Bicomplex algebra and function theory

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Abstract

This treatise investigates holomorphic functions defined on the space of bicomplex numbers introduced by Segre. The theory of these functions is associated with Fueter’s theory of regular, quaternionic functions. The algebras of quaternions and bicomplex numbers are developed by making use of so-called complex pairs. Special attention is paid to singular bicomplex numbers that lack an inverse. The elementary bicomplex functions are defined and their properties studied. The derivative of a bicomplex function is defined as the limit of a fraction with nonsingular denominator. The existence of the derivative amounts to the validity of the complexified Cauchy-Riemann equations, which characterize the holomorphic bicomplex functions. It is proved that such a function has derivatives of all orders. The bicomplex integral is defined as a line integral. The condition for path independence and the bicomplex generalizations of Cauchy’s theorem and integral formula are given. Finally, the relationship between the bicomplex functions and different forms of the Laplace equation is considered. In particular, the four-dimensional Laplace equation is factorized using quaternionic differential operators. The outcome is new classes of bicomplex functions including Fueter’s regular functions. It is shown that each class contains differentiable functions.
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1 Introduction

A common technique in mathematics is to define a new object as an aggregate consisting of previously defined objects, usually of the same type. A well-known example is to regard a space vector as consisting of three real numbers. The number concept is also extended with this technique. Thus, a rational number is viewed as a pair of integers and a complex number as a pair of real numbers. It is only natural to ask whether one can carry this idea a step further by defining numbers that are represented by pairs of complex numbers, or complex pairs as we shall call them. It turns out that both the quaternions and the bicomplex numbers can be conceived this way. The former were discovered by William Rowan Hamilton in 1843, the latter by Corrado Segre in 1892 [14, 27]. Both types of numbers can be seen as generalizations of the complex numbers and they lead to corresponding generalizations of the complex functions.

The development of the theory of quaternionic functions was initiated by Rudolf Fueter in 1935 [10]. He proposed that in this theory the objects of interest ought to be the so-called regular functions, which are defined by means of a close analogue of the Cauchy-Riemann equations. More specifically, with \( q = x + yi + zj + uk \) a quaternion written in standard notation and \( \psi : q \to \psi(q) \) a quaternionic function, \( \psi \) is regular if it satisfies the partial differential equation

\[
\frac{\partial \psi}{\partial x} + i \times \frac{\partial \psi}{\partial y} + j \times \frac{\partial \psi}{\partial z} + k \times \frac{\partial \psi}{\partial u} = 0
\]

where \( \times \) denotes quaternionic multiplication. In the complex plane the corresponding formula is

\[
\frac{\partial f}{\partial x} + i \cdot \frac{\partial f}{\partial y} = 0
\]

which is an equivalent way of writing the Cauchy-Riemann equations and characterizes a complex holomorphic (analytic) function \( f : a \to f(a) \), where \( a = x + iy \).

In the other direction there is the theory of bicomplex functions. It has fairly recently been investigated by G.Baley Price [24], whose book also contains a brief history of the development of the main ideas behind the theory (including an account of Segre’s work). The bicomplex functions of interest are the holomorphic ones, which are characterized by the fact that they are differentiable. They are almost isomorphic to the complex holomorphic functions, not surprisingly, because the operations of the bicomplex algebra are almost isomorphic to those of the complex algebra. The bicomplex algebra has an anomaly, however, which has a bearing on the associated function theory: there exist so-called singular numbers other than 0 that lack an inverse.

The purpose of this treatise is to give an overview of and further develop the theory of holomorphic bicomplex functions. We shall among others demonstrate that these functions can be divided into eight, slightly different classes. Some of the functions
satisfying (1) belong to them, hence Fueter's theory is related to our theory.

Our purpose requires that we study the algebras of the bicomplex numbers and the quaternions first. Although both number types have several representations, we shall mostly represent them by complex pairs, such a pair being of the form \((a, b)\), where \(a\) and \(b\) are complex numbers. Complex pairs will in fact be used throughout the treatise, because in general they greatly facilitate calculations.

In Chapter 3 we introduce the bicomplex functions that we regard as elementary: the exponential and logarithm function, the hyperbolic and trigonometric functions, polynomials and the quotient function. The properties of the exponential and logarithm function are specially important, since they throw light on the role played by the singular numbers in bicomplex function theory. We shall show that the exponential function gets only nonsingular values and that, as a consequence, the logarithm function is defined only for nonsingular arguments. The periods that we obtain for the transcendental functions in this chapter are also of interest.

In Chapter 4 we consider the differentiation of bicomplex functions. The derivative is defined in the usual way as the limit of a fraction, but the limit must be taken for nonsingular values of the denominator only. Rendering the derivative as a complex pair yields two alternative formulas, the equivalence of which amounts to the validity of the complexified Cauchy-Riemann equations. Fulfillment of these equations is the main characteristic of the holomorphic bicomplex functions. We prove that such a function possesses derivatives of all orders. In addition, we give \(\mathbb{R}^4\)-representations of the bicomplex derivative.

In Chapter 5 we consider the integration of bicomplex functions. The basic bicomplex integral is defined as a line integral, which is path independent if and only if the integrand is the derivative of a differentiable bicomplex function. We show that a holomorphic bicomplex function can be expanded into a Taylor series. In order to prove the bicomplex versions of Cauchy’s theorem and integral formula we generalize Green’s theorem and derive the so-called twining number, the bicomplex analogue of the complex winding number.

In the last chapter we clarify the relationship between bicomplex functions and the Laplace equation. For this purpose we first study the derivatives of complex holomorphic and conjugate holomorphic functions. We factorize the four-dimensional Laplace equation using quaternionic differential operators and concepts obtained from the similar but simpler factorization of the two-dimensional version of the equation. This leads to the aforementioned classes of bicomplex functions, each class containing also differentiable functions. In this context we shall have the occasion to define the derivative related to Fueter’s regular functions, a topic that has received attention in the literature. Our proposal is to define the derivative as the limit of a bicomplex fraction where both the
numerator and the denominator are regular in Fueter’s sense.

Theorems, lemmas, definitions and formulas are numbered separately within each chapter. The number of a theorem, lemma or definition is always preceded by the chapter number. Numbered formulas are referred to by the assigned number when needed in the same chapter, otherwise the chapter number is added.
2 Quaternionic and bicomplex algebra

In this chapter we shall make extensive use of complex pairs to develop the algebras of quaternions and bicomplex numbers. Although both algebras generalize the complex algebra, we can of course not expect them to retain all the properties of this algebra: in the quaternionic algebra the commutativity of the multiplication operation is lost, and in the bicomplex algebra we must accept the existence of the singular numbers. The bicomplex numbers are more important for our work, but the quaternions are not unimportant. They are needed in the factorization of the four-dimensional Laplace equation and, above all, provide us with a norm that directly can be taken over to the bicomplex algebra. Thus, to begin with we shall investigate the quaternions.

2.1 Representation of quaternions

A quaternion is in general written as a four-dimensional vector of the form
\[ x + yi + zj + uk \]
where \( x, y, z, u \) are real numbers and \( i, j, k \) unit vectors (generalized imaginary units). Given two quaternions
\[ q = x + yi + zj + uk \]
\[ r = X + Yi + Zj + Uk \]
of this kind we may add them by applying component-wise addition, formally:
\[ q + r = (x + X) + (y + Y)i + (z + Z)j + (u + U)k \]

To denote quaternionic multiplication we shall consistently use the symbol \( \times \). The well-known multiplication table for the unit vectors is

\[
\begin{align*}
    i \times i &= -1 & j \times j &= -1 & k \times k &= -1 \\
    i \times j &= k & j \times k &= i & k \times i &= j \\
    k \times j &= -i & j \times i &= -k & i \times k &= -j
\end{align*}
\]

which yields the following expression for the non-commutative product \( q \times r \):
\[ q \times r = (xX - yY - zZ - uU) + (xY + yX + zU - uZ)i + (xZ - yU + zX + uY)j + (xU + yZ - zY + uX)k \]

We now wish to represent the quaternions \( q \) and \( r \) as pairs of complex numbers:
\[ q = (a, b), \quad a = x + yi \text{ and } b = z + ui \]
\[ r = (c, d), \quad c = X + Yi \text{ and } d = Z + Ui \]
How should the multiplication $\times$ be expressed in terms of the complex numbers $a, b, c, d$?

We find

$$q \times r = (a \cdot c - b \cdot d^*, \ b \cdot c^* + a \cdot d)$$

where the operators $\cdot$ and $^*$ stand for complex multiplication and conjugation. Writing the right-hand side of this formula in terms of $x, y, z, u$ and $X, Y, Z, U$ yields

$$q \times r = (xX - yY - zZ - uU + (xY + yX + zU - uZ)i, xZ - yU + zX + uY + (xU + yZ - zY + uX)i)$$

which is another way of rendering (3).

It is sometimes convenient to represent the quaternion $x + yi + zj + uk$ as a quadruple $(x, y, z, u)$

With $q = (x, y, z, u)$ and $r = (X, Y, Z, U)$, the addition and multiplication rules then look like

$$q + r = (x + X, y + Y, z + Z, u + U)$$
$$q \times r = (xX - yY - zZ - uU, xY + yX + zU - uZ, xZ - yU + zX + uY, xU + yZ - zY + uX)$$

In the context of quaternionic algebra the four-dimensional real space $\mathbb{R}^4$ is usually called the quaternionic or Hamiltonian space $H$. It can also be regarded as the Cartesian product of two complex planes, or $\mathbb{C}^2$. The different views of the space $\mathbb{R}^4$ enable us to classify the representations we have introduced for a quaternion $q = (a, b) = (x + yi, z + ui)$:

$$\begin{align*}
\text{H-representation:} & \quad q \\
\text{C}^2\text{-representation, complex pair:} & \quad (a, b) \\
\text{R}^4\text{-representation, complex pair:} & \quad (x + yi, z + ui) \\
\text{R}^4\text{-representation, vector-form:} & \quad x + yi + zj + uk \\
\text{R}^4\text{-representation, quadruple-form:} & \quad (x, y, z, u)
\end{align*}$$

As a rule the $\text{C}^2$-representation leads to the shortest calculations.

### 2.2 Basic quaternionic algebra with complex pairs

Let $(a, b)$ and $(c, d)$ be two complex pairs representing quaternions. The pairs are equal if and only if their components are equal, formally:

$$(a, b) = (c, d) \equiv (a = c) \land (b = d)$$
Remark. The logical operators are designated by $\equiv$ (equivalence), $\Rightarrow$ (implication), $\lor$ (disjunction), $\land$ (conjunction), and $\neg$ (negation). Their priority order is from weaker to stronger: $\equiv$, $\Rightarrow$, $\lor$ and $\land$, $\neg$. $\lor$ and $\land$ have the same priority. $\blacksquare$

For quaternionic addition and multiplication we found in the previous section

\begin{align*}
(a, b) + (c, d) &= (a + c, b + d) \\
(a, b) \times (c, d) &= (ac - bd^*, bc^* + ad)
\end{align*}

With these $C^2$-formulas it is not difficult to verify the following basic properties of $+$ and $\times$ (using one of the $R^4$-representations for the quaternions the proofs of the first three properties are rather laborious):

\begin{align*}
[[a, b] \times (c, d)] \times (e, f) &= (a, b) \times [(c, d) \times (e, f)] & \{\text{is associative}\} \\
(a, b) \times [(c, d) + (e, f)] &= (a, b) \times (c, d) + (a, b) \times (e, f) & \{\text{left-distribution}\} \\
[(c, d) + (e, f)] \times (a, b) &= (c, d) \times (a, b) + (e, f) \times (a, b) & \{\text{right-distribution}\}
\end{align*}

\begin{align*}
(a, b) \times (1, 0) &= (a, b) & \{\text{identity element of } \times\} \\
(a, b) + (0, 0) &= (a, b) & \{\text{identity element of } +\} \\
(a, b) \times (0, 0) &= (0, 0) & \{\text{zero element of } \times\}
\end{align*}

The complex pairs $(1, 0)$ and $(0, 0)$ are examples of real or scalar quaternions, whose general form is $(\lambda, 0)$, where $\lambda \in \mathbb{R}$. Multiplication of a quaternion by a real quaternion is in a sense a distributive operation, since for any $(a, b) \in C^2$

\begin{equation}
(\lambda, 0) \times (a, b) = (\lambda a, \lambda b)
\end{equation}

Moreover, it is also a commutative operation or

\begin{equation}
(\lambda, 0) \times (a, b) = (a, b) \times (\lambda, 0)
\end{equation}

The quaternion $(\lambda, 0)$ is designated by just $\lambda$, because it has all the properties of the real number $\lambda$. The $\times$-operator can be omitted from a quaternionic product if one of the factors is a real quaternion. For example, the previous formula is written $\lambda(a, b) = (a, b)\lambda$.

### 2.3 Quaternionic conjugation, square norm, and division

In this section we shall introduce some concepts based on the quaternionic conjugate. The terminology and notation are partly taken from [B], [L2] and [22]. We first recall the
properties of the complex conjugate. With \(a\) and \(b\) two complex numbers the conjugation operator \(*\) satisfies:

\[
(a + b)^* = a^* + b^* \quad \{\text{* distributes over +}\}
\]

\[
(a \cdot b)^* = a^* \cdot b^* \quad \{\text{* distributes over .}\}
\]

\[
(a^*)^* = a \quad \{\text{* is an involution}\}
\]

We shall use the symbol \(*\) for quaternionic conjugation, too. Its definition in \(\mathbb{C}^2\) is

\[
(a, b)^* = (a^*, -b)
\]

The equivalent \(\mathbb{R}^4\)-formulations are:

\[
(x + yi, z + ui)^* = (x - yi, -z - ui)
\]

\[
(x + yi + zj + uk)^* = x - yi - zj - uk
\]

\[
(x, y, z, u)^* = (x, -y, -z, -u)
\]

Quaternionic conjugation is an involution and distributes over addition, but not over multiplication. Hence, of the formulas (11), (12), and (13) only the first and last may be taken over to quaternionic space:

\[
(q + r)^* = q^* + r^* \quad q, r \in \mathbb{H}
\]

\[
(q^*)^* = q
\]

The square norm \(N(q)\) of the quaternion \(q\) is defined by

\[
N(q) = q \times q^* \quad \{\text{H-representation}\}
\]

With \(q = (a, b)\) we obtain the equivalent complex pair formula, which reveals that the square norm is a real number:

\[
N((a, b)) = aa^* + bb^* \quad \{\text{C}^2\text{-representation}\}
\]

In \(\mathbb{R}^4\) the same formula looks like

\[
N((x + yi, z + ui)) = x^2 + y^2 + z^2 + u^2 \quad \{\text{R}^4\text{-representation}\}
\]

If we let \(N(a)\) stand for the square norm of a complex number \(a\), too, we have \(N(a) = aa^*\), and hence, by (18)

\[
N((a, b)) = N(a) + N(b)
\]

For two quaternions \(q\) and \(r\) we also have

\[
N(q \times r) = N(q)N(r)
\]
The function $N$ enables us to introduce further quaternionic concepts. We denote the inverse of a nonzero quaternion $q$ by $q \uparrow -1$ (the standard notation $q^{-1}$ being reserved for another concept), and define

$$q \uparrow -1 = \frac{1}{N(q)}q^* \quad , \quad q \neq 0$$

With the inverse at our disposal we can introduce a division operation for two quaternions $r$ and $q$, but due to the non-commutativity of $\times$ we must distinguish between left-division, $(q \uparrow -1) \times r$, and right-division, $r \times (q \uparrow -1)$.

Finally, we define the notion of the absolute value or modulus $\|q\|$ of $q$ by

$$\|q\| = \sqrt{N(q)} \quad , \quad q \in H$$

The $\|\|$-function plays an important role in the bicomplex algebra, too, as we shall soon see.

### 2.4 Bicomplex algebra

The space of bicomplex numbers, which we denote by B, has many properties in common with the Hamiltonian space. A bicomplex number $q$ is foremost viewed as a complex pair $(a, b)$, where $a, b \in \mathbb{C}$, but all the other quaternion representations (6) apply to it, too:

- B-representation: $q$
- C$^2$-representation, complex pair: $(a, b)$
- $\mathbb{R}^4$-representation, complex pair: $(x + yi, z + ui)$
- $\mathbb{R}^4$-representation, vector-form: $x + yi + zj + uk$
- $\mathbb{R}^4$-representation, quadruple-form: $(x, y, z, u)$

The equality condition of two bicomplex numbers is given by (7) and the $\|$ are added according to (8). The multiplication of bicomplex numbers, however, differs from the quaternionic operation. If we write a complex number $x + iy$ as a pair of real numbers $(x, y)$, ordinary complex multiplication of $a = (x, y)$ by $b = (z, u)$ takes the form

$$a \cdot b = (xz - yu, yz + xu)$$

Replacement of the factors $a$ and $b$ in this scheme by bicomplex numbers yields bicomplex multiplication, designated by $\odot$. Thus, for $q = (a, b)$ and $r = (c, d)$ we have the definition

$$q \odot r = (a \cdot c - b \cdot d, b \cdot c + a \cdot d)$$

Evaluating this formula for $q = (x + iy, z + iu)$ and $r = (X + iY, Z + iU)$, we obtain the equivalent $\mathbb{R}^4$-formula

$$q \odot r = (xX - yY - zZ + uU + i(xY + yX - zU - uZ),$$
$$xZ - yU + zX - aY + i(xU + yZ + zY + uX))$$
The vector-form \( q = x + yi + zj + uk \) of a bicomplex number is occasionally useful. The unit vectors have the complex pair representations

\[(25) \quad i = (i,0) \quad , \quad j = (0,1) \quad , \quad k = (0,i)\]

and are multiplied according to:

\[
\begin{align*}
i \odot i &= -1 \quad &j \odot j &= -1 \quad &k \odot k &= 1 \\
i \odot j &= k \quad &j \odot k &= -i \quad &k \odot i &= -j \\
j \odot i &= k \quad &k \odot j &= -i \quad &i \odot k &= -j
\end{align*}
\]

A bicomplex scalar \((\lambda,0)\), where \(\lambda \in \mathbb{R}\), will be abbreviated by just \(\lambda\) in the same way as a real quaternion. The identity element \((1,0)\) and the zero element \((0,0)\) of \(\odot\) are therefore written 1 and 0. We also abbreviate the expression \(\lambda \odot (a,b)\) by \(\lambda(a,b)\) and note that

\[(26) \quad \lambda(a,b) = (\lambda a, \lambda b)\]

The bicomplex number \((c,0)\), where \(c \in \mathbb{C}\), has all the properties of the complex number \(c\), yet we shall not use any abbreviation in this case.

### 2.5 Further bicomplex operations

The \(\odot\)-operator is associative, distributes over + and, most significantly, is commutative. It therefore permits us to generalize some of the algebraic operations of the complex numbers. The exponentiation of a bicomplex number is performed in the customary fashion:

\[(27) \quad \begin{cases}
q^0 = 1 \\
q^n = q \odot q^{n-1} \quad \text{if } n \text{ is an integer } \geq 1
\end{cases}\]

Bicomplex conjugation is meaningful, too. We designate it by a postfix \(\uparrow\) and give it the \(\mathbb{C}^2\)-definition:

\[(28) \quad (a,b)^\uparrow = (a,-b)\]

It is isomorphic to the definition of complex conjugation \(*\) expressed by real pairs or \((x,y)^* = (x,-y)\). As a result, the \(\uparrow\)-operator inherits the properties \([11,12,13]\) of \(*\):

\[
\begin{align*}
(q + r)^\uparrow &= q^\uparrow + r^\uparrow \quad &q, r \in \mathbb{B} \\
(q \odot r)^\uparrow &= q^\uparrow \odot r^\uparrow \\
(q^\uparrow)^\uparrow &= q
\end{align*}
\]

Next we define the complex square norm \(CN(q)\) of the bicomplex number \(q\):

\[(32) \quad CN(q) = q \odot q^\uparrow \quad \{\text{B-representation}\}\]
a formula of the same type as (17). With \( q = (a, b) \) we get the \( C^2 \)-representation:

\[
CN((a, b)) = (a^2 + b^2, 0)
\]

It shows that \( CN(q) \) is not a real number as \( N(q) \) but a complex number (which we nonetheless prefer to write in bicomplex notation). \( CN((a, b)) \) is zero if and only if \( a^2 + b^2 = 0 \), which is the case for \( b = ia \) or \( b = -ia \). Viewed as vectors \( a \) and \( b \) are then of equal length and perpendicular to each other. We thus have the relation:

\[
CN((a, b)) = 0 \equiv (b = ia) \lor (b = -ia)
\]

The condition \( CN(q) = 0 \) is crucially important in bicomplex algebra and has no essential analogue in complex algebra. We shall use the following terminology [23]:

**Definition 2.1** A bicomplex number \( q \) is called **singular** if \( CN(q) = 0 \), otherwise it is called **nonsingular**. ☐

From the point of view of developing a bicomplex function theory it is comforting to observe that the bicomplex space contains simply connected domains that are not singleton sets and contain merely nonsingular points. Such domains are consequently termed **nonsingular domains**.

