Entanglement classification of four-partite states under the SLOCC

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Abstract
We present a practical classification scheme for the four-partite entangled states under stochastic local operations and classical communication (SLOCC). By transforming a four-partite state into a triple-state set composed of two tripartite states and a bipartite state, the entanglement classification is reduced to the classification of tripartite and bipartite entanglements. This reduction method has the merit of involving only the linear constraints, and meanwhile provides an insight into the entanglement character of the subsystems.

Keywords: quantum entanglement, entanglement classification, singular value decomposition, four-partite entanglement

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement, a peculiar feature which markedly differentiates quantum theory from classical theory, is now regarded as the main physical resource in quantum information sciences [1]. By dint of entanglement, various counterintuitive and unique applications are emerging, e.g. quantum teleportation [2], super-dense coding [3, 4], and quantum cryptography protocols [5], etc. Since two states belonging to the same equivalent class under SLOCC may be employed to perform the same quantum information tasks [6], the entanglement classification plays an important role in quantum information theory. Although it has been intensively studied, we
still have very limited knowledge of the general entanglement classes under SLOCC for systems that are more complex than four-qubit entangled states.

There exist only two different kinds of genuine tripartite entangled states in pure three-qubit systems under SLOCC [6], i.e. the GHZ and W states. The number of entanglement classes increases dramatically with the increase of particles and dimensions in the entangled state. It turns out that the number of classes for a general four-qubit system is infinite, in nine different entanglement families [7]. When more particles are involved, the existing operational classification schemes are only applicable to the highly symmetric states [8]. For a general multipartite pure state, the coefficient matrix method can only identify the discrete entanglement families with different ranks, which is a rather coarse grain classification per se [9]. It is shown that the geometric relations of individual particles are capable of characterizing the different entanglement classes of multipartite states under the SLOCC [10]. Moreover, the algebraic invariants have been explored to distinguish the entanglement classes [11], where a complete set of invariants usually involves some complicated expressions, and the individuality of each particle is not explicitly manifested. Despite this progress, a practical method of verifying the SLOCC equivalence of two arbitrarily given multipartite states is highly desirable. More importantly, it is still unclear, for a multipartite entangled state, how the entanglement characters of the subsystems behave and generate the whole nature.

In this paper we present a general classification scheme for four-partite pure states of arbitrary but finite dimensions. By applying a singular value decomposition to a bipartition of the system, a four-partite state is then transformed into a triple-state set composed of two tripartite states and a bipartite state. The two four-partite quantum states are thought to be SLOCC equivalent, if and only if the quantum states in the the triple-state sets are SLOCC equivalent respectively. Our method provides a systematic procedure for reducing the entanglement classification of multipartite states to that of less partite states, and hence to distinguish the entanglement classes of the whole system through its subsystems.

2. The classification scheme

2.1. The representations of quantum states

A pure one particle quantum state may be represented by normalized complex vectors in Hilbert space, while a bipartite state of $|\Psi_I1I2\rangle = \sum_{i1,i2=1}^{I1,I2} \psi_{i1i2} |i1\rangle |i2\rangle$ may be expressed in matrix form

$$\Psi_{I1I2} = \begin{pmatrix} \psi_{11} & \psi_{12} & \cdots & \psi_{1I2} \\ \psi_{21} & \psi_{22} & \cdots & \psi_{2I2} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{I1,1} & \psi_{I1,2} & \cdots & \psi_{I1,I2} \end{pmatrix}.$$  \hspace{1cm} (1)

Two $N$-partite states $|\Psi'\rangle$ and $|\Psi\rangle$ are SLOCC equivalent, if and only if they can be connected via invertible local operators, i.e. $|\Psi'\rangle = A_1 \otimes \cdots \otimes A_N |\Psi\rangle$ [6]. For bipartite quantum states in matrix form, the SLOCC equivalence of $|\Psi'_{I1I2}\rangle = A_1 \otimes A_2 |\Psi_{I1I2}\rangle$ turns into

$$|\Psi'_{I1I2}\rangle = A_1 |\Psi_{I1I2}\rangle A_2^T.$$  \hspace{1cm} (2)

