Homogeneous Lorentzian manifolds of semisimple group

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Abstract

We describe the structure of $d$-dimensional homogeneous Lorentzian $G$-manifolds $M = G/H$ of a semisimple Lie group $G$. Due to a result by N. Kowalsky, it is sufficient to consider the case when the group $G$ acts properly, that is the stabilizer $H$ is compact. Then any homogeneous space $G/H$ with a smaller group $\tilde{H} \subset H$ admits an invariant Lorentzian metric. A homogeneous manifold $G/H$ with a connected compact stabilizer $H$ is called a minimal admissible manifold if it admits an invariant Lorentzian metric, but no homogeneous $G$-manifold $G/\tilde{H}$ with a larger connected compact stabilizer $\tilde{H} \supset H$ admits such a metric. We give a description of minimal homogeneous Lorentzian $n$-dimensional $G$-manifolds $M = G/H$ of a simple (compact or noncompact) Lie group $G$. For $n \leq 11$, we obtain a list of all such manifolds $M$ and describe invariant Lorentzian metrics on $M$. 
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1 Introduction

We discuss the problem of classification of homogeneous Lorentzian $G$-manifolds $M = G/H$ of a semisimple Lie group $G$. We say that a $G$-manifold $M$ is proper if the action of the isometry group $G$ on $M$ is proper. In contrast with the Riemannian case, there are nonproper homogeneous Lorentzian manifolds, for example, the De Sitter space $dS^n = SO_{1,n}/SO_{1,n-1}$ and the anti De Sitter space $AdS^n = SO_{2,n}/SO_{1,n-1}$.

A surprising result by Nadin Kowalsky shows that these spaces of constant curvature exhaust all nonproper homogeneous Lorentzian manifolds of a simple group $G$ (up to a local isometry).

This result had been generalized by M. Deffaf, K. Melnick and A. Zeghib to the case of a semisimple group $G$:

Any nonproper homogeneous Lorentzian manifold of a semisimple Lie group $G$ is a local product of the (anti) De Sitter space and a Riemannian homogeneous manifold. This reduces the classification of homogeneous Lorentzian manifolds $M = G/H$ of a semisimple Lie group to the case when the stabilizer $H$ is compact.

We will always assume that all considered Lie groups are connected. In particular, by a stability subgroup of an action of a Lie group on a manifold we will understand a connected stability subgroup.

We say that a proper homogeneous manifold $M = G/H$ (and the stability subgroup $H$) is **admissible** if $M$ admits an invariant Lorentzian metric. Then any homogeneous manifold $G/	ilde{H}$, where $\tilde{H} \subset H$ is a closed subgroup is admissible. We say that $M = G/H$ is a **minimal admissible** manifold (and the stabilizer $H$ is **maximal admissible**) if there is no admissible connected compact Lie subgroup $\tilde{H}$ which contains $H$ properly.

The main goal of the paper is to describe minimal admissible manifolds $M = G/H$ of a semisimple Lie group $G$ and determine invariant Lorentzian metrics on them.

In section 2, we fix notations and recall an infinitesimal description of invariant pseudo-Riemannian metrics on a homogeneous manifold $M = G/H$ in terms of the Lie algebras $\mathfrak{g}, \mathfrak{h}$ of the groups $G, H$.

In section 3, we give a necessary and sufficient conditions that a proper homogeneous manifold admits an invariant Lorentzian metric. We also give a description of minimal admissible manifolds $M = G/H$ of a group $G$ which is a direct product $G = G_1 \times G_2$. This reduces the classification of minimal admissible manifolds of a semisimple Lie group $G$ to the case of a simple group.

An explicit description of minimal admissible manifolds $M = G/H$ of a simple compact Lie group $G$ and invariant Lorentzian metrics on $M$ is given in section 4. Any such manifold $M = G/H$ is the total space on the canonical $T^1$-bundle

$$\pi : M = G/H = G/H_\alpha \to F_\alpha = G/H_\alpha \cdot T^1$$

over a minimal adjoint orbit

$$\text{Ad}_G t_\alpha = G/Z_G(t_\alpha) = G/H_\alpha \cdot T^1.$$
The minimal adjoint orbits corresponds to simple roots $\alpha$ of $G$ and are the orbits of elements $t_{\alpha}$ of a Cartan subalgebra associated with the corresponding fundamental weights. The stabilizer $H_{\alpha}$ is the semisimple part of the centralizer $Z_{G}(t_{\alpha})$. The Dynkin diagram of $H_{\alpha}$ is obtained from the Dynkin diagram of $G$ by deleting the vertex $\alpha$. Invariant Lorentzian metrics in $M = G/H_{\alpha}$ are described in terms of invariant Riemannian metrics in $F_{\alpha}$ and the invariant connection in the bundle $\pi$. If $M$ is not the total space of the sphere bundle over a compact rank one symmetric space, then they depends on $m(\alpha) + 1$ real parameters, where $m(\alpha)$ is the Dynkin mark associated with the root $\alpha$.

The section 5 is devoted to investigation of minimal homogeneous Lorentzian manifolds $M = G/H$ of a simple noncompact Lie group $G$. If $G$ has infinite center, then the stabilizer $H$ is a maximal compact subgroup of $G$. In the case of a finite center, the coset space $S = G/K$ by a maximal compact subgroup $K$ is an irreducible Riemannian symmetric space with the symmetric decomposition $g = k + p$. Let $H \subset K$ be a closed subgroup and

$$g = h + m = h + (n + p)$$

the corresponding reductive decomposition, where $\mathfrak{k} = h + n$. The subgroup $H$ is admissible if the space $m^{H} = n^{H} + p^{H}$ of $\text{Ad}_{H}$-invariant vectors is nontrivial. We say that the associated admissible manifold $M = G/H$ belongs to the class I if $n^{H} \neq 0$ and belongs to the class II if $p^{H} \neq 0$.

Geometrically, an admissible manifold $M = G/H$ belongs to the class I if it admits an invariant Lorentzian metric such that the projection $\pi : M = G/H \to S = G/K$ is a pseudo-Riemannian submersion with Lorentzian totally geodesic fibres $K/H$. In particular, the orbits of an invariant time-like vectors field on $M$ are circles. An admissible manifold $M = G/H$ belongs to the class II, if it admits an invariant Lorentzian metrics with an invariant time-like vector field which generates a noncompact 1-parameter subgroup $\mathbb{R}$.

Classification of minimal admissible manifolds $M = G/H$ of a simple noncompact Lie group $G$ reduces to description of maximal admissible subgroup $H$ of the compact Lie group $K$. This problem had been solved in section 4.

The classification of admissible manifolds of class II of a simple Lie group $G$ reduces to determination of the stabilizers $H = K_v$ of minimal orbits for the isotropy representation $j : K \to SO(p)$

of the symmetric space $S = G/K$. As an example, we determine such stabilizers $K_v$ for the group $SL_n(\mathbb{R})$ and for all simple Lie groups of real rank one and describe invariant Lorentzian metrics on the associated minimal admissible manifold $M = G/K_v$.

Starting from the list of irreducible symmetric spaces $G/K$ of dimension $m \leq 10$, by analyzing the isotropy representation $j(K)$ we derive also the list of all class II minimal admissible manifolds $M^d = G/H$ of dimension $d \leq 11$ and describe invariant Lorentzian metrics on $M^d$. 

3
2 Preliminaries

By a homogeneous manifold $M = G/H$ we will understand the homogeneous manifold of a connected Lie group $G$ modulo a closed connected subgroup $H$. We identify the tangent space $T_o M$ at the point $o = eH$ with the coset space $V = \mathfrak{g}/\mathfrak{h}$ where $\mathfrak{g} = \text{Lie}(G)$ is the Lie algebra of $G$ and $\mathfrak{h} = \text{Lie} H$ is the subalgebra associated with the subgroup $H$. We denote by $j : H \to GL(V)$ (resp., $j : \mathfrak{h} \to \mathfrak{gl}(V)$) the isotropy representation of the stability subgroup $H$ (resp., the stability subalgebra $\mathfrak{h}$). It is induced by the adjoint representation of $H$ (resp., $\mathfrak{h}$). Since the group $H$ is connected, a tensor $T$ in $V$ is $j(H)$-invariant if and only if it is $j(\mathfrak{h})$-invariant, that is $j(h)T = 0$ for all $h \in \mathfrak{h}$.

Recall the following

**Proposition 1** There is a natural bijection between $G$-invariant Riemannian (resp., Lorentzian) metrics in a homogeneous space $M = G/H$ and $j(\mathfrak{h})$-invariant Euclidean (resp., Lorentzian) scalar products $g_o$ in $V$. An invariant scalar product $g_o$ defines the metric, whose value $g_x$ at a point $x = L_o o := ao, a \in G$ is given by

$$g_x := (L_o)^* g_o = g_o((L_o)^{-1}, (L_o)^{-1}).$$

Sometimes we will identify $g_o$ and $g$ and say that $g_o$ is an invariant metric in $M$.

Recall that if the group $G$ acts effectively on a pseudo-Riemannian homogeneous manifold $M = G/H$, then the isotropy representation is exact and the stability subgroup $H$ is isomorphic to the isotropy group $j(H) \subset GL(V)$. In particular, we have

**Proposition 2** A homogeneous manifold $M = G/H$ admits an invariant Lorentzian metric if and only if the isotropy representation $j$ defines an isomorphism of the stability group $H$ onto a subgroup $L$ of the connected Lorentz group $SO_0(V)$ or, equivalently, isomorphism of the stability subalgebra $\mathfrak{h}$ onto a subalgebra $\mathfrak{l}$ of the Lorentz algebra $so(V)$.

