First-Order Gödel Logics

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Abstract
First-order Gödel logics are a family of infinite-valued logics where the sets of truth values \( V \) are closed subsets of \([0,1]\) containing both 0 and 1. Different such sets \( V \) in general determine different Gödel logics \( \mathcal{G}_V \) (sets of those formulas which evaluate to 1 in every interpretation into \( V \)). It is shown that \( \mathcal{G}_V \) is axiomatizable iff \( V \) is finite, \( V \) is uncountable with 0 isolated in \( V \), or every neighborhood of 0 in \( V \) is uncountable. Complete axiomatizations for each of these cases are given. The r.e. prenex, negation-free, and existential fragments of all first-order Gödel logics are also characterized.

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1 Introduction

1.1 Motivation

The logics we investigate in this paper, first-order Gödel logics, can be characterized in a rough-and-ready way as follows: The language is a standard first-order language. The logics are many-valued, and the sets of truth values considered are closed subsets of \([0;1]\) which contain both 0 and 1. 1 is the “designated value,” i.e., a formula is valid if it receives the value 1 in every interpretation. The truth functions of conjunction and disjunction are minimum and maximum, respectively, and quantifiers are defined by infimum and supremum over subsets of the set of truth values. The characteristic operator of Gödel logics, the Gödel conditional, is defined by:

\[
a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}
\]

Because the truth values are ordered (indeed, in many cases, densely ordered), the semantics of Gödel logics is suitable for formalization of comparisons. It is related in this respect to a more widely known many-valued logic, Łukasiewicz (or “fuzzy”) logic—yet the truth function of the Łukasiewicz conditional is defined not just using comparison, but also addition. In contrast to Łukasiewicz logic, which might be considered a logic of absolute or metric comparison, Gödel logics are logics of relative comparison. This alone makes Gödel logics an interesting subject for logical investigations.

There are other reasons why the study of Gödel logics is important. As noted, Gödel logics are related to other many-valued logics of recognized importance. Indeed, Gödel logic is one of the three basic \(t\)-norm based logics which have received increasing attention in the last 15 or so years [Haj98] (the others are Łukasiewicz and product logic). Yet Gödel logic is also closely related to intuitionistic logic: it is the logic of linearly-ordered Heyting algebras. In the propositional case, infinite-valued Gödel logic can be axiomatized by the intuitionistic propositional calculus extended by the axiom schema \((A \rightarrow B) \Rightarrow (B \rightarrow A)\). This connection extends also to Kripke semantics.
for intuitionistic logic: Gödel logics can also be characterized as logics of (classes of) linearly ordered and countable intuitionistic Kripke structures with constant domains.

One of the surprising facts about Gödel logics is that whereas there is only one infinite-valued propositional Gödel logic, there are infinitely many different infinite-valued first-order Gödel logics depending on the choice of the set of truth values. This is also the case when one considers the propositional consequence relation, and likewise when the language is extended to include quantification over propositions. For both quantified propositional and first-order Gödel logics, different sets of truth values with different order-theoretic properties result in different sets of valid formulas. Hence it is necessary to consider truth value sets other than the standard unit interval.

In the light of the result of Scarpellini [Sca62] on non-axiomatizability of infinite-valued first-order Łukasiewicz logic which can be extended to almost all linearly ordered infinite-valued logics, it is also surprising that some infinite-valued Gödel logics are recursively enumerable. Our main aim in this paper is to characterize those sets of truth values which give rise to axiomatizable Gödel logics, and those whose sets of validities are not r.e. We show that a set \( V \) of truth values determines an axiomatizable first-order Gödel logic if, and only if, \( V \) is finite, \( V \) is uncountable and 0 is isolated, or every neighborhood of 0 in \( V \) is uncountable. These cases also determine different sets of validities: the finite-valued Gödel logics \( G_n \), the logic \( G^0 \), and the “standard” infinite-valued Gödel logic \( G_R \) (based on the truth value set \( [0;1] \)).

1.2 History of Gödel logics

Gödel logics are one of the oldest families of many-valued logics. Propositional finite-valued Gödel logics were introduced by Gödel in [Göd33] to show that intuitionistic logic does not have a characteristic finite matrix. They provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logics). Dummett [Dum59] was the first to study infinite valued propositional Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom \( (A \rightarrow B) \land (B \rightarrow A) \). Hence, infinite-valued propositional Gödel logic is also sometimes called Gödel-Dummett logic or Dummett’s LC. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders.

Standard first-order Gödel logic \( G_R \)—the one based on the full interval \( [0;1] \)—has been discovered and studied by several people independently. Alfred Horn was probably the first: He discussed this logic under the name “logic with truth values in a linearly ordered Heyting algebra” [Hor69], and gave an axiomatization and the first completeness proof. Takeuti and Titani [TT84] called \( G_R \) intuitionistic fuzzy logic, and also gave an axiomatization for which they proved the completeness. This system incorporates the density rule

\[
\frac{\Gamma \land A \land (C \rightarrow p) \land (p \rightarrow B) \quad (\text{where } p \text{ is any propositional variable not occurring in the lower sequent})}{\Gamma \land A \land (C \land B)}
\]

The rule is redundant for an axiomatization of \( G_R \), as was shown by Takano [Tak87], who gave
a streamlined completeness proof of Takeuti-Titani’s system without the rule. (A syntactical proof of the elimination of the density rule was later given in [BZ00]. Other proof-theoretic investigations of Gödel logics can be found in [BC02] and [BFC03].) The density rule is nevertheless interesting: It forces the truth value set to be dense in itself (in the sense that, if the truth value set isn’t dense in itself, the rule does not preserve validity). This contrasts with the expressive power of formulas: no formula is valid only for truth value sets which are dense in themselves.

First-order Gödel logics other than $G_{\#}$ were first considered in [BLZ96b], where it was shown that $G_{\#}$, based on the truth value set $V_{\#} = \{1 = k : k \in \mathbb{N} \}$, is not r.e. Hájek [Háj05] has recently improved this result, and showed that not only is the set of validities not r.e., it is not even arithmetical. Hájek also showed that the Gödel logic $G_{\ast}$ based on $V_{\ast} = \{1 = k : k \in \mathbb{N} \}$ is $\Pi_2$-complete. Results preliminary to the results of the present paper were reported in [BPZ03, Pre02, Pre03].

1.3 Overview of the results

We begin with a preliminary discussion of the syntax and semantics of Gödel logics, including a discussion of some of the more interesting special cases of first-order Gödel logics and their relationships (Section 2). In Section 3, we present some relevant results regarding the topology of truth-value sets.

The main results of the paper are contained in Sections 4–7. We provide a complete classification of the axiomatizability of first order Gödel logics. The main results are, that a logic based on a truth value set $V$ is axiomatizable if and only if

1. $V$ is finite (Section 5.1), or
2. $V$ is uncountable and 0 is contained in the perfect kernel (Section 5.1), or
3. $V$ is uncountable and 0 is isolated (Section 5.2).

In all other cases, i.e., logics with countable truth value set (Section 5.3) and those where there is a countable neighborhood of 0 and 0 is not isolated (Section 5.3), the respective logics are not r.e.

In Section 6 we investigate the complexity of fragments of first-order Gödel logic, specifically, the prenex fragments (Section 6.1), the $\exists$-free fragments (Section 6.2), and the existential ($\forall$-free) fragments (Section 6.3). We show that the prenex fragment of a Gödel logic is axiomatizable if and only if the truth value set is finite or uncountable. This means that there are truth-value sets where the prenex fragment of the corresponding logic is r.e. even though the full logic is not. Moreover, there are axiomatizable prenex fragments coincide. This is also the case for $\exists$-free and existential fragments, but in these cases only those truth value sets determine r.e. $\exists$-free and existential fragments for which also the full logic is r.e., viz., truth value sets which are finite, uncountable with 0 isolated, and those where every neighborhood of 0 is uncountable.
2 Preliminaries

2.1 Syntax and Semantics

In the following we fix a standard first-order language \( L \) with finitely or countably many predicate symbols \( P \) and finitely or countably many function symbols \( f \) for every finite arity \( k \). In addition to the two quantifiers \( \forall \) and \( \exists \) and the constant \( \bot \) (for ‘false’); other connectives are introduced as abbreviations, in particular we let \( A \land \neg B \).

Gödel logics are usually defined using the single truth value set \([0;1]\). For propositional logic the choice of any infinite subset of \([0;1]\) leads to the same propositional logic (set of tautologies). In the first order case, where quantifiers will be interpreted as infima and suprema, a closed subset of \([0;1]\) is necessary.

**Definition 1 (Gödel set).** A Gödel set is a closed set \( V \subseteq [0;1] \) which contains 0 and 1.

The semantics of Gödel logics, with respect to a fixed Gödel set as truth value set and a fixed language \( L \) of predicate logic, is defined using the extended language \( L^U \), where \( U \) is the universe of the interpretation \( \mathfrak{I} \). \( L^U \) is \( L \) extended with constant symbols for each element of \( U \).

**Definition 2 (Semantics of Gödel logic).** Fix a Gödel set \( V \). An interpretation \( \mathfrak{I} \) into \( V \) consists of

1. a nonempty set \( U = U^\mathfrak{I} \), the ‘universe’ of \( \mathfrak{I} \),
2. for each \( k \)-ary predicate symbol \( P \), a function \( P^\mathfrak{I} : U^k \to V \),
3. for each \( k \)-ary function symbol \( f \), a function \( f^\mathfrak{I} : U^k \to U \).
4. for each variable \( v \), a value \( v^\mathfrak{I} \in U \).

Given an interpretation \( \mathfrak{I} \), we can naturally define a value \( t^\mathfrak{I} \) for any term \( t \) and a truth value \( \mathfrak{I} (A) \) for any formula \( A \) of \( L^U \). For a terms \( t = f(t_1; \ldots; t_k) \) we define \( \mathfrak{I} (t) = f^\mathfrak{I} (t_1^\mathfrak{I}; \ldots; t_k^\mathfrak{I}) \). For atomic formulas \( A \equiv P(t_1; \ldots; t_n) \), we define \( \mathfrak{I} (A) = P^\mathfrak{I} (t_1^\mathfrak{I}; \ldots; t_n^\mathfrak{I}) \). For composite formulas \( A \) we define \( \mathfrak{I} (A) \) by:

\[
\begin{align*}
\mathfrak{I} (\bot) &= 0 \\
\mathfrak{I} (A \land B) &= \min (\mathfrak{I} (A); \mathfrak{I} (B)) \\
\mathfrak{I} (A \lor B) &= \max (\mathfrak{I} (A); \mathfrak{I} (B)) \\
\mathfrak{I} (A \Rightarrow B) &= \begin{cases} 
1 & \text{if } \mathfrak{I} (A) = \mathfrak{I} (B) \\
\mathfrak{I} (B) & \text{otherwise}
\end{cases} \\
\mathfrak{I} (\forall x A (v)) &= \inf \{ \mathfrak{I} (A (u)) : u \not\in U \} \\
\mathfrak{I} (\exists x A (v)) &= \sup \{ \mathfrak{I} (A (u)) : u \not\in U \}
\end{align*}
\]

(Here we use the fact that every Gödel sets \( V \) is a closed subset of \([0;1]\) in order to be able to interpret \( \forall \) and \( \exists \) as inf and sup in \( V \).)

If \( \mathfrak{I} (A) = 1 \), we say that \( \mathfrak{I} \) satisfies \( A \), and write \( \not\models A \).
**Definition 3 (Gödel logics based on V).** For a Gödel set $V$ we define the *first order Gödel logic* $G_V$ as the set of all formulas of $\mathcal{L}$ such that $I \vDash A$ for all $V$-interpretations $I$.

It should be noted that for Gödel logics with 0 isolated, the notion of *satisfiability* for sets of formulas is not particularly interesting, since a set of formulas $\Gamma$ is satisfiable (in the sense that there is an $I$ so that $I \vDash A$ for all $A \in \Gamma$) iff it is satisfiable classically. For this reason, we take *entailment* to be the fundamental model-theoretic notion.

**Definition 4.** If $\Gamma$ is a set of formulas (possibly infinite), we say that $\Gamma$ entails $A$ in $G_V$, $\Gamma \vDash_V A$ iff for all $I$ into $V$,

$$\inf \{ I(\beta) : B \supseteq \Gamma, I(A) \};$$

and $\Gamma$ 1-entails $A$ in $G_V$, $\Gamma \vDash_V A$, iff, for all $I$ into $V$, whenever $I(\beta) = 1$ for all $B \supseteq \Gamma$, then $I(A) = 1$.

**Notation 5.** We will write $\Gamma \vDash A$ instead of $\Gamma \vDash_V A$ in case it is obvious which truth value set $V$ is meant. We will sometimes write $\Gamma \vDash \Delta \in G_V$, by which we mean that $\Gamma \vDash \Delta$. The notation $G_V \vDash A$ stands for $\emptyset \vDash A$, or $A \in G_V$.

Whether or not a formula $A$ evaluates to 1 under an interpretation $I$ depends only on the *relative ordering* of the truth values of the atomic formulas (in $\mathcal{L}$), and not directly on the set $V$ or on the values of the atomic formulas. If $V$, $W$ are both Gödel sets, and $I$ is an interpretation into $V$, then $I$ can be seen also as an interpretation into $W$, and the values $I(A)$, computed recursively using (1)–(6), do not depend on whether we view $I$ as a $V$-interpretation or a $W$-interpretation. Consequently, if $V \supseteq W$, there are more interpretations into $W$ than into $V$. Hence, if $\Gamma \vDash_W A$ then also $\Gamma \vDash_V A$ and $G_W \supseteq G_V$.

