The Flat-Sky Approximation to Galaxy Number Counts

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Abstract. In this paper we derive and test the flat sky approximation for galaxy number counts. We show that, while for the lensing term it reduces to the Limber approximation, for the standard density and redshift space distortion it is different and very accurate already at low $\ell$ while the corresponding Limber approximation completely fails. At equal redshift the accuracy of the standard terms is around 0.2% at low redshifts and 0.5% for redshift $z = 5$, even to low $\ell$. At unequal redshifts the precision is less impressive and can only be trusted for very small redshift differences, $\Delta z < \Delta z_0 \simeq 3.6 \times 10^{-4} (1 + z)^2.14$, but the lensing terms dominate for $\Delta z > \Delta z_1 \simeq 0.33 (r(z) H(z))/(z + 1)$. The Limber approximation achieves an accuracy of 0.5% above $\ell \simeq 40$ for the pure lensing term and above $\ell \simeq 80$ for the lensing-density cross-correlation. Besides being very accurate, the flat sky approximation is also very fast and can therefore be useful for data analysis and forecasts with MCMC methods.
1 Introduction

Until today, the most successful cosmological data are the anisotropies and polarisation of the cosmic microwave background (CMB), see [1] for the latest experimental results. However, in this decade there are deep and large galaxy surveys planned [2–10] which may do as well as, or in some aspects better than, CMB experiments. To make optimal use of these data, a correct analysis has to be performed. On small scales and at late times, non-linearities and baryonic effects are the most difficult challenge, while at intermediate to large scales and higher redshifts a correct relativistic treatment is most relevant.

In recent years, fully relativistic expressions for the fluctuations of the observed galaxy number counts and their spectra have been derived [11–14]. A detailed study of these spectra has shown [13–22] that, in nearly all situations, the large scale relativistic terms are very small and can be neglected for percent level accuracy. Exceptions to this are very low redshifts, \( z < 0.1 \), see [22], and very large angular scales, \( \ell < 10 \). The latter are not very relevant, at least for single tracer analyses, due to cosmic variance. Their importance is discussed in Refs. [23–26].

The remaining terms which are relevant on sub-horizon scales are the density, redshift-space distortion (RSD) and the lensing term. These are the terms which we investigate here and for which we derive simple approximations that can be computed rapidly, but are nevertheless accurate at the 0.5% level or better for equal redshift correlations. For unequal redshifts our approximations for density and RSD are much less precise, but the lensing terms, which can be computed with the Limber approximation, dominate for large redshift differences. In the past, the density and RSD terms have been computed mainly in Fourier space [27]. This is sufficient for small surveys in one redshift bin. We shall see that the flat sky approximation for density and RSD, while requiring a similar numerical effort, is not equivalent and is valid also at very large angular scales down to
While the flat sky approximation has been derived previously [28–31], its accuracy has never been analysed in any detail\(^1\). Doing this is the goal of the present paper.

The lensing term is an integral along the line of sight and cannot, without approximations, be represented in Fourier space (see [22] for an attempt). As the truly measured quantities are directions and redshifts, it is most consistent and model independent to represent the number count fluctuations as a function of direction and redshift. This is what we do in this work. When assuming a background cosmology, the redshift space correlation function can also be computed and may be more useful for the analysis of spectroscopic surveys [22, 32] within one redshift bin. However, for the very promising analysis of number counts from large photometric surveys, angular–redshift power spectra will most probably become the method of choice, since they are truly model independent. Angular and redshift fluctuations are also simple to combine with shape measurements in order to derive galaxy-galaxy lensing cross-correlation spectra, see e.g. [33, 34].

The remainder of this paper is structured as follows: In the next section we introduce the flat sky approximation for density, RSD and lensing and we compare flat sky results with CLASS results. We first present results for equal redshift correlations which are exquisitely accurate and then study unequal redshifts which are more problematic. In Section 3 we compare the flat sky and the Limber approximations and in Section 4 we summarize our findings and conclude.

1.1 Notation and Conventions

We set the speed of light \(c = 1\) throughout. We consider a Friedmann Universe with scalar perturbations only in longitudinal (Newtonian, Poisson) gauge,

\[
ds^2 = a^2(t) \left[ -(1 + 2\Psi)dt^2 + (1 - 2\Phi)\delta_{ij}dx^i dx^j \right].
\]

We denote conformal time by \(t\) and a derivative wrt. \(t\) by an overdot. The conformal Hubble parameter is denoted by \(\mathcal{H} = \dot{a}/a\) while the physical Hubble parameter is \(H = \mathcal{H}/a\).

2 The Flat Sky Approximation

2.1 Generalities

Including RSD and lensing, but neglecting large scale relativistic effects, the linear perturbation theory expressions for the number count fluctuations in direction \(\mathbf{n}\) at redshift \(z\) are [13, 14, 17]

\[
\Delta(\mathbf{n}, z) = b(z)D(r(z)\mathbf{n}, t(z)) + \frac{1}{\mathcal{H}(z)} \partial_r V_r(r(z)\mathbf{n}, t(z)) + 2(1 - \gamma(z))\kappa(\mathbf{n}, z).
\]

Here \(b(z)\) is the linear galaxy bias which depends on the class of galaxies considered in the survey, \(D\) is the density fluctuation (in comoving gauge), \(V_r\) is the radial component of the velocity field (in longitudinal gauge) and \(\kappa\) is the convergence,

\[
2\kappa(\mathbf{n}, z) = \Delta_{S^2} \int_0^{r(z)} \frac{dr' (r(z) - r')}{r(z)r'} \left( \Psi(r'\mathbf{n}, t_0 - r') + \Phi(r'\mathbf{n}, t_0 - r') \right),
\]

where \(\Delta_{S^2}\) denotes the Laplace operator on the 2-sphere, i.e. wrt. \(\mathbf{n}\). The function \(\gamma(z)\) is the luminosity bias which is given by the logarithmic derivative of the observed galaxy population at

\(^1\)More precisely, in Ref. [28] the authors claim that they find an accuracy of better than 1% for \(\Delta\nu/\nu_0 = \Delta z \simeq 10^{-3} \gg \Delta z_0 \simeq 0.06\) at redshift \(z = 10\). We roughly agree with this as we shall see later in the present paper.
the flux limit of the given survey,
\[ \gamma(z, F_{\text{lim}}) \equiv - \frac{\partial \log N(z, F > F_{\text{lim}})}{\partial \log F} \bigg|_{F_{\text{lim}}} , \]  
(2.3)
where \( N \) denotes the mean density of galaxies which are seen with a flux \( F > F_{\text{lim}} \) from redshift \( z \), i.e., the density of galaxies with luminosity \( L > L_{\text{lim}} \). The luminosity is related to the flux as usual via \( F = L/(4\pi D(z)^2) \). Clearly, this function is survey-dependent, but \( \gamma(z) \) is also directly observable and does not depend on the background cosmology (which determines e.g., \( D(z) \)).

The number count fluctuation can be expanded in spherical harmonics,
\[ \Delta(n, z) = \sum_{\ell m} a_{\ell m}(z) Y_{\ell m}(n) \]  
(2.4)
\[ a_{\ell m}(z) = \int_{S^2} \Delta(n, z) Y^*_{\ell m}(n) d\Omega_n , \]  
(2.5)
and the angular redshift power spectrum is given by
\[ \langle a_{\ell m}(z) a^*_{\ell' m'}(z') \rangle = C_\ell(z, z') \delta_{\ell \ell'} \delta_{m m'} . \]  
(2.6)
Like for the CMB, the Kronecker-deltas are a consequence of statistical isotropy.

