Vincent Guigues  
School of Applied Mathematics, FGV  
Praia de Botafogo, Rio de Janeiro, Brazil  
vguigues@fgv.br

**Abstract.** We introduce an inexact variant of Stochastic Mirror Descent (SMD), called Inexact Stochastic Mirror Descent (ISMD), to solve nonlinear two-stage stochastic programs where the second stage problem has linear and nonlinear coupling constraints and a nonlinear objective function which depends on both first and second stage decisions. Given a candidate first stage solution and a realization of the second stage random vector, each iteration of ISMD combines a stochastic subgradient descent using a prox-mapping with the computation of approximate (instead of exact for SMD) primal and dual second stage solutions. We propose two variants of ISMD and show the convergence of these variants to the optimal value of the stochastic program. We show in particular that under some assumptions, ISMD has the same convergence rate as SMD. The first variant of ISMD and its convergence analysis are based on the formulas for inexact cuts of value functions of convex optimization problems shown recently in [4]. The second variant of ISMD and the corresponding convergence analysis rely on new formulas that we derive for inexact cuts of value functions of convex optimization problems assuming that the dual function of the second stage problem for all fixed first stage solution and realization of the second stage random vector, is strongly concave. We show that this assumption of strong concavity is satisfied for some classes of problems and present the results of numerical experiments on two simple two-stage problems which show that solving approximately the second stage problem for the first iterations of ISMD can help us obtain a good approximate first stage solution quicker than with SMD.

**Keywords:** Inexact cuts for value functions and Inexact Stochastic Mirror Descent and Strong Concavity of the dual function and Stochastic Programming.

AMS subject classifications: 90C15, 90C90.

1. INTRODUCTION

We are interested in inexact solution methods for two-stage nonlinear stochastic programs of form

\[
\begin{align*}
\text{min } f(x_1) := & f_1(x_1) + Q(x_1) \\
x_1 & \in X_1
\end{align*}
\]

with \(X_1 \subset \mathbb{R}^n\) a convex, nonempty, and compact set, and \(Q(x_1) = E_{\xi_2}[\Omega(x_1, \xi_2)]\) where \(E\) is the expectation operator, \(\xi_2\) is a random vector with probability distribution \(P\) on \(\Xi \subset \mathbb{R}^k\), and

\[
\Omega(x_1, \xi_2) = \left\{ \min_{x_2} f_2(x_2, x_1, \xi_2) \right\} 
\]

\[
\left\{ \begin{array}{l}
x_2 \in X_2(x_1, \xi_2) := \{x_2 \in X_2 : Ax_2 + Bx_1 = b, g(x_2, x_1, \xi_2) \leq 0\}. 
\end{array} \right.
\]

In the problem above vector \(\xi_2\) contains in particular the random elements in matrices \(A, B,\) and vector \(b\). Problem (1.1) is the first stage problem while problem (1.2) is the second stage problem which has abstract constraints \((x_2 \in X_2)\), and linear \((Ax_2 + Bx_1 = b)\) and nonlinear \((g(x_2, x_1, \xi_2) \leq 0)\) constraints which both couple first stage decision \(x_1\) and second stage decision \(x_2\). Our solution methods are suited for the following framework:

a) first stage problem (1.1) is convex;

b) second stage problem (1.2) is convex;

c) for every realization \(\xi_2\) of \(\xi_2\), the primal second stage problem obtained replacing \(\xi_2\) by \(\tilde{\xi}_2\) in (1.2) with optimal value \(\Omega(x_1, \tilde{\xi}_2)\) and its dual are solved approximately.
There is a large literature on solution methods for two-stage risk-neutral stochastic programs. Essentially, these methods can be cast in two categories: (A) decomposition methods based on sampling and cutting plane approximations of $Q$ (which date back to [2], [7] and their variants with regularization such as [14] and (B) Robust Stochastic Approximation [12] and its variants such as stochastic Primal-Dual subgradient methods [8], Stochastic Mirror Descent (SMD) [11], [9], or Multistep Stochastic Mirror Descent (MSMD) [5]. However, for all these methods, it is assumed that second stage problems are solved exactly. This latter assumption is not satisfied when the second stage problem is nonlinear since in this setting only approximate solutions are available. On top of that, for the first iterations, we still have crude approximations of the first stage solution and it may be useful to solve inexactly, with less accuracy, the second stage problem for these iterations and to increase the accuracy of the second stage solutions computed when the algorithm progresses in order to decrease the overall computational bulk.

Therefore the objective of this paper is to fill a gap considering the situation when second stage problems are nonlinear and solved approximately (both primal and dual, see Assumption c) above). More precisely, to account for Assumption (c), as an extension of the methods from class (B) we derive an Inexact Stochastic Mirror Descent (ISMD) algorithm, designed to solve problems of form (1.1). This inexact solution method is based on an inexact black box for the objective in (1.1). To this end, we compute inexact cuts (affine lower bounding functions) for the value function of a convex optimization problem with the argument of the value function appearing in the objective and the linear and nonlinear constraints, on the basis of approximate primal and dual solutions. For this analysis, we first need formulas for exact cuts (cuts based on exact primal and dual solutions). We had shown such formulas in [3, Lemma 2.1] using convex analysis tools, in particular standard calculus on normal and tangent cones. We derive in Proposition 3.2 a proof for these formulas based purely on duality. This is an adaptation of the proof of the formulas we gave in [4] Proposition 2.7 for inexact cuts, considering exact solutions instead of inexact solutions. To our knowledge, the computation of inexact cuts for value functions has only been discussed in [4] so far (see Proposition 3.7). We propose in Section 3 new formulas for computing inexact cuts based in particular on the strong concavity of the dual function. In Section 2 we provide, for several classes of problems, conditions ensuring that the dual function of an optimization problem is strongly concave and give formulas for computing the corresponding constant of strong concavity when possible. It turns out that our results improve Theorem 10 in [15] (the only reference we are aware of on the strong concavity of the dual function) which proves the strong concavity of the dual function under stronger assumptions. The tools developed in Sections 2 and 3 allow us to build the inexact black boxes necessary for the Inexact Stochastic Mirror Descent (ISMD) algorithm and its convergence analysis presented in Section 4. Finally, in Section 5 we report the results of numerical tests comparing the performance of SMD and ISMD on two simple two-stage nonlinear stochastic programs.

- The domain $\text{dom}(f)$ of a function $f : X \to \mathbb{R}$ is the set of points in $X$ such that $f$ is finite: $\text{dom}(f) = \{x \in X : -\infty < f(x) < +\infty\}$.
- The largest (resp. smallest) eigenvalue of a matrix $Q$ having real-valued eigenvalues is denoted by $\lambda_{\text{max}}(Q)$ (resp. $\lambda_{\text{min}}(Q)$).
- The $\| \cdot \|_2$ of a matrix $A$ is given by $\|A\|_2 = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$.
- $\text{Diag}(x_1, x_2, \ldots, x_n)$ is the $n \times n$ diagonal matrix whose entry $(i, i)$ is $x_i$.
- For a linear application $A$, Ker($A$) is its kernel and Im($A$) its image.
- $f^*$ is the conjugate of function $f$.
- $\langle \cdot , \cdot \rangle$ is the usual scalar product in $\mathbb{R}^n$: $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$ which induces the norm $\|x\|_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$.

2. ON THE STRONG CONCAVITY OF THE DUAL FUNCTION OF AN OPTIMIZATION PROBLEM

The study of the strong concavity of the dual function of an optimization problem on some set is a problem of convex analysis with applications in numerical optimization. For instance, the strong concavity of the dual function and the knowledge of the associated constant of strong concavity are used by the Drift-Plus-Penalty algorithm in [15] and by the (convergence proof of) Inexact SMD algorithm presented in Section 4 when inexact cuts are computed using Proposition 3.8.
The only paper we are aware of providing conditions ensuring this strong concavity property is \cite{15}. In this section, we prove similar results under weaker assumptions and study an additional class of problems (quadratic with a quadratic constraint, see Proposition \ref{pro:2.11}).

2.1. Preliminaries. In what follows, $X \subset \mathbb{R}^n$ is a nonempty convex set.

**Definition 2.1** (Strongly convex functions). Function $f : X \to \mathbb{R} \cup \{+\infty\}$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$ if for every $x, y \in \text{dom}(f)$ we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - \frac{\alpha t(1-t)}{2} \|y - x\|^2,$$

for all $0 \leq t \leq 1$.

We can deduce the following well known characterizations of strongly convex functions $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$:

**Proposition 2.2.** (i) Function $f : X \to \mathbb{R} \cup \{+\infty\}$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$ if and only if for every $x, y \in \text{dom}(f)$ we have

$$f(y) \geq f(x) + s^T(y - x) + \frac{\alpha}{2} \|y - x\|^2, \quad \forall s \in \partial f(x).$$

(ii) Function $f : X \to \mathbb{R} \cup \{+\infty\}$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$ if and only if for every $x, y \in \text{dom}(f)$ we have

$$f(y) \geq f(x) + f'(x)(y - x) + \frac{\alpha}{2} \|y - x\|^2,$$

where $f'(x; y - x)$ denotes the derivative of $f$ at $x$ in the direction $y - x$.

(iii) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be differentiable. Then $f$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$ if and only if for every $x, y \in \text{dom}(f)$ we have

$$(\nabla f(y) - \nabla f(x))^T(y - x) \geq \alpha \|y - x\|^2.$$

(iv) Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be twice differentiable. Then $f$ is strongly convex on $X \subset \mathbb{R}^n$ with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$ if and only if $-f$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$.

**Definition 2.3** (Strongly concave functions). $f : X \to \mathbb{R} \cup \{-\infty\}$ is strongly concave with constant of strong concavity $\alpha > 0$ with respect to norm $\| \cdot \|$ if and only if $-f$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$.

The following propositions are immediate and will be used in the sequel:

**Proposition 2.4.** If $f : X \to \mathbb{R} \cup \{+\infty\}$ is strongly convex with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$ and $\ell : \mathbb{R}^n \to \mathbb{R}$ is linear then $f + \ell$ is strongly convex on $X$ with constant of strong convexity $\alpha > 0$ with respect to norm $\| \cdot \|$.

**Proposition 2.5.** Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^n$, be two nonempty convex sets. Let $A : X \to Y$ be a linear operator and let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a strongly convex function with constant of strong convexity $\alpha > 0$ with respect to a norm $\| \cdot \|_n$ on $\mathbb{R}^n$ induced by scalar product $\langle \cdot, \cdot \rangle_n$ on $\mathbb{R}^n$. Assume that $\text{Ker}(A^* \circ A) = \{0\}$. Then $g = f \circ A$ is strongly convex on $X$ with constant of strong convexity $\alpha \lambda_{\min}(A^* \circ A)$ with respect to norm $\| \cdot \|_m$.

**Proof.** For every $x, y \in X$, using Proposition \ref{pro:2.2}(ii) we have

$$f(A(y)) \geq f(A(x)) + f'(A(x); A(y - x)) + \frac{\alpha}{2} \|A(y - x)\|^2_n$$

and since $g'(x; y - x) = f'(A(x); A(y - x))$, we get

$$g(y) \geq g(x) + g'(x; y - x) + \frac{1}{2} \alpha \lambda_{\min}(A^* \circ A) \|y - x\|^2_m$$

with $\alpha \lambda_{\min}(A^* \circ A) > 0$ ($\lambda_{\min}(A^* \circ A)$ is nonnegative because $A^* \circ A$ is self-adjoint and it cannot be zero because $A^* \circ A$ is nondegenerate). \hfill \Box
In the rest of this section, we fix \( \| \cdot \| = \| \cdot \|_2 \) and provide, under some assumptions, the constant of strong concavity of the dual function of an optimization problem for this norm.\(^1\)

### 2.2. Problems with linear constraints

Consider the optimization problem

\[
(2.3) \quad \begin{cases} 
\inf f(x) \\
Ax \leq b 
\end{cases}
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, b \in \mathbb{R}^q \), and \( A \) is a \( q \times n \) real matrix.

We will use the following known fact, see for instance [13]:

**Proposition 2.6.** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a proper convex lower semicontinuous function. Then \( f^* \) is strongly convex with constant \( \alpha > 0 \) for norm \( \| \cdot \|_2 \) if and only if \( f \) is differentiable and \( \nabla f \) is Lipschitz continuous with constant \( \alpha \) for norm \( \| \cdot \|_2 \).

**Proposition 2.7.** Let \( \theta \) be the dual function of \((2.3)\) given by

\[
(2.4) \quad \theta(\lambda) = \inf_{x \in \mathbb{R}^n} \{ f(x) + \lambda^T (Ax - b) \},
\]

for \( \lambda \in \mathbb{R}^q \). Assume that the rows of matrix \( A \) are independent, that \( f \) is convex, differentiable, and \( \nabla f \) is Lipschitz continuous with constant \( L \geq 0 \) with respect to norm \( \| \cdot \|_2 \). Then dual function \( \theta \) is strongly concave on \( \mathbb{R}^q \) with constant of strong concavity \( \frac{\lambda_{\text{min}}(AA^T)}{L} \) with respect to norm \( \| \cdot \|_2 \) on \( \mathbb{R}^q \).

**Proof.** The dual function of \((2.3)\) can be written

\[
(2.5) \quad \theta(\lambda) = \inf_{x \in \mathbb{R}^n} \{ f(x) + \lambda^T (Ax - b) \} = -\lambda^T b - \sup_{x \in \mathbb{R}^n} \{ -x^T A^T \lambda - f(x) \} = -\lambda^T b - f^*(-A^T \lambda) \text{ by definition of } f^*.
\]

Since the rows of \( A \) are independent, matrix \( AA^T \) is invertible and \( \text{Ker}(AA^T) = \{0\} \). The result follows from the above representation of \( \theta \) and Propositions 2.4, 2.5, and 2.6. \( \square \)

The strong concavity of the dual function of \((2.3)\) was shown in Corollary 5 in [15] assuming that \( f \) is second-order continuously differentiable and strongly convex. Therefore Proposition 2.7 (whose proof is very short), which only assumes that \( f \) is convex, differentiable, and has Lipschitz continuous gradient, improves existing results (neither second-order differentiability nor strong convexity is required). Below, we discuss several examples. In particular, in Examples 2.9 and 2.10, \( f \) may not be strongly convex.

**Example 2.8 (Linear programs).** Let \( f : \mathbb{R}^n \to \mathbb{R} \) given by

\[
(2.6) \quad f(x) = c^T x + c_0
\]

where \( c \in \mathbb{R}^n \), \( c_0 \in \mathbb{R} \). Clearly \( f \) is convex differentiable with Lipschitz continuous gradients; any \( L \geq 0 \) being a valid Lipschitz constant. **Proposition 2.7** tells us that if the rows of \( A \) are independent then dual function \( \theta \) of \((2.3)\) given by \((2.4)\) is strongly concave on \( \mathbb{R}^q \). In this case, the strong concavity can be checked directly computing \( \theta \). Indeed, we have

\[
(2.7) \quad f^*(x) = \begin{cases} 
-c_0 & \text{if } x = c, \\
+\infty & \text{if } x \neq c,
\end{cases}
\]

and plugging this expression of \( f^* \) into \((2.5)\), we get

\[
(2.8) \quad \theta(\lambda) = \begin{cases} 
-\lambda^T b - c_0 & \text{if } A^T \lambda = -c, \\
-\infty & \text{if } A^T \lambda \neq -c.
\end{cases}
\]

Therefore if \( c \in \text{Im}(A^T) \) then there is \( \lambda \in \mathbb{R}^q \) such that

\[
(2.9) \quad A^T \lambda = -c,
\]

and if the rows of \( A \) are independent then there is only one \( \lambda \), let us call it \( \lambda_0 \), satisfying \((2.9)\). In this situation, the domain of \( \theta \) is a singleton: \( \text{dom}(\theta) = \{\lambda_0\} \), and \( \theta \) indeed is strongly convex (see Definition 2.1). If \( c \notin \text{Im}(A^T) \) then \( \text{dom}(\theta) = \emptyset \) and \( \theta \) is again strongly convex.

\(^1\)Using the equivalence between norms in \( \mathbb{R}^n \), we can derive a valid constant of strong concavity for other norms, for instance \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_1 \).

\(^2\)In this simple case, the dual function is well known and can also be obtained without using the conjugate of \( f \).
The example which follows gives a class of problems where the dual function is strongly concave on $\mathbb{R}^q$:

**Example 2.9 (Quadratic convex programs).** Consider a problem of form (2.3) where $f(x) = \frac{1}{2}x^TQ_0x + a_0^Tx + b_0$, $Q_0$ is an $n \times n$ nonsingular semidefinite positive matrix, $A$ is a $q \times n$ real matrix, $a_0 \in \text{Im}(Q_0)$, and $b_0 \in \mathbb{R}$. Clearly, $f$ is convex, differentiable, and $\nabla f$ is Lipschitz continuous with Lipschitz constant $L = \|Q_0\|_2 = \lambda_{\text{max}}(Q_0) > 0$ with respect to $\|\cdot\|_2$ on $\mathbb{R}^n$. If the rows of $A$ are independent, using Proposition 2.7 we obtain that the dual function of (2.3) is strongly concave with constant of strong concavity $\frac{\lambda_{\text{min}}(AA^T)}{\lambda_{\text{max}}(Q_0)} > 0$ with respect to norm $\|\cdot\|_2$ on $\mathbb{R}$. Observe that strong concavity holds in particular if $Q_0$ is not definite positive, in which case $f$ is not strongly convex.

