Existence and Uniqueness of Quasi-stationary Distributions for Symmetric Markov Processes with Tightness Property

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Abstract
Let $X$ be an irreducible symmetric Markov process with the strong Feller property. We assume, in addition, that $X$ is explosive and has a tightness property. We then prove the existence and uniqueness of quasi-stationary distributions of $X$.

Keywords Quasi-stationary distribution · Symmetric Markov process · Dirichlet form · Yaglom limit · Tightness

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1 Introduction
Let $E$ be a locally compact separable metric space and $m$ a positive Radon measure on $E$ with full topological support. Let $X = (\Omega, X_t, \mathbb{P}_x, \zeta)$ be an $m$-symmetric Markov process (SMP for short) on $E$. Here $\zeta$ is the lifetime of $X$. We assume that the process $X$ is irreducible and strong Feller, in addition, possesses a tightness property, i.e., for any $\epsilon > 0$, there exists a compact set $K$ such that $\sup_{x \in E} E_x (e^{\gamma \zeta (1 - K^c)}) < \infty$. Here $1_{K^c}$ is the indicator function of the complement of $K$ and $R_1$ is the 1-resolvent of $X$. In this paper, we call the family of SMPs with these three properties $\text{Class (T)}$.

We prove in [21] that if $X$ is in $\text{Class (T)}$, then for any $\gamma > 0$ there exists a compact set $K$ such that

$$\sup_{x \in E} E_x (e^{\gamma \zeta (1 - K^c)}) < \infty,$$
where $\tau_{K^c}$ is the first exit time from $K^c$. As a result, its transition operator $p_t$ is a compact operator on $L^2(E; m)$ and all its eigenfunctions have bounded continuous versions \[21, \text{Theorem 4.3, Theorem 5.4}.\] If $X$ in Class (T) is not conservative, it explodes very fast in a sense that the lifetime is exponentially integrable \[see \ (8) \ below\]. In particular, $X$ is almost surely killed, $\mathbb{P}_x(\zeta < \infty) = 1$ for all $x \in E$. The objective of this paper is to prove the existence and uniqueness of quasi-stationary distributions of explosive SMPs in Class (T).

A probability measure $\nu$ on $E$ is said to be a quasi-stationary distribution (QSD for short) of $X$, if for all $t \geq 0$ and all Borel subset $B$ of $E$

$$\nu(B) = \mathbb{P}_\nu(X_t \in B \mid t < \zeta),$$

that is, the distribution of $X_t$ conditioned to survive up to $t$ equals $\nu$ over time if the initial distribution $\nu$ is a QSD.

Let $\phi_0$ be the smallest (principal) eigenfunction of $p_t$ with eigenvalue $\lambda_0$, $p_t \phi_0 = e^{-\lambda_0 t} \phi_0$. As stated above, we can suppose that $\phi_0$ is a bounded continuous. Moreover, we can show that $\phi_0$ is strictly positive and integrable, $\phi_0 \in L^1(E; m)$ (Lemma 3.4).

Hence we can define the probability measure $\nu^{\phi_0}$ by

$$\nu^{\phi_0}(B) = \frac{\int_B \phi_0 \, dm}{\int_E \phi_0 \, dm}, \quad B \in \mathcal{B}(E),$$

where $\mathcal{B}(E)$ denotes the totality of Borel subset of $E$. Our main result is as follows (Theorem 3.1): If $X$ is in Class (T), then $\nu^{\phi_0}$ is the unique QSD of $X$.

For the proof of Theorem 3.1, the following fact is crucial: Every SMP can be transformed to an ergodic SMP by multiplicative functional. More precisely, let $X^{\phi_0} = (\Omega, X_t, \mathbb{P}^{\phi_0}_x, \zeta)$ be the process transformed by the multiplicative functional,

$$L^{\phi_0}_t = e^{\lambda_0 t} \frac{\phi_0(X_t)}{\phi_0(X_0)} 1_{\{t < \zeta\}}.$$

We then see from Lemma 6.3.2 in [8] that $X^{\phi_0}$ is an irreducible, conservative $\phi_0^2 m$-SMP on $E$. We can prove that $\nu^{\phi_0}$ is a QSD using the $\phi_0^2 m$-symmetry and conservativeness of $X^{\phi_0}$ (Corollary 3.6). Applying Fukushima’s ergodic theorem (Theorem 2.2 below) to $X^{\phi_0}$, we can prove that $\nu^{\phi_0}$ is a unique QSD of $X$. Indeed, since $\phi_0$ is strictly positive, bounded continuous as remarked above,

$$\sup_{x \in E} \frac{1}{\phi_0}(x) \leq \frac{1}{\inf_{x \in K} \phi_0(x)} < \infty$$

