Illustrations of the Relativistic Conservation Law for the Center of Energy

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Abstract

The relativistic conservation law involving the center of energy is reviewed and illustrated using simple examples from classical electromagnetic theory. It is emphasized that this conservation law is parallel to the conservation laws for energy, linear momentum, and energy, in arising from the generators of the Poincare group for electromagnetic theory; yet this relativistic law reflecting the continuous flow of energy goes virtually unmentioned in the text books. The illustrations here present situations both where external forces are present and are absent. The cases of a parallel plate capacitor, a flattened slip-joint solenoid, and two interacting charges are included.
I. INTRODUCTION

Classical electrodynamics, like any other relativistic Lagrangian field theory, is invariant under the Poincare group involving the operations of spacetime translation, spatial rotation, and proper Lorentz transformation. The associated infinitesimal generators, \( P, U, L, \) and \( U \vec{X} \), are associated with conserved quantities. The generator \( P \) of space translations is associated with conservation of linear momentum. The generator \( U \) of time translations is associated with conservation of energy. The generator \( L \) of spatial rotations is associated with conservation of angular momentum. The generator \( U \vec{X} \) of proper Lorentz transformations is associated with the uniform motion of the system center of energy. Although the conservation laws of linear momentum, angular momentum, and energy are illustrated by fine elementary examples in electromagnetism textbooks, this does not seem to be the case for the uniform motion of the center of energy. The invariant motion of the center of energy is well known but not widely known, and is rarely illustrated with examples in the electromagnetism literature. The law expresses the continuous flow of energy in relativistic systems. In this article we review the relativistic law for the invariant motion of the center of energy and then present three simple electromagnetic examples: a parallel-plate capacitor, a flattened, slip-joint solenoid, and two interacting point charges. The examples remind us that when calculating the center of energy of an electromagnetic system, relativistic particle equations of motion must be used and all the energy must be considered, including the particle rest energy and kinetic energy, and the distributed energy stored in the electromagnetic field.

II. RELATIVISTIC CONSERVATION LAWS

A. The Generators of the Poincare Group for Electromagnetism

For charged point masses \( m_i \) interacting through electromagnetic fields \( \mathbf{E} \) and \( \mathbf{B} \), the generators of the Poincare group take the forms

\[
P = \sum_i m_i \gamma_i \mathbf{v}_i + \int d^3r \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B} \quad \text{(linear momentum)} \tag{1}
\]

\[
U = \sum_i m_i \gamma_i c^2 + \int d^3r \frac{1}{8\pi} (E^2 + B^2) \quad \text{(energy)} \tag{2}
\]
\[ L = \sum_i r_i \times m_i \gamma_i v_i + \int d^3 r \times \left( \frac{1}{4\pi c} E \times B \right) \] (angular momentum) \hspace{1cm} (3)

and

\[ U \vec{X} = \sum_i r_i m_i \gamma_i c^2 + \int d^3 r \frac{1}{8\pi} (E^2 + B^2) \] (energy times center of energy) \hspace{1cm} (4)

where \( v_i = d\mathbf{r}_i/dt \) is the time-derivative of the particle displacement \( \mathbf{r}_i \), \( \gamma_i = (1 - v_i^2/c^2)^{-1/2} \), \( E \) and \( B \) represent the electric and magnetic fields, and \( \vec{X} \) is the center of energy of the system. These correspond to the generators respectively of space translation, time translation, spatial rotation, and proper Lorentz transformation. In the absence of external forces, the first three quantities are time-independent and the fourth has a constant time derivative. The electromagnetic expressions in the first three equations appear in the electromagnetism textbooks, whereas the last is usually absent. On account of this omission, we will sketch the derivation of the center of energy expression.

**B. Derivation of the Center-of-Energy Law**

The center of energy \( \vec{X} \) in Eq. (4) is analogous to the familiar center of (rest) mass \( \vec{X}_{\text{restmass}} \) of nonrelativistic mechanics

\[ M \vec{X}_{\text{restmass}} = \sum_i m_i \mathbf{r}_i, \quad M = \sum_i m_i \] \hspace{1cm} (5)

except that all energy contributes. The total energy \( U \) in Eq. (2) for a system of charged particles and electromagnetic fields is the sum of the relativistic mechanical energy of each particle \( m_i \gamma_i c^2 \) and the electromagnetic field energy found by integrating the energy density \( u = [1/(8\pi)](E^2 + B^2) \) over all space. The center-of-energy expression (4) involves weighting the displacement \( \mathbf{r} \) by the amount of the energy located at \( \mathbf{r} \). Thus a point mass of energy \( m_i \gamma_i c^2 \) contributes \( \mathbf{r}_i (m_i \gamma_i c^2) \) while the electromagnetic energy \( u \, d^3 r \) in a differential volume \( d^3 r \) contributes \( \mathbf{r} (u \, d^3 r) = \mathbf{r} [1/(8\pi)](E^2 + B^2) \, d^3 r \). Summing over the particles and integrating over all the electromagnetic fields in space, we obtain the expression (4) for the energy times the center of energy \( U \vec{X} \).

The derivation of the law for the invariant motion of the center of energy in electromagnetic theory can be given in a fashion parallel to that given for Poynting’s theorem. We
consider the integral over all space of \( \int d^3r \mathbf{r} (\mathbf{J} \cdot \mathbf{E}) \) which represents the volume-integral over the displacement \( \mathbf{r} \) weighted by \( \mathbf{J} \cdot \mathbf{E} \), the local transfer of power from electromagnetic form over to some other form due to the forces produced by electric fields on moving charges. Just as for Poynting’s theorem, we use Maxwell’s equations to rewrite this integral in terms of the electromagnetic fields alone,

\[
\int d^3r \mathbf{r} (\mathbf{J} \cdot \mathbf{E}) = \int d^3r \mathbf{r} \left[ \frac{c}{4\pi} (\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t}) \cdot \mathbf{E} \right] \\
= \int d^3r \mathbf{r} \left[ -\nabla \cdot \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right) - \frac{\partial}{\partial t} \left( \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \right) \right] \\
= -\int \mathbf{r} \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right) \cdot d\mathbf{A} + \int d^3r \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right) - \frac{d}{dt} \int d^3r \mathbf{r} \left( \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \right) \\
= \int d^3r \left( \frac{c}{4\pi} \mathbf{E} \times \mathbf{B} \right) - \frac{d}{dt} \int d^3r \mathbf{r} \left( \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \right) \\
\tag{6}
\]

where we have used the divergence theorem and have dropped the surface term assuming that the sources of electromagnetic fields are localized.