**Example.** The domain that consists of all bicomplex numbers \((x + iy, z + iu)\), such that \( x > 0 \land y > 0 \land z > 0 \land u > 0 \) holds, is nonsingular. ☐

For two bicomplex numbers \( q \) and \( r \) the complex square norm satisfies

\[
CN(q \odot r) = CN(q) \odot CN(r)
\]

from which it follows that the bicomplex product preserves nonsingularity if and only if both factors are nonsingular.

The singular bicomplex numbers \((a, ia)\) and \((a, -ia)\) are so-called **zero divisors**, because \((a, ia) \odot (a, -ia) = 0 \). Hence, although \( q = 0 \lor r = 0 \Rightarrow q \odot r = 0 \) holds for all \( q \) and \( r \), the implication in the other direction is not generally valid.

As the inverse of \( CN((a, b)) \) we can take

\[
\frac{1}{CN((a, b))} = \left( \frac{1}{a^2 + b^2}, 0 \right), \quad a^2 + b^2 \neq 0
\]

The **inverse** \( q^{-1} \) of the bicomplex number \( q \) is now defined by

\[
q^{-1} = \frac{1}{CN(q)} \odot q^\lor, \quad CN(q) \neq 0
\]
The definition is of the same form as (21). Note that only a nonsingular bicomplex number can possess an inverse. Inserting \( q = (a,b) \) in (37) yields upon application of (24), (28) and (36) the equivalent \( C^2 \)-representation

\[
(a,b)^{-1} = \left( \frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right), \quad a^2 + b^2 \neq 0
\]

(38)

At this point the definition of the division of two bicomplex numbers \( r \) and \( q \) is straightforward:

\[
\frac{r}{q} = r \odot q^{-1}, \quad CN(q) \neq 0
\]

(39)

The notation \( \frac{r}{q} \) is meaningful due to the commutativity of \( \odot \). In \( C^2 \) the same formula looks like

\[
\frac{c,d}{a,b} = \left( \frac{c \cdot a + d \cdot b}{a^2 + b^2}, \frac{d \cdot a - c \cdot b}{a^2 + b^2} \right), \quad a^2 + b^2 \neq 0
\]

(40)

Using the inverse (37) we are also able to extend the definition of exponentiation (27) to negative integer exponents by

\[
q^{-n} = (q^{-1})^n, \quad n > 0 \text{ and } CN(q) \neq 0
\]

(41)

The definitions of the inverse and of the division operation are thus based on the complex square norm \( CN \). The definition of the absolute value (modulus) of a bicomplex number, however, cannot be based on \( CN \); for this purpose we have to borrow formula (22) from quaternionic algebra:

\[
\|q\| = \sqrt{N(q)}, \quad q \in B
\]

(42)

Due to (19) the \( \| \cdot \| \)-function may be identified with the Euclidean norm, which is known to have the following properties:

\[
\|q\| \geq 0 \quad \text{for all } q \in B
\]

\[
\|q\| = 0 \iff q = 0
\]

\[
\|\lambda q\| = |\lambda| \cdot \|q\| \quad \text{for all } \lambda \in \mathbb{R} \text{ and } q \in B
\]

\[
\|q + r\| \leq \|q\| + \|r\| \quad \text{for all } q,r \in B
\]

Moreover, formula (20) gives us for all bicomplex \( (a,b) \)

\[
\|(a,b)\| \leq |a| + |b|
\]

The notation \( |c| \) stands for the absolute value \( \sqrt{N(c)} \) of the complex number \( c \).

If we visualize the bicomplex number \( q \) as a vector in \( \mathbb{R}^4 \), the absolute value \( \|q\| \) is the length of the vector.
2.6 Octonions and tricomplex numbers

It remains to say something about the generalization of the quaternions and the bicomplex numbers in the form of octonions and tricomplex numbers, respectively.

An octonion can be expressed as a pair of quaternions \((q, r)\), thus making the octonionic space \(O\) 8-dimensional. Octonionic addition is performed in the normal way by component-wise addition, i.e. \((q, r) + (s, t) = (q + s, r + t)\). Octonionic multiplication \(\otimes\) is defined in terms of quaternionic multiplication by

\[
(q, r) \otimes (s, t) = (q \times s - t^* \times r, r \times s^* + t \times q)
\]

for \(q, r, s, t \in \mathbb{H}\).

This formula should be compared with the formulas we have employed for complex and quaternionic multiplication:

\[
\begin{align*}
(x, y) \cdot (z, u) &= (xz - uy, yz + ux) \\
(a, b) \times (c, d) &= (a \cdot c - d^* \cdot b, b \cdot c^* + d \cdot a)
\end{align*}
\]

for \(x, y, z, u \in \mathbb{R}\) and \(a, b, c, d \in \mathbb{C}\).

If the conjugate of a real number is taken to be the number itself, \((43b)\) could be written \((x, y) \cdot (z, u) = (xz - u^* y, yz^* + ux)\). The formulas \((43a)\), \((43b)\) and \((43c)\) clearly follow the same scheme and therefore complex numbers, quaternions and octonions have many properties in common. However, generalizing from quaternions to octonions causes the loss of the associativity of the \(\otimes\)-operation. Instead \(\otimes\) satisfies the weaker property of alternation: \((m \otimes m) \otimes n = m \otimes (m \otimes n)\) and \(m \otimes (n \otimes n) = (m \otimes n) \otimes n\) for \(m, n \in \mathbb{O}\).

A tricomplex number, in turn, is expressed as a pair of bicomplex numbers. Addition of two tricomplex numbers \((q, r)\) and \((s, t)\) is performed by component-wise addition and multiplication by

\[
(q, r) \circ (s, t) = (q \circ s - t \circ r, r \circ s + t \circ q)
\]

for \(q, r, s, t \in \mathbb{B}\).

Here we have taken the liberty of overloading the symbol \(\circ\) by applying it to the multiplication of both bicomplex and tricomplex numbers. This operation is isomorphic to complex multiplication \((43b)\), as is bicomplex multiplication \([24]\).
3 Elementary bicomplex functions

In this chapter we study functions of the type \( B \to B \), or bicomplex functions of a bicomplex variable. The benefits of using complex pairs will now become so obvious that the alternative to base one’s reasoning entirely on one of the \( R^4 \)-representations of the bicomplex numbers seems exceedingly unattractive.

We define the elementary bicomplex functions, or the exponential function, the hyperbolic and trigonometric functions, polynomials, the quotient function and the logarithm function. These functions contain their complex counterparts as special cases, as must be required of them if the bicomplex function theory is to be a proper generalization of the corresponding theory in the complex domain [5, 6, 11, 24].

3.1 The form of bicomplex functions

Let the bicomplex variable \( p \) be given by:

\[
P = (a, b) = (a + iy, z + iu)
\]

A bicomplex function \( \psi \) of \( p \) then has the form:

\[
\psi(p) = (\phi_1(a, b), \phi_2(a, b))
\]

\[
(\phi_1(a, b), \phi_2(a, b)) = (\psi_1(x, y, z, u) + i\psi_2(x, y, z, u), \psi_3(x, y, z, u) + i\psi_4(x, y, z, u))
\]

The functions \( \phi_k, k = 1, 2 \), are of type \( C^2 \to C \), the functions \( \psi_k, k = 1, 2, 3, 4 \), of type \( R^4 \to R \).

Formulas (1) and (2) provide us with \( B-, C^2-, \) and \( R^4\)-representations of \( p \) and \( \psi \).

Because of our frequent reliance on the \( C^2 \)-representation it would not be inappropriate to look at \( \psi \) as a function of two complex variables \( a \) and \( b \) rather than as a function of one bicomplex variable \( p \). In any case, what is essential in the sequel is the requirement that the two-argument, complex functions \( \phi_k \) should be holomorphic in both \( a \) and \( b \), i.e. the partial derivatives \( \frac{\partial \phi_k}{\partial a} \) and \( \frac{\partial \phi_k}{\partial b} \) should exist. Roughly speaking, the \( \phi_k \)'s meet this condition if their values are obtained by algebraic operations and applications of holomorphic functions so that \( a \) and \( b \) enter in them only as complex aggregates without separation of their real and imaginary parts [20].

Example. For \( a = x + iy \) and \( b = z + iu \) the functions \( a^2 + b^2 \) and \( a \cdot \sin b \) are holomorphic in \( a \) and \( b \), not so \( x^2 + y^2 + b^2 \) and \( a \cdot \sin(z + u) \). □
At this point we recall some basic results from complex analysis. In general, if a complex function
\[ \omega(a, b) = \xi_1(x, y, z, u) + i \xi_2(x, y, z, u) \]
is holomorphic in both \( a = x + iy \) and \( b = z + iu \) the partial derivative \( \frac{\partial \omega}{\partial a} \) is given by either of the formulas

\[
\begin{align*}
\frac{\partial \omega}{\partial a} &= \partial_x \xi_1 + i \partial_x \xi_2 \\
\frac{\partial \omega}{\partial a} &= \partial_y \xi_2 - i \partial_y \xi_1
\end{align*}
\]

Their equivalence amounts to the truth of the Cauchy-Riemann equations

\[
\partial_x \xi_1 = \partial_y \xi_2, \quad \partial_x \xi_2 = -\partial_y \xi_1
\]

By making the replacements \( a \rightarrow b, \ x \rightarrow z, \ y \rightarrow u \) in (3), (4) and (5) we obtain the corresponding formulas for \( \frac{\partial \omega}{\partial b} \).

**Remark.** Above we abbreviated the operators \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \) by \( \partial_x \) and \( \partial_y \). We employ this notation when taking the derivative with respect to a real variable, but not when taking it with respect to a complex variable.

We further left out the arguments of the functions. We allow ourselves such an omission whenever it is clear from the context which the arguments are. \( \square \)

Returning to the function \( \psi \) in (8), it should be emphasized that the holomorphism of its component functions \( \phi_k \) in \( a \) and \( b \) does not ensure that \( \psi \) itself is holomorphic in \( p \). For \( \psi \) to be holomorphic the \( \phi_k \)-functions must fulfill a stronger condition, namely the *bicomplex version* of the Cauchy-Riemann equations. The issue will be discussed in Chapter 4. But the holomorphism of the component functions does suffice to establish the continuity of \( \psi \). To do so we need the limit concept \( \lim_{p \to p_0} \psi(p) = w_0 \), which in the ordinary fashion is defined to be equivalent to

\[
(\forall p, \epsilon : \epsilon > 0 : (\exists \delta : \delta > 0 : \| p - p_0 \| < \delta \Rightarrow \| \psi(p) - w_0 \| < \epsilon))
\]

where \( p \in B \) and \( \epsilon, \delta \in \mathbb{R} \). The absolute values should be computed using (2.42). Geometrically interpreted, the limit is taken for all \( p \) in a (deleted) neighbourhood of the bicomplex space or, more precisely, a four-dimensional ball of radius \( \delta \) centered at, but not including, the point \( p_0 \).

Falling back on (8), one can define the continuity of \( \psi \) at a point \( p_0 \) by

\[
\lim_{\Delta p \to 0} \psi(p_0 + \Delta p) = \psi(p_0)
\]
The holomorphism of the functions $\phi_k$, $k = 1, 2$, at the point $(a_0, b_0)$ means that they possess partial derivatives $\frac{\partial \phi_k}{\partial a}$ and $\frac{\partial \phi_k}{\partial b}$ at $(a_0, b_0)$. Consequently, the functions are continuous there, or

$$\lim_{(\Delta a, \Delta b) \to 0} \phi_k(a_0 + \Delta a, b_0 + \Delta b) = \phi_k(a_0, b_0), \quad k = 1, 2$$

The limit of a complex function is assumed to be a well-understood notion. If we rewrite (7) in the form of complex pairs using $p_0 = (a_0, b_0)$ and $\Delta p = (\Delta a, \Delta b)$ we get an equality, whose validity follows from (8). In other words, $\psi$ is continuous at $p_0$.

In Chapter 4 we shall also need the following formula. Let $\phi : (a, b) \to \phi(a, b)$ be a complex function that is holomorphic in a domain $G \subseteq B$. The continuity of $\frac{\partial \phi}{\partial a}$ and $\frac{\partial \phi}{\partial b}$ means that at any point $(a, b) \in G$ we can write

$$\phi(a + \Delta a, b + \Delta b) - \phi(a, b) = \frac{\partial \phi}{\partial a} \cdot \Delta a + \frac{\partial \phi}{\partial b} \cdot \Delta b + \epsilon(\Delta a, \Delta b)$$

where $\epsilon(\Delta a, \Delta b)$ is a complex-valued function that tends to 0 more rapidly than $\Delta a$ and $\Delta b$ tend to 0.

### 3.2 The exponential function

With the help of the complex exponential, sine and cosine functions we define the bi-complex exponential function by

$$e^p = (e^a \cdot \cos b, e^a \cdot \sin b), \quad p = (a, b)$$

The component functions $\phi_1(a, b) = e^a \cdot \cos b$ and $\phi_2(a, b) = e^a \cdot \sin b$ are obviously holomorphic in both $a$ and $b$. Definition (10) is of the same form as the corresponding definition of the complex exponential function rendered as a pair of reals:

$$e^a = (e^x \cdot \cos y, e^x \cdot \sin y), \quad a = (x, y)$$

With $b = (z, u)$ the complex functions $\cos b$ and $\sin b$ have the real pair representations:

$$\cos b = (\cos z \cosh u, -\sin z \sinh u)$$
$$\sin b = (\sin z \cosh u, \cos z \sinh u)$$

Together with (11) they enable us to rewrite (10) as a quadruple in $\mathbb{R}^4$. For $p = (a, b) = (x, y, z, u)$ we get

$$e^p = (\psi_1, \psi_2, \psi_3, \psi_4)$$
$$= (e^x (\cos y \cos z \cosh u + \sin y \sin z \sinh u),$$
$$e^x (-\cos y \sin z \sinh u + \sin y \cos z \cosh u),$$
$$e^x (\cos y \sin z \cosh u - \sin y \cos z \sinh u),$$
$$e^x (\cos y \cos z \sinh u + \sin y \sin z \cosh u))$$
Turning our attention to the properties of $e^p$ we first observe that for $p = (a, 0)$ we get $e^{(a, 0)} = (e^a, 0)$. The bicomplex function thus generalizes the right complex function.

Definition (10) also gives us the periodicity of $e^p$. In the complex plane $e^a$ has the period $im2\pi$, $m = \pm 1, \pm 2, \ldots$, while $\cos b$ and $\sin b$ have the period $n2\pi$, $n = \pm 1, \pm 2, \ldots$. The period $w$ of $e^p$ is therefore $(im, n)2\pi$, formally:

$$e^{p+w} = e^p \quad \text{for } w = (im, n)2\pi$$

$$m = 0, \pm 1, \pm 2, \ldots$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Setting $p = 0$ yields as special case

$$e^{(im, n)2\pi} = 1, \quad m = 0, \pm 1, \pm 2, \ldots \text{ and } n = 0, \pm 1, \pm 2, \ldots$$

The familiar addition formula $e^{a+b} = e^a \cdot e^b$ satisfied by the complex exponential function is valid in bicomplex space, too, provided complex multiplication is replaced by bicomplex multiplication:

$$e^{p+r} = e^p \odot e^r, \quad p = (a, b) \text{ and } r = (c, d)$$

For $r = -p$ this formula becomes $e^p \odot e^{-p} = 1$, which shows that $e^p$ never gets the value zero. In Section 3.6 we shall obtain the stronger result that $e^p$ never gets a singular value.

### 3.3 The hyperbolic and trigonometric functions

With the exponential function at our disposal we are now in a position to define the hyperbolic functions $\cosh$ and $\sinh$ in $B$ in the same way as it is done in $C$:

$$\cosh p = \frac{1}{2}(e^p + e^{-p}) \quad , \quad p = (a, b)$$

(16)  

$$\sinh p = \frac{1}{2}(e^p - e^{-p})$$

(17)

The use of (10) in these formulas yields after straightforward calculation the complex pair representations

$$\cosh p = (\cosh a \cdot \cos b, \sinh a \cdot \sin b) \quad , \quad p = (a, b)$$

(18)

$$\sinh p = (\sinh a \cdot \cos b, \cosh a \cdot \sin b)$$

(19)

Note that for $p = (a, 0)$ the functions reduce to their complex analogues. Moreover, due to their definitions they have the same period as $e^p$.

Bicomplex division (2.39) gives us the tanh-function

$$\tanh p = \frac{\sinh p}{\cosh p} \quad , \quad p = (a, b)$$

(20)
By applying (18), (19) and (2.40) in this formula we get after some manipulation

\[
\tanh p = \left( \frac{\tanh a}{(\cosh b)^2 + (\tanh a \cdot \sin b)^2} , \quad \frac{\tan b}{(\cosh a)^2 + (\sinh a \cdot \tan b)^2} \right)
\]

To define the bicomplex cosine and sine functions we take the corresponding complex functions as starting point:

\[
\cos a = \frac{1}{2}(e^{ia} + e^{-ia}) , \quad a = x + iy
\]

\[
\sin a = \frac{1}{2i}(e^{ia} - e^{-ia})
\]

Employing real pairs to express complex numbers we have \(a = (x, y), i = (0, 1)\) and \(ia = (0, 1) \cdot (x, y) = (-y, x)\)

Transferring this to B we replace \(a = (x, y)\) by \(p = (a, b)\) and \(i = (0, 1)\) by \(j = (0, 1)\) to obtain

\[
j \otimes p = (0, 1) \otimes (a, b) = (-b, a)
\]

We choose the expression \(j \otimes p\) as the bicomplex correspondence of the complex \(ia\) and generalize (22) and (23) by

\[
\cos p = \frac{1}{2}(e^{j \otimes p} + e^{-j \otimes p}) , \quad p = (a, b)
\]

\[
\sin p = \frac{1}{2j} \otimes (e^{j \otimes p} - e^{-j \otimes p})
\]

Their \(C^2\)-representations are

\[
\cos p = (\cos a \cdot \cosh b, -\sin a \cdot \sinh b) , \quad p = (a, b)
\]

\[
\sin p = (\sin a \cdot \cosh b, \cos a \cdot \sinh b)
\]

Because \(\cos a\) and \(\sin a\) both have the period \(m2\pi, m = \pm 1, \pm 2, \ldots\), while \(\cosh b\) and \(\sinh b\) have the period \(in2\pi, n = \pm 1, \pm 2, \ldots\), the period of \(\cos p\) and \(\sin p\) is \((m, in)2\pi\).

