On graceful and harmonious labelings of trees

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Abstract

We prove the Kotzig-Ringel and the Graham-Sloane conjectures, respectively known as the Graceful and Harmonious Labeling Conjectures. We derive from these results, the spectra of two infinite families of second order constructs.

1 Introduction

The Kotzig-Ringel [R64, Gal05] and the Graham-Sloane [GS80] conjectures, better known as the Graceful Labeling Conjecture (or GLC for short) and the Harmonious Labeling Conjecture (or HLC for short) respectively, assert that every tree admits at least one graceful labeling and at least one harmonious labeling. Both labelings have in common the fact that they assign labels to vertices so as to devise a bijection between vertex labels and induced edge labels. In the context of the GLC, induced edge labels correspond to residue classes (modulo the number of vertices) of sums of integer labels assigned to the vertices spanning each edge. In the context of the HLC, induced edge labels correspond to absolute differences of integer labels assigned to the vertices spanning each edge. For notional convenience, we consider a functional reformulation of both problems. A rooted tree on \( n \) vertices is naturally associated with a discrete function

\[
f \in ([0, n) \cap \mathbb{Z})^{(0,n)\cap \mathbb{Z}} \quad \text{subject to} \quad \left| f^{(n-1)} ([0, n) \cap \mathbb{Z}) \right| = 1
\]

(1)

where

\[
\forall i \in [0, n) \cap \mathbb{Z}, \quad f^{(0)} (i) := i \quad \text{and} \quad \forall k \geq 0, \quad f^{(k+1)} (i) = f^{(k)} (f (i)) = f \left( f^{(k)} (i) \right).
\]

arbitrary \( f \in ([0, n) \cap \mathbb{Z})^{(0,n)\cap \mathbb{Z}} \) is associated with a functional directed graph \( G_f = (V (G_f), E(G_f)) \) where

\[
V (G_f) := [0, n) \cap \mathbb{Z} \quad \text{and} \quad E(G_f) := \{(i, f (i)) : i \in [0, n) \cap \mathbb{Z}\}.
\]

We summarize some special induced edge labelings of a functional directed graph \( G_f \) associated with \( f \in ([0, n) \cap \mathbb{Z})^{(0,n)\cap \mathbb{Z}} \) as follows:

- Induced subtractive edge labels given by \( \{ |f (i) - i| : i \in [0, n) \cap \mathbb{Z} \} \) determine whether or not \( G_f \) is gracefully labeled.
- Induced additive edge labels given by \( \{ f (i) + i \mod n : i \in [0, n) \cap \mathbb{Z} \} \) determine whether or not \( G_f \) is harmoniously labeled.
- More generally, \( \tau \)-induced edge labels given by \( \{ \tau (i, f (i)) : i \in [0, n) \cap \mathbb{Z} \} \) determine whether or not \( G_f \) is \( \tau \)-Zen labeled for some function \( \tau \in ([0, n) \cap \mathbb{Z})^{(0,n)\cap \mathbb{Z} \times [0,n)\cap \mathbb{Z}}\).

Moreover a given functional directed graph \( G_f \) associated with \( f \in ([0, n) \cap \mathbb{Z})^{(0,n)\cap \mathbb{Z}} \) is said to be

- graceful if there exist \( \sigma \in S_n / Aut(G_f) \) such that \( \{ |\sigma f (i) - \sigma (i)| : i \in [0, n) \cap \mathbb{Z} \} = [0, n) \cap \mathbb{Z} \)
- harmonious if there exist \( \sigma \in S_n / Aut(G_f) \) such that \( \{ \sigma f (i) + \sigma (i) \mod n : i \in [0, n) \cap \mathbb{Z} \} = [0, n) \cap \mathbb{Z} \)

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more generally \(\tau\)-Zen if there exist \(\sigma \in S_n/\text{Aut}(G_f)\) such that \(\{\tau(\sigma(i), \sigma f(i)) : i \in [0, n) \cap \mathbb{Z}\} = [0, n) \cap \mathbb{Z}\) for some function \(\tau \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z} \times [0, n) \cap \mathbb{Z}}\).

The following two propositions respectively express the permutation reformulations of grace and harmony condition.

**Proposition 0a**: (Permutation formulation) An arbitrary functional directed graph \(G_f\) associated with a function \(f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}}\) is graceful if and only if

\[\exists \sigma \in S_n/\text{Aut}(G_f) \text{ and } \gamma \in S_n \text{ such that } \forall i \in [0, n) \cap \mathbb{Z}, \ f(i) \in \sigma \left(\sigma^{-1}(i) \pm \gamma \sigma^{-1}(i)\right)\]

**Proof**: On one hand, the proof of necessity, follows from the fact that \(G_f\) being graceful implies that

\[\exists \sigma \in S_n/\text{Aut}(G_f), \text{ such that } \{\sigma f(j) - \sigma(j) : j \in [0, n) \cap \mathbb{Z}\} = [0, n) \cap \mathbb{Z},\]

by change of variable

\[\left\{\sigma f \sigma^{-1}(i) - \sigma^{-1}(i) : i \in [0, n) \cap \mathbb{Z}\right\} = [0, n) \cap \mathbb{Z}.\]

Consequently there exist a permutation \(\gamma \in S_n\) such that

\[\forall j \in [0, n) \cap \mathbb{Z}, \ |\sigma f \sigma^{-1}(j) - j| = \gamma(j)\]

another change of variable yields the desired result

\[f(i) \in \sigma \left(\sigma^{-1}(i) \pm \gamma \sigma^{-1}(i)\right), \ \forall i \in [0, n) \cap \mathbb{Z}.\]

On the other hand the proof of sufficiency follows from the fact that if a given function \(f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}}\) is such that

\[\forall i \in [0, n) \cap \mathbb{Z}, \ f(i) \in \sigma^{-1} \left(\sigma(i) \pm \gamma \sigma(i)\right)\]

then the corresponding functional directed graph \(G_{\sigma f \sigma^{-1}}\) is isomorphic to the gracefully labeled functional directed graph \(G_{\sigma f \sigma^{-1}}\)

thereby completing the proof.

As a corollary of Proposition 0a, the assertion that \(G_f\) associated with \(f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}}\) is graceful implies that \(f\) admits a graceful expansion of the form

\[f(i) = \sigma \left(\sigma^{-1}(i) + s_f(\gamma, \sigma^{-1}(i)) \cdot \gamma \sigma^{-1}(i)\right)\]

where \(\sigma \in S_n/\text{Aut}(G_f), \gamma \in S_n\) and \(s_f \in \{-1, 1\}^{S_n \times [0, n) \cap \mathbb{Z}}\). For instance, given an identically constant function in \(([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}}\) prescribed by

\[\forall i \in [0, n) \cap \mathbb{Z}, \ f(i) = c,\]

for some fixed constant \(c \in [0, n) \cap \mathbb{Z}\) we have

\[\forall i \in [0, n) \cap \mathbb{Z}, \ f(i) = \sigma \left(\sigma^{-1}(i) + s_f(\gamma, \sigma^{-1}(i)) \cdot \gamma \sigma^{-1}(i)\right)\]

where

\[\forall i \in [0, n) \cap \mathbb{Z}, \ s_f(\gamma, i) = \begin{cases} -1 & \text{if } \gamma = \text{id} \\ 1 & \text{if } \gamma = (n - 1) - \text{id} \\ \text{undefined} & \text{otherwise} \end{cases}\]

and \(\sigma\) can be taken to be an arbitrary element of the coset of \(\text{Aut}(G_f)\) corresponding to permutations which maps \(c\) to \(0.\) We devise from Proposition 0a, a procedures for finding all graceful expansion of functional trees. Let \(f \in ([0, n) \cap \mathbb{Z})^{[0, n) \cap \mathbb{Z}}\) be given.
such that $|f^{(n-1)}([0, n) \cap \mathbb{Z})| = 1$. Let $\rho \in \mathbb{N}$ denote the length of the longest directed path in $G_f$ starting at a leaf node and terminating at the fixed point of $f$. It follows that

$$\forall i \in [0, n) \cap \mathbb{Z}, \quad f^{(\rho)}(i) = \sigma \left( \sigma^{(-1)}(i) + s_{f^{(\rho)}}(\gamma, \sigma^{(-1)}(i)) \cdot \gamma \sigma^{(-1)}(i) \right),$$

where

$$\forall i \in [0, n) \cap \mathbb{Z}, \quad s_{f^{(\rho)}}(\gamma, i) = \begin{cases} -1 & \text{if } \gamma = \text{id} \\ 1 & \text{if } \gamma = (n-1) - \text{id} \\ \text{undefined} & \text{otherwise} \end{cases}$$

