THE LATTICE OF AMOEBAS

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Abstract. We study amoebas of exponential sums as functions of the support set $A$. To any amoeba, we associate a set of approximating sections of amoebas, which we call caissons. We show that a bounded modular lattice of subspaces of a certain vector space induces a lattice structure on the set of caissons. Our results unifies the theories of lopsided amoebas and amoebas of exponential sums. As an application, we show that our theory of caissons yields improved certificates for existence of certain components of the complement of an amoeba.

1. Introduction

Amoebas were introduced by Gel’fand, Kapranov, and Zelevinsky in [7] in the context of hypergeometric functions and discriminants. They have been studied intensively over the last two decades, primarily as they form a bridge between analytic and algebraic geometry on the one hand, and tropical geometry and combinatorics on the other hand [2, 9, 10].

Determining the topology of an amoeba is a nontrivial task. Therefore, methods to approximate amoebas have been developed in, for example, [1, 5, 13]. The main tool in use is the lopsided amoeba. The term “lopsided” was introduced by Purbhoo in [13], even though the object appeared earlier in Rullgård’s thesis [14]. The available methods for approximations of amoebas have not been strong enough to tackle the main conjectures remaining in amoeba theory, for example the maximally sparse conjecture of [11] and Rullgård’s connectivity conjecture, see Section 2 and [14, p. 39]. This paper is part of an effort to obtain more refined approximations of amoebas.

In what follows, we discuss two seemingly separate approaches to the approximation problem in the generalized setting of amoebas of exponential sums. First, we investigate how amoebas behave under perturbations of their support. Second, we establish a lattice structure on the set of support sets of amoebas and introduce a new object, the $B$-caisson of an amoeba, which is a finer approximation than the lopsided amoeba, see Definition 1.1.

We show that these two approaches are essentially the same. Moreover, we demonstrate that the theory of $B$-caissons is in close analogue to the theory
of amoebas of exponential sums developed by Favorov and Silipo in [3] and [16]. Furthermore, we extend the order map for amoebas to $B$-caissons. In the end, we obtain a single theory which unifies lopsided amoebas, amoebas of exponential sums, the results by Favorov and Silipo, limits of support sets of amoebas, and functorial properties of amoebas discussed by Rullgård.

One can use $B$-caissons to certify the existence of certain connected components of the complement of an amoeba. Determining the existence of such components in dependence of the coefficients of its defining polynomial is a well-known, notoriously hard problem [11] with several applications. For example, if we consider a univariate polynomial, then the problem is to understand how the norms of its roots depend on the coefficients. This problem dates back to the late 19th century [6] and saw renewed interest in recent years [11, 15, 18] due to its impact in numerical methods, complexity theory, and applications. Using results about exponential sums supported on a barycentric circuit [17], our theory of $B$-caissons yields new results in this direction, see Corollary 6.3.

Let us explain our results in more detail. We consider an $(1+n)$-variate exponential sum

$$f(z) = \sum_{\alpha \in A} c_{\alpha} e^{\langle z, \alpha \rangle}.$$  

The set $A$ is called the support of $f$. We assume that $f(z)$ is pseudo-homogeneous in the sense that there exists a linear form $\xi \in \text{Hom}(\mathbb{R}^{1+n}, \mathbb{R})$ such that $\langle \xi, \alpha \rangle = 1$ for all $\alpha \in A$. The amoeba of an exponential sum, which was introduced by Favorov in [3], is defined by

$$A(f) = \text{Re}(V)$$

where $V = V(f) \subset \mathbb{C}^{1+n}$ is the (analytic) variety of $f$, and where the closure is taken with respect to the standard topology on $\mathbb{R}^{1+n}$.

Let $(\mathbb{C}^*)^A$ denote the family of all exponential sums (1.1) with support $A$. We often consider its closure $\mathbb{C}^A$. Let $N$ denote the cardinality of $A$. We arrange the elements of $A$ as the columns of a $(1+n) \times N$-matrix, which, by slight abuse of notation, is also denoted by $A$. The starting point of our analysis is a matrix factorization

$$A = TB,$$

where $A$ is as above and $B$ is a real, pseudo-homogeneous, $(1+m) \times N$-matrix. The matrix $T$ induces an isomorphism of $\mathbb{C}$-vector spaces $\Phi: \mathbb{C}^A \to \mathbb{C}^B$, whose inverse is given by

$$\Phi^{-1}(f)(z) = f(zT),$$

and an embedding $i: \mathbb{R}^n \to \mathbb{R}^m$ given by $x \mapsto xT$. We denote the image of $i$ by $H_T$. 

Definition 1.1. Let $A = TB$ as above and let $f \in \mathbb{C}^A$. Then, the large respectively small $B$-caissons $\mathcal{L}_B(f)$ and $\mathcal{A}_B(f)$ are defined by

$$\mathcal{L}_B(f) = \mathcal{A}(\Phi \circ f) \quad \text{and} \quad \mathcal{A}_B(f) = \mathcal{L}_B(f) \cap H_T.$$ 

Thus, $\mathcal{L}_B(f) \subset \mathbb{R}^m$, while we consider $\mathcal{A}_B(f)$ as a subset of $\mathbb{R}^n$ by taking its inverse image under the map $\iota$.

Example 1.2. We can write $A = IA$ where $I$ denotes the identity matrix of size $1 + n$. In this case $\Phi$ and $\iota$ act as the identities on $\mathbb{C}^A$ and $\mathbb{R}^{1+n}$ respectively. Thus,

$$\mathcal{A}_A(f) = \mathcal{L}_A(f) = \mathcal{A}(f).$$

Example 1.3. A second factorization of $A$ is given by $A = AI$, where $I$ denotes the identity matrix of size $N$. Note that $I$ is homogeneous with respect to the form $\xi = (1, \ldots, 1)$. In the algebraic case, $\mathcal{A}_I(f)$ coincides with the lopsided amoeba of $f$, see [5, Section 5]. We refer to $\mathcal{A}_I(f)$ as the lopsided amoeba of $f$ also in the non-algebraic setting.

