On the trigonometric Felderhof model with domain wall boundary conditions

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Received 5 February 2007
Accepted 27 February 2007
Published 16 March 2007

Abstract. We consider the trigonometric Felderhof model of free fermions in an external field, on a finite lattice with domain wall boundary conditions. The vertex weights are functions of rapidities and external fields.

We obtain a determinant expression for the partition function in the special case where the dependence on the rapidities is eliminated, but for general external field variables. This determinant can be evaluated in product form. In the homogeneous limit, it is proportional to a 2-Toda $\tau$ function.

Next, we use the algebraic Bethe ansatz factorized basis to obtain a product expression for the partition function in the general case with dependence on all variables.

Keywords: integrable spin chains (vertex models), solvable lattice models

ArXiv ePrint: math-ph/0702012
1. Introduction

In [1], Korepin introduced domain wall boundary conditions for the six-vertex model on a finite square lattice, and proposed recursion relations that determine the corresponding domain wall partition function. In [2], Izergin obtained a determinant solution of Korepin’s recursion relations. At the free fermion point, the six-vertex domain wall partition function

\[ Z_{\text{TF}}^{N \times N} \]
can be evaluated explicitly in product form [3]. In the homogeneous limit, it is proportional to a 1-Toda $\tau$ function [4].

In this work, we look for analogous results in the context of the trigonometric limit of Felderhof’s model [5], which is a vertex model of free fermions in an external field. In section 2, we recall the definition of the model in the parametrization of Deguchi and Akutsu [6], and formulate it on an $N \times N$ lattice. There are four sets of complex variables: horizontal and vertical rapidities $\{u_i, v_j\}$, and horizontal and vertical external field variables $\{\alpha_i, \beta_j\}$, where $\{i, j\} \in \{1, 2, \ldots, N\}$.

The weight $w_{ij}$ of the vertex $v_{ij}$ at the intersection of the $i$th horizontal line and $j$th vertical line depends on the difference of the rapidities, $u_i - v_j$, but depends on the external fields, $\alpha_i$ and $\beta_j$, separately.

In section 3, we impose domain wall boundary conditions and obtain an Izergin-type determinant expression for the domain wall partition function, under the restriction that the difference of any two rapidity variables is a multiple of $2\pi \sqrt{-1}$, but for general $\{\alpha_i, \beta_j\}$. This expression can be evaluated in product form. In the homogeneous limit, it is proportional to a 2-Toda $\tau$ function [7, 8].

In section 4, we use the factorized basis of the algebraic Bethe ansatz, [9, 10], to obtain a product expression for the domain wall partition function for general $\{u_i, v_j\}$ and $\{\alpha_i, \beta_j\}$.

1.1. Abbreviations

In the rest of this paper, DWBC stands for domain wall boundary conditions and DWPF stands for domain wall partition function. $Z_{\text{TF}}^{N \times N}$ is the DWPF of the trigonometric Felderhof model on an $N \times N$ lattice. $Z_{\text{6V}}^{N \times N}$ is the DWPF of the six-vertex model on an $N \times N$ lattice.

$Z_{\text{TF, res}}^{N \times N}$ is $Z_{\text{TF}}^{N \times N}$ with restrictions on the rapidities as in equation (3). $Z_{\text{TF, res, hom}}^{N \times N}$ is the homogeneous version of $Z_{\text{TF, res}}^{N \times N}$ with all horizontal external field variables equal and all vertical external field variables equal.

2. The trigonometric Felderhof model

2.1. The lattice

We work on a square lattice consisting of $N$ horizontal and $N$ vertical lines. We label the horizontal lines from top to bottom and the vertical lines from left to right.

We assign the $i$th horizontal line an orientation from left to right, a complex rapidity variable $u_i$ and a complex external field variable $\alpha_i$. We assign the $j$th vertical line an orientation from bottom to top, a complex rapidity variable $v_j$ and a complex external field variable $\beta_j$.

2.2. Vertices

Each line intersects with $N$ other lines. A line segment between two intersections is a bond. To each bond, we assign a state variable, namely an arrow that points along the orientation of that line segment or against it. The intersection of the $i$th horizontal line

\[1\] In [6], the external field variables are referred to as colour variables.
Figure 1. An $N \times N$ square lattice, with oriented lines. To each line, we attach two complex variables: a rapidity and an external field.

Figure 2. The non-zero weight vertices of the trigonometric Felderhof model. The black arrows indicate state variables. The white indicate line orientation. Notice which vertex is $a_1$ and which is $a_2$.

and the $j$th vertical line, together with the four bonds adjacent to it, and the set of arrows on these bonds, is a vertex $v_{ij}$.

