APPROXIMATION OF THE WILLMORE ENERGY BY A DISCRETE GEOMETRY MODEL

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Abstract. We prove that a certain discrete energy for triangulated surfaces, defined in the spirit of discrete differential geometry, converges to the Willmore energy in the sense of Γ-convergence. Variants of this discrete energy have been discussed before in the computer graphics literature.

1. Introduction

The numerical analysis of elastic shells is a vast field with important applications in physics and engineering. In most cases, it is carried out via the finite element method. In the physics and computer graphics literature, there have been suggestions to use simpler methods based on discrete differential geometry [MDSB03, BSSZ08]. Discrete differential geometry of surfaces is the study of triangulated polyhedral surfaces. (The epithet “simpler” has to be understood as “easier to implement”.) We mention in passing that models based on triangulated polyhedral surfaces have applications in materials science beyond the elasticity of thin shells. E.g., recently these models have been used to describe defects in nematic liquids on thin shells [CS18]. This amounts to a generalization to arbitrary surfaces of the discrete-to-continuum analysis for the XY model in two dimensions that leads to Ginzburg-Landau type models in the continuum limit [AC09, ADLGPH].

Let us describe some of the methods mentioned above in more detail. Firstly, there are the so-called polyhedral membrane models which in fact can be used for a whole array of physical and engineering problems (see e.g. the review [DP98]). In the context of plates and shells, the so-called Seung-Nelson model [SN88] is widely used. This associates membrane and bending energy to a piecewise affine map \( y : \mathbb{R}^2 \supset U \rightarrow \mathbb{R}^3 \), where the pieces are determined by a triangulation \( \mathcal{T} \) of the polyhedral domain \( U \). The bending energy is given by

\[
E^{SN}(y) = \sum_{K,L} |n(K) - n(L)|^2,
\]

where the sum runs over those unordered pairs of triangles \( K, L \) in \( \mathcal{T} \) that share an edge, and \( n(K) \) is the surface normal on the triangle \( K \). In [SN88], it has been argued that for a fixed limit deformation \( y \), the energy (1) should approximate the Willmore energy

\[
E^{W}(y) = \int_{y(U)} |Dn|^2 \, d\mathcal{H}^2
\]
when the grid size of the triangulation $T$ is sent to 0, and the argument of the discrete energy \([1]\) approximates the (smooth) map $y$. In \([2]\) above, $n$ denotes the surface normal and $\mathcal{H}^2$ the two-dimensional Hausdorff measure. These statements have been made more precise in \([SF12]\), where it has been shown that the result of the limiting process depends on the used triangulations. In particular, the following has been shown in this reference: For $j \in \mathbb{N}$, let $T_j$ be a triangulation of $U$ consisting of equilateral triangles such that one of the sides of each triangle is parallel to the $x_1$-direction, and such that the triangle size tends 0 as $j \to \infty$. Then the limit energy reads

$$E^{FS}(y) = \frac{2}{\sqrt{3}} \int_U \left( g_{11}(h_{11}^2 + 2h_{12}^2 - 2h_{11}h_{22} + 3h_{22}^2) - 8g_{12}h_{11}h_{12} + 2g_{22}(h_{11}^2 + 3h_{12}^2) \right) \left( \det g_{ij} \right)^{-1} \, dx,$$

where

$$g_{ij} = \partial_i y \cdot \partial_j y$$
$$h_{ij} = n \cdot \partial_{ij} y.$$

More precisely, if $y \in C^2(U)$ is given, then the sequence of maps $y_j$ obtained by piecewise affine interpolation of the values of $y$ on the vertices of the triangulations $T_j$ satisfies

$$\lim_{j \to \infty} E^{SN}(y_j) = E^{FS}(y).$$

Secondly, there is the more recent approach to using discrete differential geometry for shells pioneered by Grinspun et al. \([GHDS03]\). Their energy does not depend on an immersion $y$ as above, but is defined directly on triangulated surfaces. Given such a surface $T$, the energy is given by

$$E^{GHDS}(T) = \sum_{K,L} \frac{l_{KL}}{d_{KL}} \alpha_{KL}^2$$

where the sum runs over unordered pairs of neighboring triangles $K, L \in T$, $l_{KL}$ is the length of the interface between $K, L$, $d_{KL}$ is the distance between the centers of the circumcircles of $K, L$, and $\alpha_{KL}$ is the difference of the angle between $K, L$ and $\pi$, or alternatively the angle between the like-oriented normals $n(K)$ and $n(L)$, i.e. the dihedral angle.

In \([Bob05]\), Bobenko has defined an energy for piecewise affine surfaces $T$ that is invariant under conformal transformations. It is defined via the circumcircles of triangles in $T$, and the external intersection angles of circumcircles of neighboring triangles. Denoting this intersection angle for neighboring triangles $K, L$ by $\beta_{KL}$, the energy reads

$$E^B(T) = \sum_{K,L} \beta_{KL} - \pi \# \text{Vertices}(T).$$

Here Vertices($T$) denotes the vertices of the triangulation $T$, the sum is again over nearest neighbors. It has been shown in \([Bob08]\) that this energy is the same as \((3)\) up to terms that vanish as the size of triangles is sent to zero (assuming sufficient smoothness of the limiting surface). The reference \([Bob08]\) also contains an analysis of the energy for this limit. If the limit surface is smooth, and it is approximated by triangulated surfaces $T_\varepsilon$ with maximal triangle size $\varepsilon$ that satisfy a number of technical assumptions,
then the Willmore energy of the limit surface is smaller than or equal to the limit of the energies (3) for the approximating surfaces, see Theorem 2.12 in [Bob08]. The technical assumptions are

- each vertex in the triangulation $T_\varepsilon$ is connected to six other vertices by edges,
- the lengths of the sides of the hexagon formed by six triangles that share one vertex differ by at most $O(\varepsilon^4)$,
- neighboring triangles are congruent up to $O(\varepsilon^3)$.

Furthermore, it is stated that the limit is achieved if additionally the triangulation approximates a “curvature line net”.

The purpose of this present paper is to generalize this convergence result, and put it into the framework of $\Gamma$-convergence [Bra02, Dal93]. Instead of fixing the vertices of the polyhedral surfaces to lie on the limiting surfaces, we are going to assume that the convergence is weakly * in $W^{1,\infty}$ as graphs. This approach allows to completely drop the assumptions on the connectivity of vertices in the triangulations, and the assumptions of congruence – we only need to require a certain type of regularity of the triangulations that prevents the formation of small angles.

We are going to work with the energy

$$E(T) = \sum_{K,L} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2,$$

which in a certain sense is equivalent to (3) and (4) in the limit of vanishing triangle size, see the arguments from [Bob08] and Remark 1.2 (ii) below.

To put this approach into its context in the mathematical literature, we point out that it is another instance of a discrete-to-continuum limit, which has been a popular topic in mathematical analysis over the last few decades. We mention the seminal papers [BLBL02, AC04] and the fact that a variety of physical settings have been approached in this vein, such as spin and lattice systems [BG02, AC09], bulk elasticity [ACG11, BS13], thin films [ABC08, Sch08], magnetism [Sch05, SS09], and many more.

The topology that we are going to use in our $\Gamma$-convergence statement is much coarser than the one that corresponds to Bobenko’s convergence result; however it is not the “natural” one that would yield compactness from finiteness of the energy (5) alone. For a discussion of why we do not choose the latter see Remark 1.2 (i) below. Our topology is instead defined as follows:

Let $M$ be some fixed compact oriented two-dimensional $C^\infty$ submanifold of $\mathbb{R}^3$ with normal $n_M : M \to S^2$. Let $h_j \in W^{1,\infty}(M)$, $j = 1, 2, \ldots$, such that $\|h_j\|_{W^{1,\infty}} < C$ and $\|h_j\|_{\infty} < \delta(M)/2$ (where $\delta(M)$ is the radius of injectivity of $M$, see Definition 2.9 below) such that $T_j := \{x + h_j(x)n_M(x) : x \in M\}$ are triangulated surfaces (see Definition 2.2 below). We say $T_j \to T := \{x + h(x)n_M(x) : x \in M\}$ if $h_j \to h$ in $W^{1,p}(M)$ for all $1 \leq p < \infty$. Our main theorem, Theorem 1.1 below, is a $\Gamma$-convergence result in this topology. The regularity assumptions that we impose on the triangulated surfaces under considerations are “$\zeta$-regularity” and the “Delaunay property”. The definition of these concepts can be found in Definition 2.2 below.
Theorem 1.1. (o) Compactness: Let $\zeta > 0$, and let $h_j$ be a bounded sequence in $W^{1,\infty}(M)$ such that $T_j = \{x + h_j(x) n_M(x) : x \in M\}$ is a $\zeta$-regular triangulated surface and $\|h_j\|_{\infty} \leq \delta(M)/2$ for $j \in \mathbb{N}$ with $\limsup_{j \to \infty} E(T_j) < \infty$. Then there exists a subsequence $h_{j_k}$ and $h \in W^{2,2}(M)$ such that $h_{j_k} \to h$ in $W^{1,p}(M)$ for every $1 \leq p < \infty$.

(i) Lower bound: Let $\zeta > 0$. Assume that for $j \in \mathbb{N}$, $h_j \in W^{1,\infty}(M)$ with $\|h_j\|_{\infty} \leq \delta(M)/2$, $T_j := \{x + h_j(x) n_M(x) : x \in M\}$ is a $\zeta$-regular triangulated surface fulfilling the Delaunay property, and that $T_j \to S = \{x + h(x) n_M(x) : x \in M\}$ for $j \to \infty$. Then

$$\liminf_{j \to \infty} E(T_j) \geq \int_S |Dn_S|^2 \, dH^2.$$

(ii) Upper bound: Let $h \in W^{1,\infty}(M)$ with $\|h\|_{\infty} \leq \delta(M)/2$ and $S = \{x + h(x) n_M(x) : x \in M\}$. Then there exists $\zeta > 0$ and a sequence $(h_j)_{j \in \mathbb{N}} \subset W^{1,\infty}(M)$ such that $T_j := \{(x + h_j(x) n_M(x) : x \in M\}$ is a $\zeta$-regular triangulated surface satisfying the Delaunay property for each $j \in \mathbb{N}$, and we have $T_j \to S$ for $j \to \infty$ and

$$\lim_{j \to \infty} E(T_j) = \int_S |Dn_S|^2 \, dH^2.$$

Remark 1.2. (i) We are not able to derive a convergence result in a topology that yields convergence from boundedness of the energy alone. Such an approach would necessitate the interpretation of the surfaces as varifolds or currents. To the best of our knowledge, the theory of integral functionals on varifolds (see e.g. [Men14, Hut86, Man96]) is not developed to the point to allow for a treatment of this question. In particular, there does not exist a sufficiently general theory of lower semicontinuity of integral functionals for varifold-function pairs.

(ii) We can state analogous results based on the energy functionals (3, 4). To do so, our proofs only need to be modified slightly: As soon as we have reduced the situation to the graph case (which we do by assumption), the upper bound construction can be carried out as here; the smallness of the involved dihedral angles assures that the arguments from [Bob05] suffice to carry through the proof. Concerning the lower bound, we also reduce to the case of small dihedral angles by a blow-up procedure around Lebesgue points of the derivative of the surface normal of the limit surface. (Additionally, one can show smallness of the contribution of a few pairs of triangles whose dihedral angle is not small.) Again, the considerations from [Bob05] allow for a translation of our proof to the case of the energy functionals (3), (4).

