Integrable Flows of Curves/Surfaces,
Generalized Heisenberg Ferromagnet Equation
and Complex Coupled Dispersionless Equation

Guldana Bekova, Kuralay Yesmakhanova, Gaukhar Shaikhova,
Gulgassyl Nugmanova and Ratbay Myrzakulov

Eurasian International Center for Theoretical Physics and
Department of General & Theoretical Physics, Eurasian
National University, Astana, 010008, Kazakhstan

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Abstract
In the present paper, we study the Myrzakulov-XIII (M-XIII)
equation geometrically. From the geometric point of view, we estab-
lish a link of the M-XIII equation with the motion of space cur-
es in the 3-dimensional space $\mathbb{R}^3$. We also show that the complex coupled
dispersionless (CCD) equation can be derived from the geometrical
formalism such that their curve flows are formulated. Finally, the
gauge equivalence between the M-XIII equation and the CCD equa-
tion is established.

1 Introduction

One of classical nonlinear differential equations integrable by through inverse
scattering transform (IST) is the Heisenberg ferromagnet (HF) equation

$$iA_t + \frac{1}{2} [A, A_{xx}] = 0. \quad (1)$$

It describes the evolution of spin waves in nonlinear dispersive magnetic me-
dia. It admits several integrable and nonintegrable generalizations like the
Landau-Lifshitz equation, Ishimori equation and so on. It has been very suc-
cessful applications in physics and mathematics [4]. However, in the different
nonlinear regimes of spin waves and for mathematical needs, the HF equation becomes less accurate or needs some extensions [6]. There are usually some approaches to meet these requirements in the literature. For example, the first one is to add several higher-order dispersive terms to get higher-order HF equation [2]. The second one is to construct some multidimensional extensions. Several HF equations both integrable and nonintegrable have been proposed by these approaches [7, 8, 9, 10]. So that since the time the complete integrability of HF equations was discovered, many attempts for constructing its generalization have been made [22]. One of such extensions of the HF equation is the M-XIII equation which we are going to investigate in this paper.

The outline of the present paper is organized as follows. In section 2, we present the M-XIII equation. In section 3, the relation between the motion of space curves and the M-XIII equation is established. Then using this relation we found that the Lakshmanan (geometrical) equivalent counterpart of the M-XIII equation is the well-known complex coupled dispersionless (CCD) equation. The gauge equivalence between the M-XIII equation and the CCD equation is established in section 4. In Section 5, we study the relation between the M-XIII equation and differential geometry of surface. The paper is concluded by comments and remarks in section 6.

2 Myrzakulov-XIII equation

In this section, we consider the following Myrzakulov-XIII (M-XIII) equation

\[ iA_y = \frac{1}{2k} [A, A_s]_y + \frac{i}{k^2}(\rho A)_y. \]  (2)

Here \( \sigma = \pm 1, \ \ k = const, \)

\[ A = \begin{pmatrix} A_3 \\ A^+ \\ A^- \\ -A_3 \end{pmatrix}, \ \ A^2 = I, \ \ A^\pm = A_1 \pm iA_2, \ \ \sigma(A_1^2 + A_2^2) + A_3^2 = 1 \]  (3)

and

\[ \rho = \pm \sqrt{1 - \frac{\sigma k^2}{2} \text{tr}(A_s^2)} = \sqrt{1 - \sigma k^2 A_s^2}, \ \ A = (A_1, A_2, A_3), \ \ A^2 = 1. \]  (4)

The M-XIII equation (2) is integrable. Its Lax representation (LR) reads as

\[ \Phi_y = i(k - \lambda) A \Phi = U_1 \Phi, \]  (5)

\[ \Phi_s = \left[ \frac{i(k - \lambda)}{4k\lambda} \rho A + \frac{(k - \lambda)}{2\lambda} A A_s \right] \Phi = V_1 \Phi. \]  (6)


