Provably Efficient Exploration for RL with Unsupervised Learning

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Abstract

We study how to use unsupervised learning for efficient exploration in reinforcement learning with rich observations generated from a small number of latent states. We present a novel algorithmic framework that is built upon two components: an unsupervised learning algorithm and a no-regret reinforcement learning algorithm. We show that our algorithm provably finds a near-optimal policy with sample complexity polynomial in the number of latent states, which is significantly smaller than the number of possible observations. Our result gives theoretical justification to the prevailing paradigm of using unsupervised learning for efficient exploration (Tang et al., 2017; Bellemare et al., 2016).

1 Introduction

Reinforcement learning (RL) is the framework of learning to control an unknown system through trial and error. It takes as inputs the observations of the environment and outputs a policy, i.e., a mapping from observations to actions, to maximize the cumulative rewards. To learn a near-optimal policy, it is critical to sufficiently explore the environment and identify all the opportunities for high rewards. However, modern RL applications often need to deal with huge observation spaces such as those consist of images or texts, which makes it challenging or impossible (if there are infinitely many observations) to fully explore the environment in a direct way. In some work, function approximation scheme is adopted such that essential quantities for policy improvement, e.g. state-action values, can be generalized from limited observed data to the whole observation space. However, the use of function approximation alone does not resolve the exploration problem (Du et al., 2019b).

To tackle this problem, multiple empirically successful strategies are developed (Tang et al., 2017; Bellemare et al., 2016; Pathak et al., 2017; Azizzadenesheli et al., 2018; Lipton et al., 2018; Fortunato et al., 2018; Osband et al., 2016). Specifically, in Tang et al. (2017) and Bellemare et al.
(2016), the authors use state abstraction technique to reduce the problem size. They construct a mapping from observations to a small number of hidden states and devise exploration on top of the latent state space rather than the original observation space.

To construct such a state abstraction mapping, practitioners often use *unsupervised learning*. The whole process has the following steps: collect a batch of observation data, apply unsupervised learning to build a mapping, use the mapping to guide exploration and collect more data, and repeat. Empirical study evidences the effectiveness of using unsupervised learning for state abstraction mapping at addressing hard exploration problems (e.g., the infamous Montezuma’s Revenge). However, this approach has not been theoretically justified. In this paper, we aim to answer this question:

*Is exploration driven by unsupervised learning in general *provably efficient*?*

The generality includes the choice of unsupervised learning algorithms, reinforcement learning algorithms, and the condition of the problem structure.

We review some existing theoretical results on provably efficient exploration. More discussion is due to Section 2. For an RL problem with finitely many states (but unknown transition and rewards), there are many algorithms with a tabular implementation that learn to control the system efficiently. For instance, Kearns and Singh (2002); Brafman and Tennenholtz (2002); Strehl et al. (2006); Jaksch et al. (2010); Dann and Brunskill (2015); Agrawal and Jia (2017); Jin et al. (2018); Kakade et al. (2018); Zanette and Brunskill (2019) learn a near-optimal policy using a number of samples polynomially depending on the size of the state space. However, if we directly apply these algorithms to rich observations cases by treating each observation as a state, then the sample complexities are polynomial in the cardinality of the observation space. Such a dependency is unavoidable without additional structural assumptions (Jaksch et al., 2010). If structural conditions are considered, for example, observations are generated from a small number of latent states (Krishnamurthy et al., 2016; Jiang et al., 2017; Dann et al., 2018; Du et al., 2019a), then the sample complexity only scales polynomially with the number of hidden states. Unfortunately, the correctness of these algorithms often requires strict assumptions (e.g., deterministic transitions, reachability) that may not be satisfied in many real applications.

In this paper, we develop a novel framework for RL problems with rich observations that are generated from a small number of latent states. Our framework consists of an unsupervised learning algorithm and a tabular RL algorithm as sub-routines and outputs a policy. Specifically, we use unsupervised learning to learn decoding functions that can map observations to true latent states with high accuracy and then we run the tabular RL algorithm on the latent state space to guide exploration. Theoretically, we prove that as long as the input unsupervised learning and tabular RL algorithms each has a polynomial sample complexity guarantee, our framework returns a near-optimal policy with a sample complexity polynomial in the number of latent states. No other conditions on the dynamics is required. We summarize our contributions below.

- We propose a new algorithmic framework for the Block Markov Decision Process (BMDP) model (Du et al., 2019a). We combine an unsupervised learning oracle and a tabular RL algorithm in an organic way to find a near-optimal policy for a BMDP. Our framework works for almost *any* unsupervised learning algorithms and tabular RL algorithms with theoretical guarantees.

- We define a general notion, unsupervised learning oracle, which is an abstraction of methods used in Tang et al. (2017); Bellemare et al. (2016) and standard statistical generative models. We provide concrete examples of this oracle and analyze its statistical properties.
Theoretically, we prove that as long as the unsupervised learning oracle and the tabular RL algorithm each has a polynomial sample complexity, our algorithm finds a near-optimal policy with sample complexity polynomial in the number of latent states, which is significantly smaller than the number of possible observations (cf. Theorem 1). To our knowledge, this is the first provably efficient method for RL problems with huge observation spaces that uses unsupervised learning for exploration. Our result theoretically justifies the empirical paradigm used in Tang et al. (2017); Bellemare et al. (2016). Furthermore, our result does not require additional assumptions on transition dynamics as used in Du et al. (2019a).

Our Technique  Here we briefly explain our framework. We assume there is an unsupervised learning oracle (see formal definition in Section 4) which can be applied to learn decoding functions and the accuracy of learning increases as more training data are fed. Let $\mathcal{M}$ be the MDP with rich observations. We form an auxiliary MDP $\mathcal{M}'$ whose state space is the latent state space of $\mathcal{M}$. Our idea is to simulate the process of running a no-regret tabular RL algorithm $\mathcal{A}$ directly on $\mathcal{M}'$. For each episode, $\mathcal{A}$ proposes a policy $\pi$ for $\mathcal{M}'$ and expects a trajectory of running $\pi$ on $\mathcal{M}'$ for updating and then proceeds. To obtain such a trajectory, we design a policy $\phi$ for $\mathcal{M}$ as a composite of $\pi$ and some initial decoding functions. We run $\phi$ on $\mathcal{M}$ to collect observation trajectories. Although the decoding functions may be inaccurate initially, they can still help us collect observation samples for later refinement. After collecting sufficient observations, we apply the unsupervised learning oracle to retrain decoding functions and then update $\phi$ as a composite of $\pi$ and the newly-learned functions and repeat running $\phi$ on $\mathcal{M}$. After a number of iterations (proportional to the size of the latent state space), with the accumulation of training data, decoding functions are trained to be fairly accurate on recovering latent states, especially those $\pi$ has large probabilities to visit. This implies that running the latest $\phi$ on $\mathcal{M}$ is almost equivalent to running $\pi$ on $\mathcal{M}'$. Therefore, we can obtain a state-action trajectory with high accuracy as the algorithm $\mathcal{A}$ requires. Since $\mathcal{A}$ is guaranteed to output a near-optimal policy after a polynomial (in the size of the true state-space) number of episodes, our algorithm uses polynomial number of samples as well.

Organization  This paper is organized as follows. In Section 2 we review related works on provably efficient algorithms. In Section 3, we introduce the necessary background and the formal setting. In Section 4, we formalize the unsupervised learning oracle, describe our framework, and present theoretical guarantee. In Section 5, we give some examples of unsupervised learning oracles. We provide numerical experiments in Section 6 to verify the effectiveness of our framework and we conclude in Section 7.

2 Related Work

In this section, we review related provably efficient RL algorithms. We remark that we focus on environments that require explicit exploration. With certain assumptions of the environment, e.g., the existence of a good exploration policy or the distribution over the initial state is sufficiently diverse, one does not need to explicitly explore (Munos, 2005; Antos et al., 2008; Geist et al., 2019; Kakade and Langford, 2002; Bagnell et al., 2004; Scherrer and Geist, 2014; Agarwal et al., 2019; Yang et al., 2019b; Chen and Jiang, 2019). Without these assumptions, the problem can require an exponential number of samples, especially for policy-based methods (Du et al., 2019b).
Exploration is needed even in the most basic tabular setting. There is a substantial body of work on provably efficient tabular RL (Agrawal and Jia, 2017; Jaksch et al., 2010; Kakade et al., 2018; Azar et al., 2017; Kearns and Singh, 2002; Dann et al., 2017; Strehl et al., 2006; Jin et al., 2018; Simchowitz and Jamieson, 2019; Zanette and Brunskill, 2019). A common strategy is to use UCB bonus to encourage exploration in less-visited states and actions. One can also study RL in metric spaces (Pazis and Parr, 2013; Song and Sun, 2019; Yang et al., 2019a). However, in general, this type of algorithms has an exponential dependence on the state dimension.

To deal with huge observation spaces, one might use function approximation. Wen and Van Roy (2013) proposed an algorithm, optimistic constraint propagation (OCP), which enjoys polynomial sample complexity bounds for a family of $Q$-function classes, including the linear function class as a special case. But their algorithm can only handle deterministic systems, i.e., both transition dynamics and rewards are deterministic. The setting is recently generalized by Du et al. (2019c) to environments with low variance and by Du et al. (2020) to the agnostic setting. However, it is still unclear whether this type of algorithms can deal with general stochastic environments. Li et al. (2011) proposed a Q-learning algorithm which requires the Know-What-It-Knows oracle. But it is in general unknown how to implement such an oracle.

Our work is closely related to a sequence of works which assumes the transition has certain low-rank structure (Krishnamurthy et al., 2016; Jiang et al., 2017; Dann et al., 2018; Sun et al., 2019; Du et al., 2019a; Jin et al., 2019; Yang and Wang, 2019). The most related paper is Du et al. (2019a) which also builds a state abstraction map. Their sample complexity depends on two quantities of the transition probability of the hidden states: identifiability and reachability, which may not be satisfied in many scenarios. Identifiability assumption requires that the $L_1$ distance between the posterior distributions (of previous level’s hidden state, action pair) given any two different hidden states is strictly larger than some constant\footnote{Assumption 3.2 in Du et al. (2019a).}. This is an inherent necessary assumption for the method in Du et al. (2019a) as they need to use the posterior distribution to distinguish hidden states. Reachability assumption requires that there exists a constant such that for every hidden state, there exists a policy that reaches the hidden state with probability larger than this constant\footnote{Definition 2.1 in Du et al. (2019a).}. Conceptually, this assumption is not needed for finding a near-optimal policy because if one hidden state has negligible reaching probability, one can just ignore it. Nevertheless, in Du et al. (2019a), the reachability assumption is also tied with building the abstraction map. Therefore, it may not be removable if one uses the strategy in Du et al. (2019a). In this paper, we show that given an unsupervised learning oracle, one does not need the identifiability and reachability assumptions for efficient exploration.

