Groupoids and Faà di Bruno Formulae for Green Functions

Imma Gálvez  Joachim Kock  Andrew Tonks
UPC  UAB  London Metropolitan U.

October 14, 2011
The classical Faà di Bruno formula for $\text{Diff}(\mathbb{C}, 0)$

- The classical Faà di Bruno formula tells how to compute the higher derivatives of the composite of two functions in one variable.
- Consider a formal power series in one variable

$$f(z) = \sum_{n=1}^{\infty} a_n(f) z^n \in \mathbb{C}[[z]]$$

with $f(0) = 0$ and coefficients given by higher derivatives

$$a_n(f) = \frac{f^{(n)}(0)}{n!}.$$

- Traditionally $f(z)$ is regarded as the germ of a $C^\infty$ function satisfying $f(0) = 0$. It is a diffeomorphism if $a_1(f) \neq 0$, orientation-preserving if $a_1(f) > 0$ and tangent to the identity if $a_1(f) = 1$. 
Under composition of functions \( \mathbb{C}[ [ z ] ] \) becomes a (noncommutative) monoid,

\[
(g \circ f)(z) = \sum_{n=1}^{\infty} a_n(g) \left( \sum_{m=1}^{\infty} a_m(f) z^m \right)^n.
\]

with unit the identity function \( 1(z) = z \).

The coordinate functions are elements of the linear dual of this monoid,

\[
\langle a_n, f \rangle = a_n(f), \quad a_n \in \mathbb{C}[ [ z ] ]^*.
\]

The polynomial ring in the \( a_n \) has a coalgebra structure, with counit \( \varepsilon(a_n) = \langle a_n, 1 \rangle \) and comultiplication defined by

\[
\langle \Delta a_n, f \otimes g \rangle = \langle a_n, g \circ f \rangle
\]

which may be determined explicitly by expanding (1).
The Faà di Bruno bialgebra

The Faà di Bruno bialgebra

\[ \mathcal{F} = \mathbb{C}[[a_1, a_2, \ldots]] \]

is the free commutative algebra on the symbols \( a_n, n \geq 1 \), with counit and comultiplication defined above.

This is only a bialgebra. It is graded, but not connected: \( \mathcal{F}_0 \) is spanned by the powers of \( a_1 \), which are group-like.

One may impose the relation \( a_1 = 1 \) (which is easily seen to generate a bi-ideal), to obtain the classical Hopf algebra

\[ \mathcal{H} = \mathbb{C}[[a_2, a_3, \ldots]]. \]

The antipode is the classical Lagrange inversion formula.

For our purposes it will be more important to consider the bialgebra \( \mathcal{F} \), without the restriction \( a_1 = 1 \).
The Green function for $\text{Diff}(\mathbb{C}, 0)$

- The formula for $\Delta$ can be packaged into a single equation. We consider the formal series (the Green function)

$$A = \sum_{k \geq 1} \frac{A_k}{k!} = \sum_{k \geq 1} a_k \in \mathbb{C}[[a_1, a_2, a_3, \ldots]]$$

- The resulting form of the Faà di Bruno formula is the Leitmotiv of the present work:

$$\Delta(A) = \sum_{k \geq 1} A^k \otimes a_k.$$

- The values of $\Delta$ on the individual generators $a_k$ can be extracted from this formula.

- With the the obvious convention $a_0 = 0$, we may allow the sum to start at $k = 0$
Bialgebras of trees

- **Connes-Kreimer bialgebra of rooted trees**: is the free \( \mathbb{C} \)-algebra \( \mathcal{H} \) on the set of isomorphism classes of (combinatorial) trees.

\[ \Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H} \]
\[ T \mapsto \sum_c P_c \otimes S_c, \]

- The comultiplication is given on generators by

- Here the sum is over all admissible cuts of \( T \)

- \( P_c \) is the forest (interpreted as a monomial) found above the cut.

- \( S_c \) is the subtree found below the cut (or the empty forest, in case the cut is below the root).
\( \mathcal{H} \) is a connected bialgebra: the grading is by the number of nodes, and \( \mathcal{H}_0 \) is spanned by the unit.

Therefore, by general principles it acquires an antipode and becomes a Hopf algebra.
Motivation from van Suijlekom’s work

- **Walter D. van Suijlekom.**
  ‘The structure of renormalization Hopf algebras for gauge theories. I. Representing Feynman graphs on BV-algebras’. *Comm. Math. Phys.*, 290(1):291-319, 2009

- There, the Connes–Kreimer Hopf algebra of Feynman graphs is considered.

