Bethe ansatz solution of the topological Kondo model

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Abstract
Conduction electrons coupled to a mesoscopic superconducting island hosting Majorana bound states have been shown to display a topological Kondo effect with robust non-Fermi liquid correlations. With $M$ bound states coupled to $M$ leads, this is an SO($M$) Kondo problem, with the asymptotic high and low-energy theories known from bosonization and conformal field theory studies. Here we complement these approaches by analyzing the Bethe ansatz equations describing the exact solution of these models at all energy scales. We apply our findings to obtain nonperturbative results on the thermodynamics of $M \rightarrow M - 2$ crossovers induced by tunnel couplings between adjacent Majorana bound states.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Majorana fermion states are presently among the most intensively studied objects in condensed matter physics. This is mainly due to the non-Abelian anyon statistics of defects binding Majorana fermions, with promising applications to topological quantum information processing [1–6]. Majorana fermions are zero energy bound states, pairs of which form ‘topological qubits’ encoding ordinary fermion degrees of freedom in a non-local manner,
Figure 1. Schematic device setup for the topological Kondo effect with $M = 6$ Majorana fermions $\gamma_j$ coupled to normal-conducting leads. The Majorana bound states are realized as end states of spin–orbit coupled nanowires on a floating superconducting island with charging energy $E_c$. The tunnel couplings $h_{j,k}$ between pairs of Majoranas act like a Zeeman field on the ‘Majorana spin’ with components $i\gamma_j\gamma_k$.

see [7–9] for recent reviews. These exotic objects are predicted to arise in heterostructures combining simple $s$-wave superconductors and materials with strong spin–orbit coupling, and first experimental results on potential realizations based on semiconductor nanowires [10–14] or topological insulators [15, 16] are under current investigation. Much of the work aimed at detecting [17–22] and manipulating [23–28] Majorana fermions has so far been concerned with effectively noninteracting physics. However, as we have shown in a series of recent papers [29–34], Majorana fermions can also be a source of rich strongly correlated physics. Such a setup is realized by coupling a ‘floating’ mesoscopic superconducting island—which has a finite charging energy $E_c$—to normal-conducting lead electrodes via $M > 2$ Majorana modes. The device setup is sketched in figure 1 and, in the parameter regime discussed below, implies the ‘topological Kondo effect’ [29]. The Majorana fermions residing on the island thereby represent a quantum ‘impurity’ which, at low-energy scales, becomes massively entangled with the conduction electrons in the attached leads through exchange-type processes. At low temperatures, this system is predicted to display exotic non-Fermi liquid correlations, similar to but different from the well-known overscreened multi-channel Kondo effect [35–38]. While achieving such non-Fermi liquid correlations in conventional Kondo devices is usually hindered by the competition of various couplings and, in particular, by the fact that channel anisotropy is a relevant perturbation destroying the non-Fermi liquid fixed point [39], the couplings acting against the topological Kondo effect can be eliminated to exponential accuracy simply by ensuring that adjacent Majorana fermions are sufficiently far apart. We note in passing that related physics has also been predicted in junctions of transverse Ising spin chains [40–42]. However, despite of superficial similarities, the topological Kondo effect found in the Majorana device in figure 1 is substantially different and always characterized by non-Fermi liquid behavior.

The source of the topological Kondo effect is the combination of the island’s charging energy, $E_c$, and the presence of the topological qubit degrees of freedom associated to the Majoranas. The former ensures that the island has a definite number of electrons in its ground state, while the latter transforms the island into an effective non-local ‘Majorana spin’ at energies much below both $E_c$ and the gap to non-Majorana (quasi-particle) excitations. In this work we only discuss the physics on those low-energy scales. When conduction electrons in a given lead electrode (labeled by the index $j$) have a finite tunnel coupling, $t_j$, to this non-local Majorana spin via the Majorana fermion $\gamma_j$, the screening correlations ultimately responsible for non-Fermi liquid behavior arise.
As the $M$ Majorana fermions are described by operators $\gamma_j = \gamma_j^\dagger$ subject to the Clifford algebra $\{\gamma_j, \gamma_k\} = 2\delta_{jk}$ [7–9], the relevant symmetry group for this Kondo problem is SO($M$). Indeed, the $\{\gamma_j\}$ compose a spinor representation of the SO($M$) group, where the $M(M-1)/2$ different products $\gamma_j\gamma_k$ represent the different components of the ‘bare’ (i.e., uncoupled to leads) Majorana spin [33]. The Kondo limit is realized when the charging energy $E_c \gg \max(t_j^2/v)$; in what follows we set the Fermi velocity $v = 1$ and use units with $\hbar = k_B = 1$. The effective Hamiltonian describing the system at low-energy scales, i.e., below $E_c$ and the energy of non-Majorana excitations, is [29–32]

$$H = -i \sum_{j=1}^M \int_{-\infty}^{\infty} dx \, \psi_j(\gamma) \tilde{a}_j(\gamma) + \sum_{j \neq k} \lambda_{jk} \gamma_j \gamma_k \psi^\dagger_0(0) \gamma_j(0) + \sum_{j \neq k} \tilde{h}_{jk} \gamma_j \gamma_k.$$  (1)

