On some (multi)symplectic aspects of link invariants

Antonio Michele MITI and Mauro SPERA
Dipartimento di Matematica e Fisica “Niccolò Tartaglia”
Università Cattolica del Sacro Cuore
Via dei Musei 41, 25121 Brescia, Italia
antoniomichele.miti@unicatt.it, mauro.spera@unicatt.it

Abstract
In this note we construct a homotopy co-momentum map (à la Ryvkin, Wurzbacher and Zambon, RWZ) trangressing to the standard hydrodynamical co-momentum map of Arnol’d, Marsden and Weinstein and others, then generalized to a special class of Riemannian manifolds. As a byproduct, a covariant phase space interpretation of Brylinski’s manifold of mildly singular links is exhibited upon resorting to the Euler equation for perfect fluids. A semiclassical interpretation of the HOMFLYPT polynomial is also given, building on the Liu-Ricca hydrodynamical approach to the latter and on the Besana-S. symplectic approach to framing. We finally reinterpret the (Massey) higher order linking numbers in terms of conserved quantities within the RWZ multisymplectic framework and determine knot theoretic analogues of first integrals in involution.

MSC 2010: 53D50, 58D10, 53D12, 53D20, 55S30, 57M25, 76B47, 81S10.

Keywords: Knot polynomials, higher order linking numbers, symplectic and multisymplectic geometry, hydrodynamics, geometric quantization.

1 Introduction
In this note we discuss some applications of symplectic and multisymplectic techniques in a hydrodynamical context, together with some of their ramifications into knot theory. Indeed, the tight connection between fluid mechanics, electromagnetism and knot theory is already manifest from the 19th century beginnings of the theory (see e.g. [51]). The possibility of applying symplectic techniques therein ultimately comes from Arnol’d’s pioneering work culminating in the geometrization of fluid mechanics [2, 3, 4]. In particular, in this connection we may mention the paper [50], with its symplectic reinterpretation [45, 46, 47], and the general portrait depicted in [7]. The latter thread led to the symplectic approach (via Maslov theoretic ideas) to framing in knot theory developed in [8].
Here we wish to apply some recently emerged concepts in multisymplectic geometry (mostly building on [55]) and construct an explicit homotopy co-momentum map in a hydrodynamical context, leading to a multisymplectic interpretation of the so-called higher order linking numbers, viewed à la Massey ([48, 57, 24]). The construction is generalized to cover connected compact oriented Riemannian manifolds having vanishing intermediate de Rham groups. Also, as a follow-up of [5], we shall exhibit a novel interpretation of the HOMFLYPT polynomial via geometric quantization of the so-called Brylinski manifold of singular knots (and links) - the latter being in turn accommodated within the covariant phase space framework - taking inspiration from the ad hoc procedures in [38, 39].

The layout of the paper is the following. First, in Section 2, we give an example of homotopy co-momentum map in the sense of RWZ ([55, 54]) in fluid mechanics, transgressing to Brylinski’s symplectic structure on loop spaces, descending, in turn, to the manifold of mildly singular knots (or links), see [7, 5] and below for precise definitions. We briefly discuss the (non) equivariance of the above construction with respect to the group of volume preserving diffeomorphisms of 3-space (see Section 2) and we outline a generalization thereof in a Riemannian framework, signalling potential topological obstructions. In Section 3 we prepare the ground for the forthcoming applications by depicting a hydrodynamical multisymplectic portrait of basic knot theoretic objects. Furthermore (Section 4), we interpret the symplectic structure of Brylinski’s manifold as a covariant phase space one, upon resorting to the Euler equation for perfect fluids. Subsequently (Section 5), we resume the Maslov-type theory developed in [5] and enhance it to give a semiclassical wave function interpretation of the HOMFLYPT polynomial ([15, 49]) inspired by the fluid dynamical approach to the latter devised by Liu and Ricca, see [38, 39]. Finally, in Section 6, we reinterpret the Massey higher order linking numbers in multisymplectic terms: the 1-forms appearing in the hierarchical Massey construction (viewed, in turn, differential geometrically à la Chen) provide an example of first integrals in involution in a multisymplectic framework. The last section is devoted to gathering together the conclusions and to pointing out possible directions for further research. Appropriate background material is provided within the various sections in order to ease readability.

2 A hydrodynamical homotopy co-momentum map

In the present section we freely use basic material on symplectic and multisymplectic geometry tailored to our subsequent needs, prominently referring, for additional details, to [46, 51, 57, 59] for the former and to [55, 54] for the latter. For general background on symplectic geometry and (co)momentum maps we quote, among others [1, 8, 21, 22, 42, 4, 20].
2.1 Tools in multisymplectic geometry

All our objects will be smooth, unless differently specified. A (finite dimensional) multisymplectic manifold \((M,\omega)\) is a manifold equipped with a closed \((n+1)\)-form \(\omega\) (called multisymplectic form or \(n\)-plectic form) such that the map \(\alpha\) below sending vector fields to \(n\)-forms (via contraction)

\[
\mathfrak{X}(M) \ni \xi \mapsto \alpha(\xi) := \iota_\xi \omega \in \Lambda^n(M)
\]

is injective \([55]\). Dropping the last condition leads to the concept of pre-\(n\)-plectic form. The \(n = 1\) case retrieves (pre)symplectic manifolds. In the multisymplectic context, the generalization of the (co)momentum maps of the symplectic case leads to the more refined concept of homotopy co-momentum map, to be presently succinctly reviewed.

One first introduces the so called Lie \(n\)-algebra of observables \(L_\infty(M,\omega)\). Referring to \([55, 54]\) for a full coverage of the relevant apparatus, not needed to full extent here, we just point out that the latter is a graded vector space \(L\) whose degree \(i\) pieces read

\[
\Lambda^{n-1}_\text{Ham}(M), \ i = 0, \quad \Lambda^{n-1-i}(M), \ i = 1, 2, \ldots, n-1
\]

together with suitable multilinear maps denoted collectively by \(\ell\). The suffix “Ham” refers to the Hamiltonian \((n-1)\)-forms, i.e. those forms \(H\) such that

\[
\iota_X \omega + dH = \alpha(X) + dH = 0
\]

for a vector field \(X\) preserving \(\omega\) (i.e. \(\mathcal{L}_X \omega = 0\), called, in turn, a Hamiltonian vector field pertaining to \(H\).

A form \(\beta\) is said to be strictly (resp. globally, resp. locally) conserved by an \(\omega\)-preserving vector field \(X\) if \(\mathcal{L}_X \beta = 0\) (resp. \(\mathcal{L}_X \beta\) is exact, resp. closed).

Cartan’s formula immediately shows that closed forms are globally conserved; indeed, for such a form

\[
\mathcal{L}_X \beta = d\iota_X \beta + \iota_X d\beta = d\iota_X \beta
\]

Recall, from \([55]\), that a homotopy co-momentum map is an \(L_\infty\)-algebra morphism - stemming from what is called an infinitesimal action of \(\mathfrak{g}\) on \(M\) (with \(\mathfrak{g}\) being the Lie algebra of a generic Lie group \(G\), acting on \(M\) by \(\omega\)-preserving vector fields)

\[
(f) : \mathfrak{g} \to L_\infty(M,\omega)
\]

given explicitly by a sequence of linear maps

\[
(f) = \{f_i : \Lambda^i \mathfrak{g} \to \Lambda^{n-i}(M)/0 \leq i \leq n+1\}
\]

fulfilling \(f_0 = f_{n+1} = 0\) (we have tacitly set \(\Lambda^{-1}(M) = 0\)) and

\[
\text{Im} f_1 \in \Lambda^1_\text{Ham}(M) \quad (\diamond)
\]
together with (for $p \in \Lambda^k(\mathfrak{g})$):

$$-f_{k-1}(\partial p) = df_k(p) + \varsigma(k)\iota(v_p)\omega \quad (\circ\circ)$$

($k = 1, \ldots, n+1$). We explain the notation: first, if $p = \xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_k$, then $v_p = v_1 \wedge v_2 \wedge \cdots \wedge v_k$ where $v_i \equiv v_{\xi_i}$ are the fundamental vector fields associated to the action of $G$ on $M$. One sets $\iota(v_p)\omega = \iota(v_k) \ldots \iota(v_1)\omega$, $\varsigma(k) := -(-1)^{\frac{k(k+1)}{2}}$ and defines $\partial \equiv \partial_k : \Lambda^k\mathfrak{g} \to \Lambda^{k-1}\mathfrak{g}$ via

$$\partial(\xi_1 \wedge \xi_2 \wedge \cdots \wedge \xi_k) := \sum_{1 \leq i < j \leq k} (-1)^{i+j} [\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \hat{\xi}_i \wedge \cdots \hat{\xi}_j \wedge \cdots \xi_k$$

(with `denoting deletion as usual and with $\partial_0 = 0$; one has $\partial^2 = 0$).