- Trees encode nestings of Feynman graphs

- Individual graphs are not physically meaningful.

- But what is physically meaningful is to consider the *Green function* associated to them, that is, for a fixed kind of vertex \( v \)

\[
G_v = 1 + \sum_{\text{res}(\Gamma) = v} \frac{\Gamma}{\text{Aut}(\Gamma)}
\]
In van Suijlekom’s work, a Faà di Bruno formula

\[ \Delta(Y_v) = \sum_{n_1,\ldots,n_k} Y_v Y_{v_1}^{n_1} \ldots Y_{v_k}^{n_k} \otimes p_{n_1,\ldots,n_k}(Y_v) \]

appears, with \( p_{n_1,\ldots,n_k} \) is the projection onto graphs containing \( n_i \) vertices of type \( v_i \), with

\[ Y_v = \frac{G_v}{\prod_{e \in v} \sqrt{G_e}} \]

where the product runs over the edges \( e \) of the vertex \( v \).

Note that for each type of edge, one has

\[ G_e = 1 - \sum_{\text{res}(\Gamma) = e} \Gamma / \text{Aut}(\Gamma) \]

for a fixed type of edge.

Our aim was to prove a formula along the line of this one for trees. To do so, we need operadic trees.
In operad theory, the nodes represent operations, and trees are formal combinations of operations. These allow loose ends (leaves). Formal definition of operadic trees to be found in [Kock 2011, IMRN]. Here are some examples (disregard the planar aspect)

- The *leaves* are the edges that do not start in a node.
- The *root edge* does not end in a node.
- A node without incoming edge is a nullary operation.
- The (small) incoming edges drawn at every node serve to keep track of the arities of the operations.
Trees appearing in pQFT are naturally operadic. They encode nestings of 1PI Feynman graphs. Hence, have decorations by primitive 1PI graphs on nodes and by interaction labels on edges. So the graph can be recovered from the operadic (decorated) tree. Symmetries of the original Feynman graph are better dealt with by means of operadic trees. This is relevant for Green functions.
Correspondence between Feynman graphs and trees

- A Feynman graph can be reconstructed from the decorated tree.
- The decoration involves bijections encoding the exact way a small graph is substituted into the big graph.

- All this is taken care of by the theory of polynomial functors as described in last week's seminar (see also [Kock2011]).
The bialgebra of operadic trees

We will be considering

- the category of operadic trees and their morphisms.
- the category of forests and morphisms between them.

- A cut of an operadic tree is a subtree containing the root $c : S \subset T$

- If a node is in a subtree, so are all its incident edges.
- For each edge $e$ of $T$, there is an ideal tree consisting of $e$ (as the new root) and all the descendent edges and nodes.
- $P_c$ is the forest consisting of all the ideal trees generated by the leaves of $S$. 
The bialgebra of operadic trees (continued)

- $\mathcal{B}$ is the free $\mathbb{C}$-algebra on the set of isomorphism classes of operadic trees.
- A comultiplication is defined on its generators by

$$
\Delta : \mathcal{B} \longrightarrow \mathcal{B} \otimes \mathcal{B}
$$

$$
T \mapsto \sum_{c : S \subset T} P_c \otimes S,
$$

- $\mathcal{B}$ becomes a graded bialgebra.
The bialgebra of operadic trees (end)

- $\mathcal{B}$ is not connected.
- $\mathcal{B}_0$ is spanned by the trivial tree $\quad$ and all its powers.
- These are all group-like, so a connected bialgebra can be obtained by imposing the equation $1 = \quad$. 
The core of a non-trivial operadic tree $T$ is the combinatorial tree $T^\bullet$ obtained by pruning off all leaves as well as the root edge.

Taking core is functorial in root-preserving inclusions.

Hence, it induces a bialgebra homomorphism from the bialgebra of operadic trees to the Hopf algebra of combinatorial trees à la Connes–Kreimer.
Consider the power series ring completion of $B$.

The *Green function* in this ring is the series

$$G := \sum_{T} \frac{T}{|\text{Aut}(T)|}$$

where the sum runs over all isomorphism classes of operadic trees.

It is analogous to the combinatorial Green function of Feynman graphs.

If *decorated* trees are considered instead, there is a Green function for each possible decoration of the root edge.