A homogeneous manifold $M = G/H$ is called to be **reductive** if there is an $\text{Ad}_H$-invariant (reductive) decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

In this case, the complementary to $\mathfrak{h}$ subspace $\mathfrak{m}$ is identified with the tangent space $T_o M = \mathfrak{g}/\mathfrak{h}$ and the isotropy representation is identified with the restriction $\text{Ad}_H|\mathfrak{m}$ of the adjoint representation.

Any homogeneous manifold with a compact stabilizer is reductive.

3 Invariant Lorentzian metrics on a proper homogeneous $G$-manifolds

**Definition 1** An action of a Lie group $G$ on a manifold $M$ is called **proper** if the map

$$G \times M \to M \times M, \ (a, x) \mapsto (ax, x)$$

is proper.
is proper, or, equivalently, \( G \) preserves a complete Riemannian metric on \( M \). In this case \( G \)-manifold \( M \) is called proper.

The orbit space \( M/G \) of a proper \( G \)-manifold is a metric space and has a structure of a stratified manifold.

For a nonproper \( G \)-manifold, the topology of the orbit space can be very bad, for example, non-Hausdorff, see e.g. the action of the Lorentz group on the Minkowski space. On the other hand, in most cases the isometry group of a compact Lorentzian manifold is compact and, hence, acts properly. G. D’Ambra [DA] proved that the isometry group of any simply connected compact analytic Lorentzian manifold is compact (hence, it acts properly). M. Gromov [DAG] states the problem of description of all compact Lorentzian manifolds which admits a noncompact (= nonproper) isometry group. It is a special case of his more general problem of classification of geometric structures of finite order on compact manifold with a noncompact group of automorphisms. Recall the following

**Proposition 3** Let \( M = G/H \) be a homogeneous manifold with an effective action of \( G \). Then the following conditions are equivalent:

a) \( M = G/H \) is proper;

b) the stabilizer \( H \) is compact.

c) \( M \) admits an invariant Riemannian metric (which is defined by an \( H \)-invariant Euclidean metric \( g_o \) in \( T_o M_o = eH \in M \))

An \( H \)-invariant metric \( g_o \) can be constructed as the center of the ball of minimal radius in \( S^2(T^*_o M) \) (w.r.t. some Euclidean metric \( g_1 \)) which contains the orbit \( j(H)g_1 \).

### 3.1 A criterion for existence of an invariant Lorentzian metric on a proper homogeneous manifold \( M = G/H \)

**Proposition 4** A proper homogeneous manifold \( M = G/H \) admits an invariant Lorentzian metric if and only if the isotropy group \( j(H) \) preserves an 1-dimensional subspace \( L = \mathbb{R}v \subset V = g/h \).

Moreover, let \( h \) be a \( j(H) \)-invariant Euclidean scalar product and \( \eta \) is the 1-form which defines the hyperplane \( L^\perp = \ker \eta \) orthogonal to \( L \). Then one can associate with \((L, h)\) an invariant Lorentzian scalar product

\[
g_0 = h - \lambda \eta \otimes \eta
\]

where \( \lambda > 0 \) is sufficiently big number, which defines an invariant Lorentzian metric in \( M \). Any invariant Lorentzian metric can be obtained by this construction.

**Proof.** The first claim is obvious. Now we prove that any invariant Lorentzian metric \( g \) on \( M \) is obtained by this construction. The restriction \( g_o = g|_o \) is a \( j(H) \)-invariant Lorentzian scalar product in \( V = T_o M \) and \( j(H) \) is a compact subgroup of the group
\[ SO(V) = SO_{1,n-1} \] which preserves \( g_o \). Hence it belongs to a maximal compact subgroup \( O_{n-1} \subset SO_{1,n-1} \) which preserves a time-like line \( L = \mathbb{R} t \in V \). Then

\[ h := \lambda \eta \otimes \eta + g_o \]

for \( \eta := g_o(t, \cdot) \) and sufficiently big \( \lambda > 0 \) is a \( j(H) \)-invariant Euclidean metric such that \( g = -\lambda \eta \otimes \eta + h \). So the Lorentzian metric \( g \) is obtained from a Riemannian metric (associated with \( h \)) by the described construction. \( \square \)

**Corollary 1** If \( (M = G/H, g) \) be a proper homogeneous Lorentzian manifold with connected stabilizer \( H \). Then it admits an invariant time-like vector field \( T \) with \( g(T, T) = -1 \) and the formula

\[ h = \lambda g \circ T \otimes g \circ T + g \]

defines an invariant Riemannian metric for any \( \lambda > 1 \).

We will always assume in the sequel that the stability subgroup \( H \) is connected.

**Definition 2** A proper homogeneous manifold \( M = G/H \) (and the corresponding stability group \( H \) ) is called **admissible** if \( M \) admits an invariant Lorentzian metric. Moreover, a compact subgroup \( H \) is called **maximal admissible** if it is a maximal compact subgroup such that \( M = G/H \) admits an invariant Lorentzian metric. Then the manifold \( M = G/H \) is called a **minimal admissible manifold**.

**Corollary 2** A proper homogeneous manifold \( M = G/H \) with a reductive decomposition \( g = h + m \) is admissible if and only if \( m^H \neq 0 \) where \( m^H \) is the space of \( \text{Ad}_H \)-invariant vectors from \( m \).

**Proposition 5** Any closed subgroup \( H' \) of an admissible subgroup \( H \) is admissible.

**Proof.** Let \( g = h + m \) be a reductive decomposition of an admissible manifold \( M = G/H \) and \( H' \subset H \) is a subgroup with \( h' = \text{Lie}H' \). Then

\[ g = h' + m' = h' + (p + m), \]

where \( p \) is an \( \text{Ad}_{H'} \)-invariant complement to \( h' \) in \( h \), is a reductive decomposition of \( G/H' \) and

\[ m^{H'} = p^{H'} + m^{H'} \supset m^H \neq 0. \]

This shows that \( H' \) is an admissible subgroup. \( \square \)

The above observations reduce the problem of description of admissible homogeneous \( G \)-manifolds \( M = G/H \) to classification of maximal admissible subgroups \( H \) of \( G \) and a description of all closed subgroup of the (compact) maximally admissible groups \( H \). The problem of construction of all invariant Lorentzian metrics on a given admissible homogeneous manifold \( M = G/H \) with a reductive decomposition \( g = h + m \) reduces to a description of all invariant Riemannian metrics on \( M \) (or , equivalently, \( \text{ad}_h \)-invariant
Euclidean scalar products in $\mathfrak{m}$) and a description of the space $\mathfrak{m}^H$ of $\text{Ad}_H$-invariant vectors in $\mathfrak{m}$.

**Example** Let $M = G/H$ be an admissible homogeneous manifold with a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Assume that the $j(H)$-module $\mathfrak{m}$ admits a decomposition

$$\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1 + \cdots + \mathfrak{m}_k$$

where $\mathfrak{m}_0$ is a trivial module and $\mathfrak{m}_i$, $i > 0$ are non-equivalent irreducible modules. Then any invariant Lorentzian metric on $M$ is defined by a scalar product of the form

$$g = g_0 + \lambda_1 g_1 + \cdots + \lambda_k g_k$$

where $g_0$ is a Lorentzian scalar product, $g_i$ are invariant Euclidean scalar product in $\mathfrak{m}_i$, $i > 0$ and $\lambda_i$ are positive numbers.

We will use this construction in the sequel.

### 3.2 Minimal homogeneous Lorentzian $G$-manifolds where $G = G_1 \times G_2$ is a direct product

In this subsection we describe the structure of minimal admissible homogeneous $G$-manifold $M = G/H$ where $G = G_1 \times G_2$ is a direct product of two Lie groups. It reduces the classification of minimal admissible homogeneous manifolds of a semisimple Lie group $G$ to the case of simple Lie group $G$.

The reductive decomposition of $M = (G_1 \times G_2)/H$ can be written as

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{h}_1 + \mathfrak{h}_1 + \mathfrak{l}) + (\mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{l}_1)$$

where $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$, $\mathfrak{l}$ is the complementary to $\mathfrak{h}_1 + \mathfrak{h}_2$ ideal of $\mathfrak{h}$, $\mathfrak{l}_i = \pi_i(\mathfrak{l}) \simeq \mathfrak{l}$ is the projection of $\mathfrak{l}$ to $\mathfrak{h}_i$ and $\mathfrak{m}_i$ is an $\text{ad}_\mathfrak{h}$-invariant complement to the compact subalgebra $\mathfrak{h}_i + \mathfrak{l}_i$ in $\mathfrak{g}_i$, $i = 1, 2$. Assume that the space $\mathfrak{m}_1^H$ of $H$-invariant vectors in $\mathfrak{m}_1$ is not zero. Then the subalgebra $\mathfrak{h}_1 + \mathfrak{l}_1 + \mathfrak{h}_2 + \mathfrak{l}_2$ generates an admissible subgroup which, by maximality of $H$, coincides with $H$. Hence $\mathfrak{l} = 0$ and the homogeneous manifold $M$ is a direct product $M = G/H = G_1/H_1 \times G_2/H_2$. Note that a subgroup $H_1 \times H_2 \subset G_1 \times G_2$ is maximal admissible if one of the factors, say $H_1$ is a maximal admissible subgroup of $G_1$ and the other factor $H_2$ is a maximal compact subgroup of $G_2$.

Assume now that $\mathfrak{m}_i^H = 0$, $i = 1, 2$. Then the compact subalgebra $\mathfrak{l}_1$ must have a center and from the condition that $H$ is a maximal admissible subgroup we conclude that $\mathfrak{l}_i = \mathbb{R} t_i$ is an 1-dimensional subalgebra of $\mathfrak{g}_i$ and $\mathfrak{h}_i + \mathbb{R} t_i$ is its centralizer in a maximal compact subalgebra $\mathfrak{t}_i$ of $\mathfrak{g}_i$. This implies the following result.

