A Stable Supergravity Dual of Non-supersymmetric Glue

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ABSTRACT

We study non-supersymmetric fermion mass and condensate deformations of the AdS/CFT Correspondence. The 5 dimensional supergravity flows are lifted to a complete and remarkably simple 10 dimensional background. A brane probe analysis shows that when all the fermions have an equal mass a positive mass is generated for all six scalar fields leaving non-supersymmetric Yang Mills theory in the deep infra-red. We numerically determine the potential, produced by the background, in the Schroedinger equation relevant to the study of $O^{++}$ glueballs. The potential is a bounded well, providing evidence of stability and for a discrete, confined spectrum. The geometry can also describe the supergravity background around an (unstable) fuzzy 5-brane.

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1 Introduction

The possibility that there is a string description of large $N$ Yang Mills theory has been speculated on for many years [1]. The AdS/CFT Correspondence [2, 3, 4] represented the first concrete example of such a duality describing the $\mathcal{N} = 4$ super Yang Mills (SYM) theory. In this paper we deduce a IIB supergravity background that describes $\mathcal{N} = 4$ SYM deformed by mass terms for all the adjoint matter fields leaving pure non-supersymmetric Yang Mills in the deep infra-red.

In the AdS/CFT Correspondence supergravity fields behave as sources in the dual gauge theory. Expectation values of field theory operators are obtained from functional derivatives on the supergravity partition function with respect to the boundary values of the supergravity fields. It is therefore a crucial aspect of the correspondence that the supergravity partition function must be calculable in the presence of all infinitesimal sources in order that derivatives with respect to those sources are well defined. In fact, for sources which break the $\mathcal{N} = 4$ theory’s conformal symmetry, infinitesimal has no meaning since they become the only mass scale in the problem. It should therefore be possible to find gravity duals of deformed versions of the $\mathcal{N} = 4$ SYM theory, including non-supersymmetric theories.

The technology [5, 6, 7, 8] required to deform the AdS/CFT Correspondence has already been developed. In 5d supergravity one must identify the field with the appropriate symmetry properties to play the role of the field theory source and determine its action. Solutions of the classical equations of motion can then be found and are described in the literature as renormalization group flows. The 5d supergravity backgrounds are often hard to interpret in terms of the dual field theory. Pilch and Warner [9] have developed a miraculous ansatz for lifting the 5d solutions to provide the 10d metric and dilaton. It is then possible, with some work, to solve the field equations for the remaining supergravity potentials. The resulting solutions are open to the use of brane probes [2, 10, 11, 12, 13, 14, 15, 16] which, using the Dirac Born Infeld action, provide a direct translation between the gravity background and the field theory description.

Much of the early work on deformations has concentrated on supersymmetric theories such as the $\mathcal{N} = 4$ theory on moduli space [17], $\mathcal{N} = 1$ Leigh Strassler theory [18, 19], the $\mathcal{N} = 2^*$ theories [20, 21, 11, 12] and $\mathcal{N} = 1^*$ theories [22, 23, 18]. More recently interest has turned to non-supersymmetric deformations of gauge/gravity duals [24, 25, 26, 27, 28, 29, 30]. Most of these papers have focused attention [26, 27, 28, 29] on deformations of more involved $\mathcal{N} = 1$ supersymmetric constructions such as the Maldecena Nunez [31] and the Klebanov Strassler [32] backgrounds. These theories have discrete vacua and hence supersymmetry breaking perturbations will not result in an unstable background. The resulting backgrounds are though necessarily more complicated than deformations of the $\mathcal{N} = 4$ theory. In [30] we constructed the first complete 10d background of a non-supersymmetric deformation of $\mathcal{N} = 4$ involving a
mass term for a scalar operator. The resulting field theory and supergravity background shared an instability in the scalar potential. This highlights one of the most challenging problems in constructing non-supersymmetric solutions, the need to find a stable deformation.

In this paper we will deform the AdS/CFT Correspondence by including a supergravity scalar that is a source for an equal mass term for each of the four adjoint fermions of $\mathcal{N} = 4$ SYM. We solve for the 5d supergravity flows numerically. The second order equations of motion have solutions describing both a mass and a condensate for the fermion operator. It is easy to see numerically that if a condensate is present the flows are singular at finite radius of the AdS space. The mass only solution though is a unique flow and therefore much harder to numerically study since the boundary conditions must be arbitrarily fine tuned. Our analysis though suggests that this flow may also be singular in the deep infra-red. The interpretation of such singularities remains open. For example the backgrounds describing $\mathcal{N} = 4$ SYM on moduli space \cite{17} are singular but those singularities are understood to correspond to the presence of D3 branes in the solution. In the $\mathcal{N} = 2^*$ theory \cite{19,20,21,11,12} the singularities correspond to the divergence of the running gauge coupling. On the other hand the backgrounds of Klebanov Strassler \cite{32} and Maldecena Nunez \cite{31} are championed for their smooth behaviour which is indeed nice. We will not resolve this issue here, although clearly only after lifting the solution, as we do, to 10d can one hope to address the physical meaning or otherwise of singularities. We will find, encouragingly, though that the singularity does not prevent the background describing a non-singular glueball spectrum.

