GLOBAL DIMENSIONS OF SOME ARTINIAN ALGEBRAS

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INTRODUCTION

The structure of arbitrary associative commutative unital artinian algebras is well-known: they are finite products of associative commutative unital local algebras [6, pg.351, Cor. 23.12]. In the semi-simple case, we have the Artin-Wedderburn Theorem which states that any semi-simple artinian algebra (which is assumed to be associative and unital but not necessarily commutative) is a direct product of matrix algebras over division rings [6, pg.35, Par. 3.5]. Along these lines, we observe a simple classification of artinian algebras and their representations in Proposition 1.3.2 (hereby referred as the Classification Lemma) in terms of a category in which each object has a local artinian endomorphism algebra. This category is constructed using a fixed set of primitive (not necessarily central) idempotents in the underlying algebra. The Classification Lemma is a version of Freyd’s Representation Theorem [4, Sect. 5.3]: from an artinian algebra $A$ we create a category $\mathcal{C}_A$ on finitely many objects, and then the category of $A$-modules can be realized as a category of functors which admit $\mathcal{C}_A$ as their domain. This construction can also be thought as a higher dimensional analogue of the semi-trivial extensions of [10] for artinian algebras.

We use the Classification Lemma as a starting point to develop a novel effective algorithm to determine finiteness of global dimensions of a large class of artinian algebras. As a first step, in Theorem 2.3.4 (hereby referred as the Source-Sink Theorem) we get explicit lower and upper bounds for the global dimension of artinian algebras in terms of the global dimensions of a pair of artinian subalgebras. Then in the rest of Section 2 we develop our algorithm. This algorithm is based on a directed graph we construct out of a given artinian algebra and it allows us to reduce the question of finiteness of the global dimension of a given artinian algebra to finiteness of the global dimensions of a finite subset of subalgebras. The graph $\mathcal{G}_A$ we define in Definition 2.2.1 for an artinian algebra $A$ is not the Auslander-Reiten quiver of $A$ and it appears to be a simplified version of the natural quiver of an artin algebra defined in [7]. See Subsection 2.6 and Figure 1 in which we summarize the results and the algorithm we obtain in Section 2.

As an application of our algorithm, we obtain a new result in Theorem 3.1.3: we show that the quotient of a free path algebra of a cycle-free graph by a nil ideal has finite global dimension. Using the same algorithm we were also able to reproduce two results from the literature: In Proposition 3.1.2 we prove that free path algebras over cycle-free graphs are hereditary, and in Proposition 3.1.5 we prove that incidence algebras have finite global dimension. See [9, Section 9] and [2, Proposition 1.4] for the classical proofs of these Propositions.
At this point, let us make a careful distinction between an artinian algebra (an algebra which satisfies a descending chain condition on left, right or bilateral ideals) and an artin algebra (an algebra which is finitely generated as a bimodule over a unital commutative artinian center.) In this article we assume that our algebras are artinian and we make no assumption on the finiteness of the $k$-dimensions of the algebras we work with over the base field $k$. Also, it is common to restrict to the type of a presentation the base algebra has in the relevant literature, such as assuming that the base algebra is a quotient of a free path algebra of a quiver by an ideal contained in the Jacobson radical, or in its square. See for example [1]. But we make no such restrictive assumptions about presentations of our artinian algebras except in Theorem 3.1.3 where we prove finiteness of global dimensions of a class of artinian algebras which contains all incidence algebras.

**Plan of this article.** In Section 1 we either cite or prove some technical results we need in Section 2 in proving our main results. Section 2 contains our algorithm which calculates upper and lower bounds for global dimension of artinian algebras, and all other results needed to develop this algorithm. See Subsection 2.6 and Figure 1 for an overview. In Section 3 we apply the results we obtain in Section 2 to path algebras and incidence algebras, and make explicit calculations in Subsection 3.2.

**Standing assumptions and notation.** Throughout the article $k$ is a base field. We do not make any assumption about the characteristic of $k$. All unadorned tensor products are assumed to be over $k$. We assume $A$ is a unital associative algebra over $k$ which may or may not be commutative. When we say $A$ is artinian (resp. noetherian) we mean $A$ satisfies descending (resp. ascending) chain condition on both left and right ideals. In this article we heavily use idempotents (elements which satisfy $e^2 = e$), but we make no assumptions on whether these idempotents are in the center. We will use the notation $Ax$, $xA$ and $AxA$ to denote the left, right and two-sided principal ideals generated by an element $x \in A$, respectively. The multiplicative group of invertible elements in $A$ is denoted by $A^\times$. We will use $J(A)$ to denote the Jacobson Radical of $A$, i.e. the intersection of all maximal left ideals in $A$. We note that the intersection of all maximal right ideals is also $J(A)$. All $A$-modules are assumed to be left $A$-modules unless otherwise explicitly stated. We will denote the category of left $A$-modules by $A$-$\text{Mod}$ and the category of right $A$-modules by $\text{Mod}$-$A$. For a finite set $U$, we will use $|U|$ to denote the cardinality of $U$. Finally, in our graph algebras in forming products of edges we follow the categorical convention: a product of the form $fg$ in a path algebra corresponds to compositions of arrows of the form $\bullet \xleftarrow{f} \bullet \xleftarrow{g} \bullet$.

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1. **The classification lemma**

1.1. **Technical lemmata.**

Below we list few lemmata we need, without proofs. Our main references are [5, 6].

**Lemma 1.1.1.** For every $x \in A$, if $x \in J(A)$ then $1 + x \in A^\times$. 

Lemma 1.1.2. [6] Theorem 19.1] A is local if and only if $A \setminus A^\times$ is a two-sided ideal.

Lemma 1.1.3 (Nakayama Lemma). Let $M$ be a finitely generated left $A$-module. If $J(A)M = M$ then $M = 0$.

Lemma 1.1.4. Let $A$ be an algebra which splits as a direct sum of two (left) ideals $A = I \oplus J$. Then there are two orthogonal idempotents such that $1 = e + f$.

An $A$-module $M$ is called semi-simple if every submodule of $M$ is a direct summand of $M$. An algebra $A$ is called semi-simple if $A$ viewed as an $A$-module is semi-simple.

Lemma 1.1.5. Assume $A$ is an artinian $k$-algebra. Then $A$ is left semi-simple if and only if it is right semi-simple if and only if $J(A) = 0$.

Lemma 1.1.6. Let $J$ be a nil ideal. Then any idempotent in $A/J$ can be lifted to an idempotent in $A$. In other words, for every idempotent $e \in A/J$ there exists an idempotent $e \in A$ such that $e + J = e$.

Definition 1.1.7. An idempotent $e \in A$ is called primitive when for every pair of orthogonal idempotents $e_1, e_2 \in A$ if $e = e_1 + e_2$ then $e_1 = 0$ or $e_2 = 0$.

Lemma 1.1.8. [6] Proposition 21.22] Assume $A$ is an artinian algebra, $e \in A$ idempotent, $J \subset J(A)$. Then $e + J$ is a primitive idempotent of $A/J$ if and only if $e$ is a primitive idempotent of $A$.

Lemma 1.1.9. [6] Corollary 19.19] Assume $A$ is an artinian $k$-algebra. Then $A$ is local if and only if the only idempotents of $A$ are 1 and 0.

1.2. Idempotents.

Proposition 1.2.1. Assume $A$ is an artinian $k$-algebra and $e \in A$ is an idempotent. Then $e \in A$ is a primitive idempotent if and only if $eAe$ is a local ring.

Proof. Assume $e$ is a primitive idempotent. Note that, regardless of $e$ being primitive, $eAe$ is a $k$-algebra with $e \in eAe$ being its identity element. Assume $eAe$ is not local. Since $eAe$ is also an artinian algebra on its own right, there exists an idempotent $efe \in eAe$ which is not 0 or $e$ by Lemma 1.1.9.

