Fractional Analytic Perturbation Theory in QCD and Exclusive Reactions^*

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Different “analytization” procedures for the factorized pion form factor are discussed in comparison with the standard QCD perturbation theory at NLO. It is argued that demanding the analyticity of the exclusive amplitude as a whole, entails insensitivity of the results on all scheme and scale-setting parameters, including the factorization scale. This enables us to develop an approach of optimized perturbation theory within the MS scheme and to generalize the Analytic Perturbation Theory to non-integer (fractional) powers of the strong running coupling in the complex $Q^2$ plane.

1 Introduction

Quantum field theories are plagued with infinities. While those infinities related to the ultraviolet (UV) properties of the theory can be explained away by means of renormalization, singularities in the infrared (IR) are more subtle to handle. QCD is a renormalizable theory and possesses asymptotic freedom. However, the strong running coupling develops at $Q^2 = \Lambda^2_{QCD}$ an artificial singularity, termed (in one-loop) the Landau pole, that prevents the application of perturbative QCD in the low-momentum spacelike region.

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Moreover, hadronic quantities calculated at the partonic level are expressed in terms of a power-series expansion in the running coupling and are strongly affected by this IR divergence—in particular in that momentum region accessible to experiment.

Truncating this expansion, the result depends on the particular choice of the renormalization scheme and scale, though, on account of the renormalizability of QCD, all-order expressions in different schemes would be the same. Truncated series are numerically not equal and hence one has to design a scheme and specify a renormalization scale, which minimize the contribution of the discarded terms. In addition, employing a convolution approach to isolate the short-distance part of the process in question, causes beyond leading order (LO) of perturbative QCD (pQCD) a dependence of the result on the factorization scale. Much has been written about these problems, but until recently no satisfactory answers were provided.

2 Infrared-finite QCD coupling and “analytization” approaches

The situation improved dramatically during the last few years with the development of Analytic Perturbation Theory (APT) by Shirkov, Solovtsov, Milton, and Solovtsova (SSMS) \[1, 2\]. In this scheme the running coupling and its powers are replaced by singularity-free expressions in the spacelike regime using renormalization-group invariance and causality. The conversion to analytic images of the QCD coupling, $a = b_0 \alpha_s / 4\pi$, at the $l$-loop order, $A_{n}^{(l)}(L) = \left[a_{n}^{(l)}(Q^2)\right]_{an}$, is based on the dispersion relation

$$[f(Q^2)]_{an} = \int_{0}^{\infty} \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon} d\sigma$$

with the spectral density $\rho_f(\sigma) = \text{Im} \left[f(-\sigma)\right]/\pi$. The coupling (and its powers) can be analytically continued to the timelike (s-channel) region:

$$\{a^n(Q^2)\} \rightarrow \begin{cases} \{A_n(L)\}_{n \in \mathbb{N}} & L = \ln Q^2 / \Lambda^2 \quad (-q^2 = Q^2), \\ \{\mathcal{A}_n(L)\}_{n \in \mathbb{N}} & L = \ln s / \Lambda^2 \quad (q^2 = s) \end{cases}$$

But pQCD higher-order calculations (or evolution factors) entail expressions like $\left[a(L)\right]^\nu$, $\left[a(L)\right]^\nu \ln^m [a(L)]$, $a^n L^m e^{-a(L) f(x)}$ that are not covered by the SSMS “analytization” scheme. Such terms contribute to the spectral density and their analytic images are inevitably required. It was shown [3] that, using the Laplace transformation in conjunction with dispersion relations, closed-form expressions for the analytic images of the
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running coupling powers, \( a^\nu_s \), for any fractional (real) power \( \nu \) can be derived. In the spacelike region, these images can be expressed in terms of the reduced transcendental Lerch function \( F(z, \nu) \) \(^4\) (compare with \( A_1(L) \) in Eq. (2)):

\[
A_\nu(L) = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1 - \nu)}{\Gamma(\nu)},
\]

