Langevin dynamics of financial systems: 
A second-order analysis

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Abstract

We address the issue of stock market fluctuations within Langevin Dynamics (LD) and
the thermodynamics definitions of multifractality in order to study its second-order char-
acterization given by the analogous specific heat $C_q$, where $q$ is an analogous temperature
relating the moments of the generating partition function for the financial data signals.
Due to non-linear and additive noise terms within the LD, we found that $C_q$ can display
a shoulder to the right of its main peak as also found in the S&P500 historical data
which may resemble a classical phase transition at a critical point.

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1 Introduction

Simulated descriptions of at least some universal features of financial prices has recently been produced by multifractal techniques [1] and also linear [2, 3] and non-linear [4] Langevin Dynamics (LD). On one hand, multifractals reasonably describe patterns that resemble sudden (both large and small) market price oscillations, ensuing bursts of long-range volatility as in the case of stochastic multi-agent models [3]. On the other hand, non-linear [4] LD generates fat tail distributions of price differences as found in the analysis of financial time series [6, 7].

There is a vast literature on the characterization of financial data based on turbulence, multifractality, autocorrelation functions and power spectra studies (see, e.g., [8, 9, 10, 11, 12]). However the second-order derivative relating the increment of price values of assets $\Delta x(t, 1) \equiv x(t+1) - x(t)$ and given by an analogous specific heat $C_q$, similarly to the multifractal studies of diffusion-limited aggregates (DLA) [13], has only recently been analysed by the author in the case of temporal fluctuations of the S&P500 stock index [14]. The analogous specific heat is given by the second order derivative of the analogous free energy $\tau$ relating the multifractal dimension $D_q$ (c.f., Eq. (6) and (7) below). The $q$-moments of the generating partition function for the financial data signals $\Delta x(t, 1)$ corresponds to an analogous temperature.

As an illustrative example, the curve (a) in the lower Fig.1 shows the presence of an anomalous shoulder in $C_q$ when considering 3287 (S&P500) data points including the largest burst that corresponds to the so-called Black Monday crash measured in October of 1987. For comparison, in the plotted curves (b,c,d) of Fig.1 we remove the crash from the historical data and make the measurement of $C_q$ anew. Curve (b) is the result of just removing the single data point for the Black Monday (i.e., by considering 3286 points), whereas the curve (c) is the result of removing the Black Monday plus all successive data (i.e., considering 1970 initial points). On the other hand, curve (d)
is the result of taking 499 initial data points only.

Clearly, as one removes or approaches the **Black Monday** data the shoulder in $C_q$ of curve (a) does not disappear as can be seen in curves (b,c). A similar behaviour is observed analysing the data from few days prior to the S&PE500 crash. As a difference $C_q$ only displays a typical, single peaked form without anomalies for the data prior to, but far away from, the **Black Monday** (as in curve d).

In this work we make an attempt to characterize the onset of market crashes by analysing the second-order derivative relating the increment of prices of assets $\Delta x(t, 1)$ and given by $C_q$. We shall consider the LD of financial systems \[3, 4\] to show that the presence of an anomalous shoulder in the specific heats as those plotted in Fig.1 can also be a consequence of having non-linear stochastic terms within LD.

## 2 Linear and Non-linear Langevin Dynamics

We consider the differential equation with an additive noise term of the form $g_t$ times $\eta_t$:

$$ \frac{dx(t)}{dt} = h_t(x(t)) + g_t(x(t))\eta_t , $$

where as usual $\eta_t$ is assumed to be Gaussian-white whose average and variance are given by $\overline{\eta_t} = 0$ and $\overline{\eta_t \eta_t'} = 2D\delta(t-t')$, respectively.

The dynamics of simplest financial systems with random noise has been proposed in \[3, 15, 16\] by setting

$$ h_t(x(t)) \equiv -\lambda_t x(t) ; \quad g_t(x(t)) \equiv 1 , $$

with $\lambda_t$ a multiplicative noise that sets the time scale for deviations from equilibrium. In the context of economic models, $x(t)$ may represent price values, $\lambda_t$ the rate of change of the price, and $\eta_t$ some external noise of various sources. In the following we shall
refer Eq. (2) as the linear LD. We add that the scaling behaviour and the $1/f$ noise of trajectories and velocities with time within linear LD with fluctuating random forces has been reported in our previous work [17].

