Accurate determination of time delay and embedding dimension for state space reconstruction from a scalar time series

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A new and accurate method to determine the time delay and embedding dimension for state space reconstruction of a high dimensional system from a scalar time series using time delay embedding is presented. The time delay is obtained to unprecedented accuracy by evaluating the minima of a newly defined dimension deviation function. The efficacy of our method is tested by applying it to the Lorenz system and the Mackey-Glass system. A good agreement is obtained between the shape and embedding dimension of the physical system attractor(s) and the corresponding reconstruction(s) for both the systems studied. This, along with a heuristic argument provide a validation of the proposed method.

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I. INTRODUCTION

Often, as a result of experimental limitations, only one-dimensional data is available for chaotic physical systems which have higher dimensionality. The dripping faucet\cite{1, 2} is one such chaotic system. Another such system is the Rayleigh-Benard convective system\cite{3, 4}, which was experimentally realised by Castaing et al \cite{5}.

The technique of state space reconstruction is used widely in analysis of time series data. It finds applications, for example, in analysis of the time series obtained from multi-filamentation in optical beams, fiber solitons and ocean rogue waves\cite{6}. It was concluded that predictability of rogue wave phenomenon in oceans is feasible in an interval of $5\tau$, where $\tau$ is the delay time determined using linear auto correlation. It is therefore, important to determine the delay time accurately. It also finds applications in analysis of chaotic data from rainfall and other climatic systems\cite{7}. Time delay techniques are often used in analysis of financial time series and stock trends\cite{8}. Therefore, accurately understanding phase space dynamics is of paramount importance in characterizing, predicting and eventually controlling chaos.

Yet another system of particular interest is the system in \cite{9}, which yields a fifteen dimensional attractor having an intrinsic time delay of 22$\mu$s governed by delay differential equations used to predict the system time series up to several delay periods. Handling chaotic experimental data has always posed a challenge. Grassberger and Procaccia\cite{10} defined the correlation dimension, as a scalable alternative to capacity and information dimension, for finite data sets. Chaotic systems such as the Lorenz system \cite{11} and the Mackey-Glass system \cite{12} yield solutions that lie on well characterised and multi-dimensional strange attractors \cite{13}. The shape of these attractors and their dimensionality has also been well characterised. Among the differences between the Lorenz system and the Mackey-Glass system is that the latter has a well defined time-delay parameter in its governing nonlinear delay differential equation whereas the former is governed by a set of three nonlinear coupled ordinary differential equations without any explicit time delay.

II. THE METHOD

In this Letter, we present a method to reconstruct the multi-dimensional state space and strange attractor of a chaotic system using a one-dimensional time series arrived at from solution of the governing differential equations without a priori knowledge of any implicit time delay. We address the problem of accurately determining time-delay and embedding dimension for state space reconstruction of high dimensional chaotic systems using one-dimensional system data.

The first step in this direction is the Whitney embedding theorem \cite{14} which states that a map from an $n$-manifold to a $2n + 1$ dimensional Euclidean space is an embedding. Subsequently, Takens \cite{15} showed that an $n$-manifold can be recovered from a single measured quantity. It was shown \cite{15} that time delayed versions of the measured quantity $[s(t), s(t+\tau)...s(t+2n\tau)]$ would embed the $n$-manifold. However, data from physical systems do not indicate a natural choice for the delay coordinate $\tau$ and embedding dimension $2n + 1$. Figure 1 illustrates that choice of $\tau$ affects the reconstruction significantly. We are hence motivated to give a prescription to choose the delay time $\tau$ efficiently and accurately. Linear auto correlation function has popularly been used to delay time. Further, Fraser and Swinney \cite{16} have suggested the use of average mutual information to choose the delay coordinate. However, in our present case, we found that neither choice yielded an appealing reconstruction. We hence, are motivated to suggest a prescription of our own. We briefly discuss the choice of embedding dimension.

A key idea that we use in this Letter is that of fractal dimension. We hence make some elementary definitions of importance. We first denote a open ball of radius $\epsilon$ centred at the point $x$, by $B_\epsilon(x)$. We then let $\mu(S)$ denote the natural measure associated with set $S$. The point wise fractal dimension $D_p$ may be defined as \cite{17}:

$$D_p = \lim_{\epsilon \to 0} \frac{\log(\mu(B_\epsilon(x)))}{\log(\epsilon)}$$

(1)

The remarkable feature of the point wise dimension is that it is independent of the point $x$ that is chosen. A heuristic argument for this may be found in Ott \cite{17}. A more comprehensive review of fractal dimensions may be found in Farmer et al’s work\cite{18}.

We test our methodology on the well characterized Lorenz \cite{11} and Mackey-Glass \cite{12} systems. The Lorenz system is given by

$$\frac{dx}{dt} = \sigma(y-x)$$

(2)

$$\frac{dy}{dt} = x(\rho - z) - y$$

(3)

$$\frac{dz}{dt} = xy - \beta z$$

(4)
The Mackey-Glass system is given by:

\[
\frac{dx}{dt} = \frac{\beta x_r}{1 + x^n_t} - \gamma x
\]  

(5)

Where \(x_r\) represents the value of \(x\) at time \(t - \tau\). The values of \(\beta, \gamma, n\) and \(\tau\) were chosen to be 2, 1, 10 and 1500 respectively.

