Atom counting in ultra-cold gases using photoionisation and ion detection

T. Campey, C. J. Vale, M. J. Davis, N. R. Heckenberg, and H. Rubinsztein-Dunlop
School of Physical Sciences, The University of Queensland, Brisbane, QLD 4072, Australia

S. Kraft, C. Zimmermann, and J. Fortág
Physikalisches Institut der Universität Tübingen, 72076 Tübingen, Germany

We analyse photoionisation and ion detection as a means of accurately counting ultra-cold atoms. We show that it is possible to count clouds containing many thousands of atoms with accuracies better than $N^{-1/2}$ with current technology. This allows the direct probing of sub-Poissonian number statistics of atomic samples. The scheme can also be used for efficient single atom detection with high spatio-temporal resolution. All aspects of a realistic detection scheme are considered, and we discuss experimental situations in which such a scheme could be implemented.

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I. INTRODUCTION

The term “atom optics” arises from the analogy between experiments exhibiting the wave-like properties of light with those demonstrating the wave-like properties of matter \(^1\). Due to the vanishingly small de Broglie wavelength of room temperature atoms it is necessary to cool gases to microKelvin temperatures before these properties are observable. Before 1995, experiments in atom optics used clouds of laser-cooled thermal atoms that can be considered similar to thermal light sources in optics, with coherence lengths of less than a micron.

The realisation of Bose-Einstein condensation in 1995 \(^2\)\(^3\)\(^4\) provided the first intense sources of coherent matter waves, analogous to the invention of the optical laser. Many of the early experiments with BECs concentrated on their wave-like properties, which in general can be well described by the Gross-Pitaevskii equation for the classical field $\psi(x) \approx (\bar{\psi}(x))$. Some notable experiments include the interference of independently prepared condensates \(^5\) and the demonstration of atom lasers \(^6\)\(^7\)\(^8\)\(^9\). These experiments rely on the first-order coherence of the condensate, and in some sense can be considered to be coherent “classical atom optics”.

Recently there have been an increasing number of experiments focusing on the generation and measurement of non-trivial higher order coherences in matter waves that are beyond the scope of Gross-Pitaevskii mean field theory. These experiments could be said to be in the area of “quantum atom optics” \(^10\). The development of this new area bears a close resemblance to the development of quantum optics following the invention of the optical laser. Just as quantum optics relies on single photon detectors, quantum atom optics will utilise detectors with single atom resolution, such as the technique we describe in this paper.

One of the earliest experiments probing higher-order coherence in BECs was in the measurement of three-body loss from a BEC, from which the correlation function $g^{(3)}(x)$ could be inferred \(^11\). There have been several measurements made of the local second-order correlation function $g^{(2)}(x)$ for a number of systems, including a 1D strongly interacting Bose gas \(^12\)\(^13\), a quasi-condensate \(^14\), and a thermal Bose gas \(^15\). The value of $g^{(2)}(x)$ for an atom laser has been measured in ref. \(^16\).

There has also been strong interest in relative number squeezing. This is where a state is generated in which the number difference between two or more atomic samples is sub-Poissonian. Javanainen and Ivanov \(^17\) showed that splitting condensates by adiabatically turning up a tunnel barrier can produce number squeezed atomic states. Experimentally, Orzel et al. \(^18\) loaded a BEC into a one-dimensional optical lattice and observed the degradation of interference fringes. This was interpreted to be caused by increased phase-fluctuations due to reduced number fluctuations at the lattice sites. A similar observation was made in the first demonstration of a 3D Mott insulator state \(^19\) and further elucidated by other related experiments \(^20\).

Correlations and relative number squeezing have been predicted in collisions and four-wave mixing in condensates \(^21\)\(^22\)\(^23\)\(^24\), as well as in downconversion of a molecular BEC in both the spontaneous \(^25\) and stimulated \(^26\) regimes. While experiments have observed four-wave mixing processes \(^27\)\(^28\) and matter-wave amplification \(^29\), there has been no direct measurement of these correlations.

Experimentally, other high-order correlations have been directly detected by suitable analysis of absorption images from an ensemble of experiments \(^30\)\(^31\). Other schemes such as the “quantum tweezer” \(^32\) can deterministically extract sub-Poissonian atomic samples from a BEC. Optical dipole traps along with fluorescence imaging have been used to demonstrate sub-Poissonian statistics in small samples of atoms \(^33\).

In order to directly probe atom number statistics of cold quantum gases, one would ideally like an atom counter with accuracy at the single atom level. Exist-
ing single atom detection schemes can be divided into two categories: optical and contact. One of the simplest optical schemes is based on resonance fluorescence detection \[33, 34, 35\]. Assuming negligible background, the shot noise limited atom number uncertainty in a fluorescence measurement scales as \(\sqrt{N/\alpha \gamma \tau}\), where \(N\) is the number of atoms, \(\alpha\) is the collection efficiency of the detector, \(\gamma\) is the mean scattering rate and \(\tau\) is the integration time \[36\]. With a long enough integration time it is possible to measure atom numbers with sub-Poissonian precision. However, limitations such as atom heating restrict the capability of sub-Poissonian fluorescence detection to atom numbers of order \(< 100\) \[37\]. Absorption measurements offers more favourable scaling to higher atom numbers \[38\] but achieving good absolute accuracy remains technologically difficult. High finesse optical cavities offer excellent sensitivity for single atom detection \[39, 40\] and can be used for slow counting of a large number of atoms. Cavities offer an improvement in signal to noise of \(\sqrt{F}\), where \(F\) is the cavity finesse, compared to single pass measurements for a fixed amount of heating \[41, 42\]. Moderate cavities have also been shown to aid fluorescence and absorption measurements \[32\]. It is difficult however to apply cavity techniques to counting large atom numbers and long integration times are required.

