Duals of Tirilman spaces have unique subsymmetric basic sequences

Steven J. Dilworth | Denka Kutzarova | Bünyamin Sarı | Svetozar Stankov

1Department of Mathematics, University of South Carolina, Columbia, South Carolina, USA
2Department of Mathematics, University of Illinois Urbana-Champaign, Urbana, Illinois, USA
3Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Sofia, Bulgaria
4Department of Mathematics, University of North Texas, Denton, Texas, USA

Correspondence
Bünyamin Sarı, Department of Mathematics, University of North Texas, 1155 Union Circle, 311430 Denton, TX 76203-5017, USA.
Email: bunyamin.sari@unt.edu

Funding information
Simons Foundation, Grant/Award Numbers: 849142, 636954

Abstract
The Tirilman spaces $Ti(p, \gamma)$, $1 < p < \infty$, were introduced by Casazza and Shura as variations of the spaces constructed by Tzafriri. We prove that all subsymmetric basic sequences in the dual space $Ti^*(p, \gamma)$ are equivalent to its canonical subsymmetric but not symmetric basis.

MSC 2020
46B03, 46B06, 46B10, 46B25 (primary)

1 INTRODUCTION

Symmetric structures play an important role in the theory of Banach spaces. A basic sequence $(x_j)_{j=1}^{\infty}$ is symmetric if the rearranged sequence $(x_{\pi(j)})_{j=1}^{\infty}$ is equivalent to $(x_j)_{j=1}^{\infty}$ for any permutation $\pi$ of $\mathbb{N}$. Recall that a sequence $(x_j)_{j=1}^{\infty}$ is a basic sequence if it is a (Schauder) basis of its closed linear span; two basic sequences $(x_j)_{j=1}^{\infty}$ and $(y_j)_{j=1}^{\infty}$ are said to be equivalent provided a series $\sum_{j=1}^{\infty} a_j x_j$ converges if and only if $\sum_{j=1}^{\infty} a_j y_j$ does.

The class of subsymmetric basic sequences, that is, those that are unconditional and equivalent to all of their subsequences [10], is formally more general than the class of symmetric ones. For a
DUALSOFTIRILMANSPACESHAVEUNIQUESUBSYMMETRICBASICSEQUENCES

while, these two concepts were believed to be equivalent until Garling [7] provided a counterexample. Later, subsymmetric bases became important on their own within the general theory. For instance, the first arbitrarily distortable Schlumprecht space [14] has a subsymmetric basis that is not symmetric.

Albiac, Ansorena, and Wallis [3] used Garling-type spaces to provide the first example of a Banach with a unique subsymmetric basis that is not symmetric. However, as shown in a sequel paper [2], that space contains a continuum of nonequivalent subsymmetric basic sequences. Altshuler [1] (see also [10, Example 3.b.10]) constructed a space that is not isomorphic to $c_0$ or $\ell_p$, for any $1 < p < \infty$ and in which all symmetric basic sequences are equivalent to its symmetric basis. Recently, the first example of a Banach space with a unique subsymmetric basic sequence that is not symmetric is given in [4]. That answered a question posed in [9] and [2]. The space under consideration was $Su(T^*)$ [5], the subsymmetric version of $T^*$. As it became customary, $T$ is the space considered by Figiel and Johnson [6] and its dual $T^*$ is the original space constructed by Tsirelson [15], the first example of a space that does not contain an isomorphic copy of $c_0$ or $\ell_p$, $1 \leq p < \infty$.