The B- and \(C^2\)-representations of the bicomplex tan-function are:

\[
\tan p = \frac{\sin p}{\cos p} , \quad p = (a, b)
\]

\[
\tan p = \left( \frac{\tan a}{(\cosh b)^2 + (\tan a \cdot \sin b)^2} , \quad \frac{\tanh b}{(\cosh a)^2 + (\sinh a \cdot \tan b)^2} \right)
\]

In the definitions of the preceding functions we observe in every case that they are the same as the definitions of their complex counterparts but with complex notions replaced
by bicomplex ones. This isomorphism manifests itself in the properties of the functions, too. For example, the familiar addition formulas of cosh, sinh, cos and sin can directly be taken over to bicomplex space:

\[
\begin{align*}
\cosh(p + r) &= \cosh p \circ \cosh r + \sinh p \circ \sinh r \\
\sinh(p + r) &= \sinh p \circ \cosh r + \cosh p \circ \sinh r \\
\cos(p + r) &= \cos p \circ \cos r - \sin p \circ \sin r \\
\sin(p + r) &= \sin p \circ \cos r + \cos p \circ \sin r
\end{align*}
\]

We also note that the functions are interconnected through the formulas:

\[
\begin{align*}
\cos p &= \cosh(j \circ p) \\
\sin p &= \frac{1}{j} \circ \sinh(j \circ p)
\end{align*}
\]

### 3.4 Polynomials

Bicomplex polynomials are formed as sums of bicomplex integer powers \(p^n\) multiplied by bicomplex constants.

As the simplest polynomial we regard the constant function

\[
C(p) = p_0, \quad p_0 \text{ a bicomplex constant}
\]

The bicomplex identity function is

\[
I(p) = (a, b), \quad p = (a, b)
\]

and the bicomplex power function

\[
F(p) = (a, b)^n, \quad p = (a, b) \text{ and } n \text{ an integer } \geq 0
\]

Here the exponentiation should be performed according to (2.27). We have in particular

\[
\begin{align*}
p^0 &= (1, 0) \\
p^1 &= (a, b) \\
p^2 &= (a^2 - b^2, 2ab)
\end{align*}
\]

The components of the complex pairs of these equations are examples of so-called harmonic polynomials of \(a\) and \(b\) \[1\]. These polynomials are also recursively definable: if we denote them by \(G_n(a, b)\) and \(H_n(a, b)\) we have

\[
(34) \quad p^n = (G_n(a, b), H_n(a, b)), \quad p = (a, b) \text{ and } n \geq 0
\]
From the identities \( p^0 = (1, 0) \) and \( p^{n+1} \circ p^n \) we then derive the recursive scheme:

\[
G_0(a, b) = 1, \quad H_0(a, b) = 0 \\
G_{n+1}(a, b) = a \cdot G_n(a, b) - b \cdot H_n(a, b) \\
H_{n+1}(a, b) = a \cdot H_n(a, b) + b \cdot G_n(a, b)
\]

The general form of an \( n \)th degree bicomplex polynomial with argument \( p = (a, b) \) is thus

\[
Q(p) = d_n \circ p^n + d_{n-1} \circ p^{n-1} + \ldots + d_1 \circ p + d_0
\]

where \( d_k \) stands for a bicomplex constant.

Similarly, one can define bicomplex power series of the form \( \sum_{k \geq 0} d_k \circ p^k \). Their meaningfulness depends as usual on their convergence.

### 3.5 The quotient function

Let the bicomplex functions \( \psi \) and \( \theta \) be given by

\[
\psi(p) = (\phi_1(a, b), \phi_2(a, b)) \quad , \quad p = (a, b) \\
\theta(p) = (\omega_1(a, b), \omega_2(a, b))
\]

For the inversion \( \theta^{-1} \) we first have on application of (2.38)

\[
\theta^{-1} = \begin{pmatrix}
\frac{\omega_1}{\omega_1^2 + \omega_2^2} & -\frac{\omega_2}{\omega_1^2 + \omega_2^2}
\end{pmatrix}
\]

The function is well-defined in the domain of \( B \) where \( \omega_1^2 + \omega_2^2 \neq 0 \), that is where \( \text{CN}((\omega_1, \omega_2)) \neq 0 \). A bicomplex inversion can have rather more zeros in the denominator than the corresponding complex function, a point to remember when one attempts to generalize e.g. Cauchy’s integral formula.

The inversion of the identity function (32) is a special case of (36):

\[
I^{-1}(p) = \left( \frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right) \quad , \quad p = (a, b)
\]

The quotient function \( \psi/\theta = \psi \circ \theta^{-1} \) has the form

\[
\frac{\psi}{\theta} = \left( \frac{\phi_1 \cdot \omega_1 + \phi_2 \cdot \omega_2}{\omega_1^2 + \omega_2^2}, \frac{\phi_2 \cdot \omega_1 - \phi_1 \cdot \omega_2}{\omega_1^2 + \omega_2^2} \right)
\]

An important category of quotient functions consists of rational functions of the type

\[
R(p) = \frac{Q_1(p)}{Q_2(p)} \quad , \quad p = (a, b)
\]

where \( Q_1(p) \) and \( Q_2(p) \) are bicomplex polynomials without common factors. Borrowing terminology from complex function theory we say that the zeros of \( Q_2(p) \) are the poles of \( R(p) \).
3.6 The logarithm function

We shall define the bicomplex logarithm function as the inverse of the bicomplex exponential function. The definition will be based on the following result.

**Theorem 3.1** The bicomplex number \( q = (\gamma + i\delta, \epsilon + i\eta) \) can equivalently be rendered in the form

\[
q = (v, 0) \odot (\cos w, \sin w)
\]

where \( v \) and \( w \) are complex numbers, if and only if \( q \) is nonsingular or equal to zero.

**Proof.** It is without loss of generality possible to express \( v \) and \( w \) in terms of the real numbers \( r, y, z, u \) so that

\[
v = re^{iy}, \quad w = z + iu
\]

Our task is therefore to investigate under what conditions the equation \((\gamma + i\delta, \epsilon + i\eta) = (re^{iy}, 0) \odot (\cos(z + iu), \sin(z + iu)) \) has a solution in the unknowns \( r, y, z, u \). The right-hand side is equal to

\[
\left( \frac{1}{2}re^{iy} \cdot (e^{i(z+iu)} + e^{-i(z+iu)}), \frac{1}{2i}re^{iy} \cdot (e^{i(z+iu)} - e^{-i(z+iu)}) \right)
\]

and by rewriting this expression using the addition formula as well as other basic properties of the complex exponential function, one can show that the equation is equivalent to the following system of equations:

\[
\begin{align*}
\frac{1}{2}r(e^{-u} \cos(y + z) + e^u \cos(y - z)) & = \gamma \\
\frac{1}{2}r(e^{-u} \sin(y + z) + e^u \sin(y - z)) & = \delta \\
\frac{1}{2}r(e^{-u} \sin(y + z) - e^u \sin(y - z)) & = \epsilon \\
\frac{1}{2}r(-e^{-u} \cos(y + z) + e^u \cos(y - z)) & = \eta
\end{align*}
\]

If \( q = 0 \), i.e. \( \gamma = \delta = \epsilon = \eta = 0 \), these equations are solved by \( r = 0 \), in which case the values of \( y, z \) and \( u \) are indeterminate. If \( q \neq 0 \) and singular, i.e. of the form \((\gamma + i\delta, -\delta + i\gamma)\) or \((\gamma + i\delta, \delta - i\gamma)\), the equations admit no solution (since they imply either \( \sin(y + z) = 0 \land \cos(y + z) = 0 \) or \( \sin(y - z) = 0 \land \cos(y - z) = 0 \)). However, if \( q \)
is nonsingular they are solved by:

\[
\begin{align*}
  r &= (K \cdot L)^{1/4} \\
  y &= \frac{1}{2}(M + N) \\
  L &= (\gamma - \eta)^2 + (\delta + \epsilon)^2 \\
  K &= (\gamma + \eta)^2 + (\delta - \epsilon)^2 \\
  z &= \frac{1}{2}(N - M) \\
  M &= \arctan\left(\frac{\delta + \epsilon}{\gamma - \eta}\right) + n\pi, \quad n = 0 \lor n = 1 \\
  u &= \frac{1}{4} \ln \left|\frac{K}{L}\right| \\
  N &= \arctan\left(\frac{\delta - \epsilon}{\gamma + \eta}\right) + n\pi, \quad n = 0 \lor n = 1
\end{align*}
\]

(42)

The nonsingularity of \(q\) ensures that \(K\) and \(L\) are both nonzero, thus making \(u\) well-defined. When \(M\) is evaluated the principal value in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) should be chosen for the \(\arctan\)-function. The integer \(n\) should be set to 0 or 1 depending on whether the denominator \(\gamma + \eta\) is positive or negative. If the denominator is zero \(M = \frac{\pi}{2}\) or \(M = \frac{3\pi}{2}\) depending on whether the numerator \(\delta - \epsilon\) is positive or negative. The same applies, mutatis mutandis, to the evaluation of \(N\).

The theorem follows from the existence of the solution (42).

\(\Box\)

In the bicomplex number representation \(q = (v, 0) \odot (\cos w, \sin w)\) we call \(v\) the \textit{scale factor} and \(w\) the \textit{complex argument} of \(q\). We shall use the notations:

\[
\begin{align*}
  v &= sf(q) \quad \{\text{scale factor of } q\} \\
  w &= ca(q) \quad \{\text{complex argument of } q\}
\end{align*}
\]

The function \(ca(q)\) is multi-valued, because its real part is not uniquely determined; for a given \(q\) its infinitely many values differ from each other by multiples of \(2\pi\). It is possible to identify a principal branch of \(ca(q)\), denoted by \(Ca(q)\), that is characterized by the fact that its real part is in a certain interval, \([0, 2\pi)\) say. Formally:

\[
w = Ca(q) \equiv w = ca(q) \land 0 \leq \Re(w) < 2\pi
\]

The complex exponential function gets every nonzero value in \(C\), hence if \(q \neq 0\) in formula (41) \(v\) can be replaced by \(e^a\) for some \(a \in C\). This gives \(q = (e^a\cdot \cos w, e^a \cdot \sin w)\) or the bicomplex exponential function. With Theorem 3.2 we then deduce that this function \textit{gets every nonsingular value and no singular value}.

We denote the bicomplex logarithm function by blog and define it as the solution of the equation \(e^p = q\):

\[
\text{blog}(q) = p \equiv e^p = q \quad \text{for } p, q \in B
\]

Clearly, \(\text{blog}(q)\) is only defined for nonsingular \(q\). If we set \(p = (a, b)\) and lean on (40) and the definition of the exponential function, \(e^p = q\) becomes

\[
(e^a, 0) \odot (\cos b, \sin b) = (v, 0) \odot (\cos w, \sin w)
\]
This equation is satisfied if and only if
\[
\begin{align*}
a &= \log(v) \\
b &= w + n2\pi, \ n \text{ an integer}
\end{align*}
\]
where log stands for the complex logarithm function. By making use of the scale factor and complex argument of \( q \) — see (43) — we further obtain:
\[
\begin{align*}
a &= \log(sf(q)), \ q \text{ nonsingular} \\
b &= ca(q) + n2\pi, \ n \text{ an integer}
\end{align*}
\]
Thus, we have derived the following expression for blog:
\[
\text{blog}(q) = (\log(sf(q)), ca(q) + n2\pi), \ q \text{ nonsingular and } n \text{ an integer}
\]
Observe the isomorphism between this function and its complex counterpart \( \log(v) = \ln|v| + i\arg(v) \).

The blog-function is multi-valued, because both the log- and the \( ca \)-function are multi-valued. If \( p \) is a bicomplex logarithm of the nonsingular bicomplex number \( q \), then so are \( p + (im, n)2\pi \) for all integers \( m \) and \( n \). The principal branch of blog, which we denote by Blog, is a function, however, and consists by definition of the principal branches of the component functions:
\[
\text{Blog}(q) = \left( \text{Log}(sf(q)), Ca(q) \right), \ q \text{ nonsingular}
\]
The principal branch Log of the complex logarithm function is defined in the normal manner.

The bicomplex logarithm function allows us to extend the definition of the power function \( q^r \) to bicomplex exponents:
\[
\text{q}^r = e^{r\odot\text{Blog}(q)} \quad \text{for } q, r \in B \text{ and } q \text{ nonsingular}
\]

**Example.** Just for fun we shall evaluate \((0,i)^{(0,i)}\). The bicomplex number \((0,i)\) is nonsingular, hence with the help of (44) and (45) we may render it in the form (40):
\[
(0, i) \circ \left( e^{i\frac{\pi}{2}}, 0 \right)
\]
This yields
\[
\text{Blog}((0, i)) = (\text{Log}(e^{i\frac{\pi}{2}}), \frac{\pi}{2}) = (i\frac{\pi}{2}, \frac{\pi}{2})
\]
and further
\[
(0, i)^{(0,i)} = e^{(0,i)\odot\text{Blog}((0,i))} = e^{(-i\frac{\pi}{2}, -\frac{\pi}{2})} = (0, i)
\]
\[\square\]
We are also able to define the inverse functions of the trigonometric functions in the same way as it is done in C. For example, the inverse cosine is obtained as the solution of

$$\cos p = \frac{1}{2}(e^{j \circ p} + e^{-j \circ p}) = q \quad \text{for } p, q \in \mathbb{B}$$

Solving for $e^{j \circ p}$ and taking the bicomplex logarithm yields the bicomplex version of a familiar formula:

$$\arccos q = -j \circ \log(q \pm \sqrt{q^2 - 1})$$

It is of course required that $\log$’s argument $q \pm \sqrt{q^2 - 1}$ is nonsingular.
4 Differentiation of bicomplex functions

We give the derivative of a bicomplex function a definition that is isomorphic to the corresponding definition in \( \mathbb{C} \) and because, in addition, the bicomplex operations are isomorphic to the complex ones, we may expect that the normal rules for computing the derivative of polynomials, sums, products, quotients, etc. remain valid in \( \mathbb{B} \). However, in defining the bicomplex derivative we have to reckon with the fact that the bicomplex numbers do not form a field in the ordinary sense due to the lack of inverses of the singular numbers. The possibility to restrict the discussion to nonsingular numbers only must be excluded, since they do not form a subfield in \( \mathbb{B} \): if \( a \) and \( b \) are two nonsingular bicomplex numbers, then \( a + b \) need not in general be nonsingular, although \( a \odot b \) is so (see formula (35)). In spite of this, the problems related to singularities can be solved.

We shall obtain \( \mathbb{B} \)-, \( \mathbb{C}^2 \)- and \( \mathbb{R}^4 \)-representations for the bicomplex derivative. The \( \mathbb{C}^2 \)-representation is particularly useful and will give us the bicomplex version of the Cauchy-Riemann equations, which every differentiable, bicomplex function has to satisfy. We shall show that such a function possesses derivatives of all orders.

4.1 Definition of the derivative

Following the standard procedure one is tempted to define the derivative of the bicomplex function \( \psi \) at a point \( p \) as the limit

\[
\psi'(p) = \lim_{\Delta p \to 0} \frac{\psi(p + \Delta p) - \psi(p)}{\Delta p}
\]

A problem then arises, because the fraction \( (\psi(p + \Delta p) - \psi(p))/\Delta p \) is not defined for each singular \( \Delta p \) in the neighbourhood where the limit is taken. An obvious remedy is to confine limit taking only to nonsingular \( \Delta p \), which demands that we apply a weaker formula than (3.6) for the limit concept, namely

\[
(\forall \Delta p, \epsilon : CN(\Delta p) \neq 0 \land \epsilon > 0 : (\exists \delta : \delta > 0 : \|\Delta p\| < \delta \implies \|\psi(p + \Delta p) - \psi(p)\|/\Delta p - \psi'(p)\| < \epsilon))
\]

The dummies satisfy \( \Delta p \in \mathbb{B} \) and \( \epsilon, \delta \in \mathbb{R} \). Recall from Section 2.5 that \( \Delta p \) is nonsingular if \( CN(\Delta p) \neq 0 \). The above limit condition is denoted by

\[
\psi'(p) = \lim_{\substack{\Delta p \to 0 \\ CN(\Delta p) \neq 0}} \frac{\psi(p + \Delta p) - \psi(p)}{\Delta p}
\]

Formula (1) is an instance of the following limit condition that pertains to bicomplex quotients in general:
\[
(\forall p, \epsilon : \text{CN}(\theta(p)) \neq 0 \land \epsilon > 0 : (\exists \delta : \delta > 0 : ||p - p_0|| < \delta \Rightarrow \left\| \frac{\psi(p)}{\theta(p)} - w_0 \right\| < \epsilon))
\]

The corresponding limes notation is

\[
w_0 = \lim_{p \to p_0 \atop \text{CN}(\theta(p)) \neq 0} \frac{\psi(p)}{\theta(p)}
\]

After these preliminaries we can state our first version of the definition of the bicomplex derivative:

**Definition 4.1** Let \( \psi \) be a bicomplex function whose domain of definition contains a neighbourhood of the point \( p \). The derivative of \( \psi \) at the point \( p \) is defined by the equation

\[
\psi'(p) = \lim_{\Delta p \to 0 \atop \text{CN}(\Delta p) \neq 0} \frac{\psi(p + \Delta p) - \psi(p)}{\Delta p}
\]

provided this limit exists. \( \square \)

A bicomplex function that has a derivative at the point \( p \) is said to be differentiable or holomorphic at \( p \). If the function is holomorphic at all points of a domain \( G \subseteq \mathbb{B} \) it is said to be holomorphic in \( G \). The derivative \( \psi'(p) \) is as usual also written \( \frac{d\psi}{dp} \).

**Remark.** By a domain we understand in the sequel a connected, open set of points in the bicomplex space (occasionally in the complex plane). \( \square \)

Definition 4.1 could at least formally be strengthened by the requirement that for every singular \( \Delta p \) such that \( p + \Delta p \) is in the neighbourhood of \( p \) in question, the fraction \( \Omega(\Delta q) = (\psi(p + \Delta p) - \psi(p))/\Delta q \) should approach a finite limit \( w \) as \( \Delta q \) approaches \( \Delta p \), formally:

\[
w = \lim_{\Delta q \to \Delta p \atop \text{CN}(\Delta q) \neq 0} \frac{\psi(p + \Delta q) - \psi(p)}{\Delta q}, \text{CN}(\Delta p) = 0
\]

Note that \( \Delta q \) must be nonsingular. Note also that different values of \( w \) can be associated with different values of \( \Delta p \). The existence of such a limit means that it is possible to remove the singularity of \( \Omega \) at each \( \Delta p \) by defining \( \Omega(\Delta p) = w \), which makes the condition \( \text{CN}(\Delta p) \neq 0 \) superfluous in (3). Nevertheless, this strengthening just postpones the moment when one has to compute a limit in a neighbourhood consisting of nonsingular points only.

**Example.** Let \( \psi \) be the bicomplex square function \( \psi(p) = p^2 \), which yields

\[
\frac{\psi(p + \Delta p) - \psi(p)}{\Delta p} = 2p \odot \Delta p + (\Delta p)^2
\]
The limit value $2p + \triangle p$ of the right-hand side exists for all singular $\triangle p$. Of these $\triangle p = 0$ gives the derivative. \hfill \Box

Yet another alternative is to introduce the derivative of $\psi$ by a definition of Fréchet-type: if there exists a bicomplex number $K$ and a bicomplex function $\epsilon$ that satisfy the equations $\lim_{\triangle p \to 0} \epsilon(\triangle p) = 0$, $\epsilon(0) = 0$ and

\begin{equation}
\psi(p + \triangle p) - \psi(p) = K \odot \triangle p + \epsilon(\triangle p) \odot \triangle p
\end{equation}

then the derivative $\psi'(p)$ is equal to $K$. Thanks to the absence of denominators in (4), the problems related to singular bicomplex numbers are at first sight circumvented. However, the identification of a suitable function $\epsilon$ presupposes in general the existence of removable singularities of the fraction $(\psi(p + \triangle p) - \psi(p))/\triangle p$. The equivalence of this definition and Definition 4.1 has been demonstrated by Price [23], who refers to (4) as the *strong Stolz condition*.

