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COMPUTATION AND BOUNDING OF FOLKMAN NUMBERS

THESIS

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Introduction

Only simple graphs are considered in this thesis, i.e. finite, non-oriented graphs without loops and multiple edges. The vertex set and the edge set of a graph \( G \) are denoted by \( V(G) \) and \( E(G) \) respectively. The complete graph on \( n \) vertices is denoted by \( K_n \).

One fundamental problem in graph theory is the following: Let \( \mathcal{H} \) be a class of graphs. What is the minimum number of vertices of the graphs in \( \mathcal{H} \)?

\[
\min \{|V(G)| : G \in \mathcal{H}\} = ?
\]

For many important classes \( \mathcal{H} \) this problem is still unsolved. In this thesis we consider such problems. In some cases we will compute \( \min \{|V(G)|\} \) exactly, and in other cases we will obtain new bounds on \( \min \{|V(G)|\} \).

It is well known that in every coloring of the edges of the graph \( K_6 \) in two colors there is a monochromatic triangle. We will denote this property by \( K_6 \rightarrow (3, 3) \). It is clear that if \( G \) contains \( K_6 \) as a subgraph, then \( G \rightarrow (3, 3) \). In 1967 Erdős and Hajnal posed the following problem:

Does there exist a graph \( G \rightarrow (3, 3) \) which does not contain \( K_6 \) ?

Denote:
\[
\mathcal{H}_e(3, 3; q) = \left\{ G : G \rightarrow (3, 3) \text{ and } G \not\supseteq K_q \right\}.
\]

The edge Folkman number \( F_e(3, 3; q) \) is defined with:
\[
F_e(3, 3; q) = \min \{|V(G)| : G \in \mathcal{H}_e(3, 3; q)\}
\]

From \( K_6 \rightarrow (3, 3) \) and \( K_5 \not\rightarrow (3, 3) \) it follows that \( F_e(3, 3; q) = 6, \ q \geq 7 \).

The first example of a graph \( G \), such that \( G \not\supseteq K_6 \) and \( G \rightarrow (3, 3) \), was given by van Lint. Later, Graham showed that \( K_3 + C_5 \rightarrow (3, 3) \) and proved \( F_e(3, 3; 6) = 8 \).
The computation of the numbers $F_e(3, 3; 5)$ and $F_e(3, 3; 4)$ is very hard. The number $F_e(3, 3; 5)$ was finally computed in 1998 after 30 years of history. The upper bound $F_e(3, 3; 5) \leq 15$ was obtained by Nenov in [59], who constructed the first 15-vertex graph in $H_e(3, 3; 5)$. The lower bound $F_e(3, 3; 5) \geq 15$ was obtained much later with the help of a computer by Piwakowski, Radziszowski and Urbanski in [78]. Without a computer, it is impossible to prove $F_e(3, 3; 5) \geq 15$. A more detailed view on the results related to the number $F_e(3, 3; 5)$ is given in the paper [78] and the book [93].

The number $F_e(3, 3; 4)$ is not computed. It is sometimes referred to as the most wanted Folkman number. In 1970 Folkman [22] proved that $H_e(3, 3; 4) \neq \emptyset$. The graph obtained by the construction of Folkman has a very large number of vertices. Because of this, in 1975 Erdős [19] posed the problem to prove the inequality $F_e(3, 3; 4) < 10^{10}$. In 1986 Frankl and Rödl [24] almost solved this problem by showing that $F_e(3, 3; 4) < 7.02 \times 10^{11}$. In 1988 Spencer [94] proved the inequality $F_e(3, 3; 4) < 3 \times 10^9$ by using probabilistic methods. In 2008 Lu [48] constructed a 9697-vertex graph in $H_e(3, 3; 4)$, thus considerably improving the upper bound on $F_e(3, 3; 4)$. Soon after that, Lu’s result was improved by Dudek and Rödl [18], who proved $F_e(3, 3; 4) \leq 941$. The best known upper bound on this number is $F_e(3, 3; 4) \leq 786$, obtained in 2012 by Lange, Radziszowski and Xu [43].

In 1972 Lin [47] proved that $F_e(3, 3; 4) \geq 11$. The lower bound was improved by Nenov [61], who showed in 1981 that $F_e(3, 3; 4) \geq 13$. In 1984 Nenov [62] proved that every 5-chromatic $K_4$-free graph has at least 11 vertices, from which it is easy to derive that $F_e(3, 3; 4) \geq 14$. From $F_e(3, 3; 5) = 15$ [59] [78] it follows easily, that $F_e(3, 3; 4) \geq 16$. The best lower bound known on $F_e(3, 3; 4)$ was obtained in 2007 by Radziszowski and Xu [81], who proved with the help of a computer that $F_e(3, 3; 4) \geq 19$. According to Radziszowski and Xu [81], any method to improve the bound $F_e(3, 3; 4) \geq 19$ would likely be of significant interest.

A summary of the history of $F_e(3, 3; 4)$ is given in Table [1]
In the last Chapter 9 of this thesis we improve the lower bound on $F_e(3, 3; 4)$ by proving $F_e(3, 3; 4) \geq 20$. This is one of the main results in the thesis.

The computation of $F_e(a_1, \ldots, a_s; q)$ is a special case of the following more general problem:

*For given positive integers $a_1, \ldots, a_s$, $q$, $a_i \geq 2$, $i = 1, \ldots, s$, determine the minimum number of vertices of the graphs which do not contain the complete graph on $q$ vertices $K_q$ and have the following property: in every coloring of the edges in $s$ colors there exist $i \in \{1, \ldots, s\}$ such that there is a monochromatic $a_i$-clique of color $i$. This minimum is denoted by $F_e(a_1, \ldots, a_s; q)$ and is called edge Folkman number. It is known that

\begin{equation}
F_e(a_1, \ldots, a_s; q) \text{ exists } \iff q > \max \{a_1, \ldots, a_s\}.
\end{equation}

In the case $s = 2$, (1) is proved by Folkman in [22], and in the general case (1) is proved by Nešetřil and Rödl in [76].

The numbers $F_e(a_1, \ldots, a_s; q)$ are a generalization of the classic Ramsey numbers $R(a_1, \ldots, a_s)$. Furthermore, $F_e(a_1, \ldots, a_s; q) = R(a_1, \ldots, a_s)$, if $q > R(a_1, \ldots, a_s)$.\]
The vertex Folkman numbers $F_v(a_1, \ldots, a_s; q)$ are defined in the same way as the edge Folkman numbers $F_e(a_1, \ldots, a_s; q)$, but instead of coloring the edges, the vertices of the graphs are colored. Very often results for vertex Folkman numbers $F_v(a_1, \ldots, a_s; q)$ are used in the computation and bounding of the edge Folkman numbers. Let us also note, that the numbers of the form $F_v(2, \ldots, 2; q)$ determine the minimum number of vertices of the graphs with given chromatic number and given clique number (see [62], [37] and [26]).

This thesis consists of two parts. The first part is dedicated to the vertex Folkman numbers, and the second - to the edge Folkman number $F_e(3, 3; 4)$, which we discussed at the beginning of the introduction. The relation between the two parts is not obvious. The complicated computations for the proof of the inequality $F_e(3, 3; 4) \geq 20$ are done with the help of Algorithm A8. However, this algorithm is a modification of the algorithms from the first part. Therefore, to better understand the connection between the two parts, the algorithms A1, ..., A8 must be carefully followed.

The results in this thesis are obtained using both theoretical and computer methods. Some of the main theoretical results that we obtain are the results in Theorem 2.12, Theorem 3.15, and Theorem 3.19. According to these theorems, the computation of some infinite sequences of Folkman numbers is reduced to computing only the first several members. Other main theoretical result is the introduction of the modified vertex Folkman numbers. With the help of these numbers we obtain upper bounds on the vertex Folkman numbers (Theorem 1.19).

We develop eight new computer algorithms for computing and bounding Folkman numbers, which we denote by A1, ..., A8. These algorithms are optimized for high performance in terms of computational time. Using the new algorithms, we obtain results which are considered beyond the reach of existing algorithms, even if very powerful computer hardware is used. For example, the computation of the lower bound $F_e(3, 3; 4) \geq 19$ in [81] was completed in a few hours, but it is practically impossible to use the same method to further improve this bound. In comparison, using Algorithm A8 on a similarly capable computer, we obtained the result $F_e(3, 3; 4) \geq 19$ in less than a second, and we needed just several hours of computational
time to prove the new bound $F_e(3, 3; 4) \geq 20$. At first glance, some of the presented algorithms seem similar, but they have important distinctions. A particular algorithm can produce good results in the problems where it is used, and not be effective in others. Therefore, we had to develop specific algorithms for the different problems considered in this thesis.

A more precise description of the main results in each chapter follows:

**Chapter 1**

In this chapter, the necessary graph theory definitions and definitions related to vertex Folkman numbers are given.

Let $a_1, \ldots, a_s$ be positive integers. The expression $G \xrightarrow{v} (a_1, \ldots, a_s)$ means that in every coloring of $V(G)$ in $s$ colors ($s$-coloring) there exists $i \in \{1, \ldots, s\}$ such that there is a monochromatic $a_i$-clique of color $i$.

Define:

$$\mathcal{H}_v(a_1, \ldots, a_s; q) = \left\{ G : G \xrightarrow{v} (a_1, \ldots, a_s) \text{ and } G \not\supseteq K_q \right\}.$$  

The vertex Folkman numbers $F_v(a_1, \ldots, a_s; q)$ are defined by the equality:

$$F_v(a_1, \ldots, a_s; q) = \min \{|V(G)| : G \in \mathcal{H}_v(a_1, \ldots, a_s; q)\}.$$  

Folkman proved in [22] that

$$F_v(a_1, \ldots, a_s; q) \text{ exists } \iff q > \max \{a_1, \ldots, a_s\}.$$  

For arbitrary positive integers $a_1, \ldots, a_s$ the following terms are defined

$$m(a_1, \ldots, a_s) = m = \sum_{i=1}^{s} (a_i - 1) + 1 \quad \text{and} \quad p = \max \{a_1, \ldots, a_s\}.$$  

It is easy to see that $K_m \xrightarrow{v} (a_1, \ldots, a_s)$ and $K_{m-1} \not\xrightarrow{v} (a_1, \ldots, a_s)$. Therefore

$$F_v(a_1, \ldots, a_s; q) = m, \quad q \geq m + 1.$$  

In [50] it was proved that

\[ F_v(a_1, \ldots, a_s; m) = m + p. \]

Not much is known about the vertex Folkman numbers \( F_v(a_1, \ldots, a_s; m - 1) \) when \( q < m \). Thanks to the work of different authors, the exact values of all numbers of the form \( F_v(a_1, \ldots, a_s; m - 1) \) where \( \max \{ a_1, \ldots, a_s \} \leq 4 \) were obtained. The only other known number of this form is \( F_v(3, 5; 6) = 16 \), [89].

The expression \( G \rightarrow m \mid p \) means that for every choice of positive integers \( a_1, \ldots, a_s \) (\( s \) is not fixed), such that \( m = \sum_{i=1}^{s} (a_i-1)+1 \) and \( \max \{ a_1, \ldots, a_s \} \leq p \), we have \( G \rightarrow (a_1, \ldots, a_s) \).

Define:

\[ \tilde{H}_v(m \mid p; q) = \left\{ G : G \rightarrow m \mid p \text{ and } G \not\supseteq K_q \right\}. \]

The modified vertex Folkman numbers are defined by the equality:

\[ \tilde{F}_v(m \mid p; q) = \min \left\{ |V(G)| : G \in \tilde{H}_v(m \mid p; q) \right\}. \]

At the end of the chapter, we prove the following main result. For convenience, instead of \( F_v(2, \ldots, 2; m-p; p; q) \) we write \( F_v(2,m-p,p; q) \).

**Theorem 1.19.** Let \( a_1, \ldots, a_s \) be positive integers, let \( m \) and \( p \) be defined by (3), and \( q > p \). Then,

\[ F_v(2,m-p,p; q) \leq F_v(a_1, \ldots, a_s; q) \leq \tilde{F}_v(m \mid p; q). \]

Further, we will compute and bound the numbers \( F_v(a_1, \ldots, a_s; q) \) by computing and obtaining bounds on the border numbers in Theorem 1.19, \( F_v(2,m-p,p; q) \) and \( \tilde{F}_v(m \mid p; q) \).
Chapter 2

The vertex Folkman numbers of the form $F_v(a_1, \ldots, a_s; m - 1)$ where \( \max \{a_1, \ldots, a_s\} = 5 \) are considered. By \([2]\), these numbers exist when \( m \geq 7 \). In the border case \( m = 7 \), the only numbers of this form are $F_v(2, 2, 5; 6)$ and $F_v(3, 5; 6)$. It is known that $F_v(3, 5; 6) = 16 \, [89]$. We prove

**Theorem 2.7** $F_v(2, 2, 5; 6) = 16$.

With the help of the number $F_v(2, 2, 5; 6)$ we compute all other numbers in the infinite sequence $F_v(2m, 5; m - 1)$, \( m \geq 7 \), by proving

**Theorem 2.17** $F_v(2m, 5; m - 1) = m + 9$, \( m \geq 7 \).

We obtain the exact values of all modified vertex Folkman numbers in the form $\tilde{F}_v(m_{[5]}; m - 1)$:

**Theorem 2.20** The following equalities are true:

\[
\tilde{F}_v(m_{[5]}; m - 1) = \begin{cases} 
17, & \text{if } m = 7 \\
m + 9, & \text{if } m \geq 8.
\end{cases}
\]

At the end of this chapter, using Theorem 2.17 and Theorem 2.20, we complete the computation of the numbers $F_v(a_1, \ldots, a_s; m - 1)$ where \( \max \{a_1, \ldots, a_s\} = 5 \) by obtaining the following main result:

**Theorem 2.1** Let $a_1, \ldots, a_s$ be positive integers, $m = \sum_{i=1}^{s} (a_i - 1) + 1$, \( \max \{a_1, \ldots, a_s\} = 5 \) and \( m \geq 7 \). Then,

\[
F_v(a_1, \ldots, a_s; m - 1) = m + 9.
\]
Chapter 3

According to (2), the vertex Folkman numbers of the form \( F_v(a_1, ..., a_s; m - 1) \) where \( \max\{a_1, ..., a_s\} = 6 \) exist when \( m \geq 8 \). In the border case \( m = 8 \), the only numbers of this form are \( F_v(2, 2, 6; 7) \) and \( F_v(3, 6; 7) \). We compute the exact values of these numbers by showing that

Theorem 3.8 \( F_v(2, 2, 6; 7) = 17 \).

Theorem 3.9 \( F_v(3, 6; 7) = 18 \).

In [41] Nenov and Kolev pose the following question:

*Does there exist a positive integer \( p \) for which \( F_v(2, 2, p; p + 1) \neq F_v(3, p; p + 1) \)?*

Theorem 3.8 and Theorem 3.9 give a positive answer to this question, and 6 is the smallest possible value for \( p \) for which \( F_v(2, 2, p; p + 1) \neq F_v(3, p; p + 1) \).

With the help of the number \( F_v(2, 2, 6; 7) \) we compute all other numbers in the infinite sequence \( F_v(2_{m-6}, 6; m - 1), m \geq 8 \). We also use the number \( F_v(3, 6; 7) \) to compute all other numbers in the infinite sequence \( F_v(2_{m-8}, 3, 6; m - 1) \):

Theorem 3.14 \( F_v(2_{m-6}, 6; m - 1) = m + 9, m \geq 8 \).

Theorem 3.17 \( F_v(2_{m-8}, 3, 6; m - 1) = m + 10, m \geq 8 \).

We obtain the exact values of all modified vertex Folkman numbers in the form \( F_v(m|_6; m - 1) \):

Theorem 3.18 \( F_v(m|_6; m - 1) = m + 10, m \geq 8 \).

Using Theorem 3.14, Theorem 3.17, and Theorem 3.18, we complete the computation of the numbers \( F_v(a_1, ..., a_s; m - 1) \) where \( \max\{a_1, ..., a_s\} = 6 \) by obtaining the following main result:
Theorem 3.1. Let $a_1, \ldots, a_s$ be positive integers such that $2 \leq a_1 \leq \ldots \leq a_s = 6$ and $m = \sum_{i=1}^s (a_i - 1) + 1 \geq 8$. Then:

(a) $F_v(a_1, \ldots, a_s; m - 1) = m + 9$, if $a_1 = \ldots = a_{s-1} = 2$.

(b) $F_v(a_1, \ldots, a_s; m - 1) = m + 10$, if $a_{s-1} \geq 3$.

Since $F_v(a_1, \ldots, a_s; q)$ is a symmetric function of $a_1, \ldots, a_s$, in Theorem 3.1 we actually compute all numbers of the form $F_v(a_1, \ldots, a_s; m - 1)$, where $\max \{a_1, \ldots, a_s\} = 6$.

Chapter 4

In this chapter we consider the vertex Folkman numbers of the form $F_v(a_1, \ldots, a_s; m - 1)$ where $\max \{a_1, \ldots, a_s\} = 7$. By (2), these numbers exist when $m \geq 9$. We compute the number

Theorem 4.2. $F_v(2, 2, 7; 8) = 20$.

With the help of the number $F_v(2, 2, 6; 7)$ we compute all other numbers in the infinite sequence $F_v(2_{m-7}, 7; m - 1)$, $m \geq 9$:

Theorem 4.8. $F_v(2_{m-7}, 7; m - 1) = m + 11$, $m \geq 9$.

We obtain the following bounds:

Theorem 4.1. Let $a_1, \ldots, a_s$ be positive integers such that $\max \{a_1, \ldots, a_s\} = 7$ and $m = \sum_{i=1}^s (a_i - 1) + 1 \geq 9$. Then,

$$m + 11 \leq F_v(a_1, \ldots, a_s; m - 1) \leq m + 12.$$
Chapter 5

Very little is known about the vertex Folkman numbers of the form $F_v(a_1, ..., a_s; m - 2)$. The exact values of these numbers are not computed when $\max\{a_1, ..., a_s\} \geq 3$. We obtain new bounds on the two smallest unknown numbers of this form, namely $F_v(2, 2, 2; 3)$ and $F_v(2, 3, 3; 4)$:

Theorem 5.1 $20 \leq F_v(2, 2, 2; 3) \leq 22$.

Theorem 5.2 $20 \leq F_v(2, 3, 3; 4) \leq 24$.

Chapter 6

Let us remind that:

$F_v(a_1, ..., a_s; q)$ exist $\iff q > \max\{a_1, ..., a_s\}$.

The computation of the numbers of the form $F_v(a_1, ..., a_s; q)$ in the border case $q = \max\{a_1, ..., a_s\} + 1$ is very hard. The numbers $F_v(2, 2, p; p + 1)$, $p \leq 4$, and $F_v(3, p; p + 1)$, $p \leq 5$, are already computed. We prove that $F_v(2, 2, 5; 6) = 16$ (Theorem 2.7), $F_v(2, 2, 6; 7) = 17$ (Theorem 3.8), $F_v(3, 6; 7) = 18$ (Theorem 3.9), and $F_v(2, 2, 7; 8) = 20$ (Theorem 4.2). The only other computed number of this form is $F_v(2, 2, 2, 3) = 22$, [37].

The numbers $F_v(p, p; p + 1)$ are of significant interest. The only known numbers of this form are $F_v(2, 2; 3) = 5$ and $F_v(3, 3; 4) = 14$, [59], [78]. The following general lower bounds on these numbers are known:

(4) $F_v(p, p; p + 1) \geq 4p - 1$, [101].
We obtain the following bounds:

\[ F_v(p, p; p + 1) \geq F_v(2, 2, p; p + 1) + 2p - 6, \quad p \geq 3. \]  

In [65] it is proved that \( F_v(2, 2, p; p + 1) \geq 2p + 4 \). If \( F_v(2, 2, p; p + 1) = 2p + 4 \), then the inequality \( (4) \) gives a better bound for \( F_v(p, p; p + 1) \) than the inequality \( (5) \). It is interesting to note that it is not known whether the equality \( F_v(2, 2, p; p + 1) = 2p + 4 \) holds for any \( p \). If \( F_v(2, 2, p; p + 1) = 2p + 5 \), then the bounds for \( F_v(p, p; p + 1) \) from \( (4) \) and \( (5) \) coincide, and if \( F_v(2, 2, p; p + 1) > 2p + 5 \), then the inequality \( (5) \) gives a better bound for \( F_v(p, p; p + 1) \).

We improve the bounds on the numbers of the form \( F_v(p, p; p + 1) \) which follow from \( (4) \) and \( (5) \) in the cases \( p = 4, 5, 6, 7 \):

**Theorem 6.5** \( F_v(4, 4; 5) \geq F_v(2, 3, 4; 5) \geq F_v(2, 2, 4; 5) \geq 19 \).

**Theorem 6.1** \( F_v(5, 5; 6) \geq F_v(2, 2, 2, 2, 5; 6) \geq 23 \).

**Theorem 6.3** Let \( a_1, \ldots, a_s \) be positive integers such that \( \max \{a_1, \ldots, a_s\} = 6 \) and \( m = \sum_{i=1}^{s} (a_i - 1) + 1 \). Then:

(a) \( 22 \leq F_v(a_1, \ldots, a_s; 7) \leq F_v(4, 6; 7) \leq 35 \) if \( m = 9 \).

(b) \( 28 \leq F_v(a_1, \ldots, a_s; 7) \leq F_v(6, 6; 7) \leq 70 \) if \( m = 11 \).

**Theorem 6.4** If \( m \geq 13 \) and \( \max \{a_1, \ldots, a_s\} = 7 \), then

\[ F_v(a_1, \ldots, a_s; 8) \geq 3m - 10. \]

In particular, \( F_v(7, 7; 8) \geq 29 \).
Chapter 7

In this chapter the necessary definitions related to edge Folkman numbers are given.

Let $a_1, \ldots, a_s$ be positive integers. The expression $G \xrightarrow{e} (a_1, \ldots, a_s)$ means that in every coloring of $E(G)$ in $s$ colors ($s$-coloring) there exists $i \in \{1, \ldots, s\}$ such that there is a monochromatic $a_i$-clique of color $i$.

Define:

$$H_e(a_1, \ldots, a_s; q) = \{G : G \xrightarrow{e} (a_1, \ldots, a_s) \text{ and } G \nsubseteq K_q\}.$$

The edge Folkman numbers $F_e(a_1, \ldots, a_s; q)$ are defined by the equality:

$$F_e(a_1, \ldots, a_s; q) = \min \{|V(G)| : G \in H_e(a_1, \ldots, a_s; q)\}.$$

The numbers $F_e(3, 3; q)$ are of significant interest.

Chapter 8

The graph $G$ is a minimal graph in $H_e(3, 3)$ if $G \xrightarrow{e} (3, 3)$, and its every proper subgraph $H \nrightarrow (3, 3)$. All minimal graphs in $H_e(3, 3)$ with up to 9 vertices are known.

We find all minimal graphs in $H_e(3, 3)$ with up to 13 vertices. (Theorem 8.9, Theorem 8.10, Theorem 8.11 and Theorem 8.20)

We also obtain all minimal graphs $G \in H_e(3, 3)$ with $\alpha(G) \geq |V(G)| - 8$, and all minimal graphs in $G \in H_e(3, 3; 5)$ with $\alpha(G) \geq |V(G)| - 9$. Using these results, we derive the following upper bounds on the independence number of the graphs in $H_e(3, 3)$:

**Corollary 8.34.** Let $G$ be a minimal graph in $H_e(3, 3)$ and $|V(G)| \geq 27$. Then $\alpha(G) \leq |V(G)| - 9$.

**Corollary 8.36.** Let $G$ be a minimal graph in $H_e(3, 3)$ such that $G \nsubseteq K_5$ and $|V(G)| \geq 30$. Then $\alpha(G) \leq |V(G)| - 10$. 

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At the end of the chapter, we obtain the following lower bounds on the minimum degree of the minimal graphs in $\mathcal{H}_e(3, 3)$.

**Theorem 8.37.** Let $G$ be a minimal graph in $\mathcal{H}_e(3, 3; 5)$. Then $\delta(G) \geq 5$. If $v \in V(G)$ and $d(v) = 5$, then $G(v) = N_{5,i}$ for some $i \in \{1, 2, 3\}$ (see Figure 8.10).

**Theorem 8.38.** Let $G$ be a minimal graph in $\mathcal{H}_e(3, 3; 4)$. Then $\delta(G) \geq 8$. If $v \in V(G)$ and $d(v) = 8$, then $G(v) = N_{8,i}$ for some $i \in \{1, \ldots, 7\}$ (see Figure 8.11).

**Chapter 9**

At the beginning of the introduction we presented in more details the history of the edge Folkman number $F_e(3, 3; 4)$. In this chapter we obtain new lower bound on the number $F_e(3, 3; 4)$:

**Theorem 9.1.** $F_e(3, 3; 4) \geq 20$.

I was introduced to this field by my supervisor Professor N. Nenov. I would like to thank Professor Nenov for his help and support throughout the research and preparation of this thesis.
Publications

All results of the thesis are published in [2], [8], [4], [5], [3], [7], and [6]. The paper [7] is accepted for publication. All other papers are published. Preprints of the papers are published in ResearchGate and arXiv. Five of these papers are published jointly with my supervisor Professor N. Nenov.

Citations

The paper [5] is cited in [98], [97], [80], [83], and [84].
The paper [8] is cited in [99].
The paper [7] is cited in [99].
The paper [6] is cited in [99].
In [99], unpublished joint results with N. Nenov are cited as personal communications.

Approbation of the results

The results of the thesis are presented in the following conferences:
1. Spring Scientific Session of the Faculty of Mathematics and Informatics at Sofia University. Sofia, Bulgaria, March 2016.
2. 45th Spring Conference of the Union of Bulgarian Mathematicians. Pleven, Bulgaria, April 2016.
3. VI Congress of Mathematicians of Macedonia. Ohrid, Macedonia, June 2016.
4. 46th Spring Conference of the Union of Bulgarian Mathematicians. Borovets, Bulgaria, April 2017.
5. Conference dedicated to the 100th Birthday Anniversary of Professor Yaroslav Tagamlitzki. Sofia, Bulgaria, September 2017.
6. National Seminar on Coding Theory "Professor Stefan Dodunekov", Chifflika, Bulgaria, December 2017.
Author’s reference

In the author’s opinion, the main contributions of the thesis are:

1. A new method for bounding the Folkman numbers $F_v(a_1, \ldots, a_s; q)$ by $F_v(2m-p, p; q)$ and the modified Folkman numbers $\tilde{F}_v(m,p; q)$ is presented.

2. New algorithms for computation and bounding of Folkman numbers are developed (algorithms A1, \ldots, A8).

3. The following Folkman numbers are computed:
   \begin{align*}
   F_v(2, 2, 5; 6) &= 16 \text{ (Theorem 2.7)}, \\
   F_v(2, 2, 6; 7) &= 17 \text{ (Theorem 3.8)}, \\
   F_v(3, 6; 7) &= 18 \text{ (Theorem 3.9)}, \\
   F_v(2, 2, 7; 8) &= 20 \text{ (Theorem 4.2)}. \\
   \end{align*}

4. The following infinite series of Folkman numbers are computed:
   \begin{align*}
   F_v(a_1, \ldots, a_s; m-1), \text{ where } \max\{a_1, \ldots, a_s\} &= 5 \text{ (Theorem 2.1)}, \\
   F_v(a_1, \ldots, a_s; m-1), \text{ where } \max\{a_1, \ldots, a_s\} &= 6 \text{ (Theorem 3.1)}, \\
   F_v(2, \ldots, 2; m-1), \text{ where } m \geq 9 \text{ (Theorem 4.8)}. \\
   \end{align*}

5. New bounds on the following Folkman numbers are obtained:
   \begin{align*}
   20 \leq F_v(2, 2, 2, 3; 4) &\leq 22 \text{ (Theorem 5.1)}, \\
   20 \leq F_v(2, 3, 3, 4) &\leq 24 \text{ (Theorem 5.2)}. \\
   \end{align*}

6. New lower bounds on the numbers of the form $F_v(p,p;p+1)$ are obtained in the cases $p = 4, 5, 6, 7$:
   \begin{align*}
   F_v(4, 4; 5) &\geq 19 \text{ (Theorem 6.5)}, \\
   F_v(5, 5; 6) &\geq 23 \text{ (Theorem 6.1)}, \\
   F_v(6, 6; 7) &\geq 28 \text{ (Theorem 6.3)}, \\
   F_v(7, 7; 8) &\geq 29 \text{ (Theorem 6.4)}. \\
   \end{align*}
7. All minimal graphs in $\mathcal{H}_e(3, 3)$ with up to 13 vertices are obtained (Theorem 8.9, Theorem 8.10, Theorem 8.11, and Theorem 8.20).

8. New upper bounds on the independence number of the minimal graphs in $\mathcal{H}_e(3, 3)$ are obtained.

9. New lower bounds on the minimum degree of the minimal graphs in $\mathcal{H}_e(3, 3)$ are obtained.

10. The new lower bound $F_e(3, 3; 4) \geq 20$ is proved.
Part I

Vertex Folkman Numbers
Chapter 1

Definition and properties of the vertex
Folkman numbers

1.1 Graph theory notations

Only finite, non-oriented graphs without loops and multiple edges are considered in this thesis.

The following notations are used:

- $V(G)$ - the vertex set of $G$;
- $E(G)$ - the edge set of $G$;
- $\overline{G}$ - the complement of $G$;
- $\omega(G)$ - the clique number of $G$;
- $\alpha(G)$ - the independence number of $G$;
- $\chi(G)$ - the chromatic number of $G$;
- $N(v), N_G(v), v \in V(G)$ - the set of all vertices of $G$ adjacent to $v$;
- $d(v), v \in V(G)$ - the degree of the vertex $v$, i.e. $d(v) = |N(v)|$;
- $\Delta(G)$ - the maximum degree of $G$;
- $\delta(G)$ - the minimum degree of $G$;
- $G - v, v \in V(G)$ - subgraph of $G$ obtained from $G$ by deleting the vertex $v$ and all edges incident to $v$;
- $G - e, e \in E(G)$ - subgraph of $G$ obtained from $G$ by deleting the edge $e$;
- $G + e, e \in E(\overline{G})$ - supergraph of $G$ obtained by adding the edge $e$ to $E(G)$;
- $G[W]$ - subgraph of $G$ induced by $W \subseteq V(G)$;
$G(v)$ - subgraph of $G$ induced by $N_G(v)$;
$K_n$ - complete graph on $n$ vertices;
$C_n$ - simple cycle on $n$ vertices;
$G_1 + G_2$ - graph $G$ for which $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup E'$, where $E' = \{[x,y]: x \in V(G_1), y \in V(G_2)\}$, i.e. $G$ is obtained by connecting every vertex of $G_1$ to every vertex of $G_2$.

$\text{Aut}(G)$ - the automorphism group of $G$;
$R(p,q)$ - Ramsey number;
$R(p,q) = \{G: \alpha(G) < p \text{ and } \omega(G) < q\}$.
$R(p,q;n) = \{G: G \in R(p,q) \text{ and } |V(G)| = n\}$.

All undefined terms can be found in [96].

$H_v(a_1,\ldots,a_s;q)$ and $H_v(a_1,\ldots,a_s;q;n)$ - Section 1.2;
$F_v(a_1,\ldots,a_s)$ - Section 1.2;
$H_{\text{extr}}(a_1,\ldots,a_s;q)$ - Section 1.3;
$H_{\text{max}}(a_1,\ldots,a_s;q)$ and $H_{\text{max}}(a_1,\ldots,a_s;q;n)$ - Section 1.3;
$H_{\text{extr}}(a_1,\ldots,a_s;q;n)$ and $H_{\text{extr}}(a_1,\ldots,a_s;q;n)$ - Section 1.3;
$\tilde{H}_v(m\mid p;q)$ and $\tilde{H}_v(m\mid p;q;n)$ - Section 1.4;
$\tilde{F}_v(m\mid p;q)$ - Section 1.4;
$H_e(a_1,\ldots,a_s;q)$ and $H_e(a_1,\ldots,a_s;q;n)$ - Chapter 7;
$F_e(a_1,\ldots,a_s)$ - Chapter 7;
$r'_0(p) = r'_0$ - see Theorem 2.12
$r''_0(p) = r''_0$ - see Theorem 3.15
$m_0(p) = m_0$ - see Theorem 3.19

1.2 Definition of vertex Folkman numbers and some known results

Let $a_1,\ldots,a_s$ be positive integers. The expression $G \rightarrow (a_1,\ldots,a_s)$ means that in every coloring of $V(G)$ in $s$ colors ($s$-coloring) there exists $i \in \{1,\ldots,s\}$ such that there is a monochromatic $a_i$-clique of color $i$. 
In particular, $G \xrightarrow{v} (a_1)$ means that $\omega(G) \geq a_1$. Further, for convenience, instead of $G \xrightarrow{v} (2, \ldots, 2)$ we write $G \xrightarrow{v} (2_r)$ and instead of $G \xrightarrow{v} (2, a_1, \ldots, a_s)$ we write $G \xrightarrow{v} (2_r, a_1, \ldots, a_s)$. It is easy to see that

$$(1.1) \quad G \xrightarrow{v} (2_r) \iff \chi(G) \geq r + 1.$$ 

Define:

$H_v(a_1, \ldots, a_s; q) = \{G : G \xrightarrow{v} (a_1, \ldots, a_s) \text{ and } \omega(G) < q\}.$

$H_v(a_1, \ldots, a_s; q; n) = \{G : G \in H_v(a_1, \ldots, a_s; q) \text{ and } |V(G)| = n\}.$

**Remark 1.1.** In the special case $s = 1$, $G \xrightarrow{v} (a_1)$ means that $\omega(G) \geq a_1$, and therefore

$H_v(a_1; q; n) = \{G : a_1 \leq \omega(G) < q \text{ and } |V(G)| = n\}.$

The vertex Folkman numbers $F_v(a_1, \ldots, a_s; q)$ are defined by the equality:

$$F_v(a_1, \ldots, a_s; q) = \min \{|V(G)| : G \in H_v(a_1, \ldots, a_s; q)\}.$$

Folkman proves in [22] that:

$$(1.2) \quad F_v(a_1, \ldots, a_s; q) \text{ exists } \iff q > \max \{a_1, \ldots, a_s\}.$$ 

Other proofs of (1.2) are given in [17] and [49]. In the special case $s = 2$, a very simple proof of this result is given in [63] with the help of corona product of graphs.

Obviously, if $a_i = 1$, then

$$(1.3) \quad G \xrightarrow{v} (a_1, \ldots, a_i-1, a_i+1, \ldots, a_s) \Rightarrow G \xrightarrow{v} (a_1, \ldots, a_s),$$

and therefore

$$F_v(a_1, \ldots, a_s; q) = F_v(a_1, \ldots, a_i-1, a_i+1, \ldots, a_s; q).$$

It is also clear that

$$(1.4) \quad G \xrightarrow{v} (a_1, \ldots, a_s) \Rightarrow G \xrightarrow{v} (a_{\sigma(1)}, \ldots, a_{\sigma(s)})$$
for every permutation $\sigma$ of the numbers $1, \ldots, s$. Therefore, $F_v(a_1, \ldots, a_s; q)$ is a symmetric function of $a_1, \ldots, a_s$, and it is enough to consider only such Folkman numbers $F_v(a_1, \ldots, a_s; q)$ for which

\[(1.5) \quad 2 \leq a_1 \leq \ldots \leq a_s.\]

The numbers $F_v(a_1, \ldots, a_s; q)$ for which the inequalities (1.5) hold are called canonical vertex Folkman numbers.

In [50] for arbitrary positive integers $a_1, \ldots, a_s$ the following terms are defined

\[(1.6) \quad m(a_1, \ldots, a_s) = m = \sum_{i=1}^{s} (a_i - 1) + 1 \quad \text{and} \quad p = \max \{a_1, \ldots, a_s\}.\]

It is easy to see that $K_m \not\rightarrow (a_1, \ldots, a_s)$ and $K_{m-1} \not
\rightarrow (a_1, \ldots, a_s)$. Therefore

$$F_v(a_1, \ldots, a_s; q) = m, \quad q \geq m + 1.$$  

The following theorem for the numbers $F_v(a_1, \ldots, a_s; m)$ is true:

**Theorem 1.2.** Let $a_1, \ldots, a_s$ be positive integers and let $m$ and $p$ be defined by the equalities (1.6). If $m \geq p + 1$, then:

(a) $F_v(a_1, \ldots, a_s; m) = m + p$, [50], [49].

(b) If $G \in \mathcal{H}_v(a_1, \ldots, a_s; m)$ and $|V(G)| = m + p$, then $G = K_{m+p} - C_{2p+1} = K_{m-p-1} + C_{2p+1}$, [49].

The condition $m \geq p + 1$ is necessary according to (1.2). Other proofs of Theorem 1.2 are given in [65] and [67].

Little is known about the numbers $F_v(a_1, \ldots, a_s; m - 1)$. According to (1.2), we have

\[(1.7) \quad F_v(a_1, \ldots, a_s; m - 1) \text{ exists } \Leftrightarrow m \geq p + 2.\]

The following general bounds are known:

\[(1.8) \quad m + p + 2 \leq F_v(a_1, \ldots, a_s; m - 1) \leq m + 3p,\]
where the lower bound is true if \( p \geq 2 \), and the upper bound is true if \( p \geq 3 \). The lower bound is obtained in [65] and the upper bound is obtained in [42]. In the border case \( m = p + 2 \) the upper bounds in (1.8) are significantly improved in [91].

Let \( m \) and \( p \) be defined by (1.6). Then,

\[
(1.9) \quad F_v(a_1, ..., a_s; m - 1) = \begin{cases} 
  m + 4, & \text{if } p = 2 \text{ and } m \geq 6, \quad [61] \\
  m + 6, & \text{if } p = 3 \text{ and } m \geq 6, \quad [70] \\
  m + 7, & \text{if } p = 4 \text{ and } m \geq 6, \quad [70]. 
\end{cases}
\]

The remaining canonical numbers \( F_v(a_1, ..., a_s; m - 1) \), \( p \leq 4 \) are: \( F_v(2, 2, 2; 3) = 11, \quad [54] \) and [13], \( F_v(2, 2, 2, 2; 4) = 11, \quad [62] \) (see also [64]), \( F_v(2, 2, 3; 4) = 14, \quad [59] \) and [15], \( F_v(3, 3; 4) = 14, \quad [59] \) and [78]. From these facts it becomes clear that we know all Folkman numbers of the form \( F_v(a_1, ..., a_s; m - 1) \) when \( \max \{a_1, ..., a_s\} \leq 4 \). It is also known that \( F_v(3, 5; 6) = 16, \quad [89] \).

It is easy to see that in the border case \( m = p + 2 \) when \( p \geq 3 \) there are only two canonical numbers of the form \( F_v(a_1, ..., a_s; m - 1) \), namely \( F_v(2, 2, p; p + 1) \) and \( F_v(3, p; p + 1) \). Some graphs, with which upper bounds for \( F_v(3, p; p + 1) \) are obtained, can be used for obtaining general upper bounds for \( F_v(a_1, ..., a_s; m - 1) \). For example, the graph \( \Gamma_p \) from [65], which witnesses the bound \( F_v(3, p; p + 1) \leq 4p + 2, \quad p \geq 3 \), helps to obtain the upper bound in (1.8). With the help of the numbers \( F_v(2, 2, p; p + 1) \), the general lower bound in (1.8) can be improved (see Theorem 3.12). In this thesis we will clarify more thoroughly the role of the numbers \( F_v(2, 2, p; p + 1) \) in obtaining lower bounds on the numbers \( F_v(a_1, ..., a_s; m - 1) \). Thus, obtaining bounds for the numbers \( F_v(a_1, ..., a_s; m - 1) \) and computing some of them is related with computation and obtaining bounds for the numbers \( F_v(2, 2, p; p + 1) \) and \( F_v(3, p; p + 1) \). It is easy to see that \( G \twoheadrightarrow (3, p) \Rightarrow G \twoheadrightarrow (2, 2, p) \). Therefore, the following inequality holds:

\[
(1.10) \quad F_v(2, 2, 2; p + 1) \leq F_v(3, p; p + 1), \quad p \geq 3.
\]

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In [41] Nenov and Kolev pose the following problem:

**Problem 1.3.** [41] Does there exist a positive integer $p$ for which the inequality (1.10) is strict?

As we noted after (1.9), if $p = 3$ or $4$, $F_v(2, 2, p; p + 1) = F_v(3, p; p + 1)$. Further, we will prove that $F_v(2, 2, 5; 6) = F_v(3, 5; 6) = 16$ (Theorem 2.7 and Corollary 2.10), but $F_v(2, 2, 6; 7) = 17$ while $F_v(3, 6; 7) = 18$ (Theorem 3.8 and Theorem 3.9). Therefore, $p = 6$ is the smallest positive integer for which the inequality (1.10) is strict.

In the following Chapters 2 and 3 we compute the numbers $F_v(a_1, ..., a_s; m - 1)$ where $\max \{a_1, ..., a_s\} = 5$ or $6$ (Theorem 2.1 and Theorem 3.1). In Chapter 4 we compute an infinite sequence of numbers of the form $F_v(a_1, ..., a_s; m - 1)$ where $\max \{a_1, ..., a_s\} = 7$.

Not much is known about the numbers $F_v(a_1, ..., a_s; q)$ when $q \leq m - 2$. In Chapter 5 we obtain new bounds on some numbers of the form $F_v(a_1, ..., a_s; m - 2)$.

New bounds on some numbers of the form $F_v(a_1, ..., a_s; q)$ where $q = \max \{a_1, ..., a_s\} + 1$ are obtained in Chapter 6.

### 1.3 Auxiliary definitions and propositions

Let $a_1, ..., a_s$ be positive integers and $m = \sum_{i=1}^{s} (a_i - 1) + 1$.

Obviously, if $a_i \geq t \geq 2$, then

(1.11) \[ G \Rightarrow (a_1, ..., a_s) \Rightarrow G \Rightarrow (a_1, ..., a_{i-1}, t, a_i - t + 1, a_{i+1}, ..., a_s). \]

In the special case $t = 2$ we have

(1.12) \[ G \Rightarrow (a_1, ..., a_s) \Rightarrow G \Rightarrow (a_1, ..., a_{i-1}, 2, a_i - 1, a_{i+1}, ..., a_s). \]

If some of the numbers $a_1, ..., a_{i-1}, 2, a_i - 1, a_{i+1}, ..., a_s$ are greater than 2, we apply (1.12) again. Thus, by repeatedly applying (1.11) we obtain

(1.13) \[ G \Rightarrow (a_1, ..., a_s) \Rightarrow G \Rightarrow (2_{m-1}). \]
From (1.13) and (1.1) it follows

\[ G \overset{\nu}{\rightarrow} (a_1, ..., a_s) \Rightarrow \chi(G) \geq m, \]  

Simple examples of graphs for which equality in (1.14) is reached are obtained in [63] and [99].

