We propose an alternative, statistical, derivation of the Thermodynamic Bethe Ansatz based on the tree expansion of the Gaudin determinant. We illustrate the method on the simplest example of a theory with diagonal scattering and no bound states. We reproduce the expression for the free energy density and the finite size corrections to the energy of an excited state as well as the LeClair-Mussardo series for the one-point function for local operators.
1 Introduction

The finite size effects in 1+1 dimensional field theories come from the quantisation of the momenta of the physical particles, as well as from the virtual “mirror” particles winding around the space circle $R$ [1]. When $R$ is large, the exponentially small contribution from the mirror particles can be neglected and the spectrum is determined by the “asymptotic” Bethe-Yang equations, which take into account only the scattering processes between the physical particles. As it was first realised by Al. Zamolodchikov [2], for finite $R$ a powerful technique for summing up the finite size corrections is given by the Thermodynamical Bethe Ansatz, or TBA [3]. If the theory is Lorentz invariant, the finite size effects can be traded to finite temperature effects. The main idea of the TBA is that the thermal trace is dominated by a saddle point for the density of states, which is obtained as the solution of some non-linear integral equations. By analytical continuation one can obtain the “exact Bethe equations” for the spectrum of the excited states in finite volume [4].

In the last decades much attention is been focused on combining the TBA and the form factor bootstrap in order to compute the correlation functions at finite volume/temperature. This is a problem of higher complexity and in spite of the considerable progress a systematic procedure is not yet available for the higher point functions. The main difficulty is to learn how to insert efficiently the resolution of the identity between the local operators in order to split the correlation function into simpler objects, the elementary form factors at infinite volume. In other words, the saddle point analysis of the TBA is not sufficient and has to be replaced by a more subtle, field-theoretical, consideration.

Another motivation for looking at the sum over the intermediate states is the recently proposed hexagon bootstrap program in the AdS/CFT integrable model [5] which can be applied for the computation of higher point correlation functions. The proposal prescribes to insert complete sets of mirror particles between the hexagon operators. Although these effects resemble the wrapping corrections in the spectral problem, no TBA methods have yet been developed to resum them.

In this paper, we address the problem of performing the sum over the mirror states in the simplest case of a theory with diagonal scattering and no bound states. Our proposal is close in spirit to some previous works [6] [7] where the excluded volume in the sum over the intermediate states is compensated by including into the sum non-physical solutions of the asymptotic Bethe-Yang equations. The new development is that we succeeded to perform explicitly the sum over the states using a graph expansion of the Gaudin determinant which gives the integration measure over the Bethe states in the mirror channel. This graph expansion leads to a Feynman-like diagram technique which allows us to write the free energy as a sum over tree Feynman diagrams.

In section 2 we explain our method on the simplest example of a diagonal theory without bound states for which we compute the the partition function on a cylinder with circumference $R$ as the thermal trace in the mirror theory. In the rest of the text we consider two more examples, where we re-derive the formulas obtained previously by ingenuous application of the TBA. In section 3 we compute the energy of an excited state in the physical channel. In section 4 we derive the LeClair-Mussardo series for the one-point function. In all three examples we reduce the computation to a combinatorial problem involving the sum over tree graphs.
2 Integrable Quantum Field Theory on a cylinder: the partition function

2.1 Physical and mirror channels

Consider an integrable 1+1 dimensional field theory with one single type of particle excitations above the vacuum. The dispersion relation between the momentum \( p \) and the energy \( E \) of the particle is parametrised by the rapidity variable \( u \):

\[
 p = p(u), \quad E = E(u).
\]  (2.1)

We assume that there exists a transformation to the “mirror” theory in which the role of the time \( t \) and the space \( x \) are exchanged. The physical and the mirror channels are related by a “mirror” transformation \( x = -i\tilde{t}, \ t = -i\tilde{x} \) and \( E = i\tilde{p}, \ p = i\tilde{E} \). The mirror transformation can be encoded in a transformation \( \gamma : u \to \tilde{u} \) of the rapidity parameter, so that

\[
 E(\tilde{u}) = i\tilde{p}(u), \quad p(\tilde{u}) = i\tilde{E}(u).
\]  (2.2)

The square of the mirror transformation gives the crossing transformation \( \gamma^2 : u \to \bar{u} = \gamma \tilde{u} \) which relates particles to anti-particles. If the theory is Lorentz invariant, then the mirror and the physical theories are identical. The diagonal S-matrix \( S(u, v) \) is supposed to satisfy, besides the Yang-Baxter equations, unitarity \( S(u, v)S(v, u) = 1 \), crossing symmetry \( S(u, v) = S(\bar{v}, \bar{u}) \), and the condition \( S(u, u) = -1 \). We will not need to assume that the S-matrix is a function of the difference of the two rapidities.

If the theory is confined in a finite volume \( R \) with periodic boundary conditions, the eigenstates of the Hamiltonian can be constructed as superpositions of plane waves according to the Bethe Ansatz, with the spectrum of the rapidities determined by condition of periodicity. Each eigenstate from the \( N \)-particle sector is characterised by a set of rapidities \( u = \{u_1, \ldots, u_N\} \) and the energy of this state is equal to

\[
 E(u) = \sum_{j=1}^{N} E(u_j).
\]  (2.3)

When \( R \) is sufficiently large, the spectrum of the energies are determined by the Asymptotic Bethe Ansatz. The quantisation condition for the rapidities is expressed in terms of the total phase factor corresponding to a process in which one of the \( N \) particles winds once around the space circle,

\[
 \phi_j(u_1, \ldots, u_N) \equiv p(u_j)R + \frac{1}{i} \sum_{k(\neq j)}^{N} \log S(u_j, u_k) \quad (j = 1, \ldots, N).
\]  (2.4)

For periodic boundary conditions the scattering phases can take integer values modulo \( 2\pi \)

\[
 \phi_j(u_1, \ldots, u_N) = 2\pi n_j \quad \text{with} \ n_j \ \text{integer}, \quad j = 1, \ldots, N.
\]  (2.5)

In a system of units where the mass of the particle is equal to one, the asymptotic expression (3.1) for the scattering phases is true up to \( o(e^{-R}) \) terms. For finite \( R \) the Bethe-Yang equations (2.4)-(2.5) are deformed by the scattering with the virtual particles in the mirror channel which wrap the space circle [8]. One can study the finite volume effects using the TBA in the mirror channel. One can
introduce an infrared cutoff in the mirror theory by considering the cylinder as the limit of a torus obtained as the product of the space circle with a time circle with asymptotically large circumference $L$. When $L$ is large, one can construct a complete set of states in the mirror channel whose spectrum is given by the asymptotic Bethe-Yang equations. Then the partition function can be computed by taking the thermal trace in the mirror Hilbert space.

