Using Auxiliary Information More Efficiently in Population Variance Estimation - A New Family of Estimators

Kalim Ullah\(^1\), Zawar Hussain\(^1\), and Salman Arif Cheema\(^2\)
\(^1\)Department of Statistics, Quaid-i-Azam University Islamabad, Pakistan.
\(^2\)School of Mathematical and Physical Sciences, University of Newcastle, Australia.
Corresponding Authors: kailimullah@stat.qau.edu.pk

Abstract
In this article, we have suggested estimation of variance in finite population by using known values of parameter related to auxiliary information such as rank and second raw moment of auxiliary variable in stratified random sampling. The expression for the bias and mean squared error (MSE) of the suggested estimator are obtained up to first order of approximation. The proposed estimator is efficient comparatively various other estimators. A numerical and theoretical study are performed to support the suggested estimator.

Keywords: Bias, Mean squared error, Auxiliary variable, Ranks of auxiliary variable, Percentage relative efficiency. 1.5

Introduction
The estimation of finite population mean is always a laborious process due to limited resources and sensitive nature of the study variables. In literature various studies are available to estimate the finite population by the significant use of the auxiliary information.

Notations
Consider a finite population of size \(N\) denoted by \(\delta = (\delta_1, \delta_2, ..., \delta_N)\). Further, let us assume that the population consist of \(L\) homogeneous partitions (strata), each of size \(N_h\) where \(h = \{1,2, ..., L\}\), such that \(\sum_{h=1}^{L} N_h = N\). For the purpose of consistency, we define \(Y\), \(X\), \(R\) and \(U\) be the study variable, auxiliary variable, ranks and squared values of auxiliary variable taking values \(Yih\), \(Xih\), \(Rih\) and \(Uih\) respectively, on the \(ith\) unit belongs to the \(hth\) stratum, where \(i = \{1,2, ..., N_h\}\). Thus \(W_h = N_h/N\) stays as the weight of \(hth\) stratum We then draw a sample of size \(nh\) from the \(hth\) stratum using SRS without replacement scheme for the estimation of population variance ensuring that the total sample size \(n = \sum_{h=1}^{L} n_h\).

We now define population mean of study variable as \(\bar{Y}_{st} = \bar{Y} = \sum_{h=1}^{L} W_h \bar{Y}_h\) where
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population mean of $Y$ for $h$th stratum is $\bar{Y}_h = \sum_{i=1}^{N_h} Y_{ih}/N_h$. Similarly $\bar{X}_{st} = \bar{X} = \sum_{h=1}^{L} W_h \bar{X}_h$ and $\bar{R}_h = \sum_{i=1}^{N_h} R_{ih}/N_h$ are the population mean of auxiliary variable and population mean of auxiliary in $h$th stratum, respectively. Furthermore, $\bar{R}_{st} = \bar{R} = \sum_{h=1}^{L} W_h \bar{R}_h$ and $\bar{U}_h = \sum_{i=1}^{N_h} U_{ih}/N_h$ represent the population mean of ranks and mean of ranks of $h$th stratum along with $\tilde{U}_{st} = \tilde{U} = \sum_{h=1}^{L} W_h \tilde{U}_h$ and $\tilde{U}_h = \sum_{i=1}^{N_h} U_{ih}/n_h$ define as population mean of squared values of auxiliary variable and population mean of squared values in $h$th stratum, respectively. Their corresponding sample estimate are given as $s^2_{y(h)} = \hat{\lambda} = \sum_{h=1}^{L} W_h \hat{\lambda}$.

$$\tilde{y}_h = \sum_{i=1}^{n_h} Y_{ih}/n_h, \quad \tilde{x}_h = \sum_{i=1}^{n_h} X_{ih}/n_h, \quad \tilde{r}_h = \sum_{i=1}^{n_h} R_{ih}/n_h, \quad \tilde{u}_h = \sum_{i=1}^{n_h} U_{ih}/n_h.$$ Next, we define expression of population variances within stratum such that $s^2_{x,h} = \sum_{i=1}^{n_h} \left( Y_{ih} - \bar{Y}_{h} \right)^2/(N_h - 1)$, $s^2_{r,h} = \sum_{i=1}^{n_h} \left( R_{ih} - \bar{R}_{h} \right)^2/(N_h - 1)$, $s^2_{u,h} = \sum_{i=1}^{n_h} \left( U_{ih} - \bar{U}_{h} \right)^2/(N_h - 1)$, where as, covariances are given as $s_{x,y,h} = \sum_{i=1}^{n_h} \left( Y_{ih} - \bar{Y}_{h} \right) \left( X_{ih} - \bar{X}_{h} \right)/(N_h - 1)$, $s_{y,r,h} = \sum_{i=1}^{n_h} \left( Y_{ih} - \bar{Y}_{h} \right) \left( R_{ih} - \bar{R}_{h} \right)/(N_h - 1)$, $s_{y,u,h} = \sum_{i=1}^{n_h} \left( Y_{ih} - \bar{Y}_{h} \right) \left( U_{ih} - \bar{U}_{h} \right)/(N_h - 1)$ and $s_{r,u,h} = \sum_{i=1}^{n_h} \left( R_{ih} - \bar{R}_{h} \right) \left( U_{ih} - \bar{U}_{h} \right)/(N_h - 1)$.