2.3. Weights

To each vertex $v_{ij}$, we assign a weight $w_{ij}$, that depends on (1) the orientations of the four arrows on the bonds of that vertex, (2) the difference of rapidity variables flowing through the vertex and (3) the two external field variables flowing through the vertex. To satisfy the Yang–Baxter equations, only the six vertices shown in figure 2 have non-zero weights [6]. In the parametrization of [6], the non-zero weights are

$$
\begin{align*}
 a_1(\alpha_i, \beta_j, u_i, v_j) &= 1 - \alpha_i \beta_j e^{u_i - v_j}, & b_1(\alpha_i, \beta_j, u_i, v_j) &= \alpha_i - \beta_j e^{u_i - v_j}, \\
 a_2(\alpha_i, \beta_j, u_i, v_j) &= e^{u_i - v_j} - \alpha_i \beta_j, & b_2(\alpha_i, \beta_j, u_i, v_j) &= \beta_j - \alpha_i e^{u_i - v_j}, \\
 c_1(\alpha_i, \beta_j, u_i, v_j) &= \sqrt{1 - \alpha_i^2} \sqrt{1 - \beta_j^2} e^{u_i - v_j}, & c_2(\alpha_i, \beta_j, u_i, v_j) &= \sqrt{1 - \alpha_i^2} \sqrt{1 - \beta_j^2}.
\end{align*}
$$

(1)
In the following, we will drop the dependence on the variables, when that is clear from the indices. Unlike the six-vertex model, the vertex weights of the trigonometric Felderhof model are not invariant under reversing the directions of all the arrows.

2.4. DWBC

As in the six-vertex model, the DWBC are such that all arrows on the left and right boundaries point inwards, and all arrows on the upper and lower boundaries point outwards. An example is shown in figure 3.

2.5. DWPF

Given a 2-state vertex model, such as the six-vertex model or the trigonometric Felderhof model, the DWPF on an $N \times N$ lattice, $Z_{DWBC}^{N \times N}$, is defined as the sum over all weighted configurations that satisfy DWBC. The weight of each configuration is the product of the weights of the vertices

$$Z_{DWBC}^{N \times N} = \sum_{\text{configurations}} \left( \prod_{\text{vertices}} w_{ij} \right).$$

(2)

It is also possible to define DWBC and DWPFs in vertex models with more state variables [11]–[13].

3. A determinant form of the restricted DWPF

3.1. The Korepin–Izergin procedure

As a first step towards computing $Z_{TF}^{N \times N}$, we follow the Korepin–Izergin procedure: (1) specify a set of properties that fully determine $Z_{TF}^{N \times N}$, (2) conjecture a determinant expression for the required $Z_{TF}^{N \times N}$ and (3) show that the conjectured expression satisfies the required properties.

It turns out that it is not obvious how to follow the above procedure for general values of all variables. The reason is that an Izergin-type determinant solution is tightly related to the Korepin-type properties, one of which is that the DWPF is symmetric under permuting the variables on any two parallel lattice lines.

Figure 3. Domain wall boundary conditions.
In the six-vertex model, and in models discussed in [11]–[13], this condition is automatically satisfied because the vertex weights are invariant under reversing the directions of all arrows, so the two $a$-type vertices, which are involved in proving this symmetry, have the same weight. In the trigonometric Felderhof model, there is no such invariance for general values of all variables, and we need to impose restrictions on at least some of the variables.

Our plan is to restrict the variables to a point where the Korepin–Izergin prescription works. We claim that there is no Izergin-type determinant expression for $Z_{\text{TF}}^{N \times N}$ for general values of rapidities and external fields.

3.2. Restrictions

We require

$$e^{u_{i1}-u_{i2}} = e^{v_{j1}-v_{j2}} = e^{u_{i1}-v_{j1}} = 1.$$  \hspace{1cm} (3)

The restrictions in equation (3) are satisfied by choosing the difference between any two rapidity variables to be a multiple of $2\pi\sqrt{-1}$, or equivalently by simply setting all rapidities to zero. The external field variables remain free. By eliminating the dependence on the rapidities, the weights are now much simpler and can be written as

$$a_{0,ij} = a_1(\alpha_i, \beta_j, 0, 0) = a_2(\alpha_i, \beta_j, 0, 0) = 1 - \alpha_i \beta_j,$$

$$b_{0,ij} = b_1(\alpha_i, \beta_j, 0, 0) = -b_2(\alpha_i, \beta_j, 0, 0) = \alpha_i - \beta_j,$$

$$c_{0,ij} = c_1(\alpha_i, \beta_j, 0, 0) = -c_2(\alpha_i, \beta_j, 0, 0) = \sqrt{1 - \alpha_i^2} \sqrt{1 - \beta_j^2}. \hspace{1cm} (4)$$

Under these conditions, $Z_{\text{TF}}^{N \times N}$ becomes $Z_{\text{TF, res}}^{N \times N}$.

