Euclidean Geometric Objects in the Clifford Geometric Algebra of \{\text{Origin, 3-Space, Infinity}\}

Eckhard M.S. Hitzer
1 Dec. 2003 (corrections: 14 July 2004)

Abstract
This paper concentrates on the homogeneous (conformal) model of Euclidean space (Horosphere) with subspaces that intuitively correspond to Euclidean geometric objects in three dimensions. Mathematical details of the construction and (useful) parametrizations of the 3D Euclidean object models are explicitly demonstrated in order to show how 3D Euclidean information on positions, orientations and radii can be extracted.

1 Introduction

The Clifford geometric algebra of three dimensional (3D) Euclidean space with vectors
\[ p = p_1 e_1 + p_2 e_2 + p_3 e_3. \]
given in terms of an orthonormal basis \( \{e_1, e_2, e_3\} \), nicely encodes the algebra of 3D subspaces with algebraic basis
\[ \{1, e_1, e_2, e_3, i_1 = e_2 e_3, i_2 = e_3 e_1, i_3 = e_1 e_2, i = e_1 e_2 e_3\}, \]
providing geometric multivector product expressions of rotations and set theoretic operations. But in this framework line and plane subspaces always contain the origin.

The homogeneous (conformal) model of 3D Euclidean space in the Clifford geometric algebra \( \mathbb{R}^{4,1} \) provides a way out. Here positions of points, lines and planes, etc. off the origin can be naturally encoded. Other advantages are the unified treatment of rotations and translations and ways to encode point pairs, circles and spheres. The creation of such elementary geometric objects simply occurs by algebraically joining a minimal number of points in the object subspace. The resulting multivector expressions completely encode in their components positions, orientations and radii.

The geometric algebra \( \mathbb{R}^{4,1} \) can be intuitively pictured as the algebra of origin \( \mathbf{n} \), Euclidean 3D space and infinity \( \mathbf{n} \), where origin and infinity are represented by additional linearly independent null-vectors.
\[ \{\mathbf{n}, e_1, e_2, e_3, \mathbf{n}\}, \quad \mathbf{n}^2 = \mathbf{n}^* = 0. \]
2 3D Information in Homogeneous Objects

We will see how homogenous multivectors completely encode positions, directions, moments and radii of the corresponding three dimensional (3D) objects in Euclidean space. An overview of this is give in Table 1. In the rest of this section we will look at the details of extracting the encoded 3D information from each homogeneous multivector object. Where suitable, we will also give
useful alternative parametrizations of homogeneous multivector objects.

2.1 Point and Pair of Points

The Euclidean position $p$ of a conformal point

$$ P = p + \frac{1}{2}p^2n + \bar{n} \quad (4) $$

is obtained with the help of the (additive) conformal split, which is an example of a rejection: The conformal point vector $P$ is rejected off the Minkowski plane represented by the bivector $N = n \wedge \bar{n}$

$$ p = (P \wedge N)N. \quad (5) $$

Equation (4) shows how to achieve the opposite, i.e. how to get back to the conformal point $P$ from just knowing the Euclidean position $p$.

The Euclidean positions $p_1, p_2$ of a pair of points represented by the conformal bivector $V_2 = P_1 \wedge P_2$

$$ V_2 = p_1 \wedge p_2 + \frac{1}{2}(p_2^2p_1 - p_1^2p_2)n - (p_2 - p_1)\bar{n} + \frac{1}{2}(p_1^2 - p_2^2)N $$

$$ = b - \frac{1}{2}vn - u\bar{n} - \frac{1}{2} \gamma N \quad (6) $$
can be fully reconstructed from the components of $V_2$. We assume without restricting the generality, that $p_1 = \sqrt{p_1^2} \geq p_2 = \sqrt{p_2^2}$. Given any conformal bivector $V_2$ with components $b$ (a Euclidean bivector), $u$ and $v$ (Euclidean vectors of length $u = \sqrt{u^2}$ and $v = \sqrt{v^2}$), and $\gamma$ (a real scalar), the calculation works as follows

