MODIFIED SCATTERING FOR THE CRITICAL NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We consider the nonlinear Schrödinger equation

\[ iu_t + \Delta u = \lambda |u|^\alpha u \]

in all dimensions \( N \geq 1 \), where \( \lambda \in \mathbb{C} \) and \( \Im \lambda \leq 0 \). We construct a class of initial values for which the corresponding solution is global and decays as \( t \to \infty \), like \( t^{-\frac{N}{2}} \) if \( \Im \lambda = 0 \) and like \( (t \log t)^{-\frac{N}{2}} \) if \( \Im \lambda < 0 \). Moreover, we give an asymptotic expansion of those solutions as \( t \to \infty \). We construct solutions that do not vanish, so as to avoid any issue related to the lack of regularity of the nonlinearity at \( u = 0 \). To study the asymptotic behavior, we apply the pseudo-conformal transformation and estimate the solutions by allowing a certain growth of the Sobolev norms which depends on the order of regularity through a cascade of exponents.

1. Introduction

In this article, we consider the nonlinear Schrödinger equation

\[
\begin{cases}
iu_t + \Delta u = \lambda |u|^\alpha u \\
u(0, x) = u_0
\end{cases}
\tag{1.1}
\]
on \( \mathbb{R}^N \), where

\[ \alpha = \frac{2}{N} \tag{1.2} \]

and

\[ \Im \lambda \leq 0 \tag{1.3} \]

and its equivalent integral formulation

\[ u(t) = e^{it\Delta}u_0 - i\lambda \int_0^t e^{i(t-s)\Delta} |u|^{\alpha} u \, ds \tag{1.4} \]

where \( (e^{it\Delta})_{t \in \mathbb{R}} \) is the Schrödinger group.

It is well known that the Cauchy problem for (1.1)–(1.3) is globally well posed in a variety of spaces, for instance in \( H^1(\mathbb{R}^N) \), in \( L^2(\mathbb{R}^N) \), and in

\[ \Sigma = H^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N, |x|^2 \, dx). \tag{1.5} \]

See e.g. [14]. Concerning the long time asymptotic behavior of the solutions, \( \alpha = \frac{2}{N} \) is a limiting case. Indeed, for \( \alpha > \frac{2}{N} \), there is low energy scattering, i.e. a solution of (1.1) with a sufficiently small initial value (in some appropriate sense) is asymptotic as \( t \to \infty \) to a solution of the free Schrödinger equation. See [21, 7, 8, 5, 6, 16, 4]. On the other hand, if \( \alpha \leq \frac{2}{N} \), then low energy scattering cannot be expected, see [20, Theorem 3.2 and Example 3.3, p. 68] and [1].

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If $b > 0$, the existence of modified wave operators was established in [17] in dimension $N = 1$. More precisely, for all sufficiently small asymptotic state $u^+$, there exists a solution of (1.1) which behaves as $t \to \infty$ like $e^{i\varphi(t, \cdot)} e^{i\lambda t} u^+$, where the phase $\varphi$ is given explicitly in terms of $u^+$. (See also [2]. See [12, 19] for extensions in dimension $N = 2$.) Conversely, for small initial values, it was proved in [9] that the asymptotic behavior of the corresponding solution has this form when $\Im \lambda = 0$, in dimensions $N = 1, 2, 3$. (See also [15].) If $\Im \lambda < 0$, then the nonlinearity has some dissipative effect, and an extra log decay appears in the description of the asymptotic behavior of the solutions. This was established in space dimensions $N = 1, 2, 3$ in [18]. (See also [10, 11] for related results.)

Our purpose in this article is to complete the previous results for (1.1)-(1.2). In order to state our results, we introduce some notation. We consider three integers $k, m, n$ such that

$$k > \frac{N}{2}, \quad n > \max\left\{ \frac{N}{2} + 1, \frac{N}{2a} \right\}, \quad 2m \geq k + n + 1$$

(1.6)

and we let

$$J = 2m + 2 + k + n.$$ (1.7)

We consider the Banach space $\mathcal{X}$ introduced in [4, formulas (1.6) and (1.7)], i.e.

$$\mathcal{X} = \{ u \in H^J(\mathbb{R}^N); \langle x \rangle^n D^\beta u \in L^\infty(\mathbb{R}^N) \text{ for } 0 \leq |\beta| \leq 2m \}
\langle x \rangle^n D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 1 \leq |\beta| \leq 2m + 2 + k;$$

(1.8)

$$\langle x \rangle^{J-|\beta|} D^\beta u \in L^2(\mathbb{R}^N) \text{ for } 2m + 2 + k < |\beta| \leq J \}
$$

with

$$\| u \|_{\mathcal{X}} = \sum_{j=0}^{2m} \sup \| \langle x \rangle^n D^\beta u \|_{L^\infty} + \sum_{j=0}^{k+1} \sum_{\mu=0}^{n} \sup \| \langle x \rangle^{n-\mu} D^\beta u \|_{L^2}$$ (1.9)

where

$$(x) = (1 + |x|^2)^{\frac{1}{2}}.$$ (1.10)

Our main results are the following.

**Theorem 1.1.** Let $\lambda \in \mathbb{R}$. Assume (1.2), (1.6), (1.7), let $\mathcal{X}$ be defined by (1.8)-(1.9), and $\Sigma$ by (1.5). Suppose that $u_0(x) = e^{ix|x|^2} v_0(x)$, where $b \in \mathbb{R}$ and $v_0 \in \mathcal{X}$ satisfies

$$\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v_0(x)| > 0.$$ (1.10)

If $b > 0$ is sufficiently large, then there exists a unique, global solution $u$ in the class $C([0, \infty), \Sigma) \cap L^\infty((0, \infty) \times \mathbb{R}^N) \cap L^\infty((0, \infty), H^1(\mathbb{R}^N))$ of (1.4). Moreover, there exist $\delta > 0$ and $w_0 \in L^\infty(\mathbb{R}^N)$ with $\langle \cdot \rangle^n w_0 \in L^\infty(\mathbb{R}^N)$ and $h \neq 0$ such that

$$\| u(t, \cdot) - z(t, \cdot) \|_{L^2} + (1 + t)^{\frac{5}{2}} \| u(t, \cdot) - z(t, \cdot) \|_{L^\infty} \leq C(1 + t)^{-\delta}$$ (1.11)

where

$$z(t, x) = (1 + bt)^{-\frac{5}{2}} e^{i\Phi(t, \cdot)} w_0(\frac{x}{1 + bt})$$

and

$$\Phi(t, x) = \frac{b|x|^2}{4(1 + bt)} - \frac{\lambda}{b} \left| w_0(\frac{x}{1 + bt}) \right| e^{\frac{x}{b} \log(1 + bt)}.$$ (1.12)

In addition,
Theorem 1.2. Let $\lambda \in \mathbb{C}$ with $\Re \lambda < 0$. Assume (1.2), (1.6), (1.7), let $\mathcal{X}$ be defined by (1.8)-(1.9), and $\Sigma$ by (1.5). Suppose $u_0(x) = e^{i\frac{|x|^2}{4}} v_0(x)$, where $b \in \mathbb{R}$ and $v_0 \in \mathcal{X}$ satisfies (1.10). If $b > 0$ is sufficiently large, then there exists a unique, global solution $u \in C([0, \infty), \Sigma) \cap L^\infty((0, \infty) \times \mathbb{R}^N) \cap L^\infty((0, \infty), H^1(\mathbb{R}^N))$ of (1.4). Moreover, there exist $\delta > 0$ and $f_0, w_0 \in L^\infty$, with $f_0$ real valued, $\|f_0\|_{L^\infty} \leq \frac{1}{2}$, $w_0 \neq 0$ and $(\cdot)^\alpha w_0 \in L^\infty(\mathbb{R}^N)$ such that

$$
\|u(t, \cdot) - z(t, \cdot)\|_{L^2} + (1 + t)^{\frac{\delta}{2}} \|u(t, \cdot) - z(t, \cdot)\|_{L^\infty} \leq C(1 + t)^{-\delta}
$$

(1.13)

where

$$
z(t, x) = (1 + bt)^{-\frac{\delta}{2}} e^{i\Theta(t, \cdot)} \psi\left(t, \frac{x}{1 + bt}\right) w_0\left(\frac{x}{1 + bt}\right)
$$

with

$$
\Theta(t, x) = \frac{b|x|^2}{4(1 + bt)} - \frac{\Re \lambda}{3\lambda} \log\left(\psi\left(t, \frac{x}{1 + bt}\right)\right)
$$

and

$$
\psi(t, y) = \left(\frac{1 + f_0(y)}{1 + f_0(y) + \frac{2(\Re \lambda)}{N b} \|v_0(y)\|^2 \log(1 + bt)}\right)^{\frac{\delta}{2}}
$$

In addition,

$$
(t \log t)^{\frac{\delta}{2}} \|u(t)\|_{L^\infty} \xrightarrow{t \to \infty} (\Re \lambda)^{-\frac{\delta}{2}}
$$

(1.14)

Remark 1.3. Here are some comments on the above Theorems 1.1 and 1.2.

(i) The results are valid in any space dimension $N \geq 1$.

(ii) We do not require the initial value $u_0$ to have small amplitude. Instead, we require $w_0$ to be sufficiently oscillatory (in the sense that $b$ is requested to be sufficiently large). Note also that Theorems 1.1 and 1.2 do not yield any information on the behavior of the solution for $t < 0$.

(iii) It is easy to verify that $S(\mathbb{R}^N) \subset \mathcal{X}$, and that if $\rho \geq n$, then $(x)\rho \in \mathcal{X}$.

Therefore, if $v_0 = e^{i\frac{|x|^2}{4}} \phi(\alpha^{-1} x)$ and $\phi(\alpha^{-1} x) = e^{i\frac{|x|^2}{4}}$, so that (see [13]) $e^{it\Delta} = i^{-\frac{\delta}{2}} M_{1/2}^\alpha D_{1/2}^\alpha$, where $F$ is the Fourier transform. Using the relations $D_{1/2} F = F D_{1/2}$ and $M_{1/2} D_{1/2} = D_{1/2} M_{1/2}$, one obtains

$$
e^{-it\Delta} M_{\frac{1}{2}+\alpha} D_{1/2} = M_{\frac{1}{2}} e^{-i\frac{\delta}{2} \alpha^{-1} t \Delta}.
$$

Since $z$ in (1.11) can be written in the form

$$
z(t) = e^{-i\frac{\delta}{2} \|w_0\|_{L^\infty(\mathbb{R}^N)} \frac{\delta}{2} \log(1 + bt)} e^{it\Delta} e^{-it\Delta} M_{\frac{1}{2}+\alpha} D_{1/2} w_0
$$

we deduce that

$$
e^{-it\Delta} e^{i\frac{\delta}{2} \|w_0\|_{L^\infty(\mathbb{R}^N)} \frac{\delta}{2} \log(1 + bt)} z(t) \xrightarrow{t \to \infty} M_{\frac{1}{2}} e^{-i\frac{\delta}{2} \alpha^{-1} t \Delta} w_0 =: u^+
$$
in $L^2(\mathbb{R}^N)$. Therefore, (1.11) takes the form of modified scattering. In other words, $u(t)$ behaves like $e^{i\frac{1}{2}[(1+bt)^{1/2}]^2\log((1+bt)^{1/2})}e^{t\Delta}u^+$, i.e. a free solution modulated by a phase.