This is analogous to the Green function in QFT, where there is a Green function for each possible residue in the theory.
The Faà di Bruno formula for the Green function in $\mathcal{B}$

**Theorem**

Let $g_n$ be the Green function of trees with $n$ leaves in $\mathcal{B}$, so that

$$G = \sum_{n \in \mathbb{N}} g_n.$$  

Then the following Faà di Bruno formula holds

$$\Delta(G) = \sum_{n \in \mathbb{N}} G^n \otimes g_n$$

- To prove this theorem soundly we will use groupoids.
- Here $n$ is an isomorphism class of the groupoid of finite sets (of leaves). Our proof works for ‘coloured’ trees, over more general polynomial functors. The groupoid $\text{FinSet}$ is then replaced by $\text{FinSet}/I$ where $I$ is the set (or groupoid) of colours.
Groupoids

- A **groupoid** is a category in which every arrow is invertible.
- A **morphism** of groupoids is a functor.
- Intuitively, groupoids are ‘fat sets with symmetries’.
- Instead of having just a few isolated points (elements in a set) we now have large chunks of points which are equivalent, with specific arrows linking them up.
- More than one arrow can exist between two given objects, and indeed a single object can have more than one arrow to itself — these are its symmetries.
A set is considered a groupoid in which the only arrows are the identity arrows.

Conversely, a groupoid \( X \) gives rise to a set by taking its set of connected components, i.e. the set of isomorphism classes in \( X \), denoted \( \pi_0(X) \).

A group can be considered as a groupoid with only one object.

Conversely, for each object \( x \) in a groupoid \( X \) there is associated a group, the vertex group, denoted \( \pi_1(x) \) or \( \text{Aut}(x) \), which consists of all the arrows from \( x \) to itself.

The homotopy notations \( \pi_0 \) and \( \pi_1 \) reflect the fact that groupoids are a model for certain topological spaces, the homotopy 1-types.
An equivalence of groupoids is just an equivalence of categories, i.e. a functor possessing a pseudo-inverse. This is the analogue of a homotopy equivalence in topology. Equivalent groupoids have the same properties, for example the same $\pi_0$, $\pi_1$, and the same cardinality.
We will need some homotopy universal constructions.

The (homotopy) fibre of a morphism

\[ E \xrightarrow{p} B \]

over \( b \in B \) is the groupoid \( E_b \) with objects

\[ (e, \phi), \quad e \in E, \quad \phi : pe \xrightarrow{\sim} b \]

and arrows

\[ (\epsilon, \text{Id}) : (e, \phi) \rightarrow (e', \phi') \]

with \( \epsilon : e \rightarrow e' \) such that \( \phi' \circ p\epsilon = \phi \)

\[
\begin{array}{c}
  pe \xrightarrow{\sim} pe' \\
  \phi \downarrow \xRightarrow{\sim} \quad \phi' \downarrow \xRightarrow{\sim}
  b \xrightarrow{\text{Id}} \equiv b
\end{array}
\]
Weak quotients

- Whenever a group acts on a set or a groupoid $X$

\[ G \times X \to X \]

the \textit{weak quotient} $X/G$ is the groupoid obtained by gluing in a path between $x$ and $y$ for each $g \in G$ such that $gx = y$.

- The weak quotient is often denoted $X//G$ to distinguish it from the naïve quotient, but we don’t need the latter here.

- If $G$ acts on the set $\{x\}$, then the weak quotient $\{x\}/G$ is the groupoid with one object and vertex group $G$.

- For a groupoid $X$, we will be considering the groupoid $\{x\}/\text{Aut}(x)$ for each object $x \in X$. 
The equivalent skeleton of a groupoid

Every groupoid $X$ is equivalent to its skeleton:

$$X \simeq \sum_{x \in \pi_0 X} \{x\}/\text{Aut}(x)$$

where the sum sign denotes disjoint union of groupoids.
Integration formula

- Let \( f : X \to B \) be a morphism of groupoids.
- Consider the fibre over \( b \) for each \( b \in \pi_0 B \).
- The weak quotient

\[
X_b / \text{Aut}(b)
\]

gives rise to an equivalence of groupoids

\[
X \simeq \sum_{b \in \pi_0 B} X_b / \text{Aut}(b)
\]

- We will denote

\[
\int_{b \in B} X_b := \sum_{b \in \pi_0 B} X_b / \text{Aut}(b)
\]
Integration along the fibres (or: the Fubini Principle)

- Given morphisms of groupoids

\[ X \xrightarrow{f} B \xrightarrow{t} I \]

we have

\[ \sum_{b \in \pi_0 B} X_b / \text{Aut}(b) \simeq \sum_{i \in \pi_0 I} \left( \sum_{b \in \pi_0 B_i} X_b / \text{Aut}_i(b) \right) / \text{Aut}(i) \]

- In integral notation,

\[ \int_{b \in B} X_b \simeq \int_{i \in I} \left( \int_{b \in B_i} X_b \right). \]
Double Counting Lemma

Let $A, B, U$ be groupoids, together with morphisms $U^{T} \xrightarrow{\sim} U \xleftarrow{\sim} U^{S}$ for the fibres over $T \in B, S \in A$ respectively.