If we further posit by ansatz that $f^{(\rho-1)}$ admits a graceful expansion of the form

$$f^{(\rho-1)}(i) = \sigma \left( \sigma^{(-1)}(i) + s_{f^{(\rho-1)}}(\theta, \sigma^{(-1)}(i)) \cdot \theta \sigma^{(-1)}(i) \right)$$

for some permutation $\theta \in S_n$ then it follows that

$$\sigma \left( \sigma^{(-1)}f(i) + s_{f^{(\rho-1)}}(\theta, \sigma^{(-1)}f(i)) \cdot \theta \sigma^{(-1)}f(i) \right) = \sigma \left( \sigma^{(-1)}(i) + s_{f^{(\rho)}}(\gamma, \sigma^{(-1)}(i)) \cdot \gamma \sigma^{(-1)}(i) \right)$$

$$\implies \sigma^{(-1)}f(i) + s_{f^{(\rho-1)}}(\theta, \sigma^{(-1)}f(i)) \cdot \theta \sigma^{(-1)}f(i) = \sigma^{(-1)}(i) + s_{f^{(\rho)}}(\gamma, \sigma^{(-1)}(i)) \cdot \gamma \sigma^{(-1)}(i),$$

$$\implies s_{f^{(\rho-1)}}(\theta, \sigma^{(-1)}f(i)) \cdot \theta \sigma^{(-1)}f(i) = \left( \sigma^{(-1)}(i) - \sigma^{(-1)}f(i) \right) + s_{f^{(\rho)}}(\gamma, \sigma^{(-1)}(i)) \cdot \gamma \sigma^{(-1)}(i),$$

hence

$$\begin{cases} s_{f^{(\rho-1)}}(\theta, \sigma^{(-1)}f(i)) = \text{sign} \left\{ (\sigma^{(-1)}(i) - \sigma^{(-1)}f(i)) + s_{f^{(\rho)}}(\gamma, \sigma^{(-1)}(i)) \cdot \gamma \sigma^{(-1)}(i) \right\} \\ \theta \sigma^{(-1)}f(i) = \left| (\sigma^{(-1)}(i) - \sigma^{(-1)}f(i)) + s_{f^{(\rho)}}(\gamma, \sigma^{(-1)}(i)) \cdot \gamma \sigma^{(-1)}(i) \right| \end{cases}$$

Note that the graceful expansion of $f^{(\rho)}$ is prescribed modulo $\sigma \in \text{Aut}(G_{f^{(\rho)}})$ while the postulated graceful expansion of $f^{(\rho-1)}$ is prescribed only modulo $\sigma \in \text{Aut}(G_{f^{(\rho-1)}})$. Since

$$\text{Aut}(G_{f^{(\rho-1)}}) \subset \text{Aut}(G_{f^\rho})$$

both $s_{f^{(\rho-1)}}$ and $\theta$ are determined by evaluations at elements of $\text{Aut}(G_{f^{(\rho)}})$ which lie in distinct cosets of $\text{Aut}(G_{f^{(\rho)}})$. We shall refer to this evaluation process as the act of performing a symmetry breaking of $\text{Aut}(G_{f^{(\rho)}})$ by the $\text{Aut}(G_{f^{(\rho-1)}})$ symmetry. Our ansatz is thus rejected if $\theta$ is found not be one to one and onto for every valid choice of the permutation $\gamma$. On the other hand the postulated graceful expansion is validated by identifying a permutation $\theta$ which satisfy the constraints. Further more distinct choices for the permutation $\gamma$ determines distinct choices for $\theta$.

**Proposition 0b:** (Permutation reformulation) An arbitrary functional directed graph $G_f$ associated with

$$f \in (\mathbb{Z}/n\mathbb{Z})^{2/n\mathbb{Z}}$$

is harmonious if and only if

$$\exists \sigma \in S_n/\text{Aut}(G_f) \text{ and } \gamma \in S_n \text{ such that } \forall i \in \mathbb{Z}/n\mathbb{Z}, \quad f(i) \equiv \sigma^{(-1)}(\gamma \sigma(i) + (n-1) \sigma(i)) \mod n$$

**Proof:** The proof of necessity, follows from the fact that $G_f$ being harmonious implies that

$$\exists \sigma \in S_n/\text{Aut}(G_f), \text{ s.t. } \{ \sigma f(j) + \sigma(j) : j \in \mathbb{Z}/n\mathbb{Z} \} = \mathbb{Z}/n\mathbb{Z}$$

by change of variable

$$\left\{ \sigma f \sigma^{(-1)}(i) + \sigma \sigma^{(-1)}(i) : i \in \mathbb{Z}/n\mathbb{Z} \right\} = \mathbb{Z}/n\mathbb{Z}$$
Consequently there exist a permutation \( \gamma \in S_n \) such that
\[
\implies \forall j \in \mathbb{Z}/n\mathbb{Z}, \quad \sigma f^{\gamma^{-1}}(j) + j \equiv \gamma(j) \mod n
\]
a change of variable yields the desired result
\[
f(i) \equiv \sigma^{\gamma^{-1}}(\gamma \sigma(i) + (n - 1) \sigma(i)) \mod n, \quad \forall i \in \mathbb{Z}/n\mathbb{Z}
\]
On the other hand the proof of sufficiency follows from the fact that if an arbitrarily given function \( f \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}} \) is subject to the condition
\[
f(i) \equiv \sigma^{\gamma^{-1}}(\gamma \sigma(i) + (n - 1) \sigma(i)) \mod n, \quad \forall i \in \mathbb{Z}/n\mathbb{Z}
\]
then \( G_f \) is isomorphic to the harmoniously labeled functional directed graph \( G_{\sigma f^{\gamma^{-1}}} \) and thereby completes the proof.

As a corollary of Proposition 0b, the assertion that \( G_f \) associated with \( f \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}} \) is harmonious implies that \( f \) admits a harmonious expansion of the form
\[
f(i) = \sigma \left( \gamma_\sigma \sigma^{\gamma^{-1}}(i) - \sigma^{\gamma^{-1}}(i) \right)
\]
where \( \sigma \in S_n/Aut(G_f) \), \( \gamma_\sigma \in S_n \). For instance an identically constant function in \( (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}} \) prescribed by
\[
\forall i \in \mathbb{Z}/n\mathbb{Z}, \quad f(i) = c,
\]
we have
\[
\forall i \in \mathbb{Z}/n\mathbb{Z}, \quad f(i) = \sigma \left( \gamma_\sigma \sigma^{\gamma^{-1}}(i) - \sigma^{\gamma^{-1}}(i) \right)
\]
where \( \gamma_\sigma \) is the permutation of elements of \( \mathbb{Z}/n\mathbb{Z} \) prescribed by \( \gamma_\sigma = d + \text{id} \) prescribed for every permutation \( \sigma \) taken from coset of \( Aut(G_f) \) corresponding to permutations which maps \( c \) to \( d \). We also devise from Proposition 0b, a method for finding all harmonious expansion for functional trees. Let \( f \in (\mathbb{Z}/n\mathbb{Z})^{\mathbb{Z}/n\mathbb{Z}} \) be a given function such that \( |f^{(n-1)}(\mathbb{Z}/n\mathbb{Z})| = 1 \), and let \( \rho \in \mathbb{N} \) denote the edge length of the longest path in \( G_f \) whose endpoints corresponds to a leaf node and the fixed point of \( f \). It follows from these premises that
\[
f^{(\rho)}(i) = \sigma \left( \gamma_\sigma \sigma^{\gamma^{-1}}(i) - \sigma^{\gamma^{-1}}(i) \right),
\]
where \( \gamma_\sigma = d + \text{id} \). If we further posit by ansatz that \( f^{(\rho^{-1})} \) admits a graceful expansion of the form
\[
f^{(\rho^{-1})}(i) = \sigma \left( \theta_\sigma \sigma^{\gamma^{-1}}(i) - \sigma^{\gamma^{-1}}(i) \right)
\]
for some permutation \( \theta_\sigma \in S_n \) then it follows that
\[
\sigma \left( \theta_\sigma \sigma^{\gamma^{-1}}f(i) - \sigma^{\gamma^{-1}}f(i) \right) = \sigma \left( \gamma_\sigma \sigma^{\gamma^{-1}}(i) - \sigma^{\gamma^{-1}}(i) \right)
\]
\[
\implies \theta_\sigma \sigma^{\gamma^{-1}}f(i) - \sigma^{\gamma^{-1}}f(i) = \gamma_\sigma \sigma^{\gamma^{-1}}(i) - \sigma^{\gamma^{-1}}(i)
\]
\[
\implies \theta_\sigma \sigma^{\gamma^{-1}}f(i) = \gamma_\sigma \sigma^{\gamma^{-1}}(i) + \sigma^{\gamma^{-1}}f(i) - \sigma^{\gamma^{-1}}(i)
\]
Note that the harmonious expansion of \( f^{(\rho)} \) is prescribed modulo \( \sigma \in Aut(G_{f^{\rho}}) \) while the postulated graceful expansion of \( f^{(\rho^{-1})} \) is prescribed modulo \( \sigma \in Aut(G_{f^{\rho^{-1}}}) \). Since
\[
Aut(G_{f^{\rho^{-1}}}) \subset Aut(G_{f^{\rho}})
\]
\( \theta_\sigma \) is determined by performing a symmetry breaking of the \( Aut(G_{f^{\rho}}) \) symmetry by \( Aut(G_{f^{\rho^{-1}}}) \) symmetry. The ansatz is thus rejected if \( \theta_\sigma \) found not be one to one and onto. On the other hand the postulated harmonious expansion is validated by identifying a permutation \( \theta_\sigma \).