**Remark 1.4.** Here are some open questions related to Theorems 1.1 and 1.2.

(i) We do not know what happens if $\exists \lambda > 0$. Let us observe that if $\alpha < \frac{N}{2}$ and $\exists \lambda > 0$, then it follows from [3, Theorem 1.1] that every nontrivial solution of (1.1) either blows up in finite time or else is global with unbounded $H^1$ norm. The proof in [3] apparently does not apply to the case $\alpha = \frac{N}{2}$. See also Remark 4.4 below.

(ii) For equation (1.1) with $\exists \lambda > 0$, it seems that no finite time blowup result is available (for any dimension $N$ and any $\alpha > 0$). Note that for the same equation set on a bounded domain $\Omega$ with Dirichlet boundary conditions, there is no global solution for any $\alpha > 0$. See [3, Section 2].

(iii) If $\alpha < \frac{N}{2}$ and $\exists \lambda \leq 0$, it seems that no precise description of the asymptotic behavior of the solutions of (1.1) is available. When $\lambda \in \mathbb{R}$, $\lambda > 0$, it is proved in [22] that all $H^1$ solutions converge strongly to 0 in $L^p(\mathbb{R}^N)$, $2 < p < \frac{2N}{N-2}$, but even the rate of decay of these norms seems to be unknown.

For proving Theorems 1.1 and 1.2, we use the strategy of [4]. One main ingredient is the introduction of the space $X$, which is motivated by the observation that one major difficulty in studying equation (1.1)-(1.2) is the lack of regularity of the nonlinearity $|u|^\frac{\lambda}{N}u$ (except in dimension $N = 1$). However, this lack of regularity is only at $u = 0$, so it is not apparent to solutions that do not vanish. The various conditions in the definition of $X$ are here to ensure a control from below of $|u|$, provided the initial value in $X$ satisfies (1.10). See [4, Section 1]. The other main ingredient is the application of the pseudo-conformal transformation. More precisely, given any $b > 0$, $u \in C([0, \infty), \Sigma) \cap L^\infty((0, \infty) \times \mathbb{R}^N)$ is a solution of (1.1) (and its equivalent formulation (1.4)) if and only if $v \in C([0, \frac{1}{b}), \Sigma) \cap L^\infty((0, \frac{1}{b}) \times \mathbb{R}^N)$ defined by

$$u(t, x) = (1 + bt)^{-\frac{N}{2}}e^{i\frac{|x|^2}{1 + bt}}v\left(\frac{t}{1 + bt}, \frac{x}{1 + bt}\right), \quad t \geq 0, x \in \mathbb{R}^N$$

(1.15)

is a solution of the nonautonomous Schrödinger equation

$$\begin{cases}
iv_t + \Delta v = \lambda(1 - bt)^{-\frac{1}{2}}|v|^\alpha v \\
v(0) = v_0
\end{cases}$$

(1.16)

and its equivalent formulation

$$v(t) = e^{it\Delta}v_0 - i\lambda \int_0^t (1 - bs)^{-\frac{1}{2}}e^{i(t-s)\Delta}|v(s)|^\alpha v(s) \, ds$$

(1.17)

where $v_0(x) = u_0(x)e^{-i\frac{\lambda|x|^2}{2}}$. In [4, Theorem 1.3], a scattering result is established for solutions of (1.1) with $\alpha > \frac{N}{2}$. In this case, (1.15) transforms solutions of (1.1) to solutions of a nonautonomous equation similar to (1.16), but with $(1 - bt)^{-1}$ replaced by $(1 - bt)^{-\frac{N}{2}\alpha}$. Since \( \int_0^\infty (1 - bt)^{-\frac{N}{2}\alpha} \, dt = \frac{2}{k(N\alpha - 2)} \rightarrow 0 \) as $b \rightarrow \infty$, a solution $v$ can be constructed on the interval $[0, \frac{1}{b})$ by a fixed point argument, provided $b$ sufficiently large. In the present case (1.2), this argument cannot be applied since $(1 - bt)^{-1}$ is not integrable at $\frac{1}{b}$. We therefore have to modify the arguments in [4]. Crucial in our analysis is the elementary estimate

$$\int_0^t (1 - bs)^{-1-\mu} \, ds = \frac{1}{b\mu}(1 - bt)^{-\mu} - 1 \leq \frac{1}{b\mu}(1 - bt)^{-\mu}$$

(1.18)
for every $\mu > 0$ and $t < \frac{1}{b}$. It follows that if a certain norm of $e^{itm-s}\Delta v(s)^{\alpha}v(s)$ is estimated by $(1 - bs)^{-\mu}$, then the integral in (1.17) is estimated in that norm by the same power $(1 - bt)^{-\mu}$. Concretely, this means that we can control a certain growth of $v(t)$ as $t \to \frac{1}{b}$. Technically, this is achieved by introducing an appropriate cascade of exponents. See Section 4, and in particular Remark 4.2.

The rest of this paper is organized as follows. In Sections 2 and 3, we establish estimates of $e^{it\Delta}$ and $|u|^\alpha u$, which are refined versions of estimates in [4]. In Section 4, we study equation (1.16). We first obtain a local existence result with a blowup alternative. Then we show that if $b$ is sufficiently large, the solution of (1.16) exists on $[0, \frac{1}{b})$ and satisfies certain estimates as $t \uparrow \frac{1}{b}$. (Proposition 4.3.) This is the crux of the paper, which requires the estimates of Sections 2 and 3, as well as the introduction of an appropriate cascade of exponents. The asymptotics of the corresponding solutions of (1.16) as $t \uparrow \frac{1}{b}$ is determined in Section 5. Finally, the proof of Theorems 1.1 and 1.2 is completed in Section 6, by translating the results of Section 5 in the original variables via the transformation (1.15).

2. An estimate for the linear Schrödinger equation

In this section, we assume (1.6)-(1.7) (where $\alpha > 0$ is arbitrary, not necessarily given by (1.2)), and we let $X$ be defined by (1.8)-(1.9). We establish estimates for the solution of the linear, nonhomogeneous Schrödinger equation. We recall that (see [4, Proposition 1])

$$e^{it\Delta}_{t \in \mathbb{R}}$$

is a $C_0$ group on $X$

and that there exists a constant $C_1$ such that

$$\|e^{it\Delta}v\|_X \leq C_1\|v\|_X$$

and

$$\|\langle \cdot \rangle^n(e^{it\Delta}v - \psi)\|_{L^\infty} \leq tC_1\|\psi\|_X$$

for all $0 \leq t \leq 1$ and $v \in X$.

**Proposition 2.1.** There exists $C_2 \geq 1$ such that if $T > 0$, $v_0 \in X$ and $f \in C([0,T],X)$, then the solution $v$ of

$$
\begin{align*}
iv_t + \Delta v &= f \\
v(0) &= v_0
\end{align*}
$$

satisfies for all $0 \leq t \leq T$ the following estimates.

$$
\|\langle \cdot \rangle^nD^\beta v(t)\|_{L^\infty} \leq \|v_0\|_X + C_2\int_0^t (\|v(s)\|_X + \|\langle \cdot \rangle^nD^\beta f(s)\|_{L^\infty}) \, ds \tag{2.5}
$$

if $|\beta| \leq 2m$,

$$
\|\langle \cdot \rangle^{n-\mu}D^\beta v(t)\|_{L^2} \leq \|v_0\|_X + C_2\int_0^t (\|v(s)\|_X + \|\langle \cdot \rangle^{n-\mu}D^\beta f(s)\|_{L^2}) \, ds \tag{2.6}
$$

if $|\beta| = \nu + \mu + 2m + 1$ with $0 \leq \nu \leq k + 1$ and $0 \leq \mu \leq n$.

**Proof.** It follows from (2.1) that $v \in C([0,T],X)$. We first observe that if $|\beta| \leq 2m + 2$, then

$$
\|\langle \cdot \rangle^nD^\beta u\|_{L^\infty} \leq C\|u\|_X \tag{2.7}
$$

for all $u \in X$. Indeed, if $|\beta| \leq 2m$, then (2.7) follows immediately from (1.9). Moreover, if $2m + 1 \leq |\beta| \leq 2m + 2$, then by Sobolev’s inequality $\|\langle \cdot \rangle^nD^\beta u\|_{L^\infty} \leq C\|\langle \cdot \rangle^nD^\beta u\|_{L^\infty}$ since $k \geq \frac{N}{2}$ by (1.6). Applying [4, formula (2.13)] (with $s = 0$), we deduce that $\|\langle \cdot \rangle^nD^\beta u\|_{L^\infty} \leq C\|u\|_X$, and (2.7) follows.
We now prove (2.5). Let $|\beta| \leq 2m$. Applying $\langle \cdot \rangle^n D^\beta$ to equation (2.4) we obtain
\[
i(\langle \cdot \rangle^n D^\beta v) = -\langle \cdot \rangle^n D^\beta v + \langle \cdot \rangle^n D^\beta f
\] (2.8)
so that
\[
|\langle \cdot \rangle^n D^\beta v|_{L^2} \leq |\langle \cdot \rangle^n D^\beta v| + |\langle \cdot \rangle^n D^\beta f|.
\]
Integrating this last equation on $(0, t)$ with $0 < t \leq T$, we deduce that
\[
\frac{1}{2} \frac{d}{dt} \|\langle \cdot \rangle^n D^\beta v\|_{L^2}^2 
\leq \int_0^t \left( \|\langle \cdot \rangle^n D^\beta v(s)\|_{L^2} + \|\langle \cdot \rangle^n D^\beta f(s)\|_{L^2} \right) ds.
\] (2.9)

Inequality (2.5) follows by using (2.7).

Next we prove (2.6). Multiplying (2.8) by $\langle \cdot \rangle^{n-2\mu} D^\beta \varphi$ we obtain
\[
\frac{1}{2} \frac{d}{dt} \|\langle \cdot \rangle^{n-\mu} D^\beta v\|_{L^2}^2 
\leq \|\langle \cdot \rangle^{n-\mu} D^\beta v\|_{L^2} \|\langle \cdot \rangle^{n-\mu} D^\beta f\|_{L^2}.
\] (2.10)

If $\mu < n$, then using the estimate $|\nabla \langle x \rangle^{2n-2\mu}| \leq C \langle x \rangle^{2n-2\mu-1}$ (see [4, formula (A.1)]), we see that
\[
\|\langle \cdot \rangle^{n-\mu} D^\beta v\|_{L^2} \leq \|v\|_{X}\,
\] (2.11)

Since $\|\langle \cdot \rangle^{n-\mu-1} D^\beta v\|_{L^2} \leq \|v\|_{X}$, estimate (2.6) easily follows from (2.9), (2.10) and (2.11).