Here $\psi_j(x)$ is an effectively spinless right-moving fermion field describing the $j$th lead, where we unfolded from the physical lead coordinates $x < 0$ to the full line; $x = 0$ is the coordinate of the tunnel contact. The symmetric matrix $\lambda_{jk} \approx t_j t_k/E_c > 0$ is the analogue of the exchange coupling in the usual Kondo problem. The non-local couplings $\tilde{h}_{jk} = -h_{jk}$ correspond to overlaps between Majorana bound states. While they can in principle be suppressed to exponential accuracy by separating the Majoranas from each other, we include them here for two reasons. First, instead of suppressing these couplings, they can be deliberately switched on (by applying gate voltages) and used as handles to probe the physics of the Kondo screened non-local SO($M$) spin [33]. Second, their inclusion also allows one to study the eventual fate of the non-Fermi liquid physics at the lowest energy scales [33, 34]. Note that in the Kondo language, the couplings $\tilde{h}_{jk}$ act like Zeeman fields. We mention in passing that the $M$ Majoranas appearing in equation (1) might only be a subset of all Majorana bound states on the island. Regarding the complementary Majoranas which are not coupled to any lead electrode, we assume that these have no direct tunnel couplings with the $\gamma_j$.

In the absence of the Zeeman field, the weak-coupling renormalization group (RG) analysis for equation (1) indicates a flow of the exchange couplings toward a strong-coupling isotropic limit [29–32]. Taking all exchange couplings equal, $\lambda_{jk} = \lambda$, the perturbative RG approach turns out to break down on energy scales below the Kondo temperature

$$T_K \simeq E_c \exp \left(-\frac{\pi}{(M-2)\lambda}\right).$$  (2)

For temperatures $T \ll T_K$, one enters the non-Fermi liquid regime of the topological Kondo effect corresponding to an SO$_2(M)$ Wess–Zumino–Novikov–Witten boundary conformal field theory (BCFT) [38, 43]. Crucially, anisotropy in the $\lambda_{jk}$ is an irrelevant perturbation around this fixed point, in contrast to conventional overscreened multi-channel Kondo fixed points. However, anisotropy is likely to break integrability for $M > 4$.

The asymptotic low and high energy physics of the topological Kondo effect has been studied in [29–32] for $h_{jk} = 0$, while the effects of the Zeeman field components $\tilde{h}_{jk}$ have been addressed in [33]. For the $M = 3$ case, [34] has explored the full crossover from high to low energies, with and without Zeeman fields, using the numerical renormalization group technique. Here we complement those previous studies by providing analytical results obtained from exact Bethe ansatz (BA) calculations. Note that BA results for related spin chain junctions can be found in [42]. Before turning to the detailed BA solution of the topological Kondo model (1), we first summarize the main results of this paper.

2. Summary of results

In principle, the BA solution allows one to obtain very detailed information about the system, including the entire energy spectrum (for finite-length leads). This can provide access to the
full thermodynamic information for arbitrary temperature and Zeeman field. The BA solution certainly exists for \( M = 3 \), where equation (1) is equivalent to the four-channel Kondo model with spin \( S = 1/2 \) [44]. For general \( M \), we make a plausible conjecture for the BA and justify it by running various checks. In particular, we verify that the BA reproduces the results obtained by means of BCFT, including the ground-state impurity entropy. As a concrete application, we will discuss the thermodynamic response of the topological Kondo system to a Zeeman field by computing the ‘magnetization’, i.e., the expectation value of the Majorana spin,

\[
\mathcal{M}_K = (i\gamma_1\gamma_2),
\]

for which a measurement scheme has been proposed in [33]. In particular, we show that in the presence of just one Zeeman field component, say \( h_{12} > 0 \), a crossover between a Kondo problem with SO\((M)\) symmetry to another one with the symmetry group SO\((M - 2)\) is induced upon lowering the temperature. For \( M \leq 4 \), instead of another Kondo fixed point, the \( M \rightarrow M - 2 \) crossover terminates at a Fermi liquid state [45]. The BA solution discussed below allows one to address this crossover in a nonperturbative fashion. This is an attractive feature since the Zeeman field is an RG-relevant perturbation destabilizing the topological Kondo fixed point on temperatures below the temperature scale

\[
T_K \simeq T_K(h_{12}/T_K)^{M/2},
\]

which follows from simple dimensional scaling arguments [33]. Effectively, for \( T \ll T_K \), the Majorana modes \( \gamma_1 \) and \( \gamma_2 \) coupled by the Zeeman field move to finite energy and thus disappear from the low-energy theory. The remaining \( M - 2 \) Majoranas then realize a non-Fermi liquid SO\((M - 2)\) Kondo fixed point (assuming \( M > 4 \)). In this flow, the Kondo temperature \( T_K \) represents the high-energy cutoff, while the ‘effective’ Kondo temperature for the emergent low-temperature fixed point of SO\((M - 2)\) symmetry is set by \( T_{K(M-2)}^M = T_h \).

Our BA solution, explicitly constructed for \( M = 3 \) up to \( M = 6 \) below, nicely confirms this intuitive picture.

For each of the considered \( M \), we also compute the ground-state impurity entropy \( S_{\text{imp}} \). In the absence of the Zeeman field, we find that a BCFT calculation yields

\[
S_{\text{imp}} = \ln d_M, \quad d_M = \begin{cases} \sqrt{M}, & M \text{ odd,} \\ \sqrt{M/2}, & M \text{ even.} \end{cases}
\]

The quantum dimension \( d_M \) thus strongly responds to the parity of \( M \). This BCFT result follows from arguments very similar to those in [46], and the perfect agreement with our BA results provides a consistency check for the latter.

