Differentially Private Regret Minimization in Episodic Markov Decision Processes

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Abstract

We study regret minimization in finite horizon tabular Markov decision processes (MDPs) under the constraints of differential privacy (DP). This is motivated by the widespread applications of reinforcement learning (RL) in real-world sequential decision making problems, where protecting users’ sensitive and private information is becoming paramount. We consider two variants of DP – joint DP (JDP), where a centralized agent is responsible for protecting users’ sensitive data and local DP (LDP), where information needs to be protected directly on the user side. We first propose two general frameworks – one for policy optimization and another for value iteration – for designing private, optimistic RL algorithms. We then instantiate these frameworks with suitable privacy mechanisms to satisfy JDP and LDP requirements, and simultaneously obtain sublinear regret guarantees. The regret bounds show that under JDP, the cost of privacy is only a lower order additive term, while for a stronger privacy protection under LDP, the cost suffered is multiplicative. Finally, the regret bounds are obtained by a unified analysis, which, we believe, can be extended beyond tabular MDPs.

1 Introduction

Reinforcement learning (RL) is a fundamental sequential decision making problem, where an agent learns to maximize its reward in an unknown environment through trial and error. Recently, it is ubiquitous in various personalized services, including healthcare (Gottesman et al., 2019), virtual assistants (Li et al., 2016), social robots (Gordon et al., 2016) and online recommendations (Li et al., 2010). In these applications, the learning agent continuously improves its decision by learning from users’ personal data and feedback. However, nowadays people are becoming increasingly concerned about potential privacy leakage in these interactions. For example, in personalized healthcare, the private data of a patient can be sensitive informations such as her age, gender, height, weight, medical history, state of the treatment, etc. Therefore, developing RL algorithms which can protect users’ private data are of paramount importance in these applications.

Differential privacy (DP) (Dwork, 2008) has become a standard in designing private sequential decision-making algorithms both in the full information (Jain et al., 2012) and partial or bandit information (Mishra and Thakurta, 2015; Tossou and Dimitrakakis, 2016) settings. Under DP, the learning agent collects users’ raw data to train its algorithm while ensuring that its output will not reveal users’ sensitive information. This notion of privacy protection is suitable for situations, where a user is willing to share her own information to the agent in order to obtain a service specially tailored to her needs, but meanwhile she does not like to allow any third party to infer her private information seeing the output of the learning algorithm (e.g.,

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Regret (ε-JDP) \[ \tilde{O}\left(\sqrt{SAH^3T + S^2AH^3/\varepsilon}\right) \]
Regret (ε-LDP) \[ \tilde{O}\left(\sqrt{SAH^3T + S^2A\sqrt{H^5T}/\varepsilon}\right) \]

| Algorithm               | Regret (ε-JDP) | Regret (ε-LDP) |
|-------------------------|----------------|----------------|
| PO                      |                |                |
| PRIVATE-UCB-PO          | \(\tilde{O}\left(\sqrt{SAH^3T + S^2AH^3/\varepsilon}\right)\) | \(\tilde{O}\left(\sqrt{SAH^3T + S^2A\sqrt{H^5T}/\varepsilon}\right)\) |
| VI                      |                |                |
| PRIVATE-UCB-VI          | \(\tilde{O}\left(\sqrt{SAH^3T + S^2AH^3/\varepsilon}\right)\) | \(\tilde{O}\left(\sqrt{SAH^3T + S^2A\sqrt{H^5T}/\varepsilon}\right)\) |
| PUCB (Vietri et al., 2020) | \(O\left(\sqrt{SAH^3T + S^2AH^3/\varepsilon}\right)^1\) | NA |
| LDP-OBI (Garcelon et al., 2020) | NA | \(\tilde{O}\left(\sqrt{SAH^3T + S^2A\sqrt{H^5T}/\varepsilon}\right)^2\) |

Table 1: Regret comparisons for private RL algorithms on episodic tabular MDP. \(T = KH\) is total number of steps, where \(K\) is the total number of episodes and \(H\) is the number of steps per episode. \(S\) is the number of states, and \(A\) is the number of actions. \(\varepsilon > 0\) is the desired privacy level. \(\tilde{O}()\) hides polylog \((S, A, T, 1/\delta)\) factors, where \(\delta \in (0, 1]\) is the desired confidence level.

Google GBoard. However, a recent body of work (Shariff and Sheffet, 2018; Dubey, 2021) show that the standard DP guarantee is irreconcilable with sublinear regret in contextual bandits, and thus, a variant of DP, called joint differential privacy (JDP) (Kearns et al., 2014) is considered. Another variant of DP, called local differential privacy (LDP) (Duchi et al., 2013) has recently gained increasing popularity in personalized services due to its stronger privacy protection. It has been studied in various bandit settings recently (Ren et al., 2020; Zheng et al., 2020; Zhou and Tan, 2020). Under LDP, each user’s raw data is directly protected before being sent to the learning agent. Thus, the learning agent only has access to privatized data to train its algorithm, which often leads to a worse regret guarantee compared to DP or JDP.

In contrast to the vast amount of work in private bandit algorithms, much less attention are given to address privacy in RL problems. To the best of our knowledge, Vietri et al. (2020) propose the first RL algorithm – PUCB – for regret minimization with JDP guarantee in tabular finite state, finite action MDPs. On the other hand, Garcelon et al. (2020) design the first private RL algorithm – LDP-OBI – with regret and LDP guarantees. Recently, Chowdhury et al. (2021) study linear quadratic regulators under the JDP constraint. It is worth noting that all these prior work consider only value-based RL algorithms, and a study on policy-based private RL algorithms remains elusive. Recently, policy optimization (PO) has seen great success in many real-world applications, especially when coupled with deep neural networks (Silver et al., 2017; Duan et al., 2016; Wang et al., 2018), and a variety of PO based algorithms have been proposed (Williams, 1992; Kakade, 2001; Schulman et al., 2015, 2017; Konda and Tsitsiklis, 2000). The theoretical understandings of PO have also been studied in both computational (i.e., convergence) perspective (Liu et al., 2019; Wang et al., 2019a) and statistical (i.e., regret) perspective (Cai et al., 2020; Efroni et al., 2020a). Thus, one fundamental question to ask is how to build on existing understandings of non-private PO algorithms to design sample-efficient policy-based RL algorithms with general privacy guarantees (e.g., JDP and LDP), which is the main motivation behind this work. Also, the existing regret bounds in both Vietri et al. (2020) and Garcelon et al. (2020) for private valued-iteration (VI) based RL algorithms are loose. Moreover, the algorithm design and regret analysis under JDP in Vietri et al. (2020) and the ones under LDP in Garcelon et al. (2020) follow different approaches (e.g., choice of exploration bonus terms and corresponding analysis). Thus, another important question to ask is whether one can obtain tighter regret bounds for VI based private RL algorithms via a unified framework under general privacy requirements.

\(^1\)Vietri et al. (2020) claim a \(\tilde{O}\left(\sqrt{SAH^3T + S^2AH^3/\varepsilon}\right)\) regret bound for PUCB. However, to the best of our understanding, we believe the current analysis has gaps (see Section 4), and the best achievable regret for PUCB should have an additional \(\sqrt{S}\) factor in the first term.

\(^2\)Garcelon et al. (2020) consider stationary transition kernels, and show a \(\tilde{O}\left(\sqrt{S^2AH^2T + S^2A\sqrt{H^5T}/\varepsilon}\right)\) regret bound for LDP-OBI. For non-stationary transitions, as considered in this work, an additional multiplicative \(\sqrt{H}\) factor would appear in the
Contributions. Motivated by the two questions above, we make the following contributions.

- We present a general framework – PRIVATE-UCB-PO – for designing private policy-based optimistic RL algorithms in tabular MDPs. This framework enables us to establish the first regret bounds for PO under both JDP and LDP requirements by instantiating it with suitable private mechanisms – the CENTRAL-PRIVATIZER and the LOCAL-PRIVATIZER – respectively.

- We revisit private optimistic value-iteration in tabular MDPs by proposing a general framework – PRIVATE-UCB-VI – for it. This framework allows us to improve upon the existing regret bounds under both JDP and LDP constraints using a unified analysis technique.

- Our regret bounds show that for both policy-based and value-based private RL algorithms, the cost of JDP guarantee is only a lower-order additive term compared to the non-private regret. In contrast, under the stronger LDP requirement, the cost suffered is multiplicative and is of the same order. Our regret bounds and their comparison to the existing ones is summarised in Table 1.

Related work. Beside the papers mentioned above, there are other related work on differentially private online learning (Guha Thakurta and Smith, 2013; Agarwal and Singh, 2017) and multi-armed bandits (Tossou and Dimitrakakis, 2017; Hu et al., 2021; Sajed and Sheffet, 2019; Gajane et al., 2018; Chen et al., 2020). In the RL setting, in addition to Vietri et al. (2020); Garcelon et al. (2020) that focus on value-iteration based regret minimization algorithms under privacy constraints, Balle et al. (2016) considers private policy evaluation with linear function approximation. For MDPs with continuous state spaces, Wang and Hegde (2019) proposes a variant of Q-learning to protect the rewards information by directly injecting noise into value functions. Recently, a distributed actor-critic RL algorithm under LDP is proposed in Ono and Takahashi (2020) but without any regret guarantee. While there are recent advances in regret guarantees for policy optimization (Cai et al., 2020; Efroni et al., 2020a), we are not aware of any existing work on private policy optimization. Thus, our work takes the first step towards a unified framework for private policy-based RL algorithms in tabular MDPs with general privacy and regret guarantees.

2 Problem formulation

In this section, we recall the basics of episodic Markov Decision Processes and introduce the notion of differential privacy in reinforcement learning.

2.1 Learning model and regret in episodic MDPs

We consider episodic reinforcement learning (RL) in a finite horizon stochastic Markov decision process (MDP) (Puterman, 1994; Sutton, 1988) given by a tuple \((\mathcal{S}, \mathcal{A}, H, (P_h)_{h=1}^H, (c_h)_{h=1}^H)\), where \(\mathcal{S}\) and \(\mathcal{A}\) are state and action spaces with cardinalities \(|\mathcal{S}|\) and \(|\mathcal{A}|\), respectively, \(H \in \mathbb{N}\) is the episode length, \(P_h(s'|s, a)\) is the probability of transitioning to state \(s'\) from state \(s\) provided action \(a\) is taken at step \(h\) and \(c_h(s, a)\) is the mean of the cost distribution at step \(h\) supported on \([0, 1]\). The actions are chosen following some policy \(\pi = (\pi_h)_{h=1}^H\), where each \(\pi_h\) is a mapping from the state space \(\mathcal{S}\) into a probability distribution over the action space \(\mathcal{A}\), i.e. \(\pi_h(a|s) \geq 0\) and \(\sum_{a \in \mathcal{A}} \pi_h(a|s) = 1\) for all \(s \in \mathcal{S}\). The agent would like to find a policy \(\pi\) that minimizes the long term expected cost starting from every state \(s \in \mathcal{S}\) and every step \(h \in [H]\), defined first term of the bound.
as

\[ V^\pi_h(s) := \mathbb{E} \left[ \sum_{h'=h}^H c_{h'}(s_{h'}, a_{h'}) \mid s_h = s, \pi \right], \]

where the expectation is with respect to the randomness of the transition kernel and the policy. We call \( V^\pi_h \) the value function of policy \( \pi \) at step \( h \). Now, defining the \( Q \)-function of policy \( \pi \) at step \( h \) as

\[ Q^\pi_h(s, a) := \mathbb{E} \left[ \sum_{h'=h}^H c_{h'}(s_{h'}, a_{h'}) \mid s_h = s, a_h = a, \pi \right], \]

we obtain \( Q^\pi_h(s, a) = c_h(s, a) + \sum_{s' \in S} V^\pi_{h+1}(s') P_h(s' \mid s, a) \) and \( V^\pi_h(s) = \sum_{a \in A} Q^\pi_h(s, a) \pi_h(a \mid s) \).

A policy \( \pi^* \) is said to be optimal if it minimizes the value for all states \( s \) and step \( h \) simultaneously, and the corresponding optimal value function is denoted by \( V^\pi_h(s) = \min_{\pi \in \Pi} V^\pi_h(s) \) for all \( h \in [H] \), where \( \Pi \) is the set of all non-stationary policies. The agent interacts with the environment for \( K \) episodes to learn the unknown transition probabilities \( P_h(s' \mid s, a) \) and mean costs \( c_h(s, a) \), and thus, in turn, the optimal policy \( \pi^* \). At each episode \( k \), the agent chooses a policy \( \pi^k = (\pi^k_h)_{h=1}^H \) and samples a trajectory \( \{s^k_1, a^k_1, c^k_1, \ldots, s^k_H, a^k_H, c^k_H, s^k_{H+1}\} \) by interacting with the MDP using this policy. Here, at a given step \( h \), \( s^k_h \) denotes the state of the MDP, \( a^k_h \sim \pi^k_h(\cdot \mid s^k_h) \) denotes the action taken by the agent, \( c^k_h \in [0, 1] \) denotes the (random) cost suffered by the agent with the mean value \( c_h(s^k_h, a^k_h) \) and \( s^k_{h+1} \sim P_h(\cdot \mid s^k_h, a^k_h) \) denotes the next state. The initial state \( s^k_1 \) is assumed to be fixed and history independent. We measure performance of the agent by the cumulative (pseudo) regret accumulated over \( K \) episodes, defined as

\[ R(T) := \sum_{k=1}^K \left[ V^\pi_1(s^k_1) - V^*_{1}(s^k_1) \right], \]

where \( T = KH \) denotes the total number of steps. We seek algorithms with regret that is sublinear in \( T \), which demonstrates the agent’s ability to act near optimally.