In the flat sky approximation we replace the direction \( n \) by \( n = e_z + \alpha \) where \( \alpha \) lives in the plane normal to \( e_z \), the flat sky. The direction \( e_z \) is a reference direction out to the center of our survey. In the flat sky, \( \ell \) is the dimensionless variable conjugate to \( \alpha \) and the spherical harmonic transform (2.5) of an arbitrary variable \( X \) becomes a 2d Fourier transform,
\[ a_X(\ell, z) \simeq \frac{1}{2\pi} \int d^2 \alpha e^{i \ell \cdot \alpha} X(\alpha, z) , \]  
\[ X(\alpha, z) \simeq \frac{1}{2\pi} \int d^2 \ell e^{-i \ell \cdot \alpha} a_X(\ell, z) . \]  
(2.7)
Let us first consider a variable \( X(x, t) \) defined in all of space at any time with transfer function \( T_X(k, z) \) so that \( X \) is given by
\[ X(k, z) = T_X(k, z) R(k) , \]  
(2.8)
\[ X(x, z) = \frac{1}{(2\pi)^3} \int d^3 k e^{-ix \cdot k} T_X(k, z) R(k) . \]  
(2.9)
Here \( R(k) \) is the initial curvature fluctuation after inflation. Its power spectrum is defined by
\[ \langle R(k) R^*(k') \rangle = (2\pi)^3 \delta(k - k') P_R(k) \]  
(2.10)
\[ \frac{k^3}{2\pi^2} P_R(k) = P_R(k) . \]  
(2.11)
The normalization of the dimensionless power spectrum \( P_R \) is such that the correlation function of \( R \) in real space is simply
\[ \langle R(x) R(y) \rangle = \int_0^\infty dk j_0(kr) P_R(k) , \quad r = |x - y| , \]  
(2.12)
without any pre-factor. Here \( j_0 \) is the spherical Bessel function of order 0, see [35]. The scalar perturbation amplitude \( A_s \) and the scalar spectral index \( n_s \) are defined by
\[ P_R(k) = A_s(k/k_s)^{n_s-1} , \]  
(2.13)
where $k_*$ is called the pivot scale. For numerical examples in this paper we shall use the Planck values for $k_* = 0.05$/Mpc,

$$
\log(10^{10} A_s) = 3.043, \quad n_s = 0.9652. \tag{2.14}
$$

Inserting $X(\alpha, z) = X(r(z)(\mathbf{e}_z + \alpha), z)$ in (2.9) we find

$$
X(\alpha, z) = \frac{1}{(2\pi)^3} \int d^3k e^{-ir(z)(\mathbf{e}_z + \alpha) \cdot k} T_X(k, z) \mathcal{R}(k). \tag{2.15}
$$

Comparing this with (2.7) and identifying $k = k_{\parallel} \mathbf{e}_z + \ell/r(z) = k_{\parallel} \mathbf{e}_z + \mathbf{k}_{\perp}$, we find

$$
a^X(\ell, z) = \frac{1}{(2\pi)^2} \int d^3k e^{-ir(z)(\mathbf{e}_z + \alpha) \cdot k} T_X(k, z) \mathcal{R}(k) \delta(\ell - r(z) \mathbf{k}_{\perp}) \tag{2.16}
$$

$$
= \frac{1}{(2\pi)^2 r(z)^2} \int_{-\infty}^{\infty} dk_{\parallel} e^{-ir(z) k_{\parallel}} T_X(k, z) \mathcal{R}(k) \tag{2.17}
$$

where $k = k_{\parallel} \mathbf{e}_z + \ell/r(z)$ and $k = \sqrt{k_{\parallel}^2 + \ell^2/r(z)^2}$.

### 2.2 Equal redshift correlations

Let us first correlate two variables $X$ and $Y$ at the same redshift, $a^X(\ell, z)$ and $a^{Y^*}(\ell', z)$. Using (2.16) we obtain

$$
\langle a^X(\ell, z) a^{Y^*}(\ell', z) \rangle = \frac{1}{2\pi} \int d^3k P_R(k) \delta^2(\ell - r(z) \mathbf{k}_{\perp}) \delta^2(\ell' - r(z) \mathbf{k}_{\perp}) T_X(k, z) T_{Y^*}(k, z) \tag{2.18}
$$

$$
= \delta^2(\ell - \ell') \frac{1}{\pi r(z)^2} \int_{-\infty}^{\infty} dk_{\parallel} P_R(k) T_X(k, z) T_{Y^*}(k, z), \tag{2.19}
$$

$$
C^{XY}_\ell(z, z) = \frac{1}{\pi r(z)^2} \int_{0}^{\infty} dk_{\parallel} P_R(k) T_X(k, z) T_{Y^*}(k, z). \tag{2.20}
$$

The situation is more complicated when we consider different redshifts $z \neq z'$, and we defer a discussion of this case to Section 2.3.

To determine the number counts we have to apply our formalism to the density fluctuation in comoving gauge, $D$, the the radial component of the velocity, $V_r$ and to the Weyl potential, $\Psi_W = (\Psi + \Phi)/2$. The latter then has to be integrated over the lightcone in order to obtain $\kappa$:

$$
a^D(\ell, z) = \int \frac{d^3k}{(2\pi)^2} \delta(\ell - r(z) \mathbf{k}_{\perp}) e^{-ik_{\parallel} r(z)} T_D(k, z) \mathcal{R}(k) \tag{2.21}
$$

$$
a^{\text{rad}}(\ell, z) = \mathcal{H}^{-1} \int \frac{d^3k}{(2\pi)^2} \delta(\ell - r(z) \mathbf{k}_{\perp}) e^{-ik_{\parallel} r(z)} k_{\parallel}^2 T_{V}(k, z) \mathcal{R}(k), \tag{2.22}
$$

$$
a^{\epsilon}(\ell, z) = 2(1 - \gamma(z))^2 \int \frac{d^3k}{(2\pi)^2} \int_{0}^{r(z)} dr \frac{r(z) - r}{r(z)^2} \delta(\ell - r(z) \mathbf{k}_{\perp}) e^{-ik_{\parallel} r} T_{\Psi_W}(k, z(r)) \mathcal{R}(k). \tag{2.23}
$$

For Eq. (2.22) we have used that $V(k, z) = i k T_{V}(k, z) \mathcal{R}(k)$, hence $(\partial_r V_r)(k) = (k_{\parallel}^2 / k) T_{V}(k, z) \mathcal{R}(k)$. In (2.23), $r(z)$ is the redshift of the comoving distance $r$, i.e. $r(z(r)) \equiv r$.

We now consider each term, first by itself and then its correlation with the other contributions. To compute a spectrum numerically we use (2.14) for $P_R(k)$ and employ the numerical transfer function from CLASS for the variable $X$. The remaining integral (2.20) is then computed with a simple Python code. For the lensing term this would lead to triple or double integrals and we therefore perform additional simplifications, as set out in Section 2.2.3 below.
2.2.1 Density Fluctuations

![Graph showing density fluctuations](image)

**Figure 1.** Left: the density in the flat sky approximation (solid) compared to the CLASS result (dashed) at redshifts $z = 1, 2, 3, 5$ from top to bottom. Right: the relative differences in percent.

In Fig. 1 we plot the density power spectrum at redshifts $z = 1, 2, 3, 5$. The solid lines present our approximation (2.20) with $X = Y = D$ while the dashed line are the numerical result from CLASS. The agreement is clearly excellent. To be more quantitative we also show the relative differences in the right panel. The difference is about 0.4% for $z = 5$, about 0.1% for $z = 1$ and typically 0.2% for $1 < z \leq 3$. Note that 0.1% is roughly the accuracy of the CLASS code itself, hence our agreement is as good as we can expect.

2.2.2 RSD

Here we repeat the same analysis for the redshift space distortion, where we have

$$X(z, k) = k_\parallel V_r / \mathcal{H} = \frac{k}{\mathcal{H}(z)} \left( \frac{k_\parallel}{k} \right)^2 T_Y(k, z) \mathcal{R}(k).$$  \hspace{1cm} (2.24)

In Fig. 2 we plot the RSD power spectrum at redshifts $z = 1, 2, 3, 5$. The solid lines present our approximation (2.20) with $X = Y = rsd$ while the dashed line are the numerical result from CLASS. Again we see very good agreement with the numerical result from CLASS. At low $\ell$ the approximation degrades somewhat, but the error is never larger than about 1%. In general the difference is 0.5% or less.

Again at high redshift our result is slightly below the CLASS value while at low redshift it is slightly above.
2.2.3 Lensing

To obtain the lensing term we write the flat sky approximation as

$$\langle a^e(\ell, z) a^{k*}(\ell', z') \rangle = \delta^2(\ell - \ell') \frac{2\ell^4}{\pi} \int_0^\infty dk_\parallel \int_0^{r(z)} dr \int_0^{r(z')} dr' \frac{(r(z) - r)(r(z') - r')}{r(z)r'(rr')^2} e^{ik_\parallel (r-r')} P_R(k) \times T_{\Psi_W}(k, z(r)) T_{\Psi_W}^*(k, z(r')).$$  \hspace{1cm} (2.25)

Eq. (2.25) is a triple integral of a rapidly oscillating function and hence very time-consuming numerically. To simplify it, we integrate (2.25) over $k_\parallel$ neglecting the dependence of the transfer functions and the power spectrum on $k_\parallel$, i.e., simply setting $k_\parallel = 0$ in the expression $k = \sqrt{k^2_\parallel + (\ell/r)^2}$. The integral of the exponential over $k_\parallel$ then yields a $\delta$-function in the resulting expression. This corresponds to setting

$$\int dk_\parallel f(r', r, k) \exp(ik_\parallel (r - r')) \simeq 2\pi f(r, r, |k_\perp|) \delta(r - r')$$

which is a good approximation for a slowly varying function $f(r, r', k)$ and for $\ell/r \gg k_\parallel$. The integral over $r'$ then simply eliminates the $\delta$-function and we obtain

$$C^e_\ell(z, z') = 4\ell^4 \int_0^{r_{\min}} dr \frac{(r(z) - r)(r(z') - r)}{r(z)r'(rr')^2} P_R(k)|T_{\Psi_W}(k, z(r))|^2,$$  \hspace{1cm} (2.26)

where now $k = \ell/r$ and $r_{\min} = \min\{r(z), r(z')\}$. As we shall see in Section 3, (2.26) simply corresponds to the Limber approximation [36, 37] which is often used for lensing. Eq. (2.26) is a single integral of a positive definite slowly-varying function which does not pose any problem and can be calculated with a simple Python code.
In Fig. 3 we show $C^\kappa_\ell(z, z)$ for the redshifts $z = 1, 2, 3, 5$. Our approximation (2.26) is excellent for $\ell > 20$ where the relative difference are typically about 0.5% and only for $z = 1$ larger than 1%. The same is true for unequal redshifts which are shown in Fig. 6. The agreement is bad at low $\ell \leq 10$, but this is not so surprising as for small $\ell$, neglecting $k_\parallel$ wrt. $k_\perp = \ell/r$ is certainly not a good approximation. For $z = 1$ the error also raises above 1% for $20 < \ell < 50$. But lensing from $z = 1$ is very subdominant (compare the amplitudes in Figs. 1 and 3) so that this error does not contribute significantly to the total error budget.