The second category of single atom detectors is based on contact methods. These generally involve a charged particle or metastable atom with high internal energy impacting on a charged particle detector. This initiates a cascade of electron emissions, which are amplified to produce a macroscopic current pulse. This method is particularly useful for detecting metastable He atoms \[42\] where the arrival statistics of atoms released from a BEC allowed for matter wave Hanbury-Brown Twiss correlations to be observed \[45\]. While this detection method is simple, metastable atoms remain difficult to condense and most cold atom experiments use alkali atoms.

In this paper we describe an atom detection scheme based on photoionisation and ion detection. This scheme overcomes some of the difficulties associated with optical detection, can be applied to ground state alkali atoms and is capable of extending sub-Poissonian counting to much larger numbers of atoms. Photoionisation \[46\] and ionisation \[47\] have previously been considered as a means of efficient atom detection. However, these proposals have not included many factors present in realistic detectors. Firstly, we note that the best absolute detection efficiencies of room temperature charged particle detectors are of order 0.8–0.9. (Superconducting tunnel junctions can in principle provide unity detection efficiency \[45\] but are impractical for many experimental setups.) This has serious consequences for any experiment which endeavours to count a large number of atoms, but as we show in this paper, provided the detection efficiency is greater than 0.5, it is still possible to count atoms with accuracies better than \(1/\sqrt{N}\). The achievable accuracy depends on several factors including the detector calibration, background count rates, detector pulse resolution times, ionisation rates and the uncertainties in these parameters, all of which are included in our analysis.

This paper is organised as follows: in Section II we describe the experimental arrangement necessary for atom counting and how it can be applied to a cloud of trapped \(^{87}\)Rb atoms. Section III addresses the statistics associated with the detection scheme along with the requirements on the detector efficiency. In Section IV we discuss detector calibration and describe numerical simulations of the calibration including the effects mentioned above. Section V looks at the application of the scheme in the presence of losses and we follow with a discussion of potential applications. Appendix A contains a table of symbols and Appendices B and C detail the derivations of certain expressions used in the paper.

II. SCHEME

The method we propose consists of slowly photoionising a sample of atoms and accelerating the ions to a charged particle detector. Figure 1(a) shows the elements of the scheme. A cloud of cold atoms is held in an optical dipole trap where it is illuminated by the ionisation lasers. Our discussion focusses specifically on the case of \(^{87}\)Rb atoms but the scheme could be modified to detect other alkali atoms by the correct choice of lasers. Figure 1(b) shows the relevant energy levels and transitions for \(^{87}\)Rb.

The ionisation scheme consists of two stages. The first is two-photon excitation with a low intensity 778 nm laser from the \(5S_{1/2}\) state to the \(5D_{5/2}\) state. The second is ionisation from this state with a high intensity infrared laser such as a Nd:YAG laser. The \(5D_{5/2}\) state has a photoionisation cross-section approximately three orders of magnitude higher than the ground state \[48\] so ionisation can readily be achieved with a cw laser of wave-
length shorter than $\sim 1200$ nm rather than with pulsed lasers\cite{47}. Ion detection is achieved after appropriate ion collection optics using a standard ion detector such as a channel electron multiplier (CEM) or discrete dynode detector.

The atom counting scheme works best if the pulses produced by ions striking the ion detector are well resolved in time. This means the ionisation rate $\dot{N}_I$ should be much lower than the inverse of the pulse resolution time of the detection system $1/\tau_r$. In Section V we will see that this means that for typical detector efficiencies and trap lifetimes, and atom numbers of the order of $10^3$ or $10^4$, the ionisation should take place on a time scale of order several ms. In order to confine the atoms during this time, the Nd:YAG laser is focused to create a dipole trap. This localises the atoms in the highest intensity region of the beam. The intensity must be high enough to produce a trap sufficiently deep that no atoms are lost due to heating as the ionisation takes place. The major source of heating is the spontaneous decay of atoms excited to the $5D_{5/2}$ state. A high Nd:YAG laser intensity thus helps eliminate trap loss not only by creating a deep trap, but also by ensuring that an atom excited to the $5D_{5/2}$ state has a substantially higher probability of being ionised than of decaying back to the ground state. Atoms that decay from the $5D_{5/2}$ state can end up in several ground state sublevels. A benefit of the dipole trap is that it traps all of these sublevels, in contrast to magnetic fields, which can only trap a restricted class of sublevels.

The rate of ionisation can be controlled with the 778 nm laser intensity. In Section V we show that for a given atom number and trap loss rate there is an optimal ionisation rate that minimises the uncertainty in the inferred atom number. In order to hold the ionisation rate approximately constant at the ideal value, the 778 nm laser intensity can be exponentially ramped to compensate for the reduction in trapped atom number as ions are produced. However we also show that the scheme works well with constant laser intensities.