Formula $(\circ\circ)$ tells us that the closed forms

$$f_{k-1}(\partial p) + \varsigma(k)\iota(v_p)\omega$$

must actually be exact, with potential $-f_k(p)$. Closure can be quickly ascertained as follows (we use the apparatus in [40, 55], keeping in mind that $d\omega = 0$):

$$d(f_{k-1}(\partial p) + \varsigma(k)\iota(v_p)\omega) = \varsigma(k)(-1)^{k-1}\iota(v_0)p\omega - \varsigma(k-1)\iota(v_{\partial p})\omega$$

$$= [-\varsigma(k+1) - \varsigma(k-1)]\iota(v_{\partial p})\omega$$

since in general $\varsigma(k)\varsigma(k+2) = -1$. Notice that the special case $k = n+1$ asserts that the function $f_n(\cdot)$ is constant, and its value is fixed by the condition $f_n(\partial p) + \varsigma(n+1)\iota(v_p)\omega = 0$.

We shall resume this discussion in Subsection 2.3.

### 2.2 A hydrodynamical application

In this Subsection we shall introduce an explicit homotopy co-momentum map; we depart from the standard setting in that our group $G$ will be infinite dimensional. We start from the observation ([55]) that the volume form in $\mathbb{R}^3$, $\nu := dx \wedge dy \wedge dz$ can be interpreted as a multisymplectic form: in this case the map $\alpha$ is bijective (in particular, injective). In coordinates, if $\xi = (\xi^i)$, then

$$\alpha(\xi) = \iota_\xi \nu = \xi^1 dy \wedge dz + \xi^2 dz \wedge dx + \xi^3 dx \wedge dy$$

Upon introducing the Hodge $\ast$ relative to the standard Euclidean metric and the associated “musical isomorphisms”, we have ($\xi \in \mathfrak{X}(\mathbb{R}^3)$, $\beta \in \Lambda^2(\mathbb{R}^3)$):

$$\alpha(\xi) = \ast(\xi^\flat), \quad \alpha^{-1}(\beta) = \ast(\ast\beta)^\sharp$$

We denote by $\mathfrak{g}$ the (infinite dimensional) Lie subalgebra of $\mathfrak{X}(\mathbb{R}^3)$ consisting of the divergence-free vector fields on $\mathbb{R}^3$ (the “Lie algebra” of the “Lie group” $G$ of volume preserving diffeomorphisms of $\mathbb{R}^3$. We shall, as it is often done, gloss over analytic subtleties (see e.g. [2, 11, 12, 36] for more information). We
just recall here that $G$ is a \textit{regular Lie group} in the sense of Kriegl-Michor ([30], 38.4) and that its associated exponential map is not even locally surjective (a quite general phenomenon). The “hydrodynamical” bracket, equalling \textit{minus} the standard one: $[\xi_1, \xi_2] = \text{curl}(\xi_1 \times \xi_2)$ will be employed throughout.

Then we have, for $\xi \in \mathfrak{g}$ (via Cartan’s formula)

$$0 = \mathcal{L}_\xi \nu = d\iota_\xi \nu + \iota_\xi d\nu = d\iota_\xi \nu = \text{div}(\xi)\nu$$

and thus we have an isomorphism $\mathfrak{g} \cong Z^2(\mathbb{R}^3)$ (closed 2-forms on $\mathbb{R}^3$). This will be important in the sequel. The above also expresses the fact that $\nu$ is a \textit{strictly conserved} 3-form.

We shall tacitly assume that our fields \textit{rapidly vanish at infinity}, so that convergence problems are avoided and boundary terms yielded by calculations are absent.

Also, following e.g. [4], we shall consider the so-called \textit{regular dual} $\mathfrak{g}^*$ of $\mathfrak{g}$ consisting of all 1-forms modulo exact 1-forms:

$$\mathfrak{g}^* := \Lambda^1(\mathbb{R}^3)/d\Lambda^0(\mathbb{R}^3)$$

together with the standard pairing $(\omega \in \mathfrak{g}^*, \xi \in \mathfrak{g})$

$$(\omega, \xi) = \int \langle \omega(x), \xi(x) \rangle \nu$$

Nevertheless, we shall feel free to use suitable genuine distributional elements as well (i.e. \textit{currents}, in the sense of de Rham, [11]) from the full topological dual (without introducing new notation for the latter). Everything will be clear from the context.

We shall now give the promised example of homotopy co-momentum map emerging in fluid dynamics.

Define, for $b \in \mathfrak{g}$

$$f_1(b) := -B$$

where $B = \mathbf{B}^\flat$ and $\mathbf{B}$ is a vector potential for $b$, i.e. $\text{curl} \mathbf{B} = b$, chosen e.g. in such a way that $\text{div} \mathbf{B} = 0$ (Coulomb gauge).

It is immediately checked that

$$df_1(b) + \iota_b \nu = df_1(b) + \alpha(b) = 0 \quad (\diamond')$$

The above formula tells us that $f_1(b)$ is a \textit{Hamiltonian 1-form} for $b$ (and, conversely, that the vector field $b$ is a \textit{Hamiltonian vector field} pertaining to $f_1(b)$, in accordance with $(\diamond)$ in Subsection 2.1. Any $f_1(b)$ above is also a \textit{Noether current} in the sense of Gotay et al., [18]. In order to complete the definition of a homotopy co-momentum map, we just have to find $f_2$, satisfying formula $(\diamond \diamond)$ above. Indeed, for $k = 1$ we retrieve $(\diamond')$. The case $k = 2$ reads

$$-f_1(\partial p) = df_2(p) + \iota_{\nu_p} \nu \quad (\diamond \diamond \diamond)$$

Let, for $\xi_i \in \mathfrak{g}$ ($i = 1, 2$), $p = \xi_1 \wedge \xi_2$, so $\partial p = -[\xi_1, \xi_2]$.
Then one checks that (using [55], or the preceding subsection)
\[ df_1([\xi_1, \xi_2]) = d(\iota_{\xi_1} \wedge \iota_{\xi_2} \nu) = -\iota_{[\xi_1, \xi_2]} \nu. \]
(recall that \( \iota_{\xi_1} \wedge \iota_{\xi_2} \nu = \nu(\xi_1, \xi_2, \cdot) \)). Therefore, the 1-form \( \mu(\xi_1, \xi_2) := f_1([\xi_1, \xi_2]) - \iota_{\xi_1} \wedge \iota_{\xi_2} \nu \) is closed, hence exact, and \( \circ \circ \circ \) tells us that \( f_2(p) \) is a potential for it and, as such, it is determined up to a constant. Upon requiring that it vanishes at infinity, we can find it by solving the related Poisson equation; notice that, in particular, for \( q = \xi_1 \wedge \xi_2 \wedge \xi_3 \), we get the explicit formula
\[ f_2(\partial q) = \nu(\xi_1, \xi_2, \xi_3) \]
(indeed, the r.h.s. does vanish at infinity).

We define Poisson brackets via the expression
\[ \{ f_1(b), f_1(c) \}(\cdot) := \iota_c \iota_b \nu(\cdot) = \nu(b, c, \cdot) \]
We may also naturally ask the question whether the above map \( (f) \) is (infinitesimally) \( G \)-equivariant, in the sense of [55]: in particular, one should check the validity of the formula
\[ \mathcal{L}_{\xi} f_1(b) = f_1([\xi, b]) \]
for all \( \xi, b \in \mathfrak{g} \). However, working out the two sides of the above equation yields, in particular, for \( \xi = b \), the equality
\[ dB(b) = 0 \]
that is, in vector terms \( \langle B, b \rangle = c = 0 \) since \( b \) is compactly supported. However, if one considers a flux tube with non zero helicity \( \int \langle B, b \rangle \) (see [44, 5, 57] and below for further elucidation of this train of concepts), we get a contradiction. Notice that the argument does not depend on the choice of \( B \). The lack of \( G \)-equivariance is not surprising, since our construction involves Riemannian geometric features.

We may now state the following

**Theorem 2.1.** (i) The map \( f \) previously given through the above \( f_j : \Lambda^j \mathfrak{g} \to \Lambda^{2-j}(\mathbb{R}^3), \) fulfilling \( \circ \circ \circ \), yields a homotopy co-momentum map, transgressing, via the evaluation map \( \text{ev} : L\mathbb{R}^3 \times \mathbb{R} \ni (\gamma, t) \mapsto \gamma(t) \in \mathbb{R}^3 \) to the hydrodynamical co-momentum map of Arnol’d and Marsden-Weinstein (actually defined of the Brylinski manifold \( Y \) of oriented knots, to be more extensively discussed in Section 4).

