On the Morse index of harmonic maps and minimal immersions

Mohammed Benalili and Hafida Benallal

Abstract. In this paper we are concerned with harmonic maps and minimal immersions defined on compact Riemannian manifolds and with values in homogenous strongly harmonic manifolds. We show some results on the Morse index by varying these maps along suitable conformal vector fields. We obtain also that they are global maxima on some subspaces of the eigenspaces corresponding to the nonvanishing eigenvalues of the Laplacian operator on the target manifolds.

1. Introduction

Let \((M^m, g)\) and \((N^n, h)\) be Riemannian manifolds of dimension \(m\) and \(n\), respectively. If we use local coordinates, the metric tensors of \(M^m\) and \(N^n\) will be written as

\[(g_{\alpha\beta})_{\alpha,\beta=1,...,m},\]

and

\[(h_{ij})_{i,j=1,...,n}.\]

The inverse metric tensor is

\[(g^{\alpha\beta})_{\alpha\beta=1,...,m}= (g_{\alpha\beta})^{-1}_{\alpha,\beta=1,...,m},\]

and

\[|g| = \det(g_{\alpha\beta}).\]

If \(f : M^m \to N^n\) is a map of class \(C^1\), its energy density is given by

\[e(f)(x) = \frac{1}{2} g^{\alpha\beta}(x) h_{ij}(f(x)) \frac{\partial f_i(x)}{\partial x_\alpha} \frac{\partial f_j(x)}{\partial x_\beta}.\]

Keywords: harmonic map, minimal immersion, Morse index.
in local coordinates \((x_1, ..., x_m)\) on \(M^m\) and \((y_1, ..., y_n)\) on \(N^n\) where the Einstein convention summation is used. Or as an intrinsic quantity of Riemannian geometry of \(M^m\) and \(N^n\)

\[ e(f) = \frac{1}{2} \langle df, df \rangle_{T^*M^m \otimes f^{-1}TN^n} \]

where \(\langle ., . \rangle_{f^{-1}TN^n}\) is the pullback by \(f\) of the metric tensor of \(N^n\).

Then the energy of \(f\) is simply

\[ (1.2) \quad E(f) = \int_{M^m} e(f) dv_g \]

with \(dv_g = \sqrt{|g|} dx_1 \ldots dx_m\) the volume element of \(M^m\) in local coordinates.

Let \(w\) be a vector field along \(f\) that means that \(w\) is a section of \(f^{-1}TN^n\) the pullback of the tangent space \(TN^n\) by the map \(f\). In local coordinates

\[ w = w^i(x) \frac{\partial}{\partial y_i} \]

\(w\) induces a variation of \(f\) given by

\[ f_t(x) = \exp(tw) o f(x). \]

Put

\[ \phi^w_t = \exp(tw) \]

and

\[ (1.3) \quad \psi = \phi^w_t o f. \]

We get the first variation formula for the energy functional

\[ (1.4) \quad \frac{d}{dt} E(f_t) \bigg|_{t=0} = - \int_{M^m} \langle \text{trace} \nabla df \psi, w \psi \rangle_{f^{-1}TN^n} dv_g. \]

\(\tau_g(\psi) = \text{trace} \nabla df \psi\) is the so called tension field of \(\psi\), where \(\nabla\) denotes the covariant derivative on the manifold \(N^n\).

**Definition 1.** The map \(f\) is called harmonic if and only if the tensor field \(\tau(\psi) = 0\). i.e. \(f\) is a critical point of the energy functional \(E\).

**Definition 2.** The volume of an immersion \(f\), from a Riemannian manifold \((M^m, g)\) into \((N^n, h)\), denoted by \(V(f)\), is defined as the Riemannian volume of \(M^m\) endowed with the Riemannian metric \(f^* h\), the pull back of the metric \(h\).

**Definition 3.** The map \(f\) will be said minimal immersion if it satisfies

\[ \frac{d}{dt} V(f_t) \bigg|_{t=0} = 0 \]

for any vector field \(w\) on \(N^n\) along \(f\).
Many results on the Morse index of harmonic maps and minimal immersions from a compact Riemannian manifolds into the Euclidean sphere have been obtained by varying the energy functional along conformal vector fields on the Euclidean sphere. For a survey on the theory of harmonic maps and minimal immersions, we refer the reader to ([1], [4], [9]).

In ([5]), A. El Soufi shows that for every harmonic map $f$ from an $m$-dimensional compact Riemannian manifold $(M,g)$ into the Euclidean sphere $S^n$ which enjoys one of the following properties:

(i) The stress-energy tensor $S_0^g(f)$ is positive everywhere and positive definite at least at one point in $M$.

(ii) $S_0^g(f)$ is positive everywhere on $M$, $f$ is an immersion and $f(M)$ is not a totally geodesic sphere of $S^n$.

Then the Morse index of $f$, $\text{Ind}_E(f)$, $\geq n + 1$.

Also he proves

Let $f$ be a minimal immersion from a compact $m$-dimensional manifold $M^n$ into the sphere $S^n$ ($n \geq 3$). Then two possibilities hold:

(i) $\text{Ind}_V(f) = n - m$, hence $\phi(M^n)$ is a totally geodesic sphere of $S^n$.

(ii) $\text{Ind}_V(f) \geq n + 1$. Where $\text{Ind}_V(f)$ stands for the Morse index of $f$.

In this paper we extend some results on the index of harmonic maps and minimal immersions obtained in ([5]) to non necessary spherical cases. Mainly, if $L$ denotes an appropriate $n + 1$-dimensional subspace of the eigenspace $V_\lambda$ corresponding to the nonvanishing eigenvalue $\lambda$ of the Laplacian operator on the target manifold $N^n$ and $L^\perp$ is the normal component of $L$, we show that the Morse index, $\text{Ind}_E(f)$, of a harmonic map $f$ defined on a compact Riemannian manifold $M^n$ and with values in homogenous strongly harmonic manifolds $N^n$ fulfills $\text{Ind}_E(f) \geq \dim L$ provided that the sectional curvature $K$ of the target manifold $N^n$ satisfies $K \geq \kappa > 0$, where $\kappa$ is a constant, the stress-energy tensor $S_0^g(f)$ is positive definite and $\lambda \leq \frac{n^2}{2\kappa}$. We also obtain that if $f : M^n \rightarrow N^n$ is a minimal isometric immersion not totally geodesic from a compact Riemannian manifold into a homogenous strongly harmonic Riemannian manifold of dimension $n \geq 3$ with sectional curvature $K$ satisfying $K \geq \kappa > 0$, where $\kappa$ is a constant, then the Morse index, $\text{Ind}_V(f) \geq \dim(L^\perp)$ provided that $\lambda$ satisfies $\lambda \leq \frac{n^2}{2\kappa}$. Finally we prove that harmonic maps whose source manifolds are compact and the target ones are homogenous strongly harmonic are global maximum on the appropriate $(n + 1)$-dimensional subspace $L$ of the eigenspace $V_\lambda$ provided that the stress-energy tensor $S_0^g(f)$ is positive and isometric minimal immersions from compact manifolds into homogenous strongly harmonic ones are global maximum on the $L^\perp$.

2. Conformal vector fields on strongly harmonic manifolds

In this section we will construct conformal vector fields on strong harmonic manifolds along which we will vary the energy and the volume functionals.
2.1. Strongly harmonic manifolds. In this section, we will construct
conformal vector fields by means of gradients of the eigenfunctions to Lapla-
cian operator on strongly harmonic manifolds.

Definition 4. A compact Riemannian manifold $(N^n, h)$ is said to be strongly
harmonic (we shall say $SH$-manifold) if there exists a map $\Xi: R_+ \times R^*_+ \to R$
with the property that the fundamental solution of the heat equation $K$ on
$(N^n, h)$ can be written as $K(x, y, t) = \Xi(\rho(x, y), t)$ for every $x$ and $y$ in $N^n$, 
t in $R^*_+$ and $\rho$ is the distance function on $N^n$.

For an eigenvalue $\lambda_\alpha$ of the Laplacian $\Delta = -\text{div}(\nabla h)$ on $(N^n, h)$, set
$V_\alpha = \{ f : \Delta f = \lambda_\alpha f \}$ the eigenspace corresponding to $\lambda_\alpha$ and
$N_\alpha = \dim V_\alpha$.

Let $\{ \phi_\alpha^i \}_{i=1}^{N_\alpha}$ be an orthonormal basis of $V_\alpha$ with respect the global scalar
product $\langle \varphi, \psi \rangle = \int_{N^n} \varphi \psi dv$ where $dv$ denotes the Riemannian measure
Corresponding to the metric $h$.

