Multicolor Ramsey numbers for triple systems

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Abstract

Given an \( r \)-uniform hypergraph \( H \), the multicolor Ramsey number \( r_k(H) \) is the minimum \( n \) such that every \( k \)-coloring of the edges of the complete \( r \)-uniform hypergraph \( K_r^n \) yields a monochromatic copy of \( H \). We investigate \( r_k(H) \) when \( k \) grows and \( H \) is fixed. For nontrivial 3-uniform hypergraphs \( H \), the function \( r_k(H) \) ranges from \( \sqrt{6k}(1 + o(1)) \) to double exponential in \( k \).

We observe that \( r_k(H) \) is polynomial in \( k \) when \( H \) is \( r \)-partite and at least single-exponential in \( k \) otherwise. Erdős, Hajnal and Rado gave bounds for large cliques \( K_s^r \) with \( s \geq s_0(r) \), showing its correct exponential tower growth. We give a proof for cliques of all sizes, \( s > r \), using a slight modification of the celebrated stepping-up lemma of Erdős and Hajnal.

For 3-uniform hypergraphs, we give an infinite family with sub-double-exponential upper bound and show connections between graph and hypergraph Ramsey numbers. Specifically, we prove that

\[
r_k(K_3^3) \leq r_{4k}(K_3^3 - e) \leq r_{4k}(K_3) + 1,
\]

where \( K_3^3 - e \) is obtained from \( K_3^3 \) by deleting an edge.

We provide some other bounds, including single-exponential bounds for \( F_5 = \{abc, abd, cde\} \) as well as asymptotic or exact values of \( r_k(H) \) when \( H \) is the bow \( \{abc, ade\} \), kite \( \{abc, abd\} \), tight path \( \{abc, bcd, cde\} \) or the windmill \( \{abc, bde, cef, bce\} \). We also determine many new “small” Ramsey numbers and show their relations to designs. For example, the lower bound for \( r_6(kite) = 8 \) is demonstrated by decomposing the triples of [7] into six partial STS (two of them are Fano planes).

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1 Introduction, results

An r-uniform hypergraph $H$ is a pair $(V, E)$ where $V$ is a vertex set and $E \subseteq \binom{V}{r}$ is the set of edges. Let $K^r_n$ be the complete r-uniform hypergraph containing all $r$-subsets of vertices as edges. For an edge $\{v_1, v_2, \ldots, v_r\}$ we often write $v_1v_2\ldots v_r$. When $r = 2$, denote $K^r_n$ by $K_n$. We shall also use the notation $\binom{[n]}{r}$ or $\binom{V}{r}$ for the edge set of $K^r_n$. An r-uniform hypergraph $H$ is $\ell$-partite if its vertex set can be partitioned into $\ell$ parts (called partite sets) such that each edge contains at most one vertex from each part; $H$ is a complete r-partite hypergraph if each choice of $r$ vertices from distinct partite sets forms an edge, and $H$ is balanced if its partite sets differ in size by at most one. A matching is a hypergraph consisting of disjoint edges. A hypergraph $H = (V, E)$ is a subhypergraph of $F = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$. Denote by $\text{ex}(n, H)$ the maximum number of edges in an $n$-vertex $r$-uniform hypergraph containing no copy of $H$ as subhypergraph. The density of an $r$-uniform hypergraph $H = (V, E)$ on $n$ vertices is $d(H) = \frac{|E|}{\binom{n}{r}}$.

The multicolor Ramsey number for an $r$-uniform hypergraph $H$, denoted by $r_k(H)$, is the minimum $n$ such that no matter how the edges of $K^r_n$ are colored with $k$ colors, there is a monochromatic copy of $H$. While there is a number of results in the literature about $r_k(H)$ when $k$ is a small fixed number (see [4]), the case when $H$ is fixed and $k$ grows appears not to have been extensively studied. The following three results are among the few results known in this area:

**Theorem 1** (Lazebnik and Mubayi [25]). Fix integers $r, s, t \geq 2$. Let $H^r(s, t)$ be the complete $r$-partite $r$-uniform hypergraph with $r - 2$ parts of size 1, one part of size $s$ and one part of size $t$. Then

(i) $r_k(H^r(2, t + 1)) = tk^2 + O(k)$;

(ii) $r_k(H^r(s, t)) = \Theta(k^s)$, for fixed $t, s \geq 2, t > (s - 1)!$;

(iii) $r_k(H^r(3, 3)) = (1 + o(1))k^3$.

Let $M$ be a matching with two $r$-tuples. Notice that an edge-coloring of $K^r_n$ without monochromatic copies of $M$ corresponds to a proper vertex-coloring of Kneser graph $K(n, r)$, that is, the graph with vertex set $\binom{[n]}{r}$ and two $r$-sets are adjacent if and only if they are disjoint. Lovász proved that the chromatic number of $K(n, r)$ is equal to $n - 2r + 2$. Reformulating his result, we obtain the following.

**Theorem 2** (Lovász [28]). If $M$ is a matching with two $r$-tuples, then $r_k(M) = k + 2r - 1$.

Gyárfás and Raeisi observed that results of Csákány and Kahn [6] and the standard coloring of the Kneser graph imply the following.

**Proposition 3** (Gyárfás and Raeisi [14]). If $C^3_3$ is the hypergraph with edge set $\{abc, cde, efa\}$, then $k + 5 \leq r_k(C^3_3) \leq 3k + 1$. 

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In this paper, we start a systematic investigation on the growth rate of \( r_k(H) \) for some fixed \( H \) as \( k \) grows. Our first result shows that \( r_k(H) \) is polynomial in \( k \) if and only if \( H \) is \( r \)-partite.

**Proposition 4.** Let \( r \geq 2 \) be fixed and \( H \) be a connected \( r \)-uniform hypergraph. Then \( r_k(H) \) is polynomial in \( k \) if and only if \( H \) is \( r \)-partite. In particular, there are positive constants \( c \) and \( c' \), such that

(i) If \( H \) is \( r \)-partite, then \( r_k(H) = O(k^c) \)

(ii) If \( H \) is not \( r \)-partite, then \( r_k(H) \geq 2^{c'k} \).

Determining the growth rate of \( r_k(H) \) in general is known to be a very hard problem. For example, the best known bounds even for the smallest nontrivial graph case are \( c^k < r_k(K_3) < c'k! \) for some positive constants \( c \) and \( c' \) (see Chung [5] and Erdős, Szekeres [11]). Define the tower function as follows: \( t_1(n) = n \) and \( t_{i+1}(n) = 2^{t_i(n)} \) for all \( i \geq 1 \). Erdős, Hajnal and Rado gave an upper bound for all cliques and a lower bound for only large cliques.

**Theorem 5** (Erdős and Rado [10], Erdős et al. [9]). Let \( s > r \geq 2 \). There are positive integers \( c = c(s,r) \leq 3(s-r), s_0(r), \) and \( c' = c'(s,r) \) such that

\[
t_r(c'k) < r_k(K^r_s) < t_r(ck \log k)
\]

where the lower bound holds for \( s \geq s_0(r) \).

