Unitarized Heavy Baryon Chiral Perturbation Theory and the $\Delta$ Resonance in $\pi N$ Scattering

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Based on the recent phenomenological analysis done by M. Mojzis, Eur. Phys. Jour. C 2 (1998) 182, of $\pi N$ scattering within the framework of Heavy Baryon Chiral Perturbation Theory we predict, after a suitable unitarization of the amplitude, the $\Delta$ resonance at $\sqrt{s} = M_{\Delta} = 1238^{+22}_{-18}$ MeV and with a width of $\Gamma_{\Delta} = 150^{+43}_{-31}$ MeV in excellent agreement with the experimental numbers. The error bars reflect only the uncertainties in the low energy parameters as determined from an extrapolation to threshold of experimental data. We also describe satisfactorily, within errors, the phase shifts up to $\sqrt{s} \sim 1300$ MeV.

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I. INTRODUCTION

The $\Delta$ resonance plays a prominent role in intermediate energy nuclear physics \[1\]. Experimentally, it can be seen as a clear resonance of the partial amplitude $f_{2l2J}(s)$, in the $l_2l_2J = P_{33}$ channel in $\pi N$ scattering experiments, at a Center of Mass (CM) energy $\sqrt{s} = M_\Delta = 1232\text{MeV}$ and with a width $\Gamma_\Delta = 120\text{MeV}$ \[2\]. In this paper we are concerned with the possibility of generating this resonance using known information at threshold, chiral symmetry constraints and exact unitarization of the amplitude.

The practical description of resonances requires, in general, the use of some unitarization method since exactly at the resonance energy the amplitude becomes purely imaginary and takes the maximum value allowed by unitarity \[3\]. On the other hand, most unitarization methods are based on perturbation theory and hence require a resummation of some perturbative truncated expansion of the full amplitude. This obviously requires the existence of an energy region where the unitarized amplitude approximately coincides with the perturbatively expanded one. This region is typically close to threshold, since exactly at that point the amplitude is purely real, and unitarity sets no constraints on it. In other words, the amplitude is justifiably unitarizable if perturbation theory works reasonably well somewhere, for instance at threshold. If this is not the case there is nothing much one can do about it; one can formally unitarize the amplitude but the corresponding predictions at threshold will be in general very different from those obtained in perturbation theory.

The modern way to effectively incorporate chiral symmetry and departures from it, is by means of Chiral Perturbation Theory (ChPT) \[4\]. Being a perturbative Lagrangian approach it preserves exact crossing, perturbative renormalization and unitarity and allows for a bookkeeping ordering of Chiral symmetry breaking. All detailed information on higher energies or underlying microscopic detailed dynamics is effectively encoded in some low energy coefficients which, for the time being, can only be determined experimentally. For processes involving only pseudoscalar mesons the expansion parameter is taken to be $p^2/(4\pi f)^2$ with $p$ the four momentum of the pseudoscalar meson and $f$ the weak pion decay constant \[4\]. Thus, this expansion works better in the region around threshold. Away from threshold, however, the expansion breaks down and the violation of unitarity becomes increasingly strong. Recently, the use of unitarization methods for pseudoscalar mesons has been shown to work, i.e. the predicted unitarized amplitudes reproduce the threshold region, and provide definite theoretical central values and error estimates of the phase-shifts away from threshold and up to about $1\text{ GeV}$ \[4\].

