NUMERICAL ANALYSIS OF TWO NEW FINITE DIFFERENCE METHODS FOR TIME-FRACTIONAL TELEGRAPH EQUATION

XIAOZHONG YANG* AND XINLONG LIU

School of Mathematics and Physics, North China Electric Power University
Beijing, 102206, China

(Communicated by Song Jiang)

Abstract. Fractional telegraph equations are an important class of evolution equations and have widely applications in signal analysis such as transmission and propagation of electrical signals. Aiming at the one-dimensional time-fractional telegraph equation, a class of explicit-implicit (E-I) difference methods and implicit-explicit (I-E) difference methods are proposed. The two methods are based on a combination of the classical implicit difference method and the classical explicit difference method. Under the premise of smooth solution, theoretical analysis and numerical experiments show that the E-I and I-E difference schemes are unconditionally stable, with $2^{nd}$ order spatial accuracy, $2 - \alpha$ order time accuracy, and have significant time-saving, their calculation efficiency is higher than the classical implicit scheme. The research shows that the E-I and I-E difference methods constructed in this paper are effective for solving the time-fractional telegraph equation.

1. Introduction. Fractional evolution equations have profound physical background and theoretical connotation. The study of numerical solutions has important scientific significance and practical engineering value. Fractional telegraph equations are widely used in signal analysis, wave propagation, random walk theory and other engineering fields as a typical fractional diffusion wave equation [8, 26, 29]. In the fractional telegraph equation model, physically speaking, the time fractional derivative describes the physical phenomenon related to the process and is called historical dependence; the space fractional derivative describes the path dependence and global correlation of the physical process and is called global dependence. At present, numerical methods and theoretical analysis for solving fractional diffusion wave equations include finite difference method, finite element method, spectral method, finite volume method, meshless method, and matrix conversion techniques, etc. [7, 11, 17, 31].

With the continuous application of the fractional telegraph equation, the solution of the fractional telegraph equation has become an urgent research task. Because

2020 Mathematics Subject Classification. Primary: 65M06; Secondary: 65M12.
Key words and phrases. Time-fractional telegraph equation, E-I difference scheme and I-E difference scheme, unconditional stability, convergence, numerical experiments.

The work was supported in part by the Subproject of Major Science and Technology Program of China (No.2017ZX07101001-01) and the National Natural Science Foundation of China (No.11371135).

* Corresponding author: Xiaozhong Yang.
the analytical solutions of fractional telegraph equations are difficult to give explicitly, even the analytical solutions of linear fractional telegraph equations also contain special functions, such as the Mittag-Leffler function. Because the series corresponding to these functions converge slowly, the calculation of these special functions is also quite difficult in practical applications, which makes the development of efficient numerical algorithms for fractional telegraph equations particularly important [5, 13, 16].

At present, the related research on the problem of fractional telegraph equations is still very limited. R. C. Cascaval et al.(2002) [4] discussed the use of the Riemann-Liouville method to study the well-posedness and asymptotic behavior of time-fractional telegraph equations; E. Orsingher and L. Beghin(2004) [24] studied the fundamental solutions to the time-fractional telegraph equations of order $2\alpha$. For the special case $\alpha = 1/2$, they also showed that the fundamental solution is the probability density of a telegraph process with Brownian time; S. Momani(2005) [23] used the Adomian decomposition to give the analytical and approximate solutions of the space-time fractional telegraph equation; A. Yildirim(2010) [35] used homotopy perturbation method (HPM) to obtain analytic and approximate solutions of the space and time fractional telegraph equations; S. Das et al.(2011) [6] used homotopy analysis method (HAM) to solve the telegraph equation with fractional time derivative $\alpha (1 < \alpha < 2)$. By using initial values, the explicit solutions of telegraph equation for different particular cases have been derived; Z. Zhao and C. Li(2012) [36] studied the finite element method of fractional telegraph equations and gave detailed theoretical analysis; N. J. Ford et al.(2013) [10] constructed a finite difference method using Hadamard finite-part integral to solve the fractional telegraph equation and considered the stability of the difference method; L. Wei et al.(2014) [32] used the intermittent Galrekin finite element method to solve the fractional order telegraph equation, gave a detailed analysis process of prior error estimation, and numerical experiments verified the theoretical results; V. R. Hosseini et al.(2014) [14] used radial basis functions combined with finite difference methods to numerically solve the fractional telegraph equation; A. Saadatmandi and M. Mohabbati(2015) [25] proposed a calculation technique based on the orthogonal Legendre polynomials and the Tau method to find the numerical solution of the fractional telegraph equation; M. O. Mamchuev(2017) [22] obtained a general representation of the regular solution of the rectangular domain according to the appropriate Green function construction, and gave the basic solution; M. Ferreira et al.(2018) [9] used Laplace or Dirac operators to obtain the first and second fundamental solutions (FS) of the high-dimensional time-fractional equation.

This paper solves the time fractional telegraph equation. Firstly, we use the $L_1$ formula to approximate the time fractional derivative, the central difference discrete space derivative, and establish the classical explicit scheme and classical implicit scheme. Secondly, explicit-implicit (E-I) difference scheme and implicit-explicit (I-E) difference scheme are constructed on this basis. We combine the Fourier method and mathematical induction method to give the stability and convergence analysis of the difference scheme solution. Finally, numerical experiments verify the theoretical analysis conclusions, showing that the difference schemes we constructed are effective for solving the time-fractional telegraph equation. The explicit-implicit alternating schemes show the numerical advantages of symmetric difference discreteness. The two classical difference schemes which are complementary to each other are mixed. The numerical errors of the explicit scheme and the
2. E-I difference scheme for time fractional telegraph equation.

2.1. Time fractional telegraph equation. Consider the following time-fractional telegraph equation [5, 16, 29]:

\[
\begin{cases}
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \lambda \frac{\partial^\nu u(x,t)}{\partial x^\nu} = \nu \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), 0 < x < L, 0 < t < T; \\
u(0,t) = 0, \quad u(0,t) = \phi_0(x), \quad u(x,0) = \phi_1(x), 0 \leq x \leq L; \\
u(0,t) = \mu_0(t), u(L,t) = \mu_1(t), t \geq 0.
\end{cases}
\]

(1)

Where: \( f(x,t), \phi_0(x), \phi_1(x), \mu_0(t), \mu_1(t) \) are sufficiently smooth prescribed functions. \( \lambda \) is an arbitrary nonnegative constant and \( \nu \) is an arbitrary positive constant.

Also 0 < \( \alpha < 1 \). \( \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}, \frac{\partial^\nu u(x,t)}{\partial x^\nu} \) is the fractional derivative in the sense of Caputo:

\[
\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\xi)}{\partial \xi} \left( t-\xi \right)^{\alpha-1} d\xi,
\]

\[
\frac{\partial^\nu u(x,t)}{\partial x^\nu} = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{\partial^2 u(x,\xi)}{\partial \xi^2} \left( t-\xi \right)^{2\alpha-1} d\xi.
\]

2.2. Construction of E-I difference scheme. In order to obtain the difference scheme of the equation (1) to be solved, first a meshing is performed on the solution area: take the space step \( h = \frac{L}{M} \), \( x_i = ih, i = 0, 1, 2, ..., M \) and time step \( \tau = \frac{T}{N} \), \( t_n = n\tau, n = 0, 1, 2, ..., N; M, N \) are positive integers. Divide the solution area into a rectangular grid, the grid nodes are \( (x_i, t_n) \).