(ii) Moreover, we have the formula
\[ \{ f_1(b), f_1(c) \} - f_1([b, c]) = -d f_2(b \wedge c) \quad (\blacklozenge) \]
Here \( L\mathbb{R}^3 \) denotes the manifold (in the sense of [7]) consisting of all smooth loops in \( \mathbb{R}^3 \).

(iii) The map \( f \) is not \( G \)-equivariant in the sense of RWZ.
Proof. At this point we just need to observe that the relevant piece of the homotopy co-momentum map is $f_1$ which, under transgression, becomes

$$\lambda_b = -\int \gamma B$$

i.e., up to sign, the Rasetti-Regge current (RR) pertaining to $b \in \mathfrak{g}$, independent of the choice of $B$. See Section 4 as well. This is in accordance with the general result in [55] asserting that, roughly speaking, homotopy co-momentum maps transgress to homotopy co-momentum maps on loop (and even mapping) spaces. Actually, the ansatz for $f_1$ term was precisely motivated by this phenomenon.

Formula (♠) in (ii) is just a rewriting of (⋄⋄⋄).

We shall employ the above PB in Section 6. Notice that, in the standard symplectic framework, the RR-co-momentum map $b \mapsto \lambda_b$ is indeed (infinitesimally) $G$-equivariant:

$$\{\lambda_b, \lambda_c\} = \lambda_{[b,c]}$$

see e.g. [45, 46, 7, 59] and Section 4 below.

2.3 A generalization to Riemannian manifolds

We ought to notice that a hydrodynamically flavoured homotopy co-momentum map can be similarly construed also for an $(n+1)$-dimensional connected, compact, orientable Riemannian manifold $(M, g)$, upon taking its Riemannian volume form $\nu$ as a multisymplectic form and again the group $G$ of volume preserving diffeomorphism group as symmetry group. The divergence of a vector field $X$ is defined via $\text{div} X := * d * X^\flat = - \delta X^\flat$ (e.g. [64, 36]). We can indeed prove the following result:

**Theorem 2.2.** Let $(M, g)$ be a connected compact oriented Riemannian manifold of dimension $n+1$, $n \geq 1$, with multisymplectic form $\nu$ given by its Riemannian volume form, and such that all de Rham cohomology groups $H^k_{dR}(M) = 0$, $k=1,2,\ldots,n$ (one has necessarily $H^0_{dR}(M) = H^{n+1}_{dR}(M) = \mathbb{R}$). Then there exists an associated family of homotopy co-momentum maps.

**Proof.** As we have already noticed in general, the defining formula triggers a recursive construction starting from $f_1$, up to topological obstructions (we have a sequence of closed forms, which must be actually exact, together with the constraint $f_\alpha (\partial q) = (-1)^{\frac{(n+1)(n+2)}{2}} \nu(\xi_1, \ldots, \xi_{n+1})$, with $q = \xi_1 \wedge \ldots \wedge \xi_{n+1}$, for the constant function $f_\alpha()$). In the present case, a natural candidate for the $n$-form $f_1$ can be readily manufactured via Hodge theory (see e.g. [64]):

$$f_1(\xi) := - \Delta^{-1} \delta (i_\xi \nu)$$

(the direct generalization of the preceding case) after imposing $\delta f_1(\xi) = 0$ (the analogue of the Coulomb gauge condition), provided one can safely invert the
Hodge Laplacian $\Delta = d\delta + \delta d$, this being the case if $H^n_{dR}(M) = 0$. One can of course alter the above definition by addition of an exact form. The assumptions made ensure that the entire procedure goes through unimpeded.

Remark. We notice that the above result holds, in particular, for homology spheres such as, for instance, the celebrated Poincaré dodecahedral space. We point out that the case in which the intermediate homology groups are at most torsion (hence not detectable by de Rham techniques) is also encompassed: this is e.g. the case of lens spaces. Notice that $G$-equivariance cannot be expected a priori.

We plan to hopefully delve into this issue more deeply elsewhere. See e.g. [53] for a general discussion of topological constraints to existence and uniqueness of homotopy co-momentum maps.

3 A Hamiltonian 1-form for links

We may specialize the considerations in Subsection 2.2 to the case of links. As general references for knot theory we quote, among others [28, 52], together with [6] for the algebraic-topological tools employed here. We shall indifferently view Poincaré duals as genuine forms or currents in the sense of de Rham ([11]).

Building on [48, 5, 57], let $L = \bigcup_{i=1}^n L_i$ be an oriented link in $\mathbb{R}^3$ with components $L_i$, $i = 1, \ldots, n$ - required to be trivial knots and let $\omega_{L_i}$ denote the Poincaré (or Thom) dual (class) associated to $L_i$: they are 2-forms localized in a cross-section of a suitable tubular neighbourhood $T_i$ around $L_i$ - with total fibre integral equal to one - see [6], or, as currents, 2-forms which are $\delta$-like on $L_i$, see Figure 1.

![Diagram](image)

Figure 1: Tubular neighbourhoods

Then take, for each $i = 1, 2, \ldots, n$, a 1-form $v_{L_i}$ such that $dv_{L_i} = \omega_{L_i}$.
namely, \( v_{L_i} := \omega_{a_i} \) is the Poincaré dual (class) of a disc \( a_i \) bounding \( L_i \) (a Seifert surface for the trivial knot \( L_i \)). Precisely:
\[
\partial a_i = L_i, \quad dv_{L_i} = d\omega_{a_i} = \omega_{L_i} = \omega_{a_i},
\]
see Figure 2 We list, for the sake of clarity, (de Rham) cohomology and relative homology groups of \( S^3 \setminus L \) with real coefficients, respectively, reading
\[
\begin{align*}
H^0(S^3 \setminus L) & \cong H_3(S^3, L) \cong \mathbb{R} \\
H^1(S^3 \setminus L) & \cong H_2(S^3, L) \cong \mathbb{R}^n \\
H^2(S^3 \setminus L) & \cong H_1(S^3, L) \cong \mathbb{R}^{n-1} \\
H^3(S^3 \setminus L) & \cong H_0(S^3, L) \cong 0
\end{align*}
\]
The (de Rham classes of) the forms (or currents) \( v_{L_i} \) generate in fact the cohomology group \( H^1(S^3 \setminus L, \mathbb{R}) \) (or, better, that of \( S^3 \setminus T \), with \( T = \bigcup_{i=1}^n T_i \)). Their homological counterparts are given by the (classes of) the discs \( a_i \). One can also interpret the other groups: in particular, elements in \( H_1(S^3, L) \) can be represented by classes [\( \gamma_{ij} \)] of (smooth) paths \( \gamma_{ij} \) connecting two components \( L_i \) and \( L_j \), subject to the relation [\( \gamma_{ij} + \gamma_{jk} = \gamma_{ik} \)].

Now set:
\[
\omega_L := \sum_{i=1}^n \omega_{L_i}
\]
(the vorticity 2-form for the link \( L \)) together with its velocity 1-form
\[
v_L = \sum_{i=1}^n v_{L_i}, \quad dv_L = \omega_L
\]

**Proposition 3.1.** The position
\[
H = v_L
\]
produces a Hamiltonian 1-form for links.
Proof. The proof is straightforward: indeed for each component $L_i$, the Hamiltonian vector field $\xi_{L_i}$ for $v_{L_i} \equiv v_i$ is minus the vector field associated to the closed 2-form $\omega_{L_i}$ (via the map $\alpha$ of Section 2). Explicitly, one has (setting $\xi_L = \sum_{i=1}^n \xi_{L_i}$)

$$dv_L + \iota_{\xi_L} v = 0 \quad \blacklozenge$$

Remark. Inspection of the very geometry of Poincaré duality shows that the velocity 1-forms $v_i$ correspond (upon approximation of the associated Euler equation) to the so-called LIA (Linear Induction Approximation) or binormal evolution of the “vortex ring” $L_i$ (“orthogonal” to the discs $a_i$ - an easy depiction, cf. Figure 2), see [30] for more information. Formula (♣) will be the prototype for the calculations in Section 6.

Let us define the Chern-Simons (helicity) 3-form:

$$CS(L) := v_L \wedge \omega_L \equiv \mathcal{H}(L)\nu$$

The integer $\mathcal{H}(L)$ is the helicity of $L$:

$$\mathcal{H}(L) = \sum_{i,j=1}^n \ell(i,j)$$

with $\ell(i,j) = \ell(j,i)$ being the Gauss linking number of components $L_i$ and $L_j$ if $i \neq j$ and $\ell(j,j)$ is the framing of $L_j$, equal to $\ell(L_j, L'_j)$ with $L'_j$ being a section of the normal bundle of $L_j$, see e.g. [44, 66, 5, 57] and below. A regular projection of a link onto a plane produces a natural framing called the blackboard framing, see Section 5 for further developments.

