Chiral Stoner magnetism in Dirac bands

Zhiyu Dong, Leonid Levitov

Department of Physics, Massachusetts Institute of Technology, Cambridge, MA 02139

Stoner magnetism in bands endowed with Berry curvature is shown to be profoundly influenced by the coupling between spin chirality density $s \cdot (\partial_1 s \times \partial_2 s)$ and Berry’s orbital magnetization. The key effect is that carriers moving in the presence of a spin texture see it as a source of a geometric magnetic field coupled to the carrier’s orbital motion through a spin-dependent Aharonov-Bohm effect. This emergent spin-orbit interaction effect was recently predicted to enable chiral magnons propagating along system boundaries. Here we show that it also favors chiral spin textures such as skyrmions—the topologically protected objects with particle-like properties, stabilized in the ground state. The threshold for Stoner instability is found to soften, rendering chiral spin-ordered phases accessible under realistic conditions. We present a detailed analysis of the chiral effect for Bernal bilayer graphene and discuss the unique properties of skyrmion textures in graphene multilayers.

The arrival of electron systems hosting topological flat bands and strong interactions [1-5] raises fascinating questions about the impact of Berry curvature on many-body physics. Recent work has focused on graphene multilayers such as moiré graphene [6-12], where flat bands emerge when the moiré twist angle is tuned to a magic value [4]. Additionally, research has explored field-biased non-moiré bilayers and trilayers [13-17], systems in which bands are flattened by a transverse electric field [18]. The small kinetic energy of carriers in flattened bands and the strong electron interactions characteristic of graphene create a platform in which a variety of ordered many-body phases can be realized and explored. The wide variety of observed orders, including cascades of magnetic phases polarized in isospin (spin and valley) alongside the insulating and superconducting phases, prompts seeking new interactions and previously unknown many-body orders in these systems.

With this motivation, here we consider itinerant magnetic metals with spontaneously spin-polarized carriers in bands equipped with Berry curvature. We find that such systems possess a geometric coupling of the orbital and spin degrees of freedom that favors nonzero spin chirality $s \cdot (\partial_1 s \times \partial_2 s)$, where $s(x)$ is the spin density and $\partial_1, \partial_2$ are spatial derivatives. Such geometric coupling, enabled by Berry curvature and electron exchange interactions, is allowed to exist by general symmetry arguments. Below we develop a microscopic theory of this effect, which a particular focus on itinerant graphene flat-band systems in which carriers are polarized in spin and/or valley (the 1/2-metal and 1/4-metal phases) as well as partially isospin polarized (PIP) phases. The underlying physics can be understood as electromagnetic coupling $-MB$ between orbital magnetization $M$ due to the band Berry curvature and the geometric magnetic field $B$ originating from spin chirality. From a more general point of view, the effect that can be thought of as an emergent geometric spin-orbit interaction (SOI) driven by electron exchange, with the coupling strength taking a universal value mandated by geometric and topological constraints.

The geometric SOI established in this work has several notable implications. One is the existence of new collective excitations—chiral magnons propagating along the edges of the system [19]. These chiral spin waves, induced by the geometric SOI, are expected to emerge robustly in all itinerant spin-polarized phases in bands with Berry curvature. Another implication of the geometric SOI is the emergence of new magnetic phases in which spins form textures such as spin-density waves or skyrmions. Below, after demonstrating the effect microscopically, we analyze the stability of a uniformly polarized phase. We find that in the presence of geometric SOI, it becomes unstable towards formation of skyrmion textures at sufficiently low carrier densities (see text).

The interplay between orbital effects and spin de-
degrees of freedom is a topic that has been widely studied in magnetism, where microscopic spin-orbit interactions have been used to stabilize different kinds of helical and chiral orders in magnetic systems. Of special interest are the chiral magnetic phases, wherein spins wrap around the Bloch sphere. Chiral spin textures have been explored in various magnetic systems.\cite{21,22} Famously, the Dzialoshinskii-Moriya (DM) coupling favors spin textures \cite{23} such as helical spin density waves \cite{24,25} and skyrmions—the seminal topologically-protected particle-like spin configurations.\cite{24} One can therefore ask whether it is possible to achieve a chiral spin order in graphene-based systems. At first sight, this may seem problematic as staple SOI interactions that stabilize chiral magnetic orders are absent or extremely weak in graphene. Indeed, in noncentrosymmetric magnets—the bulk chiral magnetic metals \cite{28–33} and magnetic layers \cite{34,35}—skyrmions are stabilized by the DM coupling governed by the microscopic spin-orbit interactions \cite{21,22}. However, the microscopic SOI in graphene is usually negligible compared to other energy scales.\cite{36} Likewise, the mechanisms that utilize frustration \cite{31,39} are not found in graphene systems. Yet, as will be demonstrated below, the geometric SOI originating from the interplay of a band curvature and spin polarization favors chiral spin orders without invoking any extra broken symmetries, microscopic SOI or frustrations.

I. SYMMETRY OF THE CHIRAL EFFECT

The quantity that will be central to our discussion is the chirality density, or topological density, of a spin texture \( s(x) \) defined as a scalar triple product

\[

b(x) = \frac{1}{2} \epsilon_{\mu\nu} s(x) \cdot (\partial_\mu s(x) \times \partial_\nu s(x)), \tag{1}
\]

where \( \epsilon_{\mu\nu} \) a 2 \( \times \) 2 antisymmetric tensor. This quantity, which characterizes twisting in the texture, has several remarkable properties. First, despite being a function of angles which cannot be reduced to a function of the modulus \(|s|\), it is invariant under \( SU(2) \) spin rotations. Second, it is odd under time reversal and spin-reversal operations, \( t \rightarrow -t \) and \( s \rightarrow -s \). The \( SU(2) \) symmetry property suggests that chirality could emerge from spin exchange interactions that depend on the angles between interacting spins and obey \( SU(2) \) symmetry. However, being odd under time and spin reversal symmetries excludes this quantity from basic models of magnetism such as the Heisenberg model and its extensions. Yet, as we will see, this quantity is mandated by symmetry for magnetism in bands with Berry curvature.

Specifically, despite its exotic symmetry properties, this quantity appears naturally in the problem of itinerant spin magnetism in Dirac bands such as those found in graphene multilayers, moiré \cite{4,6,40} and non-moiré\cite{15,20}. In these systems carriers reside in valleys \( K \) and \( K' \) that are mapped to one another under time reversal symmetry. At the same time, the band dispersion and interactions in each valley individually are not constrained by time and spin reversal symmetries. As a result, the symmetry constraints for the quantities such chirality are relaxed. Allowed by the symmetries of the microscopic Hamiltonian, these quantities must be included in the theoretical description of the magnetic order. Furthermore, the analysis described below, which links these quantities to geometric gauge fields, predicts a universal value of the coupling constant that is large enough to significantly alter the physics of the ordered phase. As we will see, the interaction expressed as chirality density in Eq. (1) can lower the threshold for Stoner instability and stabilize chiral spin textures in the ordered phase. Figure 1 illustrates this for a quadratic Dirac band model of field-biased bilayer graphene (BBG) analyzed below.

The chiral interaction responsible for this behavior arises from the coupling between orbital magnetization due to \( k \)-space Berry curvature of Dirac carriers and position-dependent spin polarization, taking the form:

\[

\delta F = \int d^2x \sum_{i=K,K'} -(M_{i,+} - M_{i,-}) B_i(x), \tag{2}
\]

where \( M_{i,\pm} \) is the orbital magnetization of the majority-spin and minority-spin carriers in valleys \( K \) and \( K' \) (see Eq.(35)). This interaction can be viewed as an extension of the basic electromagnetic coupling of a magnetic moment and external field, \( E = -M \cdot B \), where the ‘magnetic field’ \( B_i(x) \) is defined as topological density of spin texture in each valley multiplied by the flux quantum \( \phi_0 \):

\[

B_i(x) = \frac{\phi_0}{4\pi} s_i \cdot (\partial_1 s_i \times \partial_2 s_i), \quad \phi_0 = \hbar c/e, \tag{3}
\]

with \( s_i(x) \) the unit-vector field representing spin polarization of carriers in valleys \( i = K, K' \). The quantity \( B_i(x) \) represents a geometric ‘magnetic field’ associated with the spin-dependent (geometric or chiral) Aharonov-Bohm effect. This effect originates from the fundamental properties of spin textures \( s_i(x) \) in which the position-dependent quantization axis along which carrier spins are polarized is, in general, allowed to twist in space. Spins of carriers moving through a texture and coupled to it by exchange undergo adiabatic spin rotation, which leads to geometric phase and magnetic field \( B_i(x) \) associated with the Aharonov-Bohm effect due to this phase. The magnetic field originates from carrier movement through the texture, is coupled to orbital degrees of freedom of carriers, leading to the the \(-MB\) coupling to orbital magnetization of carriers appearing in Eq.(2).

