of Snyder Algebra

With the inclusion of momentum operators, the Snyder algebra can be easily written in covariant form by working out the commutator \([x, p_x]\), where \(x\) and \(p_x\) are as given in section one, along with its companions:

\[
[x, p_x] = i \left( 1 + a^2 p_x^2 \right),
\]

\[
[t, p_t] = i \left( 1 - a^2 p_t^2 \right),
\]

\[
[x, p_y] = [y, p_x] = ia^2 p_x p_y,
\]

\[
[x, p_t] = [p_x, t] = ia^2 p_x p_t,
\]

etc.

It is easy to see that we can write the entire set of commutators (2.8) in the compact form

\[
[x_\mu, p^{\nu}] = i \left( \delta_\mu^\nu - a^2 p_\mu p^{\nu} \right).
\] (2.9)

With the set of commutators written in covariant form we are now in a position to write down by inspection a form for \(x_\mu\) which leads to (2.9). This is

\[
\tilde{x}_\mu = i \left( \partial_\mu - a^2 p_\mu (p^\rho \partial_\rho) \right)
\] (2.10)

where

\[
\partial_\mu \equiv \frac{\partial}{\partial p^\mu}
\]

and \(\tilde{x}_\mu\) is used to denote the momentum space representation of the coordinate operator. Unless otherwise stated, all differentiations will be with respect to momentum variables.

To determine the generator \(J_{\mu\nu}\) consider

\[
[x_\mu, x^{\nu}] = ia^2 J_{\mu}^{\nu}.
\] (2.11)
A simple computation using (2.10) gives

\[ [x_\mu, x^\nu] = i a^2 (p_\mu x^\nu - p^\nu x_\mu) \] (2.12)

which is exactly the desired form provided that \( p_\mu x^\nu - p^\nu x_\mu \), which has the operators in reverse order from what is usually written down, is the correct representation for \( J_{\mu \nu} \) in momentum space. That this form is equivalent to the more usual representation can be shown by using equation (2.9). We prefer the form given above since it places the derivative operators, contained in \( x_\mu \), to the right of the momentum variables.

### 2.3. Correspondence of the Snyder Algebra and Poincaré Group

Straightforward manipulation will serve to verify the following:

\[
[p_\mu, J_{\rho\sigma}] = i(g_{\mu\rho}p_\sigma - g_{\mu\sigma}p_\rho) 
\] (2.13a)

\[
[p^2, J_{\rho\sigma}] = 0
\] (2.13b)

\[
[J_{\mu \nu}, J_{\rho \sigma}] = i(g_{\nu \rho}J_{\mu \sigma} - g_{\mu \rho}J_{\nu \sigma} + g_{\mu \sigma}J_{\nu \rho} - g_{\nu \sigma}J_{\mu \rho})
\] (2.13c)

showing that the algebra in momentum space is completely consistent with that of the usual Poincaré group. The Casimir invariant operators of the Poincaré group are maintained as shown by virtue of equation (2.13b) and it is easy to see that \( W_\mu W^\mu \), where \( W_\mu = -\frac{1}{2}\epsilon_{\mu \nu \rho \sigma}J^{\nu \rho}p^\sigma \) is the Pauli–Lubanski pseudovector, must remain the second Casimir invariant. We are therefore free to characterise particles in terms of mass and spin, just as in standard field theory. We also have

\[
[x_\mu, J_{\rho \sigma}] = i(g_{\mu \rho}x_\sigma - g_{\mu \sigma}x_\rho)
\] (2.14a)

\[
[x^2, J_{\rho \sigma}] = 0
\] (2.14b)

The Lorentz invariant nature of the operator-based space-time is made apparent in (2.14b) since the light-cone operator \( x^2 \) commutes with the generators of Lorentz transformations.
3. Massless Scalar Field in Snyder Space-Time

3.1. The Geodesic Ansatz and Green’s Function

The usual operations of (coordinate space) analysis are not available in quantised spacetime since the spectra of the position operators is discrete and also cannot be simultaneously diagonalized because of the non-commuting nature of these operators. Thus it is necessary to formulate field theory in momentum space where the momenta retain a continuous spectrum. As a first step it is necessary to construct the propagator (two-point function) in momentum space using the momentum space representation of the coordinate operators.