**Theorem 1** Let $M = G/H$ be a minimal admissible homogeneous manifold of a Lie group $G = G_1 \times G_2$. 
If \( H = H_1 \times H_2 \) is consistent with the decomposition of \( G \), then one of the subgroups \( H_1, H_2 \), say \( H_1 \), is maximal admissible in \( G_1 \) and the other subgroup \( H_2 \) is maximal compact subgroup of \( G_2 \).

If \( H \) is not consistent with the decomposition, then its Lie algebra has the form

\[
h = h_1 + h_2 + \mathbb{R}(t_1 + t_2)
\]

where \( h_i + \mathbb{R}t_i = Z_{\mathfrak{k}_i}(t_i) \) is the centralizer of an element \( t_i \in \mathfrak{g}_i \) into a maximal compact subalgebra \( \mathfrak{k}_i := \text{Lie} K_i \) of \( \mathfrak{g}_i \), \( i = 1, 2 \). The reductive decomposition associated with \( M = G/H \) can be written as

\[
g = h + m = h + (m_1 + m_2 + \mathbb{R}(t_1 - t_2))
\]

where \( m_i \) is an \( \text{ad}_h \)-invariant complement to \( Z_{\mathfrak{k}_i}(t_i) \) in \( \mathfrak{g}_i \).

This theorem can be applied to the case when \( G \) is a semisimple Lie algebra and it reduces the description of admissible homogeneous manifolds of a semisimple Lie group \( G \) to the case of simple Lie groups.

4 \ Homogeneous Lorentzian manifolds of simple compact Lie group

Let \( G \) be a compact simple Lie group. The adjoint orbit \( F = \text{Ad}_G t \simeq G/Z_G(t) \) of \( G \) is called to be minimal, if the stability subgroup \( Z_G(t) \) (which is the centralizer of an element \( t \in \mathfrak{g} \)) is not contained properly in the centralizer of other non-zero element \( t' \in \mathfrak{g} \). Recall that the centralizer \( Z_G(t) \) is connected.

It is know, see, for example [Al2] that the orbit \( F \) if minimal if and only if \( Z_G(t) \) has 1-dimensional center \( T^1 = \{ \exp \lambda t \} \) and can be written as \( Z_G(t) = H \cdot T^1 \) where \( H \) is a semisimple normal subgroup. Minimal adjoint orbits (up to an isomorphism) correspond to simple roots \( \alpha \) of the Lie algebra \( \mathfrak{g} \). Moreover, the Dynkin diagram of the semisimple group \( H \) is obtained from the Dynkin diagram of \( \mathfrak{g} \) by deleting the vertex \( \alpha \). We will denote the minimal orbit associated with a simple root \( \alpha \) by \( F_\alpha \). Below we give the list of all such semisimple subgroups \( H \) for all simple Lie groups \( G \):

\[
\begin{align*}
G &= SU_n, \quad H = SU_p \times SU_q, \quad p + q = n, \quad p = 1, 2, \ldots, n - 1; \\
G &= SO_n, \quad H = SU_p \times SO_q, \quad 2p + q = n, \quad p = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor; \\
G &= Sp_n, \quad H = Sp_p \times Sp_q, \quad n = p + q, \quad p = 1, 2, \ldots, n - 1; \\
G &= G_2, \quad H = SU_{2}^{\text{short}} \times SU_{2}^{\text{long}} \\
G &= F_4, \quad H = Sp_3, \quad SU_3^{\text{short}} \times SU_2^{\text{long}}, \quad SU_2^{\text{short}} \times SU_3^{\text{long}}, \quad Spin_7; \\
G &= E_6, \quad H = Spin_{10}, \quad SU_2 \times SU_5, \quad SU_3 \times SU_3 \times SU_2, \quad SU_6; \\
G &= E_7, \quad H = E_6, \quad SU_2 \times Spin_{10}, \quad SU_3 \times SU_5, \quad SU_4 \times SU_3 \times SU_2, \quad SU_6 \times SU_2, \quad Spin_{12}, \quad SU_7; \\
G &= E_8, \quad H = E_7, \quad SU_2 \times E_6, \quad SU_3 \times Spin_{10}, \quad SU_4 \times SU_5, \quad SU_5 \times SU_3 \times SU_2, \quad SU_7 \times SU_2, \quad Spin_{14}.
\end{align*}
\]
Let \( F_\alpha = G/H \cdot T^1 \) be a minimal orbit associated with a simple root \( \alpha \). Then
\[
\pi : M_\alpha = G/H \to F_\alpha = G/H \cdot T^1
\]
is a principal fibration with the structure group \( T^1 \). Denote by
\[
\theta : TM_\alpha \to \mathbb{R} = \text{Lie}(T^1)
\]
the \( G \)-invariant principal connection defined by the condition \( \theta(t) = 1, \theta(p) = 0 \) where
\[
g = (\mathfrak{h} + \mathbb{R}t) + \mathfrak{p}
\]
is the reductive decomposition associated with the orbit \( F_\alpha = G/H \cdot T^1 \). We say that \( \pi \) is the canonical \( T^1 \) bundle with connection over the orbit \( F_\alpha \).

It is known that the tangent space \( T_oF_\alpha \simeq \mathfrak{p} \) as an \( \text{Ad}(H \cdot T^1) \)-module is decomposed into mutually non equivalent irreducible submodules
\[
\mathfrak{p} = \mathfrak{p}_1 + \cdots + \mathfrak{p}_m 
\]
and the number \( m \) of these submodules equal to the Dynkin number \( m(\alpha) = m_i \) of the corresponding simple root \( \alpha = \alpha_i \) that is the coordinate \( m_i \) over \( \alpha_i \) in the decomposition \( \mu = \sum_j m_j \alpha_j \) of the maximal root \( \mu \) with respect to the simple roots \( \alpha_1, \cdots, \alpha_r \). This implies that any invariant Riemannian metric \( g_F \) in \( F \) at the point \( o = e(H \cdot T^1) \) is given by
\[
g_o = \lambda_1 b_1 + \cdots + \lambda_m b_m
\]
where \( b_j = -B|_{\mathfrak{p}_j} \) is the restriction of the minus Killing form \( -B \) to \( \mathfrak{p}_j \) and \( \lambda_j \) are positive constants.

**Theorem 2** Any minimal admissible manifold of a simple compact Lie group \( G \) is the total space \( M_\alpha = G/H \) of the canonical fibration over a minimal orbit \( F = F_\alpha = G/H_\alpha \cdot T^1 \). Moreover, if \( M = G/H_\alpha \) is not the total space of the sphere bundle of a compact rank one symmetric space that is
\[
S(S^n) = \text{SO}_{n+1}/\text{SO}_{n-1}, \text{Spin}_7/\text{SU}_3 = S(S^7) = S^7 \times S^6, S(S^3) = \text{SU}_2 \times \text{SU}_2/T^1 = S^3 \times S^2;
\]
\[
S(\mathbb{C}P^n) = \text{SU}_{n+1}/\text{SU}_n, S(\mathbb{H}^n) = \text{Sp}_{n+1}/\text{Sp}_1 \times \text{Sp}_n/\text{Sp}_2, S(\mathbb{O}P^2) = F_4/\text{Spin}_7
\]
then any invariant Lorentz metric \( g \) on \( M \) is given by
\[
g = -\lambda \theta^2 + \pi^* g_F
\]
where \( \theta \) is the principal connection, \( g_F \) is an invariant Riemannian metric on \( F \) and \( \lambda \) is a positive number. In particular, the metric \( g \) depends on \( m(\alpha) + 1 \) positive parameters, where \( m(\alpha) \) is the Dynkin mark.

**Proof.** Let \( M = G/H \) be a minimal admissible homogeneous manifold of a simple compact Lie group \( G \) with the reductive decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \). Denote by \( t \in \mathfrak{m} \) an \( \text{Ad}_H \)-invariant non-zero vector. We can assume that \( t \) generates a closed one-parameter
subgroup since \( H \) preserves pointwise the curve \( \exp \lambda t \), hence, also its closure in \( G \). The centralizer \( \mathfrak{z}(t) \) of \( t \) in \( \mathfrak{g} \) can be decomposed into a direct sum \( \mathfrak{z}(t) = \mathfrak{h} + \mathbb{R}t \) where \( \mathfrak{h} \supset \mathfrak{h} \) is a subalgebra which generates a closed subgroup \( \tilde{H} \) of \( G \). Since \( \text{Ad}_{\tilde{H}} \) preserves \( t \), the homogeneous space \( G/\tilde{H} \) is admissible and due to minimality of \( M \) it coincides with \( M \). Hence, \( H = \tilde{H} \) and \( Z_G(t) = H \cdot T^1 \) where \( T^1 \) is the closed subgroup generated by \( t \). It is proven in [AS], that if \( M \) is not the total space of the sphere bundle of a compact rank one symmetric space, then all irreducible \( (H \cdot T^1) \)-submodules of the decomposition (1) remain irreducible and non-equivalent as \( H \)-submodules. This implies the last claim of the theorem.

\[ \square \]

5 Homogeneous Lorentzian manifolds of a simple noncompact Lie group

Now we consider minimal admissible homogeneous manifolds of a simple noncompact Lie group \( G \).