Since $f$ is convex, differentiable, its gradient being Lipschitz continuous with Lipschitz constant $L$, from Proposition 2.6 we know that $f^*$ is strongly convex with constant of strong convexity $1/\lambda_{\text{max}}(Q_0)$. This can be checked by direct computation. Indeed, let $\lambda_{\text{max}}(Q_0) = \lambda_1(Q_0) \geq \lambda_2(Q_0) \geq \ldots \geq \lambda_r(Q_0) > \lambda_{r+1}(Q_0) = \lambda_{r+2}(Q_0) = \ldots = \lambda_n(Q_0) = 0$ be the ordered eigenvalues of $Q_0$ where $r$ is the rank of $Q_0$. Let $P$ be a corresponding orthogonal matrix of eigenvectors for $Q_0$, i.e., $\text{Diag}(\lambda_1(Q_0), \ldots, \lambda_n(Q_0)) = P^TQ_0P$ with $P^TP = P^TP = I_n$. Defining

$$Q_0^+ = P \text{Diag} \left( \frac{1}{\lambda_1(Q_0)}, \ldots, \frac{1}{\lambda_r(Q_0)}, 0, \ldots, 0 \right) P^T,$$

it is straightforward to check that

$$f^*(x) = \begin{cases} -b_0 + \frac{1}{2}(x - a_0)^TQ_0^+(x - a_0) & \text{if } x \in \text{Im}(Q_0), \\ +\infty & \text{otherwise}, \end{cases}$$

and plugging expression (2.8) of $f^*$ into (2.5), we get

$$\theta(\lambda) = \begin{cases} b_0 - \lambda^Tb - \frac{1}{2}(a_0 + A^T\lambda)^TQ_0^+(a_0 + A^T\lambda) & \text{if } A^T\lambda \in \text{Im}(Q_0), \\ -\infty & \text{otherwise}. \end{cases}$$

If $x' = (x'_1, \ldots, x'_n)$ is the vector of the coordinates of $x$ in the basis $(v_1, v_2, \ldots, v_n)$ where $v_i$ is $i$th column of $P = [v_1, v_2, \ldots, v_n]$ (i.e., $(v_1, \ldots, v_r)$ is a basis of $\text{Im}(Q_0)$ and $(v_{r+1}, \ldots, v_n)$ is a basis of $\text{Ker}(Q_0)$) and writing $a_0 = \sum_{i=1}^r a'_0, v_i$, we obtain

$$f^*(x) = \begin{cases} g(P^Tx) \text{ where } g : \mathbb{R}^n \to \mathbb{R} \text{ is given by } g(x') = -b_0 + \sum_{i=1}^r \frac{(x'_i - a'_0)^2}{2\lambda_i(Q_0)} & \text{if } x \in \text{Im}(Q_0), \\ +\infty & \text{otherwise}. \end{cases}$$

Observe that for $x', y' \in \mathbb{R}^r \times \{(0, \ldots, 0)\}$ we have

$$g(y') \geq g(x') + \nabla g(x')^T(y' - x') + \frac{1}{2\lambda_1(Q_0)}\|y' - x'\|^2$$

and $g$ is strongly convex with constant of strong convexity $\frac{1}{\lambda_1(Q_0)}$ with respect to norm $\|\cdot\|_2$ on $\mathbb{R}^r \times \{(0, \ldots, 0)\}$. Recalling that $f^*(x) = g(P^Tx)$ for $x \in \text{dom}(f^*) = \text{Im}(Q_0)$, that $P^Tx \in \mathbb{R}^r \times \{(0, \ldots, 0)\}$ for $x \in \text{Im}(Q_0)$, and using Proposition 2.5 we get that $f^*$ is strongly convex with constant of strong convexity

$$\frac{\lambda_{\text{min}}(P^TP)}{\lambda_1(Q_0)} = \frac{\lambda_{\text{min}}(I_n)}{\lambda_{\text{max}}(Q_0)} = \frac{1}{\lambda_{\text{max}}(Q_0)}$$

with respect to norm $\|\cdot\|_2$.

**Example 2.10.** Let $f(x) = \sum_{k=1}^M \alpha_k f_k(x)$ for $\alpha_k \in \mathbb{R}$ and $f_k : \mathbb{R}^n \to \mathbb{R}$ convex differentiable with Lipschitz constant $L_k \geq 0$ with respect to norm $\|\cdot\|_2$ on $\mathbb{R}^n$ for $k = 1, \ldots, M$. Let $A$ be a $q \times n$ matrix with independent rows. Then dual function (2.4) of (2.3) is strongly concave on $\mathbb{R}^q$ with constant of strong concavity $\frac{\lambda_{\text{min}}(AA^T)}{\sum_{k=1}^M \alpha_k L_k}$ with respect to $\|\cdot\|_2$. 

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2.3. Problems with quadratic objective and a quadratic constraint. We now consider the following quadratic quadratically constrained optimization problem

\[(2.9)\]

\[
\begin{align*}
\inf_{x \in \mathbb{R}^n} & \ f(x) := \frac{1}{4} x^T Q_0 x + a_0^T x + b_0 \\
\text{subject to} & \ g_i(x) := \frac{1}{2} x^T Q_i x + a_i^T x + b_i \leq 0, \ i = 1, \ldots, p,
\end{align*}
\]

with \(Q_0\) definite positive and \(Q_i, i = 1, \ldots, p\), semidefinite positive. The dual function \(\theta\) of this problem is known in closed-form: for \(\mu \in \mathbb{R}^p, \mu \geq 0\), we have

\[(2.10)\]

\[\theta(\mu) = \inf_{x \in \mathbb{R}^n} \{ f(x) + \sum_{i=1}^{p} \mu_i g_i(x) \} = -\frac{1}{2} A(\mu)^T Q(\mu)^{-1} A(\mu) + B(\mu)\]

where \(A(\mu) = a_0 + \sum_{i=1}^{p} \mu_i a_i, \ Q(\mu) = Q_0 + \sum_{i=1}^{p} \mu_i Q_i, \) and \(B(\mu) = b_0 + \sum_{i=1}^{p} \mu_i b_i\).

When there is only one quadratic constraint in \((2.9)\), we can show, under some assumptions, that dual function \(\theta\) is strongly concave on some set and compute analytically the corresponding constant of strong concavity.

**Remark 2.12.** Making the change of variables \(x = y - Q_0^{-1} a_1\), we can rewrite \((2.9)\) without linear terms in \(g_1\). Therefore, we have not lost generality in Proposition 2.11 assuming \(a_1 = 0\).

2.4. General case: problems with linear and nonlinear constraints. Let us add to problem \((2.3)\) nonlinear constraints. More precisely, given \(f : \mathbb{R}^n \to \mathbb{R}, a q \times n\) real matrix \(A, b \in \mathbb{R}^q\), and \(g : \mathbb{R}^n \to \mathbb{R}^p\) with convex component functions \(g_i, i = 1, \ldots, p\), we consider the optimization problem

\[(2.11)\]

\[
\begin{align*}
\inf f(x) \\
x \in X, Ax \leq b, g(x) \leq 0.
\end{align*}
\]

Let \(v\) be the value function of this problem given by

\[(2.12)\]

\[v(c) = v(c_1, c_2) = \inf_{x \in X} f(x) \quad x \in X, Ax - b + c_1 \leq 0, g(x) + c_2 \leq 0,\]

for \(c_1 \in \mathbb{R}^q, c_2 \in \mathbb{R}^p\). In the next lemma, we relate the conjugate of \(v\) to the dual function

\[\theta(\lambda, \mu) = \inf_{x \in X} f(x) + \lambda^T (Ax - b) + \mu^T g(x)\]

of this problem:

**Lemma 2.13.** If \(v^*\) is the conjugate of the value function \(v\) then \(v^*(\lambda, \mu) = -\theta(\lambda, \mu)\) for every \((\lambda, \mu) \in \mathbb{R}_+^q \times \mathbb{R}_+^p\).
Proof. For \((\lambda, \mu) \in \mathbb{R}^q \times \mathbb{R}^p_+\), we have

\[
-v^*(\lambda, \mu) = -\sup_{(c_1, c_2) \in \mathbb{R}^q \times \mathbb{R}^p} \lambda^T c_1 + \mu^T c_2 - v(c_1, c_2)
\]

\[
= \begin{cases} 
\inf -\lambda^T c_1 - \mu^T c_2 + f(x) & x \in X, Ax - b + c_1 \leq 0, g(x) + c_2 \leq 0, \\
\in \mathbb{R}^q, c_2 \in \mathbb{R}^p, \\
\inf f(x) + \lambda^T (Ax - b) + \mu^T g(x) & x \in X, \\
\end{cases}
\]

\[
= \theta(\lambda, \mu).
\]

From Lemma 2.13 and Proposition 2.6 we obtain that dual function \(\theta\) of problem (2.11) is strongly concave with constant \(\alpha\) with respect to norm \(\| \cdot \|_2\) on \(\mathbb{R}^{p+q}\) if and only if the value function \(v\) given by (2.12) is differentiable and \(\nabla v\) is Lipschitz continuous with constant \(1/\alpha\) with respect to norm \(\| \cdot \|_2\) on \(\mathbb{R}^{p+q}\). Using Lemma 2.1 in [3] the subdifferential of the value function is the set of optimal dual solutions of (2.12). Therefore \(\theta\) is strongly concave with constant \(\alpha\) with respect to norm \(\| \cdot \|_2\) on \(\mathbb{R}^{p+q}\) if and only if the value function is differentiable and the dual solution of (2.12) seen as a function of \((c_1, c_2)\) is Lipschitz continuous with Lipschitz constant \(1/\alpha\) with respect to norm \(\| \cdot \|_2\) on \(\mathbb{R}^{p+q}\).

We now provide conditions ensuring that the dual function is strongly concave in a neighborhood of the optimal dual solution making easily verifiable assumptions.

Theorem 2.14. Consider the optimization problem

(2.13) \[
\inf_{x \in \mathbb{R}^n} \{ f(x) : Ax \leq b, g_i(x) \leq 0, i = 1, \ldots, p \}.
\]

We assume that

(A1) \(f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) is strongly convex and has Lipschitz continuous gradient;

(A2) \(g_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}, i = 1, \ldots, p\), are convex and have Lipschitz continuous gradients;

(A3) if \(x_*\) is the optimal solution of (2.13) then the rows of matrix \(\begin{pmatrix} A \\ J_g(x_*) \end{pmatrix}\) are linearly independent

where \(J_g(x)\) denotes the Jacobian matrix of \(g(x) = (g_1(x), \ldots, g_p(x))\) at \(x\);

(A4) there is \(x_0 \in \text{ri}(\{g \leq 0\})\) such that \(Ax_0 \leq b\).

Let \(\theta\) be the dual function of this problem:

(2.14) \[
\theta(\lambda, \mu) = \begin{cases} 
\inf f(x) + \lambda^T (Ax - b) + \mu^T g(x) & x \in \mathbb{R}^n. \\
\end{cases}
\]

Let \((\lambda_*, \mu_*) \geq 0\) be an optimal solution of the dual problem

\[
\sup_{\lambda \geq 0, \mu \geq 0} \theta(\lambda, \mu).
\]

Then there is some neighborhood \(\mathcal{N}\) of \((\lambda_*, \mu_*)\) such that \(\theta\) is strongly concave on \(\mathcal{N} \cap \mathbb{R}^{p+q}_+\).

Proof. Due to (A1) the optimization problem (2.14) has a unique optimal solution that we denote by \(x(\lambda, \mu)\). Assumptions (A2) and (A3) imply that there is some neighborhood \(\mathcal{V}_\varepsilon(x_*)\) of \(\{x \in \mathbb{R}^n : \|x - x_*\|_2 \leq \varepsilon\}\) of \(x_*\) for some \(\varepsilon > 0\) such that the rows of matrix \(\begin{pmatrix} A \\ J_g(x) \end{pmatrix}\) are independent for \(x\) in \(\mathcal{V}_\varepsilon(x_*)\).

We argue that \((\lambda, \mu) \to x(\lambda, \mu)\) is continuous on \(\mathbb{R}^q \times \mathbb{R}^p\). Indeed, let \((\lambda, \mu) \in \mathbb{R}^q \times \mathbb{R}^p\) and take a sequence \((\lambda_k, \mu_k)\) converging to \((\lambda, \mu)\). Take an arbitrary accumulation point \(\bar{x}\) of the sequence \(x(\lambda_k, \mu_k)\), i.e., \(\bar{x} = \lim_{k \to +\infty} x(\lambda_k, \mu_k)\) for some subsequence \(x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})\) of \(x(\lambda_k, \mu_k)\). Then by definition of \(x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})\), for every \(x \in \mathbb{R}^n\) and every \(k \geq 1\) we have

\[
f(x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})) + \lambda_{\sigma(k)}^T (Ax(\lambda_{\sigma(k)}, \mu_{\sigma(k)}) - b) + \mu_{\sigma(k)}^T g(x(\lambda_{\sigma(k)}, \mu_{\sigma(k)})) \leq f(x) + \lambda_{\sigma(k)}^T (Ax - b) + \mu_{\sigma(k)}^T g(x).
\]

Passing to the limit in the inequality above and using the continuity of \(f\) and \(g_i\) we obtain for all \(x \in \mathbb{R}^n\):

\[
f(\bar{x}) + \tilde{\lambda}^T (A\bar{x} - b) + \tilde{\mu}^T g(\bar{x}) \leq f(x) + \tilde{\lambda}^T (Ax - b) + \tilde{\mu}^T g(x),
\]
which shows that \( \bar{x} = x(\bar{\lambda}, \bar{\mu}) \). Therefore there is only one accumulation point \( \bar{x} = x(\bar{\lambda}, \bar{\mu}) \) for the sequence \( x(\lambda_k, \mu_k) \) which shows that this sequence converges to \( x(\bar{\lambda}, \bar{\mu}) \). Therefore we have shown that \( (\lambda, \mu) \rightarrow x(\lambda, \mu) \) is continuous on \( \mathbb{R}^p \times \mathbb{R}^q \). This implies that there is a neighborhood \( \mathcal{N}(\lambda_*, \mu_*) \) of \( (\lambda_*, \mu_*) \) such that for \( (\lambda, \mu) \in \mathcal{N}(\lambda_*, \mu_*) \) we have \( \|x(\lambda, \mu) - x(\lambda_*, \mu_*)\|_2 \leq \varepsilon \). Moreover, due to (A4), we have \( x(\lambda_*, \mu_*) = x_* \).

It follows that for \( (\lambda, \mu) \in \mathcal{N}(\lambda_*, \mu_*) \) we have \( \|x(\lambda, \mu) - x(\lambda_*, \mu_*)\|_2 = \|x(\lambda, \mu) - x_*\|_2 \leq \varepsilon \) which in turn implies that the rows of matrix \( \begin{pmatrix} A \\ J_g(x(\lambda, \mu)) \end{pmatrix} \) are independent. We now show that \( \theta \) is strongly concave on \( \mathcal{N}(\lambda_*, \mu_*) \cap \mathbb{R}_+^{p+q} \).