The standard TBA approach due to Yang and Yang [3] is to express the thermal trace as an integral over the density of one-particle rapidities, taking into account both the energy and the entropy of the states. The free energy is expressed as a functional of the rapidity density and the critical point of this functional gives both the thermal equilibrium state and the expression for the extensive piece $LF_0(R)$ of the free energy. In field-theoretical terms this translates to replace the sum over the intermediate states by a single “thermal state” characterised by the saddle point density. This approximation works well for evaluating the free energy and the one-point functions, where a single insertion of the identity is to be made, but it is not sufficient e.g. for the computation of the two-point functions.\footnote{There however is a class of two-point functions for which a single insertion is sufficient [9].}

### 2.2 Thermal partition function

Below we will perform a direct summation in the mirror Hilbert space. Our method is exact up to corrections exponentially small in $L$ and allows to control the whole $1/L$ expansion of the partition function. The simplest object to compute is the partition function on the torus, $Z(R, L)$, which can be evaluated as a thermal trace in the physical or in the mirror channels of the Euclidean theory,

$$Z(L, R) = \text{Tr}_{\text{phys}}[e^{-LH_{\text{phys}}}] = \text{Tr}_{\text{mir}}[e^{-RH_{\text{mir}}}], \quad (2.6)$$

Assuming that $R \ll L$, our goal is to evaluate the free energy

$$\log Z(L, R) = LF_0(R) + F_1(R) + \ldots \quad (2.7)$$

up to corrections exponentially small in $L$.

Let us stress that such an exponential accuracy is beyond the reach of the standard TBA approach which is essentially a collective field theory for the rapidity density and as such suffers from ambiguities beyond the first two terms of the expansion (2.7). The leading term in the TBA approach is determined by the saddle point of the integral over the densities, while the subleading term is produced by the gaussian fluctuations about the saddle point [10] and the normalisation of the wave function of the thermal state [11], with the two effects cancelling completely for periodic boundary conditions. Our approach does not suffer from the ambiguities of the collective theory and allows to obtain the whole series (2.7), which in the case of periodic boundary conditions consists of a single term $LF_0(R)$.

### 2.3 The partition function as a sum over mode numbers

The quantisation condition in the mirror channel is given by the Bethe-Yang equations

$$\tilde{\phi}_j = 2\pi n_j \quad \text{with } n_j \text{ integer}, \quad j = 1, \ldots, M, \quad (2.8)$$

where $\tilde{\phi}_j$ is the total scattering phase for the $j$-th mirror particle,

$$\tilde{\phi}_j(u_1, \ldots, u_M) = \bar{p}(u_j)L + \frac{1}{2} \sum_{k(\neq j)}^{M} \log \tilde{S}(u_j, u_k). \quad (2.9)$$
Here \( \hat{S}(u,v) = S(\hat{u},\hat{v}) \) denotes the S-matrix for the mirror particles. The states in the \( M \)-particle sector of the Hilbert space are labeled by \( M \) distinct mode numbers \( n_1, \ldots, n_M \) and the identity operator in this sector can be decomposed as a sum of products of normalised states

\[
\mathbb{1}_M = \sum_{n_1 < \ldots < n_M} |n_1, \ldots, n_M\rangle \langle n_1, \ldots, n_M|.
\] (2.10)

If we denote by \( \tilde{E}_M(n_1, \ldots, n_M) \) the eigenvalue of the Hamiltonian for the state \( |n_1, \ldots, n_M\rangle \), the partition function (2.6) is given by the series

\[
Z(L,R) = \sum_{M=0}^{\infty} \sum_{n_1 < n_2 < \ldots < n_M} e^{-\tilde{E}_M(n_1, \ldots, n_M)}. \] (2.11)

Our goal is to replace in the thermodynamical limit \( L \to \infty \) the discrete sums by multiple integrals. For that we have first to get rid of the ordering of the quantum numbers. For that we insert a factor which kills the configurations with coinciding quantum numbers and take the sum over non-restricted integers,

\[
Z(L,R) = \sum_{M=0}^{\infty} \frac{1}{M!} \sum_{n_1, \ldots, n_M} \prod_{j<k} (1-\delta_{n_j,n_k}) e^{-\tilde{E}_M(n_1, \ldots, n_M)}. \] (2.12)

Expanding the product of Kronecker symbols, leads to a series

\[
Z(L,R) = 1 + \sum_{n} e^{-\tilde{E}(n)} + \frac{1}{2!} \sum_{n_1,n_2} e^{-\tilde{E}(n_1,n_2)} - \frac{1}{2} \sum_{n} e^{-\tilde{E}(n,n)} + \ldots \] (2.13)

which we are going to write as an exponential. The sum in (2.13) goes over all sequences \( (n_1^{r_1}, \ldots, n_m^{r_m}) \) of positive integers \( n_j \) with multiplicities \( r_j \). For example, \( (n^2) = (n,n) \). Each such sequence defines an (unphysical) Bethe state obtained by identifying some of the momenta of a Bethe state with \( M = r_1 + \cdots + r_m \) magnons. This state is a linear combination of plane waves with momenta \( r_j \tilde{p}(u_j) \), \( j = 1, \ldots, m \) and energy

\[
\tilde{E}(n_1^{r_1}, \ldots, n_m^{r_m}) = r_1 \tilde{E}(u_1) + \cdots + r_m \tilde{E}(u_m). \] (2.14)