Since \( M_i \) describes orbital magnetization whereas \( B_i \) equals (up to a factor) to the chirality of a spin texture, the interaction in Eq.(2) can be viewed as an emergent spin-orbit interaction which originates from spin exchange and does not rely on a microscopic spin-orbital coupling. To understand the symmetry properties of this interaction, it is instructive to compare it to the atomic
spin-orbital interaction (SOI) \( E \sim \mathbf{L} \cdot \mathbf{s} \). The SOI interaction locks spin polarization \( \mathbf{s} \) to the orbital angular momentum vector \( \mathbf{L} \). As a result, it is not invariant under SU(2) spin rotations. To the contrary, the chiral interaction originates from SU(2)-invariant spin exchange interactions, which endows it with unique symmetry properties distinct from those of SOI. Indeed, the interaction in Eq. (2) is governed by the interplay of the direct-space geometric phase which gives rise to \( B_{\eta}(\mathbf{x}) \) and momentum-space Berry curvature responsible for \( M_{i,\pm} \). The quantities \( B_{\eta} \) are manifestly invariant under the SU(2) spin rotations performed separately in each valley, whereas the orbital quantities \( M_{i,\pm} \) remain invariant under such spin rotations. As a result, the chiral interaction, while describing a coupling between the spin and orbital degrees of freedom, is invariant under spin rotations. This is a high symmetry which is not known for the spin-orbit interactions originating from a microscopic spin-orbit coupling.

Moreover, the interaction in Eq. (2) also respects discrete symmetries of BBG, including the time-reversal and mirror symmetries. Indeed, the time-reversal symmetry maps \( s_K \leftrightarrow -s_{K'} \), therefore it maps \( B_K \leftrightarrow -B_{K'} \). Simultaneously, the orbital magnetization \( M_K \) is mapped to \( -M_{K'} \). Consequently, Eq. (2) is invariant under time-reversal symmetry. Likewise, the mirror symmetry maps \( M_K \leftrightarrow -M_{K'} \), whereas \( B_K \) is mapped to \( -B_{K'} \). Indeed, taking for example the mirror lying in \( yz \) plane, the gradient component \( (\partial_1, \partial_2) \) are mapped to \( (-\partial_1, \partial_2) \), leaving the chirality density in Eq. (1) unchanged. Consequently, Eq. (2) respects mirror symmetry. Same conclusion applies to the other two mirrors as they are equivalent to the \( yz \) mirror up to a \( C_3 \) rotation.

We note that in this reasoning, due to the absence of microscopic SOC, the spin and orbital degrees of freedom are decoupled and we do not need to account for the mirror symmetry action on spin degrees of freedom. However, it is straightforward to show that the conclusion would remain the same if we accounted for mirror symmetry action on spin. Indeed, the \( yz \) mirror maps \( s_K \rightarrow (s_{K'}^x, s_{K'}^y, s_{K'}^z) \rightarrow (s_{K'}^x, -s_{K'}^y, -s_{K'}^z) \), which leaves the chiral density invariant. Further, we note parenthetically that the \( C_2 \) symmetry that interchanges the top and bottom graphene layers is broken in the systems of interest due to the presence of a transverse electric field. This is a crucial condition for the system to host the chiral coupling because otherwise (in the presence of the \( C_2 \) symmetry) the orbital magnetization would not be allowed by symmetry and must vanish.

This discussion demonstrates that, while the ‘emergent SOI interaction’ given in Eq. (2) may appear exotic, it follows the form allowed by symmetry. This observation carries two important implications, one almost trivial and the other considerably less trivial. First, while the coupling in Eq. (2) obeys the continuous spin rotation symmetry and discrete time reversal and spin inversion symmetries, these symmetries may be broken in the system ground state. As illustrated by Fig. [1] this is indeed what happens in the system of interest. Second, all interactions of the form allowed by symmetry are expected to be supported by a generic realistic Hamiltonian. Therefore, regardless of the details of the microscopic derivation presented below, all conclusions of our analysis can be inferred and justified based on symmetry grounds alone. This observation makes the results of the analysis presented below applicable broadly and valid outside the specific assumptions under which the analysis will be carried out.

We note parenthetically that from the symmetry point of view, the properties of electron bands in rhombohedral multilayer graphene are the same as those in BBG. Therefore, the analysis carried out for BBG is applicable to a wider variety of multilayer graphene systems, such as rhombohedral-stacked trilayer and pentalayer graphene.

It is also worth noting that the chiral interaction that stabilizes skyrmions in our theory is distinct from the effects considered in the literature. In particular, skyrmions in isospin-polarized moiré graphene flat bands have been invoked to predict exotic superconductivity [41, 42]. The mechanism that stabilizes skyrmions in these papers is an isospin extension of quantum Hall ferromagnet physics, in which the skyrmion emerge in Landau levels spin-split by exchange interactions [22, 13, 45]. Skyrmions of this type have been predicted [46, 47] and recently observed [48, 49] in graphene at high magnetic fields.

II. ITINERANT MAGNETISM IN A BAND WITH BERRY CURVATURE

Here, we discuss Stoner magnetism in a graphene bilayer Dirac band. This analysis will set the stage for deriving microscopically the interaction between spin-chirality and orbital magnetization enabled by Berry curvature, Eq. (2). Our starting point is a fully SU(2)-invariant Hamiltonian involving a single-particle Hamiltonian and electron-electron interactions, but not involving any microscopic SOI.

\[
\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}.
\]  

We take the single-particle part \( \mathcal{H}_0 \) to be a quadratic Dirac Hamiltonian of a Bernal-stacked graphene bilayer:

\[
\mathcal{H}_0 = \sum_{\eta,p} c_{\eta,p}^\dagger H_\eta(p) c_{\eta,p}
\]

where \( H_\eta(p) \) is a 2 \times 2 Hamiltonian

\[
H_\eta(p) = \begin{pmatrix}
D & (p_1 - imp_2)^2 \\
(p_1 + imp_2)^2 & 2m
\end{pmatrix}
\]

Here, \( \eta = \pm 1 \) for the valleys \( K \) and \( K' \); the quantities \( c_{\eta,p}, c_{\eta,p}^\dagger \) are spinors with the \( A \) and \( B \) sublattice components and the ordinary spin-1/2 components. The Hamiltonian \( H_\eta(p) \) possesses particle-hole symmetry, with the
effects of particle-hole asymmetry and trigonal warping ignored for simplicity. Incorporating these terms later or generalizing to other Dirac band types would be straightforward. Estimates for realistic parameter values are provided in Sec. VII (see discussion beneath Eq. (53)).

Next, we consider the electron-electron interaction. For simplicity, we focus on the intra-valley spin exchange and ignore the exchange interaction between electrons in valleys $K$ and $K'$. The inter-valley processes are expected to be weak as they involve a large momentum transfer, and are therefore subleading to the intra-valley processes. Indeed, the electron-electron interaction predominantly arises from Coulomb interaction, which decreases as $1/p$ as a function of the momentum transfer $p$. Restricting the exchange to electrons in the same valley, we introduce a spin-exchange coupling of the form

$$H_{\text{int}} = -\frac{1}{2} \sum_{\eta, k} U(k) : s_{\eta \alpha}(k) s_{\eta \alpha}(-k) :. \quad (7)$$

Here the exchange interaction written in terms of spin density $s_{\eta \alpha}(k) = \sum_{p} c_{p,\beta,\eta, k}^{\dagger} \sigma_{\alpha} c_{p,\beta, \eta, k}$ with Pauli matrices $\sigma_{\alpha}$ representing ordinary spin-1/2 variables, $\alpha = 1, 2, 3$. Microscopically, the spin-exchange coupling originates from the Coulomb interaction. However, directly starting from microscopic interactions will significantly complicate the analysis. Here, to illustrate the physics of interest within a simplest formulation, we replace the microscopic exchange interaction with a toy-model dependence

$$U(k) = U_0 e^{-k^2 \xi^2}, \quad (8)$$

where $\xi$ is a correlation lengthscale.

Equivalently, the spin exchange Hamiltonian given in Eq. (7) can be written in coordinate representation as

$$H_{\text{int}} = \sum_{\eta} -\frac{1}{2} \sum_{x, x'} U(x - x') : s_{\eta \alpha}(x) s_{\eta \alpha}(x') :, \quad (9)$$

with a nonlocal spin-spin interaction

$$U(x - x') = 4\pi U_0 \xi^{-2} e^{-(x-x')^2/4\xi^2}, \quad (10)$$

normalized to $\int d^2 x U(x) = U_0$. As we will see, tuning the interaction radius $\xi$ provides a convenient knob for studying the spin-polarized phase, analyzing fluctuations and charting the phase diagram of skyrmion textures.

In our model, given by Eqs. (10) and (7), the electrons in two valleys are decoupled. Therefore, in the analysis below we can consider the $K$ valley alone. It should be noted that in reality the electrons in valleys $K$ and $K'$ also interact through a direct density-density interaction (the Hartree energy). However, we do not need to include this interaction in the model because this term only gives a charging energy that determines the total electron density. As such, it does not affect the spin polarization in each valley, which is quantity of interest.