### 2.2 Differential privacy in episodic RL

In the episodic RL setting described above, it is natural to view each episode \( k \in [K] \) as a trajectory associated to a specific user. To this end, we let \( U_K = (u_1, \ldots, u_K) \in \mathcal{U}^K \) to denote a sequence of \( K \) unique\(^3\) users participating in the private RL protocol with an RL agent \( \mathcal{M} \), where \( \mathcal{U} \) is the set of all users. Each user \( u_k \) is identified by the cost and state responses \( (c^k_h, s^k_{h+1})_{h=1}^H \) she gives to the actions \( (a^k_h)_{h=1}^H \) chosen by the agent. We let \( \mathcal{M}(U_k) = (a^1_h, \ldots, a^H_h) \in \mathcal{A}^{KH} \) to denote the set of all actions chosen by the agent \( \mathcal{M} \) when interacting with the user sequence \( U_k \). Informally, we will be interested in (centralized) randomized mechanisms (in this case, RL agents) \( \mathcal{M} \) so that the knowledge of the output \( \mathcal{M}(U_k) \) and all but the \( k \)-th user \( u_k \) does not reveal ‘much’ information about \( u_k \). We formalize this in the following definition.

**Definition 2.1** (Differential Privacy (DP)). For any \( \varepsilon \geq 0 \) and \( \delta \in [0, 1] \), a mechanism \( \mathcal{M} : \mathcal{U}^K \to \mathcal{A}^{KH} \) is \((\varepsilon, \delta)\)-differentially private if for all \( U_K, U'_K \in \mathcal{U}^K \) differing on a single user and for all subset of actions \( \mathcal{A}_0 \subset \mathcal{A}^{KH} \),

\[ \mathbb{P} [\mathcal{M}(U_K) \in \mathcal{A}_0] \leq \exp(\varepsilon) \mathbb{P} [\mathcal{M}(U'_K) \in \mathcal{A}_0] + \delta. \]

If \( \delta = 0 \), we call the mechanism \( \mathcal{M} \) to be \( \varepsilon \)-differentially private (\( \varepsilon \)-DP).

\(^3\)Uniqueness is assumed wlog, as for a returning user one can group her with her previous occurrences.
This is a direct adaptation of the classic notion of differential privacy (Dwork et al., 2014). However, we need to relax this definition for our purpose, because although the actions recommended to the user $u_k$ have only a small effect on the types (i.e., state and cost responses) of other users participating in the RL protocol, those can reveal a lot of information about the type of the user $u_k$. Thus, it becomes hard to privately recommend the actions to user $u_k$ while protecting the privacy of its type, i.e., its state and cost responses to the suggested actions. Hence, to preserve the privacy of individual users, we consider the notion of joint differential privacy (JDP) (Kearns et al., 2014), which requires that simultaneously for all user $u_k$, the joint distribution of the actions recommended to all users other than $u_k$ be differentially private in the type of the user $u_k$. It weakens the constraint of DP only in that the actions suggested specifically to $u_k$ may be sensitive in her type (state and cost responses). However, JDP is still a very strong definition since it protects $u_k$ from any arbitrary collusion of other users against her, so long as she does not herself make the actions suggested to her public. To this end, we let $\mathcal{M}_{-k}(U_k) := \mathcal{M}(U_k) \setminus (a_{h}^{k})_{h=1}^{H}$ to denote all the actions chosen by the agent $\mathcal{M}$ excluding those recommended to $u_k$ and formally define JDP as follows.

**Definition 2.2 (Joint Differential Privacy (JDP)).** For any $\varepsilon \geq 0$, a mechanism $\mathcal{M} : \mathcal{U}^K \rightarrow \mathcal{A}^{KH}$ is $\varepsilon$-joint differentially private if for all $k \in [K]$, for all user sequences $U_K, U'_K \in \mathcal{U}^K$ differing only on the $k$-th user and for all set of actions $\mathcal{A}_{-k} \subset \mathcal{A}^{(K-1)H}$ given to all but the $k$-th user,

$$\mathbb{P} [\mathcal{M}_{-k}(U_K) \in \mathcal{A}_{-k}] \leq \exp(\varepsilon)\mathbb{P} [\mathcal{M}_{-k}(U'_K) \in \mathcal{A}_{-k}].$$

JDP has been used extensively in private mechanism design (Kearns et al., 2014), in private matching and allocation problems (Hsu et al., 2016), in designing privacy-preserving algorithms for linear contextual bandits (Shariff and Sheffet, 2018), and it has been introduced in private tabular RL by Vietri et al. (2020).

JDP allows the agent to observe the data (i.e., the entire trajectory of state-action-cost sequence) associated with each user and the privacy burden lies on the agent itself. In some scenarios, however, the users may not even be willing to share its data with the agent directly. This motivates a stronger notion of privacy protection, called the local differential privacy (LDP) (Duchi et al., 2013). In this setting, each user is assumed to have her own privacy mechanism that can do randomized mapping on its data to guarantee privacy. To this end, we denote by $X$ a trajectory $(s_h, a_h, c_h, s_{h+1})_{h=1}^{H}$ and by $\mathcal{X}$ the set of all possible trajectories. We write $\mathcal{M}'(X)$ to denote the privatized trajectory generated by a (local) randomized mechanism $\mathcal{M}'$. With this notation, we now formally define LDP for our RL protocol.

**Definition 2.3 (Local Differential Privacy (LDP)).** For any $\varepsilon \geq 0$, a mechanism $\mathcal{M}'$ is $\varepsilon$-local differentially private if for all trajectories $X, X' \in \mathcal{X}$ and for all possible subsets $\mathcal{E}_0 \subset \{\mathcal{M}'(X)|X \in \mathcal{X}\}$,

$$\mathbb{P} [\mathcal{M}'(X) \in \mathcal{E}_0] \leq \exp(\varepsilon)\mathbb{P} [\mathcal{M}'(X') \in \mathcal{E}_0].$$

LDP ensures that if any adversary (can be the RL agent itself) observes the output of the privacy mechanism $\mathcal{M}'$ for two different trajectories, then it is statistically difficult for it to guess which output is from which trajectory. This has been used extensively in multi-armed bandits (Zheng et al., 2020; Ren et al., 2020), and introduced in private tabular RL by Garcelon et al. (2020).

### 3 Private Policy Optimization

In this section, we introduce a policy-optimization based private RL algorithm PRIVATE-UCB-PO (Algorithm 1) that can be instantiated with any private mechanism (henceforth, referred as a PRIVATIZER) satisfying a general condition. We derive a generic regret bound for PRIVATE-UCB-PO, which can be applied to obtain
Assumption 1 (Properties of private counts). For any $\varepsilon > 0$ and $\delta \in (0, 1]$, there exist functions $E_{\varepsilon, \delta, 1}, E_{\varepsilon, \delta, 2} > 0$ such that with probability at least $1 - \delta$, uniformly over all $(s, a, h, k)$, the private counts returned by the PRIVATIZER (both LOCAL and CENTRAL) satisfy: (i) $|\tilde{N}_h^k(s, a) - N_h^k(s, a)| \leq E_{\varepsilon, \delta, 1}$, (ii) $|\tilde{C}_h^k(s, a) - C_h^k(s, a)| \leq E_{\varepsilon, \delta, 1}$, and (iii) $|\tilde{N}_h^k(s, a, s') - N_h^k(s, a, s')| \leq E_{\varepsilon, \delta, 2}$.

In the following, we assume Assumption 1 holds. Then, we define, for all $(s, a, h, k)$, the private mean
Then, under Assumption 1, with probability at least \(1\), Theorem 3.2 (Algorithm 1) when instantiated with any \(P_{\text{PRIVATE}}\) (Cost of privacy) \(\text{Regret bound of } P_{\text{PRIVATE}}\) \((\text{Algorithm 1})\) is a private policy optimization (PO) algorithm based on the celebrated upper confidence bound (UCB) philosophy \((\text{Auer et al., } 2002; \text{Jaksch et al.,} 2010)\). Similar to the non-private setting \((\text{Efroni et al., } 2020a)\), it basically has two stages at each episode: \(1\) policy evaluation and \(2\) policy improvement. In the policy evaluation stage, it evaluates the policy \(\pi^k\) based on \(k - 1\) historical trajectories. In contrast to the non-private case, \(\text{PRIVATE-UCB-PO}\) relies only on the private counts \((\text{returned by the PRIVATIZER})\) to calculate the private mean empirical costs and private empirical transitions. These two along with a UCB exploration bonus term \((\text{which also depends only on private counts})\) are used to compute \(Q\)-function estimates. The \(Q\)-estimates are then truncated and corresponding value estimates are computed by taking their expectation with respect to the policy. Next, a new trajectory is rolled out by acting the policy \(\pi^k\) and the PRIVATIZER translates all non-private counts into the private ones to be used for the policy evaluation in the next episode. Finally, in the policy improvement stage, \(\text{PRIVATE-UCB-PO}\) employs a ‘soft’ update of the current policy \(\pi^k\) by following a standard mirror-descent step together with a Kullback–Leibler (KL) divergence proximity term \((\text{Beck and Teboulle, } 2003; \text{Cai et al., } 2020; \text{Efroni et al., } 2020a)\). The following theorem presents a general regret bound of \(\text{PRIVATE-UCB-PO}\) (Algorithm 1) when instantiated with any PRIVATIZER \((\text{LOCAL or CENTRAL})\) that satisfies Assumption 1.

**Theorem 3.2 (Regret bound of \(\text{PRIVATE-UCB-PO}\)).** Fix any \(\varepsilon > 0\) and \(\delta \in (0, 1]\), and set \(\eta = \sqrt{2 \log A/(H^2 K)}\). Then, under Assumption 1, with probability at least \(1 - \delta\), the cumulative regret of \(\text{PRIVATE-UCB-PO}\) is

\[
\mathcal{R}(T) = \tilde{O} \left( \sqrt{S^2 A H^3 T} + \sqrt{S^3 A^2 H^4} \right) + \tilde{O} \left( E_{\varepsilon,\delta,1} S^2 A H^2 + E_{\varepsilon,\delta,1} S A H^2 \right).
\]

**Remark 3.3 (Cost of privacy).** Theorem 3.2 shows that regret of \(\text{PRIVATE-UCB-PO}\) is lower bounded by the regret in non-private setting \((\text{Efroni et al., } 2020a, \text{Theorem 1})\), and depends directly on the privacy parameter \(\varepsilon\) through the permitted precision levels \(E_{\varepsilon,\delta,1}\) and \(E_{\varepsilon,\delta,2}\) of the PRIVATIZER. Thus, choosing \(E_{\varepsilon,\delta,1}, E_{\varepsilon,\delta,2}\) appropriately to guarantee JDP or LDP, we can obtain regret bounds under both forms of privacy. The cost of privacy, as we shall see in Section 5, is lower order than the non-private regret under JDP, and is of the same order under the stronger requirement of LDP.\(^3\)

\(^3\)The lower order terms scale with \(S^2\), which is quite common for optimistic tabular RL algorithms \((\text{Azar et al., } 2017; \text{Dann et al.,} 2017)\).
We bound Term(i) and Term(ii) by showing that
\[ \mathcal{W}(T) = \sum_{k=1}^{K} \left( V_1^{\pi}_k(s_k^k) - V_1^{\pi^*}(s_k^k) \right) = \sum_{k=1}^{K} \left( V_1^{\pi}_k(s_k^i) - \tilde{V}_1^{\pi}_k(s_k^i) + \tilde{V}_1^{\pi}_k(s_k^i) - V_1^{\pi^*}(s_k^i) \right) \]
\[ = \sum_{k=1}^{K} \left( V_1^{\pi}_k(s_k^i) - \tilde{V}_1^{\pi}(s_k^i) \right) + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \langle \tilde{Q}_h^k(s_h, \cdot), \pi_h^{k}(\cdot|s_h) - \pi_h^{\star}(\cdot|s_h) \rangle | s_k^i, \pi^* \right] \]
\[ + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \tilde{Q}_h^k(s_h, a_h) - c_h(s_h, a_h) - P_h(\cdot|s_h, a_h) \tilde{V}_{h+1}^k(s_k^i) | s_k^i, \pi^* \right] \]

