Products of Bessel functions and associated polynomials

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The symbolic method is used to get explicit formulae for the products or powers of Bessel functions and for the relevant integrals.

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I. INTRODUCTION

In a recent paper Moll and Vignat \cite{1} have considered the series expansion of powers of the modified Bessel function \( (BF) \) of first kind. They found that the expansion procedure involves a family of polynomials introduced in \cite{2} and, upon extending the relevant properties, the Authors obtained a link with the umbral formalism of ref. \cite{3}.

We reconsider here the problem addressed in ref \cite{1}, within the framework of the formalism (also of umbral nature) developed in \cite{4}, which will be reviewed in this introduction. We will prove that it is naturally suited to obtain the power series of the product of two BF’s and in the forthcoming sections we will discuss the extension to any arbitrary number.

One of the main results of \cite{4} has been the conclusion that the \( 0-th \) order cylindrical BF is the Umbral \((U-)\) image of a Gaussian function. By setting indeed that

\[
J_0 = e^{-\tilde{c}(\tilde{x})^2} \varphi_0
\]

where the \( U- \) operator \( \tilde{c} \) is defined in such a way that

\[
\tilde{c}^\nu \varphi_0 = \frac{1}{\Gamma(\nu + 1)}
\]

with \( \nu \) being not necessarily a positive real integer and, using an expression partially borrowed from field theory, \( \varphi_0 \) will be said the \( U-\) ”vacuum”. From the previous identities we recover the expansion

\[
e^{-\tilde{c}(\tilde{x})^2} \varphi_0 = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \left( \frac{x}{2} \right)^{2r} \tilde{c}^r \varphi_0 = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left( \frac{x}{2} \right)^{2r}
\]

which is the ordinary series defining the \( 0-th \) order cylindrical Bessel \cite{5}.

Let us now consider the product

\[
f(x; a, b) = J_0(ax)J_0(bx)
\]

which can be formally written as the product of two Gaussians, namely \cite{4}

\[
f(x; a, b) = e^{-(a^2 \tilde{c}_1 + b^2 \tilde{c}_2)(\tilde{x})^2} \varphi_0^{(1)} \varphi_0^{(2)}
\]

where \( \varphi_0^{(\alpha)} \) are the \( U-\) vacua on which the operators \( \tilde{c}^{\alpha} \) act. The series expansion of the exponential and the use of the previously outlined rules yield

\[
f(x; a, b) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} l_r(a^2, b^2) \left( \frac{x}{2} \right)^{2r}
\]

\[
l_r(a, b) = r! \sum_{s=0}^{r} \frac{a^{(r-s)b^s}}{(s!)^2 [(r-s)!]^2}
\]

In case of \( a = b \) the expression for \( f(x; a, b) \) is equivalent to that reported in ref \cite{3}

\[
f(x; a, a) = \sum_{r=0}^{\infty} \frac{(-1)^r}{[r!]^2} B_r^2 \left( \frac{ax}{2} \right)^{2r}
\]

where \( B_r(2) \) is calculated with a recursive formula

\[
B_n(r) = \sum_{s=0}^{n} \frac{n!}{s!(n-s)!} B_s(r-1) \quad \text{with } B_0(r) = 0; \; B_n(0) = \delta_{n,0}
\]

\section*{Footnote}

\footnotetext{1}{ Even though not explicitly stated, it is evident that in the present formalism we have }\[|J_0(x)|^2 = e^{-\tilde{c}(\tilde{x})^2} e^{-\tilde{c}(\tilde{x})^2} \varphi_0 \neq J_0 \left( \sqrt{2} x \right)\]
Leaving for the moment unspecified the nature of the polynomials \( l_r(a, b) \), we note that the function \( f(x; a, b) \) can be cast in the \( U \)-form
\[
f(x; a, b) = e^{-l(\xi)^2} \phi_0
\]
\[
l^\nu \phi_0 = l_{\nu}(a^2, b^2)
\] (9)

