UNIQUENESS OF CONSERVATIVE SOLUTIONS TO THE GENERALIZED CAMASSA-HOLM EQUATION VIA CHARACTERISTICS

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Abstract. It was showed that the generalized Camassa-Holm equation possible development of singularities in finite time, and beyond the occurrence of wave breaking which exists either global conservative or dissipative solutions. In present paper, we will further investigate the uniqueness of global conservative solutions to it based on the characteristics. From a given conservative solution \( u = u(t,x) \), an equation is introduced to single out a unique characteristic curve through each initial point. By analyzing the evolution of the quantities \( u \) and \( v = 2 \arctan u \), along each characteristic, it is obtained that the Cauchy problem with general initial data \( u_0 \in H^1(\mathbb{R}) \) has a unique global conservative solution.

1. Introduction. In this paper, we consider the generalized Camassa-Holm equation,

\[
\begin{align*}
\left\{ \begin{array}{ll}
    u_t - u_{xxt} + [g(u)]_x = \gamma (2u_x u_{xx} + uu_{xxx}), & x \in \mathbb{R}, t > 0, \\
    u(x,0) = u_0(x), & x \in \mathbb{R},
\end{array} \right.
\end{align*}
\]

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where \( g : \mathbb{R} \rightarrow \mathbb{R} \) is a given \( W^{1,\infty}_{loc}(\mathbb{R}) \)-function with \( g(0) = 0 \), and \( \gamma \) is a constant. Peakons in a particular case of equation (1) are studied in [26] and their stability is discussed in [24]. Yin established the well-posedness, global solutions for (1) in [33]. Later, Coclite, Holden and Karlsen [6] established existence of a strongly continuous semigroup of global weak solutions for equation (1), and also presented a “weak equals strong” uniqueness result. Moreover, wave breaking is the only way singularities can develop in finite time for (1) (cf.[34]), beyond the occurrence of wave breaking which exists global conservative solutions [37].

The early motivation to investigate the equation (1) is that it can be regarded as a generalization of the well-known Camassa-Holm equation (equation (1) with \( \gamma = 1, g(u) = \frac{3}{2}u^2 + ku \))

\[
\frac{u_t}{u} + 2ku + 3uu_x - u_{xxt} - 2uu_x - uu_{xxx} = 0. \tag{2}
\]

The equation (2) was first implicitly included in a bi-Hamiltonian generalization of KdV equation by Fokas and Fuchssteiner [20], and later derived as a model for unidirectional propagation of shallow water over a flat bottom by Camassa and Holm [4]. Analogous to the famous KdV equation, the Camassa-Holm equation (2) also has a bi-Hamiltonian structure [4, 20], which is completely integrable not only in the sense of the existence of a Lax pair [4], but also (by means of inverse scattering and inverse spectral theory) as an infinite-dimensional Hamiltonian flow that can be linearised in suitable action-angle variables, cf. the considerations in the papers [7, 8, 12, 13, 18]. The orbital stability of solitary waves and the stability of the peakons \( (k = 0) \) for the Camassa-Holm equation are investigated by Constantin and Strauss [14, 15]. The advantage of the CH equation in comparison with the famous KdV equation lies in the fact that the Camassa-Holm models the peculiar wave breaking phenomena, that is, the solution remains bounded but its slope becomes unbounded in finite time (cf.[5, 11]). In addition to wave breaking, one of the most interesting aspects of the equation is the existence of peakon solutions, namely, the travelling wave solutions of greatest height of the governing equations for water waves have a peak at their crest (cf.[9, 10, 36]).

Over the last three decades, there is a wide variety of patterns associated Camassa-Holm equation have been studied extensively. Such as, the hyperelasticrod wave equation (equation (1) with \( g(u) = \frac{3}{2}u^2 \) and \( \gamma \in \mathbb{R} \))

\[
\frac{u_t}{u} - u_{xxt} + 3uu_x - \gamma(2u_xu_{xx} + uu_{xxx}) = 0. \tag{3}
\]

The model (3) describes far-field, finite length, finite amplitude radial deformation waves in cylindrical compressible hyperelastic rods. The other special cases when \( g(u) = \frac{3}{2}u^2 + ku, \gamma \in \mathbb{R} \), can be found in [16, 35]. The travelling waves solution, peaked solitary wave solutions for equation (1) with \( g(u) = au^m + ku \) and some \( \alpha, m, \gamma \) were showed in [28, 27] and references therein. More related work in this fields can be found in [23, 21, 19, 22, 29, 30, 31, 32, 25].

In view of the possible development of singularities in finite time, it is natural to curious about the behavior of a solution beyond the occurrence of wave breaking. Bressan and Constantin showed that after wave breaking the solution of the solution of the Camassa-Holm equation can be continued uniquely as either global conservative or global dissipative solutions(cf.[1, 2]). Recently, the uniqueness of conservation solution to the Camassa-Holm equation was direct proved in [3]. They develop a direct method to the uniqueness of conservative solutions for the Camassa-Holm equation by using characteristics, which extended the work of generalized characteristics [17], or so-called energy variable to the Camassa-Holm
equation to separate different characteristics after singularity occurs. We known that the conservative solutions for equation (1) can be global existence (cf.\[37\]). The aim of this paper is to further prove that, the uniqueness of the global conservative weak solutions of equation (1) by suitable modifying recent result in \[3\] for the Camassa-Holm equation.

This paper is organized as follows. In Section 2, we recall basic notation and present our main theorem. In Section 3, the uniqueness of characteristic curve through each initial point is established. In Section 4, we investigate the gradient $u_x$ of a conservative solution varies a characteristic and obtain the proof of the main result.

2. Basic definitions and results. Let $G(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbb{R}$. Then $(1-\partial_x^2)^{-1}f = G * f$ for all $f \in L^2(\mathbb{R})$ and $G * m = u$. The system (1) can be rewritten as the following form

$$\begin{cases}
u_t + \gamma u_x = -\partial_x G * [g(u) + \frac{\gamma}{2}u_x^2 - \frac{\gamma}{2}u^2], & x \in \mathbb{R}, t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R},
\end{cases}$$

(4)

with initial data $u_0(x) \in H^1(\mathbb{R})$. For convenience, we set

$$P = G * [g(u) + \frac{\gamma}{2}u_x^2 - \frac{\gamma}{2}u^2].$$

If $u \in H^1(\mathbb{R})$, Young’s inequality ensure that

$$P = G * [g(u) + \frac{\gamma}{2}u_x^2 - \frac{\gamma}{2}u^2] \in H^1(\mathbb{R}).$$

