CONGRUENCES AND COORDINATE SEMIRINGS
OF TROPICAL VARIETIES

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Abstract. In this paper we present two intrinsic algebraic definitions of tropical variety motivated by
the classical Zariski correspondence, one utilizing the algebraic structure of the coordinate semiring† of
an affine supertropical algebraic set, and the second based on the layered structure. We tie them to
tropical geometry, especially in connection with the dimension of an affine variety.

1. Introduction

The goal of this paper is to study families of affine supertropical varieties in terms of their coordinate
semirings†, or equivalently certain congruences of the polynomial semiring, paying particular attention
to an algebraic formulation of tropical dimension which will match the intuitive definition obtained from
simplicial complexes. Tropical varieties have been the focus of much investigation in tropical geometry,

 founding in terms of polyhedral complexes (i.e., piecewise linear objects) satisfying the
balancing condition, but this approach, although successful for curves and hypersurfaces, is not fully
compatible with Zariski’s approach to viewing varieties as the zero locus of an ideal \( I \) of polynomials in
\( K[\lambda_1, \ldots, \lambda_n] \) over a field \( K \). A key feature of Zariski’s approach is the prime spectrum of the coordinate
ring \( K[\lambda_1, \ldots, \lambda_n]/I \), which, in classical theory, also is identified with the algebra of polynomials restricted
to the variety.

The authors have translated the tropical theory to an algebraic language more amenable to structure
theory, for example in \[8, 11, 12, 13, 14\], where an extra “ghost level” \( A' \) is adjoined to the
original max-plus algebra \( A \), and additive idempotence is replaced by supertropicality, i.e., \( a + a = a' \),
cf. \[24\]. In this framework, the algebraic set of a collection of polynomials is just the set of vectors
all taking on ghost values. Although encapsulating the definition of “corner locus” in standard tropical
geometry, this approach enables one to set up a direct algebraic approach analogous to the Zariski
correspondence.

Even so, one encounters difficulty when considering algebraic sets of polynomials: The intersection of
tropical varieties need not be a tropical variety in the usual sense (even for planar curves). For example,
the non-transversal intersection of the curves defined by \( x + y + 0 \) and \( x + y^2 + 0 \) is the union of the two rays
emanating along the axes from the origin and fails the balancing condition, as does the non-transversal
intersection of the lines defined by \( x + y + 0 \) and \( x + y + 1 \), cf. Figure \[1\] (a) and (b), respectively. Such
curves can be excluded via a requirement that curves are in generic position, but one would prefer a
theory that deals with all cases.

The approach taken in this paper is to define the coordinate semiring† of an algebraic set \( X \) as the semiring† of polynomials (over the supertropical structure \( A \cup A' \)), realized as functions, restricted
to \( X \). This enables one to study the spectrum (but now of congruences rather than ideals), and leads
to a correspondence between algebraic sets and congruences, as indicated in \[12\]. The challenge remains
of using the algebraic structure to filter out the “bad” algebraic sets of the previous paragraph, namely,
those that do not satisfy the balancing condition. The obvious way is to restrict the class of permissible
congruences defining our algebraic sets. Several options have been proffered, most notably the “bend
In this paper, we present two supertropical alternatives which are based on algebraic and topological considerations.

Our main approach, given in §5, is via the coordinate semiring† of Definition 5.1. We impose a requirement on functions whose value on a dense algebraic subset are equal, and call these algebraic sets **admissible**. This natural condition is automatic in the classical algebraic geometrical world, by virtue of the easy part of the fundamental theorem of algebra, but needs to be stipulated in the tropical world. Tropical hypersurfaces are admissible, by Proposition 5.18 whereas when we ruin the balancing condition by erasing a facet, the algebraic set becomes inadmissible, by Proposition 5.22. In this way, admissibility provides a natural generalization of the balancing condition in higher codimensions.

Once one focuses on the appropriate algebraic sets, it is not difficult to define the dimension in terms of the length of chains of admissible varieties in §6, and prove that it is well-defined and consistent with the geometric intuition (Theorem 6.4). Nevertheless, at times the theory diverges from classical algebraic geometry. For example, algebraic sets can decompose non-uniquely into varieties, as is seen in Example 5.24.

The main weakness of Definition 5.1 comes from its strength: The intersection of admissible algebraic sets need not be admissible. Indeed, in the planar scenario, we do not want the intersection of a tropical line and quadric to be admitted, since then we would have to permit all line segments as varieties, and thus they all would be reducible (except for the points). On the other hand, if one wants to define a topology whose base is the closed sets, one needs the intersection of varieties to be a variety. This leads us in our second approach to a further refinement of the supertropical structure, namely the layered structure of [11], and in §7 we present a class of congruences which is closed under intersections, taken the layering into account, and also is Noetherian by Proposition 7.15. Thus, we also have a notion of dimension here, but globally it is larger than the simplicial dimension. This discrepancy can be overcome, but requires a more detailed local treatment that is beyond the scope of this paper.

2. **Background**

We review a few notions from semigroups and semirings. As customary, \( \mathbb{N} \) denotes the positive natural numbers, \( \mathbb{Q} \) denotes the rational numbers, and \( \mathbb{R} \) denotes the real numbers.

2.1. **Semigroups and monoids.**

A **monoid** is a semigroup with a unit element \( 1_M \). For any semigroup \( M := (M, \cdot) \) we can formally adjoin the unit element \( 1_M \) by declaring that \( 1_M a = a 1_M = a \) for all \( a \in M \), so when dealing with multiplication we work with monoids.

An Abelian monoid \( M := (M, \cdot) \) is **cancellative** with respect to a subset \( S \subseteq M \) if \( as = bs \) implies \( a = b \) whenever \( a, b \in M \) and \( s \in S \). In this case, we also say that \( S \) is a cancellative subset of \( M \).

2.2. **Ordered monoids.**

**Definition 2.1.** A **partially ordered monoid** is a monoid \( M \) with a partial order satisfying

\[
a \leq b \quad \text{implies} \quad ca \leq cb,
\]

where \( a,b \in M \) and \( c \in \mathbb{R} \).
for all elements \(a, b, c \in M\). A monoid \(M\) is **ordered** if the order is total.

Note that this definition excludes ordered Abelian groups such as \((\mathbb{Q}, \cdot)\) from consideration; on the other hand, \((\mathbb{Q}, +)\) is ordered in this sense.

**Definition 2.2.** A semigroup \(M := (\mathcal{M}, \cdot)\) is called **\(\mathbb{N}\)-divisible** if \(\sqrt[n]{\alpha} \in M\) for all \(a \in M\) and all \(n \in \mathbb{N}\). A monoid is **power-cancellative** if \(a^m = b^m\) implies \(a = b\).

**Remark 2.3.** One can uniquely define rational powers of any element in an \(\mathbb{N}\)-divisible, power-cancellative semigroup \(M\); adjoining a unit element \(1_M\) to \(M\), we could define \(a^0 = 1_M\).

**Remark 2.4.** By Bourbaki [1], any strictly cancellative Abelian monoid \(M\) can be embedded into an \(\mathbb{N}\)-divisible Abelian monoid \(\tilde{M}\), which we call the **divisible closure** of \(M\). Namely, by passing to the group of fractions, cf. [1], we may assume that \(M\) is a group. We formally introduce \(\sqrt[n]{\alpha}\) for each \(a \in M\), identifying \(\sqrt[n]{\alpha}\) with \(\sqrt[n]{\beta}\) iff \(a^n = b^n\). We define the product

\[
\sqrt[n]{a} \sqrt[n]{b} = \sqrt[n]{ab}.
\]

**Lemma 2.5.** If \(M\) is partially ordered, then \(\tilde{M}\) is endowed with the partial order given by

\[
\sqrt[n]{a} \leq \sqrt[n]{b} \iff a^n \leq b^n.
\]

If \(M\) is power-cancellative, then \(\tilde{M}\) is power-cancellative.

**Proof.** The relation is well-defined, and is easily seen to be a partial order. Furthermore, if \((\sqrt[n]{a})^k = (\sqrt[n]{b})^k\), then \(a^{nk} = b^{nk}\), implying \(a^n = b^n\), and thus \(\sqrt[n]{a} = \sqrt[n]{b}\). \(\square\)

In summary, any cancellative, power-cancellative ordered Abelian monoid can be embedded into an \(\mathbb{N}\)-divisible, power-cancellative ordered Abelian group, so we usually assume these hypotheses.

### 2.3. Semirings\(^1\)

Semirings were studied by Costa [2]. A standard general reference for the structure of semirings is [6]. For reasons discussed in the introduction of [12], it is convenient to deal a semiring without a zero element, which we call a **semiring**\(^1\). Thus, a semiring\(^1\) \(\langle R, +, \cdot, 1_R \rangle\) is a set \(R\) equipped with two binary operations \(+\) and \(\cdot\), called addition and multiplication, such that:

1. \((R, +)\) is an Abelian semigroup;
2. \((R, \cdot, 1_R)\) is a monoid with identity element \(1_R\);
3. Multiplication distributes over addition;
4. There exist \(a, b \in R\) such that \(a + b = 1_R\).

Condition (4) is a very weak condition that we do not need in this paper, but is needed to develop the theory of modules in later work. It is automatic in semirings with zero since \(1_R + 0_R = 1_R\), and also is obvious in the max-plus algebra since \(1_R + b = 1_R\) for any \(b \leq 1_R\).

**Remark 2.6.** Any ordered monoid \((\mathcal{M}, \cdot)\) gives rise to a semiring\(^1\), where we define \(a + b\) to be \(\max\{a, b\}\). Indeed, associativity is clear, and distributivity follows from (2.1).

