THE LIMIT OF THE HARMONIC FLOW ON FLAT COMPLEX VECTOR BUNDLE

XI ZHANG

ABSTRACT. In this paper we study the limiting behaviour of the harmonic flow on flat complex vector bundle, and prove the limit must be isomorphic to the graded flat complex vector bundle associated to the Jordan-Hölder filtration.

1. Introduction

Let \((E, D)\) be a flat complex vector bundle of rank \(r\) over a compact Riemannian manifold \((M, g)\). We say \((E, D)\) is simple if it has no proper \(D\)-invariant sub-bundle and \((E, D)\) is semi-simple if it is a direct sum of \(D\)-invariant sub-bundles. For the general case, there is a filtration of sub-bundles

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_i \cdots \subset E_l = E,
\]

such that every sub-bundle \(E_i\) is \(D\)-invariant and every quotient bundle \((Q_i, D_i) := (E_i/E_{i-1}, D_i)\) is flat and simple, which is called the Jordan-Hölder filtration of the flat complex vector bundle \((E, D)\). It is well known that the above filtration may be not unique, but the following graded flat complex vector bundle

\[
Gr^JH(E, D) = \bigoplus^l_{i=1}(Q_i, D_i)
\]

is unique in the sense of isomorphism. By the Riemann-Hilbert correspondence, we know that there is a one-to-one correspondence between the moduli space of fundamental group representations and the moduli space of flat vector bundles. If the flat bundle \((E, D)\) is corresponding to a representation \(\tau : \pi_1(M) \rightarrow GL(r, \mathbb{C})\), the graded object \(Gr^JH(E, D)\) is corresponding to the semi-simplification of \(\tau\).

Given a Hermitian metric \(H\) on \(E\), there is a unique decomposition

\[
D = D_H + \psi_H,
\]

where \(D_H\) is a unitary connection and \(\psi_H \in \Omega^1(\text{End}(E))\) is self-adjoint with respect to \(H\). A Hermitian metric \(H\) is called harmonic on \((E, D)\) if it is a critical point of the energy functional \(\int_M |\psi_H|^2 dV_g\), i.e. it satisfies the Euler-Lagrange equation

\[
D_H^* \psi_H = 0.
\]

Under the assumption that the flat complex vector bundle \((E, D)\) is semi-simple, Corlette [3] and Donaldson [7] proved the existence of harmonic metric. Furthermore, when \((M, g)\) is a Kähler manifold, the existence of harmonic metric \(H\) implies

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that there exists a poly-stable Higgs structure \((D^{0,1}_H, \psi^{1,0}_H)\) on \(E\). On the other hand, by the work of Hitchin ([9]) and Simpson ([14]) on Donaldson-Uhlenbeck-Yau theorem for Higgs bundles, one has the non-abelian Hodge correspondence, i.e. there is an equivalence of categories between the category of poly-stable Higgs bundles with vanishing Chern numbers and the category of semi-simple flat bundles.

In order to obtain harmonic metrics, Corlette ([3]) introduced the following heat flow

\[
\frac{\partial \sigma(t)}{\partial t} \cdot (\sigma(t))^{-1} = D_{t,K}^* \psi_{t,K},
\]

where \(K\) is a fixed Hermitian metric on \((E,D)\), \(\sigma(t) \in \Gamma(\text{Aut}E)\) and

\[
D_t = \sigma(t) \{D\} = \sigma(t) \cdot D \cdot \sigma^{-1}(t) = D_{t,K} + \psi_{t,K}.
\]

The heat flow (1.5) is equivalent to the following heat flow which involves flat connections,

\[
\frac{\partial D_t}{\partial t} = -D_t \{D_{t,K}^* \psi_{t,K}\}.
\]

We call the above heat flow (1.7) the harmonic flow on the flat bundle \((E,D)\). Corlette proved the existence of long time solution \(D_t\) \((0 \leq t < \infty)\) for the heat flow (1.7). By choosing a subsequence and taking suitable unitary gauge transformations, \(D_t\) converges weakly to a flat connection \(D_\infty\) in \(L^p\). Furthermore, if \((E,D)\) is simple, Corlette showed that the limit must lie in the complex gauge orbit of \(D\), i.e. there exists \(\eta_\infty \in \Gamma(\text{Aut}E)\) such that \(D_\infty = \eta_\infty \cdot D \cdot \eta_\infty^{-1}\).

In this paper, we consider the limit of the harmonic flow (1.7) on the flat bundle \((E,D)\) which is not necessarily simple. Firstly, let’s recall the limiting behaviour of the Yang-Mills flow on holomorphic vector bundles. For the Riemann surface case, Atiyah and Bott ([1]) pointed out that the limiting holomorphic bundle should be isomorphic to the graded bundle associated to the Harder-Narasimhan-Seshadri filtration, and this conjecture has been proved by Daskalopoulos ([4]). In [2], Bando and Siu proposed an interesting question that the above Atiyah-Bott’s conjecture should still hold for reflexive sheaf \(E\) over higher dimensional Kähler manifold. When the sheaf \(E\) is locally free, this question was answered in the affirmative by Daskalopoulos and Wentworth ([5]) for Kähler surfaces case; by Jacob ([11]) and Sibley ([13]) for higher dimensional case. The general reflexive sheaves case was confirmed by Li, Zhang and the author ([12]). Inspired by this, it is natural to raise a question: should the limit of the harmonic flow (1.7) be isomorphic to the graded flat complex vector bundle associated to the Jordan-Hölder filtration? When the base manifold \((M,g)\) is Kähler, this also is conjectured by Deng ([6]) in his doctoral dissertation. In this paper, we solve this problem, i.e. we prove the following theorem.

**Theorem 1.1.** Let \((E,D)\) be a flat complex vector bundle over a compact Riemannian manifold \((M,g)\), and \(D_t\) be the long time solution of the harmonic flow (1.7) with initial data \(D\). Then the limiting flat bundle \((E,D_\infty)\) must be isomorphic to the graded flat complex vector bundle associated to the Jordan-Hölder filtration of \((E,D)\), i.e. we have:

\[
(E,D_\infty) \cong \text{Gr}^JH(E,D).
\]
This paper is organized as follows. In Section 2, we introduce some basic concepts and results about the harmonic flow on flat complex vector bundles. In Section 3, we give a proof of Theorem 1.1.

2. Preliminaries

Let \((M, g)\) be a compact Riemannian manifold of dimension \(n\), \(E\) be a complex vector bundle over \(M\) with rank \(r\). Given any connection \(D\) and Hermitian metric \(H\) on \(E\), there is a unique decomposition

\[
D = D_H + \psi_H,
\]

where \(D_H\) is an \(H\)-unitary connection, \(\psi_H \in \Omega^1(\text{End}(E))\) is \(H\)-self-adjoint, i.e. \(\psi_H^* = \psi_H\), and

\[
H(\psi_H X, Y) = \frac{1}{2}(H(DX, Y) + H(X, DY) - dH(X, Y))
\]

for any \(X, Y \in \Gamma(E)\). Suppose \(K\) is another Hermitian metric on \(E\), then we have

\[
\psi_H = \frac{1}{2} h^{-1} \circ \psi_K \circ h + \frac{1}{2} \psi_K + \frac{1}{2} (D_K - h^{-1} \circ D_K \circ h)
\]

(2.3)

and

\[
D_H = \frac{1}{2} (\psi_K - h^{-1} \circ \psi_K \circ h + D_K + h^{-1} \circ D_K \circ h)
\]

(2.4)

where \(h = K^{-1} H\).