$$ \sigma = \frac{1}{2} \gamma^2 - u \ast v, \quad \rho = \sqrt{\sigma^2 - u^2v^2}, $$

$$ p_1 = \frac{\sigma + \rho}{u}, \quad p_2 = \frac{\sigma - \rho}{u}, \quad (8) $$

$$ p_1 = \frac{p_1^2u + v}{p_1^2u + v}, \quad p_2 = \frac{p_2^2u + v}{p_2^2u + v}. \quad (9) $$

This calculation is the full solution (of two conformal points $X = P_1, P_2$) to the equation

$$ V_2 \wedge X = 0, X^2 = 0. \quad (10) $$

Dorst and Fontijne give similar parametrizations, but with the rather strong simplification, that objects are centered at the origin $\bar{n}$ or contain the origin $\bar{n}$, e.g. $C = \bar{n}$ in eq. (10), etc. The general and explicit formulas presented in the following, seem to appear nowhere else in the published literature so far.
We can further view conformal point pairs as one-dimensional circles and arrive thereby at another highly useful characterization:

\[
P_1 \wedge P_2 = 2r\{\hat{p} \wedge c + \frac{1}{2}((c^2 + r^2)\hat{p} - 2c * \hat{p} \cdot c)\cdot n + \hat{p}\bar{n} + c * \hat{p}N\},
\]

(11)

with the "radius" \( r \) defined as half the Euclidean point pair distance, \( \hat{p} \) a unit vector pointing from \( p_2 \) to \( p_1 \), and \( c \) the Euclidean midpoint (center) of the point pair:

\[
2r = |p_1 - p_2|, \quad \hat{p} = \frac{p_1 - p_2}{2r}, \quad c = \frac{p_1 + p_2}{2}.
\]

(12)

In case that the straight line defined by the point pair contains the origin, i.e. for \( \hat{p} \wedge c = 0 \ (\hat{p} \parallel c) \) we get the simplified form

\[
P_1 \wedge P_2 = 2r\{C - r^2n\}\hat{p}N.
\]

(13)

In case that the Euclidean midpoint vector \( c \) is perpendicular to \( \hat{p} \) (\( \hat{p} \perp c \)), i.e. if \( \hat{p} \cdot c = 0 \) we get

\[
P_1 \wedge P_2 = -2r\{C + r^2n\}\hat{p}.
\]

(14)

In both cases we used the conformal representation of the midpoint as

\[
C = c + \frac{1}{2}c^2n + \bar{n}.
\]

(15)

2.2 Lines

Given two conformal points \( P_1 \) and \( P_2 \) the conformal trivector

\[
V_{line} = P_1 \wedge P_2 \wedge n = p_1 \wedge p_2 \wedge n + (p_2 - p_1)N = m \cdot n + dN
\]

(16)

conveniently consists of the defining entities of the Euclidean line through \( p_1 \) and \( p_2 \). The Euclidean bivector \( m \) represents the moment and the Euclidean vector \( d \) the direction of the line. Using \( m \) and \( d \) we can give the parametric form of the line as

\[
x = (m + \alpha)d^{-1}, \quad \alpha \in \mathbb{R}.
\]

(17)

All points \( X = x + \frac{1}{2}r^2n + \bar{n} \) with the \( x \) as specified in (17) represent the full solution to the problem

\[
V_{line} \wedge X = 0, \quad X^2 = 0.
\]

(18)

The one-dimensional circle representation of point pairs (11) immediately leads to a second often useful parametrization of lines as

\[
P_1 \wedge P_2 \wedge n = 2r\hat{p} \wedge C \wedge n = 2r\{\hat{p} \wedge c \cdot n - \hat{p}N\}.
\]

(19)

It is important to note that the conformal point \( C \) in eq. (19) does not need to be the midpoint of the point pair. Any conformal point on the straight line \( P_1 \wedge P_2 \wedge n \) can take the place of \( C \) in eq. (19). \( \hat{p} \) and \( r \) are defined as in eq. (12).