It is worth noting that the lower bound in [9] was stated for the case when the number of colors, \( k \), is fixed while \( r \) grows and the bound was only for large cliques. But the proof in [9] applies naturally to our case as well, when \( k \) grows and the other parameters are fixed. Recently, an improved stepping-up lemma was proved by Conlon et al [3]. Their main result implies a lower bound for cliques of smaller sizes, but still only for \( s \geq 2r-1 \). Duffus, Lefmann and Rödl [7] took another approach, using shift graphs, and proved a lower bound for cliques of all sizes \( s > r \), but require \( k \) being fixed and \( r \gg k \). Our next result gives a proof for cliques of all sizes using a slight modification of the stepping-up lemma, due to Erdős and Hajnal (see Chapter 4.7 in [13]).

**Theorem 6.** For any \( s > r \geq 2 \) and \( k > r^{2r} \) we have

\[
r_k(K^r_s) > t_r \left( \frac{k}{2^r} \right).
\]

Our remaining results are all for 3-uniform hypergraphs and we will address the question of determining \( r_k(H) \) for most interesting \( H \)'s with 6 or fewer vertices. Let \( K^3_4 - e \) be a hypergraph obtained from \( K^3_4 \) by removing one edge. Our next theorem gives bounds on \( r_k(K^3_4 - e) \) in terms of \( r_k(K^3_3) \), showing that compared to the double-exponential bounds for \( K^3_4 \) from Theorems 5 and 6, the correct order of magnitude for \( r_k(K^3_4 - e) \) is single-exponential.
Theorem 7. For any $k \geq 2$,
\[ r_k(K_3) \leq r_{4k}(K_4^3 - e) \quad \text{and} \quad r_k(K_4^3 - e) \leq r_k(K_3) + 1. \]
Moreover $r_2(K_4^3 - e) = r_2(K_3) + 1 = 7$.

Denote by $F_5$ the hypergraph with edges $\{abc, abd, cde\}$. We show that $r_k(F_5)$ behaves similarly to $r_k(K_3)$.

Theorem 8. There is a positive constant $c$ such that, for $k \geq 4$,
\[ 2^ck \leq r_k(F_5) \leq k!, \]
and $r_2(F_5) = 6, r_3(F_5) = 7$.

The simplest non-trivial triple systems have just two edges. The kite is a 3-uniform hypergraph with two edges sharing two vertices. The bow is a 3-uniform hypergraph with two edges sharing a single vertex.

Theorem 9. Let $r_k = r_k(\text{bow})$. Then
\[ r_k = (1 + o(1))\sqrt{6k}. \]
If $k = \left(\frac{3}{n}\right)$ and $n \equiv 4, 8 \pmod{12}$, then $r_k = n + 1$. Moreover, $r_2 = 5, r_3 = r_4 = r_5 = 6, r_6 = 7, r_7 = r_8 = r_9 = r_{10} = 9, 9 \leq r_{11} \leq r_{12} \leq r_{13} \leq r_{14} \leq 10, r_{15} = 11$.

Remark. Note that $r_k(\text{bow})$ is the smallest multicolor Ramsey number among nontrivial 3-uniform hypergraphs since $r_k(H) \geq \min\{r_k(\text{bow}), r_k(\text{kite}), r_k(M)\}$, where $M$ is a matching with 2 triples. Indeed, each nontrivial 3 uniform hypergraph contains at least two edges that form one of bow, kite or $M$, and Theorem 2 gives $r_k(M) = k + 5$.

Theorem 10. Let $r_k = r_k(\text{kite})$. Then
\[ r_k = \begin{cases} k + 1, & \text{if } k \equiv 3 \pmod{6} \\ k + 1 \text{ or } k + 2, & \text{if } k \equiv 4 \pmod{6} \\ k + 2, & \text{if } k \equiv 0, 2 \pmod{6} \\ k + 3, & \text{if } k \equiv 1, 5 \pmod{6}, k \neq 5 \\ 6, & \text{if } k = 5, \\ 5, & \text{if } k = 4 \end{cases} \]

Let $a, b$ be positive integers. Denote by $F(a, b)$ the 3-uniform hypergraph with vertex set $V = A \cup B$, $A \cap B = \emptyset$, $|A| = a, |B| = b$ and edge set consisting of all triples with one vertex in $A$ and two vertices in $B$ (for example, $F(2, 2)$ is the kite).
Proposition 11. For any $a \geq 2$, we have
\[ k(a-1) < r_k(F(a,2)) \leq k(a-1) + 3. \]

In general, $r_k(F(a,b))$ grows slower than double exponential in $k$ and possibly faster than exponential in $k$. (Recall that Theorems 5 and 6 give double-exponential bounds.)

Theorem 12. Given $3 \leq a \leq b$, we have, for positive constants $c = c(a,b)$ and $c' = c(a,b)$
\[ 2c'k < r_k(F(a,b)) < r_t(K_b) + m < 2^{ck^{a+1} \log k}, \]
where $m = (a - 1)k + 1$, and $t = k(m^a)$. The windmill $W$ with center edge $abc$ is the hypergraph with six vertices and edge set \{\text{abc, abd, bce, acf}\}.

Theorem 13. \((1-o(1))3k \leq r_k(W) \leq 3k + 3.\)

It is interesting to compare Theorem 13 with Proposition 3. In fact, the upper bounds in both cases come from the corresponding Turán-type results. Indeed, $\text{ex}(n,C_3^3) = \binom{n-1}{2}$ (Frankl-Füredi [12] for large $n$, Csákány-Kahn [6] for $n \geq 6$) while $\text{ex}(n,W) \leq \binom{n}{2}$ ([12]).

The ideas giving the asymptotic of $r_k(W)$ can be also used for the tight path $P_3^3 = \{abc, bcd, cde\}$.

Theorem 14. $2k(1-o(1)) \leq r_k(P_3^3) \leq 2k + 3.$

The rest of the paper will be organized as follows. In Section 2 we give some auxiliary results and prove Proposition 4. Theorems 6 - 14 will be proved in Sections 3-6. Section 7 is devoted to exact values of Ramsey numbers for small number of colors and Section 8 contains remarks, conjectures and problems.

In some later sections we give lower bounds on Ramsey numbers based on block designs. A $t - (v,k,\lambda)$ design is a subset of $\binom{[v]}{k}$, called blocks, such that each $t$ element subset of $[v]$ is contained in exactly $\lambda$ blocks.

2 General bounds and auxiliary results

In this section we prove some general bounds on $r_k(H)$ and obtain some consequences including Proposition 4. Recall that the density of an $r$-uniform hypergraph $F$ with $n$ vertices and $e$ edges is $d(F) = \frac{e}{\binom{n}{r}}$. 

Lemma 15. Let $H$ be a fixed $r$-uniform hypergraph and $F$ be an $r$-uniform hypergraph on $n$ vertices, density $d(F) = d$, and not containing copies of $H$ as a subhypergraph. Then

(i) $r_k(H) \leq 1 + \max\{n : \binom{n}{r}/\text{ex}(n, H) \leq k\},$

(ii) If $\binom{n}{r}(1-d)^k < 1$ then $r_k(H) \geq n.$

Proof. (i) Consider a coloring of $K^r_n$ with $k$ colors and no monochromatic copy of $H$. Then each color class has at most $\text{ex}(n, H)$ edges.