Here $A_1 \in \mathbb{C}^{I1 \times I1}$ and $A_2 \in \mathbb{C}^{I2 \times I2}$ are invertible matrices, and the superscript $T$ means the matrix transposition. A tripartite state may be expressed as a tuple of the matrices [12, 13]

$$\Psi_{I1I2I3} = (\Gamma_1, \Gamma_2, \cdots, \Gamma_{I3}),$$  \hspace{1cm} (3)
where $\Gamma_i \in \mathbb{C}^{d_i \times d_i}$ for $i \in \{1, 2, \cdots, I_1\}$. In this case, the SLOCC equivalence of two tripartite states, $|\Psi_{i_1i_2i_3}\rangle = A_1 \otimes A_2 \otimes A_3 |\Psi_{i_1i_2i_3}\rangle$, may now be expressed as

$$
\Psi'_{i_1i_2i_3} = (\Gamma'_{i_1}, \Gamma'_{i_2}, \cdots, \Gamma'_{i_3}) = (A_2 \Gamma_1 A_3, A_2 \Gamma_2 A_3, \cdots, A_2 \Gamma_3 A_3) A_1^T.
$$

(4)

Here the tripartite state behaves as a row vector whose components are matrices.

For the representation of four-partite states, we first introduce two operations related to matrices, the vectorization and folding. The vectorization of an $I_1 \times I_2$ dimensional matrix $\Psi_{i_1i_2}$ with the complex elements $\psi_{ij}$ is

$$
\mathcal{V}(\Psi_{i_1i_2}) \equiv (\psi_{i_11}, \psi_{i_12}, \cdots, \psi_{i_1I_2}, \psi_{i_21}, \psi_{i_22}, \cdots, \psi_{i_2I_1})^T.
$$

(5)

We define the folding operation to be the inverse operation of the vectorization by wrapping a vector into a matrix

$$
\mathcal{W}(\vec{a})_{i_1i_2} \equiv \begin{pmatrix} a_1 & a_{h+1} & \cdots & a_{(h-1)\cdot h+1} \\ a_2 & a_{h+2} & \cdots & a_{(h-1)\cdot h+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{I_1} & a_{2\cdot I_1} & \cdots & a_{I_1\cdot I_2} \end{pmatrix},
$$

(6)

where $\vec{a} = (a_1, a_2, \cdots, a_{I_1\cdot I_2})^T$, $L = I_1 \cdot I_2$. The subscripts $I_1, I_2$ on the left-hand side of equation (6) indicate the dimensions of the obtained matrix, which may be omitted when there is no ambiguity in the matrix dimension.

Let $\Psi_{i_1i_2i_3}$ be an $I_1 \times I_2 \times I_3 \times I_4$ pure quantum state. A bipartition of the four-particle state may be expressed as $\Psi_{(i_1i_2)(i_3i_4)}$, where the four particles are grouped into two composite partitions, i.e.

$$
\Psi_{(i_1i_2)(i_3i_4)} = \begin{pmatrix} \psi_{1111} & \psi_{1112} & \cdots & \psi_{11I_4} \\ \psi_{1121} & \psi_{1122} & \cdots & \psi_{12I_4} \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{I_1i_31} & \psi_{I_1i_32} & \cdots & \psi_{I_1I_3I_4} \end{pmatrix}.
$$

(7)

The singular value decomposition of (7) goes as

$$
\Psi_{(i_1i_2)(i_3i_4)} = U \Lambda V^T,
$$

(8)

where $\Lambda$ is a diagonal matrix of rank $r$, the unitary matrices $U$ and $V$ are composed of the left and right singular vectors of $\Psi_{(i_1i_2)(i_3i_4)}$, i.e. $U = (\vec{u}_1, \vec{u}_2, \cdots)$ and $V = (\vec{v}_1, \vec{v}_2, \cdots)$, and the dimensions of vectors $\vec{u}_i$ and $\vec{v}_i$ are $I_1 \cdot I_2$ and $I_3 \cdot I_4$ respectively; $V^\dagger = (V^T)^T$ is the conjugate transpose of a matrix. It is legitimate to introduce the following triple-state set expression for the four-partite state based upon the partition $(I_1I_2)(I_3I_4)$:

$$
\Psi_{(i_1i_2)(i_3i_4)} = (\Psi_u, \Psi_\lambda, \Psi_r).
$$

(9)

