Accelerated orbits in black hole fields: the static case

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Received 11 April 2011, in final form 15 August 2011
Published 18 October 2011
Online at stacks.iop.org/CQG/28/225012

Abstract

We study non-geodesic orbits of test particles endowed with a structure, assuming the Schwarzschild spacetime as background. We develop a formalism which allows one to recognize the geometrical characterization of those orbits in terms of their Frenet–Serret parameters and apply it to explicit cases as those of spatially circular orbits which witness the equilibrium under conflicting types of interactions. In our general analysis, we solve the equations of motion offering a detailed picture of the dynamics having in mind a check with a possible astronomical setup. We focus on certain ambiguities which plague the interpretation of the measurements preventing one from identifying the particular structure carried by the particle.

PACS number: 04.20.Cv

(Some figures may appear in colour only in the online journal)

1. Introduction

Deviations from geodesic motion by a test particle in a given gravitational field are generally associated with the particle's additional properties, such as electric charge or spin, but also with external interacting fields (even test ones) added to the background. Consistency with the test hypothesis of the particle and the superposed field requires small deviations from the geodesic behavior; otherwise back reaction should be taken into account with awkward implications. Indeed, assuming the consistency condition to be satisfied, a detailed study of such deviations can easily be carried out.

The motion of charged particles as well as particles with magnetic moment in the field of a black hole when a magnetic field is also present has been analyzed in several papers [1–7]. Contrary to the previous studies, our approach enables us to solve the equations of...
motion analytically, so identifying the equilibrium solutions and the stability conditions. These properties are of clear relevance in the astrophysical context. Moreover, our method can be generalized to find the behavior of bodies endowed with whatever physical structure and even with more than one being present in the same object: this is the case of bodies with both spin and magnetic moment. The results of our analysis can be directly confronted with astrophysical observations.

In this paper, we provide an intrinsic characterization of general equatorial orbits in the Schwarzschild spacetime by studying their Frenet–Serret properties, i.e. curvature and torsions, filling a gap existing in the literature where only circular orbits are considered in detail. Here, we make explicit examples of forces which are responsible for these orbits, by considering—within a unified scenario—charged particles and magnetic dipoles moving in the given metric with a magnetic field added, spinning test bodies gravitationally interacting with the background and particles deviating from the geodesic behavior by the scattering of electromagnetic radiation.

This work can also be useful for the following reason. Nowadays progress in the instrumental sensitivity and data analysis has allowed a careful reconstruction of the orbits of a large number of astrophysical objects and hence models can be considered to fit the data. This is the case of pulsars orbiting around Sgr A*, the super-massive black hole at the center of our galaxy. For instance, it is largely believed that to measure general relativistic effects with current instruments, one needs to discover pulsars with orbital periods less than about 10 years and orbiting at a distance of about 0.01 pc from Sgr A*. Even if searches are currently underway, at least five pulsars are now known to lie within 100 pc from Sgr A* [8]. Reconstruction of the orbits directly from observations is not so far from becoming possible. Hence, if a particle deviates from geodesic motion within a properly chosen background spacetime and the deviations are measured, one may use such measurements to put limits on the particle structure or to the nature and intensity of external fields. We make this point explicit in the simple context of a Schwarzschild black hole.

Finally, another outcome of this work is the discussion of analogies in the motion of differently structured particles. Under certain conditions it may happen that either an electrically charged particle behaves in an external electromagnetic field exactly like a neutral spinning one in the given background or a spinning particle endowed with a magnetic dipole moment in an external magnetic field becomes indistinguishable from a non-spinning and neutral particle with suitable values of the above properties.

Hereafter Latin indices run from 1 to 3, whereas Greek indices run from 0 to 3 and geometrical units are assumed. The metric signature is chosen as $+2$.

2. The intrinsic properties of equatorial orbits

The Schwarzschild metric, expressed in standard coordinates $(t, r, \theta, \phi)$, is given by

$$
\text{d}s^2 = -N^2 \text{d}t^2 + N^{-2} \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2),
$$

where $N = (1 - 2M/r)^{1/2}$ is the lapse function. It is understood that the following discussion only holds outside the horizon, namely for $r > 2M$. The unit volume 4-form which assures the orientation of the spacetime is denoted by $\eta_{\alpha\beta\gamma\delta}$ and related to the Lévi-Civita alternating symbol $\epsilon_{\alpha\beta\gamma\delta}$ ($\epsilon_{0123} = 1$) by the relation

$$
\eta_{\alpha\beta\gamma\delta} = \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta},
$$

where $g$ is the determinant of the metric.
Introduce the standard orthonormal frame adapted to the static observers (in the case of metric (2.1) they coincide with ZAMOs, zero angular momentum observers), namely
e_i \equiv \eta = N^{-1} \partial_r, \quad e_\theta = N \partial_\theta, \quad e_\phi = r^{-1} \partial_\phi, \quad e_\tau = (r \sin \theta)^{-1} \partial_\phi. \tag{2.3}

Let \( U \) be the 4-velocity of a test particle,
\[
U = \gamma (n + \nu^r e_r + \nu^\theta e_\theta + \nu^\phi e_\phi), \quad \gamma = 1/\sqrt{1 - \nu^2}, \quad \nu^2 = \delta_{ab} \nu^a \nu^b, \tag{2.4}
\]
where \( \nu^a (a = r, \theta, \phi) \) are the components of the spatial velocity relative to \( n \) and \( \gamma \) is the Lorentz factor. Its 4-acceleration frame components are given by
\[
a(U)\dot{r} = \frac{d\nu^r}{dt} + \frac{\gamma^2 N}{r} \nu^2 \nu^r, \\
a(U)\dot{\theta} = \frac{d\nu^\theta}{dt} + \frac{\gamma^2 N}{r} (\nu^\theta - \nu^2 \nu^\theta), \\
a(U)\dot{\phi} = \frac{d\nu^\phi}{dt} + \frac{\gamma^2 N}{r} (\nu^\phi - \nu^2 \nu^\phi), \\
\]
where
\[
\nu^2 = \frac{1 - N^2}{2N^2} = \frac{M}{r - 2M} \tag{2.6}
\]
is the speed of a Keplerian spatially circular geodesic at radius \( r \) with the associated Lorentz factor \( \gamma_K = 1/\sqrt{1 - \nu_K^2} \). From the orthogonality condition \( U \cdot a(U) = 0 \), the following relation holds:
\[
a(U)\dot{r} = v_r a(U)\dot{r} + v_\theta a(U)\dot{\theta} + v_\phi a(U)\dot{\phi}. \tag{2.7}
\]

We shall now study the special case of a motion confined to the equatorial plane (\( \theta = \pi/2 \)), namely with \( \nu^\theta = 0 \); the above relations then simplify as follows:
\[
a(U)\dot{r} = \frac{d\nu^r}{dt} + \frac{\gamma^2 N}{r} \nu^2 \nu^r, \\
a(U)\dot{\theta} = \frac{d\nu^\theta}{dt} - \frac{\gamma^2 N}{r} (\nu^\theta - \nu^2 \nu^\theta), \\
a(U)\dot{\phi} = 0, \\
\]
with also, from equation (2.4),
\[
\frac{dr}{d\tau} = \gamma N \nu^r, \quad \frac{d\phi}{d\tau} = \frac{\gamma}{r} \nu^\phi. \tag{2.9}
\]
The general equatorial motion is thus fully described by the two equations
\[
\frac{d\nu^r}{d\tau} = \frac{1}{\gamma} \left( \frac{a(U)\dot{r}}{\gamma^2} - a(U)\dot{\theta} \nu^\theta \nu^\phi \right) + \frac{\gamma N}{r} \left[ \nu^\theta (\nu^\phi - \nu^2 \nu^\phi) \right], \\
\frac{d\nu^\phi}{d\tau} = \frac{1}{\gamma} \left( \frac{a(U)\dot{\phi}}{\gamma^2} - a(U)\dot{\theta} \nu^\theta \nu^\phi \right) - \frac{\gamma N}{r} \nu^\phi. \tag{2.10}
\]
where $\gamma_\tau = 1/\sqrt{1 - v^2}$ and $\gamma_\phi = 1/\sqrt{1 - \dot{v}^2}$. Here all the components of $a(U)$ have been re-expressed in terms of $a(U)_\tau$ and $a(U)_\phi$, by using equations (2.7) and (2.8), so that

$$a(U) = [v_\tau a(U)_{\tilde{\tau}} + \dot{v}_\phi a(U)_{\tilde{\phi}}]n + a(U)_{\tilde{\tau}} e_\tau + a(U)_{\tilde{\phi}} e_\phi,$$  