If \(D\) is a flat connection, then

\[
0 = F_D = D_H^2 + \psi_H \wedge \psi_H + D_H \circ \psi_H + \psi_H \circ D_H.
\]

(2.5)

Considering the self-adjoint and anti-self-adjoint parts of the above identity, we obtain

\[
D_H(\psi_H) = 0,
\]

(2.6)

and

\[
D_H^2 + \psi_H \wedge \psi_H = 0.
\]

(2.7)

Let \(H(t)\) be a family of Hermitian metrics on \(E\). By direct computation, one can find that

\[
\frac{\partial \psi_{H(t)}}{\partial t} = -\frac{1}{2} D_H(H^{-1} \frac{\partial H}{\partial t}) + \frac{1}{2} \psi_H \circ H^{-1} \frac{\partial H}{\partial t} - \frac{1}{2} H^{-1} \frac{\partial H}{\partial t} \circ \psi_H.
\]

(2.8)

Choosing local coordinates \(\{x^i\}_{i=1}^n\) on \(M\), we write \(g = g_{ij} dx^i \otimes dx^j\), \(\psi_H = (\psi_H)_k dx^k\) and

\[
|\psi_H|_H^2 = g^{ij} \text{tr}\{ (\psi_H)_i \circ (\psi_H)_j^*\}
\]

(2.9)
where \((\psi_H)_i \in \Gamma(\text{End}(E))\) and \((g^{ij})\) is the inverse matrix of \((g_{ij})\). After a straightforward calculation, one can check that

\[
\frac{\partial}{\partial t}|\psi_{H,ij}|_H^2 = 2Re\left(\frac{\partial \psi_{H,ij}}{\partial t}\right) - \frac{1}{2}\psi_{H,ij}H^{-1}\frac{\partial H}{\partial t} + \frac{1}{2}H^{-1}\frac{\partial H}{\partial t} \circ \psi_{H,ij}_H
\]

\[
= -Re\langle D_H(h^{-1}\frac{\partial h}{\partial t}), \psi_{H,ij}_H \rangle.
\]

Let \(K\) be a fixed metric on \(E\). Denote the group of smooth automorphisms of \(E\) (which preserve the metric \(K\)) as \(\mathcal{G}(U_K)\). Every \(\sigma \in \mathcal{G}\) acts on the connection \(D\) by

\[
\sigma(D) := \sigma \circ D \circ \sigma^{-1}.
\]

For \(\sigma \in \mathcal{G}\), i.e. \(H(X, Y) = K(\sigma X, \sigma Y)\) for any \(X, Y \in \Gamma(E)\), set \(H = K\sigma^*\sigma\). One can see that

\[
K(\psi_{(D), K}(X, Y)) = \frac{1}{2}\{K(\sigma(D)X, Y) + K(X, \sigma(D)Y) - dK(X, Y)\}
\]

\[
= \frac{1}{2}\{H(D \circ \sigma^{-1}X, \sigma^{-1}Y) + H(\sigma^{-1}X, D \circ \sigma^{-1}Y) - dH(\sigma^{-1}X, \sigma^{-1}Y)\}
\]

\[
= H(\psi_{D, H} \circ \sigma^{-1}(X), \sigma^{-1}Y)
\]

\[
= K(\sigma \circ \psi_{D, H} \circ \sigma^{-1}(X), \sigma^{-1}Y).
\]

Then

\[
\psi_{(D), K} = \sigma \circ \psi_{D, H} \circ \sigma^{-1}
\]

and

\[
\sigma(D)_K = \sigma \circ D_H \circ \sigma^{-1}.
\]

By (2.13), (2.14) and (2.13), we have (or By the definition, we have!!!!!!!)

\[
\psi_{(D), K} = (\sigma^*\sigma^{-1}) \circ \psi_{D, K} \circ \sigma^*\sigma + \frac{1}{2}\sigma \circ D \circ \sigma^{-1} - \frac{1}{2}(\sigma^*\sigma^{-1}) \circ D \circ \sigma^*\sigma
\]

and

\[
\sigma(D)_K = (\sigma^*\sigma^{-1}) \circ D_K \circ \sigma^*\sigma + \frac{1}{2}\sigma \circ D \circ \sigma^{-1} - \frac{1}{2}(\sigma^*\sigma^{-1}) \circ D \circ \sigma^*\sigma.
\]

Let \(h = \sigma^*\sigma\). There holds that

\[
\psi_{(D), K} = (\sigma^*\sigma^{-1}) \circ (\psi_{D, K} - \frac{1}{2}D(h) \circ h^{-1}) \circ \sigma^*\sigma
\]

and

\[
\sigma(D)_K = (\sigma^*\sigma^{-1}) \circ (D_K - \frac{1}{2}D(h) \circ h^{-1}) \circ \sigma^*\sigma.
\]

From the definition, it is easy to see that

\[
\langle \varphi_1, \varphi_2 \rangle_H = (\sigma \circ \varphi_1 \circ \sigma^{-1}, \sigma \circ \varphi_2 \circ \sigma^{-1})_K
\]

for any \(\varphi_1, \varphi_2 \in \Gamma(\text{End}(E))\). We know

\[
\langle \psi_{D, H}, D_H \varphi \rangle_H = \langle \psi_{D, H}, (\sigma^{-1} \circ (\sigma(D))_K \circ \sigma)(\varphi) \rangle_H
\]

\[
= \langle \psi_{D, H}, \sigma^{-1} \circ \{(\sigma(D))_K(\sigma \circ \varphi \circ \sigma^{-1})\} \circ \sigma \rangle_H
\]

\[
= (\sigma \circ \psi_{D, H} \circ \sigma^{-1}, (\sigma(D))_K(\sigma \circ \varphi \circ \sigma^{-1}))_K
\]

\[
= \langle \psi_{(D), K}, (\sigma(D))_K(\sigma \circ \varphi \circ \sigma^{-1}) \rangle_K
\]
and then
\[(2.21) \quad (\sigma(D))^*_K \psi_{\sigma(D),K} = \sigma \circ D^*_H \psi_{D,H} \circ \sigma^{-1}.\]
On the other hand, one can check that \((\sigma(D))^*_K \psi_{\sigma(D),K}\) is self-adjoint with respect to the metric \(K\).

Lemma 2.1. Let \((E, D)\) be a flat complex vector bundle on a compact Riemannian manifold \((M, g)\), and \(K\) be a Hermitian metric on \(E\). For any \(\sigma \in \mathcal{G}\), we have
\[(2.22) \quad \langle \sigma^{-1} \circ (\sigma(D))^*_K \psi_{\sigma(D),K} \circ \sigma - D^*_K \psi_{D,K}, h \rangle_K = \frac{1}{2} \Delta \tr h - \frac{1}{2} \langle D(h) \circ h^{-1}, D(h) \rangle_K,\]
\[(2.23) \quad \langle D^*_K \psi_{D,K} \circ \sigma - 1, (\sigma(D))^*_K \psi_{\sigma(D),K}, h \rangle_K = \frac{1}{8} \Delta \tr h^{-1} - \frac{1}{2} \langle h \circ D(h^{-1}), D(h^{-1}) \rangle_K\]
and
\[(2.24) \quad \langle \sigma^{-1} \circ (\sigma(D))^*_K \psi_{\sigma(D),K} \circ \sigma - D^*_K \psi_{D,K}, s \rangle_K = \frac{1}{4} \Delta |s|^2_K - \frac{1}{2} \langle D(h) \circ h^{-1}, D(s) \rangle_K,\]
where \(h = \sigma^*K \circ \sigma\) and \(s = \log h\).