### III. MEAN FIELD THEORY FOR SPIN TEXTURES

To describe spin textures, we perform a mean-field analysis in which the field describing ensemble-averaged spin polarization is allowed to vary in space. Since the exchange interactions are predominantly intravalley it will be sufficient to carry out the analysis for an individual valley and consider the role of valley degrees of freedom later. The Hubbard-Stratonovich (HS) transformation is carried out using an ordering field $h(x)$ with both the modulus and orientation being position-dependent,

$$\exp \left( \int dt \sum_{k} \frac{U(k)}{2} (s_{k} \cdot s_{-k}) \right) = \int D[h] \exp \left( \int dt \sum_{k} (h_{k}(t) s_{-k}(t) - h_{-k}(t) s_{k}(t)) \right) / 2U(k) , \quad (11)$$

where $D[h] = \prod_{k} dh_{k}(t)$. Here we introduced Fourier harmonics of the HS field and spin density $h_{k} = \int d^{2} x h(x) e^{-ikx}$, $s_{k} = \int d^{2} x s(x) e^{-ikx}$. Integrating out fermions and assuming a time-independent $h(x)$, we obtain the free energy with a nonlocal $h(x) h(x')$ interaction

$$F = \text{Tr} \log [\omega - H(p) - h_{\alpha}(x) \sigma_{\alpha}] + \sum_{k} h_{k} h_{-k} / 2U(k) , \quad (12)$$

where, for conciseness, the chemical potential $\mu$ is incorporated in $H$ and $\text{Tr}$ denotes $\sum_{x} \int dp dp' (2\pi)^{-2} \text{Tr}_{2 \times 2}$. Eq. (12) is an exact result for the fermion partition function, with no approximations made, which is applicable for any position-dependent ordering field $h(x)$.

In this framework, the behavior of the states with uniform polarization and those with a general space- and time-dependent $h(x, t)$ can be compared on equal footing. The saddle point condition $\delta F = 0$ yields a time-independent $h = |h|$, which is nothing but the Stoner mean field value

$$h = U(0)(n_{+} - n_{-}) / 2 , \quad (13)$$

where $n_{+}$ and $n_{-}$ are local densities of carriers with spins parallel and antiparallel to local spin quantization axis $h(x)$. We will call these spin species the majority and the minority spins, respectively. When the system is fully polarized, the mean field equals $h = U(0)n/2$.

Next, we consider weakly inhomogeneous $h(x)$. The term $-h_{\alpha}(x) \sigma_{\alpha}$ describes electron spins coupled to a spin texture with a position-dependent magnetization polarized along the unit vector $s(x) = h(x)/|h|$, where $|h(x)| = h$. We therefore write $h(x) = h_{0} + \delta h(x)$, where $\delta h(x) \perp h_{0}$, and approximate the dependence of the free energy on $\delta h(x)$ as a second-order functional derivative

$$\delta F = F[h(x)] - F[h_{0}] = \sum_{k} \frac{1}{2} \frac{\partial^2 F}{\partial h_{k} \partial h_{-k}} \delta h_{k} \delta h_{-k} , \quad (14)$$

where $n_{+}$ and $n_{-}$ are local densities of carriers with spins parallel and antiparallel to local spin quantization axis $h(x)$. We will call these spin species the majority and the minority spins, respectively. When the system is fully polarized, the mean field equals $h = U(0)n/2$.
where $F[h_0]$ is the free energy evaluated for spatially uniform $h(x)$. Expanding the logarithmic term in the free energy [Eq. (12)] in $\delta h(x)$ to second order yields:

$$
\delta F = \sum_k \frac{1}{2} \left( \chi_{\pm}(k) + ak^2 + \frac{1}{U(k)} \right) \delta h_k \delta h_{-k},
$$

(15)

where $\chi_{\pm}(k)$ is the Lindhard function of spin-polarized Fermi sea.

This result can be used to evaluate spin stiffness. In doing so, we expect that the $k = 0$ contributions of $\chi_{\pm}(k)$ and $1/U(k)$ cancel out, since only the spatially varying part of $h(x)$ contributes to the energy of a weakly inhomogeneous symmetry-broken state (as required by Goldstone’s theorem). Accordingly, we consider the dependence $F$ vs. $\delta h_k$ assuming that the local spin-quantization axis is slowly varying, $\xi \partial_\mu S_\mu \ll 1$. This is the case when large spin stiffness makes the short-wavelength fluctuations in $h(x)$ costly and, therefore, weak. Expanding in $k$ at second order, $U^{-1}(k) \approx U^{-1}(0)(1 + \xi^2 k^2)$, $\chi_{\pm}(k) \approx \chi_{\pm}(0) + ak^2$, gives the dependence of the free energy on $\delta h_k$:

$$
\delta F = \sum_k \frac{1}{2} \left( \chi_{\pm}(0) + ak^2 + \frac{1 + k^2 \xi^2}{U(0)} \right) \delta h_k \delta h_{-k}.
$$

(16)

Taking into account that $\chi_{\pm}(k)$ is evaluated for the spin-polarized state, we confirm that $k = 0$ contributions cancel out owing to the Stoner mean-field relation, Eq. (13). This yields the gradient expansion

$$
\delta F = \sum_x \frac{1}{2} \left( \partial_\mu S_\mu \right)^2, \quad J = \frac{\xi^2 h^2}{U(0)} + ah^2.
$$

(17)

This result indicates that the spin stiffness parameter $J$ is dominated by $U(k)$ expansion to order $k^2$ when the correlation length $\xi$ is large. For a contact interaction the stiffness is dominated by Lindhard function expansion. For a realistic short-range interaction, both contributions to stiffness are expected to be equally important.

The value and sign of $a$ are sensitive to the details of the band structure and may potentially result in a negative spin stiffness $J$. To ensure that $J$ remains positive, we focus here on the large-$\xi$ regime where $\xi^2 \gg U(0)$. In this case, the stiffness value is dominated by the first term in Eq. (17):

$$
J \approx \frac{\xi^2 h^2}{U(0)}.
$$

(18)

The positivity of $J$ is crucial because otherwise, a spin density wave pattern would emerge, rendering the approach relying on perturbation theory around the uniform saddle point invalid.

IV. SPIN-DEPENDENT GEOMETRIC MAGNETIC FIELD

Next, we extend the mean-field framework to describe spin textures. This can be done by considering the spatial dependence of the spin polarization field $h(x)$. In this section, we show that the spin texture generates a geometric spin-dependent magnetic field $B(x)$ given by Eq. (2), which couples to electrons as described in Eq. (3).

We first give a heuristic argument for the origin of the field $B(x)$. Microscopically, it arises from the adiabatic spin rotation effect for spins of electrons moving through a long-period spin texture to which they are coupled by exchange interaction. The spin rotation describes evolution of an electron spin being locked to the local spin quantization axis and tracking it along the electron trajectory. In the adiabatic regime, the effect can be described by a spin-dependent geometric phase that depends on position-dependent spin polarization in the texture. This adiabatic framework is applicable when the Stoner spin gap is large compared to $\hbar v/\ell$, where $\ell$ is the characteristic spatial lengthscale of the spin texture modulation and $v$ is the Fermi velocity. Berry curvature associated with the vector potential describing this geometric phase, defines a geometric magnetic field

$$
b(x) = \nabla_x \times a(x).
$$

(19)

Below, we identify the quantities $a$ and $b$ with spin-dependent gauge fields obtained from the microscopic Hamiltonian and demonstrate that $b$ equals to the chirality density given in Eq. (11). From this, we derive the geometric coupling between $b$ and electrons’ orbital magnetization, which is the interaction given in Eq. (3).

Next, we formally introduce a gauge field describing a position-space Berry phase for electrons in the presence of a slowly varying spin texture. In that we follow the procedure developed some time ago in the literature on quantum antiferromagnets and high-temperature superconductivity [51–53] and, more recently, in the literature on frustrated magnetic systems [55–57]. Below we present a step-by-step derivation of the gauge field picture starting with the microscopic Hamiltonian introduced in Sec. II. In doing so, it will be shown explicitly that a spin texture gives rise to an effective gauge field whose flux density is associated with the spin chirality. In our analysis below, without loss of generality, we focus on spins in valley $K$ and suppress the valley label. A spin texture is described by a position-dependent ordering field $h(r)$ introduced in Sec III through a Hubbard-Stratonovich mean field analysis, which we will write as

$$
h(x) = h s(x)
$$

(20)

where $s(x)$ is a unit-vector field, $|s(x)| = 1$. In what follows, we will ignore fluctuations of the order parameter magnitude $h$ and focus on the fluctuations of $s(x)$ orientation in spin space.