The action of the operator \( \tilde{l} \) on the corresponding \( U \)-vacuum holds for any real (positive/negative) or complex value of the exponent \( \nu \). We have concluded that the product of two cylindrical Bessel is the \( U \)-equivalent of a BF and thus the umbra of a Gaussian. Such a conclusion turns particularly useful if we are interested in the evaluation of the integrals of the function \( f(x; a, b) \), a straightforward use of the so far developed procedure yields
\[
\int_{-\infty}^{+\infty} f(x; a, b) dx = \int_{-\infty}^{+\infty} e^{-l(\xi)^2} dx \Phi_0 = 2\sqrt{\pi} l_\nu (a^2, b^2), \quad |a| > |b|
\]
\[
l_\nu(a^2, b^2) = \frac{1}{\sqrt{\pi} |a|} K \left( \frac{b}{a} \right),
\]
(10)

This result can however be viewed as an application of the Ramanujan master theorem \( \mathbb{R} \), it has, indeed, been derived by treating the \( U \)-operator \( \tilde{l} \) as an ordinary constant and then by applying the rules of the Gaussian integrals. The correctness of the result has then been checked numerically.

We have left open the question on the nature of the polynomials \( l_r(a, b) \), although we will discuss more deeply this point in the forthcoming sections, here we note that they can be viewed as a particular case of the the Jacoby polynomials \( \mathbb{J} \), as it can be inferred from the identity \( \mathbb{I} \):
\[
l_r \left( \sqrt{x - \frac{1}{2}}, \sqrt{x + \frac{1}{2}} \right) = \frac{1}{r!} P_r^{(0,0)}(x)
\]
\[
\]
(11)
\[

Furthermore, since in \( U \)-form the cylindrical Bessel functions of order \( n - th \) order read \( \mathbb{K} \)
\[
J_\nu(x) = \left( \frac{x}{2} \right)^\nu c \nu e^{-l(\xi)^2} \phi_0
\]
(12)
we obtain the following general expression for the product of two cylindrical Bessel functions of order \( \nu, \mu \) respectively
\[
f_{\nu, \mu}(x; a, b) = J_\nu(ax) J_\mu(bx) = \left( \frac{x}{2} \right)^{\nu+\mu} (a c_1)^\nu (b c_2)^\mu e^{-(a^2 \tilde{c}_1 + b^2 \tilde{c}_2)} (\tilde{\xi})^2 \phi_0^{(1)} \phi_0^{(2)} =
\]
\[
= \left( \frac{x}{2} \right)^{\nu+\mu} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} t^{(\nu, \mu)}(a^2, b^2) \left( \frac{x}{2} \right)^{2r} ;
\]
\[
l^{(\nu, \mu)}(a, b) = \sum_{s=0}^{r} \frac{a^{(r-s)+\nu} b^{s+\mu}}{\Gamma(\nu + s + 1) \Gamma(\mu + r - s + 1) s!(r-s)!}
\]
(13)

We have so far provided a first idea of how the \( U \)-formalism of ref. \( \mathbb{L} \) works and how it can be exploited to study the properties of products of (cylindrical) Bessel functions, in the forthcoming section we will take advantage from its simplicity to extend the method to arbitrary products.

II. PRODUCTS OF BESSEL FUNCTIONS

According to the tools outlined in the previous section, the product of three \( 0 - th \) order Bessel functions, can be written as
\[
f(x; a_1, a_2, a_3) = e^{-(a^2 \tilde{c}_1 + a^2 \tilde{c}_2 + a^2 \tilde{c}_3)} (\tilde{\xi})^2 \phi_0^{(1)} \phi_0^{(2)} \phi_0^{(3)}
\]
(14)
Or, in explicit form
\[
 f(x; a_1, a_2, a_3) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} I_r(a_1^2, a_2^2, a_3^2) \left(\frac{x}{2}\right)^{2r}
\]
(15)
\[
 I_r(x_1, x_2, x_3) = r! \sum_{s=0}^{r} \frac{x_{s}^{(r-s)}}{(s!)(r-s)!^2} I_s(x_1, x_2)
\]
(16)
It is evident that the extension to the case of \( n \) BF writes as in the first of eqs. (15) with
\[
 l_r(x_1, \ldots, x_n) = r! \sum_{s=0}^{r} \frac{x_s^{(r-s)}}{(s!)(r-s)!^2} l_s(x_1, \ldots, x_{n-1})
\]
in ref. [1] all the \( \alpha \) parameters (actually the variables of the \( l \) polynomials) are assumed to be 1.