3 Lakshmanan (geometrical) equivalent counterpart

Let us now find the Lakshmananan or that is same the geometrical equivalent counterpart of the M-XIII equation (2) for the case $\sigma = 1$. To do that, let us rewrite the M-XIII equation (2) in the vector form. We have several equivalent vector forms:

i)

$$
\mathbf{A}_s = \frac{1}{k} (\mathbf{A} \wedge \mathbf{A}_s)_y + \frac{1}{k^2} (\rho \mathbf{A})_y, \quad (7)
\rho_y = k \mathbf{A} \cdot (\mathbf{A}_s \wedge \mathbf{A}_y). \quad (8)
$$

ii)

$$
\mathbf{A}_s = \frac{1}{k} \mathbf{A} \wedge \mathbf{A}_{sy} + \frac{1}{k^2} \rho \mathbf{A}_y, \quad (9)
\rho_y = k \mathbf{A} \cdot (\mathbf{A}_s \wedge \mathbf{A}_y). \quad (10)
$$

iii)

$$
\mathbf{A}_s = \frac{1}{k} \mathbf{A} \wedge \mathbf{A}_{sy} + \frac{1}{k} \partial^{-1}_y [\mathbf{A} \cdot (\mathbf{A}_s \wedge \mathbf{A}_y)] \mathbf{A}. \quad (11)
$$

iv)

$$
\mathbf{A}_s = \frac{1}{k} \mathbf{A} \wedge \mathbf{A}_{sy} \pm \frac{1}{k^2} \sqrt{1 - \sigma k^2 \mathbf{A}_s^2} \mathbf{A}_y. \quad (12)
$$

Let us now we consider a curve in $R^3$ which is given by the unit vectors $\mathbf{l}_k$. These vectors obey the Frenet-Serret equations

$$
\begin{pmatrix}
\mathbf{l}_1 \\
\mathbf{l}_2 \\
\mathbf{l}_3
\end{pmatrix}_y = C
\begin{pmatrix}
\mathbf{l}_1 \\
\mathbf{l}_2 \\
\mathbf{l}_3
\end{pmatrix}, \quad
\begin{pmatrix}
\mathbf{l}_1 \\
\mathbf{l}_2 \\
\mathbf{l}_3
\end{pmatrix}_s = G
\begin{pmatrix}
\mathbf{l}_1 \\
\mathbf{l}_2 \\
\mathbf{l}_3
\end{pmatrix}. \quad (13)
$$

Here $\mathbf{l}_1, \mathbf{l}_2,$ and $\mathbf{l}_3$ are the unit tangent, normal and binormal vectors to the curve respectively, $x$ is its arclength parametrising the curve. The matrices $C$ and $G$ have the forms

$$
C = \begin{pmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{pmatrix}, \quad G = \begin{pmatrix}
0 & \gamma_3 & -\gamma_2 \\
-\gamma_3 & 0 & \gamma_1 \\
\gamma_2 & -\gamma_1 & 0
\end{pmatrix}. \quad (14)
$$

3
The curvature and torsion of the curve are given by the following formulas

\[ \kappa = \sqrt{l_{1y}^2}, \quad \tau = \frac{l_1 \cdot (l_{1y} \wedge l_{1yy})}{l_{1y}^2}. \]  

(15)

The compatibility condition of the equations (13) is given by

\[ C_s - G_y + [C, G] = 0 \]  

or in elements

\[ \kappa_s = \gamma_{3y} + \tau \gamma_2, \]  

(17)

\[ \tau_s = \gamma_{1y} - \kappa \gamma_2, \]  

(18)

\[ \gamma_{2y} = \tau \gamma_3 - \kappa \gamma_1. \]  

(19)

Now we do the following identification:

\[ A \equiv l_1. \]  

(20)

Then we have

\[ \kappa^2 = \frac{A_y^2}{A_y^2}, \]  

(21)

\[ \tau = \frac{A \cdot (A_y \wedge A_{yy})}{A_y^2}. \]  

(22)

and

\[ \gamma_1 = \frac{\kappa \rho \tau + k \kappa_s y}{k k (k + \tau)}, \]  

(23)

\[ \gamma_2 = -\frac{\kappa_s}{k}, \]  

(24)

\[ \gamma_3 = \frac{\kappa \rho}{k^2} - \frac{\kappa \rho \tau + k \kappa_s y}{k^2 (k + \tau)}. \]  

(25)

The equations for \( \kappa \) and \( \tau \) reads as

\[ \kappa_s = \left[ \frac{\kappa \rho}{k^2} - \frac{\kappa \rho \tau + k \kappa_s y}{k^2 (k + \tau)} \right]_y \frac{\tau \kappa_s}{k}, \]  

