Exact solutions of a one-dimensional mixture of spinor bosons and spinor fermions

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Received 11 May 2009; accepted 3 June 2009
Available online 6 June 2009

Abstract

The exact solutions of a one-dimensional mixture of spinor bosons and spinor fermions with δ-function interactions are studied. Some new sets of Bethe ansatz equations are obtained by using the graded nest quantum inverse scattering method. Many interesting features appear in the system. For example, the wave function has the SU(2|2) supersymmetry. It is also found that the ground state of the system is partially polarized, where the fermions form a spin singlet state and the bosons are totally polarized. From the solution of Bethe ansatz equations, it is shown that all the momentum, spin and isospin rapidities at the ground state are real if the interactions between the particles are repulsive; while the fermions form two-particle bound states and the bosons form one large bound state, which means the bosons condensed at the zero momentum point, if the interactions are attractive. The charge, spin and isospin excitations are discussed in detail. The thermodynamic Bethe ansatz equations are also derived and their solutions at some special cases are obtained analytically.

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PACS: 03.75.Mn; 03.75.Hh; 02.30.Ik

Keywords: Integrable systems; Yang–Baxter equation; Graded algebraic Bethe ansatz

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doi:10.1016/j.nuclphysb.2009.06.003
1. Introduction

Recently, tremendous experimental progresses have been taken in the research of the one-dimensional (1D) trapped cold atoms [1–12]. By means of either magnetic or optical traps, the cold atom gas has been realized. With Feshbach resonance, the scattering length and thus the couplings among atoms can be manipulated. In addition, with laser beams, one can confine particles in the valleys of periodic potential of the optical lattice. These experimental tools provide a platform to study some controllable condensed matter systems. Theoretically, many methods have been applied to study these problems. For example, the Gross–Pitaevskii (GP) theory is widely adopted in dealing with the systems of Bose–Einstein condensations with weak interactions. However, the GP theory is based on a mean field approximation and has many shortcomings. The boson-Hubbard model [13,14] are used to describe the low-energy physics of ultracold dilute gas of bosonic atoms in an optical lattice. This model cannot be solved exactly and one has to use the numerical approaches or approximate methods such as the valence bond mean field [15]. Exactly solvable models play an important role in the investigation of 1D interacting many-particle systems. The exact solutions can supply some believable results thus serve as a very good starting point to understand the new phenomena and new quantum states in trapped cold atomic systems.

Using laser beams or Feshbach resonance, the local interactions can be strongly enhanced. The low-energy behavior of trapped cold atoms can be well described by a system with δ-function potentials. Fortunately, this kind of problems can be solved exactly. The exact solutions of scalar bosons with δ-function interactions are obtained by Lieb and Liniger [16]. Li, Gu, Ying and Eckern study the two-component bosons, where the intrinsic degrees of freedom (isospin) of bosons satisfy the SU(2) symmetry [17]. They find that the ground state of the system is not an isospin singlet, but a polarized or ferromagnetic state. In fact, the Bethe ansatz equations of multi-component bosons have already been obtained by Zhou [18]. Naturally, one should consider the effects of spin exchanging interactions if the bosonic atoms have non-zero spin. An exactly solved model of bosons with spin-1 is proposed in Ref. [19], where the exchange interactions are embodied in the model Hamiltonian and the system is spin-dependent. More physical properties of bosonic cold atomic systems with δ-function potentials can be found in Refs. [20–25].

In experiments, atoms with inner degrees of freedom (hyperfine spin) are prepared by catching several hyperfine sublevels of atoms. The spins can be polarized by the external magnetic fields, thus it is easier to capture the inter state of the fermions by using the laser beams. Now, people can control the fermionic atoms steadily staying on some special inter states, and realize the fermionic atoms with multi-component hyperfine spin [9]. The theoretical model of the spin-1/2 fermions with δ-function interactions is solved by Gaudin [26] and Yang [27]. Sutherland generalizes the results to the multi-component fermionic systems [28].

Most recently, the study of ultracold Bose–Fermi mixtures become a remarkable research topic for many new matter phases may arise in these systems. Experimenters have succeeded in preparing the mixtures of 7Li–6Li, 23Na–6Li or 87Rb–40K in the optical lattices [29–37]. For example, a stable bosonic 87Rb and fermionic 40K mixture in three-dimensional optical lattices has been realized [36,37]. Theoretically, Lai and Yang obtain the exact solutions of 1D mixture of spin-1/2 fermions and scalar bosons with the δ-function interactions [38]. They calculate the ground state energy and gapless fermionic excitations in the thermodynamic limit [39]. For more studies on the boson–fermion mixtures in the optical lattices, please see Refs. [40–43].

It is natural that one should consider the mixtures of multi-component fermions and multi-component bosons. In this case, both the fermions and the bosons have the intrinsic degrees
of freedom. It is well known that the ground state of bosonic systems with intrinsic degrees of freedom can be surprisingly different from that of the scalar bosons. Furthermore, the ground state of the spin-1/2 fermionic system with $\delta$-function interactions is spin singlet, while the ground state of bosonic system is isospin polarized or ferromagnetic state. One may wonder what will happen if we mix the bosons and fermions both with some intrinsic degrees of freedom? What is the new quantum state in the mixtures? These issues are quite interesting and important nowadays due to the rapid progress in the field of cold atomic physics.

In this paper, we study a mixture of two-component fermions and two-component bosons with $\delta$-function interactions. Because the wave function of the system is supersymmetric and satisfies the $\text{SU}(2|2)$ Lie superalgebra, we use the super or graded nest quantum inverse scattering method [44–50] to derive the exact solutions of the system at 1D. We obtain the Bethe ansatz equations with different gradings. We find that the ground state of the system is partial polarized. If the interactions are repulsive ($c > 0$), all the momentum, spin and isospin rapidities in the solutions of Bethe ansatz equations are real. If the interactions are attractive ($c < 0$), the Bethe ansatz equations may have the complex solutions, where the fermions form the two-particle bounded states and the bosons are condensed at the zero momentum point at the ground state. We then study the low-lying excitations such as charge, spin and isospin excitations in detail. We also obtain the thermodynamic Bethe ansatz equations at finite temperatures and find their analytic solutions at some special limiting cases.

The integrable $\text{SU}(2|2)$-supersymmetry is a very interesting issue. The Bethe ansatz for the corresponding quantum spin chain is obtained by Kulish [44], and the continue limit and low energy behaviors are studied by Saleur [50]. In this paper, we study the corresponding continue quantum gas model, which also has many applications in the systems of cold atoms with hyperfine structure. The low-lying excitation spectrum can be measured by the scattering of phonons in experiments. Meanwhile, the cold atoms with different internal states can be prepared in experiments. The phonons scattering experiments on the bosonic cold atoms have been done several years ago [51,52]. Thus the motivation of this paper is to give a prediction on the ground state and low-lying excitation properties of the Bose–Fermi mixture in the cold atom systems, for the excitations in spin and charge sectors can show different behaviors.

The paper is organized as follows. In Section 2, we introduce the supersymmetry of the system and the generators of the corresponding Lie superalgebra. In Section 3, we derive the exact solutions of the system by using the generalized quantum inverse scattering method. We give the Bethe ansatz equations with the BBFF grading, which are the foundations of our discussions. The ground state properties are discussed in Section 4 and the low-lying excitations are discussed in Section 5. The thermodynamic Bethe ansatz equations are calculated in Section 6 and some useful limit cases are discussed in Section 7. Section 8 contains some conclusions and discussions.

2. Supersymmetry of the system

We study a 1D cold atomic system mixed by $N_{b_1}$ bosons of species 1, $N_{b_2}$ bosons of species 2, $N_{f_1}$ fermions of species 1 and $N_{f_2}$ fermions of species 2. The Hamiltonian reads

$$H = \int_0^L \sum_a \partial_x \phi_a^\dagger(x) \partial_x \phi_a(x) \, dx + \int_0^L \sum_{a\beta} g_{a\beta} \phi_a^\dagger(x) \phi_\beta^\dagger(x) \phi_\beta(x) \phi_a(x) \, dx,$$ (1)
where $\alpha, \beta = b_1, b_2, f_1, f_2$ and $\phi_\alpha$ are the bosonic or fermionic field operators. The bosonic field operators satisfy the commutation relations, $[\phi_\alpha^\dagger(x), \phi_\beta(y)] = \delta_{\alpha\beta} \delta_{xy}$, while the fermionic field operators satisfy the anticommutation relations, $\{\phi^\dagger_\alpha(x), \phi_\beta(y)\} = \delta_{\alpha\beta} \delta_{xy}$. According to the Pauli exclusion principle, the $\alpha$ and $\beta$ in Hamiltonian (1) cannot be the same species of fermions.

In this paper, we use the periodic boundary conditions. The wave function of the system (1) is supersymmetric, $\Psi(x_j, x_l) = P_{jl} \Psi(x_l, x_j)$, where $P_{jl}$ means exchanging both the coordinates and the spins (isospins) of two particles $j$ and $l$. $P_{jl} = 1$ for bosons or bosons and fermions thus the wave function is symmetric, while $P_{jl} = -1$ for fermions thus the wave function is antisymmetric.

The supersymmetry of the system (1) can be described by the $SU(2|2)$ Lie superalgebra. The superalgebra $SU(2|2)$ has fifteen generators and eight of them are fermionic [46–49]. Moreover, the pure two-component fermionic subsystem has the $SU(2)$ invariance and the pure two-component bosons also has the $SU(2)$ invariance. In order to explain the symmetry of the system more clearly, we introduce the particle creation (annihilation) operators as $a^\dagger_\sigma(x)$ ($a_\sigma(x)$), where we assume two species of fermions carrying the different spins $\sigma = \uparrow, \downarrow$. There are four kinds of states at a given position $x$,

$$
|0\rangle_x, \quad |\uparrow\rangle_x = a^\dagger_\uparrow(x)|0\rangle_x, \quad |\downarrow\rangle_x = a^\dagger_\downarrow(x)|0\rangle_x, \quad |\uparrow\downarrow\rangle_x = a^\dagger_\downarrow(x)a^\dagger_\uparrow(x)|0\rangle_x. \quad (2)
$$

The state $|0\rangle_x$ is vacuum and the state $|\uparrow\downarrow\rangle$ represents that an atom-pair is localized on a single energy level. Now, we introduce the generators of the superalgebra $SU(2|2)$. The spin operators are defined as

$$
S^+ = \int_0^L a^\dagger_\uparrow(x)a_\downarrow(x) dx, \quad S^- = \int_0^L a^\dagger_\downarrow(x)a_\uparrow(x) dx, \\
S^z = \frac{1}{2} \int_0^L [a^\dagger_\uparrow(x)a_\uparrow(x) - a^\dagger_\downarrow(x)a_\downarrow(x)] dx. \quad (3)
$$

The spin operators $S^+, S^-$ and $S^z$ form the $SU(2)$ Lie algebra, where the commutation relations between the generators are $[S^-, S^+] = 2S^z, [S^+, S^z] = S^+, [S^-, S^z] = -S^-$. The above spin operators are grassmann even (bosonic). We introduce the pairing operators, which are also bosonic generators,

$$
\eta^+ = \int_0^L a^\dagger_\uparrow(x)a^\dagger_\downarrow(x) dx, \quad \eta^- = \int_0^L a^\dagger_\downarrow(x)a_\uparrow(x) dx, \\
\eta^z = \frac{1}{2} \int_0^L [1 - a^\dagger_\uparrow(x)a_\uparrow(x) - a^\dagger_\downarrow(x)a_\downarrow(x)] dx. \quad (4)
$$