If \( a_s = p \), by applying the method for obtaining (1.13) only for the numbers \( a_1, ..., a_{s-1} \) we obtain

\[ G \overset{\nu}{\rightarrow} (a_1, ..., a_s) \Rightarrow G \overset{\nu}{\rightarrow} (2m-p, p), \]

If additionally \( a_{s-1} \geq 3 \), using the same method we obtain

\[ G \overset{\nu}{\rightarrow} (a_1, ..., a_s) \Rightarrow G \overset{\nu}{\rightarrow} (2m-p-2, 3, p). \]

It is easy to prove the following

**Proposition 1.4.** Let \( G \) be a graph, \( G \overset{\nu}{\rightarrow} (a_1, ..., a_s) \), and \( A \subseteq V(G) \) be an independent set. Then, if \( a_i \geq 2 \)

\[ G - A \overset{\nu}{\rightarrow} (a_1, ..., a_{i-1}, a_i - 1, a_{i+1}, ..., a_s). \]

We will need the following theorem which gives an upper bound on the independence number of the graphs in \( \mathcal{H}_v(a_1, ..., a_s; m - 1) \).

**Theorem 1.5.** [69] Let \( a_1, ..., a_s \) be positive integers and \( m \) and \( p \) defined by \( (1.6) \). Let \( G \in \mathcal{H}_v(a_1, ..., a_s; m - 1) \). Then,

(a) \( \alpha(G) \leq |V(G)| - m - p + 1. \)

(b) If \( |V(G)| < m + 3p \), then \( \alpha(G) \leq |V(G)| - m - p. \)

The graph \( G \) is called an extremal graph in \( \mathcal{H}_v(a_1, ..., a_s; q) \) if \( G \in \mathcal{H}_v(a_1, ..., a_s; q) \) and \( |V(G)| = F_v(a_1, ..., a_s; q) \). We denote by \( \mathcal{H}_{\text{ext}}(a_1, ..., a_s; q) \) the set of all extremal graphs in \( \mathcal{H}_v(a_1, ..., a_s; q) \).
Conjecture 1.6. If $G \in \mathcal{H}_{\text{extr}}(a_1, \ldots, a_s; m - 1)$, then $\chi(G) \leq m + 1$.

For all known examples of extremal graphs, including the extremal graphs obtained in this thesis, this inequality holds.

The graph $G$ is called a $K_q$-free graph if $\omega(G) < q$. $G$ is a maximal $K_q$-free graph if $\omega(G) < q$ and $\omega(G + e) = q, \forall e \in E(G)$.

Let $W \subseteq V(G)$. $W$ is a maximal $K_q$-free subset of $V(G)$ if $W$ does not contain a $q$-clique and $W \cup \{v\}$ contains a $q$-clique for all $v \in V(G) \setminus W$.

We say that $G$ is a maximal graph in $\mathcal{H}_v(a_1, \ldots, a_s; q)$ if $G \in \mathcal{H}_v(a_1, \ldots, a_s; q)$ but $G + e \notin \mathcal{H}_v(a_1, \ldots, a_s; q), \forall e \in E(G)$, i.e. $\omega(G + e) = q, \forall e \in E(G)$. $G$ is a minimal graph in $\mathcal{H}_v(a_1, \ldots, a_s; q)$ if $G \in \mathcal{H}_v(a_1, \ldots, a_s; q)$ but $G - e \notin \mathcal{H}_v(a_1, \ldots, a_s; q), \forall e \in E(G)$, i.e. $G - e \notin \mathcal{H}_v(a_1, \ldots, a_s), \forall e \in E(G)$.

If $G$ is both maximal and minimal graph in $\mathcal{H}_v(a_1, \ldots, a_s; q)$, then we say that $G$ is a bicritical graph in $\mathcal{H}_v(a_1, \ldots, a_s; q)$.

For convenience, we also define the following term:

Definition 1.7. The graph $G$ is called a $(+K_p)$-graph if $G + e$ contains a new $p$-clique for all $e \in E(G)$.

Obviously, $G \in \mathcal{H}_v(a_1, \ldots, a_s; q)$ is a maximal graph in $\mathcal{H}_v(a_1, \ldots, a_s; q)$ if and only if $G$ is a $(+K_p)$-graph. We denote by $\mathcal{H}_{\text{max}}(a_1, \ldots, a_s; q)$ the set of all maximal graphs in $\mathcal{H}_v(a_1, \ldots, a_s; q)$ and by $\mathcal{H}_{+K_p}(a_1, \ldots, a_s; q)$ the set of all $(+K_p)$-graphs in $\mathcal{H}_v(a_1, \ldots, a_s; q)$. We will also use the following notations:

$\mathcal{H}_{+K_p}(a_1, \ldots, a_s; q; n) = \{G \in \mathcal{H}_{+K_p}(a_1, \ldots, a_s; q) : |V(G)| = n\}$

$\mathcal{H}_{\text{max}}(a_1, \ldots, a_s; q; n) = \{G \in \mathcal{H}_{\text{max}}(a_1, \ldots, a_s; q) : |V(G)| = n\}$

$\mathcal{H}'_{+K_p}(a_1, \ldots, a_s; q; n) = \{G \in \mathcal{H}_{+K_p}(a_1, \ldots, a_s; q; n) : \alpha(G) \leq t\}$

$\mathcal{H}'_{\text{max}}(a_1, \ldots, a_s; q; n) = \{G \in \mathcal{H}_{\text{max}}(a_1, \ldots, a_s; q; n) : \alpha(G) \leq t\}$

Remark 1.8. If $a_1 \leq n \leq q - 1$ then $\mathcal{H}_{\text{max}}(a_1; q; n) = \{K_n\}$.

If $a_1 \leq q - 1 \leq n$ and $G \in \mathcal{H}_{\text{max}}(a_1; q; n)$, then $\omega(G) = q - 1$, and therefore $\mathcal{H}_{\text{max}}(a_1; q; n) = \mathcal{H}_{\text{max}}(q - 1; q; n)$.
Further, we will need the following

**Proposition 1.9.** Let $G$ be a maximal $K_q$-free graph, $A$ be an independent set of vertices of $G$, and $H = G - A$. Then $H$ is a $(+K_{q-1})$-graph.

Proof. Suppose that for some $e \in E(H)$, $e + H$ does not contain a new $(q - 1)$-clique. Then $e + G$ does not contain a $q$-clique. \[\square\]

For convenience, we will formulate the following proposition which follows directly from Proposition 1.4 and Proposition 1.9:

**Proposition 1.10.** Let $G \in \mathcal{H}_{max}(a_1, \ldots, a_s; q; n)$, $a_1 \geq 2$, and $A$ be an independent set of vertices of $G$. Then,

$$G - A \in \mathcal{H}_{+K_{q-1}}(a_1 - 1, a_2, \ldots, a_s; q; n - |A|).$$

We will use Proposition 1.10 in many of the algorithms.

We will also need the following improvement of the lower bound in (1.8)

**Theorem 1.11.** [69] Let $a_1, \ldots, a_s$ be positive integers, let $m$ and $p$ be defined by the equalities (1.6), $p \geq 3$, and $m \geq p + 2$. If $F_v(2, 2; p; p + 1) \geq 2p + 5$, then

$$F_v(a_1, \ldots, a_s; m - 1) \geq m + p + 3.$$ 

1.4 Modified vertex Folkman numbers

In this section we will define the modified vertex Folkman numbers, which help us to obtain upper bounds on the vertex Folkman numbers.

**Definition 1.12.** Let $G$ be a graph and let $m$ and $p$ be positive integers. The expression

$$G \overset{v}{\rightarrow} m|_p$$

means that for every choice of positive integers $a_1, \ldots, a_s$ ($s$ is not fixed), such that $m = \sum_{i=1}^{s} (a_i - 1) + 1$ and $\max \{a_1, \ldots, a_s\} \leq p$, we have

$$G \overset{v}{\rightarrow} (a_1, \ldots, a_s).$$
Example 1.13. Obviously, $K_m \not\rightarrow m\mid_p$, $\forall p$.

Example 1.14. [49] Let us notice that $\overline{C_{2p+1}} \not\rightarrow (p + 1)\mid_p$. Indeed, let $b_1, ..., b_s$ be positive integers such that $\sum_{i=1}^{s} (b_i - 1) + 1 = p + 1$ and $\max\{b_1, ..., b_s\} \leq p$. Assume that there exists $s$-coloring $V(G) = V_1 \cup ... \cup V_s$ such that $V_i$ does not contain a $b_i$-clique. Then $|V_i| \leq 2b_i - 2$ and $|V(G)| = \sum_{i=1}^{s} |V_i| \leq 2\sum_{i=1}^{s} (b_i - 1) = 2p$, which is a contradiction.

Further, we will need the following

Lemma 1.15. [70] Let $m_0$ and $p$ be positive integers and $G \not\rightarrow m_0\mid_p$. Then for every positive integer $m \geq m_0$ it is true that $K_{m-m_0} + G \not\rightarrow m\mid_p$.

This lemma is formulated in an obviously equivalent way and is proved by induction with respect to $m \geq m_0$ in [70] as Lemma 3.

Let $G$ be a graph and $m$ and $p$ be positive integers. If $m < p$, then from Example 1.13 it follows easily that $G \not\rightarrow m\mid_p$ if and only if $\omega(G) \geq m$. With the help of the following proposition we can significantly reduce the number of checks required to determine if a graph $G$ has the property $G \not\rightarrow m\mid_p$ (see the proofs of Theorem 2.20 and Theorem 3.18).

Proposition 1.16. Let $G$ be a graph and let $m > p$ be positive integers. If $G \not\rightarrow (a_1, ..., a_s)$ for every choice of positive integers $a_1, ..., a_s$, $s \geq 2$ ($s$ is not fixed), such that $m = \sum_{i=1}^{s} (a_i - 1) + 1$, $2 \leq a_1 \leq ... \leq a_s \leq p$, and $a_1 + a_2 - 1 > p$, then $G \not\rightarrow m\mid_p$.

Proof. Let the graph $G$ satisfy the condition of the proposition. We will prove via induction by $s$ that $G \not\rightarrow (a_1, ..., a_s)$ for every choice of $s$ positive integers $a_1, ..., a_s$ such that $m = \sum_{i=1}^{s} (a_i - 1) + 1$, $\max\{a_1, ..., a_s\} \leq p$, and $\min\{a_1, ..., a_s\} \geq 2$. Then from (1.3) it follows that $G \not\rightarrow m\mid_p$. According to (1.4), it is enough to prove that $G \not\rightarrow (a_1, ..., a_s)$ when $2 \leq a_1 \leq ... \leq a_s \leq p$.

Since $\sum_{i=1}^{s} (a_i - 1) + 1 = m > p$ and $a_1 \leq p$, it follows that $s \geq 2$. Therefore, the basis of the induction is the case $s = 2$. 27
If \( s = 2 \), then \( (a_1 - 1) + (a_2 - 1) + 1 = a_1 + a_2 - 1 = m > p \), and by the condition of the proposition, \( G \not\rightarrow (a_1, a_2) \).

Let \( s > 3 \). If \( a_1 + a_2 - 1 > p \), then by the condition of the proposition, \( G \not\rightarrow (a_1, \ldots, a_s) \). Now, consider the case \( a_1 + a_2 - 1 \leq p \). Let \( \{ b_1, \ldots, b_{s-1} \} = \{ a_1 + a_2 - 1, a_3, \ldots, a_s \} \). Then \( \sum_{i=1}^{s-1} (b_i - 1) + 1 = \sum_{i=1}^{s} (a_i - 1) + 1 = m \). By the induction hypothesis, \( G \not\rightarrow (b_1, \ldots, b_{s-1}) \). Obviously, if \( V_1 \cup V_2 \cup \ldots \cup V_s \) is an \((a_1, a_2, \ldots, a_s)\)-free coloring of \( V(G) \), i.e. \( V_i \) does not contain an \( a_i \)-clique, then \( (V_1 \cup V_2) \cup \ldots \cup V_s \) is an \((a_1 + a_2 - 1, a_3, \ldots, a_s)\)-free coloring of \( V(G) \). Therefore, from \( G \not\rightarrow (b_1, \ldots, b_{s-1}) \) it follows that \( G \not\rightarrow (a_1, \ldots, a_s) \). Thus, the proposition is proved. \( \square \)

Define:
\[
\mathcal{H}_v(m \mid p; q) = \left\{ G : G \not\rightarrow m \mid p \text{ and } \omega(G) < q \right\}.
\]
\[
\mathcal{H}_v(m \mid p; q; n) = \left\{ G : G \in \mathcal{H}_v(m \mid p; q) \text{ and } |V(G)| = n \right\}.
\]

The modified vertex Folkman numbers are defined by the equality:
\[
\tilde{F}_v(m \mid p; q) = \min \left\{ |V(G)| : G \in \mathcal{H}_v(m \mid p; q) \right\}.
\]

**Proposition 1.17.** \( \tilde{H}_v(m \mid p; q) \neq \emptyset \), i.e. \( \tilde{F}_v(m \mid p; q) \) exists \( \iff \) \( q > \min \{ m, p \} \).

**Proof.** Let \( \tilde{H}_v(m \mid p; q) \neq \emptyset \) and \( G \in \mathcal{H}_v(m \mid p; q) \). If \( m \leq p \), then \( G \not\rightarrow (m) \), and it follows \( \omega(G) \geq m \). Since \( \omega(G) < q \), we obtain \( q > m \). Let \( m > p \). Then there exist positive integers \( a_1, \ldots, a_s \), such that \( m = \sum_{i=1}^{s} (a_i - 1) + 1 \) and \( p = \max \{ a_1, \ldots, a_s \} \), for example \( a_1 = \ldots = a_{m-p} = 2 \) and \( a_{m-p+1} = p \). Since \( G \not\rightarrow (a_1, \ldots, a_s) \), it follows that \( \omega(G) \geq p \) and \( q > p \). Therefore, if \( \tilde{H}_v(m \mid p; q) \neq \emptyset \), then \( q > \min \{ m, p \} \).

Let \( q > \min \{ m, p \} \). If \( m \geq p \), then \( q > p \). According to (1.2), for every choice of positive integers \( a_1, \ldots, a_s \), such that \( m = \sum_{i=1}^{s} (a_i - 1) + 1 \) and \( \max \{ a_1, \ldots, a_s \} \leq p \), there exists a graph \( G(a_1, \ldots, a_s) \in \mathcal{H}_v(a_1, \ldots, a_s; q) \). Let \( G \) be the union of all graphs \( G(a_1, \ldots, a_s) \). It is clear that \( G \in \mathcal{H}_v(m \mid p; q) \). If \( m \leq p \), then \( m < q \), and therefore \( K_m \in \tilde{H}_v(m \mid p; q) \). \( \square \)

The following theorem for the modified Folkman numbers is true:

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Theorem 1.18. Let \( m, m_0, p \) and \( q \) be positive integers, \( m \geq m_0 \), and \( q > \min \{ m_0, p \} \). Then,

\[
\tilde{F}_v(m_p; m - m_0 + q) \leq \tilde{F}_v(m_0_p; q) + m - m_0.
\]

Proof. Let \( G_0 \in \tilde{H}_v(m_0_p; q) \), \(|V(G_0)| = \tilde{F}_v(m_0_p; q)\) and \( G = K_{m - m_0} + G_0\). According to Lemma 1.15, \( G \rightarrow m_1 \). Since \( \omega(G) = m - m_0 + \omega(G_0) < m - m_0 + q \), it follows that \( G \in \tilde{H}_v(m_p; m - m_0 + q) \). Therefore, \( \tilde{F}_v(m_p; m - m_0 + q) \leq |V(G)| = \tilde{F}_v(m_0_p; q) + m - m_0 \). \( \square \)

We see that if we know the value of one number \( \tilde{F}_v(m_0_p; q) \) we can obtain an upper bound for \( \tilde{F}_v(m_p; m - m_0 + q) \) where \( m \geq m_0 \).

From the definition of the modified Folkman numbers it becomes clear that if \( a_1, \ldots, a_s \) are positive integers and \( m \) and \( p \) are defined by (1.6), then

\[
F_v(a_1, \ldots, a_s; q) \leq \tilde{F}_v(m_p; q).
\]

(1.17)

Defining and computing the modified Folkman numbers is appropriate because of the following reasons:

1) On the left side of (1.17) there is actually a whole class of numbers, which are bound by only one number \( \tilde{F}_v(m_p; q) \).

2) The upper bound for \( \tilde{F}_v(m_p; q) \) is easier to compute than the numbers \( F_v(a_1, \ldots, a_s) \), according to Theorem 1.18.

3) As we will see further in Theorem 3.19, the computation of the numbers \( \tilde{F}_v(m_p; m - 1) \) is reduced to finding the exact values of the first several of these numbers (bounds for the number of exact values needed are given in Theorem 3.19 (c)).

The following theorem gives bounds for the numbers \( F_v(a_1, \ldots, a_s; q) \). Let us remind that \( F_v(2m - p, p; q) = F_v(\underbrace{2, \ldots, 2}_{m-p}, p, q) \).

Theorem 1.19. Let \( a_1, \ldots, a_s \) be positive integers, let \( m \) and \( p \) be defined by (1.6), and \( q > p \). Then,

\[
F_v(2m - p, p; q) \leq F_v(a_1, \ldots, a_s; q) \leq \tilde{F}_v(m_p; q).
\]
Proof. The right inequality follows from the inclusion
\[ \tilde{\mathcal{H}}_v(m \mid_p ; q) \subseteq \mathcal{H}_v(a_1, \ldots, a_s ; q) . \]
From (1.15) it follows
\[ F_v(a_1, \ldots, a_s ; q) \geq F_v(2m-p, p ; q) . \]

Remark 1.20. It is easy to see that if \( q > m \), then \( F_v(a_1, \ldots, a_s ; q) = \tilde{F}_v(m \mid_p ; q) = m \). From Theorem 1.2 it follows \( F_v(a_1, \ldots, a_s ; m) = \tilde{F}_v(m \mid_p ; m) = m + p \). If \( q = m - 1 \) and \( p \leq 4 \), according to (1.9), we also have \( F_v(a_1, \ldots, a_s ; q) = \tilde{F}_v(m \mid_p ; q) \). The first case in which the upper bound in Theorem 1.19 is not reached is \( m = 7, p = 5, q = 6 \), since \( \tilde{F}_v(7 \mid_5 ; 6) = 17 \) (see Theorem 2.20) and the corresponding numbers \( F_v(a_1, \ldots, a_s ; q) \) are not greater than 16.

Other modifications of the Folkman numbers are defined and studied in [98], [97], [84], [21], [46], [45], and [38]. These modifications of the Folkman numbers are called generalized Folkman numbers. In this thesis we do not consider generalized Folkman numbers.

Theorem 1.18 and Theorem 1.19 are published in [8].
Chapter 2

Computation of the numbers
\[ F_v(a_1, ..., a_s; m - 1) \text{ where } \max\{a_1, ..., a_s\} = 5 \]

In this chapter we will prove the following main result:

**Theorem 2.1.** Let \(a_1, ..., a_s\) be positive integers, \(m = \sum_{i=1}^{s} (a_i - 1) + 1\), \(\max\{a_1, ..., a_s\} = 5\) and \(m \geq 7\). Then,

\[ F_v(a_1, ..., a_s; m - 1) = m + 9. \]

According to (1.7), the condition \(m \geq 7\) in Theorem 2.1 is necessary.

2.1 Algorithm A1

In the border case \(m = 7\) of Theorem 2.1 there are only two canonical numbers of the form \(F_v(a_1, ..., a_s; m - 1)\), namely \(F_v(2, 2, 5; 6)\) and \(F_v(3, 5; 6)\). It is known that \(F_v(3, 5; 6) = 16\) [89], and it follows that \(F_v(2, 2, 5; 6) \leq 16\). We will prove that \(F_v(2, 2, 5; 6) = 16\) and we will also obtain all extremal graphs in \(H_v(2, 2, 5; 6)\).

The naive approach for finding all graphs in \(H_v(2, 2, 5; 6; 16)\) suggests to check all graphs of order 16 for inclusion in \(H_v(2, 2, 5; 6)\). However, this is practically impossible because the number of non-isomorphic graphs of order 16 is too large (see [51]). The method that is described uses an algorithm for effective generation of all maximal graphs in \(H_v(2, 2, 5; 6; 16)\). The remaining
graphs in $H_v(2, 2, 5; 6; 16)$ are obtained by removing edges from the maximal graphs.

The idea for Algorithm A1 comes from [78]. Similar algorithms are used in [15], [100], [44], and [89]. Also, with the help of computer, results for Folkman numbers are obtained in [37], [91], [90], and [16].

Let us remind that the notations $H_{\text{max}}$ and $H_{+K_{q-1}}$ are defined in Section 1.3.

Algorithm A1. The input of the algorithm is the set $A = H_{\text{max}}(2^{r-1}, p; q; n - k)$, where $r, p, q, n, k$ are fixed positive integers.

The output of the algorithm is the set $B$ of all graphs $G \in H_{\text{max}}(2^r, p; q; n)$ with $\alpha(G) \geq k$.

1. By removing edges from the graphs in $A$ obtain the set $A' = H_{+K_{q-1}}(2^{r-1}, p; q; n - k)$.

2. For each graph $H \in A'$:
   
   2.1. Find the family $M(H) = \{M_1, \ldots, M_l\}$ of all maximal $K_{q-1}$-free subsets of $V(H)$.

2.2. For each $k$-element multiset $N = \{M_{i_1}, \ldots, M_{i_k}\}$ of elements of $M(H)$ construct the graph $G = G(N)$ by adding new independent vertices $v_1, \ldots, v_k$ to $V(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, \ldots, k$. If $\omega(G + e) = q, \forall e \in E(G)$, then add $G$ to $B$.

3. Remove the isomorphic copies of graphs from $B$.

4. Remove from $B$ all graphs $G$ for which $G \not\rightarrow 2(r, p)$.

Theorem 2.2. After the execution of Algorithm A1, the obtained set $B$ coincides with the set of all graphs $G \in H_{\text{max}}(2r, p; q; n)$ with $\alpha(G) \geq k$.

Proof. Suppose that after the execution of Algorithm A1 the graph $G \in B$. Then $G = G(N)$ and $G - \{v_1, \ldots, v_k\} = H \in A'$, where $N$, $v_1, \ldots, v_k$, and $H$ are the same as in step 2.2. Since $H \in A'$, we have $\omega(H) < q$. Since $N_G(v_i)$ are $K_{q-1}$-free sets for all $i \in \{1, \ldots, k\}$, it follows that $\omega(G) < q$. The check at the end of step 2.2 guarantees that $G$ is a maximal $K_q$-free graph, and the check in step 4 guarantees that $G \not\rightarrow 2(r, p)$. Therefore, $G \in H_{\text{max}}(2r, p; q; n)$. Since the vertices $v_1, \ldots, v_k$ are independent, it follows that $\alpha(G) \geq k$. 

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Let $G \in \mathcal{H}_{\text{max}}(2r, p; q; n)$ and $\alpha(G) \geq k$. We will prove that, after the execution of Algorithm A1, $G \in \mathcal{B}$. Let $v_1, \ldots, v_k$ be independent vertices in $G$ and $H = G - \{v_1, \ldots, v_k\}$. Using Proposition 1.10 we derive that $H \in \mathcal{A}'$. Since $G$ is a maximal $K_q$-free graph, $N_G(v_i)$ are maximal $K_{q-1}$-free subsets of $V(H)$ for all $i \in \{1, \ldots, k\}$. Therefore, $N_G(v_i) \in \mathcal{M}(H)$ (see step 2.1). Thus, $G = G(N)$ where $N = \{N_G(v_1), \ldots, N_G(v_k)\}$, and in step 2.2 $G$ is added to $\mathcal{B}$. Clearly, after step 4 the graph $G$ remains in $\mathcal{B}$.

Remark 2.3. Note that if $G \in \mathcal{H}_{\text{max}}(2r, p; q; n)$ and $n \geq q$, then $G$ is not a complete graph and $\alpha(G) \geq 2$. Therefore, if $n \geq q$ and $k = 2$, Algorithm A1 finds all graphs in $\mathcal{H}_{\text{max}}(2r, p; q; n)$.

Theorem 2.2 is published in [8]. Algorithm A1 is a slightly modified version of Algorithm 2.2 in [8]. In the special case $p = 5$ and $k = 2$, Algorithm A1 coincides with Algorithm 6.2 in [8].

2.2 Computation of the number $F_v(2, 2, 5; 16)$

Intermediate problems, that are solved, are finding all graphs in $\mathcal{H}_v(5; 6; 10)$ and $\mathcal{H}_v(2, 5; 6; 13)$.

Theorem 2.4. $|\mathcal{H}_v(5; 6; 10)| = 1724440$.

Proof. It is clear that $\mathcal{H}_v(5; 6; 10)$ is the set of 10 vertex graphs with clique number 5. Using nauty [53] it is easy to generate all 12005168 non-isomorphic graphs of order 10. Among these graphs 1724440 have clique number 5.

Theorem 2.5. $|\mathcal{H}_v(2, 5; 6; 13)| = 20013726$.

Proof. The graphs in $\mathcal{H}_{\text{max}}(2, 5; 6; 13)$ with independence number 2 are a subset of $\mathcal{R}(3, 6; 13)$. All 275086 graphs in $\mathcal{R}(3, 6; 13)$ are known and are available on [52]. Among these graphs we find all 61 graphs in $\mathcal{H}_{\text{max}}(2, 5; 6; 13)$ with independence number 2.

Next, we find all graphs in $\mathcal{H}_{\text{max}}(2, 5; 6; 13)$ with independence number greater than 2. All 1724440 graphs in $\mathcal{H}_v(5; 6; 10)$ were obtained in Theorem
Among them there are 18 maximal graphs in $H_{\text{max}}(5; 6; 10)$. We execute Algorithm A1($n = 13; r = 1; p = 5; q = 6; k = 3$) with input the set $A = H_{\text{max}}(5; 6; 10)$ to obtain the set $B$ of all 326 graphs in $H_{\text{max}}(2, 5; 6; 13)$ with independence number greater than 2.

Thus, we obtained all 387 graphs in $H_{\text{max}}(2, 5; 6; 13)$. By removing edges from them, we obtain all 20 013 726 graphs in $H_{v}(2, 5; 6; 13)$.

**Theorem 2.6.** $|H_{v}(2, 2, 5; 6; 16)| = 147$.

**Proof.** The graphs in $H_{\text{max}}(2, 2, 5; 6; 16)$ with independence number 2 are a subset of $R(3, 6; 16)$. All 2 576 graphs in $R(3, 6; 16)$ are known and are available on [52]. Among these graphs we find all 5 graphs in $H_{\text{max}}(2, 2, 5; 6; 16)$ with independence number 2.

Next, we find all graphs in $H_{\text{max}}(2, 2, 5; 6; 16)$ with independence number greater than 2. In the proof of Theorem 2.5 we obtained all 387 graphs in $H_{\text{max}}(2, 5; 6; 13)$. We execute Algorithm A1($n = 16; r = 2; p = 5; q = 6; k = 3$) with input the set $A = H_{\text{max}}(2, 5; 6; 13)$ to obtain the set $B$ of all 32 graphs in $H_{\text{max}}(2, 2, 5; 6; 16)$ with independence number greater than 2.

Thus, we obtained all 37 graphs in $H_{\text{max}}(2, 2, 5; 6; 16)$. By removing edges from them, we obtain all 147 graphs in $H_{v}(2, 2, 5; 6; 16)$.

We denote by $G_1, \ldots, G_{147}$ the graphs in $H_{v}(2, 2, 5; 6; 16)$. The indexes correspond to the order defined in the nauty programs. In Table 2.1 are listed some properties of the graphs in $H_{v}(2, 2, 5; 6; 16)$. It is interesting to note that there are no graphs with independence number greater than 3 in $H_{v}(2, 2, 5; 6; 16)$. In our opinion this is a non-trivial fact which cannot be obtained without a computer. Among the graphs in $H_{v}(2, 2, 5; 6; 16)$ there are 37 maximal, 41 minimal, and 4 bicritical graphs, i.e. graphs that are both maximal and minimal (see Figure 2.1). Some properties of the bicritical graphs are listed in Table 2.2.

All computations were done on a personal computer. The slowest part was the step of finding all graphs in $H_{\text{max}}(2, 2, 5; 6; 16)$, which took several days to complete.
Figure 2.1: All 4 bicritical graphs in $\mathcal{H}_v(2, 5; 6; 16)$
Table 2.1: Some properties of the graphs in $H_v(2, 2, 5; 6; 16)$

| Graph | $|E(G)|$ | $\delta(G)$ | $\Delta(G)$ | $\alpha(G)$ | $\chi(G)$ | $|Aut(G)|$ |
|-------|--------|-------------|-------------|-------------|---------|---------|
| $G_{71}$ | 86    | 9          | 12          | 3           | 7       | 1       |
| $G_{78}$ | 87    | 10         | 12          | 3           | 7       | 2       |
| $G_{134}$ | 85    | 9          | 12          | 3           | 7       | 2       |
| $G_{135}$ | 85    | 9          | 12          | 3           | 7       | 1       |

Table 2.2: Some properties of the bicritical graphs in $H_v(2, 2, 5; 6; 16)$

Theorem 2.7. $F_v(2, 2, 5; 6) = 16$ and the graphs from Theorem 2.6 are all the graphs in $H_{extr}(2, 2, 5; 6)$.

Proof. From $H_v(2, 2, 5; 6; 16) \neq \emptyset$, it follows that $F_v(2, 2, 5; 6) \leq 16$. Since none of the graphs in $H_v(2, 2, 5; 6; 16)$ have an isolated vertex, we have $H_v(2, 2, 5; 6; 15) = \emptyset$, and therefore $F_v(2, 2, 5; 6) \geq 16$. Thus, the theorem is proved.

Remark 2.8. The lower bound $F_v(2, 2, 5; 6) \geq 16$ can be proved simpler in terms of computational time. With the help of Algorithm A1, we proved independently that $H_v(2, 2, 5; 6; 15) = \emptyset$, which is a lot faster.

Theorem 2.9. $H_v(3, 5; 6; 16) = \{G_{50}, G_{51}, G_{81}, G_{146}\}$, see Figure 2.2

Proof. Using $H_v(3, 5; 6; 16) \subseteq H_v(2, 2, 5; 6; 16)$, by checking with a computer which of the graphs in $H_v(2, 2, 5; 6; 16)$ belong to $H_v(3, 5; 6; 16)$, we obtain $|H_v(3, 5; 6; 16)| = 4$. The graphs in $H_v(3, 5; 6; 16)$ are shown in Figure 2.2.
Figure 2.2: All 4 graphs in $H_w(3, 5; 6; 16)$
Corollary 2.10. \( F_v(3, 5; 6) = 16 \).

Proof. From Theorem 2.7 and (1.10) we obtain \( F_v(3, 5; 6) \geq 16 \). Since \( \mathcal{H}_v(3, 5; 6; 16) \neq \emptyset \), it follows that \( F_v(3, 5; 6) \leq 16 \). \( \square \)

Some properties of the graphs in \( \mathcal{H}_v(3, 5; 6; 16) \) are listed in Table 2.3.

It is interesting to note that for all these graphs the inequality (1.14) is strict. Interesting results for Folkman graphs for which we have equality in (1.14) are obtained in [99].

The graphs \( G_{50} \) and \( G_{146} \) are maximal and the other two graphs \( G_{51} \) and \( G_{81} \) are their subgraphs and are obtained by removing one edge. In [91] the inequality \( F_v(3, 5; 6) \leq 16 \) is proved with the help of the graph \( G_{146} \). Let us note that \( |Aut(G_{146})| = 96 \), and among all graphs in \( \mathcal{H}_v(2, 2, 5; 6; 16) \) it has the most automorphisms.

Theorem 2.4, Theorem 2.5, Theorem 2.6, Theorem 2.7 and Theorem 2.9 are published in [8].

### Table 2.3: Some properties of the graphs in \( \mathcal{H}_v(3, 5; 6; 16) \)

| Graph | \( |E(G)| \) | \( \delta(G) \) | \( \Delta(G) \) | \( \alpha(G) \) | \( \chi(G) \) | \( |Aut(G)| \) |
|-------|-----------|-------------|-------------|-------------|-------------|-------------|
| \( G_{50} \) | 87 | 10 | 12 | 3 | 8 | 6 |
| \( G_{51} \) | 86 | 9 | 11 | 3 | 8 | 6 |
| \( G_{81} \) | 87 | 10 | 11 | 2 | 8 | 6 |
| \( G_{146} \) | 88 | 11 | 11 | 2 | 8 | 96 |

2.3 Bounds on the numbers \( F_v(2r, p; r + p - 1) \)

The bounds from Theorem 1.19 are useful because in general they are easier to estimate and compute than the numbers \( F_v(a_1, \ldots, a_s) \) themselves. Further, we compute the exact value of the numbers \( F_v(2m-5, 5; m - 1) \) (see Theorem 2.17) and the numbers \( \tilde{F}_v(m|_5; m - 1) \) (see Theorem 2.20). Since \( F_v(2m-5, 5; m - 1) = \tilde{F}_v(m|_5; m - 1) \), with the help of Theorem 1.19 we prove Theorem 2.1.
In this section we prove that the computation of the lower bound in Theorem 1.19 in the case $q = m - 1$, i.e. the computation of the infinite sequence of numbers $F_v(2r, p, r + p - 1)$ where $p$ is fixed, is reduced to the computation of a finite number of these numbers (Theorem 2.12). It is easy to prove that

$$G \rightarrow (a_1, \ldots, a_s) \Rightarrow K_1 + G \rightarrow (2, a_1, \ldots, a_s).$$

Therefore,

$$F_v(2r, p; r + p - 1) \leq F_v(2s, p; s + p - 1) + r - s.$$ 

\[ \text{Lemma 2.11.} \quad \text{Let} \quad 2 \leq s \leq r. \quad \text{Then,} \]

$$F_v(2r, p; r + p - 1) \leq F_v(2s, p; s + p - 1) + r - s.$$ 

\[ \text{Proof.} \quad \text{Let} \quad G \quad \text{be an extremal graph in} \quad \mathcal{H}_v(2s, p; s + p - 1). \quad \text{Consider} \quad G_1 = K_{r-s} + G. \quad \text{According to (2.1),} \quad G_1 \rightarrow (2, p). \quad \text{Since} \quad \omega(G_1) = r - s + \omega(G) < r + p - 1, \quad \text{it follows that} \quad G_1 \in \mathcal{H}_v(2r, p; r + p - 1). \quad \text{Therefore,} \]

$$F_v(2r, p; r + p - 1) \leq |V(G_1)| = F_v(2s, p; s + p - 1) + r - s. \quad \Box$$

\[ \text{Theorem 2.12.} \quad \text{Let} \quad p \quad \text{be a fixed positive integer and} \quad r'_0(p) = r'_0 \quad \text{be the} \quad \text{smallest positive integer for which} \]

$$\min_{r \geq 2} \{F_v(2r, p; r + p - 1) - r\} = F_v(2r'_0, p; r'_0 + p - 1) - r'_0.$$ 

Then:

(a) $F_v(2r, p; r + p - 1) = F(2r'_0, p; r'_0 + p - 1) + r - r'_0, \quad r \geq r'_0.$

(b) If $r'_0 = 2$, then

$$F_v(2r, p; r + p - 1) = F_v(2, 2, p; p + 1) + r - 2, \quad r \geq 2.$$ 

(c) If $r'_0 > 2$ and $G$ is an extremal graph in $\mathcal{H}(2r'_0, p; r'_0 + p - 1)$, then

$$G \rightarrow (2, r'_0 + p - 2).$$

(d) $r'_0 < F_v(2, 2, p; p + 1) - 2p.$
Proof. (a) According to the definition of \( r'_0 = r'_0(p) \), if \( r \geq 2 \), then
\[
F_v(2, r, p; r + p - 1) - r \geq F_v(2, r'_0, p; r'_0 + p - 1) - r'_0,
\]
i.e.
\[
F_v(2, r, p; r + p - 1) \geq F_v(2, r'_0, p; r'_0 + p - 1) + r - r'_0.
\]
If \( r \geq r'_0 \), according to Lemma 2.11, the opposite inequality is also true.

(b) This equality is the special case \( r'_0 = 2 \) of the equality (a).

(c) Suppose the opposite is true, and let \( G \) be an extremal graph in \( \mathcal{H}_v(2, r'_0, p; r'_0 + p - 1) \) and \( V(G) = V_1 \cup V_2 \), \( V_1 \cap V_2 = \emptyset \), where \( V_1 \) is an independent set and \( V_2 \) does not contain an \( (r'_0 + p - 2) \)-clique. We can suppose that \( V_1 \neq \emptyset \). Let \( G_1 = G[V_2] \). Then \( \omega(G_1) < r + p - 2 \), and from \( G \supseteq (2, r'_0, p) \) and Proposition 1.4 it follows \( G_1 \nrightarrow (2, r'_0 - 1, p) \). Therefore, \( G_1 \in \mathcal{H}_v(2, r'_0 - 1, p; r'_0 + p - 2) \) and
\[
|V(G_1)| \geq F_v(2, r'_0 - 1, p; r'_0 + p - 2).
\]
Since \( |V(G)| = F_v(2, r'_0, p; r'_0 + p - 1) \) and \( |V(G_1)| \leq |V(G)| - 1 \), we obtain
\[
F_v(2, r'_0 - 1, p; r'_0 + p - 2) - (r'_0 - 1) \leq F_v(2, r'_0, p; r'_0 + p - 1) - r'_0,
\]
which contradicts the definition of \( r'_0 = r'_0(p) \).

(d) According to (1.8), \( F_v(2, 2, p; p + 1) \geq 2p + 4 \). Therefore, if \( r'_0 = 2 \), the inequality (d) is obvious.

Let \( r'_0 \geq 3 \) and \( G \) be an extremal graph in \( \mathcal{H}_v(2, r'_0, p; r'_0 + p - 1) \). According to (c), \( G \in \mathcal{H}_v(2, r'_0 + p - 2; r'_0 + p - 1) \), and by Theorem 1.2 \( |V(G)| \geq 2r'_0 + 2p - 3 \). Let us notice that \( \chi(C_{2r'_0 + 2p - 3}) = r'_0 + p - 1 \), but since \( G \nrightarrow (2, r'_0, p) \), from (1.14) it follows \( \chi(G) \geq r'_0 + p = m \). Therefore, \( G \neq C_{2r'_0 + 2p - 3} \) and from Theorem 1.2 we obtain
\[
|V(G)| = F_v(2, r'_0, p; r'_0 + p - 1) \geq 2r'_0 + 2p - 2.
\]
Since \( r'_0 \geq 3 \), from the definition of \( r'_0 \) we have
\[
F_v(2, r'_0, p; r'_0 + p - 1) < F_v(2, 2, p; p + 1) + r'_0 - 2.
\]
Thus, we proved that
\[
2r'_0 + 2p - 2 < F_v(2, 2, p; p + 1) + r'_0 - 2, \text{ i.e.}
\]
\[
r'_0 < F_v(2, 2, p; p + 1) - 2p.
\]
\( \square \)

Remark 2.13. Since we suppose that \( r \geq 2 \), according to (1.2) all Folkman numbers in the proof of Theorem 2.12 exist.

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Corollary 2.14. Let $a_1, \ldots, a_s$ be positive integers, let $m$ and $p$ be defined by (1.6), $m \geq p + 2$ and $r = m - p \geq r'_0(p)$. Then,

$$F_v(a_1, \ldots, a_s; m - 1) \geq F_v(2r'_0, p; r'_0 + p - 1) + r - r'_0.$$  

In particular, if $r'_0 = 2$, then

$$F_v(a_1, \ldots, a_s; m - 1) \geq F_v(2, 2, p; p + 1) + r - 2.$$

Proof. According to Theorem 1.19

$$F_v(a_1, \ldots, a_s; m - 1) \geq F_v(2r, p; r + p - 1).$$

From this inequality and Theorem 2.12(a) we obtain the desired inequality. 

Example 2.15. From (1.9) and $F_v(2, 2, 2; 3) = F_v(2, 2, 2; 4) = 11$ it follows $r'_0(2) = 4$, and from (1.9) and $F_v(2, 2, 3; 4) = 14$ it follows $r'_0(3) = 3$. Also, from (1.9) we see that $r'_0(4) = 2$.

We suppose that the following is true:

Conjecture 2.16. Let $p \geq 4$ be a fixed integer. Then,

$$\min_{r \geq 2} \{F_v(2r, p; r + p - 1) - r\} = F_v(2, 2, p; p + 1) - 2,$$

i.e. $r'_0(p) = 2$, and

$$F_v(2r, p; r + p - 1) = F_v(2, 2, p; p + 1) + r - 2, \quad r \geq 2.$$

It is not difficult to see that Conjecture 2.16 is true if and only if for fixed $p$ the sequence $\{F_v(2r, p; r + p - 1)\}$ is strictly increasing with respect to $r \geq 2$. From (1.9) we have $r'_0(4) = 2$. Further, we prove that Conjecture 2.16 is also true in the cases $p = 5$ (Theorem 2.17), $p = 6$ (Theorem 3.14) and $p = 7$ (Theorem 4.8).

Theorem 2.12 is published in [8].
2.4 Computation of the numbers

\[ F_v(2m-5, 5; m - 1) \]

In this section we prove the following

Theorem 2.17. \( r'_0(5) = 2 \) and \( F_v(2m-5, 5; m - 1) = m + 9, \) \( m \geq 7. \)

Proof. First, we will prove that \( r'_0(5) = 2. \) From Theorem 2.12(d) we have \( r'_0(5) \leq 5. \) Therefore, we have to prove that \( r'_0(5) \neq 3, r'_0(5) \neq 4 \) and \( r'_0(5) \neq 5, \) i.e. we have to prove the inequalities \( F_v(2, 2, 5; 7) > 16, \) \( F_v(2, 2, 2, 5; 8) > 17, F_v(2, 2, 2, 2, 5; 9) > 18. \)

According to Lemma 2.11 it is enough to prove the last inequality. We will prove all three inequalities, because further we will use the results of these computations to check the correctness of Algorithm A2.

1. Proof of \( F_v(2, 2, 2, 5; 7) > 16. \)

Using nauty [53] we generate all 10-vertex non-isomorphic graphs and among them we find all graphs in \( H_{\text{max}}(5; 7; 10). \)

We execute Algorithm A1 \( (n = 12; r = 1; p = 5; q = 7; k = 2) \) with input \( A = H_{\text{max}}(5; 7; 10) \) to obtain all graphs in \( B = H_{\text{max}}(2, 2, 5; 7; 12). \) (see Remark 2.3)

We execute Algorithm A1 \( (n = 14; r = 2; p = 5; q = 7; k = 2) \) with input \( A = H_{\text{max}}(2, 5; 7; 12) \) to obtain all graphs in \( B = H_{\text{max}}(2, 2, 2, 5; 14). \)

By executing Algorithm A1 \( (n = 16; r = 3; p = 5; q = 7; k = 2) \) with input \( A = H_{\text{max}}(2, 2, 5; 7; 14) \), we obtain \( B = \emptyset. \) According to Theorem 2.2 \( H_v(2, 2, 2, 5; 7; 16) = \emptyset, \) and therefore \( F_v(2, 2, 2, 5; 7) > 16. \)

The number of graphs obtained on each step of the proof is given in Table 2.4

2. Proof of \( F_v(2, 2, 2, 5; 8) > 17. \)

Using nauty [53] we generate all 9-vertex non-isomorphic graphs and among them we find all graphs in \( H_{\text{max}}(5; 8; 9). \) Actually, it is easy to see that \( H_{\text{max}}(5; 8; 9) = \{ K_3 + K_6, C_4 + K_5 \}. \)

We execute Algorithm A1 \( (n = 11; r = 1; p = 5; q = 8; k = 2) \) with
input $A = \mathcal{H}_{\text{max}}(5; 8; 9)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}(2, 5; 8; 11)$ (see Remark 2.3).

