INCIDENCE COEFFICIENTS IN THE NOVIKOV COMPLEX FOR MORSE FORMS: RATIONALITY AND EXPONENTIAL GROWTH PROPERTIES.

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Abstract. In this paper we continue the study of generic properties of the Novikov complex, begun in the work "The incidence coefficients in the Novikov complex are generically rational functions" (dg-ga/9603006).

For a Morse map \( f : M \rightarrow S^1 \) there is a refined version of Novikov complex, defined over the Novikov completion of the fundamental group ring. We prove that for a \( C^0 \) generic \( f \)-gradient the corresponding incidence coefficients belong to the image in the Novikov ring of a (non commutative) localization of the fundamental group ring.

The Novikov construction generalizes also to the case of Morse 1-forms. In this case the corresponding incidence coefficients belong to the suitable completion of the ring of integral Laurent polynomials of several variables. We prove that for a given Morse form \( \omega \) and a \( C^0 \) generic \( \omega \)-gradient these incidence coefficients are rational functions.

The incidence coefficients in the Novikov complex are obtained by counting the algebraic number of the trajectories of the gradient, joining the zeros of the Morse form. There is V.I.Arnold's version of the exponential growth conjecture, which concerns the total number of trajectories. Namely, \( \omega \) be a Morse form on a closed manifold, \( v \) be an \( \omega \)-gradient, \( p : M' \rightarrow M \) be a free abelian covering for which \( p^*\omega = df \) with \( f : M' \rightarrow \mathbb{R} \). Let \( x \) be a critical point of \( f \) of index \( k \) and \( c \) be a real number. The conjecture says that the number of \( v \)-trajectories joining \( x \) to the critical points \( y \) of index \( k - 1 \) with \( f(y) \geq c \) grows at most exponentially with \(-c\). We confirm it for any given Morse form and a \( C^0 \) dense set of its gradients.

We give an example of explicit computation of the Novikov complex.

Introduction

A. Morse-Novikov theory. The classical Morse-Thom-Smale construction associates to a Morse function \( g : M \rightarrow \mathbb{R} \) on a closed manifold a free chain complex \( C_*(g) \) where the number \( m(C_p(g)) \) of free generators of \( C_p(g) \) equals the number of the critical points of \( g \) of index \( p \) for each \( p \). The boundary operator in this complex is defined in a geometric way, counting the trajectories of a gradient of \( g \), joining critical points of \( g \) (see [4], [7], [10], [11], [12]).

In the early 80s S.P.Novikov generalized this construction to the case of maps \( f : M \rightarrow S^1 \) (see [5]). The corresponding analog of Morse complex is a free chain complex \( C_*(f) \) over \( \mathbb{Z}[[t]][t^{-1}] \). Its number of free generators equals the number of critical points of \( f \) of index \( p \), and the homology of \( C_*(f) \) equals to the completed homology of the cyclic covering.

Fix some \( k \). The boundary operator \( \partial : C_k(f) \rightarrow C_{k-1}(f) \) is represented by a matrix, which entries are in the ring of Laurent power series. That is \( \partial_{ij} = \)
\[ \sum_{n=-N}^{\infty} a_n t^n, \quad \text{where} \quad a_n \in \mathbb{Z}. \]

Since the beginning S.P. Novikov conjectured that the power series \( \partial_{ij} \) have some nice analytic properties. In particular he conjectured that

**Generically the coefficients \( a_n \) of \( \partial_{ij} = \sum_{n=-N}^{\infty} a_n t^n \) grow at most exponentially with \( n \).**

In [9] we have proved that for a \( C^0 \) generic \( f \)-gradient the incidence coefficients above are actually rational functions. To recall the statement of the Main Theorem of [9] let \( M \) be a closed connected manifold and \( f : M \to S^1 \) a Morse map, non homotopic to zero. Denote the set of critical points of \( f \) by \( S(f) \). The set of \( f \)-gradients of the class \( C^\infty \), satisfying the transversality assumption (see §1 for terminology), will be denoted by \( \mathcal{G}(f) \). By Kupka-Smale theorem it is residual in the set of all the \( C^\infty \) gradients. Choose \( v \in \mathcal{G}(f) \). Denote by \( M \to M \) the connected infinite cyclic covering for which \( f \circ \mathcal{P} \) is homotopic to zero. Choose a lifting \( F : \tilde{M} \to \mathbb{R} \) of \( f \circ \mathcal{P} \) and let \( t \) be the generator of the structure group of \( \mathcal{P} \) such that \( F(xt) < F(x) \). The \( t \)-invariant lifting of \( v \) to \( \tilde{M} \) will be denoted by the same letter \( v \). For every critical point \( x \) of \( f \) choose a lifting \( \bar{x} \) of \( x \) to \( \tilde{M} \). Choose orientations of stable manifolds of critical points. Then for every \( x, y \in S(f) \), \( \text{ind} x = \text{ind} y + 1 \) and every \( k \in \mathbb{Z} \) the incidence coefficient \( n_k(x, y; v) \) is defined (as the algebraic number of \((-v)\)-trajectories joining \( x \) to \( y^k \)).

**Theorem [9,p.2].** In the set \( \mathcal{G}(f) \) there is a subset \( \mathcal{G}_0(f) \) with the following properties:

1. \( \mathcal{G}_0(f) \) is open and dense in \( \mathcal{G}(f) \) with respect to \( C^0 \) topology.
2. If \( v \in \mathcal{G}_0(f) \), \( x, y \in S(f) \) and \( \text{ind} x = \text{ind} y + 1 \), then \( \sum_{k \in \mathbb{Z}} n_k(x, y; v)t^k \) is a rational function of \( t \) of the form \( \frac{P(t)}{Q(t)} \), where \( P(t) \) and \( Q(t) \) are polynomials with integral coefficients, \( m \in \mathbb{N} \), and \( Q(0) = 1 \).
3. Let \( v \in \mathcal{G}_0(f) \). Let \( U \) be a neighborhood of \( S(f) \). Then for every \( w \in \mathcal{G}_0(f) \) such that \( w = v \) in \( U \) and \( w \) is sufficiently close to \( v \) in \( C^0 \) topology we have:

   \[ n_k(x, y; v) = n_k(x, y; w) \]

   for every \( x, y \in S(f) \), \( k \in \mathbb{Z} \).

In the present paper we develop the methods of [9] and apply them to the incidence coefficients with values in the Novikov completion of the fundamental group ring \( \mathbb{Z}\pi_1 M \) (§1). In §2 we consider the case of arbitrary Morse forms and free abelian coverings. We also give an example of explicit computation of the Novikov complex (§3). In §4 we prove the V.I. Arnold conjecture, concerning the total number of \((-v)\)-trajectories, joining the critical points of adjacent indices.

**B. Morse forms and Novikov rings.** To give the statement of our results, we recall some algebraic and Morse-theoretic definitions.

Let \( G \) be a group and \( \xi : G \to \mathbb{R} \) be a group homomorphism. We denote by \( (\mathbb{Z}G)^{\sim} \) the abelian group of all formal linear combinations \( \sum_{g \in G} n_g g \) (infinite in general). Recall that the Novikov ring \( \mathbb{Z}G^\xi \) is the ring of such \( \lambda \in (\mathbb{Z}G)^{\sim} \), \( \lambda = \sum_{g \in G} n_g g \), that for every \( c \in \mathbb{R} \) the set \( \text{supp} \lambda \cap \xi^{-1}([c, \infty[) \) is finite.

Let \( \omega \) be a closed 1-form on a manifold \( M \). The deRham cohomology class of \( \omega \) will be denoted by \([\omega]\) and the corresponding homomorphism \( \pi_1 M \to \mathbb{R} \) will be denoted by \([\lambda]\). We say that \( \omega \) is a Morse form, if locally it is the differential of
a Morse function. If \( f : M \to S^1 \) is a Morse map, then its differential is a Morse form, which cohomology class is in \( H^1(M, \mathbb{Z}) \). A Morse form \( \omega \) is proportional to a differential of a Morse map \( M \to S^1 \) if and only if \( \exists \lambda \in \mathbb{R} : \lambda[\omega] \in H^1(M, \mathbb{Z}) \).

The terminology of §1.A of [9] (which concerns Morse functions) is extended in an obvious way to the case of Morse forms, and we shall make free use of it. In particular we shall assume the notion of \( \omega \)-gradient. The set of all \( \omega \)-gradients, satisfying the transversality assumption will be denoted by \( \mathcal{G}(\omega) \).

Let \( \omega \) be a Morse form on a closed connected manifold \( M \) and let \( v \in \mathcal{G}(\omega) \). Let \( x, y \in S(\omega), \text{ind} x = \text{ind} y + 1 \). Choose some liftings \( \tilde{x}, \tilde{y} \) of \( x, y \) to \( \tilde{M} \) and the orientations of the stable manifolds of \( x \) and of \( y \). Then the incidence coefficient \( \tilde{n}(\tilde{x}, \tilde{y}; v) \in \mathbb{Z}(\pi_1 M)^{-1} \) is defined (for the case of Morse maps \( M \to S^1 \) see the precise definition of \( \tilde{n}(\tilde{x}, \tilde{y}; v) \) in [7]; the general case follows by the approximation procedure; see Lemma 2.6 and the discussion before it).

C. Statement of the results.

1. Morse maps \( M \to S^1 \)

Let \( \xi : G \to \mathbb{Z} \) be a group epimorphism. Denote \( \text{Ker} \xi \) by \( H \). For \( n \in \mathbb{Z} \) denote \( \xi^{-1}(n) \) by \( G_{(n)} \) and \( \{ x \in \mathbb{Z} G \mid \text{supp } x \subseteq G_{(n)} \} \) by \( \mathbb{Z}G_{(n)} \). Denote \( \xi^{-1}([-\infty, -1]) \) by \( G_{-} \) and \( \{ x \in \mathbb{Z} G \mid \text{supp } x \subseteq G_{-} \} \) by \( \mathbb{Z}G_{-} \). Choose \( \theta \in \mathbb{Z}G_{(-1)} \). It is easy to see that \( (\mathbb{Z}G)_{\xi}^{-1} \) is identified with the ring of power series of the form \( a_{-n}\theta^{n} + \ldots + a_{1}\theta + \ldots \mid a_{i} \in \mathbb{Z}H \).

Set \( \Sigma_{n} = \{ 1 + A \mid A \in \text{Mat}_{n \times n}(\mathbb{Z}G_{(-1)}) \} \). Set \( \Sigma = \bigcup_{n \geq 1} \Sigma_{n} \).

There is the corresponding localization ring \( \mathbb{Z}G_{\Sigma} \) (see [3, p.255]). Every matrix in \( \Sigma_{n} \) is invertible in \( \text{Mat}_{n \times n}(\mathbb{Z}G_{\xi}) \), the inverse of \( 1 + A \) being given by \( \sum_{n=0}^{\infty}(-1)^{n}A^{n} \), therefore the localization map \( \lambda : \mathbb{Z}G \to \mathbb{Z}G_{\Sigma} \) is injective and the inclusion \( i : \mathbb{Z}G \hookrightarrow \mathbb{Z}G_{\xi}^{-1} \) factors through a ring homomorphism \( \ell : \mathbb{Z}G_{\Sigma} \to \mathbb{Z}[\mathbb{Z}^{m}]_{\xi}^{-1} \).

Let \( M \) be a connected closed manifold and \( f : M \to S^1 \) be a Morse map, nonhomotopic to zero. Denote by \( \xi \) the induced homomorphism \( \pi_{1} M \to \mathbb{Z} \). Denote by \( p : \tilde{M} \to M \) the universal covering of \( M \).

Theorem A. In the set \( \mathcal{G}(f) \) there is a subset \( \mathcal{G}_{1}(f) \) with the following properties:

1. \( \mathcal{G}_{1}(f) \) is open and dense in \( \mathcal{G}(f) \) with respect to \( C^{0} \) topology.
2. If \( v \in \mathcal{G}_{1}(f) \) then for every \( x, y \in S(f) \) with \( \text{ind} x = \text{ind} y + 1 \) we have \( \tilde{n}(\tilde{x}, \tilde{y}; v) \in \text{Im} \ell \).
3. Let \( v \in \mathcal{G}_{1}(f) \). Let \( U \) be a neighborhood of \( S(f) \). Then for every \( w \in \mathcal{G}_{1}(f) \) such that \( w = v \) in \( U \) and \( w \) is sufficiently close to \( v \) in \( C^{0} \) topology we have: \( \tilde{n}(\tilde{x}, \tilde{y}; v) = \tilde{n}(\tilde{x}, \tilde{y}; w) \) for every \( x, y \in S(f) \).

2. Morse forms with arbitrary cohomology classes

At present we can prove the analog of the Theorem A in the case of arbitrary Morse forms only for the incidence coefficients associated with free abelian coverings.

Let \( \omega \) be a Morse form on a closed connected manifold \( M \). If \( \phi : \tilde{M} \to M \) is any regular covering with the structure group \( G \) such that \( \phi^{*}(\omega) = \theta \), then...
the homomorphism \( \{ \omega \} : \pi_1 M \to \mathbb{R} \) factors as \( \pi_1 M \to G \to \mathbb{R} \) and it is not difficult to see that the incidence coefficients \( \hat{n}(\hat{x}, \hat{y}; v) \) are defined for every \( v \in \mathcal{G}t(\omega) \) (here we suppose that \( \text{ind} x = \text{ind} y + 1 \), and that for every \( p \in S(\omega) \) a lifting \( \hat{p} \) of \( p \) to \( \hat{M} \) and an orientation of the stable manifold of \( p \) are chosen).

In particular it is the case for the maximal free abelian covering \( \pi_1 M \xrightarrow{\partial} M \) with the structure group \( H_1(M, \mathbb{Z})/\text{Tors} \approx \mathbb{Z}^m \). By abuse of notation we shall denote the corresponding homomorphism \( \mathbb{Z}^m \to \mathbb{R} \) by the same symbol as the de Rham cohomology class \([\omega]\) of \( \omega \). Assume that \([\omega] \neq 0\).

Set \( S_{[\omega]} = \{ P \in \mathbb{Z}[\mathbb{Z}^m] \mid P = 1 + Q : \supp Q \subset [\omega]^{-1}(\infty, 0)\} \).

**Theorem B.** There is a subset \( \mathcal{G}t_1(\omega) \subset \mathcal{G}t(\omega) \) with the following properties:

1. \( \mathcal{G}t_1(\omega) \) is open and dense in \( \mathcal{G}t(\omega) \) with respect to \( C^0 \) topology.
2. For every \( v \in \mathcal{G}t_1(\omega) \) and every \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1 \) we have:
   \[
   \overline{\pi}(\overline{x}, \overline{y}; v) \in S_{[\omega]}^{-1}(\mathbb{Z}[\mathbb{Z}^m]).
   \]
3. Let \( v \in \mathcal{G}t_1(\omega) \). Let \( U \) be a neighborhood of \( S(\omega) \). Then for every \( w \in \mathcal{G}t_1(\omega) \) such that \( w = v \) in \( U \) and \( w \) is sufficiently close to \( v \) in \( C^0 \) topology we have:
   \[
   \overline{\pi}(\overline{x}, \overline{y}; w) = \overline{\pi}(\overline{x}, \overline{y}; v) \text{ for every } x, y \in S(\omega), k \in \mathbb{Z}.
   \]

**3. Exponential growth estimates**

Let \( G \) be a group. For an element \( a = \sum n_g g \in \mathbb{Z} G \) we denote by \( \| a \| \) the sum \( \sum |n_g| \).

Let \( \xi : G \to \mathbb{R} \) be a homomorphism. For \( \lambda = \sum n_g g \in \mathbb{Z} G_{\xi} \) and \( c \in \mathbb{R} \) we denote by \( \lambda[c] \) the element \( \sum n_g g \) of \( \mathbb{Z} G \) and we set \( N_c(\lambda) = \| \lambda[c] \| \). We shall say that \( \lambda \) is of exponential growth if there are \( A, B \geq 0 \) such that for every \( c \in \mathbb{R} \) we have \( N_c(\lambda) \leq Ae^{-cB} \). It is easy to prove that the elements of exponential growth form a subring of \( \mathbb{Z} G_{\xi} \), which contains \( \mathbb{Z} G \).