The extension of ChPT to include both mesons and baryons as explicit degrees of freedom becomes possible if fermions are treated as heavy particles but in a covariant framework \[4\], yielding to the so called Heavy-Baryon Chiral Perturbation Theory (HBChPT) \[7–9\]. Here, the expansion to order $N = 1,2,3,\ldots$ is written in terms of a string of terms of the form $e^N/(f^{2l}M^{N+1-2l})$, with $l = 1,\ldots,[(N + 1)/2]$. Here, $f$ and $M$ are the weak pion decay constant and baryon mass respectively. The quantity $e$ is a generic parameter with dimensions of energy built up in terms of the pseudoscalar momenta and the velocity $v^a$ ($v^2 = 1$) and off-shellness $k$ of the baryons, being the latter defined as usual through the equation $p_B = \hat{M} v + k$, with $p_B$ the baryon four momentum and $\hat{M}$ the baryon mass up to corrections generated by higher orders in the HBChPT expansion. Practical calculations \[10–12\] show, however, that the convergence rate of such an expansion may not be as good
as it was in the purely mesonic case. Even at threshold, where the finite baryon mass effects should be minimal, there appear sizeable corrections to the scattering lengths for the lowest partial waves. The situation obviously gets worse as one departs from the threshold region. In the particular case of $\pi N$ scattering, a systematic calculation has been done, so far, up to third order [11,12], i.e. up to and including terms of order $1/f^2$ (first order), $1/(f^2M)$ (second order) and $1/f^4$ and $1/(f^2M^2)$ (third order). A fit to threshold properties allows to extract the low energy constants, with the result that the second order contribution turns out to be larger than the first order one. For instance, in the $P_{33}$ channel Mojzis [11] obtains that the scattering length $a_{133}^1$ changes from 34.5 GeV$^{-3}$ at first order, to 80.5 GeV$^{-3}$ at second order and to 81.4 GeV$^{-3}$ at third order. Obviously, any attempt to unitarize this amplitude based on the smallness of the second order term with respect to the first order one, will formally reproduce HBChPT but it will likely fail numerically to reproduce the corresponding phase shifts even in the vicinity of the threshold region. Thus, to be consistent with the calculation of Ref. [11] the used unitarization method should take into account the slow convergence rate of HBChPT, and ensure, to start with, that it numerically reproduces the scattering data around threshold.

The $\Delta$ resonance can be included as an explicit degree of freedom within HBChPT [13–15]. This requires the introduction of new parameters into the Chiral Lagrangian. On the other hand, the unitarization of the amplitude via the Inverse Amplitude Method (IAM) does not introduce new parameters and has recently been proposed [16], but the second order terms have been considered to be small. As a consequence, the low energy constants turn out to be very different from those found within HBChPT [11] (see also the discussion below). In particular, their value $b_{19} = -23.13$, implies a pion nucleon coupling constant $g_{\pi NN} = 19.23$, a rather odd one, compared with the experimental one $g_{\pi NN} = 13.4 \pm 0.1$. In the present work we show how the IAM should be modified in order to simultaneously i) implement exact unitarity, ii) comply with HBChPT at threshold and iii) describe the $\Delta$ resonance without having to introduce additional explicit parameters than those needed at threshold.

II. PARTIAL WAVE AMPLITUDES

We rely heavily on the notation of Ref. [11] and refer to that work for more details. The $\pi N$ scattering amplitude is given by

$$T_{\pi N}^\pm = \bar{u}(\hat{M} v + p, \sigma') [A^\pm + B^\pm \frac{\hat{g} + \hat{g}'}{2}] u(\hat{M} v + p, \sigma)$$

(1)

where $(\hat{M} v + p, q)$ and $(\hat{M} v + p', q')$ are the incoming and outgoing nucleon and pion momenta respectively. The bare nucleon mass $\hat{M}$ and the velocity $v^\mu$, ($v^2 = 1$) determine the baryon off-shellness $p$ and $p'$ and $\sigma$ and $\sigma'$ are the spin indices. The superscript “$\pm$”

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1 Both Ref. [11] and Ref. [12] make a fit of the low energy contribution either by extrapolation of close to threshold data to threshold or by extrapolation of the theory to the close to threshold region respectively, but only the first one displays the contributions of the several orders explicitly, so we will mainly refer to Ref. [11] in this regard.
corresponds to the isospin decomposition $T^{ba} = T^a \delta^{ba} + i e^{bac} T^c \tau_c$, with $a$ and $b$ the incoming and outgoing pion isospin states in the cartesian basis respectively. $A^\pm$ and $B^\pm$ are scalars depending on the Mandelstam variables $s, t$ and $u$ which for on-shell pions and nucleons verify $s + t + u = 2(m^2 + M^2)$, with $m$ the physical pion mass and $M$ the physical nucleon mass. In the CM frame, the partial wave amplitudes are written as (the subscript $\pm$ stands for $j = l \pm 1/2$),