For a system of charged particles interacting through the electromagnetic fields, we differentiate Eq. (4) to obtain

\[
\frac{d(U - \mathbf{X})}{dt} = \frac{d}{dt} \left( \sum m_i \gamma_i c^2 + \int d^3r \frac{1}{8\pi} (E^2 + B^2) \right) \\
= c^2 \sum m_i \gamma_i \mathbf{v}_i + \sum m_i \frac{d}{dt} \gamma_i c^2 + \frac{d}{dt} \int d^3r \frac{1}{8\pi} (E^2 + B^2) \\
= c^2 \sum m_i \gamma_i \mathbf{v}_i + \int d^3r \mathbf{r} (\mathbf{J} \cdot \mathbf{E}) + \frac{d}{dt} \int d^3r \frac{1}{8\pi} (E^2 + B^2) \\
= c^2 \left[ \sum m_i \gamma_i \mathbf{v}_i + \int d^3r \left( \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B} \right) \right] = c^2 \mathbf{P} \\
\tag{7}
\]

where we have used the result of Eq. (6) and the energy transfer equation for point charges

\[
\sum m_i \frac{d}{dt} \gamma_i c^2 = \int d^3r \mathbf{r} (\mathbf{J} \cdot \mathbf{E}) \\
\tag{8}
\]

where the point-charge current density is \( \mathbf{J}(\mathbf{r}, t) = \sum q_i \mathbf{v}_i \delta^3(\mathbf{r} - \mathbf{r}_i(t)) \) and \( d(m_i \gamma_i c^2)/dt = q_i \mathbf{v}_i \cdot \mathbf{E}(\mathbf{r}_i, t) \). Thus in Eq. (7) we see that the time rate of change of the quantity \{energy times the center of energy\} is equal to \( c^2 \) times the linear momentum of the system. Since the linear momentum and the energy of the system are constant in time, this means that the velocity of the center of energy is constant in time, \( d\mathbf{X}/dt = \text{const.} \).
C. Conservation Laws in the Presence of External Forces on Particles

In many cases it is convenient to consider not isolated electromagnetic systems but rather electromagnetic systems in interaction with external forces \( \mathbf{F}_{\text{ext}i} \) acting on the particles of the system. In this case the conservation ideas are changed for all the conservation laws. The sum of the external forces gives the time-rate-of-change of the system linear momentum

\[
\sum_i \mathbf{F}_{\text{ext}i} = \frac{d\mathbf{P}}{dt}
\]  

The power delivered by the external forces gives the time-rate-of-change of the system energy

\[
\sum_i \mathbf{F}_{\text{ext}i} \cdot \mathbf{v}_i = \frac{dU}{dt}
\]

The sum of the external torques gives the time-rate-of-change of the system angular momentum \( \mathbf{L} \) (about the origin)

\[
\sum_i \mathbf{r}_i \times \mathbf{F}_{\text{ext}i} = \frac{d\mathbf{L}}{dt}
\]

The law for the change in the energy times the center of energy seems unfamiliar. We can obtain the rule by using the modified equation of energy transfer

\[
d(m_i\gamma_i c^2)/dt = [q_i \mathbf{E}(\mathbf{r}_i, t) + \mathbf{F}_{\text{ext}i}] \cdot \mathbf{v}_i
\]

for the \( i \)-th particle (multiplied by \( r_i \))

\[
r_i \frac{d}{dt} (m_i\gamma_i c^2) = r_i (\mathbf{F}_{\text{ext}i} \cdot \mathbf{v}_i) + r_i (q_i \mathbf{E} \cdot \mathbf{v}_i)
\]

and summing over all the particles

\[
\sum_i r_i \frac{d}{dt} (m_i\gamma_i c^2) = \sum_i r_i (\mathbf{F}_{\text{ext}i} \cdot \mathbf{v}_i) + \int d^3r \mathbf{r} (\mathbf{J} \cdot \mathbf{E})
\]

Now using Eq. (13) for \( \int d^3r \mathbf{r} (\mathbf{J} \cdot \mathbf{E}) \) in Eq. (6) and noting the first two lines of Eq. (7), we obtain the rule for the center of energy,

\[
\sum_i (\mathbf{F}_{\text{ext}i} \cdot \mathbf{v}_i) \mathbf{r}_i = \frac{d(U \vec{X})}{dt} - c^2 \mathbf{P}
\]

Thus the power weighted by the position where the power is delivered equals the time-rate-of-change of the system energy times the center of energy minus \( c^2 \) times the system linear momentum. All of the laws (9)-(11), (14) can be integrated with respect to time so as to give integral forms. The integral form for the relativistic center-of-energy law in (14) is

\[
\sum_i \int_1^2 (d\mathbf{r}_i \cdot \mathbf{F}_{\text{ext}i}) \mathbf{r}_i = U_2 \vec{X}_2 - U_1 \vec{X}_1 - c^2 \int_1^2 dt \mathbf{P}
\]
In special relativity, the flow of energy has a continuous meaning. Thus the introduction of energy by external forces located at points in space changes the center of energy of the system.

The continuous flow of energy in space for relativistic systems is in contrast with the situation in nonrelativistic mechanics where energy can be suddenly transported from one point in space to another. Thus in nonrelativistic mechanics a long, massless, rigid pole can be used to transport energy instantaneously from one end of the pole to the other. Such poles do not exist in relativistic physics. Rather, in relativistic physics a system has a well-defined center of energy which moves through space continuously at a speed (in the absence of external forces) not exceeding the speed of light in vacuum \( c \).

It is interesting to note the nonrelativistic limit for the center-of-energy relations in Eqs. (4), (7) and (14). If we divide by a factor of \( c^2 \) and allow \( c \to \infty \), then all that remains of the energy given in Eq. (2) is the rest-mass contribution \( U/c^2 \to \sum m_i \) with no contribution from the (finite) kinetic energy or electromagnetic energy. Thus in the \( c \to \infty \) limit, Eq. (4) becomes the expression for the center of rest mass given in Eq. (5). Also, equation (7) becomes the statement that the total rest mass times the center of rest mass equals the momentum

\[
\frac{d}{dt} \left[ \left( \sum_i m_i \right) \bar{X} \right] = \sum_i m_i v_i = P \quad (c \to \infty \text{ limit})
\]