Based on above provided expressions, we now provide correlation coefficients when stratified sampling scheme is used, such as

$$\rho_{yx(st)} = \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{y,x,h} s_{y,h} s_{x,h} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda s_{y,h}^2 \sum_{h=1}^{L} W_h^2 \lambda s_{x,h}^2}}{\sum_{h=1}^{L} W_h^2 \lambda s_{y,h}^2},$$

$$\rho_{yr(st)} = \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{y,r,h} s_{y,h} s_{r,h} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda s_{y,h}^2 \sum_{h=1}^{L} W_h^2 \lambda s_{r,h}^2}}{\sum_{h=1}^{L} W_h^2 \lambda s_{y,h}^2},$$

$$\rho_{yu(st)} = \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{y,u,h} s_{y,h} s_{u,h} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda s_{y,h}^2 \sum_{h=1}^{L} W_h^2 \lambda s_{u,h}^2}}{\sum_{h=1}^{L} W_h^2 \lambda s_{y,h}^2}$$ and

$$\rho_{rx(st)} = \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{r,x,h} s_{r,h} s_{x,h} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda s_{r,h}^2 \sum_{h=1}^{L} W_h^2 \lambda s_{x,h}^2}}{\sum_{h=1}^{L} W_h^2 \lambda s_{r,h}^2},$$

$$\rho_{ru(st)} = \frac{\sum_{h=1}^{L} W_h^2 \lambda \rho_{r,u,h} s_{r,h} s_{u,h} \sqrt{\sum_{h=1}^{L} W_h^2 \lambda s_{r,h}^2 \sum_{h=1}^{L} W_h^2 \lambda s_{u,h}^2}}{\sum_{h=1}^{L} W_h^2 \lambda s_{r,h}^2}.$$ 

For further mathematical proceeding, the relative error terms are define as

$$e_{0,h} = \left( s_{y,h}^2 - s_{y(h)}^2 \right) / s_{y(h)}^2, \quad e_{1,h} = \left( s_{x,h}^2 - s_{x(h)}^2 \right) / s_{x(h)}^2, \quad e_{2,h} = \left( s_{r,h}^2 - s_{r(h)}^2 \right) / s_{r(h)}^2, \quad e_{3,h} = \left( s_{u,h}^2 - s_{u(h)}^2 \right) / s_{u(h)}^2$$ and $y_h = 1/n_h$. 

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Moreover, for \( i = 0,1,2,3 \), \( E(e_{ih}) = 0 \).

where as \( E(e_{0i}^2) = \gamma_h \beta_{2y(h)}^* \), \( E(e_{1i}^2) = \gamma_h \beta_{2x(h)}^* \), \( E(e_{2i}^2) = \gamma_h \beta_{2r(h)}^* \), \( E(e_{3i}^2) = \gamma_h \beta_{2u(h)}^* \), \( E(e_{0i}e_{1h}) = \gamma_h \lambda_{2200(h)}^* \), \( E(e_{0i}e_{2h}) = \gamma_h \lambda_{2020(h)}^* \), \( E(e_{0i}e_{3h}) = \gamma_h \lambda_{2002(h)}^* \), \( E(e_{1i}e_{2h}) = \gamma_h \lambda_{0220(h)}^* \), \( E(e_{1i}e_{3h}) = \gamma_h \lambda_{0202(h)}^* \), \( E(e_{2i}e_{3h}) = \gamma_h \lambda_{0022(h)}^* \).

Where \( \beta_{2y(h)} = \frac{\mu_{0000(h)}}{\mu_{0002(h)}} \), \( \beta_{2x(h)} = \frac{\mu_{0000(h)}}{\mu_{0022(h)}} \), \( \beta_{2r(h)} = \frac{\mu_{0004(h)}}{\mu_{0002(h)}} \), \( \beta_{2u(h)} = \frac{\mu_{0004(h)}}{\mu_{0002(h)}} \),
are the coefficient of kurtosis of \( y \), \( x \), \( r \) and \( u \) respectively.

Let \( \beta_{2y(h)}^* = \beta_{2y(h)} - 1 \), \( \beta_{2x(h)}^* = \beta_{2x(h)} - 1 \), \( \beta_{2r(h)}^* = \beta_{2r(h)} - 1 \), \( \beta_{2u(h)}^* = \beta_{2u(h)} - 1 \), \( \lambda_{2200(h)}^* = \lambda_{2200(h)} - 1 \), \( \lambda_{2020(h)}^* = \lambda_{2020(h)} - 1 \), \( \lambda_{2002(h)}^* = \lambda_{2002(h)} - 1 \), \( \lambda_{0220(h)}^* = \lambda_{0220(h)} - 1 \), \( \lambda_{0202(h)}^* = \lambda_{0202(h)} - 1 \), \( \lambda_{0022(h)}^* = \lambda_{0022(h)} - 1 \).