The first step is to perform an SU(2) spin rotation to bring all the local spin polarization to the same orientation and, in this way, generate a Hamiltonian that features a geometric spin-dependent gauge field. We start with a one-electron Hamiltonian in valley $K$, writing it
in position space:

\[
H(r) = \left( \frac{D}{(p_1 + ip_2)^2} \frac{(p_1 - ip_2)^2}{2m} - \frac{\hbar}{D} \right) 1_S - \hbar 1_L s(r) \cdot \sigma \tag{21}
\]

where \(1_S\) and \(1_L\) represent the identity matrices in the spin and sublattice subspaces, respectively. In the first term, \(p_{1,2} = -i \partial_{1,2}\) (here we set \(\hbar = 1\), restoring dimensional units later). The second term represents the effect of a position-dependent spin polarization arising after a Hubbard-Stratonovich transformation, see Eq. (12).

The coordinate-dependent spin rotation operator \(T(x)\) that rotates all spins from the local polarization direction \(s(x)\) to the \(+z\) direction is defined through

\[
|z\pm \rangle = T(x) |s(x)\rangle \pm . \tag{22}
\]

When acting with this spin rotation on the Hamiltonian in Eq. (21), the coordinate-dependent ordering field in the term \(\hbar \sigma_3 = T(x) h_\alpha(x) \sigma_\alpha T^\dagger(x)\) is transformed to a uniform field pointing in the \(+z\) direction:

\[
h_\alpha(x) T(x) h_\alpha(x) \sigma_\alpha T^\dagger(x) = \hbar \sigma_3. \tag{23}
\]

However, the simplicity comes at a price: the momentum operator in the Hamiltonian, Eq. (21), is transformed by \(T\) to a long derivative with a \(2 \times 2\) matrix gauge field. Namely, the Hamiltonian is transformed to

\[
H_\perp(x) = T(x) H(x) T^\dagger(x), \tag{24}
\]

where

\[
\Pi_\mu = -i T(x) \partial_\mu T^\dagger(x) = p_\mu + A_\mu, \\
A_\mu = -i T(x) [\partial_\mu, T^\dagger(x)], \quad \mu = 1, 2 \tag{25}
\]

Here \(A_{1,2}\) are \(2 \times 2\) matrices representing an \(SU(2)\) gauge field, and the square brackets represent commutators. The quantities \(A_\mu\) can be expressed in terms of Pauli matrices:

\[
A_\mu = \sum_{\gamma=1,2,3} a_{\mu,\gamma} \sigma_\gamma, \tag{26}
\]

where the coefficients \(a_{\mu,\gamma}\) are the scalar quantities

\[
a_{\mu,\gamma}(x) = \frac{1}{2} \text{Tr} (\sigma_\gamma A_\mu) = -i \frac{1}{2} \text{Tr} (\sigma_3 T(x) \partial_\mu T^\dagger(x)). \tag{27}
\]

This analysis, which is exact so far, simplifies in the adiabatic regime, where all spins track the spin-up and spin-down states in rotated basis. In this case, the off-diagonal components \(a_{\mu,1}\) and \(a_{\mu,2}\) describe coupling between spin states split by exchange. Such couplings only contribute at subleading order since they induce off-resonant transitions which are weak in the adiabatic limit. We can therefore retain only the diagonal \(\sigma_3\) spin components, given by Eq. (27). This gives

\[
\Pi_\mu = p_\mu + a_\mu \sigma_3, \tag{28}
\]

where from now on \(a_\mu\) will be used as a shorthand for \(a_{\mu,3}\). Plugging this back to Eq. (24), and absorbing \(\mu\) in \(H_\perp(p)\), we have the following form of free energy

\[
F = \text{Tr} [\hbar \omega - H_\perp(p + a_\mu \sigma_3) - \hbar \sigma_3] + F_h \tag{29}
\]

where \(\text{Tr}\) denotes \(\int d^2 x \sum_{\omega, p} \text{Tr} 2 \times 2\) and \(F_h\) is the spin stiffness energy given in Eq. (16).

The results indicate that the spin-up and spin-down electrons in the rotated basis, describing the majority and minority spin in the original basis, experience different \(U(1)\) gauge fields of opposite signs. After some algebra, which follows closely that in Ref. [26], one finds

\[
a_\mu = -\frac{1}{2} (1 - \cos \theta) \partial_\mu \phi \tag{30}
\]

where \(\theta\) and \(\phi\) are the spherical polar and azimuthal angles measured with respect to the \(z\) axis introduced in Eq. (28). The geometric magnetic field \(b = \nabla \times a\) is then given by

\[
b(x) = \epsilon_{\mu\nu} \partial_\nu a_\mu = \nabla_x \times a = \frac{1}{2} \epsilon_{\mu\nu} s \cdot (\partial_\mu s \times \partial_\nu s), \tag{31}
\]

which is nothing but the chirality density given in Eq. (1). The result in Eq. (30) indicates that the geometric phase picked up by an electron moving in the magnetic field \(b(x)\) is equal to 1/2 of the solid angle swept by the spin quantization axis.

So far, we have utilized the ‘geometric’ units where \(\hbar = 1\) and the units for \(p\) and \(a\) are inverse length. Consequently, the geometric field \(b\) is expressed in units of inverse length squared. It is interesting to examine how these quantities and their relationships alter when physical units are restored, setting the units for \(p_\mu\) and \(a_\mu\) to \(\hbar\) over length.

Importantly, the relationship between \(\Pi_\mu\) and \(p_\mu\) given in Eq. (28) remains unchanged upon reintroducing physical \(\hbar\) units. Furthermore, comparing this relation to the canonical electromagnetic coupling,

\[
\Pi = p - \frac{e}{h c} A, \tag{32}
\]

we observe that, while the coupling between the geometric field \(a\) and the electrons’ orbital degrees of freedom mirrors the canonical electromagnetic coupling in form, the coupling strength is quite different from the canonical coupling in Eq. (32). The coupling to geometric gauge field leads to energy scales that are independent of \(e\) and \(c\). Instead, the energy scales are governed by the length-scales describing spatial periodicity of skyrmion textures (see Eq. (30) and accompanying discussion).
V. GRADIENT EXPANSION IN $a(x)$

The discussion above implies that the geometric magnetic field $b = \nabla \times a$ is perceived by electrons as a pseudo magnetic field coupled to orbital degrees of freedom with an effective strength given by:

$$B(x) = \frac{\phi_0}{2\pi} b(x),$$

where $\phi_0 = \frac{2\pi hc}{e}$ is the flux quantum. This relationship arises from recognizing that Eq. (31) indicates that a skyrmion texture with a unit topological charge, $\pm \lambda$, can be written as a series expansion of the electronic energy in Eq. (29) in powers of $\theta = \frac{\pi}{2} \left( N_z \right)$.

One such effect is the anomalous Hall effect. One such effect is the anomalous Hall effect generated by spin textures expressed as gradients of $\delta H = \frac{\pi}{2} \left( N_z \right)$.

VI. STONER INSTABILITY IN THE ABSENCE OF CHIRAL INTERACTION

Putting everything together, and for now ignoring the chiral interaction, we can write the system energy in the absence of pseudo magnetic field $B$ as

$$F = \int d^2 x \left[ E_+ + E_- + \frac{\hbar^2}{2 U(0)} + \frac{J}{2} (\partial_a s)^2 \right].$$

Using this expression, we can seek the ground state by comparing the energies of the ordered and disorder states. To account for the effect of a long-range $1/r$ density-density interaction without incorporating it explicitly in the mean-field analysis, we consider different states at the same total carrier density $n$. This approach is valid due to the large charging energy values $E_c = \frac{1}{2} V_0 n^2$ which typically exceed other energy scales in the system. When $E_c$ is included in the analysis, the dependence of the total energy on $n$ is dominated by the following terms:

$$\frac{V_0 n^2}{2} - \mu n = \frac{V_0}{2} \left( n - \frac{\mu}{V_0} \right)^2 - \frac{\mu^2}{2 V_0}.$$  (37)

These terms pin the density to $n = \frac{\mu}{V_0}$, regardless of the order type. Therefore, comparing energies of different states at the same $\mu$ in the presence of $E_c$ is equivalent to comparing their energies at the same $n$.