---

2This characterization is e.g. very useful for investigating the full (real and virtual) meet of two circles, or of a straight line and a circle.
2.3 Circles

General conformal trivectors of the form

\[ V_3 = P_1 \wedge P_2 \wedge P_3, \]  

(20)

with conformal points \( P_1, P_2 \) and \( P_3 \) represent Euclidean circles through the corresponding Euclidean points \( p_1, p_2 \) and \( p_3 \). The equation for all points \( X \) on such a circle is again given as

\[ V_3 \wedge X = 0, \quad X^2 = 0. \]  

(21)

In order to clearly interpret and apply \( V_3 \) and its various components we will explicitly insert the three points \( P_k = p_k + \frac{1}{2} p_k^2 n + \bar{n} \), \( P_2 = p_2 + \frac{1}{2} p_2^2 n + \bar{n} \), \( P_3 = p_3 + \frac{1}{2} p_3^2 n + \bar{n} \). (22)

The conformal circle trivector becomes

\[ V_3 = p_1 \wedge p_2 \wedge p_3 
+ \frac{1}{2} (p_1^2 p_2 \wedge p_3 + p_2^2 p_3 \wedge p_1 + p_3^2 p_1 \wedge p_2) n 
+ (p_2 \wedge p_3 + p_3 \wedge p_1 + p_1 \wedge p_2) \bar{n} 
+ \frac{1}{2} \{ p_1 (p_2^2 - p_3^2) + p_2 (p_3^2 - p_1^2) + p_3 (p_1^2 - p_2^2) \} N. \]  

(23)

The Euclidean bivector component factor of \( \bar{n} \)

\[ I_c = \{ [V_3 + (V_3 \ast i) \wedge n] \wedge n \} N \]

\[ = (p_2 \wedge p_3 + p_3 \wedge p_1 + p_1 \wedge p_2) N \]

\[ = (p_1 - p_2) \wedge (p_2 - p_3) \]  

(24)

is obviously parallel to the plane (of the Euclidean circle) through \( p_1, p_2 \) and \( p_3 \). Assuming the Euclidean center vector of the circle to be \( c \) and the radius \( r \), we can rewrite (22) as

\[ P_k = c + r r_k + \frac{1}{2} c^2 + r^2 + 2 r c \ast r_k n + \bar{n}, \quad r_k^2 = 1, \quad k = 1, 2, 3. \]  

(25)

The three vectors \( r_k \) are unit length vectors pointing from the circle center \( c \) to the three points \( p_1, p_2 \) and \( p_3 \), respectively. Replacing the \( P_k \) in (23) accordingly we get after doing some algebra the simplified form

\[ V_3 = c \wedge I_c + \frac{1}{2} (c^2 + r^2) I_c - c (c \ast I_c) n + I_c \bar{n} - (c \ast I_c) N. \]  

(26)

We see that the three vectors \( r_k, k = 1, 2, 3 \) do no longer occur explicitly. They enter equation (26) only by defining the orientation of the circle plane \( I_c \) in (24).
If we assume only to know $V_3$ as outer product \( V_3 \) of three general conformal points we can now extract the radius $r$ by calculating

$$r^2 = -\frac{V_3^2}{I_c^2}. \quad (27)$$

We can decompose the center vector $c$ by way of projection and rejection into components parallel and perpendicular to the circle plane

$$c = c_\parallel + c_\perp, \quad (28)$$

$$c_\parallel = (c \land I_c^{-1})I_c = (c \land I_c)I_c^{-1} \quad (29)$$

$$c_\perp = (c \land I_c^{-1})I_c = (c \land I_c)I_c^{-1} \quad (30)$$

The Euclidean circle center vector can hence be extracted from an $V_3$ as

$$c = c_\parallel + c_\perp = -\frac{(V_3 \land n) \land n + (V_3 \ast i)i}{I_c}. \quad (31)$$