We execute Algorithm A1($n = 13; r = 2; p = 5; q = 8; k = 2$) with input $A = \mathcal{H}_{\text{max}}(2, 5; 8; 11)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}(2, 2, 5; 8; 13)$.

We execute Algorithm A1($n = 15; r = 3; p = 5; q = 8; k = 2$) with input $A = \mathcal{H}_{\text{max}}(2, 2, 5; 8; 13)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}(2, 2, 2, 5; 8; 15)$.

By executing Algorithm A1($n = 17; r = 4; p = 5; q = 8; k = 2$) with input $A = \mathcal{H}_{\text{max}}(2, 2, 2, 2, 5; 8; 15)$, we obtain $B = \emptyset$. According to Theorem 2.2, $\mathcal{H}_v(2, 2, 2, 2, 2, 2, 5; 8; 17) = \emptyset$, and therefore $F_v(2, 2, 2, 2, 2; 5; 8) > 17$.

The number of graphs obtained on each step of the proof is given in Table 2.5.

3. Proof of $F_v(2, 2, 2, 2, 2, 5; 9) > 18$.

Using nauty [53] we generate all 10-vertex non-isomorphic graphs and among them we find all graphs in $\mathcal{H}_{\text{max}}(2, 5; 9; 10)$. Actually, it is easy to see that $\mathcal{H}_{\text{max}}(2, 5; 9; 10) = \{K_3 + K_7, C_4 + K_6\}$.

We execute Algorithm A1($n = 12; r = 2; p = 5; q = 9; k = 2$) with input $A = \mathcal{H}_{\text{max}}(2, 5; 9; 10)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}(2, 2, 5; 9; 12)$ (see Remark 2.3).

We execute Algorithm A1($n = 14; r = 3; p = 5; q = 9; k = 2$) with input $A = \mathcal{H}_{\text{max}}(2, 2, 5; 9; 12)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}(2, 2, 2, 5; 9; 14)$.

We execute Algorithm A1($n = 16; r = 4; p = 5; q = 9; k = 2$) with input $A = \mathcal{H}_{\text{max}}(2, 2, 2, 5; 9; 14)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}(2, 2, 2, 2, 5; 9; 16)$.

By executing Algorithm A1($n = 18; r = 5; p = 5; q = 9; k = 2$) with input $A = \mathcal{H}_{\text{max}}(2, 2, 2, 2, 5; 9; 16)$, we obtain $B = \emptyset$. According to Theorem 2.2, $\mathcal{H}_v(2, 2, 2, 2, 2, 5; 9; 18) = \emptyset$, and therefore $F_v(2, 2, 2, 2, 2, 5; 9) > 18$.

The number of graphs obtained on each step of the proof is given in Table 2.6.

We proved that $r'_0(5) = 2$. Now, from Theorem 2.12(b) we obtain

$$F_v(2m-5, 5; m - 1) = m + 9, \ m \geq 7.$$ 

Thus, the proof of Theorem 2.17 is finished. \hfill \Box

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All computations were done on a personal computer. The slowest part was the proof of $F_v(2, 2, 2, 2, 5; 9) > 18$, which took several days to complete.

Theorem 2.17 is published in [6].

### 2.5 Algorithm A2

Algorithm A1 is related to the special Folkman numbers of the form $F_v(2_r, p; q; n)$. In this section we present Algorithm A2 with the help of which we can compute and obtain bounds on arbitrary Folkman numbers of the form $F_v(a_1, ..., a_s; q)$. This Algorithm A2 outputs all graphs $G \in \mathcal{H}_{\max}(a_1, ..., a_s; q; n)$ with $k \leq \alpha(G) \leq t$. Let us remind that the set of all graphs $G \in \mathcal{H}_{\max}(a_1, ..., a_s; q; n)$ with $\alpha(G) \leq t$ is denoted by $\mathcal{H}_{\max}^t(a_1, ..., a_s; q; n)$.  

| set                     | maximal graphs | $(+K_6)$-graphs |
|-------------------------|----------------|-----------------|
| $\mathcal{H}_v(5; 7; 10)$ | 8              | 324             |
| $\mathcal{H}_v(2, 5; 7; 12)$ | 56             | 104 283         |
| $\mathcal{H}_v(2, 2, 5; 7; 14)$ | 420            | 2 614 547       |
| $\mathcal{H}_v(2, 2, 2, 5; 7; 16)$ | 0              | 0               |

Table 2.4: Steps in the proof of $\mathcal{H}_v(2, 2, 5; 7; 16) = \emptyset$

| set                     | maximal graphs | $(+K_7)$-graphs |
|-------------------------|----------------|-----------------|
| $\mathcal{H}_v(5; 8; 9)$ | 2              | 13              |
| $\mathcal{H}_v(2, 5; 8; 11)$ | 8              | 326             |
| $\mathcal{H}_v(2, 2, 5; 8; 13)$ | 56             | 105 138         |
| $\mathcal{H}_v(2, 2, 2, 5; 8; 15)$ | 423            | 2 616 723       |
| $\mathcal{H}_v(2, 2, 2, 2, 5; 8; 17)$ | 0              | 0               |

Table 2.5: Steps in the proof of $\mathcal{H}_v(2, 2, 2, 5; 8; 17) = \emptyset$

| set                     | maximal graphs | $(+K_8)$-graphs |
|-------------------------|----------------|-----------------|
| $\mathcal{H}_v(2, 2, 5; 9; 10)$ | 2              | 13              |
| $\mathcal{H}_v(2, 2, 5; 9; 12)$ | 8              | 327             |
| $\mathcal{H}_v(2, 2, 2, 5; 9; 14)$ | 56             | 105 294         |
| $\mathcal{H}_v(2, 2, 2, 2, 5; 9; 16)$ | 423            | 2 616 741       |
| $\mathcal{H}_v(2, 2, 2, 2, 2, 5; 9; 18)$ | 0              | 0               |

Table 2.6: Steps in the proof of $\mathcal{H}_v(2, 2, 2, 2, 2, 5; 9; 18) = \emptyset$
Algorithm A2. The input of the algorithm is the set $\mathcal{A} = \mathcal{H}_{\text{max}}^t(a_1 - 1, a_2, ..., a_s; q; n-k)$, where $a_1, ..., a_s, q, n, k, t$ are fixed positive integers, $a_1 \geq 2$ and $k \leq t$.

The output of the algorithm is the set $\mathcal{B}$ of all graphs $G \in \mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$ with $\alpha(G) \geq k$.

1. By removing edges from the graphs in $\mathcal{A}$ obtain the set $\mathcal{A}' = \mathcal{H}_{\text{max}}^t(a_1 - 1, a_2, ..., a_s; q; n-k)$.

2. For each graph $H \in \mathcal{A}'$:

   2.1. Find the family $\mathcal{M}(H) = \{M_1, ..., M_l\}$ of all maximal $K_{q-1}$-free subsets of $V(H)$.

   2.2. For each $k$-element multiset $N = \{M_{i_1}, ..., M_{i_k}\}$ of elements of $\mathcal{M}(H)$ construct the graph $G = G(N)$ by adding new independent vertices $v_1, ..., v_k$ to $V(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, ..., k$. If $\alpha(G) \leq t$ and $\omega(G + e) = q, \forall e \in E(G)$, then add $G$ to $\mathcal{B}$.

3. Remove the isomorphic copies of graphs from $\mathcal{B}$.

4. Remove from $\mathcal{B}$ all graphs $G$ for which $G \not\rightarrow (a_1, ..., a_s)$.

Clearly, the special case $a_1 = ... = a_{s-1} = 2, a_s = p$ of Algorithm A2 for sufficiently large $t$ coincides with Algorithm A1. In this sense, Algorithm A2 is a generalization of Algorithm A1.

Theorem 2.18. After the execution of Algorithm A2, the obtained set $\mathcal{B}$ coincides with the set of all graphs $G \in \mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$ with $\alpha(G) \geq k$.

Proof. Naturally, the proof follows the same pattern as the proof of Theorem 2.2. Suppose that after the execution of Algorithm A2 the graph $G \in \mathcal{B}$. Then $G = G(N)$ where $N$ and the following notations are the same as in step 2.2. Since $H = G - \{v_1, ..., v_k\} \in \mathcal{A}'$, we have $\omega(H) < q$. Since $N_G(v_i)$ are $K_{q-1}$-free sets for all $i \in \{1, ..., k\}$, it follows that $\omega(G) < q$. The two checks at the end of step 2.2 guarantees that $G$ is a maximal $K_q$-free graph and $\alpha(G) \leq t$. The check in step 4 guarantees that $G \not\rightarrow (a_1, ..., a_s)$. We obtained $G \in \mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$. Since the vertices $v_1, ..., v_k$ are independent, it follows that $\alpha(G) \geq k$.

Let $G \in \mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$ and $\alpha(G) \geq k$. We will prove that, after the execution of Algorithm A2, $G \in \mathcal{B}$. Let $v_1, ..., v_k$ be independent vertices in...
G and $H = G - \{v_1, ..., v_k\}$. Using Proposition 1.10 we derive that $H \in \mathcal{A}'$. Since $G$ is a maximal $K_q$-free graph, $N_G(v_i)$ are maximal $K_{q-1}$-free subsets of $V(H)$ for all $i \in \{1, ..., k\}$, and therefore $N_G(v_i) \in \mathcal{M}(H)$, see step 2.1. Thus, $G = G(N)$ where $N = \{N_G(v_1), ..., N_G(v_k)\}$ and in step 2.2 $G$ is added to $\mathcal{B}$. Clearly, after step 4 the graph $G$ remains in $\mathcal{B}$.

\[\square\]

**Remark 2.19.** Note that if $G \in \mathcal{H}^\ell_{\max}(a_1, ..., a_s; q; n)$ and $n \geq q$, then $G$ is not a complete graph and $\alpha(G) \geq 2$. Therefore, if $n \geq q$ and $k = 2$, Algorithm $A2$ finds all graphs in $\mathcal{H}^\ell_{\max}(a_1, ..., a_s; q; n)$.

We will demonstrate the advantages of Algorithm $A2$ by giving a second proof of Theorem 2.17, which requires less computational time compared to Algorithm $A1$.

**Second proof of Theorem 2.17**

As we already showed in the first proof of Theorem 2.17, it is enough to prove the inequality $F_{\alpha}(2, 2, 2, 2, 5; 9) > 18$.

Suppose that $\mathcal{H}_{\max}(2, 2, 2, 2, 5; 9; 18) \neq \emptyset$ and let $G \in \mathcal{H}_{\max}(2, 2, 2, 2, 5; 9; 18)$. It is clear that $\alpha(G) \geq 2$. From Theorem 1.5(b) it follows that $\alpha(G) \leq 3$. Using Algorithm $A2$ we will prove separately that there are no graphs with independence number 2 and no graphs with independence number 3 in $\mathcal{H}_{\max}(2, 2, 2, 2, 5; 9; 18)$.

First, we prove that there are no graphs in $\mathcal{H}_{\max}(2, 2, 2, 2, 5; 9; 18)$ with independence number 3:

The only graph in $\mathcal{H}_{\max}(5; 9; 7)$ is $K_7$.

We execute Algorithm $A2$ with input $A = \mathcal{H}^3_{\max}(1, 5; 9; 7) = \mathcal{H}^3_{\max}(5; 9; 7) = \{K_7\}$ to obtain all graphs in $B = \mathcal{H}^3_{\max}(2, 5; 9; 9)$. (see Remark 2.19)

We execute Algorithm $A2$ with input $A = \mathcal{H}^3_{\max}(1, 2, 5; 9; 9) = \mathcal{H}^3_{\max}(2, 5; 9; 9)$ to obtain all graphs in $B = \mathcal{H}^3_{\max}(2, 2, 5; 9; 11)$.

We execute Algorithm $A2$ with input $A = \mathcal{H}^3_{\max}(1, 2, 2, 5; 9; 11) = \mathcal{H}^3_{\max}(2, 2, 5; 9; 11)$ to obtain all graphs in...
\[ B = \mathcal{H}^3_{\text{max}}(2, 2, 2, 5; 9; 13). \]

We execute Algorithm A2 \((n = 15; \ k = 2; \ t = 3)\) with input \( A = \mathcal{H}^3_{\text{max}}(1, 2, 2, 2, 5; 9; 13) = \mathcal{H}^3_{\text{max}}(2, 2, 2, 5; 9; 13) \) to obtain all graphs in \( B = \mathcal{H}^3_{\text{max}}(2, 2, 2, 5; 9; 13) \).

By executing Algorithm A2 \((n = 18; \ k = 3; \ t = 3)\) with input \( A = \mathcal{H}^3_{\text{max}}(1, 2, 2, 2, 5; 9; 15) = \mathcal{H}^3_{\text{max}}(2, 2, 2, 5; 9; 15) \), we obtain \( B = \emptyset \).

On the other hand, according to Theorem 2.18, \( B \) consists of all graphs with independence number 3 in \( \mathcal{H}^3_{\text{max}}(2, 2, 2, 2, 5; 9; 18) \). We obtained that there are no graphs in \( \mathcal{H}^3_{\text{max}}(2, 2, 2, 2, 5; 9; 18) \) with independence number 3.

It remains to be proved that there are no graphs in \( \mathcal{H}^3_{\text{max}}(2, 2, 2, 2, 5; 9; 18) \) with independence number 2:

The only graph in \( \mathcal{H}^2_{\text{max}}(5; 9; 8) \) is \( K_8 \).

We execute Algorithm A2 \((n = 10; \ k = 2; \ t = 2)\) with input \( A = \mathcal{H}^2_{\text{max}}(1, 5; 9; 8) = \mathcal{H}^2_{\text{max}}(5; 9; 8) = \{K_8\} \) to obtain all graphs in \( B = \mathcal{H}^2_{\text{max}}(2, 5; 9; 10) \). (see Remark 2.19)

We execute Algorithm A2 \((n = 12; \ k = 2; \ t = 2)\) with input \( A = \mathcal{H}^2_{\text{max}}(1, 2, 5; 9; 10) = \mathcal{H}^2_{\text{max}}(2, 5; 9; 10) \) to obtain all graphs in \( B = \mathcal{H}^2_{\text{max}}(2, 2, 5; 9; 12) \).

We execute Algorithm A2 \((n = 14; \ k = 2; \ t = 2)\) with input \( A = \mathcal{H}^2_{\text{max}}(1, 2, 2, 5; 9; 12) = \mathcal{H}^2_{\text{max}}(2, 2, 5; 9; 12) \) to obtain all graphs in \( B = \mathcal{H}^2_{\text{max}}(2, 2, 2, 5; 9; 14) \).

We execute Algorithm A2 \((n = 16; \ k = 2; \ t = 2)\) with input \( A = \mathcal{H}^2_{\text{max}}(1, 2, 2, 2, 5; 9; 14) = \mathcal{H}^2_{\text{max}}(2, 2, 2, 5; 9; 14) \) to obtain all graphs in \( B = \mathcal{H}^2_{\text{max}}(2, 2, 2, 2, 5; 9; 16) \).

By executing Algorithm A2 \((n = 18; \ k = 2; \ t = 2)\) with input \( A = \mathcal{H}^2_{\text{max}}(1, 2, 2, 2, 2, 5; 9; 16) = \mathcal{H}^2_{\text{max}}(2, 2, 2, 2, 5; 9; 16) \), we obtain \( B = \emptyset \).

On the other hand, according to Theorem 2.18, \( B \) consists of all graphs with independence number 2 in \( \mathcal{H}^2_{\text{max}}(2, 2, 2, 2, 5; 9; 18) \). We obtained that there are no graphs in \( \mathcal{H}^2_{\text{max}}(2, 2, 2, 2, 5; 9; 18) \) with independence number 2.

Thus, we proved \( \mathcal{H}^2_{\text{max}}(2, 2, 2, 2, 2, 5; 9; 18) = \emptyset \), and therefore \( F_v(2, 2, 2, 2, 2, 5; 9) > 18 \). \qed
The number of graphs obtained in each step of the proof is given in Table 2.9.

Let us note that the fact that there are no graphs in $\mathcal{H}_{\max}(2,2,2,2,5;9;18)$ with independence number greater than 3 can be proved without using Theorem 1.5(b), but by applying Algorithm A2 with $k > 3$ instead. We performed this check to test Algorithm A2.

We also used Algorithm A2 to give similar second proofs of the inequalities $F_v(2,2,2,5;7) > 16$ and $F_v(2,2,2,5;8) > 17$, see Table 2.7 and Table 2.8 respectively. We compared the results of the computations to the results obtained in the previous proofs of these inequalities to check the correctness of our implementation of Algorithm A1 and Algorithm A2. To demonstrate the advantages of Algorithm A2 we will note that the slowest step in the proof of Theorem 2.17 with Algorithm A1 is finding all 2 616 741 graphs in $\mathcal{H}_+K_8(2,2,2,2,5;9;16)$ (Table 2.6), while the slowest step using Algorithm A3 is finding all 230 370 graphs with independence number 2 in $\mathcal{H}_+K_8(2,2,2,2,5;9;16)$ (Table 2.9). Thus, the total computational time needed for the proof of Theorem 2.17 is reduced from several days to several hours.

We see that if the independence number of the emerging graphs is taken into account, we obtain a faster algorithm. In Algorithm A2 we account for the independence number with the check $\alpha(G) \leq t$ in step 2.2. In some of the following problems this is not effective enough. Therefore, in the next Algorithm A3 the check $\alpha(G) \leq t$ is replaced with other conditions (see step 2.2 of Algorithm A3). In this thesis Algorithm A2 has no significance by itself, but it helps to obtain Algorithm A3 in a natural way.

Algorithm A2 is a simplified version of Algorithm 3.7 in [7]. Theorem 2.18 follows from Theorem 3.8 published in [7].
Table 2.7: Steps in finding all maximal graphs in \( H_v(2, 2, 2, 5; 7; 16) \)

| set               | independence number | maximal graphs | \((+K_6)\)-graphs |
|-------------------|---------------------|----------------|------------------|
| \( H_v(3; 7; 5)  \) | \( \leq 3 \)         | 1              | 1                |
| \( H_v(4; 7; 7)  \) | \( \leq 3 \)         | 1              | 4                |
| \( H_v(5; 7; 9)  \) | \( \leq 3 \)         | 3              | 45               |
| \( H_v(2, 5; 7; 11) \) | \( \leq 3 \)       | 12             | 3 036            |
| \( H_v(2, 2, 5; 7; 13) \) | \( \leq 3 \)     | 14             | 1 120            |
| \( H_v(2, 2, 2, 5; 7; 16) \) | = 3             | 0              |                  |
| \( H_v(3; 7; 6)  \) | \( \leq 2 \)         | 1              | 1                |
| \( H_v(4; 7; 8)  \) | \( \leq 2 \)         | 1              | 8                |
| \( H_v(5; 7; 10) \) | \( \leq 2 \)         | 3              | 82               |
| \( H_v(2, 5; 7; 12) \) | \( \leq 2 \)       | 10             | 5 046            |
| \( H_v(2, 2, 5; 7; 14) \) | \( \leq 2 \)     | 84             | 229 077          |
| \( H_v(2, 2, 2, 5; 7; 16) \) | = 2             | 0              |                  |

Table 2.8: Steps in finding all maximal graphs in \( H_v(2, 2, 2, 5; 8; 17) \)

| set               | independence number | maximal graphs | \((+K_7)\)-graphs |
|-------------------|---------------------|----------------|------------------|
| \( H_v(4; 8; 6)  \) | \( \leq 3 \)         | 1              | 1                |
| \( H_v(5; 8; 8)  \) | \( \leq 3 \)         | 1              | 4                |
| \( H_v(2, 5; 8; 10) \) | \( \leq 3 \)       | 3              | 45               |
| \( H_v(2, 2, 5; 8; 12) \) | \( \leq 3 \)     | 12             | 3 068            |
| \( H_v(2, 2, 5; 8; 14) \) | \( \leq 3 \)     | 14             | 1 121            |
| \( H_v(2, 2, 2, 5; 8; 17) \) | = 3             | 0              |                  |
| \( H_v(4; 8; 7)  \) | \( \leq 2 \)         | 1              | 1                |
| \( H_v(5; 8; 9)  \) | \( \leq 2 \)         | 1              | 8                |
| \( H_v(2, 5; 8; 11) \) | \( \leq 2 \)       | 3              | 84               |
| \( H_v(2, 2, 5; 8; 13) \) | \( \leq 2 \)     | 10             | 5 380            |
| \( H_v(2, 2, 5; 8; 15) \) | \( \leq 2 \)     | 87             | 230 356          |
| \( H_v(2, 2, 2, 2, 5; 8; 17) \) | = 2             | 0              |                  |

Table 2.9: Steps in finding all maximal graphs in \( H_v(2, 2, 2, 2, 5; 9; 18) \)

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2.6 Computation of the numbers

\[ \widetilde{F}_v(m \mid 5; m - 1) \]

Let us remind that \( \widetilde{H}_v(m \mid p; q) \) and \( \widetilde{F}_v(m \mid p; q) \) are defined in Section 1.4. According to Proposition 1.17, we have

\[ (2.2) \quad \widetilde{F}_v(m \mid 5; m - 1) \text{ exists } \Leftrightarrow m \geq 7. \]

We prove the following

**Theorem 2.20.** The following equalities are true:

\[ \widetilde{F}_v(m \mid 5; m - 1) = \begin{cases} 17, & \text{if } m = 7 \\ m + 9, & \text{if } m \geq 8. \end{cases} \]

**Proof.** Case 1. \( m = 7 \). According to Theorem 1.19 and Theorem 2.7, \( \widetilde{F}_v(7 \mid 5; 6) \geq F_v(2, 2, 5; 6) = 16 \). With the help of the computer we check that none of the 4 graphs in \( \mathcal{H}_v(3, 5; 6; 16) \) (see Figure 2.2) belongs to \( \mathcal{H}_v(4, 4; 6; 16) \). Therefore, \( \mathcal{H}_v(7 \mid 5; 6; 16) = \emptyset \) and \( \widetilde{F}_v(7 \mid 5; 6) \geq 17 \).

By adding one vertex to the graphs in \( \mathcal{H}_v(2, 2, 5; 6; 16) \), and then removing edges from the obtained 17-vertex graphs, we find 353 graphs which belong to both \( \mathcal{H}_v(3, 5; 6; 17) \) and \( \mathcal{H}_v(4, 4; 6; 17) \). The graph \( \Gamma_1 \), given in Figure 2.3 is one of these graphs (it is the only one with independence number 4). We will prove that \( \Gamma_1 \in \widetilde{\mathcal{H}}_v(7 \mid 5; 6) \). Since \( \omega(\Gamma_1) = 5 \), it remains to be proved that if \( 2 \leq b_1 \leq ... \leq b_s \leq 5 \) (see (1.5)) are positive integers such that \( \sum_{i=1}^{s} (b_i - 1) + 1 = 7 \), then \( \Gamma_1 \not\rightarrow (b_1, ..., b_s) \). The following cases are possible:

\begin{align*}
    s &= 2: \quad b_1 = 3, b_2 = 5. \\
    s &= 2: \quad b_1 = 4, b_2 = 4. \\
    s &= 3: \quad b_1 = 2, b_2 = 2, b_3 = 5. \\
    s &= 3: \quad b_1 = 2, b_2 = 3, b_3 = 4. \\
    s &= 3: \quad b_1 = 3, b_2 = 3, b_3 = 3. \\
    s &= 4: \quad b_1 = 2, b_2 = 2, b_3 = 2, b_4 = 4. \\
    s &= 4: \quad b_1 = 2, b_2 = 2, b_3 = 3, b_4 = 3. \\
    s &= 5: \quad b_1 = 2, b_2 = 2, b_3 = 2, b_4 = 2, b_5 = 3. \\
    s &= 6: \quad b_1 = 2, b_2 = 2, b_3 = 2, b_4 = 2, b_5 = 2, b_6 = 2.
\end{align*}
According to Proposition 1.16, from $\Gamma_1 \rightarrow (3, 5)$ and $\Gamma_1 \rightarrow (4, 4)$ it follows $\Gamma_1 \rightarrow 7\mid_5$. We proved that $\Gamma_1 \in \tilde{\mathcal{H}}_v(7\mid_5; 6)$. Therefore, $	ilde{F}_v(7\mid_5; 6) \leq |V(\Gamma_1)| = 17$.

Case 2. $m = 8$. According to Theorem 1.19 and Theorem 2.17, $	ilde{F}_v(8\mid_5; 7) \geq F_v(2, 2, 5; 7) = 17$. To prove the upper bound, consider the 17-vertex graph $\Gamma_2 \in \mathcal{H}_v(4, 5; 7; 17)$, which is given in Figure 2.4. The method to obtain this graph is described below. By construction, $\omega(\Gamma_2) = 6$ and $\Gamma_2 \rightarrow (4, 5)$. According to Proposition 1.16, from $\Gamma_2 \rightarrow (4, 5)$ it follows $\Gamma_2 \rightarrow 8\mid_5$. Therefore, $\Gamma_2 \in \tilde{\mathcal{H}}_v(8\mid_5; 7)$ and $	ilde{F}_v(8\mid_5; 7) \leq |V(\Gamma_2)| = 17$.

Case 3. $m > 8$. From Theorem 1.19 and Theorem 2.17 it follows $\tilde{F}_v(m\mid_5; m-1) \geq m + 9$. From Theorem $\tilde{F}_v(m\mid_5; m-1) = 17$ it follows $\tilde{F}_v(m\mid_5; m-1) \leq m + 9$.

Obtaining the graph $\Gamma_2 \in \mathcal{H}_v(4, 5; 7; 17)$

Consider the 18-vertex graph $\Gamma_3$ (Figure 2.5). As mentioned, this is the graph with the help of which in [91] they prove the inequality $F_v(3, 6; 7) \leq 18$. With computer we check that $\Gamma_3$ is a maximal graph in $\mathcal{H}_v(4, 5; 7; 18)$. We will use the following procedure to obtain other maximal graphs in $\mathcal{H}_v(4, 5; 7; 18)$:

Procedure 2.21. Extending a set of maximal graphs in $\mathcal{H}_v(a_1, \ldots, a_s; q; n)$.

1. Let $\mathcal{A}$ be a set of maximal graphs in $\mathcal{H}_v(a_1, \ldots, a_s; q; n)$.

2. By removing edges from the graphs in $\mathcal{A}$, find all their subgraphs which are in $\mathcal{H}_v(a_1, \ldots, a_s; q; n)$. This way a set of non-maximal graphs in $\mathcal{H}_v(a_1, \ldots, a_s; q; n)$ is obtained.

3. Add edges to the non-maximal graphs to find all their supergraphs which are maximal in $\mathcal{H}_v(a_1, \ldots, a_s; q; n)$. Extend the set $\mathcal{A}$ by adding the new maximal graphs.
Starting from a set containing a single element the graph $\Gamma_3$ and executing Procedure 2.21 we find 12 new maximal graphs in $H_v(4,5;7;18)$. Again, we execute Procedure 2.21 on the new set to find 110 more maximal graphs in $H_v(4,5;7;18)$. By removing one vertex from these graphs, we obtain 17-vertex graphs, one of which is $\Gamma_2 \in H_v(4,5;7;17)$ given in Figure 2.4.

Theorem 2.20 and Procedure 2.21 are published in [8].

2.7 Proof of Theorem 2.1

Since $m \geq 7$, only the following two cases are possible:

Case 1. $m = 7$. In this case $F_v(2,2,5;6)$ and $F_v(3,5;6)$ are the only canonical vertex Folkman numbers of the form $F_v(a_1,\ldots,a_s;m−1)$. The equality $F_v(2,2,5;6) = 16$ is proved in Theorem 2.7 and the equality $F_v(3,5;6) = 16$ is proved in [89] (see also Corollary 2.10).

Case 2. $m \geq 8$. According to Theorem 2.20 and Theorem 1.19($q = m − 1$), $F_v(a_1,\ldots,a_s;m−1) \leq m + 9$. From Theorem 2.17 and Theorem 1.19($q = m − 1$) it follows that $F_v(a_1,\ldots,a_s;m−1) \geq m + 9$.

Theorem 2.1 is published in [8].
Figure 2.3: Graph $\Gamma_1 \in \mathcal{H}(3, 5; 6; 17) \cap \mathcal{H}(4, 4; 6; 17)$

Figure 2.4: Graph $\Gamma_2 \in \mathcal{H}_v(4, 5; 7; 17)$

Figure 2.5: Graph $\Gamma_3 \in \mathcal{H}_v(3, 6; 7; 18) \cap \mathcal{H}_v(4, 5; 7; 18)$ from [91]
Chapter 3

Computation of the numbers

$F_v(a_1, \ldots, a_s; m - 1)$ where $\max\{a_1, \ldots, a_s\} = 6$

In this chapter we will prove the following main result:

**Theorem 3.1.** Let $a_1, \ldots, a_s$ be positive integers such that

$$2 \leq a_1 \leq \ldots \leq a_s = 6,$$

and $m = \sum_{i=1}^{s}(a_i - 1) + 1 \geq 8$. Then:

(a) $F_v(a_1, \ldots, a_s; m - 1) = m + 9$, if $a_1 = \ldots = a_{s-1} = 2$.

(b) $F_v(a_1, \ldots, a_s; m - 1) = m + 10$, if $a_{s-1} \geq 3$.

According to (1.7), the condition $m \geq 8$ in Theorem 3.1 is necessary. Since $F_v(a_1, \ldots, a_s; q)$ is a symmetric function of $a_1, \ldots, a_s$, in Theorem 3.1 we actually compute all numbers of the form $F_v(a_1, \ldots, a_s; m - 1)$ where $\max\{a_1, \ldots, a_s\} = 6$.

### 3.1 Algorithm A3

We will present an optimization to Algorithm A2. The optimized algorithm is based on the following propositions:
Proposition 3.2. Let $G$ be a maximal $K_q$-free graph and $u \neq v$ are non-adjacent vertices of $G$. Then,

$$K_{q-2} \subseteq N_G(u) \cap N_G(v).$$

Proof. Otherwise, $G + [u, v]$ would not contain $K_q$. \hfill \Box

Proposition 3.3. Let $A$ be an independent set of vertices of $G$, $H = G - A$, and $t$ be a positive integer such that $t \geq |A|$. Then,

$$\alpha(G) \leq t \iff \alpha(H - \bigcup_{v \in A'} N_G(v)) \leq t - |A'|, \ \forall A' \subseteq A.$$

Proof. Let $\alpha(G) \leq t$. Suppose that for some $A' \subseteq A$ we have $\alpha(H - \bigcup_{v \in A'} N_G(v)) > t - |A'|$. Consequently, there exists an independent set $A''$ of vertices of $H - \bigcup_{v \in A'} N_G(v)$ such that $|A''| > t - |A'|$. We obtained that the independent set $A' \cup A''$ has more than $t$ vertices, which is a contradiction.

Now, let $\alpha(H - \bigcup_{v \in A'} N_G(v)) \leq t - |A'|, \ \forall A' \subseteq A$. Let $\tilde{A}$ be an independent set of vertices of $G$ and $|\tilde{A}| = \alpha(G)$. Then $\tilde{A} = A_1 \cup A_2$ where $A_1 \subseteq A$ and $A_2$ is an independent set in $H - \bigcup_{v \in A_1} N_G(v)$. Since $|A_2| \leq \alpha(H - \bigcup_{v \in A_1} N_G(v)) \leq t - |A_1|$, we obtain $\alpha(G) = |\tilde{A}| = |A_1| + |A_2| \leq t$. \hfill \Box

In some cases, a possible drawback of Algorithm A2 is that in step 2.2 a large number ($l^k$) of multisets $N$ are considered, and therefore for a large number of graphs $G = G(N)$ it is checked if $\alpha(G) \leq t$. In the following Algorithm A3 we modify step 2 in such a way that a smaller number of multisets $N$ are considered and it is guaranteed that the constructed graphs $G = G(N)$ satisfy the condition $\alpha(G) \leq t$.

Algorithm A3. The input of the algorithm is the set $A = \mathcal{H}_{max}^t(a_1 - 1, a_2, ..., a_s; q; n-k)$, where $a_1, ..., a_s, q, n, k, t$ are fixed positive integers, $a_1 \geq 2$ and $k \leq t$.

The output of the algorithm is the set $B$ of all graphs $G \in \mathcal{H}_{max}^t(a_1, ..., a_s; q; n)$ with $\alpha(G) \geq k$.

1. By removing edges from the graphs in $A$ obtain the set $A' = \mathcal{H}_{+K_{q-1}}^t(a_1 - 1, a_2, ..., a_s; q; n-k)$.
2. For each graph $H \in A'$:
2.1. Find the family $\mathcal{M}(H) = \{M_1, \ldots, M_l\}$ of all maximal $K_{q-1}$-free subsets of $V(H)$.

2.2. Find all $k$-element multisets $N = \{M_{i_1}, \ldots, M_{i_k}\}$ of elements of $\mathcal{M}(H)$ which fulfill the conditions:

(a) $K_{q-2} \subseteq M_{i_j} \cap M_{i_h}$ for every $M_{i_j}, M_{i_h} \in N$, $j \neq h$.

(b) $\alpha(H - \bigcup_{M_{i_j} \in N'} M_{i_j}) \leq t - |N'|$ for every subset $N'$ of $N$.

2.3. For each of the found in step 2.2 $k$-element multisets $N = \{M_{i_1}, \ldots, M_{i_k}\}$ of elements of $\mathcal{M}(H)$ construct the graph $G = G(N)$ by adding new independent vertices $v_1, \ldots, v_k$ to $V(H)$ such that $N_{G}(v_j) = M_{i_j}, j = 1, \ldots, k$. If $\omega(G + e) = q, \forall e \in E(G)$, then add $G$ to $B$.

3. Remove the isomorphic copies of graphs from $B$.

4. Remove from $B$ all graphs $G$ for which $G \not\rightarrow (a_1, \ldots, a_s)$.

We will prove the correctness of Algorithm A3 with the help of the following

**Lemma 3.4.** After the execution of step 2.3 of Algorithm A3, the obtained set $B$ coincides with the set of all maximal $K_q$-free graphs $G$ with $k \leq \alpha(G) \leq t$ which have an independent set of vertices $A \subseteq V(G), |A| = k$ such that $G - A \in \mathcal{A}'$.

**Proof.** Suppose that in step 2.3 of Algorithm A3 the graph $G$ is added to $B$. Then $G = G(N)$ and $G - \{v_1, \ldots, v_k\} = H \in \mathcal{A}'$, where $N, v_1, \ldots, v_k$, and $H$ are the same as in step 2.3. Since $v_1, \ldots, v_k$ are independent, $\alpha(G) \geq k$. From the condition (b) in step 2.2 and Proposition 3.3 it follows that $\alpha(G) \leq t$. By $H \in \mathcal{A}'$ we have $\omega(H) < q$. Since $N_{G}(v_j), j = 1, \ldots, k$, are $K_{q-1}$-free sets, it follows that $\omega(G) < q$. The check at the end of step 2.3 guarantees that $G$ is a maximal $K_q$-free graph.

Let $G$ be a maximal $K_q$-free graph, $k \leq \alpha(G) \leq t$, and $A = \{v_1, \ldots, v_k\}$ be an independent set of vertices of $G$ such that $H = G - A \in \mathcal{A}'$. We will prove that, after the execution of step 2.3 of Algorithm A3 $G \in B$. Since $G$ is a maximal $K_q$-free graph, $N_{G}(v_j), j = 1, \ldots, k$, are maximal $K_{q-1}$-free subsets of $V(H)$, and therefore $N_{G}(v_j) \in \mathcal{M}(H), j = 1, \ldots, k$, see step 2.1. Let $N = \{N_{G}(v_1), \ldots, N_{G}(v_k)\}$. By Proposition 3.2, $N$ fulfills the condition (a) in step 2.3, and by Proposition 3.3, $N$ also fulfills (b). Thus, we showed
that $N$ fulfills all conditions in step 2.2, and since $G = G(N)$ is a maximal $K_q$-free graph, in step 2.3 $G$ is added to $B$.

**Theorem 3.5.** After the execution of Algorithm $A_3$, the obtained set $B$ coincides with the set of all graphs $G \in \mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$ with $\alpha(G) \geq k$.

**Proof.** Suppose that, after the execution of Algorithm $A_3$, $G \in B$. According to Lemma 3.4, $G$ is a maximal $K_q$-free graph and $k \leq \alpha(G) \leq t$. From step 4, $G \rightarrow (a_1, ..., a_s)$, therefore $G \in \mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$.

Conversely, let $G \in \mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$ and $\alpha(G) \geq k$. Let $A \subseteq V(G)$ be an independent set of vertices of $G$, $|A| = k$, and $H = G - A$. Using Proposition 1.10 we derive that $H \in \mathcal{A}'$. From Lemma 3.4 it follows that after the execution of step 2.3, $G \in B$. Clearly, after step 4, $G$ remains in $B$.

**Remark 3.6.** Note that if $G \in \mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$ and $n \geq q$, then $G$ is not a complete graph and $\alpha(G) \geq 2$. Therefore, if $n \geq q$ and $k = 2$, Algorithm $A_3$ finds all graphs in $\mathcal{H}_{\text{max}}^t(a_1, ..., a_s; q; n)$.

We tested our implementation of Algorithm $A_3$ by reproducing the graphs obtained with the help of Algorithm $A_2$ in the second proof of Theorem 2.17 (see Table 2.7, Table 2.8 and Table 2.9). In this case, Algorithm $A_3$ is about 4 times faster than Algorithm $A_2$ and the total computational time for the proof of Theorem 2.17 is reduced about 2 times. Further, we will apply Algorithm $A_3$ to solve problems which cannot be solved in a reasonable amount of time using Algorithm $A_2$. Some examples for such problems are Theorem 3.10, Theorem 4.2, and Theorem 6.5.

At the end of this section, we will propose a method to improve Algorithm $A_3$ which is based on the following proposition:

**Proposition 3.7.** Let $G \in \mathcal{H}_v(2, 2, p; p + 1)$ and $v \in V(G)$. Then all non-neighbors of $v$ induce a graph with chromatic number greater than 2. In particular, from $G \in \mathcal{H}_v(2, 2, p; p + 1)$ it follows that $\Delta(G) \leq |V(G)| - 4$.  

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Proof. Let $W$ be the set of non-neighbors of $v$. If we assume that $W = V_1 \cup V_2$ where $V_1$ and $V_2$ are independent sets, then since $N(v)$ does not contain a $p$-clique, from $V(G) = V_1 \cup V_2 \cup N(v)$ it follows that $G \not \rightarrow (2, 2, p)$. 

As we will see further (see Table 3.1 and Table 3.2), the inequality $\Delta(G) \leq |V(G)| - 4$ is exact. In some special cases, for example the proofs of Theorem 3.8, Theorem 3.10 and Theorem 4.2 we can use the inequality $\Delta(G) \leq |V(G)| - 4$ to speed up computations in some parts of the proofs. We used this inequality only to make sure that the obtained results are correct.

Theorem 3.5 and Algorithm A3 are published in [7].

### 3.2 Computation of the numbers $F_v(2, 2, 6; 7)$ and $F_v(3, 6; 7)$

Let $a_1, \ldots, a_s$ be positive integers and let $m$ and $p$ be defined by (1.6). According to (1.7), $F_v(a_1, \ldots, a_s; m - 1)$ exists if and only if $m \geq p + 2$. In the border case $m = p + 2, \ p \geq 3$, there are only two canonical numbers in the form $F_v(a_1, \ldots, a_s; m - 1)$, namely $F_v(2, 2, p; p + 1)$ and $F_v(3, p; p + 1)$. The computation of the numbers $F_v(a_1, \ldots, a_s; m - 1)$ where $\max\{a_1, \ldots, a_s\} = 6$, i.e. the proof of Theorem 3.1 will be done with the help of the numbers $F_v(2, 2, 6; 7)$ and $F_v(3, 6; 7)$. Because of this, we will first compute these two numbers. In [91] it is proved $F_v(3, 6; 7) \leq 18$. According to Theorem 2.1, $F_v(2, 2, 2, 5; 7) = 17$. From the inclusion $\mathcal{H}_v(3, 6; 7) \subseteq \mathcal{H}_v(2, 2, 6; 7) \subseteq \mathcal{H}_v(2, 2, 2, 5; 7)$ it follows that

$$17 = F_v(2, 2, 2, 5; 7) \leq F_v(2, 2, 6; 7) \leq F_v(3, 6; 7) \leq 18.$$  

**Theorem 3.8.** $\mathcal{H}_{\text{max}}(2, 2, 6; 7; 17) = \{G_1\}$ and $F_v(2, 2, 6; 7) = 17$. Furthermore, $\mathcal{H}_{\text{extr}}(2, 2, 6; 7) = \{G_1, G_2, G_3\}$ (see Figure 3.1).
Figure 3.1: All 3 graphs in $\mathcal{H}_v(2, 2, 6; 7; 17)$
Proof. We will find all graphs in \( H_v(2, 2, 6; 7; 17) \) with the help of a computer.

Let \( G \in H_v(2, 2, 6; 7; 17) \). Clearly, \( \alpha(G) \geq 2 \), and according to Theorem 1.5(b), \( \alpha(G) \leq 3 \).

First, we prove that there are no graphs in \( H_{\text{max}}(2, 2, 6; 7; 17) \) with independence number 3:

It is clear that \( K_6 \) is the only graph in \( H_{\text{max}}(3; 7; 6) \).

We execute Algorithm A3 \((n = 8; k = 2; t = 3)\) with input \( \mathcal{A} = H_{\text{max}}^3(3; 7; 6) = \{K_6\} \) to obtain all graphs in \( \mathcal{B} = H_{\text{max}}^3(4; 7; 8) \).

We execute Algorithm A3 \((n = 10; k = 2; t = 3)\) with input \( \mathcal{A} = H_{\text{max}}^3(4; 7; 8) \) to obtain all graphs in \( \mathcal{B} = H_{\text{max}}^3(5; 7; 10) \).

We execute Algorithm A3 \((n = 12; k = 2; t = 3)\) with input \( \mathcal{A} = H_{\text{max}}^3(5; 7; 10) \) to obtain all graphs in \( \mathcal{B} = H_{\text{max}}^3(6; 7; 12) \).