for any compact set $K$. Hence by Theorem 2.2 and Corollary 2.1, we have

$$\lim_{t \to \infty} \mathbb{E}^{\phi_0}_x \left( \frac{1}{\phi_0}(X_t) \right) = \int_K \phi_0 \, dm, \quad \forall x \in E,$$

which leads us to the uniqueness of QSD (Theorem 3.1).
We know that a minimal one-dimensional diffusion process is in Class (T) if and only if no natural boundaries in Feller’s classification are present (Example 3.1). In [2,6], they treat a one-dimensional diffusion process on \([0, \infty)\) defined as the solution of the SDE:

\[dX_t = dB_t - q(X_t)dt\]

whose boundaries 0 and \(\infty\) are exit and entrance, respectively. Theorem 3.1 says that one-dimensional diffusion processes without natural boundary have a unique QSD in general.

We give two examples of multi-dimensional SMPs in Class (T), absorbing Brownian motions on domain thin at infinity and killed Brownian motions on \(\mathbb{R}^d\), which are treated in [22].

Finally, we remark that if the semigroup of an explosive symmetric Markov processes in Class (T) is intrinsic ultracontractive, \(\nu_{\phi_0}\) is a Yaglom limit: for any probability measure \(\mu\)

\[\lim_{t \to \infty} \mathbb{P}_\mu(X_t \in B \mid t < \zeta) = \nu_{\phi_0}(B).\]

For example, let \(X^D = (\mathbb{P}^D_x, X_t, \tau_D)\) be an absorbing rotationally symmetric \(\alpha\)-stable process on bounded open set \(D\), where \(0 < \alpha < 2\) and \(\tau_D\) is the first exit time from \(D\). We then see that \(X^D\) is intrinsic ultracontractive [12], and thus

\[\lim_{t \to \infty} \mathbb{P}^D_x(X_t \in B \mid t < \tau_D) = \nu_{\phi_0}(B), \quad \forall B \in \mathcal{B}(D),\]

which is an extension of a result of Pinsky [16] to absorbing symmetric \(\alpha\)-stable processes. In [14, Example 4], the give examples of open sets \(D\) such that \(m(D) = \infty\) and \(X^D\) are intrinsic ultracontractive. Applications of the intrinsic ultracontractivity to the Yaglom limit were studied in [13,15].

### 2 Ergodic Properties of SMPs

In this section, we summarize results on ergodic properties of SMPs. Let \(E\) be a locally compact separable metric space and \(E_\Delta\) the one-point compactification of \(E\) with adjoined point \(\Delta\). Let \(m\) be a positive Radon measure on \(E\) with full topological support. Let \(X = (\Omega, X_t, \mathbb{P}_x, \zeta)\) be an \(m\)-SMP. Here \(\zeta\) is the lifetime of \(X\), \(\zeta = \inf\{t > 0 : X_t = \Delta\}\). Denote by \(\{p_t; t \geq 0\}\) and \(\{R_\alpha; \alpha > 0\}\) the semigroup and resolvent of \(X\):

\[p_t f(x) = \mathbb{E}_x(f(X_t)), \quad R_\alpha f(x) = \mathbb{E}_x\left(\int_0^\infty e^{-\alpha t} f(X_t)dt\right).\]

In this section, we further assume that \(X\) is conservative, \(\mathbb{P}_x(\zeta = \infty) = 1\), and satisfies
(I) (Irreducibility) If a Borel set $A$ is $p_t$-invariant, that is, $p_t(1_A f)(x) = 1_A p_t f(x)$ $m$-a.e. for any $f \in L^2(E; m) \cap b \mathcal{B}(E)$ and $t > 0$, then $A$ satisfies either $m(A) = 0$ or $m(E \setminus A) = 0$. Here $b \mathcal{B}(E)$ is the space of bounded Borel functions on $E$.

The symmetry of $X$ enables us to strengthen the ergodic theorem as follows: Suppose $m(E) < \infty$. For $f \in L^\infty(E; m)$

$$p_t f(x) \to \frac{1}{m(E)} \int_E f(x) dm, \quad m\text{-a.e. } x. \quad (3)$$

Following the argument in [7], we will give a proof of (3).

**Theorem 2.1** Suppose $m(E) < \infty$. For any $f \in L^\infty(E; m)$, there exists a function $g$ in $L^\infty(E; m)$ such that

$$\lim_{t \to \infty} p_t f = g, \quad m\text{-a.e. and in } L^1(E; m).$$

Moreover, $g$ is $p_t$-invariant, $p_t g = g$, $m$-a.e.