(2)

where the first term corresponds to pQCD and the second one is entailed by the pole remover. This function is an entire function in the index \( \nu \) and has the properties:

\[
A_0(L) = 1, \quad A_{-m}(L) = L^m \quad \text{for} \quad m \in \mathbb{N}, \quad A_m(\pm \infty) = 0 \quad \text{for} \quad m \geq 2, \quad m \in \mathbb{N},
\]

for \(|L| < 2\pi\), it reads \( A_\nu(L) = -[1/\Gamma(\nu)] \sum_{r=0}^{\infty} \zeta(1 - \nu - r) [(-L)^r/r!] \). In the timelike region, these images are completely determined by elementary functions \(^3\):

\[
T_\nu(L) = \frac{\sin [(\nu - 1) \arccos (L/\sqrt{\pi^2 + L^2})]}{\pi (\nu - 1) (\pi^2 + L^2)^{(\nu-1)/2}}.
\]

(3)

We also defined the index derivative, needed to describe terms that contain products of coupling powers and logarithms (see Table 1). All this implements the Karanikas-Stefanis (KS) analyticity requirement \(^5\) imposed on hadronic quantities in QCD at the amplitude level and generalizes APT to Fractional Analytic Perturbation Theory (FAPT) \(^6\).

| Theory       | PT           | APT    | FAPT   |
|--------------|--------------|--------|--------|
| Space        | \( \{a^\nu\}_{\nu \in \mathbb{R}} \) | \( \{A_m\}_{m \in \mathbb{N}} \) | \( \{A_\nu\}_{\nu \in \mathbb{R}} \) |
| Series expansion | \( \sum_m f_m a^m(L) \) | \( \sum_m f_m A_m(L) \) | \( \sum_m f_m A_m(L) \) |
| Inverse powers | \( [a(L)]^{-m} \) | \( - \) | \( A_{-m}(L) = L^m \) |
| Multiplication | \( a^\mu a^\nu = a^{\mu+\nu} \) | \( - \) | \( - \) |
| Index derivative | \( a^\nu \ln^k a \) | \( - \) | \( \mathcal{D}^k A_\nu \equiv \frac{d^k}{da^k} A_\nu = \left[a^\nu \ln^k (a)\right]_{an} \) |

3 Electromagnetic pion form factor in FAPT

The “analytization” concept has been applied to the factorized part of the pion form factor in next-to-leading order (NLO) pQCD \(^7\), including also Sudakov effects due
to soft-gluon emission, in [7]. Subsequently, the power-series expansion of \( F^{\text{Fact}}_\pi(Q^2) \) in terms of the analytic coupling ("naive analytization") at NLO was traded in [8] in favor of a non-power-series (functional) expansion in terms of analytic images of the coupling and its powers, with the coefficients \( d_m \) being numbers in the \( \overline{\text{MS}} \) scheme, \( \sum_m d_m a_m(Q^2) \Rightarrow \sum_m d_m A_m(Q^2) \) ("maximal analytization"), taking into account NLO ERBL evolution and accounting for heavy-quark threshold effects. This NLO pQCD treatment has provided predictions for

\[
F^{\text{Fact}}_\pi(Q^2) = \varphi_\pi(x, \mu^2_F) \otimes T^{\text{NLO}}_H(x, y; Q^2, \mu_R^2) \otimes \varphi_\pi(y, \mu^2_F)
\]  

(4)

that are stable against changes of the renormalization scheme and associated scale settings. Here all nonperturbative information is encapsulated in the leading twist-2 pion distribution amplitude (DA):

\[
\varphi_\pi(x, \mu^2) = 6x(1-x) \left[ 1 + a_2(\mu^2) C_2^{3/2}(2x-1) + a_4(\mu^2) C_4^{3/2}(2x-1) + \ldots \right]
\]  

(5)

in terms of the Gegenbauer coefficients \( a_n \). Below, predictions are shown for a model DA, derived in [9] by means of nonlocal QCD sum rules \((\mu^2 \approx 1 \text{ GeV}^2)\).

More recently [10], the "analytization" of the electromagnetic pion form factor was performed within FAPT at two-loop order including into the "analytization" process also logarithms of the factorization scale that have non-zero spectral density. Recall (cf. Eq. (2)) that the pole remover does not contribute to the spectral density; the discontinuity is determined solely by the term \( 1/L \). This leads to the following expression for the hard-scattering amplitude—with the renormalization scale set equal to \( \mu_R^2 = \lambda_R Q^2 \):

\[
\left[ Q^2 T_H(x, y, Q^2; \mu_F^2, \lambda_R Q^2) \right]^{n_m}_{\text{KS}} = A_1^{(2)}(\lambda_R Q^2) t_H^{(0)}(x, y) + \frac{A_2^{(2)}(\lambda_R Q^2)}{4\pi} \left[ b_0 t_H^{(1,b)}(x, y; \lambda_R) + t_H^{(FG)}(x, y) + C_F t_H^{(1,F)}(x, y; \mu_R^2) \right] + \frac{\Delta_2^{(2)}(\lambda_R Q^2)}{4\pi} \left[ C_F t_H^{(0)}(x, y) (6 + 2 \ln(\bar{x} \bar{y})) \right],
\]  