3.3. Korepin-type properties

For the same reasons as in the six-vertex model, $Z_{\text{TF, res}}^{N \times N}$ is fully determined by the following properties

(1) It is symmetric in the elements of each of the sets $\{\alpha\}$, and $\{\beta\}$.

(2) It is a polynomial of degree $(N - 1)$ in $\alpha_i$, up to a factor of $\sqrt{1 - \alpha_i^2}$, where $1 \leq i \leq N$, and in $\beta_j$, up to a factor of $\sqrt{1 - \beta_j^2}$, where $1 \leq j \leq N$.

(3) It satisfies the recursion relation

$$Z_{\text{TF, res}}^{N \times N} \bigg|_{\alpha_m = \beta_n} = c_{0, mn} \left( \prod_{i=1}^{N} a_{0, in} \right) \left( \prod_{j=1}^{N} a_{0, mj} \right) Z_{\text{TF, res, (mn)}}^{(N-1) \times (N-1)}. \hspace{1cm} (5)$$

for any $m, n \in \{1, \ldots, N\}$. The subscripts $(mn)$ in $Z_{\text{TF, res, (mn)}}^{(N-1) \times (N-1)}$ indicate the variables that are not present in the reduced partition function.

(4) It satisfies the initial condition $Z_{\text{TF, res}}^{1 \times 1} = c_{0, 11}$. 

doi:10.1088/1742-5468/2007/03/P03010
3.4. Izergin-type determinant solution

Using the notation $\alpha_{[ij]} = \alpha_i - \alpha_j$, etc., the properties in section 3.3 are satisfied by

$$Z_{TF, \text{res}}^{N \times N} = \frac{\prod_{1 \leq i, j \leq N} a_{0,ij} b_{0,ij}}{\prod_{1 \leq i, j \leq N}^\alpha_{[ij]} \beta_{[ij]}} \det \left( \frac{c_{0,ij}}{a_{0,ij} b_{0,ij}} \right)_{1 \leq i, j \leq N}$$

$$= \frac{\prod_{1 \leq i, j \leq N}(1 - \alpha_i \beta_j)(\alpha_i - \beta_j)}{\prod_{1 \leq i, j \leq N}(\alpha_i - \alpha_j)(\beta_j - \beta_i)} \left( \prod_{1 \leq k \leq N} \sqrt{1 - \alpha_k^2} \sqrt{1 - \beta_k^2} \right)$$

$$\times \det \left( \frac{1}{(1 - \alpha_i \beta_j)(\alpha_i - \beta_j)} \right)_{1 \leq i, j \leq N}. \quad (6)$$

3.5. Remarks on proof of equation (6)

The proof proceeds along the same lines as Izergin’s proof, which is discussed in detail in the literature, including [12, 17] and references therein, so it suffices to outline it here. Since the four Korepin-properties in section 3.3 fully determine $Z_{TF, \text{res}}^{N \times N}$, all we need to do is to show that the right-hand side of equation (6) satisfies each of these properties.

Properties (1) and (2) are satisfied for the same reasons as in the six-vertex model [12, 17]. Property (4) is easily checked by inspection. Property (3) is checked as follows.

Expanding the determinant in the right-hand side of equation (6) along the first row, we obtain

$$Z_{TF, \text{res}}^{N \times N} = \sum_{i=1}^{N} (-1)^{i+1} c_{0,1i} \left( \prod_{j \neq i}^\alpha_{[ij]} a_{0,1j} b_{0,1j} \right) \left( \prod_{j=1}^{N} a_{0,ji} b_{0,ji} \right) Z_{TF, \text{res},(11)}^{(N-1) \times (N-1)}. \quad (7)$$

Because of the DWBC, the vertex at the upper right corner must be either $b_1$ or $c_1$. By choosing $\alpha_1 = \beta_N$, we eliminate the possibility of a $b_1$ vertex and restrict the allowed configurations as follows: (1) the upper right corner is a $c_1(\alpha_1, \beta_N, 0, 0) = c_{0,1N}$ vertex, (2) the rightmost column, apart from the upper right vertex, is a set of $a_1(\alpha_i, \beta_N, 0, 0) = a_{0,iN}$ vertices, where $\{2 \leq i \leq N\}$, and (3) the top row, apart from the upper right vertex, is a set of $a_2(\alpha_1, \beta_j, 0, 0) = a_{0,1j}$ vertices, where $\{1 \leq j \leq N - 1\}$.