The time fractional derivative can be approximated by L1 formula [18, 19, 21]:

\[
\frac{\partial^\alpha u(x_i, t_{n+1})}{\partial t^\alpha} \approx \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{n} \frac{u(x_i, t_{j+1}) - u(x_i, t_j)}{\tau} \int_{j\tau}^{(j+1)\tau} \frac{d\xi}{(t_{n+1} - \xi)^\alpha} \approx \frac{\tau^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \sum_{j=0}^{n} \frac{u(x_i, t_{n+1-j}) - u(x_i, t_{n-j})}{\tau} [(j+1)^{1-\alpha} - j^{1-\alpha}] \quad (2)
\]

Similarly,

\[
\frac{\partial^{2\alpha} u(x_i, t_{n+1})}{\partial t^{2\alpha}} \approx \frac{1}{\Gamma(3-2\alpha)} \sum_{j=0}^{n} \frac{u(x_i, t_{n+1-j}) - 2u(x_i, t_{n-j}) + u(x_i, t_{n-j-1})}{\tau^2} \int_{j\tau}^{(j+1)\tau} \frac{d\xi}{(t_{n+1} - \xi)^{2\alpha-1}} \quad (3)
\]

\[
= \frac{\tau^{2-2\alpha}}{\Gamma(3-2\alpha)} \sum_{j=0}^{n} \frac{u(x_i, t_{n+1-j}) - 2u(x_i, t_{n-j}) + u(x_i, t_{n-j-1})}{\tau^2} [(j+1)^{2-2\alpha} - j^{2-2\alpha}]
\]
Second, the classical implicit scheme is

\[ \frac{-\tau^{-2\alpha}}{\Gamma(3 - 2\alpha)} \sum_{j=0}^{n} [u(x_i, t_{n+1-j}) - 2u(x_i, t_{n-j}) + u(x_i, t_{n-j-1})][(j + 1)^{2-2\alpha} - j^{2-2\alpha}], \]

The term \( u_{i, j}^{-1} \) will appear in the above formula when \( j = 0 \) or \( j = n \), using the initial speed condition to approximate the term through the central difference formula, we get the following result \( u_{i, j}^{-1} = u_i^1 \).

Let \( u_i^n \) and \( u(x_i, t_n) \) be the numerical and exact solutions of equation (1), \( f_i^n = f(x_i, t_n) \), and suppose:

\[ \omega = \frac{\tau^{-2\alpha}}{\Gamma(3 - 2\alpha)}, \mu = \frac{\tau^{-\alpha}}{\Gamma(2 - \alpha)}, r = \frac{\mu}{h^2}, \]

\[ c_j = (j + 1)^{1-\alpha} - j^{1-\alpha}, d_j = (j + 1)^{2-2\alpha} - j^{2-2\alpha}, j = 0, 1, ..., k - 1. \]

Perform a second-order central difference approximation to \( \frac{\partial^2 u(x, t)}{\partial x^2} \) in the equation (1), for simplicity, the following notation is defined [28, 33]:

\[ \delta_x^2 u_i^n = \frac{1}{h^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n). \]

In order to construct the E-I scheme of the time fractional telegraph equation (1), the classical explicit scheme and classical implicit scheme of equation (1) are given.

At first, the classical explicit scheme is

\[ \frac{\partial^{2\alpha} u(x_i, t_{n+1})}{\partial t^{2\alpha}} + \lambda \frac{\partial^{\alpha} u(x_i, t_{n+1})}{\partial t^{\alpha}} = \nu \delta_x^2 u_i^n + f_i^n, \]

the above scheme is simplified as

\[ \omega \sum_{j=0}^{n} d_j (u_i^{n+1-j} - 2u_i^{n-j} + u_i^{n-j-1}) + \mu \sum_{j=0}^{n} c_j (u_i^{n+1-j} - u_i^{n-j}) = r(u_{i+1}^n - 2u_i^n + u_{i-1}^n) + f_i^n. \]  

(4)

Second, the classical implicit scheme is

\[ \frac{\partial^{2\alpha} u(x_i, t_{n+1})}{\partial t^{2\alpha}} + \lambda \frac{\partial^{\alpha} u(x_i, t_{n+1})}{\partial t^{\alpha}} = \nu \delta_x^2 u_i^{n+1} + f_i^{n+1}, \]

the above scheme is simplified as

\[ \omega \sum_{j=0}^{n} d_j (u_i^{n+1-j} - 2u_i^{n-j} + u_i^{n-j-1}) + \mu \sum_{j=0}^{n} c_j (u_i^{n+1-j} - u_i^{n-j}) = r(u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}) + f_i^{n+1}. \]  

(5)

The above scheme can be written as:

When \( n = 1 \), from (4),(5), we have

\[ (\omega + \mu)u_1^1 = [\omega(2 - 2d_1) + \mu - \mu c_1 - 2r]u_1^1 + ru_{i+1}^1 + ru_{i-1}^1 \]

\[ + [w(1 + 2d_1) + \mu c_1]u_0^0 + f_1^1, \]

\[ -ru_{i+1}^2 + (\omega + \mu + 2r)u_i^2 - ru_{i-1}^2 = [\omega(2 - 2d_1) + \mu - \mu c_1]u_i^1 \]

\[ + [w(1 + 2d_1) + \mu c_1]u_i^0 + f_i^{n+1}. \]
When \( n > 1 \), from \((4),(5)\), we have

\[
\begin{align*}
(\omega + \mu)u_i^{n+1} &= [\omega(2 - d_1) + \mu(1 - c_1) - 2r]u_i^n + ru_{i+1}^n + ru_{i-1}^n \\
&\quad + \sum_{j=1}^{n-2}[\omega(-d_j - 2d_j - d_j + 1) + \mu(c_j - c_j + 1)]u_i^{n-j} \\
&\quad + [\omega(-d_{n-2} + 2d_{n-1} - 2d_n) + \mu(c_{n-1} - c_n)]u_i^1 \\
&\quad + (-\omega d_{n-1} + 2\omega d_n + \mu c_n)u_i^0 + f_i^{n+1},
\end{align*}
\]

(6)

\[
-ru_i^{n+1} + (\omega + \mu + 2r)u_i^{n+1} - ru_{i-1}^{n+1} = [\omega(2 - d_1) + \mu(1 - c_1)]u_i^n \\
&\quad + \sum_{j=1}^{n-2}[\omega(-d_j + 2d_j - d_j + 1) + \mu(c_j - c_j + 1)]u_i^{n-j} \\
&\quad + [\omega(2d_{n-1} - d_{n-2} - 2d_n) + \mu(c_{n-1} - c_n)]u_i^1 \\
&\quad + (-\omega d_{n-1} + 2\omega d_n + \mu c_n)u_i^0 + f_i^{n+1}.
\]

(7)

Let

\[
\begin{align*}
q_0 &= \omega(2 - d_1) + \mu(1 - c_1), \\
q_j &= \omega(-d_j + 2d_j - d_j + 1) + \mu(c_j - c_j + 1), \\
a_n &= \omega(2d_{n-1} - d_{n-2} - 2d_n) + \mu(c_{n-1} - c_n), \\
b_n &= -\omega d_{n-1} + 2\omega d_n + \mu c_n.
\end{align*}
\]

The E-I scheme for time fractional telegraph equation is constructed as following:

The odd number layer is calculated by the explicit scheme

\[
(\omega + \mu)u_i^{2n+1} = (q_0 - 2r)u_i^{2n} + ru_{i+1}^{2n} + ru_{i-1}^{2n} \\
+ \sum_{j=1}^{2n-2} q_j u_i^{2n-j} + a_{2n} u_i^1 + b_{2n} u_i^0 + f_i^{2n},
\]

(8)

and the even number layer is calculated by the implicit scheme

\[
-ru_i^{2n+2} + (\omega + \mu + 2r)u_i^{2n+2} - ru_{i-1}^{2n+2} \\
= q_0 u_i^{2n+1} + \sum_{j=1}^{2n-1} q_j u_i^{2n+1-j} + a_{2n+1} u_i^1 + b_{2n+1} u_i^0 + f_i^{2n+2}.
\]

(9)

In summary, the E-I scheme of equation \((1)\) is as follows:

\[
\begin{align*}
(\omega + \mu)U_i^{2n+1} &= Bu_i^{2n} + \sum_{j=1}^{2n-2} q_j U_i^{2n-j} + a_{2n} U^1 + b_{2n} U^0 + F_i^{2n} + g_i^{2n} \\
AU_i^{2n+2} &= q_0 U_i^{2n+1} + \sum_{j=1}^{2n-1} q_j U_i^{2n+1-j} + a_{2n+1} U^1 + b_{2n+1} U^0 + F_i^{2n+2} + g_i^{2n+2}
\end{align*}
\]

(10)

Where

\[
U^k = (u_1^k, u_2^k, ..., u_{M-1}^k)^T, \quad (k = 0, 1, ..., N),
\]

\[
F^n = (f_1^n, f_2^n, ..., f_{M-1}^n)^T,
\]

\[
g^n = (-ru_0^n, 0, ..., 0, -ru_M^n)^T, \quad (n = 0, 1, ..., N),
\]
3. Numerical analysis of E-I difference method.

3.1. Existence and uniqueness of numerical solution of E-I difference method.

Theorem 3.1. The solution of the E-I difference scheme (10) for time-fractional telegraph equation is existing and unique.

Proof. For matrix $A$, since $\omega, \mu, r$ is greater than zero, it is easy to know that the first and last rows of the matrix have $|\omega + \mu + 2r| > |r|$, and that the other rows of the matrix have $|\omega + \mu + 2r| > 2|r| = |r| + |r|$. It can be seen that matrix $A$ is strictly diagonally dominant. That is, the coefficient matrix $A$ of the E-I difference method is a non-singular matrix. We complete the proof now.

3.2. Stability of E-I difference method. Substituting the explicit scheme (8) into the implicit scheme (9) and eliminating the solution of each odd time layer $u_{2n+1}^i$, we can obtain

$$
A = \begin{pmatrix}
\omega + \mu + 2r & -r & -r & \cdots & -r \\
-r & \omega + \mu + 2r & -r & \cdots & -r \\
-r & -r & \omega + \mu + 2r & -r & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& -r & -r & \cdots & \cdots & -r & \omega + \mu + 2r \\
\end{pmatrix}_{(M-1) \times (M-1)}
$$

$$
B = \begin{pmatrix}
q_0 - 2r & -r & -r & \cdots & -r \\
-r & q_0 - 2r & -r & \cdots & -r \\
-r & -r & q_0 - 2r & -r & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& \cdots & \cdots & \cdots & \cdots \\
& -r & -r & \cdots & \cdots & -r & q_0 - 2r \\
\end{pmatrix}_{(M-1) \times (M-1)}
$$

It can be seen that matrix $A$ is a non-singular matrix. We complete the proof now. \square
Assume that \( \hat{u}_i^n \) is the approximate solution of the E-I difference scheme and \( \varepsilon_i^n = \hat{u}_i^n - u_i^n \).

**Theorem 3.2.** The E-I difference scheme (10) of the time-fractional telegraph equation is unconditionally stable.

**Proof.** Use mathematical induction to prove the existence of a positive constant \( M \) such that \( \| E_k^\alpha \|_\infty \leq M \| E^0 \|_\infty, k = 1, 2, \ldots \). When \( n = 1 \), since \( \varepsilon_1^n = \varepsilon_0^n \), that is \( \| E^1 \|_\infty = \| E^0 \|_\infty \).

When \( n = 2 \), suppose \( \| \varepsilon_i^2 \| \leq \max_{1 \leq i \leq m-1} \| \varepsilon_i^2 \| \), we have

\[
(\omega + \mu)|\varepsilon_i^2| \leq -r|\varepsilon_{i+1}^2| + (\omega + \mu + 2r)|\varepsilon_i^2| - r|\varepsilon_{i-1}^2| \\
\leq | - r\varepsilon_{i+1}^2 + (\omega + \mu + 2r)\varepsilon_i^2 - r\varepsilon_{i-1}^2 | \\
= (\omega + \mu)|\varepsilon_i^0|.
\]

So \( |\varepsilon_i^2| \leq |\varepsilon_i^0| \), that is \( \| E^2 \|_\infty \leq \| E^0 \|_\infty \).

Assume that when \( n = 2s \), there is \( \| E^{2s} \|_\infty \leq M \| E^0 \|_\infty \), where \( M \) is a positive constant, then when \( n = 2s + 2 \), set \( \| \varepsilon_i^{2s+2} \| = \max_{1 \leq i \leq m-1} \| \varepsilon_i^{2s+2} \| \), we get

\[
(\omega + \mu)|\varepsilon_i^{2s+2}| \leq -r|\varepsilon_{i+1}^{2s+2}| + (\omega + \mu + 2r)|\varepsilon_i^{2s+2}| - r|\varepsilon_{i-1}^{2s+2}| \\
\leq | - r\varepsilon_{i+1}^{2s+2} + (\omega + \mu + 2r)\varepsilon_i^{2s+2} - r\varepsilon_{i-1}^{2s+2} | \\
= \left[ \frac{q_0}{\omega + \mu} (q_0 - 2r) + q_1 \right] \varepsilon_i^{2s} + \frac{q_0 r}{\omega + \mu} \varepsilon_i^{2s} + \frac{q_0 r}{\omega + \mu} \varepsilon_i^{2s+1} \\
+ \sum_{j=1}^{2s-2} (q_j + q_1 + \cdots + q_{2s-2} + a_{2s} + b_{2s}) \varepsilon_i^{2s-j} + \left( \frac{q_0 a_2}{\omega + \mu} + a_{2s+1} \right) \varepsilon_i^{1} \\
+ (\frac{q_0 b_2}{\omega + \mu} + b_{2s+1}) \varepsilon_i^{0} \\
\leq \left[ \frac{q_0}{\omega + \mu} (q_0 + q_1 + \cdots + q_{2s-2} + a_{2s} + b_{2s}) \\
+ (q_1 + q_2 + \cdots + q_{2s-1} + a_{2s+1} + b_{2s+1}) \right] \| E^0 \|_\infty \\
\leq (\omega + \mu) M \| E^0 \|_\infty.
\]

That is \( \| E^{2s+2} \|_\infty \leq M \| E^0 \|_\infty \).