$$f_{l \pm}^\pm(s) = e^{i \phi_{l \pm}^\pm(s)} \sin \delta_{l \pm}^\pm(s) \frac{1}{q} \frac{1}{16\pi} \sqrt{s} \left( (E + M) [A_{l \pm}^\pm(s) + (\sqrt{s} - M) B_{l \pm}^\pm(s) - (E - M) [A_{l \pm 1}^\pm(s) - (\sqrt{s} + M) B_{l \pm 1}^\pm(s)] \right)$$

(2)

where, for elastic scattering, $q = |q|$ is the CM momentum and $E = \sqrt{q^2 + M^2}$ and $\omega = \sqrt{q^2 + m^2}$ the CM nucleon and pion energies respectively. Finally, $\sqrt{s} = E + \omega$ is the total CM energy and $\delta_{l \pm}^\pm(s)$ are the corresponding phase-shifts. The projected amplitudes $A_l(s)$ and $B_l(s)$ are defined by the integrals

$$\left( A_l^\pm(s), B_l^\pm(s) \right) = \int_{-1}^{1} dz \left( A_l^\pm(s, z), B_l^\pm(s, z) \right) P_l(z)$$

(3)

with $z = \cos \theta$ and $\theta$ the CM scattering angle. In addition, $A^\pm$ and $B^\pm$ can be written as

$$A^\pm = \left( \alpha^\pm + \frac{s - u}{4} \beta^\pm \right)$$

(4)

$$B^\pm = \left( - M + \frac{t}{4M} \right) \beta^\pm$$

(5)

with $t = -2q^2(1 - \cos \theta)$ and $u = M^2 + m^2 - 2E\omega - 2q^2\cos \theta$. An expansion of both $\alpha^\pm$ and $\beta^\pm$ along the lines of HBChPT has been given in Ref. [11] up to third order, i.e. including terms of order $e/f^2$ (first order), $e^2/(f^2M)$ (second order) and $e^3/f^4$ and $e^3/(f^2M^2)$ (third order). There, however, in some cases the full nucleon mass dependence has been retained. This makes the discussion on orders a bit obscure$^2$. We have preferred to further expand any observable in terms of $1/M$ or $1/f^2$. As a consequence and in contrast to Ref. [11], there is no dependence of D,F, and higher partial waves on the low energy constants. Thus, the low energy parameters are solely determined by S and P partial waves. One thus gets the following expansion,

$$f_{l \pm}^\pm = f_{l \pm}^{(1)}^\pm + f_{l \pm}^{(2)}^\pm + f_{l \pm}^{(3)}^\pm + \cdots$$

(6)

where

$$f_{l \pm}^{(1)}^\pm = \frac{m}{f^2} t_{l \pm}^{(1,1)} \pm \left( \frac{\omega}{m} \right)$$

(7)

$$f_{l \pm}^{(2)}^\pm = \frac{m}{f^2M} t_{l \pm}^{(1,2)} \pm \left( \frac{\omega}{m} \right)$$

(8)

$$f_{l \pm}^{(3)}^\pm = \frac{m^3}{f^4} t_{l \pm}^{(3,3)} \pm \left( \frac{\omega}{m} \right) + \frac{m^3}{f^2M^2} t_{l \pm}^{(1,3)} \pm \left( \frac{\omega}{m} \right)$$

(9)

$^2$For instance, the $P_{33}$ scattering length contains some corrections $m/M$ to all orders ( see eqs.(90-92) in that reference ), instead of neglecting all pieces of order $1/(f^2M^2)$, $1/(f^4M)$ or higher as dictated by the expansion assumed in the Chiral Lagrangian.
$t_{l_{n,m}}^{(n,m)}$ are dimensionless functions of the dimensionless variable $\omega/m$, independent of $f$, $M$ and $m$, whose analytical expressions are too long to be displayed here. The convenience of the double superscript notation will be explained below; $n+1$ indicates the power in $1/f$ and $m$ the total order in the HBChPT counting. The unitarity condition $\text{Im} f_{l_{\pm}}^{(1)} = -q$ becomes in perturbation theory

$$
\text{Im} t_{l_{\pm}}^{(1,1)} = \text{Im} t_{l_{\pm}}^{(1,2)} = \text{Im} t_{l_{\pm}}^{(1,3)} = 0
$$
$$
\text{Im} t_{l_{\pm}}^{(3,3)} = \frac{q}{m} |t_{l_{\pm}}^{(1)}|^2
$$