3. A NONLINEAR ESTIMATE

Throughout this section, we consider $\alpha > 0$ (not necessarily given by (1.2)), we assume (1.6)-(1.7), and we let $X$ be defined by (1.8)-(1.9). It is proved in [4, Proposition 2] that there exists a constant $C_3$ such that if $u \in X$ and $\eta > 0$ satisfy
\[
\eta \inf_{x \in \mathbb{R}^N} \langle x \rangle^\alpha |u(x)| \geq 1
\] (3.1)

then $|u|^\alpha u \in X$ and
\[
\|u|^\alpha u\|_X \leq C_3 (1 + \eta \|u\|_X)^{2J} \|u\|_X^{\alpha + 1}.
\] (3.2)

Moreover, if both $u_1, u_2 \in X$ satisfy (3.1), then
\[
\|u_1|^\alpha u_1 - |u_2|^\alpha u_2\|_X 
\leq C_3 (1 + \eta (\|u_1\|_X + \|u_2\|_X))^{2J+1} (\|u_1\|_X + \|u_2\|_X)^\alpha \|u_1 - u_2\|_X.
\] (3.3)

We now establish a refined version of (3.2). The refinement is based on the fact that expanding $D^\beta (|u|^\alpha u)$, one obtains on the one hand a term that contains derivatives of $u$ of order $|\beta|$ and can be estimated by $C |u|^\alpha |D^2 u|$ (see (3.11)), and on the other hand terms that contain products of derivatives of $u$, all of them being of order at most $|\beta| - 1$ (see (3.12)). The refined version of (3.2) is essential in our proof of Proposition 4.3 below. (See Remark 4.2.) Given $\ell \in \mathbb{N}$, we set
\[
\|u\|_{1, \ell} = \sup_{0 \leq |\beta| \leq \ell} \|\langle \cdot \rangle^n D^\beta u\|_{L^\infty}.
\] (3.4)
Moreover, it follows from (3.1) that
\[ |u|_{2,\ell} = \begin{dcases} \sup_{2m+1 \leq |\beta| \leq \ell} \| \langle \cdot \rangle^\beta D^\beta u \|_{L^2} & \ell \geq 2m + 1 \\ 0 & \ell \leq 2m \end{dcases} \] (3.5)

and
\[ |u|_{3,\ell} = \begin{dcases} \sup_{2m+3+k \leq |\beta| \leq \ell} \| \langle \cdot \rangle^{J-\ell} D^\beta u \|_{L^2} & \ell \geq 2m + 3 + k \\ 0 & \ell \leq 2m + 2 + k \end{dcases} \] (3.6)

and we have the following estimates.

**Proposition 3.1.** There exists a constant \( C_4 \geq 1 \) such that if \( u \in X \) and \( \eta > 0 \) satisfy (3.1), then
\[ \| \langle \cdot \rangle^n D^\beta (|u|^n u) \|_{L^\infty} \leq C_4 \| u \|_{L^\infty} \| \langle \cdot \rangle^n D^\beta u \|_{L^\infty} \] (3.7)
for \( 0 \leq |\beta| \leq 1 \),
\[ \| \langle \cdot \rangle^n D^\beta (|u|^n u) \|_{L^\infty} \leq C_4 \| u \|_{L^\infty} \| \langle \cdot \rangle^n D^\beta u \|_{L^\infty} \]
\[ + C_4 \| u \|_{L^\infty} (1 + \eta \| u \|_{1,|\beta|-1})^2 |\beta| \| u \|_{1,|\beta|-1} \] (3.8)

for \( 2 \leq |\beta| \leq 2m \),
\[ \| \langle \cdot \rangle^n D^\beta (|u|^n u) \|_{L^2} \leq C_4 \| u \|_{L^\infty} \| \langle \cdot \rangle^n D^\beta u \|_{L^2} \]
\[ + C_4 (1 + \eta \| u \|_{1,2m})^{2J+\alpha} (\| u \|_{2,|\beta|-1}) \] (3.9)

for \( 2m + 1 \leq |\beta| \leq 2m + 2 + k \), and
\[ \| \langle \cdot \rangle^J - |\beta| D^\beta (|u|^n u) \|_{L^2} \leq C_4 \| u \|_{L^\infty} \| \langle \cdot \rangle^J - |\beta| D^\beta u \|_{L^2} \]
\[ + C_4 (1 + \eta \| u \|_{1,2m})^{2J+\alpha} (\| u \|_{1,2m} + \| u \|_{2,2m+2+k}) \] (3.10)

for \( 2m + 3 + k \leq |\beta| \leq J \).

**Proof.** The case \( |\beta| \leq 1 \) is immediate, so we suppose \( |\beta| \geq 2 \). We observe that
\[ D^\beta (|u|^n u) = \sum_{\gamma + \rho = \beta} c_{\gamma,\rho} D^\gamma (|u|^\rho) D^\rho u \]
with the coefficients \( c_{\gamma,\rho} \) given by Leibniz’s rule. Since \( |u|^\rho = (u^\pi)^{\frac{\rho}{2}} \) we see that the development of \( D^\beta (|u|^n u) \) contains on the one hand the term
\[ A = \left( 1 + \frac{\alpha}{2} \right) |u|^\rho D^\rho u + \frac{\alpha}{2} |u|^{\alpha-2} u^2 D^\beta \pi, \] (3.11)
and on the other hand, terms of the form
\[ B = |u|^\alpha - 2p u^{\pi \alpha} \prod_{j=1}^p D^{\gamma_{1,j}} u D^{\gamma_{2,j}} \pi \] (3.12)
where
\[ \gamma + \rho = \beta, \quad 1 \leq p \leq |\gamma|, \quad |\gamma_{1,j} + \gamma_{2,j}| \geq 1, \]
\[ \sum_{j=0}^p (\gamma_{1,j} + \gamma_{2,j}) = \gamma, \quad |\gamma_{i,j}| \leq |\beta| - 1, \quad i = 1,2. \]

It follows from (3.11) that
\[ |A| \leq (\alpha + 1) |u|^{\rho} |D^\beta u|. \] (3.13)

Moreover, it follows from (3.1) that \( |u|^{-2p} \leq \eta^{2p} (x)^{2pn} \), so that (3.12) implies
\[ |B| \leq |u|^{\alpha} \eta^{2p} (x)^{2pn} |D^\rho u| \prod_{j=1}^p |D^{\gamma_{1,j}} u| |D^{\gamma_{2,j}} u|. \] (3.14)
We begin by proving (3.8). It follows from (3.13) that
\[
\|\langle x \rangle^n A \|_{L^\infty} \leq C \|u\|_{L^\infty}^2 \|\langle x \rangle^n D^\beta u\|_{L^\infty}.
\] (3.15)
Moreover, we deduce from (3.14) and (3.4) that
\[
\|\langle x \rangle^n B \|_{L^\infty} \leq \|\langle x \rangle^n (\eta|u|_{1,|\beta| - 1})^{2p}\|_{L^1_{1,|\beta| - 1}}.
\] (3.16)
Estimate (3.8) follows from (3.15) and (3.16).

Next, we prove (3.9). It follows from (3.13) that
\[
\|\langle x \rangle^n \|_{L^\infty} \leq C \|u\|_{L^\infty}^n \|\langle x \rangle^n D^\beta u\|_{L^2}.
\] (3.17)
Now, we estimate \(\langle x \rangle^n B\). Suppose first that all the derivatives in the right-hand side of (3.14) are of order \(\leq 2m\), then each of them is estimated by \(\langle x \rangle^{-n}\|u\|_{1,2m}\).

Since \(\|u\| \leq \langle x \rangle^{-n}\|u\|_{1,2m}\), we obtain
\[
\langle x \rangle^n B \leq \langle x \rangle^{-n\alpha}\|u\|_{1,2m}^{n+1}(\eta|u|_{1,2m})^{2p}.
\] (3.18)
Moreover, \(\alpha = \frac{n}{2} > \frac{N}{2}\) by (1.6), so we deduce from (3.18) that
\[
\|\langle x \rangle^n B\|_{L^2} \leq C \|u\|_{1,2m}^{n+1}(\eta|u|_{1,2m})^{2p}.
\] (3.19)
Suppose now that one of the derivatives in the right-hand side of (3.14) is of order greater or equal to \(2m + 1\), for instance \(|\gamma_{1,1}| \geq 2m + 1\). Note that \(|\gamma_{1,1}| \leq |\beta| - 1\), so this may only occur if \(|\beta| \geq 2m + 2\). Since the sum of all derivatives has order \(|\beta| \leq 2m + 2 + k\), we have
\[
|\beta| - |\gamma_{1,1}| \leq |\beta| - (2m + 1) \leq 1 + k \leq 1 + k + n \leq 2m
\] by the last inequality in (1.6). It follows that all other derivatives have order \(\leq 2m\). Thus, (3.14) and (3.4) yield
\[
\langle x \rangle^n B \leq |u|^{\alpha}(\eta|u|_{1,2m})^{2p}(x)^{n}\{D^{\gamma_{1,1}} u\}.
\]
Since \(\|\langle x \rangle^n D^{\gamma_{1,1}} u\|_{L^2} \leq \|u\|_{2,|\beta| - 1}\) by (3.5), we see that
\[
\|\langle x \rangle^n B\|_{L^2} \leq \|\eta|u|_{1,2m}\|^{2p}\|u\|_{2,|\beta| - 1}.
\] (3.20)
Estimates (3.17), (3.19) and (3.20) imply (3.9). (Recall that \(\|u\|_{L^\infty} \leq \|\langle x \rangle^\alpha u\|_{L^\infty} \leq \|u\|_{2,m}\)).

Finally, we prove (3.10). It follows from (3.13) that
\[
\|\langle x \rangle^{J - |\beta|} A\|_{L^2} \leq C \|\langle x \rangle^{J - |\beta|} D^\beta u\|_{L^2}.
\] (3.21)
We now estimate \(\langle x \rangle^{J - |\beta|} B\). We first assume that all the derivatives in the right-hand side of (3.14) are of order \(\leq 2m\). It follows that they are estimated by \(\langle x \rangle^{-n}\|u\|_{1,2m}\), and we obtain
\[
\langle x \rangle^{J - |\beta|} |B| \leq \langle x \rangle^n |B| \leq C(\eta|u|_{1,2m})^{2p}(x)^{-\alpha n}\|u\|_{1,2m}^{n+1}.
\]
Since \(\alpha = \frac{n}{2} > \frac{N}{2}\) by (1.6), we obtain
\[
\|\langle x \rangle^{J - |\beta|} B\|_{L^2} \leq C(\eta|u|_{1,2m})^{2p}\|u\|_{1,2m}^{n+1}.
\] (3.22)
Suppose now that one of the derivatives in the right-hand side of (3.14) is of order \(\geq 2m + 1\), for example \(\gamma_{1,1} \geq 2m + 1\). Since the sum of all derivatives has order \(|\beta| \leq J = 2m + 2 + k + n\), we have
\[
|\beta| - |\gamma_{1,1}| \leq |\beta| - (2m + 1) \leq 1 + k + n \leq 2m
\] by the last inequality in (1.6). It follows that all other derivatives have order \(\leq 2m\), hence are estimated by \(\langle x \rangle^{-n}\|u\|_{1,2m}\). Therefore, (3.14) yields
\[
|B| \leq |u|^\alpha(\eta|u|_{1,2m})^{2p}|D^{\gamma_{1,1}} u|.
\] (3.23)
If $2m + 1 \leq |\gamma_{1,1}| \leq 2m + 2 + k$, we have $\|\langle \cdot \rangle^{J - \beta} D^{\gamma_{1,1}} u\|_{L^2} \leq \|\langle \cdot \rangle^n D^{\gamma_{1,1}} u\|_{L^2} \leq \|u\|_{2,2m+2+k}$, so we deduce from (3.23) that

$$\|\langle \cdot \rangle^{J - \beta} B\|_{L^2} \leq (\eta\|u\|_{1,2m})^{2p}\|u\|_{L^2}^2 \leq \|u\|_{2,2m+2+k}. \quad (3.24)$$

If $2m + 3 + k \geq |\gamma_{1,1}| \leq |\beta| - 1$, then $\|\langle x \rangle^{J - \beta} D^{\gamma_{1,1}} u\|_{L^2} \leq \|\langle x \rangle^{J - \gamma_{1,1}} D^{\gamma_{1,1}} u\|_{L^2} \leq \|u\|_{3,|\beta|-1}$, and thus

$$\|\langle \cdot \rangle^{J - \beta} B\|_{L^2} \leq (\eta\|u\|_{1,2m})^{2p}\|u\|_{L^2}^2 \leq \|u\|_{3,|\beta|-1}. \quad (3.25)$$

Estimate (3.10) follows from (3.21), (3.22), (3.24) and (3.25).

\[ \square \]

4. LOCAL AND GLOBAL EXISTENCE FOR (1.16)

Throughout this section, we assume (1.2), (1.6), (1.7) and we consider $\mathcal{X}$ defined by (1.8)-(1.9). By using the pseudo-conformal transformation (1.15), we transform equation (1.1) into the initial-value problem (1.16), or its equivalent form (1.17). We begin with a local existence result for solutions of (1.16), which follows from the results in [4].