Here, $\Psi_u = (\mathcal{W}(\vec{u}_1), \cdots, \mathcal{W}(\vec{u}_r))$ with $\mathcal{W}(\vec{u}_i) \in \mathbb{C}^{I_1 \cdot I_2}$, $\Psi_\lambda = \{\mathcal{W}(\vec{v}_1), \cdots, \mathcal{W}(\vec{v}_r)\}$ with $\mathcal{W}(\vec{v}_i) \in \mathbb{C}^{I_3 \cdot I_4}$, and $\Psi_r = \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_r)$ with $\lambda_i > 0$, $i \in \{1, \cdots, r\}$ being the non-zero singular values in $\Lambda$. In this representation, $\Psi_u$ and $\Psi_r$ may be regarded as the tripartite states of $r \times I_1 \times I_2$ and $r \times I_3 \times I_4$, and $\Psi_\lambda$ is a bipartite state of dimension $r \times r$, as shown in figure 1.

In the following we define the complementary state of a tripartite state. If a tripartite state $\Psi_{i_1i_2} = (\Gamma_1, \cdots, \Gamma_r)$, where $\Gamma_j \in \mathbb{C}^{d_j \times d_j}$ is a genuine entangled state of $r \times I_1 \times I_2$, then...
\[ V(\Gamma_i), i \in \{1, \cdots, r\} \] are linearly independent vectors. The complementary state of \( \Psi_{I_1I_2} \) is defined as
\[
\Psi_{\overline{I_1I_2}} \equiv (\Gamma_{r+1}, \cdots, \Gamma_{I_1I_2}) .
\] (10)

Here \( V(\Gamma_i), i \in \{1, \cdots, r, r+1, \cdots, I_1 \cdot I_2\} \) are linearly independent vectors. According to this definition, the complementary state of the \( r \times I_1 \times I_2 \) state \( \Psi_u \) can now be expressed as
\[
\Psi_u = (V(\bar{u}_{r+1}), \cdots, W(\bar{u}_{I_1I_2})) ,
\] (11)

which is a \((I_1 \cdot I_2 - r) \times I_1 \times I_2\) state with \( W(\bar{u}_i) \in \mathbb{C}^{I_1 \times I_2}\). Here the left-hand singular vectors are divided into two parts \( U = (U_1, U_0) \) with \( U_1 = (\bar{u}_1, \cdots, \bar{u}_r) \) and \( U_0 = (\bar{u}_{r+1}, \cdots, \bar{u}_{I_1I_2}) \). The quantum state \( \Psi_u \) is obtained by wrapping the left-hand singular vectors that correspond to the singular value zero of \( \Psi_{(I_1I_2)/(I_1I_2)} \). Similar definitions apply to \( \Psi_{\overline{I_1I_2}} \) as well.

Obviously, for a matrix \( A \in \mathbb{C}^{I_1 \cdot I_2 \times I_1 \cdot I_2} \), without loss of generality it can be expressed in the following block form:
\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1h}
A_{21} & A_{22} & \cdots & A_{2h}
\vdots & \vdots & \ddots & \vdots \\
A_{h1} & A_{h2} & \cdots & A_{hh}
\end{pmatrix} .
\] (12)

Here \( A_{ij} \) are \( I_2 \times I_2 \) submatrices. The realignment of the matrix \( A \) according to the blocks \( A_{ij} \in \mathbb{C}^{I_2 \times I_2} \) is defined to be \[14\]
\[
\mathcal{R}(A) \equiv (V(A_{11}), \cdots, V(A_{1h}), V(A_{21}), \cdots, V(A_{2h}), \cdots, V(A_{hh}))^T ,
\]
where \( \mathcal{R}(A) \in \mathbb{C}^{I_1 \cdot I_2 \times I_1 \cdot I_2} \). By means of the complementary state, the following indispensable lemma exists in the following discussion.

**Lemma 1.** The tripartite states \( \Psi_{I_1I_2} = (\Gamma'_1, \cdots, \Gamma'_r) \) and \( \Psi_{I_1I_2} = (\Gamma_1, \cdots, \Gamma_r) \) are SLOCC equivalent, if and only if there exist \( \bar{P} \equiv \begin{pmatrix} P & Y \\ 0 & \mathcal{P} \end{pmatrix} \) such that
\[
\text{rank}[\mathcal{R}(U \mathcal{P} U^T \bar{P}^{-1})] = 1 .
\] (13)

Here \( U' = (U'_1, U'_0) \) with \( U'_1 = (V(\Gamma'_1), \cdots, V(\Gamma'_r)) \), \( U'_0 = (V(\Gamma'_{r+1}), \cdots, V(\Gamma'_{I_1I_2})) \); \( U = (U_1, U_0) \) with \( U_1 = (V(\Gamma_1), \cdots, V(\Gamma_r)) \), \( U_0 = (V(\Gamma_{r+1}), \cdots, V(\Gamma_{I_1I_2})) \); \( P \in \mathbb{C}^{r \times I_1} \), and \( \mathcal{P} \in \mathbb{C}^{(I_1 \cdot I_2 - r) \times (I_1 \cdot I_2 - r)} \) are invertible matrices, and \( Y \) may be arbitrary. 