This can be generalized to embeddings between Gödel sets other than inclusion. First, we make precise which formulas are involved in the computation of the truth-value of a formula $A$ in an interpretation $I$:

**Definition 6.** The only subformula of an atomic formula $P$ in $\mathcal{L}$ is $P$ itself. The subformulas of $A \supseteq B$ for $\supseteq \in \{ \exists, \forall \}$ are the subformulas of $A$ and of $B$, together with $A \supseteq B$ itself. The subformulas of $\exists x A(x)$ and $\forall x A(x)$ with respect to a universe $U$ are all subformulas of $A(\mu u)$ for $u \in U$, together with $\exists x A(x)$ (or, $\forall x A(x)$, respectively) itself.

The set of truth-values of subformulas of $A$ under a given interpretation $I$ is denoted by

$$\text{Val}(I,A) = \{ I(\beta) : B \text{ subformula of } A \text{ w.r.t. } U^I \supseteq [ \emptyset ; 1 \emptyset ] \}.$$

If $\Gamma$ is a set of formulas, then $\text{Val}(I,\Gamma) = \bigcup \{ \text{Val}(I,\beta) : B \supseteq \Gamma, I(A) \}$.

**Lemma 7.** Let $I$ be a $V$-interpretation, and let $h : \text{Val}(I,\Gamma) \to W$ be a mapping satisfying the following properties:

1. $h(\emptyset) = 0$, $h(\{ \}) = 1$;
2. $h$ is strictly monotonic, i.e., if $a < b$, then $h(a) < h(b)$;
Proposition 11 (Downward L"owenheim-Skolem).

3. For every $X$ \( \text{Val}(I; \Gamma) \), $h(\inf X) = \inf h(X)$ and $h(\sup X) = \sup h(X)$ (provided $\inf X$, $\sup X \in \text{Val}(I; \Gamma)$).

Then the $W$-interpretation $I_h$ with universe $U^{1}$, $f^{1} = f^{1}$, and for atomic $B \in \mathcal{L}$, 1,

\[
I_h(B) = \begin{cases} 
   h(I(B)) & \text{if } I(B) \in \text{dom} h \\
   1 & \text{otherwise}
\end{cases}
\]

satisfies $I_h(A) = h(I(A))$ for all $A \in \Gamma$.

Proof. By induction on the complexity of $A$. If $A$ atomic, the claim follows from (1). If $A$ is atomic, it follows from the definition of $I_h$. For the propositional connectives the claim follows from the strict monotonicity of $h$ (2). For the quantifiers, it follows from property (3).

Remark. Note that the construction of $I_h$ and the proof of Lemma 7 also goes through without the condition $h(\emptyset) = 0$, provided that the formulas in $\Gamma$ do not contain $\mathcal{V}$, and goes through without the requirement that existing $h$’s be preserved ($h(\inf X) = \inf h(X)$ if $\inf X \in \text{Val}(I; \Gamma)$) provided they do not contain $\mathcal{W}$.

Definition 8. A $G$-embedding $h: V \rightarrow W$ is a strictly monotonic, continuous mapping between G"odel sets which preserves $0$ and $1$.

Lemma 9. Suppose $h: V \rightarrow W$ is a $G$-embedding. (a) If $I$ is a $V$-interpretation, and $I_h$ is the interpretation induced by $I$ and $h$, then $I_h(A) = h(I(A))$. (b) If $\Gamma \models W A$ then $\Gamma \models V A$ (and hence $G_W = G_V$). (c) If $h$ is bijective, then $\Gamma \models W A$ if and only if $\Gamma \models V A$ (and hence, $G_V = G_W$).

Proof. (a) $h$ satisfies the conditions of Lemma 7 for $\Gamma$ the set of all formulas. (b) If $\Gamma \models W A$, then for some $I$, $I(\emptyset) = 1$ for all $B \in \Gamma$ and $I(A) < 1$. By Lemma 7, $I_h(\emptyset) = 1$ for all $B \in \Gamma$ and $I_h(A) < 1$ (by strict monotonicity of $h$). Thus $\Gamma \models W A$. (c) If $h$ is bijective then $h^{-1}$ is also a $G$-embedding.

Definition 10 (Submodel, elementary submodel). Let $I_1$, $I_2$ be interpretations. We write $I_1 \preceq I_2$ ($I_2$ extends $I_1$) iff $U^{1}_{I_1} \subseteq U^{1}_{I_2}$, and for all $k$, all $k$-ary predicate symbols $P$ in $\mathcal{L}$, and all $k$-ary function symbols $f$ in $\mathcal{L}$ we have

\[
P^{1}_{I_1} = P^{1}_{I_2}, \quad (\mathcal{J}^{1})^{k}_{I_1} = (\mathcal{J}^{1})^{k}_{I_2}, \quad f^{1}_{I_1} = f^{1}_{I_2}, \quad (\mathcal{J}^{1})^{k}_{I_1}
\]

or in other words, if $I_1$ and $I_2$ agree on closed atomic formulas.

We write $I_1 \preceq I_2$ if $I_1 \preceq I_2$ and $I_1(A) = I_2(A)$ for all $L^{U^{1}}$-formulas $A$.

Proposition 11 (Downward L"owenheim-Skolem). For any interpretation $I$ with $U^{1}$ infinite, there is an interpretation $I^{0}$ of $I$ with a countable universe $U^{1}$.

Proof sketch. The proof is an easy generalization of the construction for the classical case. We construct a sequence of countable subsets $U_1 \subseteq U_2 \subseteq \cdots$ of $U$ simply containing $t^{1}$ for all closed terms of the original language. $U_{i+1}$ is constructed from $U_i$ by adding, for each of the (countably many) formulas of the form $\forall x A(x)$ and $\exists x A(x)$ in the language $L^{U_1}$, a countable sequence $a_j$ of elements of $U^{1}$ so that $I(A(a_j)))$ for all $I(\forall x A(x))$ or $I(\exists x A(x))$, respectively. $U^{1} = \bigcup_{i} U_i$. 

\[\square\]
Lemma 12. Let $\mathcal{I}$ be a interpretation into $V$, $w \in \{0;1\}$ and let $I_w$ be defined by

$$I_w(B) = \begin{cases} I(B) & \text{if } I(B) < w \\ 1 & \text{otherwise} \end{cases}$$

for atomic formulas $B$ in $\mathcal{L}^U$. Then $I_w$ is an interpretation into $V$. If $w \not\in \text{Val}(\mathcal{I};A)$, then $I_w(A) = I(A)$ if $I(A) < w$, and $I_w(A) = 1$ otherwise.

Proof. Let $h_w(a) = a$ if $a < w$ and $= 1$ otherwise. By induction on the complexity of formulas $B$ it is easily shown that $I_w(B) = h_w(I(B))$ for all subformulas $B$ of $A$ w.r.t. $U^I$. \hfill $\Box$

Proposition 13. $\Gamma \models A$ iff $\Gamma \vdash A$

Proof. Only if: obvious. If: Suppose that $\Gamma \not\vdash A$, i.e., there is a $V$-interpretation $\mathcal{I}$ so that $\inf \mathcal{I}(B): B \not\in \Gamma \models \mathcal{I}(A)$. By Proposition 11 we may assume that $U^I$ is countable. Hence, there is some $w$ with $I(A) < w < \inf \mathcal{I}(B): B \in \Gamma \models I(A)$. Let $I_w$ be as in Lemma 12. Then $I_w(B) = 1$ for all $B \not\in \Gamma$ and $I_w(A) < 1$. \hfill $\Box$

The coincidence of the two consequence relations is a unique feature of Gödel logics. Proposition 13 does not hold in Łukasiewicz logic, for instance. There, $A \land A \not\models A$ but $A \models A \lor B \models B$. In what follows, we will use $\not\models$ when semantic consequence is at issue; the preceding propositions shows that the results we obtain for $\not\models$ hold for as well.

Lemma 14 (Semantic deduction theorem).

$$\Gamma \vdash A \models B \iff \Gamma \vdash A \land B$$

Proof. Immediate consequence of the definition of $\not\models$ and the semantics for $\land$. \hfill $\Box$

We want to conclude this part with two interesting observations:

Relation to residuated algebras If one considers the truth value set as a Heyting algebra with $a \land b = \min(a,b), a \lor b = \max(a,b)$, and

$$a \land b = \begin{cases} 1 & \text{if } a \land b \\ b & \text{otherwise} \end{cases}$$

then $\models$ and $\land$ are residuated, i.e.,

$$(a \land b) = \sup \{x : (x \land a) \land b : \exists b \}$$
The Gödel conditional. A large class of many-valued logics can be developed from the theory of $t$-norms [Haj98]. The class of $t$-norm based logics includes not only (standard) Gödel logic, but also Łukasiewicz- and product logic. In these logics, the conditional is defined as the residuum of the respective $t$-norm, and the logics differ only in the definition of their $t$-norm and the respective residuum, i.e., the conditional. The truth function for the Gödel conditional is of particular interest as it can be ‘deduced’ from simple properties of the evaluation and the entailment relation, a fact which was first observed by G. Takeuti.

Lemma 15. Suppose we have a standard language containing a ‘conditional’ interpreted by a truth-function into $[0;1]$. Suppose further that

1. a conditional evaluates to 1 if the truth value of the antecedent is less or equal to the truth value of the consequent, i.e., if $I(A) \leq I(B)$, then $I(A \rightarrow B) = 1$;

2. $\vdash$ is defined as above, i.e., if $\Gamma \not\vdash B$, then $\min I(A) : A \not\in \Gamma \vdash I(B)$;

3. the deduction theorem holds, i.e., $\Gamma \vdash A \rightarrow B \rightarrow A$. Then is the Gödel conditional.

Proof. From (1), we have that $I(A \rightarrow B) = 1$ if $I(A) \leq I(B)$. Since $\vdash$ is reflexive, $B \vdash B$. Since it is monotonic, $B \vdash A \rightarrow B$. By the deduction theorem, $B \vdash A \rightarrow B$. By (2),

$I(B) \vdash I(A \rightarrow B)$.

From $A \not\vdash A \rightarrow B$ and the deduction theorem, we get $A \vdash B \not\vdash B$. By (2),

$\min I(A) : I(A) \not\in \vdash I(B)$.

Thus, if $I(A) > I(B)$, $I(A \rightarrow B) \vdash I(B)$. □

Note that all usual conditionals (Gödel, Łukasiewicz, product conditionals) satisfy condition (1). So, in some sense, the Gödel conditional is the only many-valued conditional which validates both directions of the deduction theorem for $\vdash$. For instance, for the Łukasiewicz conditional $A \not\vdash A \rightarrow B$. The right-to-left direction fails: $A \not\vdash A \rightarrow B$, but $A \vdash A \rightarrow B$. (With respect to $\vdash$, the left-to-right direction of the deduction theorem fails for $\not\vdash$.)

2.2 Axioms and deduction systems

In this section we introduce certain axioms and deduction systems for Gödel logics, and we will show completeness of these deduction systems subsequently. We will use a Hilbert style proof system:

Definition 16. A formula $A$ is derivable from formulas $\Gamma$ in a system $\mathcal{L}$ consisting of the axioms and the rules if there are formulas $A_0, \ldots, A_n = A$ such that for each $0 \leq i \leq n$ either $A_i \in \Gamma$, or $A_i$ is an instance of an axiom in $\mathcal{L}$, or there are indices $j_1, \ldots, j_l < i$ and a rule in $\mathcal{L}$ such that $A_{j_1}, \ldots, A_{j_l}$ are the premises and $A_i$ is the conclusion of the rule. In this case we write $\Gamma \vdash_{\mathcal{L}} A$. 
We will denote by IL the following complete axiom system for intuitionistic logic (taken from [150/7]). Rules are written as $A_1; \ldots; A_n \vdash A$.

\begin{align*}
(11) & \quad A_2 A \vdash B \wedge B \quad (12) & \quad A \vdash B_2 B \vdash C \wedge A \vdash C \\
(13) & \quad A_2 A \vdash A \wedge A \quad (14) & \quad A \vdash A \wedge B_2 A \vdash B \vdash A \\
(15) & \quad A_2 B \vdash B_2 A \wedge B \wedge B \quad (16) & \quad A \vdash B \wedge C \wedge A \vdash C \wedge B \\
(17) & \quad A \wedge B \vdash C \wedge A \quad (B \wedge C) \quad (A \wedge B) \quad C \\
(19) & \quad \neg A \\
(110) & \quad B^{(k)} \vdash A (\tau) \wedge B^{(k)} \vdash \exists x A (\tau) \\
(111) & \quad \exists x A (\tau) \vdash A \eta \\
(112) & \quad A \eta \vdash \exists x A (\tau) \\
(113) & \quad (A \tau) \vdash B^{(k)} \vdash \exists x A (\tau) \vdash B^{(k)}
\end{align*}

(where $B^{(k)}$ means that $x$ is not free in $B$).

The following axioms will play an important rôle (QS stands for ‘quantifier shift’, LIN for ‘linearity’, ISO0 for ‘isolation axiom of 0’, and FIN (η) for ‘finite with $n$ elements’):

\begin{align*}
\text{QS} & \quad \exists x (C^{(k)} \neg A (\tau)) \vdash (C^{(k)} \neg \exists x A (\tau)) \\
\text{LIN} & \quad (A \vee B) \vdash (B \vee A) \\
\text{ISO0} & \quad \exists x : A (\tau) \vdash : \exists x A (\tau) \\
\text{FIN (η)} & \quad (\forall x \neg A_1) \vdash (\forall x \neg A_2) : \vdots : (\forall x \neg A_n) \vdash (\forall x \neg A_n \eta)
\end{align*}

**Notation 17.** $\mathbf{H}$ denotes the axiom system $\mathbf{IL} + \text{QS} + \text{LIN}$.

$\mathbf{H}_n$ for $n \geq 2$ denotes the axiom system $\mathbf{H} + \text{FIN (η)}$.