The pairing operators $\eta^+, \eta^-, \eta^z$ also form a $SU(2)$ Lie algebra with the commutation relations $[\eta^-, \eta^+] = 2\eta^z, [\eta^+, \eta^z] = \eta^+, [\eta^-, \eta^z] = -\eta^-$. The eight fermionic generators are

$$
Q_{\sigma} = \int_0^L [1 - a^\dagger_\sigma(x)a_{\bar{\sigma}}(x)]a_{\sigma}(x), \quad Q^\dagger_{\bar{\sigma}} = \int_0^L [1 - a^\dagger_\bar{\sigma}(x)a_\sigma(x)]a^\dagger_\sigma(x),
$$
3. Bethe ansatz solutions of the system

In the following, we consider the case that all the coupling parameters are equal, \( g_{\alpha\beta} = c \). The system (1) has several integrable lines. If \( N_{b_2} = N_{f_1} = N_{f_2} = 0 \), the system degenerates to the scalar bosons with \( \delta \)-function interactions which is solved by Lieb and Liniger [16]. If \( N_{f_1} = N_{f_2} = 0 \), the system degenerates to the two-component \( SU(2) \) bosons and is studied by Li et al. [17]. If \( N_{b_1} = N_{b_2} = 0 \), the system degenerates to the spin-1/2 fermions and is solved by Yang [27]. If \( N_{b_2} = 0 \), the system degenerates to the mixture of scalar bosons and spin-1/2 fermions which is solved by Lai and Yang [38]. In this paper, we consider the case that all the particles numbers \( N_{b_1}, N_{b_2}, N_{f_1} \) and \( N_{f_2} \) are not zero. We fist derive the two-body scattering matrix by using the coordinate Bethe ansatz method and prove the integrability of the system. Then we determine the Bethe ansatz equations and the energy spectrum by using the nest quantum inverse scattering methods.

3.1. Coordinate Bethe ansatz

In the framework of coordinate Bethe ansatz, the wave function of the system described by a set of quasi-momenta \( \{ k_j \} \) can be written as [27,38]

\[
\Psi(x_1s_1, \ldots, x_Ns_N) = \sum_{Q,P} \theta(x_{Q_1} < \cdots < x_{Q_N})A_{s_1 \cdots s_N}(Q, P)e^{i\sum_{j=1}^{N}k_j x_{Q_j}},
\]

where \( Q = (Q_1, \ldots, Q_N) \) and \( P = (P_1, \ldots, P_N) \) are the permutations of the integers \( 1, \ldots, N \), \( N \) is the total number of particles, \( N = N_{b_1} + N_{b_2} + N_{f_1} + N_{f_2} \), \( \theta(x_{Q_1} < \cdots < x_{Q_N}) = \theta(x_{Q_N} - x_{Q_N-1}) \cdots \theta(x_{Q_2} - x_{Q_1}) \) and \( \theta(x - y) \) is the step function. The wave function is supersymmetric under permutating both the coordinates and the spins (or isospins) of two particles. The wave function is continuous but its derivative jumps when two atoms touch. With the standard coordinate Bethe ansatz procedure, we obtain the two-body scattering matrix as

\[
S_{jl}(k_j - k_l) = \frac{k_j - k_l - i\epsilon P_{jl}^s}{k_j - k_l + i\epsilon},
\]

where \( P_{jl}^s \) is the spin super permutation operator with the definition \( [P_{jl}^s]_{\alpha\beta} = (-1)^{\epsilon_a \epsilon_\mu} \delta_{\alpha\nu} \delta_{\mu\beta} \), the \( \epsilon \) and \( \mu \) are the row indices, and \( \beta \) and \( \nu \) are the column indices. Here \( \epsilon_\alpha \) is the Grassmann number, \( \epsilon_a = 0 \) for bosons and \( \epsilon_a = 1 \) for fermions. The scattering matrix satisfies the super or graded Yang–Baxter equation [44–50]

\[
S_{12}(k_1 - k_2)S_{13}(k_1 - k_3)S_{23}(k_2 - k_3) = S_{23}(k_2 - k_3)S_{13}(k_1 - k_3)S_{12}(k_1 - k_2)
\]

which ensures the integrability of the model (1). The Yang–Baxter equation (8) can also be written out explicitly as

\[
S_{12}(k_1 - k_2)^{b_1b_2}_{a_1a_3}S_{13}(k_1 - k_3)^{b_1b_3}_{a_1a_3}S_{23}(k_2 - k_3)^{c_1c_3}_{b_2b_3}(-)^{(\epsilon_{b_2} + \epsilon_{c_1})\epsilon_{b_2}}
\]
\[
S_{23}(k_2 - k_3)_{ab} b_1 b_2 S_{13}(k_1 - k_3)_{ac} b_1 c_3 S_{12}(k_1 - k_2)_{bc} c_1 c_2 (-)^{(e_{a_1} + e_{b_1})e_{b_2}}.
\] (9)

With the periodic boundary conditions of the wave function, we obtain the following eigenvalue equations

\[
S_{jN}(k_j - k_N)S_{jN-1}(k_j - k_{N-1}) \cdots S_{jj+1}(k_j - k_{j+1}) \\
\times S_{jj-1}(k_j - k_{j-1}) \cdots S_{j1}(k_j - k_1)e^{ik_j L} \xi_0 = \xi_0,
\] (10)

where \(\xi_0\) is the amplitude of initial state wave function.

### 3.2. Algebraic Bethe ansatz

Now, we derive the exact solutions of the system by using the graded nested quantum inverse scattering method [44–50]. We consider the exact solutions of the system with BBFF grading, that is the Grassmann parities for the four bases are \(\epsilon_1 = \epsilon_2 = 0\) and \(\epsilon_3 = \epsilon_4 = 1\). Please note choosing different bases is equivalent to choosing different highest weight represents when deriving the Bethe ansatz equations. The Bethe ansatz equations with different gradings can change into each others by using some transformations [48].

The matrix form of the scattering matrix \(S_{0j}(\lambda)\) in the space 0 is

\[
S_j(\lambda) = \begin{pmatrix}
  a(\lambda) - b(\lambda)e_j^{11} & -b(\lambda)e_j^{21} & -b(\lambda)e_j^{31} & -b(\lambda)e_j^{41} \\
  -b(\lambda)e_j^{12} & a(\lambda) - b(\lambda)e_j^{22} & -b(\lambda)e_j^{32} & -b(\lambda)e_j^{42} \\
  -b(\lambda)e_j^{13} & -b(\lambda)e_j^{23} & a(\lambda) + b(\lambda)e_j^{33} & b(\lambda)e_j^{43} \\
  -b(\lambda)e_j^{14} & -b(\lambda)e_j^{24} & b(\lambda)e_j^{34} & a(\lambda) + b(\lambda)e_j^{44}
\end{pmatrix},
\] (11)

where the matrix \(e_{j}^{\alpha\beta}\) acts on the \(j\)th space with its elements defined as \((e_{j}^{\alpha\beta})_{\mu\nu} = \delta_{\alpha\mu}\delta_{\beta\nu}\), \(a(\lambda) = \lambda/((\lambda + i)c)\) and \(b(\lambda) = i((\lambda + ic)/c)\). The quantity (11) is also called as the Lax operator acting on the \(j\)th space. Introduced the braid scattering matrix \(R_{12}(\lambda) = P_{12}^{s}S_{12}(\lambda)\), which satisfies the braid Yang–Baxter equation,

\[
R_{12}(\lambda - u)R_{23}(\lambda)R_{12}(u) = R_{23}(u)R_{12}(\lambda)R_{23}(\lambda - u).
\] (12)

We follow the graded nested algebraic Bethe ansatz method to solve the eigenvalue equation (10). The monodromy matrix is defined as

\[
T_N(\lambda) = S_{0j}(\lambda - k_j)S_{0N}(\lambda - k_N) \cdots S_{0j+1}(\lambda - k_{j+1})S_{0j-1}(\lambda - k_{j-1}) \cdots S_{01}(\lambda - k_1)
\]

\[
= \begin{pmatrix}
  A_{11}(\lambda) & A_{12}(\lambda) & A_{13}(\lambda) & B_1(\lambda) \\
  A_{21}(\lambda) & A_{22}(\lambda) & A_{23}(\lambda) & B_2(\lambda) \\
  A_{31}(\lambda) & A_{32}(\lambda) & A_{33}(\lambda) & B_3(\lambda) \\
  C_1(\lambda) & C_2(\lambda) & C_3(\lambda) & D(\lambda)
\end{pmatrix},
\] (13)

where 0 means the auxiliary space and \(l = 1, 2, \ldots, N\), mean the quantum spaces. From Eq. (12), we can prove that the monodromy matrix (13) satisfies the Yang–Baxter relation

\[
R_{12}(\lambda - u)\big[T_N(\lambda) \otimes_s T_N(u)\big] = \big[T_N(u) \otimes_s T_N(\lambda)\big]R_{12}(\lambda - u),
\] (14)

where \(\otimes_s\) means the super or graded tensor-product as \([A \otimes_s B]_{abcd}^{bd} = (-1)^{(e_a + e_b)e_c} A_{ab}B_{cd}\), \(a\) and \(c\) are the row indices, and \(b\) and \(d\) are the column indices. Using indices, the Yang–Baxter relation (14) can also be written as

\[
R_{12}(\lambda - u)_{bc}^{bd} T_N(\lambda)_{ba}^{bd} T_N(u)_{ab}^{bc} (-1)^{(e_{b_1} + e_{c_1})e_{b_2}}
\]

\[
= T_N(u)_{bc}^{bd} T_N(\lambda)_{ba}^{bd} R_{12}(\lambda - u)_{bc}^{bd} (-1)^{(e_{b_1} + e_{c_1})e_{b_2}}.
\]
\begin{align}
T_N(u)^{b_1}_{a_1} T_N(\lambda)^{b_2}_{a_2} R_{12}(\lambda - u)^{c_1}_{b_1}^{c_2}_{b_2} (-1)^{(e_a + e_b)} e_a e_b,
\end{align}

where all the repeated indices should be summed. The elements of scattering matrix \( S_{ij}(\lambda)^{b_1}_{a_1}^{b_2}_{a_2} \) are not zero only with the conditions (1) \( a_1 = a_2 = b_1 = b_2 \) or (2) \( a_1 = b_1, a_2 = b_2 \) or (3) \( a_1 = b_2, a_2 = b_1 \). These properties will be used in deriving the commutation relations. The transfer matrix \( t(\lambda) \) of the system is defined as the supertrace of the monodromy matrix (13) in the auxiliary space,

\begin{align}
t(\lambda) = \text{str} T_N(\lambda) = \sum_{a=1}^{4} (-1)^{e_\lambda} T_N(\lambda)^a = A_{11}(\lambda) + A_{22}(\lambda) - A_{33}(\lambda) - D_{11}(\lambda).
\end{align}

From the Yang–Baxter relation (14), we can prove that the transfer matrices with different spectral parameters commute with each other \([t(u), t(v)] = 0\). Thus the system has infinite conserved quantities and is integrable. The eigenvalue problem (10) is therefore reduced to

\begin{align}
- \text{str}_0 T_N(k_j) e^{ik_j L} \xi_0 = \xi_0.
\end{align}

We choose the local vacuum state as \(|0\rangle_j = (0, 0, 0, 1)' \) where \( t \) means the transpose. The global vacuum state is constructed as \(|0\rangle = \otimes_{j=1}^{N} |0\rangle_j \). Acting the monodromy matrix (13) on this vacuum state, we have

\begin{align}
T_N(\lambda) |0\rangle = \begin{pmatrix}
\prod_{i=1}^{N} a(\lambda - k_i) & 0 & 0 \\
0 & \prod_{i=1}^{N} a(\lambda - k_i) & 0 \\
C_1(\lambda) & C_2(\lambda) & C_3(\lambda)
\end{pmatrix} |0\rangle.
\end{align}