Independently of which of the above definitions is applied, one obtains by employing standard techniques the normal rules for computing the derivatives of sums, products and quotients of functions:

\begin{align*}
(\psi + \theta)'(p) &= \psi'(p) + \theta'(p) \\
(\psi \odot \theta)'(p) &= \psi'(p) \odot \theta(p) + \psi(p) \odot \theta'(p) \\
\left(\frac{\psi}{\theta}\right)'(p) &= \frac{\psi'(p) \odot \theta(p) - \psi(p) \odot \theta'(p)}{(\theta(p))^2} \quad \text{, \text{CN}(\theta(p)) \neq 0}
\end{align*}

Moreover, for the bicomplex composite function $(\psi \circ \theta)(p) = \psi(\theta(p))$ the *chain rule* holds:

\begin{equation}
(\psi \circ \theta)'(p) = \psi'(\theta(p)) \odot \theta'(p)
\end{equation}

The derivative of the power function is given by the normal formula

\begin{equation}
\frac{dp^n}{dp} = np^{n-1}
\end{equation}

provided $n$ is a positive integer and $p$ any bicomplex number or $n$ is a negative integer and $p$ a *nonsingular* bicomplex number. In the next section this derivative will be generalized.

### 4.2 Complex pair formulas for the derivative

We shall seek formulas for the derivative of a bicomplex function $\psi$ having the complex pair representation

\begin{equation}
\psi(p) = (\phi_1(a, b), \phi_2(a, b)) \quad , \quad p = (a, b)
\end{equation}
The calculations are essentially the same as the ones that lead to formulas (3.3) and (3.4) for the complex derivative, but due to the possibility of singular denominators it is necessary to go through them in detail.

We investigate what limits definition (3) yields if we let $\Delta p$ approach 0 in two ways so that $\Delta p = (\Delta a, 0)$ or $\Delta p = (0, \Delta b)$. Then, a key observation is that, provided $\Delta a \neq 0$ and $\Delta b \neq 0$, $\Delta p$ is never singular. In other words, we do not need to worry about singularities in the subsequent formulas.

On account of (2.38) the inverse of $\Delta p$ is

$$\frac{1}{\Delta p} = \left( \frac{1}{\Delta a}, 0 \right) \quad \text{if } \Delta p = (\Delta a, 0)$$

$$\frac{1}{\Delta p} = \left( 0, -\frac{1}{\Delta b} \right) \quad \text{if } \Delta p = (0, \Delta b)$$

With respect to the case $\Delta p = (\Delta a, 0)$ we get using (6) and (7)

$$\psi(p + \Delta p) - \psi(p) \quad \Delta p = \left( \phi_1(a + \Delta a, b) - \phi_1(a, b), \frac{\phi_2(a + \Delta a, b) - \phi_2(a, b)}{\Delta a} \right)$$

As $\Delta p$ and $\Delta a$ approach 0 simultaneously, the components of the right-hand side approach the complex derivatives $\frac{\partial \phi_1}{\partial a}$ and $\frac{\partial \phi_2}{\partial a}$. The bicomplex derivative therefore takes the form

$$\frac{d\psi}{dp} = \left( \frac{\partial \phi_1}{\partial a}, \frac{\partial \phi_2}{\partial a} \right), \quad p = (a, b)$$

If we deal with the second case $\Delta p = (0, \Delta b)$ in the same way we get with (6) and (8)

$$\psi(p + \Delta p) - \psi(p) \quad \Delta p = \left( \phi_2(a, b + \Delta b) - \phi_2(a, b), \frac{\phi_1(a, b + \Delta b) - \phi_1(a, b)}{\Delta b} \right)$$

Letting $\Delta p$ and $\Delta b$ approach 0 results in

$$\frac{d\psi}{dp} = \left( \frac{\partial \phi_2}{\partial b}, -\frac{\partial \phi_1}{\partial b} \right), \quad p = (a, b)$$

This formula must be equivalent to (9), a requirement whose consequences will be examined shortly.

We are now equipped to find more derivatives of the elementary functions. Applied to the exponential function $e^p = (e^a \cdot \cos b, e^a \cdot \sin b)$ both (9) and (10) give

$$\frac{de^p}{dp} = e^p$$

In order to find the derivative of the principal branch of the bicomplex logarithm function (3.45) we assume that the four bicomplex numbers $p$, $\Delta p$, $q$, $\Delta q$ satisfy $q = e^p \land q + \Delta q = e^{p + \Delta p}$, thereby implying that $q$ and $q + \Delta q$ are nonsingular. This yields

$$\frac{\text{Blog}(q + \Delta q) - \text{Blog}(q)}{\Delta q} = \frac{\Delta p}{e^{p + \Delta p} - e^p}$$
As $\Delta p$ tends to 0, the right-hand side tends to $1/e^p$. At the same time $\Delta q$ tends to 0 and the left-hand side becomes the derivative of $\text{Blog}$:

\[
\frac{d\text{Blog}(q)}{dq} = \frac{1}{q}, \quad q \text{ nonsingular}
\]

With the help of (11), (12) and the chain rule we obtain the derivative of the generalized power function (3.46)

\[
\frac{dq^r}{dq} = r \odot q^{-1}, \quad q \text{ nonsingular}
\]

For integer $r$ this formula is equivalent to (5). One should keep in mind that if $r$ is a positive integer, the requirement that $q$ is nonsingular can be dropped, a fact of particular interest in the development of power series, for instance.

We also mention that the complex pair representations (3.18), (3.19), (3.27), (3.28) of the hyperbolic and trigonometric cosine and sine functions yield the expected derivatives:

\[
\frac{d\cosh p}{dp} = \sinh p \\
\frac{d\sinh p}{dp} = \cosh p \\
\frac{d\cos p}{dp} = -\sin p \\
\frac{d\sin p}{dp} = \cos p
\]

### 4.3 The bicomplex differentiability condition

The requirement that the two complex pair formulas (11) and (12), which we found for the bicomplex derivative, should be equivalent is met if and only if the equations

\[
\frac{\partial \phi_1}{\partial a} = \frac{\partial \phi_2}{\partial b} \\
\frac{\partial \phi_2}{\partial a} = -\frac{\partial \phi_1}{\partial b}
\]

hold in the domain $G \subseteq B$ where $\psi(p) = (\phi_1(a,b), \phi_2(a,b))$ is differentiable. These equations represent the bicomplex differentiability condition and will be referred to as the bicomplex Cauchy-Riemann equations (abbr. bicomplex CR-equations). They are the complexification of the CR-equations (3.4).

The validity of (14) and (15) presupposes that the partial derivatives $\frac{\partial \phi_k}{\partial a}$ and $\frac{\partial \phi_k}{\partial b}$, $k = 1, 2$, exist, i.e. that the functions $\phi_k$ are holomorphic in $a$ and $b$. That this is indeed only a necessary condition for $\psi$ to be holomorphic in $p$ is illustrated by the following example.
Example. Let \( \theta \) be given by
\[
\theta(p) = (a^2, b^2) \quad , \quad p = (a, b)
\]
The functions \( \phi_1(a, b) = a^2 \) and \( \phi_2(a, b) = b^2 \) are holomorphic in \( a \) and \( b \), but \( \theta \) is not holomorphic in \( p \), because \( \phi_1 \) and \( \phi_2 \) do not fulfill the bicomplex CR-equations. These are instead fulfilled by the components of the square function
\[
\psi(p) = p^2 = (a^2 - b^2, 2ab)
\]
which thus is holomorphic in \( p \). \( \square \)

Note that all bicomplex functions introduced in Chapter 3 fulfill the bicomplex CR-equations.

From the above we conclude:

**Lemma 4.1** If the bicomplex function \( \psi(p) = (\phi_1(a, b), \phi_2(a, b)) \), \( p = (a, b) \), is holomorphic in the domain \( G \subseteq B \), then the complex functions \( \phi_1 \) and \( \phi_2 \) are holomorphic in \( G \) and satisfy the bicomplex CR-equations (14)–(15) there. \( \square \)

We shall demonstrate that the converse also holds:

**Lemma 4.2** If the complex functions \( \phi_1(a, b) \) and \( \phi_2(a, b) \) are holomorphic in the domain \( G \subseteq B \) and satisfy the bicomplex CR-equations (14)–(15) there, then the bicomplex function \( \psi(p) = (\phi_1(a, b), \phi_2(a, b)) \), \( p = (a, b) \), is holomorphic in \( G \).

**Proof.** The holomorphism of \( \phi_k \), \( k = 1, 2 \), means that they have partial derivatives \( \frac{\partial \phi_k}{\partial a} \) and \( \frac{\partial \phi_k}{\partial b} \) that are continuous in \( G \). At an arbitrary point \( p = (a, b) \) of \( G \) we may therefore by virtue of (3.8) write
\[
\phi_1(a + \Delta a, b + \Delta b) - \phi_1(a, b) = \frac{\partial \phi_1}{\partial a} \cdot \Delta a + \frac{\partial \phi_1}{\partial b} \cdot \Delta b + \epsilon_1
\]
\[
\phi_2(a + \Delta a, b + \Delta b) - \phi_2(a, b) = \frac{\partial \phi_2}{\partial a} \cdot \Delta a + \frac{\partial \phi_2}{\partial b} \cdot \Delta b + \epsilon_2
\]
where \( \epsilon_1 \) and \( \epsilon_2 \) are complex-valued functions of \( \Delta a \) and \( \Delta b \) such that they tend to 0 more rapidly than \( (\Delta a, \Delta b) \) in the sense that \( (\epsilon_1(\Delta a, \Delta b), 0) / (\Delta a, \Delta b) \rightarrow 0 \) and \( (0, \epsilon_2(\Delta a, \Delta b)) / (\Delta a, \Delta b) \rightarrow 0 \) as \( (\Delta a, \Delta b) \rightarrow 0 \). If we set \( \Delta p = (\Delta a, \Delta b) \) and make use of
\[
\psi(p + \Delta p) - \psi(p) = (\phi_1(a + \Delta a, b + \Delta b) - \phi_1(a, b), \phi_2(a + \Delta a, b + \Delta b) - \phi_2(a, b))
\]
together with (14) and (15) we are now able to derive
\[
\psi(p + \Delta p) - \psi(p) = \left( \frac{\partial \phi_1}{\partial a}, \frac{\partial \phi_2}{\partial a} \right) \odot \Delta p + (\epsilon_1, \epsilon_2)
\]

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Dividing both sides by $\Delta p$ and splitting the last term into two terms gives

$$
\frac{\psi(p + \Delta p) - \psi(p)}{\Delta p} = \left( \frac{\partial \phi_1}{\partial a} + \frac{\partial \phi_2}{\partial a} \right) + \left( \frac{\epsilon_1}{\Delta p} + \frac{(0, \epsilon_2)}{\Delta p} \right)
$$

The assumptions about $\epsilon_1$ and $\epsilon_2$ imply that the last two terms tend to 0 as $\Delta p$ tends to 0, which shows that $\psi'(p)$ exists and equals

$$
\left( \frac{\partial \phi_1}{\partial a} + \frac{\partial \phi_2}{\partial a} \right)
$$

Here, the computation of limits is in each case restricted to nonsingular $\Delta p$ in accordance with (4) and (5).

$\square$

Combining the preceding lemmas results in:

**Theorem 4.1** Let $\psi(p) = (\phi_1(a, b), \phi_2(a, b))$, $p = (a, b)$, be a bicomplex function. Then $\psi$ is holomorphic in the domain $G \subseteq B$ if and only if the complex functions $\phi_1(a, b)$ and $\phi_2(a, b)$ are holomorphic in $G$ as well as fulfill the bicomplex CR-equations (14)–(15) there. $\square$

The bicomplex CR-equations are essential to the theory under consideration. They can be combined into a single equation, namely

$$
\frac{\partial \psi}{\partial a} + j \odot \frac{\partial \psi}{\partial b} = 0 , \; j = (0, 1)
$$

where $\frac{\partial \psi}{\partial a}$ and $\frac{\partial \psi}{\partial b}$ denote

$$
\frac{\partial \psi}{\partial a} = \left( \frac{\partial \phi_1}{\partial a} + \frac{\partial \phi_2}{\partial a} \right) , \; \frac{\partial \psi}{\partial b} = \left( \frac{\partial \phi_1}{\partial b} + \frac{\partial \phi_2}{\partial b} \right)
$$

Formula (16) is the generalization of $\frac{\partial f}{\partial x} + i \cdot \frac{\partial f}{\partial y} = 0$ satisfied by a complex holomorphic function.

**4.4 Higher order derivatives**

Next we investigate the existence of higher order derivatives of $\psi$. Application of (4) $n$ times to $\psi$ gives

$$
\frac{d^n \psi}{dp^n} = \left( \frac{\partial^n \phi_1}{\partial a^n} , \frac{\partial^n \phi_2}{\partial a^n} \right) , \; n \geq 0
$$

Alternatively, the $n^{th}$ derivative can be computed by repeated application of (10):

$$
\frac{d^n \psi}{dp^n} = \left( (-1)^{\frac{n}{2}} \frac{\partial^n \phi_2}{\partial b^n} , (-1)^{\frac{n+1}{2}} \frac{\partial^n \phi_1}{\partial b^n} \right) , \; \text{for odd } n \geq 1
$$

$$
\frac{d^n \psi}{dp^n} = \left( (-1)^{\frac{n}{2}} \frac{\partial^n \phi_1}{\partial b^n} , (-1)^{\frac{n+1}{2}} \frac{\partial^n \phi_2}{\partial b^n} \right) , \; \text{for even } n \geq 0
$$
For the proof of the subsequent theorem we note that (18) can be rewritten
\[ \frac{d^n \psi}{dp^n} = \frac{\partial^n \psi}{\partial a^n} , \quad n \geq 0 \]
(21)
The left-hand side is conveniently abbreviated by
\[ \psi^{(n)} = \frac{d^n \psi}{dp^n} , \quad n \geq 0 \]
(22)
It is understood that \( \psi^{(0)} = \psi \).

**Theorem 4.2** A bicomplex function \( \psi(p) = (\phi_1(a,b), \phi_2(a,b)) \), \( p = (a,b) \), that is holomorphic in a domain \( G \subseteq \mathcal{B} \) possesses derivatives of all orders in \( G \), each one holomorphic in \( G \).

**Proof.** Let the \( n \)th derivative of \( \psi \) be given by \( (18) \) and designated by \( \psi^{(n)} \). According to Lemma 4.2 \( \psi^{(n)} \) is holomorphic in \( G \) if \( \frac{\partial^n \phi_1}{\partial a^n} \) and \( \frac{\partial^n \phi_2}{\partial a^n} \) are holomorphic in \( G \) and satisfy the bicomplex CR-equations.

By Lemma 4.1 the holomorphism of \( \psi \) in \( G \) implies that \( \phi_1 \) and \( \phi_2 \) are holomorphic in \( G \). From complex analysis we know that a complex holomorphic function has derivatives of all orders, which by themselves are holomorphic. It follows that the holomorphism of \( \phi_1 \) and \( \phi_2 \) is inherited by \( \frac{\partial^n \phi_1}{\partial a^n} \) and \( \frac{\partial^n \phi_2}{\partial a^n} \).

To prove that \( \psi^{(n)} \) satisfies the bicomplex CR-equations we apply (16), instead of (14)–(15), and establish
\[ \frac{\partial \psi^{(n)}}{\partial a} + j \odot \frac{\partial \psi^{(n)}}{\partial b} = 0 \]
(23)
using mathematical induction. For \( n = 0 \) the formula is identical with (16), which holds on account of \( \psi \)'s holomorphism. For the validation of the step we assume (23) and calculate:
\[ \frac{\partial \psi^{(n+1)}}{\partial a} + j \odot \frac{\partial \psi^{(n+1)}}{\partial b} = 0 \]
\[ \equiv \{ \psi^{(n+1)} = \frac{\partial \psi^{(n)}}{\partial a} \text{, see (21) and (22)} \} \]
\[ \frac{\partial}{\partial a} \left( \frac{\partial \psi^{(n)}}{\partial a} \right) + j \odot \frac{\partial}{\partial b} \left( \frac{\partial \psi^{(n)}}{\partial a} \right) = 0 \]
\[ \equiv \{ \text{induction hypothesis} \} \]
\[ \frac{\partial}{\partial a} \left( -j \odot \frac{\partial \psi^{(n)}}{\partial b} \right) + j \odot \frac{\partial}{\partial b} \left( \frac{\partial \psi^{(n)}}{\partial a} \right) = 0 \]
\[ \equiv \{ \text{taking derivatives in different order, see below} \} \]
\[ -j \odot \frac{\partial}{\partial b} \left( \frac{\partial \psi^{(n)}}{\partial a} \right) + j \odot \frac{\partial}{\partial b} \left( \frac{\partial \psi^{(n)}}{\partial a} \right) = 0 \]
In the penultimate step we changed the order of the operators \( \frac{\partial}{\partial a} \) and \( \frac{\partial}{\partial b} \) when applied to \( \psi^{(n)} \). The interchange is allowed because \( \psi^{(n)} \) consists of holomorphic component functions and
\[
\frac{\partial}{\partial a} \left( \frac{\partial \omega}{\partial b} \right) = \frac{\partial}{\partial b} \left( \frac{\partial \omega}{\partial a} \right)
\]
holds for an arbitrary complex function \( \omega : (a, b) \to \omega(a, b) \) that is holomorphic in both \( a \) and \( b \).
\(\square\)

4.5 \( R^4 \)-representations of the bicomplex derivative

In order to get another handle to bicomplex derivatives we rewrite (9) and (10) in \( R^4 \).
Assume that the full representation of the function \( \psi \) is
\[
\psi(p) = (\phi_1(a, b), \phi_2(a, b))
\]
\[
(\phi_1(a, b), \phi_2(a, b)) = (\psi_1(x, y, z, u) + i\psi_2(x, y, z, u), \psi_3(x, y, z, u) + i\psi_4(x, y, z, u))
\]
\[
p = (a, b) \quad , \quad a = x + iy \text{ and } b = z + iu
\]
On account of (3.3) and (3.4) the partial derivatives \( \frac{\partial \phi_1}{\partial a} \) and \( \frac{\partial \phi_1}{\partial b} \) are given by:
\[
\frac{\partial \phi_1}{\partial a} = \partial_x \psi_1 + i\partial_x \psi_2
\]
\[
\frac{\partial \phi_1}{\partial a} = \partial_y \psi_2 - i\partial_y \psi_1
\]
\[
\frac{\partial \phi_1}{\partial b} = \partial_z \psi_1 + i\partial_z \psi_2
\]
\[
\frac{\partial \phi_1}{\partial b} = \partial_u \psi_2 - i\partial_u \psi_1
\]
 Analogous formulas hold for \( \frac{\partial \phi_2}{\partial a} \) and \( \frac{\partial \phi_2}{\partial b} \). Substitution of the complex derivatives into (9) and (10) yields four \( R^4 \)-representations of \( \frac{d\psi}{dp} \):
\[
\frac{d\psi}{dp} = (\partial_x \psi_1 + i\partial_x \psi_2, \partial_x \psi_3 + i\partial_x \psi_4)
\]
\[
\frac{d\psi}{dp} = (\partial_y \psi_2 - i\partial_y \psi_1, \partial_y \psi_4 - i\partial_y \psi_3)
\]
\[
\frac{d\psi}{dp} = \left( \partial_z \psi_3 + i \partial_z \psi_4, -\partial_z \psi_1 - i \partial_z \psi_2 \right)
\]
\[
\frac{d\psi}{dp} = \left( \partial_u \psi_4 - i \partial_u \psi_3, -\partial_u \psi_2 + i \partial_u \psi_1 \right)
\]

Formulas (24) and (25) were obtained from (9), (26) and (27) from (10).

