Neutrino spin oscillations in curved space-time under the influence of external fields

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Abstract. We study neutrino spin oscillations in background matter under the influence of strong electromagnetic and gravitational fields. The neutrino spin evolution is treated quasiclassically. We derive the effective Hamiltonian governing spin oscillations of a neutrino moving in the vicinity of a black hole and interacting with a relativistic magnetized accretion disk around a black hole. Applications for the studies of spin oscillations of astrophysical neutrinos are considered.

1. Introduction
Neutrinos play an important role in the evolution of various astrophysical objects like neutron stars, supernovae, quasars, etc. In these environments, neutrinos are subject to the interaction not only with a background matter and magnetic fields, but also with a strong gravitational field. The study of astrophysical neutrinos propagating in these backgrounds becomes important owing to the recent development of the multi-messenger approach in astronomy and astrophysics, when a correlation between a neutrino signal and, e.g., the detection of gravitational waves is searched.

Neutrinos are known to be mainly left-polarized particles in the standard model. However, the interaction with external fields can change their polarization and neutrinos become invisible for a terrestrial detector. This process is known as neutrino spin oscillations. Thus, the interaction of astrophysical neutrinos with external fields, which include gravity, can significantly modify the predictions of the multi-messenger approach.

In the paper, we review our results on the description of neutrino spin oscillations in non-trivial gravitational backgrounds under the influence of various external fields. In section 2, we derive the most general quasiclassical equation for the description of the neutrino spin evolution in these external fields. In sections 3 and 4, we apply our results for the studies of neutrino spin oscillations in the vicinity of Schwarzschild and Kerr black holes (BH). Finally, in section 5, we summarize the obtained results.

2. Covariant description of the neutrino spin evolution
In this section we shall construct the quasiclassical approach for the description of the spin evolution of a Dirac neutrino moving in a background matter under the influence of
electromagnetic and gravitational fields. In particular, we generalize the neutrino spin evolution equation in matter and electromagnetic fields to include the effects of the nontrivial geometry of space-time.

The equation describing the neutrino spin evolution in matter under the influence of an external electromagnetic field in Minkowski space-time was derived in [1]. The most straightforward generalization of this equation to include a nontrivial gravitational background has the form [2],

$$\frac{DS^\mu}{D\tau} = 2\mu \left( F^{\mu\nu} S_\nu - U^\mu U_\nu F^{\nu\lambda} S_\lambda \right) + \sqrt{2} G_F E^{\mu\nu\lambda\rho} G_{\nu\rho} U_\lambda S_\mu.$$  \hspace{1cm} (1)

where $S^\mu$ and $U^\mu$ are the four vectors of a neutrino and the velocity of a neutrino, $F_{\mu\nu}$ is the skew-symmetric tensor of the electromagnetic field, $E^{\mu\nu\lambda\rho}$ is the antisymmetric tensor if curved space-time, $g = \det(g_{\mu\nu})$ is the determinant of the metric tensor $g_{\mu\nu}$, $\mu$ is the magnetic moment, and $G_F = 1.17 \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi constant. The covariant derivative $\nabla/\partial \tau$ along the world line in equation (1) is taken with respect to the interval $\tau$. Equation (1) should be completed by the geodesic equation for the evolution of the four velocity, $D U^\mu / D\tau = 0$.

The neutrino interaction with a background matter is characterized by the four vector $J^\mu_f$, which is a linear combination of hydrodynamic currents, $J^\mu_f$, and polarizations, $\Lambda^\mu_f$, of background fermions of the type $f$,

$$G^\mu = \sum_f \left( q^{(1)}_f J^\mu_f + q^{(2)}_f \Lambda^\mu_f \right).$$  \hspace{1cm} (2)

We shall express $J^\mu_f$ using the invariant number density $n_f$, i.e. the density in the rest frame of fermions, and the four velocity $U^\mu_f$ of background fermions: $J^\mu_f = n_f U^\mu_f$. The explicit expression of $\Lambda^\mu_f$ is given in [3]. The coefficients $q^{(1,2)}_f$ can be found in [1].