**Proposition 4.1.** Let $\lambda \in C$ and $b \geq 0$. If $v_0 \in \mathcal{X}$ satisfies

$$\inf_{x \in \mathbb{R}^N} |x|^n |v_0(x)| > 0, \quad (4.1)$$

then there exist $0 < T < \frac{1}{b}$ and a unique solution $v \in C([0,T], \mathcal{X})$ of (1.16) satisfying

$$\inf_{0 \leq t < T, x \in \mathbb{R}^N} \inf \langle x \rangle^n |v(t,x)| > 0. \quad (4.2)$$

Moreover, $v$ can be extended on a maximal existence interval $[0, T_{\text{max}})$ with $0 < T_{\text{max}} \leq \frac{1}{b}$ to a solution $v \in C([0, T_{\text{max}}), \mathcal{X})$ satisfying (4.2) for all $0 < T < T_{\text{max}}$, and if $T_{\text{max}} < \frac{1}{b}$, then

$$\|v(t)\|_{\mathcal{X}} + \left( \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v(t,x)| \right)^{-1} \to \infty \quad \text{as } t \to T_{\text{max}}. \quad (4.3)$$

**Proof.** Given $S > 0$, $f \in C([0,S], \mathcal{C})$ and $v_0 \in \mathcal{X}$ satisfying (4.1), we consider the equation

$$v(t) = e^{it\Delta} v_0 - i \int_0^t e^{i(t-s)\Delta} f(s)|v(s)|^\alpha v(s) \, ds. \quad (4.4)$$

We first observe that a local solution of (4.4) can be constructed by applying the method of [4, Proof of Proposition 3]. Indeed, let

$$\eta \geq 2(\inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v_0(x)|)^{-1} \quad (4.5)$$

$$M \geq 1 + C_1 \|v_0\|_{\mathcal{X}}. \quad (4.6)$$

Given $0 < T \leq S$, set

$$\mathcal{E} = \{ v \in C([0,T], \mathcal{X}); \|v\|_{L^\infty((0,T), \mathcal{X})} \leq M \text{ and } \eta \inf_{x \in \mathbb{R}^N} \langle x \rangle^n |v(t,x)| \geq 1 \text{ for } 0 < t < T \}$$

so that $\mathcal{E}$ with the distance $d(u, v) = \|u - v\|_{L^\infty((0,T), \mathcal{X})}$ is a complete metric space. Given $v \in \mathcal{E}$, we set

$$\Psi_{v_0,v}(t) = e^{it\Delta} v_0 - i \int_0^t e^{i(t-s)\Delta} f(s)|v(s)|^\alpha v(s) \, ds$$

for $0 \leq t \leq T$. It follows easily from (2.2), (2.3), (3.2) and (3.3) that if

$$T(1 + \eta)\left[ M + C_3 C_1 \|f\|_{L^\infty((0,T),\mathcal{C})}(1 + 2\eta M)^{2J+1}(2M)^{\alpha+1} \right] \leq 1 \quad (4.7)$$

then the map $v \mapsto \Psi_{v_0,v}$ is a strict contraction $\mathcal{E} \to \mathcal{E}$; and so $\Psi_{v_0,v}$ has a fixed point, which is a solution of (4.4) on $[0,T]$. (See [4, Proof of Proposition 3] for details.)
We next observe that if $v_0 \in \mathcal{X}$ satisfies (4.1), if $0 < T < \frac{1}{b}$, and if $v, w \in C([0, T], \mathcal{X})$ are two solutions of (4.4) that both satisfy (4.2), then $u = v$. This follows easily from estimates (2.2) and (3.3), and Gronwall’s inequality.

We now argue as follows. We consider $v_0 \in \mathcal{X}$ satisfying (4.1), and we first apply the local existence result for (4.4) with

$$f(t) = \lambda(1 - \beta t)^{-1}$$

where $\eta$ and $M$ are chosen sufficiently large as to satisfy (4.5) and (4.6), and then $0 < T < \frac{1}{b}$ is chosen sufficiently small so that (4.7) holds. This yields a solution $u \in C([0, T], \mathcal{X})$ of (1.17) satisfying (4.2). Next, we set

$$T_{\max} = \max\{T \in (0, \frac{1}{b})\}; \text{there exists a solution}$$

$$v \in C([0, T_{\max}], \mathcal{X}) \text{of (1.17) satisfying (4.2).} \quad (4.8)$$

It follows that $0 < T_{\max} \leq \frac{1}{b}$. Moreover, we deduce from the uniqueness property that there exists a solution $v \in C([0, T_{\max}], \mathcal{X})$ of (1.17) which satisfies (4.2) for all $0 < T < T_{\max}$. Finally, we prove the blowup alternative (4.3). Assume by contradiction that $T_{\max} \leq \frac{1}{b}$, and that there exist $B > 0$ and a sequence $(t_n)_{n \geq 1}$ such that $t_n \uparrow T_{\max}$ and

$$\|v(t_n)\|_X + \left(\inf_{x \in \mathbb{R}^N} |v(t_n, x)|\right)^{-1} \leq B. \quad (4.9)$$

We now set $\eta = 2B$ and $M = 1 + C_1 B$, so that (4.5)-(4.6) hold with $v_0$ replaced by $v(t_n)$, for all $n \geq 1$. We fix $T_{\max} < \tau < \frac{1}{b}$, then we fix $0 < T < \tau - T_{\max}$ sufficiently small so that

$$T(1 + \eta) \left[M + C_3 C_1 (1 - br)^{-1} (1 + 2\eta M)^2 r^{n+1} (2M)^{n+1}\right] \leq 1. \quad (4.10)$$

If $0 \leq t \leq T$, then $t_n - t \leq T_{\max} + T \leq \tau$, so that $\|1 - b(t_n + \cdot)\|_{L^\infty([0, T])} \leq (1 - br)^{-1}$. Thus (4.10) implies that (4.7) is satisfied with $f(t) \equiv (1 - b(t_n + t))^{-1}$ for all $n \geq 1$. It follows from the local existence result that for all $n \geq 1$ there exists $v_n \in C([0, T], \mathcal{X})$ satisfying (4.2), which is a solution of the equation

$$v_n(t) = e^{it \Delta} v(t_n) - i \int_0^t e^{i(t-s) \Delta} f(t_n + s) |v_n(s)|^\alpha v_n(s) \, ds. \quad (4.11)$$

Setting now

$$w_n(t) = \begin{cases} v(t) & 0 \leq t \leq t_n \\ v_n(t - t_n) & t_n \leq t \leq t_n + T \end{cases} \quad (4.12)$$

we see that $w_n \in C([0, t_n + T], \mathcal{X})$, that $w_n$ satisfies (4.2) with $T$ replaced by $t_n + T$, and that $w_n$ is a solution of (1.17) on $[0, t_n + T]$. Since $t_n + T > T_{\max}$ for $n$ large, we obtain a contradiction with (4.8). This completes the proof. \qed

Our next result shows that if $v_0 \in \mathcal{X}$ satisfies (4.1) and $b$ is sufficiently large, then the corresponding solution of (1.16) is defined on $[0, \frac{1}{b})$ and satisfies certain estimates as $t \uparrow \frac{1}{b}$. We first comment on the strategy of our proof in the following remark, then we introduce the required notation and state our result in Proposition 4.3.

**Remark 4.2.** We estimate derivatives of $v$, for instance $\|\langle \cdot \rangle^n D^\beta v\|_{L^\infty}$, by a contraction argument. For this, we assume that

$$\|v(t)\|_{L^\infty} \leq C \quad (4.11)$$

and

$$\|\langle \cdot \rangle^n D^\beta v(t)\|_{L^\infty} \leq C(1 - bt)^{-\mu} \quad (4.12)$$
and we want to recover \((4.11)-(4.12)\) through equation \((1.16)\). It is not too difficult to estimate \(\|v(t)\|_{L^\infty}\) by using equation \((1.16)\), estimate \((4.12)\), and the assumption \(\exists \lambda \leq 0\), so we concentrate on \((4.12)\). We use Proposition \(2.1\), and then we apply \((1.18)\). This yields an estimate of the form \((4.12)\) provided \(\|\langle \cdot \rangle^n D^\beta (|\cdot|^\alpha v)\|_{L^\infty}\) is also estimated by \(C(1 - bt)^{-\mu}\). We now apply Proposition \(3.1\) to estimate \(\|\langle \cdot \rangle^n D^\beta (|\cdot|^\alpha v)\|_{L^\infty}\). The right-hand side of \((3.8)\) contains two terms. It follows from \((4.11)-(4.12)\) that the first term is estimated by \(C(1 - bt)^{-\mu}\). Neglecting the contribution of \(\eta\), the second term in \((3.8)\) is essentially of the form \((\sup|\gamma| \leq |\beta| - 1 \|\langle \cdot \rangle^n D^\beta |\cdot|^\alpha v\|_{L^\infty})^2|\beta|^{-1}\). If we assume that \(\|\langle \cdot \rangle^n D^\beta v(t)\|_{L^\infty}\) is estimated by \(C(1 - bt)^{-\mu}\) for \(|\gamma| \leq |\beta| - 1\), then the second term in \((3.8)\) gives a contribution of the form \(C(1 - bt)^{-\mu(2|\beta|+1)}\), which is not sufficient to obtain estimate \((4.12)\). Our solution to this difficulty is to assume that derivatives of different orders are estimated by different powers of \((1 - bt)\). In other words, we assume that \(\mu\) in \((4.12)\) depends on \(|\beta|\). Therefore, we need a cascade of exponents, which we introduce below.