Let us start with the case \( M = 3 \), where the model (1) maps [44] to the four-channel SU\((2)\) Kondo problem with impurity spin \( S = 1/2 \). This correspondence allows us to use BA results obtained for the latter model [35, 37]. In particular, the ground-state magnetization \( \mathcal{M}_{12} \) can be expressed as scaling function of the variable \( Y = h_{12}/\kappa T_K \), where \( \kappa = \frac{2\pi}{\kappa} \simeq 27.21 \) and \( T_K \) is given by equation (2). For \( Y \ll 1 \) and \( T = 0 \), we find

\[
\mathcal{M}_{12} \simeq 0.868\sqrt{Y} + 0.034Y \ln Y + O(Y^3).
\]

The case \( M = 3 \) is the one where the singularities in the thermodynamic quantities and correlation functions are the strongest. Indeed, the susceptibility, \( \chi_{12} = \partial \mathcal{M}_{12}/\partial h_{12} \), has a \( Y^{-1/2} \) singularity for \( Y \to 0 \). The \( M \to M - 2 \) crossover picture here turns out to be consistent with a \( T = 0 \) fixed point describing a conventional Coulomb blockade regime, where the island ultimately decouples from the environment.

For \( M > 4 \), the \( M \to M - 2 \) crossover induced by lowering the temperature instead terminates in another Kondo fixed point with symmetry group SO\((M - 2)\). In this case, the
breakdown of perturbation theory at low temperatures $T \ll T_0$ signals in a weak singularity. For instance, for $M = 6$, we obtain a logarithmically divergent second derivative of $\mathcal{M}_{12}$,
\begin{equation}
\frac{\partial^2 \mathcal{M}_{12}}{\partial h_{12}^2} \sim T_k^{-2} \ln(T_k/h_{12}).
\end{equation}
To observe the flow $M \to M - 2$ we turn on an additional weak field component $h_{34} \ll h_{12}$. Then at $h_{34} \ll T_0$, already the first derivative of the corresponding magnetization component diverges,
\begin{equation}
\frac{\partial \mathcal{M}_{34}}{\partial h_{34}} \sim T_k^{-1} \ln(T_k/h_{34}),
\end{equation}
in a manner well known for the two-channel SU(2) Kondo model [35, 37]. This behavior can be understood from the equivalence of the SO(4) and two-channel SU(2) Kondo fixed points [30], which is also recovered from the structure of the BA equations.

The rest of the paper is organized as follows. We begin by discussing the problem for $M = 3$ and $M = 6$, including a study of the $M \to M - 2$ crossover in the latter case. In these cases, as well as for $M = 4$, the BA analysis is aided by the equivalence relations
\begin{equation}
\text{SO}(3) \sim \text{SU}(2), \quad \text{SO}(6) \sim \text{SU}(4), \quad \text{SO}(4) \sim \text{SU}(2) \times \text{SU}(2),
\end{equation}
which establish links to previously studied SU(N) Kondo models [35–37, 47]. We will then combine our $M = 6$ equations with general results, linking the group algebra to the structure of the BA equations, to suggest a generalization for arbitrary even $M$. We also propose the corresponding equations for odd $M > 3$, and discuss the $M = 5 \to 3$ crossover.

3. Bethe ansatz solution

The strategy pursued below is as follows. The general classification scheme of possible BA equations for models with Lie group symmetry is known. This allows us to address the problem with equal coupling constants $\lambda_{jk} = \bar{\lambda}$; we thus assume isotropic exchange couplings unless stated otherwise. We consider finite length $L$ of the leads with periodic boundary conditions, and take the thermodynamic limit afterwards. According to [48, 49], the general form of the BA equations is then dictated by the Dynkin diagram of the corresponding group. The next step is to determine the position of the so-called driving terms and their precise form. These are determined by representations of the bulk Hamiltonian and the impurity spin. We also check our BA equations against previously known results.

3.1. Case $M = 3$

For $M = 3$, the model (1) is equivalent to the four-channel SU(2) Kondo problem with impurity spin $S = 1/2$ [44]. To see this equivalence, let us introduce the operators $J_j = \frac{1}{2} \bar{\lambda} \epsilon_{ijk} \gamma_k \gamma_l$ (with $j = 1, 2, 3 = x, y, z$ and summation convention), which are equivalent to the components of a spin $S = 1/2$ operator [50]. Then the exchange interaction in equation (1) reads
\begin{equation}
H_K = 4i \lambda_{12} J_x \psi_1^\dagger(0) \psi_2(0) + 4i \lambda_{23} J_y \psi_2^\dagger(0) \psi_3(0) + 4i \lambda_{13} J_z \psi_3^\dagger(0) \psi_1(0) + \text{h.c.}.
\end{equation}
Expanding the bulk fermions in their real and imaginary Majorana components, $\psi_j(x) = \chi_j(x) + i \bar{\chi}_j(x)$, we get $i \bar{\epsilon}_{jkl} (\psi_k^\dagger \psi_l - \psi_l^\dagger \psi_k) = i \bar{\epsilon}_{jkl} (\chi_k \bar{\chi}_l + \bar{\chi}_l \chi_k)$, i.e., the sum of two SO(3) currents equals the SU(4) current. The result is the anisotropic spin $S = 1/2$ four-channel Kondo model, which is integrable. The BA solution for this model has been thoroughly studied [35, 37], and thereby applies also to the SO(3) topological Kondo effect. In particular one finds equation (6) for the magnetization, and $S_{\text{imp}} = \ln \sqrt{3}$ in the absence of the Zeeman field.
3.2. Case $M = 6$