One can always adjoin an additive neutral element \(0_R\) to a semiring\(^1\) to get a semiring, via the multiplicative rule

\[
0_R \cdot a = a \cdot 0_R = 0_R \quad \forall a \in R.
\]

**Definition 2.7.** A **homomorphism** of semirings\(^1\) is defined as a function \(\varphi : R \to R'\) that preserves addition and multiplication. To wit, \(\varphi\) satisfies the following properties for all \(a, b \in R\):

1. \(\varphi(a + b) = \varphi(a) + \varphi(b)\);
2. \(\varphi(a \cdot b) = \varphi(a) \cdot \varphi(b)\);
3. \(\varphi(1_R) = 1_{R'}\).
The structure theory of semirings$^\dagger$ is motivated by general considerations on universal algebra, for which we use [15] as a reference. We recall as a special case from [15, p. 61] that a congruence $\Omega$ on a semiring$^\dagger$ $R$ is an equivalence relation $\equiv$ preserving addition and multiplication, i.e., if $a_i \equiv b_i$ then $a_1 + a_2 \equiv b_1 + b_2$ and $a_1a_2 \equiv b_1b_2$. Sometimes we denote $\Omega$ as the relation $\equiv$, or, equivalently, as $\{(a, b) : a \equiv b\}$, a sub-semiring$^\dagger$ of $R \times R$.

A congruence $\Omega$ is cancellative if $ca \equiv cb$ implies $a \equiv b$; $\Omega$ is power-cancellative when $R/\Omega$ is power-cancellative (as a multiplicative monoid), i.e., if $a_1^k \equiv a_2^k$ for some $k \geq 1$ then $a_1 \equiv a_2$. (Power-cancellative congruences, also called torsion-free in [2], play the role of radical ideals.)

Any semiring$^\dagger$ homomorphism $\varphi : R \rightarrow R'$ gives rise to a congruence $\Omega_\varphi$ on $R$ given by $(a, b) \in \Omega_\varphi$ iff $\varphi(a) = \varphi(b)$; conversely, any congruence $\Omega$ gives rise to a semiring$^\dagger$ structure $R/\Omega$ on the equivalence classes, and a natural homomorphism $\varphi : R \rightarrow R/\Omega$ given by $a \mapsto [a]$.

Example 2.8. We define the trivial congruence $\Omega = \{(a, a) : a \in R\}$; in this case, $\varphi : R \rightarrow R/\Omega$ is an isomorphism.

As we shall see, the family of all congruences on the supertropical structure is too broad to support a viable geometric theory, so we restrict the family, to be specified later.

Let $\mathcal{C}(R)$ denote a given family of congruences on a given semiring$^\dagger$ $R$.

**Definition 2.9.** A congruence $\Omega \in \mathcal{C}(R)$ is $\mathcal{C}(R)$-irreducible if it cannot be written as an intersection $\Omega_1 \cap \Omega_2$ of congruences $\Omega_1$ and $\Omega_2$ in $\mathcal{C}(R)$, each properly containing $\Omega$.

$\mathcal{C}(R)$ is too broad for our purposes without a serious restriction. We say that $\mathcal{C}(R)$ is Noetherian if any ascending chain of congruences in $\mathcal{C}(R)$ terminates. Equivalently, any subset of congruences in $\mathcal{C}(R)$ has a maximal member. (For example, in classical algebra, one often takes $\mathcal{C}(R)$ to be the finitely generated congruences of the polynomial algebra.) The following observation is a standard application of Noetherian induction:

**Proposition 2.10.** Every congruence in a Noetherian family $\mathcal{C}(R)$ of congruences is a finite intersection of $\mathcal{C}(R)$-irreducible congruences.

**Proof.** Any maximal counterexample would be the intersection of two larger congruences in $\mathcal{C}(R)$, each of which by hypothesis is a finite intersection of congruences that are $\mathcal{C}(R)$-irreducible. □

There are several candidates for a working definition of $\mathcal{C}(R)$, such as [16]. In this paper we offer two: First, a traditional one using the Zariski topology, in [11] and then one in terms of the layered theory given in [17].

2.4 $\nu$-domains$^\dagger$.

Despite the elegance of Remark 2.6 the structure of the resulting semiring$^\dagger$ is too crude for some algebraic applications. To remedy this, we recall briefly the basics of supertropical algebra and generalize them in order to be able to handle functions.

**Definition 2.11.** A $\nu$-semiring$^\dagger$ is a quadruple $R := (R, \nu, \mathcal{G}, \nu)$ where $R$ is a semiring$^\dagger$, $\mathcal{T} \subset R$ is a multiplicative submonoid, $\mathcal{G} \subset R$ is a partially ordered semiring$^\dagger$ ideal, together with a map $\nu : R \rightarrow \mathcal{G}$, satisfying $\nu^2 = \nu$ as well as the conditions:

\[
\begin{align*}
    a + b &= a &\text{ whenever } \nu(a) &> \nu(b), \\
    a + b &= \nu(a) &\text{ whenever } \nu(a) &= \nu(b).
\end{align*}
\]

$R$ is called a $\nu$-domain$^\dagger$ when the multiplicative monoid $(R, \cdot)$ is commutative and cancellative with respect to $\mathcal{T}$.

If furthermore $\mathcal{T}$ (and thus also $\mathcal{G}$) is an Abelian group, we call $R$ a $\nu$-semifield$^\dagger$.

We write $a^\nu$ for $\nu(a)$. We write $a \sim_\nu b$ whenever $a^\nu = b^\nu$, and $a >_\nu b$ (resp. $a \geq_\nu b$) whenever $a^\nu > b^\nu$ (resp. $a^\nu \geq b^\nu$).

$\mathcal{T}$ is called the monoid of tangible elements, while the elements of $\mathcal{G}$ are called ghost elements and $\nu : R \rightarrow \mathcal{G}$ is called the ghost map. Intuitively, the ghost elements in $\mathcal{G}$ correspond to the original max-plus algebra, and $R$ is a cover of $\mathcal{G}$. But our interest lies in the tangible layer $\mathcal{T}$, since it captures the tropical geometry.
Definition 2.12. A supertropical domain† is a ν-domain‡ \( R := (R, \mathcal{T}, \mathcal{G}, \nu) \) for which \( \mathcal{G} := R \setminus \mathcal{T} \) is ordered and the restriction \( \nu|_\mathcal{T} : \mathcal{T} \to \mathcal{G} \) is onto. If, moreover, \( \mathcal{T} \) is an Abelian group, we call \( R \) a supertropical semifield†.

For each \( a \) in a supertropical domain† \( R \) we choose an element \( \hat{a} \in \mathcal{T} \) such that \( \hat{a}^\nu = a^\nu \). (Thus \( a \mapsto \hat{a} \) defines a section from \( \mathcal{G} \) to \( \mathcal{T} \), which we call the tangible lift.) Likewise, for \( a = (a_1, \ldots, a_n) \in R^{(n)} \), we define its tangible lift \( \hat{a} := (\hat{a}_1, \ldots, \hat{a}_n) \).

To clarify our exposition, most of the examples in this paper are presented for the supertropical semifield† \( (\mathbb{Q} \cup \mathbb{Q}^\nu, \mathbb{Q}^\nu, \nu) \), where \( 1 := 0_\mathbb{Q} \), cf. [8] and [14], built from the ordered group \( (\mathbb{Q}, +) \), whose operations are induced by the standard operations max and +. Here, \( \mathcal{T} \) is one copy of \( \mathbb{Q} \) whereas \( \mathcal{G} = \mathbb{Q}^\nu \), another copy of \( \mathbb{Q} \), and \( \nu|_\mathcal{T} : \mathcal{T} \to \mathcal{G} \) is an isomorphism; hence we can take the tangible lift simply to be \((\nu|_\mathcal{T})^{-1}\). Likewise, the same construction could be for any ordered Abelian group instead of \((\mathbb{Q}, +)\).

Tropical geometry is deeply connected to simplicial complexes, and we also need the relevant topology in this setting.

Definition 2.13. The \( \nu \)-topology on a supertropical semifield† \( F \) is defined as having the sub-base of neighborhoods \( B(a, \varepsilon) := \{ b \in F : \frac{b \wedge a}{a} \leq \varepsilon \} \), where \( a, \varepsilon \in \mathcal{T} \).

But we work in the generality of Definition 2.11 in order to handle functions, in particular polynomials, which \( \mathcal{G} \) is only partially ordered. Our structure of choice for understanding tropical geometry is the polynomial semiring† over the \( \nu \)-semifield†.

We want to describe congruences that arise with the \( \nu \)-structure.

Remark 2.14. Any congruence on a \( \nu \)-semiring† \( R \) satisfies the condition that if \( a \equiv b \) then \( a^\nu = 1_R a \equiv 1_R b = b^\nu \).

Remark 2.15. If \( \Omega \) is a congruence on a \( \nu \)-domain† \( R := (R, \mathcal{T}, \mathcal{G}, \nu) \), then \( \nu \) induces a ghost map \( [\nu] \) on \( R/\Omega = (R, \mathcal{T}/\Omega, \mathcal{G}/\Omega, [\nu]) \) via \([a]^\nu = [a^\nu]\), and when \( \nu|_\mathcal{T} \to \mathcal{G} \) is 1:1, then the restriction \([\nu] : \mathcal{T}/\Omega \to \mathcal{G}/\Omega\) also is 1:1.

We are interested in those congruences that yield \( \nu \)-domains. Towards this end, we have:

Definition 2.16. A congruence \( \Omega \) on a \( \nu \)-semiring† \( R \) is tangibly cancellative when \( ca \equiv cb \) implies \( a \equiv b \) for any \( a \in \mathcal{T} \).

3. Polynomial semirings† over supertropical domains†

Our main strategy is to define affine tropical varieties in terms of polynomials. We treat polynomials as functions that are defined logically as elementary sentences, and study their algebraic structure as a semiring†.

3.1. The function monoid and semiring†

Definition 3.1. Given a monoid \( \mathcal{M} := (\mathcal{M}, \cdot) \), we define the monoid of functions \( \text{Fun}(S, \mathcal{M}) \) to be the set-theoretic functions from \( S \) to \( \mathcal{M} \), in the usual way (via pointwise multiplication).

We say that a function \( g \in \text{Fun}(S, \mathcal{M}) \) dominates a function \( f \in \text{Fun}(S, \mathcal{M}) \) at \( a \) if \( f(a) \leq g(a) \). We take the corresponding partial order on \( \text{Fun}(S, \mathcal{M}) \) given by \( f \leq g \) iff \( f(a) \leq g(a) \) for each \( a \in S \).