Proof. By (2.17) and (2.18), and choosing local normal coordinates \(\{x^i\}_{i=1}^n\) centered at the considered point, one can easily check that
\[(2.25) \quad \sigma^{-1} \circ (\sigma(D))^*_K \psi_{\sigma(D),K} \circ \sigma = \langle D^*_K \psi_{D,K}, h \rangle_K - \frac{1}{2} \langle D^*_K (D(h) \circ h^{-1}), h \rangle_K\]
\[+ \frac{1}{2} g^{ij} \langle \partial_j (D^*_K \psi_{D,K} \circ \sigma) - \psi_{D,K} \psi_{D,K}, h \rangle_K = \frac{1}{2} \Delta \tr h - \frac{1}{2} \langle D(h) \circ h^{-1}, D(h) \rangle_K,\]
and then
\[(2.26) \quad \langle \sigma^{-1} \circ (\sigma(D))^*_K \psi_{\sigma(D),K} \circ \sigma - D^*_K \psi_{D,K}, s \rangle_K = - \frac{1}{2} \langle D^*_K (D(h) \circ h^{-1}) \circ s \rangle_K\]
\[+ \frac{1}{2} g^{ij} \langle \partial_j (D^*_K \psi_{D,K} \circ \sigma) - \psi_{D,K} \psi_{D,K} \circ s \rangle_K = - \frac{1}{2} \langle D(h) \circ h^{-1} \circ s \rangle_K\]
\[+ \frac{1}{2} g^{ij} \langle \partial_j (D^*_K \psi_{D,K} \circ \sigma) - \psi_{D,K} \psi_{D,K} \circ s \rangle_K = \frac{1}{2} \Delta |s|^2_K - \frac{1}{2} \langle D(h) \circ h^{-1}, D(h) \rangle_K.\]

Here we have used the following identity
\[(2.27) \quad \tr (D^*_K \psi_{D,K} \circ \sigma) = \tr (s \circ D^*_K \psi_{D,K} \circ \sigma).\]
Immediately (2.22) and (2.23) can be proved in a similar way. \qed
Lemma 2.2. Let \((E, \nabla)\) be a flat complex vector bundle on a compact Riemannian manifold \((M, g)\), and \(K\) be a Hermitian metric on \(E\). Assume \(\nabla K \cdot \nabla K = 0\), then \((E, \nabla)\) must be semi-simple.

Proof. Suppose that \((E, \nabla)\) is not simple, and choose a \(\nabla\)-invariant sub-bundle \(S\) which is minimal rank. Then there exists an exact sequence
\[
0 \to S \to E \to Q \to 0.
\]

Denote \(D_S\) and \(D_Q\) (respectively, \(K_S\) and \(K_Q\)) the connections (respectively, metrics) on the sub-bundle \(S\) and the quotient bundle \(Q\) induced by the connection \(\nabla\) (respectively, metric \(K\)). For the Hermitian metric \(K\) on \(E\), we have the following bundle isomorphism
\[
f_K : S \oplus Q \to E, \quad (X, [Y]) \mapsto i(X) + (\text{Id}_E - \pi_K)(Y),
\]
where \(X \in \Gamma(S)\), \(Y \in \Gamma(E)\), \(i : S \to E\) is the inclusion and \(\pi_K : E \to E\) is the orthogonal projection into \(S\) with respect to the metric \(K\). Since \(S\) is \(\nabla\)-invariant, we know
\[
\pi_K = (\pi_K)^2 = (\pi_K)^* K
\]
and
\[
(\text{Id}_E - \pi_K) \circ \nabla(\pi_K) = 0.
\]

By the definition, the pulling back metric is
\[
f_K^* (K) = \begin{pmatrix} K_S & 0 \\ 0 & K_Q \end{pmatrix},
\]
and the pulling back connection is
\[
f_K^* (\nabla) = \begin{pmatrix} D_S & \beta \\ 0 & D_Q \end{pmatrix},
\]
where \(\beta \in \Omega^1(\text{Hom}(Q, S))\) will be called the second fundamental form. One can check that
\[
\beta([Y]) = -\pi_K \circ (\nabla \pi_K)(Y),
\]
where \(Y \in \Gamma(E)\). Because \(\nabla\) is flat, we have
\[
D_S^2 = 0, \quad D_Q^2 = 0, \quad D_S \circ \beta + \beta \circ D_Q = 0.
\]
It is easy to see that
\[
f_K^* (\nabla K) = \begin{pmatrix} \psi_{DS, KS} & \frac{1}{2} \beta \\ \frac{1}{2} \beta^* & \psi_{DQ, KQ} \end{pmatrix},
\]
\[
f_K^* (\nabla K) = \begin{pmatrix} D_{KS} & \frac{1}{2} \beta \\ \frac{1}{2} \beta^* & D_{KQ} \end{pmatrix},
\]
where \(\beta^* \in \Omega^1(\text{Hom}(Q, S))\) is the adjoint of \(\beta\) with respect to the metrics \(K_S\) and \(K_Q\). In the following, we choose the normal coordinates \(\{x^i\}_{i=1}^n\) centered at the
considered point \( p \in M \). A direct calculation yields

\[
(2.38) \quad f_K^{-1} \circ \hat{D}_K^* \psi_{D,K} \circ f_K = f_K^*(\hat{D}_K)^* \{ f_K^*(\psi_{D,K}) \}
\]

\[
= \left( \frac{1}{2} D_{K,S}^* \psi_{D_S,K_S} - \frac{1}{2} g^{ij} \beta_i \circ \beta_j^* = \frac{1}{2} D_{K,Q}^* \circ S_S^* \beta - \frac{1}{2} g^{ij} \beta_i \circ \psi_{Q_j} - \psi_{S_j} \circ \beta_i \right) - \frac{1}{2} D_{K,Q}^* \psi_{D_Q,K_Q} + \frac{1}{2} g^{ij} \beta_i^* \circ \beta_j,
\]

where \( \beta_i = \beta_i(\frac{\partial}{\partial x^i}) \), \( \beta_j = \beta_j(\frac{\partial}{\partial x^j}) \), \( \psi_S \circ i = \psi_S(\frac{\partial}{\partial x^i}) \) and \( \psi_{Q,ij} = \psi_Q(\frac{\partial}{\partial x^i}). \) Due to \( \hat{D}_K^* \psi_{D,K} = 0 \), \( (2.38) \) implies

\[
(2.39) \quad D_{K,S}^* \psi_{D_S,K_S} - \frac{1}{2} g^{ij} \beta_i \circ \beta_j^* = 0,
\]

and

\[
(2.40) \quad \int_M |\beta|^2 dV_g = 2 \int_M \{ D_{K,S}^* \psi_{D_S,K_S}, \text{Id}_S \} \psi_{D,Q} dV_g = 0.
\]

So \( (E, \hat{D}) \cong (S, D_S) \oplus (Q, D_Q) \), where \( (S, D_S) \) is a simple flat bundle and \( (Q, D_Q) \) is a flat bundle with \( D_{K,Q}^* \psi_{D_Q,K_Q} = 0 \). Applying the above argument to \( (Q, D_Q) \), we obtain an isomorphism

\[
(2.41) \quad (E, \hat{D}) \cong \oplus_{i=1}^i (Q_i, D_{Q_i}),
\]

where every \( (Q_i, D_{Q_i}) \) is a simple flat bundle. \( \Box \)

Let \( \sigma(t) \) be a solution of the heat flow \( [158] \), i.e. it satisfies

\[
(2.42) \quad \frac{\partial \sigma(t)}{\partial t} \circ \sigma^{-1}(t) = (\sigma(t)\{D\})^* \psi_{\sigma(t)\{D\},K},
\]

then

\[
(2.43) \quad \frac{\partial}{\partial t} \sigma(t)\{D\} = -\sigma(t)\{D\} \frac{\partial \sigma(t)}{\partial t} \circ \sigma^{-1}(t) \quad \text{and} \quad -\sigma(t)\{D\}((\sigma(t)\{D\})^* \psi_{\sigma(t)\{D\},K}).
\]

Considering the self-adjoint and anti-self-adjoint parts of the above identity, we have

\[
(2.44) \quad \frac{\partial}{\partial t} \sigma(t)\{D\} = -[\psi_{\sigma(t)\{D\},K}, (\sigma(t)\{D\})^* \psi_{\sigma(t)\{D\},K}],
\]

and

\[
(2.45) \quad \frac{\partial}{\partial t} \psi_{\sigma(t)\{D\},K} = -\sigma(t)\{D\} [\psi_{\sigma(t)\{D\},K}, (\sigma(t)\{D\})^* \psi_{\sigma(t)\{D\},K}].
\]

Let’s recall some basic estimates of the heat flows \((1.5)\) and \((1.7)\).