(2.11)

and

$$||a(U)|| = \kappa = \sqrt{-[a(U)_\tau v^\tau + a(U)_\phi \dot{v}^\phi]^2 + a(U)_\tau^2 + a(U)_\phi^2}$$

$$= \left[ \frac{a(U)_\tau^2}{\gamma_\tau^2} + \frac{a(U)_\phi^2}{\gamma_\phi^2} - 2v^\tau \dot{v}^\phi a(U)_\tau a(U)_\phi \right]^{1/2}$$

$$= \gamma^2 \left[ \frac{v^\tau}{\gamma_\tau^2} + \frac{\dot{v}^\phi}{\gamma_\phi^2} + 2v^\tau \dot{v}^\phi \dot{v}^\phi - \frac{2N}{r} \left[ (\dot{v}^\phi - v^\phi_\phi)^2 \dot{v}^\phi - \dot{v}^\phi \dot{v}^\phi_\phi \right] \right. $$

$$\left. + \frac{N^2}{r^2} \left[ (\dot{v}^\phi - v^\phi_\phi)^2 \dot{v}^\phi + (\dot{v}^\phi - v^\phi_\phi)^2 \right] \right]^{1/2},$$

(2.12)

where a dot stands for differentiation with respect to the proper time.

In the equatorial plane, the condition for a spatially circular motion, namely $r = r_0 = const.$, $v^\phi = 0$ and $d\dot{v}^\phi/d\tau = 0$, simplifies the equations of motion which become

$$\frac{a(U)_\tau}{\gamma} + \frac{N}{r} (\dot{v}^\phi - v^\phi_\phi) = 0, \quad \frac{d\dot{v}^\phi}{d\tau} = \frac{a(U)_\phi}{\gamma^3}.$$

(2.13)

If in addition $a(U)_\phi = 0$, then $v^\phi = const.$ and $a(U)_\tau = const.$ as can be seen from the first of equation (2.13), namely

$$a(U)_\tau = -\gamma \frac{N}{r} (\dot{v}^\phi - v^\phi_\phi).$$

(2.14)

Going back to the general case, the vectors of the ZAMO frame have transport laws along $U$ given by

$$\nabla_U n = \gamma \frac{N}{r} v^\phi_\phi e_\tau, \quad \nabla_U e_\tau = \gamma \frac{N}{r} (v^\phi_\phi n + \dot{v}^\phi_\phi e_\phi), \quad \nabla_U e_\phi = -\gamma \frac{N}{r} \dot{v}^\phi_\phi e_\tau.$$

(2.15)

In order to deduce the intrinsic properties of the orbits under consideration, we shall set up a Frenet–Serret frame $\{E_i\}$ adapted to the orbit, i.e. with $E_0 = U$ and satisfying the standard relations

$$\nabla_U E_0 = \kappa E_1, \quad \nabla_U E_1 = \kappa E_0 + \tau_1 E_2, \quad \nabla_U E_2 = -\tau_1 E_1 + \tau_2 E_3, \quad \nabla_U E_3 = -\tau_2 E_2.$$

(2.16)

where $\tau_1$ and $\tau_2$ are the first and the second torsions, respectively. Recalling equations (2.11) and (2.12), we have

$$E_1 = \frac{a(U)}{||a(U)||} = \frac{1}{\kappa} \left[ (a(U)_\tau v^\tau + a(U)_\phi \dot{v}^\phi) n + a(U)_{\tilde{\tau}} e_\tau + a(U)_{\tilde{\phi}} e_\phi \right].$$

(2.17)

Following the Frenet–Serret procedure, we find

$$E_2 = \frac{\gamma}{\kappa} \left[ (\dot{v}^\phi a(U)_{\tilde{\phi}} - \dot{v}^\phi a(U)_{\tilde{\phi}}) n + (\dot{v}^\phi v^\tau a(U)_{\tilde{\tau}} - \frac{a(U)_{\tilde{\phi}}}{\gamma_\phi^2}) e_\tau \right. $$

$$\left. - (\dot{v}^\phi \dot{v}^\phi a(U)_{\tilde{\phi}} - \frac{a(U)_{\tilde{\phi}}}{\gamma_\phi^2}) e_\phi \right],$$

(2.18)

$$E_3 = -e_\phi.$$
The magnitude of the 4-acceleration is defined by equation (2.12); the first torsion is given by
\[
\kappa^2 \tau_1 = \frac{N}{r} \left[ v^\phi a(U)_\phi^2 - v^2_k v^\phi a(U)_\phi a(U)_\phi + \frac{1}{\gamma_k^2} v^\phi a(U)_\phi^2 \right] \\
+ \frac{1}{r} \gamma (a(U)_\phi \frac{da(U)_\phi}{d\tau} - a(U)_\phi \frac{da(U)_\phi}{d\tau}) + \kappa^2 (v^\phi a(U)_\phi - v^\phi a(U)_\phi) \\
= \frac{N \gamma^2}{r \gamma_k^2} \kappa^2 v^\phi - \gamma^3 \left\{ \dot{v}^\phi \left[ \ddot{v}^\phi + \frac{N \gamma}{r \gamma_k^2} v^\phi \dot{v}^\phi \right] - \dot{v}^\phi \left[ \ddot{v}^\phi - \frac{N \gamma}{r} \dot{v} \left( \dot{v}^\phi - \frac{v_k^2}{\gamma_k^2} \right) \right] \right\} \\
+ \frac{N}{r} \gamma^4 \left\{ \dot{v}^\phi \left[ \ddot{v}^\phi + 2 \dot{v}^\phi \right] + (1 + v^2_k) \dot{v}^\phi \dot{v}^\phi \right\} + \frac{N^3}{r^3} \gamma^4 \dot{v}^\phi v^\phi v^\phi (1 + 2v^2_k) \\
- \frac{N^2}{r^2} \gamma^5 \left\{ \dot{v}^\phi \dot{v}^\phi \left[ \frac{1}{\gamma_k^2} (\dot{v}^\phi - v^2_k) + \dot{v}^2 (1 - 3v^2_k - v^2_k) \right] \\
+ v^\phi \dot{v}^\phi v^\phi \left[ \frac{3}{\gamma_k^2} + \frac{v^2_k}{\gamma^2_k} \right] \right\}. \\
\tag{2.19}
\]
The second torsion \( \tau_2 \) vanishes identically due to the fact that the orbit lies in the equatorial plane and the Schwarzschild metric is reflection symmetric about that plane.

Note that if \( \dot{v}^\phi = 0, \dot{v}^\phi = \text{const.} \) (spatially circular orbits), we have [9]
\[
\tau_1 = \frac{N}{r} \gamma^2 v^\phi, \quad \kappa = \frac{N}{r} \gamma^2 |v^\phi - v^2_k|. \\
\tag{2.20}
\]
One can also consider a signed magnitude of \( \kappa \), i.e.
\[
\kappa_{(\text{sm})} \equiv - \frac{N}{r} \gamma^2 (\dot{v}^\phi - v^2_k), \\
\tag{2.21}
\]
which, once differentiated with respect to \( v^\phi \), has the following expression in terms of the first torsion:
\[
\frac{d\kappa_{(\text{sm})}}{dv} = -2 \frac{N}{r} \gamma^2 v^\phi = -2 \gamma^2 \tau_1. \\
\tag{2.22}
\]

3. Special cases of acceleration

We shall now examine some special cases.