The main effort of this paper is to use Pilch and Warner’s ansatz \cite{9,19} and the field equations to construct the full 10d background appropriate to these 5d solutions. The resulting background is remarkably simple. The stability of the solution is then tested using a brane probe. It shows that in the field theory the 6 scalars have positive masses radiatively induced by the fermion mass and hence there is no instability to the formation of a scalar vev (this means that at the 5d supergravity level there is no instability to the scalar in the 20 of SU(4)$_R$ switching on). We expect that the SO(4) symmetry acting on the fermions prevents any other elements of the 5d scalar in the 10 of SU(4)$_R$ (corresponding to the operators $\lambda_i\lambda_j$) switching on. At the level of this analysis, the background appears stable. At first sight it may seem surprising that the 10d lift has a scalar mass operator present which was not explicitly introduced at the 5d level. However, this operator is not represented by a scalar in the 5d supergravity so its presence or otherwise is not clear at the 5d level. Many of the supersymmetric deformations \cite{10,19,20,11,12,22,18} implicitly assume the presence of this operator in 5d with confirmation only coming from a brane probe in 10d as we find here. The field theory the background describes is $\mathcal{N} = 4$ SYM with masses for all the matter fields leaving pure non-supersymmetric Yang Mills in the infra-red. We call this theory Yang Mills$^*$ following the nomenclature used
for supersymmetric deformations of $\mathcal{N} = 4$ SYM.

This theory is hopefully of real use as an approximation to non-supersymmetric, pure Yang Mills theory. Of course since the $\mathcal{N} = 4$ theory is strongly coupled at all scales, the deformed theory is strongly coupled at the scale of the mass of the adjoint matter fields and in this respect differs. This situation is analogous to the thermalized 5d background of Witten [4] which also describes 4d non-supersymmetric Yang Mills in the infra-red. That theory has been used though to compute glueball masses [33] with some success, supporting the use of these geometries. It will be interesting in the future to compare the predictions of these two variants to begin to determine the size of systematic errors induced by the massive matter in each. As a step in that direction here we numerically compute the potential in the relevant Schroedinger equation [33] for the $O^{++}$ glueballs. For the mass only flow the potential is seen to be a well, from which we deduce that there is a discrete spectrum indicating that the geometry indeed describes a confining gauge theory with a mass gap.

We also note that relative to the thermalized geometry, the Yang Mills$^*$ deformation is a more systematic approach to obtaining non-supersymmetric Yang Mills and is more open to the introduction of quarks (the analysis [34] of probe D7 branes in anti de-Sitter (AdS) space appears a particularly fruitful approach). The thermalization trick would induce masses for the matter fields too.

The supergravity field we study is also capable of describing an equal bilinear condensate for each of the four adjoint fermions. As part of the analysis of the $\mathcal{N} = 1^*$ theory by Polchinski and Strassler [23] they showed that placing a fuzzy D5 brane in AdS induces, asymptotically, precisely this operator (see [35] for a review of that argument). These backgrounds then plausibly describe the supergravity theory induced around a fuzzy 5-brane. A fuzzy 5-brane is not stable unless there is some force to oppose the potential energy cost of non-commutative expansion. In the $\mathcal{N} = 1^*$ theory the 5-brane is polarized by a background 2-form potential (dual to the supersymmetry breaking mass). An alternative spin was put on the idea in [35] where the fuzzy expansion was supported by centrifugal force (corresponding to the presence of a chemical potential in the field theory). In our case there is no supporting force present and hence it is not surprising that the brane probe scalar potential is unbounded. The construction is unstable to the emission of commutative D3 branes.

In the next section we describe the Yang Mills$^*$ deformation in 5d supergravity. In section 3 we describe the oxidation process to 10d and then in section 4 we brane probe the solution. Section 5 discusses the glueball Schroedinger equation for the geometry. The full background is gathered together in the appendix for ease of reference.
2 Deformations in 5d Supergravity

According to the standard AdS/CFT Correspondence map \[3, 4\] each supergravity field plays the role of a source in the dual field theory. The simplest possibility is to consider non-trivial dynamics for a scalar field in the 5d supergravity theory. We only allow the scalar to vary in the radial direction in AdS with the usual interpretation that this corresponds to renormalization group running of the source. As is standard in the literature \[5, 24\] we look for solutions where the metric is described by

\[ ds^2 = e^{2A(r)} dx^\mu dx_\mu + dr^2 \]  

where \( \mu = 0..3 \) and \( r \) is the radial direction in AdS\(_5\). The scalar field has a Lagrangian

\[ \mathcal{L} = \frac{1}{2} (\partial \lambda)^2 - V(\lambda) \]  

There are two independent, non-zero, elements of the Einstein tensor (\( G_{00} \) and \( G_{rr} \)) giving two equations of motion plus there is the usual equation of motion for the scalar field \[5\]

\[ \lambda'' + 4 A' \lambda' = \frac{\partial V}{\partial \lambda} \]  

\[ 6A'^2 = \lambda'^2 - 2V \]  

\[ -3A'' - 6A'^2 = \lambda'^2 + 2V \]  

In fact only two of these equations are independent but it will be useful to keep track of all of them.

