Local Lipschitz bounds for solutions to certain singular elliptic equations involving one-Laplacian

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Abstract

In this paper local Lipschitz regularity of weak solutions to certain singular elliptic equations involving one-Laplacian is studied. Equations treated here also contains another well-behaving elliptic operator such as $p$-Laplacian with $1 < p < \infty$. The problem is that one-Laplacian is too singular on degenerate points, what is often called facet, which makes it difficult to obtain even Lipschitz regularity of weak solutions. This difficulty is overcome by making suitable approximation schemes, and by avoiding analysis on facet for approximated solutions. The key estimate is a local a priori uniform Lipschitz estimate for classical solutions to regularized equations, which is proved by Moser’s iteration. Another local a priori uniform Lipschitz bounds can also be obtained by De Giorgi’s truncation. Proofs of local Lipschitz estimates in this paper are rather classical and elementary in the sense that nonlinear potential estimates are not used at all.

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1 Introduction and main theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain in $n$-dimensional Euclidian space, and let $f$ be a real-valued function on $\Omega$. We fix constants $1 < p < \infty$, $0 < \beta < \infty$. The aim of this paper is to obtain local Lipschitz regularities for solutions to

$$-\beta \text{div} (\nabla u / |\nabla u|) - \text{div} \left( |\nabla u|^{p-2} \nabla u \right) \ni f \quad \text{in } \Omega,$$

(1.1)
or often simply denoted by $-\beta \Delta_1 u - \Delta_p u \ni f$ in $\Omega$. More generally, we consider equations

$$-\beta \text{div} (\nabla u / |\nabla u|) - \text{div} \nabla_z E_p(\nabla u) \ni f \quad \text{in } \Omega,$$

(1.2)

where $E_p$ is a real-valued function in $\mathbb{R}^n$, such as $|z|^p / p$ ($z \in \mathbb{R}^n$).

1.1 A typical example and our result

Consider (1.1) with $f = 0$. This equation derives from a minimizing problem of the energy functional

$$G(u) \coloneqq \beta \int_{\Omega} |\nabla u| \, dx + \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx.$$

$G$ appears as a crystal surface energy, especially for the case $p = 3$. The nonhomogeneous term $f$ can be regarded here as chemical potential for the crystal surface energy $G$, in the sense that

$$f = \frac{\delta G}{\delta u} = -\beta \text{div} (\nabla u / |\nabla u|) - \text{div} \left( |\nabla u|^{p-2} \nabla u \right).$$

For details of justifications, see [18], [33] and the references given there. Also in general, equation (1.1) comes from a minimizing problem of the energy functional

$$F(u) \coloneqq G(u) - \int_{\Omega} f u \, dx.$$ 
It is well-known that the diffusion singularity of the operator one-Laplacian, denoted by \( \Delta_1 \), appears strongly on degenerate points \( \{ \nabla u = 0 \} \), or often called facet. This singularity makes it difficult to consider a term \( \nabla u / |\nabla u| \) in classical sense over facets. Therefore in the first place, when we consider weak solutions (that is, solutions in distributional sense) to (1.1), we face to give a definition of the term \( \nabla u / |\nabla u| \), which should be mathematically valid. The definition of weak solutions is given later in Section 2.

Also, when it comes to smoothness of solutions, the problem is that elliptic regularity properties of \( \Delta_1 \) are not understood so much. It is remarkable that diffusion effect of \( \{ \nabla u = 0 \} \) is given in Appendix for the reader’s convenience.

Arguments on convergence in the paper are essentially due to Krügel’s idea [24, Theorem 3.3]. More general proof for general under the same conditions given in Theorem 1. From (1.4), we first prove Theorem 1 for \( 0 < \epsilon \leq 1 \). It is understood so much. It is remarkable that diffusion effect of \( \{ \nabla u = 0 \} \) is given in Section 2.

The definition of weak solutions is given later in Section 2.