5.1 Case when the group \( G \) has infinite center

Assume at first that \( G \) has infinite center. It is known that such group \( G \) acts transitively (and almost effectively) on a non-compact irreducible Hermitian symmetric space \( S = G/K \cdot \mathbb{R} \) with the symmetric decomposition

\[ \mathfrak{g} = (\mathfrak{k} + \mathbb{R}t) + \mathfrak{p} \]

where \( \mathbb{R}t \) is the 1-dimensional centralizer of the Lie algebra \( \mathfrak{k} \) of a maximal compact subgroup \( K \) of \( G \) and \( \text{ad}_{\mid \mathfrak{p}} \) is \( j(K \cdot \mathbb{R}) \)-invariant complex structure in the tangent space \( \mathfrak{p} = T_o S \). Obviously, we get the following

**Proposition 6** Let \( G \) be a simple non-compact Lie group and \( S = G/K \cdot \mathbb{R} \) the associated Hermitian symmetric space. Then the manifold \( M = G/K \) is the only minimal admissible \( G \)-manifold and all invariant Lorentzian metrics on \( M \) are defined by the scalar product in \( \mathfrak{m} = \mathbb{R}t + \mathfrak{p} \) of the form

\[ g = -\lambda \theta^2 + gp \]

where \( \lambda > 0 \), \( \theta \) is the 1-form dual to the vector \( t \) (such that \( \theta(t) = 0, \theta(p) = 0 \)) and \( gp \) is the invariant Euclidean scalar product in \( \mathfrak{p} \) which defines the symmetric Riemannian metric in \( S \). In particular,

\[ \pi : M = G/K \rightarrow S = G/K \cdot \mathbb{R} \]

is a pseudo-Riemannian submersion.
5.2 Duality

Now we will assume that $G$ is a simple noncompact Lie group with a finite center. Then the quotient $S = G/K$ by a maximal compact subgroup $K$ is a symmetric space of noncompact type. We will denote by $\hat{S} = \hat{G}/\hat{K}$ the dual compact symmetric space. Let

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$$

be a symmetric decomposition associated with the symmetric space $S$. Then the symmetric decomposition associated with $\hat{S}$ can be written as

$$\hat{\mathfrak{g}} = \mathfrak{h} + i\mathfrak{p}$$

where $[iX, iY] = -[X, Y]$ for $X, Y \in \mathfrak{p}$.

In particular, the dual symmetric spaces $S, \hat{S}$ have the same stabilizer $K$ and isomorphic isotropy representation $j(K) = Ad_K|_{\mathfrak{p}} \simeq Ad_K|_{i\mathfrak{p}}$. This implies the natural bijection between (maximal) admissible subgroups $H \subset K$ of the dual Lie groups $G$ and $\hat{G}$. In terms of homogeneous Lorentzian manifolds this can be reformulated as follows.

**Proposition 7** There exists a natural one-to-one correspondence between proper homogeneous Lorentzian $G$-manifolds $M = G/H$ of a simple noncompact Lie group $G$ and homogeneous Lorentzian manifolds $\hat{M} = \hat{G}/\hat{H}$ of the dual compact Lie group $\hat{G}$ such that the stabilizer $H$ belongs to the subgroup $K \subset \hat{G}$.

**Proof.** Let $M = G/H, H \subset K$ be an admissible $G$-manifold with reductive decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} := \mathfrak{h} + (\mathfrak{n} + \mathfrak{p}), \ \mathfrak{k} = \mathfrak{h} + \mathfrak{n}$$

with the invariant Lorentzian metric defined by an $Ad_H$-invariant Lorentzian scalar product $g_o$ in $\mathfrak{m} = \mathfrak{h} + \mathfrak{p}$, then the dual compact homogeneous Lorentzian manifold is the homogeneous manifold $\hat{M} = \hat{G}/\hat{H}$ with the reductive decomposition

$$\hat{\mathfrak{g}} = \mathfrak{h} + \hat{\mathfrak{m}} := \mathfrak{h} + (\mathfrak{n} + i\mathfrak{p})$$

and the metric defined by the Lorentzian scalar product in $\hat{\mathfrak{m}}$ which corresponds to the scalar product $g_o$ under the natural isomorphism

$$\hat{\mathfrak{m}} = \mathfrak{n} + i\mathfrak{p} \simeq \mathfrak{m} = \mathfrak{n} + \mathfrak{p}.$$

\[\square\]

5.3 A characterization of noncompact homogeneous Lorentzian manifolds of class I and class II

Let $M = G/H, H \subset K$ be an admissible homogeneous space of a noncompact simple Lie group $G$ with the reductive decomposition $[2]$. Then the space $\mathfrak{m}^H = \mathfrak{n}^H + \mathfrak{p}^H$ of $j(H)$-invariant vectors is not zero.
Definition 3 We say that the admissible homogeneous manifold \( M = G/H \) belongs to
the class I if \( n^H \neq 0 \) and belongs to the class II if \( p^H \neq 0 \).

Geometrically, homogeneous spaces of the class I and the class II can be characterized
as follows.

Proposition 8 An admissible \( G \)-manifold \( M = G/H \) of a simple noncompact Lie group
\( G \) belongs to the class I if it admits an invariant Lorentzian metric such that \( \pi : M = G/H \to S = G/K \) is a pseudo-Riemannian submersion with totally geodesic Lorentzian
fibres over the noncompact Riemannian symmetric space \( S = G/K \). In particular, the
invariant time-like vector field generate a compact group \( S^1 \).

An admissible manifold \( M = G/H \) belongs to the class II if it admits an invariant
Lorentzian metric with a time-like invariant vector field, which generates a noncompact
1-parameter subgroup \( \mathbb{R} \).

Proof. Assume that \( M \) belongs to the class I. Let \( t \in n^H \) be an \( H \)-invariant vector and
\( g = g_n \oplus g_p \) an Euclidean scalar product in \( m \) which is a sum of \( \text{Ad}_n \)-invariant scalar
product in \( n \) and the unique ( up to a scaling) \( \text{Ad}_K \)-invariant scalar product in \( p \). Then
the invariant Lorentzian metric in \( M \) defined by the Lorentzian scalar product of the form
\( g_{\lambda, \lambda} = g - \lambda g \circ t \otimes g \circ t \) for sufficiently big \( \lambda \) satisfies the stated property. \( \square \)

Remark It is possible that a minimal admissible \( G \)-manifold belongs to the class I
and the class II at the same time.

Let \( K \subset GL(V) \) be a linear Lie group. Recall that by the (connected) stabilizer \( K_v \) of
a vector \( v \in V \) we understand the connected component of the subgroup which preserves \( v \).

Definition 4 Let \( K \subset GL(V) \) be a linear Lie group. The orbit \( K_v \) of a vector \( v \neq 0 \) is
called a minimal orbit is the the ( connected) stabilizer \( K_v \) does not contained properly
in the (connected ) stabilizer \( K_w \) of any other non-zero vector \( w \). Then the stabilizer \( K_v \is called a maximal stabilizer.

The following obvious proposition reduces the classification of all minimal admissible
homogeneous \( G \)-manifolds \( M = G/H \) of the class I to the classification of maximal admissible
subgroups \( H \) of the maximal compact subgroup \( K \) of \( G \) and the classification of
such manifolds of the class II to the description of maximal isotropy subgroups \( K_v \) of the
isotropy representation \( \text{Ad}_K|p \) of the symmetric space \( S = G/K \).

Proposition 9 Let \( M = G/H \) be a minimal admissible homogeneous \( G \)-manifold of a
simple noncompact Lie group \( G \).

i) If \( M \) belongs to the class I, then \( H \) is a maximal admissible subgroup of a maximal
compact subgroup \( K \supset H \) of \( G \).

ii) If \( M \) belongs to the class II, then \( H = K_v \) is a maximal (connected) stabilizer of the
isotropy representation of the Riemannian symmetric space \( S = G/K \).
Let \( \mathfrak{g} = \mathfrak{k} + \mathfrak{p} \) be the symmetric decomposition of a symmetric space \( S = G/K \). For any nonzero vector \( v \in \mathfrak{p} \) we denote by \( \mathfrak{k}_v \) the stability subalgebra of the isotropy representation \( j(\mathfrak{t}) \) and by \( K_v \subset K \) corresponding connected stability subgroup.

**Definition 5** The subalgebra \( \mathfrak{k}_v \subset \mathfrak{k} \) (resp., corresponding subgroup \( K_v \subset K \)) is called a maximal stability subalgebra (resp., maximal stability subgroup) if it does not contained properly in any other stability subalgebra (resp., stability subgroup) of the isotropy representation of \( G/K \).

**Proposition 10** Let \( S = G/K \) be a symmetric space of noncompact type and \( H \subset K \) a maximal admissible subgroup of \( K \) such that the admissible manifold \( G/H \) belongs to the class II. Then \( H = H_v \) is a maximal stability subgroup of \( K \). Conversely, any maximal stability subgroup \( K_v \) of \( K \) is admissible and defines an admissible manifold \( M = G/K_v \) of the class II.

So the classification of proper homogeneous Lorentzian manifolds of a semisimple noncompact group \( G \) reduces to description of maximal stability subgroups \( K_v \) of the isotropy representation of the associated symmetric space \( S = G/K \).

Due to theorem 1, it is sufficient to describe such subgroups for simple Lie groups.

5.4 Homogeneous Lorentzian \( SL_n(\mathbb{R}) \)-manifolds

In this subsection we classify all minimal homogeneous Lorentzian \( G \)-manifolds of the class II where \( G = SL_n(\mathbb{R}) \).