This setup allows the use of two methods in the study of the topology of amoebas:

1. We can consider $B$-caissons strictly in between the amoeba $\mathcal{A}_A(f)$ and the lopsided amoeba $\mathcal{A}_I(f)$. The goal is to find a level where the structure is refined enough to capture the topology of $\mathcal{A}_A(f)$ but simple enough to be fully understood.

2. For a fixed set of coefficients, we can consider how the amoeba $\mathcal{A}(f)$ depends on the support set $A$ under (small) perturbations.

We show that these two approaches are essentially the same. The limiting object along a perturbation of the support set $A$ along a subspace of $\mathbb{C}^A$ is given by a $B$-caisson of $\mathcal{A}(f)$, where $B$ is determined by the subspace in question. Conversely, all $B$-caissons of $\mathcal{A}(f)$ arise in this fashion, see Theorems 4.1, 4.2, and 5.1.

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2. Preliminaries

In this section we introduce notation and explain key results which are necessary for an understanding of our results. For further background about amoebas and tropical geometry we recommend [10, 12] and [9] respectively.

We call the point configuration $A$ polynomial or algebraic if $A \subset \mathbb{Z}^{1+n}$. In this case, there exists a Laurent polynomial $g \in \mathbb{C}[w^{\pm 1}]$ such that $f(z) = g(e^z)$. The amoeba of the polynomial $g$ was defined in [7] as the image of
the algebraic variety $V(g)$ under the logarithmic absolute value map. That is, the amoeba of $g$ coincides with the amoeba of $f$ defined above.

Let $\mathbb{Z}[A] \subset \mathbb{R}^{1+n}$ denote the abelian group generated by the elements of $A$. We let $N$ denote the Newton polytope of $f$. That is, $N = \text{Conv}(A) \subset \mathbb{R}[A]$, where $\mathbb{R}[A]$ denotes the $\mathbb{R}$-vector space generated by the elements of $A$.

We denote the $j$th element of $A$ by $\alpha(j)$ when necessary. There are two ranks associated to $A$ which are important to us. First, let $r(A)$ denote the rank of the matrix $A$. Second, let $\rho(A)$ denote the rank of the lattice (abelian group) $\mathbb{Z}[A]$. Since $r(A)$ equals the dimension of $\mathbb{R}[A] \simeq \mathbb{R} \otimes \mathbb{Z}[A]$, it holds that $\rho(A) \geq r(A)$. This inequality can be strict. In particular, given a factorization as in (1.2), the induced group homomorphism $T: \mathbb{Z}[B] \to \mathbb{Z}[A]$ can be an isomorphism even if the induced linear transformation $T: \mathbb{R}[B] \to \mathbb{R}[A]$ is not. Here, we denote both maps by $T$ with slight abuse of notation since both maps are given by matrix multiplication by $T$.

**Remark 2.1.** In examples, it is more convenient to reduce the number of variables by dehomogenizing the exponential sum $f$. There is a standard procedure. After a change of variables, we can assume that $\xi = (1, 0, \ldots, 0)$. This implies that the top row of $A$ is the all ones vector. In this case, $f(z) = e^{z_0}g(z_1, \ldots, z_n)$ for an $n$-variate exponential sum $g$.

The order map for amoebas of polynomials, which was introduced in [3], was extend to exponential sums by Favorov in [3], where the Ronkin function for polynomial amoebas was generalized by Jessen function. In [16, Section 3.2], Silipo noted that the Ronkin function for exponential sums can be defined as follows. Let $S = \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[A], S^1)$ be the group of $S^1$-characters of $\mathbb{Z}[A]$, which is homeomorphic to $(S^1)^{\mathbb{Z}[A]}$. The Ronkin function of $f$ is given by

$$N_f(x) = \int_{\mathbb{R}} \log |f_{\chi}(x)| \, d\eta(\chi),$$

where $\eta$ denotes the Haar measure on $S$, and $f_{\chi}(x)$ is the perturbation of $f$ by $\chi$,

$$f_{\chi}(x) = \sum_{\alpha \in A} c_{\alpha} \chi(\alpha) e^{(x, \alpha)}.$$

The gradient of the Ronkin function is constant on the complement of the amoeba $\mathcal{A}(f)$, see for example [11]. Thus, the Ronkin function induces an (injunctive) map

$$\text{ord}: \pi_0(\mathbb{R}^n \setminus A) \to \mathbb{Z}[A] \cap N$$

called the order map of the amoeba $\mathcal{A}(f)$, see [3, 11].

Let $A$ be fixed, and let $\alpha \in \mathbb{Z}[A] \cap N$. For an exponential sum $f$, we denote by $E_{\alpha}(f)$ the connected component of $\mathbb{R}^n \setminus \mathcal{A}(f)$ whose order is $\alpha$. It is possible that $E_{\alpha}(f) = \emptyset$. Given a fixed $f$, we denote by $\Omega(f) \subset \mathbb{Z}[A] \cap N$ the set of all $\alpha$ such that $E_{\alpha}(f) \neq \emptyset$. Given a fixed $\alpha$, we denote by $U_{\alpha}(A) \subset \mathbb{C}^A$ the set of of all $f \in \mathbb{C}^A$ such that $E_{\alpha}(f) \neq \emptyset$. Rullgård showed in [14].
Theorem 10] that, in the algebraic case, the sets \( U_\alpha(A) \) are open and semi-algebraic, but left as an open problem whether or not the sets \( U_\alpha(A) \) are connected, which we call Rullgård’s connectivity conjecture. Silipo extended Rullgård’s results to the case of exponential sums, see [16, Theorem 2.6] and Theorem 4.2.