4 The Brylinski manifold and its covariant phase space structure

4.1 The Brylinski manifold of singular knots

In this Subsection we concisely review the geometry of the manifold $\tilde{Y}_M$ of (mildly) singular knots in a manifold $M$ introduced by Brylinski in [7] (we provisionally use a decorated notation for the sake of clarity), closely following [5].

Start from the (free) loop space $LM := C^\infty(S^1, M)$ associated to a smooth manifold $M$ of dimension $n$: it is an infinite dimensional paracompact smooth Fréchet manifold modelled on $C^\infty(S^1, \mathbb{R}^n)$. Consider the submanifold $\tilde{X}_M \subset LM$ consisting of smooth loops which are embeddings but for a finite set $A \subset S^1$, and such that the branches of the loop at any two distinct points in $A$ have finite order tangencies. The manifold of all embeddings will be denoted by $X_M$. The group $\text{Diff}^+(S^1)$ of all orientation preserving diffeomorphisms of the circle acts on $\tilde{X}_M$ in such a way that the quotient $\tilde{Y}_M := \tilde{X}_M / \text{Diff}^+(S^1)$ is
naturally a smooth paracompact Fréchet manifold modelled on $C^\infty(S^1, \mathbb{R}^{n-1})$, and $	ilde{X}_M \to \tilde{Y}_M$ becomes a principal $\text{Diff}^+(S^1)$-bundle. Accordingly, one can define $Y_M := X_M/\text{Diff}^+(S^1)$.

We shall deal with the case $M = \mathbb{R}^3$; the ensuing manifold $\tilde{Y} := \tilde{Y}_{\mathbb{R}^3}$ is called the manifold of *oriented singular knots* in $\mathbb{R}^3$, whereas $Y := Y_{\mathbb{R}^3}$ is called the manifold of *oriented knots* in $\mathbb{R}^3$. Recall that the tangent space $T_K\tilde{Y}_M$ to $K \in \tilde{Y}_M$ is intrinsically described as the space of smooth sections of the normal bundle to the normalization $\tilde{K}$ of $K$, namely, a separation of the branches of $K$ (see [7] for details). Given a volume form $\nu$ on a 3-dimensional $M$, one again gets, by transgression, a 2-form $\Omega$ on $LM$ via the formula

$$\Omega = \int_{S^1} ev^*(\nu)$$

where $ev : S^1 \times LM \to M$ given by $ev(\lambda, \gamma) := \gamma(\lambda)$ is again the evaluation map (of a loop $\gamma \in LM$ at a point $\lambda \in S^1$). More explicitly, given tangent vectors $u$ and $v$ at $\gamma$, it reads

$$\Omega_\gamma(u, v) = \int_0^1 \nu(\dot{\gamma}(\lambda), u(\lambda), v(\lambda))$$

(where we set $\dot{\gamma} = \frac{d\gamma}{d\lambda}$). The above formulae can be also synthetically cast as a Chen integral ([8, 9])

$$\Omega|_K = \int_K \nu \quad \text{or, shortly} \quad \Omega = \int \nu$$

The 2-form $\Omega$ is basic with respect to the $\text{Diff}^+(S^1)$-principal bundle $\tilde{X}_M \to \tilde{Y}_M$, namely $i_\xi \Omega = i_\xi d\Omega = 0$, with $\xi$ any vertical vector field (i.e. generating an orientation preserving reparametrization of the loop), therefore it descends to a closed, non degenerate 2-form on $\tilde{Y}_M$, i.e. a (weak) symplectic form (direct check). Also recall that, in general, the above transgression gives rise to a (degree shifting) morphism of complexes $\Lambda^* (M) \to \Lambda^{*-1}(LM)$, mapping closed (resp. exact) forms to closed (resp. exact) ones in view of the general formula (direct calculation, or see [8, 9]):

$$d \int \omega = - \int d\omega$$

where, of course, the l.h.s. differential pertains to $LM$ and the r.h.s. one pertains to $M$.

Consequently, integral cohomology classes on $M$ are mapped to integral cohomology classes on $LM$. Therefore, if $[\nu]$ is integral, then $[\Omega]$ is integral as well, this ensuring, via the Weil-Kostant theorem, the existence of a prequantum bundle $L \to LM$ (Brylinski’s line bundle), see also Section 5 for further developments. A subtle though explicit construction can be given via the integral class $[\nu] \in H^3(M, \mathbb{Z})$, defining a gerbe, see [7, 58].

11
We also notice that the weak symplectic manifold \((\hat{Y}_M, \beta)\) can be naturally equipped with a (formally) integrable compatible almost complex structure making it a Kähler manifold in an appropriate sense, see e.g. [7, 11, 15, 4].

An important observation is the following: each connected component thereof is (up to technical subtleties, see [7]) a coadjoint orbit of the group of unimodular diffeomorphisms of \(M\), i.e. those preserving a volume form, via the natural (co)momentum map hinted at above, and further elaborated on below.

We can naturally extend the above discussion to oriented links and accordingly define the symplectic structure on the generalized Brylinski space of oriented mildly singular links \(\hat{Y}\) (allowing a finite number of crossings and finite order tangencies) - no notational changes - via the same formula above, by replacing a knot \(K\) by a link \(L\):

\[
\Omega_L(\cdot, \cdot) = \left( \int_L \nu \right)(\cdot, \cdot) := \sum_{i=1}^{n} \int_{L_i} \nu(\dot{\gamma}_i, \cdot, \cdot)
\]

The manifold consisting of all bona fide oriented links in \(\mathbb{R}^3\) will be accordingly still denoted by \(Y\).

**4.2 A covariant phase space interpretation**

We are going to propose a multisymplectic interpretation of the above manifold in a wider framework which, though it not strictly needed in the sequel, ties neatly with the topics discussed in previous sections.

Start with a 4-dimensional space-time \(M = \mathbb{R}^3 \times \mathbb{R} \ni (x, y, z, t)\). Define the obvious trivial bundle \(E = M \times \mathbb{R}^3 \to M\).

Interpret \(\Sigma := \mathbb{R}^3 \ni (x, y, z)\) as a Cauchy “submanifold” of \(M\).

Any divergence-free vector field can be viewed as an initial condition \(v(x, 0)\) for the (volume-preserving) Euler evolution (at least for small times, but as we previously said, we do not insist on refined analytical nuances) \(v(x, t)\), yielding a section of \(E\). Using the 3-volume form \(\nu\), orienting fibres (notice that, when viewed on \(E\), it is only pre-3-plectic, namely closed but degenerate), and observing that we can set (abridged notation)

\[
j^1v = w \quad (:= \text{curl} v)
\]

(the natural ”covariant” jetification of the section \(v\), if we wish to look at \(v\) as the vector space counterpart of a connection 1-form, yielding a section of the jet bundle \(J^1E \to E\)) we can write down, mimicking [14], an expression (\(*\)), given below, naturally related to the Arnol’d – Marsden-Weinstein Lie-Poisson structure for \(\mathfrak{g}^*\), see e.g. [1, 37, 11, 15, 46, 47, 59, 33] (\(\{v\}\) denotes the “gauge” class of \(v\): \(\{v\} = \{v + \nabla f\}\)):

\[
\{F,G\}(\{v\}) = \int_{\Sigma = \mathbb{R}^3} \langle v, [\delta F \frac{\delta G}{\delta v}] \rangle
\]
Indeed, the variations $\frac{\delta F}{\delta v}$ and $\frac{\delta G}{\delta v}$ are vertical and divergence-free so long as we take $v$ such - see below - hence we find, successively:

$$\{F, G\}(v) = \int_{\Sigma = \mathbb{R}^3} \langle w, \frac{\delta F}{\delta v} \times \frac{\delta G}{\delta v} \rangle = \int_{\Sigma = \mathbb{R}^3} \nu(j^1 v, \frac{\delta F}{\delta v}, \frac{\delta G}{\delta v}) =: (*)$$

The expression $(*)$ can be manipulated to yield the expressive layout (with slight abuse of language)

$$(*) = \int_{\Sigma} (j^1 \ast \nu)(v, B, C) = \int_{\Sigma} \nu(j^1 v, j^1 B, j^1 C) = \int_{\Sigma} \nu(w, b, c)$$

where, again, $b = \text{curl } B$ et cetera; then recall that $\frac{\delta F}{\delta v} = \text{curl } (\frac{\delta F}{\delta w})$ and take $F = \lambda_a$ (see e.g. in particular [46, 59]).