Take \( (\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \mathcal{N}(\lambda_*, \mu_*) \cap \mathbb{R}_+^{p+q} \) and denote \( x_1 = x(\lambda_1, \mu_1) \) and \( x_2 = x(\lambda_2, \mu_2) \). The optimality conditions give

\[
\nabla \theta(\lambda, \mu) = \begin{pmatrix} Ax(\lambda, \mu) - b \\ g(x(\lambda, \mu)) \end{pmatrix}
\]

and we obtain, using the notation \( \langle x, y \rangle = x^T y \):

\[
\begin{aligned}
\nabla \theta(\lambda_2, \mu_2) - \nabla \theta(\lambda_1, \mu_1), & \quad \left( \begin{array}{c} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{array} \right) \\
\langle x_2 - x_1, A^T x_1 \rangle + & \quad \langle J_g(x_1) \rangle^T \mu_1 - J_g(x_2) \rangle^T \mu_2, x_2 - x_1).
\end{aligned}
\]

By convexity of constraint functions we can write for \( i = 1, \ldots, p \) :

\[
\begin{aligned}
g_i(x_2) & \geq g_i(x_1) + \langle \nabla g_i(x_1), x_2 - x_1 \rangle \\
g_i(x_1) & \geq g_i(x_2) + \langle \nabla g_i(x_2), x_1 - x_2 \rangle.
\end{aligned}
\]

Multiplying (2.17)-(a) by \( \mu_1(i) \geq 0 \) and (2.17)-(b) by \( \mu_2(i) \geq 0 \) we obtain

\[
\begin{aligned}
\langle x_2 - x_1, A^T x_1 \rangle + & \quad \langle J_g(x_1) \rangle^T \mu_1 - J_g(x_2) \rangle^T \mu_2, x_2 - x_1).
\end{aligned}
\]

Recalling (A1), we can find \( 0 \leq L(f) < +\infty \) such that for all \( x, y \in \mathbb{R}^n \):

\[
\|\nabla f(y) - \nabla f(x)\|_2 \leq L(f)\|y - x\|_2.
\]

Using (2.16) and (2.18) and denoting by \( \alpha > 0 \) the constant of strong convexity of \( f \) with respect to norm \( \| \cdot \|_2 \) we get:

\[
\langle x_2 - x_1, A^T x_1 \rangle + \langle J_g(x_1) \rangle^T \mu_1 - J_g(x_2) \rangle^T \mu_2, x_2 - x_1).
\]

Now recall that for every \( x \in \mathcal{V}_c(x_*) \) the rows of the matrix \( \begin{pmatrix} A \\ J_g(x) \end{pmatrix} \) are independent and therefore the matrix \( \begin{pmatrix} A \\ J_g(x) \end{pmatrix} \) is invertible. Moreover, the function \( x \rightarrow \lambda_{\min} \left( \begin{pmatrix} A \\ J_g(x) \end{pmatrix} \right)^T \) is continuous (due to (A2)) and positive on the compact set \( \mathcal{V}_c(x_*) \). It follows that we can define

\[
\tilde{\lambda}_c(x_*) = \min_{x \in \mathcal{V}_c(x_*)} \lambda_{\min} \left( \begin{pmatrix} A \\ J_g(x) \end{pmatrix} \right)^T,
\]

and \( \tilde{\lambda}_c(x_*) > 0 \). Since \( x_2 \in \mathcal{V}_c(x_*) \), we deduce that

\[
\|a\|_2 \geq \sqrt{\tilde{\lambda}_c(x_*)} \left( \begin{array}{c} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{array} \right)_2.
\]
Recalling that \((\lambda_1, \mu_1)\) is in \(N(\lambda_*, \mu_*)\), we obtain for \(|\mu_1|_1\) the bound
\[
(2.22) \quad |\mu_1|_1 \leq U_\varepsilon(x_*) := |\mu_*|_1 + \frac{\varepsilon\alpha}{L}.
\]
Due to (A2), there is \(L(g) \geq 0\) such that for every \(x, y \in \mathbb{R}^n\), we have
\[
\|\nabla g_i(y) - \nabla g_i(x)\|_2 \leq L(g)\|y - x\|_2, 1, \ldots, p.
\]
Combining this relation with \((2.22)\), we get
\[
(2.23) \quad \|b\|_2 \leq |\mu_1|_1 L(g)\|x_2 - x_1\|_2 \leq L(g)U_\varepsilon(x_*)\|x_2 - x_1\|_2.
\]
Therefore
\[
\|a + b\|_2 \geq |\mu_1|_1 \|x_2 - x_1\|_2 \geq \sqrt{A_\varepsilon(x_*)}\left\|\left(\begin{array}{c} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{array}\right) - L(g)U_\varepsilon(x_*)\right\|_2
\]
and combining this relation with \((2.20)\) we obtain
\[
\|x_2 - x_1\|_2 \geq \frac{1}{L(f)} \left[ \sqrt{A_\varepsilon(x_*)}\left\|\left(\begin{array}{c} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{array}\right) - L(g)U_\varepsilon(x_*)\right\|_2 \right].
\]
This gives
\[
(2.24) \quad \|x_2 - x_1\|_2 \geq \frac{\sqrt{A_\varepsilon(x_*)}}{L(f) + L(g)U_\varepsilon(x_*)}\left\|\left(\begin{array}{c} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{array}\right)\right\|_2.
\]
Plugging \((2.24)\) into \((2.20)\) we get
\[
- \left(\nabla \theta(\lambda_2, \mu_2) - \nabla \theta(\lambda_1, \mu_1), \left(\begin{array}{c} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{array}\right)\right) \geq \frac{a\lambda,(x_*)}{(L(f) + L(g)U_\varepsilon(x_*))^2}\left\|\left(\begin{array}{c} \lambda_2 - \lambda_1 \\ \mu_2 - \mu_1 \end{array}\right)\right\|_2^2.
\]
Using Proposition 2.2 (iii), the relation above shows that \(\theta\) is strongly concave on \(N(\lambda_*, \mu_*) \cap \mathbb{R}^{p+q}\) with constant of strong concavity \(\frac{a\lambda,(x_*)}{(L(f) + L(g)U_\varepsilon(x_*))^2}\) with respect to norm \(\|\cdot\|_2\).

The local strong concavity of the dual function of \((2.13)\) was shown recently in Theorem 10 in [15] assuming (A3), assuming instead of (A1) that \(f\) is strongly convex and second-order continuously differentiable (which is stronger than (A1)), and assuming instead of (A2) that \(g_i, i = 1, \ldots, p\), are convex second-order continuously differentiable, which is stronger than (A2)\(^3\). Therefore Theorem 2.14 gives a new proof of the local strong concavity of the dual function and improves existing results.

3. Computing \(\varepsilon\)-subgradients for value functions of convex optimization problems

3.1. Preliminaries. Let \(Q : X \to \mathbb{R} \cup \{+\infty\}\) be the value function given by
\[
(3.25) \quad Q(x) = \left\{ \inf_{y \in \mathbb{R}^n} f(y, x) \mid y \in S(x) := \{y \in Y : Ay + Bx = b, g(y, x) \leq 0\}. \right.
\]
Here, and in all this section, \(X \subseteq \mathbb{R}^m\) and \(Y \subseteq \mathbb{R}^n\) are nonempty, compact, and convex sets, and \(A\) and \(B\) are respectively \(q \times n\) and \(q \times m\) real matrices. We will make the following assumptions which imply, in particular, the convexity of \(Q\) given by \((3.25)\):

\(\text{(H1)}\) \(f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) is lower semicontinuous, proper, and convex.

\(\text{(H2)}\) For \(i = 1, \ldots, p\), the \(i\)-th component of function \(g(y, x)\) is a convex lower semicontinuous function \(g_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}\).

In what follows, we say that \(C\) is a cut for \(Q\) on \(X\) if \(C\) is an affine function of \(x\) such that \(Q(x) \geq C(x)\) for all \(x \in X\). We say that the cut is exact at \(\bar{x}\) if \(Q(\bar{x}) = C(\bar{x})\). Otherwise, the cut is said to be inexact at \(\bar{x}\).

In this section, our basic goal is, given \(\bar{x} \in X\) and \(\varepsilon\)-optimal primal and dual solutions of \((3.25)\), written for \(x = \bar{x}\), to derive an inexact cut \(C(x)\) for \(Q\) at \(\bar{x}\), i.e., an affine lower bounding function for \(Q\) such that the distance \(Q(\bar{x}) - C(\bar{x})\) between the values of \(Q\) and of the cut at \(\bar{x}\) is bounded from above by a known function of the problem parameters. Of course, when \(\varepsilon = 0\), we will check that \(Q(\bar{x}) = C(\bar{x})\).

\(^3\)Note that we used (A4) to ensure that \(x(\lambda_*, \mu_*) = x_*\), which is also used in the proof of Theorem 10 in [15].
We first provide in Proposition 3.2 below a characterization of the subdifferential of value function \( Q \) at \( \bar{x} \in X \) when optimal primal and dual solutions for \((3.25)\) written for \( x = \bar{x} \) are available (computation of exact cuts).

Consider for problem \((3.25)\) the Lagrangian dual problem
\[
(3.26) \quad \sup_{(\lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^p_{+}} \theta_x(\lambda, \mu)
\]
for the dual function
\[
(3.27) \quad \theta_x(\lambda, \mu) = \inf_{y \in Y} L_x(y, \lambda, \mu)
\]
where
\[
L_x(y, \lambda, \mu) = f(y, x) + \lambda^T(Ay + Bx - b) + \mu^T g(y, x).
\]
We denote by \( \Lambda(x) \) the set of optimal solutions of the dual problem \((3.26)\) and we use the notation
\[
\text{Sol}(x) := \{y \in S(x) : f(y, x) = Q(x)\}
\]
to indicate the solution set to \((3.25)\).

**Lemma 3.1** (Lemma 2.1 in [3]). Consider the value function \( Q \) given by \((3.25)\) and take \( \bar{x} \in X \) such that \( S(\bar{x}) \neq \emptyset \). Let Assumptions (H1) and (H2) hold and assume the Slater-type constraint qualification condition: there exists \( (x_\star, y_\star) \in X \times \text{ri}(Y) \) such that \( Ay_\star + Bx_\star = b \) and \( (y_\star, x_\star) \in \text{ri}(\{g \leq 0\}) \).

Then \( s \in \partial Q(\bar{x}) \) if and only if
\[
(0, s) \in \partial f(\bar{y}, \bar{x}) + \left\{ [AT; BT] \lambda : \lambda \in \mathbb{R}^q \right\} + \left\{ \sum_{i \in I(\bar{y}, \bar{x})} \mu_i \partial g_i(\bar{y}, \bar{x}) : \mu_i \geq 0 \right\} + \mathcal{N}_Y(\bar{y}) \times \{0\},
\]
where \( \bar{y} \) is any element in the solution set \( \text{Sol}(\bar{x}) \) and with
\[
I(\bar{y}, \bar{x}) = \left\{ i \in \{1, \ldots, p\} : g_i(\bar{y}, \bar{x}) = 0 \right\}.
\]
In particular, if \( f \) and \( g \) are differentiable, then
\[
(3.29) \quad \partial Q(\bar{x}) = \left\{ \nabla_x f(\bar{y}, \bar{x}) + BT \lambda + \sum_{i \in I(\bar{y}, \bar{x})} \mu_i \nabla_x g_i(\bar{y}, \bar{x}) : (\lambda, \mu) \in \Lambda(\bar{x}) \right\}.
\]

The proof of Lemma 3.1 is given in [3] using calculus on normal and tangent cones. In Proposition 3.2 below, we show how to obtain a subgradient of \( Q \) at \( \bar{x} \in X \) using convex duality when \( f \) and \( g \) are differentiable.

**Proposition 3.2.** Consider the value function \( Q \) given by \((3.25)\) and take \( \bar{x} \in X \) such that \( S(\bar{x}) \neq \emptyset \). Let Assumptions (H1) and (H2) hold and assume the following constraint qualification condition: there exists \( y_0 \in \text{ri}(Y) \cap \text{ri}(\{g(\cdot, \bar{x}) \leq 0\}) \) such that \( Ay_0 + Bx = b \). Assume that \( f \) and \( g \) are differentiable on \( Y \times X \). Let \((\bar{\lambda}, \bar{\mu})\) be an optimal solution of dual problem \((3.26)\) written with \( x = \bar{x} \) and let
\[
(3.30) \quad s(\bar{x}) = \nabla_x f(\bar{y}, \bar{x}) + BT \bar{\lambda} + \sum_{i \in I(\bar{y}, \bar{x})} \bar{\mu}_i \nabla_x g_i(\bar{y}, \bar{x}),
\]
where \( \bar{y} \) is any element in the solution set \( \text{Sol}(\bar{x}) \) and with
\[
I(\bar{y}, \bar{x}) = \left\{ i \in \{1, \ldots, p\} : g_i(\bar{y}, \bar{x}) = 0 \right\}.
\]
Then \( s(\bar{x}) \in \partial Q(\bar{x}) \).

**Proof.** The constraint qualification condition implies that there is no duality gap and therefore
\[
(3.31) \quad f(\bar{y}, \bar{x}) = Q(\bar{x}) = \theta_x(\bar{\lambda}, \bar{\mu}).
\]
Moreover, \( \bar{y} \) is an optimal solution of \( \inf\{L_x(y, \bar{\lambda}, \bar{\mu}) : y \in Y\} \) which gives
\[
\langle \nabla_y L_x(y, \bar{\lambda}, \bar{\mu}), y - \bar{y} \rangle \geq 0 \quad \forall y \in Y,
\]
and therefore
\[(3.32) \quad \min_{y \in Y} \langle \nabla_y L_x(y, \bar{\lambda}, \bar{\mu}), y - \bar{y} \rangle = 0.\]

Using the convexity of the function which associates to \((x, y)\) the value \(L_x(y, \bar{\lambda}, \bar{\mu})\) we obtain for every \(x \in X\) and \(y \in Y\) that
\[(3.33) \quad L_x(y, \bar{\lambda}, \bar{\mu}) \geq L_x(\bar{y}, \bar{\lambda}, \bar{\mu}) + \langle \nabla_x L_x(\bar{y}, \bar{\lambda}, \bar{\mu}), x - \bar{x} \rangle + \langle \nabla_y L_x(\bar{y}, \bar{\lambda}, \bar{\mu}), y - \bar{y} \rangle.\]

By definition of \(\theta_x\), for any \(x \in X\) we get
\[Q(x) \geq \theta_x(\bar{\lambda}, \bar{\mu})\]
which combined with (3.33) gives
\[Q(x) \geq L_x(\bar{y}, \bar{\lambda}, \bar{\mu}) + \langle \nabla_x L_x(\bar{y}, \bar{\lambda}, \bar{\mu}), x - \bar{x} \rangle + \langle \nabla_y L_x(\bar{y}, \bar{\lambda}, \bar{\mu}), y - \bar{y} \rangle\]
where the last equality follows from \(3.31\). \(\bar{y} + B\bar{x} = b\) (feasibility of \(\bar{y}\)), \((\bar{\mu}, g(\bar{y}, \bar{x})) = 0\), and \(\bar{\mu}_i = 0\) if \(i \notin I(\bar{y}, \bar{x})\)(complementary slackness for \(\bar{y}\)). \(\square\)

3.2. Inexact cuts with fixed feasible set. As a special case of (3.25), we first consider value functions where the argument only appears in the objective of optimization problem (3.25):
\[(3.34) \quad Q(x) = \inf_{y \in \mathbb{R}^n} f(y, x) \quad \text{ for } y \in Y.\]

We fix \(\bar{x} \in X\) and denote by \(\bar{y} \in Y\) an optimal solution of (3.34) written for \(x = \bar{x}\):
\[(3.35) \quad Q(\bar{x}) = f(\bar{y}, \bar{x}).\]

If \(f\) is differentiable, using Proposition 3.2 we have that \(\nabla_x f(\bar{y}, \bar{x}) \in \partial Q(\bar{x})\) and
\[C(x) := Q(\bar{x}) + \langle \nabla_x f(\bar{y}, \bar{x}), x - \bar{x} \rangle\]
is an exact cut for \(Q\) at \(\bar{x}\). If instead of an optimal solution \(\bar{y}\) of (3.34) we only have at hand an approximate \(\varepsilon\)-optimal solution \(\hat{y}(\varepsilon)\) Proposition 3.3 below gives an inexact cut for \(Q\) at \(\bar{x}\):

**Proposition 3.3** (Proposition 2.2.1.2 in [4]). Let \(\bar{x} \in X\) and let \(\hat{y}(\varepsilon) \in Y\) be an \(\varepsilon\)-optimal solution for problem (3.34) written for \(x = \bar{x}\) with optimal value \(Q(\bar{x})\), i.e., \(Q(\bar{x}) \geq f(\hat{y}(\varepsilon), \bar{x}) - \varepsilon\). Assume that \(f\) is convex and differentiable on \(Y \times X\). Then setting \(\eta(\varepsilon, \bar{x}) = \ell_1(\hat{y}(\varepsilon), \bar{x})\) where \(\ell_1 : Y \times X \to \mathbb{R}_+\) is the function given by
\[(3.36) \quad \ell_1(\hat{y}, \bar{x}) = -\min_{y \in Y} \langle \nabla_y f(\hat{y}, \bar{x}), y - \hat{y} \rangle = \max_{y \in Y} \langle \nabla_y f(\hat{y}, \bar{x}), \hat{y} - y \rangle,
\]
the affine function
\[(3.37) \quad C(x) := f(\hat{y}(\varepsilon), \bar{x}) - \eta(\varepsilon, \bar{x}) + \langle \nabla_x f(\hat{y}(\varepsilon), \bar{x}), x - \bar{x} \rangle\]
is a cut for \(Q\) at \(\bar{x}\), i.e., for every \(x \in X\) we have \(Q(x) \geq C(x)\) and the quantity \(\eta(\varepsilon, \bar{x})\) is an upper bound for the distance \(Q(\bar{x}) - C(\bar{x})\) between the values of \(Q\) and of the cut at \(\bar{x}\).

**Remark 3.4.** If \(\varepsilon = 0\) then \(\hat{y}(\varepsilon)\) is an optimal solution of problem (3.34) written for \(x = \bar{x}\), \(\eta(\varepsilon, \bar{x}) = \ell_1(\hat{y}(\varepsilon), \bar{x}) = 0\) and the cut given by Proposition 3.3 is exact. Otherwise it is inexact.