(iii) As we will show in Chapter 5, we need to require the Delaunay property in order to obtain the lower bound statement. Without this requirement, we will show that a hollow cylinder can be approximated by triangulated surfaces with arbitrarily low energy, see Proposition 5.7.

(iv) Much more general approximations of surfaces by discrete geometrical objects have recently been proposed in [BLM17, BLM18, BLM19], based on tools from the theory of varifolds.

Plan of the paper. In Section 2 we will fix definitions and make some preliminary observations on triangulated surfaces. The proofs of the compactness and lower bound...
part will be developed in parallel in Section 3. The upper bound construction is carried out in Section 4 and in Section 5 we demonstrate that the requirement of the Delaunay property is necessary in order to obtain the lower bound statement.

2. Definitions and preliminaries

2.1. Some general notation.

**Notation 2.1.** For a two-dimensional submanifold $M \subset \mathbb{R}^3$, the tangent space of $M$ in $x \in M$ is denoted by $T_xM$. For functions $f : M \to \mathbb{R}$, we denote their gradient by $\nabla f \in T_xM$; the norm $| \cdot |$ on $T_xM \subset \mathbb{R}^3$ is the Euclidean norm inherited from $\mathbb{R}^3$. For $1 \leq p \leq \infty$, we denote by $W^{1,p}(M)$ the space of functions $f \in L^p(M)$ such that $\nabla f \in L^p(M; \mathbb{R}^3)$, with norm

$$\|h\|_{W^{1,p}(M)} = \|f\|_{L^p(M)} + \|\nabla f\|_{L^p(M)}.$$  

For $U \subset \mathbb{R}^n$ and a function $f : U \to \mathbb{R}$, we denote the graph of $f$ by

$$\text{Gr } f = \{(x, f(x)) : x \in U\} \subset \mathbb{R}^{n+1}.$$  

For $x_1, \ldots, x_m \subset \mathbb{R}^k$, the convex hull of $\{x_1, \ldots, x_m\}$ is denoted by

$$[x_1, \ldots, x_m] = \left\{ \sum_{i=1}^{m} \lambda_i x_i : \lambda_i \in [0, 1] \text{ for } i = 1, \ldots, m, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$  

We will identify $\mathbb{R}^2$ with the subspace $\mathbb{R}^2 \times \{0\}$ of $\mathbb{R}^3$. The $d$-dimensional Hausdorff measure is denoted by $\mathcal{H}^d$, the $k$-dimensional Lebesgue measure by $\mathcal{L}^k$. The symbol “C” will be used as follows: A statement such as “$f \leq C(\alpha)g$” is shorthand for “there exists a constant $C > 0$ that only depends on $\alpha$ such that $f \leq Cg$”. The value of $C$ may change within the same line. For $f \leq Cg$, we also write $f \lesssim g$.

2.2. Triangulated surfaces: Definitions.

**Definition 2.2.** (i) A **triangle** is the convex hull $[x, y, z] \subset \mathbb{R}^3$ of three points $x, y, z \in \mathbb{R}^3$. A **regular** triangle is one where $x, y, z$ are not colinear, or equivalently $\mathcal{H}^2([x, y, z]) > 0$.

(ii) A **triangulated surface** is a finite collection $\mathcal{T} = \{K_i : i = 1, \ldots, N\}$ of regular triangles $K_i = [x_i, y_i, z_i] \subset \mathbb{R}^3$ so that $\bigcup_{i=1}^{N} K_i \subset \mathbb{R}^3$ is a topological two-dimensional manifold with boundary; and the intersection of two different triangles $K, L \in \mathcal{T}$ is either empty, a common vertex, or a common edge.

We identify $\mathcal{T}$ with its induced topological manifold $\bigcup_{i=1}^{N} K_i \subset \mathbb{R}^3$ whenever convenient. We say that $\mathcal{T}$ is **flat** if there exists an affine subspace of $\mathbb{R}^3$ that contains $\mathcal{T}$.

(iii) The **size** of the triangulated surface, denoted $\text{size}(\mathcal{T})$, is the maximum diameter of all its triangles.

(iv) The triangulated surface $\mathcal{T}$ is called **$\zeta$-regular**, with $\zeta > 0$, if the minimum angle in all triangles is at least $\zeta$ and $\min_{K \in \mathcal{T}} \text{diam}(K) \geq \zeta \text{size}(\mathcal{T})$.

(v) The triangulated surface satisfies the **Delaunay** property if for every triangle $K = [x, y, z] \in \mathcal{T}$ the following property holds: Let $B(q, r) \subset \mathbb{R}^3$ be the smallest
ball such that \( \{x, y, z\} \subset \partial B(q, r) \). Then \( B(q, r) \) contains no vertex of any triangle in \( T \). The point \( q = q(K) \in \mathbb{R}^3 \) is called the circumcenter of \( K \), \( B(q, r) \) its circumball with circumradius \( r(K) \), and \( \partial B(q, r) \) its circumsphere.

Note that triangulated surfaces have normals defined on all triangles and are compact and rectifiable. For the argument of the circumcenter map \( q \), we do not distinguish between triples of points \( (a, b, c) \in \mathbb{R}^3 \times 3 \) and the triangle \([a, b, c]\) (presuming \([a, b, c]\) is a regular triangle).

**Notation 2.3.** If \( T = \{K_i : i = 1, \ldots, N\} \) is a triangulated surface, and \( g : T \to \mathbb{R} \), then we identify \( g \) with the function \( \bigcup_{i=1}^N K_i \to \mathbb{R} \) that is constant on the (relative) interior of each triangle \( K \), and equal to 0 on \( K \cap L \) for \( K \neq L \in T \). In particular we may write in this case \( g(x) = g(K) \) for \( x \in \text{int} \ K \).

**Definition 2.4.** Let \( T \) be a triangulated surface and \( K, L \in T \). We set

\[
\begin{align*}
l_{KL} &:= \mathcal{H}^1(K \cap L) \\
d_{KL} &:= |q(K) - q(L)|
\end{align*}
\]

If \( K, L \) are adjacent, i.e. if \( l_{KL} > 0 \), we may define \( |n(K) - n(L)| \in \mathbb{R} \) as the norm of the difference of the normals \( n(K), n(L) \in S^2 \) which share an orientation, i.e. \( 2 \sin \frac{\alpha_{KL}}{2} \), where \( \alpha_{KL} \) is the dihedral angle between the triangles, see Figure 1a. The discrete bending energy is then defined as

\[
E(T) = \sum_{K, L \in T} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2.
\]

Here, the sum runs over all unordered pairs of triangles. If \( |n(K) - n(L)| = 0 \) or \( l_{KL} = 0 \), the energy density is defined to be 0 even if \( d_{KL} = 0 \). If \( |n(K) - n(L)| > 0 \), \( l_{KL} > 0 \) and \( d_{KL} = 0 \), the energy is defined to be infinite.

**Figure 1.** (A) The dihedral angle \( \alpha_{KL} \) for triangles \( K, L \). It is related to the norm of the difference between the normals via \( |n(K) - n(L)| = 2 \sin \frac{\alpha_{KL}}{2} \). (B) Definitions of \( d_{KL}, l_{KL} \).
Notation 2.5. Let $H$ be an affine subplane of $\mathbb{R}^3$. For triangles $K, L \subset H$ that share an edge and $v \in \mathbb{R}^3$ parallel to $H$, we define the function $\mathbb{I}^v_{KL} : H \to \{0, 1\}$ as $\mathbb{I}^v_{KL}(x) = 1$ if and only if $[x, x + v] \cap (K \cap L) \neq \emptyset$. If the intersection $K \cap L$ does not consist of a single edge, then $\mathbb{I}^v_{KL} \equiv 0$. Furthermore, we let $\nu_{KL} \in \mathbb{R}^3$ denote the unit vector parallel to $H$ orthogonal to the shared edge of $K, L$ pointing from $K$ to $L$ and

$$\theta^v_{KL} = \frac{\nu_{KL} \cdot v}{|v|}.$$ 

See Figure 2 for an illustration of Notation 2.5.

We collect the notation that we have introduced for triangles and triangulated surfaces for the reader’s convenience in abbreviated form: Assume that $K = [a, b, c]$ and $L = [b, c, d]$ are two regular triangles in $\mathbb{R}^3$. Then we have the following notation:

- $q(K)$: center of the smallest circumball for $K$
- $r(K)$: radius of the smallest circumball for $K$
- $d_{KL} = |q(K) - q(L)|$
- $l_{KL}$: length of the shared edge of $K, L$
- $n(K)$: unit vector normal to $K$

The following are defined if $K, L$ are contained in an affine subspace $H$ of $\mathbb{R}^3$, and $v$ is a vector parallel to $H$:

- $\nu_{KL}$: unit vector parallel to $H$ orthogonal to the shared edge of $K, L$ pointing from $K$ to $L$
- $\theta^v_{KL} = \frac{\nu_{KL} \cdot v}{|v|}$
- $\mathbb{I}^v_{KL}$: function defined on $H$, with value one if $[x, x + v] \cap (K \cap L) \neq \emptyset$, zero otherwise.
2.3. Triangulated surfaces: Some preliminary observations. For two adjacent triangles \( K, L \in \mathcal{T} \), we have \( d_{KL} = 0 \) if and only if the vertices of \( K \) and \( L \) have the same circumsphere. The following lemma states that for noncospherical configurations, \( d_{KL} \) grows linearly with the distance between the circumsphere of \( K \) and the opposite vertex in \( L \).

**Lemma 2.6.** The circumcenter map \( q : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^3 \) is \( C^1 \) and Lipschitz when restricted to \( \zeta \)-regular triangles. For two adjacent triangles \( K = [x, y, z], L = [x, y, p] \), we have that

\[
d_{KL} \geq \frac{1}{2} |q(K) - p| - r(K)|.
\]

**Proof.** The circumcenter \( q = q(K) \in \mathbb{R}^3 \) of the triangle \( K = [x, y, z] \) is the solution to the linear system

\[
\begin{align*}
(q - x) \cdot (y - x) &= \frac{1}{2} |y - x|^2 \\
(q - x) \cdot (z - x) &= \frac{1}{2} |z - x|^2 \\
(q - x) \cdot ((z - x) \times (y - x)) &= 0.
\end{align*}
\]

Thus, the circumcenter map \( (x, y, z) \mapsto q \) is \( C^1 \) when restricted to \( \zeta \)-regular \( K \). To see that the map is globally Lipschitz, it suffices to note that it is 1-homogeneous in \((x, y, z)\). For the second point, let \( s = q(L) \in \mathbb{R}^3 \) be the circumcenter of \( L \). Then by the triangle inequality, we have

\[
\begin{align*}
|p - q| &\leq |p - s| + |s - q| = |x - s| + |s - q| \leq |x - q| + 2|s - q| = r + 2d_{KL}, \\
|p - q| &\geq |p - s| - |s - q| = |x - s| - |s - q| \geq |x - q| - 2|s - q| = r - 2d_{KL}.
\end{align*}
\]

This completes the proof. \( \square \)

**Lemma 2.7.** Let \( \zeta > 0 \), and \( a, b, c, d \in \mathbb{R}^2 \) such that \( K = [a, b, c] \) and \( L = [b, c, d] \) are \( \zeta \)-regular.

