Conformal Ward–Takahashi Identity at Finite Temperature

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Abstract We study conformal Ward–Takahashi identities for two-point functions in $d(\geq 3)$-dimensional finite-temperature conformal field theory. We first show that the conformal Ward–Takahashi identities can be translated into the intertwining relations of conformal algebra $\mathfrak{so}(2,d)$. We then show that, at finite temperature, the intertwining relations can be translated into the recurrence relations for two-point functions in complex momentum space. By solving these recurrence relations, we find the momentum-space two-point functions that satisfy the Kubo–Martin–Schwinger thermal equilibrium condition.

1 Introduction

It is widely believed that conformal symmetry is always broken at finite temperature. This comes from the naive argument that finite-temperature field theory necessarily contains one particular scale—the temperature—and hence must break scale and conformal invariance. Contrary to this popular belief, however, finite temperature and conformal invariance can in fact be compatible with each other: If conformal field theory (CFT) is thermalized via the Unruh effect, conformal symmetry remains intact even at finite temperature. The purpose of this paper is to report our recent work on this subject [11] and to see how the conformal symmetry determines finite-temperature two-point functions in momentum space. The key is the intertwining relations of conformal algebra $\mathfrak{so}(2,d)$ [4, 6, 10, 14], which follow from the conformal Ward–Takahashi identities for two-point functions. We shall show that, at finite temperature, the intertwining relations are recast into the recurrence relations in complex momentum space. These recurrence relations can be

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regarded as the conformal Ward–Takahashi identities at finite temperature, from which we can deduce the possible forms of momentum-space two-point functions.

The rest of the paper is organized as follows: In Section 2 we first introduce the intertwining operator, which is defined as an integral transform whose kernel is the two-point function. We then discuss that the conformal Ward–Takahashi identities are rewritten as the intertwining relations. In Section 3 we introduce the $d(\geq 3)$-dimensional Rindler wedge, light-cone, and diamond, all of which are subspaces of Minkowski spacetime and conformal to $\mathbb{H}^1 \times \mathbb{H}^{d-1}$. These subspaces are the whole universes of our finite-temperature CFT and possess the global timelike conformal Killing vectors associated with the subgroup $SO(1, 1) \subset SO(2, d)$. In Section 4 we study the intertwining relations in the basis in which the $SO(1, 1)$ generator becomes diagonal. We shall see that in this basis the intertwining relations reduce to the recurrence relations for momentum-space two-point functions. We also give two minimal solutions that correspond to the positive- and negative-frequency two-point Wightman functions and satisfy the Kubo–Martin–Schwinger (KMS) thermal equilibrium condition.

Throughout the paper we work with the metric signature $(-, +, \cdots, +)$.

2 From Conformal Ward–Takahashi Identities to Intertwining Relations

To begin with, let us consider a scalar primary operator $\mathcal{O}_\Delta(x)$ of scaling dimension $\Delta$. Let $g \in SO(2, d)$ be an element of the conformal group and $x \mapsto x_g$ be the associated conformal transformation. Then the scalar primary operator transforms as follows:

$$U(g)\mathcal{O}_\Delta(x)U^{-1}(g) = \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} \mathcal{O}_\Delta(x_g), \tag{1}$$

where $U$ is a unitary representation of the conformal group and $|\partial x_g/\partial x|$ stands for the Jacobian of the conformal transformation.

Let us next consider a two-point function $G_\Delta(x, y)$ of $\mathcal{O}_\Delta$. For example, one may consider this to be the positive- or negative-frequency two-point Wightman functions, $\langle 0|\mathcal{O}_\Delta(x)\mathcal{O}_\Delta^\dagger(y)|0 \rangle$ or $\langle 0|\mathcal{O}_\Delta^\dagger(y)\mathcal{O}_\Delta(x)|0 \rangle$, where $|0 \rangle$ stands for the conformally-invariant vacuum state that satisfies $U(g)|0 \rangle = |0 \rangle$ for any $g \in SO(2, d)$. Then $G_\Delta(x, y)$ satisfies the following identity:

$$G_\Delta(x, y) = \left| \frac{\partial x_g}{\partial x} \right|^{\Delta/d} \left| \frac{\partial y_g}{\partial y} \right|^{\Delta/d} G_\Delta(x_g, y_g). \tag{2}$$
As is well-known, this identity—the finite form of conformal Ward–Takahashi identity—fully determines the possible forms of two-point functions. For example, up to the $i\epsilon$ prescription the Wightman functions must be of the form $G_\Delta(x, y) \propto [(x - y)^2]^{-\Delta}$.