In Proposition 3.3 below, we derive inexact cuts with an additional assumption of strong convexity on \(f\):

- \((H3)\) \(f\) is convex and differentiable on \(Y \times X\) and for every \(x \in X\) there there exists \(\alpha(x) > 0\) such that the function \(f(\cdot, x)\) is strongly convex on \(Y\) with constant of strong convexity \(\alpha(x) > 0\) for \(\|\cdot\|_2\):
\[f(y_2, x) \geq f(y_1, x) + (y_2 - y_1)^T \nabla_y f(y_1, x) + \frac{\alpha(x)}{2} \|y_2 - y_1\|_2^2, \quad \forall x \in X, \forall y_1, y_2 \in Y.\]

We will also need the following assumption, used to control the error on the gradients of \(f\):

- \((H4)\) For every \(y \in Y\) the function \(f(y, \cdot)\) is differentiable on \(X\) and for every \(x \in X\) there exists \(0 \leq M_1(x) < +\infty\) such that for every \(y_1, y_2 \in Y\), we have
\[\|\nabla_x f(y_2, x) - \nabla_x f(y_1, x)\|_2 \leq M_1(x) \|y_2 - y_1\|_2.\]
Proposition 3.5. Let $\bar{x} \in X$ and let $\hat{y}(\varepsilon) \in Y$ be an $\varepsilon$-optimal solution for problem (3.34) written for $x = \bar{x}$ with optimal value $Q(\bar{x})$, i.e., $Q(\bar{x}) \geq f(\hat{y}(\varepsilon), \bar{x}) - \varepsilon$. Let Assumptions (H3) and (H4) hold. Then setting

$$
\eta(\varepsilon, \bar{x}) = \varepsilon + M_1(\bar{x}) \text{Diam}(X) \sqrt{\frac{2\varepsilon}{\alpha(\bar{x})}},
$$

the affine function

$$
C(x) := f(\hat{y}(\varepsilon), \bar{x}) - \eta(\varepsilon, \bar{x}) + \langle \nabla_x f(\hat{y}(\varepsilon), \bar{x}), x - \bar{x} \rangle
$$

is a cut for $Q$ at $\bar{x}$, i.e., for every $x \in X$ we have $Q(x) \geq C(x)$ and the distance $Q(\bar{x}) - C(\bar{x})$ between the values of $Q$ and of the cut at $\bar{x}$ is at most $\eta(\varepsilon, \bar{x})$, or, equivalently, $\nabla_x f(\hat{y}(\varepsilon), \bar{x}) \in \partial_{\eta(\varepsilon, \bar{x})} Q(\bar{x})$.

Proof. For short, we use the notation $\hat{\lambda}$ instead of $\hat{y}(\varepsilon)$. Using the fact that $\hat{y} \in Y$, the first order optimality conditions for $\hat{y}$ imply $(\hat{y} - \hat{y})^T \nabla_y f(\hat{y}, \bar{x}) \geq 0$, which combined with Assumption (H3), gives

$$
f(\hat{y}, \bar{x}) \geq f(\hat{y}, \bar{x}) + (\hat{y} - \hat{y})^T \nabla_y f(\hat{y}, \bar{x}) + \frac{\alpha(\bar{x})}{2} \|\hat{y} - \hat{y}\|_2^2 \geq Q(\bar{x}) + \frac{\alpha(\bar{x})}{2} \|\hat{y} - \hat{y}\|_2^2,
$$
yielding

$$
\|\hat{y} - \hat{y}\|_2 \leq \sqrt{\frac{2}{\alpha(\bar{x})} \left( f(\hat{y}, \bar{x}) - Q(\bar{x}) \right)} \leq \sqrt{\frac{2\varepsilon}{\alpha(\bar{x})}}.
$$

Now recalling that $\nabla_x f(\hat{y}, \bar{x}) \in \partial Q(\bar{x})$, we have for every $x \in X$,

$$
\begin{aligned}
Q(x) &\geq f(\hat{y}, \bar{x}) - \varepsilon + (x - \bar{x})^T \nabla_x f(\hat{y}, \bar{x}) \\
&\geq f(\hat{y}, \bar{x}) - \varepsilon + (x - \bar{x})^T \left( \nabla_x f(\hat{y}, \bar{x}) - \nabla_x f(\hat{y}, \bar{x}) \right) \\
&\geq f(\hat{y}, \bar{x}) - \varepsilon + (x - \bar{x})^T \nabla_x f(\hat{y}, \bar{x}) - M_1(\bar{x}) \|\hat{y} - \hat{y}\|_2 \|x - \bar{x}\|_2 \\
&\geq f(\hat{y}, \bar{x}) - \varepsilon - M_1(\bar{x}) \text{Diam}(X) \sqrt{\frac{2\varepsilon}{\alpha(\bar{x})}} + (x - \bar{x})^T \nabla_x f(\hat{y}, \bar{x}),
\end{aligned}
$$

where for the third inequality we have used Cauchy-Schwartz inequality and Assumption (H4). Finally, observe that $C(\bar{x}) = f(\hat{y}, \bar{x}) - \eta(\varepsilon, \bar{x}) \geq Q(\bar{x}) - \eta(\varepsilon, \bar{x})$. \qed

Remark 3.6. As expected, if $\varepsilon = 0$ then $\eta(\varepsilon, \bar{x}) = 0$ and the cut given by Proposition 3.5 is exact. Otherwise it is inexact. The error term $\eta(\varepsilon, \bar{x})$ is the sum of the upper bound $\varepsilon$ on the error on the optimal value and of the error term $M_1(\bar{x}) \text{Diam}(X) \sqrt{\frac{2\varepsilon}{\alpha(\bar{x})}}$ which accounts for the error on the subgradients of $Q$.

3.3. Inexact cuts with variable feasible set. For $x \in X$, recall that for problem (3.25), the Lagrangian function is

$$
L_x(y, \lambda, \mu) = f(y, x) + \lambda^T (Bx + Ay - b) + \mu^T g(y, x),
$$
and the dual function is given by

$$
\theta_x(\lambda, \mu) = \inf_{y \in Y} L_x(y, \lambda, \mu).
$$

Define $\ell_2 : Y \times X \times \mathbb{R}^q \times \mathbb{R}^p_+ \rightarrow \mathbb{R}_+$ by

$$
\ell_2(\hat{y}, \hat{\lambda}, \hat{\mu}) = -\min_{y \in Y} (\nabla_y L_x(\hat{y}, \hat{\lambda}, \hat{\mu}), y - \hat{y}) = \max_{y \in Y} (\nabla_y L_x(\hat{y}, \hat{\lambda}, \hat{\mu}), y - \hat{y}).
$$

We make the following assumption which ensures no duality gap for (3.25) for any $x \in X$:

(H5) if $Y$ is polyhedral then for every $x \in X$ there exists $y_x \in Y$ such that $Bx + Ay_x = b$ and $g(y_x, x) < 0$ and if $Y$ is not polyhedral then for every $x \in X$ there exists $y_x \in \text{ri}(Y)$ such that $Bx + Ay_x = b$ and $g(y_x, x) < 0$.

The following proposition, proved in [3], provides an inexact cut for $Q$ given by (3.25):
Proposition 3.7. [Proposition 2.7 in [1]] Let \( \bar{x} \in X \), let \( \hat{y}(\varepsilon) \) be an \( \varepsilon \)-optimal feasible primal solution for problem (3.25) written for \( x = \bar{x} \) and let \((\hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon))\) be an \( \varepsilon \)-optimal feasible solution of the corresponding dual problem, i.e., of problem (3.26) written for \( x = \bar{x} \). Let Assumptions (H1), (H2), and (H5) hold. If additionally \( f \) and \( g \) are differentiable on \( Y \times X \) then setting \( \eta(\varepsilon, \bar{x}) = \ell_2(\hat{y}(\varepsilon), \bar{x}, \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)) \), the affine function

\[
C(x) := L_{\bar{x}}(\hat{y}(\varepsilon), \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)) - \eta(\varepsilon, \bar{x}) + (\nabla_x L_{\bar{x}}(\hat{y}(\varepsilon), \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)), x - \bar{x})
\]

with

\[
\nabla_x L_{\bar{x}}(\hat{y}(\varepsilon), \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)) = \nabla_x f(\hat{y}(\varepsilon), \bar{x}) + B^T \hat{\lambda}(\varepsilon) + \sum_{i=1}^p \hat{\mu}_i(\varepsilon) \nabla_x g_i(\hat{y}(\varepsilon), \bar{x}),
\]

is a cut for \( Q \) at \( \bar{x} \) and the distance \( Q(\bar{x}) - C(\bar{x}) \) between the values of \( Q \) and of the cut at \( \bar{x} \) is at most \( \varepsilon + \ell_2(\hat{y}(\varepsilon), \bar{x}, \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)) \).

In Proposition 3.8 below, we derive another formula for inexact cuts with an additional assumption of strong convexity:

(H6) Strong concavity of the dual function: for every \( x \in X \) there exists \( \alpha_D(x) > 0 \) and a set \( D_x \) containing the set of optimal solutions of dual problem (3.26) such that the dual function \( \theta_x \) is strongly concave on \( D_x \) with constant of strong concavity \( \alpha_D(x) \) with respect to \( \| \cdot \|_2 \).

We refer to Section 2 for conditions on the problem data ensuring Assumption (H6).

If the constants \( \alpha(\bar{x}) \) and \( \alpha_P(\bar{x}) \) in Assumptions (H3) and (H6) are sufficiently large and \( n \) is small then the cuts given by Proposition 3.8 are better than the cuts given by Proposition 3.7 i.e., \( Q(\bar{x}) - C(\bar{x}) \) is smaller. We refer to Section 3.4 for numerical tests comparing the cuts given by Propositions 3.7 and 3.8 on quadratic programs.

To proceed, take an optimal primal solution \( \bar{y} \) of problem (3.25) written for \( x = \bar{x} \) and an optimal dual solution \((\hat{\lambda}, \hat{\mu})\) of the corresponding dual problem, i.e., problem (3.26) written for \( x = \bar{x} \).

With this notation, using Proposition 3.2 we have that \( \nabla_x L_{\bar{x}}(\bar{y}, \lambda, \mu) \in \partial Q(\bar{x}) \). Since we only have approximate primal and dual solutions, \( \hat{y}(\varepsilon) \) and \((\hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon))\) respectively, we will use the approximate subgradient \( \nabla_x L_{\bar{x}}(\hat{y}(\varepsilon), \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)) \) instead of \( \nabla_x L_{\bar{x}}(\bar{y}, \lambda, \mu) \). To control the error on this subgradient, we assume differentiability of the constraint functions and that the gradients of these functions are Lipschitz continuous. More precisely, we assume:

(H7) \( g \) is differentiable on \( Y \times X \) and for every \( x \in X \) there exists \( 0 \leq M_2(x) < +\infty \) such that for all \( y_1, y_2 \in Y \), we have

\[
\| \nabla_x g_i(y_1, x) - \nabla_x g_i(y_2, x) \|_2 \leq M_2(x) \| y_1 - y_2 \|_2, \quad i = 1, \ldots, p.
\]

If Assumptions (H1)-(H7) hold, the following proposition provides an inexact cut for \( Q \) at \( \bar{x} \):

Proposition 3.8. Let \( \bar{x} \in X \), let \( \hat{y}(\varepsilon) \) be an \( \varepsilon \)-optimal feasible primal solution for problem (3.25) written for \( x = \bar{x} \) and let \((\hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon))\) be an \( \varepsilon \)-optimal feasible solution of the corresponding dual problem, i.e., of problem (3.26) written for \( x = \bar{x} \). Let Assumptions (H1), (H2), (H3), (H4), (H5), (H6), and (H7) hold. Let

\[
U = \max_{i = 1, \ldots, p} \| \nabla_x g_i(\hat{y}(\varepsilon), \bar{x}) \|_2.
\]

Let also \( L_{\bar{x}} \) be any lower bound on \( Q(\bar{x}) \). Define

\[
U_{\bar{x}} = \frac{f(y_{\bar{x}, \bar{x}}) - L_{\bar{x}}}{\min(-g_i(y_{\bar{x}, \bar{x}}), i = 1, \ldots, p)}
\]

and

\[
\eta(\varepsilon, \bar{x}) = \varepsilon + \left( (M_1(\bar{x}) + M_2(\bar{x})U_{\bar{x}}) \sqrt{\frac{2}{\alpha(\bar{x})}} + 2 \max(\|B^T\|_1, \sqrt{U}) \right) \text{Diam}(X) \sqrt{\varepsilon}.
\]

Then

\[
C(x) := f(\hat{y}(\varepsilon), \bar{x}) - \eta(\varepsilon, \bar{x}) + (\nabla_x L_{\bar{x}}(\hat{y}(\varepsilon), \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)), x - \bar{x})
\]

where

\[
\nabla_x L_{\bar{x}}(\hat{y}(\varepsilon), \hat{\lambda}(\varepsilon), \hat{\mu}(\varepsilon)) = \nabla_x f(\hat{y}(\varepsilon), \bar{x}) + B^T \hat{\lambda}(\varepsilon) + \sum_{i=1}^p \hat{\mu}_i(\varepsilon) \nabla_x g_i(\hat{y}(\varepsilon), \bar{x}),
\]
is a cut for $Q$ at $\bar{x}$ and the distance $Q(\bar{x}) - C(\bar{x})$ between the values of $Q$ and of the cut at $\bar{x}$ is at most $\eta(\epsilon, \bar{x})$.

**Proof.** For short, we use the notation $\hat{y}, \hat{\lambda}, \hat{\mu}$ instead of $\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon)$. Since $\nabla_x L_x(\bar{y}, \hat{\lambda}, \hat{\mu}) \in \partial Q(\bar{x})$, we have

$$Q(x) \geq Q(\bar{x}) + \langle \nabla_x L_x(\bar{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle \geq f(\bar{y}, \bar{x}) - \epsilon + \langle \nabla_x L_x(\bar{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle. \quad (3.47)$$

Next observe that

$$\|\nabla_x L_x(\bar{y}, \hat{\lambda}, \hat{\mu}) - \nabla_x L_x(\hat{y}, \hat{\lambda}, \hat{\mu})\| \leq M_1(\bar{x})\|\bar{y} - \hat{y}\| + \|B^T\|\|\hat{\lambda} - \bar{\lambda}\|
+ \|\sum_{i=1}^p \hat{\mu}(i) \left( \nabla_x g_i(\hat{y}, \hat{\lambda}) - \nabla_x g_i(\bar{y}, \bar{\lambda}) \right) \|
+ \|\sum_{i=1}^p \left( \hat{\mu}(i) - \hat{\mu}(i) \right) \nabla_x g_i(\hat{y}, \hat{\lambda})\|
\leq M_1(\bar{x})\|\bar{y} - \hat{y}\| + \|B^T\|\|\hat{\lambda} - \bar{\lambda}\| + M_2(\bar{x})\|\hat{\mu}\|_1\|\bar{y} - \hat{y}\| + U\sqrt{p}\|\hat{\mu} - \bar{\mu}\|
\leq (M_1(\bar{x}) + M_2(\bar{x})\|\hat{\mu}\|_1\|\bar{y} - \hat{y}\| + \sqrt{2}\max(\|B^T\|, U\sqrt{p})\sqrt{\|\bar{\lambda} - \hat{\lambda}\|^2 + \|\hat{\mu} - \bar{\mu}\|^2}. \quad (3.48)$$

Using Remark 2.3.3, p.313 in [3] and Assumption (H5) we have for $\|\hat{\mu}\|_1$ the upper bound

$$\|\hat{\mu}\|_1 \leq \frac{f(y, \bar{x}) - Q(\bar{x})}{\min(-g_i(y, \bar{x}), i = 1, \ldots, p)} \leq U_\epsilon. \quad (3.49)$$

Using Assumptions (H3) and (H6), we also get

$$\|\bar{y} - \hat{y}\|^2 \leq \frac{2\epsilon}{\alpha(x)} \text{ and } \|\bar{\lambda} - \hat{\lambda}\|^2 + \|\hat{\mu} - \bar{\mu}\|^2 \leq \frac{2\epsilon}{\alpha_D(\bar{x})}. \quad (3.50)$$

Combining (3.48), (3.49), and (3.50), we get

$$\|\nabla_x L_x(\bar{y}, \hat{\lambda}, \hat{\mu}) - \nabla_x L_x(\hat{y}, \hat{\lambda}, \hat{\mu})\| \leq \frac{\eta(\epsilon, \bar{x}) - \epsilon}{\text{Diam}(X)} \quad (3.51)$$

Plugging the above relation into (3.47) and using Cauchy-Schwarz inequality, we get

$$Q(x) \geq f(\hat{y}, \bar{x}) - \epsilon + \langle \nabla_x L_x(\hat{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle + \langle \nabla_x L_x(\hat{y}, \hat{\lambda}, \hat{\mu}) - \nabla_x L_x(\bar{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle
\geq f(\hat{y}, \bar{x}) - \epsilon - \|\nabla_x L_x(\hat{y}, \hat{\lambda}, \hat{\mu}) - \nabla_x L_x(\bar{y}, \hat{\lambda}, \hat{\mu})\|\text{Diam}(X) + \|\nabla_x L_x(\hat{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x}\|
\geq f(\hat{y}, \bar{x}) - \eta(\epsilon, \bar{x}) + \langle \nabla_x L_x(\hat{y}, \hat{\lambda}, \hat{\mu}), x - \bar{x} \rangle. \quad (3.52)$$

Finally, since $\hat{y} \in S(\bar{x})$ we check that $Q(\bar{x}) - C(\bar{x}) = Q(\bar{x}) - f(\hat{y}, \bar{x}) + \eta(\epsilon, \bar{x}) \leq \eta(\epsilon, \bar{x})$, which achieves the proof of the proposition. \qed

Observe that the ”slope” $\nabla_x L_x(\hat{y}(\epsilon), \hat{\lambda}(\epsilon), \hat{\mu}(\epsilon))$ of the cut given by Proposition 3.7 is the same as the ”slope” of the cut given by Proposition 3.8.

