INVARIANT CONNECTIONS AND INVARIANT
HOLOMORPHIC BUNDLES ON HOMOGENEOUS MANIFOLDS

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Abstract. Let \( X \) be a differentiable manifold endowed with a transitive action \( \alpha : A \times X \longrightarrow X \) of a Lie group \( A \). Let \( K \) be a Lie group. Under suitable technical assumptions, we give explicit classification theorems, in terms of explicit finite dimensional quotients, of three classes of objects:

1. equivalence classes of \( \alpha \)-invariant \( K \)-connections on \( X \),
2. \( \alpha \)-invariant gauge classes of \( K \)-connections on \( X \), and
3. \( \alpha \)-invariant isomorphism classes of pairs \((Q, P)\) consisting of a holomorphic \( K^C \)-bundle \( Q \rightarrow X \) and a \( K \)-reduction \( P \) of \( Q \) (when \( X \) has an \( \alpha \)-invariant complex structure).

1. Introduction. Invariant gauge classes of connections

Hermitian holomorphic bundles on the upper half-plane, which are \( \text{SL}(2, \mathbb{R}) \)-invariant up to isomorphism, can be classified [Bi1]. Our starting point is the observation of the second author that this problem can be reformulated and generalized using ideas from gauge theory.

Let \( X \) be a connected manifold, \( A \) a connected Lie group, and \( \alpha : A \times X \longrightarrow X \) a smooth action. For any \( a \in A \), the diffeomorphism \( x \mapsto \alpha(a, x) \) of \( X \) will be denoted by \( f_a \). Let \( K \) be a Lie group with Lie algebra \( \mathfrak{k} \), and let \( p : P \longrightarrow X \) be a principal \( K \)-bundle over \( X \). Since \( A \) is connected, it follows that \( f_a^*(P) \simeq P \) for every \( a \in A \). This follows from the homotopy invariance of pull-backs (see [Hu] Theorem 9.9). Therefore the isomorphism type of \( P \) is \( \alpha \)-invariant.

Definition 1. A connection \( \Gamma \) on \( P \) will be called gauge \( \alpha \)-invariant if for every \( a \in A \) there is an id\(_X\)-covering principal \( K \)-bundle isomorphism \( \phi : P \longrightarrow f_a^*(P) \) such that \( \phi^*(f_a^*(\Gamma)) = \Gamma \).

The above condition depends only on the image of \( A \) in the diffeomorphism group \( \text{Aut}(X) \), so \( \Gamma \) is gauge \( \alpha \)-invariant if and only if it is invariant with respect to the natural action of the image of \( A \) in this group. Note that an id\(_X\)-covering bundle isomorphism \( P \longrightarrow f_a^*(P) \) can be regarded as an \( f_a \)-covering bundle isomorphism \( P \longrightarrow P \).

There exists an interesting alternative interpretation of this definition using ideas from gauge theory: Let \( p : P \longrightarrow X \) be an arbitrary principal \( K \)-bundle on a manifold \( X \) and \( \mathcal{K}_P \) its gauge group, i.e., the group \( \text{Aut}(P) \) of id\(_X\)-covering bundle
automorphisms of $P$ commuting with the action of $K$. This group can be identified with the group of $C^\infty$ sections of the group bundle $P \times_{\text{Ad}} K$, where
\[ \text{Ad} : K \to \text{Aut}(K) \]
is the adjoint action $(k, l) \mapsto klk^{-1}$ of $K$ on itself.

We recall that two connections $\Gamma, \Gamma'$ on $P$ are called gauge equivalent if there exists an $\varphi \in K_P$ such that $\Gamma' = \varphi^*(\Gamma)$, meaning they are conjugate modulo the natural action of the gauge group $K_P$ on the affine space $A(P)$ of connections on $P$. The moduli space of connections on $P$ is the quotient
\[ B(P) := A(P)/K_P, \]
edowed with the quotient topology of the $C^\infty$-topology on the affine space $A(P)$.

Note that if two principal $K$-bundles $p : P \to X$ and $q : Q \to X$ are isomorphic (as principal $K$-bundles over $X$), then there exists a canonical identification
\[ B(P) = B(Q), \]
because any two $id_X$-covering isomorphisms $P \to Q$ differ by the composition with a gauge transformation of $P$. Therefore, we can define in a coherent way the moduli space $B(\mathfrak{P})$, where $\mathfrak{P}$ is an isomorphism class of principal $K$-bundles over $X$. Formally $B(\mathfrak{P})$ is the disjoint union $\bigsqcup_{\mathfrak{P} \in B(P)} B(P)$ factorized by the equivalence relation generated by the canonical identifications $B(P) = B(Q)$ mentioned above.

Since $f_a^*(P) \simeq P$ for every $a \in A$, the isomorphism type $\mathfrak{P}$ of $P$ is a fixed point under that natural action of $A$ on the set of isomorphism types of $K$-bundles on $X$. So the moduli space $B(\mathfrak{P})$ associated with the isomorphism type of $P$ comes with a well-defined right $A$-action given by
\[ (a, [\Gamma]) \mapsto [f_a^*(\Gamma)] \in B(f_a^*(P)) = B(\mathfrak{P}), \]
where $\Gamma \in A(P)$. Definition\[\Box\] can be reformulated as follows:

**Remark 2.** A connection $\Gamma$ on $P$ is gauge $\alpha$-invariant if and only if its gauge class $[\Gamma] \in B(P) = B(\mathfrak{P})$ is a fixed point with respect to the natural $A$-action on $B(\mathfrak{P})$.

Our first goal is to describe the set of all classes of gauge $\alpha$-invariant connections on $K$-bundles on $X$. More precisely let $B_K(X)$ be the union
\[ B_K(X) := \bigsqcup_{\mathfrak{P} \text{ isomorphism type of } K\text{-bundles on } X} B(\mathfrak{P}), \]
so it is the moduli space of all gauge equivalent classes of connections in $K$-bundles on $X$. This moduli space comes with a natural $A$-action given by
\[ A \times B(\mathfrak{P}) \ni (a, [\Gamma]) \mapsto [f_a^*(\Gamma)] \in B(f_a^*(\mathfrak{P})) \]

Using this formalism, we see that our first goal is to describe the space of $A$-invariant elements in $B_K(X)$, in other words, to describe the fixed point set:
\[ F_{\alpha,K} := \{ \gamma \in B_K(X) | f_a^*(\gamma) = \gamma \forall a \in A \} \subset B_K(X). \]

The elements of the above set correspond bijectively to the equivalence classes of pairs $(P, \Gamma)$, where $P$ is a principal $K$-bundle on $X$, and $\Gamma$ is a gauge $\alpha$-invariant connection on $P$; the equivalence relation is defined by $id_X$-covering bundle isomorphisms. The elements of this set will be called $\alpha$-invariant gauge classes of $K$-connections on $X$. When $A$ is a subgroup of the diffeomorphism group $\text{Aut}(X)$
(or, equivalently, when the action \( A \) is effective), we will also say \( A \)-invariant gauge classes of \( K \)-connections on \( X \).

We will study this problem in detail in the particular case where \( K \) is compact and \( A \) acts transitively on \( X \).