Indeed,

$$\|P(t, x)\|_{L^2(\mathbb{R})} \leq C\|g(u)\|_{L^2(\mathbb{R})} + \left\|\frac{1}{2}e^{-|x|} * \left(\frac{\gamma}{2}u_x^2 - \frac{\gamma}{2}u^2\right)\right\|_{L^2(\mathbb{R})} \leq C\|g(u)\|_{L^2(\mathbb{R})} + \frac{\gamma}{2}\frac{1}{2}e^{-|x|}\|u_x^2 + u^2\|_{L^1(\mathbb{R})} \leq C\|g(u)\|_{L^2(\mathbb{R})} + \frac{\gamma}{2}\|u\|_{H^1(\mathbb{R})}^2.$$

Due to the Sobolev’s equality $\|u\|_{L^\infty(\mathbb{R})} \leq \|u\|_{H^1(\mathbb{R})}$ and $g \in W_{loc}^2(\mathbb{R})$ (g(0) = 0), we deduce that

$$|g(u(x))| \leq \sup_{|y| \leq \|u\|_{L^\infty(\mathbb{R})}} |g'(y)||u(x)| \leq C\|u\|_{H^1(\mathbb{R})}|u(x)|.$$

Thus, $P_x \in L^2(\mathbb{R})$. Analogously, we can obtain $P_x \in L^2(\mathbb{R})$, that is, $P_x \in L^2(\mathbb{R})$.

Definition 2.1. We called $u(t, x)$ is a solution of the Cauchy problem (4) on $(0, T)$, if a Hölder continuous function $u(t, x)$ defined on $(0, T) \times \mathbb{R}$ with the following properties:

(i) At each fixed $t$ we have $u(\cdot, t) \in H^1(\mathbb{R})$.

(ii) the map $t \mapsto u(\cdot, t)$ is Lipschitz continuous from $(0, T]$ to $L^2(\mathbb{R})$, satisfying the initial condition $u_0(x) \in H^1(\mathbb{R})$ together with

$$u_t + \gamma u_x = -P_x$$

(5)

for a.e. t. Here (5) is understood as an equality between functions in $L^2(\mathbb{R})$. 
For smooth solutions, we claim that the total energy
\[ E(t) = \int_R u^2 + u_x^2 \, dx \]  
(6)
is constant in time. In fact, by using \( \partial^2_x G \ast f = G \ast f - f \) and differentiating the equation (4) with respect to \( x \), we have
\[ u_{xt} + \gamma uu_{xx} + \gamma u_x^2 = g(u) + \frac{\gamma}{2} u_x^2 - \frac{\gamma}{2} u^2 - P. \]  
(7)

Multiplying the first equation (4) by \( u \), and the equation (7) by \( u_x \), we can get the following equations
\[ \frac{(u^2)}{2}_t + \left( \frac{\gamma u^3}{3} \right)_x + uP_x = 0, \]  
(8)
\[ \frac{(u_x^2)}{2}_t + \gamma \frac{(u u_x^2)}{2} = (g(u) - \frac{\gamma}{2} u^2 - P) u_x. \]  
(9)

(8)-(9) imply that
\[ \frac{d}{dt} E(t) = \frac{d}{dt} \int_R (u^2 + u_x^2) \, dx = 0. \]  
(10)

Therefore, the conservation law given by
\[ \| u \|_{H^1} = \left( \int_R (u^2 + u_x^2) \, dx \right)^{\frac{1}{2}}. \]  
(11)

Next, we present the definition of the conservative solutions for the equation (4).

**Definition 2.2.** A solution \( u = u(t, x) \) is conservative if \( w = u_x^2 \) provides a distributional solution to the following balance law
\[ w_t + (\gamma uw)_x = 2(g(u) - \frac{\gamma}{2} u^2 - P) u_x, \]  
(12)
namely,
\[ \int_0^\infty \int w_\varphi_t + \gamma uw_\varphi_x + 2(g(u) - \frac{\gamma}{2} u^2 - P) u_x \, dx dt + \int \varphi_0 (0, x) dx = 0 \]  
(13)
for every test function \( \varphi \in C^1_c(\mathbb{R}) \).

In [37], we showed the existence of global conservation for (1) as follows.

**Theorem 2.3.** Let \( u_0 \in H^1(\mathbb{R}) \), the generalized Camassa-Holm equation (1) has a global conservative solution \( u = u(t, x) \). If there exists a family of Radon measure \( \{ \mu(t) ; t \in \mathbb{R} \} \), depending continuously on time w.r.t. the topology of weak convergence of measures, such that the following properties hold.

(i) The function \( u \) provides a solution to the Cauchy problem (4) in the sense of Definition 2.1.

(ii) There exists a null set \( \mathcal{N} \subset \mathbb{R} \) with meass\( \mathcal{N} = 0 \) such that for \( t \notin \mathcal{N} \) the measure \( \mu(t) \) is absolutely continuous and has density \( u^2_x(t, \cdot) \) w.r.t. Lebesgue measure.

(iii) The family \( \{ \mu(t) ; t \in \mathbb{R} \} \) provides a measure-valued solution \( w \) to the linear transport equation with source
\[ w_t + (\gamma uw)_x = 2u_x(g(u) - \frac{\gamma}{2} u^2 - P). \]  
(14)
The measure $\mu(t)$ has a nontrivial singular part for for a time $t \in \mathcal{N}$. Owing to the conservative solution $u$ is not smooth, generally, we only know that the energy $E$ in (6) coincides a.e. with a constant, that is,

$$E(t) = E(0) \quad \text{for } t \notin \mathcal{N}, \quad E(t) < E(0) \quad \text{for } t \in \mathcal{N}.$$  

The main purpose of this paper is to add the uniqueness in the above solution.

**Theorem 2.4.** Given any initial data $u_0 \in H^1(\mathbb{R})$, the Cauchy problem (4) has a unique conservative solution.

### 3. Uniqueness of characteristics

Set $u = u(t, x)$ be a solution of Cauchy problem (4) with the additional balance law (13). By introducing the new coordinates $(t, \beta)$, related to the original coordinates $(t, x)$ from the following integral relation

$$x(t, \beta) + \int_{-\infty}^{x(t, \beta)} u_2^2(t, \xi) d\xi = \beta.$$  

Here, we define $x(t, \beta)$ to be the unique point $x$ such that, for any time $t$ and $\beta \in \mathbb{R}$,

$$x(t, \beta) + \mu(t)(\{(-\infty, x)\} \leq \beta \leq x(t, \beta) + \mu(t)(\{(-\infty, x)\}).$$  

Note that at every time where $\mu(t)$ is absolutely continuous with density $u_2^2$ w.r.t. Lebesgue measure. Next, combing the following Lemma 3.1 with Lemma 3.3, we establish the Lipschitz continuity of $x$ and $u$ as functions of the variables $t, \beta$.