We complete the proof. \( \square \)

3.3. **Convergence of E-I difference method.** We will analyze the accuracy of the E-I difference method. The scheme (8) and the scheme (9) are expanded as the Taylor series at the point \( u_i^{n+1} \). The truncation errors are \( T_1(\tau, h), T_2(\tau, h) \), respectively.

\[
\frac{\partial^\alpha u(x_i, t_{n+1})}{\partial t^\alpha} = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=0}^{n} [u(x_i, t_{n+1-j}) - u(x_i, t_{n-j})] \\
\cdot [(j + 1)^{1-\alpha} - j^{1-\alpha}] + e_i^{n+1}.
\]

The approximation of first order truncation error \( e_i^{n+1} \) bound is given in [3, 20] as \( |e_i^{n+1}| \leq C_1 \tau^{2-\alpha} \), where \( C_1 \) is a constant.
The truncation error in the explicit scheme is

\[
\frac{\partial^{2\alpha}u(x_i, t_{n+1})}{\partial t^{2\alpha}} = \frac{\tau^{-2\alpha}}{\Gamma(3-2\alpha)} \sum_{j=0}^{n} [u(x_i, t_{n+1-j}) - 2u(x_i, t_{n-j}) + u(x_i, t_{n-j-1})] \\
\cdot [(j + 1)^{2-2\alpha} - j^{2-2\alpha}] + r_i^{n+1},
\]

The approximation of second order truncation error \( r_i^{n+1} \) bound is given in [18] as \(|r_i^{n+1}| \leq C_2 \tau^{2-\alpha} \), where \( C_2 \) is a constant.

We also suppose that \( u_{tt}, \ u_{xxxx} \) are continuous over the intervals \([0, L]\) and \([0, T]\), and that there is a positive constant \( F \), such that

\[
|u_{tt}| \leq F, \ |u_{xxxx}| \leq F. \tag{12}
\]

The truncation error in the explicit scheme is

\[
T_1(\tau, h) = \frac{\partial^{2\alpha}u(x_i, t_{n+1})}{\partial t^{2\alpha}} + \frac{\partial^{\alpha}u(x_i, t_{n+1})}{\partial t^{\alpha}} - \frac{1}{h^2}(u_i^{n-1} - 2u_i^n + u_i^{n+1})
\]

As

\[
\frac{\partial^{\alpha+1}u(x_i, t_{n+1})}{\partial t^{\alpha+1}} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial^2u(x_i, \xi)}{\partial \xi^2} \left( (t_{n+1} - \xi)^\alpha \right) \frac{d\xi}{(t_{n+1} - \xi)^\alpha}
\]

\[
\leq \frac{1}{\Gamma(1-\alpha)} \max_{0 \leq t \leq t_{n+1}} \left( \left| \frac{\partial^2u(x, t)}{\partial t^2} \right| \right) \sum_{j=0}^{n} \int_{j\tau}^{(j+1)\tau} \frac{d\xi}{(t_{n+1} - \xi)^\alpha}
\]

\[
= C_V \frac{\Gamma(1-\alpha) \sum_{j=0}^{n} \int_{j\tau}^{(j+1)\tau} \frac{d\xi}{(t_{n+1} - \xi)^\alpha}}{\Gamma(1-\alpha) \sum_{j=0}^{n} \int_{j\tau}^{(j+1)\tau} \frac{d\xi}{(t_{n+1} - \xi)^\alpha}}
\]

\[
= \frac{C_V \tau^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)} \cdot \sum_{j=0}^{n} [(n + 1 - j)^{1-\alpha} - (n - j)^{1-\alpha}]
\]

\[
\leq \frac{(n + 1)^{1-\alpha} C_V \tau^{1-\alpha}}{\Gamma(2-\alpha)},
\]

where \( C_V = \max_{0 \leq t \leq t_{n+1}} \left( \left| \frac{\partial^2u(x, t)}{\partial t^2} \right| \right) \), we have \( \tau \frac{\partial^{\alpha+1}u(x_i, t_{n+1})}{\partial t^{\alpha+1}} \leq C \tau^{2-\alpha} \).

The \( \tau u_{xx} \) in \( T_1(\tau, h) \) is the same as the \( \tau u_{xx} \) in \( T_2(\tau, h) \), but with the opposite sign. So alternating explicit scheme and implicit scheme can counteract partial error and these terms will disappear. Therefore, we can obtain that the accuracy of the E-I difference method is 2nd order in space and 2 \(-\alpha\) order in time.
Suppose that and the 2-analysis, we know that the accuracy of the E-I scheme is the 2nd order in space.

Proof. We use mathematical induction to prove, and according to the previous

\[ R_k = \frac{q_0}{\omega + \mu} (q_0 - 2r) |e_i^k| + \frac{q_0 r}{\omega + \mu} e_i^{2k-2}. \]

\[ d_{k-2} M \tau^\alpha (\tau^{2-\alpha} + h^2), \]

where \|\|_\infty = \max_{1 \leq m \leq 1} |\|_1 |, M is a constant.

Lemma 3.3. Suppose that \( u(x, t) \) satisfies the smoothness condition (12), and \( e_i^k = u(x, t_k) - u_i^k, i = 0, 1, ..., M; k = 1, 2, ..., N \), we have \|e_i^k\|_\infty \leq d_{k-2} M \tau^\alpha (\tau^{2-\alpha} + h^2), \]

suppose that when \( k \leq 2s \), here is \( \|e_i^{2s}\|_\infty \leq d_{2s-1} M \tau^\alpha (\tau^{2-\alpha} + h^2). \) Then when \( k = 2s + 2 \), set \( |e_i^{2s+2}| = \max_{1 \leq m \leq 1} |e_i^{2s+2}|, \) we obtain

\[ \|e_i^{2s+2}\|_\infty = \|e_i^{2s+2}\|_\infty \leq -r|e_i^{2s+2}| + (\omega + \mu + 2r)|e_i^{2s+2}| - r|e_i^{2s+2}| \]

\[ \leq | - r|e_i^{2s+2}| + (\omega + \mu + 2r)|e_i^{2s+2}| - r|e_i^{2s+2}| \]

\[ = d_{2s-1} M \tau^\alpha (\tau^{2-\alpha} + h^2). \]

So when \( k = 2s + 2 \), we can get \( \|e_i^{2s+2}\|_\infty \leq d_{2s-1} M \tau^\alpha (\tau^{2-\alpha} + h^2). \)

Therefore, the proof of lemma is complete.\]
Theorem 3.4. Suppose that $u(x,t)$ satisfies the smoothness condition (12), the E-I difference scheme (10) of the time-fractional telegraph equation is convergent, and the convergence order is $O(\tau^{2-\alpha} + h^2)$.

Proof. Due to
\[ \lim_{n \to \infty} \frac{d_k^{-1}}{k^\alpha} = \lim_{n \to \infty} \frac{k^{-\alpha}}{k^{1-\alpha} - (k-1)^{1-\alpha}} = \lim_{n \to \infty} \frac{k^{-1}}{1 - (1 - \frac{1}{k})^{1-\alpha}} = \frac{1}{1 - \alpha}, \]
so there is a constant $C > 0$ make
\[ \|e^k\|_\infty \leq d_k^{-1} M \tau^\alpha (\tau^{2-\alpha} + h^2) \]
\[ \leq k^\alpha C \tau^\alpha (\tau^{2-\alpha} + h^2) \]
\[ = (k\tau)^\alpha C (\tau^{2-\alpha} + h^2). \]
Since $k\tau \leq T$ is finite, then exists $\tilde{C} = CT^\alpha$, so that $|u(x_i, t_k) - u^k| \leq \tilde{C}(\tau^{2-\alpha} + h^2)$.
Where $i = 0, 1, ..., M_i; k = 1, 2, ..., N$. \qed

It is worth noting that when $u(x,t)$ is nonsmooth, this rate of convergence is clearly lower than the $O(\tau^{2-\alpha})$. In [27], the authors considered the advection diffusion problem with Caputo time derivative, the solutions of such problems were shown to be nonsmooth in general near the initial time $t = 0$. And this paper proved that the rate of convergence of its solutions is $O(\tau^\alpha + h^2)$ when $u(x,t)$ is nonsmooth.