**Theorem C.** Let \( v \) be an \( \omega \)-gradient, belonging to \( \mathcal{G}t_1(\omega) \). Let \( x, y \in S(\omega), \text{ind} x = \text{ind} y + 1 \). Then \( \overline{n}(\overline{x}, \overline{y}; v) \in \mathbb{Z}[\pi_1 M]_{\{\omega\}} \) is of exponential growth.

**4. An example**

In the subsection 3 we construct a three-manifold \( M \), a Morse map \( f : M \to S^1 \) and an \( f \)-gradient \( v \) such that \( n_0(\hat{x}, \hat{y}; v) = 0 \) and for \( k \geq 0 \) we have \( n_{k+1}(\hat{x}, \hat{y}; v) = \frac{4}{\sqrt{5}} \left( \left( \frac{3 + \sqrt{5}}{2} \right)^k - \left( \frac{3 - \sqrt{5}}{2} \right)^k \right) \).

**5. Exponential estimates of absolute number of trajectories: Morse maps \( M \to S^1 \)**

We assume here the terminology of Subsection A. The set of all \( f \)-gradients of class \( C^\infty \) will be denoted by \( G_f \). Recall from [8 §2B] that an \( f \)-gradient \( \omega \) is called
good if for every \( p, q \in S(f) \) we have

\[
\left( \text{ind} p \leq \text{ind} q + 1 \right) \Rightarrow \left( D(p, v) \pitchfork D(q, -v) \right)
\]

The set of all good \( f \)-gradients will be denoted by \( Gd(f) \). For \( v \in G(f) \) we denote by the same letter \( v \) the \( t \)-invariant lifting of \( v \) to \( \bar{M} \). Choose a lifting \( F : \bar{M} \to \mathbb{R} \) of \( f \) to \( \bar{M} \). It is easy to prove that for \( p, q \in S(F) \), \( \text{ind} p = \text{ind} q + 1 \) and for \( v \in Gd(f) \) the set of \((-v)\)-trajectories, joining \( p \) to \( q \) is finite. The liftings of critical points of \( f \) to \( \bar{M} \) being chosen, denote by \( N_k(x, y; v) \) the number of \((-v)\)-trajectories joining \( \bar{x} \) to \( \bar{y}t^k \) (where \( \text{ind} x = \text{ind} y + 1 \)).

**Theorem D.** In the set \( G(f) \) there is a subset \( G_0(f) \) with the following properties:

1. \( G_0(f) \) is \( C^0 \) dense in \( G(f) \) and \( G_0(f) \subset Gd(f) \).
2. Let \( v \in G_0(f) \). Then there are constants \( C, D > 0 \) such that for every \( x, y \in S(f) \) with \( \text{ind} x = \text{ind} y + 1 \) and for every \( k \in \mathbb{Z} \) we have \( N_k(x, y; v) \leq C \cdot D^k \).

6. Exponential estimates of absolute number of trajectories: Morse forms

Let \( \omega \) be a Morse form on a closed connected manifold \( M \). Let \( \mathcal{P} : \bar{M} \to M \) be the maximal free abelian covering of \( M \) with structure group \( \mathbb{Z}^m \); we identify the cohomology class \([\omega]\) of \( \omega \) with the corresponding homomorphism \( \mathbb{Z}^m \to \mathbb{R} \). We denote by \( G(\omega) \) the set of all \( \omega \)-gradients of class \( C^\infty \) and by \( Gd(\omega) \) the set of all good \( \omega \)-gradients of class \( C^\infty \). Let \( v \in Gd(\omega) \). For every zero \( x \) of \( \omega \) choose a lifting \( \bar{x} \) of \( x \) to \( \bar{M} \) and an orientation of the stable manifold of \( x \). Then for every \( g \in \mathbb{Z}^m \) and every \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1 \) the set of \((-v)\)-trajectories joining \( \bar{x} \) to \( \bar{y}g \) is finite and we denote its cardinality by \( N(\bar{x}, \bar{y}, g; v) \). For \( c \in \mathbb{R} \) we denote by \( N_{\geq c}(\bar{x}, \bar{y}; v) \) the sum \( \sum_{g : [\omega](g) \geq c} N(\bar{x}, \bar{y}, g; v) \).

**Theorem E.** In the set \( G(\omega) \) there is a subset \( G_0(\omega) \) with the following properties:

1. \( G_0(\omega) \) is dense in \( G(\omega) \) with respect to \( C^0 \) topology; \( G_0(\omega) \subset Gd(\omega) \).
2. Let \( v \in G_0(\omega) \). There are constants \( C, D > 0 \) such that for every \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1 \) and every \( c \in \mathbb{R} \) we have \( N_{\geq \lambda}(x, y; v) \leq C \cdot D^{-\lambda} \).
§1. Morse maps \( M \rightarrow S^1 \)

A. Algebraic preliminaries.

We accept here the terminology of Subsection C.1 of Introduction. Further, we say that an element \( \xi \in Z(\pi_1 M) \) is of type \((\mathcal{L})\), if

\((\mathcal{L})\) There are \( r, q \in G \), a natural number \( m \), an \( m \times m \)-matrix \( A = (a_{ij})_{1 \leq i, j \leq m} \) where \( a_{ij} \in ZG(-1) \) and vectors \( (X_i)_{1 \leq i \leq m}, (Y_i)_{1 \leq i \leq m}, X_i, Y_i \in ZH \), such that

\[
\xi = r \left( \sum_{s \geq 0} \sum_{1 \leq i \leq m} Y_i a_{ij}^{(s)} X_j \right)^q
\]

where \( a_{ij}^{(s)} \) are the entries of \( A^s \). The next lemma follows from the definition of the homomorphism \( \ell \).

**Lemma 1.1.** The elements of type \((\mathcal{L})\) are contained in \( \text{Im} \ell \). \( \square \)

**Proposition 1.2.** The elements of type \((\mathcal{L})\) are of exponential growth .

*Proof.* It suffices to prove that every matrix entry of the matrix series \( u = \sum_{s \geq 0} A^s \), where \( A = (a_{ij}) \) and \( a_{ij} \in ZG(-1) \), is of exponential growth . For an \((m \times m)\)-matrix \( B = (b_{ij}) \) we denote by \( \|B\| \) the number \( \max ||b_{ij}|| \). It is easy to check that \( ||BC|| \leq ||B|| \cdot ||C|| \cdot m \). Let \( A \) be \((m \times m)\)-matrix. Then \( ||A^s|| \leq ||A||^s \cdot m^{s-1} \leq ||A||^s \cdot m^s \). Let \( 1 \leq i, j \leq m \). Write \( u_{ij} = \sum n_g \). Then for \( k \geq 0 \) we have \( \sum_{\xi(g) = -k} |n_g| \leq (m \cdot ||A||)^k \). If \( m||A|| \leq 1 \) this gives \( \sum_{\xi(g) = -k} |n_g| \leq k + 1 \leq e^k \). If \( m||A|| > 1 \) we have \( \sum_{\xi(g) \geq -k} |n_g| \leq \frac{(m||A||)^{k-1}}{(m||A||)^{-1}} < D(m||A||)^k \).

Therefore in any case there are \( A, B > 0 \) such that for \( k > 0, k \in Z \) we have \( \sum_{\xi(g) \geq k} |n_g| \leq A \cdot B^{-k} \). For \( k \geq 0 \) it is true obviously and this implies that \( u \) is of exponential growth . \( \square \)

B. Statement of Theorem 1.3.

Theorem A follows immediately from the next theorem.

**Theorem 1.3.** In the set \( \mathcal{G}t(f) \) there is a subset \( \mathcal{G}t_1(f) \) with the following properties:

1. \( \mathcal{G}t_1(f) \) is open and dense in \( \mathcal{G}t(f) \) with respect to \( C^0 \) topology.
2. If \( v \in \mathcal{G}t_1(f) \) then for every \( x, y \in S(f) \) with \( \text{ind}x = \text{ind}y + 1 \) we have: \( \tilde{n}(\tilde{x}, \tilde{y}; v) \) satisfies \((\mathcal{L})\).
3. Let \( v \in \mathcal{G}t_1(f) \). Let \( U \) be a neighborhood of \( S(f) \). Then for every \( w \in \mathcal{G}t_1(f) \) such that \( w = v \) in \( U \) and \( w \) is sufficiently close to \( v \) in \( C^0 \) topology we have: \( \tilde{n}(\tilde{x}, \tilde{y}; v) = \tilde{n}(\tilde{x}, \tilde{y}; w) \) for every \( x, y \in S(f) \) such that \( \text{ind}x = \text{ind}y + 1 \).

C. Generalities on intersection indices. Let \( M \) be a manifold without boundary, \( Q : \hat{M} \rightarrow M \) be a regular covering (not necessarily connected) with structure group \( H \). We say, that a submanifold \( N \) of \( M \) is lifted-oriented (resp. lifted-cooriented) if a lifting \( \hat{i} : N \hookrightarrow \hat{M} \) of the inclusion map \( i : N \hookrightarrow M \) is fixed, and \( N \) is oriented (resp. cooriented). We shall denote \( \hat{N} \) by \( \hat{N} \).
Let \( X \subset M \) and let \( N \) be a lifted-oriented submanifold of \( M \) such that \( N \setminus \text{Int} X \) is compact. Then \( \hat{N} \setminus \text{Int} Q^{-1}(X) \) is compact, and the homology class \( [\hat{N}]_{\hat{M},Q^{-1}(X)} \in H_n(\hat{M},Q^{-1}(X)) \) is defined, where \( n = \dim N \) (see [9,§4]). We shall denote it by \([\hat{N}]_{M,X} \), or simply by \([\hat{N}] \) if there is no possibility of confusion. Let \( L \) be a compact lifted-cooriented submanifold without boundary of \( M \). Then there is the coorientation class \([\hat{L}] \in H_m^{-1}(\hat{M},\hat{M} \setminus \hat{L})\), where \( l = \dim L, m = \dim M \).

Assume that \( X \cap L = \emptyset, N \cap L \) and \( n + l = m \). Then \( \hat{N} \cap \hat{L}, \) and \( \hat{N} \cap \hat{L} \) is finite, and the intersection index \( \hat{N} \sharp \hat{L} \in \mathbb{Z} \) is defined. Denote by \( j \) the inclusion \( (\hat{M},Q^{-1}(X)) \hookrightarrow (\hat{M},\hat{M} \setminus \hat{L}) \). The next lemma is standard.

**Lemma 1.4.** \( \hat{N} \sharp \hat{L} = j^*(|[\hat{L}]|[\hat{N}]) \). \( \square \)

**D. Ranging systems.** Let \( f : W \to [a,b] \) be a Morse function on a compact riemanian cobordism, \( f^{-1}(b) = V_1, f^{-1}(a) = V_0, \) be an \( f \)-gradient. Let \( Q : \hat{W} \to W \) be a regular covering with a structure group \( H \). The lifting of \( v \) to \( \hat{W} \) will be denoted by \( \hat{v} \). If \( x \in W \) and \( \gamma(x,t;v) \) is defined on \([0,a], \) and \( \hat{x} \in Q^{-1}(x), \) then the lifting to \( \hat{W} \) of \( \gamma(x,\cdot;v) \), starting at \( \hat{x} \) is the \( \hat{v} \)-trajectory \( \gamma(x,\cdot;\hat{v}) \). It is easy to define with the help of this lifting procedure a diffeomorphism \( \hat{v}^\sim : Q^{-1}(V_1 \setminus K(-v)) \to Q^{-1}(V_0 \setminus K(v)) \). For \( X \subset V_1 \) we denote by abuse of notation \( \hat{v}^\sim(X \setminus K(-v)) \) by \( \hat{v}^\sim(X) \).

Let \( N \) be a oriented-lifted submanifold of \( V_1 \). Then it is easy to see that \( \hat{v}(N) \) is an oriented-lifted submanifold of \( V_0 \).

Now let \( \{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda} \) be a ranging system for \( (f,v) \) (see [9,§4 Subsection B] for definitions). Let \( N \) be an oriented-lifted submanifold of \( V_1 \setminus B_b \) such that \( N \setminus \text{Int} A_b \) is compact. Then Prop. 4.6 of [9] implies that \( \hat{v}(N) \) is an oriented-lifted submanifold of \( V_0 \setminus B_a \) such that \( \hat{v}(N) \setminus \text{Int} A_a \) is compact.

The following proposition is a generalization of the Proposition 4.7 of [9], which can be considered as a particular case \( (H = \{1\}) \) of the Proposition 1.5. The proof of 1.5 is carried out along the lines of [9, §4, Subsection B]. We recommend to the reader to consult [9, §4, Subsection B] for the basic definitions (such as the definition of ranging system, cited above ). We present below the main steps of proof.

**Proposition 1.5.** There is a homomorphism \( \hat{H}(v) : H_*(Q^{-1}(V_1 \setminus B_b), Q^{-1}(A_b)) \to H_*(Q^{-1}(V_0 \setminus B_a), Q^{-1}(A_a)) \) of right \( \mathbb{Z}H \)-modules, such that:

1. If \( N \) is an oriented-lifted submanifold of \( V_1 \setminus B_b \), such that \( N \setminus \text{Int} A_b \) is compact, then \( \hat{H}(v)([\hat{N}]) = [\hat{v}(N)^-] \).

2. There is an \( \epsilon > 0 \) such that for every \( f \)-gradient \( w \) with \( \|w - v\| < \epsilon \) we have \( \hat{H}(v) = \hat{H}(w) \).

**Proof.** An easy induction argument shows that it is sufficient to prove the proposition in the case \( \text{card} \Lambda = 1 \). Let \( S(f) = S1(f) \sqcup S2(f), \) where for every \( p \in S1(f), \) resp. \( p \in S2(f) \) the i), resp. ii) of (RS2) of Definition 4.3 in [9,p.23] holds. Pick Morse functions \( \phi_1, \phi_2 : W \to [a,b], \) adjusted to \( (f,v), \) such that there are regular values \( \mu_1, \mu_2 \) of \( \phi_1, \mu_2 \) of \( \phi_2 \) satisfying: (1) for every \( p \in S1(f) \) we have: \( \phi_1(p) < \mu_1 \) and \( \phi_2(p) > \mu_2. \) (2) for every \( p \in S2(f) \) we have: \( \phi_1(p) > \mu_1 \) and \( \phi_2(p) < \mu_2. \)

For \( \delta > 0 \) denote by \( D1_\delta(v) \), resp. by \( D1_\delta(-v) \), the intersection with \( V_0, \) resp. with \( V_1 \) of \( \text{Int} D_\delta(p,v) \), resp. of \( \text{Int} D_\delta(p,-v) \). By abuse of notation, we define \( \hat{H}(v) \) by the following formula:

\[
\hat{H}(v)([\hat{N}]) = [\hat{v}(N)^-].
\]
the intersection of $\cup_{p \in S1(f)} D(p, v)$ with $V_0$ will be denoted by $D_1\delta(v)$. Denote 
$D_2\delta(-v) \cup (-v)^\sim B_0$ by $D_\delta(v) \cup v(A_b)$ by $\nabla(\delta, v)$. The similar notation 
like $D_2\delta(-v)$, $D_1\delta(v)$, $D_2\delta_0(v)$ etc. are now clear without special definition. For $\delta > 0$ sufficiently small we have

\[
\begin{align*}
&D_1\delta(p) \subset \phi^{-1}_1([a, \mu_1]) \quad D_\delta(p) \subset \phi^{-1}_2([\mu_2, b]) \\
&D_2\delta(p) \subset \phi^{-1}_1([\mu_2, b]) \quad D_\delta(p) \subset \phi^{-1}_2([a, \mu_2])
\end{align*}
\]

It is easy to prove that for $\delta > 0$ sufficiently small we have:

\[
\begin{align*}
\nabla(\delta, v) &\subset \text{Int } A_a, \quad \Delta(\delta, -v) \subset \text{Int } B_b, \\
\Delta(\delta, -v) \cap D_1\delta(-v) &\subseteq \emptyset
\end{align*}
\]

Fix some $\delta > 0$ satisfying (D1) and (D2).