(26)

\[ \tau_s = \left[ \frac{\kappa \rho \tau + k \kappa_s y}{k k (k + \tau)} \right]_y + \frac{(\kappa^2)_s}{2 k}. \]  

(27)

Now we introduce a new function \( v \) as

\[ v = k \int \frac{\gamma_3}{\kappa} ds. \]  

(28)

4
It is not difficult to verify that the functions $\kappa$ and $v$ are solutions of the following equations

$$
\kappa_{sy} = \kappa v_y v_s - 0.5 \partial_y^{-1} [(|q|^2)_s] \kappa, \quad (29)
$$
$$
v_{sy} = \frac{\kappa s v_y}{\kappa y v_s}. \quad (30)
$$

Next we introduce a new complex function $q(s, y)$ as

$$
q = \kappa e^{iv}. \quad (31)
$$

As result, the function $q$ satisfies the following equation

$$
q_{sy} + 0.5 \partial_y^{-1} [(|q|^2)_s] q = 0. \quad (32)
$$

Let us rewrite this equation as

$$
q_{sy} - \rho q = 0, \quad (33)
$$
$$
\rho_y + 0.5(|q|^2)_s = 0. \quad (34)
$$

It is nothing but the focusing complex coupled dispersionless (CCD) system $[37]-[38]$. Thus we have proved the Lakshmanan (geometrical) equivalence between the M-XIII equation (2) and the CCD equation (33)-(34). Finally, some comments in order. From (33)-(34) follows that $[37]-[38]$

$$
\rho^2 + \sigma |q_s|^2 = const \quad (35)
$$

or for simplicity

$$
\rho^2 + \sigma |q_s|^2 = 1. \quad (36)
$$

Thus

$$
\rho = \pm \sqrt{1 - \sigma |q_s|^2}. \quad (37)
$$

Finally instead of the set (33)-(34) we have

$$
q_{sy} \mp \sqrt{1 - \sigma |q_s|^2} q = 0. \quad (38)
$$

On the face of it, the set of equations (33)-(34) contains two dependent variables $q$ and $\rho$. But as follows from (36), in fact we have only one dependent variable namely $q(y, s)$. 

5
4 Gauge equivalent counterpart

In the section 2, we have proved that the M-XIII equation (2) and the CCD equation (33)-(34) is the lakshmanan/geometrically equivalent each to other. Let us we now show that these equations are gauge equivalent each to other. Consider the gauge transformation $\Phi = g^{-1}\Psi$, where $g = \Psi|_{\lambda=k}$. Then it is not difficult to show that the function $\Psi$ satisfies the following set of equations

$$\Psi_y = U_2\Psi,$$

$$\Phi_s = V_2\Phi,$$

where

$$U_2 = -i\lambda\sigma_3 + 0.5 \begin{pmatrix} 0 & -q \\ q & 0 \end{pmatrix}, \quad V_2 = \frac{i}{4\lambda}[\rho\sigma_3 + \begin{pmatrix} 0 & q_s \\ q_s & 0 \end{pmatrix}].$$

The compatibility condition of the equations (39)-(40) is equivalent to the CCD equation (33)-(34). It means that between the M-XIII equation (2) and the CCD equation (33)-(34) takes place the gauge equivalence.

5 Integrable surfaces related with the M-XIII equation

In this section, our aim is to establish the link between the M-XIII equation (2) and differential geometry of surface.

5.1 Case 1: $A \equiv r_y$

Consider the identification

$$A \equiv r_y,$$

where $r(y,s)$ is the position vector of the curve embedded on the surface. In terms of $r$, the M-XIII equation (2) converted to the equations

$$r_s = \frac{1}{k}r_y \wedge r_{sy} + \frac{1}{k^2}\rho r_y,$$

$$\rho_y = kA \cdot (A_s \wedge A_y).$$

This set can be rewritten as

$$r_{ys} + kr_y \times r_s = 0.$$
Note that for the case \(k = -1\), the last equation was obtained in [38] and studied in detail. As integrable system, Eq.(43)-(44) admits the following LR

\[
F_y = 0.5i(k - \lambda)yF = U_3F, \tag{46}
\]
\[
F_s = \left[ \frac{i(k - \lambda)}{2k\lambda} \rho y + \frac{(k - \lambda)}{2\lambda} r_y r_{ys} \right] F = V_3F. \tag{47}
\]