**Proof** Define $\mathcal{G}_t = \{X_s \mid s \geq t\}$ and $Y_t = \mathbb{E}_m(f(X_0)|\mathcal{G}_t)$, where $\mathbb{P}_m(\cdot) = \int_E \mathbb{P}_x(\cdot) dm(x)$. By the time reversibility of $X_t$ with respect to $\mathbb{P}_m$, $Y_t = p_t f(X_t)$, $\mathbb{P}_m$-a.e., and so

$$\mathbb{E}_m(Y_t|\mathcal{F}_0) = \mathbb{E}_m(p_t f(X_t)|\mathcal{F}_0) = p_{2t} f(X_0), \quad \mathbb{P}_m\text{-a.e.}$$

Here $\mathcal{F}_0 = \{X_0\}$. Since $f(X_0) \in L^1(\mathbb{P}_m)$ and $Y_t$ is a reversed martingale,

$$\lim_{t \to \infty} Y_t = \mathbb{E}_m(f(X_0)| \cap_{t>0} \mathcal{G}_t), \quad \mathbb{P}_m\text{-a.e. and in } L^1(\mathbb{P}_m)$$

(cf. [17, Theorem:II.51.1]). Put $Z = \mathbb{E}_m(f(X_0)| \cap_{t>0} \mathcal{G}_t)$. Noting that $|Y_t| \leq \|f\|_\infty$, $\mathbb{P}_m$-a.e. by the definition of $Y_t$, we see from the conditional bounded convergence theorem (cf. [17, II.40.41, (41)(g)]) that

$$\lim_{t \to \infty} p_{2t} f(X_0) = \lim_{t \to \infty} \mathbb{E}_m(Y_t|\mathcal{F}_0) = \mathbb{E}_m(Z|\mathcal{F}_0) = \mathbb{E}_X(Z), \quad \mathbb{P}_m\text{-a.e. and in } L^1(\mathbb{P}_m). \quad (4)$$

Put $g(x) = \mathbb{E}_x(Z)$. We then see from (4) that $\lim_{t \to \infty} p_t f = g$, $m$-a.e. and in $L^1(E; m)$. The $p_t$-invariance of $g$ follows from

$$p_t g = \lim_{s \to \infty} p_t(p_s f) = \lim_{s \to \infty} p_{t+s} f = g, \quad m\text{-a.e.},$$

which completes the proof. \hfill $\square$

**Theorem 2.2** [7] Assume $m(E) < \infty$. If the Markov process $X$ is irreducible and conservative, then for $f \in L^\infty(E; m)$

$$\lim_{t \to \infty} p_t f(x) = \frac{1}{m(E)} \int_E f dm, \quad m\text{-a.e. and in } L^1(E; m) \quad (5)$$
Proof By combining Theorem 2.1 with [4, Theorem 2.1.11](see also [10, Theorem 1]), we see \( \lim_{t \to \infty} p_t f \) is constant \( m \)-a.e. Since \( (p_t f, 1)_m = \int_E f \, dm \), the constant is equal to the right-hand side of (5).

Remark 2.1 Suppose that \( X \) satisfies the absolute continuity condition:

\( (\text{AC}) \ p_t(x, dy) = p_t(x, y)m(dy), \forall t > 0, \forall x \in E \).

Then “\( m \)-a.e. \( x \)” in Theorem 2.2 can be strengthened to “all \( x \)” Indeed, for any \( x \in E \)

\[
\lim_{t \to \infty} p_t f(x) = \lim_{t \to \infty} \int_E p_1(x, y) \left( \int_E p_t - 1(y, z) f(z) \, dm(z) \right) \, dm(y) = \int_E p_1(x, y) \left( \frac{1}{m(E)} \int_E f \, dm \right) \, dm(y) = \frac{1}{m(E)} \int_E f \, dm.
\]

Corollary 2.1 Suppose the assumptions of Theorem 2.2 hold. Assume, in addition, \( (\text{AC}) \) and the ultracontractivity, \( \| p_t \|_{1, \infty} \leq c_t < \infty \). Here \( \| \cdot \|_{1, \infty} \) is the operator norm from \( L^1(E; m) \) to \( L^\infty(E; m) \). Then for \( f \in L^1(E; m) \)

\[
\lim_{t \to \infty} p_t f(x) = \frac{1}{m(E)} \int_E f \, dm, \forall x \in E. 
\]

Proof For \( f \in L^1(E; m) \), \( p_1 f \in L^\infty(E; m) \) by the ultracontractivity. Hence

\[
\lim_{t \to \infty} p_t f(x) = \lim_{t \to \infty} p_{t - 1}(p_1 f)(x) = \frac{1}{m(E)} \int_E p_1 f \, dm, \forall x \in E
\]

by Theorem 2.2 and Remark 2.1. By the symmetry with respect to \( m \) and conservativeness of \( p_1 \)

\[
\int_E p_1 f \, dm = \int_E p_1 \cdot f \, dm = \int_E f \, dm,
\]

and (6) is proved.