Ultimately, we reached the following conclusion:

**Theorem 4.1.** (i) The Poisson manifold $\mathfrak{g}^*$ can be naturally be interpreted as a (generalized) covariant phase space pertaining to the volume preserving Euler evolution: the latter indeed preserves the symplectic leaves of $\mathfrak{g}^*$ given by the $G$-coadjoint orbits $O[V]$.

(ii) The above construction reproduces the symplectic structure of $\hat{Y}$ upon taking singular vorticities, concentrated on a knot (singular Poincaré duals), with the coadjoint orbits labelled by the equivalence types of knots (via ambient isotopies), by virtue of a result of Brylinski, see [7]: the covariant phase space picture is fully retrieved upon passing to a 2-dimensional space-time $S^1 \times \mathbb{R} \hookrightarrow (\lambda, t)$, with $\lambda \in S^1 \equiv \Sigma$ being a knot parameter (and staying of course with the same $\nu$). Links are also readily accommodated within this picture.

**Remarks.** 1. We stress the fact that we did not literally follow the standard multisymplectic recipe developed e.g. in [29, 18, 14, 68, 10]; in fact we directly took the volume form $\nu$ in $\mathbb{R}^3$ as a pre-3-plectic structure. This neatly matches the result of Brylinski quoted above and fits with the stance long advocated, among others, by Rasetti and Regge and Goldin (see e.g. [50, 16, 19, 17], and [59] as well) pinpointing the special and ubiquitous role played by the group $G$.

2. In line with the preceding remark, notice that the above portrait can, in principle, be generalized to any volume form (on an orientable manifold), with its attached group $G$. The covariant phase space picture should basically persist in the sense that one might construct, in greater generality, an n-plectic structure out of an (n+1)-plectic one via an expression akin to $(*)$. The (non) $G$-equivariance issue should be relevant in this context. We hope to be able to tackle this problem elsewhere.

5 A symplectic approach to the HOMFLYPT polynomial

Pursuing the analysis initiated in the preceding section, we resume the theory developed in [5], see also [57], closely following these papers. We refer, for full details, to [21, 43, 25] or to [42].
5.1 Lagrangian submanifolds revisited

Recall that a Lagrangian submanifold of a symplectic manifold is defined by the property that the symplectic form vanishes thereupon, and it is of maximal dimension (i.e. the tangent space at any point is a maximal isotropic subspace with respect to the symplectic form, i.e. it coincides with its symplectic complement). If $M$ is a smooth manifold of dimension $n$, then its cotangent space $T^*M$ is a symplectic manifold (equipped with a canonical symplectic form). A Lagrangian submanifold $\Lambda \subset T^*M$ in general position can be described in the following way (Maslov-Hörmander Morse family theorem, see e.g. [43], [25], [21], [42]): there exists (locally) a smooth function $\phi = \phi(q, a), (q, a) \in M \times \mathbb{R}^k$ (for some $k$: $\mathbb{R}^k$ is a space of auxiliary parameters) and a submanifold $C_\phi = \{(q, a) \in M \times \mathbb{R}^k \mid d_a \phi = 0\}$ with $d(d_a)$ of maximal rank thereon (here $d = d_q + d_a$) such that the map

$$C_\phi \to T^*M$$

$$(q, a) \mapsto (q, d_q \phi)$$

is an immersion with image $\Lambda$. If the Hessian $H_a$ (with respect to the auxiliary variables $a$) is non degenerate, one can solve $a = a(q)$ and define the phase function $F = F(q) := \phi(q, a(q))$, with $(q, dF(q)) \in \Lambda$. The covector $dF(q) = p(q)$ is the momentum at $q$.

This fails at the singular points of the obvious projection $\Lambda \to M$, but the singular locus $Z$ (the Maslov cycle) turns out to be orientable and of codimension 1 in $\Lambda$ with $\partial Z$ of codimension $\geq 3$.

Taking a good open cover $\{V_i\}_{i \in I}$ of $\Lambda$, and letting $\sigma_i$ be the signature of the Hessian $H_a$ on $V_i \setminus Z$, one readily manufactures the so-called Maslov cocycle, see the above references. This situation is general for a symplectic manifold, as a consequence of a result by Weinstein (65).

In [5] we developed a similar ad hoc but rigorous set-up for knot theory - aimed at placing the construction of the (Abelian) Witten invariant in [66] on firm ground by avoiding the use of path integrals - which will be recalled and applied below, with appropriate en-route modifications.

5.2 Geometric quantization and HOMFLYPT

Observe that the volume form $\nu$ can be portrayed as

$$\nu = dx \wedge dy \wedge dz = d(z \, dx \wedge dy) = d\theta$$

in terms of the symplectic potential $\theta$. Then, on planes $z = c$, $\theta \equiv 0$. Within the manifold $\tilde{Y}$, the submanifold $\Lambda$ consisting of the loops on a plane (with indentations keeping track of crossings), see [5], is a Lagrangian one. Now, observe that links in $\mathbb{R}^3$ can be viewed as solutions of the Euler-Lagrange equations pertaining to a Chern-Simons Lagrangian, with source given by the link itself,
which appears to be the singular curvature (vorticity $\omega \leftrightarrow F_A$ curvature of $A$) pertaining to an Abelian connection $A \leftrightarrow v$ velocity:

$$\Phi = \Phi(A, L) := \frac{k}{8\pi} \int A \wedge dA + \int_A \equiv \frac{k}{8\pi} \int A \wedge dA + T_L(A)$$

This CS Lagrangian is then taken, as in [5], as our Morse family, with the auxiliary parameters given by Abelian connections. Solving the ensuing Euler-Lagrange equation leads to:

$$\frac{k}{4\pi} F_A + T_L = \frac{k}{4\pi} dA + T_L = 0$$

i.e. we are looking for a connection (viewed as a current) whose curvature is concentrated (i.e. $\delta$-like) on $L$. The solution can be given in standard vector calculus terms (with a so-called Coulomb gauge fixing, $\text{div} A = 0$, or, Hodge theoretically, $\delta A = 0$), also cf. Section 2. Call $A_L$ the (singular) connection with $dA_L = T_L$ and $\delta A_L = 0$. The solution can be compactly written in the form

$$A_L = -\frac{4\pi}{k} \Delta^{-1} \delta T_L$$

where $\Delta$ is the Hodge Laplacian on 1-forms, acting component-wise as the ordinary Laplacian (up to a negative constant), since we are in flat space. Existence, in the sense of currents, follows, e.g., from the Hörmander-Lojasiewicz theorem, see e.g. [63]. Notice that if we want to insert $A_L$ into $\Phi$, we are forced to consider ordinary links. In this case the current $T_L$ may be written in terms of a singular Poicaré dual $\eta_L$ form and represented by a 2-form concentrated on $L$:

$$T_L(A) = \int_L A = \int_{\mathbb{R}^3} A \wedge \eta_L$$

Proceeding as in [5] we get, for the local phase $\phi$, the expression

$$\phi(L) = -\frac{2\pi}{k} \mathcal{H}(L) \equiv 2\pi \lambda \mathcal{H}(L)$$

(i.e. $\lambda := -1/k$, with $k$ a non zero integer or any non zero real number). Helicity can be interpreted, as in [4], as a regularised signature as well (and, as such, it enters Maslov theory).

Upon loop transgression, the symplectic potential of Brylinski’s form can be taken equal to zero, so the phase, i.e. the helicity, is (locally) constant, being a topological invariant. The Lagrangian submanifold $\Lambda$ is thence locally given by the graph

$$(L, d\mathcal{H}(L)) = (L, 0)$$

($d\mathcal{H}(L) = 0$ is the so-called eikonal equation, see [5, 59]).

Now, given a prequantizable symplectic manifold $(M, \omega)$, i.e. (Weil-Kostant, see e.g. [55, 31, 56, 7, 61, 67]), $\omega \in H^2(M, \mathbb{Z})$ - so that there exists a complex line bundle $\mathcal{L} \rightarrow M$ (prequantum bundle), equipped with a Hermitian metric
and compatible connection $\nabla$ with curvature $\Omega_{\nabla} = -2\pi i \omega$ - and Lagrangian submanifold $\Lambda$ of $M$, the symplectic 2-form $\omega$ vanishes upon restriction to $\Lambda$ by definition, and any (local) symplectic potential $\vartheta$ (i.e. a 1-form such that $d\vartheta = \omega$) becomes a closed form thereon, giving a (local) connection form pertaining to the restriction of the prequantum connection $\nabla$, denoted by the same symbol. The latter is a flat connection, and a global covariantly constant section of the (restriction of) the prequantum line bundle exists if and only if it has trivial holonomy, that is, otherwise stated, the induced character $\chi : \pi_1(\Lambda) \to U(1)$ (with a base point tacitly understood) is trivial (see e.g. [61]). In our context the assumptions of the Weil-Kostant theorem are fulfilled and a covariantly constant section (also called WKB wave function) is just a locally constant function on $Y$ of the form

$$e^{2\pi i \lambda H(L)}$$

i.e. a (regular isotopy, i.e. up to the first Reidemeister move) link invariant, in adherence to the theory of Vassiliev [62, 34] (cf. [5] and also [38, 39] and below for further comments). The generic value taken by $\lambda$ (in particular, it can be taken to be rational, namely a root of $\pm 1$) avoids trivialities.