**Remark 3.9.** If $\hat{y}(\epsilon)$ and $\hat{\lambda}(\epsilon), \hat{\mu}(\epsilon)$ are respectively optimal primal and dual solutions, i.e., $\epsilon = 0$, then Proposition 3.8 gives, as expected, an exact cut for $Q$ at $\bar{x}$.

As shown in Corollary 3.10, the formula for the inexact cuts given in Proposition 3.8 can be simplified depending if there are nonlinear coupling constraints or not, if $f$ is separable (sum of a function of $x$ and of a function of $y$) or not, and if $g$ is separable.

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Corollary 3.10. Consider the value functions $Q : X \rightarrow \mathbb{R}$ where $Q(x)$ is given by the optimal value of the following optimization problems:

\[
\begin{align*}
\text{(a)} & \quad \min_y f(y, x) \\
& \quad A y + B x = b, \\
& \quad h(y) + k(x) \leq 0, \\
& \quad y \in Y, \\
& \quad \min_y f(y, x) \\
\text{(d)} & \quad \min_y f(y, x) \\
& \quad g(y, x) \leq 0, \\
& \quad y \in Y, \\
& \quad \min_y f(y, x) \\
\text{(g)} & \quad \min_y f_0(y) + f_1(x) \\
& \quad h(y) + k(x) \leq 0, \\
& \quad y \in Y, \\
& \quad \min_y f(y, x) \\
\text{(b)} & \quad \min_y f_0(y) + f_1(x) \\
& \quad A y + B x = b, \\
& \quad g(y, x) \leq 0, \\
& \quad y \in Y, \\
& \quad \min_y f(y, x) \\
\text{(e)} & \quad \min_y f(y, x) \\
& \quad h(y) + k(x) \leq 0, \\
& \quad y \in Y, \\
& \quad \min_y f(y, x) \\
\text{(h)} & \quad \min_y f(y, x) \\
& \quad A y + B x = b, \\
& \quad y \in Y, \\
& \quad \min_y f(y, x) \\
\text{(c)} & \quad \min_y f_0(y) + f_1(x) \\
& \quad A y + B x = b, \\
& \quad h(y) + k(x) \leq 0, \\
& \quad y \in Y, \\
& \quad \min_y f(y, x) \\
\text{(f)} & \quad \min_y f(y, x) \\
& \quad g(y, x) \leq 0, \\
& \quad y \in Y, \\
& \quad \min_y f(y, x) \\
\text{(i)} & \quad \min_y f(y, x) \\
& \quad A y + B x = b, \\
& \quad y \in Y.
\end{align*}
\]

For problems (b), (c), (f), (g), (i) above define $f(y, x) = f_0(y) + f_1(x)$ and for problems (a), (e), (h) define $g(y, x) = h(y) + k(x)$. With this notation, assume that $(H1)$, $(H2)$, $(H3)$, $(H4)$, $(H5)$, $(H6)$, and $(H7)$ hold for these problems. If $g$ is defined, let $L_x(y, \lambda, \mu) = f(y, x) + \lambda^T(Bx + Ay - b) + \mu^T g(y, x)$ be the Lagrangian and define

\[
U = \max_{i=1,\ldots,p} \| \nabla_x g_i(\hat{y}(\varepsilon), \bar{x}) \| \quad \text{and} \quad U_x = \frac{f(y, \bar{x}) - L_x}{\min(-g_i(y, \bar{x}), i = 1, \ldots, p)}
\]

where $L_x$ is any lower bound on $Q(x)$. If $g$ is not defined, define $L_x(y, \lambda) = f(y, x) + \lambda^T(Bx + Ay - b)$.

Let $\bar{x} \in X$, let $\bar{y}$ be an $\varepsilon$-optimal feasible primal solution for problem [(3.25)] written for $x = \bar{x}$ and let $(\hat{\lambda}, \hat{\mu})$ be an $\varepsilon$-optimal feasible solution of the corresponding dual problem, i.e., of problem [(3.26)] written for $x = \bar{x}$.

Then $C(x) = f(\bar{y}, \bar{x}) - \eta(\varepsilon, \bar{x}) + \langle \bar{s}(\varepsilon), x - \bar{x} \rangle$ is an inexact cut for $Q$ at $\bar{x}$ where the formulas for $\eta(\varepsilon, \bar{x})$ and $\bar{s}(\varepsilon)$ in each of cases (a)-(i) above are the following:

\[
\begin{align*}
\text{(a)} & \quad \eta(\varepsilon, \bar{x}) = \varepsilon + \left( M_1(\bar{x}) \frac{1}{\sqrt{\Delta(x)}} + \sqrt{2} \max(\|B^T\|, \|pU\}) \frac{1}{\sqrt{\Delta(x)}} \right) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f(\bar{y}, \bar{x}) + B^T\bar{\lambda} + \sum_{i=1}^p \mu_i \nabla_x k_i(\bar{x}), \\
& \quad \eta(\varepsilon, \bar{x}) = \varepsilon + \left( M_2(\bar{x}) \frac{1}{\sqrt{\Delta(x)}} + \sqrt{2} \max(\|B^T\|, \|pU\}) \frac{1}{\sqrt{\Delta(x)}} \right) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f_1(\bar{x}) + B^T\hat{\lambda} + \sum_{i=1}^p \mu_i \nabla_x g_i(\bar{y}, \bar{x}), \\
& \quad \eta(\varepsilon, \bar{x}) = \varepsilon + 2 \max(\|B^T\|, \|pU\}) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f_1(\bar{x}) + B^T\hat{\lambda} + \sum_{i=1}^p \mu_i \nabla_x k_i(\bar{x}), \\
& \quad \eta(\varepsilon, \bar{x}) = \varepsilon + \left( M_1(\bar{x}) \frac{1}{\sqrt{\Delta(x)}} + \sqrt{2} \max(\|B^T\|, \|pU\}) \frac{1}{\sqrt{\Delta(x)}} \right) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f(\bar{y}, \bar{x}) + \sum_{i=1}^p \mu_i \nabla_x g_i(\bar{y}, \bar{x}), \\
& \quad \eta(\varepsilon, \bar{x}) = \varepsilon + \left( M_2(\bar{x}) \frac{1}{\sqrt{\Delta(x)}} + \sqrt{2} \max(\|B^T\|, \|pU\}) \frac{1}{\sqrt{\Delta(x)}} \right) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f_1(\bar{x}) + B^T\hat{\lambda} + \sum_{i=1}^p \mu_i \nabla_x k_i(\bar{x}), \\
& \quad \eta(\varepsilon, \bar{x}) = \varepsilon + \left( M_1(\bar{x}) \frac{1}{\sqrt{\Delta(x)}} + \sqrt{2} \max(\|B^T\|, \|pU\}) \frac{1}{\sqrt{\Delta(x)}} \right) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f_1(\bar{x}) + B^T\hat{\lambda}, \\
& \quad \eta(\varepsilon, \bar{x}) = \varepsilon + \left( \sqrt{2} \max(\|B^T\|, \|pU\}) \frac{1}{\sqrt{\Delta(x)}} \right) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f_1(\bar{x}) + B^T\hat{\lambda}, \\
& \quad \eta(\varepsilon, \bar{x}) = \varepsilon + \left( \sqrt{2} \max(\|B^T\|, \|pU\}) \frac{1}{\sqrt{\Delta(x)}} \right) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f_1(\bar{x}) + B^T\hat{\lambda}, \\
& \quad \eta(\varepsilon, \bar{x}) = \varepsilon + \left( \sqrt{2} \max(\|B^T\|, \|pU\}) \frac{1}{\sqrt{\Delta(x)}} \right) \text{Diam}(X)\sqrt{2\varepsilon}, \\
& \quad s(\bar{x}) = \nabla_x f_1(\bar{x}) + B^T\hat{\lambda}.
\end{align*}
\]
Proof. It suffices to follow the proof of Proposition 3.8 specialized to cases (a)-(i). For instance, let us check the formulas in case (g). For (g), \( s(\bar{x}) = \nabla_x L_2(\hat{y}, \hat{\mu}) = \nabla_x f_1(\bar{x}) + \sum_{i=1}^n \bar{\mu}_i \nabla_x k_i(\bar{x}) \) and

\[
\|\nabla_x L_2(\hat{y}, \hat{\mu}) - \nabla_x L_2(\bar{y}, \bar{\mu})\| = \|\sum_{i=1}^n (\bar{\mu}_i - \hat{\mu}_i) \nabla_x k_i(\bar{x})\| \leq 2 \|\mu - \bar{\mu}\|_1 \leq U \|\mu - \bar{\mu}\|_1 \leq U \sqrt{F} \|\mu - \bar{\mu}\| \leq U \sqrt{F} \sqrt{\frac{2\epsilon}{\alpha(x)}}.
\]

It then suffices to combine (3.47) and (3.55). \(\square\)

3.4. Numerical results.

3.4.1. Argument of the value function in the objective only. Let \( S = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix} \) be a definite positive matrix, let \( c_1, c_2 \in \mathbb{R}^n \) be vectors of ones, and let \( Q \) be the value function given by

\[
Q(x) = \begin{cases} \min_{y \in \mathbb{R}^n} & \left( \begin{array}{c} x \\ y \end{array} \right)^T S \left( \begin{array}{c} x \\ y \end{array} \right) + \left( \begin{array}{c} c_1 \\ c_2 \end{array} \right)^T \left( \begin{array}{c} x \\ y \end{array} \right) \\ \text{subject to} & y \in Y := \{ y \in \mathbb{R}^n : y \geq 0, \sum_{i=1}^n y_i = 1 \}, \end{cases}
\]

(3.56)

Clearly, Assumption (H3) is satisfied with \( \alpha(x) = \lambda_{\min}(S_3) \), and

\[
\|\nabla_x f(y_2, x) - \nabla_x f(y_1, x)\| = \|S_2(y_2 - y_1)\|_2 \leq \|S_2\|_2 \|y_2 - y_1\|_2
\]

implying that Assumption (H4) is satisfied with \( M_1(\bar{x}) = \|S_2\|_2 = \sigma(S_2) \) where \( \sigma(S_2) \) is the largest singular value of \( S_2 \). We take \( X = Y \) with \( \text{Diam}(X) = \max_{x_1, x_2 \in X} \|x_2 - x_1\|_2 \leq \sqrt{2} \). With this notation, if \( \hat{y} \) is an \( \epsilon \)-optimal solution of (3.56) written for \( x = \bar{x} \), we compute at \( \bar{x} \) the cut \( C(x) = f(\hat{y}, \bar{x}) - \eta(\epsilon, \bar{x}) + \langle \nabla_x f(\hat{y}, \bar{x}), x - \bar{x} \rangle = f(\hat{y}, \bar{x}) - \eta(\epsilon, \bar{x}) + \langle c_1 + S_1 \bar{y} + S_2 \hat{y}, x - \bar{x} \rangle \) where

- \( \eta(\epsilon, \bar{x}) = \eta_1(\epsilon, \bar{x}) = \epsilon + 2M_1(\bar{x}) \sqrt{\frac{\alpha(\bar{x})}{\epsilon}} \) using Proposition 3.5
- \( \eta_2(\epsilon, \bar{x}) \) is given by

\[
\eta(\epsilon, \bar{x}) = \eta_2(\epsilon, \bar{x}) = \begin{cases} \max \langle \nabla_y f(\bar{y}, \bar{x}), \bar{y} - y \rangle & y \geq 0, \sum_{i=1}^n y_i = 1, \\ \max \langle c_2 + S_2^T \bar{y} + S_3 \hat{y}, \bar{y} - y \rangle & y \geq 0, \sum_{i=1}^n y_i = 1, \end{cases}
\]

using Proposition 3.3.

We compare in Table 1 the values of \( \eta_1(\epsilon, \bar{x}) \) and \( \eta_2(\epsilon, \bar{x}) \) for several values of \( m = n, \epsilon \), and \( \alpha(\bar{x}) \). In these experiments \( S \) is of the form \( A \Lambda A^T + \lambda I_{2n} \) for some \( \lambda > 0 \) and \( A \) has random entries in \([-20, 20]\). Optimization problems were solved using Mosek optimization toolbox [4], setting Mosek parameter MSK_DPAR_INTPTN_TOL_GAP which corresponds to the relative error \( \varepsilon_r \) on the optimal value to 0.1, 0.5, and 1. In each run, \( \varepsilon \) was estimated computing the duality gap (the difference between the approximate optimal values of the dual and the primal). Though \( \eta_1(\epsilon, \bar{x}) \) does not depend on \( \bar{x} \) (because on this example \( \alpha \) and \( M_1 \) do not depend on \( \bar{x} \)), the absolute error \( \varepsilon \) depends on the run (for a fixed \( \varepsilon_r \), different runs corresponding to different \( \bar{x} \) yield different errors \( \varepsilon, \eta_1(\epsilon, \bar{x}) \) and \( \eta_2(\epsilon, \bar{x}) \)). Therefore, for each fixed \( (\varepsilon_r, \alpha(\bar{x}), n) \), the values \( \varepsilon, \eta_1(\epsilon, \bar{x}), \) and \( \eta_2(\epsilon, \bar{x}) \) reported in the table correspond to the mean values of \( \varepsilon, \eta_1(\epsilon, \bar{x}), \) and \( \eta_2(\epsilon, \bar{x}) \) more much conservative on nearly all combinations of parameters, except on three of these combinations when \( n = 10 \) and more \( \alpha(\bar{x}) = 10^8 \) is very large.

3.4.2. Argument of the value function in the objective and constraints. We close this section comparing the error terms in the cuts given by Propositions 3.7 and 3.8 on a very simple problem with a quadratic objective and a quadratic constraint.

Let \( S = \begin{pmatrix} S_1 & S_2 \\ S_2^T & S_3 \end{pmatrix} \) be a definite positive matrix, let \( c_1, c_2 \in \mathbb{R}^n \), and let \( Q : X \to \mathbb{R} \) be the value function given by

\[
Q(x) = \min_{y \in \mathbb{R}^n} \{ f(y, x) : g_1(y, x) \leq 0 \},
\]

(3.57)
Table 1. Values of $\eta(\varepsilon, \bar{x}) = \eta_1(\varepsilon, \bar{x})$ (resp. $\eta(\varepsilon, \bar{x}) = \eta_2(\varepsilon, \bar{x})$) for the inexact cuts given by Proposition 3.5 (resp. Proposition 3.3) for value function (3.56) for various values of $n$ (problem dimension), $\alpha(\bar{x}) = \lambda_{\min}(S_3)$, and $\varepsilon$.

| $\varepsilon$ | $\alpha(\bar{x})$ | $n$ | $\eta_1$ | $\eta_2$ | $\varepsilon$ | $\alpha(\bar{x})$ | $n$ | $\eta_1$ | $\eta_2$ |
|---------------|----------------|-----|----------|----------|---------------|----------------|-----|----------|----------|
| 0.0024        | 102.9          | 10  | 1.76     | 0.025    | 0.0061        | 190.2          | 10  | 2.73     | 0.026    |
| 0.0080        | 10 087         | 10  | 0.86     | 0.054    | 0.0024        | 10$^p$         | 10  | 0.076    | 0.354    |
| 0.016         | 129.0          | 10  | 9.81     | 0.047    | 0.0084        | 174.5          | 10  | 4.85     | 0.037    |
| 0.029         | 10054          | 10  | 2.49     | 0.128    | 0.0020        | 10$^p$         | 10  | 0.09     | 0.342    |
| 0.008         | 112.3          | 10  | 8.07     | 0.043    | 0.0088        | 150.0          | 10  | 6.36     | 0.022    |
| 0.018         | 10 090         | 10  | 1.29     | 0.078    | 0.0019        | 10$^p$         | 10  | 0.06     | 0.442    |
| 0.15          | 531.9          | 100 | 175.6    | 0.3      | 0.18          | 665.3          | 100 | 183.5    | 0.3      |
| 0.23          | 10 687         | 100 | 44.5     | 0.2      | 0.03          | 10$^p$         | 100 | 2.1      | 0.9      |
| 0.17          | 676.2          | 100 | 185.7    | 0.2      | 0.09          | 734.3          | 100 | 106.5    | 0.2      |
| 0.11          | 10 638         | 100 | 37.9     | 0.2      | 0.02          | 10$^p$         | 100 | 1.7      | 0.3      |
| 0.05          | 660            | 100 | 106.7    | 0.2      | 0.40          | 777            | 100 | 253.8    | 0.4      |
| 0.07          | 10 585         | 100 | 32.6     | 0.2      | 0.02          | 10$^p$         | 100 | 1.3      | 0.4      |
| 6.78          | 6017.9         | 1000| 4177.8   | 9.5      | 2.69          | 5991.4         | 1000| 2778.8   | 6.8      |
| 8.12          | 15 722         | 1000| 3059.5   | 11.1     | 0.99          | 10$^p$         | 1000| 132.1    | 3.2      |
| 7.40          | 5799           | 1000| 4160.2   | 9.8      | 7.83          | 6020          | 1000| 4590.7   | 9.3      |
| 12.5          | 15860          | 1000| 4001.6   | 14.6     | 1.3           | 10$^p$         | 1000| 153.6    | 3.47     |
| 9.9           | 6065           | 1000| 4996.4   | 11.8     | 8.3           | 5955          | 1000| 4034.9   | 8.3      |
| 7.2           | 15 895         | 1000| 2564.3   | 3.4      | 9.7           | 10$^p$         | 1000| 117.2    | 1.8      |

where

$$f(y, x) = \frac{1}{2} \begin{pmatrix} x \\ y \end{pmatrix}^T S \begin{pmatrix} x \\ y \end{pmatrix} + \left( \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \right)^T \begin{pmatrix} x \\ y \end{pmatrix}$$