To determine the propagator in the quantised spacetime, first consider the fundamental Green functions for massless (free) scalar fields in ordinary quantum field theory. The different Green’s functions (retarded, advanced, Feynman, etc.) for a massless scalar field can all be represented by the formula

\[
G(x, x') = \frac{1}{(2\pi)^n} \int \! d^n p \, e^{-ip(x-x')} \tilde{G}(p) \tag{3.1}
\]

where \(\tilde{G}(p) = 1/p^2\) plus the prescription made for the avoidance of the singularity in the complex momentum plane (see Birrell and Davies [23]). Carrying out the integration explicitly for the retarded and advanced Green’s functions, we obtain in 4-D

\[
G_{\text{retadv}}(x) = \frac{1}{2\pi} \theta(\pm x_0) \delta(x^2) \tag{3.2}
\]

where \(\theta(x)\) is the unit step function. Thus support is concentrated on the forward light-cone in the case of the retarded Green’s function, and concentrated on the backward light-cone for the advanced Green’s function. We can thus see that the product of the light cone geodesic with the retarded or advanced Green’s functions (in 4-D) vanishes,

\[
x^2 G_{\text{retadv}}(x) = 0. \tag{3.3}
\]

To extend this idea to the quantised space-time, consider the state \(|G\rangle\) representing the Green functions. We then begin with a (complete) set of states \(|x\rangle \equiv |\lambda_2, \lambda_1, \lambda_2\rangle\) which are eigenstates of the set of commuting operators \(x_3\) (i.e. one of the four coordinate operators) and the two casimir invariants \(C_1\) and \(C_2\) from (2.5). In particular,
\[ \hat{x}_3 |x\rangle = \lambda_z |x\rangle \]

where \( \hat{x}_3 \) denotes the operator and \( \lambda_z \) denotes the eigenvalue. Then it is evident that \( \langle x|G \rangle \) is a c-number function of the eigenvalues \( \lambda_z, \lambda_1, \lambda_2 \). The ansatz is then made that the expectation value of the light cone operator with the Green’s functions vanishes in the quantised space-time. With this ansatz, we have the fundamental constraint \( \langle x|\hat{x}^2G \rangle = 0 \), leading to

\[
\langle x|\hat{x}^2G \rangle = \int \frac{d^4p}{f(p)} \langle x|p \rangle \langle p|\hat{x}^2G \rangle
\]

\[
= \int \frac{d^4p}{f(p)} \langle x|p \rangle \hat{x}^2 \langle p|G \rangle = \int \frac{d^4p}{f(p)} \langle x|p \rangle \tilde{x}^2 G(p) = 0
\]

where \( \tilde{x}^2 \) represents the Lorentz invariant light-cone operator, \( \hat{x}^2 \) is the momentum-space representation of this operator, and the momentum space Green function \( \tilde{G}(p) = \langle p|G \rangle \) is a function of the continuous momentum eigenvalues \( p_\mu \).

The function \( f(p) \) is a measure required for symmetries of (2.1) and is also necessary for the operator \( \tilde{x} \) to be Hermitian in the space of functions of momentum \([4]\).

To determine the measure, consider the following analysis.

\[
\langle \varphi|x_\mu|\Psi \rangle = \int \frac{d^4p}{f(p)} \varphi^*(p) \tilde{x}_\mu \Psi(p)
\]

\[
= \int \frac{d^4p}{f(p)} \varphi^* i(\partial_\mu - a^2 p_\mu p \cdot \partial) \Psi
\]

\[
= \int \frac{d^4p}{f(p)} (\tilde{x}_\mu \Phi)^* \Psi - i \int d^4p \varphi^* \left( \partial_\mu \frac{1}{f(p)} - \partial_\rho \left( \frac{a^2 p_\mu p_\rho}{f(p)} \right) \right) \Psi
\]

The operator will be Hermitian if \( f(p) \) can be found such that the second term in the last line above vanishes. Thus we obtain a differential equation for \( f(p) \),

\[
i \left( \partial_\mu - a^2 p_\mu (p \cdot \partial - 5) \right) f(p) = 0.
\]

with solution

\[
f(p) = (1 - a^2 p^2)^{5/2}
\]

This result is notably identical to the volume element of the curved momentum space considered by Gol’fand \([4]\).