In this paper, we give more examples of spaces with a subsymmetric but not symmetric basis that contain, up to equivalence, a unique subsymmetric basic sequence. These examples are based on Tzafriri spaces. Tzafriri [16] had constructed (counter)-examples of spaces with (symmetric bases) showing that the notions of equal-norm type $p$ and equal-norm-cotype $q$ are not equivalent to the notions of type $p$ and cotype $q$ for $p, q \neq 2$, respectively. The Tirilmans spaces $Ti(p, \gamma)$, where $1 < p < \infty$ and $0 < \gamma < 1$, are modified Tzafriri spaces, which were introduced and studied by Casazza and Shura [5]. They were named after Tzafriri’s Romanian surname. We prove that for $1 < p < \infty$ and sufficiently small $0 < \gamma < 1$, the dual space $Ti^*(p, \gamma)$, whose canonical basis is subsymmetric but not symmetric contains, up to equivalence, a unique subsymmetric basic sequence. That is, all the subsymmetric basic sequences are equivalent to the canonical basis. The method of our proof is parallel to the one in [4]: Although the normalized block bases $(x_j)$ of the canonical basis of $Su(T^*)$ with the property $\|x_j\|_\infty \to 0$ are shown to be asymptotic-$c_0$ sequences, we show that the similar block bases in $Ti^*(p, \gamma)$ yield asymptotic-$\ell_q$ sequences, where $\frac{1}{p} + \frac{1}{q} = 1$. Moreover, unlike its dual $Ti(p, \gamma)$ has continuum many nonequivalent subsymmetric basic sequences. This follows immediately from [4, Theorem 21] that states that if a subsymmetric basis $(e_i)$ is not equivalent to the unit vector basis of $c_0$ or $\ell_p$, then either $(e_i)$ or $(e_i^*)$ admits a continuum of nonequivalent subsymmetric block bases.

2 SPACES WITH A UNIQUE SUBSYMMETRIC BASIC SEQUENCE

Given two basic sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ in Banach spaces $X$ and $Y$, respectively, we say that $(x_n)_{n=1}^\infty$ dominates $(y_n)_{n=1}^\infty$ if the bounded linear operator $T(x_n) = y_n$ from $[(x_n)_{n=1}^\infty]$ to $[(y_n)_{n=1}^\infty]$ has norm $\|T\| \leq K$. We say that $(x_n)_{n=1}^\infty$ dominates $(y_n)_{n=1}^\infty$ if $(x_n)_{n=1}^\infty$ K-dominates $(y_n)_{n=1}^\infty$ for some $K < \infty$. A block basis with respect to a basic sequence $(x_n)_{n=1}^\infty$ is a sequence $(y_n)_{n=1}^\infty$ of nonzero vectors of the form $y_n = \sum_{k=p_n+1}^{p_{n+1}} a_k x_k$ where $p_1 < p_2 < \cdots$ is an increasing sequence of natural numbers. For a vector $x$ in the closed linear span of $(x_n)_{n=1}^\infty$, its support (with respect to $(x_n)_{n=1}^\infty$) is the set of indices of its nonzero coefficients. For finite sets of natural numbers $E$ and $F$ we say that $E < F$ if max($E$) < min($F$). For a natural number $n$, we say $n < x$, resp., $n \leq x$, if $n < \min($supp$(x))$, resp., $n \leq \min($supp$(x)$). A basic sequence $(x_n)$ is called 1-subsymmetric if it is 1-unconditional and isometrically equivalent to its subsequences.
A basic sequence \((x_j)_{j=1}^\infty\) is called (strongly) asymptotic-\(\ell_p\), \(1 \leq p < \infty\) if there exist a constant \(C > 0\) such that for every \(m \in \mathbb{N}\) there is an \(M \in \mathbb{N}\) such that for every normalized block basis \((y_j)_{j=1}^M\) of \((x_j)_{j=M}^\infty\) and any set of real numbers \((a_i)\), we have

\[
\frac{1}{C} \left( \sum_{i=1}^m |a_i|^p \right)^{\frac{1}{p}} \leq \left\| \sum_{i=1}^m a_i y_i \right\| \leq C \left( \sum_{i=1}^m |a_i|^p \right)^{\frac{1}{p}}.
\]

Although we will drop the term “strongly” when referring to asymptotic-\(\ell_p\) sequences, it is important to note this is a stronger version of the original definition from [12] that was given in a more general setting.

Let \(1 < p < \infty\) and \(0 < \gamma < 1\). As in the case of Tsirelson space, the norm is defined via an implicit equation. For all \(a = (a_i) \in c_{00}\), the linear space of finitely supported real-valued sequences, define

\[
\|a\| = \max \left\{ \|a\|_\infty, \gamma \sup \sum_{j=1}^n \|E_j a\| \right\},
\]

where \(\frac{1}{p} + \frac{1}{q} = 1\) and the inner supremum is taken over all finite consecutive sets of natural numbers \(1 \leq E_1 < \cdots < E_n\) and all \(n\) and \(E_j a\) denotes the restriction of \(a\) to the set \(E_j\). This norm can be computed via the limit of a recursive sequence of norms. We refer to [5, section X.d.5] for more details. The Tirilmans space \(Ti(p, \gamma)\) is the completion of \((c_{00}, \| \cdot \|)\). It follows from the definition that the unit vectors \((e_i)_{i=1}^\infty\) form a 1-subsymmetric basis for \(Ti(p, \gamma)\). We shall summarize some of their known properties. The first one is the obvious analogue of [5, Proposition X.d.8] that was proved for \(Ti(2, \gamma)\).