$\mathbf{H}_0$ denotes the axiom system $\mathbf{H} + \text{ISO0}$.

**Theorem 18 (Soundness).** Suppose $\Gamma$ contains only closed formulas, and all axioms of $\mathbf{H}$ are valid in $\mathbf{G}_V$. Then, if $\Gamma \vdash_A \neg A$ then $\Gamma \vdash_V A$.

**Proof.** By induction on the complexity of proofs. By assumption, all axioms of $\mathbf{H}$ are valid in $\mathbf{G}_V$, hence $\Gamma \vdash_V A_i$ if $A_i$ is an axiom. If $A_i \in \Gamma$, then obviously $\Gamma \vdash_V A_i$. It remains to show that the rules of inference preserve consequence. We consider this for modus ponens (11) and existential generalization (113), the other cases are analogous.

Suppose $\Gamma \vdash_V A \wedge B$ and consider a $V$-interpretation $I$. Let $\eta = \inf \{ \mathcal{C} \in C : 2 \Gamma \eta \downarrow \}$. If $I(\eta) \vdash I(\mathcal{B})$, then we have $\eta \vdash I(\mathcal{B})$ because $\eta \vdash I(\eta)$. If $I(\eta) \vdash I(\mathcal{B})$, then $\eta \vdash I(\mathcal{B})$ because $I(\mathcal{B}) = I(\eta \wedge \mathcal{B})$.

Suppose $\Gamma \vdash_V A (\tau) \wedge B$ and $x$ does not occur free in $B$. Let $I$ be a $V$-interpretation, and let $w = \sup \{ \mathcal{C} : u \in U \mathcal{B} \}$. Let $I_u$ be the interpretation resulting from $I$ by assigning $u$ to $x$. Since the formulas in $\Gamma$ are all closed and $B$ does not contain $x$ free, $I_u(\mathcal{C}) = I(\mathcal{C})$ for all $C \in \Gamma \setminus \{ \mathcal{B}_2 \text{ and } u \in U \mathcal{B} \}$. Now suppose $w \supset I(\exists x A (\tau) \wedge B)$. In this case, $I(\exists x A (\tau)) \supset I(\mathcal{B})$. But then, for some $u \in U \mathcal{B}$, $I_u(\exists x A (\tau)) > I(\mathcal{B})$ and we’d have $w > I_u(A (\tau) \wedge B)$, contradicting $\Gamma \vdash_V A (\tau) \wedge B$. The case for (110) is analogous.

Note that the restriction to closed formulas in $\Gamma$ is essential: $A (\tau) \vdash_V \exists x A (\tau)$ but obviously $A (\tau) \not\vdash_V \exists x A (\tau)$.
2.3 Relationships between Gödel logics

The relationships between finite and infinite valued propositional Gödel logics are well understood. Any choice of an infinite set of truth-values results in the same propositional Gödel logic, viz., Dummett’s LC. LC was defined using the set of truth-values $V_\#$ (see below). Furthermore, we know that LC is the intersection of all finite-valued propositional Gödel logics, and that it is axiomatized by intuitionistic propositional logic IPL plus the schema $(A \to B) \to (B \to A)$. IPL is contained in all Gödel logics.

In the first-order case, the relationships are somewhat more interesting. First of all, let us note the following fact corresponding to the end of the previous paragraph:

**Proposition 19.** Intuitionistic predicate logic IL is contained in all first-order Gödel logics.

**Proof.** The axioms and rules of IL are sound for the Gödel truth functions.

As a consequence of this proposition, we will be able to use any intuitionistically sound rule and intuitionistically true formula when working in any of the Gödel logics. We can consider special truth value sets which will act as prototypes for other logics. This is due to the fact that the logic is defined extensionally as the set of formulas valid in this truth value set, so the Gödel logics on different truth value sets may coincide.

$$
V_R = \{0, 1\}
$$

$$
V_\# = \{k : k \leq 1\}
$$

$$
V_* = \{k : k \leq 1\}
$$

$$
V_m = \{k : 1 \leq k \leq m\}
$$

The corresponding Gödel logics are $G_R$, $G_\#$, $G_*$, $G_m$, $G_\infty$ is the standard Gödel logic. The logic $G_\#$ also turns out to be closely related to some temporal logics [BLZ96b, BLZ96a]. $G_*$ is the intersection of all finite-valued first-order Gödel logics as shown in Theorem 23.

**Proposition 20.** $G_R = \prod V G_V$, where $V$ ranges over all Gödel sets.

**Proof.** If $G_V \models A$ for every $V$, then also for $V = \{0, 1\}$. Conversely, if there is some Gödel set $V$ and a $V$-interpretation $I$ with $I \models A$, then $I$ is also a $\{0, 1\}$-interpretation and hence $G_R \models A$.

**Proposition 21.** The following strict containment relationships hold:

1. $G_m \supset G_{m+1}$.
2. $G_m \supset G_* \supset G_R$.
3. $G_m \supset G_\# \supset G_R$. 


Proof. The only non-trivial part is proving that the containments are strict. For this note that

\[ (A_1 \land A_2) \land \cdots \land (A_m \land A_{m+1}) \]

is valid in \( G_m \) but not in \( G_{m+1} \). Furthermore, let

\[ C_a = \exists x \left( A(x) \land \exists y A(y) \right) \] and
\[ C_b = \exists x \left( \exists y A(y) \land A(x) \right) : \]

\( C_b \) is valid in all \( G_m \) and in \( G^* \) and \( G^*_3 \); \( C_a \) is valid in all \( G_m \) and in \( G^* \), but not in \( G^*_3 \); neither is valid in \( G_R \) ([BLZ96b], Corollary 2.9). \( \square \)

The formulas \( C_a \) and \( C_b \) are of some importance in the study of first-order infinite-valued Gödel logics. \( C_a \) expresses the fact that every infimum in the set of truth values is a minimum, and \( C_b \) states that every supremum (except possibly 1) is a maximum. The intuitionistically admissible quantifier shifting rules are given by the following implications and equivalences:

\[
\begin{align*}
(\forall x A(x) \land B) &\quad 8x (A(x) \land B) \\
(\exists x A(x) \land B) &\quad 9x (A(x) \land B) \\
(\forall x A(x) \land \neg B) &\quad 8x (A(x) \land \neg B) \\
(\exists x A(x) \land \neg B) &\quad 9x (A(x) \land \neg B) \\
(\forall B \land \exists x A(x)) &\quad 8x (B \land A(x)) \\
(\exists B \land \exists x A(x)) &\quad 9x (B \land A(x)) \\
(\forall x A(x) \land \neg B) &\quad 9x (A(x) \land \neg B) \\
(\exists x A(x) \land \neg B) &\quad 8x (A(x) \land \neg B)
\end{align*}
\]

The remaining three are:

\[
\begin{align*}
(\forall x A(x) \land \neg B) &\quad 8x (A(x) \land \neg B) \quad (S_1) \\
(\exists B \land \exists x A(x)) &\quad 9x (B \land A(x)) \quad (S_2) \\
(\exists x A(x) \land \neg B) &\quad 9x (A(x) \land \neg B) \quad (S_3)
\end{align*}
\]

Of these, \( S_1 \) is valid in any Gödel logic. \( S_2 \) and \( S_3 \) imply and are implied by \( C_b \) and \( C_a \), respectively (take \( \exists y A(y) \) and \( \exists y A(y) \), respectively, for \( B \)). \( S_2 \) and \( S_3 \) are, respectively, both valid in \( G^*_3 \), invalid and valid in \( G^*_b \), and both invalid in \( G_R \). Thus we obtain

**Corollary 22.** \( G^* \) is the only Gödel logic where every formula is equivalent to a prenex formula with the same propositional matrix.

We now also know that \( G^* \nsubseteq G^*_b \). In fact, we have \( G^*_b \nsubseteq G^* \); this follows from the following theorem.

**Theorem 23.**

\[
G^* = \bigwedge_{m \geq 2} G_m
\]

**Proof.** By Proposition [21] \( G^* \supseteq \bigwedge_{m \geq 2} G_m \). We now prove the reverse inclusion. Assume that there is an interpretation \( \mathcal{I} \) such that \( \mathcal{I} \models A \), we want to give an interpretation \( \mathcal{I}^0 \) such that \( \mathcal{I}^0 \models A \) and \( \mathcal{I}^0 \) is a \( G_m \) interpretation for some \( m \).
Suppose there is an interpretation $I$ such that $I \models A$, let $I(A) = 1 \models k$. Let $w$ be somewhere between $1 \models k$ and $1 \models (k + 1)$. Then the interpretation $I_w$ given in Lemma 12 also is a counterexample for $A$. Since there are only finitely many truth values below $w$ in $V^*$, $I_w$ is a $G_{k+1}$ interpretation with $I_w \not\models A$. This completes the proof of the theorem.

Corollary 24. $G_m \models \frac{T}{m} G_m = G^* \models G^*_k \models G_R$

As we will see later, the axioms $\text{FIN}(n)$ axiomatize exactly the finite-valued Gödel logics. In these logics the quantifier shift axiom $QS$ is not necessary. Furthermore, all quantifier shift rules are valid in the finite valued logics. Since $G^*$ is the intersection of all the finite ones, all quantifier shift rules are valid in $G^*$. Moreover, any infinite-valued Gödel logic other than $G^*$ is defined by some $V$ which either contains an infimum which is not a minimum, or a supremum (other than 1) which is not a maximum. Hence, in $V$ either $C_\epsilon$ or $C_\delta$ will be invalid, and therewith either $S_3$ or $S_2$. We have:

Corollary 25. $G^*$ is the only Gödel logic with infinite truth value set which admits all quantifier shift rules.

3 Topology and Order

3.1 Perfect sets

All the following notations, lemmas, theorems are carried out within the framework of Polish spaces, which are separable, completely metrizable topological spaces. For our discussion it is only necessary to know that $\mathbb{R}$ and all its closed subsets are Polish spaces (hence, every Gödel set is a Polish space). For a detailed exposition see [Mos80, Kec95].

Definition 26 (limit point, perfect space, perfect set). A limit point of a topological space is a point that is not isolated, i.e. for every open neighborhood $U$ of $x$ there is a point $y \in U$ with $y \neq x$. A space is perfect if all its points are limit points. A set $P \subseteq \mathbb{R}$ is perfect if it is closed and together with the topology induced from $\mathbb{R}$ is a perfect space.

It is obvious that all (non-trivial) closed intervals are perfect sets, also all countable unions of (non-trivial) intervals. But all these sets generated from closed intervals have the property that they are ‘everywhere dense’, i.e., contained in the closure of their inner component. There is another very famous set which is perfect but is nowhere dense, the Cantor set:

Example (Cantor Set). The set of all numbers in the unit interval which can be expressed in triadic notation only by digits 0 and 2 is called Cantor set.

A more intuitive way to obtain this set is to start with the unit interval, take out the open middle third and restart this process with the lower and the upper third. Repeating this you get exactly the Cantor set because the middle third always contains the numbers which contain the digit 1 in their triadic notation.

This set has a lot of interesting properties, the most important one is that it is a perfect set:
Proposition 27. The Cantor set is perfect.

It is possible to embed the Cauchy space into any perfect space, yielding the following proposition:

Proposition 28. If $X$ is a nonempty perfect Polish space, then the cardinality of $X$ is $2^\aleph_0$ and therefore, all nonempty perfect subsets, too, have cardinality of the continuum.

It is possible to obtain the following characterization of perfect sets (see Win99):

Proposition 29 (Characterization of perfect sets in $\mathbb{R}$). For any perfect subset of $\mathbb{R}$ there is a unique partition of the real line into countably many intervals such that the intersections of the perfect set with these intervals are either empty, the full interval or isomorphic to the Cantor set.

So we see that intervals and Cantor sets are prototypical for perfect sets and the basic building blocks of more complex perfect sets.

Every Polish space can be partitioned into a perfect kernel and a countable rest. This is the well known Cantor-Bendixon Theorem:

Theorem 30 (Cantor-Bendixon). Let $X$ be a Polish space. Then $X$ can be uniquely written as $X = P \setminus C$, with $P$ a perfect subset of $X$ and $C$ countable and open. The subset $P$ is called the perfect kernel of $X$ (denoted with $\mathcal{V}_\infty$).

As a corollary we obtain that any uncountable Polish space contains a perfect set, and therefore, has cardinality $2^\aleph_0$.

3.2 Relation to Gödel logics

The following lemma was originally proved in Pre03, where it was used to extend the proof of recursive axiomatizability of `standard’ Gödel logics (those with $V = \{0,1\}$) to Gödel logics with a truth value set containing a perfect set in the general case. The following more simple proof is inspired by BGP.

Lemma 31. Suppose that $M = \{0;1\}$ is countable and $P = \{0;1\}$ is perfect. Then there is a strictly monotone continuous map $h: M \rightarrow P$ (i.e., infima and suprema already existing in $M$ are preserved). Furthermore, if $\inf M \geq M$, then one can choose $h$ such that $h(\inf M) = \inf P$.