(i) We have that

\[
\int_{\mathbb{R}^2} \mathbf{1}^v_{KL}(x) \, dx = |v| l_{KL} \theta_{KL}.
\]

(ii) Let \( \delta > 0, v, w \in \mathbb{R}^2, \bar{v} = (v, v \cdot w) \in \mathbb{R}^3, \bar{a} = (a, a \cdot w) \in \mathbb{R}^3 \) and \( \bar{b}, \bar{c}, \bar{d} \in \mathbb{R}^3 \) defined analogously. Let \( \bar{K} = [\bar{a}, \bar{b}, \bar{c}], \bar{L} = [\bar{b}, \bar{c}, \bar{d}] \). Then

\[
\int_{\mathbb{R}^2} \mathbf{1}^v_{KL}(x) \, dx = \frac{|\bar{v}|}{\sqrt{1 + |w|^2}} l_{KL} \theta_{KL}.
\]

**Proof.** The equation (i) follows from the fact that \( \mathbf{1}^v_{KL} \) is the characteristic function of a parallelogram, see Figure 2. To prove (ii) it suffices to observe that \( \int_{\mathbb{R}^2} \mathbf{1}^v_{KL}(x) \sqrt{1 + w^2} \, dx \) is the volume of the parallelogram from (i) pushed forward by the map \( h(x) = (x, x \cdot w) \), see Figure 3. \( \square \)
2.4. Graphs over manifolds.

**Assumption 2.8.** We assume $M \subset \mathbb{R}^3$ is an oriented compact two-dimensional $C^\infty$-submanifold of $\mathbb{R}^3$.

This manifold will be fixed in the following. We denote the normal of $M$ by $n_M : M \to S^2$, and the second fundamental form at $x_0 \in M$ is denoted by $S_M(x_0) : T_{x_0}M \to T_{x_0}M$.

**Definition 2.9.** The radius of injectivity $\delta(M) > 0$ of $M$ is the largest number such that the map $\phi : M \times (-\delta(M), \delta(M)) \to \mathbb{R}^3$, $(x, h) \mapsto x + hn_M(x)$ is injective and the operator norm of $\delta(M)S_M(x) \in L(T_{x_0}M)$ is at most 1 at every $x \in M$.

We define a graph over $M$ as follows:

**Definition 2.10.**

(i) A set $M_h = \{x + h(x)n_M(x) : x \in M\}$ is called a graph over $M$ whenever $h : M \to \mathbb{R}$ is a continuous function with $\|h\|_\infty \leq \delta(M)/2$.

(ii) The graph $M_h$ is called a ($Z$-)Lipschitz graph (for $Z > 0$) whenever $h$ is ($Z$-)Lipschitz, and a smooth graph whenever $h$ is smooth.

(iii) A set $N \subset B(M, \delta(M)/2)$ is said to be locally a tangent Lipschitz graph over $M$ if for every $x_0 \in M$ there exists $r > 0$ and a Lipschitz function $h : (x_0 + T_{x_0}M) \cap B(x_0, r) \to \mathbb{R}$ such that the intersection of $N$ with the cylinder $C(x_0, r, \delta(M)/2)$ over $(x_0 + T_{x_0}M) \cap B(x_0, r)$ with height $\delta(M)$ in both directions of $n_M(x_0)$, where

$$C(x_0, r, s) := \{x + tn_M(x_0) : x \in (x_0 + T_{x_0}M) \cap B(x_0, r), t \in [-s, s]\},$$

is equal to the graph of $h$ over $T_{x_0}M \cap B(x_0, r)$,

$$N \cap C\left(x_0, r, \frac{\delta(M)}{2}\right) = \{x + h(x)n_M(x_0) : x \in (x_0 + T_{x_0}M) \cap B(x_0, r)\}.$$  

**Lemma 2.11.** Let $N \subset B(M, \delta(M)/2)$ be locally a tangent Lipschitz graph over $M$. Then $N$ is a Lipschitz graph over $M$.

**Proof.** By Definition 2.10 (iii), we have that for every $x \in M$, there exists exactly one element

$$x' \in N \cap (x + n_M(x_0)|[-\delta(M), \delta(M)]).$$

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**Figure 3.** The parallelogram pushed forward by an affine map $x \mapsto (x, x \cdot w)$. 
We write $h(x) := (x' - x) \cdot n_M(x)$, which obviously implies $N = M_h$. For every $x_0 \in M$ there exists a neighborhood of $x_0$ such that $h$ is Lipschitz continuous in this neighborhood by the locally tangent Lipschitz property and the regularity of $M$. The global Lipschitz property for $h$ follows from the local one by a standard covering argument. \hfill \Box

Lemma 2.12. Let $h_j \in W^{1,\infty}(M)$ with $\|h_j\|_\infty \leq \delta(M)/2$ and $h_j \rightharpoonup h \in W^{1,\infty}(M)$ for $j \to \infty$. Then for every point $x \in M$, there exists a neighborhood $V \subset x + T_xM$, a Euclidean motion $R$ with $U := R(x + T_xM) \subset \mathbb{R}^2$, functions $\tilde{h}_j : U \to \mathbb{R}$ and $\tilde{h} : U \to \mathbb{R}$ such that $\tilde{h}_j \rightharpoonup \tilde{h}$ in $W^{1,\infty}(U)$ and

$$R^{-1}\text{Gr} \tilde{h}_j \subset M_{h_j}$$
$$R^{-1}\text{Gr} \tilde{h} \subset M_h.$$

Proof. This follows immediately from our assumption that $M$ is $C^2$ and the boundedness of $\|\nabla h_j\|_{L^\infty}$. \hfill \Box

3. Proof of compactness and lower bound

Notation 3.1. If $U \subset \mathbb{R}^2$, $T$ is a flat triangulated surface $T \subset U$, $h : U \to \mathbb{R}$ is Lipschitz, and $K = [a, b, c] \in T$, then we write

$$h_\ast K = [(a, h(a)), (b, h(b)), (c, h(c))].$$

We denote by $h_\ast T$ for the triangulated surface defined by

$$K \in T \iff h_\ast K \in h_\ast T.$$ 

For an illustration Notation 3.1, see Figure 4.

![Figure 4](image)

**Figure 4.** Definition of the push forward of a triangulation $T \subset \mathbb{R}^2$ by a map $h : \mathbb{R}^2 \to \mathbb{R}$.

Lemma 3.2. Let $U \subset \mathbb{R}^2$, let $T$ be a flat triangulated surface with $U \subset T \subset \mathbb{R}^2$, let $h$ be a Lipschitz function $U \to \mathbb{R}$ that is affine on each triangle of $T$, $T^\ast = h_\ast T$, let $g$ be a function that is constant on each triangle of $T$, $v \in \mathbb{R}^2$, $U^v = \{x \in \mathbb{R}^2 : [x, x + v] \subset U\}$, and $W \subset U^v$.  

10
Using these estimates and Lemma 2.7 (i), we obtain

\[ \int_W |g(x + v) - g(x)|^2 dx \]

\[ \leq |v| \left( \sum_{K,L \in \mathcal{T}} \frac{l_{K,L^*}}{d_{K,L^*}} |g(K) - g(L)|^2 \right) \max_{x \in W} \sum_{K,L \in \mathcal{T}} \mathbf{1}^v_{KL}(x) \frac{\theta^v_{KL} I_{KL}^d L_{K,L^*}}{l_{K,L^*}}, \]

where we have written \( K^* = h_* K, L^* = h_* L \) for \( K, L \in \mathcal{T} \).

(ii) Let \( w \in \mathbb{R}^2 \), and denote by \( K, L \) the triangles \( K, L \) pushed forward by the map \( x \mapsto (x, x \cdot w) \). Then

\[ \int_W |g(x + v) - g(x)|^2 dx \]

\[ \leq \frac{|v|}{\sqrt{1 + |w|^2}} \left( \sum_{K,L \in \mathcal{T}} \frac{l_{K,L^*}}{d_{K,L^*}} |g(K) - g(L)|^2 \right) \max_{x \in W} \sum_{K,L \in \mathcal{T}} \mathbf{1}^v_{KL}(x) \frac{\theta^v_{KL} I_{KL}^d L_{K,L^*}}{l_{K,L^*}} \]

**Proof.** By the Cauchy-Schwarz inequality, for \( x \in W \), we have that

\[ |g(x + v) - g(x)|^2 \leq \left( \sum_{K,L \in \mathcal{T}} \mathbf{1}^v_{KL}(x)|g(K) - g(L)| \right)^2 \]

\[ \leq \left( \sum_{K,L \in \mathcal{T}} \frac{l_{K,L^*}}{d_{K,L^*}} \frac{l_{K,L^*}}{d_{K,L^*}} \mathbf{1}^v_{KL}(x)|g(K) - g(L)|^2 \right) \times \left( \sum_{K,L \in \mathcal{T}} \mathbf{1}^v_{KL}(x) \frac{\theta^v_{KL} I_{KL}^d L_{K,L^*}}{l_{K,L^*}} \right). \]

Using these estimates and Lemma 2.7 (i), we obtain

\[ \int_{U^v} |g(x + v) - g(x)|^2 dx \]

\[ \leq \int_{U^v} \left( \sum_{K,L \in \mathcal{T}} \mathbf{1}^v_{KL}(x) \frac{l_{K,L^*}}{d_{K,L^*}} \frac{l_{K,L^*}}{d_{K,L^*}} |g(K) - g(L)|^2 \right) \times \left( \sum_{K,L \in \mathcal{T}} \mathbf{1}^v_{KL}(x) \frac{\theta^v_{KL} I_{KL}^d L_{K,L^*}}{l_{K,L^*}} \right) dx \]

\[ \leq |v| \left( \sum_{K,L \in \mathcal{T}} \frac{l_{K,L^*}}{d_{K,L^*}} |g(K) - g(L)|^2 \right) \max_{x \in U^v} \sum_{K,L \in \mathcal{T}} \mathbf{1}^v_{KL}(x) \frac{\theta^v_{KL} I_{KL}^d L_{K,L^*}}{l_{K,L^*}}. \]

This proves (i). The claim (ii) is proved analogously, using \( \frac{\theta^v_{KL} I_{KL}^d L_{K,L^*}}{l_{K,L^*}} \) instead of \( \frac{l_{K,L^*}}{d_{K,L^*}} \) in the Cauchy-Schwarz inequality, and then Lemma 2.7 (ii). \( \square \)
In the following proposition, we will consider sequences of flat triangulated surfaces \( T_j \) with \( U \subseteq T_j \subseteq \mathbb{R}^2 \) and sequences of Lipschitz functions \( h_j : U \to \mathbb{R} \). We write \( T_j^* = (h_j)_* T_j \), and for \( K \in T_j \), we write
\[
K^* = (h_j)_* K.
\]

**Proposition 3.3.** Let \( U, U' \subseteq \mathbb{R}^2 \) be open, \( \zeta > 0 \), \((T_j)_{j \in \mathbb{N}} \) a sequence of flat \( \zeta \)-regular triangulated surfaces with \( U \subseteq T_j \subseteq U' \) and size \( (T_j) \to 0 \). Let \((h_j)_{j \in \mathbb{N}} \) be a sequence of Lipschitz functions \( U \to \mathbb{R} \) with uniformly bounded gradients such that \( h_j \) is affine on each triangle of \( T_j \) and the triangulated surfaces \( T_j^* = (h_j)_* T_j \) satisfy the Delaunay property.