Where \( \lambda_{pqrs(h)} = \frac{\mu_{pqrs(h)}}{\mu_{0002(h)}^{1/2} \mu_{0000(h)}^{1/2} \mu_{0200(h)}^{1/2} \mu_{0022(h)}^{1/2}} \), \( \mu_{pqrs(h)} = \frac{1}{\sqrt{N_h}} \sum_{i=1}^{N_h} (X_{hi} - \bar{X})^p (Y_{hi} - \bar{Y})^q (U_{hi} - \bar{U})^r (R_{hi} - \bar{R})^s \).

\[
\rho_{s_{y(h)}^2 s_{x(h)}^2} = \frac{\lambda_{2200(h)}^*}{\beta_{2y(h)}^* \beta_{2x(h)}^*}, \quad \rho_{s_{y(h)}^2 s_{r(h)}^2} = \frac{\lambda_{2020(h)}^*}{\beta_{2y(h)}^* \beta_{2r(h)}^*}, \quad \rho_{s_{y(h)}^2 s_{u(h)}^2} = \frac{\lambda_{2002(h)}^*}{\beta_{2y(h)}^* \beta_{2u(h)}^*}
\]

\[
\rho_{s_{x(h)}^2 s_{r(h)}^2} = \frac{\lambda_{0220(h)}^*}{\beta_{2x(h)}^* \beta_{2r(h)}^*}, \quad \rho_{s_{x(h)}^2 s_{u(h)}^2} = \frac{\lambda_{0202(h)}^*}{\beta_{2x(h)}^* \beta_{2u(h)}^*}, \quad \rho_{s_{r(h)}^2 s_{u(h)}^2} = \frac{\lambda_{0022(h)}^*}{\beta_{2r(h)}^* \beta_{2u(h)}^*}
\]

see Garcia and Cebrian (1996)

**Existing Estimators of Variance in Stratified Random Sampling (VStRS)**

We consider the existing estimators for the population variance \( s_{y(h)}^2 \). Recall that the variance of the conventional unbiased variance estimator \( s_{y}^2 \) (the sample variance), the variance is given by:

1. \( \text{Var}(\hat{s}_{y(h)}^2) = \sum_{h=1}^{H} W_h^4 \gamma_h^2 s_{y(h)}^4 \beta_{2y(h)}^* \).

2. The Isaki (1983) introduced the following ratio estimator for population variance is given by:

\[
\hat{s}_{R(h)}^2 = s_{y(h)}^2 \left( \frac{s_{y(h)}^2}{s_{x(h)}^2} \right)
\]

The MSE of \( \hat{s}_{R(h)}^2 \) up to first degree of approximation, is defined as:

\[
\text{MSE}(\hat{s}_{R(h)}^2) = \sum_{h=1}^{H} W_h^4 \gamma_h^2 s_{y(h)}^4 \left[ \beta_{2y(h)}^* + \beta_{2x(h)}^* - 2 \lambda_{2200(h)}^* \right].
\]

3. The traditional regression estimator defined by Isaki (1983) \( \hat{s}_{Reg(h)}^2 \) is given by

\[
\hat{s}_{Reg(h)}^2 = s_{y(h)}^2 + b_h \left( s_{x(h)}^2 - s_{x(h)}^2 \right)
\]

Where \( b_h \) is the sample regression coefficient of \( y_h \) on \( x_h \) with corresponding population regression coefficient given by \( \beta_h = \frac{s_{y(h)}^2 \lambda_{2200(h)}^*}{s_{x(h)}^2 \beta_{2x(h)}^*} \) in stratum \( h \). The variance of \( \hat{s}_{Reg(h)}^2 \), to the
first degree of approximation, is given by
\[ \text{Var}(\hat{S}_{\text{Reg}(st)}) = \sum_{h=1}^{k} W_h Y_h^{-2} S_{Y(h)}^2 \beta_{2y(h)}^2 \left( 1 - \beta_{2y(h)}^2 \right). \] (5)

4. Singh (1988) difference estimator is given by
\[ \hat{S}_{D(st)}^2 = w_1 S_{Y(h)}^2 + w_2 (S_{X(h)}^2 - S_{X(h)}^2), \] (6)
where \( w_1 \) and \( w_2 \) are suitable constants having optimum values are
\[ w_1(\text{opt}) = \frac{1}{1 + \gamma_h \beta_{2y(h)}^2 (1 - \beta_{2y(h)}^2 S_{X(h)}^2)} \] and \( w_2(\text{opt}) = w_1(\text{opt}) \frac{S_{Y(h)}^2 \beta_{2y(h)}^2}{S_{X(h)}^2 \beta_{y(h)}^2}. \)

The MSE of \( \hat{S}_{D(st)}^2 \), to the first degree of approximation is given by
\[ \text{MSE}_{\text{min}}(\hat{S}_{D(st)}^2) \equiv \sum_{h=1}^{k} W_h Y_h^4 \frac{S_{Y(h)}^2}{1 + \gamma_h \beta_{2y(h)}^2 (1 - \beta_{2y(h)}^2 S_{X(h)}^2)}. \] (7)

5. The Bahl and Tuteja (1991) exponential ratio estimator is given by
\[ \hat{S}_{\text{BT,R}(st)}^2 = S_{Y(h)}(\exp \left( \frac{S_{X(h)}^2 - S_{X(h)}^2}{S_{X(h)}^2 + S_{X(h)}^2} \right). \] (8)

The MSE of \( \hat{S}_{\text{BT,R}(st)}^2 \), up to first degree of approximation, is given by
\[ \text{MSE}(\hat{S}_{\text{BT,R}(st)}^2) \equiv \sum_{h=1}^{k} W_h Y_h^4 \frac{S_{Y(h)}^2}{1 + \gamma_h \beta_{2y(h)}^2 (1 - \beta_{2y(h)}^2 S_{X(h)}^2)}. \] (9)

