The growth and zeros of Bargmann functions

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Abstract. The Bargmann formalism is presented with emphasis on the growth and zeros of Bargmann functions. Hadamard’s theorem is expressed in a physical language and it is used to construct quantum states, with given zeros in their Bargmann function. Various examples are discussed. The time evolution of Bargmann functions and the motion of their zeros, for various Hamiltonians, is studied.

1. Introduction
Various analytic representations have been used in quantum mechanics[1, 2]. The Bargmann function in the complex plane [3] has been used in conjunction with the coherent states formalism. There are deep theorems which relate the growth of entire functions and the density of their zeros[4]. In the context of Bargmann functions, they lead to results on the over-completeness or under-completeness of sequences of coherent states[5, 6, 7, 8]. Refs [9, 10, 11, 12] have studied the time evolution of quantum states for various Hamiltonians, through the paths of the zeros of their Bargmann functions (sometimes they are called ‘Husimi zeros’ or ‘stellar representation’).

In this paper we present the Bargmann formalism with emphasis on Hadamard’s theorem which constructs entire functions with a given set of zeros. We express Hadamard’s theorem for Bargmann functions in a physical language and construct quantum states whose Bargmann function has a a given set of zeros (with density smaller than a certain value)[8]. The time evolution of Bargmann functions and the motion of their zeros, for various Hamiltonians, is studied. During the time evolution some zeros might cease to exist or new zeros might appear (we call this ‘creation and annihilation of zeros’).

The aim of the paper is to present these ideas, which involve advanced topics from the theory of analytic functions, in a pedagogical and physical language. The general theory is complemented with several examples.

2. Bargmann functions
We consider a harmonic oscillator and let $a^\dagger, a$ be the creation and annihilation operators. $D(z)$ are displacement operators, $|z\rangle$ are coherent states and $|N\rangle$ are number eigenstates:

$$|z\rangle = D(z)|0\rangle = \exp \left(-\frac{1}{2}|z|^2\right) \sum_{N=0}^{\infty} z^N (N!)^{-1/2} |N\rangle; \quad D(z) = \exp(z a^\dagger - z^* a) \quad (1)$$
Let $|f\rangle$ be an arbitrary normalized ket state

$$
|f\rangle = \sum_{N=0}^{\infty} f_N |N\rangle; \quad \sum_{N=0}^{\infty} |f_N|^2 = 1
$$

We use the notation

$$
\langle f | = \sum_{N=0}^{\infty} f_N^* \langle N|; \quad |f^*\rangle = \sum_{N=0}^{\infty} f_N^* |N\rangle; \quad \langle f^*| = \sum_{N=0}^{\infty} f_N \langle N|
$$

In the Bargmann representation the state $|f\rangle$ is represented by the entire function

$$
f(z) = \exp\left(\frac{1}{2} |z|^2\right) \langle f^*| z \rangle = \exp\left(\frac{1}{2} |z|^2\right) \langle z^*| f \rangle = \sum_{N=0}^{\infty} f_N z^N \langle N| \rangle^{-1/2}
$$

The scalar product is given by

$$
\langle f|g\rangle = \int_{C} [f(z)]^* g(z) \exp(-|z|^2) d\mu(z) = \sum_{N} f_N^* g_N; \quad d\mu(z) = \frac{dz R dz I}{\pi}
$$

The indices $R, I$ in a complex number, indicate its real and imaginary parts, correspondingly. As examples, we give the Bargmann function for number eigenstates and for coherent states:

$$
|N\rangle \rightarrow f(z) = \frac{z^N}{N!}; \quad |\zeta\rangle \rightarrow f(z) = \exp\left[\zeta z - \frac{|\zeta|^2}{2}\right]
$$

We also consider the squeezed state:

$$
|w; r, \theta, \lambda\rangle_{sq} = S(r, \theta, \lambda) |w\rangle
$$

$$
S(r, \theta, \lambda) = \exp\left[-\frac{1}{4} re^{-i\theta} (a^\dagger)^2 + \frac{1}{4} re^{i\theta} a^2\right] \exp[i\lambda a^\dagger a]
$$