We execute Algorithm A3 \((n = 14; k = 2; t = 3)\) with input \( \mathcal{A} = H_{\text{max}}^3(6; 7; 12) \) to obtain all graphs in \( \mathcal{B} = H_{\text{max}}^3(2, 6; 7; 14) \).

By executing Algorithm A3 \((n = 17; k = 3; t = 3)\) with input \( \mathcal{A} = H_{\text{max}}^3(1, 6; 7; 13) = H_{\text{max}}^3(6; 7; 14) \), we obtain \( \mathcal{B} = \emptyset \). According to Theorem 3.5, there are no graphs in \( H_{\text{max}}(2, 2, 6; 7; 17) \) with independence number 3.

It remains to find all graphs in \( H_{\text{max}}(2, 2, 6; 7; 17) \) with independence number 2:

It is clear that \( K_7 - e \) is the only graph in \( H_{\text{max}}(3; 7; 7) \).

We execute Algorithm A3 \((n = 9; k = 2; t = 2)\) with input \( \mathcal{A} = H_{\text{max}}^2(3; 7; 7) = \{K_7 - e\} \) to obtain all graphs in \( \mathcal{B} = H_{\text{max}}^2(4; 7; 9) \).

We execute Algorithm A3 \((n = 11; k = 2; t = 2)\) with input \( \mathcal{A} = H_{\text{max}}^2(4; 7; 9) \) to obtain all graphs in \( \mathcal{B} = H_{\text{max}}^2(5; 7; 11) \).

We execute Algorithm A3 \((n = 13; k = 2; t = 2)\) with input \( \mathcal{A} = H_{\text{max}}^2(5; 7; 11) \) to obtain all graphs in \( \mathcal{B} = H_{\text{max}}^2(6; 7; 13) \).

We execute Algorithm A3 \((n = 15; k = 2; t = 2)\) with input \( \mathcal{A} = H_{\text{max}}^2(6; 7; 13) \) to obtain all graphs in \( \mathcal{B} = H_{\text{max}}^2(2, 6; 7; 15) \).
By executing Algorithm $A_3(n = 17; k = 3; t = 2)$ with input $A = \mathcal{H}_{\max}^2(1, 2, 6; 7; 15) = \mathcal{H}_{\max}^2(2, 6; 7; 15)$, we obtain $\mathcal{B} = \{G_1\}$, where the graph $G_1$ is shown in Figure 3.1. According to Theorem 3.5, $G_1$ is the only graph in $\mathcal{H}_{\max}(2, 2, 6; 7; 17)$ with independence number 2. Since there are no graphs in $\mathcal{H}_{\max}(2, 2, 6; 7; 17)$ with independence number greater than 2, we proved that $\mathcal{H}_{\max}(2, 2, 6; 7; 17) = \{G_1\}$.

The number of maximal $K_7$-free graphs and $(+K_6)$-graphs obtained in each step of the proof is given in Table 3.3. By removing edges from $G_1$ we find that there are only two other graphs in $\mathcal{H}_v(2, 2, 6; 7; 17)$, which we will denote by $G_2$ and $G_3$ (see Figure 3.1). We proved that $\mathcal{H}_{\extr}(2, 2, 6; 7) = \{G_1, G_2, G_3\}$. Thus, we finish the proof of Theorem 3.8.

Let us also note, that $G_1 \supset G_2 \supset G_3$, and for the graphs $G_1$, $G_2$, and $G_3$ the inequality (1.14) is strict (see Conjecture 1.6). It is clear that $G_3$ is the only minimal graph in $\mathcal{H}_v(2, 2, 6; 7; 17)$. Some properties of the graphs $G_1$, $G_2$, and $G_3$ are listed in Table 3.1.

**Theorem 3.9.** $F_v(3, 6; 7) = 18$.

**Proof.** According to (3.1), it remains to be proved that $F_v(3, 6; 7) \neq 17$. From (1.11) we have $\mathcal{H}_v(3, 6; 7) \subseteq \mathcal{H}_v(2, 2, 6; 7)$. By Theorem 3.8 $\mathcal{H}_v(2, 2, 6; 7) = \{G_1, G_2, G_3\}$ (see Figure 3.1). We check with a computer that none of the graphs $G_1$, $G_2$, and $G_3$ belongs to $\mathcal{H}_v(3, 6; 7; 17)$, and therefore we obtain $\mathcal{H}_v(3, 6; 7; 17) = \emptyset$. Since the only maximal graph in $\mathcal{H}_v(2, 2, 6; 7)$ is the graph $G_1$, it is enough to check that $G_1 \notin \mathcal{H}_v(3, 6; 7)$.

Using Algorithm $A_3$ we were also able to obtain all graphs in $\mathcal{H}_v(2, 2, 6; 7; 18)$:

**Theorem 3.10.** $|\mathcal{H}_v(2, 2, 6; 7; 18)| = 76515$. 61
Proof. Similarly to the proof of Theorem 3.8, we will find all graphs in $\mathcal{H}_v(2,2,6;7;18)$ with the help of a computer. Some of the graphs that we obtain in the steps of this proof were already obtained in the proof of Theorem 3.8 (compare Table 3.3 to Table 3.4).

Let $G \in \mathcal{H}_v(2,2,6;7;18)$. Clearly, $\alpha(G) \geq 2$, and according to Theorem 1.5(b), $\alpha(G) \leq 4$.

First, we prove that there are no graphs in $\mathcal{H}_{\max}(2,2,6;7;18)$ with independence number 4:

The only graph in $\mathcal{H}_{\max}(3;7;6)$ is $K_6$.

We execute Algorithm $A_3(n = 8; k = 2; t = 4)$ with input $\mathcal{A} = \mathcal{H}_{\max}^4(3;7;6) = \{K_6\}$ to obtain all graphs in $\mathcal{B} = \mathcal{H}_{\max}^4(4;7;8)$. (see Remark 3.6)

We execute Algorithm $A_3(n = 10; k = 2; t = 4)$ with input $\mathcal{A} = \mathcal{H}_{\max}^4(4;7;8)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}_{\max}^4(5;7;10)$.

We execute Algorithm $A_3(n = 12; k = 2; t = 4)$ with input $\mathcal{A} = \mathcal{H}_{\max}^4(5;7;10)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}_{\max}^4(6;7;12)$.

We execute Algorithm $A_3(n = 14; k = 2; t = 4)$ with input $\mathcal{A} = \mathcal{H}_{\max}^4(1,6;7;12) = \mathcal{H}_{\max}^4(6;7;12)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}_{\max}^4(2,6;7;14)$.

By executing Algorithm $A_3(n = 18; k = 4; t = 4)$ with input $\mathcal{A} = \mathcal{H}_{\max}^4(1,2,6;7;14) = \mathcal{H}_{\max}^4(2,6;7;14)$, we obtain $\mathcal{B} = \emptyset$. According to Theorem 3.5 there are no graphs in $\mathcal{H}_{\max}(2,2,6;7;18)$ with independence number 4.

Next, we find all graphs in $\mathcal{H}_{\max}(2,2,6;7;18)$ with independence number 3:

The only graph in $\mathcal{H}_{\max}(3;7;7)$ is $K_7 - e$

We execute Algorithm $A_3(n = 9; k = 2; t = 3)$ with input $\mathcal{A} = \mathcal{H}_{\max}^3(3;7;7) = \{K_7 - e\}$ to obtain all graphs in $\mathcal{B} = \mathcal{H}_{\max}^3(4;7;9)$. (see Remark 3.6)

We execute Algorithm $A_3(n = 11; k = 2; t = 3)$ with input $\mathcal{A} = \mathcal{H}_{\max}^3(4;7;9)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}_{\max}^3(5;7;11)$.

We execute Algorithm $A_3(n = 13; k = 2; t = 3)$ with input $\mathcal{A} = \mathcal{H}_{\max}^3(5;7;11)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}_{\max}^3(6;7;13)$.
\(\mathcal{H}^3_{max}(5; 7; 11)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}^3_{max}(6; 7; 13)\).

We execute Algorithm \(A3(n = 15; k = 2; t = 3)\) with input \(A = \mathcal{H}^3_{max}(1; 6; 7; 13) = \mathcal{H}^3_{max}(6; 7; 13)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}^3_{max}(2; 6; 7; 15)\).

By executing Algorithm \(A3(n = 18; k = 3; t = 3)\) with input \(A = \mathcal{H}^3_{max}(1, 2; 6; 7; 15) = \mathcal{H}^3_{max}(2, 6; 7; 15)\), we obtain all 308 graphs in \(\mathcal{B}\). According to Theorem 3.5, the graphs in \(\mathcal{B}\) are all graphs in \(\mathcal{H}^3_{max}(2, 2; 6; 7; 18)\) with independence number 3.

The last step, is to find all graphs in \(\mathcal{H}^3_{max}(2, 2; 6; 7; 18)\) with independence number 2:

It is easy to see that \(\mathcal{H}^3_{max}(3; 7; 8) = \{K_3 + K_5, C_4 + K_4\}\), and therefore \(C_4 + K_4\) is the only graph in \(\mathcal{H}^3_{max}(3; 7; 8)\) with independence number 2.

We execute Algorithm \(A3(n = 10; k = 2; t = 2)\) with input \(A = \mathcal{H}^2_{max}(3; 7; 8) = \{C_4 + K_4\}\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}^2_{max}(4; 7; 10)\). (see Remark 3.6)

We execute Algorithm \(A3(n = 12; k = 2; t = 2)\) with input \(A = \mathcal{H}^2_{max}(4; 7; 10)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}^2_{max}(5; 7; 12)\).

We execute Algorithm \(A3(n = 14; k = 2; t = 2)\) with input \(A = \mathcal{H}^2_{max}(5; 7; 12)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}^2_{max}(6; 7; 14)\).

We execute Algorithm \(A3(n = 16; k = 2; t = 2)\) with input \(A = \mathcal{H}^2_{max}(1, 6; 7; 14) = \mathcal{H}^2_{max}(6; 7; 14)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}^2_{max}(2, 6; 7; 16)\).

By executing Algorithm \(A3(n = 18; k = 2; t = 2)\) with input \(A = \mathcal{H}^2_{max}(1, 2, 6; 7; 16) = \mathcal{H}^2_{max}(2, 6; 7; 16)\), we obtain all 84 graphs in \(\mathcal{B}\). According to Theorem 3.5, the graphs in \(\mathcal{B}\) are all graphs in \(\mathcal{H}^2_{max}(2, 2, 6; 7; 18)\) with independence number 2.

Thus, we obtained all 392 graphs in \(\mathcal{H}_{max}(2, 2, 6; 7; 18)\). By removing edges from these graphs we find all 76 515 graphs in \(\mathcal{H}_{v}(2, 2, 6; 7; 18)\). Some properties of these graphs are listed in Table 3.2. The number of maximal \(K_7\)-free graphs and \((+K_6)\)-graphs obtained in each step of the proof is given in Table 3.4. Because of the large number of graphs in \(\mathcal{H}_{v}^2(2, 6; 7; 16)\), we needed about two weeks to complete the computations. \(\square\)
Table 3.2: Some properties of the graphs in $H_v(2, 2, 6; 7; 18)$

We check with a computer that among the 76 515 graphs in $H_v(2, 2, 6; 7; 18)$, only the graph $\Gamma_3$ (see Figure 2.5) belongs to $H_v(3, 6; 7; 18)$. This is the graph that gives the upper bound $F_v(3, 6; 7) \leq 18$ in [91]. Now, from Theorem 3.9 it follows

**Theorem 3.11.** $H_{extr}(3, 6; 7) = H_v(3, 6; 7; 18) = \{\Gamma_3\}$ (see Figure 2.5).

Let us note that $\chi(\Gamma_3) = 9$, and for this graph the inequality (1.14) is strict. However, from Theorem 3.11 it follows that in this special case Conjecture 1.6 is true.

There are two 13-regular graphs in $H_v(2, 2, 6; 7; 18)$, one of them being $\Gamma_3$. The graph $\Gamma_3$ is the only vertex transitive graph in $H_v(2, 2, 6; 7; 18)$ and it has 36 automorphisms. The other 13-regular graph has 24 automorphisms.

Let us also note that 2 467 of the graphs in $H_v(2, 2, 6; 7; 18)$ do not contain a subgraph in $H_v(2, 2, 6; 7; 17)$. We obtained the remaining 74048 graphs in $H_v(2, 2, 6; 7; 18)$ in another way by adding one vertex to the graphs in $H_v(2, 2, 6; 7; 17)$. This also testifies to the correctness of our programs.

Theorem 3.8, Theorem 3.9, Theorem 3.10 and Theorem 3.11 are published in [7].

| $|E(G)|$ | $\# \delta(G)$ | $\# \Delta(G)$ | $\# \alpha(G)$ | $\# \chi(G)$ | $|\text{Aut}(G)|$ |
|--------|----------|----------|----------|----------|----------|
| 106    | 1        | 0        | 3        | 2        | 8        | 1        |
| 107    | 4        | 1        | 20       | 3        | 76       | 9        | 2        |
| 108    | 19       | 2        | 124      |          | 4        |          |          |
| 109    | 88       | 3        | 571      |          | 8        |          |          |
| 110    | 369      | 4        | 1 943    |          | 10       |          |          |
| 111    | 1 240    | 5        | 4 986    |          | 16       |          |          |
| 112    | 3 303    | 6        | 9 826    |          | 20       |          |          |
| 113    | 6 999    | 7        | 14 896   |          | 24       |          |          |
| 114    | 11 780   | 8        | 17 057   |          | 36       |          |          |
| 115    | 15 603   | 9        | 14 288   |          | 40       |          |          |
| 116    | 15 956   | 10       | 8 397    |          |          |          |          |
| 117    | 12 266   | 11       | 3 504    |          |          |          |          |
| 118    | 6 575    | 12       | 876      |          |          |          |          |
| 119    | 2 044    | 13       | 24       |          |          |          |          |
| 120    | 261      |          |          |          |          |          |          |
| 121    | 7        |          |          |          |          |          |          |
| set            | independence number | maximal graphs | (+K₆)-graphs |
|---------------|---------------------|----------------|--------------|
| $\mathcal{H}_v(3; 7; 6)$ | $\leq 3$            | 1              | 2            |
| $\mathcal{H}_v(4; 7; 8)$ | $\leq 3$            | 2              | 12           |
| $\mathcal{H}_v(5; 7; 10)$ | $\leq 3$            | 6              | 274          |
| $\mathcal{H}_v(6; 7; 12)$ | $\leq 3$            | 37             | 78 926       |
| $\mathcal{H}_v(2, 6; 7; 14)$ | $\leq 3$            | 20             | 5 291        |
| $\mathcal{H}_v(2, 2, 6; 7; 17)$ | $\leq 3$            | 0              |              |
| $\mathcal{H}_v(3; 7; 7)$ | $\leq 2$            | 1              | 3            |
| $\mathcal{H}_v(4; 7; 9)$ | $\leq 2$            | 2              | 22           |
| $\mathcal{H}_v(5; 7; 11)$ | $\leq 2$            | 5              | 468          |
| $\mathcal{H}_v(6; 7; 13)$ | $\leq 2$            | 24             | 97 028       |
| $\mathcal{H}_v(2, 6; 7; 15)$ | $\leq 2$            | 473            | 10 018 539   |
| $\mathcal{H}_v(2, 2, 6; 7; 17)$ | $\leq 2$            | 1              |              |
| $\mathcal{H}_v(2, 2, 6; 7; 17)$ | $\equiv 3$          | 1              |              |

Table 3.3: Steps in finding all maximal graphs in $\mathcal{H}_v(2, 2, 6; 7; 17)$

| set            | independence number | maximal graphs | (+K₆)-graphs |
|---------------|---------------------|----------------|--------------|
| $\mathcal{H}_v(3; 7; 6)$ | $\leq 4$            | 1              | 2            |
| $\mathcal{H}_v(4; 7; 8)$ | $\leq 4$            | 2              | 13           |
| $\mathcal{H}_v(5; 7; 10)$ | $\leq 4$            | 7              | 317          |
| $\mathcal{H}_v(6; 7; 12)$ | $\leq 4$            | 50             | 102 387      |
| $\mathcal{H}_v(2, 6; 7; 14)$ | $\leq 4$            | 20             | 5 293        |
| $\mathcal{H}_v(2, 2, 6; 7; 18)$ | $\equiv 4$          | 0              |              |
| $\mathcal{H}_v(3; 7; 7)$ | $\leq 3$            | 1              | 4            |
| $\mathcal{H}_v(4; 7; 9)$ | $\leq 3$            | 3              | 45           |
| $\mathcal{H}_v(5; 7; 11)$ | $\leq 3$            | 12             | 3 071        |
| $\mathcal{H}_v(6; 7; 13)$ | $\leq 3$            | 168            | 4 691 237    |
| $\mathcal{H}_v(2, 6; 7; 15)$ | $\leq 3$            | 1627           | 70 274 176   |
| $\mathcal{H}_v(2, 2, 6; 7; 18)$ | $\equiv 3$          | 308            |              |
| $\mathcal{H}_v(3; 7; 8)$ | $\leq 2$            | 1              | 8            |
| $\mathcal{H}_v(4; 7; 10)$ | $\leq 2$            | 3              | 82           |
| $\mathcal{H}_v(5; 7; 12)$ | $\leq 2$            | 10             | 5 057        |
| $\mathcal{H}_v(6; 7; 14)$ | $\leq 2$            | 96             | 2 799 416    |
| $\mathcal{H}_v(2, 6; 7; 16)$ | $\leq 2$            | 7509           | 920 112 878  |
| $\mathcal{H}_v(2, 2, 6; 7; 18)$ | $\equiv 2$          | 84             |              |
| $\mathcal{H}_v(2, 2, 6; 7; 18)$ | $\equiv 3$          | 392            |              |

Table 3.4: Steps in finding all maximal graphs in $\mathcal{H}_v(2, 2, 6; 7; 18)$
3.3 Computation of the numbers

\[ F_v(2_{m-6}; 6; m - 1) \]

In support to Conjecture 2.16 we will prove the following theorem:

**Theorem 3.12.** Let \( F_v(2, 2, p; p + 1) \leq 2p + 5 \). Then \( r'_0(p) = 2 \) and

\[ F_v(2_r, p; r + p - 1) = F_v(2, 2, p; p + 1) + r - 2, \quad r \geq 2. \]

**Proof.** From Theorem 2.12(b) it follows that it is enough to prove the equality \( r'_0(p) = 2 \). According to (1.8), \( F_v(2, 2, p; p + 1) \geq 2p + 4 \). Therefore, only the following two cases are possible:

**Case 1.** \( F_v(2, 2, p; p + 1) = 2p + 4 \). According to (1.8), \( F_v(2_r, p; r + p - 1) \geq m + p + 2 = r + 2p + 2 \). Therefore,

\[ F_v(2_r, p; r + p - 1) - r \geq 2p + 2 = F_v(2, 2, p; p + 1) - 2, \quad r \geq 2, \]

and we have \( r'_0(p) = 2 \).

**Case 2.** \( F_v(2, 2, p; p + 1) = 2p + 5 \). From Theorem 1.11 we have \( F_v(2_r, p; r + p - 1) \geq r + 2p + 3, \quad r \geq 2 \). From this inequality we obtain

\[ F_v(2_r, p; r + p - 1) - r \geq 2p + 3 = F_v(2, 2, p; p + 1) - 2, \quad r \geq 2. \]

Therefore, in this case we also have \( r'_0(p) = 2 \). \( \square \)

**Remark 3.13.** It is unknown whether the first case is possible, i.e. if \( F_v(2, 2, p; p + 1) = 2p + 4 \) for some \( p \). If \( p \leq 7 \) this equality is not true.

**Theorem 3.14.** \( r'_0(6) = 2 \) and \( F_v(2_{m-6}; 6; m - 1) = m + 9, \quad m \geq 8 \).

**Proof.** By Theorem 3.8 \( F_v(2, 2; 6; 7) = 17 \). From this fact and Theorem 3.12 we obtain \( r'_0(6) = 2 \) and the equality \( F_v(2_{m-6}; 6; m - 1) = m + 9, \quad m \geq 8 \). \( \square \)

Theorem 3.12 and Theorem 3.14 are published in [7].
3.4 Proof of Theorem 3.1(a)

Since \( a_1 = \ldots = a_{s-1} = 2 \) and \( a_s = 6 \), we have \( m = s + 5 \), and therefore \( F_v(a_1, \ldots, a_s; m - 1) = F_v(2s-1, 6; m - 1) = F_v(2m-6, 6; m - 1) \).

From Theorem 3.14 it now follows that \( F_v(a_1, \ldots, a_s; m - 1) = m + 9. \)

Theorem 3.1 is published in [7].

The lower bound \( F_v(a_1, \ldots, a_s; m - 1) \geq m + 9 \) in Theorem 3.1 is first proved and published in [8].

3.5 Bounds on the numbers \( F_v(2r, 3, p; r + p + 1) \)

The proof of Theorem 3.1(b) is more complex. First, we will compute the numbers \( F_v(2m-8, 3, 6; m - 1), m \geq 8 \) (Theorem 3.17). For the computation of the numbers \( F_v(2m-8, 3, 6; m - 1), m \geq 8 \) we will need the following theorem:

**Theorem 3.15.** Let \( p \) be a fixed positive integer and \( r''_0(p) = r''_0 \) be the smallest positive integer for which

\[
\min_{r \geq 0} \{ F_v(2r, 3, p; r + p + 1) - r \} = F_v(2r''_0, 3, p; r''_0 + p + 1) - r''_0.
\]

Then:

(a) \( F_v(2r, 3, p; r + p + 1) = F(2r''_0, 3, p; r''_0 + p + 1) + r - r''_0, \ r \geq r''_0. \)

(b) If \( r''_0 = 0 \), then \( F_v(2r, 3, p; r + p + 1) = F_v(3, p; p + 1) + r, \ r \geq 0. \)

(c) If \( r''_0 > 0 \) and \( G \) is an extremal graph in \( \mathcal{H}(2r''_0, 3, p; r''_0 + p + 1) \), then \( G \not\rightarrow (2, r''_0 + p). \)

(d) \( r''_0 < F_v(3, p; p + 1) - 2p - 2. \)
Proof. \((a)\) According to the definition of \(r''_0 = r''_0(p)\) we have
\[ F_v(2, 3, p; r + p + 1) \geq F_v(2, r''_0, 3, p; r''_0 + p + 1) + r - r''_0, \ r \geq 0.\]
Now we will prove that if \(r \geq r''_0\) the opposite inequality is also true. Let \(G \in \mathcal{H}_{extv}(2, r''_0, 3, p; r''_0 + p + 1)\). Then from (2.1) it follows that \(K_{r-r''_0} + G \in \mathcal{H}_v(2, 3, p; r + p + 1), \ r \geq r''_0\). Therefore
\[ F_v(2, 3, p; r + p + 1) \leq |V(K_{r-r''_0} + G)| = F_v(2, r''_0, 3, p; r''_0 + p + 1) + r - r''_0, \ r \geq r''_0.\]
Thus, \((a)\) is proved.

\((b)\) If \(r''_0(p) = 0\), then obviously the equality \((b)\) follows from \((a)\).

\((c)\) Assume the opposite is true and let \(G\) be an extremal graph in \(\mathcal{H}_v(2, r''_0, 3, p; r''_0 + p + 1)\) such that \(V(G) = V_1 \cup V_2\) where \(V_1\) is an independent set and \(V_2\) does not contain \((r''_0 + p)\)-clique. We can assume that \(V_1 \neq \emptyset\). Let \(G_1 = G[V_2] = G - V_1\). Then \(\omega(G_1) < r''_0 + p\) and since \(r''_0 \geq 1\), from Proposition 1.4 it follows that \(G_1 \rightarrow (2, r''_0 - 1, 3, p)\). Therefore, \(G_1 \in \mathcal{H}_v(2, r''_0 - 1, 3, p; r''_0 + p)\) and
\[ |V(G)| - 1 \geq |V(G_1)| \geq F_v(2, r''_0 - 1, 3, p; r''_0 + p).\]
Since \(|V(G)| = F_v(2, r''_0, 3, p; r''_0 + p + 1)\), we obtain
\[ F_v(2, r''_0 - 1, 3, p; r''_0 + p) - (r''_0 - 1) \leq F_v(2, r''_0, 3, p; r''_0 + p + 1) - r''_0,\]
which contradicts the definition of \(r''_0\).

\((d)\) According to (1.8) \(F_v(3, p; p + 1) \geq 2p + 4\), and therefore in the case \(r''_0 = 0\) the inequality holds. Let \(r''_0 > 0\) and \(G\) be an extremal graph in \(\mathcal{H}_v(2, r''_0, 3, p; r''_0 + p + 1)\). According to (1.14).
\[ (3.2) \quad \chi(G) \geq r''_0 + p + 2.\]
According to \((c)\), \(G \in \mathcal{H}_v(2, r''_0 + p; r''_0 + p + 1)\), and by Theorem 1.2
\[ |V(G)| \geq 2r''_0 + 2p + 1.\]
Since \(\chi(\overline{C_{2r''_0 + 2p + 1}}) = r''_0 + p + 1\), from (3.2) it follows \(G \neq \overline{C_{2r''_0 + 2p + 1}}\). By Theorem 1.2(b),
\[ |V(G)| = F_v(2, r''_0, 3, p; r''_0 + p + 1) \geq 2r''_0 + 2p + 2.\]
Since \( \gamma_0'' > 0 \), we have
\[
F_v(2\gamma_0', 3, p; \gamma_0'' + p + 1) - \gamma_0'' < F_v(3, p; p + 1).
\]
From the last two inequalities it follows that
\[
\gamma_0'' < F_v(3, p; p + 1) - 2p - 2.
\]

Since \( F_v(3, 3; 4) = 14 \), from (1.9) we obtain \( \gamma_0''(3) = 1 \). Also from (1.9) we see that \( \gamma_0''(4) = 0 \). From Theorem 2.1 it follows that \( \gamma_0''(5) = 0 \). We suppose the following conjecture is true

**Conjecture 3.16.** Let \( p \geq 4 \) be a fixed integer. Then,
\[
\min_{r \geq 0} \{F_v(2r, 3, p; r + p - 1) - r\} = F_v(3, p; p + 1),
\]
i.e. \( \gamma_0''(p) = 0 \), and
\[
F_v(2r, 3, p; r + p + 1) = F_v(3, p; p + 1) + r.
\]

It is not difficult to see that Conjecture 3.16 is true if and only if for fixed \( p \) the sequence \( \{F_v(2r, 3, p; r + p + 1)\} \) is strictly increasing with respect to \( r \). We will prove that when \( p = 6 \) Conjecture 3.16 is also true. Theorem 3.1 (b) follows easily from this fact.

Theorem 3.15 is published in [7].

### 3.6 Computation of the numbers

\[
F_v(2m-8, 3, 6; m - 1)
\]

**Theorem 3.17.** \( \gamma_0''(6) = 0 \) and \( F_v(2m-8, 3, 6; m - 1) = m + 10, m \geq 8 \).

**Proof.** From Theorem 3.15 (d) we obtain \( \gamma_0''(6) < 4 \). Therefore, we have to prove \( \gamma_0''(6) \neq 1, \gamma_0''(6) \neq 2, \) and \( \gamma_0''(6) \neq 3 \). Since \( F_v(3, 6; 7) = 18 \), we
have to prove the inequalities $F_v(2, 3, 6; 8) > 18$, $F_v(2, 2, 3, 6; 9) > 19$, and $F_v(2, 2, 3, 6; 10) > 20$. We will prove these inequalities with the help of a computer. From (2.1) ($t = 1$) it is easy to see that $F_v(2r−1, 3; p) + 1 ≥ F_v(2r, 3; p + 1)$, and therefore it is enough to prove $F_v(2, 2, 3, 6; 10) > 20$.

We will present the proof of this inequality only, but we also proved the other two inequalities in the same way with a computer, since the obtained additional information is interesting and useful.

Similarly to the proof of Theorem 3.8, we will use Algorithm A3 to prove that $H_v(2, 2, 3, 6; 10; 20) = \emptyset$. According to Theorem 1.5(b), there are no graphs in $H_v(2, 2, 3, 6; 10; 20)$ with independence number greater than 3.

First, we prove that there are no graphs in $H_{\text{max}}(2, 2, 3, 6; 10; 20)$ with independence number 3:

The only graph in $H_{\text{max}}(6; 10; 9)$ is $K_9$.

We execute Algorithm A3($n = 11; k = 2; t = 3$) with input $A = H_{\text{max}}^3(1, 6; 10; 9) = H_{\text{max}}^3(6; 10; 9) = \{K_9\}$ to obtain all graphs in $B = H_{\text{max}}^3(2, 6; 10; 11)$. (see Remark 3.6)

We execute Algorithm A3($n = 13; k = 2; t = 3$) with input $A = H_{\text{max}}^3(2, 6; 10; 11)$ to obtain all graphs in $B = H_{\text{max}}^3(3, 6; 10; 13)$.

We execute Algorithm A3($n = 15; k = 2; t = 3$) with input $A = H_{\text{max}}^3(1, 3, 6; 10; 13) = H_{\text{max}}^3(3, 6; 10; 13)$ to obtain all graphs in $B = H_{\text{max}}^3(2, 3, 6; 10; 15)$.

We execute Algorithm A3($n = 17; k = 2; t = 3$) with input $A = H_{\text{max}}^3(1, 2, 3, 6; 10; 15) = H_{\text{max}}^3(2, 3, 6; 10; 15)$ to obtain all graphs in $B = H_{\text{max}}^3(2, 2, 3, 6; 10; 17)$.

By executing Algorithm A3($n = 20; k = 3; t = 3$) with input $A = H_{\text{max}}^3(1, 2, 2, 3, 6; 10; 17) = H_{\text{max}}^3(2, 2, 3, 6; 10; 17)$, we obtain $B = \emptyset$. According to Theorem 3.5, there are no graphs in $H_{\text{max}}(2, 2, 3, 6; 10; 20)$ with independence number 3.

It remains to be proved that there are no graphs in $H_{\text{max}}(2, 2, 2, 3, 6; 10; 20)$ with independence number 2:
The only graph in $\mathcal{H}_{\text{max}}(6; 10; 10)$ is $K_{10} - e$.

We execute Algorithm $A3(n = 12; k = 2; t = 2)$ with input $A = \mathcal{H}^2_{\text{max}}(1, 6; 10; 10) = \mathcal{H}^2_{\text{max}}(6; 10; 10) = \{K_{10} - e\}$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^2_{\text{max}}(2, 6; 10; 12)$. (see Remark 3.6)

We execute Algorithm $A3(n = 14; k = 2; t = 2)$ with input $A = \mathcal{H}^2_{\text{max}}(2, 6; 10; 12)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^2_{\text{max}}(3, 6; 10; 14)$.

We execute Algorithm $A3(n = 16; k = 2; t = 2)$ with input $A = \mathcal{H}^2_{\text{max}}(1, 3, 6; 10; 14) = \mathcal{H}^2_{\text{max}}(3, 6; 10; 14)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^2_{\text{max}}(2, 3, 6; 10; 16)$.

We execute Algorithm $A3(n = 18; k = 2; t = 2)$ with input $A = \mathcal{H}^2_{\text{max}}(1, 2, 3, 6; 10; 16) = \mathcal{H}^2_{\text{max}}(2, 3, 6; 10; 16)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^2_{\text{max}}(2, 2, 3, 6; 10; 18)$.

By executing Algorithm $A3(n = 20; k = 2; t = 2)$ with input $A = \mathcal{H}^2_{\text{max}}(1, 2, 2, 3, 6; 10; 18) = \mathcal{H}^2_{\text{max}}(2, 2, 3, 6; 10; 18)$, we obtain $\mathcal{B} = \emptyset$. According to Theorem 3.5, there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 2, 3, 6; 10; 20)$ with independence number 2.

Thus, we proved $\mathcal{H}_{\text{max}}(2, 2, 2, 3, 6; 10; 20) = \emptyset$, and therefore $F_v(2, 2, 2, 3, 6; 10) > 20$ and $r''_0(6) = 0$.

From $r''_0(6) = 0$ and Theorem 3.15(b) we obtain

$$F_v(2_{m-8}, 3, 6; m-1) = m + 10, \ m \geq 8.$$ 

Thus, Theorem 3.17 is proved. \hfill \Box

The number of graphs obtained in each step is given in Table 3.7 (see also Table 3.5 and Table 3.6).

Theorem 3.17 is published in [7].
Table 3.5: Steps in finding all maximal graphs in $H_v(2, 3, 6; 8; 18)$

| set                  | independence number | maximal graphs | (+$K_7$)-graphs |
|----------------------|---------------------|----------------|-----------------|
| $H_v(4; 8; 7)$        | $\leq 3$            | 1              | 2               |
| $H_v(5; 8; 9)$        | $\leq 3$            | 2              | 12              |
| $H_v(6; 8; 11)$       | $\leq 3$            | 6              | 276             |
| $H_v(2, 6; 8; 13)$    | $\leq 3$            | 37             | 79 749          |
| $H_v(3, 6; 8; 15)$    | $\leq 3$            | 21             | 3 458           |
| $H_v(2, 3, 6; 8; 18)$ | $\equiv 3$          | 0              |                 |
| $H_v(4; 8; 8)$        | $\leq 2$            | 1              | 3               |
| $H_v(5; 8; 10)$       | $\leq 2$            | 2              | 22              |
| $H_v(6; 8; 12)$       | $\leq 2$            | 5              | 489             |
| $H_v(2, 6; 8; 14)$    | $\leq 2$            | 25             | 119 126         |
| $H_v(3, 6; 8; 16)$    | $\leq 2$            | 509            | 3 582 157       |
| $H_v(2, 3, 6; 8; 18)$ | $\equiv 2$          | 0              |                 |

Table 3.6: Steps in finding all maximal graphs in $H_v(2, 3, 6; 9; 19)$

| set                  | independence number | maximal graphs | (+$K_9$)-graphs |
|----------------------|---------------------|----------------|-----------------|
| $H_v(5; 9; 8)$        | $\leq 3$            | 1              | 2               |
| $H_v(6; 9; 10)$       | $\leq 3$            | 2              | 12              |
| $H_v(2, 6; 9; 12)$    | $\leq 3$            | 6              | 277             |
| $H_v(3, 6; 9; 14)$    | $\leq 3$            | 37             | 79 901          |
| $H_v(2, 3, 6; 9; 16)$ | $\leq 3$            | 21             | 3 459           |
| $H_v(2, 2, 3, 6; 9; 19)$ | $\equiv 3$      | 0              |                 |
| $H_v(5; 9; 9)$        | $\leq 2$            | 1              | 3               |
| $H_v(6; 9; 11)$       | $\leq 2$            | 2              | 22              |
| $H_v(2, 6; 9; 13)$    | $\leq 2$            | 5              | 496             |
| $H_v(3, 6; 9; 15)$    | $\leq 2$            | 25             | 121 499         |
| $H_v(2, 3, 6; 9; 17)$ | $\leq 2$            | 512            | 3 585 530       |
| $H_v(2, 2, 3, 6; 9; 19)$ | $\equiv 2$      | 0              |                 |

Table 3.7: Steps in finding all maximal graphs in $H_v(2, 2, 3, 6; 10; 20)$

| set                  | independence number | maximal graphs | (+$K_9$)-graphs |
|----------------------|---------------------|----------------|-----------------|
| $H_v(6; 10; 9)$      | $\leq 3$            | 1              | 2               |
| $H_v(2, 6; 10; 11)$  | $\leq 3$            | 2              | 12              |
| $H_v(3, 6; 10; 13)$  | $\leq 3$            | 6              | 277             |
| $H_v(2, 3, 6; 10; 15)$ | $\leq 3$      | 37             | 79 934          |
| $H_v(2, 2, 3, 6; 10; 17)$ | $\leq 3$      | 21             | 3 459           |
| $H_v(2, 2, 2, 3, 6; 10; 20)$ | $\equiv 3$      | 0              |                 |
| $H_v(6; 10; 10)$     | $\leq 2$            | 1              | 3               |
| $H_v(2, 6; 10; 12)$  | $\leq 2$            | 2              | 22              |
| $H_v(3, 6; 10; 14)$  | $\leq 2$            | 5              | 498             |
| $H_v(2, 3, 6; 10; 16)$ | $\leq 2$      | 25             | 121 864         |
| $H_v(2, 2, 3, 6; 10; 18)$ | $\leq 2$      | 512            | 3 585 546       |
| $H_v(2, 2, 2, 3, 6; 10; 20)$ | $\equiv 2$      | 0              |                 |

Table 3.8: Steps in finding all maximal graphs in $H_v(2, 2, 2, 3, 6; 10; 20)$
3.7 Computation of the numbers

\( \tilde{F}_v(m\mid_6; m - 1) \)

Let us remind that \( \tilde{H}_v(m\mid_p; q) \) and \( \tilde{F}_v(m\mid_p; q) \) are defined in Section 1.4.

According to Proposition 1.17, we have

\[
(3.3) \quad \tilde{F}_v(m\mid_6; m - 1) \text{ exists } \iff m \geq 8.
\]

We will prove the following

**Theorem 3.18.** \( \tilde{F}_v(m\mid_6; m - 1) = m + 10, \ m \geq 8. \)

**Proof.** The lower bound \( \tilde{F}_v(m\mid_6; m - 1) \geq m + 10 \) follows from Theorem 3.17 and Theorem 1.19.

To prove the upper bound consider the 18-vertex graph \( \Gamma_3 \) (Figure 2.5) with the help of which in [91] the authors proved the inequality \( F_v(3,6;7) \leq 18. \) In addition to the property \( \Gamma_3 \rightarrow (3,6) \), the graph \( \Gamma_3 \) also has the property \( \Gamma_3 \rightarrow (4,5) \). According to Proposition 1.16 from \( \Gamma_3 \rightarrow (3,6) \) and \( \Gamma_3 \rightarrow (4,5) \) it follows \( \Gamma_3 \rightarrow 8\mid_6. \) Since \( \omega(\Gamma_3) = 6, \) we obtain \( \Gamma_3 \in \tilde{H}_v(8\mid_6; 7) \) and \( \tilde{F}_v(8\mid_6; 7) \leq |V(\Gamma_3)| = 18. \) From this inequality and Theorem 1.18 \((m_0 = 8, p = 6; q = 7) \) it follows \( \tilde{F}_v(m\mid_6; m - 1) \leq m + 10, \ m \geq 8. \)

The numbers \( \tilde{F}_v(m\mid_6; m - 1) \) can be computed directly with the help of the following Theorem 3.19. This is done in [1].

**Theorem 3.19.** Let \( m_0(p) = m_0 \) be the smallest positive integer for which

\[
\min_{m \geq p + 2} \left\{ \tilde{F}_v(m\mid_p; m - 1) - m \right\} = \tilde{F}_v(m_0\mid_p; m_0 - 1) - m_0.
\]

Then:

(a) \( \tilde{F}_v(m\mid_p; m - 1) = \tilde{F}_v(m_0\mid_p; m_0 - 1) + m - m_0, \ m \geq m_0. \)

(b) If \( m_0 > p + 2 \) and \( G \) is an extremal graph in \( \tilde{H}_v(m_0\mid_p; m_0 - 1) \), then \( G \rightarrow (2, m_0 - 2). \)

(c) \( m_0 < \tilde{F}_v((p + 2)\mid_p; p + 1) - p. \)

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Theorem 3.19 is published in [4]. The proof is similar to the proof of Theorem 2.12 and Theorem 3.15. According to Theorem 3.19, the computation of the numbers $\tilde{F}_v(m_{6}; m - 1)$ is reduced to finding the exact values of the first several of these numbers. In [4], we use a modification of Algorithm A2 to prove that $\tilde{F}_v(8_{6}; 7) = 18$, $\tilde{F}_v(9_{6}; 8) > 18$, $\tilde{F}_v(10_{6}; 9) > 19$, and $\tilde{F}_v(11_{6}; 10) > 20$. Thus, with the help of Theorem 3.19, we obtain $m_0(6) = 8$ and $\tilde{F}_v(m_{6}; m - 1) = m + 10$, $m \geq 8$. We will not present this proof in detail.

3.8 Proof of Theorem 3.1(b)

According to Theorem 1.19 and Theorem 3.17

$F_v(a_1, \ldots, a_s; m - 1) \geq F_v(2m - 8, 3, 6; m - 1) = m + 10.$

From Theorem 1.19 and Theorem 3.18 we obtain

$F_v(a_1, \ldots, a_s; m - 1) \leq \tilde{F}_v(m_{6}; m - 1) = m + 10.$

Theorem 3.1 is published in [7]. The upper bound $F_v(a_1, \ldots, a_s; m - 1) \leq m + 10$ in Theorem 3.1 is first proved and published in [8].
Chapter 4

Numbers of the form \( F_v(a_1, ..., a_s; m - 1) \) where \( \max\{a_1, ..., a_s\} = 7 \)

In this chapter we obtain the bounds:

**Theorem 4.1.** Let \( a_1, ..., a_s \) be positive integers such that \( \max\{a_1, ..., a_s\} = 7 \) and \( m = \sum_{i=1}^{s}(a_i - 1) + 1 \geq 9 \). Then:

\[
m + 11 \leq F_v(a_1, ..., a_s; m - 1) \leq m + 12.
\]

According to (1.7), the condition \( m \geq 9 \) in Theorem 4.1 is necessary.

First, with the help of Algorithm A3 we will find the exact value of the number \( F_v(2, 2, 7; 8) \). Then, we compute the numbers \( F_v(2m - 7, 7; m - 1) \). For the computation of these numbers we will need a new improved Algorithm A4.

4.1 Computation of the number \( F_v(2, 2, 7; 8) \)

Regarding the number \( F_v(2, 2, 7; 8) \) we know that \( 18 \leq F_v(2, 2, 7; 8) \leq 22 \). The lower bound follows from (1.8) and the upper bound follows from \( F_v(3, 7; 8) \leq 22 \), [91].

**Theorem 4.2.** \( F_v(2, 2, 7; 8) = 20 \).

**Proof.** 1. Proof of the lower bound \( F_v(2, 2, 7; 8) \geq 20 \).

We will prove that \( \mathcal{H}_v(2, 2, 7; 8; 19) = \emptyset \) using a similar method to the proof of Theorem 3.8. Suppose that \( G \in \mathcal{H}_v(2, 2, 7; 8; 19) \). Clearly, \( \alpha(G) \geq 2 \), and according to Theorem 1.5(b), \( \alpha(G) \leq 3 \). It remains to be proved that there are no graphs in \( \mathcal{H}_{max}(2, 2, 7; 8; 19) \) with independence number 2 or 3.
First, we prove that there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 7; 8; 19)$ with independence number 3:

The only graph in $\mathcal{H}_{\text{max}}(4; 8; 8)$ is $K_8 - e$.