Lemma 3.2. If the monoid \( \mathcal{M} \) is cancellative, then so is \( \text{Fun}(S, \mathcal{M}) \).

Proof. Easy componentwise verification, given in [12] Lemma 7.3. \( \square \)

Remark 3.3. When \( \mathcal{M} \) is a semiring†, then \( \text{Fun}(S, \mathcal{M}) \) is also a semiring† in the usual way (via pointwise addition).

As customary, we write \( f|_U \) for the restriction of a function \( f \in \text{Fun}(S, \mathcal{M}) \) to a nonempty subset \( U \subset S \). Although failing to satisfy bipotence, \( \text{Fun}(S, R) \) does satisfy the weaker property for a semiring† \( R \):

Remark 3.4. If \( R \) is idempotent then so is \( \text{Fun}(S, R) \), as seen by pointwise verification.
Remark 3.5. Given any sets $S' \subseteq S$, there is a natural onto homomorphism $\text{Fun}(S, R) \to \text{Fun}(S', R)$ given by $f \mapsto f|_{S'}$. Our main interest in this paper is to study chains of these homomorphisms. For any homomorphism $\varphi : R \to R'$ and $a \in S$, we can define the **evaluation homomorphism**

$$\psi_{a, \varphi} : \text{Fun}(S, R) \to R', \quad f \mapsto \varphi(f(a)).$$

The point of using $\nu$-domains is in the following observation:

**Remark 3.6.** Given a $\nu$-domain $R := (R, \mathcal{T}, G, \nu)$, define

$$\text{Fun}_{abtng}(S, R) := \{f \in \text{Fun}(S, R) : f(a) \in G \text{ for all } a \in S\},$$

$$\text{Fun}_{abtng}(S, R) := \{f \in \text{Fun}(S, R) : f(a) \in \mathcal{T} \text{ for all } a \in S\}.$$  \hfill (3.1)

Then $(\text{Fun}(S, R), \text{Fun}_{abtng}(S, R), \text{Fun}_{abtng}(S, R), \nu)$ becomes a $\nu$-domain, the main object of this paper, where we define $f^\nu$ by $f^\nu(a) := f(a)^\nu$. If $R$ is a supertropical domain, then so is $\text{Fun}(S, R)$, since $\nu$ induces an onto map $\text{Fun}_{abtng}(S, R) \to \text{Fun}_{abtng}(S, R)$.

**Example 3.7.** The functions of interest to us are the **polynomials** in $\Lambda := \{\lambda_1, \ldots, \lambda_n\}$, defined by formulas in the elementary language under consideration. $R[\Lambda]$ denotes the usual polynomials over the semiring $R$. In our examples, $1_R = 0$, so we write $\lambda$ for $0_\lambda$.

If we adjoin the symbol $-1$ (for multiplicative inverse), then we have the **Laurent polynomials** $R[\Lambda^\pm] := R[\lambda_1, \ldots, \lambda_n, \lambda_1^{-1}, \ldots, \lambda_n^{-1}]$. If our language also includes the symbol $\sqrt{-1}$, i.e., if we are working over a power-cancellative, divisibly closed monoid, then we may consider the polynomials $R[\Lambda]_{\text{rat}}$ with rational powers.

We need to study polynomials (in the appropriate context) and their roots, but viewed in the above context as functions under the natural map given by sending a polynomial $f$ to the function $a \mapsto f(a)$. Thus, $\text{Pol}(S, R)$ denotes the image in $\text{Fun}(S, R)$ of $R[\Lambda]$, $\text{Laur}(S, R)$ denotes the image of $R[\Lambda^\pm]$, and $\text{Rat}(S, R)$ denotes the image of $R[\Lambda]_{\text{rat}}$.

When $R$ is a supertropical domain, $\text{Pol}(S, R)$, $\text{Laur}(S, R)$, and $\text{Rat}(S, R)$ are sub-$\nu$-domains of $\text{Fun}(S, R)$. (But their structure differs from that of $\text{Fun}(S, R)$ because of the issue of tangibility, as we shall see.)

3.2. **Decompositions of polynomials.**

We assume throughout the remainder of this paper that $F = (F, \mathcal{T}, G, \nu)$ is a supertropical-semifield, with $\nu$ 1:1 and onto, and we have a given tangible lift $G \to \mathcal{T}$ given by $\nu^{-1}$, and $S \subseteq F^{(n)}$ is given. $\mathcal{R}$ denotes $\text{Pol}(S, F)$, $\text{Laur}(S, F)$, or $\text{Rat}(S, F)$, and monomials and polynomials are taken in the appropriate context. Namely any monomial has the form $h = \alpha \lambda_1^{i_1} \cdots \lambda_n^{i_n}$ for $\alpha \in R$ and each $i_j \in \mathbb{N}$, $\mathbb{Z}$ or $\mathbb{Q}$ respectively. We call $\lambda_1^{i_1} \cdots \lambda_n^{i_n}$ the **pure part** of $h$. Note that if $h_1 = \alpha_1 \lambda_1^{i_1} \cdots \lambda_n^{i_n}$ and $h_2 = \alpha_2 \lambda_1^{i_1} \cdots \lambda_n^{i_n}$ have the same pure part, then $h_1 + h_2 = (\alpha_1 + \alpha_2) \lambda_1^{i_1} \cdots \lambda_n^{i_n}$ is also a monomial.

**Remark 3.8.** Customarily one takes $\mathcal{R} = \text{Pol}(S, F)$, but it is easy to check via localization at the $\lambda_i$ that the definitions provide the same results for $\mathcal{R} = \text{Laur}(S, F)$.

**Definition 3.9.** A **decomposition** of $f \in \mathcal{R}$ is a sum $f := \sum_i h_i$ of monomials whose pure parts are distinct. (In other words, the number of monomials that are summands of $f$ is minimal.) The monomial $h_i$ is **essential** in $f$ at $a$ if $f|_{U} \neq (\sum_{j \neq i} h_j)|_{U}$ for some open neighborhood $U$ of $a$. A monomial $h_i$ is **essential** in $f$ if it is essential in $f$ at $a$ for some point $a$. A polynomial is **essential** if each monomial in its decomposition is essential.

Thus, a polynomial $f$ is a tangible monomial iff it has no proper decomposition. (In fact, this is an intrinsic way to define monomial.) We also need to handle the case in which a monomial is not essential anywhere, but does contribute to $f$ by taking on the same value at some point.

**Definition 3.10.** Decomposing a polynomial $f := \sum h_i$ as a sum of monomials, we say that an inessential monomial $h_i$ of $f$ is **quasi-essential** at $a$ if $f|_{U} \equiv \gamma \nu h_i(a)$. An inessential monomial $h_i$ is **quasi-essential** in $f$ if it is quasi-essential in $f$ at $a$ for some point $a$.

The **support** $\text{supp}_a(f)$ of $f$ is $\sum h_i$ at the point $a \in S$ is the set of monomials $h_i$ which dominate $f$ at $a$. The **support** $\text{supp}(f)$ of $f$ is $\bigcup_{a \in S} \text{supp}_a(f)$.

The **shell** of the decomposition of $f$ is the sum of the essential monomials $h_i$ in $\text{supp}(f)$. 


Example 3.11. The polynomial $f = \lambda^2 + 6$ has the obvious decomposition as written, and is its own shell. For the polynomial $f = \lambda^2 + 3\lambda + 6$, the monomial $h = 3\lambda$ is quasi-essential, since $f(3) = 6′$, whereas $h(3) = 6$.

Example 3.12. The polynomial $g = 2\lambda_1^2 + 2\lambda_2^2 + 0$ is the shell of $f = 2\lambda_1^2 + 2\lambda_2^2 + \lambda_1\lambda_2 + 0$, because $\lambda_1\lambda_2$ is dominated by $2\lambda_1^2 + 2\lambda_2^2$.

Example 5.20 below shows how a monomial can be quasi-essential at one point but essential somewhere else.

Lemma 3.13. Any monomial $h$ is multiplicative along any line, in the sense that

$$h(a^t b^{1-t}) = h(a)^t h(b)^{1-t}$$

for all $t \in \mathbb{R}$.

Proof. Write $h = a \lambda_1^{i_1} \cdots \lambda_n^{i_n}$, $a = (a_1, \ldots, a_n)$, and $b = (b_1, \ldots, b_n)$. Then

$$h(a^t b^{1-t}) = (a_1^t b_1^{1-t})^{i_1} \cdots (a_n^t b_n^{1-t})^{i_n} = a_1^{i_1 t} \cdots a_n^{i_n t} \alpha_1^{1-t} b_1^{1-t} \cdots b_n^{1-t} = h(a)^t h(b)^{1-t}.$$

□

Proposition 3.14. If two monomials $h_1$ and $h_2$ are equal at two points $a$ and $b$ then they are equal at every point in the line connecting $a$ and $b$.

Proof. Follows at once from the lemma. □

Proposition 3.15. If a monomial $h_1$ dominates $h_2$ at two points $a$ and $b$ then $h_1$ dominates $h_2$ at every point in the line connecting $a$ and $b$.

Proof. Each point can be written as $a^t b^{1-t}$ for $0 \leq t \leq 1$, and so

$$h_1(a^t b^{1-t}) = h_1(a)^t h_1(b)^{1-t} \geq_\nu h_2(a)^t h_2(b)^{1-t} = h_2(a^t b^{1-t}).$$

□

Definition 3.16. A polynomial $f \in \mathcal{R}$ is tangible when all of the coefficients of its essential monomials are tangible. $\mathcal{R}_\text{tng}$ denotes the monoid of tangible polynomials, and $\mathcal{R}_\text{gh}$ denotes the ideal of polynomials whose essential monomials have ghost coefficients.

Remark 3.17. This does not quite match the definition of $\text{Fun}_{\text{abtng}}(S, \mathcal{R})$ in Remark 3.6. For example, taking $f = \lambda + 2$ we have $f(2) = 2''$. Later on, we cope with this difficulty by considering evaluations on dense subsets, cf. Definition 4.11 below. This problem does not arise for monomials, so we can refer to tangible monomials without ambiguity.

