ON MAXIMUM, TYPICAL, AND GENERIC RANKS

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Abstract. We show that for several notions of rank including tensor rank, Waring rank, and generalized rank with respect to a projective variety, the maximum value of rank is at most twice the generic rank. We show that over the real numbers, the maximum value of the real rank is at most twice the smallest typical rank, which is equal to the (complex) generic rank.

1. Introduction

In many areas of applied mathematics, machine learning, and engineering, the notion of shortest decomposition of a vector into simple vectors is of prime importance. See for example [CM96, BCMT10, BBCM11, Lan12] and [DSS09, Chapter 4]. The length of the shortest decomposition is usually called the rank of the vector.

In this article we consider the rank of a vector with respect to a variety over an arbitrary field F, but we will highlight the real and complex situations later on. Let $X \subset \mathbb{F}P^n$ be a projective variety and let $\hat{X} \subset \mathbb{F}^{n+1}$ be the affine cone over $X$. The variety $X$ is called nondegenerate if $X$ (or equivalently $\hat{X}$) is not contained in a hyperplane. In this case, for any vector $v \in \mathbb{F}^{n+1}$, $v \neq 0$, we can define the rank of $v$ with respect to $X$ (X-rank of $v$ for short) as follows:

$$\text{rank}_X(v) = \min r \text{ such that } v = \sum_{i=1}^{r} x_i \text{ where } x_i \in \hat{X},$$

i.e., the X-rank of $v$ is the length of the shortest decomposition of $v$ into elements of $\hat{X}$. We will assume that the variety $X$ is irreducible, as is the case in the applications of interest.

For example, tensor rank (real or complex) is rank with respect to the Segre variety, symmetric tensor rank (also called Waring rank) is rank with respect to the Veronese variety, and anti-symmetric tensor rank is rank with respect to the Grassmannian variety. See section 3 for more discussion of examples.

A rank $r$ is called generic if the vectors of X-rank $r$ contain a Zariski open subset of $\mathbb{F}^{n+1}$. For this we assume $\mathbb{F}$ is infinite, to avoid trivialities. It is well-known that over any algebraically closed field there is a unique generic X-rank for a nondegenerate variety $X$. Over $\mathbb{C}$, we can equivalently define rank $r$ to be generic if the set of vectors of rank $r$ contains an open subset of $\mathbb{C}^{n+1}$ with respect to the standard product topology. A significant effort has gone into the calculation of the generic rank for various varieties $X$, and the generic rank is fairly well understood for various tensor ranks over $\mathbb{C}$, see section 3.

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Much less is known about the maximum $X$-rank. One of our main results shows that the generic rank and the maximum rank cannot be too far apart:

**Theorem 1.** Let $X \subset \mathbb{F}^n$ be an irreducible nondegenerate variety over an algebraically closed field $\mathbb{F}$. Let $r_{\text{max}}$ be the maximum value of rank with respect to $X$ and let $r_{\text{gen}}$ be the generic value of rank with respect to $X$. Then

$$r_{\text{max}} \leq 2r_{\text{gen}}.$$  

While the above bound is elementary, we will show in Section 3 that it strongly improves many existing bounds on the maximum tensor rank. Also, due to its generality it applies to any notion of tensor rank: regular, symmetric and antisymmetric. We show in Theorem 6 that the above bound can be slightly improved to $r_{\text{max}} \leq 2r_{\text{gen}} - 1$ in the special case where the Zariski closure of the vectors of rank $r_{\text{gen}} - 1$ forms a hypersurface in $\mathbb{F}^{n+1}$ and $\mathbb{F}$ has characteristic 0.

Over the real numbers a rank $r$ is called **typical** if the set of vectors of rank $r$ contains an open subset of $\mathbb{R}^{n+1}$ with respect to the Euclidean topology. Unlike over the complex numbers, there may exist more than one typical rank. However, the lowest typical rank is equal to the generic rank over the complexification of $X$. While this is probably well-known to the experts, we did not find it explicitly stated in the literature in this generality:

**Theorem 2.** Let $X \subset \mathbb{R}^n$ be an irreducible nondegenerate real projective variety whose real points are Zariski dense. Let $r_0$ be the minimum typical rank with respect to $X$ and let $r_{\text{gen}}$ be the generic rank with respect to the complexification $X_\mathbb{C} = X \otimes \mathbb{C}$. Then

$$r_0 = r_{\text{gen}}.$$  

We are able to show a very similar and elementary bound for maximum $X$-rank with respect to real varieties as well:

**Theorem 3.** Let $X \subset \mathbb{R}^n$ be an irreducible nondegenerate real projective variety whose real points are Zariski dense. Let $r_0$ be the minimum typical rank with respect to $X$, and let $r_{\text{max}}$ be the maximum value of rank with respect to $X$. Then

$$r_{\text{max}} \leq 2r_0.$$  

Again, despite being elementary the above theorem provides the best known bounds on maximum real tensor rank in many cases. In fact, much less is known about the maximum real rank, so the bound of Theorem 3 is even stronger.

## 2. Proofs of Main Theorems

We begin by working over an algebraically closed field, then we work over $\mathbb{R}$.

### 2.1. Rank over algebraically closed fields.

Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate variety over an algebraically closed field $\mathbb{F}$. Since $X$ spans $\mathbb{P}^n$, choosing a basis from points of $X$ shows that every point has rank at most $n + 1$. The best general bound on maximum rank with respect to $X$ is to our knowledge, in characteristic 0:

$$r_{\text{max}}(X) \leq n + 1 - \text{dim}(X),$$  

see [Ger96], remarks following Theorem 7.6, and [LT10] Prop. 5.1. This fails in positive characteristic. For example, if $X$ is a smooth plane conic in characteristic 2 and $P$ is its
strange point, meaning that every line through $P$ is tangent to $X$, then $r(P) = 3$. However see [Bal11] for the result $r_{\text{max}}(X) \leq n + 2 - \dim(X)$ in positive characteristic.

We now prove Theorem 1 which usually provides a much better bound on the maximum rank:

Proof of Theorem 1. Every point is a sum of two general points, each having rank $r_{\text{gen}}$, and therefore the maximum rank is at most $2r_{\text{gen}}$.

Explicitly, let $U$ be a Zariski dense open subset of points of rank exactly $r_{\text{gen}}$. Let $q \in \mathbb{P}^n$ be any point and let $p$ be any point in $U$. The line $L$ through $q$ and $p$ intersects $U$ at another point $p'$ (in fact, at infinitely many more points). Since $p$ and $p'$ span $L$, $q$ is a linear combination of $p$ and $p'$. Since $p$ and $p'$ each have rank $r_{\text{gen}}$, $q$ has rank at most $2r_{\text{gen}}$. □

We now describe a slight improvement to this theorem in certain cases. First we recall the definition of the secant variety.

Definition 4. The $r$-th secant variety $\sigma_r(X)$ is the Zariski closure of the set of points of rank at most $r$.

Since points of $X$ span $\mathbb{P}^n$ the $(n + 1)$-st secant variety $\sigma_{n+1}(X)$ certainly fills $\mathbb{P}^n$. Note that the generic rank with respect to $X$ is the least $r$ such that $\sigma_r(X) = \mathbb{P}^n$.

The following map will be useful. Let $\hat{X}$ be the affine cone over $X$ and let $\Sigma_{r,X} : \hat{X}^r \to \mathbb{A}^{n+1}$ be the map

$$\Sigma_{r,X}(x_1, \ldots, x_r) = x_1 + \cdots + x_r.$$

The image of $\Sigma_{r,X}$ is precisely the affine cone over the set of points of rank $r$ or less. The secant variety $\sigma_r(X)$ is the Zariski closure of the projectivization of $\Sigma_{r,X}(\hat{X}^r)$. (See for example [SS06] where a slight variant of this map is used to describe the defining ideal of $\sigma_r(X)$.) If $X$ is irreducible then so is $\sigma_r(X)$ and if $r < r_{\text{gen}}(X)$ then $\dim \sigma_r(X) < \dim \sigma_{r+1}(X)$ [Adl87, 1.2].

The set of points of rank $r_{\text{gen}}$ contains a dense Zariski open subset of $\sigma_{r_{\text{gen}}}(X) = \mathbb{P}^n$. For $r < r_{\text{gen}}$ the set of points of rank $\leq r$ contains a dense subset of $\sigma_r(X)$; this dense subset can be taken to be open, and to consist of points of rank equal to $r$:

Lemma 5. Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate variety over an algebraically closed field. Let $r \leq r_{\text{gen}}$. The set of points of rank equal to $r$ contains a dense Zariski open subset of $\sigma_r(X)$.