The scattering lengths $a_{l_{\pm}}$ and effective ranges $b_{l_{\pm}}$ are defined by

$$
\text{Re} f_{l_{\pm}} = q^2 (a_{l_{\pm}} + q^2 b_{l_{\pm}} + \cdots)
$$

Obviously, the expansion of Eq.(6) for the amplitudes carries over to the threshold parameters, $a_{l_{\pm}}$ and $b_{l_{\pm}}$ (see below).

### III. RE-FITTING THE LOW ENERGY CONSTANTS

As we have said, in Ref. [11], a systematic expansion for the fixed-$t$ amplitudes $\alpha_{\pm}$ and $\beta_{\pm}$ was undertaken, but some higher order terms were retained in the partial wave amplitudes. This makes the discussion about orders a bit unclear, since different orders are mixed. In all our following considerations we further expand the amplitudes, and consequently the threshold and close to threshold parameters in the spirit of HBChPT. For the scattering lengths we have

$$
a_{2l_{2,1}} m^{2l+1} = \frac{m}{f^2} \alpha_{(1,1)} + \frac{m^2}{f^2 M} \alpha_{(1,2)} + \frac{m^3}{f^2 M^2} \alpha_{(1,3)} + \frac{m^3}{f^4} \alpha_{(3,3)} + \cdots
$$

and a similar expression for $b_{2l_{2,1}} m^{2l+3}$. With this expansion we may reanalyze the fit of Ref. [11]. For a better comparison we do so for the same experimental data, and with the same input values of the parameters, $M = 939$ MeV, $m = 138$ MeV, $f = 93$ MeV and $g_A = 1.26$. The result of the fit is presented in Table 1. As we see from the table the changes in the parameters are small, and are compatible within two standard deviations. This was to be expected since the difference between Mojzis’s parameters and ours is of higher order. It is also noteworthy that the resulting $\chi^2 = 10.68$ is mainly made out of $b_{0,+}^+$ and $\sigma$ which contribute with 3.3 and 7.3 respectively to the total $\chi^2$. The bad result concerning the $\sigma$-term is substantiated by the findings of Refs. [10–12].