**Proposition 1.** For every \(1 < p < \infty\) and \(0 < \gamma < 1\), the canonical basis \((e_i)_{i=1}^\infty\) is 1-dominated by every normalized block basis of \((e_i)_{i=1}^\infty\).

Some further properties of \(Ti(p, \gamma)\) that were proved in [5] for \(Ti(2, \gamma)\) were listed in [13, Theorem 6.1].

**Proposition 2.** Let \(1 < p < \infty\). Then for sufficiently small \(0 < \gamma < 1\), the following hold for \(Ti(p, \gamma)\).

(i) For any normalized successive blocks \((x_j)_{j=1}^\infty\) of the basis \((e_i)\), we have

\[
\gamma n^{\frac{1}{p}} \leq \left\| \sum_{j=1}^n x_j \right\| \leq 3^{\frac{1}{q}} n^{\frac{1}{p}}.
\]

(ii) \(Ti(p, \gamma)\) does not contain isomorphs of any \(\ell_r\), \(1 \leq r < \infty\) or of \(c_0\). In particular, \(Ti(p, \gamma)\) is reflexive.

**Remark.** We shall apply the above proposition for \(\gamma < 3^{-\frac{1}{q}}\).

Actually, we need the more general version of the right-hand inequality of (i), which is the \(p\)-analogue of [5, Lemma X.d.4].
Proposition 3. If $0 < \gamma < 3^{-\frac{1}{q}}$ and $(x_j)_{j=1}^n$ are block vectors in $T_i(p, \gamma)$ with consecutive supports, $n \in \mathbb{N}$, then

$$\left\| \sum_{j=1}^n x_j \right\| \leq 3^{\frac{1}{q}} \left( \sum_{j=1}^n \|x_j\|^p \right)^{\frac{1}{p}}.$$ 

As an immediate corollary, we obtain the following

Lemma 4. Let $0 < \gamma < 3^{-\frac{1}{q}}$. Let $(x^*_j)$ be a normalized block basis of $(e^*_j)$ in the dual space $T^*_i(\gamma, p)$. Then for every $n$ and every choice of real numbers $(a_j)_{j=1}^n$, we have

$$\left\| \sum_{j=1}^n a_j x^*_j \right\| \geq \frac{1}{3^{\frac{1}{q}}} \left( \sum_{j=1}^n |a_j|^q \right)^{\frac{1}{q}}.$$ 

Proof. For any $1 \leq j \leq n$ choose an $x_j \in T_i(\gamma, p)$ with $\|x_j\| = 1$ and $x^*_j(x_j) = 1$. Let $(a_j)_{j=1}^n$ be a set of real numbers. By 1-unconditionality, we may assume that $a_j \geq 0$ and supp $x_j \subseteq$ supp $x^*_j$. Then by duality,

$$\sum_{j=1}^n a^q_j = \sum_{j=1}^n a_j x^*_j(a^p_j x_j) = \left( \sum_{j=1}^n a_j x^*_j \right) \left( \sum_{j=1}^n a^q_j x_j \right) \leq \left\| \sum_{j=1}^n a_j x^*_j \right\| \left\| \sum_{j=1}^n a^q_j x_j \right\| \leq \left\| \sum_{j=1}^n a_j x^*_j \right\| 3^{\frac{1}{q}} \left( \sum_{j=1}^n a^q_j \right)^{\frac{1}{q}},$$

which gives the needed inequality. \hfill \Box

Proposition 5 [13]. Let $1 < p < \infty$ and let $\gamma > 0$ be sufficiently small. Then $T_i(p, \gamma)$ contains no symmetric basic sequence.

Remark. It was proved in [8] that $c_0$ is finitely representable in $T_i(2, \frac{1}{2})$ (disjointly with respect to $(e_j)$) that provides an alternative proof that $(e_j)$ is not symmetric.