We see that the elements \( A_{11}(\lambda), A_{22}(\lambda), A_{33}(\lambda) \) and \( D(\lambda) \) acting on this vacuum state give the eigenvalues. The elements \( B_a(\lambda) \) acting on the vacuum state are zero. The elements \( C_a(\lambda) \) acting on the vacuum state give nonzero values and can be regarded as the creation operators. We assume the eigenstates of the system (1) are obtained by applying the creation operators \( C_a \) on the vacuum state as

\begin{align}
|\lambda_1, \ldots, \lambda_{N_1} | F \rangle = C_{a_1}(\lambda_1) \cdots C_{a_{N_1}}(\lambda_{N_1}) |0\rangle F^{a_{N_1} \cdots a_1},
\end{align}

where \( F^{a_{N_1} \cdots a_1} \) is a function of the spectral parameters \( \lambda_j \) and \( N_1 \) is the number of creation operators. When the transfer matrix acting on the Bethe states (19), we need the commutation relations between \( A_{11}, A_{22}, A_{33}, D \) and \( C_a \). From the Yang–Baxter relation (15) and using the properties of the \( R \) matrix, we find following commutation relations

\begin{align}
D(u) C_c(\lambda) &= \frac{1}{a(\lambda - u)} C_c(\lambda) D(u) - \frac{b(u - \lambda)}{a(\lambda - u)} C_c(u) D(\lambda),
\end{align}

\begin{align}
A_{ab}(u) C_c(\lambda) &= (-1)^{e_a e_c + e_a e_b} R_{BBF}^{(1)}(u - \lambda)^{c_b}_{d_c} a(u - \lambda) C_c(\lambda) A_{ad}(u) \\
&\quad - (-1)^{(e_a + 1)(e_b + 1)} \frac{b(u - \lambda)}{a(u - \lambda)} C_b(u) A_{ac}(\lambda),
\end{align}

\begin{align}
C_{a_1}(u) C_{a_2}(\lambda) &= R_{FFB}^{(1)}(u - \lambda)^{a_{21}}_{b_{21}} C_{b_2}(\lambda) C_{b_1}(u),
\end{align}

where all the indices take values 1, 2 and 3. The first nesting \( R \) matrices are defined as

\begin{align}
R_{FFB}^{(1)}(u) = b(u) + a(u) P_{FFB}^{(1)}(u), \quad R_{BBF}^{(1)}(u) = -b(u) + a(u) P_{BBF}^{(1)}(u),
\end{align}
where $P_{F B B}^{(1)}$ and $P_{B B F}^{(1)}$ are the $9 \times 9$ super permutation matrices for the grading $\epsilon_1 = \epsilon_2 = 1$, $\epsilon_3 = 0$ and $\epsilon_1 = \epsilon_2 = 0$, $\epsilon_3 = 1$, respectively.

Acting the transfer matrix (17) on the assumed eigenstate (19), applying repeatedly the commutation relations (20)–(22) and using the result (18), we have

$$t(u)|\lambda_1, \ldots, \lambda_{N_1}|F\rangle = \left\{ \prod_{j=1}^{N_1} \frac{1}{a(u - \lambda_j)} \right\} \prod_{l=1}^{N} a(u - k_l) t^{(1)}(u)$$

$$\quad - \left\{ \prod_{j=1}^{N_1} \frac{1}{a(\lambda_j - u)} \right\} |\lambda_1, \ldots, \lambda_{N_1}|F\rangle + \text{u.t.}, \quad (24)$$

where $t^{(1)}(u)$ is the first nesting transfer matrix and u.t. means the unwanted terms. If the unwanted terms cancel with each other, which gives following Bethe ansatz equations

$$\prod_{j=1, \neq \alpha}^{N_1} a(\lambda_{\alpha} - \lambda_j) \prod_{l=1}^{N} \frac{1}{a(\lambda_{\alpha} - k_l)} F^{b_{N_1} \cdots b_1}_{a_{1} \cdots a_{N_1}} = t^{(1)}(u) a_{a_1 \cdots a_{N_1}}^{b_1 \cdots b_{N_1}} F^{a_{N_1} \cdots a_1},$$

$$\alpha = 1, 2, \ldots, N_1, \quad (25)$$

then the assumed states (19) are the eigenstates of the transfer matrix $t(u)$ and the corresponding eigenvalues are given by the first term in Eq. (24).

Now, seeking the eigenvalues of $t(u)$ becomes seeking the eigenvalues of $t^{(1)}(u)$. The elements of first nested transfer matrix $t^{(1)}(u)$ can be written out explicitly

$$t^{(1)}(u, \{\lambda\})^{b_1 \cdots b_{N_1}}_{a_1 \cdots a_{N_1}}$$

$$= (-1)^{\epsilon_0} S_{0_{N_1}}^{(1)}(u - \lambda_{N_1})^{c_{N_1} - 1 a_{N_1}}_{c_0 a_{N_1}} S_{0_{N_1}}^{(1)}(u - \lambda_{N_1 - 1})^{c_{N_1 - 1} a_{N_1}}_{c_0 b_{N_1 - 1}} \cdots S_{0_{N_1}}^{(1)}(u - \lambda_1)_{c_1 b_1}$$

$$\times (-1)^{\epsilon_0} \sum_{i=1}^{N_1} \epsilon_i \epsilon_{i+1}.$$  \quad (26)

Here all the indices $c_i$ are summed over and $S^{(1)}(u) = P_{B B F}^{(1)} R_{B B F}^{(1)}(u)$. In order to interpret $t^{(1)}(u)$ as the supertrace of the monodromy matrix, we define a new graded tensor product $[F \otimes G^d_{a c}] = F^a_{\bar{a}} G^d_{c (-1)^{(\epsilon_a + \epsilon_c)} (\epsilon_{a c} + 1)}$. This new graded tensor-product switches even and odd Grassmann parities. Meanwhile, the first nesting monodromy matrix is defined as

$$T^{(1)}_{N_1}(u) = S_{0_{N_1}}^{(1)}(u - \lambda_{N_1}) \otimes S_{0_{N_1 - 1}}^{(1)}(u - \lambda_{N_1 - 1}) \otimes \cdots \otimes S_{0_{1}}^{(1)}(u - \lambda_1)$$

$$= \begin{pmatrix}
A^{(1)}_{11}(u) & A^{(1)}_{12}(u) & B^{(1)}_{1}(u) \\
A^{(1)}_{21}(u) & A^{(1)}_{22}(u) & B^{(1)}_{2}(u) \\
C^{(1)}_{1}(u) & C^{(1)}_{2}(u) & D^{(1)}(u)
\end{pmatrix}, \quad (27)$$

which satisfies the graded Yang–Baxter relation,

$$\hat{r}(u - v) [T^{(1)}_{N_1}(u) \otimes T^{(1)}_{N_1}(v)] = [T^{(1)}_{N_1}(v) \otimes T^{(1)}_{N_1}(u)] \hat{r}(u - v),$$

where the $\hat{r}$-matrix is $\hat{r}^{b d}_{a c} = -b(u) \delta_{a b} \delta_{c d} + a(u) \delta_{a d} \delta_{b c} (-1)^{\epsilon_a + \epsilon_c + \epsilon_{a c}}$. Then the transfer matrix (26) is the supertrace of the first nesting monodromy matrix (27)

$$t^{(1)}(u, \{\lambda\})^{b_1 \cdots b_{N_1}}_{a_1 \cdots a_{N_1}} = \text{str} T^{(1)}_{N_1}(u) = A^{(1)}_{11}(u) + A^{(1)}_{22}(u) - D^{(1)}(u). \quad (29)$$
We choose $|0\rangle^{(1)}_{j} = (0, 0, 1)^{t}$ as the local reference state for the first nesting. The global reference state is $|0\rangle^{(1)} = \bigotimes_{j=1}^{N_{1}} |0\rangle^{(1)}_{j}$. Acting the first nesting monodromy matrix (27) on this reference state, we have

$$T^{(1)}_{N_{1}}(u)|0\rangle^{(1)} = \left( \prod_{j=1}^{N_{1}} a(u - \lambda_{j}) \begin{pmatrix} 0 & 0 \\ \prod_{j=1}^{N_{1}} a(u - \lambda_{j}) & 0 \end{pmatrix} \right) |0\rangle^{(1)}.$$

Assume the eigenstates of the first nesting transfer matrix are

$$|\lambda_{1}^{(1)}, \ldots, \lambda_{N_{2}}^{(1)}\rangle = C_{b_{1}}^{(1)}(\lambda_{1}^{(1)}) \cdots C_{b_{N_{2}}}^{(1)}(\lambda_{N_{2}}^{(1)}) |0\rangle^{(1)} G^{b_{N_{2}} \cdots b_{1}},$$

where $N_{2}$ is the number of creation operators. From the Yang–Baxter relation (28), we obtain the following commutation relations

$$D^{(1)}(u) C^{(1)}_{e}(\lambda) = \frac{1}{a(\lambda - u)} C^{(1)}_{e}(\lambda) D^{(1)}(u) - \frac{b(\lambda - u)}{a(\lambda - u)} C^{(1)}_{e}(u) D^{(1)}(\lambda),$$

$$A^{(1)}_{ab}(u) C^{(1)}_{e}(\lambda) = \frac{R_{bb}^{(2)}(u - \lambda) a^{b} C^{(1)}_{e}(\lambda)}{a^{(1)(u - \lambda)}} A^{(1)}_{ab}(u) + \frac{b(\lambda - u)}{a(\lambda - u)} C^{(1)}_{b_{1}}(u) A^{(1)}_{ac}(\lambda),$$

$$C^{(1)}_{b_{1}}(u) C^{(1)}_{b_{2}}(\lambda) = R_{FF}^{(2)}(u - \lambda) C^{(1)}_{b_{2}}(\lambda) C^{(1)}_{c_{1}}(u).$$

Here, all the indices take values 1 and 2. The second nesting $R$ matrices are $R_{FF}^{(2)}(u) = b(u) + a(u) P_{FF}^{(2)}$ and $R_{BB}^{(2)}(u) = -b(u) + a(u) P_{BB}^{(2)}$, where $P_{FF}^{(2)}$ and $P_{BB}^{(2)}$ are the 4 × 4 super permutation matrices for the grading $\epsilon_{1} = \epsilon_{2} = 1$ and $\epsilon_{1} = \epsilon_{2} = 0$, respectively. $[P_{FF}^{(2)}]_{ab} = -\delta_{ab} \delta_{bc}$, $[P_{BB}^{(2)}]_{ab} = -\delta_{ab} \delta_{bc}$. Acting the first nesting transfer matrix (29) on the assumed states (31), we have

$$t^{(1)}(u) |\lambda_{1}^{(1)}, \ldots, \lambda_{N_{2}}^{(1)}\rangle = \left\{ \prod_{j=1}^{N_{2}} \frac{1}{a(u - \lambda_{j}^{(1)})} \prod_{l=1}^{N_{1}} a(u - \lambda_{l}) t^{(2)}(u) \right\} |\lambda_{1}^{(1)}, \ldots, \lambda_{N_{2}}^{(1)}\rangle + \text{u.t.},$$

where $t^{(2)}(u)$ is the second nesting transfer matrix

$$t^{(2)}(u) = \text{str} S^{(2)}_{0N_{2}}(u - \lambda_{N_{2}}^{(1)}) S^{(2)}_{0N_{2}-1}(u - \lambda_{N_{2}-1}^{(1)}) \cdots S^{(2)}_{01}(u - \lambda_{1}^{(1)}).$$

If the unwanted terms cancel with each other, which give following Bethe ansatz equations

$$\prod_{j=1, \neq \alpha}^{N_{2}} a(\lambda_{\alpha}^{(1)} - \lambda_{j}^{(1)}) \prod_{l=1}^{N_{1}} \frac{1}{a(\lambda_{\alpha}^{(1)} - \lambda_{l})} G^{b_{N_{2}} \cdots b_{1}} = t^{(2)}(\lambda_{\alpha}^{(1)} \lambda_{a_{1} \cdots a_{N_{2}} = a_{1} \cdots a_{N_{2}}} G^{a_{N_{2}} \cdots a_{1}},$$

the assumed states (31) are the eigenstates of the first nesting transfer matrix (29). Now, the remnant problem is finding the eigenvalues of second nesting transfer matrix $t^{(2)}(u)$.