We next define the dimension deviation function, \(f\) as,

\[
\theta\left(\int_{\epsilon} \sum_{i=1}^{N} \sqrt{\sum_{j=1}^{m-1} \left[s(i + k\tau) - s(j + k\tau)\right]^2}\right)
\]  

(6)

The measure \(\mu\) is;

\[
\mu = \frac{1}{N} \sum_{j=1}^{N} \theta(\epsilon - r_{ij})
\]  

(7)

where \(\theta\) is the unit step function, \(N\) the total number of state space points obtained, and \(\epsilon\), an arbitrarily small number. It is then easy to see that the following equation for \(D_p\) would correspond to the expression for the pointwise dimension taken centred about point “i”

\[
D_p(i, \tau) = \lim_{\epsilon \to 0} \frac{\log(\mu(i, \tau, \epsilon))}{\log(\epsilon)}
\]  

(8)

We next define the dimension deviation function, \(f\) as,

\[
f(\tau) = \frac{1}{N} \sum_{i=1}^{N} (D_p(i, \tau) - \bar{D}_p)^2
\]  

(9)

FIG. 1: (a) shows the original Lorenz attractor (b)-(h) show the reconstructions of the Lorenz attractor from a time series for successively larger values of delay coordinate. The time series was generated by using the x coordinates of \(10^5\) successive points. The numerical solution of the Lorenz equation was obtained using the euler method from the parameters \(\sigma = 10, \beta = 2.667\) and \(\rho = 28\), with initial condition \(x = -10.4, y = -20.6, z = 30.5\). The high degree of similarity between fig:1a and fig:1e vindicates the method used for reconstruction.

We now propose a new prescription for the choice of delay coordinate in a reconstruction. However, we would first require to make a guess for the embedding dimension \(m\). But, in principle, once \(m\) is determined by the prescriptions suggested later, the process may be repeated. We let \(s(i)\) denote the \(i^{th}\) entry of the time series. Then we write the euclidean distances as:

\[
r_{ij} = \sqrt{\sum_{k=0}^{m-1} [s(i + k\tau) - s(j + k\tau)]^2}
\]  

(6)

The measure \(\mu\) is;

\[
\mu = \frac{1}{N} \sum_{j=1}^{N} \theta(\epsilon - r_{ij})
\]  

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\[
f(\tau) = \frac{1}{N} \sum_{i=1}^{N} (D_p(i, \tau) - \bar{D}_p)^2
\]  

(9)
We now claim that the minima of $f(\tau)$ is a good choice for $\tau$.

We motivate this claim, with the following argument. We first notice that $\mu(i, \tau, \epsilon)$ is the measure associated with an open ball or radius $\epsilon$ centred around a point in phase space, labelled $i$, reconstructed with delay time $\tau$. From Equation 8, it follows that $D_p(i, \tau)$ is the pointwise dimension of the attractor. Hence $f(\tau)$ is the standard deviation in the pointwise dimension. Should the reconstruction be a one that recovers most of the attracting set dynamics, we would expect to obtain zero standard deviation (since the pointwise dimension is invariant with respect to the point chosen, for an attracting set [17]). Hence a minima in the standard deviation would definitely occur if the attractor is fully recovered, since the standard deviation is necessarily a positive quantity.

We may further argue that only if there exists a set of points $S_1$ where the measure remains invariant and positive and another disjoint set $S_2$ with another value of the measure would we see non zero or relatively larger values of standard deviation in the pointwise dimension. However the measure on the set $S_1 \cup S_2$ would not be ergodic, and hence cannot correspond to an attracting set of a smooth map. An ergodic measure $\mu$ cannot be decomposed into two measures, $\mu_1$ and $\mu_2$, such that [17];

$$\mu = p\mu_1 + (1 - p)\mu_2$$  \hspace{1cm} (11)

Where $p$ is any real which lies in the interval $(0,1)$.

It may however prove tricky to actually compute the pointwise dimension from a finite quantity of data, since no finite amount of data can give an accurate estimate of measure. It is therefore recommended to use very small values of $\epsilon$ in Equation 8, but allow only those values that contain at least two points within the open ball, to avoid outliers. A regression fit of $\log(\epsilon)$ against $\log(\mu)$ ought to give a reliable estimate of the the pointwise dimension. Further, computations can be cut down by choosing to use a large representative set of points, as the centers for the computation of the pointwise dimension, rather than the entire data set.

In the case of the Lorenz attractor, we observe that the fist local minima of the dimension deviation function obtained at $\tau = 1805$ gives the best reconstruction observed visually. Fig 2 shows the dimension deviation as a function of $\tau$.

Figure 3 shows dimension deviation function for the Mackey-Glass attractor. The first minima was obtained at $\tau = 1605$ units, while the delay used in the underlying attractor was $\tau = 1500$ units. This high degree of accuracy indicates that the first local minima of the dimension deviation is indeed a good choice for the delay coordinate.
FIG. 3: Figure shows the dimension-deviation function against delay time for the Mackey-Glass attractor that was studied. The first minima was found at $\tau = 1600$. The delay coordinate chosen in the underlying delay differential equation is $\tau = 1500$. Time is in normalised units.

