FOURIER MULTIPLIERS ON WEIGHTED $L^p$ SPACES

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ABSTRACT. The paper provides a complement to the classical results on Fourier multipliers on $L^p$ spaces. In particular, we prove that if $q \in (1, 2)$ and a function $m : \mathbb{R} \to \mathbb{C}$ is of bounded $q$-variation uniformly on the dyadic intervals in $\mathbb{R}$, i.e., $m \in V_q(\mathcal{D})$, then $m$ is a Fourier multiplier on $L^p(\mathbb{R}, wdx)$ for every $p \geq q$ and every weight $w$ satisfying Muckenhoupt’s $A_{p/q}$-condition. We also obtain a higher dimensional counterpart of this result as well as of a result by E. Berkson and T.A. Gillespie including the case of the $V_q(\mathcal{D})$ spaces with $q > 2$. New weighted estimates for modified Littlewood-Paley functions are also provided.

1. Introduction and Statement of Results

For an interval $[a, b]$ in $\mathbb{R}$ and a number $q \in [1, \infty)$ denote by $V_q([a, b])$ the space of all functions $m : [a, b] \to \mathbb{C}$ of bounded $q$-variation over $[a, b]$, i.e.,

$$\|m\|_{V_q([a, b])} := \sup_{x \in [a, b]} |m(x)| + \|\text{Var}_q([a, b])m\| < \infty,$$

where $\|m\|_{\text{Var}_q([a, b])} := \sup\{\sum_{i=0}^{n-1} |m(t_{i+1}) - m(t_i)|^{q/2}\}$ and the supremum is taken over all finite sequences $a =: t_0 < t_1 < \ldots < t_n := b$ ($n \in \mathbb{N}$). We write $\mathcal{D}$ for the dyadic decomposition of $\mathbb{R}$, i.e., $\mathcal{D} := \{\pm(2^k, 2^{k+1}) : k \in \mathbb{Z}\}$, and set

$$V_q(\mathcal{D}) := \left\{ m : \mathbb{R} \to \mathbb{C} : \sup_{I \in \mathcal{D}} \|m_I\|_{V_q(I)} < \infty \right\} \quad (q \in [1, \infty)).$$

Moreover, let $A_p(\mathbb{R})$ ($p \in [1, \infty)$) be the class of weights on $\mathbb{R}$ which satisfy the Muckenhoupt $A_p$ condition. Denote by $[w]_{A_p}$ the $A_p$-constant of $w \in A_p(\mathbb{R})$. If $w \in A_\infty(\mathbb{R}) := \cup_{p \geq 1} A_p(\mathbb{R})$ we write $M_p(\mathbb{R}, w)$ for the class of all multipliers on $L^p(\mathbb{R}, w)$ ($p > 1$), i.e.,

$$M_p(\mathbb{R}, w) := \{ m \in L^\infty(\mathbb{R}) : T_m \text{ extends to a bounded operator on } L^p(\mathbb{R}, w) \}.$$

Here $T_m$ stands for the Fourier multiplier with the symbol $m$, i.e., $(T_m f) \hat{f} = m \hat{f}$ ($f \in S(\mathbb{R})$). Note that $M_p(\mathbb{R}, w)$ becomes a Banach space under the norm $\|m\|_{M_p(\mathbb{R}, w)} := \|T_m\|_{L^p(L^p(\mathbb{R}, w))}$ ($m \in M_p(\mathbb{R}, w)$).

The main result of the paper is the following complement to results due to D. Kurtz [18], R. Coifman, J.-L. Rubio de Francia, S. Semmes [8], and E. Berkson, T. Gillespie [3].

Theorem A. (i) Let $q \in (1, 2]$. Then, $V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w)$ for every $p \geq q$ and every Muckenhoupt weight $w \in A_{p/q}(\mathbb{R})$.

(ii) Let $q > 2$. Then, $V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w)$ for every $2 \leq p < (\frac{1}{2} - \frac{1}{q})^{-1}$ and every Muckenhoupt weight $w \in A_{p/2}$ with $s_w > (1 - p(\frac{1}{2} - \frac{1}{q}))^{-1}$.

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Here, for every \( w \in A_\infty(\mathbb{R}) \), we set \( s_w := \sup\{ s \geq 1 : w \in RH_s(\mathbb{R}) \} \) and we write \( w \in RH_s(\mathbb{R}) \) if
\[
\sup_{a < b} \left( \frac{1}{b-a} \int_a^b w(x)^s \, dx \right)^{1/s} \left( \frac{1}{b-a} \int_a^b w(x) \, dx \right)^{-1} < \infty.
\]
Recall that, by the reverse Hölder inequality, \( s_w \in (1, \infty] \) for every Muckenhoupt weight \( w \in A_\infty(\mathbb{R}) \).

For the convenience of the reader we repeat the relevant material from the literature, which we also use in the sequel.

Recall first that in [18] D. Kurtz proved the following weighted variant of the classical Marcinkiewicz multiplier theorem.

**Theorem 1** ([18] Theorem 2]). \( V_1(\mathcal{D}) \subset M_p(\mathbb{R}, w) \) for every \( p \in (1, \infty) \) and every Muckenhoupt weight \( w \in A_p(\mathbb{R}) \).

As in the unweighted case, Theorem 1 is equivalent to a weighted variant of the Littlewood-Paley decomposition theorem, which asserts that for the square function \( S^p \) corresponding to the dyadic decomposition \( \mathcal{D} \) of \( \mathbb{R} \), \( \| S^p f \|_{p, w} \approx \| f \|_{p, w} \) (\( f \in L^p(\mathbb{R}, w) \)) for every \( p \in (1, \infty) \) and \( w \in A_p(\mathbb{R}) \); see [18] Theorem 1, and also [18] Theorem 3.3. Here and subsequently, if \( \mathcal{I} \) is a family of disjoint intervals in \( \mathbb{R} \), we write \( S^\mathcal{I} \) for the Littlewood-Paley square function corresponding to \( \mathcal{I} \), i.e., \( S^\mathcal{I} f := (\sum_{I \in \mathcal{I}} |S_If|^2)^{1/2} \) (\( f \in L^2(\mathbb{R}) \)).

Recall also that in [26] J.-L. Rubio de Francia proved the following extension of the classical Littlewood-Paley decomposition theorem.

**Theorem 2** ([26] Theorem 6.1]). \( 2 < p < \infty \) and \( w \in A_{p/2}(\mathbb{R}) \). Then for an arbitrary family \( \mathcal{I} \) of disjoint intervals in \( \mathbb{R} \) the square function \( S^\mathcal{I} \) is bounded on \( L^p(\mathbb{R}, wdx) \).

Applying Rubio de Francia’s inequalities, i.e. Theorem 2 R. Coifman, J.-L. Rubio de Francia, and S. Semmes [8] proved the following extension and improvement of the classical Marcinkiewicz multiplier theorem. (See Section 2 for the definition of \( R_2(\mathcal{D}) \).)

**Theorem 3** ([8] Théorème 1 and Lemme 5]). \( 2 \leq q < \infty \). Then, \( V_q(\mathcal{D}) \subset M_p(\mathbb{R}) \) for every \( p \in (1, \infty) \) such that \( \frac{1}{p} - \frac{1}{2} < \frac{1}{q} \).

Furthermore, \( R_2(\mathcal{D}) \subset M_2(\mathbb{R}, w) \) for every \( w \in A_1(\mathbb{R}) \).

Subsequently, a weighted variant of Theorem 3 was given by E. Berkson and T. Gillespie in [4]. According to our notation their result can be formulated as follows.

**Theorem 4** ([4] Theorem 1.2]). Suppose that \( 2 \leq p < \infty \) and \( w \in A_{p/2}(\mathbb{R}) \). Then, there is a real number \( s > 2 \), depending only on \( p \) and \( [w]_{A_{p/2}} \), such that \( \frac{1}{s} > |\frac{1}{p} - \frac{1}{2}| \) and \( V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w) \) for all \( 1 \leq q < s \).