4. I-E difference method for time fractional telegraph. Similar to the method of constructing the E-I scheme, we give the I-E scheme of the time fractional telegraph equation (1): The odd number layers is calculated by the implicit scheme:
\[ - ru_{i+1}^{2n+1} + (\omega + \mu + 2r)u_{i+1}^{2n+1} - ru_{i-1}^{2n+1} \]
\[ = q_0 u_i^{2n} + \sum_{j=1}^{2n-2} q_j u_i^{2n+1-j} + a_{2n} u_i^{1} + b_{2n} u_i^0 + f_i^{2n+1}. \] (13)

The even number layers is calculated by the explicit scheme:
\[ \quad (\omega + \mu)u_{i}^{2n+2} = (q_0 - 2r)u_{i+1}^{2n+1} + ru_{i+1}^{2n+1} + ru_{i-1}^{2n+1} \]
\[ + \sum_{j=1}^{2n-1} q_j u_i^{2n+1-j} + a_{2n+1} u_i^{1} + b_{2n+1} u_i^0 + f_i^{2n+1}. \] (14)

In summary, the I-E scheme of the time fractional telegraph equation (1) is as follows:
\[
\begin{cases}
AU^{2n+1} = q_0 U^{2n} + \sum_{j=1}^{2n-2} q_j U^{2n-j} + a_{2n} U^1 + b_{2n} U^0 + F^{2n+1} + g^{2n+1} \\
(\omega + \mu)U^{2n+2} = BU^{2n+1} + \sum_{j=1}^{2n-1} q_j U^{2n+1-j} + a_{2n+1} U^1 + b_{2n+1} U^0 + F^{2n+1} + g^{2n+1}
\end{cases}
\] (15)

Where $A, B, g^n, F^n$ is defined as (10).

The analysis similar to the E-I scheme (10) proves that there are the following theorems:

Theorem 4.1. Suppose that $u(x,t)$ satisfies the smoothness condition (12), the I-E difference scheme (15) of the time-fractional telegraph equation is uniquely solvable, unconditionally stable and convergent, and the convergence order is $O(\tau^{2-\alpha} + h^2)$.
The E-I scheme and the I-E scheme are both two-step schemes, in which one of the steps is calculated in implicit scheme, and the other is calculated in explicit scheme. The difference between them is in the order of using explicit and implicit schemes. Therefore, the E-I scheme and the I-E scheme are equivalent in computation.

Substituting (13) into (14), we get

\[(\omega + \mu)u_i^{2n+2} = (\omega + \mu + q_0 - 2r)u_i^{2n+1} + q_0 u_i^{2n} + \sum_{j=1}^{2n-1} q_j(u_i^{2n+1-j} - u_i^{2n-j}).\] (16)

this is Richardson scheme of the time fractional telegraph equation (1). However, it should be pointed out that the calculation process using the I-E scheme (15) and the Richardson scheme (16) process alone are inconsistent. The Richardson scheme cannot calculate the value of \(u_i^{2n+2}\) from the value of \(u_i^{2n}\). Because \(u_i^{2n+1}\) is also an unknown number, the process of joint use (15) is equivalent to using the implicit scheme (13) to calculate \(u_i^{2n+1}\) from \(u_i^{2n}\), and then using the explicit scheme (14) to calculate \(u_i^{2n+2}\) from \(u_i^{2n}, u_i^{2n+1}\). So it’s a completely different calculation process. Therefore, the use of Richardson scheme (16) alone is unstable, but alternately using (13) and (14) is stable.

It needs to be pointed out why the combination of “explicit-implicit” or “implicit-explicit” has such great advantages. First, the classical implicit hidden “potential” stability, which is not needed in calculation, but when used alternately, this potential stability just offsets the shortcomings of the stability of the explicit scheme. The other is that each two steps of the explicit or implicit scheme produce two basic error components which are opposite to each other, and they partially cancel each other, so as to get more accurate results [2, 12, 34].

5. Numerical experiments. Numerical experiments will be done in Matlab R2014a, based on the Intel Core i5-2400 CPU.

5.1. Example. We consider the following time fractional telegraph equation [1, 15]:

\[
\begin{align*}
\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} &= \frac{1}{2} \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \\
0 < x < 1, 0 < t < 1, \frac{1}{2} < \alpha \leq 1, \\
u(x,0) &= 0, \frac{\partial u(x,0)}{\partial t} = 0, 0 \leq x \leq 1, \\
u(0,t) = t^2, u(1,t) = e^t, 0 \leq t \leq 1.
\end{align*}
\] (17)

Where

\[f(x,t) = \frac{2e^x}{\Gamma(3-2\alpha)}t^{2-2\alpha} + \frac{2e^x}{\Gamma(3-\alpha)}t^{2-\alpha} - \frac{1}{2} t^2 e^x.\]

The exact solution is

\[u(x,t) = t^2 e^x.\]

We compare the numerical solutions of the E-I scheme (10) and I-E scheme (15) of equation (17) in this paper with the exact solution and the classical implicit scheme solution at \(t = 0.1, \alpha = 0.8\). We take \(N = 300, M = 300\), that is \(h = \frac{1}{M} = \frac{1}{300}, \tau = \frac{1}{N} = \frac{1}{300}\). The calculation results are shown in Table 1.
TWO DIFFERENCE METHODS FOR TIME-FRACTIONAL TELEGRAPH EQUATION

Table 1. Comparison of numerical and exact solutions

| M  | Exact solution | Implicit scheme solution | E-I scheme solution | I-E scheme solution |
|----|----------------|--------------------------|--------------------|--------------------|
| 21 | 0.032816       | 0.032727                 | 0.032238           | 0.032168           |
| 41 | 0.038767       | 0.038662                 | 0.037960           | 0.038002           |
| 61 | 0.045800       | 0.045673                 | 0.044992           | 0.044895           |
| 81 | 0.054104       | 0.053957                 | 0.053152           | 0.053037           |

In Table 1, from the point of view of calculation accuracy, the numerical solutions of E-I scheme and I-E scheme are very close to the exact solution. Therefore, the E-I difference method and the I-E difference method are a high-precision difference method for solving the time fractional telegraph equation.

We will compare the exact solution surface, the implicit scheme solution surface, the E-I scheme solution surface and the I-E scheme solution surface of equation (17). Take $M = 300, N = 100, \alpha = 0.8$.

Figures 1, 2, 3 and 4 are the exact solution surface, the implicit scheme solution surface, the E-I scheme solution surface and the I-E scheme solution surface, respectively. The exact solution surface graph of the time fractional telegraph equation (17) and the three numerical solution surface graphs are smooth and the shape remains the same as a whole.