Before solving the defining equation of free Green’s functions \( \tilde{x}^2 \tilde{G}(p) = 0 \) in the quantised space-time, it helps to further develop the geodesic analogy between quantised and standard space-time. Consider the product \( x^2 G(x) \), where \( G(x) \) is the
general Green’s function defined in equation (3.1). In a manner exactly analogous to (3.4) we take $x_\mu = i \frac{\partial}{\partial p^\mu}$ as an operator and with simple manipulation we can show that

$$x^2 G(x) = \frac{-1}{(2\pi)^n} \int d^n p e^{-ipx} \Box \tilde{G}(p)$$

and further, using the definition of the general propagator, we can easily see that

$$\Box \tilde{G}(p) = \Box \frac{1}{p^2} = \frac{8 - 2n}{(p^2)^2} = 0, \quad p^2 \neq 0$$

which vanishes when the number of dimensions $n$ is equal to 4. Therefore, $1/p^2$ is a solution of the differential equation $\Box \tilde{G} = 0$, in four dimensions. Thus in standard massless scalar quantum field theory we can write

$$\int \frac{d^4 p}{(2\pi)^4} e^{ipx} x^2 \tilde{G}(p) = 0,$$

where $x^2$ is treated as an operator.

Therefore, in quantised space-time, the equation of free Green’s functions is obtained from (3.4) as

$$\int \frac{d^4 p}{(2\pi)^4} \frac{\langle x | p \rangle}{(1 - a^2 p^2)^{5/2}} x^2 \tilde{G}(p) = 0.$$  \hfill (3.11)

Note for clarity that $\tilde{x}^2$ is not acting on the measure, since it is considered to multiply the operator from the left. Thus $\tilde{x}^2 \tilde{G}(p) = 0$ in momentum space. Writing the coordinate operators explicitly then gives the differential equation

$$i(\partial_\mu - a^2 p_\mu \partial_\rho) i(\partial^\mu - a^2 p^\mu \partial^\sigma) \tilde{G}(p) = 0$$

$$i(\partial_\mu - a^2 p_\mu \partial_\rho) \tilde{G}(p) = 0. \quad (3.12)$$

From symmetry considerations we expect that $\tilde{G}$ will be a function of $p^2$. The partial differential equation is then converted to the ordinary equation

$$a^2 \left(4y(1 - y) \frac{d^2}{dy^2} - 2(3y - 4) \frac{d}{dy} \right) \tilde{G}(y) = 0$$

where $y = a^2 p^2$. This equation can be solved for all real values of $y$ subject to the conditions that $\tilde{G}(y) \to 0$ as $y \to \pm \infty$, $y\tilde{G}(y) \to a^2$ as $y \to 0$. This latter condition ensures that $\tilde{G}(p^2) \to 1/p^2$ as $a \to 0$, corresponding to the usual propagator for massless scalar fields. If $y < 0$ the solution to (3.13) subject to the above conditions is given by

$$\tilde{G}(p) = \frac{\sqrt{1 - a^2 p^2}}{p^2} - a^2 \coth^{-1} \left( \frac{\sqrt{1 - a^2 p^2}}{a} \right).$$

$$\tilde{G}(p) = \frac{\sqrt{1 - a^2 p^2}}{p^2} - a^2 \coth^{-1} \left( \frac{\sqrt{1 - a^2 p^2}}{a} \right). \quad (3.14)$$
We obtain a momentum-space representation of this propagator by continuing this solution to complex momentum, utilizing standard prescriptions for avoiding singularities in the momentum plane:

\[
\begin{align*}
\text{Feynman} & : p^2 \to p^2 + i\epsilon; \\
\text{retarded} & : p^2 \to (p_0 + i\epsilon)^2 - p^2; \\
\text{advanced} & : p^2 \to (p_0 - i\epsilon)^2 - p^2.
\end{align*}
\] (3.15)

Subject to these restrictions, we obtain the Fourier representation

\[
G(x) = \int \frac{d^4p}{(2\pi)^4} \frac{\langle x|p \rangle}{(1 - a^2p^2)^{5/2}} \left( \frac{\sqrt{1 - a^2p^2}}{p^2} - a^2 \coth^{-1}\left( \frac{\sqrt{1 - a^2p^2}}{p} \right) \right)
\] (3.16)

which yields the usual scalar field Feynman propagator

\[
G_F(p) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ipx}}{p^2}
\] (3.17)

in the \(a \to 0\) limit. To understand this, note that the transformation Kernel \(\langle x|p \rangle\) must satisfy relations following from the property of the state \(|x\rangle\) as eigenstates of the operators \(\hat{x}_3\), \(C_1\) and \(C_2\).