Then we introduce a locally Minkowskian frame by considering the vierbein $e^a_\mu$, $a = 0, \ldots, 3$, satisfying the relations, $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ and $\eta_{ab} = e^a_\mu e^b_\nu g_{\mu\nu}$, where $e^a_\mu$ is the inverse vierbein and $\eta_{ab} = \text{diag}(1, -1, -1, -1)$ is the metric tensor in the Minkowski space-time. Decomposing all the quantities in equation (1) in the vierbein basis, we get the following equation for the neutrino spin $s^a = e^a_\mu S^\mu$:

$$\frac{ds^a}{d\tau} = \frac{1}{\gamma} \left[ G^{ab} s_b + 2\mu \left( f^{ab} s_b - u^a u_b f^{bc} s_c \right) + \sqrt{2} G_F e^{abcd} g_{bc} u_c s_d \right],$$  \hspace{1cm} (3)

where $G^{ab} = \eta^{ac} \eta^{bd} \gamma_{cde} u^e$ is the antisymmetric tensor accounting for the gravitational interaction of neutrinos, $\gamma_{abc} = \eta_{ad} e^d_\mu e^e_\nu e^c_\mu$ are the Ricci rotation coefficients (the semicolon stays for the covariant derivative), $\gamma = U^0$, $g^a = e^a_\mu G^\mu = (g^0, g)$ is the effective potential of the matter interaction in the vierbein frame, $f_{ab} = e^a_\mu e^b_\nu F_{\mu\nu} = (e, b)$ is the electromagnetic field tensor in the vierbein frame, and $u^a = e^a_\mu U^\mu = (u^0, u)$ is the neutrino velocity in the vierbein basis. We should also reformulate the dynamics of $U^\mu$ in the vierbein frame. One can obtain the following evolution equation for $u^a$:

$$\frac{du^a}{d\ell} = \frac{1}{\gamma} G^{ab} u_b.$$  \hspace{1cm} (4)

The details of the derivation of equations (3) and (4) can be found in [2, 4].

The three vector of the neutrino polarization $\zeta$ is defined as

$$s^a = \left( \zeta \cdot u, \frac{u(\zeta \cdot u)}{1 + u^0} \right).$$  \hspace{1cm} (5)
Using equations $[5]$ and $[3]$ one finds the evolution equation for $\zeta$ as

$$\frac{d\zeta}{dt} = \frac{2}{\gamma}[\zeta \times \mathbf{G}],$$

(6)

where

$$\mathbf{G} = \frac{1}{2} \left[ \mathbf{b}_g + \frac{1}{1 + u^0} (\mathbf{e}_g \times \mathbf{u}) \right] + \frac{G\mathbf{v}}{\sqrt{2}} \left[ \mathbf{u} \left( \frac{g^0}{1 + u^0} \right) - \mathbf{g} \right] + \mu \left[ u^0 \mathbf{b} - \frac{\mathbf{u}(\mathbf{ub})}{1 + u^0} + (\mathbf{e} \times \mathbf{u}) \right].$$

(7)

Here $\mathbf{e}_g$ and $\mathbf{b}_g$ are the components of the tensor $G_{ab}: G_{ab} = (\mathbf{e}_g, \mathbf{b}_g)$. To derive equation $[7]$ we use the fact that $u_a u^a = 1$.

3. Neutrino spin oscillations in Schwarzschild metric

Let us apply general equations $[6]$ and $[7]$ to describe the neutrino spin evolution in the vicinity of a non-rotating BH. In this section, we shall omit the influence of the background matter and the electromagnetic field on neutrino oscillations.

When we study the gravitational field of a non-rotating BH, the interval is known to be

$$dt^2 = A^2 dt^2 - A^{-2} dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(8)

where $A = \sqrt{1 - r_g/r}$, $r_g = 2M$ is the Schwarzschild radius, and $M$ is the BH mass. In Eq. $[8]$ we use the spherical coordinates $(r, \theta, \phi)$.

The neutrino spin precession is determined by the vector $\mathbf{\Omega} = \mathbf{G}/\gamma$. The components of this vector can be found on the basis of equation $[7]$ (see also $[4]$),

$$\Omega_1 = \frac{1}{2} v_\phi \cos \theta, \quad \Omega_2 = v_\phi \sin \frac{1}{2} \left( -A + \frac{\gamma r_g}{(1 + \gamma A) 2r} \right), \quad \Omega_3 = v_\theta \frac{1}{2} \left( A - \frac{\gamma r_g}{(1 + \gamma A) 2r} \right),$$

(9)

where $\mathbf{v} = (v_r, v_\theta, v_\phi)$ are the components of the world velocity.