Inserting the decomposition $c = c_\parallel + c_\perp$ we get the following expression for the circle trivector

$$V_3 = c_\perp I_c + \left[ \frac{1}{2}(r^2 - c^2)I_c + cc_\perp I_c \right]n + I_c \bar{n} - c_\parallel I_c N$$

$$= \{c_\perp N + \left[ \frac{1}{2}(r^2 - c^2) + cc_\perp \right]n - \bar{n} - c_\parallel \} I_c N$$

$$= \{-c_\parallel - \frac{1}{2}c^2 n - \bar{n} + \frac{1}{2}r^2 n + c_\perp N + \left[ -\frac{1}{2}c^2 + cc_\perp \right]n \} I_c N \quad (32)$$

In the case that the circle plane includes the origin ($c_\perp = 0$) we are left with

$$V_3 = -[C - \frac{1}{2}r^2 n]I_c N, \quad (33)$$

and can extract the conformal center

$$C = c + \frac{1}{2}c^2 n + \bar{n} \quad (34)$$

simply as

$$C = -\frac{V_3}{I_c N} + \frac{1}{2}r^2 n. \quad (35)$$

### 2.4 Planes

Given three conformal points $P_1, P_2$ and $P_3$ as in \( V_3 \) the conformal 4-vector

$$V_{plane} = P_1 \land P_2 \land P_3 \land n$$

$$= p_1 \land p_2 \land p_3 \land n$$

$$= -(p_2 \land p_3 + p_3 \land p_1 + p_1 \land p_2)N \quad (36)$$
represents the plane through the Euclidean points \( p_1, p_2 \) and \( p_3 \). The Euclidean bivector component factor of \( N \)
\[ I_p = -(V_{plane} \cdot n) \mathbf{L} \cdot \mathbf{n} = p_2 \wedge p_3 + p_3 \wedge p_1 + p_1 \wedge p_2 = (p_1 - p_2) \wedge (p_2 - p_3) \] gives the orientation of the plane in the Euclidean space. This allows us to rewrite \( V_{plane} \) as
\[ V_{plane} = (d \wedge I_p) n - I_p N = d I_p n - I_p N, \] where \( d \) represents the Euclidean distance vector from the origin to the plane, itself perpendicular to the plane. The Euclidean distance vector can be extracted from \( V_{plane} \) by
\[ d = (V_{plane} \wedge \mathbf{n}) I_p^{-1} n. \] The equation for all points \( X \) on the plane is again given as
\[ V_{plane} \wedge X = 0, \quad X^2 = 0. \]

Somewhat in analogy of the relation of point pairs as one-dimensional circles and the resulting alternative parametrization of lines, an alternative parametrization of planes by means of a general conformal point \( C \) on the plane is possible
\[ P_1 \wedge P_2 \wedge P_3 \wedge n = C \wedge I_c \wedge n = c \wedge I_c n - I_c N \] For \( c \wedge I_c = 0 \) (origin \( \mathbf{n} \) in plane) we get
\[ P_1 \wedge P_2 \wedge P_3 \wedge n = -I_c N. \]

### 2.5 Spheres

General conformal 4-vectors of the form
\[ V_4 = P_1 \wedge P_2 \wedge P_3 \wedge P_4 \] with conformal points
\[ P_k = p_k + \frac{1}{2} p_k^2 n + \mathbf{n}, \quad k = 1, 2, 3, 4 \] represent Euclidean spheres through the corresponding Euclidean points \( p_k, k = 1, 2, 3, 4 \). The equation for all points \( X \) on the sphere is again given as
\[ V_4 \wedge X = 0, \quad X^2 = 0. \]