Then there are equivalences of groupoids

$$\int_{T \in B} T U \xrightarrow{\sim} U \xleftarrow{\sim} \int_{S \in A} U_{S}.$$
Cardinality

A groupoid $X$ is called *compact* when $\pi_0 X$ is a finite set, and for each object $x \in X$ the fundamental group $\text{Aut}(x)$ is a finite group.

The *cardinality* of a compact groupoid (a.k.a. *groupoid cardinality* or *homotopy cardinality*) is the rational number

$$|X| := \sum_{x \in \pi_0 X} \frac{1}{|\text{Aut}(x)|}$$

where $|\text{Aut}(x)|$ denotes the order of the vertex group at $x$. 
If $X$ is a finite set considered as a groupoid, then the groupoid cardinality coincides with the set cardinality.

If $G$ is a group considered as a one-object groupoid, then the groupoid cardinality is the inverse of the order of the group.

Groupoid cardinality is compatible with the sum, product and powers of groupoids:

\[
\begin{align*}
|X + Y| &= |X| + |Y| \\
|X \times Y| &= |X| \times |Y| \\
|\text{Grpd}(S, X)| &= |X|^{|S|} \quad (S \in \text{FinSet})
\end{align*}
\]

just as for finite sets.
Let $S$ be a compact groupoid and $G$ a finite group. Given any action of $G$ on $S$, we have

$$|S/G| = |S|/|G|.$$
Formal cardinality

- Let $B$ be a groupoid such that $\text{Aut}(b)$ is finite for each $b \in B$.
- Let $X \to B$ be a groupoid morphism with compact fibres.
- Consider the completed vector space spanned by the symbols $\delta_b$ for $b \in \pi_0(B)$.
- The \textit{formal cardinality of $X$ over $B$} is the element in that space given by

\[
|X|_B := \sum_{b \in \pi_0(B)} |X_b| / |\text{Aut}(b)| \cdot \delta_b.
\]

- If $B = B_1 \times B_2$ is a product groupoid we write the symbol

\[
\delta_{(b_1,b_2)} \quad \text{as} \quad b_1 \otimes b_2
\]
Let $T$ and $F$ be the groupoids of trees and forests respectively.

Taking the set of leaves or the set of roots gives morphisms $L : \text{FinSet} \to F$ and $R : F \to \text{FinSet}$.

The fibres (of the first three) are denoted $nF$, $F_n$, $nT$ as usual.

Let $C$ be the groupoid of trees and cuts, with

- **Objects**: root preserving inclusions $S \hookrightarrow T$ of trees
- **Morphisms**: isomorphisms of such arrows
Double Counting Lemma for Trees and Cuts

There are two projections $T \xleftarrow{m} C \xrightarrow{r} T$

\[
\begin{pmatrix}
T & \xrightarrow{\sim} & T' \\
\downarrow & & \downarrow \\
S & \xrightarrow{\sim} & S'
\end{pmatrix}
\xleftarrow{m} \begin{pmatrix}
T & \xrightarrow{\sim} & T' \\
\downarrow & & \downarrow \\
S & \xrightarrow{\sim} & S'
\end{pmatrix}
\xrightarrow{r} \begin{pmatrix}
S & \xrightarrow{\sim} & S'
\end{pmatrix}
\]

There are equivalences of groupoids

\[
\int_{T \in T} T C \xlongsim C \xlongsim \int_{S \in T} C_S
\]

where $T C$, $C_S$ are the fibres of $m, r$ over $T, S \in T$.

For each tree $T$ the fibre $T C$ is a discrete groupoid: it is equivalent to the set $\text{cut}(T)$ of cuts of $T$. 

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Groupoids and Faà di Bruno Formulae for Green Functions
The pullback groupoid \( \mathbf{F} \times \text{FinSet} \mathbf{T} \)

- Recall the groupoid homomorphisms

\[
R : \mathbf{F} \to \text{FinSet}, \quad L : \mathbf{T} \to \text{FinSet}
\]

\(R(P)\) = set of roots of a forest \(P\), \(L(S)\) = set of leaves of a tree \(S\).