Then $e = efe + (e - efe)$ and observe that since $efe(e - efe) = (e - efe)efe = 0$, we write $e$ as a sum of two orthogonal idempotents in $A$ which is a contradiction since $e$ is primitive. So, $e$ is primitive implies $eAe$ is a local artinian $k$-algebra. Conversely, assume $eAe$ is local artinian. Then we know that the only idempotents of $eAe$ are $e$ and 0. Assume that $e$ can be written as a sum of two orthogonal idempotents in $A$ as $e = u + v$. Then we see that $eu = ue = u^2 = u$ and $ev = ve = v^2 = v$ and therefore $e = u + v = eue + eve$ in $eAe$. Since the only idempotents in $eAe$ are 0 and 1 we see that $eue = u = 0$ or $eve = v = 0$, i.e. that $e$ is primitive. Thus we conclude that if $eAe$ is local artinian then $e$ is primitive. \[ \]

Proposition 1.2.2. (Compare this with [6] Theorem 23.6]. Note that any artinian ring is semi-perfect.) Let $A$ be an artinian $k$-algebra. Then the unit $1 \in A$ is a finite sum of pairwise orthogonal primitive idempotents.
Proof. Since $A$ is artinian, so is $A' := A/J(A)$ and since the Jacobson radical of $A'$ is trivial we see that $A'$ is (left) semi-simple by Lemma 1.1.3. So, $A'$ splits as finite direct sum of minimal (left) ideals $\bigoplus_{i=1}^{n} I_i$. Then by Lemma 1.1.4 there exists pairwise orthogonal idempotents $e'_i \in A'$ such that $I_i = A'e'_i$ for $i = 1, \ldots, n$, and $1_{A'} = \sum_{i=1}^{n} e'_i$. Now to show that $e'_i$ is primitive, assume $e'_i = f_1 + f_2$ for some orthogonal idempotents $f_1, f_2$ of $A'$. Clearly, $A'e'_i \subset A'f_1 \oplus A'f_2$. For the converse inclusion, observe that any $x = r_1 f_1 + r_2 f_2 \in A'f_1 \oplus A'f_2$, $x e'_i = x \in A'e'_i$. Then $A'e'_i = A'f_1 \oplus A'f_2$. Since, $A'e'_i$ is a minimal left ideal, we get $A'f_1 = 0$ or $A'f_2 = 0$. So $f_1 = 0$ or $f_2 = 0$. Any idempotent in $A'$ can now be lifted to an idempotent in $A$ by Lemma 1.1.6 and by Lemma 1.1.8 the lifts can be chosen from primitive idempotents in $A$. \hfill \qed

**Proposition 1.2.3.** (Compare this with [6 Exercise 21.17]) Assume $A$ is an artinian $k$-algebra. If $E$ and $F$ are two finite sets of pairwise orthogonal primitive idempotents such that $1 = \sum_{e \in E} e = \sum_{f \in F} f$ then $|E| = |F|$. 

Proof. We consider the right ideal $fA$ of $A$ for a fixed $f \in F$. One can see that $\text{End}_A(fA)$ the $k$-algebra of right $A$-module endomorphisms of $fA$ is a quotient of the local artinian $k$-algebra $fAf$ which acts by multiplication on the left, thus itself is local artinian. We will denote $L_x$ the endomorphism of the right $A$-module $fA$ given by any $x \in fAf$. Then the identity morphism $L_f$ is split as $L_f = \sum_{e \in E} L_{f,e}$ since $f = \sum_{e \in E} f,e,f$. Now, there must be at least one $e \in E$ such that $L_{f,e} \in (fAf)\times$ otherwise the element $L_{f,e}$ would have been in the unique maximal ideal of $fAf$ for every $e \in E$ and $L_f = \sum_{e \in E} L_{f,e}$ would have not been invertible. Consider the sequence of morphisms of right $A$-modules $fA \xrightarrow{L_{e}} eA \xrightarrow{L_{f}} fA$. Since $L_{f,e} = L_{f,e}$ is invertible, we see that $fA$ must be a direct summand of $eA$ as a right $A$-module. However, $eAe$ is also local artinian, and therefore has no idempotents other than 0 and 1 by Lemma 1.1.9. This means $eA$ is indecomposable. Then $eA$ and $fA$ must be isomorphic as right $A$-modules. Thus get that $L_{e}$ has a two-sided inverse $L_{f,x} : eA \rightarrow fA$ for some $fx$ which satisfies $f = fxe$ and $e = efx$. Then we get $fe = fxe^2 = fxe = f$ and $ef = efxe = e^2 = e$. These equalities imply that for each $e \in E$ the idempotent $f \in F$ is unique and vice versa. This gives us $|E| = |F|$. \hfill \qed

### 1.3. The classification lemma.

**Definition 1.3.1.** For an artinian $k$-algebra $A$ we associate a canonical $k$-linear category $\mathcal{C}_A$ as follows. The set of objects of $\mathcal{C}_A$ is a finite set $E$ of orthogonal primitive idempotents which split $1_A$ as $1_A = \sum_{e \in E} e$. This set is nonempty by Proposition 1.2.2. For any $e, f \in \text{Ob}(\mathcal{C}_A)$ we let

$$\text{Hom}_{\mathcal{C}_A}(e, f) = fAe$$

and the composition is defined by the multiplication operation in $A$.

**Proposition 1.3.2** (The Classification Lemma). Let $\text{Hom}_k(\mathcal{C}_A, k\text{-Mod})$ be the category of $k$-linear functors from $\mathcal{C}_A$ to $k\text{-Mod}$ and their natural transformations. Then the category $\text{Hom}_k(\mathcal{C}_A, k\text{-Mod})$ is isomorphic to the category $A\text{-Mod}$ of left $A$-modules.
Proof. Using decomposition of the identity element $1 = \sum_{e \in E} e$ in terms of a set of primitive idempotents, we split any left $A$-module as a direct sum of $k$-modules $M = \bigoplus_{e \in E} eM$ and for any $x \in fAe$, the left action by $x$ defines a $k$-linear morphism $L_x : eM \to fM$ for any $e, f \in E$. In short, every left $A$-module defines a functor $\Phi(M) : C_A \to k\text{-Mod}$ which is defined by $\Phi(M)(e) = eM$ on the level of objects, and for any $fxe \in \text{Hom}_{C_A}(e, f)$ we let $\Phi(M)(fxe) := L_{fxe}$ the $k$-linear operator defined on $M$ by left action of $fxe \in fAe$. Conversely, for every functor $M : C_A \to k\text{-Mod}$ we define a left $A$-module $\Psi(M) := \bigoplus_{e \in E} M(e)$ where the left action of $A$ on $\Psi(M)$ is defined by using the decomposition $A = \bigoplus_{e, f \in E} fAe$, i.e. $fxe \cdot m = 0$ unless $m \in M(e)$ and then $fxe \cdot m$ is defined as $M(fxe)(m)$, the evaluation of $m$ under the $k$-linear morphism $M(fxe) : M(e) \to M(f)$. One can easily see that $\Psi \Phi$ is the identity functor on the category of $A\text{-Mod}$, and conversely $\Phi \Psi$ is the identity functor on $\text{Hom}_k(C_A, k\text{-Mod})$. □

Corollary 1.3.3. The category $C_A$ we defined in Definition 1.3.1 for a given artinian algebra $A$ is independent (up to an isomorphism) of the set of primitive idempotents splitting the identity.

Proof. The categories $C_A$ one can define for $E$ and for $F$ are both isomorphic to the category of left $A$-modules, and therefore, isomorphic to each other. □

Remark 1.3.4. One can view our Classification Lemma 1.3.2 as a generalization of semi-trivial extensions developed in [10] for artinian algebras. The Classification Lemma, in fact, allows one to reduce an artinian algebra to a $n \times n$ generalized matrix ring where $n = |E|$ is the number of primitive idempotents splitting the identity. Semi-trivial extensions are the $2 \times 2$ examples of this construction (not just for artinian algebras) in which one does not require primitivity of the idempotents. Since we work with artinian algebras, and we reduce our algebras to the submodules $eAf$ using idempotents $e$ and $f$ until they can not be further reduced, (i.e. until idempotents are primitive) we end up with a larger matrix ring.