The second nesting scattering matrix is $S^{(2)}(u) = a(u) - b(u) P_{BB}^{(2)}$. The second nesting monodromy matrix is

$$T^{(2)}_{N_{2}}(u) = S^{(2)}_{0N_{2}}(u - \lambda_{N_{2}}^{(1)}) S^{(2)}_{0N_{2}-1}(u - \lambda_{N_{2}-1}^{(1)}) \cdots S^{(2)}_{01}(u - \lambda_{1}^{(1)})$$
We choose $N$ if the unwanted terms cancel, the assumed states (42) are the eigenstates of the second nesting state, we have

$$\text{Assume the eigenstates of the second nesting transfer matrix}$$

$$\frac{2}{1} = \frac{2}{1} \times \frac{2}{1}$$

The second nesting transfer matrix is the supertrace of the corresponding monodromy matrix

$$t^{(2)} (u) = \text{str} T^{(2)}_{N_2} (u) = A^{(2)} (u) + D^{(2)} (u).$$

We choose $|0 \rangle^{(2)} = (0, 1)^t$ as the local vacuum state for the second nesting. The global vacuum state is $|0 \rangle^{(2)} \otimes_{j=1}^{N_2} |0 \rangle^{(2)}$. Acting the second nesting monodromy matrix (38) on this vacuum state, we have

$$T^{(2)}_{N_2} (u) |0 \rangle^{(2)} = \left( \prod_{i=1}^{N_2} a^{(1)} (u - \lambda^{(1)}_i) \right) C^{(2)} (u) \left( \prod_{i=1}^{N_2} [a^{(1)} (u - \lambda^{(1)}_i) - b^{(1)} (u - \lambda^{(1)}_i)] \right) |0 \rangle^{(2)}.$$ (41)

Assume the eigenstates of the second nesting transfer matrix $t^{(2)} (u)$ are

$$| \lambda^{(2)}_1, \ldots, \lambda^{(2)}_{N_3} \rangle = C^{(2)} (\lambda^{(2)}_1) \cdots C^{(2)} (\lambda^{(2)}_{N_3}) |0 \rangle^{(2)},$$ (42)

where $N_3$ is the number of creation operators. From the Yang–Baxter relation (39), we obtain the following commutation relations

$$D^{(2)} (u) C^{(2)} (\lambda) = \frac{a(\lambda - u) - b(\lambda - u)}{a(\lambda - u)} C^{(2)} (\lambda) D^{(2)} (u) + \frac{b(\lambda - u)}{a(\lambda - u)} C^{(2)} (u) D^{(2)} (\lambda),$$ (43)

$$A^{(2)} (u) C^{(2)} (\lambda) = \frac{a(u - \lambda) - b(u - \lambda)}{a(u - \lambda)} C^{(2)} (\lambda) A^{(2)} (u) + \frac{b(u - \lambda)}{a(u - \lambda)} C^{(2)} (u) A^{(2)} (\lambda),$$ (44)

$$C^{(2)} (u) C^{(2)} (\lambda) = C^{(2)} (\lambda) C^{(2)} (u).$$ (45)

The second nesting transfer matrix $t^{(2)} (u)$ acting the assumed states (42) gives

$$t^{(2)} (u) C^{(2)} (\lambda^{(2)}_1) \cdots C^{(2)} (\lambda^{(2)}_{N_3}) |0 \rangle^{(2)}$$

$$= \left\{ \prod_{j=1}^{N_3} a(u - \lambda^{(2)}_j) - b(u - \lambda^{(2)}_j) \right\} \left( \prod_{l=1}^{N_2} a(u - \lambda^{(1)}_l) \right) + \left\{ \prod_{j=1}^{N_3} a(\lambda^{(2)}_j - u) - b(\lambda^{(2)}_j - u) \right\} \frac{a(\lambda^{(2)}_j - u)}{a(\lambda^{(2)}_j - u)}$$

$$\times \left( \prod_{l=1}^{N_2} [a(u - \lambda^{(1)}_l) - b(u - \lambda^{(1)}_l)] \right) | \lambda^{(2)}_1, \ldots, \lambda^{(2)}_{N_3} \rangle + \text{u.t.}$$ (46)

If the unwanted terms cancel, the assumed states (42) are the eigenstates of the second nesting transfer matrix $t^{(2)} (u)$, which gives the following Bethe ansatz relations

$$\prod_{j=1}^{N_3} a(\lambda^{(2)}_j - \lambda^{(2)}_j) - b(\lambda^{(2)}_j - \lambda^{(2)}_j) a(\lambda^{(2)}_j - \lambda^{(2)}_j)$$

$$= \prod_{l=1}^{N_2} a(\lambda^{(2)}_l - \lambda^{(1)}_l) - b(\lambda^{(2)}_l - \lambda^{(1)}_l).$$ (47)
where \( \alpha = 1, 2, \ldots, N_3 \). Now, the eigenvalues of the transfer matrix \( t(u) \) can be calculated directly by synthetically considering Eqs. (24), (35) and (46).

The forth set of Bethe ansatz equations are obtained from the eigen-equation (10) as

\[
e^{-ik_j L} = \prod_{a=1}^{N_1} a(\lambda_a - k_j), \quad j = 1, \ldots, N.
\]

(48)

Put \( \lambda_j \rightarrow \lambda_j - ic/2, \lambda_j^{(1)} \rightarrow \lambda_j^{(1)} - ic \) and \( \lambda_j^{(2)} \rightarrow \lambda_j^{(2)} - ic/2 \), then the Bethe ansatz equations (25), (37), (47) and (48) can be written out explicitly as

\[
e^{ik_j L} = \prod_{a=1}^{N_1} k_j - \lambda_a + \frac{i}{2} c, \quad j = 1, \ldots, N, \quad a = 1, \ldots, N_1, \quad \beta = 1, \ldots, N_2, \quad \gamma = 1, \ldots, N_3,
\]

(49)\(\quad\) (50)\(\quad\) (51)\(\quad\) (52)

where \( N = N_{b_1} + N_{b_2} + N_{j_1} + N_{j_2}, \quad N_1 = N_{b_1} + N_{b_2} + N_{f_1}, \quad N_2 = N_{b_1} + N_{b_2}, \quad N_3 = N_{b_1} \). Taking the logarithm of Eqs. (49)–(52), we arrive at

\[
k_j L = 2\pi I_j - \sum_{b=1}^{N_1} \Delta_{1/2}(k_j - \lambda_b), \quad j = 1, \ldots, N,
\]

(53)

\[
2\pi J_a = \sum_{l=1}^{N} \Delta_{1/2}(\lambda_a - k_l) - \sum_{b=1}^{N_1} \Delta_{1}(\lambda_a - \lambda_b) + \sum_{c=1}^{N_2} \Delta_{1/2}(\lambda_a - \lambda_c^{(1)}) + \sum_{c=1}^{N_2} \Delta_{1/2}(\lambda_a - \lambda_c^{(2)}), \quad a = 1, \ldots, N_1, \quad a_1 = 1, \ldots, N_2,
\]

(54)\(\quad\) (55)

\[
2\pi J_{a_1}^{(1)} = \sum_{b=1}^{N_1} \Delta_{1/2}(\lambda_{a_1}^{(1)} - \lambda_b) - \sum_{c=1}^{N_2} \Delta_{1/2}(\lambda_{a_1}^{(1)} - \lambda_c^{(2)}), \quad a_1 = 1, \ldots, N_2, \quad a_2 = 1, \ldots, N_3,
\]

(56)

where \( \Delta_m(x) = 2\tan^{-1}\{x/(mc)\}, I_j, J_a, J_{a_1}^{(1)} \) and \( J_{a_2}^{(2)} \) are integer or half-odd quantum numbers. Here we have used the formula \( \ln[(k + ic)/(k - ic)] = i[\pi - 2\tan^{-1}(k/c)] \). These equations are the special case of that in Ref. [18]. If we set \( N_{b_1} = N_{b_2} = 0 \), the system (1) degenerates to the spin-1/2 fermions model with \( \delta \)-function potentials and our Bethe ansatz equations (49)–(52) are the same as the ones obtained by Yang [27]. If we set \( N_{b_2} = 0 \), the system (1) degenerates to the Bose–Fermi mixture considered by Lai and Yang [38], and our Bethe ansatz equations are the same as their results.

The eigenvalues of the Hamiltonian (1) are \( E = \sum_{j=1}^{N} k_j^2 \), where possible values of the momentum \( k_j \) are determined by the Bethe ansatz equations (53)–(56).
3.3. Corresponding lattice model

From the transfer matrix $t(u)$, we can construct the corresponding lattice model by using the standard integrable theory in the statistic physics. The Hamiltonian can be obtained by taking the derivative of the logarithmic form of the transfer matrix at the zero spectral parameter point [53]. After some calculations, we find that the $\partial \ln t(u)/\partial u|_{u=0}$ gives the Hamiltonian studied by Essler, Korepin and Schoutens [46–49] up to a constant. The system (1) is the continue case of Essler–Korepin–Schoutens (EKS) model. The two-body scattering matrix (11) of present model has the same structure as that of EKS model. In such a sense, the spin dynamics of present model keeps some similarities to that of EKS model.

We note that the integrable $SU(n|m)$-supersymmetric quantum spin chain has been studied by Kulish [44]. The corresponding Hamiltonian can be obtained directly from the transfer matrix. If the charge rapidities tend to zero, our results degenerate into that obtained by Kulish. The thermodynamics and low-energy limit of the $SU(n|m)$-invariant spin chain are studied by Saleur [50]. These results are also valid for the present model in the spin sector.

4. Ground state of the system

We first consider the case of $c > 0$, which means the interactions among the particles are repulsive. We use the BBFF grading as a demonstration. Above we have confined the particles in a finite 1D box with the length $L$. Some useful properties can be obtained by the analysis of the poles or zeros of the Bethe ansatz equations in the thermodynamic limit, that is the system size $L$, particles numbers $N,N_1,N_2$ and $N_3$ tend to infinity, but the ratios $N/L,N_1/L,N_2/L$ and $N_3/L$ keep finite. For example, if some $k_j$ are in the upper complex plane, then the left-hand side of Eq. (49) tends to zero when the system size tends to infinity. Thus the right-hand side of Eq. (49) should go to zero too. From further analysis of the Bethe ansatz equations (49)–(52), we find that the momentum $k_j$, rapidities $\lambda$ and $\lambda^{(1)}$ are real at the ground state. The particle number $N_{b_2} = 0$ thus the corresponding rapidity $\lambda^{(2)}$ is zero too. That is to say only one species of bosons are left or the bosons are totally polarized at the ground state. The ground state for the bosons is a ferromagnetic state. Meanwhile, the particle numbers of two species of fermions are equal, $N_{f_1} = N_{f_2}$ and they form the spin singlet states. The ground state for the fermions is a antiferromagnetic state. Therefore, the ground state of the system (1) is partial polarized.