From a formal point of view the use of the multinomial expansion allows to define the previous family of polynomials as
\[
 l_r(x_1, \ldots, x_n) = (x_1 \partial_1 + \cdots + x_n \partial_n)^r x_1 \cdots x_n \partial_0 \varphi_0^{(1)} \cdots \varphi_0^{(n)}
\]
(17)
and the use of the multinomial expansion yields
\[
 l_r(x_1, \ldots, x_n) = \sum_{k_1+\cdots+k_n=r} \binom{r}{k_1 \cdots k_r} \frac{x_1^{k_1}}{(k_1!)^2} \cdots \frac{x_n^{k_n}}{(k_n!)^2}
\]
(18)
Going back to the two variable case it is easy to check that they satisfy the differential equation
\[
 \partial_{x_1} x_1 \partial_{x_2} l_r(x_1, x_2) = \partial_{x_2} x_2 \partial_{x_1} l_r(x_1, x_2) = r l_{r-1}(x_1, x_2)
\]
(19)
With \( \partial_x x \partial_x \) being the so called Laguerre derivative [8]. The Laguerre polynomials can indeed cast in the form of eq. (16)
\[
 L_n(x, y) = (y - \partial_1 x)^n \varphi_0^{(1)}
\]
(20)
To obtain the extension to the product of arbitrary cylindrical Bessel, it will be sufficient to replace in the previous equations the function \( l_r(a_1^2, \ldots, a_n^2) \) with \( l_r(\nu_1, \ldots, \nu_n)(a_1^2, \ldots, a_n^2) \)
\[
 l_r(\nu_1, \ldots, \nu_n)(x_1, \ldots, x_n) = \sum_{k_1+\cdots+k_n=r} \binom{r}{k_1 \cdots k_r} \frac{x_1^{k_1}}{(k_1!)\Gamma(\nu_1+k_1)} \cdots \frac{x_n^{k_n}}{(k_n!)\Gamma(\nu_n+k_n)}
\]
(21)
Thus getting
\[
 \prod_{s=1}^{n} J_{\nu_s}(a_s x) = \left(\frac{x}{2}\right)^{n \nu_s} \sum_{k_1+\cdots+k_n=r} \binom{r}{k_1 \cdots k_n} \frac{(-1)^r}{r!} l_r(\nu_1, \ldots, \nu_n)(a_1^2, \ldots, a_n^2) \left(\frac{x}{2}\right)^{2r}
\]
(22)
In the case of modified BF of first kind the procedure is the same, the function can be formally expressed as a quadratic exponential and we can recover the results of ref. [1], by noting that the functions actually used in that paper the BF are given by
\[
 \tilde{I}_\nu(x) = \sum_{r=0}^{\infty} \frac{\Gamma(\nu + 1)}{r! \Gamma(r + \nu + 1)} \left(\frac{x}{2}\right)^{2r} = \Gamma(\nu + 1) \tilde{c}_\nu e^{\tilde{c}(\frac{x}{2})^2} \varphi_0
\]
(23)
According to our formalism the relevant \( k - th \) power reads
\[
 (\tilde{I}_\nu(x))^k = \Gamma(\nu + 1) k \sum_{r=0}^{\infty} \frac{1}{r!} l_r(\nu, \ldots, \nu)(1, \ldots, 1) \left(\frac{x}{2}\right)^{2r}
\]
(24)
The polynomials defined in [1] are expressible in terms of our \( l^{(\nu_1,\ldots,\nu_n)}_r(x_1,\ldots,x_n) \) as

\[
B^{(\nu)}(k) = \Gamma(\nu + 1)^{k-1} \Gamma(r + \nu + 1) l^{(\nu)}_r(k) \]
\[
l^{(\nu)}(1,\ldots,1) = l^{(\nu)}_r(k) \]
\[
l^{(\nu)}_{r+1}(k) = \sum_{j=1}^{k} l^{(\nu+j)}_r(k) \]
\[
\{\nu + 1, j, k\} = (\nu,\ldots,\nu + 1,\ldots,\nu)
\]

The nature of the polynomials \( l^{(\nu)}_r(k) \) will be further discussed in the forthcoming concluding section.