To analyze the ordering described by Eq. (36), we proceed in two steps: First analyze the Stoner instability while temporarily ignoring $B$. Next, we consider the dependence on $B$ and the transition from a uniform magnetic order to a twisting order.
In the absence of $B$, Eq. (36) describes the standard Stoner instability—a transition from a disordered state to a uniformly polarized state. Since the density of states in the quadratic Dirac band monotonically decreases as a function of energy, the ground state configuration is either fully spin polarized or spin unpolarized, depending on the band parameters and interaction strength. [For a more general band dispersion, partial spin polarization can also occur.] The energy density of a fully polarized phase with $n_+ = n$ and $n_- = 0$ is given by

$$F_{\text{fp}} = E_{\text{tot}}(n) - \frac{U_0 n^2}{2}$$  \hspace{1cm} (38)

where $n$ is a given total carrier density. We have used Eq. (13). Here

$$E_{\text{tot}}(n) = \int_0^{\sqrt{4\pi n}} \frac{d^2 k}{4\pi^2} \epsilon_k$$  \hspace{1cm} (39)

represents the total kinetic energy of electrons of density $n$ in one spin one valley in the absence of interaction. Similarly, the energy of unpolarized state where $n_+ = n_- = n/2$ is given by

$$F_{\text{unp}} = 2E_{\text{tot}}(n/2)$$  \hspace{1cm} (40)

Here, we have used $\hbar = 0$ in unpolarized phase. For our quadratic Dirac band, $E_{\text{tot}}$ takes the following form:

$$E_{\text{tot}}(n) = \frac{m D^2}{4\pi} \left( \log(x + \sqrt{1 + x^2}) + x \sqrt{1 + x^2} \right),$$  \hspace{1cm} (41)

where $x = \frac{2\pi n}{mD}$ is a dimensionless density parameter. The regime of interest is that of strong exchange interaction, which corresponds to low values $n \ll 2mD$. In this case, we can use power-series expansion $E_{\text{tot}}(n) = \frac{mD^2}{2\pi}(x + \frac{5}{12}x^3 + ...).$ This allows a direct comparison of the energies of polarized and unpolarized states. Simple algebra then predicts the fully polarized state to win when

$$\frac{n}{D} < \frac{2m^2 U_0}{5\pi^2}.$$  \hspace{1cm} (42)

Therefore, the phase boundary is a straight line on the $D$-$n$ phase diagram (see Fig. 1). This phase transition is first-ordered since the full polarization occurs abruptly.

VII. CHIRAL INTERACTION AND ENERGETICS OF SPIN TEXTURES

Next, we consider spin textures $s(x)$ and derive the condition for skyrmion proliferation in the presence of chiral interaction. From Eqs. (35), (36), we see that system energy depends on $s$ as $E_s = \int d^2 x \left[ \frac{1}{2} (\partial_{\mu} S_{\mu})^2 + \Delta MB + \frac{\chi}{2} B^2 \right]$. Therefore, the spin texture enters the energetics in two ways: through pseudo magnetic field $B$ which is proportional to the spin chirality density $E_s$ (31), and also through the spin stiffness energy $\frac{1}{2} J (\partial_{\mu} s)^2$. However, the latter contribution has a lower bound associated with spin chirality

$$\frac{1}{2} \int d^2 x (\partial_{\mu} s)^2 \geq \frac{1}{2} \int d^2 x [\epsilon_{\mu\nu} s \cdot (\partial_{\mu} s \times \partial_{\nu} s)].$$

This relation follows from the well-known identity [24]:

$$\int d^2 x [\epsilon_{\mu\nu} s \cdot (\partial_{\mu} s \times \partial_{\nu} s)] = \frac{1}{2} \int d^2 x (\partial_{\mu} s \pm \epsilon_{\mu\nu} s \times \partial_{\nu} s)^2 \geq 0,$$  \hspace{1cm} (44)

Expressing the stiffness energy through $|B|$ gives

$$E_s[B] = \int d^2 x \left[ -\Delta MB + \frac{2 J e}{\hbar c} |B| + \frac{\chi}{2} B^2 \right],$$  \hspace{1cm} (45)

Obviously, the quantity $E_s[B]$ is only well-defined when spin polarization occurs. Therefore, below we focus on the effect of $E_s[B]$ on the fully spin-polarized state.

It is now straightforward to derive the condition for nonzero chirality to be favored. The free energy in Eq. (45) gives the threshold for nucleating chiral spin textures in the ground state:

$$\Delta M \geq 2Je/\hbar c.$$  \hspace{1cm} (46)

As a reminder, $\Delta M = M_+ - M_-$. $M_{\pm} = \mu (\mu \pm h)$ in one particular valley. Below, without loss of generality, we focus on $K$ valley.

The net orbital magnetization of all electrons in one valley is governed by the Dirac band and its Berry curvature. This quantity, evaluated in our particle-hole-symmetric Dirac model, takes a simple form (see Ref. [59] and Appendix A):

$$M_K(\mu) = \begin{cases} \frac{eD}{2\pi\hbar c}, & \mu > D, \\ \frac{e\mu}{2\pi \hbar c}, & -D < \mu < D, \\ \frac{eD}{2\pi\hbar c}, & \mu < -D. \end{cases}$$  \hspace{1cm} (47)

taking opposite values in valleys $K$ and $K'$. This result is derived following the approach in Ref. [59], where a similar result was established for a monolayer graphene with a staggered sublattice potential. Ref. [59] derived the valley-dependent magnetization in a gapped Dirac band, arriving at a relation between the orbital magnetization (per spin) and the band Berry curvature $\Omega(k)$:

$$M = \frac{e}{\hbar} \int d^2 k \frac{2\mu}{(2\pi)^2} \Omega(k) f(\epsilon_k)$$  \hspace{1cm} (48)

where $\Omega(k)$ is of opposite signs in the particle and hole bands, $f(\epsilon_k)$ is the Fermi-Dirac distribution. However, as argued in Ref. [59], this relation between $M$ and $\Omega(k)$ holds for a generic band with particle-hole symmetry.

Applying this formalism to quadratic Dirac band model yields the orbital magnetization in Eq. (47). We note that the dependence in Eq. (47), with $M$ being constant in each band, is a unique property of the quadratic
Eq. (47) yields:

\[ \mu + h = \sqrt{D^2 + (\frac{4\pi n}{\xi})^2}, \]

where the terms \( O(n^2) \) were ignored since we are interested in the low-density regime. Plugging this into Eq. (47) yields:

\[ M_\mp = M(\mu + h) = (D - U(0)n)\frac{e}{2\pi \hbar c}. \]  

As a result, the quantity \( \Delta M = M_+ - M_- \) equals

\[ \Delta M = U(0)n\frac{e^2}{2\pi \hbar c}. \]  

Plugging this into Eq. (46) and using an estimate for spin stiffness obtained above, \( J \sim c^2k^2/(\hbar^0) \), we find a condition for the transition from a fully polarized state to a chiral spin state:

\[ \frac{eU(0)n}{2\pi \hbar c} \geq \frac{2e^2U(0)n^2}{4\hbar^0}. \]

This condition can be expressed in terms of carrier density and the correlation length \( \xi \) introduced in Sec. II:

\[ n\xi^2 \leq \frac{1}{\pi}. \]

Since our mean-field analysis works for \( \xi \) exceeding the Fermi wavelength \( \lambda_F \), the condition in Eq. (53) is marginally met. The threshold Eq. (53) can be further softened in multilayer graphene (such as trilayer, quadrilayer, or pentalayer) since \( \Delta M \) is proportional to the valley Chern number. For \( N \)-layer graphene, in a simplest model, \( C \) can take values that scale with the number of layers, \( C = N/2 \). As a result, the threshold softens to

\[ n\xi^2 \leq \frac{C}{\pi}. \]

Using these results, we can predict schematic phase diagram in which the chiral phase (skyrmions) coexists with a uniformly spin-polarized phase, as shown in Fig. 1.

Applying the condition for instability toward skyrmion texture to realistic systems requires estimating the values of \( \xi \) obtained from microscopic parameters. In the interest of system of interest, the carrier density \( n_c \) at the onset of Stoner transition can be estimated from Eq. (42). For interaction strength \( U(0) = 5 \times 10^3 \text{ meV } \text{ nm}^{-2} \), a band mass \( m = 0.03m_e \) and a typical high value \[ D = 100 \text{ meV} \], it predicts Stoner instability at \( n_c \approx 3 \times 10^{11} \text{ cm}^{-2} \). This carrier density indeed lies within the density range where the Stoner instability is observed \[ 13 \text{ – } 17 \]. Eq. (53) then predicts that to nucleate skyrmions, then predicts that to nucleate skyrmions the characteristic length \( \xi \) must satisfy \( \xi \lesssim 15 \text{ nm} \), a value comparable to the screening length of Coulomb interaction in realistic settings. To further stabilize the chiral spin texture, one can reduce the spin stiffness \( J \), which can be achieved by suppressing the correlation length of the spin-spin exchange interaction.

A phase diagram describing the competition between the orders described above is shown in Fig. 1. The transition line from uniformly polarized phase to skyrmion phase is given by Eq. (53). We note that, compared to the transition line between the uniformly polarized phase and unpolarized phase, the transition line between skyrmion phase and unpolarized phase is pushed slightly into the unpolarized phase. This is because when skyrmion condenses, the energy contribution from pseudo magnetic field Eq. (51) is always negative and tends to stabilize ordered state. This phase boundary is a first-order phase transition because the translation symmetry and the spin SU(2) symmetry are simultaneously broken on this line.

We note that the condition for skyrmion instability Eq. (53) can be softened in Dirac bands with larger valley Chern numbers, since \( \Delta M \) is proportional to the total Hall conductivity in the lower band. Large valley Chern numbers can be achieved in graphene multilayers, such as trilayer, quadrilayer, or pentalayer. Another appealing system is moiré graphene, where valley-Chern minibands \[ 10 \] give rise to a doping-dependent orbital magnetization, potentially leading to a skyrmion instability triggered by spin polarization onset.

