Generic Evolutionary Quantum Universe

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We consider a Schrödinger quantum dynamics for the gravitational field associated to a
generic cosmological model and then we solve the corresponding eigenvalue problem. We
show that, from a phenomenological point of view, an Evolutionary Quantum Cosmology
overlaps the Wheeler-DeWitt approach.

1. Introduction

The canonical approach in Quantum Gravity is characterized by the so-called frozen
formalism, i.e. the absence of a time evolution for the wave functional.\textsuperscript{1,2} It has
been proposed\textsuperscript{3,4} that such feature disappears as soon as the impossibility of a
physical slicing without frame fixing is recognized for a quantum spacetime. In this
work\textsuperscript{5} we start with a Schrödinger dynamics for the gravitational field using Planck
mass particle as a “clock” for the system; and we will analyze the meaning of the
corresponding spectrum (we deal with a new energy density) in the framework of a
generic inhomogeneous Universe.

2. Evolutionary Quantum Gravity

In this section we briefly analyze the implication of a Schrödinger formulation of
the quantum dynamics for the gravitational field. We require the theory to evolve
along the spacetime slicing so that $\Psi = \Psi(t, \{h_{ij}\})$; so the quantum evolution is
governed by a smeared Schrödinger equation

$$i\partial_t \Psi = \hat{\mathcal{H}} \Psi \equiv \int_{\Sigma_t} d^3 x \left( N \hat{H} \right) \Psi$$

(1)
being $\hat{H}$ the super-Hamiltonian operator and $N$ the lapse function. If we now take the right expansion for the wave functional, the Schrödinger dynamics is reduced to an eigenvalues problem of the form

$$\hat{H}\chi = \epsilon\chi, \quad \hat{H}_i\chi = 0,$$

which outlines the appearance of a non-zero super-Hamiltonian eigenvalue. Is not difficult to show that the classical limit (in the sense of WKB approximation) of the above model is characterized by the appearance of a new matter contribution, which admits the following energy density:

$$T_{00} = -\frac{\epsilon(x) x^i}{\sqrt{h}}, \quad h = \det h_{ij}.\quad (3)$$

The explicit form of (3) is that of a dust fluid co-moving with the slicing 3-hypersurfaces, i.e. we deal with $T_{\mu\nu} = \rho n_{\mu} n_{\nu}$.

3. The Generic Quantum Universe and its Spectrum

We now apply the Schrödinger approach to a Quantum Universe that has to be described by a generic inhomogeneous model, which has a dynamics summarized, asymptotically to the Big-Bang, by the following variational principle

$$\delta S = \delta \int_{\Sigma} dt d^3x (p_\alpha \partial_t q^\alpha - NH) \quad (4)$$

where adopting Misner-like variables $R, \beta_\pm$ ($R$ is the scale factor and $\beta_\pm$ describes the anisotropies), the super-Hamiltonian has the structure

$$H(x) = \kappa \left[ \frac{\rho H}{R} + \frac{1}{R^3} \left( p_+^2 + p_-^2 \right) \right] + \frac{3}{8\pi} \frac{p_+^2}{R^3} \frac{R^3}{4\pi l_p^4} V(\beta_\pm) + R^3 (\rho_{ur} + \rho_{pg}), \quad (5)$$

where $\kappa = 8\pi l_p^2$. We have added to the dynamics of the system an ultrarelativistic energy density ($\rho_{ur} = \mu^2/R^4$), a perfect gas contribution ($\rho_{pg} = \sigma^2/R^5$) and a scalar field $\phi$ (a free inflaton field).

Performing the canonical quantization of this model we obtain the following eigenvalue problem (2), with the right normal ordering:

$$\left\{ \kappa \frac{\partial R}{R} \frac{1}{R^3} (\partial_+^2 + \partial_-^2) - \frac{3}{8\pi} \frac{p_+^2}{R^3} \frac{R^3}{4\pi l_p^4} V(\beta_\pm) + R^3 (\rho_{ur} + \rho_{pg}) \right\} \chi = \epsilon \chi. \quad (6)$$

The appropriate boundary condition for this problem are: i) $\chi(R = 0, \beta_\pm, \phi) < \infty$ that relies on the idea that the quantum Universe is singularity-free and ii) $\chi(R \to \infty, \beta_\pm, \phi) = 0$ that ensures a physical behavior at “large” scale factor.

In order to study the previous eigenvalue problem we expand the wave function as $\chi(R, \beta_\pm, \phi) = \int \theta_K(R) F_K(R, \beta_\pm, \phi) dK$, and then performing an adiabatic approximation ($|\partial_R F| \ll |\partial_R \theta|$) we obtain the following reduced problems:

$$\kappa \frac{d}{dR} \left( \frac{1}{R} d\theta \right) + \left( \kappa \frac{K^2}{R^3} + R^3 (\rho_{ur} + \rho_{pg}) - \epsilon \right) \theta = 0, \quad (7)$$
\[-(\partial_r^2 + \partial_\phi^2 + \frac{3}{8\pi\kappa} \partial_\phi^2) F + \frac{R_0^6}{4\kappa^2 l_{in}^2} V(\beta_\pm) F = K^2(R) F. \quad (8)\]

The function \(F\) is a plane wave as soon as we neglect the potential term in (8) for same \(R^* \ll 1\). The solution of (7) is a series in \(R\) multiplied by a Gaussian function peaked around \(R = c l_p^2/16\pi\). Since we required the wave function to decay at large scale factor \(R\) we have to terminate the series and obtain the spectrum of the super-Hamiltonian:

\[\epsilon_{n,\gamma} = \frac{\sigma^2}{l_P^2 (n + \gamma - 1/2)}, \quad (9)\]

so that the ground state \(n = 0\) eigenvalue, for \(\gamma < 1/2\), is negative; therefore is associated via (3) to a positive dust energy density.

4. Phenomenology of the Dust Fluid

In order to analyze the cosmological implication of this new matter contribution, we have to impose a cut-off length in our model, requiring that the Planck length \(l_P\) is the minimal physical length accessible by an observer \((l \geq l_P)\). So, from the thermodynamical relation for the perfect gas, we obtain a constraint on the \(\rho_{pg}\) and then on the super-Hamiltonian eigenvalue:

\[l^3 \equiv \frac{V}{N} = \frac{3}{2} \frac{l_P^2}{\rho_{pg} \lambda^2} \geq l_P^3 \quad \Rightarrow \quad \rho_{pg} \leq O(1/l_P^4), \quad (10)\]

where \(l\) is the length per particle and \(\lambda\) the corresponding thermal length \((\lambda = l_P)\). Therefore we get \(\sigma^2 \leq O(l_P)\) and so \(|\epsilon_0| \leq (1/l_P)\): the spectrum is limited by below.

The contribution of our dust fluid to the actual critical parameter is

\[\Omega_{dust} \sim \frac{\rho_{dust}}{\rho_{today}} \sim O \left(10^{-60}\right). \quad (11)\]

Such a parameter is much less then unity and so no phenomenology can came out (today) from our dust fluid. In other words an Evolutionary Quantum Cosmology overlaps the Wheeler-DeWitt approach. Finally we face the question of the classical limit of the spectrum in the sense of large occupation numbers \(n \rightarrow \infty\). As we can see from (11) the eigenvalue approaches zero as \(1/n\). Therefore for very large \(n\), our quantum dynamics would overlap the Wheeler-DeWitt approach.

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