Lemma 3.18. $\mathcal{R}_\text{tng}$ is a monoid, and $(\mathcal{R}, \mathcal{R}_\text{tng}, \mathcal{R}_\text{gh}, \nu)$ is a supertropical domain.

Proof. For $f, g \in \mathcal{R}_\text{tng}$, the essential monomials of $fg$ are products of essential monomials and thus tangible. Clearly $\mathcal{R}$ is a $\nu$-domain, seen by restricting Remark 3.6 and $\nu_\mathcal{R}_\text{tng}$ is onto, by inspection. □

Given a monomial $h = a \lambda_1^{i_1} \cdots \lambda_n^{i_n}$, we write $\hat{h}$ for $\hat{a} \lambda_1^{i_1} \cdots \lambda_n^{i_n}$, and for the decomposition $f = \sum_i h_i$ we write $\hat{f}$ for $\sum_i \hat{h}_i$ — the tangible lift of $f$.

4. Supertropical $\mathfrak{A}(\mathcal{R})$-varieties

We work over a $\nu$-semifield $\mathcal{F} = (F, T, \mathcal{G}, \nu)$, and fix a subset $S \subseteq F^{(\nu)}$. Recall that $\mathcal{R}$ denotes $\text{Pol}(S, F)$, $\text{Laur}(S, F)$, or $\text{Rat}(S, F)$, and monomials are taken in the appropriate context. In principle, we want to designate a family $\mathfrak{A}(\mathcal{R})$ of tropical algebraic subsets of $S$ with respect to elements of $\mathcal{R}$. An algebraic set then is $\mathfrak{A}(\mathcal{R})$-irreducible if it cannot be written as the proper union of two $\mathfrak{A}(\mathcal{R})$-algebraic sets, and $\mathfrak{A}(\mathcal{R})$ is Noetherian if every descending chain of $\mathfrak{A}(\mathcal{R})$-algebraic sets stabilizes. In this section we deal with the supertropical version.
4.1. Supertropical algebraic sets.

**Definition 4.1.** Take some set $S \subseteq F^{(n)}$. An element $a \in S$ is a corner root of $f \in R$ if $\hat{f}(a) \in \mathcal{G}$. The (affine) corner locus of $f$ with respect to the set $S$ is
\[ Z_{\text{corn}}(f; S) := \{ a \in S : a \text{ is a corner root of } f \} . \]
We write $Z_{\text{corn}}(f)$ for $Z_{\text{corn}}(f; F^{(n)})$. The total locus of $f$ is
\[ Z(f; S) := \{ a \in S : f(a) \in \mathcal{G} \} . \]

**Definition 4.2.** The (affine) corner algebraic set and the (affine) algebraic set of a non-empty subset $I \subseteq R$, with respect to the set $S$, are respectively
\[ Z_{\text{corn}}(I; S) := \bigcap_{f \in I} Z_{\text{corn}}(f; S), \quad Z(I; S) := \bigcap_{f \in I} Z(f; S) . \]
When $S$ is unambiguous (usually $F^{(n)}$), we write $Z_{\text{corn}}(I)$ and $Z(I)$ for $Z_{\text{corn}}(I; S)$ and $Z(I; S)$ respectively.

**Example 4.3.** Given $a = (a_1, \ldots, a_n) \in F^{(n)}$, the corner algebraic set of the non-empty subset $\{ \lambda_1 + a_1, \ldots, \lambda_n + a_n \} \subseteq R$ consists of all vectors $\nu$-equivalent to $a$, i.e., the $\nu$-fiber of $a$, and could be considered as the $\nu$-analog of a point. These are the minimal corner algebraic sets in $F^{(n)}$.

As usual, a hypersurface is the algebraic set of a single polynomial. A facet of a hypersurface $X = Z(f)$, $f = \sum h_i$ is a decomposition, is a maximal (with respect to inclusion) connected subset of $X$ contained in the hypersurface $Z(h_i + h_j)$ for some $h_i, h_j$ or $Z(h_i)$ (for a ghost monomial $h_i$). A face is a nonempty intersection of facets. A facet of an algebraic set $X = Z(I) = \bigcap_{f \in I} Z(f)$ is a maximal connected subset $W \subseteq X$ contained in an intersection of facets of $Z(f), f \in I$.

We want our varieties to be the irreducible algebraic sets, and these should correspond to the irreducible congruences. But there are subtleties that have to be dealt with. For $S \subseteq F^{(n)}$ we write $S|_{\text{ng}}$ for $S \cap T^{(n)}$, the tangible part of $S$.

**Example 4.4.** Let $X_1$ be the tropical line defined by the polynomial $\lambda_1 + 1\lambda_2 + 1$ and $X_2$ be the tropical curve defined by the polynomial $\lambda_1\lambda_2 + \lambda_1 + 0$, see Fig. 2. (This can be viewed as the curve of the Laurent polynomial $\lambda_1^{-1} + \lambda_2 + 0$, which is a flip of the tropical line, cf. Remark 3.8.) Then $(X_1 \cap X_2)|_{\text{ng}}$ is just the segment $[0, 1]$ on the $\lambda_1$-axis, so we see that any segment can be obtained as a corner algebraic set. This means that we will not have irreducible algebraic sets other than points, unless we make a serious restriction on the algebraic sets that we admit!

![Figure 2](image)

Likewise, any congruence defines its algebraic set:

**Definition 4.5.** An element $a \in S$ is a corner root of a pair $(f, g)$ (for $f, g \in R$) modulo a congruence $\Omega$, if $\hat{f}(a) \equiv \hat{g}(a) \in \mathcal{G}$. The (affine) corner locus of $f \in R$ with respect to the set $S$, modulo $\Omega$, is
\[ Z_{\text{corn}}((f, g); S)_{\Omega} := \{ a \in S : a \text{ is a corner root of } (f, g) \} . \]
We write $Z_{\text{corn}}(f)_{\Omega}$ for $Z_{\text{corn}}(f; F^{(n)})_{\Omega}$. The total locus of $(f, g)$, modulo $\Omega$, is
\[ Z((f, g); S)_{\Omega} := \{ a \in S : f(a) \equiv g(a) \in \mathcal{G} \} . \]
Definition 4.6. The (affine) corner algebraic set and the (affine) algebraic set of a non-empty subset $A \subseteq R \times R$ modulo a congruence $\Omega$, with respect to the set $S$, are respectively

$$Z_{\text{corn}}(A; S) \Omega := \bigcap_{(f, g) \in A} Z_{\text{corn}}((f, g); S) \Omega, \quad Z(A; S) \Omega := \bigcap_{(f, g) \in A} Z((f, g); S) \Omega.$$  

When $S$ is unambiguous (usually $F^{(n)}$), we write $Z_{\text{corn}}(A) \Omega$ and $Z(A) \Omega$ for $Z_{\text{corn}}(A; S) \Omega$ and $Z(A; S) \Omega$ respectively.

Note that any (corner) algebraic set of a set $A \subseteq R$ is a (corner) algebraic set of $A$ modulo the trivial congruence. Thus Definition 4.6 encompasses Definition 4.2.

Definition 4.7. Given a family $C(R)$ of congruences on $R$, we define a $C(R)$-(corner) algebraic set to be a (corner) algebraic set modulo some congruence in $C(R)$. A $C(R)$-(corner) algebraic set is $C(R)$-irreducible if it cannot be written as the union of two $C(R)$-(corner) algebraic sets. A $C(R)$- (corner) variety is an irreducible $C(R)$-(corner) algebraic set.

The $C(R)$-varieties are the basis for tropical geometry, under the appropriate choice of $C(R)$.

4.2. The Zariski topology.

We continue with the appropriate version of the Zariski topology. Each essential monomial of a polynomial defines an open set comprised of the points at which it dominates the other monomials. Let us formalize this notion.

Definition 4.8. For any decomposition $f = \sum_i h_i$ of a polynomial $f \in R$, define the component $D_{f,i}$ to be

$$D_{f,i} := \{ a \in S : \hat{f}(\hat{a}) = \hat{h}_i(\hat{a}) \}. \quad (4.1)$$

A component $D_{f,i}$ is tangible if the monomial $h_i$ is tangible, i.e., the $h_i(\hat{a}) \in T$ for all $a \in D_{f,i}$.

We call $h_i$ the dominant summand of $f$ on $D_{f,i}$. The weak topology is comprised of the tangible open sets generated by the components.

(Note that these are open, because the dominant monomials change at the closure.) But this is not the topology that we want to work with, since open sets need not be dense.

Definition 4.9. We define the principal corner open sets to be

$$D_{\text{corn}}(f; S) = S \setminus Z_{\text{corn}}(f; S) = \bigcup_{i \in I} D_{f,i},$$

taken over all components.

Put another way,

$$D(f; S)_{\text{corn}} = \{ a \in S : \hat{f}(\hat{a}) = \hat{h}_i(\hat{a}) \text{ for some unique monomial } h_i \text{ of } f \}.$$  

The principal corner open sets form a base for a topology on $S$, which we call the corner Zariski topology, whose closed sets are affine corner algebraic sets.

We quote [12] Proposition 9.4:

Proposition 4.10. The intersection of two principal corner open sets contains a nonempty principal corner open set. Hence, the principal corner open sets form a base of a topology on $R$, in which every open set is dense.

From now on, we use this topology, and its relative topology on any subset $S$ of $F^{(n)}$.

4.3. Tangible polynomials.

The naive choice for tangibles, $R_{\text{abtng}}$, cf. Remark 3.10 would not include polynomials (except tangible constants) since they all have corner roots and thus are not in $R_{\text{abtng}}$. The Zariski topology gives us a better $\nu$-structure for polynomials, which matches Definition 3.10.

Definition 4.11. A function $f \in \text{Fun}(S, R)$ is tangible over $S$ if $\{ a \in S : f(\hat{a}) \in T \}$ is dense under the relative Zariski topology on $S$ induced from $R$. $R_{\text{tng}}$ is the set of tangible polynomials of $R$, and

$$R_{\text{gh}} := \{ f \in R : f(\hat{a}) \in G \text{ for all } a \in S \}$$

is the set of ghost elements of $R$. 