1. Assume that
\[
h_j \overset{\lambda}{\to} h \quad \text{in } W^{1,\infty}(U'),
\]
and \( \liminf_{j \to \infty} \sum_{K, L \in T_j} \frac{l_{K^* L^*}}{d_{K^* L^*}} |n(K^*) - n(L^*)|^2 < \infty \). Then \( h \in W^{2,2}(U) \).

2. Let \( U = Q = (0, 1)^2 \), and let \((g_j)_{j \in \mathbb{N}} \) be a sequence of functions \( U' \to \mathbb{R} \) such that \( g_j \) is constant on each triangle in \( T_j \). Assume that
\[
h_j \to h \quad \text{in } W^{1,2}(U'),
\]
g \( \to g \) in \( L^2(U') \),

where \( h(x) = w \cdot x \) and \( g(x) = u \cdot x \) for some \( u, w \in \mathbb{R}^2 \). Then we have
\[
u^T (1_{2 \times 2} + w \otimes w)^{-1} u \sqrt{1 + |w|^2} \leq \liminf_{j \to \infty} \sum_{K, L \in T_j} \frac{l_{K^* L^*}}{d_{K^* L^*}} |g_j(K) - g_j(L)|^2.
\]

**Proof of (i).** We write
\[
E_j := \sum_{K, L \in T_j} \frac{l_{K^* L^*}}{d_{K^* L^*}} |n(K^*) - n(L^*)|^2.
\]

Fix \( v \in B(0, 1) \subseteq \mathbb{R}^2 \), write \( U^v = \{ x \in \mathbb{R}^2 : [x, x + v] \subseteq U \} \), and fix \( k \in \{1, 2, 3\} \). Define the function \( N_j^k : U \to \mathbb{R}^3 \) by requiring \( N_j^k(x) = n(K^*) \cdot e_k \) for \( x \in K \in T_j \). By Lemma 3.2 with \( g_j = N_j^k \), we have that
\[
\int_{U^v} |N_j^k(x + v) - N_j^k(x)|^2 \, dx \leq |v| \left( \max_{x \in U^v} \sum_{K, L \in T_j} 1_{K^* L^*} \frac{\theta_{K^* L^*}}{l_{K^* L^*}} \frac{l_{K^* L^*}}{l_{K^* L^*}} \right) E_j.
\]

Since \( h_j \) is uniformly Lipschitz, there exists a constant \( C > 0 \) such that
\[
\frac{l_{K^* L^*}}{l_{K^* L^*}} < C d_{K^* L^*}.
\]

We claim that
\[
\max_{x \in U^v} \sum_{K, L \in T_j} 1_{K^* L^*} \theta_{K^* L^*} d_{K^* L^*} \lesssim |v| + C \text{size}(T_j).
\]
Indeed, let $K_0, \ldots, K_N \in T_j$ be the sequence of triangles so that there is $i : [0, 1] \to \{1, \ldots, N\}$ non-decreasing with $x + tv \in K_{i(t)}$. We have that for all pairs $K_i, K_{i+1} \in T_j$,

$$\theta_{K_i, K_{i+1}} d_{K_i, K_{i+1}} = \left| \frac{q(K_{i+1}) - q(K_i)}{v} \right|,$$

which yields the last estimate in (10). Inserting in (9) yields

$$\int_{U_v} |N_j^k(x + v) - N_j^k(x)|^2 \, dx \leq C|v|(|v| + C \text{size}(T_j))E_j.$$

By passing to the limit $j \to \infty$ and standard difference quotient arguments, it then follows that the limit $N^k = \lim_{j \to \infty} N_j^k$ is in $W^{1,2}(U)$. Since $h$ is also in $W^{1,\infty}(U)$ and $(N^k)_{k=1,2,3} = (\nabla h, -1)/\sqrt{1 + |\nabla h|^2}$ is the normal to the graph of $h$, it follows that $h \in W^{2,2}(U)$.

\[ \square \]

**Proof of (ii).** We write

$$E_j := \sum_{K, L \in T_j} \frac{|K \cdot L|}{d_{K \cdot L}} |g_j(K) - g_j(L)|^2$$

and may assume without loss of generality that $\liminf_{j \to \infty} E_j < \infty$.

Fix $\delta > 0$. Define the set of bad triangles as

$$B^\delta := \{ K \in T_j : |\nabla h_j(K) - w| > \delta \}.$$

Fix $v \in B(0, 1)$, and write $Q^v = \{ x \in \mathbb{R}^2 : [x, x + v] \subset Q \}$. Define the set of good points as

$$A^\delta_v := \left\{ x \in Q^v : \#\{ K \in B^\delta : K \cap [x, x + v] \neq \emptyset \} \leq \frac{\delta|v|}{\text{size}(T_j)} \right\}.$$

We claim that

$$\mathcal{L}^2(Q^v \setminus A^\delta_v) \to 0 \quad \text{for } j \to \infty.$$

Indeed, let $v^\perp = (-v_2, v_1)$, and let $P_{v^\perp} : \mathbb{R}^2 \to v^\perp \mathbb{R}$ denote the projection onto the linear subspace parallel to $v^\perp$. Now by the definition of $A^\delta_v$, we may estimate

$$\int_{Q^v} |\nabla h_j - w|^2 \, dx \gtrsim \#B^\delta_j (\text{size } T_j)^2 \delta \gtrsim \frac{\mathcal{L}^2(Q \setminus A^\delta_v)}{|v| \text{size } T_j} \frac{\delta|v|}{\text{size } T_j} (\text{size } T_j)^2 \delta \gtrsim \mathcal{L}^2(Q^v \setminus A^\delta_v) \delta^2|v|,$$

and hence (13) follows by $h_j \to h$ in $W^{1,2}(Q)$. For the push-forward of $v$ under the affine map $x \mapsto (x, h(x))$, we write

$$\bar{v} = (v, v \cdot w) \in \mathbb{R}^3.$$

Also, for $K = [a, b, c] \in T_j$, we write

$$\bar{K} = [(a, a \cdot w), (b, b \cdot w), (c, c \cdot w)] = h_* K.$$
By Lemma 3.2, we have that

\[
\int_{A_j^{\delta,v}} |g_j(x + v) - g_j(x)|^2 \, dx 
\]

\[
\leq \frac{|\bar{v}|}{\sqrt{1 + |w|^2}} \left( \max_{x \in A_j^{\delta,v}} \sum_{K,L \in T_j} \mathds{1}_{KL}(x) \frac{\theta^0_{KL} l_{KL} d_{K^*L^*}}{l_{K^*L^*}} \right) E_j. 
\]

(14)

We claim that

\[
\max_{x \in A_j^{\delta,v}} \sum_{K,L \in T_j} \mathds{1}_{KL}(x) \frac{\theta^0_{KL} l_{KL} d_{K^*L^*}}{l_{K^*L^*}} \leq (1 + C\delta) \left( |\bar{v}| + C \text{size}(T_j) \right). 
\]

(15)

Indeed, Let \(K_0, \ldots, K_N \in T_j\) be the sequence of triangles so that there is \(i : [0,1] \to \{1, \ldots, N\}\) non-decreasing with \(x + tv \in K_i(t)\).

For all pairs \(K_i, K_{i+1} \in T_j\) we have

\[
\theta^0_{K_iK_{i+1}} d_{K_iK_{i+1}} = (q(K_{i+1}) - q(K_i)) \cdot \frac{\bar{v}}{|\bar{v}|}. 
\]

(16)

Also, we have that for \(K_i, K_{i+1} \in T_j \setminus B_j^\delta\),

\[
l_{K_i^*K_{i+1}^*} d_{K_i^*K_{i+1}^*} \leq 1 + C\delta.
\]

Hence

\[
\sum_{i : \{K_i, K_{i+1}\} \cap B_j^\delta = \emptyset} \frac{\theta^0_{K_iK_{i+1}} l_{K_iK_{i+1}^*} d_{K_i^*K_{i+1}^*}}{l_{K_i^*K_{i+1}^*}} \leq (1 + C\delta) \sum_{i : \{K_i, K_{i+1}\} \cap B_j^\delta = \emptyset} \left( (q(K_{i+1}) - q(K_i)) \cdot \frac{\bar{v}}{|\bar{v}|} \right). 
\]

(17)

If one of the triangles \(K_i, K_{i+1}\) is in \(B_j^\delta\), then we may estimate

\[
\left| (q(K_{i+1}) - q(K_i)) \cdot \frac{\bar{v}}{|\bar{v}|} \right| \leq C \text{size } T_j.
\]

Since there are few bad triangles along \([x, x + v]\), we have, using \(x \in A_j^{\delta,v}\),

\[
\sum_{i : \{K_i, K_{i+1}\} \cap B_j^\delta \neq \emptyset} \frac{\theta^0_{K_iK_{i+1}} l_{K_iK_{i+1}^*} d_{K_i^*K_{i+1}^*}}{l_{K_i^*K_{i+1}^*}} - (q(K_{i+1}) - q(K_i)) \cdot \frac{\bar{v}}{|\bar{v}|} \leq C \# \{K \in B_j^\delta : K \cap [x, x + v] \neq \emptyset\} \text{size}(T_j) \leq C\delta|\bar{v}|. 
\]

(18)
Combining (17) and (18) yields
\[
N - 1 \sum_{i=0}^{N-1} \frac{l_{K_i} K_{i+1}^* l_{K_{i+1}^*}}{l_{K_i} K_{i+1}^*} \leq (1 + C \delta) \sum_{i=0}^{N-1} (q(K_{i+1}) - q(K_i)) \cdot \frac{\bar{v}}{|\bar{v}|} + C \delta |\bar{v}|
\]
\[
= (1 + C \delta)(q(K_N) - q(K_0)) \cdot \frac{\bar{v}}{|\bar{v}|} + C \delta |\bar{v}|
\]
\[
\leq (1 + C \delta) (|\bar{v}| + C \text{size}(T_j)).
\]
This proves (15).