The emergence of skyrmions through the mechanism discussed above can lead to different ground states depending on the interactions between skyrmions and the strength of the spin order parameter zero-point or thermal fluctuations. Strong repulsive interactions would stabilize a chiral skyrmion crystal state, whereas strong fluctuations would lead to a chiral skyrmion liquid. Overall, these phases are expected to have properties similar to those of the vortex lattice and vortex liquid phases in superconductors \[ 60 \text{ – } 63 \]. Which of the two states—skyrmion crystal or skyrmion liquid—wins in the true ground state is an interesting topic for future work.

These two phases can be readily distinguished by transport measurements. In the presence of a valley polarization, such as the one seen in Bernal bilayer graphene as well as rhombohedral multilayer graphene systems and moiré graphene systems, we expect quantized topological Hall effect (QHE) in both states in the absence of an applied magnetic field \[ 57 \text{ – } 64 \]. This QHE contribution will appear in addition to (non-quantized) valley Hall effect.

So far we discussed skyrmions in one valley. If, however, the system is valley-unpolarized but each valley exhibits Stoner spin-polarized order, skyrmions can occur
in both valleys. The valleys $K$ and $K'$ are related by time reversal symmetry, which requires opposite signs of the band Berry curvature and associated orbital magnetization, $M_K = -M_{K'}$. This imposes a peculiar relation between skyrmion chiralities in the two valleys, which can be understood from the chiral interaction in Eq.\(^2\),

$$\delta F = \int d^2x \left[ -\Delta M_K B_K - \Delta M_{K'} B_{K'} \right],$$  \hspace{1cm} (55)

where $\Delta M_K = M_{K,+} - M_{K,-}$ and $\Delta M_{K'} = M_{K,+} - M_{K,-}$, with the plus and minus indicating contributions of majority-spin and minority-spin carriers. Ignoring, at first, the intervalley exchange interaction, the system can be viewed as two identical Stoner problems with Berry curvatures of opposite signs in the two valleys. In this case, spin polarization directions in the two valleys are completely decoupled. Then, time-reversal symmetry of the Hamiltonian predicts equal degrees of spin polarization in valleys $K$ and $K'$, and requires $M_{K,+} = -M_{K,-}$ and $M_{K',+} = -M_{K',-}$, namely, $\Delta M_+ = -\Delta M_-$. Indeed, under time reversal the majority (minority) spins in valley $K$ are always mapped to the majority (minority) spins in valley $K'$. As a result, the SU(2)$_K$ $\otimes$ SU(2)$_{K'}$ symmetry of the microscopic Hamiltonian in the absence of intervalley exchange interactions, enforces the property $\Delta M_+ = -\Delta M_-$ regardless of whether spin polarization directions in the two valleys are parallel, antiparallel or canted relative to one another. Therefore, in this case, the system will favor spin textures of opposite chiralities in the two valleys.

In realistic systems, however, the spins in the two valleys are weakly coupled by intervalley exchange interactions, which are much weaker than the intravalley interactions. The intervalley exchange is expected to be of a ferromagnetic sign in the regime of low carrier density. To optimize this intervalley interaction, spin textures in valleys $K$ and $K'$ would need to be of identical direction and sign, i.e. $s_K(x) = s_{K'}(r)$. However, this would result in equal-sign chirality densities in valleys $K$ and $K'$. Such spin textures do not optimize the chiral interaction in Eq.\(^5\) which favors opposite chiral densities in valleys $K$ and $K'$. Therefore, in this case, our system is expected to exhibit a frustration effect — the energies of all terms in the Hamiltonian cannot be simultaneously optimized. As a result, many states can be envisaged as candidates for the ground state. One possibility is a two-valley chiral liquid where spin long-range order is washed out, but the chirality densities being of opposite signs in two valleys and exhibiting a long-range order. Another interesting option is a staggered skyrmion lattice, in which skyrmions formed by electrons in valleys $K$ and $K'$ are arranged in two interpenetrating lattices. Understanding such frustrated skyrmion phases represents an interesting direction for future work.

When time-reversal is not spontaneously broken (no valley polarization), one expects a quantized topological valley Hall effect but no charge Hall effect, since the time-reversal symmetry requires skyrmions in valleys $K$ and $K'$ to have opposite chiralities. In addition, the longitudinal transport will be very different in the two phases—vanishing for skyrmion crystal and nonzero for skyrmion liquid, dual to that of superconducting vortex crystals and liquids.

VIII. SOME IMPLICATIONS OF SKYRMION TEXTURES

In conclusion, this work predicts a geometric spin-orbit coupling that arises in spin-polarized bands endowed with Berry curvature. The mechanism underpinning this coupling is that a spin of an electron moving through a spin texture is rotated in spin space. This spin rotation effect, arising due to an electron spin being locked to the local spin quantization axis and tracking it along the electron trajectory, is described by a spin-dependent geometric phase. The adiabatic regime in which the geometric phase picture applies occurs when the Stoner spin gap is large compared to $hv/\ell$, where $\ell$ is the characteristic spatial lengthscale of the spin texture modulation and $v$ is Fermi velocity.

This coupling leads to an instability of a uniformly spin-polarized state towards skyrmions. Occurring in an itinerant magnetic system, the skyrmions have several interesting properties. One is that they act on electrons as a geometric pseudomagnetic field, such that each skyrmion effectively generates one flux quantum of the field. The effective strength of this field is proportional to skyrmion density $n_s$ and can be expressed as

$$B_{\text{eff}} = 4.13 \cdot 10^{-11} \times n_s [\text{cm}^{-2}] \text{Tesla}. \hspace{1cm} (56)$$

For skyrmion density of $n_s \approx 10^{10} \text{cm}^{-2}$ this predicts $B_{\text{eff}}$ on the order 0.5 Tesla. The field $B_{\text{eff}}$ grows rapidly as $n_s$ increases. The Landau levels induced by $B_{\text{eff}}$ introduce a new energy scale which governs the skyrmion-induced topological gap in the system spectrum.

The geometric field $B_{\text{eff}}$ would result in a topological Hall effect, manifested through a nonvanishing Hall conductivity occurring in the absence of an applied magnetic field. Because the geometric magnetic field has opposite signs for carriers with opposite spins, this Hall conductivity will exhibit a characteristic dependence on spin polarization, distinguishing it from the typical charge Hall conductivity. At carrier densities corresponding to $\nu$ electrons per skyrmion, where $\nu$ is an integer, the system will host $\nu$ fully filled skyrmion-induced Landau levels, leading to a quantized Hall conductivity

$$\sigma_H = \nu e^2 / h. \hspace{1cm} (57)$$

For skyrmion crystal or liquid of a high density such that the number of electrons per skyrmion is small, this scenario predicts a state with large $B_{\text{eff}}$ and a large topological gap.

In the extreme limit, when the skyrmion density achieved through this mechanism is large enough to be
close to the density of itinerant carriers, the system can spontaneously adopt an insulating ground state where all electrons occupy one or several lowest Landau levels. This scenario may potentially apply to the quantized Hall phases recently observed in pentagonal graphene in the absence of a magnetic field\cite{29, 30}. The predicted dependence on the number of layers, implying that more layers tend to lower the threshold for the emergence of spin chirality, potentially explains why this effect occurs in pentagonal rather than bilayer and trilayer systems.

This work greatly benefited from discussions with Eli Zeldov, Steven Kivelson and Patrick Lee. We acknowledge support from the Science and Technology Center for Integrated Quantum Materials, National Science Foundation Grant No. DMR1231319. ZD is currently affiliated with California Institute of Technology, Physics Department.