We know from (2.1) that the fundamental solution of the heat equation $K$, on the manifold $(N^n, h)$ writes as

\begin{equation}
K(x, y, t) = \sum_\alpha e^{-\lambda_\alpha t} \sum_{i=1}^{N_\alpha} \phi_\alpha^i(x) \phi_\alpha^i(y) \text{ for every } x, y \text{ in } N^n \text{ and } t \text{ in } R^*_+.
\end{equation}

From (2.1), we deduce that the compact manifold $(N^n, h)$ is $SH$ if for every
eigenvalue $\lambda_\alpha$ there exists a map $\Xi_\alpha: R_+ \to R$ with

\begin{equation}
\sum_{i=1}^{N_\alpha} \phi_\alpha^i(x) \phi_\alpha^i(y) = \Xi_\alpha(\rho(x, y)) \text{ for every } x, y \text{ in } N^n.
\end{equation}

Example 1. An important class of homogenous $SH$-manifold is the compact symmetric spaces of rank one which we denote by $\text{CROSS}$. Among the $\text{CROSS}$ spaces we quote the real projective spaces $\mathbb{R}P^n$, and the Euclidean spheres $S^n$ (see [3]).

Let $(N^n, h)$ be an homogeneous $SH$-manifold and put for any $y \in N^n$, 
$\Lambda(y) = (\varphi_1^y(y), ..., \varphi_{N_\alpha}^y(y))$. By the relation (2.2) the values of $\Lambda$ are in
the Euclidean sphere $S^{N_\alpha-1}$ centred at the origin of $R^{N_\alpha}$ and of radius
$(\Xi_\alpha(0)Vol(N, g))^{\frac{1}{2}}$. We quote the following lemma

Lemma 1. [3] $\Lambda$ is an immersion and its image $\Lambda(N^n)$ is an $n$-dimensional submanifold of the Euclidean sphere $S^{N_\alpha-1}(0, R)$ in $R^{N_\alpha}$ of radius $R = (\Xi_\alpha(0)Vol(N, g))^{\frac{1}{2}}$.

As a corollary of Lemma 1 we have

Corollary 1. For an $n$-dimensional $SH$-manifold every eigenvalue of the
Laplacian has multiplicity greater or equal to $n + 1$. 

For any integer $2 \leq n$ we proceed by recurrence. In the case of the matrix $\det(\Xi_\alpha)$, consider the lemma 2. The matrix $(\Xi_\alpha(\rho(m_j, m_k)))_{1 \leq j,k \leq n+1}$ is invertible.

Proof. Let us first show that the determinant of the matrix $(\Xi_\alpha(\rho(m_j, m_k)))_{1 \leq j,k \leq n+1}$ enjoys
\begin{equation}
\det(\Xi_\alpha(\rho(m_j, m_k))) = \frac{1}{(n+1)!} \sum_{i_1, i_2, \ldots, i_{n+1}} \det \left( \begin{array}{c}
\varphi^{\alpha}_{i_1}(m_1) & \cdots & \varphi^{\alpha}_{i_1}(m_{n+1}) \\
\vdots & \ddots & \vdots \\
\varphi^{\alpha}_{i_{n+1}}(m_1) & \cdots & \varphi^{\alpha}_{i_{n+1}}(m_{n+1})
\end{array} \right)^2.
\end{equation}

We proceed by recurrence. In the case $n+1 = 2$, the determinant of the matrix $(\Xi_\alpha(\rho(m_j, m_k)))_{1 \leq j,k \leq 2}$ is given by
\begin{equation}
\det(\Xi_\alpha(\rho(m_j, m_k))) = \sum_{i_1, i_2} \varphi^{\alpha}_{i_1}(m_1)^2 \varphi^{\alpha}_{i_2}(m_2)^2 - \left( \sum_{i_1, i_2} \varphi^{\alpha}_{i_1}(m_1) \varphi^{\alpha}_{i_2}(m_2) \right)^2
\end{equation}
\begin{align*}
&= \sum_{i_1 \neq i_2} \varphi^{\alpha}_{i_1}(m_1)^2 \varphi^{\alpha}_{i_2}(m_2)^2 - 2 \sum_{i_1 < i_2} \varphi^{\alpha}_{i_1}(m_1) \varphi^{\alpha}_{i_2}(m_2) \varphi^{\alpha}_{i_1}(m_1) \varphi^{\alpha}_{i_2}(m_2) \\
&= \sum_{1 \leq i_1 < i_2 \leq N_\alpha} \left( \varphi^{\alpha}_{i_1}(m_1) \varphi^{\alpha}_{i_2}(m_2) - \varphi^{\alpha}_{i_1}(m_2) \varphi^{\alpha}_{i_2}(m_1) \right)^2
\end{align*}
\begin{equation}
\text{(2.5)}
= \frac{1}{2} \sum_{i_1, i_2} \det \left( \begin{array}{cc}
\varphi^{\alpha}_{i_1}(m_1) & \varphi^{\alpha}_{i_1}(m_2) \\
\varphi^{\alpha}_{i_2}(m_1) & \varphi^{\alpha}_{i_2}(m_2)
\end{array} \right)^2.
\end{equation}

For any integer $2 \leq p < n + 1$, supposing that
\begin{equation}
\det(\Xi_\alpha(\rho(m_j, m_k))) = \frac{1}{p!} \sum_{i_1, \ldots, i_p} \det \left( \begin{array}{cccc}
\varphi^{\alpha}_{i_1}(m_1) & \cdots & \varphi^{\alpha}_{i_1}(m_p) \\
\vdots & \ddots & \vdots \\
\varphi^{\alpha}_{i_p}(m_2) & \cdots & \varphi^{\alpha}_{i_p}(m_p)
\end{array} \right)^2
\end{equation}
We have
\[
\det (\Xi_\alpha (\rho(m_j, m_k))) = \sum_{\sigma \in S_{p+1}} sgn(\sigma) \Xi_\alpha (\rho(m_1, m_{\sigma(1)}) \cdots \Xi_\alpha (\rho(m_{p+1}, m_{\sigma(p+1)})
\]
\[
= \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} \sum_{\sigma \in S_{p+1}} sgn(\sigma) \varphi_1^\alpha (m_1) \varphi_2^\alpha (m_{\sigma(1)}) \cdots \varphi_{i_{p+1}}^\alpha (m_{p+1}) \varphi_{i_{p+1}}^\alpha (m_{\sigma(p+1)})
\]
\[
= \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} \varphi_1^\alpha (m_1)^2 \sum_{\sigma \in S_{p+1}} sgn(\sigma) \varphi_2^\alpha (m_2) \varphi_2^\alpha (m_{\sigma(2)}) \cdots \varphi_{i_{p+1}}^\alpha (m_{p+1}) \varphi_{i_{p+1}}^\alpha (m_{\sigma(p+1)})
\]
(2.6)
\[\]
\[+ \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} \sum_{\sigma \in S_{p+1}, \sigma(1) \neq 1} sgn(\sigma) \varphi_1^\alpha (m_1) \varphi_2^\alpha (m_{\sigma(1)}) \cdots \varphi_{i_{p+1}}^\alpha (m_{p+1}) \varphi_{i_{p+1}}^\alpha (m_{\sigma(p+1)})
\]
where \(S_{p+1}\) denotes the finite cyclic group of cardinal \(p+1\) and \(sgn(\sigma)\) stands for the sign of the permutation \(\sigma\).

By the recurrent hypothesis the first term of the right hand side of the equality (2.6) becomes
\[
\sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} \varphi_1^\alpha (m_1)^2 \sum_{\sigma \in S_{p+1}} sgn(\sigma) \varphi_2^\alpha (m_2) \varphi_2^\alpha (m_{\sigma(2)}) \cdots \varphi_{i_{p+1}}^\alpha (m_{p+1}) \varphi_{i_{p+1}}^\alpha (m_{\sigma(p+1)}) =
\]
\[
= \frac{1}{p!} \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} \varphi_1^\alpha (m_1)^2 \det \left( \begin{array}{ccc} \varphi_2^\alpha (m_2) & \cdots & \varphi_2^\alpha (m_{p+1}) \\ \vdots & \ddots & \vdots \\ \varphi_{i_{p+1}}^\alpha (m_2) & \cdots & \varphi_{i_{p+1}}^\alpha (m_{p+1}) \end{array} \right)^2
\]
\[
= \frac{1}{(p+1)!} \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} \varphi_1^\alpha (m_1)^2 \left( \det \left( \begin{array}{ccc} \varphi_1^\alpha (m_2) & \cdots & \varphi_1^\alpha (m_{p+1}) \\ \varphi_2^\alpha (m_2) & \cdots & \varphi_2^\alpha (m_{p+1}) \\ \vdots & \ddots & \vdots \\ \varphi_{i_{p+1}}^\alpha (m_2) & \cdots & \varphi_{i_{p+1}}^\alpha (m_{p+1}) \end{array} \right)^2 + \cdots \right.
\]
(2.7)
\[\]
\[
+ \varphi_{i_{p+1}}^\alpha (m_1)^2 \det \left( \begin{array}{ccc} \varphi_1^\alpha (m_2) & \cdots & \varphi_1^\alpha (m_{p+1}) \\ \varphi_2^\alpha (m_2) & \cdots & \varphi_2^\alpha (m_{p+1}) \\ \vdots & \ddots & \vdots \\ \varphi_{i_{p+1}}^\alpha (m_2) & \cdots & \varphi_{i_{p+1}}^\alpha (m_{p+1}) \end{array} \right)^2
\]
where the writing \(\widehat{x}\) means that \(x\) does not appear.