Any polynomial $f \in \mathcal{R}_{\text{tng}}$ is tangible over $F^{(n)}$. Conversely, when $f$ is tangible over $F^{(n)}$, its essential monomials all must have tangible coefficients, since any quasi-essential monomial is dominated by the other monomials on a dense set.

The next observation explains why we can exclude the inessential monomials (even when quasi-essential) in the shell of $f$.

**Lemma 4.13.** Suppose $f = \sum h_i \in \mathcal{R}$, written as a sum of monomials, and, for $a \in S$, let

$$f_a := \sum \{ h_i : h_i \text{ is essential at } a \}.$$ 

Then $f(a) = f_a(a)$ in either of the following cases:

1. $a$ is an interior point in the $\nu$-topology, or
2. $f_a(a) \in \mathcal{G}$.

**Proof.** (i) Otherwise, $f(a) \neq f_a(a)$ would imply $f_a(a)$ would be tangible, i.e., there would be only one monomial $h_i$ essential at $a$, for which $h_i(a) = f(a)$. But the assumption that $a$ is an interior point implies that any quasi-essential monomial $h$ at $a$ satisfies $h(b) >_\nu h_i(b)$ for some $b$ in a neighborhood of $a$, and taking $b$ near enough to $a$ yields $f(b) = h_i(b) \leq_\nu h(b)$, contrary to the definition of quasi-essential.

(ii) Either $f(a) = f_a(a)$ or $f(a) = f_a(a)^\nu = f_a(a)$.

5. The coordinate semiring $^\dagger$

We return from tropical geometry to algebra via the coordinate semiring $^\dagger$, just as in classical algebraic geometry.

**Definition 5.1.** The coordinate semiring $^\dagger$ of an affine algebraic set $X \subseteq F^{(n)}$, denoted $F[X]$, is the image of the semiring $^\dagger$ map $\text{Pol}(F^{(n)}, F) \rightarrow \text{Fun}(X, F)$ given by the natural restriction $f \mapsto f|_X$. The Laurent coordinate semiring $^\dagger$ $F[X^\pm]$ is the image of $\text{Laur}(F^{(n)}, F)$ in $\text{Fun}(X, F)$. (Similarly, we could define $F[X]_{\text{rat}}$ to be image of $\text{Rat}(F^{(n)}, F)$ in $\text{Fun}(X, F)$.)

**Proposition 5.2.** Any polynomial $f \in F[X]$ has the same image on the interior of $X$ as its shell in $\text{Fun}(X, F)$.

**Proof.** We use Lemma 4.13 to remove all the inessential monomials.

We have a $\nu$-structure induced by functions. Define $F[X]_{\text{tng}}$ to be those polynomials which are tangible in the sense of Definition 4.11 and $F[X]_{\text{gh}}$ to be the restriction of $\mathcal{R}_{\text{gh}}$ to $X$.

**Lemma 5.3.** $F[X]_{\text{tng}}$ is a monoid, and $(F[X], F[X]_{\text{tng}}, F[X]_{\text{gh}}, \nu)$ is a $\nu$-semiring $^\dagger$. Likewise for $F[X^\pm]$ and $F[X]_{\text{rat}}$.

**Proof.** For $f, g \in F[X]_{\text{tng}}$, $\{ a \in S : f(a), g(a) \in \mathcal{T} \}$ is the intersection of two dense sets and thus is dense, implying $fg \in F[X]_{\text{tng}}$. The last assertion is clear by restricting Remark 3.6.

**Example 4.4.** $F[X]_{\text{tng}}$ is not supertropical, since $\nu$ no longer is onto. Indeed, let $X$ be the supertropical line, e.g., consider the algebraic set of the polynomial $f = \lambda_1 + \lambda_2 + 0$. The restriction of $f$ to $X$ is ghost by definition, and any tangible lift $\hat{f}$ would have to include either $\lambda_1 + \lambda_2$ or 0, seen by considering the vertical and horizontal rays. But then the (tangible) diagonal ray must include $\lambda_1 + \lambda_2$ or $0 + \lambda_1$ for $i = 1$ or $i = 2$, and then $\hat{f}$ produces a ghost value on one of the rays, contrary to it being tangible on $X$.

Note that $\lambda_1^2 + \lambda_2^2 + 0 = f^2$ as a function) does have the tangible lift $\lambda_1 \lambda_2 + 0$ on $X$. Likewise, $f$ has the tangible lift $\lambda_1 + \lambda_2^2 + 0$ on $F[X]_{\text{rat}}$.

**Remark 5.5.** When $X \subseteq Y$ we have a natural homomorphism $\text{Fun}(Y, F) \rightarrow \text{Fun}(X, F)$ obtained by restricting the domain of the function from $Y$ to $X$. This induces natural homomorphisms $F[Y] \rightarrow F[X]$,

$F[Y^\pm] \rightarrow F[X^\pm]$, and $F[Y]_{\text{rat}} \rightarrow F[X]_{\text{rat}}$.

The restriction map gives rise to a congruence $\Omega$ on $F[Y]$, for which $F[X] \cong F[Y]/\Omega$. Conversely, we say that a congruence $\Omega$ on $F[Y]$ is geometric if $F[Y]/\Omega \cong F[X]$ for some $X \subseteq Y$. Then we have a 1:1 correspondence between geometric congruences and coordinate semirings $^\dagger$.
Remark 5.6. Any geometric congruence $\Omega$ on $F[X]$ is a power-cancellative congruence which is cancellative with respect to the tangible polynomials.

Definition 5.7. Given a subset $X \subset S$, the congruence $\Omega_X$ on $F[X]$, called the congruence of $X$, is defined by the relation

$$f \equiv_X g \iff f(a) = g(a) \text{ for all } a \in X,$$

which we call a polynomial relation on $X$.

Example 5.8. If a monomial $h_i$ dominates $f$ and a tangible monomial $h'_j$ dominates $g$ on some subset $W$ of $X$, then the polynomial relation on that subset is given by $h_i(a) = h'_j(a)$ for all $a \in W$, which can be viewed as a Laurent relation $\frac{h_i}{h'_j}(a) = 1_F$ on $W$, and can be used in $W$ to eliminate any one variable appearing nontrivially.

Remark 5.9. $X$ is an algebraic set precisely when $\Omega_X$ is a geometric congruence. Thus we have a 1:1 correspondence between algebraic sets and geometric congruences.

An example of a non-geometric set precisely when $\Omega_X$ is a geometric congruence:

Example 5.10. Define the congruence $\Omega_1$ on $F[X]$ generated by

$$\Omega_1 := \{(f, g) : f \text{ and } g \text{ both lack constant terms}\}.$$

Then the images in $F[X]/\Omega_1$ of all constants are distinct, and we also have the classes of $\lambda + \alpha$ for each $\alpha \in F$. $F[X]/\Omega_1$ contains one more class, comprised of all polynomials lacking constant terms.

Next, define the congruence $\Omega_2$ on $F[X]$ generated by $\Omega_1$ and $\{(\alpha, \beta) : \alpha, \beta \in F\}$. Then $F[X]/\Omega_2$ has only three elements: The classes of $1_F$, $\lambda$, and $\lambda + 1_F$.

Example 5.11. Define $\Omega$ on $F[X]$ to be the congruence generated by some pair $(f, g)$ where $f$ and $g$ both have the same leading monomial in $\lambda_1$. For example, take $(f, g) = (\lambda_1^2 + \lambda_2, \lambda_1^2 + \lambda_2 \lambda_3) \in \Omega$. Then $\Omega$ restricts to the trivial equivalence wherever $\lambda_2, \ldots, \lambda_n$ are specialized to elements small enough in relation to $\lambda_1$.

The familiar correspondence between coordinate semirings$^1$ and algebraic sets is discussed in [12]. Lemma 3.2 shows that the coordinate semirings$^1$ all are $\nu$-domains$^1$. We want to single out those coordinate semirings$^1$ corresponding to algebraic sets that have tropical significance, and use these to define tropical dimension. This is an extremely delicate issue, since various natural candidates for tropical varieties fail to satisfy the celebrated “balancing condition” [7]. For example, as is well known, the intersection of the (standard) tropical lines defined by the polynomials $\lambda_1 + \lambda_2 + 0$ and $\lambda_1 + \lambda_2 + a$ for $a > 0$ is just the ray given by $\lambda_1 = \lambda_2$ starting at $(a, a)$. Thus, if we were to define a variety as the intersection of tropical curves, we would have to cope with line segments of arbitrary length. Likewise, the intersection of the curves defined by $\lambda_1 + \lambda_2^k + 0$ over $k \in \mathbb{N}$ is just two perpendicular rays. So we need conditions to identify such degeneracies, preferably in terms of polynomials.

Remark 5.12. By Proposition 3.14, if two monomials agree on a dense subset of $X$, then they agree on $X$. It follows that if two polynomials $f$ and $g$ agree on a dense subset of $X$ then $f(a) \equiv_\nu g(a)$ for all $a \in X$; in other words, their only difference is in being ghost or not.

Definition 5.13. Two polynomials $f$ and $g$ essentially agree on $X \subseteq F^{(n)}$ if there is an open dense subset $U$ of $X$ (in the relative topology obtained from the Zariski topology) for which $f|_U = g|_U$. The coordinate semiring$^1$ $F[X]$ is admissible if it is a $\nu$-domain$^1$ satisfying the following condition:

- Any two polynomials $f$ and $g$ that essentially agree on $X$ are equal.

We now get to our main objective.

Definition 5.14. An admissible (corner) algebraic set is a (corner) algebraic set whose coordinate semiring$^1$ is an admissible $\nu$-domain$^1$.

$\mathcal{C}(\mathbb{R})_{adm}$ is the set of geometric congruences corresponding to admissible (corner) algebraic sets.