Inserting (44) explicitly in \( V_4 \) yields
\[ V_4 = \frac{1}{2} (p_1^2 p_{234} + p_2^2 p_{314} + p_3^2 p_{124} + p_4^2 p_{132}) n \\
- (p_{234} + p_{314} + p_{124} + p_{132}) \mathbf{n} \\
+ \frac{1}{2} \{(p_2^2 - p_3^2) p_{14} + (p_3^2 - p_1^2) p_{24} + (p_1^2 - p_2^2) p_{34} \\
+ (p_1^2 - p_4^2) p_{23} + (p_2^2 - p_4^2) p_{31} + (p_3^2 - p_4^2) p_{12}\} N, \]
with the abbreviations

\[ \mathbf{p}_{kl} = \mathbf{p}_k \wedge \mathbf{p}_l, \quad \mathbf{p}_{klm} = \mathbf{p}_k \wedge \mathbf{p}_l \wedge \mathbf{p}_m, \quad k, l, m \in \{1, 2, 3, 4\}. \tag{47} \]

The \( \mathbf{n} \) factor component

\[
\begin{align*}
i_s &= -(V_4 \wedge \mathbf{n})N \\
&= -(\mathbf{p}_{234} + \mathbf{p}_{314} + \mathbf{p}_{124} + \mathbf{p}_{132}) \\
&= (\mathbf{p}_1 - \mathbf{p}_2) \wedge (\mathbf{p}_2 - \mathbf{p}_3) \wedge (\mathbf{p}_3 - \mathbf{p}_4) \tag{48}
\end{align*}
\]

is a Euclidean pseudoscalar, i.e. proportional to \( i \). Similar to the discussion of the circle, assuming the Euclidean center vector of the sphere to be \( \mathbf{c} \) and the radius \( r \), we can rewrite (44) as

\[
P_k = \mathbf{c} + r \mathbf{r}_k + \frac{1}{2} \left( \mathbf{c}^2 + r^2 + 2r \mathbf{c} \mathbf{r}_k \right) \mathbf{n} + \mathbf{n}, \quad r_k^2 = 1, \quad k = 1, 2, 3, 4. \tag{49}
\]

Replacing the \( P_k, k = 1, 2, 3, 4 \) in (46) accordingly we get after doing lots of algebra

\[
\begin{align*}
V_4 &= \frac{1}{2} (r^2 - c^2) i_s \mathbf{n} + i_s \mathbf{n} + c i_s N \\
&= (\mathbf{c} + \frac{1}{2} c^2 \mathbf{n} + \mathbf{n} - \frac{1}{2} r^2 \mathbf{n}) i_s N \\
&= (C - \frac{1}{2} r^2 \mathbf{n}) i_s N, \tag{50}
\end{align*}
\]

where \( C \) represents the conformal center of the sphere. An important relationship used in the derivation of (50) is

\[
i_s = (\mathbf{p}_1 - \mathbf{p}_2) \wedge (\mathbf{p}_2 - \mathbf{p}_3) \wedge (\mathbf{p}_3 - \mathbf{p}_4) = r^3 (\mathbf{r}_1 - \mathbf{r}_2) \wedge (\mathbf{r}_2 - \mathbf{r}_3) \wedge (\mathbf{r}_3 - \mathbf{r}_4). \tag{51}
\]

The elegant form (50) of \( V_4 \) makes it easy to extract the radius and the center from any general conformal (sphere) 4-vector:

\[
r^2 = \frac{V_4^2}{(V_4 \wedge \mathbf{n})^2}, \quad C = \frac{1}{2} r^2 \mathbf{n} + \frac{V_4}{-V_4 \wedge \mathbf{n}}. \tag{52}
\]

3 Conclusions

We explained how to algebraically construct conformal (homogeneous) subspaces with very intuitive Euclidean interpretations.