Another simple financial agent model has been introduced in [4] in analogy with the mean field approach to the Ising model for a magnetic system characterized by

$$h_t(x(t)) \equiv Jx(t) + bx^2(t) - cx^3(t) \quad ; \quad g_t(x(t)) \equiv e + x(t) \ ,$$

with $J$, $b$ and $c$ constants associated with higher-order agent interactions leading to power laws tails. The parameter $e$ allows this agent model to take up the simple Langevin form when $x(t) \to 0$. The dynamics of time series from stochastic processes governed by a similar non-linear LD has also been reported in [18].

3 Multifractality and Analogous Specific Heat

In accordance with the standard of economic notation only relative changes $\Delta x(t, 1)$ are usually relevant. From these fluctuations, we follow [14] and construct the generating partition function $Z$ in terms of the normalized measures $\mu_t > 0$, and its moments $q$, by the scaling

$$Z(q, N) = \sum_{t=1}^{N} \mu_t^q \sim N^{-\tau(q)} \ ,$$

where

$$\mu_t \equiv \frac{|\Delta x(t, 1)|}{\sum_{t=1}^{N} |\Delta x(t, 1)|} \ .$$

In the above we have divided the 1D system of length $L$ into $N$ lines of length $\ell$ (i.e., $N \sim L/\ell$), and have associated this $N$ with the measured discrete $x(t)$ sequences [19].
In order to get the thermodynamics interpretation of multifractality (see, e.g. [13]), we then consider standard definitions

\[ \tau(q) \equiv [q - 1]D_q , \]  

where \( \tau \) represents an analogous free energy and \( D_q \) the multifractal dimension. Of particular interest here is to consider the analogous specific heat given by the second order derivative of \( \tau \), namely

\[ -C_q \equiv \frac{\partial^2 \tau(q)}{\partial q^2} \approx \tau(q + 1) - 2\tau(q) + \tau(q - 1) . \]  

4 Discussion

In the upper curves of Fig.2 we show some typical time evolution of \( \Delta x(t, 1) \) governed by the linear LD with \( h_t \) and \( g_t \) given in Eq.(2) for different values of \( \eta_t = (1 - 2r_n)v \) and \( \lambda_t = (1 - 2r_n)u \) with \( v = 0.1 \), \( u = -0.005, -0.02 \) and \(-0.1\), and using the random numbers \( 0 \leq r_{n,n'} \leq 1 \). In the lower curves of Fig.2 we display the non-linear LD fluctuations of \( \Delta x(t, 1) \) with \( h_t \) and \( g_t \) given in Eq.(3) with \( J = -1, b = 0.01, c = 0.001, e = 0.005 \) and for different values of \( \eta_t = (1 - 2r_{n'})n \) such that \( n = 1.2, 1.5 \) and \( 1.8 \) and using random numbers \( 0 \leq r_{n''} \leq 1 \).

From the curves in Fig.2 it can be seen that \( \Delta x(t, 1) \) has relatively small finite values, and exhibits intermittent bursts due to the weak noise terms. The presence of peaked narrow fluctuations can increase with both the multiplicative and additive noise strength. However, wider volatility clusters are formed over large periods of time when decreasing, in the linear LD, the amplitude \( u \) of the multiplicative \( \lambda_t \) noise as compared to the case of increasing \( n \) of the additive noise \( \eta_t \) within the non-linear LD. It is this difference of behaviour that will lead \( C_q \) to display an anomalous shoulder to the right of its main peak to be shown below.
Let us first see if the generalized dimensions $D_q$ for both linear and non-linear LD, which are extracted by considering the definition in Eq. (6) in conjunction with the scaling of Eq. (4), are sufficiently smooth and multifractal for $C_q$ of Eq. (7) to be meaningful [20, 21]. The discrete values of $q$ used range from $-5$ to $+5$ at increments of $0.033$.

In Fig. 3 we display the plots of $D_q$ for the different values of $u$ and $n$ as used in Fig. 2 together with the plots of the limit $1 - D_q \rightarrow \infty$ for the linear and non-linear LD (upper and lower curves, respectively). If $q < -1$, we observe that both results for $D_q$ follow a typical convergent behaviour according to multifractal physics. In the range $u > -0.02$ and $n < 1.5$, $D_q$ is fully multifractal-like.