Let us discuss a neutrino orbiting BH. For simplicity we consider only circular orbits with the radius $R$. We may restrict ourselves to the consideration of the orbits lying only in the equatorial plane ($\theta = \pi/2$ or equivalently $v_\theta = 0$) because of the spherical symmetry of the gravitational field. In this case $\Omega_1 = \Omega_3 = 0$ in equation $[9]$. The expressions for the neutrino angular velocity and $\gamma$ have the form,

$$v_\phi = \frac{d\phi}{dt} = \sqrt{\frac{r_g}{2R^2}}, \quad \gamma^{-1} = \frac{d\tau}{dt} = \sqrt{1 - \frac{3r_g}{2R}}.$$  

(10)

It should be noted that the vierbein four velocity now takes the form, $u^a = (\gamma A, 0, 0, \gamma v_\phi r)$. One can verify that $u^a u_a = 1$ with help of equation $[10]$. We also mention that equation $[10]$ is also identically satisfied since $G^{ab} u_b = 0$. Therefore a neutrino has a constant four velocity with respect to the vierbein frame.

Using equation $[1]$, we can rewrite the remaining nonzero component of $\mathbf{\Omega}$ in the more simple form, $\Omega_2 = -v_\phi/2\gamma$. We suppose that initially a neutrino is left-handed, i.e. its initial spin vector is antiparallel to the particle’s velocity. According to equations $[9]$ and $[10]$, the neutrino spin rotates around the second axis. Therefore we can derive the effective Schrödinger equation for neutrino spin oscillations in the gravitational field of a non-rotating BH,

$$i \frac{d\nu}{dt} = H_{\text{eff}} \nu, \quad H_{\text{eff}} = -(\mathbf{\sigma} \cdot \mathbf{\Omega}) = \begin{pmatrix} 0 & -i\Omega_2 \\ i\Omega_2 & 0 \end{pmatrix}. $$

(11)
Figure 1. Neutrino spin oscillations frequency versus the radius of the neutrino orbit.

where \( \nu^T = (\nu_R, \nu_L) \) is the effective neutrino wave function, \( H_{\text{eff}} \) is the effective Hamiltonian, and \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices.

Using equation (11), we obtain the expression for the transition probability for neutrino spin oscillations,

\[
P(t) = \sin^2(\Omega_2 t).
\]

One can see that there is a full mixing in our case, and thus, the neutrino transition probability can achieve a unit value. Let us plot the frequency of neutrino spin oscillations versus the radius of the orbit. It is possible to see in figure 1 that \( |\Omega_2| = 0 \) at \( R = 1.5r_g \) and \( |\Omega_2| \to 0 \) at \( R \to \infty \). One can also conclude that \( |\Omega_2| \) has its maximal value, which is equal to \( 6.25 \times 10^{-2}r_g^{-1} \), at \( R = 2r_g \). Let us evaluate the number of revolutions \( N \), that a neutrino should make, necessary for the total spin flip. Using equation (10) we can find that \( N = \gamma = 2 \) at \( R = 2r_g \). It is interesting to evaluate the characteristic period of the neutrino spin oscillations, i.e. the oscillations length. For \( M = 10M_\odot \) at \( R = 2r_g \) we get for \( T = \pi/|\Omega_2| \approx 4.94 \times 10^{-3} \) s.

4. Neutrino propagation in the vicinity of a rotating black hole

Now let us describe neutrino spin oscillations in matter and a magnetic field when a particle moves in the vicinity of a rotating BH. As in section 3 we shall consider circular neutrino motion and derive the effective Schrödinger equation as well as find the transition probability for spin oscillations.