Next we discuss the case $M = 6$. We demonstrate that a Zeeman field drives the system from $M = 6$ to $M = 4$, which in turn is equivalent to the SU(2) Kondo effect \cite{30}. We restrict our analysis to Zeeman fields $h_M$ that couple to commuting pairs of Majoranas forming a Cartan subalgebra. These pairs could be chosen arbitrarily but should not overlap. For $M$ wires, we then have $[M/2]$ pairs ($\lfloor x \rfloor$ is the integer part of $x$). For $M = 6$, the non-vanishing Zeeman couplings are taken as

$$H_1 = h_{12}, \quad H_2 = h_{34}, \quad H_3 = h_{56}. \quad (11)$$

For $M = 6$, the BA equations have the same general form as for the SU(2) Kondo model because the corresponding Dynkin diagrams coincide. Since creation and annihilation operators of the bulk fermions transform according to the vector representation of the O(6) group, this representation is isomorphic to the representation of the SU(4) group, where the Young tableaux have one column containing two boxes. The suggested BA equations are, with rapidities $x_a^{(j)}$, given by

$$e_1 \left( x_a^{(1)} - \frac{1}{\lambda} \right) \prod_{b=1}^{M_2} e_1 \left( x_a^{(1)} - x_b^{(2)} \right) = \prod_{b=1}^{M_1} e_2 \left( x_a^{(1)} - x_b^{(1)} \right),$$

$$\left[ e_2 \left( x_a^{(2)} \right) \right]^{M_2} \prod_{b=1}^{M_2} e_1 \left( x_a^{(2)} - x_b^{(1)} \right) \prod_{b=1}^{M_1} e_1 \left( x_a^{(2)} - x_b^{(3)} \right) = \prod_{b=1}^{M_1} e_2 \left( x_a^{(2)} - x_b^{(2)} \right),$$

$$\prod_{b=1}^{M_2} e_1 \left( x_a^{(3)} - x_b^{(2)} \right) = \prod_{b=1}^{M_1} e_2 \left( x_a^{(3)} - x_b^{(3)} \right). \quad (12)$$

where the energy follows as

$$E = \sum_{a=1}^{M_1} \frac{1}{21} \ln e_2 \left( x_a^{(2)} \right), \quad e_n(x) = \frac{x - in/2}{x + in/2}. \quad (13)$$

Here $N$ is the number of particles in the Fermi sea, and $M_{1,2,3}$ are integer numbers equal to linear combinations of eigenvalues of the Cartan operators of the group \cite{49}. The driving term for the bulk is located in the second equation, as it fits the vector representation of the O(6) group. Since the impurity spin is in the spinor representation, its driving term is in the first equation.

As a first step in the derivation of the thermodynamic Bethe ansatz (TBA) equations, we then classify solutions of the bare BA equations (13). These solutions are complex, but in the thermodynamic limit, $L \to \infty$ with $N/L$ and $M_{1,2,3}/L$ finite, their imaginary parts are simple: they group into clusters with common real part $X_a^{(j,n)}$, the so-called 'strings', where the rapidities are given by

$$x_a^{(j)} = X_a^{(j,n)} + \frac{i}{2} (n + 1 - 2p) + O(e^{-ca_0}), \quad (14)$$

with $n = 1, 2, \ldots, p = 1, \ldots, n$, and $c_0 > 0$. As next step, we introduce distribution functions for rapidities of string centers, $\rho_n^{(j)}(x)$, and unoccupied spaces, $\tilde{\rho}_n^{(j)}(x)$. The discrete equations (13) are thereby transformed into integral equations relating $\tilde{\rho}$ and $\rho$,

$$\tilde{\rho}_n^{(j)} + A_{n,2} \ast C^{(j)} \star \rho_n^{(j)} = A_{n,2} \ast s(x) \delta^{(j,2)} + \frac{1}{N} a_n (x - 1/\lambda) \delta^{(j,1)}. \quad (15)$$
where \( n = 1, 2, \ldots, j = 1, 2, 3 \), we use the summation convention, convolutions are denoted by a star, i.e., \( f \ast g(x) = \int dy f(x-y)g(y) \), and

\[
A_{nm}(\omega) = \coth(|\omega|/2) \left( e^{-|m-n|\omega/2} - e^{-(n+m)|\omega|/2} \right),
\]

\[
C_{nm}(\omega) = \delta_{mn} - s(\omega)(\delta_{n,m-1} + \delta_{n,m+1}),
\]

\[
s(\omega) = \frac{1}{2\cosh(\omega/2)}, \quad a_n(\omega) = e^{-n|\omega|/2}.
\]

The TBA equations now follow by minimization of the generalized free energy, \( F = E - TS \), subject to the constraints imposed by equation (15) and with the entropy

\[
S = N \sum_{n=0}^{\infty} \sum_{j} \int dx \left[ (\rho_{n}^{(j)} + \tilde{\rho}_{n}^{(j)}) \ln (\rho_{n}^{(j)} + \tilde{\rho}_{n}^{(j)}) - \rho_{n}^{(j)} \ln \rho_{n}^{(j)} - \tilde{\rho}_{n}^{(j)} \ln \tilde{\rho}_{n}^{(j)} \right].
\]