Its Bargmann function is [8, 2]

$$
f(z) = (1 - |\alpha|^2)^{1/4} \exp\left(\frac{\alpha}{2} z^2 + \beta z + \epsilon\right); \quad \alpha = -\tanh\left(\frac{1}{2} r\right) e^{-i\theta}
$$

$$
\beta = re^{i\lambda}(1 - |\alpha|^2)^{1/2}; \quad \epsilon = -\frac{\alpha^*}{2} w^2 e^{2i\lambda} - \frac{1}{2} |w|^2
$$

An operator $\Theta$ is represented with the following function

$$
\Theta \rightarrow \Theta(z, \zeta^*) = \exp\left[\frac{1}{2}(|\zeta|^2 + |z|^2)\right] \langle z^*| \Theta |\zeta^*\rangle = \sum_{N,M} z^M \Theta_{MN} \zeta_N^*
$$

where $\Theta_{MN} = \langle M|\Theta|N\rangle$. It acts on the Bargmann function $f(z)$ of the state $|f\rangle$ as follows:

$$
\Theta |f\rangle \rightarrow \int_{C} d\mu(z) \exp(-|\zeta|^2) \Theta(z, \zeta^*) f(z)
$$

As examples, we consider the creation and annihilation operators for which

$$
a^\dagger \rightarrow z \exp(z\zeta^*); \quad a \rightarrow \zeta^* \exp(z\zeta^*)
$$

This can be proven using Eq.(9). In addition to this ‘integral representation’, the creation and annihilation operators can also be represented with the differential operators

$$
a^\dagger \rightarrow z; \quad a \rightarrow \partial_z
$$

This can be proved by acting with these operators on the Bargmann function $z^N/(N!)$ of the number eigenstate $|N\rangle$ and then confirm that $a^\dagger |N\rangle = (N+1)^{1/2} |N+1\rangle$ and $a |N\rangle = N^{1/2} |N-1\rangle$. 

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3. Growth and density of zeros of Bargmann functions

The growth of an analytic function \( f(z) \) is described with two non-negative numbers, the order \( \rho \) and the type \( \sigma \). We refer to the literature [4] for the formal definitions. For our purposes let

\[
M(R) = \max_{|z| = R} (|f(z)|)
\]

Then \( M \approx \exp(\sigma R^\rho) \) as \( R \to \infty \).

A growth \((\rho, \sigma)\) is smaller than \((\rho', \sigma')\) if \( \rho < \rho' \) (in which case we do not need to compare the types) or if both \( \rho = \rho' \) and \( \sigma < \sigma' \). Below we use the notation \( \mathcal{H}(\rho, \sigma) \) for the space of functions with growth smaller than \((\rho, \sigma)\).

Convergence of the scalar product of Eq.(5), implies that Bargmann functions should have growth smaller than \((\rho = 2, \sigma = 1/2)\) (i.e., the Bargmann space is \( \mathcal{H}(2, 1/2) \)).

The zeros of analytic functions are isolated. We consider a general analytic function \( f(z) \) with an infinite number of zeros \( \zeta_N \) (i.e., \( f(\zeta_N) = 0 \)), which we label them as:

\[
0 < |\zeta_1| \leq |\zeta_2| \leq |\zeta_3| \leq \ldots;
\]

Let \( n(R) \) be the number of terms of this sequence within the circle \( |z| = R \). The density of the sequence \( \{\zeta_N\} \) is described with two non-negative numbers, the convergence exponent \( \eta \) and \( \delta \). We refer to the literature [4] for the formal definitions, but for our purposes it is sufficient to say that \( n(R) \approx \delta R^\eta \) as \( R \to \infty \). It is known that \( \eta \) is the infimum of positive numbers \( \kappa \) such that

\[
\sum_{N=1}^{\infty} \frac{1}{|\zeta_N|^\kappa} < \infty
\]

We say that the density \((\eta, \delta)\) is smaller than the density \((\eta', \delta')\) if \( \eta < \eta' \) (in which case we do not compare \( \delta, \delta' \)) or if both \( \eta = \eta' \) and \( \delta < \delta' \).