6. Shabbir and Gupta (2007) proposed the following estimator for \( \hat{S}_{SG(st)}^2 \) is given by
\[ \hat{S}_{SG(st)}^2 = (w_3 S_{Y(h)}^2 + w_4 (S_{X(h)}^2 - S_{X(h)}^2) \exp \left( \frac{S_{X(h)}^2 - S_{X(h)}^2}{S_{X(h)}^2 + S_{X(h)}^2} \right). \]
where \( w_3 \) and \( w_4 \) are suitably chosen constants having optimum values given by
\[ w_3(\text{opt}) = \frac{\gamma_h \beta_{2x(h)}^2}{8 - \gamma_h \beta_{2x(h)}^2 + \gamma_h \beta_{2y(h)}^2 \beta_{2x(h)}^2 - \lambda_{2200(h)}^2} \] and
\[ w_4(\text{opt}) = \frac{\gamma_h \beta_{2x(h)}^2}{8 S_{X(h)}^2} \left[ 8 \beta_{2x(h)}^2 + 8 \beta_{2y(h)}^2 S_{X(h)}^2 - \gamma_h \beta_{2x(h)}^2 + \gamma_h \beta_{2y(h)}^2 \beta_{2x(h)}^2 - \lambda_{2200(h)}^2. \]
The MSE of \( \hat{S}_{SG(st)}^2 \), up to first degree of approximation, is given by
\[ \text{MSE}(\hat{S}_{SG(st)}^2) = \sum_{h=1}^{k} W_h Y_h^4 \frac{S_{Y(h)}^2}{64} \left[ \frac{\gamma_h (-\beta_{2x(h)}^2 - 16 \beta_{2y(h)}^2 (1 - \beta_{2y(h)}^2 (1 + \beta_{2y(h)}^2) \beta_{2y(h)}^2 - 4))}{1 + \gamma_h \beta_{2y(h)}^2 (1 - \beta_{2y(h)}^2 (1 + \beta_{2y(h)}^2) \beta_{2y(h)}^2 - 4))} \right]. \] (10)

7. Singh (2014a) proposed an improved estimator for \( \hat{S}_{SM(st)}^2 \), is given by
\[ \hat{S}_{SM(st)}^2 = S_{Y(h)}^2 \left[ w_5 + w_6 (S_{X(h)}^2 - S_{X(h)}^2) \right] \exp \left\{ \frac{a S_{X(h)}^2 - S_{X(h)}^2}{a S_{X(h)}^2 + S_{X(h)}^2 + b} \right\}, \] (11)
where \( w_5 \) and \( w_6 \) are suitably constants, \( \Psi \) takes values +1 and −1 for ratio and product estimators and \( a, b \) be the known population parameters of the auxiliary variables. The minimum MSE of \( \hat{S}_{SM(st)}^2 \), up to first degree of approximation, at optimum values \( g = \Psi = 1 \),
\[ w_5(\text{opt}) = \frac{1}{4} \left[ \frac{-12 \beta_{2x(h)}^2 \lambda_{2200(h)}^2 + 3 \beta_{2x(h)}^2 \lambda_{2200(h)}^2 - 8 \beta_{2x(h)}^2 - 2 \beta_{2x(h)}^2}{\beta_{2x(h)}^2 - 4 \beta_{2x(h)}^2 \lambda_{2200(h)}^2 + 8 \beta_{2x(h)}^2 \lambda_{2200(h)}^2 - 2 \beta_{2x(h)}^2 - 2 \beta_{2x(h)}^2} \right]. \]
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and

\[ w_{6(\text{opt})} = \frac{1}{4 S^2_x(h)} \left[ -6 \beta^*_{2x(h)} \lambda^*_{2200(h)} + \beta^*_{2x(h)} + 8 \lambda^*_{2200(h)} - 4 \beta^*_{2x(h)} + 8 \lambda^*_{2200(h)} - 8 \beta^*_{2x(h)} \lambda^*_{2200(h)} + 4 \beta^*_{2x(h)} \beta^*_{2y(h)} \right] \]

is given

\[
\text{MSE}(\hat{S}^2_{\text{SM}(st)})_{\text{min}} = \sum_{h=1}^{L} W^4_h Y^3_h \left[ \frac{\beta^*_{2x(h)} \left[ \frac{\beta^*_{2x(h)}(\beta^*_{2x(h)} + 8 \lambda^*_{2200(h)} + 16(\beta^*_{2x(h)} - 4 \text{Var}(\hat{S}^2_{\text{Reg}(st)}))) + 16 \lambda^*_{2200(h)} \lambda^*_{2200(h)} \beta^*_{2x(h)})}{-\beta^*_{2x(h)}(1 + \beta^*_{2y(h)} + 2 \lambda^*_{2200(h)}) + 4 \lambda^*_{2200(h)}} \right]}{64} \right]^{(12)}
\]

8. Yaqub and Shabbir (2016) proposed an improved class of estimators for \( \hat{S}^2_{\text{YS}(st)} \), is given by

\[
\hat{S}^2_{\text{YS}(st)} = S^2_Y(w_7 + w_9(S^2_{x(h)} - S^2_{x(h)})) \left[ \frac{a(S^2_{x(h)} - S^2_{x(h)})}{a(S^2_{x(h)} + S^2_{x(h)}) + 2b} \right]^{\frac{1}{2}} + \text{exp} \left( \frac{a(S^2_{x(h)} - S^2_{x(h)})}{a(S^2_{x(h)} + S^2_{x(h)}) + 2b} \right) \right]^{(13)}
\]

where \( w_7 \) and \( w_9 \) are suitably constants and \( a \) and \( b \) be the known population parameters of the auxiliary variables. The minimum MSE of \( \hat{S}^2_{\text{YS}(st)} \), up to first degree of approximation, at optimum values \( a = 1, b = 0 \),