**Lemma 2.3.** \([3]\) Let \((E, D)\) be a flat complex vector bundle on a compact Riemannian manifold \((M, g)\), and \( K \) be a Hermitian metric on \( E \). If \( \sigma(t) \) is a solution of the heat flow \((1.5)\), then we have

\[
(2.46) \quad \frac{d}{dt} \| \psi_{\sigma(t)\{D\},K} \|^2_{L^2} = -2 \| (\sigma(t)\{D\})^* \psi_{\sigma(t)\{D\},K} \|^2_{L^2},
\]

\[
(2.47) \quad (\Delta - \frac{\partial}{\partial t}) \| \psi_{\sigma(t)\{D\},K} \|^2_K = 2 |\nabla (\sigma(t)\{D\})^* \psi_{\sigma(t)\{D\},K} |^2_K + 2 \langle \psi_{\sigma(t)\{D\},K} \circ \text{Ric}, \psi_{\sigma(t)\{D\},K} \rangle_K + 2 |\psi_{\sigma(t)\{D\},K}, \psi_{\sigma(t)\{D\},K} |^2_K.
\]
and
\begin{equation}
(\Delta - \frac{\partial}{\partial t})\vert_{\sigma(t)} (\sigma(t)\{D\})_{K} \psi_{\sigma(t)} \{D\}, K_{K}^{2}K \\
= 2|\sigma(t)\{D\})_{K} (\sigma(t)\{D\})_{K} \psi_{\sigma(t)} \{D\}, K_{K}|^{2}K + 2|\psi_{\sigma(t)} \{D\}, K_{K}^{2}K|^{2}K.
\end{equation}

**Proposition 2.1.** ([3]) Let \((E, D)\) be a flat complex vector bundle on a compact Riemannian manifold \((M, g)\), and \(K\) be a Hermitian metric on \(E\). The harmonic flow \(f\) has a long time solution \(\sigma(t)\) for \(t \in [0, \infty)\). Furthermore, for every sequence \(t_{i} \rightarrow \infty\) there exists a subsequence \(t_{j}\) such that \(\sigma(t_{j})\{D\} = (\sigma(t_{j})\{D\})_{K} + \psi_{\sigma(t_{j})}\{D\}, K_{K}\) converges weakly, modulo \(K\)-unitary gauge transformations, to a flat connection \(D_{\infty} = D_{\infty, K} + \psi_{\infty, K}\) in \(L_{2}^{1}\)-topology, and \(D_{\infty, K}\psi_{\infty, K} = 0\).

In the following, we will show that the above convergence can be strengthened to in \(C^{\infty}\)-topology. Using (2.47), we deduce
\begin{equation}
(\Delta - \frac{\partial}{\partial t})|_{\sigma(t)} (\sigma(t)\{D\})_{K} \geq -C_{1}|\psi_{\sigma(t)} \{D\}, K_{K}^{2}K,
\end{equation}
equivalently
\begin{equation}
(\Delta - \frac{\partial}{\partial t})(e^{-C_{1}t}|\psi_{\sigma(t)} \{D\}, K_{K}^{2}K) \geq 0,
\end{equation}
where \(C_{1}\) is a positive constant depending only on the Ricci curvature of \((M, g)\). Let \(f(x, t) = \int_{M} \chi(x, y, t - t_{0})e^{-C_{1}t_{0}|\psi_{\sigma(t_{0})}\{D\}, K_{K}^{2}K(y)dV_{g}(y)\), where \(\chi\) is the heat kernel of \((M, g)\). Of course (2.50) implies:
\begin{equation}
(\Delta - \frac{\partial}{\partial t})(e^{-C_{1}t}|\psi_{\sigma(t)} \{D\}, K_{K}^{2}K - f(x, t)) \geq 0
\end{equation}
and
\begin{equation}
f(t, t_{0}) = e^{-C_{1}t_{0}|\psi_{\sigma(t_{0})}\{D\}, K_{K}^{2}K}.
\end{equation}
From the maximum principle and (2.46), for any \(t_{0} \geq 0\), it follows that
\begin{equation}
\max_{M} e^{-C_{1}(t_{0} + 1)|\psi_{\sigma(t_{0} + 1)}\{D\}, K_{K}^{2}K} \leq \max_{M} f(x, t_{0} + 1)
\end{equation}
\begin{equation}
= \int_{M} \chi(x, y, 1)e^{-C_{1}t_{0}|\psi_{\sigma(t_{0})}\{D\}, K_{K}^{2}K(y)dV_{g}(y)
\end{equation}
\begin{equation}
\leq C_{2}e^{-C_{1}t_{0}} \int_{M} |\psi_{\sigma(t_{0})\{D\}, K_{K}^{2}K}dV_{g}
\end{equation}
\begin{equation}
\leq C_{2}e^{-C_{1}t_{0}} \int_{M} |\psi_{D, K_{K}^{2}K}dV_{g},
\end{equation}
and then
\begin{equation}
\max_{M} |\psi_{\sigma(t_{0} + 1)}\{D\}, K_{K}^{2}K \leq C_{2}e^{C_{1}} \int_{M} |\psi_{D, K_{K}^{2}K}dV_{g},
\end{equation}
where \(C_{2}\) is a positive constant depending only on the upper bound of \(\chi(x, y, 1)\). Hence we know that \(\sup_{M} |\psi_{\sigma(t)}\{D\}, K_{K}^{2}K\) is uniformly bounded.
Choosing local normal coordinates \( \{ x^i \}_{i=1}^n \) centered at the considered point, we have
\[
\Delta |\nabla^\sigma \psi_{(t)\{D\}} , K|^2_K = 2 |\nabla^\sigma \nabla^\sigma \psi_{(t)\{D\}} , K|^2_K
\]
\[
+ 2 \text{Re} \{ g^{ij} \left( \frac{\partial}{\partial x_i} \nabla^\sigma \psi_{(t)\{D\}} , K \right) \left( \frac{\partial}{\partial x_j} \nabla^\sigma \psi_{(t)\{D\}} , K \right) \},
\]
where \( \nabla^\sigma \) is the covariant derivative induced by the connection \( (\sigma(t)\{D\})_K \) and the Levi-Civita connection \( \nabla \) of \( (M,g) \). Denote
\[
\nabla^\sigma \psi_{(t)\{D\}} , K = \psi_{m,i} dx^m \otimes dx^i,
\]
and we have
\[
g^{ij} \psi_{m,i} \psi_{n,j} = g^{ij} \psi_{m,i} dx^m \otimes dx^j,
\]
where \( \psi_{m,i} \psi_{n,j} \) denotes the component of the covariant derivative. According to the Ricci identity, one can check that
\[
g^{ij} \psi_{m,i} \psi_{n,j} = g^{ij} \left( \psi_{j,iml} + \psi_{a,l} R^a_{jmi} + \psi_{a,l} R^a_{mlj} + \psi_{a,m} R^a_{jli} + \psi_{a,m} R^a_{mlj} + [F_{im,i}, \psi_{j}] + [F_{im}, \psi_{j,l}] + [F_{jl}, \psi_{j,m}] + [F_{jl,i}, \psi_{m}] + [F_{jl}, \psi_{m,i}] \right),
\]
where \( R \) denotes the Riemannian curvature of \( g \) and \( F \) denotes the curvature of the connection \( (\sigma(t)\{D\})_K \). By the harmonic flow \((1.5)\), we have
\[
\frac{\partial}{\partial t} \nabla^\sigma \psi_{(t)\{D\}} , K |^2_K = -2 \text{Re} \left( \nabla^\sigma (\sigma(t)\{D\}) (\sigma(t)\{D\})_K \nabla^\sigma \psi_{(t)\{D\}} , K \right) \nabla^\sigma \psi_{(t)\{D\}} , K
\]
\[- 2 \text{Re} \left( \nabla^\sigma \psi_{(t)\{D\}} , K \nabla^\sigma \psi_{(t)\{D\}} , K \right) \psi_{(t)\{D\}}_K \nabla^\sigma \psi_{(t)\{D\}} , K |^2_K.
\]
On the other hand, it is straightforward to check that
\[
\nabla^\sigma ((\sigma(t)\{D\})_K (\sigma(t)\{D\})_K) = - g^{ij} \psi_{j,iml} dx^m \otimes dx^i.
\]
From \((2.55)\), \((2.58)\), \((2.59)\) and \((2.60)\), we get
\[
\Delta - \frac{\partial}{\partial t} |\nabla^\sigma \psi_{(t)\{D\}} , K|^2_K \geq 2 |\nabla^\sigma \nabla^\sigma \psi_{(t)\{D\}} , K|^2_K
\]
\[- C_3 (|Rm| + |\psi_{(t)\{D\}} , K|^2_K) \nabla^\sigma \psi_{(t)\{D\}} , K |^2_K
\]
\[- C_4 (|\psi_{(t)\{D\}} , K|_K (\sigma(t)\{D\})_K |K + |\psi_{(t)\{D\}} , K|_K |Rm|) |\nabla^\sigma \psi_{(t)\{D\}} , K |_K,
\]
where \( C_3 \) and \( C_4 \) are uniform constants depending only on \( \text{dim}(M) \) and \( \text{rank}(E) \). \((2.47)\) and \((2.54)\) yields
\[
\int_{M \times [t_0,t_0+3]} |\nabla^\sigma \psi_{(t)\{D\}} , K |^2_K dV_g dt \leq C_5
\]
for all \( t_0 \geq 0 \), where \( C_3 \) is a uniform constant. Making use of \((2.61)\) and the Moser’s parabolic estimate, one can derive
\[
\sup_{M \times [t_0+1,t_0+2]} |\nabla^\sigma \psi_{(t)\{D\}} , K |^2_K \leq C_6.
\]
Furthermore, it holds that
\[
\Delta - \frac{\partial}{\partial t} |(\nabla^\sigma)^\alpha \psi_{(t)\{D\}} , K |^2_K \geq 2 |(\nabla^\sigma)^{\alpha+1} \psi_{(t)\{D\}} , K |^2_K
\]
\[- C_5 |(\nabla^\sigma)^\alpha \psi_{(t)\{D\}} , K |^2_K - C_6 |(\nabla^\sigma)^\alpha \psi_{(t)\{D\}} , K |_K,
\]
We will show that such that every sub-bundle $E$ and repeating the above argument, we have the following uniform estimates
\[
\sup_{|0, \infty| \times M} (|(\nabla^\alpha F(\sigma(t)(D))|_K|^2 + |(\nabla^\alpha)^{-1} \psi_\sigma(t)(D)|_K|^2 \leq \tilde{C}_\alpha
\]
for every $0 \leq \alpha < \infty$. Then, applying a result of Donaldson-Kronheimer (Theorem 2.3.7 in [8]) and Hong-Tian’s argument (Proposition 6 in [10]), we obtain the following proposition.