\begin{thebibliography}{99}

[1] J. M. B. Lopes dos Santos, N. M. R. Peres, and A. H. Castro Neto, Graphene Bilayer with a Twist: Electronic Structure, Phys. Rev. Lett. 99, 256802 (2007)
[2] E. J. Mele, Commensuration and interlayer coherence in twisted bilayer graphene, Phys. Rev. B 81, 161405(R) (2010)
[3] E. S. Morell, J. D. Correa, P. Vargas, M. Pacheco, and Z. Bartaicic, Flat bands in slightly twisted bilayer graphene: Tight-binding calculations, Phys. Rev. B 82, 121407 (R) (2010)
[4] R. Bistritzer and A. H. MacDonald, Moiré bands in twisted double-layer graphene, PNAS 108 (30) 12233-12237 (2011)
[5] F. Guinea and N. R. Walet, Electrostatic effects, band distortions, and superconductivity in twisted graphene bilayers, PNAS 115 (52) 13174-13179 (2018)
[6] Andrei, E.Y. and MacDonald, A.H., 2020. Graphene bilayers with a twist. Nature materials, 19(12), pp.1265-1275.
[7] Cao, Y., Fatemi, V., Fang, S., Watanabe, K., Taniguchi, T., Kaxiras, E. and Jarillo-Herrero, P., 2018. Unconventional superconductivity in magic-angle graphene superlattices. Nature, 556(7699), pp.43-84.
[8] Cao, Y., Fatemi, V., Demir, A., Fang, S., Tomarken, S.L., Luo, J.Y., Sanchez-Yamagishi, J.D., Watanabe, K., Taniguchi, T., Kaxiras, E. and Ashoori, R.C., 2018. Correlated insulator behaviour at half-filling in magic-angle graphene superlattices. Nature, 556(7699), pp.80-84.
[9] Cao, Y., Rodan-Legrain, D., Rubies-Bigorda, O., Park, J.M., Watanabe, K., Taniguchi, T. and Jarillo-Herrero, P., 2020. Tunable correlated states and spin-polarized phases in twisted bilayer-bilayer graphene. Nature, 583(7815), pp.215-220.
[10] Zondiner, U., Rozen, A., Rodan-Legrain, D., Cao, Y., Queiroz, R., Taniguchi, T., Watanabe, K., Oreg, Y., von Oppen, F., Stern, A. and Berg, E., 2020. Cascade of phase transitions and Dirac revivals in magic-angle graphene. Nature, 582(7811), pp.203-208.
[11] Wong, D., Nuckolls, K.P., Oh, M., Lian, B., Xie, Y., Jeon, S., Watanabe, K., Taniguchi, T., Bernevig, B.A. and Yazdani, A., 2020. Cascade of electronic transitions in magic-angle twisted bilayer graphene. Nature, 582(7811), pp.198-202.
[12] Saito, Y., Yang, F., Ge, J., Liu, X., Taniguchi, T., Watanabe, K., Li, J.I.A., Berg, E. and Young, A.F., 2021. Isospin Pomeranchuk effect in twisted bilayer graphene. Nature, 592(7853), pp.220-224.
[13] Seiler, A.M., Geisenhof, F.R., Winterer, F., Watanabe, K., Taniguchi, T., Xu, T., Zhang, F. and Weitz, R.T., 2022. Quantum cascade of correlated phases in trigonally warped bilayer graphene. Nature, 608(7922), pp.298-302.
[14] Zhou, H., Holleis, L., Saito, Y., Cohen, L., Huynh, W., Patterson, C.L., Yang, F., Taniguchi, T., Watanabe, K. and Young, A.F. Isospin magnetism and spin-polarized superconductivity in Bernal bilayer graphene. Science, 375(6582), pp.774-778 (2022).
[15] de la Barrera, S.C., Aronson, S., Zheng, Z., Watanabe, K., Taniguchi, T., Ma, Q., Jarillo-Herrero, P. and Ashoori, R., 2022. Cascade of isospin phase transitions in Bernal-stacked bilayer graphene at zero magnetic field. Nature Physics, pp.1-5.
[16] Zhou, H., Xie, T., Taniguchi, T., Watanabe, K. and Young, A.F., 2021. Superconductivity in rhombohedral trilayer graphene. Nature, 598(7881), pp.434-438.
[17] Zhou, H., Xie, T., Ghazaryan, A., Holder, T., Ehrets, J.R., Spanton, E.M., Taniguchi, T., Watanabe, K., Berg, E., Serbyn, M. and Young, A.F., 2021. Half-and quarter-metals in rhombohedral trilayer graphene. Nature, 598(7881), pp.429-433.
[18] E. McCann, and M. Koshino, The electronic properties of bilayer graphene. Reports on Progress in physics, 76(5), p.056503 (2013).
[19] Z. Dong, O. Ogumina, and L. Levitov, Collective excitations in chiral Stoner magnets. Physical Review Letters, 130(20), p.206701 (2023).
[20] Castro, E.V., Peres, N.M.R., Stauber, T. and Silva, N.A.P., Low-density ferromagnetism in biased bilayer graphene. Phys. Rev. Lett., 100(18), 186803 (2008).
[21] A. Fert, N. Reyren, V. Cros, Magnetic skyrmions: advances in physics and potential applications, Nature Reviews Materials, 2, 17031 (2017)
[22] N. Nagaosa, Y. Tokura, Topological properties and dynamics of magnetic skyrmions. Nature Nanotech 8, 375 (2013).
[23] A. N. Bogdanov, U. K. Rössler, Chiral symmetry breaking in magnetic thin films and multilayers. Phys. Rev. Lett., 87, 037203 (2001).
[24] A. M. Polyakov, Gauge Fields and Strings (Harwood Acad. Publ. 1987).
[25] Binz, B., Vishwanath, A. and Aji, V., Theory of the helical spin crystal: a candidate for the partially ordered state of MnSi. Phys. Review Lett., 96(20), 207202 (2006).
[26] Nakanishi, O., Yanase, A., Hasegawa, A. and Kataoka, M., 1980. The origin of the helical spin density wave in twist double-layer graphene, PNAS 108 (30) 1223-12237 (2011).
\end{thebibliography}
Skyrmions without Sigma Models in Quantum Hall Ferromagnets, Phys. Rev. Lett. 76, 2153 (1996).

Fertig, H. A., Brey, L., Coté, R., MacDonald, A. H., Karlhede, A., and Sondhi, S. L., Hartree-Fock theory of Skyrmions in quantum Hall ferromagnets, Phys. Rev. B 55, 10671 (1997).

Nomura, K. and MacDonald, A.H. Quantum Hall ferromagnetism in graphene. Phys. Rev. Lett., 96(25), p.256602. (2006).

Yang, K., Das Sarma, S., and MacDonald, A. H., Collective modes and skyrmion excitations in graphene $SU(4)$ quantum Hall ferromagnets, Phys. Rev. B 74, 075423 (2006).

Zhou, H., Polshyn, H., Taniguchi, T. et al. Solids of quantum Hall skyrmions in graphene. Nat. Phys. 16, 154–158 (2020).

Liu, X., Farahi, G., Chiu, C.L., Papic, Z., Watanabe, K., Taniguchi, T., Zaletel, M.P. and Yazdani, A. Visualizing broken symmetry and topological defects in a quantum Hall fermagnet. Science, 375(6578) (2022).

Coleman, P., Introduction to Many-Body Physics, Cambridge University Press (2015)

Baskaran, G. and Anderson, P. W., Gauge theory of high-temperature superconductors and strongly correlated Fermi systems, Phys. Rev. B 37.580 (1988).

Wiegmann, P. B., Superconductivity in strongly correlated electronic systems and confinement versus deconfinement phenomenon, Phys. Rev. Lett. 60.821 (1988).

Schulz, H. J., Effective action for strongly correlated fermions from functional integrals, Phys. Rev. Lett. 65.2462 (1990).

Ioffe, L. B. and Kalmeyer, V. and Wiegmann, P. B., Hall coefficient of the doped Mott insulator: A signature of parity violation, Phys. Rev. B 43.1219 (1991).

Kenya Ohgushi, Shuichi Murakami, and Naoto Nagaosa. Spin anisotropy and quantum Hall effect in the lagomé lattice: Chiral spin state based on a ferromagnet. Phys. Rev. B 62, R6065 (2000).

Fujita, T., Jalil, M.B.A., Tan, S.G. and Murakami, S. Gauge fields in spintronics. Journal of applied physics, 110(12), p.17 (2011).

K. Hamamoto, M. Ezawa, and N. Nagaosa, Quantized topological Hall effect in skyrmion crystal, Phys. Rev. B 92, 115417 (2015).

Y. Taguchi, O. Oohara, H. Yoshizawa, N. Nagaosa, and Y. Tokura. “Spin chirality, Berry phase, and anomalous Hall effect in a frustrated ferromagnet.” Science 291, no. 5513 (2001): 2573-2576.

Xiao, D., Yao, W. and Niu, Q. Valley-contrast physics in graphene: magnetic moment and topological transport. Phys. Rev. Lett., 99(23), p.236809 (2007).

Huberman, B.A. and Doniach, S. Melting of two-dimensional vortex lattices. Phys. Rev. Lett., 43(13), p.950 (1979).

Fisher, D.S., Fisher, M.P. and Huse, D.A. Thermal fluctuations, quenched disorder, phase transitions, and transport in type-II superconductors. Phys. Rev. B, 43(1), 130 (1991).

Huberman, B.A. and Doniach, S. Melting of two-dimensional vortex lattices. Phys. Rev. Lett., 43(13), p.950 (1979).

Fisher, D.S., Fisher, M.P. and Huse, D.A. Thermal fluctuations, quenched disorder, phase transitions, and transport in type-II superconductors. Phys. Rev. B, 43(1), 130 (1991).
tional topological Hall effect in skyrmion crystals caused by the topology of the lattice. Phys. Rev. B, 95(9), 094413 (2017).

[65] Koshino, M. and Ando, T., Anomalous orbital magnetism in Dirac-electron systems: Role of pseudospin paramagnetism. Phys. Rev. B, 81(19), 195431 (2010).

[66] McCann, E. and Fal’ko, V.I., Landau-Level Degeneracy and Quantum Hall Effect in a Graphite Bilayer, Phys. Rev. Lett. 96, 086805 (2006).

[67] Halperin, B.I., Lee, P.A. and Read, N., 1993. Theory of the half-filled Landau level. Physical Review B, 47(12), p.7312.