Lemma 6. Let $(e_i)$ be a 1-unconditional basis of a reflexive Banach space $X$ that is $K$-dominated by its normalized block bases, where $K \geq 1$. Then $(e^*_i)$ $K$-dominates all normalized block bases of $(e^*_i)$ in the dual space $X^*$.

Proof. Let $(x^*_i)$ be a normalized block-basis of $(e^*_i)$ and let $(a_i)_{i=1}^n, n \in \mathbb{N}$, be an arbitrary set of real numbers. $(e^*_i)$ is also 1-unconditional, so we may assume that $a_i \geq 0$ for all $1 \leq i \leq n$. Pick a norming element $w \in X$, $\|w\| = 1$, $(\sum_{i=1}^n a_i x^*_i)(w) = \| \sum_{i=1}^n a_i x^*_i \|$. Denote $A_i = \text{supp}(x^*_i)$. 

The 1-unconditionality of \((e_i)\) allows us to assume that
\[
\text{supp}(w) \subseteq \bigcup_{i=1}^{n} A_i.
\]

Let \(w_i = w|_{A_i}\) be the restriction of \(w\) to the set \(A_i\). Denote \(\|w_i\| = c_i\) and \(B = \{1 \leq i \leq n : c_i \neq 0\}\). By 1-unconditionality, \(c_i \leq 1, 1 \leq i \leq n\). For each \(i \in B\), let \(z_i = \frac{w_i}{c_i}\). Clearly, \((z_i)_{i=1}^{n}\) is a normalized block-basis of \((e_i)_{i=1}^{\infty}\) and
\[
w = \sum_{i \in B} c_i z_i.
\]

Then,
\[
\left\| \sum_{i=1}^{n} a_i x_i^* \right\| = \left( \sum_{i=1}^{n} a_i x_i^* \right) \left( \sum_{i \in B} c_i z_i \right)
= \sum_{i \in B} a_i c_i x_i^* (z_i) \leq \sum_{i \in B} a_i c_i
= \left( \sum_{i \in B} a_i e_i^* \right) \left( \sum_{i \in B} c_i e_i \right) \leq \left\| \sum_{i \in B} a_i e_i^* \right\| \cdot \left\| \sum_{i \in B} c_i e_i \right\|
\]

By the \(K\)-domination,
\[
\left\| \sum_{i \in B} c_i e_i \right\| \leq K \left\| \sum_{i \in B} c_i z_i \right\| = K.
\]

Thus,
\[
\left\| \sum_{i=1}^{n} a_i x_i^* \right\| \leq K \left\| \sum_{i \in B} a_i e_i^* \right\| \leq K \left\| \sum_{i=1}^{n} a_i e_i^* \right\|.
\]

\[\square\]

**Lemma 7.** For any \(n\) and any sequence of normalized blocks \((x_j^*)_{j=1}^{n}\) of \((e_j^*)_{j=1}^{\infty}\) in \(T_i^*(p, \gamma)\),
\[
\left\| \sum_{j=1}^{n} x_j^* \right\| \leq \frac{n^{\frac{1}{q}}}{\gamma}.
\]

**Proof.** By the previous Lemma 6 and Proposition 1, \((x_j^*)_{j=1}^{n}\) is 1-dominated by \((e_j^*)_{j=1}^{n}\), so
\[
\left\| \sum_{j=1}^{n} x_j^* \right\| \leq \left\| \sum_{j=1}^{n} e_j^* \right\|.
\]

The vector \(\frac{\gamma}{n^{\frac{1}{q}}} \sum_{j=1}^{n} e_j^*\) belongs to the unit ball of \(T_i^*(p, \gamma)\), see, for example, [11], so \(\left\| \sum_{j=1}^{n} e_j^* \right\| \leq \frac{n^{\frac{1}{q}}}{\gamma}\).

\[\square\]
Lemma 8. $Ti^*(p, \gamma)$ does not contain an isomorphic copy of $\ell_q$ $(\frac{1}{p} + \frac{1}{q} = 1)$.

**Proof.** Assume the contrary. Without loss of generality, we may assume that a normalized block basis $(x_j^*)$ of $(e_j^*)$ is C-equivalent to the unit vector basis of $\ell_q$. Denote $I_j = \text{supp}(x_j^*)$. Choose norming elements $x_j \in Ti(p, \gamma)$, $\|x_j\| = 1$, $x_j^*(x_j) = 1$. By the 1-unconditionality we may assume that $\text{supp}(x_j) \subseteq I_j \subset \mathbb{N}$ for all $j \in \mathbb{N}$. Clearly, $I_1 < I_1 < \ldots$ and denote by $P_j$ the projection on $I_j$.