Even though this approximation is much better than the flat sky approximation for unequal redshifts which we discuss below, it cannot capture the behavior of the lensing term at low $\ell$. Nevertheless, we shall see that the low $\ell$ contribution from the lensing terms is subdominant so that we can still achieve a good approximation for the power spectrum of the full number count $C^\Delta_\ell(z, z)$. Note that while density and RSD decrease with increasing redshift, the situation is reversed for the integrated lensing term. While at $z = 1$, the lensing is about 100 times smaller than the density term, at redshift $z = 5$ it is only about twice smaller at low $\ell$.

### 2.2.4 Cross-Correlations

The number count expression (2.1) implies that the full number count power spectrum is given by

$$C^\Delta_\ell(z, z') = b(z)b(z')C^D_\ell(z, z') + b(z)C^{\text{r,sd}, D}_\ell(z, z') + b(z')C^{\text{r,sd}, D}_\ell(z, z') + C^{\text{r,sd}}_\ell(z, z')$$

$$+ b(z)(1 - \gamma(z'))C^{\text{r,sd, r}}_\ell(z, z') + (1 - \gamma(z))b(z')C^{\kappa, D}_\ell(z, z') + (1 - \gamma(z'))C^{\text{r,sd, r}, \kappa}_\ell(z, z')$$

$$+ (1 - \gamma(z))C^{\kappa, \text{r,sd}}_\ell(z, z') + (1 - \gamma(z))(1 - \gamma(z'))C^{\kappa}_\ell(z, z')$$  \hspace{1cm} (2.27)

For simplicity and in order to be as model independent as possible, we set $b(z) = 1$ and $\gamma(z) = 0$ in the following, but they are easily re-introduced for any specific example. In this section we show numerical examples of the correlation spectra, $C^X_\ell(z, z)$ for $X \neq Y$. As a consequence of the
continuity equation, the velocity transfer function is very simply related to the density transfer function via
\[ T_V(k,z) = f(z)T_D(k,z), \quad f(z) = -\frac{d\log D_1(z)}{d\log(1+z)}, \] (2.28)
where \( D_1(z) \) is the growth function of density perturbations. In a matter-dominated universe \( D_1 \propto 1/(1+z) \), while during dark energy domination \( D_1 \) grows slower and tends to a constant as \( \Omega_m \to 0 \). With this the correlation \( C^{D,\text{rsd}}_\ell(z,z) \) simply becomes
\[ C^{D,\text{rsd}}_\ell(z,z) \simeq \frac{f(z)}{\pi r(z)^2H(z)} \int_0^\infty dk \frac{k^2}{k} P_R(k)T_D(k,z)T^*_D(k,z). \] (2.29)

**Figure 4.** Left: the density-RSD cross spectrum in the flat sky approximation (solid) compared to the CLASS result (dashed) at redshifts \( z = 1, 2, 3, 5 \) from top to bottom. Right: the relative differences.

In Figure 4 we show some examples of the density-RSD cross-correlation spectrum for equal redshifts. Not surprisingly, the errors are like the ones for density or RSD terms, i.e. for \( \ell \geq 10 \) never larger than 0.5% and largest for high redshift and low \( \ell \). More precisely, the largest error at \( \ell = 2 \) and \( z = 5 \) is 0.7%.

Let us consider the lensing-density cross-correlation next. Inserting (2.21) and (2.23) in (2.20) we obtain
\[ C^{D,\kappa}_\ell(z,z') \simeq -\frac{\ell^2}{\pi} \int_{\infty}^{-\infty} dk_\parallel \int_0^{r(z')} \frac{r(z')-r}{r(z')rR^2} P_R(k)T_D(k,z)T^*_\Psi_W(k/z(r)) \exp(ik_\parallel(r(z)-r)). \] (2.30)
Here \( R = \sqrt{r(z)r} \), hence we cannot take \( P_R(k) \) in front of the \( r \)-integration, as \( k = \sqrt{k_\parallel^2 + \ell^2/R^2} \).

We perform the same simplification as in the lensing integral. We neglect the dependence on \( k_\parallel \) in \( k \) and integrate the exponential over \( k_\parallel \) which then yields \( 2\pi \delta(r(z)-r) \) so that we end up with
\[ C^{D,\kappa}_\ell(z,z') \simeq \begin{cases} 
-2\ell^2 \frac{r(z')-r(z)}{r(z')r(z)} P_R(\ell/r(z))T_D(\ell/r(z),z)T^*_\Psi_W(\ell/r(z),z) & \text{if } z < z' \\
0 & \text{if } z \geq z'.
\end{cases} \] (2.31)
Thus, for equal redshifts in this approximation, the contribution from the lensing-density cross-correlation vanishes. This is well-justified as the spectrum for this term is between two and three orders of magnitude smaller than the density or RSD autocorrelations individually, and can safely be neglected for equal redshifts.

\[ C_{\ell}^{\Delta \kappa}(z, z') \approx 0. \]  

In Figure 5 we show the total power spectrum result. The precision is always better than about 0.5%. Hence, neglecting the lensing-density and lensing-RSD terms does not degrade the accuracy. At the highest redshift, \( z = 5 \), lensing contributes about 10% to the total result, while at \( z = 1 \) it drops to about 0.1% and thus below the accuracy of our approximation.

### 2.3 Unequal redshift correlations

Let us also consider unequal redshifts. For lensing and lensing-density cross-correlations we simply use approximations (2.26) and (2.31). The results are shown in Figs. 6 and 7. While the lensing-lensing term is growing with redshift, the amplitude of the density-lensing term is more complex: the density decreases with increasing redshift while lensing increases. These two competing effects lead to a maximal signal (in amplitude, the sign of this term is always negative) at \((z, z') = (1, 1.5)\). At \((z, z') = (3, 4)\) the signal is smallest, while \((z, z') = (1, 1.1)\) yields the second smallest signal (for \( \ell > 40 \)) and \((z, z') = (2, 3)\) is the second largest signal. More precisely, the signals for \((z, z') = (1, 1.5)\) and \((z, z') = (2, 3)\) cross at \( \ell \approx 30 \). Note also that for \( z \geq 2 \), the positive lensing-lensing term always dominates over the negative density-lensing term, while for \((z, z') = (1, 1.5)\) the density-lensing term is larger for \( \ell \gtrsim 100 \) and for \((z, z') = (1, 1.1)\) it dominates already for \( \ell \gtrsim 60 \). This can be seen in Figure 8, which shows the sum of these two terms.
The precision of the lensing-lensing approximation at different redshifts is as good as the one at equal redshifts, namely on the order of 1% for $10 \leq \ell \leq 100$ and around 0.5% for $\ell > 100$. For high redshifts, $z \geq 3$ this precision is reached already at lower $\ell$.

Figure 6. Left: the lensing term in the flat sky approximation (solid) compared to the CLASS result (dashed) at redshifts $(z, z') = (1, 1.1), (z, z') = (1, 1.5), (z, z') = (1, 2), (z, z') = (3, 4)$ from bottom to top. Right: the relative differences.

Figure 7. Left: the density-lensing cross spectrum in our approximation (solid) compared to the CLASS result (dashed) at redshifts $(z, z') = (1, 1.1), (z, z') = (1, 1.5), (z, z') = (1, 2), (z, z') = (3, 4)$ from top to bottom. Right: the relative differences.

The density-lensing cross-correlations are less accurate than the pure lensing term, at low $\ell$. The reason here is that Limber approximation is less accurate for this case. This is especially true...
at low redshift, $z = 1$ where for $\ell < 60$ the error is larger than 1% and it exceeds 4% for $\ell < 20$. For higher redshifts the accuracy is better and an accuracy of 0.5% can be achieved for $\ell > 50$. For $\ell > 50$, the density-lensing cross-correlation is as accurate as the pure lensing term. Comparing Figs 6 and 7, we see that the pure lensing term actually dominates in all the cases presented in these figures. Only at very low redshift, $z < 1$, does the density-lensing term become larger. We have checked that the RSD-lensing term always remains very subdominant and we neglect it in our approximation.