Now, let us consider another scalar primary operator $O_{d-\Delta}(x)$ of scaling dimension $d - \Delta$. Once $O_{d-\Delta}(x)$ and $G_\Delta(x, y)$ are given, we can define an operator $G_\Delta$ through the following integral transform:

$$G_\Delta : O_{d-\Delta}(x) \mapsto (G_\Delta O_{d-\Delta})(x) := \int d^d y\ G_\Delta(x, y)O_{d-\Delta}(y). \quad (3)$$

It is easy to check that thus defined operator $(G_\Delta O_{d-\Delta})(x)$ satisfies the transformation law (1) and hence is a primary operator of scaling dimension $\Delta$. Conversely, one can start from $O_{d}(x)$ and $G_{d-\Delta}(x, y)$ and then define an operator $G_{d-\Delta}$ through the integral $(G_{d-\Delta}O_{\Delta})(x) := \int d^d y\ G_{d-\Delta}(x, y)O_{\Delta}(y)$. In this case $(G_{d-\Delta}O_{\Delta})(x)$ becomes a primary operator of scaling dimension $d - \Delta$. In short, $G_\alpha$ is a map from one primary operator to another, where $\alpha \in \{\Delta, d - \Delta\}$. In the literature [5] $(G_\alpha O_{d-\alpha})(x)$ is called the shadow operator of $O_{d-\alpha}(x)$.

Let us now turn to the infinitesimal conformal invariance. If $g \in SO(2, d)$ is infinitesimally close to the identity element, (1) is recast into the following commutation relations:

$$[J^{ab}, O_\Delta(x)] = -J^{ab}_\Delta(x, \partial_x)O_\Delta(x). \quad (4)$$

Likewise, (2) becomes the following identities (the infinitesimal form of conformal Ward–Takahashi identities):

$$(J^{ab}_\Delta(x, \partial_x) + J^{ab}_\Delta(y, \partial_y)) G_\Delta(x, y) = 0. \quad (5)$$

Here $J^{ab} = -J^{ba}$ $(a, b = 0, 1, \cdots, d + 1)$ are the generators of $SO(2, d)$ and satisfy the following commutation relations of the Lie algebra $so(2, d)$:

$$[J^{ab}, J^{cd}] = i(\eta^{ac} J^{bd} - \eta^{ad} J^{bc} - \eta^{bc} J^{ad} + \eta^{bd} J^{ac}), \quad (6)$$

where $\eta_{ab} = \eta^{ab} = \text{diag}(-1, +1, \cdots, +1, -1)$. On the other hand, $J^{ab}_\Delta(x, \partial_x)$ are the following differential representations of $J^{ab}$:

$$J^{ab}_\Delta(x, \partial_x) = i \left(k^{\mu ab}(x) \partial_\mu + \frac{\Delta}{d}(\partial_\mu k^{\mu ab})(x) \right), \quad (7)$$

where $k^{ab}_\Delta(x) = -k^{ba}_\Delta(x)$ are the conformal Killing vectors given by

$$k^{\mu \nu \lambda}(x) = \eta^{\mu \nu} x^\lambda - \eta^{\mu \lambda} x^\nu, \quad k^{\mu \nu d}(x) = \frac{\ell^2 - x \cdot x}{2\ell} \eta^{\mu \nu} + \frac{x^{\mu} x^{\nu}}{\ell}, \quad (8)$$

$$k^{\mu \nu, d+1}(x) = \frac{\ell^2 + x \cdot x}{2\ell} \eta^{\mu \nu} - \frac{x^{\mu} x^{\nu}}{\ell}, \quad k^{\mu d, d+1}(x) = -x^{\mu} \quad (9)$$
Here \( \ell > 0 \) is an arbitrary reference length scale which needs to be introduced to adjust the length dimensions of the equations. Note that these vectors satisfy the conformal Killing equations

\[
\partial_\mu k_{\nu}{}^{ab} + \partial_\nu k_{\mu}{}^{ab} = 2\eta_{\mu\nu}\partial_\rho k^{\rho ab}.
\]