**Lemma 3.1.** Assume that $u = u(t, x)$ be a conservative solution of (4). Then the maps $\beta \mapsto x(t, \beta)$ and $\beta \mapsto u(t, \beta) \equiv u(t, x(t, \beta))$ implicitly defined by (15) are Lipschitz continuous with Lipschitz constant 1, for every $t \geq 0$. The map $t \mapsto x(t, \beta)$ is also Lipschitz continuous with a constant depending only on $\|u_0\|_{H^1}$.

**Proof of Lemma 3.1.** The proof of Lemma 3.1 consist of several steps.

**Step 1.** Fix any time $t \geq 0$. Then the map

$$x \mapsto \beta(t, x)$$

is right continuous and strictly increasing. Hence it has well defined, continuous, nondecreasing inverse $\beta \mapsto x(t, \beta)$. If $\beta_1 < \beta_2$, then

$$x(t, \beta_2) - x(t, \beta_1) + \mu(t)(\{(x(t, \beta_2), x(t, \beta_1))\}) \leq \beta_2 - \beta_1.$$  

This gives

$$x(t, \beta_2) - x(t, \beta_1) \leq \beta_2 - \beta_1,$$  

implying that the map $\beta \mapsto x(t, \beta)$ is a contraction.

**Step 2.** We claim that the map $\beta \mapsto u(t, \beta)$ is Lipschitz continuity as $\beta_1 < \beta_2$. Indeed, from (17), we get

$$|u(t, x(t, \beta_2)) - u(t, x(t, \beta_1))| \leq \int_{x(t, \beta_1)}^{x(t, \beta_2)} |u_2| dx$$

$$\leq \int_{x(t, \beta_1)}^{x(t, \beta_2)} \frac{1}{2} (1 + u_2^2) dy$$

$$\leq \frac{1}{2} \left[ x(t, \beta_2) - x(t, \beta_1) + \mu(t)(\{(x(t, \beta_2), x(t, \beta_1))\}) \right]$$

$$\leq \frac{1}{2} (\beta_2 - \beta_1).$$  

(19)
exists a unique Lipschitz continuous map satisfies both

\[
\|2u_x(g(u) - \frac{\gamma}{2} u^2 - P)\|_{L^1} \leq 2\left(\|g(u)\|_{L^2} + \frac{\gamma}{2} \|u\|_{L^\infty} \|u\|_{L^2} + \|P\|_{L^2}\right)\|u_x\|_{L^2} \leq C_s,
\]

where the constant $C_s$ depending only on the $H^1(\mathbb{R})$ of norm and $\gamma$. Therefore, we have

\[
\mu(t)\{(-\infty, y - C_s(t - \tau))\} \leq \mu(t,\tau)\{(-\infty, y)\} + C_s(t - \tau).
\]

Defining $y^{-}(t) \doteq y - (C_s + C_s)(t - \tau)$, we obtain

\[
y^{-}(t) + \mu(t)\{(-\infty, y^{-}(t))\} \leq y - (C_s + C_s)(t - \tau) + \mu(t)\{(-\infty, y)\} + C_s(t - \tau) \leq y + \mu(t)\{(-\infty, y)\} \leq \beta.
\]

This yields $x(t, \beta) \geq y^{-}(t)$ for all $t > \tau$. A similar argument implies

\[
x(t, \beta) \leq y^+(t) \doteq y + (C_s + C_s)(t - \tau),
\]

proving the uniform Lipschitz continuity of the map $t \mapsto x(t, \beta)$.

**Step 3.** We prove the Lipschitz continuity of the map $t \mapsto x(t, \beta)$. Assume $x(\tau, \beta) = y$. We note that the family of measure $\mu(t)$ satisfies the balance law (12), where for each $t$ drift $u$ and the source term $2u_x(g(u) - \frac{\gamma}{2} u^2 - P)$ satisfy

\[
\|2u_x(g(u) - \frac{\gamma}{2} u^2 - P)\|_{L^1} \leq 2\left(\|g(u)\|_{L^2} + \frac{\gamma}{2} \|u\|_{L^\infty} \|u\|_{L^2} + \|P\|_{L^2}\right)\|u_x\|_{L^2} \leq C_s,
\]

where the constant $C_s$ depending only on the $H^1(\mathbb{R})$ of norm and $\gamma$. Therefore, we have

\[
\mu(t)\{(-\infty, y - C_s(t - \tau))\} \leq \mu(t,\tau)\{(-\infty, y)\} + C_s(t - \tau).
\]

Defining $y^{-}(t) \doteq y - (C_s + C_s)(t - \tau)$, we obtain

\[
y^{-}(t) + \mu(t)\{(-\infty, y^{-}(t))\} \leq y - (C_s + C_s)(t - \tau) + \mu(t)\{(-\infty, y)\} + C_s(t - \tau) \leq y + \mu(t)\{(-\infty, y)\} \leq \beta.
\]

This yields $x(t, \beta) \geq y^{-}(t)$ for all $t > \tau$. A similar argument implies

\[
x(t, \beta) \leq y^+(t) \doteq y + (C_s + C_s)(t - \tau),
\]

proving the uniform Lipschitz continuity of the map $t \mapsto x(t, \beta)$.

The next result shows that characteristics can be uniquely determined by an integral equation combining the characteristic equation and balance law of $\mu(t)$.

**Lemma 3.2.** Assume that $u = u(t, x)$ be conservative solution of (4). Then, there exists a unique Lipschitz continuous map $t \mapsto x(t)$ for any $y \in \mathbb{R}$, which the map satisfies both

\[
\frac{d}{dt} x(t) = \gamma u(t, x), \quad x(0) = \tilde{y}
\]

and

\[
\frac{d}{dt} \int_{-\infty}^{x(t, \beta)} u_x^2 \, dx = \int_{-\infty}^{x(t, \beta)} 2(g(u) - \frac{\gamma}{2} u^2 - P)u_x \, dx.
\]

Additionally, for any $0 \leq \tau \leq t$ one has

\[
u(t, x(t)) - u(\tau, x(\tau)) = -\int_{t}^{\tau} P_x(s, x(s)) \, ds.
\]

**Proof.** **Step 1.** Taking advantage of the adapted coordinates $(t, \beta)$, we write the characteristic beginning with $y$ in the form $t \mapsto x(t) = (t, \beta(t))$, where $\beta(\cdot)$ is a map to be determined. Combing the two equation (21) and (22) integrating and w.r.t. time we obtain for $t \in \mathcal{N}$,

\[
x(t) + \int_{-\infty}^{x(t)} u_x^2(t, y) \, dy = \tilde{y} + \int_{-\infty}^{\tilde{y}} u_x^2 \, dy
\]

\[
+ \int_{0}^{t} \left(\gamma u(s, x(s)) + \int_{-\infty}^{x(s)} [2g(u)u_x - \gamma u_x^2 - 2Pu_x] \, dy \right) ds.
\]