(ii) Consider $k$ copies of hypergraph $F$ obtained by mapping its vertices randomly to a given set $V$ of $n$ vertices. Here, we choose vertex permutations uniformly. Assign the edges of the $i$th copy of $F$ color $i$, $i = 1, \ldots, k$. If an edge belongs to several copies of $F$, assign the smallest available label. We claim that with positive probability, each edge of $K^r_n$ belongs to some copy of $F$. Indeed, the probability that a given edge of $K^r_n$ uncovered is $(1-d)^k$. Thus, the probability that there is an uncovered edge of $K^r_n$ is at most $\binom{n}{r}(1-d)^k < 1$. Therefore, with positive probability, all edges are covered and the resulting coloring of $K^r_n$ contains no monochromatic copy of $H$.

Proof of Proposition 4. (i) The proposition follows from Lemma 15(i) by using the fact that $\text{ex}(n, H) < n^{r-c}$ for some positive constant $c = c(H)$, when $H$ is $r$-partite, see [8]. So, $k \geq \binom{n}{r}/\text{ex}(n, F) \geq C n^r/n^{r-c} = C n^c$, for a constant $C = C(r)$. Thus $n \leq C^{-1/c}k^{1/c}$.

(ii) Let $H$ be non-$r$-partite. Apply Lemma 15(ii) with $F$ being a complete $r$-uniform $r$-partite balanced hypergraph on $n = 2c/k$ vertices (and $r|n$). Clearly $H$ is not contained in $F$ as a subgraph. Moreover, $d(F) \geq \frac{(n/r)^r}{\binom{n}{r}} > \frac{(n/r)^r}{(en/r)^r} = e^{-r}.$ Hence for $k = c \log n$ and $c > e^r(r+1)$,

$$\binom{n}{r}(1-d)^k \leq \binom{n}{r}(1-d)^{c\log n} < n^r e^{-cd\log n} = e^{c(r-cd)\log n} < 1.$$

The trace of a 3-uniform hypergraph $H$ at vertex $v$ is the graph on vertex set $V(H) - \{v\}$ and with edge set $\{e - \{v\} : e \in H, v \in e\}$. A transversal of a hypergraph is a set of vertices non-trivially intersecting each edge.

Lemma 16. Let $H$ be a 3-uniform hypergraph with a single-vertex transversal $\{v\}$. Let $G$ be a trace of $H$ with respect to $v$. Then $r_k(H) \leq r_k(G) + 1.$

Proof. Given a $k$-coloring $c$ of $\binom{n}{3}$ with no monochromatic $H$, let $c'$ be the $k$-coloring of $\binom{n-1}{2}$ defined by $c'(ij) = c(ijn)$. Then $c'$ has no monochromatic $G$ and consequently $r_k(G) \geq r_k(H) - 1$ as required.

3 $K^r_s$ for $s > r \geq 2$

In this section we prove Theorem 6 using a variant of the stepping-up lemma of Erdős and Hajnal.
Proof of Theorem 6. It suffices to prove the result for \( s = r + 1 \) since \( r_k(K_t^n) \geq r_k(K_{r+1}^r) \) for any \( s > r \). We use induction on \( r \) to show that \( r_k(K_{r+1}^r) \geq t_r(k/2^{r-2} - 2r) \) for all \( k \geq 2r^r \). Since \( k \geq 2r^r \), we have \( k/2^{r-2} - 2r \geq k/2^r \) and the result follows.

The base case \( r = 2 \) is given by \( r_k(K_3^2) > 2^k > 2^{k-4} = t_2(k - 4) \). Assume the result holds for some \( r \geq 2 \) and let \( n = r_k(K_{r+1}^r) - 1 \). By the inductive hypothesis

\[
n \geq t_r(k/2^{r-2} - 2r) - 1.
\]

Let \( \phi : ([n]) \to [k] \) be a coloring with no monochromatic \( K_{r+1}^r \). We will construct a coloring \( \psi : ([2^n]) \to [2k + 2r - 4] \) with no monochromatic \( K_{r+2}^{r+1} \). This shows that

\[
r_{2k+2r-4}(K_{r+2}^{r+1}) \geq 1 + 2^n \geq 1 + \frac{1}{2} t_{r+1}(k/2^{r-2} - 2r).
\]

Now suppose we are given \( k' \geq (r + 1)2^{r+1} \). If \( k' - 2r + 4 \) is odd, then let \( k'' = k' - 1 \) and if \( k' - 2r + 4 \) is even then let \( k'' = k' \). Set \( k = (k'' - 2r + 4)/2 \) (which is an integer) and observe that \( k \geq 2r^r \) and \( k'' = 2k + 2r - 4 \). Then

\[
k/2^{r-2} - 2r \geq k''/2^{r-1} - 2(r + 1) + 1
\]

and \( r_{k'}(K_{r+2}^{r+1}) \) is at least

\[
r_{k''}(K_{r+2}^{r+1}) \geq 1 + \frac{1}{2} t_{r+1}(k''/2^{r-1} - 2(r + 1)) > t_{r+1}(k'2^{r-1} - 2(r + 1)).
\]

Now we shall construct a coloring \( \psi \) of \( ([2^n]) \) using the coloring \( \phi \) of \( ([n]) \) that has no monochromatic \( K_{r+1}^r \). Represent the elements of \([2^n]\) with 0-1-sequences on \( n \) coordinates. For a vertex \( u \) and integer \( i \), we denote \( u(i) \) the \( i \)th coordinate of \( u \) in this representation. Given two vertices \( u, v \in [2^n] \), say that \( u < v \) if \( u(i) < v(i) \) and \( u(j) = v(j) \) for \( j < i \). Denote such an \( i \) by \( f(uv) \). Given any \( u_1 < \cdots < u_{r+1} \), let \( f_i := f(u_iu_{i+1}) \), for every \( 1 \leq i \leq r \). Observe crucially that

1. \( f_i \neq f_{i+1} \), for every \( 1 \leq i \leq r - 1 \);
2. \( f(u_1u_{r+1}) = \min_{1 \leq i \leq r} \{f_i\} \) and the minimum is reached by a unique \( i \).