This state selective scheme also represents an efficient single atom detector\cite{48}. The ionisation lasers can be tightly focussed to detect atoms at a well defined spatial location. The two photon excitation rate is proportional to the squared intensity of the 778 nm laser so that scattered light is less likely to heat atoms not in the detection region. Additionally, for single atom detection the spatial intensity profiles of the two lasers can be chosen so that the repulsive optical dipole potential of the blue-detuned 778 nm laser can be compensated for by the attractive potential of the high power red-detuned ionising laser. This leads to minimal perturbation of undetected atoms which may for instance be magnetically detuned or trapped in a different internal state. These factors, and the relatively quick ionisation times, make this a flexible scheme well suited to single atom detection with excellent spatio-temporal resolution.

### III. COUNTING WITH IMPERFECT DETECTORS

We now consider how accurately an atom number can be determined using a scheme in which ions are counted with a non-unity efficiency detector. The overall ion detection efficiency is

$$\eta_i = \eta_{det} \eta_{pr},$$

where $\eta_{det}$ is the efficiency of the ion detection system and is the product of the ion detector efficiency $\eta_{det}^c$ and the collection efficiency $\eta_{coll}$. The quantity $\eta_{pr}$ is the probability that an ion is temporally resolvable from the other ions detected, and is ionisation rate dependent.

If $N_I$ ions are produced, the number of ions counted, $N_i$, is described by a binomial distribution. Provided $\eta_i(1-\eta_i)N_I$ is greater than $\sim 5$, this is well approximated by a normal distribution with mean $\eta_iN_I$ and standard deviation $\sqrt{\eta_i(1-\eta_i)N_I}$\cite{49}. In a given experiment, the number of atoms ionised can be inferred to be

$$N_{I,inf} = \frac{N_i}{\eta_i},$$

with an uncertainty

$$\sigma_{N_{I,inf}} = \frac{\sqrt{N_i(1-\eta_i)}}{\eta_i^2} + \frac{N_i^2 \sigma_{\eta_i}^2}{\eta_i^4},$$

where $\sigma_{\eta_i}$ is the uncertainty in $\eta_i$.

In order to compare this uncertainty to Poissonian fluctuations, we define the normalised count uncertainty:

$$\kappa \equiv \frac{\sigma_{N_{I,inf}}}{N_{I,inf}}.$$  \hspace{1cm} (4)

The quantity $\kappa$ is related to the measurement Fano factor discussed in Section VI but includes an additional systematic contribution due to the uncertainty in the detection efficiency (the second term under the square root in Eq. 3). $\kappa = 1$ implies the count uncertainty is equal to the uncertainty inherent in Poissonian statistics. In order to count ions with error less than $\sqrt{N_I}$, $\kappa$ must be less than one.

In Fig. 2 contours of the normalised count uncertainty $\kappa$ are plotted as a function of $\eta_i$ and $\sigma_{\eta_i}$ for $N_I = 10^3$. From this figure it can be seen that for a detection system in which the detection efficiency is known exactly, i.e. $\sigma_{\eta_i} = 0$, the overall ion detection efficiency $\eta_i$ need only be greater that 50\% to achieve $\kappa$ less than 1. As the uncertainty in the detection efficiency increases, the required detection efficiency to achieve $\kappa \leq 1$ also increases. Thus, to achieve a low value of $\kappa$, we require a high efficiency detector that is well calibrated. In the next section we examine how this can be done.

### IV. DETECTOR CALIBRATION

To infer the number of atoms ionised and its uncertainty from an ion count, we need to know the overall ion
The ion detection efficiency $\eta_i$ and its uncertainty $\sigma_{\eta_i}$. From Eq. (1), this means we need to know the ion detection system efficiency $\eta_{det}$ and the ion temporal resolution probability $p_r$, each with their respective uncertainties. We first consider how to measure $\eta_{det}$ to high accuracy, or in other words, how to calibrate the ion detection system.

Accurate calibration can be achieved by means of a scheme in which an additional detector is used to count the electrons produced by ionisation [51]. If $N_f$ atoms are ionised in a calibration, and the overall ion and electron detection efficiencies are $\eta_i$ and $\eta_e$ respectively (where analogous to Eq. (1), $\eta_e = \eta_{det,eP_r,e}$, then

$$N_i = \eta_i N_f,$$

$$N_e = \eta_e N_f,$$

$$N_c = \eta_i \eta_e N_f,$$

where $N_i$ is the number of ions counted, $N_e$ is the number of electrons counted and $N_c$ is the number of ion-electron coincidences counted. Therefore

$$\eta_i = \frac{N_c}{N_e},$$

which shows that only the coincidence and electron counts are required to find the ion detection efficiency; we don’t need to know the efficiency of the electron detector.

In order to count the number of coincidences, it is first necessary to specify how coincidences are identified. In general there will be a distribution of times taken for ions to move from the point of ionisation to striking the detector and producing a pulse. The width of this distribution will depend on the range of initial velocities of the ions and the variation in electric field at different locations within the atom cloud. The same is true for the electrons. There will therefore be a distribution of intervals between ions and their corresponding electrons. With modern pulse counting electronics it is straightforward to produce lists of the arrival times of electrons and ions with precisions on the order of 100 ps [51] and these can be used to generate a list of intervals between electrons and the following ions. If a histogram of these intervals is plotted, true coincidences will appear as a peak against a small background of false coincidences. This peak can then be used to define a window for true coincidences.

The coincidence window may also contain false coincidences, which occur when an unpartnered ion (i.e. one for which the corresponding electron was not detected) arrives within the window of an unpartnered electron. Also, due to the finite size of the window, some true coincidences may not be counted. The mean net number of false coincidences $\bar{N}_f$ and its uncertainty $\sigma_f$ are derived in Appendix [B].