Our second goal concerns the case when \( X \) has an \( A \)-invariant complex structure. In this case we will be interested in the subset \( \mathcal{F}_{A,\alpha}^{1,1} \subset \mathcal{F}_{A,\alpha} \) of \( \alpha \)-invariant gauge classes of type \((1,1)\) connections on \( K \)-bundles on \( X \). We recall that, for a compact Lie group \( K \), a connection on a principal \( K \)-bundle \( P \) on a complex manifold \( X \) is of type \((1,1)\) if its curvature \( F_\Gamma \) is of Hodge type \((1,1)\). This condition is equivalent to the condition that the almost complex structure on the complexified bundle \( Q := P \times_K K^C \) associated with \( \Gamma \) via the Chern correspondence (see for instance \([LT\) Section 7.1]) is integrable. In other words, denoting by \( G \) the complex reductive group \( K^C \), we see that the elements of \( \mathcal{F}_{A,\alpha}^{1,1} \) correspond bijectively to equivalence classes of holomorphic principal \( G \)-bundles \( Q \) on \( X \) endowed with a \( K \)-reduction, with the equivalence relation defined by \( \text{id}_X \)-covering holomorphic isomorphisms which respect the \( K \)-reductions. Equivalently, two pairs \((Q, P)\) and \((Q', P')\) consisting of holomorphic \( G \)-bundles endowed with \( K \)-reductions are considered equivalent if they are \emph{holomorphically isometric}, meaning there is a holomorphic isomorphism \( Q \rightarrow Q' \) of principal \( G \)-bundles that takes \( P \) to \( P' \).

2. Equivariant bundles and invariant connections

Let \( \alpha : A \times X \rightarrow X \) be a smooth action of a connected Lie group \( A \) on a connected smooth manifold \( X \). For \( a \in A \), denote by \( f_a : X \rightarrow X \) the diffeomorphism \( x \mapsto \alpha(x,a) \). Let \( K \) be a connected Lie group.

**Definition 3.** A principal \((K,\alpha)\)-bundle over \( X \) is a pair \((P, \beta)\) consisting of a principal \( K \)-bundle \( p : P \rightarrow X \) on \( X \) and an action \( \beta : A \times P \rightarrow P \) such that for every \( a \in A \), the corresponding diffeomorphism \( \beta_a : P \rightarrow P \), \( z \mapsto \beta(a, z) \), is an \( f_a \)-covering isomorphism of principal \( K \)-bundles.

In other words, \( \beta \) is an \( \alpha \)-covering action by principal \( K \)-bundle isomorphisms. According to the terminology used in the literature, a pair \((P, \beta)\) as in Definition 3 is also called an \( \alpha \)-equivariant (or an \( A \)-equivariant) principal \( K \)-bundle over the \( A \)-manifold \((X, \alpha)\).

**Definition 4.** Let \((P, \beta)\) and \((P', \beta')\) be two principal \((K,\alpha)\)-bundles over \( X \). An isomorphism

\[
(P, \beta) \rightarrow (P', \beta')
\]

is an \( \text{id}_X \)-covering \( K \)-bundle isomorphism that commutes with the \( A \)-actions \( \beta \) and \( \beta' \) on \( P \) and \( P' \) respectively.

**Definition 5.** Let \((P, \beta)\) be a principal \((K,\alpha)\)-bundle. A connection \( \Gamma \) on \( P \) is called invariant if \( \beta_a^*(\Gamma) = \Gamma \) for every \( a \in A \). In this case, we will also say that \( \Gamma \) is a \( \beta \)-invariant connection on \( P \).

An \( \alpha \)-invariant \( K \)-connection on \( X \) is a triple \((P, \beta, \Gamma)\), where \((P, \beta)\) is a principal \((K,\alpha)\)-bundle on \( X \), and \( \Gamma \) is an invariant connection on \( P \).

Let \((P, \beta)\) be a principal \((K,\alpha)\)-bundle. Let \( \mathcal{A}(P) \) be the space of all connections on \( P \) and \( \mathcal{A}(P)^\beta \subset \mathcal{A}(P) \) the invariant connections on \((P, \beta)\). Let \( \mathcal{A}^1(\text{ad}(P))^\beta \) denote the \( \beta \)-invariant \( \text{ad}(P) \)-valued 1-forms. We will need an explicit description of the space \( \mathcal{A}(P)^\beta \), supposing that it is non-empty. The difference \( \Gamma' - \Gamma \) of two
invariant connections is an element of $A^1(\text{ad}(P))$. Conversely, for $\Gamma \in \mathcal{A}(P)$ and $\omega \in A^1(\text{ad}(P))$, clearly $\Gamma + \omega \in \mathcal{A}(P)$. Therefore, if non-empty, the space $\mathcal{A}(P)$ has a natural affine space structure over $A^1(\text{ad}(P))$. We recall that, denoting by $\mathfrak{t}$ the Lie algebra of $K$, the space $A^1(\text{ad}(P))$ can be identified with the space $A^1_{\text{ad}}(P, \mathfrak{t})$ of $\mathfrak{t}$-valued tensorial forms of type ad on $P$ (defined in [KN, p. 75]; see [KN, p. 76, Example 5.2] for this identification.

Fix now $x_0 \in X$ and denote by $H_0$ the stabilizer of $x_0$ in $A$. Choose $y_0 \in P_{x_0}$, and note that $H_0$ acts on the fiber $P_{x_0}$ via $\beta$. We define the map $\chi_{y_0} : H_0 \to K$ by

$$\phi_h(y_0) := \beta(h, y_0) = y_0(\chi_{y_0}(h)).$$

This map is a group homomorphism because for $h, k \in H_0$ we have

$$y_0(\chi_{y_0}(kh)) = \beta(kh, y_0) = \beta(k, \beta(h, y_0)) = \beta(k, y_0(\chi_{y_0}(h))) = \beta(k, y_0)(\chi_{y_0}(h)) = y_0(\chi_{y_0}(k))(\chi_{y_0}(h)).$$

It is easy to check that, replacing $y_0$ by $y_0k$ for an element $k \in K$, we have

$$\chi_{y_0k} = k^{-1}\chi_{y_0}k = \iota_k \circ \chi_{y_0},$$

so $\chi_{y_0}$ is well defined up to conjugation by an element of $K$. In this formula we used the notation $\iota_k$ for the inner automorphism defined by $k$.