2. Global dimension of artinian algebras

2.1. Reductions.

Remark 2.1.1. Assume $M$ is a module in $\text{Mod}-A$. Define the flat dimension (also known as the weak dimension) of $M$ as

$$\text{fl.dim}_A(M) = \sup\{n \in \mathbb{N} \mid \text{Tor}^A_n(M, N) \neq 0, N \in \text{Ob}(A\text{-Mod})\}$$

where the quantity of the right side is defined to be $\infty$ if the subset is unbounded in $\mathbb{N}$. Similarly we define the projective dimension of $M$ as

$$\text{pr.dim}_A(M) = \sup\{n \in \mathbb{N} \mid \text{Ext}^A_n(M, N) \neq 0, N \in \text{Ob}(\text{Mod}-A)\}$$

Now we define the global and the weak global dimension of $A$ as the quantities

$$\text{gl.dim}(A) = \sup\{\text{pr.dim}_A(M) \in \mathbb{N} \cup \{\infty\} \mid M \in \text{Ob}(\text{Mod}-A)\}$$
and

\[ \text{wgl.dim}(A) = \sup \{ \text{fl.dim}_A(M) \in \mathbb{N} \cup \{ \infty \} \mid M \in \text{Ob} (\text{Mod}-A) \} \]

Since \( A \) is not necessarily commutative, technically we should define a left and a right version of these dimensions. But because of Hopkins-Levitzki Theorem, if \( A \) is artinian it is also noetherian, and therefore, the left and right global dimensions and the global weak dimensions agree. (cf. [5, Cor. 5.60])

**Definition 2.1.2.** The two sided bar complex of an algebra \( A \) is defined as the differential graded \( A \)-bimodule \( CB_*(A) := \bigoplus_{n \geq 0} A^{\otimes n+2} \) together with the differentials

\[
d_n(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^n (-1)^i (a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1})
\]

We also define another differential graded module \( CB'_*(A) \) which is defined as a graded module as \( \bigoplus_{n \geq 0} A^{\otimes n+1} \) with similar differentials.

**Remark 2.1.3.** The two sided bar complex \( CB_*(A) \) gives us a projective resolution of \( A \) viewed as an \( A \)-bimodule. Thus the modified bar complex \( CB'_*(A) \) is acyclic. For a right \( A \)-module \( X \) and a left \( A \)-module \( Y \), the differential graded \( k \)-module

\[
C_*(X; A; Y) := X \otimes_A CB_*(A) \otimes_A Y
\]

together with the induced differentials give us the Tor-groups \( \text{Tor}^A_*(X,Y) \) in homology. We also define a second complex

\[
C'_*(X; A; Y) := X \otimes_A CB'_*(A) \otimes_A Y
\]

together with the induced differentials. Since \( C_*(X; A; A) \) is a projective resolution of \( X \), we see that \( C'_*(X; A; A) \) is acyclic. The same is true for \( C'_*(A; A; Y) \).

**Proposition 2.1.4.** Consider the subcomplex

\[
S_*(X; A; Y) := \bigoplus \bigoplus X e_0 \otimes e_0 A e_1 \otimes \cdots \otimes e_{n-1} A e_n \otimes e_n Y
\]

of the complex \( C_*(X; A; Y) \). Then the injection \( S_*(X; A; Y) \to C_*(X; A; Y) \) is a quasi-isomorphism.

**Proof.** Using the Pierce decomposition of \( A \) as \( \bigoplus_{e,f \in E} e A f \) we can split \( C_*(X; A; Y) \) as

\[
\bigoplus_{p \geq 0} \bigoplus e_i \in E X e_0 \otimes e_1 A e_2 \otimes \cdots \otimes e_{2p-1} A e_{2p} \otimes e_{2p+1} Y
\]

One can easily see that

\[
C_*(X; A; Y) = S_*(X; A; Y) \oplus \bigoplus_{e \neq f} C'_*(X; A; A e) \otimes C'_*(f A; A; Y)
\]

by counting the number of idempotents \( e_{2i} \neq e_{2i+1} \) in the full decomposition of \( C_*(X; A; Y) \). The result follows from the fact that \( C'_*(X; A; A e) \) and \( C'_*(f A; A; Y) \) are acyclic. \( \square \)

2.2. The directed graph of an artinian algebra.
Definition 2.2.1. Let us define a directed graph $G_A$ using an artinian algebra $A$ and a complete set of primitive idempotents $E$ splitting the identity. The set of vertices of $G_A$ is $E$. Two idempotents $e$ and $f$ are connected with a directed edge $f \leftarrow e$ if and only if the $k$-vector subspace $fAe$ is a non-zero.

Lemma 2.2.2. $G_A$ is independent of the choice of any complete set of primitive idempotents splitting identity.

Proof. Assume $E$ and $F$ are two sets of primitive idempotents splitting the identity, and let $G_A^E$ and $G_A^F$ be the directed graphs defined in Definition 2.2.1 for these sets. By Proposition 1.2.3, we have a bijection $\omega: E \rightarrow F$ which satisfies the identities

\[
e \cdot \omega(e) = e \quad \text{and} \quad \omega(e) \cdot e = \omega(e)
\]

for every $e \in E$. These identities also give us an isomorphism between the right $A$-modules $eA$ and $\omega(e)A$. Now, for every $e, e' \in E$ we have an edge from $e$ to $e'$ if and only if $e'Ae$ is non-zero. The $k$-vector subspace $e'Ae$ is also $\text{Hom}_A(eA, e'A)$ the $k$-vector space of $A$-module morphisms from $eA$ to $e'A$. Thus we obtain that $\text{Hom}_A(eA, e'A)$ is non-zero if and only if $\text{Hom}_A(\omega(e)A, \omega(e')A)$ is non-zero. This proves that the graphs $G_A^E$ and $G_A^F$ we write using $E$ and $F$ are isomorphic. □

Remark 2.2.3. For the rest of the article, we will fix an artinian algebra $A$ and a set $E$ of primitive idempotents splitting the identity $1_A$ of $A$.

Let $\pi_0(G_A)$ be the set of connected components of $G_A$. Note that since $E$ is finite, $\pi_0(G_A)$ is also finite. For every $\alpha \in \pi_0(G_A)$ we let $\chi_\alpha := \{ e \in E | \text{the vertex } e \text{ appears in } \alpha \}$. Note that for each $\alpha \in \pi_0(G_A)$

\[
e_\alpha := \sum_{e \in \chi_\alpha} e
\]

is a central idempotent in $A$ and also is the identity element of the subalgebra $A(\chi_\alpha)$. Moreover, the set $\{e_\alpha | \alpha \in \pi_0(G_A)\}$ forms a set of pairwise orthogonal central idempotents.

Definition 2.2.4. For a right $A$-module $X$ we define the support of $X$ as

\[E_X := \{ e \in E | Xe \neq 0 \}\]

The support of a left $A$-module is defined similarly.

Lemma 2.2.5. Assume $X$ and $Y$ are right $A$-modules. If there are no paths starting in $E_Y$ and ending in $E_X$ in the directed graph $G_A$ then $\text{Ext}^n_A(X, Y) = 0$ for every $n \geq 1$. The analogous result for $\text{Tor}^n_A(X, N)$ is true for every left $A$-module $N$.

Proof. The proof for the analogous result for $\text{Tor}^n_A(X, N)$ follows immediately from Proposition 2.1.4. Here we give a proof for $\text{Ext}^n_A(X, Y)$. Use the projective resolution of $X$ given by $S_*(X; A; A)$. Then the homology of the complex

\[
\bigoplus_{n \geq 0} \bigoplus_{e_i \in E} \text{Hom}_A(Xe_0 \otimes e_0Ae_1 \otimes \cdots e_{n-1}Ae_n \otimes e_nA, Y)
\]
with the induced differential, calculates $\text{Ext}^*_A(X, Y)$. On the other hand, we have $\text{Hom}_A(Ze \otimes eA, T) \cong \text{Hom}_k(Ze, Te)$ for every $e \in E$ and for all right $A$-modules $Z$ and $T$. So, we can rewrite the complex above as

$$\bigoplus_{n \geq 0} \bigoplus_{e_i \in E} \text{Hom}_k(Xe_0 \otimes e_0 Ae_1 \otimes \cdots e_{n-1} Ae_n, Ye_n)$$

this time with the differentials

$$(d\phi)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i \phi(\cdots \otimes a_i a_{i+1} \otimes \cdots) + (-1)^n \phi(a_0 \otimes \cdots \otimes a_{n-1}) a_n$$

for every $\phi \in \bigoplus_{e_i \in E} \text{Hom}_k(Xe_0 \otimes e_0 Ae_1 \otimes \cdots e_{n-1} Ae_n, Ye_n)$ and homogeneous tensor $a_0 \otimes a_1 \otimes \cdots \otimes a_n \in Xe_0 \otimes e_0 Ae_1 \otimes \cdots e_{n-1} Ae_n$. The result follows.