The relevance of such states has been already pointed out by Woynarovich [7] and by Dorey et al in . The rapidities \( u_1, \ldots, u_m \) are determined by the Bethe-Yang equations (2.8) with \( M = r_1 + \cdots + r_m \). The phase \( \tilde{\phi}_j \) is acquired by the wave function if to one of the \( r_j \) particles with rapidity \( u_j \) winds once around the time circle,

\[
\tilde{\phi}_j = \tilde{p}(u_j)L + \frac{1}{i} \sum_{k(\neq j)} r_k \log \hat{S}(u_j,u_k) + \pi(r_j-1) = 2\pi n_j \quad (j = 1, \ldots, m). \] (2.15)

The term \( \pi(r_j-1) \) originates in the scattering of the probe particle with the \( r_j - 1 \) particles with the same rapidity \( u_j \).

The full series (2.13) has the form

\[
Z(L,R) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{n_1, \ldots, n_m} \sum_{r_1, \ldots, r_m} (-1)^{r_1+\cdots+r_m} C_{r_1 \ldots r_m} e^{-\tilde{E}(n_1^{r_1}, \ldots, n_m^{r_m})}. \] (2.16)
where the coefficients $C_{r_1...r_m}$ are purely combinatorial. They can be fixed from the expansion of the thermal partition function when the quasiparticles are free fermions, $S(u_i, u_j) = -1$ and $\tilde{E}(n_1, ..., n_M) = \tilde{E}(n_1) + \cdots + \tilde{E}(n_M)$. In the occupation numbers representation, the partition function for free fermions can be written as an infinite product

$$Z_{\text{free fermions}} = \prod_{n \in \mathbb{Z}} \left( 1 + e^{-R \tilde{E}(n)} \right) = \exp \sum_{n \in \mathbb{Z}} \sum_{r=1}^{\infty} \frac{(-1)^{r-1}}{r} e^{-rR \tilde{E}(n)}$$

(2.17)

Comparing with (2.16) we find for the combinatorial coefficients

$$C_{r_1...r_m} = \frac{1}{r_1...r_m}.$$  

(2.18)

In the case of free fermions, the multiplicities $r_j$ have obvious meaning. The vacuum energy is a sum of all fermionic loops including those winding $r$ times around the space circle. The weight of an $r$-winding loop consists of a Boltzmann factor $e^{-rRE_u}$, a sign $(-1)^r$ due to the Fermi statistics and a combinatorial factor $1/r$ counting for the $Z_r$ cyclic symmetry. It is natural to interpret the multiplicities $r_j$ as winding, or wrapping, numbers also in the case of non-trivial scattering, which we are going to do in the following.

2.4 From mode numbers to rapidities

The discrete sum over the allowed values of the phases $\tilde{\phi}_j(u_1, r_1; \ldots, u_m, r_m)$ for given wrapping numbers can be replaced, up to exponentially small in $L$ terms, by an integral,

$$\sum_{n_1, \ldots, n_m} = \int \frac{d\tilde{\phi}_1}{2\pi} \cdots \frac{d\tilde{\phi}_m}{2\pi}$$

(2.19)

Since the energy takes a simple form as a function of the rapidities, eq. (2.14), we are going to change the variables from scattering phases $\phi_j$ to rapidities $u_j$,

$$Z(L, R) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{r_1, \ldots, r_m} \frac{(-1)^{r_1+\cdots+r_m}}{r_1 \cdots r_m} \int \frac{du_1}{2\pi} \cdots \frac{du_m}{2\pi} e^{-r_1 \tilde{E}(u_1)} \cdots e^{-r_m \tilde{E}(u_m)}.$$ 

(2.20)

The change of variables brings a volume-dependent Jacobian (the Gaudin determinant)

$$\tilde{G} = \det \tilde{G}_{kj}, \quad \tilde{G}_{kj} = \frac{\partial}{\partial u_k} \tilde{\phi}_j(u_1^{r_1}, \ldots, u_m^{r_m}),$$

(2.21)

which gives the density of the particle states in the rapidity space. The explicit form of the Gaudin matrix $\tilde{G}_{kj}$ is

$$\tilde{G}_{kj} = \left( L\tilde{p}'(u_j) + \sum_{l=1}^{m} r_lK(u_j, u_l) \right) \delta_{jk} - r_kK(u_k, u_j),$$

(2.22)

where $K(u, v) = \frac{1}{4} \partial_u \log \tilde{S}(u, v)$. 

6
2.5 Graph expansion of the Gaudin determinant

Let us denote for brevity
\[
\tilde{p}_j' \equiv \tilde{p}(u_j) \quad K_{jk} \equiv K(u_j, u_k).
\] (2.23)