**Example 2.2.** Consider the factorization \( A = TB \) given by

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & \pi \\
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\times
\begin{bmatrix}
1 & 1 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
\]

On the one hand, we have that \( \rho(A) = \rho(B) = 3 \), so \( \mathbb{Z}[A] \simeq \mathbb{Z}[B] = \mathbb{Z}^3 \) where the isomorphism is given by the matrix \( T \). On the other hand, we have that \( r(A) = 2 \) while \( r(B) = 3 \). That is, \( \mathbb{R}[A] \simeq \mathbb{R}^2 \) and \( \mathbb{R}[B] \simeq \mathbb{R}^3 \).

To describe the sets \( \mathbb{Z}[A] \cap \mathcal{N}(A) \) and \( \mathbb{Z}[B] \cap \mathcal{N}(B) \), we dehomogenize as in Remark 2.1 (we proceed in an analogous manner in all later examples). The Newton polytope \( \mathcal{N}(B) \) is the standard simplex in \( \mathbb{R}^2 \). The set \( \mathbb{Z}[B] \cap \mathcal{N}(B) \) contains three points: the three vertices of the simplex. The Newton polytope \( \mathcal{N}(A) \) is the interval \([0, \pi]\). The set \( \mathbb{Z}[A] \cap \mathcal{N}(A) \) is infinite. It follows from [16], however, that the image of the order map is still finite, see Section 4.

The spine \( S \) of an algebraic amoeba was introduced in [11]. While it is possible to extend the concept of spines to amoebas of exponential sums, it suffices for our purposes to consider spines of polynomial amoebas. The spine \( S \) is a tropical variety which is a strong deformation retract of the (algebraic) amoeba [11, Theorem 1]. In particular, there is an bijection \( \pi_0(\mathbb{R}^n \setminus A) \to \pi_0(\mathbb{R}^n \setminus S) \), implying that the order map is well-defined also for the spine \( S \). The spine, as any tropical variety, has a dual triangulation of the Newton polytope \( \mathcal{N} \), see [11, Definition 2].

3. **The Poset Lattice**

In this section we describe the lattice structure induced by the relationship (1.2). This lattice structure is the theoretical framework within which we establish a lattice structure of amoebas.

Let us denote by \( M_N = M_N(\mathbb{R}) \) the set of real matrices \( A \) with \( N \) columns and any number of rows, such that there exists a linear form \( \xi \) with \( \xi A = 1 \), where \( 1 = (1, \ldots, 1) \). That is, the vector \( 1 \) belongs to the real row span \( \text{Row}(A) \) of \( A \).

**Definition 3.1.** We define an equivalence relation \( \sim \) on \( M_N \) by

\[
A \sim B \iff \text{Row}(A) = \text{Row}(B).
\]

We denote the quotient \( M_N / \sim \) by \( M_N \), and for \( A \in M_N \) we denote the corresponding equivalence class by \([A] \in M_N\). We write \( \text{Row}([A]) \) for the row space defined by any representative of \([A]\).
Remark 3.2. Let $L \subseteq \mathbb{R}^N$ be a subspace of dimension $d$ containing $1$. By choosing a basis of $L$ we obtain an $d \times N$-matrix $A$ with $\text{Row}(A) = L$. We make two remarks. First, $L$ is realized as the row space of some $[A] \in \mathcal{M}_N$. Second, each equivalence class $[A] \in \mathcal{M}_N$ has a representative $A$ which is an $(1+n) \times N$-matrix where $1+n \leq N$.

**Definition 3.3.** We define a partial order on $\mathcal{M}_N(\mathbb{R})$ by 
$$[B] \subseteq [A] \iff \text{Row}([A]) \subseteq \text{Row}([B]).$$
We explain the reversal of the inequalities in Section 4.1. \( \diamond \)

Notice that $A \sim B$ if and only if there are matrices $S$ and $T$ such that $A = TB$ and $B = SA$. And, $[B] \subseteq [A]$ if and only if for any choice of representatives $A$ and $B$ there is a matrix $T$ with $A = TB$.

**Remark 3.4.** The rank function $r(A)$ is constant on the classes $[A]$, as it equals the dimension of $\text{Row}(A)$. Hence, it induces a function $r: \mathcal{M}_N \to \mathbb{N}$. To adjust for the reversal of the inequalities in Definition 3.3, we define $\hat{r}([A]) = N - r([A])$. Notice that $\hat{r}([A])$ is the rank of the orthogonal complement of $\text{Row}(A)$, and since $\text{Span}(1) \subseteq \text{Row}(A)$ we have that $\hat{r}([A]) \in \{0, \ldots, N-1\}$.

**Theorem 3.5.** The space $(\mathcal{M}_N, \subseteq)$ with the rank function $\hat{r}$ is isomorphic to the bounded modular lattice of all subspaces of $\mathbb{R}^{N-1}$.

**Proof.** The space of all subspaces of $\mathbb{R}^{N-1}$ is a bounded modular lattice with respect to inclusion, where the grading is given by the vector space dimension. The isomorphism is given by the map $[A] \mapsto \text{Row}([A])$. It is injective by definition, and surjective by Remark 3.2. It is straightforward to check the remaining conditions of a lattice. \( \square \)

Consider an exponential sum $f$. The matrix $A$ is obtained by choosing an ordering of the elements of the set $A$. It can happen that the matrices one obtains from distinct choices of orderings define distinct equivalence classes in $\mathcal{M}_N$. One could enlarge the equivalence classes by allowing also permutations of columns. This would not introduce any additional difficulty, but yields a less clear notation and no further insights. Thus, we have chosen not to take this approach.