In the large \( r \) limit, where the solution will return to AdS\(_5\) at first order and \( \lambda \to 0 \) and \( V \to \frac{m^2}{2} \lambda^2 \), only the first equation survives with solution

\[ \lambda = ae^{-\Delta r} + be^{-(a-\Delta)r} \]

\( a \) and \( b \) are constants and

\[ m^2 = \Delta(\Delta - 4) \]

\( a \) is interpreted as a source for an operator and \( b \) as the vev of that operator since \( e^r \) has conformal dimension 1.

If the solution retains some supersymmetry then the potential can be written in terms of a superpotential
and the second order equations reduce to first order

\[ V = \frac{1}{8} \left| \frac{\partial W}{\partial \lambda} \right|^2 - \frac{1}{3} |W|^2 \]  

(8)

\[ \lambda' = \frac{1}{2} \frac{\partial W}{\partial \lambda}, \quad A' = -\frac{1}{3} W \]  

(9)

The deformation we will consider will break supersymmetry completely and therefore not have such a description.

2.1 A Fermionic Operator

Let us now make a particular choice for the scalar field we will consider. We take a scalar from the multiplet in the 10 of SO(6). These operators have been identified \[22\] as playing the role of source and vev for the fermionic operator \(\psi_i \bar{\psi}_j\) in the field theory. In particular we will choose the scalar corresponding to the operator

\[ \mathcal{O} = \sum_{i=1}^4 \psi_i \bar{\psi}_i \]  

(10)

The potential for the scalar can be obtained from the \(N = 1^*\) solution of \[22\] by setting their two scalars equal (to be precise one must set their \(m = \sqrt{3/4\lambda}\) and \(\sigma = \sqrt{1/4\lambda}\) to maintain a canonically normalized kinetic term)

\[ V = -\frac{3}{2} \left( 1 + \cosh^2 \lambda \right) \]  

(11)

In this case \(m^2 = -3\) and the ultra-violet solutions are

\[ \lambda = M e^{-r} + Ke^{-3r} \]  

(12)

The field theory operators have dimension 1 and 3. Thus in what follows \(M = 0\) corresponds to a solution with just bi-fermion vevs while \(K = 0\) corresponds to the purely massive case.

2.2 Numerical Solutions

We have not been able to solve the second order equations of motion analytically but they are easily solved numerically. The evolution of \(\lambda\) as a function of \(r\) for a variety of different initial conditions on \(\lambda'\) is shown in Fig 1. The mass only and condensate only cases are highlighted. The functions \(A(r)\) evolve as \(A(r) \sim r\) except in the very deep infra-red. Note that \(\lambda\) typical diverges at finite \(r\) with the \(K = 0\) solution lying on the boundary between solutions that blow up positively and negatively. In the right hand plot in Fig. 1 we show solutions close to the
mass only solution displaying this behaviour in the infrared \((r < 0)\). The uniqueness of the mass only solution makes it very hard to study in the deep infra-red because one must arbitrarily fine tune the initial conditions on \(\lambda\). Hence we can not specify the final fate of the mass only solution in the very deep infra-red although Fig. 1 may suggest that the solution is diverging.

Whilst analyzing this flow we also analyzed the flow where an operator corresponding to a vev or a mass for just three of the four fermions is switched on (in this case the supergravity scalar potential is given by \(V = -3/8(3 + \cosh^2(2\lambda/\sqrt{3}) + 4\cosh(2\lambda/\sqrt{3}))\). The mass only solution of this case is one of the \(\mathcal{N} = 1^*\) solutions \([22]\) which is known analytically. The behaviour of the second order solutions is very similar to Fig. 1 with the mass only flow lying, in a similar fashion, between clearly divergent flows. In that case the analytic, mass only flow does diverge suggesting further that our mass only flow diverges. As discussed in the introduction the interpretation of the singularity is a delicate issue and remains to be determined.

