EPR, Bell, Schrodinger’s cat, and the Monty Hall Paradox

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The purpose of this manuscript is to provide a short pedagogical explanation why “quantum collapse” is not a metaphysical event, by pointing out the analogy with a “classical collapse” which is associated with the Monty Hall Paradox.

This manuscript constitutes a short self-contained version of some selected sections taken from “lecture notes in Quantum Mechanics” [quant-ph/0605180] (∼250p). In particular section 47.4 regarding the notion of collapse has attracted some attention. In this section I suggest to use the “Monty Hall Paradox” as a pedagogical introduction to the discussion of “quantum collapse”. From my experience it is the most effective way to convince students and other non-experts that “quantum collapse” is not a metaphysical event.

[8] Quantum States

[8.1] Is the world classical? (EPR, Bell)

We would like to examine whether the world we live in is “classical” or not. The notion of classical world includes mainly two ingredients: (i) realism (ii) determinism. By realism we means that any quantity that can be measured is well defined even if we do not measure it in practice. By determinism we mean that the result of a measurement is determined in a definite way by the state of the system and by the measurement setup. We shall see later that quantum mechanics is not classical in both respects: In the case of spin 1/2 we cannot associate a definite value of \( \hat{\sigma}_y \) for a spin which has been polarized in the \( \hat{\sigma}_x \) direction. Moreover, if we measure the \( \hat{\sigma}_y \) of a \( \hat{\sigma}_x \) polarized spin, we get with equal probability \( \pm 1 \) as the result.

In this section we would like to assume that our world is "classical". Also we would like to assume that interactions cannot travel faster than light. In some textbooks the latter is called "locality of the interactions" or "causality". It has been found by Bell that the two assumptions lead to an inequality that can be tested experimentally. It turns out from actual experiments that Bell’s inequality are violated. This means that our world is either non-classical or else we have to assume that interactions can travel faster than light.

If the world is classical it follows that for any set of initial conditions a given measurement would yield a definite result. Whether or not we know how to predict or calculate the outcome of a possible measurement is not assumed. To be specific let us consider a particle of zero spin, which disintegrates into two particles going in opposite directions, each with spin 1/2. Let us assume that each spin is described by a set of state variables.

\[
\begin{align*}
\text{state of particle A} &= x_1^A, x_2^A, ... \\
\text{state of particle B} &= x_1^B, x_2^B, ...
\end{align*}
\]

The number of state variables might be very big, but it is assumed to be a finite set. Possibly we are not aware or not able to measure some of these “hidden” variables.

Since we possibly do not have total control over the disintegration, the emerging state of the two particles is described by a joint probability function \( \rho(x_1^A, x_1^B, ...) \). We assume that the particles do not affect each other after the disintegration (“causality” assumption). We measure the spin of each of the particles using a Stern-Gerlach apparatus. The measurement can yield either 1 or −1. For the first particle the measurement outcome will be denoted as \( a \), and for the second particle it will be denoted as \( b \). It is assumed that the outcomes \( a \) and \( b \) are determined in a deterministic fashion. Namely, given the state variables of the particle and the orientation \( \theta \) of the apparatus we have

\[
\begin{align*}
a &= a(\theta_A) = f(\theta_A, x_1^A, x_2^A, ...) = \pm 1 \\
b &= b(\theta_B) = f(\theta_B, x_1^B, x_2^B, ...) = \pm 1
\end{align*}
\]
where the function \( f() \) is possibly very complicated. If we put the Stern-Gerlach machine in a different orientation then we will get different results:

\[
\begin{align*}
a' &= a(\theta_A') = f(\theta_A', x_1^A, x_2^A, ...) = \pm 1 \\
b' &= b(\theta_B') = f(\theta_B', x_1^B, x_2^B, ...) = \pm 1
\end{align*}
\]

We have following innocent identity:

\[
\begin{align*}
ab + ab' + a'b - a'b' &= \pm 2
\end{align*}
\]

The proof is as follows: if \( b = b' \) the sum is \( \pm 2a \), while if \( b = -b' \) the sum is \( \pm 2a' \). Though this identity looks innocent, it is completely non trivial. It assumes both "reality" and "causality". This becomes more manifest if we write this identity as

\[
\begin{align*}
a(\theta_A)b(\theta_B) + a(\theta_A)b(\theta_B') + a(\theta_A')b(\theta_B) - a(\theta_A')b(\theta_B') &= \pm 2
\end{align*}
\]

The realism is reflected by the assumption that both \( a(\theta_A) \) and \( a(\theta_A') \) have definite values, though it is clear that in practice we can measure either \( a(\theta_A) \) or \( a(\theta_A') \), but not both. The causality is reflected by assuming that \( a \) depends on \( \theta_A \) but not on the distant setup parameter \( \theta_B \).

