Torsional Responses and Liouville Anomaly in Weyl Semimetals with Dislocations

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Weyl nodes in three-dimensional Weyl semimetals break the Liouville equation, leading to the Liouville anomaly. Here we present a new approach to derive the semiclassical action and equations of motion for Weyl fermions in the presence of electromagnetic fields and torsions from the quantum field theory: combining the Wigner transformation with band projection operation. It has been shown that the Liouville anomaly, including a new pure torsion anomaly term, a mixing term between the electromagnetic fields and torsions as well as the conventional chiral anomaly, entirely differs from the counterpart of axial gauge fields. We find various torsional responses and reproduce the chiral vortical effect and the torsional chiral magnetic effect. A new torsion modified anomalous Hall effect due to the mixing term in the Liouville anomaly is predicted and its implementation is also discussed. Therefore, our work not only provides new insights into the torsional responses for Weyl fermions but also acts as a starting point to investigate their topological responses.

Introduction.—Quantum anomalies, the breaking of classical symmetries by quantum fluctuations, have attracted much attention in condensed matter physics due to the deep connection with topology [1,2]. Many exotic responses of topological phases of matter can be understood in the language of quantum anomalies, including topological insulators [3,4] and topological semimetals [5,6]. Recently, the chiral anomaly in the context of three dimensional Weyl and Dirac semimetals has led to rich physical phenomena [7,8], such as the chiral magnetic effect (CME) [9,12,20], the negative longitudinal magnetoresistance [21], the nonlocal transport [22], the giant planar Hall effect [23,24], and the unconventional collective excitations [25,26], some of which have been confirmed experimentally [27,28].

Historically, the chiral anomaly was first derived by use of the perturbation method [29,30], later by the Fujikawa’s path integral method [1] and from the transport of the chiral zeroth Landau level [31]. Recently, the Berry curvature modified semiclassical equations of motion are used to derive the equation of the chiral anomaly [32]. Within the framework of the semiclassical equations of motion [11], the chiral anomaly equation manifests itself as breaking the conservation of the phase-space current or the Liouville equation, which is also dubbed as the Liouville anomaly [33,34,35]. Compared with Fujikawa’s method, the absence of ultraviolet cut-off in the Liouville anomaly can be traced back to the charge pumping between Weyl nodes with opposite chirality. In the presence of dislocations or temperature gradients, gravity with torsion would emerge. A non-vanishing torsion can contribute a new term to the chiral anomaly equation, the Nieh-Yan term [36], which gives rise to novel geometrical responses for Weyl fermions [15,16].

In fact, the construction of the semiclassical equations of motion for torsions is highly nontrivial. Both the wave-packet approach [47] and the chiral kinetic theory [40] are based on the Hamiltonian mechanics, while the Hamiltonian from the curved-spacetime Dirac equation is tortured by the Hermiticity problem [38]. Although a Hermitian Hamiltonian can be obtained from some careful manipulations [39], it turns out to be too cumbersome for our purposes. In addition, semiclassical chiral kinetic theory can also be derived from quantum field theory, but it only keeps valid in the homogeneous limit [50]. Therefore, a new method for deriving the semiclassical equations of motion with torsions is highly desirable and crucial in investigating the related topological responses.

In this letter, we develop a new formalism to derive the semiclassical action and equations of motion for Weyl fermions in the presence of electromagnetic fields and torsions from the quantum field theory: combining the Wigner transformation with band projection operation. The relevant Liouville anomaly consists of a new pure torsion anomaly term, and a term mixing the electromagnetic fields and torsions, in addition to the conventional chiral anomaly. Various novel responses are obtained, such as the chiral vortical effect and the newly proposed torsional chiral magnetic effect. Meanwhile, we find a new torsion modified anomalous Hall effect from the mixed term in the Liouville anomaly and discuss its implementation in Weyl semimetals with broken time reversal symmetry.