One must then accommodate passage through a “caustic”, i.e. through the Maslov cycle $Z$, given in our case by the (mildly) singular links possessing exactly one singular point causing a sudden jump of writhe (helicity) (see again [5]), and, what is crucial in the link context, we must take into due account the fact that removal of a crossing changes the number of components of a given link and thus places the new link in a different connected component of the space $Y$.

Explicitly, denote, as usual, by $L_+, L_-$ and $L_0$ three links (regularly projected onto a plane, $z = 0$, say) differing at a single crossing ((1)-crossing, no crossing, respectively), see Figure 3. Then, inspired by the Liu-Ricca (LR) approach ([38, 39]), introduce the “figures of eight” $E_{\pm}$, that is trivial knots with (1)-writhe: $H(E_{\pm}) = 1$. Starting, for instance, from $L_0$, one can “add” $E_+$ to the two coherently oriented parallel strands of $L_0$ in such a way that $E_+$ comes with the opposite orientation: a partial cancellation occurs and the net result is $L_+$. Conversely, proceeding backwards we can, by adding appropriately an $E_-$, produce $L_0$ from $L_+$ and so on. Therefore, addition of $E_{\pm}$ allows one to pass from one local configuration to the other, see Figure 4.

Now set:

$$\alpha := e^{2\pi i \lambda H(E_+)} = e^{2\pi i \lambda}, \quad \alpha^{-1} = e^{-2\pi i \lambda} = e^{2\pi i \lambda H(E_-)}$$

This is the local contribution to the semiclassical (WKB) wave function upon addition of an eight figure (or “curl”), which can be applied to a single branch (first Reidemeister move) as well.

We now wish to assemble these local objects so as to produce a genuine link invariant. Precisely, let $\Psi$ be a covariantly constant wave function stemming from application of the GQ-procedure, normalised in such a way that $\Psi(\bigcirc) = 1$ ($\bigcirc$ being the unknot): $\Psi$ can be made to depend naturally on two parameters, the above $\alpha$ and $z$, below. Notice in fact that the removal of a positive (negative)
crossing via addition of a negative (positive) writhe produces, via superposition, a wave function which we require, when evaluated at $L_0$, to match $\Psi(L_0)$, up to a (universal) constant $z$. Therefore, $\Psi$ satisfies the skein relation (and normalization) for the HOMFLYPT polynomial $P$ ([13 49] - here $\alpha^{-1}$ is LR’s $\alpha$)

$$\alpha^{-1}P_+ - \alpha P_- = zP_0; \quad P(\bigcirc) = 1$$

and the latter acquires, in turn, a vivid quantum mechanical significance.
The skein relation can be equivalently written in the form

\[ P_- = \alpha^{-2} P_+ - z \alpha^{-1} P_0 \]

which tells us that \( P_- \) can be obtained by superposing \( P_+ \), corrected by a Maslov type transition (local surgery via \( \alpha^{-2} \) - one has the same number of link components) and \( P_0 \), corrected by a “component transition” \( \alpha^{-1} \) (and multiplied by an extra coefficient \( z \)). The latter contribution was absent in [5] since that paper dealt with knots only. Notice that upon setting \( z = \alpha^{-1} - \alpha \) and letting \( \alpha \to 1 \), we get the trivial invariant \( \Psi \equiv 1 \).

**Remarks.** 1. In this way we essentially recover the hydrodynamical portrait of Liu and Ricca [38, 39], essentially stating that “\( P = t^H \)” albeit with a different (and more conceptual) interpretation. In particular, the two parameters used in HOMFLYPT are not quite the same. The local surgery operation involves helicity, as in LR, but we portray the latter as a local phase function, governing a component transition (or Maslov, upon squaring it), as in [5].

2. Passage from \( L_\pm \) to \( L_0 \) (and conversely) in \( \hat{Y} \) - abutting, as already remarked, at a change in the number of the link components - involves coalescence of two crossings into one and corresponding tangent alignment. This is a sort of “higher order” contribution beyond the Maslov one.

3. The CS form can be interpreted, in adherence to [58], as a connection 3-form for a 2-gerbe, having zero curvature. The wave function \( \exp 2\pi i \lambda H(L) \) then becomes the “parallel transport” of this connection “along” \( \mathbb{R}^3 \).

The upshot of the discussion carried out in this section is the following:

**Theorem 5.1.** The HOMFLYPT polynomial \( P = P(\alpha, z) \) can be recovered from the geometric quantization procedure applied to the Brylinski manifold \( Y \) and to its Lagrangian subspace \( \Lambda \), namely, it coincides (after normalization) with a suitable covariantly constant section \( \Psi = \Psi(\alpha, z) \) thereby obtained. The coefficient \( \alpha \) of \( P \) is a phase factor related to the helicity of a standard “eight-figure” and \( z \) comes from the possibility of varying the number of components of a link.

**Remark.** Our approach can be compared with the Jeffrey-Weitsman one ([26, 27]), providing a rigorous framework for the Jones-Witten theory, [66, 32]. The latter, though again based on geometric quantization, is much more sophisticated. In our setting, no reference to Lie groups is made and, as in LR, everything is based on helicity only, at the cost of relying on the Maslov-Hörmander approach of [5], together with an appropriate semiclassical interpretation of the skein relation. This leads directly to the HOMFLYPT (hence, in particular, to the Jones) polynomial.
6 A multisymplectic interpretation of Massey products

In this section we resume the techniques developed in Sections 2 and 3 above and propose a reformulation of the so-called higher order linking numbers in multisymplectic terms. Ordinary and higher order linking numbers provide, among others, a quite useful tool for the investigation of Brunnian phenomena in knot theory: recall that a link is *almost trivial* or *Brunnian* if upon removing any component therefrom one gets a trivial link. They can be defined recursively in terms of Massey products, or equivalently, Milnor invariants, by the celebrated Turaev-Porter theorem (see [13, 48, 57, 24]). We are going to review, briefly and quite concretely, the basic steps of the Massey procedure, read differentially geometrically as in [48, 57, 24], presenting at the same time our novel multisymplectic interpretation thereof.

Let \( L \) be an oriented link with three or more components \( L_j \). The cohomological reinterpretation of the ordinary linking number \( \ell(1, 2) \) of two components \( L_1 \) and \( L_2 \), say, starts from consideration of the closed 2-form

\[
\Omega_{12} := v_1 \wedge v_2
\]

yielding the (integral) de Rham class

\[
\langle L_1, L_2 \rangle := [\Omega_{12}] \in H^2(S^3 \setminus L)
\]

The linking number \( \ell(1, 2) \) is non zero precisely when \( \langle L_1, L_2 \rangle \), which, in \( H_1(S^3, L) \) equals \( \ell(1, 2)[\gamma_{12}] \), is non trivial. If the latter class vanishes (i.e. \( \Omega_{12} \) is exact), we have

\[
dv_{12} + v_1 \wedge v_2 = dv_{12} + \Omega_{12} = 0
\]  

(*)

for some 1-form \( v_{12} \). Now, assuming that all the ordinary mutual linking numbers of the components under consideration vanish, one can manufacture the (closed, direct check) 2-form (Massey product)

\[
\Omega_{123} = v_1 \wedge v_3 + v_{12} \wedge v_3
\]

yielding a *third order linking number* (as a class):

\[
\langle L_1, L_2, L_3 \rangle := [\Omega_{123}] \in H^2(S^3 \setminus L)
\]

If the latter class vanishes, we find a 1-form \( v_{123} \) such that

\[
dv_{123} + v_1 \wedge v_2 + v_{12} \wedge v_3 = dv_{123} + \Omega_{123} = 0
\]  

(♣♣♣)