We then analysed in detail how the joining of conformal points yields explicit expressions for points, pairs of points, lines, circles, planes and spheres. After that we showed how the Euclidean 3D information of positions, orientations and

---

They are e.g. implemented, together with algebraic expressions for arbitrary translations and rotations, and for subspace operations of union (join), intersection (meet), projection and rejection as methods in the GeometricAlgebra Java package. [8, 9, 16]
radii, etc. can be extracted. In some cases useful alternative parametrizations were given. Applications of these alternative parametrizations can e.g. be found in [15].

Acknowledgement

Soli Deo Gloria. The author does want to thank his wife and children. He further thanks S. Krausshaar and P. Leopardi for organizing the ICIAM 2003 Clifford mini-symposium.

References

[1] D. Hestenes, New Foundations for Classical Mechanics (2nd ed.), Kluwer, Dordrecht, 1999.
[2] C. Doran, A. Lasenby, J. Lasenby Conformal Geometry, Euclidean Space and Geometric Algebra, in J. Winkler (ed.), Uncertainty in Geometric Computations, Kluwer, 2002.
[3] G. Sommer (ed.), Geometric Computing with Clifford Algebras, Springer, Berlin, 2001.
[4] D. Hestenes, G. Sobczyk, Clifford Algebra to Geometric Calculus, Kluwer, Dordrecht, reprinted with corrections 1992.
[5] G. Sobczyk, Clifford Geometric Algebras in Multilinear Algebra and Non-Euclidean Geometries, Lecture at Computational Noncommutative Algebra and Applications, July 6-19, 2003, http://www.prometheus-inc.com/asi/algebra2003/abstracts/sobczyk.pdf
[6] L. Dorst, The Inner Products of Geometric Algebra, in L. Dorst et. al. (eds.), Applications of Geometric Algebra in Computer Science and Engineering, Birkhaeuser, Basel, 2002.
[7] D. Hestenes, H. Li, A. Rockwood, New Algebraic Tools for Classical Geometry, in G. Sommer (ed.), Geometric Computing with Clifford Algebras, Springer, Berlin, 2001.
[8] E. Hitzer, KamiWaAi - Interactive 3D Sketching with Java based on Cl(4,1) Conformal Model of Euclidean Space, Advances in Applied Clifford Algebras 13(1) pp. 11-45 (2003). Compare also [9].
[9] KamiWaAi Java application and GeometricAlgebra Java package websites: http://sinai.mech.fukui-u.ac.jp/gej/software/KamiWaAi/index.html and G. Utama (admin.) http://sourceforge.net/projects/kamiwaai/

---

These formulas precisely yield the optimal mathematical structure of the related Java methods each geometric object is to have e.g. in the GeometricAlgebra Java package implementation. [4]
[10] C. Perwass, http://www.perwass.de/cbup/clu.html

[11] D. Fontijne, T. Bouma, L. Dorst, Gaigen: a Geometric Algebra Implementation Generator, July 28, 2002, preprint. http://www.science.uva.nl/ga/gaigen/files/20020728.gaigen.pdf

[12] L. Dorst et. al., GAVviewer: interactive geometric algebra with OpenGL visualization, http://www.science.uva.nl/ga/viewer/

[13] E. Hitzer, The geometric product and derived products, preprint. http://sinai.mech.fukui-u.ac.jp/gala2/GAtopics/axioms.pdf

[14] L. Dorst, D. Fontijne, An Algebraic Foundation for Object-Oriented Euclidean Geometry, To appear in R. Nagaoka, H. Ishi, E. Hitzer (eds.), Proc. of Innovative Teaching of Mathematics with Geometric Algebra 2003, Nov. 20-22, Kyoto University, Japan, printed by the Research Institute for Mathematical Sciences, Kyoto University, July 2004.

[15] E. Hitzer, Conic Sections in Geometric Algebra, Lecture at the International Workshop on Geometric Invariance and Applications in Engineering, May 24-28, 2004, Xian, China.

[16] E. Hitzer, G. Utama, Homogeneous Model and Java Implementation, Preprint for the Mem. Fac. Eng. Fukui Univ. 52(2) (2004).