We define coloring \( \psi \) as follows:

\[
\psi(u_1\ldots u_{r+1}) = \begin{cases} 
(\phi(f_1, \ldots, f_r), 1) & \text{if } (f_1, \ldots, f_r) \text{ is an increasing sequence}, \\
(\phi(f_1, \ldots, f_r), 2) & \text{if } (f_1, \ldots, f_r) \text{ is a decreasing sequence}, \\
(i, 3) & \text{if } f_1 < f_2 < \cdots < f_i > f_{i+1}, 2 \leq i \leq r - 1, \text{ for } r \geq 3, \\
(i, 4) & \text{if } f_1 > f_2 > \cdots > f_i < f_{i+1}, 2 \leq i \leq r - 1, \text{ for } r \geq 3.
\end{cases}
\]

Suppose to the contrary that there is a monochromatic copy of \( K_{r+2}^{r+1} \) under \( \psi \) on vertex set \( U = \{u_1, \ldots, u_{r+2}\} \) with \( u_1 < \cdots < u_{r+2} \). Without loss of generality, we distinguish two cases.
Case 1: The second coordinate of $\psi$ on each $(r+1)$-tuple is 1. First notice that the second coordinate of $\psi$ on $u_1,\ldots,u_{r+1}$ and $u_2,\ldots,u_{r+2}$ being 1 implies $f_1 < f_2 < \cdots < f_r < f_{r+1}$ and together with (2), we have $f(u_1u_i) = f(u_1u_2) = f_1$ for all $3 \leq i \leq r + 2$. Similarly from $u_2,\ldots,u_{r+2}$, we have that for every $2 \leq p < q \leq r + 2$, $f(u_pu_q) = f_p$. Recall that the color of the $(r+1)$-set $\{u_1,\ldots,u_{r+2}\} - \{u_i\}$ under $\psi$ is determined by the color of the $r$-set $\{f_1,\ldots,f_{r+1}\} - \{f_i\}$ under $\phi$. Let $F := \{f_1,\ldots,f_{r+1}\}$ and $U = \{u_1,\ldots,u_{r+2}\}$. Let us denote the above implication by

$$U \setminus \{u_i\} \Rightarrow F \setminus \{f_i\}.$$ 

Thus a monochromatic $K_{r+1}^{r+1}$ on $U$ under $\psi$ yields a monochromatics $K_{r+1}^r$ on $F$ under $\phi$, a contradiction.

Case 2: Each $(r+1)$-tuple gets color $(i,3)$ for some $i$ with $2 \leq i \leq r - 1$. Then $\psi(u_1,\ldots,u_{r+1}) = (i,3)$ implies $f_i > f_{i+1}$. On the other hand, $\psi(u_2,\ldots,u_{r+2}) = (i,3)$ implies $f_i < f_{i+1}$, a contradiction.

If the second coordinate is 2 or 4 the arguments are almost identical to those in Case 1 or 2.

## 4 $K_4^3 - e$ and $F_5$

Notice that in contrast to the double-exponential growth for $K_4^3$, $r_k(K_4^3 - e)$ is single-exponential in the number of colors $k$. Indeed, since $K_4^3 - e$ is not 3-partite, Proposition 4 yields $r_k(K_4^3 - e) > 2^{ck}$. For the upper bound, one can use a variation of the classical Erdős-Rado pigeonhole argument to obtain $r_k(K_4^3 - e) < 2^{(k+1)\log k}$. We will, however, use a different approach to prove this fact, which also shows some connection between the multicolor Ramsey number of $K_4^3 - e$ and the multicolor Ramsey number of a triangle.

**Proof of Theorem 7.** For the lower bound, let $n = r_k(K_3) - 1$ and $\phi : \binom{[n]}{2} \to k$ be a $k$-coloring of $\binom{[n]}{2}$ with no monochromatic triangles. We will construct a coloring $\psi$ of $\binom{[n]}{3}$ with $4k$ colors with no monochromatic $K_4^3 - e$. This then would imply that $r_{4k}(K_4^3 - e) \leq n + 1 = r_k(K_3)$ as desired. Let $\psi$ be the following coloring of the triples $i < j < k$. If $P$ is a path with vertices $i, j, k$, denote by $\phi(P)$ the color under $\phi$ of the edge in $\{i, j, k\}$ that is not in $P$. For such a path $P$, let the type of $P$, $t(P) = 1, 2,$ or $3$ if $i, j$ or $k$ is its center, respectively. If $\{i, j, k\}$ is a rainbow triangle, let $\psi(ijk) = (0, \phi(jk))$. If $\{i, j, k\}$ induces a monochromatic path $P$, let $\psi(ijk) = (t(P), \phi'(P))$.

Suppose there is a monochromatic copy $K = \{abc, abd, acd\}$ of $K_4^3 - e$, we will show a contradiction when the first coordinate is 0, namely all three triples $\{abc, abd, acd\}$ span rainbow triangles under $\phi$. The cases when the first coordinate is 1, 2 or 3, can be proved using a similar argument. Notice that when the first coordinate is 0, by the definition of $\psi$, the color of a triple depends on the color, under $\phi$, of the edge spanned by the two largest elements in that triple. Since $b, c, d$ play a symmetric role, we can assume that $b < c < d$. 

\[\text{True for the lower bound, let } n = r_k(K_3) - 1 \text{ and } \phi : \binom{[n]}{2} \to k \text{ be a } k\text{-coloring of } \binom{[n]}{2} \text{ with no monochromatic triangles. We will construct a coloring } \psi \text{ of } \binom{[n]}{3} \text{ with } 4k \text{ colors with no monochromatic } K_4^3 - e. \text{ This then would imply that } r_{4k}(K_4^3 - e) \leq n + 1 = r_k(K_3) \text{ as desired. Let } \psi \text{ be the following coloring of the triples } i < j < k. \text{ If } P \text{ is a path with vertices } i, j, k, \text{ denote by } \phi(P) \text{ the color under } \phi \text{ of the edge in } \{i, j, k\} \text{ that is not in } P. \text{ For such a path } P, \text{ let the type of } P, t(P) = 1, 2, \text{ or } 3 \text{ if } i, j \text{ or } k \text{ is its center, respectively. If } \{i, j, k\} \text{ is a rainbow triangle, let } \psi(ijk) = (0, \phi(jk)). \text{ If } \{i, j, k\} \text{ induces a monochromatic path } P, \text{ let } \psi(ijk) = (t(P), \phi'(P)). \text{ Suppose there is a monochromatic copy } K = \{abc, abd, acd\} \text{ of } K_4^3 - e, \text{ we will show a contradiction when the first coordinate is 0, namely all three triples } \{abc, abd, acd\} \text{ span rainbow triangles under } \phi. \text{ The cases when the first coordinate is 1, 2 or 3, can be proved using a similar argument. Notice that when the first coordinate is 0, by the definition of } \psi, \text{ the color of a triple depends on the color, under } \phi, \text{ of the edge spanned by the two largest elements in that triple. Since } b, c, d \text{ play a symmetric role, we can assume that } b < c < d.\]
If $a$ is the smallest, then $\psi(abc) = \psi(abd) = \psi(acd)$ implies $\phi(bc) = \phi(bd) = \phi(cd)$, i.e. $bcd$ is monochromatic under $\phi$. Thus $b$ is the smallest. But then $\psi(abc) = \psi(abd)$ implies $\phi(ac) = \phi(ad)$, which means $acd$ is not a rainbow triangle under $\phi$, a contradiction.

For the upper bound, simply notice that $K_{4}^{3} - e = \{abc, abd, acd\}$ has a single vertex transversal $\{a\}$, and the trace of $a$ is a triangle on $\{b, c, d\}$. Thus the upper bound follows from Lemma 16. The case with 2 colors is treated in Section 7.