Inspecting the expansion of the Gaudin determinant for \( m = 1, 2, 3 \)
\[
\tilde{G}(u^r) = L \tilde{p}', \\
\tilde{G}(u_1^r, u_2^r) = L^2 \tilde{p}_1^r \tilde{p}_2^r + L \tilde{p}_1^r r_1 K_{21} + L \tilde{p}_2^r r_2 K_{12}, \\
\tilde{G}(u_1^r; u_2^r, u_3^r) = L^3 \tilde{p}_1^r \tilde{p}_2^r \tilde{p}_3^r \\
+ L^2 \tilde{p}_2^r \tilde{p}_3^r r_2 K_{12} + L^2 \tilde{p}_2^r \tilde{p}_3^r r_3 K_{13} + L^2 \tilde{p}_1^r \tilde{p}_3^r r_1 K_{21} \\
+ L^2 \tilde{p}_1^r \tilde{p}_3^r r_3 K_{23} + L^2 \tilde{p}_1^r \tilde{p}_2^r r_1 K_{31} + L^2 \tilde{p}_1^r \tilde{p}_2^r r_2 K_{32} \\
+ \tilde{p}_3^r L r_1 r_3 K_{21} + \tilde{p}_3^r L r_2 r_3 K_{12} K_{23} + \tilde{p}_2^r L r_3^r K_{13} K_{23} \\
+ \tilde{p}_2^r L r_1^r K_{12} K_{31} + \tilde{p}_1^r L r_2 r_3 K_{13} K_{31} + \tilde{p}_1^r L r_3^r K_{23} K_{31} \\
+ \tilde{p}_1^r L r_2^r K_{21} K_{32} + \tilde{p}_2^r L r_2^r K_{12} K_{32} + \tilde{p}_2^r L r_3^r K_{13} K_{32},
\] (2.24)
we see that there are no cycles of the type \( K_{12} K_{21} \) or \( K_{12} K_{23} K_{31} \). We will see below that this property hold for general order \( m \). To evaluate the Gaudin determinant for general state \( \{ u_1^r, \ldots, u_m^r \} \), we will consider in the following a slightly modified Gaudin matrix, \( \hat{G}_{kj} = \tilde{G}_{kj} r_j \). The determinants of the two matrices are simply related,
\[
\hat{G} = \frac{\det \hat{G}_{jk}}{\prod_{j=1}^m r_j}, \quad \hat{G}_{kj} \equiv \hat{G}_{kj} r_j.
\] (2.25)

The the modified Gaudin matrix has the advantage that it is a sum of a diagonal matrix \( \hat{D}_j \delta_{jk} \) and a Laplacian matrix \( \hat{K}_{kj} \) (a matrix with zero row sums):
\[
\hat{G}_{kj} = \hat{D}_k \delta_{kj} - \hat{K}_{kj}
\]

with \( \hat{D}_j = L r_j \tilde{p}'(u_j) \) and \( \hat{K}_{k,j} = r_k r_j K(u_k, u_j) - \delta_{kj} \sum_{l=1}^m r_j r_l K(u_j, u_l) \) (2.26)

According to the Matrix-Tree Theorem (see e.g. \[12, 13\]), the determinant of the matrix \( \hat{G}_{ij} \) can be expanded as a sum of graphs called directed spanning forests. A directed forest spanning the graph \( \Gamma \) is an oriented subgraph \( \mathcal{F} \) fulfilling the following three conditions:
(i) \( \mathcal{F} \) contains all vertices of \( \Gamma \);
(ii) \( \mathcal{F} \) does not contain cycles;
(iii) For any vertex of \( \Gamma \) there is at most one oriented edge of \( \mathcal{F} \) ending at this vertex.

The vertices with no incoming lines are called roots. Any forest \( \mathcal{F} \) is decomposed into connected components called directed trees. Each tree contains one and only one root. The Matrix-Tree Theorem states that the determinant of the matrix \( \hat{G} \) is a sum of all directed forests \( \mathcal{F} \) spanning the totally connected graph with vertices labeled by \( j = 1, \ldots, m \):
\[
\det_{m \times m} \left( \hat{D}_j \delta_{jk} - \hat{K}_{jk} \right) = \sum_{\mathcal{F}} \prod_{v_i \in \text{roots}} \hat{D}_i \prod_{l_{jk} \in \mathcal{F}} \hat{K}_{kj}.
\] (2.27)
The weight of a forest $F$ is a product of factors $\hat{D}_k$ associated with the roots and factors $\hat{K}_{kj}$ associated with the oriented edges $\ell_{jk} = (v_j \rightarrow v_k)$ of the $F$. The expansion in spanning forests for $m = 1, 2, 3$ is depicted in Fig. 1.

Applying the above graph expansion to the Jacobian, we write the partition function as

$$
Z(L, R) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{r_1, r_2, \ldots, r_m} \int \prod_{j=1}^{m} \frac{du_j}{2\pi} \frac{\left[ -e^{-R\tilde{E}(u_j)} \right] r_j}{r_j^2}
\times \sum_{\mathcal{F}} \prod_{j \in \text{roots}} L r_j \hat{p}'(u_j) \prod_{\ell_{ij} \in \mathcal{F}} r_i r_j K(u_j, u_i).
$$

The next step is to invert the order of the sum over graphs and the integral/sum over the coordinates $(u_j, r_j)$ assigned to the vertices. As a result we obtain a sum over the ensemble of abstract oriented tree graphs, with their symmetry factors, embedded in the space $\mathbb{R} \times \mathbb{N}$ where the coordinates $u, r$ of the vertices take values. The embedding is free, in the sense that the sum over the positions of the vertices is taken without restriction. As a result, the sum over the embedded tree graphs is the exponential of the sum over connected ones. One can think of these graphs as tree level Feynman diagrams obtained by applying the following Feynman rules:

$$
\begin{align*}
(u, r) = \frac{(-1)^{r-1}}{2} e^{-rR\tilde{E}(u)} \\
(u, r) = L p'(u) \frac{(-1)^{r-1}}{r} e^{-rR\tilde{E}(u)} \\
(u_1, r_1) (u_2, r_2) = r_1 r_2 K(u_2, u_1)
\end{align*}
$$
In this way we can write the free energy as

\[
\log Z(L, R) = L \int \frac{du}{2\pi} \tilde{\rho}(u) \sum_{r=1}^{\infty} r \tilde{Y}_r(u),
\]

(2.30)

where \( \tilde{Y}_r(u) \) is the partition sum of all \textit{connected} directed rooted trees with root at the point \((u, r)\), fig. 2.

\[
\bigcirc = \sum_{(u, r)}
\]

Figure 2: The generating function \( \tilde{Y}_r(u) \) of the directed trees with root at \((u, r)\). The weight of each tree in the sum is a product of factors associated with its vertices and edges according to the Feynman rules (2.29). The root is denoted by a black dot because here it has the same weight as the rest of the vertices of the tree.