Now, let \( G_{\Delta}(x, y) \) satisfy the infinitesimal conformal Ward–Takahashi identities (5). Then, upon integration by parts one can prove the following identities:

\[
\int d^d y J^{ab}_\Delta(x, \partial_x) G_{\Delta}(x, y) \mathcal{O}_{d-\Delta}(y) = \int d^d y G_{\Delta}(x, y) J^{ab}_{d-\Delta}(y, \partial_y) \mathcal{O}_{d-\Delta}(y),
\]

or, more compactly,

\[
(J^{ab}_{\Delta} G_{\Delta} \mathcal{O}_{d-\Delta})(x) = (G_{\Delta} J^{ab}_{d-\Delta} \mathcal{O}_{d-\Delta})(x),
\]

where \( (J^{ab}_{\alpha} \mathcal{O}_{\alpha})(x) := J^{ab}_{\alpha}(x, \partial_x) \mathcal{O}_{\alpha}(x), \alpha \in \{\Delta, d-\Delta\} \). Since this holds for arbitrary \( \mathcal{O}_{d-\Delta} \) we get the following operator identities:

\[
J^{ab}_{\Delta} G_{\Delta} = G_{\Delta} J^{ab}_{d-\Delta}.
\]

These are the intertwining relations, and in this respect \( G_{\Delta} \) is called the intertwining operator. As is evident from the above discussions the intertwining relations are essentially the same as the conformal Ward–Takahashi identities. There is, however, a big advantage of using (12): The operator identities (12) are basis independent and hence easy to manipulate in an algebraic language. In the rest of the paper we shall apply the intertwining relations to a certain (improper) basis for a representation space of conformal algebra. In other words, we shall apply (12) to a mode function \( f_{\alpha,p}(x) \) in terms of which the operator \( \mathcal{O}_{\alpha}(x) \) is expanded as \( \mathcal{O}_{\alpha}(x) = \int \frac{d^d p}{(2\pi)^d} \tilde{\mathcal{O}}_{\alpha}(p) f_{\alpha,p}(x) \). In zero-temperature CFT such mode function is just the plane wave \( e^{ip \cdot x} \). In this case the intertwining relations just result in the well-known momentum-space conformal Ward–Takahashi identities at zero temperature. In finite-temperature CFT thermalized via the Unruh effect, on the other hand, \( f_{\alpha,p}(x) \) becomes a quite nontrivial function. In a more algebraic language, \( f_{\alpha,p}(x) \) is chosen to be an eigenfunction for the generator of one-parameter subgroup \( SO(1, 1) \subset SO(2, d) \). Before going to study the intertwining relations in the \( SO(1, 1) \) diagonal basis, let us first recall the significance of \( SO(1, 1) \) for finite-temperature CFT.

3 Timelike Conformal Killing Vectors Associated with the Subgroup \( SO(1, 1) \subset SO(2, d) \)

Let us start with the KMS condition [7]. The KMS condition is a thermal equilibrium condition for quantum systems and expressed as an analytic condition for positive- and negative-frequency two-point Wightman functions
G^+(t) and G^−(t). It demands that (i) $G^+(t)$ ($G^−(t)$) should be an analytic function on the strip $-\beta < \text{Im} \, t < 0$ ($0 < \text{Im} \, t < \beta$); and (ii) $G^+(t)$ and $G^−(t)$ should satisfy the following boundary conditions on the strips:

$$
G^+(t) = G^−(t + i\beta) \quad \& \quad G^−(t) = G^+(t - i\beta), \quad \forall t \in \mathbb{R},
$$

(13)

where $\beta = 1/T$ is the inverse temperature. (For the moment we will suppress the spatial coordinates.) The advantage of using the KMS condition is that these analytic conditions remain valid even after the thermodynamic limit. (Note that the extensive property of the free energy $F = -(1/\beta) \log \text{Tr} \, e^{-\beta H}$ would render the density matrix $\rho = e^{-\beta(H-F)}$ ill-defined in the thermodynamic limit.) For a full account of the KMS condition we refer to [7, 8].

