AN OVERVIEW OF THE $\epsilon$ EXPANSION AND THE ELECTROWEAK PHASE TRANSITION

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In this talk I’m going to summarize work done with Larry Yaffe [1] on applying $\epsilon$ expansion techniques to the electroweak phase transition. As I shall review, analysis based on the perturbative expansion of the effective potential, which has already been discussed in this conference, is only valid (in the minimal standard model) when the zero-temperature Higgs mass is small compared to the zero-temperature W mass. The goal of my talk will be to discuss what can be done in the experimentally more realistic case when this condition fails. I shall discuss how one might compute various parameters of the transitions such as the correlation length at the critical temperature, the latent heat of the transition, the bubble nucleation rate, and the baryon violation rate after the transition. Note that these are all quasi-equilibrium quantities. In particular, I shall modestly not attempt to tackle the much harder problems directly associated with the generation of baryon asymmetry on the expanding bubble walls.

Throughout this talk I will work in a simple toy model: the minimal standard model with a single Higgs doublet. I do this because the model has relatively few unknown parameters, and so it is relatively easy to analyze and results are not obscured by complicated parameter dependence. I call it a toy model because I don’t care to step into the debate of whether the minimal model has enough CP violation[2, 3]. For simplicity, I will also be ignoring the Weinberg angle and focusing on a pure SU(2) Higgs theory.

Why life is simple when $m(\text{Higgs})_{T=0} \ll M(\text{W})_{T=0}$

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Consider the classical Mexican-hat potential for the Higgs:

\[ V_0 \sim -\mu^2 \phi^2 + \lambda \phi^4 \].

(1)

And now consider the free energy of the system at finite temperature in the background of a Higgs field \( \phi \). One contribution will be from the classical energy \( V_0 \) above for that \( \phi \). But there will also be contributions from the free energy of the real particles present in the plasma. Focus on the contribution from W bosons. This contribution (at one-loop) is nothing more than a formula you can look up in your old graduate statistical mechanics book; it is the free energy of an ideal Bose gas:

\[ \Delta F \sim T \int d^3k \ln \left( 1 - e^{-\beta E_k} \right) , \]

where the energy of a relativistic particle is

\[ E_k = \sqrt{k^2 + M_W^2} \sim \sqrt{k^2 + g^2 \phi^2} . \]

(3)

Note that the W gets its mass from the background Higgs field, so that \( M_W \sim g \phi \). Thus \( \Delta F \) above depends on the background value of \( \phi \). At high temperatures, one can expand \( \Delta F \) in powers of \( M_W/T \), and one gets a high-temperature expansion of the form

\[ \Delta F \sim \# T^4 + \# M_W^2 T^2 - \# M_W^3 T + \cdots , \]

(4)

where \( \# \) indicates numerical constants that I’m not going to bother writing down.

Putting the classical potential \( V_0 \) together with the W contribution to \( \Delta F \) gives

\[ F \sim V_0 + \Delta F \sim \text{const} + ( -\mu^2 + g^2 T^2 ) \phi^2 - g^3 \phi^3 T + \lambda \phi^4 + \cdots \]

(5)

(where I’m no longer going to bother even putting in the \( \# \) signs). The \( g^2 T^2 \phi^2 \) term comes from the \( M^2 T^2 \) term of the high-temperature expansion of \( \Delta F \). This term is responsible for the phase transition by turning the curvature of the potential at the origin from negative at low temperature to positive at high temperature. The \( g^3 \phi^3 T \) term, which comes from the \( M^3 T^2 \) term, is responsible for making the phase transition first-order as shown in fig. 1. Specifically, there exists a temperature \( T_c \) at which there are two degenerate minima of the potential, and the expectation of \( \phi \) changes discontinuously as one cools through the transition. I shall ignore the generally small effects of subsequent terms in the high-temperature expansion (4).