It is then easy to devise a general pattern, giving rise to forms \( v_I, \Omega_I \) (\( I \) being a general multiindex). Actually, everything can be organised - via Chen’s calculus of iterated path integrals [8, 9] - in terms of sequences of nilpotent connections \( v^{(k)} \), \( k = 1, 2 \ldots \) on a trivial vector bundle over \( S^3 \setminus L \) and their attached curvature
forms $w^{(k)}$ (ultimately, the $\Omega_I$, [48 57 60 23]), everything stemming from the Cartan structure equation

$$d\nu^{(k)} + \nu^{(k)} \wedge v^{(k)} = w^{(k)}$$

together with the ensuing Bianchi identity

$$dw^{(k)} + v^{(k)} \wedge w^{(k)} - w^{(k)} \wedge v^{(k)} = 0$$

(the latter implying closure of the forms $\Omega_I$). In order to give a flavour of the general argument, start from the nilpotent connection $v^{(1)}$ with its corresponding curvature $w^{(1)}$:

$$v^{(1)} = \begin{pmatrix} 0 & v_1 & 0 & 0 \\ 0 & 0 & v_2 & 0 \\ 0 & 0 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad w^{(1)} = \begin{pmatrix} 0 & 0 & \Omega_{12} = v_1 \wedge v_2 & 0 \\ 0 & 0 & 0 & \Omega_{23} = v_2 \wedge v_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Then proceed similarly with

$$v^{(2)} = \begin{pmatrix} 0 & v_1 & v_{12} & 0 \\ 0 & 0 & v_2 & v_{23} \\ 0 & 0 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad w^{(2)} = \begin{pmatrix} 0 & 0 & 0 & \Omega_{123} = v_1 \wedge v_{23} + v_{12} \wedge v_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(we made use of $dv_{12} + \Omega_{12} = dv_{23} + \Omega_{23} = 0$), and so on.

Also recall that all forms $\Omega_I$ can be neatly interpreted, via Poincaré duality, as auxiliary (trivial) knots $L_I$, and $v_I$ as discs bounding them, in adherence to the considerations in Section 3, see [48 57] for more details and worked out examples, including the Whitehead link (involving fourth order linking numbers - with repeated indices) and the Borromean rings (exhibiting a third order linking number). Just notice here that, for instance, formula (∗) becomes, intersection theoretically

$$\partial a_{12} + a_1 \cap a_2 = 0,$$

see Figure 5. Formula (♣♣) can be rewritten as

$$dv_{123} + \iota_{\xi_{123}} \nu = 0$$

Figure 5: Starting the Chen procedure
where $\xi \equiv \xi_{123} = \alpha^{-1}(\Omega_{123})$. The above ("vorticity") vector field $\xi_{123}$ can be thought of as being concentrated on the knot corresponding to $\xi_{123}$, or, alternatively, in a thin tube around it, when considering a bona fide Poincaré dual, cf. (♣).

This tells us that $v_{123}$ is a Hamiltonian 1-form in the sense of [55] and the formula

$$L_\xi \Omega_{123} = d\iota_\xi \Omega_{123} + \iota_\xi d\Omega_{123} = d\iota_\xi \Omega_{123}$$

expresses the fact that $\Omega_{123}$ is a globally conserved 2-form, and the same holds for $\Omega_{12}$ and, in general, for $\Omega_I$, with their corresponding vector fields $\xi_I$. Specifically, we have the following:

**Proposition 6.1.** (i) The volume form $\nu$ and all Massey 2-forms are globally conserved.

(ii) The 1-forms $v_I = f_1(\xi_I)$ are Hamiltonian with respect to the volume form.

**Proof.** Ad (i). This is clear since the mentioned forms are closed.

Ad (ii). The previous discussion can be carried out verbatim for a general multiindex $I$:

$$dv_I + \iota_\xi dv_I = 0$$

(an extension of (♣)), this yielding the second conclusion.

The following is the main result of this section.

**Theorem 6.2.** With the above notation:

The 1-forms $v_I$ are first integrals in involution with respect to the flow generated by the Hamiltonian vector field $\xi_L$, namely

$$L_{\xi_L} v_I = 0$$

(i.e. the $v_I$’s are strictly conserved) and

$$\{v_I, v_J\} = 0$$

(for multiindices $I$ and $J$).

**Proof.** Using Cartan’s formula, we get

$$L_{\xi_L} v_I = d\iota_{\xi_L} v_I + \iota_{\xi_L} dv_I = d\iota_{\xi_L} v_I - \iota_{\xi_L} \iota_\xi \nu,$$

but the second summand vanishes in view of the general expression

$$\{v_\xi, v_\eta\}(\cdot) = \nu(\xi, \eta, \cdot)$$

and of the peculiar structure of the vector fields involved (they either partially coincide or have disjoint supports). By the same argument, one gets $\iota_{\xi_L} v_I = 0$, in view of the Poincaré dual interpretation of $v_I$ (cf. Section 3), together with the second assertion; a crucial point to notice is that the auxiliary links obtained
via Chen’s procedure may be suitably split from their ascendants, this leading to
\[ \iota_{\xi L} v_I = 0 \]
The consequent strict conservation of the \( v_I \)'s is then immediate.

Notice that, in particular, from
\[ \iota_{\xi L} v_L = 0 \]
(Poincaré dual interpretation again) we also get
\[ \mathcal{L}_{\xi L} v_L = 0 \]
(this is not to be expected a priori in multisymplectic geometry, cf. [55]).

We ought to remark that, upon altering the \( v_I \)'s by an exact form, we may lose strict conservation, but in any case global conservation is assured (the PB is an exact form, by (♠) in Section 2 and in view of commutativity of the vector fields \( \xi_I \) and \( \xi_J \)).

Ultimately, we can draw the conclusion that the Massey invariant route to ascertain the Brunnian character of a link can be mechanically understood as a recursive test of a kind of knot theoretic integrability: the Massey linking numbers provide obstructions to the latter.

Thus, somewhat curiously, higher order linking phenomena receive an interpretation in terms of multisymplectic geometry, which is a sort of higher order symplectic geometry. Also, integrability comes in with a twofold meaning: first, higher order linking numbers emerge from the construction of a sequence of flat, i.e. integrable nilpotent connections; second, this very process yields first integrals in involution in a mechanical sense.

7 Conclusions and outlook

In this note we applied techniques from symplectic and multisymplectic geometry, with a strong hydrodynamical flavour, together with geometric quantization, to the study of specific link invariants, such as the the Massey higher order linking numbers and the HOMFLYPT polynomial - motivated by the ingenious Liu-Ricca approach to the latter - leading to a possibly vivid and clearcut mechanical interpretation thereof (classical and quantum, respectively). We have also exhibited a covariant phase space interpretation of the geometrical framework of the Euler equation for perfect fluids. The multisymplectic approach appears to be very promising for further advancement in this area. Also, the notion of integrability cropping up in our analysis of Massey products may deserve further scrutiny in a general multisymplectic context. Finally, we also noticed that some of our constructions actually make sense in wider contexts, this possibly calling for deeper elucidation.

Acknowledgements. The authors, both members of the GNSAGA group of INDAM, acknowledge support from Unicatt local D1-funds (ex MIUR 60% funds). They are also grateful to Marcello Spera for help with graphics.
References

[1] Abraham R. and Marsden J., *Foundations of Mechanics*, Benjamin/Cummings, Reading, MA, 1978.

[2] Arnołd V.I., Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits, *Ann. Inst. Fourier* (Grenoble) 16 (1966) fasc. 1, 319-361.

[3] Arnołd V.I., *Mathematical Methods of Classical Mechanics*, Springer, New York, 1989.

[4] Arnołd V.I. and Khesin B., *Topological Methods in Hydrodynamics*, Springer, Berlin, 1998.

[5] Besana A. and Spera M., On some symplectic aspects of knots framings, *J. Knot Theory Ram.* 15 (2006), 883-912.

[6] Bott R. and Tu L., *Differential Forms in Algebraic Topology*, Springer, Berlin, 1982.

[7] Brylinski J.-L., *Loop Spaces, Characteristic Classes and Geometric Quantization*, Modern Birkhäuser Classics, Basel, 1993.

[8] Chen K.-T., Iterated path integrals, *Bull.Am.Math.Soc.* 83 (1977), 831-879.

[9] Chen K.-T., *Collected Papers of K.-T. Chen* (eds P. Tondeur and R. Hain), Contemporary Mathematicians, Birkhäuser, Boston, MA, 2001.

[10] Crnković Ć., Symplectic geometry of the covariant phase space, *Classical and Quantum Gravity* 5 (1988), 1557-1575.

[11] de Rham G., *Variétés différentiables*, Hermann, Paris, 1954.

[12] Ebin D. and Marsden J., Groups of diffeomorphisms and the motion of incompressible fluids, *Ann.Math.* 92 (1970), 102-163.

[13] Fenn R.A., *Techniques of geometric topology*, London Mathematical Society, Lecture Notes Series 57, Cambridge University Press, Cambridge, 1983.