Homomorphism $\bar{H}(v; \mu', \mu; U) : H_*(Q^{-1}(V_1 \setminus B_b), Q^{-1}(A_b)) \to H_*(Q^{-1}(V_0 \setminus B_a), Q^{-1}(A_a))$]

Let $0 \leq \mu' < \mu \leq \delta$. Let $U$ be any subset of $V_1$ such that 

\[
\Delta(0, -v) \subset U \subset B_b \quad \text{and} \quad U \cap D_1\delta(-v) = \emptyset
\]

(for example $U = \Delta (\delta, -v)$ will do). Denote by $\bar{H}(v; \mu', \mu; U)$ the following sequence of homomorphisms

\[
\begin{align*}
H_*(Q^{-1}(V_1 \setminus B_b), Q^{-1}(A_b)) &\xrightarrow{\bar{I}} H_*(Q^{-1}(V_1 \setminus U), Q^{-1}(A_b \cup D_1\mu(-v))) \xrightarrow{\text{Exc}^{-1}} \\
H_*(Q^{-1}(V_1 \setminus (U \cup D_1\mu'(-v))), Q^{-1}((A_b \cup D_1\mu(-v)) \setminus D_1\mu'(-v))) &\xrightarrow{\bar{\mu}^\sim} \\
H_*(Q^{-1}(V_0 \setminus B_a), Q^{-1}(A_a)).
\end{align*}
\]

Here $\bar{I}$ is the corresponding inclusion. Note that the last arrow is well defined since 

$(-v)^\sim B_0 \subset U$ and $D_0(0, -v) \cap V_1 \subset D_2\delta(0, -v) \cup D_1\mu(-v)$. All the three arrows are homomorphisms of right $\mathbb{Z}H$-modules; first two - by the obvious reasons, the last because $\bar{\mu}^\sim$ commutes with the right action of $H$.

The composition $\bar{\mu}^\sim \circ \text{Exc}^{-1} \circ \bar{I}$ of this sequence will be denoted by $\bar{H}(v; \mu', \mu; U)$.

The reasoning similar to [9], page 25 shows that this homomorphism does not depend on the choice of $U$, neither on the choice of $\mu', \mu$ or $\delta$, or on the choice of presentation $S(f) = S1(f) \sqcup S2(f)$ (if there is more then one such presentation). Therefore this homomorphism is well determined by $v$, the ranging system 

$\{(A_\lambda, B_\lambda)\}_{\lambda \in \Lambda}$ and the covering $Q : \hat{W} \to W$. We shall denote it by $\hat{H}(v)$. The proof of properties 1) and 2) of $\hat{H}(v)$ is similar of the proof of [9, 4.7 (1,2)] and will be omitted. \hfill \Box

E. Equivariant ranging systems and the proof of Theorem 1.3. We return here to the terminology of Subsection C.1 of the Introduction. We begin by some algebraic preliminaries. Let $M, N$ be right $\mathbb{Z}H$-modules and $f : M \to N$ be a homomorphism of abelian groups. We say that $f$ is $\theta$-semilinear, if we have $f(xh) = f(x)\theta h \theta^{-1}$ for every $x \in M$. If $M, N$ are free finitely generated $\mathbb{Z}H$-modules with bases $(e_j), (d_i)$, one can associate to each $\theta$-semilinear homomorphism $f : M \to N$ a matrix $M(f) = (m_{ij})$ by the following rule: $f(e_j) = \sum d_i m_{ij}$.
If $M$ is a right $\mathbb{Z}H$-module and $f : M \to \mathbb{Z}$ is a homomorphism of abelian groups, then we shall say that $f$ is of finite type, if for every $m \in M$ the set of $h \in H$, such that $f(mh) \neq 0$ is finite. If $f : M \to \mathbb{Z}$ is a homomorphism of a finite type, then we define a homomorphism $\tilde{f} : M \to \mathbb{Z}H$ of $\mathbb{Z}H$-modules by $\tilde{f}(m) = \sum_{h \in H} f(mh)h^{-1}$.

Returning to Morse maps, we assume moreover that $f : M \to S^1$ belongs to an indivisible cohomology class in $H^1(M, \mathbb{Z})$. Further, denote by $\pi : \widetilde{M} \to M$ the (unique) infinite cyclic covering, such that $f \circ \pi \sim 0$. The universal covering $p : \tilde{M} \to M$ factors as $p = \pi \circ \mathcal{Q}$ where $\mathcal{Q} : \tilde{M} \to \tilde{M}$ is a covering with structure group $H = \text{Ker } \xi$. Let $u$ be an $f$-gradient and let $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ be a $t$-equivariant ranging system for $(F, u)$ (see [9,Def. 4.14] for definition).

For $\nu, \mu \in \Sigma, \nu < \mu$ denote by $\tilde{H}_{(\nu, \mu]}(u)$ the homomorphism $\tilde{H}(u \mid F^{-1}(\nu, \mu))$, associated by virtue of Proposition 1.5 to the ranging system $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma, \nu \leq \sigma \leq \mu}$ and the covering $\mathcal{Q}$ (restricted to the cobordism $F^{-1}(\nu, \mu)$). Denote by $\tilde{H}_{(\nu, \mu]}(u)$ the identity homomorphism of $H_\ast(\mathcal{Q}^{-1}(F^{-1}(\mu) \setminus B_\mu), \mathcal{Q}^{-1}(A_\mu))$ to itself.

It follows from the construction that for every $g \in G$ with $\xi(g) = k \in \mathbb{Z}$ we have $\tilde{H}_{(\nu, \mu]}(u) = R(g) \circ \tilde{H}_{(\nu, \mu]}(u) \circ R(g^{-1})$. We have also $\tilde{H}_{[\nu, \theta]}(u) \circ \tilde{H}_{(\nu, \mu]}(u) = \tilde{H}_{[\nu, \theta]}(u)$. For $\nu \in \Sigma$ denote $R(\theta^{-1}) \circ \tilde{H}_{[\nu, \nu-1]}(u)$ by $\tilde{h}_\nu(u)$. It is a $\theta$-semilinear endomorphism of $H_\ast(\mathcal{Q}^{-1}(F^{-1}(\nu) \setminus B_\nu), \mathcal{Q}^{-1}(A_\nu))$. We have obviously $\tilde{H}_{(\nu, \nu-k]}(u) = R(g^k) \circ \tilde{h}_\nu(u))^k$. The next lemma follows directly from 1.5.

Lemma 1.6. Let $\nu, \mu \in \Sigma, \nu \leq \mu; \text{let } k \in \mathbb{N}$. Let $N$ be oriented-lifted submanifold of $F^{-1}(\mu) \setminus B_\mu$ such that $N \setminus \text{Int } A_\mu$ is compact. Let $L$ be a cooriented-lifted compact submanifold of $F^{-1}(\nu) \setminus A_\nu$. Assume that $\text{dim } N + \text{dim } L = \text{dim } M - 1$. Then:

1. $N'_k = u^{-1}_{[\nu, \nu-k]}(N)$ is an oriented-lifted submanifold of $F^{-1}(\nu - k) \setminus B_{\nu - k}$ such that $N'_k \setminus \text{Int } A_{\nu - k}$ is compact. If $N'_k \cap L^k$, then $N'_k \cap L^k$ is finite and $\tilde{N}'_k \cap \tilde{L}^k = \tilde{i}^s(\tilde{L}^k(\tilde{h}_\nu(u)))$, where $i$ stands for the inclusion map $(\mathcal{Q}^{-1}(F^{-1}(\nu) \setminus B_\nu), \mathcal{Q}^{-1}(A_\nu)) \hookrightarrow \mathcal{Q}^{-1}(F^{-1}(\nu), \mathcal{Q}^{-1}(F^{-1}(\nu) \setminus \tilde{L}))$.

2. For every $f$-gradient $w$, sufficiently close to $u$ in $C^0$-topology, $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ is also a $t$-equivariant ranging system for $(F, w)$ and $\tilde{h}_\nu(w) = \tilde{h}_\nu(u)$.

Proof of Theorem 1.3 It is easy to see that it suffices to prove our theorem for the case of indivisible homotopy class $[f] \in H^1(M, \mathbb{Z})$ and we make this assumption up to the end of this subsection.

Fix first two points $x, y \in S(f), \text{ind}x = \text{ind}y + 1$. Recall that we have fixed a lifting $\tilde{x} \in \tilde{M}$ for every $x \in S(f)$; denote $\mathcal{Q}(\tilde{x})$ by $\tilde{x}$. We can assume that $F(\tilde{y}) < F(\tilde{x}) \leq F(\tilde{y}) + 1$. Denote $\text{dim } M$ by $n$; denote $\text{ind}x$ by $l + 1$, then $\text{ind}y = l$. Choose some set $\Sigma$ of regular values of $F$, satisfying (S) of [9, Def. 4.14].

Denote by $\theta$ the maximal element of $\Sigma$ with $\theta < F(\tilde{x})$ and by $N(\nu)$ the intersection $D(\tilde{x}, \nu) \cap F^{-1}(\theta) \setminus N(\nu)$ is an oriented submanifold of $F^{-1}(\theta)$, diffeomorphic to $S^l$. Denote by $\eta$ the minimal element of $\Sigma$, satisfying $\eta > F(\tilde{y})$, then $\eta \leq \theta < \eta + 1$. Denote by $L(-\nu)$ the intersection $D(\tilde{y}, -\nu) \cap F^{-1}(\eta) \setminus L(-\nu)$ is a cooriented submanifold of $F^{-1}(\eta)$, diffeomorphic to $S^{n-1}$. Denote by $W$ the cobordism $F^{-1}(\{\eta, \eta + 1\})$. Note that $\tilde{x} \in W^\circ$. Denote $F^{-1}(\eta)$ by $V_0$, $F^{-1}(\eta + 1)$ by $V_1, \Sigma \cap [\eta, \eta + 1]$ by $\Lambda$. 


Denote by $\mathcal{G}_1(f; x, y)$ the subset of $\mathcal{G}_t(f)$, consisting of all the $f$-gradients $v$, such that there is an equivariant ranging system $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ for $(F, v)$ satisfying

(1.1) \[ N(v) \cap B_\theta = \emptyset, \quad L(-v) \cap A_\eta = \emptyset, \]

(1.2) \[ \{F^{-1}(\eta) \setminus B_\eta, A_\eta\} \text{ has a homotopy type of a finite CW-pair} \]

having only 1-dimensional cells.

Now we shall prove 3 properties of the set $\mathcal{G}_1(f; x, y)$.

(1). $\mathcal{G}_1(f; x, y)$ is an open and dense subset of $\mathcal{G}_t(f)$ with respect to $C^0$ topology.

This is proved exactly in the same way as the open-and-dense property of $\mathcal{G}_0(f; x, y)$ in [9,p.29,30].

(2). If $v \in \mathcal{G}_1(f; x, y)$ then $\tilde{n}(\bar{x}, \bar{y}; v)$ satisfies the condition $(\mathcal{L})$.

The liftings $x \mapsto \bar{x}$ and $y \mapsto \bar{y}$ define a lifting $\tilde{N}(v)$ of $N(v)$ to $Q^{-1}(F^{-1}(\theta))$ and $\tilde{L}(-v)$ of $L(-v)$ to $Q^{-1}(F^{-1}(\eta))$. The $ZH$-module $\mathcal{H} = H_1((Q^{-1}(F^{-1}(\eta)) \setminus B_\eta), Q^{-1}(A_\eta))$ is free; choose some basis $e_1, ..., e_m$ of this module. The homomorphism $\tilde{h}_\eta(v)$ of this module is $\theta$-semilinear. Denote by $B = (b_{ij})$ its matrix, and denote by $A$ the matrix $(b_{ij})$. Let $a_i^{(s)}$ be the coefficients of $A^s$. Consider the element $\xi = \tilde{H}_{[\theta, \eta]}(v)(\tilde{N}(v))$ of $\mathcal{H}$; let $\xi = \sum e_i X_i$ with $X_i \in ZH$. Consider $\beta = \tilde{f}^*(\tilde{L}(-v))$ as a homomorphism of $\mathcal{H}$ to $ZH$. It is of finite type; denote $\beta(e_j)$ by $Y_j$; then $Y_j \in ZH$. We claim that

(*) \[ \tilde{n}(\bar{x}, \bar{y}; v) = \sum_{s \geq 0} Y_i \cdot a_i^{(s)} \cdot X_j \]

To prove it write $\tilde{n}(\bar{x}, \bar{y}; v) = \sum_{s \geq 0} \left( \sum_{h \in H} \nu(\bar{x}, \bar{y}h\theta^s) \cdot h\theta^s \right)$ (here $\nu(\bar{x}, \bar{y}h\theta^s)$ stands for the algebraic number of $(-v)$-trajectories, joining $\bar{x}$ with $\bar{y}h\theta^s$; note that since $F(\bar{x}) \leq F(\bar{y}) + 1$, there are no $(-v)$-trajectories joining $\bar{x}$ to $\bar{y}g$ if $\xi(g) > 0$).

To make the following computation more easy to comprehend, we make the following terminology conventions (valid only here). The homomorphism $\tilde{h}_\eta(v) : \mathcal{H} \to \mathcal{H}$ will be denoted by $\mu$. We identify the cohomology classes in $H^*(Q^{-1}((F^{-1}(\nu), Q^{-1}(F^{-1}(\nu) \setminus \tilde{L})))$ with their images in $H^*(Q^{-1}(F^{-1}(\nu) \setminus B_\nu), Q^{-1}(A_\nu))$ (thus suppressing $\tilde{f}$ in the notation). We have:

$\tilde{n}(\bar{x}, \bar{y}; v) = \sum_{s \geq 0} \left( \sum_{h \in H} ([\tilde{L} \cdot h]([\mu^s(\xi)])h) \right) \theta^s$

(by 1.6). The latter expression equals to

$\sum_{s \geq 0} \left( \sum_{h \in H} ([\tilde{L}([\mu^s(\xi) \cdot h^{-1}])]h) \right) \theta^s = \sum_{s \geq 0} \left( \sum_{h \in H} \beta([\mu^s(\xi) \cdot h^{-1}])h \right) \theta^s = \sum_{s \geq 0} \beta([\mu^s(\xi)] \cdot \theta^s$

To obtain from this expression the formula (*) we need only a lemma, allowing to calculate $\mu^s(\xi)$ in terms of the coordinates of $\xi$ and the matrix of $\mu$ (the expression is slightly different from the standard one in linear algebra since $\mu$ is $\theta$-semilinear).
Lemma 1.7. Let $F$ be a free $\mathbb{Z}H$-module with a basis $e_1, \ldots, e_m$ and $\mu : F \to F$ be a $\theta$-semilinear homomorphism of $F$. Let $m_{ij}$ be its matrix. Denote by $M$ the $m \times m$ matrix $(m_{ij} \cdot \theta)$. Let $\xi \in F, \xi = \sum e_j \xi_j$.

Then for every natural $s \geq 0$ we have

$$\mu^s(\xi) = \sum_{i,j} e_i [M^s]_{ij} \xi_j \theta^{-s}$$

Proof. Induction in $s$. We have

$$\mu^{s+1}(\xi) = \mu(\mu^s(\xi)) = \mu\left( \sum_i e_i \left( \sum_j [M^s]_{ij} \xi_j \theta^{-s} \right) \right) = \sum_i \mu(e_i) \cdot \left( \theta \sum_j [M^s]_{ij} \xi_j \theta^{-s-1} \right)$$

$$= \sum_i \left( \sum_k e_k m_{ki} \right) \cdot \left( \theta \sum_j [M^s]_{ij} \xi_j \theta^{-s-1} \right) = \sum_k e_k \left( \sum_i (m_{ki} \theta) \cdot [M^s]_{ij} \right) \xi_j \theta^{-s-1}. \quad \square$$

Now substitute the expression for $\mu^s(\xi)$ into the above formula, and get (*).