We execute Algorithm $A_3(n = 10; k = 2; t = 3)$ with input $\mathcal{A} = \mathcal{H}^3_{\text{max}}(4; 8; 8) = \{K_8 - e\}$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^3_{\text{max}}(5; 8; 10)$. (see Remark 3.6)

We execute Algorithm $A_3(n = 12; k = 2; t = 3)$ with input $\mathcal{A} = \mathcal{H}^3_{\text{max}}(5; 8; 10)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^3_{\text{max}}(6; 8; 12)$.

We execute Algorithm $A_3(n = 14; k = 2; t = 3)$ with input $\mathcal{A} = \mathcal{H}^3_{\text{max}}(6; 8; 12)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^3_{\text{max}}(7; 8; 14)$.

We execute Algorithm $A_3(n = 16; k = 2; t = 3)$ with input $\mathcal{A} = \mathcal{H}^3_{\text{max}}(1, 7; 8; 14) = \mathcal{H}^3_{\text{max}}(7; 8; 14)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^3_{\text{max}}(2, 7; 8; 16)$.

By executing Algorithm $A_3(n = 19; k = 3; t = 3)$ with input $\mathcal{A} = \mathcal{H}^3_{\text{max}}(1, 2, 7; 8; 16) = \mathcal{H}^3_{\text{max}}(2, 7; 8; 16)$, we obtain $\mathcal{B} = \emptyset$. According to Theorem 3.5, there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 7; 8; 19)$ with independence number 3.

It remains to prove that there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 7; 8; 19)$ with independence number 2:

It is easy to see that $\mathcal{H}_{\text{max}}(4; 8; 9) = \{K_3 + K_6, C_4 + K_5\}$, and therefore $C_4 + K_5$ is the only graph in $\mathcal{H}_{\text{max}}(4; 8; 9)$ with independence number 2.

We execute Algorithm $A_3(n = 11; k = 2; t = 2)$ with input $\mathcal{A} = \mathcal{H}^2_{\text{max}}(4; 8; 9) = \{C_4 + K_5\}$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^2_{\text{max}}(5; 8; 11)$. (see Remark 3.6)

We execute Algorithm $A_3(n = 13; k = 2; t = 2)$ with input $\mathcal{A} = \mathcal{H}^2_{\text{max}}(5; 8; 11)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^2_{\text{max}}(6; 8; 13)$.

We execute Algorithm $A_3(n = 15; k = 2; t = 2)$ with input $\mathcal{A} = \mathcal{H}^2_{\text{max}}(6; 8; 13)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^2_{\text{max}}(7; 8; 15)$.

We execute Algorithm $A_3(n = 17; k = 2; t = 2)$ with input $\mathcal{A} = \mathcal{H}^2_{\text{max}}(1, 7; 8; 15) = \mathcal{H}^2_{\text{max}}(7; 8; 15)$ to obtain all graphs in $\mathcal{B} = \mathcal{H}^2_{\text{max}}(2, 7; 8; 17)$.

By executing Algorithm $A_3(n = 19; k = 2; t = 2)$ with input $\mathcal{A} = \mathcal{H}^2_{\text{max}}(1, 2, 7; 8; 17) = \mathcal{H}^2_{\text{max}}(2, 7; 8; 17)$, we obtain $\mathcal{B} = \emptyset$. According
| set                        | independence number | maximal graphs | (+$K_7$)-graphs |
|----------------------------|---------------------|----------------|-----------------|
| $H_v(4; 8; 8)$             | $\leq 3$            | 1              | 4               |
| $H_v(5; 8; 10)$            | $\leq 3$            | 3              | 45              |
| $H_v(6; 8; 12)$            | $\leq 3$            | 12             | 3 104           |
| $H_v(7; 8; 14)$            | $\leq 3$            | 169            | 4 776 518       |
| $H_v(2, 7; 8; 16)$         | $\leq 3$            | 34             | 22 896          |
| $H_v(2, 2, 7; 8; 19)$      | $= 3$               | 0              |                 |
| $H_v(4; 8; 9)$             | $\leq 2$            | 1              | 8               |
| $H_v(5; 8; 11)$            | $\leq 2$            | 3              | 84              |
| $H_v(6; 8; 13)$            | $\leq 2$            | 10             | 5 394           |
| $H_v(7; 8; 15)$            | $\leq 2$            | 102            | 4 984 994       |
| $H_v(2, 7; 8; 17)$         | $\leq 2$            | 2760           | 380 361 736     |
| $H_v(2, 2, 7; 8; 19)$      | $= 2$               | 0              |                 |
| $H_v(2, 2, 7; 8; 19)$      |                     | 0              |                 |

Table 4.1: Steps in finding all maximal graphs in $H_v(2, 2, 7; 8; 19)$

To Theorem 3.5, there are no graphs in $H_{\text{max}}(2, 2, 7; 8; 19)$ with independence number 2.

Thus, we proved $H_{\text{max}}(2, 2, 7; 8; 19) = \emptyset$, and therefore $F_v(2, 2, 7; 8) \geq 20$.

The number of graphs obtained in each step of the proof is given in Table 4.1. The computer needed almost two weeks to complete the computations, because of the large number of graphs in $H_2^2 + K_7(2, 7; 8; 17)$.

2. Proof of the upper bound $F_v(2, 2, 7; 8) \leq 20$.

We need to construct a 20-vertex graph in $H_v(2, 2, 7; 8; 20)$. All vertex transitive graphs with up to 31 vertices are known and can be found in [87]. With the help of a computer we check which of these graphs belong to $H_v(2, 2, 7; 8)$. This way, we find 4 24-vertex graphs in $H_v(2, 2, 7; 8)$.

By removing one vertex from the 24-vertex transitive graphs in $H_v(2, 2, 7; 8)$, we obtain 3 23-vertex graphs in $H_v(2, 2, 7; 8)$, and by removing two vertices, we obtain 8 22-vertex graphs in $H_v(2, 2, 7; 8)$. We add edges to one of the 8 22-vertex graphs to obtain one graph in $H_{\text{max}}(2, 2, 7; 8; 22)$. Using Procedure 2.21, we find 1696 more graphs in $H_{\text{max}}(2, 2, 7; 8; 22)$.

By removing one vertex from the obtained graphs in $H_{\text{max}}(2, 2, 7; 8; 22)$, we find 22 21-vertex graphs in $H_v(2, 2, 7; 8)$. We add edges to these graphs to obtain 22 graphs in $H_{\text{max}}(2, 2, 7; 8; 21)$. Then we apply Procedure 2.21 twice to obtain 15259 more graphs in $H_{\text{max}}(2, 2, 7; 8; 21)$.
By removing one vertex from the obtained graphs in $\mathcal{H}_{\text{max}}(2, 2, 7; 8; 21)$, we find 9 20-vertex graphs in $\mathcal{H}_v(2, 2, 7; 8)$. Again, by successively applying Procedure 2.21 we obtain 39 graphs in $\mathcal{H}_{\text{max}}(2, 2, 7; 8; 20)$. One of these graphs is the graph $G_{2,2,7}$ shown in Figure 4.1. Further, we will use the graph $G_{2,2,7}$ in the proof of Theorem 4.1. The 20-vertex graphs in $\mathcal{H}_{\text{max}}(2, 2, 7; 8; 20)$ that we found are obtained from one of the 24-vertex transitive graphs. However, at the beginning of the proof we use all 4 graphs, since we do not know in advance which one will be useful. In [7] we omitted to note that all 4 24-vertex transitive graphs are used.

We proved that $F_v(2, 2, 7; 20) \leq 20$, which finishes the proof of Theorem 4.2. \qed

Theorem 4.2 is published in [7].
4.2 Algorithm A4

In this section we present one optimization to Algorithm A3.

We will use the following term:

**Definition 4.3.** We say that \( v \) is a cone vertex in the graph \( G \) if \( v \) is adjacent to all other vertices in \( G \), i.e. \( G = K_1 + H \).

**Proposition 4.4.** Let \( G = K_1 + H \in \mathcal{H}_{\max}(a_1, ..., a_s; q; n) \) and \( a_1 \geq 2 \). Then \( H \in \mathcal{H}_{\max}(a_1 - 1, a_2, ..., a_s; q - 1; n - 1) \).

**Proof.** According to Proposition 1.4, \( H \rightarrow (a_1 - 1, a_2, ..., a_s) \). It is clear that \( \omega(H) = \omega(G) - 1 \), and therefore \( H \in \mathcal{H}_v(a_1 - 1, a_2, ..., a_s; q - 1; n - 1) \). Since \( G \in \mathcal{H}_{\max}(a_1, ..., a_s; q; n) \), it follows that \( H \in \mathcal{H}_{\max}(a_1 - 1, a_2, ..., a_s; q - 1; n - 1) \). \( \square \)

According to Proposition 4.4, if we know all the graphs in \( \mathcal{H}_{\max}(a_1 - 1, a_2, ..., a_s; q - 1; n - 1) \) we can easily obtain the graphs in \( \mathcal{H}_{\max}(a_1, ..., a_s; q; n) \) which have a cone vertex. We will use this fact to modify Algorithm A3 and make it faster in the cases where all graphs in \( \mathcal{H}_{\max}(a_1 - 1, a_2, ..., a_s; q - 1; n - 1) \) are already known. To find the graphs in \( \mathcal{H}_{\max}(a_1, ..., a_s; q; n) \) which have no cone vertices we will need the following:

**Proposition 4.5.** Let \( G \in \mathcal{H}_{\max}(a_1, ..., a_s; q; n) \) be a graph without cone vertices and \( A \) be an independent set in \( G \) such that \( G - A \) has a cone vertex, i.e. \( G - A = K_1 + H \). Then \( G = \overline{K}_{k+1} + H \), where \( k = |A| \), \( H \) has no cone vertices, and \( K_1 + H \in \mathcal{H}_{\max}(a_1, ..., a_s; q; n - k) \).

**Proof.** Let \( A = \{v_1, ..., v_k\} \) be an independent set in \( G \) and \( G - A = K_1 + H = \{u\} + H \). Since \( G \) has no cone vertices, there exist \( v_i \in A \) such that \( v_i \) is not adjacent to \( u \). Then \( N_G(v_i) \subseteq N_G(u) \), and since \( G \) is a maximal \( K_q \)-free graph, we obtain \( N_G(v_i) = N_G(u) = V(H) \). Hence, \( u \) is not adjacent to any of the vertices in \( A \), and therefore \( N_G(v_j) = N_G(u) = V(H), j = 1, ..., k \).

We derived \( G = \overline{K}_{k+1} + H \). The graph \( H \) has no cone vertices, since any cone vertex in \( H \) would be a cone vertex in \( G \). It is easy to see that from \( \overline{K}_{k+1} + H \rightarrow (a_1, ..., a_s) \) it follows \( K_1 + H \rightarrow (a_1, ..., a_s) \). It is clear that \( K_1 + H \in \mathcal{H}_{\max}(a_1, ..., a_s; q; n - k) \). \( \square \)
Now we present the optimized algorithm:

**Algorithm A4.** The input of the algorithm are the set \( A_1 = \mathcal{H}_{max}(a_1 - 1, a_2, ..., a_s; q; n - k) \) and the set \( A_2 = \mathcal{H}_{max}(a_1 - 1, a_2, ..., a_s; q - 1; n - 1) \), where \( a_1, ..., a_s, q, n, k, t \) are fixed positive integers, \( a_1 \geq 2 \) and \( k \leq t \).

The output of the algorithm is the set \( B \) of all graphs \( G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n) \) with \( \alpha(G) \geq k \).

1. By removing edges from the graphs in \( A_1 \), obtain the set
   \[ A'_1 = \left\{ H \in \mathcal{H}_{+K_{q-1}}(a_1 - 1, a_2, ..., a_s; q; n - k) : \text{H has no cone vertices} \right\} \]
2. Repeat step 2 of Algorithm A3.
3. Repeat step 3 of Algorithm A3.
4. Repeat step 4 of Algorithm A3.
5. If \( t > k \), find the subset \( A''_1 \) of \( A_1 \) containing all graphs with exactly one cone vertex. For each graph \( K_1 + H \in A''_1 \), if \( K_1 + H \not\rightarrow (a_1, ..., a_s) \), then add \( K_{k+1} + H \) to \( B \).
6. For each graph \( H \) in \( A_2 \) such that \( \alpha(H) \geq k \), if \( K_1 + H \not\rightarrow (a_1, ..., a_s) \), then add \( K_1 + H \) to \( B \).

**Theorem 4.6.** After the execution of Algorithm A4, the obtained set \( B \) coincides with the set of all graphs \( G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n) \) with \( \alpha(G) \geq k \).

**Proof.** Suppose that after the execution of Algorithm A4, \( G \in B \). If after step 4 \( G \in B \), then according to the proof of Theorem 3.5, \( G \in \mathcal{H}_{max}(a_1, ..., a_s; q; n) \) and \( \alpha(G) \geq k \).

If \( G \) is added to \( B \) in step 5, then \( G = \overline{K}_{k+1} + H \) and from \( K_1 + H \not\rightarrow (a_1, ..., a_s) \) it follows \( G \not\rightarrow (a_1, ..., a_s) \). Since \( \alpha(H) \leq t \) and \( t > k \), we have \( \alpha(G) \leq t \). In this case it is clear that \( \alpha(G) \geq k + 1 \). Since \( K_1 + H \in \mathcal{H}_{max}(a_1, ..., a_s; q; n - k) \) and \( K_1 + H \) is not a complete graph, it follows that \( G = \overline{K}_{k+1} + H \in \mathcal{H}_{max}(a_1, ..., a_s; q; n) \).

If \( G \) is added to \( B \) in step 6, then \( G = K_1 + H \not\rightarrow (a_1, ..., a_s) \), where \( H \in A_2 \) and \( \alpha(H) \geq k \). Since \( k \leq \alpha(H) \leq t \), we have \( k \leq \alpha(G) \leq t \). From \( H \in \mathcal{H}_{max}(a_1 - 1, a_2, a_s; q - 1; n - 1) \) it follows that \( G = K_1 + H \in \mathcal{H}_{max}(a_1, ..., a_s; q; n) \).
Now let $G \in \mathcal{H}_{\text{max}}^{t}(a_1, ..., a_s; q; n)$ and $\alpha(G) \geq k$. If $G = K_1 + H$ for some graph $H$, then, according to Proposition 4.4 $H \in \mathcal{A}_2$ and in step 6 $G$ is added to $\mathcal{B}$.

Suppose that $G$ has no cone vertices. Let $A \subseteq V(G)$ be an independent set such that $|A| = k$.

If $G - A$ has a cone vertex, i.e. $G - A = K_1 + H$, then, according to Proposition 4.5 $G = \overline{K}_{k+1} + H$, where $K_1 + H \in \mathcal{A}_1''$. In this case it is clear that $t > k$, and therefore in step 5 $G$ is added to $\mathcal{B}$.

If $G - A$ has no cone vertices, then according to Proposition 1.10 $G - A \in \mathcal{A}_1'$. From the proof of Theorem 3.5 it follows that after the execution of step 4, $G \in \mathcal{B}$.

Remark 4.7. Note that if $G \in \mathcal{H}_{\text{max}}^{t}(a_1, ..., a_s; q; n)$ and $n \geq q$, then $G$ is not a complete graph and $\alpha(G) \geq 2$. Therefore, if $n \geq q$ and $k = 2$, Algorithm $\mathcal{A}_4$ finds all graphs in $\mathcal{H}_{\text{max}}^{t}(a_1, ..., a_s; q; n)$.

We will use Algorithm $\mathcal{A}_4$ to prove Conjecture 2.16 in the case $p = 7$ (Theorem 4.8). We could prove Theorem 4.8 using Algorithm $\mathcal{A}_3$, but it would take us more than a month of computational time, while the presented proof was completed in just one day. To test our implementation of Algorithm $\mathcal{A}_4$ we proved again in a different way Conjecture 2.16 in the case $p = 5$ (Theorem 2.17) and Conjecture 3.16 in the case $p = 6$ (Theorem 3.14), and compared the graphs obtained in each step of the proof to the graphs obtained in the proofs of these theorems.

Theorem 4.6 is published in [6]. Algorithm $\mathcal{A}_4$ is published in [6] as Algorithm 3.9 (due to a typographical error, in step 5 of Algorithm 3.9 in [6] it is written $H \in \mathcal{A}_1''$ instead of the correct $K_1 + H \in \mathcal{A}_1'$).
4.3 Computation of the numbers

\[ F_v(2m-7, 7; m-1) \]

Let us remind that the numbers \( r'_0(p) \) are defined in the formulation of Theorem 2.12.

We will prove that Conjecture 2.16 is true in the case \( p = 7 \).

**Theorem 4.8.** \( r'_0(7) = 2 \) and \( F_v(2m-7, 7; m-1) = m + 11, \ m \geq 9 \).

**Proof.** Since \( F_v(2, 2, 7; 8) = 20 \), according to Theorem 2.12(d), to prove that \( r'_0(7) = 2 \) we have to prove the inequalities \( F_v(2, 2, 2, 7; 9) > 20, \ F_v(2, 2, 2, 7; 10) > 21, \) and \( F_v(2, 2, 2, 2, 7; 11) > 22 \). By Lemma 2.11, it is enough to prove only the last of the three inequalities. Using Algorithm A3, it is possible to prove that \( F_v(2, 2, 2, 2, 7; 11) > 22 \), but it would take a lot of time. Instead, we will prove the three inequalities successively with Algorithm A4. Only the proof of the first inequality is presented in details, since the proofs of the other two are very similar. We will show that \( \mathcal{H}_v(2, 2, 7; 8; 19) = \emptyset \). The proof uses the graphs \( \mathcal{H}^3_{\text{max}}(4; 8; 8), \mathcal{H}^3_{\text{max}}(5; 8; 10), \mathcal{H}^3_{\text{max}}(6; 8; 12), \mathcal{H}^3_{\text{max}}(7; 8; 14), \mathcal{H}^3_{\text{max}}(2, 7; 8; 16), \mathcal{H}^3_{\text{max}}(2, 2, 7; 8; 19), \mathcal{H}^2_{\text{max}}(4; 8; 9), \mathcal{H}^2_{\text{max}}(5; 8; 11), \mathcal{H}^2_{\text{max}}(6; 8; 13), \mathcal{H}^2_{\text{max}}(7; 8; 15), \mathcal{H}^2_{\text{max}}(2, 7; 8; 17), \mathcal{H}^2_{\text{max}}(2, 2, 7; 8; 19) \) obtained in the proof of the lower bound \( F_v(2, 2, 7; 8) \geq 20 \) (see Table 4.1).

Suppose that \( G \in \mathcal{H}_v(2, 2, 7; 9; 20) \). According to Theorem 1.5(b), \( \alpha(G) \leq 3 \). It is clear that \( \alpha(G) \geq 2 \). Therefore, it is enough to prove that there are no graphs with independence number 2 or 3 in \( \mathcal{H}_{\text{max}}(2, 2, 7; 9; 20) \).

First we prove that there are no graphs in \( \mathcal{H}_{\text{max}}(2, 2, 7; 9; 20) \) with independence number 3:

It is clear that \( K_7 \) is the only graph in \( \mathcal{H}_{\text{max}}(4; 9; 7) \).
We execute Algorithm A4\((n = 9; k = 2; t = 3)\) with inputs \(A_1 = \mathcal{H}_\text{max}^3(4; 9; 7) = \{K_7\}\) and \(A_2 = \mathcal{H}_\text{max}^3(4; 8; 8)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}_\text{max}^3(5; 9; 9)\) (see Remark 4.7).

We execute Algorithm A4\((n = 11; k = 2; t = 3)\) with inputs \(A_1 = \mathcal{H}_\text{max}^3(5; 9; 9)\) and \(A_2 = \mathcal{H}_\text{max}^3(5; 8; 10)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}_\text{max}^3(6; 9; 11)\).

We execute Algorithm A4\((n = 13; k = 2; t = 3)\) with inputs \(A_1 = \mathcal{H}_\text{max}^3(6; 9; 11)\) and \(A_2 = \mathcal{H}_\text{max}^3(6; 8; 12)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}_\text{max}^3(7; 9; 13)\).

We execute Algorithm A4\((n = 15; k = 2; t = 3)\) with inputs \(A_1 = \mathcal{H}_\text{max}^3(1, 7; 9; 13) = \mathcal{H}_\text{max}^3(7; 9; 13)\) and \(A_2 = \mathcal{H}_\text{max}^3(7; 8; 14)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}_\text{max}^3(2, 7; 9; 15)\).

We execute Algorithm A4\((n = 17; k = 2; t = 3)\) with inputs \(A_1 = \mathcal{H}_\text{max}^3(1, 2, 7; 9; 15) = \mathcal{H}_\text{max}^3(2, 7; 9; 15)\) and \(A_2 = \mathcal{H}_\text{max}^3(1, 2, 7; 8; 16) = \mathcal{H}_\text{max}^3(2, 7; 8; 16)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}_\text{max}^3(2, 2, 7; 9; 17)\).

By executing Algorithm A4\((n = 20; k = 3; t = 3)\) with inputs \(A_1 = \mathcal{H}_\text{max}^3(1, 2, 2, 7; 9; 17) = \mathcal{H}_\text{max}^3(2, 2, 7; 9; 17)\) and \(A_2 = \mathcal{H}_\text{max}^3(1, 2, 2, 7; 8; 19) = \mathcal{H}_\text{max}^3(2, 2, 7; 8; 19)\), we obtain \(\mathcal{B} = \emptyset\). According to Theorem 4.6, there are no graphs in \(\mathcal{H}_\text{max}(2, 2, 2, 7; 9; 20)\) with independence number 3.

It remains to be proved that there are no graphs in \(\mathcal{H}_\text{max}(2, 2, 2, 7; 9; 20)\) with independence number 2.

It is clear that \(K_8\) is the only graph in \(\mathcal{H}_\text{max}(4; 9; 8)\).

We execute Algorithm A4\((n = 10; k = 2; t = 2)\) with inputs \(A_1 = \mathcal{H}_\text{max}^2(4; 9; 8) = \{K_8\}\) and \(A_2 = \mathcal{H}_\text{max}^2(4; 8; 9)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}_\text{max}^2(5; 9; 10)\) (see Remark 4.7).

We execute Algorithm A4\((n = 12; k = 2; t = 2)\) with inputs \(A_1 = \mathcal{H}_\text{max}^2(5; 9; 10)\) and \(A_2 = \mathcal{H}_\text{max}^2(5; 8; 11)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}_\text{max}^2(6; 9; 12)\).

We execute Algorithm A4\((n = 14; k = 2; t = 2)\) with inputs \(A_1 = \mathcal{H}_\text{max}^2(6; 9; 12)\) and \(A_2 = \mathcal{H}_\text{max}^2(6; 8; 13)\) to obtain all graphs in \(\mathcal{B} = \mathcal{H}_\text{max}^2(7; 9; 14)\).
We execute Algorithm $A_4(n = 16; k = 2; t = 2)$ with inputs $A_1 = H_{\max}^2(1, 7; 9; 14) = H_{\max}^2(7; 9; 14)$ and $A_2 = H_{\max}^2(7; 8; 15)$ to obtain all graphs in $B = H_{\max}^3(2, 7; 9; 16)$.

We execute Algorithm $A_4(n = 18; k = 2; t = 2)$ with inputs $A_1 = H_{\max}^2(1, 2, 7; 9; 16) = H_{\max}^2(2, 7; 9; 16)$ and $A_2 = H_{\max}^2(1, 2, 7; 8; 17) = H_{\max}^2(2, 7; 8; 17)$ to obtain all graphs in $B = H_{\max}^3(2, 2, 7; 9; 18)$.

By executing Algorithm $A_4(n = 20; k = 2; t = 2)$ with inputs $A_1 = H_{\max}^2(1, 2, 2, 7; 9; 18) = H_{\max}^2(2, 2, 7; 9; 18)$ and $A_2 = H_{\max}^2(1, 2, 2, 7; 8; 19) = H_{\max}^2(2, 2, 7; 8; 19)$, we obtain $B = \emptyset$. According to Theorem 4.6, there are no graphs in $H_{\max}^2(2, 2, 2, 7; 9; 20)$ with independence number 2.

We proved that $H_{\max}^2(2, 2, 2, 7; 9; 20) = \emptyset$ and $F_v(2, 2, 2, 7; 9) > 20$.

In the same way, the graphs obtained in the proof of the inequality $F_v(2, 2, 2, 7; 9) > 20$ are used to prove $F_v(2, 2, 2, 2, 7; 10) > 21$, and the graphs obtained in the proof of the inequality $F_v(2, 2, 2, 2, 7; 10) > 21$ are used to prove $F_v(2, 2, 2, 2, 2, 7; 11) > 22$. The number of graphs obtained in each step of the proofs is given in Table 4.2, Table 4.3, and Table 4.4. Notice that the number of graphs without cone vertices is relatively small, which reduces the computation time significantly.

Thus, we proved that $r_0^*(7) = 2$. From Theorem 2.12(b) we obtain

\[ F_v(2m-7, 7; m - 1) = m + 11, \quad m \geq 9. \]

\[ \square \]

Theorem 4.8 is published in [6].
### Table 4.2: Steps in finding all maximal graphs in $H_v(2, 2, 2, 7; 9; 20)$

| set                  | independence number | maximal graphs | maximal graphs no cone v | (+$K_9$)-graphs | (+$K_{10}$)-graphs |
|----------------------|---------------------|----------------|--------------------------|------------------|---------------------|
| $H_v(4, 9; 7)$        | $\leq 3$            | 1              | 0                        | 1                | 0                   |
| $H_v(5, 9; 9)$        | $\leq 3$            | 1              | 0                        | 4                | 0                   |
| $H_v(6, 9; 11)$       | $\leq 3$            | 3              | 0                        | 45               | 0                   |
| $H_v(7, 9; 13)$       | $\leq 3$            | 12             | 0                        | 3 113            | 9                   |
| $H_v(2, 7, 9; 15)$    | $\leq 3$            | 169            | 0                        | 4 783 615        | 7 097               |
| $H_v(2, 2, 2, 7; 9, 17)$ | $\leq 3$            | 36             | 2                        | 22 918           | 22                  |
| $H_v(2, 2, 2, 7; 9, 20)$ | $= 3$              | 0              | 0                        | 0                | 0                   |
| $H_v(4, 9; 8)$        | $\leq 2$            | 1              | 0                        | 1                | 0                   |
| $H_v(5, 9; 10)$       | $\leq 2$            | 1              | 0                        | 8                | 0                   |
| $H_v(6, 9; 12)$       | $\leq 2$            | 3              | 0                        | 85               | 1                   |
| $H_v(7, 9; 14)$       | $\leq 2$            | 10             | 0                        | 5 474            | 80                  |
| $H_v(2, 7, 9; 16)$    | $\leq 2$            | 103            | 1                        | 5 346 982        | 361 988             |
| $H_v(2, 2, 7, 9; 18)$ | $\leq 2$            | 2845           | 85                       | 387 948 338      | 7 586 602           |
| $H_v(2, 2, 2, 7; 9, 20)$ | $= 2$              | 0              | 0                        | 0                | 0                   |

### Table 4.3: Steps in finding all maximal graphs in $H_v(2, 2, 2, 7; 10; 21)$

| set                  | independence number | maximal graphs | maximal graphs no cone v | (+$K_9$)-graphs | (+$K_{10}$)-graphs |
|----------------------|---------------------|----------------|--------------------------|------------------|---------------------|
| $H_v(5, 10; 8)$      | $\leq 3$            | 1              | 0                        | 1                | 0                   |
| $H_v(6, 10; 10)$     | $\leq 3$            | 1              | 0                        | 4                | 0                   |
| $H_v(7, 10; 12)$     | $\leq 3$            | 3              | 0                        | 45               | 0                   |
| $H_v(2, 7, 10; 14)$  | $\leq 3$            | 12             | 0                        | 3 113            | 2                   |
| $H_v(2, 2, 7, 10; 16)$ | $\leq 3$            | 169            | 0                        | 4 784 483        | 868                 |
| $H_v(2, 2, 2, 7; 10; 18)$ | $\leq 3$            | 36             | 0                        | 22 919           | 1                   |
| $H_v(2, 2, 2, 2, 7; 10; 21)$ | $= 3$              | 0              | 0                        | 0                | 0                   |
| $H_v(5, 10; 9)$      | $\leq 2$            | 0              | 0                        | 0                | 0                   |
| $H_v(6, 10; 11)$     | $\leq 2$            | 0              | 0                        | 0                | 0                   |
| $H_v(7, 10; 13)$     | $\leq 2$            | 0              | 0                        | 0                | 0                   |
| $H_v(2, 7, 10; 15)$  | $\leq 2$            | 0              | 0                        | 0                | 0                   |
| $H_v(2, 2, 7, 10; 17)$ | $\leq 2$            | 0              | 0                        | 0                | 0                   |
| $H_v(2, 2, 2, 7; 10; 19)$ | $\leq 2$            | 0              | 0                        | 0                | 0                   |
| $H_v(2, 2, 2, 2, 7; 10; 21)$ | $= 2$              | 0              | 0                        | 0                | 0                   |

### Table 4.4: Steps in finding all maximal graphs in $H_v(2, 2, 2, 2, 7; 11; 22)$

| set                  | independence number | maximal graphs | maximal graphs no cone v | (+$K_9$)-graphs | (+$K_{10}$)-graphs |
|----------------------|---------------------|----------------|--------------------------|------------------|---------------------|
| $H_v(6, 11; 9)$      | $\leq 3$            | 1              | 0                        | 1                | 0                   |
| $H_v(7, 11; 11)$     | $\leq 3$            | 1              | 0                        | 8                | 0                   |
| $H_v(2, 7, 11; 13)$  | $\leq 3$            | 3              | 0                        | 85               | 0                   |
| $H_v(2, 2, 7, 11; 15)$ | $\leq 3$            | 12             | 0                        | 3 116            | 1                   |
| $H_v(2, 2, 2, 7; 11; 17)$ | $\leq 3$            | 169            | 0                        | 4 784 638        | 155                 |
| $H_v(2, 2, 2, 2, 7; 11; 19)$ | $\leq 3$            | 36             | 0                        | 22 919           | 0                   |
| $H_v(2, 2, 2, 2, 2, 7; 11; 22)$ | $= 3$              | 0              | 0                        | 0                | 0                   |
| $H_v(6, 11; 10)$     | $\leq 2$            | 1              | 0                        | 1                | 0                   |
| $H_v(7, 11; 12)$     | $\leq 2$            | 1              | 0                        | 8                | 0                   |
| $H_v(2, 7, 11; 14)$  | $\leq 2$            | 3              | 0                        | 85               | 0                   |
| $H_v(2, 2, 7, 11; 16)$ | $\leq 2$            | 10             | 0                        | 5 502            | 7                   |
| $H_v(2, 2, 2, 7; 11; 18)$ | $\leq 2$            | 103            | 0                        | 5 374 143        | 2 492               |
| $H_v(2, 2, 2, 2, 7; 11; 20)$ | $\leq 2$            | 2848           | 0                        | 387 968 676      | 18                  |
| $H_v(2, 2, 2, 2, 2, 7; 11; 22)$ | $= 2$              | 0              | 0                        | 0                | 0                   |
| $H_v(2, 2, 2, 2, 2, 7; 11; 23)$ | $= 2$              | 0              | 0                        | 0                | 0                   |
4.4 Proof of Theorem 4.1

The lower bound follows from Theorem 4.8 and Theorem 1.19.

To prove the upper bound, we will use the graph \( \Gamma_4 \in H_v(3, 7; 8) \cap H_v(4, 6; 8) \cap H_v(5, 5; 8) \) (see Figure 4.2) obtained by adding one vertex to the graph \( G_{2,2,7} \in H_v(2, 2, 7; 20) \) (see Figure 4.1). According to Proposition 1.16, from \( \Gamma_4 \rightarrow (3, 7) \), \( \Gamma_4 \rightarrow (4, 6) \), and \( \Gamma_4 \rightarrow (5, 5) \) it follows \( \Gamma_4 \rightarrow 9|_{\gamma} \). Therefore, \( \Gamma_4 \in \tilde{H}_v(9|_{\gamma}; 8) \) and \( \tilde{F}_v(9|_{\gamma}; 8) \leq 21 \). Now, from Theorem 1.19 and Theorem 1.18 we derive

\[
F_v(a_1, ..., a_s; m - 1) \leq \tilde{F}_v(m|_{\gamma}; m - 1) \leq \tilde{F}_v(9|_{\gamma}; 8) + m - 9 \leq m + 12.
\]

Thus, the theorem is proved.

Regarding the number \( F_v(3, 7; 8) \), the following bounds were known:

\[
18 \leq F_v(3, 7; 8) \leq 22.
\]

The lower bound is true according to (1.8) and the upper bound was proved in [91]. We improve these bounds by proving the following

**Theorem 4.9.** \( 20 \leq F_v(3, 7; 8) \leq 21 \).

**Proof.** The upper bound is true according to Theorem 4.1 and the lower bound follows from \( F_v(3, 7; 8) \geq F_v(2, 2, 7; 8) = 20 \).

Theorem 4.1 and Theorem 4.9 are published in [6].
Figure 4.2: Graph $\Gamma_4 \in \mathcal{H}_v(3,7;8;21) \cap \mathcal{H}_v(4,6;8;21) \cap \mathcal{H}_v(5,5;8;21)$
Chapter 5

Numbers of the form $F_v(a_1, \ldots, a_s; m - 2)$

Not much is known about the numbers $F_v(a_1, \ldots, a_s; q)$ when $q \leq m - 2$. In the case $q = m - 2$ we know all the numbers where $p = \max \{a_1, \ldots, a_s\} = 2$, i.e. all the numbers of the form $F_v(2_r; r - 1)$. Nenov proved in [61] that $F_v(2_r; r - 1) = r + 7$ if $r \geq 8$, and later in [73] he proved that the same equality holds when $r \geq 6$. Other proof of the same equality was given in [71]. By a computer aided research, Jensen and Royle [37] obtained the result $F_v(2_4; 3) = 22$. All extremal graphs in $H_v(2_4; 3)$, as well as other interesting graphs, are available in [9]. The last remaining number of this form is $F_v(2_5; 4) = 16$. The upper bound was proved by Nenov in [73] and the lower bound was proved by Lathrop and Radziszowski in [11] with the help of a computer. Also in [11] it is proved that there are exactly two extremal graphs in $H_v(2_5; 4)$. The number $F_v(2_5; 3)$ is also of significant interest. The newest bounds for this number are $32 \leq F_v(2_5; 3) \leq 40$ obtained by Goedgebeur [26].

The exact values of the numbers $F_v(a_1, \ldots, a_s; m - 2)$ for which $p \geq 3$ are unknown and no good general bounds have been obtained. According to [1.2], $F_v(a_1, \ldots, a_s; m - 2)$ exists if and only if $m \geq p + 3$. If $p = 3$, it is easy to see that in the border case $m = 6$ there are only two numbers of the form $F_v(a_1, \ldots, a_s; m - 2)$, namely $F_v(2, 2, 2, 3; 4)$ and $F_v(2, 3, 3; 4)$. Since $G \not\rightarrow (2, 3, 3) \Rightarrow G \not\rightarrow (2, 2, 2, 3)$, it follows that

$$F_v(2, 2, 2, 3; 4) \leq F_v(2, 3, 3; 4).$$
Computing and obtaining bounds on the numbers $F_v(2, 2, 2, 3; 4)$ and $F_v(2, 3, 3; 4)$ has an important role in relation to computing and obtaining bounds on the other numbers of the form $F_v(a_1, ..., a_s; m - 2)$ for which $p = 3$ (see [3] for more details).

Shao, Xu, and Luo [90] proved in 2009 that

$$18 \leq F_v(2, 2, 2, 3; 4) \leq F_v(2, 3, 3; 4) \leq 30.$$ 

In 2011 Shao, Liang, He, and Xu [88] raised the lower bound on $F_v(2, 3, 3; 4)$ to 19.

Our interest in the number $F_v(2, 3, 3; 4)$ is also motivated by our conjecture that the inequality $F_v(2, 3, 3; 4) \leq F_e(3, 3; 4)$ is true. If this inequality holds, then there would be another possible way to prove a lower bound on the number $F_e(3, 3; 4)$ by bounding the number $F_v(2, 3, 3; 4)$.

We improve the bounds on the numbers $F_v(2, 2, 2, 3; 4)$ and $F_v(2, 3, 3; 4)$ by proving the following theorems:

**Theorem 5.1.** $20 \leq F_v(2, 2, 2, 3; 4) \leq 22$.

**Theorem 5.2.** $20 \leq F_v(2, 3, 3; 4) \leq 24$.

### 5.1 Algorithm A5

According to Proposition 1.10 if $G \rightarrow (2, 3, 3)$ and the graph $H$ is obtained by removing an independent set of vertices from $G$, then $H \rightarrow (3, 3)$. Shao, Liang, He, and Xu showed in [88] that if $G \in \mathcal{H}_v(2, 3, 3; 4; 18)$, then $G$ can be obtained by adding 4 independent vertices to 111 of all the 153 graphs in $\mathcal{H}_v(3, 3; 4; 14)$ obtained in [78]. In this way, with the help of a computer, the authors proved that $\mathcal{H}_v(2, 3, 3; 4; 18) = \emptyset$, and therefore $F_v(2, 3, 3; 4) \geq 19$. Similarly, if $G \in \mathcal{H}_v(2, 2, 2, 3; 4; 18)$, then $G$ can be obtained by adding 4 independent vertices to 12064 of the 12227 graphs in $\mathcal{H}_v(2, 2, 3; 4; 14)$, which were obtained in [15]. Proving the bound $F_v(2, 2, 2, 3; 4) \geq 19$ is harder because of the larger number of graphs that have to be extended.
Definition 5.3. We say that $G$ is a Sperner graph if $N_G(u) \subseteq N_G(v)$ for some pair of vertices $u, v \in V(G)$.

Proposition 5.4. If $G \in \mathcal{H}_v(2_r, 3; 4; n)$ is a Sperner graph and $N_G(u) \subseteq N_G(v)$, then $G - u \in \mathcal{H}_v(2_r, 3; 4; n - 1)$.

In the special case when $G$ is a maximal Sperner graph in $\mathcal{H}_v(2_r, 3; 4; n)$, from $N_G(u) \subseteq N_G(v)$ we derive $N_G(u) = N_G(v)$, and it follows that $G - u$ is a maximal graph in $\mathcal{H}_v(2_r, 3; 4; n - 1)$. Thus, we proved the following proposition:

Proposition 5.5. Every maximal Sperner graph in $\mathcal{H}_v(2_r, 3; 4; n)$ is obtained by duplicating a vertex in some of the maximal graphs in $\mathcal{H}_v(2_r, 3; 4; n - 1)$.

Proposition 5.6. Let $G$ be a non-Sperner graph, $A \subseteq V(G)$ be an independent set of vertices of $G$, and $H = G - A$. Then,

$$N_G(u) \not\subseteq N_H(v), \quad \forall u \in A \text{ and } \forall v \in V(H).$$

Proof. If we suppose the opposite is true, it follows that $N_G(v) \supseteq N_G(u)$, which is a contradiction.

We will use the following specialized version of Algorithm A3 to efficiently generate all non-Sperner graphs $G \in \mathcal{H}_{\text{max}}(2_r, 3; 4; n)$ with $\alpha(G) = k$:

Algorithm A5. The input of the algorithm is the set $A = \mathcal{H}_{\text{max}}^k(2_{r-1}, 3; 4; n - k)$, where $r, n, k$ are fixed positive integers.

The output of the algorithm is the set $B$ of all non-Sperner graphs $G \in \mathcal{H}_{\text{max}}(2_r, 3; 4; n)$ with $\alpha(G) = k$.

1. By removing edges from the graphs in $A$ obtain the set $A' = \mathcal{H}_{+K_3}^k(2_{r-1}, 3; 4; n - k)$.
2. For each graph $H \in A'$:
   2.1. Find the family $\mathcal{M}(H) = \{M_1, \ldots, M_l\}$ of all maximal $K_3$-free subsets of $V(H)$.
   2.2. Find all $k$-element subsets $N = \{M_{i_1}, \ldots, M_{i_k}\}$ of $\mathcal{M}(H)$ which fulfill the conditions:
(a) $M_{ij} \neq N_H(v)$ for every $v \in V(H)$ and for every $M_{ij} \in N$.
(b) $K_2 \subseteq M_{ij} \cap M_{ih}$ for every $M_{ij}, M_{ih} \in N, j \neq h$.
(c) $\alpha(H - \bigcup_{M_{ij} \in N'} M_{ij}) \leq k - |N'|$ for every $N' \subseteq N$.

2.3. For each of the found in step 2.2 $k$-element subsets $N = \{M_{i1}, ..., M_{ik}\}$ of $\mathcal{H}(H)$ construct the graph $G = G(N)$ by adding new independent vertices $v_1, ..., v_k$ to $V(H)$ such that $N_G(v_j) = M_{ij}, j = 1, ..., k$.

If $G$ is not a Sperner graph and $\omega(G + e) = 4, \forall e \in E(G)$, then add $G$ to $\mathcal{B}$.

3. Remove the isomorphic copies of graphs from $\mathcal{B}$.

4. Remove from $\mathcal{B}$ all graphs $G$ for which $G \not\rightarrow (2_r, 3)$.

We will prove the correctness of Algorithm $A5$ with the help of the following

Lemma 5.7. After the execution of step 2.3 of Algorithm $A5$, the obtained set $\mathcal{B}$ coincides with the set of all maximal $K_4$-free non-Sperner graphs $G$ with $\alpha(G) = k$ which have an independent set of vertices $A \subseteq V(G), |A| = k$ such that $G - A \in \mathcal{A}'$.

Proof. The proof follows the proof of Lemma 3.4. Suppose that in step 2.3 of Algorithm $A3$ the graph $G$ is added to $\mathcal{B}$. In the same way as in the proof of Lemma 3.4 we obtain $G \in \mathcal{H}_{\max}(2_r, 3; 4; n)$ and $\alpha(G) = k$. The check at the end of step 2.3 guarantees that $G$ is not a Sperner graph.

Let $G$ be a maximal $K_4$-free non-Sperner graph, $\alpha(G) = k$, and $A = \{v_1, ..., v_k\}$ be an independent set of vertices of $G$ such that $H = G - A \in \mathcal{A}'$. We will prove that, after the execution of step 2.3 of Algorithm $A5$, $G \in \mathcal{B}$. Let $N = \{N_G(v_1), ..., N_G(v_k)\}$. In the same way as in the proof of Lemma 3.4 we show that $N$ fulfills the conditions (b) and (c) in step 2.2. Since $G$ is not a Sperner graph, from Proposition 5.6 it follows that $N$ fulfills (a). Thus, we showed that $N$ fulfills all conditions in step 2.2, and since $G = G(N)$ is a maximal $K_4$-free non-Sperner graph, in step 2.3 $G$ is added to $\mathcal{B}$.

Theorem 5.8. After the execution of Algorithm $A5$, the obtained set $\mathcal{B}$ coincides with the set of all non-Sperner graphs with independence number $k$ in $\mathcal{H}_{\max}(2_r, 3; 4; n)$.
Proof. Suppose that, after the execution of Algorithm $A_5$, $G \in B$. According to Lemma 5.7, $G$ is a maximal $K_4$-free non-Sperner graph with independence number $k$. From step 4 it follows that $G \in H_{\text{max}}(2r, 3; 4; n)$.