\[
w_{9(\text{opt})} = \frac{\beta^*_{2x(h)}}{2} \left[ \frac{1 + 7(1 - \beta^*_{2x(h)})}{\beta^*_{2x(h)} + 4 \beta^*_{2x(h)}(1 - \beta^*_{2x(h)}) + 4 \beta^*_{2y(h)} \beta^*_{2x(h)} - 4 \lambda^*_{2200(h)}} \right]
\]

and

\[
w_{10(\text{opt})} = \frac{S^2_{Y(h)} \left[ \frac{\lambda^*_{2200(h)} + 7 \lambda^*_{2200(h)}(1 - \beta^*_{2x(h)}) - 8 \beta^*_{2x(h)}(1 - \beta^*_{2x(h)}) + 8 \beta^*_{2y(h)} \beta^*_{2x(h)} - 8 \lambda^*_{2200(h)}}{\beta^*_{2x(h)} + 4 \beta^*_{2x(h)}(1 - \beta^*_{2x(h)}) + 4 \beta^*_{2y(h)} \beta^*_{2x(h)} - 4 \lambda^*_{2200(h)}} \right]}{2 S^2_{x(h)}} \]

is given by

\[
\text{MSE}(\hat{S}^2_{\text{YS}(st)})_{\text{min}} = \sum_{h=1}^{L} W^4_h Y^3_h \left[ \frac{\lambda^*_{2200(h)} + 7 \lambda^*_{2200(h)}(1 - \beta^*_{2x(h)}) - 8 \beta^*_{2x(h)}(1 - \beta^*_{2x(h)}) + 8 \beta^*_{2y(h)} \beta^*_{2x(h)} - 8 \lambda^*_{2200(h)}}{\beta^*_{2x(h)} + 4 \beta^*_{2x(h)}(1 - \beta^*_{2x(h)}) + 4 \beta^*_{2y(h)} \beta^*_{2x(h)} - 4 \lambda^*_{2200(h)}} \right]^{(14)}
\]

9. Muneer (2018) proposed an ratio-product type exponential estimator for \( \hat{S}^2_{M(st)} \), is given by

\[
\hat{S}^2_{M(st)} = S^2_{Y(h)} \left[ w_{11} \left( \frac{S^2_{x(h)}}{S^2_{x(h)}} \right) + w_{12} \left( \frac{S^2_{x(h)}}{S^2_{x(h)}} \right) \right] \exp \left( \frac{S^2_{x(h)} - S^2_{x(h)}}{S^2_{x(h)} + S^2_{x(h)}} \right), \quad \text{(15)}
\]

where \( w_{11} \) and \( w_{12} \) are unknown constants whose values are

\[
w_{11(\text{opt})} = \frac{1}{8} \left[ \frac{16 \lambda^*_{2200(h)} + 16 \lambda^*_{2200(h)} \beta^*_{2y(h)} - 24 \lambda^*_{2200(h)} \beta^*_{2x(h)} - 16 \beta^*_{2y(h)} \beta^*_{2x(h)} - \beta^*_{2x(h)} - 16 \lambda^*_{2200(h)} - 8 \beta^*_{2x(h)}}{16 \lambda^*_{2200(h)} - 16 \lambda^*_{2200(h)} \beta^*_{2x(h)} - 4 \beta^*_{2y(h)} \beta^*_{2x(h)} + \beta^*_{2x(h)} - 4 \beta^*_{2x(h)}} \right]
\]

and
The minimum MSE of $\hat{S}_{Y_{(st)}}^2$, up to first degree of approximation, is given by

$$\text{MSE}_{\text{min}}(\hat{S}_{Y_{(st)}}^2) \approx \frac{1}{\sum_{h=1}^{k} W_h^4 \gamma_h^2 \left( \begin{array}{c} 48\lambda^2_{2200(h)} - 16\lambda^2_{2200(h)} \beta_{2y(h)} - 72\lambda^2_{2200(h)} \beta^*_{2x(h)} + 16\beta^*_{2y(h)} \beta^*_{2x(h)} + 21\beta^*_{2x(h)} \\ + 16\lambda^*_{2200(h)} - 24\beta^*_{2x(h)} \\ - 16\lambda^*_{2200(h)} - 16\beta^*_{2y(h)} \beta^*_{2x(h)} - 4\beta^*_{2y(h)} \beta^*_{2x(h)} + \beta^*_{2x(h)} - 4\beta^*_{2x(h)} \\ 16\lambda^*_{2200(h)} - 16\beta^*_{2y(h)} \beta^*_{2x(h)} - 4\beta^*_{2y(h)} \beta^*_{2x(h)} + \beta^*_{2x(h)} - 4\beta^*_{2x(h)} \\ \end{array} \right) \left( \frac{64\lambda^2_{2200(h)} \beta^*_{2y(h)} - 48\lambda^2_{2200(h)} \beta^*_{2x(h)} - 128\lambda^*_{2200(h)} \beta^*_{2y(h)} \beta^*_{2x(h)} + 64 \beta^*_{2y(h)} \beta^*_{2x(h)} + 9\beta^*_{2x(h)}}{16\lambda^*_{2200(h)} - 16\lambda^*_{2200(h)} \beta^*_{2x(h)} - 4\beta^*_{2y(h)} \beta^*_{2x(h)} + \beta^*_{2x(h)} - 4\beta^*_{2x(h)}} \right) \right) \right) \cdot (16).$$