For general analytic functions it is known that [4] \( \eta < \rho \) or both \( \eta = \rho \) and \( \delta \leq \sigma \rho \). Bargmann functions have smaller growth than \((\rho = 2, \sigma = 1/2)\) and this implies that \( \eta < 2 \) or both \( \eta = 2 \) and \( \delta \leq 1 \). Therefore the density of zeros of Bargmann functions is smaller that \((\eta = 2, \delta = 1)\). The zeros of Bargmann functions have an interesting physical meaning. Eq.(4) shows that if \( \zeta \) is a zero of \( f(z) \), then the coherent state \( |\zeta\rangle \) is orthogonal to \( |f^*\rangle \).

4. Hadamard’s theorem for Bargmann functions

The Weierstrass factors are defined as

\[
E(z, 0) = 1 - z; \quad E(z, p) = (1 - z) \exp \left[ z + \frac{z^2}{2} + \ldots + \frac{z^p}{p} \right]
\]

where \( p \) is a positive integer. The exponential terms in the Weierstrass factors in Eq.(16) are important for the convergence of the infinite products below.

The Weierstrass canonical product corresponding to the sequence \( \{\zeta_N\} \) is

\[
\Pi_p(z) = z^m \prod_{N=1}^{\infty} E \left( \frac{z}{\zeta_N}, p \right).
\]

It converges absolutely and it has growth with order \( \rho \) equal to the convergence exponent \( \eta \) of the set of zeros \( \{\zeta_N\} \).

Let \( f(z) \) be an entire function of order \( \rho \) with zeros \( \{\zeta_N\} \). Hadamard’s theorem states that an entire function of order \( \rho \) can be factorised as

\[
f(z) = z^m \prod_{N=1}^{\infty} E \left( \frac{z}{\zeta_N}, p \right) \exp[Q_\eta(z)]
\]
Here \( p \leq \rho \), \( Q_q(z) \) is a polynomial of degree \( q \leq \rho \) and \( m \) is the multiplicity of the zero at the origin. For Bargmann functions \( \rho \leq 2 \) and therefore \( p, q = 0, 1, 2 \).

We discuss the physical meaning of Eq.(18), starting with the term \( \exp[Q_q(z)] \). Using the notation \( |Q_q\rangle \) for the state represented by the Bargmann function \( \exp[Q_q(z)] \), we get (up to a normalisation constant)

\[
\exp[Q_0(z)] = \exp(u) \rightarrow |Q_0\rangle = \langle 0 | \\
\exp[Q_1(z)] = \exp(Az + u) \rightarrow |Q_1\rangle = |A\rangle_{coh} \\
\exp[Q_2(z)] = \exp\left( \frac{\alpha}{2} z^2 + \beta z + \epsilon \right) \rightarrow |Q_2\rangle = |A; r, \theta, \lambda\rangle_{sq}
\]

where the relation between \( \alpha, \beta, \epsilon \) and \( A, r, \theta, \lambda \) is given in Eq.(8). Therefore, the \( \exp[Q_q(z)] \) with \( q = 0, 1, 2 \) is the Bargmann function for the vacuum state, a coherent state and a squeezed state, correspondingly.

We next discuss the physical meaning of the Weierstrass terms. Using Eq.(12) we express the Weierstrass factors in terms of creation operators as

\[
E\left( \frac{a^\dagger}{\xi}, p \right) = \left( 1 - \frac{a^\dagger}{\xi} \right) \exp \left[ \frac{a^\dagger}{\xi} z + \frac{(a^\dagger)^2}{2\xi^2} z^2 + \cdots + \frac{(a^\dagger)^p}{p!\xi^p} z^p \right]; \quad p = 0, 1, 2
\]

and the state corresponding to Eq.(18) as

\[
|f\rangle = \mathcal{N}(a^\dagger)^m \prod_{N=1}^{\infty} E\left( \frac{a^\dagger}{\xi_N}, p \right) |Q_q\rangle; \quad p, q = 0, 1, 2
\]

where \( \mathcal{N} \) is a normalization constant. Given any sequence of complex numbers \( \{\xi_N\} \) with density smaller than \( (\eta = 2, \delta = 1) \) we construct with Eq.(21) a state whose Bargmann function has these zeros. Below we discuss several examples.