Let \( S = SL_n(\mathbb{R})/SO_n \). We identify \( S \) with the codimension one orbit \( SL_n(\mathbb{R})g_0 \) of the Euclidean metric \( g_0 \) in the space \( S^2V^* \) of symmetric bilinear forms in \( V = \mathbb{R}^n \) (or with the space of symmetric matrices). In particular, the tangent space \( T_{g_0}S = T_0S \) is identified with the space of \( S_0^2(V^*) \) of traceless (w.r.t. \( g_0 \)) bilinear forms. Let \( V = U + W \) be a decomposition of \( V \) into a \( g_0 \)-orthogonal sum of subspaces of dimension \( p \) and \( q \), respectively, and \( H = SO(U) \times SO(W) = SO_p \times SO_q \) the connected subgroup of \( SO(V) = SO_n \) which preserves this decomposition. Consider the homogeneous manifold

\[
M_{p,q} = G/H := SL_n(\mathbb{R})/SO_p \times SO_q, \quad p + q = n.
\]

It has the natural fibration

\[
M_{p,q} = SL_n/(SO_p \times SO_q) \to S = SL_n/SO_n
\]
over the symmetric space $S = SL_n/SO_n$ with the Grassmannian $Gr_p(\mathbb{R}^n) = SO_n/SO_p \times SO_q$ as a fibre. The Grassmannian is an irreducible symmetric manifold with the symmetric decomposition
$$\mathfrak{so}_n = \mathfrak{so}(V) = (\mathfrak{so}(U) + \mathfrak{so}(W)) + U \wedge W.$$ Then the reductive decomposition of the homogeneous manifold
$$M_{p,q} = SL(V)/SO(U) \times SO(W) = SL_n(\mathbb{R})/SO_p \times SO_q$$
can be written as
$$\mathfrak{g} := \mathfrak{sl}(V) = \mathfrak{h} + \mathfrak{m} = (\mathfrak{so}(U) + \mathfrak{so}(W)) + (\mathbb{R}b + U^* \wedge W^* + S_0^2U^* + S_0^2W^* + U^* \vee W^*)$$
where $\vee$ is the symmetric product, $b := qg_0|_U - pg_0|_W$ and $S_0^2U^*, S_0^2W^*$ are irreducible submodules of traceless bilinear forms. As a $j(H)$-module, the tangent space $\mathfrak{m}$ is isomorphic to
$$\mathfrak{m} = \mathbb{R}b + (U \otimes V) \otimes \mathbb{R}^2 + S_0^2U + S_0^2W.$$ In particular,
$$\mathfrak{m}^H = \mathbb{R}b \neq 0.$$ We get

**Proposition 11** The homogeneous manifold $M_{p,q}$ is an admissible manifold. Any invariant Lorentzian metric on it is defined by the scalar product of the form
$$g = -\lambda_1 b^* \otimes b^* + g_1 \otimes g_{\mathbb{R}^2} + \lambda_2 g_2 + \lambda_3 g_3$$
where $\lambda_i, i = 1, 2, 3$ are positive constants, $g_1, g_2, g_3$ are the Euclidean scalar products in $U \otimes V, S_0^2U$ and $S_0^2W$ respectively, induced by the metric $g_0$ and $g_{\mathbb{R}^2}$ is an Euclidean scalar product in $\mathbb{R}^2$.

The following theorem shows that the spaces $M_{p,q}$ exhaust all minimal homogeneous Lorentzian $SL_n(\mathbb{R})$-manifolds of the class II.

**Theorem 3** A minimal admissible homogeneous $SL_n(\mathbb{R})$-manifold $M$ of class II is isomorphic to the manifold $M_{p,q} = SL_n/(SO_p \times SO_q)$ for some $p, q$ with $p + q = n$.

**Proof.** The isotropy representation $j$ of the symmetric space $S = SL_n(\mathbb{R})/SO_n$ is the standard representation of $K = SO_n$ in the space $T_0S = S_0^2\mathbb{R}^n$ of traceless symmetric matrices. The stability subgroups of $j(SO_n)$ are $SO_{p_1} \times \cdots \times SO_{p_s}$ and maximal admissible subgroups are $SO_p \times SO_q$. They defines manifolds $M_{p,q}$. \qed

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5.5 Homogeneous Lorentzian $G$-manifolds where $G$ is a simple Lie group of real rank one

In this subsection we describe minimal homogeneous Lorentzian manifolds $M = G/H$ of the class II for all simple Lie group $G$ of real rank 1. The isotropy group $j(K)$ of the associated rank one symmetric space $S = G/K$ acts transitively on the unit sphere in $T_oS$ and the stability subgroups $K_v$ of a point $0 \neq v \in T_oS$ is unique ( up to a conjugation), hence, maximal.

The list of all noncompact rank one symmetric space $S = G/K$ is given below, see [H].

List of rank one noncompact symmetric spaces $S = G/K$.

- $\mathbb{R}H^n = SO_{1,n}/SO_n$, $\mathbb{C}H^n = SU_{1,n}/U_n$, $\mathbb{H}H^n = Sp_{1,n}/Sp_1 \times Sp_n$, $\mathbb{O}P^2 = F_4/Spin_9$.

We describe corresponding minimal admissible manifolds $M = G/H = G/K_v$ of the class II for each of these groups together with the reductive decomposition $g = h + m$ and the decomposition of the tangent space $m$ into irreducible $j(H)$-modules. It allows to give an explicit description of all invariant Lorentzian metrics on $M$.

5.5.1 Case of the group $G = SO_{1,n}^0$

Let $V = \mathbb{R}^{1,n}$ is the Minkowski vector space and $V = \mathbb{R}e_0 + E$ its decomposition where $e_0, e_0^2 = -1$, is a unit time-like vector and $E = e_0^\perp$. The hyperbolic space is the orbit $\mathbb{R}H^n = G/K = SO_{1,n}^0e_0$ and $E = T_{e_0}\mathbb{R}H^n$ is the tangent space with the standard action of the isotropy group $SO_n = SO(E)$. We will identify the Lie algebra $\mathfrak{so}_{1,n} = \mathfrak{so}(V)$ with the space $\Lambda^2V$ of bivectors. Then the reductive decomposition of $G/K$ is given by

$$g = \mathfrak{h} + \mathfrak{p} = \Lambda^2E + e_0 \wedge E.$$ 

The stability subalgebra $\mathfrak{h} = \mathfrak{k}_{e_1}$ of a unit vector $e_1 \in E$ is $\mathfrak{so}(W) = \Lambda^2W$ where $W = e_1^\perp$ is the orthogonal complement of $e_1$ in $E$. This implies

Proposition 12 The only class II minimal admissible manifold of the group $G = SO_{1,n}$ is the manifold $M = SO_{1,n}^0/\text{SO}_{n-1}$. It has the reductive decomposition

$$\mathfrak{so}_{1,n} = \mathfrak{so}(V) = \mathfrak{so}(W) + (\mathbb{R}(e_0 \wedge e_1) + e_0 \wedge W + e_1 \wedge W)$$

where

$$\mathbb{R}^{1,n} = V = \mathbb{R}e_0 + \mathbb{R}e_1 + W$$

is an orthogonal decomposition of the Minkowski space $V$. In particular, $m^H = \mathbb{R}(e_0 \wedge e_1)$. 

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5.5.2 Case of the group $G = SU_{1,n}$

Let $\mathbb{C}^{1,n} = V$ be the complex pseudo-Hermitian space with the Hermitian scalar product $\langle \cdot, \cdot \rangle$ of complex signature $(1, n)$ and

$$V = \mathbb{C}e_0 + E = \mathbb{C}e_0 + \mathbb{C}e_1 + W$$

an orthogonal decomposition, such that

$$\langle e_0, e_0 \rangle = -1, \langle e_1, e_1 \rangle = 1.$$

The complex hyperbolic space is the orbit

$$\mathbb{C}H^n = SU_{1,n}[e_0] = SU_{1,n}/U_n$$

of the point $[e_0] := \mathbb{R}e_0 \in PV$ in the projective space $PV = \mathbb{CP}^{n+1}$. The tangent space $T_{[e_0]}\mathbb{C}H^n$ is identified with $E = \mathbb{C}e_1 + W$. In matrix notations (with respect to an orthonormal basis $e_0, e_1, \cdots, e_n$ of $V$) the reductive decomposition of $\mathbb{C}H^n$ can be written as

$$G = h + p = u_n + C^n,$$

$$u_n = \left\{ \begin{pmatrix} -\alpha & 0 \\ 0 & A \end{pmatrix} \mid A \in u_n, \alpha = \text{tr} A \right\}, \quad p = \{ X := \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix} \mid X \in C^n, X^* := \bar{X}^t \}.$$

The stability subalgebra $\mathfrak{k} = \mathfrak{su}_n \oplus \mathbb{R}z_0$, where

$$z_0 = i \text{diag} (1, -\frac{1}{n} \text{Id}_n).$$

We identify the tangent space $p = T_{e_0}\mathbb{C}H^n = E$ with the space $C^n$ of columns. Then the subalgebra $\mathfrak{su}_n$ acts in $p = C^n$ by the matrix multiplication and $z_0$ as the multiplication by $-\frac{n-1}{n}i$.

The element $v = (1, 0, \cdots, 0)^t \in C^n = T_{e_0}\mathbb{C}H^n = m$ corresponds to the matrix

$$v = e_1 \otimes e_0^* - e_0 \otimes e_1^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The stabilizer $H = K_v \simeq U_{n-1}$ has the Lie algebra

$$\mathfrak{h} = \mathfrak{k}_v = \mathfrak{su}_{n-1} \oplus \mathbb{R}z$$

where $\mathfrak{su}_{n-1} = \mathfrak{su}(W)$ acts trivially on $e_0, e_1$ and with respect to the decomposition $V = \mathbb{C}e_0 + \mathbb{C}e_1 + W$ the matrix $z \in \mathfrak{h} \subset \mathfrak{su}_{1,n}$ is given by

$$z = i \text{diag} (1, 1, -\frac{2}{n-1} \text{Id}_W).$$

The stability subalgebra $\mathfrak{h} = \mathfrak{su}_{n-1} \oplus \mathbb{R}z$ annihilates the 2-dimensional space

$$\mathbb{C}v = \{ cv = ce_1 \otimes e_0^* - e_0 \otimes (ce_1)^* \} \subset m.$$
The Lie algebra $\mathfrak{su}_{n-1} \subset \mathfrak{h}$ acts in the standard way on the complementary subspace $\mathfrak{p}' = \{ w \otimes e_0^* - e_0 \otimes w^*, w \in W \} \subset \mathfrak{p}$ isomorphic to $W$. The element $z$ acts on $\mathfrak{p}'$ as a multiplication by $-\frac{n+1}{n-1}i$.