**Case 1.** Spatially radial acceleration: \( a(U)_\phi \neq 0, a(U)_\phi = 0. \)

In the case \( a(U)_\phi = 0 \), i.e.
\[
a(U) = a(U)_\phi (v^\phi n + e_\phi), \quad \kappa = \frac{|a(U)_\phi|}{\gamma_r}, \\
\tag{3.1}
\]
the Frenet–Serret frame (2.17) and (2.18) reduces to
\[
E_1 = \epsilon_\phi \gamma_r (v^\phi n + e_\phi), \quad \epsilon_\phi = \text{sgn}[a(U)_\phi], \\
E_2 = \epsilon_\phi \gamma_r \left[ v^\phi (n + v^\phi e_\phi) + \frac{1}{\gamma_r^2} e_\phi \right], \\
E_3 = -e_\phi,
\]
and the first torsion (2.19) becomes
\[
\tau_1 = v^\phi \left[ \gamma_r^2 \frac{N}{r} - a(U)_\phi \right]. \\
\tag{3.3}
\]

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This particular situation also includes the case of spatially circular orbits where in addition \( v_r = 0 \) and \( \gamma_\phi = 1 \), so that

\[
a(U) = a(U)_{\tau} e_\tau, \quad \kappa = |a(U)_{\tau}|, \quad \tau_1 = v^\phi \left( \frac{N}{r} - a(U)_{\tau} \right),
\]

and

\[
E_1 = \epsilon e_\tau, \quad E_2 = \epsilon \gamma (v^\phi n + e_\phi), \quad E_3 = -e_\phi.
\]

**Case 2. Spatially tangential acceleration: \( a(U)_{\tau} \neq 0, a(U)_{\phi} = 0 \).**

In the special case \( a(U)_{\phi} = 0 \), i.e.

\[
a(U) = a(U)_{\phi} (v^\phi n + e_\phi), \quad \kappa = \frac{|a(U)_{\phi}|}{\gamma_{\phi}},
\]

the Frenet–Serret relations reduce to

\[
E_1 = \epsilon_{\phi} \gamma_{\phi} (v^\phi n + e_\phi), \quad \epsilon_{\phi} = \text{sgn}[a(U)_{\phi}],
\]

\[
E_2 = -\epsilon_{\phi} \gamma_{\phi} \left[ v^\phi (n + v^\phi e_\phi) + \frac{1}{\gamma^2_{\phi}} e_\phi \right],
\]

\[
E_3 = -e_\phi,
\]

and

\[
\tau_1 = \gamma_{\phi}^2 v^\phi \frac{N}{r \gamma^2_{\phi}} + v^\phi a(U)_{\phi}.
\]

**Case 3. Spatial acceleration: \( a(U)_{\tau} = 0 \).**

In the special case \( a(U)_{\phi} = 0 \), namely \( v^\tau a(U)_{\tau} + v^\phi a(U)_{\phi} = 0 \), we have

\[
E_1 = \frac{1}{\kappa} (a(U)^\tau e_\tau + a(U)^\phi e_\phi) = \frac{a(U)^\phi}{\kappa v^\phi} (-v^\phi e_\tau + v^\phi e_\phi),
\]

\[
E_2 = -\frac{\gamma}{v} \left[ v^\phi n + v^\phi e_\phi + v^\phi e_\phi \right],
\]

\[
E_3 = -e_\phi,
\]

where \( \nu = \sqrt{v^2 + v^2_{\phi}} \) and

\[
\kappa = \sqrt{a(U)^2_{\tau} + a(U)^2_{\phi}} = \frac{v}{|v|} |a(U)_{\phi}|, \quad \tau_1 = \frac{N v^2_{\phi}}{r \nu^2} + \frac{a(U)_{\phi}}{v^\phi}.
\]

Evidently in the special cases here considered, we have not specified the physical source of the acceleration nor the type of orbit. In what follows we shall discuss explicit examples.
4. Explicit examples

4.1. Charged particles in external magnetic fields

The solution of an electromagnetic field added to the Schwarzschild background was found by Bičák and Janis [10] in 1985 and is presented in the appendix. Here we limit our consideration to an electromagnetic field $F$ stemming from a vector potential $A$ given by

$$A = A\langle 0 \rangle = \frac{1}{2} B_0 r^2 \sin^2 \theta \, d\phi, \quad F = dA = B_0 r \sin \theta (\sin \theta \, dr + r \cos \theta \, d\theta) \wedge d\phi,$$

where $B_0$ is an arbitrary constant.

The electric and magnetic fields relative to a given observer $u$ are generally defined as

$$E(u) = F \mathsf{u}, \quad B(u) = {}^* F \mathsf{u},$$

where $\mathsf{u}$ denotes right contraction and $^*$ the spacetime duality operation, specifically

$${}^* F_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\rho\gamma\delta} F^{\rho\gamma\delta}.$$

It should be kept in mind that the components of the electric field always appear with the specification of the observer; therefore they should not be confused with the legs of a tetrad.

With respect to a static observer $u = n$, the electromagnetic field (4.1) is a purely magnetic one, i.e., the electric field vanishes ($E(n) = 0$) and

$$B(n) = B_0 [\pm \cos \theta e_t + N \sin \theta e_\theta].$$

On the equatorial plane $\theta = \pi/2$, $B(n)$ reduces to

$$B(n) = B_0 Ne_\theta.$$

We recall that we consider only motion in the equatorial plane $\theta = \pi/2$.

The force on a charged test particle $U$ (with mass $m$ and charge $q$) due to the given electromagnetic field is

$$f_{\text{em}}(U) = qE(U) = qF_{\alpha\beta} U^\beta = q \eta^\alpha [\nu \times B(n)]_{\beta} = \eta^\alpha n_{\rho\delta\gamma} \nu^\delta B(n)^\gamma,$$

where $\eta(n)_{\rho\delta\gamma} = \eta_{\rho\delta\gamma}$. The components of the electric field $E(U)$ are

$$E(U) = -\nu \nu^\delta B(n)^\gamma, \quad E(U) = 0, \quad E(U) = \nu \nu^\delta B(n)^\gamma,$$

so that

$$f_{\text{em}}(U) = q \eta N B_0 [\pm \nu^\delta e_t + \nu^\delta e_\theta] = m \zeta_0 \nu N [\pm \nu^\delta e_t + \nu^\delta e_\theta],$$

where we have introduced for convenience the ‘cyclotron frequency’

$$\zeta_0 = \frac{qB_0}{m}. $$

The motion of the particle under the effect of this force is governed by

$$ma(U) = f_{\text{em}}(U),$$

and since $f_{\text{em}}(U)_{\gamma} = 0$, it is described by case 3 of the previous section. The equations of motion are then, from equations (2.10),

$$\frac{d\nu^\gamma}{d\tau} = N \left\{ -\zeta_0 \nu^\gamma + \frac{\nu^\delta \nu^\gamma}{\tau} \left[ \nu^\delta \nu^\gamma - \nu^2 (1 - \nu^2) \right] \right\},$$

$$\frac{d\nu^\phi}{d\tau} = -N \nu^\phi \left( -\zeta_0 + \frac{\nu^\phi}{\tau} \nu^\phi \right).$$
together with
\[ \frac{dr}{d\tau} = \gamma N \nu', \quad \frac{d\phi}{d\tau} = \frac{\gamma}{r} \nu'. \] (4.12)

We then proceed to the analysis of these orbits, considering their intrinsic properties, the equilibrium condition and its stability and the actual form of the orbit obtained by a direct numerical integration of equations (4.11) and (4.12).