Proof. The projectivization of the image of $\Sigma_{r,X}$ is dense in $\sigma_r(X)$ and constructible by Chevalley’s theorem. Every dense constructible set contains a dense open subset. The set of points of rank less than $r$ is contained in $\sigma_{r-1}(X)$, which has strictly lower dimension than $\sigma_r(X)$. Removing it leaves a nonempty dense open subset of the points of rank equal to $r$. □

Now we can give a slight improvement to Theorem 1 if the secant variety $\sigma_{r_{\text{gen}}-1}(X)$ is a hypersurface, in characteristic zero.

Theorem 6. Let $X \subset \mathbb{P}^n$ be an irreducible nondegenerate variety over an algebraically closed field of characteristic zero. Suppose $X$ is not a hypersurface, but for some $r$, $\sigma_r(X)$ is a hypersurface; necessarily $r = r_{\text{gen}} - 1$. Then $r_{\text{max}} \leq 2r + 1 = 2r_{\text{gen}} - 1$. If the hypersurface $\sigma_r(X)$ has no points of multiplicity equal to $\deg(\sigma_r(X)) - 1$ then $r_{\text{max}} \leq 2r = 2r_{\text{gen}} - 2$. 


Proof. Let $d = \deg(\sigma_r(X))$. First suppose $q$ is any point at which $\sigma_r(X)$ has multiplicity strictly less than $d - 1$, including multiplicity 0 if $q$ lies off of $\sigma_r(X)$. Let $p \in \sigma_r(X)$ be a general point and let $L$ be the line through $p$ and $q$. By Bertini’s theorem \cite[Thm.~6.3]{Jou83} $L$ intersects $\sigma_r(X)$ with multiplicity 1 at $p$, and with multiplicity strictly less than $d - 1$ at $q$ (if $q$ lies on $\sigma_r(X)$). So $L$ intersects $\sigma_r(X)$ in at least one more point $p'$ distinct from $p$ and $q$. Since $p$ is general, so is $p'$. Then $q$ is a linear combination of $p$ and $p'$, so $r(q) \leq r(p) + r(p') = 2r$.

Next suppose $q$ is a point at which $\sigma_r(X)$ has multiplicity $d$. Then $\sigma_r(X)$ is a cone with vertex $q$. Let $p \in \sigma_r(X)$ be a general point and $L$ the line through $p$ and $q$; then $L \subset \sigma_r(X)$ and $L$ intersects the open subset of points of rank $r$, so in fact $L$ has infinitely many points of rank $r$. Choosing $p'$ as before, once again $r(q) \leq 2r$.

Finally suppose $q$ is a point at which $\sigma_r(X)$ has multiplicity equal to $d - 1$. Let $p \in \mathbb{P}^n$ be a general point and let $L$ be the line through $p$ and $q$. Since $p$ is general, $L$ intersects $\sigma_r(X)$ with multiplicity $d - 1$ at $q$, so $L$ intersects $\sigma_r(X)$ at another point $p'$. Since $p$ is general, $p' \in \sigma_r(X)$ is general. Then $q$ is a linear combination of $p$ and $p'$, so $r(q) \leq r(p) + r(p') = 2r + 1$. □

We mention a simple generalization of Theorem 6 and (1):

Proposition 7. Suppose $\mathbb{F}$ has characteristic 0 and $\sigma_k(X)$ has codimension $c$. Let $s$ be the maximum rank of points on $\sigma_k(X)$. Then $r_{\text{max}} \leq \max\{s, (c + 1)k\}$.

Proof. Let $q \in \mathbb{P}^N$. If $q \in \sigma_k(X)$ then $r_X(q) \leq s$. Otherwise, a general $c$-plane through $q$ is spanned by its intersection with $\sigma_k(X)$, which is reduced by Bertini’s theorem \cite[Thm.~6.3]{Jou83}, giving $q$ as a linear combination of $c + 1$ general points on $\sigma_k(X)$ which each have rank $k$.