Note that the part (i) of Theorem A fills a gap which occurs in Theorem 1 and the weighted part of Theorem 3. The part (ii) identifies the constant \( s \) in Berkson-Gillespie’s result, i.e., Theorem 3 as \( \left( \frac{1}{2} - \frac{1}{p} \right)^{-1} \), where \( s_w' := \frac{1}{s_w - 1} \), and in general, this constant is best possible.

Except for some details, the proofs given below reproduce well-known arguments from the Littlewood-Paley theory; in particular, ideas which have been
presented in [18], [8], [26], and [29]. A new point of our approach is the following result on weighted estimates for modified Littlewood-Paley functions $S_q^f(·) := \left(\sum_{I \in \mathcal{I}} |S_I(·)|_{q'}^{1/q'} \right)^{1/q'}$ ($q \in (1,2]$), which may be of independent interest.

**Theorem B.** (i) Let $q \in (1,2)$, $p > q$, and $w \in A_{p/q}(R)$. Then, there exists a constant $C > 0$ such that for any family $\mathcal{I}$ of disjoint intervals in $R$

$$\|S_q^f\|_{p,w} \leq C\|f\|_{p,w} \quad (f \in L^p(R,wdx)).$$

Moreover, for every $q \in (1,2)$, $p > q$ and $V \subset A_{p/q}(R)$ with $\sup_{w \in V}[w]_{A_{p,q}} < \infty$

$$\sup \{\|S_q^f\|_{p,w} : w \in V, \mathcal{I} \text{ a family of disjoint intervals in } R, \|f\|_{p,w} = 1\} < \infty.$$

(ii) Let $q \in (1,2)$. For any family $\mathcal{I}$ of disjoint intervals in $R$ and every Muckenhoupt weight $w \in A_1(R)$, the operator $S_q^f$ maps $L^q(R,wdx)$ into weak-$L^{q'}(R,wdx)$. Moreover, if $V \subset A_1(R)$ with $\sup_{w \in V}[w]_{A_1} < \infty$ then

$$\sup \{\|S_q^f\|_{L^{q',w}} : w \in V, \mathcal{I} \text{ a family of disjoint intervals in } R, \|f\|_{L^q} = 1\} < \infty.$$
to keep the pattern of the proof of the main result of the paper, Theorem A, more transparent. Therefore, we postpone the proof of Theorem B(ii) to Section 3.

2. Proofs of Theorems B(i) and A

We first introduce auxiliary spaces which are useful in the proof of Theorem A. Let \( q \in [1, \infty) \). If \( I \) is an interval in \( \mathbb{R} \) we denote by \( \mathcal{E}(I) \) the family of all step functions from \( I \) into \( \mathbb{C} \). If \( m := \sum_{J \in \mathcal{I}} a_J \chi_J \), where \( \mathcal{I} \) is a decomposition of \( I \) into subintervals and \( \{a_J\} \subset \mathbb{C} \), write \( [m]_q := (\sum_{J \in \mathcal{I}} |a_J|^q)^{1/q} \). Set \( \mathcal{R}_q(I) := \{m \in \mathcal{E}(I) : [m]_q \leq 1\} \) and

\[
\mathcal{R}_q(D) := \{ m : \mathbb{R} \to \mathbb{C} : m_I \in \mathcal{R}_q(I) \text{ for every } I \in D \}.
\]

Moreover, let

\[
R_q(I) := \left\{ \sum_j \lambda_j m_j : m_j \in \mathcal{R}_q(I), \sum_j |\lambda_j| < \infty \right\}
\]

and

\[
\|m\|_{R_q(I)} := \inf \left\{ \sum_j |\lambda_j| : m = \sum_j \lambda_j m_j, \ m_i \in \mathcal{R}_q(I) \right\} \quad (m \in R_q(I)).
\]

Note that \( (R_q(I), \| \cdot \|_{R_q(I)}) \) is a Banach space. Set

\[
R_q(D) := \left\{ m : \mathbb{R} \to \mathbb{C} : \sup_{I \in D} \|m_I\|_{R_q(I)} < \infty \right\} \quad (q \in [1, \infty)).
\]

In the sequel, if \( \mathcal{I} \) is a family of disjoint intervals in \( \mathbb{R} \), we write \( S^r_I f := (\sum_{J \in \mathcal{I}} |S_I(f)|^r)^{1/r}, \ (r \in (1, 2], f \in L^r(\mathbb{R})) \) and \( S^r_I f := \sup_{t \in \mathcal{I}} |S_I(f)| (f \in L^1(\mathbb{R})) \).

We next collect main ingredients of the proof of Theorem B(i), which provides crucial vector-valued estimates for weighted multipliers in the proof of Theorem A; see e.g. \([3]\).

Lemma 5 ([25 Theorem 2.1, Part III]). Let \( s \in (1, \infty) \) and \( w \in A_s(\mathbb{R}) \). Then, there exists a constant \( C > 0 \) such that for any family \( \mathcal{I} \) of disjoint intervals in \( \mathbb{R} \)

\[
\|S^r_I f\|_{s,w} \leq C\|f\|_{s,w} \quad (f \in L^s(\mathbb{R}, wdx)).
\]

Moreover, for every \( s > 1 \) and every set \( \mathcal{V} \subset A_s(\mathbb{R}) \) with \( \sup_{w \in \mathcal{V}} [w]_{A_s} < \infty \)

\[
\sup \{ \|S^r_I\|_{s,w} : w \in \mathcal{V}, \mathcal{I} \text{ a family of disjoint intervals in } \mathbb{R} \} < \infty.
\]

Remark 6. The second statement of Lemma 5 can be obtained from a detailed analysis of the constants involved in the results which are used in the proof of [24 Theorem 2.1(a) \Rightarrow (b), Part III], i.e., the weighted version of the Fefferman-Stein inequality and the reverse Hölder inequality.

Recall the weighted version of the Fefferman-Stein inequality, which in particular says that for every \( p \in (1, \infty) \) and every Muckenhoup weight \( w \in A_p(\mathbb{R}) \) there exists a constant \( C_{p,w} > 0 \), which depends only on \( p \) and \([w]_{A_p} \), such that

\[
\int_{\mathbb{R}} M f(t)^p w(t) \, dt \leq C_{p,w} \int_{\mathbb{R}} M^\# f(t)^p w(t) \, dt \quad (f \in L^p(\mathbb{R}) \cap L^p(\mathbb{R}, w)),
\]

(1)
where $M$ and $M'$ denote the Hardy-Littlewood maximal operator and the Fefferman-Stein sharp maximal operator, respectively; see [15, Theorem, p.41], or [14, Theorem 2.20, Chapter IV]. We emphasize here that the constant $C_{p,w}$ on the right-hand side of this inequality is not given explicitly in the literature, but it can be obtained from a detailed analysis of the constants involved in the results which are used in the proof of (1), $\sup_{w \in V} C_{p,w} < \infty$ for every subset $V \subset A_p(\mathbb{R})$ with $\sup_{w \in V} [w]_{A_p} < \infty$.

Furthermore, it should be noted that if $V \subset A_p(\mathbb{R})$ with $\sup_{w \in V} [w]_{A_p} < \infty$, then there exists $\epsilon > 0$ such that $V \subset A_{p-\epsilon}(\mathbb{R})$ and $\sup_{w \in V} [w]_{A_{p-\epsilon}} < \infty$. It can be directly obtained from a detailed analysis of the constants involved in main ingredients of the proof of the reverse Hölder inequality. Cf., e.g., [20, Lemma 2.3].

We refer the reader to [13, Chapter IV] and [12, Chapter 7] for recent expositions of the results involved in the proof of the reverse Hölder inequality and the Fefferman-Stein inequality, which originally come from [7], and [22, 23].

The next lemma is a special variant of Rubio de Francia’s extrapolation theorem; see [26, Theorem 3]. For the convenience of the reader we rephrase [20, Theorem 3] here in the context of Muckenhoupt weights merely.