![Figure 1](image_url)

Figure 1: Plots of \(\lambda\) vs \(r\) for a variety of initial conditions on \(\lambda'\). In the left hand figure the marked regions correspond to initial conditions: I positive mass and condensate; II negative mass, positive condensate; III positive mass, negative condensate. The right hand figure shows a close up of initial conditions close to the mass only solution in the IR \((r < 0)\)

## 3 The 10d Background

To lift the solution to a 10d background requires us to find a solution of the full set of IIB supergravity equations of motion. As was found in \([18, 19]\) where a fermion mass term was introduced in a supersymmetric context all the supergravity fields will be non-zero. We first summarize the field equations taken from \([36]\):

- The Einstein equations:

\[
R_{MN} = T_{MN}^{(1)} + T_{MN}^{(3)} + T_{MN}^{(5)}
\]  

\((13)\)
where the energy momentum tensor contributions from the dilaton, 2-form potential and 4-form potential are given by

\[ T_{MN}^{(1)} = P_M P_N^* + P_N P_M^* \]  
\[ T_{MN}^{(3)} = \frac{1}{8}(G^{PQ}_M G_{PQN}^* + G^{*PQ}_M G_{PQN} - \frac{1}{6} g_{MN} G^{PQR} G_{PQR}) \]  
\[ T_{MN}^{(5)} = \frac{1}{6} F^{PQRS}_M F_{PQRS} \]

The dilaton is written in unitary gauge where

\[ P_M = f^2 \partial_M B, \quad Q_M = f^2 \text{Im}(B \partial_M B^*), \quad f = \frac{1}{(1 - BB^*)^{1/2}} \]

The more familiar dilaton-axion field is given by

\[ a + i e^\Phi = i \frac{(1 - B)}{(1 + B)} \]

and the 3-form field strength is defined by

\[ G_{(3)} = f(F_{(3)} - BF_{(3)}^*) \]

• The Maxwell equations:

\[ (\nabla^P - iQ^P)G_{MNP} = P^P G_{MNP}^* - \frac{2}{3} i F_{MNPQR} G^{PQR} \]  
\[ (\nabla^M - 2iQ^M)P_M = -\frac{1}{24} G^{PQR} G_{PQR} \]

• The self-dual equation:

\[ F_{(5)} = \ast F_{(5)} \]

• Bianchi identities:

\[ F_{(3)} = dA_{(2)}, \quad dF_{(5)} = -\frac{1}{8} \text{Im}(F_{(3)} \wedge F_{(3)}^*) \]
3.1 The UV limit

Let us first concentrate on lifting the ultra-violet ($r \to \infty$) limit of the 5d flow. The supergravity scalar lifts to the 2-form potential in 10d \[37\]. To determine its form we can use the group theory technique in \[23, 35\].

Parameterize the 6d space perpendicular to the D3 branes of the construction as

$$z_1 = \frac{w^1 + iy^1}{\sqrt{2}}, \quad z^2 = \frac{w^2 + iy^2}{\sqrt{2}}, \quad z_3 = \frac{w^3 + iy^3}{\sqrt{2}} \quad (24)$$

Under the SO(2) rotation subgroups of the 6 dimensional representation of SU(4), $z^i \to e^{i\phi_i}z_i$, the 4 dimensional representation transforms as

$$\lambda_1 \to e^{i(\phi_1 - \phi_2 - \phi_3)/2}\lambda_1, \quad \lambda_2 \to e^{i(-\phi_1 + \phi_2 - \phi_3)/2}\lambda_2,$$

$$\lambda_3 \to e^{i(-\phi_1 - \phi_2 + \phi_3)/2}\lambda_3, \quad \lambda_4 \to e^{i(\phi_1 + \phi_2 + \phi_3)/2}\lambda_4. \quad (25)$$

We can thus construct a 3-form field strength with the symmetry properties of a fermion mass or condensate

$$\langle \lambda_1 \lambda_4 \rangle dz^1 \wedge dz^2 \wedge dz^3 + \langle \lambda_2 \lambda_3 \rangle d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + \langle \lambda_1 \lambda_3 \rangle dz^1 \wedge \bar{d}z^2 \wedge \bar{d}z^3 + \langle \lambda_1 \lambda_2 \rangle d\bar{z}^1 \wedge dz^2 \wedge dz^3 \quad (26)$$

setting all the operators equal gives

$$F_3 = dw^1 \wedge dw^2 \wedge dw^3 + idy^1 \wedge dy^2 \wedge dy^3 \quad (27)$$

It will be useful to write the 6d space in terms of two $S^2$ (with metrics $d\Omega^2_{\pm} = d\theta^2_{\pm} + \cos^2 \theta_{\pm} d\phi^2_{\pm}$) corresponding to the $w$ and $y$ spaces, a radial direction $r$, and an angular coordinate between the spheres, $\alpha$. The appropriate 2-form is then

$$A_2 = \cos^3 \alpha \cos \theta_+ d\theta_+ d \wedge \phi_+ + i \sin^3 \alpha \cos \theta_- d\theta_- \wedge d\phi_- \quad (28)$$

A survey of the field equations reveals that only the 2-form’s Maxwell equation is of leading order in the perturbing field $\lambda$. We find that

$$A_2 = 2\lambda (i \cos^3 \alpha \cos \theta_+ d\theta_+ \wedge d\phi_+ - \sin^3 \alpha \cos \theta_- d\theta_- \wedge d\phi_-) \quad (29)$$

indeed reproduces the asymptotic form of the 5d field equation

$$\lambda'' + 4\lambda' = -3 \quad (30)$$

when substituted into that Maxwell equation.