**Proposed Estimator**

We propose a difference ratio type estimator of the population variance $\hat{S}_{Y_{(st)}}^2$ in stratified random sampling as

$$\hat{S}_{P_{(st)}}^2 = \{w_{13} \gamma_{y(h)}^2 + w_{14} (\hat{S}_{x(h)}^2 - \gamma_{x(h)}^2) + w_{15} (\hat{S}_{r(h)}^2 - \gamma_{r(h)}^2) + w_{16} (\hat{S}_{u(h)}^2 - \gamma_{u(h)}^2)\} \cdot \exp\left(\frac{a(\gamma_{y(h)}^2 - \gamma_{x(h)}^2)}{a(\gamma_{x(h)}^2 + \gamma_{u(h)}^2) + 2b}\right), \quad (17)$$

where $w_{13}$, $w_{14}$, $w_{15}$ and $w_{16}$ are unknown constants, whose values to be determined for optimality. After rewriting the proposed estimator $\hat{S}_{P_{(st)}}^2$ in terms of relative error. Moreover, $a$ and $b$ can take varying values and thus provide different members of our proposed family of estimators. Table 1 below, presents various values of $a$ and $b$ and resultant estimators.

$$\hat{S}_{P_{(st)}}^2 = \{w_{13} \gamma_{y(h)}^2 + w_{14} \gamma_{y(h)}^2 e_{0h} w_{13} - \gamma_{y(h)}^2 e_{1h} w_{14} - \gamma_{y(h)}^2 e_{2h} w_{15} - \gamma_{y(h)}^2 e_{3h} w_{16}\} \cdot (18)$$

solving further the right hand side of (23), and keeping terms only up to second order in $e_{ih}s$, we can write

$$\left(\hat{S}_{P_{(st)}}^2 - S_{y(h)}^2\right) = S_{y(h)}^2 (w_{13} - 1) + S_{y(h)}^2 e_{0h} w_{13} - S_{x(h)}^2 e_{1h} w_{14} - S_{r(h)}^2 e_{2h} w_{15} - S_{u(h)}^2 e_{3h} w_{16} - \frac{1}{2} \theta S_{y(h)}^2 e_{1h} w_{13} + \frac{3}{8} \theta S_{y(h)}^2 e_{1h} w_{14} - \frac{3}{8} \theta S_{y(h)}^2 e_{1h} w_{15} + \frac{1}{2} \theta S_{y(h)}^2 e_{1h} w_{16} + \frac{1}{2} \theta S_{r(h)}^2 e_{1h} w_{13} + \frac{1}{2} \theta S_{u(h)}^2 e_{1h} w_{16}. \quad (19)$$

By applying the expectation on both sides on (24), we obtain the bias of an estimator $\hat{S}_{P_{(st)}}^2$ as

$$\text{Bias}(\hat{S}_{P_{(st)}}) = \sum_{h=1}^{k} W_h^2 \left[ S_{y(h)}^2 (w_{13} - 1) + \frac{3}{8} \gamma_{h}^2 \theta^2 S_{y(h)}^2 e_{1h} w_{13} - \frac{1}{2} \gamma_{h}^2 \theta S_{y(h)}^2 e_{1h} w_{13} + \frac{1}{2} \gamma_{h}^2 \theta S_{r(h)}^2 e_{1h} w_{15} + \frac{1}{2} \gamma_{h}^2 \theta S_{u(h)}^2 e_{1h} w_{16}. \quad (20)$$

Squaring and then taking expectation of both sides of (24) we get the MSE of an estimator.
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\( \hat{s}_{Pr(st)}^2 \), up to first degree of approximation, as

\[
\text{MSE}(\hat{s}_{Pr(st)}^2) = \left[ \sum_{h=1}^{L} W_h^4 \right] \left( \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} \right) \left( \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} \right) \left( \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} \right) \left( \frac{1}{4} + \frac{1}{3} + \frac{1}{2} + \frac{1}{1} \right)
\]

By minimizing (26) we have the optimum values of \( w_{13}, w_{14}, w_{15} \) and \( w_{16} \), respectively, given by

\[
w_{13}(opt) = \frac{1}{\left( -A_4 \lambda_{2020}^2 + A_5 \lambda_{2020}^2 + A_7 - 2 A_8 + A_9 \lambda_{2022}^2 \right)} \left[ -A_4 \lambda_{2022}^2 + A_5 \lambda_{2022}^2 + A_7 - 2 A_8 + A_9 \lambda_{2022}^2 \right]
\]

\[
w_{14}(opt) = \frac{1}{2 \lambda_{2022}^2} \left[ -A_4 \lambda_{2022}^2 + A_5 \lambda_{2022}^2 + A_7 - 2 A_8 + A_9 \lambda_{2022}^2 \right]
\]

\[
w_{15}(opt) = \frac{1}{\left( -A_4 \lambda_{2022}^2 + A_5 \lambda_{2022}^2 + A_7 - 2 A_8 + A_9 \lambda_{2022}^2 \right)} \left[ -A_4 \lambda_{2022}^2 + A_5 \lambda_{2022}^2 + A_7 - 2 A_8 + A_9 \lambda_{2022}^2 \right]
\]

and

\[
w_{16}(opt) = \frac{1}{\left( -A_4 \lambda_{2022}^2 + A_5 \lambda_{2022}^2 + A_7 - 2 A_8 + A_9 \lambda_{2022}^2 \right)} \left[ -A_4 \lambda_{2022}^2 + A_5 \lambda_{2022}^2 + A_7 - 2 A_8 + A_9 \lambda_{2022}^2 \right]
\]