Let
\[
0 < \sigma < (4J + 2\alpha + 1)^{-J}
\]
and set
\[
\sigma_j = \begin{cases} 0 & j = 0 \\ (4J + 2\alpha + 1)^{\frac{j}{\sigma}} & 1 \leq j \leq 2m \leq J \end{cases}
\]
so that
\[
0 = \sigma_0 < \sigma < \sigma_j < \sigma_k \leq \sigma_J < 1, \quad 1 \leq j < k \leq J.
\]
Given \(0 < T < \frac{1}{b}\) and \(v \in C([0, T], X)\) satisfying \((4.2)\), we define
\[
\Phi_{1,T} = \sup_{0 \leq t < T} \sup_{0 \leq j \leq 2m} (1 - bt)^{\sigma_j} \|v\|_{1,j}
\]
\[
\Phi_{2,T} = \sup_{0 \leq t < T} \sup_{0 \leq j \leq 2m + 2 + k} (1 - bt)^{\sigma_j} \|v\|_{2,j}
\]
\[
\Phi_{3,T} = \sup_{0 \leq t < T} \sup_{0 \leq j \leq 2m + 3 + k} (1 - bt)^{\sigma_j} \|v\|_{3,j}
\]
\[
\Phi_{4,T} = \sup_{0 \leq t < T} \inf_{x \in R^N} \left(1 - bt\right)^{\sigma_1} \|v(t, x)\|_{L^\infty}
\]
where the norms \(\|\cdot\|_{j,\ell}\) are defined by \((3.4)-(3.6)\), and we set
\[
\Psi_T = \max\{\Phi_{1,T}, \Phi_{2,T}, \Phi_{3,T}, \Phi_{4,T}\} \quad \Phi_T = \max\{\Phi_{1,T}, \Phi_{2,T}, \Phi_{3,T}, \Phi_{4,T}\} = \max\{\Phi_T, \Phi_{4,T}\}.
\]

Note that \((3.4)-(3.6)\) imply
\[
\Phi_{1,T} = \sup_{0 \leq t < T} \sup_{0 \leq \beta \leq 2m} (1 - bt)^{\sigma_\beta} \|\langle \cdot \rangle^n D^\beta v\|_{L^\infty}
\]
\[
\Phi_{2,T} = \sup_{0 \leq t < T} \sup_{2m + 1 \leq \beta \leq 2m + 2 + k} (1 - bt)^{\sigma_\beta} \|\langle \cdot \rangle^n D^\beta v\|_{L^2}
\]
\[
\Phi_{3,T} = \sup_{0 \leq t < T} \sup_{2m + 3 + k \leq \beta \leq J} (1 - bt)^{\sigma_\beta} \|\langle \cdot \rangle^{J-|\beta|} D^\beta v\|_{L^2}.
\]
Moreover, one verifies easily that
\[
\Phi_T \leq \|v\|_{L^\infty((0, T), X)}
\]
\[
\Phi_T \geq \|\langle \cdot \rangle^n v\|_{L^\infty((0, T) \times R^N)} + \frac{1}{C_5} (1 - bT)^{\sigma_j} \|v\|_{L^\infty((0, T), X)}
\]
where the constant \(C_5 \geq 1\) is independent of \(T\).
Proposition 4.3. Suppose $\exists \lambda \leq 0$. Given any $K > 0$, there exists $b_0 > 1$ such that if $v_0 \in X$ satisfies
\[
\|v_0\|_X + \left( \inf_{x \in \mathbb{R}^N} (x)^n |v_0(x)| \right)^{-1} \leq K \tag{4.23}
\]
then for every $b \geq b_0$ the corresponding solution $v \in C([0, T_{\max}), X)$ of (1.17) given by Proposition 4.1, satisfies $T_{\max} = \frac{1}{b}$ and
\[
\sup_{0 < T < \frac{1}{b}} \Psi_T \leq 4K \tag{4.24}
\]
where $\Psi_T$ is defined by (4.17).

Proof. Since $v \in C([0, T_{\max}), X)$, we see that $\|v\|_{L^\infty([0, T), X)} \to \|v_0\|_X$ as $T \downarrow 0$. Therefore, it follows from (4.21) and (4.23) that $\|v\|_T \leq 2K$ if $T \in (0, T_{\max})$ is sufficiently small, where $K$ is given by (4.23). Moreover, from (4.23) and the property $v \in C([0, T_{\max}), X)$, we deduce that
\[
\sup_{0 < T \leq T_{\max}} \left( \inf_{x \in \mathbb{R}^N} (x)^n |v(t, x)| \right)^{-1} \leq 2K
\]
if $T \in (0, T_{\max})$ is sufficiently small. Therefore, if we set
\[
T^* = \sup \{ 0 < T < T_{\max}; \Psi_T \leq 4K \} \tag{4.25}
\]
then we see that $0 < T^* \leq T_{\max}$. We claim that if $b$ is sufficiently large, then
\[
T^* = T_{\max}. \tag{4.26}
\]
Assuming (4.26), the conclusion of the theorem follows. Indeed, (4.17) and (4.22) imply that
\[
\Psi_T \geq (1 - bT)^{-\sigma_j} \max \left\{ \sup_{0 < t < T} \left( \inf_{x \in \mathbb{R}^N} (x)^n |v(t, x)| \right)^{-1}, \frac{1}{C_5} \|v\|_{L^\infty([0, T), X)} \right\} \tag{4.27}
\]
If (4.26) holds and $T_{\max} < \frac{1}{b}$, then it follows from (4.27) that
\[
\limsup_{t \uparrow T_{\max}} \|v(t)\|_X + \left( \inf_{x \in \mathbb{R}^N} (x)^n |v(t, x)| \right)^{-1} \leq 4K (1 + C_5) (1 - b T_{\max})^{-\sigma_j} < \infty
\]
which contradicts the blowup alternative (4.3). Therefore, we have $T^* = T_{\max} = \frac{1}{b}$, from which the desired conclusion easily follows.