Note also that these unequal-redshift correlations are always at least one or two orders of magnitude smaller than the full equal redshift result, which is dominated by the density and RSD terms. Therefore, the signal to noise of individual unequal-redshift terms is always very small. On the other hand, their number scales quadratic with the number of redshift bins and they can become relevant if we consider more than 10 bins. In Fig. 8 we plot the sum of the two lensing terms. Since the lensing-lensing term is positive while the lensing density term is negative, there can be significant cancellation. Especially, for $(z, z') = (1, 1.1)$, the total signal nearly vanishes for $\ell > 100$. For $(z, z') = (2, 3)$ the same cancellation is effective for $\ell > 200$. This second cancellation is especially relevant as at this redshift difference the lensing term usually dominates.

For the density and RSD, the so-called standard terms, unequal redshifts are much less straightforward. The first difficulty is the following fact: In real space, the correlation function for unequal redshifts is a function of $r\alpha - r'\alpha'$ where $r = r(z)$ and $r' = r(z')$. For $z \neq z'$ this is not proportional to $\alpha - \alpha'$. Therefore, upon Fourier transforming, we will not obtain a delta function $\delta(\ell - \ell')$, since we break the flat sky analog of statistical isotropy. The reason for this is that in principle we now consider correlations of functions that live on two different skies: one at comoving distance $r(z)$ and the other at $r(z')$. In order to restore statistical isotropy we have to...
project them onto one sky at some fiducial common distance $R$.

To do this we introduce the angles $\tilde{\alpha} = \alpha r/R$ and $\tilde{\alpha}' = \alpha' r'/R$, so that $r\alpha - r'\alpha' = R(\tilde{\alpha} - \tilde{\alpha}')$. Isotropy is now equivalent to translation invariance in the $\tilde{\alpha}$ plane. Furthermore, we can write

$$a^X(\ell, z) \simeq \frac{1}{2\pi} \int d^2\tilde{\alpha} e^{-i\ell \cdot \tilde{\alpha}} X(\tilde{\alpha}, z)$$

$$= \int d^2\tilde{\alpha} \int \frac{d^3k}{(2\pi)^3} e^{i(\ell - Rk_\perp) \cdot \tilde{\alpha}} e^{-i\delta r(z)} T_X(k, z) R(k).$$

(2.33)

The integration of $d^2\tilde{\alpha}$ just generates a Dirac-$\delta$ function (times $(2\pi)^2$) so that

$$a^X(\ell, z) = \int \frac{d^3k}{(2\pi)^2} \delta(\ell - Rk_\perp) e^{-i\delta r(z)} T_X(k, z) R(k).$$

(2.34)

This corresponds to Eq. (2.16) except that now, in the Dirac-$\delta$ function, $r(z)$ is replaced by $R$. Correlating two variables $X$ and $Y$ at redshifts $z$ and $z'$ now yields

$$\langle a^X(\ell, z) a^{Y*}(\ell', z') \rangle = \frac{1}{2\pi} \int d^3k P_R(k) \delta^2(\ell - Rk_\perp) \delta^2(\ell' - Rk_\perp) e^{-i\delta r(z) - i\delta r(z')}$$

$$\times T_X(k, z) T_Y^*(k, z')$$

$$= \delta^2(\ell - \ell') \frac{1}{R^2} \int_0^\infty dk_\parallel P_R(k) T_X(k, z) T_Y^*(k, z') \cos(k_\parallel (r - r')).$$

(2.35)

where $k = \sqrt{\ell^2/R^2 + k_\parallel^2}$. The power spectrum at unequal redshift is therefore given by

$$C^XY(\ell, \ell') \simeq \frac{1}{R^2} \int_0^\infty dk_\parallel P_R(k) T_X(k, z) T_Y^*(k, z') \cos(k_\parallel (r - r'))$$

$$= 2\pi \int_0^\infty \frac{dk_\parallel}{k_\parallel^2} P_R(k) T_X(k, z) T_Y^*(k, z') \cos(k_\parallel (r - r')).$$

(2.36)

This expression has two problems. First of all, the result depends on the choice of $R$ via $k$ and via the pre-factor $1/R^2$. If $z = z'$, we can simply choose $R = r(z)$ which is the true physical distance of the flat sky. However, for $z \neq z'$, there are different possibilities. The simplest choice is $R = \sqrt{rt}$. We shall see below that this is also the choice motivated by the exact expression. The second problem is that the integrand is now rapidly oscillating and, in what concerns the numerical computation, nothing is really gained wrt. the exact calculation performed by CLASS.

Therefore, let us go back and consider a term on the surface at redshift $z$, for example the density perturbation. Writing the exponential as a sum of Legendre polynomials and spherical Bessel functions it is easy to obtain the exact standard result [13] for two local (not integrated) variables $X$ and $Y$:

Replacing in (2.15) $e_z + \alpha = n$ by $n$ and expanding the exponential in Legendre polynomials and spherical Bessel functions, we obtain the exact expression

$$X(n, z) = \int \frac{d^3k}{(2\pi)^3} e^{-ir(z) n \cdot k} T_X(k, z) R(k)$$

$$= \sum_{\ell} \ell^2 (2\ell + 1) \int \frac{d^3k}{(2\pi)^3} P_\ell(n \cdot \hat{k}) j_\ell(kr(z)) T_X(k, z) R(k),$$

(2.40)

(2.41)
where \( P_\ell \) is the Legendre polynomial of degree \( \ell \). Applying the addition theorem for spherical harmonics, we find for the correlation function

\[
\langle X(n, z)Y(n', z') \rangle = \frac{1}{2\pi^2} \sum_\ell (2\ell + 1) P_\ell(n \cdot n') \int_0^\infty dk k^2 j_\ell(kr(z)) j_\ell(kr(z')) T_X(k, z) T_Y^*(k, z') P_\mathcal{R}(k),
\]

so that

\[
C_{\ell}^{XY}(z, z') = 4\pi \int_0^\infty \frac{dk}{k} j_\ell(kr(z)) j_\ell(kr(z')) T_X(k, z) T_Y^*(k, z') P_\mathcal{R}(k). \tag{2.42}
\]

For the last equation we used (2.11) and the fact that the correlation function is related to the power spectrum by

\[
\langle X(n, z)Y(n', z') \rangle = \frac{1}{4\pi} \sum_\ell (2\ell + 1) P_\ell(n \cdot n') C_{\ell}^{XY}(z, z'). \tag{2.43}
\]

So far, no approximation has been made and CLASS actually calculates the power spectra using expression (2.42).

We now use the following approximation for the Bessel functions, see [38]:

\[
j_\ell(x) \simeq \begin{cases} 
0, & x < L, \quad L = \ell + 1/2 \\
\cos[\sqrt{x^2 - L^2} - \arccos(\frac{L}{x}) - \pi/4] / \sqrt{(x^2 - L^2)^{1/4}}, & x > L
\end{cases} \tag{2.44}
\]

This approximation has a singularity at \( x \to L \), but is an excellent approximation for \( x \gtrsim L + 1 \). Inserting it into (2.42) we obtain

\[
C_{\ell}^{XY}(z, z') \simeq 4\pi \int_{r_{\min}}^{\ell+1/2} \frac{dk}{k \sqrt{k_- k_+}} P_\mathcal{R}(k) T_X(k, z) T_Y^*(k, z') \cos \left( r k_\parallel - r k_\perp \arccos \left( \frac{k_+}{k} \right) - \pi/4 \right) \times \cos \left( r' k'_\parallel - r' k'_\perp \arccos \left( \frac{k'_+}{k} \right) - \pi/4 \right) \tag{2.45}
\]

\[
= 2\pi \int_{r_{\min}}^{\ell+1/2} \frac{dk}{k \sqrt{k_- k_+}} P_\mathcal{R}(k) T_X(k, z) T_Y^*(k, z') \left\{ \cos \left[ r k_\parallel - r' k'_\parallel - r k_\perp \left( \arccos \left( \frac{k_+}{k} \right) - \arccos \left( \frac{k'_+}{k} \right) \right) \right] \right. \\
+ \sin \left[ r k_\parallel + r' k'_\parallel - r k_\perp \left( \arccos \left( \frac{k_+}{k} \right) - \arccos \left( \frac{k'_+}{k} \right) \right) \right] \right\}. \tag{2.46}
\]

Here \( r_{\min} = \min\{r, r'\} \) and we have defined

\[
k_\perp = \frac{\ell + 1/2}{r}, \quad k'_\perp = \frac{\ell + 1/2}{r'}, \quad k_\parallel = \sqrt{k^2 - k_\perp^2}, \quad k'_\parallel = \sqrt{k'^2 - k'_\perp^2}. \tag{2.47}
\]

Note \( r k_\perp = r' k'_\perp = \ell + 1/2 \), but \( r k_\parallel \neq r' k'_\parallel \). For \( z = z' \), after replacing \( \ell \to \ell + 1/2 \), neglecting the rapidly oscillating sin-term and making the variable transform \( kdk = k_\parallel dk_\parallel \), this becomes exactly our flat sky approximation (2.20).
However, when $z \neq z'$, the case of interest here, the original flat sky approximation (2.38) for $z \neq z'$ is obtained for $R^2 = r r'$ only if we neglect the sin-term, set $k'_\parallel = k_\parallel$ and drop fully the contributions in the argument of the cos which are proportional to $r k_\perp$ which corresponds to setting $k'_\perp = k_\perp$. We have found that while keeping the sin term is not crucial, the differences between $k_\parallel$ and $k'_\parallel$, as well as between $k_\perp$ and $k'_\perp$, are.