Let $\eta \in A^1_{\text{ad}}(P, \mathfrak{t})$ be a $\beta$-invariant tensorial form of type ad. Its restriction $\eta_{y_0}$ to $T_{y_0}(P)$ descends to a linear map $\eta_{y_0} : T_{x_0}(X) \to \mathfrak{t}$ given by $\eta_{y_0}(\xi) := \eta_{y_0}(\xi)$, where $\xi$ is an arbitrary lift of $\xi$ in $T_{y_0}P$. For a tangent vector $\xi \in T_{x_0}(X)$ and a lift $\xi \in T_{y_0}(P)$ of $\xi$, we have $p_*(\beta_{h*}(\xi)) = f_{h*}(\xi)$ for all $h \in H_0$. So $\beta_{h*}(\xi)$ (and every right translation of it) is a lift of $f_{h*}(\xi)$. Since $\eta$ is $\beta$-invariant and ad-tensorial we obtain

$$\eta_{y_0}(\xi) = \eta_{y_0}(\xi) = \eta_{y_0}(\phi_{h*}(\xi)) = \eta_{y_0}(\chi_{y_0}(h))((\phi_{h*}(\xi))) =$$

$$= R_{\chi_{y_0}(h)}^*(\eta)(R_{\chi_{y_0}(h)}^{-1}((\phi_{h*}(\xi))) = \text{ad}_{\chi_{y_0}(h)}(\eta_{y_0}(R_{\chi_{y_0}(h)}^{-1}((\phi_{h*}(\xi)))) =$$

$$= \text{ad}_{\chi_{y_0}(h)}(\eta_{y_0}(f_{h*}(\xi))),$$

where (as in [KN]) $R_k$ stands for the right translation $P \to P$ associated with an element $k \in K$. Therefore $\eta_{y_0}$ must satisfy the identity

$$\eta_{y_0}(f_{h*}(\xi)) = \text{ad}_{\chi_{y_0}(h)}(\eta_{y_0}(\xi)) \forall \xi \in T_{x_0}(X), \forall h \in H_0. \ (3)$$

Composing $\eta_{y_0}$ with the derivative $\alpha_{x_0*} : a \to T_{x_0}(X)$ at $e \in A$ of the map $a \mapsto f_a(x_0)$ we get a linear map

$$\mu^0 := \eta^0 \circ \alpha_{x_0*} : a \to \mathfrak{t}$$

satisfying the properties

$$\mu^0 \big|_{y_0} = 0, \ (1)$$

$$\mu^0 \circ \text{ad}_h = \text{ad}_{\chi_{y_0}(h)} \circ \mu^0 \forall h \in H_0. \ (2)$$

The following result is a consequence of Wang’s classification theorem for invariant connections on a principal bundle with respect to a “fibre-transitive” action (see [W], [KN] p. 106). We include a short proof for completeness.
Lemma 6. Let \((P, \beta)\) be a \((K, \alpha)\)-bundle. Choose \(x_0 \in X, y_0 \in P_{x_0}\), and let \(\chi_{y_0} : H_0 \rightarrow K\) be the associated group morphism. If \(\alpha\) is transitive, then the above map

\[ s_{y_0} : \eta \mapsto \mu^y := \eta^y \circ \alpha_{x_0, y} \]

defines an isomorphism between the space \(A^1_{\text{ad}}(P, \mathfrak{k})\) of \(\beta\)-invariant tensorial 1-forms of type \(\text{ad}\) on \(P\), and the subspace

\[ S_{y_0} := \{ \mu \in \text{Hom}(\mathfrak{a}, \mathfrak{k}) \mid \mu|_{\mathfrak{h}_0} = 0, \mu \circ \text{ad}_h = \text{ad}_{\chi_{y_0}(h)} \circ \mu \forall h \in H_0 \} \subset \text{Hom}(\mathfrak{a}, \mathfrak{k}) \]

For fixed \(x_0\), the space \(S_{y_0}\) and the isomorphism \(s_{y_0} : A^1_{\text{ad}}(P, \mathfrak{k}) \rightarrow S_{y_0}\) depend on \(y_0 \in P_{x_0}\) according to the formula

\[ S_{y_0} = \text{ad}_{k^{-1}}(S_{y_0}) , \quad s_{y_0} = \text{ad}_{k^{-1}} \circ s_{y_0} \forall k \in K \]

Proof. Given a linear map \(\mu : \mathfrak{a} \rightarrow \mathfrak{k}\) satisfying the above two conditions we obtain easily a linear map \(\eta_{y_0} : T_y(P) \rightarrow \mathfrak{k}\) vanishing on the vertical tangent space at \(y_0\). Using the right \(K\)-equivariance and left \(A\)-invariance property of the tensorial forms of type \(\text{ad}\), and the transitivity assumption, we can extend this form to all tangent spaces \(T_y(P)\). The two properties

\[ \mu|_{\mathfrak{h}_0} = 0, \mu \circ \text{ad}_h = \text{ad}_{\chi_{y_0}(h)} \circ \mu \forall h \in H_0 \]

ensure that this extension is well-defined, meaning for \(y \in P\), the resulting linear map \(T_y(P) \rightarrow \mathfrak{k}\) does not depend on the representation \(y = \phi_{\alpha}(y_0)k\), with \(\alpha \in A\) and \(k \in K\). \(\square\)

Remark 7. The condition \(\mu|_{\mathfrak{h}_0} = 0\) means that the linear map \(\mu : \mathfrak{a} \rightarrow \mathfrak{k}\) descends to a linear map \(\mathfrak{a}/\mathfrak{h}_0 \rightarrow \mathfrak{k}\). This map will also be denoted by \(\mu\). The condition

\[ \mu \circ \text{ad}_h = \text{ad}_{\chi_{y_0}(h)} \circ \mu \]

in the definition of the space \(S_{y_0}\) has a very natural interpretation: it means that \(\mu\) is a morphism of \(H_0\) spaces, where \(\mathfrak{a}/\mathfrak{h}_0\) is considered as a \(H_0\)-space via the adjoint representation of the subgroup \(H_0 \subset A\), and \(\mathfrak{k}\) is considered a \(H_0\)-space via \(\text{ad} \circ \chi_{y_0}\).

Corollary 8. Let \((P, \beta)\) be a \((K, \alpha)\)-bundle. Suppose that \(\alpha\) is transitive and that \(P\) admits a \(\beta\)-invariant connection. Choose \(x_0 \in X, y_0 \in P_{x_0}\), and let

\[ \chi_{y_0} : H_0 \rightarrow K \]

be the associated group morphism. Then the space \(\mathcal{A}(P)^\beta\) of invariant connections on \((P, \beta)\) is naturally an affine space over the finite dimensional space \(S_{y_0} \subset \text{Hom}(\mathfrak{a}/\mathfrak{h}_0, \mathfrak{k})\).

Our next goal is the classification of \(\beta\)-invariant connections on different bundles of type \((K, \alpha)\) up to equivalence.

Definition 9. Two \(\alpha\)-invariant connections \((P, \beta, \Gamma), (P', \beta', \Gamma')\) on \(X\) are called equivalent if there is an isomorphism \(\phi : (P, \beta) \rightarrow (P', \beta')\) of \((K, \alpha)\)-bundles such that \(\phi'(\Gamma') = \Gamma\).

We denote by \(\Phi_{\alpha, K}\) the set of isomorphism classes of \(\alpha\)-invariant connections. Since the isomorphism class of a \(\alpha\)-invariant connection \(\Gamma\) is preserved by gauge transformations commuting with \(\alpha\), we obtain an obvious comparison map

\[ \rho_{\alpha, K} : \Phi_{\alpha, K} \rightarrow \mathcal{F}_{\alpha, K}, \ [P, \beta, \Gamma] \mapsto [\Gamma] \]
which will be used in the next section to understand the set $\mathcal{F}_{\alpha,K}$ in $[1]$. Intuitively, the comparison map $\rho_{\alpha,K}$ relates equivalence classes of invariant connections to invariant equivalence (gauge) classes of connections. Whereas the right hand set $\mathcal{F}_{\alpha,K}$ depends only on the image of $A$ in $\text{Aut}(X)$, the left hand set $\Phi_{\alpha,K}$ depends effectively on the action $\alpha$. In particular, replacing $\alpha$ by the induced action $\bar{\alpha}$ of the universal cover $\bar{A}$ of $A$ we will get a comparison map

$$\rho_{\bar{\alpha},K} : \Phi_{\bar{\alpha},K} \longrightarrow \mathcal{F}_{\bar{\alpha},K} = \mathcal{F}_{\alpha,K}$$

which will play an important role in the next section.