The ultra-violet solution for $A_2$ provides a useful check at each stage of the computation of the full lift which we come to next.
3.2 The Metric

Pilch and Warner [9, 18] have provided an ansatz for the lift of a 5d supergravity flow to 10d (note that, although they study supersymmetric flows, their ansatz is not restricted to the supersymmetric solution of the second order equations of motion). In particular the ansatz provides us with the metric and dilaton. We can find the lifts we want as a limit of the metrics in [18]; that lift is of the $\mathcal{N} = 1^*$ GPPZ flows [22] in which three of the fermions are given a mass and the fourth develops a bilinear condensate. Setting these scalars equal (again to be precise one must set their $m = \sqrt{3/4 \lambda}$ and $\sigma = \sqrt{1/4 \lambda}$ to maintain a canonically normalized kinetic term) gives the metric we require

$$ds^{10}_2 = \xi^{\frac{1}{2}} ds^{2}_{1,4} + \xi^{-\frac{3}{2}} ds^{2}_5$$

The metric of the deformed five sphere in their coordinates $(u^i, v^i, i = 1..3)$ is given by

$$ds^2_5 = c^2 du^i du_i - 4s^2 u.v du_i dv_i + c^2 dv^i dv_i + c^2 s^2 d(u.v)^2$$

where

$$c = \cosh \lambda, \quad s = \sinh \lambda$$

This metric is subject to the constraint

$$u^2 + v^2 = 1$$

The warp factor is given by

$$\xi^2 = c^4 - 4s^4 (u.v)^2$$

We must move to more appropriate coordinates for our problem. The metric can be diagonalized by the change of coordinates

$$U^i_\pm = \frac{1}{\sqrt{2}[u^i \pm v^i]}$$

$$ds^2_5 = (c^2 - s^2[U^2_+ - U^2_-]) dU^i_+ dU^i_+ + (c^2 + s^2[U^2_+ - U^2_-]) dU^i_- dU^i_- + 4c^2 s^2 U^i_+ U^i_- dU^i_+ dU^i_-$$

The constraint can now be applied using the coordinates used in the UV limit above $(r, \alpha$ and two $S^2$ parameterized by $\theta_\pm, \phi_\pm$)

$$U^1_+ = \cos \theta_+ \cos \phi_+ \cos \alpha, \quad U^2_+ = \sin \theta_+ \cos \phi_+ \cos \alpha, \quad U^3_+ = \sin \theta_+ \cos \alpha$$

$$U^1_- = \cos \theta_- \cos \phi_- \sin \alpha, \quad U^2_- = \sin \theta_- \cos \phi_- \sin \alpha, \quad U^3_- = \sin \theta_- \sin \alpha$$
The metric then takes the form

\[ ds^2_5 = \cos^2 \alpha (c^2 - s^2 \cos 2\alpha) d\Omega^2_+ + \sin^2 \alpha (c^2 + s^2 \cos 2\alpha) d\Omega^2_- + \xi^2 d\alpha^2 \]  

(39)

and the warp factor becomes

\[ \xi^2 = c^4 - s^4 \cos^2 2\alpha \]  

(40)

### 3.3 The Ricci Tensor

The calculation of the Ricci tensor is carried out by computer and the second order 5d flow equations for the scalar field are used throughout to simplify the expressions. The results are lengthy but we note that the non-zero components of the Ricci tensor are

\[ R_{00} = R_{11} = R_{22} = R_{33}, \quad R_{rr}, \quad R_{a\alpha}, \quad R_{r\alpha} = R_{\alpha r}, \quad R_{66} = R_{77}, \quad R_{88} = R_{99} \]  

(41)

### 3.4 The Dilaton

The dilaton can again be extracted from [18] where they provide

\[ \mathcal{M} = SS^T = \frac{1}{\xi} \begin{pmatrix} \cosh^2 \lambda & \sinh^2 \lambda \cos 2\alpha \\ \sinh^2 \lambda \cos 2\alpha & \cosh^2 \lambda \end{pmatrix} \]  

(42)

where (in unitary gauge)

\[ S = f \begin{pmatrix} 1 & B \\ B^* & 1 \end{pmatrix}, \quad f = \frac{1}{(1 - |B|^2)^{1/2}} \]  

(43)

we thus find

\[ f = \frac{1}{\xi^{1/2}} \sqrt{\frac{\cosh^2 \lambda + \xi}{2}}, \quad B = \frac{\sinh^2 \lambda \cos 2\alpha}{\cosh^2 \lambda + \xi} \]  

(44)

Note that \( B \) is a real function and therefore from [18] the axion is zero for this flow. The \( r \) dependence implies the gauge coupling runs although finding the correct coordinate system to match it to the gauge theory will be difficult.