Where \( B_1 = \left( -\lambda_{2022}^2 \lambda_{2200}^2 + (\lambda_{2020}^2 \lambda_{2020}^2) \lambda_{0022}^2 - \beta_{2r}^* \lambda_{2020}^2 \lambda_{2020}^2 \right)\)
Substituting the optimum values of $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$, and $\beta_6$, in (21), we get the minimum MSE of $\tilde{S}^2_{Pr(st)}$, is given by

$$
MSE_{\min}(\tilde{S}^2_{Pr(st)}) \approx \sum_{h=1}^{L} W_h Y_h^3 \left[ \begin{array}{c}
\frac{1}{16} B_e \gamma_h \theta^4 \beta_2^2 y(h) - \gamma_h \theta^2 \beta_2^2 y(h) \frac{1}{16} A_2 \theta^2 + A_3 \\
+ A_4 \beta^2_{2u(h)} + \gamma_h \theta^2 \beta_2^2 y(h) + G + B_e \lambda^2_{2020(h)} \\
+ 4(A_3 + A_4 \beta^2_{2u(h)}) - 4(G + B_e \lambda^2_{2020(h)}) \\
\end{array} \right]^{\frac{1}{4}}$$

Members of the suggested families of estimators

| $a$ | $b$ | $\tilde{S}^2_{Pr(st)}$ |
|-----|-----|---------------------|
| 1   | $C_x(st)$ | $\tilde{S}^2_{Pr(st)}^{(1)}$ |

$$
\tilde{S}^2_{Pr(st)}^{(1)} = \frac{(w_1 S^2_y(h) + w_1 s^2_x(h)) + w_{15}(S^2_{r(h)} - s^2_{r(h)}) + w_{16}(S^2_u(h))}{\left(\frac{(S^2_x(h) - s^2_x(h))}{(S^2_x(h) + s^2_x(h)) + 2C_x(st)} \right)}
$$
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| $\beta_2(st)(x)$ | $C_{x(st)}$ | $S_{Pr(st)}^{2(3)}$ | $\beta_2(st)(x)$ | $C_{x(st)}$ | $S_{Pr(st)}^{2(4)}$ | $\beta_2(st)(x)$ | $C_{x(st)}$ | $S_{Pr(st)}^{2(5)}$ | $\rho_{yx(st)}$ | $C_{x(st)}$ | $S_{Pr(st)}^{2(6)}$ | $\rho_{yx(st)}$ | $C_{x(st)}$ | $S_{Pr(st)}^{2(7)}$ | $\rho_{yx(st)}$ | $\beta_2(st)(x)$ | $S_{Pr(st)}^{2(8)}$ | $\rho_{yx(st)}$ | $\beta_2(st)(x)$ | $S_{Pr(st)}^{2(9)}$ | $\rho_{yx(st)}$ | $NS_x^2$ | $S_{Pr(st)}^{2(10)}$ |
|-----------------|-------------|---------------------|-----------------|-------------|---------------------|-----------------|-------------|---------------------|-----------------|-------------|---------------------|-----------------|-------------|---------------------|-----------------|-----------------|---------------------|-----------------|-------------|---------------------|-----------------|-------------|
| $\hat{s}_{u(h)}^2$ | $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ | $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ |
| $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ | $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ | $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ |
| $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ | $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ |
| $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ | $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ |
| $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ | $\exp\left(f\left(x_{(h)}^2-s^2_{x(h)}\right)\right)$ |

Performance Comparison

In this section, we advance by comparing the efficiency of our proposed family with Muneer et al.
(2018), we need to show $\text{MSE}_{\text{min}}(\hat{S}_{M(st)}^2) - \text{MSE}_{\text{min}}(\hat{S}_{\text{pr(st)}}^2) > 0$, which on simplification provides the general efficiency condition such as,
\[
64\lambda_{2200(h)}^2\beta_{2y(h)}^* - 48\lambda_{2200(h)}^*\beta_{2x(h)}^* - 128\lambda_{2200(h)}\beta_{2y(h)}^*\beta_{2x(h)}^* + 48\lambda_{2200(h)}\beta_{2x(h)}^*\beta_{2x(h)}^* + 64\beta_{2y(h)}^*\beta_{2x(h)}^2 + 9\beta_{2x(h)}^3
\]
\[
+64\lambda_{2200(h)}^2 - 64\lambda_{2200(h)}\beta_{2x(h)}^*
\]
\[
= \left(-G - A_2^2\lambda_{0022(h)}^2 - 2\beta_{2x(h)}^*\lambda_{2020(h)}^*\lambda_{2002(h)}^*\lambda_{0022(h)}^2 + (A_2^2\lambda_{2u(h)}^* + \beta_{2x(h)}^*\lambda_{2002(h)}^*\beta_{2r(h)}^* + \beta_{2u(h)}^*\lambda_{2020(h)}^*\gamma + A_2 + A_1\beta_{2x(h)}^* \right)
\]
\[
-4\left[16\lambda_{2200(h)}^2 - 16\lambda_{2200(h)}\beta_{2x(h)}^* - 4\beta_{2y(h)}^*\beta_{2x(h)}^* + \beta_{2x(h)}^2 - 4\beta_{2x(h)}^2\right]
\]
\[
\left[\frac{-1}{16}A_2\gamma_2h^2\beta_{2x(h)}^2 - \gamma_hh^2\beta_{2x(h)}^2(\frac{1}{16}A_2\theta^2 + A_3) + A_4\beta_{2u(h)}^* + \gamma_hh^2\beta_{2x(h)}^* + G + A_1\lambda_{2200(h)}^2 \right]
\]
\[
+4(A_3 + A_4\beta_{2u(h)}^*) - 4(G + A_1\lambda_{2200(h)}^2)
\]

**Empirical Study**

For numerical comparisons of the proposed and existing estimators, a dataset is considered. The newly proposed estimators are compared in terms of PREs and the results are reported in Table 1. The detail of the data set is given below.