(3). Let $v \in G_{t_1}(f; x, y)$. Let $U$ be a neighborhood of $S(f)$. Then there is $\epsilon > 0$ such that for every $w \in G_{t_0}(f; x, y)$ with $\|w - v\| < \epsilon$ and $w|U = v|U$ we have:

$$\tilde{n}(\tilde{x}, \tilde{y}; v) = \tilde{n}(\tilde{x}, \tilde{y}; w).$$

Let $w$ be an $f$-gradient, sufficiently close to $v$. Then $\{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma}$ is still a $t$-equivariant ranging system for $(F, w)$, satisfying (1.1), (1.2). It is not difficult to see that $[\tilde{N}(w)] = [\tilde{N}(v)], [\tilde{L}(-w)] = [\tilde{L}(-v)]$. Then Lemma 1.5 together with the formula (*) above finishes the proof.

Set now $G_{t_1}(f)$ to be the intersection of all the $G_{t_1}(f; x, y)$ where $x, y \in S(f), \text{ind}x = \text{ind}y + 1$, and the Theorem 1.3 is proved. \quad \square

Remark 1.8. There is an obvious analog of Theorem 1.3 for any regular covering $\phi : \tilde{M} \to M$ such that $\phi^* [\omega] = 0$. 
§2. Morse forms with arbitrary cohomology classes

A. Algebraic preliminaries.

We shall need some lemmas about the ring $\mathbb{Z}[\mathbb{Z}^m]$ and its completions and localizations.

Let $\eta : \mathbb{Z}^m \to \mathbb{R}$ be a non-zero homomorphism. We extend it to a linear map $\mathbb{R}^m \to \mathbb{R}$, which will be denoted by the same letter. We say that a set $Z \subset \mathbb{R}^m$ is an $\eta$-cone, if there is a compact convex nonempty set $K \subset \eta^{-1}(1)$ such that $Z = \{ \lambda z \mid \lambda \in \mathbb{R}, \lambda \geq 0, z \in K \}$. We say, that $Z$ is $(\xi, \eta)$-cone if $Z$ is $\xi$-cone and $\eta$-cone. We say that a set $Z \subset \mathbb{R}^m$ is an integral $\eta$-cone if there are $e_1, ..., e_k \in \mathbb{Z}^m$, such that

1. $\text{rk} (e_1, ..., e_k) = m$
2. $\eta(e_i) < 0$
3. $Z = \{ \lambda_1 e_1 + ... + \lambda_k e_k \mid \lambda_i \geq 0 \}$

We shall also write $Z = Z(e_1, ..., e_k)$. We say, that $Z$ is an integral $(\xi, \eta)$-cone if the vectors $e_1, ..., e_k$ above satisfy $\xi(e_i) < 0, \eta(e_i) < 0$ for all $i$. Note that an integral $(\xi, \eta)$-cone is a $(\xi, \eta)$-cone.

Lemma 2.1. Let $Z$ be an $(\eta_1, \eta_2)$-cone. Then there is an integral $(\eta_1, \eta_2)$-cone $Z_0 \supset Z$.

Proof. We assume that $\eta_1, \eta_2$ are linearly independent; the other case is considered similarly. Denote $\eta_1^{-1}(1)$ by $H$, and the set $H \cap \{ \eta_2(x) < 0 \}$ by $H_0$; then $H_0$ is an open halfspace of $H$, containing $Z \cap H$. Denote by $\mathcal{L}$ the set $\{ \lambda x \mid \lambda \geq 0, x \in \mathbb{Z}^m \}$. $\mathcal{L}$ is everywhere dense in $H$ and in $H_0$. It is not difficult to prove that there is a finite subset $\mathcal{L}_0 \subset \mathcal{L}$ such that $\mathcal{L}_0 \subset H_0$ and $\langle \mathcal{L}_0 \rangle \supset Z \cap H$. We can choose $\mathcal{L}_0$ so that $\text{rk} \mathcal{L}_0 = m$ and the lemma is proved. \qed

Lemma 2.2. Let $\xi : \mathbb{R}^m \to \mathbb{R}$ be a non-zero linear form. Let $\epsilon > 0$. Then there is a finite set $I$ of linear forms $\theta_i : \mathbb{R}^m \to \mathbb{R}, i \in I$ with $\|\theta_i\| < \epsilon$ such that

1. the set $\Gamma = \{ x \in \mathbb{R}^m \mid (\xi + \theta_i)(x) \leq 0 \}$ is a $\xi$-cone.
2. There is an integral $\xi$-cone $\Gamma_0$ such that for every family $\{ A_i \}_{i \in I}$ of real numbers there is $b \in \mathbb{Z}^m$ with the property:

\[ \{ x \in \mathbb{R}^m \mid (\xi + \theta_i)(x) \leq A_i \} \subset \Gamma_0 + b \]

Proof. 1) Pick any $m$ linearly independant linear forms $\alpha_1, ..., \alpha_m : \mathbb{R}^m \to \mathbb{R}$ with $\|\alpha_i\| < \min(\epsilon, \|\xi\|/2)$. I claim that the finite family $\{ \alpha_1, -\alpha_1, ..., \alpha_m, -\alpha_m \}$ of linear forms satisfy our conclusions. Note first that

$$Z = \xi^{-1}(-1) \cap \{ x \mid \forall i : (\xi + \alpha_i)(x) \leq 0, (\xi - \alpha_i)(x) \leq 0 \}$$

is non empty and compact. (Indeed, let $x$ be a vector such that $|x| = 1$ and $\xi(x) = \|\xi\|$. Then $a = -\frac{x}{\xi(x)} \in Z$. Further, if $Z$ is not bounded, then there is a sequence $x_n \in Z$ such that $|x_n| \to \infty$. Consider the sequence $x_n/|x_n|$. We can assume that it converges to some $v, \|v\| = 1$. Since $\xi(x_n) = -1$, we have $\xi(v) = 0$. Further, for every $i$ we have $(\xi + \alpha_i)(v) \leq 0, (\xi - \alpha_i)(v) \leq 0$, therefore $\alpha_i(v) = 0$ and $v = 0$.) Further, $\Gamma \subset \xi^{-1}(\{ -\infty, 0 \})$ and the intersection $\Gamma \cap \xi^{-1}(0)$ consists of 0. Therefore $x \in \Gamma, x \neq 0$ implies $x = \lambda y, y \in Z$. 2) Choose a vector $x_0 \in \mathbb{Z}^m$ such that $\xi(x_0) < 0$ and for every $1 \leq i \leq m$ we have $(\xi + \alpha_i)(x_0) < 0, (\xi - \alpha_i)(x_0) < 0$.
(such a vector exists; it suffices to note that \((\xi + \alpha_i)(a) = -1 \mp \alpha_i(x) < 0\), and to approximate \(a\) by an element \(y_0 \in \mathbf{Q}^m\).

Denote \(|(\xi + \alpha_i)(x_0)|\) by \(\beta_i^\pm > 0\). Now \((\xi + \alpha_i)(x) \leq A_i\) implies that \((\xi + \alpha_i)(x + px_0) \leq A_i - p\beta_i^\pm\). Therefore if \(p\) is sufficiently big, we obtain (*) with \(b = -px_0\), or, equivalently, \(\{x \in \mathbf{R}^m \mid (\xi + \theta_i)(x) \leq A_i\} \subset \Gamma + b\). Then apply 2.1 to obtain an integral \(\xi\)-cone \(\Gamma\), such that \(\Gamma \subset \Gamma_0\). \(\square\)

Next we shall recall some facts from [6]. Let \(\eta : \mathbf{Z}^m \to \mathbf{R}\) be a homomorphism and \(Z = Z\langle e_1,...,e_k\rangle\) be an integral-\(\xi\)-cone. Denote \(Z^m \cap Z\) by \([Z]\), and the set \(\{n_1e_1 + ... + n_ke_k \mid n_i \in N\}\) by \([\hat{Z}]\). Note that \([Z]\) and \([\hat{Z}]\) are submonoids of \(Z^m, [\hat{Z}] \subset [Z]\), and \([Z]\) is finitely generated over \([\hat{Z}]\). Consider the group rings \(Z[Z]\) and \(Z[\hat{Z}]\) of the monoids \(Z\) and \(\hat{Z}\); since \(Z[\hat{Z}]\) is an image of the ring \(Z[t_1,...,t_k]\), it is noetherian. The ring \(Z[Z]\) is finitely generated as a left module over \(Z[\hat{Z}]\), therefore it is also noetherian.

Consider the submonoid \(\{x \in Z \mid x \neq 0\}\) of \(Z\) and denote its group ring by \(m_Z\). It is an ideal of \(Z[Z]\). Denote by \(Z[Z]\) the \(m_Z\)-completion of \(Z[Z]\). Denote by \(S_Z\) the multiplicative set \(1 + m_Z \subset Z[Z]\). Then \((S_Z^{-1}Z[Z])^\sim = ([Z[Z]]\)^{\sim}\) (see [2], Ch 3,§3, Prop. 12). The ring \((Z[Z])^\sim\) is easily identified with the ring of all the elements \(\lambda \in (Z[Z])^\sim\) such that \(\text{supp} \lambda \subset Z\) (this latter ring is obviously a subring of \(Z[Z]_{\eta}^\sim\)). The ring \((Z[Z])^\sim\) is faithfully flat over \(S_Z^{-1}Z[Z]\) (ibid. Prop. 9). Therefore, if \(P, Q \in Z[Z]\) and there is \(x \in (Z[Z])^\sim\) such that \(P = Qx\), then \(x \in S_Z^{-1}Z[Z]\) (ibid. Ch 1,§3,p.7).

Set \(\sigma_Z = \{ t^M \mid I \in Z\}\); it is a multiplicative subset of \(Z[Z]\). It is easy to see that \(\sigma_Z^{-1}Z[Z] = Z[Z^m]\) and \(\sigma_Z^{\sim 1}(Z[Z])^\sim = \{ S \in (Z[Z])^\sim \mid \exists x \in Z^m : \text{supp} S \subset Z + x\}\). The faithful flatness property cited above implies immediately that if \(P, Q \in Z[Z^m]\) and there is \(x \in \sigma_Z^{\sim 1}(Z[Z])^\sim\) such that \(P = Qx\), then \(x \in \sigma_Z^{-1}S_Z^{-1}Z[Z]\).

For a linear form \(\eta\), consider the multiplicative subset \(\{1 + Q \mid \text{supp} Q \subset \eta^{-1}([-\infty,0])\}\).

**Lemma 2.3.** Let \(\alpha : \mathbf{Z}^m \to \mathbf{Z}\) be an indivisible homomorphism. Denote \(\text{Ker} \eta\) by \(H\). Let \(A\) be \((k \times k)\)-matrix, such that \(a_{ij} \in \mathbf{Z}[\mathbf{Z}^m]_{(-1)}\). Let \(\xi = (\xi_1,...,\xi_k, \eta)\) be vectors in \((\mathbf{Z}H)^k\). Denote by \(a_{ij}^{(s)}\) the \((ij)\)-coefficient of the matrix \(A^s\).

Then \((1 - \det A) \sum_{s \geq 0} (\sum_{i,j} \xi_i a_{ij}^{(s)} \eta_j)\) belongs to \(Z[Z^m]\).

**Proof.** It suffices to prove that every coefficient of the \((k \times k)\)-matrix \((1 - \det A) \left( \sum_{s \geq 0} A^s \right)\) belongs to \(Z[Z^m]\). Consider the matrix \(1 - A\). It is invertible in the ring \(S_o^{-1}Z[Z^m]\) and the Cramer rules imply \((1 - \det A)(1 - A)^{-1} \in \text{Mat}(Z[Z^m])\). On the other hand \((1 - A)^{-1} = \sum_{s \geq 0} A^s\). \(\square\)

**B. Preliminaries on Morse forms and their gradients.**

In this subsection we assume the terminology of §1 of [9]. Moreover, we assume the definition of \(\omega\)-chart-system and, respectively, of \(\omega\)-gradient (where \(\omega\) is a Morse form), which are completely similar to those of \(f\)-chart- and \(f\)-gradient, see Definition 1.1 of [9].

So let \(\omega\) be a Morse form on a closed connected manifold \(M\). Let \(p : \widetilde{M} \to M\) be the universal covering and let \(\overline{F} : \widetilde{M} \to \mathbf{R}\) be a Morse function such that \(d\overline{F} = p^*\omega\). Note that \(\overline{F}(x_0) = \overline{F}(x) + [c](a)\), where \(c \in \pi_1 M\). Choose a riemannian metric
on $M$. Then $\tilde{M}$ obtains a $\pi_1 M$-invariant riemannian metric. If $v$ is an $\omega$-gradient we denote by the same letter $v$ its liftings to $\tilde{M}$ and $\overline{M}$, since there is no possibility of confusion. The length of a curve $\gamma$ will be denoted by $l(\gamma)$. The following simple and useful lemma is known since the early 80s. I knew it from J.-Cl.Sikorav.

Assume that for every $x \in S(f)$ a lifting $\tilde{x}$ of $x$ to $\tilde{M}$ is fixed.

**Lemma 2.4.** Let $v$ be an $\omega$-gradient. Then there are constants $A,B > 0$ such that:

For every $g \in \pi_1 M$ and every $(-\nu)$-trajectory $\gamma$, joining $\tilde{x}$ and $\tilde{y} \cdot g$ we have:

\[
l(\gamma) \leq A - B \{\omega\}(g).
\]

**Proof.** Choose any $\delta > 0$ less than the injectivity radius of $M$ and less than $\min_{p,q \in S(\omega)} \rho(p,q)$. Then any nontrivial piecewise smooth loop in $M$ is longer than $\delta$. Therefore for any $x \in \tilde{M}$, $1 \neq g \in \pi_1 M$, any piecewise smooth path in $\tilde{M}$, joining a point of $D(x, \delta/3)$ with a point of $D(xg, \delta/3)$ has the length $\geq \delta/3$. Also for $x,y \in \tilde{M}$ with $p(x),p(y) \in S(\omega)$ any piecewise smooth path joining joining a point of $D(x, \delta/3)$ with a point of $D(y, \delta/3)$ has the length $\geq \delta/3$.

Now let $\{\Phi_p : U_p \to B^n(0,r)\}_{p \in S(\omega)}$ be an $\omega$-chart-system such that $G(U_p, \Phi_p) \leq C$ for every $p \in S(\omega)$ and that $rC < \delta/12$. This condition implies in particular that $U_p \subset D(p, \delta/12)$.

Choose some liftings $\tilde{U}_p$ of neighborhoods $U_p$, extending $x \mapsto \tilde{x}$. Let $D > 0$ be less than $\min_{x \neq \cup U_p} \omega(\nu(x))$. Denote $\|v\|$ by $E$.

Now let $\gamma$ be a $v$-trajectory, joining $\tilde{x}$ with $\tilde{y}g$, and let $A_0 = \tilde{x}, A_1, ..., A_n, \tilde{y} = A_{n+1}$ be the points in $p^{-1}(S(\omega))$ such that $\gamma$ intersects $\tilde{U}_{A_i}$. Then $(n+1) \cdot \delta/3 \leq l(\gamma)$.

The length of the part of $\gamma$ inside of $\bigcup_{i=0}^{n+1} \tilde{U}_{A_i}$ is not more than $2rC(n + 1)$.