Inserting (15) in (14), and passing to the limits \(j \to \infty\) and \(\delta \to 0\), we obtain
\[
|v \cdot u|^2 \leq \liminf_{j \to \infty} E_j.
\]

Now let
\[
u := (I_{2 \times 2}, w)^T (I_{2 \times 2} + w \otimes w)^{-1} u.
\]
Then we have \(|\nu \cdot \bar{v}| = |u \cdot v|\) and hence
\[
|\nu|^2 = \sup_{v \in \mathbb{R}^2 \setminus \{0\}} \frac{|u \cdot \bar{v}|^2}{|\bar{v}|^2} \leq \frac{1}{\sqrt{1 + |w|^2}} \liminf_{j \to \infty} E_j.
\]
This proves the proposition. \(\square\)

3.1. Proof of compactness and lower bound in Theorem 1.1

Proof of Theorem 1.1 (i). For a subsequence (no relabeling), we have that \(h_j \rightharpoonup h\) in \(W^{1,\infty}(M)\). By Lemma 2.12, \(T_j\) may be locally represented as the graph of a Lipschitz function \(\tilde{h}_j : U \to \mathbb{R}\), and \(M_h\) as the graph of a Lipschitz function \(\tilde{h} : U \to \mathbb{R}\), where \(U \subset \mathbb{R}^2\) and \(\tilde{h}_j \rightharpoonup \tilde{h}\) in \(W^{1,\infty}(U)\).

It remains to prove that \(\tilde{h} \in W^{2,2}(U)\). Since the norm of the gradients are uniformly bounded, \(\|\nabla \tilde{h}_j\|_{L^\infty(U)} < C\), we have that the projections of \(T_j\) to \(U\) are (uniformly) regular flat triangulated surfaces. Hence by Proposition 3.3 (i), we have that \(\tilde{h} \in W^{2,2}(U)\). \(\square\)

Proof of Theorem 1.1 (ii). Let \(\mu_j = \sum_{K,L \in T_j} \frac{1}{d_{KL}} |n(K) - n(L)|^2 \mathcal{H}^1|_{K \cap L} \in \mathcal{M}_+(\mathbb{R}^3)\). Note that either a subsequence of \(\mu_j\) converges narrowly to some \(\mu \in \mathcal{M}_+(M_h)\) or there is nothing to show. We will show in the first case that
\[
(19) \quad \frac{d\mu}{d\mathcal{H}^2}(z) \geq |Dn_{M_h}|^2(z)
\]
at \(\mathcal{H}^2\)-almost every point \(z \in M_h\) which implies in particular the lower bound.

By Lemma 2.12 we may reduce the proof to the situation that \(M_{\tilde{h}_j}\), \(M_h\) are given as graphs of Lipschitz functions \(\tilde{h}_j : U \to \mathbb{R}\), \(\tilde{h} : U \to \mathbb{R}\) respectively, where \(U \subset \mathbb{R}^2\) is some open bounded set.
We have that \( \tilde{h}_j \) is piecewise affine on some (uniformly in \( j \)) regular triangulated surface \( \tilde{T}_j \) that satisfies

\[(\tilde{h}_j)_* \tilde{T}_j = T_j. \]

Writing down the surface normal to \( M_h \) in the coordinates of \( U \),

\[N(x) = \frac{(-\nabla \tilde{h}, 1)}{\sqrt{1 + |\nabla \tilde{h}|^2}},\]

we have that almost every \( x \in U \) is a Lebesgue point of \( \nabla N \). We write \( N^k = N \cdot e_k \) and note that (19) is equivalent to

\[
\begin{align*}
\frac{d\mu}{dH^2}(z) \geq \sum_{k=1}^3 \nabla N^k(x) \cdot \left( \mathbb{1}_{2 \times 2} + \nabla \tilde{h}(x) \otimes \nabla \tilde{h}(x) \right)^{-1} \nabla N^k(x),
\end{align*}
\]

where \( z = (x, \tilde{h}(x)) \). Also, we define \( N^k_j : U \to \mathbb{R}^3 \) by letting \( N^k_j(x) = n((\tilde{h}_j)_* K) \cdot e_k \) for \( x \in K \in \tilde{T}_j \). (We recall that \( n((\tilde{h}_j)_* K) \) denotes the normal of the triangle \( (\tilde{h}_j)_* K) \).

Let now \( x_0 \in U \) be a Lebesgue point of \( \nabla \tilde{h} \) and \( \nabla N \). We write \( z_0 = (x_0, \tilde{h}(x_0)) \). Combining the narrow convergence \( \mu_j \to \mu \) with the Radon-Nikodym differentiation Theorem, we may choose a sequence \( r_j \downarrow 0 \) such that

\[
\begin{align*}
\liminf_{j \to \infty} \frac{\mu_j(Q^{(3)}(x_0, r_j))}{r_j} = \frac{d\mu}{dH^2}(z_0) \sqrt{1 + |\nabla \tilde{h}(x_0)|^2},
\end{align*}
\]

where \( Q^{(3)}(x_0, r_j) = x_0 + [-r_j/2, r_j/2]^2 \times \mathbb{R} \) is the cylinder over \( Q(x_0, r_j) \).

Furthermore, let \( \tilde{N}_j, \tilde{h}_j, \tilde{N}, \tilde{h} : Q \to \mathbb{R} \) be defined by

\[
\begin{align*}
\tilde{N}_j^k(x) &= \frac{N_j(x_0 + r_j x) - N_j(x_0)}{r_j}, \\
\tilde{N}^k(x) &= \nabla N^k(x_0) \cdot (x - x_0), \\
\tilde{h}_j(x) &= \frac{\tilde{h}_j(x_0 + r_j x) - \tilde{h}_j(x_0)}{r_j}, \\
\tilde{h}(x) &= \nabla \tilde{h}(x_0) \cdot (x - x_0).
\end{align*}
\]

We recall that by assumption we have that \( N^k \in W^{1,2}(U) \). This implies in particular that (unless \( x_0 \) is contained in a certain set of measure zero, which we discard), we have that

\[
\begin{align*}
\tilde{N}^k \to \tilde{N}^k \quad \text{in } L^2(Q) .
\end{align*}
\]

Also, let \( T_j \) be the blowup map

\[
T_j(x) = \frac{x - x_0}{r_j}.
\]

and let \( T'_j \) be the triangulated surface one obtains by blowing up \( \tilde{T}_j \), defined by

\[
\tilde{K} \in \tilde{T}_j \iff T_j \tilde{K} \in T'_j.
\]
Now let $S_j$ be the smallest subset of $T_j'$ (as sets of triangles) such that $Q \subset S_j$ (as subsets of $\mathbb{R}^2$). Note that size $S_j \to 0$, $\bar{N}_j^k$ is constant and $\bar{h}_j$ is affine on each $K \in S_j$. Furthermore, for $x \in K \in \bar{T}_j$, we have that
\[ \nabla \bar{h}_j(x) = \nabla \bar{h}_j(T_jx) \]
This implies in particular
\[ (22) \quad \bar{h}_j \to \bar{h} \quad \text{in } W^{1,2}(Q). \]
Concerning the discrete energy functionals, we have for the rescaled triangulated surfaces $(T_j')^* = (\bar{h}_j)_* T_j'$, with $K^* = (\bar{h}_j)_* K$ for $K \in T_j'$,
\[ (23) \quad \liminf_{j \to \infty} \sum_{K,L \in T_j'} \frac{l_{K^*L^*}}{d_{K^*L^*}} |\bar{N}_j(K) - \bar{N}_j(L)|^2 \leq \liminf_{j \to \infty} r_j^{-2} \mu_j(Q^{(3)}(x_0,r_j)). \]
Thanks to (21), (22), we may apply Proposition 3.3 (ii) to the sequences of functions $(\bar{h}_j)_{j \in \mathbb{N}}, (\bar{N}_j^k)_{j \in \mathbb{N}}$. This yields (after summing over $k \in \{1,2,3\}$)
\[ |Dn_{M_{h_{\bar{f}}}}|^2(z_0) \sqrt{1 + |\nabla \bar{h}(x_0)|^2} \]
\[ = \nabla N(x_0) \cdot \left(1 + \nabla \bar{h}(x_0) \otimes \nabla \bar{h}(x_0)\right)^{-1} \nabla N(x_0) \sqrt{1 + |\nabla \bar{h}(x_0)|^2} \]
\[ \leq \liminf_{j \to \infty} \sum_{K,L \in T_j'} \frac{l_{K^*L^*}}{d_{K^*L^*}} |\bar{N}_j(K) - \bar{N}_j(L)|^2, \]
which in combination with (23) yields (20) for $x = x_0$, $z = z_0$ and completes the proof of the lower bound.

4. Surface triangulations and upper bound

Our plan for the construction of a recovery sequence is as follows: We shall construct optimal sequences of triangulated surfaces first locally around a point $x \in M_h$. It turns out the optimal triangulation must be aligned with the principal curvature directions at $x$. By a suitable covering of $M_h$, this allows for an approximation of the latter in these charts (Proposition 4.1). We will then formulate sufficient conditions for a vertex set to supply a global approximation (Proposition 4.4). The main work that remains to be done at that point to obtain a proof of Theorem 1.1 (ii) is to add vertices to the local approximations obtained from Proposition 4.1 such that the conditions of Proposition 4.4 are fulfilled.

4.1. Local optimal triangulations.

**Proposition 4.1.** There are constants $\delta_0, C > 0$ such that for all $U \subset \mathbb{R}^2$ open, convex, and bounded; and $h \in C^3(U)$ with $\|\nabla h\|_{\infty} =: \delta \leq \delta_0$, the following holds:
Let $\varepsilon > 0$, $C\varepsilon^2 < |\theta| \leq \frac{1}{2}$, and define $X := \{(ek + \theta l, \varepsilon l, h(ek + \theta l, \varepsilon l)) \in U \times \mathbb{R} : k,l \in \mathbb{Z}\}$. Then any Delaunay triangulated surface $T$ with vertex set $X$ and maximum
\[
\sum_{K,L \in T} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2 
\leq (1 + C(|\theta| + \delta + \varepsilon)) L^2(U) \times 
\left( \max_{x \in U} |\partial_1 h(x)|^2 + \max_{x \in U} |\partial_2 h(x)|^2 + \frac{1}{|\theta|} \max_{x \in U} |\partial_{12} h(x)|^2 \right) + C\varepsilon.
\]

Proof. We assume without loss of generality that \( \theta > 0 \). We consider the projection of \( X \) to the plane, 
\[ \bar{X} := \{(\varepsilon k + \theta \varepsilon l, \varepsilon l) \in U : k, l \in \mathbb{Z}\}. \]
Let \( \bar{T} \) be the flat triangulated surface that consists of the triangles of the form 
\[ \varepsilon [ke_1 + l(\theta e_1 + e_2), (k + 1)e_1 + l(\theta e_1 + e_2), ke_1 + (l + 1)(\theta e_1 + e_2)] \]
or 
\[ \varepsilon [ke_1 + l(\theta e_1 + e_2), (k + 1)e_1 + l(\theta e_1 + e_2), ke_1 + (l - 1)(\theta e_1 + e_2)], \]
with \( k, l \in \mathbb{Z} \) such that the triangles are contained in \( U \), see Figure 5.

Figure 5. The flat triangulated surface \( \bar{T} \).