![Figure 1. A four-partite state \( \Psi_{I_1I_2I_3I_4} \) is factored into a triple-state set according to the bipartition \((I_1I_2)/(I_3I_4)\), which includes two tripartite pure states, \( \Psi_u \) and \( \Psi_v \), and one bipartite pure state \( \Psi_A \).](image)
Proof. If $\Psi'_{\text{th,b}}$ is SLOCC equivalent to $\Psi_{\text{th,b}}$, then
\[
(A_1^{-1} \otimes A_2^{-1})(\mathcal{V}(\Gamma'_1), \ldots, \mathcal{V}(\Gamma'_r)) = (\mathcal{V}(\Gamma_1), \ldots, \mathcal{V}(\Gamma_r))P,
\]
where $A_1 \in \mathbb{C}^{h \times h}$, $A_2 \in \mathbb{C}^{l \times l}$, and $P \in \mathbb{C}^{r \times r}$ are all invertible matrices. Because the column vectors $(A_1^{-1} \otimes A_2^{-1})\mathcal{V}(\Gamma'_i), i \in \{r + 1, \ldots, l_1 \cdot l_2\}$ are linearly independent and belong to the complementary vector spaces of the column vectors of $(\mathcal{V}(\Gamma_1), \ldots, \mathcal{V}(\Gamma_r))P$, therefore, there exists an invertible matrix $P$ such that
\[
(A_1^{-1} \otimes A_2^{-1})(U'_1, U'_0) = (U_1, U_0) \begin{pmatrix} P & Y \\ 0 & P \end{pmatrix}.
\]
Here $U'_1 = (\mathcal{V}(\Gamma'_1), \ldots, \mathcal{V}(\Gamma'_r))$, $U'_0 = (\mathcal{V}(\Gamma'_{r+1}), \ldots, \mathcal{V}(\Gamma_{l_1 \cdot l_2}))$, $U_1 = (\mathcal{V}(\Gamma_1), \ldots, \mathcal{V}(\Gamma_r))$ and $U_0 = (\mathcal{V}(\Gamma_{r+1}), \ldots, \mathcal{V}(\Gamma_{l_1 \cdot l_2}))$. Therefore $A_1^{-1} \otimes A_2^{-1} = UPU'^{-1}$, and $\text{rank}(R(UPU'^{-1})) = 1$, as the realignment of a matrix is rank one if and only if it is a direct product of two matrices [15, 16]. The converse is quite straightforward. \qed

2.2. The SLOCC equivalence of four-partite states

For two four-partite quantum states $\Psi'$ and $\Psi$ with the triple-state forms of $(\Psi_{u'}, \Psi_X, \Psi_{v'})$ and $(\Psi_u, \Psi_X, \Psi_v)$, we have the following theorem:

**Theorem 1.** Two quadripartite quantum states $\Psi$ and $\Psi'$ are SLOCC equivalent, if and only if the states in their corresponding triple-state sets are SLOCC equivalent in the following form:
\[
|\Psi_{u'}\rangle = P \otimes A_1 \otimes A_2 |\Psi_u\rangle, \quad |\Psi_{v'}\rangle = Q \otimes A_3^* \otimes A_4^* |\Psi_v\rangle. P \otimes Q^* |\Psi_{v'}\rangle = |\Psi_{u'}\rangle,
\]
where $A_1, A_2, A_3, A_4, P$ and $Q$ are all invertible matrices.