Model.—In the presence of dislocations, the corresponding deformation of media is described by the displacement vectors \(u^a(x)\), where the superscript \(a = 0, 1, 2, 3\) denotes locally flat spacetime coordinates with the metric tensor \(\eta_{ab} = \text{diag}(1, -1, -1, -1)\). That is, under a lattice deformation, lattice coordinates are shifted, i.e., \(x \rightarrow x + u\). So there is \(\partial_\mu = e^a_\mu \partial_a\), where \(e^a_\mu\) is the vielbein, i.e. \(e^a_\mu = \delta^a_\mu + \partial_\mu u^a\), and \(\mu = 0, 1, 2, 3\) denotes the curved spacetime coordinates (or lab coordinates) with the metric tensor \(g_{\mu
u} = e^a_\mu \eta_{ab} e^b_\nu\). The action has the
form \[^{51, 52}\]

\[
S = \frac{1}{2} \int d^3x \left| \text{det} e^a_{\mu} \right| \left[ \Psi e^{\mu \gamma a} (i \partial_\mu \Psi) - (i \partial_\mu \Psi) e^{\mu \gamma a} \Psi \right],
\]

where \( \gamma^a \) are the 4 by 4 gamma matrices and \( e^a_\mu \) is the inverse of \( e^a_{\mu} \). The action is written in this way to ensure Hermiticity locally. The torsions or torsional electromagnetic fields are defined as \( T^a_{\mu \nu} = \partial_\nu e^a_\mu - \partial_\mu e^a_\nu \). Hereafter we assume \( u^a \) is small compared to the crystal constants.

**Band-projected Green's function and Wigner's transformation**—The Green’s function for the right-handed Weyl fermions can be read off from the action above directly. One can see that this Green’s function depends on both momentum and position. It is well-known that in quantum physics, the position operator and the momentum operator do not commute with each other. To develop a semiclassical theory described by both momentum and position, one needs to utilize the Wigner transformation and has the Wigner-transformed Green’s function \[^{53}\]

\[
iG^{-1} = (1 + w) p_\mu \partial^\mu a^a - u^a \partial_\mu p_\mu a^a + O (u^2),
\]

where \( a^a = (1, \sigma^a) \), \( w^\mu = \delta^\mu_\rho \partial_\rho \), \( w = \delta^\mu_\rho \), and \( w^\mu \) is defined by one of the determinant \( \left| \text{det} e^a_\mu \right| \). One can see that, up to the linear-order terms in \( u \), this Green’s function is equivalent to \( iG^{-1} \approx (1 + w) p_\mu \partial^\mu a^a \). Thus, \( w^\mu \) couples to Weyl fermions with a coupling charge \( p \) in a way similar to the electromagnetic gauge fields.

In order to derive the semiclassical action, one shall project the two-band Green’s function in Eq. \[^{2}\] onto its positive-energy bands, i.e. \( G^{\downarrow \uparrow}_+ = \langle u_+ | a_+ \rangle = \langle u_+ | a_+ \rangle \) where \( |u_+\rangle \) is the positive-energy eigenstates: \( p \cdot \sigma |u_+\rangle = |p\rangle |u_+\rangle \). The Moyal star product, \( * = \exp \left( -\frac{i}{2} \left[ \partial_\mu \partial_\nu - \partial_\nu \partial_\mu \right] \right) \), is from the Wigner transformation. After lengthy calculations, one gets the projected Green’s function \[^{53}\]

\[
iG^{\downarrow \uparrow}_+ = iG^{-1} - a_+ \partial_q iG^{-1} + \xi_{\text{viel}},
\]

