Critical Behavior of the Two-Dimensional Random Quantum Ising Ferromagnet

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(March 24, 2022)

We study the quantum phase transition in the two-dimensional random Ising model in a transverse field by Monte Carlo simulations. We find results similar to those known analytically in one-dimension: the dynamical exponent is infinite and, at the critical point, the typical correlation function decays with a stretched exponential dependence on distance. Away from the critical point, there may be different exponents for the divergence of the average and typical correlation lengths, again as in one-dimension, but the evidence for this is less strong.

1. INTRODUCTION

Though classical phase transitions occurring at finite temperature are very well understood, our knowledge of quantum transitions at $T = 0$ is relatively poor, at least for systems with quenched disorder. There is, however, considerable interest in these systems since they (i) exhibit new universality classes, and (ii) display “Griffiths-McCoy” singularities even away from the critical point, due to rare regions with stronger than average interactions.

Just as the simplest model with a classical phase transition is the regular Ising model, the simplest random model with a quantum transition is arguably the Ising model in a transverse field whose Hamiltonian is given by

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} \sigma^z_i \sigma^z_j - \sum_i h_i \sigma^x_i.$$  

Here the $\{\sigma^a\}$ are Pauli spin matrices, and the nearest neighbor interactions $J_{ij}$ and transverse fields $h_i$ are both independent random variables. Naturally this model has been quite extensively studied and many analytical results are available for the case of the one-dimensional chain. Among the surprising predictions for the critical behavior in 1-d are:

1. The dynamic critical exponent, $z$, is infinite. Instead of a characteristic time scale $\xi_\tau$ varying as a power of a characteristic length scale $\xi$ according to $\xi_\tau \sim \xi^z$, one has instead an exponential relation

$$\xi_\tau \sim \exp(\text{const.} \, \xi^\mu),$$

where $\mu = 1/2$. This is called activated dynamical scaling.

2. Distributions of the equal-time $\sigma^z - \sigma^z$ correlations are very broad. As a result average and typical correlations behave rather differently, since the average is dominated by a few rare (and hence atypical) points. At the critical point, for example, the average correlation function falls off with a power of the distance $r$ as $C_{av}(r) \sim r^{-\lambda}$, where $\lambda = 2 - \phi \simeq 0.38$ with $\phi$ the golden mean, whereas the typical value falls off much faster, as a stretched exponential

$$C_{typ}(r) \sim \exp(\text{const.} \, r^\sigma),$$

with $\sigma = 1/2$.

3. Away from the critical point the average and typical correlation lengths both diverge but with different exponents, i.e. $\xi_{av} \sim \delta^{\nu_{av}}$; $\xi_{typ} \sim \delta^{\nu_{typ}}$, where $\delta$ is the deviation from criticality and

$$\nu_{av} = 2; \quad \nu_{typ} = 1.$$  

Unfortunately, the analytical approach is only valid in 1-d, and very little is known about the critical behavior in higher dimensions. An important question is whether these striking analytical results are a special feature of 1-d, or whether they are valid more generally. Here, we make a first step in investigating this question by performing large-scale Monte Carlo simulations of the random transverse field Ising ferromagnet in two dimensions. Because the ferromagnet has no frustration we are able to use highly efficient cluster algorithms which considerably reduce critical slowing down. Consequently, we are able to study larger sizes and get much better statistics than in earlier work on quantum spin glasses. Furthermore, cluster algorithms have an improved estimator for correlation functions, which greatly reduces the statistical errors when the mean is small, so we are able to probe the distribution of correlation functions out to relatively large distances.

Our main conclusion is that the behavior of the 2-d system at the critical point is very similar to that of the 1-d case, see points 1 and 2 above. There are some difficulties in interpreting our data away from the critical point but, nonetheless, here too there is some evidence for behavior similar to that in 1-d, see point 3 above. Further details and additional results will be presented elsewhere.