3 Preliminaries

Notations Given a set $\mathcal{A}$, we denote by $|\mathcal{A}|$ the cardinality of $\mathcal{A}$, $\mathcal{P}(\mathcal{A})$ the set of all probability distributions over $\mathcal{A}$, and $\text{Unif}(\mathcal{A})$ the uniform distribution over $\mathcal{A}$. We use $[h]$ for the set $\{1, 2, \ldots, h\}$ and $f_{[h]}$ for the set of functions $\{f_1, f_2, \ldots, f_h\}$. Given two functions $f : \mathcal{X} \rightarrow \mathcal{Y}$ and $g : \mathcal{Y} \rightarrow \mathcal{Z}$, their composite is denoted as $g \circ f : \mathcal{X} \rightarrow \mathcal{Z}$. Given an event $\mathcal{E}$, we denote by $\Pr(\mathcal{E})$ the probability that $\mathcal{E}$ occurs. Given a random variable $X$, we denote by $X \sim q(\cdot)$ if $X$ follows a distribution with density $q(\cdot)$. We use $\mathcal{O}$ and $\Theta$ to represent the leading order in upper and minimax bound, respectively. We use $\text{poly}(\cdot)$ for the polynomial dependency.
Block Markov Decision Process

We consider a Block Markov Decision Process (BMDP), which is first formally introduced in Du et al. (2019a). A BMDP is described by a tuple \( \mathcal{M} := (\mathcal{S}, \mathcal{A}, \mathcal{X}, \mathcal{P}, r, f_{[H+1]}, H) \). \( \mathcal{S} \) is a finite unobservable latent state space, \( \mathcal{A} \) is a finite action space, and \( \mathcal{X} \) is a possibly infinite observable context space. \( \mathcal{X} \) can be partitioned into \( |\mathcal{S}| \) disjoint blocks \( \{\mathcal{X}_s\}_{s \in \mathcal{S}} \), where each block \( \mathcal{X}_s \) corresponds to a unique state \( s \). \( \mathcal{P} \) is the collection of the state-transition probability \( p_{[H]}(s'|s, a) \) and the context-emission distribution \( q(x|s) \) for all \( s, s' \in \mathcal{S}, a \in \mathcal{A}, x \in \mathcal{X} \). \( r : [H] \times \mathcal{S} \times \mathcal{A} \rightarrow [0, 1] \) is the reward function. \( f_{[H+1]} \) is the set of decoding functions, where \( f_h \) maps every observation at level \( h \) to its true latent state. Finally, \( H \) is the length of horizon. When \( \mathcal{X} = \mathcal{S} \), we obtain the usual Episodic Markov Decision Process (Puterman, 2014).

For each episode, the agent starts at level 1 with the initial state \( s_1 \) and takes \( H \) steps to the final level \( H + 1 \). We denote by \( S_h \) and \( X_h \) the set of possible states and observations at level \( h \in [H + 1] \), respectively. At each level \( h \in [H + 1] \), the agent has no access to the true latent state \( s_h \) but an observation \( x_h \sim q(\cdot|s_h) \). An action \( a_h \) is then selected following some policy \( \phi : [H] \times \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A}) \). As a result, the environment evolves into a new state \( s_{h+1} \sim p_h(\cdot|s_h, a_h) \) and the agent receives an instant reward \( r(h, s_h, a_h) \). A trajectory has such a form: \( \{s_1, x_1, a_1, \ldots, s_H, x_H, a_H, s_{H+1}, x_{H+1}\} \), where all state components are unknown.

Policy

Given a BMDP \( \mathcal{M} := (\mathcal{S}, \mathcal{A}, \mathcal{X}, \mathcal{P}, r, f_{[H+1]}, H) \), there is a corresponding MDP \( \mathcal{M}' := (\mathcal{S}, \mathcal{A}, \mathcal{P}, r, H) \), which we refer to as the underlying MDP in the later context. A policy on \( \mathcal{M} \) has a form \( \phi : [H] \times \mathcal{X} \rightarrow \mathcal{P}(\mathcal{A}) \) and a policy on \( \mathcal{M}' \) has a form \( \pi : [H] \times \mathcal{S} \rightarrow \mathcal{P}(\mathcal{A}) \). Given a policy \( \pi \) on \( \mathcal{M}' \) and a set of functions \( \hat{f}_{[H+1]} \) where \( \hat{f}_h : \mathcal{X}_h \rightarrow \mathcal{S}_h, \forall h \in [H + 1] \), we can induce a policy on \( \mathcal{M} \) as \( \pi \circ \hat{f}_{[H+1]} := \phi \) such that

\[ \phi(h, x_h) = \pi(h, \hat{f}_h(x_h)), \forall x_h \in \mathcal{X}_h, h \in [H]. \]

If \( \hat{f}_{[H+1]} = f_{[H+1]} \), then \( \pi \) and \( \phi \) are equivalent in the sense that they induce the same probability measure over the state-action trajectory space.

Given an MDP, the value of a policy \( \pi \) (starting from \( s_1 \)) is defined as

\[ V_1^\pi = \mathbb{E}^\pi \left[ \sum_{h=1}^H r(h, s_h, a_h) | s_1 \right] = \sum_{h=1}^H \sum_{s \in S_h} \sum_{a \in \mathcal{A}} P_h^\pi(s|s_1) \pi(a|h, s) r(h, s, a), \]

where the expectation is taken over the whole trajectory following \( \pi \), \( P_h^\pi(s|s_1) \) denotes the probability of reaching state \( s \) at level \( h \) starting from \( s_1 \) following \( \pi \), and \( \pi(a|h, s) \) is the probability of selecting action \( a \) at state \( s \) and level \( h \). A policy that has the maximal value is an optimal policy and the optimal value is denoted by \( V_1^* \), i.e., \( V_1^* = \max_\pi V_1^\pi \). Given \( \varepsilon > 0 \), we say \( \pi \) is \( \varepsilon \)-optimal if \( V_1^* - V_1^\pi \leq \varepsilon \).

Similarly, given a BMDP, we define the value of a policy \( \phi \) (starting from \( s_1 \)) as:

\[ V_1^\phi = \mathbb{E}^\phi \left[ \sum_{h=1}^H r(h, s_h, a_h) | s_1 \right] = \sum_{h=1}^H \sum_{s \in S_h} \sum_{x \in \mathcal{X}_s} \sum_{a \in \mathcal{A}} P_h^\phi(s|s_1) q(x|s) \phi(a|h, x) r(h, s, a), \]

where the expectation is taken following \( \phi \), \( P_h^\phi(s|s_1) \) is the reaching probability to \( s \) at level \( h \) following \( \phi \), and \( \phi(a|h, x) \) denotes the probability of selecting action \( a \) at level \( h \) with observation \( x \). The notions of optimallity and \( \varepsilon \)-optimality are similar to MDP. In particular, if \( \pi^* \) is optimal for the underlying MDP, then \( \pi^* \circ f_{[H+1]} \) is optimal for the BMDP, since the reward function \( r \) depends on states rather than observations.
4 A Unified Framework for Unsupervised Reinforcement Learning

4.1 Unsupervised Learning Oracle and No-regret Tabular MDP Algorithm

In this paper, we consider RL on a BMDP. The goal is to find a near-optimal policy with sample complexity polynomial to the cardinality of the latent state space. We assume no knowledge of $\mathcal{P}$, $r$, and $f[H+1]$, but the access to an unsupervised learning oracle $\mathcal{ULO}$ and an $(\varepsilon, \delta)$-correct episodic no-regret algorithm. We give the definitions below.

**Definition 1 (Unsupervised Learning Oracle $\mathcal{ULO}$).** There exists a function $g(n, \delta)$ such that for any fixed $\delta > 0$, $\lim_{n \to \infty} g(n, \delta) = 0$. Given a distribution $\mu$ over $S$, and $n$ samples from $\sum s \in S q(\cdot|s) \mu(s)$ such that with probability at least $1 - \delta$ over $n$ training data, we can find a function $\hat{f}: \mathcal{X} \to S$ such that

$$\mathbb{P}_{s \sim \mu, x \sim q(\cdot|s)}(\hat{f}(x) = \alpha(s)) \geq 1 - g(n, \delta)$$

for some unknown permutation $\alpha: S \to S$.

In Definition 1, we assume $f$ is the true decoding function i.e., $\mathbb{P}_{s \sim \mu, x \sim q(\cdot|s)}(\hat{f}(x) = s) = 1$ and we call the permutation $\alpha$ as a good permutation between $f$ and $\hat{f}$. For the function $g(n, \delta)$, we introduce a corresponding inverse function:

$$g^{-1}(\epsilon, \delta) := \min\{N \mid \text{for all } n > N, g(n, \delta) < \epsilon\}.$$ 

Since $\lim_{n \to \infty} g(n, \delta) = 0$, $g^{-1}(\epsilon, \delta)$ is well-defined. We assume that $g^{-1}(\epsilon, \delta)$ is a polynomial in terms of $1/\epsilon, \log(\delta^{-1})$ and possibly problem-dependent parameters. Later in Section 5 we will give specific examples of $\mathcal{ULO}$.

**Definition 2 (($\varepsilon, \delta$)-correct Episodic No-regret Algorithm).** Let $\varepsilon > 0$ and $\delta > 0$. $\mathcal{A}$ is an $(\varepsilon, \delta)$-correct episodic no-regret algorithm if for any MDP $\mathcal{M}' := (S, A, \mathcal{P}, r, H)$ with the initial state $s_1$, $\mathcal{A}$

- runs for at most $C(\varepsilon, \delta) = \text{poly}(|S|, |A|, H, 1/\varepsilon, \log(\delta^{-1}))$ episodes (which we call as the sample complexity of $\mathcal{A}$);
- proposes a policy $\pi^k$ at the beginning of episode $k$ and collects a sample trajectory of $\mathcal{M}'$ following $\pi^k$;
- outputs a policy $\pi$ at the end such that with probability at least $1 - \delta$, $\pi$ is $\varepsilon$-optimal.