Let \( \text{GrL}(G_f) \) and \( \text{HaL}(G_f) \) respectively denote the set of distinct graceful and harmonious relabelings of the functional directed graph \( G_f \). The induced edge label sequence of a graph refers to the non-decreasing sequence of induced edge labels. For instance the function in Figure 1
\[
f : \{0, 1, 2, 3, 4, 5\} \rightarrow \{0, 1, 2, 3, 4, 5\}
\]
defined by
\[ f(0) = 0, \ f(1) = 0, \ f(2) = 0, \ f(3) = 0, \ f(4) = 3, \ f(5) = 3, \]
is a functional spanning subtree of the complete graph (or functional tree for short) on 6 vertices. The attractive fixed point condition from Eq. (1) is met since
\[ f^5(\{0, 1, 2, 3, 4, 5\}) = \{0\}. \]
The edge set of \( G_f \) is \( \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 3), (5, 3)\} \), the corresponding induced subtractive edge label sequence is equal to the corresponding induced additive edge label sequence and given by
\[ (0, 1, 1, 2, 2, 3). \]

The GLC and HLC are easily verified for the families of functional stars such as identically constant functions. This is seen from the fact that the constant zero function
\[ f : [0, n) \cap \mathbb{Z} \rightarrow [0, n) \cap \mathbb{Z} \]
such that
\[ f(i) = 0, \ \forall i \in [0, n) \cap \mathbb{Z} \]
is simultaneously gracefully and harmoniously labeled. In particular
\[ |\text{GrL}(G_f)| = 2 \quad \text{and} \quad |\text{HaL}(G_f)| = n. \]

Our main results are proofs of Composition Lemmas, from which proofs of the GLC, the strong GLC \cite{GW18} and the HLC follow as corollaries. We conclude the paper by showing that the strong GLC and HLC provide concrete illustrations of spectra for two infinite families of constructs as introduced in \cite{GG18}.

This article is accompanied by an extensive SageMath\cite{S18} graceful graph package which implements the symbolic constructions described here. The package is made available at the link:
https://github.com/gnang/Graceful-Graphs-Package

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2 Determinantal Certificate of Gracefulness and Harmony

Given two multivariate polynomials \( F(x), G(x) \in \mathbb{C}[x_0, \cdots, x_{n-1}] \) each of which splits into linear factors over \( \mathbb{C} \) of the form
\[
F(x) = \prod_{0 \leq i < m} (P_i(x))^{\alpha_i}, \quad G(x) = \prod_{0 \leq i < m} (P_i(x))^{\beta_i},
\]
for some non-negative integers \( \{\alpha_i, \beta_i\}_{0 \leq i < m} \). If each non-identically constant factor \( P_i(x) \) is a multivariate polynomial of degree at most 1 in each individual variable and has no common root in the field of fractions \( \mathbb{C}(x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n-1}) \) with any
other factor when viewed as a polynomial over \( \mathbb{C} [x_0, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n-1}] [x_k] \) for each \( 0 \leq k < n \) then recall that
\[
\prod_{0 \leq i < m} (P_i(x))^{\max(n_i, \beta_i)} = \operatorname{LCM} \{ F(x), G(x) \}
\]
\[
\prod_{0 \leq i < m} (P_i(x))^{\min(n_i, \beta_i)} = \operatorname{GCD} \{ F(x), G(x) \}
\]
(2)

Also recall that for an arbitrary multivariate polynomial \( H(x) \in \mathbb{C} [x_0, \cdots, x_{n-1}] \) we define the canonical representative of the residue class \( H(x) \mod \{ x_i^n \}_{0 \leq i < n} \) (also called the remainder) to be the polynomial interpolant of individual variable degree at most \( (n-1) \) given by
\[
\left( H(x) \mod \{ x_i^n \}_{0 \leq i < n} \right) := \sum_{f \in (\mathbb{Z} \cap [0, n])^n} H(f(0), \cdots, f(n-1)) \prod_{k \in [0, n) \cap \mathbb{Z}} \left( \prod_{j_k \in [0, n) \cap \mathbb{Z} \setminus f(k)} \left( \frac{x_k - j_k}{f(k) - j_k} \right) \right).
\]
(3)

For notational convenience we used above the falling factorial shorthand notation
\[
x_i^\underline{n} := \prod_{j \in [0, n) \cap \mathbb{Z}} (x_i - j).
\]

Consequently, the remainder
\[
H(x) \mod \{ x_i^n \}_{0 \leq i < n}
\]
is obtainable via Lagrange interpolation as prescribed in Eq. (3) or alternatively by performing Euclidean divisions of successive remainders obtained where the divisors are taken from the set of univariate polynomials in the set \( \{ x_i^n \}_{0 \leq i < n} \) irrespective of the order.

The following proposition describes a determinental construction for certifying that a given functional directed graph \( G_f \) associated with an arbitrary \( f \in (\mathbb{Z} \cap [0, n)]^n \) is graceful.

**Proposition 1a : (Determinental gracefulness certificate)** A functional directed graph \( G_f \) associated with \( f \in (\mathbb{Z} \cap [0, n)]^n \) is such that
\[
\max_{\sigma \in S_n} \left\{ \left| \sigma^{(-1)} f \sigma(i) - i \right| : i \in [0, n) \cap \mathbb{Z} \right\} = m
\]
for some \( m \leq n \) if and only if there exist \( L \subseteq [0, n) \cap \mathbb{Z} \) subject to \( |L| = m \) such that
\[
0 \neq \operatorname{LCM} \left\{ \prod_{u < v} (x_v - x_u), \prod_{i < j} \left( (x_{f(i)} - x_j)^2 - (x_{f(j)} - x_i)^2 \right) \right\} \mod \{ x_k^6 : k \in L \cup f(L) \}
\]

**Proof :** As a result of the fact that we are moding by the algebraic relations
\[
\{ x_k^6 = 0 : k \in L \cup f(L) \}
\]
the remainder of each LCM is completely determined by evaluations over the integer lattice \((\mathbb{Z} \cap [0, n)]^n \) as prescribed in Eq. (3). This ensure a discrete set of roots for the resulting polynomial. It suffices to prove the claim when \( L = [0, n) \cap \mathbb{Z} \). A particular LCM therefore vanishes at a particular point of the integer lattice \((\mathbb{Z} \cap [0, n)]^n \) only if one of the factors vanishes at that point. Furthermore, a factor of the multivariate polynomial construction vanishes at a particular point of the integer lattice \((\mathbb{Z} \cap [0, n)]^n \) if two vertices are assigned the same label (from the vertex Vandermonde determinant factor) or alternatively if two distinct edges
are assigned the same label (from the edge Vandermonde determinant factor). The proof of necessity follows from the observation that the only possible roots to the multivariate polynomial

$$\text{LCM} \left\{ \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq i < j < n} \left( (x_f(j) - x_j)^2 - (x_f(i) - x_i)^2 \right) \right\} \mod \{x_i^n\}_{0 \leq i < n}$$

arise from vertex label assignments \(x \in ([0, n) \cap \mathbb{Z})^n\) for which either distinct vertex variables are assigned the same label or distinct edges are assigned the same induced subtractive edge label. Consequently, the congruence identity

$$0 \equiv \text{LCM} \left\{ \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq i < j < n} \left( (x_f(j) - x_j)^2 - (x_f(i) - x_i)^2 \right) \right\} \mod \{x_i^n\}_{0 \leq i < n}$$

implies that \(G_f\) admits no graceful labeling. On the other hand the proof of sufficiency follows from the fact that every graceful labeling of \(G_f\) yields an assignment to the vertex variable entries of \(x\) such that

$$\prod_{0 \leq i < j < n} (x_j - x_i) \left( (x_f(j) - x_j)^2 - (x_f(i) - x_i)^2 \right) \in \left\{ \pm \prod_{0 \leq i < j < n} \left( j-i \right)^2 (j+i) \right\},$$

thus completing the proof.\(\square\)

Note that the polynomial construction above is determinental since

$$\text{det}(V) = \prod_{0 \leq i < j < n} (x_j - x_i) \left( (x_f(j) - x_j)^2 - (x_f(i) - x_i)^2 \right),$$

where

$$V[i, j] = \sum_{0 \leq k < n} x_i^k (x_f(j) - x_j)^{2k} = \frac{1 - \left( x_i (x_f(j) - x_j)^{2} \right)^n}{1 - x_i (x_f(j) - x_j)^{2}}, \quad \forall 0 \leq i, j < n.$$  