We then bound each of the three terms. In particular, \( T_2 \) and \( T_3 \) have the same bounds as in the non-private case. Specifically, by setting \( \eta = \sqrt{2 \log A / (H^2 K)} \), we can show that \( T_2 \leq \sqrt{2H^2 K \log A} \) via a standard online mirror descent analysis, because \( \tilde{Q}_h^k \in [0, H] \) by design. Furthermore, due to Lemma 3.1 and our choice of bonus terms, we have \( T_3 \leq 0 \). The key is to bound \( T_1 \). By the update rule of \( Q \)-estimate and the choice of bonus terms in \textsc{Private-UCB-PO}, we can bound \( T_1 \) by the sum of expected bonus terms, i.e.,
\[ T_1 \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ 2\beta_k^{h,c}(s_h, a_h) | s_k^i, \pi^k \right] + H \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ 2\beta_k^{h,p}(s_h, a_h) | s_k^i, \pi^k \right] \]

Now, by Assumption 1 and the definition of exploration bonus \( \beta_k^{h,c}(s, a) \) in Lemma 3.1, we have
\[ \text{Term(i)} \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{L_c(\delta)}{\max\{N_k^h(s_h, a_h), 1\}} + \frac{3E_{\epsilon, \delta, 1}}{\max\{N_k^h(s_h, a_h), 1\}} | s_k^i, \pi^k \right]. \]

Note the presence of an additive privacy dependent term. Similarly, we obtain
\[ \text{Term(ii)} \leq H \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{L_p(\delta)}{\max\{N_k^h(s_h, a_h), 1\}} + \frac{S_{\epsilon, 0, 2} + 2E_{\epsilon, \delta, 1}}{\max\{N_k^h(s_h, a_h), 1\}} | s_k^i, \pi^k \right]. \]

We bound Term(i) and Term(ii) by showing that
\[ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{\max\{1, N_k^h(s_h, a_h)\}} | \mathcal{F}_{k-1} \right] = O \left( SAH \log T + H \log(H/\delta) \right), \]
\[ \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{\max\{1, N_k^h(s_h, a_h)\}} | \mathcal{F}_{k-1} \right] = O \left( \sqrt{SAHT} + SAH \log T + H \log(H/\delta) \right) \]

with high probability. These are generalization of results proved under stationary transition model (Efroni et al., 2019; Zanette and Brunskill, 2019) to our non-stationary setting (similar results were stated in (Efroni et al., 2020b,a), but without proofs). Finally, putting everything together, we complete the proof.

\[ \Box \]

4 Private UCB-VI Revisited

In this section, we turn to investigate value-iteration based private RL algorithms. It is worth noting that private valued-based RL algorithms have been studied under both JDP and LDP requirements (Vietri et al.,
2020; Garcelon et al., 2020). However, to the best of our understanding, the regret analysis of the JDP algorithm presented in Vietri et al. (2020) has gaps and does not support the claimed result. Under LDP, the regret bound presented in Garcelon et al. (2020) is sub-optimal in the cardinality of the state space and as the authors have remarked, it is possible to achieve the optimal scaling using a refined analysis. Motivated by this, we revisit private value iteration by designing an optimistic algorithm PRIVATE-UCB-VI (Algorithm 2) that can be instantiated with a PRIVATIZER (CENTRAL and LOCAL) to achieve both JDP and LDP.

**PRIVATE-UCB-VI algorithm.** Our algorithm design principle is again based on the UCB philosophy, the private estimates defined in (1) and a value-aware concentration result for the estimates stated in Lemma 4.1 below. Similar to the non-private setting (Azar et al., 2017), PRIVATE-UCB-VI (Algorithm 2) follows the procedure of optimistic value iteration. Specifically, at each episode $k$, using the private counts and a private UCB bonus term, it first compute private $Q$-estimates and value estimates using optimistic Bellman recursion. Next, a greedy policy $\pi^k$ is obtained directly from the estimated $Q$-function. Finally, a trajectory is rolled out by acting the policy $\pi^k$ and then PRIVATIZER translates all non-private statistics into private ones to be used in the next episode.

**Lemma 4.1 (Refined concentration of private estimates).** Fix any $\varepsilon > 0$ and $\delta \in (0, 1)$. Then, under Assumption 1, with probability at least $1 - 3\delta$, uniformly over all $(s, a, s', h, k)$,

\[
|c^k_h(s, a) - \tilde{c}^k_h(s, a)| \leq \beta^{k,c}_h(s, a),
\]

\[
\left|{P^k_h - P_h}V^*_h(s, a)\right| \leq \beta^{k,pv}_h(s, a),
\]

\[
|P^k_h(s'|s, a) - \tilde{P}^k_h(s'|s, a)| \leq C \sqrt{\frac{L'(\delta)P^k_h(s'|s, a)}{\max\{1, N^k_h(s, a) + E_{\varepsilon, \delta, 1}\}}} + \frac{CL'(\delta) + 2E_{\varepsilon, \delta, 1} + E_{\varepsilon, \delta, 2}}{\max\{1, N^k_h(s, a) + E_{\varepsilon, \delta, 1}\}},
\]

where $\beta^{k,c}_h(s, a)$ and $L_c(\delta)$ is as defined in Lemma 3.1, $(PV_{h+1})(s, a) := \sum_{s'} P(s'|s, a)V_{h+1}(s')$, $\beta^{k,pv}_h(s, a) := \frac{H(\varepsilon, \delta, 2)}{\sqrt{\max\{1, N^k_h(s, a) + E_{\varepsilon, \delta, 1}\}} + \frac{H(SE_{\varepsilon, \delta, 2} + 2E_{\varepsilon, \delta, 1})}{\max\{1, N^k_h(s, a) + E_{\varepsilon, \delta, 1}\}}}$, $C > 0$ is some constant, and $L'(\delta) := \log \left(\frac{6SAH}{\delta}\right)$.

The bonus term $\beta^{k,pv}_h$ in PRIVATE-UCB-VI does not have the factor $\sqrt{S}$ in the leading term compared to $\beta^{k,pv}_h$ in PRIVATE-UCB-PO. This is achieved by following a similar idea in UCB-PO (Azar et al., 2017). That is, instead of bounding the transition dynamics as in Lemma 3.1, we maintain a confidence bound directly over the optimal value function (the second result in Lemma 4.1). Due to this, we have an extra term in the regret bound, which can be carefully bounded by using a Bernstein-type inequality (the third result in Lemma 4.1). These two steps enable us to obtain an improved dependence on $S$ in the regret bound compared to existing private value-based algorithms (Vietri et al., 2020; Garcelon et al., 2020) under both JDP and LDP. This is stated formally in the next theorem, which presents a general regret bound of PRIVATE-UCB-VI (Algorithm 2) when instantiated with any PRIVATIZER (LOCAL or CENTRAL).

**Theorem 4.2 (Regret bound for PRIVATE-UCB-VI).** Fix any $\varepsilon > 0$ and $\delta \in (0, 1)$. Then, under Assumption 1, with probability $\geq 1 - \delta$, the regret of PRIVATE-UCB-VI is

\[
R(T) = \tilde{O}\left(\sqrt{SATH^3T} + S^2AH^3\right) + \tilde{O}\left(S^2AH^2E_{\varepsilon, \delta, 1} + S^2AH^2E_{\varepsilon, \delta, 2}\right).
\]
Algorithm 2: PRIVATE-UCB-VI

Input: Number of episodes $K$, time horizon $H$, privacy level $\varepsilon > 0$, a PRIVATIZER (LOCAL or CENTRAL) and confidence level $\delta \in (0, 1]$

1. Initialize private counts $\tilde{C}^k_h(s, a) = 0$, $\tilde{N}^k_h(s, a) = 0$ and $\tilde{N}^{k+1}_h(s, a, s') = 0$ for all $(s, a, s', h)$
2. Set precision levels $E_{\varepsilon, \delta, 1}, E_{\varepsilon, \delta, 2}$ of the PRIVATIZER

3. for $k = 1, \ldots, K$ do

4. Initialize private value estimates: $\tilde{V}^k_{H+1}(s) = 0$
5. for $h = H, H - 1, \ldots, 1$ do

6. Compute $\tilde{c}^k_h(s, a)$ and $\tilde{P}^k_h(s'|s, a) \forall (s, a, s')$ as in (1) using the private counts
7. Set exploration bonus using Lemma 4.1: $\beta^k_h(s, a) = \beta^{k,c}_h(s, a) + \beta^{k,pv}_h(s, a) \forall (s, a)$
8. Compute: $\forall (s, a)$, $\tilde{Q}^k_h(s, a) = \min\{H - h + 1, \max\{0, \tilde{c}^k_h(s, a) + \sum_{s'\in S} \tilde{V}^k_{h+1}(s') \tilde{P}^k_h(s'|s, a) - \beta^k_h(s, a)\}\}$
9. Compute private value function: $\forall s$, $\tilde{V}^k_h(s) = \min_{a \in A} \tilde{Q}^k_h(s, a)$
10. Compute policy: $\forall (s, h)$, $\pi^k_h(s) = \arg\min_{a \in A} \tilde{Q}^k_h(s, a)$ (with breaking ties arbitrarily)
11. Roll out a trajectory $(s^k_1, a^k_1, c^k_1, \ldots, s^k_{H+1})$ by acting the policy $\pi^k = (\pi^k_h)_{h=1}^H$
12. Receive private counts $\tilde{C}^k_{H+1}(s, a)$, $\tilde{N}^{k+1}_h(s, a)$, $\tilde{N}^{k+1}_h(s, a, s')$ from the PRIVATIZER

Remark 4.3 (Cost of privacy). Similar to PRIVATE-UCB-PO, the regret of PRIVATE-UCB-VI is lower bounded by the regret in non-private setting (see Azar et al. (2017, Theorem 1)), and the privacy parameter appear only in the lower order terms.

Remark 4.4 (VI vs. PO). The regret bound of PRIVATE-UCB-VI is a $\sqrt{\delta}$ factor better in the leading privacy-independent term compared to PRIVATE-UCB-PO. This follows the same pattern as in the non-private case, i.e., UCB-VI (Azar et al., 2017) vs. OPPO (Efroni et al., 2020a).

5 Privacy and regret guarantees

In this section, we instantiate PRIVATE-UCB-PO and PRIVATE-UCB-VI using a CENTRAL-PRIVATIZER and a LOCAL-PRIVATIZER, and derive corresponding privacy and regret guarantees.

5.1 Achieving JDP using CENTRAL-PRIVATIZER

The CENTRAL-PRIVATIZER runs a private $K$-bounded binary-tree mechanism (counter) (Chan et al., 2010) for each count $N^k_h(s, a)$, $C^k_h(s, a)$, $N^k_h(s, a, s')$, i.e. it uses $2SAH + S^2AH$ counters in total. Let us focus on the counters – there are $SAH$ many of them – for the number of visited states $N^k_h(s, a)$. Each counter takes as input the data stream $\sigma_h(s, a) \in \{0, 1\}^K$, where the $j$-th bit $\sigma^j_h(s, a) := \mathbb{1}\{s^j_h = s, a^j_h = a\}$ denotes whether the pair $(s, a)$ is encountered or not at step $h$ of episode $j$, and at the start of each episode $k$, release a private version $\tilde{N}^k_h(s, a)$ of the count $N^k_h(s, a) := \sum_{j=1}^{H-1} \sigma^j_h(s, a)$. Let us now discuss how private counts are computed. To this end, we let $N^{i,j}_h(s, a) = \sum_{k=i}^{j} \sigma^k_h(s, a)$ to denote a partial sum (P-sum) of the counts in episodes $i$ through $j$, and consider a binary interval tree, each leaf node of which represents an episode.

\[^6\]In the non-private setting, Azar et al. (2017) assume stationary transition kernels $P_h = P$ for all $h$. We consider non-stationary kernels, which adds a multiplicative $\sqrt{H}$ factor in our non-private regret.
Furthermore, with probability at least $\tilde{1}$ (i.e., the tree has $k$ leaves at the start of episode $k$). Each interior node of the tree represents the range of episodes covered by its children. At the start of episode $k$, first a noisy $P$-sum corresponding to each node in the tree is released by perturbing it with an independent Laplace noise $\text{Lap}(\frac{1}{\epsilon})$, where $\epsilon' > 0$ is a given privacy parameter. Then, the private count $\tilde{N}_h^k(s, a)$ is computed by summing up the noisy $P$-sums released by the set of nodes — which has cardinality at most $O(\log k)$ — that uniquely cover the range $[1, k - 1]$. Observe that, at the end of episode $k$, the counter only needs to store noisy $P$-sums required for computing private counts at future episodes, and can safely discard $P$-sums those are no longer needed.

The counters corresponding to empirical rewards $C_h^k(s, a)$ and state transitions $N_h^k(s, a, s')$ follow the same underlying principle to release the respective private counts $\tilde{C}_h^k(s, a)$ and $\tilde{N}_h^k(s, a, s')$. The next lemma sums up the properties of the CENTRAL-PRIVATIZER.

