Helical magnetic fields from Riemann coupling

Ashu Kushwaha

Department of Physics, IIT Bombay

Based on the work with S. Shankaranarayanan,
PRD 102,103528 (2020) [ arXiv:2008.10825]

DAE-BRNS HIGH ENERGY PHYSICS SYMPOSIUM 2020

December 17, 2020
Observations of magnetic fields in universe

Picture source: Max Planck Institute for Radio Astronomy

Micro-Gauss strength magnetic field over 10\(kpc\) coherence length scale is present in galaxies.

\[1pc = 2.1 \times 10^5 \text{AU} = 3.1 \times 10^{16} \text{m}\]
Two kinds of fields

- Electromagnetic field has two transverse degrees of freedom which can be associated with Left circular and right circular polarization.
- For massless particle helicity is the projection of the direction of spin (clockwise or anti-clockwise) along the direction of propagation. Hence giving $+1, -1$ for right handed and left handed helicity modes.
- Same propagation (speed or dispersion relation) of both polarization modes lead to non-helical, and differently propagating modes lead to helical fields.
- If both the polarization modes propagate differently $\rightarrow$ Helicity Imbalance

How to create helicity imbalance?
Lorentz force, $\vec{F} = m\frac{d\vec{v}}{dt} = \vec{E} + \vec{v} \times \vec{B}$ implies that under parity transformation (changing the sign of coordinate system):
$\vec{E} \rightarrow -\vec{E}, \vec{B} \rightarrow \vec{B}$.

Because standard EM action, $F_{\mu\nu}F^{\mu\nu} = B^2 - E^2$, is quadratic in $\vec{E}$ and $\vec{B}$, it is invariant under parity symmetry.

$F_{\mu\nu}\tilde{F}^{\mu\nu} = -4\vec{E} \cdot \vec{B}$ is parity non-invariant, where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$.

Hence $F_{\mu\nu}\tilde{F}^{\mu\nu}$ can create the Helicity imbalance.
Vorticity is defined as $\mathbf{\Omega} = \nabla \times \mathbf{v}$, where $\mathbf{v}$ is velocity field.
Magnetic helicity

- Magnetic helicity ($\mathcal{H}_M$) is defined as:
  \[ \int d^3x \vec{A} \cdot \vec{B} \text{ and } \vec{B} \cdot \nabla \times \vec{B}. \]

- It is a measure of twist and linkage of magnetic field lines.

\[ \mathcal{H}_M = \int d^3x \vec{A} \cdot \vec{B} = 2 V_1 \cdot V_2 \]
Why helical magnetic fields are interesting?

- Helical magnetic fields leave a very distinct signature as they violate parity symmetry which leads to observable effects, e.g. correlations between the anisotropies in the temperature and B-polarisation or in the E- and the B-polarisations in the CMB. Kahniashvili (2006)

- One of the interests in primordial magnetic helicity is that it can be a direct indication of parity violation (CP violation) in the early Universe. Vachaspati (2001)
How to generate magnetic fields?
Problem with magnetic field generation during inflation

- EM action for an arbitrary 4-D metric

\[ S_{\text{em}} = -\frac{1}{4} \int d^4x \sqrt{-g} g^{\alpha\mu} g^{\beta\nu} F_{\alpha\beta} F_{\mu\nu} \]

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( A^\mu \) is electromagnetic four vector.

- Under conformal transformation \( \tilde{g}_{\mu\nu} = \omega^2(x)g_{\mu\nu} \)

\[ \tilde{S}_{\text{em}} = -\frac{1}{4} \int d^4x \sqrt{-\tilde{g}} \tilde{g}^{\alpha\mu} \tilde{g}^{\beta\nu} F_{\alpha\beta} F_{\mu\nu} = S_{\text{em}} \]

- EM action is conformally invariant.

- Because Flat FRW metric is conformally equivalent to Minkowski spacetime, \( B \sim \frac{1}{a^2} \).

We need to break the conformal invariance of EM action!
Scalar field coupled models: \( f(\phi)F_{\mu\nu}\tilde{F}^{\mu\nu} \)

where \( f(\phi) \) is time-dependent coupling function.