For lower values of $u < -0.02$ and greater values of $n > 1.5$, we find a non-monotonous decreasing behaviour of $D_q$, in correspondence with the double-peaked form of the respective $C_q$ displayed in Fig. 4 which relates the presence of the intermittent bursts in $\Delta x(t, 1)$ shown in Fig. 2. Such a non-monotonous behaviour of $D_q$ is more evident in the case of non-linear LD. From the $D_q$ data for the linear LD case, the multifractality strength of the $\Delta x(t, T) \equiv x(t + T) - x(t)$ sequences at different integer time lags $T \geq 1$, i.e., $1 - D_q \rightarrow \infty$, does not seem to follow a power-law scaling for $1 < T < 10$ within all $u$ considered. As a consequence of the complicated behaviour of the non-linear form of Eq. (3) coupled to $\eta_t$, which influences the (thinner) volatility clusters shown in Fig. 2 (lower curves), it is not straightforward that $1 - D_q \rightarrow \infty$ in this case follows a power scaling as in [21].

Let us see next how this type of $D_q$ behaviour given in Fig. 3 influences the analogous specific heat of Eq. (7) and how it characterizes the temporal large and small intermittent bursts representing variations in financial prices. In Fig. 4 we plot $C_q$ for the LD parameter set used in Figs. 2 and 3. For the linear LD case, we find that the main peak of our numerical $C_q$ curves for $u > -0.02$ resembles a classical first-order phase transition at a critical point $q = -1$. However, the sharp peak turns slightly asymmetric around the
critical point for \( u = -0.1 \). Surprisingly, this asymmetry becomes evident within the non-linear LD. The analogous specific heat in this case displays a higher shoulder to the right of the main peak as a function of \( n \geq 1.5 \) which is in qualitative agreement with the \( C_q \) curve for the S&P500 index data shown in Fig.1 (lower curves). The presence of the shoulder in \( C_q \) is then associated to the non-monotonous behaviour of \( D_q \) as in Fig.3.

The same \( C_q \) shape at a phase transition as in Curve (d) of Fig.1 and Fig.4, has been reported in the case of DLA [13]. A non monotonic specific heat shape with a superimposed second peak (resembling the shoulder of the type of Curve (a) in Fig.1 and Fig.4 lower curves) has been found in the Hubbard model on a cluster of magnetic sites [22, 23] and, most recently, in a uniform spin model on a fractal [24].

When decreasing \( n \leq 1.2 \) in the non-linear LD, the sudden intermittent bursts in \( \Delta x(t, 1) \) as well as the shoulder in \( C_q \) tend to vanish. It is this latter feature that make us argue that the stochastic price behaviour of financial assets may be characterized by an analogous \( C_q \) which resembles the phase transition features as measured in multifractal physics.

5 Conclusion

We have made a novel attempt to characterize the onset of market crashes via an analogous specific heat using the thermodynamics definitions of multifractality. We presented an study of generalized dimensions \( D_q \) of Eq.(6), and the analogous specific heat \( C_q \) of Eq.(7), characterizing the absolute moments of the increments of two LD, a linear one (c.f., Eq.(2)) and a non-linear one (c.f., Eq.(3)), within the context of a multifractal approach to financial time series analysis [14].

Tunning parameters in these dynamics introduces wide clusters and bursts of volatility in the processes. This behaviour was shown to mimics characteristics of S&P500
data which were measured according to an analogous phase transition in the behaviour of $C_q$.

In this work, the second-order derivative relating the increment of prices of assets $\Delta x(t, 1)$, and given by $C_q$, follows similar Legendre transforms for the thermodynamics analysis of (DLA) [13]. Our findings may relate the results of the kind of those described in [25] where a phase transition-like appears in the presence of two distinct phases when analysing the behaviour of $D_q$. In the latter, these two phases are built out of rare events on the one hand and some ”background” (e.g., bursts of volatility in the processes) on the other hand. We have found that a shoulder appears in the analogous $C_q$ due to the presence of wide volatility clusters as found in the S&P500 historical data, or in the non-linear LD.

The message here is that due to non-linear and additive noise terms within the LD, $C_q$ can display a shoulder to the right of its main peak data as also found in the S&P500 historical data, which may resemble a classical phase transition at a critical point.

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References

[1] B.B. Mandelbrot, Sci. Am., p.50 Feb 1999.

[2] J.-P. Bouchaud and R. Cont, Eur. J Phys. B 6, 543 (1998).

[3] H. Nakao, Phys. Rev. E 58, 1591 (1998).