**Lemma 7 (20, Theorem 3).** Let $\lambda$ and $r$ be fixed with $1 \leq \lambda \leq r < \infty$, and let $S$ be a family of sublinear operators which is uniformly bounded in $L^p(\mathbb{R}, v dx)$ for each $w \in A_{r,\lambda}(\mathbb{R})$, i.e.,

$$\int |Sf|^r v dx \leq C_{r,w} \int |f|^r v dx \quad (S \in S, \ w \in A_{r,\lambda}(\mathbb{R})).$$

If $\lambda < p, \alpha < \infty$ and $w \in A_{p,\alpha}(\mathbb{R})$, then $S$ is uniformly bounded in $L^p(\mathbb{R}, w dx)$ and even more:

$$\left(\sum_j |S_j f_j|^\alpha\right)^{p/\alpha} v dx \leq C_{p,\alpha,w} \left(\sum_j |f_j|^\alpha\right)^{p/\alpha} w dx \quad (f_j \in L^p(\mathbb{R}, w dx), \ S_j \in S).$$

Combining Lemma 5 with Theorem 2, we get the intermediate weighted estimates for operators $S_q^2$ ($q \in (1, 2)$) stated in Theorem B(i).

For the background on the interpolation theory we refer the reader to [9]; in particular, see [3, Chapter 4 and Section 5.5].

**Proof of Theorem B(i).** Fix $q \in (1, 2)$ and $w \in A_{2/q}(\mathbb{R})$. By the reverse Hölder inequality, $w \in A_{2/r}(\mathbb{R})$ for some $r \in (q, 2)$. Note that there exist $p \in (2, q')$ and $s > 1$ such that $\frac{1}{q} + \frac{1}{q'} = \frac{1}{s}$. Therefore, combining Theorem 2 with Lemma 5 by complex interpolation, the operator $S_q^2_{2q/(p')}$ is bounded on $L^r(\mathbb{R}, v)$ for every $v \in A_1(\mathbb{R})$. Since $p > 2$, the same conclusion holds for $S_q^2$.

By Rubio de Francia’s extrapolation theorem, Lemma 7, we get that $S_q^2$ is bounded on $L^2(\mathbb{R}, v)$ for every $v \in A_{2/r}(\mathbb{R})$. According to our choice of $r$, we get the boundedness of $S_q^2$ on $L^2(\mathbb{R}, w)$.

Since the weight $w$ was taken arbitrarily, we can again apply Rubio de Francia’s extrapolation theorem, Lemma 7, to complete the proof of the first statement.

The second statement follows easily from a detailed analysis of the first one. For a discussion on the character of the dependence of constants in Rubio de Francia’s iteration algorithm, we refer the reader to [11, or 10 Section 3.4]. See also the comment on the reverse Hölder inequality in Remark 6.

Note that $R_q(I) \subset V_q(I)$ for every interval $I$ in $\mathbb{R}$ and $q \in [1, \infty)$. However, the following reverse inclusions hold for these classes.
Lemma 8 ([8] Lemma 2]). Let $1 \leq q < p < \infty$. For every interval $I$ in $\mathbb{R}$, $V_q(I) \subset R_p(I)$ with the inclusion norm bounded by a constant independent of $I$.

The patterns of the proofs of the parts $(i)$ and $(ii)$ of Theorem A are essentially the same. Therefore, we sketch the proof of the part $(ii)$ below.

Proof of Theorem A. $(i)$ We only give the proof for the more involved case $q \in (1,2)$; the case $q = 2$ follows simply from Theorem [3] and interpolation arguments presented below; see also Remark [9] below.

Fix $q \in (1,2)$. We first show that for every subset $\mathcal{V} \subset A_{2/q}(\mathbb{R})$ such that $\sup_{w \in \mathcal{V}} [w]_{A_{2/q}} < \infty$ we have

$$\sup \{ \| T_{m_{\chi I}} \|_{2,w} : m \in R_q(\mathcal{D}), \| m \|_{R_q(\mathcal{D})} \leq 1, w \in \mathcal{V}, I \in \mathcal{D} \} < \infty.$$ 

Fix $\mathcal{V} \subset A_{2/q}(\mathbb{R})$ with $\sup_{w \in \mathcal{V}} [w]_{A_{2/q}} < \infty$. Note that, by the definition of the $R_q$-classes, it is sufficient to prove the claim with $R_q(\mathcal{D})$ replaced by $R_q(\mathcal{D})$. Fix $m \in R_q(\mathcal{D})$ and set $m_{\chi I} := \sum_{j \in \mathbb{Z}} a_{1,j} \chi_{I}$ for every $I \in \mathcal{D}$, where $\mathcal{I}_q = \mathcal{I}_{1,m}$ is a decomposition of $I$ and $(a_{1,j})_{j \in I} \subset \mathbb{C}$ is a sequence with $\sum_{j \in I} |a_{1,j}|^q \leq 1$. Note that $T_{m_{\chi I}} f = \sum_{j \in \mathbb{Z}} a_{1,j} S_j f$ and $\| T_{m_{\chi I}} f \|_{2,w} \leq \| S_j f \|_{2,w}$ for every $I \in \mathcal{D}$, $w \in \mathcal{V}$ and $f \in L^2(\mathbb{R}, w)$. Therefore, by Lemma [5], our claim holds.

By interpolation argument, we next sharpen this claim and prove that for every subset $\mathcal{V} \subset A_{2/q}(\mathbb{R})$ with $\sup_{w \in \mathcal{V}} [w]_{A_{2/q}} < \infty$ there exists $\alpha = \alpha(q, \mathcal{V}) > 1$ such that

$$\sup \{ \| T_{m_{\chi I}} \|_{2,w} : m \in R_q(\mathcal{D}), \| m \|_{R_q(\mathcal{D})} \leq 1, w \in \mathcal{V}, I \in \mathcal{D} \} < \infty. \quad (2)$$

Note that, by the reverse Hölder inequality, see also Remark [6], there exists $\alpha > 1$ such that $w^\alpha \in A_{2/q}(\mathbb{R})$ ($w \in \mathcal{V}$) and $\sup_{w \in \mathcal{V}} [w^\alpha]_{A_{2/q}} < \infty$. From what has already been proved and Plancherel’s theorem, for every $I \in \mathcal{D}$ and $w \in \mathcal{V}$ the bilinear operators

$$R_q(I) \times L^2(\mathbb{R}, w^\alpha dx) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R}, w^\alpha dx)$$

$$L^\infty(\mathbb{R}) \times L^2(\mathbb{R}) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R})$$

are well-defined and bounded uniformly with respect to $w \in \mathcal{V}$ and $I \in \mathcal{D}$. Therefore, by complex interpolation, $(R_q(I), L^\infty(\mathbb{R}))_{\frac{1}{\alpha}} \subset M_2(\mathbb{R}, w)$. However, it is easy to check that $R_q(I) \subset (R_q(I), L^\infty(\mathbb{R}))_{\frac{1}{\alpha}}$ with the inclusion norm bounded by a constant independent of $I \in \mathcal{D}$. We thus get (2).

In consequence, by Lemma [8] it follows that

$$\sup \{ \| T_{m_{\chi I}} \|_{2,w} : m \in V_q(\mathcal{D}), \| m \|_{V_q(\mathcal{D})} \leq 1, w \in \mathcal{V}, I \in \mathcal{D} \} < \infty \quad (3)$$

for every subset $\mathcal{V} \subset A_{2/q}(\mathbb{R})$ with $\sup_{w \in \mathcal{V}} [w]_{A_{2/q}} < \infty$.