In order to take false coincidences and background ions and electrons into account in the ion detection efficiency, we define corrected ion, electron and coincidence counts as $N_i' \equiv N_i - N_{ib}$, $N_e' \equiv N_e - N_{eb}$ and $N_c' \equiv N_c - N_f$, respectively. $N_{ib}$ and $N_{eb}$ are respectively the mean numbers of background ions and electrons counted in the calibration period and can be measured by counting ions and electrons in the absence of the calibration atoms. The overall ion detection efficiency is then

$$\eta_i = \frac{N_c'}{N_e'},$$

(9)

The ion detection efficiency given by Eq. (9) is the value that applies in a calibration and is valid for the ionisation rate used in the calibration. To find the system ion detection efficiency (which is ionisation rate independent) we need to correct $\eta_i$ for the value of the ion temporal resolution probability $p_r$ that applies in the calibration. From Eqs (1) and (9), the system ion detection efficiency is

$$\eta_{det} = \frac{N_c'}{N_e' p_r},$$

(10)

and from Appendix [C]

$$p_r = 1 - \tau_r N_i - \frac{1}{2}(\tau_r N_i)^2,$$

(11)

where $\tau_r$ is the minimum time interval for which two ions can be resolved from each other, and $N_i$ is the average rate of ion detection in the calibration.

We also need to determine the uncertainty in $\eta_{det}$. This uncertainty arises mainly from the fact that for a given number of atoms ionised, the number of ion, electron and coincidence counts are described by binomial distributions. An additional contribution comes from the uncertainties in the number of background electrons and false coincidences counted during the calibration. Taking all of these effects into account, and for values of $p_r$ close to unity, the uncertainty in the ion detection system

![FIG. 2: Contours of the normalised count uncertainty $\kappa$ for an atom counting system with ion detection efficiency $\eta_i \pm \sigma_{\eta_i}$, for ion number $N_i = 10^3$.](image-url)
efficiency is

$$\sigma_{\eta_{det}} = \frac{N'_c}{N_e} \left( \frac{1}{N'_e} - \frac{1}{N_e} \right) + \frac{\sigma_{eb}^2}{N'_e} + \frac{\sigma_f^2}{N_e^2}, \quad (12)$$

where \(\sigma_{eb}\) is the uncertainty in the number of background electrons counted and is equal to \(\sqrt{N_{eb}}\), assuming the detection of background electrons is a Poisson process.

To minimise the uncertainty in the ion detection efficiency, it is desirable to minimise background counts. Background electrons and ions arise from three main potential sources. The vacuum gauges and ion getter pumps commonly used in ultra-cold atom experiments work by ionising atoms and hence some of the ionisation products may find their way to the detector. While the vacuum gauge can simply be turned off during the experiment, ion counts from the pump can be reduced by adding appropriate shielding. The third potential source of background ions and electrons is the ionisation by stray laser light of the Rb atoms covering the interior of the vacuum chamber.

Background ions and electrons can be prevented from reaching the detectors by using suitable ion optics, such as the apparatus presented in [15]. In this apparatus four electrodes are used to accelerate and focus ions and guide them to the detector. The ion trajectories were calculated using standard software and it was found that only a volume of \(\sim 1 \text{ cm}^3\) is imaged onto the detector. Ions produced outside of this volume are not detected and it is estimated that background counts can be almost entirely suppressed.

From Appendix B, the uncertainty in the number of false coincidences can also be made very small. From Eqs (10) and (12), neglecting background counts and false coincidences, and for \(p_r \approx 1\), the average relative uncertainty in the ion detection efficiency is

$$\frac{\sigma_{\eta_{det}}}{\eta_{det}} = \sqrt{\frac{1 - \eta_{det}}{\eta_{det} \eta_{det,e} N_I}}. \quad (13)$$

For calibration parameters \(\hat{N}_I = 10^5 \text{ s}^{-1}\), \(\eta_{det} = \eta_{det,e} = 0.8\) and a calibration time of \(\tau_{cal} = 10 \text{ s}\), this expression is accurate to within 1% for background electron and ion rates of up to \(10^3 \text{ s}^{-1}\). Equation (13) shows that this calibration method works best for large atom numbers and high overall ion and electron detection efficiencies.

In order to test the values for \(\eta_{det}\) and \(\sigma_{\eta_{det}}\), predicted by the calibration equations, numerical simulations of calibration experiments were carried out. These simulations included all the relevant features of a real calibration, such as

- ionisation following Poissonian statistics
- detection of ions and electrons subject to their respective detection efficiencies
- background electrons and ions
- resolution times of the detectors
- generation of a coincidence window from the lists of electron and ion arrival times
- use of the coincidence window to determine the number of coincidences

Each simulation generated electron, ion and coincidence counts for both “real” and background electrons and ions. Then, as would be done in a real calibration experiment, the corrected counts were calculated using the estimated numbers of background electrons, background ions and false coincidences. An inferred value and an inferred uncertainty of the ion detection system efficiency \(\eta_{det}\) were then found using Eqs (10) and (12).

Excellent agreement was found between the theory and simulation for a wide range of values. Figure 3 shows a histogram of the inferred ion detection efficiency for 1000 runs of the simulation with \(\eta_{det} = 0.9\), electron detection system efficiency \(\eta_{det,e} = 0.9\), ionisation rate \(\hat{N}_I = 10^4 \text{ s}^{-1}\), calibration time \(\tau_{cal} = 10 \text{ s}\), pulse resolution time \(\tau_r = 10 \text{ ns}\) and, background electron and ion detection rates \(\hat{N}_{ib} = N_{eb} = 10^2 \text{ s}^{-1}\) respectively.