Now let $t_i$, resp. $\tau_i$ be the moment when $\gamma$ enters $\tilde{U}_{A_i}$, resp. quits $\tilde{U}_{A_i}$. We have

\[
\int_{\tau_i}^{t_{i+1}} \omega(\nu) dt = -\int_{\tau_i}^{t_{i+1}} \omega(v) dt = \tilde{F}(\gamma(t_{i+1})) - \tilde{F}(\gamma(\tau_i)).
\]

Therefore the total time which $\gamma$ can spend outside $\bigcup_p \tilde{U}_{A_i}$ is not more than $|\tilde{F}(\tilde{y}g) - \tilde{F}(\tilde{x})|/D$, and the length of the corresponding part of the curve is $\leq E/D|\tilde{F}(\tilde{y}g) - \tilde{F}(\tilde{x})|$. Since $\gamma$ joins $\tilde{x}$ to $\tilde{y}g$, the last expression is $\leq E/D \cdot (\tilde{F}(\tilde{x}) - \tilde{F}(\tilde{y}) - \{\omega\}(g))$. Therefore

\[
l(\gamma) \leq 2rC(n + 1) + E/D(\tilde{F}(\tilde{x}) - \tilde{F}(\tilde{y})) - E/D\{\omega\}(g) \leq \frac{6rC}{\delta} l(\gamma) + \frac{A}{2} - \frac{B}{2} \{\omega\}(g)
\]

where $A$ is chosen such that $A/2 \geq |\tilde{F}(\tilde{x}) - \tilde{F}(\tilde{y})|$ for every $x,y \in S(f)$ and $B = 2E/D$. Then the inequality $l(\gamma) \leq A - B \{\omega\}(g)$ easily follows. $\square$

Let $\omega$ be a Morse form, let $\{\Phi_p : U_p \to B^n(0, r_p)\}_{p \in S(\omega)}$ be an $\omega$-chart-system. Choose a basis $a_1, ..., a_m$ in $H_1(M, \mathbb{Z})/\text{Tors}$. Choose and fix closed 1-forms $\lambda_1, ..., \lambda_m$.

\[\text{Recall from [9,p.3] that for a chart } \Phi : U \to V \subset \mathbb{R}^n \text{ of } M \text{ and } x \in U \text{ we denote by } G(x, \Phi) \text{ the number } \sup_{x \in M, h \in T_x M, h \neq 0} \max \{\|h\|/\|\Phi_* h\|, |\Phi_* h|/\|h\|\}, \text{ where } \rho \text{ stands for the metric on } M \text{ and } e \text{ for the euclidean metric in } \mathbb{R}^n.\]
on $M$, such that $\langle [\lambda_i], a_j \rangle = \delta_{ij}$ and $\text{supp } \lambda_i \cap U_p = \emptyset$ for every $i$ and every $p \in S(\omega)$. 
(To prove that we can satisfy the second condition, let $\theta$ be any closed 1-form. Let $\{ \tilde{\Phi}_p : U'_p \to B^n(0, r'_p) \}_{p \in S(\omega)}$ be some standard extension of $\{ \Phi_p : U_p \to B^n(0, r_p) \}_{p \in S(\omega)}$ and let $\phi_p$ be a $C^\infty$ function which equals to 1 in a neighborhood of $U'_p$ and $\text{supp } \phi_p \subset U'_p$. Let $F_p$ be a function on $U'_p$, such that $dF_p = \theta$. Consider the form $\theta' = \theta - \sum d(\phi_p F_p)$. We have $[\theta'] = [\theta]$ and $\theta(x) = 0$ in every $U'_p$.)

For $\epsilon = (\epsilon_1, ..., \epsilon_m) \in \mathbb{R}^m$ we denote by $\epsilon \cdot \lambda$ the form $\sum_{i=1}^m \epsilon_i \lambda_i$, by $\omega_{\epsilon}$ the form $\omega + \epsilon \cdot \lambda$. For $\epsilon > 0$ set

$$\Omega_\epsilon = \{ \omega_{\epsilon} \mid |\epsilon| = \max_i |\epsilon_i| \leq \epsilon \}.$$

We shall say that $\Omega_\epsilon$ is a Morse family, if for every $\epsilon$ with $|\epsilon| \leq \epsilon$ the form $\omega_{\epsilon}$ is a Morse form and $S(\omega_{\epsilon}) = S(\omega)$. Let $\Omega_{\epsilon}$ be a Morse family, and $v$ be a vector field. We say, that $v$ is an $\omega_{\epsilon}$-gradient, if $v$ is an $\omega$-gradient for each $\omega_{\epsilon} \in \Omega_{\epsilon}$.

**Lemma 2.5.**
1. There is $\epsilon > 0$, such that $\Omega_\epsilon$ is a Morse family.
2. Let $v$ be an $\omega$-gradient. Then there is $\epsilon > 0$ such that $v$ is an $\Omega_{\epsilon}$-gradient.
3. Let $v$ be an $\Omega_{\epsilon}$-gradient. Then there is $\delta > 0$ such that every $\omega$-gradient $u$ with $\|u - v\| < \delta$ is an $\Omega_{\epsilon}$-gradient.

**Proof.** 1) Denote $\sup_{1 \leq i \leq m} |\lambda_i(x)|$ by $\lambda$ and $\min_{x \in M \setminus \cup_p U_p} \|\omega(x)\|$ by $\eta$. Then $\epsilon = \frac{\eta}{2m\lambda}$ will do.

2) Denote $\min_{x \in M \setminus \cup_p U_p} \omega(v(x))$ by $\bar{\eta}$ and $\sup_{1 \leq i \leq m} |\lambda_i(v)(x)|$ by $\bar{\lambda}$. Then $\epsilon = \frac{\bar{\eta}}{2m\lambda}$ will do.

3) Denote $\sup_{1 \leq i \leq m} |\lambda_i(x)|$ by $\lambda$. Denote by $Q$ the compact set $(M \setminus \cup_p U_p) \times [-\epsilon, \epsilon]^m$ and by $F : Q \to \mathbb{R}$ the map $F : (x, \nu_1, ..., \nu_m) \mapsto \omega_p(v)(x)$. Since $\text{Im } F \subset [0, \infty[, there is $\beta > 0$ such that $\text{Im } F \subset [\beta, \infty]$. Let $u$ be any $\omega$-gradient such that $\|u - v\| \cdot (m\lambda \varepsilon + \|\omega\|) \leq \beta/2$.

We claim that $u$ is an $\Omega_{\epsilon}$-gradient. Indeed, note first that $\omega_{\epsilon}(u)(x) > 0$ for any $x \in (\cup_p U_p \setminus S(\omega))$ and any $|\epsilon| \leq \epsilon$. Further, if $x \in (M \setminus \cup_p U_p)$, we have

$$|\omega_{\epsilon}(u)(x) - \omega_{\epsilon}(v)(x)| = |(\omega + \sum_i \epsilon_i \lambda_i)(u - v)(x)| \leq \|u - v\| \cdot (\|\omega\| + \epsilon \lambda m) \leq \beta/2$$

therefore $\omega_{\epsilon}(x) > 0$. Finally, $u$ has a standard form with respect to some $\omega$-chart-system.

The suitable restriction of this system will be an $\omega_{\epsilon}$-chart-system for any $\epsilon$. □

Now we can define the incidence coefficients with respect to the universal cover. The preceding lemma implies that $v$ is an $\omega$-gradient for some 1-form $\omega$ which cohomology class is rational. Therefore (see [7]) for every $g \in \pi_1(M)$ there is at most finite set of $(-v)$-trajectories joining $\tilde{x}$ with $\tilde{y}g$ if $\text{ind } x = \text{ind } y + 1$. Choose orientations of descending discs. For each such trajectory we denote by $e(\gamma)$ the sign of intersection of $D(\tilde{x}, v)$ with $D(\tilde{y}g, -v)$ along $\gamma$. The element $\sum_{\gamma} e(\gamma)$ is denoted by $\nu(\tilde{x}, \tilde{y}g)$ and we set $\tilde{\nu}(\tilde{x}, \tilde{y}; v) = \sum_{g \in G} \nu(\tilde{x}, \tilde{y}g)g$. This is an element of the abelian group $(\mathbb{Z}[\pi_1 M])^\wedge$ of all the formal linear combinations (infinite in general) of the elements of $G$. 
Lemma 2.6. \( \tilde{n}(\tilde{x}, \tilde{y}; v) \in (\mathbb{Z}[\pi_1 M])_{[\omega]}. \)

Proof. Recall that Novikov ring \( \mathbb{Z}([\pi_1 M])_{[\omega]} \) consists of all \( \lambda \in (\mathbb{Z}[\pi_1 M])^\sim \) such that for every \( c \in \mathbb{R} \) we have: \( \text{supp} \lambda \cap \{\omega\}^{-1}([c, \infty]) \) is finite. Then our lemma follows from Lemma 2.4. \( \square \)

Remark 2.7. Note that the analogs of Lemmas 2.4 and 2.6 are obviously true for any regular covering \( p' : \tilde{M} \to M \) such that \( (p')^*[\omega] = 0. \)

C. Incidence coefficients with respect to a free abelian cover.

Next we pass to free abelian coverings. We assume here the terminology of Subsection C.2 of Introduction. The universal covering factors then as \( p = P \circ Q; \) \( Q : \tilde{M} \to \overline{M}. \) The epimorphism \( \pi_1(M) \to H_1(M, \mathbb{Z})/ \text{Tors} \) will be denoted by \( Q. \) The deRham cohomology class \( [\omega] \) of \( \omega \) defines a homomorphism \( H^1(M, \mathbb{Z})/ \text{Tors} \to \mathbb{R}, \) which will be denoted by the same letter \( [\omega]. \) Note that \( \{\omega\} = [\omega] \circ Q. \) Since \( P^*(H^1(M, \mathbb{R})) = 0, \) there is a Morse function \( \overline{F} : \overline{M} \to \mathbb{R}, \) such that \( d\overline{F} = P^* \omega, \) and we shall assume that \( \overline{F} = \overline{F} \circ Q. \) We have chosen a riemannian metric on \( M; \) therefore the manifold \( \overline{M} \) obtain a riemannian metric, which is \( \mathbb{Z}^m \)-invariant. We have chosen a basis \( (a_1, ..., a_m) \) in \( H_1(M, \mathbb{Z})/ \text{Tors}. \) Therefore this group is identified with \( \mathbb{Z}^m, \) and the vector space \( H^1(M, \mathbb{R}) \) with the dual space of linear forms \( \mathbb{R}^m \to \mathbb{R}. \) We choose the \( L_1 \)-norm in \( \mathbb{R}^m; \) then the dual space obtains the \( \sup \)-norm. (That is \( \| \sum \alpha_i a_i \| = \sum |\alpha_i| \) and \( \| \sum \beta_i a_i^* \| = \max_j |\beta_j|, \) where \( \{a_i^*\} \) is the base dual to \( \{a_i\}. \)

Corollary 2.8. Let \( v \) be an \( \omega \)-gradient. There is such an \( \epsilon > 0, \) that every linear form \( \eta : \mathbb{R}^m \to \mathbb{R} \) with \( ||[\omega] - \eta|| \leq \epsilon \) is a cohomology class of a Morse form \( \omega(\eta) \) such that \( v \) is an \( \omega \)-gradient.

Proof. Every linear form \( \eta : \mathbb{R}^m \to \mathbb{R} \) with \( ||[\omega] - \eta|| \leq \epsilon \) can be written as \( \eta = [\omega] + \sum \eta_i a_i^* \) where \( \eta_i \in \mathbb{R}, |\eta_i| \leq \epsilon. \) Now let \( \epsilon > 0 \) be so small that \( \Omega_\epsilon \) is a Morse family and \( v \) is an \( \Omega_\epsilon \)-gradient and set \( \omega(\eta) = \omega + \sum \eta_i \lambda_i. \) \( \square \)

For two critical points \( x, y \in S(\omega) \) and an \( \omega \)-gradient \( v \) we set \( I(x, y; v) = \{ g \in \pi_1(M) \mid \text{there is a } (v) \text{-trajectory } \gamma, \text{joining } \tilde{x} \text{ to } \tilde{y} \cdot g \}. \) If the set of \( (v) \)-trajectories joining \( \tilde{x} \) to \( \tilde{y} \cdot g \) is finite, we denote by \( N(\tilde{x}, \tilde{y}, g; v) \) its cardinality. (We identify here two trajectories which differ by a parameter change.)

Remark 2.9. If \( v \) is a good \( \omega \)-gradient and \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1, \) then \( \text{supp } (\tilde{n}(\tilde{x}, \tilde{y}; v)) \subset I(x, y; v) \) and for every \( g \in \pi_1 M \) the set \( N(\tilde{x}, \tilde{y}, g; v) \) is finite. \( \triangle \)

Lemma 2.10. There is an integral \([\omega]\)-cone \( \Gamma \) and a vector \( a \in \mathbb{Z}^m \) such that \( Q(I(x, y; v)) \subset \Gamma + a. \)

Proof. Note that if \( v \) is a \( \omega \)-gradient for some Morse form \( \omega, \) then \( Q(I(x, y; v)) \subset [\omega]^{-1}([-\infty, A]) \) for some \( A. \) By Corollary 2.8 there is \( \epsilon > 0, \) such that every linear form \( \eta : \mathbb{R}^m \to \mathbb{R} \) with \( ||[\omega] - \eta|| \leq \epsilon \) is the cohomology class of a Morse form \( \omega(\eta) \) such that \( v \) is an \( \omega(\eta) \)-gradient. Choose then the linear forms \( \eta_i \) so as to satisfy Lemma 2.2 and obtain the conclusion. \( \square \)

Lemma 2.11. Let \( \gamma, \gamma' \) be two Morse forms with \( [\gamma] \neq 0, [\gamma'] \neq 0. \) Assume that a vector field \( v \) satisfying the transversality assumption is \( \gamma \)-gradient and \( \gamma' \)-gradient. Let \( x, y \in S(\gamma), \) \( \text{ind} x = \text{ind} y + 1. \)
Then

1) If \( [\gamma] = \alpha[\gamma'] \) with \( \alpha < 0 \), then \( I(x, y; v) \) is finite.

2) If there is no \( \alpha < 0 \) with \( [\gamma] = \alpha[\gamma'] \), then there is an integral \( ([\gamma], [\gamma']) \)-cone \( \Delta \) and \( b \in \mathbb{Z}^m \) such that \( Q(I(x, y; v)) \subset \Delta + b \).

Proof. 1) Obvious. 2) If \( [\gamma] \) and \( [\gamma'] \) are linearly dependant, then \( [\gamma] = \alpha[\gamma'] \) with \( \alpha > 0 \) and our lemma follows from 2.10. Therefore we can assume that \( [\gamma] \) and \( [\gamma'] \) are linearly independant, which imply that there is \( h \in \mathbb{Z}^m \) with \( [\gamma](h), [\gamma'](h) < 0 \). From 2.10 we know that there are integral \( [\gamma] \)-cone \( \Gamma_1 \) and \( a_1 \in \mathbb{Z}^m \) such that \( Q(I(x, y; v)) \subset \Gamma_1 + a_1 \). Also there are integral \( [\gamma'] \)-cone \( \Gamma_2 \) and \( a_2 \in \mathbb{Z}^m \) such that such that \( Q(I(x, y; v)) \subset \Gamma_2 + a_2 \). Adding to the generators of \( \Gamma_1 \) and \( \Gamma_2 \) some other integral vectors we can assume that \( h \in \text{Int} \Gamma_1, h \in \text{Int} \Gamma_2 \). Then there exists \( N \in \mathbb{N} \) such that \( \Gamma_1 + a_1 \subset \Gamma_1 - Nh \) and \( \Gamma_2 - Nh \supset \Gamma_2 + a_2 \). Thus \( Q(I(x, y; v)) \subset (\Gamma_1 - Nh) \cap (\Gamma_2 - Nh) = \Gamma_1 \cap \Gamma_2 - Nh \). The set \( \Gamma_1 \cap \Gamma_2 \) is a \([\gamma]\)-cone, and by 2.1 there is an integral \( ([\gamma], [\gamma']) \)-cone \( \Delta \) such that \( \Gamma_1 \cap \Gamma_2 \subset \Delta \) therefore \( Q(I(x, y; v)) \subset \Delta - Nh \). \( \square \)

By the remark 2.7 the incidence coefficient \( \overline{\nu}(\overline{x}, \overline{y}; v) \in (\mathbb{Z}[\mathbb{Z}^m])_{[\omega]} \) is defined. We shall assume that the liftings \( \overline{x} \) of points \( x \in S(\omega) \) are chosen so that \( Q(\overline{x}) = \overline{x} \). Note that obviously \( \text{supp} (\overline{\nu}(\overline{x}, \overline{y}; v)) \subset Q(\text{supp} \overline{\nu}(\overline{x}, \overline{y}; v)) \).