Define the projection

$$P(x^*) = \sum_{j=1}^{\infty} \langle P_j(x^*), x_j \rangle x_j^*.$$ 

Then

$$\|P(x^*)\| \leq C \left( \sum_{j=1}^{\infty} |\langle P_j(x^*), x_j \rangle|^q \right)^{\frac{1}{q}} \leq C \left( \sum_{j=1}^{\infty} \|P_j(x^*)\|^q \right)^{\frac{1}{q}} \leq 3^q C \|x^*\|.$$ 

Thus, the subspace generated by $(x_j^*)_{j=1}^{\infty}$ is complemented in $Ti^*(p, \gamma)$ which implies that $Ti(p, \gamma)$ contains an isomorphic copy of $\ell_p$, a contradiction. \hfill $\square$

By Lemmas 4 and 7 for all $n$ and all normalized block sequences $(u_i)_{i=1}^{n}$ in $Ti^*(p, \gamma)$, we have $\| \sum_{i=1}^{n} u_i \|_K \sim n^{1/q}$ for some $K$. In [8], it was shown that spaces with such a property are saturated by asymptotic-$\ell_q$ sequences. An inspection of their proof (of Theorem 3.7) shows that any block sequence $(x_i)$ with $\|x_i\|_{\infty} \to 0$ is asymptotic-$\ell_q$. Thus, the next proposition follows from the proof of [8, Theorem 3.7]. We reproduce the proof for completeness, which is slightly easier in our case.

**Proposition 9.** Let $1 < p < \infty$ and $0 < \gamma < 3^{-1/q}$. Every normalized block sequence $(x_i)_{i=1}^{\infty}$ in $Ti^*(p, \gamma)$ satisfying $\|x_i\|_{\infty} \to 0$ is an asymptotic $\ell_q$ basic sequence where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** Let $m \in \mathbb{N}$, $m \geq 2$. Choose $\varepsilon, \delta > 0$, and $\delta'$ satisfy

$$0 < \varepsilon < \frac{1}{4m^{31/q}}, \quad \delta = \frac{\varepsilon}{6\gamma^{-1}m}, \quad 0 < \delta' < \frac{\delta^{q+1}}{\gamma^{-q}m}.$$ 

(1)

Let $M \in \mathbb{N}$ be such that $\|x_i\|_{\infty} < \delta'$ for all $i \geq M$. Let $(y_i)_{i=1}^{m}$ be a normalized block basis of $(x_i)_{i \geq M}$. We will show that for all scalars $(a_i)_{i=1}^{m}$ with $\sum_{i=1}^{m} |a_i|^q = 1$ we have

$$\frac{1}{3^{1/q}} \leq \left\| \sum_{i=1}^{m} a_i y_i \right\| \leq 3^{q+1} \gamma^{-q}.$$ 

(2)

Fix $(a_i)_{i=1}^{m}$. The left-hand side inequality holds for all normalized block vectors and was shown in Lemma 4.
For each $i$, write $a_i y_i = \sum_{i=1}^{n_i+1} y_{i,j}$ where $y_{i,j}$'s are successive blocks with $\delta \leq \|y_{i,j}\| < \delta + \delta'$ and $\|y_{i,n_i+1}\| < \delta$. Then by Lemma 4

$$|a_i| = \|a_i y_i\| \geq 3^{-1/q} \left( \sum_{i=1}^{n_i+1} \|y_{i,j}\|^{q} \right)^{1/q} \geq 3^{-1/q} \delta n_i^{1/q}.$$  

Thus for all $1 \leq i \leq m$,

$$n_i \leq \frac{3|a_i|^q}{\delta q}.$$  

(3)

Moreover, by shrinking each $y_{i,j}$ to have norm exactly $\delta$ at a cost of $\delta'$ we have by Lemma 7 that

$$\|a_i y_i\| \leq \gamma^{-1} \delta n_i^{1/q} + n_i \delta' + \delta \leq \gamma^{-1} \delta n_i^{1/q} + 2\delta$$

as $n_i \delta' + \delta \leq \frac{3}{\delta q} \delta' + \delta \leq \frac{3}{\delta q} \frac{\delta^{q+1}}{\gamma q m} + \delta \leq \frac{\delta}{m} + \delta < 2\delta$.