Proof of Theorem 8. The cases $k = 2, 3$ are treated in Section 7. The general lower bound follows from Proposition 4(ii), since $F_5$ is not 3-partite.

The upper bound follows by induction with basis $k = 4$. Suppose that the edges of $K_{24}^{3}$ with vertex set $V$ can be 4-colored so that there is no monochromatic $F_5$. There are 22 triples $uvx$ containing a fixed pair $uv$. Assume that $uvx_1, uvx_2$ are red triangles. Then $x_1x_2y$ cannot be red for $y \in Y = V - \{u, v, x_1, x_2\}$. Thus we have a set $S$, $S \subseteq Y$, $|S| \geq \lceil (|V| - 4)/3 \rceil = 7$ and $x_1x_2y$ are blue triples for all $y \in S$. Therefore, no triple in $S$ is colored blue, and thus $\binom{|S|}{3}$ uses $k - 1 = 3$ colors. Applying Theorem 25 to the 3-colored subhypergraph spanned by $S$, we get a contradiction.

The inductive step is simply repeating the argument above in general. Suppose we already know $r_k(F_5) \leq k!$ for some $k \geq 4$ and we have a $K_{n}^{3}$ with a $(k + 1)$-coloring such that there is no monochromatic $F_5$. Selecting $u, v, x_1, x_2$ as above and applying the same argument, we get $n - 4 \leq k(k! - 1) < (k + 1)! - k$, thus $n \leq (k + 1)! - k + 4 \leq (k + 1)!$. This implies $r_{k+1}(F_5) \leq (k + 1)!$.

Remark. The above results slightly suggests that $r_k(F_5) \leq r_k(K_3)$ might hold. Although the bound $r_k(F_5) \leq k!$ in Theorem 8 can be improved slightly, this improvement still does not show that $r_k(F_5) \leq r_k(K_3)$.

5 Bow, Kite, $F(a, b)$

The next lemma (without the statements on the extremal configurations) is referred in [27] as an unpublished remark of Erdős and Sós.

Lemma 17.

$$ex(n, bow) = \begin{cases} 
n & \text{if } n \equiv 0 \pmod{4} \\
n - 1 & \text{if } n \equiv 1 \pmod{4} \\
n - 2 & \text{if } n \equiv 2, 3 \pmod{4}. 
\end{cases}$$

When $n \equiv 0, 1 \pmod{4}$, the extremal configurations are unique, all components are $K_4^3$-s, (apart from a possible one vertex component). When $n \equiv 2 \pmod{4}$, the extremal configuration is either $\frac{n+2}{4}$ copies $K_4^3$-s and two isolated vertices or any number of $K_4^3$-s and one star component. Similarly, when $n \equiv 3 \pmod{4}$, the extremal configuration is either $\frac{n-3}{4}$ copies $K_4^3$-s and component with a single edge or any number of $K_4^3$-s and one star component.
Proof. Suppose $C$ is the vertex set of a nontrivial connected component of a 3-uniform hypergraph without a bow. Then either $C$ spans only one edge or there are two edges $e_1, e_2$ in $C$, intersecting in two vertices, $u,v$. Suppose that $|C| > 4$. Then every edge $f$ that is not covered by $e_1 \cup e_2$ and intersecting $e_1 \cup e_2$ must contain $u,v$ and a vertex $w$ not covered by $e_1 \cup e_2$. It is easy to see that these vertices $w$ cover $C$ and $C$ has no other edges, thus $C$ has $|C| - 2$ edges, all containing $u,v$. Such a component is called a star component.

On the other hand, if $|C| = 4$ then we have two, three or four edges in $C$. From this analysis the lemma follows.

Lower bounds of $r_k(bow)$ follow from the existence of resolvable designs. A $3-(n,4,1)$ design is a set of 4-element subsets (blocks) of an $n$-element set $V$ such that each 3-element subset of $V$ is in precisely one block. Hanani [15] showed that $3-(n,4,1)$ designs exist if and only if $n \equiv 2, 4 \pmod{6}$. A $3-(n,4,1)$ design is called resolvable if its blocks can be grouped so that each group (parallel class) gives a partition of $V$. Resolvable $3-(n,4,1)$ designs exist if and only if $n \equiv 4, 8 \pmod{12}$, see [18, 19], and [21].

Proof of Theorem 9. When $n \equiv 4, 8 \pmod{12}$, $k = \frac{n}{3}$, $ex(n,bow) = n$, thus Lemma 15(i) gives $r_k \leq n+1$. This is sharp, since $K_3^n$ can be partitioned into $k$ matchings. The statement $r_k(bow) \approx \sqrt{6k}$ follows from considering the construction for largest $n$, $n \equiv 4, 8 \pmod{12}$, $k \geq \frac{n}{3}$ for the lower bound and applying the Lemma 15(i) for the upper bound. The statements about the small values are proved in Section 7.

Proof of Theorem 10. Let $H = F(2,2)$ be the kite. Then $ex(n,H)$ corresponds to the maximum number of triples on $n$ elements such that any two triples intersect in at most one element, i.e. the maximum number of edges in a linear 3-uniform hypergraph. A well-known result of Schönhem [36] and others (the cases $n \equiv 0, 1, 2, 3 \pmod{6}$) go back even to Kirkman [22]) is $ex(n,H) = \lfloor \frac{n}{3} \lfloor \frac{n-1}{2} \rfloor \rfloor - \epsilon$, where $\epsilon = 1$ for $n \equiv 5 \pmod{6}$, otherwise $\epsilon = 0$. Lemma 15(i) gives, after some calculations, the upper bounds.

The lower bound for the cases $k \equiv 3, 4 \pmod{6}$ is easy. Given $K_3^n = (V,E)$, consider $V = Z_n$ and color triple $ijk$ with color $i + j + k \pmod{n}$. Clearly this coloring yields no monochromatic $H$, hence $r_k(H) > k$.

For the cases $k \equiv 0, 1, 2, 5 \pmod{6}$ the (difficult) constructions of J. X. Lu [29, 30] finished by Teirlinck [38] are needed: for $n > 7, n \equiv 1, 3 \pmod{6}$, $K_3^n$ can be partitioned into $n-2$ Steiner triple systems (called a large set of STS).

Indeed, for $k \equiv 0, 2 \pmod{6}$ we need a kite-free $k$-coloring of $K_3^{k+1}$ i.e. $(n-1)$-coloring of $K_3^n$ when $n \equiv 1, 3 \pmod{6}$. This can be done even with $n-2$ colors according to the cited result of Lu. However, the case $k = 6$ is exceptional because Lu’s theorem does not hold for $n = 7$. Nevertheless, there is a 6-coloring of $K_3^7$ without a monochromatic kite as shown in Proposition 22. Similarly, for $k \equiv 1, 5 \pmod{6}$ we need a kite-free $k$-coloring of $K_3^{k+2}$ i.e. $(n-2)$-coloring of $K_3^n$ when $n \equiv 3, 1 \pmod{6}$. This is provided by Lu’s theorem, apart from the case $k \equiv 5 \pmod{6}$ which is indeed exceptional, in Proposition 22 we prove that
Proof of Proposition 11. In an $F(a, 2)$-free coloring of $K^3_n$ any pair of vertices is in at most $a - 1$ edges of the same color. Thus $n \leq 2 + k(a - 1)$, proving the upper bound. (One can also use Lemma 16 and the multicolor Ramsey number for stars (see [1]): $r_k(K_{1,a}) \leq k(a - 1) + 2$.)