where \( iG^{-1} = (1 + w) p_\mu \partial^\mu a^a - u^a \partial_\mu p_\mu a^a + O (u^2) \) is the Berry connection for electrons in the conduction band. The bold alphabet here is used for vectors in Euclidean space, e.g. \( q_\mu = (q^0, -\vec{q}) \) and \( \partial_q = \partial / \partial q^0 \) is the derivative with respect to coordinates \( q \). In Eq. \[^{3}\], the first two terms can be regarded as first-order Taylor’s expansion of \( G^{-1} (q - a_+) \). Hence, compared to electromagnetic fields, the Berry connection is like gauge fields in the momentum space. In addition, \( G^{-1} \) is the next-lowest-order expansion of \( \left| \text{det} e^a_\mu \right| p_\mu e^{\mu \gamma a} \). \( \hat{\rho} \hat{\omega}_a \) originates from \( e^{\mu \gamma a} (u_+ | \sigma^a | u_+ \rangle \), where \( e^{\mu \gamma a} \) links the locally flat spacetime to the lab coordinates. Because, for right-handed Weyl fermions, the velocity operator is \( v^a = \partial H / \partial p_a = \sigma^a, \hat{\rho} \hat{\omega}_a \) is expected to link to velocity in lab coordinate, which is true as we shall shown. Finally, the energy correction \( \xi_{\text{viel}} = e^{\alpha \beta \rho} \hat{\rho} \hat{\omega}_a \) describes the coupling between the orbital magnetic moment \( \hat{\rho} / 2 |\hat{\rho}| \) and the spatial components of the torsion tensor \( T^a_{\mu \nu} \). The torsional magnetic field \( \mathbf{T}^a \) is defined as the Hodge dual of the torsion tensor \( (\mathbf{T}^a)^b_k = \partial_k (e^a)^b - \partial_b (e^a)^k ) \), i.e. \( (\mathbf{T}^a)^i = \frac{i}{2} \epsilon^{ijk} (\mathbf{T}^a)^j_k \).

It should be noted that, since Weyl fermions are massless, the non-diagonal components of Green’s function \( \langle G_{+ -} \rangle \) are generally small. In this paper, we would like to focus on the semiclassical region \( |p| \gg \sqrt{\mathbf{B}} \), in which the Fermi level crosses many Landau levels such that \( \hat{\omega} \) becomes negligible \[^{40}\].

**Semiclassical action and equations of motion**—The dispersion relation for the positive-energy particles can be obtained by solving the equation \( \hat{G}^{-1} = (1) \). That is, the on-shell particles are located at poles of Green’s function. By keeping terms up to order \( u \) and restoring the electromagnetic fields, one can straightforwardly find the solution to Eq. \[^{3}\], leading to following semiclassical action

\[
L = \mathbf{k} \cdot \mathbf{q} - (|\mathbf{k}| - \xi_{\text{viel}} - \xi_{\text{em}}) + (w^a)_\mu k_a - A_\mu \partial_q \mathbf{k},
\]

where \( k_a = (|\mathbf{k}|, -\mathbf{k}), \hat{\rho} \hat{\omega}_a = (1, \hat{\omega}_a), A_\mu = (\phi, -A^\alpha) \) is the electromagnetic gauge potential and \( \xi_{\text{em}} = e^{\alpha \beta \rho} \hat{\omega}_a \) stems from the orbital magnetic moment of electrons. It is clear that dislocations modify the semiclassical action through two ways: the shift of the gauge potential and the correction of the energy dispersion, which implies that \( (w^a)_\mu \) does behave like the electromagnetic gauge fields \( A_\mu \) but with a coupling charge \( k_a \). Note that we have changed variable from canonical momentum \( p \) to mechanical momentum \( \mathbf{k} = p + (w^0) |p| + (w^\alpha) p^\alpha - \mathbf{A} \). The corresponding equations of motion from the semiclassical action in Eq. \[^{4}\],

\[
D q_j = \left\{ [1 - \partial_k (k_a w^a)^n] \delta_{ij} + \partial_k (k_a w^a)^i \right\} v^j - e^{ijk} \left[ (\hat{\omega}_k - \hat{\omega}_j) \hat{\rho} \hat{\omega}_a \right] (\partial_q \mathbf{E} - T^{\text{ele}}) - \left( \hat{\omega} \cdot \mathbf{v} \right) T^{\text{mag}},
\]