**Proposition 2.2.6.** $\text{gl.dim}(A) = \max_{\alpha \in \pi_0(G_A)} \text{gl.dim}(A(\chi_\alpha))$

**Proof.** Either we use Lemma 2.2.5 or we observe that one can see that the differential graded $k$-module $S_*(X; A; Y)$ splits as a direct sum of differential graded $k$-modules as

$$S_*(X; A; Y) = \bigoplus_{\alpha \in \pi_0(G_A)} S_*(X; A(\chi_\alpha); Y)$$

and then use the fact that weak global dimension and global dimensions agree when $A$ is artinian. \qed

**Remark 2.2.7.** We will call an artinian algebra $A$ connected if $G_A$ is connected, i.e. has only one connected component. An immediate corollary of Proposition 2.2.6 is that the problem of calculating the global dimension of an artinian algebra reduces to calculating the global dimensions of the connected subalgebras constructed out of connected components of $G_A$. Therefore, we can assume, without any loss of generality, that $A$ is connected.

### 2.3. The Source-Sink Theorem.

**Definition 2.3.1.** For any $U, V \subseteq E$ nonempty subset, we define

$$UAV := \bigoplus_{u \in U} \bigoplus_{v \in V} uAv$$

and we also define a new (non-unital) subalgebra $A(U)$ of $A$ by

$$A(U) := UAU = \bigoplus_{u, u' \in U} uAu'$$

**Definition 2.3.2.** Assume $\Gamma$ is a directed graph with no loops or multiple edges, and $U$ is an arbitrary subset of vertices. The quotient graph $\Gamma/U$ is the new graph obtained from $\Gamma$ by contracting every vertex in $U$ to a single vertex, which is again denoted by $U$, and then deleting all loops and oriented multiple edges. A vertex is $v$ in $\Gamma$ is called a sink (resp. a source) if there are no directed edges leaving $v$ (resp. ending at $v$.)

**Proposition 2.3.3.** Assume that there exists a proper subset of primitive idempotents $U \subset E$ with the property that the collapsed vertex $U$ in the quotient graph $G_A/U$ is a source or a sink. Then we
have
\[
\max\{\text{gl.dim}(A(U)), \text{gl.dim}(A(U^c))\} \leq \text{gl.dim}(A)
\]
In particular, if \(A(U)\) has infinite global dimension then so does \(A\).

Proof. We will use the fact that \(A\) is artinian, and therefore, the global and weak dimensions in both left and right variations all agree. Assume without generality that \(U\) is a sink in \(G/U\), and that \(M\) is a left \(A\)-module. One can easily see that \(A\) splits as
\[
A = UAU \oplus UAU^c \oplus U^cAU \oplus U^cAU^c
\]
However, we assumed that \(U\) is a sink in \(G/U\). This means \(U^cAU\) is 0. Therefore, given any free \(A\)-module \(F = A^\oplus I\) one has a splitting of the form
\[
F = A^\oplus I = (UAU)^\oplus I \oplus (U^cA)^\oplus I
\]
Hence any free \(A\)-module, restricted to a left \((A(U))\)-module becomes a flat left \((A(U))\)-module because \(U^cA\) is a flat left \((A(U))\)-module. The same also holds for projective \(A\)-modules. This means any \(A\)-projective resolution \(P_\bullet\) when restricted to a \((A(U))\)-module becomes \(UP_\bullet\) which is a flat \((A(U))\)-module. Therefore for every left \((A(U))\)-module \(M\) we get
\[
\text{pr.dim}_A(M) \geq \text{fl.dim}_{A(U)}(UM)
\]
which implies
\[
\text{gl.dim}(A) \geq \text{wgl.dim}(A(U)) = \text{gl.dim}(A(U))
\]
If we repeat the same proof for right modules, we get the same inequality for \(\text{gl.dim}(A(U^c))\). \(\square\)

Theorem 2.3.4 (The Source-Sink Theorem). Assume \(A\) is an artinian algebra such that there is a proper subset of primitive idempotents \(U \subset E\) with the property that the collapsed vertex \(U\) in the quotient graph \(G_A/U\) is a source or a sink. Then
\[
\max\{\text{gl.dim}(A(U)), \text{gl.dim}(A(U^c))\} \leq \text{gl.dim}(A) \leq 1 + \text{gl.dim}(A(U)) + \text{gl.dim}(A(U^c))
\]
Moreover, if \(A\) is flat over \((A(U))\) and \((A(U^c))\) then
\[
\max\{\text{gl.dim}(A(U)), \text{gl.dim}(A(U^c))\} \leq \text{gl.dim}(A) \leq \max\{1, \text{gl.dim}(A(U)), \text{gl.dim}(A(U^c))\}
\]
Proof. We proved that the lower bounds hold in Proposition 2.3.3. Now, without loss of generality we assume \(U\) is a sink in \(G_A/U\). Otherwise we switch \(U\) and \(U^c\), because \(U\) is a sink in \(G_A/U\) if and only if \(U^c\) is a source in \(G_A/U^c\). We define a filtration
\[
L^p_n = \sum_{q < p} \bigoplus_{n_0 + \cdots + n_q = n - q} \bigoplus_{\beta_i \in \{U,U^c\}} X_{\beta_0} \otimes A(\beta_0) \otimes \cdots \otimes A(\beta_q - 1) \otimes A(\beta_q) \otimes Y
\]
We consider the sequence of idempotents in each component \(X_{e_0} \otimes e_0 A e_1 \otimes \cdots e_{n-1} A e_n \otimes e_n Y\) as an oriented path in \(G_A\). The filtration counts the number of times each path enters in an out of \(U\) and \(U^c\). When we consider the spectral sequence associated with the filtration we see that \(E^{0}_{p,q} = L^p_{p+q}/L^{p-1}_{p+q}\).
is given by
\[
\bigoplus_{q_0 + \cdots + q_p = q, \beta_i \neq \beta_{i+1}} X \beta_0 \otimes A(\beta_0)^{\otimes q_0} \otimes \beta_0 A \beta_1 \otimes \cdots \otimes A(\beta_{p-1})^{\otimes q_{p-1}} \otimes \beta_{p-1} A \beta_p \otimes A(\beta_p)^{\otimes q_p} \otimes \beta_p Y
\]

Consider the subalgebra \( B = A(U) \oplus A(U^c) \). Then \( E^0_{p,q} \) represents a (multi-)product in \( D^+(B) \) the derived category of \( B \)-modules
\[
(1) \quad \bigoplus_{\beta_i \neq \beta_{i+1}} X \beta_0 \otimes_B R \beta_0 A \beta_1 \otimes_B R \cdots \otimes_B R \beta_{p-1} A \beta_p \otimes_R B \beta_p Y
\]

Notice that if \( A \) is flat over \( A(U) \) and \( A(U^c) \) then the columns in the \( E^1 \)-page collapse at \( q = 0 \) line for \( p > 0 \). In any case, if \( U \) were not a source and a sink, we would have had paths where the idempotents on the \( q = 0 \) line which would alternate between idempotents in \( U \) and \( U^c \). The first column \( E^0_{0,q} \), regardless of \( A \) is flat over \( A(U) \) and \( A(U^c) \), is represented by the product
\[
XU \otimes_B^R U^c Y \oplus XU^c \otimes_B^R U^c Y
\]
in the derived category while the second column \( E^0_{1,q} \) is given by
\[
XU \otimes_B^R UAU^c \otimes_B^R U^c Y
\]
and the rest of the terms are zero since \( U \) is a sink. In the first page \( E^1_{*,*} \) the height of the first column (after taking supremum over all \( X \) and \( Y \)) is bounded by the maximum of \( \text{gl.dim}(A(U)) \) and \( \text{gl.dim}(A(U^c)) \). The height of the second column \( E^1_{1,q} \) is bounded above by \( \text{gl.dim}(A(U)) + \text{gl.dim}(A(U^c)) \). Since this is the second column, its contribution to the homological dimension of \( A \) is shifted by 1. In the flat case, we see that \( E^1_{p,q} = 0 \) for \( p > 1 \) and \( q > 1 \). The only non-zero term on \( q = 0 \) line is at \( E^1_{1,0} = \sum_{u \in U} \sum_{v \in U^c} Xu \otimes A(U) uAv \otimes_A(U^c) vY \) and we have \( E^1_{p,0} = 0 \) for \( p > 1 \).