Example 5.15. Consider the surface $X := Z_{corn}(f)$ of the polynomial $f = \lambda_1 + \lambda_2 + \lambda_3 + 0$ in $F^{(3)}$, where we erase the facets contained in the hyperplanes determined by $\lambda_1 = \lambda_2$ and $\lambda_3 = 0$, and take the closure. Then the functions $\lambda_1 + \lambda_2$ and $\lambda_3 + 0$ are the same on all points except $(\alpha, \alpha, \alpha)$ for $\alpha > 0$ and...
(0, β, 0) for β < α, 0, where α, β ∈ T, for which one side is ghost and the other tangible. Thus, λ₁ + λ₂ and λ₃ + 0 essentially agree on X, and X is not admissible.

Example 5.16. Let X be the hypersurface defined by the tangible polynomial \( f = \sum_i h_i \), written as a sum of at least 3 monomials, and let \( X' \) be obtained by erasing the set \( \{ a \in X : h_i(a) >_\nu h_i(a), i \geq 3 \} \) and taking the closure. (Renumbering the \( h_i \) if necessary, we may assume that this set is nonempty.) Let \( f_k = \sum_{i \neq k} h_i \), for \( k = 1, 2 \). Then \( f_k \) is ghost on every facet of \( X' \) except those defined by \( h_1 + h_k \), and furthermore \( f_1 \mid X' \equiv \nu f_2 \mid X' \) since, by definition, we are left with segments in which some \( h_i \) dominate for \( h_i \neq h_1, h_2 \). Hence, \( f_1 \) and \( f_2 \) essentially agree on X, and X is not admissible. Note that \( f_1 f_2 \) is ghost on \( X' \).

In this way, we exclude intersections of algebraic set in which a facet is eliminated. We also must cope with examples such as the intersection of the planar curves defined by \( \lambda_1 + \lambda_2 + 0 \) and \( \lambda_1 + \lambda_2 + 1 \).

Example 5.17. Let \( X_{a_i} \) be the curve defined by the polynomial \( f = \lambda_1 + \lambda_2 + a_i \), for \( a_i \in F \). If \( a_1 <_\nu a_2 \) then the tangible part of the intersection \( X := X_{a_1} \cap X_{a_2} \) is the ray \( \{(b, b) : b \geq_\nu a_2, b \in T \} \subset T(2) \). The functions \( f_1 = \lambda_1 \) and \( f_2 = \lambda_1 + a_2 \) agree for every \( b >_\nu a_2 \) in \( T \), but \( f_1((a_2, -)) = a_2 \) whereas \( f_2((a_2, -)) = a_2 \). Hence, X is not admissible.

Clearly admissibility can be checked locally, i.e., at each neighborhood of each point \( a \in S \), so the next observation is the key.

Proposition 5.18. Any hypersurface defined by a tangible polynomial is an admissible algebraic set.

Proof. We need to show that the coordinate semiring \( \mathbb{Z}_{\text{corn}}(f) \) defined by the polynomial \( f = \sum_i f_i \) is admissible. Suppose that polynomials \( g_1 = \sum_j h'_j \) and \( g_2 = \sum_k h''_k \) essentially agree on X. We want to check that they agree on any given point \( a \) of \( X \). By hypothesis they agree on some dense subset of some small open set \( U \subset X \) whose closure contains \( a \). We replace \( g_1 \) and \( g_2 \) by their essential parts on \( U \). Since \( g_1(a) \equiv \nu g_2(a) \) by Remark 5.12, we are done unless say \( g_1(a) \in T \) whereas \( g_2(a) \in \mathcal{G} \). Thus \( g_1 \) has only one dominant monomial \( h'_j \) at \( a \), whereas \( g_2(a) \) has at least two essential monomials \( h''_1, h''_2, \ldots, h''_t \) at \( a \). By hypothesis, there are facets \( C_1, C_2, \ldots, C_t \) of \( U \), defined by binomials of \( f \), for which \( h''_j_{|C_k} = h''_k_{|C_k}, k = 1, 2, \ldots, t \). It is convenient to work with Laurent polynomials, cf. Remark 5.3 since then we can divide out by some given \( h'_j \) and assume that \( h'_j \) is the constant monomial \( 1_F \).

Likewise, we may normalize \( f \) as a Laurent polynomial to assume that one of the essential monomials of \( f \) at \( a \) is \( 1_F \), and \( C_1 \) is given by \( f_1 + 1_F \). Then \( C_2 \) is given by \( f_2 + f_3 \) where \( f_2 \neq 1_F \), and \( f_1 + f_2 \) defines another facet, on which \( 1_F = h''_k \), a contradiction. \( \square \)

In particular, the coordinate semiring \( \mathbb{Z}_{\text{corn}}(f) \) of a tropical line is admissible. The proposition fails for nontangible polynomials, since the neighborhood of a point might not have enough components to get the contradiction in the previous proof.

Example 5.19. Let \( f = \lambda^2 + a^\nu \lambda + ab \) for \( b < a \), for whose algebraic set \( X = \mathbb{Z}(f) \) the tangible part is the interval \( X_{|_{\text{ghost}}} = [b, a] \). Then \( \lambda \) and \( \lambda + b \) agree on \( X \setminus \{ b \} \) but not on \( b \), since \( b + b = b^\nu \neq b \). Hence, X is not admissible.

Example 5.20. Here is an example of how a monomial can be quasi-essential at one point of a hypersurface X but essential at another portion of X.

Let X be the hypersurface defined by the polynomial \( \lambda_1 + \lambda_2 + 2\lambda_3 + \lambda_3 \lambda_4 \), and let \( f = 2\lambda_4^2 + 2\lambda_3^2 + \lambda_1 \lambda_2 \lambda_4 \). When \( \lambda_3 \) takes a small value with respect to the substitutions of \( \lambda_1, \lambda_2 \), and \( \lambda_4 \), X becomes the algebraic set of \( \lambda_1 + \lambda_2 \), for which the monomial \( \lambda_1 \lambda_2 \lambda_4 \) can be essential in f. But when \( \lambda_3 \) takes on a large value, with respect to the substitutions of \( \lambda_1, \lambda_2, X \) becomes the algebraic set of \( 2 + \lambda_4 \), i.e., \( \lambda_4 = 2 \), where \( \lambda_1 \lambda_2 \lambda_4 \) is only quasi-essential in f.

Example 5.21. We consider some familiar examples from tropical geometry, viewed in the supertropical context.

(i) Let \( f_k = \lambda_1 + \lambda_k^2 + 0 \). Its corner locus X is admissible, by Proposition 5.14.
Suppose $f = \lambda_1^2 + 3\lambda_1 + \lambda_2^2 + 4\lambda_1\lambda_2 + 5$. Specializing $\lambda_2$ to some small value sends the algebraic set of $f$ to an algebraic set in which $\lambda_1 = 3$ or $\lambda_1 = 2$; i.e., the algebraic set has become disconnected and reducible. The same effect can be applied to tropical elliptic curves.

Proposition 5.22. Suppose $X = \mathcal{Z}_{\text{corn}}(f)$ where $f = \sum h_i \subset F[\lambda_1, \lambda_2]$ is essential, and erase the tropical facet given by $\mathcal{Z}_{\text{corn}}(h_1 + h_2)$. The ensuing curve is not admissible.

Proof. This was considered in Example 5.19. We claim that the polynomial $(\sum_{i \neq 1} h_i)(\sum_{i \neq 2} h_i)$ agrees with $g := f \sum_{i \neq 1, 2} h_i$ on a dense subset of $X$. This is seen from by considering each segment in turn, defined by $h_i + h_j$. If $i, j > 2$ the assertion is obvious, so we may assume that $i > 2 \geq j$. Then on the interior of this segment we have $(h_i + h_j)h_i$ on both sides, proving the claim.

On the other hand, at the intersection in which $h_1, h_2$ agree but not with any other $h_i$, we have $h_1(a), h_2(a)$ tangible, but not $h_1(a) + h_2(a)$. □

Remark 5.23. An admissible algebraic set $X$ is $\mathcal{E}(R)_{\text{adm}}$-irreducible iff the corresponding geometric congruence is $\mathcal{E}(R)_{\text{adm}}$-irreducible.

As opposed to the classical situation, a reducible algebraic set can be the union of irreducible algebraic sets in several different ways (because of non-unique factorization), and thus a congruence can be the intersection of irreducible congruences in several different ways.
Example 5.24.

1. The algebraic set $Z_{\text{corn}}((\lambda_1 + \lambda_2 + 0)(\lambda_1 \lambda_2 + \lambda_1 \lambda_2 + 0 \lambda_2))$ can be viewed as the union of the tropical line $Z_{\text{corn}}(\lambda_1 + \lambda_2 + 0)$ and conic $Z_{\text{corn}}(\lambda_1 \lambda_2 + \lambda_1 \lambda_2 + 0 \lambda_2)$ (see Fig. 4(a)), as well as the three curves $Z_{\text{corn}}(\lambda_1 + 0), Z_{\text{corn}}(\lambda_2 + 0),$ and $Z_{\text{corn}}(\lambda_1 + \lambda_2)$ (see Fig. 4(b)).

2. Although in (1), we could say that the two decompositions differ at the multiplicity of the point $(0,0)$, and thus could be detected in the layered congruence, Sheiner [18, Example 5.7] found the following example in which even the multiplicities match:

\[
(\lambda_2 + \lambda_1 + \lambda_2^0 + (-1)\lambda_1^0)(\lambda_2 + 0 + \lambda_2^1 + (-2)\lambda_1^1) = \\
(\lambda_2 + \lambda_1 + \lambda_2^2 + (-2)\lambda_1^2)(\lambda_2 + 0 + \lambda_2^2 + (-1)\lambda_1^2).
\]

So far we have two basic ways of initiating a homomorphism on a coordinate semiring $\dagger$: Either restrict its algebraic set, or put in new relations among the indeterminates of $\Lambda$. By binomial relation we mean a relation of the form $h|_W = h'|_W$, where $h, h'$ are different monomials and $W \subseteq X$ is nonempty.

Lemma 5.25. Suppose $X \subset Y$ are algebraic sets. Then the induced map $\Phi : F[Y] \to F[X]$ involves an extra binomial relation on each facet of $Y$ not contained in a facet of $X$.