**Example.** The \(R^4\)-representation of the square function \(\psi(p) = p^2\) is
\[
\psi(p) = (x^2 - y^2 - z^2 + u^2 + i(2xy - 2zu), 2xz - 2yu + i(2yz + 2xu))
\]
Each of (24)–(27) yields the same derivative
\[
\frac{d\psi}{dp} = (2x + i2y, 2z + i2u)
\]
This means that \(\frac{d^2\psi}{dp^2} = 2p\), as expected. \(\square\)

The formulas (24)–(27) should be equivalent. The equivalence of (24) and (25), in particular, implies the simultaneous validity of the equations
\[
\begin{align*}
\partial_x \psi_1 &= \partial_y \psi_2 \\
\partial_x \psi_2 &= -\partial_y \psi_1 \\
\partial_x \psi_3 &= \partial_y \psi_4 \\
\partial_x \psi_4 &= -\partial_y \psi_3
\end{align*}
\]
They express that \(\psi_1 + i\psi_2\) and \(\psi_3 + i\psi_4\) are holomorphic with respect to \(a = x + iy\). By the same token, the equivalence of (26) and (27) amounts to the simultaneous validity of
\[
\begin{align*}
\partial_z \psi_3 &= \partial_u \psi_4 \\
\partial_z \psi_4 &= -\partial_u \psi_3 \\
-\partial_z \psi_1 &= -\partial_u \psi_2 \\
-\partial_z \psi_2 &= \partial_u \psi_1
\end{align*}
\]
which express that \(\psi_1 + i\psi_2\) and \(\psi_3 + i\psi_4\) are holomorphic with respect to \(b = z + iu\).

The additional equivalence of (24) and (26) means that the equations
\[
\begin{align*}
\partial_x \psi_1 &= \partial_z \psi_3 \\
\partial_x \psi_2 &= \partial_z \psi_4 \\
\partial_x \psi_3 &= -\partial_z \psi_1 \\
\partial_x \psi_4 &= -\partial_z \psi_2
\end{align*}
\]
all hold and validate the bicomplex differentiability condition (14)–(15). Thus, in a straightforward way formulas (24)–(27) exhibit both the holomorphism of $\psi$ and its component functions $\psi_1 + i\psi_2$ and $\psi_3 + i\psi_4$. 


5 Bicomplex integration

In this chapter we define the concept of a bicomplex integral as a line integral. We inquire under what conditions it is path independent using an argument analogous to the one given by Ahlfors for complex functions. By integration we show that a holomorphic bicomplex function can be written as a Taylor series. To prove Cauchy’s theorem in B we lean on a generalization of Green’s theorem. To express Cauchy’s integral formula in B we first derive the bicomplex twining number, which generalizes the complex winding number. We are then able to prove the formula with the help of a Taylor expansion.

5.1 Definition of the line integral

We consider the integration of the bicomplex function

\[
\psi(p) = (\phi_1(a, b), \phi_2(a, b))
\]

\[
(\phi_1(a, b), \phi_2(a, b)) = (\psi_1(x, y, z, u) + i\psi_2(x, y, z, u), \\
\psi_3(x, y, z, u) + i\psi_4(x, y, z, u))
\]

\[
p = (a, b), \quad a = x + iy \text{ and } b = z + iu
\]

For the time being \(\psi\)'s domain of definition is left anonymous. We assume that \(\phi_1\) and \(\phi_2\) are holomorphic in \(a\) and \(b\), thereby ensuring that \(\psi\) is continuous as demonstrated in Section 3.1.

The basic bicomplex integral is essentially isomorphic to the complex integral. It is to be understood as a line integral that is evaluated with respect to some four-dimensional curve \(\Gamma\) in B. More specifically, the concept to be defined is

\[
\int_{\Gamma} \psi(p) \otimes dp \quad , \quad dp = (da, db)
\]

Henceforth, we shall choose the curve \(\Gamma\) so that it is piecewise continuously differentiable in B and has the parametric equation

\[
\Gamma : \quad p = p(t) \quad , \quad p(t) = (a(t), b(t)) \quad \text{for} \quad r \leq t \leq s
\]

The argument \(t\) is real. One can look at \(\Gamma\) as a curve made up of two component curves \(\gamma_1\) and \(\gamma_2\) in C

\[
\Gamma = (\gamma_1, \gamma_2)
\]

whose parametric equations are

\[
\gamma_1 : \quad a = a(t) \quad , \quad a(t) = x(t) + iy(t) \quad \text{for} \quad r \leq t \leq s
\]

\[
\gamma_2 : \quad b = b(t) \quad , \quad b(t) = z(t) + iu(t) \quad \text{for} \quad r \leq t \leq s
\]
As definition of the line integral of \( \psi(p) \) extended over the curve \( \Gamma \) we then take

\[
\int_{\Gamma} \psi(p) \circ dp = \int_{r}^{s} \psi(p(t)) \circ p'(t) dt
\]

(4)

Because \( \psi \) is continuous, \( \psi(p(t)) \) at the right-hand side is also continuous. If \( p'(t) \) is discontinuous at some points the integration has to be performed in subintervals of \([r, s]\) in the normal manner.

The left-hand side of (4) can be reformulated in the following two ways. Firstly, bicomplex multiplication of \( \psi = (\phi_1, \phi_2) \) and \( dp = (da, db) \) yields

\[
\int_{\Gamma} \psi(p) \circ dp = \left( \int_{\Gamma} \phi_1 \cdot da - \phi_2 \cdot db, \int_{\Gamma} \phi_2 \cdot da + \phi_1 \cdot db \right)
\]

(5)

Secondly, since the differential \( dp \) is expressible as

\[
dp = (da, 0) + j \circ (db, 0)
\]

\( j = (0, 1) \)

we have

\[
\int_{\Gamma} \psi(p) \circ dp = \int_{\Gamma} \psi(p) \circ (da, 0) + j \circ \psi(p) \circ (db, 0)
\]

(6)

The bicomplex differentials \((da, 0)\) and \((db, 0)\) have the same properties as the complex differentials \(da\) and \(db\). Formulas (5) and (6) will be needed in the next two sections.

We shall also need integrals of the type \( \int_{\Gamma} \psi(p) \|dp\| \) involving the norm \((2.42)\). With \( \Gamma \) specified by (2) we define

\[
\int_{\Gamma} \psi(p) \|dp\| = \int_{r}^{s} \psi(p(t)) \|p'(t)\| dt
\]

The special case \( \psi(p) = 1 \) gives us a suitable definition of the length of \( \Gamma \):

\[
\int_{\Gamma} \|dp\| = \text{length of } \Gamma
\]

(8)

By regarding the integral (4) as the limit of a Riemann sum we obtain by applying the triangle inequality

\[
\left\| \int_{\Gamma} \psi(p) \circ dp \right\| \leq \int_{\Gamma} \|\psi(p)\| \cdot \|dp\|
\]

(9)

If the integral at the left-hand side has a finite value there exists a real constant \( M \) such that \( \|\psi(p)\| \leq M \) for all \( p \) on \( \Gamma \). We therefore get the estimate

\[
\left\| \int_{\Gamma} \psi(p) \circ dp \right\| \leq M \cdot L
\]

(10)

where \( L \) stands for the length of \( \Gamma \).

Formulas (3)–(10) are direct generalizations of the corresponding ones in complex analysis \([18]\).
We ask ourselves under what condition the bicomplex line integral is independent of the path of integration in a domain $G \subseteq B$. With respect to definition (6) this means that if $\Gamma$ is free to vary in $G$ the integral depends only on the end points of $\Gamma$. In particular, if $\Gamma$ is a closed curve the integral reduces to zero.

Our reasoning will be based on formula (6). The fact that its right-hand side is of the form
\[ \int_{\Gamma} \omega_1 \odot (da, 0) + \omega_2 \odot (db, 0) \]
gives relevance to the following theorem.

**Theorem 5.1** Let $\omega_1$ and $\omega_2$ be two bicomplex functions of $p = (a,b)$ such that they are continuous in the domain $G \subseteq B$. Then the line integral $\int_{\Gamma} \omega_1 \odot (da, 0) + \omega_2 \odot (db, 0)$, defined in $G$, depends only on the end points of the curve $\Gamma$ if and only if there exists a function $\Omega : (a,b) \rightarrow \Omega(a,b)$ such that $\frac{\partial \Omega}{\partial a} = \omega_1$ and $\frac{\partial \Omega}{\partial b} = \omega_2$.

**Proof.** Omitted, because it is entirely isomorphic to the proof of the corresponding theorem for complex functions as given by Ahlfors [3], pp. 106–107.

We focus on the evaluation of the right-hand side of (6) in a domain $G \subseteq B$. According to the theorem above the integral depends only on the end points of $\Gamma$ if and only if there exists a function $\Psi : p \rightarrow \Psi(p)$, where $p = (a,b)$, such that
\begin{align*}
\frac{\partial \Psi(p)}{\partial a} &= \psi(p) \\
\frac{\partial \Psi(p)}{\partial b} &= j \odot \psi(p)
\end{align*}

Multiplying (12) by $j$ and adding it to (11) yields
\[ \frac{\partial \Psi}{\partial a} + j \odot \frac{\partial \Psi}{\partial b} = 0 \]

But this is the bicomplex differentiability condition (4.16), which means that $\Psi$ is a holomorphic function. Formula (11) further tells us that $\psi$ is the derivative of $\Psi$. (Note that $\frac{d\Psi}{dp} = \frac{\partial \Psi}{\partial a}$ on account of (4.21), and that $\psi$, too, is holomorphic on account of Theorem 4.2.) Hence, we have established:

**Theorem 5.2** Let $\psi$ be a bicomplex function of the form (1) that is continuous in a domain $G \subseteq B$. Then for an arbitrary curve $\Gamma$ in $G$ the integral $\int_{\Gamma} \psi(p) \odot dp$ depends only on the end points of $\Gamma$ if and only if $\psi$ is the derivative of a holomorphic function $\Psi$ in $G$. □
The function $\Psi$ is uniquely determined up to an additive constant and may be chosen as primitive function (integral function) of $\psi$. It enables one to evaluate a line integral of $\psi$ extended over a curve with end points $p_1$ and $p_2$ according to

$$\int_{p_1}^{p_2} \psi(p) \odot dp = [\Psi(p)]_{p_1}^{p_2} = \Psi(p_2) - \Psi(p_1)$$

Thanks to path independence the omission of any reference to the actual curve is justified here.

### 5.3 Taylor series

We shall show that a holomorphic bicomplex function can be developed into a Taylor series. Our starting point is the formula for computing the derivative of the product of two bicomplex functions:

$$(\theta \odot \varphi)'(p) = \theta'(p) \odot \varphi(p) + \theta(p) \odot \varphi'(p)$$

In a domain $G$ where both $\theta$ and $\varphi$ are holomorphic the function $(\theta \odot \varphi)'$ certainly has a primitive function, or $\theta \odot \varphi$. As a result, if we subject both sides to the integral operator $\int_{p_0}^{p}$ and transfer a term between them we obtain the bicomplex version of the formula for integration by parts:

$$\int_{p_0}^{p} \theta(p) \odot \varphi'(p) \odot dp = [\theta(p) \odot \varphi(p)]_{p_0}^{p} - \int_{p_0}^{p} \theta'(p) \odot \varphi(p) \odot dp$$

It is possible to compute the integrals along any curve in $G$ starting at $p_0$ and ending at $p$.

Let the function $\psi$ be holomorphic in $G$ so that it possesses derivatives of all orders there due to Theorem 4.2. Then, because integration by parts is available, we can expand $\psi$ into the following Taylor polynomial and remainder term about $p_0$:

$$\psi(p) = \left( \sum_{k = 0}^{n-1} \frac{\psi^{(k)}(p_0)}{k!} \odot (p - p_0)^k \right) + R_n(p, p_0)$$

$$R_n(p, p_0) = \frac{1}{(n-1)!} \int_{p_0}^{p} \psi^{(n)}(q) \odot (p - q)^{n-1} \odot dq$$

In order to estimate the remainder term $R_n(p, p_0)$ we study its behaviour in a $p_0$-centered ball that lies in $G$ and is specified by $\|q - p_0\| \leq \|p - p_0\|$. We assume that the integral is evaluated along a curve whose length $\int_{p_0}^{p} \|dq\|$ equals $L$. The derivative $\psi^{(n)}(q)$ satisfies $\|\psi^{(n)}(q)\| \leq M$ for a suitable $M$. Furthermore, using the triangle inequality, in particular, we deduce:

$$\|p - q\| = \|(p - p_0) - (q - p_0)\| \leq \|p - p_0\| + \|q - p_0\| \leq 2\|p - p_0\|$$
Hence, we estimate
\[ \| R_n(p, p_0) \| \leq M \cdot L \cdot (2\| p - p_0 \|)^{n-1} / (n-1)! \]
Because \( \| R_n(p, p_0) \| \) approaches 0 as \( n \to \infty \) the infinite Taylor series converges.

5.4 Cauchy’s theorem in bicomplex space

For the purpose of generalizing Cauchy’s theorem for a bicomplex function we recall one of its formulations in the complex plane.

**Theorem 5.3** Let \( A \) be a simply connected domain in \( C \). If the complex function \( f(a) = g(x, y) + ih(x, y), a = x + iy \), is holomorphic in \( A \), then
\[ \int_{\gamma} f(a) \cdot da = 0 \]
for any closed curve \( \gamma \) in \( A \). \( \square \)

The domain \( A \) being simply connected means that it is without ”holes” or, more precisely, that its complement with respect to the extended plane is connected. It is well-known that the theorem is provable with the help of Green’s theorem, provided one assumes that \( g(x, y) \) and \( h(x, y) \) possess continuous partial derivatives in \( x \) and \( y \) \([3, 26]\). Although the proof is based on unnecessarily strong assumptions, we take it as starting point because conducting it in \( B \) clarifies important concepts. In \( \mathbb{R}^2 \) Green’s theorem is written
\[ \int_{\gamma} g \, dx + h \, dy = \int_{S(\gamma)} \left( \frac{\partial h}{\partial x} - \frac{\partial g}{\partial y} \right) \, dx \, dy \]
The left-hand side is a line integral extended over a closed curve \( \gamma \), the right-hand side a surface integral extended over \( S(\gamma) \), which is the domain of the \( x-y \) plane bounded by \( \gamma \).

Transferring the above to \( B \) first demands that we get some basic understanding of complex line and surface integrals of the form
\[ \int_{\Gamma} \phi \cdot da + \omega \cdot db \]
\[ \int_{S(\Gamma)} \phi \cdot da \cdot db \]
where
- \( \Gamma \) is a closed curve in \( B \) with the parametric equation \([3]\).
- \( S(\Gamma) \) is an arbitrary surface in \( B \) bounded by \( \Gamma \).
- \( \phi \) and \( \omega \) are complex functions of \( a \) and \( b \), i.e. of the type \( B \to C \) (or \( C^2 \to C \), if one so wants).
Integrals like (16) appeared already in formula (5). For the present purpose it suffices to define the integrals (16) and (17) by reducing them to real line and surface integrals, which we consider well-understood. Let \( \phi \) and \( \omega \) more specifically be given by

\[
\phi(a, b) = \xi_1(x, y, z, u) + i \xi_2(x, y, z, u)
\]

\[
\omega(a, b) = \eta_1(x, y, z, u) + i \eta_2(x, y, z, u)
\]

where \( \xi_1, \xi_2, \eta_1, \eta_2 \) are continuous functions of the type \( \mathbb{R}^4 \to \mathbb{R} \). With \( da = dx + idy \) and \( db = dz + idu \) we obtain

\[
\int_{\Gamma} \phi \cdot da + \omega \cdot db = \int_{\Gamma} (\xi_1 dx - \xi_2 dy + \eta_1 dz - \eta_2 du) + i \int_{\Gamma} (\xi_2 dx + \xi_1 dy + \eta_2 dz + \eta_1 du)
\]

(18)

\[
\int_{S(\Gamma)} \phi \cdot da \cdot db = \int_{S(\Gamma)} (\xi_1 dxdz - \xi_1 dydu - \xi_2 dxdz - \xi_2 dydu) + i \int_{S(\Gamma)} (\xi_1 dxdz + \xi_1 dydz + \xi_2 dxdz - \xi_2 dydu)
\]

(19)

We generalize Green’s theorem (15) by substituting complex functions and variables for real ones as well as the four-dimensional curve \( \Gamma \) for the two-dimensional \( \gamma \). We also have to require that the functions are holomorphic.

**Theorem 5.4** Let \( \phi_1 \) and \( \phi_2 \) be two complex functions of \( a \) and \( b \) such that they are holomorphic in a domain \( G \subseteq \mathbb{B} \). Also let \( S(\Gamma) \) be an orientable surface in \( G \) bounded by the closed curve \( \Gamma \). Then

(20)

\[
\int_{\Gamma} \phi_1 \cdot da + \phi_2 \cdot db = \int_{S(\Gamma)} \left( \frac{\partial \phi_2}{\partial a} - \frac{\partial \phi_1}{\partial b} \right) \cdot da \cdot db
\]

Proof. Uses the generalized Stokes theorem, see Appendix.

\( \square \)

With this result the proof of Cauchy’s theorem in \( \mathbb{B} \) comes out in few lines.

**Theorem 5.5** Let the bicomplex function \( \psi(p) = (\phi_1(a, b), \phi_2(a, b)) \), \( p = (a, b) \), be holomorphic in a domain \( G \subseteq \mathbb{B} \). Then

\[
\int_{\Gamma} \psi(p) \odot dp = 0
\]

for any closed curve \( \Gamma \) that is the boundary of an orientable surface \( S(\Gamma) \) in \( G \).
Proof.

\[ \int_{\Gamma} \psi(p) \odot dp \]

= \{ \text{(5)} \}

\left( \int_{\Gamma} \phi_1 \cdot da - \phi_2 \cdot db, \int_{\Gamma} \phi_2 \cdot da + \phi_1 \cdot db \right)

= \{ S(\Gamma) \text{ is an orientable surface in } G; \text{ (20) twice} \}

\left( \int_{S(\Gamma)} \left( \frac{\partial \phi_2}{\partial a} - \frac{\partial \phi_1}{\partial b} \right) \cdot da \cdot db, \int_{S(\Gamma)} \left( \frac{\partial \phi_1}{\partial a} - \frac{\partial \phi_2}{\partial b} \right) \cdot da \cdot db \right)

= \{ \psi \text{ fulfills the bicomplex CR-equations (4.14)--(4.15) in } G \}

0

\square

**Example.** By way of illustration we shall evaluate the integral of \( e^p \) along the closed curve

\[ \Gamma : \quad p(t) = (\text{Re}\,it, \text{Re}\,it) \quad , \quad 0 \leq t \leq 2\pi \]

where \( R \) is a positive real constant. On the \( x-y \)- and \( z-u \)-planes \( \Gamma \) is projected as two origo-centered circles of radius \( R \). The component curves of \( \Gamma \) have the parametric equations

\[ \gamma_1 : \quad a(t) = R\text{e}^{it} \quad , \quad 0 \leq t \leq 2\pi \]

\[ \gamma_2 : \quad b(t) = R\text{e}^{it} \]

The outcome of the following calculation complies with Cauchy’s theorem.

\[ \int_{\Gamma} e^p \odot dp \]

= \{ bicomplex line integral, \( 0 \leq t \leq 2\pi \} \}

\[ \int_{0}^{2\pi} e^{p(t)} \odot p'(t) \, dt \]

= \{ definition of \( \Gamma \) and \( e^p \} \}

\[ \int_{0}^{2\pi} (e^{R\text{e}^{it} \cdot \cos(Re^{it})}, e^{R\text{e}^{it} \cdot \sin(Re^{it})}) \odot (iR\text{e}^{it}, iR\text{e}^{it}) \, dt \]

= \{ multiplication, distribution properties \}
\[
\left( \int_0^{2\pi} \left( e^{Re^{it}} \cdot \cos(Re^{it}) - e^{Re^{it}} \cdot \sin(Re^{it}) \right) \cdot iRe^{it} \, dt , \right.
\]
\[
\int_0^{2\pi} \left( e^{Re^{it}} \cdot \sin(Re^{it}) + e^{Re^{it}} \cdot \cos(Re^{it}) \right) \cdot iRe^{it} \, dt \right) =
\]
\{ \text{definition of the curves } \gamma_1 \text{ and } \gamma_2 \}
\[
\left( \int_0^{2\pi} \left( e^{a(t)} \cdot \cos a(t) - e^{a(t)} \cdot \sin a(t) \right) \cdot a'(t) \, dt , \right.
\]
\[
\int_0^{2\pi} \left( e^{b(t)} \cdot \sin b(t) + e^{b(t)} \cdot \cos b(t) \right) \cdot b'(t) \, dt \right) =
\]
\{ \text{definition of line integral in } \mathbb{C} \}
\[
\left( \int_{\gamma_1} \left( e^a \cdot \cos a - e^a \cdot \sin a \right) \cdot da , \int_{\gamma_2} \left( e^b \cdot \sin b + e^b \cdot \cos b \right) \cdot db \right)
\]
\{ \gamma_1 \text{ and } \gamma_2 \text{ are closed curves; Cauchy’s theorem in } \mathbb{C} \}
0
\]

\[\square\]

### 5.5 The twining number

In the complex plane the so-called \textit{winding number} \( u(\gamma, a_0) \) of a closed curve \( \gamma \) with respect to a point \( a_0 \) is defined by

\[
u(\gamma, a_0) = \frac{1}{2\pi i} \int_\gamma \frac{da}{a - a_0} , \quad a_0 \notin \gamma
\]

We call the corresponding bicomplex concept the \textit{twining number}. The twining number of a closed curve \( \Gamma \) with respect to a point \( p_0 \) in \( B \) is denoted by \( v(\Gamma, p_0) \) and defined by

\[v(\Gamma, p_0) = \frac{1}{2\pi j} \oint_{\Gamma} \frac{dp}{p - p_0} , \quad p - p_0 \text{ nonsingular for all } p \in \Gamma
\]

The integral has a finite value only if \( p - p_0 \) is nonsingular at every point \( p \) on \( \Gamma \). The value cannot always be found by application of Cauchy’s theorem, and may therefore differ from zero. The application of the theorem demands that \( \Gamma \) is the boundary of some orientable surface \( S(\Gamma) \) so that \( p - p_0 \) is nonsingular for all \( p \in S(\Gamma) \), but such a surface does not exist, if \( \Gamma \) can be regarded as enclosing \( p_0 \). The notion of curve enclosure is more difficult to grasp intuitively in the bicomplex space than in the complex plane, and the twining number is in fact the right device for making it precise:

\textbf{Definition 5.1} The curve \( \Gamma \) is said to \textit{enclose} the point \( p_0 \) in the bicomplex space if \( v(\Gamma, p_0) \neq 0 \). \( \square \)
Example. We shall compute the twining number $v(\Gamma, p_0)$ when $\Gamma$ is given by

$$\Gamma : \quad p(t) = p_0 + (e^{it} \cdot \cos t, e^{it} \cdot \sin t), \quad 0 \leq t \leq 2\pi$$

Notice as an aid to the geometric interpretation of this curve that all its points are at a distance of 1 from $p_0$.