Proof of Theorem B. For a Morse form \( \xi \) such that \( [\xi] \in H^1(M, \mathbb{Q}), [\xi] \neq 0 \) we denote by \( \xi_0 \) the (unique) Morse form, such that \( [\xi_0] \) is an indivisible class in \( H^1(M, \mathbb{Z}) \) and that \( \xi_0 = \mu \xi \) with \( \mu > 0 \). The map \( M \rightarrow S^1 \), corresponding to \( \xi_0 \), will be denoted by \( f_0(\xi) \).

Define now \( Gt_1(\omega) \) as the set of all \( \omega \)-gradients \( v \in Gt(\omega) \), satisfying the following property:

\[ (C) \text{ There is } \epsilon : 0 < \epsilon \leq \|\omega\|/2 \text{ such that } v \text{ is an } \Omega_\epsilon \text{-gradient and there is a Morse form } \xi \in \Omega_\epsilon \text{ with } [\xi] \in H^1(M, \mathbb{Q}) \text{ such that } v \in Gt_1(f_0(\xi)). \]

We shall now prove the properties of \( Gt_1(\omega) \).

1) \( Gt_1(\omega) \) is \( C^0 \)-open in \( Gt(\omega) \).

Indeed, if \( v \) satisfies \((C)\) then every \( \omega \)-gradient \( u \), sufficiently close to \( v \), is also an \( \Omega_\epsilon \)-gradient (by 2.5) and \( u \in Gt_1(f_0(\xi)) \) since \( Gt_1(f_0(\xi)) \) is \( C^0 \)-open in \( Gt(f_0(\xi)) \).

2) \( Gt_1(\omega) \) is \( C^0 \)-dense in \( Gt(\omega) \).

Indeed, if \( v \in Gt(\omega) \), then there is an \( \epsilon > 0 \) such that \( v \) is an \( \Omega_\epsilon \)-gradient. Choose any form \( \omega' \in \Omega_\epsilon \) with \( [\omega'] \in H^1(M, \mathbb{Q}) \). Then by Theorem 1.3 arbitrarily close to \( v \) there is an \( \omega' \)-gradient \( u \in Gt_1(f_0(\omega')) \). By 2.5 \( u \) is also an \( \Omega_\epsilon \)-gradient.

3) If \( v \in Gt_1(\omega) \), then \( \overline{\nu}(\overline{x}, \overline{y}; v) \in S^{-1}[\omega]Z[Z^m] \).

Indeed, \( v \) is an \( \omega \)-gradient and a \( \xi \)-gradient for some \( \xi \in \Omega_\epsilon \) with \( [\xi] \in H^1(M, \mathbb{Q}) \). Note that if \( [\xi] \) and \( [\omega] \) are linearly dependant, then \( [\xi] = \alpha[\omega] \) with \( \alpha > 0 \). (Indeed, \( [\xi] = \alpha[\omega] \) with \( \alpha < 0 \) would imply \((1 - \alpha)[\omega] + \sum \epsilon_i a_i^* = 0 \), which contradicts \( \epsilon \leq \|\omega\|/2 \). Therefore 2.11 implies that there is an integral \( ([\xi], [\omega]) \)-cone \( \Delta \), such that \( \text{supp} \overline{\nu}(\overline{x}, \overline{y}; v) \subset \Delta + b \) for some \( b \in \mathbb{Z}^m \). Consider \( \overline{\nu}(\overline{x}, \overline{y}; v) \) as an element of \( (Z[Z^m])_{[\xi]} \). Lemma 2.3 together with Theorem 1.3 imply that it belongs to the localization \( S^{-1}[\xi]Z[Z^m] \), therefore there are \( P, Q \in \mathbb{Z}[Z^m] \) such that \( P = Q \cdot \overline{\nu}(\overline{x}, \overline{y}; v) \). Since \( \overline{\nu}(\overline{x}, \overline{y}; v) \in \sigma^1_\Delta [\xi] \) the faithful flatness property imply \( \overline{\nu}(\overline{x}, \overline{y}; v) \in \sigma^1 S^{-1}Z[Z^m] \). \( \square \)
D. Incidence coefficients with respect to the universal cover: the exponential estimate.

Passing to the exponential estimate, we need first a lemma.

Lemma 2.12. Let \( \xi, \eta : \mathbb{R}^m \to \mathbb{R} \) be non zero linear forms, and let \( \Gamma \) be a \((\xi, \eta)\)-integral cone. Then there is \( A > 0 \) such that for every \( b \in \mathbb{R}^m \) there is \( B \) such that for every \( c \in \mathbb{R} \) we have:

\[
(\Gamma + b) \cap \xi^{-1}([c, \infty]) \subset (\Gamma + b) \cap \eta^{-1}([Ac + B, \infty])
\]

Proof. Abbreviate \( \xi^{-1}([c, \infty]) \) by \( \{ \xi \geq c \} \). It is sufficient to prove that there is \( A \geq 0 \) such that

\[
(*) \quad \Gamma \cap \{ \xi \geq c \} \subset \Gamma \cap \{ \eta \geq Ac \}
\]

(the case of general \( b \in \mathbb{R}^m \) follows then with \( B = \eta(b) - A\xi(b) \)). To prove (*) let \( e_i \) be the generators of \( \Gamma \), and choose \( A > 0 \) such that \( \eta(e_i) \geq A\xi(e_i) \) for all \( i \). Then \( x \in \Gamma \cap \{ \xi \geq c \} \) means: \( x = \sum \lambda_i e_i \) where \( \lambda_i \geq 0 \) and \( \sum \lambda_i \xi(e_i) \geq c \). This implies \( \eta(x) = \sum \lambda_i \eta(e_i) \geq \sum \lambda_i A\xi(e_i) \geq A\xi(x) \).

Let \( \lambda = \sum g \mu g \in Z[\pi_1 M]^{-} \) and \( c \in \mathbb{R} \). Denote by \( \lambda[c] \) the element \( \sum_{\xi(g) \geq c} n_g \cdot g \) of \( Z[\pi_1 M] \), and by \( N_c(\lambda) \) the norm of \( \lambda[c] \), that is \( N_c(\lambda) = \sum_{\xi(g) \geq c} |n_g| \).

Proof of Theorem C. By the definition of \( Gt_1(\omega) \) there is \( \epsilon \in \{0, \frac{1}{4} \|[\omega]\| \} \), and a Morse form \( \xi \in \Omega_\epsilon \), such that \( [\xi] \in H^1(M, Q) \) and \( \nu \in Gt_1(f_0(\xi)) \). Therefore there is an integral \((\xi, [\omega])\)-cone \( \Delta \) and \( b \in \mathbb{Z}^m \), such that \( Q(\text{supp } n(\bar{x}, \bar{y}; v)) \subset \Delta + b \), and, therefore, 2.12 implies that the set \( \text{supp } \tilde{n}(\bar{x}, \bar{g}; v) \cap ([\omega])^{-1}([c, \infty]) \subset \text{supp } \tilde{n}(\bar{x}, \bar{g}; v) \cap ([\xi])^{-1}([Ac + B, \infty]) \), and since \( v \) is a \( \xi \)-gradient, our estimate follows from Theorem 1.3 and Proposition 1.2.

E. On the Novikov complex for a Morse form. In this paper we do not use the notion of Novikov complex, working only with the incidence coefficients. The latter were introduced however in [5] as the matrix entries of the boundary operators in the Novikov complex. In this Subsection we use the results of Subsection B and Subsection C to give a simple proof of the fact that \( \partial^2 = 0 \) in this complex (reducing it to the corresponding statement about rational Morse forms, proved in [7]).

We assume here the terminology of Subsection B of the Introduction. Moreover, if \( \Delta \) is an integral \([\omega]\)-cone, we denote by \( \Lambda_{\Delta} \) the subset of \( Z[\pi_1 M]^{-\omega} \), defined by

\[
\Lambda_{\Delta} = \{ \lambda \exists b \in \mathbb{Z}^m \text{ such that } Q(\text{supp } \lambda) \subset \Delta + b \}
\]

It is not difficult to see that \( \Lambda_{\Delta} \) is a subring of \( Z[\pi_1 M]^{-\omega} \).

Now let \( b \) an \( \omega \)-gradient satisfying transversality assumption and \( \epsilon > 0 \) so small that \( \Omega_\epsilon \) is a Morse family and \( \nu \) an \( \Omega_\epsilon \)-gradient. Let \( \xi \in \Omega_\epsilon \) be a Morse form such that \( [\xi] \in H^1(M, Q) \). Then it follows from 2.11 that there is an integral \((\xi, [\omega])\)-cone \( \Delta \), such that for every \( x, y \in S(\omega) \) with \( \text{ind} x = \text{ind} y + 1 \) we have:

\[
\tilde{n}(\bar{x}, \bar{y}; v) \subset \Delta_{\omega}. \quad \text{Therefore the homomorphism } \partial : C_{*}(\omega, v) \to C_{*-1}(\omega, v) \text{ is defined actually over the ring } \Lambda_{\Delta} \text{ and to verify that } \partial^2 = 0 \text{ it is sufficient to verify it over the ring } Z[\pi_1 M]_{\text{fin}}, \text{ which is done in [7].}
§3. An example

In this section we shall construct a Morse map \( M \to S^1 \) on a closed 3-manifold \( M \), having two critical points: \( x \) of index 2 and \( y \) of index 1, such that

\[
n(\bar{x}, \bar{y}; v) = \sum_{k \geq 0} n_k t^k \quad \text{with} \quad n_k \sim \alpha \cdot \beta^k \quad \text{with} \quad \alpha < 0, \beta > 0.
\]

For any \( f \)-gradient \( w \) sufficiently close to \( v \) in \( C^0 \) topology, we shall have:

\[
n_k(\bar{x}, \bar{y}; w) = n_k(\bar{x}, \bar{y}; v).
\]

The construction is illustrated on the fig.1 and we invite the reader to consult it.\(^2\) We start with a torus \( T^2 \), move away from it two open discs and obtain a surface \( S \) with two components of boundary. Choose and fix a parallel \( \beta \) and a meridian \( \alpha \) of this twice punctured torus. The copies of this surface (resp. the copies of \( \alpha \), \( \beta \), etc.) will be denoted by the same letter \( S \) (resp. \( \alpha \), \( \beta \), etc.) adding indices in order to distinguish between them. The corresponding discs will be denoted by \( D_1 \), \( D_2 \).

We glue a copy of \( S \), denoted by \( S(1,1) \), to a copy of \( S \), denoted by \( S(1,2) \) and close the boundary by two discs \( D_1(1), D_1(2) \) (as it is shown on the figure 1). The resulting surface is called \( N \).

We glue three copies of \( S \) successively, close the boundary and obtain a closed surface \( L = D_1(1/2) \cup S(1/2,1) \cup S(1/2,0) \cup S(1/2,2) \cup D_2(1/2) \).

The surface \( K = D_1(0) \cup S(0,1) \cup S(0,2) \cup D_2(0) \) is constructed similarly to \( N \).

Let \( W_1 \) be the cobordism between \( N \) and \( L \), corresponding to the surgery on the circle \( \beta(1/2,2) \). Introduce on \( W_1 \) the corresponding Morse function \( F_1: W_1 \to [1/2,1] \) with one critical point \( x \) of index 2, and such that \( F_1^{-1}(1) = N \), \( F_1^{-1}(1/2) = L \). We can find an \( F_1 \)-gradient \( v_1 \) such that

1. \( (v_1)_{[1/2,1]} \) restricted to \( D_1(1/2) \cup S(1/2,1) \cup S(1/2,2) \) is a diffeomorphism of this surface onto \( D_1(1) \cup S(1,1) \cup S(1,2) \) which identifies \( S(1/2,0) \) with \( S(1,2) \) and \( S(1/2,1) \) with \( S(1,1) \), slightly diminished from the left, so that the image of \( D_1(1/2) \) contains \( D_1(1) \) in its interior (and therefore \( (v_1)_{[1/2,1]}(D_1(1)) \subset Int D_1(1/2) \)).

2. \( (v_1)_{[1/2,1]}(S(1/2,2) \cup D_2(1/2)) = D_2(1) \); \( (v_1)_{[1/2,1]}(D_2(1/2)) \subset Int D_2(1) \);

and for some \( \delta > 0 \) the \( (v_1)_{[1/2,1]} \)-image of a \( \delta \)-tubular neighborhood of \( \beta(1/2,2) \) is in \( Int D_2(1) \) (this image equals to \( D_\delta(v_1) \cap F_1^{-1}(1) \)).

Let \( W_0 \) be the cobordism between \( K \) and \( L \), corresponding to the surgery on the circle \( \beta(1/2,1) \). Introduce on \( W_0 \) the corresponding Morse function \( F_0: W_0 \to [0,1/2] \) with one critical point \( y \) of index 1, and such that \( F_0^{-1}(1/2) = L \), \( F_0^{-1}(0) = K \). We can find an \( F_0 \)-gradient \( v_0 \) such that

1. \( (v_0)_{[1/2,0]} \) restricted to \( S(1/2,0) \cup S(1/2,2) \cup D_2(1/2) \) is a diffeomorphism of this surface onto \( D_1(0) \cup S(0,1) \cup S(0,2) \) which identifies \( S(1/2,0) \) with \( S(0,1) \) and \( S(1/2,2) \) with \( S(0,2) \) slightly diminished from the right so that the image of \( D_2(1/2) \) contains \( D_2(0) \) in its interior (and therefore \( (v_0)_{[1/2,0]}(D_2(0)) \subset Int D_2(1/2) \)).

2. \( (v_0)_{[1/2,0]}(S(1/2,1) \cup D_1(1/2)) = D_1(0) \); \( (v_0)_{[1/2,0]}(D_1(1/2)) \subset Int D_1(1) \);

and for some \( \delta > 0 \) the \( (v_0)_{[1/2,0]} \)-image of a \( \delta \)-tubular neighborhood of \( \beta(1/2,2) \) (this image equals to \( D_\delta(v_0) \cap F_0^{-1}(0) \)) is in \( Int D_1(0) \).

Glue together \( W_0 \) and \( W_1 \) along \( L \), denote the resulting cobordism by \( W \); we have

\(^2\)The author does not have a TEX-version of the figure. The figure is available on request from the author.
\[ \partial W = K \cup N. \] Glue the functions \( F_1 \) and \( F_0 \) to a Morse function \( F : W \to [0, 1] \) and the vector fields \( v_1, v_0 \) to an \( F \)-gradient \( v \). We shall now define a ranging system for \((F, v)\). Consider the set \( \Lambda = \{0, 1/2, 1\} \) of regular values, and set

\[
\begin{align*}
A_1 &= D_1(1); & B_1 &= D_2(1); \\
A_{1/2} &= D_1(1/2); & B_{1/2} &= D_2(1/2); \\
A_0 &= D_1(0); & B_0 &= D_2(0);
\end{align*}
\]

The fact that \( \{(A_\lambda, B_\lambda), \lambda \in \Lambda \} \) is a ranging system for \((F, v)\) follows immediately from the properties of \( v_0, v_1 \) cited above.

It is not difficult to compute the homomorphism \( H(v) : H_1(N \setminus D_2(1), D_1(1)) \to H_1(K \setminus D_2(0), D_1(0)) \). Namely, \( [\alpha(1, 1)] \mapsto 0, \quad [\beta(1, 1)] \mapsto 0, \quad [\alpha(1, 2)] \mapsto [\alpha(0, 1)], \quad [\beta(1, 2)] \mapsto [\beta(0, 1)] \).

It is also obvious that the homology class of \( D(x, v) \cap K \) equals to \([\beta(0, 2)] \) and the homology class of \( D(y, -v) \cap N \) equals to \([\beta(1, 1)] \).

Consider the embedded curves \( \alpha(0, 1), \beta(0, 1), \alpha(0, 2), \beta(0, 2) \) in \( K \). Their homology classes form a symplectic basis in \( K \). Consider the isomorphism of \( H_1(K, \mathbb{Z}) \approx \mathbb{Z}^4 \) given in this basis by the matrix

\[
\begin{pmatrix}
0 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
-1 & 3 & 0 & -2
\end{pmatrix}.
\]

It preserves the intersection form, therefore it can be realized by a diffeomorphism \( \Phi : K \to K \) and it is easy to see that we can assume that \( \Phi(x) = x \) for \( x \in D_1(0) \cup D_2(0) \). Denote by \( \Psi \) the composition of \( \Phi \) with the subsequent identification of \( K \) with \( N \).