Defining the function

\[
G(t, \beta) = \int_{-\infty}^{x(s)} [\gamma u_x + 2g(u)u_x - \gamma u_x^2 - 2Pu_x] \, dx
\]

(25)
and the constant
\[ \tilde{\beta} = \tilde{y} + \int_{-\infty}^{\tilde{y}} u_{0,x}(y)dy. \] (26)

Then we can rewrite the equation (24) in the following form:
\[ \beta(t) = \tilde{\beta} + \int_0^t G(s,\beta(s))ds, \text{ for all } t > 0. \] (27)

**Step 2.** For each fix \( t \geq 0 \), since the maps \( x \mapsto u(t,x) \) and \( x \mapsto P(t,x) \) are both in \( H^1(\mathbb{R}) \), the function \( \beta \mapsto G(t,\beta) \) defined in (25) is uniformly bounded and absolutely continuous. Furthermore, we have
\[ G_\beta = \left[ \gamma u_x + 2g(u)u_x - \gamma u^2 u_x - 2Pu_x \right] x_\beta \]
for some constant \( C \) depending only on the \( H^1 \) norm of \( u \). Hence the function \( G \) in (27) is uniformly Lipschitz continuous w.r.t. \( \beta \).

**Step 3.** By the Lipschitz continuity of the function \( G \), the existence of a unique solution to the integral (27) can be proved by a standard fixed point argument. More details can be found in [3].

![Figure 1: The Lipschitz continuous text function \( \varphi^\epsilon \) introduced at (32).](image)

**Step 4.** Thanks to the previous construction, the map \( t \mapsto x(t) \) provides the unique solution to (24). Since the Lipschitz continuity of \( \beta(t) \) and \( x(t) = x(t,\beta(t)) \), \( \beta(t) \) and \( x(t) \) are differentiable almost everywhere, so we only have to consider the time where \( x(t) \) is differentiable. It suffices to show that (21) holds at almost every time. Suppose, on the contrary, that \( \dot{x}(\tau) = \gamma u(x(\tau),x(\tau)) \).

Without loss of generality, let
\[ \dot{x}(\tau) = \gamma u(x(\tau),x(\tau)) + 2\epsilon_0 \] (29)
for some \( \epsilon_0 > 0 \). The case \( \epsilon_0 < 0 \) is entirely similar. To obtain a contradiction we find that, for all \( t \in (\tau,\tau + \delta] \), with \( \delta > 0 \) small enough one has
\[ x^+(t) = x(\tau) + (t-\tau)[\gamma u(x(\tau),x(\tau)) + \epsilon_0] < x(t). \] (30)

We also observe that if \( \varphi \) is Lipschitz continuous with compact support then (13) is still true.
For any $\epsilon > 0$, we can still use the blow test functions used in [3].

Define

$$\rho^\epsilon(s, y) = \begin{cases} 0 & \text{if } y \leq -\epsilon^1, \\ (y + \epsilon^1) & \text{if } -\epsilon^1 \leq y \leq 1 - \epsilon^1, \\ 1 & \text{if } 1 - \epsilon^1 \leq y \leq x^+(s), \\ 1 - \epsilon^1(y - x(s)) & \text{if } x^+(s) \leq y \leq x^+(s) + \epsilon, \\ 0 & \text{if } y \geq x^+(s) + \epsilon, \end{cases}$$

and

$$\chi^\epsilon(s, y) = \begin{cases} 0 & \text{if } s \leq \tau - \epsilon, \\ \epsilon^1(s - \tau + \epsilon) & \text{if } \tau - \epsilon \leq s \leq \tau, \\ 1 & \text{if } \tau \leq s \leq t, \\ 1 - \epsilon^1(s - t) & \text{if } t \leq s < t + \epsilon, \\ 0 & \text{if } s \geq t + \epsilon. \end{cases}$$

Define

$$\varphi^\epsilon(s, y) = \min\{\rho^\epsilon(s, y), \chi^\epsilon(s)\}. \quad (32)$$

Using $\varphi^\epsilon$ as test function in (13) we obtain

$$\int \int \left[ u_x^2 \varphi^\epsilon_t + \gamma uu_x^2 \varphi^\epsilon_x + 2\left[ g(u) - \frac{\gamma}{2} u^2 - P \right] u_x \varphi^\epsilon \right] dxdt = 0 \quad (33)$$

If $t$ is sufficiently close to $\tau$, one has

$$\lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{x^+(s) - \epsilon}^{x^+(s) + \epsilon} u_x^2(\varphi^\epsilon_t + \gamma u \varphi^\epsilon_x) dyds \geq 0. \quad (34)$$

In fact, for $s \in [\tau + \epsilon, t - \epsilon]$, we have

$$0 = \varphi^\epsilon_t + \gamma u(\gamma(x) + \epsilon_0) \varphi^\epsilon_x \leq \varphi^\epsilon_t + \gamma u(s, x) \varphi^\epsilon_x,$$

since $\gamma u(s, x) < \gamma(\gamma(s, x) + \epsilon_0)$ and $\varphi^\epsilon_x \leq 0$.

Due to the family of measures $\mu(t)$ depends continuously on $t$ in the topology of weak convergence, taking the limit of (33) as $\epsilon \to 0$, we obtain

$$0 = \int_{-\infty}^{x(\tau)} u_x^2(\gamma, y) dy - \int_{-\infty}^{x(t)} u_x^2(y, t) dy + \int_{\tau}^{t} \int_{-\infty}^{x(s)} 2\left[ g(u) - \frac{\gamma}{2} u^2 - P \right] u_x dyds$$

$$+ \lim_{\epsilon \to 0} \int_{\tau}^{t} \int_{x^+(s) - \epsilon}^{x^+(s) + \epsilon} u_x^2(\varphi^\epsilon_t + \gamma u \varphi^\epsilon_x) dyds$$

$$\geq \int_{-\infty}^{x(\tau)} u_x^2(\gamma, y) dy - \int_{-\infty}^{x(t)} u_x^2(y, t) dy + \int_{\tau}^{t} \int_{-\infty}^{x(s)} 2\left[ g(u) - \frac{\gamma}{2} u^2 - P \right] u_x dyds.$$ 