Now, let us take a closer look at the boundary conditions (13). These conditions are best understood in statistical mechanics for finite degrees of freedom in a finite box. Let $O(t) = e^{iHt}O(0)e^{-iHt}$ be a Heisenberg operator. Then we have

$$
\langle O(t)O^\dagger(t') \rangle = \frac{1}{Z} \text{Tr} \left( e^{-\beta H}O(t)O^\dagger(t') \right) = \frac{1}{Z} \text{Tr} \left( e^{-\beta H}O(t)e^{\beta H}e^{-\beta H}O^\dagger(t') \right)
$$

$$
= \frac{1}{Z} \text{Tr} \left( e^{-\beta H}O(t)e^{-\beta H}O^\dagger(t') \right)
$$

$$
= \frac{1}{Z} \text{Tr} \left( e^{-\beta H}O(t)e^{\beta H}O(t + i\beta) \right) = \langle O^\dagger(t')O(t + i\beta) \rangle,
$$

(14)

where $Z = \text{Tr} \, e^{-\beta H}$ is the partition function. The second line follows from the cyclic property of trace and the last line the identity $e^{izH}O(t)e^{-izH} = O(t+z)$ with $z = i\beta$. Setting $t' = 0$ we get the condition $G^+(t) = G^−(t + i\beta)$. Likewise, one can prove $G^−(t) = G^+(t - i\beta)$ in a similar manner.

The above discussion is based on the expectation value with respect to the density matrix $\rho = e^{-\beta H}/Z$. However, the boundary conditions (13) themselves can be formulated without recourse to the density matrix. Suppose that there exist a state $|\Omega\rangle$ and an antiunitary operator $J$ such that the following identity holds:

$$
Je^{-\frac{z}{2}H}O(t)|\Omega\rangle = O^\dagger(t)|\Omega\rangle,
$$

(15)

where $O(t)$ is an arbitrary Heisenberg operator and $H$ is assumed to satisfy $H|\Omega\rangle = 0$. Once we have the identity (15), we can prove that the Wightman functions with respect to the state $|\Omega\rangle$ satisfy (13). Indeed, by using the inner
produce notation \((\ast, \ast)\) we have (see also Chapter 5 of [13])

\[
\langle \Omega | \mathcal{O}(t) \mathcal{O}^\dagger(t') | \Omega \rangle = (\langle \Omega |, \mathcal{O}(t) \mathcal{O}^\dagger(t') | \Omega \rangle) = (\mathcal{O}^\dagger(t') |, \mathcal{O}(t) | \Omega \rangle)
\]

where the second line follows from the assumption (15), the third line the antiunitarity of \(J\) (i.e., \((\langle \Psi |, J \Phi \rangle) = (\langle \Phi |, \Phi \rangle)\)), and the fifth line the relations \(e^{-\beta H} \mathcal{O}(t) e^{\beta H} = \mathcal{O}(t + i\beta)\) and \(e^{-\beta H} |\Omega\rangle = |\Omega\rangle\). Setting \(t' = 0\) we get the condition \(G^+ (t) = G^-(t + i\beta)\). Likewise, one can prove \(G^-(t) = G^+(t - i\beta)\). These mean that, if (15) holds, the Wightman functions with respect to the state \(|\Omega\rangle\) are nothing but the thermal Wightman functions at temperature \(T = 1/\beta\) (except the question of analyticity on the strips).

The above discussion, though simplified, captures the essence of the interplay between the KMS condition and the Bisognano–Wichmann theorem [1, 2]. In the mid-1970s Bisognano and Wichmann showed that there exists an identity (15) in generic Poincaré-invariant quantum field theories. There, the state \(|\Omega\rangle\) is the vacuum state \(|0\rangle\) for inertial observers, \(J\) is the CPT conjugate (with a partial reflection), and \(\beta H\) is the generator of Lorentz boost. The temporal coordinate \(t\) is proportional to the dimensionless Lorentz boost parameter \(\theta\) and identified as \(\theta = (2\pi/\beta) t\). Physically speaking, \(t\) is identical to the proper time for uniformly accelerating observers and the proportional coefficient \(2\pi/\beta\) is identical to the proper acceleration \(a\), from which we can deduce the Unruh temperature \(T = a/(2\pi)\). This is the physical content of Bisognano–Wichmann theorem, which provides a nonperturbative proof for the thermality of Wightman functions with respect to the vacuum [12].