The last term of the right hand side of (2.6) writes
\[
\sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} \sum_{\sigma \in S_{p+1}, \sigma(1) \neq 1} sgn(\sigma) \varphi_1^\alpha (m_1) \varphi_1^\alpha (m_{\sigma(1)}) \cdots \varphi_{i_{p+1}}^\alpha (m_{p+1}) \varphi_{i_{p+1}}^\alpha (m_{\sigma(p+1)}) =
\]
Now, by permuting the indices \((i_1, \ldots, i_{p+1})\)
\[
\frac{N_\alpha}{p_1} \cdots \frac{N_\alpha}{p_{p+1}} \left\{ -\varphi_{i_2}(m_1) \sum_{\sigma \in S_{p+1}, \sigma(1) \neq 1} sgn(\sigma) \varphi_{i_1}(m_{\sigma(1)}) \varphi_{i_{p+1}}(m_{\sigma(p+1)}) + \cdots \\
\right.
\]
\[
+ (-1)^p \varphi_{i_{p+1}}(m_1) \sum_{\sigma \in S_{p+1}, \sigma(1) \neq 1} sgn(\sigma) \varphi_{i_1}(m_{\sigma(1)}) \varphi_{i_{p+1}}(m_{\sigma(p+1)}) \right\} =
\]
\[
= \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} \varphi_{i_1}(m_1) \varphi_{i_{p+1}}(m_{p+1}) \left\{ -\varphi_{i_1}(m_1) \det \begin{pmatrix}
\varphi_{i_2}(m_2) & \cdots & \varphi_{i_{p+1}}(m_{p+1}) \\
\varphi_{i_2}(m_2) & \cdots & \varphi_{i_{p+1}}(m_{p+1}) \\
\vdots & \vdots & \vdots \\
\varphi_{i_{p+1}}(m_2) & \cdots & \varphi_{i_{p+1}}(m_{p+1})
\end{pmatrix} + \cdots \\
\right.
\]
\[
+ (-1)^p \varphi_{i_{p+1}}(m_1) \det \begin{pmatrix}
\varphi_{i_1}(m_2) & \cdots & \varphi_{i_{p+1}}(m_{p+1}) \\
\varphi_{i_1}(m_2) & \cdots & \varphi_{i_{p+1}}(m_{p+1}) \\
\vdots & \vdots & \vdots \\
\varphi_{i_{p+1}}(m_2) & \cdots & \varphi_{i_{p+1}}(m_{p+1})
\end{pmatrix} \right\} =
\]

By the definition of the determinant, we get
\[
= \frac{1}{p!} \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} -\varphi_{i_1}^\alpha(m_1) \varphi_{i_2}^\alpha(m_1) \sum_{\sigma \in S_{p}, \sigma(2) \neq 1} \varphi_{i_2}^\alpha(m_{\sigma(2)}) \cdots \varphi_{i_{p+1}}^\alpha(m_{\sigma(p+1)})
\]
\[
\times \det \begin{pmatrix}
\varphi_{i_1}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_1}^\alpha(m_{\sigma(p+1)}) \\
\varphi_{i_2}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_2}^\alpha(m_{\sigma(p+1)}) \\
\vdots & \vdots & \vdots \\
\varphi_{i_{p+1}}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_{p+1}}^\alpha(m_{\sigma(p+1)})
\end{pmatrix} + \cdots
\]
\[
+ \frac{1}{p!} \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} (-1)^p \varphi_{i_1}^\alpha(m_1) \varphi_{i_{p+1}}^\alpha(m_1) \sum_{\sigma \in S_{p}, \sigma(p) \neq 1} \varphi_{i_2}^\alpha(m_{\sigma(2)}) \cdots \varphi_{i_{p+1}}^\alpha(m_{\sigma(p+1)})
\]
\[
\times \det \begin{pmatrix}
\varphi_{i_1}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_1}^\alpha(m_{\sigma(p+1)}) \\
\varphi_{i_{p+1}}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_{p+1}}^\alpha(m_{\sigma(p+1)}) \\
\varphi_{i_{p+1}}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_{p+1}}^\alpha(m_{\sigma(p+1)})
\end{pmatrix}
\]

Now, by permuting the indices \((i_1, \ldots, i_{p+1})\) and noting that it turns out to permute \((m_2, \ldots, m_{p+1})\), we deduce
\[
= \frac{1}{p!} \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} -\varphi_{i_1}^\alpha(m_1) \varphi_{i_2}^\alpha(m_1) \sum_{\sigma \in S_{p}, \sigma(2) \neq 1} sgn(\sigma) \varphi_{i_2}^\alpha(m_{\sigma(2)}) \cdots \varphi_{i_{p+1}}^\alpha(m_{\sigma(p+1)})
\]
\[
\times \det \begin{pmatrix}
\varphi_{i_1}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_1}^\alpha(m_{\sigma(p+1)}) \\
\varphi_{i_2}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_2}^\alpha(m_{\sigma(p+1)}) \\
\vdots & \vdots & \vdots \\
\varphi_{i_{p+1}}^\alpha(m_{\sigma(2)}) & \cdots & \varphi_{i_{p+1}}^\alpha(m_{\sigma(p+1)})
\end{pmatrix} + \cdots
\]
\[
\frac{1}{p!} \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} (-1)^p \varphi_{i_1}^\alpha (m_1) \varphi_{i_{p+1}}^\alpha (m_1) \sum_{\sigma \in S_p, \sigma(p+1) \neq 1} \text{sgn}(\sigma) \varphi_{i_2}^\alpha (m_{\sigma(2)}) \cdots \varphi_{i_{p+1}}^\alpha (m_{\sigma(p+1)}) \times \det \begin{pmatrix}
\varphi_{i_1}^\alpha (m_2) & \cdots & \varphi_{i_1}^\alpha (m_{p+1}) \\
\vdots & & \vdots \\
\varphi_{i_p}^\alpha (m_2) & \cdots & \varphi_{i_p}^\alpha (m_{p+1}) \\
\varphi_{i_{p+1}}^\alpha (m_2) & \cdots & \varphi_{i_{p+1}}^\alpha (m_{p+1})
\end{pmatrix}
\]

\[
= \frac{1}{p!} \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} (-1)^p \varphi_{i_1}^\alpha (m_1) \varphi_{i_{p+1}}^\alpha (m_1) \det \begin{pmatrix}
\varphi_{i_1}^\alpha (m_2) & \cdots & \varphi_{i_1}^\alpha (m_{p+1}) \\
\varphi_{i_2}^\alpha (m_2) & \cdots & \varphi_{i_2}^\alpha (m_{p+1}) \\
\vdots & & \vdots \\
\varphi_{i_{p+1}}^\alpha (m_2) & \cdots & \varphi_{i_{p+1}}^\alpha (m_{p+1})
\end{pmatrix}
\]

\[
= \frac{1}{(p-1)!} \sum_{i_1, \ldots, i_{p+1}=1}^{N_\alpha} (-1)^p \varphi_{i_1}^\alpha (m_1) \varphi_{i_{p+1}}^\alpha (m_1) \det \begin{pmatrix}
\varphi_{i_1}^\alpha (m_2) & \cdots & \varphi_{i_1}^\alpha (m_{p+1}) \\
\varphi_{i_2}^\alpha (m_2) & \cdots & \varphi_{i_2}^\alpha (m_{p+1}) \\
\vdots & & \vdots \\
\varphi_{i_{p+1}}^\alpha (m_2) & \cdots & \varphi_{i_{p+1}}^\alpha (m_{p+1})
\end{pmatrix}
\]
and since we get there is a cofactor independent. The gradient vector fields Lemma 3. Reporting (2.7) and (2.8) in (2.6), we obtain the equality (2.4).

\[
\frac{2}{(p+1)!} \sum_{i_1, \ldots, i_{p+1} = 1}^{N_\alpha} \sum_{1 \leq j < l \leq p+1} (-1)^{j+l} \varphi_{i_j}^\alpha(m_1) \varphi_{i_l}^\alpha(m_1) \det \begin{pmatrix}
\varphi_{i_1}^\alpha(m_2) & \cdots & \varphi_{i_1}^\alpha(m_{p+1}) \\
\vdots & \ddots & \vdots \\
\varphi_{i_{p+1}}^\alpha(m_2) & \cdots & \varphi_{i_{p+1}}^\alpha(m_{p+1})
\end{pmatrix}
\]

\[
\times \det \begin{pmatrix}
\varphi_{i_1}^\alpha(m_2) & \cdots & \varphi_{i_1}^\alpha(m_{p+1}) \\
\vdots & \ddots & \vdots \\
\varphi_{i_{p+1}}^\alpha(m_2) & \cdots & \varphi_{i_{p+1}}^\alpha(m_{p+1})
\end{pmatrix}
\]

Reporting (2.7) and (2.8) in (2.6), we obtain the equality (2.4).

