Asymmetric energy flow in coupled nonlinear LC transmission line

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We constructed a coupled LC transmission line and studied the propagation of waves in it. We found asymmetric energy flow when we changed the driving conditions at the boundary. We analyzed this change and believe that it occurs because of the band-pass characteristics of the LC transmission line and high-order harmonic waves induced by nonlinearities. The LC transmission line could be used to simulate a microscopic crystal lattice. Therefore, we hope to observe thermal rectification in the system. We investigated the dependence of the system on different parameters, and then discussed the multi-frequency condition to aid in experimental verification.

1 Coupled LC transmission line

The coupled LC transmission line that we studied contains two sub-lines (Lines a and b), which have the same structure but different parameters. They are coupled by an inductor, L_int. This is shown in Figure 1. Each cell of the transmission lines has the usual form [6], which consists of one capacitor and two inductors. In Line a, the capacitor is nonlinear. A variable capacitance diode, BB112, working at a DC voltage near $V_d=2.0$ V, serves as the nonlinear capacitor. If $|V_n|<2.0$ V, the capacitance is related to the AC voltage $V_n$ as follows [6]:

$$C_n(V_n) = C_0(1 - 2\alpha V_n + 3\beta V_n^2),$$

(1)
where the parameters are \( C_0 = 320 \text{ pF} \), \( \alpha = 0.21 \text{ V}^{-1} \) and \( \beta = 0.0197 \text{ V}^{-2} \). In Line b, the capacitor is linear, and \( C_0 = 320 \text{ pF} \). The series inductors \((L_a, L_b)\) and the parallel inductors \((L_{pa}, L_{pb})\) are linear. The dissipation of these elements is taken into account. However, we only consider the dissipation contributions from the capacitor and the parallel inductor (main factor [7]), which are represented by the conductivity \((g)\). Furthermore, we assume that the dissipation is the same in the two sub-lines. In each line, according to Kirchhoff’s law, we have

\[
\frac{d}{dt}(C_0 \frac{dV}{dt}) = \frac{1}{L_a}(V_{n+1} + V_{n+2} - 2V_n) - \frac{1}{L_a} V_n - g \frac{dV}{dt}, (i = a, b).
\]

(2)

If the voltage is small \((V_n \ll 1)\) and the dissipation is weak, we can neglect the high order terms and dissipation terms in the equation. Consequently, we get the linearized equation:

\[
\frac{d^2}{dt^2} V_n = \frac{1}{L_a C_0} (V_{n+1} + V_{n+2} - 2V_n) - \frac{1}{L_a C_0} V_n, (i = a, b).
\]

(3)

In the uncoupled infinite-line condition, we can get the linear dispersion relation solution for plain waves:

\[
\omega_n^2 = \omega_0^2 + 4\omega_0^2 \sin^2 \frac{k}{2}, (i = a, b),
\]

(4)

where \( \omega_0 = \sqrt{\frac{1}{L_a C_0}} \), \( u_0 = \sqrt{\frac{k}{L_a C_0}} \) \((i = a, b)\).

According to eq. (4), we get a lower cut-off (angular) frequency of \( \omega_{a, \text{min}} = \omega_{b, \text{min}} \) and an upper cut-off frequency of \( \omega_{a, \text{max}} = \sqrt{\omega_0^2 + 4u_0^2} \). In the linear approximation, waves at frequency \( \omega \) can propagate in the LC line if \( \omega_{a, \text{min}} \leq \omega \leq \omega_{a, \text{max}} \); otherwise they decay exponentially. This leads to the band-pass characteristics of the line. Both the lower and upper cut-off frequencies can be adjusted by changing these elementary parameters.

To investigate asymmetric energy flow in the coupled transmission line, we designed two sub-lines, with different passbands \([\omega_{a, \text{min}}, \omega_{a, \text{max}}]\) and \([\omega_{b, \text{min}}, \omega_{b, \text{max}}]\). This is shown in Figure 2.

The two sub-lines are coupled using a large inductor, \( L_{\text{int}} \) \((L_{\text{int}} > L_a, L_b)\). Because \( L_{\text{int}} \) is large, the coupling is weak, and we expect that the two sub-lines will retain their original characteristics, for example, their passbands. At the interface, the voltage satisfies

\[
L_{\text{int}} \frac{dI}{dt} = V_{n+1/2} - V_{n-1/2},
\]

(5)

where \( I_{\text{int}} \) is the current passing through \( L_{\text{int}} \).

A mono-frequency voltage, \( V_0 \sin \omega t \), serving as the driving frequency at one end the frequency, \( \omega \), satisfies

\[
\omega_{a, \text{min}} \leq \omega \leq \omega_{a, \text{max}}
\]

(6)

and

\[
\omega < \omega_{b, \text{min}} \leq 2\omega \leq \omega_{b, \text{max}}.
\]

(7)

Namely, \( \omega \) is in the shaded area of Figure 2.