**Remark 2.3.5.** One can obtain a version of Proposition 2.3.3 using Corollary 4.3 of [3], and a version of Theorem 2.3.4 as a consequence of Corollary 3 of [11] for artinian algebras.

**Remark 2.3.6.** One can use Theorem 2.3.4 recursively and reduce the calculation of lower and upper bounds for the global dimension of \( A \) into calculations of the global dimensions of certain subalgebras \( A(U) \) and \( A(U^c) \) provided that the distinguished vertex \( U \) in the quotient graph \( G_A/U \) is a source or a sink at each recursion step. The recursion tree will split the graph \( G_A \) into full subgraphs and corresponding artinian subalgebras. Since \( E \) is finite, the recursion terminates and we obtain a partition \( U_1, \ldots, U_m \) of \( E \) where every subset \( U' \subset U_i \) is neither source nor a sink in \( G_{A(U_i)}/U' \) for each \( i = 1, \ldots, m \). The partition \( U_1, \ldots, U_m \), or the number of elements in the partition, need not be unique but for any such sequence we have the inequality
\[
\max\{\text{gl.dim}(A(U_1)), \ldots, \text{gl.dim}(A(U_m))\} \leq \text{gl.dim}(A) \leq (m-1) + \text{gl.dim}(A(U_1)) + \cdots + \text{gl.dim}(A(U_m))
\]

From this inequality we deduce that \( A \) has finite global dimension if and only if all of these terminal subalgebras have finite global dimension. We will further refine this algorithm in the next section. In the current version and in the best possible case we obtain the following Corollary.
Definition 2.4.2. One can define a graded multiplication structure on \( R \). Assume Corollary 2.3.7. Then

\[
\ker \left( \bigoplus_{u \in U} A^{\mathbb{Z}} \otimes uA \otimes uA \right) = 0
\]

Furthermore, if \( A \) is flat over \( A(U) \) for every non-empty \( U \subseteq E \) then

\[
\max \{ \text{gl.dim}(eA) \mid e \in E \} \leq \text{gl.dim}(A) \leq |E| - 1 + \sum_{e \in E} \text{gl.dim}(eA)
\]

2.4. Further reductions.

Definition 2.4.1. Fix a non-empty proper subset \( U \subseteq E \) and define a new differential graded \( k \)-module \( R^U \) by letting \( R^U_0 = A \) and \( R^U_1 = \bigoplus_{u \in U} A^{\mathbb{Z}} \otimes uA \) and then

\[
R^U_n = \bigoplus_{u_1 \in U} A^{\mathbb{Z}} \otimes u_1A^{\mathbb{Z}} \otimes u_2A \otimes \cdots \otimes u_{n-1}A \otimes u_nA
\]

Notice that \( R^U_\ast \) is a differential graded submodule of \( CB^\ast(A) \). We define \( R^U_\ast(X) \) and \( R^U_\ast(Y) \) similarly for a right \( A \)-module \( X \) and a left \( A \)-module \( Y \). Now, let us compute the homology \( H_n(R^U_\ast) \). If we consider the brutal truncation \( R^U_{\ast > 0} \)

\[
\bigoplus_{u \in U} A^{\mathbb{Z}} \otimes uA \xleftarrow{d_2} \bigoplus_{u_1, u_2 \in U} A^{\mathbb{Z}} \otimes u_1A^{\mathbb{Z}} \otimes u_2A \xleftarrow{d_4} \bigoplus_{u_1, u_2, u_3 \in U} A^{\mathbb{Z}} \otimes u_1A^{\mathbb{Z}} \otimes u_2A \otimes u_3A
\]

we obtain \( S_\ast(A; A(U); A) \). So, it is clear that

\[
H_n(R^U_\ast) = \text{Tor}_n^{A(U)}(A, A)
\]

for every \( n \geq 2 \). For lower degrees we must consider

\[
A \xleftarrow{d_1} \bigoplus_{u \in U} A^{\mathbb{Z}} \otimes uA \xleftarrow{d_2} \bigoplus_{u_1, u_2 \in U} A^{\mathbb{Z}} \otimes u_1A^{\mathbb{Z}} \otimes u_2A
\]

Since \( d_1 \) is really the multiplication morphism, for \( n = 0 \) we will get

\[
H_0(R^U_\ast) = \frac{A}{\sum_{u \in U} uA}
\]

Then \( \ker(d_1) = \ker \left( \bigoplus_{u \in U} A^{\mathbb{Z}} \otimes uA \to A \right) \) and

\[
H_0S_\ast(A; A(U); A) = \sum_{u \in U} A^{\mathbb{Z}} \otimes A(U) \ uA
\]

we get

\[
H_1(R^U_\ast) = \ker \left( \sum_{u \in U} A^{\mathbb{Z}} \otimes A(U) \ uA \to A \right)
\]

We have similar results for \( R^U_\ast(X) \) and \( R^U_\ast(Y) \), here written only for \( X \) as follows:

\[
H_nR^U_\ast(X) = \begin{cases} X/(\sum_{u \in U} XuA) & \text{if } n = 0 \\ \ker(\sum_{u \in U} Xu \otimes A(U) uA \to X) & \text{if } n = 1 \\ \text{Tor}_{n-1}^{A(U)}(X, A) & \text{if } n \geq 2 \end{cases}
\]

Definition 2.4.2. One can define a graded multiplication structure on \( R^U_\ast \) as follows: for any \( x = (x_0 \otimes \cdots \otimes x_n) \in R^U_n \) and \( y = (y_0 \otimes \cdots \otimes y_m) \in R^U_m \) we define \( x \cdot y \in R^U_{n+m} \) as

\[
x \cdot y := (x_0 \otimes \cdots \otimes x_{n-1} \otimes x_ny_0 \otimes y_1 \otimes \cdots \otimes y_m)
\]
Note that for every \((a_0u \otimes u a_1) \in R_*^U\) we have
\[
(a_0u \otimes u a_1) = (a_0 \otimes u)a_1 = a_0(u \otimes u a_1)
\]
and for \(n \geq 2\) and \((a_0u_1 \otimes u_1 a_1 u_2 \otimes \cdots \otimes u_{n-1} a_{n-1} u_n \otimes u_n a_n) \in R_n^U\) we see
\[
(a_0u_1 \otimes u_1 a_1 u_2 \otimes \cdots \otimes u_{n-1} a_{n-1} u_n \otimes u_n a_n) = (a_0u_1 \otimes u_1) \cdots (a_{n-1} u_n \otimes u_n)a_n
\]
In other words, \(R_*^U\) is generated (not necessarily freely) by elements of degree 1.

**Proposition 2.4.3.** \(R_*^U\) is a differential graded \(k\)-algebra. Moreover, if \(X\) is a right \(A\)-module then \(R_*^U(X)\) is a differential graded right \(R_*^U\)-module. A similar statement holds for a left \(A\)-module \(Y\) and \(R_*^U(Y)\).