\[
\begin{align*}
\langle x|\hat{x}_3|p \rangle &= \lambda_2 \langle x|p \rangle = -\left( \partial_{p_z} - a^2 p_z \cdot \partial \right) \langle x|p \rangle \quad (3.18a) \\
\langle x|C_1|p \rangle &= \lambda_1 \langle x|p \rangle \quad (3.18b) \\
\langle x|C_2|p \rangle &= \lambda_2 \langle x|p \rangle \quad (3.18c)
\end{align*}
\]

Then when \(a \to 0\), the casimirs become \(C_1 \to \hat{x}^2\), \(C_2 \to 0\) and \(\hat{x}_\mu \to \partial/\partial p_\mu\) which gives \(\langle x|p \rangle \to e^{-ipx}\) in the \(a \to 0\) limit.

As usual the free Green function is associated with a time ordered product:

\[
\langle 0|T(\varphi(x_1)\varphi(x_2))|0 \rangle = iG(x_1 - x_2)
\] (3.19)

with \(G(x)\) given by (3.16), with the time ordered product implying that the Feynman prescription for the avoidance of singularities in the complex momentum plane is to be employed. In this manner we have arrived at a definition of the propagator of our theory, through an ansatz which involves the geodesic equation of the light cone. An alternative propagator, based on the solution of (3.13) for all real \(y\), is discussed in the appendix.

It is important that the propagator has analyticity properties consistent with causality, and a singularity and branch cut structure which does not prevent one
from making the usual Wick rotation into Euclidean space. Also, since the integral measure is closely associated with the propagator, it must be included in this analysis.

It is evident that the integrand of (3.16) with \( p^2 \to p^2 + i\epsilon \) has singularities in the complex \( p_0 \) plane located at

\[
p_0 = \pm(|\vec{p}| - i\epsilon) \quad p_0 = \pm(\sqrt{p^2 + 1/a^2} - i\epsilon)
\]

(3.20)

The second term of (3.16) also has branch cuts from both the integration measure and the inverse hyperbolic function. The integration-measure branch cut occurs when \([1 - a^2(p^2 + i\epsilon)]\) is negative and real, corresponding to two branch cuts in the complex \( p_0 \) plane.

\[
Im(p_0) = -i\epsilon, \quad Re(p_0) > \sqrt{p^2 + 1/a^2}
\]

(3.21a)

\[
Im(p_0) = +i\epsilon, \quad Re(p_0) < -\sqrt{p^2 + 1/a^2}
\]

(3.21b)

The branch cut in the inverse hyperbolic cotangent begins when its argument is real and less than one, corresponding to positive real values of \( y = a^2(p^2 + i\epsilon) \). In the complex \( p_0 \) plane, this cut adds the following two segments to the branch cuts of (3.21):

\[
Im(p_0) = -i\epsilon, \quad |\vec{p}| < Re(p_0) < \sqrt{p^2 + 1/a^2}
\]

(3.22a)

\[
Im(p_0) = +i\epsilon, \quad -\sqrt{p^2 + 1/a^2} < Re(p_0) < -|\vec{p}|
\]

(3.22b)

As is evident from the above analysis, integration along the real \( p_0 \) axis can be performed along the Wick-rotated contour without encountering any singularities, consistent with the analyticity (and causality) properties of the Feynman propagator expected from standard quantum field theory.