Letting $\alpha_1 = \beta_N$ in equation (7), we obtain

$$Z_{TF, \text{res}}^{N \times N} \bigg|_{\alpha_1 = \beta_N} = (-)^{N+1} c_{0,1N} \left( \prod_{j=1}^{N-1} a_{0,1j} b_{0,1j} \right) \left( \prod_{j=2}^{N} a_{0,jN} b_{0,jN} \right) Z_{TF, \text{res},(1N)}^{(N-1) \times (N-1)}$$

$$= c_{0,1N} \left( \prod_{j=1}^{N-1} a_{0,1j} \right) \left( \prod_{j=2}^{N} a_{0,jN} \right) Z_{TF, \text{res},(1N)}^{(N-1) \times (N-1)}. \quad (8)$$

which is the recursion relation of equation (5), as we expect. Thus the determinant expression on the right-hand side of equation (6) satisfies all Korepin properties.
3.6. Further check on equation (6)

The proof of equation (6) outlined above made use of a recursion relation obtained by freezing the upper right corner. We could have also chosen to freeze the upper left corner. Let us check that equation (6) satisfies that second recurrence relation as well.

Because of the DWBC, the vertex at the upper left corner must be either a $a_2$ or $c_1$. By choosing $\alpha_1\beta_1 = 1$, we eliminate the possibility of an $a_2$ vertex, and restrict the allowed configurations as follows: (1) the upper left corner is a $c_1(\alpha_1, \beta_1, 0, 0) = c_{0,11}$ vertex, (2) the leftmost column, apart from the upper left vertex, is a set of $b_2(\alpha_i, \beta_1, 0, 0) = -b_{0,1i}$ vertices, where $\{2 \leq i \leq N\}$, and (3) the top row, apart from the upper left vertex, is a set of $b_1(\alpha_1, \beta_j, 0, 0) = b_{0,1j}$ vertices, where $\{2 \leq j \leq N\}$.

Letting $\alpha_1\beta_1 = 1$ in equation (7), we obtain

$$Z_{TF, res}^{N \times N}_{\alpha_1\beta_1 = 1} = c_{0,11} \frac{\left( \prod_{j=2}^{N} a_{0,1j} b_{0,1j} \right) \left( \prod_{j=2}^{N} a_{0,j1} b_{0,j1} \right)}{\prod_{j=2}^{N} a_{1[j]} \prod_{j=2}^{N} \beta_{1j}} Z_{TF, res}^{(N-1) \times (N-1)}$$

$$= c_{0,11} \left( \prod_{j=2}^{N} b_{0,1j} \right) \left( \prod_{j=2}^{N} (-b_{0,j1}) \right) Z_{TF, res}^{(N-1) \times (N-1)}$$

(9)

which is what we expect. Equation (9) is a second recursion relation for $Z_{TF, res}^{N \times N}$ and provides an independent check of equation (6).

3.7. On a determinant form with more general parameters

It is natural to look for a determinant expression for the DWPF with less restrictions on the rapidities than in equation (3). We were unable to find any such expression, even for the simplest variations on the conditions of equation (3), such as allowing only one rapidity, such as $u_1$, to be free, and so forth. This, of course, is not a proof that no such generalization exists but only that, if there is one, it is unlikely to be of the Izergin form of equation (6).

3.8. The homogeneous limit

In the homogeneous limit $\alpha_i \to \alpha$ and $\beta_j \to \beta$, a standard procedure gives

$$Z_{TF, res, hom}^{N \times N} = \frac{(-1)^{N(N-1)/2}}{\left( \prod_{n=1}^{N-1} n! \right)^2} \left( (\alpha - \beta)(1 - \alpha \beta) \right)^{N^2} \left( \sqrt{1 - \alpha^2} \right)^{N} \left( \sqrt{1 - \beta^2} \right)^{N}$$

$$\times \det \left( \begin{array}{c} \left( \frac{\partial}{\partial \alpha} \right)^i \left( \frac{\partial}{\partial \beta} \right)^j \left( \frac{1}{(\alpha - \beta)(1 - \alpha \beta)} \right) \end{array} \right)_{1 \leq i, j \leq N}.$$  

(10)

3.9. 2-Toda $\tau$-function

Because the determinant in equation (10) is bi-Wronskian, with partial derivatives in two complex variables, it is straightforward to show [8], using the Jacobi identity for
determinants, that it is a \( \tau \)-function of the 2-Toda partial differential equation

\[
\frac{\partial^2}{\partial \alpha \partial \beta} \log(\tau_N) = \frac{\tau_{N+1} \tau_{N-1}}{\tau_N^2}.
\]  

(11)

As mentioned earlier, the homogeneous limit of Izergin’s determinant expression of \( Z_{6V}^{N \times N} \) is proportional to a bi-Wronskian with partial derivatives in one complex variable, and therefore is a \( \tau \)-function of the 1-Toda partial differential equation [14].