### 3.5 The 2-form and 4-form Potentials

We now move on to determining the potentials in the solution. Motivated by the UV limit we make an ansatz for the 2-form potential of the form
\[ A_{(2)} = i A_+ (\lambda(r), \alpha) \cos^3 \alpha \cos \theta_+ d\theta_+ \wedge d\phi_+ - A_- (\lambda(r), \alpha) \sin^3 \alpha \cos \theta_- d\theta_- \wedge d\phi_- \quad (45) \]

where \( A_+ \) and \( A_- \) are arbitrary functions that become \( \lambda(r) \) in the UV.

The non-vanishing components of the three form energy momentum tensor are then

\[ T^{(3)0} = T^{(3)1} = T^{(3)2} = T^{(3)3} = -\frac{1}{8}[A + B + C + D] \quad (46) \]

\[ T^{(3)r} = \frac{1}{2}[A + C] + T^{(3)0} \quad (47) \]

\[ T^{(3)\alpha} = \frac{1}{2}[B + D] + T^{(3)0} \quad (48) \]

\[ T^{(3)}_6 = T^{(3)}_7 = \frac{1}{2}[A + B] + T^{(3)0} \quad (49) \]

\[ T^{(3)}_8 = T^{(3)}_9 = \frac{1}{2}[C + D] + T^{(3)0} \quad (50) \]

\[ T^{(3)r}_\alpha = \frac{1}{2}[G^{\alpha 67} G_{\alpha 67} + G^{\alpha 89} G_{\alpha 89}] \quad (51) \]

where the functions \( A, B, C \) and \( D \) are given by

\[ A = g^{rr} g^{66} g^{77} |G_{r 67}|^2, \quad B = g^{\alpha \alpha} g^{66} g^{77} |G_{\alpha 67}|^2, \]

\[ C = g^{rr} g^{88} g^{99} |G_{r 89}|^2, \quad D = g^{\alpha \alpha} g^{88} g^{99} |G_{\alpha 89}|^2 \quad (52) \]

We now turn to the 4-potential. The self duality condition of the five form field strength is satisfied by construction using the ansatz

\[ F_{(5)} = F + *F, \quad F = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\omega(r, \alpha) \quad (53) \]

The non-vanishing components of the five form energy momentum tensor are

\[ T^{(5)0}_r = T^{(5)1}_r = T^{(5)2}_r = T^{(5)3}_r = -T^{(5)6}_r = -T^{(5)7}_r = -T^{(5)8}_r = -T^{(5)9}_r = X + Y \quad (54) \]

\[ T^{(5)r}_\alpha = -T^{(5)\alpha}_r = X - Y \quad (55) \]

where the functions \( X \) and \( Y \) are given by
\[ \mathcal{X} = \frac{1}{2} g^{00} g^{11} g^{22} g^{33} g^{rr} \left( \frac{\partial \omega}{\partial r} \right)^2, \quad \mathcal{Y} = \frac{1}{2} g^{00} g^{11} g^{22} g^{33} g^{\alpha\alpha} \left( \frac{\partial \omega}{\partial \alpha} \right)^2 \] (56)

and

\[ T_{(5)}^{\alpha} = \frac{1}{2} g^{00} g^{11} g^{22} g^{33} g^{rr} \left( \frac{\partial \omega}{\partial \alpha} \right) \left( \frac{\partial \omega}{\partial \alpha} \right) \] (57)

To solve the supergravity equations we need to disentangle the contributions from the 2-form potential and the 4-form. Furthermore, we need to separate the function \( A_+ \) from \( A_- \). We achieve this with the following combinations in which the 4-form cancels

\[ R_7^r - R_9^r + 2R_r^r + 2R_\alpha^r - 2T_{(1)}^r - 2T_{(5)}^\alpha = A + B \] (58)

\[ R_9^r - R_7^r + 2R_r^r + 2R_\alpha^r - 2T_{(1)}^r - 2T_{(5)}^\alpha = C + D \] (59)

Since \( \lambda \) is the only function of \( r \) in the solution we can separate out the pieces proportional to \( \lambda^2 \) and those not, to distinguish between, for example, \( A \) and \( B \). A lengthy calculation, in which the 5d field equations are used repeatedly to simplify expressions, yields the simple result

\[ A_{\pm}(r, \alpha) = \frac{\sinh 2 \lambda}{\cosh^2 \lambda \pm \cos 2\alpha \sinh^2 \lambda} \] (60)

The remaining function in the 4-form potential \( \omega(r, \alpha) \) can now be found using the equations

\[ R_0^r + R_r^r - T_{(1)}^r - T_{(3)}^r - T_0^0 = \mathcal{X} \] (61)

\[ R_0^r - R_r^r + T_{(1)}^r + T_{(3)}^r - T_0^0 = \mathcal{Y} \] (62)