3 Quasi-stationary Distribution

In this section, we consider the existence and uniqueness of quasi-stationary distributions. We assume that \( X \) possesses the next three properties:

(I) (Irreducibility)

(II) (Strong Feller Property) For each \( t > 0, p_t(b \mathcal{B}(E)) \subset bC(E), \) where \( bC(E) \) is the space of bounded continuous functions on \( E \).
(III) **(Tightness)** For any \( \epsilon > 0 \), there exists a compact set \( K \) such that

\[
\sup_{x \in E} R_1 1_{K^c}(x) \leq \epsilon.
\]

A SMP with three properties above is said to be in **Class \((T)\)**. Note that Condition \((\text{II})\) implies \((\text{AC})\).

We see that if \( X \) is not conservative, the tightness property implies a fast explosion in a sense that the lifetime \( \zeta \) is exponentially integrable. In particular, \( X \) is almost surely killed, \( \mathbb{P}_x(\zeta < \infty) = 1 \) for any \( x \in E \). Indeed, let \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) be the Dirichlet form on \( L^2(E; m) \) generated by \( X \):

\[
\begin{aligned}
\mathcal{D}(\mathcal{E}) &= \left\{ u \in L^2(E; m) \mid \lim_{t \to 0} \frac{1}{t} (u - T_t u, u)_m < \infty \right\} \\
\mathcal{E}(u, v) &= \lim_{t \to 0} \frac{1}{t} (u - T_t u, v)_m.
\end{aligned}
\]

We define

\[
\lambda_0 = \inf\{ \mathcal{E}(u, u) \mid u \in \mathcal{D}(\mathcal{E}), \|u\|_2 = 1 \},
\]

where \( \| \cdot \|_2 \) is the \( L^2(E; m) \)-norm. We then see in [18, Corollary 3.8] that \( \lambda_0 > 0 \) and for \( 0 < \gamma < \lambda_0 \)

\[
\sup_{x \in E} \mathbb{E}_x(e^{\gamma \zeta}) < \infty.
\]

In the sequel, we assume that \( X \) is an explosive SMP in **Class \((T)\)**.

A probability measure \( \nu \) on \( E \) is said to be **quasi-stationary distribution** (**QSD** for short) of \( X \) if for all \( t \geq 0 \) and all Borel set \( B \in \mathcal{B}(E) \),

\[
\nu(B) = \mathbb{P}_\nu(X_t \in B \mid t < \zeta) = \left( \frac{\mathbb{P}_\nu(X_t \in B)}{\mathbb{P}_\nu(X_t \in E)} \right).
\]

where \( \mathbb{P}_\nu(\cdot) = \int_E \mathbb{P}_\nu(\cdot) d\nu(x) \). QSDs capture the long-time behavior of surely killed process \( X \) when \( X \) is conditioned to survive.

A function \( \phi_0 \) on \( E \) is called a **ground state** of \((\mathcal{E}, \mathcal{D}(\mathcal{E}))\) if \( \phi_0 \in \mathcal{D}(\mathcal{E}) \), \( \|\phi_0\|_2 = 1 \) and \( \lambda_0 = \mathcal{E}(\phi_0, \phi_0) \). The ground state \( \phi_0 \) exists because the embedding of \((\mathcal{E}_1, \mathcal{D}(\mathcal{E}))\) into \( L^2(E; m) \) is compact [19, Theorem 2.1]. Here \( \mathcal{E}_1 = \mathcal{E} + (\cdot, \cdot)_m \).

**Lemma 3.1** *For a Borel set \( B \subset E \) with \( m(B) > 0 \), define

\[
\lambda_0^B = \inf \left\{ \mathcal{E}(u, u) + \int_B u^2 dm \mid u \in \mathcal{D}(\mathcal{E}), \|u\|_2 = 1 \right\}.
\]

Then it holds that \( \lambda_0^B > \lambda_0 \).*
Proof. There exists a minimizer $\phi_0^B$ attaining the infimum in (9) by Takeda [19, Theorem 2.1]. Hence

$$\lambda_0^B = \mathcal{E}(\phi_0^B, \phi_0^B) + \int_B (\phi_0^B)^2 dm > \mathcal{E}(\phi_0^B, \phi_0^B) \geq \mathcal{E}(\phi_0, \phi_0) = \lambda_0.$$  

$\square$

**Proposition 3.1** [21] The ground state $\phi_0$ has a bounded continuous version with $\phi_0(x) > 0$ for any $x \in E$.