This just represents an algorithm that has polynomial sample guarantee for tabular episodic MDPs. The instances of algorithms satisfy Definition 2 are vivid in literature, e.g., Kearns and Singh (2002); Strehl et al. (2006); Azar et al. (2017); Agrawal and Jia (2017); Jaksch et al. (2010); Kakade et al. (2018); Jin et al. (2018).

4.2 A Unified Framework

With a $\mathcal{ULO}$ and an $(\varepsilon, \delta)$-correct episodic no-regret algorithm $\mathcal{A}$, we propose a unified framework in Algorithm 1 to solve an unsupervised RL problem on a BMDP. The main challenge of this problem is that the true states are unobservable. Hence $\mathcal{A}$ cannot be run on the underlying MDP. If one resorts to directly viewing the BMDP as a giant MDP with state space $\mathcal{X}$, then the dimension
Algorithm 1: A Unified Framework for Unsupervised RL

1: Input: BMDP $\mathcal{M}$; $\mathcal{ULO}$; $(\varepsilon, \delta)$-correct episodic no-regret algorithm $\mathcal{A}$; batch size $B > 0$; $\varepsilon \in (0, 1)$; $\delta \in (0, 1)$; $N := \lceil \log(2/\delta) / 2 \rceil$; $L := \lceil 9H^2/(2\varepsilon^2) \log(2N/\delta) \rceil$.
2: for $n = 1$ to $N$ do
3:  Clear the memory of $\mathcal{A}$ and restart;
4:  for $k = 1$ to $K$ do
5:   Obtain $\pi_k$ from $\mathcal{A}$;
6:   Obtain a trajectory: $\tau^k, f_{k[H+1]}^k \leftarrow \text{TSR}(\mathcal{ULO}, \pi_k, B)$;
7:   Update the algorithm: $\mathcal{A} \leftarrow \tau^k$;
8:  end for
9:  Obtain $\pi_{K+1}$ from $\mathcal{A}$;
10: Finalize the decoding functions: $\tau^{K+1}, f_{K[H+1]}^{K+1} \leftarrow \text{TSR}(\mathcal{ULO}, \pi_{K+1}, B)$;
11: Construct a policy for $\mathcal{M}$: $\phi^n \leftarrow \pi_{K+1} \circ f_{K[H+1]}^{K+1}$.
12: end for
13: Run each $\phi^n (n \in [N])$ for $L$ episodes and get the average rewards per episode $\bar{V}_1^{\phi^n}$.
14: Output a policy $\phi = \arg\max_{\phi \in \Phi[N]} \bar{V}_1^\phi$.

of the problem is significantly large. To circumvent this issue, our solution is to unsupervised learn decoding functions $\hat{f}_{[H+1]}$ with observation samples. Specifically, for each episode, we use the policy $\pi$ proposed by $\mathcal{A}$ for the underlying MDP together with certain decoding functions $\hat{f}_{[H+1]}$ to generate a policy $\pi \circ \hat{f}_{[H+1]}$ for the BMDP and then collect observation samples by running $\pi \circ \hat{f}_{[H+1]}$. As more samples are collected, we refine the decoding functions using the $\mathcal{ULO}$. As long as we collect sufficient samples, we can simulate a trajectory as if using the true decoding functions (up to some permutations) and therefore, as if running the policy $\pi$ directly on the underlying MDP as $\mathcal{A}$ required. We continue this procedure until the algorithm $\mathcal{A}$ halts. Note that this procedure is essentially what practitioners use Tang et al. (2017); Bellemare et al. (2016), as we have discussed in Section 1.

We now describe in more detail our algorithm. Suppose the algorithm $\mathcal{A}$ runs for $K$ episodes. At the beginning of each iteration $k \in [K]$, $\mathcal{A}$ proposes a policy $\pi_k : [H] \times S \to A$ for the underlying MDP. We use the Trajectory Sampling Routine $\text{TSR}$ to generate a trajectory $\tau^k$ given $\pi_k$ and then feed $\tau^k$ to $\mathcal{A}$. After $K$ iterations, we obtain a policy $\pi_{K+1}$ from $\mathcal{A}$ and a set of decoding functions $f_{[H+1]}^{K+1}$ from $\text{TSR}$. We claim that with high probability, $\pi_{K+1} \circ f_{[H+1]}^{K+1}$ is a near-optimal policy for the BMDP.

The detailed description of $\text{TSR}$ is displayed in Algorithm 2. We here briefly explain the idea. Note that, at episode $k$ of $\mathcal{A}$, our goal is to simulate a trajectory of $\pi_k$ running on the underlying MDP. As discussed in Section 3, if the true decoding functions $f_{[H+1]}$ are known, then the induced policy $\pi_k \circ f_{[H+1]}$ for BMDP is equivalent to $\pi_k$ and we can simply run $\pi_k \circ f_{[H+1]}$ on the BMDP to generate such a trajectory. Now, without $f_{[H+1]}$, we need to learn the correct labels. $\text{TSR}$ achieves the aforementioned goal in an iterative fashion: it starts with $\pi_k$ and a set of existing decoding functions $f_{[H+1]}^{k,0} := f_{[H+1]}^{k-1}$; for each iteration $i$, it collects a batch of observations by playing $\pi_k \circ f_{[H+1]}^{k,i-1}$; then updates $f_{[H+1]}^{k,i-1}$ to $\hat{f}_{[H+1]}^{k,i}$ by running $\mathcal{ULO}$ on all existing observations. Note that $\mathcal{ULO}$ may output labels inconsistent with previously trained decoding functions. We further match labels of $\hat{f}_{[H+1]}^{k,i}$
Algorithm 2 Trajectory Sampling Routine TSR ($\mathcal{ULO}, \pi, B$)

1: **Input:** $\mathcal{ULO}$; a policy $\pi : [H] \times S \rightarrow A$; batch size $B > 0$; $\epsilon \in (0,1); \delta_1 \in (0,1); J := (H + 1)|S| + 1$.
2: **Data:**
   - all training data $\mathcal{D}$ from previous runs;
   - label standard data $\mathcal{Z} := \{Z_1, Z_2, \ldots, Z_{H+1}\}$, $\mathcal{Z}_h := \{\mathcal{D}_{h,s_1}, \mathcal{D}_{h,s_2}, \ldots\}$;
   - present decoding functions $f_i^{H+1}$.
3: for $i = 1$ to $J$ do
   4: Generate $B$ trajectories of training data $\mathcal{D}'$ and $B$ trajectories of testing data $\mathcal{D}''$ with $\pi \circ f_i^{H+1}$;
   5: Combine data: $\mathcal{D} \leftarrow \mathcal{D} \cup \mathcal{D}'$;
   6: Train with $\mathcal{ULO}$: $\hat{f}_i^{H+1} \leftarrow \mathcal{ULO}(\mathcal{D})$;
   7: Match labels: $\hat{f}_i^{H+1} \leftarrow \text{FixLabel}(\hat{f}_i^{H+1}, \mathcal{Z})$;
   8: for $h \in [H + 1]$ do
      9: Let $\mathcal{D}_{h,s}' := \{x \in \mathcal{D}_h' : \hat{f}_i(x) = s, s \in S_h\}$;
     10: Update label standard set: if $\mathcal{D}_{h,s} \notin \mathcal{Z}_h$ and $|\mathcal{D}_{h,s}'| \geq 3\epsilon \cdot B \log(\delta_1^{-1})$, then let $\mathcal{Z}_h \leftarrow \mathcal{Z}_h \cup \{\mathcal{D}_{h,s}'\}$
   11: end for
12: end for
13: Run $\pi \circ \hat{f}_i^{H+1}$ to obtain a trajectory $\tau$;
14: Renew $\hat{f}_i^{H+1} \leftarrow \hat{f}_i^{H+1}$;
15: **Output:** $\tau, \hat{f}_i^{H+1}$.

with the former ones by calling the FixLabel routine (Algorithm 3) at Line 7 of Algorithm 2. Note that we store a set of observation examples for each state $s$ in the set $\mathcal{Z}$ and we use $\mathcal{Z}$ for label matching in Algorithm 3. After the matching step, we get $\hat{f}_i^{H+1}$. Continuously running for $J$ iterations, we stop and use $\pi \circ \hat{f}_i^{H+1}$ to obtain a trajectory.

We now present our main theoretical result.

**Theorem 1.** Suppose in Definition 1, $g^{-1}(\epsilon, \delta_1) = \text{poly}(|S|, 1/\epsilon, \log(\delta_1^{-1}))$ for any $\epsilon, \delta_1 \in (0,1)$ and $\mathcal{A}$ is $(\epsilon, \delta_2)$-correct with sample complexity $\text{poly}(\log((|S|, |A|, H, 1/\epsilon, \log(\delta_2^{-1})))$ for any $\epsilon, \delta_2 \in (0,1)$. Then Algorithm 1 outputs a policy $\phi$ such that with probability at least $1 - \delta$, $\phi$ is an $\epsilon$-optimal policy for the BMDP, using at most $\text{poly}(\log((|S|, |A|, H, 1/\epsilon, \log(\delta^{-1})))$ trajectories.

Theorem 1 formally justifies what we claimed in Section 1 that as long as the sample complexity of $\mathcal{ULO}$ is polynomial and $\mathcal{A}$ is a no-regret tabular RL algorithm, polynomial number of trajectories suffices to find a near-optimal policy. To our knowledge, this is the first result that proves unsupervised learning can guide exploration in RL problems with a huge observation space and therefore, we theoretically justify the empirical paradigm used in Tang et al. (2017); Bellemare et al. (2016). In the following, we give a formal proof of our main result.