To better verify the stability of E-I scheme (10) and I-E scheme (15), we give the
change of relative error with time. We take the numerical solution of $\tilde{u}_i^j$ the scheme is a perturbation solution, and the exact solution $u_i^j$ is the control solution. The relative error is defined as SRET (the sum of relative error for every level):

$$SRET(j) = \sum_{i=1}^{M} \frac{|\tilde{u}_i^j - u_i^j|}{u_i^j}$$

As shown in Figure 5, when $N = 300, M = 300, \alpha = 0.8$, it can be seen that the SRET of the E-I scheme solution and I-E scheme solution are larger before $M = 20$. As the time layer increases, the SRET of the implicit scheme solution, E-I scheme solution and I-E scheme solution decrease rapidly and are bounded and tend to 0. Therefore, Figure 5 shows that the E-I scheme and I-E scheme of the time fractional telegraph equation are computationally stable.

In order to verify the accuracy of the E-I scheme (10) and I-E scheme (15), considering the distribution of the errors on the spatial grid points, the error energy DTE (the difference total energy) is defined:

$$DTE(i) = \frac{1}{2} \sum_{j=1}^{N} (\tilde{u}_i^j - u_i^j)^2.$$  

When $N = 600, M = 300, \alpha = 0.8$, it can be seen from Figure 6 that the DTE of the implicit scheme, E-I scheme and I-E scheme solutions are controlled from 0 to $6e - 3$, it shows that the solutions of implicit scheme, E-I scheme and I-E scheme are very close to exact solutions, so the accuracy of E-I scheme, I-E scheme and implicit scheme of time fractional telegraph equation are approximate, and they all have good accuracy.

In order to better compare the calculation efficiency of the implicit scheme, E-I scheme and I-E scheme, let $\alpha = 0.8$, fix the time layer to $N = 30$. Select the number of spatial grids $M = 100, 200, 300, 400, 500, 600, 700, 800, 900, 1000$, as the calculation result as follows:
Alternate use of explicit scheme and implicit scheme can reduce or avoid the operation time of implicit scheme calculation. The E-I scheme and I-E scheme reduce the number of implicit solution layers by 50%, thereby saving about 25% of the number of multiplication and division operations [12, 34]. It can be seen from Figure 7 that for the time fractional telegraph equation, the calculation time of the difference scheme increases with the increase of the number of grids. After $M = 300$, as the number of spatial grids increases, the calculation time of the implicit scheme increases rapidly and the rate of change is large. The E-I scheme and I-E scheme constructed in this paper have small changes in calculation time and a small change rate. When $M = 1000$, the calculation efficiency of E-I scheme and I-E scheme is about 78% higher than that of implicit scheme. Compared with the paper [1], the method in this paper has higher calculation efficiency in E-I scheme and I-E scheme.

We will give the numerical experiments of time convergence order and spatial convergence order of implicit scheme, E-I scheme and I-E scheme, and analyze the
convergence order in spatial direction (Order1) and time direction (Order2). Let E-I scheme, I-E scheme and implicit scheme numerical solution \( \tilde{u}^i_j \) be the perturbation solution, the exact solution \( u^i_j \) be the control solution. The definition of \( E_2(h, \tau) \), Order1 and Order2 are as follows \[28, 30\]:

\[
E_2(h, \tau) = \max_{0 \leq n \leq N} \| \tilde{u}^n_i - u^n_i \|_2,
\]

\[
\text{Order1} = \log_2 \left( \frac{E_2(2h, \tau)}{E_2(h, \tau)} \right), \text{Order2} = \log_2 \left( \frac{E_2(h, 2\tau)}{E_2(h, \tau)} \right).
\]

Table 2. Spatial convergence orders and numerical errors of implicit scheme, E-I scheme and I-E scheme (\( \tau = h^2 \))

| \( \alpha \) | \( M \) | \begin{tabular}{c|c|c|c|c|c} Implicit scheme & E-I scheme & I-E scheme \\ \hline \end{tabular} |
|---|---|---|
| \( 0.6 \) | 6 | \begin{tabular}{c|c|c|c|c|c} \hline \( E_2(h, \tau) \) & Order1 & \( E_2(h, \tau) \) & Order1 & \( E_2(h, \tau) \) & Order1 \\ \hline \end{tabular} |
| 0.601261438 | ——– | 0.670908875 | ——– | 0.676867797 | ——– |
| 0.17131703 | 1.811200301 | 1.854724481 | 1.85742355 | 1.865571243 | ——– |
| 0.027626239 | 1.861354661 | 1.880049324 | 1.885472448 | 1.888092124 | ——– |
| \( 0.7 \) | 6 | \begin{tabular}{c|c|c|c|c|c} \hline \( E_2(h, \tau) \) & Order1 & \( E_2(h, \tau) \) & Order1 & \( E_2(h, \tau) \) & Order1 \\ \hline \end{tabular} |
| 0.572754131 | ——– | 0.629245364 | ——– | 0.632403830 | ——– |
| 0.154051855 | 1.894499914 | 1.902779364 | 1.917706812 | 1.914904807 | ——– |
| 0.043125187 | 1.968989813 | 1.954591537 | 1.942269721 | 1.953396613 | ——– |
| \( 0.8 \) | 6 | \begin{tabular}{c|c|c|c|c|c} \hline \( E_2(h, \tau) \) & Order1 & \( E_2(h, \tau) \) & Order1 & \( E_2(h, \tau) \) & Order1 \\ \hline \end{tabular} |
| 0.331546455 | ——– | 0.384936257 | ——– | 0.392207783 | ——– |
| 0.093281787 | 1.829553269 | 0.000168779 | 0.000592926 | 0.963099311 | ——– |
| 0.025799014 | 1.887189262 | 0.042627245 | 0.042269721 | 1.953396613 | ——– |
| \( 0.9 \) | 6 | \begin{tabular}{c|c|c|c|c|c} \hline \( E_2(h, \tau) \) & Order1 & \( E_2(h, \tau) \) & Order1 & \( E_2(h, \tau) \) & Order1 \\ \hline \end{tabular} |
| 0.312856198 | ——– | 0.351172503 | ——– | 0.354347245 | ——– |
| 0.079460269 | 1.977194091 | 0.084117921 | 0.084447694 | 2.069038990 | ——– |
| 0.020221416 | 1.978099907 | 0.021169310 | 0.021427164 | 2.056110578 | ——– |

First verify the spatial accuracy of the implicit scheme, E-I scheme and I-E scheme. Let \( M = 6, 12, \tau = h^2(N = M^2) \) and \( \alpha = 0.6, 0.7, 0.8, 0.9 \). From Table 2 and Order1 = \( \log_2 \left( \frac{E_2(2h, \tau)}{E_2(h, \tau)} \right) \approx 2 \), we can see that the spatial accuracy of implicit scheme is second-order and the spatial accuracy of the E-I scheme and I-E scheme is second-order. Then calculate the time accuracy of the implicit scheme, E-I scheme and I-E scheme. Fixing the spatial step size \( h = \frac{1}{10} \), that is to say, \( M = 40 \), and taking \( N = 300, 600 \), \( \alpha = 0.6, 0.7, 0.8, 0.9 \), respectively. From Table 3 and Order2 = \( \log_2 \left( \frac{E_2(h, 2\tau)}{E_2(h, \tau)} \right) \approx 2 - \alpha \), we can see that the time accuracy of the implicit scheme is \( 2 - \alpha \) order, and the time accuracy of the E-I scheme and I-E scheme is \( 2 - \alpha \) order. Therefore, the spatial and temporal accuracy of the E-I scheme, I-E scheme and implicit scheme constructed in this paper are equivalent, and the numerical test results are consistent with the theoretical analysis.
5.2. Example. Consider the following time-fractional telegraph equation [16, 17]:

$$
\begin{cases}
\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{2} u(x,t)}{\partial x^{2}} + f(x,t), \\
0 < x < 1, 0 < t < 1, 0 < \alpha < 1,
\end{cases}
$$

(18)

where $f(x,t)$ is appropriate for the exact solution $u(x,t) = x(x - 1)t^{1.5}$.