**Problems with these models:**

- **Strong coupling** - Coupling between charged particles and the EM field is so strong that theory can not be treated perturbatively.

- **Back-reaction** - Overproduction of gauge fields affect the background inflationary dynamics

Because magnetic fields are produced near the end of inflation, strength of the fields generated depends on the reheating scale.

To resolve strong coupling and back-reaction problem \( f(\phi) \) is assumed to increase during inflation and decrease back to its initial value post inflation. Sharma et al.(2018)
Helical magnetic fields from Riemann coupling
Motivation

- Non-minimal coupling to the Riemann tensor generates sufficient primordial helical magnetic fields at all observable scales.

- One of the helical states decay while the other helical mode increases, leading to a net non-zero helicity $\rightarrow$ helicity imbalance

- **Necessary condition**: Conformal invariance breaking $+$ parity violation

$$S = -\frac{M_P^2}{2} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right] - \frac{1}{4} \int d^4x \sqrt{-g} F_{\mu\nu} F^{\mu\nu} - \frac{\sigma}{M^2} \int d^4x \sqrt{-g} \tilde{R}^{\mu\nu\alpha\beta} F_{\alpha\beta} F_{\mu\nu} \quad (1)$$

where $\tilde{R}^{\mu\nu\alpha\beta} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} R_{\rho\sigma}{}^{\alpha\beta}$ and $M$ is the energy scale, which sets the scale for the breaking of conformal invariance.
Electromagnetic energy density

To identify whether these modes lead to back-reaction on the metric, we define $R$, which is the ratio of the total energy density of the fluctuations and background energy density during inflation: Talebian et al. (2020)

$$R = \frac{(\rho_B + \rho_E)|_{k^* \sim H}}{6M_P^2 H^2}$$

| $\alpha$        | $\rho$ (in GeV$^4$) | $R$  |
|-----------------|---------------------|------|
| $-\frac{1}{2} - \epsilon$ | $\sim 10^{64}$       | $\sim 10^{-4}$   |
| $-\frac{3}{4}$  | $\sim 10^{62}$       | $\sim 10^{-6}$   |
| $-1$            | $\sim 10^{61}$       | $\sim 10^{-7}$   |
| $-3$            | $\sim 10^{59}$       | $\sim 10^{-9}$   |

No back-reaction on the background metric.
Estimating the strength of helical magnetic fields

- Assuming **instantaneous reheating**, and the Universe becomes radiation dominated after inflation. Due to **flux conservation**, the magnetic energy density will decay as $1/a^4$: Subramanian (2016)

- Using the fact that the **relevant modes exited Hubble radius around 30 e-foldings of inflation**, with energy density $\rho_B \approx 10^{64} \text{GeV}^4$, the primordial helical fields at **Gpc scales** is:

  $$ B_0 \approx 10^{-20} \text{G} $$  \hspace{1cm} (3)

- Helical magnetic fields that re-entered the horizon at two different epochs:

  $$ B|_{50 \text{ Mpc}} \sim 10^{-18} \text{G} \ (z \sim 20) ; \quad B|_{1 \text{ Mpc}} \sim 10^{-14} \text{G} \ (z \sim 1000) $$
Our model does not require the coupling of the electromagnetic field with the scalar field. Hence, there are no extra degrees of freedom and will not lead to a strong-coupling problem.

Since the curvature is large in the early Universe, the coupling term will introduce non-trivial corrections to the electromagnetic action.

Power spectrum of the helical fields generated has a slight red-tilt for slow-roll inflation which is different compared to the scalar field coupled models where the power-spectrum has a blue-tilt.