### 4.3 Proofs

We first give a sketch of the proof. Note that if TSR always correctly simulates a trajectory of $\pi_k$ on the underlying MDP, then by the correctness of $\mathcal{A}$, the output policy of $\mathcal{A}$ in the end is near-optimal.
Algorithm 3 \text{FixLabel}([f_{[H+1]}], Z)

1: \textbf{Input}: a set of decoding functions \(\tilde{f}_{[H+1]}\); a set of label standard data \(Z := \{Z_1, Z_2, \ldots, Z_{H+1}\}\), \(Z_h := \{D_{h,s_1}, D_{h,s_2}, \ldots\}\).
2: \textbf{for } \(h \in [H + 1]\) \textbf{do}
3: \hspace{1em} \textbf{for } \(D_{h,s} \in Z_h\) \textbf{ do}
4: \hspace{2em} \textbf{if } \(s \in S_h\) \textbf{ and } |\{x \in D_{h,s} : \tilde{f}_h(x) = s'\}| > 3/5|D_{h,s}| \textbf{ then}
5: \hspace{3em} Swap the output of \(s'\) with \(s\) in \(\tilde{f}_h\);
6: \hspace{2em} \textbf{end if}
7: \hspace{1em} \textbf{end for}
8: \textbf{end for}
9: \textbf{Output}: \(\tilde{f}_{[H+1]}\)

with high probability. If in TSR, \(f_{[H+1]}^{k,j}\) decodes states correctly (up to a fixed permutation, with high probability) for every observation generated by playing \(\pi^k \circ f_{[H+1]}^{k,j}\), then the obtained trajectory (on \(S\)) is as if obtained with \(\pi^k \circ f_{[H+1]}\) which is essentially equal to playing \(\pi^k\) on the underlying MDP. Let us now consider \(\pi^k \circ f_{[H+1]}^{k,i}\) for some intermediate iteration \(i \in [J]\). If there are many observations from a previously unseen state, \(s\), then \(U\overline{LCO}\) guarantees that all the decoding functions in future iterations will be correct with high probability of identifying observations of \(s\). Since there are at most \(|S|\) states to reach for each level following \(\pi^k\), after \((H + 1)|S|\) iterations, TSR is guaranteed to output a set of decoding functions that are with high probability correct under policy \(\pi^k\). With this set of decoding functions, we can simulate a trajectory for \(S\) as if we know the true latent states.

In the full description of TSR (Algorithm 2), we denote the decoding functions of episode \(k\) and iteration \(i\) as \(f_{[H+1]}^{k,i}\) (Line 7) and \(\tilde{f}_{[H+1]}^{k,i}\) as the decoding functions before fixing labels (Line 6). We denote the training and testing dataset \(\mathcal{D}'\) and \(\mathcal{D}''\) generated by \(\pi^k \circ f_{[H+1]}^{k,i-1}\) as \(\{\mathcal{D}_{k,i,h}^{H+1}\}_{h=1}^{H+1}\) and \(\{\mathcal{D}_{k,i,h}''\}_{h=1}^{H+1}\), respectively (Line 4). To formally prove the correctness of our framework, we first present the following lemma, showing that whenever some policy \(\pi\) with some decoding functions visits a state \(s\) with relatively high probability, all the decoding functions of later iterations will correctly decode the observations from \(s\) with high probability.

Lemma 1. Suppose for some \(s^* \in S_h\), \((k,i)\) is the earliest pair such that \(|\{x \in \mathcal{D}_{k,i,h}'' : f_{h}^{k,i}(x) = \alpha_{h}(s^*)\}| \geq 3 \cdot B \log(\delta_1^{-1})\) and \(\{x \in \mathcal{D}_{k,i,h}'' : f_{h}^{k,i}(x) = \alpha_{h}(s^*)\}\) is added into \(Z_h\) as \(Z_h, \alpha_{h}(s^*)\) at line 10 Algorithm 2, where \(\alpha_{h}\) is a good permutation between \(f_{h}^{k,i}\) and \(f_{h}\). Then for each \((k', i') > (k,i)\) (in lexical order), with probability at least \(1 - O(\delta_1)\),

\[
\Pr_{x \sim q(\cdot|s^*)} [f_{h}^{k', i'}(x) \neq \alpha_{h}(s^*)] \leq \epsilon
\]

provided \(0 < \epsilon \log(\delta_1^{-1}) \leq 0.1\) and \(B \geq B_0\). Here \(B_0\) is some constant to be determined later and \(\alpha_{h}^*\) is some fixed permutation on \(S_h\).

Proof of Lemma 1. At Line 4 of Algorithm 2, we have collected \(B\) training samples \(\mathcal{D}_{k,i,h}^{'}\) from the distribution \(\mu_{k,i,h}(\cdot)\) which is the reaching distribution to level \(h\) by playing \(\pi^k \circ f_{[H+1]}^{k,i-1}\). We have
also collected a set of same-sized testing samples $\mathcal{D}_{k,i,h}^\prime$. For future iterations $(k', i') \geq (k, i)$, the function $\tilde{g}_{k', i'}$ is obtained by applying $\mathcal{ULC}$ on the dataset generated by

$$\mu' := \text{Unif}\{(\mu_{k'', i'', h})_{(k'', i'') < (k', i')}\}$$

and the dataset has size $((k' - 1) \cdot J + i' - 1) \cdot B = \Theta(k'J \cdot B)$. Thus, with probability at least $1 - \delta_1$, for some permutation $\alpha'_h$,

$$\Pr_{s \sim \mu', x \sim q(\cdot)}[\tilde{f}_{k', i'}(x) \neq \alpha'_h \circ f_h(x)] \leq g(\Theta(k'J \cdot B), \delta_1). \quad (1)$$

By taking

$$B_0 := \Theta\left(\frac{g^{-1}(\epsilon^2/(K \cdot J), \delta_1)}{K \cdot J}\right) \quad (2)$$

we can have $g(\Theta(k'J \cdot B), \delta_1) \leq \epsilon^2/(K \cdot J)$ for all $k' \in [K]$. Later, in Proposition 1, we will show that $B_0 = \text{poly}(|S|, |A|, H, 1/\epsilon)$. Now we consider $f_{k,i}^{k,i}$. Since the FixLabel routine (Algorithm 3) does not change the accuracy ratio, from Equation (1), it holds with probability at least $1 - \delta_1$

$$\Pr_{s \sim \mu_{k,i,h}, x \sim q(\cdot)}[f_{k,i}^{k,i}(x) \neq \alpha_h \circ f_h(x)] \leq k \cdot J \cdot g(\Theta(kJ \cdot B), \delta_1) \leq \epsilon.$$

Therefore, by Chernoff bound, with probability at least $1 - O(\delta_1)$,

$$|\{x \in \mathcal{D}_{k,i,h}^\prime : f_h(x) \neq s \text{ and } f_{k,i}^{k,i}(x) = \alpha_h(s)\}| < \epsilon \cdot B \log(\delta_1^{-1}).$$

Since $|\{x \in \mathcal{D}_{k,i,h}^\prime : f_{k,i}^{k,i}(x) = \alpha_h(s^*)\}| \geq 3 \epsilon \cdot B \log(\delta_1^{-1})$, we have that

$$|\{x \in \mathcal{D}_{k,i,h}^\prime : f_h(x) = s^* \text{ and } f_{k,i}^{k,i}(x) = \alpha_h(s^*)\}| \geq \frac{2}{3} \cdot |\{x \in \mathcal{D}_{k,i,h}^\prime : f_{k,i}^{k,i}(x) = \alpha_h(s^*)\}| \quad (3)$$

$$\geq 2 \epsilon \cdot B \log(\delta_1^{-1}).$$

Thus, with probability at least $1 - O(\delta_1)$, $\mu_{k,i,h}(s^*) \geq \epsilon \cdot \log(\delta_1^{-1})$. Also note that $f_{k,i}^{k,i}$ is the first function that has confirmed on $s^*$ (i.e., no $\mathcal{D}_{k,i,h}(s^*)$ exists in $Z_h$ of line 8 at iteration $(k, i)$). By Line 9 and Line 10, for later iterations, in $Z_h$, $\mathcal{D}_{k,i,h}(s^*) = \{x \in \mathcal{D}_{k,i,h}^\prime : f_{k,i}^{k,i}(x) = \alpha_h(s^*)\}$.

Next, for another $(k', i') > (k, i)$, we let the corresponding permutation be $\alpha'_h$ for $\tilde{f}_{k', i'}$. Since $\mu'(s') \geq \mu_{k,i,h}(s')/(k' \cdot J)$, with probability at least $1 - \delta_1$,

$$\Pr_{s \sim \mu_{k,i,h}, x \sim q(\cdot)}[\tilde{f}_{k', i'}(x) \neq \alpha'_h \circ f_h(x)] \leq k' \cdot J \cdot g(\Theta(k'J \cdot B), \delta_1).$$

Notice that

$$\Pr_{s \sim \mu_{k,i,h}, x \sim q(\cdot)}[\tilde{f}_{k', i'}(x) \neq \alpha'_h \circ f_h(x)] = \sum_{s' \in S_h} \mu_{k,i,h}(s') \Pr_{x \sim q(\cdot)}[\tilde{f}_{k', i'}(x) \neq \alpha'_h \circ f_h(x)]$$

$$\geq \mu_{k,i,h}(s^*) \Pr_{x \sim q(\cdot)}[\tilde{f}_{k', i'}(x) \neq \alpha'_h \circ f_h(x)]$$

$$\geq \epsilon \cdot \log(\delta_1^{-1}) \Pr_{x \sim q(\cdot)}[\tilde{f}_{k', i'}(x) \neq \alpha'_h \circ f_h(x)].$$
Thus, with probability at least $1 - \delta_1$, 
\[
\Pr_{x \sim q(\cdot|s^*)}[\tilde{f}^{k',i'}_h(x) \neq \alpha^*_h \circ f_h(x)] \leq \frac{k' \cdot J \cdot g(\Theta(k'|S|B), \delta_1)}{\epsilon \cdot \log(\delta_1^{-1})} \leq \epsilon
\]
by the choice of $B_0$ in Equation (2). Let $s' := \alpha^*_h(s^*)$. Conditioning on $\mathcal{U} \mathcal{L} \mathcal{O}$ being correct on $\tilde{f}^{k',i'}_{[H+1]}$ and $f^{k,i}_{[H+1]}$, by Chernoff bound and Equation (3), with probability at least $1 - O(\delta_1)$, we have 
\[
\left|\{x \in \mathcal{D}_{h,\alpha_h(s^*)} : \tilde{f}^{k',i'}_h(x) = s'\}\right| \geq \left|\{x \in \mathcal{D}_{h,\alpha_h(s^*)} : f_h(x) = s^*, \tilde{f}^{k',i'}_h(x) = s'\}\right|
\geq (1 - \epsilon \cdot \log(\delta_1^{-1})) \cdot \frac{2}{3} \cdot \left|\mathcal{D}_{h,\alpha_h(s^*)}\right| > \frac{3}{5}\left|\mathcal{D}_{h,\alpha_h(s^*)}\right|,
\]
where we use the fact that $\mathcal{D}'_{k,i,h}$ are independent from the training dataset. By our label fixing procedure, we find a permutation that swaps $s'$ with $s$ for $\tilde{f}^{k',i'}_h$ to obtain $f^{k,i}_h$. By the above analysis, with probability at least $1 - O(\delta_1)$, 
\[
\Pr_{x \sim q(\cdot|s^*)}[\tilde{f}^{k',i'}_h(x) \neq \alpha_h(s^*)] \leq \epsilon
\]
as desired. Consequently, we let $\alpha^*_h(s^*) = \alpha_h(s^*)$, which satisfies the requirement of the lemma. \hfill \blacksquare

Next, by the definition of our procedure of updating the label standard dataset (Line 10, Algorithm 2), we have the following corollary.