Let us assume that we have conducted this experiment many times. Since we have a joint probability distribution \( \rho \), we can calculate average values, for instance:

\[
\langle ab \rangle = \int \rho(x_1^A, ..., x_1^B, ...) f(\theta_A, x_1^A, ...) f(\theta_B, x_1^B, ...)
\]

Thus we get that the following inequality should hold:

\[
|\langle ab \rangle + \langle ab' \rangle + \langle a'b \rangle - \langle a'b' \rangle| \leq 2
\]

This is called Bell’s inequality. Let us see whether it is consistent with quantum mechanics. We assume that all the pairs are generated in a singlet (zero angular momentum) state. It is not difficult to calculate the expectation values. The result is

\[
\langle ab \rangle = -\cos(\theta_A - \theta_B) \equiv C(\theta_A - \theta_B)
\]

we have for example

\[
C(0^\circ) = -1, \quad C(45^\circ) = -\frac{1}{\sqrt{2}}, \quad C(90^\circ) = 0, \quad C(180^\circ) = +1.
\]

If the world were classical the Bell’s inequality would imply

\[
|C(\theta_A - \theta_B) + C(\theta_A - \theta_B') + C(\theta_A' - \theta_B) + C(\theta_A' - \theta_B')| \leq 2
\]

Let us take \( \theta_A = 0^\circ \) and \( \theta_B = 45^\circ \) and \( \theta_A' = 90^\circ \) and \( \theta_B' = -45^\circ \). Assuming that quantum mechanics holds we get

\[
\left| \left( -\frac{1}{\sqrt{2}} \right) + \left( -\frac{1}{\sqrt{2}} \right) + \left( \frac{1}{\sqrt{2}} \right) - \left( \frac{1}{\sqrt{2}} \right) \right| = 2\sqrt{2} > 2
\]

It turns out, on the basis of celebrated experiments that Nature has chosen to violate Bell’s inequality. Furthermore it seems that the results of the experiments are consistent with the predictions of quantum mechanics. Assuming that we do not want to admit that interactions can travel faster than light it follows that our world is not classical.
[8.2] The four Postulates of Quantum Mechanics

The 18th century version classical mechanics can be derived from three postulates: The three laws of Newton. The better formulated 19th century version of classical mechanics can be derived from three postulates: (1) The state of classical particles is determined by the specification of their positions and its velocities; (2) The trajectories are determined by a minimum action principle. (3) The form of the Lagrangian of the theory is determined by symmetry considerations, namely Galilei invariance in the non-relativistic case. See the Classical Mechanics book of Landau and Lifshitz for details.

Quantum mechanically requires four postulates: Two postulates define the notion of quantum state, while the other two postulates, in analogy with classical mechanics, are about the laws that govern the evolution of quantum mechanical systems. [The rest of this section can be found in the lecture notes].

[8.3] What is a Pure State

"Pure states" are states that have been filtered. The filtering is called "preparation". For example: we take a beam of electrons. Without "filtering" the beam is not polarized. If we measure the spin we will find (in any orientation of the measurement apparatus) that the polarization is zero. On the other hand, if we "filter" the beam (e.g. in the left direction) then there is a direction for which we will get a definite result (in the above example, in the right/left direction). In that case we say that there is full polarization - a pure state. The "uncertainty principle" tells us that if in a specific measurement we get a definite result (in the above example, in the right/left direction), then there are different measurements (in the above example, in the up/down direction) for which the result is uncertain. The uncertainty principle is implied by the first postulate.

[8.4] What is a Measurement

In contrast with classical mechanics, in quantum mechanics measurement only has meaning in a statistical sense. We measure "states" in the following way: we prepare a collection of systems that were all prepared in the same way. We make the measurement on all the "copies". The outcome of the measurement is an event \( \hat{x} = x \) that can be characterized by a distribution function. The single event has no statistical meaning. For example, if we measured the spin of a single electron and get \( \hat{\sigma}_z = 1 \), it does not mean that the state is polarized "up". In order to know if the electron is polarized we must measure a large number of electrons that were prepared in an identical way. If only 50% of the events give \( \hat{\sigma}_z = 1 \) we should conclude that there is no definite polarization in the direction we measured!