The TBA equations determine ratios of the distribution functions, which are collected in \( \phi_{n}^{(j)} \) functions according to

\[
\tilde{\rho}_{n}^{(j)}(x)/\rho_{n}^{(j)}(x) = e^{-\phi_{n}^{(j)}(x)}.
\]

For \( M = 6 \), the TBA equations in the scaling limit coincide with those of the SU\(_2\)(4) Kondo (or Coqblin–Schrieffer) model, with the impurity in the fundamental (single box) representation [47],

\[
F_{\text{imp}} = -T \sum_{j=1}^{3} \int dx f_{j} \left[ x + \frac{\ln(T_{k}/T)}{\pi} \right] \ln(1 + e^{\phi_{n}^{(j)}(x)}),
\]

\[
\ln(1 + e^{\phi_{n}^{(j)}}) - A_{jl} \ast C_{nm} \ast \ln(1 + e^{\phi_{m}^{(l)}}) = \delta_{n,2} \sin(\pi j/4)e^{-\pi x/2},
\]

where \( j, l = 1, 2, 3 \) and \( n, m = 1, 2, \ldots \). The Zeeman fields \( H_{j} \) in equation (11) enter through the constraint

\[
\lim_{n \to \infty} \frac{\phi_{n}^{(j)}}{n} = H_{j}/T,
\]

and the Fourier transforms of the above kernels are given by

\[
f_{j}(\omega) = \frac{\sinh((2-j)/2)\omega)}{\sinh(2\omega)},
\]

\[
A_{jl} = [C^{-1}]_{jl} = 2 \coth(\omega/2) \frac{\sinh[(2 - \max(j, l))/2]\omega]}{\sinh(2\omega)} \frac{\sinh[\min(j, l)/2]\omega/2]}{\sinh(2\omega)}.
\]

The ground-state impurity entropy is then determined by the asymptotics of \( \phi_{n}^{(j)}(-\infty) \). In the absence of the Zeeman field, the solution for the general SU\(_2\)(N) case is

\[
1 + e^{\phi_{n}^{(j)}(-\infty)} = \sin \left[ \frac{\pi (n+j)}{2N} \right] \sin \left[ \frac{\pi (n-j)}{2N} \right].
\]

Substituting this into the above equations and putting \( n = 1, N = 4 \), and \( k = 2 \), as is appropriate for our SO\(_2\)(6) problem (see equation (9)), we obtain \( S_{\text{imp}} = \ln \sqrt{3} \) in accordance with equation (5).

Below we consider the thermodynamics at \( T = 0 \), such that the equations for the ground-state root densities suffice. All roots of the BA equations are in \( n = 2 \) strings of different ‘colours’, \( j = 1, 2, 3 \). In that case, equations (15) are reduced to a set of Wiener–Hopf equations,

\[
s(\delta^{j/2} + \frac{\delta_{k}}{N}[s \ast s](x - 1/\lambda) = [A_{2,3}]^{-1} \tilde{\rho}_{2}^{(2)}(x) + C_{j} \ast \rho_{2}^{(j)}(x).
\]
Let us then isolate the terms proportional to $1/N$, which are associated with the impurity,
\begin{equation}
\rho = \rho_n + \frac{1}{N} \rho_{\text{imp}},
\end{equation}
where $\rho = \rho_2$ or $\tilde{\rho}_2$. In the ground state, the $j$th-order roots fill the interval $(-\infty, B_j)$, where the limits $B_j$ are determined by the Zeeman fields $H_j$ in equation (11),
\begin{equation}
\chi H_j = \int_{B_j}^{\infty} dx [\tilde{\rho}^{(j)}(x)]_0,
\end{equation}
with the bulk susceptibility $\chi = 1/(2\pi)$ (we use $v = 1$). To proceed from this point on, we need to specify the precise Zeeman field configuration.

3.2.1. All $B_j$ equal. With equation (11), we consider the Zeeman field configuration with
\begin{equation}
H_j = h_0 \sin(\pi j/4), \quad j = 1, 2, 3,
\end{equation}
where all $B_j$ are equal. In this case, equations (23) describe the vicinity of the SU2(4) fixed point for small $h_0$. The system of Wiener–Hopf equations (23) with equal limits of integration is then solved by
\begin{equation}
\rho^{(j,+)}(\omega = 0) = \frac{1}{16\pi i} \sum_{l=1}^3 \sin(\pi j l/4)
\end{equation}
\begin{equation}
\times \int \frac{d\omega}{\omega - i0^+} f^{(-)}_l(\omega) \exp \left[ \frac{i\omega}{\pi} \ln(h_0/[f^{(-)}_l(-i\pi/2)\mathcal{T}_K]) \right],
\end{equation}
\begin{equation}
f^{(-)}_l(\omega) = \frac{\Gamma(1 + i\frac{\omega}{2\pi}) \Gamma(1 - i\frac{\omega}{2\pi})}{\Gamma(\frac{1}{4} + i\frac{\omega}{2\pi}) \Gamma(1 + \frac{1}{4} + i\frac{\omega}{2\pi})},
\end{equation}
with the Gamma function $\Gamma$. For small Zeeman fields, $h_0 \ll T_K$, the result is dominated by the linear term but acquires a non-Fermi liquid correction,
\begin{equation}
\rho^{(j,+)}(\omega = 0) = \sin(\pi j/4) \frac{h_0}{2\pi T_K} + b_j \frac{h_0^2}{T_K} \ln(T_K/h_0) + \cdots,
\end{equation}
where $b_j$ is a numerical coefficient. The second (non-Fermi liquid) term originates from the double pole in the integrand at $\omega = -i\pi$. From here on it is straightforward to obtain equation (7) for the magnetization, which has been quoted in section 2.