Hence, we can apply a truncation argument based on Kurtz’ weighted variant of Littlewood-Paley’s inequality. Namely, fix $w \in A_{2/q}(\mathbb{R})$, $m \in V_q(\mathcal{D})$ with $\| m \|_{V_q(\mathcal{D})} \leq 1$, and $f \in L^2(\mathbb{R}) \cap L^2(\mathbb{R}, w)$, $g \in L^2(\mathbb{R}, w) \cap L^2(\mathbb{R}, w^{-1})$. Note that $gw \in L^2(\mathbb{R})$ and $A_{2/q}(\mathbb{R}) \subset A_2(\mathbb{R})$. Therefore, combining the Cauchy-Schwarz
Lemma 7, yields \( V \) and \( L \). Hölder inequality and a density argument show that the reverse Hölder inequality for \( w \) exists uniformly with respect to \( I \). Finally, it is easy to see that for every \( 2 < s > r \), \( q > r \), we get:

\[
| (T_m f, g)_{L^2(\mathbb{R}, w)} | = \left| \sum_{I \in \mathcal{D}} \int_{I} S_I(T_m f) S_I(g) w dx \right| \\
\leq C \left\| \left( \sum_{I \in \mathcal{D}} |T_m x_I| S_I f |^2 \right)^{1/2} \right\|_{L^2(w)} \left\| \left( \sum_{I \in \mathcal{D}} |S_I(g)|^2 \right)^{1/2} \right\|_{L^2(w^{-1})} \\
\leq C \| f \|_{L^2(w)} \| g \|_{L^2(w)},
\]

where \( C \) is an absolute constant independent of \( m, f \) and \( g \). Now the converse of Hölder inequality and a density argument show that \( m \in M_2(\mathbb{R}, w) \).

Consequently, \( V_q(\mathcal{D}) \subset M_q(\mathbb{R}, w) \), and Rubio de Francia’s extrapolation theorem, Lemma \ref{lemma} yields \( V_q(\mathcal{D}) \subset M_q(\mathbb{R}, w) \) for every \( p > q \) and every Muckenhoupt weight \( w \in A_{p/q}(\mathbb{R}) \).

It remains to prove that \( V_q(\mathcal{D}) \subset M_q(\mathbb{R}, w) \) for every \( w \in A_1(\mathbb{R}) \). Fix \( m \in V_q(\mathcal{D}) \) and \( w \in A_1(\mathbb{R}) \). Then, by Theorem \ref{thm} (see also Remark \ref{rem}), \( T_m \) is bounded on \( L^r(\mathbb{R}, w) \) for every \( r \in (1, \infty) \). From what has already been proved, \( T_m \) is bounded on \( L^r(\mathbb{R}, w) \) for every \( r > q \). Therefore, the boundedness of \( T_m \) on \( L^q(\mathbb{R}, w) \) follows by the reverse Hölder inequality for \( w \) and a similar interpolation argument as before. This completes the proof of the part \((i)\).

\[(i)\] Fix \( q > 2 \) and \( s > \frac{q}{2} \). Let \( \mathcal{V}_s := \{ w \in A_1(\mathbb{R}) : w \in RH_s(\mathbb{R}) \} \). Note that there exists \( r = r_s > q \) such that \( \frac{1}{2} + \frac{1}{r} < \frac{1}{q} \).

Fix \( w \in \mathcal{V}_s \). By Theorem \ref{thm} the bilinear operators

\[
R_r(\mathcal{D}) \times L^2(\mathbb{R}) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R})
\]

\[
R_2(\mathcal{D}) \times L^2(\mathbb{R}, w^s) \ni (m, f) \mapsto T_m f \in L^2(\mathbb{R}, w^s)
\]

are well-defined and bounded. By interpolation, it follows that

\[
M_2(\mathbb{R}, w) \supset (R_2(\mathcal{D}), R_r(\mathcal{D})) \big|_{(\frac{1}{2}, \frac{1}{r})} \supset R_{\alpha q}(I)
\]

uniformly with respect to \( I \in \mathcal{D} \), where \( \alpha = \alpha_s := (\frac{1}{2} + \frac{1}{r})^{-1}/q > 1 \).

As in the corresponding part of the proof of \((i)\), by truncation and duality arguments, we get \( R_{\alpha q}(\mathcal{D}) \subset M_2(\mathbb{R}, w) \).

Consequently, since \( \alpha_s > 1 \) for every \( s > \frac{q}{2} \), by Lemma \ref{lemma}

\[
V_q(\mathcal{D}) \subset M_2(\mathbb{R}, w) \quad \text{for every } w \in \bigcup_{s > \frac{q}{2}} \mathcal{V}_s(\mathbb{R}). \tag{4}
\]

Note that this is precisely the assertion of \((ii)\) for \( p = 2 \).

We can now proceed by extrapolation. Since for every \( s > \frac{q}{2} \) we can rephrase \( \mathcal{V}_s \) as \( A_{\frac{1}{2}}(\mathbb{R}) \cap RH(\frac{1}{w^s}) \), by [11] Theorem 3.31, we get

\[
V_q(\mathcal{D}) \subset M_p(\mathbb{R}, w) \quad \text{for every } s > \frac{q}{2}, \quad 2 < p < 2s', \quad \text{and } w \in A_{\frac{1}{2}}(\mathbb{R}) \cap RH(\frac{1}{w^s}) \quad \text{for every } w \in \bigcup_{s > \frac{q}{2}} \mathcal{V}_s(\mathbb{R}). \tag{5}
\]

Finally, it is easy to see that for every \( 2 \leq p < \frac{1}{2} - \frac{1}{q} = 2(\frac{q}{2})' \) and \( w \in A_{p/2}(\mathbb{R}) \) with \( s_w > (1 - p(\frac{1}{2} - \frac{1}{q}'))^{-1} = (\frac{q}{2} )' \) there exists \( s = s_{p, w} > \frac{q}{2} \) such that \( p < 2s' \) and \( w \in RH(\frac{1}{w^s}) \). Therefore, \ref{4} completes the proof of \((ii)\). \hfill \square
One always has $1 \leq p \leq q \leq \infty$, see for example [2, Proposition 5.13, p.149], where the Boyd indices are defined as the reciprocals with respect to our definitions.
Let $w$ be a weight in $A_\infty(\mathbb{R})$. Then we can associate with $E$ and $w$ a rearrangement invariant Banach function space over $(\mathbb{R}, wd\mathbf{x})$ as follows

$$E_w = \{ f : \mathbb{R} \to \mathbb{C} \text{ measurable} : f^*_w \in E \},$$

and its norm is $\| f \|_w = \| f^*_w \|$, where $f^*_w$ denotes the decreasing rearrangement of $f$ with respect to $w \mathbf{x}$.

For further purposes, recall also that examples of rearrangement Banach function spaces are the Lorentz spaces $L^{p,q}$ ($1 \leq p, q \leq \infty$). Note that $L^{p,\infty} = \text{weak}^* - L^p(\mathbb{R}, w)$ for every $p \in (1, \infty)$ and $w \in A_\infty(\mathbb{R})$. The Boyd indices can be computed explicitly for many examples of concrete rearrangement invariant Banach function spaces, see e.g. [2, Chapter 4]. In particular, we have $p = q = p$ for $E := L^{p,q}$ ($1 < p < \infty, 1 \leq q \leq \infty$); see [2, Theorem 4.6].

**Lemma 10.** Let $E$ be a rearrangement invariant Banach function space on $(\mathbb{R}, dx)$ such that $1 < p, q < \infty$. Then the following statements hold.

(i) For every Muckenhoupt weight $w \in A_{\text{p.e.}}(\mathbb{R})$ there exists a constant $C_{w}$, such that for any family $I$ of disjoint bounded intervals in $\mathbb{R}$

$$C_{-1,w}^{-1} \| S^2 f \|_w \leq \| S^{W^2} f \|_w \leq C_{,w} \| S^2 f \|_w$$

and

$$\| M f \|_w \leq C_{,w} \| M^2 f \|_w$$

for every $f \in E_w$.

Moreover, if $V \subset A_{\text{p.e.}}(\mathbb{R})$ with $\sup_{w \in V} |w| A_{\text{p.e.}} < \infty$, then $\sup_{w \in V} C_{,w} < \infty$.