Now consider the \( T = T_c \) curve in fig. 1. The shape comes from the interplay of the \( \phi^2 \), \( \phi^3 \) and \( \phi^4 \) terms in the free energy (5): the \( \phi^2 \) term turns it up at the origin, the \( \phi^3 \) term turns it down again, and the \( \phi^4 \) term turns it up at large \( \phi \). In particular, in the region of \( \phi \) near the symmetry-breaking minimum (where \( \phi \neq 0 \)) or near the hump in the potential, these three terms must be the same order of magnitude, so that

\[ ( -\mu^2 + g^2 T^2 ) \phi^2 \sim g^3 \phi^3 T \sim \lambda \phi^4 . \]

(6)

From the last relation in particular, one finds that this region of \( \phi \) is characterized by

\[ \phi \sim \frac{g^3}{\lambda} T . \]

(7)

Now consider the order of the loop expansion parameter if we perturbatively analyze the physics of the phase transition. Each loop costs a factor of \( g^2 \). However, \[ ^1 \text{Of course, there are also scalar and other contributions to } \Delta F. \text{ I’m just focusing on the W contributions here for the sake of brevity and pedagogy.} \]
at high temperatures, each loop is also associated with a factor of the temperature $T$, so that the cost of a loop is actually $g^2 T$. (One way to see this factor of $T$ is to remember that, at finite temperature, Euclidean time becomes periodic with period $\beta = 1/T$. As a result, zero-temperature loop momentum integrals $\int d^4p$ are replaced by finite Fourier sums $T \sum_{p_0} \int d^3p$ over the frequencies, and a factor of $T$ then appears explicitly.) Finally, by dimensional analysis, $g^2 T$ must be divided by the mass scale $M_W$ of the problem to determine the final, dimensionless cost of adding a loop. The loop expansion parameter (after resummation) is then

$$\frac{g^2 T}{M_W} \sim \frac{g T}{\phi} \sim \frac{\lambda}{g^2},$$

(8)

where I have used $M_W \sim g \phi$ and (7). $\lambda$ and $g^2$, however, are related to the zero-temperature Higgs and W masses, so that the loop expansion parameter can be rewritten as

$$\frac{\lambda}{g^2} \sim \frac{m^2_{(\text{Higgs})_{T=0}}}{M^2_{(W)_{T=0}}}.\tag{9}$$

So we see that the perturbative loop expansion is only valid when the zero-temperature Higgs mass is small compared to the zero-temperature W mass. All of this power counting is reviewed in ref. [4].

**Why life is not simple**

One of the constraints of electroweak baryogenesis is that there must not be any significant amount of baryon number violation after the completion of the phase transition. Otherwise, any baryon asymmetry generated during the transition would be washed away to zero. The rate of baryon number violation is exponentially sensitive to the sphaleron mass:

$$\text{rate} \sim e^{-\beta E(\text{sphaleron})} \sim e^{-\#M_W/g^2T} \sim e^{-\#g^2/\lambda}.\tag{10}$$

The second equality above comes because the sphaleron mass is of order $M_W/\alpha_w$, and the final equality because the exponent $M_W/g^2T$ is (not coincidentally) nothing other
than the inverse of the loop expansion parameter \( \beta \), which we previously determined for the broken-symmetry phase. You can now see that the rate will be small after the transition only if the zero-temperature Higgs mass (and hence \( \lambda \)) is small. The inverse relationship between the rate exponent and the zero-temperature Higgs mass is shown schematically in fig. 2. Requiring that the rate be small compared to the expansion rate of the universe puts a lower bound on \( \beta E(\text{sphaleron}) \) and hence an upper bound on the Higgs mass. A careful analysis, based on the one-loop effective potential, yields an upper bound on the Higgs mass of 35–40 GeV due to this constraint\(^5\). This seems to rule out electroweak baryogenesis in the minimal standard model.

But now it’s time to wonder whether a one-loop analysis can be trusted. In particular, is a Higgs mass of 35 GeV in fact small compared to the W mass of 80 GeV? That depends on all the factors of 2 left out of the order-of-magnitude analysis of loop corrections that I presented above. To settle this issue, the two-loop potential was computed in ref. \( ^4 \) (and the most dominant pieces independently in ref. \( ^6 \)). Fig. 3 shows the difference between the one-loop and two-loop potentials at the critical temperature. Whereas the VEV changed only by about 20\%\(^6\), the height of the hump changed by a factor of three! The moral is that, for some quantities, \((35 \text{ GeV})/(80 \text{ GeV})\) is not a small number. And so it is not clear, in particular, whether higher-loop corrections to the baryon number violation rate will be small.