4.1.1. Intrinsic analysis of the orbits. A Frenet–Serret frame adapted to the particle orbit \( U \) and satisfying equation (2.16) is given, from equation (3.9), by
\[ E_1 = \frac{1}{\nu}(-\nu' e_r + \nu^2 e_\phi), \quad E_2 = \frac{\gamma}{\nu}(\nu^2 n + \nu' e_r + \nu^2 e_\phi), \quad E_3 = -e_\theta, \] (4.13)
modulo a sign choice. The magnitude of the acceleration \( \kappa \) and the first torsion \( \tau_1 \) turn out to be
\[ \kappa = \frac{1}{m} \|f_{(\text{em})}(U)\| = \gamma N \zeta_0 \nu, \quad \tau_1 = -\gamma N \zeta_0 - \frac{M}{N \nu^2} \nu'. \] (4.14)

4.1.2. Equilibrium solution and stability. The charged particle \( U \) moves under the combined effect of the background metric and the external electromagnetic field. When these two effects balance each other, we have an equilibrium solution for the equatorial orbit.

For the system (4.11) and (4.12), the equilibrium solution is associated with a spatially circular orbit such that \( \nu' = 0 \) and
\[ -\zeta_0 \nu + \frac{\gamma}{r}(\nu^2 - \nu^2 \zeta_0^2) = 0, \] (4.15)
with \( \gamma = (1 - \nu^2)^{-1/2} \). Hereafter we shall assume \( B_0 > 0 \) without any loss of generality; for particles moving in the equatorial plane of the Schwarzschild spacetime, this means a magnetic field aligned with the negative \( z \) direction (i.e. aligned with \( \partial_\theta \)).

Equation (4.15) determines the equilibrium radius \( r_0 \) in terms of the orbital speed \( \nu_0 = \pm \nu_0 \) (\( \nu_0 > 0 \)) at that radius (once \( \nu_0 \) is re-expressed in terms of \( r_0 \), namely \( \nu_0 = \sqrt{\nu^2/(r_0^2 - 2M)} \)). A solution for \( r_0 \) is easily found if we introduce the rapidity parameter \( \nu_0 = \tanh \alpha \). In fact, equation (4.15) becomes
\[ (r_0 - 3M) \sinh^2 \alpha = \pm r_0 \zeta_0 (r_0 - 2M) \sinh \alpha + M = 0, \] (4.16)
and it can be solved for \( \sinh \alpha \). In the limiting case of a flat spacetime (\( M = 0 \)), the above relation reduces to
\[ \sinh \alpha (\sinh \alpha = \mp r_0 \zeta_0) = 0, \] (4.17)
and the corresponding solutions in terms of \( \nu_0 \) are given by
\[ \nu_0 = 0, \quad \nu_0 = \frac{r_0 |\zeta_0|}{\sqrt{1 + r_0^2 \zeta_0^2}}. \] (4.18)
In the nonrelativistic limit of \( r_0 \zeta_0 \ll 1 \), equation (4.18) implies the known result
\[ \nu_0 = r_0 |\zeta_0|. \] (4.19)
Note that in the weak-field slow-motion limit (\( M/r_0 \ll 1, \nu_0 \ll 1 \)), the equilibrium condition (4.16) implies
\[ -\zeta_0 \nu_0 = \frac{M}{r_0^2} - \frac{\nu_0^2}{r_0} \] (4.20)
or, more explicitly, the balance between Lorentz, gravitational and centripetal forces

\[
\frac{m v_0^2}{r_0} = \frac{mM}{r_0^2} - qB_0 v_0.
\]

The stability of the equilibrium orbit can be studied by looking for the solution of the perturbative problem

\[
r = r_0 + r_1(\tau), \quad \phi = \phi_0(\tau) + \phi_1(\tau), \quad \nu^j = \nu_1^j(\tau), \quad \nu^\phi = \pm v_0 + \nu_1^\phi(\tau).
\]

From the general equations of motion (4.11) and (4.12), the first-order perturbations satisfy the following equations:

\[
\frac{d\nu_1^i}{d\tau} = y_0 N \nu_1^i,
\]

\[
\frac{d\phi_1}{d\tau} = -\frac{y_0}{r_0} \left[ \pm v_0 \frac{r_1}{r_0} - y_0^2 \nu_1^\phi \right],
\]

\[
\frac{d\nu_1^j}{d\tau} = -\frac{y_0 N}{r_0} \left[ (\nu_0^2 - v_K^2 (1 + v_K^2)) \frac{r_1}{r_0} - \frac{y_0^2}{(\pm v_0)} [v_0^2 + v_K^2 - 2 v_0^2 v_K^2] \nu_1^\phi \right],
\]

\[
\frac{d\nu_1^\phi}{d\tau} = -\frac{N v_K^2}{r_0 \gamma_0 (\pm v_0)} \nu_1^i,
\]

that is, formally, a linear system of the type

\[
\frac{dX^\alpha}{d\tau} = A^\alpha_\beta X^\beta,
\]

where \( X = [r_1, \phi_1, \nu_1^j, \nu_1^\phi] \). The eigenvalues associated with the stability matrix \( A \) are \( \lambda_1 = 0 = \lambda_2, \lambda_3 = -\lambda_4 = i\Lambda \), with

\[
\Lambda = \frac{\gamma_0 N r_0 |v_0|}{y_0 \nu_0} \sqrt{(\nu_0^2 - v_K^2)^2 + v_0^2 v_K^2 (1 - 4 v_K^2)},
\]

for \( r_0 > 6M \), which vanishes at

\[
v_0 = v_0^\pm = v_K \left[ \frac{1}{2} + 2 v_K^2 \pm \frac{1}{2} \sqrt{(4 v_K^2 - 1)(4 v_K^2 + 3)} \right]^{1/2}.
\]

Therefore, the solution of the perturbed system exhibits an oscillating behavior with proper frequency \( \Lambda \) for \( r_0 > 6M \), which assures the existence of a stability regime in that region. In fact, the second term in the square root in equation (4.25) is always positive there. It vanishes at \( r_0 = 6M \) (where \( \tilde{n}_0^+ = \tilde{n}_0^- \)), so that the eigenvalues all vanish for the circular geodesics and \( v_0 = 1/2 = v_K |_{r_0=6M} \). For \( r_0 < 6M \) instead an instability region appears corresponding to negative values of the argument of the square root in equation (4.25), implying that the nonvanishing eigenvalues are both real and of different sign, i.e. \( \lambda_1 = 0 = \lambda_2, \lambda_3 = -\lambda_4 = -|\Lambda| \).

Figure 1 shows typical orbits for a small value of the parameter \( M\zeta_0 \) and initial conditions close to the equilibrium solution. The orbit spirals either inward or outward depending on whether the initial value of the azimuthal velocity is less or greater than the critical equilibrium one. The corresponding behaviors of the signed magnitude of the acceleration and first torsion are shown in figure 2. Equilibrium orbits and their associated stability region are shown in figure 3.

Further examples of orbits with corresponding curvature and torsions are shown in figures 4 and 5. For every choice of \( r(0)/M \) and \( M\zeta_0 \), there exist in general two equilibrium
values $v^\phi(0) = v_0^-$, $v^\phi(0) = v_0^+$, where $r(0)$ and $v^\phi(0)$ are the initial values of the radius and azimuthal speed at $\tau = 0$. This is evident from figure 3, where the equilibrium azimuthal velocity $v^\phi$ is plotted as a function of $r_0/M$ for fixed values of $M\zeta_0$ and the values $v_0^\pm$ can be located once the equilibrium radius is fixed. As a general feature for $v^\phi(0) < v_0^-$ and $v^\phi(0) > v_0^+$, the orbits exhibit an oscillating behavior around a mean value $\tilde{r} > r(0)$; for $v_0^- < v^\phi(0) < v_0^+$ instead the oscillations are around a mean value $\tilde{r} < r(0)$ which may cause the particle to fall down to the horizon. The maximum amplitude in both cases is $|\tilde{r} - r(0)|$.