The numerical solutions of E-I, I-E scheme of equation are compared with exact solutions and implicit scheme solutions. We take $N = 300, M = 100$. The calculation results are shown in Table 4.

5.3. Example. Consider the following time-fractional telegraph equation [16, 17]:

$$
\begin{cases}
\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = \frac{\partial^{2} u(x,t)}{\partial x^{2}} + f(x,t), \\
0 < x < 1, 0 < t < 1, 0 < \alpha < 1,
\end{cases}
$$

(18)

where $f(x,t)$ is appropriate for the exact solution $u(x,t) = x(x - 1)t^{1.5}$.

The numerical solutions of E-I, I-E scheme of equation are compared with exact solutions and implicit scheme solutions. We take $N = 300, M = 100$. The calculation results are shown in Table 4.
Figures 8, 9, 10 and Figure 11 show the exact solution, implicit solution, E-I solution and I-E solution, respectively. It can be seen that they have the same shape.

To better verify the stability of E-I scheme (10) and I-E scheme (15), we give the change of relative error with time (SRET).
As shown in Figure 12, when \( N = 300, M = 300, \alpha = 0.5 \), it can be seen that the SRET of the E-I scheme solution and I-E scheme solution are larger before \( M = 20 \). As the time layer increases, the SRET of E-I scheme solution and I-E scheme solution decrease rapidly and are bounded and tend to 0. Therefore, Figure 5 shows that the E-I scheme and I-E scheme of the time fractional telegraph equation are computationally stable.

In order to verify the accuracy of the E-I scheme (10) and I-E scheme (15), considering the distribution of the errors on the spatial grid points (DTE).

![Figure 13. The distribution of DTE in spatial grid](image)

When \( N = 300, M = 150, \alpha = 0.5 \), it can be seen from Figure 13 that the DTE of E-I scheme and I-E scheme solutions are controlled from 0 to 0.09, it shows that the solutions of E-I scheme and I-E scheme are very close to exact solutions, therefore, the E-I method and the I-E method are equally applicable to fractional telegraph equation for solving low-regular artificial solutions.

Since the E-I and I-E methods are very close, we only consider the E-I method. The numerical experiment of the spatial convergence order of the implicit scheme and E-I scheme are given below, and the spatial convergence order is analyzed.

Table 5 shows that the spatial accuracy of the implicit scheme is almost reached 2nd order, and the spatial accuracy of the E-I scheme is almost reached 2nd order. It can be seen from the calculation data that the convergence order is consistent with the theoretical analysis results. Although \( \alpha \) changes, the lack of convergence results is almost unchanged. And numerical experiment shows that the space convergence speed is independent. Table 6 shows that the time accuracy of the E-I scheme is almost reached \( \alpha \) order, and it is better than the time accuracy of the implicit scheme. We can see that under the premise of nonsmooth solution, this rate of convergence \( O(\tau^\alpha) \) is lower than the \( O(\tau^{2-\alpha}) \) proved by Theorem 3.4. In view of the results of numerical analysis, it shows that the E-I difference method and I-E difference method can solve the fractional telegraph equation well.

6. Conclusion. In order to be widely used, the computational efficiency of the difference scheme is very important. In this paper, the E-I difference method and I-E difference method for the time fractional telegraph equation are proposed. The
existence, uniqueness, unconditional stability and convergence of the solutions of these two difference methods are theoretically analyzed. Numerical experiments verify the theoretical analysis. The accuracy of numerical solutions of E-I and I-E schemes is equal to that of classical implicit scheme. Under the premise of consistent accuracy, E-I scheme and I-E scheme have higher computational efficiency than classic implicit scheme.

The use of explicit scheme and implicit scheme alternately reduces or avoids the operation time of implicit calculation. The E-I difference scheme and the I-E difference scheme again illustrate the numerical advantages of symmetric difference dispersion, and illustrate the effectiveness and calculation accuracy of the E-I difference method and the I-E difference method in solving the time fractional telegraph equation. Based on the above reasons, the E-I method and I-E method have a wide range of applications.

### Table 5. Spatial convergence orders and numerical errors of implicit scheme and E-I scheme (τ = h²)

| α    | M   | Implicit scheme          | E-I scheme          |
|------|-----|--------------------------|---------------------|
|      |     | $E_2(h, \tau)$ Order1 | $E_2(h, \tau)$ Order1 |
| 0.2  | 16  | 0.001926582 1.940537664 | 0.001786161 1.955606884 |
|      | 32  | 0.000486619 1.985178414 | 0.000447580 1.996642445 |
|      | 64  | 0.000121970 1.996358831 | 0.000110261 2.004338515 |
| 0.3  | 16  | 0.001926975 1.944911724 | 0.001748227 1.944911724 |
|      | 32  | 0.000486657 1.996642453 | 0.000440517 1.988623313 |
|      | 64  | 0.000121970 1.996374655 | 0.000110261 1.998266445 |
| 0.4  | 16  | 0.001927758 1.941830430 | 0.001736738 1.938186701 |
|      | 32  | 0.000486747 1.985359045 | 0.000438647 1.988623313 |
|      | 64  | 1.988623313 1.996518415 | 0.000109876 1.996547989 |
| 0.5  | 16  | 0.001929358 1.942987829 | 0.001735124 1.935839557 |
|      | 32  | 0.000486961 1.986241089 | 0.000438424 1.984638177 |
|      | 64  | 0.000122001 1.996814720 | 0.000109876 1.996459048 |
Table 6. Time convergence orders and numerical errors of implicit scheme and E-I scheme (h = 1/40)

| α   | N   | Implicit scheme     | E-I scheme     |                 |                   |
|-----|-----|---------------------|----------------|----------------|------------------|
|     |     | $E_2(h, τ)$ Order2  | $E_2(h, τ)$ Order2 |
| 150 | 300 | 0.003247111         | 0.003127877    |                 |                  |
| 0.2 | 300 | 0.0038902781        | 0.260726667    | 0.003593341     | 0.200141963      |
| 600 |     | 0.0046878951        | 0.26969826     | 0.000447580     | 0.211580117      |
| 150 | 300 | 0.003053496         | 0.002880601    |                 |                  |
| 0.3 | 300 | 0.003751789         | 0.297116513    | 0.0036603101    | 0.32869023       |
| 600 |     | 0.004664256         | 0.31469826     | 0.004535791     | 0.32115801       |
| 150 | 300 | 0.002847062         | 0.002645836    |                 |                  |
| 0.4 | 300 | 0.003604515         | 0.340331208    | 0.003490906     | 0.399877794      |
| 600 |     | 0.004639253         | 0.359191478    | 0.004623847     | 0.405493914      |
| 150 | 300 | 0.002620013         | 0.002391991    |                 |                  |
| 0.5 | 300 | 0.003442873         | 0.394038915    | 0.003330367     | 0.477464645      |
| 600 |     | 0.004561314         | 0.405837085    | 0.004690116     | 0.493943176      |

Acknowledgments. We are grateful to Dr. Lifei Wu of North China Electric Power University for many useful discussions. The authors would like to express their sincere thanks to the editor and anonymous referees for insightful comments and suggestions which have led to improvements in presentation of this manuscript.