Currently we are looking at the effect of this helical field on baryon asymmetry during the early universe.
Thank you
Backup slides
Conformal transformation

$$\tilde{g}_{\mu\nu} = \omega^2(x)g_{\mu\nu} \implies \tilde{\Gamma}_{\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda + C_{\mu\nu}^\lambda$$  \hspace{1cm} (4)

where $C_{\mu\nu}^\lambda = \omega^{-1} \left( \delta_\mu^\lambda \nabla_\nu \omega + \delta_\nu^\lambda \nabla_\mu \omega - g_{\mu\nu}g^{\rho\lambda} \nabla_\rho \omega \right)$

$$F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu = \partial_\mu A_\nu - \Gamma_{\mu\nu}^\lambda A_\lambda - \partial_\nu A_\mu + \Gamma_{\nu\mu}^\lambda A_\lambda = \partial_\mu A_\nu - \partial_\nu A_\mu$$  \hspace{1cm} (5)

$$\tilde{R}_{\sigma\mu\nu}^\lambda = R_{\sigma\mu\nu}^\lambda + \nabla_\mu C_{\nu\sigma}^\lambda - \nabla_\nu C_{\mu\sigma}^\lambda + C_{\mu\rho}^\lambda C_{\nu\sigma}^\rho - C_{\nu\rho}^\lambda C_{\mu\sigma}^\rho$$  \hspace{1cm} (6)

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \left[ 2\delta_\mu^\alpha \delta_\nu^\beta + g_{\mu\nu}g^{\alpha\beta} \right] \omega^{-1} (\nabla_\alpha \nabla_\beta \omega)$$

$$+ \left[ 4\delta_\mu^\alpha \delta_\nu^\beta - g_{\mu\nu}g^{\alpha\beta} \right] \omega^{-2} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$  \hspace{1cm} (7)

$$\tilde{R} = \omega^{-2} R - -6g^{\alpha\beta} \omega^{-3} (\nabla_\alpha \nabla_\beta \omega)$$  \hspace{1cm} (8)

$$\tilde{\nabla}_\mu \tilde{\nabla}_\nu \phi = \nabla_\mu \nabla_\nu \phi - \left( \delta_\mu^\alpha \delta_\nu^\beta + \delta_\mu^\beta \delta_\nu^\alpha \right) \omega^{-1} (\nabla_\alpha \omega)(\nabla_\beta \omega)$$  \hspace{1cm} (9)
Energy densities

Gauge field decomposition:

\[ A^i(\vec{x}, \eta) = \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda=1,2} \varepsilon^i_\lambda \left[ A_\lambda(k, \eta) b_\lambda(\vec{k}) e^{ik\cdot x} + A^*_\lambda(k, \eta) b^\dagger_\lambda(\vec{k}) e^{-ik\cdot x} \right] \]  

(10)

The EM energy densities with respect to the comoving observer are:

\[ \rho_B(\eta, k) \equiv -\frac{1}{2} \langle 0 | B_\mu B^\mu | 0 \rangle = \int \frac{dk}{k} \frac{1}{(2\pi)^2} \frac{k^5}{a^4} \left( |A_+ (\eta, k)|^2 + |A_- (\eta, k)|^2 \right) \]  

(11)

\[ \rho_E(\eta, k) \equiv -\frac{1}{2} \langle 0 | E_\mu E^\mu | 0 \rangle = \int \frac{dk}{k} \frac{1}{(2\pi)^2} \frac{k^3}{a^4} \left( |A'_+ (\eta, k)|^2 + |A'_- (\eta, k)|^2 \right) \]  

(12)

\[ \rho_h(\eta, k) \equiv -\langle 0 | A_\mu B^\nu | 0 \rangle = \int \frac{dk}{k} \frac{1}{2\pi^2} \frac{k^4}{a^3} \left( |A_+ (\eta, k)|^2 - |A_- (\eta, k)|^2 \right) \]  

(13)

where spectral energy density is given by \( \frac{d\rho_\gamma}{dk} \) for \( \gamma \in (B, E, h) \).
Evolution equation

In Flat FRW universe: $ds^2 = a^2(\eta) (d\eta^2 - \delta_{ij}dx^i dx^j)$. In the Coulomb gauge ($A^0 = 0, \partial_i A^i = 0$), equation of motion is

$$A''_i + \frac{4 \epsilon_{ijl}}{M^2} \left( \frac{a'''}{a^3} - 3 \frac{a'' a'}{a^4} \right) \partial_j A_l - \partial_j \partial_j A_i = 0 \quad (14)$$