### 4.2. Spinning particles

The motion of a test-spinning particle in a gravitational field has been extensively investigated for the relevance of rotation among the astrophysical bodies. Limiting one’s interest to the pole–dipole approximation, the corresponding equations of motion are given by the Mathisson–Papapetrou equations [11, 12]. To become a close set, however, the above equations need supplementary conditions, specifically denoted as Papapetrou–Corinaldesi (PC) [13], Pirani (P) [14] and Tulczyjew (T) [15], each of one leading to different solutions. In a series of papers
Figure 2. Charged particle in an external magnetic field. The behaviors of the signed magnitude $\kappa$ of the 4-acceleration and the first torsion $\tau_1$ are shown as functions of the proper time $\tau$ for the same choice of parameters and initial conditions as in figure 1. Part (a) corresponds to the orbit spiraling outward, whereas (b) to that falling down to the hole. The dashed lines correspond to the constant values $\kappa \approx 0.006$ and $\tau_1 \approx 0.096$ of the equilibrium solution.

[16–20], the effects of each supplementary condition on equatorial orbits in various spacetime metrics have been studied and the main physically relevant conclusion was that the P and T conditions are the only relevant ones although no clear reason for selecting one instead of the other can be theoretically motivated. Most important is the result that, in the limit of small spin, the above two conditions are equivalent. Since these conditions are considered here, we assume them without further notice.

The motion of a test-spinning body is driven by the force

$$f_{(\text{spin})}(U)^\mu = -\frac{1}{2} P^\mu_{\nu\rho\sigma} U^{\nu\rho} S^{\sigma\beta},$$

so that the equations of motion write as

$$ma(U)^\mu = f_{(\text{spin})}(U)^\mu.$$  \hspace{1cm} (4.27)

Let us recall that the particle is moving in the equatorial plane ($\theta = \pi/2$, $\hat{v}^\theta = 0$) and assume that its spin vector is constant and orthogonal to the motion plane, i.e.

$$S = -s e^\theta,$$  \hspace{1cm} (4.29)

with $s$ being the signed magnitude of the spin vector. We will also use the spin dimensionless quantity

$$\hat{s} = \frac{s}{mM},$$  \hspace{1cm} (4.30)

which should be small in order to avoid back-reaction effects.

The nonvanishing components of the spin force are given by [16]

$$f_{(\text{spin})}(U)^t = \frac{3M}{r^3} s y^2 \hat{v}^\phi \hat{v}^\phi,$$

$$f_{(\text{spin})}(U)^r = -\frac{3M}{r^3} s y^2 \hat{v}^\phi,$$

\hspace{1cm} (4.31)
Figure 3. Charged particle in an external magnetic field. The equilibrium azimuthal velocity $\nu^\phi$ is plotted as a function of $r_0/M$ for fixed values of $M\zeta = [0 \text{ (black)}, -0.1 \text{ (red)}, -0.5 \text{ (blue)}, -1 \text{ (green)}, -5 \text{ (brown)}]$. The corresponding equilibrium orbits are stable outside the shaded region. For every fixed value of the equilibrium radius $r_0/M$, there exist in general two values of the azimuthal velocity corresponding to co-rotating and counter-rotating orbits.

that is

$$f_{\text{spin}}(U) = -\frac{3M}{r^3} s\gamma^2 \nu^\phi (\nu^\phi n + e_r).$$

as in case 1 of section 3.

The equations of motion (2.10) are then

$$\frac{d\nu^\phi}{dr} = -\frac{3M^2}{r^3} \gamma^2 \nu^\phi + \frac{NY}{r^2} \left[\nu^2 - \nu^2 (1 - \nu^2)\right],$$

$$\frac{d\nu^r}{dr} = \frac{3M^2}{r^3} \gamma \nu^\phi \nu^r - \frac{NY}{r^2} \nu^\phi \nu^r.$$ (4.33)

Because of their awkwardness, these can only be studied numerically except for very special cases. It results that for small values of $\delta$, the resulting orbits spiral either inward or outward depending on whether the initial value of the azimuthal velocity is less or greater than the critical equilibrium one. The behavior is very similar to the one illustrated in figure 1. Equilibrium orbits and their associated stability region are shown in figure 6.

4.2.1. Intrinsic analysis of the orbits. A Frenet–Serret frame satisfying equation (2.16) can be built up with the triad

$$E_1 = \gamma (\nu^\phi n + e_r), \quad E_2 = \frac{\gamma}{\gamma^\phi} \left(\nu^\phi e^\phi + \gamma \nu^\phi U\right), \quad E_3 = -e_\delta,$$

with the associated curvature and first torsion

$$\kappa = \frac{1}{m} ||f_{\text{spin}}(U)|| = -\frac{3M^2}{r^3} \gamma^2 \nu^\phi, \quad \tau_1 = \nu^\phi \left(\frac{\gamma N}{r} - \kappa\right).$$ (4.35)
Figure 4. Charged particle in an external magnetic field. (a) The orbit corresponding to $M_\zeta_0 = -0.01$ for the choice of initial conditions $r(0) = 5M$, $\phi(0) = 0$, $\nu^r(0) = 0$ and $\nu^\phi(0) = 0.6$. The equilibrium values of the azimuthal velocity are $\nu^\phi(0) \approx [-0.598, 0.557]$. (b) The corresponding behaviors of the signed magnitude $\kappa$ and the first torsion $\tau_1$ as functions of the proper time $\tau$.

Figure 5. Charged particle in an external magnetic field. (a) The orbit corresponding to $M_\zeta_0 = -1$ for the following choices of initial conditions: $r(0) = 5M$, $\phi(0) = 0$, $\nu^r(0) = 0$ and $\nu^\phi(0) = [-0.8 \text{ (green)}, -0.5 \text{ (blue)}, 0.066 \text{ (black)}, 0.3 \text{ (red)}]$. For the selected values of $r(0)/M$ and $M_\zeta_0$, the equilibrium values of the azimuthal velocity are $\nu^\phi(0) \approx [-0.991, 0.066]$. (b) The behaviors of the signed magnitude $\kappa$ and the first torsion $\tau_1$ as functions of the proper time $\tau$ for the same choice of parameters and initial conditions as in (a). The critical values corresponding to equilibrium are $\kappa \approx -0.051$ and $\tau_1 \approx -0.007$. 
4.2.2. Equilibrium solution and stability. An equilibrium circular orbit solution exists on the equatorial plane at \( r = r_0 \) only if \( \nu^\gamma = 0 \) and \( \nu^\phi = \pm \nu_0 \), implying

\[
N \left( \nu_0^2 - \nu_K^2 \right) = \pm \frac{3M}{r_0^2} \frac{s}{\nu_0},
\]

which can be easily solved for \( \nu_0 \). In the weak-field and slow-motion limit, the above relation admits the classical limit

\[
m \frac{\nu_0^2}{r_0} = \frac{mM}{r_0^2} \pm \frac{3M}{r_0^2} \nu_0.
\]

In this approximated situation, one may compare equation (4.37) with the equilibrium condition (4.20) for a massive and charged particle subjected to a magnetic field; the equations coincide if

\[
|\xi_0| = \left| \frac{qB_0}{m} \right| \rightarrow \frac{3M^2}{r_0^3} |\hat{s}|.
\]

This shows that in the case of a spinning particle, the role of the dimensionless spin \( \hat{s} \) is played by the electric charge. Moreover, we clearly see that there exists a value of the average radius \( r_0 \) of the orbit for each 'equivalent' triplet \([q, B_0, m]\) specifying the particle for which the two cases coincide.
Finally, one may consider the stability of this orbit looking for the solution of the perturbative problem

\[ r = r_0 + r_1(\tau), \quad \phi = \phi_0(\tau) + \phi_1(\tau), \quad \nu^i = \nu^i_0(\tau), \quad \nu^\phi = \pm v_0 + v^\phi_1(\tau). \] (4.39)