We will now give an explicit description of the set $\Phi_{\alpha,K}$ in the special case where $\alpha$ is transitive and the principal $H_0$-bundle $q : A \longrightarrow X$, $a \longmapsto \alpha(a, x_0)$, where $H_0$ is the stabilizer of $x_0$, has an invariant connection. We will see that this condition has a simple interpretation and is satisfied for a large class of interesting examples.

From now on, throughout this section, we will suppose that $\alpha$ is transitive. We will regard the composition $\lambda : A \times A \longrightarrow A$ as a left-translation action of $A$ on itself. Note that $(A, \lambda)$ is a principal $(H_0, \alpha)$-bundle over $X$, i.e., $\lambda$ is an $\alpha$-covering action by bundle isomorphisms (see Definition 3). This pair should be regarded as a tautological equivariant bundle over $X$, because it was constructed using only the pointed manifold $(X, x_0)$ and the transitive action $\alpha$ on $X$.

This tautological equivariant bundle has an important role in our constructions, because we will see that any $(K, \alpha)$-bundle $(P, \beta)$ over $X$ (for an arbitrary Lie group $K$) can be regarded, in an essentially well defined way, as a bundle associated with the tautological equivariant bundle $(A, \lambda)$ over $X$. This simple remark will allow us to construct invariant connections on every $(K, \alpha)$-bundle $(P, \beta)$ over $X$, starting with an invariant connection on this tautological equivariant bundle.

Let $(P, \beta)$ be a $(K, \alpha)$-bundle over $X$. Choose $y_0 \in P_{x_0}$, and consider the homomorphism $\chi_{y_0} : H \longrightarrow K$ in $[2]$. The map

$$\psi_{y_0} : A \longrightarrow P$$

given by $\psi_{y_0}(a) := \beta(a, y_0)$ is an id$_X$-covering principal bundle morphism of type $\chi_{y_0} : H_0 \longrightarrow K$, because for $a \in A$ and $h \in H_0$ we have

$$\psi_{y_0}(ah) = \beta(ah, y_0) = \beta(a, \beta(h, y_0)) = \beta(a, y_0 \chi_{y_0}(h))$$

$$= \beta(a, y_0)(\chi_{y_0}(h)) = \psi_{y_0}(a)(\chi_{y_0}(h))$$

by the definition of $\chi_{y_0}$. We refer to [KN] for the concept of bundle morphism and for the transformation of connections via bundle morphisms.

Suppose that the equivariant tautological bundle $(A, \lambda)$ over $X$ has an invariant connection $\Gamma_0$. We obtain a connection $(\psi_{y_0})_*(\Gamma_0)$ on $P$ which will be $\beta$-invariant because, for any $a \in A$ we have

$$(\phi_a)_*((\psi_{y_0})_*(\Gamma_0)) = (\psi_{y_0})_*((l_a)_*(\Gamma_0)) = (\psi_{y_0})_*(\Gamma_0).$$

Moreover, the invariant connection $(\psi_{y_0})_*(\Gamma_0)$ on $P$ does not depend on the choice of $y_0 \in P_{x_0}$, because $\psi_{y_0,k} = R_k \circ \psi_{y_0}$, so the horizontal spaces of the connections $(\psi_{y_0})_*(\Gamma_0)$ and $(\psi_{y_0,k})_*(\Gamma_0)$ coincide.

Using Lemma $[3]$ and Corollary $[3]$ we obtain:

**Proposition 10.** Suppose that $\alpha$ is transitive. Fix $x_0 \in X$ with stabilizer $H_0$, and let $\Gamma_0$ be an invariant connection on the tautological equivariant $H_0$-bundle $(A, \lambda)$. Then
(1) Any \((K, \alpha)\)-bundle \((P, \beta)\) over \(X\) has a canonical invariant connection.

(2) After choosing a point \(y_0 \in P_{x_0}\), there is a canonical identification between the space \(\mathcal{A}(P)\) of \(\beta\)-invariant connections on \(P\) and the space 

\[
S_{y_0} := \{ \mu \in \text{Hom}(a/h_0, \mathfrak{k}) | \mu \circ \text{ad}_h = \text{ad}_{\chi_{y_0}} \circ \mu \forall h \in H_0 \} \subset \text{Hom}(a/h_0, \mathfrak{k}).
\]

Now we can prove the main result of this section.

**Definition 11.** We introduce the moduli space \(\mathcal{M}(A, H_0, K)\) by

\[
\mathcal{M}(A, H_0, K) := \{ (\chi, \mu) \in \text{Hom}(H_0, K) \times \text{Hom}(a/h_0, \mathfrak{k}) | \mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu \forall h \in H_0 \}/K,
\]

where \(K\) acts by conjugation on the set of pairs \((\chi, \mu)\).

**Theorem 12.** Suppose that \(\alpha\) is transitive. Fix \(x_0 \in X\) with stabilizer \(H_0\), and suppose that the tautological equivariant \(H_0\)-bundle \((A, \lambda)\) over \(X\) has a \(\lambda\)-invariant connection \(\Gamma_0\). Let \(K\) be a Lie group. Let \(\Phi_{a,K}\) be the set of equivalence classes of \(\alpha\)-invariant connections, i.e., the set of triples \((P, \beta, \Gamma)\), where \((P, \beta, \Gamma)\) is a \((K, \alpha)\)-bundle and \(\Gamma\) a \(\beta\)-invariant connection on \(P\), up to equivalence. There exists a natural bijection

\[
C_{x_0} : \Phi_{a,K} \xrightarrow{\sim} \mathcal{M}(A, H_0, K).
\]

**Proof.** Consider a triple \((P, \beta, \Gamma)\), where \((P, \beta)\) is a \((K, \alpha)\)-bundle and \(\Gamma\) a \(\beta\)-invariant connection on \(P\). Choosing a point \(y_0 \in P_{x_0}\) and using the construction explained above we obtain a group morphism \(\chi_{y_0} : H_0 \rightarrow K\), an equivariant bundle map \(\psi_{y_0} : (A, \lambda) \rightarrow (P, \beta)\) over \(X\), and a \(\beta\)-invariant connection \((\psi_{y_0})_*(\Gamma_0)\). By Corollary 6 the difference \(\Gamma - (\psi_{y_0})_*(\Gamma_0)\) can be regarded as an element \(\mu_{y_0} \in S_{y_0}\). We define the map \(C_{x_0}\) by

\[
(P, \beta, \Gamma) \rightarrow (\chi_{y_0}, \mu_{y_0}).
\]

Using the equivariance properties proved above, we see that \((\chi_{y_0}, \mu_{y_0})\) is independent of \(y_0\) up to conjugation. The proof uses the fact that the reference connection \((\psi_{y_0})_*(\Gamma_0)\) is independent of \(y_0\) (this was shown above), so \(\Gamma - (\psi_{y_0})_*(\Gamma_0)\) is a well defined \(\beta\)-invariant tensorial form of type \(\text{ad}\). Moreover, two equivalent triples \((P, \beta, \Gamma), (P', \beta', \Gamma')\) define obviously the same element in the quotient

\[
\{ (\chi, \mu) \in \text{Hom}(H_0, K) \times \text{Hom}(a/h_0, \mathfrak{k}) | \mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu \forall h \in H_0 \}/K.
\]