This LR gives

\[
ir_{sy} = \frac{1}{4k}[r_y, r_{sy}] + \frac{i}{4k^2} \rho y, \tag{48}
\]
\[
\rho_y = -iktr(r_y \cdot [r_{yy}, r_{sy}]). \tag{49}
\]

It is just the matrix form of the equation (43)-(44). Also we note that

\[
\rho = kr_y \cdot r_s = k \cos \theta, \tag{50}
\]
so that \(\theta\) represents the angle between the vectors \(r_y\) and \(r_s\). In what follows for simplicity we assume that \(k = 1\). In the \(\sigma = 1\), from (36) follows that

\[
\rho = \cos \theta, \quad q_s = \sin \theta e^{-i\omega}, \tag{51}
\]

where \(\theta\) and \(\omega\) are some real functions. These formulas give us [38]

\[
q = (\theta y - i\omega y \tan \theta)e^{-i\omega}. \tag{52}
\]

Thus finally for LR (39)-(40) we obtain the expressions [38]

\[
U_2 = -i\lambda \sigma_3 + \left( \begin{array}{cc}
0 & -\frac{1}{2}(\theta y - i\omega y \tan \theta)e^{-i\omega} \\
\frac{1}{2}(\theta y + i\omega y \tan \theta)e^{i\omega} & 0
\end{array} \right), \tag{53}
\]
\[
V_2 = \frac{i}{4\lambda} \left( \begin{array}{cc}
\cos \theta & \sin \theta e^{-i\omega} \\
\sin \theta e^{i\omega} & -\cos \theta
\end{array} \right). \tag{54}
\]

Let us we return to the surface. Its fundamental forms read as [38]

\[
I = dy^2 + 2 \cos \theta dy ds + ds^2, \tag{55}
\]
\[
II = (\tan \theta)\omega_y dy^2 + 2 \sin \theta dy ds + (\sin \theta)\omega_s ds^2. \tag{56}
\]

Now we are ready to write the Gauss-Weingarten equations of the surface. We have [38]

\[
\begin{align*}
\boldsymbol{r}_{yy} & = (\cot \theta)\theta y \boldsymbol{r}_y - (\csc \theta)\theta y \boldsymbol{r}_s - (\tan \theta)\omega_y \boldsymbol{N}, \tag{57} \\
\boldsymbol{r}_{ys} & = \sin \theta \boldsymbol{N}, \tag{58} \\
\boldsymbol{r}_{ss} & = -(\csc \theta)\theta s \boldsymbol{r}_y + (\cot \theta)\theta s \boldsymbol{r}_s + (\sin \theta)\omega_s \boldsymbol{N}, \tag{59} \\
\boldsymbol{N}_y & = (\cot \theta + \csc \theta \sec \omega_y) \boldsymbol{r}_y - (\csc \theta \omega_y + \csc \theta) \boldsymbol{r}_s, \tag{60} \\
\boldsymbol{N}_s & = -(\csc \theta - \cot \theta \omega_s) \boldsymbol{r}_y + (\cot \theta + \csc \theta \omega_s) \boldsymbol{r}_s. \tag{61}
\end{align*}
\]
The compatibility conditions of these equations gives us the following Mainardi-Codazzi equation

\[(\omega_s \cos \theta)_y = (\omega_y)_{s \cos \theta}.\]  

(62)

At the same time, the Gaussian curvature of the surface reads as

\[K = -\frac{(\tan \theta)\omega_s \omega_s + \sin \theta}{\sin \theta}.\]  

(63)

The important formula follows from the Liouville-Beltrami form of the Theorema egregium and has the form \[38\]

\[\theta_{ys} - \sin \theta - (\tan \theta)\omega_y \omega_s = 0.\]  

(64)

Let us write the position vector on the surface in the component form as

\[r = (r_1, r_2, r_3)\]  

(65)

or in the matrix form

\[r = r_1 e_1 + r_2 e_2 + r_3 e_3.\]  

(66)

Now following \[38\] we introduce new three matrices of the forms

\[T = \Phi^{-1} e_3 \Phi, \quad N = \Phi^{-1} e_2 \Phi, \quad B = \Phi^{-1} e_1 \Phi,\]  

(67)

where

\[e_j = \frac{1}{2i} \sigma_j.\]  

(68)

It follows from the following well-known formula

\[r_y = \Phi^{-1} U_\lambda \Phi \bigg|_{\lambda=1},\]  