- We consider the (homotopy) pullback groupoid

\[
\begin{array}{ccc}
\mathbf{F} \times \text{FinSet} \mathbf{T} & \xrightarrow{\sim} & \mathbf{F} \times \text{FinSet} \mathbf{T} \\
\downarrow R & & \downarrow R \\
\mathbf{F}_{LS} & \xrightarrow{\sim} & \mathbf{F} \\
\end{array}
\]

- **Objects**: \((P, S, R(P) \xrightarrow{\lambda} L(S))\) with \(P\) a forest and \(S\) a tree,

- **Morphisms** \((P, S, \lambda) \to (P', S', \lambda')\): pairs \((P \xrightarrow{\pi} P', S \xrightarrow{\sigma} S')\) compatible with the bijections \(\lambda, \lambda'\), that is, \(L\sigma \circ \lambda = \lambda' \circ R\pi\).

- We will also need the **fibres** over a fixed tree \(S\).
The Key Lemma

Lemma
There is an equivalence of groupoids

\[ \mathbb{C} \cong \mathbb{F} \times_{\text{FinSet}} \mathbb{T} \]

\[(S \mapsto T) \begin{array}{c} \text{prune} \\ \text{graft} \end{array} (P = (P_\rho)_{\rho \in \mathbb{R}P}, S, \lambda : \mathbb{R}P \cong \mathbb{L}S) \]

Idea of Proof

\[ S \mapsto T \]

\[ \begin{array}{c} \ell_1 \\ \ell_2 \end{array} \begin{array}{c} \ell_3 \ell_4 \ell_5 \end{array} \]

\[ P = (P_\rho) : \rho_i \leftarrow^\lambda \ell_i \]
The fibres over a fixed subtree $S$

- We have identified the fibres of the (homotopy) pullback

$$\left( \mathbf{F} \times_{\text{FinSet}} \mathbf{T} \right)_S \simeq \mathbf{F}_{LS}$$

- Hence, by the Key Lemma,

$$\mathbf{C}_S \simeq \mathbf{F}_{LS}$$

- By the double counting lemma,

$$\int_T \text{cut}(T) \simeq \int_T T \mathbf{C} \simeq \mathbf{C} \simeq \int_{S \in T} \mathbf{C}_S \simeq \int_{S \in T} \mathbf{F}_{LS}$$
Application of the Fubini principle

- Now we can split into different fibres, according to the composition

\[ \mathbf{C} \to \mathbf{T} \xrightarrow{L} \text{FinSet} \]

- Therefore one has

\[
\int T \text{cut}(T) \simeq \mathbf{C} \simeq \int S \mathbf{C}_S \quad \text{double counting}
\]

\[
\simeq \int S \mathbf{F}_{LS} \quad \text{Key Lemma}
\]

\[
\simeq \int_n \int_{S \in_n T} \mathbf{F}_n \quad \text{Fubini}
\]

\[
\simeq \int_n \mathbf{F}_n \times n \mathbf{T} \quad \text{integration of constant}
\]

- This is the groupoid version of the Faà di Bruno Theorem.

- It is an equivalence of groupoids over \( \mathbf{F} \times \mathbf{T} \).
Towards the Faà di Bruno Formula

Hence, we have proved the following

**Theorem**

\[
\int_{T \in T} \text{cut}(T) \simeq \int_{n \in \text{FinSet}} F_n \times_n T
\]

- Both sides are groupoids over \( F \times T \).
- The formal cardinality of the set \( \text{cut}(T) \) is

\[
|\text{cut}(T)| = \sum_{c \in \text{cut}(T)} P_c \otimes S_c
\]

- On the other hand, the formal cardinality of \( F_n \times_n T \) is

\[
|F_n \times_n T| = |T|^n \otimes |nT| = G^n \otimes G_n
\]
Theorem: The Faà di Bruno Formula

Therefore

\[ \sum_{T \in \pi_0 T} \sum_{c \in \text{cut}(T)} P_c \otimes S_c / |\text{Aut}(T)| = \sum_{n \in \pi_0 \text{FinSet}} G^n \otimes G_n / |\text{Aut}(n)| \]

That is,

\[ \sum_{T \in \pi_0 T} \Delta(T) / |\text{Aut}(T)| = \sum_{n} G^n \otimes g_n \]

So that we have proved

Theorem: The Faà di Bruno Formula

\[ \Delta(G) = \sum_{n} G^n \otimes g_n \]