[14] Forger M. and Vieira Romero S., Covariant Poisson brackets in geometric field theory, *Commun.Math.Phys.* 256 (2005), 375-410.

[15] Freyd P., Yetter D., Hoste J., Lickorish W.B.R., Millet K. and Ocneanu A., A new polynomial invariant of knots and links, *Bull.Am.Math.Soc.* 12 (1985), 239-246.

[16] Goldin G., Non-relativistic current algebras as unitary representations of groups, *J. Math. Phys.* 12 (1971), 462-488.
[17] Goldin G., Diffeomorphism groups and nonlinear quantum mechanics, J.Phys.: Conference Series **343** (2012), 012006.

[18] Gotay M.J., Isenberg J., Marsden J.E. and Montgomery R., Momentum Maps and Classical Fields Part I: Covariant Field Theory [arxiv:physics/9801019v2 [math-ph], 1998, Part II: arxiv: math-ph/0411036 [math-ph], 2004,

[19] G. Goldin, Parastatistics, θ-statistics, and Topological Quantum Mechanics from Unitary Representations of Diffeomorphism Groups, *Proceedings of the XV International Conference on Differential Geometric Methods in Physics*, H.D Doebner and J.D.Henning (eds) (World Scientific, Singapore, 1987), 197-207.

[20] Guillemin V., Ginzburg V. and Kashon Y., *Moment maps, cobordisms and Hamiltonian group actions*, Mathematical Surveys and Monographs 98, AMS, Providence, RI, 2002.

[21] Guillemin V. and Sternberg S., *Geometric Asymptotics* AMS, Providence, RI, 1977.

[22] Guillemin V. and Sternberg S., *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge, 1984.

[23] Hain R., The Geometry of the Mixed Hodge Structure on the Fundamental Group *Proc.Symp.Pure Math.* **46** (1987), 247-282.

[24] Hebda J.J. and Tsau C.M., An approach to higher order linking invariants through holonomy and curvature, *Trans.Amer.Math.Soc.* **364** (2012), 4283-4301.

[25] Hörmander L., Fourier Integral operators I, *Acta Math.* **127** (1971), 79-183.

[26] Jeffrey L. and Weitsman J., Bohr-Sommerfeld Orbits in the Moduli Space of Flat Connections and the Verlinde dimension Formula, *Comm.Math.Phys.* **58** (1992), 593-630.

[27] Jeffrey L. and Weitsman J., Half-density quantization of the moduli space of flat connections and Witten’s semiclassical invariants, *Topology* **32** (1993), 509-529.

[28] Kauffman L.H., *Knots and Physics*, 3rd Edition, World Scientific, Singapore, 2001.

[29] Kijowski J. and Szczyrba V., A canonical structure for classical field theories, *Commun.Math.Phys* **46** (1976), 183-206.

[30] Khesin B., The vortex filament equation in any dimension, *Topological Fluid Dynamics: Theory and Applications*, *Procedia IUTAM* **7** (2013) 135-140.
[31] Kirillov A., Geometric Quantization, *Dynamical Systems IV*, 139–176, Encyclopaedia Math. Sci. 4, Springer, Berlin, 2001.

[32] Kohno T., *Conformal Field Theory and Topology* AMS, Providence, RI, 2002.

[33] Kolev B., Poisson brackets in hydrodynamics, *Discrete and continuous dynamical systems* 19 n.3 (2007), 555-574.

[34] Kontsevich M., Vassiliev’s knot invariants *Adv.Sov.Math.* 16, Part 2 (1993), 137-150, AMS, Providence, RI.

[35] Kostant B., Quantization and unitary representations, in: *Lectures in Modern Analysis and Applications*, Lecture Notes in Mathematics, Vol. 170, Springer, Berlin, 1970, pp. 87-208.

[36] Kriegl A. and Michor P.W., *The convenient setting of global analysis*, volume 53 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.

[37] Kuznetsov E.A. and Mikhailov A.V., On the topological meaning of canonical Clebsch variables, *Phys. Lett A* 77 (1980), 37-38.

[38] Liu X. and Ricca R.L., The Jones polynomial for fluid knots from helicity, *J. Phys A: Math. Theor.* 45 (2012), 205501 (14pp).

[39] Liu X. and Ricca R.L., On the derivation of the HOMFLYPT polynomial invariant for fluid knots, *J. Fluid Mechanics* 773 (2015), 34-48.

[40] Madsen T.B. and A. Swann A., Closed forms and multi-moment maps. *Geom. Dedicata* 165 1, (2013), 25-52.

[41] Marsden J. and Weinstein A., Coadjoint orbits, vortices, Clebsch variables for incompressible fluids, *Physica* 7 D (1983), 305-323.

[42] McDuff D. and Salamon D., *Introduction to Symplectic Topology*, Clarendon Press, Oxford, 1998.

[43] Maslov V.P., *Théorie des Perturbations et Méthodes Asymptotiques* Ed. de l’Université de Moscou, 1965 (en russe). Traduction française: Dunod - Gauthier-Villars, Paris, 1971.

[44] Moffatt H.K. and Ricca R.L., Helicity and the Călugăreanu invariant *Proc.R.Soc.Lond. A* 439 (1992), 411-429.

[45] Penna V. and Spera M., A geometric approach to quantum vortices *J.Math.Phys.* 30 (1989), 2778-2784.

[46] Penna V. and Spera M., On coadjoint orbits of rotational perfect fluids *J.Math.Phys.* 33 (1992), 901-909.
[47] Penna V. and Spera M., String limit of vortex current algebra Phys.Rev.B 62 (2000), 14547-14553.

[48] Penna V. and Spera M., Higher order linking numbers, curvature and holonomy, J.Knot Theory Ram. 11 (2002), 701-723.

[49] Przytycki J.H. and Traczyk P., Conway algebras and skein equivalence for links, Proc.Am.Math.Soc. 100, 740-748.

[50] Rasetti M. and Regge T., Vortices in He-II, current algebras and quantum knots, Physica A 80 (1975), 217-233.

[51] Ricca R.L. and Nipoti B., Gauss' linking number revisited J. Knot Theory and Its Ram. 20 (2011), 1325-1343.

[52] Rolfsen D., Knots and Links Publish or Perish, Berkeley, 1976.

[53] Ryvkin L. and Wurzbacher T., Existence and unicity of co-moments in multisymplectic geometry, Differential geometry and its applications 41 (2015), 1-11.

[54] Ryvkin L. and Wurzbacher T., An invitation to multisymplectic geometry arXiv:1804.02553v1 [math.DG] 7 Apr. 2018

[55] Ryvkin L., Wurzbacher T. and Zambon M., Conserved quantities on multisymplectic manifolds, arXiv: 1610.05592 v1. [math.SG] 18 Oct. 2016

[56] Souriau, J-M., Structure des systèmes dynamiques, Dunod, Paris, 1970.

[57] Spera M., A survey on the differential and symplectic geometry of linking numbers, Milan J.Math. 74 (2006), 139-197.

[58] Spera M., A note on n-gerbes and transgressions, Portugaliae Mathematica 68 (2011), 381-387.

[59] Spera M., Moment map and gauge geometric aspects of the Schrödinger and Pauli equations, Int.J.Geom.Meth.Mod.Phys. 13 (4) (2016), 1630004 (1-36).

[60] Tavares J.N., Chen Integrals, Generalized Loops and Loop Calculus, Int.J.Mod.Phys.A 9 (1994), 4511-4548.

[61] Tyurin A., Quantization, Classical and Quantum Field Theory and Theta Functions, CRM Monograph Series 21, AMS, Providence, RI, 2003.

[62] Vassiliev V. Complements of Discriminants of smooth maps: Topology and Applications, AMS, Providence, 1992.

[63] Vladimirov V. Distributions en Physique Mathématique (French translation) Éd.MIR, Moscou, 1974.
[64] Warner F. *Foundations of Differentiable Manifolds and Lie Groups*, GTM 94, Springer, Berlin, Heidelberg, 1983.

[65] Weinstein A., Symplectic manifolds and their Lagrangian submanifolds *Adv.Math.* 6 (1971), 329-346.

[66] Witten E., Quantum field theory and the Jones polynomial, *Commun.Math.Phys.* 121 (1989), 351-399.

[67] Woodhouse N., *Geometric Quantization*, Oxford University Press, Oxford, 1992.

[68] Zuckerman G.J., Action Principles and Global Geometry *Proc. Conference on Mathematical Aspects of String Theory*, 21 July - 2 Aug 1986. San Diego, California, S.T. Yau (Ed.), 259-284, World Scientific, Singapore, 1987.