(3.58)

$$g_1(y, x) = \frac{1}{2} \|y - y_0\|_2^2 + \frac{3}{4} \|x - x_0\|_2^2 - \frac{R^2}{2}$$

$$X = \{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq 1\}.$$
with \( a_0, b_0, b_1 \) given by (3.60) and setting
\[
\alpha_D(\bar{x}) = a_0(\bar{x})^T(S_3 + U_2 I_n)^{-3} a_0(\bar{x}),
\]
if \( a_0(\bar{x}) \neq 0 \) then \( \theta_\varepsilon \) is strongly concave on the interval \([0, U_\varepsilon]\) with constant of strong concavity \( \alpha_D(\bar{x}) \) where \( U_\varepsilon \) is given by (3.59). Let \( \hat{y} \) be an \( \varepsilon \)-optimal primal solution of (3.57) written for \( x = \bar{x} \) and let \( \hat{\mu} \) be an \( \varepsilon \)-optimal solution of its dual. If \( a_0(\bar{x}) \neq 0 \), using Corollary 3.10(e) with \( p = 1, U = \|x - x_0\| \), we obtain for \( Q \) the cut
\[
(3.62)\quad C_1(x) = \begin{cases} f(\hat{y}, \bar{x}) - \eta_1(\varepsilon, \bar{x}) + \langle \nabla_x L_\bar{\varepsilon}(\hat{y}, \hat{\mu}), x - \bar{x} \rangle & \text{where} \nabla_x L_\bar{\varepsilon}(\hat{y}, \hat{\mu}) = S_1 \bar{x} + c_1 + S_2 \hat{y} + \hat{\mu}(\bar{x} - x_0). \\ \eta_1(\varepsilon, \bar{x}) = \varepsilon + D(X) \sqrt{\varepsilon} \left( \frac{\eta_2(x)}{\sqrt{\alpha(x)}} + \frac{\|\pi - \bar{x}\|}{\sqrt{\alpha_D(x)}} \right) & \text{with} \ D(X) = 2, M_1(\bar{x}) = \|S_2\|_2, \alpha(\bar{x}) = \lambda_{\min}(S_3), \end{cases}
\]

We now apply Proposition 3.7 to obtain another inexact cut for \( Q \) at \( \bar{x} \in X \) rewriting (3.57) under the form (3.25) with \( Y \) the compact set \( Y = \{y \in \mathbb{R}^n : \|y - y_0\|_2 \leq R\} \).

(3.63)\quad Q(x) = \min_{y \in \mathbb{R}^n} \{ f(y, x) : g_1(y, x) \leq 0, \|y - y_0\|_2 \leq R \}.

Applying Proposition 3.7 to reformulation (3.63) of (3.57), we obtain for \( Q \) the inexact cut \( C_2 \) at \( \bar{x} \) where
\[
\begin{align*}
C_2(x) &= f(\hat{y}, \bar{x}) - \eta_2(\varepsilon, \bar{x}) + \langle \nabla_x L_\bar{\varepsilon}(\hat{y}, \hat{\mu}), x - \bar{x} \rangle \\
\eta_2(\varepsilon, \bar{x}) &= -\min \{ \langle \nabla_y L_\bar{\varepsilon}(\hat{y}, \hat{\mu}), y - \bar{y} \rangle : \|y - y_0\|_2 \leq R \}, \\
\nabla_x L_\bar{\varepsilon}(\hat{y}, \hat{\mu}) &= S_1 \bar{x} + c_1 + S_2 \hat{y} + \hat{\mu}(\bar{x} - x_0), \\
\nabla_y L_\bar{\varepsilon}(\hat{y}, \hat{\mu}) &= S_3 \hat{y} + \hat{\mu}(\bar{x} - x_0).
\end{align*}
\]

As in the previous example, we take \( S \) of form \( S = A A^T + \lambda_2 I_n \) where the entries of \( A \) are randomly selected in the range \([-20, 20]\). We also take \( c_i(1) = c_2(i) = 1, i = 1, \ldots, n \). For 8 values of the pair \((n, \lambda)\), namely \((n, \lambda) \in \{(1, 1), (10, 1), (100, 1), (1000, 1), (1, 100), (10, 100), (100, 100), (1000, 100)\}\), we generate a matrix \( S \) of form \( A A^T + \lambda_2 I_n \) where the entries of \( A \) are realizations of independent random variables with uniform distribution in \([-20, 20]\). In each case, we select randomly \( \bar{x} \in X \) and solve (3.57), (3.58) and its dual written for \( x = \bar{x} \) using Mosk interior point solver. The value of \( \alpha(\bar{x}) = \lambda_{\min}(S_3) \), the dual function \( \theta_\varepsilon(\cdot) \), and the dual iterates computed along the iterations are reported in Figure 2 in the Appendix. Figure 2 shows the plots of \( \eta_1(\varepsilon_k, \bar{x}) \) and \( \eta_2(\varepsilon_k, \bar{x}) \) as a function of iteration \( k \) where \( \varepsilon_k \) is the duality gap at iteration \( k \).

The cuts computed by Proposition 3.8 are more conservative than cuts given by Proposition 3.7 on nearly all instances and iterations. We also see that, as expected, the error terms \( \eta_1(\varepsilon_k, \bar{x}) \) and \( \eta_2(\varepsilon_k, \bar{x}) \) go to zero when \( \varepsilon_k \) goes to zero (see the proof of Theorem 4.1 for a proof of this statement).

4. INEXACT STOCHASTIC MIRROR DESCENT FOR TWO-STAGE NONLINEAR STOCHASTIC PROGRAMS

The algorithm to be described in this section is an inexact extension of SMD [1] to solve (1.1). Let \( \|\cdot\| \) be a norm on \( \mathbb{R}^n \) and let \( \omega : X_1 \to \mathbb{R} \) be a distance-generating function. This function should

- be convex and continuous on \( X_1 \),
- admit on \( X_1^o = \{x \in X_1 : \partial \omega(x) \neq \emptyset\} \) a selection \( \omega'(x) \) of subgradients, and
- be compatible with \( \|\cdot\| \), meaning that \( \omega(\cdot) \) is strongly convex with constant of strong convexity \( \mu(\omega) > 0 \) with respect to the norm \( \|\cdot\| \):
\[
(\omega'(x) - \omega'(y))^T(x - y) \geq \mu(\omega) \|x - y\|^2 \forall x, y \in X_1^o.
\]

We also define

1. the \( \omega \)-center of \( X_1 \) given by \( x_1,\omega = \arg \min_{x \in X_1} \omega(x) \in X_1^o; \)
2. the Bregman distance or prox-function
\[
V_\omega(y) = \omega(y) - \omega(x) - (y - x)^T \omega'(x),
\]
for \( x \in X_1^o, y \in X_1; \)
3. the \( \omega \)-radius of \( X_1 \) defined as
\[
D_{\omega,X_1} = \sqrt{2 \max_{x \in X_1} \omega(x) - \min_{x \in X_1} \omega(x)}.
\]
The proximal mapping

\[(4.67) \quad \text{Prox}_\varepsilon (\zeta) = \arg\min_{y \in X_1} \{ \omega (y) + y^T (\zeta - \omega'(x)) \} \quad \text{for} \; x \in X_1^o, \zeta \in \mathbb{R}^n,\]

taking values in \(X_1^o\).

We describe below ISMD, an inexact variant of SMD for solving problem (1.1) in which primal and dual second stage problems are solved approximately.

We use the following notation: let \(D_{X_1} = \max_{x,y \in X_1} \| y - x \| \) be the diameter of \(X_1\), let \(x_2(x_1,\xi_2,\varepsilon)\) be an \(\varepsilon\)-optimal primal solution of (1.2), let

\[L_{x_1,\xi_2}(x_2,\lambda,\mu) = f_2(x_2, x_1, \xi_2) + \langle \lambda, Ax_2 + Bx_1 - b \rangle + \langle \mu, g(x_2, x_1, \xi_2) \rangle,\]

let \(\theta_{x_1,\xi_2}\) be the dual function

\[(4.68) \quad \theta_{x_1,\xi_2}(\lambda,\mu) = \min_{x_2 \in X_2} L_{x_1,\xi_2}(x_2,\lambda,\mu),\]

and let \((\lambda(x_1,\xi_2,\varepsilon),\mu(x_1,\xi_2,\varepsilon))\) be an \(\varepsilon\)-optimal solution of the dual problem

\[(4.69) \quad \begin{cases} \max \theta_{x_1,\xi_2}(\lambda,\mu) \\
\mu \geq 0, \lambda = Ax_2 + Bx_1 - b, x_2 \in \text{Aff}(X_2). \end{cases}\]

Denoting by \(s_{f_1}(x_1)\) a subgradient of \(f_1\) at \(x_1\), we define

\[H(x_1,\xi_2,\varepsilon) = \nabla x_1 f_2(x_2(x_1,\xi_2,\varepsilon), x_1,\xi_2) + B^T \lambda(x_1,\xi_2,\varepsilon) + \sum_{i=1}^p \mu_i(x_1,\xi_2,\varepsilon) \nabla x_1 g_i(x_2(x_1,\xi_2,\varepsilon), x_1,\xi_2),\]

\[G(x_1,\xi_2,\varepsilon) = s_{f_1}(x_1) + H(x_1,\xi_2,\varepsilon).\]

Inexact Stochastic Mirror Descent (ISMD) for risk-neutral two-stage nonlinear stochastic problems.

Initialization. Take \(x_1^1 = x_{1\omega}\). Fix the number of iterations \(N - 1\) and positive deterministic stepsizes \(\gamma_1, \ldots, \gamma_N\).

Loop. For \(t = 1, \ldots, N - 1\), sample a realization \(\xi_2^t\) of \(\xi_2\) (with corresponding realizations \(A^t\) of \(A\), \(B^t\) of \(B\), and \(b^t\) of \(b\)), compute an \(\varepsilon_t\)-optimal solution \(x_2^t\) of the problem

\[(4.70) \quad \Omega(x_1^t,\xi_2^t) = \begin{cases} \min_{x_2} f_2(x_2, x_1^t, \xi_2^t) \\
A^t x_2 + B^t x_1^t = b^t, \\
g(x_2, x_1^t, \xi_2^t) \leq 0, \\
x_2 \in X_2, \end{cases}\]

and an \(\varepsilon_t\)-optimal solution \((\lambda^t,\mu^t) = (\lambda(x_1^t,\xi_2^t,\varepsilon_t),\mu(x_1^t,\xi_2^t,\varepsilon_t))\) of the dual problem

\[(4.71) \quad \begin{cases} \max \theta_{x_1^t,\xi_2^t}(\lambda,\mu) \\
\mu \geq 0, \lambda = A^t x_2 + B^t x_1^t - b^t, x_2 \in \text{Aff}(X_2) \end{cases},\]

used to compute \(G(x_1^t,\xi_2^t,\varepsilon_t)\). Compute

\[(4.72) \quad x_1^{t+1} = \text{Prox}_{\varepsilon_t}(\gamma_t G(x_1^t,\xi_2^t,\varepsilon_t)).\]

Outputs:

\[(4.73) \quad x_1(N) = \frac{1}{N} \sum_{t=1}^N \gamma_t x_1^t, \quad \tilde{f}_N = \frac{1}{N} \sum_{t=1}^N \gamma_t \left( f_1(x_1^t) + f_2(x_2^t, x_1^t, \xi_2^t) \right) \quad \text{with} \quad \Gamma_N = \sum_{t=1}^N \gamma_t.\]

Convergence of Inexact Stochastic Mirror Descent for solving (1.1) can be shown when noises \((\varepsilon_t)\) asymptotically vanish:

**Theorem 4.1** (Convergence of ISMD). Consider problem (1.1) and assume that (i) \(X_1\) and \(X_2\) are nonempty, convex, and compact, (ii) \(f_1\) is convex, finite-valued, and has bounded subgradients on \(X_1\), (iii) for every \(x_1 \in X_1\) and \(x_2 \in X_2\), \(f_2(x_2, x_1, \cdot)\) and \(g_i(x_2, x_1, \cdot)\), \(i = 1, \ldots, p\), are measurable, (iv) for every \(\xi_2 \in \Xi\) the functions \(f_2(\cdot,\cdot,\xi_2)\) and \(g_i(\cdot,\cdot,\xi_2)\), \(i = 1, \ldots, p\), are convex and continuously differentiable on \(X_2 \times X_1\), (v) for \(\exists \kappa > 0\) and \(r > 0\) such that for all \(x_1 \in X_1\), for all \(\xi_2 \in \Xi\), there exists \(x_2 \in X_2\) such that \(B(x_2, r) \cap \text{Aff}(X_2) \neq \emptyset\),
\[ \hat{A}x_2 + \hat{B}x_1 = \hat{b}, \text{ and } g(x_2, x_1, \xi_2) < -\epsilon e \text{ where } e \text{ is a vector of ones}. \] If \( \gamma_t = \frac{\theta}{\sqrt{N}} \) for some \( \theta > 0 \), if the support \( \Xi \) of \( \xi_2 \) is compact, and if \( \lim_{t \to \infty} \epsilon_t = 0 \), then
\[
\lim_{N \to +\infty} \mathbb{E}[f(x_1(N))] = \lim_{N \to +\infty} \mathbb{E}[f_N] = f_*
\]
where \( f_* \) is the optimal value of (1.1).