The above inequality implies

$$\int_{-\infty}^{x(t)} u_x^2(y, t) dy \geq \int_{-\infty}^{x(\tau)} u_x^2(\gamma, y) dy + \int_{\tau}^{t} \int_{-\infty}^{x(s)} 2\left[ g(u) - \frac{\gamma}{2} u^2 - P \right] u_x dyds$$

$$= \int_{-\infty}^{x(\tau)} u_x^2(y, t) dy + \left[ 2\int_{\tau}^{t} \int_{-\infty}^{x(s)} 2\left[ g(u) - \frac{\gamma}{2} u^2 - P \right] u_x dyds + o_1(t - \tau). \right]$$
Note that the last term is higher order infinitesimal, satisfying $\frac{o_1(t-\tau)}{t-\tau} \to 0$ as $t \to \tau$. Indeed

$$|o_1(t-\tau)| = \left| \int_{\tau}^{t} \int_{x(t)}^{x(t)+\varepsilon} 2[g(u) - \frac{\gamma}{2} u^2 - P] u_x dy ds \right|$$

$$\leq ||2[g(u) - \frac{\gamma}{2} u^2 - P]||_{L^\infty} \int_{\tau}^{t} \int_{x(t)}^{x(t)+\varepsilon} |u_x| dy ds$$

$$\leq ||2[g(u) - \frac{\gamma}{2} u^2 - P]||_{L^\infty} \int_{\tau}^{t} (x(s) - x(t)+\varepsilon) \frac{1}{2} \|u_x(s, \cdot)\|_{L^2} ds$$

$$\leq C \cdot (t-\tau)^{\frac{3}{2}}.$$

On the other hand, using (25) and (27) a linear approximation we obtain

$$\beta(t) = \beta(t) + (t-\tau) \left[ \gamma u(\tau), x(\tau) + \int_{-\infty}^{x(\tau)} 2[g(u) - \frac{\gamma}{2} u^2 - P] u_x dy \right] + o_2(t-\tau), \quad (36)$$

with $\frac{o_2(t-\tau)}{t-\tau}$ as $t \to \tau$. For $t$ sufficiently close to $\tau$, we have

$$\beta(t) = x(t) + \int_{-\infty}^{x(t)} (u_x^2(t, y)) dy$$

$$> x(\tau) + (t-\tau) [\gamma u(\tau), x(\tau)] + \varepsilon_0 + \int_{-\infty}^{x(t)} u_x^2(t, y) dy$$

$$\geq x(\tau) + (t-\tau) [\gamma u(\tau), x(\tau)] + \varepsilon_0 + \int_{-\infty}^{x(\tau)} u_x^2(t, y) dy$$

$$+ \int_{\tau}^{t} \int_{-\infty}^{x(s)} 2[g(u) - \frac{\gamma}{2} u^2 - P] u_x dy ds + o_1(t-\tau). \quad (37)$$

Relying on (36) and (37), observe that

$$\beta(t) + (t-\tau) \left[ \gamma u(\tau), x(\tau) + \int_{-\infty}^{x(\tau)} 2[g(u) - \frac{\gamma}{2} u^2 - P] u_x dy \right] + o_2(t-\tau)$$

$$\geq \left[ x(\tau) + \int_{-\infty}^{x(\tau)} u_x^2(t, y) dy \right] + (t-\tau) [\gamma u(\tau), x(\tau)] + \varepsilon_0$$

$$+ \int_{\tau}^{t} \int_{-\infty}^{x(s)} 2[g(u) - \frac{\gamma}{2} u^2 - P] u_x dy ds + o_1(t-\tau). \quad (38)$$

Subtracting common terms, dividing both sides by $t-\tau$ and letting $t \to \tau$, we get a contradiction, namely, (21) must hold.

**Step 5.** Now, we prove (22). From (5), one has

$$\int_{0}^{\infty} \int \left[ u \phi_t + \frac{\gamma}{2} u^2 \phi_x + P_x \phi \right] dx dt + \int u_0(x) \phi(0, x) dx = 0. \quad (39)$$

for every test function $\phi \in C_c^\infty(\mathbb{R})$. Let $\phi = \varphi_x$ which $\varphi \in C_c^\infty$ is given. Owing to the map $x \mapsto u(t, x)$ is absolutely continuous, integrating by part w.r.t. $x$ and get

$$\int_{0}^{\infty} \int \left[ u_x \phi_t + \gamma u u_x \phi_x + P_x \phi_x \right] dx dt + \int u_{0,x}(x) \phi(0, x) dx = 0. \quad (40)$$
By an approximation argument, the identity (40) still holds for any test function $\varphi$ which is Lipschitz continuous with compact support. We now consider the functions
\[
\eta(s, y) = \begin{cases} 
0 & \text{if } y \leq -\epsilon^{-1}, \\
(y + \epsilon^{-1}) & \text{if } -\epsilon^{-1} \leq y \leq 1 - \epsilon^{-1}, \\
1 & \text{if } 1 - \epsilon^{-1} \leq y \leq x(s), \\
1 - \epsilon^{-1}(y - x(s)) & \text{if } x(s) \leq y \leq x(s) + \epsilon, \\
0 & \text{if } y \geq x(s) + \epsilon,
\end{cases}
\]
for any $\epsilon > 0$ is sufficiently small. Define
\[
\psi^\varepsilon(s, y) = \min\{\eta^\varepsilon(s, y), \chi^\varepsilon(s)\},
\]
with $\chi^\varepsilon(s)$ defined in (31). We then use the test function $\varphi = \psi^\varepsilon$ in (40) and let $\epsilon \to 0$. Noticing that the function $P_x$ is continuous, we obtain that
\[
\int_{-\infty}^{x(t)} u_x(t, y)dy = \int_{-\infty}^{x(\tau)} u_x(\tau, y)dy - \int_{\tau}^{t} P_x(s, x(s))ds \\
+ \lim_{\epsilon \to 0} \int_{\tau}^{t+\epsilon} \int_{x(s)}^{x(s)+\epsilon} u_x^2(\psi^\varepsilon + \gamma u \psi^\varepsilon)dyds.
\]
To complete the proof it suffices to show that the last term on the right hand side of (42) vanishes. The Cauchy’s inequality implies
\[
\left| \int_{\tau}^{t} \int_{x(s)}^{x(s)+\epsilon} u_x(\varphi^\varepsilon + \gamma u \psi^\varepsilon)dyds \right| \\
\leq \int_{\tau}^{t} \left( \int_{x(s)}^{x(s)+\epsilon} |u_x|^2dy \right)^{\frac{1}{2}} \left( \int_{x(s)}^{x(s)+\epsilon} (\psi^\varepsilon + \gamma u \psi^\varepsilon)^2dy \right)^{\frac{1}{2}} ds,
\]
since $u_x \in L^2$. Consider the function
\[
\zeta_\varepsilon(s) = \left( \sup_{x \in \mathbb{R}} \int_{x}^{x(s)} u_x^2(s, y)dy \right)^{\frac{1}{2}},
\]
for each $\epsilon > 0$. Observe that all functions $\zeta_\varepsilon$ are uniformly bounded. Furthermore, we have $\zeta_\varepsilon(t) \to 0$ pointwise at a.e. time $t$ as $\epsilon \to 0$. Therefore, in view of the dominated convergence theorem, we have
\[
\lim_{\epsilon \to 0} \int_{\tau}^{t} \left( \int_{x(s)}^{x(s)+\epsilon} |u_x(s, y)|^2dx \right)^{\frac{1}{2}} ds \leq \lim_{\epsilon \to 0} \int_{\tau}^{t} \zeta_\varepsilon(s)ds = 0.
\]
Otherwise, for every time $s \in [\tau, t]$ by construction we have
\[
\psi^\varepsilon(s, y) = \epsilon^{-1}, \quad \psi^\varepsilon(s, y) + \gamma u(s, x(s))\psi^\varepsilon(s, y) = 0,
\]
for $x(s) < y < x(s) + \epsilon$. This yields
\[
\int_{x(s)}^{x(s)+\epsilon} |\psi^\varepsilon(s, y) + \gamma u(s, x(s))\psi^\varepsilon(s, y)|^2dy = \gamma^2 \epsilon^{-2} \int_{x(s)}^{x(s)+\epsilon} |u(s, y) - u(s, x(s))|^2dy \\
\leq \gamma^2 \epsilon^{-1} \cdot \left( \max_{x(s) \leq y \leq x(s) + \epsilon} |u(s, y) - u(s, x(s))| \right)^2 \\
\leq \gamma \epsilon^{-1} \cdot \left( \int_{x(s)}^{x(s)+\epsilon} |u_x(s, y)|dy \right)^2 \\
\leq \gamma \epsilon^{-1} (\epsilon^2 \cdot \|u_x(s)\|_{L^2})^2 \leq \gamma \|u(s)\|_{H^1}.
\]
Combining (45) and (46), this prove that integral in (43) approaches zero as $\epsilon \to 0$. We now estimate the integral near the corners of the domain:

$$
\left| \int_{\tau + \epsilon}^{\tau + \epsilon} \int_{x(s)}^{x(s) + \epsilon} \left[ u_x(\psi_t^\epsilon + \gamma u \psi_x^\epsilon) \right] ds \right|
$$

$$
\leq \left( \int_{\tau - \epsilon}^{\tau + \epsilon} \int_{x(s)}^{x(s) + \epsilon} \left| u_x \right|^2 dx \right)^{\frac{1}{2}} \left( \int_{x(s)}^{x(s) + \epsilon} \left( \psi_t^\epsilon + \gamma u \psi_x^\epsilon \right)^2 dx \right)^{\frac{1}{2}} ds \quad (47)
$$

$$
\leq 2\epsilon \cdot \left\| u(s) \right\|_{L^1} \left( \int_{x(s)}^{x(s) + \epsilon} 4\epsilon^{-2} (1 + \gamma \left\| u \right\|_{L^\infty})^2 dx \right)^{\frac{1}{2}} \leq C\epsilon^4 \to 0
$$

as $\epsilon \to 0$. The above analysis shows that

$$
\lim_{\epsilon \to 0} \int_{\tau - \epsilon}^{\tau + \epsilon} \int_{x(s)}^{x(s) + \epsilon} u_x^2(\psi_t^\epsilon + \gamma u \psi_x^\epsilon) dx ds = 0. \quad (48)
$$

Therefore, we achieve (20) by (42).

**Step 6.** The uniqueness of the solution $t \to x(t)$ now becomes clear. Relying on (23) we can show Lipschitz continuity of $u$ w.r.t. $t$, in the auxiliary coordinate system.

**Lemma 3.3.** Let $u = u(t, x)$ be a conservative solution of equation (4). Then the map $(t, \beta) \mapsto u(t, \beta) \equiv u(t, x(t, \beta))$ is Lipschitz continuous, with a constant depending only on the norm $\left\| u_0 \right\|_{H^1}$.

**Proof.** From (19), (23) and (27), we obtain

$$
\left| u(t, x(t, \tilde{\beta})) - u(t, \tilde{\beta}) \right|
$$

$$
\leq \left| u(t, x(t, \tilde{\beta})) - u(t, x(t, \beta(t))) \right| + \left| u(t, x(t, \beta(t))) - u(t, x(\tau, \beta(\tau))) \right| \quad (49)
$$

$$
\leq \frac{1}{2} |\beta(t) - \tilde{\beta}| + C(t - \tau) \leq C(t - \tau),
$$

where $C$ is a constant depending only on the $\left\| u_0 \right\|_{H^1}$.

The next result shows that the solutions $\beta(\cdot)$ of (27) depend Lipschitz continuously on the initial data.

**Lemma 3.4.** Let $u$ be a conservative solution to the equation (4). Call $t \mapsto \beta(t; \tau, \tilde{\beta})$ the solution to the integral equation

$$
\beta(t) = \tilde{\beta} + \int_{\tau}^{t} G(\tau, \beta(\tau)) d\tau, \quad (50)
$$

the $G$ is defined in (25). Then there exists a constant $C$ such that, for any two initial data $\tilde{\beta}_1, \tilde{\beta}_2$ and any $t, \tau \geq 0$ the corresponding solutions satisfy

$$
|\beta(t; \tau; \tilde{\beta}_1) - \beta(t; \tau; \tilde{\beta}_2)| \leq C(t - \tau) |\tilde{\beta}_1 - \tilde{\beta}_2|. \quad (51)
$$

The proof is easy because of Lipschitz continuity of $G$ with respect to $\beta$. We leave it to reader.

**Lemma 3.5.** Suppose that $u \in H^1(\mathbb{R})$. Then $P_x$ is absolutely continuous and satisfies

$$
P_{xx} = P - \left( g(u) + \frac{\gamma}{2} u_x^2 - \frac{\gamma}{2} u^2 \right). \quad (52)$$
Finally, combing (53)-(55), we obtain the following O. D. E. systems:

On the other hand, differentiating w.r.t. \( \tilde{\beta} \)

We seek an ODE describing how the quantities

Step 2. \( \beta \)

Lemma 3.3. An entirely similar argument implies the map \( (t, \beta) \mapsto G(t, \beta) = G(t, x(\beta)) \) and \( \beta \mapsto P_x(t, \beta) = P_x(t, x(\beta)) \) are also Lipschitz continuous. In view of the Rademacher’s theorem, the partial derivatives \( x_t, x_\beta, u_t, u_\beta \) and \( P_{x,\beta} \) exist almost everywhere. In addition, for these derivatives, a.e. point \( (t, \beta) \) is a Lebesgue point. Calling \( t \mapsto \beta(t, \tilde{\beta}) \) the unique solution to the integral equation (27), from Lemma 3.4 for a.e. \( \tilde{\beta} \) the following holds.