These are familiar results in Galilean-invariant (nonrelativistic) mechanics. On dividing Eq. (14) by \( c^2 \) and allowing \( c \to \infty \), the left-hand side involving external forces vanishes entirely and the right-hand side involves simply the same statement in Eq. (16) obtained from the \( c \to \infty \) limit of Eq. (7). Within nonrelativistic physics, there is a continuous flow of rest mass but not of energy. Thus in nonrelativistic physics there is no separate law regarding the location where energy is introduced into the system.
III. ILLUSTRATIONS OF THE CENTER-OF-ENERGY CONSERVATION LAW

A. Quasi-static Changes for Stationary Systems

1. A Single Point Mass

As the simplest possible example of the relativistic conservation laws for stationary systems, we consider a single point mass \( m \) at rest at displacement \( \mathbf{r} \) in some inertial frame. The conserved quantities associated with Poincare invariance then involve energy \( U = mc^2 \), linear momentum \( P = 0 \), angular momentum about the origin \( L = 0 \), and energy times center of energy \((mc^2)\mathbf{x} = mc^2\mathbf{r}\). We now use an external force \( \mathbf{F}_{\text{ext}} \) to move the mass from \( \mathbf{r} \) to \( \mathbf{r}' \) quasi-statically. Since the external force can be chosen arbitrarily small, there is no linear impulse delivered, no net work done, no angular impulse, and no moment-of-work done. The only conservation law with some non-vanishing terms is the fourth involving the center of energy. Here the system linear momentum has a nonvanishing time-integral so that the integral form of the law in Eq. (15) gives

\[
0 = (mc^2\mathbf{r}') - (mc^2\mathbf{r}) - c^2 \int_{1}^{2} P \, dt \tag{17}
\]

which is consistent with the momentum of a particle

\[
P = \frac{m}{\sqrt{1 - [(dr/dt)/c]^2}} \frac{dr}{dt} \approx m \frac{dr}{dt} \quad \text{(quasi-static)} \tag{18}
\]

We notice that even though the linear momentum \( P \) can be made as small as desired by taking the external forces sufficiently small, the time integral of the linear momentum gives a finite non-zero value independent of the magnitude of the small external force in the limit \( dr/dt \to 0 \). The change in the position of the system center of energy was associated with a flow of momentum as required by special relativity.

In this simplest case where all of the energy is rest-mass energy, we could actually have divided Eq. (17) through by \( c^2 \) and have obtained a result valid in nonrelativistic physics where the linear momentum is given by exactly \( \mathbf{p} = m\mathbf{v} \). In nonrelativistic physics, the change in rest mass position is continuous and is associated with the flow of linear momentum.
2. Parallel Plate Capacitor

A parallel-plate capacitor provides a simple illustration of the conservation law for the center of energy when electrostatic energy is involved. The electrostatic energy contributes to the center of energy of the system in relativistic physics, whereas it does not contribute to the center of rest mass which appears in nonrelativistic physics. We consider a capacitor consisting of two parallel conducting plates, each of dimension \( L \times L \), the left-hand plate of mass \( m \) in the plane with \( x \)-coordinate \( x \), and the right-hand plate of mass \( M \) in the plane with \( x \)-coordinate \( X \). In this section discussing quasi-static displacement, we will take the masses \( m \) and \( M \) as negligible. The plates are centered so that the \( x \)-axis passes through the center of each plate. Plate \( m \) is charged with total charge \(+Q\) and plate \( M \) with charge \(-Q\). It is assumed that the plates form a parallel-plate capacitor of small separation \( 0 < X - x << L \) with an electric field given by the electrostatic expression

\[
E = \frac{\hat{i}4\pi Q}{L^2}
\]  

(19)

between the plates. There is no magnetic field present and it is assumed that we may neglect the fringing fields outside the plates.

In order to maintain the capacitor plates at rest, there must be external forces

\[
F_{\text{ext}m} = -Q(E + 0)/2 = -2\pi Q^2/L^2 = -F_{\text{ext}M}
\]

on the left-hand plate of negligible rest mass \( m \) at \( x \) and on the right-hand plate of negligible rest mass \( M \) at \( X \) respectively. The illustration of energy conservation for this situation is easily carried out.

Thus if the two plates are displaced quasi-statically from \( x \) to \( x' \) and from \( X \) to \( X' \) respectively, the work done by the external forces of constraint is found to equal the change in electrostatic energy

\[
F_{\text{ext}m}(x' - x) + F_{\text{ext}M}(X' - X) = \frac{2\pi Q^2}{L^2}[-(x' - x) + (X' - X)]
\]

\[
= \frac{1}{8\pi} \left( \frac{4\pi Q}{L^2} \right)^2 L^2(X' - x') - \frac{1}{8\pi} \left( \frac{4\pi Q}{L^2} \right)^2 L^2(X - x)
\]

(20)

However, in contrast to the work-energy law, the relativistic center-of-energy law in Eq. (15) usually goes unmentioned. On quasi-static displacement of the plates, there is no magnetic field generated in the region between the plates and therefore no electromagnetic field momentum between the plates. If the plates are displaced quasi-statically from \( x \) to
and from \( X \) to \( X' \), then the left-hand side of Eq. (15) gives

\[
\sum_i \int dx_i F_{ext,x_i} = \int_x^{x'} \left( \frac{-2\pi Q^2}{L^2} \right) x'' + \int_x^{X'} \left( \frac{2\pi Q^2}{L^2} \right) X''
\]

\[
= -\pi Q^2 (x''^2 - x^2) + \frac{\pi Q^2 (X''^2 - X^2)}{L^2}
\]

(21)

while the right-hand side of Eq. (15) gives

\[
U_2X_2 - U_1X_1 - c^2 \int_1^2 dt P_x = \frac{\pi Q^2 (X'^2 - x'^2)}{L^2} - \frac{\pi Q^2 (X^2 - x^2)}{L^2} - 0
\]

(22)

After rearrangement, equations (21) and (22) are seen to involve the same quantities on the right-hand sides. Thus indeed moving the capacitor plates illustrates the relativistic center-of-energy law (15) with external forces. Thus the energy introduced by the external forces at the plates provides not only the change in electrostatic energy but also the continuous motion of the center of electrostatic energy.

3. Flattened, Slip-Joint Solenoid

It was pointed out recently \[9\] that energy calculations for a solenoid can be made analogous to those for a parallel-plate capacitor by flattening the solenoid and fitting it with slip joints which allow relative motion of the front and back current sheets while maintaining the continuity of the circulating surface currents. Here we will use this solenoidal configuration to carry out calculations for a solenoid which are analogous to those given above for a capacitor.