Proof. Write $F[X] \cong F[Y]/\Omega$. On each facet of $Y$ we have some pair $(f, g) \in \Omega$ and we take their dominant monomials $(f_i, g_j)$ on this facet. Then $(f_i, g_j)$ is the extra binomial relation that we want, and we are done unless always $f_i = g_j$, which means that $\Phi$ is the identity on our facet of $Y$, which then is embedded in a facet of $X$.

Lemma 5.26. Suppose $X \subset Y$ are algebraic sets for which $F[Y]$ is obtained from $F[X]$ by adjoining one polynomial relation $f = g$. Then this polynomial relation arises from a binomial relation that dominates $Y$.

Lemma 5.27. Suppose $F[X]$ is an admissible coordinate semiring $\dagger$ which is defined by a set of polynomials $f_1, \ldots, f_m$, and two of these polynomial functions $f_1$ and $f_2$ coalesce at an interior point $a$ of some facet $W$ of $X$. Then $f_1$ and $f_2$ agree on all of the facet $W$.

Proof. They agree via their leading monomials on some open subset containing $a$ and thus on all rays emanating from $a$ in $W$, by Proposition 6.14. Suppose some other monomial $h_1$ of $f_1$ dominates them elsewhere on $W$. Then $h_1(a') \equiv_{\nu} f_1(a') \equiv_{\nu} f_2(a')$ at some point $a' \in W$, which is by definition on the boundary of $W$. This would mean $f_1(a') = f_2(a')$, contradicting admissibility unless $f_2(a') \in \mathcal{G}$, i.e., $f_2$ has a monomial $h_2$ such that $h_1(a') \equiv_{\nu} h_2(a')$. Continue on an open neighborhood of $a'$, and apply this argument throughout $W$.

6. Dimensions of Admissible Corner Varieties

Binomials play a key role in defining corner algebraic sets, since corner algebraic sets are defined “piecewise” by binomials. Localizing $F[X]$ at the tangible monomials enables us to pass to the Laurent coordinate semiring $\dagger F[X]$, which then is viewed inside $F[X]_{\text{rat}}$. 
Theorem 6.2. If \( X \subset Y \) are \( \mathcal{C}(R) \)-corner varieties, then the induced map \( \Phi : F[Y] \to F[X] \), if not 1:1, involves extra binomial relations which dominate \( Y \).

Proof. For any new element \((f,g)\) of a congruence, using Proposition \ref{prop:correspondence} we obtain a new binomial relation \( h|_W = h'|_W \) on some facet \( W \). Taking its tangible lift and localizing, we may assume \( h' = 1_F \) and write \( h = \alpha \lambda_1 \cdots \lambda_m \), for each \( i_j \in \mathbb{Z} \), not all 0. Reindexing the indeterminates, we may assume that \( \lambda_n \neq 0 \). By Lemma \ref{lemma:indetelimination} we get a new binomial relation, which enables us to solve

\[
\lambda_n \mapsto \alpha + \frac{1}{\lambda_1} \frac{1}{\lambda_m} \cdots \frac{1}{\lambda_{n-1}}
\]

in terms of the indeterminates \( \lambda_1, \ldots, \lambda_{n-1} \) (working in \( F[X]_{\text{rat}} \), on this facet. This provides an inductive procedure on each of our finitely many facets, which must terminate when we eliminate all of the indeterminates in each facet. \( \square \)

There are several possible definitions of dimension which can be garnered from the coordinate semiring\(^\dagger\). We take the algebraic one. This is close to the approach of Perri \[17\].

Definition 6.3. The dimension \( \dim X \) of an irreducible admissible corner algebraic set (i.e., of a \( \mathcal{C}(R)_{\text{adm}} \)-corner variety) is the maximal length of a chain of \( \mathcal{C}(R)_{\text{adm}} \)-subvarieties of \( X \), i.e., the maximal length \( m \) of a chain of \( \mathcal{C}(R)_{\text{adm}} \)-irreducible coordinate semirings\(^\dagger\)

\[
F[X] = F[X_0] \to \cdots \to F[X_m]
\]

(where \( X_m \) is the \( \nu \)-fiber of a point, as in Example \ref{example:admissiblefiber}).

Theorem 6.4. If there is a chain of homomorphisms \( F[\Lambda] \to F[X_1] \to \cdots \to F[X_m] \), where \( \Lambda = \{\lambda_1, \ldots, \lambda_m\} \) and each \( X_i \) is a \( \mathcal{C}(R)_{\text{adm}} \)-corner variety, then \( m \leq n \). Furthermore, for any such chain of maximal length, \( m = n \).

Proof. We review the proof Theorem \ref{thm:cornersemiring} with some extra care. These homomorphisms \( F[X_i] \to F[X_{i+1}] \) are obtained at each facet by new binomial relations, say \( 1_F + h \), where \( h \) is a Laurent monomial. Without loss of generality, assume that \( \lambda_n \) appears in \( h \). At any facet for which \( h \) is essential, \( h \) which can be used to eliminate the indeterminate \( \lambda_n \) in terms of the others. We claim that this can be done at most \( n \) times at any given facet, at which stage any polynomial is locally constant. But the constants are the same since polynomials are continuous (and \( X_i \) is connected in view of Proposition \ref{prop:connectedness}), so by assumption the polynomial is constant after at most \( n \) steps, which means that we cannot continue the chain further.

The only difficulty with this argument is that some of the reductions might be trivial, along the lines of Example \ref{example:trivialreduction}. In other words, \( h \) might be dominated by \( 1_F \). But now we appeal to an idea of Tal Perri in his dissertation \[17\]. In order to make \( h = \alpha \lambda_1 \cdots \lambda_m \) inessential, we must have \( \alpha \lambda_1 \cdots \lambda_m \in \nu \cdot \mathbb{Z}_F \). This yields a new inequality among the indeterminates, involving \( \lambda_n \), which Perri calls an order relation. This can only happen if \( \lambda_n \) appears in one of the essential monomials defining the facet, so again we can substitute for \( \lambda_n \) and eliminate it.

At each step we eliminate one more indeterminate, and so the process must terminate after \( n \) steps. This proves that \( m \leq n \). When \( m < n \), there remain “free” variables in each facet; since the facets can be viewed locally as hypersurfaces, we conclude with Proposition \ref{prop:hyperlocal} \( \square \)

In conclusion:

Corollary 6.5. Any chain of irreducible admissible corner algebraic subsets of \( F^{(n)} \) can be refined to a chain of irreducible admissible corner algebraic subsets of \( F^{(n)} \) of length \( n \).

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\(^\dagger\) For a definition of \( \mathcal{C}(R)_{\text{adm}} \)-corner varieties and \( \mathcal{C}(R)_{\text{adm}} \)-subvarieties, see Section 6.2.
7. The layered approach to varieties

Although algebraic sets (Definition 5.14) rely heavily on the use of the \( \nu \)-structure, applications concerning multiple roots rely on more refined layerings, so we briefly present the foundations for this alternative.

7.1. Layered domains and semifields

We recall the main example in [12].

**Construction 7.1.** Suppose \( \mathcal{T} \) is a cancellative monoid, and \( L \) is a semiring to be used as an index set. We define the \( L \)-layered domain \( \mathcal{A}(L, \mathcal{T}) \) (or layered domain\(^\dagger\) for short, when \( L \) is understood) to be set-theoretically \( L \times \mathcal{T} \), where for \( k, \ell \in L \), and \( a, b \in \mathcal{T} \), we write \([k]a\) for \((k, a)\) and define multiplication componentwise, i.e.,

\[
[k]a \cdot [\ell]b = [k\ell]ab, \tag{7.1}
\]

and addition from the rules:

\[
[k]a + [\ell]b = \begin{cases} [k]a & \text{if } a > b, \\ [\ell]b & \text{if } a < b, \\ [k+\ell]a & \text{if } a = b. \end{cases} \tag{7.2}
\]

\( \mathcal{A}(L, \mathcal{T}) \) is equipped with the sort map given by \( s([k]a) = k \), and maps

\[
\nu_{\ell,k} : (k, \mathcal{T}) \to (\ell, \mathcal{T}), \quad k \leq \ell, \quad k, \ell \in L,
\]

given by \([k]b \mapsto [\ell]b\). We define the \( \ell \)-layer \( R_{\ell} := \{ [\ell]a : a \in \mathcal{T} \} \), and write \( R := \mathcal{A}(L, \mathcal{T}) \) as the disjoint union

\[
\mathcal{A}(L, \mathcal{T}) = \bigcup_{\ell \in L} R_{\ell}.
\]

We also define \( e_{\ell} := [\ell]1_{\mathcal{T}}. \)

We write \( a \equiv_\nu b \) (resp. \( a \succ_\nu b \)) for \( a \in R_k \) and \( b \in R_\ell \), whenever \( \nu_{m,k}(a) = \nu_{m,\ell}(b) \) (resp. \( \nu_{m,k}(a) > \nu_{m,\ell}(b) \)) in \( R_m \) for some \( m \geq k, \ell \). (This notation is used generically: we write \( a \equiv_\nu b \) even when the sort transition maps \( \nu_{m,\ell} \) are notated differently.)

This construction is put into a more formal context in [12] [13]. In order not to be distracted here from the impact of the algebra on geometric considerations, we take the sort semiring\(^\dagger\) \( L \) to be a totally ordered (commutative) semiring\(^\dagger\), perhaps with an absorbing element \( 0 = 0_L \) adjoined.

\( R_1 \) is called the set of tangible elements of \( R \), and plays a key role in the theory. It is convenient for \( 1 = 1_L \) to be the minimal sorting index in \( L \). Towards this end, for any layered semiring\(^\dagger\) \( R \), we may replace \( R \) by \( \bigcup_{\ell \geq 1} R_{\ell} \), a sub-semiring\(^\dagger\) of \( R \). The tangible lift is given by \([\ell]a \mapsto [1]a\). We bear in mind the examples \( L = \mathbb{N} \), and \( L = \mathbb{Q}_{\geq 1} \), each with the usual order. Let \( \bar{L} = \{1, \infty\} \). We then have a semiring\(^\dagger\) homomorphism \( L \to \bar{L} \) sending \( k \mapsto \bar{k} \), where \( 1 = 1 \) and \( \bar{k} = \infty \) for each \( 1 < k \in L \).