For the lower bound, set $n = k(a - 1)$ and consider $K^3_n = (V, E)$ with $V = \mathbb{Z}_n$. Color a each edge with the sum of its vertices mod $k$. Then a monochromatic copy of $F(a, 2)$ would require that for some $y, z \in V$, $y + z + x_1, ..., y + z + x_a$ are all equal (mod $k$) i.e. we have $a$ different positive $x_s$, all equal (mod $k$), which is impossible. Hence $r_k(F(a, 2)) > k(a-1)$.

Proof of Theorem 12. For the upper bound, let $N = r_t(K_b) + m$. Consider a $k$-coloring $\phi$ of the triples of $K_N$. Fix a set $S$ of $m$ vertices and define a $t$-coloring $c$ on the pairs of the remaining $N - m$ vertices as follows. Let $c(xy) = (\phi(xys_i), s_1, s_2, ..., s_a)$, where $\phi(xys_i)$ is the majority color on triples containing $x$ and $y$, and $s_1, s_2, ..., s_a \in S$ is the lexicographically first $a$-tuple in $S$ such that $\phi(xys_i) = \phi(xys_j)$ for every $1 \leq j \leq a$ (by the choice of $m$ there is such an $a$-tuple). Since $c$ is a $t$-coloring of a complete graph on $N - m = r_t(K_b)$ vertices, there is monochromatic $K_b$ in $c$, which gives a monochromatic $F(a,b)$ in $\phi$.

A lower bound for $r_k(F(a,b))$ is obtained from Proposition 4 (i) since $F(a,b)$ is not 3-partite, for $b \geq 3$.

6 Windmill and tight path

The following result (conjectured by Kalai) is a special case of a theorem of Füredi and Frankl ([12], Theorem 3.8). We give their proof also, since it is extremely short in this special case.

Theorem 18. $\text{ex}(n,W) \leq \binom{n}{2}$ with equality for every $n \equiv 1, 5 \pmod{20}$.

Proof. The lower bound comes from the following construction. Let $n \equiv 1, 5 \pmod{20}$ and consider a Steiner system $S$, a $2 - (n, 5, 1)$ design, i.e., a set of 5-element blocks on $n$ elements such that every pair lies in precisely one block. Its existence is proved by Hanani [16, 17]. Then the number of blocks is $\binom{n}{2}/10$. Now place 10 triples inside each block of $S$. The resulting triple system, $H$, has $\binom{n}{2}$ triples and is $W$-free. Indeed, a copy of $W$ would have to be contained in one of the blocks, but each block has less vertices than the number of vertices in $W$.

To prove the upper bound, suppose that $H$ is a 3-uniform hypergraph with no $W$. For $x, y \in V(H)$, the codegree $d(x, y)$ is the number of edges of $H$ containing both $x, y$. Let $a, b, c$ be codegrees of three pairs of vertices from a edge of $H$, $1 \leq a \leq b \leq c$. If $a = 2$, $b \geq 3$ and $c \geq 4$, then $H$ contains a copy of $W$. Thus either $a = 1$ or $a = b = 2$ or $a = 2, b = 3, c = 3$. 

\[ r_5(\text{kite}) = 6 \] (together with the case $k = 4$). \[ \square \]
In each of these cases we have that $1/a + 1/b + 1/c \geq 1$. For each edge $e = uvw$ of $H$, let

$$w(e) = \frac{1}{d(u,v)} + \frac{1}{d(v,w)} + \frac{1}{d(u,w)}.$$ 

We see that $w(e) \geq 1$. Let $s = \sum_{e \in H} w(e)$. Notice that $s \leq \binom{n}{2}$, since a term $\frac{1}{d(u,v)}$ appears exactly $d(u,v)$ times for each pair $uv$ that belongs to at least one edge of $H$. Now, $|H| \leq |H| \min_{e \in H} w(e) \leq s \leq \binom{n}{2}$. \hfill \Box

For the next proof we need the following decomposition result:

**Theorem 19** (Pippenger and Spencer [33]). Let $r$ be fixed and $D$ be sufficiently large. Let $H$ be an $r$-uniform hypergraph with $d(v) = (1 + o(1))D$ for every $v \in V(H)$ and codegree of $o(D)$ for every pair $\{u,v\} \subseteq V(H)$. Then $E(H)$ can be partitioned into $(1 + o(1))D$ matchings.

**Proof of Theorem 13.** To prove the lower bound, let $S$ be a $3 - (n,5,1)$ design, i.e. a set of 5-element blocks of an $n$-element set such that each 3-element set is in precisely one block. The existence of such designs are known for infinitely many $n$, for example for $n = 4^s + 1, s \geq 2$ [20], see also [32]. Construct an auxiliary 10-uniform hypergraph $H$ where $V(H)$ is the set of $\binom{5}{2}$ pairs in $V(S)$, and ten of these pairs form an edge of $H$ if and only if they are the ten pairs in a block of $S$. Since every pair in $V(S)$ is in exactly $(n-2)/3$ blocks of $S$, $H$ is an $(n-2)/3$-regular hypergraph. On the other hand, the codegree of any two vertices in $H$ is at most one. Indeed, any two vertices in $H$ (two pairs in $V(S)$) contain at least three vertices in $V(S)$, and they can be in at most one block of $S$. With large enough $n$, and with $r = 10, D = n/3$, the conditions of Theorem 19 hold so we can decompose $E(H)$ into $m = (1 + o(1))n/3$ matchings $M_i, i = 1, 2, \ldots, m$. Each $M_i$ corresponds to a subset of blocks $S_i$ of $S$ and any two blocks in $S_i$ share at most one element in $V(S)$. The set of triples covered by the blocks of any $S_i$ form a $W$-free triple system (the center edge of a windmill $W$ in a block $B \in S_i$ would force the other three edges of $W$ to $B$, similarly as in Theorem 18). Thus $K^3_n$ is decomposed into $m = (1 + o(1))n/3$ $W$-free triple systems, showing $r_k(W) \geq (1 - o(1))3k$.

The upper bound follows from Theorem 18, in a $k$-coloring of $K^3_n$ with no monochromatic $W$, each color class has at most $\text{ex}(n,W) = \binom{n}{2}$ edges. Thus $\binom{n}{2}/k \leq \binom{n}{2}$, implying $n \leq 3k + 2$. So by Lemma 15(i), $r_k(W) \leq 3k + 3$. \hfill \Box

We need the following result for tight path.