This observation was used in [14] to study the free energy of the six-vertex model in the presence of DWBC. Since \( Z_{TF}^{N \times N} \) can be computed explicitly in product form using the algebraic Bethe ansatz, as we will see below, the free energy can also be computed explicitly, and the relationship with 2-Toda remains a curious observation.

3.10. On enumeration

As is well known, \( Z_{6V}^{N \times N} \) can be used to enumerate alternating sign matrices (ASMs) [15, 16]. At the free fermion point, \( Z_{TF}^{N \times N} \) 2-enumerates ASMs [16]

\( Z_{TF}^{N \times N} \) can also be used to enumerate ASMs, but because the model is yet again a free fermion model, one can easily show that here too one obtains 2-enumerations.

3.11. A product form for \( Z_{TF, \text{res}}^{N \times N} \)

The determinant in equation (6) factorizes

\[
\det(M_{N \times N}) = \left( \prod_{i<j} (1 - \alpha_i \alpha_j)(1 - \beta_i \beta_j) \right) \prod_{1 \leq i < j \leq N} (\alpha_i - \alpha_j)(\beta_j - \beta_i) \prod_{1 \leq i, j \leq N} (1 - \alpha_i \beta_j)(\alpha_i - \beta_j).
\]

(12)

The simplest way to see this is to notice that there is a change of variables that allows one to rewrite the determinant in equation (6) in Cauchy form\(^2\).

This is reminiscent of the factorization of Izergin’s determinant in the six-vertex model, at the free fermion point [3]. We attribute the factorization of equation (12) to the fact that the trigonometric Felderhof model is a free fermion model.

From equations (6) and (12), we obtain

\[
Z_{TF, \text{res}}^{N \times N} = \prod_{1 \leq i, j \leq N} \sqrt{1 - \alpha_i \alpha_j} \sqrt{1 - \beta_i \beta_j}.
\]

(13)

The simple form of the factorized result in equation (13) suggests that a similar result may hold in the general case with dependence on all parameters. This will be the topic of the next section.

4. Product form for general \( Z_{TF}^{N \times N} \)

Unlike \( Z_{TF, \text{res}}^{N \times N} \), \( Z_{TF}^{N \times N} \) is not invariant under permuting adjacent variables. All expressions in this section are valid only for the ordering shown in figure 1. Expressions of \( Z_{TF}^{N \times N} \) with different orderings are related by factors of vertex weights.

\(^2\) This remark is due to Ch Krattenthaler.

doi:10.1088/1742-5468/2007/03/P03010
4.1. Definitions

Consider weights which depend only on the vertical variables \{\beta\} and \{v\}
\[\begin{align*}
\tilde{a}_{1,ij} &= 1 - \beta_i \beta_j e^{v_i - v_j}, \\
\tilde{b}_{1,ij} &= \beta_i - \beta_j e^{v_i - v_j}, \\
\tilde{c}_{1,ij} &= \sqrt{1 - \beta_i^2} \sqrt{1 - \beta_j^2} e^{v_i - v_j},
\end{align*}\]

4.1. Definitions

The expression in equation (16) is not easy to evaluate directly, since the creation operators are sums containing \(2^N\) terms, and each term is a tensor product acting in all of the spaces \(1, \ldots, N\).

4.2. A factorizing matrix

Following [9], we define an initial factorizing \(F\)-matrix, \(F_{1,2\ldots N}\), by
\[F_{1,2\ldots N}(\beta_1; \beta_2, \ldots, \beta_N; v_1, v_2, \ldots, v_N) = e^{(11)}_1 + e^{(22)}_1 T_{1,2\ldots N}(\beta_1; v_1)\]
\[= \begin{pmatrix} 1 & 0 \\ C_{2\ldots N}(\beta_1, v_1) & D_{2\ldots N}(\beta_1, v_1) \end{pmatrix}_1.\]