Proposition 2.2. Let $(E, D)$ be a flat complex vector bundle over a compact Riemannian manifold $(M, g)$, and $K$ be a Hermitian metric on $E$. If $\sigma(t)$ is a longtime solution of the heat flow (1.5), then for every sequence $t_i \to \infty$ there exists a subsequence $t_j$ such that $\sigma(t_j)(D) = (\sigma(t_j)(D))_K + \psi_\sigma(t_j)(D)$ converges, modulo $K$-unitary gauge transformations, to a flat connection $D_\infty = D_K, \infty + \psi_\infty$ in $C^\infty$-topology, and $D^*_K, \infty \psi_\infty = 0$.

3. Proof of Theorem 1.1

The following proposition about the existence of the Jordan-Hölder filtration and the uniqueness of the graded flat complex vector bundle should be well known to experts, and we give a proof here for the reader’s convenience.

Proposition 3.1. Let $(E, D)$ be a flat complex vector bundle. There is a filtration of sub-bundles
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_i \cdots \subset E_l = E,
\]
such that every sub-bundle $E_i$ is $D$-invariant and every quotient bundle $(Q_i, D_{Q_i}) := (E_i/E_{i-1}, D_{Q_{i-1}})$ is flat and simple. Furthermore, the graded flat complex vector bundle $\oplus_{i=1}^l (Q_i, D_{Q_i})$ is unique in the sense of isomorphism.

Proof. Suppose that $(E, D)$ is not simple. Let $E_1$ be a $D$-invariant sub-bundle of $E$ with minimal rank. Then we have the following exact sequence
\[
0 \to E_1 \to E \xrightarrow{i} P \to Q \to 0,
\]
and $(E_1, D_{E_1})$ is simple. By induction, we can assume that there is a Jordan-Hölder filtration of the flat bundle $(Q, D_Q)$, i.e.
\[
0 = Q_0 \subset Q_1 \subset \cdots \subset Q_{i-1} \subset \hat{Q}_{i-1} = Q.
\]
Choosing a Hermitian metric $K$ on $E$, we get a bundle isomorphism $P^*K : Q \to E_1^\perp$, where $P^*K$ is the adjoint of the projection $P$ with respect to the metric $K$. Set
\[
E_i = E_1 \oplus P^*K(\hat{Q}_{i-1})
\]
for all $1 < i \leq l$. One can easily find that every $E_i$ is $D$-invariant and $(E_i/E_{i-1}, D_{Q_i})$ is simple. So, we obtain a Jordan-Hölder filtration of $(E, D)$.

Suppose that there is another Jordan-Hölder filtration of $(E, D)$,
\[
0 = \tilde{E}_0 \subset \tilde{E}_1 \subset \cdots \subset \tilde{E}_i \cdots \subset \tilde{E}_l = E.
\]
We will show that
\[
\oplus_{i=1}^l (Q_i, D_{Q_i}) \cong \oplus_{i=1}^l (\hat{Q}_i, D_{\hat{Q}_i}).
\]
It is straightforward to check that there exist bundle isomorphisms $f : \bigoplus_{i=1}^{I} Q_i \to E$ and $\tilde{f} : \oplus_{\alpha=1}^{J} \tilde{Q}_\alpha \to E$ such that

$$f^*(D) = \begin{pmatrix}
D_{Q_1} & \cdots & \beta_{1I} \\
& \ddots & \vdots \\
0 & \cdots & D_{Q_i}
\end{pmatrix},$$

and

$$\tilde{f}^*(D) = \begin{pmatrix}
D_{\tilde{Q}_1} & \cdots & \gamma_{1\tilde{i}} \\
& \ddots & \vdots \\
0 & \cdots & D_{\tilde{Q}_i}
\end{pmatrix}.$$