Proof. Let $\sigma$ be the mapping which scales and shifts $M$ into $\{0;1\}$ i.e. the mapping $x \mapsto (\sigma(x) = \inf M) = \sup M \inf M$ (assuming that $M$ contains more than one point). Let $w$ be an injective monotone map from $\sigma(M)$ into $2^\omega$, i.e. $w(m)$ is a fixed binary representation of $m$. For dyadic rational numbers (i.e. those with different binary representations) we fix one possible.

Let $i$ be the natural bijection from $2^\omega$ (the set of infinite $0;1$-sequences, ordered lexicographically) onto $\mathbb{D}$, the Cantor set. $i$ is an order preserving homeomorphism. Since $P$ is perfect, we can find a continuous strictly monotone map $c$ from the Cantor set $\mathbb{D} = \{0,1\}$ into $P$, and $c$ can be chosen so that $c(0) = \inf P$. 

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Now \( h = c \cdot w \) \( \sigma \) is also a strictly monotone map from \( M \) into \( P \), and \( h \downarrow M = \inf P \), if \( \inf M \not\in M \). Since \( c \) is continuous, existing infima and suprema are preserved. \( \square \)

**Corollary 32.** A Gödel set \( V \) is uncountable iff it contains a non-trivial dense linear subordering.

**Proof.** If: Every countable non-trivial dense linear order has order type \( \eta, 1 + \eta, \eta + 1 \), or \( 1 + \eta + 1 \) [Ros82 Corollary 2.9], where \( \eta \) is the order type of \( \emptyset \). The completion of any ordering of order type \( \eta \) has order type \( \lambda \), the order type of \( R \) [Ros82 Theorem 2.30], thus the truth value set must be uncountable.

Only if: By Theorem 30, \( V^\infty \) is non-empty. Take \( M = \emptyset \setminus \{0,1\} \) and \( P = V^\infty \) in Lemma 31. The image of \( M \) under \( h \) is a non-trivial dense linear subordering in \( V \). \( \square \)

**Theorem 33.** Suppose \( V \) is a truth value set with non-empty perfect kernel \( P \), and let \( W = V \setminus \{\inf P;1\} \). Then \( \dot{\gamma}_V = \dot{\gamma}_W \), i.e. \( \Gamma \dot{\gamma}_V A \) iff \( \Gamma \dot{\gamma}_W A \). Thus also the logics induced by \( V \) and \( W \) are the same, i.e., \( G_V = G_W \).

**Proof.** As \( V \) \( W \) we have \( \dot{\gamma}_W \dot{\gamma}_V \) (cf. the Remark preceding Definition 3). Now assume that \( I \) is a \( W \)-interpretation which shows that \( \Gamma \dot{\gamma}_W A \) does not hold, i.e., \( \inf f(\emptyset) : B \subset 2 \Gamma g > I(A) \). By Proposition 11 we may assume that \( U^I \) is countable. The set \( \text{Val}(\emptyset;I^A) \) has cardinality at most \( R_0 \), thus there is a \( b \in \emptyset \setminus \{0,1\} \) such that \( b \not\in \text{Val}(\emptyset;I^A) \) and \( I(A) < b < 1 \). By Lemma 31 \( I_b(A) < b < 1 \). Now consider \( M = \text{Val}(\emptyset_b;I^A) \): these are all the truth values from \( W = V \setminus \{\inf P;1\} \) required to compute \( I_b(A) \) and \( I_b(\emptyset) \) for all \( B \subset 2 \Gamma \). We have to find some way to map them to \( V \) so that the induced interpretation is a counterexample to \( \Gamma \dot{\gamma}_V A \).

Let \( M_0 = M \setminus \emptyset \cap \{\inf P\} \) and \( M_1 = (M \setminus \{\inf P;1\}) \setminus \{\inf P;2\} \). By Lemma 31, there is a strictly monotone continuous (i.e. preserving all existing infima and suprema) map \( h \) from \( M_1 \) into \( P \). Furthermore, we can choose \( h \) such that \( h(\inf M_1) = \inf P \).

We define a function \( g \) from \( \text{Val}(\emptyset_b;I^A) \) to \( V \) as follows:

\[
\begin{align*}
g(\emptyset) & = 0 & x & = 0 \inf P \\
g(\emptyset) & = 1 & x & = 1
\end{align*}
\]

Note that there is no \( x < 2 \) \( \emptyset \) \( \text{Val}(\emptyset_b;I^A) \) with \( b < x < 1 \). This function has the following properties: \( g(\emptyset) = 0 \), \( g(1) = 1 \), \( g \) is strictly monotone and preserves existing infima and suprema. Using Lemma 7 we obtain that \( I_g \) is a \( V \)-interpretation with \( I_g(\emptyset) = \{ g(\emptyset_b;C) \} \) for all \( C \subset 2 \Gamma^A \), thus also \( \inf f(I_g(\emptyset)) : B \subset 2 \Gamma g > I_g(A) \). \( \square \)

### 4 Countable Gödel sets

In this section we show that the first-order Gödel logics where the set of truth values does not contain a dense subset are not axiomatizable. We establish this result by reducing the classical validity of a formula in all finite models to the validity of a formula in Gödel logic (the set of these formulas is not r.e. by Trakhtenbrot’s Theorem).
Definition 34. A formula is called crisp if all occurrences of atomic formulas are either negated or double-negated.

Lemma 35. If A and B are crisp and classically equivalent, then also $G_R \models A \equiv B$. Specifically, if A (x) and B are crisp, then

\[ \models \exists x A (x) \land B \equiv \exists x (A (x) \land B) \text{ and } \]

\[ \models \exists x A (x) \land \neg B \equiv \exists x (A (x) \land \neg B) \]  

Proof. Given an interpretation $\mathcal{I}$, define $I^0 (\mathcal{C}) = 1$ if $\mathcal{I} (\mathcal{C}) > 0$ and $0$ if $\mathcal{I} (\mathcal{C}) = 0$ for atomic $C$. It is easily seen that if $A, B$ are crisp, then $\mathcal{I} (A) = I^0 (A)$ and $\mathcal{I} (B) = I^0 (B)$. But $I^0$ is a classical interpretation, so by assumption $I^0 (A) = I^0 (B)$.

Theorem 36. If $V$ is countably infinite, then $G_V$ is not recursively enumerable.

Proof. By Theorem 32, $V$ is countably infinite iff it is infinite and does not contain a non-trivial densely ordered subset. We show that for every sentence $A$ there is a sentence $A^e$ s.t. $A^e$ is valid in $G_V$ iff $A$ is true in every finite (classical) first-order structure.

We define $A^e$ as follows: Let $P$ be a unary and $L$ be a binary predicate symbol not occurring in $A$ and let $Q_1, \ldots, Q_n$ be all the predicate symbols in $A$. We use the abbreviations $x \leq y : L (x, y)$ and $x \neq y : (\neg P (x) \land P (y))$. Note that for any interpretation $\mathcal{I}$, $\mathcal{I} (x \leq y)$ is either 0 or 1, and as long as $\mathcal{I} (P (x)) < 1$ for all $x$ (in particular, if $\mathcal{I} (\exists z P (z)) < 1$), we have $\mathcal{I} (x \neq y) = 1$ if $\mathcal{I} (P (x)) < \mathcal{I} (P (y))$. Let $A^e$

\[ \begin{align*}
S \land c_1 \land c_2 \land c_1 \land \\
8i \land 8x y z D_{\geq} \land 8x : (\exists z (x \neq y)) & \equiv (A^0 \land \neg \exists u P (u))
\end{align*} \tag{7} \]

where $S$ is the conjunction of the standard axioms for 0, successor and , with double negations in front of atomic formulas,

\[ D \equiv (\exists i \leq x \land j \leq k \land i \leq y \land k \leq x \land y) \land \\
(\exists i \leq z x \land z \leq z \land y) \]

and $A^0$ is $A$ where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R (\emptyset)$ $\neg \exists i (x \neq i)$.

Intuitively, $L$ is a predicate that divides a subset of the domain into levels, and $x \leq i$ means that $x$ is an element of level $i$. If the antecedent is true, then the true standard axioms $S$ force the domain to be a model of PA, which could be either a standard model (isomorphic to $\mathbb{N}$) or a non-standard model ($\mathbb{N}$ followed by copies of $\mathbb{Z}$). $P$ orders the elements of the domain which fall into one of the levels in a subordering of the truth values.

The idea is that for any two elements in a level $i$ there is an element in a non-empty level $j$ which lies strictly between those two elements in the ordering given by $\leq$. If this condition cannot be satisfied, the levels above $i$ are empty. Clearly, this condition can be satisfied in an interpretation $\mathcal{I}$ only for finitely many levels if $V$ does not contain a dense subset, since if more than finitely many levels are non-empty, then $d \exists \exists (\mathcal{I} (P \emptyset))$ gives a dense subset. By relativizing the quantifiers in $A$ to
the indices of non-empty levels, we in effect relativize to a finite subset of the domain. We make this more precise:

Suppose $A$ is classically false in some finite structure $I$. W.l.o.g. we may assume that the domain of this structure is the naturals $0, \ldots, n$. We extend $I$ to a $G_V$-interpretation $I^V$ with domain $\mathbb{N}$ as follows: Since $V$ contains infinitely many values, we can choose $c_1, c_2, L$ and $P$ so that $9x (x \geq i)$ is true for $i = 0, \ldots, n$ and false otherwise, and so that $I^V (\exists x P(x)) < 1$. The number-theoretic symbols receive their natural interpretation. The antecedent of $A^g$ clearly receives the value 1, and the consequent receives $I^V (\exists x P(x)) < 1$, so $I^V \not\models A^g$.

Now suppose that $I \models A^g$. Then $I (\exists x P(x)) < 1$. In this case, $I (x \geq y) = 1$ iff $I (P(k)) < I (P(y))$, so $P$ defines a strict order on the domain of $I$. It is easily seen that in order for the value of the antecedent of $A^g$ under $I$ to be greater than that of the consequent, it must be $1$ (the values of all subformulas are either $I (\exists x P(x))$ or $= 1$). For this to happen, of course, what the antecedent is intended to express must actually be true in $I$, i.e., that $x \geq i$ defines a series of levels and any level $i > 0$ is either empty, or for all $x$, and $y$ occurring in some smaller level there is a $z$ with $x < z < y$ and $z \geq i$.

To see this, consider the relevant part of the antecedent, $B = s \Theta y z b k z D$ _ $\exists x: (x \geq i)$. If $I (\Theta) = 1$, then for all $i$, either $I (\exists x y z b k z D) = 1$ or $I (\exists x: (x \geq i)) = 1$. In the first case, we have $I (\exists D) = 1$ for all $x, y, j$, and $k$. Now suppose that for all $z$, $I (\Theta) < 1$, yet $I (\exists D) = 1$. Then for at least some $z$ the value of that formula would have to be $> I (\exists z P(z))$, which is impossible. Thus, for every $x, y, j, k$, there is a $z$ such that $I (\Theta) = 1$. But this means that for all $x, y$ s.t. $x \geq i$ and $y \geq k$ with $j, k = i$ and $x \leq y$ there is a $z$ with $x < z < y$ and $z \geq i + 1$.

In the second case, where $I (\exists x: (x \geq i)) = 1$, we have that $I (x \geq i) = 1$ for all $x$, hence $I (x \geq i) = 0$ and level $i$ is empty.

Note that the non empty levels can be distributed over the whole range of the non-standard model, but since $V$ contains no dense subset, the total number of non empty levels is finite. Thus, $A$ is false in the classical interpretation $I^c$ obtained from $I$ by restricting $I$ to the domain $\ell i : 9x (x \geq i)g$ and $I^c (Q) = I (x : Q)$ for atomic $Q$.

This shows that no infinite-valued Gödel logic whose set of truth values does not contain a dense subset, i.e., no countably infinite Gödel logic is axiomatizable. We strengthen this result in Section 6.1 to show that the prenex fragments are likewise not axiomatizable.

5 Uncountable Gödel sets

5.1 0 is contained in the perfect kernel

If $V$ is uncountable, and $0$ is contained in $V^\omega$, then $G_V$ is axiomatizable. Indeed, Theorem 23 showed that the sets of validities of all such $V$ coincide. Thus, it is only necessary to establish completeness of the axioms system $H$ with respect to $G_R$. This result has been shown by several people over the years. We give here a generalization of the proof of Takano [Tak87].
Theorem 37 (Strong completeness of Gödel logic \cite{Tak87}). If $\Gamma \not\vdash A$ in $G_B$, then $\Gamma \vdash_H A$.

Proof. Assume that $\Gamma \not\vdash A$, we construct an interpretation $I$ in which $I(A) = 1$ for all $B \not\in \Gamma$ and $I(A) < 1$. Let $y_1, y_2, \ldots$ be a sequence of free variables which do not occur in $\Gamma \setminus A$, let $T$ be the set of all terms in the language of $\Gamma \setminus A$ together with the new variables $y_1, y_2, \ldots,$ and let $F = \{F_1; F_2; \ldots; z\}$ be an enumeration of the formulas in this language in which $y_i$ does not appear in $F_1, \ldots, F_i$ and in which each formula appears infinitely often.

If $\Delta$ is a set of formulas, we write $\Gamma \vdash \Delta$ if for some $A_1, \ldots, A_n \in \Gamma$, and some $B_1, \ldots, B_m \not\in \Delta$, $\Delta \vdash_H (\Delta \cup \{A_1, \ldots, A_n\})$ (and $\gamma$ if this is not the case). We define a sequence of sets of formulas $\Gamma_n, \Delta_n$ such that $\Gamma_n \vdash \Delta_n$ by induction. First, let $\Gamma_0 = \Gamma$ and $\Delta_0 = \{A\}$. By the assumption of the theorem, $\Gamma_0 \vdash \Delta_0$.