But what can we do when mean field theory (\( i.e. \) perturbation theory) fails—that is, when fluctuations around the mean field \( \phi \) are large and cannot be treated perturbatively. To phrase it another way, what do we do when the correlation length \( 1/M_W \) becomes very large at \( T_c \) so that the loop expansion parameter \( \beta \) is large? Condensed matter physicists have long dealt with this problem in the context of second-order phase transitions, where the correlation length is infinite at \( T_c \). Our strategy is to borrow one of their techniques—the \( \epsilon \) expansion—which has been very successful for some systems\(^7\).

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\(^5\) Beyond leading order the VEV is an unphysical and gauge-dependent quantity. The result shown here is for Landau gauge.

\(^6\) The idea of applying the \( \epsilon \) expansion to the electroweak phase transition has been suggested previously by Gleisser and Kolb \( ^3 \), March-Russel \( ^8 \), and probably by others that I’m not familiar with.
The $\epsilon$ expansion

The $\epsilon$ expansion consists of the following steps.

Step 1. Recall that equilibrium quantities can generally be expressed in terms of the partition function $Z = \text{tr} e^{-\beta H}$. The exponential $e^{-\beta H}$ looks just like an evolution operator $e^{iHt}$ when the time $t = i\beta$ is imaginary. Just like the evolution operator, $Z$ has a path-integral representation, and the only difference with standard zero-temperature path-integrals is that time only extends for Euclidean time $\beta$:

$$Z = \int [D\phi] \exp \left( -\int_0^\beta d\tau L_E \right). \quad (11)$$

Also, the trace is implemented by integrating over all configurations where the initial state is the same as the final state, which translates into the periodic boundary condition $\phi(0,\vec{x}) = \phi(\beta,\vec{x})$ on the path integral. (Fermions have anti-periodic boundary conditions.)

Step 2. Since the Euclidean time dimension only has extent $\beta$, it disappears in the high temperature limit $\beta \to 0$. That is, if one is studying (equilibrium) physics at distance scales $l$ large compared to $1/T$, then the Euclidean time dimension decouples and the four-dimensional theory is reduced to an effective three-dimensional theory of the static modes ($p_0 = 0$). The effects of fluctuations ($p_0 \neq 0$) in the Euclidean time direction decouple, as powers of $\beta/l$, except for renormalizations of relevant operators. For example, the three dimensional mass and couplings are related to the original four-dimensional ones by

$$m_3^2 = m_4^2 + g^2 T^2 + \text{higher-order}, \quad (12)$$

$$g_3^2 = g_4^2(T) + \text{higher-order}. \quad (13)$$

The $g^2 T^2$ term in the mass is just the term I discussed earlier that turns the curvature of the potential at $\phi = 0$ from concave down at low temperature to concave up at high temperature, and so drives the phase transition. In four-dimensional language, the phase transition is achieved by varying $T$. In three-dimensional language, that corresponds to varying the mass $m_3^2$ from negative values up through positive values.
Step 3. As we shall see, the three-dimensional theory is difficult to directly solve in a well-defined systematic fashion. The trick of the $\epsilon$ expansion is to solve instead a related class of theories by generalizing the 3 spatial dimensions to $4 - \epsilon$ spatial dimensions, replacing the action in the usual way by

$$S(d=3) \to \int d^{4-\epsilon}x \left[ (D\phi)^2 + m^2 \phi^2 + \mu^\epsilon \lambda \phi^4 + \cdots \right]. \quad (14)$$

(Note that I am defining my couplings, such as $\lambda$, to be dimensionless.)

Step 4. For $\epsilon \ll 1$, the theory turns out to be solvable perturbatively in $\epsilon$ by using renormalization-group (RG) improved perturbation theory. I’ll discuss this in more detail in a moment.

Step 5. After computing to a given order in $\epsilon$, return to the world of three spatial dimensions by taking $\epsilon \to 1$.