### III. OTHER METHODS

Some other methods for determining the optimal delay have been proposed in literature\[16\]\[19\]. The first method uses the linear auto correlation function. The linear auto correlation is defined by the following relationship.

$$C_l(\tau) = \frac{1}{N} \sum_{m=1}^{N} [s(m + \tau) - \bar{s}] [s(m) - \bar{s}]$$  \hspace{1cm} (12)

where $\bar{s}$ is the average value of the time series. We immediately remark that this definition follows from finding the best fit function $C_l(\tau)$ for the linear relationship,

$$s(n + \tau) - \bar{s} = C_l(\tau) [s(n) - \bar{s}]$$  \hspace{1cm} (13)

The prescription often used for the choice of $\tau$ is the first zero of the auto-correlation function defined above $[19]$. It is easy to see that, the linear auto-correlation function, may yield a bad choice for $\tau$ for nonlinear systems, since minimising the linear dependence of terms separated by a time-span of $\tau$, does not necessarily minimise the overall dependence that arises from the non linear terms. Further, minimising the dependence of terms separated by $\tau$ may not be the best strategy, since we are looking for an intermediate $\tau$ such that terms separated by a distance of $\tau$ are neither statistically independent, nor nearly overlapping. In the study of the Lorenz attractor, we found that that the auto correlation had its first zero, far from the point where the best visual reconstruction was found. This demonstrates the failure of this prescription for nonlinear systems. Yet another prescription, used often is the mutual information function. It is a generalization of the linear auto-correlation function, and relates the information content in one set to the information content in another. It was first proposed by Gallaghar$[20]$. In the context of time series analysis, we measure the mutual information content, of terms separated by a distance $\tau$. We define mutual information between terms separated by a distance $\tau$ to be;

$$I(\tau) = \sum_n P[n, n+\tau] \log \left( \frac{P[n]P[n+\tau]}{P[n, n+\tau]} \right)$$  \hspace{1cm} (14)

where $P[n]$ is the probability measure for the occurrence of $s(n)$ and $P[n, n+\tau]$ represents the joint probability of their occurrence. The prescription often suggested is the use of the first minima of the average mutual information as an appropriate choice of $\tau$ $[16]$. However, since $I(\tau)$ also measures the information between two terms separated, by $\tau$, we expect that its first minima is close to the zero of the linear auto-correlation function. $P(s(n))$ in determined by the relative frequency of the occurrence of the value $s(n)$ and $P(s(n+\tau))$ is determined likewise. $P(s(n), s(n+\tau))$ is determined by the relative frequency of occurrence of the the pair of numbers $s(n)$ and $s(n+\tau)$, separated by exactly a time-span of $\tau$.

In the present study of the Lorenz attractor, the plot of the average mutual information against $\tau$ yielded a minima
that was far from the delay time used in the best visual reconstruction. However, it was closer to the optimal value of delay time, as compared to that predicted by the linear auto-correlation. Hence, this method too yields a value that is far off in the present case.

IV. EMBEDDING DIMENSION

While there exist many methods [21–23] that one may use to determine the optimal value of the embedding dimension, we suggest one that is along the lines of the method of false nearest neighbours listed in Abarbanel et al’s [19] work. We rewrite Equation 7 however now making it a function of \(m\), keeping \(\tau\) fixed.

\[
\nu(i, m, \epsilon) = \sum_{j=1}^{N} \theta(\epsilon - r_{ij})
\]

Here \(\nu\) indicates the total number of nearest neighbours. We now observe that at a low dimension the number of false neighbours at every point would be higher. However, when embedded in any dimensionality higher than the optimal embedding dimension, the number of neighbours would remain nearly the same. Hence looking for the point of saturation of the total number of neighbours, against the embedding dimension, would give us a good estimate of the optimal embedding dimension.

In the present case, for the Lorenz attractor, using the plot of embedding dimension against the number of neighbours it was found to saturate at a embedding dimension value of 5.

V. SUMMARY

To summarise, we have tested the methods for delay coordinate choice given in literature and found that they have not succeeded in our case study of the Lorenz and Mackey-Glass systems. We further developed an alternative prescription for the choice of delay coordinate, modelled after the deviation in the pointwise dimension. It worked significantly better for our particular case.

We then used the same definition to write down a prescription for the choice in embedding dimension as well. The major shortcoming of the proposed methodology lies in an arbitrary initial choice in embedding dimension that has to be made to accurately determine the delay time. To circumvent this, an arbitrary and high choice of the embedding dimension can be made to determine the optimal delay, then determine the optimal embedding dimension and further redo the calculation for \(\tau\) in the new embedding dimension.

Our method has a time complexity of \(O(n^2)\), while mutual information, has a algorithm with time complexity \(O(n\log n)\) [16]. However, one may average over a representative set of points rather than the whole set and obtain the standard deviation to reduce computation time by any desirable factor.

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