Now, since by construction the matrix \( \left( \varphi_{j}^\alpha(m_k) \right)_{1 \leq j \leq N_\alpha, 1 \leq k \leq n+1} \) is of rank \( n+1 \), there is a cofactor \( \left( \varphi_{j}^\alpha(m_k) \right)_{1 \leq j, k \leq n+1} \) with determinant \( \det \left( \varphi_{j}^\alpha(m_k) \right) \neq 0 \), and from the relation (2.4), we deduce that \( \det (\Xi_\alpha(\rho(m_j, m_k))) \neq 0 \).

**Lemma 3.** The gradient vector fields \( \nabla u_j, \ j = 1, \ldots, n+1 \) are linearly independent.

**Proof.** Let \( \xi_j \in R \) be real numbers such that \( \sum_{j=1}^{n+1} \xi_j \nabla u_j = 0 \). So

\[
0 = \sum_{j=1}^{n+1} \xi_j \int_{N^n} \langle \nabla u_j, \nabla u_k \rangle dv_h = \sum_{j=1}^{n+1} \xi_j \int_{N^n} \Delta u_j u_k dv_h
\]

\[
= \lambda_\alpha \sum_{j=1}^{n+1} \xi_j \int_{N^n} u_j u_k dv_h
\]

\[
= \lambda_\alpha \sum_{i=1}^{n+1} \xi_j \sum_{i=1}^{N_\alpha} \int_{N^n} \varphi_{i_1}^\alpha(m_j) \varphi_{i_1}^\alpha(m_k) dv_h
\]

\[
= \lambda_\alpha \sum_{i=1}^{N_\alpha} \xi_j \sum_{i=1}^{n+1} \varphi_{i_1}^\alpha(m_j) \varphi_{i_1}^\alpha(m_k)
\]

and since

\[
\sum_{i=1}^{N_\alpha} \varphi_{i_1}^\alpha(m_j) \varphi_{i_1}^\alpha(m_k) = \Xi(\rho(m_j, m_k))
\]

we get

\[
\sum_{j=1}^{n+1} \xi_j \Xi_\alpha(\rho(m_j, m_k)) = 0
\]
so if the matrix $\Xi(\rho(m_j, m_k))_{1 \leq j, k \leq n+1}$ is invertible, we get that $\xi_j = 0$ for $j = 1, ..., n + 1$.

**Lemma 4.** The gradient vector fields $\nabla u_j$, $j = 1, ..., n + 1$ are conformal vector fields on $(N^n, h)$.

**Proof.** Let $X, Y$ any orthogonal vector fields on $N^n$ i.e. $h(X, Y) = 0$ and $Z = \nabla u_j$. We have to show that the Lie derivative $L_Z(h)$ of the tensor metric $h$ with respect to $Z$ satisfies

$$(L_Z h)(X, Y) = 0.$$  

But

$$(L_Z h)(X, Y) = h(\nabla_X Z, Y) + h(\nabla_Y Z, X)$$

and since the manifold $(N^n, h)$ is $SH$, there is a function $\Xi_\alpha: R_+ \to R$ with

$$(2.9) \quad \sum_{i=1}^{N_\alpha} \varphi_i^\alpha(x) \varphi_i^\alpha(y) = \Xi_\alpha(\rho(x, y))$$

for every $x, y$ in $N^n$.

Putting $x = y$ in (2.10) and differentiating twice we get

$$(2.11) \quad \sum_{i=1}^{N_\alpha} d\varphi_i^\alpha(x) \otimes d\varphi_i^\alpha(y) + \sum_{i=1}^{N_\alpha} \varphi_i^\alpha(y) Hess(\varphi_i^\alpha)(y) = 0.$$  

Now differentiating (2.10) twice with respect to $x$ with $y$ fixed we obtain

$$(2.12) \quad \sum_{i=1}^{N_\alpha} \varphi_i^\alpha(y) Hess(\varphi_i^\alpha)(x) = Hess(\Xi_\alpha(\rho(\cdot, y))(x)$$

and evaluating at $x = y$ we get

$$Hess(\Xi(\rho(\cdot, y)))(y) = \Xi''_\alpha(0)h.$$  

By taking account of (2.11), we obtain

$$\sum_{i=1}^{N_\alpha} d\varphi_i^\alpha(x) \otimes d\varphi_i^\alpha(x) = -\Xi''_\alpha(0)h.$$  

To compute $\Xi''_\alpha(0)$, we take the traces in (2.12) and infer that

$$\Xi''_\alpha(0)(y)n = \sum_{i=1}^{N_\alpha} \varphi_i^\alpha(y) \Delta \varphi_i^\alpha(y) = -\lambda_\alpha \sum_{i=1}^{N_\alpha} (\varphi_i^\alpha(y))^2$$

$$= -\lambda_\alpha \Xi_\alpha(\rho(y, y)) = -\lambda_\alpha \Xi_\alpha(0)$$

where $n$ is the dimension of the manifold $N^n$.

So

$$\Xi''_\alpha(0)(y) = -\frac{\lambda_\alpha}{n} \Xi_\alpha(\rho(y, y))$$
and
\[ \sum_{i=1}^{N_n} d\varphi_i^\alpha \otimes d\varphi_i^\beta = \frac{\lambda_\alpha}{n} \Xi_\alpha(0) h. \]

By (2.11), we get
\[ (2.13) \quad \sum_{i=1}^{N_n} \varphi_i^\alpha(y) \left( \text{Hess}(\varphi_i^\alpha)(y) + \frac{\lambda_\alpha}{n} \varphi_i^\alpha(y) h \right) = 0. \]

On the other hand, for the functions \( u_j \) defined by (2.4) and for any \( y \in N^n \), we have
\[ \text{Hess}(u_j)(y) + \frac{\lambda_\alpha}{n} u_j(y) h = \sum_{i=1}^{N_n} \varphi_i^\alpha(m_j) \left( \text{Hess}(\varphi_i^\alpha)(y) + \frac{\lambda_\alpha}{n} \varphi_i^\alpha(y) h \right) \]
and by the homogeneity of the manifold \( N^n \), for any \( y \in N^n \), there is an isometry \( \sigma \) on \( N^n \) such that \( \sigma(y) = m_j \), consequently
\[ \sigma^* \left( \text{Hess}(u_j)(.) + \frac{\lambda_\alpha}{n} u_j(.) h \right)(y) = \sigma^* \left( \sum_{i=1}^{N_n} \varphi_i^\alpha(m_j) \left( \text{Hess}(\varphi_i^\alpha)(.) + \frac{\lambda_\alpha}{n} \varphi_i^\alpha(.) h \right) \right)(y) \]
\[ = \sum_{i=1}^{N_n} \varphi_i^\alpha(m_j) \left( \text{Hess}(\varphi_i^\alpha)(m_j) + \frac{\lambda_\alpha}{n} \varphi_i^\alpha(m_j) h \right) \]
and from the relation (2.13), we get
\[ (2.14) \quad \text{Hess}(u_j)(y) + \frac{\lambda_\alpha}{n} u_j(y) h = 0 \]
so by the equality (2.9), we deduce
\[ (2.15) \quad (L_Z h)(X,Y) = 2 \text{Hess}(u_j)(X,Y) = -2(\frac{\lambda_\alpha}{n} u_j) h(X,Y) \]
and since \( h(X,Y) = 0 \), we obtain
\[ (L_Z h)(X,Y) = 0. \]

\[ \square \]