Now we have come to the point. From a group theoretical viewpoint the most important thing in the Bisognano–Wichmann theorem is that the time-translation generator \(H\) is given by the Lorentz boost generator—the generator of one-parameter subgroup \(SO(1, 1)\) of the Poincaré group \(ISO(1, d - 1)\). In Poincaré-invariant quantum field theories the Lorentz boost is the only way to realize the group \(SO(1, 1)\) as a coordinate transformation. However, there emerge several options if the theory enjoys conformal invariance. Typical examples are the following [11]:

\[
SO(1, 1) : x^\mu \mapsto x'^\mu(\theta) = A^\mu_\nu x^\nu, \tag{17}
\]

\[
SO(1, 1) : x^\mu \mapsto x'^\mu(\theta) = e^{-\theta} x^\mu, \tag{18}
\]

\[
SO(1, 1) : x^\mu \mapsto x'^\mu(\theta) = e^{-\varphi} \frac{x^\mu - b^\mu(x \cdot x)}{1 - 2(b \cdot x) + (b \cdot b)(x \cdot x)} + a^\mu, \tag{19}
\]
where \( A = \begin{pmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{pmatrix} \), \( \varphi = 2 \log \cosh \frac{\theta}{2} \), \( b^\mu = (\frac{1}{\ell} \tanh \frac{\theta}{2}, 0, \cdots, 0) \), and \( a^\mu = (\ell \tanh \frac{\theta}{2}, 0, \cdots, 0) \). Note that (17) is a Lorentz boost on the \((x^0, x^1)\)-plane, (18) is a dilatation, and (19) is a special conformal transformation followed by a dilatation followed by a translation. Note also that these transformations are the solutions to the following flow equations generated by the conformal Killing vectors \( k^{\mu 10} \), \( k^{\mu d,d+1} \), and \( k^{\mu d,0} = - k^{\mu 0d} \):

\[
\dot{x}^\mu (\theta) = \eta^\mu x^0 (\theta) - \eta^0 x^1 (\theta),
\]

\[
\dot{x}^\mu (\theta) = - x^\mu (\theta),
\]

\[
\dot{x}^\mu (\theta) = - \frac{\ell^2 - x(\theta) \cdot x(\theta) \eta^{\mu 0} - x^\mu (\theta) x^0 (\theta)}{\ell},
\]

where dot stands for the derivative with respect to \( \theta \).

Now we wish to identify the parameter \( \theta \) with the temporal coordinate \( t \) (up to the factor \( 2\pi/\beta \)). To justify this, the above conformal Killing vectors must be timelike; that is, \( \dot{x}(\theta) \cdot \dot{x}(\theta) < 0 \). It is a straightforward exercise to classify their timelike domains. The results are as follows (see also Figure 1):

- **Rindler wedge.** The Killing vector (20) becomes timelike in the following domains:

\[
W_{\pm} = \{ x^\mu : \pm x^1 > |x^0| \},
\]

which are nothing but the right and left Rindler wedges. The coordinate systems in which (17) yields the time-translation are given by

\[
x^0 = \pm \ell \frac{\sinh(t/\ell)}{H^0 + H^1}, \quad x^1 = \pm \ell \frac{\cosh(t/\ell)}{H^0 + H^1}, \quad x^i = \ell \frac{H^i}{H^0 + H^1},
\]

where \( H^\mu = (H^0, H^1, \cdots, H^{d-1}) \) describes the upper half of two-sheeted hyperbolic space \( \mathbb{H}^{d-1} \) and is subject to the conditions \( H \cdot H = -(H^0)^2 + (H^1)^2 + \cdots + (H^{d-1})^2 = -1 \) and \( H^0 \geq 1 \). The induced metrics on \( W_{\pm} \) are

\[
ds^2_{W_{\pm}} = \frac{-dt^2 + \ell^2 dH \cdot dH}{(H^0 + H^1)^2}.
\]

- **Light-cone.** The conformal Killing vector (21) becomes timelike in the following domains:

\[
V_{\pm} = \{ x^\mu : \pm x^0 > |x| \},
\]

which are nothing but the future and past light-cones. The coordinate systems in which (18) yields the time-translation are given by

\[
x^\mu = \pm \ell e^{-t/\ell} H^\mu,
\]

where \( H^\mu \) is the same as above. The induced metrics on \( V_{\pm} \) are

\[
ds^2_{V_{\pm}} = e^{-2t/\ell} ( -dt^2 + \ell^2 dH \cdot dH ).
\]
• **Diamond.** The conformal Killing vector \((22)\) becomes timelike in the following domain:\(^1\)

\[ D = \{ x^\mu : |x| + |x^0| < \ell \}, \quad (29) \]

which is nothing but the diamond (or double cone). The coordinate system in which \((19)\) yields the time-translation is given by

\[ x^0 = \ell \frac{\sinh(t/\ell)}{\cosh(t/\ell) + H^0}, \quad x^i = \ell \frac{H^i}{\cosh(t/\ell) + H^0}, \quad (30) \]

where \(H^\mu\) is the same as above. The induced metric on \(D\) is

\[ ds_D^2 = -\frac{dt^2 + \ell^2 dH \cdot dH}{(\cosh(t/\ell) + H^0)^2}. \quad (31) \]