The upper bound (1) is given by $k = s = 1$. Theorem 6 is the case $c = 1$. We believe that the best bounds resulting from this proposition are just these previously observed extreme cases, and intermediate values of $k$ and $c$ probably do not give interesting new bounds. Perhaps if some secant variety of $X$ is highly degenerate, this bound might be interesting.

Remark 8. The $X$-rank of $q$ is the least length of a reduced zero-dimensional subscheme of $X$ whose span includes $q$. A zero-dimensional subscheme of $X$ whose span includes $q$ is called an apolar scheme to $q$. Related quantities include the cactus rank (or scheme length), the least length of any apolar scheme; the smoothable rank, the least length of any smoothable apolar scheme; the curvilinear rank, similarly; and so on. See \cite{BBM12} for a thorough treatment and comparison of these and several other notions. In \cite{Bal13} it is shown that the cactus rank, smoothable rank, and curvilinear rank with respect to a Veronese variety are bounded by twice the generic rank. As all these quantities are less than or equal to the $X$-rank, we recover these results.

2.2. Real varieties. Now we consider rank with respect to a real variety. As before, let $X \subset \mathbb{R}P^n$ be an irreducible nondegenerate variety. We assume the real points of $X$ are Zariski dense in $X$, that is, $X$ is the Zariski closure of its set of real points; equivalently, $X$ has a smooth real point.

Let $X_\mathbb{C} = X \otimes \mathbb{C}$ be the complexification of $X$, i.e. the variety in $\mathbb{C}P^n$ defined by the same equations that define $X \subset \mathbb{R}P^n$. Since $X_{\mathbb{C}}$ is still the Zariski closure of the real points of $X$, $X_{\mathbb{C}}$ is irreducible. (Otherwise, if $f, g$ are complex polynomials such that $fg$ vanishes on $X_{\mathbb{C}}$ but neither $f$ nor $g$ does, then $|f|^2$ and $|g|^2$ are real polynomials inducing a decomposition of the real points of $X$.) A priori, the real rank of $v \in \mathbb{R}^{n+1}$ with respect to $X$ may be...
strictly greater than the (complex) rank of the same point \( v \in \mathbb{C}^{n+1} \) with respect to the complexification \( X_C \). Moreover, the maximum real rank with respect to \( X \) may be strictly greater than the maximum (complex) rank with respect to \( X_C \). See for example \cite{Rez13b}.

An integer \( r \) is called a **typical rank** if it is the rank of every point in some nonempty open subset of \( \mathbb{R}^{n+1} \) in the Euclidean topology. In contrast to the closed field case, there may be more than one typical rank. See for example \cite{Ble12}.

We now show Theorem 2. It is proved for triple tensor products \( \mathbb{C}^\ell \otimes \mathbb{C}^m \otimes \mathbb{C}^n \) in \cite[Thm. 7.1]{Fri12}. (See also results and references in \cite[§7]{Fri12} regarding the maximum typical rank.)

**Proof of Theorem 2.** Certainly \( r_0 \geq r_{\text{gen}} \): \( \sigma_{r_{\text{gen}}-1}(X_C) \) is contained in a hypersurface and it is defined over \( \mathbb{R} \). Therefore \( \sigma_{r_{\text{gen}}-1}(X_C) \) is contained in a hypersurface defined over \( \mathbb{R} \), and hence so is \( \sigma_{r_{\text{gen}}-1}(X) \). Thus \( r_{\text{gen}} - 1 \) is not a typical rank, nor is any rank less than \( r_{\text{gen}} - 1 \).

On the other hand, \( r = r_{\text{gen}} \) is a typical rank. Since \( \hat{X} \) and \( \hat{X}^r \) are semialgebraic sets and the map \( \Sigma_{r,X} \) is a linear projection, the image \( S = \Sigma_{r,X}(\hat{X}^r) \) is a semialgebraic set by the Tarski-Seidenberg theorem \cite{BCR98}. We can write \( S \) as a finite union \( B_1 \cup \cdots \cup B_t \) of basic semialgebraic sets, where each \( B_i \) is nonempty and defined by a finite set of real polynomial equations \( f(x) = 0 \) and inequalities \( f(x) > 0 \) \cite{BCR98}. If the definition of \( B_i \) includes an equation then \( B_i \) is contained in a hypersurface. Since the Zariski closure of \( S \) is \( \sigma_{r_{\text{gen}}}(X) = \mathbb{RP}^n \), \( S \) is Zariski dense, so there must be at least one \( B_i \) whose definition consists solely of inequalities. Removing the closure of points of rank less than \( r_{\text{gen}} \) if necessary, this \( B_i \) is an open set in the Euclidean topology consisting of points of rank \( r_{\text{gen}} \). This shows \( r_{\text{gen}} \) is a typical rank. \( \square \)