Denote by \( M \) the 3-manifold obtained by gluing of \( K \) to \( N \) by means of \( \Psi \), and let \( f : M \to S^1 \) be the Morse function obtained from \( F \). The corresponding cyclic covering \( \tilde{M} \) is a union of countably many copies of \( W \), denoted by \( \tilde{W}[i], i \in \mathbb{Z} \), glued together by the diffeomorphisms \( \Psi : K[i] \to N[i - 1] \) of the components of boundaries.

The identification \( W[k] \to W \) will be denoted by \( J_k \). For \( X \subset W \) we denote \( J^{-1}_k(X) \) by \( X[k] \). Define a lifting \( \tilde{F} : \tilde{M} \to \mathbb{R} \) of \( f \) by setting \( \tilde{F}|W[k] = F \circ J_k + k \).

Set \( \Sigma = \{n/2 \mid n \in \mathbb{Z}\} \); for \( n \in \mathbb{Z} \) set \( A_n = D_1(0)[n], B_n = D_2(0)[n] \); for \( n = k + 1/2, k \in \mathbb{Z} \) set \( A_n = D_1(1/2)[k], B_n = D_2(1/2)[k] \). It is obvious that \( \{(A_\sigma, B_\sigma)\}_{\sigma \in \Sigma} \) is a \( t \)-equivariant ranging system for \((\tilde{F}, v)\). Set \( \tilde{x} = J_0^{-1}(x), \tilde{y} = J_0^{-1}(y) \).

We have: \( n_0(\tilde{x}, \tilde{y}; v) = 0, n_1(\tilde{x}, \tilde{y}; v) = \Psi_*([\beta_0(0, 2)]) \neq [\beta(1, 1)] = 0 \). Denote by \( D \) the matrix of \( \Psi_* \circ H(v) \); then

\[
D = \begin{pmatrix}
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 3
\end{pmatrix}.
\]

Then for \( k \geq 1 \) we have \( n_{k+1}(\tilde{x}, \tilde{y}; v) = (\Psi_* \circ H(v))^k(\Psi_*([\beta_0(0, 2)]) \neq [\beta(1, 1)] \).

To abbreviate the notation we denote \([\alpha(1, 1)] \) by \( a_1 \), \([\alpha(1, 2)] \) by \( a_2 \), \([\beta(1, 1)] \) by \( b_1 \), \([\beta(1, 2)] \) by \( b_2 \). Then \( n_{k+1}(\tilde{x}, \tilde{y}; v) \) is the coefficient of \( a_2 \) in \( D^k(b_1 - 2b_2) = D^{k-1} + D^{k-2} + \ldots + D + 1 \)
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\((-2)D^k(b_2)\). To find \(D^k(b_2)\) assume by induction that 
\[D^k(b_2) = \alpha_k a_1 + \gamma_k a_2 + \beta_k b_2.\] Then 
\[D^{k+1}(b_2) = -\gamma_k b_2 + \beta_l(2a_1 + 3\beta_2 + a_2) = 2\beta_k a_1 + (3\beta_k - \gamma_k)b_2 + \beta_k a_2.\] Therefore the vectors \((\beta_k, \gamma_k)\) satisfy

\[
\begin{pmatrix}
\beta_{k+1} \\
\gamma_{k+1}
\end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_k \\
\gamma_k \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \beta_0 \\
\gamma_0 \end{pmatrix} = \begin{pmatrix} 1 \\
0 \end{pmatrix}.
\]

The coefficient \(\alpha_{k+1} = 2\beta_k\).

Denote 
\[
\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix}
\]
by \(C\); then the explicit computation shows

\[
C^n = \frac{1}{\sqrt{5}} \begin{pmatrix} \lambda_1^{n+1} - \lambda_2^{n+1} & \lambda_1^n - \lambda_1^n \\
\lambda_1^n - \lambda_2^n & \lambda_2^{n-1} - \lambda_1^{n-1} \end{pmatrix}
\]

where \(\lambda_1 = \frac{3+\sqrt{5}}{2}, \lambda_2 = \frac{3-\sqrt{5}}{2}\). Therefore

\[
n_{k+1}(\bar{x}, \bar{y}; v) = -\frac{4}{\sqrt{5}} \left( \left( \frac{3+\sqrt{5}}{2} \right)^k - \left( \frac{3-\sqrt{5}}{2} \right)^k \right).
\]
§4. Exponential estimates of the absolute number of trajectories.

A. Bunches of ranging systems.

Let \( f : W \to [a, b] \) be a Morse function on a cobordism \( W, f^{-1}(b) = V_1, f^{-1}(a) = V_0, \dim W = n \). Let \( v \) be an \( f \)-gradient, \( \Lambda = \{ \lambda_0, ..., \lambda_k \} \) be a set of regular values of \( f \), such that \( \lambda_0 = a < \lambda_1 < ... < \lambda_{k-1} < \lambda_k = b \) and between any two adjacent values \( \lambda_i \) and \( \lambda_{i+1} \) there is exactly one critical value of \( f \).

Definition 4.1. Assume that for each integer \( s : 0 \leq s \leq n \) and every \( \lambda \in \Lambda \) there are given compacts \( A^{(s)}_\lambda, B^{(s)}_\lambda \) in \( f^{-1}(\lambda) \). We shall say that \( \{(A^{(s)}_\lambda, B^{(s)}_\lambda)\}_{\lambda \in \Lambda, 0 \leq s \leq n} \) is a bunch of ranging systems for \((f, v)\) (abbreviation: BRS for \((f, v)\)) if the following conditions hold:

1. \( A^{(0)}_\lambda = B^{(0)}_\lambda = \emptyset \)
2. \( r \leq s \Rightarrow \left( A^{(r)}_\lambda \subset A^{(s)}_\lambda, B^{(r)}_\lambda \subset B^{(s)}_\lambda \right) \)
3. \( A^{(r)}_\lambda \cap B^{(s)}_\lambda = \emptyset \) for \( r + s \leq n \)
4. If \( \lambda, \mu \in \Lambda \) are adjacent, then \( v_{\mu, \lambda}^{-1}(A^{(r)}_\mu) \subset \text{Int} A^{(r)}_\lambda \) and \( (-v)^{-1}_{\lambda, \mu}(B^{(s)}_\lambda) \subset \text{Int} B^{(s)}_\mu \) for every \( r, s \).
5. Let \( \lambda, \mu \in \Lambda \) be adjacent. Then for every \( p \in S(f) \cap f^{-1}([\lambda, \mu]) \) we have:
   - i) \( D(p, v) \cap f^{-1}(\lambda) \subset \text{Int} A^{(\text{indp})}_\lambda \)
   - ii) \( D(p, -v) \cap f^{-1}(\mu) \subset \text{Int} B^{(\text{indp})}_\mu \).

Let \( \{(A^{(s)}_\lambda, B^{(s)}_\lambda)\}_{\lambda \in \Lambda, 0 \leq s \leq n} \) be a BRS for \((f, v)\). The following properties are either obvious or proved similarly to the corresponding properties of ranging systems (see §4 of [9]).

i) For every \( s : 0 \leq s \leq n \) the family \( \{(A^{(s)}_\lambda, B^{(n-s-1)}_\lambda)\}_{\lambda \in \Lambda} \) is a ranging system for \((f, v)\).

ii) \( v \) is an almost good \( f \)-gradient

iii) There is \( \epsilon > 0 \) such that for every \( f \)-gradient \( w \) with \( \|w - v\| < \epsilon \) the system \( \{(A^{(s)}_\lambda, B^{(s)}_\lambda)\}_{\lambda \in \Lambda, 0 \leq s \leq n} \) is a BRS for \((f, w)\).

iii) For every \( \delta > 0 \) sufficiently small the following strengthening of 5) holds:

\[
\begin{align*}
D_\delta(p, v) & \cap f^{-1}(\lambda) \subset \text{Int} A^{(\text{indp})}_\lambda \\
D_\delta(p, -v) & \cap f^{-1}(\mu) \subset \text{Int} B^{(\text{indp})}_\mu
\end{align*}
\]

For \( p \in S(f) \) we shall denote by \( \lambda_p \) resp. \( \mu_p \) the maximal resp. minimal element of \( \Lambda \) with the property \( \lambda < f(p) \)(resp. \( \mu > f(p) \)).

Let \( \{(A^{(s)}_\lambda, B^{(s)}_\lambda)\}_{\lambda \in \Lambda, 0 \leq s \leq n} \) be a BRS for \((f, v)\). Choose \( \rho > 0 \) so small that for every \( p \in S(f) \) we have: \( \lambda_p < \lambda_p + \rho < f(p) - \rho < f(p) + \rho < \mu_p - \rho \). Let \( \delta > 0 \) satisfy (5δ) above and assume further that

\[
\left\{ \begin{array}{ll}
(1) & \text{If } p, q \in S(f) \text{ and } f(p) = f(q) \text{ then } D_\delta(p, v) \cap D_\delta(q, v) \cap f^{-1}([\lambda_p, \mu_p]) = \emptyset \\
& \text{and } D_\delta(p, -v) \cap D_\delta(q, -v) \cap f^{-1}([\lambda_p, \mu_p]) = \emptyset.
\end{array} \right.
\]

(2) For every \( p \in S(f) \) we have \( D_\delta(p) \subset f^{-1}([f(p) - \rho, f(p) + \rho]) \).

Let \( \epsilon > 0 \) satisfy (iii) above. The manifold \( S_p = D(p, -v) \cap f^{-1}(\mu_p) \) is an embedded sphere of dimension \( n - \text{ind}_m - 1 \). The normal bundle to \( S_p \) in \( f^{-1}(\mu) \) is trivial.
Chose an embedding \( \psi : S_p \times B^{\text{indp}}(0, \theta) \to f^{-1}(\mu_p) \) (where \( \theta > 0 \)) which is a diffeomorphism onto its image, and such that \( \psi|(S_p \times \{0\}) = \text{id} \).

For \( \varepsilon \in [0, \theta] \) the set \( \psi(S_p \times B^{\text{indp}}(0, \varepsilon)) \) will be denoted by \( \text{Tub}(S_p, \varepsilon) \) and for \( \eta \in B^{\text{indp}}(0, \theta) \) the embedded sphere \( \psi(S_p \times \{\eta\}) \) by \( S_p(\eta) \).

Assume that \( \theta \) is so small that \( \text{Im} \ \psi \subset B_\delta(p, -v) \). Let \( \tau > 0 \) and denote by \( \Psi : [-\tau, 0] \times S_p \times B^{\text{indp}}(0, \theta) \to W \) the map \( (t, s, h) \mapsto \gamma(\psi(s, h), t; v) \). It is a diffeomorphism onto its image. We shall assume that \( \tau \) is so small that \( \text{Im} \ \Psi \subset f^{-1}([\mu_p - \rho, \mu_p]) \).

The vector field \( \Psi^{-1}(v) \) equals to \( (1, 0, 0) \).

Let \( \xi \in B^{\text{indp}}(0, \theta/2) \) and let \( H_\tau(\xi), t \in [-\tau, 0] \) be an isotopy of \( B^{\text{indp}}(0, \theta) \) having the following properties:

1. \( \frac{d}{dt}H_\tau(\xi)(x) = 0 \) for \( t \in [-\tau, -\frac{2\tau}{3}] \cup [-\frac{1}{3}\tau, 0] \).
2. \( H_\tau(\xi)(x) = x \) for \( x \in B^{\text{indp}}(0, \theta) \setminus B^{\text{indp}}(0, \theta/2) \), and for \( t \in [-\tau, -\frac{\tau}{2}] \).
3. \( H_0(\xi)(0) = \xi \).

Then \( \frac{d}{dt}H_{\tau}(\xi) \) is a time-dependant vector field on \( B^{\text{indp}}(0, \theta) \).

Define a vector field \( w(\xi) \) on \( [-\tau, 0] \times S_p \times B^{\text{indp}}(0, \theta) \) by \( w(\xi)(t, x, y) = (1, 0, \frac{d}{dt}H_t(\xi)) \).

Note that

(A) \( \forall x \in S_p \times B^{\text{indp}}(0, \theta/2) \) we have \( \gamma(x, -\tau; w(\xi)) \in \{ -\tau \} \times S_p \times B^{\text{indp}}(0, \theta/2) \).

Choose \( \alpha \in [0, \theta/2] \) so small that for each \( \xi \in B^{\text{indp}}(0, \alpha) \) there is an isotopy \( H_\tau(\xi) \) such that

\[ \|v - \Psi_\tau(w(\xi))\| < \epsilon. \]

We perform this construction for every \( p \in S(f) \). We distinguish between the notation of the corresponding objects by adding in the index the notation of the point, e.g. \( \Psi_p, \xi_p, w_p(\xi_p), \ldots \). We assume that \( \theta_p, \alpha_p, \tau_p \) are chosen to be independent of \( p \), and we denote them simply by \( \theta, \alpha, \tau \).

Chose now for every \( p \in S(f) \) a vector \( \xi_p \in B^{\text{indp}}(0, \alpha) \), and the corresponding vector field \( w_p(\xi_p) \), satisfying (*) . The set \( \{\xi_p\}_{p \in S(f)} \) will be denoted by \( \tilde{\xi} \).

Define a new vector field \( u = v(\tilde{\xi}) \) setting

\[
\begin{align*}
  u(x) = v(x) & \quad \text{if } x \in W \setminus \bigcup_p \text{Im} \ \Psi_p \\
  u(x) = (\Psi_p)_*(w_p(\xi_p))(x) & \quad \text{for } x \in \text{Im} \ \Psi_p
\end{align*}
\]

It follows from the construction that for every \( \tilde{\xi} \) the vector field \( v(\tilde{\xi}) \) is an \( f \)-gradient . Note also that for \( p \in S(f) \) we have \( D(p, -v(\tilde{\xi})) \cap f^{-1}(\mu_p) = S_p(\xi_p) \). The condition (*) implies that \( \{(A^{(1)}_\lambda, B^{(s)}_\lambda)\}_{\lambda \in \Lambda, 0 \leq s \leq n} \) is a BRS for \( (f, v(\tilde{\xi})) \); therefore \( v(\tilde{\xi}) \) is an almost good \( f \)-gradient . It will be called regular perturbation of \( v \) corresponding to \( \tilde{\xi} \).

**Proposition 4.2.**

1. Let \( p, q \in S(f) \), \( \text{indp} = \text{indq} + 1 \). The set of \( -v(\tilde{\xi}) \)-trajectories joining \( p \) with \( q \) is in a bijective correspondence with \( (D(p, v) \cap f^{-1}(\mu_q)) \setminus S_q(\xi_q) \).

2. The vector field \( v(\tilde{\xi}) \) is a good \( f \)-gradient if for every \( p, q \in S(f) \) with \( \text{indp} = \text{indq} + 1 \) the submanifold \( D(p, v) \cap f^{-1}(\mu_q) \) of \( f^{-1}(\mu_q) \) is transversal to \( S_q(\xi_q) \).

\[ \text{We identify here two trajectories which differ by a parameter change.} \]
Proof. 1) Let \( \gamma \) be a \((-v(\xi))-\)trajectory, joining \( p \) with \( q \). Let \( \gamma(t_0) \in f^{-1}(\mu_{q}) \), where \( t_0 \in \mathbb{R} \). I claim that for \( t < t_0 \) we have \( \gamma(t) \notin \text{supp} \left( v-v(\xi) \right) \). Indeed, if the opposite is true, let \( t_1 < t_0 \) be the first moment when \( \gamma \) intersect \( \text{supp} \left( v-v(\xi) \right) \). Then there is \( s \in S(f) \) such that \( \gamma(t_1) \in \text{Tub}(S_{s},\theta/2) \). Note that \( \text{Tub}(S_{s},\theta/2) \subset D_{\delta}(s,-v) \cap f^{-1}(\mu_{s}) \subset \text{Int} B_{\mu_{s}}^{(n-\text{inds})} \), therefore \( \gamma(t_1) \) is in the intersection of \( A_{\mu_{s}}^{(\text{inds})} \) with \( B_{\mu_{s}}^{(n-\text{inds})} \); and \( \text{inpd} \geq \text{inds} + 1 \).