2 Simulation

The parameters of inductors are set to be \( L_{pa} = L_{0} = 220 \mu \text{H} \), \( L_{pb} = 0.25 L_0 \), \( L_a = 2.0 L_0 \) and \( L_b = 1.0 L_0 \). The capacitor parameters are the same as those in eq. (1). Therefore, Line a’s passband is \([\omega_0, \sqrt{3}\omega_0]\) , and Line b’s passband is \([2\omega_0, 2\sqrt{2}\omega_0]\) with \( \omega_0 = \frac{1}{\sqrt{L_a C_0}} \) (see Figure 2). The coupling inductance is \( L_{\text{int}} = 4L_0 = 880 \mu \text{H} \), and the conductivity is \( g = 1.2 \times 10^{-5} \text{ \Omega}^{-1} \). To approximate reasonable experimental conditions, we set the total number of cells to \( N = 40 \), and each sub-line consists of 20 cells.

The driving frequency is \( \omega = 1.2 \omega_0 \), which satisfies eqs. (6) and (7). To ensure the validity of eq. (1), we use a small amplitude \( V_0 = 0.6 \text{ V} \) to avoid unphysical results. The driving frequency is the input at one end of the transmission line, and the other end is connected to the ground. Therefore, if
we drive at End A, the boundaries satisfy (Ends A, B correspond to \( n=0, N+1 \))

\[
V_n(t) = V_0 \sin \omega t, \quad (8)
\]

\[
V_{n+1}(t) = 0. \quad (9)
\]

If we drive at End B,

\[
V_n(t) = 0, \quad (10)
\]

\[
V_{n+1}(t) = V_0 \sin \omega t. \quad (11)
\]

The fourth-order Runge-Kutta method was used, with an integration step of \( \Delta t = \pi/500\omega \) (1000 steps per period). We care about the energy distribution after the system reaches steady state. If higher-order energy terms are neglected, \( \{V_n^2\} \) is proportional to the mean energy stored in the capacitor (\( \langle \ldots \rangle \) indicates a temporal average). In the harmonic approximation, the mean energy of the capacitor is equal to the sum of the mean energy of all the inductors in each cell. Therefore, we only need to consider \( \{V_n^2\} \) by dividing by the mean square of the driving voltage, \( V_0^2/2 \). This yields the dimensionless mean energy:

\[
E_n = 2 \{V_n^2\} / V_0^2. \quad (12)
\]

The results when the driving input is at End A are shown by the solid curve in Figure 3. It can be seen that the energy flows pass the interface and enter Line b. For driving at End B, attenuation occurs in Line b, and Line a receives no energy as a result (see the dotted curve in Figure 3).

By comparison, it can be seen that the energy flow is asymmetric across the two directions. If we treat energy flow through the interface as conduction, the system conducts when the driving is input at End A, but insulates when driven at End B. The entire coupled LC transmission line serves as a rectifier.

Physically, when driving at End A and \( \omega \) satisfies eq. (6), the fundamental wave can propagate in Line a. However, high-order harmonic waves, mainly the second-order harmonic wave, are generated through nonlinearity. The second-order harmonic wave, at frequency \( 2\omega \), satisfies eq. (7). Therefore, it can propagate in Line b after passing through \( L_{ab} \), which results in a non-zero energy flux through the interface. When the input end is changed, \( \omega \) is outside of Line b’s passband, and the fundamental wave decays exponentially, which prevents energy flux through the interface. This is very similar to Liang et al.’s acoustic diode [8] in principle.

To verify the physical mechanism mentioned above, we calculated the power spectral density for the voltages of the 20th and 40th cells for driving input at End A.

From Figure 4, it can be seen that there are harmonic waves present (second and third harmonics) in \( V_{20} \), and essentially the second harmonic is the only one present in \( V_{40} \). At the 40th node, the fundamental wave is not present (see Figure 4(b)) because it is outside the passband of Line b.

The third harmonic is also not present for the same reason. Only the second harmonic wave passes through Line b, which leads to conduction for the whole system.

The occurrence of asymmetric energy flow is dependent on the driving frequency. Furthermore, we calculated the energy distribution in the coupled line for four driving frequencies: \( \omega = 0.9\omega_1, \omega = 1.3\omega_1, \omega = 1.6\omega_1 \) and \( \omega = 2.2\omega_1 \).

The driving voltage, \( \omega = 0.9\omega_1 \), is outside the either passband for both transmission lines. Therefore, it cannot propagate in either line, and the system is an insulator in both directions. The driving frequency, \( \omega = 1.3\omega_1 \), satisfies eqs. (6) and (7), and the system behaves as a rectifier (solid curve in Figure 5). Although the waves at \( \omega = 1.6\omega_1 \) and \( \omega = 2.2\omega_1 \) can propagate in Lines a and b, respectively, neither allows for conduction across the entire system. We have marked the frequencies that satisfy eqs. (6) and (7) by the shaded region in Figure 2.