We now prove the claim (4.26), and we assume by contradiction that
\[
T^* < T_{\max}. \tag{4.28}
\]
It easily follows from (4.25) and (4.28) that
\[
\Psi_T = 4K. \tag{4.29}
\]
We will use the elementary estimate (1.18), as well as the following consequence of (4.22) and (4.29),
\[
\int_0^t \|v(s)\|_X \leq 4KC_5 \int_0^t (1 - bs)^{-\sigma_j} \leq \frac{4KC_5}{b(1 - \sigma_j)}. \tag{4.30}
\]
Next, we set
\[
\eta(t) = 4K (1 - bt)^{-\sigma_j}, \tag{4.31}
\]
so that by (4.29)
\[
\eta(t) \inf_{x \in \mathbb{R}^N} (x)^n |v(t, x)| \geq 1 \tag{4.32}
\]
for all $0 \leq t \leq T^*$. Moreover, it follows from (4.29) that for all $0 \leq t \leq T^*$
\[
\|v(t)\|_{p,q} \leq 4K (1 - bt)^{-\sigma_v} \quad \text{if} \quad \begin{cases} 
0 \leq q \leq 2m & p = 1 \\
0 \leq q \leq 2m + 2 + k & p = 2 \\
0 \leq q \leq J & p = 3.
\end{cases} \tag{4.33}
\]
If $0 \leq j \leq 2m$, then by (4.31) and (4.33) yield
\[ 1 + \eta(t)\|v(t)\|_{1,J} \leq 1 + (4K)^2(1 - bt)^{-\sigma_{1} - \sigma_{J}} \leq 2(4K)^2(1 - bt)^{-\sigma_{1} - \sigma_{J}}, \tag{4.34} \]
since $K \geq 1$. Consider now
\[
\begin{align*}
0 & \leq \rho \leq 2J + \alpha \\
0 & \leq j \leq 2m
\end{align*}
\]
and $\ell, q$ such that
\[
\begin{cases}
\max\{2, j + 1\} \leq \ell \leq 2m & p = 1 \\
\max\{2, j + 1\} \leq \ell \leq 2m + 2 + k & p = 2 \\
\max\{2, j + 1\} \leq \ell \leq J & p = 3.
\end{cases}
\]
Using the properties $\sigma_{1} \leq \sigma_{\ell - 1}$, $\sigma_{J} \leq \sigma_{\ell - 1}$, and $(4J + 2\alpha + 1)\sigma_{\ell - 1} \leq \sigma_{\ell}$ (see (4.14)), we deduce from (4.33) (with $q = \ell - 1$) and (4.34) that
\[
(1 + \eta(t)\|v(t)\|_{1,J})^{p}\|v(t)\|_{p,\ell-1} \leq 2^{2\ell+\alpha}(4K)^{\ell+2\alpha}(1 - bt)^{-(2\ell+\alpha)(\sigma_{1} + \sigma_{J}) - \sigma_{\ell - 1}} \leq (8K)^{\ell+2\alpha}(1 - bt)^{-\sigma_{\ell}}
\tag{4.35}
\]
for all $0 \leq t \leq T^*$. We now estimate $\Phi_{4,T^*}$. It follows from (1.16) that (recall that $|v| > 0$ on $[0,T^*] \times \mathbb{R}^N$)
\[ |v|_t = L + 3\lambda(1 - bt)^{-1}|v|^\alpha + 1 \tag{4.36} \]
where
\[ L(t, x) = i\frac{(\nabla v - v\nabla v)}{2|v|} = -\frac{3(\nabla v)}{|v|}. \tag{4.37} \]
It follows from (4.36) that
\[ -\frac{1}{\alpha} \frac{\partial}{\partial t}(|v|^{-\alpha}) = |v|^{-\alpha - 1}L + 3\lambda(1 - bt)^{-1}. \tag{4.38} \]
Setting
\[ w(t, x) = \langle x \rangle^\alpha|v(t, x)| \]
we deduce from (4.38) that
\[ -\frac{1}{\alpha} \frac{\partial}{\partial t}(w^{-\alpha}) = \langle x \rangle^\alpha w^{-\alpha - 1}L + 3\lambda(1 - bt)^{-1}\langle x \rangle^{-\alpha}. \tag{4.39} \]
We note that for $0 \leq t < T^*$
\[ \langle x \rangle^\alpha|L| \leq \|\langle x \rangle^{\alpha}\Delta v\|_{L^\infty} \leq (1 - bt)^{-\sigma_{2}}\Psi_{T^*} \leq 4K(1 - bt)^{-\sigma_{2}} \tag{4.40} \]
by (4.29). Integrating (4.39) in $t$, and applying (4.40), we obtain
\[ \frac{1}{w(t, x)^\alpha} \leq \frac{1}{w(0, x)^\alpha} + 4\alpha K \int_{0}^{t} \frac{ds}{(1 - bs)^{\sigma_{2}}w(s, x)^{\alpha + 1}} + \alpha|\lambda| \int_{0}^{t} \frac{ds}{1 - bs}. \]
Since $\frac{1}{w(0, x)} \leq K$ by (4.23) and $\frac{1}{w(t, x)} \leq 4K(1 - bt)^{-\sigma_{2}}$ by (4.31)-(4.32), the above estimate implies
\[ \frac{1}{w(t, x)^\alpha} \leq K^\alpha + \alpha(4K)^{\alpha + 2}\int_{0}^{t} \frac{ds}{(1 - bs)^{\sigma_{2} + (\alpha + 1)\sigma_{1}}} + \alpha|\lambda| |\log(1 - bt)|. \tag{4.41} \]
Note that $\sigma_{2} + (\alpha + 1)\sigma_{1} \leq 2\sigma_{2} \leq \sigma_{J}$ by (4.14), so that (4.41) yields
\[ \frac{1}{w(t, x)^\alpha} \leq K^\alpha + \alpha(4K)^{\alpha + 2} \frac{\alpha|\lambda|}{b(1 - \sigma_{J})} |\log(1 - bt)|, \]
from which it follows that
\[
\Phi_{4,T} \leq \sup_{0 \leq t < T} \left( (1 - bt)^{\sigma_{t}} \left( K^{\alpha} + \frac{\alpha(4K)^{\alpha+2}}{b(1 - \sigma_{j})} + \frac{\alpha|3\lambda|}{b} \right) \right) \nonumber
\]
\[
\leq \left( K^{\alpha} + \frac{\alpha(4K)^{\alpha+2}}{b(1 - \sigma_{j})} + \frac{\alpha|3\lambda|}{b} \sup_{0 \leq t < T} [(1 - bt)^{\sigma_{t}} \log(1 - bt)] \right) \frac{1}{b} \nonumber
\]
\[
= \left( K^{\alpha} + \frac{\alpha(4K)^{\alpha+2}}{b(1 - \sigma_{j})} + \frac{\alpha|3\lambda|}{b} \sup_{0 \leq t < T} [(1 - bt)^{\sigma_{t}} \log t] \right) \frac{1}{b}. \tag{4.42}
\]
We next estimate \( \| \langle \cdot \rangle^{n} v \|_{L^{\infty}} \). It follows from (4.36) that
\[
|v|_{t} \leq |\Delta v|. \tag{4.43}
\]
Note that by (4.30)
\[
\int_{0}^{t} \| (x)^{n} \Delta v(s) \|_{L^{\infty}} \leq \int_{0}^{t} \| v(s) \|_{x} \leq \frac{4KC_{5}}{b(1 - \sigma_{j})}. \tag{4.44}
\]
Applying (4.43), (4.44) and (4.23), we obtain
\[
\| \langle \cdot \rangle^{n} v(t) \|_{L^{\infty}} \leq \| \langle \cdot \rangle^{n} v_{0} \|_{L^{\infty}} + \int_{0}^{t} \| \langle \cdot \rangle^{n} \Delta v(s) \|_{L^{\infty}} \leq K + \frac{4KC_{5}}{b(1 - \sigma_{j})}. \tag{4.45}
\]
We now estimate \( \| \langle \cdot \rangle^{n} D^{\beta} v \|_{L^{\infty}} \) for \( 1 \leq |\beta| \leq 2m \), and we use the estimates of Propositions 2.1 and 3.1. Applying (2.5), (4.32), (3.7) and (3.8), we deduce that
\[
\| \langle \cdot \rangle^{n} D^{\beta} v \|_{L^{\infty}} \leq \| \langle \cdot \rangle^{n} v_{0} \|_{L^{\infty}} + C_{2} \int_{0}^{t} \left( \| v(s) \|_{x} + \| v(s) \|_{x} + |\lambda|(1 - bs)^{-1} \| \langle \cdot \rangle^{n} D^{\beta} (\langle \cdot \rangle^{n} v(s)) \|_{L^{\infty}} \right) \nonumber
\]
\[
\leq \| \langle \cdot \rangle^{n} v_{0} \|_{L^{\infty}} + C_{2} \int_{0}^{t} \left( \| v(s) \|_{x} + |\lambda|C_{2}C_{4} \int_{0}^{t} (1 - bs)^{-1} \| v(s) \|_{L^{\infty}} \| \langle \cdot \rangle^{n} D^{\beta} v \|_{L^{\infty}} \right) \nonumber
\]
\[
+ |\lambda|C_{2}C_{4} \int_{0}^{t} (1 - bs)^{-1} \| v(s) \|_{L^{\infty}} (1 + \eta(s)) \| v_{1,|\beta|-1} \|_{L^{\infty}}^{2|\beta|} \| v_{1,|\beta|-1} \|_{L^{\infty}} \right) \nonumber
\]
with \( \kappa = 0 \) if \( |\beta| = 1 \) and \( \kappa = 1 \) if \( |\beta| \geq 2 \). Moreover, \( \| \langle \cdot \rangle^{n} D^{\beta} v \|_{L^{\infty}} \leq \| v \|_{1,|\beta|} \), so that by (4.29)
\[
\| v \|_{L^{\infty}} \| \langle \cdot \rangle^{n} D^{\beta} v \|_{L^{\infty}} \leq (1 - bs)^{-|\sigma_{|\beta|}}(4K)^{\alpha+1}. \tag{4.47}
\]
We deduce from (4.47) and (1.18) that
\[
\int_{0}^{t} (1 - bs)^{-1} \| v \|_{L^{\infty}} \| \langle \cdot \rangle^{n} D^{\beta} v \|_{L^{\infty}} \leq \frac{(4K)^{\alpha+1}}{b|σ_{|\beta|}} (1 - bt)^{-|\sigma_{|\beta|}}. \tag{4.48}
\]
Next, assuming \( |\beta| \geq 2 \), we apply (4.35) with \( j = |\beta| - 1 \), \( \rho = 2|\beta| \), \( p = 1 \) and \( \ell = |\beta| \), to obtain
\[
(1 + \eta(s)) \| v(s) \|_{1,|\beta|-1}^{2|\beta|} \| v(s) \|_{1,|\beta|-1} \leq \left( 8K \left( 4f + 3\alpha + 1 \right) (1 - bs)^{-|\sigma_{|\beta|}} \right) \| v(s) \|_{L^{\infty} (1 + \eta(s)) \| v \|_{L^{\infty}} \| v \|_{1,|\beta|-1}^{2|\beta|} \| v \|_{1,|\beta|-1} \leq (8K)^{4f + 3\alpha + 1} (1 - bs)^{-|\sigma_{|\beta|}}. \nonumber
\]
Applying (1.18), we deduce that
\[
\int_{0}^{t} (1 - bs)^{-1} \| v \|_{L^{\infty}} (1 + \eta(s)) \| v \|_{1,|\beta|-1}^{2|\beta|} \| v \|_{1,|\beta|-1} \leq \left( 8K \left( 4f + 3\alpha + 1 \right) (1 - bt)^{-|\sigma_{|\beta|}} \right). \tag{4.49}
\]
It follows from (4.46), (4.23), (4.30), (4.48) and (4.49) that
\[
\| \langle \cdot \rangle^{n} D^{\beta} v \|_{L^{\infty}} \leq K + \frac{4KC_{5}C_{2}}{b(1 - \sigma_{j})} \left( 2|\lambda|C_{2}C_{4}(8K)^{4f + 3\alpha + 1} (1 - bt)^{-|\sigma_{|\beta|}}. \tag{4.50}
\]
We next estimate \( \| \langle \cdot \rangle^n D^\beta v \|_{L^2} \) for \( 2m + 1 \leq |\beta| \leq 2m + 2 + k \). Estimates (2.6) (with \( \mu = 0 \) and \( \nu = |\beta| - 2m - 1 \)), (4.32) and (3.9) imply
\[
\| \langle \cdot \rangle^n D^\beta v \|_{L^2} \leq \| v_0 \|_\infty + C_2 \int_0^t \| v \|_\infty + |\lambda| (1 - bs)^{-1} \| \langle \cdot \rangle^n D^\beta (v^\alpha v) \|_{L^2}
\]
\[
\leq \| v_0 \|_\infty + C_2 \int_0^t \| v \|_\infty + |\lambda| C_2 C_4 \int_0^t (1 - bs)^{-1} \| v \|_{L^\infty}^2 \| \langle \cdot \rangle^n D^\beta v \|_{L^2} 
\]
\[
+ C_2 C_4 |\lambda| \int_0^t (1 - bs)^{-1} (1 + \eta \| v \|_{L^{1,2m}})^{2J+\alpha} (\| v \|_{L^{1,2m}} + \| v \|_{L^{2,|\beta|-1}}).
\]
We have \( \| \langle \cdot \rangle^n D^\beta v \|_{L^2} \leq \| v \|_{L^{2,|\beta|}} \), so that by (4.29)
\[
\| v \|_{L^\infty}^2 \| \langle \cdot \rangle^n D^\beta v \|_{L^2} \leq (4K)^{\alpha+1} (1 - bs)^{-|\beta|}.
\]
Applying (1.18), we deduce that
\[
\int_0^t (1 - bs)^{-1} \| v \|_{L^\infty}^2 \| \langle \cdot \rangle^n D^\beta v \|_{L^2} \leq \frac{(4K)^{\alpha+1}}{b|\sigma|} (1 - bt)^{-|\beta|}.
\]
Next, we have by applying (4.35) with \( j = 2m, \rho = 2J + \alpha \), and successively \( p = 1 \) and \( \ell = 2m + 1 \), then \( p = 2 \) and \( \ell = |\beta| \)
\[
(1 + \eta \| v \|_{L^{1,2m}})^{2J+\alpha} (\| v \|_{L^{1,2m}} + \| v \|_{L^{2,|\beta|-1}}) \leq 2 (8K)^{4J+2\alpha+1} (1 - bs)^{-|\beta|}.
\]
It then follows from (1.18) that
\[
\int_0^t (1 - bs)^{-1} (1 + \eta \| v \|_{L^{1,2m}})^{2J+\alpha} (\| v \|_{L^{1,2m}} + \| v \|_{L^{2,|\beta|-1}})
\]
\[
\leq \frac{2 (8K)^{4J+2\alpha+1}}{b\sigma|\beta|} (1 - bt)^{-|\beta|}.
\]
Applying (4.23), (4.30), (4.52) and (4.53), we deduce from (4.51) that
\[
\| \langle \cdot \rangle^n D^\beta v \|_{L^2} \leq K + \frac{4KC_0 C_2}{b(1 - \sigma)^J} + \frac{3|\lambda| C_2 C_4 (4K)^{4J+2\alpha+1}}{b|\sigma|} (1 - bt)^{-|\beta|}.
\]
Now, we estimate \( \| \langle \cdot \rangle^{J-|\beta|} D^\beta v \|_{L^2} \) for \( m + 3 \leq |\beta| \leq J \). It follows from (2.6) (with \( \mu = -|\beta| + n - J \) and \( \nu = k + 1 \)), (4.32), and (3.10) that
\[
\| \langle \cdot \rangle^{J-|\beta|} D^\beta v \|_{L^2} \leq \| v_0 \|_\infty + C_2 \int_0^t \| v \|_\infty
\]
\[
+ |\lambda| C_2 C_4 \left( \int_0^t (1 - bs)^{-1} \| v \|_{L^\infty}^2 \| \langle \cdot \rangle^{J-|\beta|} D^\beta v \|_{L^2} \right)
\]
\[
+ \int_0^t (1 - bs)^{-1} (1 + \eta \| v \|_{L^{1,2m}})^{2J+\alpha} (\| v \|_{L^{1,2m}} + \| v \|_{L^{2,2m+2+k}} + \| v \|_{L^{3,|\beta|-1}}).
\]
We have \( \| \langle \cdot \rangle^{J-|\beta|} D^\beta v \|_{L^2} \leq \| v \|_{L^{3,|\beta|}} \), hence
\[
\| v \|_{L^\infty}^2 \| \langle \cdot \rangle^{J-|\beta|} D^\beta v \|_{L^2} \leq (4K)^{\alpha+1} (1 - bt)^{-|\beta|}
\]
by (4.29). Applying (1.18), we obtain
\[
\int_0^t (1 - bs)^{-1} \| v \|_{L^\infty}^2 \| \langle \cdot \rangle^{J-|\beta|} D^\beta v \|_{L^2} \leq \frac{(4K)^{\alpha+1}}{b|\sigma|} (1 - bt)^{-|\beta|}.
\]
Next, we apply (4.35) with \( j = 2m, \rho = 2J + \alpha \), and successively \( p = 1 \) and \( \ell = 2m + 1 \), then \( p = 2 \) and \( \ell = 2m + 3 + k \), then \( p = 3 \) and \( \ell = |\beta| \)
\[
(1 + \eta \| v \|_{L^{1,2m}})^{2J+\alpha} (\| v \|_{L^{1,2m}} + \| v \|_{L^{2,2m+2+k}} + \| v \|_{L^{3,|\beta|-1}})
\]
\[
\leq 3 (8K)^{4J+2\alpha+1} (1 - bs)^{-\sigma}.
\]
Therefore, we deduce from (1.18) that
\[
\int_0^t (1 - bs)^{-1}(1 + \eta\|v\|_{1,2m})^{2J+\alpha}(\|v\|_{1,2m} + \|v\|_{2,2m+2+k} + \|v\|_{3,|\beta|-1}) \leq \frac{3b(8K)^{4J+2\alpha+1}}{(1 - bt)^{-\sigma|\beta|}}
\]  
(4.58)