From that, the full \(F\)-matrix, \(F_{1\ldots N}\), is defined recursively by
\[F_{1\ldots N}(\beta_1, \ldots, \beta_N; v_1, \ldots, v_N) = F_{2\ldots N} F_{1,2\ldots N} = \cdots = F_{(N-1)N} F_{(N-2),(N-1)N} \cdots F_{1,2\ldots N}\]

where
\[F_{(N-1)N}(\beta_{(N-1)}, \beta_N; v_{(N-1)}, v_N) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \tilde{c}_{2,(N-1)N} & \tilde{b}_{2,(N-1)N} & 0 \\ 0 & 0 & 0 & \tilde{a}_{2,(N-1)N} \end{pmatrix}_{(N-1)N} \]
4.3. A twisted monodromy matrix

The full $F$-matrix, $F_{1...N}$, is now used to construct a twisted monodromy matrix

$$\tilde{T}_{0,1...N}(\alpha; \beta_1, \ldots, \beta_N; u; v_1, \ldots, v_N) = F_{1...N} T_{0,1...N}(\alpha; \beta_1, \ldots, \beta_N; u; v_1, \ldots, v_N) F_{1...N}^{-1}. \quad (20)$$

4.4. Twisted creation operators

We define twisted versions of the Bethe ansatz operators as follows: $\tilde{A}_{1...N}(\alpha, u) = F_{1...N} A_{1...N}(\alpha, u) F_{1...N}^{-1}$, etc. Using the notation $a_{1,0j} = 1 - \alpha \beta_j e^{u-v_j}$, $a_{2,0j} = e^{u-v_j} - \alpha \beta_j$, etc., where the label 0 indicates dependence on the horizontal variables $\{\alpha\}$, one can show that

$$\tilde{A}_{1...N}(\alpha, u) = \bigotimes_{j=1}^{N} \begin{pmatrix} a_{1,0j} & 0 \\ 0 & \frac{a_{2,0j}a_{2,0k}}{b_{2,0l}} \end{pmatrix}_j + \tilde{B}_{1...N}(\alpha, u) \tilde{D}_{1...N}(\alpha, u) \tilde{C}_{1...N}(\alpha, u) \quad (21)$$

$$\tilde{B}_{1...N}(\alpha, u) = \sum_{l=1}^{N} \bigotimes_{j<l} \begin{pmatrix} b_{2,0j} & 0 \\ 0 & \frac{b_{2,0j}b_{2,0k}}{a_{2,0l}} \end{pmatrix}_j \begin{pmatrix} c_{1,0l} & 0 \\ 0 & 0 \end{pmatrix}_l \bigotimes_{j>l} \begin{pmatrix} \tilde{a}_{1,0j}b_{2,0j} & 0 \\ 0 & \frac{\tilde{a}_{2,0j}\tilde{a}_{2,0k}}{b_{2,0l}} \end{pmatrix}_j \quad (22)$$

$$\tilde{C}_{1...N}(\alpha, u) = \sum_{l=1}^{N} \bigotimes_{j<l} \begin{pmatrix} \tilde{a}_{1,0j}b_{2,0j} & 0 \\ 0 & \frac{\tilde{a}_{2,0j}b_{2,0k}}{a_{2,0l}} \end{pmatrix}_j \begin{pmatrix} c_{2,0l} & 0 \\ 0 & 0 \end{pmatrix}_l \bigotimes_{j>l} \begin{pmatrix} \tilde{b}_{2,0j} & 0 \\ 0 & \frac{\tilde{b}_{2,0j}b_{2,0k}}{a_{2,0l}} \end{pmatrix}_j \quad (23)$$

$$\tilde{D}_{1...N}(\alpha, u) = \bigotimes_{j=1}^{N} \begin{pmatrix} b_{2,0j} & 0 \\ 0 & a_{2,0j} \end{pmatrix}_j. \quad (24)$$

4.5. Remarks on proof of equations (21)–(24)

One first verifies by direct computation that the above formulae hold for $\tilde{A}_{12}(\alpha, u)$, $\tilde{B}_{12}(\alpha, u)$, $\tilde{C}_{12}(\alpha, u)$ and $\tilde{D}_{12}(\alpha, u)$. This becomes the basis for a proof by induction, in which each formula is proven individually. For example, to prove the formula for the twisted operator $\tilde{B}_{1...N}$, we observe that, by construction, the untwisted operator $B_{1...N}$ satisfies

$$B_{1...N}(\alpha, u) = A_{2...N}(\alpha, u) \begin{pmatrix} 0 & 0 \\ c_{1,01} & 0 \end{pmatrix}_1 + B_{2...N}(\alpha, u) \begin{pmatrix} b_{2,01} & 0 \\ 0 & a_{2,01} \end{pmatrix}_1. \quad (25)$$