In order to prove that \(C_{x_0}\) is bijective, we will construct an inverse map. For a pair \((\chi, \mu) \in \text{Hom}(H_0, K) \times \text{Hom}(a/h_0, \mathfrak{k})\) with \(\mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu\), we define a triple \((P, \beta, \Gamma)\) by

\[
P := A \times \chi K, \quad \beta(a', [a, k]) := [a' a, k], \quad \Gamma = \psi_{x_0}(\Gamma_0) + \eta_\mu,
\]

where \(\psi : A \rightarrow P\) is the obvious map defined by \(\psi(a) := [a, e]\), and \(\eta_\mu\) denotes the \(\beta\)-invariant tensorial 1-form of type \(\text{ad}\) associated with \(\mu\). This map is a morphism of principal bundle of type \(\chi\) over \(X\) (see [KN]), and it is equivariant with respect to the left \(A\)-actions.

Taking into account Remark 4 we get

**Remark 13.** The condition on \(\mu\) in the definition of the moduli space \(\mathcal{M}(A, H_0, K)\) has a natural interpretation: \(\mu\) is a \(H_0\)-equivariant linear map \(a/h_0 \rightarrow \mathfrak{k}\) with \(\mathfrak{k}\) regarded as a \(H_0\)-space via \(\text{ad} \circ \chi\).
Remark 14. Suppose that \( \alpha \) is transitive. Fix \( x_0 \in X \) with stabilizer \( H_0 \). Then the map
\[
\Gamma \mapsto \Gamma_c \subset \mathfrak{a}
\]
defines a bijection between the space of \( \lambda \)-invariant connections on the \( H_0 \)-bundle \( A \to X \) and the space of \( \text{ad}_{H_0} \)-invariant complements of \( \mathfrak{t}_0 \) in \( \mathfrak{a} \). In particular the tautological \((H_0, \alpha)\)-bundle \((A, \lambda)\) over \( X \) admits an invariant connection if the pair \((A, H_0)\) satisfies the condition
\[
(4) \quad \text{The subalgebra } \mathfrak{h}_0 \subset \mathfrak{a} \text{ admits an } \text{ad}_{H_0} \text{-invariant complement in } \mathfrak{a}.
\]
If \( \mathfrak{h}_0 \) has an \( \text{ad}_{H_0} \)-invariant complement \( \mathfrak{s} \) in \( \mathfrak{a} \), then the tautological \((H_0, \alpha)\)-bundle \((A, \lambda)\) over \( X \) has a unique invariant connection \( \Gamma_0 \) whose horizontal space in \( e \in A \) is \( \mathfrak{s} \). This follows directly from Proposition XVIII in [GHV, p. 285] and is also implicitly used in Theorem 11.7 [KN]. The connection \( \Gamma_0 \) corresponding to a complement \( \mathfrak{s} \) is flat if and only if \( \mathfrak{s} \) is a Lie subalgebra.

Note the condition (1) plays an important role in the theory of homogeneous spaces. If it is satisfied, the homogeneous space \( A/H_0 \) (or more precisely the pair \((A, H_0)\)) is called reductive [Ya, p. 30]. The following remark shows that this condition is always satisfied when \( H_0 \) is compact, and that it is compatible with covers \( \tilde{A} \to A \):

Lemma 15. Suppose again that \( \alpha \) is transitive. Fix \( x_0 \in X \) with stabilizer \( H_0 \).

1. If \( H_0 \) is compact, then the pair \((A, H_0)\) satisfies the condition (1).
2. If \((A, H_0)\) satisfies the condition (4) and \( c : \tilde{A} \to A \) is a cover of \( A \), then the pair \((\tilde{A}, c^{-1}(H_0))\) also satisfies the condition (4).

Proof. The first statement is Corollary III p. 286 in [GHV].

The subgroup \( c^{-1}(H_0) \) will be denoted by \( \widetilde{H}_0 \). For any \( \tilde{a} \in \tilde{A} \) one has \( \text{ad}_{\tilde{a}} = \text{ad}_{c(\tilde{a})} \in \text{GL}(\mathfrak{a}) = \text{GL}(\tilde{\mathfrak{a}}) \). So for any \( \text{ad}_{\tilde{H}_0} \)-invariant complement \( \mathfrak{s} \) of \( \mathfrak{h}_0 \) in \( \mathfrak{a} \), the pull-back \( c^{-1}_*(\mathfrak{s}) \) is an \( \text{ad}_{\tilde{H}_0} \)-invariant complement of \( \tilde{\mathfrak{h}}_0 \) in \( \tilde{\mathfrak{a}} \). Hence the second statement follows. \qed

Corollary 16. Suppose that the action \( \alpha \) is transitive. Fix \( x_0 \in X \), and suppose that the pair \((A, H_0)\) satisfies the condition (4). Then Theorem 12 applies to \( \alpha \) and to the induced action \( \tilde{\alpha} : \tilde{A} \times X \to X \) associated with any cover \( c : \tilde{A} \to A \). Therefore, for any such cover \( c \), we get an identification
\[
\Phi_{\tilde{\alpha}, K} \cong \mathcal{M}(\tilde{A}, \tilde{H}_0, K),
\]
where \( \tilde{H}_0 := c^{-1}(H_0) \).

In the particular case when \( \tilde{H}_0 \) is simply connected, the right hand quotient in Corollary 14 can be described using Lie algebra morphisms \( \chi \) instead of Lie group morphisms:

Remark 17. If \( \tilde{H}_0 := c^{-1}(H_0) \) is simply connected, then
\[
\mathcal{M}(\tilde{A}, \tilde{H}_0, K) = \{ (\chi, \mu) \in \text{Hom}_{\text{Lie-Alg}}(\mathfrak{h}_0, \mathfrak{t}) \times \text{Hom}(\mathfrak{a}/\mathfrak{h}_0, \mathfrak{t}) \mid \mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu \forall h \in \mathfrak{h}_0 \}/K,
\]
which is the quotient of a \( K \)-invariant real algebraic affine subvariety of the vector space \( \text{Hom}(\mathfrak{h}_0, \mathfrak{t}) \times \text{Hom}(\mathfrak{a}/\mathfrak{h}_0, \mathfrak{t}) \) on which \( K \) acts by linear automorphisms.
3. Comparing isomorphism classes of invariant connections with invariant gauge classes of connections

Let \( p : P \rightarrow X \) be a principal \( K \)-bundle on \( X \). As we have seen in Section 1, the elements of the gauge group \( \mathcal{G}_P \) of \( P \) can be interpreted as sections of the group bundle \( P \times_{\text{Ad}} K \). The fiber \( K_x \) of \( P \times_{\text{Ad}} K \) at a point \( x \in X \) is identified with the group of automorphisms \( P_x \rightarrow P_x \) which commute with the right action of \( K \) on \( P_x \) (this follows from the fact that the group of diffeomorphisms of \( K \) commuting with the right translation action of \( K \) on itself is precisely the left translations). Therefore, \( K_x \cong K \) (unique up to an inner automorphism) for every \( x \in X \), and a point \( y \in P_x \) defines an isomorphism \( i_y : K_x \rightarrow K \) which is given by the formula \( y(i_y(g)) = g(y) \) for any \( g \in K_x \).