Applying (4.23), (4.30), (4.57) and (4.58), we deduce from (4.55) that
\[
\|\langle\cdot\rangle^{-|\beta|}D^\beta v\|_{L^2} \leq K + \frac{4KC_\alpha C_2}{b(1 - \sigma_j)} + \frac{4|\lambda|C_\alpha C_4(8K)^{4J+2\alpha+1}}{b\sigma|\beta|}(1 - bt)^{-\sigma|\beta|}.
\]  
(4.59)

It follows from (4.18)–(4.20), (4.45), (4.50), (4.54), and (4.59) that
\[
\Phi_{T^*} \leq K + \frac{4KC_\alpha C_2}{b(1 - \sigma_j)} + \frac{4|\lambda|C_\alpha C_4(8K)^{4J+3\alpha+1}}{b\sigma|\beta|}.
\]  
(4.60)

Finally, we assume that \(b_0\) is sufficiently large so that
\[
\frac{4KC_\alpha C_2}{b_0(1 - \sigma_j)} + \frac{4|\lambda|C_\alpha C_4(8K)^{4J+3\alpha+1}}{b_0\sigma|\beta|} \leq K
\]  
(4.61)

and
\[
\left(\frac{K^\alpha + \frac{4|\lambda|}{b_0(1 - \sigma_j)} \sup_{0 \leq t < 1} [\langle\cdot\rangle t^\alpha |\log t|]^{1/\sigma}}{b_0} \right)^{1/\sigma} \leq 2K.
\]  
(4.62)

We deduce from (4.42) and (4.62), that if \(b \geq b_0\), then
\[
\Phi_{4T^*} \leq 2K.
\]  
(4.63)

Moreover, we deduce from (4.60) and (4.61), that if \(b \geq b_0\), then
\[
\Phi_{T^*} \leq 2K.
\]  
(4.64)

Inequalities (4.63) and (4.64) yield \(\Psi_{T^*} \leq 2K\), which contradicts (4.29), thus completing the proof. \(\square\)

**Remark 4.4.** Note that the only place in the proof of Proposition 4.3 where we use the assumption \(3\lambda \leq 0\) is estimate (4.43). Yet, the conclusion of Proposition 4.3 fails if \(3\lambda > 0\). More precisely, if \(v_0 \in X\) satisfies (4.23) and \(b > 0\), then there is no solution \(v \in C([0, \frac{1}{b}])\) satisfying (1.17) satisfying (4.24). Indeed, suppose that \(v \in C([0, \frac{1}{b}])\) satisfies (1.17) and (4.24). Applying identity (4.39) with \(x = 0\) and integrating in \(t\) yields
\[
0 \geq w(t, 0)^{-\alpha} = -w(0, 0)^{-\alpha} + \alpha \int_0^t w(s, 0)^{-\alpha-1}L(s, 0)\,ds + \frac{\alpha}{b} 3\lambda |\log(1 - bt)|.
\]

Since the integral on the right-hand side of the above inequality is bounded as \(t \uparrow \frac{1}{b}\) by (4.24), we obtain a contradiction by letting \(t \uparrow \frac{1}{b}\).

5. **Asymptotics for (1.16)**

We now turn to the study of the asymptotic of the solution \(v\) as \(t \to \frac{1}{b}\). We prove the following:

**Proposition 5.1.** Suppose \(3\lambda \leq 0\). Assume (1.2), (1.6), (1.7) and let \(X\) be defined by (1.8)-(1.9). Let \(K \geq 1\), and let \(b_0\) be given by Proposition 4.3. Suppose \(b \geq b_0\), let \(v_0 \in X\) satisfy (4.23), and let \(v \in C([0, \frac{1}{b}])\) be the solution of (1.16) given by Proposition 4.3. There exists \(b_1 \geq b_0\) such that if \(b \geq b_1\), then there exist \(f_0, w_0 \in L^\infty\), with \(f_0\) real valued, \(\|f_0\|_{L^\infty} \leq \frac{3}{2}\), \(w_0 \neq 0\) and \((\cdot)^nw_0 \in L^\infty(\mathbb{R}^N)\) such that
\[
\|\langle\cdot\rangle^n(v(t, \cdot) - w_0(\cdot)\psi(t, \cdot)e^{-\theta(t)}))\|_{L^\infty} \leq C(1 - bt)^{1-\sigma_j}
\]  
(5.1)
for all $0 \leq t < \frac{1}{b}$, where
\[
\psi(t, x) = \left( \frac{1 + f_0(x)}{1 + f_0(x) + \frac{\alpha |\Im\lambda|}{b} |v_0(x)|^\alpha |\log(1 - bt)|} \right)^\frac{1}{\alpha}
\] (5.2)
and
\[
\theta(t, x) = \frac{\Re\lambda}{b} \int_0^{|v_0(x)|^\alpha |\log(1 - bt)|} \frac{d\tau}{1 + f_0(x) + \frac{\alpha |\Im\lambda|}{b}}.
\] (5.3)
In addition, if $\Im\lambda = 0$, then
\[
\psi(t, x) \equiv 1 \text{ and } \theta(t, x) = \frac{\lambda}{b} |w_0(x)|^\alpha |\log(1 - bt)|.
\] (5.4)
Furthermore,
\[
\|v(t)\|_{L^\infty} \to \|w_0\|_{L^\infty}
\] (5.5)
if $\Im\lambda = 0$ and
\[
|\log(1 - bt)| \frac{1}{\alpha} \|v(t)\|_{L^\infty} \to \left( \frac{b}{\alpha |\Im\lambda|} \right)^\frac{1}{\alpha}
\] (5.6)
if $\Im\lambda < 0$.

**Proof.** We first determine the asymptotic behavior of $|v|$. Integrating equation (4.38) on $(0, t)$ with $0 \leq t < \frac{1}{b}$, we obtain
\[
\frac{1}{\alpha |v|^\alpha} = \frac{1}{\alpha |v_0|^\alpha} + \frac{|\Im\lambda|}{b} |\log(1 - bt)| - \int_0^t |v|^{-\alpha - 1} L,
\] (5.7)
where $L$ is defined by (4.37), so that
\[
|v|^\alpha = \frac{|v_0|^\alpha}{1 + f + \frac{2|\Im\lambda|}{b} |v_0|^\alpha |\log(1 - bt)|}
\] (5.8)
with
\[
f(t, x) = -\alpha \int_0^t |v_0(x)|^\alpha |v(s, x)|^{-\alpha - 1} L(s, x) ds.
\] (5.9)
Since $\|v_0\|_{L^\infty} \leq K$, we have $\langle x \rangle \langle |v_0(x)| \rangle \leq K$. Moreover,
\[
\langle x \rangle \langle |v(s, x)| \rangle^{-\alpha - 1} \leq (4K)^{\alpha + 1} (1 - bs)^{-\alpha - 1} \sigma_1
\]
\[
\langle x \rangle \langle |L(s, x)| \rangle \leq \langle x \rangle \langle |\Delta v(s, x)| \rangle \leq 4K (1 - bs)^{-\sigma_2}
\]
by (4.24). Since $(\alpha + 1) \sigma_1 + \sigma_2 \leq (\alpha + 2) \sigma_2 \leq \sigma_3 < 1$ by (4.14), we deduce that
\[
|v_0(x)|^\alpha |v(s, x)|^{-\alpha - 1} |L(s, x)| \leq K(4K)^{\alpha + 2} (1 - bs)^{-\alpha - 1} \sigma_1
\]
\[
\leq K(4K)^{\alpha + 2} (1 - bs)^{-\sigma_3}.
\] (5.10)
Thus we see that the integral in (5.9) is convergent in $L^\infty(\mathbb{R}^N)$ as $t \uparrow \frac{1}{b}$. It follows that $f$ can be extended to a continuous function $\left[0, \frac{1}{b}\right] \to L^\infty(\mathbb{R}^N)$ and we set
\[
f_0 = f\left(\frac{1}{b}\right) = -\alpha \int_0^{\frac{1}{b}} |v_0(x)|^\alpha |v(s, x)|^{-\alpha - 1} L(s, x) ds.
\] (5.11)
We note that by (5.9), (5.10) and (5.11),
\[
\|f(t)\|_{L^\infty} \leq \frac{\alpha K(4K)^{\alpha + 2}}{b(1 - \sigma_3)}
\]
and
\[
\|f(t) - f_0\|_{L^\infty} \leq \frac{\alpha K(4K)^{\alpha + 2}}{b(1 - \sigma_3)} (1 - bt)^{1 - \sigma_3}.
\]
for $0 \leq t \leq \frac{1}{b}$. In particular, if $b_1 \geq b_0$ is sufficiently large and $b \geq b_1$, then

$$\|f(t)\|_{L^\infty} \leq \frac{1}{2}$$

(5.12)

$$\|f(t) - f_0\|_{L^\infty} \leq (1 - bt)^{1-\sigma_3}$$

(5.13)

for all $0 \leq t \leq \frac{1}{b}$. Therefore, $1 + f_0 > 0$ by (5.12), and it follows from formula (5.2) that

$$0 \leq \psi \leq 1.$$  

(5.14)

Moreover, $1 - f(t) \geq \frac{1}{2}$ so that

$$\left\| \frac{1}{1 + f(t) + \frac{\alpha|\Im \lambda|}{b}|v_0|^\alpha \log(1 - bt)} \right\|_{L^\infty} \leq 2$$

(5.15)

for all $0 \leq t < \frac{1}{b}$. We set

$$\tilde{v}(t, x) = \left(1 + f_0(x) + \frac{|v_0(x)|^\alpha}{1 + f_0(x) + \frac{\alpha|\Im \lambda|}{b}|v_0(x)|^\alpha \log(1 - bt)}\right)^{\frac{1}{b}}.$$  