For a compact set $K$ with $m(K) > 0$, define

$$p_t^K f(x) = \mathbb{E}_x \left( e^{-\int_0^t 1_K(X_s) ds} f(X_t) \right), \quad t \geq 0,$$

$$R_{\beta}^{\lambda_0, K} f(x) = \mathbb{E}_x \left( \int_0^\infty e^{-\beta t + \lambda_0 t} p_t^K f(x) dt \right), \quad \beta \geq 0.$$  

We denote $R_{\lambda_0, K}$ for $R_{\lambda_0, K}^0$ simply.

**Lemma 3.2** It holds that $\sup_{x \in E} R_{\lambda_0, K} 1(x) < \infty$.

Proof. By the $L^p$-independence of the growth bound of $p_t^K$ [5, Theorem 1.3], for any $\delta > 0$ there exists a positive constant $C(\delta)$ such that

$$\sup_{x \in E} p_t^K 1(x) = \|p_t^K 1\|_\infty \leq C(\delta) e^{-(\lambda_0^K - \delta)t}.$$  

Since $\lambda_0^K > \lambda_0$ by Lemma 3.1, for $0 < \delta < \lambda_0^K - \lambda_0$

$$\|R_{\lambda_0, K} 1\|_\infty \leq \int_0^\infty e^{\lambda_0^K t} \sup_{x \in E} p_t^K 1(x) dt \leq C(\delta) \int_0^\infty e^{-(\lambda_0^K - \lambda_0 - \delta)t} dt$$

$$= \frac{C(\delta)}{\lambda_0^K - \lambda_0 - \delta} < \infty.$$  

$\square$

We define symmetric bilinear forms on $L^2(E; m)$: For $u \in \mathcal{D}(\mathcal{E})$

$$\mathcal{E}^{\lambda_0}(u, u) = \mathcal{E}(u, u) - \lambda_0 \int_E u^2 dm,$$

$$\mathcal{E}^{\lambda_0, K}(u, u) = \mathcal{E}(u, u) - \lambda_0 \int_E u^2 dm + \int_E u^2 1_K dm.$$  

For a general symmetric bilinear form $\mathcal{A}$, $\mathcal{A}_\beta$ denotes $\mathcal{A} + \beta(\cdot, \cdot)_m$.

**Lemma 3.3** The ground state $\phi_0$ satisfies $\phi_0(x) = R_{\lambda_0, K}^{\phi_0 1_K}(x)$ for all $x \in E$.  

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Proof For $\varphi \in b\mathcal{B}_0^+(E)$, the set of nonnegative bounded functions with compact support,

$$
\mathcal{E}_{\lambda,0}^K \left( R_{\beta}^{\lambda_0,K} \varphi, R_{\beta}^{\lambda_0,K} \varphi \right) \leq \mathcal{E}_{\beta}^{\lambda_0,K} \left( R_{\beta}^{\lambda_0,K} \varphi, R_{\beta}^{\lambda_0,K} \varphi \right) = \int_E \varphi R_{\beta}^{\lambda_0,K} \varphi dm \leq \int_E \varphi R_{\lambda_0,K} \varphi dm < \infty
$$

by Lemma 3.2. Since $R_{\beta}^{\lambda_0,K} \varphi \uparrow R_{\lambda_0,K} \varphi$ as $\beta \downarrow 0$, the function $R_{\lambda_0,K} \varphi$ belongs to the extended Schrödinger space $\mathcal{D}_e(\mathcal{E}_{\lambda_0,K})$ (For the definition of extended Schrödinger space, see [20, Section 2]).

By the definition of $\mathcal{E}_{\lambda_0,K}$,

$$
\mathcal{E}_{\lambda_0,K} \left( \phi_0, R_{\beta}^{\lambda_0,K} \varphi \right) = \mathcal{E}_{\lambda_0} \left( \phi_0, R_{\beta}^{\lambda_0,K} \varphi \right) + \int_E 1_K \phi_0 R_{\beta}^{\lambda_0,K} \varphi dm.
$$

Noting that $\mathcal{D}_e(\mathcal{E}_{\lambda_0,K}) \subset \mathcal{D}_e(\mathcal{E}_{\lambda_0})$ because $\mathcal{E}_{\lambda_0}(u, u) \leq \mathcal{E}_{\lambda_0,K}(u, u)$, we have

$$
\mathcal{E}_{\lambda_0,K} \left( \phi_0, R_{\lambda_0,K} \varphi \right) = \mathcal{E}_{\lambda_0} \left( \phi_0, R_{\lambda_0,K} \varphi \right) + \int_E 1_K \phi_0 R_{\lambda_0,K} \varphi dm \quad (10)
$$

as $\beta \to 0$.

