BLACK HOLE PARTITION FUNCTION AND AIRY EQUATION∗

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Abstract

We have used the Topological String Theory partition function in the scaling limit, to relate the Black Hole partition function and Topological String Theory partition function.

1 Introduction

In this article, we will first introduce the Topological String Theory partition function and add non-holomorphic corrections to it. Then we will employ a scaling limit which will convert the expression for partition function into an Airy equation. We will close the article by expressing Black Hole partition function in terms of a function that is solution of the Airy equation.

2 Topological String Theory partition function

The topological string partition function is given by a perturbative expansion in the topological string coupling \( \lambda \), summing the free energies \( F^{(g)} \) over the world sheet genera \( g \):

\[
Z_{\text{top}} = \exp \sum \lambda^{2g-2} F^{(g)}.
\]

The perturbative free energies \( F^{(g)} \) satisfy following recursive DEs.

\[
\partial_i F^{(g)} = \frac{1}{2} \left( C^{jk}_i D_j F^{(g-1)} + \sum_{r=1}^{g-1} D_j F^{(g-r)} D_k F^{(r)} \right)
\]

That were solved using Feynman diagrams but the number of Feynman diagrams grow fast at higher genus and so become unyielding. This issue was

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resolved due to provision of a polynomial structure of the higher genus free energies in finitely many generators. It is useful to introduce the following propagators $S, S^i, S^{ij}$,

$$S \in \Gamma(\mathcal{M}, \mathcal{L}^{-2}), \quad S^i \in \Gamma(\mathcal{M}, \mathcal{L}^{-2} \otimes T\mathcal{M}), \quad S^{ij} \in \Gamma(\mathcal{M}, \mathcal{L}^{-2} \otimes \text{Sym}^2(T\mathcal{M})).$$

### 2.1 Non-holomorphic corrections

In order to study Black Holes in string theory, a guiding principle is that string theory dualities should remain preserved. Let $\mathcal{M}$ be the moduli space of a CY threefold $Y$ which is a special Kähler Manifold. The moduli space contains information about coupling constants, among other objects. The free energy $F$, in general, is a polynomial in terms of holomorphic and non-holomorphic generators. In some cases, it is possible to consider only the contribution from holomorphic terms to $F$. But in case of higher genus mirror symmetry, the free energy $F$ depends not only on holomorphic generators but also on non-holomorphic generators. Also, in order to preserve the dualities, it is important to consider the non-holomorphic dependence of the coupling constants on the moduli. In particular we consider the non-holomorphic generator $S^{zz}$. This and other generators are also called propagators in order to make connection with Feynman diagrams explicit. So from the expression for $F$, we pick only the terms of the form $f(z)(S^{zz})^{3g-3}$, where $f(z)$ is a rational function of the modulus $z$. Those terms turn out to be of the form $a_g C_{zzz}^{2g-2}$, $a_g \in \mathbb{Q}, C_{zzz}$ are structure constants that are normalised B-model Yukawa couplings.

### 2.2 Scaling limit

The idea of scaling limit is similar to the very important notion of holomorphic limit. Though the following expression contains both holomorphic and non-holomorphic terms, the holomorphic limit allows us to obtain a holomorphic section from the total free energy. Consider the total free energy

$$F = \sum \lambda^{2g-2} F(g).$$

We would like to select the terms $a_g C_{zzz}^{2g-2} (S^{zz})^{3g-3}$ from $F$. To this end, we rescale the generators $S^{zz}$ and $\lambda$ such that only such terms survive.

### 3 Airy equation

The equation for the partition function $Z_{top,s} = \exp F_s$ in the scaling limit becomes

$$\left( \lambda s e^{ \frac{1}{3\lambda^2 s}} \right)^2 \left( \lambda s e^{ \frac{1}{3\lambda^2 s}} \right)^2 - \left( \frac{1}{3\lambda^2 s} \right) - \frac{1}{9} = 0, \lambda s e^{ \frac{1}{3\lambda^2 s}} Z_{top,s} = 0.$$
where subscript $s$ denotes scaling limit. Above equation has two singularities. But evidently $\lambda_s = \infty$ is a regular singularity so perturbative expansion is valid for strong coupling. This can then be further reduced to an Airy equation, via a simple change of variables:

$$(\partial_z^2 - z)v(z) = 0.$$ 

Finally the topological string partition function can be put as

$$Z_{\text{top},s} = 2^{\frac{1}{3}} e^{-\frac{1}{3\lambda_s^2}} \lambda_s^{-\frac{1}{3}} v\left(\left(\frac{1}{2\lambda_s^2}\right)^\frac{2}{3}\right).$$ 

The function $v$ is a solution of Airy equation. That is a classic DE with two linearly independent solutions. As already pointed out, both solutions can be expanded around strong coupling.

### 3.1 Black Hole Partition Function

This is well known\cite{1} that black hole partition function $Z_{BH}$ is related to the partition function of topological string

$$Z_{BH} \approx |Z_{\text{top}}|^2.$$ 

In the special case of scaling limit, this leads to

$$Z_{BH} \approx \left|2^{\frac{1}{3}} e^{-\frac{1}{3\lambda_s^2}} \lambda_s^{-\frac{1}{3}} v\left(\left(\frac{1}{2\lambda_s^2}\right)^\frac{2}{3}\right)\right|^2.$$ 

### 4 Conclusion and Outlook

In this article, we have expressed the Black Hole partition function in terms of a function that satisfies Airy equation. It was made possible due to scaling limit approach to Topological String Theory partition function. Further implications and importance of this result remain to be seen. For instance, the Topological String Theory partition function has a mathematical interpretation as a quantisation of the third cohomology of a CY threefold. What does this mean for $Z_{BH}$? Since Black Hole physics is relatively better understood, such results aid in understanding of the Topological String Theory, despite the fact that we have made use of Topological String Theory to obtain an expression for the Black Hole partition function.

### References

[1] M. Alim, S.-T. Yau and J. Zhou, *Lett. Math. Phys.* **106** (2016) 719, [arXiv:1506.01375 [hep-th]]
[2] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Commun. Math. Phys. 165 (1994) 311, arXiv:hep-th/9309140 [hep-th]

[3] S. Yamaguchi and S.-T. Yau, JHEP 07 (2004) 047, arXiv:hep-th/0406078 [hep-th]

[4] T. Mohaupt, Springer Proc. Phys. 134 (2010) 165, arXiv:0812.4239 [hep-th]

[5] A. Kanazawa and J. Zhou, arXiv e-prints (Sep 2014) arXiv:1409.4105, arXiv:1409.4105 [math.AG]

[6] M. Alim and J. D. Lange, JHEP 10 (2007) 045, arXiv:0708.2886 [hep-th]

[7] H. Ooguri, A. Strominger and C. Vafa, Phys. Rev. D70 (2004) 106007, arXiv:hep-th/0405146 [hep-th]