Obviously the flat triangulated surface \( \bar{T} \) has vertex set \( \bar{X} \). Also, we have that 
\[ |x - y| \leq |(x, h(x)) - (y, h(y))| \leq (1 + C\delta)|x - y| \]
for all \( x, y \in \bar{X} \). We claim that for \( \delta \) chosen small enough, we have the implication 
\[ h_* K = [(x, h(x)), (y, h(y)), (z, h(z))] \in T \quad \Rightarrow \quad K = [x, y, z] \in \bar{T}. \]
Indeed, if \( K \notin \bar{T} \), then either \( r(K) > \frac{3}{2}\varepsilon \) or there exists \( w \in X \) with \( |w - q(K)| < (1 - C\theta)r(K) \). In the first case, \( r(h_* K) > (1 - C\delta)\frac{3}{2}\varepsilon \) by \( (25) \) and hence \( h_* K \notin \bar{T} \) for \( \delta \) small enough. In the second case, we have by \( (25) \) and Lemma 2.6 that 
\[ |(w, h(w)) - q(h_* K)| < (1 + C\delta)(1 - C\theta)r(h_* K), \]
and hence \( h_* K \) does not satisfy the Delaunay property for \( \delta \) small enough. This proves \( (26) \).

Let \( [x, y] \) be an edge with either \( x, y \in X \) or \( x, y \in \bar{X} \). We call this edge horizontal if \( (y - x) \cdot e_2 = 0 \), vertical if \( (y - x) \cdot (e_1 - \theta e_2) = 0 \), and diagonal if \( (y - x) \cdot (e_1 + (1 - \theta)e_2) = 0 \).
By its definition, $\tilde{T}$ consists only of triangles with exactly one horizontal, vertical, and diagonal edge each. By what we have just proved, the same is true for $T$.

To calculate the differences between normals of adjacent triangles, let us consider one fixed triangle $K \in T$ and its neighbors $K_1, K_2, K_3$, with which $K$ shares a horizontal, diagonal and vertical edge respectively, see Figure 6.

![Figure 6. Top view of a triangle $K \in T$ with its horizontal, diagonal and vertical neighbors $K_1, K_2, K_3$.](-figure6.png)

We assume without loss of generality that one of the vertices of $K$ is the origin. We write $x_0 = (0, 0), x_1 = \varepsilon (1-\theta, -1), x_2 = \varepsilon (1, 0), x_3 = \varepsilon (1+\theta, 1), x_4 = \varepsilon (\theta, 1), x_5 = \varepsilon (\theta-1, 1),$ and $y_i = (x_i, h(x_i))$ for $i = 0, \ldots, 5$. With this notation we have $K = [y_0, y_2, y_4], K_1 = [y_0, y_1, y_2], K_2 = [y_2, y_3, y_4]$ and $K_3 = [y_4, y_5, y_0]$. See Figure 6. As approximations of the normals, we define

$$v(K) = \varepsilon^{-2}y_2 \wedge y_4$$
$$v(K_1) = \varepsilon^{-2}y_1 \wedge y_2$$
$$v(K_2) = \varepsilon^{-2}(y_3 - y_2) \wedge (y_4 - y_2)$$
$$v(K_3) = \varepsilon^{-2}y_4 \wedge y_5.$$  

Note that $v(L)$ is parallel to $n(L)$ and $|v(L)| \geq 1$ for $L \in \{K, K_1, K_2, K_3\}$. Hence for $i = 1, 2, 3$, we have that

$$|n(K) - n(K_i)|^2 \leq |v(K) - v(K_i)|^2.$$  

For each $x_i$, we write

$$h(x_i) = x_i \cdot \nabla h(0) + \frac{1}{2} x_i \nabla^2 h(0) x_i^T + O(\varepsilon^3),$$

where $O(\varepsilon^3)$ denotes terms $f(\varepsilon)$ that satisfy $\limsup_{\varepsilon \to 0} \varepsilon^{-3} |f(\varepsilon)| < \infty$. By an explicit computation we obtain that

$$|v(K) - v(K_1)|^2 = \varepsilon^2 |(\theta - 1)\partial_{11} h + 2(\theta - 1)\partial_{12} h + \partial_{22} h|^2 + O(\varepsilon^3)$$
$$|v(K) - v(K_2)|^2 = \varepsilon^2 \left( |(\theta)\partial_{11} h + (\theta - 1)\partial_{12} h|^2 + |(\theta - 1)\partial_{11} h + (\theta - 1)\partial_{12} h|^2 \right) + O(\varepsilon^3)$$
$$|v(K) - v(K_3)|^2 = \varepsilon^2 \left( \theta^2 |(\theta - 1)\partial_{11} h + \partial_{12} h|^2 + |(\theta - 1)\partial_{11} h + \partial_{12} h|^2 \right) + O(\varepsilon^3),$$

19
We say that $X = (x, y, z)$ is $\delta$-protected if whenever $x, y, z \in X$ form a regular triangle $[x, y, z]$ with circumball $B(q, r)$ satisfying $r \leq D(X, N)$, then $|p - q| - r \geq \delta$ for any $p \in X \setminus \{x, y, z\}$.

**Theorem 4.3.** [BCKO08] Let $\alpha > 0$. Let $X \subset \mathbb{R}^2$ be finite and not colinear. Define

$$\Omega := \text{conv}(X).$$

Assume that

$$\min_{x \neq y \in X} |x - y| \geq \alpha D(X, \Omega),$$

and that $X$ is $\delta D(X, \Omega)$-protected for some $\delta > 0$. Then there exists a unique maximal Delaunay triangulated surface $\mathcal{T}$ with vertex set $X$, given by all regular triangles $[x, y, z]$, $x, y, z \in X$, with circumsphere $B(q, r)$ such that $B(q, r) \cap X = \emptyset$. 

4.2. **Global triangulations.** We are going to use a known fact about triangulations of point sets in $\mathbb{R}^2$, and transfer them to $\mathbb{R}^3$. We first cite a result for planar Delaunay triangulations, Theorem 4.3 below, which can be found in e.g. [BCKO08, Chapter 9.2]. This theorem states the existence of a Delaunay triangulated surface associated to a protected set of points.

**Definition 4.2.** Let $N \subset \mathbb{R}^3$ be compact, $X \subset N$ a finite set of points and

$$D(X, N) = \max_{x \in X} \min_{y \in X} |x - y|.$$

We say that $X$ is $\delta$-protected if whenever $x, y, z \in X$ form a regular triangle $[x, y, z]$ with circumball $B(q, r)$ satisfying $r \leq D(X, N)$, then $|p - q| - r \geq \delta$ for any $p \in X \setminus \{x, y, z\}$. 

**Theorem 4.3.** Let $\alpha > 0$. Let $X \subset \mathbb{R}^2$ be finite and not colinear. Define $\Omega := \text{conv}(X)$. Assume that

$$\min_{x \neq y \in X} |x - y| \geq \alpha D(X, \Omega),$$

and that $X$ is $\delta D(X, \Omega)$-protected for some $\delta > 0$. Then there exists a unique maximal Delaunay triangulated surface $\mathcal{T}$ with vertex set $X$, given by all regular triangles $[x, y, z]$, $x, y, z \in X$, with circumsphere $B(q, r)$ such that $B(q, r) \cap X = \emptyset$. 

20
The triangulated surface $\mathcal{T}$ forms a partition of $\Omega$, in the sense that
\[ \sum_{K \in \mathcal{T}} 1_K = 1_{\Omega} \quad \mathcal{H}^2 \text{almost everywhere}, \]
where $1_A$ denotes the characteristic function of $A \subset \mathbb{R}^3$. Further, any triangle $K \in \mathcal{T}$ with $\text{dist}(K, \partial \Omega) \geq 4D(X, \Omega)$ is $c(\alpha)$-regular, and $d_{KL} \geq \frac{2}{5}D(X, \Omega)$ for all pairs of triangles $K \neq L \in \mathcal{T}$.

We are now in position to formulate sufficient conditions for a vertex set to yield a triangulated surface that serves our purpose.

**Proposition 4.4.** Let $N \subset \mathbb{R}^3$ be a 2-dimensional compact smooth manifold, and let $\alpha, \delta > 0$. Then there is $\varepsilon = \varepsilon(N, \alpha, \delta) > 0$ such that whenever $X \subset N$ satisfies
\begin{align*}
\text{(a)} & \quad D(X, N) \leq \varepsilon, \\
\text{(b)} & \quad \min_{x, y \in X} |x - y| \geq \alpha D(X, N), \\
\text{(c)} & \quad X \text{ is } \delta D(X, N) \text{-protected;}
\end{align*}
then there exists a triangulated surface $\mathcal{T}(X, N)$ with the following properties:
\begin{enumerate}[(i)]
\item $\text{size}(\mathcal{T}(X, N)) \leq 2D(X, N)$,
\item $\mathcal{T}(X, N)$ is $c(\alpha)$-regular.
\end{enumerate}

We note that the protection property (c) is vital to the argument. A very similar result to Proposition 4.4 was proved in [BDG13], but we present a self-contained proof here.

**Proof of Proposition 4.4.** We construct the triangulated surface $\mathcal{T}(X, N)$ as follows: Consider all regular triangles $K = [x, y, z]$ with $x, y, z \in X$ such that the Euclidean Voronoi cells $V_x, V_y, V_z$ intersect in $N$, i.e. there is $\tilde{q} \in N$ such that $|\tilde{q} - x| = |\tilde{q} - y| = |\tilde{q} - z| \leq |\tilde{q} - p|$ for any $p \in \tilde{X} \setminus \{x, y, z\}$.

**Proof of (i):** Let $[x, y, z] \subset \mathcal{T}(X, N)$. Let $\tilde{q} \in V_x \cap V_y \cap V_z \cap N$, set $\tilde{r} := |\tilde{q} - x|$. Then $\tilde{r} = \min_{p \in X} |\tilde{q} - p| \leq D(X, N)$, and because $[x, y, z] \subset \overline{B(\tilde{q}, \tilde{r})}$ we have $\text{diam}([x, y, z]) \leq 2\tilde{r} \leq 2D(X, N)$.

**Proof of (ii):** Let $\overline{B(q, r)}$ denote the Euclidean circumball of $[x, y, z]$. Then $r \leq \tilde{r}$ by the definition of the circumball. Thus $\min(|x - y|, |x - z|, |y - z|) \geq \alpha r$, and $[x, y, z]$ is $c(\alpha)$-regular by the following argument: Rescaling such that $r = 1$, consider the class of all triangles $[x, y, z]$ with $x, y, z \in S^1$, $\min(|x - y|, |x - z|, |y - z|) \geq \alpha$. All these triangles are $\zeta$-regular for some $\zeta > 0$, and by compactness there is a least regular triangle in this class. That triangle’s regularity is $c(\alpha)$.

**Proof of (iii):** Because of (ii), $N \cap \overline{B(q, r)}$ is a $C(\alpha, N)\varepsilon$-Lipschitz graph over a convex subset $U$ of the plane $x + \mathbb{R}(y - x) + \mathbb{R}(z - x)$, say $N \cap \overline{B(q, r)} = U_h$. It follows that $\tilde{q} - q = h(\tilde{q})u$. Because $h(x) = 0$, it follows that $|\tilde{q} - q| = |h(\tilde{q})| \leq C(\alpha, N)D(X, N)^2$. Thus, for $D(X, N) \leq \delta(2C(\alpha, N))^{-1}$, we have that $|\tilde{q} - q| \leq \frac{\delta}{2}D(X, N)$. This together with (c) suffices to show the Delaunay property of $\mathcal{T}(X, N)$: Assume there exists $p \in \mathcal{T}(X, N)$.
Assume \( r \) and show that for \( x, y, z \) and \( K \) we have \( d_{KL} \geq \frac{\delta}{2} D(X, N) \).