(69)

\[r_s = \Phi^{-1} V_\lambda \Phi \bigg|_{\lambda=1}.\]  

(70)

Thus we finally have

\[r_y = \Phi^{-1} e_3 \Phi = T,\]  

(71)

\[r_s = (\cos \theta)T + (\sin \theta \cos \omega)N + (\sin \theta \sin \omega)B,\]  

(72)

where \(y\) plays a role of arc length of the curve. These equations give us the following equation for the position vector \(r\) \[38\]:

\[r_s = \rho \rho r_y + r_y \wedge r_{ys}.\]  

(73)

It coincide with the \(r\)-form of the M-XIII equation (45) after some transformation. Finally we note that

\[r_s^2 = r_y^2 = 1\]  

(74)

or in the matrix form

\[r_s^2 = r_y^2 = I.\]  

(75)
5.2 Case 2: $\mathbf{A} \equiv \mathbf{r}_s$

Now let us consider the another type identification namely the following one

$$\mathbf{A} \equiv \mathbf{r}_s,$$

(76)

where $\mathbf{r}(y, s)$ is the position vector of the curve embedded on the surface and $s$ is the arclength parameter of the curve. Then the M-XIII equation (9)-(10) takes the form

$$\mathbf{r}_{ss} = k^{-1} \mathbf{r}_s \wedge \mathbf{r}_{yss} + k^{-2} \rho \mathbf{r}_{sy},$$

(77)

$$\rho_y = k \mathbf{r}_s \cdot (\mathbf{r}_{ss} \wedge \mathbf{r}_{sy}).$$

(78)

This equation defines some integrable surface in $\mathbb{R}^3$. Note that Eq.(77)-(78) is integrable with the following LR

$$F_y = 0.5i(k - \lambda)\mathbf{r}_s F = U_4 F,$$

(79)

$$F_s = \left[ \frac{i(k - \lambda)}{2k\lambda} \rho \mathbf{r}_s + \frac{(k - \lambda)}{2\lambda} \mathbf{r}_s \mathbf{r}_{ss} \right] F = V_4 F.$$

(80)

The compatibility condition of the equations (79)-(80) gives

$$i\mathbf{r}_{ss} = 0.5k^{-1} [\mathbf{r}_s, \mathbf{r}_{ss}]_y + ik^{-2} \rho \mathbf{r}_{sy},$$

(81)

$$\rho_y = -0.5iktr(\mathbf{r}_s \cdot [\mathbf{r}_{ys}, \mathbf{r}_{ss}]).$$

(82)

It is just the matrix form of the equation (77)-(78).

6 Soliton solutions of the M-XIII equation

Let us now present the simple 1-soliton solution of the M-XIII equation (2). We use the corresponding solution of the CCD equation (33)-(34). Let the seed solution of the last equation has the form

$$q = 0, \quad \rho = 1.$$

(83)

Then we have

$$U_2 = -i\lambda y \sigma_3, \quad V_2 = \frac{i}{4\lambda} s \sigma_3.$$

(84)

The corresponding Sym-Tafel formula is given by

$$r = \Phi^{-1} \Phi_\lambda |_{\lambda = 1}.$$  

(85)
Then the 1-soliton surface reads as \[38\]

\[ r_1 = \frac{b}{(1-a)^2 + b^2} \text{sech} R \cos W, \quad (86) \]

\[ r_2 = \frac{b}{(1-a)^2 + b^2} \text{sech} R \sin W, \quad (87) \]

\[ r_3 = \frac{b}{(1-a)^2 + b^2} \tanh R + y + s, \quad (88) \]

where

\[ R = by + \frac{b}{a^2 + b^2} s, \quad W = (1-a)y + \left(1 + \frac{a}{a^2 + b^2}\right) s. \]

Finally we can write the 1-soliton solution of the M-XIII equation (2) as

\[ A_1 = r_{1y}, \quad A_2 = r_{2y}, \quad A_3 = r_{3y}. \quad (89) \]

7 Conclusions

In this paper, we have established the relation between the M-XIII equation (2) and the CCD equation (33)-(34). We have shown that the M-XIII equation (2) and the CCD equation (33)-(34) is the geometrically equivalent each to other. Also the gauge equivalence between these equations is proved. Our results are significant for the deep understand of integrable spin systems and their relations with differential geometry of curves and surfaces.

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