Theorem 1. Let \( u \) be a solution to (1.1) in weak sense. Then we have
\[
\| \nabla u \|_{L^\infty(B_R)} \leq C(n, p, q, \beta, \theta) \left( 1 + \| f \|_{L^\infty(B_R)}^{1/(p-1)} + R^{-n/p} \| \nabla u \|_{L^p(B_R)} \right)
\]
for any fixed closed ball \( B_R \subset \Omega \) with its radius \( 0 < R \leq 1 \), any \( 2 \leq n < q \leq \infty \) and \( 0 < \theta < 1 \).

This type of gradient bound estimate has already been given in [24] and [38], where nonhomogeneous terms are controlled by \( L^\infty \)-datum. The novelty of Theorem 1 is that the nonhomogeneous term \( f \) is controlled by an \( L^q \)-datum with \( n < q \leq \infty \). We also note that local Lipschitz estimate is valid for any \( n \geq 2 \) and \( 1 < p < \infty \), whereas the two previous works need to restrict conditions on \( n \) and \( p \).

After this work was completed, we were informed of a recent excellent work of Beck and Mingione [3]. In their paper, they established general theorems on local Lipschitz regularity, especially for solutions to nonuniformly elliptic equations. From some of their estimates [4, Theorem 1.9 and 1.11], we are able to obtain a more sophisticated estimate than that of Theorem 1. Although our basic strategy written in Section 1.2 below seems to be similar to theirs, our individual methods are rather classical, direct and elementary. Moreover, the details are quite different from theirs. For more detailed comparison, see Section 1.4 and Remark 5-6 in Section 4.

1.2 Our strategy

From a viewpoint of comparing \( \Delta_1 \) with \( \Delta_p \) in Section 1.1, we describe our strategy briefly. We first get over the problem how to define the term \( \nabla u / |\nabla u| \) by regarding it as a subdifferential operator. Subdifferential operators often play important roles in many fields of mathematical analysis, including convex analysis [30], [32] and nonlinear semigroup theory [8], [9].

The type of definition has already been taken by Xu [38] under the Neumann boundary condition. Our strategy for Theorem 1 is to make suitable approximation schemes, and to avoid analysis on facet. Here we illustrate our approaches for local Lipschitz regularity.

For an approximation to (1.1), we consider classical solutions to regularized equations
\[
-\beta \text{div} \left( \frac{\nabla u^\epsilon}{\sqrt{\epsilon^2 + |\nabla u^\epsilon|^2}} \right) - \text{div} \left( (\epsilon^2 + |\nabla u^\epsilon|^2)^{p/2-1} \nabla u^\epsilon \right) = f
\]
for \( 0 < \epsilon \leq 1 \). From [25], Chapter IV and V], if \( f \in C^\infty(\Omega) \), then for each fixed \( 0 < \epsilon \leq 1 \), \( u^\epsilon \) admits \( C^\infty \)-inner regularity. The key estimate in this paper is the following local a priori estimate;
\[
\sup_{B_R} |\nabla u^\epsilon| \leq C(n, p, q, \beta, \theta) \left( 1 + \| f \|_{L^\infty(B_R)}^{1/(p-1)} + R^{-n/p} \| \nabla u^\epsilon \|_{L^p(B_R)} \right)
\]
under the same conditions given in Theorem 1. From [14], we first prove Theorem 1 for \( f \in C^\infty(\Omega) \). We extend our proof for general \( f \in L^q(\Omega)(n < q \leq \infty) \) by density argument and the Hölder inequality. To justify this argument, we need some basic properties of solutions to (1.1), including the minimizing property of solutions and the stability estimate of solutions. Also, we should make an appropriate justification of convergence \( \nabla u^\epsilon \to \nabla u \) as \( \epsilon \to 0 \). Arguments on convergence in the paper are essentially due to Krügel’s idea [24, Theorem 3.3]. More general justification is given in Appendix for the reader’s convenience.
The proof of the key estimate [1.4] is similar to that of [10 Proposition 3.3], but the significant difference is that we have to choose test functions so carefully that their support does not contain any facet of approximated solutions. We obtain local a priori Lipschitz estimates for solutions to regularized equations in two ways. The first is by Moser’s iteration and the second is by De Giorgi’s truncation, both of which are used for local boundedness of weak solutions to uniformly elliptic equations. For materials on local boundedness, we refer the reader to [2 Chapter 3.6], [17 Chapter 8.3, 8.4], [24 Chapter 4.2] and [31 Chapter 7.1]. By testing suitable functions which are supported in a certain regular set of \( \nabla u^\varepsilon \), we prove local boundedness of \( \nabla u^\varepsilon \), uniformly for an approximation parameter \( 0 < \varepsilon \leq 1 \).