4.1. Example: the Mittag-Leffler function

An example of a state whose Bargmann function has growth with a given order \( \rho \) and given type \( \sigma \) (which may or may not be integers) is [8, 2]

\[
|\rho, \sigma\rangle = \sum_{N=0}^{\infty} f_N |N\rangle; \quad f_N = \mathcal{N} \frac{\epsilon^{iN\theta} \sigma^{N/\rho} (N!)^{1/2}}{\Gamma\left( \frac{N}{\rho} + 1 \right)}
\]

where \( \mathcal{N} \) is the normalisation constant

\[
\mathcal{N} = \left[ \sum_{N=0}^{\infty} \frac{\sigma^{2N/\rho} N!}{\Gamma\left( \frac{N}{\rho} + 1 \right)^2} \right]^{-1/2}
\]

\( \mathcal{N} \) is finite when \( 0 \leq \rho < 2 \); and also when \( \rho = 2 \) and \( \sigma < \frac{1}{2} \). The Bargmann function of this state is \( \mathcal{N} E_{1/\rho}(\epsilon^{i\theta} \sigma^{1/\rho} z) \) where \( E_{\nu}(\zeta) \) is the Mittag-Leffler function (e.g.,[13]).

4.2. Example: the Riemann \( \xi(Az) \) function

As an example we consider the Riemann \( \xi(z) \) function [14, 15]

\[
\xi(z) = \frac{1}{2} z(z-1) \pi^{-z/2} \Gamma\left( \frac{z}{2} \right) \zeta(z).
\]
where $\zeta(z)$ is the Riemann zeta function

$$
\zeta(z) = \sum_{N=1}^{\infty} \frac{1}{N^z}, \tag{25}
$$

The Riemann zeta function has been used in quantum mechanics in [16, 17].

$\xi(z)$ is an analytic function in the complex $z$-plane and satisfies the functional relation

$$
\xi(z) = \xi(1-z). \tag{26}
$$

Therefore if $w_n$ is a zero of $\xi(z)$ then $1-w_n$ is also a zero. It can be proved that all zeros are in the strip $0 \leq \Re(w_n) \leq 1$. The Riemann hypothesis is that all zeros are in the line $\Re(w_n) = 1/2$.

The function $\xi(z)$ can be written in the product form[14, 15]

$$
\xi(z) = \xi(0) \prod_{w_n} \left(1 - \frac{z}{w_n} \right) = \xi(0) \prod_{\Im(w_n) > 0} \left[1 - \frac{z(1-z)}{w_n(1-w_n)} \right] \tag{27}
$$

The order of its growth is $\rho = 1$ and therefore $\xi(Az)$ qualifies as a Bargmann function. The corresponding ket state is

$$
|A\text{Rie}\rangle = N\xi(0) \prod_{w_n} \left(1 - \frac{a^\dagger}{A^{-1}w_n} \right) |0\rangle = N\xi(0) \prod_{\Im(w_n) > 0} \left[1 - \frac{Aa^\dagger - A^2(a^\dagger)^2}{w_n(1-w_n)} \right] |0\rangle \tag{28}
$$

where $N$ is a normalization constant.

4.3. Example: the function $\sin(\pi Az)$ (Schrödinger cat)

The entire function $\sin(\pi Az)$, where $A$ is a complex number ($A \neq 0$), is factorized as [4, 18]

$$
\sin(\pi Az) = \pi Az \frac{\Gamma(Az)}{\Gamma(-Az)} = \pi \frac{\Gamma(Az)}{\Gamma(1-Az)} \tag{29}
$$