Combining the above result with Theorem 1 shows that the maximum complex rank with respect to \( X \) is at most twice the lowest typical rank. But the maximum real rank may be greater than the maximum complex rank. So in Theorem 3 we show, analogously to Theorem 1 that the maximum real rank with respect to \( X \) is also bounded by twice the least typical rank.

**Proof of Theorem 3.** Let \( B \subset \mathbb{R}^{n+1} \) be a small open ball in which every point has rank \( r_0 \). Then \( B - B \) is an open neighborhood of the origin in which every point \( p \) is a sum (difference) of two points of rank \( r_0 \), so \( r(p) \leq 2r_0 \). But every nonzero point has a scalar multiple in \( B - B \) and rank is invariant under scalar multiplication. \( \square \)

### 3. Applications to Tensor Rank

We now apply our bound on the maximum rank with respect to \( X \) to various tensor ranks over \( \mathbb{C} \) and \( \mathbb{R} \). We also discuss the relation between our bound and previously known bounds on the maximum tensor rank. Note that there seems to be relatively little known about upper bounds for real tensor rank. With rare exceptions previously known upper bounds on maximum rank are over \( \mathbb{C} \), while our Theorems also give the same upper bounds over \( \mathbb{R} \).

#### 3.1. Symmetric Tensor Rank

Symmetric tensors correspond to homogeneous polynomials (forms). The symmetric tensor rank of a homogeneous form \( F \) of degree \( d \) in \( n \) variables (equivalently \( n \)-variate symmetric tensor of order \( d \)) is the least number \( r \) of terms needed to write \( F \) as a linear combination of \( d \)th powers of linear forms,

\[
F = c_1 \ell_1^d + \cdots + c_r \ell_r^d.
\]
This corresponds precisely to the rank of a form $F$ with respect to the $d$-th Veronese variety $\nu_d(\mathbb{P}^{n-1})$. This is also known as the Waring rank of $F$. For example, since $xy = \frac{1}{4}(x + y)^2 - \frac{1}{4}(x - y)^2$, the Waring rank of $xy$ is 2 (as long as the characteristic of the field is not 2). See [CM96, IK99, Lan12] for introductions to Waring rank. We limit our discussion to the fields $\mathbb{R}$ and $\mathbb{C}$.

We denote the maximum Waring rank $r_{\text{max}}(n, d)$. Classically, $r_{\text{max}}(n, 2) = n$ and $r_{\text{max}}(2, d) = d$ are well-known, but to our knowledge, only two other values of maximum rank over $\mathbb{C}$ are known: $r_{\text{max}}(3, 3) = 5$ [Yer32, CM96, LT10] and $r_{\text{max}}(3, 4) = 7$ [Kle99, Par13].

The vector space of forms of degree $d$ in $n$ variables has dimension $\binom{n+d-1}{n-1}$, so trivially $r_{\text{max}}(n, d) \leq \binom{n+d-1}{n-1}$ (by taking a basis consisting of powers of linear forms). Several improvements are known: $r_{\text{max}}(n, d) \leq \binom{n+d-1}{n-1} - n + 1$ [Ger96, LT10]; better, $r_{\text{max}}(n, d) \leq \binom{n+d-2}{n-1}$ [BBS08]; and up till now the best known upper bound for complex Waring rank is $r_{\text{max}}(n, d) \leq \binom{n+1}{n-3}$.

(2) $r_{\text{max}}(n, d) \leq \binom{n+d-2}{n-1} - \binom{n+d-6}{n-3}.