For redshifts that are sufficiently well-separated, terms of the above form become very small. In this case the spectrum is dominated by the integrated lensing and lensing differences and also the redshift difference above which they can be neglected with respect to the density terms which cannot be neglected, and we must use Eq. (2.46) to calculate them. Henceforth we refer to the collection of the terms $(C^D, C^\text{rd}, C^{D, \text{rd}})$ as the standard terms. We want to estimate their contribution for small redshift differences and also the redshift difference above which they can be neglected with respect to the lensing and lensing×density contributions.

For this we examine (2.46) in the particular case when $r$ and $r'$ are close,

$$r' = r(1 + \epsilon) \quad 0 < \epsilon \ll 1.$$  \hfill (2.48)

Without loss of generality we assume $r < r'$ and hence $\epsilon > 0$. Expanding the argument of the cos-term in (2.46), let us call it $a_-$, to second order in $\epsilon$ we find

$$a_- = r k_\parallel - r' k'_\parallel - r k_\perp \left( \arccos \left( \frac{k_\perp}{k} \right) - \arccos \left( \frac{k'_\perp}{k} \right) \right) \simeq -\epsilon r k_\parallel - \frac{\epsilon^2 k_\perp^2 r}{2 k_\parallel} + \mathcal{O}(\epsilon^3). \quad (2.49)$$

Considering only the term $\propto \epsilon$ results exactly in approximation (2.38). Expanding the argument of the sin-term, let us call it $a_+$, to second order in $\epsilon$ we find

$$a_+ = r k_\parallel + r' k'_\parallel - r k_\perp \left( \arccos \left( \frac{k_\perp}{k} \right) + \arccos \left( \frac{k'_\perp}{k} \right) \right) \simeq 2r k_\parallel - (2\ell + 1) \arccos \left( \frac{k_\perp}{k} \right) + \epsilon r k_\parallel + \frac{\epsilon^2 k_\perp^2 r}{2 k_\parallel} + \mathcal{O}(\epsilon^3). \quad (2.50)$$

The sin-term oscillates with a frequency of the order of $2\ell$ which is very rapid and we can neglect it when $k_\parallel$ is not very small so that $k > k_\perp$. The cos term oscillates slowly for small $\epsilon$ and we should take it into account. Nevertheless, our expansion in $\epsilon$ cannot be trusted in the regime of very small $k_\parallel$ since the $\epsilon^2$ terms diverge when $k_\parallel \to 0$. Furthermore, in this limit the term multiplied by $(2\ell + 1)$ in the sin tends to $\arccos(1) = 0$ and is not rapidly oscillating anymore. Before we can believe the approximation we therefore must request that the higher order terms in $\epsilon$ become smaller as the expansion proceeds. The series expansion is of the general form

$$a_- = \frac{x_\parallel^3}{L^2} \sum_{n=1}^\infty \alpha_n \left[ \left( \frac{L^2}{x_\parallel^2} \right) (1 + \mathcal{O}(x_\parallel^2/L^2)) \epsilon^3 \right]^n$$  \hfill (2.51)

$$a_+ = 2x_\parallel + \frac{x_\parallel^3}{L^2} \sum_{n=1}^\infty \beta_n \left[ \left( \frac{L^2}{x_\parallel^2} \right) (1 + \mathcal{O}(x_\parallel^2/L^2)) \epsilon^3 \right]^n$$  \hfill (2.52)

with coefficients $|\alpha_n| \leq 1$ and $|\beta_n| \leq 1$. Here we have introduced $x = k r$, $x' = k r' = x(1 + \epsilon)$, $x_\perp = k_\perp r = k'_\perp r' = x_\perp' = \ell + 1/2 \equiv L$. Note that

$$x_\parallel' = \sqrt{x_\parallel^2(1 + \epsilon)^2 + (2\epsilon + \epsilon^2)L^2}.$$  \hfill (2.53)
The terms in square brackets in the series are small when \( \frac{L^2}{x^3} \epsilon < 1 \) but diverge for \( x \parallel \to 0 \). We therefore must request that \( x \parallel > L\sqrt{\epsilon} \) at least, for these series to converge. On the other hand, even when \( x \parallel = L\sqrt{\epsilon} \), for small \( L \) the arguments become

\[
\begin{align*}
  a_-(x \parallel) &= L\sqrt{\epsilon} \sim L\sqrt{\epsilon^3}(1 + O(\epsilon)) \quad (2.54) \\
  a_+(x \parallel) &= L\sqrt{\epsilon} \sim L\sqrt{\epsilon^3}(1 + O(\epsilon)),
\end{align*}
\]

which may still be smaller than 1, especially for \( a_- \). We can neglect the integrals only when they start oscillating (rapidly). For \( x \parallel > 1/\epsilon \) we find

\[
a_-(x \parallel) = 1/\epsilon \sim -1 - \frac{L^2 \epsilon^3}{2} (1 + O(\epsilon)) \quad \text{while} \quad a_+(x \parallel) = 1/\epsilon \sim \frac{2}{\epsilon}(1 + O(\epsilon)).
\]  

Hence, we choose an \( x \parallel^{max} \equiv 10 \times \max\{L\sqrt{\epsilon}, 1/\epsilon\} \) above which both arguments of the cosine can become large and the contributions can be neglected.

Numerical testing shows that while the second (the sin) term does improve the overall amplitude of the approximate spectrum somewhat (largest effect for large epsilon) the integration introduces additional oscillations in \( \ell \), and increases the calculation time. Therefore we neglect the second term and examine the resulting approximation. Our the final expression for the approximation is:

\[
C_{\ell,XY}^{z,z'} \simeq \frac{2\pi}{r^2 r'} \left\{ \int_{0}^{x \parallel^{max}} dx \parallel T(k, z, z') \frac{\sqrt{x \parallel}}{x^2 \sqrt{x'}} \times \cos \left[ x \parallel - x' - (\ell + 1/2) \left( \arccos \left( \frac{\ell + 1/2}{x} \right) - \arccos \left( \frac{\ell + 1/2}{x'} \right) \right) \right] \right\}, \quad (2.57)
\]

where \( T(k, z, z') = \mathcal{P}_R(k)T_X(k, z)T_Y(k, z') \).

Finally, one can convert \( \epsilon \) into the redshift difference \( \Delta z = z' - z \). At first order \( \epsilon \) and \( \Delta z \) are related by

\[
r' = r(1 + \epsilon) = r + \frac{dr}{dz} \Delta z = r + \frac{1}{H(z)} \Delta z \quad \text{or} \quad \epsilon = \frac{\Delta z}{r(z)H(z)}. \quad (2.58)
\]

For \( C_{\ell,XY}^{z,z'} \) we have also converted the integral over \( x = rk \) into an integral over \( x \parallel \) using \( x \parallel dx \parallel = xdx \).