3. Harmonic maps

Let \( \left\{ \frac{\partial}{\partial x_\alpha} \right\}_{\alpha=1,\ldots,m} \) be an orthonormal basis in a neighborhood of a point \( x \in M^m \), we have, for the mapping \( \psi \) defined by (1.3)
\[ \text{trace} \nabla d\psi(x) = \sum_{\alpha=1}^{m} \left( \frac{\nabla^w \psi}{\partial x_\alpha} d\psi \left( \frac{\partial}{\partial x_\alpha} \right) - \psi \left( \frac{\partial}{\partial x_\alpha} \right) \right) \]
\[ = \sum_{\alpha=1}^{m} \left( \frac{\nabla^w \psi}{\partial x_\alpha} d\phi^w_{\psi,\alpha} - \psi \left( \frac{\partial}{\partial x_\alpha} \right) \right) \]
\[
\begin{align*}
&= \sum_{\alpha=1}^{m} \text{Hess}_{t_0}^{w} \left( df(\frac{\partial}{\partial x_{\alpha}}), df(\frac{\partial}{\partial x_{\alpha}}) \right) + \text{Hess}_{t_0}^{w} \left( \sum_{\alpha=1}^{m} \nabla^{f^{-1}T^{n}} \frac{\partial}{\partial x_{\alpha}} df(\frac{\partial}{\partial x_{\alpha}}) \right) \\
&\quad - d\phi_{t_0}^{w} df \left( \sum_{\alpha=1}^{m} \nabla^{m}_{\frac{\partial}{\partial x_{\alpha}}} \frac{\partial}{\partial x_{\alpha}} \right) \\
&= \sum_{\alpha=1}^{m} \text{Hess}_{t_0}^{w} \left( df(\frac{\partial}{\partial x_{\alpha}}), df(\frac{\partial}{\partial x_{\alpha}}) \right) + d\phi_{t_0}^{w} (\tau_{g}(f))
\end{align*}
\]

where
\[
\text{Hess}_{t_0}^{w}(X,Y) = \nabla^{(\phi_{t_0}^{w})^{-1}T^{n}} d\phi_{t_0}^{w}(Y) - d\phi_{t_0}^{w}(\nabla^{n}X).
\]

So if \( f \) is harmonic i.e. \( \tau_{g}(f) = 0 \), we get
\[
(3.1) \quad \tau_{g}(\psi) = \sum_{\alpha=1}^{m} \text{Hess}_{t_0}^{w} \left( df(\frac{\partial}{\partial x_{\alpha}}), df(\frac{\partial}{\partial x_{\alpha}}) \right).
\]

3.1. Lower bound of the index. Let \( \Gamma(f) \) be the space of vector fields along the map \( f : M^{m} \to N^{n} \) i.e. the sections of the pulled back bundle on \( M^{m} \) induced by \( f \) from the tangent \( T^{n} \) bundle on \( N^{n} \).

The general formula of the second variation of the Energy functional in the direction of the vector fields \( w \) writes
\[
\begin{align*}
\frac{d^2 E(f_t)}{dt^2} |_{t=0} &= \int_{M^{m}} \left( \frac{\|\nabla w\|^2}{f^{-1}T^{n}} - \text{trace}_{M} \left( R^{n} (df, w) w, df \right)_{f^{-1}T^{n}} \right) dv_g \\
&\quad + \int_{M^{m}} \left( \nabla \frac{\partial f}{\partial t}, \text{trace} \nabla df \right)_{f^{-1}T^{n}} dv_g
\end{align*}
\]

and if \( f \) is harmonic, we obtain
\[
(3.3) \quad \frac{d^2 E(f_t)}{dt^2} |_{t=0} = \int_{M^{m}} \left( \frac{\|\nabla w\|^2}{f^{-1}T^{n}} - \text{trace}_{M} \left( R^{n} (df, w) w, df \right)_{f^{-1}T^{n}} \right) dv_g.
\]

For any vector field \( w \) on the target manifold \( N^{n} \) along \( f \), we associate the following quadratic form
\[
(3.4) \quad Q_{f}(w) = \frac{d^2 E_{g}(f_t)}{dt^2} |_{t=0}
\]

where \( f_{t}(x) = \exp(tw) \alpha f(x) \). The Morse index of the harmonic map \( f \) is defined as the integer
\[
\text{Ind}(f) = \text{Sup} \{ \dim F; F \subset \Gamma(f) \text{ such that } Q_{f} \text{ is negative defined on } F \}.
\]

Now let \( S_{g}(f) \) be the stress-energy tensor introduced by Baird and Eells (see [1]).
\[
(3.5) \quad S_{g}(f) = e_{g}(f) g - f^{*} h
\]

For every \( x \in M^{m} \), we put
\[
S_{g}^{0}(f)(x) = \text{Inf} \{ S_{g}(f)(X,X) : X \in T_{x}M^{m} \text{ and } g(X,X) = 1 \}.
\]
The tensor will be said positive at $x$ (resp. positive definite) if we have $S^0_g(f)(x) \geq 0$ (resp. $S^0_g(f)(x) > 0$).

First we state the following theorem

**Theorem 1.** Let $(M^m, g)$ be a Riemannian compact $m-$dimensional manifold and $(N^n, h)$ be a homogeneous strongly harmonic Riemannian manifold of dimension $n \geq 3$ with sectional curvature $K$ satisfying $K \geq \kappa > 0$ where $\kappa$ is a positive constant. Let $f : M^m \to N^n$ be a non constant harmonic map. Suppose that the stress-energy tensor of $f$ is positive definite everywhere on $M^m$ and the nonvanishing eigenvalue of the Laplacian operator $\lambda$ satisfies $\lambda \leq \frac{n^2}{2}\kappa$. Then the index of $f$, $\text{Ind}_E(f) \geq \dim L$, where $L$ is a $n + 1$-dimensional subspace of the eigenspace $V_\lambda$ corresponding to $\lambda$.

**Proof.** At each point $x \in M^m$, we denote respectively by $w^T(x)$ and $w^\perp(x)$ the tangential and the normal projections of $w(x)$ on the space $df(T_xM)$ and $df(T_xM)^\perp$. Let $\{e_1, ..., e_m\}$ be an orthonormal basis of $T_xM^m$ which diagonalizes $f^*h$ such that $\{df(e_1), ..., df(e_l)\}$ be a basis of $df(T_xM^m)$. If $e_g(f)(x) \neq 0$, then at the point $x$ we have

$$\|w^T(x)\|_h^2 = \sum_{i=1}^l \|df(e_i)\|_h^{-2} \langle w(x), df(e_i) \rangle_h^2$$

on the other hand we have for each $i \leq l$

$$\|df(e_i)(x)\|_h^2 = e_g(f)(x) - S_g(f)(x)(e_i, e_i) \leq e_g(f)(x) - S_g(f)(x).$$

So we deduce that

$$(e_g(f) - S_g^0(f))(x) \|w^T(x)\|_h^2 \geq \sum_{i=1}^l h(w(x), df(e_i))^2$$

and

$$\sum_{i=1}^l h(w(x), df(e_i))^2 - e_g(f) \|w^T(x)\|_h^2 \leq -S_g^0(f)(x) \|w^T(x)\|_h^2$$

then

$$\sum_{i=1}^l h(w(x), df(e_i))^2 - e_g(f) \|w(x)\|_h^2$$

$$= \sum_{i=1}^l h(w(x), df(e_i))^2 - e_g(f) \|w^T(x)\|_h^2 - e_g(f) \|w^\perp(x)\|_h^2$$

$$\leq -S_g^0(f)(x) \|w^T(x)\|_h^2 - e_g(f) \|w^\perp(x)\|_h^2$$

$$\leq -S_g^0(f)(x) \|w(x)\|_h^2.$$
Consequently

\[
\sum_{i=1}^{l} h(w(x), df(e_i))^2 - e_g(f) \|w(x)\|^2 \leq -S_g^0(f(x)) \|w(x)\|^2.
\]

Now we let

\[
w_j(x) = -gradu_j(f(x))
\]

where \( u_j = \sum_{i=1}^{N_\alpha} \varphi_i^\alpha(m_j) \varphi_i^\alpha \) is an eigenfunction of the Laplacian operator on \( N^\alpha \) defined previously by \( (2.3) \). Then

\[
\|df\|^2_{f^{-1}T_{N^\alpha}^n} \|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} - \sum_{i=1}^{l} \langle df(e_i), \text{grad}u_j \rangle_{f^{-1}T_{N^\alpha}^n} \geq S_g^0(f) \|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} + e_g(f) \|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n}.
\]

So, we have

\[
\|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} - \sum_{i=1}^{l} \langle R^N_\alpha(df(e_i), \text{grad}u_j)df(e_i), \text{grad}u_j \rangle_{f^{-1}T_{N^\alpha}^n} \leq
\]

\[
\|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} - \kappa \sum_{i=1}^{l} \left( \|df(e_i)(x)\|^2_{\|h\|^2} \|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} - \langle df(e_i), \text{grad}u_j \rangle_{f^{-1}T_{N^\alpha}^n} \right) \leq
\]

\[
\|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} - \kappa \left( e_g(f) \|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} + S_g^0(f) \|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} \right).
\]