Our approaches given above are valid even for generalized equations (1.2), if \( E_p : \mathbb{R}^n \to \mathbb{R} \) admits some reasonable properties. These will be stated in Section 1.3 below.

It is still left open whether solutions to (1.1) are always \( C^{1,\alpha} \) for \( f \in C^\infty(\Omega) \), or more generally for \( f \in L^q(\Omega)(n < q \leq \infty) \). To solve this problem, we will probably need more analysis over facet.

1.3 General result

Here we describe proper conditions for equations, and state our main theorem, which covers Theorem 1.

For regularities, we only require \( f \in L^q(\Omega)(n < q \leq \infty) \) and \( E_p \in C^1(\mathbb{R}^n) \). However, we also assume that \( E_p \) is strictly convex and admits a family of strictly convex functions \( \{ E_p^\varepsilon \}_{0 < \varepsilon \leq 1} \subset C^\infty(\mathbb{R}^n) \), and that there exists constants \( 0 < c_1 \leq c_2 < \infty \), independent of \( 0 < \varepsilon \leq 1 \), such that

\[
c_1 |z|^{p} \leq E_p^\varepsilon(z_0) \leq c_2 |z|^{p}, \tag{1.5}
\]

\[
c_1 \left( \varepsilon^2 + |z_0|^2 \right)^{p/2} \leq \langle \nabla^2 E^\varepsilon(z_0) \zeta | \zeta \rangle, \tag{1.6}
\]

\[
|\langle \nabla^2 E^\varepsilon(z_0) \zeta | \omega \rangle| \leq c_2 \left( \varepsilon^2 + |z_0|^2 \right)^{p/2} |\zeta||\omega|, \tag{1.7}
\]

\[
E_p^\varepsilon(z_0) \leq E_p^\varepsilon(z_0), \quad E_p(z_0) = \lim_{\varepsilon \to 0} E_p^\varepsilon(z_0), \tag{1.8}
\]

\[
\nabla E_p^\varepsilon(z_0) = \lim_{\varepsilon \to 0} \nabla E_p^\varepsilon(z_0) \tag{1.9}
\]

for all \( z_0, \zeta, \omega \in \mathbb{R}^n \). Here \( \langle \cdot | \cdot \rangle \) denotes the canonical inner product in \( \mathbb{R}^n \). For a sufficiently smooth functional \( E : \mathbb{R}^n \to \mathbb{R} \), we also write \( \nabla E(z_0) \) and \( \nabla^2 E(z_0) \) as the gradient and the Hessian matrix at \( z_0 \in \mathbb{R}^n \) in classical sense respectively.

A typical example is

\[
E_p(z) \coloneqq \frac{1}{p} |z|^p \quad \text{and} \quad E_p^\varepsilon(z) \coloneqq \frac{1}{p} \left( \varepsilon^2 + |z|^2 \right)^{p/2} \quad (0 < \varepsilon \leq 1). \tag{1.10}
\]

It is easy to check that they satisfy (1.5)–(1.9) with \( c_1 \coloneqq \min\{ p - 1, 1/p \} \), \( c_2 \coloneqq \max\{ p - 1, 1 \} \). For the special case \( \{1.10\}, \{1.2\} \) becomes \( \{1.1\} \).