Review of a success story: pure scalar theory

Forget about gauge theories for a moment and consider a theory of a single, real scalar field:

$$\mathcal{L}_E \sim (\partial \phi)^2 + m^2 \phi^2 + \mu^\epsilon \lambda \phi^4. \quad (15)$$

The phase transition of the tree-level potential is depicted in Fig. 4, which shows a 2nd-order transition. The 1-loop potential (appropriately resummed) for this model turns out to predict a 1st-order phase transition, but the loop expansion parameter turns out to be order 1 and so mean field theory results cannot be trusted.\footnote{For a review, see section II.A of ref. [4].} In fact, this scalar theory goes by a familiar name near its critical point: it is the Ising model, and the Ising model is known to have a 2nd-order phase transition. (More accurately, the scalar theory is in the same universality class as the Ising model. The field $\phi$ roughly corresponds to the Ising spin averaged over a volume of size $1/\mu^3$.)

In a 2nd-order phase transition, the correlation length becomes infinite. To study such a transition, one should therefore examine the system on larger and larger distance

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{potential.pdf}
\caption{The classical potential for scalar theory, as a function of the mass of the effective 3 dimensional theory.}
\end{figure}
scales and try to determine its scaling behavior. But, as particle physicists, we're all quite familiar with how to change scale in $4 - \epsilon$ dimensions. We use the renormalization group:

$$\mu \partial_\mu \lambda = -\epsilon \lambda + c\lambda^2 + O(\lambda^2).$$  \hspace{1cm} (16)

If one sets $\epsilon$ to zero, this would just be the usual four-dimensional renormalization group for a pure scalar theory, with the one-loop $\beta$-functions depicted explicitly as $c\lambda^2$. $c$ is a numerical constant that is easily computed. Slightly away from four dimensions, the additional term $-\epsilon \lambda$ on the right-hand side simply represents the classical scaling of the interaction $\lambda \phi^4$ due to its dimension. For example, at tree level, a change in $\mu$ by a factor of 2 can be compensated in $\mu$ by changing $\lambda$ by a factor of $2^{-\epsilon}$. This trivial scaling is the sole content of the $-\epsilon \lambda$ term in (16). It is the other terms which contain the physics of integrating out degrees of freedom as the scale $\mu$ is changed.

The flow of the RG equation (16) is shown in Fig. 5. The arrows depict flow into the infrared ($\mu \to 0$). At small $\lambda$, the $-\epsilon \lambda$ term dominates and drives $\lambda$ bigger; at large $\lambda$, the $+c\lambda^2$ term dominates and drives it smaller. There is an infrared-stable fixed point when the two terms balance at $\lambda_* = \epsilon/c$. The long-distance behavior of the system will be described by this fixed point.

When $\epsilon$ is small, the fixed point $\lambda_* = \epsilon/c$ will be at small coupling, and therefore our neglect of two-loop and higher-order contributions to the renormalization group in (16) will be justified. More generally, the loop expansion in $\lambda$ at the fixed point is equivalent to an expansion in $\epsilon$. We know, however, that the loop expansion does not converge but is asymptotic, blowing up at order $\lambda_*^n$ where $n \sim 1/\lambda_*$. That means that the expansion in $\epsilon$ will also be asymptotic, blowing up at order $\epsilon^n$ where $n \sim 1/\epsilon$. If we’re lucky, $n \sim 1/\epsilon$ will mean 3 or 4 when $\epsilon \to 1$ and the first few terms of the expansion will be useful in three dimensions, giving a reasonable approximation for quantities of interest. If we’re unlucky, $n \sim 1/\epsilon$ will mean zero when $\epsilon \to 1$, and the asymptotic series will start blowing up immediately after the first term and be useless in three dimensions.

Are we lucky or unlucky? Let’s look at some examples. The first is the susceptibility exponent $\gamma$ defined by $\chi \sim |T - T_c|^\gamma$ where $\chi$ measures the response of $\langle \phi \rangle$ to an external source $J\phi$. The result is

$$\gamma = 1 + 0.167\epsilon + 0.077\epsilon^2 - 0.049\epsilon^3 + \cdots,$$  \hspace{1cm} (17)