Now it is obvious that these subspaces of the flat Minkowski spacetime \(\mathbb{R}^{1,d-1}\) are all conformal to \(\mathbb{H}^1 \times \mathbb{H}^{d-1} \supset (t,H^\mu)\). Hence the correlation functions on \(\mathbb{H}^1 \times \mathbb{H}^{d-1}\) with respect to the inertial vacuum \(|0\rangle\) are just given by conformal transformations of those in the Cartesian coordinate system. For example, the positive- and negative-frequency two-point Wightman functions \(\langle 0|\mathcal{O}_\Delta(t,H)\mathcal{O}_\Delta^\dagger(t',H')|0\rangle\) and \(\langle 0|\mathcal{O}_\Delta^\dagger(t',H')\mathcal{O}_\Delta(t,H)|0\rangle\) are given by

\[ \left[ \frac{2\pi^2 T^2}{-\cosh(2\pi T(t - t' \mp i\epsilon)) - H \cdot H'} \right]^\Delta, \quad (32) \]

where \(T = 1/(2\pi \ell)\). It can be shown that these Wightman functions indeed satisfy the KMS condition and hence give the thermal correlation functions on \(\mathbb{H}^1 \times \mathbb{H}^{d-1}\) at temperature \(T\) \(^{11}\). We note that there also exist theorems \([3, 9]\) which generalize the Bisognano–Wichmann theorem and consider the conformal Killing vectors \((21)\) and \((22)\) and their timelike domains.

So far we have considered correlation functions in position space. For practical applications, however, we often need to know momentum-space correlators. A standard approach to momentum-space correlators is the Fourier transform of position-space correlators. However, the Fourier transform of correlation functions is generally hard to carry out. In fact, the Fourier transform of \((32)\) is (though not impossible) quite complicated and requires a lot of integration techniques. Hence in this paper we shall take another route to momentum-space correlators: the intertwining relations. As we will see below these enable us to deduce momentum-space two-point functions in a purely Lie-algebraic fashion.

\(^1\) In fact, as depicted in Figure 1 the conformal Killing vector \((22)\) becomes timelike also in the domains \(K = \{ x^\mu : |x| - |x^0| > \ell \}\) and \(V_\pm = \{ x^\mu : \pm x^0 > |x| + \ell \}\).
4 Intertwining Relations in the $SO(1, 1)$ Basis

Let us finally move on to the intertwining relations in the $SO(1, 1)$ diagonal basis—the conformal Ward–Takahashi identities at finite temperature. We emphasize that this section is rather sketchy. For more details we refer to our paper [11]. In what follows we shall set $2\pi T = 1/\ell = 1$ for simplicity. The temperature dependence is easily restored by dimensional analysis.

To begin with, let us consider the quadratic Casimir operator of the Lie algebra $\mathfrak{so}(2, d)$, which is given by

$$C_2[\mathfrak{so}(2, d)] = \frac{1}{2} J_{ab} J^{ab}. \quad (33)$$

We wish to identify the $SO(1, 1)$ generator as the time-translation generator $H$. In group theoretical language, this means that we need to work with the basis where the following subgroup becomes diagonal:

$$SO(1, 1) \times SO(1, d - 1) \subset SO(2, d). \quad (34)$$

Correspondingly, the quadratic Casimir operator is decomposed as follows:

$$C_2[\mathfrak{so}(2, d)] = -H(H + id) - \eta_{ab} E^{+a} E^{-b} + C_2[\mathfrak{so}(1, d - 1)], \quad (35)$$

where $E^{\pm a}$ are certain linear combinations of $J^{ab}$ and $C_2[\mathfrak{so}(1, d - 1)]$ is the quadratic Casimir operator of the subalgebra $\mathfrak{so}(1, d - 1)$. For example, in the case of Rindler wedge we have $H = J^{10}$, $E^{+a} = J^{0a} \pm J^{1a}$, and $C_2[\mathfrak{so}(1, d - 1)] = (1/2) J_{ab} J^{ab}$, where $a$ and $b$ run through 2 to $d + 1$.