Let \(\mathbb{S}_n/\mathbb{A}_{\text{AutG}_f}\) denote the set of representative of the equivalence classes of permutation orbits of the form \(\{\sigma, (n-1) - \sigma\} \subset \mathbb{S}_n/\mathbb{A}_{\text{AutG}_f}\). The following combinatorial resolvent construction follows as a corollary of Prop. (1a)

$$\forall \gamma \in \mathbb{S}_n, \prod_{\sigma \in \mathbb{S}_n/\mathbb{A}_{\text{AutG}_f}} \left[ 1 - \prod_{0 \leq i < j < n} \left( \frac{(x_{\sigma^{-1}} f \sigma(j) - x_j)^2 - (x_{\sigma^{-1}} f \sigma(i) - x_i)^2}{j^2 - i^2} \right)^2 \right] \equiv 1 \mod \{x_i - \gamma(i)\}_{0 \leq i < n}$$

if the functional graph \(G_f\) admits no graceful labeling and otherwise whenever the functional graph \(G_f\) is graceful we have

$$\prod_{\sigma \in \mathbb{S}_n/\mathbb{A}_{\text{AutG}_f}} \left[ 1 - \prod_{0 \leq i < j < n} \left( \frac{(x_{\sigma^{-1}} f \sigma(j) - x_j)^2 - (x_{\sigma^{-1}} f \sigma(i) - x_i)^2}{j^2 - i^2} \right)^2 \right] = \sum_{0 \leq i < n} (x_i - i) \ g_{i, f}(x)$$

for some \(\{g_{i, f}(x) : i \in [0, n) \cap \mathbb{Z}\} \subset \mathbb{C}[x_0, \cdots, x_{n-1}]\). Symmetries of the polynomials in \(\{g_{i, f}(x) : i \in [0, n) \cap \mathbb{Z}\}\) relative to permutation of the variables determine the vertices which can be assigned the labeled 0 in some graceful labeling of \(G_f\).

We now describe a similar determinental construction for certifying that a functional directed graph is harmonious. Recall from [GS80] that a functional directed graph \(G_f\) associated with \(f \in ([0, n) \cap \mathbb{Z})^2\) is harmonious if

$$\forall 0 \leq i < j < n, \ (f(j) + j) \neq (f(i) + i) \mod n.$$ 

Note that induced additive edge labels are more simply obtained from products of \(n\)-th roots of unity assigned to the vertices spanning each edge.
**Proposition 1b:** (Determinantal harmony certificate) A functional directed graph $G_f$ associated with $f \in ([0, n] \cap \mathbb{Z})^{[0, n] \cap \mathbb{Z}}$ is such that

$$\max_{\sigma \in S_n} \left\{ \sigma^{(-1)} f \sigma \left( i + i \mod n : i \in [0, n] \cap \mathbb{Z} \right) \right\} = m$$

for some $m \leq n$ if and only if there exist $L \subseteq [0, n] \cap \mathbb{Z}$ subject to $|L| = m$ such that

$$0 \neq \text{LCM} \left\{ \prod_{u < v} (x_u - x_v), \prod_{i < j}^{0 \leq i < j < n} (x_{f(i)j} x_j - x_{f(i)i} x_i) \right\} \mod \{x_k^n - 1 : k \in [0, n] \cap \mathbb{Z}\}$$

**Proof:** It suffices to prove the claim when $|L| = n$. Quite similarly to the argument used to prove Prop. (1a), the proof of sufficiency follows from the observation that the only possible roots to the multivariate polynomial

$$\text{LCM} \left\{ \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq i < j < n}^{0 \leq i < j < n} (x_{f(j)j} x_j - x_{f(i)i} x_i) \right\} \mod \{x_k^n - 1 : 0 \leq i < n\},$$

arise from vertex label assignments for which either distinct vertex variables are assigned the same label or distinct edges are assigned the same induced additive edge label. Consequently the congruence identity

$$0 \equiv \text{LCM} \left\{ \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq i < j < n}^{0 \leq i < j < n} (x_{f(j)j} x_j - x_{f(i)i} x_i) \right\} \mod \{x_k^n - 1 : 0 \leq i < n\},$$

implies that $G_f$ admits no harmonious labeling. On the other hand the proof of necessity follows from the fact that every harmonious labeling of $G_f$ yields an assignment to the vertex variables $\{x_i\}_{0 \leq i < n}$ such that

$$\prod_{0 \leq i < j < n} (x_j - x_i) (x_{f(j)j} x_j - x_{f(i)i} x_i) \in \left\{ \pm \prod_{0 \leq i < j < n} \left( \omega_n^j - \omega_n^i \right)^2 \right\},$$

where $\omega_n := \exp\left( \frac{2\pi \sqrt{-1}}{n} \right)$. Thus completing the proof. □

Note that the polynomial construction above is also determinental since

$$\det(W) = \prod_{0 \leq i < j < n} (x_j - x_i) (x_{f(j)j} x_j - x_{f(i)i} x_i),$$

where

$$W[i, j] = \frac{1 - (x_i x_j x_{f(j)})^n}{1 - x_i x_j x_{f(j)}}, \quad \forall 0 \leq i, j < n.$$

The following combinatorial resolvent construction follows as a corollary of Prop. (1b)

$$\forall \gamma \in S_n, \prod_{\sigma \in S_n/\text{Aut} G_f} \left[ 1 - \prod_{0 \leq i < j < n} \left( \frac{(x_j x_{\sigma^{-1} f \sigma(j)}) - (x_i x_{\sigma^{-1} f \sigma(i)})}{\omega_n^j - \omega_n^i} \right)^2 \right] \equiv 1 \mod \left\{ x_i - \omega_n^{\gamma(i)} \right\}_{0 \leq i < n}$$

if $G_f$ is not harmonious and otherwise if $G_f$ is harmonious

$$\prod_{\sigma \in S_n/\text{Aut} G_f} \left[ 1 - \prod_{0 \leq i < j < n} \left( \frac{(x_{\sigma^{-1} f \sigma(j)}) - (x_{\sigma^{-1} f \sigma(i)})}{\omega_n^j - \omega_n^i} \right)^2 \right] = \sum_{0 \leq i < n} (x_i - \omega_n^i) h_{i, f}(x)$$

for some $\{h_{i, f}(x) : i \in [0, n] \cap \mathbb{Z}\} \subseteq \mathbb{C}[x_0, \ldots, x_{n-1}]$. 8
3 Composition Lemmas.

We state and prove here the first of two weak Composition Lemmas.

**Lemma 2a**: (weak Graceful Composition Lemma) For every \( f \in ([0, n] \cap \mathbb{Z})^{0,n} \cap \mathbb{Z} \)

\[
\max_{\sigma \in S_n} \left\{ \left| \left( \sigma f \sigma^{-1} \right) (i) - i \right| : i \in [0, n] \cap \mathbb{Z} \right\} \leq \max_{\sigma \in S_n} \left\{ \left| \sigma f \sigma^{-1} (i) - i \right| : i \in [0, n] \cap \mathbb{Z} \right\}.
\]

**Proof**: Note that for all \( g \in ([0, n] \cap \mathbb{Z})^{0,n} \cap \mathbb{Z} \) subject to the fixed point condition \( |g^{(n-1)}([0, n] \cap \mathbb{Z})| = 1 \), the explicit expression of the LCM

\[
\pm \text{LCM} \left\{ \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq i < j < n} \left( (x_{g(j)} - x_j)^2 - (x_{g(i)} - x_i)^2 \right) \right\} \mod \{x_n^i\}_{0 \leq i < n}
\]

is given by

\[
\pm \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq i < j < n} \left( (x_{g(j)} - x_j)^2 - (x_{g(i)} - x_i)^2 \right) \times
\]

\[
\prod_{d(i, g^2(i)) = 3} (x_{g^3(i)} - x_{g^2(i)})^2 - (x_{g(i)} - x_i)^2 \prod_{d(i, g^2(i)) = 2} (x_{g^2(i)} + x_i - 2x_{g(i)}) \prod_{0 \leq i < j < n} (2x_{g(j)} - x_j - x_i),
\]

where \( d(u, v) \) denotes the undirected non-loop edge distance separating vertex \( u \) from vertex \( v \) in \( G_f \). We prove the claim by contradiction. Assume for the sake of establishing a contradiction that

\[
m = \max_{\sigma \in S_n} \left\{ \left| \left( \sigma f \sigma^{-1} \right) (i) - i \right| : i \in [0, n] \cap \mathbb{Z} \right\} > \max_{\sigma \in S_n} \left\{ \left| \sigma f \sigma^{-1} (i) - i \right| : i \in [0, n] \cap \mathbb{Z} \right\},
\]

for some \( m \leq n \). On the one hand the premise that

\[
\max_{\sigma \in S_n} \left\{ \left| \left( \sigma f \sigma^{-1} \right) (i) - i \right| : i \in [0, n] \cap \mathbb{Z} \right\} = m
\]

implies that there exist \( L \subset [0, n] \cap \mathbb{Z} \) subject to \( |L| = m \) such that

\[
0 \neq \prod_{u < v} (x_v - x_u) \prod_{0 \leq i < j < n} \left( (x_{f^{(2)}(j)} - x_{f^{(2)}(i)})^2 - (x_{f^{(2)}(i)} - x_i)^2 \right) \mod \{x_k^m : k \in L \cup f(L) \cup f^{(2)}(L)\}
\]

and crucially for every set \( S \subset [0, n] \cap \mathbb{Z} \) subject to \( |S| > |L| \) we have

\[
0 = \prod_{u < v} (x_v - x_u) \prod_{0 \leq i < j < n} \left( (x_{f^{(2)}(j)} - x_{f^{(2)}(i)})^2 - (x_{f^{(2)}(i)} - x_i)^2 \right) \mod \{x_k^m : k \in S \cup f(S) \cup f^{(2)}(S)\}.
\]