In this example the zeros are $w_{2N} = -NA^{-1}$, $w_{2N+1} = NA^{-1}$ (where $N$ is a natural number). The origin is also a zero (i.e., $m = 1$). Also $Q_q(z) = 0$. Its growth has order $\rho = 1$ and $\sin(\pi Az)$ qualifies as a Bargmann function. The corresponding state is the following superposition of two coherent states (Schrödinger cat):

$$
|A\rangle_{\sin} = N\pi Az \prod_{N=1}^{\infty} \left(1 - \frac{(a^\dagger)^2}{N^2 A^{-2}} \right) |0\rangle
\quad = [2 - 2 \exp(-2|\pi A|^2)]^{-1/2}[|i\pi A\rangle - |-i\pi A\rangle] \tag{30}
$$

The index ‘sin’ in the notation indicates that the corresponding Bargmann function is sinusoidal. In this example, the zeros can be related to interference between the coherent states $|i\pi A\rangle$ and $|-i\pi A\rangle$.

4.4. Example: the function $\frac{1}{\Gamma(Az)}$

Another example which is related to the previous one but has only half of its zeros, is the $\frac{1}{\Gamma(Az)}$, where $\Gamma(z)$ is the Gamma function and $A$ is a complex number ($A \neq 0$). The relationship is seen in the following formula:

$$
\sin(\pi Az) = -\frac{\pi}{Az \Gamma(Az) \Gamma(-Az)} = \frac{\pi}{\Gamma(Az) \Gamma(1-Az)} \tag{31}
$$
The entire function \( \frac{1}{\Gamma(Az)} \) is factorized as follows:

\[
\frac{1}{\Gamma(Az)} = Az \prod_{N=1}^{\infty} \left( 1 + \frac{z}{NA-1} \right) \exp \left( -\frac{z}{NA-1} \right) \exp(\gamma Az)
\]

where \( \gamma = 0.577 \) is the Euler constant\[4, 18\]. The growth of \( 1/\Gamma(Az) \) has order \( \rho = 1 \) and therefore it qualifies as Bargmann function. The corresponding ket state, for which we use the notation \( |A\rangle_{\text{Gam}} \), is

\[
|A\rangle_{\text{Gam}} = \mathcal{N} \exp \left( \frac{\gamma^2|A|^2}{2} \right) (Aa^\dagger) \prod_{N=1}^{\infty} \left( 1 + \frac{a^\dagger}{NA-1} \right) \exp \left( -\frac{a^\dagger}{NA-1} \right) |\gamma A\rangle_{\text{coh}}
\]

where \( \mathcal{N} \) is a normalization factor.

4.5. Example: the Weierstrass sigma function
We consider the Weierstrass sigma function,

\[
\sigma(z|2^{-1}A, i2^{-1}A) = z \prod_{M,N} \left[ 1 - \frac{z}{w_{MN}} \right] \exp \left( \frac{z}{w_{MN}} + \frac{z^2}{2w_{MN}^2} \right); \quad w_{MN} = A(M + iN)
\]

where \( A \) is a positive number, \( M, N \) are integers, and the product includes all \((M, N) \neq (0, 0)[4, 18]\). The density of the zeros \( w_{MN} \) has \( \eta = 2 \) and \( \delta = \pi/A^2 \). Therefore, it can be used as a Bargmann function only if \( A > \pi^{1/2} \).

The Weierstrass sigma function has growth with order \( \rho = 2 \) and type \( \sigma = \pi/(2A^2) \). For \( A > \pi^{1/2} \) we have \( \sigma < 1/2 \) and the Weierstrass sigma function qualifies as Bargmann function. The corresponding state, for which we use the notation \( |A\rangle_{\text{Wei}} \), is

\[
|A\rangle_{\text{Wei}} = \mathcal{N}(a^\dagger) \prod_{M,N} \left( 1 - \frac{a^\dagger}{w_{MN}} \right) \exp \left( \frac{a^\dagger}{w_{MN}} + \frac{(a^\dagger)^2}{2w_{MN}^2} \right) |0\rangle; \quad A > \pi^{1/2}
\]

where \( \mathcal{N} \) is a normalization constant. We note that ref.[19] used the Weierstrass sigma function in the context of the completeness of coherent states.

5. Time evolution: motion of the zeros
In this section we consider a system with a Hamiltonian \( H \), which at time \( t = 0 \) is in some initial state. We study the time evolution of this state, and the motion of the zeros of its Bargmann function.