The space-time of a rotating BH can be described by the Kerr metric. In Boyer-Lindquist coordinates \( x^\mu = (t, r, \theta, \phi) \), this metric has the form,

\[
dr^2 = \left( 1 - \frac{r_g}{\Sigma} \right) dt^2 + 2\frac{rr_g a \sin^2 \theta}{\Sigma} dtd\phi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \frac{\Xi}{\Sigma} \sin^2 \theta d\phi^2,
\]

where

\[
\Delta = r^2 - rr_g + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad \Xi = (r^2 + a^2) \Sigma + rr_g a^2 \sin^2 \theta.
\]

The angular momentum of BH is \( J = Ma \). Coordinates \( x^\mu \) in equations (12) and (13) correspond to the world frame rather than to the vierbein frame.

We shall discuss BH surrounded by matter forming an accretion disk in equatorial plane. In this case only \( U^0_f \) and \( U^\phi_f \) are nonzero. Moreover we shall consider a stationary accretion.

By the symmetry reasons, the quantities \( n_f \), \( U^0_f \), and \( U^\phi_f \) can be functions of \( r \) and \( \theta \) only. Assuming that the temperature of background fermions is sufficiently high, we can take that the
The vector velocity in the vierbein frame \( u^a \) where \( x \) analytically. Assuming that initially only left-polarized neutrinos are present, i.e. \( \nu \) characterized by the four vector, \( \xi \) background matter is unpolarized. In this case, the effective neutrino interaction with matter is of the type \( F \) (see, e.g., [2]), the electric and magnetic field strengths can be calculated as components of the vector potential. On the basis of the explicit form of the vector potential, \( \pi r \) accounted for, we present the dependence \( \xi + \Omega \) oscillations reaches its maximal value \( \Omega \approx \Omega_{osc} N_{osc} + 1/2 \). It can be seen that for retrograde orbits the maximal value of \( \Omega \) versus \( x = r/g \) for different values of \( \alpha \) in figure 2. It can be seen that for retrograde orbits the maximal value of \( \Omega \) decreases with \( \alpha \), whereas for direct orbits this value increases with \( \alpha \).

If we discuss the maximal possible angular momentum of BH corresponding to \( \alpha = 1/2 \), we get from figure 2 that for a direct neutrino orbit with \( r \approx r_g \), the frequency of neutrino spin oscillations reaches its maximal value \( \Omega_2^+ \approx 0.12r_g^{-1} \). For a retrograde orbit with \( r \approx 2.60r_g \), we get the maximal frequency \( \Omega_2^{-} \approx 0.05r_g^{-1} \). Considering BH with \( M = 10M_{\odot} \), we obtain the corresponding oscillations lengths, \( L_{osc} \approx 7.85 \times 10^7 \) cm and \( L_{osc} \approx 1.88 \times 10^8 \) cm. On the basis of these oscillations lengths one gets the times of the neutrino spin-flip, \( T_{sf} \approx L_{osc}/2, \) as \( T_{nf} \approx 1.31 \times 10^{-3} \) s and \( T_{nf} \approx 3.13 \times 10^{-3} \) s.

Note that for \( \alpha = 1/2 \) the minimal possible radius of a stable retrograde orbit is equal to \( r_{min} = 4.5r_g \). Therefore the obtained \( \Omega_2^+ \) corresponds to an unstable orbit. In case of a direct orbit \( r_{min} \rightarrow r_g/2 \).

\[
g^a = n_{eff} \left( rU^0 \sqrt{\frac{\Delta}{\Xi_0}}, 0, 0, \frac{U^0_{\phi} - arr_g U^0_i}{r\sqrt{\Xi_0}} \right), \quad n_{eff} = \sum_f q_f^{(1)} n_f, \tag{14}
\]

where \( \Xi_0 = \Xi(\theta = \pi/2) = r^4 + a^2(r + r_g) \). In equation (14) we assume that all types of background fermions have the same four velocity.

Now let us specify the structure of the electromagnetic field. We suggest that, at relatively great distance from BH, there is a constant and uniform magnetic field parallel to the rotation axis of BH and having the strength \( B_0 \). This magnetic field can be created by the plasma motion in the accretion disk. The structure of the axially symmetric electromagnetic field in a curved space-time having such an asymptotics was studied in [5], where one can find the nonzero components of the vector potential. On the basis of the explicit form of the vector potential (see, e.g., [2]), the electric and magnetic field strengths can be calculated as \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \).