**Corollary 1.** Consider Algorithm 2. Let $\mathcal{Z}_{k,i,h}$ be the label standard dataset at episode $k$ before iteration $i$ for $\mathcal{S}_h$. Then, with probability at least $1 - O(H|S|\delta_1)$, 
\[
\text{for all } k, i \text{ and } \mathcal{D}_{h,s} \in \mathcal{Z}_{k,i,h}, |\{x \in \mathcal{D}_{h,s} : \alpha^*_h \circ f_h(x) = s, s \in \mathcal{S}_h\}| > 2/3|\mathcal{D}_{h,s}|.
\]

At episode $k$ and iteration $i$ of the algorithm TSR, let $\mathcal{E}_{k,i}$ be the event that for all $h \in [H+1], \mathcal{D}_{h,s} \in \mathcal{Z}_{k,i,h}$, 
\[
\Pr_{x \sim q(\cdot|s)}[\tilde{f}^{k,i}_h(x) \neq \alpha^*_h \circ f_h(x)] \leq \epsilon.
\]
We have the following corollary as a consequence of Lemma 1 by taking the union bound over all states.

**Corollary 2.** $\forall k, i : \quad \Pr[\mathcal{E}_{k,i}] \geq 1 - O(H|S|\delta_1)$.

The next lemma shows that after $(H + 1)|S| + 1$ iterations of the TSR subroutine, the algorithm outputs a trajectory for the algorithm $\mathcal{A}$ as if it knows the true mapping $f_{[H+1]}$.

**Lemma 2.** Suppose in an episode $k$, we are running algorithm TSR. Then after $J = (H + 1)|S| + 1$ iterations, we have, for every $j \geq J$, with probability at least $1 - O(H|S|\delta_1)$, 
\[
\text{for all } h \in [H + 1], \quad \Pr_{s \sim \mu_{k,j+1,h}, x \sim q(\cdot|s)}[f^{k,j}_h(x) \neq \alpha^*_h \circ f_h(x)] \leq \epsilon'
\]
for some $\epsilon' > 8H \cdot \epsilon \cdot |S|$, provided $B \geq B_0$ as defined in Lemma 1.

**Proof of Lemma 2.** For $i < J$, there are two cases:

1. there exists an $h \in [H + 1]$ such that $\Pr_{s \sim \mu_{k,i+1,h}, x \sim q(\cdot|s)}[f^{k,i}_h(x) \neq \alpha_h \circ f_h(x)] > \epsilon' / (2H)$;
2. for all $h \in [H + 1], \Pr_{s \sim \mu_{k,i+1,h}, x \sim q(\cdot|s)}[f^{k,i}_h(x) \neq \alpha_h \circ f_h(x)] \leq \epsilon' / (2H)$,
where $\alpha_h$ is some good permutations between $f_{h}^{k,i}$ and $f_h$. If case 1 happens, then there exists a state $s^* \in S_h$ such that

$$\Pr_{x \sim q(|s^*|)} [f_{h}^{k,i}(x) \neq \alpha_h \circ f_h(x)] \cdot \mu_{k,i+1,h}(s^*) > \frac{\epsilon'}{2H|S|}. \quad (4)$$

If $D_{h,\alpha_h(s^*)} \in Z_{k,i,h}$, by Lemma 1, with probability at least $1 - O(\delta_1)$,

$$\Pr_{x \sim q(|s^*|)} [f_{h}^{k,i}(x) \neq \alpha_h^* \circ f_h(x)] \leq \epsilon$$

and $\alpha_h^*(s^*) = \alpha_h(s^*)$. Thus, $\mu_{k,i+1,h}(s^*) > \frac{\epsilon'}{2H|S|}/\epsilon > 1$, a contradiction with $\mu_{k,i+1,h}(s^*) \leq 1$. Therefore, there is no $D_{h,\alpha_h(s^*)} \in Z_{k,i,h}$. Then, due to $\Pr_{x \sim q(|s^*|)} [f_{h}^{k,i}(x) \neq \alpha_h \circ f_h(x)] \leq 1$, by Equation (4), we have

$$\mu_{k,i+1,h}(s^*) > \frac{\epsilon'}{2H|S|}.$$ 

Since $f_{h}^{k+1,i}$ is trained on $\text{Unif}(|\{\mu_{k',i',h}\}_{(k',i') \leq (k,i+1)}|)$, by Definition of $\mathcal{U}LO$, with probability at least $1 - \delta_1$,

$$\Pr_{s \sim \mu_{k,i+1,h}, x \sim q(|s^*|)} [f_{h}^{k+1,i}(x) \neq \alpha_h'(s^*)] \leq K \cdot J \cdot g(\Theta(k|S|B), \delta_1) \leq \epsilon^2 < \epsilon,$$

by our choice of $B$ in Equation (2) and $\alpha_h'$ is some good permutation between $f_{h}^{k+1,i}$ and $f_h$. Thus, by Chernoff bound, with probability at least $1 - O(\delta_1)$,

$$|\{x \in D_{k,i+1,h}^o : f_{h}^{k+1,i}(x) = \alpha_h'(s^*)\}| \geq 3\epsilon \cdot B \log(\delta_1^{-1}).$$

Therefore, if case 1 happens, one state $s$ will be confirmed after iteration $i + 1$ and $\alpha_h^*(s^*) = \alpha_h'(s^*)$ is defined.

To analyze case 2, we first define sets $\{G_{k,i+1,h}\}_{i = 1}^{H+1}$, where $G_{k,i+1,h} := \{s \in S_h \mid D_{h,s} \in Z_{k,i+1,h}\}$. If case 2 happens, we further divide the situation into two subcases:

a) for all $h \in [H+1]$ and all $s \in G_{k,i+1,h}^c$, $\mu_{k,i+1,h}(s) \leq \epsilon'/(8H|S|);$

b) there exists an $h \in [H+1]$ and a state $s^* \in G_{k,i+1,h}^c$ such that $\mu_{k,i+1,h}(s^*) \leq \epsilon'/(8H|S|)$.

First notice that for every $h \in [H+1]$ and $j > i$, since $f_{h}^{k,j}$ is trained on $\text{Unif}(|\{\mu_{k',i',h}\}_{(k',i') \leq (k,j)}|)$, by Definition of $\mathcal{U}LO$ and our choice of $B$ in Equation (2), with probability at least $1 - \delta_1$, we have

$$\Pr_{s \sim \mu_{k,i+1,h}, x \sim q(|s|)} [f_{h}^{k,j}(x) \neq \alpha_h'(s)] \leq \epsilon^2, \quad (5)$$

$$\Rightarrow \sum_{s \in G_{k,i+1,h}} \mu_{k,i+1,h}(s) \Pr_{x \sim q(|s|)} [f_{h}^{k,j}(x) \neq \alpha_h'(s)] + \sum_{s \notin G_{k,i+1,h}} \mu_{k,i+1,h}(s) \Pr_{x \sim q(|s|)} [f_{h}^{k,j}(x) \neq \alpha_h'(s)] \leq \epsilon^2,$$

where $\alpha_h'$ is some good permutation between $f_{h}^{k,j}$ and $f_h$.

If subcase a) happens, note that for $s \in G_{k,i+1,h}$, due to the FixLabel routine (Algorithm 3), $\alpha_h'(s) = \alpha_h^*(s)$, for $f_{h}^{k,j}$ ($j > i$) we have

$$\sum_{s \in S_h} \mu_{k,i+1,h}(s) \Pr_{x \sim q(|s|)} [f_{h}^{k,j}(x) \neq \alpha_h^*(s)]$$
Then for each iteration of the outer loop of Algorithm 1, the policy
BMDP with probability at least 
Therefore, if case 2 and subcase a) happens, the desired result is obtained.
we have
\[
\begin{align*}
\Pr_{s \sim \mu_{k,j+1,h},x \sim q(\cdot |s)} [f_h^{k,j}(x) = \alpha_h^*(s)]
\geq & \Pr_{s \sim \mu_{k,j+1,h},x \sim q(\cdot |s)} [f_h^{k,j}(x) = \alpha_h^*(s) = f_h^{k,i}(x)] \\
\geq & \Pr_{s \sim \mu_{k,j+1,h},x \sim q(\cdot |s)} [\text{for all } h', f_{h'}^{k,j}(x) = \alpha_{h'}^*(s) = f_{h'}^{k,i}(x)] \\
= & \Pr_{s \sim \mu_{k,j+1,h},x \sim q(\cdot |s)} [\text{for all } h', f_{h'}^{k,j}(x) = \alpha_{h'}^*(s) = f_{h'}^{k,i}(x)] \\
\geq & 1 - (\epsilon'/(2H) + \epsilon'/(4H)) \cdot H \geq 1 - \epsilon'.
\end{align*}
\]
Taking a union bound over all \( f_{[H+1]}^{k,j} \), we have that for any \( h \in [H+1] \), with probability at least
\[ 1 - \mathcal{O}(H\delta_1), \]
\[
\Pr_{s \sim \mu_{k,j+1,h},x \sim q(\cdot |s)} [f_h^{k,j}(x) = \alpha_h^*(s)] \geq \Pr_{s \sim \mu_{k,j+1,h},x \sim q(\cdot |s)} [f_h^{k,j}(x) = \alpha_h^*(s) = f_h^{k,i}(x)] \\
\geq \Pr_{s \sim \mu_{k,j+1,h},x \sim q(\cdot |s)} [\text{for all } h', f_{h'}^{k,j}(x) = \alpha_{h'}^*(s) = f_{h'}^{k,i}(x)] \\
= \Pr_{s \sim \mu_{k,j+1,h},x \sim q(\cdot |s)} [\text{for all } h', f_{h'}^{k,j}(x) = \alpha_{h'}^*(s) = f_{h'}^{k,i}(x)] \\
\geq 1 - (\epsilon'/(2H) + \epsilon'/(4H)) \cdot H \geq 1 - \epsilon'.
\]
Therefore, if case 2 and subcase a) happens, the desired result is obtained.