III. FINAL COMMENTS

The link between the polynomials \( l^{(\nu_1,\ldots,\nu_n)}_r(x_1,\ldots,x_n) \) and the Jacobi family has been recognized in the introductory section in the case of two variables, the multivariable extension does not modify this result.

We find in particular that the relevant generating function is expressible in terms of product of Bessel like functions, namely

\[
\sum_{r=0}^{\infty} \frac{t^r}{r!} l^{(\nu_1,\ldots,\nu_n)}_r(x_1,\ldots,x_n) = \prod_{j=1}^{n} C_{\nu_j}(tx_j)
\]

\[
C_{\nu}(x) = \sum_{r=0}^{\infty} \frac{x^r}{r! \Gamma(\nu + r + 1)}
\]

where \( C_{\nu}(x) \) denotes the Bessel Tricomi function \( \tilde{\Gamma} \) of order \( \nu \).

The polynomials \( l^{(\nu)}_r(k) \) are something else

\[
l^{(\nu)}_r(k) = \tilde{c}^{\nu}_1 \cdots \tilde{c}^{\nu}_k (\tilde{c}_1 + \cdots + \tilde{c}_k)^r \varphi^{(1)}_0 \cdots \varphi^{(k)}_0
\]

We find that

\[
l^{(\nu)}_r(k + 1) = \tilde{c}^{\nu}_{k+1} \cdots \tilde{c}^{\nu}_k (\tilde{c}_1 + \cdots + \tilde{c}_k + \tilde{c}_{k+1})^r \varphi^{(1)}_0 \cdots \varphi^{(k)}_0 \varphi^{(k+1)}_0 = \]
\[
= \tilde{c}^{\nu}_{k+1} \sum_{j=0}^{r} \left( \begin{array}{c} r \\ j \end{array} \right) \tilde{c}^{\nu-j}_{k+1} \varphi^{(k)}_0 = \sum_{j=0}^{r} \left( \begin{array}{c} r \\ j \end{array} \right) \frac{1}{\Gamma(r - j + \nu + 1)} \tilde{r}^{(r)}_j(k)
\]

The various identities reported in [1] follow from the above equation, which can be generalized in various ways, as e.g.

\[
l^{(\nu)}_r(k + s) = \tilde{c}^{\nu}_{k+1} \cdots \tilde{c}^{\nu}_{k+s} \tilde{c}^{\nu}_0 (\tilde{c}_1 + \cdots + \tilde{c}_k + \tilde{c}_{k+1} + \cdots + \tilde{c}_{k+s})^{r-s} \varphi^{(1)}_0 \cdots \varphi^{(k)}_0 \varphi^{(k+s)}_0 = \]
\[
= \sum_{j=0}^{r-s} \left( \begin{array}{c} r \\ j \end{array} \right) l^{(\nu)}_{r-j}(s) \tilde{r}^{(r-j)}_j(k)
\]

The use of the \( U \)–formalism has been proven to be a powerful tool allowing a very quick understanding of the various
technicalities underlying the handling of products or powers of Bessel functions.

We have noted in the introductory section that the use of straightforward algebraic manipulations allows the
derivation of an expression yielding the integral of the product of two cylindrical Bessel functions. We have checked

\[ C_{\nu}(x) = (\frac{x}{2})^{-\frac{\nu}{2}} J_{\nu}(2\sqrt{x}) \]
that the extension to the products of three or more is anyway feasible.

Regarding the case of three we obtain

$$
\int_{-\infty}^{+\infty} f(x; a_1, a_2, a_3) dx = 2\sqrt{\pi} l_{-\frac{1}{2}} (a_1^2, a_2^2, a_3^2), \quad |a_3| > |a_2| > |a_1| 
$$

$$
l_{-\frac{1}{2}}(a_1, a_2, a_3) = \Gamma \left( \frac{1}{2} \right) \sum_{s=0}^{\infty} \frac{a_3^{-(\frac{1}{2}+s)}}{(s!)\Gamma \left( \frac{1}{2} - s \right)} l_s(a_1, a_2) \tag{30}\]