FIG. 3: Histogram of the inferred values of the ion detection system efficiency \(\eta_{det}\) for 1000 runs of the calibration simulation with \(\eta_{det} = 0.9\), electron detection system efficiency \(\eta_{det,e} = 0.9\), ionisation rate \(\hat{N}_I = 10^4 \text{ s}^{-1}\), calibration time \(\tau_{cal} = 10 \text{ s}\), pulse resolution time \(\tau_r = 10 \text{ ns}\) and, background electron and ion detection rates \(\hat{N}_{ib} = N_{eb} = 10^2 \text{ s}^{-1}\) respectively.
at a constant rate in atoms per second, $p_r$ is given by Eq. (C7) in Appendix C. If ionisation occurs at a constant per atom rate (i.e. ionising laser powers are held constant), we use the effective ion resolution probability $p_r$, given by Eq. (C11). It should be noted that for the calibration to be valid, the ion collection efficiency $\eta_{coll}$ must remain constant between calibration and experiment. This could be achieved by calibrating using cold atoms in the same trap as used in the experiment. To achieve the lowest possible uncertainty in the atom count, $\eta_{coll}$ should be unity, which can in principle be achieved with well-designed ion optics.

V. TRAP LOSS AND OPTIMAL IONISATION RATE

In an atom counting experiment it is important to know how the rate of ionisation affects the normalised count uncertainty $\kappa$. The greater the rate of ionisation, the smaller the value of $p_r$ and the larger its uncertainty. This in turn decreases $\eta_i$ and increases $\sigma_{\eta_i}$, resulting in an increased value of $\kappa$. However in most experiments it would generally be desirable to ionise a cloud as quickly as possible without overly affecting the count accuracy. Figure 4 shows how $\kappa$ is affected by the ionisation time $\tau_I \equiv N_I/N_f$, for $N_I = 10^3$ and $\tau_r = 5$ ns, assumed to be known with a 10% uncertainty. The crosses correspond to an infinitely long ionisation (in which case $\eta_i = \eta_{det}$) and the circles and triangles correspond to ionisation times of 2 and 1 ms respectively. It can be seen that down to a certain ionisation time, $\kappa$ is close to its minimum value. However as the ionisation time decreases, $\kappa$ begins to increase rapidly.

If there is negligible trap loss, the number of atoms ionised $N_I$ is the same as the number of atoms originally in the trap $N$, and the inferred atom number $N_{inf}$ is the same as the inferred ion number $N_{I,inf}$. If there is trap loss however, we have to correct the inferred ion number to obtain an inferred atom number. In the remainder of this section we show how this is achieved and how the uncertainty in the inferred atom number is affected by trap loss. We also find the optimal ionisation rate which minimises the uncertainty in the inferred atom number due to both trap loss and imperfect ion temporal resolution.

To calculate the atom count corrected for trap loss, we assume a constant loss rate $R_l$ and that ionisation occurs at a constant rate $R_I$ (both rates are per atom). The respective time dependent probabilities of an atom being in the trap and of having been ionised follow

$$\dot{P}_T = -(R_I + R_l)P_T,$$
$$\dot{P}_I = R_I P_T.$$  

The probability that an atom is ionised rather than lost is

$$p_I = \int_0^\infty \dot{P}_I dt = \frac{R_I}{R_I + R_l}. \tag{16}$$

The total number of atoms ionised $N_I$ is described by a binomial distribution, which again can be approximated by a normal distribution with mean $p_IN$ and standard deviation $\sqrt{p_I(1-p_I)N}$. The original number of atoms in the trap can then be inferred to be

$$N_{inf} = \left(1 + \frac{R_l}{R_I}\right)N_{I,inf}, \tag{17}$$

with a statistical uncertainty of

$$\sigma_{N_{inf,stat}} = \sqrt{\frac{R_l}{R_I} \left(1 + \frac{R_l}{R_I}\right) N_{I,inf}}. \tag{18}$$

Due to uncertainties in $R_I$, $R_l$ and $N_{I,inf}$, we get extra terms in the uncertainty in $N_{inf}$ and we find

$$\sigma_{N_{inf}}^2 = \left[\frac{N_{I,inf} \sigma_{R_I}}{R_I}\right]^2 + \left[\frac{R_l N_{inf} \sigma_{R_l}}{R_I^2}\right]^2 + \left[\left(1 + \frac{R_l}{R_I}\right) \sigma_{N_{inf}} + \frac{R_l}{R_I} \left(1 + \frac{R_l}{R_I}\right) N_{I,inf}\right]^2, \tag{19}$$

with $N_{I,inf}$ and $\sigma_{N_{I,inf}}$ given by Eqs (2) and (4). Thus in order to infer the atom number and its uncertainty, we need to know $R_I$ and $R_l$ and their uncertainties in addition to $N_i$, $\eta_i$, and $\sigma_{\eta_i}$. We now consider how these values can be measured in a given experiment.

The sum of $R_I$ and $R_l$ can be found by plotting $\ln[N_i - N_i(t)]$ against time, where $N_i(t)$ is the cumulative ion count. The slope is then $-(R_I + R_l)$. The value of $R_I$ (and hence of $R_I$) may be found by plotting $R_I + R_l$ against the intensity of the 778 nm laser for a number of
runs. $R_l$ is given by the extrapolated $y$-intercept of this plot. (Extrapolation is required because $R_l$ cannot be measured by counting ions if no ionisation occurs. However, $R_l$ could be measured directly by absorption imaging at successive times in the absence of 778 nm light.)