**Proof.** Since \(R_*^U\) is generated by elements of degree 1, we must check whether the Leibniz rule is satisfied only for elements of degree 0 and generators of degree 1. We have
\[
d((a_0u \otimes u)a_1) = a_0ua_1 = d(a_0u \otimes u)a_1
\]
and
\[
d(a_0(u \otimes u a_1)) = a_0ua_1 = a_0d(u \otimes u a_1)
\]
for any \((a_0u \otimes u a_1) \in R_1^U\). For two generators of degree 1 we check
\[
d((au \otimes u)(bv \otimes v)) = d(au \otimes ubv \otimes v) = (aubv \otimes v) - (au \otimes ubv)
\]
On the other hand
\[
d(au \otimes u)(bv \otimes v) - (au \otimes u)d(bv \otimes v) = (aubv \otimes v) - (au \otimes ubv)
\]
are equal. So, the Leibniz rule
\[
d(x \cdot y) = d(x) \cdot y + (-1)^{\deg(x)} x \cdot d(y)
\]
holds. \(\square\)

**Proposition 2.4.4.** If \(H_0(R_*^U) = A/(\sum_{u \in U} AuA) = 0\) then \(A\) is a flat \(A(U)\)-module and \(H_n(R_*^U(X)) = 0\) for every \(A\)-module \(X\) and for every \(n \geq 0\). In that case \(A\) is Morita equivalent to \(A(U)\), and therefore we get \(\text{gl.dim}(A) = \text{gl.dim}(A(U))\).

**Proof.** We have \(H_0(R_*^U) = 0\) if and only if \(A = \sum_{u \in U} AuA\). In particular, there exists \(\alpha_u, \beta_u \in A\) such that \(1 = \sum_{u \in U} \alpha_u u \beta_u\). Since \(1 = \sum_{u \in U} \alpha_u u \beta_u\), and we see that the action morphism \(XU \otimes A(U)UA \to X\) is surjective for every right \(A\)-module \(X\), i.e. \(H_0(R_*^U(X)) = 0\). Let \(\sum_{u \in U} x_u u \otimes u a_u\) be in the kernel of the same morphism, i.e. let \(0 = \sum_{u \in U} x_u u a_u\). Then
\[
\sum_u x_u u \otimes u a_u = \sum_{u, u'} x_u u \otimes u a_u \alpha_{u'} u' \beta_{u'} = \sum_{u, u'} x_u u a_u \alpha_{u'} u' \otimes u' \beta_{u'} = 0
\]
since the tensor product is over \(A(U)\). In other words, the action morphism is also injective, i.e. \(H_1(R_*^U(X)) = 0\). Put \(P = AU := \sum_{u \in U} Au\) and \(Q = UA := \sum_{u \in U} uA\) and then we see that \(Q \otimes_A P \cong A(U)\). Then \(P \otimes_{A(U)} Q\) is isomorphic to \(A\) as a \(A\)-bimodule via the multiplication morphism.
The result follows that $A$ is Morita equivalent to $A(U)$. This means $UA$ and $AU$ are projective $A(U)$-modules. Then $A$ is a flat $A(U)$-module. This gives us $H_n(R^U_n(X)) = \text{Tor}_n^{A(U)}(X, A) = 0$ for $n \geq 1$. \hfill $\square$

**Remark 2.4.5.** The result we obtain in Proposition 2.4.4 shows obvious similarities with [10] for artinian algebras if one considers Remark 2, Theorem 2 and Remark 3 of *ibid.* together. However, our proof is specific to artinian algebras, and there are differences in assumptions such as using flatness as opposed to projectivity. Also, we realize the quotient $A/\left( \sum_{u \in U} AuA \right)$ as the 0-th homology of a differential graded algebra which acts on the (co)homology groups in a crucial way, as opposed to a separate invariant.

### 2.5. Flatness of $A$ as a bimodule over its subalgebras.

**Lemma 2.5.1.** Assume that there exists a subset $U \subset E$ such that $A$ viewed as a (bi-)module over $A(U)$ is flat. Then $\text{gl.dim}(A(U)) \leq \text{gl.dim}(A)$

This lemma appears as Lemma 1 of [8]. The proof we present below is different.

**Proof.** Assume $M$ is a left $A(U)$-module. Then the induced module $\text{Ind}^A_{A(U)}(M) := \bigoplus_{u \in U} Au \otimes_{A(U)} M$ is a left $A$-module. Let $P_*$ be a $A$-projective resolution of the induced $A$-module $\text{Ind}^A_{A(U)}(M)$ of length $\text{pr.dim}_A(\text{Ind}^A_{A(U)}(M)) \leq \text{gl.dim}(A)$. Since $\bigoplus_{u \in U} uA$ is right $A$-projective and left $A(U)$-flat, and since the functor $\bigoplus_{u \in U} uA \otimes_A (\cdot)$ sends projective $A$-modules to flat $A(U)$-modules, we see that

$$\bigoplus_{u \in U} uP_* = \bigoplus_{u \in U} uA \otimes_A P_*$$

is right $A(U)$-flat resolution of

$$M = \bigoplus_{u \in U} \sum_{u' \in U} uA \otimes_A u' \otimes_{A(U)} u'M$$

Then we easily see that $\text{wgl.dim}(A(U)) \leq \text{gl.dim}(A)$. The result follows since $A(U)$ is also artinian. \hfill $\square$

**Definition 2.5.2.** Assume we have a directed graph $G$, and a 2-coloring of vertices, i.e. the set of vertices is split as a union of two disjoint subsets. In this set-up an oriented path $\alpha$ is called *alternating* if any two consecutive vertices have opposite colors.

**Proposition 2.5.3.** Assume there exists a proper subset $U \subset E$ with the property that (i) $U$ is not a source or a sink in $G_A/U$, (ii) $H_0(R^U_0) := A/\left( \sum_{u \in U} AuA \right)$ is non-zero, (iii) $A$ is flat over both $A(U)$ and $A(U^c)$, and (iv) the global dimension $\text{gl.dim}(A)$ and $\text{gl.dim}(A(U^c))$ are both finite. In that case $\text{gl.dim}(A)$ is infinite if and only if $G_A$ has at least one alternating cycle with respect to the coloring $E = U \cup U^c$.

**Proof.** Assume $X = Y = H_0(R^U_0) := A/\left( \sum_{u \in U} AuA \right)$ is non-trivial. Consider $S_\ast(X; A; Y)$ and the filtration we defined in the proof of Theorem 2.3.4. Note that since $U$ is neither a source nor a sink in $G_A/U$, we see that $UAU^c$ and $U^c AU$ are both non-zero. The first column of the associated spectral sequence in the $E^1$-page yields $E^1_{0, q} = \text{Tor}_q^{A(U^c)}(X, Y)$ and in particular

$$E^1_{0, 0} = XU^c \otimes_{A(U^c)} U^c Y$$
This is because $X = Y$ are $A$-modules with the property that action of the subalgebra $A(U)$ is given by 0. We also see that $E_{1,0}^1$ is

$$XU \otimes_{A(U)} UAU^c \otimes A(U^c) U^c Y \oplus XU^c \otimes A(U^c) U^c AU \otimes A(U) UY$$

and it is trivial. Moreover, since we assumed $A$ is flat over $A(U)$ and $A(U^c)$, the spectral sequence collapses on two non-zero axis. Since $U$ is not a source or a sink in $G_{A/U}$, one can write paths of arbitrarily large lengths alternating between $U$ and $U^c$ if and only if $G_A$ has at least one alternating path. This means on the $q = 0$ row of the the $E^1$-page all odd dimensional vector spaces $E^1_{2m+1,0}$ are zero and there are non-trivial term of arbitrarily large even dimension if and only if $G_A$ has at least one alternating path. The phenomenon is repeated in homology since the differentials are necessarily 0. This means in the $E^2$-page on $q = 0$ row we have non-zero terms with arbitrarily large degrees if and only if $G_A$ has at least one alternating path. The result follows because the $p = 0$ column has bounded height due to the fact that $A(U^c)$ has finite global dimension. \[\square\]

2.6. The algorithm.

![Figure 1. An algorithm to estimate the global dimension of $A$.](image)