**Proof.** First, suppose two four-partite states $\Psi'$ and $\Psi$ are SLOCC equivalent, then $\Psi' = (A_1 \otimes A_2)\Psi(A_3 \otimes A_4)^T$ and
\[
U'\Lambda'V'^T = (A_1 \otimes A_2)U\Lambda V^T (A_3 \otimes A_4)^T.
\]
The QR factorizations [17] of $(A_1 \otimes A_2)U$ and $(A_3^* \otimes A_4^*)V$ are
\[
(A_1 \otimes A_2)U = Q_u R_u, \quad (A_3^* \otimes A_4^*)V = Q_v R_v.
\]
Here $Q_u$ and $Q_v$ are unitary matrices and $R_u$ and $R_v$ are upper triangular matrices. Taking equations (18) into (17), we have
\[
U'\Lambda'V'^T = Q_u R_u \Lambda R_v^T Q_v^* = Q_u X \Lambda' Y^T Q_v^*.
\]
where $R_u \Lambda R_v^T = X \Lambda' Y^T$ is the singular value decomposition of $R_u \Lambda R_v^T$. This leads to the following relations:
\[
U' = Q_u X (\oplus u_i), \quad V' = Q_v Y (\oplus u_i).
\]
Here $u_i$ are unitary matrices with dimensions that are conformal to the degeneracies of the singular values in $\Lambda'$. Considering equation (18), we get
\[
U' = (A_1 \otimes A_2)UR_u^{-1}X(\oplus u_i) = (A_1 \otimes A_2)U\tilde{P}.
\]
Theorem 1 turns the SLOCC equivalence of a four-partite state into tripartite and bipartite ones. For the SLOCC equivalence of tripartite states, the following corollary holds.

**Corollary 1.** Two tripartite states $\Psi'_{A_{12}^t}$ and $\Psi_{A_{12}^t}$ are SLOCC equivalent if and only if there exists an invertible matrix $\tilde{P} = \begin{pmatrix} P & Y \\ 0 & P \end{pmatrix}$ such that for arbitrary $I_1 \cdot I_2$ vectors $\tilde{a}$ we have $\text{rank}[\mathcal{W}(\tilde{U}'\tilde{P}U'{-1}\tilde{a})] = \text{rank}[\mathcal{W}(\tilde{a})]$, where $P \in \mathbb{C}^{r \times r}$, $Y \in \mathbb{C}^{r' \times r''}$ are invertible matrices, and $U$ and $U'$ are matrices composed of $\Psi'_{A_{12}^t}$, $\Psi_{A_{12}^t}$ and their complementary states.

**Proof.** If $\Psi_{u'}$ and $\Psi_u$ are SLOCC equivalent, then according to equation (15) we have

$$U' = (A_1 \otimes A_2)U\tilde{P}, \quad V' = (A_3 \otimes A_4)^*V\tilde{Q}, \quad \tilde{P}\Lambda\tilde{Q}^\dagger = \Lambda,'$$

where $\tilde{P}$ and $\tilde{Q}$ have the form of equation (25). Therefore

$$\Psi' = U'\Lambda'V' = (A_1 \otimes A_2)U\tilde{P}\Lambda'\tilde{Q}^\dagger\Lambda V(A_3 \otimes A_4)^T$$

$$= (A_1 \otimes A_2)U\Lambda V^\dagger(A_3 \otimes A_4)^T$$

$$= (A_1 \otimes A_2)\Psi(A_3 \otimes A_4)^T. \quad (27)$$

This means $\Psi'$ and $\Psi$ are SLOCC equivalent. (Superscripts of the transposition on $P$ and $Q$ in equation (16) may be needed for consistency, which have no influence on the proof.)

**Corollary 1.** Two tripartite states $\Psi'_{A_{12}^t}$ and $\Psi_{A_{12}^t}$ are SLOCC equivalent if and only if there exists an invertible matrix $\tilde{P} = \begin{pmatrix} P & Y \\ 0 & P \end{pmatrix}$ such that for arbitrary $I_1 \cdot I_2$ vectors $\tilde{a}$ we have $\text{rank}[\mathcal{W}(\tilde{U}'\tilde{P}U'{-1}\tilde{a})] = \text{rank}[\mathcal{W}(\tilde{a})]$, where $P \in \mathbb{C}^{r \times r}$, $Y \in \mathbb{C}^{r' \times r''}$ are invertible matrices, and $U$ and $U'$ are matrices composed of $\Psi'_{A_{12}^t}$, $\Psi_{A_{12}^t}$ and their complementary states.