The first-order quantities satisfy the following system of equations:

\[
\begin{align*}
\frac{dr_1}{d\tau} &= \gamma_0 N v_1^r, \\
\frac{d\phi_1}{d\tau} &= -\frac{\gamma_0}{r_0} \left[ \pm v_0 \frac{r_1}{r_0} - \gamma_0 v_1^\phi \right], \\
\frac{dv_1^r}{d\tau} &= \frac{\gamma_0 N}{r_0} \left[ (v_K^r - v_0^2) (1 - v_0^2) + 2v_0^2 \right] \frac{r_1}{r_0} + \frac{1}{r_0} (\pm v_0) (v_0^2 + v_K^2) v_1^\phi, \\
\frac{dv_1^\phi}{d\tau} &= -\frac{N(\pm v_0)}{r_0 \gamma_0} v_1^r.
\end{align*}
\] (4.40)

The associated eigenvalues are

\[ \lambda_1 = 0 = \lambda_2, \lambda_3 = -\lambda_4 = \Lambda_s, \]

with

\[
\Lambda_s = \frac{\gamma_0 N}{r_0} \sqrt{(v_0^2 + v_K^2)^2 + v_0^2 - 2v_K^2}.
\] (4.41)

where

\[ \bar{v}_0^{\pm2} = -v_K^2 - \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 12v_K^2}. \] (4.42)

Therefore, the equilibrium is stable only in the region wherein the argument of the square root is negative, i.e., for \(-\bar{v}_0^2 < v_0 < \bar{v}_0^2\). The solution of the perturbed system exhibits there an oscillating behavior with proper frequency \(\Lambda_s\).

On the basis of the above analysis in the case of equilibrium solutions, one cannot distinguish between the equatorial motion of a charged particle in an external electromagnetic field and a spinning neutral particle in the given background.

### 4.3. Particles with a magnetic dipole moment

When a magnetic field is added to the Schwarzschild background as in equation (4.5), we can also discuss the motion of a magnetic dipole \([21, 22, 6, 7]\) with magnetic moment \(\mu^\alpha = -(\mu/r)\delta_0^\alpha\). The force driving the motion in this case is given by

\[
\begin{align*}
f_{(\text{dip})}(U)^\sigma &= \eta^{\alpha\beta\gamma\delta} U_\alpha \mu_\beta \nabla^\gamma F_{\gamma\delta}, \quad \text{that is}\end{align*}
\] (4.43)

\[
\begin{align*}
f_{(\text{dip})}(U) &= \frac{2\mu M y B_0}{r^2} (v^j n + e_r).
\end{align*}
\] (4.44)

This force is very similar to the one experienced by a spinning particle (see equation (4.32)); hence, we show only numerical integration of the orbits. In this case, the equilibrium solution \(r = r_0, v = v_0\) is given by

\[
\beta \frac{M}{r_0} + N\gamma_0 (v_0^2 - v_K^2) = 0, \] (4.45)
where $\beta = 2\mu B_0 / m$. In this situation, one may compare equation (4.45) with the corresponding equilibrium condition for a spinning particle (4.37); the equations coincide if

$$2\mu B_0 = \frac{3}{r_0} s - \frac{v_0}{\sqrt{1 - v_0^2}}.$$  \hspace{1cm} (4.46)

This shows that, at the equilibrium, one cannot distinguish between a magnetic dipole moving in an external magnetic field and a spinning particle subjected to gravitation only.

### 4.4. Superposed magnetic field and motion of a spinning particle also endowed with a magnetic dipole

In view of the various astrophysical applications, it is also worth considering the combined effects of a magnetic dipole which is also spinning. The force which drives the motion becomes

$$f'(U) = f_{(dip)}(U) + f_{(spin)}(U) = \left(2\mu B_0 - 3\frac{s y' v^\phi}{r} \right) \frac{M y}{r^2} (v^r n + e_r),$$  \hspace{1cm} (4.47)

where $f_{(dip)}(U)$ is given by equation (4.44) and $f_{(spin)}(U)$ by equation (4.32). The equations of motion are given by

$$\frac{d\nu^r}{d\tau} = \frac{a(U)_r}{\gamma y^2} + \frac{y N}{r} \left[ v^\phi 2 - v^2 K \right],$$

$$\frac{d\nu^\phi}{d\tau} = -v^r v^\phi \left( \frac{a(U)_r}{\gamma} + \frac{y N}{y^2 r} \right),$$  \hspace{1cm} (4.48)

with

$$a(U)_r = \left(2\mu B_0 - 3\frac{s y' v^\phi}{r} \right) \frac{M y}{m r^2}.$$  \hspace{1cm} (4.49)

In this case, similar to those discussed above, an equilibrium spatially circular orbit at $r = r_0$ exists, i.e. with $\nu^r = 0$ ($\gamma r = 1$) and $\nu^\phi = \text{const.}$, such that

$$\left(2\mu B_0 - 3\frac{s y' v^\phi}{r_0} \right) \frac{M y}{r_0^2} + \frac{y^2 N}{r_0} \left[ v^\phi 2 - v^2 K \right] = 0,$$  \hspace{1cm} (4.50)

that is

$$\left( \beta - 3\frac{\tilde{s} M y v^\phi}{r_0} \right) \frac{M}{r_0} + y N (v^\phi 2 - v^2 K) = 0.$$  \hspace{1cm} (4.51)

This equation is actually a polynomial relation in $v^\phi$ of the fourth order and can be solved exactly. There is also the possibility for the azimuthal speed to keep the geodesic value, i.e. $v^\phi = v_K$, at $r = r_0$ and then to behave as a neutral non-spinning body, moving along a geodesic. In this case, the effects on the particle’s motion by the magnetic dipole and the intrinsic spin with respect to a background gravitational field coupled with a uniform magnetic field compensate each other so to annul their combined effects. The critical condition for ‘hiding’ the structures of the body is given by

$$\beta = 3\gamma r v^2 K,$$  \hspace{1cm} (4.52)

where, we recall, $\beta = 2\mu B_0 / m$, $\tilde{s} = s/(mM)$ with $m$ and $M$ the masses of the orbiting body and of the source of the background field, $\mu$ is the body’s magnetic dipole moment and $B_0$ the added external magnetic field.
Let us see whether equation (4.52) provides astrophysically plausible conditions. Let us first convert the latter equation into conventional units and denote the corresponding quantities by a tilde,

\[ M = \frac{G\tilde{M}}{c^2}, \quad \nu = \frac{\tilde{\nu}}{c}, \quad \mu = \sqrt{\frac{G}{4\pi \epsilon_0 c^2}} \tilde{\mu}, \quad B_0 = \sqrt{\frac{4\pi \epsilon_0 G}{c^2}} \tilde{B}_0, \quad s = \frac{G}{c^3} \tilde{s}, \]

(4.53)

so that equation (4.52) writes as

\[ \tilde{\mu} \tilde{B}_0 = \frac{3}{2r_0} \tilde{s} \nu K \tilde{\nu} K. \]

(4.54)

Recalling that \( \tilde{s} \sim MR^2/T \), where \( T \) is the period of rotation and that for a neutron star on average, we have \( \tilde{M} \sim M_\odot \sim 2 \times 10^{30} \) kg, \( R \sim 10^4 \) m, \( T \sim 1 \) s, \( \tilde{\nu} K \sim 5 \times 10^6 \) ms\(^{-1} \) and, in the case of Sgr A\( ^* \), \( \tilde{r}_0 \sim 2 \times 10^{13} \) m,

\[ \tilde{\mu} \tilde{B}_0 \sim 1.5 \times 10^{32} \) J;

(4.55)

hence, with a value of \( \tilde{B}_0 \sim 1 \) T (average interstellar magnetic field close to the black hole location), we find \( \tilde{\mu} \sim 10^{32} \) J T\(^{-1} \), a value which is not too far from what is expected for a magnetized neutron star.