Multiplying equation (25) from the left by $F_{1...N}$, and from the right by $F_{2...N}^{-1}$, we find that $\tilde{B}_{1...N}(\alpha, u)$ satisfies

$$\tilde{B}_{1...N}(\alpha, u) F_{1...N} F_{2...N}^{-1} = F_{1...N} F_{2...N}^{-1} \begin{pmatrix} \tilde{A}_{2...N}(\alpha, u) \begin{pmatrix} 0 & 0 \\ c_{1,01} & 0 \end{pmatrix}_1 + \tilde{B}_{2...N}(\alpha, u) \begin{pmatrix} b_{2,01} & 0 \\ 0 & a_{2,01} \end{pmatrix}_1 \) \quad (26)$$

doi:10.1088/1742-5468/2007/03/P03010
From equation (18), we have \( F_{1...N}F_{2...N}^{-1} = F_{2...N}F_{1,2...N}F_{2...N}^{-1} \), so equation (26) becomes the following matrix equation in space 1

\[
\tilde{B}_{1...N}(\alpha, u) \left( \begin{array}{cc} 1 & 0 \\ \tilde{C}_{2...N}(\beta_1, v_1) & \tilde{D}_{2...N}(\beta_1, v_1) \end{array} \right)_1 = \left( \begin{array}{cc} 1 & 0 \\ \tilde{C}_{2...N}(\beta_1, v_1) & \tilde{D}_{2...N}(\beta_1, v_1) \end{array} \right)_1 \\
\times \left\{ \tilde{A}_{2...N}(\alpha, u) \left( \begin{array}{cc} 0 & 0 \\ c_{1,01} \end{array} \right) + \tilde{B}_{2...N}(\alpha, u) \left( \begin{array}{cc} b_{2,01} & 0 \\ 0 & a_{2,01} \end{array} \right) \right\}.
\]

(27)

This is a recursion relation for \( \tilde{B}_{1...N}(\alpha, u) \) in terms of twisted operators over the \( N-1 \) spaces \( 2, \ldots, N \). We use it to prove equation (22) inductively, as follows.

First, we postulate that the expressions of equations (21)–(24) hold over the \( N-1 \) spaces \( 2, \ldots, N \). Next, we substitute them into equation (27). From that, the expression in equation (22) for \( \tilde{B}_{1...N} \) is seen to be the unique solution to equation (27). Repeating this procedure for the other twisted operators, we prove the postulate over the \( N \) spaces \( 1, \ldots, N \).

4.6. A recursion relation

Since \( |1\rangle F_{1...N} = \prod_{j<k} \tilde{a}_{2,jk} |1\rangle \) and \( F_{1...N}^{-1} |0\rangle = |0\rangle \), we can rewrite \( Z_{TF}^{N \times N} \) in terms of the twisted creation operators

\[
\sum_{\alpha, \beta, v_i} \left\{ \prod_{j<k} \tilde{a}_{2,jk} \right\} Z_{TF}^{N \times N} = \prod_{j<k} \tilde{a}_{2,jk} \left\{ \prod_{j<k} \tilde{a}_{2,jk} \right\} Z_{TF}^{N \times N}.
\]

Following [10], we use the expression in equation (28), together with the explicit expression for the twisted \( B \)-operator, to derive the recursion relation

\[
Z_{TF}^{N \times N} = \sum_{\alpha, \beta, v_i} \left\{ \prod_{j<k} \tilde{a}_{2,jk} \right\} Z_{TF}^{N \times N} = \prod_{j<k} \tilde{a}_{2,jk} \left\{ \prod_{j<k} \tilde{a}_{2,jk} \right\} Z_{TF}^{N \times N}
\]

where the subscripts in \( Z_{TF}^{N \times N} \) indicate the omission of the variables \( \{\alpha_N, u_N\} \) and \( \{\beta_i, v_i\} \).

4.7. Remarks on proof of equation (29)

Acting with \( \tilde{B}_{1...N}(\alpha_N, u_N) \) on \( |0\rangle \), in equation (28), we immediately find

\[
Z_{TF}^{N \times N} = \sum_{\alpha, \beta, v_i} \left\{ \prod_{j<k} \tilde{a}_{2,jk} \right\} Z_{TF}^{N \times N} = \prod_{j<k} \tilde{a}_{2,jk} \left\{ \prod_{j<k} \tilde{a}_{2,jk} \right\} Z_{TF}^{N \times N}
\]

\[
+ \prod_{j<k} \tilde{a}_{2,jk} \left\{ \prod_{j<k} \tilde{a}_{2,jk} \right\} Z_{TF}^{N \times N}.
\]

(30)