After performing this re-fitting procedure we have found instructive to separate the contributions to the scattering lengths and effective ranges of the lowest partial waves, as done in Table 2. A clear distinctive pattern emerging from Table 2 is that the contribution of order $1/(f^2 M^2)$ is always rather small. Only in some cases, however, is the contribution of order $1/f^4$ also small. This is so in the $P_{33}$ channel in particular. Incidentally, let us note that for this channel the smallness of the third order contribution is due to the smallness of both the $1/f^4$ and the $1/(f^2 M)$ terms and does not stem from a cancelation between large contributions. So, it seems that close to threshold the $1/f^2$ expansion converges faster than
the $1/M$ expansion. Actually, the terms in $1/f^2$ and $1/(f^2 M)$ are of comparable importance. From here, it is clear that any unitarization method will only be consistent with the HBChPT approach, if it treats both the first and the second order as equally important.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
 & Mojzis (Ref. [11]) & This Work \\
\hline
 & Set I & Set II \\
\hline
$a_1$ & $-2.60 \pm 0.03$ & $-2.72 \pm 0.03$ \\
$a_2$ & $1.40 \pm 0.05$ & $1.39\pm0.05$ \\
$a_3$ & $-1.00 \pm 0.06$ & $-1.11\pm0.06$ \\
$a_5$ & $2.20\pm0.05$ & $2.3\pm0.07$ \\
$b_1$ & $1.8\pm0.0$ & $1.9\pm0.0$ \\
$b_3$ & $3.4\pm0.0$ & $3.5\pm0.0$ \\
$b_6$ & $3.0\pm0.0$ & $3.1\pm0.0$ \\
$b_{16} - b_{15}$ & $0.7\pm0.0$ & $0.8\pm0.0$ \\
$b_{19}$ & $-2.4\pm0.0$ & $-2.5\pm0.0$ \\
\hline
\end{tabular}
\caption{Low energy constants obtained from fitting the threshold parameters $a^\pm_{0,1}$, $b^\pm_{0,1}$, $a^\pm_{1,2}$ found in the present work and in Ref. [11], the Goldberger-Treiman discrepancy and the nucleon $\sigma$-term to the HBChPT predictions. We take $M = 939$ MeV, $m = 138$ MeV, $f = 93$ MeV and $g_A = 1.26$. Experimental data are taken from Ref. [17]. The first column refers to the work of Mojzis [11], where the fixed $t$ amplitudes $a^\pm$ and $b^\pm$ have been expanded using HBChPT but not the threshold parameters. In our case, we have further expanded the threshold parameters (see text for details) and fitted to the same data. The dependence of the Goldberger-Treiman discrepancy and the nucleon $\sigma$-term on the fitted parameters can be found in Ref. [11]. There is only one degree of freedom. Errors in any parameter have been calculated by changing the corresponding parameter until that the minimized $\chi^2$ with respect to the remaining parameters varies by one unit. The fit yields a $\chi^2_{min}=10.68$ which is basically made out of $b^+_{0,1}$ and $\sigma$ which give 3.3 and 7.3 contribution to the total $\chi^2$ respectively. The value of the total $\chi^2$ in Ref. [11] is not given.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
 & $1/f^2$ & $1/(f^2 M)$ & $1/(f^2 M^2)$ & $1/f^4$ & Total \\
\hline
$a_{1,2}^0$ & $-0.63$ & $-0.08$ & $+0.07$ & $-0.11$ & $-0.75$ \\
$a_{1,2}^1$ & $1.27$ & $-0.36$ & $-0.07$ & $0.41$ & $1.25$ \\
$a_{1,3}^3$ & $35.3$ & $48.1$ & $-1.6$ & $-0.3$ & $81.5$ \\
$a_{1,3}^1$ & $-17.6$ & $14.3$ & $-3.0$ & $-5.2$ & $-11.5$ \\
$a_{1,3}^5$ & $-17.6$ & $11.8$ & $-1.7$ & $-9.6$ & $-17.1$ \\
$a_{1,1}^6$ & $-70.6$ & $86.9$ & $-3.6$ & $-43.1$ & $-30.4$ \\
\hline
\end{tabular}
\caption{HBChPT, lowest partial S and P wave- scattering lengths, $a^+_{1,2,3}$ (in GeV$^{-2l-1}$ units) for $\pi N$ scattering in the in HBChPT, decomposed as a sum of terms of first order $1/f^2$, second order, $1/(f^2 M)$, and third order, $1/(f^2 M^2)$ and $1/(f^4)$. The sum of all the terms yields the total scattering length parameter. The experimental value obtained in Ref. [17] is also quoted.}
\end{table}
IV. UNITARIZATION METHOD FOR THE $P_{33}$ CHANNEL

Our unitarization method is based on assuming, as suggested by the perturbative calculation, that the chiral expansion in terms $1/f^2$, has a stronger convergence rate than the finite nucleon mass $1/M$ corrections. Such a hypothesis can only be supported by confrontation with experimental data, and indeed there are some recent theoretical attempts [18,19] to define a relativistic power counting not requiring the heavy baryon idea somehow retaking the spirit of older relativistic studies [20]. Indeed, we show that a slightly modified version of the well-known IAM approach turns out to describe the $P_{33}$ phase-shift satisfactorily, just using the low energy parameters determined at threshold, i.e., without re-fitting them in the unitarized case. The idea is to consider the expansion of the inverse amplitude in terms of $m^2/f^2$. The expansion of the amplitude can be written as

$$f^{1}_{33}(\omega, m, f, M) = \frac{m}{f^2} t^{(1)}(\omega/m, m/M) + \frac{m^3}{f^4} t^{(3)}(\omega/m, m/M) + \ldots$$