[68] You, Y.Z. and Vishwanath, A., Kohn-Luttinger su-

[69] pereconductivity and intervalley coherence in rhombohe-

dral trilayer graphene. Phys. Rev. B, 105(13), p.134524 (2022).

[70] Han, T., Lu, Z., Scuri, G., Sung, J., Wang, J., Han, T., Watanabe, K., Taniguchi, T., Park, H. and Ju, L., 2024. Correlated insulator and Chern insulators in pentalayer rhombohedral-stacked graphene. Nature Nanotechnology, 19(2), pp.181-187.

[71] Koshino, M. and Ando, T., Anomalous orbital magnetism in Dirac-electron systems: Role of pseudospin paramagnetism. Phys. Rev. B, 81(19), 195431 (2010).

Appendix A: Valley-dependent orbital magnetization in graphene bilayer

To gain more insight into the physics of the orbital magnetization, Eq. (47), here we rederive this known result by a method that does not explicitly use Berry curvature. Our plan is to calculate the orbital magnetization in an individual graphene valley using thermodynamic relation:

\[ M_K = -\frac{\partial \Xi_K}{\partial B}, \]  

(A1)

where \( \Xi_K \) is the thermodynamic potential of electrons in this valley, defined as

\[ \Xi_K = \sum_{\alpha} (\epsilon_{\alpha} - \mu) f(\epsilon_{\alpha}), \quad f(\epsilon) = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} \]  

(A2)

where \( \epsilon_{\alpha} \) are the Landau level energies in the particle and hole bands, labeled by \( \alpha = \{\pm, n\} \).

In order to obtain the magnetization at \( B = 0 \) we first calculate the Landau level energies \( \epsilon_{\alpha} \) and, by using the Euler-Maclaurin summation formula, extract the part of the sum over \( \alpha \) in Eq. (A2) which is linear in \( B \) at small \( B \). As we will see, the contribution linear in \( B \) is equal to that originating from the anomalous Landau levels reduced by a factor of two, as discussed below. We will end this section by discussing the general character of this result and its relation to the spectral flow.

The Landau level energies can be derived directly from the BBG Hamiltonian \[^{[13]}^{[66]}\]. For illustration, here we do it for a simplified form of the Hamiltonian involving no trigonal warping terms:

\[ H_K(p) = \left( D + \frac{p^2}{2m_0} + \frac{p^2}{2m_n} - \frac{(p_1 + ip_2)^2}{2m} \right) . \]  

(A3)

Magnetic field can be incorporated in the Hamiltonian through the substitution \( p \rightarrow p - eB \). We will first carry out the analysis ignoring the terms \( p^2/2m_0 \) and \( p^2/2m_n \). This is justified because these two terms are subleading for a realistic BBG band. For the same reason we ignore the trigonal warping term (not shown in Eq. (A3)). To illustrate the generality of our results, we will subsequently present the analysis for the full Hamiltonian in Eq. (A3), finding that the quadratic terms \( p^2/2m_0 \) and \( p^2/2m_n \) do not affect the result.

Next we consider the Landau levels formed in the presence of a \( B \) field, at first excluding the quadratic terms in the diagonal elements. As is well known, in each valley — \( K \) or \( K' \) — the Hamiltonian in Eq. (A3), with the quadratic terms excluded, in the presence of a magnetic field generates three groups of Landau levels: (i) a pair of anomalous Landau levels at the edges of the hole band for valley \( K \) and particle band for valley \( K' \), and (ii) two sequences of Landau levels in the particle and hole bands that are related by particle-hole symmetry. The energies of these Landau levels in valley \( K \) can be written as

\[ \epsilon_{\pm, n} = \epsilon_{\pm}(x_n) = \pm \sqrt{x_n^2 - \frac{1}{4} \hbar^2 \omega_c^2 + D^2}, \quad n \geq 2, \]

\[ \epsilon_{0, 1} = -D, \quad x_n = \hbar \omega_c \left(n - \frac{1}{2}\right), \]  

(A4)

where \( \omega_c = eB/mc \) is the cyclotron frequency. For valley \( K' \) similar expressions arise, however the anomalous Landau levels are positioned at the particle band edge, \( \epsilon_{0, 1} = D \).

Accordingly, the thermodynamic potential \( \Xi_K \) in the presence of a \( B \) field is a sum of three contributions

\[ \Xi_K = \Xi_+ + \Xi_- + \Xi_{01}, \]  

(A5)

where

\[ \Xi_{\pm} = \frac{eB}{\hbar c} \sum_n (\epsilon_{\pm, n} - \mu) f(\epsilon_{\pm, n}), \]

\[ \Xi_{01} = \frac{2eB}{\hbar c} (-D - \mu) f(-D), \]  

(A6)

(A7)

with \( eB/\hbar c \) representing the numbers of electrons in each Landau level per unit area.

Magnetization is given by the linear \( (O(B)) \) term in \( \Xi_K(B) \). The \( O(B) \) contribution from the anomalous levels in each valley is already clearly written in Eq. (A7). To calculate the \( O(B) \) contribution from the anomalous levels in each valley is already clearly written in Eq. (A7). To calculate the \( O(B) \) contribution from the anomalous levels in each valley is already clearly written in Eq. (A7)}. To calculate the \( O(B) \) contribution from the anomalous levels in each valley is already clearly written in Eq. (A7).
have
\[
\Xi_+ = \frac{eB}{\hbar c} \left[ \frac{1}{\hbar \omega_c} \int_{x_{n=2}}^{\infty} dx (\epsilon(x) - \mu) f(\epsilon(x)) \right] (A8)
+ \frac{1}{2} (\epsilon(x_{n=2}) - \mu) f(\epsilon(x_{n=2})) + O(B^2),
\]
where \(x_{n=2} = \frac{3}{2} \hbar \omega_c\), see Eq. (A4). Here we have used \(\epsilon(\infty)f(\infty) = 0\). Working out the integral gives
\[
\Xi_+ = -\frac{eB}{\hbar c} (D - \mu) f(D) + O(B^2). \tag{A9}
\]

Similarly, the \(O(B)\) contribution of the lower-band Landau levels is given by
\[
\Xi_- = \frac{eB}{\hbar c} (-D - \mu) f(-D) + O(B^2). \tag{A10}
\]

After plugging Eqs. (A7), (A9) and (A10) into Eqs. (A1) and (A5), we arrive at
\[
M_K(\mu) = \begin{cases} 
2eD, & \mu > D \\
\frac{\epsilon(\mu+D)}{2e}\hbar c, & -D < \mu < D \\
0, & \mu < -D
\end{cases} \tag{A11}
\]

We note that this dependence differs by a constant shift of \(\Delta M_K = \frac{2eD}{2e}\hbar c\) from the result in Eq. (A7) that was inferred from the general expression for orbital magnetization obtained in Ref. [59]. This constant shift arises from the way the contribution of the deep-lying levels is cut off, which is different from the conventional way [59]. However, this difference is immaterial because the deep-lying states, due to their uncertain valley character and identical occupancies for opposite spins, are not expected to affect physical observables.

Indeed, at the bottom of the graphene band the carrier states cannot be unambiguously identified with the \(K\) and \(K'\) valleys. Therefore the ambiguity arising from the cutoff is a matter of convention rather than a physical effect. Furthermore, the quantity that matters for the physics of interest is the difference of the contributions from the spin-up and spin-down bands, \(\Delta M = M_{K,\uparrow} - M_{K,\downarrow}\). The bands for opposite spins are filled equally at the bottom, such that the contributions of the deep-lying states to \(M_{K,\uparrow}\) and \(M_{K,\downarrow}\) cancel each other.

The meaning of the resulting dependence \(M_K(\mu)\), in which \(M_K\) is constant when the Fermi level lies within one of the bands, can be understood in terms of a spectral flow induced by a variation of \(B\). Namely, the role of the Landau levels moving up and down is merely to cancel half of the contribution to magnetization \(M_K\) of the anomalous Landau levels in the corresponding bands. As a result, there is no \(\mu\) dependence when the Fermi level lies outside the gap. In that each anomalous level contributes a half of the ‘nominal value’ of a single Landau level. This contribution comes with a plus sign or a minus sign depending on whether an anomalous Landau level is present or absent for the band and valley in question. The resulting dependence of orbital magnetization is identical for the \(K\) and \(K'\) valleys up to a sign reversal, \(M_K(\mu) = -M_{K'}(\mu)\).

This analysis can be applied to a realistic model of Bernal bilayer graphene, where the band Hamiltonian takes a more complicated form [18]. Here we show that adding the two quadratic terms given in Eq. (A3), that were neglected temporarily, does not alter the result for \(M_K(\mu)\).

The term \(p^2/2m_o\) is an \(\sigma_3\) sublattice structure. Such a term does affect the wavefunctions, and yet, this term alone does not break the particle-hole symmetry. Also, this term does not affect the energy of the lowest Landau level in the particle band. As a result, the two conditions necessary for the reasoning above [from Eq. (A5) to Eq. (A10)] — the particle-hole symmetry and the presence of two anomalous Landau levels — remain valid. As a result, the answer for magnetization given above remains unchanged.