In related work, Rieger and Kawashima have studied the same model but with continuous imaginary time, concentrating on Griffiths-McCoy singularities in the paramagnetic phase, which are characterized by a continuously varying dynamical exponent, $z(\delta)$. Kawashima and
Riegler find that $z(\delta)$ tends to infinity as the critical point is approached ($\delta \to 0$), again as in 1-d.

As is standard, we represent the $d$-dimensional quantum Hamiltonian in Eq. (2) by an effective classical “Hamiltonian” in $(d+1)$-dimensions, where the extra dimension, imaginary time, is of size $\beta \equiv 1/T$ and is divided up into $L_\tau$ intervals each of width $\Delta \tau = \beta/L_\tau$.

The effective classical Hamiltonian is given by

$$\beta_c \mathcal{H}_{cl} = - \sum_{\langle i,j \rangle, \tau} K_{ij} S_i(\tau) S_j(\tau) - \sum_{i, \tau} \tilde{K}_i S_i(\tau) S_i(\tau + 1),$$ \hspace{1cm} (5)

where, $\tau' = \tau + \Delta \tau$, $S_i(\tau) = \pm 1$, $K_{ij} = \Delta \tau J_{ij}$, $\exp(-2\tilde{K}_i) = \tanh(\Delta \tau h_i)$, and $\beta_{cl} \equiv 1/T_{cl}$ where $T_{cl}$ is an effective “classical” temperature (not equal to the real temperature which is the inverse of the size in the time direction). Note that the disorder is quenched and so the interactions are independent of $\tau$. Correlations of the classical Ising spins, $S_i(\tau)$, correspond to $\sigma^- \sigma^+$ correlations in the original quantum model. These are the correlations that we are interested in here.

In order to capture the random quantum critical behavior in the intermediate size systems that we can simulate, we wish the disorder to be effectively quite strong. In particular, we would like some of the fields to be stronger than the bonds in their vicinity and vice-versa. In particular, we would like some of the fields to be stronger than the bonds in their vicinity and vice-versa. This is captured by having distributions for both the fields and interactions with a finite weight at the origin.

For fixed $L$, $g_{av}$ has a peak as a function of $L$. At the critical point, $T_c$, the peak height is independent of $L$ and the values of $L_\tau$ at the maximum, $L^{\text{max}}$, vary as $L^z$. Furthermore, a plot of $g_{av}$ against $L_\tau/L^{\text{max}}$ at the critical point, which has the advantage of not needing a value for $z$, should collapse the data. We see in Figs. 1 and 2 that this does not happen either in 1-d or 2-d. Rather the curves clearly become broader for larger sizes. This is easy to understand since we know that for 1-d $z$ is infinite and it is the log of the characteristic time which scales with a power of the length scale. This suggests that the scaling variable should be $\ln L_\tau/\ln L^{\text{max}}$ with $\ln L^{\text{max}} \sim L^\mu$, say. The inset to Fig. 1 shows that this works moderately well for 1-d (though not perfectly for this range of sizes) with the expected value $\mu = 1/2$. In 2-d, the data collapse for sizes $L \geq 12$, shown in the inset to Fig. 2 is quite good for $\mu = 0.42$, and not quite so good with the 1-d value, $\mu = 1/2$, though we would not claim that $\mu = 1/2$ is ruled out. The close similarity of the data for 1-d and 2-d, particularly the broadening of the data in the main part of the figures, suggests that $z$ is infinite also in 2-d.