(ii) For every $r \in (1, \infty)$ and every Muckenhoupt weight $w \in A_{\text{p.e.}}(\mathbb{R})$ there exists a constant $C_{r,w}$ such that for any family $I$ of disjoint intervals in $\mathbb{R}$

$$\left\| \left( \sum_{I \in \mathcal{I}} |S_I f_i|^r \right)^{1/r} \right\|_w \leq C_{r,w} \left\| \left( \sum_{I \in \mathcal{I}} |f_i|^r \right)^{1/r} \right\|_w$$

for every $(f_i)_{i \in \mathcal{I}} \subset E_w(I^r(\mathcal{I}))$.

The proof follows the idea of the proof of [26, Lemma 6.3], i.e., it is based on the iteration algorithm of the Rubio de Francia extrapolation theory. We refer the reader to [10] for a recent account of this theory; in particular, see the proofs of [10, Theorems 3.9 and 4.10]. We provide below main supplementary observations which should be made.

**Proof of Lemma 10.** Note that we can restrict ourself to finite families $I$ of disjoint bounded intervals in $\mathbb{R}$. The final estimates obtained below are independent of $I$, and a standard limiting argument proves the result in the general case.

According to [18, Theorem 3.1], for every Muckenhoupt weight $w \in A_2(\mathbb{R})$ there exists a constant $C_{2,w}$ such that

$$C_{2,w}^{-1} \| S^2 f \|_{L^2(\mathbb{R}, w)} \leq \| S^{W^2} f \|_{L^2(\mathbb{R}, w)} \leq C_{2,w} \| S^2 f \|_{L^2(\mathbb{R}, w)}$$

($f \in L^2(\mathbb{R}, w)$). (9)

Moreover, one can show that $\sup_{w \in V} C_{2,w} < \infty$ for every subset $V \subset A_2(\mathbb{R})$ with $\sup_{w \in V} |w| A_2 < \infty$.

Therefore, we are in a position to adapt the extrapolation techniques from $A_2$ weights; see for example the proof of [10, Theorem 4.10, p. 76]. Fix $E$ and $w \in
A_{p_k}(\mathbb{R})$ as in the assumption. Let $E'_w$ be the associate space of $E_w$, see [2] Definition 2.3, p. 9. Let $\mathcal{R} = \mathcal{R}_w : E_w \to E_w$ and $\mathcal{R}' = \mathcal{R}'_w : E'_w \to E'_w$ be defined by

$$\mathcal{R}h(t) = \sum_{j=0}^{\infty} \frac{M^j h(t)}{2^j M^j}_w, \quad 0 \leq h \in E_w,$$

and

$$\mathcal{R}'h(t) = \sum_{j=0}^{\infty} \frac{S^j h(t)}{2^j S^j}_w, \quad 0 \leq h \in E'_w,$$

where $Sh := M(hw)/w$ for $h \in E'_w$. As in the proof of [10] Theorem 4.10, p. 76] the following statements are easily verified:

(a) For every positive $h \in E_w$ one has

$$h \leq \mathcal{R}h \quad \text{and} \quad \|\mathcal{R}h\|_w \leq 2\|h\|_w,$$

and

$$\mathcal{R}h \in A_1 \quad \text{with} \quad [\mathcal{R}h]_{A_1} \leq 2\|M\|_w.$$

(b) For every positive $h \in E'_w$ one has

$$h \leq \mathcal{R}'h \quad \text{and} \quad \|\mathcal{R}'h\|_w \leq 2\|h\|_w,$$

and

$$([\mathcal{R}'h]w, w \in (A_1).$$

The last lines in (a) and (b) follow from the estimates $M(\mathcal{R}h) \leq 2\|M\|_w \mathcal{R}h$ and $M((\mathcal{R}'h)w) \leq 2\|S\|_w (\mathcal{R}'h)w)$, respectively, which in turn follow from the definitions of $\mathcal{R}$ and $\mathcal{R}'$.

Note that $f \in L^2(\mathbb{R}, w[f], h)$ for every $f \in E_w$ and every positive $h \in E'_w$, where $w[g,h] := (\mathcal{R}g)^{-1}(\mathcal{R}'h)w$ for every $0 \leq g \in E_w$ and $0 \leq h \in E'_w$. Moreover, by Boyd’s interpolation theorem, the Hilbert transform is bounded on $E_w$. Therefore, by the well-known identity relating partial sum operators $S_I$ and the Hilbert transform, since $I$ is finite, we get that $S^2 f \in E_w$ for every $f \in E_w$. Similarly, combining Kurtz’ inequalities, [18] Theorem 3.1], with Boyd’s interpolation theorem, we conclude that $S^{2\gamma} f \in E_w (I \in I)$, and consequently $S^{2\gamma} f \in E_w$ for every $f \in E_w$.

Finally, a close analysis of the proof of [10] Theorem 4.10] shows that we can take

$$C_{w,w} := 4\sup\{C_{2,w,g,w} : 0 \leq g \in E_w, 0 \leq h \in E'_w, \|g\|_w \leq 2, \|h\|_w = 1\}.$$

Recall that for every $p \in (1, \infty)$ there exists a constant $C_p > 0$ such that $\|M\|_{L^p_{loc}} \leq C_p[w]_{A_p}^{p/p}$ for every Muckenhoupt weight $w \in A_p(\mathbb{R})$; see [5]. A detailed analysis of Boyd’s interpolation theorem shows that $\sup_{w \in V} \max(\|M\|_w, \|S\|_w) < \infty$ for every $V \subset A_p(\mathbb{R})$ with $\sup_{w \in V} [w]_{A_p} < \infty$. By the so-called reverse factorization (or by Hölder’s inequality; see e.g. [12] Proposition 7.2), and by properties (a) and (b), we obtain that $w_{g,h} \in A_2(\mathbb{R})$ and

$$[w_{g,h}]_{A_2} \leq [Rg]_{A_1} (\mathcal{R}'h)_w]_{A_1} \leq 4\|M\|_w \|S\|_w,$$

for every $0 \leq g \in E_w$ and $0 \leq h \in E'_w$. Therefore, on account of the remark on the constants $C_{2,w}$ in [9], we get the desired boundedness property of constants $C_{w,w}$. This completes the proof of [10].

Note that, by the weighted Fefferman-Stein inequality, see Remark 6 and the basic inequality $M^2 f \leq 2Mf (f \in L^1_{loc}(\mathbb{R}))$, the analogous reasoning as before yields [7].

For the proof of the part (ii), for fixed $r \in (1, \infty)$ it is sufficient to apply Rubio de Francia’s extrapolation algorithm from $A_r$ weights in the same manner as above.
To proceed, following [20, Section 3], consider the smooth version of $S^W$, $G = G^W$, defined as follows: let $\phi$ be an even, decreasing, smooth function such that $\hat{\phi}(\xi) = 1$ on $\xi \in [-\frac{1}{2}, \frac{1}{2}]$ and supp $\phi \subset [-1, 1]$. Let $\phi_I(x) := e^{2\pi ic_I \cdot x} |I| \phi(|I|x)$ $(x \in \mathbb{R}^d)$, where $c_I$ stands for the center of an interval $I \in \mathcal{W}$ and $|I|$ for its length. Then,

$$Gf := G^W f := \left( \sum_{I \in \mathcal{W}} |\phi_I * f|^2 \right)^{1/2} \quad (f \in L^2(\mathbb{R})).$$

Since $\hat{\phi}_I(\xi) = 1$ for $\xi \in I$, and $\hat{\phi}_I(\xi) = 0$ for $\xi \notin 2I$, by Plancherel’s theorem, $\|G\|_{L^2} \leq 5$.

Recall that the crucial step of the proof of [20, Theorem 6.1] consists in showing that the Hilbert space-valued kernel related with $G$ satisfies weak-$(D_2)$ condition (see [20, Part IV(E)] for the definition). This leads to the following pointwise estimates for $G$:

$$M^p(Gf)(x) \leq CM(|f|^2)(x)^{1/2} \quad (\text{a.e. } x \in \mathbb{R}) \quad (10)$$

for every $f \in L^{\infty}(\mathbb{R})$ with compact support. In particular, $G$ is bounded on $L^p(\mathbb{R}, w)$ for every $p > 2$ and every Muckenhoupt weight $w \in A_{p/2}(\mathbb{R})$.