We reiterate that the first term in (3.16) contains no branch cuts of its own, in which case the branch structure of the entire propagator is dictated by the second term. In the complex \( p^2 \) plane, the integration-measure branch cut begins at \( p^2 = 1/a^2 \) and extends to \(+\infty\), indicative of particle creation at energies sufficiently large to probe the structure of the quantised spacetime. The branch segment (3.22), which covers all positive values of \( p^2 \) below \( 1/a^2 \), is proportional to \( a^2 \) and must be considered to be a weak effect of the quantised spacetime occurring at all energy scales in the theory.
3.2. The Perturbation Expansion

To construct the perturbative expansion of field theory in quantised space-time, we first define the extended fields \( \varphi_e \) as superpositions of extended plane wave solutions \[10\]

\[
G(\hat{p})^{-1/2} \langle p|x \rangle \quad (3.23)
\]

where \( G \) is given by (3.16), and \( \hat{p} \) is formally the momentum operator in coordinate space. One can then write down a Lagrangian density in terms of these extended fields in the usual way

\[
\mathcal{L} = \frac{1}{2} \varphi_e G(\hat{p})^{-1} \varphi_e + \mathcal{L}_{\text{int}}(\varphi_e).
\]

Since commutators of field operators remain c-numbers, and time- and normal-ordering remain well-defined since time and momentum space are continuous, the S-matrix can be written in terms of these extended field objects as

\[
S = : \exp \left\{ \sum_z M(z) \varphi_{\text{in}}(z) G(z)^{-1} \frac{\delta}{\delta J(z)} \right\} : Z[J] \Big|_{J=0} \quad (3.24)
\]

where \( Z \) is the generating functional of Green functions, \( J \) is the usual Schwinger field source and \( M(z) \) is a possible coordinate space measure, analogous to the momentum space measure discussed previously. The sum/integral over the eigenvalues \( \lambda_z, \lambda_1, \lambda_2 \) of the states \( |x\rangle \) are heuristically denoted by \( \sum_z \). Thus standard perturbation theory is valid in terms of the extended fields \( \varphi_e \).

Calculations in field theory are generally carried out in momentum space due to the difficulty of working in coordinate space. Here we are lacking any definite method of computation in coordinate space and rely completely on the momentum space formulation of the theory. The formal transformation of the S-matrix elements into momentum space is straightforward, provided that we have

\[
\sum_x M(x) \langle p|x \rangle \langle x|q \rangle = \delta(p - q) \quad (3.25a)
\]

\[
\langle x - y|p \rangle = \langle x|p \rangle \langle p|y \rangle. \quad (3.25b)
\]

Equation (3.25b) is easy to show from the defining equation (3.18a) provided we make the natural identification of the eigenvalue \( \lambda_x - \lambda_y \) with the eigenvalue \( \lambda_{x-y} \). However,
(3.25a) is more problematic. Lacking definite solutions of (3.18), except in the $a \to 0$ limit, we will assume that this condition is satisfied for $a \neq 0$. This amounts to the assumption of conservation of momentum.

4. One-Loop Calculations

It is now possible to formulate the one-loop processes in $g\varphi^4$ theory. Using (3.24) we have the two point function as

$$
\langle O| T(\varphi_e(x)\varphi_e(0))|O \rangle = D(x) = \langle x|D \rangle
$$

$$
= g \sum_y M(y)\langle O| T(\varphi_e(x)\varphi_e(y))|O \rangle \langle O|\varphi_e(y)\varphi_e(y)|O \rangle \langle O| T(\varphi_e(y)\varphi_e(0))|O \rangle
$$

$$
= g \sum_y M(y)G(x-y)G(0)G(y)
$$

then

$$
D(p) = \langle p|D \rangle = \sum_x M(x)\langle p|x \rangle \langle x|D \rangle
$$

$$
= gG(0) \sum_x M(x)\langle x|p \rangle \sum_y M(y)G(y-x)G(y)
$$

$$
= gG(0) \sum_x M(x)\langle x|p \rangle \sum_y M(y) \int \frac{d^4l}{M(l)} \int \frac{d^4s}{M(s)} \langle l|y-x \rangle \langle s|y \rangle \tilde{G}(l)\tilde{G}(s)
$$

$$
= gG(0) \sum_x \sum_y \int \frac{d^4l}{f(l)} \int \frac{d^4s}{f(s)} M(x)\langle l|x \rangle \langle x|p \rangle M(y)\langle l|y \rangle \langle y|s \rangle \tilde{G}(l)\tilde{G}(s)
$$

$$
= g \frac{\tilde{G}(p)}{f(p)} G(0) \frac{\tilde{G}(p)}{f(p)} = \tilde{G}(p) \frac{G(p)}{f(p)} g \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k) \frac{\tilde{G}(p)}{f(p)}
$$

where we have used (3.25). The external propagators and their associated measures are, of course, removed to give the truncated Green function. The four point function can be handled the same way.
4.1. Particle Self-Energy to One Loop in $g \phi^4$