This circumstance makes a patent ambiguity plausible and realistic in the observation of stellar fields around massive black holes.

4.5. Poynting–Robertson effect

Special attention has recently been given to the case of a radiation field superposed to a Schwarzschild spacetime. Scattering (absorbing and consequent re-emitting) of such radiation by moving particles causes a drag force which acts on the particles determining deviations from geodesic motion termed Poynting–Robertson effect. Details can be found in [23, 24] so we give here only a brief account.

A (null) radiation field added to a Schwarzschild spacetime is described by the energy–momentum tensor

\[ T = \Phi^2 k \otimes k, \quad k^\alpha k_\alpha = 0, \quad k^\alpha \nabla_\alpha k^\beta = 0, \]

(4.56)

where \( k \) is a null and geodesic vector of the background, while the flux \( \Phi \) is determined by \( \nabla_\beta T^{\alpha\beta} = 0 \). The force acting on a massive particle with 4-velocity \( U \) can be written as

\[ f_{(\text{rad})\alpha} = -\tilde{\sigma} P(U)_{\alpha\beta} T^{\beta\mu} U^\mu, \]

(4.57)

where \( P(U) = g + U \otimes U \) projects orthogonally to \( U \) and \( \tilde{\sigma} \) is a coefficient modeling the absorption and consequent re-emission of radiation by the particle. Let us consider photons moving on the equatorial plane of the Schwarzschild spacetime, i.e. with momentum \( k \) given by

\[ k = E(n)(n + \hat{v}_k), \quad \hat{v}_k \cdot \hat{v}_k = 1, \]

(4.58)

with \( E(n) \) being the energy of the photons relative to ZAMOS and \( \hat{v}_k \) being their (spacelike, unitary) direction of propagation

\[ \hat{v}_k = \sin \beta e_\tau + \cos \beta e_\phi. \]

(4.59)

The photon energy as measured by an observer comoving with the particle is given by

\[ E(U) \equiv -U \cdot k = \gamma E(n)(1 - \hat{v}_k \cdot \hat{v}) = \gamma E(n)(1 - \sin \beta \nu^\tau - \cos \beta \nu^\phi). \]

(4.60)
whereas the relative energy of the photons with respect to ZAMOs is

\[ E(n) = -k \cdot n = \frac{E}{N}. \]  

(4.61)

Here \( E = -k_0 > 0 \) is a constant of the motion representing the conserved energy associated with the timelike Killing vector field; \( L = k_0 \) is another constant of the motion representing the conserved angular momentum associated with the rotational Killing vector field. We use the notation

\[ b \equiv \frac{L}{E}. \]  

(4.62)

for the photon impact parameter [25] so that

\[ \cos \beta = \frac{bN}{r} \Rightarrow N|b \tan \beta| = \sqrt{r^2 - b^2N^2}. \]  

(4.63)

We will restrict ourselves to the case of photons with \( E > 0 \) so that \( E(n) > 0 \) and \( k \) is a future-directed vector.

The case \( \sin \beta > 0 \) corresponds to outgoing photons (increasing \( r \)) and \( \sin \beta < 0 \) to ingoing photons (decreasing \( r \)). The case \( \sin \beta = 0 \) for spatially circular geodesic motion of the photons can only take place at \( r = 3M \), so we exclude it.

Since \( k \) is completely determined, the coordinate dependence of the quantity \( \Phi \) follows from the conservation equations \( \nabla_T T^{\alpha\beta} = 0 \). The result is as follows [24]:

\[ \Phi^2 = \frac{\Phi_0^2}{rN|b \tan \beta|} = \frac{\Phi_0^2}{\sqrt{r^2 - b^2N^2}}, \]  

(4.64)

where \( \Phi_0 \) is a constant.

The motion of a massive particle under the effect of this force is governed by the equations

\[ ma(U) = f_{(rad)}(U), \]

(4.65)

and apart from very special situations, the analysis could only be performed numerically. However, since \( f_{(rad)}(U) \cdot U = 0 \), we have

\[ f_{(rad)}(U) \hat{y} = v_f f_{(rad)}(U) \hat{y} + v_\phi f_{(rad)}(U) \hat{\phi} \]  

(4.66)

with

\[ f_{(rad)}(U) \hat{y} = \sigma \Phi^2 E(U)E(n)[\sin \beta - \gamma^2(1 - \sin \beta v^\hat{y} - \cos \beta v^\hat{\phi})v^\hat{y}], \]

\[ f_{(rad)}(U) \hat{\phi} = \sigma \Phi^2 E(U)E(n)[\cos \beta - \gamma^2(1 - \sin \beta v^\hat{y} - \cos \beta v^\hat{\phi})v^\hat{\phi}]. \]  

(4.67)

If we make explicit \( \Phi \) as in equation (4.64) and \( E(U) \) and \( E(n) \) as in equations (4.60) and (4.61), respectively, the expression of the force depends on the single constant \( m \Lambda \equiv \sigma \Phi_0^2 E^2 \).

4.5.1. Equilibrium solution. The general equations of motion (2.10) admit in this case an equilibrium solution at a fixed radius \( r = r_0 \) with

\[ v^\hat{y} = 0, \quad v^\hat{\phi} = \pm v_0, \quad \gamma = 1/\sqrt{1 - v_0^2}, \]

(4.68)

in fact, recalling that \( v^\phi = \text{const.} \). at the equilibrium, equations (2.10) simplify as

\[ \frac{1}{m} f^\hat{y}_{(rad)} = \frac{1}{m} [\sigma \Phi^2 E(U)E(n) \sin \beta_0] = -\frac{\gamma_0^2 N}{r_0} (v_0^2 - v_0^2), \]

(4.69)

\[ \frac{1}{m} f^\hat{\phi}_{(rad)} = \frac{1}{m} [\sigma \Phi^2 E(U)E(n)\gamma_0^2 (\cos \beta_0 \mp v_0)] = 0. \]
and are satisfied by
\[ \pm \nu_0 = \cos \beta_0 = \frac{bN}{r_0} \quad \rightarrow \quad \gamma_0 = 1/|\sin \beta_0|, \] (4.70)
and
\[ -\frac{\gamma_0^2 N}{r_0} (\nu_0^2 - \nu_k^2) = \frac{\sigma \Phi_0^2 \varepsilon^2}{m} \quad \frac{\gamma_0 \sin^3 \beta_0}{r_0 N^2 \sqrt{r_0^2 - b^2 N^2}}, \] (4.71)
being now
\[ \hat{u}_k \cdot \nu = \cos^2 \beta_0 \quad \rightarrow \quad \mathcal{E}(U) = \gamma_0 \varepsilon(n) \sin^2 \beta_0. \] (4.72)
The equilibrium condition (4.71) can then be rewritten as
\[ N \gamma_0^3 \left( 1 - \frac{\nu_0^2}{\nu_k^2} \right) = \text{sgn}[\sin \beta_0] \frac{A}{M}. \] (4.73)
Clearly, when \( b = 0 \) (i.e. \( \nu_0 = 0, \gamma_0 = 1 \)) and \( \sin \beta_0 > 0 \), i.e. in the case of purely radial outward photon motion, equation (4.73) reduces to
\[ MN = A \quad \rightarrow \quad r_0 = \frac{2M}{1 - A^2/M^2}. \] (4.74)

The stability of these orbits has been studied in detail in [24], to which we also refer for further analysis.

5. Analogies between different kinds of situations

Consider the equilibrium circular orbit associated with different kinds of particles as discussed in section 4.