and

\[
D k_i = \left\{ [1 - \partial_k (k_a w^a)^n] \delta_{ij} + \partial_k (k_a w^a)^i \right\} (\partial_q \mathbf{E} - T^{\text{ele}}) + e^{ijk} v^j \left[ -\delta_{ij} + \partial_k (k_a w^a)^n \right] T^{\text{mag}} + \Omega^{ij} \left( \hat{\rho} \hat{\omega}_a \right) T^{\text{ele}}.
\]

where \( v^i = \partial_{q_i} (\mathbf{E} - u^a (q^0) k_a) \) is the velocity with \( \partial_{q_i} = \partial / \partial q^i \) being the derivative with respect to momentum \( \mathbf{k} \). \( \Omega^{ij} = \frac{1}{2} \epsilon^{ijk} \mathbf{E} \) is the Hodge dual of the Berry curvature \( \Omega^{ij} = \partial_k a^k_j - \partial_k a^k_j \). In addition to the torsional
magnetic fields, there also exists the torsional electric fields \( (T_{0i}^{\text{mag}}) = -T_{0i}^{\text{ele}} \), which links to the thermal transport \([64][67]\). Due to the common role played by torsions and the electromagnetic fields, we could define \( T_{\text{mag}} = -B^i + k_a (T^a)^i \) and \( T_{\text{ele}} = E^i + k_a (T_0^a)^i \).

The modified density of states is given as \( D/(2\pi)^3 \) with \( D = 1 - \Omega \cdot T_{\text{mag}} + \hat{\Omega} \partial_k (k_a w^a) T_{\text{mag}} \). If there is no torsion, then \( D \) reduces to \( 1 + \hat{\Omega} \cdot B \), which is well-known in semiclassical physics \([28][63]\). Interestingly, the torsion coupling charge \( k \) leads to an extra term in \( D \). Eqs. \([4][5]\) are part of the main results in this work.

Let us now turn to the physics encoded in Eq. \([5]\). The terms in the first line show that the velocity is modified by torsions. The terms in the second line correspond to the anomalous Hall effect. Because temperature gradient can be defined as \( T_0^a \) with \( e_{0}^a \) coupling with the energy current, the anomalous thermoelectric effect is also included. The term in the last line contains both the chiral magnetic effect and the torsional chiral magnetic effect. To be more specific, the current caused by torsional magnetic fields is \(-\int \frac{e_0^a}{e_0} f_n \left( \frac{1}{2\pi} \int n \left( \hat{\Omega}_n \cdot v_n \right) (T^a k_a) \right) \), where index \( n \) denotes bands and chirality. \( f_n \) is the Fermi-Dirac distribution function. Because \( f_n \left( \hat{\Omega}_n \cdot v_n \right) (T^a k_a) \) is an odd function of momentum, this current should vanish unless a pair of Weyl nodes with opposite chirality located at different position in energy-momentum space, which can be implemented through breaking either time reversal symmetry or inversion symmetry.

Meanwhile, the terms in the first line of Eq. \([6]\) are the electric force and the counterpart from torsions. Similarly, those in the second line are the Lorentz force and the counterpart from torsions. Symmetry or inversion symmetry.

The terms in the first line of Eq. \([6]\) are located at \( \mu = \lambda_\mu + \lambda_\sigma \). Eqs. \([7][8]\) are part of the main results in this work.