REFERENCES

[1] T. Akram, M. Abbas, A. I. Ismail, N. Hj. M. Ali and D. Baleanu, Extended cubic B-splines in the numerical solution of time fractional telegraph equation, *Adv. Differ. Equ.*, (2019), Paper No. 365, 20 pp.

[2] D. J. Arrigo and S. G. Krantz, *Analytical Techniques for Solving Nonlinear Partial Differential Equations*, Morgan & Claypool Publishers, 2019.

[3] A. Atangana, On the stability and convergence of the time-fractional variable order telegraph equation, *J. Comput. Phys.*, 293 (2015), 104–114.

[4] R. C. Cascaval, E. C. Eckstein, C. L. Frota and J. A. Goldstein, Fractional telegraph equations, *J. Math. Anal. Appl.*, 276 (2002), 145–159.

[5] J. Chen, F. Liu and V. Anh, Analytical solution for the time-fractional telegraph equation by the method of separating variables, *J. Math. Anal. Appl.*, 338 (2008), 1364–1377.

[6] S. Das, K. Vishal, P. K. Gupta and A. Yildirim, An approximate analytical solution of time-fractional telegraph equation, *Appl. Math. Comput.*, 217 (2011), 7405–7411.

[7] W. Deng and Z. Zhang, *High Accuracy Algorithm for the Differential Equations Governing Anomalous Diffusion, Algorithm and Models for Anomalous Diffusion*, World Scientific, Singapore, 2019.

[8] K. Diethelm, *The Analysis of Fractional Differential Equations*, Springer, 2010.
[9] M. Ferreira, M. M. Rodrigues and N. Vieira, First and second fundamental solutions of the time-fractional telegraph equation with Laplace or Dirac operators, *Adv. Appl. Clifford Algebr.*, 28 (2018), Art. 42, 14 pp.

[10] N. J. Ford, M. M. Rodrigues, J. Xiao and Y. Yan, Numerical analysis of a two-parameter fractional telegraph equation, *J. Comput. Appl. Math.*, 249 (2013), 95–106.

[11] B. Guo, X. Pu and F. Huang, *Fractional Partial Differential Equations and Their Numerical Solutions*, Science Press, Beijing, 2011, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.

[12] L. Hervé and L. Brigitte, *Partial Differential Equations: Modeling, Analysis and Numerical Approximation*, Springer International Publishing, Switzerland, 2016.

[13] M. H. Heydari, M. R. Hooshmandasl and F. Mohammadi, Two-Dimensional legendre wavelets for solving time-fractional telegraph equation, *Adv. Appl. Math. Mech.*, 6 (2014), 247–260.

[14] V. R. Hosseini, W. Chen and Z. Avazzadeh, Numerical solution of fractional telegraph equation by using radial basis functions, *Eng. Anal. Bound. Elem.*, 38 (2014), 31–39.

[15] W. Jiang and Y. Lin, Representation of exact solution for the time-fractional telegraph equation in the reproducing kernel space, *Commun. Nonlinear Sci. Numer. Simul.*, 16 (2011), 3639–3645.

[16] K. Kumar, R. K. Pandey and S. Yadav, Finite difference scheme for a fractional telegraph equation with generalized fractional derivative terms, *Physica A*, 535 (2019), Art. 122271, 15 pp.

[17] C. Li and F. Zeng, *Numerical Methods for Fractional Calculus*, CRC Press, Boca Raton, 2015.

[18] C. Li, Z. Zhao and Y. Chen, Numerical approximation of nonlinear fractional differential equations with subdiffusion and superdiffusion, *Comput. Math. Appl.*, 62 (2011), 855–875.

[19] Y. Lin and C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, *J. Comput. Phys.*, 225 (2007), 1533–1552.

[20] F. Liu, P. Zhuang, V. Anh, I. Turner and K. Burrage, Stability and convergence of the difference methods for the space-time fractional advection-diffusion equation, *Appl. Math. Comput.*, 191 (2007), 17–20.

[21] F. Liu, P. Zhuang and Q. Liu, *Numerical Solutions of Fractional Order Partial Differential Equations and its Applications*, Science Press, Beijing, 2015. (in Chinese)

[22] M. O. Mamchuev, Solutions of the main boundary value problems for the time-fractional telegraph equation by the green function method, *Fract. Calc. Appl. Anal.*, 20 (2017), 190–211.

[23] S. Momani, Analytic and approximate solutions of the space- and time-fractional telegraph equations, *Appl. Math. Comput.*, 170 (2005), 1126–1134.

[24] E. Orsingher and L. Beghin, Time-fractional telegraph equations and telegraph processes with brownian time, *Probab. Theory Relat. Fields*, 128 (2004), 141–160.

[25] A. Saadatmandi and M. Molabahati, Numerical solution of fractional telegraph equation via the tau method, *Math. Rep.*, 17 (2015), 155–166.

[26] J. Sabatier, O. P. Agrawal and J. A. Tenreiro Machado (Editors), *Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering*, World Book Incorporated Beijing, 2014.

[27] M. Stynes, E. O’ Riordan and J. L. Gracia, Error analysis of a finite difference method on graded meshes for a time-fractional diffusion equation, *SIAM J. Numer. Anal.*, 55 (2017), 1057–1079.

[28] Z. Sun and G. Gao, *Finite Difference Methods for Fractional Differential Equations*, Science Press, Beijing, 2015. (in Chinese)

[29] V. V. Uchaikin, *Fractional Derivatives for Physicists and Engineers: Volume II Applications*, Higher Education Press; Springer, Heidelberg, Beijing, 2013.

[30] S. Vong, P. Lyu and Z. Wang, A compact difference scheme for fractional sub-diffusion equations with the spatially variable coefficient under neumann boundary conditions, *J. Sci. Comput.*, 66 (2016), 725–739.

[31] H. Wang, A. Cheng and K. Wang, Fast finite volume methods for space-fractional diffusion equations, *Discrete Contin. Dyn. Syst. Ser. B.*, 20 (2015), 1427–1441.

[32] L. Wei, H. Dai, D. Zhang and Z. Si, Fully discrete local discontinuous Galerkin method for solving the fractional telegraph equation, *Calcolo*, 51 (2014), 175–192.
[33] P. Xanthoules and G. E. Zouraris, A linearly implicit finite difference method for a Klein-Gordon–Schrödinger system modeling electron-ion plasma waves, *Discrete Contin. Dyn. Syst. Ser. B.*, **10** (2008), 239–263.

[34] X. Yang, L. Wu, S. Sun and X. Zhang, A universal difference method for time-space fractional Black-Scholes equation, *Adv. Differ. Equ.*, (2016), Paper No. 71, 14 pp.

[35] A. Yildirim, He’s homotopy perturbation method for solving the space and time fractional telegraph equations, *Int. J. Comput. Math.*, **87** (2010), 2998–3006.

[36] Z. Zhao and C. Li, Fractional difference/finite element approximations for the time-space fractional telegraph equation, *Appl. Math. Comput.*, **219** (2012), 2975–2988.

Received December 2019; 1st revision April 2020; final revision July 2020.

*E-mail address:* yxiaozh@ncepu.edu.cn

*E-mail address:* 1063503220@qq.com