3.2.2. Case $H_1 \gg H_2, 3$, and $M \to M - 2$ flow. Consider next a Zeeman field where the amount of holes (unoccupied spaces) in the $j = 2$ equation, see equation (23), strongly exceeds their amount in the $j = 1, 3$ equations, such that $B_2 \ll B_1, 3$. This situation is realized when one of the Zeeman field components by far exceeds the others, for instance, $H_1 \gg H_2, H_3$ in equation (11). The $H_1$ field then generates the temperature scale $T_h$ in equation (4), below which the physics is expected to be determined by the SO(4) $\sim$ SU2(2) Kondo effect. In this limit, we can neglect $\tilde{\rho}^{(1,3)}$ and rewrite equation (23) as
\begin{equation}
K \star \tilde{\rho}^{(1)} + \rho^{(1)} = s \ast \rho^{(2)} + \frac{1}{N} [s \ast s](x - 1/\lambda),
\end{equation}
\begin{equation}
K \star \tilde{\rho}^{(3)} + \rho^{(3)} = s \ast \rho^{(2)},
\end{equation}
Substituting this into equation (33), we get
\[ \rho^{(2)} + K \ast A_{2,2} \ast \rho^{(2)} = A_{2,2} \ast s + \frac{1}{N} A_{2,2} \ast [s \ast s](x - 1/\lambda), \]
\[ K(\omega) = [A_{2,2}]^{-1} = \frac{1}{1 + e^{-2|\omega|}}. \]  

(31)

The densities \(\tilde{\rho}^{(3)}\) and \(\rho^{(3)}\) do not contribute to the impurity thermodynamics. In the scaling limit, we have to keep only the asymptotics of the bulk driving term, and the explicit form of equation (31) is
\[ \rho^{(2)}(x) + \int_{R_c}^{\infty} dy K(x - y) \tilde{\rho}^{(2)}(y) = \frac{1}{\sqrt{2}} e^{-\pi s/2} + \frac{1}{N} e^{-\pi x} \frac{1}{2 \cosh[\pi(x - 1/\lambda)/2]}. \]  

(32)

Equations (29) and (30) determine the magnetization components \(\mathcal{M}_{jk}\) with \((j, k) \neq (1, 2)\), which are not directly affected by the large Zeeman field \(H_1 = h_{12}\). Since \(B_2 \ll B_{1,3}\), one can approximate \(s \ast \rho^{(2)}\) by the asymptotic expression
\[ s \ast \rho^{(2)} \approx \left( A + \frac{1}{N} A' \right) e^{-\pi x}, \]
\[ A + \frac{1}{N} A' = \int_{-\infty}^{R_c} dy e^{\pi y} \rho^{(2)}(y). \]  

(33)

Then equation (29) coincides with the equation for the ground-state root density of the SU(2) Kondo model, and equation (30) coincides with the equation for the SU(1) Kondo model in the Fermi liquid limit. Indeed, the impurity part \(\sim 1/N\) of equation (33) is also \(\sim e^{-\pi x}\), and we have
\[ K \ast \tilde{\rho}^{(1)} + \rho^{(1)} = A e^{-\pi x} + \frac{1}{N} [s \ast s](x - 1/\lambda), \]
\[ K \ast \tilde{\rho}^{(3)} + \rho^{(3)} = A e^{-\pi x} + \frac{1}{N} A' e^{-\pi x}. \]  

(34)

(35)

Eliminating the prefactor \(A\) in equation (34) by a shift of \(x\), we bring equation (34) to the canonical form for the SU(2) model, with the renormalized Kondo temperature for the emergent \(M = 4\) Kondo model,
\[ T_k^{(4)} = A^{-1} e^{-\pi/\lambda}. \]  

(36)

The factor \(A\) can now be extracted from the solution of the Wiener–Hopf equation (32),
\[ \rho^{(+)}_2(\omega) = \frac{2}{\pi G^{(-)}(-i\pi/2)G^{(+)}(\omega)} e^{-\pi B_i/2} + \frac{1}{N} \frac{i}{2\pi} G^{(+)}(\omega) \]
\[ \times \int \frac{d\omega'}{\omega - \omega + 10^\pi} e^{-i\omega'(B_i - 1/\lambda)}, \]
\[ G^{(+)}(\omega) = G^{(-)}(i\omega) = \frac{\Gamma(1/2 + i\omega/\pi)}{\sqrt{\pi}} e^{-\omega^2/4} e^{-i\omega/\pi}. \]  

(37)

The bulk part of this expression yields the bulk magnetic moment \(\chi H_1\), and thus determines the value of \(B_2\). In fact, we find
\[ \chi H_1 = \frac{\sqrt{8} e^{-\pi B_i/2}}{\pi G^{(-)}(-i\pi/2)}. \]  

(38)