In eq.(10) we have recognized that the series defining \(l_{-\frac{1}{2}}(a, b)\) can be recognized as that defining a quarter period elliptic integral, in this case we obtain

$$
l_{-\frac{1}{2}}(a_1, a_2, a_3) = \frac{1}{\sqrt{\pi |a_3|}} F(a_1, a_2, a_3) 
$$

$$
F(a_1, a_2, a_3) = \sum_{s=0}^{\infty} \left[ \frac{(2s)!}{2^s (s!)^2} \right]^2 \frac{l_s(a_1, a_2)}{a_3^s} = 2F_1 \left( \frac{1}{2}, \frac{1}{2}; 1; \frac{\hat{f}}{a_3} \right) \chi_0 \tag{31}\]

\[\hat{f} \chi_0 = sll s(a_1, a_2)\]

namely, we have reduced the series at least formally to the same hypergeometric defining the elliptic integral period. This result can be easily generalized to the case of an arbitrary product.

A further element of interest concerns the fact that, since, as already remarked, by replacing \(\tilde{f}\) with \(\hat{c}\) the functions defining the product of Bessel and the Bessel functions are \(U\)–equivalent, we can take advantage from the formalism to establish e.g. the \(n\)–th derivative of the \(f(x; a, b)\) functions. By noting again that it is formally written as a Gaussian, we use the following property \([5]\)

$$
\hat{D}_x^n e^{ax^2} = H_n(2ax, a) e^{ax^2} 
$$

$$
H_n(x, y) = n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \frac{x^{n-2r} y^r}{(n-2r)! r!} \tag{32}\]

we can write the \(n – th\) derivative of the product of two Bessel functions in terms of the two variable Hermite polynomials \(H_n(x, y)\) as

$$
\hat{D}_x^n f(x; a, b) = \hat{D}_x^n e^{-l(x)^2} \Phi_0 = 
$$

$$
= (-1)^n H_n \left( \frac{\hat{f}}{2}, \frac{n}{4} \right) e^{-l(x)^2} \Phi_0 \tag{33}\]

The use of the properties of the \(-\hat{l}\) operator finally yields the explicit result as

$$
\hat{D}_x^n f(x; a, b) = \frac{(-1)^n}{2^n} n! \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r x^{n-2r} \frac{r!(n-2r)!}{r!} (n-r) f(x; a, b) 
$$

$$
= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} l_{r+s}(a, b) \left( \frac{x}{2} \right)^{2r} \tag{34}\]

The method we have proposed here has further elements of flexibility which should be carefully examined. A more general problem, which will be just touched here and treated in a dedicated paper is the application of the formalism to the theory of multi-index Bessel functions. We remind that the Humbert functions \([9]\) within the present formalism are defined as

$$
I_{m_1, m_2}(x) = \hat{c}_1^{m_1} \hat{c}_2^{m_2} e^{\hat{c}_1 \hat{c}_2 x} \phi_1(0) \phi_2(0) = \sum_{s=0}^{\infty} \frac{x^r}{r!(m_1 + r)!(m_2 + r)!} \tag{35}\]
The relevant properties are easily deduced, for example we find
\[ \tilde{D}_x I_{m_1,m_2}(x) = \tilde{c}_1^{m_1+1} \tilde{c}_2^{m_2+1} e^{\tilde{c}_1 \tilde{c}_2 x} \varphi_1(0) \varphi_2(0) = I_{m_1+1,m_2+1}(x) \] (36)

Or, by applying the same integration procedure as before, we obtain
\[ \int_{-\infty}^{+\infty} I_{0,0}(x) e^{-\beta x^2} dx = \sqrt{\frac{\pi}{\beta}} I_{0,0} \left( \frac{1 + \beta |x|}{2} \right) \]
\[ I_{m_1,m_2}(x \mid k) = \sum_{r=0}^{\infty} \frac{x^r}{r! \Gamma(kr + 1 + m_1) \Gamma(kr + 1 + m_2)} \] (37)

The second of eq. (37) is a two index Bessel-Wright equation and the Gaussian integral in the first of eq. (37) can be viewed as the integral transform adopted for their definition.

In this paper we have shown that a formalism of umbral nature can be exploited to simplify in a significant way the technicalities underlying the theory of Bessel functions and of their manipulations leading to combinations or to the introduction of new forms. In forthcoming investigations we will further extend the method.

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