**Definition 7.2.** \( \mathcal{A}(L, \mathcal{T}) \) is a **layered 1-semifield**\(^\dagger\) if \( \mathcal{T} \) is a group.

**Remark 7.3.** \( R_1 \) is a cancellative submonoid isomorphic to \( \mathcal{T} \). Localizing \( R := \mathcal{A}(L, \mathcal{T}) \) at \( R_1 \) yields a layered 1-semifield\(^\dagger\), whose \( 1 \)-layer is an ordered group iff \( \mathcal{T} \) is an ordered monoid.

7.2. Function and polynomial semirings of layered semirings

As in [12], we can pass the layered structure from \( R \) to \( \text{Fun}(S, R) \), at the expense of enlarging the layering set from \( L \) to \( \text{Fun}(S, L) \).

**Example 7.4.** As noted in [11] Remark 5.4, when \( R \) is an \( L \)-layered semiring\(^\dagger\), then \( \text{Fun}(S, R) \) is layered with respect to \( \text{Fun}(S, L) \), where \( \text{Fun}(S, R) \) has the sort map \( s \) given by

\[
s(f)(a) = s(f(a)),
\]

for \( a := (a_1, \ldots, a_n) \) in \( S \).

We can extend \( \geq_\nu \) to a partial order on \( \text{Fun}(S, R) \) as follows:

(i) \( f \equiv_\nu g \iff f(a) \equiv_\nu g(a), \forall a \in S; \)

(ii) \( f \prec_\nu g \iff f(a) \prec_\nu g(a), \forall a \in S; \)

(iii) \( f \preceq_\nu g \iff f(a) \preceq_\nu g(a), \forall a \in S; \)

(iv) \( f \supseteq_\nu g \iff f(a) \supseteq_\nu g(a), \forall a \in S. \)
(ii) $f >_\nu g$ if $f(a) >_\nu g(a)$, $\forall a \in S$.

Although not totally ordered, $\text{Fun}(S, R)$ satisfies the weaker properties:

If $f >_\nu g$, then $f + g = f$; $2f \equiv_\nu f$ with $s(2f) = 2s(f)$, seen by pointwise verification.

The construction and definition were generalized in [11, 12] and [13], but we work with the more specific case here in order to avoid further complications.

Our main interest is in the case where $\mathcal{R} := F[\Lambda]$ in commuting indeterminates $\Lambda = \{\lambda_1, \ldots, \lambda_n\}$ over a layered 1-semifield$^\dagger$ $F$. We want to understand the homomorphic images of $\mathcal{R}$ by specializing certain $\lambda_i$ in terms of extensions of $F$, in order to prepare the groundwork for a layered version of affine geometry. The main idea is that in specializing $\lambda_1, \ldots, \lambda_n$ to elements of $F$, we also obtain a homomorphism $L[\lambda_1, \ldots, \lambda_n] \to L$ and thus recover the original sorting set $L$.

**Lemma 7.5.** If $R = \mathcal{A}(L, T)$, then

$$\text{Pol}(\Lambda, R) = \mathcal{A}(\text{Pol}(\Lambda, L), \text{Pol}(\Lambda, T)),$$

and

$$\text{Laur}(\Lambda, R) = \mathcal{A}(\text{Laur}(\Lambda, L), \text{Laur}(\Lambda, T)).$$

**Proof.** The unit element of the monoid $\text{Pol}(\Lambda, R)$ is the constant function sending all elements to $1_R$, and its corresponding layer in $\text{Pol}(\Lambda, L)$ is clearly a monoid. The same argument holds for Laurent polynomials and rational polynomials.

In this way, we can replace $R$ by $\text{Fun}(S, R)$ in the theory described above, but at the cost of replacing the original sorting set $L$ by a more complicated sorting set.

The reason we used the supertropical and not the layered structure in our definition of $\mathcal{C}(\mathcal{R})$-variety is because of the following sort of example. To ease notation, we write $a$ for $[1]a$, and $\lambda$ for $0\lambda$.

**Example 7.6.** Consider the corner algebraic set of the polynomial $f = \lambda^2 + 0$ in $F$. The function $g = \lambda^2 + \lambda + 0$ agrees with $f$ at all points except $a = 0$. Over the supertropical structure, $f(0) = 0^\nu = g(0)$, and the corner algebraic set is easily seen to be admissible. But the analogous property fails with respect to the layered structure, since $f(0) = [2]0$ whereas $g(0) = [3]0$.

Such an example could not interfere with the supertropical theory, because of Lemma [11, 13](ii). Nonetheless, the layered approach enables one to cope better with different multiplicities of roots, and gives us the following alternative approach.

### 7.3. Layered algebraic sets.

Let $F = \mathcal{A}(L, T)$ be a layered 1-semifield$^\dagger$. Recall [11] Definition 5.7:

**Definition 7.7.** The **layering map** of a function $f \in \text{Fun}(S, F)$ is the map $\vartheta_f : S \to L$ given by

$$\vartheta_f(a) := s(f(a)), \quad \forall a \in S.$$

The **layering map** of a set of functions $A \subseteq \text{Fun}(S, F)$ is given by

$$\vartheta_A(a) := \inf_{f \in A} \vartheta_f(a).$$

Thus, $\vartheta_A \in \text{Fun}(S, L)$.

**Definition 7.8.** The **layering** $\mathcal{L}(A)$ of a subset $A \subseteq \text{Fun}(S, F)$ is the set

$$\mathcal{L}(A) := \{(a, \vartheta_A(a)) : a \in S\}.$$

The **layered algebraic set** $X := X_A$ is the subset

$$X_A := \{(a, \vartheta_A(a)) \in \mathcal{L}(A) : \vartheta_A(a) > 1\}.$$

We write $X$ for the projection of $X$ onto $S$, which is $\{a \in S : \vartheta_A(a) > 1\}$. This matches our definition of algebraic set but also records the jump in multiplicity. Thus $A$, although not always notated, is intrinsic in the definition of $X$, and the second coordinate $\vartheta_A(a)$ plays a key role.
Definition 7.9. As in Definition [7] given two layered algebraic sets $X = X_A$ and $Y = Y_B$, we define $X \vee Y = \{ (a, \max\{ \vartheta_A(a), \vartheta_B(a) \}) : a \in S, \ \vartheta_A(a) > 1 \text{ or } \vartheta_B(a) > 1 \};$ $X \wedge Y = \{ (a, \min\{ \vartheta_A(a), \vartheta_B(a) \}) : a \in S, \ \vartheta_A(a) > 1 \text{ and } \vartheta_B(a) > 1 \}.$

We say that $X \preceq Y$ if $X \wedge Y = X$, i.e., if $X \subseteq Y$ and $\vartheta_A(a) \leq \vartheta_B(a)$ for each $a \in S$.

Remark 7.10. $X \vee Y = X \cup Y, \ X \wedge Y = X \cap Y$.

Repeating Definition 5.7 where now $F$ is layered, we now call $\Omega_X$ a layered congruence of $X$, and define $\mathcal{C}(\mathcal{R})_{lay}$ to be the set of layered congruences on $\mathcal{R} := F[A]$.

Definition 7.11. A layered algebraic set $X$ is $\mathcal{C}(\mathcal{R})_{lay}$-irreducible if it cannot be written as $X_1 \vee X_2$ for layered algebraic sets $X_1, X_2, \neq X$, and $\mathcal{C}(\mathcal{R})_{lay}$ is Noetherian if every descending chain of layered algebraic sets (under $\preceq$) stabilizes.

Remark 7.12. As in Remark 5.23, A layered algebraic set $X$ is $\mathcal{C}(\mathcal{R})_{lay}$-irreducible iff the corresponding geometric congruence is $\mathcal{C}(\mathcal{R})_{lay}$-irreducible.

Example 7.13.

(i) Let us view Example 4.4 from this perspective. Let $L_\alpha := X_{f_\alpha}$ be the tropical line defined by the polynomials $f_\alpha = \lambda_1 + \lambda_2 + \alpha$ and $X := X_\beta$ be the tropical curve defined by the polynomial $g = \lambda_1 \lambda_2 + \lambda_2 + \alpha$, and let $X_\alpha = L_\alpha \wedge X$. Then $(L_\alpha \cap X)_{F(2)}$ is just the segment $[0, \alpha]$ on the $\lambda_2$-axis, but $\vartheta_{f_\alpha}(0) = 2 = \vartheta(g)$ whereas $\vartheta_{f_\alpha}(\alpha) = 3 = \vartheta(g)(0)$. In other words, when $\alpha < \beta$ we do not have $X_\alpha \preceq X_\beta$ even though $X_\alpha \subseteq X_\beta$.

(ii) Likewise, let $L_1$ be the tropical line defined by the polynomial $\lambda_1 + \lambda_2 + 0$ and $X_2$ be the tropical curve defined by the polynomial $\lambda_1^2 + \lambda_2 + 0$. Now $(L_1 \wedge X_2)_{F(2)}$ still is the union of two rays (the lower $\lambda_1$ and $\lambda_2$ axes), which is properly contained in $L_1$. We can get the third ray by intersecting the hyperplanes of $\lambda_1 + \lambda_2$ and $\lambda_1 + 0$, but the level at $(0,0)$ is only 2, not 3. Thus, $L_1$ is irreducible with respect to $\preceq$.

Let us formalize Example 7.13(i).

Remark 7.14. By definition, the layering function is constant on any facet. Hence, if $X \preceq Y$, every facet of $X$ is contained in the corresponding facet of $Y$.

Proposition 7.15. The class of layered algebraic subsets of $F(n)$ is Noetherian.

Proof. There are only a finite number of facets, and each increase of the congruence decreases the level of some facet (since by Remark 7.14 it cannot “cut” a facet).

Note that the layered dimension of the tropical line, defined in terms of a maximal descending chain of irreducible layered algebraic sets would be 3, not 1, since the ray along an axis (as well as the union of the two semi-axes) is a layered algebraic set. This discrepancy could be resolved by further restricting our class of congruences along the lines of [6].

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