We summarize results we proved in this section in Figure 1 as an algorithm. This algorithm allows us to decide whether the global dimension of an artinian algebra is finite or infinite for a large class of algebras. However, there are also cases in which our algorithm fails to produce a definitive answer. We start with a proper subset $U \subset E$. If $U$ we chose is a source or a sink in the quotient graph $G_{A/U}$, we use Theorem 2.3.4 and reduce the calculation of the global dimension of $A$ to computing the global
dimensions of the subalgebras $A(U)$ and $A(U^c)$. Then we apply the algorithm to the subalgebras $A(U)$ and $A(U^c)$ recursively. If the subset $U$ we chose satisfies the condition that $A = \sum_{u \in U} AuA$ then we observe that $A$ and $A(U)$ have the same global dimension by Proposition 2.4.4. In that case we replace $A$ by $A(U)$ and proceed recursively. If it happens that that $U$ is not a source or a sink in the quotient graph $G_A/U$, and that $A$ is not equal $\sum_{u \in U} AuA$ we use a combination of Lemma 2.5.1 and Proposition 2.5.3 to proceed. The algorithm will reduce the calculation of upper and lower bounds for global dimensions of an artinian algebra $A$ at hand until it can no longer reduce $A$ using the graph $G_A$. The type of artinian algebras that can not be further reduced is called non-recursive whose definition we give below. The algorithm we described step by step in this section will determine the finiteness of global dimension of $A$ in terms of the global dimensions of some subset (which is not unique and will depend on the choices made) of possibly non-recursive artinian subalgebras.

**Definition 2.6.1.** An artinian algebra $A$ is called non-recursive if it satisfies the following conditions for every $U \subseteq E$:

1. $U$ is not a source or a sink in $G_A/U$
2. $A \neq \sum_{u \in U} AuA$
3. $A$ is not flat over $A(U)$, or $A$ is not flat over $A(U^c)$ and $\text{gl.dim}A(U)$ is finite.

Otherwise, we will call $A$ as recursive.

## 3. Incidence algebras and Quotients of free path algebras

### 3.1. Definitions and basic properties.

Assume $Q$ is a finite directed graph. The path algebra $kQ$ of the finite directed graph $Q$ is the vector space spanned by all the paths in $Q$. The multiplication on the paths is given by the concatenation of the sequences of edges of the paths $p$ and $q$ if source of $p$ is the same as the range of $q$ and zero otherwise. The set of all vertices in $Q$ form a complete set of orthogonal idempotents. Moreover, each vertex in $Q$, is a primitive idempotent of the path algebra, and vice versa. Hence, the directed graph $G_{kQ}$ of the path algebra $kQ$ in the sense of Definition 2.2.1 has the same vertex set as of $Q$. For any non-zero path $\alpha$ from vertex $e$ to vertex $f$, $\alpha = f\alpha e$ is in $f(kQ)e$ which is a non-zero $k$-vector space of $kQ$. So there is an edge connecting $e \rightarrow f$ in $G_{kQ}$ and the directed graph $G_{kQ}$ is an extension of the directed graph $Q$. If $Q$ does not have any oriented cycles, neither does $G_{kQ}$.

**Lemma 3.1.1.** Let $A := kQ$ be the free path algebra over a finite directed graph $Q$. Then $A$ is flat over $A(U)$ for every subset $U$ of vertices in $Q$.

**Proof.** Let $U$ be a subset of vertices in $Q$. Then $UA := \sum_{u \in U} uA$, the left ideal of paths ending at a vertex in $U$, has a basis over $A(U)$. This basis is the set of all paths starting at a vertex in $U^c$, ending at a vertex in $U$ and consist of no cycles within $U$. It is clear that this is a generating set, let us show that it is a basis. Assume we have distinct paths $\beta_1, \ldots, \beta_n$ in this generating set such that there are elements $\alpha_1, \ldots, \alpha_n$ in $A(U)$ with the property that $\sum \alpha_i \beta_i = 0$. Then for every path $\gamma$ appearing in $\alpha_1$ there is a path $\beta_j$ and a path $\gamma'$ in $A(U)$ appearing in $\alpha_j$ such that $\gamma \beta_1 = \gamma' \beta_j$. Without loss
of generality, assume the length of $\beta_1$ is less than or equal to the length of $\beta_j$. Then, $\beta_1$ is a part of $\beta_j$, which forces $\beta_j$ to go through the vertex set $U$ at least twice. This is a contradiction. Then the lengths of $\beta_1$ and $\beta_j$ are the same, and therefore, they are the same. This is also a contradiction. So, we conclude that the elements $\beta_1, \ldots, \beta_n$ are linearly independent which means $UA$ is free over $A(U)$, and therefore flat. Now, $A$ splits as a direct sum $A = UA \oplus U^cA$. The action of $A(U)$ on $U^cA$ is by zero making it flat over $A(U)$. Therefore, $A$ is flat over $A(U)$. The proof for $A$ viewed as a right module $A(U)$ is similar. □

Proposition 3.1.2. [2, Proposition 1.4] Assume $Q$ is a finite directed graph with no oriented cycles. Then the free path algebra $kQ$ of $Q$ is a hereditary $k$-algebra.

Proof. Since $kQ$ is a finite dimensional $k$-algebra, it is artinian. Moreover, $Q \subset G_{kQ}$ has no oriented cycles and $e(kQ)e = k$ for every vertex in $Q$ is of global dimension 0. The result follows using Corollary 2.3.7. □

Theorem 3.1.3. Let $A$ be the free path algebra of a cycle-free directed graph $Q = (Q_0, Q_1)$ and $I$ be a nil ideal of $A$. Then the artinian algebra $B := A/I$ has finite global dimension.

Proof. Let $\pi : A \rightarrow B$ be the canonical quotient morphism. Let $E_B$ denote a set of complete primitive idempotents of the artinian algebra $B$. Since $I$ is nil, for each $u \in E_B$ we can pick a primitive idempotent $\hat{u} \in A$ such that $\pi(\hat{u}) = u$ by Lemma 1.1.6 and Lemma 1.1.8. Let $e = \sum_{u \in E_B} \hat{u}$ the sum of all of these primitive idempotents, and let $f = 1_A - e$. Then $fAf$ is an artinian algebra with unit $f$, and therefore, has its own set of primitive idempotents splitting its unit $f$. Thus we completed the lifted set of primitive idempotents $\{\hat{u} \in A| u \in E_B\}$ to a full set of primitive idempotents $E_A$ splitting $1_A$. This set of primitive idempotents might not be necessarily the set $Q_0$, but one can find a bijection as in Proposition 1.2.2 and the resulting graph is isomorphic to $G_A$ by Lemma 2.2.2. Now, assume by way of contradiction that the directed graph $G_B$ of $B$ has a cycle $\alpha$ which starts and ends in an idempotent $\hat{u} \in E_A$. This is a contradiction since $Q$, and therefore $G_A$, has no cycles. Now, by Corollary 2.3.7 we get the finite upper bound $|E| - 1$ because $eBe \cong k$ is of global dimension 0 for every $e \in E$. □

Remark 3.1.4. Note that if $I$ is a nil ideal then it contains no idempotents other than 0. In particular, it contains no elements from the set of primitive idempotents.

For any partially ordered set $X$, define the corresponding incidence algebra $I(X)$ over $k$ as the set of all functions $f : X \times X \rightarrow k$ with $f(x, y) = 0$ unless $x \leq y$, together with the operations

$$(f + g)(x, y) = f(x, y) + g(x, y)$$

$$fg(x, y) = \sum_{x \leq z \leq y} f(x, z)g(z, y)$$

$$(rf)(x, y) = rf(x, y)$$
for any $r \in k$ and $f, g \in I(X)$ and $x, y \in X$. However, since the partially ordered sets we consider are all finite, we can identify our incidence algebras as algebras generated by symbols $E_{xy}$ for every $x \leq y$ subject to the relations

1. $E_{xt}E_{yz} = 0$ for $x, y, z, t \in X$ when $x \leq t$ and $y \leq z$, and finally $t \neq y$.
2. $E_{xz} = E_{xy}E_{yz}$ for $x, y, z \in X$ when $x \leq y \leq z$.
3. $E_{xy} = E_{xx}E_{xy} = E_{xy}E_{yy}$ for $x, y \in X$ when $x \leq y$.