The reductive decomposition of the sphere $K/H = U_n/U_{n-1}$ has the form

$$\mathfrak{k} = \mathfrak{h} + \mathfrak{n} = (\mathfrak{su}_{n-1} + \mathbb{R}z) + (\mathbb{R}z' + \mathfrak{n}')$$

where $z' := \text{diag}(1, -1, 0_{n-1})$ and

$$\mathfrak{n}' := \{ w \otimes e_0^* - e_0 \otimes w^*, w \in W \}.$$

The $j(H)$- invariant subspace $\mathfrak{n}^H = \mathbb{R}z'$ and $j(z)$ acts on $\mathfrak{n}' \cong \mathbb{C}^{n-1}$ as multiplication by $-\frac{n+1}{n-1}i$. We get

**Proposition 13** The only minimal admissible $SU_{1,n}$- manifold of the class II is the manifold $M = SU_{1,n}/U_{n-1}$ with the reductive decomposition

$$\mathfrak{su}_{1,n} = (\mathfrak{su}_{n-1} + \mathbb{R}z) + (\mathbb{R}z' + \mathfrak{n}')$$

(We indicate the action of the central element $z \in \mathfrak{h}$ on the corresponding irreducible subspaces.)

Since $\mathfrak{n}^H = \mathbb{R}z' \neq 0$, the manifold $M$ belongs also to the the class I. The next proposition, which follows from Theorem 2 and Theorem 3, describe all minimal admissible $SU_{1,n}$-manifolds of the class I. Let $\mathfrak{g}_n = \mathbb{R}z_0 + \mathfrak{su}_n$ be the Lie algebra of the group $U_n$ and $a \in \mathfrak{su}_n$ an element such that $\mathbb{R}(z_0 + a)$ generate a closed subgroup $T^1_a$ of $U_n$.

**Proposition 14** Any class I minimal admissible $SU_{1,n}$-manifold is isomorphic to one of the manifolds :

a) $SU_{1,n}/SU_n$,

b) $SU_{1,n}/T^1_a \cdot Z_{SU_n}(a), \ 0 \neq a \in \mathfrak{su}_n$ or

c) $SU_{1,n}/T^1_0 \cdot H'$ where $H'$ is a maximal admissible subgroup of $SU_n$.

**Proof.** We have to describe maximal admissible subgroups $H$ of $U_n$. If the Lie algebra $\mathfrak{h}$ of $H$ contains the center $\mathfrak{z} = \mathbb{R}z_0$, we get c). If the projection of $\mathfrak{h}$ on $\mathfrak{z}$ is trivial, then $\mathfrak{h} = \mathfrak{su}_n$ and we get a). If the projection is non trivial, then $\mathfrak{h} = \mathbb{R}(z + a) \oplus \mathfrak{h}'$ for some non-zero $a \in \mathfrak{su}_n$, where $\mathfrak{h}'$ is a subalgebra of $\mathfrak{su}_n$. The reductive decomposition of $\mathfrak{u}_n$ can be written as

$$\mathfrak{u}_n = \mathfrak{h} + (\mathbb{R}z + \mathfrak{m}')$$

where $\mathfrak{su}_n = (\mathbb{R}a + \mathfrak{h}') + \mathfrak{m}'$ is a reductive decomposition of $\mathfrak{su}_n$. The maximally of $\mathfrak{h}$ implies that $\mathbb{R}(z + a) = \mathfrak{z}\mathfrak{su}_n(a)$ and we get b), where $T^1_a$ is the 1-parameter subgroup generated by $z + a$. □
5.5.3 Case of the group $G = Sp_{1,n}$

Let $V = \mathbb{H}^{1,n}$ be the quaternionic vector space with a Hermitian form $<.,.>$ of quaternionic signature $(1,n)$ and

$$V = \mathbb{H}e_0 + E = \mathbb{H}e_0 + \mathbb{H}e_1 + W$$

its orthogonal decomposition with $<e_0,e_0> = <e_1,e_1> = -1$. The quaternionic hyperbolic space $\mathbb{H}P^n = G/K = SU_{1,n}/Sp_1 \cdot Sp_n$ is the orbit $\mathbb{H}H^n = SU_{1,n}[e_0]$ in the quaternionic projective space $\mathbb{H}P^{n+1}$. The tangent space $T_{[e_0]}\mathbb{H}H^n = E$. In terms of an orthonormal basis $e_0,e_1,\cdots,e_n$ of $\mathbb{H}^{1,n}$, the reductive decomposition of $\mathbb{H}H^n$ is given by

$$\mathfrak{sp}_{1,n} = \mathfrak{h} + \mathfrak{p}$$

$\mathfrak{h} = \{(\alpha, 0, 0, A), \alpha \in \text{Im} \mathbb{H} = \mathfrak{sp}_1, A \in \mathfrak{sp}_n\}$, $\mathfrak{p} = \{(0, X^t, 0), X \in \mathbb{H}^n\}$.

Under identification $T_{[e_0]}\mathbb{H}H^n = E = \mathfrak{p}$, the vector $e_1$ is identified with the matrix

$$v = e_1 \otimes e_0^* - e_0 \otimes e_1^* = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}.$$

The stabilizer $H = K_v$ of the vector $v = e_1 \in E = T_{[e_0]}\mathbb{H}H^n$ has the Lie algebra

$$\mathfrak{h} = \mathfrak{sp}_1 + \mathfrak{sp}_{n-1} = \{(\alpha, A) := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & A \end{pmatrix}, \alpha \in \text{Im} \mathbb{H}, A \in \mathfrak{sp}_{n-1}\}.$$

The action $j(\alpha, A)$ on the space

$$\mathfrak{p} = \mathbb{H}v + \mathfrak{p}' = \{(x, X) := \begin{pmatrix} 0 & x^* & X^* \\ x & 0 & 0 \\ X & 0 & 0 \end{pmatrix}, x \in \mathbb{H}, X \in \mathbb{H}^{n-1}, X^* = X^t\}$$

is given by

$$j(\alpha, A)(x, X) = (\alpha x - x \alpha, AX - X\alpha).$$

The complementary subspace $\mathfrak{n}$ to $\mathfrak{h}$ in $\mathfrak{k}$ is given by

$$\mathfrak{n} = \text{Im} H + \mathfrak{n}' = \{(y', Y) := \begin{pmatrix} -y' & 0 & 0 \\ 0 & y' & -Y^* \\ 0 & Y & 0_{n-1} \end{pmatrix}, y' \in \text{Im} \mathbb{H}, Y \in \mathbb{H}^{n-1}\}.$$

The action $j(\alpha, A) \in j(\mathfrak{h})$ on $(y', Y) \in \mathfrak{n}$ is given by

$$j(\alpha, A)(y', Y) = (\alpha y' - y' \alpha, AY = Y\alpha).$$

These formulas implies the following proposition.
Proposition 15 The minimal admissible $SU_{1,n}$-manifold of the class II is the manifold $M = Sp_{1,n+1}/Sp_1 \times Sp_{n-1}$ with the reductive decomposition
\[
\mathfrak{sp}_{1,n} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{sp}_1 + \mathfrak{sp}_{n-1}) + (\text{Im}\mathbb{H} + \mathfrak{n}' + \mathbb{R}v + (\text{Im}\mathbb{H})v + p').
\]
where $\mathfrak{n}' \simeq p' \simeq \mathbb{H}^{n-1}$.
In particular, the space $\mathfrak{m}^H = \mathbb{R}v$ is one-dimensional. As $(\mathfrak{sp}_1 + \mathfrak{sp}_{n-1})$-module, the tangent space $\mathfrak{m}$ is isomorphic to
\[
\mathfrak{m} = \mathbb{R}v + \mathfrak{sp}_1 \otimes \mathbb{R}^2 + \mathbb{H}^{n-1} \otimes \mathbb{R}^2
\]
with the natural action of $\mathfrak{h} = \mathfrak{sp}_1 + \mathfrak{sp}_{n-1}$. Any invariant Lorentzian metric in $M$ is defined by the scalar product of the form
\[
g = -\lambda v^* \otimes v^* + g_1 \otimes h_1 + g_2 \otimes h_2
\]
where $g_1, g_2$ are invariant Euclidean scalar products on $\mathfrak{sp}_1, \mathbb{H}^{n-1}$, respectively, $h_1, h_2$ are any invariant Euclidean scalar products in $\mathbb{R}^2$, and $\lambda$ is a positive constant.

5.5.4 Case of the group $G = F_4$

We consider the noncompact exceptional Lie group $F_4$ with maximal compact subgroup $K = Spin_9$. The symmetric space $O\mathbb{H}^2 = G/K = F_4/Spin_9$ is dual to the octonion plane. The isotropy group $j(K)$ acts transitively on the unit sphere $S^{15}$ in the tangent space $T_0O\mathbb{H}^2 = \mathfrak{m}$ with stability subgroup $Spin_7$. The irreducible spinor $Spin_9$-module $\mathfrak{p} \simeq \mathbb{R}^{16}$ as a $Spin_7$-module is decomposed into the following irreducible $Spin_7$-submodules
\[
\mathfrak{m} = \mathbb{R}v + \mathfrak{m}_1^8 + \mathfrak{m}_2^7
\]
where $\mathfrak{spin}_7 + \mathfrak{m}_2^7 \simeq \mathfrak{spin}_8 \simeq so_8$ and $\mathfrak{m}_1$ is 8-dimensional spinor $Spin_7$-module. We get

Proposition 16 The minimal admissible $F_4$-manifold is the manifold $M = F_4/Spin_7$ with the reductive decomposition
\[
f_4 = \mathfrak{spin}_7 + \mathfrak{m} = \mathfrak{spin}_7 + (\mathbb{R}v + \mathfrak{m}_1^8 + \mathfrak{m}_2^7).
\]
Any invariant Lorentzian metric is given by
\[
g = -\lambda_0 v^* \otimes v^* + \lambda_1 g_1 + \lambda_2 g_2
\]
where $g_1, g_2$ are some fixed Euclidean invariant scalar products in $\mathfrak{m}_1^8$ and $\mathfrak{m}_2^7$ and $\lambda_i > 0, i = 0, 1, 2$.