We denote the generic complex Waring rank—the Waring rank of a general form in $n$ variables of degree $d$, that is, one with general coefficients—by $r_{\text{gen}}(n, d)$. Its value is given by the Alexander–Hirschowitz theorem [AH95]: $r_{\text{gen}}(n, d) = \lceil \frac{1}{n} \binom{n+d-1}{n-1} \rceil$, except if $(n, d) = (n, 2), (3, 4), (4, 4), (5, 4), (5, 3)$. In the first exceptional case, $r_{\text{gen}}(n, 2) = n$. For the rest, $r_{\text{gen}}(n, d) = \lceil \frac{1}{n} \binom{n+d-1}{n-1} \rceil + 1$, and the secant variety $\sigma_{r_{\text{gen}}-1}(\nu_d(\mathbb{P}^{n-1}))$ is a hypersurface.

We immediately obtain the following Corollary:

**Corollary 9.** The maximum real Waring rank of a real form of degree $d \geq 3$ in $n$ variables is at most:

$$r_{\text{max}}(n, d) \leq 2 \left\lfloor \frac{1}{n} \binom{n+d-1}{n-1} \right\rfloor,$$

except $r_{\text{max}}(3, 4) \leq 11$, $r_{\text{max}}(4, 4) \leq 19$, $r_{\text{max}}(5, 4) \leq 29$, $r_{\text{max}}(5, 3) \leq 15$. The same upper bound holds for the complex Waring rank.

Asymptotically, Jelisiejew’s upper bound is $nd/(n+d-1)$ times the generic rank. Our bound is asymptotically better than Jelisiejew’s upper bound, but worse for some small cases. In the following table, $r_{\text{max}}^J$ denotes Jelisiejew’s upper bound (2) and $r_{\text{max}}^*$ denotes our upper bound, $r_{\text{max}}^* = 2r_{\text{gen}}$. The exact maximum is listed in the two cases where it is known.

| $n$ | $d$ | $r_{\text{gen}}$ | $r_{\text{max}}^J$ | $r_{\text{max}}^*$ | $r_{\text{max}}$ |
|-----|-----|-----------------|----------------------|----------------------|----------------------|
| 3   | 3   | 5               | 8                    | 5                    | 5                    |
| 3   | 4   | 9               | 11                   | 7                    | 7                    |
| 3   | 5   | 14              | 14                   | 14                   | 14                   |
| 3   | 6   | 20              | 20                   | 20                   | 20                   |
| 3   | 7   | 27              | 24                   | 24                   | 24                   |
| 3   | 8   | 35              | 30                   | 30                   | 30                   |
| 4   | 3   | 9               | 10                   | 10                   | 10                   |
| 4   | 4   | 18              | 19                   | 19                   | 19                   |
| 4   | 5   | 32              | 28                   | 28                   | 28                   |
| 4   | 6   | 52              | 42                   | 42                   | 42                   |
| 4   | 7   | 79              | 60                   | 60                   | 60                   |
| 4   | 8   | 114             | 84                   | 84                   | 84                   |

Other than the exceptional cases of the Alexander–Hirschowitz theorem, the hypersurface condition of Theorem 6 happens for Veronese varieties $X = \nu_d(\mathbb{P}^{n-1})$ if and only if

$$(r_{\text{gen}} - 1)n - 1 = \binom{n+d-1}{n-1} - 2$$
This happens if and only if \((n+d-1) \equiv 1 \pmod{n}\). For example, if \(n = 2\) and \(d\) is even then \((n+d-1) = d + 1\) is odd; other instances include \((8) \equiv 1 \pmod{3}\), \((11) \equiv 1 \pmod{4}\), \((14) \equiv 1 \pmod{5}\).

**Example 10.** For binary \(d\)-forms the maximum rank (real or complex) is \(r_{\text{max}} = d\) and the generic rank is \(r_{\text{gen}} = \left\lfloor \frac{d+1}{2} \right\rfloor\). See [BCG11, Rez13b] for real binary forms with real (but not complex) Waring rank \(d\). In particular \(r_{\text{max}} = 2r_{\text{gen}} - 2\) if \(d\) is even, \(r_{\text{max}} = 2r_{\text{gen}} - 1\) if \(d\) is odd, so the upper bound of Theorem 1 is almost sharp.

It is known that in the case \(d\) is even, the \((r_{\text{gen}} - 1)\)st secant variety is a hypersurface, defined by the vanishing of the determinant of the middle—that is, \((d/2)\)th—catalecticant [Syl51]. Theorem 6 then gives the upper bound \(2r_{\text{gen}} - 1\), which is still not sharp.