The flattened solenoid consists of two large perfectly-conducting plates of size \( L \times l \) with negligible masses \( m \) and \( M \) located in the planes \( x \) and \( X \) respectively and connected through short perfectly-conducting sides parallel to the \( yz \)-plane which are fitted with slip joints. The slip joints maintain the continuity of the electrical circuit while allowing the plates to move along the \( x \)-axis, which passes through the centers of the plates. The surface current \( K \) is always perpendicular to the \( \hat{k} \)-direction and flows around the solenoid, in the \( +\hat{j} \) direction in the \( M \) plate and in the \( -\hat{j} \) direction in the \( m \) plate. The surface current \( K \) causes a magnetic field

\[
B = \hat{k}4\pi K/c
\]

(23)
parallel to the \( z \)-axis within the flattened solenoid. The magnetic flux \( \Phi \) through the solenoid is given by the magnitude of \( B \) times the cross-sectional area

\[
\Phi = BL(\text{X} - x) = 4\pi KL(\text{X} - x)/c \quad (24)
\]

We assume that the separation between the plates is very small \( 0 < \text{X} - x << L, l \) compared to the other dimensions so that we can neglect the fringing fields.

Here we are interested in the case where external mechanical forces on the left and right current sheets of the flattened solenoid allow these to change location quasi-statically from \( x \) to \( x' \) and from \( \text{X} \) to \( \text{X}' \) respectively. We assume that there is no ohmic resistance in the sheets nor any batteries present, so that the currents of the solenoid flow in such a fashion as to maintain the total magnetic flux \( \Phi \) through the solenoid as constant in time. The external forces needed to balance the magnetic forces on the current sheets at \( x \) and \( \text{X} \) are given by

\[
F_{\text{extx}} = i\frac{KLl}{c} = \frac{i}{8\pi} B^2 Ll = -\frac{i}{8\pi} L(\text{X} - x)^2
\]

which confirms the energy conservation law.

It is also possible to verify the law in Eq. (14) for the relativistic center of energy. Now since the cross-sectional area of the solenoid is changing, it follows that the magnetic field must be changing and this means that electric fields must be induced. Induced electric fields together with the solenoid magnetic field will lead to electromagnetic field linear momentum and hence to a contribution in Eq. (14) from \( c^2 \mathbf{P} \). In order to find the electric field induced when the current sheets are moved apart, we consider a single current sheet seen in a new Lorentz frame. If we consider a current sheet normal to the \( x \)-axis with a current \( \mathbf{K} \) flowing in the \( \mathbf{j} \) direction, then there is a magnetic field \( \mathbf{B} = \pm\mathbf{k}(2\pi/c)\mathbf{K} \), the factor of \( 2\pi \) rather than \( 4\pi \) since only a single current sheet is involved. Under Lorentz transformation to a
new inertial frame moving with velocity \( v = c\beta \) along the x-axis, one finds a uniform electric field 
\[
E = \hat{j} \gamma \beta B \cong \pm \hat{j} \gamma (v/c)(2\pi K/c)
\]
Applying this to both plates of the capacitor, we find that in the region between the moving current sheets, there is a net electric field
\[
E = \hat{j} \gamma m \frac{2\pi K}{c^2} \frac{dx}{dt} + \hat{j} \gamma M \frac{2\pi K}{c^2} \frac{dX}{dt}
\]
(28)

Thus inside the flattened solenoid, there is an electromagnetic linear momentum
\[
P = \frac{1}{4\pi c} E \times B L l (X - x)
\]
\[
= \frac{1}{4\pi c} \hat{i} \left( \sqrt{1 - [(dx/dt)/c]^2} \frac{dx}{dt} + \sqrt{1 - [(dX/dt)/c]^2} \frac{dX}{dt} \right) \frac{2\pi K}{c^2} \left( \frac{4\pi K}{c} \right) L l (X - x)
\]
(29)

In the quasi-static limit, we drop the terms in \([(dx/dt)/c]^2\) and rewrite the expression for \(P\) in terms of the constant magnetic flux \(\Phi\), giving
\[
c^2 P = \frac{1}{8\pi} \hat{i} \left( \frac{dx}{dt} + \frac{dX}{dt} \right) \frac{\Phi^2 L}{L (X - x)}
\]
\[
= \frac{1}{8\pi} \frac{\Phi^2 L}{(X - x)^2} \left( (X - x) \frac{dx}{dt} + (X - x) \frac{dX}{dt} \right)
\]
(30)

If the masses \(m\) and \(M\) of the plates supporting the current sheets are regarded as negligible, the position of the center of energy is at the middle of the solenoid volume
\[
U \hat{x} = \hat{i} \left( \frac{1}{8\pi} B^2 L l (X - x) \right) \frac{(x + X)}{2} = \hat{i} \frac{1}{16\pi} \frac{\Phi^2 l (x + X)}{L (X - x)}
\]
(31)

Then the time-rate-of-change of the energy times the center of energy gives
\[
\frac{d}{dt} (U \hat{x}) = \frac{d}{dt} \left( \frac{1}{16\pi} \frac{\Phi^2 l (x + X)}{L (X - x)} \right) = \frac{1}{8\pi} \frac{\Phi^2 l}{L (X - x)^2} \left( X \frac{dx}{dt} - x \frac{dX}{dt} \right)
\]
(32)

The position-weighted power required on the left-hand side of Eq. (14) is
\[
(F_{ext} \cdot \hat{v}) \hat{i} x + F_{ext} \hat{X} \cdot \hat{v} \hat{i} X = \hat{i} \left( \frac{1}{8\pi} \frac{\Phi^2 l}{L (X - x)^2} \frac{dx}{dt} \right) x + \hat{i} \left( -\frac{1}{8\pi} \frac{\Phi^2 l}{L (X - x)^2} \frac{dX}{dt} \right) X
\]
(33)

Now combining Eqs. (30) and (32), we see that the sum of the right-hand sides matches the right-hand side of Eq. (33). Indeed the quasi-static expansion of a solenoid satisfies the relativistic law Eq. (14) for the center of energy.
4. Two Point Charges at Rest

The final quasi-static example involves two point charges, one of mass \( m \) charge \( q \) and the other of mass \( M \) and charge \( Q \), both at rest in some inertial frame. Again the analysis involves both electromagnetic field energy and also electromagnetic field momentum as these charges are displaced quasi-statically from \( \mathbf{r} \) to \( \mathbf{r}' \) and from \( \mathbf{R} \) to \( \mathbf{R}' \) respectively. In the limit of quasi-static motion, there is no radiation emission on changing the electrostatic configuration and so the center-of-energy theorem can be verified exactly.