If subcase b) happens, we consider the function \( f_{k,i+1}^h \). By Equation (5),
\[
\begin{align*}
\mu_{k,i+1,h}(s^*) \Pr_{x \sim q(\cdot |s^*)} [f_h^{k,i+1}(x) \neq \alpha_h^*(s^*)] & \leq \epsilon^2 \\
\Rightarrow \Pr_{x \sim q(\cdot |s^*)} [f_h^{k,i+1}(x) \neq \alpha_h^*(s^*)] & \leq \epsilon^2/(\epsilon'/(8H|S|)) \leq \epsilon,
\end{align*}
\]
where \( \alpha_h^* \) here is some good permutation between \( f_{k,i+1}^h \) and \( f_h \). Thus, by Chernoff bound, with probability at least
\[ 1 - \mathcal{O}(\delta_1), \]
\[
|\{ x \in D_{k,i+1,h}^\mu : f_h^{k,i+1}(x) = \alpha_h^*(s^*) \}| \geq 3 \epsilon \cdot B \log(\delta_1^{-1}).
\]
Therefore, the state \( s^* \) will be confirmed and \( \alpha_h^*(s^*) = \alpha_h^*(s^*) \) is defined.

In conclusion, for each iteration, there are three scenarios, either the desired result in Lemma 2 holds already or a new state will be confirmed or a new state will be confirmed for the next iteration. Since there are in total \( \sum_{h=1}^{H+1} |S_h| \leq (H+1)|S| \) states, after \( J := (H+1)|S|+1 \) iterations, by Lemma 1, with probability at least
\[ 1 - \mathcal{O}(H|S|\delta_1), \]
for every \( j \geq J \), for all \( h \in [H+1] \) and all \( s \in S_h \), we have
\[
\Pr_{x \sim q(\cdot |s)} [f_h^{k,j}(x) \neq \alpha_h^*(s)] \leq \epsilon. \]
Therefore, it holds that for
\[
\Pr_{\mu_{k,j+1,h},x \sim q(\cdot |s)} [f_h^{k,j}(x) \neq \alpha_h^*(s)] \leq \epsilon \leq \epsilon'.
\]
\[ \blacksquare \]

**Proposition 1.** Suppose in Definition 1, \( g^{-1}(\epsilon, \delta_1) = \poly(1/\epsilon, \log(\delta_1^{-1})) \) for any \( \epsilon, \delta_1 \in (0,1) \) and \( \mathcal{A} \) is \( (\epsilon, \delta_2) \)-correct with sample complexity \( \poly(|S|, |A|, H, 1/\epsilon, \log(\delta_2^{-1})) \) for any \( \epsilon, \delta_2 \in (0,1) \). Then for each iteration of the outer loop of Algorithm 1, the policy \( \phi^* \) is an \( \epsilon/3 \)-optimal policy for the BMDP with probability at least 0.99, using at most \( \poly(|S|, |A|, H, 1/\epsilon) \) trajectories.
Proof of Proposition 1. We first show that the trajectory obtained by running \( \pi^k \) with the learned decoding functions \( f^{k,J}_{[H+1]} \) matches, with high probability, that from running \( \pi^k \) with \( \alpha^*_{[H+1]} \circ f_{[H+1]} \). For each episode \( k \in [K] \), where \( K = C(\varepsilon/4, \delta_2) \) is the total number of episodes played by \( \mathcal{A} \), let the trajectory of observations be \( \{x^k_h\}_{h=1}^{H+1} \). We define event
\[
\mathcal{E}_k := \{ \forall h \in [H + 1], f^{k,J}_h(x^k_h) = \alpha^*_h(f_h(x^k_h)) \},
\]
where \( J = (H + 1)|S| + 1 \). Note that on \( \mathcal{E}_k \), the trajectory of running \( \pi^k \circ \alpha^*_{[H+1]} \circ f_{[H+1]} \) equals running \( \pi^k \circ f^{k,J}_{[H+1]} \). We also let the event \( \mathcal{F} \) be that \( \mathcal{UO} \) succeeds on every iteration (satisfies Lemma 2). Thus,
\[
\Pr[\mathcal{F}] \geq 1 - K \cdot J \cdot \delta_1 = 1 - \text{poly}(|S|, |A|, H, 1/\varepsilon, \log(\delta^{-1})) \cdot \delta_1.
\]
Furthermore, each \( x^k_h \) is obtained by the distribution \( \sum_s \mu_{k,J+1,h}(s)q(\cdot|s) \). On \( \mathcal{F} \), by Lemma 2, we have
\[
\Pr[f^{k,J}_h(x^k_h) = \alpha^*_h(f_h(x^k_h))] \leq \varepsilon^0
\]
by the choice of \( B \). Therefore,
\[
\Pr[\mathcal{E}_k | \mathcal{F}] \geq 1 - (H + 1)\varepsilon^0.
\]
Overall, we have
\[
\Pr[\mathcal{E}_k, \forall k \in [K] | \mathcal{F}] \geq 1 - K(H + 1)\varepsilon^0.
\]
Thus, with probability at least \( 1 - \delta_2 - \text{poly}(|S|, |A|, H, 1/\varepsilon, \log(\delta^{-1})) \cdot (\varepsilon^0 + \delta_1) \), \( \mathcal{A} \) outputs a policy \( \pi \), that is \( \varepsilon/4 \)-optimal for the underlying MDP with state sets \( \{S^h_{k=1} \} \) permuted by \( \alpha^*_{[H+1]} \), which we denote as event \( \mathcal{E}^n \). Conditioning on \( \mathcal{E}^n \), since on a high probability event \( \mathcal{E}^n \) with \( \Pr[\mathcal{E}^n] \geq 1 - (H + 1)\varepsilon^0 \), \( \pi \circ f^{K,J}_{[H+1]} \) and \( \pi \circ \alpha^*_{[H+1]} \circ f_{[H+1]} \) have the same trajectory, the value achieved by \( \pi \circ f^{K,J}_{[H+1]} \) and \( \pi \circ \alpha^*_{[H+1]} \circ f_{[H+1]} \) differ by at most \( (H + 1)^2 \varepsilon^0 \). Thus, with probability at least \( 1 - \delta_2 - \text{poly}(|S|, |A|, H, 1/\varepsilon, \log(\delta^{-1})) \cdot (\varepsilon^0 + \delta_1) \), the output policy \( \pi \circ f^{K,J}_{[H+1]} \) is at least \( \varepsilon/4 + \mathcal{O}(H^2\varepsilon^0) \) accurate, i.e.,
\[
V^*_1 - V^*_1 \pi \circ f^{K,J}_{[H+1]} \leq V^*_1 - V^*_1 \pi \circ f^{K,J}_{[H+1]} \leq \varepsilon/4 + \mathcal{O}(H^2 \varepsilon^0).
\]
Setting \( \varepsilon^0 \), \( \delta_1 \), and \( \delta_2 \) properly, \( V^*_1 - V^*_1 \pi \circ f^{K,J}_{[H+1]} \leq \varepsilon/3 \) with probability at least 0.99. Since \( 1/\delta_1 = \text{poly}(|S|, |A|, H, 1/\varepsilon) \) and \( 1/\varepsilon = \text{poly}(|S|, |A|, H, 1/\varepsilon, \log(\delta^{-1})) \), \( B_0 \) in Lemma 1 and Lemma 2 is \( \text{poly}(|S|, |A|, H, 1/\varepsilon) \). The desired result is obtained.

Finally, based on Proposition 1, we establish Theorem 1.

Proof of Theorem 1. By Proposition 1 and taking \( N = \lceil \log(2/\delta)/2 \rceil \), with probability at least \( 1 - \delta/2 \), there exists a policy in \( \{\phi^n\}_{n=1}^N \) that is \( \varepsilon/3 \)-optimal for the BMDP. For each policy \( \phi^n \), we take \( L := \lceil 9H^2/(2\varepsilon^2) \log(2N/\delta) \rceil \) episodes to evaluate its value. Then by Hoeffding’s inequality, with probability at least \( 1 - \delta/2 \),
\[
|\bar{V}^n_1 - V^n_1| \leq \varepsilon/3.
\]
By taking the union bound and selecting the policy \( \phi \in \arg\max_{\phi \in [N]} \bar{V}^n_1 \), with probability at least \( 1 - \delta \), it is \( \varepsilon \)-optimal for the BMDP. In total, the number of needed trajectories is \( N \cdot K \cdot J \cdot (2B) + N \cdot L = \text{poly}(|S|, |A|, H, 1/\varepsilon, \log(\delta^{-1})) \). We complete the proof.
5 Examples of Unsupervised Learning Oracle

In this section, we give some examples of UŁO. Recall that in Definition 1, observations are generated by first sampling a state $s$ following certain finite distribution $\mu(\cdot)$ and then sampling an observation $x$ following a distribution $q(\cdot|s)$ determined by the selected state. Such a generation mechanism is termed as the mixture model in statistics (McLachlan and Basford, 1988; McLachlan and Peel, 2004), which has a wide range of applications from image segmentation to genomic analysis (see e.g., Bouguila and Fan (2020)). Mixture models have been generally adopted in the study of clustering and unsupervised learning and may quantitative results are available. Therefore, we have vivid options for UŁO. In the sequel, we list examples of mixture models and some theoretically guaranteed algorithms as candidates of UŁO.

Gaussian Mixture Models The first class we consider is the Gaussian Mixture Model (GMM). GMM is the most intensively studied subarea of mixture models. Suppose our states and observations are points in $\mathbb{R}^d$. In the setting of GMM, $q(\cdot|s) = \mathcal{N}(s, \sigma_s^2)$, i.e., observations are hidden states plus some mean zero Gaussian noise. This is a natural assumption considering that noised data is ubiquitous. When the noises are (truncated) Gaussian, under certain conditions e.g. states are well-separated, we are still able to identify the latent states with high accuracy. A series work (Sanjeev and Kannan, 2001; Vempala and Wang, 2004; Achlioptas and McSherry, 2005; Dasgupta and Schulman, 2000; Regev and Vijayaraghavan, 2017) proposed algorithms that can return arbitrarily close estimations of the true means with a polynomial number of samples.