Since \( A \) is connected the isomorphism type of \( P \) is \( \alpha \)-invariant; let \( \Gamma \) be a gauge \( \alpha \)-invariant connection on \( P \) (see Definition 1). We denote by \( U \) the stabilizer of \( \Gamma \) in the gauge group \( \mathcal{G}_P \) of \( P \). The elements of \( U \) correspond bijectively to \( \Gamma \)-parallel sections of the associated bundle \( P \times_{\text{Ad}} K \) endowed with the connection induced by \( \Gamma \). This proves that, for any fixed \( x \in X \), the group \( U \) can be identified with the closed subgroup of \( K_x \) consisting of the elements that commute with the holonomy group of the connection \( \Gamma \) on \( P \) (the holonomy group is a subgroup of \( K_x \) obtained by taking parallel translations of \( P_x \) along loops based at \( x \)). Note that \( U \) does not need to be connected.

Following [Bi1] we define an extension of \( A \) by \( U \) by

\[
V := \{ (\phi, a) \mid a \in A, \phi : P \rightarrow P \text{ is an } f_a\text{-covering bundle isom. with } \phi^*(\Gamma) = \Gamma \}.
\]

It is easy to see that \( V \) has a Lie group structure such that the natural monomorphism \( j : U \rightarrow V \) identifies the stabilizer \( U \) with a closed subgroup of \( V \), and such that the natural projection \( \pi : V \rightarrow A \) becomes a Lie group epimorphism. Therefore we obtain a Lie group exact sequence

\[
(5) \quad 1 \rightarrow U \xrightarrow{j} V \xrightarrow{\pi} A \rightarrow 1.
\]

**Proposition 18.** Suppose that the Lie algebra \( a \) is semi-simple and \( A \) is simply connected. Then the following statements hold:

1. There exists a Lie group homomorphism \( s : A \rightarrow V \) such that \( \pi \circ s = \text{id}_A \).
2. If moreover \( K \) is compact and all simple summands of \( a \) are non-compact\(^{1}\), then \( s \) is unique.

**Proof.** Statement (1): Using the terminology and [CE, p. 122, Theorem 24.4] we see that when \( a \) is semisimple, the Lie algebra extension

\[
0 \rightarrow u \xrightarrow{j} v \xrightarrow{\pi} a \rightarrow 0
\]

associated with \( \mathfrak{g} \) is inessential. Therefore there exists a homomorphism of Lie algebras \( \sigma : a \rightarrow v \) such that \( \pi_s \circ \sigma = \text{id}_a \). If \( A \) is simply connected then \( \sigma \) is associated with a group homomorphism \( s : A \rightarrow V \). This homomorphism clearly satisfies the condition \( \pi \circ s = \text{id}_A \).

Statement(2): Since \( K \) is compact, its closed subgroup \( U \) is also compact. As \( A \) is

---

\(^{1}\)By compact Lie algebra we mean a Lie algebra \( \mathfrak{g} \) which is the Lie algebra of a compact Lie group, or equivalently, a Lie algebra which admits an inner product which is ad-invariant, in the sense that the endomorphisms \( \text{ad}(X), \ X \in \mathfrak{g} \) are skew-symmetric (see [BM] p. 194, Theorem 6.6).
connected, the adjoint representation Ad of $V$ defines via $s$ a group homomorphism 
$r : A \rightarrow \text{Aut}_0(U)$ in the connected component $\text{Aut}_0(U)$ of the automorphism
 group $\text{Aut}(U)$ of $U$. Since $U$ is a compact Lie group it follows that $L := \text{Aut}_0(U)$ is
 a compact Lie group (see [HM, p. 264, Theorem 6.66]). We obtain an induced Lie
 algebra homomorphism $r_* : a \rightarrow \mathfrak{l}$, which must vanish, because the Lie algebra
 $\mathfrak{l}$ is compact and all the simple summands of $a$ are non-compact. Therefore the
 homomorphism $r$ is trivial, implying that the elements of $A' := s(A)$ commute
 with the elements of $U' := j(U)$.

Therefore, the map $v \mapsto v(s\pi(v))^{-1}$ is a group homomorphism $\theta : V \rightarrow U$
 whose kernel is $A'$. For another group homomorphism $s_1 : A \rightarrow V$ with
 $\pi \circ s_1 = \text{id}_A$, we obtain a group homomorphism $\theta \circ s_1 : A \rightarrow U$, which is also
 trivial because $U$ is compact and all the simple summands of $a$ are non-compact.
 Therefore $\text{im}(s_1) \subset A'$, which shows that $s_1(a) = (\pi|_{U'})^{-1}(a) = s(a)$ for every
 $a \in A$.

Note that a group homomorphism $s : A \rightarrow V$ with $\pi \circ s = \text{id}_A$ can be
 regarded as an action $\beta : A \times P \rightarrow P$ by bundle isomorphisms leaving $\Gamma$
 invariant. Therefore:

**Corollary 19.** Suppose that $a$ is semi-simple, and denote by $\tilde{\alpha} : \tilde{A} \times X \rightarrow X$
 the induced action of the universal cover $\tilde{A}$ of $A$. Then for every gauge $\alpha$-invariant
 connection $\Gamma \in \mathcal{A}(P)$, there exists an $\tilde{\alpha}$-covering action $\beta : \tilde{A} \times X \rightarrow P$
 by bundle isomorphisms such that $\Gamma$ is $\beta$-invariant. If, moreover, $K$ is compact and
 all simple summands of $a$ are non-compact, the action $\beta$ is unique.

In the case when we have uniqueness of action preserving $\Gamma$, we will write $\beta_T$
 instead of $\beta$.

Suppose now that we have two gauge $\alpha$-invariant connections $\Gamma \in \mathcal{A}(P)$ and
 $\Gamma' \in \mathcal{A}(P')$ and an $\text{id}_X$-covering bundle isomorphism $\phi : P \rightarrow P'$ such that
 $\phi^*(\Gamma') = \Gamma$. Using the uniqueness of action in Proposition 15 we see that $\phi$
 is equivariant with respect to the two actions $\beta_T, \beta_{T'}$. In other words, in this case one
 can assign in a well defined way to every $\alpha$-invariant gauge class $[\Gamma]$ of $K$-connections
 an equivalence class $[P, \beta_T, \Gamma]$ of $\tilde{\alpha}$-invariant connections.

Therefore, recalling that the set $\mathcal{F}_{\alpha,K}$ in (11) depends only on the image of $A$ in
 $\text{Aut}(X)$, we obtain:

**Corollary 20.** If $a$ is semisimple then the comparison map

$$
\rho_{\tilde{\alpha},K} : \Phi_{\tilde{\alpha},K} \rightarrow \mathcal{F}_{\tilde{\alpha},K} = \mathcal{F}_{\alpha,K} : [P, \beta, \Gamma] \mapsto [\Gamma]
$$

is surjective. If, moreover, $K$ is compact and all the simple summands of $a$ are non-compact, this map is bijective.