Eq. (2.30) gives the free energy up to \( e^{-L} \) terms, hence the subleading terms in the expansion (2.7) vanish. Of course this is true only for periodic boundary conditions. Eq. (2.30) gives the free energy up to \( e^{-L} \) terms, hence the subleading terms in the expansion (2.7) vanish. Of course this is true only for periodic boundary conditions.

\[
\bigcirc = \sum_{(u, r)}
\]

Figure 3: The non-linear equation for the generating function \( \tilde{Y}_r(u) \) of the trees with root at \((u, r)\)

2.6 Performing the sum over trees

As any partition sum of trees, \( \tilde{Y}_r(u) \) satisfies a simple non-linear equation (a Schwinger-Dyson equation in the QFT language) depicted in Fig. 3,

\[
\tilde{Y}_r(u) = \frac{(-1)^{r-1}}{r^2} e^{-rR\tilde{E}(u)} \sum_{n=1}^{\infty} \frac{1}{n!} \left( \sum_s \int \frac{dv}{2\pi} rsK(v, u)\tilde{Y}_s(v) \right)^n
\]

(2.31)
In particular for $r = 1$

$$
\tilde{Y}_1(u) = e^{-R\tilde{E}(u)} e^{\sum_s \int \frac{dv}{2\pi} sK(v,u)\tilde{Y}_s(v)}.
$$

(2.32)

Substituting the rhs of (2.32) in the square brackets in the second line of eq. (2.31), we express all $Y_r$ in terms of $Y_1$,

$$
\tilde{Y}_r(u) = \left(\frac{-1}{r^2}\right)^{r-1} [\tilde{Y}_1(u)]^r, \quad r = 1, 2, 3, \ldots
$$

(2.33)

Now we can express the rhs of (2.30) and the exponent in on the rhs of (2.32) in terms of $Y_1$ only,

$$
\sum_r r\tilde{Y}_r(v) = \log \left[1 + \tilde{Y}_1(v)\right].
$$

(2.34)

Now eq. (2.32) becomes a closed equation for $Y_1$,

$$
\tilde{Y}_1(u) = e^{-R\tilde{E}(u)+\int \frac{dv}{2\pi} K(v,u)\log[1+\tilde{Y}_1(v)]},
$$

(2.35)

which determines completely the free energy

$$
\log Z(L,R) = L \int \frac{du}{2\pi} \tilde{p}'(u) \log \left[1 + \tilde{Y}_1(v)\right] + o(e^{-L}).
$$

(2.36)

In this way we reproduced, by summing up the tree expansion of the free energy, the TBA equation for the pseudoenergy $\epsilon(u) = -\frac{1}{L}\log \tilde{Y}_1(v)$. The expression (2.36) for the free energy is true in all orders in $1/L$. In particular, there is no $O(1)$ piece, in accord with the TBA based computation in [11].

3 The energy of an excited state

In this section we will apply the tree expansion to the case of an excited state $|u\rangle$ in the physical channel characterised by a set of rapidities $u = \{u_1, \ldots, u_N\}$. We assume that the excited state is an eigenstate of the Hamiltonian with energy given by eq. (2.3).

For large $R$ the wrapping phenomena can be neglected and the rapidities $u$ satisfy the asymptotic Bethe equations (2.8)-(2.4). In order to determine the exact energy and the exact values of the rapidities for finite $R$, we again introduce a cutoff $L$ by compactifying the cylinder into a torus obtained as the product of a space-like circle $R$-circle and a time-like $L$-circle, with a projector $|u\rangle\langle u|$ inserted in the physical channel. The phases of the mirror particles now contain an extra piece which comes from the scattering with the physical particles:

$$
\tilde{\phi}_j(v_1, \ldots, v_M) \equiv \tilde{p}(v_j)L + \frac{1}{i} \sum_{k=1}^N \log S(\tilde{v}_j, u_k) + \frac{1}{i} \sum_{l(\neq j)}^M \log S(\tilde{v}_j, \tilde{v}_l), \quad j = 1, \ldots, M.
$$

(3.1)

The computation of the partition function then follows strictly the argument of the previous section, with the only difference that the mirror energy is modified by the scattering with the physical particles. We have to replace

$$
e^{-L\tilde{E}(v)} \rightarrow \tilde{Y}_1^{\alpha}(v) \equiv e^{-L\tilde{E}(v)} \prod_{k=1}^M S(\tilde{v}, u_k).$$

(3.2)
Furthermore we have to add to the free energy the contribution from the physical particles that go directly to the opposite edge without scattering,

$$\log Z(L, R, u) = -L \sum_{j=1}^{N} E(u_j) + L \int \frac{du}{2\pi} p'(u) \log \left[ 1 + Y_1(v) \right] + O(e^{-L}).$$ (3.3)

with the function $Y(u)$ satisfying non-linear integral equation which slightly generalises eq. (2.35),

$$\tilde{Y}_1(v) = Y_1(v) e^{\int \frac{du}{2\pi} \log(1+Y_1(u))K(u,v)}. \quad (3.4)$$

The rapidities of the physical particles are no longer determined by the asymptotic Bethe-Yang equations but by the “exact Bethe equations” which take into account all virtual excitations in the mirror channel. The exact Bethe equations are formulated in terms of the function $\tilde{Y}_1$. In order to avoid confusion we introduce the $Y$-function in the physical channel, which is related to $\tilde{Y}$ by

$$\tilde{Y}_1(v) = Y_1(\tilde{v}). \quad (3.5)$$

The exact Bethe equations are obtained by the following requirement. Let $Z_j(R, L)$ be the partition function with the $j$-th physical particle winding once around the space circle before winding around the time circle. The configurations that contribute to $Z(R, L)$ and $Z_j(R, L)$ are depicted in Figs. 4a and 4b.

Figure 4: The configurations that lead to the exact Bethe equation. The physical magnon winding once around the space circle has the same effect, up to a factor $(-1)$, as a physical magnon going straight in presence of a mirror magnon with rapidity $u_j$.