Bernoulli Mixture Models The second class we consider is the Bernoulli Mixture Model (BMM), which is also a widely adopted genre, especially in binary image processing (J. Vidal and Juan (2004)) and texts classification (Juan and Vidal (2002)). In BMM, every observation $x$ is a point in $\{0,1\}^d$. A true state $s$ determines a frequency vector $p^s \in [0,1]^d$ such that $x_i \sim \text{Bernoulli}(p^s_i)$, $i \in [d]$ and therefore, $q(x|s) = \prod_{i=1}^d (p^s_i)^{x_i}(1-p^s_i)^{1-x_i}$. In Najafi et al. (2020), the authors proposed a reliable clustering algorithm for BMM data with polynomial sample complexity guarantee (see Theorem 1 in Najafi et al. (2020) for details).

Subspace Clustering In some cases, each state is a set of vectors $\{v^s_1, v^s_2, \ldots, v^s_k\}$, where each vector is in $\mathbb{R}^d$ and $k$ is normally much smaller than $d$. The corresponding observations are points in the subspace spanned by $s$ and $q(\cdot|s)$ is a probability measure over the subspace. Suppose for different states, the basis vector sets differ under certain metric, then recovering the latent state is equivalent to subspace clustering. Subspace clustering is an effective technique to deal with high-dimensional data (Parsons et al. (2004)). A variety of applications are related to subspace clustering such as face clustering in computer vision, community clustering in social networks, and DNA sequence analysis (Wallace et al. (2015); Vidal (2011); Elhamifar and Vidal (2013)). Proper algorithms to serve as UŁO can be found in e.g., Wang et al. (2013); Soltanolkotabi et al. (2014).

Miscellaneous In addition to the aforementioned models, other reasonable settings are Categorical Mixture Models Bontemps and Toussile (2013), Poisson Mixture Models Li and Zha (2006), Dirichlet Mixture Models Dahl (2006), etc.

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3To make the model satisfy the disjoint block assumption in Section 3, we need some truncation of the Gaussian noise so that each observation only corresponds to a unique hidden state.

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6 Numerical Experiments

Environments We conduct numerical experiments in two environments: LockBernoulli and LockGaussian. These environments are studied in Du et al. (2019a). Both environments share the same latent state structure with $H$ levels, 3 states per level and 4 actions. At level $h$, from states $s_{1,h}$ and $s_{2,h}$ one action leads with probability $1 - \alpha$ to $s_{1,h+1}$ and with probability $\alpha$ to $s_{2,h+1}$, another has the flipped behavior, and the remaining two lead to $s_{3,h+1}$. All actions from $s_{3,h}$ lead to $s_{3,h+1}$. Non-zero reward is only achievable if the agent can reach $s_{1,H+1}$ or $s_{2,H+1}$ and the reward follows Bernoulli(0.5). The good actions are randomly assigned for every state. We consider three cases: $\alpha = 0$, $\alpha = 0.2$, and $\alpha = 0.5$.

In LockBernoulli, the observation space is $\{0, 1\}^{H+3}$ where the first 3 coordinates are reserved for the one-hot encoding of the latent state and the last $H$ coordinates are drawn i.i.d from Bernoulli(0.5). LockBernoulli meets our requirements as a BMDP. In LockGaussian, the observation space is $\mathbb{R}^{H+3}$. Every observation is constructed by first letting the first three coordinates be the one-hot encoding of the latent state, then adding i.i.d Gaussian noises $\mathcal{N}(0, \sigma^2)$ to all $H + 3$ coordinates. We consider $\sigma = 0.1$ and $\sigma = 0.2$. LockGaussian is not a BMDP. We use this environment to evaluate the robustness of our method to violated assumptions.

The environments are designed to be hard for exploration. There are in total $4^H$ choices of actions of one episode, but only $2^H$ of them lead to non-zero reward in the end. So random exploration requires exponentially many trajectories. Also, with a larger $H$, the difficulty of learning accurate decoding functions increases and makes exploration with observations a more challenging task.

Algorithms and Hyperparameters We compare 4 algorithms: OracleQ, the Optimistic tabular Q-Learning algorithm of Jin et al. (2019); QLearning, the tabular Q-Learning with $\epsilon$-greedy exploration; URL, our method; and PCID, the algorithm proposed in Du et al. (2019a). For OracleQ and QLearning, we implement them in two ways: 1. they have direct access to the latent states (OracleQ-lat and QLearning-lat); 2. they only have observation data (OracleQ-obs and QLearning-obs). For URL and PCID, only observations are available. Note that since there is no access to the true states in reality, OracleQ-lat and QLearning-lat are not for practical use but only set as a near-optimal skyline and a sanity-check baseline to measure the efficiency of observation-only algorithms. OracleQ-obs and QLearning-obs are only tested in LockBernoulli environment since there are infinitely many observations for LockGaussian environment.

For OracleQ, there are two hyperparameters that we tune, the learning rate and a confidence parameter; for QLearning, we tune the learning rate and the exploration parameter $\epsilon$; for PCID, we follow the code provided in Du et al. (2019a), tune the number of clusters for $k$-means and the number of trajectories $n$ to collect in each outer iteration, and finally select the better result between linear function and neural network implementation.

In our method, we use OracleQ as the tabular RL algorithm to operate on the decoded state space and try three unsupervised learning approaches to learn the decoding functions: 1. first conduct principle component analysis (PCA) on the observation data and then use $k$-means (KMeans) to cluster; 2. first apply PCA, then use Density-Based Spatial Clustering of Applications with Noise (DBSCAN) for clustering, and finally use support vector machine to fit a classifier; 3. employ Gaussian Mixture Model (GMM) to fit the observation data then generate a label predictor. We call the python library sklearn for all these methods. During unsupervised learning, we do not separate

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4https://github.com/Microsoft/StateDecoding
observations by levels but add level information in decoded states. Besides the hyperparameters for OracleQ and the unsupervised learning oracle, we also tune the batch size $B$ adaptively in Algorithm 2. Indeed, when the learning of decoding functions turns stable after some episodes, we set $B = 0$ and directly use the latest learned functions for all future decoding. In our tests, it takes 100 episodes to train the decoding functions when $H = 5$, $500 \sim 1000$ episodes when $H = 10$, and $1000 \sim 2500$ episodes when $H = 20$, which is a fairly fast procedure.

### Results

The results are shown in Figure 1, 2, and 3, where $x$-axis is the number of running episodes and $y$-axis is the average rewards per episode. All lines are mean values of 50 tests and the shaded areas depict the standard deviation. The title for each subfigure records the length of the horizon, switch parameter $\alpha$ in actions, and the unsupervised learning method we apply for URL. In LockBernoulli, OracleQ-obs and QLearning-obs are far from being optimal even for small-horizon cases. URL is mostly as good as the skyline (OracleQ-lat) and much better than the baseline (QLearning-lat) especially when $H = 20$. URL outperforms PCID in most cases. When $H = 20$, we observe a probability of 80% that URL returns near-optimal values for $\alpha = 0.2$ and 0.5. In LockGaussian, OracleQ-obs and QLearning-obs are omitted due to infinitely many observations. Similarly, in most cases URL outperforms PCID. For $H = 20$, we observe a probability of $> 75\%$ that URL returns a near-optimal policy for $\alpha = 0.2$ and 0.5.

In summary, we preliminarily show the effectiveness and flexibility of our framework in terms of sample complexity and options of unsupervised learning methods. By selecting proper unsupervised learning approaches (especially when prior knowledge is available), we believe our framework is competitive and practical for RL with rich observations.

### 7 Conclusion

The current paper gave a general framework that turns an unsupervised learning algorithm and a no-regret tabular RL algorithm into an algorithm for RL problems with huge observation spaces. We provided theoretically analysis to show it is provably efficient. We also conducted numerical experiments to show the effectiveness of our framework in practice. This result complements empirical findings that unsupervised learning can guide exploration. An interesting future theoretical direction is to characterize the optimal sample complexity under our assumptions.

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### References

Achlioptas, D. and McSherry, F. (2005). On spectral learning of mixtures of distributions. In *International Conference on Computational Learning Theory*, pages 458–469. Springer.

Agarwal, A., Kakade, S. M., Lee, J. D., and Mahajan, G. (2019). Optimality and approximation with policy gradient methods in markov decision processes. *arXiv preprint arXiv:1908.00261*. 

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Figure 1: Performances for LockBernoulli. $x$-axis is the number of running episodes and $y$-axis is average rewards per episode. The lines are mean values of 50 tests and the shaded areas represent the standard deviations. The title for each subfigure includes the length of horizon, switch parameter in actions, and the unsupervised learning method we apply for URL.

Agrawal, S. and Jia, R. (2017). Posterior sampling for reinforcement learning: worst-case regret bounds. In *NIPS*.

Antos, A., Szepesvári, C., and Munos, R. (2008). Learning near-optimal policies with bellman-residual minimization based fitted policy iteration and a single sample path. *Machine Learning*, 71(1):89–129.

Azar, M. G., Osband, I., and Munos, R. (2017). Minimax regret bounds for reinforcement learning. In *Proceedings of the 34th International Conference on Machine Learning-Volume 70*, pages 263–272. JMLR. org.

Azizzadenesheli, K., Brunskill, E., and Anandkumar, A. (2018). Efficient exploration through bayesian deep Q-networks. In *2018 Information Theory and Applications Workshop (ITA)*, pages 1–9.

Bagnell, J. A., Kakade, S. M., Schneider, J. G., and Ng, A. Y. (2004). Policy search by dynamic programming. In *Advances in neural information processing systems*, pages 831–838.

Bellemare, M., Srinivasan, S., Ostrovski, G., Schaul, T., Saxton, D., and Munos, R. (2016). Unifying count-based exploration and intrinsic motivation. In *Advances in Neural Information Processing Systems*, pages 1471–1479.
Figure 2: Performances for LockGaussian with $\sigma = 0.1$. OracleQ-lat and QLearning-lat have direct access to the latent states, which are not for practical use. URL and PCID only have access to the observations. OracleQ-obs and QLearning-obs are omitted due to infinitely many observations.

Bontemps, D. and Toussile, W. (2013). Clustering and variable selection for categorical multivariate data. *Electronic Journal of Statistics*, 7:2344–2371.

Bouguila, N. and Fan, W. (2020). *Mixture models and applications*. Springer.