Conversely, let $G$ be an arbitrary non-Sperner graph with independence number $k$ in $H_{\text{max}}(2r, 3; 4; n)$. Let $A \subseteq V(G)$ be an independent set of vertices of $G$, $|A| = k$ and $H = G - A$. According to Proposition 1.10, $H \in A'$, and from Lemma 5.7 it follows that, after the execution of step 2.3, $G \in B$. Clearly, after step 4, $G$ remains in $B$. □

We performed various tests to our implementation of Algorithm $A_5$. For example, we used the Algorithm $A_5$ to reproduce all 12227 graphs in $H_v(2, 2, 3; 4; 14)$, which were obtained in [15].

Theorem 5.8 is published in [3]. Algorithm $A_5$ is a slightly modified version of Algorithm 2.4 in [3].

5.2 Proof of Theorem 5.1 and Theorem 5.2

Proof of Theorem 5.1

1. Proof of the inequality $F_v(2, 2, 2, 3; 4) \geq 19$.

It is enough to prove that $H_v(2, 2, 2, 3; 4; 18) = \emptyset$. From $R(4, 4) = 18$ it follows that there are no graphs with independence number less than 4 in $H_v(2, 2, 2, 3; 4; 18)$, and from Proposition 1.10 and $F_v(2, 2, 3; 4) = 14$ [15] we derive that there are no graphs with independence number more than 4 in this set. From $F_v(2, 2, 2, 3; 4) \geq 18$ and Proposition 5.4 it follows that there are no Sperner graphs in $H_v(2, 2, 2, 3; 4; 18)$. It remains to be proved that there are no non-Sperner graphs with independence number 4 in $H_{\text{max}}(2, 2, 2; 3; 4; 18)$.

We execute Algorithm $A_5$ ($n = 18$; $r = 3$; $k = 4$) with input the set $A$ of all 584 graphs in $H_{\text{max}}^4(2, 2, 3; 4; 14)$. Let us remind, that all 12227 graphs in $H_v(2, 2, 3; 4; 14)$ were obtained in [15]. After the execution of step 3, 130923 graphs remain in the set $B$. None of these graphs belong to $H_v(2, 2, 2, 3; 4)$, and therefore after step 4 we have $B = \emptyset$. Now, from Theorem 5.8 we
conclude that there are no non-Sperner graphs with independence number 4 in $H_{\text{max}}(2, 2, 3; 4; 18)$, which finishes the proof.

2. Proof of the inequality $F_v(2, 2, 3; 4) \geq 20$.

It is enough to prove that $H_v(2, 2, 3; 4; 19) = \emptyset$. Again, from $R(4, 4) = 18$ it follows that there are no graphs with independence number less than 4 in $H_v(2, 2, 3; 4; 19)$, and from Proposition 1.10 and $F_v(2, 2, 3; 4) = 14$ [15] we derive that there are no graphs with independence number more than 5 in this set. From $F_v(2, 2, 3; 4) \geq 19$ and Proposition 5.4 it follows that there are no Sperner graphs in $H_v(2, 2, 3; 4; 19)$. It remains to be proved that there are no non-Sperner graphs with independence number 4 or 5 in $H_{\text{max}}(2, 2, 3; 4; 19)$:

First we prove that there are no non-Sperner graphs with independence number 5 in $H_{\text{max}}(2, 2, 3; 4; 19)$:

We execute Algorithm $\text{A5}$ ($n = 19; \ r = 3; \ k = 5$) with input the set $\mathcal{A}$ of all 591 graphs in $H_{\text{max}}^5(2, 2, 3; 4; 14)$ known from [15]. After the execution of step 3, 2743657 graphs remain in the set $\mathcal{B}$. Since none of these graphs are in $H_v(2, 2, 3; 4)$, after step 4 we have $\mathcal{B} = \emptyset$. Now, from Theorem 5.8 we conclude that there are no non-Sperner graphs with independence number 5 in $H_{\text{max}}(2, 2, 3; 4; 19)$.

It remains to be proved that there are no non-Sperner graphs with independence number 4 in $H_{\text{max}}(2, 2, 3; 4; 19)$:

Using the $\text{nauty}$ program [53] we generate all 11-vertex non-isomorphic graphs and among them we find all 353 graphs in $H_{\text{max}}^4(2, 3; 4; 11)$. We execute Algorithm $\text{A5}$ ($n = 15; \ r = 2; \ k = 4$) with input $\mathcal{A} = H_{\text{max}}^4(2, 3; 4; 11)$. By Theorem 5.8 we obtain all 165614 non-Sperner graphs in $H_{\text{max}}(2, 2, 3; 4; 15)$ with independence number 4.

According to Proposition 5.5 all Sperner graphs in $H_{\text{max}}(2, 2, 3; 4; 15)$ are obtained by duplicating a vertex in the graphs in $H_{\text{max}}(2, 2, 3; 4; 14)$. In this way, we find all 4603 Sperner graphs with independence number 4 in $H_{\text{max}}(2, 2, 3; 4; 15)$.
There are exactly 640 15-vertex $K_4$-free graphs with independence number less than 4, which are available on [52]. Among them there are 35 graphs in $\mathcal{H}_{\max}(2, 2, 3; 4; 15)$.

Thus, we found all 170252 graphs in $\mathcal{H}_{\max}^4(2, 2, 3; 4; 15)$.

We execute Algorithm $A_5$ ($n = 19$; $r = 3$; $k = 4$) with input $\mathcal{A} = \mathcal{H}_{\max}^4(2, 2, 3; 4; 15)$. After the execution of step 3, 347307340 graphs remain in the set $\mathcal{B}$. Similarly to the previous case, none of these graphs are in $\mathcal{H}_v(2, 2, 2, 3; 4)$, and after step 4 we have $\mathcal{B} = \emptyset$. By Theorem [5.8], there are no non-Sperner graphs with independence number 4 in $\mathcal{H}_{\max}(2, 2, 2, 3; 4; 19)$, which finishes the proof.

All computations were completed in about a week on a personal computer.

3. Proof of the inequality $F_v(2, 2, 2, 3; 4) \leq 22$.

We need to construct a 22-vertex graph in $\mathcal{H}_v(2, 2, 2, 3; 4)$. First, we find a large number of 24-vertex and 23-vertex maximal graphs in $\mathcal{H}_v(2, 2, 2, 3; 4)$.

All 352366 24-vertex graphs $G$ with $\omega(G) \leq 3$ and $\alpha(G) \leq 4$ were found by McKay, Radziszowski and Angeltveit (see [52]). Among these graphs there are 3903 maximal graphs in $\mathcal{H}_v(2, 2, 2, 3; 4; 24)$.

By removing one vertex from the obtained 24-vertex maximal graphs we find 6 graphs in $\mathcal{H}_v(2, 2, 2, 3; 4; 23)$. Let us note that the removal of two vertices from the 24-vertex maximal graphs does not produce any graphs in $\mathcal{H}_v(2, 2, 2, 3; 4; 22)$. Out of the 6 obtained graphs in $\mathcal{H}_v(2, 2, 2, 3; 4; 23)$, 5 are maximal. One more maximal graph is obtained by adding one edge to the 6th graph, thus we have 6 maximal graphs in $\mathcal{H}_v(2, 2, 2, 3; 4; 23)$. By applying Procedure [2.21] to these graphs, we find 192 more maximal graphs in $\mathcal{H}_v(2, 2, 2, 3; 4; 23)$.

By removing one vertex from the obtained 23-vertex maximal graphs we find a maximal graph in $\mathcal{H}_v(2, 2, 2, 3; 4; 22)$, which is shown in Figure [5.1]. By removing one edge from the maximal graph in $\mathcal{H}_v(2, 2, 2, 3; 4; 22)$, we obtain two more graphs in $\mathcal{H}_v(2, 2, 2, 3; 4; 22)$.

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Proof of Theorem \(5.2\)

1. Proof of the inequality \(F_v(2, 3, 3; 4) \geq 20\).
   Since \(F_v(2, 3, 3; 4) \geq F_v(2, 2, 2, 3; 4)\), the lower bound \(F_v(2, 3, 3; 4) \geq 20\) follows from Theorem \(5.1\).

2. Proof of the inequality \(F_v(2, 3, 3; 4) \leq 24\).
   All vertex transitive graphs with up to 31 vertices are known and can be found in [87]. With the help of a computer we check which of these graphs belong to \(H_v(2, 3, 3; 4)\). In this way, we find one 24-vertex graph, one 28-vertex graph and six 30-vertex graphs in \(H_v(2, 3, 3; 4)\). The only 24-vertex transitive graph in \(H_v(2, 3, 3; 4)\) is given in Figure \(5.2\). It does not have proper subgraphs in \(H_v(2, 3, 3; 4)\), but by adding edges to this graph we obtain 18 more graphs in \(H_v(2, 3, 3; 4; 24)\), two of which are maximal \(K_4\)-free graphs.

   Theorem \(5.1\) is published in [3]. Theorem \(5.2\) is published in [5]. The lower bound \(F_v(2, 3, 3; 4) \geq 20\) in Theorem \(5.2\) follows from Theorem \(5.1\) but it is proved first in [5] without using Theorem \(5.1\).
Figure 5.1: 22-vertex graph in $H_v(2, 2, 2, 3; 4)$

Figure 5.2: 24-vertex transitive graph in $H_v(2, 3, 3; 4)$
Chapter 6

Bounds on some numbers of the form

\[ F_v(a_1, \ldots, a_s; q) \] where \( q = \max \{a_1, \ldots, a_s\} + 1 \)

The computation of the numbers of this form is very hard. The numbers \( F_v(2, 2, p; p + 1), \ p \leq 4 \), and \( F_v(3, p; p + 1), \ p \leq 5 \), are already computed (see the text after (1.9)). We proved that \( F_v(2, 2, 5; 6) = 16 \) (Theorem 2.7), \( F_v(2, 2, 6; 7) = 17 \) (Theorem 3.8), \( F_v(3, 6; 7) = 18 \) (Theorem 3.9), and \( F_v(2, 2, 7; 8) = 20 \) (Theorem 4.2). The only other known number of this form is \( F_v(2, 2, 2; 3) = 22 \). [37]

We also obtained the bounds \( 20 \leq F_v(3, 7; 8) \leq 21 \) (Theorem 4.9). In the previous chapter we proved that \( 20 \leq F_v(2, 2, 2, 3; 4) \leq 22 \) (Theorem 5.1) \( 20 \leq F_v(2, 3, 3; 4) \leq 24 \) (Theorem 5.2).

The numbers \( F_v(p, p; p + 1) \) are of significant interest. Obviously, \( F_v(2, 2; 3) = 5 \). It is known that \( F_v(3, 3; 4) = 14 \), [59] and [78], and in [100] it is proved that \( 17 \leq F_v(4, 4; 5) \leq 23 \). For now, there is no good upper bound for \( F_v(5, 5; 6) \). In [101] it is proved that

\[(6.1) \quad F_v(p, p; p + 1) \geq 4p - 1.\]

Let us pose the following question:

Is it true, that the sequence \( F_v(p, p; p + 1), \ p \geq 2 \), is increasing?

In this chapter we will obtain new lower bounds on the numbers \( F_v(4, 4; 5), F_v(5, 5; 6), F_v(6, 6; 7), \) and \( F_v(7, 7; 8) \).

Let \( G \in \mathcal{H}_v(2_r, p; p + 1) \) and \( A \subseteq V(G) \) be an independent set. Then
obviously, $G - A \in \mathcal{H}_v(2r-1, p; p + 1)$, and therefore

$$F_v(2, r; p + 1) \geq F_v(2r-1, p; p + 1) + \alpha(r, p), \ r \geq 2,$$

where $\alpha(r, p) = \max \{\alpha(G) : G \in \mathcal{H}_{ext}(2r, p; p + 1)\}$.

From (6.2) it follows easily

$$F_v(2, r; p + 1) \geq F_v(2, 2; p + 1) + \sum_{i=3}^{r} \alpha(i, p), \ r \geq 3.$$

Since $\alpha(i, p) \geq 2$, from (6.3) we obtain

$$F_v(2, r; p + 1) \geq F_v(2, 2; p + 1) + 2(r - 2), \ r \geq 3.$$

From (6.4) and Theorem 1.19 we see that

$$F_v(p, p; p + 1) \geq F_v(2p - 1, p; p + 1) \geq F_v(2, 2; p + 1) + 2p - 6, \ p \geq 3.$$  

According to (1.8), $F_v(2, 2; p + 1) \geq 2p + 4$. If $F_v(2, 2; p; p + 1) = 2p + 4$, then the inequality (6.1) gives a better bound for $F_v(p, p; p + 1)$ than the inequality (6.5). It is interesting to note that it is not known whether the equality $F_v(2, 2; p; p + 1) = 2p + 4$ holds for any $p$. If $F_v(2, 2; p; p + 1) = 2p + 5$, then the bounds for $F_v(p, p; p + 1)$ from (6.1) and (6.5) coincide, and if $F_v(2, 2; p; p + 1) > 2p + 5$, then the inequality (6.5) gives a better bound for $F_v(p, p; p + 1)$.

In the case $p = 5$, using the graphs from Theorem 2.6 an even better bound for $F_v(5, 5; 6)$ can be obtained. From $F_v(2, 2, 5; 6) = 16$ and (6.4) it follows that $F_v(2, 5; 6) \geq 18, \ r \geq 3$. Since the Ramsey number $R(3, 6) = 18$, we have $\alpha(r, 5) \geq 3, \ r \geq 3$. Now, from (6.3) we derive

$$F_v(2, 5; 6) \geq F_v(2, 2, 2, 5; 6) + 3(r - 2) = 10 + 3r.$$  

From this inequality we see that $F_v(2, 2, 2, 5; 6) \geq 19$. We will prove that $F_v(2, 2, 2, 5; 6) \geq 20$. Suppose that $G \in \mathcal{H}_v(2, 2, 2, 5; 6)$. Since $R(3, 6) = 18$, we have $\alpha(G) \geq 3$. From $F_v(2, 2, 5; 6) = 16$ and Proposition 1.4 it follows that $\alpha(G) \leq 3$. It remains to be proved that there are no graphs with independence number 3 in $\mathcal{H}_{max}(2, 2, 2, 5; 6; 19)$. Let us remind, that in
the proof of Theorem 2.6 we found all 37 graphs in $H_{\text{max}}(2, 2, 5; 6; 16)$. From Table 2.1 we see that $H_{\text{max}}(2, 2, 5; 6; 16) = H_{\text{max}}^3(2, 2, 5; 6; 16)$. By executing Algorithm A3 ($n = 19; k = 3; t = 3$) with input $A = H_{\text{max}}^3(2, 2, 5; 6; 16)$, we obtain $B = \emptyset$. According to Theorem 3.5, there are no graphs in $H_{\text{max}}(2, 2, 2, 5; 6; 19)$ with independence number 3. Thus, we proved $H_v(2, 2, 2, 5; 6; 19) = \emptyset$, and therefore $F_v(2, 2, 2, 5; 6) \geq 20$. Since $\alpha(4, 5) \geq 3$, from (6.2) ($k = 4, p = 5$) and Theorem 1.19 we obtain

Theorem 6.1. $F_v(5, 5; 6) \geq F_v(2, 2, 2, 2, 5; 6) \geq 23$.

Now we proceed to bound the number $F_v(6, 6; 7)$. We will prove the following more general result:

**Theorem 6.2.** Let $a_1, \ldots, a_s$ be positive integers such that $\max \{a_1, \ldots, a_s\} = 6$ and $m = \sum_{i=1}^{s} (a_i - 1) + 1 \geq 9$. Then,

$$F_v(a_1, \ldots, a_s; 7) \geq F_v(2m-6, 6; 7) \geq 3m - 5.$$  

In particular, $F_v(6, 6; 7) \geq 28$.

**Proof.** According to Theorem 1.19 from this paper, $F_v(a_1, \ldots, a_s; 7) \geq F_v(2m-6, 6; 7)$. We will prove by induction that $F_v(2m-6, 6; 7) \geq 3m - 5, m \geq 9$.

The base case is $m = 9$, i.e. we have to prove that $F_v(2, 2, 2, 6; 7) \geq 22$. We will show that $H_v(2, 2, 2, 6; 7; 21) = \emptyset$. From $F_v(2, 2, 6; 7) = 17$ and Proposition 1.4 it follows that there are no graphs in $H_v(2, 2, 2, 6; 7; 21)$ with independence number greater than 4.

Let us remind that all graphs in $H_v(2, 2, 6; 7; 17)$ were obtained in the proof of Theorem 3.8 and all graphs in $H_v(2, 2, 6; 7; 18)$ were obtained in the proof of Theorem 3.10.

According to Theorem 3.8 $H_{\text{max}}(2, 2, 6; 7; 17) = \{G_1\}$ from Figure 3.1. By executing Algorithm A3 ($n = 21; k = 4; t = 4$) with input $A = H_{\text{max}}^4(1, 2, 2, 6; 7; 17) = H_{\text{max}}^4(2, 2, 6; 7; 17) = \{G_1\}$, we obtain $B = \emptyset$. According to Theorem 3.5, there are no graphs in $H_{\text{max}}(2, 2, 2, 6; 7; 21)$ with independence number 4.
In the proof of Theorem 3.10 we found all 392 graphs in $\mathcal{H}_{\text{max}}(2, 2, 6; 7; 18)$. From Table 3.2 we see that $\mathcal{H}_{\text{max}}(2, 2, 6; 7; 18) = \mathcal{H}_{\text{max}}^3(2, 2, 6; 7; 18)$. By executing Algorithm A3 ($n = 21; k = 3; t = 3$) with input $A = \mathcal{H}_{\text{max}}^3(1, 2, 2, 6; 7; 18) = \mathcal{H}_{\text{max}}^3(2, 2, 6; 7; 18)$, we obtain $B = \emptyset$. According to Theorem 3.5, there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 2; 7; 18)$ with independence number 3.

It remains to be proved that there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 2; 7; 21)$ with independence number 2. All 21-vertex graphs $G$ for which $\alpha(G) < 3$ and $\omega(G) < 7$ are known and are available on [52]. There are 1 118 436 such graphs $G$, and with the help of the computer we check that none of these graphs belong to $\mathcal{H}_{c}(2, 2, 2; 7; 21)$.

Thus, we proved $\mathcal{H}_{c}(2, 2, 2; 6; 7; 21) = \emptyset$ and $F_{c}(2, 2, 2; 6; 7) \geq 22$.

Now, suppose that for all $m'$ such that $9 \leq m' < m$ we have $F_{c}(2m' - 6, 6; 7) \geq 3m' - 5$. Let $G \in \mathcal{H}_{c}(2m' - 6, 6; 7)$ and $|V(G)| = F_{c}(2m' - 6, 6; 7)$. From the base case it follows that $F_{c}(2m' - 6, 6; 7) > 22$. Since the Ramsey number $R(3, 7) = 23$, we have $\alpha(G) \geq 3$. According to (6.2),

$$F_{c}(2m' - 6, 6; 7) = |V(G)| \geq F_{c}(2m' - 7, 6; 7) + 3 \geq 3(m' - 1) - 5 + 3 = 3m' - 5.$$

**Theorem 6.3.** Let $a_1, \ldots, a_s$ be positive integers such that $\max\{a_1, \ldots, a_s\} = 6$ and $m = \sum_{i=1}^{s}(a_i - 1) + 1$. Then:

(a) $22 \leq F_{c}(a_1, \ldots, a_s; 7) \leq F_{c}(4, 6; 7) \leq 35$, if $m = 9$.

(b) $28 \leq F_{c}(a_1, \ldots, a_s; 7) \leq F_{c}(6, 6; 7) \leq 70$, if $m = 11$.

**Proof.** The lower bounds in (a) and (b) follow from Theorem 6.2. It is easy to see that $F_{c}(a_1, \ldots, a_s; 7) \leq F_{c}(4, 6; 7)$ if $m = 9$, and $F_{c}(a_1, \ldots, a_s; 7) \leq F_{c}(6, 6; 7)$ if $m = 11$. To finish the proof we will use the following inequality proved by Kolev in [39]:

$$F_{c}(a_1, \ldots, a_s; q + 1).F_{c}(b_1, \ldots, b_s; t + 1) \geq F_{c}(a_1b_1, \ldots, a_s b_s; qt + 1),$$

where $q \geq \max\{a_1, \ldots, a_s\}$ and $t \geq \max\{b_1, \ldots, b_s\}$. 

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From $F_v(2, 3; 4) = 7$ (Theorem 1.2) and $F_v(2, 2; 3) = 5$ it follows
$F_v(4, 6; 7) \leq F_v(2, 2; 3).F_v(2, 3; 4) = 35$.
Since $F_v(2, 2; 3) = 5$ and $F_v(3, 3; 4) = 14$, [52] and [78], it follows that
$F_v(6, 6; 7) \leq F_v(2, 2; 3).F_v(3, 3; 4) = 70$. □

Let $a_1, ..., a_s$ be positive integers and $m$ and $p$ be defined by (1.6). If
$p = 7$, by Theorem 1.19 we have
(6.6) $F_v(a_1, ..., a_s; 8) \geq F_v(2m - 8, 7; 8), \ m \geq 8$.

Since $F_v(2, 2, 7; 8) = 20$, from (6.6) and (6.4) we obtain
(6.7) $F_v(a_1, ..., a_s; 8) \geq 2m + 2$.

In particular, when $m = 13$ we have $F_v(a_1, ..., a_s; 8) \geq 28$. Since the Ramsey
number $R(3, 8) = 28$, it follows that $\alpha(i, 7) \geq 3$, when $i \geq 6$. Now, from
(6.3) it follows easily that

**Theorem 6.4.** If $m \geq 13$, and $\max\{a_1, ..., a_s\} = 7$, then

$F_v(a_1, ..., a_s; 8) \geq 3m - 10$.

In particular, $F_v(7, 7; 8) \geq 29$.

It is clear that when $3m - 10 \geq R(4, 8)$ these bounds for $F_v(a_1, ..., a_s; 8)$
can be improved considerably. In comparison, note that by (6.1), we have
$F_v(7, 7; 8) \geq 27$, and according to (6.7), $F_v(7, 7; 8) \geq 28$.

At the end of this chapter we will prove the following theorem for the
number $F_v(4, 4; 5)$:

**Theorem 6.5.** $F_v(4, 4; 5) \geq F_v(2, 3, 4; 5) \geq F_v(2, 2, 2, 4; 5) \geq 19$.

**Proof.** The inequalities $F_v(4, 4; 5) \geq F_v(2, 3, 4; 5) \geq F_v(2, 2, 2, 4; 5)$ follow
from (1.11). It remains to be proved that $F_v(2, 2, 2, 4; 5) \geq 19$. Suppose
that $H_{\max}(2, 2, 2, 4; 5; 18) \neq \emptyset$ and let $G \in H_{\max}(2, 2, 2, 4; 5; 18)$. Since
the Ramsey number $R(3, 5) = 14$, $\alpha(G) \geq 3$. In [68] it is proved that
$F_v(2, 2, 4; 5) = 13$. From Proposition 1.4 and the equality $F_v(2, 2, 4; 5) = 13$
it follows that $\alpha(G) \leq 5$. It remains to be proved that there are no graphs with independence number 3, 4, or 5 in $\mathcal{H}_{\text{max}}(2, 2, 2, 4; 5; 18)$.

In [100] it is proved $\mathcal{H}_v(2, 2, 4; 5; 13) = \{Q\}$, where $Q$ is the unique 13-vertex $K_5$-free graph with independence number 2.

By executing Algorithm $A_3(n = 18; k = 5; t = 5)$ with input $A = \mathcal{H}_{\text{max}}^5(1, 2, 2, 4; 5; 13) = \mathcal{H}_{\text{max}}^5(2, 2, 4; 5; 13) = \{Q\}$, we obtain $B = \emptyset$. According to Theorem 3.5 there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 2, 4; 5; 18)$ with independence number 5.

Now we will prove that there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 2, 4; 5; 18)$ with independence number 4:

Using the nauty program [53] we generate all 8-vertex non-isomorphic graphs and among them we find all 7 graphs in $\mathcal{H}_{\text{max}}^4(3; 5; 8)$

We execute Algorithm $A_3(n = 10; k = 2; t = 4)$ with input $A = \mathcal{H}_{\text{max}}^4(3; 5; 8)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}^4(4; 5; 10)$. (see Remark 3.6).

We execute Algorithm $A_3(n = 12; k = 2; t = 4)$ with input $A = \mathcal{H}_{\text{max}}^4(1, 4; 5; 10) = \mathcal{H}_{\text{max}}^4(4; 5; 10)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}^4(2, 4; 5; 12)$.

We execute Algorithm $A_3(n = 14; k = 2; t = 4)$ with input $A = \mathcal{H}_{\text{max}}^4(1, 2, 4; 5; 12) = \mathcal{H}_{\text{max}}^4(2, 4; 5; 12)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}^4(2, 2, 4; 5; 14)$.

By executing Algorithm $A_3(n = 18; k = 4; t = 4)$ with input $A = \mathcal{H}_{\text{max}}^4(1, 2, 2, 4; 5; 14) = \mathcal{H}_{\text{max}}^4(2, 2, 4; 5; 14)$, we obtain $B = \emptyset$. According to Theorem 3.5 there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 2, 4; 5; 18)$ with independence number 5.

The last step is to prove that there are no graphs in $\mathcal{H}_{\text{max}}(2, 2, 2, 4; 5; 18)$ with independence number 3:

Using the nauty program [53] we generate all 11-vertex non-isomorphic graphs and among them we find all 11 graphs in $\mathcal{H}_{\text{max}}^3(3; 5; 9)$

We execute Algorithm $A_3(n = 11; k = 2; t = 3)$ with input $A = \mathcal{H}_{\text{max}}^3(3; 5; 9)$ to obtain all graphs in $B = \mathcal{H}_{\text{max}}^3(4; 5; 11)$. (see Remark 3.6).

We execute Algorithm $A_3(n = 13; k = 2; t = 3)$ with input $A = \mathcal{H}_{\text{max}}^3(1, 4; 5; 11) = \mathcal{H}_{\text{max}}^3(4; 5; 11)$ to obtain all graphs in
Table 6.1: Steps in finding all maximal graphs in $H_v(2, 2, 2, 4; 5; 18)$

| set                          | independence number | maximal graphs | (+$K_4$)-graphs |
|------------------------------|---------------------|----------------|-----------------|
| $H_v(3; 5; 8)$               | ≤ 4                 | 7              | 274             |
| $H_v(4; 5; 10)$              | ≤ 4                 | 44             | 65 422          |
| $H_v(2, 4; 5; 12)$           | ≤ 4                 | 1 059          | 18 143 174      |
| $H_v(2, 2, 4; 5; 14)$        | ≤ 4                 | 13             | 71              |
| $H_v(2, 2, 2, 4; 5; 18)$     | = 4                 | 0              |                 |
| $H_v(3; 5; 9)$               | ≤ 3                 | 11             | 2 252           |
| $H_v(4; 5; 11)$              | ≤ 3                 | 135            | 1 678 802       |
| $H_v(2, 4; 5; 13)$           | ≤ 3                 | 11 439         | 2 672 047 607   |
| $H_v(2, 2, 4; 5; 15)$        | ≤ 3                 | 1 103          | 78 117          |
| $H_v(2, 2, 2, 4; 5; 18)$     | = 3                 | 0              |                 |
| $H_v(2, 2, 2, 4; 5; 18)$     |                     | 0              |                 |

$B = H_{max}^3(2, 4; 5; 13)$.

We execute Algorithm $\text{A3}(n = 15; k = 2; t = 3)$ with input $\mathcal{A} = H_{max}^3(1, 2, 4; 5; 13) = H_{max}^3(2, 4; 5; 13)$ to obtain all graphs in $B = H_{max}^3(2, 2, 4; 5; 15)$.

By executing Algorithm $\text{A3}(n = 18; k = 3; t = 3)$ with input $\mathcal{A} = H_{max}^3(1, 2, 2, 4; 5; 15) = H_{max}^3(2, 2, 4; 5; 15)$, we obtain $B = \emptyset$. According to Theorem 3.5, there are no graphs in $H_{max}(2, 2, 2, 4; 5; 18)$ with independence number 3.

We proved that $H_{max}(2, 2, 2, 4; 5; 18) = \emptyset$, and therefore $F_v(2, 2, 2, 4; 5) \geq 19$. \hfill \qed

The number of graphs obtained in each step of the proof is given in Table 6.1. Because of the large number of graphs in $H_{+K_4}^3(2, 4; 5; 13)$, the computer needed about a month to complete the computations.

The upper bound $F_v(4, 4; 5) \leq 23$ is proved in [100] with the help of a 23-vertex transitive graph. We were not able to obtain any other graphs in $H_v(4, 4; 5; 23)$, which leads us to believe that $F_v(4, 4; 5) = 23$. We did find a large number of 23-vertex graphs in $H_v(2, 2, 2, 4; 5)$, but so far we have not obtained smaller graphs in this set.

Theorem 6.1 is published in [8]. Theorem 6.2 and Theorem 6.3 are published in [7]. Theorem 6.4 and Theorem 6.5 are published in [6].

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Part II

Edge Folkman Numbers
Chapter 7

Definition of the edge Folkman numbers and some known results

The expression \( G \xrightarrow{e} (3,3) \) means that in every coloring of the edges of the graph \( G \) in two colors there is a monochromatic triangle.

It is well known that \( K_6 \xrightarrow{e} (3,3) \).

Denote:
\[
\mathcal{H}_e(3,3) = \{ G : G \xrightarrow{e} (3,3) \}
\]
\[
\mathcal{H}_e(3,3; q) = \{ G : G \xrightarrow{e} (3,3) \text{ and } \omega(G) < q \}
\]
\[
\mathcal{H}_e(3,3; q; n) = \{ G : G \in \mathcal{H}_e(3,3; q) \text{ and } |V(G)| = n \}
\]

The edge Folkman number \( F_e(3,3; q) \) is defined with
\[
F_e(3,3; q) = \min \{|V(G)| : G \in \mathcal{H}_e(3,3; q)\}
\]

The equality \( R(3,3) = 6 \) means that \( K_6 \xrightarrow{e} (3,3) \) and \( K_5 \xrightarrow{e} (3,3) \). It follows that \( F_e(3,3; q) = 6, q \geq 7 \).

In [20] Erdős and Hajnal posed the following problem:

*Does there exist a graph \( G \xrightarrow{e} (3,3) \) with \( \omega(G) < 6 \)?*

The first example of a graph which gives a positive answer to this question was given by van Lint. The complement of this graph is shown in Figure 7.1.

Van Lint did not publish this result himself, but the graph was included in [31]. Later, Graham [29] constructed the smallest possible example of such a graph, namely \( K_3 + C_5 \). Thus, he proved that \( F_e(3,3; 6) = 8 \). It is easy to see that the van Lint graph contains \( K_3 + C_5 \) (it is the subgraph induced by the black vertices in Figure 7.1).
Nenov [59] constructed a 15-vertex graph in $\mathcal{H}_e(3,3;5)$ in 1981, thus proving $F_e(3,3;5) \leq 15$. This graph is obtained from the graph $\Gamma$ shown in Figure 7.2 by adding a new vertex which is adjacent to all vertices of $\Gamma$. In 1999 Piwakowski, Radziszowski, and Urbanski [78] completed the computation of the number $F_e(3,3;5)$ by proving with the help of a computer that $F_e(3,3;5) \geq 15$. In [78] they also obtained all graphs in $\mathcal{H}_e(3,3;5;15)$.
Folkman constructed a graph $G \rightarrow (3,3)$ with $\omega(G) = 3$ [22]. The exact value of the number $F_e(3,3;4)$ is not known. It is known that $19 \leq F_e(3,3;4) \leq 786$, [81] [43]. In Chapter 9 we improve the lower bound on this number by proving $F_e(3,3;4) \geq 20$.

A more detailed view on results related to the numbers $F_e(3,3;q)$ is given in the book [93], and also in the papers [81], [30], [43] and [82].

Let $a_1, \ldots, a_s$ be positive integers. The expression $G \rightarrow (a_1, \ldots, a_s)$ means that in every coloring of $E(G)$ in $s$ colors ($s$-coloring) there exists $i \in \{1, \ldots, s\}$ such that there is a monochromatic $a_i$-clique of color $i$.

Define:

$\mathcal{H}_e(a_1, \ldots, a_s) = \left\{ G : G \rightarrow (a_1, \ldots, a_s) \right\}$.

$\mathcal{H}_e(a_1, \ldots, a_s; q) = \left\{ G : G \rightarrow (a_1, \ldots, a_s) \text{ and } \omega(G) < q \right\}$.

$\mathcal{H}_e(a_1, \ldots, a_s; q; n) = \left\{ G : G \in \mathcal{H}_e(a_1, \ldots, a_s; q) \text{ and } |V(G)| = n \right\}$.

The edge Folkman numbers $F_e(a_1, \ldots, a_s; q)$ are defined by the equality:

$$F_e(a_1, \ldots, a_s; q) = \min \{|V(G)| : G \in \mathcal{H}_e(a_1, \ldots, a_s; q)\}.$$

In general, very little is known about the numbers $F_e(a_1, \ldots, a_s; q)$.

In Chapter 8 we study the minimal (inclusion-wise) graphs in $\mathcal{H}_e(3,3)$. In Chapter 9 we obtain the new lower bound $F_e(3,3;4) \geq 20$. 

In general, very little is known about the numbers $F_e(a_1, \ldots, a_s; q)$. 

In Chapter 8 we study the minimal (inclusion-wise) graphs in $\mathcal{H}_e(3,3)$. In Chapter 9 we obtain the new lower bound $F_e(3,3;4) \geq 20$. 

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Chapter 8

Minimal graphs in $\mathcal{H}_e(3, 3)$

Obviously, if $H \xrightarrow{e} (p, q)$, then its every supergraph $G \xrightarrow{e} (p, q)$.

**Definition 8.1.** We say that $G$ is a minimal graph in $\mathcal{H}_e(p, q)$ if $G \xrightarrow{e} (p, q)$ and $H \not\xrightarrow{e} (p, q)$ for each proper subgraph $H$ of $G$.

It is easy to see that $K_6$ is a minimal graph in $\mathcal{H}_e(3, 3)$ and there are no minimal graphs in $\mathcal{H}_e(3, 3)$ with 7 vertices. The only minimal 8-vertex graph is the Graham graph $K_3 + C_5$, and there is only one minimal 9-vertex graph, obtained by Nenov [55] (see Figure 8.1).

![Figure 8.1: 9-vertex minimal graph in $\mathcal{H}_e(3, 3)$](image1)

![Figure 8.2: 10-vertex minimal graph in $\mathcal{H}_e(3, 3)$](image2)
For each pair of positive integers \( p \geq 3, q \geq 3 \) there exist infinitely many minimal graphs in \( \mathcal{H}_e(p, q) \) [11], [23]. The simplest infinite sequence of minimal graphs in \( \mathcal{H}_e(3, 3) \) are the graphs \( K_3 + C_{2r+1}, r \geq 1 \). This sequence contains the already mentioned graphs \( K_6 \) and \( K_3 + C_5 \). It was obtained by Nenov and Khadzhiivanov in [74]. Later, this sequence was reobtained in [12], [25], and [95].

Three 10-vertex minimal graphs in \( \mathcal{H}_e(3, 3) \) are known. One of them is \( K_3 + C_7 \) from the sequence \( K_3 + C_{2r+1}, r \geq 1 \). The other two were obtained by Nenov in [58] (the second graph is given in Figure 8.2 and the third is a subgraph of \( K_1 + C_5 \)).

In the following sections, we obtain new minimal graphs in \( \mathcal{H}_e(3, 3) \). We also obtain some general bounds for the graphs in \( \mathcal{H}_e(3, 3) \).

We will need the following results:

**Theorem 8.2.** [11][23] Let \( G \) be a minimal graph in \( \mathcal{H}_e(p, p) \). Then \( \delta(G) \geq (p - 1)^2 \). In particular, when \( p = 3 \), we have \( \delta(G) \geq 4 \).

**Proposition 8.3.** If \( G \) is a minimal graph in \( \mathcal{H}_e(p, q) \), then \( G \) is not a Sperner graph.

**Proof.** Suppose the opposite is true, and let \( u, v \in V(G) \) be such that \( N_G(u) \subseteq N_G(v) \). We color the edges of \( G - u \) with two colors in such a way that there is no monochromatic \( p \)-clique of the first color and no monochromatic \( q \)-clique of the second color. After that, for each vertex \( w \in N_G(u) \) we color the edge \([u, w]\) with the same color as the edge \([v, w]\). We obtain a 2-coloring of the edges of \( G \) with no monochromatic \( p \)-cliques of the first color and no monochromatic \( q \)-cliques of the second color.

Since \( F_e(3, 3; 5) = 15 \) [59][78], every graph \( G \in \mathcal{H}_e(3, 3) \) with no more than 14 vertices contains a 5-clique. There exist 14-vertex graphs in \( \mathcal{H}_e(3, 3) \) containing only a single 5-clique, an example of such a graph is given in Figure 8.3. The graph in Figure 8.3 is obtained with the help of the only 15-vertex bicritical graph in \( \mathcal{H}_e(3, 3) \) with clique number 4 from [78]. First, by removing a vertex from the bicritical graph, we obtain 14-vertex graphs without 5 cliques. After that, by adding edges to the obtained graphs, we...
find a 14-vertex graph in $\mathcal{H}_e(3, 3)$ with a single 5-clique whose subgraph is the minimal graph in $\mathcal{H}_e(3, 3)$ in Figure 8.3. Let us note that in [78] the authors obtain all 15-vertex graphs in $\mathcal{H}_e(3, 3)$ with clique number 4, and with the help of these graphs, one can find more examples of 14-vertex minimal graphs in $\mathcal{H}_e(3, 3)$.

Figure 8.3: 14-vertex minimal graph in $\mathcal{H}_e(3, 3)$ with a single 5 clique

**Theorem 8.4.** [47] Let $G \rightarrow (p, q)$. Then $\chi(G) \geq R(p, q)$. In particular, if $G \rightarrow (3, 3)$, then $\chi(G) \geq 6$.

**Corollary 8.5.** Let $G \rightarrow (3, 3)$, $v_1, ..., v_s$ be independent vertices of $G$, and $H = G - \{v_1, ..., v_s\}$. Then $\chi(H) \geq 5$.

**Theorem 8.6.** Let $G$ be a minimal graph in $\mathcal{H}_e(3, 3)$. Then for each vertex $v \in V(G)$ we have $\alpha(G(v)) \leq d(v) - 3$.

**Proof.** Suppose the opposite is true, and let $A \subseteq N_G(v)$ be an independent set in $G(v)$ such that $|A| = d(v) - 2$. Let $a, b \in N_G(v) \setminus A$. Consider a 2-coloring of the edges of $G - v$ in which there are no monochromatic triangles. We color the edges $[v, a]$ and $[v, b]$ with the same color in such a way that there is no monochromatic triangle (if $a$ and $b$ are adjacent, we chose the color of $[v, a]$ and $[v, b]$ to be different from the color of $[a, b]$, and if $a$ and $b$ are not adjacent, then we chose an arbitrary color for $[v, a]$ and $[v, b]$). We color the remaining edges incident to $v$ with the other color, which is different from
the color of \([v, a]\) and \([v, b]\). Since \(N_G(v) \setminus \{a, b\} = A\) is an independent set, we obtain a 2-coloring of the edges of \(G\) without monochromatic triangles, which is a contradiction.

**Corollary 8.7.** Let \(G\) be a minimal graph in \(\mathcal{H}_e(3, 3)\) and \(d(v) = 4\) for some vertex \(v \in V(G)\). Then \(G(v) = K_4\).

Theorem 8.6 is published in [2].

### 8.1 Algorithm A6

The following algorithm is appropriate for finding all minimal graphs in \(\mathcal{H}_e(3, 3)\) with a small number of vertices.

**Algorithm A6.** Let \(n\) be a fixed positive integer such that \(7 \leq n \leq 14\). The output of the algorithm is the set \(B\) of all \(n\)-vertex minimal graphs in \(\mathcal{H}_e(3, 3)\).

1. Generate all \(n\)-vertex non-isomorphic graphs with minimum degree at least 4, and denote the obtained set by \(B\).
2. Remove from \(B\) all Sperner graphs.
3. Remove from \(B\) all graphs with clique number not equal to 5.
4. Remove from \(B\) all graphs with chromatic number less than 6.
5. Remove from \(B\) all graphs which are not in \(\mathcal{H}_e(3, 3)\).
6. Remove from \(B\) all graphs which are not minimal graphs in \(\mathcal{H}_e(3, 3)\).

**Theorem 8.8.** Fix \(n \in \{7, \ldots, 14\}\). Then after executing Algorithm A6, \(B\) coincides with the set of all \(n\)-vertex minimal graphs in \(\mathcal{H}_e(3, 3)\).

**Proof.** From step 1 it becomes clear, that the set \(B\) contains only \(n\)-vertex graphs. Step 6 guaranties that \(B\) contains only minimal graphs in \(\mathcal{H}_e(3, 3)\).

Let \(G\) be an arbitrary \(n\)-vertex minimal graph in \(\mathcal{H}_e(3, 3)\). We will prove that, after the execution of Algorithm A6, \(G \in B\). By Theorem 8.2, \(\delta(G) \geq 4\), and by Proposition 8.3, \(G\) is not a Sperner graph. From \(|V(G)| \geq 7\) it follows that \(G \nsubseteq K_6\). Since \(|V(G)| \leq 14\), by \(F_e(3, 3; 5) = 15\), \([59][78]\), we obtain \(\omega(G) = 5\). By Theorem 8.4, \(\chi(G) \geq 6\). Therefore, after step 4, \(G \in B\).
We will use Algorithm A6 to obtain all minimal graphs in $\mathcal{H}_e(3,3)$ with up to 12 vertices. Algorithm A6 is not appropriate in the cases $n \geq 13$, because the number of graphs generated in step 1 is too large. To find the 13-vertex minimal graphs in $\mathcal{H}_e(3,3)$, we will use Algorithm A7-M, which is defined below.

Theorem 8.8 and Algorithm A6 are published in [2].

### 8.2 Minimal graphs in $\mathcal{H}_e(3,3; n)$ for $n \leq 12$

We execute Algorithm A6 for $n = 7, 8, 9, 10, 11, 12$, and we find all minimal graphs in $\mathcal{H}_e(3,3)$ with up to 12 vertices except $K_6$. In this way, we obtain the known results: there is no minimal graph in $\mathcal{H}_e(3,3)$ with 7 vertices, the Graham graph $K_3 + C_5$ is the only minimal 8-vertex graph, and there exists only one minimal 9-vertex graph, the Nenov graph from [55] (see Figure 8.1). We also obtain the following new results:

**Theorem 8.9.** There are exactly 6 minimal 10-vertex graphs in $\mathcal{H}_e(3,3)$. These graphs are given in Figure 8.12, and some of their properties are listed in Table 8.1.