(GC) For a.e. \( t > 0 \), except a measure zero set \( \mathcal{N} \subseteq \mathbb{R}^+ \), the point \( (t, \beta(t, \tilde{\beta})) \) is a Lebesgue point for the partial derivatives \( x_t, x_\beta, u_t, u_\beta, G_\beta, P_{x,\beta} \). Moreover, \( x_\beta(t, \beta(t, \tilde{\beta})) > 0 \) for a.e. \( t > 0 \).

If the above condition holds, we say that \( t \mapsto \beta(t, \tilde{\beta}) \) is a good characteristic.

Step 2. We seek an ODE describing how the quantities \( u_\beta \) and \( x_\beta \) vary along a good characteristic. As in Lemma 3.4, we denote by \( t \mapsto \beta(t, \tau, \tilde{\beta}) \) the solution to (50). If \( \tau, t \notin \mathcal{N} \), supposing \( \beta(\cdot; \tau, \tilde{\beta}) \) is a good characteristic, differentiating (50) w.r.t. \( \beta \) we find

Next, differentiating w.r.t. \( \tilde{\beta} \) the identity

we obtain

On the other hand, differentiating w.r.t. \( \tilde{\beta} \) the identity (23), we obtain

Finally, combing (53)-(55), we obtain the following O. D. E. systems:

\[
\begin{cases}
\frac{d}{dt} [x_\beta(t, \beta(t, \tilde{\beta}))] = G_\beta(t, x_\beta(t, \beta(t, \tilde{\beta}))) \cdot \frac{d}{d\tilde{\beta}} \beta(t, \beta(t, \tilde{\beta})), \\
\frac{d}{dt} [u_\beta(t, \beta(t, \tilde{\beta}))] = \gamma u_\beta(t, \beta(t, \tilde{\beta})) \cdot \frac{d}{d\tilde{\beta}} \beta(t, \beta(t, \tilde{\beta})), \\
\frac{d}{dt} [u_\beta(t, \beta(t, \tilde{\beta}))] = P_{x,\beta}(t, \beta(t, \tilde{\beta})) \cdot \beta(t, \beta(t, \tilde{\beta})).
\end{cases}
\]
we obtain
\[ \frac{d}{dt} x_{\beta} + G_{\beta} x_{\beta} = \gamma u_{\beta}, \]
\[ \frac{d}{dt} u_{\beta} + G_{\beta} u_{\beta} = \left[ g(u) + \frac{\gamma}{2} u^2 - \frac{\gamma}{2} u^2 - P \right] x_{\beta} \]
\[ = \left[ g(u) - \frac{\gamma}{2} u^2 + \frac{\gamma}{2} (\frac{1}{x_{\beta}} - 1) - P \right] x_{\beta} \]
\[ = \left[ g(u) - \frac{\gamma}{2} u^2 - P - \frac{\gamma}{2} \right] x_{\beta} + \frac{\gamma}{2}. \]  

(57)

**Step 3.** We now revert to the original \((t, x)\) coordinates and deduce an evolution equation for the partial derivative \(u_x\) along a “good” characteristic curve. Fix a point \((\tau, \tilde{x})\) with \(\tau \not\in N\). Suppose that \(\tilde{x}\) is a Lebesgue point for the map \(x \mapsto u_x(t, x)\). Let \(\tilde{\beta}\) be such that \(\tilde{x} = x(\tau, \tilde{\beta})\) and assure that \(t \mapsto \beta(t; \tau, \tilde{\beta})\) is a good characteristic, so that (GC) is valid. We observe that

\[ u_x^2(\tau, x) = \frac{1}{x_{\beta}(\tau, \tilde{\beta})} - 1 \geq 0 \quad x_{\beta}(\tau, \tilde{\beta}) > 0. \]

If only \(x_{\beta} > 0\), along the characteristic though \((\tau, \tilde{x})\) the partial derivative \(u_x\) can be calculated as

\[ u_x(t, x(\beta(t; \tau, \tilde{\beta}))) = \frac{u_x(t, \beta(t; \tau, \tilde{\beta}))}{x_{\beta}(t, \beta(\tau; \tau, \tilde{\beta}))), \]  

(58)

From the two ODEs (54)-(55) representing through the evolution of \(u_{\beta}\) and \(x_{\beta}\), we obtain that the map \(t \mapsto u_x(t, x(\beta(t; \tau, \tilde{\beta})\)) is absolutely continuous (as long as \(x_{\beta} \neq 0\) and satisfies

\[ \frac{d}{dt} u_x(t, x(\beta(t; \tau, \tilde{\beta}))) = \frac{d}{dt} \left( \frac{u_{\beta}}{x_{\beta}} \right) \]
\[ = x_{\beta} \left[ g(u) - \frac{\gamma}{2} u^2 - P - \frac{\gamma}{2} x_{\beta} + \frac{\gamma}{2} - u_{\beta} G_{\beta} \right] - \frac{\gamma u_{\beta}^2 - x_{\beta} + \gamma u_{\beta} - x_{\beta} G_{\beta}}{x_{\beta}} \]
\[ = g(u) - \frac{\gamma}{2} u^2 - P - \frac{\gamma}{2} x_{\beta} + \frac{\gamma}{2} - \gamma u_{\beta}^2 - x_{\beta} + \gamma u_{\beta} - x_{\beta} G_{\beta} \]
\[ = g(u) - \frac{\gamma}{2} u^2 - P - \gamma \frac{\gamma}{2} x_{\beta} - \gamma u_{\beta}^2 \frac{x_{\beta}}{x_{\beta}}. \]  

(59)

In turn, as long as \(x_{\beta} > 0\) this means

\[ \frac{d}{dt} \arctan(u_x(t, x(\beta(t; \tau, \tilde{\beta})))) = \frac{1}{1 + u_x^2(t, x(\beta(t; \tau, \tilde{\beta})))} \frac{d}{dt} u_x \]
\[ = \left( g(u) - \frac{\gamma}{2} u^2 - P - \frac{\gamma}{2} x_{\beta} + \gamma \frac{\gamma}{2} x_{\beta} \right) x_{\beta} \]
\[ = \left( g(u) - \frac{\gamma}{2} u^2 - P - \gamma \frac{\gamma}{2} \right) x_{\beta} + \frac{\gamma}{2} \]
\[ = \left( g(u) - \frac{\gamma}{2} u^2 - \gamma u_{\beta}^2 - P - \gamma \frac{\gamma}{2} \right) x_{\beta} + \frac{\gamma}{2} \]
\[ = \left( g(u) - \frac{\gamma}{2} u^2 - P + \frac{\gamma}{2} \right) x_{\beta} - \frac{\gamma}{2} \]  