**Proof.** If $\Psi_{u'}$ and $\Psi_u$ are SLOCC equivalent, then according to equation (15) we have

$$(A_1^{-1} \otimes A_3^t)U' = \tilde{U}\tilde{P},$$

and

$$\mathcal{W}(U'\tilde{P}U'{-1}\tilde{a}) = \mathcal{W}[(A_1^{-1} \otimes A_2^{-1})\tilde{a}] = (A_1^{-1})\mathcal{W}(\tilde{a})(A_2^{-1}) . \quad (28)$$

Hence the ranks of $\mathcal{W}(U'\tilde{P}U'{-1}\tilde{a})$ and $\mathcal{W}(\tilde{a})$ are equal for arbitrary $\tilde{a}$.

Second, the invertible matrix $\Phi = U'\tilde{P}U'{-1}$ acting on a vector induces a linear map $\varphi : \mathbb{C}^{h_{12}} \rightarrow \mathbb{C}^{h_{12}}$ for the wrapping operation

$$\mathcal{W}(\Phi\tilde{a}) = \varphi[\mathcal{W}(\tilde{a})]. \quad (29)$$
Because we have $\text{rank}[\mathcal{W}(\Psi \hat{\alpha})] = \text{rank}[\mathcal{W}(\hat{\alpha})] = \text{rank}[\mathcal{W}(\alpha)]$ for all $\alpha$, the linear map on matrices $\mathcal{W}(\hat{\alpha}) = A_1 \mathcal{W}(\hat{\alpha}) A_2$ follows, where $A_1$ and $A_2$ are invertible matrices according to theorem 3.1 of [18]. (Note, when the dimensions $I_1 = I_2$, the linear map may be $\mathcal{W}(X) = A_1X^T A_2$, where the two states are SLOCC equivalent up to a permutation of particles.)

Decomposing a four-partite state into tripartite and bipartite states not only greatly reduces the complexity of the entanglement classification of multipartite states, it also provides a way of studying the multipartite entanglement of the whole system via that of the subsystems. In practice, if we want to verify the SLOCC equivalence of two four-partite states, the equivalence of the two tripartite states should be clarified first. However, lemma 1 and corollary 1 provide effective ways of verifying the SLOCC equivalence of tripartite states. It is quite clear that detailed information of the connecting matrices $A_1, A_2, A_3$ and $A_4$ is not a prerequisite for verifying the SLOCC equivalence of two arbitrary four-partite entangled states. We shall show this by the following examples.

2.3. Examples

**Example 1.** Consider the four-qubit GHZ and W states, i.e. $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$ and $|\Psi'\rangle = \frac{1}{2}(|0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle)$. According to the partition $(12)(34)$, we have $\Psi_{(12)(34)} = (\Psi_u, \Psi_\lambda, \Psi_v)$ where

$$\Psi_u = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right), \Psi_\lambda = \left( \begin{array}{cc} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{array} \right), \Psi_v = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right),$$

and $\Psi'_{(12)(34)} = (\Psi'_u, \Psi'_\lambda, \Psi'_v)$, where

$$\Psi'_u = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right), \Psi'_\lambda = \left( \begin{array}{cc} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{array} \right), \Psi'_v = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right).$$

According to theorem 1 and corollary 1, $\Psi'$ and $\Psi$ are SLOCC equivalent if and only if

$$\text{rank}[\mathcal{R}(\mathcal{P}\mathcal{U}\mathcal{U}^\dagger)] = 1, \text{rank}[\mathcal{R}(\mathcal{V}\mathcal{Q}\mathcal{V}^\dagger)] = 1,$$

where the submatrices $P$ and $Q$ in $\mathcal{P}$ and $\mathcal{Q}$ should further satisfy $Q = \Psi_\lambda^{-1} P \Psi_\lambda$. Equation (32) induces only linear equations on the matrix elements, and we can easily find that $\mathcal{P} = 0$ and $\mathcal{Q} = 0$, which indicates that the GHZ and W states are SLOCC inequivalent.