Let $\eta = f^{-1} \circ \tilde{f} = (\eta_{i\alpha})$, where $\eta_{i\alpha} \in \text{Hom}(\tilde{Q}_j, Q_i)$. (3.7) and (3.8) mean

$$\eta \circ \begin{pmatrix}
D_{Q_1} & \cdots & \gamma_{1\tilde{i}} \\
& \ddots & \vdots \\
0 & \cdots & D_{\tilde{Q}_i}
\end{pmatrix} = \begin{pmatrix}
D_{Q_1} & \cdots & \beta_{1I} \\
& \ddots & \vdots \\
0 & \cdots & D_{Q_i}
\end{pmatrix} \circ \eta.$$

Assume that $\eta_{11} = 0, \cdots, \eta_{(k+1)1} = 0$ and $\eta_{k1} \neq 0$. We express $\eta$ as a partitioned matrix

$$\eta = \begin{pmatrix}
\tilde{\eta}_{11} & \tilde{\eta}_{12} \\
\eta_{k1} & \eta_{32}
\end{pmatrix},$$

where $\tilde{\eta}_{11} = \left(\begin{array}{c}
\eta_{11} \\
\vdots \\
\eta_{(k-1)1}
\end{array}\right)$, $\tilde{\eta}_{12} = \left(\begin{array}{c}
\eta_{12} \\
\vdots \\
\eta_{(k-1)2}
\end{array}\right)$, $\tilde{\eta}_{22} = \left(\begin{array}{c}
\eta_{k2} \\
\vdots \\
\eta_{(k-1)2}
\end{array}\right)$,

$\tilde{\eta}_{32} = \left(\begin{array}{c}
\eta_{l2} \\
\vdots \\
\eta_{(k-1)l}
\end{array}\right)$. Write:

$$\begin{pmatrix}
D_{\tilde{Q}_1} & \cdots & \gamma_{1\tilde{i}} \\
& \ddots & \vdots \\
0 & \cdots & D_{\tilde{Q}_i}
\end{pmatrix} = \begin{pmatrix}
D_{\tilde{Q}_1} & \tilde{\gamma}_{12} \\
0 & D_{\tilde{Q}_{2l}}
\end{pmatrix},$$

and

$$\begin{pmatrix}
D_{Q_1} & \cdots & \beta_{1I} \\
& \ddots & \vdots \\
0 & \cdots & D_{Q_i}
\end{pmatrix} = \begin{pmatrix}
\hat{D}_{Q_{1(k-1)}} & \hat{\beta}_{12} & \hat{\beta}_{13} \\
0 & D_{Q_k} & \hat{\beta}_{23} \\
0 & 0 & D_{Q_{(k+1)l}}
\end{pmatrix},$$

where $\gamma_{12} = (\gamma_{12} \cdots \gamma_{1\tilde{i}})$, $\hat{D}_{\tilde{Q}_{2l}} = \left(\begin{array}{c}
\gamma_{2\tilde{i}} \\
\vdots \\
0 \\
\vdots \\
\gamma_{2l}
\end{array}\right)$, $\hat{D}_{Q_{1(k-1)}} = \left(\begin{array}{c}
\beta_{1(k-1)} \\
\vdots \\
0 \end{array}\right)$,

$\hat{\beta}_{12} = \left(\begin{array}{c}
\beta_{1k} \\
\vdots \\
\beta_{(k-1)k}
\end{array}\right)$, $\hat{\beta}_{13} = \left(\begin{array}{c}
\beta_{1(k+1)} \\
\vdots \\
\beta_{(k-1)(k+1)} \\
\beta_{(k-1)l}
\end{array}\right)$, $\hat{\beta}_{23} = \left(\begin{array}{c}
\beta_{k(k+1)} \\
\vdots \\
\beta_{kl}
\end{array}\right)$.
\[ D_{Q_{(k+1)\ell}} = \begin{pmatrix} D_{Q_{k+1}} & \cdots & \beta_{(k+1)\ell} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{Q_\ell} \end{pmatrix} \]  

(3.13) tells us that 

\[ \eta_{k_1} \circ D_{Q_{1\ell}} = D_{Q_k} \circ \eta_{k_1}. \]

Since \((Q_1, D_{Q_1})\) and \((Q_k, D_{Q_k})\) are simple, \(\eta_{k_1} : (Q_1, D_{Q_1}) \to (Q_k, D_{Q_k})\) is an isomorphism. Denote

\[ A := \begin{pmatrix} \text{Id}_{Q_1} \oplus \cdots \oplus \text{Id}_{Q_{k-1}} & -\hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \\ 0 & \text{Id}_{Q_k} \\ 0 & 0 \end{pmatrix} \]

and

\[ B := \begin{pmatrix} \text{Id}_{Q_1} & -(\eta_{k_1})^{-1} \circ \hat{\eta}_{22} \\ 0 & \text{Id}_{Q_2} \oplus \cdots \oplus \text{Id}_{Q_\ell} \end{pmatrix}. \]

By direct calculation, we have:

\[ A \circ \eta \circ B = \begin{pmatrix} 0 & \hat{\eta}_{12} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\eta}_{22} \\ \eta_{k_1} & 0 \\ 0 & \hat{\eta}_{32} \end{pmatrix}. \]

(3.9) is equivalent to the following formula

\[ A \circ \eta \circ B \circ B^{-1} \begin{pmatrix} D_{Q_1} & \cdots & \gamma_{1f} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{Q_\ell} \end{pmatrix} = A \circ \begin{pmatrix} D_{Q_1} & \cdots & \beta_{1f} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & D_{Q_\ell} \end{pmatrix} A^{-1} \circ A \circ \eta \circ B, \]

and then

\[ \left( \begin{array}{c} 0 \\ \eta_{k_1} \circ D_{Q_1} \\ 0 \end{array} \right) \begin{pmatrix} (\hat{\eta}_{12} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\eta}_{22}) \circ \hat{D}_{Q_{2\ell}} \\ \hat{\eta}_{22} \circ \hat{D}_{Q_{2\ell}} - D_{Q_k} \circ \hat{\eta}_{22} + \eta_{k_1} \circ \hat{\gamma}_{12} \\ \hat{\eta}_{32} \circ \hat{D}_{Q_{2\ell}} \end{pmatrix} \]

\[ = \begin{pmatrix} \hat{D}_{Q_{(k-1)\ell}} \circ \hat{\eta}_{11} - \hat{\eta}_{11} \circ \hat{D}_{Q_{1\ell}} + \hat{\beta}_{12} \circ \eta_{k_1} \circ \hat{D}_{Q_{(k-1)\ell}} \circ (\hat{\eta}_{12} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\eta}_{22}) + \hat{\alpha} \circ \hat{\eta}_{32} \\ 0 \\ \hat{D}_{Q_k} \circ \eta_{k_1} \end{pmatrix} \begin{pmatrix} \hat{D}_{Q_{(k-1)\ell}} \circ \hat{\eta}_{11} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\eta}_{22} + \hat{\beta}_{12} \circ \eta_{k_1} \circ \hat{D}_{Q_{(k-1)\ell}} \circ (\hat{\eta}_{12} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\eta}_{22}) + \hat{\alpha} \circ \hat{\eta}_{32} \\ \hat{D}_{Q_{(k-1)\ell}} \circ \hat{\eta}_{11} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\eta}_{22} + \hat{\beta}_{12} \circ \eta_{k_1} \circ \hat{D}_{Q_{(k-1)\ell}} \circ (\hat{\eta}_{12} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\eta}_{22}) + \hat{\alpha} \circ \hat{\eta}_{32} \\ 0 \end{pmatrix}, \]

where \(\hat{\alpha} = \hat{\beta}_{13} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\beta}_{22}.\) Set

\[ \bar{\eta} = \left( \begin{array}{c} \hat{\eta}_{12} - \hat{\eta}_{11} \circ (\eta_{k_1})^{-1} \hat{\eta}_{22} \\ \hat{\eta}_{32} \end{array} \right). \]