**Proposition 3.6.** The function $\rho: \mathcal{M}_N \to \mathbb{Z}$ given by $[A] \mapsto \text{rank}(\mathbb{Z}[A])$ is well-defined and decreasing.

**Proof.** Let $A = TB$ for some $T$. It follows that $\mathbb{Z}[A]$ is the image of $\mathbb{Z}[B]$ under the group homomorphism induced by $T$. In particular, any set of generators of $\mathbb{Z}[B]$ projects by $T$ onto a set of generators of $\mathbb{Z}[A]$, so $\rho(B) \geq \rho(A)$. \( \square \)

**Definition 3.7.** We call a matrix $A$ with $\rho(A) = r(A)$ a **rational point configuration**, and the corresponding class $[A]$ a **rational** class. Let $[A] \in \mathcal{M}_N$. We call a rational class $[B]$ satisfying $[B] \subseteq [A]$ a **rational lift** of $[A]$. 
If, in addition, \( \rho([B]) = \rho([A]) \) then we call \([B]\) a **minimal rational lift** of \([A]\).

The following universal property of minimal rational lifts shows that any rational lift of \([A]\) is a composition of the unique minimal rational lift and integer (algebraic) transformation. This decomposition is one of our main tools to understand the relationship between an amoeba and its caissons.

**Theorem 3.8.** Each \([A] \in M_N\) has a unique minimal rational lift, denoted as \(\hat{[A]}\), fulfilling the universal property that every rational lift of \([A]\) is a lift of \(\hat{[A]}\).

**Proof.** An element \([B] \in M_N\) is rational if and only if Row\((B)\) is rational, i.e., it has a basis consisting of rational vectors. Given two rational subspaces Row\((B_1)\) and Row\((B_2)\) of \(\mathbb{R}^N\) containing Row\((A)\), then Row\((B_1) \cap \text{Row}(B_2)\) is also a rational subspace containing Row\((A)\). Hence, if Row\((B_1)\) is minimal, then Row\((B_1) = \text{Row}(B_1) \cap \text{Row}(B_2)\), implying both claims of the theorem.

\[\Box\]

**4. The Relationship between an Amoeba and its Caissons**

In this section, we apply the results about the poset lattice from Section 3 to understand the relationship between an amoeba and its associated caissons. We obtain a lattice structure on the set of amoebas, and an associated lattice structure on the set of sets of orders.

**Theorem 4.1.** Let \([B] \sqsubseteq [A]\). Then, for any \(f \in C^A\), it holds that \(A \subset A_B\) and \(\iota(A) \subset \mathcal{L}_B\).

**Proof.** It is sufficient to show the second inclusion. Let \(g = \Phi \circ f \in C^B\). Let \(x \in A\), so that \(x = \text{Re}(z)\) for some \(z\) with \(f(z) = 0\). Then, \(g(z^T) = 0\), implying that \(\text{Re}(z^T) \in \mathcal{L}_B\). Therefore, \(x^T = \text{Re}(z)T = \text{Re}(z^T) \in \mathcal{L}_B\), where the second equality holds since \(T\) is a real matrix.

\[\Box\]

In Example 1.2, the fundamental inclusion \(A \subset A_B\) was an equality. It is an interesting problem to determine when such an equality holds. We include Silipo’s Theorem [16, Theorem 2.6] here, as it is the fundamental result on amoebas of exponential sums. We formulate the theorem in the language of Section 3.

**Theorem 4.2 (Silipo).** Let \([B]\) be the minimal rational lift of \([A]\), and let \(f \in C^A\). Then, \(A(f) = A_B(f)\).

To describe the relationship between \(A\) and \(A_B\) in more detail, we study the relationship between the associated order maps. Indeed, Theorem 4.1 implies that there exists a well-defined map

\[\pi_0(\mathbb{R}^n \setminus A_B) \rightarrow \pi_0(\mathbb{R}^n \setminus A),\]

given by inclusion of subsets of \(\mathbb{R}^n\).
Theorem 4.3. Let $[B] \subseteq [A]$, and let $f \in \mathbb{C}^A$ with $g = \Phi \circ f \in \mathbb{C}^B$. If $E_\beta(g)$ meets $H_T$, so that $E_\beta(g) \cap H_T \subset E$ for some $E \in \pi_0(\mathbb{R}^n \setminus A)$, then $\text{ord}_f(E) = T \beta$.

Proof. In the case that $[A]$ and $[B]$ are both rational, this follows from Rullgård’s trick to compute orders from [13, p. 20]. In the case that $[B]$ is a minimal rational lift of $[A]$, this follows from Silipo’s analysis in [16, Section 3.2] (see in particular the proof of [16, Lemma 3.6]). The general case can always be reduced to a composition of these two cases, using Theorem 3.8, as follows. Let $\hat{A}$ and $\hat{B}$ be the minimal rational lifts of $A$ and $B$. By universality of $\hat{A}$, we obtain a commutative diagram of abelian groups

$$
\begin{array}{ccc}
\mathbb{Z}[\hat{B}] & \xrightarrow{\hat{T}} & \mathbb{Z}[\hat{A}] \\
\downarrow{S_B} & & \downarrow{S_A} \\
\mathbb{Z}[B] & \xrightarrow{T} & \mathbb{Z}[A]
\end{array}
$$

We find that $\beta = S_B(\gamma)$ for some $\gamma \in \mathbb{Z}[\hat{B}]$ by Silipo’s results mentioned above, and by using both Rullgård’s and Silipo’s results we find that the order of $E$ is equal to

$$\text{ord}_f(E) = (S_A \circ \hat{T})(\gamma) = (T \circ S_B)(\gamma) = T(\beta). \quad \square$$

Theorem 4.4. Let $[B] \subseteq [A]$. Then, we obtain an injective map $\pi_0(\mathbb{R}^n \setminus A_B) \to \pi_0(\mathbb{R}^n \setminus A)$ given by inclusion.