(14)

where we have explicitly used that the functions $t^{(2n+1)}$ are dimensionless, and hence depend only on dimensionless variables, such as $\omega/m$ and $m/M$. For the sake of a lighter notation, the quantum numbers $l I$ and $J$ have been purposely suppressed. Perturbative unitarity in this expansion requires at lowest order,

$$\text{Im} t^{(1)}(\omega/m, m/M) = 0$$

(15)

$$\text{Im} t^{(3)}(\omega/m, m/M) = \frac{q}{m} |t^{(1)}(\omega/m, m/M)|^2$$

(16)

which in turn imply an infinite number of conditions in the $1/M$ expansion. The functions $t^{(1)}$ and $t^{(3)}$ are only known in a further $m/M$ expansion,

$$t^{(2n+1)}(\omega/m, m/M) = t^{(2n+1,2n+1)}(\omega/m) + \frac{m}{M} t^{(2n+1,2n+2)}(\omega/m) + \left(\frac{m}{M}\right)^2 t^{(2n+1,2n+3)}(\omega/m) \ldots$$

(17)

yielding Eq.(3) and Eq.(4) after a suitable isospin projection. For the inverse amplitude we get then

$$\frac{1}{f^{1}_{33}(s, m, f, M)} = \frac{f^2}{s} \frac{1}{t^{(1)}(\omega/m, m/M)} - \frac{m^3}{f^4} t^{(3)}(\omega/m, m/M) + \ldots$$

(18)

Obviously, expanding in $m/f^2$ may be justified provided these corrections are small somewhere. As we have shown, for the $P_{33}$ channel, they are small precisely at threshold. So in the threshold region the unitarization scheme will, approximately, reproduce the perturbative result. At the same time, unitarity is exactly implemented. However, the $1/M$ terms are not small corrections at threshold, so we refrain from further “expanding the denominator”. In this way we keep the first mass corrections as equally important. The state of the art

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3 The relation between $f^{I}_{2I,2I}$ and $f^{I}_{1,1}$ is given by $f^{I}_{3,2I+1} = f^{I}_{I,1} - f^{I}_{I,1}$ and $f^{I}_{1,2I+1} = f^{I}_{I,1} + 2f^{I}_{I,1}$. 
in HBChPT calculations is such that only \( t^{(1,1)} \), \( t^{(1,2)} \), \( t^{(1,3)} \) and \( t^{(3,3)} \) are known. To ensure exact elastic unitarity we should keep only \( t^{(1,1)} \) in the denominator of the second term in Eq. (18) above. This induces a tiny error thanks to the smallness of \( t^{(3,3)} \) (correction \( 1/f^4 \)) in the \( P_{33} \) channel. Of course, it would be highly desirable to compute at least \( t^{(3,4)} \) and \( t^{(3,5)} \) (orders \( 1/(f^4M) \) and \( 1/(f^4M^2) \) respectively) in order to be able to keep also \( t^{(1,2)} \) in the denominator of the second term of the mentioned equation. After these remarks, we have

\[
\frac{1}{f_{33}^+(s, m, f, M)}_{\text{Unitarized}} = \frac{f^2}{m} \frac{t^{(1,1)}(\omega/m) + \frac{m}{M} t^{(1,2)}(\omega/m) + \left(\frac{m}{M}\right)^2 t^{(1,3)}(\omega/m)}{-m \frac{t^{(3,3)}(\omega/m)}{t^{(1,1)}(\omega/m)^2}} \tag{19}
\]