**Example 2.** Consider the entangled states $\Psi_{abcd}$ of the first entanglement family in [7], i.e.

$$\Psi_{abcd} = \frac{1}{2} \begin{pmatrix} a+d & 0 & a-d \\ 0 & b+c & b-c \\ 0 & b-c & b+c \end{pmatrix} = \mathcal{U} \Lambda \mathcal{U}^\dagger$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}^\dagger.$$
Here $V = U$ because $\Psi_{abcd}$ is transposition symmetric. Another quantum state $\Psi_{a'b'c'd'} = U\Lambda'U^\dagger$ is SLOCC equivalent to $\Psi_{abcd}$ if and only if

$$\text{rank}[\mathcal{R}(U^\dagger P)] = 1, \quad \text{rank}[\mathcal{R}(U\Lambda^{-1}P\Lambda'U^\dagger)] = 1,$$

where $P$ shall have the same solution in the two equations. $\tilde{P} = P$ in equation (34) because $\Lambda$ has the full rank of 4. As the identity matrix is a solution to the first equation of $P$ in equation (34), we get $\Psi_{abcd}$ and $\Psi_{a'b'c'd'}$ are SLOCC equivalent if $\frac{a}{d} = \frac{b}{d} = \frac{c}{d} = \frac{1}{d}$ or $\frac{a}{d} = \frac{b}{d} = -\frac{1}{d}$ from the second equation in equation (34). Other solutions of $P$ would induce more symmetries for the entanglement family parameterized by $a, b, c, d$.

**Example 3.** Cluster or graph states are highly entangled multiqubit states which are a key resource in measurement base quantum computation [19] and various quantum error correction codes [20]. Consider the following four-qubit cluster states

$$|C^{(1)}\rangle = \frac{1}{2}(|0000\rangle + |0101\rangle + |1010\rangle - |1111\rangle),$$

$$|\Psi^{(2)}\rangle = a|0000\rangle - b|0111\rangle - c|1010\rangle + d|1101\rangle.$$  

Here $|C^{(1)}\rangle$ and $|\Psi^{(2)}\rangle$ are 1D and 2D lattice cluster states respectively [21]. Because all the coefficient matrices have the same rank, we do not know whether these two states are SLOCC equivalent or not using the technique of [9]. Here we demonstrate the SLOCC equivalence of $|C^{(1)}\rangle$ and $|\Psi^{(2)}\rangle$ based on theorem 1.

The singular value decomposition according to the bipartition (12)(34) gives

$$\Psi^{(2)} = U\Lambda V^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$C^{(1)} = U'\Lambda'V'^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

from which we may get $(\Psi_u, \Psi_3, \Psi_v)$ and $(\Psi_u', \Psi_3', \Psi_v')$. The SLOCC equivalence of the resulting $\Psi_u$ and $\Psi_u'$, and $\Psi_v$ and $\Psi_v'$ may be verified by exploring lemma 1. Similarly, as in equation (34) we have that the following two matrices must have rank 1

$$\mathcal{R}(U^\dagger P) = \begin{pmatrix} p_{11} & -p_{21} & p_{12} & -p_{22} \\ -p_{31} & p_{41} & -p_{32} & p_{42} \\ p_{13} & -p_{23} & -p_{14} & p_{24} \\ -p_{33} & p_{43} & p_{34} & -p_{44} \end{pmatrix},$$

$$\mathcal{R}(V\Lambda^{-1}P\Lambda'U^\dagger) = \frac{1}{2} \begin{pmatrix} p_{11} & a & p_{12} & a \\ p_{21} & c & p_{22} & c \\ p_{31} & d & p_{32} & d \\ p_{41} & a & p_{42} & a \end{pmatrix}.$$
Here \( p_{ij} \) are the matrix elements of \( P \) and we have used the relation \( V \Lambda^{-1} P \Lambda V^\dagger = (V \Lambda V^\dagger)^{-1} \). The solutions for equations (39) and (40) having rank 1 are

\[
\begin{pmatrix}
p_{11} & p_{12} & xp_{11} & -xp_{12} \\
p_{21} & p_{22} & xp_{21} & -xp_{22} \\
-yp_{11} & -yp_{12} & -zp_{11} & zp_{12} \\
-yp_{21} & -yp_{22} & -zp_{21} & zp_{22}
\end{pmatrix}
\]

where \( x, y, z \) and \( \alpha, \beta, \gamma \) are nonzero parameters. A consistent solution of matrix \( P \) is

\[
p_{12} = p_{21} = 0, \quad x = \alpha = -\frac{\gamma}{\beta}, \quad y = -\frac{c\beta}{a} = -\frac{d}{b\beta}, \quad z = -\frac{c\gamma}{a} = \frac{d\alpha}{b\beta}.
\]