Proof. By Theorems 3.8 and 4.2 it suffices to show that $\pi_0((\mathbb{R}^n \setminus A_B) \cap H) \to \pi_0((\mathbb{R}^n \setminus A) \cap H)$ is injective whenever $A$ and $B$ are rational and $H$ is an arbitrary affine subspace of $\mathbb{R}^n$. By convexity of the components of the complement of an amoeba, it suffices to show that $\pi_0(\mathbb{R}^n \setminus A_B) \to \pi_0(\mathbb{R}^n \setminus A)$ is injective whenever $A$ and $B$ are rational.

This last statement follows from the existence of spines of amoebas. Let $T$ be such that $A = TB$, and consider the caisson $\mathcal{L}_B \subset \mathbb{R}^m$. By definition of $A_B$ and $\mathcal{L}_B$, it suffices to show that $T$ is injective when restricted to the set of orders of $\pi_0((\mathbb{R}^m \cap \mathcal{L}_B) \cap H_T)$. It is equivalent to show the same statement when $\mathcal{L}_B$ is replaced by its spine $S_B$. Since the connected components of the complement of $S_B$ are open and convex there is no restriction in assuming that $H_T$ is rational and in general position with respect to the spine $S_B$ (i.e., its intersection with any cell of $S_B$ is an affine space of expected dimension). Then, $S = S_B \cap H_T$ is a tropical variety in $\mathbb{R}^n$. Hence, the vertices of the dual triangulation to $S$, which are all distinct, correspond bijectively to the set of projections $T \beta$ where $\beta$ ranges over the set $\pi_0((\mathbb{R}^m \cap S_B) \cap H_T). \quad \square$

Definition 4.5. Let $[B] \subseteq [A]$. We define the order map of the $B$-caisson $A_B$ to be the composition of the map $\pi_0(\mathbb{R}^n \setminus A_B) \to \pi_0(\mathbb{R}^n \setminus A)$ from
Theorem 4.4 and the order map of the amoeba $\mathcal{A}$. We denote by $\Omega_B(f) \subset \mathbb{Z}[A] \cap N$ the image of the order map for the $B$-caisson $\mathcal{A}_B(f)$. 

Remark 4.6. It is clear that $\Omega_B(f)$ only depends on the class $[[B]]$. Indeed, if $B$ and $B'$ represent the same class, then we obtain injections

$$\pi_0(\mathbb{R}^n \setminus \mathcal{A}_B) \hookrightarrow \pi_0(\mathbb{R}^n \setminus \mathcal{A}_{B'}) \hookrightarrow \pi_0(\mathbb{R}^n \setminus \mathcal{A}_B).$$

Since moreover both sets are finite, see for example [16, (1.15)], they coincide.

4.1. An Interpretation as Lattice Morphisms. Let us, in this subsection, fix the point configuration $A$. For any $f \in \mathbb{C}^A$, we have $\Omega(f) \subset \mathbb{Z}[A] \cap N(A)$. We could, in what follows, take our starting point in the set $\mathbb{Z}[B] \cap N(B)$. However, this set is larger than necessary, as, for example, it might be infinite even though $\Omega(f)$ is always finite. Following Silipo, let us consider the following construction. Let $[[B]]$ be a minimal rational lift of $[[A]]$, with the induced isomorphism $T: \mathbb{Z}[B] \rightarrow \mathbb{Z}[A]$. Since $[[B]]$ is rational, the set $\mathbb{Z}[B] \cap N(B)$ is finite. Let $\omega(A) \subset \mathbb{Z}[A] \cap N(A)$ denote the set $T(\mathbb{Z}[B] \cap N(B))$, which is finite as $T$ is an isomorphism of abelian groups. Let $\mathcal{O}(A)$ denote the bounded modular lattice of all subsets of $\omega(A)$. Then, Theorem 4.4 can be restated as follows.

**Theorem 4.7.** Each $f \in \mathbb{C}^A$ induces a morphism $\zeta(f): \mathcal{M}_N(A) \rightarrow \mathcal{O}(A)$ of bounded modular lattices, where $\mathcal{M}_N(A) \subset \mathcal{M}_N$ denotes the lattice of all lifts of the class $[[A]]$. \hfill \Box

5. Continuity of Amoebas as Functions of the Support

Amoebas have been considered in various contexts with respect to their configuration spaces $\mathbb{C}^A$ for a fixed support set $A$. It is a natural question to ask, how amoebas depend on the choice of support set. Particularly, considering amoebas of exponential sums one would like to understand how an amoeba behaves under a limit process of support sets. In this section we show that limits of amoebas with respect to their support sets are caissons.

Let us consider the deformation of the exponential sum $f(z)$ as in (1.1) given by

$$f_\lambda(z, t) = \sum_{\alpha \in A} c_\alpha e^{\langle \alpha, z \rangle + \lambda \langle \kappa_\alpha, t \rangle}$$

where $t = (t_1, \ldots, t_k) \in \mathbb{C}^k$ are additional variables, the vectors $\kappa_\alpha \in \mathbb{R}^k$, and $\lambda$ is a real parameter. We do not exclude the case that some $\kappa_\alpha = 0$. We wish to consider the limit

$$\lim_{\lambda \rightarrow 0} A(f_\lambda),$$

which, by definition, is the (closure of) the set of all limit points of sequences $(z_k, t_k) \in A(f_\lambda)$ where $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$ and $\lambda_k \neq 0$ for all $k$. It is a consequence of our proof of the forthcoming Theorem 5.1 that this limit does not depend on the choice of the sequence $\{\lambda_k\}_{k=0}^\infty$. 