However it is not hard to show (and seems to be known to the experts) that this hypersurface, which has degree \((d/2) + 1\), has points of multiplicity \(d/2\) precisely along the Veronese \(\nu_{d}(\mathbb{P}^1)\), the set of points of rank 1. Thus we recover the sharp bound \(r_{\text{max}} \leq 2r_{\text{gen}} - 2\).

### 3.2. Tensor Rank.

**Tensor rank of a tensor of order \(d\) in \(n\) variables corresponds precisely to the rank with respect to the Segre variety \(\text{Seg}(\mathbb{P}^n \times \cdots \times \mathbb{P}^n)\), with \(d\) factors.** The generic tensor rank has been well-studied and it is known for several families of tensors.

**Example 11** (Tensors of format \(2 \times \cdots \times 2\)). The generic rank of tensors in \((\mathbb{C}^2)^{\otimes n} = \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2\) is \(\left\lfloor \frac{n^2}{n+1} \right\rfloor\) [CGG11]. Therefore the maximum rank (real or complex) is at most \(2\left\lfloor \frac{n^2}{n+1} \right\rfloor\). For \(n > 7\) this is better than the bound \(2^{n-2}\) given in [SSM13, Cor. 6.2] (see also [Sta12] for the bound \(3 \cdot 2^{n-3}\)).

**Example 12** (Tensor rank in triple products). It is known that the generic rank of tensors in \(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n\) is 5 when \(n = 3\), or \(\left\lfloor \frac{n^2}{5n-2} \right\rfloor\) when \(n > 3\) [Lic85, Lan12 Thm. 3.1.4.3]. Therefore by Theorem 1 and Theorem 3 the maximum rank of tensors in \(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n\) and the maximum real rank of tensors in \(\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n\) are at most 10 when \(n = 3\), or \(2\left\lfloor \frac{n^2}{5n-2} \right\rfloor \approx \frac{2}{3}n^2\) when \(n > 3\).

However this is not as good as previously known bounds. It is known that the maximum rank of tensors in \(\mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n\) and the maximum real rank of tensors in \(\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n\) are at most \(\left\lfloor \frac{n+1}{2} \right\rfloor\) [SMS10, Thm. 3.4, Thm. 3.7] and [AS79, AL80]. (See also [Lan12, Cor. 3.1.2.1], [BL13, Cor. 3.5] and [Fri12, Cor. 6.7].)

### 3.3. Waring problem with higher degree terms.

A variant of Waring rank is the number of terms needed to write a homogeneous polynomial of degree \(kd\) as a linear combination of \(k\)th powers of \(d\)-forms. See [FOSI12, Rez13a, CO13]. This is given by rank with respect to the projection of \(\nu_{k}(\mathbb{P}(\text{Sym}^d \mathbb{F}^n))\) to \(\mathbb{P}(\text{Sym}^{kd} \mathbb{F}^n)\) given by the multiplication map \(\text{Sym}^k(\text{Sym}^d \mathbb{F}^n) \to \text{Sym}^{kd} \mathbb{F}^n\). In [FOSI12] it is shown that a generic complex \(kd\)-form in \(n+1\) variables is a sum of at most \(k^n\) \(k\)th powers of \(d\)-forms, and no fewer when \(d\) is sufficiently large. Therefore every complex \(kd\)-form is a sum of at most \(2k^n\) \(k\)th powers of \(d\)-forms, and every real \(kd\)-form is a real linear combination (possibly including negative coefficients) of at most \(2k^n\) \(k\)th powers of real \(d\)-forms.

### 3.4. Antisymmetric Tensor Rank.

The rank of an alternating tensor \(T \in \bigwedge^k \mathbb{C}^n\) is the least number of terms needed to write \(T\) as a linear combination of simple wedges. It is given by the rank with respect to a Grassmannian in its Plücker embedding. In [AOP12] it is shown that the generic rank of an alternating tensor in \(\bigwedge^3 \mathbb{C}^n\) is asymptotically \(\frac{n^2}{18}\). Therefore the maximum rank of such a tensor is asymptotically less than or equal to \(\frac{n^2}{9}\).
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