which, for $\epsilon = 1$, gives $1.195 + \cdots$. This compares favorably to the result $1.2405 \pm 0.0015$ which comes from more sophisticated techniques (the $\epsilon$ expansion combined with Pade resummation, or renormalization group studies directly in three dimensions). It is also consistent with lattice results, which have larger error bars. It is also consistent with experiments, which have yet larger error bars. I should also give a much more marginal example, which is the correlation exponent $\eta$ defined by $\langle \phi(r)\phi(0) \rangle \sim r^{2-d-\eta}$ as $r \to \infty$. $\eta$ is simply twice the anomalous dimension of $\phi$ and is given by

$$\eta = 0.0185\epsilon^2 + 0.0187\epsilon^3 - 0.0083\epsilon^4 + 0.0359\epsilon^5 + \cdots.$$  \hspace{1cm} (18)
If one adds the first three terms for $\epsilon = 1$, and ignores the fourth where the series is clearly blowing up, one gets 0.0289. The actual result is believed to be about 0.035. So the naive application of the $\epsilon$ expansion gives an approximation to within about 20%.

**Electroweak theory**

The one-loop renormalization group equations for the couplings of electroweak theory are similar to the case of scalar theory:

$$
\mu \partial_\mu g^2 = -\epsilon g^2 + \beta_0 g^4, \quad (19)
$$

$$
\mu \partial_\mu \lambda = -\epsilon g^2 + (ag^4 + bg^2\lambda + c\lambda^2). \quad (20)
$$

The scalar $\beta$-function is a little more complicated now because there are terms involving $g^2$ as well as $\lambda$.

When one computes the numerical coefficients in these equations and plots the renormalization group flow, one gets Fig. 6. The $g^2 = 0$ line is similar to our previous result for pure scalar theory. However, because of the flow in $g^2$, the fixed point is no longer infrared stable. As observed by Ginsparg [9], there are in fact no infrared stable fixed points seen in the $\epsilon$ expansion. This suggests that the phase transition might always be 1st-order. Another, extremely hand-waving, suggestion of this comes from the observation that all theories flow into the region $\lambda < 0$ and, at tree level, $\lambda < 0$ would be an unstable theory. This suggests that there is something suspect about flowing to arbitrarily large distances, as one would desire if the transition were 2nd-order.

The fact that the transition is 1st-order can be made rigorous, for small $\epsilon$, by remembering that we know how to solve the theory when $\lambda \ll g^2$. When $\lambda \ll g^2$, straightforward perturbation theory is under control and predicts a 1st-order transition. So now consider a theory initially defined by the couplings $g_1^2$ and $\lambda_1$ as depicted in Fig. 6. The renormalization group allows us to flow to larger distance scales and finds a completely equivalent theory where $\lambda \ll g^2$, also depicted in the figure. This equivalent theory can then be solved with straightforward perturbation theory and predicts a 1st-order transition. In summary, our method is as follows.

**Step 1.** Start with initial couplings $g_1^2, \lambda_1$ at scale $\mu_1 \sim T$ (which is the scale where the Euclidean time fluctuations first decouple).

**Step 2.** Use the renormalization group to flow to an equivalent theory with $\lambda \ll g^2$. 

![Figure 6: The renormalization group flow, into the infrared, of electroweak theory.](image-url)
Step 3. We can then perturbatively compute the effective potential or whatever other quantities are of interest.

Step 4. Expand all result in $\epsilon$.

This procedure is closely related to the work of Rudnick [10] and Chen, Lubensky and Nelson [11].

In the first step, we started with initial couplings $g_1^2$ and $\lambda_1$. We know what the initial couplings should be in the three dimensional theory; they are perturbatively related to the couplings in the original 3+1 dimensional theory as in (13). But in the $\epsilon$ expansion we have generalized the 3-dimensional theory to a $4-\epsilon$ dimensional theory. What couplings $g_1^2$ and $\lambda_1$ in the $4-\epsilon$ dimensional theory most naturally extrapolate to those of the 3-dimensional theory as $\epsilon \rightarrow 1$?