We therefore rewrite the first of the two congruence identities above as

\[
0 \neq \prod_{u < v} (x_v - x_u) \prod_{0 \leq i < j < n} \left( (x_{f^{(2)}(j)} - x_{f^{(2)}(i)} - x_i)^2 - (x_{f^{(2)}(i)} - x_f(i) - x_i)^2 \right) \mod \{x_k^m\}_{k}
\]
On the other hand the premise that
this latter identity asserts that the congruence identity is invariant under performing the following linear transformation

\[ W \text{ derive from both identities the following congruence identity} \]

\[ 0 \equiv \prod_{u < v} (x_v - x_u) \prod_{i < j} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right)^{1 - \delta_{ij}} \left( x_j - x_{f(j)} \right)^2 - \left( x_i - x_{f(i)} \right)^2 \mod \left\{ x_k^m \mid t \in L \cup f(L) \cup f^2(L) \right\}. \]

On the other hand the premise that
\[ \max_{\sigma \in S_n} \left\{ \left| \sigma f \sigma^{-1}(i) - i \right| \mid i \in [0, n] \cap \mathbb{Z} \right\} < m \]
implies that

\[ 0 \equiv \prod_{u < v} (x_v - x_u) \prod_{i < j} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right)^{1 - \delta_{ij}} \left( x_j - x_{f(j)} \right)^2 - \left( x_i - x_{f(i)} \right)^2 \mod \left\{ x_k^m \mid t \in L \cup f(L) \cup f^2(L) \right\}. \]

We therefore rewrite the congruence identity as

\[ 0 \equiv \prod_{u < v} (x_v - x_u) \prod_{i < j} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right)^{1 - \delta_{ij}} \left( x_j - x_{f(j)} \right)^2 - \left( x_i - x_{f(i)} \right)^2 \mod \left\{ x_k^m \mid t \in L \cup f(L) \cup f^2(L) \right\}. \]

It follows by construction that the right hand side of the latter expression must vanish for any assignment of \( x \) on the integer lattice \( ([0, n] \cap \mathbb{Z})^n \) for which

\[ \left| \left\{ (x_{f(i)} - x_i) \mid i \in [0, n] \cap \mathbb{Z} \right\} \right| < m. \]

We derive from both identities the following congruence identity

\[ \pm \prod_{u < v} (x_v - x_u) \sum_{k_{ij} \in \{0, 1\}} \prod_{0 = \prod_{i < j} k_{ij}} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right)^{1 - \delta_{ij}} \left( x_j - x_{f(j)} \right)^2 - \left( x_i - x_{f(i)} \right)^2 \mod \left\{ x_k^m \mid t \in L \cup f(L) \cup f^2(L) \right\}. \]

This latter identity asserts that the congruence identity is invariant under performing the following linear transformation

\[ \{(x_{f(i)} - x_i) \leftrightarrow (x_{f(i)} - x_i) \mid i \in L \}. \]
to the variables in the edge Vandermonde factor. This in turn leads to a contradiction

\[ \prod_{u < v} (x_v - x_u) \prod_{i < j} \left( (x_{f(ij)} - x_j)^2 - (x_{f(ij)} - x_i)^2 \right) \equiv 0, \quad u, v \in L \cup f(L) \cup f^2(L) \]

\[ \prod_{u < v} (x_v - x_u) \prod_{i < j} \left( (x_{f(j)} - x_j)^2 - (x_{f(i)} - x_i)^2 \right) \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^2(L) \right\}. \]

We therefore conclude that

\[ \max_{\sigma \in S_n} \left\{ \left| \left( \sigma^{-1} f \sigma \right)^2(i) - i \right| : i \in [0, n] \cap \mathbb{Z} \right\} \leq \max_{\sigma \in S_n} \left\{ \left| \sigma^{-1} f \sigma(i) + i \right| : i \in [0, n] \cap \mathbb{Z} \right\}. \]

We now state and prove a similar Harmonious Composition Lemma.

**Lemma 2b**: (Harmonious Composition Lemma) For every \( f \in (\mathbb{Z}/n\mathbb{Z})^2 \)

\[ \max_{\sigma \in S_n} \left\{ \left| \left( \sigma^{-1} f \sigma \right)^2(i) - i \right| : i \in [0, n] \cap \mathbb{Z} \right\} \leq \max_{\sigma \in S_n} \left\{ \left| \sigma^{-1} f \sigma(i) + i \right| : i \in [0, n] \cap \mathbb{Z} \right\} \]

**Proof**: Note that for all \( g \in (\mathbb{Z}/n\mathbb{Z})^2 \) subject to \( |g^{(n-1)}(\mathbb{Z}/n\mathbb{Z})| = 1 \), the explicit expression of the LCM is given by

\[ \pm \text{LCM}_g = \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq u < v < n} (x_{g(j)} x_j - x_{g(i)} x_i) \prod_{d(i, g(i)) = 3} (x_{g^{(3)}}(i) x_{g^{(2)}}(i) - x_{g(i)} x_i) \times \left( \prod_{d(i, g^{(2)}(i)) = 2} x_{g(i)} \right) \left( \prod_{0 \leq i < j < n} x_{g(j)} \right). \]

We prove the claim by contradiction. Assume for the sake of establishing a contradiction that

\[ m = \max_{\sigma \in S_n} \left\{ \left| \left( \sigma^{-1} f \sigma \right)^2(i) + i \right| : i \in [0, n] \cap \mathbb{Z} \right\} > \max_{\sigma \in S_n} \left\{ \left| \sigma^{-1} f \sigma(i) + i \right| : i \in [0, n] \cap \mathbb{Z} \right\}, \]

for some \( m \leq n \). On the one hand the premise that

\[ \max_{\sigma \in S_n} \left\{ \left| \left( \sigma^{-1} f \sigma \right)^2(i) + i \right| : i \in [0, n] \cap \mathbb{Z} \right\} = m \]

implies that there exist \( L \subset [0, n] \cap \mathbb{Z} \) subject to \( |L| = m \) such that

\[ 0 \neq \prod_{u < v} (x_v - x_u) \prod_{i < j} (x_{f(ij)} x_j - x_{f^{(2)}} x_i) \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}. \]
and crucially for every set \( S \subset [0, n) \cap \mathbb{Z} \) subject to \(|S| > |L|\) we have

\[
0 \equiv \prod_{u < v} (x_v - x_u) \prod_{i < j} (x_{f^{(2)}(j)}x_j - x_{f^{(2)}(i)}x_i) \mod \left\{ x_k^n - 1 : k \in S \cup f(S) \cup f^{(2)}(S) \right\}.
\]

We therefore rewrite the first of the two congruence identity as

\[
0 \equiv \prod_{u < v} (x_v - x_u) \prod_{i < j} \left[x_j \left(x_{f^{(2)}(j)} - x_{f(j)}\right) - x_i \left(x_{f^{(2)}(i)} - x_{f(i)}\right) + x_jx_{f(j)} - x_ix_{f(i)}\right] \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]

\[
\Rightarrow 0 \not\equiv \prod_{u < v} (x_v - x_u) \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]

\[
\sum_{k_{ij} \in \{0, 1\}} \prod_{i < j} \left[x_j \left(x_{f^{(2)}(j)} - x_{f(j)}\right) - x_i \left(x_{f^{(2)}(i)} - x_{f(i)}\right)\right]^{1-k_{ij}} \left(x_jx_{f(j)} - x_ix_{f(i)}\right)^{k_{ij}} \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]

On the other hand the premise

\[
\max_{\sigma \in S_n} \left| \left\{ \sigma^{-1}f\sigma(i) + i : i \in [0, n) \cap \mathbb{Z} \right\} \right| < m
\]

implies that

\[
0 \equiv \prod_{u < v} (x_v - x_u) \prod_{i < j} (x_i x_{f(i)} - x_j x_{f(j)}) \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]

We therefore rewrite the congruence identity as

\[
0 \equiv \prod_{u < v} (x_v - x_u) \prod_{i < j} \left(x_j \left(x_{f(j)} - x_{f^{(2)}(j)}\right) - x_i \left(x_{f(i)} - x_{f^{(2)}(i)}\right) + x_jx_{f(j)} - x_ix_{f(i)}\right) \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]

\[
\Rightarrow 0 \not\equiv \prod_{u < v} (x_v - x_u) \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]

\[
\sum_{k_{ij} \in \{0, 1\}} \prod_{i < j} \left[x_j \left(x_{f(j)} - x_{f^{(2)}(j)}\right) - x_i \left(x_{f(i)} - x_{f^{(2)}(i)}\right)\right]^{1-k_{ij}} \left(x_jx_{f^{(2)}(j)} - x_ix_{f^{(2)}(i)}\right)^{k_{ij}} \mod \left\{ x_k^n - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]
It follows by construction that the expression must vanish for any assignment of $x$ on the integer lattice $([0, n] \cap \mathbb{Z})^n$ for which
\[
\left| \{ x_{f^{(2)}(i)} x_i : i \in [0, n] \cap \mathbb{Z} \} \right| < m.
\]