**Lemma 5.1 (Properties of CENTRAL-PRIVATIZER).** For any $\epsilon > 0$, CENTRAL-PRIVATIZER with parameter $1/\epsilon' = \frac{3H\log K}{\epsilon}$ is $\epsilon$-DP. Furthermore, for any $\delta \in (0, 1]$, it satisfies Assumption 1 with $E_{\epsilon, \delta, 1} = \frac{3H}{\epsilon} \sqrt{8 \log^3 K \log(6SAT/\delta)}$, and $E_{\epsilon, \delta, 2} = \frac{3H}{\epsilon} \sqrt{8 \log^3 K \log(6S^2AT/\delta)}$.

Lemma 5.1 follows from the privacy guarantee of the Laplace mechanism, and the concentration bound on the sum of i.i.d. Laplace random variables (Dwork et al., 2014). Using Lemma 5.1, as corollaries of Theorem 3.2 and Theorem 4.2, we obtain the regret and privacy guarantees for PRIVATE-UCB-PO and PRIVATE-UCB-VI with the CENTRAL-PRIVATIZER.

**Corollary 5.2 (Regret under JDP).** For any $\epsilon > 0$ and $\delta \in (0, 1]$, both PRIVATE-UCB-PO and PRIVATE-UCB-VI, if instantiated using CENTRAL-PRIVATIZER with parameter $1/\epsilon' = \frac{3H\log K}{\epsilon}$, satisfy $\epsilon$-JDP. Furthermore, with probability at least $1 - \delta$, we obtain the regret bounds:

$$R_{\text{PRIVATE-UCB-PO}}(T) = \tilde{O}\left(\sqrt{S^2AH^3T} + S^2AH^3/\epsilon\right),$$

$$R_{\text{PRIVATE-UCB-VI}}(T) = \tilde{O}\left(\sqrt{SAH^3T} + S^2AH^3/\epsilon\right).$$

We prove the JDP guarantees using the billboard model (Hsu et al., 2016, Lemma 9) which, informally, states that an algorithm is JDP if the output sent to each user is a function of the user’s private data and a common quantity computed using a standard DP mechanism. Note that by Lemma 5.1 and the post-processing property of DP (Dwork et al., 2014), the sequence of policies $(\pi^k)_k$ are $\epsilon$-DP. Therefore, by the billboard model, the actions $(a_h^k)_{h,k}$ suggested to all the users are $\epsilon$-JDP.

**Remark 5.3.** Corollary 5.2, to the best of our understanding, provides the first regret bound for private PO, and a correct regret bound for private VI as compared to Vietri et al. (2020), under the requirement of JDP.

### 5.2 Achieving LDP using LOCAL-PRIVATIZER

The LOCAL-PRIVATIZER, at each episode $k$, release the private counts by injecting Laplace noise into the aggregated statistics computed from the trajectory generated in that episode. Let us discuss how private counts for the number of visited states are computed. At each episode $j$, given privacy parameter $\epsilon' > 0$, LOCAL-PRIVATIZER perturbs $\sigma_h^j(s, a)$ with an independent Laplace noise $\text{Lap}(\frac{1}{\epsilon})$, i.e. it makes $SAH$ noisy perturbations in total. The private counts for the $k$-th episode are computed as $\tilde{N}_h^k(s, a) = \sum_{j=1}^{k-1} \tilde{\sigma}_h^j(s, a)$, where $\tilde{\sigma}_h^j(s, a)$ denotes the noisy perturbations. The private counts corresponding to empirical rewards $C_h^k(s, a)$ and state transitions $N_h^k(s, a, s')$ are computed similarly. The next lemma sums up the properties of the LOCAL-PRIVATIZER.

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$^7$A random variable $X \sim \text{Lap}(b)$, with scale parameter $b > 0$, if $\forall x \in \mathbb{R}$, it’s p.d.f. is given by $f_X(x) = \frac{1}{2b} \exp(-|x|/b)$.  

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Lemma 5.4 (Properties of LOCAL-PRIVATIZER). For any $\varepsilon > 0$, LOCAL-PRIVATIZER with parameter $1/\varepsilon' = \frac{3H}{\varepsilon}$ is $\varepsilon$-LDP. Furthermore, for any $\delta \in (0, 1]$, it satisfies Assumption 1 with $E_{\varepsilon, \delta, 1} = \frac{3H}{\varepsilon} \sqrt{8K \log(6SAT/\delta)}$ and $E_{\varepsilon, \delta, 2} = \frac{3H}{\varepsilon} \sqrt{8K \log(6S^2AT/\delta)}$.

Corollary 5.5 (Regret under LDP). For any $\varepsilon > 0$ and $\delta \in (0, 1]$, instantiating PRIVATE-UCB-PO and PRIVATE-UCB-VI using LOCAL-PRIVATIZER with parameter $1/\varepsilon' = \frac{3H}{\varepsilon}$, we obtain, with probability $\geq 1 - \delta$, the regret bounds:

$$R_{\text{PRIVATE-UCB-PO}}(T) = \tilde{O} \left( \sqrt{S^2AH^3T} + S^2A\sqrt{H^5T/\varepsilon} \right),$$

$$R_{\text{PRIVATE-UCB-VI}}(T) = O \left( \sqrt{SAH^3T} + S^2A\sqrt{H^5T/\varepsilon} \right).$$

Remark 5.6. Corollary 5.5, to the best of our knowledge, provides the first regret guarantee for private PO, and an improved regret bound for private VI as compared to Garcelon et al. (2020), under the requirement of LDP.

Remark 5.7 (JDP vs. LDP). The noise level in the private counts is $O(\log k)$ under JDP and $O(k)$ under LDP. Due to this, the privacy cost for LDP is $\tilde{O}(\sqrt{T}/\varepsilon)$, whereas for JDP it is only $O(1/\varepsilon)$.

Remark 5.8 (Alternative LDP mechanisms). Other than the Laplace noise, one can also use Bernoulli and Gaussian noise in the LOCAL-PRIVATIZER to achieve LDP (Kairouz et al., 2016; Wang et al., 2019b). Thanks to Theorem 3.2 and Theorem 4.2, the regret bounds are readily obtained by plugging in the corresponding $E_{\varepsilon, \delta, 1}$ and $E_{\varepsilon, \delta, 2}$.

6 Experiments

In this section, we conduct simple numerical experiments to verify our theoretical results for both policy-based and value-based algorithms.

6.1 Settings

Our experiment is based on the standard tabular MDP environment RiverSwim (Strehl and Littman, 2008; Osband et al., 2013), illustrated in Fig. 1. It consists of six states and two actions ‘left’ and ‘right’, i.e., $S = 6$ and $A = 2$. It starts with the left side and tries to reach the right side. At each step, if it chooses action ‘left’, it will always succeed (cf. the dotted arrow). Otherwise it often fails (cf. the solid arrow). It only receives a small amount of reward if it reaches the leftmost side while obtaining a larger reward once it arrives at the rightmost state. Thus, this MDP naturally requires a sufficient exploration to obtain the optimal policy. Each episode is reset every $H = 20$ steps.

6.2 Results

We evaluate both PRIVATE-UCB-PO and PRIVATE-UCB-VI under different privacy budget $\varepsilon$ and also compare them with the corresponding non-private algorithms UCB-VI (Azar et al., 2017) and OPPO (Efroni et al., 2020a), respectively. We set all the parameters in our proposed algorithms as the same order as the theoretical results and tune the learning rate $\eta$ and the scaling of the confidence interval. We run 20 independent experiments, each consisting of $K = 2 \cdot 10^4$ episodes. We plot the the average cumulative regret along with standard deviation for each setting, as shown in Fig. 2.
As suggested by theory, in both PO and VI cases, we can see that the cost of privacy under the JDP requirement becomes negligible as number of episodes increases (since the cost is only a lower order additive term). But, under the stricter LDP requirement, the cost of privacy remains high (since the cost is multiplicative and is of the same order). Furthermore, it is worth noting that the cost of privacy increases with increasing protection level, i.e. with decreasing $\varepsilon$.

7 Conclusions

In this work, we presented the first private policy-optimization algorithm in tabular MDPs with regret guarantees under both JDP and LDP requirements. We also revisited private value-iteration algorithms by improving the regret bounds of existing results. These are achieved by developing a general framework for algorithm design and regret analysis in private tabular RL settings. Though we focus on statistical guarantees of private RL algorithms, it will be helpful to understand these from a practitioner’s perspective. We leave this as a possible future direction. Another important direction is to apply our general framework to MDPs with function approximation, e.g., linear MDPs (Jin et al., 2019), kernelized MDPs (Chowdhury and Gopalan, 2019) and generic MDPs (Ayoub et al., 2020).
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Appendix

A Proofs for Section 3

We first define the non-private mean empirical costs and empirical transition probabilities as follows.

\[ c_h^k(s, a) := \frac{C_h^k(s, a)}{N_h^k(s, a) \vee 1}, \quad \bar{P}_h^k(s'|s, a) := \frac{N_h^k(s, a, s')}{N_h^k(s, a) \vee 1}. \]

Based on them, we define the following events

\[ F^c = \left\{ \exists s, a, h, k : |c_h(s, a) - \bar{c}_h^k(s, a)| \geq \sqrt{\frac{2 \ln \frac{4SAT}{\delta^2}}{N_h^k(s, a) \vee 1}} \right\}, \]

\[ F^p = \left\{ \exists s, a, h, k : \|P_h(\cdot|s, a) - \bar{P}_h^k(\cdot|s, a)\|_1 \geq \sqrt{\frac{4S \ln \frac{6SAT}{\delta^2}}{N_h^k(s, a) \vee 1}} \right\}, \]

and \( \bar{G} := F^c \cup F^p \), which is the complement of the good event \( G \). The next lemma states that the good event happens with high probability.

**Lemma A.1.** For any \( \delta \in (0, 1], \mathbb{P}[G] \geq 1 - \delta. \)

**Proof.** We first have \( \mathbb{P}[F^c] \leq \delta/2 \). This follows from Hoeffding’s inequality and union bound over all \( s, a \) and all possible values of \( N_h^k(s, a) \) and \( k \). Note that when \( N_h^k(s, a) = 0 \), this bound holds trivially. We also have \( \mathbb{P}[F^p] \leq \delta/2 \). This holds by (Weissman et al., 2003, Theorem 2.1) along with the application of union bound over all \( s, a \) and all possible values of \( N_h^k(s, a) \) and \( k \). Note that when \( N_h^k(s, a) = 0 \), this bound holds trivially. \( \square \)

Now, we are ready to present the proof for Lemma 3.1.

**Proof of Lemma 3.1.** Assume that both the good event \( G \) and the event in Assumption 1 hold. We first study the concentration of the private cost estimate. Note that under the event in Assumption 1,

\[ \left| \frac{\bar{C}_h^k(s, a)}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} - \frac{C_h^k(s, a)}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} \right| \leq \frac{E_{\varepsilon, \delta, 1}}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1}, \]

since \( \bar{N}_h^k(s, a) + E_{\varepsilon, \delta} \geq N_h^k(s, a) \geq 0 \) and \( \left| \bar{C}_h^k(s, a) - C_h^k(s, a) \right| \leq E_{\varepsilon, \delta, 1} \). Moreover, we have

\[
\begin{align*}
&\left| \frac{C_h^k(s, a)}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} - c_h(s, a) \right| \\
&\leq c_h(s, a) \left( \frac{N_h^k(s, a) \vee 1}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} - 1 \right) + \left( \frac{1}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} - \frac{N_h^k(s, a) \vee 1}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} \right) \left( \frac{C_h^k(s, a)}{N_h^k(s, a) \vee 1} - c_h(s, a) \right) \\
&\leq c_h(s, a) \left( 1 - \frac{N_h^k(s, a) \vee 1}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} \right) + \frac{N_h^k(s, a) \vee 1}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} \sqrt{\frac{L_c(\delta)}{\bar{N}_h^k(s, a)}} \\
&\leq \frac{2E_{\varepsilon, \delta, 1}}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1} + \frac{L_c(\delta) \sqrt{\bar{N}_h^k(s, a)}}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1},
\end{align*}
\]
where (a) holds by the concentration of the cost estimate under good event $G$ with $L_c(\delta) := \sqrt{2 \ln \frac{4S \ln T}{\delta}}$.