The optimal ionisation rate is that which gives the lowest value of $\kappa$ for given values of $R_l$, $\eta_{\text{det}}$, $\tau_r$ and $N$. In Fig. 5 the optimal value of $1/R_l$ is plotted as a function of $N$ for various values of $1/R_l$, for $\eta_{\text{det}} = 0.9$, $\sigma_m = 0$, $\tau_r = 10$ ns and assuming a 10% uncertainty in the measured values of $R_l$ and $R_T$. The corresponding minimum values of $\kappa/\kappa_{\text{min}}$ are plotted in Fig. 6 where $\kappa_{\text{min}}$ is the value of $\kappa$ for zero trap loss. It can be seen that reasonably sized atom clouds ($10^3 - 10^4$ atoms) can be counted in times on the order of 10 ms with close to the theoretical minimum count uncertainties for realistic trap lifetimes.

An improvement on the uncertainties resulting from ionising with constant per atom probability would be obtained by increasing the 778 nm laser power during the ionisation in such a way as to obtain a constant ionisation rate in atoms/s. This would mean that the cloud would be ionised faster with less trap loss. The time dependent rate equations describing this scheme could be integrated numerically to determine the probability that an atom is ionised rather than lost, from which the original atom number and its uncertainty could be inferred.

VI. DISCUSSION

We now consider the application of our atom counting scheme to the characterisation of unknown atomic samples. To determine the number of atoms in a single cloud, one would simply count a certain number of ions $N_i$ and use the known calibration data to obtain an inferred atom number and uncertainty $N_{\text{inf}} \pm \sigma_{N_{\text{inf}}}$ from Eqs. (17) and (19). In the case that trap losses are not negligible, Eqs. (17) and (19) are also required. If however we wish to know the atom number variance in a specific situation we need to analyse a number of repeated measurements. As an example, consider a sample of $n$ measurements on atom clouds generated in an identical manner which, under ideal (noise free) circumstances, would yield identical numbers of atoms (for example using a quantum tweezer to extract atoms from a BEC). Each experiment yields a measure $N_{\text{inf},j} \pm \sigma_{N_{\text{inf},j}}$ where $j = 1, 2, ..., n$. The quantities of interest are the mean atom number, the atom number variance and the uncertainties in these values.

The population mean can be estimated from the mean of all inferred atom numbers $\bar{N}$ and has an uncertainty of $\sqrt{s_T^2/n + (\sigma_{\eta}/\eta)^2}$ where $s_T^2$ is the bias-corrected sample total variance and is given by $s_T^2 = 1/(n-1)\sum(N_{\text{inf},j} - \bar{N})^2$. The true atom number variance $\sigma_A^2$ is estimated by the difference of the total variance of the measurements $s_T^2$ and the variance associated with the binomial statistics of the ion counting $\sigma_m^2 \equiv \bar{N}(1-\eta)/\eta$. Note that the expression for $\sigma_m^2$ does not include a term due to $\sigma_{\eta}$ as in Eq. (5), since we assume that any error in the detection efficiency (calibration) will be the same for each measurement. This leads to a systematic error in the atom count and does not contribute to the variance $\sigma_A^2$.

The degree of number squeezing of the atomic population is described by the Fano factor $F_A = \sigma_A^2/N$. A Poissonian sample has $F_A = 1$ and a sub-Poissonian (number-squeezed) sample has $F_A < 1$. We can define a total Fano factor $F_T = s_T^2/N$ and measurement Fano factor $F_m = \sigma_m^2/N$, which accounts for the fluctuations due to the binominal counting of ions. The population

\[
\kappa = \frac{1}{\tau_r} \ln \left( \frac{N}{N_0} \right)
\]

\[
\eta_{\text{det}} = \frac{N_{\text{inf}}}{N}
\]

\[
\sigma_m = \sqrt{\frac{N^2}{N_{\text{inf}}}}
\]

\[
\kappa_{\text{min}} = \sqrt{\frac{N}{N_{\text{inf}}}}
\]

\[
\sigma_A = \sqrt{\frac{\sigma_m^2 + \sigma_{\eta}^2}{\eta}}
\]

\[
F_A = \frac{\sigma_A^2}{N}
\]

\[
F_T = \frac{s_T^2}{N}
\]

\[
F_m = \frac{\sigma_m^2}{N}
\]

\[
\bar{N} = \frac{1}{n} \sum_{j=1}^{n} N_{\text{inf},j}
\]

\[
\sigma_{N_{\text{inf}}} = \sqrt{\frac{1}{n} \sum_{j=1}^{n} \sigma_{N_{\text{inf},j}}^2}
\]

\[
\bar{N} = \frac{1}{n} \sum_{j=1}^{n} N_{\text{inf},j}
\]

\[
\sigma_{N_{\text{inf}}} = \sqrt{\frac{1}{n} \sum_{j=1}^{n} \sigma_{N_{\text{inf},j}}^2}
\]

\[
\sigma_A = \sqrt{\frac{\sigma_m^2 + \sigma_{\eta}^2}{\eta}}
\]

\[
F_A = \frac{\sigma_A^2}{N}
\]

\[
F_T = \frac{s_T^2}{N}
\]

\[
F_m = \frac{\sigma_m^2}{N}
\]

\[
\bar{N} = \frac{1}{n} \sum_{j=1}^{n} N_{\text{inf},j}
\]

\[
\sigma_{N_{\text{inf}}} = \sqrt{\frac{1}{n} \sum_{j=1}^{n} \sigma_{N_{\text{inf},j}}^2}
\]
Fano factor is then given by \( F_A = F_T - F_m \), and it can be shown that for \( \sigma_n < 5\% \) and \( N > 5 \) the uncertainty in \( F_A \) is well approximated by \( \sigma_{F_A} = \sqrt{2/n(F_A + F_m)} \). Knowing \( F_m \) from the calibration it is possible to determine how many measurements are required to obtain a suitably small uncertainty \( \sigma_{F_A} \) in the measured (normalised) atomic variance \( F_A \).