Substituting this into equation (33), we get \(A = (\chi H_1)^3/\pi\). We now plug this result back into equation (36) and take into account that the \(M = 6\) Kondo temperature \(T_k^{(6)}\) (defined for \(h_{jk} = 0\)) is given by equation (2). With an extra factor two in the exponent because of the
different normalization of SO(6) and SU(4) generators, the $M = 6$ Kondo temperature here reads

$$T_k^{(6)} = \chi^{-1} \exp \left( -\frac{\pi}{2\chi} \right),$$

and therefore we finally get the Kondo temperature of the effective low-energy SO(4) model, realized at $T \ll T_h$, in the form

$$T_k^{(4)} = \frac{1}{\pi} T_k^{(6)} (H_1 / T_k^{(6)})^3.$$

Up to a prefactor of order unity, this scale coincides with the crossover scale $T_h$ in equation (4). We have thus shown that $T_h$ acts as the Kondo temperature for the emergent SO($M - 2$) topological Kondo effect.

The well-known result for the magnetization of the two-channel Kondo model [35],

$$\langle S^z \rangle \sim \frac{H_1}{2\pi T_K} \ln(T_K / H_1),$$

then dictates the $T = 0$ magnetization behavior announced above in equation (8). Moreover, the results of [37] yield the ground-state impurity entropy for the SU(2) case corresponding to $M = 4$, $S_{\text{imp}} = \ln \sqrt{2}$, which is again in agreement with equation (5).

Now that we have successfully run the checks for $M = 6$, we can write down the BA equations for arbitrary even $M$. Putting $M = 2K$, they read

$$e_1(x_a^{(K-1)} - 1/\lambda) \prod_{b=1}^{M_{K-2}} e_1(x_a^{(K-1)} - x_b^{(K-2)}) = \prod_{b=1}^{M_{K-1}} e_2(x_a^{(K-1)} - x_b^{(K-1)}),$$

$$\prod_{b=1}^{M_{K-2}} e_1(x_a^{(K-2)} - x_b^{(K-1)}) \prod_{b=1}^{M_K} e_1(x_a^{(K-2)} - x_b^{(K)}) = \prod_{b=1}^{M_{K-2}} e_2(x_a^{(K-2)} - x_b^{(K-2)}),$$

$$\prod_{b=1}^{M_{K-3}} e_1(x_a^{(K-3)} - x_b^{(K-2)}) \prod_{b=1}^{M_K} e_1(x_a^{(K-3)} - x_b^{(K-3)}) = \prod_{b=1}^{M_{K-3}} e_2(x_a^{(K-3)} - x_b^{(K-3)}),$$

$$\prod_{b=1}^{M_{K-2}} e_1(x_a^{(p-1)} - x_b^{(p)}) \prod_{b=1}^{M_p} e_1(x_a^{(p)} - x_b^{(p+1)}) = \prod_{b=1}^{M_{K-2}} e_2(x_a^{(p)} - x_b^{(p)}), \quad p = 2, \ldots, K - 1,$n

$$\prod_{b=1}^{M_{K-1}} e_1(x_a^{(K-1)} - x_b^{(K-1)}) = \prod_{b=1}^{M_{K-1}} e_2(x_a^{(K-1)} - x_b^{(K-1)}),$$

$$E = \sum_{a=1}^{M_1} \frac{1}{2M} \ln e_2(x_a^{(1)}).$$

With these equations, see also [42], one can obtain thermodynamic observables in an exact manner for arbitrary even $M$.

### 3.3. Odd $M$. Detailed description of $M = 5$

For the SO($M = 2K + 1$) group, the BA equations (up to the driving terms) can be extracted, for instance, from [49]. The positions of the bulk and the impurity driving terms are determined
by the same logic as before, that is by representation theory considerations and the \( M \to M + 2 \) flow. For the \( SO_2(2K + 1) \) model, we suggest the following bare BA equations, see also [42],

\[
e_{1/2}(x_a^{(K)} - 1/\lambda) \prod_{b=1}^{M_{K-1}} e_1(x_a^{(K)} - x_b^{(K-1)}) = \prod_{b=1}^{M_{K-1}} e_1(x_a^{(K)} - x_b^{(K)}),
\]

\[
\prod_{b=1}^{M_{p-1}} e_1(x_a^{(p)} - x_b^{(p-1)}) \prod_{b=1}^{M_{p+1}} e_1(x_a^{(p)} - x_b^{(p+1)}) = \prod_{b=1}^{M_p} e_2(x_a^{(p)} - x_b^{(p)}), \quad p = 2, \ldots, K - 1,
\]

\[
[e_2(x_a^{(1)})]^{M_1} \prod_{b=1}^{M_1} e_1(x_a^{(1)} - x_b^{(2)}) = \prod_{b=1}^{M_1} e_2(x_a^{(1)} - x_b^{(1)}),
\]

\[
E = \frac{1}{2T} \sum_a \ln [e_2(x_a^{(1)})].
\]

The impurity is in the spinor representation, and its driving term is in the first equation. For \( M = 5 \), we shall see that this is consistent with the flow \( M = 5 \to M = 3 \) driven by a single-component Zeeman field.