**Proof.** Let \( x_1^* \) be an optimal solution of (1.1). Standard computations on the proximal mapping give
\[
\sum_{\tau=1}^{N} \gamma_{\tau} G(x_1^*, \xi_2^*, \epsilon_\tau)^{T}(x_1^* - x_1) \leq \frac{1}{2} D_{\omega, x_1}^2 + \frac{1}{2\mu(\omega)} \sum_{\tau=1}^{N} \gamma_{\tau}^2 \|G(x_1^*, \xi_2^*, \epsilon_\tau)\|_{2}^2.
\]

Next using Proposition 3.7 we have
\[
\mathcal{Q}(x_1^*, \xi_2^*) \geq \mathcal{Q}(x_1^*, \xi_2^*) - \eta_{\xi_2^*}(\epsilon_\tau, x_1^*) + \langle H(x_1^*, \xi_2^*, \epsilon_\tau), x_1^* - x_1^* \rangle
\]
where
\[
\eta_{\xi_2^*}(\epsilon_\tau, x_1^*) = \left\{ \begin{array}{ll}
\max & \langle \nabla x_2^* L x_1^*, \xi_2^*(x_2^*, 0^r, \mu^r), x_2^* - x_2 \rangle \\
& \text{if } x_2 \in \mathcal{X}_2 \\
0 & \text{if } x_2 \notin \mathcal{X}_2 \end{array} \right.
\]

Setting \( \xi_2^{1:1} = (\xi_2^1, \ldots, \xi_2^{1:1}) \) and taking the conditional expectation \( \mathbb{E}_{\xi_2} \left[ |\xi_2^{1:1} | \right] \) on each side of (4.75) we obtain almost surely
\[
\mathcal{Q}(x_1^*) \geq \mathcal{Q}(x_1^*) - \mathbb{E}_{\xi_2} [\eta_{\xi_2^*}(\epsilon_\tau, x_1^*) | \xi_2^{1:1}] + \left[ \mathbb{E}_{\xi_2} [H(x_1^*, \xi_2^*, \epsilon_\tau) | \xi_2^{1:1}] \right] \langle x_1^* - x_1^* \rangle.
\]

Combining (4.73), (4.77), and using the convexity of \( f \) we get
\[
0 \leq \mathbb{E}[f(x_1(N)) - f(x_1^*)] \leq \frac{1}{\Gamma_N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[f(x_1^*) - f(x_1^*)]
\]

\[
\leq \frac{1}{\Gamma_N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[\eta_{\xi_2^*}(\epsilon_\tau, x_1^*)] + \frac{1}{2\Gamma_N} \left[ D_{\omega, x_1}^2 + \frac{1}{\mu(\omega)} \sum_{\tau=1}^{N} \gamma_{\tau}^2 \|G(x_1^*, \xi_2^*, \epsilon_\tau)\|_{2}^2 \right].
\]

We now show by contradiction that
\[
\lim_{\tau \to \infty} \eta_{\xi_2^*}(\epsilon_\tau, x_1^*) = 0 \text{ almost surely.}
\]

Take an arbitrary realization of ISMD. We want to show that
\[
\lim_{\tau \to \infty} \eta_{\xi_2^*}(\epsilon_\tau, x_1^*) = 0
\]
for that realization. Assume that (4.80) does not hold. Let \( x_2^* \) (resp. \( \bar{x}_2^* \)) be an optimal solution of (4.70) (resp. (4.76)). Then there is \( \epsilon_0 > 0 \) and \( \sigma_1 : N \to N \) increasing such that for every \( \tau \in N \), we have
\[
\langle \nabla x_2 f_2(x_2^*(\sigma_1(\tau)), x_1^*(\sigma_1(\tau)), \xi_2^*(\sigma_1(\tau))) + (A^r)^T \lambda^r(\tau) + \sum_{i=1}^{p} \mu_{i}^r \nabla x_2 g_i(x_2^*(\sigma_1(\tau)), x_1^*(\sigma_1(\tau)), \xi_2^*(\sigma_1(\tau))), x_2^*(\tau) - \bar{x}_2^*(\tau) \rangle \geq \epsilon_0.
\]

By \( \epsilon_1 \)-optimality of \( x_2^* \) we obtain
\[
f_2(x_2^*, x_1, \xi_2) \leq f_2(x_2^*, x_1^*, \xi_2) \leq f_2(x_2^*, x_1, \xi_2) + \epsilon_t.
\]

Using Assumptions (i), (iii), (iv), and Proposition 3.1 in [3] we get that the sequence \( \langle \lambda^r(\tau), \mu^r(\tau) \rangle \) is almost surely bounded. Let \( D \) be a compact set to which this sequence belongs. By compactness, we can find \( \sigma_2 : N \to N \) increasing such that setting \( \sigma = \sigma_1 \circ \sigma_2 \) the sequence \( (x_2^*(\tau), x_1^*(\tau), \lambda^*(\tau), \mu^*(\tau), \xi_2^*(\tau)) \) converges to some \( (\bar{x}_2, x_1^*, \lambda^*, \mu_*, \xi_2) \in \mathcal{X}_2 \times \mathcal{X}_1 \times D \times \Xi \). We will denote by \( A_1, B_1, b_1 \) the values of \( A, B, \) and \( b \) in \( \xi_2 \). By continuity arguments there is \( \eta_0 \) in \( N \) such that for every \( \tau \geq \eta_0 \):
\[
\left| \langle \nabla x_2 f_2(x_2^*(\tau), x_1^*(\tau), \xi_2^*(\tau)) + (A^r)^T \lambda^r(\tau) + \sum_{i=1}^{p} \mu_{i}^r \nabla x_2 g_i(x_2^*(\tau), x_1^*(\tau), \xi_2^*(\tau)), x_2^*(\tau) - \bar{x}_2^*(\tau) \rangle - \langle \nabla x_2 f_2(\bar{x}_2, x_1^*, \xi_2), A^r \lambda_*, + \sum_{i=1}^{p} \mu_{i}(*, t) \nabla x_2 g_i(\bar{x}_2, x_1^*, \xi_2), \bar{x}_2 - \bar{x}_2^*(\tau) \rangle \right| \leq \epsilon_0/2.
\]

\( \text{4The proof is similar to the proof of Proposition 4.6 in [3].} \)
We deduce from (4.81) and (4.83) that for all $\tau \geq \tau_0$

\begin{equation}
(4.84) \quad \left( \nabla x_2 f_2(\bar{x}_2, x_{1*}, \xi_2) + A^T \lambda \right) + \sum_{i=1}^{p} \mu_i(i) \nabla x_2 g_i(\bar{x}_2, x_{1*}, \xi_2) \geq \varepsilon_0 / 2 > 0.
\end{equation}

Assumptions (i)-(iv) imply that primal problem (4.70) and dual problem (4.71) have the same optimal value and for every $x_2 \in \mathcal{X}_2$ and $\tau \geq \tau_0$ we have:

\begin{align*}
&f_2(x_2^*(\tau), \xi_2^*(\tau)) + \langle A^* x_2^*(\tau), \xi_2^*(\tau) \rangle + \langle B^* x_2^*(\tau) - b^*(\tau), \lambda^*(\tau) \rangle + \langle \mu^*(\tau), g(x_2^*(\tau), x_{1*}, \xi_2^*(\tau)) \rangle \\
&\leq f_2(x_2^*(\tau), \xi_2^*(\tau)) + \varepsilon_{\sigma_{\tau}}(\tau) \quad \text{by definition of } x_2^*(\tau) \text{ and since } \mu^*(\tau) \geq 0, x_2^*(\tau) \in X_2(\xi_2^*(\tau), \lambda^*(\tau)), \\
&\leq \theta x_2^*(\tau), \xi_2^*(\tau) \text{ and there is no duality gap,}
\end{align*}

\begin{align*}
&\leq f_2(x_2, x_{1*}, \xi_2^*(\tau)) + \langle A^* x_2 + B^* x_{1*} - b^*(\tau), \lambda^*(\tau) \rangle + \langle \mu^*(\tau), g(x_2, x_{1*}, \xi_2^*(\tau)) \rangle + 2 \varepsilon_{\sigma_{\tau}}(\tau)
\end{align*}

where in the last relation we have used the definition of $\theta x_2^*(\tau), \xi_2^*(\tau)$. Taking the limit in the above relation as $\tau \to +\infty$, we get for every $x_2 \in \mathcal{X}_2$:

\begin{align*}
f_2(\bar{x}_2, x_{1*}, \xi_2) + \langle A^* \bar{x}_2 + B^* x_{1*} - b^*, \lambda \rangle + \langle \mu^*, g(\bar{x}_2, x_{1*}, \xi_2) \rangle \\
\leq f_2(x_2, x_{1*}, \xi_2) + \langle A^* x_2 + B^* x_{1*} - b^*, \lambda \rangle + \langle \mu^*, g(x_2, x_{1*}, \xi_2) \rangle.
\end{align*}

Recalling that $\bar{x}_2 \in \mathcal{X}_2$ this shows that $\bar{x}_2$ is an optimal solution of

\begin{equation}
(4.85) \quad \left\{ \begin{array}{l}
\min f_2(x_2, x_{1*}, \xi_2) + \langle A^* x_2 + B^* x_{1*} - b^*, \lambda \rangle + \langle \mu^*, g(x_2, x_{1*}, \xi_2) \rangle \\
\forall x_2 \in \mathcal{X}_2.
\end{array} \right.
\end{equation}

The first order optimality conditions for $\bar{x}_2$ can be written

\begin{equation}
(4.86) \quad \left( \nabla x_2 f_2(\bar{x}_2, x_{1*}, \xi_2) + A^T \lambda \right) + \sum_{i=1}^{p} \mu_i(i) \nabla x_2 g_i(\bar{x}_2, x_{1*}, \xi_2) \geq 0
\end{equation}

for all $x_2 \in \mathcal{X}_2$. Specializing the above relation for $x_2 = x_2^*(\tau_0) \in \mathcal{X}_2$, we get

\begin{equation}
\left( \nabla x_2 f_2(\bar{x}_2, x_{1*}, \xi_2) + A^T \lambda \right) + \sum_{i=1}^{p} \mu_i(i) \nabla x_2 g_i(\bar{x}_2, x_{1*}, \xi_2) \geq 0,
\end{equation}

but the left-hand side of the above inequality is $-\varepsilon_0 / 2 < 0$ due to (4.84) which yields the desired contradiction. Therefore we have shown (4.79) and since the sequence $\eta_{\kappa}(\varepsilon_{\tau}, x_{\tau})$ is almost surely bounded, this implies $\lim_{\tau \to +\infty} \mathbb{E}[\eta_{\kappa}(\varepsilon_{\tau}, x_{\tau})] = 0$ and consequently $\lim_{N \to +\infty} \frac{1}{N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[\eta_{\kappa}(\varepsilon_{\tau}, x_{\tau})] = 0$. Using the boundedness of the sequence $(\lambda^\tau, \mu^\tau)$ and Assumption (ii) we get that $\| G(x_{\tau}^*, \xi_{\tau}^*, \varepsilon_{\tau}) \|^2$ is almost surely bounded. Combining these observations with relation (4.78) and using the definition of $\gamma_{\tau}$ we have $\lim_{N \to +\infty} \mathbb{E}[f(x_1(N))] = f_{1*}$. Finally, recalling relation (4.78), to show $\lim_{N \to +\infty} \mathbb{E}[f_N] = f_{1*}$ we have to show is

\begin{equation}
(4.87) \quad \lim_{N \to +\infty} \frac{1}{N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[Q(x_{\tau}^\tau) - f_2(x_{\tau}^\tau, x_{\tau}^\tau)] = 0.
\end{equation}

The above relation immediately follows from

\begin{equation}
(4.88) \quad \mathbb{E}[Q(x_{\tau}^\tau)] = \mathbb{E}[Q(x_{\tau}^\tau) - [Q(x_{\tau}^\tau)] \mathbb{E}[Q(x_{\tau}^\tau)] \mathbb{E}[Q(x_{\tau}^\tau)] \mathbb{E}[Q(x_{\tau}^\tau)] \mathbb{E}[Q(x_{\tau}^\tau)] + \varepsilon_{\tau} \mathbb{E}[Q(x_{\tau}^\tau)] + \varepsilon_{\tau} \mathbb{E}[Q(x_{\tau}^\tau)] + \varepsilon_{\tau} \mathbb{E}[Q(x_{\tau}^\tau)]
\end{equation}

which holds since $Q(x_{\tau}^\tau, \xi_{\tau}^\tau) \leq f_2(x_{\tau}^\tau, x_{\tau}^\tau, \xi_{\tau}^\tau) \leq Q(x_{\tau}^\tau, \xi_{\tau}^\tau, \varepsilon_{\tau})$ by definition of $x_{\tau}^\tau$.

**Remark 4.2.** Output $f_N$ of ISMD is a computable approximation of the optimal value $f_{1*}$ of optimization problem (1.1).

**Theorem 4.3.** Consider problem (1.1) and assume that Assumptions (i)-(iv) of Theorem 4.1 are satisfied. We also make the following assumptions:

(a) $\exists \alpha > 0$ such that for every $\xi_2 \in \Xi$, for every $x_1 \in X_1$, for every $y_1, y_2 \in \mathcal{X}_2$ we have

\begin{equation}
f_2(y_2, x_1, \xi_2) \geq f_2(y_2, x_1, \xi_2) + \langle y_2 - y_1 \rangle^T \nabla x_2 f_2(y_2, x_1, \xi_2) + \alpha \| y_2 - y_1 \|^2;
\end{equation}

(b) there is $0 < M_1 < +\infty$ such that for every $\xi_2 \in \Xi$, for every $x_1 \in X_1$, for every $y_1, y_2 \in \mathcal{X}_2$ we have

\begin{equation}
\| \nabla x_1 f_2(y_2, x_1, \xi_2) - \nabla x_1 f_2(y_1, x_1, \xi_2) \| \leq M_1 \| y_2 - y_1 \|;
\end{equation}
(c) there is $0 < M_2 < +\infty$ such that for every $\xi_2 \in \Xi$, for every $x_1 \in X_1$, for every $i = 1, \ldots, p$, for every $y_1, y_2 \in X_2$, we have

$$\|\nabla_{x_1} g_i(y_2, x_1, \xi_2) - \nabla_{x_1} g_i(y_1, x_1, \xi_2)\|_2 \leq M_2 \|y_2 - y_1\|_2;$$

(d) $\exists \alpha_D > 0$ such that for every $x_1 \in X_1$, for every $\xi_2 \in \Xi$, dual function $\theta_{x_1, \xi_2}$ given by (4.68) is strongly concave on $D_{x_1, \xi_2}$ with constant of strong concavity $\alpha_D$ where $D_{x_1, \xi_2}$ is a set containing the set of solutions of second stage dual problem (4.69) such that $(\lambda^t, \mu^t) \in D_{x_1, \xi_2}$.

(e) There are functions $G_0, M_0$ such that for every $x_1 \in X_1$, for every $\xi_2 \in \Xi$, we have

$$\max(\|B_{T}^T\|, \sqrt{p} \max_{i=1,\ldots,p} \|\nabla_{x_1} g_i(x_2, x_1, \xi_2)\|_2) \leq G_0(\xi_2) \text{ and } \|\nabla_{x_1} f_2(x_2, x_1, \xi_2)\|_2 \leq M_0(\xi_2)$$

with $E[G_0(\xi_2)]$ and $E[M_0(\xi_2)]$ finite;

(f) There are functions $\tilde{f}_2, f_2$ such that for all $x_1 \in X_1, x_2 \in X_2$ we have

$$f_2(\xi_2) \leq f_2(x_2, x_1, \xi_2) \leq \tilde{f}_2(\xi_2)$$

with $E[\tilde{f}_2(\xi_2)]$ and $E[f_2(\xi_2)]$ finite.

(g) There exists $0 < L(f_2) < +\infty$ such that for every $\xi_2 \in \Xi$, for every $x_1 \in X_1$, function $f_2(\cdot, x_1, \xi_2)$ is Lipschitz continuous with Lipschitz constant $L(f_2)$.

Let $A$ such that matrix $A$ in $\xi_2$ almost surely belongs to $A$ and let $M_3 < +\infty$ such that $\|s_{f_1}(x_1)\|_2 \leq M_3$ for all $x_1 \in X_1$. Let $V_{X_2}$ be the vector space $V_{X_2} = \{x - y : x, y \in \text{Aff}(X_2)\}$. Define the functions $\rho$ and $\rho_\ast$ by

$$\rho(A, z) = \begin{cases} \max t \|z\| & t \geq 0, t z \in A(B(0, r) \cap V_{X_2}), \\ 0 & \|z\| = 1, z \in AV_{X_2}. \end{cases} \quad \rho_\ast(A) = \min \rho(A, z)$$

Assume that $\gamma_t = \frac{\theta_1}{\sqrt{N}}$ and $\varepsilon_t = \frac{\theta_2}{r^2}$ for some $\theta_1, \theta_2 > 0$. Let

$$U_1 = \frac{(E[\tilde{f}_2(\xi_2)] - E[f_2(\xi_2)])}{\kappa}, \quad U_2(r, \xi_2) = \frac{\tilde{f}_2(\xi_2) - f_2(\xi_2) + \theta_2 L(f_2)}{\min(\rho_\ast, r/2)} \quad \text{with } \rho_\ast = \min_{A \in A} \rho_\ast(A),$$

$$U = \frac{(M_1 + M_2 U_1) \sqrt{\frac{2}{\alpha}} + \frac{2E[G_0(\xi_2)]}{\sqrt{\alpha_D}}} \sqrt{\text{Diam}(X_2)},$$

$$M_\ast(r) = \sqrt{E[M_3 + M_0(\xi_2)] + \sqrt{2 \text{Diam}(r, \xi_2) G_0(\xi_2)}^2].$$

Let $\hat{f}_N$ computed by ISMD. Then there is $r_0 > 0$ such that

$$f_1^\ast \leq E[\hat{f}_N] \leq f_1^\ast + \frac{2\theta_2 + U \sqrt{\frac{V_{X_2}}{N}}}{\kappa} + \frac{U \sqrt{\frac{\ln(N)}{N}}}{\kappa} + \frac{D^2_{X_1} + \theta_1 M^2_{\ast}(r)}{\mu(\varepsilon_t) 2\sqrt{N}}$$

where $f_1^\ast$ is the optimal value of (1.1).