Furthermore, we have

$$\frac{L_c(\delta) \sqrt{N_h^k(s, a) \lor 1}}{(\hat{N}_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} \leq \frac{L_c(\delta) \sqrt{(\hat{N}_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1}}{(\hat{N}_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} = \frac{L_c(\delta)}{\sqrt{(\hat{N}_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1}}.$$

Putting everything together, yields

$$|c_h(s, a) - \hat{c}_h^k(s, a)| \leq \frac{3E_{\epsilon, \delta, 1}}{(\hat{N}_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} + \frac{L_c(\delta)}{\sqrt{(\hat{N}_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1}}.$$

Now, we turn to bound the transition dynamics. The error between the true transition probability and the private estimate can be decomposed as

$$\sum_{s'} |P_h(s'|, s, a) - \hat{P}_h^k(s'|s, a)| = \sum_{s'} \left| \frac{\hat{N}_h^k(s, a, s')}{(\hat{N}_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} - P_h(s'|s, a) \right| \leq \sum_{s'} \left| \frac{N_h^k(s, a, s')}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} - P_h(s'|s, a) \right| + \sum_{s'} \left| \frac{\hat{N}_h^k(s, a, s') - N_h^k(s, a, s')}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} \right|.$$

For $\mathcal{P}_1$, we have

$$\mathcal{P}_1 = \sum_{s'} \left| \frac{N_h^k(s, a, s')}{N_h^k(s, a) \lor 1} \frac{N_h^k(s, a) \lor 1}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} - P_h(s'|s, a) \right| \leq \frac{N_h^k(s, a) \lor 1}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} \left\| \hat{P}_h^k(\cdot|s, a) - P_h(\cdot|s, a) \right\|_1 + \sum_{s'} \left( P_h(s'|s, a) \frac{2E_{\epsilon, \delta, 1}}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} \right)$$

$$(a) \leq \frac{L_p(\delta)}{\sqrt{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1}} + \frac{2E_{\epsilon, \delta, 1}}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} \leq \frac{L_p(\delta)}{\sqrt{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1}} + \frac{2E_{\epsilon, \delta, 1}}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1},$$

where (a) holds by concentration of transition probability under good event $G$ with $L_p(\delta) := \sqrt{4S \ln \frac{6S \ln T}{\delta}}$.

For $\mathcal{P}_2$, we have

$$\mathcal{P}_2 \leq \sum_{s'} \frac{|E_{\epsilon, \delta, 2}|}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1} = \frac{SE_{\epsilon, \delta, 2}}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \lor 1}.$$
Putting together $\mathcal{P}_1$ and $\mathcal{P}_2$, yields
\[
\|P_h(\cdot|s,a) - \tilde{P}_h^k(\cdot|s,a)\|_1 \leq \frac{L_p(\delta)}{\sqrt{(\tilde{N}_h^k(s,a) + E_{\varepsilon,1}) \vee 1}} + \frac{SE_{\varepsilon,\delta,2} + 2E_{\varepsilon,\delta,1}}{(\tilde{N}_h^k(s,a) + E_{\varepsilon,\delta,1}) \vee 1}.
\]

Finally, applying union bound over good event $G$ and the event in Assumption 1, yields the required result in Lemma 3.1.

We turn to present the proof for Theorem 3.2 as follows.

**Proof of Theorem 3.2.** As in the non-private case, we first decompose the regret by using the extended value difference lemma (Efroni et al., 2020a, Lemma 17).

\[
\mathcal{R}(T) = \sum_{k=1}^{K} \left( V_1^{\pi_k}(s_k^1) - V_1^{\pi^*}(s_1^k) \right) = \sum_{k=1}^{K} \left( (V_1^{\pi_k}(s_k^1) - \tilde{V}_1^k(s_k^1) + \tilde{V}_1^k(s_k^1) - V_1^{\pi^*}(s_k^1)) \right)
\]

\[
= \sum_{k=1}^{K} \left( V_1^{\pi_k}(s_k^1) - \tilde{V}_1^k(s_k^1) \right) + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ (\tilde{Q}_h^k(s_h, \cdot), \pi_h^k(\cdot|s_h) - \pi_h^k(\cdot|s_h)) | s_k^1, \pi^* \right] + \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \tilde{Q}_h^k(s_h, a_h) - c_h(s_h, a_h) - P_h(\cdot|s_h, a_h) \tilde{V}_{h+1}^k(s_1^1, \pi^*) \right].
\]

We then need to bound each of the three terms.

**Analysis of $\mathcal{T}_2$.** To start with, we can bound $\mathcal{T}_2$ by following standard mirror descent analysis under KL divergence. Specifically, by (Efroni et al., 2020a, Lemma 17), we have for any $h \in [H]$, $s \in \mathcal{S}$ and any policy $\pi$

\[
\sum_{k=1}^{K} (\tilde{Q}_h^k(s, \cdot), \pi_h^k(\cdot|s) - \pi_h^k(\cdot|s)) \leq \frac{\log A}{\eta} + \frac{\eta}{2} \sum_{k=1}^{K} \sum_{a} \pi_h^k(a|s)(Q_h^k(s, a))^2 \leq \frac{\log A}{\eta} + \frac{\eta H^2 K}{2},
\]

where (a) holds by $Q_h^k(s, a) \in [0, H]$, which follows from the truncated update of $Q$-value in Algorithm 1 (line 9). Thus, we can bound $\mathcal{T}_2$ as follows.

\[
\mathcal{T}_2 = \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ (\tilde{Q}_h^k(s_h, \cdot), \pi_h^k(\cdot|s_h^k) - \pi_h^k(\cdot|s_h^k)) | s_1^k, \pi^* \right] \leq \frac{H \log A}{\eta} + \frac{\eta H^3 K}{2}.
\]

Choosing $\eta = \sqrt{2 \log A/(H^2 K)}$, yields

\[
\mathcal{T}_2 \leq \sqrt{2H^4 K \log A}.
\]

**Analysis of $\mathcal{T}_3$.** First, by the update rule of $Q$-value in Algorithm 1 and $P_h(\cdot|s,a)V_{h+1} = \sum_{s'} P_h(s'|s,a)V_{h+1}(s')$,
we have
\[
\bar{Q}_h^k(s, a) = \min\{H, \max\{0, c_h^k(s, a) + \sum_{s' \in S} \bar{V}_{h+1}^k(s')\bar{P}_h^k(s'|s, a) - \beta_h^k(s, a)\}\} \\
= \min\{H, \max\{0, c_h^k(s, a) + \bar{P}_h(|s_h^k, a_h^k)\bar{V}_{h+1}^k - \beta_h^k(s, a)\}\} \\
\leq \max\left\{0, c_h^k(s, a) - \beta_h^{k,c}(s, a) + \bar{P}_h(|s_h^k, a_h^k)\bar{V}_{h+1}^k - H\beta_h^{k,p}(s, a)\right\} \\
\overset{(a)}{=} \max\left\{0, c_h^k(s, a) - \beta_h^{k,c}(s, a)\right\} + \max\left\{0, \bar{P}_h(|s_h^k, a_h^k)\bar{V}_{h+1}^k - \beta_h^{k,p}(s, a)\right\}
\]
where (a) holds since for any \(a, b, \max\{a + b, 0\} \leq \max\{a, 0\} + \max\{b, 0\}\). Thus, for any \((k, h, s, a)\), we have
\[
\bar{Q}_h^k(s, a) - c_h(s, a) - P_h(|s, a)\bar{V}_{h+1}^k \\
\leq \max\left\{0, c_h^k(s, a) - \beta_h^{k,c}(s, a)\right\} + \max\left\{0, \bar{P}_h(|s, a)\bar{V}_{h+1}^k - H\beta_h^{k,p}(s, a)\right\} - c_h(s, a) - P_h(|s, a)\bar{V}_{h+1}^k \\
= \max\left\{-c_h(s, a), c_h^k(s, a) - c_h(s, a) - \beta_h^{k,c}(s, a)\right\} \\
+ \max\left\{-P_h(|s, a)\bar{V}_{h+1}^k, \bar{P}_h(|s, a)\bar{V}_{h+1}^k - P_h(|s, a)\bar{V}_{h+1}^k - H\beta_h^{k,p}(s, a)\right\} \\
\leq \max\left\{0, c_h^k(s, a) - c_h(s, a) - \beta_h^{k,c}(s, a)\right\} + \max\left\{0, \bar{P}_h(|s, a)\bar{V}_{h+1}^k - P_h(|s, a)\bar{V}_{h+1}^k - H\beta_h^{k,p}(s, a)\right\}. 
\]

We are going to show that both (3) and (4) are less than zero for all \((k, h, s, a)\) with high probability by Lemma 3.1. First, conditioned on the first result in Lemma 3.1, we have (3) is less than zero. Further, we have conditioned on the second result in Lemma 3.1
\[
\bar{P}_h^k(|s, a)|\bar{V}_{h+1}^k - P_h(|s, a)\bar{V}_{h+1}^k - H\beta_h^{k,p}(s, a) \\
\overset{(a)}{=} \|\bar{P}_h^k(|s, a) - P_h(|s, a)|\|_1 \|\bar{V}_{h+1}^k\|_\infty - H\beta_h^{k,p}(s, a) \\
\overset{(b)}{=} H \|\bar{P}_h^k(|s, a) - P_h(|s, a)|\|_1 - H\beta_h^{k,p}(s, a) \\
\overset{(c)}{=} 0
\]
where (a) holds by Holder’s inequality; (b) holds since \(0 \leq \bar{V}_{h+1}^k \leq H\) based on our update rule; (c) holds by Lemma 3.1. Thus, we have shown that
\[
\mathcal{T}_3 \leq 0. 
\]

**Analysis of \(\mathcal{T}_1\).** Assume the good event \(G\) and the event in Assumption 1 hold (which implies the
concentration results in Lemma 3.1. We have

$$T_1 = \sum_{k=1}^{K} V_{t}^{\pi_{k}}(s_1) - \tilde{V}_{t+1}^{\pi_{k}}(s_1)$$

(a) $$= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ c_h(s_h, a_h) + P_h(\cdot|s_h, a_h) \tilde{V}_{h+1}^{k} - \tilde{Q}_{h+1}^{k}(s_h, a_h)|s_1^{k}, \pi_k \right]$$

(b) $$= \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ c_h(s_h, a_h) + P_h(\cdot|s_h, a_h) \tilde{V}_{h+1}^{k} |s_1^{k}, \pi_k \right]$$

$$- \sum_{k=1}^{K} \sum_{h=1}^{H} \left[ \max \left\{ \tilde{c}_h^{k}(s_h, a_h) - \beta^{k,c}_h(s_h, a_h) + \tilde{P}_h^{k}(\cdot|s_h, a_h) \tilde{V}_{h+1}^{k} - H \beta^{k,p}_h(s_h, a_h), 0 \right\} |s_1^{k}, \pi_k \right]$$

where (a) holds by the extended value difference lemma (Efroni et al., 2020a, Lemma 1); (b) holds by the update rule of Q-value in Algorithm 1. Note that here we can directly remove the truncation at $H$ since by Lemma 3.1, $\tilde{c}_h^{k}(s_h, a_h) - \beta^{k,c}_h(s_h, a_h) + \tilde{P}_h^{k}(\cdot|s_h, a_h) \tilde{V}_{h+1}^{k} - H \beta^{k,p}_h(s_h, a_h) \leq c_h(s, a) + P_h(\cdot|s, a) \tilde{V}_{h+1}^{k} \leq 1 + H - h \leq H$.

Now, observe that for any $(k, h, s, a)$, we have

$$c_h(s, a) + P_h(\cdot|s, a) \tilde{V}_{h+1}^{k} - \max \left\{ \tilde{c}_h^{k}(s_h, a_h) - \beta^{k,c}_h(s_h, a_h) + \tilde{P}_h^{k}(\cdot|s_h, a_h) \tilde{V}_{h+1}^{k} - H \beta^{k,p}_h(s_h, a_h), 0 \right\} \leq c_h(s, a) - \tilde{c}_h^{k}(s, a) + \beta^{k,c}_h(s, a) + P_h(\cdot|s, a) \tilde{V}_{h+1}^{k} - \tilde{P}_h^{k}(\cdot|s, a) \tilde{V}_{h+1}^{k} + H \beta^{k,p}_h(s, a)$$

(a) $$\leq 2\beta^{k,c}_h(s, a) + 2H \beta^{k,p}_h(s, a),$$

where (a) holds by Lemma 3.1 and a similar analysis as in (5). Plugging (8) into (7), yields

$$T_1 \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ 2\beta^{k,c}_h(s_h, a_h)|s_1^{k}, \pi_k \right] + H \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ 2\beta^{k,p}_h(s_h, a_h)|s_1^{k}, \pi_k \right]$$

By the definition of $\beta^{k,c}_h$ and $\beta^{k,p}_h$ in Lemma 3.1 and Assumption 1, we have with probability $1 - 2\delta$,

Term(i) $$\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{L_{c}(\delta)}{\max\{N^k_h(s_h, a_h), 1\}} \right] + \frac{3E_{\epsilon,\delta,1}}{\max\{N^k_h(s_h, a_h), 1\}} |s_1^{k}, \pi_k \right]$$

Term(ii) $$\leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{L_{p}(\delta)}{\max\{N^k_h(s_h, a_h), 1\}} \right] + \frac{SE_{\epsilon,\delta,1} + 2E_{\epsilon,\delta,1}}{\max\{N^k_h(s_h, a_h), 1\}} |s_1^{k}, \pi_k \right]$$

We bound the two terms above by using the following lemma. These are generalization of results proved under stationary transition model Efroni et al. (2019); Zanette and Brunskill (2019) to our non-stationary setting. The proof is given at the end of this section.