6 Homogeneous Lorentzian class II manifolds of dimension $d \leq 11$ of a simple noncompact Lie group

Here we describe noncompact minimal admissible class II manifolds $M = G/H$ of dimension $d \leq 11$ with a simple Lie group $G$. The stability subgroup $H$ is the stability subgroup $H = K_v$ of a minimal orbit $j(K)v$ of the isotropy representation
\[
j : K \rightarrow GL(\mathfrak{p})
\]
of the corresponding noncompact symmetric space $S = G/K$ of dimension $m \leq 10$. Since we already treated the case of $G = SL_n(\mathbb{R})$ and the case of real rank one, it is sufficient to consider simple Lie groups $G \neq SL_n(\mathbb{R})$ of real rank greater then one. Any such manifold $G/H$ admits a fibration over a noncompact symmetric space of dimension $m \leq 10$. Due to section 5.4, we may assume that $G \neq SL_n(\mathbb{R})$.

**List of symmetric spaces $S = G/K$ of dimension $m \leq 10$, where $G \neq SL_n(\mathbb{R})$ is a simple noncompact group of real rank $> 1$ (up to a local isomorphism)**

| $m$  | $G_r^k(\mathbb{C}) = SU_{2,2}/SU_2 \times SU_2$ | $p = \mathbb{C}^2 \times \mathbb{C}^2$ |
|------|-----------------------------------------------|------------------------------------------|
| 8    | $Gr_2^3(\mathbb{C}^4) = SU_{2,2}/SU_2 \times SU_2$ | $p = \mathbb{R}^2 \times \mathbb{R}^2$ |
| $2p,q$ | $Gr_2^{2q}(\mathbb{R}^{2+q}) = SO_{2,q}/SO_2 \times SO_q$ | $p = \mathbb{R}^3 \times \mathbb{R}^3$ |
| $2p,3$ | $Gr_3^9(\mathbb{R}^6) = SO_{3,3}/SO_3 \times SO_3$ | $p = \mathbb{C}^2 \times \mathbb{C}^2$ |

**Remark** Here we take into account the local isomorphism of the following symmetric spaces:

- $SU_{1,1}/U_1 \simeq SO_4/U_2 \simeq Sp_1(\mathbb{R})/U_1 \simeq SL_2(\mathbb{R})/SO_2 = \mathbb{R}H^2$,
- $Sp_{1,1}/Sp_1 \times Sp_1 \simeq SO_{1,4}/SO_4 = \mathbb{R}H^4$,
- $SO_{p}^q/U_3 \simeq SU_{1,3}/U_3 = \mathbb{C}H^3$,
- $Sp_2(\mathbb{R})/U_2 \simeq SO_{2,3}/SO_2 \times SO_3$.

Recall that local isomorphism means the isomorphism of the universal covering and we consider all homogeneous spaces up to a covering.

### 6.1 Case of the group $G = SO_{p,q}$

The isotropy representation of the symmetric space $SO_{p,q}/SO_p \times SO_q$ is the standard representation of $K = SO_p \times SO_q = SO(U) \times SO(W)$, $U = \mathbb{R}^p$, $W = \mathbb{R}^q$ in the space $V = p = U \otimes W$. Any element $v \in V$ belongs to the $K_v$-invariant subspace $U(v) \times W(v)$ where

$$U(v) := i_W v, \quad W(v) = i_V v$$

are supports of $v$. Note that $\dim U(v) = \dim W(v) = r$, where $r$ is the rank of $v$. This reduces the classification of $K$-orbits in $V$ to the case when $\dim U = \dim V = r$, that is to the classification of the orbits of nondegenerate $r \times r$ matrices $v \in Mat_r$ with respect to the natural action of the group $K = SO_r \times SO_r$. Since any matrix can be decomposed into a product of an orthogonal matrix and a symmetric matrix and any symmetric matrix is conjugated by element from $SO_r$ to a diagonal matrix, we get

**Lemma 1** Any $K = SO_r \times SO_r$-orbit in the space $Mat_r$ contains a diagonal matrix. The orbit of a nondegenerate matrix is minimal if it is the orbit of the diagonal matrix of the form $\lambda D_k$, where

$$D_k = \text{diag} (\text{Id}_{r-k}, -\text{Id}_k)$$
The stability subgroup of the identity matrix $D_0$ is the diagonal subgroup $K_{D_0} = SO^{\text{diag}}_r \subset K = SO_r \times SO_r$. The stability subgroup $K_{D_k} \simeq SO_r$ is a twisted diagonal subgroup of $K$ with the Lie algebra

$$\mathfrak{e}_{D_k} = \{ \left( \begin{array}{cc} A_{11} & A_{12} \\ -A^t_{12} & A_{22} \end{array} \right), \left( \begin{array}{cc} -A_{11} & A_{12} \\ -A^t_{12} & -A_{22} \end{array} \right) \}$$

Using this lemma, one can easily describe all class II minimal admissible manifolds $M^m = SO_{p,q}/H$ of dimension $m \leq 11$. To state the final result, we fix some notations.

We denote by $e_i, i = 1, \cdots, p$ an orthonormal basis of $U = \mathbb{R}^p$ and by $f_1, \cdots, f_q$ an orthonormal basis of $W = \mathbb{R}^q$ and we use the identifications

$$\mathfrak{so}_p = \mathfrak{so}(U) = \Lambda^2 U, \mathfrak{so}_q = \mathfrak{so}(W) = \Lambda^2 W.$$

Now we describe the minimal admissible manifolds $M = SO_{p,q}/H = SO_{p,q}/K_v$ associated with minimal orbits $j(K)v$ of different diagonal elements $v \in V = U \otimes W$. We indicate also the stability subalgebra $\mathfrak{h} = \mathfrak{e}_v \subset \mathfrak{so}(U) + \mathfrak{so}(W)$ and the reductive decomposition

$$\mathfrak{so}_{p,q} = \mathfrak{h} + \mathfrak{m} = \mathfrak{h} + (n + p)$$

and the subspace $m^H$ of invariant vectors. We set

$$U' = e_1^1, \ W' = f_1^1, \ U'' = \text{span}(e_1, e_2), \ W'' = \text{span}(f_1, f_2),$$

$$E = \text{span}(e_1, e_2), \ F = \text{span}(f_1, f_2).$$

a) $v = e_1 \otimes f_1$.

$$H = K_v = SO(U') \times SO(W'),$$

$$\mathfrak{h} = \mathfrak{so}(U') + \mathfrak{so}(W'),$$

$$\mathfrak{n} = (e_1 \wedge U' + f_1 \wedge W'),$$

$$\mathfrak{p} = (\mathbb{R}v + e_1 \otimes W' + U' \otimes f_1 + U' \otimes W').$$

$$m^H = \mathfrak{p}^H = \mathbb{R}v.$$

b) $v = e_1 \otimes f_1 \pm e_2 \otimes f_2$.

$$K_v = SO_2^{\text{diag}} \times SO(U'') \times SO(W '').$$

$$\mathfrak{h} = \mathbb{R}(e_1 \wedge e_2 \pm f_1 \wedge f_2) + \mathfrak{so}(U'') + \mathfrak{so}(W''),$$

$$\mathfrak{n} = \mathbb{R}(e_1 \wedge e_2 \mp f_1 \wedge f_2) + E \wedge W'' + U'' \wedge F,$$

$$\mathfrak{p} = \mathbb{R}v + \mathbb{R}(e_1 \otimes f_2 \mp e_2 \otimes f_1) + \text{span}(e_1 \otimes f_2 \pm e_2 \otimes f_1, e_1 \otimes f_1 \mp e_2 \otimes f_2) +$$

$$E \otimes W'' + U'' \otimes F + U'' \otimes W''$$

$$n^H = \mathbb{R}(e_1 \wedge e_2 \mp f_1 \wedge f_2),$$

$$p^H = \mathbb{R}v.$$

c) $v_\pm = e_1 \otimes f_1 \pm e_2 \otimes f_2 \pm e_3 \otimes f_3$.

We assume for simplicity that $p = q = 3$.

$$K_{v_\pm} = SO_3^{\text{diag}} \subset K = SO_3 \times SO_3,$$

$$\mathfrak{h} = \mathfrak{e}_{v_\pm} = \text{span}(e_i \wedge e_j + f_i \wedge f_j, i, j = 1, 2, 3),$$

$$\mathfrak{n} = \text{span}(e_i \wedge e_j - f_i \wedge f_j),$$

$$\mathfrak{p} = \mathbb{R}v_\pm + \mathfrak{sl}_3(\mathbb{R}) = \mathbb{R}v_\pm + \Lambda^2(\mathbb{R}^3) + S_0^3(\mathbb{R}).$$

$$m^H = p^H = \mathbb{R}v.$$
Remark i) The group $K_v = SO_3$ acts in the space $p = Mat_3 = \mathfrak{gl}_3(\mathbb{R})$ by conjugation and its preserves the 1-dimensional space $\mathbb{R}v_+$ of scalar matrices and acts irreducibly on the space $\Lambda^2(\mathbb{R}^3)$ of skew-symmetric matrices and on the space $S^2_0(\mathbb{R}^3)$ of traceless symmetric matrices.

ii) The case of the minimal orbit of the vector $v_-$ is similar, but the description of the reductive decomposition is more complicated and it is omitted.