The strategy described in Section 1.2 yields main theorem in the paper, which states local Lipschitz regularity of solutions to (1.2).

**Theorem 2.** Let \( u \) be a solution to (1.2) in weak sense. Then we have

\[
\|\nabla u\|_{L^p(B_R)} \leq C(n, p, q, \beta, c_1, c_2, \theta) \left( 1 + \|f\|_{L^p(B_R)}^{1/(p-1)} + R^{-n/p} \|u\|_{L^q(B_R)} \right) \tag{1.11}
\]

for any fixed closed ball \( B_R \subset \Omega \) with its radius \( 0 < R \leq 1 \), any \( 2 \leq n < q \leq \infty \) and \( 0 < \theta < 1 \). Here \( 0 < c_1 \leq c_2 < \infty \) are constants satisfying (1.5)–(1.7).

Clearly Theorem 2 covers Theorem 1.
1.4 Literature overview

Here we describe previous relevant researches, especially on regularities for solutions to (1.1), in short.

Elliptic regularity of p-Laplacian, especially $C^{1,\alpha}$-regularity of $p$-harmonic functions, has been proved by many excellent mathematicians. As a series of papers, we refer the reader to, for instance, Uhlenbeck [36] and Evans [12] for $2 \leq p < \infty$ and DiBenedetto [10], Tolksdorff [35] and Wang [37] for $1 < p < \infty$. Among them the most related work is one by DiBenedetto [10] in 1983. There he discussed $C^{1,\alpha}$-regularity of solutions to equations, including

$$-\div E_p(\nabla u) = 0 \quad \text{in } \Omega.$$  

In [10] Proposition 3.3], he showed local a priori gradient bounds for solutions to certain regularized equations

$$-\div \nabla \nabla^p(\nabla u^\varepsilon) = 0,$$

uniformly for $0 \leq \varepsilon \leq 1$. Our proofs of local a priori gradient bounds in Section 4 are essentially obtained by a modification of his arguments. The difference is that we have to make analysis only for regular points, whereas DiBenedetto did make analysis for both degenerate and regular points.

Some mathematical properties of the equation (1.1) with $f = \text{const.}$ were discussed in Krügel’s thesis in 2013 [24]. On local Lipschitz regularity, inspired by the paper [15], Krügel proved a local a priori uniform Lipschitz estimate for regularized equations [13], by Moser’s iteration [28]. Despite Krügel’s claim that the estimate is valid for any $n \geq 2$ and $1 < p < \infty$, it seems that there need more arguments or modifications especially for $1 < p < 2$ (for details, see Remark 7 in Section 4.2). Also, the nonhomogeneous term $f = \text{const.}$ is controlled by an $L^\infty$-datum in the proof. Our first proof of a local a priori Lipschitz bound (Proposition 2) is similar to [24, Lemma 4.9], but our proof works for general $1 < p < \infty$ and $n < q \leq \infty$. A justification of convergence for approximation schemes was also discussed in the thesis, the results of which are organized more generally in Appendix of this paper.

Recently in 2019, Xu [38] studied a homogeneous Neumann boundary value problem for a certain nonlinear fourth order equation. There he showed a local Lipschitz estimate for solutions to equations of the type

$$-\beta \div \left( \frac{\nabla u^\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u^\varepsilon|^2}} \right) - \div \left( (\varepsilon^2 + |\nabla u^\varepsilon|^2)^{p/2-1} \nabla u^\varepsilon \right) = f^\varepsilon \quad \text{with } \sup_{0 < \varepsilon \leq 1} \|f^\varepsilon\|_{L^\infty} < \infty \quad (1.12)$$