This question is easily answered if one observes that the one-loop renormalization group equations are independent of $\epsilon$ if appropriately rescaled. Specifically, take

$$g^2 \rightarrow \epsilon g^2, \quad \lambda \rightarrow \epsilon \lambda, \quad \mu \rightarrow \tilde{\mu}^{-1/\epsilon}. \quad (21)$$

The one-loop RG equations (20) then become

$$\mu \partial_\mu \bar{g}^2 = -\bar{g}^2 + \beta_0 \bar{g}^4, \quad (22)$$

$$\mu \partial_\mu \bar{\lambda} = -\bar{g}^2 + (a \bar{g}^4 + b \bar{g}^2 \bar{\lambda} + c \bar{\lambda}^2) \quad (23)$$

and are independent of $\epsilon$. Thus, if I had simply plotted $g^2/\epsilon$ vs. $\lambda/\epsilon$ in Fig. 6 rather than $g^2$ vs. $\lambda$, the trajectories would have been independent of $\epsilon$. The natural generalization of any particular theory from 3 to 4 $-\epsilon$ dimensions is then clear: just keep the same place on the same trajectory. So the relationship between the couplings is

$$(g_{4-\epsilon}^2, \lambda_{4-\epsilon}) = (\epsilon g_3^2, \epsilon \lambda_3). \quad (24)$$

Note that, even though the trajectories are independent of $\epsilon$ once the couplings are scaled by $\epsilon$, the rate at which those trajectories are traversed is exponentially sensitive to $\epsilon$ as a result of the scaling of $\mu$ in (21). In particular, the trajectories are traversed exponentially slowly as $\epsilon \rightarrow 0$.

Example

I’ll now very roughly and schematically outline a sample calculation. Consider the classical Higgs potential

$$V_\text{cl} \sim m^2 \phi^2 + \mu \lambda \phi^4. \quad (25)$$

Suppose I’ve used the renormalization group to run to $\lambda \ll g^2$. In particular, it’s computationally convenient to run to $\lambda = 0$. I can now compute the 1-loop potential, and the result for $\epsilon \rightarrow 0$ is the familiar Coleman-Weinberg potential:

$$V^{(1\text{-loop})} \sim m^2 \phi^2 + g^4 \phi^4 \ln \left( \frac{g \phi}{\mu} \right). \quad (26)$$

The dependence of this potential on the effective mass $m$ gives a 1st-order phase transition, as shown in Fig. 7. To study the system at the critical temperature, I should find the critical value of $m$ where the two minima are degenerate. I can then compute,

\footnote{There is a long story about why the $\epsilon$-expansion prediction that the phase transition is always 1st-order has, historically, been believed to be incorrect in the U(1) case, where the theory models the phase transition of superconductors. For a discussion of problems with and loopholes to this belief, see section I.E of ref. [1].}
for example, the scalar correlation length $\xi$ in the asymmetric phase at the critical temperature, which at this order is related to the curvature of the potential. One finds

$$\xi^2 \sim \frac{1}{V''(\phi_{\text{min}})} \sim \frac{1}{g^2 \mu^2}. \quad (27)$$

[The dependence $1/g^2 \mu^2$ is easy to obtain from the form of (26) by simple scaling arguments.] Now recall from the previous section that $g^2$ is $O(\epsilon)$ and that the amount of running required to run to $\lambda = 0$ is exponentially long as $\epsilon \to 0$, so that

$$\mu \sim e^{O(1/\epsilon)} T. \quad (28)$$

As a result, the dependence of (27) on $\epsilon$ is

$$\xi^2 \sim \frac{1}{\epsilon} e^{\#/\epsilon} \frac{1}{T^2}, \quad (29)$$

where $\#$ is computed from the renormalization group equations. If one works to higher orders in $\epsilon$, one finds that the prefactor of the exponential is a series in $\epsilon$:

$$\xi^2 T^2 \sim \frac{1}{\epsilon} e^{\#/\epsilon}(1 + \#\epsilon + \#\epsilon^2 + \cdots). \quad (30)$$

When I later refer to orders of the $\epsilon$ expansion, I shall be referring to the expansion of the prefactor. So “leading-order” will refer to a calculation that includes the first term of the prefactor, “next-to-leading order” includes the second term, and so forth.