Since $\phi_0$ is the eigenfunction corresponding to $\lambda_0$, $\mathcal{E}_{\lambda_0}(\phi_0, R_{\beta}^{\lambda_0,K} \varphi) = 0$ for any $\beta > 0$, and so $\mathcal{E}_{\lambda_0}(\phi_0, R_{\lambda_0,K} \varphi) = 0$. Hence, by (10) and the symmetry of $R_{\lambda_0,K}$ with respect to $m$

$$
\mathcal{E}_{\lambda_0,K} \left( \phi_0, R_{\lambda_0,K} \varphi \right) = \int_E 1_K \phi_0 R_{\lambda_0,K} \varphi dm = \int_E R_{\lambda_0,K}(1_K \phi_0) \varphi dm. \quad (11)
$$

On the other hand,

$$
\mathcal{E}_{\lambda_0,K} \left( \phi_0, R_{\beta}^{\lambda_0,K} \varphi \right) = \mathcal{E}_{\beta}^{\lambda_0,K} \left( \phi_0, R_{\beta}^{\lambda_0,K} \varphi \right) - \beta \int_E \phi_0 R_{\beta}^{\lambda_0,K} \varphi dm \quad (12)
$$

$$
= \int_E \phi_0 \varphi dm - \beta \int_E \phi_0 R_{\beta}^{\lambda_0,K} \varphi dm.
$$

Since

$$
\int_E \phi_0 R_{\beta}^{\lambda_0,K} \varphi dm = \int_E R_{\beta}^{\lambda_0,K} \phi_0 \varphi dm \leq \|\phi_0\|_\infty \|R_{\lambda_0,K} 1\|_\infty \int_E \varphi dm < \infty,
$$

by Lemma 3.2, we have from (12)

$$
\mathcal{E}_{\lambda_0,K} \left( \phi_0, R_{\lambda_0,K} \varphi \right) = \int_E \phi_0 \varphi dm. \quad (13)
$$

by letting $\beta \to 0$. 
By (11) and (13),
\[ \int_E R^{\lambda_0, K}(\phi_0 1_K) \varphi \, dm = \int_E \varphi \phi_0 \, dm, \ \forall \varphi \in bB_0^+(E) \]
and thus
\[ \phi_0 = R^{\lambda_0, K}(\phi_0 1_K), \ m\text{-a.e.} \]
By the continuity of both functions, “m-a.e. \(x\)” can be strengthened to “all \(x\)”.

\[ \begin{array}{c}
\text{Lemma 3.4} \\
The ground state \(\phi_0\) belongs to \(L^1(E; m)\).
\end{array} \]

**Proof** By Lemma 3.3 and the symmetry of \(R^{\lambda_0, K}\) with respect to \(m\), we see
\[ \int_E \phi_0 \, dm = \int_E R^{\lambda_0, K}(1_K \phi_0) \, dm = \int_E 1_K \phi_0 R^{\lambda_0, K} \, 1 \, dm. \]
The right hand side is finite by Proposition 3.1 and Lemma 3.2.

\[ \begin{array}{c}
\text{Lemma 3.5} \\
Let \(\mu\) be a QSD. Then \(\mu\) is absolutely continuous with respect to \(m\).
\end{array} \]

**Proof** If \(m(B) = 0\), then
\[ \mathbb{P}_\mu(X_t \in B) = \int_E \left( \int_B p_t(x, y) \, dm(y) \right) \, d\mu = 0, \]
and thus \(\mu(B) = \mathbb{P}_\mu(X_t \in B)/\mathbb{P}_\mu(t < \zeta) = 0\).

We define the space \(D^+(A)\) by
\[ D^+(A) = \left\{ R_\alpha f \mid \alpha > 0, \ f \in L^2(E; m) \cap bC^+(E), \ f \not\equiv 0 \right\}. \]

Here \(bC^+(E)\) is the set of nonnegative bounded continuous functions. For \(\phi = R_\alpha g \in D^+(A)\) define the multiplicative functional \(L_\phi^\alpha\) by
\[ L_\phi^\alpha_t = \frac{\phi(X_t)}{\phi(X_0)} \exp \left( -\int_0^t A\phi/\phi(X_s) \, ds \right) 1_{[t < \zeta]}, \ A\phi = \alpha \phi - g. \quad (14) \]