**Proof of Theorem B(ii).** We can assume that $\mathcal{I}$ is a finite family of bounded intervals in $\mathbb{R}$. By a standard limiting arguments we easily get the general case.

We start with the proof of the statement of Theorem B(ii) for $q = 2$. Recall that $p = q = 2$ for $\mathcal{E} := L^{2,\infty}$; see [2, Theorem 4.6]. Fix $w \in A_1(\mathbb{R})$ and $f \in L^{\infty}(\mathbb{R})$ with compact support. Note that the classical Littlewood-Paley theory shows that $G^W_\mathcal{I}$ is bounded on $L^2_w$ for every $I \in \mathcal{I}$. Consequently, $G = G^W_\mathcal{I}$ maps $L^2_w$ into itself.

Therefore, combining Lemma [11], Lebesgue’s differentiation theorem and (10) we get

$$\|S^2 f\|_{L^2_w} \leq C_w \|S^2 f\|_{L^2_w} \leq C_w \|Gf\|_{L^2_w} \leq C \|M(Gf)\|_{L^2_w} \leq C \|M^2(Gf)(x)^{1/2}\|_{L^2_w} \leq C \|f\|_{L^2_w},$$

where $C_w$ is an absolute constant independent on $\mathcal{I}$ and $f$. The last inequality follows from the fact that the Hardy-Littlewood maximal operator $M$ is of weak (1, 1) type. Furthermore, one can show that for every subset $\mathcal{V} \subset A_1(\mathbb{R})$ with $\sup_{w \in \mathcal{V}} w A_1 < \infty$ we have $\sup_{w \in \mathcal{V}} C_w < \infty$. Since $S^2$ is continuous on $L^2_w$ and the space of all functions in $L^{\infty}(\mathbb{R})$ with compact support is dense in $L^2_w$ we get the desired boundedness for $S^2$. This completes the proof of the statement of Theorem B(ii) for $q = 2$. Note also that the analogous conclusion holds for the operators $G^W_\mathcal{I}$ instead of $S^2$.

We now proceed by interpolation to show the case of $q \in (1, 2)$. First, it is easily seen that $|\phi_I * f| \leq (\int |\diamond f|) Mf$ for every $f \in L^1_{\text{loc}}(\mathbb{R})$ and $I \in \mathcal{W}$. Therefore, for every $w \in A_1(\mathbb{R})$ the operators

$$L^1_w \ni f \mapsto (\phi_I * f)_{I \in \mathcal{W}} \in L^1_w \left( \mathcal{I}^\infty \right)$$

$$L^2_w \ni f \mapsto (\phi_I * f)_{I \in \mathcal{W}} \in L^2_w \left( \mathcal{I}^2 \right)$$
are bounded with the norms bounded by a constant independent on $I$. Fix $q \in (1,2)$. By interpolation arguments, we conclude that the operator

$$L^q_w \ni f \mapsto (\phi_I * f)_{I \in \mathcal{W}^q} \in L^{q,\infty}_w(I)$$

is well-defined and bounded with the norm bounded by a constant independent on $I$. To show it one can proceed analogously to the proof of a relevant result, see [24, Lemma 3.1]. Therefore, we omit details here.

Since $p = q = q$ for $\mathbb{E} := L^{q,\infty}$, see [24, Theorem 4.6], by Lemma 10 (i) and (ii), for every $w \in A_1(\mathbb{R})$ we get

$$\|S^{\mathbb{E}} f\|_{L^{q,\infty}_w} \leq C_{q,w} \|S^{\mathbb{W}} f\|_{L^{q,\infty}_w} \leq C_{q,w} \left( \sum_{I \in \mathcal{W}^q} |S_I(\phi_I * f)|^{q'} \right)^{1/q'}_{L^{q,\infty}_w} \leq C_{q,w} \|f\|_{L^{q,\infty}_w},$$

where $C_{q,w}$ is an absolute constant independent on $I$. Moreover, the constants $C_{q,w}$ ($w \in A_1(\mathbb{R})$) have the desired boundedness property, i.e., $\sup_{w \in \mathcal{V}} C_{q,w} < \infty$ for every subset $\mathcal{V} \subset A_1(\mathbb{R})$ with $\sup_{w \in \mathcal{V}} [w]_{A_1} < \infty$.

**Remark 11.** We conclude with the relevant result on $A_2$-weighted $L^2$-estimates for square functions $S^\mathcal{I}$ corresponding to arbitrary families $\mathcal{I}$ of disjoint intervals in $\mathbb{R}$, i.e., $\|S^\mathcal{I}\|_{L^2,w} \leq C \|f\|_{L^2,w}$ ($f \in L^2_w$). According to [25, Part IV(E)(ii)], these weighted endpoint estimates can be reached by interpolation provided that $\mathcal{I}$ is a family such that $S^\mathcal{I}$ admits an extension to a bounded operator on (unweighted) $L^p(\mathbb{R})$ for some $p < 2$. This observation leads to a natural question: for which partitions $\mathcal{I}$ of $\mathbb{R}$ do there exist local variants of the Littlewood-Paley decomposition theorem, i.e., there exists $r \geq 2$ such that $S^\mathcal{I}$ is bounded on $L^p(\mathbb{R})$ for all $\frac{1}{r} \leq \frac{1}{2}$.

Recall that L. Carleson, who first noted the possible extension of the classical Littlewood-Paley inequality for other types of partitions of $\mathbb{R}$, proved in the special case $\mathcal{I} := \{[n,n+1) : n \in \mathbb{Z}\}$ that the corresponding square function $S^\mathcal{I}$ is bounded on $L^p(\mathbb{R})$ only if $p \geq 2$; see [6]. Moreover, it should be noted that such lack of the boundedness of the square function $S^\mathcal{I}$ on $L^p(\mathbb{R})$ for some $p < 2$ occurs in the case of decompositions of $\mathbb{R}$ determined by sequences which are in a sense not too different from lacunary ones. Indeed, applying the ideas from [13, Section 8.5], we show below that even in the case of the decomposition $\mathcal{I}$ of $\mathbb{R}$ determined by a sequence $(a_j)_{j=0}^\infty \subset (0,\infty)$ such that $a_{j+1} - a_j \sim \lambda^{\phi(j)}$, where $\lambda > 1$ and $\phi(j) \to 0^+$ arbitrary slowly as $j \to \infty$, the square function $S^\mathcal{I}$ is not bounded on $L^p(\mathbb{R})$ for every $p < 2$.

If $I$ is a bounded interval in $\mathbb{R}$, set $f_I$ for the function with $\hat{f}_I = \chi_I$. Then, $|f_I| = \left| \frac{\sin((I/\pi) \cdot)}{\pi(I)} \right|$, and for every $p > 2$ and every $\epsilon > 0$ there exists $c > 0$ such that

$$\frac{1}{c} |I|^{1/p'} \leq \|f_I\|_p \leq c |I|^{1/p'}$$

for all intervals $I$ with $|I| > \epsilon$. This simply observation allows to express [13, Theorem 8.5.4] for decompositions of $\mathbb{R}$ instead of $\mathbb{Z}$. Namely, if $a = (a_j)_{j=0}^\infty \subset (0,\infty)$ is an increasing sequence such that $a_j - a_{j-1} \to \infty$ as $j \to \infty$, and $\mathcal{I}_a := \{(-a_0, a_0) \cup \pm [a_{j-1}, a_j) \}_{j \geq 1}$, then the boundedness of $S^{\mathcal{I}_a}$ on $L^p(\mathbb{R})$ for some
Moreover, it is straightforward to adapt the idea of the proof of [13 Corollary 8.5.5] to give the following generalization. Let \( \delta = (\delta_j)_{j=1}^\infty \subset (0, \infty) \) be an increasing sequence such that \( \delta_{j+1} - \delta_j \sim \lambda^{\psi(j)} \), where \( \lambda > 1 \), the function \( \psi \in C^1([0, \infty)) \) is increasing and satisfies the condition: \( \psi(s)/s \to 0 \) and \( \psi'(s) \to 0 \) as \( s \to \infty \). If the square function \( S_{T^a} \) were bounded on \( L^{p'}(\mathbb{R}) \) for some \( p > 2 \), then (11) yields

\[
\left( \int_0^{k-1} \lambda^{\psi(s)} ds \right)^{2/p'} \leq C_p \int_0^{k+1} \lambda^{\psi(s)2/p'} ds \quad (k \geq 1).
\]

However, this leads to a contradiction with the assumptions on \( \psi \).