Here again, the only interesting aspects of the conservation laws involve the energy and the center-of-energy. The external forces needed to move the charges quasi-statically simply balance the electrostatic forces between the charges

\[
\mathbf{F}_{\text{extm}} = \frac{qQ (\mathbf{R} - \mathbf{r})}{|\mathbf{R} - \mathbf{r}|^3} = -\mathbf{F}_{\text{extM}}
\]

while the total energy is the rest-mass energy plus the electrostatic energy

\[
U = mc^2 + Mc^2 + \frac{qQ}{|\mathbf{R} - \mathbf{r}|}
\]

The energy conservation law (10) in the quasi-static limit takes the familiar form

\[
\mathbf{F}_{\text{extm}} \cdot \mathbf{v} + \mathbf{F}_{\text{extM}} \cdot \mathbf{V} = \frac{qQ (\mathbf{R} - \mathbf{r})}{|\mathbf{R} - \mathbf{r}|^3} \cdot (\mathbf{v} - \mathbf{V}) = \frac{d}{dt} \left( \frac{qQ}{|\mathbf{R} - \mathbf{r}|} \right) = \frac{dU}{dt}
\]

where \( \mathbf{v} = d\mathbf{r}/dt \) and \( \mathbf{V} = d\mathbf{R}/dt \), and \( U = qQ/|\mathbf{R} - \mathbf{r}| \) is the electrostatic energy associated with the two point charges. Although this law for energy conservation is familiar, the relativistic law (14) for the center of energy is not. The center of the electromagnetic energy, by symmetry or by direct integration of the interference energy between the point-charge fields \( [1/(8\pi)] \int d^3r \ 2\mathbf{E}_m \cdot \mathbf{E}_M \), is located half-way between the two charges so that the energy times the center of energy is given by

\[
U \bar{X} = mc^2 \mathbf{r} + Mc^2 \mathbf{R} + \frac{qQ}{|\mathbf{R} - \mathbf{r}|} \left( \mathbf{r} + \frac{\mathbf{R}}{2} \right)
\]

The evaluation of the position-weighted power on the left-hand side of (14) involves

\[
(F_{\text{extm}} \cdot \mathbf{v}) \mathbf{r} + (F_{\text{extM}} \cdot \mathbf{V}) \mathbf{R} = \frac{qQ}{|\mathbf{R} - \mathbf{r}|^3} \{ ([\mathbf{R} - \mathbf{r}] \cdot \mathbf{v}) \mathbf{r} - ([\mathbf{R} - \mathbf{r}] \cdot \mathbf{V}) \mathbf{R} \}
\]

In the low-velocity limit appropriate for quasi-static changes, the linear momentum of two point charges is given by the sum of the mechanical linear momentum and the linear momentum in the electromagnetic field

\[
\mathbf{P} \approx m \mathbf{v} + m \mathbf{V} + \frac{qQ}{2c^2 |\mathbf{R} - \mathbf{r}|} \left( \mathbf{v} + \mathbf{V} + \frac{([\mathbf{R} - \mathbf{r}] \cdot \mathbf{v})(\mathbf{R} - \mathbf{r}) + ([\mathbf{R} - \mathbf{r}] \cdot \mathbf{V})(\mathbf{R} - \mathbf{r})}{|\mathbf{R} - \mathbf{r}|^2} \right)
\]
For the quasi-static displacement, the time rate of change of the energy times the center of
energy in (37) is
\[
\frac{d}{dt}(U\vec{X}) = mc^2v + Mc^2V + \frac{qQ}{2|\vec{R} - \vec{r}|}(v + V) - \frac{qQ}{2|\vec{R} - \vec{r}|^3}[(\vec{R} - \vec{r}) \cdot (\vec{V} - \vec{v})](\vec{r} + \vec{R})
\] (40)

Then combining Eqs. (39) and (40), we find
\[
\frac{d}{dt}(U\vec{X}) - c^2p = \frac{qQ}{2|\vec{R} - \vec{r}|} \left( \frac{[(\vec{R} - \vec{r}) \cdot \vec{v}](2\vec{r}) - [(\vec{R} - \vec{r}) \cdot \vec{V}](2\vec{R})}{|\vec{R} - \vec{r}|^2} \right)
\] (41)

This agrees exactly with the position-weighted power expression on the right-hand side of
Eq. (38). Hence indeed the relativistic law for the center of energy is illustrated in this
case; the Coulomb potential between two point charges fits with the low-velocity limit of
electromagnetic theory so as to give continuous motion for the center of energy under quasi-
static displacements by external forces.

**B. Systems Involving Acceleration**

In the examples above, we have tried to illustrate how considerations of momentum and
electromagnetic energy enter into the relativistic law for the center of energy when treating
quasi-static changes of stationary systems. Here we wish to note the role of relativistic
energy and momentum for particles. The simplest example seems to be that discussed
above in Section A2 involving a parallel plate capacitor where now the masses \(m\) and \(M\)
of the plates are no longer treated as negligible and where the external forces providing
a static configuration are removed. In this case, the parallel plates \(m\) and \(M\) of the
solenoid accelerated toward each other under electrostatic attraction. We will verify all
of the conservation laws for the quantities in Eqs. (1)-(4), and we will note just where it
is that the distinction between nonrelativistic and relativistic particle mechanics becomes
important.

The parallel plate capacitor example of Section A2 involves motion along only the \(x\)-axis.
Newton’s equations of motion for the plates along the \(x\)-axis take the form
\[
\frac{dp_m}{dt} = Q\frac{(E + 0)}{2} = \frac{2\pi Q^2}{L^2} = -\frac{dp_M}{dt}
\] (42)

where the electrostatic force on each plate is due to the average field across the plate or is
regarded as due to the electric field due to the other plate. In the approximation of large
parallel plates with small separation, there is no magnetic field present even if the plates are moving with finite velocity, so that there is no electromagnetic linear momentum for the system. Therefore the system linear momentum is simply the mechanical momentum of the particles

\[ \mathbf{P} = \hat{\mathbf{i}} \mathbf{p}_m + \hat{\mathbf{i}} \mathbf{p}_M \]  

(43)

The angular momentum about the origin vanishes

\[ \mathbf{L} = 0 \]  

(44)

since the \( x \)-axis passes through the center of each plate. The energy of the system includes the mechanical particle energies \( U_m \) and \( U_M \) and the energy in the electric field \( \mathbf{E} = \hat{\mathbf{i}} 4\pi Q/L^2 \) between the plates

\[ U = U_m + U_M + U_{em} = U_m + U_M + \frac{1}{8\pi} E^2 L^2 (X - x) = U_m + U_M + \frac{2\pi Q^2 (X - x)}{L^2} \]  