We assume that at time \( t = 0 \) the system is in the state \( |f\rangle \) and has Bargmann function \( f(z) \) with zeros \( \{w_N\} \). At time \( t \) the system is in the state \( |f(t)\rangle = \exp(itH)|f\rangle \) and has Bargmann function \( f(z,t) \) with zeros \( \{w_N(t)\} \).

The number of zeros corresponding to \( |f(t)\rangle \) may change as a function of time. This is obvious from the fact that any two states can be related with a unitary transformation. Below we give examples which show this explicitly.

5.1. Systems with the Hamiltonian \( H_1 = \kappa a^\dagger + \kappa^*a \)
In the case of systems with Hamiltonian \( H_1 = \kappa a^\dagger + \kappa^*a \) the evolution operator is \( \exp(itH_1) = D(it\kappa) \). It is easily seen that the Bargmann function \( f(z,t) \) of the state \( |f(t)\rangle \) is given by

\[
f(z; t) = f(z + it\kappa^*) \exp \left( -\frac{1}{2}|t\kappa|^2 + it\kappa z \right)
\]
From this it follows that the motion of the zeros of the Bargmann function, is given by
\[ w_N(t) = w_N - it\kappa^* \] (37)
In this case the relative position of the zeros does not change with time.

5.2. Systems with the Hamiltonian \( H_2 = \Omega a^\dagger a \)
In the case of systems with Hamiltonian \( H_2 = \Omega a^\dagger a \) we prove that the Bargmann function \( f(z; t) \) of the state \( |f(t)\rangle \) is
\[ f(z; t) = f[z\exp(i\Omega t)] \] (38)
From this it follows that the motion of the zeros of the Bargmann function, is given by
\[ w_N(t) = w_N \exp(-i\Omega t) \] (39)
In this case also the relative position of the zeros does not change with time.

5.3. Creation and annihilation of zeros
We consider a system described by the evolution operator [20]
\[ U(t) = \exp[i\gamma_N(t)]|N\rangle\langle N|; \quad U(t)[U(t)]^\dagger = 1 \] (40)
where \( \gamma_N(t) \) are real functions of time (such that \( \gamma_N(0) = 0 \)). At \( t = 0 \) the system is in the coherent state \( |A\rangle_{\text{coh}} \) and the corresponding Bargmann function has no zeros. At time \( t \) the system evolves into the state
\[ |f\rangle = \exp\left( -\frac{|A|^2}{2} \right) \sum_{N=0}^{\infty} \frac{A^N \exp[i\gamma_N(t)]}{(N!)^{1/2}} |N\rangle \] (41)
and its Bargmann function is
\[ f(z) = \exp\left( -\frac{|A|^2}{2} \right) \sum_{N=0}^{\infty} \frac{(Az)^N \exp[i\gamma_N(t)]}{N!} \] (42)
This function can have zeros. For example, we assume that at some time \( t_0 \)
\[ \gamma_{2N}(t_0) = \pi N; \quad \gamma_{2N+1}(t_0) = \pi \] (43)
Then
\[ f(z) = \exp\left( -\frac{|A|^2}{2} \right) [\cos(Az) - \sinh(Az)] \] (44)
It is easily seen that this function has at least one zero. Therefore the number of zeros may change as a function of time.

6. Discussion
Given any sequence of complex numbers with density less than \( (\eta = 2, \delta = 1) \), we have used Hadamard’s theorem to construct in Eq.(21), a state whose Bargmann function has these zeros. Examples which use as Bargmann functions the Riemann \( \xi(Az) \) function, the \( \sin(\pi Az) \), and the \( \frac{1}{\Gamma(Az)} \) were discussed.

The time evolution of Bargmann functions, has been studied. It has been shown that the Hamiltonians \( H_1, H_2 \) preserve the zeros in all Bargmann functions, and they also preserve the relative position of these zeros. We also gave an example where creation and annihilation of zeros occurs.

The work is a contribution to the study of quantum systems through the zeros of their Bargmann functions.
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