Since

\[
\|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} = g^\alpha_\beta \left\langle \nabla_{\partial x^\alpha} \text{grad}u_j, \nabla_{\partial x^\beta} \text{grad}u_j \right\rangle_{f^{-1}T_{N^\alpha}^n}
\]

\[
= g^\alpha_\beta \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \left\langle \nabla_{\partial x^\sigma} \text{grad}u_j, \nabla_{\partial x^{\sigma}} \text{grad}u_j \right\rangle_h
\]

\[
= g^\alpha_\beta \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \text{hess}(u_j(f(x))) \left\langle \frac{\partial}{\partial f^i}, \nabla_{\partial f^k} \text{grad}u_j \right\rangle_h
\]

where the Einstein convention summation is used, and taking account of the formula \( (2.11) \)

\[
= -g^\alpha_\beta \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \frac{\lambda}{n} u_j(f(x)) \left\langle \frac{\partial}{\partial f^i}, \nabla_{\partial f^k} \text{grad}u_j \right\rangle_h
\]

\[
= g^\alpha_\beta \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^k}{\partial x^\beta} \frac{\lambda^2}{n^2} u^2_{ij} = 2 \frac{\lambda^2}{n^2} e_g(f) u^2_j.
\]

On the other hand, we get from formula \( (2.11) \)

\[
\int_{M^m} \|\text{grad}u_j\|^2_{f^{-1}T_{N^\alpha}^n} dv_g = \int_{M^m} \|du_j\|^2_{f^{-1}T_{N^\alpha}^n} dv_g = \int_{M^m} \text{trace}(du_j \otimes du_j)(f(x)) dv_g
\]

\[
= \lambda \int_{M^m} u^2_j(f(x)) dv_g
\]
so
\[
\int_{M^m} \| \nabla \text{grad} u_j \|^2_{f^{-1}T_{N^n}} \, dv_g = 2 \frac{\lambda}{n^2} \int_{M^m} e_g(f) \| \text{grad} u_j \|^2_h \, dv_g.
\]

Consequently
\[
\int_{M^m} \left( \| \nabla \text{grad} u_j(f(x)) \|^2_h - \sum_{i=1}^m \langle R^{N^n}(df(e_i), \text{grad} u_j)df(e_i), \text{grad} u_j(f(x)) \rangle_h \right) \, dv_g \leq
\]
\[
\int_{M^m} \left( 2 \frac{\lambda}{n^2} e_g(f) \| \text{grad} u_j(f(x)) \|^2_h - \kappa e_g(f) \| \text{grad} u_j(f(x)) \|^2_h + S^o_g(f) \| \text{grad} u_j(f(x)) \|^2_h \right) \, dv_g =
\]
\[
(2 \frac{\lambda}{n^2} - \kappa) \int_{M^m} e_g(f) \| \text{grad} u_j(f(x)) \|^2_h \, dv_g - \kappa \int_{M^m} S^o_g(f) \| \text{grad} u_j(f(x)) \|^2_h \, dv_g
\]
Hence
\[
\frac{d^2E(f_t)}{dt^2} \big|_{t=0} \leq
\]
\[
\int_{M^m} e_g(f)(2 \frac{\lambda}{n^2} - \kappa) \| \text{grad} u_j(f(x)) \|^2_h \, dv_g - \kappa \int_{M^m} S^o_g(f)(x) \| \text{grad} u_j(f(x)) \|^2_h \, dv_g
\]
which shows that if the eigenvalue \( \lambda \) of the Laplacian \( \Delta_{N^n} \) operator on \( N^n \) satisfies
\[
\lambda \leq \frac{n^2 \kappa}{2}
\]
then
\[
\frac{d^2E(f_t)}{dt^2} \big|_{t=0} \leq -\kappa \int_{M^m} S^o_g(f)(x) \| \text{grad} u_j(f(x)) \|^2_h \, dv_g
\]
so
\[
\text{Ind}_E(f) \geq \dim L
\]
where \( L \) is the \( n + 1 \) -dimensional subspace spanned by the gradient vector fields \( \text{grad} u_j, j = 1, \ldots, n + 1. \) \( \Box \)

4. Harmonic map as a global maximum

In this section, we prove the following global theorem

**Theorem 2.** Let \((M^m, g)\) be a compact Riemannian \( m \)-dimensional manifold, \((N^n, h)\) be a homogenous strongly harmonic Riemannian manifold of dimension \( n \geq 3 \). If the stress-energy tensor of the harmonic map \( f : M^m \to N^n \) is positive everywhere on \( M^m \), then the map \( f \) is a global maximum of the energy functional on the \( n + 1 \)-dimensional subspace \( L \) of the eigenspace \( V_\lambda \) corresponding to a nonvanishing eigenvalue \( \lambda \) of the Laplacian operator on the target manifold.
Now we compute the gradient of the function $\sigma = -df e$ eigenfunction $u = 2$ since for simplicity $w = u_j$ defined previously as minus the gradient of the connections is then given by: for any vector field $X$, $Y$ on the manifold $N^n$ of $\sigma$ of, $\sigma$, where $\tau$ is some function on the manifold $M$, $\tau$ and such that $(df(O\frac{\partial}{\partial x_1}), ..., df(O\frac{\partial}{\partial x_m}))$ is a basis of $df(T_x M)$. Then

$$d\phi^w_t (\tilde{\nabla}^N_X Y) = \nabla_{d\phi^w_t(X)} d\phi^w_t(Y)$$

where for simplicity $w = w_j$ given by (2.3), so

$$\tau_g(\psi) = \sum_{a=1}^{m} Hess \phi^w_t \left( df(O\frac{\partial}{\partial x_a}), df(O\frac{\partial}{\partial x_a}) \right)$$

$$= \sum_{a=1}^{m} d\phi^w_t \left( \nabla^{-1}_x Hess df(O\frac{\partial}{\partial x_a}) - \nabla^{-1}_x Hess df(O\frac{\partial}{\partial x_a}) \right).$$

Since $w$ is a conformal infinitesimal transformation i.e. $(\phi^w_t)^* h = e^{2\alpha} h$, where $\alpha$ is some function on the manifold $N^n$, the conformal change of connections is then given by: for any vector field $X, Y$ on the manifold $N^n$

$$\tilde{\nabla} X Y - \nabla_X Y = X(\sigma) Y + Y(\sigma) X - \langle X, Y \rangle_h \nabla \sigma$$

where $\langle ., . \rangle_h = h$ and $\nabla \sigma$ is the gradient vector field of the function $\sigma$. Hence

$$\tau_g(\psi) = d\phi^w_t \left( \sum_{a=1}^{m} \left( \nabla \sigma of, df(O\frac{\partial}{\partial x_a}) \right)_h df(O\frac{\partial}{\partial x_a}) - \sum_{a=1}^{m} \left( df(O\frac{\partial}{\partial x_a}), df(O\frac{\partial}{\partial x_a}) \right)_h \nabla \sigma of \right)$$

$$= 2 d\phi^w_t \left( \sum_{a=1}^{m} \left( \nabla \sigma of, df(O\frac{\partial}{\partial x_a}) \right)_h df(O\frac{\partial}{\partial x_a}) - e_g(f) \nabla \sigma of \right)$$

where $\psi = \phi^w_t \sigma$. Consequently

$$\int_M \langle \tau_g(\psi), w \phi^w \rangle_h dv_g =$$

$$= 2 \int_M \langle \phi^w_t \rangle^* \left( \sum_{a=1}^{m} \left( \nabla \sigma of, df(O\frac{\partial}{\partial x_a}) \right)_h \left( df(O\frac{\partial}{\partial x_a}), w \phi^w \right)_h - e_g(f) \langle \nabla \sigma of, w \phi^w \rangle_h \right) dv_g$$

$$= 2 \int_M e^{2\alpha \phi^w(x)} \left\{ \sum_{a=1}^{m} \left( \nabla \sigma of, df(O\frac{\partial}{\partial x_a}) \right)_h \left( df(O\frac{\partial}{\partial x_a}), w \phi^w \right)_h - e_g(f) \langle \nabla \sigma of, w \phi^w \rangle_h \right\} dv_g$$