**Torsional responses and Liouville Anomaly.** The anomaly term on the right-hand side in Eq. \([5]\) contains a mixing term between the electromagnetic fields and torsions. But the Berry curvature always leads to a Dirac delta function in the anomaly equation. Coupling charge of torsion is the momentum, so we require Weyl nodes not to locate at the origin so as to produce nontrivial results. Hence, we assume that Weyl nodes with chirality \( s (s = \pm 1) \) are located at \( \lambda_{\mu a} \), and there is an extra term in Eq. \([1]\), i.e. \( \Sigma_s \int | \det e_\mu^a | \hat{\Psi} (-\lambda_{\mu a} P_s) \gamma^a \hat{\Psi} \). \( P_s = \frac{1}{2} (1 + s \gamma^5) \) and \( \lambda_{\mu a} = (\lambda_{\mu 0} - \lambda_{\mu} s) \). Compared to results in last section, we have restored chirality index back. The semiclassical action becomes

\[
L_s = k \cdot \dot{q} - (|k - \lambda_s| + \lambda_{s 0} - \xi_{\text{vsi}} - \xi_{\text{em}})
+ (w^a)^l k_a \dot{q}^l - A_l \dot{q}^l - a_{X s} \cdot k,
\]

where \( a_{X s} = a_{X s} (k - \lambda_s) \) is a function of \( (k - \lambda_s) \). The energy correction from the orbital magnetic moment term becomes \( \xi_{\text{vsi}} + \xi_{\text{em}} = \hbar \delta_{03} \beta \gamma^5 \frac{1}{2} (k_0 T_0^a + e_0 a_s) k - \lambda_{\sigma 0} \). Note that, up to lowest-order dependence upon external fields, the dispersion relation is \( k_0 = |k - \lambda_s| + \lambda_{s 0} \).

In this setup, the current stemming from \( \hat{\Omega} \cdot \nu \) \( T_{\text{mag}}^a k_a \) and \( \xi_{\text{vsi}} \) in Eq. \([5]\) is given as

\[
\dot{J} = \frac{\Lambda \lambda_\mu}{2\pi^2} T^a + \frac{\mu \lambda_\mu}{2\pi^2} \left( 1 + \frac{1}{3} \right) T^a + \frac{\mu \lambda_\mu}{2\pi^2} \left[ \mu_5 \left( \frac{1}{2} + \frac{1}{3} \right) - \frac{\lambda_{\mu 0}}{6} \right] T_0^a,
\]

where \( \Lambda \) is the energy cut-off rather than the momentum cut-off and is actually from the distribution function for negative-energy particles: \( f_{-\nu} = \{ \exp [\beta (|k| + \lambda_{s 0} - \mu_s)] + 1 \}^{-1} \). The chemical potential for s-Weyl fermions is \( \mu_s = \mu + s \mu_5 \) and \( \mu_5 \) is the chiral chemical potential induced by the chiral anomaly. In addition, we have assumed \( \lambda_{\mu 0} = s \lambda_\mu \) hereafter.

The first term on the right-hand side of Eq. \([6]\) is the torsional CME. The relevant current is proportional to energy cut-off \( \Lambda \), which is actually from the distribution function for negative-energy particles. Note that the \( \Lambda \)-dependent current had been tested numerically in a tight-binding model \([65]\).

The coefficients of 1 and 1/3 in the second term come from \( \hat{\Omega} \cdot \nu \) \( T_{\text{mag}}^a k_a \) and \( \xi_{\text{vsi}} \), respectively, which are first obtained in the present work. The second term means that the torsional magnetic fields can induce currents proportional to the chemical potential rather than the chiral chemical potential. Physically, in the presence of an external magnetic field, Weyl fermions with different chirality would move oppositely and the net current is thus proportional to the chiral chemical potential, which gives rise to the CME \([29]\). On the other hand, for torsional magnetic fields, \( \lambda_{s 0} \) provides an extra minus sign, so the current turns out to be proportional to the chemical potential \( \mu \). As we shall shown later, the current from

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**Figure 1.** Schematic picture of the Landau levels due to screw dislocations. The red lines refer to the zeroth chiral Landau level for the right-handed Weyl fermions near \( p_z = \lambda_\mu \) and the left-handed Weyl fermions near \( p_z = -\lambda_\mu \), respectively.
\((\Omega \cdot \mathbf{v}) \mathbf{T}_{\text{mag}}\) is closely related to the Liouville anomaly in Eq. [7]. Compared with the chiral pseudomagnetic effect [16] [66] [67], both currents are proportional to the chemical potential. However, the extra minus sign in the chiral pseudomagnetic effect comes from the opposite coupling between the axial gauge fields and the right-handed or left-handed Weyl fermions.