Lemma A.2. With probability $1 - 2\delta$, we have

$$\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{\max\{1, N^k_h(s_h, a_h)\}} |\mathcal{F}_{k-1} \right] = O \left( SAH \ln KH + H \ln(H/\delta) \right),$$

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and
\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{\sqrt{\max\{1, N^k_h(s_h, a_h)\}}} |F_{k-1}| = O \left( \sqrt{SAH^2K} + SAH \ln KH + H \ln(H/\delta) \right),
\]
where the filtration \( F_k \) includes all the events until the end of episode \( k \).

Therefore, by Lemma A.2 (since \( \pi^k \) is determined by \( F_{k-1} \)), we have with probability at least \( 1 - 4\delta \),
\[
\text{Term(i)} \leq \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{L_c(\delta)}{\sqrt{\max\{N^k_h(s_h, a_h), 1\}}} \frac{3E_{\varepsilon,\delta,1}}{\max\{N^k_h(s_h, a_h), 1\}} \right] \leq \tilde{O} \left( \sqrt{SAH^2K} + SAH + E_{\varepsilon,\delta,1}SAH \right).
\]

and
\[
\text{Term(ii)} \leq H \sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{L_p(\delta)}{\sqrt{\max\{N^k_h(s_h, a_h), 1\}}} \frac{SE_{\varepsilon,\delta,2} + 2E_{\varepsilon,\delta,1}}{\max\{N^k_h(s_h, a_h), 1\}} \right] \leq \tilde{O} \left( \sqrt{S^2AH^4K} + \sqrt{S^3A^2H^4} + E_{\varepsilon,\delta,2}S^2AH^2 + E_{\varepsilon,\delta,1}SAH^2 \right).
\]

Plugging (10) and (11) into (9), yields \( (T = KH) \)
\[
T_1 = O \left( \left( \sqrt{S^2AH^3T} + \sqrt{S^3A^2H^4} + E_{\varepsilon,\delta,2}S^2AH^2 + E_{\varepsilon,\delta,1}SAH^2 \right) \log(S, A, T, 1/\delta) \right)
\]
Finally, putting the bounds on \( T_1, T_2 \) and \( T_3 \) together, completes the proof.

We are left to present the proof for Lemma A.2. In the case of a stationary transition, Efroni et al. (2019); Zanette and Brunskill (2019) resort to the method of properly defining a ‘good’ set of episodes (cf. (Zanette and Brunskill, 2019, Definition 6)). We prove our results in the non-stationary case via a different approach. In particular, inspired by Jin et al. (2020), we will use the following Bernstein-type concentration inequality for martingale as our main tool, which is adapted from Lemma 9 in Jin et al. (2020).

**Lemma A.3.** Let \( Y_1, \ldots, Y_K \) be a martingale difference sequence with respect to a filtration \( \mathcal{F}_0, \mathcal{F}_1, \ldots, \mathcal{F}_K \). Assume \( Y_k \leq R \) a.s. for all \( i \). Then, for any \( \delta \in (0, 1) \) and \( \lambda \in [0, 1/R] \), with probability \( 1 - \delta \), we have
\[
\sum_{k=1}^{K} Y_k \leq \lambda \sum_{k=1}^{K} \mathbb{E} [Y^2_k|\mathcal{F}_{k-1}] + \frac{\ln(1/\delta)}{\lambda}.
\]

Now, we are well-prepared to present the proof of Lemma A.2.

**Proof of Lemma A.2.** Let \( T^k_h(s, a) \) be the indicator so that \( \mathbb{E} [T^k_h(s, a)|\mathcal{F}_{k-1}] = w^k_h(s, a) \), which is the
probability of visiting state-action pair \((s, a)\) at step \(h\) and episode \(k\). First note that

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{\max\{1, N_k^k(s_h, a_h)\}} \right] | \mathcal{F}_{k-1} \\
= \sum_{k=1}^{K} \sum_{h,s,a} w_k^k(s, a) \frac{1}{\max\{1, N_k^k(s, a)\}} \\
= \sum_{k=1}^{K} \sum_{h,s,a} \frac{T_k^k(s, a)}{\max\{1, N_k^k(s, a)\}} + \sum_{k=1}^{K} \sum_{h,s,a} \frac{w_k^k(s, a) - T_k^k(s, a)}{\max\{1, N_k^k(s, a)\}}.
\]

The first term can be bounded as follows.

\[
\sum_{k=1}^{K} \sum_{h,s,a} \frac{T_k^k(s, a)}{\max\{1, N_k^k(s, a)\}} \leq \sum_{h,s,a} \sum_{k=1}^{K} \frac{1}{\max\{1, N_k^k(s, a)\}} \\
= \sum_{h,s,a} \sum_{i=1}^{N_k^k(s, a)} \frac{1}{i} \\
\leq c' \sum_{h,s,a} \ln(N_k^k(s, a)) \\
\leq c'SAH \ln \left( \sum_{s,a,h} N_k^k(s, a) \right) \\
= O(SAH \ln(KH)).
\]

To bound the second term, we will use Lemma A.3. In particular, consider \(Y_{k,h} := \sum_{s,a} \frac{w_k^k(s, a) - T_k^k(s, a)}{\max\{1, N_k^k(s, a)\}} \leq 1\), \(\lambda = 1\), and the fact that for any fixed \(h\),

\[
\mathbb{E} \left[ Y_{k,h}^2 | \mathcal{F}_{k-1} \right] \leq \mathbb{E} \left[ \left( \sum_{s,a} \frac{T_k^k(s, a)}{\max\{1, N_k^k(s, a)\}} \right)^2 | \mathcal{F}_{k-1} \right] \\
= \mathbb{E} \left[ \sum_{s,a} \frac{T_k^k(s, a)}{\max\{1, (N_k^k(s, a))^2\}} | \mathcal{F}_{k-1} \right] \\
\leq \sum_{s,a} \frac{w_k^k(s, a)}{\max\{1, N_k^k(s, a)\}}.
\]

Then, via Lemma A.3, we have with probability at least \(1 - \delta\),

\[
\sum_{k=1}^{K} \sum_{h,s,a} \frac{w_k^k(s, a) - T_k^k(s, a)}{\max\{1, N_k^k(s, a)\}} = \sum_{h=1}^{H} \sum_{k=1}^{K} Y_{k,h} \leq \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{s,a} \frac{w_k^k(s, a)}{\max\{1, N_k^k(s, a)\}} + H \ln(H/\delta) \\
= O(SAH \ln(KH) + H \ln(H/\delta)),
\]

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which completes the proof of the first result in Lemma A.2. To show the second result, similarly, we decompose it as

$$
\sum_{k=1}^{K} \sum_{h=1}^{H} \mathbb{E} \left[ \frac{1}{\sqrt{\max\{1, N^k_h(s_h, a_h)\}}} | \mathcal{F}_{k-1} \right]
$$

$$
= \sum_{k=1}^{K} \sum_{h,s,a} w^k_h(s,a) \frac{1}{\sqrt{\max\{1, N^k_h(s,a)\}}}
$$

$$
= \sum_{k=1}^{K} \sum_{h,s,a} \frac{\mathcal{I}^k_h(s,a)}{\sqrt{\max\{1, N^k_h(s,a)\}}} + \sum_{k=1}^{K} \sum_{h,s,a} \frac{w^k_h(s,a) - \mathcal{I}^k_h(s,a)}{\sqrt{\max\{1, N^k_h(s,a)\}}}
$$

The first term can be bounded as follows.

$$
\sum_{k=1}^{K} \sum_{h,s,a} \frac{\mathcal{I}^k_h(s,a)}{\max\{1, N^k_h(s,a)\}} \leq \sum_{h,s,a} \sum_{k=1}^{K} \frac{1}{\sqrt{\max\{1, N^k_h(s,a)\}}} = \sum_{h,s,a} \sum_{i=1}^{N^k_h(s,a)} \frac{1}{\sqrt{i}} \leq c' \sum_{h,s,a} \sqrt{N^k_h(s,a)}
$$

$$
\leq c' \sqrt{\left( \sum_{h,s,a} 1 \right) \left( \sum_{h,s,a} N^k_h(s,a) \right)}
$$

$$
= O \left( \sqrt{SAH^2} \right)
$$

To bound the second term, we apply Lemma A.3 again. Consider $Y_{k,h} := \sum_{s,a} w^k_h(s,a) - \mathcal{I}^k_h(s,a) \leq 1$, $\lambda = 1$ and the fact that for any fixed $h$,

$$
\mathbb{E} \left[ Y^2_{k,h} | \mathcal{F}_{k-1} \right] \leq \mathbb{E} \left[ \left( \sum_{s,a} \frac{\mathcal{I}^k_h(s,a)}{\max\{1, N^k_h(s,a)\}} \right)^2 | \mathcal{F}_{k-1} \right]
$$

$$
= \mathbb{E} \left[ \sum_{s,a} \frac{\mathcal{I}^k_h(s,a)}{\max\{1, N^k_h(s,a)\}} | \mathcal{F}_{k-1} \right]
$$

$$
= \sum_{s,a} \frac{w^k_h(s,a)}{\max\{1, N^k_h(s,a)\}}.
$$

Then, via Lemma A.3, we have with probability at least $1 - \delta$,

$$
\sum_{k=1}^{K} \sum_{h,s,a} \frac{w^k_h(s,a) - \mathcal{I}^k_h(s,a)}{\sqrt{\max\{1, N^k_h(s,a)\}}} = \sum_{h=1}^{H} \sum_{k=1}^{K} Y_{k,h} \leq \sum_{h=1}^{H} \sum_{k=1}^{K} \sum_{s,a} \frac{w^k_h(s,a)}{\max\{1, N^k_h(s,a)\}} + H \ln(H/\delta)
$$

$$
= O \left( SAH \ln(KH) + H \ln(H/\delta) \right)
$$

Putting the two bounds together, yields the second result and completes the proof. \qed
B Proofs for Section 4

In this section, we present proof for Lemma 4.1 and Theorem 4.2. We also discuss the gaps in the regret analysis of Vietri et al. (2020) at the end of this section. To start with, we present the proof of Lemma 4.1 as follows.

Proof of Lemma 4.1. Assume the event in Assumption 1 hold. The first result in Lemma 4.1 follows the same analysis as in the proof of Lemma 3.1. In particular, we have with probability at least $1 - \delta/2$,

$$|c_h(s, a) - \hat{c}_h(s, a)| \leq \beta_h^{c}(s, a).$$

To show the second result, we first note that $V^*$ is fixed and $V_h^*(s) \leq H$ for all $h$ and $s$. This enables us to use standard Hoeffding’s inequality. Specifically, we have

$$(\hat{P}_h - P_h)V_{h+1}^*(s, a)$$

$$= \sum_{s'} \hat{P}_h(s'|s, a)V_{h+1}^*(s') - P_h(s'|s, a)V_{h+1}^*(s')$$

$$\leq \sum_{s'} \left(\frac{N_h^k(s, a, s')}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1} - P_h(s'|s, a)\right)V_{h+1}^*(s') + \sum_{s'} \left(\frac{\hat{N}_h^k(s, a, s') - N_h^k(s, a, s')}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1} V_{h+1}^*(s')\right).$$

We are going to bound the two terms respectively. For the first term, we have with probability at least $1 - \delta/2$

$$\left|\sum_{s'} \left(\frac{N_h^k(s, a, s')}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1} - P_h(s'|s, a)\right)\right| V_{h+1}^*(s')$$

$$\leq \sum_{s'} V_{h+1}^*(s') \left(\frac{N_h^k(s, a, s')}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1} - P_h(s'|s, a)\right) + \sum_{s'} V_{h+1}^*(s') P_h(s'|s, a) \left(\frac{N_h^k(s, a) \vee 1}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1} - 1\right)$$

$$\leq \frac{N_h^k(s, a) \vee 1}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1} H \sqrt{\frac{L(\delta)}{N_h^k(s, a) \vee 1}} + \frac{\hat{N}_h^k(s, a, s') - N_h^k(s, a, s')}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1} V_{h+1}^*(s')$$

where in (a) we use standard Hoeffding inequality with $L(\delta') := 2 \ln \frac{4SA_T}{\delta}$ and $V_{h+1}^*(s') \leq H$.

For the second term, we have

$$\left|\sum_{s'} \left(\frac{\hat{N}_h^k(s, a, s') - N_h^k(s, a, s')}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1} V_{h+1}^*(s')\right)\right|$$

$$\leq H \sqrt{\frac{L(\delta)}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1}} + \frac{2H E_{\epsilon, \delta, 1}}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1}$$

$$\leq H \sqrt{\frac{L(\delta)}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1}} + \frac{2H E_{\epsilon, \delta, 1}}{(N_h^k(s, a) + E_{\epsilon, \delta, 1}) \vee 1}$$

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Putting the two bounds together, we have
\[
\left| (\tilde{P}_h^k - P_h)V_{h+1}^*(s, a) \right| \leq H \sqrt{\frac{L(\delta')}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1}} + H(SE_{\varepsilon, \delta, 2} + 2E_{\varepsilon, \delta, 1}) \frac{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1}
\]
Noting that \( L(\delta) = \sqrt{L(\delta)} \), we have obtained the second result in Lemma 4.1.