Now let $|\Delta, \omega, j; \sigma \rangle$ be a simultaneous eigenstate of $C_2[\mathfrak{so}(2, d)]$, $H$, and $C_2[\mathfrak{so}(1, d - 1)]$ that satisfies the following eigenvalue equations:

$$C_2[\mathfrak{so}(2, d)] |\Delta, \omega, j; \sigma \rangle = \Delta(\Delta - d) |\Delta, \omega, j; \sigma \rangle, \quad (36)$$
$$H |\Delta, \omega, j; \sigma \rangle = \omega |\Delta, \omega, j; \sigma \rangle, \quad (37)$$
$$C_2[\mathfrak{so}(1, d - 1)] |\Delta, \omega, j; \sigma \rangle = j(j - d + 2) |\Delta, \omega, j; \sigma \rangle, \quad (38)$$

where $\sigma$ stands for eigenvalues of other simultaneously commuting generators which are irrelevant in the following discussion. Below we shall focus on the case $j(j - d + 2) < -(d - 2)^2/4$ and parameterize $j$ as follows:

$$j = \frac{d - 2}{2} + ik, \quad k \in (0, \infty). \quad (39)$$

In other words, we shall focus on the principal series representation of $\mathfrak{so}(1, d - 1)$. Note that $j(j - d + 2) = -k^2 - (d - 2)^2/4$ is real though $j$ is complex. Physically, $k$ plays the role of the modulus of spatial momentum. From now on we shall write the eigenstate as $|\Delta, \omega, k; \sigma \rangle$. 


Now there are two important things for the following discussion. The first is that the eigenvalue $\Delta (\Delta - d)$ is invariant under the exchange $\Delta \to d - \Delta$, which means that the vectors $|\Delta, \omega, k; \sigma \rangle$ and $|d - \Delta, \omega, k; \sigma \rangle$ share the same eigenvalue of $C_2[so(2, d)]$. These two vectors are mapped to each other by the intertwining operators and satisfy the following relations:

$$G_\alpha |d - \alpha, \omega, k; \sigma \rangle = \tilde{G}_\alpha (\omega, k) |\alpha, \omega, k; \sigma \rangle, \quad \alpha \in \{\Delta, d - \Delta\},$$  \hspace{1cm} (40)

where the proportional coefficients $\tilde{G}_\alpha (\omega, k)$ are the momentum-space two-point functions. From now on $J_\alpha^{ab}$, $H_\alpha$, $E_\alpha^{ab}$, etc. denote the $SO(2, d)$ generators that act on the representation space spanned by the vectors $\{|\alpha, \omega, k; \sigma \rangle\}$. For example, their differential representations are given in (7).

The second important thing is the set of generators $E_\alpha^{\pm a}$. One can show that there exist certain linear combinations $E_\alpha^{\pm}$ of these generators that satisfy the following ladder equations:

$$E_\alpha^{\pm} |\alpha, \omega, k; \sigma \rangle = A^{\pm} \left[ \alpha - \frac{d - 2}{2} \mp i(\omega \pm k) \right] |\alpha, \omega \pm i, k \mp i; \sigma \rangle + B^{\pm} \left[ \alpha - \frac{d - 2}{2} \mp i(\omega \mp k) \right] |\alpha, \omega \pm i, k \mp i; \sigma \rangle. \hspace{1cm} (41)$$

For example, in the case of Rindler wedge they are given by $E_\alpha^{\pm} = E_\alpha^{\pm d} + E_\alpha^{\pm (d+1)}$. Note that $A^{\pm}$ and $B^{\pm}$ are $\alpha$-independent irrelevant factors.