We derive from both identities the following congruence identity
\[
\pm \prod_{u, v \in L \cup f(L) \cup f^{(2)}(L)} (x_j - x_i) \sum_{k_{ij} \in \{0, 1\}} \prod_{i < j} (x_j x_{f^{(2)}(j)} - x_i x_{f^{(2)}(i)}) k_{ij} [x_j (x_{f^{(2)}(j)} - x_{f(j)}) - x_i (x_{f^{(2)}(i)} - x_{f(i)})]^{1 - k_{ij}}
\]
\[
\equiv \prod_{u, v \in L \cup f(L) \cup f^{(2)}(L)} (x_j - x_i) \times
\sum_{k_{ij} \in \{0, 1\}} \prod_{i < j} (x_j x_{f^{(2)}(j)} - x_i x_{f^{(2)}(i)}) k_{ij} [x_j (x_{f(j)} - x_{f(j)_{i}}) - x_i (x_{f(i)} - x_{f(i)_{i}})]^{1 - k_{ij}}
\]
\mod \left\{ x^2_k - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]

This latter identity asserts that the congruence class is invariant under performing the transformation
\[
\{ (x_{f^{(2)}(i)} x_i) \leftrightarrow (x_{f(i)} x_i) : i \in L \},
\]
to the variables in the edge Vandermonde factor which leads to a contradiction
\[
\prod_{u, v \in L \cup f(L) \cup f^{(2)}(L)} (x_v - x_u) \prod_{i < j} (x_{f^{(2)}(j)} x_j - x_{f^{(2)}(i)} x_i) \equiv
\prod_{u, v \in L \cup f(L) \cup f^{(2)}(L)} (x_v - x_u) \prod_{i < j} (x_{f(i)} x_i - x_{f(j)} x_j) \mod \left\{ x^2_k - 1 : k \in L \cup f(L) \cup f^{(2)}(L) \right\}.
\]

We therefore conclude that
\[
\max_{\sigma \in S_n} \left| \left\{ (\sigma^{-1} f \sigma)^{(2)}(i) + i : i \in \mathbb{Z} / n \mathbb{Z} \right\} \right| \leq \max_{\sigma \in S_n} \left| \left\{ (\sigma^{-1} f \sigma)(i) + i : i \in \mathbb{Z} / n \mathbb{Z} \right\} \right|.
\]

4 The Graceful and Harmonious Labeling Theorems.

The Graceful Labeling Theorem follows from the weak graceful Composition Lemma as follows.

**Theorem 3a** (Graceful Labeling Theorem) All trees are graceful.

**Proof**: For a functional directed graph $G_f$ associated with a function $f \in ([0, n] \cap \mathbb{Z})^{[0, n] \cap \mathbb{Z}}$ subject to $|f([0, n] \cap \mathbb{Z})| = 1$, we have
\[
\left( \lim_{t \to \infty} f(t) \right) = f^{(n-1)}.
\]
and for such functions
\[
0 \neq \text{LCM} \left\{ \prod_{0 \leq i < j \leq n} (x_j - x_i), \prod_{0 \leq i < j \leq n} (x_{f^{(n-1)}(i)} - x_j)^2 - (x_{f^{(n-1)}(i)} - x_i)^2 \right\} \mod \left\{ x^2_k : i \in [0, n] \cap \mathbb{Z} \right\}.
\]
In fact the explicit expression for the remainder is given by

\[
\prod_{0 \leq i < j < n} (j^2 - i^2) \left( \sum_{\sigma \in S_n} \text{sgn} \sigma \prod_{0 \leq i < n} \prod_{0 \leq j < n, j \neq i} (x_i - \sigma(j)) / (\sigma(i) - \sigma(j)) \right) + \\
\sum_{\sigma \in S_n} (-1)^{\binom{n}{2}} \text{sgn} \sigma \prod_{0 \leq i < n} \prod_{0 \leq j < n, j \neq i} (x_i - \sigma(j)) / (\sigma(i) - \sigma(j)) \right).
\]

The desired result therefore follows by repeatedly applying the contrapositive of the weak graceful Composition Lemma. □

Similarly, we derive as a corollary of the harmonious Composition Lemma a proof of the Harmonious Labeling Conjecture as follows.

**Theorem 3b** : (Harmonious Labeling Theorem) All trees are harmonious.

*Proof*: For a functional directed graph \( G_f \) associated with a function \( f \in (\mathbb{Z}/n\mathbb{Z})^2/\mathbb{Z} \) subject to \( |f(\mathbb{Z}/n\mathbb{Z})| = 1 \), we have

\[
\lim_{t \to \infty} f^{(t)} = f^{(n-1)}.
\]

Furthermore, for such functions

\[
0 \not\equiv \text{LCM} \left\{ \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq u < v < n} (x_f^{(n-1)}(v)x_v - x_f^{(n-1)}(u)x_u) \right\} \mod \{ x_i^n - 1 \}_{0 \leq i < n},
\]

\[
\text{LCM} \left\{ \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq u < v < n} (x_f^{(n-1)}(v)x_v - x_f^{(n-1)}(u)x_u) \right\} = (x_f^{(n-1)}(0))^{\binom{n}{2}} \prod_{0 \leq i < j < n} (x_j - x_i)
\]

from which the explicit expression of the remainder of the LCM is given by

\[
\sum_{\sigma \in S_n} \text{sgn}(\sigma) \left( x_f^{(n-1)}(0) \right)^{\binom{n}{2}} \prod_{0 \leq i < j < n} (x_i)^{\sigma(i)}
\]

The desired result therefore follows by repeatedly applying the contrapositive of the harmonious Composition Lemma. □

### 5 Strengthening the Composition Lemma.

We state here and prove by contradiction a strong Composition Lemma which establishes as a corollary the strong GLC first proposed in [GW18]. For notational convenience, let

\[
\text{LCM}_{\ell,f} := \text{LCM} \left\{ \prod_{0 \leq i < j < n} (x_j - x_i), \prod_{0 \leq i \neq f(i) < n} (x_f(i) - x_i)^2 - j^2 \right\} \times
\]

\[
\prod_{n - \ell \leq j < n} (x_f(i) - x_i)^2 - j^2 \right\}
\]
\[
\prod_{0 \leq u < v < w < n} (x_{f(w)} - x_w)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^0 + (x_{f(v)} - x_v)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^1 + (x_{f(u)} - x_u)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^2
\]

The polynomial construction above is a variant of the determinantal certificate described in Prop. (1a). The main difference between the two certificates is that the polynomial \( \text{LCM}_{\ell,f} \) does not vanish identically modulo the algebraic relations

\[
\{ x_i^n = 0 : i \in [0, n) \cap \mathbb{Z} \}
\]

if and only if the functional directed graph \( G_f \) admits a labeling in which no two distinct vertices are assigned the same label, no three distinct edges are assigned the same subtractive induced edge label and none of the induced edge labels is greater than \( n - \ell - 1 \).

**Lemma 4**: (strong Graceful Composition Lemma) For all \( f \in ([0,n) \cap \mathbb{Z})^{(0,n)\cap\mathbb{Z}} \) subject to Eq. \([0]\) and some integer \( 0 \leq \ell < \left\lceil \frac{n-1}{2} \right\rceil \)

\[
0 \not\equiv \text{LCM}_{\ell,f(2)} \mod \{x_i^n\}_{0 \leq i < n} \quad \text{and} \quad 0 \not\equiv \text{LCM}_{\ell,f(2)} \mod \{x_i^n\}_{0 \leq i < n}.
\]

**Proof**: Assume for the sake of establishing a contradiction that

\[
0 \equiv \text{LCM}_{\ell,f} \mod \{x_i^n\}_{0 \leq i < n} \quad \text{and} \quad 0 \not\equiv \text{LCM}_{\ell,f(2)} \mod \{x_i^n\}_{0 \leq i < n}.
\]

On the one hand the premise that \( 0 \equiv \text{LCM}_{\ell,f} \mod \{x_i^n\}_{0 \leq i < n} \) implies that

\[
0 \equiv \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq i < n} \left( (x_{f(i)} - x_i)^2 - j^2 \right) \times \prod_{n - \ell \leq j < n} (x_f(w) - x_w)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^0 + (x_{f(v)} - x_v)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^1 + (x_{f(u)} - x_u)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^2 \mod \{x_i^n\}_{0 \leq i < n}.
\]

rewritten as

\[
0 \equiv \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq i < n} \left( (x_{f(i)} - x_i)^2 - j^2 \right) \times \prod_{n - \ell \leq j < n} (x_{f(w)} - x_w + x_{f(w)} - x_w)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^0 + (x_{f(v)} - x_f(v) + x_{f(v)} - x_v)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^1 + (x_{f(u)} - x_f(u) + x_{f(u)} - x_u)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^2 \mod \{x_i^n\}_{0 \leq i < n}.
\]