From (3.16) and (3.18), it is not hard to see that \(\bar{\eta} : \hat{Q}_2 \oplus \cdots \oplus \hat{Q}_\ell \to Q_1 \oplus \cdots \oplus \hat{Q}_{k-1} \oplus Q_{k+1} \oplus \cdots \oplus \hat{Q}_\ell\) is a bundle isomorphism and satisfies

\[ \bar{\eta} \circ \hat{D}_{Q_{2\ell}} = \begin{pmatrix} \hat{D}_{Q_{(k-1)\ell}} & \hat{\alpha} \\ 0 & \hat{D}_{Q_{(k+1)\ell}} \end{pmatrix} \circ \bar{\eta}. \]

According to (3.13), (3.20) and induction, we can prove there must exist an isomorphism between \(\oplus_{i=1}^l (Q_\ell, D_{Q_\ell})\) and \(\oplus_{\alpha=1}^l (\hat{Q}_\alpha, D_{Q_\alpha}).\) □
Proposition 3.2. Let $(\hat{E}, \hat{D})$ be a flat complex vector bundle over a compact Riemannian manifold $(M, g)$, $K$ be a Hermitian metric on $\hat{E}$ and $i_0 : \hat{S} \to \hat{E}$ be a $\hat{D}$-invariant sub-bundle of $\hat{E}$. Suppose that there is a sequence of gauge transformation $\hat{\sigma}_l$ such that $\hat{D}_l := \hat{\sigma}_l(\hat{D}) \to \hat{D}_\infty$ weakly in $L^1_2$-topology as $l \to +\infty$. Furthermore, $\|\hat{D}^*_l K \hat{\psi}_{\hat{D}_l, K}\|_{L^\infty}$ and $\|\hat{\psi}_{\hat{D}_l, K}\|_{L^\infty}$ are uniformly bounded. Then there is a subsequence of $\eta_l := \hat{\sigma}_l \circ i_0$, up to rescaling, converges weakly to a nonzero map $\eta_\infty : \hat{S} \to E$ satisfying $\eta_\infty \circ D_\hat{S} = \hat{D}_\infty \circ \eta_\infty$ in $L^2_2$-topology, where $D_\hat{S} = \hat{D}|_\hat{S}$ is the induced flat connection on $\hat{S}$.

Proof. With respect to the Hermitian metric $K$, we have the following decomposition

$$\hat{D}_l = \hat{D}_{l,K} + \psi_{\hat{D}_l, K},$$

where $\hat{D}_{l,K}$ is a $K$-unitary connection, $\psi_{\hat{D}_l, K} \in \Omega^1(\text{End}(E))$ is $K$-self-adjoint. Under the condition $\hat{D}_l \to \hat{D}_\infty$ weakly in $L^1_2$-topology, we know that

$$\hat{D}_{l,K} \to \hat{D}_{\infty,K}, \quad \text{and} \quad \psi_{\hat{D}_l, K} \to \psi_{\hat{D}_\infty, K}$$

weakly in $L^p_1$-topology as $l \to \infty$. Choose local coordinates $\{x^i\}_{i=1}^n$ on $M$, and write $g = g_{ij}dx^i \otimes dx^j$. Note that $\hat{S}$ is a $\hat{D}$-invariant sub-bundle, and we have

$$\eta_l \circ D_\hat{S} = \hat{D}_l \circ \eta_l,$$

(3.23)

$$\Delta_{K, \eta_l} = g^{ij}(\overline{\nabla^K_{x^i}}(\hat{D}_{l,K} \circ \eta_l - \eta_l \circ D_{S,K}))((\partial / \partial x^j)) = \hat{D}^*_l K \psi_{\hat{D}_l, K} \circ \eta_l - \eta_l \circ D^*_S K \psi_{\hat{D}_l, K} - 2g^{ij}\psi_{\hat{D}_l, K}(\partial / \partial x^i) \circ \eta_l \circ \psi_{D, K}(\partial / \partial x^j) + g^{ij} \eta_l \circ \psi_{D, K}(\partial / \partial x^i) \circ \eta_l + g^{ij} \eta_l \circ \psi_{D, K}(\partial / \partial x^j) \circ \eta_l$$

and

(3.24)

$$\Delta |\eta_l|^2 \geq -\tilde{C}_0(\|\hat{D}^*_l K \psi_{\hat{D}_l, K}\|_{L^\infty} + \|D^*_S K \psi_{\hat{D}_l, K}\|_{L^\infty} + \|\psi_{\hat{D}_l, K}\|_{L^2}^2 + \|\psi_{D, K}\|_{L^2}^2) |\eta_l|^2,$$

where $\tilde{C}_0$ is a constant depending only on the dimension of $M$. Set $\tilde{\eta}_l = \eta_l / |\eta_l|_{L^2}$. From (3.25) and the Moser’s iteration, we see that there exists a uniform constant $\tilde{C}_1$ such that

(3.26)

$$\|\tilde{\eta}_l\|_{L^\infty} \leq \tilde{C}_1$$

for all $l$. By the above uniform $C^0$-estimate (3.26), the equation (3.24) and the assumption that $\hat{D}_l \to \hat{D}_\infty$ weakly in $L^p_2$-topology, the elliptic theory gives us that there exists a subsequence of $\eta_l$ which converges weakly in $L^2_2$-topology to a map $\eta_\infty$ such that $\eta_\infty \circ D_\hat{S} = \hat{D}_\infty \circ \eta_\infty$. On the other hand, the fact that $\|\tilde{\eta}_l\|_{L^2} = 1$ for all $l$ implies that the map $\eta_\infty$ is non-zero.

Theorem 3.1. Let $(E, D)$ be a rank $r$ flat complex vector bundle over a compact Riemannian manifold $(M, g)$, $K$ be a Hermitian metric on $E$. Suppose that there is a sequence of gauge transformation $\sigma_j$ such that $D_j := \sigma_j(D) \to D_\infty$ weakly in $L^1_2$-topology with $D^*_\infty K \psi_{D_\infty, K} = 0$. Furthermore, $\|D^*_j K \psi_{D_j, K}\|_{L^\infty}$ and $\|\psi_{D_j, K}\|_{L^\infty}$ are uniformly bounded. Then, we have:

(3.27)

$$(E, D_\infty) \cong Gr^{\text{JH}}(E, D),$$
where $Gr^JH(E,D)$ is the graded flat complex vector bundle associated to the Jordan-Hölder filtration of $(E,D)$.

Proof. We prove this by induction. Let’s assume that the conclusion of this theorem is true for $\text{rank}(E) < r$. If $(E, D)$ is simple, Proposition 3.2 implies that there exists an isomorphic map between $(E, D)$ and $(E, D_\infty)$. Suppose $(E, D)$ is not simple, and then we have the following Jordan-Hölder filtration of sub-bundles

\begin{equation}
0 = E_0 \subset E_1 \subset \cdots \subset E_i \cdots \subset E_l = E,
\end{equation}

such that every sub-bundle $E_i$ is $D$-invariant and every quotient bundle $(Q_i, D_i) := (E_i/E_{i-1}, D_i)$ is flat and simple. Let $S = E_1$ and $Q = E/E_1$. We consider the following exact sequence

\begin{equation}
0 \to S \overset{i_0}{\to} E \overset{\pi}{\to} Q \to 0.
\end{equation}