**Proposition 20.** $\text{ex}(n, P^3_3) \leq \frac{n(n-1)}{3}$ with equality for $n \equiv 1, 4 \pmod{12}$.

**Proof of Proposition.** For a $P^3_3$-free hypergraph $T$ on $n$ vertices and a vertex $v$, the degree $d(v) \leq \text{ex}(n-1, P_4) \leq n - 1$. Thus $3|E(T)| = \sum_v d(v) \leq n(n-1)$. The statement for equality comes from a $2 - (n,4,1)$ design by replacing all blocks by $K^3_4$-s. \hfill \Box
Proof of Theorem 14. Observe that the trace of $P^3_3$ at its transversal vertex is $P_4$, the path on four vertices. The upper bound can be obtained in two ways.

Applying and Lemma 15 (i) with proposition 20, we have $r_k(P^3_3) \leq 2k + 3$. On the other hand, we may apply Lemma 16 as well: $r_k(P^3_3) \leq r_k(P_4) + 1 \leq 2k + 3$ (34).

For the lower bound we start with a $3-(n,4,1)$ design $F$ (already used in the proof of Theorem 9) and follow the construction in the proof of Theorem 13. Consider the 6-uniform hypergraph $H$ with vertex set being the set of pairs of vertices of $F$ and edges formed by the sets of pairs within the blocks of $F$. The degree of any vertex in $H$ is $d = (n - 2)/2$, the codegree of any pair of vertices is at most one, so the conditions for Pippenger-Spencer Theorem are satisfied, giving a decomposition of $H$ into $(1 + o(1))d = (1 + o(1))n/2$ matchings, $M_i$. Each $M_i$ corresponds to a set $F_i$ of blocks of $F$, intersecting each other in at most one element. Let $T_i$ be the set of triples covered by the blocks of $F_i$. The $T_i$-s provide the required $P^3_3$-free coloring of $K^3_n$ with $(1 + o(1))n/2$ colors.

\[\square\]

7 Small Ramsey numbers

The only known non-trivial classical Ramsey number for triples is $r_2(K^3_4) = 13$, due to McKay and Radziszowski [31].

It is proven in (34) that $13 \leq r_3(K^3_4 - e) \leq 16$ and stated as an easy fact without a proof that $r_2(K^3_4 - e) = 7$. Here we prove this for completeness.

Proposition 21. $r_2(K^3_4 - e) = 7$.

Proof. Consider the following coloring $C$ of $K^3_6$. Fix the set of five vertices, $V$, and let $c$ be the 2-coloring of $K_5$ on vertex set $V$ with two monochromatic $C_5$’s. Let $v$ be the remaining vertex of $K^3_6$. For any triple containing $v$, let $C$($\{v, u, w\}$) = $c$($uw$).

For each triple $xyz$, not containing $v$, let $C$($\{x, y, z\}$) be the color different from $c$($V - \{x, y, z\}$). Under coloring $C$, there are two triples of each color in every 4-set, hence there is no monochromatic $K^3_4 - e$.

The following proposition determines the small undecided cases from Theorem 10. A hypergraph is linear if every two edges share at most one vertex.

Proposition 22. $r_4(kite) = 5, r_5(kite) = 6, r_6(kite) = 8$.

Proof. It is obvious that $r_4(kite) > 4$. The fact that $r_4(kite) \leq 5$ follows by observing that any 4-coloring of the edges of $K^3_3$ contains three edges of the same color.

Coloring the triple $ijk$, $1 \leq i < j < k \leq 5$ by color $i + j + k \ (mod \ 5)$ gives $r_5(kite) > 5$. To show that $r_5(kite) \leq 6$, we need the result of Cayley [2], stating that the maximum number
of pairwise disjoint Fano planes in $K_7^3$ is 2. Suppose $K_6^3$ on vertex set $V$ is 5-colored so that each color class $i$ is a linear hypergraph $P_i$. Since the average number of edges in a color class is four and no linear hypergraphs on 6 vertices can have more than four edges, it follows that each $P_i$ must be a Pasch configuration. Therefore the pairs uncovered by the triples of $P_i$ form a matching $M_i$ in the complete graph on $V$. The $M_i$'s must form a factorization on $V$ otherwise some pair in $V$ would be covered by at most three $P_i$'s instead of the required four. These $P_i$'s can be extended by a new vertex to a decomposition of $K_7^3$ into five Fano planes, contradicting Cayley's theorem stated above.

The upper bound $r_6(kite) \leq 8$ is already proved (see the proof of Theorem 10). For the lower bound we need a partition of $K_7^3$ into six linear hypergraphs, see Figure 1. Set $V = [7]$ and let $F_1, F_2$ be the two Fano planes generated by shifts of $124, 134 \pmod 7$. The next two sets are isomorphic to a Fano plane from which one line is deleted:

$$F_3 = \{135, 167, 236, 257, 347, 456\}, F_4 = \{123, 146, 247, 256, 345, 367\}$$

and $F_6 = \{127, 136, 145, 246, 567\}$ (Fano plane from which two lines are deleted), $F_7 = \{125, 147, 234, 357\}$ (a Pasch configuration).

Figure 1: Partition of $K_7^3$ into two Fano, two Fano $-e$, Fano $-2e$, Pasch
Proposition 23. Set $r_k = r_k(bow)$, then $r_1 = r_2 = 5, r_3 = r_4 = r_5 = 6, r_6 = 7, r_7 = r_8 = r_9 = r_{10} = 9, 9 \leq r_{11} \leq r_{12} \leq r_{13} \leq r_{14} \leq 10, r_{15} = 11$.

Proof. All but one upper bounds are obtained from Lemma 15(i). The exceptional case is when Lemma 15(i) gives $r_5(bow) \leq 7$. Here we improve it as follows. Suppose $K_6^3$ is 5-colored without monochromatic bow. From Lemma 17 each color class is either a $K_4^3$ (type A) or four triples pairwise intersecting in the same base pair (type B). There are at most three type A colors. The base pairs for different type B colors must be vertex disjoint. Thus there are at least two type A color classes, w.l.o.g. $abcd, cdef$. But then only the base pairs $ae, af, be, bf$ are available for type B colors. Therefore we have two type B and three type A colors, the third is the $K_4^3$ spanned by $abef$. Now there is no base pair available for type B color classes since every pair of vertices is covered by a type A $K_4^3$.

Lower bounds should be exhibited for $r_1, r_3, r_6, r_7, r_{15}$ only. Coloring all triples of $K_4^3$ with the same color, $r_1 > 4$ follows. Coloring the triples of $\{1, 2, 3, 4\}$ with color 1, the triples 125, 135, 235 with color 2, the triples 145, 245, 345 with color 3, $r_3 > 5$ follows. Then $r_6 > 6$ comes from the following 6-coloring with color classes $(1, 2, 3, 4), (1, 3, 4, 5, 6), (1, 4, 5, 6), (2, 4, 5, 6) \setminus \{4, 5, 6\}, (1, 2, 3, 5) \setminus \{1, 2, 3\}, (1, 2, 3, 6) \setminus \{1, 2, 3\}$. The 7-coloring of $K_8^3$ is the 7 parallel classes of the unique $3 - (8, 4, 1)$ design. Finally, the 15-coloring of $K_{10}^3$ comes from the unique $3 - (10, 4, 1)$ design whose 30 blocks can be partitioned into 15 disjoint pairs. □

Proposition 24. $r_2(F_5) = 6$.