**Theorem 8.10.** There are exactly 73 minimal 11-vertex graphs in $\mathcal{H}_e(3,3)$. Some of their properties are listed in Table 8.2. Examples of 11-vertex minimal graphs in $\mathcal{H}_e(3,3)$ are given in Figure 8.13 and Figure 8.14.

**Theorem 8.11.** There are exactly 3041 minimal 12-vertex graphs in $\mathcal{H}_e(3,3)$. Some of their properties are listed in Table 8.3. Examples of 12-vertex minimal graphs in $\mathcal{H}_e(3,3)$ are given in Figure 8.15 and Figure 8.16.

We will use the following enumeration for the obtained minimal graphs in $\mathcal{H}_e(3,3)$:
- $G_{10.1}, ..., G_{10.6}$ are the 10-vertex graphs;
- $G_{11.1}, ..., G_{11.73}$ are the 11-vertex graphs;
- $G_{12.1}, ..., G_{12.3041}$ are the 12-vertex graphs;
Table 8.1: Some properties of the 10-vertex minimal graphs in $\mathcal{H}_e(3, 3)$

| $|E(G)|$ | $\# \delta(G)$ | $\# \Delta(G)$ | $\# \alpha(G)$ | $\# \chi(G)$ | $|\text{Aut}(G)|$ |
|------|----------|----------|----------|----------|----------|
| 30   | 1        | 1        | 9        | 6        | 6        | 4        |
| 31   | 1        | 5        | 4        | 2        | 3        | 8        |
| 32   | 2        | 6        | 1        | 3        | 3        | 16       |
| 33   | 1        |          |          |          |          | 84       |
| 34   | 1        |          |          |          |          |          |

Table 8.2: Some properties of the 11-vertex minimal graphs in $\mathcal{H}_e(3, 3)$

| $|E(G)|$ | $\# \delta(G)$ | $\# \Delta(G)$ | $\# \alpha(G)$ | $\# \chi(G)$ | $|\text{Aut}(G)|$ |
|------|----------|----------|----------|----------|----------|
| 35   | 6        | 4        | 5        | 8        | 1        | 2        |
| 36   | 13       | 5        | 58       | 10       | 72       | 4        |
| 37   | 23       | 6        | 10       | 3        | 66       | 4        |
| 38   | 25       |          |          | 4        | 3        | 6        |
| 39   | 5        |          |          |          |          | 8        |
| 41   | 1        |          |          |          |          | 12       |
| 42   | 1        |          |          |          |          | 16       |
| 43   | 1        |          |          |          |          | 24       |

Table 8.3: Some properties of the 12-vertex minimal graphs in $\mathcal{H}_e(3, 3)$

| $|E(G)|$ | $\# \delta(G)$ | $\# \Delta(G)$ | $\# \alpha(G)$ | $\# \chi(G)$ | $|\text{Aut}(G)|$ |
|------|----------|----------|----------|----------|----------|
| 38   | 5        | 4        | 129      | 8        | 43       | 2        |
| 39   | 27       | 5        | 2 178    | 9        | 1 196    | 3        |
| 40   | 144      | 6        | 611      | 11       | 1 802    | 4        |
| 41   | 418      | 7        | 123      | 5        | 1        | 6        |
| 42   | 1 014    |          |          |          |          | 8        |
| 43   | 459      |          |          |          |          | 12       |
| 44   | 224      |          |          |          |          | 16       |
| 45   | 351      |          |          |          |          | 24       |
| 46   | 299      |          |          |          |          | 32       |
| 47   | 84       |          |          |          |          | 36       |
| 48   | 16       |          |          |          |          | 96       |
| 49   | 418      |          |          |          |          | 108      |

Table 8.4: Some properties of the 13-vertex minimal graphs in $\mathcal{H}_e(3, 3)$

| $|E(G)|$ | $\# \delta(G)$ | $\# \Delta(G)$ | $\# \alpha(G)$ | $\# \chi(G)$ | $|\text{Aut}(G)|$ |
|------|----------|----------|----------|----------|----------|
| 41   | 4        | 4        | 13 725   | 8        | 16       | 2        |
| 42   | 44       | 5        | 191 504  | 9        | 61 678   | 3        |
| 43   | 220      | 6        | 85 932   | 10       | 175 108  | 4        |
| 44   | 1 475    | 7        | 15 391   | 12       | 69 833   | 5        |
| 45   | 7 838    | 8        | 83       | 6        | 2        | 6        |
| 46   | 28 805   |          |          |          |          | 8        |
| 47   | 33 810   |          |          |          |          | 12       |
| 48   | 26 262   |          |          |          |          | 16       |
| 49   | 39 718   |          |          |          |          | 24       |
| 50   | 62 390   |          |          |          |          | 32       |
| 51   | 59 291   |          |          |          |          | 36       |
| 52   | 34 132   |          |          |          |          | 40       |
| 53   | 18 878   |          |          |          |          | 48       |
| 54   | 1 680    |          |          |          |          | 72       |
| 55   | 86       |          |          |          |          | 96       |
| 56   | 2        |          |          |          |          | 144      |
The indexes correspond to the order of the graphs’ canonical labels defined in nauty \[53\].

Detailed data for the number of graphs obtained at each step of the execution of Algorithm \[A6\] is given in Table \[8.5\].

Theorem \[8.9\], Theorem \[8.10\], and Theorem \[8.11\] are published in \[2\].

### 8.3 Algorithm A7

In order to present the next algorithms, we will need the following definitions and auxiliary propositions:

We say that a 2-coloring of the edges of a graph is \((3,3)\)-free if it has no monochromatic triangles.

**Definition 8.12.** Let \(G\) be a graph and \(M \subseteq V(G)\). Let \(G_1\) be a graph which is obtained by adding a new vertex \(v\) to \(G\) such that \(N_{G_1}(v) = M\). We say that \(M\) is a marked vertex set in \(G\) if there exists a \((3,3)\)-free 2-coloring of the edges of \(G\) which cannot be extended to a \((3,3)\)-free 2-coloring of the edges of \(G_1\).

It is clear that if \(G \not\rightarrow (3,3)\), then there are no marked vertex sets in \(G\).

The following proposition is true:

**Proposition 8.13.** Let \(G\) be a minimal graph in \(\mathcal{H}_e(3,3)\), let \(v_1, \ldots, v_s\) be independent vertices of \(G\), and \(H = G - \{v_1, \ldots, v_s\}\). Then \(N_G(v_i), i = 1, \ldots, s\), are marked vertex sets in \(H\).
Proof. Suppose the opposite is true, i.e. $N_G(v_i)$ is not a marked vertex set in $H$ for some $i \in \{1, \ldots, s\}$. Since $G$ is a minimal graph in $\mathcal{H}_e(3, 3)$, there exists a (3,3)-free 2-coloring of the edges of $G - v_i$, which induces a (3,3)-free 2-coloring of the edges of $H$. By supposition, we can extend this 2-coloring to a (3,3)-free 2-coloring of the edges of the graph $H_i = G - \{v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_s\}$. Thus, we obtain a (3,3)-free 2-coloring of the edges of $G$, which is a contradiction.

Definition 8.14. Let $\{M_1, \ldots, M_s\}$ be a family of marked vertex sets in the graph $G$. Let $G_i$ be a graph which is obtained by adding a new vertex $v_i$ to $G$ such that $N_{G_i}(v_i) = M_i$, $i = 1, \ldots, s$. We say that $\{M_1, \ldots, M_s\}$ is a complete family of marked vertex sets in $G$, if for each (3,3)-free 2-coloring of the edges of $G$ there exists $i \in \{1, \ldots, s\}$ such that this 2-coloring can not be extended to a (3,3)-free 2-coloring of the edges of $G_i$.

Proposition 8.15. Let $v_1, \ldots, v_s$ be independent vertices of the graph $G$, and $H = G - \{v_1, \ldots, v_s\}$. If $\{N_G(v_1), \ldots, N_G(v_s)\}$ is a complete family of marked vertex sets in $H$, then $G \not\rightarrow (3,3)$.

Proof. Consider a 2-coloring of the edges of $G$ which induces a 2-coloring with no monochromatic triangles in $H$. According to Definition 8.14, this 2-coloring of the edges of $H$ can not be extended in $G$ without forming a monochromatic triangle.

It is easy to prove the following strengthening of Proposition 8.13:

Proposition 8.16. Let $G$ be a minimal graph in $\mathcal{H}_e(3, 3)$, let $v_1, \ldots, v_s$ be independent vertices of $G$, and $H = G - \{v_1, \ldots, v_s\}$. Then $\{N_G(v_1), \ldots, N_G(v_s)\}$ is a complete family of marked vertex sets in $H$. Furthermore, this family is a minimal complete family, in the sense that it does not contain a proper complete subfamily.

Before presenting Algorithm A7 we will prove

Proposition 8.17. For every fixed positive integer $k$ there exist at most a finite number of minimal graphs $G \in \mathcal{H}_e(3, 3)$ such that $\alpha(G) \geq |V(G)| - k \geq 1$. 

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Proof. Let \( G \) be a minimal graph in \( \mathcal{H}_e(3, 3) \) such that \( \alpha(G) \geq |V(G)| - k \geq 1 \). Let \( s = |V(G)| - k \), \( v_1, ..., v_s \) be independent vertices of \( G \), and \( H = G - \{v_1, ..., v_s\} \). According to Proposition 8.3, \( N_G(v_i) \not\subseteq N_G(v_j) \), \( i \neq j \). Thus, the vertices \( v_1, ..., v_s \) are connected to different subsets of \( V(H) \). We derive that each \( k \)-vertex graph \( H \) can be extended to at most finitely many possible graphs \( G \). \( \square \)

The following algorithm finds all minimal graphs \( G \in \mathcal{H}_e(3, 3) \) for which \( \alpha(G) \geq |V(G)| - k \geq 1 \), where \( k \) is fixed (but \( |V(G)| \) is not fixed).

**Algorithm A7.** Let \( q \) and \( k \) be fixed positive integers.

The input of the algorithm is the set \( \mathcal{A} \) of all \( k \)-vertex graphs \( H \) for which \( \omega(H) < q \) and \( \chi(H) \geq 5 \). The output of the algorithm is the set \( \mathcal{B} \) of all minimal graphs \( G \in \mathcal{H}_e(3, 3; q) \) for which \( \alpha(G) \geq |V(G)| - k \geq 1 \).

1. For each graph \( H \in \mathcal{A} \):
   1.1. Find all subsets \( M \) of \( V(H) \) which have the properties:
      (a) \( K_{q-1} \not\subseteq M \), i.e. \( M \) is a \( K_{(q-1)} \)-free subset.
      (b) \( M \not\subseteq N_H(v) \), \( \forall v \in V(H) \).
      (c) \( M \) is a marked vertex set in \( H \) (see Definition 8.12).

   Denote by \( \mathcal{M}(H) \) the family of subsets of \( V(H) \) which have the properties (a), (b) and (c). Enumerate the elements of \( \mathcal{M}(H) \): \( \mathcal{M}(H) = \{M_1, ..., M_l\} \).

   1.2. Find all subfamilies \( \{M_{i_1}, ..., M_{i_s}\} \) of \( \mathcal{M}(H) \) which are minimal complete families of marked vertex sets in \( H \) (see Definition 8.14). For each such found subfamily \( \{M_{i_1}, ..., M_{i_s}\} \) construct the graph \( G = G(M_{i_1}, ..., M_{i_s}) \) by adding new independent vertices \( v_1, ..., v_s \) to \( V(H) \) such that \( N_G(v_j) = M_{i_j}, j = 1, ..., s \). Add \( G \) to \( \mathcal{B} \).

   2. Remove the isomorphic copies of graphs from \( \mathcal{B} \).

   3. Remove from \( \mathcal{B} \) all graphs which are not minimal graphs in \( \mathcal{H}_e(3, 3) \).

**Remark 8.18.** It is clear, that if \( G \) is a minimal graph in \( \mathcal{H}_e(3, 3) \) and \( \omega(G) \geq 6 \), then \( G = K_6 \). Obviously there are no graphs in \( \mathcal{H}_e(3, 3) \) with clique number less than 3. Therefore, we will use Algorithm A7 only for \( q \in \{4, 5, 6\} \).
Theorem 8.19. After executing Algorithm A7, the set $B$ coincides with the set of all minimal graphs $G \in \mathcal{H}_e(3,3;q)$ for which $\alpha(G) \geq |V(G)| - k \geq 1$.

Proof. From step 1.2 it becomes clear that every graph $G$ which is added to $B$ is obtained by adding the independent vertices $v_1, ..., v_s$ to a graph $H \in A$. Therefore, $\alpha(G) \geq s = |V(G)| - |V(H)| = |V(G)| - k$. From $\omega(H) < q$ and $K_{q-1} \not\subseteq N_G(v_i), i = 1, ..., s$, it follows $\omega(G) < q$. According to Proposition 8.15, after step 1.2, $B$ contains only graphs in $\mathcal{H}_e(3,3;q)$, and after step 3, $B$ contains only minimal graphs in $\mathcal{H}_e(3,3;q)$.

Consider an arbitrary minimal graph $G \in \mathcal{H}_e(3,3;q)$ for which $\alpha(G) \geq |V(G)| - k \geq 1$. We will prove that $G \in B$.

Denote $s = |V(G)| - k \geq 1$. Let $v_1, ..., v_s$ be independent vertices of $G$, and $H = G - \{v_1, ..., v_s\}$. By Corollary 8.5, $\chi(H) \geq 5$. Therefore, $H \in A$.

From $\omega(G) < q$ it follows that $K_{q-1} \not\subseteq N_G(v_i), i = 1, ..., s$. By Proposition 8.3, $G$ is not a Sperner graph, and therefore $N_G(v_i) \not\subseteq N_H(v), \forall v \in V(H)$. According to Proposition 8.13, $N_G(v_i)$ are marked vertex sets in $H$. Therefore, after executing step 1.1, $N_G(v_i) \in \mathcal{M}(H), i = 1, ..., s$.

From Proposition 8.16 it becomes clear that $\{N_G(v_1), ..., N_G(v_s)\}$ is a minimal complete family of marked vertex sets in $H$. Therefore, in step 1.2 the graph $G$ is added to $B$. 

In order to find the 13-vertex minimal graphs in $\mathcal{H}_e(3,3)$ we will use the following modification of Algorithm A7 in which $n = |V(G)|$ is fixed:

Algorithm A7-M. Modification of Algorithm A7 for finding all minimal graphs $G \in \mathcal{H}_e(3,3;q;n)$ for which $\alpha(G) \geq n - k \geq 1$, where $q$, $k$, and $n$ are fixed positive integers.

In step 1.2 of Algorithm A7 add the condition to consider only minimal complete subfamilies $\{M_{i_1}, ..., M_{i_s}\}$ of $\mathcal{M}(H)$ in which $s = n - k$.

Theorem 8.19, Algorithm A7 and Algorithm A7-M are published in [2].
8.4 Minimal graphs in $H_e(3,3;13)$

The method that we will use to find all minimal graphs in $H_e(3,3;13)$ consists of two parts:

1. All minimal graphs in $H_e(3,3;13)$ with independence number 2 are a subset of $R(3,6;13)$. All 275 086 graphs in $R(3,6;13)$ are known and are available on [52]. With a computer we find that exactly 13 of these graphs are minimal graphs in $H_e(3,3;13)$.

2. It remains to find the 13-vertex minimal graphs in $H_e(3,3)$ with independence number at least 3. We generate all 10-vertex non-isomorphic graphs using the nauty program [53], and among them we find the set $A$ of all 1 923 103 graphs $H$ with 10 vertices for which $\omega(H) \leq 5$ and $\chi(H) \geq 5$. By executing Algorithm $A7-M(n=13;k=10;q=6)$ with input $A$ we obtain the set $B$ of all 306 622 minimal 13-vertex graphs in $H_e(3,3)$ with independence number at least 3.

Finally, we obtain

**Theorem 8.20.** There are exactly 306 635 minimal 13-vertex graphs in $H_e(3,3)$. Some of their properties are listed in Table 8.4. Examples of 13-vertex minimal graphs in $H_e(3,3)$ are given in Figure 8.4, Figure 8.18 and Figure 8.19.

We enumerate the obtained minimal 13-vertex graphs in $H_e(3,3)$:

$G_{13.1}, \ldots, G_{13.306635}$.

All graphs in $R(3,6)$ are known, and since $R(3,6) = 18$, these graphs have at most 17 vertices. With the help of a computer we check that there are no minimal graphs in $H_e(3,3)$ with independence number 2 and more than 13 vertices. Thus, we prove

**Theorem 8.21.** Let $G$ be a minimal graph in $H_e(3,3)$ and $\alpha(G) = 2$. Then $|V(G)| \leq 13$. There are exactly 145 minimal graphs $G$ in $H_e(3,3)$ for which $\alpha(G) = 2$. 

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- 8-vertex: 1 \((K_3 + C_5)\);
- 9-vertex: 1 (see Figure 8.1);
- 10-vertex: 3 \((G_{10,3}, G_{10,5}, G_{10,6}, \text{see Figure 8.12})\);
- 11-vertex: 4 \((G_{11,46}, G_{11,47}, G_{11,54}, G_{11,69}, \text{see Figure 8.14})\);
- 12-vertex: 124;
- 13-vertex: 13 (see Figure 8.19);

By executing Algorithm A7-M\((n = 10, 11, 12; k = 7, 8, 9; q = 6)\), we find all minimal graphs in \(\mathcal{H}_e(3, 3)\) with 10, 11, and 12 vertices and independence number greater than 2. In this way, with the help of Theorem 8.21, we obtain a new proof of Theorem 8.9, Theorem 8.10, and Theorem 8.11.

Theorem 8.20 and Theorem 8.21 are published in [2].

8.5 Properties of the minimal graphs in \(\mathcal{H}_e(3, 3; n)\) for \(n \leq 13\)

Minimum and maximum degree

By Theorem 8.2, if \(G\) is a minimal graph in \(\mathcal{H}_e(3, 3)\), then \(\delta(G) \geq 4\). Via very elegant constructions, in [11] and [23] it is proved that the bound \(\delta(G) \geq (p - 1)^2\) from Theorem 8.2 is exact. However, these constructions are not very economical in the case \(p = 3\). For example, the minimal graph \(G \in \mathcal{H}_e(3, 3)\) from [23] with \(\delta(G) = 4\) is not presented explicitly, but it is proved that it is a subgraph of a graph with 17,577 vertices. From the next theorem we see that the smallest minimal graph \(G \in \mathcal{H}_e(3, 3)\) with \(\delta(G) = 4\) has 10 vertices:

**Theorem 8.22.** Let \(G\) be a minimal graph in \(\mathcal{H}_e(3, 3)\) and \(\delta(G) = 4\). Then \(|V(G)| \geq 10\). There is only one 10-vertex minimal graph \(G \in \mathcal{H}_e(3, 3)\) with \(\delta(G) = 4\), namely \(G_{10,2}\) (see Figure 8.12). Furthermore, \(G_{10,2}\) has only a single vertex of degree 4. For all other 10-vertex minimal graphs \(G \in \mathcal{H}_e(3, 3)\), \(\delta(G) = 5\).
Let $G$ be a graph in $\mathcal{H}_e(3, 3)$. By Theorem 8.4, $\chi(G) \geq 6$ and from the inequality $\chi(G) \leq \Delta(G) + 1$ (see [32]) we obtain $\Delta(G) \geq 5$. From the Brooks’ Theorem (see [32]) it follows that if $G \neq K_6$, then $\Delta(G) \geq 6$. The following related question arises naturally:

Are there minimal graphs in $\mathcal{H}_e(3, 3)$ which are 6-regular?

(i.e. $d(v) = 6, \forall v \in V(G)$)

From the obtained minimal graphs in $\mathcal{H}_e(3, 3)$ we see that the following theorem is true:

**Theorem 8.23.** Let $G$ be a regular minimal graph in $\mathcal{H}_e(3, 3)$ and $G \neq K_6$. Then $|V(G)| \geq 13$. There is only one regular minimal graph in $\mathcal{H}_e(3, 3)$ with 13 vertices, and this is the graph given in Figure 8.4, which is 8-regular.

Regarding the maximum degree of the minimal graphs in $\mathcal{H}_e(3, 3)$ we obtain the following result:

**Theorem 8.24.** Let $G$ be a minimal graph in $\mathcal{H}_e(3, 3)$. Then:

(a) $\Delta(G) = |V(G)| - 1$, if $|V(G)| \leq 10$.

(b) $\Delta(G) \geq 8$, if $|V(G)| = 11, 12, \text{ or } 13$. 

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Chromatic number

By Theorem 8.4, if $G \in \mathcal{H}_e(3,3)$, then $\chi(G) \geq 6$.

From the obtained minimal graphs in $\mathcal{H}_e(3,3)$ we derive the following results:

**Theorem 8.25.** Let $G$ be a minimal graph in $\mathcal{H}_e(3,3)$ and $|V(G)| \leq 12$. Then $\chi(G) = 6$.

**Theorem 8.26.** Let $G$ be a minimal graph in $\mathcal{H}_e(3,3)$ and $|V(G)| \leq 14$. Then $\chi(G) \leq 7$. The smallest 7-chromatic minimal graphs in $\mathcal{H}_e(3,3)$ are the 13 minimal graphs on 13 vertices and independence number 2, given in Figure 8.19.

**Proof.** Suppose the opposite is true, i.e. $\chi(G) \geq 8$. Then, according to [72], $G = K_1 + Q$, where $Q$ is the graph shown in Figure 8.5. The graph $K_1 + Q$ is a graph in $\mathcal{H}_e(3,3)$, but it is not minimal. By Theorem 8.25, there are no 7-chromatic minimal graphs in $\mathcal{H}_e(3,3)$ with less than 13 vertices. The graphs in Figure 8.19 are 13-vertex minimal graphs in $\mathcal{H}_e(3,3)$ with independence number 2, and therefore these graphs are 7-chromatic. With a computer we find that among the 13-vertex graphs in $\mathcal{H}_e(3,3)$ with independence number greater than 2 there are no 7-chromatic graphs.

![Figure 8.5: Graph $\overline{Q}$](image)
Multiplicities

**Definition 8.27.** Denote by $M(G)$ the minimum number of monochromatic triangles in all 2-colorings of $E(G)$. The number $M(G)$ is called a $K_3$-multiplicity of the graph $G$.

In [27] the $K_3$-multiplicities of all complete graphs are computed, i.e. $M(K_n)$ is computed for all positive integers $n$. Similarly, the $K_p$-multiplicity of a graph is defined [33]. The following works are dedicated to the computation of the multiplicities of some concrete graphs: [35], [36], [86], [10], [77].

With the help of a computer, we check the $K_3$-multiplicities of the obtained minimal graphs in $\mathcal{H}_e(3,3)$ and we derive the following results:

**Theorem 8.28.** If $G$ is a minimal graph in $\mathcal{H}_e(3,3)$, $|V(G)| \leq 13$, and $G \neq K_6$, then $M(G) = 1$.

We suppose the following hypothesis is true:

**Hypothesis 8.29.** If $G$ is a minimal graph in $\mathcal{H}_e(3,3)$ and $G \neq K_6$, then $M(G) = 1$.

In support to this hypothesis we prove the following:

**Proposition 8.30.** If $G$ is a minimal graph in $\mathcal{H}_e(3,3)$, $G \neq K_6$ and $\delta(G) \leq 5$, then $M(G) = 1$.

*Proof.* Let $v \in V(G)$ and $d(v) \leq 5$. Consider a 2-coloring of $E(G - v)$ without monochromatic triangles. We will color the edges incident to $v$ with two colors in such a way that we will obtain a 2-coloring of $E(G)$ with exactly one monochromatic triangle. To achieve this, we consider the following two cases:

**Case 1.** $d(v) = 4$. By Corollary 8.7, $G(v) = K_4$. Let $N_v = \{a, b, c, d\}$ and suppose that $[a, b]$ is colored with the first color. Then $[c, d]$ is also colored with the first color (otherwise, by coloring $[v, a]$ and $[v, b]$ with the second color and $[v, c]$ and $[v, d]$ with the first color, we obtain a 2-coloring of $E(G)$ without monochromatic triangles). Thus, $[a, b]$ and $[c, d]$ are colored in the
first color. We color \([v, a]\) and \([v, b]\) with the first color and \([v, c]\) and \([v, d]\) with the second color. We obtain a 2-coloring of \(E(G)\) with exactly one monochromatic triangle \([v, a, b]\).

**Case 2.** \(d(v) = 5\). Since \(\omega(G) \leq 5\), in \(N_G(v)\) there are two non-adjacent vertices \(a\) and \(b\). From \(G \Rightarrow (3, 3)\) it follows easily that in \(G(v) - \{a, b\}\) there is an edge of the first color and an edge of the second color. Therefore, we can suppose that in \(G(v) - \{a, b\}\) there is exactly one edge of one of the colors, say the first color. We color \([v, a]\) and \([v, b]\) with the second color and the other three edges incident to \(v\) with the first color. We obtain a 2-coloring of \(E(G)\) with exactly one monochromatic triangle.

In the end, also in support to the hypothesis, let us note that \(M(K_3 + C_{2r+1}) = 1\), \(r \geq 2\) \cite{74}.

**Automorphism groups**

Denote by \(\text{Aut}(G)\) the automorphism group of the graph \(G\). We use the \textit{nauty} programs \cite{53} to find the number of automorphisms of the obtained minimal graphs in \(H_e(3, 3)\) with 10, 11, 12, and 13 vertices. Most of the obtained graphs have small automorphism groups (see Table 8.1, Table 8.2, Table 8.3, and Table 8.4). We list the graphs with at least 60 automorphisms:

- The graphs in the form \(K_3 + C_{2r+1}\): \(|\text{Aut}(K_3 + C_5)| = 60\), \(|\text{Aut}(K_3 + C_7)| = 84\), \(|\text{Aut}(K_3 + C_9)| = 108\);

- \(|\text{Aut}(G_{12.2240})| = 96\) (see Figure 8.16);

- \(|\text{Aut}(G_{13.255653})| = 144\), \(|\text{Aut}(G_{13.248305})| = 96\), \(|\text{Aut}(G_{13.304826})| = 96\), \(|\text{Aut}(G_{13.113198})| = 72\), \(|\text{Aut}(G_{13.175639})| = 72\), \(|\text{Aut}(G_{13.302168})| = 72\) (see Figure 8.18);

Theorem 8.22, Theorem 8.23, Theorem 8.24, Theorem 8.25, and Theorem 8.26 are published in \cite{2}. 

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8.6 Upper bounds on the independence number of the minimal graphs in $\mathcal{H}_e(3,3)$

Regarding the maximal possible value of the independence number of the minimal graphs in $\mathcal{H}_e(3,3)$, the following theorem holds:

**Theorem 8.31.** [57] If $G$ is a minimal graph in $\mathcal{H}_e(3,3)$, $G \neq K_6$ and $G \neq K_3 + C_5$, then $\alpha(G) \leq |V(G)| - 7$. There is a finite number of graphs for which equality is reached.

According to Theorem 8.19, by executing Algorithm $A7(q = 6; k = 8)$, we obtain the set $B$ of all minimal graphs $G \in \mathcal{H}_e(3,3; 6)$ with $\alpha(G) \geq |V(G)| - 8 \geq 1$. Since the only minimal graphs in $\mathcal{H}_e(3,3)$ with less than 9 vertices are $K_6$ and $K_3 + C_5$, from Theorem 8.31 it follows that $B$ consists of all minimal graphs $G \in \mathcal{H}_e(3,3)$ for which $\alpha(G) = |V(G)| - 7$ or $\alpha(G) = |V(G)| - 8$. Thus, we derive the following additions to Theorem 8.31:

**Theorem 8.32.** There are exactly 11 minimal graphs $G \in \mathcal{H}_e(3,3)$ for which $\alpha(G) = |V(G)| - 7$:
- 9-vertex: 1 (Figure 8.1);
- 10-vertex: 3 ($G_{10.1}$, $G_{10.2}$, $G_{10.4}$, see Figure 8.12);
- 11-vertex: 3 ($G_{11.1}$, $G_{11.2}$, $G_{11.21}$, see Figure 8.13);
- 12-vertex: 1 ($G_{12.163}$, see Figure 8.15);
- 13-vertex: 2 ($G_{13.1}$, $G_{13.2}$, see Figure 8.17);
- 14-vertex: 1 (see Figure 8.6);

**Theorem 8.33.** There are exactly 8633 minimal graphs $G \in \mathcal{H}_e(3,3)$ for which $\alpha(G) = |V(G)| - 8$. The largest of these graphs has 26 vertices, and it is given in Figure 8.7. There is only one minimal graph $G \in \mathcal{H}_e(3,3)$ for which $\alpha(G) = |V(G)| - 8$ and $\omega(G) < 5$, and it is the 15-vertex graph $K_1 + \Gamma$ from [59] (see Figure 7.2).

**Corollary 8.34.** Let $G$ be a minimal graph in $\mathcal{H}_e(3,3)$ and $|V(G)| \geq 27$. Then $\alpha(G) \leq |V(G)| - 9$. 

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Figure 8.6: 14-vertex minimal graph in $\mathcal{H}_e(3, 3)$ with independence number 7

Figure 8.7: 26-vertex minimal graph in $\mathcal{H}_e(3, 3)$ with independence number 18
Figure 8.8: 29-vertex minimal graph in \( \mathcal{H}_e(3,3) \) with independence number 20

According to Theorem 8.19, by executing Algorithm A7\((q = 5; k = 9)\), we obtain the set \( \mathcal{B} \) of all minimal graphs \( G \in \mathcal{H}_e(3,3;5) \) for which \( \alpha(G) \geq |V(G)| - 9 \geq 1 \). Since \( F_e(3,3;5) = 15 \) \[59\] \[78\], all graphs in \( \mathcal{H}_e(3,3;5) \) have at least 15 vertices. We already proved that the only minimal graph \( G \in \mathcal{H}_e(3,3;5) \) with \( \alpha(G) \geq |V(G)| - 8 \) is the graph \( K_1 + \Gamma \) from \[59\] (see Theorem 8.33). Therefore, \( \mathcal{B} \) consists of \( K_1 + \Gamma \) and all minimal graphs \( G \in \mathcal{H}_e(3,3;5) \) for which \( \alpha(G) = |V(G)| - 9 \). Thus, we proved the following theorem:

**Theorem 8.35.** There are exactly 8903 minimal graphs \( G \in \mathcal{H}_e(3,3) \) for which \( \omega(G) < 5 \) and \( \alpha(G) = |V(G)| - 9 \). The largest of these graphs has 29 vertices, and it is given in Figure 8.8.

**Corollary 8.36.** Let \( G \) be a minimal graph in \( \mathcal{H}_e(3,3) \) such that \( \omega(G) < 5 \) and \( |V(G)| \geq 30 \). Then \( \alpha(G) \leq |V(G)| - 10 \).

Theorem 8.32, Theorem 8.33, and Theorem 8.35 are published in [2].

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8.7 Lower bounds on the minimum degree of the minimal graphs in $\mathcal{H}_e(3, 3)$

According to Proposition 8.13, if $G$ is a minimal graph in $\mathcal{H}_e(3, 3)$, then for each vertex $v$ of $G$, $N_G(v)$ is a marked vertex set in $G - v$, and therefore $N_G(v)$ is a marked vertex set in $G(v)$.

It is easy to see that if $W \subseteq V(G)$ and $|W| \leq 3$, or $|W| = 4$ and $G[W] \neq K_4$, then $W$ is not a marked vertex set in $G$. A $(3, 3)$-free 2-coloring of $K_4$ which cannot be extended to a $(3, 3)$-free 2-coloring of $K_5$ is shown in Figure 8.9. Therefore, the only 4-vertex graph $N$ such that $V(N)$ is a marked vertex set in $N$ is $K_4$.

With the help of a computer, we derive that there are exactly 3 graphs $N$ with 5 vertices such that $K_4 \not\subset N$ and $V(N)$ is a marked vertex set in $N$. Namely, they are the graphs $N_{5,1}$, $N_{5,2}$, and $N_{5,3}$ given in Figure 8.10. Let us note that $N_{5,1} \subset N_{5,2} \subset N_{5,3}$. From these results we derive

**Theorem 8.37.** Let $G$ be a minimal graph in $\mathcal{H}_e(3, 3; 5)$. Then $\delta(G) \geq 5$. If $v \in V(G)$ and $d(v) = 5$, then $G(v) = N_{5,i}$ for some $i \in \{1, 2, 3\}$ (see Figure 8.10).

The bound $\delta(G) \geq 5$ from Theorem 8.37 is exact. For example, the graph $K_1 + \Gamma$ from [59] (see Figure 7.2) has 7 vertices $v$ such that $d(v) = 5$ and $G(v) = N_{5,3}$.

Also with the help of a computer, we derive that the smallest graphs $N$ such that $K_3 \not\subset N$ and $V(N)$ is a marked vertex set in $N$ have 8 vertices, and there are exactly 7 such graphs. Namely, they are the graphs $N_{8,i}$, $i = 1, \ldots, 7$ given in Figure 8.11. Among them the minimal graphs are $N_{8,1}$, $N_{8,2}$, and $N_{8,3}$, and the remaining 4 graphs are their supergraphs. Thus, we derive the following

**Theorem 8.38.** Let $G$ be a minimal graph in $\mathcal{H}_e(3, 3; 4)$. Then $\delta(G) \geq 8$. If $v \in V(G)$ and $d(v) = 8$, then $G(v) = N_{8,i}$ for some $i \in \{1, \ldots, 7\}$ (see Figure 8.11).

Theorem 8.37 and Theorem 8.38 are published in [2].
Figure 8.9: $(3, 3)$-free 2-coloring of the edges of $K_4$

Figure 8.10: The graphs $N_{5,1}, N_{5,2}, N_{5,3}$

Figure 8.11: The graphs $N_{8,i}, i = 1, ..., 7$
Figure 8.12: All 6 10-vertex minimal graphs in $H_e(3, 3)$

| $G_{10.1}$ | $G_{10.2}$ | $G_{10.3}$ |
|------------|------------|------------|
| 0 0 0 0 1 0 1 1 1 1 | 0 0 0 0 0 1 1 0 1 1 | 0 0 1 1 0 0 1 0 1 1 |
| 0 0 0 0 0 1 1 1 1 1 | 0 0 1 0 0 1 0 1 1 1 | 0 0 0 1 1 0 1 1 1 1 |
| 0 0 0 1 1 0 1 1 1 1 | 0 1 0 0 0 1 0 1 1 1 | 1 0 0 0 1 1 1 1 1 1 |
| 0 0 1 0 1 1 0 1 1 1 | 0 0 0 1 0 1 1 1 1 1 | 1 0 0 1 0 1 1 1 1 1 |
| 0 1 1 0 0 1 0 1 1 1 | 0 0 1 0 0 1 1 1 1 1 | 1 1 0 0 1 1 1 1 1 1 |
| 1 1 0 1 1 0 0 1 1 1 | 1 0 1 1 1 0 0 1 1 1 | 0 1 1 1 1 1 1 1 1 1 |
| 1 1 1 0 0 1 0 1 1 1 | 0 1 1 1 1 0 0 1 1 1 | 1 1 1 1 1 1 0 0 1 1 |
| 1 1 1 1 1 1 1 1 1 0 | 1 1 1 1 1 1 1 1 1 0 | 1 1 1 1 1 1 1 1 1 0 |

| $G_{10.4}$ | $G_{10.5}$ | $G_{10.6}$ |
|------------|------------|------------|
| 0 1 1 0 0 0 0 1 1 1 | 0 1 1 0 0 0 1 1 1 1 | 0 1 1 1 0 0 1 1 1 1 |
| 1 0 0 0 1 0 1 1 1 1 | 1 0 0 1 0 1 1 1 1 1 | 1 0 0 1 1 0 1 1 1 1 |
| 1 0 0 0 0 1 1 1 1 1 | 1 0 0 1 1 0 1 1 1 1 | 1 0 1 0 0 1 1 1 1 1 |
| 0 0 0 1 1 0 1 1 1 1 | 0 1 0 0 1 1 1 1 1 1 | 0 1 0 0 1 1 1 1 1 1 |
| 0 0 1 0 0 0 1 1 1 1 | 0 0 1 1 0 1 1 1 1 1 | 0 1 0 1 1 0 1 1 1 1 |
| 0 1 0 1 0 0 0 1 1 1 | 0 0 1 1 1 0 1 1 1 1 | 0 1 1 1 1 1 0 0 1 1 |
| 0 0 1 1 0 1 1 0 1 1 | 1 1 1 0 1 1 1 0 1 1 | 1 0 1 1 1 1 0 0 1 1 |
| 1 1 1 1 1 1 1 0 1 1 | 1 1 1 0 1 1 1 0 1 1 | 0 1 1 1 1 1 1 0 1 1 |
| 1 1 1 1 1 1 1 1 1 0 | 1 1 1 1 1 1 1 1 1 0 | 1 1 1 1 1 1 1 1 1 0 |

Figure 8.13: 11-vertex minimal graphs in $H_e(3, 3)$ with independence number 4

| $G_{11.1}$ | $G_{11.2}$ | $G_{11.21}$ |
|------------|------------|------------|
| 0 0 0 0 0 1 1 1 1 1 | 0 0 0 1 0 0 1 1 1 1 | 0 0 1 0 0 0 1 1 1 1 |
| 0 0 0 0 1 1 0 1 1 1 | 0 0 0 1 0 1 1 1 1 1 | 0 0 0 1 0 1 1 1 1 1 |
| 0 0 0 1 1 0 0 1 1 1 | 0 0 0 0 1 1 1 1 1 1 | 0 0 0 1 0 1 1 1 1 1 |
| 0 0 1 1 0 0 0 1 1 1 | 0 0 0 1 1 0 1 1 1 1 | 0 1 0 0 1 1 1 1 1 1 |
| 0 1 1 1 0 0 0 1 1 1 | 1 0 0 0 1 1 1 1 1 1 | 0 1 0 1 1 1 1 1 1 1 |
| 0 1 1 1 1 1 0 1 1 1 | 1 0 1 1 1 0 0 1 1 1 | 0 0 1 1 1 1 1 1 1 1 |
| 1 0 0 1 1 0 0 1 1 1 | 0 0 0 1 1 1 0 0 1 1 | 0 0 1 1 1 1 1 1 1 1 |
| 1 1 1 0 0 0 0 1 1 1 | 1 1 1 0 0 0 1 1 1 1 | 1 1 1 0 1 1 1 1 1 1 |
| 1 1 1 1 1 1 1 1 1 0 | 1 1 1 1 1 1 1 1 1 0 | 1 1 1 1 1 1 1 1 1 0 |

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Figure 8.14: 11-vertex minimal graphs in $H_e(3, 3)$ with independence number 2

Figure 8.15: 12-vertex minimal graphs in $H_e(3, 3)$ with independence number 5

Figure 8.16: 12-vertex minimal graphs in $H_e(3, 3)$ with 96 automorphisms
Figure 8.17: 13-vertex minimal graphs in $H_e(3, 3)$ with independence number 6

| $G_{13.1}$ | $G_{13.2}$ |
|------------|------------|
| 0 0 0 0 0 | 0 0 0 0 0 |
| 0 0 0 0 0 | 1 0 1 0 1 |
| 0 0 0 0 0 | 1 0 1 0 1 |
| 0 0 0 0 0 | 1 0 1 0 1 |
| 0 0 0 0 0 | 1 0 1 0 1 |
| 1 0 1 0 1 | 0 0 1 0 0 |
| 0 1 0 1 0 | 0 0 0 1 1 |
| 0 1 0 1 0 | 0 0 0 1 1 |
| 1 0 1 0 1 | 0 0 0 1 1 |
| 0 1 0 1 0 | 0 0 0 1 1 |
| 1 0 1 0 1 | 0 0 0 1 1 |
| 1 0 1 0 1 | 0 0 0 1 1 |

Figure 8.18: 13-vertex minimal graphs in $H_e(3, 3)$ with a large number of automorphisms

| $G_{13.13198}$ | $G_{13.175639}$ | $G_{13.248305}$ |
|----------------|----------------|----------------|
| 0 0 1 0 0 | 0 0 0 0 1 | 0 1 0 0 0 |
| 0 0 0 1 0 | 1 1 1 0 0 | 0 0 1 0 0 |
| 0 0 0 1 0 | 1 1 1 0 0 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |
| 0 0 1 0 0 | 0 0 0 1 1 | 0 0 1 0 0 |

| $G_{13.255653}$ | $G_{13.302168}$ | $G_{13.304826}$ |
|----------------|----------------|----------------|
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |
| 0 1 0 0 0 | 0 0 0 0 1 | 0 0 0 0 1 |

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Figure 8.19: 13-vertex minimal graphs in $\mathcal{H}_c(3, 3)$ with independence number 2
Chapter 9

New lower bound on $F_e(3, 3; 4)$

The graph $G$ obtained by the construction of Folkman \[22\], for which $G \not\rightarrow (3, 3)$ and $\omega(G) = 3$, has a very large number of vertices. Because of this, in 1975 Erdős \[19\] posed the problem to prove the inequality $F_e(3, 3; 4) < 10^{10}$. In 1986 Frankl and Rödl \[24\] almost solved this problem by showing that $F_e(3, 3; 4) < 7.02 \times 10^{11}$. In 1988 Spencer \[94\] proved the inequality $F_e(3, 3; 4) < 3 \times 10^9$ by using probabilistic methods. In 2008 Lu \[48\] constructed a 9697-vertex graph in $H_e(3, 3; 4)$, thus considerably improving the upper bound on $F_e(3, 3; 4)$. Soon after that, Lu’s result was improved by Dudek and Rödl \[18\], who proved $F_e(3, 3; 4) \leq 941$. The best known upper bound on this number is $F_e(3, 3; 4) \leq 786$, obtained in 2012 by Lange, Radziszowski, and Xu \[43\]. Exoo conjectured that the 127-vertex graph $G_{127}$, used by Hill and Irwing \[34\] to prove the bound $R(4, 4, 4) \geq 128$, has the property $G_{127} \not\rightarrow (3, 3)$. This conjecture was studied in \[81\] and \[82\]. It is still unknown whether $G_{127} \not\rightarrow (3, 3)$.

In 1972 Lin \[47\] proved that $F_e(3, 3; 4) \geq 11$. The lower bound was improved by Nenov \[61\], who showed in 1981 that $F_e(3, 3; 4) \geq 13$. In 1984 Nenov \[62\] proved that every 5-chromatic $K_4$-free graph has at least 11 vertices, from which it is easy to derive that $F_e(3, 3; 4) \geq 14$. From $F_e(3, 3; 5) = 15$ \[59\] \[78\] it follows easily, that $F_e(3, 3; 4) \geq 16$. The best lower bound known on $F_e(3, 3; 4)$ was obtained in 2007 by Radziszowski and Xu \[81\], who proved with the help of a computer that $F_e(3, 3; 4) \geq 19$. 