Brafman, R. I. and Tennenholtz, M. (2002). R-max—a general polynomial time algorithm for near-optimal reinforcement learning. *Journal of Machine Learning Research*, 3(Oct):213–231.

Chen, J. and Jiang, N. (2019). Information-theoretic considerations in batch reinforcement learning. *arXiv preprint arXiv:1905.00360*.

Dahl, D. B. (2006). Model-based clustering for expression data via a dirichlet process mixture model. *Bayesian inference for gene expression and proteomics*, 4:201–218.

Dann, C. and Brunskill, E. (2015). Sample complexity of episodic fixed-horizon reinforcement learning. In *Advances in Neural Information Processing Systems*, pages 2818–2826.

Dann, C., Jiang, N., Krishnamurthy, A., Agarwal, A., Langford, J., and Schapire, R. E. (2018). On polynomial time PAC reinforcement learning with rich observations. *arXiv preprint arXiv:1803.00606*.

Dann, C., Lattimore, T., and Brunskill, E. (2017). Unifying PAC and regret: Uniform PAC bounds for episodic reinforcement learning. In *Proceedings of the 31st International Conference on Neural Information Processing Systems*, NIPS'17, pages 5717–5727, USA. Curran Associates Inc.
Figure 3: Performances for LockGaussian with $\sigma = 0.2$. OracleQ-lat and QLearning-lat have direct access to the latent states, which are not for practical use. URL and PCID only have access to the observations. OracleQ-obs and QLearning-obs are omitted due to infinitely many observations.

Dasgupta, S. and Schulman, L. J. (2000). A two-round variant of em for gaussian mixtures. In Proceedings of the Sixteenth conference on Uncertainty in artificial intelligence, pages 152–159.

Du, S., Krishnamurthy, A., Jiang, N., Agarwal, A., Dudik, M., and Langford, J. (2019a). Provably efficient rl with rich observations via latent state decoding. In International Conference on Machine Learning, pages 1665–1674.

Du, S. S., Kakade, S. M., Wang, R., and Yang, L. F. (2019b). Is a good representation sufficient for sample efficient reinforcement learning? arXiv preprint arXiv:1910.03016.

Du, S. S., Lee, J. D., Mahajan, G., and Wang, R. (2020). Agnostic Q-learning with function approximation in deterministic systems: Tight bounds on approximation error and sample complexity. arXiv preprint arXiv:2002.07125.

Du, S. S., Luo, Y., Wang, R., and Zhang, H. (2019c). Provably efficient Q-learning with function approximation via distribution shift error checking oracle. In Advances in Neural Information Processing Systems, pages 8058–8068.

Elhamifar, E. and Vidal, R. (2013). Sparse subspace clustering: Algorithm, theory, and applications. IEEE transactions on pattern analysis and machine intelligence, 35(11):2765–2781.

Fortunato, M., Azar, M. G., Piot, B., Menick, J., Hessel, M., Osband, I., Graves, A., Mnih, V., Munos, R., Hassabis, D., Pietquin, O., Blundell, C., and Legg, S. (2018). Noisy networks for exploration. In International Conference on Learning Representations.
Geist, M., Scherrer, B., and Pietquin, O. (2019). A theory of regularized markov decision processes. In International Conference on Machine Learning, pages 2160–2169.

Jaksch, T., Ortner, R., and Auer, P. (2010). Near-optimal regret bounds for reinforcement learning. Journal of Machine Learning Research, 11(Apr):1563–1600.

Jiang, N., Krishnamurthy, A., Agarwal, A., Langford, J., and Schapire, R. E. (2017). Contextual decision processes with low bellman rank are PAC-learnable. In Proceedings of the 34th International Conference on Machine Learning—Volume 70, pages 1704–1713. JMLR. org.

Jin, C., Allen-Zhu, Z., Bubeck, S., and Jordan, M. I. (2018). Is Q-learning provably efficient? In Advances in Neural Information Processing Systems, pages 4863–4873.

Jin, C., Yang, Z., Wang, Z., and Jordan, M. I. (2019). Provably efficient reinforcement learning with linear function approximation. arXiv preprint arXiv:1907.05388.

Juan, A. and Vidal, E. (2002). On the use of bernoulli mixture models for text classification. Pattern Recognition, 35(12):2705–2710.

Juan, A. and Vidal, E. (2004). Bernoulli mixture models for binary images. In Proceedings of the 17th International Conference on Pattern Recognition, 2004. ICPR 2004., volume 3, pages 367–370. IEEE.

Kakade, S. and Langford, J. (2002). Approximately optimal approximate reinforcement learning. In ICML, volume 2, pages 267–274.

Kakade, S., Wang, M., and Yang, L. F. (2018). Variance reduction methods for sublinear reinforcement learning. arXiv preprint arXiv:1802.09184.

Kearns, M. and Singh, S. (2002). Near-optimal reinforcement learning in polynomial time. Machine learning, 49(2-3):209–232.

Krishnamurthy, A., Agarwal, A., and Langford, J. (2016). PAC reinforcement learning with rich observations. In Advances in Neural Information Processing Systems, pages 1840–1848.

Li, J. and Zha, H. (2006). Two-way poisson mixture models for simultaneous document classification and word clustering. Computational Statistics & Data Analysis, 50(1):163–180.

Li, L., Littman, M. L., Walsh, T. J., and Strehl, A. L. (2011). Knows what it knows: a framework for self-aware learning. Machine learning, 82(3):399–443.

Lipton, Z. C., Li, X., Gao, J., Li, L., Ahmed, F., and Deng, L. (2018). BBQ-networks: Efficient exploration in deep reinforcement learning for task-oriented dialogue systems. In AAAI.

McLachlan, G. J. and Basford, K. E. (1988). Mixture models: Inference and applications to clustering, volume 38. M. Dekker New York.

McLachlan, G. J. and Peel, D. (2004). Finite mixture models. John Wiley & Sons.

Munos, R. (2005). Error bounds for approximate value iteration. In Proceedings of the National Conference on Artificial Intelligence, volume 20, page 1006. Menlo Park, CA; Cambridge, MA; London; AAAI Press; MIT Press; 1999.
Najafi, A., Motahari, S. A., and Rabiee, H. R. (2020). Reliable clustering of bernoulli mixture models. Bernoulli, 26(2):1535–1559.

Osband, I., Van Roy, B., and Wen, Z. (2016). Generalization and exploration via randomized value functions. In Proceedings of the 33rd International Conference on International Conference on Machine Learning - Volume 48, ICML’16, pages 2377–2386. JMLR.org.

Parsons, L., Haque, E., and Liu, H. (2004). Subspace clustering for high dimensional data: a review. Acm Sigkdd Explorations Newsletter, 6(1):90–105.

Pathak, D., Agrawal, P., Efros, A. A., and Darrell, T. (2017). Curiosity-driven exploration by self-supervised prediction. In International Conference on Machine Learning (ICML), volume 2017.

Pazis, J. and Parr, R. (2013). PAC optimal exploration in continuous space markov decision processes. In Proceedings of the Twenty-Seventh AAAI Conference on Artificial Intelligence, AAAI’13, pages 774–781. AAAI Press.

Puterman, M. L. (2014). *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons.

Regev, O. and Vijayaraghavan, A. (2017). On learning mixtures of well-separated gaussians. In 2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS), pages 85–96. IEEE.

Sanjeev, A. and Kannan, R. (2001). Learning mixtures of arbitrary gaussians. In Proceedings of the thirty-third annual ACM symposium on Theory of computing, pages 247–257.

Scherrer, B. and Geist, M. (2014). Local policy search in a convex space and conservative policy iteration as boosted policy search. In Joint European Conference on Machine Learning and Knowledge Discovery in Databases, pages 35–50. Springer.

Simchowitz, M. and Jamieson, K. G. (2019). Non-asymptotic gap-dependent regret bounds for tabular mdps. In Advances in Neural Information Processing Systems, pages 1151–1160.

Soltanolkotabi, M., Elhamifar, E., Candes, E. J., et al. (2014). Robust subspace clustering. The Annals of Statistics, 42(2):669–699.

Song, Z. and Sun, W. (2019). Efficient model-free reinforcement learning in metric spaces. arXiv preprint arXiv:1905.00475.

Streth, A. L., Li, L., Wiewiora, E., Langford, J., and Littman, M. L. (2006). PAC model-free reinforcement learning. In Proceedings of the 23rd international conference on Machine learning, pages 881–888. ACM.

Sun, W., Jiang, N., Krishnamurthy, A., Agarwal, A., and Langford, J. (2019). Model-based rl in contextual decision processes: PAC bounds and exponential improvements over model-free approaches. In Conference on Learning Theory, pages 2898–2933.

Tang, H., Houthooft, R., Foote, D., Stooke, A., Chen, O. X., Duan, Y., Schulman, J., DeTurck, F., and Abbeel, P. (2017). # Exploration: A study of count-based exploration for deep reinforcement learning. In Advances in Neural Information Processing Systems, pages 2753–2762.
Vempala, S. and Wang, G. (2004). A spectral algorithm for learning mixture models. *Journal of Computer and System Sciences*, 68(4):841–860.

Vidal, R. (2011). Subspace clustering. *IEEE Signal Processing Magazine*, 28(2):52–68.

Wallace, T., Sekmen, A., and Wang, X. (2015). Application of subspace clustering in dna sequence analysis. *Journal of Computational Biology*, 22(10):940–952.

Wang, Y.-X., Xu, H., and Leng, C. (2013). Provable subspace clustering: When lrr meets ssc. In *Advances in Neural Information Processing Systems*, pages 64–72.

Wen, Z. and Van Roy, B. (2013). Efficient exploration and value function generalization in deterministic systems. In *Advances in Neural Information Processing Systems*, pages 3021–3029.

Yang, L. F., Ni, C., and Wang, M. (2019a). Learning to control in metric space with optimal regret. *arXiv preprint arXiv:1905.01576*.

Yang, L. F. and Wang, M. (2019). Sample-optimal parametric Q-learning using linearly additive features. In *International Conference on Machine Learning*, pages 6995–7004.

Yang, Z., Xie, Y., and Wang, Z. (2019b). A theoretical analysis of deep Q-learning. *arXiv preprint arXiv:1901.00137*.

Zanette, A. and Brunskill, E. (2019). Tighter problem-dependent regret bounds in reinforcement learning without domain knowledge using value function bounds. In *International Conference on Machine Learning*, pages 7304–7312.