Next we consider the behavior of the equal time correlations at the critical point using the improved estimator $\bar{\rho}(\bar{J})$. Figs. 3 and 4 show data for the average and typical correlations for spins separated by $r = L/2$ in 1-d and $\bar{r} = (L/2, 0)$ in 2-d. For each value of $L$, we took $L_\tau$ such that $g_{av}$ is close to the peak. According to finite size scaling, the dependence on $L$ for a finite system should be the same as the dependence on $r$ in a bulk system. Fig. 3 shows that in 1-d, the average correlation falls off with a power law while the typical value decays with an $\exp(-\text{const.} \ L^{1/2})$ variation, as expected. Note, though, that the exponent for the average, $-\lambda$, obtained by fitting the data gives $\lambda = 0.50$, somewhat different from the expected value of 0.38. We will discuss this discrepancy further below. The data in Fig. 3 shows that the behavior in 2-d is quite similar: the average falls off with a power law, with $\lambda$ about 1.5, while the typical value falls off faster than a power law, (because of the downward curvature)
consistent with a stretched exponential behavior of the form \( \exp(-\text{const. } L^\sigma) \), with \( \sigma \approx 1/3 \). The statistical errors (shown) are generally smaller than the size of the points, so the downward curvature is statistically significant. The value of \( \sigma \) seems to be different from the 1-d result of 1/2 (the data is curved on the appropriate plot) but we are not able to determine it with great precision.

Finally we briefly describe our data away from the critical point, keeping keep the same values of \( L \) and \( L_T \) as at the critical point. In 1-d, \( g_{av} \) scales well with the expected value \( \nu_{av} = 2 \) and the corresponding plot in 2-d also works well, this time with \( \nu_{av} \approx 1.5 \). Plots of the typical correlation function (divided by the value at criticality) give an exponent close to the expected value of \( \nu_{typ} = 1 \) in 1-d and around 0.7 in 2-d, i.e. in each case much smaller than the exponent for the average correlation length obtained from \( g_{av} \). However, attempts to determine \( \nu_{av} \) directly by scaling data for the average correlation function away from criticality were less successful. In 1-d, the fit, which works quite well, gives \( \nu_{av} \approx 1.2 \), much less than the expected value of 2, while in 2-d we found \( \nu_{av} \approx 0.75 \), not significantly different from \( \nu_{typ} \).

This discrepancy for results from average correlation function may be due to difficulties in taking enough samples to probe the rare regions which dominate the average. This may also be why the exponent for the decay of the average correlation at criticality in 1-d was not found accurately. Another concern with the data away from the critical point is that we do not understand the way that the critical singularities go over to Griffiths-McCoy singularities, which have a continuously variable \( z(\delta) \). Perhaps the latter give big corrections to scaling, thus possibly making the size of the critical region quite small.

To conclude, we have found a strong similarity between the critical behavior of the random transverse field Ising model in one and two dimensions. The evidence that \( z = \infty \) and that the typical correlations at the critical point decay with a stretched exponential function of distance are quite strong, but the evidence for different exponents for the average and typical correlations in 2-d is weaker because of discrepancies involving the data for the average correlation function.

After this work was completed, we heard that S.-C. Mau, O. Motrunich and D. A. Huse (private communication) have implemented numerically for \( d > 1 \) the renormalization group approach used in Ref. 3, finding a flow to the infinite disorder critical fixed point, just as in \( d = 1 \).
FIG. 3. The main figure shows the average and typical correlations between spins $L/2$ apart at the critical point in one dimension. As expected, the average decays with a power law (though the fit to the data gives a slope, $-\lambda$, with $\lambda = 0.50$ compared with the analytical value of 0.38), while the typical value decays faster than a power, as shown by the downward curvature in the plot. The inset shows that the typical correlation function decays as the stretched exponential, $C_{\text{typ}}(L/2) \sim \exp(-\text{const}. L^{1/2})$, as expected.

ACKNOWLEDGMENTS

We would like to thank D. S. Fisher, D. A. Huse, H. Rieger, N. Kawashima, and R. N. Bhatt for helpful discussions, and O. Narayan for a critical reading of the manuscript. This work was supported by the National Science Foundation under grant DMR 9713977 and the Deutsche Forschungsgemeinschaft (DFG) under contract Pi 337/1-2.

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