Using the identities \( (\downarrow_i) \sigma^- = (\sigma^-)^2 = 0 \), one obtains

\[
(\downarrow_i) \tilde{B}_{1...N}(\alpha_1, u_1) \ldots \tilde{B}_{1...N}(\alpha_{(N-1)}, u_{(N-1)}) (\downarrow_i)
\]

\[
(\downarrow_i) \tilde{B}_{1...N}(\alpha_1, u_1) \ldots \tilde{B}_{1...N}(\alpha_{(N-1)}, u_{(N-1)}) (\downarrow_i)
\]

\[
= \left( \prod_{j=1}^{N-1} a_{2,ji} \right) \left( \prod_{k<i} \tilde{a}_{2,ik} \right) \left( \prod_{k<i} \tilde{a}_{2,ki} \right)
\]

\[
\times \tilde{B}_{1...(i-1)(i+1)...N}(\alpha_1, u_1) \ldots \tilde{B}_{1...(i-1)(i+1)...N}(\alpha_{(N-1)}, u_{(N-1)}).
\]

(31)

doi:10.1088/1742-5468/2007/03/P03010
Equation (29) comes from substituting equation (31) in equation (30), and using the fact that

\[
Z_{TF, (N)}^{(N-1)\times(N-1)} = \frac{\prod_{j<i} \tilde{a}_{2,ji} \prod_{i<k} \tilde{a}_{2,ik}}{\prod_{j<k} \tilde{a}_{2,jk}} \prod_{j \neq i} (\tilde{B}_i) \times \prod_{j \neq i} (\tilde{B}_j) \times \prod_{i<k} (\tilde{B}_i) \times \prod_{j \neq i} (\tilde{B}_j) \times \prod_{i<k} (\tilde{B}_i) \times \prod_{j \neq i} (\tilde{B}_j).
\]

(32)

4.8. Solution of recursion equation

It can be shown that the following product expression satisfies equation (29)

\[
Z_{TF}^{N \times N} = \left( \prod_{k=1}^{N} e^{k(u_k-v_k)} \sqrt{1 - \alpha_k^2} \sqrt{1 - \beta_k^2} \right) \left( \prod_{1 \leq j \leq k \leq N} (e^{u_j-u_k} - \alpha_j \alpha_k) (e^{v_k-v_j} - \beta_k \beta_j) \right)
\]

(33)

4.9. Remarks on proof of equation (33)

From equation (33), we have

\[
Z_{TF, (N)}^{(N-1)\times(N-1)} = \frac{\left( e^{-N u_N + iv_i} \prod_{k=i+1}^{N} e^{v_k} \right)}{\left( \prod_{j=1}^{N-1} a_2(\alpha_j, \alpha_N, u_j, u_N) \prod_{j<i} a_2(\beta_j, \beta_i, v_j, v_i) \right) \times \frac{1}{c_2(a_N, \beta_i)} Z_{TF}^{N \times N}}.
\]

(34)

Using equation (34), and after considerable manipulation, one recovers

\[
\sum_{i=1}^{N} c_{1,Ni} \prod_{j=1}^{N-1} a_2(\alpha_j, \beta_i, u_j, v_i) \prod_{k \neq i} b_{2,ik} \prod_{k<i} \tilde{a}_{2,ik} \times Z_{TF, (N)}^{(N-1)\times(N-1)}
\]

(35)

Finally, we observe that

\[
\sum_{i=1}^{N} \left( \prod_{j=1}^{N-1} a_2(\alpha_j, \beta_i, u_j, v_i) \right) \left( \prod_{k \neq i} b_{2}(\beta_k, \alpha_N, v_k, u_N) \right) \left( \prod_{j \neq k} b_2(\beta_j, \beta_k, v_j, v_k) \right)
\]

(36)
which can be checked by noticing that both sides are polynomials of degree \(N - 1\) in the variable \(\alpha_N\), and furthermore the equality is satisfied at the \(N\) points \(\alpha_N = \beta_j e^{\psi_j - u_N}\), where \(j = 1, \ldots, N\). Equation (36) means that

\[
\sum_{i=1}^{N} \left( \prod_{j=1}^{N-1} \frac{a_2(\alpha_j, \beta_i, u_j, v_i)}{a_2(\alpha_j, \alpha_N, u_j, u_N)} \right) \left( \prod_{k \neq i} b_2(\beta_k, \alpha_N, v_k, u_N) \right) = 1
\]

which, when substituted in equation (35), causes it to collapse to the recursion relation equation (29), as required.

Acknowledgments

We thank T Deguchi, N Kitanine and Ch Krattenthaler for useful comments. AC, MW and MZ are supported by Australian Postgraduate Awards.

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doi:10.1088/1742-5468/2007/03/P03010