The momentum $k_j$, rapidities $\lambda$ and $\lambda^{(1)}$ at the ground state satisfy the following coupled equations

$$k_j L = 2\pi I_j - \sum_{b=1}^{N_1} \xi_1/2(k_j - \lambda_b),$$  \hspace{1cm} (57)

$$2\pi J_a = \sum_{l=1}^{N} \xi_1/2(\lambda_a - k_l) - \sum_{b=1}^{N_1} \xi_1(\lambda_a - \lambda_b) + \sum_{c=1}^{N_2} \xi_1/2(\lambda_a - \lambda^{(1)}_c),$$  \hspace{1cm} (58)

$$2\pi J_{a_1}^{(1)} = \sum_{b=1}^{N_1} \xi_1/2(\lambda^{(1)}_{a_1} - \lambda_b).$$  \hspace{1cm} (59)

The quantum numbers $I_j$ take integer (half-odd integer) values if $N_{b_1} + N_{f_1}$ is even (odd), $J_a$ take integer (half-odd integer) values if $N_{b_1} + N_{f_2}$ is even (odd) and $J_{a_1}^{(1)}$ take integer (half-odd integer) values if $N_{b_1} + N_{f_1}$ is even (odd). Then quantum number configuration of the ground
state are \( \{ I_j \} = \{ -(N - 1)/2, -(N - 3)/2, \ldots, (N - 1)/2 \}, \{ J_{\alpha} \} = \{ -(N_1 - 1)/2, -(N_1 - 3)/2, \ldots, (N_1 - 1)/2 \} \) and \( \{ J_{\alpha}^{(1)} \} = \{ -(N_2 - 1)/2, -(N_2 - 3)/2, \ldots, (N_2 - 1)/2 \} \), which are symmetrically centered around the origin if \( N_{f_1} = N_{f_2} \) is odd and \( N_{b_1} = N_{b_2} \) is even.

In the thermodynamic limit, the summations become integrations. The quantum number \( I_j, J_{\alpha} \) and \( J_{\alpha}^{(1)} \) become continue functions of the spectral parameters \( k, \lambda \) and \( \lambda^{(1)} \), respectively. Denote the densities of momentum \( k \), rapidities \( \lambda \) and \( \lambda^{(1)} \) by \( \rho(k), \rho_1(\lambda) \) and \( \rho_1^{(1)}(\lambda^{(1)}) \), respectively. Then we have \( \rho(k) = dI_j/(Ldk) \), \( \rho_1(\lambda) = dJ_{\alpha}/(Ld\lambda) \) and \( \rho_1^{(1)}(\lambda^{(1)}) = dJ_{\alpha}^{(1)}/(Ld\lambda^{(1)}) \). Taking the derivative of Eqs. (57)–(59), we obtain the densities of states at the ground state as

\[
\rho(k) = \frac{1}{2\pi} + \frac{1}{\pi} \int_{-B}^{B} \frac{2c\rho_1(\lambda) d\lambda}{c^2 + 4(k - \lambda)^2},
\]

\[
\rho_1(\lambda) = \frac{1}{\pi} \int_{-Q}^{Q} \frac{2c\rho(k) dk}{c^2 + 4(\lambda - k)^2} - \frac{1}{\pi} \int_{-B}^{B} \frac{c\rho_1(\lambda') d\lambda'}{c^2 + (\lambda - \lambda')^2} + \frac{1}{\pi} \int_{-D}^{D} \frac{2c\rho_1^{(1)}(\lambda^{(1)}) d\lambda^{(1)}}{c^2 + 4(\lambda^{(1)} - \lambda)^2},
\]

\[
\rho_1^{(1)}(\lambda^{(1)}) = \frac{1}{\pi} \int_{-B}^{B} \frac{2c\rho_1(\lambda) d\lambda}{c^2 + 4(\lambda^{(1)} - \lambda)^2}.
\]

The integral limits \( Q, B \) and \( D \) are determined by

\[
N/L = \int_{-Q}^{Q} \rho(k) dk, \quad N_{f_1}/L = \int_{-B}^{B} \rho_1(\lambda) d\lambda, \quad N_{f_2}/L = \int_{-D}^{D} \rho_1^{(1)}(\lambda^{(1)}) d\lambda^{(1)}.
\]

The quantity \( Q \) is the Fermi surface of the system. The densities of energy and momentum at the ground state are

\[
E/L = \int_{-Q}^{Q} k^2\rho(k) dk, \quad P/L = \int_{-Q}^{Q} k\rho(k) dk.
\]

The magnetization of the fermions \( S_j^z \) and the magnetization of bosons \( T_b^z \) are

\[
S_j^z = N_{f_1} - N_{f_2} = 0, \quad T_b^z = N_{b_1} - N_{b_2} = N_{b_1}.
\]

In the case of \( c < 0 \), besides real solutions, Eqs. (49)–(52) also have complex solutions which are usually called as string solutions. After some algebraic calculations, we find that at the ground state, the momentum \( k_j \) have the following 2-string and \( N_{b_1} \)-string solutions,

\[
k_j = \Lambda_j + \frac{ic}{2} (3 - 2j) + o(e^{-\delta L}), \quad j = 1, 2,
\]

\[
k_l = \frac{ic}{2} (N_{b_1} + 1 - 2l) + o(e^{-\delta'L}), \quad l = 1, 2, \ldots, N_{b_1},
\]

where \( \Lambda_j \) is a real parameter, \( \delta \) and \( \delta' \) are some positive constants. Eqs. (61) and (62) means that the fermions form the spin singlet states and the bosons condensed at the zero momentum point.
5. Low-lying excitation

In this section, we consider the low-lying excitations in the system. We will follow the methods proposed by Takahashi [54–56], Essler and Korepin [49] very closely. The low-lying excitations are very rich due to the complicated solutions of the Bethe ansatz equations. Let us consider them one by one.

5.1. Charge–hole excitation

The simplest excitation is obtained by removing a quantum number $I_j$ from the sequence $\{I_j\}$ and putting it outside the sequence, i.e.,

$$\{I_j\} = \left\{ \frac{N-1}{2}, \ldots, m-1, m+1, \ldots, \frac{N-1}{2}, I_n \right\},$$

where $I_n = (N-1)/2 + n$, and keep the other two quantum number sequence $\{J_a, J_{a_1}\}$ unchanged. We call this excitation the charge–hole excitation since a “hole” is created under the Fermi surface and a particle outside the surface. The charge–hole excitation spectra are shown in Fig. 1. The dispersion relations of charge and hole are shown in Fig. 2.

In the thermodynamic limit, the densities of states at this excited state read

$$\rho(k) = \frac{1}{2\pi} + \frac{1}{\pi} \int_{-}\frac{2c\rho_1(\lambda) d\lambda}{c^2 + 4(k - \lambda)^2} - \frac{1}{L} \delta(k - k_h),$$

$$\rho_1(\lambda) = \frac{1}{\pi} \int_{-Q}^{Q} \frac{2c\rho(k) dk}{c^2 + 4(\lambda - k)^2} - \frac{1}{\pi} \int_{-}^{B} \frac{c\rho_1(\lambda') d\lambda'}{c^2 + (\lambda' - \lambda')^2}$$

$$+ \frac{1}{\pi} \int_{-D}^{D} \frac{2c\rho_1^{(1)}(\lambda^{(1)}) d\lambda^{(1)}}{c^2 + 4(\lambda - \lambda^{(1)})^2} + \frac{1}{\pi L} \frac{2c}{c^2 + 4(\lambda - k_{\rho})^2},$$
\[ \rho^{(1)}(\lambda^{(1)}) = \int_{-B}^{B} \frac{2c \rho_1(\lambda)}{c^2 + 4(\lambda^{(1)} - \lambda)^2} \, d\lambda, \]  

(63)

where \( k_h \) is the momentum of the hole and \( k_p \) represent the momentum of quasi-particles. We use the same notations \( Q, D \) and \( B \) to present the new integral limits.

Now, we calculate the excitation energy \( E_{\text{ex}} = E - E_{\text{GS}} \), where \( E_{\text{GS}} \) is the ground state energy. Follow the methods proposed by Takahashi [54–56], Ebler and Korepin [49], we define the differences of the densities of states between the ground state and the excited state as

\[ \sigma_1(k) = L\left[ \rho(k) - \rho_{\text{GS}}(k) \right], \quad \sigma_2(\lambda) = L\left[ \rho_1(\lambda) - \rho_{1,\text{GS}}(\lambda) \right], \]

\[ \sigma_3(\lambda^{(1)}) = L\left[ \rho^{(1)}(\lambda^{(1)}) - \rho^{(1)}_{\text{GS}}(\lambda^{(1)}) \right], \]  

(64)

where \( \rho_{\text{GS}}, \rho_{1,\text{GS}} \) and \( \rho^{(1)}_{\text{GS}} \) are the corresponding densities at the ground state. The corrections to the densities are

\[ \varphi_1 = -\delta(k - k_h), \quad \varphi_2 = \frac{2c}{\pi c^2 + 4(\lambda - k_p)^2}, \quad \varphi_3 = 0. \]  

(65)

Define the bare energies as

\[ \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \end{pmatrix} = \begin{pmatrix} 1 & -\hat{a}_1^B & 0 \\ -\hat{a}_1^Q & 1 + \hat{a}_2^B & -\hat{a}_1^Q \\ 0 & -\hat{a}_1^B & 1 \end{pmatrix}^{-1} \begin{pmatrix} k^2 \\ 0 \\ 0 \end{pmatrix}, \]  

(66)

where the integral operator \( \hat{a}_n^A(x) \) satisfies

\[ \hat{a}_n^A(x) = \frac{1}{\pi} \frac{2nc}{n^2c^2 + 4x^2}, \quad \hat{a}_n^A(x) * f = \int_{-A}^{A} \frac{1}{\pi} \frac{2nc}{n^2c^2 + 4(x - y)^2} \, f(y) \, dy. \]  

(67)

We obtain the excitation energy as

\[ E_{\text{ex}} = \sum_{\alpha=1}^{3} a_{\alpha} \epsilon_\alpha(\mu) \varphi_\alpha \, d\mu = \epsilon(k_p) - \epsilon(k_h), \]  

(68)
where $\epsilon(k)$ is the dressed energy

$$
\epsilon(k) = k^2 + \frac{1}{\pi} \int_{-B}^{B} \frac{2ce_n(\lambda) d\lambda}{c^2 + 4(k - \lambda)^2},
$$

(69)

$$
\epsilon_n(\lambda) = \frac{1}{\pi} \int_{-Q}^{Q} \frac{2nc(\lambda) d\lambda}{c^2 + 4(\lambda - k)^2} - \frac{1}{\pi} \int_{-B}^{B} A_{1n}(\lambda - \lambda') \epsilon_1(\lambda') d\lambda' + \frac{1}{\pi} \int_{-D}^{D} \frac{2nc\epsilon_1^{(1)}(\lambda^{(1)}) d\lambda^{(1)}}{n^2c^2 + 4(\lambda - \lambda^{(1)})^2},
$$

(70)

$$
\epsilon_1^{(1)}(\lambda^{(1)}) = \frac{1}{\pi} \int_{-B}^{B} \frac{2c\epsilon_1(\lambda) d\lambda}{c^2 + 4(\lambda^{(1)} - \lambda)^2}.
$$

(71)

The dress energies at zero temperature are calculated from the thermodynamic limit. The detailed derivations and the further explanations can be found in the next section.