[8.5] Random Variables

A random variable is an object that can have any numerical value. In other words \( \hat{x} = x \) is an event. Let’s assume, for example, that we have a particle that can be in one of five sites: \( x = 1, 2, 3, 4, 5 \). An experimentalist could measure \( \text{Prob}(\hat{x} = 3) \) or \( \text{Prob}(\hat{p} = 3(2\pi/5)) \). Another example is a measurement of the probability \( \text{Prob}(\hat{\sigma}_z = 1) \) that the particle will have spin up.

The collection of values of \( x \) is called the spectrum of values of the random variable. We make the distinction between random variables with a discrete spectrum, and random variables with a continuous spectrum. [The rest of this section can be found in the lecture notes].

[8.6] Quantum Versus Statistical Mechanics

Quantum mechanics stands opposite classical statistical mechanics. A particle is described in classical statistical mechanics by a probability function (for presentation purpose we treat \( \hat{x} \) and \( \hat{p} \) as having discrete spectrum):

\[
\rho(x, p) = \text{Prob}\{\hat{x} = x, \hat{p} = p\}
\]
The expectation value of a random variable $\hat{A} = A(\hat{x}, \hat{p})$ is calculated using the definition:

$$\langle \hat{A} \rangle = \sum_{x,p} \rho(x,p) A(x,p) \equiv \text{trace}(\rho A)$$

(13)

In particular we can write:

$$\rho(x,p) = (\delta(\hat{p} - p) \delta(\hat{x} - x))$$

(14)

From the definition of the expectation value follows the linear relation $\langle \alpha \hat{A} + \beta \hat{B} \rangle = \alpha \langle \hat{A} \rangle + \beta \langle \hat{B} \rangle$ for any pair of observables. This linear relation is a trivial result of classical probability theory. It assumes that the joint probability function Eq.(12) can be defined. But in quantum mechanics we cannot define a “quantum state” using a joint probability function, as implied by the observation that our world is not “classical”. For this reason, we have to use a more sophisticated approach. Loosely speaking one may say that Quantum Mechanics takes Eq.(14) as the definition of $\rho$, and use the linear relation of the expectation values as a second postulate in order to deduce Eq.(13).

[The rest of this section an be found in the lecture notes].

## [47] Theory of Quantum Measurements

### [47.4] Measurements, the notion of collapse

In elementary textbooks the quantum measurement process is described as inducing “collapse” of the wavefunction. Assume that the system is prepared in state $\rho_{\text{initial}} = |\psi\rangle\langle\psi|$ and that one measures $\hat{P} = |\varphi\rangle\langle\varphi|$. If the result of the measurement is $\hat{P} = 1$ then it is said that the system has collapsed into the state $\rho_{\text{final}} = |\varphi\rangle\langle\varphi|$. The probability for this “collapse” is given by the projection formula $\text{Prob}(\varphi|\psi) = |\langle\varphi|\psi\rangle|^2$.

If one regard $\rho(x,x')$ or $\psi(x)$ as representing physical reality, rather than a probability matrix or a probability amplitude, then one immediately gets into puzzles. Recalling the EPR experiment this world imply that once the state of one spin is measured at Earth, then immediately the state of the other spin (at the Moon) would change from unpolarized to polarized. This would suggest that some spooky type of “interaction” over distance has occurred.

In fact we shall see that the quantum theory of measurement does not involve any assumption of spooky “collapse” mechanism. Once we recall that the notion of quantum state has a statistical interpretation the mystery fades away. In fact we explain (see below) that there is “collapse” also in classical physics! To avoid potential miss-understanding it should be clear that I do not claim that the classical “collapse” which is described below is an explanation of the the quantum collapse. The explanation of quantum collapse using a quantum measurement (probabilistic) point of view will be presented in a later section. The only claim of this section is that in probability theory a correlation is frequently mistaken to be a causal relation: “smokers are less likely to have Alzheimer” not because cigarettes help to their health, but simply because their life span is smaller. Similarly quantum collapse is frequently mistaken to be a spooky interaction between well separated systems.