(45)

The energy times the center of energy for the system is

\[ U \bar{\mathbf{X}} = \hat{\mathbf{i}} \left( U_m x + U_M X + U_{em} \frac{x + X}{2} \right) = \hat{\mathbf{i}} \left( U_m x + U_M X + \frac{2\pi Q^2 (X - x) (x + X)}{L^2} \right) \]  

(46)

The conservation laws can easily be verified for this system by using the equations of motion. The system linear momentum is constant in time

\[ \frac{d\mathbf{P}}{dt} = \hat{\mathbf{i}} \frac{dp_m}{dt} + \hat{\mathbf{i}} \frac{dp_M}{dt} = 0 \]  

(47)

as follows from Eq. (42) since the forces on the plates are equal in magnitude and opposite in direction. The system energy is constant in time

\[ \frac{dU}{dt} = \frac{dU_m}{dt} + \frac{dU_M}{dt} + \frac{dU_{em}}{dt} = \left( \frac{dp_m}{dt} - \frac{2\pi Q^2}{L^2} \right) v + \left( \frac{dp_M}{dt} + \frac{2\pi Q^2}{L^2} \right) V = 0 \]  

(48)

as follows from the equations of motion in (42) when multiplied by \( v = dx/dt \) and by \( V = dX/dt \) and then added. Here it is crucial to note that for both nonrelativistic and relativistic particle energy

\[ \frac{dU_{mech}}{dt} = \frac{d\mathbf{P}_{mech}}{dt} \cdot \mathbf{v} \]  

(49)

Thus for the nonrelativistic kinetic energy

\[ \frac{d}{dt} U_{mech-nonrel} = \frac{d}{dt} \left( \frac{1}{2} m v^2 \right) = (m \mathbf{v}) \cdot \frac{d\mathbf{v}}{dt} = \frac{d(\mathbf{mv})}{dt} \cdot \mathbf{v} = \frac{d\mathbf{p}_{mech}}{dt} \cdot \mathbf{v} \]  

(50)
while for the relativistic energy

\[
\frac{d}{dt} U_{\text{mech-rel}} = \frac{d}{dt} \left( \frac{mc^2}{\sqrt{1 - (v/c)^2}} \right) = \frac{m}{\sqrt{1 - (v/c)^2}} \frac{dv}{dt}
\]

(51)

\[
= \frac{d}{dt} \left[ \frac{mv}{\sqrt{1 - (v/c)^2}} \right] \cdot v = \frac{dP_{\text{mech}}}{dt} \cdot v
\]

(52)

The system angular momentum is a constant at \( L = 0 \) for all time. Thus all of the conservation laws treated so far, linear momentum, energy, and angular momentum, have not required the specification of nonrelativistic or relativistic particle mechanics in this electromagnetic system. However, the relativistic law for the center of energy is different; this involves the generator of proper Lorentz transformations and it requires a fully relativistic treatment. Thus the time-rate-of-change of the system energy times the center of energy follows from Eqs. (46) and (49) as

\[
\frac{d}{dt} (U \vec{X}) = \hat{i} \left( \frac{dp_m}{dt} v_x + \frac{dp_M}{dt} VX + U_m v + U_M V + \frac{2\pi Q^2}{L^2} (XV - xv) \right)
\]

\[
= \hat{i} \left( U_m v + U_M V + \left[ \frac{dp_m}{dt} - \frac{2\pi Q^2}{L^2} \right] v_x + \left[ \frac{dp_M}{dt} + \frac{2\pi Q^2}{L^2} \right] VX \right)
\]

\[
= \hat{i} (U_m v + U_M V)
\]

(53)

where the terms in square brackets vanish because of the equations of motion in (42). We obtain the correct relativistic law (7) only provided

\[
\frac{d}{dt} (U \vec{X}) = U_m v + U_M V = c^2 P
\]

(54)

However, as we see from Eq. (43) this requires that \( U_m v = c^2 p_m \) and \( U_M V = c^2 p_M \). This is not true for nonrelativistic particle energy and momentum. It is true only for the relativistic mechanical energy and momentum where

\[
U_{\text{mech-rel}} = \frac{mc^2}{\sqrt{1 - (v/c)^2}} \quad \text{and} \quad p_{\text{mech-rel}} = \frac{mv}{\sqrt{1 - (v/c)^2}}
\]

(55)

Thus provided that we use the exact relativistic expressions for particle energy and momentum as well as the exact results of electromagnetic theory, the relativistic center-of-energy law (7) is indeed satisfied for a parallel-plate capacitor where there are no external forces present and the plates are free to accelerate. We notice that the contributions from both the relativistic mechanical energy and the electromagnetic energy are absolutely necessary for the validity of the center-of-energy law.
It might seem that the other examples involving a flattened slip-joint solenoid and two charged particles can be carried over to the situation allowing accelerations when no external forces are present. However, these extensions fail because the electromagnetic behavior is not correctly treated for situations of finite velocity and acceleration. Although the electromagnetic field expressions for a capacitor, in the large-plate-small-separation approximation, do not change at finite velocity and acceleration, this is not true for the a flattened slip-joint solenoid or point charges. The expressions used in these quasi-static analyses are valid only in the low-velocity limit and can be extended to the situation of accelerating particles only in this low-velocity limit. The complications involved are clearly evident in the case of two charged particles. The Darwin Lagrangian\(^\text{[10]}\) correctly describes the interaction of point charges through order \(v^2/c^2\). Even in this order, the particle equations of motion, can be exceedingly complex,\(^\text{[11]}\) and beyond this order one requires the full Maxwell’s equations to describe the electromagnetic field. These situations involve radiation emission and do not seem to lend themselves to simple examples.