In fact there is no angular dependence in \( \omega \) and we find again the simple result

\[ \frac{\partial \omega}{\partial \alpha} = 0, \quad \frac{\partial \omega}{\partial r} = e^{4A(r)} V(r) \] (63)

And hence

\[ \omega(r) = e^{4A(r)} A'(r) \] (64)

This completes the solution. The remaining equations of motion act as a check of the solution.
4 Brane Probing

The most successful technique for connecting backgrounds and their dual field theories has been brane probing \[2, 11, 12, 10, 13, 14, 15, 16\] which converts the background to the U(1) theory on the probe’s surface. We thus substitute the background into the Born-Infeld action which, since the 2-form field is entirely orthogonal to the probe directions, takes the form \[11, 12\]

\[
S_{\text{probe}} = -r_3 \int_{\mathcal{M}_4} d^4x \det[G_{ab}^{(E)} + 2\pi\alpha' e^{-\Phi/2}F_{ab}]^{1/2} + \mu_3 \int_{\mathcal{M}_4} C_4, \tag{65}
\]

where \(C_4\) is the pull back of the 4-form potential on to the brane which corresponds here to the function \(w\) above. The resulting scalar potential is given by

\[
V_{\text{probe}} = e^{4A} \left[ \xi - A' \right]. \tag{66}
\]

\[\begin{array}{c}
\text{Figure 2: Plots of the probe potential in the infra-red for the mass only solution showing the stability of the solution (I) and the condensate only solution which is unstable (II).}
\end{array}\]

4.1 Yang Mills* Boundary Conditions

It is illuminating to evaluate this potential at leading order in the ultra-violet with

\[
\lambda = \mathcal{M} e^{-r} + ..., \quad A = r + ...
\]

(67)

We find

\[
V = \mathcal{M}^2 e^{2r} + ...
\]

(68)
Remembering that $e^r$ has conformal dimension of mass this is an equal mass term for each of the 6 scalar fields. The field theory at large scalar vevs is bounded suggesting the set up is stable. Note that the potential’s dependence on the angle $\alpha$ in $\xi$ is subleading in the ultra-violet. The infra-red behaviour can be found numerically by solving (3) and (4) and a sample plot is shown in Fig 2. The potential is largely independent of $\alpha$ in the infra-red too. The plot supports the hypothesis that the scalar potential pins the probe at the origin of the space.

This background therefore appears to be the dual of a stable non-supersymmetric gauge theory in which all the adjoint matter fields are massive. The deep infra-red physics is pure Yang Mills.

4.2 Fuzzy Sphere Boundary Conditions

Alternatively if we look at the other possible asymptotic solution

$$\lambda = K e^{-3r} + ..., \quad A = r + ... \quad (69)$$

We find

$$V = K^2 e^{-2r} + ... \quad (70)$$

a condensate leaves a runaway potential. Again the infra-red behaviour can be found numerically (Fig 2) and shows the same behaviour as the asymptotic solution. This configuration, which asymptotically looks like the $F_{(3)}$ field we would expect a D5 in AdS to generate, is unstable to the emission of probe like D3 branes. This is not surprising since there is no force supporting the expansion of the D3s into a fuzzy D5 brane.

The other possible solutions with both a mass and a condensate present interpolate between the two forms of solution we’ve seen. In the infra-red they are unstable whilst in the ultra-violet the mass term dominates. In between there is a minimum of the probe potential. However, given the instability of the core structure there is probably little physics associated with this minimum.

5 Glueballs and Confinement

We can make an initial investigation of the infra-red properties of the gauge theory described by our geometry as follows. The $O^{++}$ glueballs of the theory have been identified with excitations of the dilaton field of the form

$$\delta \Phi = \psi(r) e^{-ikx}, \quad k^2 = -M^2 \quad (71)$$
This deformation must be a solution of the 5d dilaton field equation

\[ \partial_{\mu}(\sqrt{-g} g^{\mu\nu} \partial_{\nu}) \delta \Phi = 0 \]  \hspace{1cm} (72)

If we make the change of coordinates \( (r \to z) \) such that

\[ \frac{dz}{dr} = e^{2A} \]  \hspace{1cm} (73)

and rescale

\[ \psi \to e^{-3A/2} \psi \]  \hspace{1cm} (74)

Then the dilaton field equation takes a Schrödinger form

\[ (-\partial^2_z + V(z)) \psi(z) = M^2 \psi(z) \]  \hspace{1cm} (75)

where

\[ V = \frac{3}{2} A'' + \frac{9}{4} (A')^2 \]  \hspace{1cm} (76)

In these coordinates the equations of motion become

\[ \lambda'' + 3 \lambda' A' = e^{2A} \frac{\partial V}{\partial \lambda} \]  \hspace{1cm} (77)

\[ 6A'^2 = \lambda'^2 - 2 e^{2A} V \]  \hspace{1cm} (78)

It is now a simple matter to solve these equations numerically with UV boundary conditions

\[ \lambda \simeq \mathcal{M} z + \mathcal{K} z^3, \quad A \simeq - \log z \]  \hspace{1cm} (79)

By suitably fine tuning the boundary conditions close to the mass only solution we can numerically determine the potential relevant to the Schrödinger equation. We display the results in Fig. 3. Note that if there is any fermion condensate present then the potential is unbounded but as we tune onto the mass only solution we find a bounded potential well. This behaviour is again analogous to that in the case where a mass is given to three of the four fermions as discussed in section 2.2 - there the mass only case is known analytically and the potential is known to be fully bounded \[38\].