Consider the thought experiment which is known as the “Monty Hall Paradox”. There is a car behind one of three doors. The car is like a classical ”particle”, and each door is like a ”site”. The initial classical state is such that the car has equal probability to be behind any of the three doors. You are asked to make a guess. Let us say that you peak door #1. Now the organizer opens door #2 and you see that there is no car behind it. This is like a measurement. Now the organizer allows you to change your mind. The naive reasoning is that now the car has equal probability to be behind either of the two remaining doors. So you may claim that it does not matter. But it turns out that this simple answer is very very wrong! The car is no longer in a state of equal probabilities: Now the probability to find it behind door #2 has increased. A standard calculation reveals that the probability to find it behind door #3 is twice large compared with the probability to find it behind door #2. So we have here an example for a classical collapse.

If the reader is not familiar with this well known ”paradox”, the following may help to understand why we have this collapse (I thank my colleague Eitan Bachmat for providing this explanation). Imagine that there are billion doors. You peak door #1. The organizer opens all the other doors except door #234123. So now you know that the car is either behind door #1 or behind door #234123. You want the car. What are you going to do? It is quite obvious that the car is almost definitely behind door #234123. It is also clear that the collapse of the car into site #234123 does not imply any physical change in the position of the car.
[47.5] Quantum measurements, Schroedinger’s cat

What do we mean by quantum measurement? In order to clarify this notion let us consider a system and a detector which are prepared independently as

$$\Psi = \left[ \sum_a \psi_a |a\rangle \right] \otimes |q = 0\rangle \quad (15)$$

As a result of an interaction we assume that the detector correlates with the system as follows:

$$\hat{U}_{\text{measurement}} \Psi = \sum \psi_a |a\rangle \otimes |q = a\rangle \quad (16)$$

We call such type of unitary evolution "ideal measurement". If the system is in a definite $a$ state, then it is not affected by the detector. Rather, we gain information on the state of the system. One can think of $q$ as representing a memory device in which the information is stored. This memory device can be of course the brain of a human observer. Form the point of view of the observer, the result at the end of the measurement process is to have a definite $a$. This is interpreted as a "collapse" of the state of the system. Some people wrongly think that "collapse" is something that goes beyond unitary evolution. But in fact this term just makes over dramatization of the above unitary process.

The concept of measurement in quantum mechanics involves psychological difficulties which are best illustrated by considering the "Schroedinger’s cat" experiment. This thought experiment involves a radioactive nucleus, a cat, and a human being. The half life time of the nucleus is an hour. If the radioactive nucleus decays it triggers a poison which kills the cat. The radioactive nucleus and the cat are inside an isolated box. At some stage the human observer may open the box to see what happens with the cat... Let us translate the story into a mathematical language. A time $t = 0$ the state of the universe (nucleus$\otimes$cat$\otimes$observer) is

$$\Psi = |\uparrow\rangle \otimes |q = 1 = \text{alive}\rangle \otimes |Q = 0 = \text{ignorant}\rangle \quad (17)$$

where $q$ is the state of the cat, and $Q$ is the state of the memory bit inside the human observer. If we wait a very long time the nucleus would definitely decay, and as a result we will have a definitely dead cat:

$$U_{\text{waiting}} \Psi = |\downarrow\rangle \otimes |q = -1 = \text{dead}\rangle \otimes |Q = 0 = \text{ignorant}\rangle \quad (18)$$

If the observer opens the box he/she would see a dead cat:

$$U_{\text{seeing}} U_{\text{waiting}} \Psi = |\uparrow\rangle \otimes |q = -1 = \text{dead}\rangle \otimes |Q = -1 = \text{shocked}\rangle \quad (19)$$

But if we wait only one hour then

$$U_{\text{waiting}} \Psi = \frac{1}{\sqrt{2}} \left[ |\uparrow\rangle \otimes |q = +1\rangle + |\downarrow\rangle \otimes |q = -1\rangle \right] \otimes |Q = 0 = \text{ignorant}\rangle \quad (20)$$

which means that from the point of view of the observer the system (nucleus+cat) is in a superposition. The cat at this stage is neither definitely alive nor definitely dead. But now the observer open the box and we have:

$$U_{\text{seeing}} U_{\text{waiting}} \Psi = \frac{1}{\sqrt{2}} \left[ |\uparrow\rangle \otimes |q = +1\rangle \otimes |Q = +1 = \text{happy}\rangle + |\downarrow\rangle \otimes |q = -1\rangle \otimes |Q = -1 = \text{shocked}\rangle \right] \quad (21)$$

We see that now, form the point of view of the observer, the cat is in a definite(!) state. This is regarded by the observer as “collapse” of the superposition. We have of course two possibilities: one possibility is that the observer sees a definitely dead cat, while the other possibility is that the observer sees a definitely alive cat. The two possibilities
"exist" in parallel, which leads to the "many worlds" interpretation. Equivalently one may say that only one of the two possible scenarios is realized from the point of view of the observer, which leads to the "relative state" concept of Everett. Whatever terminology we use, "collapse" or "many worlds" or "relative state", the bottom line is that we have here merely a unitary evolution.