[4] P. Richmond, in proceedings 2nd EPS Conference on "Application of Physics in Financial Analysis", held in Liège (Belgium) July 2000 - To appear in Eur. J. Phys. B (2001).

[5] T. Lux and M. Marchesi, Nature 397, 498 (1999).

[6] R.N. Mantegna and H.E. Stanley, Nature 376, 46 (1995).

[7] D. Sornette, P. Simonetti and J.V. Andersen, Phys. Rep. 335, 19 (2000).

[8] F. Schmitt, D. Schertzer and S. Lovejoy, in "Chaos, Fractals and Models 96", Eds. F.M. Guindani and G. Salvadori (Italian University Press, p.150-157, 1998); see also Appl. Stochastic Models Data Anal. 15, 29 (1999), Int. J. Th. Appl. Finance, in press 2000.

[9] N. Vandewalle and M. Ausloos, Eur. J Phys. B 4, 257 (1998).

[10] K. Ivanova and M. Ausloos, Eur. J Phys. B 8, 665 (1999).

[11] J.-P. Bouchaud, M. Potters and M. Meyer, Eur. J Phys. B 13, 595 (2000).

[12] H. Katsuragi, Phys. A 278, 275 (2000).

[13] J. Lee and H.E. Stanley, Phys. Rev. Lett 61, 2945 (1988); see also "Fractals and Disordered Systems", Eds. A. Bunde and S. Havlin (Springer-Verlag, p.15, 1991).

[14] E. Canessa, J. Phys. A: Math. Gen. 33, 3637 (2000).
[15] H. Takayasu, A. Sato and M. Takayasu, Phys. Rev. Lett. 79, 966 (1997).

[16] A. Sato and H. Takayasu, Phys. A 250, 231 (1998).

[17] E. Canessa and V.L. Nguyen, Phys. Stat. Sol. (b) 175, 319 (1993).

[18] J. Gradišek. S. Siegert, R. Friedrich and I. Grabec, Phys. Rev. E 62, 3146 (2000).

[19] A. Bershadskii, Phys. Rev. E 58, 2660 (1998).

[20] R. Riedi, J. Math. Anal. Appl. 189, 462 (1995).

[21] R. Pastor-Satorras, Phys. Rev. E 56, 5284 (1997).

[22] A. Georges and W. Krauth, Phys. Rev. B 48, 7167 (1993).

[23] D. Vollhardt, Phys. Rev. Lett. 78, 1307 (1997).

[24] J.C. Lessa and R.F.S. Andrade, Phys. Rev. E 62, 3083 (2000).

[25] P. Grassberger, R. Badii and A. Politi, J. Stat. Phys. 51, 135 (1988).
Figure captions

- **Fig.1**: Time evolution $\Delta x(t, 1)$ of the S&P500 stock index for the period 1980-1992 as a function of trading time lags (in a.u.) and corresponding analogous specific heat $C_q$ obtained using Eq.(7). The lower curve (a) shows the presence of an anomalous shoulder in $C_q$ when considering 3287 (S&P500) data points including the largest burst that corresponds to the crash measured in October of 1987. Curve (b) is the result of removing this Black Monday crash data point, whereas curve (c) is the result of removing all successive data. Curve (d) is the result of taking 499 initial data points only.

- **Fig.2**: Time evolution of $\Delta x(t, 1)$ governed by the LD of Eq.(1) using the linear terms in Eq.(2) (upper curves) and non-linear terms of Eq.(3) (lower curves).

- **Fig.3**: Generalized dimensions $D_q$ and multifractality strength $1 - D_{q\to\infty}$ for different time lags $T$ for the linear LD using Eq.(2) (upper curves) and the non-linear LD using Eq.(3) (lower curves). The $1 - D_{q\to\infty}$ curves have been evaluated by considering $\Delta x(t, T) \equiv x(t + T) - x(t)$.

- **Fig.4**: Analogous specific heat $C_q$ of the linear (upper curve) and non-linear (lower curve) LD derived using Eq.(7).
$X(t+1) - X(t)$

C(q)

S&P 500

$C(q)$
This figure "fig2.jpg" is available in "jpg" format from:

http://arxiv.org/ps/cond-mat/0104412v1
Linear LD

Non-linear LD

\[ C(q) \]

- \( u = -0.005 \)
- \( u = -0.02 \)
- \( u = -0.1 \)

- \( n = 1.2 \)
- \( n = 1.5 \)
- \( n = 1.8 \)