If $\Gamma_n \vdash \Delta_n = \{F_n g\}$, then $\Gamma_{n+1} = \Gamma_n = \{F_n g\}$ and $\Delta_{n+1} = \Delta_n$. In this case, $\Gamma_{n+1} \vdash \Delta_{n+1}$, since otherwise we would have $\Gamma_{n+1} \vdash \{F_n g\}$ and $\Gamma_n \vdash \{F_n g\} \vdash \Delta_n$. But then, we’d have that $\Gamma_n \vdash \Delta_n$, which contradicts the induction hypothesis (note that $\Delta \vdash_H \{A\}$).

If $\Gamma_n \vdash \Delta_n = \{F_n g\}$, then $\Gamma_{n+1} = \Gamma_n \setminus \{F_n g\}$ and $\Delta_{n+1} = \{\{F_n g\} \vdash \{F_n g\}\}$ as well. But then $\Gamma_{n+1} \vdash \Delta_{n+1}$, since each $\Delta_n$ is closed under provable implication, since if $\Gamma_{n+1} \vdash \Delta_{n+1}$, then $\Gamma_{n+1} \setminus \{F\} \vdash \Delta_{n+1}$.

Let $\Gamma = \Gamma_n \setminus \{F\}$ and $\Delta = \{\Gamma_n \vdash \Delta_n\}$. We have:

1. $\Gamma \vdash \Delta$, for otherwise there would be a $k$ so that $\Gamma_k \vdash \Delta_k$.
2. $\Gamma \vdash \Delta$ (by construction).
3. $\Gamma = \{\gamma\}$, since each $F_n$ is either in $\Gamma_{n+1} \setminus \{F\}$, and for some $n$, $F_n \not\in \Gamma \setminus \Delta$, there would be a $k$ so that $F_k \not\in \Gamma \setminus \Delta$, which is impossible since $\Gamma_k \vdash \Delta_k$.
4. $\Gamma \vdash B_i, \ldots, B_m$, then $B_i \not\in \Gamma \setminus \Delta$, and hence, by (3), $B_i \not\in \Delta$. But then $\Gamma \vdash \Delta$, contradicting (1).
5. If $\Gamma \vdash B_i \not\in \Gamma \setminus \Delta$, then $\exists x B (x) \not\in \Gamma$, and so there is some $n$ so that $\exists x B (x) = F_n$ and $\Delta_n+1$ contains $\exists x B (x)$, contradicting (1).
6. $\Gamma$ is closed under provable implication, since if $\Gamma \vdash \Delta$, then $\Delta \vdash \Delta$ and so, again by (3), $\Delta \vdash \Delta$. In particular, if $\Delta \vdash_H A$, then $\ Gamma \vdash_H A$.

Define relations $\vdash_H$ and $\vdash_H$ on $F$ by

$$B \vdash C, B \vdash C \quad \text{and} \quad B \vdash C, B \vdash C \quad \text{and} \quad B \vdash C \vdash C \vdash C \vdash B.$$ 

Then $\vdash_H$ is reflexive and transitive, since for every $B$, $\vdash_B B$ and so $B \vdash B$, and so $B \not\in \Gamma \setminus A$, and if $B \in \Gamma \setminus A$, then $B \not\in \Gamma \setminus A$, and if $B \in \Gamma \setminus A$, then $B \not\in \Gamma \setminus A$, and if $B \not\in \Gamma \setminus A$, then $B \not\in \Gamma \setminus A$, and if $B \not\in \Gamma \setminus A$.

Recall (6) above. Hence, $\vdash_B$ is an equivalence relation on $F$. For every $B$ in $F$ we let $\Delta \vdash_B$ be the equivalence class under $\vdash_H$ to which $B$ belongs, and $F$ is the set of all equivalence classes. Next we define the relation $\vdash_H$ on $F$ by

$$\Delta \vdash_B \Delta \vdash_B C, B \vdash B \vdash C \vdash B \vdash C \vdash B \vdash C \vdash C \vdash B.$$ 

Obviously, $\vdash_H$ is independent of the choice of representatives $A, B$. 

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Lemma 38. $\mathcal{F} = \{ \} \text{ is a countably linearly ordered structure with distinct maximal element } \not\vDash \not\exists j \text{ and minimal element } \not\vDash \exists j$. 

Proof. Since $\mathcal{F}$ is countably infinite, $\mathcal{F} = \{ \}$ is countable. For every $B$ and $C$, $\not\vDash (B \land C)$ by 1IN, and so either $B ! \not\vDash 2 \Gamma$ or $C ! \not\vDash 2 \Gamma$ (by (4)), hence $\vDash$ is linear. For every $B$, $\not\vDash B^! > B$, and so $B ! > 2 \Gamma$ and ? $B ! 2 \Gamma$, hence $\vDash \not\exists j$ if $\not\exists j$ are the maximal and minimal elements, respectively. Pick any $A$ in $\Delta$. Since $\not\exists j \vDash A$ and $A \not\vDash \Gamma, \not\exists j \vDash \exists j$. We abbreviate $\not\vDash \exists j$ by $1$ and $\not\vDash \exists j$ by $0$.

Lemma 39. The following properties hold in $\mathcal{F} = \{ \}$:

1. $B \vDash 1$, $B \not\vDash 2 \Gamma$.
2. $B \land C \vDash \min f B \vDash C$.
3. $B \land C \vDash \max f B \vDash C$.
4. $B \not\vDash 1$ if $B \vDash C \not\vDash B ! C \vDash 0$ otherwise.
5. $\vDash B \vDash 1$ if $B \vDash 0$; $\vDash B \vDash 0$ otherwise.
6. $\exists x B (x) \vDash \sup f B (x)$: $\exists x T$.
7. $\exists x B (x) \vDash \inf f B (x)$: $\exists x T$.

Proof. (1) If $B \vDash 1$, then $B \not\vDash 2 \Gamma$, and hence $B \not\vDash 2 \Gamma$. And if $B \not\vDash 2 \Gamma$, then $B \not\vDash 2 \Gamma$ since $B \not\vDash ! 2 \Gamma$. So $\vDash \exists x B$ if also $\exists x B \not\vDash \exists x B$. (2) From $B \land C ! B \land C$ and $D ! B ; D ! C$ it follows that $B \land C \vDash \inf f B \vDash C \vDash 0$. (3) is proved analogously.

(4) If $B \not\vDash 1$, then $B ! 2 \Gamma$, and since $B \not\vDash 2 \Gamma$, it follows that $B \not\vDash 2 \Gamma$ since $B \not\vDash ! 2 \Gamma$. So $\vDash B \not\vDash \exists x B$ if also $\exists x B \not\vDash \exists x B$. (5) If $B \vDash 0$, then $B \not\vDash ! 2 \Gamma$, and hence $\exists x B \vDash 1$ by (1). Otherwise, $B \not\vDash \exists x B$ and so by (4), $\exists x B \not\vDash \exists x B$.

(6) Since $\not\vDash (B \not\vDash C \not\vDash ! x B (x); \not\vDash B \not\vDash ! x B (x)$ for every $\exists x T$. On the other hand, for every $D$ without $x$ free,

$\exists x B (x) \vDash 1$ for every $\exists x T$

$\exists x B (x) \vDash 2 \Gamma$ for every $\exists x T$

$\exists x B (x) \vDash ! D \not\vDash \exists x B (x)$ for every $\exists x T$

$\exists x B (x) \vDash D$ for every $\exists x T$

$\exists x B (x) \vDash ! D \not\vDash \exists x B (x)$ since $\exists x B (x) \vDash ! D$.

(7) is proved analogously.

$\mathcal{F} = \{ \}$ is countable, let $0 = a_0, 1 = a_1, a_2, \ldots$ be an enumeration. Define $h (0) = 0$, $h (1) = 1$, and define $h (\psi_n)$ inductively for $n > 1$: Let $a_n = \max f a_i : i < n$ and $a_i < a_{n+1}$ and $a_i \not\vDash \exists x B (x)$ for every $\exists x T$. Then $h (\psi_n) = h (\psi_n) + h (\psi_n)^2 = 2$ (thus, $a_2 = 0$ and $a_3 = 1$ as $0 = a_0 < a_2 < a_1 = 1$, hence $h (\psi_2) = \frac{1}{2}$). Then $h : \mathcal{F} = \{ \}$.
exists a $G$-embedding $h^0$ from $\mathcal{Q}$ \( \downarrow \{0, 1\} \) into $\mathcal{B}$ \( \downarrow \{0, 1\} \) which is also strictly monotone and preserves infs and sups. Put $\mathfrak{I}(B) = h^0(h(B))$ for every atomic $B \vDash \Gamma$ and we obtain a $V^0$-interpretation.

Note that for every $B$, $\mathfrak{I}(B) = 1$ iff $\mathcal{Q} \vDash B$ iff $\mathcal{Q} \vDash B \vDash \Gamma$. Hence, we have $\mathfrak{I}(B) = 1$ for all $B \vDash \Gamma$ while if $A \vDash \Gamma$, then $\mathfrak{I}(A) < 1$, so $\mathcal{Q} \vDash A$. Thus we have proven that on the assumption that if $\mathcal{Q} \vDash A$, then $\mathcal{Q} \vDash A$. Hence, we have $\mathfrak{I}(B) = 1$ for all $B \vDash \Gamma$ while if $A \vDash \Gamma$, then $\mathfrak{I}(A) < 1$, so $\mathcal{Q} \vDash A$. Thus we have proven that on the assumption that if $\mathcal{Q} \vDash A$, then $\mathcal{Q} \vDash A$.

As already mentioned we obtain from this completeness proof together with the soundness theorem (Theorem 18) and Theorem 33 the characterization of recursive axiomatizability:

**Theorem 40.** Let $V$ be a Gödel set with $0$ contained in the perfect kernel of $V$. Suppose that $\Gamma$ is a set of closed formulas. Then $\mathcal{Q} \vDash A$ iff $\Gamma \vDash V^0 A$.

**Corollary 41 (Deduction theorem for Gödel logics).** Suppose that $\Gamma$ is a set of formulas, and $A$ is a closed formula. Then

$$\Gamma \vDash \mathcal{Q} A \iff \Gamma \vDash V^0 A$$

**Proof.** Use the soundness theorem (Theorem 18), completeness theorem (Theorem 40) and the semantic deduction theorem 14. Another proof would be by induction on the length of the proof. See [Haj98], Theorem 2.2.18.

### 5.2 0 is isolated

In the case where $0$ is isolated, and thus also not contained in the perfect kernel, we will transform a counter example in $\mathcal{G}_R$ for $\Gamma; \Pi \vDash A$, where $\Pi$ is a set of sentences stating that every infimum is a minimum, into a counter example in $\mathcal{G}_V$ for $\Gamma \vDash A$.

**Lemma 42.** Let $x \vec{y}$ be the free variables in $A$.

$$\mathcal{Q} \vDash \mathfrak{I}(8 \vec{y} : 8x A(x; \vec{y}) \iff 9x : A(x; \vec{y}))$$

**Proof.** It is easy to see that in all Gödel logics the following weak form of the law of excluded middle is valid: $\mathcal{Q} : A(\bar{a}) \iff \mathcal{Q} A(\bar{a})$. By quantification we obtain $\mathcal{Q} 8x : A(\bar{a}) \iff \mathcal{Q} 9x : A(\bar{a})$ and by valid quantifier shifting rules $\mathcal{Q} : 8x A(\bar{a}) \iff \mathcal{Q} 9x : A(\bar{a})$. From the intuitionistically valid $\mathcal{Q} : A \iff \mathcal{Q} A$ we can prove $\mathcal{Q} 8x A(\bar{a}) \iff \mathcal{Q} 9x : A(\bar{a})$. A final quantification of the free variables concludes the proof.

**Theorem 43.** Let $V$ be an uncountable Gödel set where $0$ is isolated. Suppose $\Gamma$ is a set of closed formulas. Then $\mathcal{Q} \vDash A$ iff $\Gamma \vDash V^0 A$.

**Proof.** If: Follows from soundness (Theorem 18) and the observation that $\text{ISO}_0$ is valid for any $V$ where $0$ is isolated.

Only if: We already know from Theorem 33 that the entailment relation of $V$ and $V \downarrow \text{inf} P \downarrow 1$ coincide, where $P$ is the perfect kernel of $V$. So we may assume without loss of generality that $V$ already is of this form, i.e. that $\mathcal{Q} = \text{inf} P \downarrow 1 \uparrow \{\lambda, 1\} \downarrow \{\lambda, 1\}$. Let $V^0 = \{0; 1\}$. Define

$$\Pi = f \exists \bar{y} : 8x A(x; \vec{y}) \iff 9x : A(x; \vec{y})) : A(x; \vec{y}))$$

formulag
where \( A(\bar{v}) \) ranges over all formulas with free variables \( x \) and \( \bar{y} \). We consider the
entailment relation in \( V \). Either \( \Pi \models \Gamma \models_A \) or \( \Pi \not\models \Gamma \models_A \). In the
former case we know from the strong completeness of \( H \) for \( G_R \) that there are finite subsets \( \Pi^0 \) of \( \Pi \)
and \( \Gamma \), respectively, such that \( \Pi^0 \models \Gamma^0 \models_H A \). Since all the sentences in \( \Pi \) are provable in
\( H_0 \) (see Lemma 42) we obtain that \( \Gamma^0 \models_{H_0} A \). In the latter case there is an interpretation
\( \Gamma^0 \) such that

\[
\inf \exists \Pi^0(G) : G \geq \Pi \models \Gamma \models \Pi^0(A) :.
\]

It is obvious from the structure of the formulas in \( \Pi \) that their truth value will always be either 0 or 1. Combined with the above we know that for all \( G \geq \Pi \), \( \Gamma^0(G) = 1 \). Next we define a function \( f(\bar{v}) \) which maps values from \( \text{Val}(\Pi^0 \models \Pi \models A \Gamma) \) into \( V \):

\[
f(\bar{v}) = \begin{cases} 
0 & x = 0 \\
\lambda + \lambda = \lambda & x > 0 
\end{cases}
\]

We see that \( f \) satisfies conditions (1) and (2) of Lemma 7, but we cannot use Lemma 7
directly, as not all existing infima and suprema are necessarily preserved.