Equation (42) predicts an invertible matrix \( P \), and therefore \( |C^{(1)}\rangle \) and \( |\Psi^{(2)}\rangle \) are SLOCC equivalent. After getting \( P \), we may also easily get

\[
A_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & \frac{1}{a} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & \frac{1}{p_{12}} \end{pmatrix}, \quad A_3 = \frac{1}{2} \begin{pmatrix} 1 & \beta \\ \alpha & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} \frac{c\beta}{a} & 0 \\ 0 & \frac{d\beta}{a} \end{pmatrix},
\]

which connect the matrices \( |C^{(1)}\rangle = A_1 \otimes A_2 \otimes A_3 \otimes A_4 |\Psi^{(2)}\rangle \). The SLOCC equivalence of \( |C^{(1)}\rangle \) and \( |\Psi^{(2)}\rangle \) indicates that different dimensional (1D and 2D) cluster states may be equivalent in realizing quantum computation tasks, which is important for the study of measurement-based quantum computation models.

The above examples indicate that the new method works effectively for finite dimensional four-partite systems. Different choices of partitions, i.e. (13)(24) and (12)(34) for four-partite states, do not influence the application of the method. However, one must choose the same partition for two four-partite states when verifying their SLOCC equivalence. It should be noted that the entangled quantum states may be infinite dimensional [22]. Although it is still unclear whether our method is applicable to this situation or not, the reduction method is nevertheless inspiring for the study of continuous-variable entanglement.

3. Conclusions

We proposed a practical classification scheme for the four-partite entangled state, in which the neat mathematical trick introduced defines a virtual system with subsystems that are different from the original one, whose entangled structure, however, faithfully represents the SLOCC relations of the original system. According to this scheme, a prerequisite for connecting matrices between two four-partite states is unnecessary, which greatly reduces the complexity in the usual procedure for verifying the SLOCC equivalence. According to the reduction method of this work, the relation between a high-partite entangled state, the four-partite state in this work, and its subsystems and bridges between them turns out to be manifest. It is notable that the method developed here opens a hopeful door to the general multipartite entanglement SLOCC classification. Furthermore, it is tempting to think that the high-order singular value decomposition technique in the local unitary (LU) classification of the multipartite entangled state is worthy of [23].
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References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Information (Cambridge: Cambridge University Press)
[2] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels Phys. Rev. Lett. 70 1895–99
[3] Bennett C H and Wiesner S J 1992 Communication via one- and two-particle operators on Einstein–Podolsky–Rosen states Phys. Rev. Lett. 69 2881–84
[4] Mattle K, Weinfurter H, Kwiat P G and Zeilinger A 1996 Dense coding in experimental quantum communication Phys. Rev. Lett. 76 4656–9
[5] Ekert A K 1991 Quantum cryptography based on Bell’s theorem Phys. Rev. Lett. 67 661–3
[6] Dür W, Vidal G and Cirac J I 2000 Three qubits can be entangled in two inequivalent ways Phys. Rev. A 62 062314
[7] Bastin T, Krins S, Mathonet P, Godefroid M, Lamata L and Solano E 2009 Operational families of entanglement classes for symmetric N-qubit states Phys. Rev. Lett. 103 070503
[8] Li X and Li D 2012 Classification of general n-qubit states under stochastic local operations and classical communication in terms of the rank of coefficient matrix Phys. Rev. Lett. 108 180502
[9] Walter M, Doran B, Gross D and Christandl M 2013 Entanglement polytopes: multiparticle entanglement from single-particle information Science 340 1205–8
[10] Van Loan C F 2000 The ubiquitous Kronecker product J. Comput. Appl. Math. 123 85–100
[11] Sun L-L, Li J-L, Li X and Qiao C-F 2015 Criterion for SLOCC equivalence of multipartite quantum states J. Phys. A: Math. Theor. A 48 205301
[12] Horn R A and Johnson C R 2013 Matrix Analysis (Cambridge: Cambridge University Press)
[13] Bai Y-K and Wang Z D 2008 Multipartite entanglement in four-qubit cluster-class states Phys. Rev. Lett. 100 050501
[14] Braunstein S L and van Loock P 2005 Quantum information with continuous variables Rev. Mod. Phys. 77 513–77
[15] Luo X-L and Yang C-F 2012 Local unitary classification of arbitrary dimensional multipartite pure states Phys. Rev. Lett. 108 050501