Which in helicity basis can be written as:

$$A''_h + \left[ k^2 - \frac{4kh}{M^2} \Gamma(\eta) \right] A_h = 0 \quad (15)$$

where,

$$\Gamma(\eta) = \frac{a'''}{a^3} - 3 \frac{a'' a'}{a^4} = \frac{1}{a^2} \left( \mathcal{H}'' - 2 \mathcal{H}^3 \right) \quad (16)$$

which vanishes for de-sitter case.
Helical magnetic field generation

- For power law inflation: \( a(\eta) = \left( \frac{-\eta}{\eta_0} \right)^{\beta+1} \), de-sitter \( \beta = -2 \), we have

\[
A''_h + \left[ k^2 - \frac{8 k h}{M^2} \frac{\beta (\beta + 1)(\beta + 2)}{\eta_0^3} \left( \frac{-\eta_0}{\eta} \right)^{(2\beta+5)} \right] A_h = 0 \quad (17)
\]

- Sub-horizon mode \( | - k\eta | \gg 1 \) solution is: \( A_h = \frac{1}{\sqrt{k}} e^{-ik\eta} \)

- For super-horizon mode \( | - k\eta | \ll 1 \), with dimensionless variable, \( \tau = \left( \frac{-\eta_0}{\eta} \right)^{\alpha} \) and \( \alpha = \beta + \frac{3}{2} \)

\[
A_+(\tau, k) = \tau^{-\frac{1}{2\alpha}} J_{\frac{1}{2\alpha}} \left( \frac{\varsigma \sqrt{k}}{\alpha} \tau \right) C_1 + \tau^{-\frac{1}{2\alpha}} Y_{\frac{1}{2\alpha}} \left( \frac{\varsigma \sqrt{k}}{\alpha} \tau \right) C_2 \quad (18a)
\]

\[
A_-(\tau, k) = \tau^{-\frac{1}{2\alpha}} J_{\frac{1}{2\alpha}} \left( -i \frac{\varsigma \sqrt{k}}{\alpha} \tau \right) C_3 + \tau^{-\frac{1}{2\alpha}} Y_{\frac{1}{2\alpha}} \left( -i \frac{\varsigma \sqrt{k}}{\alpha} \tau \right) C_4 \quad (18b)
\]
Taking $\mathcal{H} \sim \eta_0^{-1} \sim 10^{14}\text{GeV}$, and $M \sim 10^{17}\text{GeV}$ gives

$$|C_1| \approx |C_3| \approx 10^{-17/2}\text{GeV}^{-1/2}, \quad \text{and} \quad |C_2| \approx |C_4| \approx 10^{-11/2}\text{GeV}^{-1/2}. \quad (19)$$

**Figure**: Figure showing the behaviour of positive and negative helicity mode for $\alpha = -0.53$ and $\alpha = -1$. $\tilde{\tau} = 10^{-6\frac{3}{2}} \tau$ and the vertical axis is in GeV$^{-1/2}$.

**We can ignore the negative helicity mode.**
Using the fact that we can approximate the super-horizon modes by power law, we have

\[ A_+(\tau, k) = C \, k^{\frac{1}{4\alpha}} - C_2 \frac{\mathcal{F}^{-1}}{\pi} \Gamma \left( \frac{1}{2\alpha} \right) k^{-\frac{1}{4\alpha}} \tau^{-\frac{1}{\alpha}} \]  

(20)

where

\[ \mathcal{F}(\tau) = F(\tau) \left( \frac{\varsigma}{2\alpha} \right)^{\frac{1}{2\alpha}}, \]  

(21)

\[ C(\tau) = F(\tau) \left( \frac{\varsigma}{2\alpha} \right)^{\frac{1}{2\alpha}} \left[ \frac{C_1}{\Gamma(1 + \frac{1}{2\alpha})} - \frac{C_2}{\pi} \Gamma \left( -\frac{1}{2\alpha} \right) \cos \left( \frac{\pi}{2\alpha} \right) \right], \]  

(22)

and the approximate values are

\[ |\mathcal{F}| \sim 10^{-\frac{5}{\alpha}} \text{ GeV}^{-1/4\alpha}, \quad |C| \sim 10^{-\frac{5}{\alpha}-\frac{11}{2}} \text{ GeV}^{-\frac{1}{4\alpha}-\frac{1}{2}}. \]