If $|a_i| \geq \varepsilon$, then $n_i \neq 0$ and from above $\varepsilon \leq \|a_i y_i\| \leq \gamma^{-1} \delta n_i^{1/q} + 2\delta \leq 3\gamma^{-1} \delta n_i^{1/q}$ because $\gamma^{-1} n_i^{1/q} > 1$. Thus,

$$n_i^{1/q} > \frac{\varepsilon \gamma}{3\delta} = 2m.$$

Let

$$N = \sum_{\{i:|a_i| \geq \varepsilon\}} n_i.$$

Then $2m\delta < \delta n_i^{1/q} \leq \delta N^{1/q}$, and by above $N \delta' < \delta$. We have, using Lemma 7 again,

$$\left\| \sum_{\{i:|a_i| \geq \varepsilon\}} a_i y_i \right\| \leq \gamma^{-1} \delta N^{1/q} + N \delta' + m \delta$$

$$\leq \gamma^{-1} \delta N^{1/q} + \delta + m \delta$$

$$\leq \gamma^{-1} \delta N^{1/q} + 2m \delta$$

$$\leq \gamma^{-1} \delta N^{1/q} + \delta N^{1/q}.$$

Thus,

$$\left\| \sum_{\{i:|a_i| \geq \varepsilon\}} a_i y_i \right\| \leq 2\gamma^{-1} \delta N^{1/q}.$$  

(4)

On the other hand, by Lemma 4 we have

$$\left\| \sum_{\{i:|a_i| \geq \varepsilon\}} a_i y_i \right\| \geq 3^{-1/q} \left( \sum_{\{i:|a_i| \geq \varepsilon\}} |a_i|^q \right)^{1/q} \geq 3^{-1/q} (1 - \varepsilon m)^{1/q} \geq \frac{1}{2} 3^{-1/q},$$  

(5)
and
\[ \left\| \sum_{\{i: |a_i| < \varepsilon\}} a_i y_i \right\| < m \varepsilon \leq \frac{1}{4} 3^{-1/q} \sum_{\{i: |a_i| \geq \varepsilon\}} a_i y_i < \gamma^{-1} \delta N^{1/q}. \]

Thus, by the triangle inequality
\[ \left\| \sum_{i=1}^{m} a_i y_i \right\|^q < 3^q \gamma^{-q} \delta^q N \]
\[ \leq 3^q \gamma^{-q} \delta^q \sum_{\{i: |a_i| \geq \varepsilon\}} n_i \]
\[ \leq 3^{q+1} \gamma^{-q} \sum_{\{i: |a_i| \geq \varepsilon\}} |a_i|^q \quad \text{by (3)} \]
\[ \leq 3^{q+1} \gamma^{-q}. \]

**Theorem 10.** Let \( 1 < p < \infty \) and \( \gamma > 0 \) be sufficiently small. Every subsymmetric basic sequence in the dual space \( T_i^*(p, \gamma) \) is equivalent to the subsymmetric canonical basis \( (e_j^*)_{j=1}^\infty \) that is not symmetric.

**Proof.** By Proposition 5, \( (e_j^*)_{j=1}^\infty \) is not symmetric.

Let \( (x_j^*)_{j=1}^\infty \) be a normalized subsymmetric basic sequence in \( T_i^*(p, \gamma) \). By passing to a subsequence we may assume that \( (x_j^*)_{j=1}^\infty \) is a block basis of \( (e_j^*)_{j=1}^\infty \). If we suppose that \( \lim_{j \to \infty} \|x_j^*\|_\infty = 0 \), then by combining Lemma 4, Lemma 6, and Proposition 9, we obtain that \( (x_j^*)_{j=1}^\infty \) is an asymptotic \( \ell_q^* \) basic sequence. Then the subsymmetry would imply that \( (x_j^*)_{j=1}^\infty \) is equivalent to the unit vector basis of \( \ell_q^* \) that contradicts Lemma 8.

Thus, by passing again to a subsequence, we may assume that for all \( j \in \mathbb{N}, \|x_j^*\|_\infty \geq c \) for some \( c > 0 \). Then \( (x_j^*)_{j=1}^\infty \) \( c \)-dominates \( (e_j^*)_{j=1}^\infty \). On the other hand, by Lemma 6 \( (x_j^*)_{j=1}^\infty \) is \( 1 \)-dominated by \( (e_j^*)_{j=1}^\infty \) and therefore, they are equivalent.