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According to Radziszowski and Xu [81], any method to improve the bound $F_e(3, 3; 4) \geq 19$ would likely be of significant interest.

We improve the lower bound on the number $F_e(3, 3; 4)$ by proving

**Theorem 9.1.** $F_e(3, 3; 4) \geq 20$.

### 9.1 Algorithm A8

Let $G \in \mathcal{H}_e(3, 3; 4; n)$, $A \subseteq V(G)$ be an independent set of vertices of $G$, $|A| = k$, and $H = G - A$. Then obviously, $G$ is a subgraph of $\overline{K}_k + H$, therefore $\overline{K}_k + H \in \mathcal{H}_e(3, 3; 5; n)$, and it is easy to see that $K_1 + H \in \mathcal{H}_e(3, 3; 5; n - k + 1)$. By this reasoning, in [81] it is proved, but not explicitly formulated (see the proofs of Theorem 2 and Theorem 3), the following

**Proposition 9.2.** Let $G \in \mathcal{H}_e(3, 3; 4; n)$, $A \subseteq V(G)$ be an independent set of vertices of $G$, and $H = G - A$. Then $K_1 + H \in \mathcal{H}_e(3, 3; 5; n - |A| + 1)$.

With the help of Proposition 9.2 in [81] the authors prove that every graph in $\mathcal{H}_e(3, 3; 4; 18)$ can be constructed by adding 4 independent vertices to a 14-vertex graph $H$ such that $K_1 + H \in \mathcal{H}_e(3, 3; 5; 15)$. All 659 graphs in $\mathcal{H}_e(3, 3; 5; 15)$ were obtained in [78], and among them 153 graphs are of the form $K_1 + H$. With the help of a computer in [81] it is proved that, by extending the 14-vertex graphs $H$ with 4 independent vertices, it is not possible to obtain a graph in $\mathcal{H}_e(3, 3; 4; 18)$. By Proposition 9.2 $\mathcal{H}_e(3, 3; 4; 18) = \emptyset$ and $F_e(3, 3; 4) \geq 19$.

This method is not suitable for proving Theorem 9.1 because not all graphs in $\mathcal{H}_e(3, 3; 5; 16)$ are known and their number is too large. Because of this, we first prove that if $G \in \mathcal{H}_e(3, 3; 4; 19)$, then $G$ can be obtained by adding 4 independent vertices to some of 1 139 033 appropriately selected 15-vertex graphs, which we obtain in advance, or by adding 5 independent vertices to some of the 14-vertex graphs known from [78], mentioned above. With the help of a new computer algorithm we check that these extensions do not lead to the construction of a graph in $\mathcal{H}_e(3, 3; 4; 19)$ and we derive that $\mathcal{H}_e(3, 3; 4; 19) = \emptyset$ and $F_e(3, 3; 4) \geq 20$. 

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For convenience, we will use the following notations:

\[ L(n; p) = \left\{ G : |V(G)| = n, \omega(G) < 4 \text{ and } K_p + G \not\rightarrow (3,3) \right\} \]

\[ L(n; p; k) = \{ G \in L(n; p) : \alpha(G) = k \} \]

From \( R(3,4) = 9 \) it follows that

\[ (9.1) \quad L(n; p; k) = \emptyset, \text{ if } n \geq 9 \text{ and } k \leq 2, \]

and from \( R(4,4) = 18 \) it follows that

\[ (9.2) \quad L(n; p; k) = \emptyset, \text{ if } n \geq 18 \text{ and } k \leq 3. \]

In [78] it is proved that \( L(n; 1) \neq \emptyset \) if and only if \( n \geq 14 \), and all 153 graphs in \( L(14; 1) \) are found. Further, we will use the following fact:

\[ (9.3) \quad L(14; 1; k) \neq \emptyset \iff k \in \{4, 5, 6, 7\}, [78]. \]

From Theorem 8.4 it follows

\[ (9.4) \quad G \in L(n; p) \Rightarrow \chi(G) \geq 6 - p. \]

Obviously, \( \mathcal{H}_e(3,3; 4; n) = L(n; 0) \). Let \( G \in \mathcal{H}_e(3,3; 4) \). From the equality \( F_e(3,3; 5) = 15 \) [59] [78] and Proposition 9.2 it follows that \( \alpha(G) \leq |V(G)| - 14 \). Therefore, either \( |V(G)| \geq 20 \) or \( \alpha(G) \leq 5 \). From \( R(4,4) = 18 \) it follows that \( \alpha(G) \geq 4 \). Thus, we obtain

\[ (9.5) \quad \mathcal{H}_e(3,3; 4; 19) = L(19; 0) = L(19; 0; 4) \cup L(19; 0; 5). \]

We will need the following proposition, which follows easily from Proposition 9.2:

**Proposition 9.3.** Let \( G \in \mathcal{L}(n; p) \), \( A \subseteq V(G) \) be an independent set of vertices of \( G \), and \( H = G - A \). Then \( H \in \mathcal{L}(n - |A|; p + 1) \).

We denote by \( \mathcal{L}_{\text{max}}(n; p; k) \) the set of all maximal \( K_4 \)-free graphs in \( \mathcal{L}(n; p; k) \), i.e. the graphs \( G \in \mathcal{L}(n; p; k) \) for which \( \omega(G + e) = 4 \) for every \( e \in \bar{E}(G) \). Since every graph in \( \mathcal{L}(19; 0) \) is contained in a maximal \( K_4 \)-free graph in \( \mathcal{L}(19; 0) \), according to (9.5) to prove the Theorem 9.1 it is enough to
prove that $\mathcal{L}_{\text{max}}(19; 0; 4) = \emptyset$ and $\mathcal{L}_{\text{max}}(19; 0; 5) = \emptyset$. In the proofs of these inequalities we will use Algorithm A8 formulated below.

We denote by $\mathcal{L}_{+K_3}(n; p; k)$ the set of all $(+K_3)$-graphs in $\mathcal{L}(n; p; k)$ (see Definition 1.7). Let $G \in \mathcal{L}_{\text{max}}(n; p; k)$. Let $A \subseteq V(G)$ be an independent set of vertices of $G$, $|A| = k$ and $H = G - A$. According to Proposition 9.3 it follows that $H$ is $(+K_3)$-graph. From $\alpha(G) = k$ it follows that $\alpha(H) \leq k$. Therefore, $H \in \mathcal{L}_{+K_3}(n - k; p + 1; k')$ for some $k' \leq k$. Thus, we proved the following

**Proposition 9.4.** Let $G \in \mathcal{L}_{\text{max}}(n; p; k)$. Let $A \subseteq V(G)$ be an independent set of vertices of $G$, $|A| = k$, and $H = G - A$. Then,

$$H \in \bigcup_{K' \leq k} \mathcal{L}_{+K_3}(n - k; p + 1; k').$$

If $G \in \mathcal{L}(n; p; k)$ is a Sperner graph, i.e. $N_G(u) \subseteq N_G(v)$ for some $u, v \in V(G)$, then $G - u \in \mathcal{L}(n - 1; p; k')$, for $k - 1 \leq k' \leq k$. Therefore, every Sperner graph $G \in \mathcal{L}(n; p; k)$ is obtained by adding one vertex to some graph $H \in \mathcal{L}(n - 1; p; k')$, $k - 1 \leq k' \leq k$. In the special case, when $G$ is a Sperner graph and $G \in \mathcal{L}_{\text{max}}(n; p; k)$, from $N_G(u) \subseteq N_G(v)$ it follows that $N_G(u) = N_G(v)$. Therefore $G - u \in \mathcal{L}_{\text{max}}(n - 1; p; k')$, $k - 1 \leq k' \leq k$, i.e. $G$ is obtained by duplicating a vertex in some graph $H \in \mathcal{L}_{\text{max}}(n - 1; p; k')$. All non-Sperner graphs in $\mathcal{L}_{\text{max}}(n; p; k)$ are obtained very efficiently with the help of Algorithm A8 formulated below, which is based on Proposition 9.4 and the following

**Proposition 9.5.** Let $A$ be an independent set of vertices of $G$, $|A| = k$, and $H = G - A$. Then,

$$\alpha(G) = k \iff \alpha(H - \bigcup_{v \in A'} N_G(v)) \leq k - |A'|, \forall A' \subseteq A.$$

**Proof.** Proposition 9.5 is the special case $t = |A| = k$ of Proposition 3.3.

Now we formulate:

**Algorithm A8.** Let $n$, $p$ and $k$ be positive integers.

The input of the algorithm is the set $\mathcal{A} = \bigcup_{K' \leq k} \mathcal{L}_{\text{max}}(n - k; p + 1; k').$

The output of the algorithm is the set $\mathcal{B}$ of all non-Sperner graphs in $\mathcal{L}_{\text{max}}(n; p; k)$.

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1. By removing edges from the graphs in $A$ obtain the set
   
   $A' = \bigcup_{k' \leq k} L_{+K_3}(n - k; p + 1; k')$.

2. For each graph $H \in A'$:

   2.1. Find the family $\mathcal{M}(H) = \{M_1, ..., M_l\}$ of all maximal $K_3$-free subsets of $V(H)$.

   2.2. Find all $k$-element subsets $N = \{M_{i_1}, ..., M_{i_k}\}$ of $\mathcal{M}(H)$ which fulfill the conditions:

   (a) $M_{i_j} \neq N_H(v)$ for every $v \in V(H)$ and for every $M_{i_j} \in N$.

   (b) $K_2 \subseteq M_{i_j} \cap M_{i_h}$ for every $M_{i_j}, M_{i_h} \in N$.

   (c) $\alpha(H - \bigcup_{M_{i_j} \in N'} M_{i_j}) \leq k - |N'|$ for every $N' \subseteq N$.

   2.3. For each of the found in step 2.2 $k$-element subsets $N = \{M_{i_1}, ..., M_{i_k}\}$ of $\mathcal{M}(H)$ construct the graph $G = G(N)$ by adding new independent vertices $v_1, ..., v_k$ to $V(H)$ such that $N_G(v_j) = M_{i_j}, j = 1, ..., k$.

   If $G$ is not a Sperner graph and $\omega(G + e) = 4, \forall e \in E(G)$, then add $G$ to $B$.

3. Remove the isomorphic copies of graphs from $B$.

4. Remove from $B$ all graphs with chromatic number less than $6 - p$.

5. Remove from $B$ all graphs $G$ for which $K_p + G \not\rightarrow (3, 3)$.

We will prove the correctness of Algorithm $A_8$ with the help of the following

Lemma 9.6. After the execution of step 2.3 of Algorithm $A_8$, the obtained set $B$ coincides with the set of all maximal $K_4$-free non-Sperner graphs $G$ with $\alpha(G) = k$ which have an independent set of vertices $A \subseteq V(G), |A| = k$ such that $G - A \in A'$.

Proof. Suppose that in step 2.3 of Algorithm $A_8$ the graph $G$ is added to $B$. Then $G = G(N)$ and $G - \{v_1, ..., v_k\} = H \in A'$, where $N, v_1, ..., v_k$, and $H$ are the same as in step 2.3. By $H \in A'$, we have $\omega(H) < 4$. Since $N_G(v_j), j = 1, ..., k$, are $K_3$-free sets, it follows that $\omega(G) < 4$. From the condition (c) in step 2.2 and Proposition 9.5 it follows that $\alpha(G) = k$. The two checks at the end of step 2.3 guarantee that $G$ is a maximal $K_4$-free non-Sperner graph.

Let $G$ be a maximal $K_4$-free non-Sperner graph, $\alpha(G) = k$, and $A = \{v_1, ..., v_k\}$ be an independent set of vertices of $G$ such that $H = G - A \in A'$. 

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We will prove that, after the execution of step 2.3 of Algorithm $A_8$, $G \in \mathcal{B}$.

Since $G$ is a maximal $K_4$-free graph, $N_G(v_i), i = 1, \ldots, k,$ are maximal $K_3$-free subsets of $V(H)$, and therefore $N_G(v_i) \in \mathcal{M}(H), i = 1, \ldots, k$ (see step 2.1).

Let $N = \{N_G(v_1), \ldots, N_G(v_k)\}$. Since $G$ is not a Sperner graph, $N$ is a $k$-element subset of $\mathcal{M}(H)$, and according to Proposition 5.6, $N$ fulfills the condition (a) in step 2.2. By Proposition 3.2, $N$ fulfills the condition (b), and by Proposition 9.5, $N$ also fulfills (c). Thus, we showed that $N$ fulfills all conditions in step 2.2, and since $G = G(N)$ is a maximal $K_4$-free non-Sperner graph, in step 2.3 $G$ is added to $\mathcal{B}$.

**Theorem 9.7.** After the execution of Algorithm $A_8$, the obtained set $\mathcal{B}$ coincides with the set of all non-Sperner graphs in $\mathcal{L}_{max}(n; p; k)$.

**Proof.** Suppose that, after the execution of Algorithm $A_8$, $G \in \mathcal{B}$. According to Lemma 9.6, $G$ is a maximal $K_4$-free non-Sperner graph and $\alpha(G) = k$.

Now, from step 5 it follows that $G \in \mathcal{L}_{max}(n; p; k)$.

Conversely, let $G$ be an arbitrary non-Sperner graph in $\mathcal{L}_{max}(n; p; k)$. Let $A \subseteq V(G)$ be an independent set of vertices of $G$, $|A| = k$, and $H = G - A$.

According to Proposition 9.4, $H \in \mathcal{A}'$. Now, from Lemma 9.6 we obtain that, after the execution of step 2.3, the graph $G$ is included in the set $\mathcal{B}$. By (9.4), after the execution of step 4, $G$ remains in $\mathcal{B}$. It is clear that after step 5, $G$ also remains in $\mathcal{B}$.

**Remark 9.8.** Since $\mathcal{L}(18; 0) = \emptyset$, from Theorem 8.38 it follows easily that for each graph $G \in \mathcal{L}(19, 0)$ we have $\delta(G) \geq 8$. Using this result we can improve Algorithm $A_8$ in the case $n = 19, p = 0$ in the following way:

1. In step 1 we remove from the set $\mathcal{A}'$ the graphs with minimum degree less than $8 - k$.

2. In step 2.2 we add the following conditions for the subset $N$:

   (d) $|M_{i_j}| \geq 8$ for every $M_{i_j} \in N$.

   (e) If $N' \subseteq N$, then $d_H(v) \geq 8 - k + |N'|$ for every $v \not\in \bigcup_{M_{i_j} \in N'} M_{i_j}$.

   This way it is guaranteed that in step 2.3 only graphs $G$ for which $\delta(G) \geq 8$ are added to the set $\mathcal{B}$. 

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Theorem 9.7 is published in [5]. Algorithm A8 is a slightly modified version of Algorithm 2.7 in [5].

9.2 Proof of Theorem 9.1

According to (9.5), it is enough to prove that $L_{\text{max}}(19; 0; 5) = \emptyset$ and $L_{\text{max}}(19; 0; 4) = \emptyset$.

1. Proof of $L_{\text{max}}(19; 0; 5) = \emptyset$.

With a computer check we find all 8 maximal graphs among the graphs in $L(14; 1)$, which are known from [78]. All of these graphs have independence number 4. Denote 

$$A_3 = L_{\text{max}}(14; 1; 4) = \bigcup_{k' \leq 5} L_{\text{max}}(14; 1; k').$$

We execute Algorithm A8 ($n = 19$, $p = 0$, $k = 5$) with the set $A = A_3$ as an input. In step 1 we obtain all 85 graphs in $L_{+K_3}(14; 1; 4)$ and all 28 graphs in $L_{+K_3}(14; 1; 5)$. In step 2.3, 502 901 graphs are added to the set $B$, 251 244 of which remain in $B$ after the isomorph rejection in step 3. After step 4, 31 graphs with chromatic number 6 remain in $B$. In the end, after executing step 5 we obtain $B = \emptyset$. Since $L(18; 0) = \emptyset$, there are no Sperner graphs in $L(19; 0)$, and by Theorem 9.7 we obtain $L_{\text{max}}(19; 0; 5) = \emptyset$.

2. Proof of $L_{\text{max}}(19; 0; 4) = \emptyset$.

Using nauty [53] we generate all 11-vertex graphs with a computer and among them we find all 102 graphs in $L_{\text{max}}(11; 2; 3)$ and all 270 graphs in $L_{\text{max}}(11; 2; 4)$. Let us denote $A_1 = L_{\text{max}}(11; 2; 3) \cup L_{\text{max}}(11; 2; 4)$. By (9.1),

$$A_1 = \bigcup_{k' \leq 4} L_{\text{max}}(11; 2; k').$$

We execute Algorithm A8 ($n = 15$, $p = 1$, $k = 4$) with the set $A = A_1$ as an input. In step 1 we obtain all 362 439 graphs in $L_{+K_3}(11; 2; 3)$ and all 7 949 015 graphs in $L_{+K_3}(11; 2; 4)$. According to Theorem 9.7 after the execution of the algorithm we find all 5750 non-Sperner graphs in $L_{\text{max}}(15; 1; 4)$. Among the graphs in $L(14; 1)$, which are known from [78], there are 8 maximal $K_4$-free graphs and they all have independence number 4. By adding a new vertex to each of these 8 graphs which duplicates some of their
vertices, we obtain all 20 non-isomorphic Sperner graphs in $\mathcal{L}_{\text{max}}(15; 1; 4)$ (see Proposition 5.5). Thus, all 5770 graphs in $\mathcal{L}_{\text{max}}(15; 1; 4)$ are obtained. All graphs $G$ for which $\omega(G) < 4$ and $\alpha(G) < 4$ are known and can be found in [52]. There are 640 such 15-vertex graphs, among which we find the only 2 graphs in $\mathcal{L}_{\text{max}}(15; 1; 3)$. Let $\mathcal{A}_2 = \mathcal{L}_{\text{max}}(15; 1; 3) \cup \mathcal{L}_{\text{max}}(15; 1; 4)$. By (9.1),

$\mathcal{A}_2 = \bigcup_{k' \leq 4} \mathcal{L}_{\text{max}}(15; 1; k')$.

We execute Algorithm $\mathcal{A}_8$ ($n = 19$, $p = 0$, $k = 4$) with the set $\mathcal{A} = \mathcal{A}_2$ as an input. In step 1 we obtain all 1 139 023 graphs in $\mathcal{L}_{+K_3}(15; 1; 4)$ and all 5 graphs in $\mathcal{L}_{+K_3}(15; 1; 3)$. In step 2.3, 2 551 314 graphs are added to the set $\mathcal{B}$, 2 480 352 of which remain in $\mathcal{B}$ after the isomorph rejection in step 3. After step 4, 2 597 graphs with chromatic number 6 remain in $\mathcal{B}$. In the end, after executing step 5 we obtain $\mathcal{B} = \emptyset$. Since $\mathcal{L}(18, 0) = \emptyset$, there are no Sperner graphs in $\mathcal{L}(19, 0)$, and by Theorem 9.7 we obtain $\mathcal{L}_{\text{max}}(19; 0; 4) = \emptyset$.

Remark 9.9. It is easy to see that if $G \xrightarrow{v} (3, 3)$, then $K_1 + G \xrightarrow{e} (3, 3)$. Thus, we derive

$\mathcal{H}(3, 3; 4; n) \subseteq \mathcal{L}(n; 1)$.

Let us note that $\mathcal{H}(3, 3; 4; 14) = \mathcal{L}(14; 1)$ [73], but $\mathcal{H}(3, 3; 4; 15) \neq \mathcal{L}(15; 1)$. We found all 2 081 234 graphs $\mathcal{L}(15; 1)$. In the proof of Theorem 9.1 we already found the only 2 graphs in $\mathcal{L}_{\text{max}}(15; 1; 3)$ and all 5770 graphs in $\mathcal{L}_{\text{max}}(15; 1; 4)$. With the help of Algorithm $\mathcal{A}_8$ similarly to the case $k = 4$, we find all graphs in $\mathcal{L}_{\text{max}}(15; 1; k)$, $k \geq 5$. We obtain all 826 graphs $\mathcal{L}_{\text{max}}(15; 1; 5)$, all 12 graphs in $\mathcal{L}_{\text{max}}(15; 1; 6)$, and $\mathcal{L}_{\text{max}}(15; 1; k) = \emptyset$, $k \geq 7$. Thus, we obtain all 6 610 maximal $K_4$-free graphs in $\mathcal{L}(15; 1)$. By removing edges from them, we find all 2 081 234 graphs in $\mathcal{L}(15; 1)$. Some properties of these graphs are listed in Table 9.1. Among the graphs in $\mathcal{L}(15; 1)$ there are exactly 20 graphs, which are not in $\mathcal{H}(3, 3; 4; 15)$. Properties of these 20 graphs are given in Table 9.2, and one of these graphs (which has 51 edges) is given in Figure 9.1.

Theorem 9.1 is published in [5].
Figure 9.1: Example of a graph in $L(15; 1) \setminus H_{i}(3, 3; 4; 15)$
| $|E(G)|$ | $\# \delta(G)$ | $\# \Delta(G)$ | $\# \alpha(G)$ | $|\text{Aut}(G)|$ | $\#$ |
|-----|-------------|-------------|-------------|----------------|---------|
| 42  | 1           | 0           | 7           | 65             | 3       | 5       | 1     | 2 052 543 |
| 43  | 4           | 1           | 1 629       | 8              | 675 118 | 4       | 1 300 452 | 2     | 27 729   |
| 44  | 44          | 2           | 10 039      | 9              | 1 159 910 | 5     | 747 383 | 3     | 9       |
| 45  | 334         | 3           | 34 921      | 10             | 165 612 | 6       | 32 618 | 4     | 850     |
| 46  | 2 109       | 4           | 649 579     | 11             | 80 529 | 7       | 766   | 6     | 22      |
| 47  | 9 863       | 5           | 1 038 937   | 8              |          |         |       |        |         |
| 48  | 35 812      | 6           | 339 395     | 10             |          |         |       |        |         |
| 49  | 101 468     | 7           | 6 581       |                |          |         | 12    | 11     |
| 50  | 223 881     |             |             |                |          |         | 14    | 4      |
| 51  | 378 614     |             |             |                |          |         | 16    | 4      |
| 52  | 478 582     |             |             |                |          |         | 20    | 1      |
| 53  | 436 693     |             |             |                |          |         | 24    | 4      |
| 54  | 273 824     |             |             |                |          |         |       |        |         |
| 55  | 110 592     |             |             |                |          |         |       |        |         |
| 56  | 26 099      |             |             |                |          |         |       |        |         |
| 57  | 3 150       |             |             |                |          |         |       |        |         |
| 58  | 160         |             |             |                |          |         |       |        |         |
| 59  |             |             |             |                |          |         |       |        |         |

Table 9.1: Some properties of the graphs in $\mathcal{L}(15;1)$

| $|E(G)|$ | $\# \delta(G)$ | $\# \Delta(G)$ | $\# \alpha(G)$ | $|\text{Aut}(G)|$ | $\#$ |
|-----|-------------|-------------|-------------|----------------|---------|
| 47  | 2           | 4           | 7           | 8              | 20      | 4       | 5       | 1     | 5       |
| 48  | 5           | 5           | 10          | 5              | 15      | 2       | 12      |
| 49  | 7           | 6           | 3           |                | 4       | 3       |
| 50  | 3           |             |             |                |          |         |         |         |
| 51  | 1           |             |             |                |          |         |         |         |
| 52  | 2           |             |             |                |          |         |         |         |

Table 9.2: Some properties of the graphs in $\mathcal{L}(15;1) \setminus \mathcal{H}_v(3,3;4;15)$
Computations

All computations were done on a personal computer. Using one processing core, the time needed to execute Algorithm A8 in the case \((n = 19, p = 0, k = 5)\) is about half a minute, and in the case \((n = 19, p = 0, k = 4)\) it is about one hour. Note that in the first case among the 31 6-chromatic graphs obtained after step 4 of Algorithm A8 11 have a minimum degree of 8 or more. In the second case, the total number of 6-chromatic graphs is 2597, 794 of which have a minimum degree of 8 or more. Using the improvements of Algorithm A8 described in Remark 9.8, the time needed for computations is reduced more than 10 times in the first case and almost 2 times in the second case.

Let us note that the result \(L_{\text{max}}(19; 0; 5) = \emptyset\) can be obtained with the algorithm from [81], but a lot more computations have to be performed by the computer.

We performed various tests to check the correctness of our implementation of Algorithm A8. One such test was to reproduce all 153 graphs in \(L(14; 1)\), which are known from [78]. First, by executing Algorithm A8 with \(n = 14, p = 1, k \geq 3\), we obtain the 8 graphs in \(L_{\text{max}}(14; 1; 4)\), and \(L_{\text{max}}(14; 1; k) = \emptyset, k \neq 4\). By removing edges from the 8 maximal graphs, we obtain all 153 graphs in \(L(14; 1)\). Among the 2 081 234 graphs in \(L(15; 1)\), there are exactly 153 graphs with one isolated vertex (see Remark 9.9 and Table 9.1). Thus, we obtain once again, indirectly, all graphs in \(L(14; 1)\).

We used Algorithm A8 to give another proof of the bound \(F_e(3, 3; 4) \geq 19\). Similarly to (9.5), it is easy to see that \(H_e(3, 3; 4; 18) = L(18; 0) = L(18; 0; 4)\). By executing Algorithm A8 \((n = 18, p = 0, k = 4)\) with input \(A = L_{\text{max}}(14; 1; 4) = \bigcup_{k' \leq 4} L_{\text{max}}(14; 1; k')\), we obtain \(B = \emptyset\). Since there are no non-Sperner graphs in \(L(18; 0)\), by Theorem 9.7 we derive \(L_{\text{max}}(18; 0; 4) = \emptyset\), and therefore \(F_e(3, 3; 4) \geq 19\). The computation time in this case is less than a second.
Bibliography

[1] A. Bikov. A computer research on critical Ramsey graphs (in Bulgarian), 2013. Master Thesis, Sofia University “St. Kliment Ohridski”.

[2] A. Bikov. Small minimal (3, 3)-ramsey graphs. Ann. Univ. Sofia Fac. Math. Inform., 103:123–147, 2016. Preprint: arxiv:1604.03716, April 2016.

[3] A. Bikov. New bounds on the vertex Folkman number $F_v(2, 2, 2, 3; 4)$. Mathematics and Education. Proceedings of the 46th Spring Conference of the Union of Bulgarian Mathematicians, 46:137–144, 2017. Preprint arXiv:1611.06418, November 2016.

[4] A. Bikov and N. Nenov. Modified vertex Folkman numbers. Mathematics and Education. Proceedings of the 45th Spring Conference of the Union of Bulgarian Mathematicians, 45:113–123, 2016. Preprint: arxiv:1511.02125, November 2015.

[5] A. Bikov and N. Nenov. The edge Folkman number $F_e(3, 3; 4)$ is greater than 19. GEOMBINATORICS, 27(1):5–14, 2017. Preprint: arxiv:1609.03468, September 2016.

[6] A. Bikov and N. Nenov. Lower bounding the Folkman numbers $F_v(a_1, ..., a_s; m − 1)$. Ann. Univ. Sofia Fac. Math. Inform., 104:39–53, 2017. Preprint: arxiv:1711.01535, November 2017.

[7] A. Bikov and N. Nenov. On the vertex Folkman numbers $F_v(a_1, ..., a_s; m − 1)$ when $\max\{a_1, ..., a_s\} = 6$ or 7. To appear in the
[8] A. Bikov and N. Nenov. The vertex Folkman numbers \( F_v(a_1, \ldots, a_s; m - 1) = m + 9 \), if \( \max\{a_1, \ldots, a_s\} = 5 \). *Journal of Combinatorial Mathematics and Combinatorial Computing*, 103:171–198, 2017. Preprint: arxiv:1503.08444, August 2015.

[9] G. Brinkmann, K. Coolsaet, J. Goedgebeur, and H. Mélot. House of Graphs, https://hog.grinvin.org/.

[10] J. Brown and V. Rödl. A new construction of \( k \)-Folkman graphs. *Ars Comb.*, 29:265–269, 1990.

[11] S. Burr, P. Erdős, and L. Lovász. On graphs of Ramsey type. *Ars. Combinatoria*, 1(1):167–190, 1976.

[12] S. Burr and V. Rosta. On the Ramsey multiplicities of Graphs - Problems and recent results. *Journal of Graph Theory*, 4:347–361, 1980.

[13] V. Chvátal. The minimality of the Mycielski graph. *Lecture Notes in Mathematics*, 406:243–246, 1979.

[14] J. Coles. Algorithms for bounding Folkman numbers, 2005. Master Thesis, RIT, https://ritdml.rit.edu/bitstream/handle/1850/2765/JColesThesis2004.pdf.

[15] J. Coles and S. Radziszowski. Computing the Folkman number \( F_v(2, 2, 3; 4) \). *Journal of Combinatorial Mathematics and Combinatorial Computing*, 58:13–22, 2006.

[16] F. Deng, M. Liang, Z. Shao, and X. Xu. Upper bounds for the vertex Folkman number \( F_v(3, 3, 3; 4) \) and \( F_v(3, 3, 3; 5) \). *ARS Combinatoria*, 112:249–256, 2013.

[17] A. Dudek and V. Rödl. New upper bound on vertex Folkman numbers. *Lecture Notes in Computer Science*, 4557:473–478, 2008.
[18] A. Dudek and V. Rödl. On the Folkman number $F(2,3,4)$. *Experimental Mathematics*, 17:63–67, 2008.

[19] P. Erdös. Problems and results on finite and infinite graphs. In *Recent Advances in Graph Theory, Proc. Second Czechoslovak Symp.*., pages 183–192, April 1975.

[20] P. Erdös and A. Hajnal. Research problem 2-5. *J. Combin. Theory*, 2:104, 1967.

[21] G. Exoo and J. Goedgebeur. Bounds for the smallest k-chromatic graphs of given girth. Preprint: arxiv:1805.06713, May 2018.

[22] J. Folkman. Graphs with monochromatic complete subgraphs in every edge coloring. *SIAM Journal on Applied Mathematics*, 18:19–24, 1970.

[23] J. Fox and K. Lin. The minimum degree of Ramsey-minimal graphs. *J. of Graph Theory*, 54(2):167–177, 2006.

[24] P. Frankl and V. Rödl. Large triangle-free subgraphs in graphs without $K_4$. *Graphs and Combinatorics*, 2:135–144, 1986.

[25] A. Galuccio, M. Simonovits, and G. Simonyi. On the structure of co-critical graphs. *Graph theory, combinatorics and algorithms*, Vol 1, 2 (Kalamazoo, MI, 1992):1053–1071, 1995.

[26] J. Goedgebeur. On minimal triangle-free 6-chromatic graphs. Preprint: arxiv:1707.07581, August 2017.

[27] A. Goodman. On sets of acquaintances and strangers at any party. *Amer. Math. Monthly*, 66:778–783, 1959.

[28] R. Graham and S. Butler. *Rudiments of Ramsey Theory: Second Edition*. AMS and CBMS, 2015.

[29] R. L. Graham. On edgewise 2-colored graphs with monochromatic triangles containing no complete hexagon. *J. Combin. Theory*, 4:300, 1968.
[30] R. L. Graham. Some graph theory problems i would like to see solved. In *SIAM My Favorite Graph Theory Conjectures*, 2012.

[31] R. L. Graham and J. H. Spencer. On small graphs with forced monochromatic triangles. *Lecture Notes in Math.*, 186:137–141, 1971. Recent Trends in Graph Theory.

[32] F. Harary. *Graph Theory*. Addison - Wesley, 1969.

[33] F. Harary and G. Prins. Generalized Ramsey theory for graphs. IV: The Ramsey multiplicity of a graph. *Networks*, 4:163–173, 1974.

[34] R. Hill. and R.W. Irwing. On group partitions associated with lower bounds for symmetric Ramsey numbers. *European Journal of Combinatorics*, 3:35–50, 1982.

[35] M. Jacobson. A note on Ramsey multiplicity. *Discrete Math.*, 29:201–203, 1980.

[36] M. Jacobson. A note on Ramsey multiplicity for stars. *Discrete Math.*, 42:63–66, 1982.

[37] T. Jensen and G. Royle. Small graphs with chromatic number 5: a computer research. *Journal of Graph Theory*, 19:107–116, 1995.

[38] J. Kaufmann, H. Wickus, and S. Radziszowski. On some edge Folkman numbers small and large. In preparation, 2017.

[39] N. Kolev. A multiplicative inequality for vertex Folkman numbers. *Discrete Mathematics*, 308:4263–4266, 2008.

[40] N. Kolev and N. Nenov. The Folkman number $F_v(3, 4; 8)$ is equal to 16. *Comptes rendus de l’Academie bulgare des Sciences*, 59(1):25–30, 2006.

[41] N. Kolev and N. Nenov. New recurrent inequality on a class of vertex Folkman numbers. In *Proceedings of the 35th Spring Conference of the Union of Bulgarian Mathematicians*, pages 164–168, April 2006.
[42] N. Kolev and N. Nenov. New upper bound for a class of vertex Folkman numbers. *The Electronic Journal of Combinatorics*, 13, 2006.

[43] A. Lange, S. Radziszowski, and X. Xu. Use of MAX-CUT for Ramsey arrowing of triangles. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 88:61–71, 2014.

[44] J. Lathrop and S. Radziszowski. Computing the Folkman number $F_v(2, 2, 2, 2; 4)$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 78:213–222, 2011.

[45] Y. Li and Q. Lin. On generalized Folkman numbers. *Taiwanese J. Math.*, 21:1–9, 2017.

[46] Q. Lin and Y. Li. A Folkman linear family. *SIAM J. Discrete Math.*, 29:1988–1998, 2015.

[47] S. Lin. On Ramsey numbers and $K_r$-coloring of graphs. *J. Combin. Theory Ser. B*, 12:82–92, 1972.

[48] L. Lu. Explicit construction of small Folkman graphs. *SIAM Journal on Discrete Mathematics*, 21:1053–1060, 2008.

[49] T. Łuczak, A. Ruciński, and S. Urbański. On minimal vertex Folkman graphs. *Discrete Mathematics*, 236:245–262, 2001.

[50] T. Łuczak and S. Urbański. A note on restricted vertex Ramsey numbers. *Periodica Mathematica Hungarica*, 33:101–103, 1996.

[51] B.D. McKay. Combinatorial data, [http://users.cecs.anu.edu.au/~bdm/data/](http://users.cecs.anu.edu.au/~bdm/data/).

[52] B.D. McKay. Ramsey graphs, [http://users.cecs.anu.edu.au/~bdm/data/ramsey.html](http://users.cecs.anu.edu.au/~bdm/data/ramsey.html).

[53] B.D. McKay and A. Piperino. Practical graph isomorphism, II. *J. Symbolic Computation*, 60:94–112, 2013. Preprint version at [arxiv.org](http://arxiv.org).
[54] J. Mycielski. Sur le coloriage des graphes. *Colloquium Mathematicum*, 3:161–162, 1955.

[55] N. Nenov. Up to isomorphism there exist only one minimal $t$-graph with nine vertices. (in Russian). *God. Sofij. Univ. Fak. Mat. Mekh.*, 73:169–184, 1979.

[56] N. Nenov. On the existence of a minimal $t$-graph with a given number of vertices. (in Russian). *Serdica*, 6:270–274, 1980.

[57] N. Nenov. On the independence number of minimal $t$-graphs. (in Russian). In *Mathematics and education in mathematics, Proc. 9th Spring Conf. Union Bulg. Math*, pages 74–78, Sunny Beach / Bulgaria, 1980.

[58] N. Nenov. *Ramsey graphs and some constants related to them*. (Bulgarian). PhD thesis, University of Sofia, 1980.

[59] N. Nenov. An example of a 15-vertex Ramsey $(3, 3)$-graph with clique number 4. (in Russian). *C. A. Acad. Bulg. Sci.*, 34:1487–1489, 1981.

[60] N. Nenov. Certain remarks on Ramsey multiplicities. (in Russian). In *Mathematics and education in mathematics, Proc. 10th Spring Conf. Union Bulg. Math*, pages 176–179, Sunny Beach / Bulgaria, 1981.

[61] N. Nenov. On the Zykov numbers and some its applications to Ramsey theory. *Serdica Bulgariaceae Mathematicae*, 9:161–167, 1983. (in Russian).

[62] N. Nenov. The chromatic number of any 10-vertex graph without 4-cliques is at most 4. *Comptes rendus de l’Academie bulgare des Sciences*, 37:301–304, 1984. (in Russian).

[63] N. Nenov. Application of the corona-product of two graphs in Ramsey theory. *Ann. Univ. Sofia Fac. Math. Inform.*, 79:349–355, 1985. (in Russian).
[64] N. Nenov. On the small graphs with chromatic number 5 without 4-cliques. *Discrete Mathematics*, 188:297–298, 1998.

[65] N. Nenov. On a class of vertex Folkman graphs. *Ann. Univ. Sofia Fac. Math. Inform.*, 94:15–25, 2000.

[66] N. Nenov. Computation of the vertex Folkman numbers $F(2, 2, 2, 3; 5)$ and $F(2, 3, 3; 5)$. *Ann. Univ. Sofia Fac. Math. Inform.*, 95:71–82, 2001.

[67] N. Nenov. A generalization of a result of Dirac. *Ann. Univ. Sofia Fac. Math. Inform.*, 95:59–69, 2001.

[68] N. Nenov. On the 3-coloring vertex Folkman number $F(2, 2, 4)$. *Serdica Mathematical Journal*, 27:131–136, 2001.

[69] N. Nenov. Lower bound for a number of vertices of some vertex Folkman graphs. *Comptes rendus de l'Academie bulgare des Sciences*, 55(4):33–36, 2002.

[70] N. Nenov. On a class of vertex Folkman numbers. *Serdica Mathematical Journal*, 28:219–232, 2002.

[71] N. Nenov. On the vertex Folkman numbers $F_v(2, ..., 2; r)$. *Serdica Mathematical Journal*, 35:251–272, 2009. Preprint: arXiv:0903.3812 March 2009.

[72] N. Nenov. Chromatic number of graphs and edge Folkman numbers. *C. A. Acad. Bulg. Sci.*, 63(8):1103–1110, 2010.

[73] N. Nenov. On the vertex Folkman numbers $F_v(2, ..., 2; r - 1)$ and $F_v(2, ..., 2; r - 2)$. *Ann. Univ. Sofia Fac. Math. Inform.*, 101:5–17, Submitted in 2007, 2013. Preprint: arXiv:0903.3151 March 2009.

[74] N. Nenov and N. Khadzhiivanov(Hadziivanov). On edgewise 2-colored graphs containing a monochromatic triangle. (in Russian). *Serdica*, 5:303–305, 1979.
[75] N. Nenov and N. Khadzhiiivanov(Hadziivanov). Every Ramsey graph without 5-cliques has more than 11 vertices. (in Russian). Serdica, 11:341–356, 1985.

[76] J. Nesetril and V. Rödl. The Ramsey property for graphs with forbidden complete subgraphs. J. Combin. Theory, Ser. B, 20:243–249, 1976.

[77] K. Piwakowski and S. Radziszowski. The Ramsey Multiplicity of $K_4$. Ars. Combinatorica, LX:131–136, 2001.

[78] K. Piwakowski, S. Radziszowski, and S. Urbanski. Computation of the Folkman number $F_e(3, 3; 5)$. Journal of Graph Theory, 32:41–49, 1999.

[79] S. Radziszowski. Small Ramsey numbers. The Electronic Journal of Combinatorics, Dynamic Survey revision 14, January 12 2014.

[80] S. Radziszowski. Computers in Ramsey Theory. Testing, Constructions and Nonexistence. Computers in Scientific Discovery 8 Mons, Belgium, August 24, 2017.

[81] S. Radziszowski and X. Xiaodong. On the Most Wanted Folkman Graph. Geombinatorics, XVI(4):367–381, 2007.

[82] S. Radziszowski and X. Xu. On some open questions for Ramsey and Folkman numbers. Graph Theory, Favorite Conjectures and Open Problems, 1:43–62, 2016.

[83] S. Radziszowski, X. Xu, and M. Liang. Some Folkman problems. Chromatic vertex Folkman numbers, Existence and non-existence, Computational challenges. CanaDAM, Toronto, 13 June, 2017.

[84] S. Radziszowski, X. Xu, and M. Liang. Some Folkman problems. Existence and non-existence of generalized Folkman numbers, Computational challenges. GGTW, Ghent, 16 August, 2017.

[85] P. Ramsey. On a problem of formal logic. Proc. London Math. Soc., 30:264–268, 1930.
[86] V. Rosta and L. Suranyi. A note on the Ramsy-multiplicity of the circuit. *Period. Math. Hung.*, 7:223–227, 1977.

[87] G. Royle. Combinatorial data, http://staffhome.ecm.uwa.edu.au/~00013890/data.html

[88] Z. Shao, M. Liang, J. He, and X. Xu. New lower bounds for two multicolor vertex Folkman numbers. In *International Conference on Computer and Management (CAMAN)*, pages 1–3, 2011.

[89] Z. Shao, M. Liang, L. Pan, and X. Xu. Computation of the Folkman number $F_v(3, 5; 6)$. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 81:11–17, 2012.

[90] Z. Shao, X. Xu, and H. Luo. Bounds for two multicolor vertex Folkman numbers. *Application Research of Computers*, 3:834–835, 2009. (in Chinese).

[91] Z. Shao, X. Xu, and L. Pan. New upper bounds for vertex Folkman numbers $F_v(3, k; k + 1)$. *Utilitas Mathematica*, 80:91–96, 2009.

[92] Z. Shao, X. Xu, and L. Pan. Computation of the vertex Folkman number $F_v(3, 5; 6)$. *J. of Comb. Math. and Comb. Computing*, 81:11–18, 2012.

[93] A. Soifer. *The Mathematical Coloring Book*. Springer, 2008.

[94] J. Spencer. Three hundred million points suffice. *Journal of Combinatorial Theory, Series A*, 49:210–217, 1988. Also see erratum by M. Hovey in 50:323.

[95] T. Szabó. On nearly regular co-critical graphs. *Discrete Math.*, 160:279–281, 1977.

[96] D. West. *Introduction to Graph Theory*. Prentice Hall, Inc., Upper Saddle River, 2 edition, 2001.

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[97] X. Xu, M. Liang, and S. Radziszowski. A note on upper bounds for some generalized Folkman numbers. *Discussiones Mathematicae Graph Theory*, in press. Preprint: arxiv:1708.00125, August 2017.

[98] X. Xu, M. Liang, and S. Radziszowski. On the nonexistence of some generalized Folkman numbers. *Graphs Combin.*, to appear. Preprint: arxiv:1705.06268, May 2017.

[99] X. Xu, M. Liang, and S. Radziszowski. Chromatic vertex Folkman numbers. Preprint: arxiv:1612.08136v2, May 2018.

[100] X. Xu, H. Luo, and Z. Shao. Upper and lower bounds for $F_v(4,4;5)$. *Electronic Journal of Combinatorics*, 17, 2010.

[101] X. Xu and Z. Shao. On the lower bound for $F_v(k,k;k+1)$ and $F_e(3,4;5)$. *Utilitas Mathematica*, 81:187–192, 2010.