The application of the Feynman rules to the vacuum diagram [Fig. 4.1] gives us the following integral:

$$g \int \frac{d^4p}{(2\pi)^4} \frac{1}{(1 - i\epsilon - a^2p^2)^{5/2}} \left( -\frac{\sqrt{1 - i\epsilon - a^2p^2}}{p^2 + i\epsilon} + a^2 \coth^{-1}\left(\sqrt{1 - i\epsilon - a^2p^2}\right) \right).$$

(4.3)

The result for the first order contribution to the self-energy is†

$$-i\Sigma(0) = G(0) = \frac{ig}{24\pi^2a^2} [1 + 2 \log 2]$$

(4.4)

Note that $\lim_{a \to 0} G(0)$ is unbounded, as these integrals normally are in field theory without regularisation or renormalisation. The quantised space-time thus regulates the UV divergences of the two-point function.

To determine how the self energy shifts the poles in the propagator away from the bare values, we write the complete propagator $G_c$ as

$$iG_c(p) = iG(p) \left( \frac{1}{1 + i\Sigma(p)iG(p)} \right)$$

(4.5)

Where $G(p)$ is the momentum space propagator of the present theory. Putting in the complete expression for $G(p)$ gives

$$iG_c(p) = \frac{i \left( -\sqrt{1 - a^2p^2} + a^2p^2 \coth^{-1}\left(\sqrt{1 - a^2p^2}\right) \right)}{p^2(1 - a^2p^2)^{5/2} - \Sigma(p) \left( -\sqrt{1 - a^2p^2} + a^2p^2 \coth^{-1}\left(\sqrt{1 - a^2p^2}\right) \right)}$$

(4.6)

† MAPLE was found useful for several computations. Symbolic Computation Group, MAPLE (a computer program) version V, Computer Science Dept., University of Waterloo, Waterloo, Ontario, 1990
Figure 4.2: Four-point diagram.

Nothing essential has been altered. The points \( p^2 = 0 \) and \( p^2 = 1/a^2 \) remain branch points, and thus the same resonances exist here as in the bare propagator. The physics is not altered in any essential way.

This is analogous to standard \( g\phi^4 \) field theory in dimensional regularisation, where no shifting of the mass shell for a massless particle occurs to one-loop because the massless tadpoles are zero. In this case, however, rather than dealing with a mass shell, we have a resonance resulting from the quantised space-time.

The vacuum diagram considered here is known to be the highest-degree primitive divergence in \( g\phi^4 \) field theory. Since we have shown that this theory renders this diagram finite, we can thus argue that all diagrams in \( g\phi^4 \) are rendered ultraviolet finite by this approach, thus making the entire theory ultraviolet finite. The asymptotic behaviour of the propagator in equation (3.16) is \( \sim 1/p^2 \). This takes into account the four powers of the integration variable in the numerator as well as the integral measure. We thus have, in the worst case, quadratic ultraviolet convergence.

The infrared behaviour is also convergent, for this self-energy diagram, since a sufficient number of powers of the loop momentum in the numerator force the inverse hyperbolic cotangent and the \( 1/p^2 \) behaviour of the first term to zero as \( p^2 \to 0 \).

4.2. The Four-Point Function

Application of the Feynman rules to the diagram of Fig. 4.2, which represents one of three possible channels for this process, results in the integral

\[
\frac{g^2}{(2\pi)^4} \int \frac{d^4q}{(1 - a^2q^2)^{5/2} (1 - a^2(p - q)^2)^{5/2}} \left( \frac{\sqrt{1 - a^2q^2}}{q^2} - a^2 \coth^{-1} \left( \sqrt{1 - a^2q^2} \right) \right)
\]
each of the channels is calculated through a similar integration.

Regarding the IR finiteness of the integrals over the Feynman parameters for \(D\) regularisation is simply an intermediate step in this calculation. This procedure is carried out for each of \(I_1, I_2, I_3, \text{ and } I_4\) and the integral obtained by referring to existing compilations of such integrals \[24,25\].