Now, we turn to focus on the third result in Lemma 4.1. Note that
\[
\sum_{s'} |P_h(s'|, s, a) - \tilde{P}_h^k(s'|, s, a)| \leq C \sqrt{\frac{L'(\delta)P_h(s'|s, a)}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1}} + C \frac{L'(\delta) + 2E_{\varepsilon, \delta, 1} + E_{\varepsilon, \delta, 2}}{(N_h^k(s, a) + E_{\varepsilon, \delta, 1}) \vee 1}.
\]
Finally, applying union bound over all the events, yields the required results in Lemma 4.1.
Now, we turn to establish Theorem 4.2. First, the next lemma establishes that the value function maintained in our algorithm is optimistic.

Lemma B.1. Fix $\delta \in (0, 1]$, with probability at least $1 - 3\delta$, $\bar{V}^k_h(s) \leq V^*_h(s)$ for all $(k, h, s)$.

Proof. For a fixed $k$, consider $h = H + 1, H, \ldots, 1$. In the base case $h = H + 1$, it trivially holds since $\bar{V}^k_{H+1}(s) = 0 = V^*_H(s)$. Assume that $\bar{V}^k_h(s) \leq V^*_h(s)$ for all $s$. Then, by the update rule, we have

$$\bar{Q}^k_h(s, a) = \min\{H, \max\{0, \bar{c}^k_h(s, a) + (P^k_h\bar{V}^k_{h+1})(s, a) - \beta^k_h(s, a)\}\}$$

First, we would like to show that the truncation at $H$ does not affect the analysis. To see this, first observe that under Lemma 4.1

$$\bar{c}^k_h(s, a) + (P^k_h\bar{V}^k_{h+1})(s, a) - \beta^k_h(s, a)\]

where (a) holds by the first result in Lemma 4.1; (b) holds by induction; (c) holds by the second result in Lemma 4.1. This directly implies that

$$\bar{Q}^k_h(s, a) = \max\{0, \bar{c}^k_h(s, a) + (P^k_h\bar{V}^k_{h+1})(s, a) - \beta^k_h(s, a)\}$$

Hence, if the maximum is attained at zero, then $\bar{Q}^k_h(s, a) \leq Q^*_h(s, a)$ trivially holds since $Q^*_h(s, a) \in [0, H]$. Otherwise, by Eq. (12), we also have $\bar{Q}^k_h(s, a) \leq Q^*_h(s, a)$. Therefore, we have $\bar{Q}^k_h(s, a) \leq Q^*_h(s, a)$, and hence $\bar{V}^k_h(s) \leq V^*_h(s)$.

Based on the result above, we are now able to present the proof of Theorem 4.2.

Proof of Theorem 4.2. By the optimistic result in Lemma B.1, we have

$$\mathcal{R}(K) = \sum_{k=1}^K (V^\pi_k(s_1) - V^*_1(s_1)) \leq \sum_{k=1}^K (V^\pi_k(s_1) - \bar{V}^k_1(s_1))$$

Now, we turn to upper bound $V^\pi_k(s^k_h) - \bar{V}^k_h(s^k_h)$ by a recursive form. First, observe that

$$(V^\pi_k - \bar{V}^k_h)(s^k_h) = (Q^k_h - \bar{Q}^k_h)(s^k_h, a^k_h),$$

which holds since the action executed by $\pi_k$ at step $h$, and the action used to update $\bar{V}^k_h$ is the same. Now, to
bound the $Q$-value difference, we have
\[
(Q^k - \tilde{Q}^k)(s^k, a^k)
\]
\[
\leq 2\beta_{\delta h}^k(s^k, a^k) + (P_h V^k_{h+1} - \tilde{P}_h \tilde{V}^k_{h+1})(s^k, a^k) + \beta_{\delta h}^{k,pv}(s, a)
\]
\[
= \left[(P_h - \tilde{P}_h)\tilde{V}^k_{h+1}\right](s^k, a^k) + \left[P_h(V^k_{h+1} - \tilde{V}^k_{h+1})\right](s^k, a^k) + 2\beta_{\delta h}^{k,c}(s^k, a^k) + \beta_{\delta h}^{k,pv}(s, a)
\]
\[
= \left[(P_h - \tilde{P}_h)V^*_{h+1}\right](s^k, a^k) + \left[\tilde{P}_h - P_h\right](V^*_{h+1} - \tilde{V}^k_{h+1}\right)(s^k, a^k)
\]
\[
+ \left[P_h(V^k_{h+1} - \tilde{V}^k_{h+1})\right](s^k, a^k) + 2\beta_{\delta h}^{k,c}(s^k, a^k) + \beta_{\delta h}^{k,pv}(s, a)
\]
\[
\leq \left[(P_h - \tilde{P}_h)(V^*_{h+1} - \tilde{V}^k_{h+1})\right](s^k, a^k) + \left[P_h(V^k_{h+1} - \tilde{V}^k_{h+1})\right](s^k, a^k)
\]
\[
+ 2\beta_{\delta h}^{k,c}(s^k, a^k) + 2\beta_{\delta h}^{k,pv}(s^k, a^k),
\]
(14)

where (a) we have used the cost concentration result in Lemma 4.1; (b) holds by the transition concentration result in Lemma 4.1. Thus, so far we have arrived at
\[
(V^k_{h+1} - \tilde{V}^k_{h+1})(s^k, a^k) \leq \left[(P_h - \tilde{P}_h)(V^*_{h+1} - \tilde{V}^k_{h+1})\right](s^k, a^k) + \left[P_h(V^k_{h+1} - \tilde{V}^k_{h+1})\right](s^k, a^k)
\]
\[
+ 2\beta_{\delta h}^{k,c}(s^k, a^k) + 2\beta_{\delta h}^{k,pv}(s^k, a^k).
\]
(15)

We will first carefully analyze the first term. In particular, let $G := (V^*_{h+1} - \tilde{V}^k_{h+1})$, we have
\[
\left[P_h(s'|s^k, a^k) - \tilde{P}_h(s'|s^k, a^k)\right]\ G(s')
\]
\[
\leq c \sum_{s'} \left(\sqrt{\frac{L'(\delta)P_h(s'|s^k, a^k)}{(N_h^k(s, a) + E_{\epsilon,\delta,1} \vee 1)}} + \frac{L'(\delta)}{(N_h^k(s, a) + E_{\epsilon,\delta,1} \vee 1)} + \frac{2E_{\epsilon,\delta,1} + E_{\epsilon,\delta,2}}{(N_h^k(s, a) + E_{\epsilon,\delta,1} \vee 1)}\right) G(s')
\]
\[
\leq c \sum_{s'} \left(\frac{P_h(s'|s^k, a^k) H}{cH L'(\delta)}\right) G(s') + c \sum_{s'} \left(\frac{2E_{\epsilon,\delta,1} + E_{\epsilon,\delta,2}}{(N_h^k(s, a) + E_{\epsilon,\delta,1} \vee 1)}\right) G(s')
\]
\[
\leq c \sum_{s'} \left(\frac{P_h(s'|s^k, a^k) H}{cH^2 L'(\delta)}\right) G(s') + c \sum_{s'} \left(\frac{2E_{\epsilon,\delta,1} + E_{\epsilon,\delta,2}}{(N_h^k(s, a) + E_{\epsilon,\delta,1} \vee 1)}\right) G(s'),
\]
(16)

where (a) holds by the third result in Lemma 4.1 and $c$ is some absolute constant; (b) holds by $\sqrt{xy} \leq x + y$ for positive numbers $x, y$; (c) holds since $G(s') \leq 2H$ by the boundedness of $V$-value. Now, plugging the
We are only left to bound each of them. To start with, we focus on the bonus term. We focus on $\beta$ where (a) holds by definitions $\beta$ particular since it upper bounds the term $\chi$ where $V := k \pi \geq V + h \geq 1$. Plugging (17) into (15), yields the following recursive formula.

$$
\left( V^\pi_h - \bar{V}^\pi_h \right)(s_k^h) \leq \left( 1 + \frac{1}{H} \right) \left[ P_h(V^\pi_h - \bar{V}^\pi_h) \right] \left( s_k^h, a_k^h \right) + \xi_k^h + \zeta_k^h + 2\beta_k^h
$$

where in (a), we let $\beta_k^k := \beta_k^k(s_k^h, a_k^h)$ for notation simplicity; (b) holds since $V_{h+1}^\pi \geq V_{h+1}^\pi$. Based on this, we have the following bound on $\left( \bar{V}^\pi - V^\pi \right)(s^k_1)$,

$$
\left( \bar{V}^\pi - V^\pi \right)(s^k_1) \leq \left( 1 + \frac{1}{H} \right) \left( (\chi_1^k + \xi_1^k + \zeta_1^k + 2\beta_1^k) + \left( 1 + \frac{1}{H} \right)^2 (\chi_2^k + \xi_2^k + \zeta_2^k + 2\beta_2^k) + \ldots + \left( 1 + \frac{1}{H} \right)^H (\chi_H^k + \xi_H^k + \zeta_H^k + 2\beta_H^k) \right)
$$

$$
\leq 3 \sum_{h=1}^{H} (\chi_h^k + \xi_h^k + \zeta_h^k + 2\beta_h^k).
$$

Therefore, plugging (18) into (13), we have the regret decomposition as follows.

$$
\mathcal{R}(K) \leq 3 \sum_{k=1}^{K} \sum_{h=1}^{H} (\chi_h^k + \xi_h^k + \zeta_h^k + 2\beta_h^k)
$$

We are only left to bound each of the term. To start with, we focus on the bonus term. We focus on $\beta_k^k(s, a)$ in particular since it upper bounds the term $\beta_k^k(s, a)$ as shown in Lemma 4.1. By definition, we have

$$
\sum_{k=1}^{K} \sum_{h=1}^{H} \beta_k^k(s_k^h, a_k^h) = H \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{L_k(\delta)}{\sqrt{(N_k^h(s_k^h, a_k^h) + E_{\delta, 1}) \lor 1}} + H \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{(SE_{\delta, 2} + 2E_{\delta, 1})}{\sqrt{(N_k^h(s_k^h, a_k^h) + E_{\delta, 1}) \lor 1}}.
$$
The first term can be upper bounded as follows \((T := KH)\) under Assumption 1.

\[
T_1 \leq HL_c(\delta) \sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\frac{1}{N_h^K(s, a) \lor 1}}
\]

\[
= HL_c(\delta) \sum_{h, s, a} \sum_{i=1}^{N_h^K(s, a)} \frac{1}{\sqrt{i}}
\]

\[
\leq c' HL_c(\delta) \sum_{h, s, a} \sqrt{N_h^K(s, a)}
\]

\[
\leq c' HL_c(\delta) \left[ \left( \sum_{h, s, a} 1 \right) \left( \sum_{h, s, a} N_h^K(s, a) \right) \right]
\]

\[
= \tilde{O} \left( \sqrt{H^3 SAT} \right).
\]

The second term can be upper bounded as follows under Assumption 1.

\[
T_2 \leq cH(Se_{\delta, 2} + E_{\varepsilon, \delta, 1}) \sum_{k=1}^{K} \sum_{h=1}^{H} \frac{1}{N_h^K(s_h^k, a_h^k) \lor 1}
\]

\[
= cH(Se_{\delta, 2} + E_{\varepsilon, \delta, 1}) \sum_{h, s, a} \sum_{i=1}^{N_h^K(s, a)} \frac{1}{i}
\]

\[
\leq c'H(Se_{\delta, 2} + E_{\varepsilon, \delta, 1}) \sum_{h, s, a} \ln(N_h^K(s, a))
\]

\[
= \tilde{O} \left( H^2 S^2 A E_{\varepsilon, \delta, 2} + H^2 S A E_{\varepsilon, \delta, 1} \right).
\]

Putting them together, we have the following bound on the summation over \(\beta_h^k\).

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \beta_h^k = \tilde{O} \left( \sqrt{H^3 SAT} + H^2 S^2 A E_{\varepsilon, \delta, 2} + H^2 S A E_{\varepsilon, \delta, 1} \right)
\]

By following the same analysis as in \(T_2\), we can bound the summation over \(\xi_h^k = \frac{c'SHL'(\delta)}{(N_h^K(s, a) + E_{\varepsilon, \delta, 1}) \lor 1}\) as follows.