5.2. Spin wave excitation

The second class excitation is flipping one “spin”, which means add two holes in the distribution of the quantum integer series $\{J^{(1)}_\alpha\}$. Then the quantum numbers change from integer to half-odd-integer or vice versa. We denote the spectral parameters corresponding to the missing integers in the $\{J^{(1)}_\alpha\}$ by $\lambda_1^h$ and $\lambda_2^h$. In the thermodynamic limit we obtain following coupled integral equations for a state with two holes

$$
\rho(k) = \frac{1}{2\pi} + \frac{1}{\pi} \int_{-B}^{B} \frac{2c\rho_1(\lambda) d\lambda}{c^2 + 4(k - \lambda)^2},
$$

$$
\rho_1(\lambda) = \frac{1}{\pi} \int_{-Q}^{Q} \frac{2c\rho(\lambda) d\lambda}{c^2 + 4(\lambda - k)^2} - \frac{1}{\pi} \int_{-B}^{B} \frac{c\rho_1(\lambda') d\lambda'}{c^2 + (\lambda - \lambda')^2} + \frac{1}{\pi} \int_{-D}^{D} \frac{2c\rho_1^{(1)}(\lambda^{(1)}) d\lambda^{(1)}}{c^2 + 4(\lambda - \lambda^{(1)})^2} - \frac{1}{L} \sum_{j=1}^{2} \delta(\lambda - \lambda_j^h),
$$

$$
\rho^{(1)}(\lambda^{(1)}) = \frac{1}{\pi} \int_{-B}^{B} \frac{2c\rho_1(\lambda) d\lambda}{c^2 + 4(\lambda^{(1)} - \lambda)^2}.
$$

(72)

The differences of the densities of states and the bare energies are defined as before. The corrections to the densities at present case are

$$
\varphi_1 = 0, \quad \varphi_2 = -\sum_{j=1}^{2} \delta(\lambda - \lambda_j^h), \quad \varphi_3 = 0.
$$

(73)
Fig. 3. The spinon–spinon excitation spectra calculated for $L = 98, N_{f_1} = N_{f_2} = 25, N_{b_1} = 48$, and the coupling $c = 1.0$ (left) and $c = 10$ (right).

We obtain the excitation energy as

$$E_{\text{ex}} = E - E_{\text{GS}} = \sum_{\alpha=1}^{3} \int_{a_{\alpha}}^{d_{\alpha}} \varepsilon_{\alpha}(\mu) \varphi_{\alpha} d\mu = -\varepsilon(\lambda_{1}^{h}) - \varepsilon(\lambda_{2}^{h}).$$

(74)

This excitation is gapless. The total $S = 1$ and the total $S^z = 1$, thus it is a spin triplet excitation.

Another spin excitation is the spin singlet excitation. This case means digging two holes in the quantum number sequence $\{J_{\alpha}^{(1)}\}$ and constructing one $\lambda$-string of length 2 (with the quantum number $J_{\alpha}^{(2)}$). Thus in the rapidities $\{\lambda\}$, two of them form a 2-string ($\lambda_{1}^{(2)} \pm i c/2$) and the rest are real. The string center $\lambda_{1}^{(2)}$ is determined by

$$\sum_{j=1}^{N} \Xi_{1}(\lambda_{1}^{(2)} - k_{j}) = 2\pi J_{\alpha}^{(2)} + \sum_{\beta=1}^{N_{b_1} + N_{f_2} - 2} \Xi_{1/2}(\lambda_{1}^{(2)} - \lambda_{1}^{(2)} - \lambda_{\beta}^{1}) + \sum_{\gamma=1}^{N_{b_1}} \Xi_{1}(\lambda_{\gamma}^{1} - \lambda_{1}^{(2)}).$$

(75)

If the system-size keeps finite, the Bethe ansatz equations can be solved numerically. The excitation spectra and dispersion relation are shown in Figs. 3 and 4, respectively. If the system-size tends to infinity, we obtain the following coupled integral equations for a state with two holes and one 2-string

$$\rho(k) = \frac{1}{2\pi} + \frac{1}{\pi} \int_{-B}^{B} \frac{2c \rho_{1}(\lambda) d\lambda}{c^2 + 4(k - \lambda)^2} + \frac{1}{\pi L} \frac{c}{c^2 + (k - \lambda)^2},$$

$$\rho_{1}(\lambda) = \frac{1}{\pi} \int_{-Q}^{Q} \frac{2c \rho_{1}(\lambda) d\lambda}{c^2 + 4(\lambda - k)^2} - \frac{1}{\pi} \int_{-B}^{B} \frac{c \rho_{1}(\lambda') d\lambda'}{c^2 + (\lambda - \lambda')^2} + \frac{1}{\pi} \int_{-D}^{D} \frac{2c \rho_{1}(\lambda^{(1)} d\lambda^{(1)}{}}{c^2 + 4(\lambda - \lambda^{(1)})^2}$$

$$- \frac{1}{L} \sum_{j=1}^{2} \delta(\lambda - \lambda_{j}^{(2)}) - \frac{1}{\pi L} \left( \frac{2c}{c^2 + 4(\lambda - \lambda_{1}^{(2)})^2} + \frac{6c}{9c^2 + 4(\lambda - \lambda_{1}^{(2)})^2} \right).$$
\[
\rho^{(1)}(\lambda^{(1)}) = \frac{1}{\pi} \int_{-B}^{B} \frac{2c \rho_1(\lambda) d\lambda}{c^2 + 4(\lambda^{(1)} - \lambda)^2} + \frac{1}{\pi L} \frac{c}{c^2 + (\lambda^{(1)} - \lambda)^2}. \tag{76}
\]

The corrections to the densities of states at present case are
\[
\varphi_1 = \frac{1}{\pi} \frac{c}{c^2 + (k - \lambda)^2}, \quad \varphi_3 = \frac{1}{\pi} \frac{c}{c^2 + (\lambda^{(1)} - \lambda)^2},
\]
\[
\varphi_2 = -\sum_{j=1}^{2} \delta(\lambda - \lambda_j) - \frac{1}{\pi} \left( \frac{2c}{c^2 + 4(\lambda - \lambda^{(1)}_j)^2} + \frac{6c}{9c^2 + 4(\lambda - \lambda^{(2)}_j)^2} \right). \tag{77}
\]

The excitation energy is
\[
E_{\text{ex}} = \sum_{\alpha=1}^{3} \int_{-a_\alpha}^{a_\alpha} \epsilon_\alpha(\mu) \varphi_\alpha d\mu.
\]
In order to calculate the excitation energy, we recall the definition (70) of the dressed energy \(\epsilon_n(\lambda)\) of \(\lambda\)-string with the length \(n\). It is easy to prove that \(\epsilon_2(\lambda) = 0\) for the present case, then we obtain the excitation energy as
\[
E_{\text{ex}} = -\epsilon(\lambda^{(1)}_1) - \epsilon(\lambda^{(1)}_2). \tag{78}
\]
We see that the excitation energy (78) is the same as that of the excitation with only two holes and no strings.

### 5.3. Isospin excitation

The third excitation is replacing \(p\) holes in the ground state distribution of the quantum number series \(\{J^{(2)}_\alpha\}\) by a \(\lambda^{(2)}\)-string with the length \(p\), \(\lambda^{(2)}_j = \kappa + ic(p + 1 - 2j)/2\), where \(\kappa\) is a real number and \(j = 1, \ldots, p\). The \(p\)-string solution describes an excitation of \(p\) particles bound state. This excitation is spinless. The quantum numbers \(I\) and \(J_\alpha\) are the same as that at the ground state, while the quantum number \(J^{(1)}_\alpha\) jumps from half-odd integer to integer. The allowed range of integer for the \(\lambda^{(2)}\)-string is \(|J^{(2)}| \leq \frac{1}{2}(N_{b_1} - 2p)\). The center \(\kappa\) of the \(\lambda^{(2)}\)-string is determined by
\[
2\pi J^{(2)p} = \sum_{\gamma=1}^{N_{b_1}} \Xi_{p/2}(\kappa - \lambda^{(1)}_\gamma). \tag{79}
\]

![Fig. 4. The dispersion relation of the spinon excitation.](image-url)
Please see the next section for further explanations. The densities of states at this excitation are

\[ \rho(k) = \frac{1}{2\pi} + \frac{1}{\pi} \int_{-B}^{B} \frac{2c\rho_1(\lambda) \, d\lambda}{c^2 + 4(k - \lambda)^2}, \]

\[ \rho_1(\lambda) = \frac{1}{\pi} \int_{-Q}^{Q} \frac{2c\rho(k) \, dk}{c^2 + 4(\lambda - k)^2} - \frac{1}{\pi} \int_{-B}^{B} \frac{c\rho_1(\lambda') \, d\lambda'}{c^2 + (\lambda - \lambda')^2} + \frac{1}{\pi} \int_{-D}^{D} \frac{2c\rho_1^{(1)}(\lambda^{(1)}) \, d\lambda^{(1)}}{c^2 + 4(\lambda - \lambda^{(1)})^2}, \]

\[ \rho^{(1)}(\lambda^{(1)}) = \frac{1}{\pi} \int_{-B}^{B} \frac{2c\rho_1(\lambda) \, d\lambda}{c^2 + 4(\lambda^{(1)} - \lambda)^2} - \frac{1}{\pi} \frac{2pc}{p^2c^2 + 4(\lambda^{(1)} - \kappa)^2}. \]  

(80)

Thus the corrections to the densities are

\[ \varphi_1 = \varphi_2 = 0, \quad \varphi_3 = -\frac{1}{\pi} \frac{2pc}{p^2c^2 + 4(\lambda^{(1)} - \kappa)^2}. \]  

(81)

Using the similar method, we obtain the excitation energy as

\[ E_{\text{ex}} = -\frac{1}{\pi} \int_{-B}^{B} \frac{2c}{c^2 + 4(\lambda^{(1)} - \kappa)^2} \epsilon^{(1)}(\lambda^{(1)}) \, d\lambda^{(1)} = \epsilon^{(2)}_p(\kappa). \]  

(82)

Another isospin excitation is the \( \lambda^{(1)} - \lambda^{(2)} \)-string excitation. The quantum number series \( \{I_j\} \), \( \{J_\alpha\} \) and \( \{J^{(1)}_\alpha\} \) are filled symmetrically around the zero in this excitation. In the rapidities \( \{\lambda^{(1)}\} \), two of them form the \( \lambda^{(1)} - \lambda^{(2)} \)-string, \( \lambda^{(1)} = \lambda^{(2)} \pm ic/2 \) and the rest are real. The allowed range of integer for the \( \lambda^{(1)} - \lambda^{(2)} \)-string is \( |J_2^{(1)}| \leq \frac{1}{2}(N_{b1} - 1) \). The center of the \( \lambda^{(1)} - \lambda^{(2)} \)-string \( \kappa = \lambda^{(2)} \) is determined by

\[ 2\pi J_2^{(1)} = \sum_{\alpha=1}^{N_{b1}+N_{j2}} \mathcal{E}_1(\kappa - \lambda_\alpha) - \sum_{\gamma=1}^{N_{b1}-1} \mathcal{E}_{1/2}(\kappa - \lambda_\gamma^{(1)}). \]  

(83)

The Bethe ansatz equations and energy spectrum for the finite system-size case can be solved numerically. From that, we obtain the isospinon–isospinon excitation spectra and corresponding dispersion relation, which are shown in Figs. 5 and 6, respectively. Comparing Figs. 4 and 6, we see that the excitation spectra of spinon is linear while that of isospinon is quadratic for the small momentum.