**VII. CONCLUSION**

We have analysed photoionisation and ion detection as a means of achieving accurate counting of ultra-cold atoms. The scheme relies on a three-photon, cw ionisation process and can also be used for efficient single atom detection with high spatio-temporal resolution. To count large numbers of atoms, one of the lasers acts as an optical dipole trap for the target atoms. We have shown that a well calibrated ion detector with realistic efficiencies could allow direct probing of atom number correlations in molecular down conversion and four wave mixing experiments which are predicted to show number difference squeezing.

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**APPENDIX A: TABLE OF SYMBOLS**

| Symbol | Description |
|--------|-------------|
| \( N \) | number of atoms originally in the trap |
| \( N_{inf} \) | inferred no. of atoms originally in the trap |
| \( \sigma_{N_{inf}} \) | uncertainty in the inferred atom number |
| \( \sigma_{N_{inf,stat}} \) | statistical uncertainty in the inferred atom no. |
| \( N_t \) | number of atoms ionised |
| \( \hat{N}_{inf} \) | inferred number of atoms ionised |
| \( \sigma_{\hat{N}_{inf}} \) | uncert. in the inferred no. of atoms ionised |
| \( N_{inf} \) | ionisation rate (s\(^{-1}\)) |
| \( \eta_{i} \) | overall ion detection efficiency |
| \( \eta_{det} \) | ion detection system efficiency |
| \( \sigma_{\eta_{det}} \) | uncert. in the ion detection system eff. |
| \( \eta_{det,e} \) | electron detection system efficiency |
| \( \eta_{coll} \) | ion collection efficiency |
| \( p_t \) | ion pulse temporal resolution probability |
| \( p_t' \) | effective ion pulse temporal resolution prob. |
| \( p_{e,c} \) | electron pulse temporal resolution probability |
| \( \kappa \) | normalised count uncertainty |
| \( \kappa_{min} \) | normalised count uncertainty for zero trap loss |
| \( \tau_r \) | pulse resolution time (s) |
| \( \tau_{cal} \) | calibration time (s) |
| \( \tau_I \) | ionisation time (s) |
| \( \tau_{win} \) | coincidence window length (s) |
| \( \tau \) | interval between successive ion detections (s) |
| \( N_i \) | number of ions counted |
| \( N_i(t) \) | cumulative ion count |
| \( N_i^* \) | ion detection rate (s\(^{-1}\)) |
| \( N_i^* \) | ion detection rate as \( \tau \rightarrow 0 \) (s\(^{-1}\)) |
| \( N_{i,cal} \) | ion detection rate in a calibration (s\(^{-1}\)) |
| \( N_c \) | number of electrons counted |
| \( N'_{e} \) | corrected electron count |
| \( N'_{e} \) | corrected coincidence count |
| \( N_{eb} \) | mean no. of background electrons counted |
| \( \sigma_{eb} \) | uncertainty in the background electron count |
| \( N_{ib} \) | mean net no. of false coincidences counted |
| \( \sigma_{f} \) | uncertainty in the false coincidence count |
| \( p_f \) | probability of a given unpartnered ion being in the window of an unpartnered electron |
| \( \sigma_{int} \) | probability of a true coincidence being outside the coincidence window |
| \( R_l \) | standard deviation of the time intervals between electrons and their corresponding ions |
| \( \sigma_{R_l} \) | trap loss rate per atom (s\(^{-1}\)) |
| \( R_l \) | uncertainty in \( R_l \) (s\(^{-1}\)) |
| \( \sigma_{R_b} \) | ionisation rate per atom (s\(^{-1}\)) |
| \( R_b \) | uncertainty in \( R_b \) (s\(^{-1}\)) |
| \( P_{th} \) | probability of an atom being in the trap as a function of time |
| \( P_l \) | probability of an atom being ionised as a function of time |
| \( p_{th} \) | probability of an atom being ionised as \( t \rightarrow \infty \) |
| \( f_{int}(\tau) \) | probability density function of the time interval between successive detections |
| \( f_{th}(t) \) | probability density function of arrival times of ions at the detector |
| \( \beta \) | \( \tau_{th} N R_l \) |
| \( n \) | number of atom number measurements |
| \( \bar{N} \) | mean of inferred atom numbers |
| \( s^2 \) | bias-corrected sample atom number total variance |
| \( N_{inf,j} \) | inferred atom number in the \( j^{th} \) measurement |
| \( \sigma_{N_{inf,j}} \) | uncertainty in the inferred atom number in the \( j^{th} \) measurement |
| \( \sigma_A \) | atom number population variance |
| \( \sigma_m \) | variance in the atom counts due to the binomial statistics of the measurement |
| \( F_A \) | Fano factor of a population of atomic clouds |
| \( \sigma_{F_A} \) | uncertainty in the Fano factor of a population of atomic clouds |
| \( F_T \) | Fano factor of a sample of inferred atom numbers |
| \( F_m \) | measurement Fano factor |
APPENDIX B: NET NUMBER OF FALSE COINCIDENCES