At threshold, our formula yields a modified scattering length

\[
\frac{1}{a_{\text{Unitarized}}} = \frac{f^2}{m} \left( \frac{\alpha^{(1,1)}}{[\alpha^{(1,1)}]^2} + \frac{\alpha^{(1,2)}}{[\alpha^{(1,1)}]^2} + \left(\frac{m}{M}\right)^2 \frac{\alpha^{(1,3)}}{[\alpha^{(1,1)}]^2} - m \frac{\alpha^{(3,3)}}{[\alpha^{(1,1)}]^2} \right) - \frac{m}{[\alpha^{(1,1)}]^2} \tag{20}
\]

\[
= \frac{1}{a_{\text{HPChPT}} - \left(\frac{m}{M}\right) \alpha^{(3,3)}} - m \frac{\alpha^{(3,3)}}{[\alpha^{(1,1)}]^2} \tag{21}
\]

which on view of Table 2 yields, for the \( P_{33} \) channel, the value \( a_{3,3}^1|_{\text{Unitarized}} = 80.2 \text{GeV}^{-3} \), to be compared to \( a_{3,3}^1|_{\text{HBChPT}} = 81.0 \text{GeV}^{-3} \), a compatible value with the experimental one. Notice that if we had expanded the denominator considering the second order contribution (1/(\text{\( f^2M \)}) to be small we would have obtained strictly the IAM method as used in Ref. [16],

\[
\frac{1}{a_{\text{IAM}}} = \frac{f^2}{m} \left\{ \frac{1}{\alpha^{(1,1)}} - \frac{\alpha^{(1,2)}}{[\alpha^{(1,1)}]^2} - \left(\frac{m}{M}\right)^2 \frac{\alpha^{(1,3)}}{[\alpha^{(1,1)}]^2} + \left(\frac{m}{M}\right)^2 \frac{\alpha^{(1,3)}}{[\alpha^{(1,1)}]^2} \right\} - m \frac{\alpha^{(3,1)}}{[\alpha^{(1,1)}]^2} \tag{22}
\]

yielding \( a_{3,3}^1|_{\text{IAM}} = 22.8 \text{GeV}^{-3} \), a completely odd result. This explains why the low energy parameters recently found in Ref. [16] are so different from those found by Mojzis [11]. Actually, their value of \( b_{19} = -23.13 \) would yield a pion-nucleon coupling constant \( g_{\pi NN} = 19.2 \), completely out of question.

V. NUMERICAL RESULTS

From our previous discussion it is clear that at threshold our unitarized amplitude will reproduce very accurately and within error bars the HBChPT results and hence the experimental data. It is thus tempting to extend the \( P_{33} \) phase shift up to the resonance region and propagate the errors of the low energy parameters. We show in Fig.1 and Fig.2 the corresponding phase shifts as the outcome of our unitarization method, Eq. (19), and compare them to the experimental \( \pi N \) data [2]. We use both parameter sets given in Table 1. In Fig.2 we use the parameters determined in Ref. [11] (set I) and in Fig.1 those determined in the present work (set II).
FIG. 1. $P_{33}$ phase shifts as a function of the total CM energy $\sqrt{s}$. Experimental data are from Ref. [2]. The upper solid curve is the result of our approach (Eq. 19), with the corresponding propagated error bars (dashed-dotted curve). The intermediate solid curve represents the static limit prediction, $M \to \infty$, also with error bands (dashed-dotted). The bottom solid line is the prediction of the conventional IAM method of Ref. [16]. In all cases, the low energy parameters are those labeled as set II in Table 1. The position and width of the resonance for the upper curve are $M_\Delta = 1267^{+35}_{-25}$ MeV and $\Gamma_\Delta = 213^{+54}_{-54}$ MeV respectively.

As one can see from both figures, the prediction of Unitarized Heavy Baryon Chiral Perturbation Theory agrees rather well within errors with the data. It is fair to say, however, that for the parameters determined in Ref. [11] the agreement seems much better, describing the data for larger CM energy values. We do not ascribe any particular significance to a very accurate agreement at larger energies since there is certainly more physics to be taken into account, like inelasticities. In both Fig.1 and Fig.2, the error bars reflect only the uncertainty in the low energy parameters and have been obtained by means of a Monte Carlo gaussian sampling of the low energy parameters for any given CM energy value. The location and width of the $\Delta$ resonance come out to be

$$
M_\Delta = 1238^{+22}_{-15}$ MeV (set I), $1267^{+35}_{-25}$ MeV (set II), (exp. $1232 \pm 2$ MeV)
\Gamma_\Delta = 150^{+43}_{-31}$ MeV (set I), $213^{+54}_{-54}$ MeV (set II), (exp. $120 \pm 10$ MeV)

(23)
$$
in agreement within two standard deviations with the experimental numbers, although with much larger errors. Given the good quality of our description from threshold up to the $\Delta$ resonance, one might even re-fit the parameters of our unitarized amplitude, and hence reduce the errors of the low energy constants entering the definition of the amplitude in the $P_{33}$ channel. In this way we would reduce errors quoted in Eq.( 23) as suggested in Ref. [12].