**Tests**

Now that I’ve described how to go about computing things with the $\epsilon$ expansion, it would be nice to find some cases where we can check to see if the $\epsilon$ expansion is working. One case where we *already* know how to compute results is in the perturbative regime where the initial $\lambda_1 / g_1^2$ is $\ll 1$. As a first check, we can compute the results of the $\epsilon$ expansion in the same limit. Table [A] shows the ratio of $\epsilon$ expansion results to the correct answer in the limit $\lambda_1 / g_1^2 \to 0$. Note that leading-order in $\epsilon$ is generally within
| observable ratio                      | LO  | NLO |
|--------------------------------------|-----|-----|
| asymmetric phase correlation length  | 1.14| 0.94|
| symmetric phase correlation length   | 1.62| 0.92|
| latent heat                          | 0.77| 1.04|
| surface tension of a domain wall     | 0.60| 0.98|
| $\Delta F(T_0)$                      | 0.24| 0.56|

Table 1: The ratio of the $\epsilon$-expansion results, computing prefactors through leading order (LO) and next-to-leading order (NLO) in $\epsilon$, to the corresponding three-dimensional result when $\lambda_1 \ll q_1^2$.

roughly 50% and next-to-leading order within roughly 5-10%. (The exception is the last quantity, which is the difference of free energies of the two ground states at the temperature where the $\phi = 0$ ground state is just loosing its meta-stability.)

We have also tested the $\epsilon$ expansion against large $N$ results where $N$ is the number of scalar fields in the theory. I didn’t have time to discuss this in Sintra. The result is that the $\epsilon$ expansion works qualitatively but fails quantitatively at leading-order in $\epsilon$. However, the $\epsilon$-expansion self-diagnoses its own quantitative failure because a computation of next-to-leading order corrections in $\epsilon$ shows that they are large compared to the leading-order results.

The perturbative test discussed above is only of limited interest because, after all, we already knew how to solve the theory in the perturbative regime. The large $N$ tests are of limited interest because we want to know what happens when $N$ is small. (Indeed, the large $N$ theory is very different. For example, it is not asymptotically free.) What we really want is a test of the $\epsilon$ expansion when $\lambda_1 \gtrsim q_1^2$. A natural thing to do is to check whether the $\epsilon$ expansion diagnoses its own failure—that is, whether next-to-leading order corrections are small or large compared to leading-order results. Computing results at next-to-leading order in this regime is involved, and we only had the stamina to make one test. Our test quantity is related to the specific heat. More specifically, for technical reasons, it is

$$(\text{asymmetric phase correlation length})^2 \times (\text{latent heat}). \quad (31)$$

The calculation to next-to-leading order requires the 2-loop renormalization group and a 2-loop computation of the potential. Fig. 8 shows the results.

From this figure, we conclude that the $\epsilon$ expansion may be quantitatively useful for $m_h$ below 150 GeV or so, where the correction is within $\pm 30\%$. The expansion may be useful qualitatively, but probably not quantitatively, for larger Higgs mass. You should be cautioned, however, that we have only tested one quantity in this way, and perhaps it’s better behaved than most.

**Qualitative results for B violation**

The original motivation for this study was to examine the bounds on electroweak baryogenesis, and so I should say a few words about the implications of the $\epsilon$ expansion for the rate of B violation at the completion of the phase transition. In particular, one can compare the results of renormalization-group improved perturbation theory in $4-\epsilon$
dimensions to the results one would get with naive, unimproved perturbation theory in $4-\epsilon$ dimensions. At one loop, we find that RG improvement predicts a stronger phase transition than the unimproved one-loop result, consistent with the nature of the 2-loop correction of Fig. 3 discussed earlier. One might assume that a stronger phase transition implies a smaller rate of B violation after the transition. Instead, we find a larger rate of B violation in $4-\epsilon$ dimensions. This conclusion is, as always, only rigorous for $\epsilon \ll 1$ and may or may not hold qualitatively when $\epsilon = 1$. (Ideally, one should compute next-to-leading order corrections and see if they are under control.) However, our result offers an important warning. If one studies the transition on a lattice, and finds a transition much stronger than expected by perturbation theory, that does not necessarily mean that baryogenesis is more viable in the model being studied.

I should conclude with a final caveat, however. The calculation of the B violation rate in the $\epsilon$ expansion is beset by additional technical perils arising from the intrinsically 3-dimensional nature of the sphaleron. And so the extrapolation of $\epsilon \to 1$ may be even more suspect than usual. See ref. [1] for details.

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