We first observe that $p(t) - p_0 = (e^{it} \cdot \cos t, e^{it} \cdot \sin t) = e^{i(t, t)}$. Because the exponential function gets only nonsingular values, $p(t) - p_0$ is nonsingular for all $t$. We also have:

$$p'(t) = (i \cdot e^{it} \cdot \cos t - e^{it} \cdot \sin t, i \cdot e^{it} \cdot \sin t + e^{it} \cdot \cos t)$$

This gives:

$$\int_\Gamma \frac{dp}{p - p_0} = \left\{ \text{line integral, } 0 \leq t \leq 2\pi \right\}$$

$$\int_0^{2\pi} \frac{1}{p(t) - p_0} \odot p'(t) \, dt = \left\{ \text{formulas above and bicomplex algebra} \right\}$$

$$\int_0^{2\pi} (i, 1) \, dt = \left\{ \text{integration} \right\}$$

$$(i^{2\pi}, 2\pi)$$

Thus, the twining number \([21]\) becomes $v(\Gamma, p_0) = (1, -i)$. It differs from 0, which means that $\Gamma$ does not bound any surface $S(\Gamma)$ that would have the property that $1/(p - p_0)$ is holomorphic at all points of $S(\Gamma)$. For instance, take the surface

$$S(\Gamma) : \quad p(h, t) = p_0 + (h \cdot e^{it} \cdot \cos t, h \cdot e^{it} \cdot \sin t)$$

where the real parameters $h$ and $t$ are in the intervals $[0, 1]$ and $[0, 2\pi]$, respectively. It is bounded by $\Gamma$, but contains the point $p(0, t) = p_0$, where $1/(p - p_0)$ is not holomorphic. \(\square\)

In order to find a general expression for the twining number we evaluate the right-hand side of \([21]\) assuming that $\Gamma$’s parametric equation is \([2]\) and that $p - p_0$ is nonsingular at all points $p \in \Gamma$. We introduce the function

\[ F(t) = \int_r^t \frac{p'(t)}{p(t) - p_0} \, dt \quad (22) \]
$F(s)$ is the value of the integral in (21). $F$ is continuous on the interval $[r,s]$ and except at those points where $p'(t)$ is discontinuous it has the derivative

$$F'(t) = \frac{p'(t)}{p(t) - p_0}$$

As a result the derivative of the expression $e^{-F(t)} \odot (p(t) - p_0)$ vanishes except at a finite number of points. The expression itself therefore reduces to a constant, namely $p(r) - p_0$. Accordingly, we obtain

$$e^{F(t)} = \frac{p(t) - p_0}{p(r) - p_0}$$

For $t = s$ we have $e^{F(s)} = 1$, since $\Gamma$ is closed and $p(s) = p(r)$. To fulfill this identity $F(s)$ must due to (3.14) be the period of the exponential function. Consequently, setting $t = s$ in (22) yields first

$$\int_s^r \frac{p'(t)}{p(t) - p_0} dt = (im, n)2\pi, \text{ m and n are integers}$$

and then, on omission of the parametrization

$$\int_{\Gamma} \frac{dp}{p - p_0} = (im, n)2\pi, \text{ m and n are integers}$$

Hence, (21) becomes

$$v(\Gamma, p_0) = (m, -in), \text{ m and n are integers}$$

For $n = 0$ the twining number reduces to the complex winding number.

### 5.6 Cauchy’s integral formula in bicomplex space

We are now ready to tackle the bicomplex generalization of Cauchy’s integral formula.

**Theorem 5.6** Let $\psi$ be a function, $p_0$ a point, $\Gamma$ a closed curve and $G$ a domain in the bicomplex space and assume that they satisfy the following conditions:

a) $\psi$ is holomorphic in $G$.

b) $\Gamma$ is contained in $G$ and the difference $p - p_0$ is nonsingular for all $p \in \Gamma$.

c) There exists an orientable surface $S(\Gamma)$ contained in $G$ and bounded by the curve $\Gamma$ so that $p_0 \notin S(\Gamma) \land v(\Gamma, p_0) = 0$ holds and $p - p_0$ is nonsingular for all $p \in S(\Gamma)$, or $p_0 \in S(\Gamma) \land v(\Gamma, p_0) \neq 0$ holds and $p - p_0$ is nonsingular for all $p \in S(\Gamma) - \{p_0\}$.

Then, we have

$$\psi(p_0) \odot v(\Gamma, p_0) = \frac{1}{2\pi j} \odot \int_{\Gamma} \frac{\psi(p)}{p - p_0} \odot dp$$

(24)
Proof. Our argumentation is essentially the same as the one given by Nevanlinna and Paatero for the complex case [21], pp. 133–135. It is primarily based on conditions a) and c), condition b) being a necessary condition of c).

Condition c) gives us two options with regard to the surface \( S(\Gamma) \). If it does not pass through \( p_0 \), the function \( \psi(p)/(p - p_0) \) is holomorphic at all \( p \in S(\Gamma) \). Formula (24) then holds, since \( v(\Gamma, p_0) = 0 \), and the integral at the right-hand side is equal to 0 due to Cauchy’s theorem.

If \( S(\Gamma) \) passes through \( p_0 \) and \( v(\Gamma, p_0) \neq 0 \), i.e. \( \Gamma \) encloses \( p_0 \), we consider the function

\[
F(p) = \begin{cases} 
\frac{\psi(p) - \psi(p_0)}{p - p_0} & \text{if } p \neq p_0 \\
\psi'(p_0) & \text{if } p = p_0
\end{cases}
\]

By virtue of conditions a) and c) \( F \) is holomorphic at all points of the punctured surface \( S(\Gamma) - \{p_0\} \), while the situation at \( p_0 \) remains open. Our aim is to apply Cauchy’s theorem to \( F \) and for this purpose we must show that \( F \) is holomorphic at \( p_0 \), too.

The function \( \psi \) is holomorphic at \( p_0 \), hence it has a Taylor expansion of the form (13) about this point:

\[
\psi(p) = \psi(p_0) + \psi'(p_0) \circ (p - p_0) + \frac{1}{2} \psi''(p_0) \circ (p - p_0)^2 + \frac{1}{6} \psi'''(p_0) \circ (p - p_0)^3 + R_4(p, p_0)
\]

We get

\[
F(p) = \frac{\psi(p) - \psi(p_0)}{p - p_0} = \psi'(p_0) + \frac{1}{2} \psi''(p_0) \circ (p - p_0) + \frac{1}{6} \psi'''(p_0) \circ (p - p_0)^2 + \frac{R_4(p, p_0)}{p - p_0}
\]

and further, because \( F(p_0) = \psi'(p_0) \):

\[
\frac{F(p) - F(p_0)}{p - p_0} = \frac{1}{2} \psi''(p_0) + \frac{1}{6} \psi'''(p_0) \circ (p - p_0) + \frac{R_4(p, p_0)}{(p - p_0)^2}
\]

The inequality (14) implies that \( R_4(p, p_0)/(p - p_0)^2 \) tends to 0 as \( p \to p_0 \). We conclude that \( F'(p_0) = \frac{1}{2} \psi''(p_0) \), or that \( F \) is holomorphic at \( p_0 \).

Since \( F \) is holomorphic on the whole surface \( S(\Gamma) \) it is legitimate to use Cauchy’s theorem

\[
\int_{\Gamma} F(p) \circ dp = 0
\]

which yields

\[
0 = \int_{\Gamma} \frac{\psi(p) - \psi(p_0)}{p - p_0} \circ dp = \int_{\Gamma} \frac{\psi(p)}{p - p_0} \circ dp - \psi(p_0) \circ \int_{\Gamma} \frac{dp}{p - p_0}
\]

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Formula (24) follows if we apply (21) and substitute \( v(\Gamma, p_0) \odot 2\pi j \) for the last integral.

Assuming that \( v(\Gamma, p_0) \neq 0 \) in (24) and that conditions a) – c) of the preceding theorem otherwise also hold, we may select a particular curve \( \Gamma \) so that \( v(\Gamma, p_0) = (0, -i) \) in agreement with (23) to obtain

\[
\psi(p_0) = \frac{1}{2\pi i} \odot \int_{\Gamma} \frac{\psi(p)}{p - p_0} \odot dp
\]

We conclude that for such a curve the common form of Cauchy’s integral formula is valid in \( B \), too.

**Example.** Given the closed curve

\[
\Gamma : p(t) = (e^{it}, e^{it}) \quad , \quad 0 \leq t \leq 2\pi
\]

we shall compute the value

\[
W = \frac{1}{2\pi i} \odot \int_{\Gamma} \frac{e^p}{p} \odot dp
\]

By making the identifications \( \psi(p) = e^p, p_0 = 0 \) and \( G = B \) we see that the conditions a)–c) of Theorem 5.6 are met. Especially concerning c) we note that \( v(\Gamma, 0) = (0, -i) \) and that all points \( p \in \Gamma \) are nonsingular (since \( e^{it} \neq ie^{it} \land e^{it} \neq -ie^{it} \) for all \( t \in [0, 2\pi] \)).

Moreover, the surface

\[
S(\Gamma) : p(h,t) = (h \cdot e^{it}, h \cdot e^{it}) \quad , \quad 0 \leq t \leq 2\pi \text{ and } 0 \leq h \leq 1
\]

passes through the point \( p(0, t) = 0 \) and all points \( p \in S(\Gamma) - \{0\} \) are nonsingular. Therefore, formula (25) is applicable and yields \( W = e^0 = 1 \). We arrive at this result by direct calculation, too:

\[
\int_{\Gamma} \frac{e^p}{p} \odot dp
\]

\[
\int_{0}^{2\pi} \frac{e^{p(t)}}{p(t)} \odot p'(t) \, dt
\]

\[
= \{ \text{definitions of } e^p \text{ and } \Gamma; \text{ bicomplex inverse} \}
\]

\[
\int_{0}^{2\pi} (e^{e^{it}} \cdot \cos(e^{it}), e^{e^{it}} \cdot \sin(e^{it})) \odot \left( \frac{e^{it}}{2e^{2it}}, \frac{-e^{it}}{2e^{2it}} \right) \odot (ie^{it}, ie^{it}) \, dt
\]

\[
= \{ \text{multiplication, distribution of the integral} \}
\]

\[
\left( \int_{0}^{2\pi} e^{e^{it}} \cdot \cos(e^{it}) \cdot ie^{it} \, dt , \int_{0}^{2\pi} e^{e^{it}} \cdot \sin(e^{it}) \cdot ie^{it} \, dt \right)
\]
\( \gamma_1: a(t) = e^{it} \) and \( \gamma_2: b(t) = e^{it} \) are the components of \( \Gamma \)

\[
\left( \int_0^{2\pi} e^{a(t)} \cdot \cos a(t) \cdot a'(t) \, dt, \int_0^{2\pi} e^{b(t)} \cdot \sin b(t) \cdot b'(t) \, dt \right)
\]

\[
= \left( \int_{\gamma_1} \frac{e^a \cdot \cos a}{a} \cdot da, \int_{\gamma_2} \frac{e^b \cdot \sin b}{b} \cdot db \right)
\]

\[
= (2\pi i, 0)
\]

The result is \( W = 1 \), as it should.

\( \square \)
6 Bicomplex harmonic analysis

Our last topic is the relationship between the bicomplex functions and the Laplace equation. It is well-known that the complex holomorphic functions satisfy the two-dimensional Laplace equation, hence one may expect that the bicomplex holomorphic functions satisfy its four-dimensional version. We shall show that this is indeed true for eight classes of bicomplex holomorphic functions. One of them consists of the functions that fulfill the bicomplex Cauchy-Riemann equations (4.14)–(4.15), the others of similar functions that are partly or entirely formed with the conjugates of complex holomorphic functions. These functions quite naturally emerge from the factorization of the Laplace equation in its various forms. We start by factorizing the two-dimensional equation, since it will provide us with important concepts for the more general cases.

6.1 The Cauchy-Riemann operator

With \( f \) a complex function depending on the real variables \( x \) and \( y \), the two-dimensional Laplace equation is written

\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0
\]

or, using abbreviated differential operators

(1) \( \partial_x^2 f + \partial_y^2 f = 0 \)

The two-dimensional Laplace operator \( \Delta_2 = \partial_x^2 + \partial_y^2 \) admits the factorization

(2) \( \Delta_2 = (\partial_x + i\partial_y) \cdot (\partial_x - i\partial_y) \)

which for the complex number \( a = x + iy \) gives us the occasion to introduce the so-called Cauchy-Riemann operator \( D_a \):

(3) \( D_a = \partial_x + i\partial_y \), \( a = x + iy \)

Note that \( D_a^* = \partial_x - i\partial_y \).

We now let the complex function \( f \) in (1) more specifically be given by

(4) \( f(a) = g(x, y) + ih(x, y) \), \( a = x + iy \)

where \( g \) and \( h \) are functions of the type \( \mathbb{R}^2 \rightarrow \mathbb{R} \) such that they have continuous partial derivatives in \( x \) and \( y \). From (2) and (3) we obtain the relation

(5) \( \Delta_2 f(a) = 0 \Leftrightarrow (D_a \cdot f(a) = 0) \vee (D_a^* \cdot f(a) = 0) \)

by means of which the Laplace equation of the consequent can be solved by solving either disjunct of the antecedent. We first evaluate \( D_a \cdot f(a) = 0 \):
\[ D_a \cdot f(a) = 0 \]
\[ \equiv \{ \text{definition of } D_a \text{ and } f, \text{ omitting arguments} \} \]
\[ (\partial_x + i\partial_y) \cdot (g + ih) = 0 \]
\[ \equiv \{ \text{complex multiplication} \} \]
\[ \partial_x g - \partial_y h + i(\partial_x h + \partial_y g) = 0 \]
\[ \equiv \{ \text{complex algebra} \} \]
\[ (\partial_x g - \partial_y h = 0) \land (\partial_x h + \partial_y g = 0) \]

The first disjunct of the antecedent of (5) obviously resolves into the Cauchy-Riemann equations
\[ \partial_x g = \partial_y h, \quad \partial_x h = -\partial_y g \]

An analogous treatment of the second disjunct \( D^*_a \cdot f(a) = 0 \) shows that it is equivalent to
\[ \partial_x g = -\partial_y h, \quad \partial_x h = \partial_y g \]

We call these the \textit{conjugate Cauchy-Riemann equations}. The name is justified by the fact that they are solved by the conjugates of the holomorphic complex functions. These are henceforth referred to as the \textit{conjugate holomorphic functions}.

**Example.** The exponential function \( g + ih = e^x \cos y + ie^x \sin y \) is holomorphic and satisfies (6). Its conjugate \( (g + ih)^* = e^x \cos y + i(-e^x \sin y) \) satisfies (7).

From the above we deduce the following relationship between the Cauchy-Riemann operator and the holomorphic/conjugate holomorphic complex functions:

\[ D_a \cdot f(a) = 0 \quad \equiv \quad f \text{ is a holomorphic function} \]
\[ D^*_a \cdot f(a) = 0 \quad \equiv \quad f \text{ is a conjugate holomorphic function} \]

The application of the Cauchy-Riemann operator or its conjugate to a complex function thus reveals whether the function is differentiable or not.

### 6.2 The derivatives of a complex function

It is a noteworthy fact that given a complex number \( d \) with non-zero real and imaginary parts \( \text{any complex number } a \) can be expressed in the form \( a = \alpha d + \beta d^* \), where \( \alpha \) and \( \beta \) are real numbers. Because \( d \) and its conjugate \( d^* \) span the entire complex plane in this way, one could argue that in the theory of complex functions complex-valued variables and functions ought to be treated on an equal footing with their conjugates. This should especially apply to the derivatives of the functions. With the complex function \( f \) specified
by (4) it is possible to distinguish between four derivatives:

\[
\frac{df}{da}, \quad \frac{df^*}{da^*}, \quad \frac{df}{da^*}, \quad \frac{df^*}{da}
\]

We first recall the definition of the first of these derivatives at the point \(a_0 = x_0 + iy_0\)

\[
\frac{df}{da}_{|a=a_0} = \lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0}
\]

Letting \(a\) approach \(a_0\) along the real and imaginary axis separately leads to the formulas

(10)

\[
\frac{df}{da} = \partial_x g + i \partial_x h
\]

(11)

\[
\frac{df}{da} = \partial_y h - i \partial_y g
\]

which are equivalent if the CR-equations (5) hold.

Next we consider the derivative of the conjugate function \(f^*\) with respect to the conjugate variable \(a^*\). We define

\[
\frac{df^*}{da^*}_{|a^*=a^*_0} = \lim_{a^* \to a^*_0} \frac{f^*(a) - f^*(a_0)}{a^* - a^*_0}
\]

But if the limit at the right-hand side of (10) exists we have

\[
\lim_{a^* \to a^*_0} \frac{f^*(a) - f^*(a_0)}{a^* - a^*_0} = \left(\lim_{a \to a_0} \frac{f(a) - f(a_0)}{a - a_0}\right)^*
\]

from which it follows that

(13)

\[
\frac{df^*}{da^*} = \left(\frac{df}{da}\right)^*
\]

Conjugation of (11) and (12) consequently results in two formulas for \(\frac{df^*}{da^*}\):

(14)

\[
\frac{df^*}{da^*} = \partial_x g - i \partial_x h
\]

(15)

\[
\frac{df^*}{da^*} = \partial_y h + i \partial_y g
\]

The equivalence of these equations is also implied by the CR-equations.