Consider as in Lemma 7 the interpretation \( \Gamma_f(\bar{B}) = f(\Pi^0(\bar{B})) \) for atomic subformulas of \( \Pi \models \Pi \models A \Gamma \). We want to show that the identity \( \Gamma_f(\bar{B}) = f(\Pi^0(\bar{B})) \) extends
to all subformulas of \( \Pi \models \Pi \models A \Gamma \). For propositional connectives and the existentially quantified formulas this is obvious. The important case is \( \exists x A(\bar{v}) \). First assume that
\( \Gamma^0(\exists x A(\bar{v})) > 0 \). Then it is obvious that \( \Gamma_f(\exists x A(\bar{v})) = f(\Pi^0(\exists x A(\bar{v}))) \). In the case
where \( \Gamma^0(\exists x A(\bar{v})) = 0 \) we observe that \( A(\bar{v}) \) contains a free variable and therefore
\( \exists x A(\bar{v}) \models \Pi \models A(\bar{v}) \models_2 \Pi \), thus \( \Gamma^0(\exists x A(\bar{v})) = 1 \). This implies that there
is a witness \( c \) such that \( \Gamma^0(A(c)) = 0 \). Using the induction hypothesis we know that
\( \Gamma_f(A(c)) = 0 \), too. We obtain that \( \Gamma_f(\exists x A(\bar{v})) = 0 \), concluding the proof.

Thus we have shown that \( \Gamma_f \) is a counterexample to \( \Gamma \not\models \models V \) which completes the
proof of the theorem.

\[\square\]

### 5.3 0 not isolated but not in the perfect kernel

In the preceding sections, we gave axiomatizations for the logics based on those uncountably infinite Gödel sets \( V \) where 0 is either isolated or in the perfect kernel of \( V \). It remains to determine whether logics based on uncountable Gödel sets where 0 is neither isolated nor in the perfect kernel are axiomatizable. The answer in this case is negative. If 0 is not isolated in \( V \), 0 has a countably infinite neighborhood. Furthermore, any sequence \( \langle a_n \rangle_{n \in \mathbb{N}} \) such that, for sufficiently large \( n \), \( V \not\models \langle a_n \rangle \) is countable and hence, by the proof of Theorem 32, contains no densely ordered subset. This fact is the basis for the following non-axiomatizability proof, which is a variation on the proof of Theorem 36.

**Theorem 44.** If \( V \) is uncountable, 0 is not isolated in \( V \), but not in the perfect kernel of \( V \), then \( G_V \) is not axiomatizable.

**Proof.** We show that for every sentence \( A \) there is a sentence \( A^b \) s.t. \( A^b \) is valid in \( G_V \)
iff \( A \) is true in every finite (classical) first-order structure.

The definition of \( A^b \) mirrors the definition of \( A^f \) in the proof of Theorem 36 except that the construction there is carried out infinitely many times for \( V \not\models \langle a_n \rangle \), where
Note that $f$ be a binary and the quantifiers are relativized to the predicate $A$.

As before, for a fixed $\lambda$, provided $\mathcal{I}(\exists x P(x;\lambda)) < 1$, $\mathcal{I}(x \cdot y) = 1$ iff $P(x;\lambda) < P(y;\lambda)$, and $\mathcal{I}(x \cdot y)$ is always either 0 or 1. We also need a binary predicate symbol $Q(\cdot)$ to give us the descending sequence $(a_n)_{n\in\mathbb{N}}$.

Note that $\mathcal{I}(\exists y Q(y;\lambda)) = 1$ iff $\inf\mathcal{I}(Q(y;\lambda)) = 0$ and $\mathcal{I}(\exists y Q(y;\lambda)) = 1$ iff $0 \not\in \mathcal{I}(Q(y;\lambda)) = 2$.

Let $A^h$ be the least $\lambda$.

where $S$ is the conjunction of the standard axioms for 0, successor and , with double negations in front of atomic formulas,

$$E \left( j \mid i^x 2, j^k i^x 2, j^x y, y^x, y \right)$$

and $A^0$ is $A$ where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(\cdot)$.

The idea here is that an interpretation $\mathcal{I}$ will define a sequence $(a_n)_{n\in\mathbb{N}}$ by $a_n = \mathcal{I}(Q^n(\cdot))$ where $a_n > a_{n+1}$ and $a_n < 1$ for all $n$. Let $L^i = f(x) = \mathcal{I}(x \cdot i)g$ be the $i$-th level. $P(x;\lambda)$ orders the set $\lambda$, $L^i = f(x) = \mathcal{I}(x \cdot i)g$ is a subordering of $V \setminus \{a_0, \ldots, a_n\}$ with $f(x, y) = 1$. Again we force that whenever $x,y \in L^i$ with $x < y$, there is a $z \leq L^i$ with $x < z < y$, or, if no possible such $z$ exists, $L^i = \emptyset$. Let $r(\cdot)$ be the least $i$ so that $L^i$ is empty, or $\infty$ otherwise. If $r(\cdot) = \infty$ then there is a densely ordered subset of $V \setminus \{a_0, \ldots, a_n\}$. So if $0$ is not in the perfect kernel, for some sufficiently large $L$, $r(\cdot) < \infty$ for all $\lambda \in L$. $\mathcal{I}(R(\cdot)) = 1$ iff $r(\cdot) = \infty$ hence $E^t: \mathcal{I}(R(\cdot)) = 1g$ is finite whenever the interpretations of $P, Q, L$ and $Q$ are as intended.

Now if $A$ is classically false in some finite structure $\mathcal{I}$, we can again choose a $G_{\mathcal{G}}$-interpretation $\mathcal{I}^h$ in which the interpretations of $P, Q, L$ are as intended, the number theoretic predicates and functions receive their standard interpretation, there are as many $\lambda$ with $\mathcal{I}^h(R(\cdot)) = 1$ as there are elements in the domain of $\mathcal{I}$, and the predicates of $A$ behave on $E^t: \mathcal{I}(R(\cdot)) = 1g$ just as they do on $\mathcal{I}^h$. $\mathcal{I}^h A^h$.

On the other hand, if $\mathcal{I} \not\models A^h$, then the value of the consequent is $< 1$. Then as required, for all $x, y, \mathcal{I}(P(x;\lambda)) < 1$ and $\mathcal{I}(Q(\cdot)) < 1$. Since the antecedent, as before, must be $= 1$, this means that $x < y$ expresses a strict ordering of the elements of $L^i$, and $\mathcal{I}(\exists y Q(y;\lambda)) = 1$ for all $\lambda$ guarantees that $\mathcal{I}(Q(x;\lambda)) = a_{n+1} < a_n = \mathcal{I}(Q(\cdot))$. The other conditions are likewise seen to hold as intended, so that we can extract a finite countermodel $A$ based on the interpretation of the predicate symbols of $A$ on $E^t: \mathcal{I}(R(\cdot)) = 1g$, which must be finite.
6 Fragments

6.1 Prenex fragments

One interesting restriction of the axiomatizability problem is the question whether the
prenex fragment of $G_V$, i.e., the set of prenex formulas valid in $G_V$, is axiomatizable.
This is non-trivial, since in general in Gödel logics, arbitrary formulas are not equivalent
to prenex formulas. Thus, so far the proofs of non-axiomatizability of the logics
treated in Sections 4 and 5 do not establish the non-axiomatizability of their prenex
fragments, nor do they exclude the possibility that the corresponding prenex fragments
are r.e. We investigate this question in this section, and show that the prenex fragments
of all finite and uncountable Gödel logics are r.e., and that the prenex fragments of
all countably infinite Gödel logics are not r.e. The axiomatizability result is obtained
from a version of Herbrand’s Theorem for finite and uncountably-valued Gödel logics,
which is of independent interest. The non-axiomatizability of countably infinite Gödel
logics is obtained as a corollary of Theorem 5.6.

Let $V$ be a Gödel set which is either finite or uncountable. Let $G_V$ be a Gödel
logic with such a truth value set. We show how to effectively associate with each
prenex formula $A$ a quantifier-free formula $A$ which is valid in $G_V$ if and only if $A$ is
a tautology. The axiomatizability of the prenex fragment of $G_V$ then follows from
the axiomatizability of $LC$ (in the infinite-valued case) and propositional $G_m$ (in the
finite-valued case).

Definition 45 (Herbrand form). Given a prenex formula $A \quad Q_1x_1 \ldots Q_n x_n B (\forall) \quad (B
quantifier free), the Herbrand form $A^H$ of $A$ is $\exists x_{i_1} \ldots \exists x_{m_1} B \, \forall \, j \quad (j
max \, i)$ is the set of existentially quantified variables in $A$, and $i_i$ is $x_{j_i}$ if $i = i_j$,
or is $f_j (x_{i_1} \ldots x_{i_2})$ if $x_{i_1}$ is universally quantified and $k = max \, i \, j$. We will write
$B \, \forall \, j \quad (j \leq k)$ if we want to emphasize the free variables.

Lemma 46. If $A$ is prenex and $G_V \not\models A$, then $G_V \not\models A^H$.

Proof. Follows from the usual laws of quantification, which are valid in all Gödel
logics.

Our next main result will be Herbrand’s theorem for $G_V$ for $V$ uncountable or finite.
The Herbrand universe $HU (\beta^F)$ of $B^F$ is the set of all variable-free terms which can
be constructed from the set of function symbols occurring in $B^F$. To prevent $HU (\beta^F)$
from being finite or empty we add a constant and a function symbol of positive arity
if no such symbols appear in $B^F$. The Herbrand base $HB (\beta^F)$ is the set of atoms
constructed from the predicate symbols in $B^F$ and the terms of the Herbrand universe.
In the next theorem we will consider the Herbrand universe of a formula $\exists \beta B^F (\exists)$. We
fix a non-repetitive enumeration $C_1, C_2, \ldots$ of $HB (\beta^F)$, and let $X := \exists ? \, C_1; \ldots ; C_i > g$
(we may take $>$ to be a formula which is always $=$ 1). $B^F (\exists)$ is an $\exists$-instance of $B^F (\exists)$
if the atomic subformulas of $B^F (\exists)$ are in $X$.

Definition 47. An $\exists$-constraint is a non-strict linear ordering of $X$, s.t. $?$ is minimal
and $>$ is maximal. An interpretation $I$ fully satisfies the $\exists$-constraint provided for all
$C \in C^0 \, X$, $C \in C^0$ if $I (C) \not\models I (C)$. We say that the constraint $\exists^0$ on $X + 1$ extends
if for all

\[ C \in C^0 \, X, \quad C \in C^0 \text{iff } C \not\models 0 C^0. \]

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Lemma 3 showed that if \( h : V \to W \) is a \( G \)-embedding and \( \mathcal{I} \) is a \( V \)-interpretation, then \( h (\mathcal{I} (\mathcal{A})) = \mathcal{I}_h (\mathcal{A}) \) for any formula \( \mathcal{A} \). If no quantifiers are involved in \( \mathcal{A} \), this also holds without the requirement of continuity. For the following proof we need a similar notion. Let \( V \) be a Gödel sets, \( X \) a set of atomic formulas, and suppose there is an order-preserving, strictly monotone \( h : \mathcal{I} (\mathcal{C}) : C \to X \) which is so that \( h (1) = 1 \) and \( h (\emptyset) = 0 \). Call any such \( h \) a truth value injection on \( X \). Now suppose \( B \) is a quantifier-free formula, and \( X \) its set of atomic subformulas. Two interpretations \( \mathcal{I}, \mathcal{J} \) are compatible on \( X \) if \( \mathcal{I} (\mathcal{C}) = \mathcal{J} (\mathcal{C}) = \mathcal{C} (0) \) for all \( C \in \mathcal{C} X \).

**Proposition 48.** Let \( B^F \) be a quantifier free formula, and \( X \) its set of atomic subformulas together with \( >, ?, \). If \( \mathcal{I}, \mathcal{J} \) are compatible on \( X \), then there is a truth value injection \( h \) on \( X \) with \( h (\mathcal{I} (\mathcal{C})) = \mathcal{J} (\mathcal{C}) \).

**Proof.** Let \( h (\mathcal{I} (\mathcal{C})) = \mathcal{J} (\mathcal{C}) \) for \( B \in \mathcal{C} X \). Since \( \mathcal{I}, \mathcal{J} \) are compatible on \( X \), \( \mathcal{I} (\mathcal{C}) = \mathcal{J} (\mathcal{C}) \) iff \( \mathcal{J} (\mathcal{C}) = \mathcal{C} (0) \), and hence \( \mathcal{I} (\mathcal{C}) = \mathcal{J} (\mathcal{C}) \) iff \( h (\mathcal{I} (\mathcal{C})) = \mathcal{J} (\mathcal{C}) \) and \( h \) is strictly monotonic. The conditions \( h (\emptyset) = 0 \) and \( h (1) = 1 \) are satisfied by definition, since \( >, ?, \in \mathcal{C} X \). We get \( h (\mathcal{J} (\mathcal{C})) = \mathcal{J} (\mathcal{C}) \) by induction on the complexity of \( \mathcal{C} \).