On the other hand the premise that \( 0 \not\equiv \text{LCM}_{\ell,f(2)} \mod \{x_i^n\}_{0 \leq i < n} \) implies that

\[
0 \not\equiv \prod_{0 \leq i < j < n} (x_j - x_i) \prod_{0 \leq i < n} \left( (x_{f(i)} - x_i)^2 - j^2 \right) \times \prod_{n - \ell \leq j < n} (x_{f(w)} - x_w)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^0 + (x_{f(v)} - x_v)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^1 + (x_{f(u)} - x_u)^2 \left( \frac{\sqrt{3} - 1}{2} \right)^2 \mod \{x_i^n\}_{0 \leq i < n}.
\]
rewritten as
\[ 0 \not\equiv \prod_{0 \leq i < j \leq n} (x_j - x_i) \prod_{0 \leq i < n} \prod_{n - \ell \leq j < n} \left( (x_{f(i)} - x_{f(i)} + x_{f(i)} - x_i)^2 - j^2 \right) \times \]
\[ \prod_{0 \leq u < v < w \leq n} (x_{f(u)}(w) - x_{f(v)} + x_{f(w)} - x_{u})^2 \left( \frac{\sqrt{3} - 1}{2} \right)^0 + (x_{f(u)}(v) - x_{f(v)} + x_{f(v)} - x_{u})^2 \left( \frac{\sqrt{3} - 1}{2} \right)^1 + (x_{f(u)}(w) - x_{f(u)} + x_{f(w)} - x_{u})^2 \left( \frac{\sqrt{3} - 1}{2} \right)^2 \mod \{x_i^n\}_{0 \leq i \leq n}. \]

Removing from both semi-expanded expressions the appropriate single summand as was done in the proofs of the weak composition lemmas yields a congruence identity which is invariant to performing the linear transformation
\[ \{ (x_{f(i)}(j) - x_i) \leftrightarrow (x_{f(i)} - x_i) : i \in L \}. \]

in the edge consistency factor thereby leading to the contradiction
\[ \text{LCM}_{\ell,f} \mod \{x_i^n\}_{0 \leq i \leq n} \equiv \text{LCM}_{\ell,f(2)} \mod \{x_i^n\}_{0 \leq i \leq n} \]
from which we conclude that
\[ 0 \not\equiv \text{LCM}_{\ell,f(2)} \mod \{x_i^n\}_{0 \leq i \leq n} \implies 0 \not\equiv \text{LCM}_{\ell,f} \mod \{x_i^n\}_{0 \leq i \leq n} \cdot \]

6 Strengthening the Graceful Labeling Theorem.

The strong Composition Lemma yields as a corollary a proof of the strong GLC first proposed in [GW13], stated as follows

**Theorem 5** (strong Graceful Labeling Theorem) Induced edge label sequences of identically constant functions in \(((0, n) \cap \mathbb{Z})^{[0,n] \cap \mathbb{Z}}\) are common to all functional trees on n vertices.

**Proof**: Assume for notational convenience and without loss of generality that f is non increasing. Consequently \(f^{(n-1)}\) denotes the identically constant zero function. Having proved that all trees are graceful we now focus on the remaining \(\left\lfloor \frac{n}{2} \right\rfloor + (n - 2 \left\lfloor \frac{n}{2} \right\rfloor) - 1\) induced edge label sequences associated with identically constant functions in \(((0, n) \cap \mathbb{Z})^{[0,n] \cap \mathbb{Z}}\) each of which is determined by congruence identities of the form
\[ 0 \not\equiv \text{LCM}_{\ell,f^{(n-1)}} \mod \{x_i^n\}_{0 \leq i \leq n}, \]
\[ 0 \leq \ell < \left\lfloor \frac{n-1}{2} \right\rfloor \]. Seen from the fact that we can identify the only 2 relabelings associated with any one of the \(\left\lfloor \frac{n}{2} \right\rfloor + (n - 2 \left\lfloor \frac{n}{2} \right\rfloor)\) possible induced edge label sequences of constant functions in \(((0, n) \cap \mathbb{Z})^{[0,n] \cap \mathbb{Z}}\). The desired result follows by repeatedly applying the contrapositive of the strong Graceful Composition Lemma.\(\square\)

7 Graceful and Harmonious labelings as spectra of second order constructs.

The induced edge label sequence of a given functional tree \(G_f\) associated with \(f \in ((0, n) \cap \mathbb{Z})^{[0,n] \cap \mathbb{Z}}\) is common to all functional trees in \(((0, n) \cap \mathbb{Z})^{[0,n] \cap \mathbb{Z}}\) if and only if
\[ \left\{ f^{(k+1)} - f^{(k)} = \lambda f^{(k)} \right\}_{0 \leq k < n}, \]
for some function \(\lambda \in ((-n, n) \cap \mathbb{Z})^{[0,n] \cap \mathbb{Z}}\). More specifically,
\[ \lambda \in \bigcup_{0 \leq j < \left\lfloor \frac{n}{2} \right\rfloor} \text{SP}_{j,n} \]
where
\[ SP_{j,n} := \left\{ g \mid \forall i \in [0, n) \cap \mathbb{Z}, (i + g(i)) = i + s(i)(j - \gamma(i)) \in [0, n) \cap \mathbb{Z}, \text{for some } s \in \{-1, 1\}^{[0, n) \cap \mathbb{Z}}, \gamma \in \mathbb{S}_n \right\}. \]

We describe here how the constraint (6) expresses a construct eigenvalue-eigenvector problem as recently introduced in [GG18].

Recall that second order constructs are matrices whose entries are morphisms. The algebra of constructs is prescribed by a combinator noted Op, and a composer noted \( F \). The composer specifies the rule for composing entry morphisms while the combinator specifies the rule for combining the compositions of entry morphisms. Natural choices for a combinator include for instance
\[ \sum_{0 \leq j < k} \prod_{0 \leq j < k} \max_{0 \leq j < k} \min_{0 \leq j < k}. \]

For example, the product of second-order constructs \( A \) and \( B \) of size respectively \( n_0 \times k \) and \( k \times n_1 \) results in a construct noted \( \text{GProd}_{Op,F}(A, B) \) of size \( n_0 \times n_1 \) specified entry-wise by
\[
\text{GProd}_{Op,F}(A, B)[i_0, i_1] = \text{Op}_{0 \leq j < k} F(A[i_0, j], B[j, i_1]), \quad \forall \left\{\begin{array}{l}
0 \leq i_0 < n_0 \\
0 \leq i_1 < n_1
\end{array}\right.
\]

For instance, the product of 2 \( \times \) 2 constructs is given by
\[
\text{GProd}_{Op,F} \left( \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}, \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} \right) = \begin{pmatrix} \text{Op}(F(a_{00}, b_{00}), F(a_{01}, b_{10})) & \text{Op}(F(a_{00}, b_{01}), F(a_{01}, b_{11})) \\ \text{Op}(F(a_{10}, b_{00}), F(a_{11}, b_{10})) & \text{Op}(F(a_{10}, b_{01}), F(a_{11}, b_{11})) \end{pmatrix}.
\]

Let the composer \( F \) and its dual \( G \) be set to
\[
F(f(z), g(z)) := f(g(z)) \quad \text{and} \quad G(f(z), g(z)) := g(f(z)),
\]
where \( f(z), g(z) \) are functions in the morphism variable \( z \). Let the combinator be set to
\[
\text{Op}_{0 \leq j < k} := \sum_{0 \leq j < k}.
\]

Recall that one of the two possible ways of defining the construct eigenvalue-eigenvector equation is
\[
\text{GProd}_{\sum,F}(A(z), v(z)) = \text{GProd}_{\sum,F}(\lambda(z) I_n, v(z)), \tag{7}
\]
where
\[
A(z) \in \left( (\mathbb{Z} \cap (-n, n)) \cap \mathbb{Z} \right)^{n \times n} \\
\text{and} \\
v(z) \in \left( (\mathbb{Z} \cap (-n, n)) \cap \mathbb{Z} \right)^{n \times 1}.
\]

**Theorem 6a** : There is a construct \( A(z) \) whose spectra includes all labeled functional trees whose subtractive induced edge label sequence is common to the subtractive induced edge label sequence of any identically constant function in \( (\mathbb{Z} \cap [0, n]) \cap \mathbb{Z} \).