Using Corollary 16 and Remark 17 we obtain:

**Corollary 21.** Suppose that $a$ is semisimple, $K$ is compact, all the simple sum-
 mands of $a$ are non-compact, the action $\alpha$ is transitive, and the pair $(A, H_0)$ satis-
 fies the condition (11) (which holds automatically when $H_0$ is compact). Then the
 set $\Phi_{\tilde{\alpha},K} \simeq \mathcal{F}_{\tilde{\alpha},K}$ can be identified with $\mathcal{M}(\tilde{A}, H_0, K)$ (see Definition 77),
 where $H_0 := c^{-1}(H_0)$. If, moreover, the pull-back $\tilde{H}_0$ is simply connected, then the set
 $\Phi_{\tilde{\alpha},K}$ can be identified with the quotient

$$
\left\{ (\chi, \mu) \in \text{Hom}(\text{LieAlg}(h_0, \mathfrak{t}) \times \text{Hom}(a/h_0, \mathfrak{t}) \mid \mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu \ \forall \ h \in h_0 \right\}/K,
$$
which is the quotient of a $K$-invariant real algebraic affine subvariety of the vector space $\text{Hom}(h_0, \mathfrak{t}) \times \text{Hom}(\mathfrak{a}/h_0, \mathfrak{t})$ on which $K$ acts by linear automorphisms.

4. Examples

4.1. Invariant connections over the half-plane. The main result in [Bi1] can be recovered as a special case of our general results. The upper half-plane $\mathbb{H}$ can be identified with the homogeneous manifold $\text{PSL}(2)/H_0$, where $H_0 = \text{SO}(2)/\{\pm 1\}$, whose Lie algebra is

$$h_0 = \mathbb{R}h_0, \quad h_0 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$ 

Since the pull-back $\tilde{H}_0$ in $\tilde{\text{PSL}}(2)$ is simply connected, we obtain

$$\mathcal{M}(\tilde{\text{PSL}}(2), \tilde{H}_0, K) = \{ (\chi, \mu) \in \text{Hom}_{\text{LieAlg}}(h_0, \mathfrak{t}) \times \text{Hom}(sl(2)/h_0, \mathfrak{t}) | \mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu \quad \forall \ h \in h_0 \}/K.$$ 

The space $\text{Hom}(sl(2, \mathbb{R})/h_0, \mathfrak{t})$ can be identified with $\text{Hom}(h_0^\perp, \mathfrak{t})$, where $h_0^\perp$ is the complement

$$h_0^\perp := \langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle$$

of $h_0$, which is $\text{ad}_{H_0}$-invariant. Putting

$$B := \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C := \mu \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we see that the condition $\mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu \quad \forall \ h \in h_0$ is equivalent to

$$[\chi(h_0), B] = 2C, \quad [\chi(h_0), C] = -2B,$$

so, denoting $A := C + iB \in \mathfrak{t} \otimes \mathbb{C}$, this can be written as

$$(6) \quad [\chi(h_0), A] = 2iA.$$

On the other hand $\chi$ is obviously determined by the vector $\chi_0 := \chi(h_0) \in \mathfrak{t}$. Therefore, the moduli space $\mathcal{M}(\text{PSL}(2), \tilde{H}_0, K)$ can be identified with the quotient

$$\mathcal{M}_K := \{ (\chi_0, A) \in \mathfrak{t} \times (\mathfrak{t} \otimes \mathbb{C}) | [\chi_0, A] = 2iA \}/K.$$ 

Denoting by $\alpha_{\mathbb{H}}, \tilde{\alpha}_{\mathbb{H}}$ the standard actions of $\text{PSL}(2)$, respectively $\tilde{\text{PSL}}(2)$ on $\mathbb{H}$, and using Corollary 21 we obtain

**Corollary 22.** The set $\Phi_{\tilde{\alpha}_{\mathbb{H}}, K}$ of equivalence classes of $\tilde{\alpha}_{\mathbb{H}}$-invariant $K$-connections on $\mathbb{H}$ can be naturally identified with the moduli space $\mathcal{M}_K$. If $K$ is compact, then the same moduli space classifies

1. $\alpha_{\mathbb{H}}$-invariant gauge classes of $K$-connections,
2. $\alpha_{\mathbb{H}}$-invariant isomorphism classes of pairs $(Q, P)$ consisting of a holomorphic $K^\mathbb{C}$-bundle $Q$ and a differentiable $K$-reduction $P$ of $Q$. 

4.2. The case of complex homogeneous complex manifolds. We begin with the following important

**Definition 23.** Let $K$ be compact Lie group and $X$ a complex manifold. A Hermitian holomorphic $K$-bundle is a pair $(Q, P)$ consisting of a holomorphic $K$-bundle $Q \to M$ and a $K$-reduction $P$ of $Q$. An isomorphism (or an isometric biholomorphic isomorphism) of Hermitian holomorphic $K$-bundles $(Q, P)$, $(Q', P')$ is a holomorphic isomorphism $f : Q \to Q'$ such that $f(P) = P'$.

Suppose that the conditions of Corollary 21 are satisfied, $K$ is compact, and $X$ possesses an $A$-invariant complex structure. The classification of isomorphism classes of Hermitian holomorphic $K$-bundles $(Q, P)$ on $X$ reduces to the classification of $\alpha$-invariant gauge classes of $K$-connections which are of type $(1, 1)$ (defined in Section 4.1). The condition of being type $(1, 1)$ produces an explicit algebraic equation on the space of pairs $(\chi, \mu)$ (as in Corollary 21). This equation has a simple form when the tautological equivariant $H_0$-bundle $(A, \lambda)$ over $X$ has a $\lambda$-invariant connection $\Gamma_0$, which is itself of type $(1, 1)$. This is the case when $X$ is an irreducible Hermitian symmetric space of non-compact type (see [B2]).

Suppose that the pair $(A, H_0)$ is reductive (i.e., it satisfies condition (11)). Let $\mathfrak{s}$ be a $H_0$-invariant complement of $\mathfrak{h}_0$ in $\mathfrak{a}$ and $\Gamma_0$ the corresponding invariant connection on the $H_0$-bundle $A \to X$. When $\Gamma_0$ is of type $(1, 1)$, all the induced connections $(\psi_y)_* (\Gamma_0)$ (see the proof of Theorem 12) will also be of type $(1, 1)$. For a pair $(\chi, \mu) \in \text{Hom}(H_0, K) \times \text{Hom}(\mathfrak{a}/\mathfrak{h}_0, \mathfrak{t})$ with $\mu \circ \text{ad}_h = \text{ad}_h \chi(h) \circ \mu \forall h \in H_0$, the condition that the associated connection is of type $(1, 1)$ reduces to the following condition:

\begin{equation}
(D_0(\eta_\mu) + \frac{1}{2}[\eta_\mu \wedge \eta_\mu])^2 + (D_0(\eta_\mu) - \frac{1}{2}[\eta_\mu \wedge \eta_\mu])^{0,2} = 0,
\end{equation}

where $D_0$ is the exterior covariant derivative [K] associated with the type $(1, 1)$ connection $(\psi)_* (\Gamma_0)$, and $\eta_\mu$ is the $\beta$-invariant tensorial 1-form of type $\text{ad}$ associated with $\mu$ (see the proof of Theorem 12). The tensorial 2-form $D_0(\eta_\mu) + \frac{1}{2} [\eta_\mu \wedge \eta_\mu]$ is determined by its value at $y_0 = [(e, e)] \in A \times K$, which is an anti-symmetric bilinear map $T_{y_0}(P) \times T_{y_0}(P) \to \mathfrak{t}$ whose pull-back via $\psi_{y_0}$ is a bilinear map $\mathfrak{a} \times \mathfrak{a} \to \mathfrak{t}$ vanishing when one of the arguments belongs to $\mathfrak{h}_0$.