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \xi_h^k = \tilde{O} \left( H^3 S^2 A \right)
\]

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \zeta_h^k = \tilde{O} \left( H^2 S^2 A(E_{\varepsilon, 2} + E_{\varepsilon, 1}) \right).
\]

Finally, we are going to bound the summation over \(\chi_h^k := (P_h(V^k_{h+1} - \tilde{V}_{h+1}^k))(s_h^k, a_h^k) - (V_{\pi_h}^{k} - \tilde{V}_{h+1}^k)(s_{h+1}^k)\), which turns out to be a martingale difference sequence. In particular, we define a filtration \(\mathcal{F}_h^k\)
that includes all the randomness up to the \( k \)-th episode and the \( h \)-th step. Then, we have \( \mathcal{F}_1^1 \subset \mathcal{F}_1^2 \ldots \subset \mathcal{F}_H^1 \subset \mathcal{F}_1^2 \subset \mathcal{F}_2^2 \ldots \). Also, we have \((\tilde{V}_{h+1}^k - V_{h+1}^{\pi_h}) \in \mathcal{F}_1^k \subset \mathcal{F}_h^k\) since they are decided by data collected up to episode \( k - 1 \). A bit abuse of notation, we define \( X_{h+1}^k := \chi_h^k \). Then, we have
\[
\mathbb{E} \left[ X_{h+1}^k | \mathcal{F}_h^k \right] = 0.
\]
This holds since the expectation only captures randomness over \( s_h^k \). Thus, \( X_{h+1}^k \) is a martingale difference sequence. Moreover, we have \(|X_{h+1}^k| \leq 4H\) a.s. By Azuma-Hoeffding inequality, we have with probability at least \( 1 - \delta \)
\[
\sum_{k=1}^{K} \sum_{h_1=1}^{H} \chi_h^k = \sum_{k=1}^{K} \sum_{h_1=1}^{H} X_{h+1}^k = c' \sqrt{H^2 T \ln(2/\delta)} = \tilde{O} \left( \sqrt{H^2 T} \right)
\]
Putting everything together, and applying union bound on all high-probability events, we have shown that with probability at least \( 1 - \delta \),
\[
\mathcal{R}(T) = O \left( \left( \sqrt{SAH^3 T} + S^2 AH^3 + S^2 AH^2 E_{\epsilon, \delta, 1} + S^2 AH^2 E_{\epsilon, \delta, 2} \right) \log(S, A, T, 1/\delta) \right). \tag{20}
\]

\[\Box\]

### B.1 Discussions

We end this section by comparing our results with existing works on private value-iteration RL, i.e., Garcelon et al. (2020) on LDP and Vietri et al. (2020) on JDP.

- In Garcelon et al. (2020), the privacy-independent leading term has a dependence on \( S \) rather than \( \sqrt{S} \) in our result (i.e., the first term in (20)). This is because they directly bound the transition term \( \left\| P_h(\cdot | s, a) - \tilde{P}_h^k(\cdot | s, a) \right\|_1 \), which incurs the additional \( \sqrt{S} \). In particular, after step (a) in (14), they directly bound \( \tilde{P}_h^k V_{h+1}^k \) by \( P_h \tilde{V}_{h+1}^k + H \beta_h^k \) and then recursively expand the term. Note that \( \beta_h^k \) has an additional factor \( \sqrt{S} \), which directly leads to the dependence \( S \) in the final result. In contrast, we handle (a) in (14) by following the idea in Azar et al. (2017). That is, we first extract the term \( (P_h - \tilde{P}_h^k)V_{h+1}^* \), which can be bounded by standard Hoeffding’s inequality since \( V_{h+1}^* \) is fixed and hence no additional \( \sqrt{S} \) is introduced. Due to this extraction, we have an additional ‘correction’ term, i.e., \( (P_h - \tilde{P}_h^k)(V_{h+1}^k - V_{h+1}^*) \). To bound it, we use Bernstein’s-type inequality to bound \( (P_h(s'|s, a) - \tilde{P}_h^k(s'|s, a)) \) in (16). This allows us to obtain the final recursive formula.

- In Vietri et al. (2020), although the claimed result has the same regret bound as ours, its current analysis has gaps. First, to derive the regret decomposition in Lemma 18 therein, the private estimates were incorrectly used as the true cost and transition functions. This lead to a simpler but incorrect regret decomposition since it omits the ‘error’ term between the private estimates and true values. Second, even if we add the omitted ‘error’ term (between private estimates and true values) into the regret decomposition, its current analysis cannot achieve the same result as ours. This is due to a similar argument in bullet one. That is, in order to use its current confidence bound \( \text{conf}_T \) to avoid the additional \( \sqrt{S} \) factor, it needs to use Bernstein’s-type inequality to bound the ‘correction’ term. They fail to consider this since the regret decomposition does not have the ‘error’ term as it was incorrectly omitted in Lemma 18.
C Proofs for Section 5

First, we present the proof of Lemma 5.1. Corollary 5.2 is a direct consequence of Lemma 5.1 and Theorems 3.2 and 4.2.

Proof of Lemma 5.1. We start with the privacy guarantee of the CENTRAL-PRIVATIZER. First, consider the counters for number of visited states \( N_h^k(s, a) \). Note that there are \( SAH \) many counters, and each counter is a \( K \)-bounded binary mechanism of Chan et al. (2010). Now, consider the counter corresponding to a fixed tuple \((s, a, h) \in S \times A \times [H]\). Note that, at every episode \( k \in [K] \), the private count \( \tilde{N}_h^k(s, a) \) is the sum of at most \( \log K \) noisy \( P \)-sums, where each \( P \)-sum is corrupted by an independent Laplace noise \( \text{Lap}\left(\frac{3H \log K}{\varepsilon}\right) \).

Therefore, by (Chan et al., 2010, Theorem 3.5), the private counts \( \{\tilde{N}_h^k(s, a)\}_{k \in [K]} \) satisfy \( \frac{\varepsilon}{3H} \)-DP.

Now, observe that each counter takes as input the data stream \( \sigma_h(s, a) \in \{0, 1\}^K \), where the \( j \)-th bit \( \sigma_h^j(s, a) := \mathbb{I}\{s_h^j = s, a_h^j = a\} \) denotes whether the pair \((s, a)\) is encountered or not at step \( h \) of episode \( j \). Consider, some other data stream \( \sigma'_h(s, a) \in \{0, 1\}^K \) which differs from \( \sigma_h(s, a) \) only in one entry. Then, we have \( \|\sigma_h(s, a) - \sigma'_h(s, a)\|_1 = 1 \). Furthermore, since at every episode at most \( H \) state-action pairs are encountered, we obtain

\[
\sum_{(s, a, h) \in S \times A \times [H]} \|\sigma_h(s, a) - \sigma'_h(s, a)\|_1 \leq H,
\]

Therefore, by (Hsu et al., 2016, Lemma 34), the composition of all these \( SAH \) different counters, each of which is \( \frac{\varepsilon}{3H} \)-DP, satisfies \( \frac{\varepsilon}{3} \)-DP.

Using similar arguments, one can show that composition of the counters for empirical rewards \( C_h^k(s, a) \) and state transitions \( N_h^k(s, a, s') \) satisfy \( \frac{\varepsilon}{3} \)-DP individually. Finally, employing the composition property of DP Dwork et al. (2014), we obtain that the CENTRAL-PRIVATIZER is \( \varepsilon \)-DP.

Let us now focus on the utility of the CENTRAL-PRIVATIZER. First, fix a tuple \((s, a, h) \in S \times A \times [H]\), and consider the private counts \( \tilde{N}_h^k(s, a) \) corresponding to number of visited states \( N_h^k(s, a) \). Note that, at each episode \( k \in [K] \), the cost of privacy \( |\tilde{N}_h^k(s, a) - N_h^k(s, a)| \) is the sum of at most \( \log K \) i.i.d. random variables \( \text{Lap}\left(\frac{3H \log K}{\varepsilon}\right) \). Therefore, by (Chan et al., 2010, Theorem 3.6), we have

\[
\mathbb{P}\left[|\tilde{N}_h^k(s, a) - N_h^k(s, a)| \leq \frac{3H}{\varepsilon} \sqrt{8 \log^3 K \log(6/\delta)} \right] \geq 1 - \delta/3.
\]

Now, by a union bound argument, we obtain

\[
\mathbb{P}\left[\forall (s, a, k, h), \; |\tilde{N}_h^k(s, a) - N_h^k(s, a)| \leq \frac{3H}{\varepsilon} \sqrt{8 \log^3 K \log(6S^2AT/\delta)} \right] \geq 1 - \delta/3.
\]

Using similar arguments, one can show that the private counts \( \tilde{C}_h^k(s, a) \) and \( \tilde{N}_h^k(s, a, s') \) corresponding rewards \( C_h^k(s, a) \) and state transitions \( N_h^k(s, a, s') \), respectively, satisfy

\[
\mathbb{P}\left[\forall (s, a, k, h), \; |\tilde{C}_h^k(s, a) - C_h^k(s, a)| \leq \frac{3H}{\varepsilon} \sqrt{8 \log^3 K \log(6SAT/\delta)} \right] \geq 1 - \delta/3,
\]

\[
\mathbb{P}\left[\forall (s, a, s', k, h), \; |\tilde{N}_h^k(s, a, s') - N_h^k(s, a, s')| \leq \frac{3H}{\varepsilon} \sqrt{8 \log^3 K \log(6SAT^2/\delta)} \right] \geq 1 - \delta/3.
\]

Combining all the three guarantees together using a union bound, we obtain that CENTRAL-PRIVATIZER satisfies Assumption 1. \( \square \)
Next, we present the proof of Lemma 5.4. Corollary 5.5 is a direct consequence of Lemma 5.4 and Theorems 3.2 and 4.2.

Proof of Lemma 5.4. We start with the utility guarantee of the LOCAL-PRIVATIZER. First, fix a tuple \((s, a, h) \in S \times A \times [H]\), and consider the private counts \(\tilde{N}^k_h(s, a)\) for the number of visited states \(N^k_h(s, a)\). Note that, at each episode \(k \in [K]\), the cost of privacy \(|\tilde{N}^k_h(s, a) - N^k_h(s, a)|\) is the sum of at most \(K\) i.i.d. random variables Lap \(\left(\frac{3H}{\varepsilon}\right)\). Therefore, by (Dwork et al., 2014, Corollary 12.4), we have

\[
P\left[|\tilde{N}^k_h(s, a) - N^k_h(s, a)| \leq \frac{3H}{\varepsilon} \sqrt{8K \log(6/\delta)}\right] \geq 1 - \delta/3.
\]

Now, by a union bound argument, we obtain

\[
P \left[\forall (s, a, k, h), \ |\tilde{N}^k_h(s, a) - N^k_h(s, a)| \leq \frac{3H}{\varepsilon} \sqrt{8K \log(6SAT/\delta)}\right] \geq 1 - \delta/3,
\]

Using similar arguments, one can show that the private counts \(\tilde{C}^k_h(s, a)\) and \(\tilde{N}^k_h(s, a, s')\) corresponding rewards \(C^k_h(s, a)\) and state transitions \(N^k_h(s, a, s')\), respectively, satisfy

\[
P \left[\forall (s, a, k, h), \ |\tilde{C}^k_h(s, a) - C^k_h(s, a)| \leq \frac{3H}{\varepsilon} \sqrt{8K \log(6SAT/\delta)}\right] \geq 1 - \delta/3,
\]

\[
P \left[\forall (s, a, s', k, h), \ |\tilde{N}^k_h(s, a, s') - N^k_h(s, a, s')| \leq \frac{3H}{\varepsilon} \sqrt{8K \log(6S^2AT/\delta)}\right] \geq 1 - \delta/3.
\]

Combining all the three guarantees together using a union bound, we obtain that LOCAL-PRIVATIZER satisfies Assumption 1.

Now, we turn towards the privacy guarantee of the LOCAL-PRIVATIZER. First, we fix an episode \(k \in [K]\). Now, we fix a tuple \((s, a, h) \in S \times A \times [H]\), and consider the private version \(\tilde{\sigma}^k_h(s, a)\) of the bit \(\sigma^k_h(s, a)\) denoting whether the pair \((s, a)\) is encountered or not at step \(h\) of episode \(k\). Note that \(\tilde{\sigma}^k_h(s, a)\) is obtained from \(\sigma^k_h(s, a)\) via a Laplace mechanism with noise level \(\frac{3H}{\varepsilon}\). Since, the sensitivity of the input function is 1, the Laplace mechanism is \(\frac{\varepsilon}{3}\)-DP, as well as it is \(\frac{\varepsilon}{3}\)-LDP (Dwork et al., 2014). Furthermore, since at every episode at most \(H\) state-action pairs are encountered, by (Hsu et al., 2016, Lemma 34), the composition of all these \(SAH\) different Laplace mechanisms are \(\varepsilon/3\)-LDP.

Using similar arguments, one can show that composition of the corresponding Laplace mechanisms for empirical rewards and state transitions satisfy \(\varepsilon/3\)-LDP individually. Finally, employing the composition property of DP (Dwork et al., 2014), we obtain that the LOCAL-PRIVATIZER is \(\varepsilon\)-LDP. \qed