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4 We obtain $M_\Delta$ from the condition $\delta_{3,3}^{1}(M_\Delta^2) = \pi/2$, and the width $\Gamma_\Delta$ from $1/\Gamma_\Delta = M_\Delta \left. \frac{dd^1_{3,3}(s)}{ds} \right|_{s=M_\Delta^2}$.
FIG. 2. The same as in Fig.1 but with the low energy parameters labeled as set I. The position and width of the resonance for the upper curve are $M_\Delta = 1238^{+22}_{-18}$ MeV and $\Gamma_\Delta = 150^{+43}_{-31}$ MeV respectively.

To have an idea of the convergence rate of our calculation, we have also depicted in both figures the prediction in the static limit ($M \rightarrow \infty$). As we see, there also appears a resonant behaviour, but at higher energies $\sqrt{s} \sim 1450$ MeV. Thus, though the bulk of the dynamics is contained in the static limit, the finite mass corrections, particularly the $1/(f^2 M)$ contribution, are important to achieve an accurate description.

Finally we also show the results obtained within the conventional IAM approach, see for details Ref. [16], for both sets of parameters of Table 1. As we already anticipated, the description is extremely poor mainly due to improper treatment of the $1/(f^2 M)$ corrections. Thus within this framework, a fit to the data becomes possible only if huge changes in the low energy parameters are allowed. For instance, the authors of that reference get $b_{16} - \tilde{b}_{15} = 68.9$, $b_{19} = -23.13$, $a_5 = 27.76$, $a_1 = 9.1$, $a_3 = -8.7 \cdots$ which strongly differ from the values of Table 1.

The situation in other $l_{212J}$ channels might be somehow different and definitely deserves further study, though the results of the present paper regarding the $P_{33}$ channel, will remain unchanged. We anticipate, as can be already seen from Table 2, that the agreement of our unitarized threshold parameters with those stemming from HBChPT, would get a bit spoiled in some of the remaining channels, since the $1/f^4$ is not a small contribution due to a, perhaps accidental, but effective cancelation of the leading, $1/f^2$, and the next to leading, $1/(f^2 M)$, order terms. The situation might improve, however, if the low energy constants would be allowed to vary as to provide a better description of the scattering data from threshold up to the $\Delta$ resonance region. A systematic and detailed investigation of this topic will be presented elsewhere [21].
VI. CONCLUSIONS AND OUTLOOK

We summarize our results. Heavy Baryon Chiral Perturbation Theory provides definite predictions for the $\pi N$ scattering amplitudes in the threshold region, but it violates exact unitarity if the perturbative expansion is truncated to some finite order. Hence it is unable to describe the $\Delta$ resonance in the $P_{33}$ channel. The analysis up to third order shows that the leading finite nucleon mass correction, which is second order, is of comparable size to the static approximation and in fact it dominates the corrections at threshold. This sets conditions on the unitarization method suggesting an expansion in inverse powers of the weak pion decay constant but without making the heavy baryon expansion. Such an idea is supported by recent theoretical attempts to redefine a relativistic chiral counting for baryons. Our unitarization method is a slight but important variant of the IAM and provides a prediction for the phase shifts in the $P_{33}$ channel, in terms of low energy parameters and their errors, as determined from HBChPT. The agreement with experimental data is good within uncertainties. It is also clear that with our unitarization method, the combination of parameters which enter the $P_{33}$ amplitude, might be determined to a better accuracy than that obtained by only looking at threshold, since the central predicted values of the $P_{33}$ phase shift, fall mainly on top of the central experimental data.

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