### 4. Higher Dimensional Analogue of Theorem A

The higher dimensional extension of the results due to Coifman, Rubio de Francia and Semmes [8] was established essentially by Q. Xu in [29]; see also M. Lacey [19, Chapter 4].

We start with higher dimensional counterparts of some notions from previous sections. Here and subsequently, we consider only bounded intervals with sides parallel to the axes.

Let \( q \geq 1 \) and \( d \in \mathbb{N} \). For \( h > 0 \) and \( 1 \leq k \leq d \) we write \( \Delta_h^{(k)} \) for the difference operator, i.e.,

\[
(\Delta_h^{(k)} m)(x) := m(x + h e_k) - m(x) \quad (x \in \mathbb{R}^d)
\]

for any function \( m : \mathbb{R}^d \to \mathbb{C} \), where \( e_k \) is the \( k \)-th coordinate vector. Suppose that \( J \) is an interval in \( \mathbb{R}^d \) and set \( \mathcal{J} = \{i_{d+1}[a_i, a_i + h_i]\} \) with \( h_i > 0 \) (\( 1 \leq i \leq d \)). We write

\[
(\Delta_J m) := (\Delta_{h_1}^{(1)} \cdots \Delta_{h_d}^{(d)} m)(a),
\]

where \( a := (a_1, \ldots, a_d) \) and \( m : \mathbb{R} \to \mathbb{C} \). Moreover, for an interval \( I \) in \( \mathbb{R}^d \) and a function \( m : \mathbb{R}^d \to \mathbb{C} \) we set

\[
\|m\|_{\text{Var}_q(I)} := \sup_{\mathcal{J}} \left( \sum_{J \in \mathcal{J}} |\Delta_J m|^q \right)^{1/q},
\]

where \( \mathcal{J} \) ranges over all decompositions of \( I \) into subintervals.

Following Q. Xu [29], see also [19 Section 4.2], the spaces \( V_q(I) \) for intervals in \( \mathbb{R}^d \) are defined inductively as follows.

The definition of \( V_q(I) \) (\( q \in [1, \infty) \)) for one-dimensional intervals is introduced in Section 1. Suppose now that \( d \in \mathbb{N} \setminus \{1\} \) and fix an interval \( I = I_1 \times \ldots \times I_d \) in \( \mathbb{R}^d \). For a function \( m : \mathbb{R}^d \to \mathbb{C} \), we write \( m \in V_q(I) \) if

\[
\|m\|_{V_q(I)} := \sup_{x \in I} |m(x)| + \sup_{x_1 \in I_1} |m(x_1)| \|V_q(I_2 \times \ldots \times I_d) + \|m\|_{\text{Var}_q(I)} < \infty.
\]

Subsequently, \( \mathcal{D}_d \) stands for the family of the dyadic intervals in \( \mathbb{R}^d \). The definition of the spaces \( (V_{q,d}, \| \cdot \|_{V_q(\mathcal{D}_d)}) \) (\( d \geq 2 \), \( q \in [1, \infty) \)) is quite analogous to the corresponding ones in the case of \( d = 1 \) from Section 1.
For a Banach space $X$, an interval $I$ in $\mathbb{R}$ and $q \geq 1$, we consider below the vector-valued variants $V_q(I;X)$, $R_q(I;X)$, and $R_q(I;X)$ of the spaces $V(I)$, $R_q(I)$, and $R(I)$, respectively. Note that $V_q(I;X) \subset R_q(I;X)$ for any $1 \leq q < p$ and any interval $I$ in $\mathbb{R}$ with the inclusion norm bounded by a constant depending only on $p$ and $q$; see [28 Lemma 2]. Moreover, higher dimensional counterparts of these spaces we define inductively as follows: let $I := \Pi_{i=1}^d I_i$ be a closed interval in $\mathbb{R}^d$ ($d \geq 2$). Set $\tilde{R}_q(I) := R_q(I_1; \tilde{R}_q(I_2 \times \ldots \times I_d))$ and $\tilde{V}_q(I) := V_q(I_1; \tilde{V}_q(I_2 \times \ldots \times I_d))$, where $\tilde{R}_q(I_d) := R(I_d)$ and $\tilde{V}_q(I_d) := V_q(I_d)$. Recall also that for any $1 \leq q < p$ and any interval $I$ in $\mathbb{R}^d$ ($d \geq 1$) we have

$$V_q(I) \subset \tilde{V}_q(I) \subset \tilde{R}_q(I)$$  \hspace{1cm} (12)

with the inclusion norm bounded by a constant independent of $I$.

Finally, we denote by $A^*_p(\mathbb{R}^d)$ ($p \in [1, \infty)$) the class of weights on $\mathbb{R}^d$ which satisfy the strong Muckenhoupt $A_p$ condition. Note that, in the case of $d = 1$, $A^*_p(\mathbb{R})$ is the classical Muckenhoupt $A_p(\mathbb{R})$ class ($p \in [1, \infty)$). We refer the reader, e.g., to [18 or 14 Chapter IV.6] for the background on $A^*_p$-weights.

The following complement to [28 Theorem (i))] is the main result of this section.

**Theorem C.** Let $d \geq 2$ and $q \in (1, 2]$. Then, $V_q(\mathbb{R}^d) \subset M_q(\mathbb{R}^d, w)$ for every $p \geq q$ and every weight $w \in A^*_{p/q}(\mathbb{R}^d)$. 

(ii) Let $d \geq 2$ and $q > 2$. Then, $V_q(\mathbb{R}^d) \subset M_q(\mathbb{R}^d, w)$ for every $2 \leq p < (\frac{1}{q} - \frac{1}{q})^{-1}$ and every weight $w \in A^*_{p/q}(\mathbb{R}^d)$ with $s_w > (1 - p(\frac{1}{2} - \frac{1}{q}))^{-1}$.

**Lemma 12.** For every $d \in \mathbb{N}$, $q \in (1, 2]$, $p > q$, and every subset $V \subset A^*_{p/q}(\mathbb{R}^d)$ with $\sup_{w \in V} [w] A^*_{p/q}(\mathbb{R}^d) < \infty$ we have $\tilde{R}_q(\mathbb{R}^d) \subset M_q(\mathbb{R}^d, w)$ ($w \in V$) and

$$\sup \left\{ \|T_{m\chi_I}\|_{p,w} : m \in \tilde{R}_q(\mathbb{R}^d), \|m\|_{\tilde{R}_q(\mathbb{R}^d)} \leq 1, w \in V, I \in \mathcal{D}^d \right\} < \infty.$$

Here $\tilde{R}_q(\mathbb{R}^d)$ ($q \geq 1$) stands for the space of all functions $m$ defined on $\mathbb{R}^d$ such that $m\chi_I \in \tilde{R}_q(I)$ for every $I \in \mathcal{D}^d$ and $\sup_{I \in \mathcal{D}^d} \|m\chi_I\|_{\tilde{R}_q(I)} < \infty$. Define $\tilde{V}_q(\mathbb{R}^d)$ similarly.

The classes $\tilde{R}_q(\mathbb{R}^d)$ and $A^*_p(\mathbb{R}^d)$ are well adapted to iterate one-dimensional arguments from the proof of Theorem A(i). Therefore, below we give only main supplementary observations should be made.