[47.6] Measurements, formal treatment

In this section we describe mathematically how an ideal measurement affects the state of the system. First of all let us write how the $U$ of a measurement process looks like. The formal expression is

$$
\hat{U}_{\text{measurement}} = \sum_a \hat{P}(a) \otimes \hat{D}(a)
$$

where $\hat{P}(a) = |a\rangle \langle a|$ is the projection operator on the state $|a\rangle$, and $\hat{D}(a)$ is a translation operator. Assuming that the measurement device is prepared in a state of ignorance $|q = 0\rangle$, the effect of $\hat{D}(a)$ is to get $|q = a\rangle$. Hence

$$
\hat{U}\psi = \left[ \sum_a \hat{P}(a) \otimes \hat{D}(a) \right] \left( \sum_a \psi_a |a\rangle \otimes |q = 0\rangle \right) = \sum_a \psi_a |a\rangle \otimes \hat{D}(a)|q = 0\rangle = \sum_a \psi_a |a\rangle \otimes |q = a\rangle
$$

A more appropriate way to describe the state of the system is using the probability matrix. Let us describe the above measurement process using this language. After "reset" the state of the measurement apparatus is $\sigma^{(0)} = |q = 0\rangle \langle q = 0|$. The system is initially in an arbitrary state $\rho$. The measurement process correlates that state of the measurement apparatus with the state of the system as follows:

$$
\hat{U}\rho \otimes \sigma^{(0)}\hat{U}^\dagger = \sum_{a,b} \hat{P}(a) \rho \hat{P}(b) \otimes [\hat{D}(a)]^{\dagger} \sigma^{(0)} [\hat{D}(b)] = \sum_{a,b} \hat{P}(a) \rho \hat{P}(b) \otimes |q = a\rangle \langle q = b|
$$

Tracing out the measurement apparatus we get

$$
\rho^{\text{system}} = \sum_a \hat{P}(a) \rho^{\text{preparation}} \hat{P}(a) = \sum_a p_a \rho^{(a)}
$$

Where $p_a$ is the trace of the projected probability matrix $\hat{P}(a) \rho \hat{P}(a)$, while $\rho^{(a)}$ is its normalized version. We see that the effect of the measurement is to turn the superposition into a mixture of $a$ states, unlike unitary evolution for which $\rho^{\text{system}} = U_{\text{system}} \rho^{\text{preparation}} U_{\text{system}}^\dagger$. So indeed a measurement process looks like a non-unitary process: it turns a pure superposition into a mixture. A simple example is in order. Let us assume that the system is a spin 1/2 particle. The spin is prepared in a pure polarization state $\rho = |\psi\rangle \langle \psi|$ which is represented by the matrix

$$
\rho_{ab} = \psi_a \psi_b^* = \begin{pmatrix}
|\psi_1|^2 & \psi_1 \psi_2^* \\
\psi_2 \psi_1^* & |\psi_2|^2
\end{pmatrix}
$$

where 1 and 2 are (say) the "up" and "down" states. Using a Stern-Gerlach apparatus we can measure the polarization of the spin in the up/down direction. This means that the measurement apparatus projects the state of the spin using

$$
P^{(1)} = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
$$

and

$$
P^{(2)} = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
$$

leading after the measurement to the state

$$
\rho^{\text{system}} = P^{(1)} \rho^{\text{preparation}} P^{(1)} + P^{(2)} \rho^{\text{preparation}} P^{(2)} = \begin{pmatrix}
|\psi_1|^2 & 0 \\
0 & |\psi_2|^2
\end{pmatrix}
$$

Thus the measurement process has eliminated the off-diagonal terms in $\rho$ and hence turned a pure state into a mixture. It is important to remember that this non-unitary non-coherent evolution arise because we look only on the state of the system. On a universal scale the evolution is in fact unitary.