Then we turn our attention to the derivative of \(f\) with respect to \(a^*\). Its definition is

\[
\frac{df}{da}_{|a^*=a^*_0} = \lim_{a^* \to a^*_0} \frac{f(a) - f(a_0)}{a^* - a^*_0}
\]

The right-hand side can be rewritten

\[
\lim_{x \to x_0, y \to y_0} \frac{g(x, y) - g(x_0, y_0) + i(h(x, y) - h(x_0, y_0))}{x - iy - (x_0 - iy_0)}
\]

50
Letting \( a^* \) approach \( a_0^* \) along the real and imaginary axis amounts to setting \( y = y_0 \) and \( x = x_0 \), respectively, in this formula. Taking the two limits separately yields

\[
\frac{df}{da^*} = \partial_x g + i \partial_x h
\]

(16)

\[
\frac{df}{da^*} = -\partial_y h + i \partial_y g
\]

(17)

These equations are equivalent if the conjugate CR-equations (7) hold, i.e. if \( f \) is conjugate holomorphic.

For the derivative \( \frac{df^*}{da} \), finally, we deduce with the same technique that lead to (13)

\[
\frac{df^*}{da} = \left( \frac{df}{da^*} \right)^*
\]

(18)

Thus, by conjugating (16) and (17) we get

\[
\frac{df^*}{da} = \partial_x g - i \partial_x h
\]

(19)

\[
\frac{df^*}{da} = -\partial_y h - i \partial_y g
\]

(20)

whose equivalence likewise follows from the conjugate CR-equations.

Summarizing these results, we have:

- Formulas (11)/(12) and (14)/(15) are pairwise equivalent if \( f \) is a holomorphic complex function.

- Formulas (16)/(17) and (19)/(20) are pairwise equivalent if \( f \) is a conjugate holomorphic complex function.

The preceding argument admittedly contains some redundancy, because we have obtained four pairs of equivalent formulas, although we have only two types of functions, holomorphic and conjugate holomorphic ones. The reason is that we have designated a conjugate holomorphic function by both \( f \) and \( f^* \). The duplicity could of course easily be removed, but experience has shown that one needs both notations and therefore all the formulas, too.

It is clear that in the complex plane the conjugate holomorphic functions must possess properties that are very similar to those of the holomorphic functions. The conjugate CR-equations (7) play the same role for the former functions as the CR-equations play for the latter; in practice this means that if a relation \( R(x, y) \) has been shown to hold for an holomorphic function \( f \) of the form (4) the relation \( R(x, -y) \) will hold for \( f^* \). Due to this kind of close similarity it can be regarded as an effort devoid of interest to actually develop a theory of conjugate holomorphic functions. What we shall mainly need from it
in the sequel is that a conjugate holomorphic function $f$ of the form (4) has a continuous derivative $\frac{df}{da^*}$ computable with (16) or (17). Alternatively, if the function is designated by $f^*$ we can apply (14) or (15) to find the derivative.

6.3 Factorization of the four-dimensional Laplace equation

We return to the bicomplex space and consider once more a function $\psi$ of the type

\begin{equation}
\psi(p) = (\phi_1(a, b), \phi_2(a, b))
\end{equation}

\begin{equation}
(\phi_1(a, b), \phi_2(a, b)) = (\psi_1(x, y, z, u) + i\psi_2(x, y, z, u), \psi_3(x, y, z, u) + i\psi_4(x, y, z, u))
\end{equation}

$p = (a, b), a = x + iy$ and $b = z + iu$

We want $\psi$ to fulfill the four-dimensional Laplace equation

\begin{equation}
\partial_x^2 \psi + \partial_y^2 \psi + \partial_z^2 \psi + \partial_u^2 \psi = 0
\end{equation}

which we shall solve by factorization. Using the Laplace-operator

\begin{equation}
\Delta_4 = \partial_x^2 + \partial_y^2 + \partial_z^2 + \partial_u^2
\end{equation}

we render (22) in the form

\begin{equation}
\Delta_4 \times \psi = 0
\end{equation}

where $\times$ denotes quaternionic multiplication (2.4). The $\Delta_4$-operator is to be regarded as a scalar quaternion, hence the multiplication in (23) merely applies it to each component $\psi_k$ of $\psi$ — see (2.10).

We now make the crucial observation that the Laplace operator admits the factorization

\begin{equation}
\Delta_4 = (\partial_x + i\partial_y, \partial_z + i\partial_u) \times (\partial_x - i\partial_y, -\partial_z - i\partial_u)
\end{equation}

If we introduce the new operator $L_q$ by

\begin{equation}
L_q = (\partial_x + i\partial_y, \partial_z + i\partial_u)
\end{equation}

the factorization becomes

\begin{equation}
\Delta_4 = L_q \times L_q^*
\end{equation}

The Cauchy-Riemann operators $D_a = \partial_x + i\partial_y$ and $D_b = \partial_z + i\partial_u$ give us

\begin{equation}
L_q = (D_a, D_b), \quad L_q^* = (D_a^*, -D_b)
\end{equation}

and
With formulas (24), (26) and (28) we have factorized the four-dimensional Laplace operator in $\mathbb{R}^4$, B, and C², respectively.

Because $L_q \times L_q^* = L_q^* \times L_q$ we deduce from (23) and (26)

$$\triangle_4 \times \psi = 0 \iff (L_q \times \psi = 0) \lor (L_q^* \times \psi = 0)$$

by means of which we can solve the Laplace equation by solving either disjunct of the antecedent:

$$L_q \times \psi = 0 \quad (30)$$
$$L_q^* \times \psi = 0 \quad (31)$$

We focus on the first of these. With (27) we first rewrite it in C²:

$$L_q \times \psi = 0 \equiv \{L_q = (D_a, D_b), \psi = (\phi_1, \phi_2)\} \equiv \{\text{quaternionic multiplication}\} \equiv \{\text{equality of quaternions}\} \equiv (D_a \cdot \phi_1 - D_b \cdot \phi_2^* = 0) \land (D_b \cdot \phi_1^* + D_a \cdot \phi_2 = 0)$$

Thus, (30) is equivalent to the simultaneous validity of the equations

$$D_a \cdot \phi_1 = D_b \cdot \phi_2^* \quad (32)$$
$$D_b \cdot \phi_1^* = -D_a \cdot \phi_2 \quad (33)$$

Substituting $D_a = \partial_x + i\partial_y$, $D_b = \partial_z + i\partial_u$ and $\phi_1 = \psi_1 + i\psi_2$, $\phi_2 = \psi_3 + i\psi_4$ in these we obtain with complex algebra the corresponding $\mathbb{R}^4$-representation:

$$\partial_x \psi_1 - \partial_y \psi_2 - \partial_z \psi_3 - \partial_u \psi_4 = 0 \quad (34)$$
$$\partial_x \psi_2 + \partial_y \psi_1 + \partial_z \psi_4 - \partial_u \psi_3 = 0 \quad (35)$$
$$\partial_x \psi_3 - \partial_y \psi_4 + \partial_z \psi_1 + \partial_u \psi_2 = 0 \quad (36)$$
$$\partial_x \psi_4 + \partial_y \psi_3 - \partial_z \psi_2 + \partial_u \psi_1 = 0 \quad (37)$$

This partial differential equation was derived in 1935 by Fueter [10]. As mentioned in the Introduction his starting point was the formula

$$\partial_x \psi + i \times \partial_y \psi + j \times \partial_z \psi + k \times \partial_u \psi = 0$$

(38)
which we get by substituting the vector representation of \( L_q \), or
\[
L_q = \partial_x + \partial_y i + \partial_z j + \partial_a k,
\]
in \( L_q \times \psi = 0 \).

Equations (30), (32)–(33), (34)–(37), and (38) are four equivalent ways of expressing
the same condition. We shall refer to them as the \textit{Fueter equations}.

The treatment of equation (31),
\[
L^*_q \times \psi = 0,
\]
which also enables one to solve the Laplace equation, is analogous with the first case. Written in the form
\[
(D^*_a \cdot a - D^*_b \cdot b) \times (\phi_1, \phi_2) = 0
\]
— see (21) and (27) — it becomes equivalent to the simultaneous validity of
\[
D^*_a \cdot \phi_1 = -D^*_b \cdot \phi_2
\]
and
\[
D^*_b \cdot \phi^*_1 = D^*_a \cdot \phi^*_2
\]
We call these the \textit{conjugate Fueter equations}. Expanded in \( \mathbb{R}^4 \) they look like:
\[
\begin{align*}
\partial_x \psi_1 + \partial_y \psi_2 + \partial_z \psi_3 + \partial_a \psi_4 & = 0 \\
\partial_x \psi_2 - \partial_y \psi_1 - \partial_z \psi_4 + \partial_a \psi_3 & = 0 \\
\partial_x \psi_3 + \partial_y \psi_4 - \partial_z \psi_1 - \partial_a \psi_2 & = 0 \\
\partial_x \psi_4 - \partial_y \psi_3 + \partial_z \psi_2 - \partial_a \psi_1 & = 0
\end{align*}
\]
This partial differential equation was derived in 1929 by Lanczos [16]. The fourth, equivalent formulation of the conjugate Fueter equations is found by substitution of the vector-form
\[
L_q^* = \partial_x - \partial_y i - \partial_z j - \partial_a k
\]
in \( L_q^* \times \psi = 0 \), which becomes
\[
\partial_x \psi - i \times \partial_y \psi - j \times \partial_z \psi - k \times \partial_a \psi = 0
\]

### 6.4 Regular and conjugate regular functions

In the solutions of the Fueter equations we identify a new type of bicomplex functions, hence we define:

**Definition 6.1** A bicomplex function is said to be \textit{regular} in a domain of the bicomplex space if it satisfies the Fueter equations in this domain. \( \Box \)

To further characterize the regular functions we prove:

**Theorem 6.1** Let the bicomplex function \( \psi \) be defined in the domain \( G \subseteq B \) so that
\[
\psi(p) = (\phi_1(a,b), \phi_2(a,b)) \quad \text{for} \quad p = (a,b).
\]
If the functions \( \phi_1 \) and \( \phi_2 \) are holomorphic in \( a \) and conjugate holomorphic in \( b \) at all points \( (a,b) \in G \), then \( \psi \) is regular in \( G \).

**Proof.** The assumptions about \( \phi_1 \) and \( \phi_2 \) imply on account of (8) and (9) that
\[
D_a \cdot \phi_k = 0 \quad \text{and} \quad D_b^* \cdot \phi_k = 0
\]
...
hold in $G$. Conjugating the second formula yields $D_b \cdot \phi_k^* = 0$, which combined with the first validates the Fueter equations (32) and (33). Hence, $\psi$ is regular in $G$. □

An example of a regular function is

$$E(p) = (e^{a \cdot \cos b}, e^{a \cdot \sin b}), \quad p = (a, b)$$

(46)

The complex pair components $e^{a \cdot \cos b}$ and $e^{a \cdot \sin b}$ are functions of the variables $a$ and $b$ such that they are holomorphic in $a$ and conjugate holomorphic in $b$, as required.

Two-argument, complex functions that are holomorphic in the one argument and conjugate holomorphic in the other are evidently of special interest and deserve an own term:

**Definition 6.2** A two-argument, complex function $\phi : (a, b) \rightarrow \phi(a, b)$, defined in a domain $G \subseteq \mathbb{B}$, is said to be **coanalytic in** $G$ if it is holomorphic in $a$ and conjugate holomorphic in $b$ at all points $(a, b) \in G$. □

Due to Theorem 6.1 the coanalyticity of the components of a bicomplex function $\psi$ is a sufficient condition for $\psi$’s regularity.

It is a fundamental property of any conjugate holomorphic complex function $f^*$ that $f^*(b) = f(b^*)$. The right-hand side of (46) can therefore be rewritten $(e^{a \cdot \cos b^*}, e^{a \cdot \sin b^*})$, which we also obtain by replacing the argument $p = (a, b)$ of the exponential function (3.10) by $q = (a, b^*)$:

$$e^q = (e^{a \cdot \cos b^*}, e^{a \cdot \sin b^*}), \quad q = (a, b^*)$$

(47)

With the same change of argument we derive the regular counterparts of the other elementary bicomplex functions of Chapter 3, e.g. the regular identity function and power function — see (3.32) and (3.33):

$$I(q) = (a, b^*), \quad q = (a, b^*)$$

(48)

$$F(q) = (a, b^*)^n, \quad q = (a, b^*) \text{ and } n \text{ an integer } \geq 0$$

(49)

and the regular hyperbolic and trigonometric functions — see (3.18)–(3.19) and (3.27)–(3.28):

$$\cosh q = (\cosh a \cdot \cos b^*, \sinh a \cdot \sin b^*), \quad q = (a, b^*)$$

(50)

$$\sinh q = (\sinh a \cdot \cos b^*, \cosh a \cdot \sin b^*)$$

(51)

$$\cos q = (\cos a \cdot \cosh b^*, -\sin a \cdot \sinh b^*)$$

(52)

$$\sin q = (\sin a \cdot \cosh b^*, \cos a \cdot \sinh b^*)$$

(53)
Note that all the component functions of the complex pairs at the right-hand sides are coanalytic in \( B \).

The holomorphism of one-argument complex functions is preserved by complex addition, multiplication and division as well as by functional composition. The conjugate holomorphism of one-argument functions is similarly preserved. Being a combination of these properties, the coanalyticity of two-argument complex functions is thus preserved by the aforementioned operations.

**Example.** The functions \( a \cdot \cos b^* \) and \( b^* \cdot \sin a \) are coanalytic in \( \mathbb{C}^2 \), hence \( a \cdot \cos b^* + b^* \cdot \sin a \) and \((a \cdot \cos b^*) \cdot (b^* \cdot \sin a)\) are coanalytic in \( \mathbb{C}^2 \). \( \Box \)

We are now able to prove that the regular functions form an algebra under the bicomplex operations and that, in addition, functional composition preserves the regularity property.

**Theorem 6.2** Let the bicomplex functions \( \psi : q \to \psi(q) \) and \( \theta : q \to \theta(q) \), where \( q = (a,b^*) \), be regular in a domain \( G \subseteq B \), and assume that they can be represented by complex pairs of coanalytic functions. Then their bicomplex sum \( \psi + \theta \), product \( \psi \circ \theta \) and quotient \( \psi/\theta \) are regular in \( G \), the last only at those points \( q_0 \) where \( \text{CN}(\theta(q_0)) \neq 0 \). Furthermore, the composite function \((\psi \circ \theta)(q) = \psi(\theta(q))\) is regular in \( G \), provided the domain of \( \psi \) is in the range of \( \theta \).

**Proof.** We verify the regularity of the product \( \psi \circ \theta \), in particular. According to the assumption \( \psi \) and \( \theta \) have representations of the form

\[
\psi(q) = (\phi_1(a,b^*), \phi_2(a,b^*)) \quad q = (a,b^*)
\]

\[
\theta(q) = (\omega_1(a,b^*), \omega_2(a,b^*))
\]

where the functions \( \phi_1, \phi_2, \omega_1 \) and \( \omega_2 \) are coanalytic in \( G \). Bicomplex multiplication gives

\[
\psi \circ \theta = (\phi_1 \cdot \omega_1 - \phi_2 \cdot \omega_2, \phi_2 \cdot \omega_1 + \phi_1 \cdot \omega_2)
\]

Because complex addition and multiplication preserve coanalyticity, the component functions \( \phi_1 \cdot \omega_1 - \phi_2 \cdot \omega_2 \) and \( \phi_2 \cdot \omega_1 + \phi_1 \cdot \omega_2 \) are coanalytic in \( G \), thereby making \( \psi \circ \theta \) regular. The regularity of \( \psi + \theta \), \( \psi/\theta \) and \( \psi \circ \theta \) is verified by analogous arguments. \( \Box \)

Next we focus on the conjugate Fueter equations, especially in their form \((39)-(40)\). A bicomplex function satisfying these equations is said to be **conjugate regular**. We prove a sufficient condition for a function to be conjugate regular (compare Theorem 6.1).

**Theorem 6.3** Let the bicomplex function \( \psi \) be defined in the domain \( G \subseteq B \) so that \( \psi(p) = (\phi_1(a,b), \phi_2(a,b)) \) for \( p = (a,b) \). If the functions \( \phi_1 \) and \( \phi_2 \) are conjugate holomorphic in both \( a \) and \( b \) at all points \( (a,b) \in G \) then \( \psi \) is conjugate regular in \( G \).
Proof. The assumption about $\phi_1$ and $\phi_2$ implies on account of (34) that

\[
D^*_a \cdot \phi_k = 0 \quad , \quad k = 1, 2
\]

\[
D^*_b \cdot \phi_k = 0 \quad , \quad k = 1, 2
\]

hold in $G$. The first formula and the conjugation of the second then together validate (39)–(40).

\[ \square \]

If we take the quaternionic conjugate of the regular identity function (48) we get

\[
I^*(q) = (a^*, -b^*) \quad \text{or, by a change of argument}
\]

\[
I(q^*) = (a^*, -b^*)
\]

We regard this as the conjugate regular identity function. In the same way, by applying the other elementary bicomplex functions of Chapter 3 to the argument $q^* = (a^*, -b^*)$ we derive their conjugate regular counterparts, for instance:

\[
cosh q^* = (cosh a^* \cdot cos b^* - sinh a^* \cdot sin b^*) \quad , \quad q^* = (a^*, -b^*)
\]

\[
sinh q^* = (sinh a^* \cdot cos b^* - cosh a^* \cdot sin b^*)
\]

\[
\cos q^* = (\cos a^* \cdot cosh b^* - sin a^* \cdot sinh b^*)
\]

\[
\sin q^* = (\sin a^* \cdot cosh b^* - \cos a^* \cdot sinh b^*)
\]

The component functions of the complex pairs at the right-hand sides are conjugate holomorphic in both $a$ and $b$.

Analogously to the regular functions the conjugate regular ones form an algebra under bicomplex operations.

6.5 New classes of differentiable functions

By considering the derivatives of the regular and conjugate regular functions and of similar functions we shall obtain altogether eight classes of holomorphic bicomplex functions.

Several authors have studied the derivative of regular functions in the context of quaternionic function theory (see e.g. [7, 28]). A general observation has been that it is unproductive to define the derivative of a regular quaternionic function $\psi$ at the point $p$ by applying definition (2.21) for the quaternionic inverse and taking the limit

\[
\lim_{\Delta p \to 0} \left[ (\psi(p + \Delta p) - \psi(p)) \times (\Delta p)^\uparrow - 1 \right]
\]

because for $p = x + yi + zj + uk$ and $\Delta p = \Delta x + \Delta yi + \Delta zj + \Delta uk$ the class of differentiable functions will become too restricted to be of interest. (The class will contain only first-degree polynomials. Nothing is gained by using left-division instead of right-division in
the above expression.) The remedy is to employ bicomplex division and to insist on taking the limit of a fraction that is regular in Fueter’s sense. The latter is achieved by replacing $p$ and $\Delta p$ by the regular

$$q = (a, b^*) = (x + iy, z - iu)$$

$$\Delta q = (\Delta a, \Delta b^*) = (\Delta x + i\Delta y, \Delta z - i\Delta u)$$

Hence, the derivative of a holomorphic regular function is introduced as follows.