**Proof of Lemma 12** We proceed by induction on $d$. The proof of the statement of Lemma 12 for $d = 1$ and $p = 2$ is provided in the proof of Theorem A(i). The general case of $d = 1$ and $p > q$ follows from this special one by means of Rubio de Francia’s extrapolation theorem; see Lemma 7.

Assume that the statement holds for $d \geq 1$: we will prove it for $d + 1$. Let $m \in \tilde{R}_q(\mathbb{R}^{d+1})$ with $\|m\|_{\tilde{R}_q(\mathbb{R}^{d+1})} \leq 1$. By approximation, we can assume that $m_I \in R_q(I_1; \tilde{R}_q(I_2 \times \ldots \times I_{d+1}))$ for every $I := I_1 \times \ldots \times I_{d+1} \in \mathcal{D}^{d+1}$. Set $m_I := \sum_{J \in I_1} \gamma_I \cdot a_I \cdot \chi_J$, where $\gamma_I \cdot J \geq 0$ with $\sum_J \gamma_I \cdot J \leq 1$ and $a_I \cdot J \in \tilde{R}_q(I_2 \times \ldots \times I_{d+1})$ with $\|a_I \cdot J\|_{\tilde{R}_q(I_2 \times \ldots \times I_{d+1})} = 1$ for every $I \in \mathcal{D}^{d+1}$. Here $I_1$ stands for a decomposition of $I_1$ corresponding to $m_I$.

Let $q \in (1, 2]$, $p \geq q'$ and $V_{q,p} \subset A^*_{p/q}(\mathbb{R}^{d+1})$ with $\sup_{w \in V} [w] A^*_{p/q}(\mathbb{R}^{d+1}) < \infty$. By Lebesgue’s differentiation theorem, for every $w \in A^*_p(\mathbb{R}^{d+1})$ ($r > 1$) one can easily
show that \( w(\cdot, \cdot) \in A_{p/q}(\mathbb{R}), \) \( w(x, \cdot) \in A_{p}^{*}(\mathbb{R}^d), \) and \([w(\cdot, y)]_{A_{p/q}(\mathbb{R})}, [w(x, \cdot)]_{A_{p}^{*}(\mathbb{R}^d)} \leq [w]_{A_{p}^{*}(\mathbb{R}^{d+1})} \) for almost every \( y \in \mathbb{R}^d \) and \( x \in \mathbb{R}; \) see e.g. \([18, \text{Lemma 2.2}]\). Therefore, by induction assumption, for every \( q \in (1, 2] \) and \( p \in [q', \infty) \setminus \{2\} \) there exists a constant \( C_{q,p} > 0 \) independent of \( m \) and \( w \in V_{q,p} \) such that for every \( w \in V_{q,p} \):

\[
\sup \left\{ \|T_{a_{I,j}}\|_{p,w(x,\cdot)} : J \in \mathcal{I}, I \in \mathcal{D}^{d+1} \right\} \leq C_{q,p} \quad \text{for a.e. } x \in \mathbb{R}. \tag{13}
\]

Let \( f(x,y) := \phi(x)\rho(y) ((x,y) \in \mathbb{R}^{d+1}), \) where \( \phi \in \mathcal{S}(\mathbb{R}) \) and \( \rho \in \mathcal{S}(\mathbb{R}^d). \) Note that the set of functions of this form is dense in \( L^{q'}(\mathbb{R}^{d+1}, w). \) Indeed, by the strong doubling and open ended properties of \( A_{p}^{*}\)-weights, we get \((1 + | \cdot |)^{-d}w \in L^{1}(\mathbb{R}^d)\) \((r > 1, w \in A_{r}^{*}(\mathbb{R}^d));\) see e.g. \([28, \text{Chapter IX, Proposition 4.5}]\). Hence, this claim follows from the standard density arguments. Moreover, we have \( T_{m,f} = \sum_{J} \gamma_{I,j}S_{J}\varphi T_{a_{I,j}}\rho. \) In the sequel, we consider the case of \( q \in (1, 2) \) and \( q = 2 \) separately. For \( q \in (1, 2), \) by Fubini’s theorem, we get

\[
\|T_{m,f}\|_{q,w}^{q'} = \sum_{J \in \mathcal{I}} \int_{\mathbb{R}} |S_{J}\varphi|^{q'} \int_{\mathbb{R}^d} |T_{a_{I,j}}\rho|^{q} w dy dx \quad (w \in V_{q,q'}, I \in \mathcal{D}^{d+1}).
\]

Therefore, by Lemma \([5] \text{and } [13], \) we conclude that

\[
\sup \left\{ \|T_{m,f}\|_{q',w} : w \in V_{q,q'}, m \in \tilde{R}_{q}(\mathcal{D}^{d+1}), \|m\|_{\tilde{R}_{q}(\mathcal{D}^{d+1})} \leq 1, I \in \mathcal{D}^{d+1} \right\} < \infty.
\]

Consequently, by Rubio de Francia’s extrapolation theorem, see \([27, \text{Theorem 3}] \text{or } [10, \text{Chapter 3}], \) the same conclusion holds for all \( p > q. \)

For \( q = 2, \) by Fubini’s theorem and Minkowski’s inequality, we conclude that

\[
\|T_{m,f}\|_{p,w} \leq \int_{\mathbb{R}} |S_{I}\varphi(x)|^p \left( \sum_{J \in \mathcal{I}} \gamma_{I,j}^{2} \|T_{a_{I,j}}\rho\|_{p,w(x,\cdot)}^2 \right)^{\frac{1}{2}} dx \quad (w \in V_{2,p}, I \in \mathcal{D}^{d+1}).
\]

for every \( p > 2. \) Hence, by Theorem \([2] \text{and } [13], \) we get the statement of Lemma \([12] \text{also for } q = 2. \)

**Proof of Theorem C.** Note first that for every \( V \subset A_{1}^{*}(\mathbb{R}^{d}) \) with \( \sup_{w \in V}|w|_{A_{1}^{*}} < \infty, \) by the reverse Hölder inequality, there exists \( s > 1 \) such that \( w^{*} \in A_{s}^{*}(\mathbb{R}^{d}) \) \((w \in V) \) and \( \sup_{p \geq 2, w \in V}|w^{*}|_{A_{p/2}^{*}} < \infty. \) Thus, by Lemma \([12] \text{and an interpolation argument similar to that in the proof of Theorem A(i)}, \) we get

\[
\sup \left\{ \|T_{m_{x_{I}}}\|_{2,w} : w \in V, m \in \tilde{R}_{2}(\mathcal{D}^{d}), \|m\|_{\tilde{R}_{2}(\mathcal{D}^{d})} \leq 1, I \in \mathcal{D}^{d} \right\} < \infty.
\]

Therefore, as in the proof of Theorem A(i), one can show that for every \( q \in (1, 2] \) and every subset \( V \subset A_{2/q}^{*}(\mathbb{R}^{d}) \) with \( N := \sup_{w \in V}|w|_{A_{2/q}^{*}} < \infty, \) there exists a constant \( \alpha = \alpha(d, q, N) > 1 \) such that \( \tilde{R}_{q}(\mathcal{D}^{d}) \subset M_{2}(\mathcal{D}^{d}, w) \) \((w \in V) \) and

\[
\sup \left\{ \|T_{m_{x_{I}}}\|_{2,w} : m \in \tilde{R}_{q}(\mathcal{D}^{d}), \|m\|_{\tilde{R}_{q}(\mathcal{D}^{d})} \leq 1, w \in V, I \in \mathcal{D}^{d} \right\} < \infty.
\]

Now, by means of \([12], \) Kurtz’ weighted variant of Littlewood-Paley’s inequalities, \([18, \text{Theorem 1}] \text{and Rubi} \) bo de Francia’s extrapolation theorem, \([27, \text{Theorem 3}] \), the rest of the proof of (i) runs analogously to the corresponding part of the proof of Theorem A(i).

Consequently, by (i), the proof of the part (ii) follows the lines of the proof of Theorem A(ii). \( \square \)
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