To illustrate the case of odd \( M \), we now analyze equations (43) for \( K = 2 \), i.e., for the group \( SO(5) \). The corresponding equations for the densities are

\[
s * A_{n,2} = \tilde{\sigma}_n + A_{2,n} * \sigma_m - [s(\omega/2) * A_{2n,m}(\omega/2)] * \rho_m,
\]

\[
\frac{1}{N}[s(\omega/2)] e^{i\omega/\lambda} = -[s(\omega/2) * A_{2n,m}(\omega/2)] * \sigma_m + \tilde{\rho}_n + [A_{2n,m}(\omega/2)] * \rho_m.
\]

From equation (45), we can then derive the TBA equations. There are two types of energies, where \( \phi_n(\xi_n) \) is related to \( \rho_n(\sigma_n) \). With \( n = 0, 1, 2, \ldots \), we obtain

\[
\phi_{2n+1} = s_{1/2} * \ln(1 + e^{\phi_n})(1 + e^{\phi_{2n+1}}),
\]

\[
\phi_{2n} = -s_{1/2} * e^{-2\pi x/3} + s_{1/2} * \ln[(1 + e^{\phi_{2n-1}})(1 + e^{\phi_{2n+1}})]
\]

\[
- \frac{s_{1/2}}{1 - s} * [s_{1/2} C_{2n,m} * \ln(1 + e^{\phi_n}) + C_{2n,m} * \ln(1 + e^{\phi_{2n-1}})]
\]

\[
- \ln(1 + e^{-\phi_n}) = -\frac{C_{nm}}{1 - s} * \ln(1 + e^{\phi_n}) - \frac{s_{1/2} * C_{nm}}{1 - s} * \ln(1 + e^{\phi_{2n-1}}) - \delta_{n,2} e^{-2\pi x/3}.
\]

The impurity free energy reads

\[
F_{imp} = -T \int dx \left[ s_{3/2} \left[ x + \frac{3}{2\pi} \ln(T_k/T) \right] \ln(1 + e^{\phi(x)})
\right.
\]

\[
+ s_{1/2} \left[ x + \frac{3}{2\pi} \ln(T_k/T) \right] \ln(1 + e^{\phi(x)})
\]

\[
+ s_{1/2} * s_{3/2} \left[ x + \frac{3}{2\pi} \ln(T_k/T) \right] \ln(1 + e^{\phi(x)}) \right],
\]

where \( s_{n/2} = s(\omega_n/2) \).

We now address the flow \( M = 5 \to M = 3 \) with just one non-zero Zeeman field component \( h_{12} = h_0 \), putting \( T = 0 \) for simplicity. The nonvanishing densities are \( \sigma_2, \tilde{\sigma}_2, \rho_4, \) and \( \tilde{\rho}_4 \), with the corresponding equations

\[
\frac{1}{2 \cosh(\omega/2)} = [A_{2,2}^{-1}] \ast \tilde{\sigma}_2 + \sigma_2 - \cosh(\omega/4) / \cosh(\omega/2) \ast \rho_4.
\]

\[
\frac{e^{i\omega/2} \tanh(\omega/4)}{N} = [A_{4,4}(\omega/2)]^{-1} \ast \tilde{\rho}_4 + \rho_4 - \frac{1}{2 \cosh(\omega/4)} \ast \sigma_2.
\]
In the absence of the Zeeman field, $\sigma_2, \rho_4 \neq 0$ on the entire real axis, and $\tilde{\sigma}_4 = \tilde{\sigma}_2 = 0$. In the opposite case of large $h_0$, however, $\tilde{\sigma}_2$ is significant and corresponds to the progressive emptiness of $\sigma_2$. Now $\sigma_2(x) \neq 0$ only at $x < B$, where $B$ is determined by the Zeeman field. As a result, the asymptotics of $\rho_4(x)$ at $+\infty$, which implies the low-energy behavior of the free energy, is determined by equation (45). Here we can approximate

$$\left(\frac{1}{2 \cosh(\nu/2)}\right)^{\ast} \sigma_2 \approx A e^{-2\pi x}, \quad A = \int_B^{+\infty} dy e^{2\pi y} \sigma_2(y).$$

(50)

In the end, we find that equation (49) coincides with the equation for the SU$_4(2)$ Kondo model with impurity spin $S = 1/2$, corresponding to the SO(3) topological Kondo effect. This once more illustrates the flow $M \rightarrow M - 2$ induced by lowering temperature below $T_h$.

Equation (43) also allows one to calculate the ground-state impurity entropy. Solving equations (46) for $x \rightarrow -\infty$, we find that $\xi_2, \phi_4 \rightarrow -\infty$. As a consequence, the equations for $\phi_{1,2,3}$ and $\xi_1$ decouple from the rest. Their solution is given by

$$e^{\phi_1} = e^{\phi_3} = \frac{3}{2}, \quad 1 + e^{\phi_2} = e^{2\phi_1}, \quad e^{\xi_1} = \frac{1}{3}.$$  

(51)

Substituting this into equation (47), we find $S_{\text{imp}} = \ln \sqrt{5}$, in accordance with the result quoted in section 2.

4. Conclusions

To conclude, we have formulated a Bethe ansatz solution for the SO($M$) topological Kondo problem realized by a mesoscopic superconducting island coupled to external leads via $M > 2$ Majorana fermions. In our previous paper [33], we reported that in this model, the Majorana spin non-locally encoded by the Majorana fermions exhibits rich and observable dynamics characterized by nonvanishing multi-point correlations and nonperturbative crossovers between different non-Fermi liquid Kondo fixed points. The Bethe ansatz results provided in the present work support these conclusions and provide a nonperturbative approach to the model spectrum and its thermodynamics.

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