As in the case of Schwarzschild metric, studied in section 3 here one can show that the four velocity in the vierbein frame \( u^a \) is constant if we study circular neutrino orbits. Using this fact, we can reformulate the neutrino spin dynamics using the effective Schrödinger equation (11).

The vector \( \Omega \), which defines the neutrino spin precession, has the following nonzero components:

\[
\Omega_2 = \frac{1}{2 \gamma r_g} \left\{ \pm \frac{1}{\sqrt{2} x^{3/2}} - \mu B_0 r_g \frac{2\sqrt{2} x^2(x-1) \pm \alpha \sqrt{x(x-1)} + \sqrt{2} \alpha^2}{x^{3/2} \sqrt{2 x^3 - 3 x^2 \pm 2 \sqrt{2} \alpha x^{3/2}}} \right\},
\]

\[
\Omega_3 = \frac{G_F n_{eff}}{\sqrt{2} \gamma} \frac{\pm U^0_i - r_g U^0_{\phi} (\sqrt{2} x^{3/2} + \pm \alpha)}{\sqrt{2 x^3 - 3 x^2 \pm 2 \sqrt{2} \alpha x^{3/2}}}, \tag{15}
\]

where \( x = r/r_g \) and \( \alpha = a/r_g \). The upper sign refers to the direct orbits, i.e. when a neutrino corotates with BH, and the lower sign corresponds to retrograde ones (a particle counter-rotates).

Supposing that \( B_0 \) and \( n_{eff} \) do not depend on time, we can solve the Schrödinger equation analytically. Assuming that initially only left-polarized neutrinos are present, i.e. \( \nu(0) = (0, 1) \), we get the probability to detect a right-polarized neutrino in the form,

\[
P(t) = P_{max} \sin^2 \left( \frac{\pi L_{osc}}{T_{osc}} t \right), \quad P_{max} = \frac{\Omega_2^+}{\Omega_2^+ + \Omega_3^+}, \quad L_{osc} = \frac{\pi}{\sqrt{\Omega_2^+ + \Omega_3^+}}. \tag{16}
\]

Here \( L_{osc} \) is the length of oscillations.
5. Conclusion

In this work, we developed the quasiclassical approach for the description of neutrino spin oscillations in background matter under the influence of electromagnetic and gravitational fields. As applications of the obtained results, we considered spin evolution of neutrinos orbiting Schwarzschild and Kerr BH. We also discussed the possibility of the neutrino interaction with an ultrarelativistic magnetized accretion disk around BH.

Recently, the dynamics of the fermion spin in gravitational fields was studied in [6]. In that work, the spin evolution was based on the analysis of the Dirac equation in curved space-time. Although the results of [6] are analogous to that of our works [2, 4], where we used the quasiclassical approximation, we can describe the neutrino spin evolution in an arbitrary gravitational field without referring to the Dirac equation. This fact makes the approach in [2, 4] to be more attractive for possible applications.

I am thankful to the organizers of DSPIN-17 for the invitation and a financial support, as well as to the Tomsk State University Competitiveness Improvement Program and RFBR (research project No. 15-02-00293) for a partial support.

References

[1] Dvornikov M and Studenikin A 2002 Neutrino spin evolution in presence of general external fields J. High Energy Phys. JHEP09(2002)016 (Preprint hep-ph/0202113)

[2] Dvornikov M 2013 Neutrino spin oscillations in matter under the influence of gravitational and electromagnetic fields J. Cosmol. Astropart. Phys. JCAP06(2013)015 (Preprint arXiv:1306.2659)

[3] Lobanov A E and Studenikin A I 2001 Neutrino oscillations in moving and polarized matter under the influence of electromagnetic fields Phys. Lett. B 515 94–98

[4] Dvornikov M 2006 Neutrino spin oscillations in gravitational fields Int. J. Mod. Phys. D 15 1017–1034 (Preprint hep-ph/0601095)

[5] Wald R M 1974 Black hole in a uniform magnetic field Phys. Rev. D 10 1680–1685

[6] Obukhov Yu N, Silenko A J and Teryaev O V 2017 General treatment of quantum and classical spinning particles in external fields Preprint arXiv:1708.05601