Further, \( f(s) > f(q) \) and \( \lambda_{s} \geq \mu_{q} \). Denote by \( t_2 \) the moment when \( \gamma \) intersects \( f^{-1}(\lambda_{s}) \). Then \( t_0 \geq t_2 > t_1 + \tau \), and it is easy to see that \( \gamma \) does not intersect \( \text{supp} \left( v-v(\xi) \right) \). The property (A) implies that \( \gamma(t_1+\tau) \in D_{\overline{s}}(s,-v) \), and, therefore, \( \gamma(t_2) \in D_{\overline{s}}(s,v) \cap f^{-1}(\lambda_{s}) \subset \text{Int} A_{\lambda_{s}}^{(\text{inds})} \), which implies \( \gamma(t_0) \in \text{Int} A_{\mu_{q}}^{(\text{inds})} \). Therefore \( A_{\mu_{q}}^{(\text{inds})} \cap D_{\overline{s}}(q,-v) \neq \emptyset \) and, further, \( A_{\mu_{q}}^{(\text{inds})} \cap B_{\mu_{q}}^{(n-\text{indq})} \neq \emptyset \), which implies \( \text{inds} \geq \text{indq} \) and \( \text{inpd} \geq \text{inds} + 1 > \text{indq} + 1 \). Contradiction.

Therefore \( \gamma \) is a \((-v)-\)trajectory. In the same way one can show that every \((-v)-\)trajectory joining \( p \) with a point of \( D_{\delta}(q,-v) \cap f^{-1}(\mu_{q}) \) does not intersect \( \text{supp} \left( v-v(\xi) \right) \). That proves 1).

2) Note that \( v(\xi) \) is almost good. Therefore \( v(\xi) \) is good if and only if

\[
\text{inpd} = \text{indq} + 1 \Rightarrow \left( D(p,v(\xi)) \cap D(q,-v(\xi)) \right.
\]

Let \( p, q \in S(f) \), \( \text{inpd} = \text{indq} + 1 \). It suffices to prove that \( D(p,v(\xi)) \cap f^{-1}(\mu_{q}) \) is transversal to \( S_{q}(\xi_{q}) \). Let \( x \) be a point in the intersection of these manifolds. In the part 1) we have proved that there is a \((-v)-\)trajectory, joining \( p \) with \( x \) and not intersecting \( \text{supp} \left( v-v(\xi) \right) \). Then a small neighborhood of this trajectory does not intersect \( \text{supp} \left( v-v(\xi) \right) \) and the transversality sought follows from \( \left( D(p,v) \cap f^{-1}(\mu_{q}) \right) \cap S_{q}(\xi_{q}) \). \( \square \)

B. Volume estimates.

We assume here the terminology of the previous subsection. Let \( \{(A_{\lambda}^{(s)},B_{\lambda}^{(s)})\}_{\lambda \in \Lambda,0 \leq s \leq n} \) be a BRS for \((f,v)\). Assume that \( \delta > 0 \) satisfies (\( \delta \)) from Subsection A. Fix an integer \( s : 0 \leq s \leq n \). Let \( \lambda < \mu \) be adjacent elements of \( \Lambda \). The set

\[
D_{\mu}^{(s)} = f^{-1}(\mu) \setminus \left( \text{Int} A_{\mu}^{(s)} \cup \text{Int} B_{\mu}^{(n-s-1)} \cup \left( \bigcup_{p \in S'} B_{\delta}(p,-v) \cap f^{-1}(\mu) \right) \right)
\]

(where \( S' \) stands for the subset of all \( p \in S(f) \) such that \( f(p) \in ]\lambda,\mu[ \) and \( \text{inpd} \leq s \)) is a compact subset of the domain of definition of \( v_{[\mu,\lambda]}^{(s)} \). Denote by \( N_{\mu}^{(s)} \) the order of the derivative of \( v_{[\mu,\lambda]}^{(s)} \) restricted to \( D_{\mu}^{(s)} \) (that is \( N_{\mu}^{(s)} = \sup_{x \in D_{\mu}^{(s)}} \| (v_{[\mu,\lambda]}^{(s)})'(x) \| \)) and by \( A \) the maximum of \( (N_{\mu}^{(s)})^{k} \) over all \( \mu \in \Lambda,0 \leq s \leq n \), and \( 0 \leq k \leq n \).

For \( p \in S(f) \) denote by \( B_{(p)} \) the norm of the derivative of \( \psi_{p}^{-1} |_{\psi_{p}(S_{p} \times D_{\text{inpd}}^{(0,\theta/2)})} \) (where the manifold \( S_{p} \times B_{\text{inpd}}^{(0,\theta)} \)) is endowed with the product riemannian metric). Denote by \( B \) the maximum of \( B_{(p)} \) over all \( p \in S(f) \).

Lemma 4.3. Let \( \lambda,\mu \in \Lambda, \lambda < \mu \). Let \( s \) be an integer, \( 0 \leq s \leq n \). Let \( N \) be a submanifold of \( f^{-1}(\mu) \setminus B_{\mu}^{(n-s-1)} \) such that \( N \cap A^{(s)} \) is compact. Denote by \( h \) the
number of elements of \( \Lambda \) belonging to \([\lambda, \mu]\). Then \( N' = v_{[\mu, \lambda]}(N) \) is a submanifold of \( f^{-1}(\lambda) \setminus B_{\lambda}^{(n-s-1)} \), such that \( N' \setminus A^{(s)}_{\lambda} \) is compact, and

1) \( \text{vol}(N' \setminus A^{(s)}_{\lambda}) \leq A^{k-1} \cdot \text{vol}(N \setminus A^{(s)}_{\mu}) \);

2) If \( p \in S(f) \) with \( \text{ind} p = s \) and \( \mu_p = \lambda \), then

\[
\text{vol}(\psi_p^{-1}(N' \cap \text{Tub}(S_p, \theta/2))) \leq BA^{k-1} \cdot \text{vol}(N \setminus A^{(s)}_{\mu})
\]

Proof. For the proof of 1) note that it suffices to consider the case \( k = 2 \). For this case we have obviously \( N' \setminus \bar{\Sigma}(\lambda) \subset \bar{v}(N \cap \mathcal{D}_{\mu}^{(s)}) \). 2) follows from 1) since \( \text{Tub}(S_p, \theta/2) \subset B_{\delta}(p, -v) \) does not intersect \( A^{(s)}_{\lambda} \). \( \square \)

C. Bunches of equivariant ranging systems.

We assume here the terminology of Subsection A of Introduction. Further, we assume that the homotopy class of \( f \) is indivisible, that is \( F(xt) = F(x) - 1 \).

**Definition 4.4.** Let \( \Sigma \) be the set of regular values of \( F \) such that

1. \( \sigma \in \Sigma \Rightarrow \sigma + n \in \Sigma \) for every \( n \in \mathbb{Z} \).
2. If \( \sigma_1, \sigma_2 \in \Sigma \) are adjacent, there is exactly one critical value of \( F \) in \( ]\sigma_1, \sigma_2[ \).
3. For every \( A, B \in \mathbb{R} \) the set \( \Sigma \cap [A, B] \) is finite.

Assume that for every integer \( s : 0 \leq s \leq n \) and every \( \sigma \in \Sigma \) there are given compacts \( A^{(s)}_{\sigma}, B^{(s)}_{\sigma} \) in \( F^{-1}(\sigma) \). We shall say that \( \{(A^{(s)}_{\sigma}, B^{(s)}_{\sigma}) \}_{\sigma \in \Sigma, 0 \leq s \leq n} \) is a bunch of \( t \)-equivariant ranging systems (abbreviation: BERS for \((F, v)\)) if:

1. For every \( \mu, \nu \in \Sigma, \mu < \nu \) the system \( \{(A^{(s)}_{\sigma}, B^{(s)}_{\sigma}) \}_{\sigma \in \Sigma, \mu \leq \sigma \leq \nu} \) is a BRS for \((F|F^{-1}([\mu, \nu]), v)\).
2. \( A^{(s)}_{\sigma-n} = A^{(s)}_{\sigma} \cdot t^n, \quad B^{(s)}_{\sigma-n} = B^{(s)}_{\sigma} \cdot t^n \) for every \( n \in \mathbb{Z} \). \( \triangle \)

Up to the end of this subsection we assume that \((F, v)\) has a BERS \( \{(A^{(s)}_{\sigma}, B^{(s)}_{\sigma}) \}_{\sigma \in \Sigma, 0 \leq s \leq n} \). Choose any \( \sigma \in \Sigma \), denote by \( W \) the cobordism \( F^{-1}([\sigma, \sigma + 1]) \); denote \( \Sigma \cap [\sigma, \sigma + 1] \) by \( \Lambda \). We apply the constructions of Subsections A,B to \( W \) and then extend the results to the whole of \( M \) in the \( t \)-invariant way, thus obtaining the sets \( \text{Tub}(S_q, \alpha) \subset F^{-1}(\mu_q) \) for every \( q \in S(F) \). Let \( p, q \in S(F) \); assume that \( F(p) \geq F(q) \) and \( \text{ind} p = \text{ind} q + 1 \). Denote by \( N_{p,q} \) the submanifold of dimension \( \text{ind} q \) of \( S_q \times B^{\text{ind} q}(0, \theta/2) \), defined by

\[
N_{p,q} = \psi_q^{-1}\left( (D(p, v) \cap F^{-1}(\mu_q)) \cap (\text{Tub}(S_q, \theta/2)) \right).
\]

The next lemma follows from 4.3.

**Lemma 4.5.** There are constants \( C, D > 0 \) such that for every \( p, q \in S(F) \) with \( \text{ind} p = \text{ind} q + 1 \) and \( F(p) > F(q) \) we have: \( \text{vol}(N_{p,q}) \leq C \cdot D^{F(p) - F(q)} \). \( \square \)

The next lemma is a direct consequence of Sard Theorem.

**Lemma 4.6.** Let \( q \in S(F) \). Then there is a residual subset \( Q \subset B^{\text{ind} q}(0, \alpha) \) such that for every \( \xi \in Q \) and every \( p \in S(F) \) with \( \text{ind} p = \text{ind} q + 1 \) we have: \( D(p, v) \cap S_q(\xi) \). \( \square \)

The next proposition follows from 4.5 by the argument due to V.I.Arnold (see [1, p.81]).
Proposition 4.7. Let $q \in S(F)$. Then there is a subset $Q \subset B^{indq}(0, \alpha)$ of full measure such that for every $\xi \in Q$ we have:

For every $p \in S(F)$ with $\text{ind} p = \text{ind} q + 1$ there are constants $K, R > 0$, such that for every integer $l \geq 0$ we have:

$$\#(N_{pt-l,q} \cap (S_q \times \{\xi\})) \leq K \cdot R^l. \quad \square$$

Corollary 4.8. Let $\nu > 0$. Then there is a good $f$-gradient $u$ with $\|u - v\| < \nu$ and constants $K, L > 0$ such that for every $p, q \in S(f)$ with $\text{ind} p = \text{ind} q + 1$ we have $N_l(p, q; v) \leq L \cdot K^l$.

Proof. The construction of regular perturbation, described in Subsection A, applied to $v|W$, gives for every $\xi = \{\xi_p\}_{p \in S(F) \cap W}, \xi_p \in B^{indp}(0, \alpha)$ an $F|W$-gradient $v(\xi)$. Since $\text{supp} (v - v(\xi))$ does not intersect $\partial W$, we can extend $v(\xi)$ $t$-invariantly to $M$ and obtain a $t$-invariant $F$-gradient, which will be denoted by the same symbol $v(\xi)$. Note that since $\{(A_{\sigma}^{(s)}, B_{\sigma}^{(s)})\}_{\sigma \in \Sigma, 0 \leq s \leq n}$ is a BERS for $(F, v(\xi))$, the vector field $v(\xi)$ is an almost good $F$-gradient.

If all the $\xi_q \in B^{indq}(0, \alpha)$ are sufficiently small, we shall have $\|u - v(\xi)\| < \nu$. Propositions 4.6, 4.7 imply that we can choose the components $\xi_q$ of $\xi$ in such a way that

1. For every $p \in S(F), \text{ind} p = \text{ind} q + 1$ we have: $D(p, v) \cap F^{-1}(\mu_q) \ni S_q(\xi_q)$.
2. There are $K, R > 0$ such that for every $p \in S(F)$ with $\text{ind} p = \text{ind} q + 1$ we have: $\#(D(pt^{-1}, v) \cap S_q(\xi_q)) \leq K \cdot R^l$.

The proposition 4.2 implies then that $u = v(\xi)$ satisfy the conclusion. \square

D. Proof of Theorem D. We can assume that $S(f) \neq 0$ and that the homotopy class of $f$ is indivisible. In view of Corollary 4.8 it is sufficient to prove that the set of $f$-gradients, having a BERS, is $C^0$ dense in $G(f)$. To prove that, let $v$ be any $f$-gradient and $\epsilon > 0$. Choose any regular value $\lambda$ of $F$ and denote by $W$ the cobordism $F^{-1}([\lambda, \lambda + 1])$. Choose any set of regular values of $F$, satisfying $(\Sigma 1) - (\Sigma 3)$ of Def. 4.4. Theorem 4.13 of [9] imply that there is an almost good $F|W$-gradient $u$ and a ranging pair $(\mathcal{V}_0, \mathcal{V}_1)$ for $(F|W, u)$ such that $\|v - u\| \leq \epsilon$ and $v = u$ near $\partial W$. We can also assume that $\mathcal{V}_{0f} = \mathcal{V}_1$. Then the procedure, described in Construction 4.10 of [9] define a bunch of ranging systems for $(F|W, u)$. Extend $u$ to a $t$-invariant $F$-gradient. The BRS constructed is easily extended to a BERS for $(F, u)$. \square

We mention here an obvious corollary of Theorem D which will be of use in the proof of Theorem E.

For a good $f$-gradient $v, c \in \mathbb{R}$ and $x, y \in S(f)$ with $\text{ind} x = \text{ind} y + 1$ we denote by $N_{\geq c}(x, y; v)$ the sum $\sum_{-k \geq c} N_k(x, y; v)$.

Corollary 4.9. Let $v \in G_0(f)$. Then there are constants $R, Q > 0$ such that for every $x, y \in S(f)$ with $\text{ind} x = \text{ind} y + 1$ and $c \in \mathbb{R}$ we have: $N_{\geq c}(x, y; v) \leq R \cdot Q^{-c}$. \square

E. Proof of the Theorem E.

By Lemma 2.5 find $\epsilon > 0$ and $\delta > 0$ such that $\Omega_\epsilon$ is a Morse family, $v$ is an $\Omega_\epsilon$-gradient and every $\omega$-gradient $u$ with $\|u - v\| < \delta$ is an $\Omega_\epsilon$-gradient. Choose some $\xi \in \Omega_\epsilon$ with $[\xi] \in H^1(M, Q)$ and choose (by Theorem D) a good $\xi$-gradient $u$, satisfying 2) of Theorem D and such that $\|u - v\| < \delta$; $u$ is then an $\Omega_\epsilon$-gradient. By
Lemma 2.11 for every \( x, y \in S(\omega) \) there is an integral \(([\xi], [\omega])\)-cone \( \Delta \) and \( b \in \mathbb{Z}^m \) such that \( Q(I(\tilde{x}, \tilde{y}; u)) \subset \Delta + b \). Lemma 2.12 imply that there are \( A, B \) such that for every \( \lambda \in \mathbb{R} \) we have \( I(\tilde{x}, \tilde{y}; u) \cap \{\omega\}^{-1}(\lambda, \infty] \subset I(\tilde{x}, \tilde{y}; u) \cap \{\xi\}^{-1}(A\lambda + B, \infty] \), and this together with 4.9 implies the conclusion. \( \square \)

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