Now we compute the gradient of the function $\sigma$ to get

$$\langle \nabla \sigma of, w \phi^w \rangle_h(x) = \frac{\langle \nabla w \phi^w(x), w \phi^w(x) \rangle_h}{\|w \phi^w(x)\|^2_h} - \frac{\langle \nabla w \phi^w(x), w \phi^w(x) \rangle_h}{\|w \phi^w(x)\|^2_h}$$

$$= - \frac{Hess(u(\phi^w_t(f)(x))) (w(\phi^w_t(f)(x)), w(\phi^w_t(f)(x)))_h}{\|w(\phi^w_t(f)(x))\|^2_h} + \frac{Hess(u(f(x))) (w(f(x)), w(f(x)))_h}{\|w(f(x))\|^2_h}.$$
Using the inequality \((2.14)\), we get that
\[
\langle \nabla \sigma_{\text{of}}, w \sigma_{\text{of}} \rangle_h (x) = \frac{\lambda}{n} \left( u(\phi_{t_0}^w(f(x)) - u((f(x))) \right)
\]
and also, we have
\[
\begin{align*}
H & \text{ess}(u(\phi_{t_0}^w(f(x)))) \left( \phi_{t_0}^w(d \frac{\partial}{\partial x_\alpha}), w(\phi_{t_0}^w(f(x))) \right)_h \quad + \quad H & \text{ess}(u(f(x))) \left( d \frac{\partial}{\partial x_\alpha}, w(f(x)) \right)_h \\
& = \frac{\lambda}{n} \left( u(\phi_{t_0}^w(f(x))) \left( \phi_{t_0}^w(d \frac{\partial}{\partial x_\alpha}), w(\phi_{t_0}^w(f(x))) \right)_h \right) \\
& \quad - \frac{\lambda}{n} \left( u((f(x))) \left( d \frac{\partial}{\partial x_\alpha}, w(f(x)) \right)_h \right) \\
& = \frac{\lambda}{n} \left( u(\phi_{t_0}^w(f(x)) - u((f(x))) \right) \left( d \frac{\partial}{\partial x_\alpha}, w(f(x)) \right)_h
\end{align*}
\]
consequently
\[
\begin{align*}
\left( \nabla \sigma_{\text{of}}, d \frac{\partial}{\partial x_\alpha} \right)_h (x) \left( d \frac{\partial}{\partial x_\alpha}, w \sigma_{\text{of}} \right)_h (x) = \frac{\lambda}{n} \left( u(\phi_{t_0}^w(f(x)) - u((f(x))) \right) \left( d \frac{\partial}{\partial x_\alpha}, w(f(x)) \right)_h
\end{align*}
\]
and
\[
\sum_{\alpha=1}^{m} \left( \nabla \sigma_{\text{of}}, d \frac{\partial}{\partial x_\alpha} \right)_h (x) \left( d \frac{\partial}{\partial x_\alpha}, w \sigma_{\text{of}} \right)_h (x) - e_g(f) \langle \nabla \sigma_{\text{of}}, w \sigma_{\text{of}} \rangle_h (x) =
\]
\[
= \frac{\lambda}{n} \left( u(\phi_{t_0}^w(f(x)) - u((f(x))) \right) \sum_{\alpha=1}^{m} \left( d \frac{\partial}{\partial x_\alpha}, w(f(x)) \right)_h^2 - e_g(f) \left\| w(f(x)) \right\|_h^2
\]
Now, if at each point \( x \in M^m \), \( w^T(x) \) and \( w^\perp(x) \) denote the tangential and the normal projections of \( w(x) \) on the space \( df(\mathcal{T}_x M) \) and \( df(\mathcal{T}_x M)^\perp \) and \( S_g^o(f) \) is the stress-energy tensor of \( f \), we have
\[
(e_g(f) - S_g^o(f)) \left\| w^T(f(x)) \right\|_h^2 = (e_g(f) - S_g^o(f)) \sum_{\alpha=1}^{l} \left( d \frac{\partial}{\partial x_\alpha}, \frac{\left\| w(f(x)) \right\|}{d \frac{\partial}{\partial x_\alpha}} \right)_h^2
\]
\[
\geq \sum_{\alpha=1}^{l} \left( d \frac{\partial}{\partial x_\alpha}, w(f(x)) \right)_h^2
\]
so
\[
\sum_{\alpha=1}^{m} \left( d \frac{\partial}{\partial x_\alpha}, w(f(x)) \right)_h^2 - e_g(f) \left\| w(f(x)) \right\|_h^2 \leq \sum_{\alpha=l+1}^{m} \left( d \frac{\partial}{\partial x_\alpha}, w(f(x)) \right)_h^2 - S_g^o(f) \left\| w^T(f(x)) \right\|_h^2 - e_g(f) \left\| w^\perp(f(x)) \right\|_h^2
\]
of the linear forms on the Euclidean space $\mathbb{R}^n$ endowed with canonical metric then for every function $u$ the eigenfunctions, corresponding to the first non zero eigenvalue of the Laplacian on the sphere $S\text{Conf}$, Lie algebra of $S\text{Conf}$, are the restrictions of $\sum_{i=1}^{N_\alpha} \varphi_i(m_j) \varphi_i$ corresponding to a non vanishing eigenvalue $\lambda$, and since $S_\alpha^0(f) \leq e_g(f)$, we get

$$\frac{d}{dt} \big|_{t=t_0} u(\phi^w_t(f(x))) = -\|w(\phi^w_{t_0}(f(x)))\| \leq 0$$

that is the function $t \rightarrow u(\phi^w_t(f(x)))$ is decreasing.

Consequently

$$\sum_{\alpha=1}^m \left< \nabla \sigma f, df \left( \frac{\partial}{\partial x_\alpha} \right) \right>_h (x) \left< df \left( \frac{\partial}{\partial x_\alpha} \right), wo f \right>_h (x) - e_g(f) \left< \nabla \sigma f, wo f \right>_h (x) \geq -\frac{\lambda}{n} \left( u(\phi^w_{t_0}(f(x))) - u((f(x))) S_\alpha^0(f) \right)$$

and finally we obtain that

$$\frac{d}{dt} E(f_t) \big|_{t=t_0} = -\int_{M^n} \langle \tau_g(\psi), wo \psi \rangle f^{-1} T_N dv_g$$

$$\leq \frac{\lambda}{n} \int_{M^n} e^{2\sigma f(x)} \left( u(\phi^w_{t_0}(f(x))) - u((f(x))) \right) S_\alpha^0(f) dv_g$$

So if the stress-energy tensor $S_\alpha^0(f)$ is positive, the energy functional is decreasing that means that the harmonic map $f$ is a global maximum of the energy functional $E(\cdot)$. □

Let $\text{Isom}(S^n)$ and $\text{Conf}(S^n)$ be respectively the isometric and the conformal Lie group on the standard unit sphere $S^n$. Denote by $\text{conf}(S^n)$ the Lie algebra of $\text{Conf}(S^n)$ and let $w$ be the negative gradient of the eigenfunction $u_j = \sum_{i=1}^{N_\alpha} \varphi_i(m_j) \varphi_i$ corresponding to a non vanishing eigenvalue of the Laplacian on the sphere $S^n$, given by (2.4), we have

**Corollary 2.** *If the target manifold is the standard unit $n$-sphere $S^n$ ($n \geq 3$) endowed with canonical metric then for every $w \in \text{conf}(S^n)$*

$$E(wo f) \leq E(f)$$

*provided that the stress-energy tensor is positive.*

**Proof.** We know that

$$\dim(\text{Conf}(S^n)/\text{Isom}(S^n)) = n + 1.$$ 

Since the eigenfunctions, corresponding to the first non zero eigenvalue $\lambda_1 = n$ of the Laplacian operator on the standard $n$-sphere $S^n$, are the restrictions of the linear forms on the Euclidean space $\mathbb{R}^{n+1}$ to $S^n$ and their gradients are
conformal vector fields, it follows that the set of these gradients is nothing than $conf(S^n)$ and by Theorem we get for every $w \in conf(S^n)$: $E(wf) \leq E(f)$.

Remark 1. In the particular case where the target manifold is the standard $n$-sphere, we have $dim V_\lambda = n+1$. So we get the result in (15).

5. Minimal Immersions

5.1. Morse index of minimal immersions. With notations of the previous section, if the manifold $M^m$ is a minimal submanifold of the manifold $N^n$, the second variation with respect to the vector field $w$ along $f$ is given by the following integral

$$
\frac{d^2}{dt^2} V(f_t) \big|_{t=0} =
\int_{M^m} \left( \| \nabla f^{-1} T N^n w o f \|_h^2 - \| \sigma(wf) \|_h^2 - trace_{M^m} \langle R^{N^n}(df, w^\perp of), df \rangle_h \right) dv_g
$$

where $trace_{M^m} \langle R^{N^n}(df, w)w, df \rangle = \sum_{i=1}^m \left\langle R^{N^n}(df(\frac{\partial}{\partial x_i}), w)w, df(\frac{\partial}{\partial x_i}) \right\rangle_h$

and $\sigma(w)$ denotes the second fundamental form relative to $w$.