**Dataset: Source: Murthy (1967)**

- $y$: Output of the factory and $x$: number of workers.
- $N = 80, \ n = 10, \ \bar{Y} = 5182.64, \ \bar{x} = 285.125, \ \bar{R} = 40.5, \ \bar{U} = 153514.2, \ Sy = 1835.659, \ Sx = 270.4294, \ Sr = 23.23749, \ Su = 256931.1,$
- $\rho_{yx} = 0.5334603, \ \rho_{yr} = 0.8305842, \ \rho_{yu} = 0.4397924, \ \rho_{xr} = 0.4284402, \ \rho_{ux} = 0.9329431, \ \rho_{ru} = 0.285485, \ \rho_{yu} = 0.354194, \ \rho_{xu} = 0.948459, \ \rho_{xy} = 0.573765, \ \rho_{ux} = 1.673663, \ \beta_{2(y)} = 2.238321, \ \beta_{2(x)} = 3.536017, \ \beta_{2(r)} = 1.776874, \ \beta_{2(u)} = 7.196021, \ \lambda_{2002} = 2.294326, \ \lambda_{2200} = 1.893889, \ \lambda_{0022} = 2.172258,$
- $\gamma = 0.1.$

$\text{PREs} = Var(\hat{S}_i^2)/\text{MSE}(\hat{S}_i^2) \times 100; \ \ i = y,R,Reg,RD,GK,AH,Pr.$

**Table 1:** The PREs of estimators for different choices of $a$ and $b$

| S.No | $a$   | $b$   | $\hat{S}_{\text{gk}}^2$ | $\hat{S}_{\text{ah}}^2$ | $\hat{S}_{\text{pr}}^2$ |
|------|-------|-------|--------------------------|--------------------------|--------------------------|
| 1    | 1     | $\beta_{2(x)}$ | 212.5869 | 419.8807 | 992.5903 |
| 1    | 1     | $C_x$ | 212.5887 | 419.8847 | 992.6066 |
|      | $\beta_{2(x)}$ | $C_x$ |                        |                          |                          |
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|                | $c_x$ | $p_{yx}$ | $\beta_{2(x)}$ | $\rho_{yx}$ | $c_x$ | $p_{yx}$ | $\beta_{2(x)}$ | $\rho_{yx}$ | $\beta_{2(x)}$ | $\rho_{yx}$ | $c_x$ | $p_{yx}$ | $\beta_{2(x)}$ | $\rho_{yx}$ |
|----------------|-------|----------|-----------------|-------------|-------|----------|-----------------|-------------|-----------------|-------------|-------|----------|-----------------|-------------|
|                |       |          | $\delta_2^2$    |             |       |          | $\delta_2^2$    |             | $\delta_2^2$    |             |       |          | $\delta_2^2$    |             |
|                | 212.5891 | 419.8858 | 992.6109        |             |       |          | 212.5867        | 419.8804    | 992.5891        |             |       |          | 212.5877        | 419.8826    |
|                | 212.5876 | 419.8824 | 992.5973        |             |       |          | 212.5877        | 419.8826    | 992.5973        |             |       |          | 212.5895        | 419.8855    |
|                | 212.5889 | 419.8852 | 992.6085        |             |       |          | 212.5895        | 419.8855    | 992.6099        |             |       |          | 212.5895        | 419.8855    |
|                | 212.5876 | 419.8824 | 992.5973        |             |       |          | 212.5876        | 419.8824    | 992.5973        |             |       |          | 212.5895        | 419.8855    |
|                | 212.5889 | 419.8852 | 992.6085        |             |       |          | 212.5889        | 419.8852    | 992.6085        |             |       |          | 212.5895        | 419.8855    |
|                | 212.5876 | 419.8824 | 992.5973        |             |       |          | 212.5876        | 419.8824    | 992.5973        |             |       |          | 212.5895        | 419.8855    |
|                | 212.5889 | 419.8852 | 992.6085        |             |       |          | 212.5889        | 419.8852    | 992.6085        |             |       |          | 212.5895        | 419.8855    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |
|                | 192.4908 | 380.7184 | 861.3078        |             |       |          | 212.5883        | 419.8838    | 992.6028        |             |       |          | 212.5883        | 419.8838    |

From the results contained in table 1, it is much clear that the $PRES$ of the proposed estimators are significantly greater than those with their competitors, which shows the appropriateness of the new estimators.

**Conclusion**

On the basis of theoretical as well as numerical results, it is inferred that the suggested estimator is more efficient as compared to the existing mean, product, classical regression, ratio, exponential-ratio, Rao (1991), Smarandache et al. (2009), Shabbir and Gupta (2010), Grover and Kaur (2011), Grover and Kaur (2014) and Haq et al. (2017) estimators.

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