(5.16)

It follows from (4.23) and (5.15) that

$$\|\langle \cdot \rangle^\alpha \tilde{v}(t, \cdot)\|_{L^\infty} \leq 2\tilde{K}. $$

(5.17)

In addition, we deduce from (5.8), (5.16), (5.13) and (5.15) (with $t$ and with $t = \frac{1}{\lambda}$) that

$$\|\langle \cdot \rangle^\alpha (|v(t, \cdot)|^\alpha - \tilde{v}(t, \cdot)^\alpha)\|_{L^\infty} \leq 4\|\langle \cdot \rangle^\alpha v_0\|_{L^\infty} (1 - bt)^{1-\sigma_3} \leq 4K^\alpha(1 - bt)^{1-\sigma_3}$$

(5.18)

for $0 \leq t < 1/b$. Next, we introduce the decomposition

$$v(t, x) = w(t, x)\psi(t, x)e^{-i\theta(t, x)}$$

(5.19)

where $\psi$ and $\theta$ are defined by (5.2) and (5.3). Differentiating (5.19) with respect to $t$, we obtain

$$iw_t = i\frac{e^{i\theta}}{\psi}v_t - i\frac{\psi_t}{\psi} - \theta_t.$$  

(5.20)

Moreover, it follows from (5.2) and (5.16) that

$$\frac{\psi_t}{\psi} = -|\Im \lambda|(1 - bt)^{-1}\tilde{\psi} = \Im \lambda(1 - bt)^{-1}\tilde{v}$$

and from (5.3) and (5.16) that

$$\theta_t = \Re \lambda(1 - bt)^{-1} \tilde{v}.$$  

Thus we see that

$$-iw\frac{\psi_t}{\psi} - \theta_t = -\lambda(1 - bt)^{-1}\tilde{v}w = -\frac{e^{i\theta}}{\psi}\lambda(1 - bt)^{-1}\tilde{v}.$$  

(5.21)

Formulas (5.20), (5.21) and (1.16) yield

$$iw_t = \frac{e^{i\theta}}{\psi}(iv_t - \lambda(1 - bt)^{-1}\tilde{v}v)$$

(5.22)

$$= \frac{e^{i\theta}}{\psi}(-\Delta v + \lambda(1 - bt)^{-1}(|v|^\alpha - \tilde{v}^\alpha)v).$$

It follows that

$$\|\langle \cdot \rangle^\alpha w_t\|_{L^\infty} \leq \|\psi^{-1}\|_{L^\infty}\|\langle \cdot \rangle^\alpha \Delta v\|_{L^\infty}$$

$$ + \|\psi^{-1}\|_{L^\infty}|\lambda|(1 - bt)^{-1}\| |v|^\alpha - \tilde{v}^\alpha\|_{L^\infty}\|\langle \cdot \rangle^\alpha v\|_{L^\infty}.$$  

(5.23)
Note that by (5.2)

\[
\frac{1}{\psi} = \left(1 + \frac{\alpha|\lambda|v_0^{|\alpha|} \log(1 - bt)}{1 + f_0}\right)^{-\frac{1}{\psi}}.
\]

Since \(\|f_0\|_{L^\infty} \leq \frac{1}{2}\) by (5.12), we deduce that

\[
\|\psi^{-1}\|_{L^\infty} \leq \left(1 + 2 \frac{\alpha|\lambda|}{b} \|v_0\|_{L^\infty}^{|\alpha|} \log(1 - bt)\right)^{-\frac{1}{\psi}}.
\] (5.24)

Moreover, \(\|\langle \cdot \rangle \Delta v\|_{L^\infty} \leq 4K(1 - bt)^{-\sigma_2}\) and \(\|\langle \cdot \rangle^n v\|_{L^\infty} \leq 4K\) by (4.24). Therefore, it follows from (5.23), (5.24) and (5.18) that

\[
\|\langle \cdot \rangle^n w_t\|_{L^\infty} \leq C(1 + |\log(1 - bt)|)^{\frac{1}{\psi}}[(1 - bt)^{-\sigma_2} + (1 - bt)^{-\sigma_3}] \leq C(1 - bt)^{-\sigma_j}
\]

since \(\sigma_2 < \sigma_3 < \sigma_J\). We deduce that

\[
\|\langle \cdot \rangle^n (w(t) - w(s))\|_{L^\infty} \leq C(1 - bt)^{1-\sigma_j}
\]

for all \(0 \leq s < t < \frac{1}{b}\), so that there exists \(w_0\) such that \(\langle \cdot \rangle^n w_0 \in L^\infty(\mathbb{R}^N)\) and

\[
\|\langle \cdot \rangle^n (w(t) - w_0)\|_{L^\infty} \leq C(1 - bt)^{1-\sigma_j}
\] (5.25)

for all \(0 \leq t < \frac{1}{b}\). It follows from (5.19), (5.14), and (5.25) that

\[
\|\langle \cdot \rangle^n v(t, \cdot) - w_0(\cdot)\psi(t, \cdot)e^{-\theta(t, \cdot)}\|_{L^\infty} \leq C(1 - bt)^{1-\sigma_j}
\] (5.26)

which yields (5.1). We next prove that \(w_0 \neq 0\). (Note that if \(3\lambda = 0\), this is obvious by conservation of the \(L^2\) norm.) Assuming by contradiction that \(w_0 = 0\), we deduce from (5.26) and the property \(n > \frac{\sigma_j}{2}\) that

\[
\|v(t)\|_{L^2} + \|v(t)\|_{L^\infty} \leq C(1 - bt)^{1-\sigma_j}.
\] (5.27)

On the other hand, it follows from equation (1.16) that

\[
\frac{1}{2} \frac{d}{dt}\|v(t)\|^2_{L^2} = -\frac{1}{2} \frac{\alpha|\lambda|}{1 - bt} \int_{\mathbb{R}^N} |v|^\alpha + \frac{c}{(1 - bt)^{1-\alpha(1-\sigma_j)}} \|v(t)\|^2_{L^2}
\]

for some \(c > 0\), by using the \(L^\infty\) estimate of (5.27). Therefore,

\[
\|v(t)\|^2_{L^2} \geq \|v_0\|^2_{L^2} \exp\left(-2c \int_0^t \frac{ds}{(1 - bs)^{1-\alpha(1-\sigma_j)}}\right) > 0.
\]

This is absurd, since \(\|v(t)\|_{L^2} \to 0\) as \(t \uparrow \frac{1}{b}\) by the \(L^2\) estimate of (5.27).

We now prove (5.4), so we assume \(3\lambda = 0\). The first identity is an immediate consequence of (5.2). Moreover, it follows from (5.3) that

\[
\theta(t, x) = \frac{\lambda}{b} \log(1 - bt) \frac{|v_0(x)|^\alpha}{1 + f_0(x)}.
\] (5.28)

On the other hand, we deduce from (5.16) that \(\tilde{v}(t, x) = (1 + f_0(x))^{-\frac{1}{\psi}}|v_0(x)|\), so that (5.18) yields

\[
|v(t, \cdot)|^\alpha \longrightarrow \frac{|v_0(\cdot)|^\alpha}{1 + f_0(\cdot)}
\]

in \(L^\infty(\mathbb{R}^N)\). Since \(|v(t, x)| = |w(t, x)|\) by (5.19) and the first identity in (5.4), and \(|w(t, x)| \to |w_0(x)|\), we conclude that

\[
|w_0(\cdot)|^\alpha = \frac{|v_0(\cdot)|^\alpha}{1 + f_0(\cdot)}.
\]

The second identity in (5.4) now follows from (5.28).

If \(3\lambda = 0\), then (5.5) is an immediate consequence of (5.1) and (5.4). Assuming now \(3\lambda < 0\), we deduce from (5.16) that

\[
|\log(1 - bt)| \tilde{v}^\alpha = \frac{|v_0|^\alpha \log(1 - bt)}{1 + f_0 + \frac{2|\alpha|\lambda|v_0|^\alpha \log(1 - bt)}}.
\] (5.29)
Since $1 + f_0 \geq 0$ by (5.12), it follows in particular that
\[
|\log(1 - bt)| \|\tilde{\varphi}^\alpha\|_{L^\infty} \leq \frac{b}{\alpha|3\lambda|}.
\]  
(5.30)

Moreover, since $1 + f_0 \leq 2$, we deduce from (5.29) that
\[
|\log(1 - bt)| \tilde{\varphi}^\alpha(t, 0) \geq \frac{|v_0(0)|^\alpha |\log(1 - bt)|}{2 + \frac{|v_0(0)|^\alpha |\log(1 - bt)|}.}
\]

Since $|v_0(0)| > 0$ by (4.23), it follows that
\[
\liminf_{t \downarrow \frac{b}{\alpha}} |\log(1 - bt)| \tilde{\varphi}^\alpha(t, 0) \geq \frac{b}{\alpha|3\lambda|},
\]  
(5.31)

Inequalities (5.30) and (5.31) yield
\[
|\log(1 - bt)| \|\tilde{\varphi}^\alpha(t, \cdot)\|_{L^\infty} \to \frac{b}{\alpha|3\lambda|}
\]
and (5.6) follows by applying (5.18). This completes the proof. \qed

6. Proof of Theorems 1.1 and 1.2

Let $v_0 \in \mathcal{X}$ satisfy (1.10), let $K > 0$ be sufficiently large so that (4.23) holds, and let $b_1$ be given by Proposition 5.1. Given $b \geq b_1$, let $v \in C([0, b_1), \mathcal{X})$ be the corresponding solution of (1.17) given by Proposition 4.3. It is easy to verify that $u$ given by the pseudo-conformal transformation (1.15) satisfies $u \in C((0, \infty), \Sigma) \cap L^\infty((0, \infty) \times \mathbb{R}^N)$, and is a solution of (1.4) with $u_0(x) = e^{i\frac{3\lambda}{2}x} v_0(x)$. Moreover, it follows easily from (4.24) and formula (1.15) that $u \in L^\infty((0, \infty), H^1(\mathbb{R}^N))$. (Here we use the property $\frac{n - 1}{\alpha} > \frac{N}{2}$.)

We now apply Proposition 5.1 and, since $\frac{n - 1}{\alpha} > \frac{N}{2}$, we deduce from (5.1) that
\[
\|v(t, \cdot) - u_0(\cdot) e^{i\theta(t, \cdot)}\|_{L^\infty \cap L^2} \leq C(1 - bt)^{1 - \sigma_j}.
\]  
(6.1)

If $\Im \lambda = 0$, then (1.12) follows from (5.5) and (1.15); and (1.11) follows from (6.1), (5.4), and formula (1.15). This proves Theorem 1.1.

If $\Im \lambda < 0$, then (1.14) follows from (5.6) and (1.15). Moreover, it follows from (5.3) and (5.2) that
\[
\theta(t, x) = \frac{\Re \lambda}{\alpha|3\lambda|} \log(\psi(t, x)^{-\alpha}) = \frac{\Re \lambda}{3\lambda} \log(\psi(t, x)).
\]  
(6.2)

Estimate (1.13) follows from (6.1), (6.2), and formula (1.15). This proves Theorem 1.2.

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