(6)

where \( \bar{x} \equiv 1-x \). The deviation from its counterpart within the maximal "analytization" procedure of [8] is encoded in the term [10]

\[
\Delta_2^{(2)}(Q^2) = \mathcal{L}_2^{(2)}(Q^2) - A_2^{(2)}(Q^2) \ln \left[ Q^2/\Lambda^2 \right]
\]  

(7)
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with

\[ \mathcal{L}^{(2)}_2(Q^2) = \left[ \frac{\alpha_s^{(2)}(Q^2)}{\alpha_s^{(1)}(Q^2)} \right]^{an} \ln \left( \frac{Q^2}{\Lambda^2} \right)_{KS} = \frac{4\pi}{b_0} \left[ \frac{\alpha_s^{(2)}(Q^2)}{\alpha_s^{(1)}(Q^2)} \right]^{an} \ln \left( \frac{Q^2}{Q^2/\Lambda^2} \right). \]  

(8)

Here \( \ln(Q^2/\mu^2_R) = \ln(\lambda R Q^2/\Lambda^2) - \ln(\lambda R \mu^2_F/\Lambda^2) \). Performing the “analytization” one finds

\[ \mathcal{L}^{(2)}_2(Q^2) = \frac{4\pi}{b_0} \left[ A_1^{(2)}(Q^2) + c_1 \frac{4\pi}{b_0} f_L(Q^2) \right], \]

(9)

where

\[ f_L(Q^2) = \sum_{n \geq 0} \left[ \psi(2) \zeta(-n - 1) - \frac{d\zeta(-n - 1)}{dn} \right] \left[ -\ln \left( \frac{Q^2/\Lambda^2}{\Lambda^2} \right) \right]^n \Gamma(n + 1) \]

(10)

and \( \zeta(z) \) is the Riemann zeta-function.

The main upshot of the FAPT analysis is that the dependence of the prediction for \( F^{\text{Fact}}_\pi(Q^2) \) on all perturbative scheme and scale settings is diminished already at NLO. In addition to the renormalization-scale stability already achieved with the “maximal

Figure 1: Left: Difference corresponding to Eqs. (9) and (10) between “maximal” and KS “analytization” in \( Q^2 T_{\text{NLO}}^N \) (solid line). The other curves are approximations explained in [10]. Right: Results for the factorized pion form factor, scaled with \( Q^2 \), and setting \( \mu^2_F = Q^2, \mu^2_R = 5.76 \text{ GeV}^2 \) in pQCD (dashed line), dash-dotted line—naïve APT; solid line—maximal APT. \( [Q^2 F^{\text{Fact}}_\pi(Q^2)]_{KS} \) almost coincides with \( [Q^2 F^{\text{Fact}}_\pi(Q^2)]_{\text{max}} \) (not shown).
analytization” in [8], the prediction now becomes insensitive also to the variation of the factorization scale. This offers the possibility to calculate perturbatively hadronic processes in QCD with high theoretical accuracy in a wide range of momenta from sub-asymptotic values down to a few hundred MeV. In Fig. 1 we show the contribution to $Q^2 T_R^{NLO}$ induced by the KS “analytization” (left panel), whereas the right panel shows the predictions for $Q^2 F_{\pi}^{Facet}(Q^2)$, using pQCD, naive APT, and maximal APT.

4 Conclusions

The “analytization” scheme at the amplitude level—technically realized by means of FAPT—has so far only been used in fully worked out detail in the calculation of the factorized pion’s electromagnetic form factor at NLO [10]. But the concept [5] and the developed mathematical apparatus [3], underlying this specific calculation, is not limited to that case. Moreover, the fact that the predictions derived from it show minimal sensitivity to both the factorization and the renormalization scale, and also to the associated scheme-setting procedure—be it within the MS or the $\alpha_V$ scheme, attaches to it fundamental importance.

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