Expansion of \((4.7)\) using \(\text{coth}^{-1}(x) = x \int_0^1 \frac{du}{x^2-u^2}\) gives us the four integrals

\[
I_1 = \frac{g^2 \mu^8}{(2\pi)^4} \int^1_0 \frac{d^4q}{q^2 (\mu^2 - q^2)^2 (p-q)^2 (\mu^2 - (p-q)^2)^2}
\]

\[
I_2 = \frac{-g^2 \mu^8}{(2\pi)^4} \int^1_0 \frac{du}{u^4} \int^1_0 \frac{d^4q}{q^2 (\mu^2 - q^2)^2 (p-q)^2 (\mu^2 - (p-q)^2)^2 (\sigma^2 - (p-q)^2)}
\]

\[
I_3 = \frac{-g^2 \mu^8}{(2\pi)^4} \int^1_0 \frac{du}{u^4} \int^1_0 \frac{dv}{v^4} \int^1_0 \frac{d^4q}{(p-q)^2 (\mu^2 - (p-q)^2)^2 (\mu^2 - q^2)^2 (\sigma^2 - q^2)}
\]

\[
I_4 = \frac{g^2 \mu^8}{(2\pi)^4} \int^1_0 \frac{du}{u^4} \int^1_0 \frac{dv}{v^4} \int^1_0 \frac{d^4q}{(\mu^2 - q^2)^2 (\mu^2 - (p-q)^2)^2 (\sigma^2 - q^2) (\xi^2 - (p-q)^2)}
\]

where we have put \(\mu^2 = a^{-2}\), \(\sigma^2 = \mu^2(1-u^2)\) and \(\xi^2 = \mu^2(1-v^2)\). It is obvious that \(I_3 = I_2\) with change of variable. Integrals \(I_1, I_2, \text{ and } I_4\) can be computed through decomposition into partial fractions, for example

\[
\frac{1}{q^2(\mu^2 - q^2)} = \frac{1}{\mu^4 q^2} + \frac{1}{\mu^4 (\mu^2 - q^2)} + \frac{1}{\mu^2 (\mu^2 - q^2)},
\]

thus for \(I_4\) we obtain nine integrals of the form

\[
\frac{g^2}{(2\pi)^4} \int^1_0 \frac{du}{u^4} \int^1_0 \frac{dv}{v^4} \int^1_0 \frac{d^4q}{(\mu^2 - q^2)(\mu^2 - (p-q)^2)}
\]

each of which is rewritten in a dimensionally regularised form, i.e., for \((4.10)\)

\[
\frac{g^2}{v^{2\epsilon}} \int^1_0 \frac{du}{u^4} \int^1_0 \frac{dv}{v^4} \int\frac{d^{D}q}{(2\pi)^{D} (q^2 - \mu^2 + i\eta)((p-q)^2 - \mu^2 + i\eta)}
\]

and the integral obtained by referring to existing compilations of such integrals \[24,25\]. This procedure is carried out for each of \(I_1, I_2, \text{ and } I_4\). Dimensional regularisation is simply an intermediate step in this calculation.

It is found that for each integral the divergent parts of the sub-integrals vanish for \(D \rightarrow 4\), as one would expect for a finite integral. A certain question remains regarding the IR finiteness of the integrals over the Feynman parameters \(u\) and \(v\) in the case of \(I_2\) (and \(I_3\) and \(I_4\)). One can see that in the example, \((4.11)\), these integrals are divergent. The final result for the four-point function is sufficiently complex that
it precludes a direct analysis. We can, however, expand each of the nine integrals of $I_2$ and $I_4$ in a power series in $u$ and $v$ to determine the behaviour as $u$ and $v$ approach zero. For each integral the divergent terms in the series completely vanish. We can therefore conclude that the behaviour of the four point function near $u = v = 0$ is finite, and therefore there are no IR divergences.

4.3. Comparison of Quantised-Space Propagators

In this section we will compare the propagator of the present theory, obtained in section 3.2, to that obtained in the related theory of Namsrai [9]. We wish to note the dissimilarity in the convergence behaviour of each of these propagators as a function of increasing (Euclidean) momentum.

From Namsrai (op cit) we have the massless free scalar propagator of the stochastic theory as

$$N(p) = \frac{1}{p^2 \cosh^2 \left( a \sqrt{-p^2} \right)}$$  \hspace{1cm} (4.12)

In the energy regime $p^2 < 1/a^2$, the two propagators are very nearly identical, however their asymptotic behaviour is quite different. In the case of (4.12) we have exponential suppression, whereas for (3.16) there is power-law decay. This indicates that the stochastic and non-stochastic formulations are distinct.