Let us construct a matrix $B$ as the $(n + k) \times N$ matrix
\[
B = \begin{bmatrix}
\alpha_1 & \alpha_2 & \cdots & \alpha_N \\
\kappa_1 & \kappa_2 & \cdots & \kappa_N
\end{bmatrix},
\]
and note that $[B]$ is a lift of $[A]$ as $A = TB$ where, in block form, $T = (I, 0)$.

**Theorem 5.1.** We have that
\[
\lim_{\lambda \to 0} A(f_\lambda) = A_B(f) \times \mathbb{R}^k.
\]

Before we turn to the proof of Theorem 5.1, let us introduce some additional notation. Consider the auxiliary exponential sum
\[
g(z, t) = \sum_{i=1}^{N} c_i e^{(\alpha_i z + \kappa_i t)}.
\]
We have that $f_\lambda(z, t) = g(z, \lambda t)$. By definition, $A_B(f) = A(g) \cap H_T$.

**Proof of Theorem 5.1.** Fix a sequence $\{\lambda_k\}_{k=0}^{\infty}$ such that $\lambda_k \to 0$ as $k \to \infty$. Consider an arbitrary associated convergent sequence
\[
\{(z_k, t_k)\}_{k=0}^{\infty} \to (z^*, t^*)
\]
with $\text{Re}(z_k, t_k) \in A(f_\lambda)$ for all $k$. It follows that $\text{Re}(z_k, \lambda_k t_k) \in A(g)$. Since the sequence $\{t_k\}_{k=0}^{\infty}$ converges, it holds that
\[
\lim_{k \to \infty} \lambda_k t_k = 0.
\]
Thus, we have $\lim_{k \to \infty} \text{Re}(z_k, \lambda_k t_k) \in H_T$. Since $A(g)$ is closed, we know moreover that the limit is contained in $A(g)$ and we can conclude that
\[
\text{Re}(z^*, 0) = \lim_{k \to \infty} \text{Re}(z_k, \lambda_k t_k) \in A(g) \cap H_T = A_B(f).
\]
In particular, since $t^* \in \mathbb{R}^k$,
\[
\lim_{k \to \infty} A(f_\lambda) \subset A_B(f) \times \mathbb{R}^k.
\]

For the converse inclusion, let $\{\lambda_k\}_{k=0}^{\infty}$ be a sequence such that $\lambda_k \to 0$, and let $(z^*, t^*) \in A_B(f) \times \mathbb{R}^k$. As $A_B(f) \times \mathbb{R}^k$ is a regular closed set (i.e., it is the closure of its interior) we can find a sequence $\{(z_m, t_m)\}_{m=0}^{\infty}$ contained in the interior of $A_B(f) \times \mathbb{R}^k$ such that
\[
\lim_{m \to \infty} (z_m, t_m) = (z^*, t^*).
\]
In particular, $z_m$ is an interior point of $A_B(f)$ and $A_B(f) = A(g) \cap H_T$. We have for every $z_m$ that
\[
\lim_{k \to \infty} (z_m, \lambda_k t_m) = (z_m, 0),
\]
and thus, since $z_m$ is an interior point of $A_B(f)$, for each $m \in \mathbb{Z}$ there exists a $K(m) \in \mathbb{Z}$ such that $(z_m, \lambda_k t_m) \in A(g)$ for $k \geq K(m)$. Choose an increasing sequence $\{k_m\}_{m=0}^{\infty}$ such that $k_m \geq K(m)$; by taking a subsequence
of \( \{\lambda_k\}_{k=0}^{\infty} \) we can assume that \( k_m = m \). Finally, \((z_m, \lambda_m t_m) \in A(g)\) is equivalent to \((z_m, t_m) \in A(f_{\lambda_m})\) as \( \lambda_m \neq 0 \). It follows that

\[
(z^*, t^*) = \lim_{m \to \infty} (z_m, t_m) \in \lim_{m \to \infty} A(f_{\lambda_m}),
\]

which concludes the proof. \( \square \)

6. New Certificates for the Existence of Components of the Complement of Amoebas

In this section we show that our approach allows to certify the existence of specific components of the complement of certain amoebas in dependence of the coefficients of its defining exponential sum. These certificates improve previous certificates obtained via lopsided amoebas.

The main problem is to find a semi-algebraic description of a set \( U_\alpha(A) \). If \( \alpha \) is a vertex of \( \text{Conv}(A) \), then it follows by [7, Corollary 1.8, p. 196] that \( U_\alpha(A) = \mathbb{C}^A \). If \( \alpha \) is not a vertex of \( \text{Conv}(A) \) a semi-algebraic description of \( U_\alpha(A) \) is unknown in all cases except when \( A \) is a barycentric circuit.

**Definition 6.1.** We call a support \( A = \{\alpha(0), \ldots, \alpha(n), \gamma\} \) of cardinality \( N = n + 2 \) a barycentric circuit if \( \alpha(0), \ldots, \alpha(n) \) are the vertices of an \( n \)-dimensional simplex and \( \gamma \) is the barycenter of the simplex. If an exponential sum is supported on a barycentric circuit, we say that \( f \) is a barycentric circuit. Every barycentric circuit \( f \) is of the form

\[
f(z) = \sum_{j=0}^{n} c_j e^{\langle z, \alpha(j) \rangle} - c_\gamma e^{\langle z, \gamma \rangle}
\]

For a barycentric circuit \( f \) we define \( \text{eq}(f) \in \mathbb{R}^n \) as the real part of the unique point where all terms \( c_j e^{\langle z, \alpha(j) \rangle} \) for \( j = 0, \ldots, n \) are in equilibrium, that is, where they attain the same absolute value.