In Fig. 9 we plot the power spectrum of the standard terms for five different redshift pairs \((z, z')\). The solid lines present our approximation (2.57) with \( X = Y = D + \partial_r V_r / H \). Clearly the accuracy is much worse than for equal redshifts, when the redshift difference is not very small, but note also that the amplitude is small, especially for the problematic terms with the two largest redshift separations. As we argue below, for significant redshift differences the unequal redshift terms are largely dominated by the lensing terms. We use adjusted relative differences to determine the error. These are given by \( (C_{\ell,\text{app}} - C_{\ell,\text{ex}})/\bar{C}_{\ell,\text{ex}} \), where the \( \ell \)-band average \( \bar{C}_{\ell,\text{ex}} \) is defined by

\[
\bar{C}_{\ell,\text{ex}} = \sqrt{\sum_{\ell=50}^{50}(C_{\ell,\text{ex}})^2} \quad \text{for} \quad \ell > 50, \quad \text{and for smaller} \quad \ell \text{'s} \quad \text{as} \quad \text{many} \quad \text{neighbouring} \quad \text{points} \quad \text{as} \quad \text{are} \quad \text{available} \quad \text{are} \quad \text{used}. \quad \text{This} \quad \text{effectively} \quad \text{removes} \quad \text{large} \quad \text{spikes} \quad \text{in} \quad \text{the} \quad \text{errors} \quad \text{caused} \quad \text{by} \quad \text{zero-crossings} \quad \text{of} \quad \text{the} \quad \text{spectra}. \quad \text{The} \quad \text{redshift} \quad \text{pairs} \quad \text{are} \quad \text{chosen} \quad \text{as} \quad \text{follows}: \quad \text{The} \quad \text{(blue)} \quad \text{pair} \quad (z, z') = (1, 1.001) \quad \text{demonstrates} \quad \text{the} \quad \text{accuracy} \quad \text{of} \quad \text{the} \quad \text{approximation} \quad \text{in} \quad \text{the} \quad \text{limit} \quad \text{tending} \quad \text{to} \quad \text{the} \quad \text{equal} \quad \text{redshift} \quad \text{case}. \quad \text{The} \quad \text{pairs}
(z, z′) = (2, 2.004) and (z, z′) = (5, 5.015) are also chosen where the approximation is still accurate to within 10% or better. The remaining two pairs (z, z′) = (2, 2.2) and (z, z′) = (2, 3) show the invalidity of the approximation when the separation in redshift increases. However, the signal at these redshift differences is also very small, which makes this problem less relevant. In the figure the signal is enhanced by a factor of 50 for better visibility, while the error is reduced by a factor 10^{-4}, so that the maxima at 40% indicate an error of a factor of 4000. Our approximation has high and low frequency oscillations in ℓ which are not present in the standard result. If one averages the approximation over a rather large band of ∆ℓ ∼ 100, the approximation becomes better but it is still not better than the correct order of magnitude.

The approximation for the standard term at larger redshift separations is in general worse. However, since at large separations the lensing contribution dominates, this problem is not very severe in the total spectra, see Fig.11. More precisely, comparing the amplitudes of the lensing terms with those of the standard terms, we find that for redshift differences

\[ |z - z'| = \Delta z > \Delta z_1 \simeq 0.33 \frac{r(z)H(z)}{1 + z} \text{ and } \ell \sim 100 \]

the standard terms contribute less than 1% to the total result. In most cases, this is then true for all ℓ ≥ 20, but there are exceptions as we shall discuss.

The approximation for the standard terms at unequal redshifts is only valid (to within 10%) for small redshift separations. We find that this second critical value, below which the redshift will
have at most 10% error, follows a power law: \( \Delta z_0 \simeq 3.6 \times 10^{-4}(1 + z)^{2.14} \).

In Fig. 10 we show these critical separations \( \Delta z_0 \) and \( \Delta z_1 \) as functions of redshift.

![Figure 10](image)

**Figure 10.** The critical redshift differences, \( \Delta z_1 \simeq 0.33r(z)H(z)/(1 + z) \) (orange) and \( \Delta z_0 = 3.6 \times 10^{-4}(1 + z)^{2.14} \) (blue) respectively showing the values: above which the standard terms can be neglected (for mean values between \( \ell = 90 \) and \( \ell = 110 \), \( C^{STD}_\ell = 1\%C^{TOT}_\ell \)) and below which the approximation is accurate to 10% or less, in the unequal time correlators.

For redshift differences larger than \( \Delta z_1 \) we neglect the standard contribution, as we can again obtain a very good accuracy (1% or better) when including only the lensing terms. However, our approximation for the standard terms can only be trusted for redshift differences smaller than \( \Delta z_0 \), where it is accurate to 10% at least for \( z \neq z' \) and better than 0.5% for \( z = z' \). Therefore, for redshift differences \( |z - z'| \in [\Delta z_0, \Delta z_1] \) we have no satisfactory approximation and these cases need to be computed with the CLASS (or CAMB) code.

In Fig. 11 we show the total power spectrum for unequal redshifts. Again, the adjusted relative differences are used, and the redshift pairs are chosen to illustrate: the accuracy of the approximation in the limit approaching equal redshifts with separations close to \( \Delta z_0 \) \( ((z, z') = (1, 1.001), (z, z') = (2, 2.004) \) and \( (z, z') = (5, 5.015) \)) and more widely separated pairs with \( \Delta z > \Delta z_1 \) \( ((z, z') = (2, 3) \) and \( (z, z') = (3, 4) \)). At the redshift differences \( 3 - 2 = 1 \) and \( 4 - 3 = 1 \) which are larger than \( \Delta z_1(2) \) and \( \Delta z_1(3) \) correspondingly, we neglect the contribution from the standard terms which is inaccurate in this regime. At sufficiently small separations (the blue, orange and purple curves), the additional contribution of the lensing terms is small but the error, mainly due to the standard term, remains below 10%. For large separations, on the other hand, we neglect the standard terms and include only the approximation of the remaining lensing terms. The approximation for \( (z, z') = (3, 4) \) is very good, especially above \( \ell \approx 30 \). However for \( (z, z') = (2, 3) \) and \( \ell > 200 \), the error actually increases to about 15%. This is a very special redshift pair where the negative definite lensing-density correlation and the positive definite lensing-lensing correlation
Figure 11. Left: the total power spectrum result in the flat sky approximation (solid) compared to the CLASS result (dashed) at redshifts \((z, z') = (1, 1.001), (z, z') = (2, 2.004), (z, z') = (2, 2.2), (z, z') = (2, 3), (z, z') = (5, 5.015)\) from top to bottom. The 2 pairs at highest redshift separation \((z, z') = (2, 2.2)\) and \((z, z') = (2, 3)\) have been enhanced by a factor of 300 to facilitate interpretation of the figure. Right: the adjusted relative differences.

nearly cancel for \(\ell > 200\) (this can be seen in Fig. 8) so that the standard terms which we neglect here contribute up to 15%. We have checked this fact with CLASS which produces the same difference when neglecting the standard terms for this redshift pair. For higher redshifts the positive lensing-lensing term dominates, while for lower redshifts the negative density-lensing term dominates. In the pair \((3, 4)\), above the critical separation \(\Delta z_1(3) \simeq 0.5\), the lensing-lensing and lensing-density contributions for this redshift pair no longer cancel. The lensing-lensing term is truly the dominant contribution. As a result we see that, while the error increases for low \(\ell\), it remains on the order of a few percent for all \(\ell > 50\), and does not exhibit the particular structure that we see for the redshift pair \((z, z') = (2, 3)\). We have also tested (but not plotted) the pair \((2, 2.2)\), still below the critical separation \(\Delta z_1(2) \simeq 0.3\) where lensing contributes more than 99% to the signal, but much larger than \(\Delta z_0(2) \simeq 0.03\). For this case the error is very large at low \(\ell\) but reaches a level below 5% for \(\ell > 130\). For lower \(\ell\) the negative contribution of the standard terms to the total spectrum cannot be neglected, but is also not well modelled by our approximation. If such cases are relevant, they have to be computed with CLASS. As the separation increases even further (above the critical separation), the lensing terms, which are well approximated, make up a sufficiently significant portion of the total spectrum such that the standard terms can be neglected and the results has an error below 5%. This is true also for \((z, z') = (2, 2.2)\) and \(\ell > 130\).

Hence, there is an interplay between the accuracy of the approximation of the standard terms (which decreases for increasing separation) and their magnitude relative to the total power spectrum (which also decreases for increasing separation). If the separation is sufficiently small (e.g. blue, orange or purple curves), our approximation of the total spectra is very accurate. It degrades as the separation increases, but then it improves again once the lensing terms begin to
dominate (red curve). The approximation is worst for an intermediate value of the separation, \( \Delta z_0 < \Delta z < \Delta z_1 \) (e.g. the example not plotted, with \((z, z') = (2, 2.2)\) while \(\Delta z_1(2) \simeq 0.3\) and \(\Delta z_0(2) \simeq 0.004\), which attains sub-5% error only for \(\ell > 130\)), where our approximation of the standard terms is not accurate and we include only the lensing terms. Finally there is the special case \((z, z') = (2, 3)\) where the lensing terms nearly cancel for \(\ell > 200\) which degrades the approximation and induces an error of up to 15% (green curve in Fig. 11). There are also other redshift pairs at lower \(z\), e.g. \((z, z') = (1, 1.1)\) where the lensing terms nearly cancel, but these redshift separations are smaller and the standard terms are expected to be the most relevant contribution there. For \(z > 2.2\) no significant cancellation occurs anymore and the lensing term remains positive.