In order to compute the partition function $Z_j(R, L)$ we notice that the configurations in Fig. 4b can be simulated by pulling one of the mirror particles out of the thermal ensemble giving to its rapidity a physical value $u_j$. Indeed, since $S(u_j, u_j) = -1$, the partition function in presence of such extra mirror particle is $-Z_j(R, L)$. In this way $Z_j(R, L)$ is given by the sum over all trees, with one extra tree having a root $\tilde{v} = u_j$ and $r = 1$. The generating function for such trees is $Y_1(u_j)$, while the contribution of the “vacuum” trees give the partition function: $Z_j = -Y_1(u_j) Z$. The periodicity in the space direction requires that $Z_j = Z$, which gives the exact Bethe-Yang equations

$$Y_1(u_j) = -1, \quad j = 1, \ldots, N. \quad (3.6)$$
4 One-point functions at finite volume/temperature

In this section we will apply the tree expansion to compute the diagonal matrix elements of a local operator at finite volume $R$. The LeClair-Mussardo conjecture \cite{14} gives an expression for the exact finite temperature one-point functions. In terms of infinite-volume diagonal connected form factors, and densities of mirror states determined by the TBA equation. The conjecture was proven for operators representing densities of conserved quantities in \cite{15} and for general local operator in \cite{16}. The proof of \cite{16} concerns the formula about the diagonal form factors in asymptotically large volume conjectured by Pozsgay and Takacs \cite{17}, which is equivalent to the L-M formula. The Pozsgay-Takacs formula, which generalises a result by Saleur \cite{15}, gives an expansion of the diagonal matrix elements of a local operator in terms of the infinite-volume form factors with the same or lower number of particles.

4.1 The one-point function in terms of connected diagonal form factors

In order to simplify the notations, in this section we assume that the physical Hilbert space is associated with the $L$-circle and the mirror Hilbert space is associated with the $R$-circle. In infinite volume, all matrix elements of a local operator $O$ can be expressed, with the help of the crossing formula, in terms of the elementary form factors

$$F_n^O(u_1, \ldots, u_n) = \langle 0 | O | u_1, \ldots, u_n \rangle_\infty. \quad (4.1)$$

The elementary form factors for local operators satisfy the Watson equations

$$F_n(u_1, \ldots, u_j, u_{j+1}, \ldots, u_n) = S(u_j, u_{j+1}) F_n(u_1, \ldots, u_{j+1}, u_j, \ldots, u_n) \quad (4.2)$$

and have kinematical singularities

$$F(v, u, u_1, \ldots, u_n) = \frac{i}{\bar{v} - u} \left( 1 - \prod_{j=1}^{n} S(u, u_j) \right) F_n(u_1, \ldots, u_n) + \text{regular}, \quad (4.3)$$

where $\bar{v}$ is obtained from $v$ by a crossing transformation. Here it is assumed that the infinite volume states are normalised as $\langle u | v \rangle = 2\pi \delta(u - v)$.

The diagonal limit of the form factors for local operators is ambiguous\footnote{In the case of the non-local operators the situation is even worse: their diagonal limit diverges as $L^M$ where $M$ is the number of the particle pairs.} and there are two prescriptions for evaluating the finite piece, the symmetric and the connected one \cite{17}. The connected diagonal form factor $F_{2n}^c(u_1, \ldots, u_n)$ is obtained by performing the simultaneous limit $\varepsilon_1, \ldots, \varepsilon_n \to 0$ of the elementary form factor $F_{2n}(u_1, \ldots, u_{2n})$ defined by eq. (4.1), with $u_{2n-j+1} = \bar{u}_j + i\varepsilon_j$. The limit is not uniform and depends on the prescription, which in this case is to retain only the $\varepsilon$-independent part:

$$F_{2n}^c(\bar{u}_n + i\varepsilon_n, \ldots, \bar{u}_1 + i\varepsilon_1, u_1, \ldots, u_n) = F_{2n}^c(u_1, \ldots, u_n) + \varepsilon\text{-dependent terms}. \quad (4.4)$$

The Saleur-Pozsgay-Takacs formula \cite{15, 17} relates the diagonal matrix elements in asymptotically large but finite volume $L$ to the connected diagonal form-factors. The formula reads

$$\langle u | O | u \rangle_L \approx \sum_{\alpha_\lambda, \lambda = \bar{\alpha}} F_{2n}^c(\alpha) \times \det_{j,k \in \bar{\alpha}} G_{jk} + O(e^{-L}), \quad (4.5)$$
where the sum goes over all partitions of the rapidities \( \mathbf{u} = \{u_1, \ldots, u_n\} \) in to two complementary sets \( \alpha \) and \( \bar{\alpha} \), and \( G_{jk} = \partial_{u_j} \phi_k \) is the Gaudin matrix for the \( n \) rapidities. It is assumed that \( F_0^c = 0 \), so there is no term with \( \alpha = \emptyset \). The formula is written for the normalisation with the Gaudin norm

\[
\langle \mathbf{u} | \mathbf{u} \rangle = \det_{j,k \in \mathbf{u}} G_{jk}. \tag{4.6}
\]

The determinants on the rhs are the minors of the Gaudin determinant obtain by deleting the lines and the columns that belong to the subset \( \alpha \). It is shown \([11, 18]\) that the expansion (4.5) is equivalent to the Leclair and Mussardo series for the one-point function of a local operator \([14]\)

\[
\langle \mathcal{O} \rangle_R = \sum_{n=1}^{\infty} \frac{1}{n!} \int \prod_{j=1}^{n} \frac{du_j}{2\pi} f(u_j) F_{2n}^c(u_1, \ldots, u_n), \quad f(u) = \frac{Y_1(u)}{1 + Y_1(u)}. \tag{4.7}
\]

Below we will derive the Leclair-Mussardo formula from the tree expansion method. In particular, we will reproduce the result obtained by Saleur \([15]\) for the one-point function of a conserved charge. For that we will need the diagonal matrix elements also for the multi-wrapping states \( |u_1^{r_1}, \ldots, u_M^{r_M}\rangle \).