Proposition 17 All class II minimal admissible $SO_{p,q}$-manifolds $M = SO_{p,q}/K_v$ of dimension $m \leq 11$ belong to the following list:

- $M^5 = SO_{2,2}/SO_2^{\text{diag}}$ for $v = e_1 \otimes f_1 + e_2 \otimes f_2$
- $M^5_1 = SO_{2,2}/\{e\} \times SO_2$ for $v = e_1 \otimes f_1 - e_2 \otimes f_2$
- $M^9 = SO_{2,3}/SO_2^{\text{diag}}$ for $v = e_1 \otimes f_1 \pm e_2 \otimes f_2$.

Proof. The proof follows from given above description of the stability subgroup $K_v$ of diagonal elements of the form

$$v = e_1 \otimes f_1, e_1 \otimes f_1 \pm e_2 \otimes f_2, e_1 \otimes f_1 + e_2 \otimes f_2 \pm e_3 \otimes f_3$$

and calculation of the dimension of the corresponding manifold $SU_{p,q}/K_v$. □

6.2 Case of the group $G = G_2$

The isotropy action of the symmetric space $G_2/SU_2 \times SU_2$ is the standard action of $K = SU_2 \times SU_2$ in the space $p = C^2 \otimes C^2 = \mathfrak{gl}_2(C)$ of complex matrices. The manifold $M = G_2/K_v$ has dimension $\leq 11$ if $\dim K_v \geq 3$. There is the only one such stability subgroup, the diagonal subgroup $SU_2^{\text{diag}}$, which is the stabilizer of the identity matrix. The group $SU_2^{\text{diag}}$ acts irreducibly on the subspace $\text{Herm}_2^0 \subset \mathfrak{gl}(\mathbb{C})$ of Hermitian matrices with zero trace and on the space $i\text{Herm}_2^0(\mathbb{C}) = \mathfrak{su}_2$ of skew-Hermitian matrices. We get

Proposition 18 The only class II minimal admissible $G_2$-manifold is the manifold $M^{11} = G_2/SU_2^{\text{diag}}$. It has the following reductive decomposition

$$\mathfrak{g}_2 = \mathfrak{gsu}_2^{\text{diag}} + (\mathfrak{su}_2^{\text{diag}} + \mathbb{C} \text{Id} + \text{Herm}_2^0 + i\text{Herm}_2^0)$$

where $\mathfrak{su}_2^{\text{diag}}$ is the anti-diagonal subspace, such that

$$\mathfrak{su}_2 + \mathfrak{su}_2 = \mathfrak{su}_2^{\text{diag}} + \mathfrak{su}_2^{\text{diag}}.$$ 

In particular, $\mathfrak{m}^H = \mathbb{C} \text{Id} \simeq \mathbb{R}^2$ and $\mathfrak{su}_2^{\text{diag}}$-module $\mathfrak{m} \simeq \mathbb{R}^2 + 3\mathfrak{su}_2$.

6.3 The main theorem

Combining all obtained results, we get the following theorem.
Theorem 4 All minimal admissible class II manifolds \( M^d = G/H \) of dimension \( d \leq 11 \) where \( G \) is a simple noncompact Lie group are described in the Table I. There are also indicated the maximal compact subgroup \( K \) of \( G \) and the space \( m = T_0G/K \) of its isotropy representation, the dimension \( m \) of the symmetric space \( G/K \) and the fibre \( K/H \) of the natural \( G \)-equivariant fibration \( M = G/H \rightarrow S = G/K \) over the symmetric space \( S = G/K \).

Table I.

| \( d \) | \( M^d \)       | \( K \)       | \( m \) | \( m \) | \( K/H \) |
|-------|-----------------|---------------|--------|--------|----------|
| 3     | \( SL_2(\mathbb{R}) \) | \( SO_2 \) | \( \mathbb{R}^2 \) | 2      | \( S^1 \) |
| 5     | \( SO_{1,3}/SO_2 \) | \( SO_3 \) | \( \mathbb{R}^3 \) | 3      | \( S^2 \) |
| 7     | \( SL_3(\mathbb{R})/SO_2 \) | \( SO_3 \) | \( S_0^2(\mathbb{R}^4) \) | 5      | \( S^2 \) |
| 7     | \( SU_{1,2}/U_1 \) | \( U_2 \) | \( \mathbb{C}^2 \) | 4      | \( S^3 \) |
| 7     | \( SO_{1,4}/SO_3 \) | \( SO_4 \) | \( \mathbb{R}^4 \) | 4      | \( S^3 \) |
| 9     | \( SO_{1,5}/SO_4 \) | \( SO_5 \) | \( \mathbb{R}^5 \) | 5      | \( S^4 \) |
| 11    | \( SU_{1,3}/U_2 \) | \( U_3 \) | \( \mathbb{C}^3 \) | 6      | \( S^5 \) |
| 11    | \( SO_{1,6}/SO_5 \) | \( SO_6 \) | \( \mathbb{R}^6 \) | 6      | \( S^6 \) |
| 11    | \( G_2/SU_2^{\text{sing}} \) | \( SU_3 \times SU_2 \) | \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) | 8      | \( S^3 \) |

References

[A] Adams S., Dynamics on Lorentzian manifolds. Workd Sci. Publishing, 2001,

[A1] Adams S., Transitive actions on Lorentzian manifolds with noncompact stabilizer, Geom. Dedicata, 98,2003, 1–45,

[A2] Adams S., Orbit nonproper actions on Lorentz manifolds, GAFA, Geom. Funct. Anal. 11,n 2, 2001, 201–243,

[A3] Adams S., Orbit nonproper dynamics on Lorentz manifolds , Illinois J. Math. 45,n 4, 2001, 1191–1245,

[A4] Adams S., Dynamics of semisimple Lie groups on Lorentz manifolds, Geom. Ded. 105, 2004, 1–12,

[A5] Adams S., Induction of geometric actions, Geom. Ded. 88, 2001, 91–112,

[A6] Adams S., Locally free actions on Lorentz manifolds, Geom. Funct. Anal. 10, n.3, 2000, 453-515,

[AS] Adams S., Stuck G., The isometry group of a compact Lorentz manifold I, Inv. Math. 129, n 2, 1997, 239–261,

[AS1] Adams S., G. Stuck G., The isometry group of a compact Lorentz manifold II, Inv. Math. 129, n 2 1997, 263–287,

[AS2] Adams S., Stuck G., Isometric actions of \( SL_n(R) \times \mathbb{R}^n \) on Lorentz manifolds, Israel J. Math. 121, 2001, 93-111,
[Al] Alekseevsky D.V., *Space-homogeneous Lorentzian manifolds*, Tartu Univ. Toime-tised, v.940, 1992, 17–20,

[All] Alekseevsky D.V., *Lorentzian homogeneous manifolds with completely reducible isotropy*, Pure and Applied Differential Geometry, PAGE 2007, Shaker Verlag, ed., F. Dillen and I. Van de Woestyne, 2007, 7–13,

[Al2] Alekseevsky D.V., *Flag manifolds*, Zbornik Radova, book 6 (14), 11 Yugoslav. Seminar, Beograd, 1997,3–35,

[AS] Alekseevsky D.V., Spiro A., *Invariant CR structures on compact homogeneous manifolds*, Hokk. Math. J, v. 32, no.2, 209–276, 2003,

[ADZ] Arouche A., Deffaf M., Zeghib A., *On Lorentz dynamics: From group actions to warped products via homogeneous spaces*, Trans. Amer. Math. Soc. 359, 2007, 1253–1263,

[BZ] Barbot T., Zeghib A., *Group actions on Lorentz spaces, mathematical aspects: a survey*, in "The Einstein Equations and the Large-Scale Behavior of Gravitational Fields" (P. Chrusciel, H. Friedrich, eds.) Birkhäuser, Basel, 2004, 401–439,

[DA] D’Ambra G., *Isometry groups of Lorentzian manifolds*, Invent. Math., 92, 1988, 555–565,

[DAG] D’Ambra G., Gromov M., *Lectures on transformation groups: geometry and dynamics*, in "Surveys in Diff. Geom.” Cambridge, MA. 1990, 19–111,

[DMZ] Deffaf M., Melnick K., Zeghib A., *Actions of noncompact semisimple groups on Lorentz manifolds*, Geom. Funct. Anal. 18, n 2, 2008, 463–488,

[DK] Doubrov B., Komrakov B., *Low-dimensional Pseudo-Riemannian Homogeneous Spaces*, Preprint University of Oslo, No. 13, March 1995,

[H] Helgason S., *Differential geometry, Lie groups and symmetric spaces*, Academic Press, 1978, 628p.,

[Ko] Komrakov K., *Four-dimensional Pseudo-Riemannian Homogeneous Spaces. Classification of real pairs*, Preprint University of Oslo, No. 32 June, 1995,

[K] Kowalsky N., *Noncompact simple automorphism groups of Lorentz manifolds*, Ann. Math. 144, 1997, 611–640,

[P] Patrangenaru V., *Lorentz manifolds with the three largest degrees of symmetry*, Geom. Dedicata 102, 2003, 25–33,

[W] Witte D., *Homogeneous Lorentzian manifolds with simple isometry group*, Beiträge Alg. Geom., 42, 2001, 451–461,

[Z] Zeghib A., *Sur les espaces-temps homogènes*, The Epstein birthday schrift, 551-576 , Geom. Topol.Monogr., 1, Geom. Topol. Publ., Coventry, 1998.