3 Dependence of parameters

Here, we define a variable called the mean transmission energy density, \( E_{p,i} (i=A, B) \). Its value is equal to the total mean energy passing through the interface. If the system is driven at End A,

\[
E_{p,A} = \frac{1}{n_h} \sum_{n=1}^{n_h} E_n, \quad (13)
\]
where \( n_b \) is the total number of cells in sub-line b, and \( E_n \) is from eq. (12). If the system is driven at End B,
\[
E_{p,B} = \frac{1}{n_a} \sum_{n} E_{n},
\]
with \( n_a \) being the total number of cells in sub-line a. For the conducting condition \((E_{p}>0)\), the larger \( E_{p,A} \) is, the larger the energy flux is. For the insulating condition \((E_{p}=0)\) the energy flux is zero. Using the mean transmission energy density, we investigated the dependence of the asymmetric energy flow on the parameters of the system, such as the driving amplitude and coupling strength.

### 3.1 Driving amplitude

Allowing the driving frequency to satisfy eqs. (6) and (7), we set \( \omega=1.2\omega_1 \). The numerical calculation shows that \( E_{p,A} \) increases with the amplitude (see Figure 6). The nonlinear effect becomes stronger as the amplitude increases, which leads to more energy being transferred to the higher-order harmonic waves from the fundamental wave. The frequency of the second harmonic wave is in the passband of Line b, which forms a finite energy flux through the coupled line. Therefore, the higher the amplitude of the second harmonic wave, the larger the energy flux is. However, \( E_{p,B} \) remains zero.

![Figure 6](image_url)  
**Figure 6** \( E_{p,A} \) versus \( V_0 \). Main parameters: \( N=40, L_{int}=4.0L_0, g=1.2\times10^{-5} \Omega^{-1}, \omega=1.2\omega_1 \).

### 3.2 Coupling strength

During thermal rectification, the coupling strength is critical to the asymmetric energy flow. In strong and weak conditions, the coupled Frenkel-Kontorova chains behave differently \([1,9]\). While \( E_{p,B} \) keeping at zero, we calculated the dependence of \( E_{p,A} \) on \( L_{int} \) (Figure 7). As \( L_{int} \) increases, \( E_{p,A} \) initially increases, then reaches the maximal when \( L_{int}=2.5L_0 \), and finally decreases. When \( L_{int} \) is small, the coupling is strong; the two sub-lines interfere with each other’s passband. In this case, our analysis, which is based on two uncoupled infinite lines, is no longer valid. However, if \( L_{int} \) is too large, the coupling will be too weak, and the energy flux across the interface will be negligible. In practice, weak coupling is the most common situation.

### 3.3 Cell numbers

Here, we discuss the effect of the cell number. Our calculation shows that, \( E_{p,A} \) decreases with the cell number, except when \( N=60 \) (Figure 8). It is likely that the increase of total dissipation is the main cause. We do not consider \( E_{p,B} \) because it is nearly zero. A sufficient number of cells to accurately model the bandpass are needed. We find that the asymmetry weakens with increasing cell number, when dissipation is considered. However, we do not consider the effect of the cell number at a non-dissipation condition here.
4 Discussion and conclusion

We have studied energy flow in a coupled nonlinear LC transmission line excited by a mono-frequency voltage at one end. Because of nonlinearity, high-order harmonic waves form in the first sub-line in one direction. The second-order harmonic propagates thorough the other sub-line, which leads to energy transmission. However, when the system is driven in the other direction, the fundamental wave decays exponentially in the first sub-line and no harmonic wave forms, which means that no energy transmission occurs. The energy flow is asymmetric. The overlap of the frequency and the passband of the sub-line determines the propagation or attenuation of the wave. If the driving frequency is in the shaded area shown in Figure 2, asymmetric energy flow may occur. The dependence of the system on various parameters was studied in our numerical simulation. We found that the asymmetry increases with the driving amplitude. The coupling inductance should be properly chosen to yield the greatest asymmetry. The energy flow will be attenuated if the coupling inductance is too large. However, if the coupling inductance is too small, the energy flow will also be attenuated.

For thermal flux, the cell energy would correspond to the temperature. In this case, we would connect the end of the coupled line to a heat bath. This corresponds to a complex voltage including a number of frequencies, instead of only one. As long as the frequency band of the complex voltage contains frequencies that satisfy eqs. (6) and (7) (the shaded area in Figure 2), an asymmetric energy flux will be possible. However, frequencies outside the shaded area will not form energy flows in the coupled line in either direction. Therefore, asymmetric energy flux can still be observed when a complex voltage serves as the driving voltage. We aim to provide the theoretical foundation for experimental verification, which is currently being performed.

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