If the system-size tends to infinity, we obtain the densities of states at this excitation as

\[ \rho(k) = \frac{1}{2\pi} + \frac{1}{\pi} \int_{-B}^{B} \frac{2c\rho_1(\lambda) \, d\lambda}{c^2 + 4(k - \lambda)^2}, \]

\[ \rho_1(\lambda) = \frac{1}{\pi} \int_{-Q}^{Q} \frac{2c\rho(k) \, dk}{c^2 + 4(\lambda - k)^2} - \frac{1}{\pi} \int_{-B}^{B} \frac{c\rho_1(\lambda') \, d\lambda'}{c^2 + (\lambda - \lambda')^2} 

+ \frac{1}{\pi} \int_{-D}^{D} \frac{2c\rho_1^{(1)}(\lambda^{(1)}) \, d\lambda^{(1)}}{c^2 + 4(\lambda - \lambda^{(1)})^2} + \frac{1}{\pi} \frac{c}{p^2c^2 + (\lambda - \kappa)^2}, \]
Fig. 5. The isospinon–isospinon excitation spectra calculated for $L = 98$, $N_f = 50$, $N_b = 48$, and the coupling $c = 1.0$ (left) and $c = 10$ (right).

Fig. 6. The dispersion relation of the elementary excitation: isospinon.

$$
\rho^{(1)}(\lambda^{(1)}) = \frac{1}{\pi} \int_{-B}^{B} \frac{2c\rho_1(\lambda) d\lambda}{c^2 + 4(\lambda^{(1)} - \lambda)^2} - \frac{1}{\pi L} \frac{2c}{c^2 + 4(\lambda^{(1)} - \kappa)^2}.
$$

Thus the corrections to the densities are

$$
\varphi_1 = 0, \quad \varphi_2 = \frac{1}{\pi} \frac{c}{c^2 + (\lambda - \kappa)^2}, \quad \varphi_3 = -\frac{1}{\pi} \frac{2c}{c^2 + 4(\lambda - \kappa)^2}.
$$

Using the similar method, we obtain the excitation energy as

$$
E_{\text{ex}}(\kappa) = \frac{1}{\pi} \int_{-D}^{D} \frac{c}{c^2 + (\lambda - \kappa)^2} \epsilon_1(\lambda) d\lambda - \frac{1}{\pi} \int_{-B}^{B} \frac{2c}{c^2 + 4(\lambda^{(1)} - \kappa)^2} \epsilon^{(1)}(\lambda^{(1)}) d\lambda^{(1)} = \epsilon^{(1)}_2(\kappa).
$$
6. Thermodynamics of the system

The finite-temperature properties of the system can be studied based on the solutions of the Bethe ansatz equations in the thermodynamic limit. The solutions of the Bethe ansatz equations are a little bit complicated. Besides real solutions, the Bethe ansatz equations also have complex solutions. Generally, the complex solutions are determined by the poles or zeros of the Bethe ansatz equations in the thermodynamic limit. Which gives us a hint to determine the structures of the solutions of the Bethe ansatz equations. In the following, we only consider the case \( c > 0 \), because the bosons with attractive interactions \( c < 0 \) do not have the thermodynamics [54].

After some analysis, we find that the structures of the solutions of the Bethe ansatz equations (49)–(52) are: (1) All the momentums \( k_j \) are real. (2) The rapidities \( \lambda \) form the \( m \)-string, \( \lambda_j = \lambda_{k_j} + (m + 1 - 2j)ic/2 + o(e^{-\delta L}), \) where \( \lambda_{k_j} \) are real and \( j = 1, \ldots, m \). (3) Some rapidities \( \lambda^{(1)} = \lambda^{(1)r} \) are real. (4) Some rapidities \( \lambda^{(1)} \) form the \( \lambda^{(1)} - \lambda^{(2)} \)-strings, \( \lambda^{(1)}_\alpha = \lambda^{(1)}_\beta \pm ic/2 + o(e^{-\delta L}) \) and \( \lambda^{(2)} = \lambda^{(1)} \), where \( \lambda^{(1)}_\beta \) are real. (5) Some rapidities \( \lambda^{(2)} \) are real, which are the real part of the \( \lambda^{(1)} - \lambda^{(2)} \)-strings. (6) Some \( \lambda^{(2)} \) form the \( p \)-strings, \( \lambda^{(2)}_p = \lambda^{(2)}_q + (p + 1 - 2l)ic/2 + o(e^{-\delta L}), \) where \( \lambda^{(2)}_r \) are real and \( l = 1, \ldots, p \).

At the temperature \( T \), the system (1) arrive at the thermal equilibrium. We denote the density of momentum \( k \) by \( \rho(k) \), the density of \( \lambda \)-strings with length \( n \) by \( \rho_n(\lambda) \), the density of real rapidity \( \lambda^{(1)} \) by \( \rho^{(1)}_1(\lambda^{(1)}) \), the density of \( \lambda^{(1)} - \lambda^{(2)} \)-strings by \( \rho^{(1)}_2(\lambda^{(1)}) \) and the density of \( \lambda^{(2)} \)-strings by \( \rho^{(2)}_p(\lambda^{(2)}) \). Meanwhile, the notations \( \rho^h(k), \rho^h_n(\lambda), \rho^{(1)h}_1(\lambda^{(1)}), \rho^{(2)h}_p(\lambda^{(2)}) \) represents the densities of corresponding holes. From the Bethe ansatz equations (49)–(52), these densities should satisfy

\[
\rho(k) + \rho^h(k) = \frac{1}{2\pi} + \sum_{n=1}^{\infty} a_n \ast \rho_n(k),
\]

\[
\rho_n(\lambda) + \rho^h_n(\lambda) = a_n \ast \rho(\lambda) - \sum_{m=1}^{\infty} A_{n,m} \ast \rho_m(\lambda) + (A_{n,1} - \delta_{n,1}) \ast \rho^{(1)}_2(\lambda) + a_n \ast \rho^{(1)}_1(\lambda),
\]

\[
\rho^{(1)}_1(\lambda^{(1)}) + \rho^{(1)h}_1(\lambda^{(1)}) = \sum_{n=1}^{\infty} a_n \ast \rho_n(\lambda^{(1)}) - \sum_{p=1}^{\infty} a_p \ast \rho^{(2)}_p(\lambda^{(1)}) - a_1 \ast \rho^{(1)}_1(\lambda^{(1)}),
\]

\[
\rho^{(2)}_p(\lambda^{(2)}) + \rho^{(2)h}_p(\lambda^{(2)}) = -\sum_{q=1}^{\infty} A_{p,q} \ast \rho^{(2)}_q(\lambda^{(2)}) + a_p \ast \rho^{(1)}_1(\lambda^{(2)}),
\]

where the integral operators \( a_n(x) \) and \( A_{nm}(x) \) are

\[
a_n(x) = \frac{1}{\pi} \frac{2nc}{n^2c^2 + 4x^2}, \quad a_0(x) = \delta(x),
\]

\[
A_{n,m}(x) = \begin{cases} a_{\lfloor n-m \rfloor}(x) + 2a_{\lfloor n-m \rfloor+2}(x) + \cdots + 2a_{n+m-2}(x) + a_{n+m}(x), & \text{if } n \neq m, \\ 2a_2(x) + 2a_4(x) + \cdots + 2a_{2n-2}(x) + a_{2n}(x), & \text{if } n = m. \end{cases}
\]

The convolution is defined as \( a \ast b(x) = \int a(x-y)b(y) \, dy \).
The Gibbs free energy at a given temperature $T$, chemical potential $A$ and external magnetic field $h$ is given by

$$G(T, A, h) = E - AN - h(M^f + M^b) - TS.$$  \hspace{1cm} (89)

The densities of energy and particles number are $E/L = \int_{-Q}^{Q} k^2 \rho(k) dk$, $N/L = \int_{-Q}^{Q} \rho(k) dk$. The magnetization for fermions and that for bosons are

$$M^f = Nf_1 - Nf_2 = -M^{f(1)}_1 - 2M^{f(2)}_1 + 2 \sum_{n=1}^{\infty} nM_n - N,$$  \hspace{1cm} (90)\nn
$$M^b = g(Nb_1 - Nb_2) = 2g \sum_{p=1}^{\infty} pM^{(2)}_p - gM^{(1)}_1,$$  \hspace{1cm} (91)\n
where $g$ is the Landau $g$-factor. The entropy of the system is $S/L = \int[(\rho + \rho^h) \ln(\rho + \rho^h) - \rho \ln \rho - \rho^h \ln \rho^h] d\lambda + \int[(\rho^{f(1)})^h \ln(\rho^{f(1)})^h + \rho^{f(1)}(1) \ln(\rho^{f(1)}(1)) - \rho^{f(1)}(1) \ln(\rho^{f(1)}(1))h] d\lambda^{(1)} + \int[(\rho^{b(1)})^h \ln(\rho^{b(1)})^h + \rho^{b(1)}(1) \ln(\rho^{b(1)}(1)) - \rho^{b(1)}(1) \ln(\rho^{b(1)}(1))h] d\lambda^{(1)} + \sum_{p=1}^{\infty} \int[(\rho^{(2)}_p + \rho^{(2)}_p)^h \ln(\rho^{(2)}_p + \rho^{(2)}_p) - \rho^{(2)}_p \ln(\rho^{(2)}_p - \rho^{(2)}_p) \ln(\rho^{(2)}_p)] d\lambda^{(2)}$.

For convenience, we introduce following notations $\xi = \rho^h / \rho$, $\alpha_n = \rho_n^h / \rho_n$, $\beta_1 = \rho_1^{(1)}h / \rho_1(1)$, $\beta_2 = \rho_2^{(1)}h / \rho_2(1)$ and $\gamma_p = \rho_p^{(2)}h / \rho_p(2)$. At the thermodynamic equilibrium state, the free energy must be minimized. From the variation of free energy is zero, we obtain following coupled non-linear integration equations

$$\ln \xi(k) = \frac{k^2 - A + h}{T} - \sum_{n=1}^{\infty} a_n * \ln(1 + \alpha_n^{-1}(k)),$$  \hspace{1cm} (92)\n
$$\ln \alpha_n(\lambda) = -\frac{2nh}{T} + \sum_{m=1}^{\infty} A_{m,n} * \ln(1 + \alpha_m^{-1}(\lambda)) - a_n(\lambda) \ln(1 + \xi^{-1}(\lambda))$$

$$+ a_n(\lambda) \ln(1 + \beta_1^{-1}(\lambda)) - (A_{n-1,1} - \delta_{n,1}) * \ln(1 + \beta_2^{-1}(\lambda)),$$  \hspace{1cm} (93)\n
$$\ln \beta_1(\lambda^{(1)}) = \frac{(1+g)h}{T} - \sum_{n=1}^{\infty} a_n(\lambda) \ln(1 + \alpha_n^{-1}(\lambda^{(1)}))$$

$$+ a_1 \ln(1 + \beta_2^{-1}(\lambda^{(1)})) - \sum_{p=1}^{\infty} a_p \ln(1 + \gamma_p^{-1}(\lambda^{(1)})),$$  \hspace{1cm} (94)\n
$$\ln \beta_2(\lambda^{(1)}) = \frac{2h}{T} - \sum_{n=1}^{\infty} (A_{n,1} - \delta_{n,1}) * \ln(1 + \alpha_n^{-1}(\lambda^{(1)}))$$

$$+ a_2 \ln(1 + \beta_2^{-1}(\lambda^{(1)})) + a_1 \ln(1 + \beta_1^{-1}(\lambda^{(1)})),$$  \hspace{1cm} (95)\n
$$\ln \gamma_p(\lambda^{(2)}) = -\frac{2pgh}{T} + \sum_{q=1}^{\infty} A_{q,p} \ln(1 + \gamma_q^{-1}(\lambda^{(2)})) + a_p \ln(1 + \beta_1^{-1}(\lambda^{(2)})),$$  \hspace{1cm} (96)\n
where $*$ denotes the convolution.
7. Special limits

In principle, the thermodynamic Bethe ansatz equations cannot be solved analytically. One has to use the numerical simulations or the approximate methods. However, at some special limit cases, the thermodynamic Bethe ansatz equations can be solved exactly.