**Lemma 24.** The restriction to $\mathfrak{s} \times \mathfrak{s}$ of the anti-symmetric bilinear map $\mathfrak{a} \times \mathfrak{a} \to \mathfrak{t}$ induced by $D_0(\eta_\mu) + \frac{1}{2} [\eta_\mu \wedge \eta_\mu]$ via $\psi_{y_0}$ is given by

\[(\xi, \zeta) \mapsto -\mu([\xi, \zeta]) + [\mu(\xi), \mu(\zeta)].\]

**Proof.** The exterior covariant derivative $D_0(\eta_\mu)$ with respect to the connection $\psi_* (\Gamma_0)$ of $\eta_\mu$ corresponds to the exterior covariant derivative with respect to $\Gamma_0$ of the pull-back of $\eta_\mu$ via the bundle morphism $\psi$. This pull-back is a tensorial 1-form of type $\chi \circ \text{ad}$ on the total space $A$ of the $H_0$-bundle $A \to X$, and coincides with the left invariant $\mathfrak{t}$-valued 1-form $\tilde{\mu}$ on $A$ whose restriction to $\mathfrak{a}$ is $\mu$. It suffices to compute $(D_{\Gamma_0}(\tilde{\mu}))(\xi, \zeta)$ for two tangent vectors $\xi, \zeta \in \mathfrak{s}$. Let $\xi, \zeta$ be the left invariant vector fields on $A$ determined by $\xi, \zeta$. Since $\xi$ and $\zeta$ are $\Gamma_0$-horizontal, and $\tilde{\mu}(\xi), \tilde{\mu}(\zeta)$ are constant, we get

\[D_{\Gamma_0}(\tilde{\mu})(\xi, \zeta) = (d\tilde{\mu})(\xi, \zeta) = \xi(\tilde{\mu}(\zeta)) - \tilde{\mu}(\xi(\zeta)) - \tilde{\mu}(\xi, \eta) = -\tilde{\mu}(\xi, \zeta).\]
This shows that $D_{\xi}(\mu)(\xi, \zeta) = -\mu([\xi, \zeta])$ where $[\xi, \eta]$ denotes the Lie bracket of $\xi$, $\eta$ in the Lie algebra $\mathfrak{a}$. Note that $\mu([\xi, \eta])$ depends only on the $s$-component (the horizontal component) of $[\xi, \eta]$.

Using Lemma 24 and our results about the classification of invariant connections we obtain:

**Theorem 25.** Let $s$ be a $H_0$-invariant complement of $h_0$ in $a$ endowed with a complex structure $\mathcal{J}$ such that

1. The invariant almost complex structure determined by $\mathcal{J}$ on $X = A/H_0$ is integrable.
2. The curvature of the connection $\Gamma_0$ on the $H_0$-bundle $A \to X = A/H_0$ is of Hodge type $(1,1)$.

Let $K$ be a compact Lie group. Then the equivalences classes of triples $(Q, P, \beta)$ consisting of a holomorphic $K$-bundle $(Q, P)$ on $X$, and an $\alpha$-lifting action $\beta$ by holomorphic $K$-bundle isomorphisms, correspond bijectively to the points of the quotient $\mathcal{M}(A, H_0, K, s, J) \subset \mathcal{M}(A, H_0, K)$ defined by

$$\mathcal{M}(A, H_0, K, s, J) := \left\{(\chi, \mu) \in \text{Hom}(H_0, K) \times \text{Hom}(a/h_0, t) \mid \mu \circ \text{ad}_h = \text{ad}_{\chi(h)} \circ \mu \forall h \in H_0, \quad \mathfrak{J}(\mu) = 0 \right\}/K$$

where the map $\mathfrak{J}_J : \text{Hom}(a/h_0, t) \to \text{Alt}^2(s, t)$ is defined by

$$\mathfrak{J}_J(\mu)(\xi, \zeta) := -\mu([\xi, \zeta]) + [\mu(\xi), \mu(\zeta)] + \mu([J\xi, J\zeta]) - [\mu(J\xi), \mu(J\zeta)].$$

If, moreover, $a$ is semisimple and all the simple summands of $a$ are of non-compact type, then the $\alpha$-invariant isomorphism classes of holomorphic $K$-bundles on $X$ correspond bijectively to the points of the quotient $\mathcal{M}(\tilde{A}, \tilde{H}_0, K, s, J)$, where $c : \tilde{A} \to A$ is the universal cover of $A$ and $\tilde{H}_0 := c^{-1}(H_0)$.

The main result of [12] can be recovered as a special case of this general theorem:

**Remark 26.** Taking for $A$ a simple Lie group of non-compact type and for $H_0$ a maximal compact subgroup of $A$, we obtain the classification of $\alpha$-invariant classes of holomorphic $K$-bundles $(Q, P)$ on any irreducible symmetric Hermitian space of non-compact type. The condition of being type $(1,1)$ required in Theorem 25 is satisfied by the results of [Ra], so in this case the $\alpha$-invariant equivalence classes of pairs $(Q, P)$ correspond bijectively to the moduli space $\mathcal{M}(A, H_0, K, s, J)$.

4.3. **Non-transitive actions.** Let now $\alpha$ be a smooth action of a Lie group $A$ on a manifold $X$. Restricting an $\alpha$-invariant $K$-connection $(P, \beta, \Gamma)$ on $X$ to an orbit $Y \subset X$ of $\alpha$, one obtains an $\alpha_Y$-invariant $K$-connection on $Y$ endowed with the induced transitive action $\alpha_Y$.

If $A$ acts on $X$ with compact stabilizers, then Theorem 12 can be applied to all these transitive actions, and one obtains for every orbit $Y$ an explicit description of the set $\Phi_{\alpha_Y, K}$ of equivalence classes of $\alpha_Y$-invariant $K$-connections on $Y$ in terms of a moduli space $\mathcal{M}_Y$, which is a $K$-quotient of a finite dimensional space. A natural problem is to endow the union $\mathcal{M} := \bigsqcup_{Y \in X/A} \mathcal{M}_Y$ with a natural topology such that the projection

$$r : \mathcal{M} \to X/A$$
is continuous, and such that every $\alpha$-invariant $K$-connection $(P, \beta, \Gamma)$ on $X$ defines a continuous section of $r$. We believe that a natural strategy for understanding the set $\Phi_{\alpha,K}$ of equivalence classes of $\alpha$-invariant $K$-connections on $X$ is to study the map $R$ which associates to every $\alpha$-invariant $K$-connection $(P, \beta, \Gamma)$ on $X$ the section of the fibration $r$ obtained by restriction to the orbits. The first step in this direction would be to describe explicitly the topology of the total space $\mathcal{M}$, and the image and the fibers of $R$.

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