Denote that $D_S$ and $D_Q$ are the induced connections on $S$ and $Q$, $H_j = K_{\sigma_j} \circ \sigma_j$, $\pi_{\infty}^i$ is the orthogonal projection onto $E_1$ with respect to the metric $H_j$. Set $\pi_i = \sigma_j \circ \pi_i^\infty \circ \sigma_j^{-1}$. It is easy to see that

\begin{equation}
(\pi_i^\infty)^*K = \pi_i^2 = (\pi_i^2)^2
\end{equation}

and

\begin{equation}
(\text{Id}_E - \pi_i) \circ D_j \pi_i = 0.
\end{equation}

According to (2.31), (2.34), (2.38), (2.21) and the conditions of the theorem, we derive

\begin{align}
\int_M |D_j \pi_i^\infty|_{H_j}^2 dV_g &= \int_M |\pi_i \circ D_j \pi_i^\infty|_{H_j}^2 dV_g = \int_M |\pi_i^\infty \circ D_j \pi_i^\infty|_{H_j}^2 dV_g \\
&= 2 \int_M \langle D_S \pi_i^\infty \circ D_j \pi_i^\infty, H_j \rangle dV_g \\
&= -2 \int_M \langle \pi_i^\infty \circ D_j \pi_i^\infty, H_j \rangle dV_g \\
&\leq 2 \int_M \langle D_j \pi_i^\infty, H_j \rangle dV_g \\
&= 2 \int_M |D_j \pi_i^\infty, H_j \rangle dV_g \to 0.
\end{align}

On the other hand, $|\pi_i^\infty|_{H_j} = \text{rank}(S)$. After going to a subsequence, one can obtain $\pi_i^\infty \to \pi_i^{\infty}$ strongly in $L^p \cap L^2$, and

\begin{equation}
D_\infty \pi_i^\infty = 0.
\end{equation}

We know that $\pi_i^\infty$ determines a $D_\infty$-invariant sub-bundle $E_1^\infty$ of $(E, D_\infty)$ with $\text{rank}(E_1^\infty) = \text{rank}(E_1)$, and

\begin{equation}
(E, D_\infty) \cong (E_1^\infty, D_1, \infty) \oplus (Q_\infty, D_{Q_\infty}),
\end{equation}

where $Q_\infty = (E_1^\infty)^{\perp K}$, $D_1, \infty$ and $D_{Q_\infty}$ are the induced connections on $E_1^\infty$ and $Q_\infty$ by the connection $D_\infty$.

Proposition 3.2 yields that there is a subsequence of $\eta_j := \frac{\sigma_j \circ \eta_0}{\|\sigma_j \circ \eta_0\|_E^2}$, up to rescale, converges to a nonzero map $\eta_\infty : S \to E$ satisfying $\eta_\infty \circ D_S = D_\infty \circ \eta_\infty$.

Due to $\pi_i^\infty \circ \sigma_j \circ i_0 = \sigma_j \circ i_0$, we have:

\begin{equation}
\pi_i^\infty \circ \eta_\infty = \eta_\infty.
\end{equation}
The condition $D_{\infty,K}^{*}\psi_{D,\infty,K} = 0$ implies that $D_{\infty}$ is smooth, and then $\eta_{\infty}$ is also smooth. Because $E_{1}$ is simple, it is easy to see that $\eta_{\infty}$ is an isomorphic map between $(E_{1},D_{S})$ and $(E_{1}^{\infty},D_{1,\infty})$.

Let $\{e_{\alpha}\}$ be a local frame of $E_{1}$, and $H_{j,\alpha\beta} = (\eta_{j}(e_{\alpha}),\eta_{j}(e_{\beta}))_{K}$. We write

$$
\pi_{1}^{j}(Y) = (Y,\eta_{j}(e_{\beta}))_{K}H_{j}^{\alpha\beta} \eta_{j}(e_{\alpha})
$$

for any $Y \in \Gamma(E)$, where $(H_{j}^{\alpha\beta})$ is the inverse of the matrix $(H_{j,\alpha\beta})$. Since $\eta_{j} \to \eta_{\infty}$ weakly in $L_{2}^{p}$-topology, and $\eta_{\infty}$ is injective, we know that $\pi_{1}^{j} \to \pi_{1}^{\infty}$ weakly in $L_{2}^{p}$-topology. Here, $\pi_{1}^{\infty}: E \to E$ is just the projection onto $E_{1}^{\infty}$ with respect to the metric $K$.

Using Lemma 5.12 in [3], we can choose a sequence of $K$-unitary gauge transformations $u_{j}$ such that $\pi_{1}^{j} = u_{j} \circ \pi_{1}^{\infty} \circ u_{j}^{-1}$ and $u_{j} \to \text{Id}_{E}$ weakly in $L_{2}^{p}$-topology as $j \to \infty$. It is straightforward to check that $u_{j}(Q_{\infty}) = u_{j}((E_{1}^{\infty})^{1,K}) = (\pi_{1}^{j}(E))^{1,K}$, and the $K$-unitary gauge transformation $u_{0}$ satisfies $u_{0}((E_{1}^{\infty})^{1,K}) = E_{1}^{\infty}$. Set

$$
D_{j}^{Q} = (P^{*}K)^{-1} \circ u_{0} \circ (\pi_{1}^{\infty})^{1,K} \circ u_{j}^{-1} \circ D_{j} \circ u_{j} \circ (\pi_{1}^{\infty})^{1,K} \circ u_{0}^{-1} \circ P^{*}K,
$$

and

$$
\hat{\sigma}_{j} = (P^{*}K)^{-1} \circ u_{0} \circ (\pi_{1}^{\infty})^{1,K} \circ u_{j}^{-1} \circ \sigma_{j} \circ P^{*}K
$$

and

$$
\hat{\sigma}_{j}^{-1} = (P^{*}K)^{-1} \circ (\pi_{1}^{\infty})^{1,K} \circ \sigma_{j}^{-1} \circ u_{j} \circ u_{0} \circ P^{*}K.
$$

One can find that

$$
D_{j}^{Q} = \hat{\sigma}_{j} \circ D_{Q} \circ \hat{\sigma}_{j}^{-1}
$$

and

$$
D_{j}^{Q} \to D_{\infty}^{Q} = (P^{*}K)^{-1} \circ u_{0} \circ D_{Q_{\infty}} \circ u_{0}^{-1} \circ P^{*}K
$$

weakly in $L_{1}^{p}$-topology. From (2.36), (2.38) and (3.32), it follows that $\|\psi_{D_{j}^{Q},K}\|_{L^{\infty}}$ and $\|(D_{j}^{Q,\psi})^{*}\|_{L^{\infty}}$ are uniformly bounded, and $D_{Q_{\infty},K}^{*}\psi_{D_{Q_{\infty},K}} = 0$. According to the induction hypothesis, we have

$$
(Q_{\infty},D_{Q_{\infty}}) \cong (Q,D_{Q}) \cong \text{Gr}^{JH}(Q,D_{Q}).
$$

This completes the proof of the theorem.

**Proof of Theorem 1.1.** Thanks to Proposition 2.1 ([3]), we know that the harmonic flow $\{t\}$ has a long time solution $\sigma(t)$ for $t \in [0,\infty)$, and there exists a sequence $t_{j} \to \infty$ such that $\sigma(t_{j})\{D\}$ converges weakly, modulo $K$-unitary gauge transformations, to a flat connection $D_{\infty}$ in $L_{2}^{p}$-topology, and $D_{\infty,K}^{*}\psi_{\infty,K} = 0$. (2.53) and (2.48) imply that $\|\psi_{\sigma(t)}\{D\}\|_{L^{\infty}}$ and $\|\sigma(t)\{D\}\|_{K}^{*}\psi_{\sigma(t)}\{D\}\|_{L^{\infty}}$ are uniformly bounded. By Theorem 3.1, we deduce

$$
(E,D_{\infty}) \cong \text{Gr}^{JH}(E,D).
$$
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Xi Zhang, School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, P. R. China,
Email address: mathzx@ustc.edu.cn