Reflexivity of \( T_i(p, \gamma) \) and duality yield the following

**Corollary 11.** Let \( 1 < p < \infty \) and \( \gamma > 0 \) be sufficiently small. Every subsymmetric basis of a quotient space of \( T_i(p, \gamma) \) is equivalent to the canonical basis \( (e_j)_{j=1}^\infty \).

**Proposition 12** [4]. Let \( (e_j^*) \) be a subsymmetric basis that is not equivalent to the unit vector basis of \( \ell_\infty \) or \( c_0 \). Then either \( (e_j) \) or \( (e_j^*) \) admits a continuum of nonequivalent subsymmetric block bases.

This, together with Theorem 10, give us the following

**Corollary 13.** For \( 1 < p < \infty \) and sufficiently small \( \gamma \), the basis \( (e_j) \) of \( T_i(p, \gamma) \) has a continuum many nonequivalent subsymmetric block bases.
ACKNOWLEDGEMENTS
The first author was supported by Simons Foundation Collaboration (Grant Number: 849142). The second author was supported by Simons Foundation Collaboration (Grant Number: 636954). The first three authors thank the Workshop in Analysis and Probability at Texas A&M University (2022) for support where this work was initiated.

JOURNAL INFORMATION
The Bulletin of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID
Bünyamin Sarı © https://orcid.org/0000-0002-8497-8967

REFERENCES
1. Z. Altshuler, A Banach space with a symmetric basis which contains no $\ell_p$ or $c_0$, and all its symmetric basic sequences are equivalent, Compos. Math. 35 (1977), no. 2, 189–195.
2. F. Albiac, J. L. Ansorena, S. J. Dilworth, and D. Kutzarova, A dichotomy for subsymmetric basic sequences with applications to Garling spaces, Trans. Amer. Math. Soc. 374 (2021), no. 3, 2079–2106.
3. F. Albiac, J. L. Ansorena, and B. Wallis, Garling sequence spaces, J. Lond. Math. Soc. (2) 98 (2018), no. 1, 204–222.
4. P. Casazza, S. J. Dilworth, D. Kutzarova, and P. Motakis, On uniqueness and plentitude of subsymmetric sequences, Israel J. Math, to appear.
5. P. G. Casazza and T. Shura, Tsirelson’s space, Lecture Notes in Mathematics, vol. 1363, Springer, Berlin, 1989.
6. T. Figiel and W. B. Johnson, A uniformly convex Banach space which contains no $\ell_p$, Compos. Math. 29 (1974), 179–190.
7. D. J. H. Garling, Symmetric bases of locally convex spaces, Studia Math. 30 (1968), 163–181.
8. M. Junge, D. Kutzarova, and E. Odell, On asymptotically symmetric Banach spaces, Studia Math. 173 (2006), no. 3, 203–231.
9. D. Kutzarova, A. Manoussakis, and A. Pelczar-Barwacz, Isomorphisms and strictly singular operators in mixed Tsirelson spaces, J. Math. Anal. Appl. 388 (2012), no. 2, 1040–1060.
10. J. Lindenstrauss and L. Tzafriri, Classical Banach spaces. I, Springer, Berlin-New York, 1977.
11. A. Manoussakis, On the structure of a certain class of mixed Tsirelson spaces, Positivity 5 (2001), no. 3, 193–238.
12. B. Maurey, V. D. Milman, and N. Tomczak-Jaegermann, Asymptotic infinite-dimensional theory of Banach spaces, Oper. Theory: Adv. Appl. 77 (1995), 149–175.
13. B. Sari, Envelope functions and asymptotic structures in Banach spaces, Studia Math. 164 (2004), no. 3, 283–306.
14. Th. Schlumprecht, An arbitrarily distortable Banach space, Israel J. Math. 76 (1991), no. 1–2, 81–95.
15. B. S. Tsirelson, It is impossible to imbed $\ell_p$ or $c_0$ into an arbitrary Banach space, Funkcional. Anal. i Priložen. 8 (1974), no. 2, 57–60 (Russian).
16. L. Tzafriri, On the type and cotype of Banach spaces, Israel J. Math. 32 (1979), no. 1, 32–38.