For the last term, the coefficient of \(\mu_5/2\) comes from \((\Omega \cdot \mathbf{v}) \mathbf{T}_{\text{mag}}^0 k_0\), both the coefficients \(\mu_5/3\) and \(\lambda_0/6\) come from \(\xi_{\text{v}}\). Because \(\mathbf{T}^0\) links to the background rotation, \(\mu_5/2 \mathbf{T}^0 / 2\pi^2\) corresponds to the chiral vortical effect [12] [40] [69]. In addition, in analogy to the dynamical CME [70] [73], \(-\lambda_0/\mu/6\pi^2\) can be regarded as the dynamical chiral vortical effect, which stems from the orbital magnetic moment of electrons on the Fermi surface as well.

The Liouville theorem states that the phase-space volume does not change under evolution. If we define \(\Omega_{\text{VL}}\) as the volume form in the extended phase space (position, momentum and time), then the Liouville theorem is equivalent to \(\mathcal{L}_V \Omega_{\text{VL}} = 0\). \(V = q^i \partial_i + \mathbf{k}_j \partial_j + \partial_t\) is a vector relates to translation along time: for an arbitrary function \(g(q, k, t)\), \(\mathcal{L}_V g = \frac{d}{dt} g\). \(\mathcal{L}_V\) is the Lie derivative of vector \(V\). Then, in the presence of the Berry connection, it can be shown that \(\mathcal{L}_V \Omega_{\text{VL}} \propto d \Omega \times (\ldots)\) [43], where \(\Omega^{jk}\) is the Berry curvature, \(d \Omega = \partial_a \Omega^{ij} dk_i \wedge dk_j \wedge dk_k\). That is, the Liouville equation no longer holds because that the Berry curvature is singular at the Weyl nodes. Hence, the Liouville anomaly originates from the infrared physics. But the Nieh-Yan term states that \(\hat{\theta}_j \mathbf{j}_n = \frac{\Lambda_r^2}{16\pi^2} \epsilon^{\mu
u\rho\sigma} \partial_\mu \xi_{n\rho} \partial_\nu \xi_{n\sigma}\), where \(\Lambda_r\) is the energy-momentum cut-off. So it is aware of cut-off and thus conflicts with the picture from semiclassical physics. From the Liouville equation in the collisionless limit, one reaches the anomaly equation in the presence of both the electromagnetic fields and torsions [69]

\[
\partial_\mu j_\mu = -\frac{\epsilon^{\mu\nu\rho\sigma}}{32\pi^2} \left( F_{\mu\nu} F_{\rho\sigma} + \lambda_{sa} \lambda_{ab} T_{\mu\nu}^a T_{\rho\sigma}^b - 2 \lambda_{sa} F_{\mu\nu} T_{\rho\sigma}^a \right),
\]

where \(j_\mu = (j^0, \mathbf{j})\) is the current of Weyl fermions with chirality \(s\). This new Liouville anomaly equation is another main result in our work. It is clear that the last two terms explicitly depend on the positions of Weyl nodes in energy-momentum space thus are significantly different from the counterpart of axial gauge fields [68]. According to Ref. [74], the coupling charge between the torsional electric fields (or temperature gradient) and particles is \(k_0 - \mu_s\) rather than \(k_0\), where terms proportional to temperature are neglected for simplicity. This shift of the coupling charge leads to some extra terms in Eq. (9), one of which proportional to \(-\epsilon_{\mu\nu\rho\sigma} \left( \mathbf{T}_\text{mag}^0 \right)^T \mathbf{B} \) can provide an intuitive explanation to the recently observed negative magnetothermal resistance in the Weyl semimetal NbP [73].