7.1. Zero temperature limit

We first consider the zero temperature limit. The dressed energies $\epsilon$, $\epsilon_n$, $\epsilon_1^{(1)}$, $\epsilon_2^{(1)}$ and $\epsilon_p^{(2)}$ are defined as $\xi = \exp(\epsilon/T)$, $\alpha_n = \exp(\epsilon_n/T)$, $\beta_1 = \exp(\epsilon_1^{(1)}/T)$, $\beta_2 = \exp(\epsilon_2^{(1)}/T)$ and $\gamma_p = \exp(\epsilon_p^{(2)}/T)$. Substituting them into the thermodynamic Bethe ansatz equations (92)–(96) and letting the temperature tends to zero, we obtain the analytic formulas for the dressed energies

$$
\epsilon(k) = k^2 + \frac{1}{\pi} \int_{-B}^{B} \frac{2c\epsilon\lambda}{{c^2} + 4(k - \lambda)^2},
$$

$$
\epsilon_n(\lambda) = \frac{1}{\pi} \int_{-Q}^{Q} \frac{2nc\epsilon(k)}{c^2 + 4(\lambda - \lambda_k)^2} - \frac{1}{\pi} \int_{-B}^{B} A_{1,n}(\lambda - \lambda')\epsilon_1(\lambda')d\lambda',
$$

$$
\epsilon_1^{(1)}(\lambda^{(1)}) = \frac{1}{\pi} \int_{-B}^{B} \frac{2c_1(\lambda)}{c^2 + 4(\lambda^{(1)} - \lambda)^2},
$$

$$
\epsilon_2^{(1)}(\lambda^{(1)}) = \frac{1}{\pi} \int_{-B}^{B} \frac{c_1(\lambda)}{c^2 + (\lambda^{(1)} - \lambda)^2} - \frac{1}{\pi} \int_{-D}^{D} \frac{2c_1(\lambda)}{c^2 + 4(\lambda^{(1)} - \lambda)^2},
$$

$$
\epsilon_p^{(2)}(\lambda^{(2)}) = -\frac{1}{\pi} \int_{-B}^{B} \frac{2p_1\epsilon_1^{(1)}(\lambda^{(1)})d\lambda^{(1)}}{c^2 + 4(\lambda^{(2)} - \lambda^{(1)})^2}.
$$

These equations have been used in calculating the low-lying excitation energies.

7.2. High temperature limit

If the temperature $T$ tends to infinity, the rapidities $\alpha_n$, $\beta_1$, $\beta_2$ and $\gamma_p$ become constants and the thermodynamic Bethe ansatz equations become a set of coupled algebraic equations. The thermodynamic Bethe ansatz equations (93) and (96) read

$$
\ln \alpha_1 = \frac{1}{2} \ln \left[ \frac{1 + \alpha_2}{1 + \beta_1^{-1}} \right],
\ln \alpha_2 = \frac{1}{2} \ln \left[ \frac{(1 + \alpha_1)(1 + \alpha_3)}{1 + \beta_2^{-1}} \right],
\ln \alpha_n = \frac{1}{2} \ln \left[ (1 + \alpha_{n-1})(1 + \alpha_{n+1}) \right], \quad n \geq 3,
$$
\[
\ln \gamma_1 = \frac{1}{2} \ln \left[ (1 + \gamma_2)(1 + \beta_1^{-1}) \right], \\
\ln \gamma_p = \frac{1}{2} \ln \left[ (1 + \gamma_{p-1})(1 + \gamma_{p+1}) \right], \quad p \geq 2.
\]

The solutions of Eq. (102) are
\[
\alpha_n = g^2(n) - 1, \quad n \geq 2, \quad \alpha_1 = g(2)/f(0), \\
\gamma_p = f^2(p) - 1, \quad p \geq 1, \quad \beta_1 = 1/[f^2(0) - 1],
\]
where \( g(n) = (a^n - a^{-n})/(a - a^{-1}) \), \( f(p) = (bd^p - b^{-1}d^{-p})/(d - d^{-1}) \) and the parameters \( a, b, d \) are determined by
\[
\ln \xi = \frac{k^2 - A + h T}{T} - \sum_{n=1}^{\infty} \ln(1 + \alpha_n^{-1}), \\
\ln(1 + \alpha_1) = -\frac{2h}{T} - 2 \sum_{n=1}^{\infty} \ln(1 + \alpha_n^{-1}) - \ln(1 + \xi^{-1}) - \ln(1 + \beta_1^{-1}) - \ln(1 + \beta_2^{-1}), \\
\ln \beta_1 = \frac{(1 + g)h}{T} - \sum_{n=1}^{\infty} \ln(1 + \alpha_n^{-1}) + \ln(1 + \beta_2^{-1}) - \sum_{p=1}^{\infty} \ln(1 + \gamma_p^{-1}), \\
\ln \beta_2 = \frac{2h}{T} - 2 \sum_{n=1}^{\infty} \ln(1 + \alpha_n^{-1}) + \ln(1 + \alpha_1^{-1}) + \ln(1 + \beta_1^{-1}) + \ln(1 + \beta_2^{-1}), \\
\ln(1 + \gamma_1) = -\frac{2gh}{T} + 2 \sum_{p=1}^{\infty} \ln(1 + \gamma_p^{-1}) + \ln(1 + \beta_1^{-1}), \\
\lim_{n \to \infty} \frac{\ln \alpha_n}{n} = \frac{2h}{T}, \quad \lim_{p \to \infty} \frac{\ln \gamma_p}{p} = \frac{2gh}{T}.
\]

7.3. Weak coupling limit

If the coupling constant \( c \) tends to zero, the thermodynamic Bethe ansatz equations can be simplified as
\[
\ln \xi(k) = \frac{k^2 - A}{T} - \frac{1}{2} \ln \left[ (1 + \alpha_1(k))(1 + \xi^{-1}(k))(1 + \beta_1^{-1}(k))(1 + \beta_2^{-1}(k)) \right], \\
\ln \alpha_1(\lambda) = \frac{1}{2} \left[ \ln(1 + \alpha_2(\lambda)) - \ln(1 + \beta_1^{-1}(\lambda)) \right], \\
\ln \alpha_2(\lambda) = \frac{1}{2} \left[ \ln(1 + \alpha_1(\lambda)) + \ln(1 + \alpha_3(\lambda)) - \ln(1 + \beta_1^{-1}(\lambda)) \right], \\
\ln \alpha_n(\lambda) = \frac{1}{2} \ln \left[ (1 + \alpha_{n-1}(\lambda))(1 + \alpha_{n+1}(\lambda)) \right], \quad n \geq 3, \\
\ln \beta_1(\lambda^{(1)}) = -\frac{1}{2} \ln \left[ (1 + \alpha_1(\lambda^{(1)}))(1 + \gamma_1(\lambda^{(1)}))(1 + \xi^{-1}(\lambda^{(1)}))(1 + \beta_1^{-1}(\lambda^{(1)})) \right], \\
\ln \beta_2(\lambda^{(1)}) = -\ln \left[ \alpha_1(\lambda^{(1)})(1 + \xi^{-1}(\lambda^{(1)})) \right], \\
\ln \gamma_1(\lambda^{(2)}) = \frac{1}{2} \ln \left[ (1 + \gamma_2(\lambda^{(2)})) - \ln(1 + \beta_1^{-1}(\lambda^{(2)})) \right],
\]
\[
\ln \gamma_p(\lambda^{(2)}) = \frac{1}{2} \ln \left[ (1 + \gamma_{p-1}(\lambda^{(2)}))(1 + \gamma_{p+1}(\lambda^{(2)})) \right], \quad p \geq 2,
\]

\[
\lim_{n \to \infty} \frac{\ln \alpha_n(k)}{n} = \frac{2h}{T}, \quad \lim_{p \to \infty} \frac{\ln \gamma_p(\lambda^{(2)})}{p} = \frac{2gh}{T}.
\]

7.4. Strong coupling limit

If the coupling constant \( c \) tends to infinity, the momentum rapidities are completely decoupled with other rapidities. Meanwhile, the solutions for \( \alpha_n, \beta_1, \beta_2 \) and \( \gamma_p \) turn into constants and are independent of the variables \( \lambda, \lambda^{(1)} \) and \( \lambda^{(2)} \). The thermodynamic Bethe ansatz equations can be solved analytically in this case. The solutions of Eqs. (93) and (96) still take the form of (103), where the boundary conditions (104) become \( \alpha_1 = \beta_2^{-1} \) and \( \ln \beta_1 = -\frac{1}{2} \ln(1 + \gamma_1) \). Then the analytic solutions of thermodynamic Bethe ansatz equations are

\[
\xi = e^{\frac{g^2 - A}{T}} \left( 2 \cosh \frac{gh}{T} + 2 \cosh \frac{h}{T} \right)^{-1}, \quad \alpha_n = \left( \frac{\sinh \frac{nh}{T}}{\sinh \frac{h}{T}} \right)^2 - 1, \quad n \geq 2,
\]

\[
\alpha_1 = \frac{\cosh \frac{h}{T}}{\cosh \frac{gh}{T}} = \beta_2^{-1}, \quad \beta_1 = \frac{1}{4 \cosh^2 \frac{gh}{T} - 1},
\]

\[
\gamma_p = \left( \frac{\sinh \left( \frac{(p+2)gh}{T} \right)}{\sinh \frac{gh}{T}} \right)^2 - 1, \quad p \geq 1.
\]

8. Conclusions

In summary, we study the exact solutions of 1D mixture of spinor bosons and spinor fermions with \( \delta \)-function interactions. The wave function for the bosonic parties is symmetric while for the fermionic parties is anti-symmetric. The global wave function of the system is supersymmetric. After obtaining the two-body scattering matrix by using the coordinate Bethe ansatz method, we prove that the system is integrable. Then we derive the energy spectrum and the Bethe ansatz equations with different gradings by using the graded nest quantum inverse scattering or algebraic Bethe ansatz method. Based on the solutions of the Bethe ansatz equations, we discuss the ground state properties of the system. We find that if the interactions are repulsive, the fermions form spin singlet states and the bosons are polarized, thus the global ground state is partial polarized. If the interactions are attractive, the bosons condensed at the zero momentum point. We discuss the charge–hole excitation, spin wave excitation, isospin wave excitation and corresponding excitation energies very detailed. We also obtain the thermodynamic Bethe ansatz equations at finite temperature and find their analytic solutions at some special limit cases.

Acknowledgements

This work is supported by the Earmarked Grant for Research from the Research Grants Council of HKSAR, China (Project Nos. CUHK 402107 and CUHK 401108), NSF of China, and the national program for basic research of MOST under Grant No. 2006CB921300. J. Cao acknowledges the financial support from the C.N. Yang Foundation.
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