In the case the \( A \) is algebraic we abuse notation and write

\[
f(w) = \sum_{j=0}^{n} c_j w^{\alpha(j)} - c_\gamma w^{\gamma}.
\]

**Theorem 6.2 ([17, Theorem 6.1]).** Let \( A \) be an algebraic barycentric circuit. Then, the following statements are equivalent.

i) \( f \in U_\gamma(A) \)

ii) \( c_\gamma \) is not contained in the region

\[
\left\{ \prod_{j=0}^{n} |c_j|^{1/n} \cdot \sum_{j=0}^{n} e^{i(\arg(c_j) + \langle \alpha(j) - \beta, \phi \rangle)} : \phi \in \mathbb{T}^n \right\},
\]

where \( \mathbb{T} = [0, 2\pi) \).

iii) \( \text{eq}(f) \in E_\gamma(f) \),

where \( E_\gamma(f) \) is the \( \gamma \)-effective amoeba of \( f \).
The region (6.3) contains the origin and is bounded by a hypocycloid, see [17, Section 6] for further details. With the results of this article we obtain the following corollary of Theorem 6.2.

Corollary 6.3. Let $B$ be a rational lift of $A$ which is an algebraic barycentric circuit. Let $f \in C^A$ and let $g = \Phi \circ f \in C^B$. If $eq(g) \in H_T$ and $c_\gamma$ is not contained in the region (6.3), then $E_T \gamma(f) \neq \emptyset$.

Proof. If $c_\gamma$ is not contained in the region defined in (6.3), then $E \gamma(g) \neq \emptyset$ by Theorem 6.2. Thus, the statement follows from Theorems 4.1 and 4.3. □

In what follows we provide two examples demonstrating Corollary 6.3.

Example 6.4. Consider the family of univariate polynomials $f(w) = 1 + w^3 + c w^4 + w^9$, where $c \in \mathbb{C}$. We associate to $f$ its homogeneous support set $A$ (see Remark 2.1) and rational lift $B$ given by

\begin{equation}
A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 4 & 9 \\ 0 & 6 & 2 & 0 \end{bmatrix},
\end{equation}

see Figure 1. Set $g = \Phi \circ f$, so that $g(w) = 1 + w^3w_2^6 + c w_4^4w_2^2 + w_1^9$. For every fixed $c$, let $a_1, \ldots, a_9 \in \mathbb{C}$ denote the roots of $f(w)$ ordered with respect to magnitude. The set of norms of these roots is the image of $A(f)$ under the exponential map. We would like to determine for which $c$ we have that $|a_4| = |a_5|$. The lopsidedness certificate in this example implies that $|a_4| \neq |a_5|$ if

\[ c > \min\{w^{-4} + w^{-1} + w^5 : w \in \mathbb{R}\} = 3. \]

It follows from Corollary 6.3 that $|a_4| < |a_5|$ if $c \in \mathbb{C}$ is not contained in the region (6.3). Let us write $c = re^{i \theta}$. Then, by applying a Gröbner basis computation, the boundary of (6.3) is given by the hypocycloid $h(r, \theta) = 0$

\begin{equation}
h(r, \theta) = -27 + 18r^2 + r^4 - 8r^3 \cos(3\theta).\end{equation}

In particular, if $h(r, \theta) > 0$, then $|a_4| < |a_5|$. Notice that $r > 3$ implies that $h(r, \theta) > 0$, so our certificate is an improvement of the lopsidedness certificate. See Figure 1 for a comparison.

For example, if we require that $|c| > 1.5$ then we obtain the following numerical intervals in the argument of $c$ which ensures that $|a_4| < |a_5|$: \[ \arg(c) \in [-0.25\pi, 0.42\pi] \cup [0.91\pi, -0.91\pi] \cup [-0.42\pi, -0.25\pi]. \]

Similarly, if $|c| > 2.5$ then we obtain the following numerical intervals: \[ \arg(c) \in [0.32\pi, 0.34\pi] \cup [0.99\pi, -0.99\pi] \cup [-0.34\pi, -0.32\pi]. \]
Example 6.5. Consider the family of bivariate polynomials
\[ f(w_1, w_2) = 1 + w_1^2 w_2^2 + c w_1^3 w_2^3 + w_1^4 w_2^6 + w_1^6 w_2^4, \]
where \( c \in \mathbb{C} \). We associate to \( f \) its homogeneous support set \( A \) (see Remark 2.1) and a rational lift \( B \) given by
\[
A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 6 \\ 0 & 2 & 3 & 6 & 4 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 6 \\ 0 & 2 & 3 & 6 & 4 \\ 0 & 4 & 1 & 0 & 0 \end{bmatrix},
\]
see Figure 2. Set \( g = \Phi \circ f \), so that
\[ g(w_1, w_2, w_3) = 1 + w_1^2 w_2^2 w_3^4 + c w_1^3 w_2^3 w_3 + w_1^4 w_2^6 + w_1^6 w_2^4. \]
By the lopsidedness criterion we can conclude that \( A(f) \) has a bounded component of order \( (3, 3) \) if \( |c| > 4 \). It follows from Corollary 6.3 that \( A(f) \) has a bounded component of order \( (3, 3) \) if \( c \in \mathbb{C} \) is not contained in the region \( (6.3) \). Again, let us write \( c = re^{i \theta} \). Then, the boundary of \( (6.3) \) is in this case given by the hypocycloid \( h(r, \theta) = 0 \) where
\[
h(r, \theta) = -4096 + 768r^2 + 6r^4 + r^6 - 54r^4 \cos(4 \theta).
\]
In particular, if \( h(r, \theta) > 0 \), then \( A(f) \) has a bounded component of order \( (3, 3) \). For example, if we require that \( |c| > 2.5 \) then we obtain the following numerical intervals in the argument of \( c \) which ensures the existence of a component of order \( (3, 3) \):
\[
\arg(c) \in [0.08 \pi, 0.42 \pi] \cup [0.58 \pi, 0.92 \pi] \cup [-0.92 \pi, -0.58 \pi] \cup [-0.42 \pi, -0.08 \pi].
\]
Similarly, if \( |c| > 3.5 \) then we obtain the following numerical intervals:
\[
\arg(c) \in [0.01 \pi, 0.49 \pi] \cup [0.51 \pi, 0.99 \pi] \cup [-0.99 \pi, -0.51 \pi] \cup [-0.49 \pi, -0.01 \pi].
\]