Figure 2. An illustration of global charge conservation for a pair of dislocations with equal and opposite Burgers vectors \(D_{1,2}\). The sum of Burgers vectors of \(D_1\) and \(D_2\) is \(-f_{C_3} x^\mu \epsilon_{\mu}^a = -f_{C_1} \epsilon_{\mu}^a x^\mu - f_{C_2} \epsilon_{\mu}^a x^\mu\) [78] [79]. Loop \(C_4\) does not contain any dislocation axis, thus \(\oint_{C_4} x^\mu \epsilon_{\mu}^a = 0\).

It is straightforward to derive the axial current [64]

\[
\partial_\mu j^5_\mu = -\frac{\epsilon^{\mu\nu\rho\sigma}}{16\pi^2} \left( F_{\mu\nu} F_{\rho\sigma} + \lambda_{a} \lambda_{b} T_{\mu\nu}^a T_{\rho\sigma}^b \right),
\]

and the continuity equations for Weyl fermions,

\[
\partial_\mu j^\mu = \frac{\epsilon^{\mu\nu\rho\sigma}}{8\pi^2} \left( \lambda_{a} F_{\mu\nu} T_{\rho\sigma}^a \right),
\]

which can be understood from the chiral zeroth Landau level (see Fig. 1 [70].)

The first term in Eq. (10) corresponds to the conventional chiral anomaly [74] [85]. Unlike the Nieh-Yan term, the second term specifically depends on the locations of Weyl nodes but is independent of the cut-off. A finite chiral chemical potential could be developed by a time dependent dislocation even without any external electromagnetic fields and would be crucial to the anomalous transport phenomena for Weyl semimetals [71].

Interestingly, Eq. (11) involves a mixing term between the electromagnetic fields and torsions, which seems to violate the charge conservation. However, the charge conservation is broken by dislocations locally, but not globally. To demonstrate this point, we assume that currents and electric fields are homogeneous. Dislocations are static, along the \(z\)-axis and infinitely long. The materials \(M\) with boundary \(\partial M\) (radius \(R\)) is a cylinder along the \(z\)-axis with \(z \in (-\infty, +\infty)\), azimuthal coordinates \(\theta \in [0, 2\pi]\) and radial coordinates \(r \in [0, \infty]\) (see Fig. 2). After integrating over spatial coordinates, the left-handed side of Eq. (11) is \(\partial j_5\), with \(Q = \int d^3 x j^0\) being the total charge. While, the right-handed side of Eq. (11) becomes an integral of boundary,

\[
\frac{1}{4\pi} \oint_{\partial M} dS_1 \epsilon^{i j k} \lambda_a \epsilon_{i j k} F_{a b},
\]

which can be further written as

\[
\frac{1}{4\pi} \int_{0}^{2\pi} d\theta \epsilon_{\theta}^a \int dx R(\lambda_a F_{\theta}),
\]

due to \(\oint_{C_4} x^\mu \epsilon_{\mu}^a = 0\) as shown in Fig. 2. It means that the variation of total charge is proportional to the sum of Burgers vectors, \((\oint_{0}^{2\pi} d\theta \epsilon_{\theta}^a)\). To reduce the total elastic energy, the dislocations with equal and opposite Burgers vectors should
appear in pairs in realistic solids [30], so the sum of Burgers vectors is zero, for example, the two dislocations $D_{1,2}$ in Fig. 2. Hence, charges are conserved globally. Eq. (11) describes the charge pumping between dislocations with opposite Burgers vectors. On the other hand, in the confined systems (nanowires or thin films) with a single dislocation, the local nonconservation of electric charges might be recovered though the charge pumping between the bulk and the surface.