It is clear from the potential that there are stable, discrete glueball states in the Yang Mills* geometry. This immediately implies that the theory is confining and has a mass gap which is
encouraging. The boundedness of the well also provides further support for the claim that the
geometry and field theory are stable. The instability of solutions with a condensate for the
fermion operator matches nicely with the instability of the brane probe in these cases. We leave
further investigation of the infra-red dynamics for future work.

Figure 3: Plots of the Schroedinger potential relevant to the $O^{++}$ glueballs for three flows
progressively fine tuned towards the mass only (Yang Mills*) geometry. The presence of a
condensate leads to an unstable potential, as can be seen at the right hand side of the well,
but as it is reduced a clearly bounded potential well emerges.

6 Summary

In this paper we have studied deformations of the AdS/CFT Correspondence which are bi-
fermion masses or condensates in the field theory. The major challenge has been to lift the
solutions to a complete 10d IIB supergravity background. The resulting background, summa-
rized in the appendix, is surprisingly simple. We have brane probed the solution in order to
study the field theory scalar potential. For the mass only solution the probe potential is stable
and the scalars massive. This theory is therefore non-supersymmetric Yang Mills theory in the
infra-red. We hope that this geometry will provide a new tool for studying Yang Mills theory.
As a first step in this direction we have determined the Schroedinger potential relevant to the
study of $O^{++}$ glueballs in the geometry, which is a bounded well. One can hence deduce that
the geometry indeed describes a confined spectrum in the infra-red. It should also be possible in
the future to include probe D7 branes in the geometry and study the fermionic quark potential
for chiral symmetry breaking. Whether this Yang Mills* theory is a good approximation to pure
Yang Mills remains to be seen.

Any solution with a fermion condensate present generates an unstable probe and glueball
potential. The asymptotic form of the solution suggests there is a D5 brane in the core of the
geometry. We have interpreted these solutions as the geometries around a fuzzy D5 brane with
no force supporting the non-commutative expansion.
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7 Appendix - Summary of the Yang Mills* Geometry

The geometry is

\[ ds_{10}^2 = \xi^4 ds_{1,4}^2 + \xi^{-2} ds_5^2 \]  

(80)

\[ \xi^2 = e^4 - s^4 \cos^2 2\alpha, \quad c = \cosh \lambda, \quad s = \sinh \lambda \]  

(81)

\[ ds_{1,4}^2 = e^{2A} dx_+/d + dr^2 \]  

(82)

\[ ds_5^2 = \cos^2 (c^2 - s^2 \cos 2\alpha)d\Omega^2_+ + \sin^2 (c^2 + s^2 \cos 2\alpha)d\Omega^2_- + \xi^2 d\alpha^2 \]  

(83)

\[ d\Omega^2_{\pm} = d\theta^2_{\pm} + \cos^2 \theta_{\pm} d\phi^2_{\pm} \]  

(84)

The dilaton in unitary gauge is described by the functions

\[ f = \frac{1}{\xi^{1/2}} \sqrt{\frac{\cosh^2 \lambda + \xi}{2}}, \quad B = \frac{\sinh^2 \lambda \cos 2\alpha}{\cosh^2 \lambda + \xi} \]  

(85)

The 2-form potential is given by

\[ A_{(2)} = iA_+ (r, \alpha) \cos^3 \alpha \cos \theta_+ d\theta_+ \wedge d\phi_+ - A_- (r, \alpha) \sin^3 \alpha \cos \theta_- d\theta_- \wedge d\phi_- \]  

(86)

\[ A_\pm (r, \alpha) = \frac{\sinh 2\lambda}{\cosh^2 \lambda \pm \cos 2\alpha \sinh^2 \lambda} \]  

(87)

The 4-form potential is given by

\[ F = F + \ast F, \quad F = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge d\omega \]  

(88)

\[ \omega (r) = e^{4A(r)} A'(r) \]  

(89)

The functions \( A \) and \( \lambda \) are solutions of

\[ \lambda'' + 4A' \lambda' = \frac{\partial V}{\partial \lambda} \]  

(90)

\[ 6A'^2 = \lambda'^2 - 2V \]  

(91)

with

\[ V = -\frac{3}{2} \left( 1 + \cosh^2 \lambda \right) \]  

(92)
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