**Proof of (iv):** It follows also from (c) and Lemma 2.6 that for all adjacent \( K, L \in T(X, N) \) we have \( d_{KL} \geq \frac{\delta}{2} D(X, N) \).

**Proof of (v) and (vi):** Let \( \eta > 0 \) to be fixed later. There is \( s > 0 \) such that for every \( x_0 \in N \), the orthogonal projection \( \pi : \mathbb{R}^3 \to x_0 + T_{x_0} N \) is an \( \eta \)-isometry when restricted to \( N \cap B(x_0, s) \), in the sense that \( |D\pi - \text{id}_{T_N}| \leq \eta \).

Let us write \( X_\pi = \pi(X \cap B(x_0, s)) \). This point set fulfills all the requirements of Theorem 4.3 (identifying \( x_0 + T_{x_0} N \) with \( \mathbb{R}^2 \), except for possibly protection. We will prove below that

\[
X_\pi \text{ is } \frac{\delta}{4} D(X, N) \text{-protected.}
\]

We will then consider the planar Delaunay triangulated surface \( T' := T(X_\pi, x_0 + T_{x_0} N) \), and show that for \( x, y, z \in B(x_0, s/2) \) we have

\[
K := [x, y, z] \in T(X, N) \iff K_\pi := [\pi(x), \pi(y), \pi(z)] \in T'.
\]

If we prove these claims, then (v) follows from Theorem 4.3, while (vi) follows from Theorem 4.3 and Lemma 2.11.

We first prove (27). Let \( \pi(x), \pi(y), \pi(z) \in X_\pi \), write \( K_\pi = [\pi(x), \pi(y), \pi(z)] \), and assume \( r(K_\pi) \leq D(X_\pi, \text{conv}(X_\pi)) \). For a contradiction, assume that \( \pi(p) \in X_\pi \setminus \{\pi(x), \pi(y), \pi(z)\} \) such that

\[
||q(K_\pi) - \pi(p)|| - r(K_\pi) < \frac{\delta}{4} D(X, N).
\]

Using again \( |D\pi - \text{id}_{T_N}| < \eta \) and Lemma 2.6, we obtain, with \( K = [x, y, z] \),

\[
||q(K) - p|| - r(K) < (1 + C\eta) \frac{\delta}{4} D(X, N).
\]

Choosing \( \eta \) small enough, we obtain a contradiction to (c). This completes the proof of (27).

Next we show the implication \( K \in T \Rightarrow K_\pi \in T' \): Let \( p \in X \cap B(x_0, s) \setminus \{x, y, z\} \). Assume for a contradiction that \( \pi(p) \) is contained in the circumball of \( K_\pi \),

\[
||p - q(K_\pi)|| \leq r(K_\pi).
\]

Then by \( |D\pi - \text{id}_{T_N}| < \eta \) and Lemma 2.6,

\[
||p - q(K)|| \leq r(K) + C(\alpha)\eta D(X, N).
\]

Choosing \( \eta < \frac{\delta}{2(2C(\alpha))} \), we have by (c) that

\[
||p - q(K)|| \leq r(K) - \delta D(X, N),
\]

which in turn implies \( |p - q| < r \). This is a contradiction to \( \tilde{q} \) is closer to \( q \) than any of \( x, y, z \). This shows \( K_\pi \in T' \).

Now we show the implication \( K_\pi \in T' \Rightarrow K \in T \): Let \( x, y, z \in X \cap B(x_0, s/2) \) with \( [\pi(x), \pi(y), \pi(z)] \in T' \). Let \( p \in X \cap B(x_0, s) \setminus \{x, y, z\} \). Assume for a contradiction that

\[
|p - \tilde{q}| \leq \tilde{r}.
\]

Then again by Lemma 2.6 we have

\[
|p - q| < \tilde{r} \Rightarrow |p - q| < r + \delta D(X, N) \Rightarrow |p - q| \leq r - \delta D(X, N) \Rightarrow |\pi(p) - q'| < r'.
\]
Figure 7. The global triangulation of a smooth surface is achieved by first covering a significant portion of the surface with the locally optimal triangulation, then adding additional points in between the regions, and finally finding a global Delaunay triangulation.

Here again we used (c) and the fact that \( D(X, N) \) is small enough. The last inequality is a contradiction, completing the proof of (28), and hence the proof of the present proposition.

Remark 4.5. A much shorter proof exists for the case of the two-sphere, \( N = S^2 \).

Here, any finite set \( X \subset S^2 \) such that no four points of \( X \) are coplanar and every open hemisphere contains a point of \( X \) admits a Delaunay triangulation homeomorphic to \( S^2 \), namely \( \partial \text{conv}(X) \).

Because no four points are coplanar, every face of \( \partial \text{conv}(X) \) is a regular triangle \( K = [x, y, z] \). The circumsphere of \( K \) lies on \( S^2 \) and \( q(K) = n(K) |q(K)| \), where \( n(K) \in S^2 \) is the outer normal. (The case \( q(K) = -|q(K)| n(K) \) is forbidden because the hemisphere \( \{ x \in S^2 : x \cdot n(K) > 0 \} \) contains a point in \( X \).) To see that the circumball contains no other point \( p \in X \setminus \{ x, y, z \} \), we note that since \( K \subset \partial \text{conv}(X) \) we have \( (p-x) \cdot n(K) < 0 \), and thus \( |p-q(K)|^2 = 1+1-2p \cdot q(K) > 1+1-2x \cdot q(K) = |x-q(K)|^2 \).

Finally, \( \partial \text{conv}(X) \) is homeomorphic to \( S^2 \) since \( \text{conv}(X) \) contains a regular tetrahedron.

We are now in a position to prove the upper bound of our main theorem, Theorem 1.1 (ii).

Proof of Theorem 1.1 (ii). We first note that it suffices to show the result for \( h \in C^3(M) \) with \( \|h\|_\infty < \frac{\delta(M)}{2} \). To see this, we approximate in the general case \( h \in W^{2,2}(M) \cap W^{1,\infty}(M) \), \( \|h\|_\infty \leq \frac{\delta(M)}{2} \) by smooth functions \( h_\beta := H_\beta h \), where \( (H_\beta)_{\beta \geq 0} \) is the heat semigroup. Clearly \( H_\beta h \in C^\infty(M) \), and \( \nabla H_\beta h \to \nabla h \) uniformly, so that \( \|h\|_\infty \leq \frac{\delta}{2} \) and \( \|\nabla h_\beta\|_\infty < \|\nabla h\|_\infty + 1 \) for \( \beta \) small enough.

Then
\[
\int_M f(x, h_\beta(x), \nabla h_\beta(x), \nabla^2 h_\beta) \, d\mathcal{H}^2 \to \int_M f(x, h(x), \nabla h(x), \nabla^2 h) \, d\mathcal{H}^2
\]
for \( \beta \to 0 \) whenever
\[
f : M \times [-\delta(M)/2, \delta(M)/2] \times B(0, \|\nabla h\|_\infty + 1) \times (TM \times TM) \to \mathbb{R}
\]
is continuous with quadratic growth in $\nabla^2 h$. The Willmore functional
\[
h \mapsto \int_{M_h} |Dn_{M_h}|^2 d\mathcal{H}^2,
\]
which is our limit functional, may be written in this way. This proves our claim that we may reduce our argument to the case $h \in C^3(M)$, since the above approximation allows for the construction of suitable diagonal sequences in the strong $W^{1,p}$ topology, for every $p < \infty$.

For the rest of the proof we fix $h \in C^3(M)$. We choose a parameter $\delta > 0$. By compactness of $M_h$, there is a finite family of pairwise disjoint closed open sets $(Z_i)_{i \in I}$ such that
\[
\mathcal{H}^2 \left( M_h \setminus \bigcup_{i \in I} Z_i \right) \leq \delta
\]
and such that, after applying a rigid motion $R_i : \mathbb{R}^3 \to \mathbb{R}^3$, the surface $R_i(M_h \cap Z_i)$ is the graph of a function $h_i \in C^2(U_i)$ for some open sets $(U_i)_{i \in I}$ with $\|\nabla h_i\|_\infty \leq \delta$ and $\|\nabla^2 h_i - \text{diag}(\alpha_i, \beta_i)\|_\infty \leq \delta$.

We can apply Proposition 4.1 to $R_i(M_h \cap Z_i)$ with global parameters $\theta := \delta$ and $\varepsilon > 0$ such that $\text{dist}(Z_i, Z_j) > 2\varepsilon$ for $i \neq j$, yielding point sets $X_{i,\varepsilon} \subset M_h \cap B_i$. The associated triangulated surfaces $T_{i,\varepsilon}$ (see [7]) have the Delaunay property, have vertices $X_{i,\varepsilon}$ and maximum circumball radius at most $\varepsilon$. Furthermore, we have that

\[
\sum_{i \in I} \sum_{K,L \in T_{i,\varepsilon}} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2
\leq (1 + C(\delta + \varepsilon)) \sum_{i \in I} \mathcal{L}^2(U_i) \times
\left( \max_{x \in U_i} |\partial_{11} h_i(x)|^2 + \max_{x \in U_i} |\partial_{22} h_i(x)|^2 + \delta^{-1} \max_{x \in U_i} |\partial_{12} h_i(x)|^2 \right) + C\varepsilon
\leq (1 + C(\delta + \varepsilon)) \sum_{i \in I} \int_{M_h \cap Z_i} |Dn_{M_h}|^2 d\mathcal{H}^2 + C(\varepsilon + \delta),
\]

where in the last line we have used $\|\nabla h_i\|_\infty \leq \delta$, $\|\nabla^2 h_i, \text{diag}(\alpha_i, \beta_i)\|_\infty \leq \delta$, and the identity

\[
\int_{M_h \cap Z_i} |Dn_{M_h}|^2 d\mathcal{H}^2 = \int_{U_i} |Dn_{U_i} h_i|^2 d\mathcal{H}^2
= \int_{U_i} \left( |1_{2 \times 2} + \nabla h_i \otimes \nabla h_i|^{-1} \nabla^2 h_i \right)^2 (1 + |\nabla h_i|^2)^{-1/2} dx.
\]

We shall use the point set $Y_{0,\varepsilon} := \bigcup_{i \in I} X_{i,\varepsilon}$ as a basis for a global triangulated surface. We shall successively augment the set by a single point $Y_{n+1,\varepsilon} := Y_{n,\varepsilon} \cup \{p_{n,\varepsilon}\}$ until the construction below terminates after finitely many steps. We claim that we can choose the points $p_{n,\varepsilon}$ in such a way that for every $n \in \mathbb{N}$ we have

(a) $\min_{x,y \in Y_{n,\varepsilon}, x \neq y} |x - y| \geq \frac{\varepsilon}{2}.$
(b) Whenever \(x, y, z, p \in Y_{n, \varepsilon}\) are four distinct points such that the circumball \(B(q, r)\) of \([x, y, z]\) exists and has \(r \leq \varepsilon\), then

\[
||p - q| - r| \geq \frac{\delta}{2}\varepsilon.
\]

If at least one of the four points \(x, y, z, p\) is not in \(Y_{0, \varepsilon}\), then

\[
||p - q| - r| \geq c\varepsilon,
\]

where \(c > 0\) is a universal constant.