IV. ILLUSTRATING THE CENTER OF ENERGY LAW IN OTHER INERTIAL FRAMES

Since the energy times the center of energy is the generator of proper Lorentz transformations, it is natural to wish to see the form taken by the examples in various inertial frames. The example involving the acceleration of the capacitor plates retains its form under any Lorentz transformation along the \(x\)-axis. Indeed, the electric field between the plates remains \(E = \hat{\mathbf{i}}4\pi Q/L^2\) in any such inertial frame and the expressions for the mechanical energy and momentum are unchanged so that the entire analysis is identical for any such Lorentz-transformed frame. However, if a Lorentz transformation is made in another direction, then the situation becomes distinctly more complicated. The parallel plate capacitor requires forces of constraint for its stability. Provided these forces of constraint do no work in a Lorentz-transformed frame, they will not disrupt the conservation laws, just as they did not in our calculations above. However, in any inertial frame where the forces of constraint do work, there must be a flow of energy, and hence also of momentum, which invalidate any conservation laws which do not take account of these flows.\(^\text{[12]}\) The parallel plates in our examples have finite extent and therefore must have forces of constraint in the
and $z$-directions, which forces prevent the charged plates from flying apart. Thus our conservation analysis will hold in any inertial frame moving with finite velocity in the $x$-direction since the forces in the $y$- and $z$-directions do no work. The examples involving the flattened slip-joint solenoid and two charged particles relatively at rest also require forces of constraint which must be analyzed carefully. [13]

**V. DISCUSSION**

Nonrelativistic mechanics is invariant under the group of Galilean transformations. Electrodynamics is invariant under the Poincare group. However, often nonrelativistic particle mechanics is joined with Maxwell’s electromagnetic theory in describing physical phenomena. Indeed, in classes where the elementary examples above are assigned as homework, students invariably use nonrelativistic equations of particle motion unless explicitly required to calculate with the relativistic forms. Using nonrelativistic equations of particle motion, students have no trouble with the conservation laws for linear momentum, angular momentum, and energy. Clearly both nonrelativistic particle dynamics and electromagnetism contain these conservation laws, and the examples involve simply the transfer of these quantities from one system to the other through forces. It is only in the invariance of the velocity of the center of energy that we become aware that Poincare invariance enforces strong restrictions on the theory. Nonrelativistic particle equations of motion fail to yield the invariant motion of the center of energy when electromagnetic energy and particle kinetic energy are included.

The three examples of the relativistic conservation laws which we have given here all involve classical electromagnetism which is invariant under the Poincare group. The example of the accelerating plates of a parallel-plate capacitor illustrates that mixtures of nonrelativistic and relativistic physics still lead to the conservation laws for linear momentum, angular momentum, and energy while only fully relativistic systems satisfy the law for the center of energy. Calculation of the center-of-energy motion forces us to notice the distinction between relativistic physics and the alternatives. Within relativistic physics, it is not at all clear that particles can interact through any arbitrary potential function $V(|\mathbf{r} - \mathbf{R}|)$; the $1/r$ Coulomb or Kepler potential appears as part of the relativistic theories of electromagnetism and gravitation. Indeed it seems fascinating that the generator of the O(4) symmetry associated with the Runge-Lenz vector of the nonrelativistic $1/r$ Kepler problem [14] is precisely
the nonrelativistic limit of the generator $U\vec{\Lambda}$ for proper Lorentz transformations obtained from the Darwin Lagrangian for the $v^2/c^2$-interaction of two charged particles.\[15\]

The relativistic conservation laws associated with Poincare invariance require the use of relativistic physics for both the interactions and the mechanical energy and momentum.\[16\] However, both the textbook and research literature in physics contain many examples where nonrelativistic and relativistic aspects are mixed together. This arrangement maintains the conservation laws of linear momentum, energy, and angular momentum, but not the relativistic law for the center of energy. For the most part this does not lead to significant difficulties in one-step calculations when the particle mechanics is taken as nonrelativistic in the presence of fixed electromagnetic fields and only the particle motion is of interest.\[17\] However, there are multi-step calculations where the charged particles respond with nonrelativistic motion to electromagnetic fields and in turn the electromagnetic fields arising from the nonrelativistically-moving particles are of interest; these calculations lead to questionable conclusions. Thus, for example, the Aharonov-Bohm phase shift involves the $v^2/c^2$-interaction of a point charge and a solenoid; yet the response of the solenoid to the charged particle’s fields is often treated using nonrelativistic physics.\[18\] One instance of the paradoxical and erroneous descriptions which can arise from such a treatment of the charged particle-solenoid interaction is discussed by Coleman and Van Vleck;\[1\] other discussions have also been given.\[19\] A second example involves the scattering of random classical radiation by a mechanical scatterer so as to obtain the equilibrium spectrum corresponding to thermal (blackbody) radiation. It is common practice\[20\] to use nonrelativistic mechanical behavior for the scattering charges despite the fact that the electromagnetic fields arising from the nonrelativisticly-moving particles are of crucial interest in obtaining radiation equilibrium. In some instances,\[21\] relativistic particle mechanics has been combined with nonrelativistic potential functions in an attempt to discuss classical radiation equilibrium. In all these instances, the relativistic center-of-energy law is violated because the systems do not satisfy Poincare invariance. Yet relativistic transformations are clearly crucial in understanding blackbody radiation since the Planck spectrum can be obtained by Lorentz transformations associated with uniform (proper) acceleration through Lorentz-invariant zero-point radiation.\[22\]

In this article we have given several elementary examples of the relativistic center-of-energy law. There are very few examples of the law presented in the textbook literature
and the relativistic restrictions associated with the law seem to be unnoticed in some of the physics research literature. Thus a century after Einstein’s striking work on special relativity, there are still elementary aspects of Lorentz invariance which go unmentioned in the textbooks and unappreciated in the research literature.

[1] S. Coleman and J. H. Van Vleck, "Origin of ‘Hidden Momentum Forces’ on Magnets," Phys. Rev. 171, 1370-1375 (1968).

[2] Perhaps in part because there are so few simple examples involving the center of energy, there is a variety of terminology in the literature. Here we have chosen to speak of the "center of energy," following the usage of Coleman and Van Vleck in Ref. 1. However, E. F. Taylor and J. A. Wheeler in Spacetime Physics (Freeman, San Francisco, 1966), p. 143, speak of the "center of mass" with the understanding that "mass" means "mass-energy" as befits a relativistic theory. L. D. Landau and E. M. Lifshitz in The Classical Theory of Fields, 4th ed. (Pergamon, New York, 1985), p. 168, speak of the "center of inertia." J. L. Anderson, "Principles of Relativity Physics" (Academic Press, New York 1967), p. 207-208 writes, "We can call \( Z \) the center of energy of the system of particles, in analogy to the center of mass as defined in Newtonian mechanics. According to Eq.(7-5.7), this point moves like a free particle with a velocity \( V \), given by Eq. (7-5.9) \( P_\mu P^\mu = M^2 + 0 \), we can always perform a mapping so that \( P_\mu = (M, 0) \). The corresponding reference frame is called variously, the center of mass, or center of momentum, or center of energy frame. We prefer the latter terminology. In this frame \( V = 0 \)." All of these designations mean the same thing. Here the "center of energy" terminology has been used so as to impress upon the reader that this is a change in point of view from the nonrelativistic "center of (rest) mass" concept where there is no role for electromagnetic energy or even particle kinetic energy and only particle rest masses are involved.