2.4 Overall Performance

Comparing Figs. 5 and 11 we see that there is a large difference in the accuracy of the approximations for equal and unequal redshifts. While for equal redshifts in the range of redshifts considered, \(z \in [1, 5]\), the agreement of our approximation with the CLASS result is always better than \(\sim 0.5\%\), the error of the unequal redshift correlators is smaller than a few percent only for small or large redshift differences, \(\Delta z \leq \Delta z_0\) or \(\Delta z \geq \Delta z_1\). For intermediate redshift differences, \(\Delta z_0 < \Delta z < \Delta z_1\) and \(\ell < 100\), our approximation cannot be trusted at all. The approximation for unequal redshifts can only be used with confidence for redshifts that either are very close, \(\Delta z \leq \Delta z_0\) or in the other extreme where the separation is sufficiently large, \(\Delta z \geq \Delta z_1\), such that the standard terms may be safely neglected. We also show the exception to this rule, namely the redshift pair \((z, z') = (2, 3)\). For \(z = 2\) and \(z' > 2.5\), the positive lensing-lensing term and the negative density-lensing term nearly cancel for \(\ell > 200\) so that the approximation neglecting the standard terms becomes worse again and the error increases to nearly 15\% for some values of \(\ell\).

In a real observation we cannot measure the \(C_\ell(z, z')\) at exact redshifts \(z, z'\) without error. First of all, even for spectroscopic surveys the redshift accuracy is finite, of order \(\Delta z \sim 10^{-4}(1+z)\). For photometric surveys redshift determination is much less accurate, of order \(\Delta z \sim 0.05(1+z)\) in the most optimistic case. But even if redshift accuracy is very high, in a too slim redshift bin there are few galaxies, and shot noise will prevent the determination of the \(C_\ell\)’s especially at high \(\ell\). We therefore have also investigated windowed \(C_\ell\)’s defined by

\[
C_\ell(z, \Delta z) = \int dz_1 dz_2 C_\ell(z_1, z_2) W_{\Delta z}(z, z_1) W_{\Delta z}(z, z_2)
\]

(2.59)

where \(W_{\Delta z}(z, z')\) denotes a (normalized) window function of full width \(\Delta z\) centred at \(z\). Typically one chooses a Gaussian or a tophat window. In the windowed \(C_\ell\)’s unequal redshift correlators always enter and our reduced accuracy for them therefore affects the windowed \(C_\ell\)’s. If we choose a slim window, \(\Delta z \lesssim 2\Delta z_0\), as shown in Fig. 12, the approximation is good with an error of 4.5\% or less for \(z \leq 3\). For \(z = 5\), the error decreases to less than 4\%.

However, if we choose a photometric window width, \(\Delta z \gtrsim 0.05(1+z)\), the accuracy degrades significantly, up to 17\%, see Fig. 13.

The reason for this is clear, we enter the regime \(\Delta z_0 < \Delta z < \Delta z_1\) for which we have no good approximation. Neglecting the standard terms already for \(\Delta z > \Delta z_0\) is also not a good approximation. The result then underestimated the true windowed \(C_\ell\)’s be nearly a factor of 10. This somewhat surprising finding shows that the standard terms do contribute significantly (about 90\% in total) also for \(z \neq z'\) in the interval \(\Delta z_0 < \Delta z < \Delta z_1\). Also multiplying the result from the \(\Delta z_0\)-window by a factor \(\Delta z/\Delta z_0\) does not give an accurate approximation. Therefore, a windowed
Figure 12. Left: the windowed full power spectrum, $C^{\Delta}(z, \Delta z)$ in the flat sky approximation (solid) compared to the CLASS result (dashed) at redshifts $z = 1, 2, 3, 5$ from top to bottom. A top-hat window function of width $\Delta z = 2\Delta z_0$ has been applied. Right: the relative differences.

Figure 13. Left: the windowed full power spectrum, $C^\Delta(z, z)$ in the flat sky approximation (solid) compared to the CLASS result (dashed) at redshifts $z = 1, 2, 3, 5$ from top to bottom. A top-hat window function of width $\Delta z = 0.05(1 + z)$ has been applied. Right: the relative differences. Clearly the approximation is not satisfactory.
correlation function cannot be determined with an accuracy better than 17% with the flat sky approximation except for very narrow redshift bins.

3 Comparison to the Limber Approximation

An approximation which is well-known, especially for lensing, is the so-called Limber approximation [36, 37]. We shall see that while this approximation is equivalent to the flat sky one for lensing and density-lensing correlations, it is very different and actually a bad approximation for the density and RSD terms. The fact that the Limber approximation does not work for density and RSD has already been noted in Refs. [39, 40].

We start with the exact expression (2.42) which is also used in CLASS to calculates the power spectra. The Limber approximation now makes use of the fact that, for a sufficiently slowly varying function \( f(x) \) one can approximate

\[
\int x^2 f(x) j_\ell(yx) j_\ell(y'x) dx \simeq \frac{\pi}{2y^2} \delta(y - y') f \left( \frac{\ell + 1/2}{y} \right) \tag{3.1}
\]

This equation is exact if \( f \) is a constant. Using it in (2.42) for \( X = Y = D \) yields

\[
C^D_\ell (z, z') = \frac{\delta(r - r')}{r(z)^2} P_R \left( \frac{\ell + 1/2}{r(z)} \right) \left| T_D \left( \frac{\ell + 1/2}{r(z)}, z \right) \right|^2
\]

\[
= \frac{\delta(z - z') H(z)}{r(z)^2} P_R \left( \frac{\ell + 1/2}{r(z)} \right) \left| T_D \left( \frac{\ell + 1/2}{r(z)}, z \right) \right|^2. \tag{3.3}
\]

Up to \( \ell \to \ell + 1/2 \), this is obtained from (2.38) if we neglect \( k_\parallel \) in \( k \), i.e. we set \( k \simeq \ell/R \) in \( P_R(k) \) and \( T_D(k, z) \) and integrate the cosine over \( k_\parallel \). Hence, the Limber approximation is analogous to the flat sky approximation which we used for the lensing terms. The \( \delta \)-function pre-factor means that for a physically sensible result we have to introduce a window function such that

\[
C^D_\ell (z, z', \Delta z) = \int dz_1 dz'_1 W_{\Delta z}(z, z_1) W_{\Delta z}(z', z'_1) C^D_\ell (z_1, z'_1)
\]

\[
= \int dz_1 W_{\Delta z}(z, z_1) W_{\Delta z}(z', z_1) \frac{H(z_1)}{r(z_1)^2} P_R \left( \frac{\ell + 1/2}{r(z_1)} \right) T_D^2 \left( \frac{\ell + 1/2}{r(z_1)}, z_1 \right). \tag{3.4}
\]

In other words, density fluctuations are only considered correlated only if they are at equal redshift and for unequal redshifts correlations are due entirely to overlapping window functions.

For the lensing spectra we find

\[
C^\kappa_\ell (z, z') = 4[(\ell + 1)] \int_0^{r_{\text{min}}} \frac{dr}{r^2} \frac{(r(z) - r)(r(z') - r)}{r(z)r(z')r^2} P_R \left( \frac{\ell + 1/2}{r} \right) \times
\]

\[
\left| T_{\Psi W} \left( \frac{\ell + 1/2}{r}, z(r) \right) \right|^2, \tag{3.5}
\]

\[
C^{D,\kappa}_\ell (z, z') = \begin{cases} 
-2\ell(\ell + 1) \frac{(z'_z - r(z))}{r(z)'} P_R \left( \frac{\ell + 1/2}{r(z)} \right) T_D \left( \frac{\ell + 1/2}{r(z)}, z \right) T_{\Psi W} \left( \frac{\ell + 1/2}{r(z)}, z \right) \\
0 \quad \text{if } z < z', \\
& \text{if } z \geq z', 
\end{cases} \tag{3.6}
\]

\[
C^{R,\kappa}_\ell (z, z') = \begin{cases} 
-2\ell(\ell + 1) \frac{(z'_z - r(z))}{r(z)'} P_R \left( \frac{\ell + 1/2}{r(z)} \right) T_D \left( \frac{\ell + 1/2}{r(z)}, z \right) T_{\Psi W} \left( \frac{\ell + 1/2}{r(z)}, z \right) \\
0 \quad \text{if } z < z', \\
& \text{if } z \geq z', 
\end{cases} \tag{3.7}
\]
Contrary to the Limber approximation of the density, for $z = z'$ these are identical to the simplified flat sky approximations given in the previous section (up to $\ell \to \ell + 1/2$).

The RSD terms are more subtle. In Appendix A we show that RSD terms are suppressed in the Limber approximation. It is well known that for very narrow windows, RSD can be nearly as large as the density term and therefore this approximation can at best hold for sufficiently large windows where RSD are in any case suppressed.

In Fig. 14 we compare the numerical result with the Limber approximation for the density contribution for $z = 1$ and different window widths $\Delta z$. Clearly, the Limber approximation cannot be trusted even at quite high $\ell$ if $\Delta z$ is not very large.