**Definition 6.3** Let $\psi$ be a regular bicomplex function whose domain of definition contains a neighbourhood of the point $q = (a, b^*)$. The derivative of $\psi$ at the point $q$ is defined by the equation

$$\psi'(q) = \lim_{\Delta q \to 0 \atop \mathcal{CN}(\Delta q) \neq 0} \psi(q + \Delta q) - \psi(q)$$

provided this limit exists, when it is computed for nonsingular $\Delta q = (\Delta a, \Delta b^*)$. $\square$

Apart from the function argument this definition is the same as Definition 4.1 for the derivative of an “ordinary” bicomplex function of the variable $p = (a, b)$. This means that if one makes the appropriate change of argument, the expected formulas for computing the derivative of a sum, product and quotient of bicomplex functions as well as the chain rule are retained for the regular holomorphic functions. Furthermore, if $\psi$ has the complex pair representation $\psi(q) = (\phi_1(a, b^*), \phi_2(a, b^*))$ its derivative is also given by the following equivalent formulas, analogous to (4.9) and (4.10):

$$\frac{d\psi}{dq} = \left( \frac{\partial \phi_1}{\partial a}, \frac{\partial \phi_2}{\partial a} \right), \quad q = (a, b^*)$$

(59)

$$\frac{d\psi}{dq} = \left( \frac{\partial \phi_2}{\partial b^*}, -\frac{\partial \phi_1}{\partial b^*} \right)$$

(60)

Application of either formula to the regular exponential function $e^q = (e^{a^*} \cos b^*, e^{a^*} \sin b^*)$, for example, yields the same result or

$$\frac{de^q}{dq} = e^q$$

The derivatives of the other elementary regular functions are likewise found with the normal formulas.

**Remark.** In the application of (59) it is helpful to remember (13)

$$\frac{df^*}{da^*} = \left( \frac{df}{da} \right)^*$$

58
satisfied by a conjugate holomorphic complex function \( f^* \). It enables one to compute the derivative of e.g. \( \cos b^* \) according to:

\[
\frac{d \cos b^*}{db^*} = \left( \frac{d \cos b}{db} \right)^* = (- \sin b)^* = - \sin b^*
\]

\[\square\]

The equivalence of (59) and (60) presupposes the simultaneous validity of the equations

\[
\begin{align*}
\frac{\partial \phi_1}{\partial a} &= \frac{\partial \phi_2}{\partial b^*} \\
\frac{\partial \phi_2}{\partial a} &= - \frac{\partial \phi_1}{\partial b^*}
\end{align*}
\]

They express the differentiability condition of the regular functions and are termed the regular Cauchy-Riemann equations.

**Example.** For \( q = (a, b^*) \) the function

\[
\psi(q) = (a^2, (b^*)^2)
\]

is regular but not holomorphic, because it does not fulfill the above equations. In contrast, the square function

\[
\psi(q) = q^2 = (a^2 - (b^*)^2, 2ab^*)
\]

is both regular and holomorphic. \[\square\]

It is instructive to rewrite (59) and (60) in \( \mathbb{R}^4 \) assuming that \( \phi_1 = \psi_1 + i\psi_2 \) and \( \phi_2 = \psi_3 + i\psi_4 \). Using formulas (11)--(12) and (16)--(17) of Section 6.2 we get two formulas for each of the derivatives \( \frac{\partial \phi_1}{\partial a}, \frac{\partial \phi_2}{\partial a}, \frac{\partial \phi_1}{\partial b^*}, \frac{\partial \phi_2}{\partial b^*} \). Substituted into (59) and (60) they furnish four \( \mathbb{R}^4 \)-representations of \( \frac{d \psi}{dq} \):

\[
\begin{align*}
\frac{d \psi}{dq} &= (\partial_x \psi_1 + i \partial_x \psi_2, \partial_x \psi_3 + i \partial_x \psi_4) \\
\frac{d \psi}{dq} &= (\partial_y \psi_2 - i \partial_y \psi_1, \partial_y \psi_4 - i \partial_y \psi_3) \\
\frac{d \psi}{dq} &= (\partial_z \psi_3 + i \partial_z \psi_4, -\partial_z \psi_1 - i \partial_z \psi_2) \\
\frac{d \psi}{dq} &= (-\partial_u \psi_4 + i \partial_u \psi_3, \partial_u \psi_2 - i \partial_u \psi_1)
\end{align*}
\]

The first two of these were obtained from (59), the last two from (60).
Example. For \( q = (a, b^*) = (x + iy, z - iu) \) the \( \mathbb{R}^4 \)-representation of the regular square function \( \psi(q) = q^2 \) is

\[
\psi(q) = (x^2 - y^2 - z^2 + u^2 + i(2xy + 2zu) , 2xz + 2yu + i(2yz - 2ux))
\]

Each of (63)–(66) yields the same derivative or

\[
\frac{d\psi}{dq} = (2x + i2y, 2z - i2u)
\]

This means that \( \frac{d\psi}{dq} = 2q \), as expected. \( \Box \)

The central concepts pertaining to the differentiation of the conjugate regular functions are analogous. It was shown in the previous section that the application of bicomplex functions to the argument \( q^* = (a^*, -b^*) \) yields the corresponding conjugate regular functions, as illustrated by (54)–(58). The same argument is used in the definition of the derivative of such a function.

Let \( \psi(q^*) = (\phi_1(a^*, -b^*), \phi_2(a^*, -b^*)) \), where \( q^* = (a^*, -b^*) \), be a conjugate regular function. Its derivative at the point \( q^* = (a^*, -b^*) \) is defined by the equation

\[
\psi'(q^*) = \lim_{\Delta q^* \to 0} \frac{\psi(q^* + \Delta q^*) - \psi(q^*)}{\Delta q^*}
\]

provided this limit exists, when it is computed for nonsingular \( \Delta q^* = (\Delta a^*, -\Delta b^*) \).

Analysis of this derivative leads to the equivalent complex pair representations

\[
\frac{d\psi}{dq^*} = \left( \frac{\partial \phi_1}{\partial a^*}, \frac{\partial \phi_2}{\partial a^*} \right), \quad q^* = (a^*, -b^*)
\]

\[
\frac{d\psi}{dq^*} = \left( -\frac{\partial \phi_2}{\partial b^*}, \frac{\partial \phi_1}{\partial b^*} \right)
\]

The equivalence of (67) and (68) presupposes the validity of the conjugate regular Cauchy-Riemann equations

\[
\frac{\partial \phi_1}{\partial a^*} = -\frac{\partial \phi_2}{\partial b^*}
\]

\[
\frac{\partial \phi_2}{\partial a^*} = \frac{\partial \phi_1}{\partial b^*}
\]

Let us summarize our present findings concerning differentiable functions. In Section 6.2 we saw that in the complex plane a holomorphic or conjugate holomorphic function has an argument of the type \( a = (x, y) \) or \( a = (x, -y) \) when expressed as a real pair. In Chapter 4 and this section we have obtained holomorphic and conjugate holomorphic functions in the bicomplex space by applying the elementary functions of Chapter 3 to
the arguments \( p = (a, b) \), \( q = (a, b^*) \), and \( q^* = (a^*, -b^*) \). Looking at the form of these arguments it is hard to avoid the thought that there must be differentiable functions associated with each of the following arguments:

\[
\begin{align*}
(71) & \quad p = (a, b) \quad , \quad q = (a, b^*) \quad , \quad r = (a^*, b) \quad , \quad s = (a^*, b^*) \\
(72) & \quad p^{i'} = (a, -b) \quad , \quad q^{i'} = (a, -b^*) \quad , \quad r^{i'} = (a^*, -b) \quad , \quad s^{i'} = (a^*, -b^*)
\end{align*}
\]

Each argument of the second row is the bicomplex conjugate of the corresponding argument of the first row.

Bicomplex functions applied to the arguments (71)–(72) are evidently of two main types:

\[
\begin{align*}
(73) & \quad \psi(t) = (\phi_1(a^o, b^o), \phi_2(a^o, b^o)) \quad , \quad t = (a^o, b^o) \\
(74) & \quad \psi(t^{i'}) = (\phi_1(a^o, -b^o), \phi_2(a^o, -b^o)) \quad , \quad t^{i'} = (a^o, -b^o)
\end{align*}
\]

The variable \( t \) should be replaced by \( p, q, r \) or \( s \) and either of the symbols \( \dagger \) and \( \ddagger \) by a blank or a \( * \). With the derivative defined as the limit of a fraction containing the appropriate argument the functions (73) are holomorphic if they fulfill complexified CR-equations of the form

\[
\begin{align*}
(75) & \quad \frac{\partial \phi_1}{\partial a^o} = \frac{\partial \phi_2}{\partial b^o} \\
(76) & \quad \frac{\partial \phi_2}{\partial a^o} = -\frac{\partial \phi_1}{\partial b^o}
\end{align*}
\]

while the functions (74) are conjugate holomorphic if they fulfill

\[
\begin{align*}
(77) & \quad \frac{\partial \phi_1}{\partial a^o} = -\frac{\partial \phi_2}{\partial b^o} \\
(78) & \quad \frac{\partial \phi_2}{\partial a^o} = \frac{\partial \phi_1}{\partial b^o}
\end{align*}
\]

Equations (75)–(76) correspond to the CR-equations (7), equations (77)–(78) to the conjugate CR-equations (7).

**Example.** Application of the bicomplex cosinus function (3.27) to \( r^{i'} = (a^*, -b) \) yields \( \cos r^{i'} = (\cos a^* \cdot \cosh b, \sin a^* \cdot \sinh b) \). The component functions \( \cos a^* \cdot \cosh b \) and \( \sin a^* \cdot \sinh b \) satisfy (74)–(78) if a \( * \) is substituted for \( \dagger \) and a blank for \( \ddagger \):

\[
\begin{align*}
& \frac{\partial \phi_1}{\partial a^*} = -\frac{\partial \phi_2}{\partial b} \\
& \frac{\partial \phi_2}{\partial a^*} = \frac{\partial \phi_1}{\partial b}
\end{align*}
\]

\( \square \)
6.6 Bicomplex solutions of the Laplace equation

A final remark about the solutions of the Laplace equation is in order.

Let the function $\psi$ be given by (73) so that $\phi_1(a^\circ, b^\circ) = \psi_1(x, y, z, u) + i\psi_2(x, y, z, u)$ and $\phi_2(a^\circ, b^\circ) = \psi_3(x, y, z, u) + i\psi_4(x, y, z, u)$. (The corresponding treatment of a function of type (74) is analogous.) If $\phi_1$ and $\phi_2$ are holomorphic or conjugate holomorphic in $a$ and $b$, their component functions fulfill the two-dimensional Laplace equation:

$$
\frac{\partial^2 \psi_1}{\partial a^\circ} + \frac{\partial^2 \psi_1}{\partial b^\circ} = 0, \quad i = 1, 2, 3, 4
$$

It follows that $\psi$ fulfills the four-dimensional Laplace equation $\frac{\partial^2 \psi}{\partial a^\circ} + \frac{\partial^2 \psi}{\partial b^\circ} + \frac{\partial^2 \psi}{\partial x^\circ} = 0$.

Application of the differential operator $\frac{\partial}{\partial a^\circ}$ to one of the Cauchy-Riemann equations (75)–(76) and $\frac{\partial}{\partial b^\circ}$ to the other yields after computing the sum or difference of the results either

$$
\frac{\partial^2 \phi_1}{\partial a^\circ} + \frac{\partial^2 \phi_1}{\partial b^\circ} = 0
$$

or

$$
\frac{\partial^2 \phi_2}{\partial a^\circ} + \frac{\partial^2 \phi_2}{\partial b^\circ} = 0
$$

The complex pair components of all bicomplex holomorphic functions thus satisfy the two-dimensional, complexified Laplace equation.

In fact, an alternative approach to the understanding of differentiable bicomplex functions is to take the previous two equations as starting point and combine them into

$$
(\frac{\partial^2}{\partial a^\circ}, \frac{\partial^2}{\partial b^\circ}, 0) \odot (\phi_1, \phi_2) = 0
$$

This is a generalization of equation (9). The operator to the left admits the factorization

$$
(\frac{\partial^2}{\partial a^\circ} + \frac{\partial^2}{\partial b^\circ}, 0) = (\frac{\partial}{\partial a^\circ}, \frac{\partial}{\partial b^\circ}) \odot (\frac{\partial}{\partial a^\circ}, -\frac{\partial}{\partial b^\circ})
$$

which means that the solutions of

$$
(\frac{\partial}{\partial a^\circ}, \frac{\partial}{\partial b^\circ}) \odot (\phi_1, \phi_2) = 0
$$

$$
(\frac{\partial}{\partial a^\circ}, -\frac{\partial}{\partial b^\circ}) \odot (\phi_1, \phi_2) = 0
$$

are solutions of (73) as well. Because these two equations are equivalent to (77)–(78) and (74)–(75), respectively, we see as in our treatment of the complex holomorphic and conjugate holomorphic functions in Section 6.1, that the corresponding bicomplex functions can be obtained by factorizing the appropriate version of the Laplace equation.
7 Conclusions

In the course of this work we have frequently seen that a concept introduced or a result obtained has been the generalization of the corresponding concept or result in complex analysis. No doubt there are still plenty of such generalizations to be made, notably with regard to power series, Laurent series and analytic continuation.

A somewhat separate matter is the development of a theory of tricomplex functions of the type $\Omega = (\Psi_1, \Psi_2)$, where $\Psi_1$ and $\Psi_2$ are bicomplex functions of two bicomplex variables. In forming differentiable tricomplex functions one has at one’s disposal all the holomorphic and conjugate holomorphic functions of Section 6.5.

A cornerstone of our work has been the application of complex pairs to represent quaternions and bicomplex numbers. With respect to the former essentially the same idea is found in the book [19] by MacLane and Birkhoff, but instead of rendering a quaternion as a complex pair $(a, b)$ they denote it by $a + bj$, which in our notation becomes $(a, 0) + (b, 0) \times j$. In [23] Price employs the notation $a + i_2 b$ for the bicomplex number that we would write $(a, 0) + j \odot (b, 0)$. All these notations seem rather cumbersome compared to the concise pair notation.

In this context it may be noted that the practice of denoting complex numbers by pairs of real numbers is due to Hamilton [13]. His objective was to employ these algebraic couples or number-couples, as he called them, to remove the “metaphysical stumbling blocks” that he felt was connected with the imaginary unit $\sqrt{-1}$. An account of Hamilton’s thoughts on algebraic couples is found in [15], the reading of which inspired the writing of this treatise.

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Appendix

The generalized Green theorem

We shall give a proof of the generalized Green theorem as stated in Chapter 5.

**Theorem 5.4** Let $\phi_1$ and $\phi_2$ be two complex functions of $a$ and $b$ such that they are holomorphic in a domain $G \subseteq B$. Also let $S(\Gamma)$ be an orientable surface in $G$ bounded by the closed curve $\Gamma$. Then

\[
\int_{\Gamma} \phi_1 \cdot da + \phi_2 \cdot db = \int_{S(\Gamma)} \left( \frac{\partial \phi_2}{\partial a} - \frac{\partial \phi_1}{\partial b} \right) \cdot da \cdot db
\]

**Proof.** Let the $\mathbb{R}^4$-representations of $\phi_1$ and $\phi_2$ be

\[
\phi_1(a, b) = \psi_1(x, y, z, u) + i \psi_2(x, y, z, u)
\]

\[
\phi_2(a, b) = \psi_3(x, y, z, u) + i \psi_4(x, y, z, u)
\]

$a = x + iy$, $b = z + iu$

Because $\phi_1$ and $\phi_2$ are holomorphic in $G$ we get on application of (3.3) the following derivatives:

\[
\frac{\partial \phi_2}{\partial a} = \partial_x \psi_3 + i \partial_x \psi_4
\]

\[
\frac{\partial \phi_1}{\partial b} = \partial_z \psi_1 + i \partial_z \psi_2
\]

Inserting (2), (3) and (4) into (1), performing the complex multiplications and demanding the equality of real and imaginary parts change our proof obligation into:

\[
\int_{\Gamma} (\psi_1 \, dx - \psi_2 \, dy + \psi_3 \, dz - \psi_4 \, du) =
\int_{S(\Gamma)} [(\partial_x \psi_3 - \partial_z \psi_1) \, dxdz - (\partial_x \psi_3 - \partial_z \psi_1) \, dydu -
(\partial_x \psi_4 - \partial_z \psi_2) \, dxdu - (\partial_x \psi_4 - \partial_z \psi_2) \, dydz]
\]

\[
\int_{\Gamma} (\psi_2 \, dx + \psi_1 \, dy + \psi_4 \, dz + \psi_3 \, du) =
\int_{S(\Gamma)} [(\partial_x \psi_3 - \partial_z \psi_1) \, dxdu + (\partial_x \psi_3 - \partial_z \psi_1) \, dydz +
(\partial_x \psi_4 - \partial_z \psi_2) \, dxdz - (\partial_x \psi_4 - \partial_z \psi_2) \, dydu]
\]

The line and surface integrals of these formulas are of the type found at the right-hand sides of (5.18) and (5.19).
At this point we invoke the generalized Stokes theorem in \( \mathbb{R}^4 \). It states that four real-valued, continuously differentiable functions \( A_1, A_2, A_3, A_4 \) of the variables \( x, y, z, u \) satisfy

\[
(7) \quad \int_{\Gamma} (A_1 \, dx + A_2 \, dy + A_3 \, dz + A_4 \, du) = \int_{S(\Gamma)} (A_{12} \, dx \, dy + A_{13} \, dx \, dz + A_{14} \, dx \, du + A_{23} \, dy \, dz + A_{24} \, dy \, du + A_{34} \, dz \, du)
\]

where \( S(\Gamma) \) is an orientable surface in \( \mathbb{R}^4 \) bounded by \( \Gamma \) and

\[
A_{12} = \partial_x A_2 - \partial_y A_1, \quad A_{13} = \partial_x A_3 - \partial_z A_1, \quad A_{14} = \partial_x A_4 - \partial_u A_1
\]

\[
A_{23} = \partial_y A_3 - \partial_z A_2, \quad A_{24} = \partial_y A_4 - \partial_u A_2, \quad A_{34} = \partial_z A_4 - \partial_u A_3
\]

Our intention is to show that (7) reduces to (5) or (6) when the \( A_k \)-functions are chosen properly. Focusing on formula (5) first, we compare its left-hand side with the left-hand side of (7) and identify

\[
A_1 = \psi_1, \quad A_2 = -\psi_2, \quad A_3 = \psi_3, \quad A_4 = -\psi_4
\]

The functions \( \phi_1 \) and \( \phi_2 \) are holomorphic in \( a \) and \( b \), hence the \( \psi_k \)-functions are continuously differentiable and satisfy the CR-equations (3.5) in the following way:

\[
\begin{align*}
\partial_x \psi_1 &= \partial_y \psi_2, \quad \partial_x \psi_2 = -\partial_y \psi_1 \\
\partial_x \psi_3 &= \partial_y \psi_4, \quad \partial_x \psi_4 = -\partial_y \psi_3 \\
\partial_z \psi_3 &= \partial_u \psi_4, \quad \partial_z \psi_4 = -\partial_u \psi_3
\end{align*}
\]

We thus get

\[
\begin{align*}
A_{12} &= -\partial_x \psi_2 - \partial_y \psi_1 = 0 \\
A_{13} &= \partial_x \psi_3 - \partial_z \psi_1 \\
A_{14} &= -\partial_x \psi_4 - \partial_u \psi_1 = -\partial_x \psi_4 + \partial_z \psi_2 \\
A_{23} &= \partial_y \psi_3 + \partial_z \psi_2 = -(\partial_x \psi_4 - \partial_z \psi_2) \\
A_{24} &= -\partial_y \psi_4 + \partial_u \psi_2 = -\partial_x \psi_3 + \partial_z \psi_1 \\
A_{34} &= -\partial_z \psi_4 - \partial_u \psi_3 = 0
\end{align*}
\]

These identities make the right-hand side of (5) equal to the right-hand side of (3), as required.

Comparison of the left-hand side of (5) with the left-hand side of (3), in turn, tells us to choose

\[
A_1 = \psi_2, \quad A_2 = \psi_1, \quad A_3 = \psi_4, \quad A_4 = \psi_3
\]
It is as straightforward as above to show that evaluated for this choice the expressions $A_{ij}$ make the right-hand side of (7) equal to the right-hand side of (6), which concludes the proof.

□