**Proposition 49.** (a) If \( \mathcal{I} \) extends \( \mathcal{J} \), then every \( \mathcal{I} \) which fulfills \( \mathcal{I} \) also fulfills \( \mathcal{J} \). (b) If \( \mathcal{I} \), \( \mathcal{J} \) fulfill the \( \mathcal{C} \)-constraint and there is a truth value injection \( h \) on \( X \) with \( h (\mathcal{I} (\mathcal{C})) = \mathcal{J} (\mathcal{C}) \) for all \( \mathcal{I} \)-instances \( B^F \mathcal{C} \) of \( B^F \mathcal{C} \); in particular, \( \mathcal{I} (\mathcal{C} (0)) = 1 \) if \( h (\mathcal{I} (\mathcal{C})) = \mathcal{J} (\mathcal{C}) \).

**Proof.** (a) Obvious. (b) Follows from Proposition 48 together with the observation that \( \mathcal{I} \) and \( \mathcal{J} \) both fulfill \( \mathcal{C} \)-constraint.

**Lemma 50.** Let \( B^F \) be a quantifier-free formula, and let \( V \) be a finite or uncountably infinite Gödel set. If \( \mathcal{G}_V \models \exists \mathcal{I} \mathcal{B}^F \mathcal{C} \mathcal{T} \) then there are tuples \( \mathcal{T}_1, \ldots, \mathcal{T}_n \) of terms in \( U \mathcal{B}^F \mathcal{T} \), such that \( \mathcal{G}_V \models \mathcal{I}_h \mathcal{B}^F \mathcal{T}_i \).

**Proof.** Suppose first that \( V \) is uncountable. By Theorem 52 \( V \) contains a dense linear subordering. We construct a “semantic tree” \( \mathcal{T} \); i.e., a systematic representation of all possible order types of interpretations of the atoms \( \mathcal{C}_i \) in the Herbrand base. \( \mathcal{T} \) is a rooted tree whose nodes appear at levels. Each node at level \( \mathcal{C} \) is labelled with an \( \mathcal{C} \)-constraint.

\( \mathcal{T} \) is constructed in levels as follows: At level \( 0 \), the root of \( \mathcal{T} \) is labelled with the constraint \( \mathcal{C} \). Let \( v \) be a node added at level \( \mathcal{C} \) with label \( \mathcal{C} \), and let \( \mathcal{T} \) be the set of terms occurring in \( \mathcal{X} \). Let \( (*) \) be: For every interpretation \( \mathcal{I} \) which fulfills \( \mathcal{I} \), there is some \( \mathcal{C} \)-instance \( B^F \mathcal{T} \) so that \( \mathcal{I} (B^F \mathcal{T}) = 1 \). If \( (*) \) obtains, \( v \) is a leaf node of \( \mathcal{T} \), and no successor nodes are added at level \( \mathcal{C} \).

Note that by Proposition 49(b), any two interpretations which fulfill \( (*) \) make the same \( \mathcal{C} \)-instances of \( B^F \mathcal{T} \) true; hence \( v \) is a leaf node if and only if there is an \( \mathcal{C} \)-instance \( A \mathcal{T} \) s.t. \( \mathcal{I} (A \mathcal{T}) = 1 \) for all interpretations \( \mathcal{I} \) that fulfill \( (*) \).

If \( (*) \) does not obtain, for each \( \mathcal{C} \)-constraint \( \mathcal{C} \)-extending \( \mathcal{T} \) we add a successor node \( v' \) labelled with \( \mathcal{C} \)-extension \( \mathcal{T} \) at level \( \mathcal{C} \).

We now have two cases:

1. \( \mathcal{T} \) is finite. Let \( v_1, \ldots, v_m \) be the leaf nodes of \( \mathcal{T} \) of levels \( \mathcal{C}_1, \ldots, \mathcal{C}_m \), each labelled with a constraint \( \mathcal{C}_1, \ldots, \mathcal{C}_m \). By \( (*) \), for each \( j \) there is an \( \mathcal{C}_j \)-instance \( B^F \mathcal{T}_j \)
with \( I(\varphi^E_\bar{v}) = 1 \) for all \( I \) which fulfill \( j \). It is easy to see that every interpretation fulfills at least one of the \( j \). Hence, for all \( I, I(\varphi^E_\bar{v}_1)_-^{\ldots}_-^{\ldots}_-^{\ldots}_-^m B^E \bar{v}_m) = 1 \), and so \( G^V \upharpoonright \bar{v} \upharpoonright B^E \bar{v} \).

(2) \( T \) is infinite. By König’s lemma, \( T \) has an infinite branch with nodes \( v_0, v_1, v_2, \ldots \) where \( v_n \) is labelled by \( \downarrow \) and is of level \( n \). Each \( v_n \) extends \( v_{n-1} \), hence we can form \( \bar{v}_n \). Let \( V_0 \) be a non-trivial densely ordered subset of \( V \), let \( V_0 \upharpoonright 3 c < 1 \), and let \( V_0 = V_0 \upharpoonright \{ \varphi \} \). \( V_0 \) is clearly also densely ordered. Now let \( V_c \) be \( V_0 \upharpoonright \{ \varphi \} \), and let \( h : B(\varphi) \mid [ ? ] \uparrow \gamma \). \( V_c \) be an injection which is so that, for all \( A_1 \varphi A_j \neq B(\varphi) \), \( h(\varphi_i) = h(\varphi_j) \) iff \( A_i = A_j \), \( h(\varphi) = 0 \) and \( h(\varphi) = 1 \). We define an interpretation \( I \) by: \( f^I \varphi_1 \ldots \varphi_n = f \varphi_1 \ldots \varphi_n \) for all \( \gamma \)-ary function symbols \( f \) and \( P^I \varphi_1 \ldots \varphi_n = h(\varphi_1 \ldots \varphi_n) \) for all \( \gamma \)-ary predicate symbols \( P \) (clearly then, \( I(\varphi) = h(\varphi) \)). By definition, \( I \) \( \gamma \)-fulfills \( \gamma \) for all \( \gamma \). By (*) \( I(\bigcap I) < 1 \) for all \( \gamma \)-instances \( A(\varphi) \) of \( A \), and by the definition of \( V_c \), \( I(\bigcap I) < c \). Since every \( A(\varphi) \) with \( T \upharpoonright U(\varphi) \) is an \( \gamma \)-instance of \( A \) for some \( \gamma \), we have \( I(\exists ! x A(\varphi)) \downarrow \gamma = 1 \). This contradicts the assumption that \( G^V \uparrow \bigcap \exists ! x A(\varphi) \).

If \( V \) is finite, the proof is the similar, except simpler. Suppose \( V \uparrow \gamma = n \). Call a constraint \( \gamma \)-admissible if there is some \( V \)-interpretation \( I \) which fulfills it. Such have no more than \( n \) equivalence classes under the equivalence relation \( C \equiv C \bigcup C \bigcup C \ldots \bigcup C \). In the construction of the semantic tree above, replace each mention of \( \gamma \)-constraints by \( \gamma \)-admissible \( \gamma \)-constraints. The argument in the case where the resulting tree is finite is the same. If \( T \) is infinite, then the resulting order \( \gamma \)-admissible, since all \( \gamma \) are. Let \( c = \max \{ \varphi \} \downarrow \gamma \) \( V \downarrow \gamma = 1 \) and \( V_c = V \). The rest of the argument goes through without change.

\[ \text{Lemma 51.} \quad \text{Let } \bigcap B^E(\varphi) \text{ be the Herbrand form of the prenex formula } A, \text{ and let } \bar{v}_1 \ldots \bar{v}_m \text{ be tuples of terms in } \text{HU}(\varphi^E). \text{ If } G^V \uparrow \bigcap_{v_1} B^E \bar{v}_m, \text{ then } G^V \uparrow A. \]

**Proof.** For any Gödel set \( V \), the following rules are valid in \( G^V \):

1. \( A \uparrow \gamma \) \( \downarrow \bigcup B \uparrow \gamma A \).
2. \( \langle A \uparrow \gamma \rangle \downarrow \bigcup C \downarrow \bigcup A \uparrow \bigcup \varphi \downarrow \bigcup C \).
3. \( \langle A \uparrow \gamma \rangle \downarrow \bigcup B \downarrow \bigcup A \uparrow \bigcup B \).
4. \( A \uparrow \gamma \) \( \downarrow \bigcup 8x A \uparrow \gamma \).
5. \( A \uparrow \gamma \) \( \downarrow \bigcup 9x A \uparrow \gamma \).
6. \( 8x \langle A \uparrow \gamma \rangle \downarrow \bigcup 8x A \uparrow \gamma \).
7. \( 9x \langle A \uparrow \gamma \rangle \downarrow \bigcup 9x A \uparrow \gamma \).

\( (x \) is not free in \( B. \) \) The result follows from [BCF01], Lemma 6, and are also easily verified directly.

\[ \text{Theorem 52.} \quad \text{Let } A \text{ be prenex, } \bigcap B^E(\varphi) \text{ its Herbrand form, and let } V \text{ be a finite or uncountably infinite Gödel set. Then } G^V \uparrow A \text{ if and only if there are tuples } \bar{v}_1 \ldots \bar{v}_m \text{ of terms in } \text{HU}(\varphi^E), \text{ such that } G^V \uparrow \bigcap_{v_1} B^E \bar{v}_m. \]

**Proof.** If: This is Lemma 51. Only if: By Lemma 56 and Lemma 50.

**Remark.** An alternative proof of Herbrand’s theorem can be obtained using the analytic calculus \( \text{HIF} \) (“Hypersequent calculus for Intuitionistic Fuzzy logic”) [BZ00].
Theorem 53. The prenex fragment of a Gödel logic based on a truth value set \( V \) which is either finite or uncountable infinite is axiomatizable. An axiomatization is given by the standard axioms and rules for \( \text{LC} \) extended by the rules (4)–(7) of the proof of Lemma 52. For the \( m \)-valued case add the characteristic axiom for \( G_m \) and
\[
W_1 m_{j=1}^{m+1} (A_j) \Leftrightarrow (A_j \vee A_i)
\]
Proof. Completeness: Let \( \overline{\psi} \) be a prenex formula valid in \( G_V \). By Theorem 52, a Herbrand disjunction \( n_{i=1}^n B_i \overline{\xi_i} \) is a tautology in \( G_V \). Hence, it is provable in \( \text{LC} \) or \( \text{LC} + G_m \) [Got01, Chapter 10.1]. \( \overline{\psi} \) is provable by Lemma 51.

Soundness: The rules in the proof of Lemma 51 are valid in \( G_V \). In particular, note that \( \exists x (A (x) \land B) \) and \( \exists A (x) \) with \( x \) not free in \( B \) is valid in all Gödel logics, and \( \exists x (A (x) \land B) \) is intuitionistically valid.

In Theorem 56 we showed that for every first-order formula \( A \), there is a formula \( A^8 \) which is valid in \( G_V \) for \( V \) countably infinite if \( A \) is valid in every finite classical interpretation. We now strengthen this result to show that the prenex fragment of \( G_V \) (for \( V \) countably infinite) is likewise not axiomatizable. This is done by showing that if \( A \) is prenex, then there is a formula \( A^G \) which is also prenex and which is valid in \( G_V \) if \( A \) is. Note that not all quantifier shifting rules are generally valid in Gödel logics, so we have to show that for the particular case of formulas of the form of \( A^8 \), there is a prenex formula which is valid in \( G_V \) if \( A^8 \) is.

Theorem 54. If \( V \) is countably infinite, the prenex fragment of \( G_V \) is not r.e.

Proof. By the proof of Theorem 56, a formula \( A \) is true in all finite models if \( G_V \models A^8 \). \( A^8 \) is of the form \( B \) (\( A^0 \land \forall u P (u) \)). We show that \( A^8 \) is validity-equivalent in \( G_V \) to a prenex formula.

From Lemma 35, we see that each crisp formula is equivalent to a prenex formula; let \( A_0 \) be a prenex form of \( A \). Since all quantifier shifts for conjunctions are valid, the antecedent of \( B \) of \( A^8 \) is equivalent to a prenex formula \( \forall x_1 \ldots \forall x_n B_0 (x_1, \ldots, x_n) \). Hence, \( A^8 \) is equivalent to \( \forall \forall x B_0 (\overline{\xi}) \) (\( A_0 \land \forall u P (u) \)).

Let \( \overline{\psi} \) be of the form \( \forall Q_1 \ldots \forall Q_j \forall B_0 (\overline{\xi}) \) (\( \forall B_0 (\overline{\xi}) \) is equivalent to \( \forall \forall x B_0 (\overline{\xi}) \) by induction on \( n \). Let \( \forall \forall x B_0 (Q_1 x_1 \ldots Q_j x_i B_0 (\overline{\xi}) \overline{d} x_c) \). Since quantifier shifts for 9 in the antecedent of a conditional are valid, we only have to consider the case \( Q_1 = 9 \). Suppose \( \exists (x_0 B_1 (\overline{\xi}) x_i) \) \( \not\models \exists (x_0 B_1 (\overline{\xi}) x_i) \) \( (C) \). This only happen if \( \exists (x_0 B_1 (\overline{\xi}) x_i) \) \( \models (C) < 1 \) but \( \exists (B_1 (\overline{\xi}) x_i) \) \( > (C) \) \( \forall c \). However, it is easy to see by inspecting \( B \) that \( \exists (B_1 (\overline{\xi}) x_i) \) is either \( 1 \) or \( c \).