To the immersion $f$, we assign the quadratic form associated to the second variation of the volume functional defined on $\Gamma(f)$ by

$$
H_f(w) = \frac{d^2}{dt^2} V(f_t) \big|_{t=0}
$$

and since this latter depends only on the normal component of elements of $\Gamma(f)$, we consider only the restriction of $H_f$ to the normal projection $\Gamma^{N^n}(f)$ of the space $\Gamma(f)$. The Morse index of $f$, denoted $Ind_V(f)$, is defined as the dimension of the maximal subspace of $\Gamma^{N^n}(f)$ on which $H_f$ is negative-definite.

In this section, we state the following theorem

**Theorem 3.** Let $(M^m, g)$ be a Riemannian compact $m$-dimensional manifold, $(N^n, h)$ be a homogenous strongly harmonic Riemannian manifold of dimension $n \geq 3$ with sectional curvature $K$ satisfying $K \geq \kappa > 0$, where $\kappa$ is a constant, $f : M^m \to N^n$ be a minimal isometric immersion not totally geodesic and $\lambda$ be a non vanishing eigenvalue of the Laplacian operator on the target manifold $N^n$. Suppose that $\lambda$ satisfies $\lambda \leq \frac{m^2}{2} \kappa$. Then $Ind_V(f) \geq dim(L^\perp)$ where $L^\perp$ denotes the normal component of an $n+1$-dimensional subspace $L$ of the eigenspace $V_\lambda$ corresponding to $\lambda$.

**Proof.** $H_f(w)$ can be written as

$$
H_f(w) = \int_{M^m} \left( \| \nabla f^{-1} T N^n w^\perp of \|_h^2 - trace_{M^m} \langle R^{N^n}(df, w^\perp of), df \rangle_h \right) dv_g
$$
where \( w = -\text{grad}u_j \) and \( w^\perp \) is the orthogonal projection of \( w \) on \( df(T_xM^m) \).

Now taking account of formula (3.7), we obtain

\[
H_f(w) = \int_M 2e_g(f) \left( \frac{\lambda}{n^2} - \frac{1}{2}\kappa \right) \left\| w^\perp of \right\|_h^2 dv_g - \int_M \left\| \sigma(w^\perp of) \right\|_h^2 dv_g.
\]

so, since \( \lambda \leq \frac{\kappa n^2}{2} \) and \( \sigma(w^\perp) \neq 0 \)
we obtain

\[
H_f(w) < 0.
\]

\[\square\]

5.2. Minimal Immersion as a global maximum. In this subsection, we establish the following global theorem

**Theorem 4.** Let \((M^m,g)\) be a Riemannian \(m\)-dimensional compact manifold, \((N^n,h)\) be a strongly harmonic Riemannian manifold of dimension \(n \geq 3\). If \(f : M^m \to N^n\) is a minimal isometric immersion, then \(f\) is a global maximum of the volume functional on the normal component \(L^\perp\) of an \(n+1\)-dimensional subspace \(L\) of the eigenspace \(V_\lambda\) corresponding to a nonvanishing eigenvalue of the Laplacian operator \(\lambda\) on the target manifold.

**Proof.** Let \(w = -\text{grad}u_j\), where \(u_j\) is the eigenfunction of the Laplacian operator on \(N^n\) given in section 2 by (2.3). Let \(\psi = \phi^w_t of\) where \(\phi^w_t\) denotes the flow generated by the vector field \(w\). The first variation formula reads as

\[
\frac{d}{dt}V(f_t) = -\int_{\psi(M)} \left\langle H_{f(x)}^{\psi(M^m)}, w(f(x)) \right\rangle_h dv_g
= -\int_{M^m} \left\langle H_{\psi(x)}^{\psi(M)}, w(\psi(x)) \right\rangle_h e^{n\sigma} dv_g
\]

where \(H_{\psi(x)}^{\psi(M)}\) denotes the mean curvature of the submanifold \(\psi(M^m)\) in \(N^n\). The variation depends only on the normal component \(w^\perp\) of the vector field \(w\). Let \(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_m}\) be an orthonormal basis at the point \(x \in M\), then

\[
\left\langle H_{\psi(x)}^{\psi(M)}, w(\psi(x)) \right\rangle_h = \sum_{i=1}^m \left\langle \nabla_{\psi_* \frac{\partial}{\partial x_i}} \psi_* \frac{\partial}{\partial x_i}, w^\perp(\psi_t(x)) \right\rangle_h .
\]

Since \(\phi^w_t : (N^n, e^{2\sigma}h) \to (N^n, h)\) is an isometry, it follows that

\[
(\phi^w_t)^* \left( \nabla_{f_* \frac{\partial}{\partial x_i}} f_* \frac{\partial}{\partial x_i} \right) = \nabla_{(\psi_t)^* \frac{\partial}{\partial x_i}} (\psi_t^w)^* \frac{\partial}{\partial x_i}.
\]
where $\nabla^N_n$ is the connection on the manifold $(N^n, e^{2\sigma} h)$ and $\psi^w = \phi^w of$. From the conformal change of connections formula, we obtain

$$
\langle H^{\psi(x)} \psi(x), w(\psi(x)) \rangle = \sum_{i=1}^m (\phi_i^w)^* \left\langle \nabla^N_n \phi_i^w \frac{\partial}{\partial x_i}, w^+(f(x)) \right\rangle 
$$

$$
= \sum_{i=1}^m (\phi_i^w)^* \left( \nabla^N_n f \cdot \frac{\partial}{\partial x_i} f + 2 \left\langle f \frac{\partial}{\partial x_i}, \text{grad}(\sigma) of \right\rangle f \cdot \frac{\partial}{\partial x_i} \phi_i^w \right) \right.
$$

$$
- \left\langle f \frac{\partial}{\partial x_i}, f \frac{\partial}{\partial x_i} \right\rangle \frac{\partial}{\partial x_i} \text{grad}(\sigma) of, w^+ of \right\rangle
$$

and since $\left\langle f \frac{\partial}{\partial x_i}, w^+ of \right\rangle = 0$, we obtain

$$
\langle H^{\psi(x)} \psi(x), w(\psi(x)) \rangle = (\phi_i^w)^* \left\langle H^{f(x)} - m \text{grad}(\sigma) of, w of \right\rangle
$$

$$
= -m (\phi_i^w)^* \left\langle \text{grad}(\sigma) of, w of \right\rangle , \text{since f is minimal.}
$$

Consequently

$$
\frac{d}{dt} |_{t=t_0} V(f_t) = m \int_{M^m} e^{2\sigma of(x)} \langle \text{grad}(\sigma) of, w of \rangle dv_g; \ t_o > 0
$$

and taking into account the relation (2.14)

$$
\frac{d}{dt} |_{t=t_o} V(f_t) = \frac{m\lambda}{n} \int_{M^m} e^{2\sigma of(x)} \left( u(\phi^w_{2t_0} f(x)) - u(\phi^w_{t_0} f(x)) \right) dv_g.
$$

Since the vector field $w$ is minus the gradient of the eigenfunction $u_j$, we have

$$
\frac{d}{dt} u_j(\phi^w_{t_0} f(x)) |_{t=t_0} = - \| w(\phi^w_{t_0} f(x)) \|_h^2 \leq 0
$$

so

$$
\frac{d}{dt} V(f_t) \leq 0 \ \text{that is to say}
$$

$$
V(f_t) \leq V(M^m).
$$

\[\square\]

References

[1] Baird, P., Eells, J., A conservation law for harmonic maps, Lecture Notes in Math. 894 (1981), 1-25.

[2] Berger, M., Gauduchon, P., Mazet, E.: Le spectre d’une variété Riemannienne. Lecture Notes Vol. 194. Berlin-Heidelberg-New-York: Springer 1971.

[3] Besse, A. L., Manifolds all of whose Geodesics are Closed. Berlin-Heidelberg-New-York: Springer 1978.

[4] J. Eells, L. Lemaire, A report on harmonic maps. Bull. London Math. Soc. 10 (1978) 1-68.

[5] El Soufi, A., Applications harmoniques, immersions minimales et transformations conformes de la sphère, Compo.Math. 85 (1993), 281-298.

[6] El Soufi, S. Ilias, Riemannian manifolds admitting isometric immersion by their first eigenfunctions. Pacific J. Math., 195 (2000) 91-99.

[7] Karcher, H. Riemannian Comparison construction, Preprint Bonn 1987.

[8] Leug, P.F., On the stability of harmonic maps, Lecture Notes in Math. 949(1980),122-129.
[9] Simons, J., Minimal varieties in Riemannian manifolds, Ann. of Math. 88(2) (1968), 62-105.

University AbouBakr Belkaid, Faculty of Sciences, Dept. of Mathematics, B.P. 119, Tlemcen Algeria

E-mail address: m_benalili@mail.univ-tlemcen.dz

E-mail address: hafedabenallal@yahoo.fr