We will make a very natural conjecture about this action, which turns out to be compatible with the correct formula (4.7), namely

\[
\langle u_M^{r_M}, \ldots, u_1^{r_1} | \mathcal{O} | u_1^{r_1}, \ldots, u_M^{r_M} \rangle_L = \sum_{\alpha \cup \bar{\alpha} = \{u_1, \ldots, u_M\}} \prod_{j \in \alpha} r_j F_{2|\alpha|}^c(\alpha) \times \det_{j,k \in \bar{\alpha}} G_{jk}. \tag{4.8}
\]

The logic behind this conjecture is that the action of the operator on a multi wrapping particle is the same as if it were single wrapping particle. The only difference is that the \( r \)-wrapping particle appears \( r \) times in the same time slice, the operator acts on each copy, which brings an overall factor of \( r \).

We should mention here that a discussion about the “multi-diagonal” matrix elements was presented \([19]\).

### 4.2 LeClair-Mussardo series from the tree expansion

Repeating the argument from the beginning of section 2.4, we can perform the sum over the complete set of states in the thermal expectation value of the operator \( \mathcal{O} \)

\[
\langle \mathcal{O} \rangle_R = \sum_{M=0}^{\infty} \sum_{n_1 < n_2 < \cdots < n_M} e^{-RE(n_1, \ldots, n_M)} \langle n_1, \ldots, n_M | \mathcal{O} | n_M, \ldots, n_1 \rangle \tag{4.9}
\]

by inserting the expansion (4.5) in each term of the sum and proceeding as in Section 2.3. The expansion analogous to the formula (2.20) for the partition function is

\[
\langle \mathcal{O} \rangle_R = \frac{1}{Z(L, R)} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \sum_{r_1, \ldots, r_m} \int \frac{du_1}{2\pi} \ldots \frac{du_m}{2\pi} \frac{e^{-Lr_1E(u_1) - Lr_mE(u_m)}}{r_1 \ldots r_m} \tag{4.10}
\]

\[
\times \sum_{\alpha \cup \bar{\alpha} = \{u_1, \ldots, u_m\}} \prod_{j \in \alpha} r_j F_{2|\alpha|}^c(\alpha) \frac{\det_{j,k \in \bar{\alpha}} \hat{G}_{jk}}{\prod_{i \in \bar{\alpha}} r_i},
\]

where the matrix \( \hat{G}_{jk} \) is defined by eq. (2.26) with \( \bar{p} \) replaced by \( p \) and the scattering kernel defined as \( K(u, v) = \frac{1}{2} \partial_u \log S(u, v) \).

The next step is to apply the matrix-tree theorem for the diagonal minors of the Gaudin determinant in the last factor in the integrand in (4.10). A minor obtained by removing all edges and all
columns from the subset $\alpha \subset \{1, \ldots, m\}$ of the matrix $\hat{G}_{jk}$ defined in eq. (2.26) has the following expansion,

$$
\det_{j,k \in \bar{\alpha}} \hat{G}_{jk} = \sum_{F \in F_{\alpha,\bar{\alpha}}} \prod_{\text{roots} \in \bar{\alpha}} \hat{D}_i \prod_{\ell \in F} \hat{K}_{kj}.
$$

(4.11)

The spanning forests $F \in F_{\alpha,\bar{\alpha}}$ are subjected to conditions (i)–(iii) of section 2.5, with the additional restriction that all vertices belonging to $\alpha$ are roots. The weight of these roots is one. An example is given in fig. 5.

![Figure 5:](image)

The expansion (4.11) follows directly from the expansion (2.27) of the previous section which corresponds to the particular case $\alpha = \emptyset, \bar{\alpha} = \{u_1, \ldots, u_m\}$. Indeed, the rhs of (4.11) by retaining only the terms in the rhs of (2.27) that contain the factor $\prod_{j \in \alpha} \hat{D}_j$ and then dividing the sum by this factor.

Now we can proceed similarly to what we have done in the computation of the partition function, where rearranging of the order of summation allowed us to rewrite the sum as a series of tree Feynman diagrams. This time there will be two kinds of Feynman graphs: the “vacuum trees” and diagrams representing a vertex $F^c_{2n}$ with $n$ lines and a tree attached to each line. The weight of such tree is the same as the weight of the vacuum trees except for a factor of $r^2$ associated with the root. This factor becomes obvious if one writes the dependence of the integrand/summand of (4.10) on the wrapping numbers $r_1, \ldots, r_m$ as

$$
\frac{1}{r_1^2 \ldots r_m^2} \prod_{j \in \alpha} r_j^2.
$$

The sum over the vacuum trees cancels with the partition function and the sum over the surviving terms has the same structure as (4.7), which is depicted in Fig. (6). The factor $f(u)$ is obtained as the sum of all trees with a root at the point $u$, with extra weight $r^2$ associated with the root:

$$
\sum_r r^2 Y_r(u) = \sum_r (-1)^{r-1} [Y_1(u)]^r = \frac{Y_1(u)}{1 + Y_1(v)} = f(u).
$$

(4.12)

The difference of the sum over trees in the factor $f(u)$ compared with the sum over vacuum trees (2.34) is that there is an extra factor $r$ associated with the root reflecting the breaking of the $Z_r$ symmetry of the corresponding wrapping process.

### 4.3 The case of a conserved charge

The simplest local operator $\mathcal{O}$ is of the type of conserved charge, such as the energy or the momentum. Such operators act diagonally on multi-particle states with one-particle values $o(u)$. The matrix
elements of the operator on a multi-particle state at zero temperature are

\[ \mathcal{O} = L^{-1} \int dx \mathcal{O}(x), \quad \langle u_n, \ldots, u_1 | \mathcal{O} | u_1, \ldots, u_n \rangle = \frac{1}{L} \sum_{j=1}^{n} o(u_j). \quad (4.13) \]

By direct computation one obtains [15]

\[ F_{2n}^c(u_1, \ldots, u_n) = p' \left( u_1 \right) K(u_2, u_1) K(u_3, u_2) \cdots K(u_n, u_{n-1}) o(u_n) \]

+ permutations,

\[ (4.14) \]

to be substituted in the LeClair-Mussardo series (4.7).