(i) Particles with charge \( q \) in an external magnetic field:
\[ \gamma (\nu \hat{\phi}^2 - \nu_k^2) = r_0 \zeta_0 \nu \hat{\phi}, \quad \zeta_0 = qB_0/m. \] (5.1)

(ii) Particles with a magnetic dipole in an external magnetic field:
\[ \gamma (\nu \hat{\phi}^2 - \nu_k^2) = -\beta \frac{M}{r_0 \bar{N}}, \quad \beta = 2\mu B_0/m. \] (5.2)

(iii) Particles with spin in the background geometry:
\[ (\nu \hat{\phi}^2 - \nu_k^2) = \frac{3M^2}{r_0^2 \bar{N}} \tilde{s} \hat{\phi}, \quad \tilde{s} = s/(mM). \] (5.3)

(iv) Neutral particles in a given radiation field:
\[ N \gamma^3 \left( 1 - \frac{\nu_0^2}{\nu_k^2} \right) = \text{sgn}[\sin \beta_0] \frac{A}{M}, \quad A = \sigma \Phi_0^2 \varepsilon^2/m. \] (5.4)

In all these cases (as well as in cases which are combinations of these), which originate in different contexts, deviations from circular geodesic motion are given by
\[ \nu \hat{\phi} = \pm \nu_k + \Delta \nu, \] (5.5)
where, with obvious meaning of notation,

\[
\Delta v_0 = \pm \frac{r_0 \delta_0}{2\gamma K}, \quad \Delta v_\mu = \pm \frac{3}{2} \left(\frac{M}{r_0}\right)^{3/2} \nu K \hat{s}, \quad \Delta v_\nu = -\frac{\beta}{2\gamma K} \left(\frac{M}{r_0}\right)^{1/2}, \quad \Delta v_A = \frac{1}{2\gamma K} \left(\frac{M}{r_0}\right)^{1/2} \mathrm{sgn}[\sin \beta_0] \frac{A}{M}
\]

(5.6)

If these spatially circular orbits which mark the equilibrium were the object of a measurement, the uncertainty of the spatial velocity would mirror the uncertainty about the structure of the particle. In the weak field limit, the identification of the corrections about a fixed \( r_0 \) implies the following kinds of ambiguities, which should always be taken into account.

(i) One cannot distinguish between a particle with a magnetic dipole \( \mu \) moving on a mean radius \( r_0 \) and a one with electric charge \(|q| = 2\mu M^{1/2} r_0^{-3/2}\), for any mass \( m \) and a magnetic (test) field \( B_0 \).

(ii) One cannot distinguish between a neutral particle moving on a mean radius \( r_0 \) with spin \( s \) and a particle having a magnetic dipole \( \mu \) in a magnetic field \( B_0 \) with \(|\mu B_0| = (3/2)\nu K \gamma K (s/r_0)\), for any mass \( m \).

(iii) One cannot distinguish between a spinning particle with spin \( s \) on a mean radius \( r_0 \) and a charged particle in a magnetic field \( B_0 \) with \(|q B_0| = 3M^{1/2} r_0^{-5/2} \gamma K \nu K s \). The latter case is complementary to the previous ones. It is then clear that a measurement of the correction to any given geodesic property is not sufficient by itself alone to identify the structure of the particle under consideration. Only combined measurements of different kinds can overcome this ambiguity.

(iv) One cannot distinguish between a spinning particle with spin \( s \) also endowed with a magnetic dipole moment (e.g. a pulsar) and neutral non-spinning and not magnetized geodesic particle. The discussion about this point has been made explicitly in section 4.

6. Concluding remarks

We have studied the geometrical (Frenet–Serret intrinsic) properties of generally non-geodesic orbits of test particles with structure, moving in the equatorial plane of the Schwarzschild spacetime. The analysis of the motion has been performed either numerically by a direct integration of the corresponding equations leading to complicate patterns or by studying their analytical solutions in the special cases of equilibrium circular orbits. In detail, we have studied the conditions which guarantee the existence of stable spatially circular orbits with non-Keplerian velocities maintained by particular particle’s structures embedded in a black hole spacetime with added test fields. We have explicitly considered the cases of charged particles as well as those with a magnetic dipole in an external (test) magnetic field, and of spinning particles as well as the ones undergoing the Poynting–Robertson effect due to scattering of electromagnetic radiation moving along such orbits. Finally, we have also considered the combined effect of an intrinsic spin and of a magnetic dipole moment in an external magnetic field as expected by a pulsar-like objects. If deviations from geodesic behavior are the results of astrophysical measurements within a clearly specified gravitational background, then an ambiguity may arise to explain the origin of such deviations. This complication can only be overcome by combining different kinds of measurements.
Acknowledgments

DB and AG thank Professors R T Jantzen, O Semeják and L Stella for useful discussions on the Poynting–Robertson effect around black holes. All the authors thank ICRANET for support.

Appendix. Superposed electromagnetic fields to the Schwarzschild spacetime

In this appendix, we briefly reproduce the results found by Bičák and Janis [10] in 1985 for the solution of a general magnetic field superposed to the Schwarzschild background. In terms of a vector potential, such a solution is conveniently written as

\[ A = A_{(0)} + A_{(1)}, \]  

(A.1)

where

\[ A_{(0)} = \frac{1}{2} B_0 r^2 \sin^2 \theta \, d\phi, \]

\[ A_{(1)} = -B_1 [\sin \theta \cos \theta (r - M) (\sin \phi \, dr + r \cos \phi \, d\phi) + r \sin \phi [(r - 2M) \cos^2 \theta - M] \, d\theta], \]

(A.2)

with \( B_0 \) and \( B_1 \) being arbitrary constants. The associated electromagnetic field \( F = dA \) can be represented in terms of electric and magnetic fields relative to a given observer \( u \); in fact, denoting such fields as

\[ E(u) = F \wedge u, \quad B(u) = {}^*F \wedge u, \]

(A.3)

where \( \wedge \) denotes right contraction and \( {}^* \) the spacetime duality operation, specifically

\[ {}^*F_{\alpha\beta} = \frac{1}{2} \eta_{\alpha\beta\gamma\delta} F^{\gamma\delta}, \]

(A.4)

we have

\[ F = u \wedge E(u) + {}^*[u \wedge B(u)]. \]

(A.5)

As measured by a static observer \( u = n \) in the Schwarzschild background, this electromagnetic field results a purely magnetic one, i.e. the electric field vanishes \( (E(n) = 0) \) and

\[ B(n) = B_0 [\cos \theta e_\theta + N \sin \theta e_\phi] - B_1 [\sin \theta \cos \phi e_\phi + N \cos \theta \cos \phi e_\theta - N \sin \phi e_\phi]. \]

(A.6)

In this paper, we have limited our considerations to the simpler case \( B_1 = 0 \), which assures the motion to be confined to the equatorial plane.

The force on a charged test particle \( U = \gamma [n + v] \) as in equation (2.4) and due to the above external electromagnetic field is given by \( f_{(em)}(U) = qE(U) \), where

\[ E(U)_\hat{\alpha} = F_{\hat{\alpha}\hat{\beta}} U^{\hat{\beta}} = \gamma [v \times B(n)]_\hat{\alpha} = \gamma \eta(n)_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \hat{\nu}^\hat{\beta} B(n)^\hat{\delta}; \]

(A.7)

here \( \eta(n)_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} = \eta_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \). In components

\[ E(U)_\hat{t} = \gamma (\hat{v}^\hat{\phi} B(n)^\hat{\phi} - \hat{v}^\hat{\phi} B(n)^\hat{\phi}), \]

\[ E(U)_\hat{\theta} = \gamma (\hat{v}^\hat{\phi} B(n)^\hat{\phi} - \hat{v}^\hat{\phi} B(n)^\hat{\phi}), \]

\[ E(U)_\hat{\phi} = \gamma (\hat{v}^\hat{\phi} B(n)^\hat{\phi} - \hat{v}^\hat{\phi} B(n)^\hat{\phi}). \]

(A.8)

The motion of a particle (with mass \( m \) and charge \( q \)) under the effect of this force is governed by the equations

\[ ma(U) = f_{(em)}(U), \]

(A.9)

where the components of \( a(U) \) are given explicitly by equation (2.5).
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