Let \(X^\phi = (\Omega, X_t, \mathbb{P}_x^{\phi, \xi})\) the transformed process of \(X\) by \(L_\phi^\alpha\) and denote by \(p_t^\phi\) its semigroup, \(p_t^\phi f(x) = \mathbb{E}_x(L_t^\phi f(X_t))\). We then see from Lemma 6.3.2 in [8] that \(X^\phi\) is an irreducible, conservative \(\phi^m\)-SMP on \(E\), \((p_t^\phi f, g)_{\phi^m} = (f, p_t^\phi g)_{\phi^m}\) (In [3] this fact is extended to \(\phi \in D(E)\)). Since \(\phi_0 \in D^+(A)\) and \(A\phi_0 = -\lambda_0 \phi_0\), \(L_{\phi_0}^\alpha\) in (14) is simply written as
\[ L_{\phi_0}^\alpha = e^{\lambda_0 t} \frac{\phi_0(X_t)}{\phi_0(X_0)} 1_{[t < \zeta]}. \quad (15) \]
Hence the following equalities hold:

\[ p_t^{\phi_0} f(x) = e^{\lambda_0 t} \frac{1}{\phi_0(x)} \mathbb{E}_x (\phi_0(X_t) f(X_t)) = e^{\lambda_0 t} \frac{1}{\phi_0(x)} p_t(\phi_0 f)(x) \]

and so

\[ p_t f(x) = e^{-\lambda_0 t} \phi_0(x) p_t^{\phi_0} \left( \frac{f}{\phi_0} \right)(x). \tag{16} \]

We see from Lemma 3.4 that the probability measure \( \nu^{\phi_0} \) can be defined by

\[ \nu^{\phi_0}(B) = \frac{\int_B \phi_0 \, dm}{\int_E \phi_0 \, dm}. \tag{17} \]

**Lemma 3.6** The measure \( \nu^{\phi_0} \) is a QSD of \( X \).

**Proof** By (16)

\[ \mathbb{P}^{\nu^{\phi_0}}(X_t \in B) = \frac{\int_E \mathbb{P}_x^t (X_t \in B) \phi_0(x) \, dm}{\int_E \phi_0(x) \, dm} = e^{-\lambda_0 t} \int_E \mathbb{E}_x^{\phi_0} ((1_B/\phi_0)(X_t)) \phi_0^2(x) \, dm \frac{\int_E \phi_0(x) \, dm}{\int_E \phi_0(x) \, dm}. \]

Since \( X^{\phi_0} \) is \( \phi_0^2m \)-symmetric and conservative, \( p_t^{\phi_0} 1 = 1 \),

\[ \int_E \mathbb{E}_x^{\phi_0} ((1_B/\phi_0)(X_t)) \phi_0^2(x) \, dm = \int_E p_t^{\phi_0} (1_B/\phi_0)(x) \phi_0^2(x) \, dm \]

\[ = \int_E (1_B/\phi_0)(x) p_t^{\phi_0} 1(x) \phi_0(x)^2 \, dm = \int_B \phi_0 \, dm. \]

Hence we see

\[ \mathbb{P}^{\nu^{\phi_0}}(X_t \in B \mid X_t \in E) = \frac{\mathbb{P}^{\nu^{\phi_0}}(X_t \in B)}{\mathbb{P}^{\nu^{\phi_0}}(X_t \in E)} = \frac{\int_B \phi_0 \, dm}{\int_E \phi_0 \, dm} = \nu^{\phi_0}(B). \]

\( \square \)

**Theorem 3.1** Assume that \( X \) is an explosive SMP in Class \((T)\). Then the measure \( \nu^{\phi_0} \) defined in (17) is the unique QSD of \( X \).

**Proof** Let \( \mu \) is a QSD, i.e.,

\[ \mu(B) = \mathbb{P}^{\mu}(X_t \in B \mid t < \xi) = \frac{\mathbb{P}^{\mu}(X_t \in B)}{\mathbb{P}^{\mu}(X_t \in E)}. \]

\( \square \)
For compact sets \( K, F \subset E \),

\[
\mu(K) = \frac{\mathbb{P}_\mu(X_t \in K)}{\mathbb{P}_\mu(X_t \in E)} \leq \frac{\mathbb{P}_\mu(X_t \in K)}{\mathbb{P}_\mu(X_t \in F)} = \frac{\int_E p_t 1_K \, d\mu}{\int_E p_t 1_F \, d\mu}.
\]

By (16), the right hand side equals

\[
\int_E \phi_0 p_t \frac{1_K}{\phi_0} \, d\mu = \int_E \phi_0 p_t \frac{1_F}{\phi_0} \, d\mu.
\]

Since

\[
\frac{1_K(x)}{\phi_0(x)} \leq \frac{1}{\inf_{x \in K} \phi_0(x)} < \infty,
\]

\( 1_K/\phi_0 \) belongs to \( L^\infty(E; m) \). Noting \( X^{\phi_0} \) satisfies (AC) by definition, we see from Remark 2.1 that

\[
\lim_{t \to \infty} \mathbb{P}_\nu(X_t \in B | t < \zeta) = \nu(\phi_0)(B),
\]

for any probability measure \( \nu \).