Now we show that \( \exists (B_0 (\overline{\xi}) \exists (A_0 \land \forall u P (u))) \not\models (A_0 \land \forall u P (u))) \). If \( A_0 = 0 \), then both sides equal \( 1 \). If \( A_0 = 0 \), then \( \exists (A_0 \land \forall u P (u)) = v \). The only case where the two sides might differ is if \( \exists (B_0 (\overline{\xi}) x_i) \models (V (P (\overline{\psi})) \not\models (P (\overline{\psi})) \not< \forall c \). But inspection of \( B_0 \) shows that \( (B_0 (\overline{\xi}) x_i) = 1 \) or \( (P (\overline{\psi}) \not< \forall c \) for some \( d (\overline{\xi}) x_i \) which do not appear negated are of the form \( c (\overline{\xi}) x_i \). Hence, if \( \exists (B_0 (\overline{\xi}) x_i) = v \), then for some \( e \), \( x_i < (P (\overline{\psi})) \).

Last we consider the quantifiers in \( A_0 \). Since \( A_0 \) is crisp, \( \exists (B_0 (\overline{\xi}) \exists (A_1 \land \forall u P (u))) \) \( \exists (A_1 \land \forall u P (u))) \) (for all \( d \)). To see this, first note that shifting quantifiers across \( \land \) and shifting universal quantifiers out of the consequent of
a conditional is always possible. Hence it suffices to consider the case of \( \exists y A(2) \) is either = 0 or = 1. In the former case, both sides equal \( I(\forall l(\mathcal{B}) ! \ P l) \), in the latter, both sides equal 1.

In summary, we obtain the following characterization of axiomatizability of prenex fragments of Gödel logics:

**Theorem 55.** The prenex fragment of \( G_V \) is axiomatizable if and only if \( V \) is finite or uncountable. The prenex fragments of any two \( G_V \) where \( V \) is uncountable coincide.

### 6.2 \( ? -free fragments

In the following we will denote the \( ? -free fragment of \( G_V \) with \( G_V^? \). \( G_V^? \) is the set of all \( G_V \)-valid formulas which do not contain \( ? \) (and hence also no \( : \)). First we show that the only candidates for r.e. fragments are the \( ? -free fragments of \( G_V \) where \( V \) is uncountable and either 0 2 \( V^\infty \) or 0 is isolated \( V \).

**Lemma 56.** If \( G_V \) is not r.e., then \( G_V^? \) is also not r.e.

Define \( A^b \) as the formula obtained from \( A \) by replacing all occurrences of \( ? \) with the new propositional variable \( b \) (a 0-place predicate symbol). Then define \( A \) as

\[
A = \bigwedge_{P \in A} \forall \overline{x} (P \overline{x}) \rightarrow A^b
\]

where \( P \in A \) means that \( P \) ranges over all predicate symbols occurring in \( A \). We will first prove a lemma relating \( A \) and \( A^b \):

**Lemma 57.**

\( G_V \not\models A \iff G_V^? \not\models A \)

**Proof.** If: Replace \( b \) by \( ? \).

Only if: Suppose \( G_V^? \not\models A \). Thus, there is an interpretation \( I_0 \) such that \( I_0(A) < 1 \). By Proposition 11 and Lemma 12 there is an interpretation \( I \) such that \( I(A^b) < 1 \) and \( I\left(\bigwedge_{P \in A} \forall \overline{x} (P \overline{x}) \rightarrow A^b\right) = 1 \). Because of the latter, for every atomic subformula \( B \) of \( A \), \( I(B) = v \). Define \( I^0(B) \) for atomic subformulas \( B \) of \( A \) by

\[
I^0(B) = \begin{cases} 0 & I(B) = v \\ I(B) & I(B) > v \end{cases}
\]

(and arbitrary for other atomic formulas). It is easily seen by induction that \( I^0(B) = I(B) \) if \( I(B) > v \), and if \( I(B) = v \), then \( I^0(B) = I^0(A^b) = I^0(A) \). In particular, \( I^0(A^b) < 1 \). But, of course, \( I^0(? ) = I^0(? ) = 1 \), and hence \( I^0(A^b) = I^0(A) \).

**Proof of Lemma 57.** If \( G_V^? \) were recursively enumerable, then by Lemma 57, \( G_V \) would also be recursively enumerable.

\( \square \)
Thus, by Theorem 36, we only have two candidates for axiomatizable ?-free fragments: both truth-value sets have a non-empty perfect kernel \( P \), and in the one case \( 0 \notin P \) and in the other \( 0 \notin P \) but 0 is isolated. The prototypical Gödel sets for these cases are \( V_1 = \{ 0; 1 \} \) and \( V_2 = \{ 0 \} \). We will show that the ?-free fragments of these two logics coincide, thus in fact proving that there is only one axiomatizable ?-free fragment.

**Lemma 58.** Let \( V_1 = \{ 0; 1 \} \) and \( V_2 = \{ 0 \} \). The ?-free fragments of \( G_{V_1} \) and \( G_{V_2} \) coincide, i.e.

\[
G_{V_1}^\emptyset \not\vdash A \iff G_{V_2}^\emptyset \not\vdash A
\]

**Proof.** Only if: obvious, since a counter-example in \( V_2 \) actually also is a counter-example in \( V_1 \).

If: Suppose that \( G_{V_1}^\emptyset \not\vdash A \), i.e., there is an \( I_1 \) such that \( I_1 (A) < 1 \). Define \( I_2 \) for all atomic subformulas \( B \) of \( A \) by \( I_2 (B) = 1 = 2 (1 + I_1 (B)) \). By Lemma 7 and the remark following it we see that the definition of \( I_2 \) extends to all formulas. \( \square \)

**Theorem 59.** The ?-free fragment of \( G_V \) is recursively axiomatizable if and only if \( V \) is finite or uncountable and either 0 belongs to \( V^\infty \) or is isolated. The ?-free fragment of any two such \( V \) coincide.

**Proof.** From Lemma 58, Lemma 55, and Theorem 33 for the uncountable case. The finite case is obvious as the additional axioms \( \text{FIN} (\emptyset) \) do not contain ?.

### 6.3 \( \emptyset \)-free fragments

In the following we will denote the \( \emptyset \)-fragment of \( G_V \) with \( G_V^\emptyset \). It is the set of all formulas valid in \( G_V \) which do not contain \( \emptyset \).

First we show, as in the case of the ?-free fragment, that the only candidates for axiomatizable fragments are the two uncountable ones, \( 0 \notin P \) and 0 isolated. We will do this by showing that the formulas used to reduce validity in the other cases to Trachtenbrodt’s Theorem are validity-equivalent to \( \emptyset \)-free formulas.

**Lemma 60.** If \( A (\emptyset) \) and \( B \) are \( \emptyset \)-free, then

\[
\not\vdash \exists x A (\emptyset) \iff \not\vdash \exists x (A (\emptyset) \lor B)
\]

**Proof.** If: This is a valid quantifier shift rule.

Only if: Suppose that \( \not\exists x (A (\emptyset) \lor B) \), i.e., there is an interpretation \( I \) such that \( I (\exists x (A (\emptyset) \lor B)) < 1 \). But this implies that

\[
\emptyset u \ni U \quad I (A (\emptyset)) \ni I (B) \quad \text{if } I (Q) \ni I (B)\quad \text{if } I (Q) \ni I (B)
\]

Now define \( I^\emptyset (Q) \) for atomic subformulas \( Q \) of \( A \) by

\[
I^\emptyset (Q) = \begin{cases} 
I (Q) & \text{if } I (Q) \ni I (B) \\
1 & \text{if } I (Q) \ni I (B)
\end{cases}
\]
Then (i) If \( C \) is \( 8 \)-free and \( I(C) > I(B) \), then \( I^0(C) = 1 \), and if \( I(C) < I(B) \), then \( I^0(C) = I(C) \); and (ii) \( I^0(\exists x A(\kappa)) = 1 \)

(i) For atomic \( C \) this is the definition of \( I^0 \). The cases for \( , , \), and \( ! \) are trivial.

Now let \( C \equiv \exists x D(\kappa) \). If \( I(\exists x D(\kappa)) > I(B) \), then for some \( u \in U^1 \), \( I(\exists D(\kappa)) > I(B) \). By induction hypothesis, \( I^0(\exists D(\kappa)) = 1 \) and hence \( I^0(\exists x D(\kappa)) = 1 \). Otherwise, \( I(\exists x D(\kappa)) \equiv I(B) \), in which case \( I^0(\exists D(\kappa)) = I^0(\exists D(\kappa)) \) for all \( u \). (ii) By [2], for all \( u \in U \), \( I(A(\kappa)) > I(B) \), hence, by (i), \( I^0(A(\kappa)) = 1 \).

By (i) and (ii) we have that \( I^0(\exists x A(\kappa)) = 1 \) and \( I^0(B) = I(B) < 1 \), thus \( I^0(\exists x A(\kappa) \lor B) < 1 \), i.e., \( 2 \exists x A(\kappa) \lor B \).

\[ \square \]

Note that in the preceding Lemma we can replace the prefix of \( A(\kappa) \) by a string of universal quantifiers and the same proof will work.

**Lemma 61.** If \( G_V \) is not recursively enumerable, then also \( G_V^0 \).

**Proof.** It is sufficient to show that Formula [7] for \( A^F \) as given on page 16 and Formula [8] for \( A^h \) as given on page 22 are validity-equivalent to \( 8 \)-free formulas.

If we only consider the quantifier structure of these formulas and apply valid quantifier shifting rules, including the shifting rule for crisp formulas given in Lemma 55 we obtain in both cases formulas which are of the form

\[ 8 \exists A(\bar{\kappa}) \lor B \]

where \( A(\bar{\kappa}) \) and \( B \) are \( 8 \)-free. By Lemma 60 we see that both formulas are validity equivalent to \( 8 \)-free formulas.

\[ \square \]

As for the \( ? \)-free fragments, it turns out that the two prototypical examples of Gödel sets create the same \( 9 \)-fragment:

**Lemma 62.** Let \( V_1 = \{ 0; 1 \} \) and \( V_2 = \{ \varnothing; \{ 1 \} \} \). The \( 9 \)-fragments of \( G_{V_1} \) and \( G_{V_2} \) coincide, i.e.

\[ G_{V_1}^9 \models A \iff G_{V_2}^9 \models A \]

**Proof.** Only if: obvious, since a counter-example in \( V_2 \) actually also is a counter-example in \( V_1 \).

If: Suppose that \( G_{V_1}^9 \models A \), i.e., there is an \( I_1 \) such that \( I_1(A) < 1 \). Define \( I_2 \) for all atomic subformulas \( B \) of \( A \) by \( I_2(B) = 1 \) if \( I_1(B) > 0 \) and \( = 0 \) if \( I_1(B) = 0 \). By Lemma 7 and the remark following it we see that the definition of \( I_2 \) extends to all formulas.

\[ \square \]

**Theorem 63.** The \( 9 \)-fragment of \( G_V \) is r.e. if and only if \( V \) is finite or uncountable and either \( 0 \) belongs to \( V^\omega \) or is isolated. The \( 9 \)-fragment of any two such \( V \) coincide.

**Proof.** From Lemma 61, Lemma 62 and Theorem 53 for the uncountable case. The finite case is obvious as the additional axioms \( \text{FIN} \{ \eta \} \) do not contain universal quantifiers.

\[ \square \]
7 Conclusion

In the preceding sections, we have given a complete characterization of the r.e. and non-r.e. first-order Gödel logics. Our main result is that there are two distinct r.e. infinite-valued Gödel logics, viz., $G_R$ and $G^0$. What we have not done, however, is investigate how many non-r.e. Gödel logics there are. It is known that there are continuum-many different propositional consequence relations and continuum-many different propositional quantified Gödel logics [BV00]. In forthcoming work [BGP], it is shown that there are only countably many first-order Gödel logics. Although this result goes some way to clarifying the situation, a criterion of identity of Gödel logics using some topological property of the underlying truth value set is a desideratum. We have only given (Lemma 9) a sufficient condition: if there is a continuous bijection between $V$ and $V^0$, then $G_V = G_V^0$. But this condition is not necessary: any pair of non-isomorphic uncountable Gödel sets with 0 contained in the perfect kernel provides a quick counterexample (as any two such sets determine $G_R$ as their logic). Such a topological characterization of first-order infinite valued Gödel logics could then be used to obtain a more fine-grained analysis of the complexity of the non-r.e. Gödel logics. As noted already, these also differ in the degree to which they are non r.e. [Haj05]

Another avenue for future research would be to carry out the characterization offered here for extensions of the language. Candidates for such extensions are the addition of the projection modalities ($4_a = 0$ if $a = 1$ and $= 1$ if $a < 1$), of the globalization operator of [TT86], or of the involutive negation ($a = 1 - a$). It is known that $G_R$ with the addition of these operators is still axiomatizable. The presence of the projection modality, in particular, disturbs many of the nice features we have been able to exploit in this paper, for instance, in the presence of 4 the crucial Lemma 12 and Proposition 13 no longer hold. Thus, not all of our results go through for the extended language and new methods will have to be developed.

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