Proof. The lower bound is obvious, color triples of $K_5^3$ containing a fixed vertex with color 1 and other triples by color 2. For the upper bound, consider a 2-colored $K_5^3$ on vertex set $\{1, 2, 3, 4, 5, 6\}$ and its 2-colored trace $K = K_5^2$ with respect to vertex 6. There is a monochromatic, say red odd cycle $C$ in $V(K) - \{6\}$. If $C = 1, 2, 3, 1$ then either there is a red triple in $K$ with two vertices on $C$ and one vertex not in $C$ or all such triples are blue. The former gives a red, the latter a blue $F_5$. If $C = 1, 2, 3, 4, 5, 1$ then either there is a red triple with vertices non-consecutive on $C$ or all the five such triples are blue. Again, the former gives a red, the latter a blue $F_5$. □

Theorem 25. $r_3(F_5) = 7$.

Proof. For the lower bound, color the triples of $K_6^3$ containing $v$ with color 1, color uncoupled triples containing vertex $w \neq v$ with color 2 and color all other edges with color 3.

To prove the upper bound, call a graph $G$ nice if for every triple $T = \{v_1, v_2, v_3\}$ of vertices at least one of the following holds:

1. There are two vertex disjoint edges of $G$, such that one of them is in $T$ and the other meets $T$. 2. There is a path of length two in $G$ connecting two vertices of $T$ with midpoint not in $T$.

Observation 26. If $H$ is an $F_5$-free 3-uniform hypergraph, such that the trace of $v$ for a vertex $v$ is a nice graph, then all edges of $H$ within $V(G) \cup \{v\}$ contain $v$. 15
Indeed, otherwise from the definition of a nice graph we find $F_5$ in $H$. Thus finding a large nice subgraph in a trace one can reduce the number of colors. More generally, a graph is $i$-nice if the property holds for all but at most $i$ triples of vertices.

We need a lemma on 6-vertex graphs. Since its proof is routine but lengthy, we state it without proof.

**Lemma 27.** Suppose $G$ has six vertices. If $|E(G)| \geq 9$ then $G$ is nice. If $|E(G)| = 8$ then $G$ is 1-nice, if $|E(G)| = 7$ then $G$ is 2-nice. If $|E(G)| = 6$ then $G$ is 5-nice, except in one case, when $G$ is $K_{2,3}$ plus an isolated vertex (in this case it is 6-nice).

With these preparations we are ready to prove the upper bound. The majority color, say red in a 3-colored $K_7^3$, has at least 12 edges. Some vertex $v$ has red degree at least 6. Let $G$ be the trace of a red hypergraph at $v$. We get a contradiction from Lemma 27 (and from the fact that we have 12 edges) except when $G$ has exactly six edges and the trace is $K_{2,3} + w$. This case implies that the red color class has 12 edges forming $K_{2,2,3}$, a complete 3-partite hypergraph with parts of sizes 2, 2, and 3. However, among the $35 - 12 = 23$ edges of other colors, one color, say blue, has at least 12 edges. Repeating the argument for the blue hypergraph, we conclude that the blue hypergraph is also a $K_{2,2,3}$. However, as one can easily check, there is no way to place two edge disjoint $K_{2,2,3}$-s on 7 vertices.

### 8 Concluding remarks

We determined, for 3-uniform hypergraphs, $r_k$ ranges from $\sqrt{k}$ to double exponential in $k$, and showed a jump in $r_k$ when $H$ changes from $r$-partite to non-$r$-partite. This leads to the following question.

**Problem 28.** For which 3-uniform hypergraphs $F$, is $r_k(F)$ double exponential? Are there other jumps that the Ramsey function $r_k$ exhibits?

The ramsey-numbers $r_k(\text{bow}), r_k(\text{kite})$ are closely connected to block designs. In case of the kite the only uncertainty is whether $r_k(\text{kite})$ is $k + 1$ or $k + 2$ when $k \equiv 4 \pmod{6}$. This leads to the following problem.

**Problem 29.** Suppose $n \equiv 5 \pmod{6}$. Is it possible to partition the triples of an $n$-element set into $n - 1$ partial triple systems, i.e. into parts so that distinct triples in each part intersect in at most one vertex? By Theorem 10, this is not possible for $n = 5$ but perhaps for large enough $n$ (possibly for $n \geq 11$) such partitions exist.

In case of the bow, the problems related to sharper bounds of $r_k(\text{bow})$ are not purely design theoretic, since color classes can be star components as well. We state just one of those problems.
Problem 30. Suppose \( n \equiv 6, 10 \pmod{12} \). Is it possible to partition the triples of an \( n \)-element set into \( \frac{n(n-1)}{2} \) classes so that each class is the union of some disjoint \( K_3^3 \)-s and at most one star component? (Any color class has \( n-2 \) triples.) For \( n = 6 \) there is no solution.

Concerning \( r_k(K_3 - e) \) the most challenging (perhaps difficult) problem is to decrease the upper bound of Theorem 7 by one.

Problem 31. \( r_k(K_3^3 - e) < r_k(K_3) + 1 \) for every \( k \geq 3 \)?

A challenging open problem is to improve the estimates of \( r_k(P) \) (and/or \( ex(n, P) \)) where \( P \) is the Pasch configuration with edges \{\( abc, bde, cef, adf \)\}. (It can be obtained from the Fano plane by deleting a vertex.) Presently only the following is known.

Proposition 32. For positive constants \( c, c' \),
\[
c \left( \frac{k}{\log k} \right)^2 < r_k(P) < c' k^4.
\]

Proof. The lower bound is based on the following \( P \)-free hypergraph, showing that \( ex(n, P) = \Omega(n^{5/2}) \). \[26\]. Take an incidence graph \( G \) of a projective plane with \( n \) points and \( n \) lines. It has \( \Omega(n^{3/2}) \) edges. Add \( n \) new vertices \( x_1, \ldots, x_n \) and add all triples of the form \( x_i \cup e \), where \( e \) is an edge of \( G \). The resulting 3-uniform hypergraph, call it \( H \), has \( 3n \) vertices and \( \Omega(n^{5/2}) \) edges.

Notice that the edge-density of \( H \) is \( d(H) = cn^{-1/2} \) for some constant \( c > 0 \). From Lemma 15(ii) we see that there is a coloring of \( K_n^3 \) with \( (c' n^{1/2} \log n) \) colors and no monochromatic \( P \). Thus \( r_k(P) > n \) with \( k = c' n^{1/2} \log n \). Expressing \( n \) in terms of \( k \) gives the desired lower bound.

The upper bound follows from Lemma 15(i) and the fact that \( ex(n, P) = O(n^{11/4}) \) \[26\]. This is based on the claim that \( ex(n, K(2, 2, 2)) = O(n^{11/4}) \) proved by Erdős \[8\], where \( K(2, 2, 2) \) is the complete 3-partite 3-uniform hypergraph with two vertices in each part. \( \square \)

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