From the anomaly equation of Eq. (11), one finds the following solution [81]

$$
\mathbf{j} = -\frac{\lambda_\mu}{2\pi^2} (\mathbf{w}^a \times \mathbf{E} - w_0^a \mathbf{B}). \quad (12)
$$

In this work, we mainly focus on the static dislocations and would like to neglect the term proportional to $w_0^a$. $\mathbf{j} = -\frac{\lambda_\mu}{2\pi^2} \mathbf{w}^a \times \mathbf{E}$ is first discovered here and can be dubbed as the torsion modified anomalous Hall effect. This resulting anomalous Hall current is still perpendicular to the electric field, but can be parallel to the momentum spacing between the two Weyl nodes with opposite chirality $\lambda$. To show this point, let us set both screw dislocations and $\lambda$ along the $z$-axis, i.e. $\mathbf{w}^a = (0, 0, 0, \mathbf{u}^3(x, y))$. In addition, the only non-vanishing $\mathbf{w}^a$ is $\mathbf{w}^3 = \left((\mathbf{w}^3)^1, (\mathbf{w}^3)^2, 0 \right)$. When the electric field is along the $x$-axis, the responses current is along the $z$-axis, i.e. $j^z = \frac{\lambda_\mu}{2\pi^2} (\mathbf{w}^3)^2 \mathbf{E}$ [1]. Note that the mixing term in Eq. (11) also underlies the terms of $\frac{a^0_i}{2\pi^2} \mathbf{T}^i$ in Eq. [8]. In the thermal field theory, the chemical potential would couple to $j^0$, i.e. $\int j^0 \mu$. So by keeping terms to order in $\mu$, the effective action from this mixing term is $\frac{a^0_i}{4\pi^2} \int \mu A_i \left(\lambda_\mu T^{j_0}_3 \right)$. Thus, the response current is given as $\mathbf{j} = \frac{\mu}{\pi^2} \lambda \mathbf{T}$. [52] In summary, we have presented a formalism to construct the semiclassical action and equations of motion for Weyl fermions in the presence of both electromagnetic fields and torsions from the quantum field theory. It has been shown that the torsional electromagnetic fields make the Liouville anomaly equation essentially different from the counterpart of axial gauge fields. Our results could give rise to various torsional responses and reproduce the torsional CME and the chiral vortical effect. In addition, a new torsion modified anomalous Hall effect originating from the mixed term in the Liouville anomaly is predicted and its implementation in Weyl semimetals lacking time reversal symmetry is discussed as well.

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Intuitively, electrons under dislocations might form the Landau levels [53]. Let us consider Weyl semimetals with screw dislocations along the z-axis. The Weyl nodes with chirality s locate at $s\lambda_0$ with $s = \pm 1$. For simplicity, we set $\lambda_0 = (0, 0, -\lambda_z) > 0$. The displacement vector is assumed to be along the z-axis, so only one component of the vielbeins survives $e_0^\mu = \frac{1}{s}(0, -Ty, Tz, 0)$. Note that the surface density of the Burgers vector fields rather than itself is constant. Consequently, the zeroth Landau levels near $p_\perp = s\lambda_0$ are $p_\perp - s\lambda_0$, as shown in Fig. 1. Turning on an electric field along the z-axis, charges are pumped up from the Dirac sea and extra particles are "produced". Specifically, the level degeneracy is roughly $\frac{1}{2}\pi \lambda_0$ in the vicinity of $p_\perp = s\lambda_0$ and the variation of momentum is, $\Delta p_\perp = E\Delta z$. Hence, the total variation of charge density is given by $\frac{1}{2}\pi \lambda_0 TE$, whose covariant form is Eq. (11) in the main text.

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The lowest-order solution of Eq. (11) is $j = \frac{1}{2\pi} (-\lambda \times E + \lambda_0 B)$. The first term is the anomalous Hall effect and the second term would be canceled by terms from energy cut-off (see Refs. [19, 20] for details). In addition, we have neglected the contributions from polarization and magnetization.

The definition of Weyl-node’s position in energy space (\lambda_0) in the framework of quantum field theory is a bit subtle [19] and its effect is not the key focus of this paper.

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