Corollary 6.3 is applicable also when \( \text{rank}(A) < \text{rank}(B) + 1 \). We leave the task of constructing an example for the curious reader, to keep the article at a reasonable length.
Figure 2. Left, the amoeba of \( f(w_1, w_2) \) from Example 6.5 for \( c = 3.5 e^{0.63 \pi i} \). Middle, the Newton polytope of \( B \) with its projection onto \( A \). Right, the region (6.3) bounded by the hypocycloid \( h(r, \theta) = 0 \) in red.

The fact that Theorem 6.2 requires a barycentric circuit is a serious restriction. For non-barycentric circuits the only known certificates are certain upper and lower bounds on the norms of the coefficients, see [17, Theorems 4.1 and 4.4]. Unfortunately, the lower bounds yields no new certificates in conjunction with Theorem 4.1. The upper bounds are equivalent to the lopsidedness criterion, see [17, Theorem 5.3].

We observed experimentally that a non-barycentric circuit polynomial with \( n \geq 2 \) has a solid amoeba if and only if the coefficient of the term whose exponent corresponds to the interior point is not contained in a region bounded by a hypocycloid determined by the remaining coefficients. We believe that a similar statement as Theorem 4.1 holds for all circuits, this is the subject of an ongoing investigation.

References

1. M. Avendano, R. Kogan, M. Nisse, and J. Maurice Rojas, Metric estimates and membership complexity for archimedean amoebae and tropical hypersurfaces, arXiv:1307.3681, 2013.
2. T. de Wolff, Amoebas and their tropicalizations - a survey, Analysis Meets Geometry: The Mikael Passare Memory Volume (M. Andersson, J. Boman, C. Kiselman, P. Kurasov, and R. Sigurdsson, eds.), Trends in Mathematics, Birkhäuser, Basel, 2017, pp. 157–190.
3. S. Y. Favorov, Holomorphic almost periodic functions in tube domains and their amoebas, Comput. Methods Funct. Theory 1 (2001), no. 2, [On table of contents: 2002], 403–415.
4. M. Forsberg, M. Passare, and A. Tsikh, Laurent determinants and arrangements of hyperplane amoebas, Adv. Math. 151 (2000), no. 1, 45–70.
5. J. Forsgård, L. Matusievich, N. Mehlhop, and T. de Wolff, Lopsided approximation of amoebas, 2016, to appear in Math. Comp., see arXiv:1608.08663.
6. M. Fujiwara, Über die obere Schranke des absoluten Betrages der Wurzeln einer algebraischen Gleichung., Tohoku Math. J. 10 (1916), 167–171 (English; Japanese).
7. I. M. Gel’fand, M. M. Kapranov, and A. V. Zelevinsky, Discriminants, resultants and multidimensional determinants, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1994.
8. E. Landau, Sur quelques généralisations du théorème de M. Picard, Ann. Sci. École Norm. Sup. (3) 24 (1907), 179–201, in French.
9. D. Maclagan and B. Sturmfels, *Introduction to Tropical Geometry*, Amer. Math. Soc., Providence, R.I., 2015.
10. G. Mikhalkin, *Amoebas of algebraic varieties and tropical geometry*, Different Faces of Geometry (S. K. Donaldson, Y. Eliashberg, and M. Gromov, eds.), Kluwer, New York, 2004, pp. 257–300.
11. M. Passare and H. Rullgård, *Amoebas, Monge-Ampère measures and triangulations of the Newton polytope*, Duke Math. J. **121** (2004), no. 3, 481–507.
12. M. Passare and A. Tsikh, *Amoebas: their spines and their contours*, Idempotent mathematics and mathematical physics, Contemp. Math., vol. 377, Amer. Math. Soc., 2005, pp. 275–288.
13. K. Purbhoo, *A Nullstellensatz for amoebas*, Duke Math. J. **14** (2008), no. 3, 407–445.
14. H. Rullgård, *Topics in geometry, analysis and inverse problems*, Ph.D. thesis, Stockholm University, 2003.
15. D. Schleicher and R. Stoll, *Newton’s method in practice: Finding all roots of polynomials of degree one million efficiently*, Theoret. Comput. Sci. **681** (2017), 146–166.
16. J. Silipo, *The Ronkin number of an exponential sum*, Math. Nachr. **285** (2012), no. 8-9, 1117–1129.
17. T. Theobald and T. de Wolff, *Amoebas of genus at most one*, Adv. Math. **239** (2013), 190–213.
18. T. Theobald and T. de Wolff, *Norms of roots of trinomials*, Math. Ann. (2015), 1–29.

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