**Proof.** Let $x^t_1$ be an optimal solution of (1.1). Under our assumptions, we can apply Proposition 3.8 to value function $Q(x^t_1, \xi^t_2)$ and $\bar{x} = x^t_1$, which gives

$$Q(x^t_1, \xi^t_2) = f_2(x^t_1, x_1, \xi^t_2) + \langle H(x^t_1, \xi^t_2, \varepsilon_t), x^t_1 - x^t_1 \rangle - \eta_{\xi^t_2}(\varepsilon_t, x^t_1),$$

where

$$\eta_{\xi^t_2}(\varepsilon_t, x^t_1) = \varepsilon_t + \left(M_1 + \frac{M_2}{\kappa}(f_2(\bar{x}^t_2, x^t_1, \xi^t_2) - f_2(\xi^t_2))\right) \sqrt{\frac{2\varepsilon_t}{\alpha}} \text{Diam}(X_2),$$

$$+ 2\max \left(\|B_{T}^T\|, \sqrt{p} \max_{i=1,\ldots,p} \|\nabla_{x_1} g_i(x^t_2, x^t_1, \xi^t_2)\|_2\right) \text{Diam}(X_2) \sqrt{\frac{\varepsilon_t}{\alpha_D}},$$

for some $\bar{x}^t_2 \in X_2$ depending on $\xi^t_{2^t}$. Taking the conditional expectation $E\xi^t_2|\xi^{t+1}_{2^t}]$ in (4.90) and using (e)-(f), we get

$$Q(x^t_1) \geq E\xi^t_2[f_2(x^t_2, x^t_1, \xi^t_2)|\xi^{t+1}_{2^t}] + E\xi^t_2[(H(x^t_1, \xi^t_2, \varepsilon_t), x^t_1 - x^t_1)|\xi^{t+1}_{2^t}] - (\varepsilon_t + U \sqrt{\varepsilon_t}).$$

Summing (4.91) with the relation

$$f_1(x^t_1) \geq f_1(x^t_1) + \langle s_{f_1}(x^t_1), x^t_1 - x^t_1 \rangle$$

and taking the expectation operator $E\xi^t_{2^t-1}$ on each side of the resulting inequality gives

$$f(x^t_1) \geq E[f_2(x^t_2, x^t_1, \xi^t_2) + f_1(x^t_1)] + E[(G(x^t_1, \xi^t_2, \varepsilon_t), x^t_1 - x^t_1)] - (\varepsilon_t + U \sqrt{\varepsilon_t}).$$
From (4.92), we deduce

\[
(4.93) \quad \mathbb{E}[\hat{f}_N - f_1] \leq \frac{1}{N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[f(x_{1}^\tau)] + \frac{1}{N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[(G(x_1^\tau, \xi_2^\tau, \varepsilon_{\tau}), x_1^\tau - x_1^\tau)].
\]

Using Proposition 3.1 in [4] and our assumptions, we can find \( r_0 > 0 \) such that \( M_2^2(r_0) \) is an upper bound for \( \mathbb{E}[\|G(x_1^\tau, \xi_2^\tau, \varepsilon_{\tau})\|^2] \). Using this observation, (4.93), and (4.90) (which still holds), we get

\[
(4.94) \quad \mathbb{E}[\hat{f}_N - f_1] \leq \frac{1}{N} \left( \theta_2 \left( 1 + \int_1^N \frac{dx}{x^2} \right) + U \sqrt{2 \gamma_2} \left( 1 + \int_1^N \frac{dx}{x} \right) \right) + \frac{1}{2 \theta_1 \sqrt{N}} \left( D_{\omega} x_1 + \frac{M_2^2(r_0) \theta_1^2}{\mu(\omega)} \right)
\]

Finally

\[
0 \leq \frac{1}{N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[f(x_{1}^\tau)] - f_1
\]

(4.95)

\[
= \frac{1}{N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[f_1(x_{1}^\tau) + Q(\xi_{1}^\tau)] - f_1
\]

(4.96)

\[
\leq \frac{1}{N} \sum_{\tau=1}^{N} \gamma_{\tau} \mathbb{E}[f_1(x_{1}^\tau) + f_2(x_{1}^\tau, x_{1}^\tau, \xi_{2}^\tau)] - f_1 = \mathbb{E}[\hat{f}_N - f_1].
\]

Combining (4.94) and (4.95) we obtain (4.89). \( \square \)

5. Numerical Experiments

We compare the performances of SMD and ISMD on two simple two-stage quadratic stochastic programs which satisfy the assumptions of Theorems 4.1 and 4.3.

The first two-stage program is

\[
(5.96) \quad \begin{cases} \min c^T x_1 + \mathbb{E}[\Omega(x_1, \xi_2)] \\ x_1 \in \{ x_1 \in \mathbb{R}^n : x_1 \geq 0, \sum_{i=1}^{n} x_1(i) = 1 \} \end{cases}
\]

where the second stage recourse function is given by

\[
(5.97) \quad \Omega(x_1, \xi_2) = \begin{cases} \min_{x_2 \in \mathbb{R}^n} \frac{1}{n} \left( x_1 \begin{array}{c} x_2 \\ \xi_2 \end{array} \right)^T \left( \begin{array}{c} \xi_2 \\ \lambda \end{array} \right) + \xi_2 \begin{array}{c} x_1 \\ x_2 \end{array} \\ x_2 \geq 0, \sum_{i=1}^{n} x_2(i) = 1. \end{cases}
\]

The second two-stage program is

\[
(5.98) \quad \begin{cases} \min c^T x_1 + \mathbb{E}[\Omega(x_1, \xi_2)] \\ x_1 \in \{ x_1 \in \mathbb{R}^n : \|x_1 - x_0\|_2 \leq 1 \} \end{cases}
\]

where cost-to-go function \( \Omega(x_1, \xi_2) \) has nonlinear objective and constraint coupling functions and is given by

\[
(5.99) \quad \Omega(x_1, \xi_2) = \begin{cases} \min_{x_2 \in \mathbb{R}^n} \frac{1}{n} \left( x_1 \begin{array}{c} x_2 \\ \xi_2 \end{array} \right)^T \left( \begin{array}{c} \xi_2 \\ \lambda \end{array} \right) + \xi_2 \begin{array}{c} x_1 \\ x_2 \end{array} \\ \frac{1}{2} \|x_2 - y_0\|^2 + \frac{1}{2} \|x_1 - x_0\|^2 - \frac{\theta_1}{2} \leq 0. \end{cases}
\]

For both problems, \( \xi_2 \) is a Gaussian random vector in \( \mathbb{R}^{2n} \) and \( \lambda > 0 \). We consider two instances of these problems: \( n = 200 \) for the first instance and \( n = 400 \) for the second. For each instance, the components of \( \xi_2 \) are independent with means and standard deviations randomly generated in respectively intervals [5, 25] and [5, 15]. We fix \( \lambda = 2 \) while the components of \( c \) are generated randomly in interval [1, 3]. For problem (5.98), (5.99) we also take \( R = 5 \) and \( x_0(i) = y_0(i) = 10, i = 1, \ldots, n. \)
In SMD and ISMD, we take \( \omega(x) = \sum_{i=1}^{n} x_i \ln(x_i) \) for problem (5.96)-(5.97). For this distance generating function, \( x_+ = \text{Prox}_x(\zeta) \) can be computed analytically for \( x \in \mathbb{R}^n \) with \( x > 0 \) (see [11, 5] for details): defining \( z \in \mathbb{R}^n \) by \( z(i) = \ln(x(i)) \) we have \( x_+ = \exp(z_+(i)) \) where

\[
x_+ = w - \ln \left( \sum_{i=1}^{n} e^{w(i)} \right) \mathbf{1} \text{ with } w = z - \zeta - \max_i [z(i) - \zeta(i)],
\]

and with \( \mathbf{1} \) a vector in \( \mathbb{R}^n \) of ones.

For problem (5.98)-(5.99), SMD and ISMD are run taking distance generating function \( \omega(x) = \frac{1}{2} \|x\|_2^2 \) (in this case, SMD is just the Robust Stochastic Approximation). For this choice of \( \omega \), if \( x_+ = \text{Prox}_x(\zeta) \) we have

\[
x_+ = \begin{cases} 
  x - \zeta & \text{if } \|x - \zeta - x_0\|_2 \leq 1, \\
  x_0 + \frac{x - \zeta - x_0}{\|x - \zeta - x_0\|_2} & \text{otherwise.}
\end{cases}
\]

In SMD and ISMD, the interior point solver of the Mosek Optimization Toolbox [1] is used at each iteration to solve the quadratic second stage problem (given first stage decision \( x_1^t \) and realization \( \xi_2^t \) of \( \xi_2 \) at iteration \( t \)) and constant steps are used: if there are \( N \) iterations, the step \( \gamma_2^t \) for iteration \( t \) is \( \gamma_2^t = \frac{1}{N} \).

For the two instances (the first one with \( n = 200 \) and the second one with \( n = 400 \)) of both problems (stochastic problem (5.96)-(5.97) and stochastic problem (5.98)-(5.99)) we do the following experiment: to obtain a good approximation of the optimal value, we run SMD with a (large) sample, of size \( N \) (stochastic problem (5.96)-(5.97) and stochastic problem (5.98)-(5.99)) we do the following experiment: to obtain a good approximation of the optimal value, we run SMD with a (large) sample, of size \( N \) of both problems (the first with \( n = 200 \) and the second with \( n = 400 \)) of both problems (the first with \( n = 200 \) and the second with \( n = 400 \)) of both problems (the first with \( n = 200 \) and the second with \( n = 400 \)).

We then fix the level of inexactness of the second stage primal and dual solutions in ISMD as follows.

For each instance, we run SMD for \( N = N_1 = 80 \) iterations 50 times (on 50 independent samples of \( \xi_2 \) of size 80) and observed that on these runs Mosek interior point solver required between 6 and 14 iterations to compute second stage solutions for problem (5.96)-(5.97) (both for \( n = 200 \) and \( n = 400 \)) and between 9 and 23 iterations to compute second stage solutions for problem (5.98)-(5.99) (both for \( n = 200 \) and \( n = 400 \)).

For ISMD, we therefore limit the number of iterations of the interior point solver used to solve second stage problems for the first iterations and increase this maximal number of iterations along the iterations of the method, in such a way that for the last iterations, SMD and ISMD use the same maximal number of iterations (14 for problem (5.96)-(5.97) and 23 for problem (5.98)-(5.99) for the quadratic solver used to compute second stage solutions. More precisely, the maximal number of iterations for Mosek interior point solver used to solve second stage problems as a function of the iteration number \( i \) is at most 10% distant to the optimal value of the problem, estimated using a large sample, of size \( N = 5000 \) for the instances where \( n = 200 \) and of size \( N = 600 \) for the instances where \( n = 400 \), as we have just discussed. With our problem data, we find \( N_1 = 80 \) for all instances, i.e., the two instances (the first with \( n = 200 \) and the second with \( n = 400 \)) problem (5.96)-(5.97) and the two instances (the first with \( n = 200 \) and the second with \( n = 400 \)) problem (5.98)-(5.99).

For ISMD run for \( N_1 \) iterations (estimated computing the empirical estimation of the optimal value obtained running SMD 50 times with \( N_1 \) iterations) is at most 10% distant to the optimal value of the problem, estimated using a large sample, of size \( N = 5000 \) for the instances where \( n = 200 \) and of size \( N = 600 \) for the instances where \( n = 400 \), as we have just discussed. With our problem data, we find \( N_1 = 80 \) for all instances, i.e., the two instances (the first with \( n = 200 \) and the second with \( n = 400 \)) problem (5.96)-(5.97) and the two instances (the first with \( n = 200 \) and the second with \( n = 400 \)) problem (5.98)-(5.99).

As was done for SMD, we then look for the smallest sample size \( N_1 \) (starting from \( N_1 = 10 \) and increasing \( N_1 \) by 10 until finding an appropriate sample size) such that the expected optimal value with ISMD run for \( N_1 \) iterations (estimated computing the empirical estimation of the optimal value obtained running ISMD 50 times with \( N_1 \) iterations) is at most 10% distant to the optimal value of the problem, estimated as explained before. With our problem data, we find \( N_1 = 80 \) for all instances.

For each problem instance, we run SMD and ISMD 50 times with \( N = N_1 = 80 \) iterations on each run. The time required to solve the two problem instances (with \( n = 200 \) and \( n = 400 \)) on these 50 runs is reported in Figure 1 for problem (5.96)-(5.97) and in Figure 2 for problem (5.98)-(5.99). It can be seen that on most runs ISMD is quicker and for \( n = 400 \) the standard deviation (s.d.) of the time required to solve the problem is much less for ISMD (see also Table 3 for the mean and s.d. of the CPU time).

---

5We proposed in Theorem 4.3 specific sequences for error terms \( \varepsilon_t \). However, according to Mosek documentation, it is not currently possible to fix arbitrary levels of inaccuracy with Mosek and the specified errors are relative and not absolute errors. Therefore, we obtained inexact second stage solutions for the first iterations of ISMD limiting the number of iterations of Mosek quadratic solver.
Table 2. Maximal number of iterations for Mosek interior point solver used to solve second stage problems as a function of the iteration number $i = 1, \ldots, N$ of ISMD to solve problem (5.96)-(5.97) (top two tables) and problem (5.98)-(5.99) (bottom two tables). For instance for $N = 80$ and problem (5.96)-(5.97), for iterations $0.4 \times 80 + 1, \ldots, 0.55 \times 80 = 33, \ldots, 44$, Mosek interior point solver is run to solve second stage problems limiting the maximal number of iterations to 10.

Table 3. Empirical mean and standard deviation (computed over $M$ runs) of the time required to solve (5.96)-(5.97) and (5.98)-(5.99) with SMD and ISMD for instances with $n = 200$ and $n = 400$.

In Figures 1 and 2 we also report the mean (computed over the 50 runs), the mean plus the standard deviation (s.d., computed over the 50 runs), and the mean minus the standard deviation of the estimation of the optimal value of problem (5.96)-(5.97) (for Figure 1) and of problem (5.98)-(5.99) (for Figure 2) along the $N = 80$ iterations of SMD and ISMD. For the instance with $n = 200$ these values are close while for $n = 400$ we observe a behavior which is what we would expect in general from ISMD: for the first iterations, since only approximate primal and dual second stage solutions are computed, we obtain worse approximations of the optimal value with ISMD and the approximations of the optimal values with SMD and ISMD get closer when the number of iteration increases.

The advantage of ISMD over SMD on our runs can also be seen in Table 3 which compares the mean and standard deviation of the CPU time needed to solve our problem instances with SMD and ISMD. On these runs, ISMD is in general quicker on average and in general exhibits a smaller standard deviation of the CPU time. However, the performance of ISMD clearly depends on the choice of the levels of inexactness of the second stage solutions computed, either choosing absolute error terms $\varepsilon_t$ or, as we did for our experiments, limiting the number of iterations of the algorithm used to solve the second stage problems for the first iterations. For problem (5.98)-(5.99) with $n = 200$, the average times reported for SMD and ISMD in Table 3 are the same and the s.d. of the CPU time is worse (slightly larger) with ISMD.

6. Conclusion

We introduced an inexact variant of SMD called ISMD to solve (general) nonlinear two-stage stochastic programs. The method and convergence analysis was based on two studies of convex analysis.
Figure 1. Left plots: time needed to solve two instances of problem (5.96)-(5.97) (with $n = 200$ and $n = 400$) on 50 runs of SMD (red dotted lines) and ISMD (black solid lines). Right plots: mean (computed over the 50 runs), mean plus standard deviation (s.d. computed over the 50 runs), and mean minus standard deviation of the estimation of the optimal value of (5.96)-(5.97) along the $N = 80$ iterations of SMD (red dotted lines) and ISMD (black solid lines).

(a) the computation of inexact cuts for value functions of a large class of convex optimization problems having nonlinear objective and constraints which couple the argument of the value function and the decision variable;

(b) the study of the strong concavity of the dual function of an optimization problem (used to derive one of our formulas for inexact cuts).

It is worth mentioning that the formulas we derived for inexact cuts could also be used to propose inexact level methods [10] to solve nonlinear two-stage stochastic programs (1.1)-(1.2), when primal and dual second stage problems are solved approximately (inexactly).

It would also be interesting to test ISMD and the aforementioned inexact level methods on several relevant instances of nonlinear two-stage stochastic programs.

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Figure 2. Left plots: time needed to solve two instances of problem (5.98)-(5.99) on 50 (resp. 20) runs of SMD (red dotted lines) and ISMD (black solid lines) for \( n = 200 \) (resp. \( n = 400 \)). Right plots: mean (computed over the 50 runs for \( n = 200 \) and the 20 runs for \( n = 400 \)), mean plus standard deviation (s.d. computed over the 50 runs for \( n = 200 \) and the 20 runs for \( n = 400 \)), and mean minus standard deviation of the estimation of the optimal value of (5.98)-(5.99) along the \( N = 80 \) iterations of SMD (red dotted lines) and ISMD (black solid lines).

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**APPENDIX**
Figure 3. Dual function $\theta_{\bar{x}}$ of problem (3.57) for some $\bar{x}$ randomly drawn in ball $\{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq 1\}$, $S = AA^T + \lambda I_{2n}$ for some random matrix $A$ with random entries in $[-20, 20]$, and several values of the pair $(n, \lambda)$. The dual iterates are represented by red diamonds.
Figure 4. Plots of $\eta_1(\varepsilon_k, \bar{x})$ and $\eta_2(\varepsilon_k, \bar{x})$ as a function of iteration $k$ where $\varepsilon_k$ is the duality gap at iteration $k$ for problem (3.57) for some $\bar{x}$ randomly drawn in ball $\{x \in \mathbb{R}^n : \|x - x_0\|_2 \leq 1\}$, $S = AA^T + \lambda I_{2n}$ for some random matrix $A$ with random entries in $[-20, 20]$, and several values of the pair $(n, \lambda)$. 