Lemma 3.7 If \( X \) is intrinsic ultracontractive, then

\[
\lim_{t \to \infty} \mathbb{P}_\nu(X_t \in B | t < \zeta) = \nu(\phi_0)(B)
\]

for any probability measure \( \nu \).
Proof Since
\[ P_\nu(X_t \in B) = e^{-\lambda_0 t} \int_E \phi_0 p_t^{\phi_0} \left( \frac{1_B}{\phi_0} \right) d\nu, \]
we have
\[ P_\nu(X_t \in B \mid t < \zeta) = \frac{\int_E \phi_0 p_t^{\phi_0} \left( \frac{1_B}{\phi_0} \right) d\nu}{\int_E \phi_0 p_t^{\phi_0} \left( \frac{1}{\phi_0} \right) d\nu}. \]
Noting that \( 1/\phi_0 \in L^1(\phi_0^2 m) \), we have this lemma by Corollary 2.1.

Example 3.1 Let us consider a one-dimensional diffusion process \( X = (X_t, \mathbb{P}_x, \zeta) \) on an open interval \( I = (r_1, r_2) \) such that \( \mathbb{P}_x(X_{\zeta^-} = r_1 \text{ or } r_2, \ zeta < \infty) = \mathbb{P}_x(zeta < \infty), \ x \in I, \) and \( \mathbb{P}_a(\sigma_b < \infty) > 0 \) for any \( a, b \in I \). The diffusion \( X \) is symmetric with respect to its canonical measure \( m \) and it satisfies I and II. The boundary point \( r_i \) of \( I \) is classified into four classes: regular boundary, exit boundary, entrance boundary and natural boundary [9, Chapter 5]:

(a) If \( r_2 \) is a regular or exit boundary, then \( \lim_{x \rightarrow r_2} R_1 l (x) = 0. \)
(b) If \( r_2 \) is an entrance boundary, then \( \lim_{x \rightarrow r_2} \sup_{x \in l} R_1 l (r_2) (x) = 0. \)
(c) If \( r_2 \) is a natural boundary, then \( \lim_{x \rightarrow r_2} R_1 l (r_2) (x) = 1 \) and thus \( \sup_{x \in (r_1, r_2)} R_1 l (r_2) (x) = 1. \)

Therefore, the tightness property III is fulfilled if and only if no natural boundaries are present. For the intrinsic ultracontractivity of one-dimensional diffusion processes, refer to [23].

Example 3.2 Let \( \mathcal{D} \) be the family of open sets in \( \mathbb{R}^d \). We set
\[ \mathcal{D}_0 = \left\{ D \in \mathcal{D} \mid \lim_{x \in D, |x| \rightarrow \infty} m(D \cap B(x, 1)) = 0 \right\}, \]
where \( m \) denotes the Lebesgue measure on \( \mathbb{R}^d \) and \( B(x, 1) \) the open ball with center \( x \) and radius 1. Let \( X \) be the symmetric \( \alpha \)-stable process, the Markov process generated by \( (-\Delta)^{\alpha/2} \) \( (0 < \alpha \leq 2) \). We can show by the same argument as in [22, Lemma 3.3] that if an open set \( D \) belongs to \( \mathcal{D}_0 \), then the absorbing process \( X^D \) on \( D \) is in Class (T). For the intrinsic ultracontractivity of \( X^D \), refer to [1, 12, 14]. In particular, it is shown in [12] that for \( 0 < \alpha < 2 \) \( X^D \) is intrinsic ultracontractive for any bounded open set \( D \). As a result,
\[ \lim_{t \rightarrow \infty} \mathbb{P}^D_x (X_t \in B \mid t < \tau_D) = \nu^{\phi_0} (B), \ \forall B \in \mathcal{B}(D). \]
In [14, Example 4], the author gives an example of open set \( D \) such that \( m(D) = \infty \) and \( X^D \) is intrinsic ultracontractive.
Example 3.3 Let $V$ be a positive function in the local Kato class. If

$$\lim_{|x| \to \infty} m(\{x \in \mathbb{R}^d \mid V(x) \leq M\}) = 0 \text{ for any } M > 0,$$

then the subprocess of the BM by $\exp \left( - \int_0^t V(B_s) \, ds \right)$ is in Class (T) [22]. For the intrinsic ultracontractivity of Schrödinger semigroups, refer to [11].

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