5. Conclusions

A quantum scalar field is constructed in a space-time generated by an operator algebra originally proposed by Snyder [4]. This algebra introduces a fundamental length scale, leading to quantised spatial and continuous temporal coordinates. The Snyder algebra contains the Poincaré group with its Casimir invariants, allowing particles to be described in terms of spin and mass.

The retarded and advanced Green’s functions in standard scalar quantum field theory have support on the light cone in the massless case. This is used to construct Green’s functions for the massless scalar field in the quantised space-time directly in momentum space, through the momentum space ansatz that an exactly analogous equation holds in the quantised space-time. The measure that must be introduced
into the theory from the symmetries of the space-time algebra not only ensures the space-time coordinate operators are Hermitian, but also corresponds to the volume element in the constant-curvature momentum space considered by Gol’fand.

We find that the propagator has a branch structure indicating the presence of resonance thresholds. There are two main effects. The first results in the creation of particles of mass \( m = 1/a \) at energies sufficient to probe the structure of the quantised space-time. The other is a much weaker threshold at \( p^2 = 0 \), proportional to \( a^2 \), which is an effect due to the quantised space-time throughout the energy range of the theory.

The self-energy and the four-point function in \( g\phi^4 \) were calculated to one-loop and shown to be ultraviolet finite, demonstrating that the fundamental length scale of the space-time algebra regulates ultraviolet divergences. These one-loop processes were also found to be infrared finite.

Throughout the calculations undertaken, it is emphasised that in the limit as the fundamental length of the space-time vanishes, all results make sense in terms of standard quantum field theory. In particular, the self-energy and four point function are singular as \( a \to 0 \) and the free propagator (3.16) reduces to its continuous space-time counterpart.

Also worthy of note is the position-momentum uncertainty relation derivable from equation (2.9). So-called generalised uncertainty relations have been of recent interest in superstring theory and quantum gravity \(^{[2,26]}\) and seem to arise naturally from these theories. It is possible that quantised space-times and more general uncertainty relations imply one another.

It is hoped that this non-stochastic approach to massless QFT in a quantised space-time can be extended to massive fields in future research.

Acknowledgements

We are grateful for the financial support of the Natural Sciences and Engineering Research Council of Canada.
Appendix

Alternate Solution for the Propagator

There exists a second solution to the propagator equation, \((3.16)\). In the course of solving \((3.13)\) we obtain the solution in the form of the integral

\[
\tilde{G}(y) = C \int dy \frac{\sqrt{|1-y|}}{y^2},
\]  

(5.1) where \(y = a^2p^2\). This equation can be solved for all real values of \(y\), subject to the conditions stated prior to \((3.14)\), as well as the further assumption that purely \(1/p^2\) behaviour occurs in the immediate neighbourhood of \(p^2 = 0\). (\(1/p^2 + \text{constant}\) is in principle compatible with the condition \(y\tilde{G}(y) \to a^2\) as \(y \to 0^+\)). The integrand of the resulting solution

\[
G(x) = \int \frac{d^4p \langle x|p \rangle}{(2\pi)^4(1-a^2p^2)^{5/2}} \begin{cases} 
\frac{\sqrt{1-a^2p^2}}{p^2} - a^2 \coth^{-1} \left( \frac{\sqrt{1-a^2p^2}}{a^2} \right) & p^2 < 0 \\
\frac{\sqrt{1-a^2p^2}}{p^2} - a^2 \tanh^{-1} \left( \frac{\sqrt{1-a^2p^2}}{a^2} \right) & 0 \leq a^2p^2 < 1 \\
0 & a^2p^2 \geq 1
\end{cases}
\]  

(5.2)

is defined only for real values of \(p^2\). Thus the \(p_0\) integration in \(d^4p\) must be performed over the real \(p_0\) axis; a Wick rotation is not permitted because the solution does not have a unique analytic continuation for complex \(p_0\). An interesting feature of \((5.2)\) is the occurrence of a cut-off in the invariant mass. Thus particles more massive that \(1/a^2\) cannot occur, consistent with the explicit assumption of others working in curved momentum spaces \([7]\).
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