[3] See, for example, D. J. Griffiths, Introduction to Electrodynamics, 3rd ed. (Prentice Hall, Upper Saddle River, NJ. 1999). Chapter 8 is devoted to conservation laws in electromagnetic theory— including conservation of charge, energy, linear momentum, and angular momentum. There is no mention of the invariant motion of the center of energy. Indeed, it was Griffiths’ clear organization of the conservation laws which made me acutely aware of the absence of
the last conservation law of Poincare invariance.

[4] I am not aware of any simple examples of the center-of-energy theorem in electromagnetism textbooks. L. D. Landau and E. M. Lifshitz in The Classical Theory of Fields, 2nd ed. (Pergamon, Oxford, 1962), p. 194, give as a problem the determination of the "center of inertia" for a collection of interacting point charges. This problem is repeated on page 168 of the 4th edition listed in Ref. 2 above. J. D. Jackson in Classical Electrodynamics 2nd ed. (Wiley, New York, 1975), p. 617, has problem 12.16 to derive the uniform motion of the "center of mass" for an arbitrary, localized distribution of source-free electromagnetic fields. The question is repeated as problem 12.19 in the 3rd edition of 1999.

[5] The basic idea appears in the early work of A. Einstein, "Prinzip von der Erhaltung der Schw- erpunktsbewegung und die Tr¨ agheit der Energie," Annalen der Physik 20, 626-633 (1906). It is also given by E. Bessel-Hagen, "¨Uber die Erhaltungss¨ atze der Elektrodynamick," Mathematisch Annalen 84, 259-276 (1921). Bessel-Hagen analyzes the conservation laws associated with the conformal group satisfied by Maxwell's equations.

[6] See any standard text on electromagnetic theory; for example, Griffiths' Section 8.1.2 "Poynting's Theorem" in Ref. 3.

[7] I am not aware of any place where the center-of-energy law with external forces appears in the physics literature.

[8] The forces on each plate can be regarded as due to the average electric field across the plate, or as due to the electric field of the other plate, or as due to the pressure of the electromagnetic field. (See, for example, D. J. Griffiths in Ref. 3, p. 102, Eq. (2.50), or E. M. Purcell, Electricity and Magnetism, 2nd ed. (McGraw-Hill, New York, 1985), pp. 30 and 31.)

[9] See, for example, T. H. Boyer, "Electric and magnetic forces and energies for a parallel-plate capacitor and a flattened, slip-joint solenoid," Am. J. Phys. 69, 1277-1279 (2001).

[10] See, for example, J. D. Jackson, Classical Electrodynamics, 2nd ed. (Wiley, New York 1975), Section 12.7 "Lowest-Order Relativistic Corrections to the Lagrangian for Interacting Charged Particles, the Darwin Lagrangian."

[11] See, for example, the fields given by L. Page and N. I. Adams, "Action and reaction between moving charges," Am. J. Phys. 13, 141-147 (1945). These electromagnetic fields follow from the Darwin Lagrangian.

[12] See, for example, the discussion in the introduction of the article by T. H. Boyer, "Example
of mass-energy relation: Classical hydrogen atom accelerated or supported in a gravitational
field,” Am. J. Phys. 66, 872-876 (1998).

[13] See, for example, T. H. Boyer, ”Lorentz-transformation properties for energy and momentum
in electromagnetic systems,” Am. J. Phys. 53, 167-171 (1985).

[14] See, for example, H. Goldstein, Classical Mechanics 2nd ed. (Addison-Wesley, Reading, Mas-
sachusetts 1980), Section 3-9 ”The Laplace-Runge-Lenz Vector.”

[15] J. P. Dahl, ”Physical origin of the Runge-Lenz vector,” J. Phys. A: Math. Gen. 30, 6831-6840
(1997).

[16] F. Rohrlich in Classical Charged Particles (Addison-Wesley, Reading, MA 1965), p. 210,
emphasizes that the combination of nonrelativistic particle mechanics and electromagnetic
fields is ”inconsistent” in the sense that the combination satisfies neither Galilean invariance
nor Lorentz invariance.

[17] See, for example, Ref. 3, Example 5.2, where Griffiths discusses ”Cycloid Motion” of a non-
relativistic charged particle in electric and magnetic fields.

[18] See, for example, the calculations by M. Peshkin, I. Talmi, and L. J. Tassie, ”The Quantum
Mechanical Effects of Magnetic Fields Confined to Inaccessible Regions,” Ann. Phys. (N.Y.)
12, 426-435 (1961), especially Section V, ”A Mechanical Model.”

[19] See, for example, T. H. Boyer, ”Classical Electromagnetic Interaction of a Point Charge and
a Magnetic Moment: Considerations Related to the Aharonov-Bohm Phase Shift,” Found.
Phys. 32, 1-39 (2002).

[20] See, for example, J. H. Van Vleck, ”The Absorption of Radiation by Multiply Periodic Or-
bits, and its Relation to the Correspondence Principle and the Rayleigh-Jeans Law. Part II
Calculation of Absorption by Multiply Periodic Orbits,” Phys. Rev. 24, 347-365 (1924) and
T. H. Boyer, ”Equilibrium of random classical electromagnetic radiation in the presence of a
nonrelativistic nonlinear electric dipole oscillator,” Phys. Rev. 13, 2832-2845 (1976).

[21] See R. Blanco, L. Pesquera, and E. Santos, ”Equilibrium between radiation and matter for
classical relativistic multiperiodic systems. Derivation of Maxwell-Boltzmann distribution from
Rayleigh-Jeans spectrum,” Phys. Rev. D 27, 1254-1287 (1983); ”Equilibrium between radia-
tion and matter for classical relativistic multiperiodic systems II. Study of radiative equilib-
rium with Rayleigh-Jeans radiation,” Phys. Rev. D 29, 2240-2254 (1984).

[22] P. C. W. Davies, “Scalar particle production in Schwarzschild and Rindler metrics,” J. Phys.
A 8, 609-616 (1975); W. G. Unruh, “Notes on black-hole evaporation,” Phys. Rev. D 14, 871-892 (1976); T. H. Boyer, “Thermal effects of acceleration for a classical dipole oscillator in classical electromagnetic zero-point radiation,” Phys. Rev. D 29, 1089–1095 (1984); D. C. Cole, “Properties of a classical charged harmonic oscillator accelerated through classical electromagnetic zero-point radiation,” Phys. Rev. D 31, 1972–1981 (1985).