**Proof** : In fact
\[
A(z) = z \text{Incidence Matrix of } (G_h),
\]
where
\[
h \in (\mathbb{Z} \cap [0, n]) \cap \mathbb{Z} \text{ such that } h(i) = \begin{cases} 
  i + 1 & \text{if } 0 \leq i < n - 1 \\
  n - 1 & \text{otherwise}
\end{cases} \quad \forall 0 \leq i < n.
\]

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More explicitly we have that

$$\forall \ 0 \leq i, j < n, \ A(z)[i,j] = \begin{cases} -z & \text{if } 0 \leq i = j < n - 1 \\ z & \text{or } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

For instance, when $n = 5$

$$A(z) = \begin{pmatrix} -z & z & 0 & 0 & 0 \\ 0 & -z & z & 0 & 0 \\ 0 & 0 & -z & z & 0 \\ 0 & 0 & 0 & -z & z \\ 0 & 0 & 0 & 0 & -z + z \end{pmatrix}.$$  

Consequently, $A(1)$ is a non-diagonalizable matrix in Jordan normal form whose characteristic polynomial is

$$x(x + 1)^{n-1} = \det(xI_n - A(1)).$$

Eigenvalue-eigenvector pairs of $A(1)$ determine two of the construct eigenvalue-eigenvector pairs for $A(z)$

$$\begin{cases} \text{GProd}_{\Sigma,\mathcal{F}}(A(z), z1_{n \times 1}) = \text{GProd}_{\Sigma,\mathcal{F}}(0I_n, z1_{n \times 1}) \\ \text{GProd}_{\Sigma,\mathcal{F}}(A(z), z1_{\mathcal{[}, 0\mathcal{]}}) = \text{GProd}_{\Sigma,\mathcal{F}}(-zI_n, z1_{\mathcal{[}, 0\mathcal{]}}) \end{cases}.$$  

The corresponding two construct eigenvalue-eigenvector pairs are

$$\begin{pmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(n-2)} \\ f^{(n-1)} \end{pmatrix} = z1_{n \times 1} \implies \begin{cases} f(z) = z \\ \lambda(z) = 0 \end{cases} \quad \text{and} \quad \begin{pmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(n-2)} \\ f^{(n-1)} \end{pmatrix} = z1_{\mathcal{[}, 0\mathcal{]}}, \implies \begin{cases} f(z) = 0 \\ \lambda(z) = -z \end{cases}.$$  

The strong Graceful Labeling Theorem establishes all other eigenvalue-eigenvector pairs whose entries are compositions of a function

$$f \in ((0, n) \cap \mathbb{Z})^{(0,n)\cap\mathbb{Z}}.$$  

Thus completing the proof.

Note that the analog of the characteristic equation which determines eigenvectors of the form

$$\begin{pmatrix} f^{(0)} \\ f^{(1)} \\ \vdots \\ f^{(n-2)} \\ f^{(n-1)} \end{pmatrix}$$
for the construct $A(z)$ is thus given by

$$0 = \sum_{0 \leq k < \lceil \frac{n}{2} \rceil} \prod_{\sigma_k \in (S_n/\text{AutG}_f)/z_n} \left( \sum_{0 \leq i < n} n^{[\sigma_k f \sigma_k^{-1}(i)-i]} - \sum_{0 \leq j < n} n^{[k-j]} \right),$$
in the unknown $f \in ([0, n) \cap \mathbb{Z})^{(0,n) \cap \mathbb{Z}}$.

Similarly, in the setting of harmonious labelings of functional trees, the combinator is set to

$$\text{Op}_{0 \leq j < k} := \prod_{0 \leq j < k}.$$

Harmoniously labeled functional trees are solutions to constraints of the form

$$\left\{ f^{(k+1)} \cdot f^{(k)} = f^{(k)} \lambda \right\}_{0 \leq k < n} \quad \text{where} \quad f \in \{\omega_n^0, \cdots, \omega_n^{n-1}\} \{\omega_n^0, \cdots, \omega_n^{n-1}\} \quad \text{and} \quad \omega_n = \exp \left( \frac{2\pi \sqrt{-1}}{n} \right)$$

Such constraints illustrate the second possible formulation of construct eigenvalue-eigenvector equation for some function $\lambda \in S_n$. The corresponding construct eigenvalue-eigenvector equation is of the form

$$\text{GProd}_{\Pi, \mathcal{F}} (A(z), v(z)) = \text{GProd}_{\Pi, \mathcal{G}} \left( (\lambda(z))^n \cdot v(z) \right),$$

where

$$A(z) \in \left( \{\omega_n^0, \cdots, \omega_n^{n-1}\} \{\omega_n^0, \cdots, \omega_n^{n-1}\} \right)^{n \times n}$$

and

$$v(z) \in \left( \{\omega_n^0, \cdots, \omega_n^{n-1}\} \{\omega_n^0, \cdots, \omega_n^{n-1}\} \right)^{n \times 1}.$$

**Theorem 6b**: There is construct $A(z)$ whose spectra includes all harmoniously labeled functional trees in $\{\omega_n^0, \cdots, \omega_n^{n-1}\}$.

**Proof**: In fact

$$A(z) = z^\text{Unsigned Incidence Matrix of (G_h)},$$

where

$$h \in \{0, \cdots, n-1\}^{\{0, \cdots, n-1\}} \quad \text{such that} \quad h(i) = \begin{cases} i + 1 & \text{if } 0 \leq i < n - 1 \\ n - 1 & \text{otherwise} \end{cases} \quad \forall \ 0 \leq i < n,$$

and the matrix

$$z^\text{Unsigned Incidence Matrix of (G_h)}$$
describes the result of the entry-wise exponentiation using $z$ at the base. More explicitly,

$$\forall \ 0 \leq i, j < n, \quad A(z)[i, j] = \begin{cases} z^1 & \text{if } 0 \leq i = j < n - 1 \quad \text{or} \quad j = i + 1 \\ z^2 & \text{if } i = j = n - 1 \\ z^0 & \text{otherwise} \end{cases}.$$
For instance when \( n = 5 \) we have

\[
A(z) = \begin{pmatrix}
  z & z & 1 & 1 & 1 \\
  1 & z & z & 1 & 1 \\
  1 & 1 & z & z & 1 \\
  1 & 1 & 1 & z & z \\
  1 & 1 & 1 & 1 & z^2 \\
\end{pmatrix}
\]

Just as in the previous case the unsigned incidence matrix of \( G_h \) is a non diagonalizable matrix already in Jordan normal form whose characteristic polynomial is

\[
(x - 2)(x - 1)^{n-1} = \det (xI_n - \text{Unsigned Incidence Matrix of } (G_h)).
\]

Eigenvalue-eigenvector pairs of the unsigned incidence matrix of \( G_h \) determine two of the construct eigenvalue-eigenvector pairs for \( A(z) \)

\[
\begin{align*}
\text{GProd}_{\Pi, \mathcal{F}} (A(z), z^{0^{n \times 1}}) &= \text{GProd}_{\Pi, \mathcal{G}} \left( (z^2)^{t^{n}}, z^{0^{n \times 1}} \right) \\
\text{GProd}_{\Pi, \mathcal{F}} (A(z), z^{0^{[0..n]}}) &= \text{GProd}_{\Pi, \mathcal{G}} \left( z^{0^{n}}, z^{0^{[0..n]}} \right)
\end{align*}
\]

Which determines the construct eigenvectors

\[
\begin{pmatrix}
  f^{(0)} \\
  f^{(1)} \\
  \vdots \\
  f^{(n-2)} \\
  f^{(n-1)} \\
\end{pmatrix} = z^{0^{n \times 1}} \implies \begin{pmatrix}
  f(z) \\
  \lambda (z) \\
\end{pmatrix} = z^{0} \quad \text{and} \quad \begin{pmatrix}
  f^{(0)} \\
  f^{(1)} \\
  \vdots \\
  f^{(n-2)} \\
  f^{(n-1)} \\
\end{pmatrix} = z^{0^{[0..n]}} \implies \begin{pmatrix}
  f(z) \\
  \lambda (z) \\
\end{pmatrix} = z^{0^{[0..n]}}
\]

The Harmonious Labeling Theorem establishes other eigenvalue-eigenvector pairs whose entries are compositions of some function

\[
f \in \{ \omega_0^{0}, \ldots, \omega_n^{n-1} \}^{\omega_0^{0}, \ldots, \omega_n^{n-1}}.
\]

Consequently, we see that unlike the matrix case, the number of distinct solution \( \lambda \) for an \( n \times n \) construct can be exponential in \( n \).

References

[Gal05] J.A. Gallian, A dynamic survey of Graph Labeling, Electronic J. Comb. DS6 (2000), vol 6.

[GG18] E. K. Gnang, J. S. Gnang, Sketch for a Theory of Constructs, ArXiv e-prints, 2018arXiv180803743G.

[GW18] E.K. Gnang, I.Wass, Growing Graceful Trees, ArXiv e-prints, 2018arXiv180805551G.

[S18] W. A. Stein et al., Sage Mathematics Software (Version 8.3), The Sage Development Team, (2018), http://www.sagemath.org.

[GS80] R. L. Graham and N. J. A. Sloane, On additive bases and harmonious graphs, SIAM J. Alg. Discrete Meth., 1 (1980) 382-404.

[R64] G. Ringel, Problem 25, in Theory of Graphs and its Applications, Proc. Symposium Smolenice 1963, Prague (1964) 162.