In this appendix we derive the required correction to a coincidence count to account for false coincidences and uncounted true coincidences. In a given calibration there will be $N_i - N_c$ unpartnered ions and $N_e - N_c$ unpartnered electrons. (We assume negligible background ion and electron numbers.) The probability of a given unpartnered ion being in the window of an unpartnered electron is well approximated by

$$p_f = (N_e - N_c) \frac{\tau_{\text{win}}}{\tau_{\text{cal}}}, \quad \text{(B1)}$$

where $\tau_{\text{win}}$ is the window length and $\tau_{\text{cal}}$ is the duration of the calibration.

The probability of a true coincidence being outside the window is

$$p_w = 1 - \text{erf} \left( \frac{\tau_{\text{win}}}{2 \sqrt{2} \sigma_{\text{int}}} \right), \quad \text{(B2)}$$

where $\sigma_{\text{int}}$ is the the standard deviation of the time intervals between electrons and their corresponding ions.

The net number of false coincidences counted in a given calibration is then

$$N_f = \bar{N}_f \pm \sigma_f, \quad \text{(B3)}$$

where

$$\bar{N}_f = p_f (N_i - N_e) - p_w N_e, \quad \text{(B4)}$$

and

$$\sigma_f = \sqrt{p_f (1 - p_f) (N_i - N_e) + p_w (1 - p_w) N_e} \quad \text{(B5)}$$

APPENDIX C: CALCULATION OF $p_r$ AND $p'_r$

As the overall ion detection efficiency $\eta$ is ionisation rate dependent, relating the ion detection efficiency measured in a calibration to the ion detection efficiency which applies in an experiment requires finding an expression for $p_r$ in terms of the ionisation rate $[\text{see Eq. (1)}]$. We consider two cases: constant ionisation rate in atoms per second, and constant per atom ionisation rate. We look first at the former.

We assume the ion detection system has some resolution time $\tau_r$, such that if an ion arrives at the detector within $\tau_r$ of the previous ion it will not be counted. We define $\bar{N}_i^*$ as the average rate (ions/s) at which ions are detected as $\tau_r \to 0$. Assuming a constant average detection rate, the detections follow Poissonian statistics, and the time interval between successive detections $\tau$, is described by a negative exponential probability density function $[49]:$

$$f_{\text{int}}(\tau) = \bar{N}_i^* e^{-\bar{N}_i^* \tau} \quad \text{for } \tau \geq 0. \quad \text{(C1)}$$

The probability that an ion arriving at the detector is resolvable from the previous ion is

$$p_r = 1 - \int_0^{\tau_r} f_{\text{int}}(\tau) d\tau. \quad \text{(C2)}$$

From Eqs (C1) and (C2) we find

$$p_r = \exp(-\tau_r \bar{N}_i^*). \quad \text{(C3)}$$

The rate at which ions are actually detected is

$$\bar{N}_i^* = p_r \bar{N}_i^* \quad \text{(C4)}$$

The number of ions counted during a calibration period $\tau_{\text{cal}}$ is described by a Poisson distribution: $N_i = \bar{N}_i^* \tau_{\text{cal}} \pm \sqrt{\bar{N}_i^* \tau_{\text{cal}}}$. If $N_i$ ions are counted in time $\tau_{\text{cal}}$, the ion detection rate in the calibration is inferred to be

$$\bar{N}_i_{\text{cal}} = \frac{N_i}{\tau_{\text{cal}}} \pm \frac{\sqrt{N_i}}{\tau_{\text{cal}}}. \quad \text{(C5)}$$

From Eqs (C3) and (C4) we find

$$\bar{N}_i^* = (1 + \tau_r \bar{N}_i^*) \bar{N}_i + O \left( (\tau_r \bar{N}_i)^3 \right). \quad \text{(C6)}$$

Substituting Eq. (C6) into Eq. (C3) and neglecting the third order term (which results in a relative error in $p_r$ of less than $10^{-9}$ for $\tau_r \bar{N}_i < 10^{-3}$) we get

$$p_r = 1 - \tau_r \bar{N}_i - \frac{1}{2} (\tau_r \bar{N}_i)^2. \quad \text{(C7)}$$

Now we consider the case of ionisation with a constant per atom ionisation rate. In this case we need to use an effective ion resolution probability $p'_r$, because by ionising with a constant per atom rate, the ionisation rate in ions per second is a function of time and hence so is $p_r$. The effective ion resolution probability is given by

$$p'_r = \int_0^{\infty} p_r(t) f_a(t) dt, \quad \text{(C8)}$$

where from Eqs (C3), (13) and (15)

$$p_r(t) = \exp \left( -\beta e^{-(R_1 + R_2) t} \right), \quad \text{(C9)}$$

where $\beta = \tau_r \eta_{\text{det}} N_R L$ and

$$f_a(t) = (R_1 + R_2) e^{-(R_1 + R_2) t} \quad \text{(C10)}$$

is the probability density function of arrival times of ions at the detector. It can be shown that, neglecting terms of $O(\beta^3)$,

$$p'_r = 1 - \frac{1}{2} \beta + \frac{1}{3} \beta^2. \quad \text{(C11)}$$