First, we note that both (a) and (b) are true for \(Y_{0, \varepsilon}\).

Now, assume we have constructed \(Y_{n, \varepsilon}\). If there exists a point \(x \in M_h\) such that \(B(x, \varepsilon) \cap Y_{n, \varepsilon} = \emptyset\), we consider the set \(A_{n, \varepsilon} \subset M_h \cap B(x, \varepsilon)\) consisting of all points \(p \in M_h \cap B(x, \varepsilon)\) such that for all regular triangles \([x, y, z]\) with \(x, y, z \in Y_{n, \varepsilon}\) and circumball \(B(q, r)\) satisfying \(r \leq 2\varepsilon\), we have \(||p - q| - r| \geq c\varepsilon\).

Seeing as how \(Y_{n, \varepsilon}\) satisfies (a), the set \(A_{n, \varepsilon}\) is nonempty if \(c > 0\) is chosen small enough, since for all triangles \([x, y, z]\) as above we have

\[
\mathcal{H}^2\left(\left\{p \in B(x, \varepsilon/2) \cap M_h : ||p - q| - r| < c\varepsilon\right\}\right) \leq 4c\varepsilon^2,
\]

and the total number of regular triangles \([x, y, z]\) with \(r \leq 2\varepsilon\) and \(B(q, r) \cap B(x, \varepsilon) \neq \emptyset\) is universally bounded as long as \(Y_{n, \varepsilon}\) satisfies (a).

We simply pick \(p_{n, \varepsilon} \in A_{n, \varepsilon}\), then clearly \(Y_{n+1, \varepsilon} := Y_{n, \varepsilon} \cup \{p_{n, \varepsilon}\}\) satisfies (a) by the triangle inequality. We now have to show that \(Y_{n+1, \varepsilon}\) still satisfies (b).

This is obvious whenever \(p = p_{n, \varepsilon}\) by the definition of \(A_{n, \varepsilon}\). If \(p_{n, \varepsilon}\) is none of the points \(x, y, z, p\), then (b) is inherited from \(Y_{n, \varepsilon}\). It remains to consider the case \(p_{n, \varepsilon} = x\). Then \(x\) has distance \(c\varepsilon\) to all circumspheres of nearby triples with radius at most \(2\varepsilon\). We now assume that the circumball \(B(q, r)\) of \([x, y, z]\) has radius \(r \leq \varepsilon\) and that some point \(p \in Y_{n, \varepsilon}\) is close to \(\partial B(q, r)\). To this end, define

\[
\eta := \frac{||p - q| - r|}{\varepsilon}.
\]

We show that \(\eta \geq \eta_0\) for some uniform constant. To this end, we set

\[
p_t := (1 - t)p + t\left(q + r\frac{p - q}{||p - q||}\right)
\]

(see Figure 3) and note that if \(\eta \leq \eta_0\), all triangles \([y, z, p_t]\) are uniformly regular.

Define the circumcenters \(q_t := q(y, z, p_t)\), and note that \(q_1 = q\). By Lemma 2.6, we have \(|q_1 - q_0| \leq C|p_1 - p_0| = Cn\varepsilon\) if \(\eta \leq \eta_0\). Thus the circumradius of \([y, z, p_0]\) is bounded by

\[
|y - q_0| \leq |y - q| + |q - q_0| \leq (1 + C\eta)\varepsilon \leq 2\varepsilon
\]

if \(\eta \leq \eta_0\). Because \(x \in Y_{n+1, \varepsilon} \setminus Y_{n, \varepsilon} \subset A_{n, \varepsilon}\), we have, using (30),

\[
\varepsilon \leq ||x - q_0| - |p - q_0|| \leq ||x - q| - |p - q|| + 2|q - q_0| \leq (1 + 2C)\eta\varepsilon,
\]

i.e. that \(\eta \geq \frac{\varepsilon}{1 + 2C}\). This shows (b).

Since \(M_h\) is compact, this construction eventually terminates, resulting in a set \(X_{\varepsilon} := Y_{N(\varepsilon), \varepsilon} \subset M_h\) with the properties (a), (b), and \(D(X_{\varepsilon}, M) \leq \varepsilon\).
Consider a Lipschitz function \( g : M_h \rightarrow \mathbb{R} \). Since \( M_h \) is a \( C^2 \) surface, we have that for \( \| g \|_{W^{1,\infty}} \) small enough, \( (M_h)_g \) is locally a tangent Lipschitz graph over \( M \), see Definition 2.10 (iii). By Lemma 2.11, this implies that \( (M_h)_g \) is a graph over \( M \).

Invoking Proposition 4.4 yields a Delaunay triangulated surface \( T_{\varepsilon} := T(X_{\varepsilon}, M_h) \) with vertex set \( X_{\varepsilon} \) that is \( \zeta_0 \)-regular for some \( \zeta_0 > 0 \), and \( \bigcup_{K \in T_{\varepsilon}} (M_h)_{g_{\varepsilon}} \) with \( \| g_{\varepsilon} \|_{W^{1,\infty}} \leq C(\delta)\varepsilon \).

By the above, there exist Lipschitz functions \( h_{\varepsilon} : M \rightarrow \mathbb{R} \) such that \( (M_h)_{g_{\varepsilon}} = (M_h)_{h_{\varepsilon}} \), with \( h_{\varepsilon} \rightarrow h \) in \( W^{1,\infty} \), \( \| h_{\varepsilon} \|_{\infty} \leq \frac{\| h \|_{W^{1,\infty}}}{2} \) and \( \| \nabla h_{\varepsilon} \| \leq \| \nabla h \|_{\infty} + 1 \).

It remains to estimate the energy. To do so, we look at the two types of interfaces appearing in the sum

\[
\sum_{K,L \in T_{\varepsilon}} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2.
\]

First, we look at pairwise interactions where \( K, L \in T(X_{i,\varepsilon}) \) for some \( i \). These are bounded by (29).

Next, we note that if \( \varepsilon < \min_{i\neq j\in I} \text{dist}(B_i, B_j) \), it is impossible for \( X_{i,\varepsilon} \) and \( X_{j,\varepsilon}, i \neq j \), to interact.

Finally, we consider all interactions of neighboring triangles \( K, L \in T_{\varepsilon} \) where at least one vertex is not in \( Y_{0,\varepsilon} \). By (30), these pairs all satisfy \( \frac{l_{KL}}{d_{KL}} \leq C \) for some universal constant \( C \) independent of \( \varepsilon, \delta \), and \( |n(K) - n(L)| \leq C\varepsilon \) because \( T \) is \( \zeta_0 \)-regular and \( M_h \) is \( C^2 \). Further, no points were added inside any \( B_j \). Thus

\[
\sum_{K,L \in T_{\varepsilon} : \text{at least one vertex is not in } Y_{0,\varepsilon}} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2 
\]

\[
\leq C H^2 \left( M_h \setminus \bigcup_{i \in I} B(x_i, r_i - 2\varepsilon) \right) 
\]

\[
\leq C\delta + C(\delta)\varepsilon.
\]
Choosing an appropriate diagonal sequence $\delta(\varepsilon) \to 0$ yields a sequence $T_\varepsilon = M_{h_\varepsilon}$ with $h_\varepsilon \to h$ in $W^{1,\infty}(M)$ with

$$\limsup_{\varepsilon \to 0} \sum_{K,L \in T_\varepsilon} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2 \leq \int_{M_h} |Dn_{M_h}|^2 \, d\mathcal{H}^2.$$ 

□

5. Necessity of the Delaunay property

We now show that without the Delaunay condition, it is possible to achieve a lower energy. In contrast to the preceding sections, we are going to choose an underlying manifold $M$ with boundary (the “hollow cylinder” $S^1 \times [-1,1]$). By “capping off” the hollow cylinder one can construct a counterexample to the lower bound in Theorem 1.1, where it is assumed that $M$ is compact without boundary.

**Proposition 5.1.** Let $M = S^1 \times [-1,1] \subset \mathbb{R}^3$ be a hollow cylinder and $\zeta > 0$. Then there are $\zeta$-regular triangulated surfaces $T_j \subset \mathbb{R}^3$ with size$(T_j) \to 0$ and $T_j \to M$ for $j \to \infty$ with

$$\limsup_{j \to \infty} \sum_{K,L \in T_j} \frac{l_{KL}}{d_{KL}} |n(K) - n(L)|^2 < c(\zeta) \int_M |Dn_M|^2 \, d\mathcal{H}^2,$n

where the positive constant $c(\zeta)$ satisfies

$$c(\zeta) \to 0 \quad \text{for} \quad \zeta \to 0.$$ 

**Proof.** For every $\varepsilon = 2^{-j}$ and $s \in \{2\pi j^{-1} : j = 3, 4, 5, \ldots \}$, we define a flat triangulated surface $T_j \subset \mathbb{R}^2$ with size$(T_j) \leq \varepsilon$ as follows: As manifolds with boundary, $T_j = [0, 2\pi] \times [-1,1]$ for all $j$; all triangles are isosceles, with one side a translation of $[0, \varepsilon]e_2$ and height $s\varepsilon$ in $e_1$-direction. We neglect the triangles close to the boundary $[0, 2\pi] \times \{\pm1\}$, and leave it to the reader to verify that their contribution will be negligible in the end.
We then wrap this triangulated surface around the cylinder, mapping the corners of triangles onto the surface of the cylinder via \((\theta, t) \mapsto (\cos \theta, \sin \theta, t)\), to obtain a triangulated surface \(\tilde{T}_j\). Obviously, the topology of \(\tilde{T}_j\) is \(S^1 \times [-1, 1]\). Then we may estimate all terms \(\frac{L_{KL}}{d_{KL}} |n(K) - n(L)|^2\). We first find the normal of the reference triangle \(K \in \tilde{T}_j\) spanned by the points \(x = (1, 0, 0)\), \(y = (1, 0, \varepsilon)\), and \(z = (\cos(s\varepsilon), \sin(s\varepsilon), \varepsilon/2)\). We note that

\[
n(K) = \frac{(y - x) \times (z - x)}{|(y - x) \times (z - x)|} = \frac{(-s\varepsilon \sin(s\varepsilon), s\varepsilon(\cos(s\varepsilon) - 1), 0)}{s\varepsilon(2 - 2\cos(s\varepsilon))} = (1, 0, 0) + O(s\varepsilon).
\]

We note that the normal is the same for all translations \(K + te_3\) and for all triangles bordering \(K\) diagonally. The horizontal neighbor \(L\) also has \(n(L) = (1, 0, 0) + O(s\varepsilon)\). However, we note that the dimensionless prefactor satisfies \(\frac{L_{KL}}{d_{KL}} \leq \frac{2s}{\varepsilon/s} = s\). Summing up the \(O(s^{-1}\varepsilon^{-2})\) contributions yields

\[
\sum_{K,L \in \tilde{T}_j} \frac{L_{KL}}{d_{KL}} |n(K) - n(L)|^2 \leq Cs^3 \varepsilon^2 = Cs^2.
\]

This holds provided that \(\varepsilon\) is small enough. Letting \(s \to 0\), we see that this energy is arbitrarily small. \(\square\)

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