(60)

In particular, the quantities within square brackets on the left hand sides of (56) are absolutely continuous. From (56), using Lemma 3.5 along a good characteristic
Step 4. Now we consider the function
\[ v = \begin{cases} 
2 \arctan u_x & \text{if } 0 < x_\beta \leq 1, \\
\pi & \text{if } x_\beta = 0.
\end{cases} \] (61)
This implies
\[ x_\beta = \frac{1}{1 + u_x^2} = \cos^2 \frac{v}{2}, \quad u_x = \frac{\frac{1}{2} \sin v}{1 + u_x^2} = \frac{\frac{1}{2} \sin v}{2} = \frac{\sin^2 v}{2} \] (62)
Then, \( v \) will be regarded as map taking values in the unit circle \( \Omega \equiv [-\pi, \pi] \) with endpoints identified. We claim that, along each good characteristic, the map \( t \rightarrow x(t) = x(t, \beta(t); \tau(\beta)) \) is absolutely continuous and satisfies
\[ \frac{d}{dt} v(t) = (2g(u) - \gamma u^2 - 2P + \gamma) \cos^2 \frac{v}{2} - \gamma. \] (63)
In fact, denote by \( x_\beta(t), u_\beta(t) \) and \( u_x(t) = \frac{u_\beta(x(t), \beta(t))}{x_\beta(x(t), \beta(t))} \) the values of \( x_\beta, u_\beta \) and \( u_x \) along this particular characteristic. By (GC) we have \( x_\beta(t) > 0 \) for a.e. \( t > 0 \). Assume that \( \tau \) is any time where \( x_\beta(\tau) > 0 \), we can find a neighborhood \( I = [\tau - \delta, \tau + \delta] \) such that \( x_\beta(\tau) > 0 \) on \( I \). From (60) and (62), \( v = 2 \arctan(\frac{u}{2}) \) is absolutely continuous restricted to \( I \) and satisfies (63). To prove our claim, there remains to verify that \( t \mapsto v(t) \) is continuous on the null set \( N \) of times where \( x_\beta(t) = 0 \). Let \( x_\beta(t_0) = 0 \), by the following identity
\[ u_x^2(t) = \frac{1 - x_\beta(t)}{x_\beta(t)}, \] (64)
valid as long as \( x_\beta > 0 \), it is clear that \( u_x^2 \rightarrow \infty \) as \( t \rightarrow t_0 \) and \( x_\beta(t) \rightarrow 0 \). This indicates \( v(t) = 2 \arctan u_x(t) \rightarrow \pm \pi \). Since we identify the points \( \pm \pi \) in \( \Omega \), this establishes the continuity of \( v \) for all \( t \geq 0 \), proving our claim.

Step 5. Let now \( u = u(t, x) \) be a conservation solution. As shown by the previous analysis, in terms of the variables \( t, \beta \) the quantities \( x, u, v \) satisfy the semilinear system
\[ \begin{align*}
\frac{d}{dt} \beta(t, \tilde{\beta}) &= G(t, \beta(t, \tilde{\beta})), \\
\frac{d}{dt} x(t, \beta(t, \tilde{\beta})) &= \gamma u(t, \beta(t, \tilde{\beta})), \\
\frac{d}{dt} u(t, \beta(t, \tilde{\beta})) &= -P_x(t, \beta(t, \tilde{\beta})), \\
\frac{d}{dt} v(t, \beta(t, \tilde{\beta})) &= (2g(u) - \gamma u^2 - 2P + \gamma) \cos^2 \frac{v}{2} - \gamma.
\end{align*} \] (65)
We recall that \( P \) and \( G \), the function \( P \) is a representation in term of the variable \( \beta \), namely
\[ P(x(\beta)) = \frac{1}{2} \int_{-\infty}^\infty \exp \left\{ -\int_{\beta}^{\beta'} \cos^2 \frac{v(s)}{2} ds \right\} \left( (g(u(\beta')) - \gamma u(\beta')^2) \cos^2 \frac{v(\beta')}{2} + \gamma \sin^2 \frac{v(\beta')}{2} \right) d\beta', \] (66)
\[ P_x(x(\beta)) = \frac{1}{2} \left( \int_{0}^\infty - \int_{-\infty}^0 \right) \exp \left\{ -\int_{\theta}^{\beta} \cos^2 \frac{v(s)}{2} ds \right\} \left( (g(u(\beta')) - \gamma u(\beta')^2) \cos^2 \frac{v(\beta')}{2} + \gamma \sin^2 \frac{v(\beta')}{2} \right) d\beta'. \] (67)
For every $\beta \in \mathbb{R}$ we have the initial condition
\begin{align}
\beta(0, \beta) &= \beta, \\
x(0, \beta) &= x(0, \beta), \\
u(0, \beta) &= u_0(x(0, \beta)), \\
v(0, \beta) &= 2 \arctan u_0(x(0, \beta)).
\end{align}
\tag{68}

Taking advantage of the Lipschitz continuity of coefficients, the Cauchy problem (65) with initial data in (68) has a unique solution, globally defined for all $t \geq 0, x \in \mathbb{R}$.

**Step 6.** To complete the proof of uniqueness, consider two conservative solutions $u, \bar{u}$ of the equation (4) with the same initial data $u_0 \in H^1(\mathbb{R})$. For a.e. $t > 0$ the corresponding Lipschitz continuous maps $\beta \mapsto x(t, \beta)$, $\beta \mapsto \bar{x}(t, \beta)$ are strictly increasing. Therefore, they have continuous inverses, say $x \mapsto \beta^*(t, x)$, $x \mapsto \bar{\beta}^*(t, x)$. In view of the previous analysis, the map $(t, \beta) \mapsto (x(t, \beta), u(t, \beta), v(t, \beta))$ is uniquely resolved by the initial data $u_0$. Hence
\[x(t, \beta) = \bar{x}(t, \beta), \quad u(t, \beta) = \bar{u}(t, \beta)\]

In turn, for a.e. $t > 0$ this yields
\[u(t, x) = u(t, \beta^*(t, x)) = \bar{u}(t, \bar{\beta}^*(t, x)) = \bar{u}(t, x)\]

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