This formula can be readily obtained from the tree expansion using only the definition (4.13). We start with the series for the partition function (2.28), with \( \tilde{p} \) and \( \tilde{E} \) replaced by \( p \) and \( E \), and multiply each term by the eigenvalue of the operator \( \mathcal{O} \), which acts on the states \( |u_1^{r_1}, \ldots, u_m^{r_m}\rangle \) as

\[ \mathcal{O} |u_1^{r_1}, \ldots, u_m^{r_m}\rangle = \frac{1}{L} \sum_{j} r_j o(u_j) |u_1^{r_1}, \ldots, u_m^{r_m}\rangle. \quad (4.15) \]

After expanding the Gaudin norm in trees, one of the trees will acquire an extra factor \( r_j o(u_j) \) associated with one of its vertices. The sum over the vacuum gives the partition function which is to be stripped off and one is left with the sum over connected trees with one marked point,

\[ \langle \mathcal{O} \rangle_{L,R} = \int \frac{du_1}{2\pi} \int \frac{du_2}{2\pi} \sum_{r_1, r_2} L r_1 p' \left( u_1 \right) Y(u_1, r_1; u_2, r_2) \frac{1}{L} r_2 o(u_2) \quad (4.16) \]

where \( Y(u_1, r_1; u_2, r_2) = \delta(u_1 - u_2)\delta_{r_1, r_2} Y_{r_1}(u_1) + \ldots \) is the partition function of all directed trees with root at \( (u_1, r_1) \) and a marked vertex at \( (u_2, r_2) \). Any such tree can be decomposed into a backbone consisting of the edges connecting the root and the marked point, and a collection of trees rooted at the vertices along the backbone. We will associate a factor \( K_{jk} \) with the edge \( e_{kj} \) of the backbone, while the factors \( r_k \) and \( r_j \) will be absorbed into the weights of the trees rooted at the vertices \( k \) and \( j \). In this way the trees rooted at the point \( j \) of the backbone contain a factor \( r_j^2 \) coming from the two adjacent edges. The sum of such trees gives the factor \( f(u) \), eq. (4.12). The net result is

\[ \langle \mathcal{O} \rangle_R = \sum_{n=1}^{\infty} \prod_{j=1}^{n} \frac{du_j}{2\pi} p' \left( u_1 \right) f(u_1) K(u_2, u_1) f(u_2) K(u_3, u_2) \cdots K(u_n, u_{n-1}) f(u_n) o(u_n) \quad (4.17) \]

which is illustrated by fig. 7.
Another way to obtain the one-point function of a conserved charge is by replacing the energy \( E(u) \) in the thermal factors with \( E(u) - \alpha o(u) \). In this way the problem is reduced to the problem of the computation of the thermal partition function, but with slightly changed form of the energy. Since the computation of the partition function does not depend on the specific form of the energy, we can use the formulas of the previous section where \( Y_1(u) \) is replaced by \( Y_1(u, \alpha) \) determined by the non-linear integral equation
\[
\log Y(u, \alpha) = -RE(u) + \alpha o(u) + \int \frac{dv}{2\pi} K(v, u) \log [1 + Y_1(v, \alpha)].
\] (4.18)
The one-point function is given by the derivative
\[
\langle O \rangle_R = \frac{\partial}{\partial \alpha} \int du \frac{d'p(u)}{2\pi} \log(1 + Y_1(u, \alpha)) \bigg|_{\alpha=0} = \int du \frac{d'p(u)}{2\pi} f(u) \tilde{o}(u),
\] (4.19)
with \( \tilde{o}(u) \) satisfying a linear integral equation obtained by differentiating (2.35),
\[
\tilde{o}(u) = o(u) + \int \frac{dv}{2\pi} K(v, u) f(v) \tilde{o}(v).
\] (4.20)
This gives again the series (4.16).

5 Conclusion

We proposed a method for computing the finite volume (or finite temperature for the mirror theory) observables in (1+1)-dimensional field theories with factorised diagonal scattering and no bound states. The method is based on an exact treatment of the sum over a complete set of eigenstates of the Hamiltonian of the mirror theory using a graph expansion of the Gaudin measure using the Matrix-Tree Theorem. The free energy and the observables are expressed in terms of tree Feynman graphs. The vertices of such a graph correspond to virtual particles winding multiple times around the compact dimension and the oriented propagators correspond to scattering kernels. The method generalises trivially to the case of a theory with bound states. It is very natural to conjecture that the method can be generalised to theories with non-diagonal scattering.

The tree expansion derived here does not use relativistic invariance, hence the scattering matrix is not necessarily of difference form. Our principal motivation comes from AdS/CFT, where the world sheet (1+1)-dimensional field theory is not Lorentz invariant. We believe that after being generalised
for a theory with non-diagonal scattering and bound states, our construction will help to give a renormalised formulation of the hexagon proposal of [5] for computation of correlation functions of trace operators.

Another exercise would be to re-derive the \( g \)-functions in the case of integrable boundaries [20, 21]. The exact \( g \)-function for diagonal scattering is known [11] but the extension to non-diagonal scattering is still out of reach. The method might be also relevant for the one-point functions in AdS/dCFT [22, 23].

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Note added

After the completion of this work we learned about the earlier papers by G. Kato and M. Wadati [24–27], where the expression for the free energy of the Lieb-Liniger model and the XXX Heisenberg ferromagnetic has been obtained by a direct combinatorial method which is essentially identical to the one we are proposing here. We thank Balázs Pozsgay for bringing these works to our knowledge.

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