In Fig. 14 we see that the Limber approximation of the density term is extremely bad for slim windows, but gets increasingly more accurate as the windows widen. The accuracy is within $\sim 6\%$ above $\ell = 100$ for the widest window with $\Delta z = 0.3$. As expected the Limber approximation also improves with increasing $\ell$. For $\Delta z = 0.1$, an accuracy better than $\sim 6\%$ is achieved only above $\ell = 200$.

4 Discussion and Conclusions

In this paper we have investigated the 'flat space' approximation for the galaxy number counts, which allows the calculation of the $C^\Delta_\ell(z, z')$ with a simple, not heavily oscillating 1d numerical integral once the transfer function is known. For the lensing terms, the flat sky approximation is equivalent to the Limber approximation, but for density and RSD it is very different. For the
density term, the Limber approximation becomes good to about 5% above $\ell \sim 100$ only for very wide redshift bins of $\Delta z \gtrsim 0.3$ for $z = 1$.

For equal redshifts our approximation for the standard terms, i.e., density and redshift space distortions, up to $z = 3$ is accurate within about 0.3% which is close to the accuracy of CLASS itself, for all values of $\ell \geq 2$. Including the lensing terms and going to $z = 5$, for $\ell > 10$, the approximation is still excellent, better than 0.5%. This is our first main result.

However, for unequal redshifts, while the approximation for the lensing terms remains very good, the approximation of density and RSD degrades rapidly with growing redshift difference. For sufficiently large $\Delta z$, i.e. $\Delta z > \Delta z_1 \approx 0.33 r(z) H(z)/(z+1)$, the lensing terms in general dominate and the unequal redshift $C_\ell$'s can be approximated well by neglecting the standard terms. Hence also for large redshift differences we have a good approximation with errors of about 0.5% or less for $\ell \gtrsim 60$. For redshift differences beyond $\Delta z_0 \approx 3.6 \times 10^{-4} (1+z)^{2.14}$, the error of the standard term approximation becomes larger than 10% and the flat sky approximation deviates by several orders of magnitude for $\ell > 100$ once we reach $\Delta z_1$. This is our second main result: for the standard terms, the flat sky approximation is very bad for unequal redshifts.

These results are valid for Dirac-$\delta$ windows in redshift space, which is equivalent to redshift errors of less than about $10^{-4}$, i.e. spectroscopic redshifts. The approximation remains good, within a few percent, for windows slimmer than $\Delta z = 2\Delta z_0$. If we want to study photometric redshift bins, we have to include a window function of width $\Delta z = 0.05 (1+z)$ or more. Doing this will always include redshift differences for which the flat sky approximation is not valid. We have found that the approximation underestimates the true result by up to 17%. This somewhat disappointing result is shown in Fig. 13. It is our third main result: for photometric windows, the unequal redshift standard terms are sufficiently important to degrade the the flat sky approximation considerably.

Clearly, the flat sky approximation cannot be used to estimate the $C_\ell$'s at equal redshifts with photometric bin width. As a next step we plan to find an approximation that works also for unequal redshifts where we clearly have to go beyond both, the flat-space and the Limber approximations. This is needed to obtain a useful approximation also for photometric redshifts. It is also surprising that including only correlations of standard terms with redshift differences up to $\Delta z_0$ underestimates the windowed $C_\ell$'s by as much as a factor 10 for bin widths of $\Delta z = 0.05 (1+z)$.

For spectroscopic redshifts it is often more useful to employ the correlation function since very narrow redshifts bins are plagued by shot noise on the one hand and by non-linearities in the radial direction on the other hand [31]. It will therefore be useful to investigate how the flat-sky approximation can be used for the correlation function as calculated e.g. in Ref. [32].

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A The Limber approximation for the redshift space distortion

The exact expression for the RSD is not of the form (3.1) but

$$\int x^2 F(x) j''_\ell(y x) j''_\ell(y' x) dx.$$  \hspace{1cm} (A.1)
More precisely, from the expressions in [13] one finds

\[ C_{\ell}^{\text{rsd}}(z_1, z_2) = 4\pi f(z_1)f(z_2) \int dk k^2 j_\ell'(kr_1) j_{\ell}'(kr_2) P_R(k) T_D(k, z_1) T_D(k, z_2) \quad \text{and} \quad \text{(A.2)} \]

\[ C_{\ell}^{D,\text{rsd}}(z_1, z_2) = 4\pi f(z_2) \int dk k^2 j_\ell'(kr_1) j_{\ell}'(kr_2) P_R(k) T_D(k, z_1) T_D(k, z_2). \quad \text{(A.3)} \]

Here \( f(z) \) is the growth function defined in (2.28). In a \( \Lambda \)CDM universe one has [13]

\[ f(z) = 1 + 1/2 \frac{\dot{T}_\Psi}{T_\Psi}. \quad \text{(A.4)} \]

In order to find an approximation for integrals of the form (A.1), we use the identity

\[ j_\ell'(x) = \frac{\ell^2 - \ell - x^2}{x^2} j_\ell(x) + \frac{2}{x} j_{\ell+1}(x). \quad \text{(A.5)} \]

To perform the required integrals we not only need a formula for integrals with equal \( \ell \)'s but also with \( \ell \) and \( \ell + 1 \). One might be tempted to neglect the latter, but a numerical study actually shows that both contribution to the above integral are of the same order. We therefore follow [40] and use the following crude approximation for the spherical Bessel functions to obtain integrals of unequal \( \ell \)

\[ j_\ell(x) \sim \sqrt{\frac{\pi}{2\ell + 1}} \delta(\ell + 1/2 - x) \quad \text{which yields} \quad \text{(A.6)} \]

\[ \frac{2}{\pi} \int_0^\infty dk k^2 F(k) j_\ell(kr_1) j_{\ell+1}(kr_2) \sim \frac{2\ell + 1}{2\ell + 3} P_\ell \left( \frac{\ell + 1/2}{r_1} \right) \frac{\delta \left( \frac{r_1}{2r_2} - 1 \right)}{r_1^2}. \quad \text{(A.7)} \]

Inserting (A.7) and (A.5) in (A.2) we obtain \((r_1 = r(z_1) \text{ and } r_2 = r(z_2))\)

\[ C_{\ell}^{\text{rsd}}(z_1, z_2) \sim \frac{4f^2(z_1)}{\ell^2} \left[ C_{\ell}^{D}(z_1, z_2) + C_{\ell+1}^{D}(z_1, z_2) - \left\{ \frac{\delta \left( \frac{r_1}{2r_2} - 1 \right)}{r(z)^2} P_R \left( \frac{\ell + 1/2}{r_1} \right) \right\} \right] \times \]

\[ T_D \left( \frac{\ell + 1/2}{r_1}, z_1 \right) T_D \left( \frac{\ell + 1/2}{r_1}, z_1 \right) + (r_1 \leftrightarrow r_2) \}. \quad \text{(A.8)} \]

For the second term we have used that both \( f(z) \) and \( T_D(k, z) \) are slowly varying functions and we have neglected the difference between \( z_1 \) and \( z_2 \) in them. We have also neglected higher order terms in \( 1/\ell \). Note that if we neglect the difference between \( \ell + 3/2 \) and \( \ell + 1/2 \), the second term just cancels the first term. Numerical evaluation also has shown, that for \( F \approx \text{constant} \), the integral (A.7) as a function of \( r_2 \) peaks even closer to \( r_2 = r_1 \) than to \( r_2 = r_1 \frac{2\ell + 3}{2\ell + 1} \). Therefore, RSD are strongly suppressed in the Limber approximation. Numerically one finds that, like for density perturbations, the Limber approximation is valid only in very wide windows where redshift space distortions are indeed strongly suppressed.

Inserting (A.5) in (A.3), we obtain for the density-RSD correlations in the Limber approximation

\[ C_{\ell}^{D,\text{rsd}}(z_1, z_2) \sim -\frac{2f(z_1)}{\ell} \left[ C_{\ell}^{D}(z_1, z_2) - \frac{\delta \left( \frac{r_1}{2r_2} - 1 \right)}{r(z)^2} P_R \left( \frac{\ell + 1/2}{r_1} \right) \right] \times \]

\[ T_D \left( \frac{\ell + 1/2}{r_1}, z_1 \right) T_D \left( \frac{\ell + 1/2}{r_1}, z_1 \right) \}. \quad \text{(A.9)} \]
As for the pure RSD term, when neglecting the difference between $r_1$ and $r_2$, this term vanishes and therefore, for sufficiently high $\ell$, where the Limber approximation is applicable, it is negligible.

Like for the density term, the RSD Limber approximation has to be integrated over a window with some finite width $\Delta z$ to become physically meaningful. But for large window sizes, where the Limber approximation for the density becomes reasonably accurate, the contribution from RSD can actually be neglected. Also the Limber approximation of lensing–RSD is always very small. Therefore, the Limber approximation for RSD is either very bad (for slim redshift bins) or too small to be relevant. For the wide redshift bins where the Limber approximation for the dominant density term can be sufficiently accurate, the RSD contribution is never relevant, and neglecting it is a good approximation.

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