A complete set of primitive idempotents of $I(X)$ splitting the identity is $\{E_{xx} : x \in X\}$. Hence, the set of all vertices of $G_{I(X)}$ is exactly $X$ and there is an edge of the form $x \leadsto y$ in $G_{I(X)}$ if and only if $x \leq y$ in $X$.

**Corollary 3.1.5.** [9] Section 9] The incidence algebra of any finite poset has finite global dimension.

**Proof.** Any finite poset $X$ determines a directed graph $G_X = (X, E)$ where edges are defined by elements which are in relation. Then the incidence algebra $I(X)$ over $X$ is isomorphic to a quotient of the free path algebra $kG_X$ of this graph as follows. One can define an algebra epimorphism by mapping each path $\alpha$ in $kX$ to the edge $E_{xy}$ which is an element in $I(X)$ where $x$ is the terminal point (target) of the path $\alpha$ and $y$ is the initial point (source) of $\alpha$. The kernel of this epimorphism is the subalgebra generated by the difference of paths with the same source and target. Then the result follows from Theorem 3.1.3. \qed

### 3.2. Calculations.

**Example 3.2.1.** Consider the directed graph $Q$

$$
\begin{array}{c}
\bullet \quad x \\
\bullet \quad y \\
\bullet \quad f
\end{array}
$$

and the free path algebra $A := kQ$ generated by this graph over our base field $k$. Let $U = \{e\}$ and $U^c = \{f\}$. We see that $A(U) = \text{Span}_k(e, xy, (yx)^2, (yx)^3, \ldots)$ and $A(U^c) = \text{Span}_k(f, xy, (xy)^2, (xy)^3, \ldots)$ are isomorphic to polynomial algebras over one indeterminate over $k$ which are commutative and of global dimension 1. Moreover, $A$ splits as an $A(U)$ module

$$A = A(U) \oplus xA(U) \oplus A(U)y \oplus xA(U)y \oplus \text{Span}_k(f)$$

and as an $A(U^c)$-module

$$A(U^c) \oplus yA(U^c) \oplus A(U^c)x \oplus yA(U^c)x \oplus \text{Span}_k(e)$$

This means $A$ is flat over $A(U)$ and $A(U^c)$. Moreover, we have $H_0(R^U_s) \cong H_0(R^U_{sc}) \cong k$ both are non-zero. Then by Proposition 2.5.3 we see that $\text{gl.dim}(kQ)$ is infinite.

**Example 3.2.2.** Consider the same directed graph $Q$ and we define $A := kQ/\langle xy, yx \rangle$. Then $A$ is a finite dimensional algebra $A = \text{Span}_k(e, f, x, y)$. Let $U = \{e\}$ and $U^c = \{f\}$. We see that $A(U) \cong k \cong A(U^c)$ which are semi-simple, and therefore, of global dimension 0. This means $A$ is
flat over \( A(U) \) and \( A(U^c) \). Moreover, we have \( H_0(R^U_0) \cong H_0(R^U_0^c) \cong k \) both are non-zero. Then by Proposition 2.5.3 we see that \( \text{gl.dim}(A) \) is infinite.

**Example 3.2.3.** Consider the same directed graph \( Q \) and let \( A := kQ/ \langle xy \rangle \). We get a finite dimensional algebra

\[
A = \text{Span}_k(e, f, x, y, xy)
\]

Again, let \( U = \{e\} \) and \( U^c = \{f\} \). We immediately see that \( A(U^c) = \text{Span}_k(f) \cong k \) is of global dimension 0. On the other hand, \( A(U) = \text{Span}_k(e, yx) \) of infinite global dimension because it is isomorphic to \( k[t]/ \langle t^2 \rangle \), and \( A \) splits as a \( A(U) \)-bimodule as

\[
A = A(U) \oplus \text{Span}_k(f, x, y)
\]

Then \( A \) is flat over \( A(U) \), and therefore, by Lemma 2.5.1 we conclude that \( A = kQ/ \langle xy \rangle \) is of infinite global dimension.

**Example 3.2.4.** Notice that Example 3.6 in [1] considers the path algebra \( A \) over \( Q \)

\[
\begin{array}{c}
\bullet \quad f_0 \\
\downarrow x_1 & \downarrow x_2 & \downarrow x_3 \\
\bullet \quad e_1 & \bullet \quad e_2 \\
\downarrow y_1 & \downarrow y_2 & \downarrow y_3 \\
\bullet \quad g_0 & \bullet \quad y_1 & \bullet \quad y_2 \end{array}
\]

subject to the relations \( x_2x_1 = 0 = y_2y_1, x_3x_2 = y_3y_2 \) and computes the \( \text{gl.dim}(A) = 2 \). Here we use category theoretic composition convention when we multiply elements. With the procedure defined above we get \( G_A \) as:

\[
\begin{array}{c}
\bullet \quad g_0 \\
\downarrow x_1 & \downarrow x_2 & \downarrow x_3 \\
\bullet \quad e_1 & \bullet \quad e_2 \\
\downarrow z_2 & \downarrow z_3 & \downarrow z_1 \\
\bullet \quad f_0 & \bullet \quad f_1 & \bullet \quad g_1 \\
\end{array}
\]

Let \( U = \{f_0, g_0\} \), then \( A(U) = f_0Af_0 \oplus g_0Ag_0 \) which is isomorphic to \( k \oplus k \), So \( \text{gl.dim}(A(U)) = 0 \). Let \( B = A(U^c) \). Now, pick new \( U = \{e_1, f_1, g_1\} \), then \( B(U) \) is a free path algebra that is not semi-simple and \( B(U^c) = e_2Be_2 \cong k \). Hence, \( \text{gl.dim}(A) \leq 3 \) is an upper bound which does not contradict the actual dimension.
Example 3.2.5. Consider the directed graph $Q$,

![Diagram of a directed graph]

and let $A$ be the path algebra over $Q$ subject to the relations

$$x_{2k}x_{2k-1} = y_{2k}y_{2k-1}, \quad x_{2k+1}x_{2k} = 0 = y_{2k+1}y_{2k}$$

for $k = 1, 2, \ldots, n$ with the same composition convention as above.

Then $G_A$ will be:

![Diagram of $G_A$]

Let $U = \{e_0, f_0, g_0\}$, then $A(U) = e_0Ae_0 \oplus e_0Af_0 \oplus e_0Ag_0 \oplus f_0Af_0 \oplus g_0Ag_0$. Since $A(U)$ is a free path algebra, it is hereditary. Moreover, $A(U)$ is algebra isomorphic to $\begin{bmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & k \end{bmatrix}$ which is not semi-simple, as the submodule $\begin{bmatrix} k & k & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is not a direct summand. So $\text{gl.dim}(A(U)) = 1$. By Theorem 2.3.3

$$\text{gl.dim}(A) \leq 1 + \text{gl.dim}(A(U)) + \text{gl.dim}(A(U^c)) = 2 + \text{gl.dim}(A(U^c))$$

Notice that if we call $A = A_n$ and $Q = Q_n$, then $A(U^c)$ is the path algebra on $Q_{n-1}$ subject to the relations $x_{2k}x_{2k-1} = y_{2k}y_{2k-1}, \ x_{2k+1}x_{2k} = 0 = y_{2k+1}y_{2k}$ for $k = 2, 3, \ldots, n$. Use the same procedure for the algebra $A_{n-1}$. Pick $U = \{e_1, f_1, g_1\}$. We get,

$$\text{gl.dim}(A_{n-1}) \leq 2 + \text{gl.dim}(A_{n-2})$$

If we continue in a similar manner, $A_0$ becomes $A_0 = e_0Ae_0$ which is isomorphic to $k$, so $\text{gl.dim}(A_0) = 0$. Hence, $\text{gl.dim}(A) \leq 2n$.

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