Now we have almost done. Let us finally consider the intertwining relations $E_\Delta^{\pm} G_\Delta = G_\Delta E_{d-\Delta}^{\pm}$. Applying these to the state $|d - \Delta, \omega, k; \sigma \rangle$ we get

$$E_\Delta^{\pm} G_\Delta |d - \Delta, \omega, k; \sigma \rangle = G_\Delta E_{d-\Delta}^{\pm} |d - \Delta, \omega, k; \sigma \rangle. \hspace{1cm} (42)$$

It follows from (40) and (41) that the identities (42) result in the following nontrivial functional equations in complex momentum space:

$$\tilde{G}_\Delta (\omega \pm i, k \mp i) = \frac{\Delta - \frac{d - 2}{2} \mp i(\omega + k)}{\Delta - \frac{d - 2}{2} \mp i(\omega + k)} \tilde{G}_\Delta (\omega, k), \hspace{1cm} \text{ (43)}$$

$$\tilde{G}_\Delta (\omega \pm i, k \mp i) = \frac{\Delta - \frac{d - 2}{2} \mp i(\omega - k)}{\Delta - \frac{d - 2}{2} \mp i(\omega - k)} \tilde{G}_\Delta (\omega, k), \hspace{1cm} \text{ (44)}$$

where $\tilde{\Delta} = d - \Delta$ is the scaling dimension of the shadow operator. Since these are kind of recurrence relations, we can guess the solution by iteration.

"Minimal" solutions to the recurrence relations are as follows:

$$\tilde{G}_\Delta^{\pm} (\omega, k) \propto e^{\pm \pi \omega} \left| \Gamma \left( \Delta - \frac{d - 2}{2} + i(\omega + k) \right) \right|^2 \left| \Gamma \left( \Delta - \frac{d - 2}{2} + i(\omega - k) \right) \right|^2, \hspace{1cm} \text{ (45)}$$

which can be interpreted as the positive- and negative-frequency Wightman functions. Indeed, these satisfy the KMS condition in momentum space, $\tilde{G}_\Delta^{\pm} (\omega, k) = e^{2\pi \omega} \tilde{G}_\Delta^{\pm} (\omega, k)$. One can also check that the solutions (45) exactly
coincide with the Fourier transform of (32) [11]. Note that $T$ can be restored by the replacements $\omega \rightarrow \omega/(2\pi T)$ and $k \rightarrow k/(2\pi T)$.

To summarize, we have seen that the intertwining relations, which are just the conformal Ward–Takahashi identities in disguise, result in the recurrence relations (43) and (44) when applied to the $SO(1,1)$ diagonal basis. These are the conformal Ward–Takahashi identities at finite temperature and give us nontrivial constraints on momentum-space two-point functions. Though may need a bit of experience, one can deduce the momentum-space correlators from these constraints without recourse to the notoriously complicated Fourier transform. We think this is a big step toward the understanding of real-time momentum-space correlators in $d(\geq 3)$-dimensional finite-temperature CFT, because these have not been studied in the literature. In fact, for $d \geq 3$ and at nonzero temperature, even the momentum-space two-point functions of scalar primary operators have been unknown. It would be quite interesting to generalize our approach to thermal spinning correlators.

References

1. J.J. Bisognano, E.H. Wichmann, J. Math. Phys. 16 (1975) 985–1007
2. J.J. Bisognano, E.H. Wichmann, J. Math. Phys. 17 (1976) 303–321
3. D. Buchholz, On the structure of local quantum fields with nontrivial interaction, in Operator Algebras, Ideals and their Applications in Theoretical Physics (Teubner, Leipzig, 1978), pp. 146–153
4. V.K. Dobrev, G. Mack, V.B. Petkova, S.G. Petrova, I.T. Todorov, Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory (Springer, Berlin–Heidelberg, 1977)
5. S. Ferrara, A.F. Grillo, G. Parisi, R. Gatto, Lett. Nuovo Cim. 4 (1972) 115–120
6. E.S. Fradkin, M.Y. Palchik, Phys. Rept. 44 (1978) 249–349
7. R. Haag, N.M. Hugenholtz, M. Winnink, Commun. Math. Phys. 5 (1967) 215–236
8. R. Haag, Local Quantum Physics: Fields, Particles, Algebras, 2nd edn. (Springer, Berlin–Heidelberg, 1996)
9. P.D. Hislop, R. Longo, Commun. Math. Phys. 84 (1982) 71–85
10. K. Koller, Commun. Math. Phys. 40 (1975) 15–35
11. S. Ohya, Int. J. Mod. Phys. A32 (2017) 1750006, arXiv:1611.00763 [hep-th]
12. G.L. Sewell, Annals Phys. 141 (1982) 201–224
13. S. Takagi, Prog. Theor. Phys. Suppl. 88 (1986) 1–142
14. I.T. Todorov, M.C. Mintchev, V.B. Petkova, Conformal Invariance in Quantum Field Theory (Edizioni della Normale, Pisa, 1978)