by De Giorgi’s levelset argument [9] and analysis on regular points. From this he proved that there exists a solution to the nonlinear fourth order Neumann problem with global Lipschitz continuity under some suitable conditions. In the proof of uniform Lipschitz bounds for solutions to (1.12) by Xu, the condition $n = 2$ cannot be removed. This is basically due to the fact that his argument is an adaptation of those given in [19] Chapter 12.2, where elliptic equations in two variables are especially treated. His proof also requires another condition $p > 4/3$ for technical reasons related to estimates for levelsets, and arguments for $2 \leq p < \infty$ are almost omitted. On local a priori Lipschitz bounds for classical solutions, our two proofs are totally different from that given by Xu [38] Claim 4.1. In the first place, the weak formulation (4.5) in this paper is different from the one used in his paper. While most of Xu’s computations are valid only for $n = 2$, our proofs of a priori estimates are valid for general $n \geq 2$.

On local Lipschitz regularities, our proofs of local a priori estimates given in Section 4 are more general than those from two previous researches by Krügel and Xu, in the sense that our methods are valid for any $1 < p < \infty$, $n \geq 2$ and that the nonhomogeneous term $f$ is controlled by an $L^q$-datum with $n < q \leq \infty$. This advantage directly yields our main result of local gradient bounds (Theorem 1.2) for any $n \geq 2, 1 < p < \infty, n < q \leq \infty$. It is remarkable that the condition $n < q \leq \infty$ is optimal for Lipschitz regularity (see 3.2 Section 3).

As mentioned in Section 1, a recent paper [4] gives us more general results on local Lipschitz regularity for minimizers of variational integrals, especially nonuniformly elliptic ones. These Lipschitz bounds are proved by sophisticated estimates from nonlinear potential theory. Remarkably, in [4] Section 1.3], the external term $f$ is assumed to be only in a Lorentz space $L(n,1)$

$$\|f\|_{L(n,1)(\Omega)} := \int_0^\infty L^n(\{x \in \Omega \mid |f(x)| > A\})^{1/n} dA < \infty$$

for the case $n \geq 3$, and for the case $n = 2$ only in an Orlicz space $L^2(\log L)^\alpha (\alpha > 2)$, i.e.,

$$\int_\Omega |f|^2 \log^\alpha (1 + |f|) dx < \infty$$
for some \( \alpha > 2 \). As a special case of [4] Theorem 1.9], we are able to conclude that

\[
\sup_{B_R/2} |\nabla u|^\epsilon \leq C(n, p, \beta) \left( 1 + \|f\|_L^{1/p-1} + R^{-n/p} \|\nabla u\|^\epsilon_{L^p(B_R)} \right) \quad \text{for any } n \geq 3 \text{ and } B_R \subset \Omega,
\]

where \( u^\epsilon \in W^{1,p}(\Omega) \) is a weak solution to (1.1) with \( f \in L(n, 1)(\Omega) \) (for details, see Remark 5 in Section 4). We recall that continuous and strict inclusions \( L^{2+\epsilon} \subset L^n(1, \Omega) \) hold true for any \( \epsilon > 0 \), and the assumption \( f \in L(n, 1) \) can be regarded as critical from previous researches on elliptic regularity for solutions to \( -\Delta u = f \) (see [8], [34]). A sharp estimate for \( n = 2, f \in L^2(\log L)^\alpha (\alpha > 2) \) can also be deduced from [4] Theorem 1.11] (we note that continuous and strict inclusions \( L^{2+\epsilon} \subset L^2(\log L)^\alpha \subset L^2 \) hold true for any \( \alpha > 0 \) and \( \epsilon > 0 \)).

Their strategy for the proof of local Lipschitz bounds [4] Theorem 1.9 and 1.11 broadly consist four parts; construction of approximation schemes [4] Section 4.1], a Caccioppoli-type estimate for approximated solutions, an iteration [4] Section 3.1, 4.2 and 4.3], and justification of the convergence [4] Section 4.4]. It seems that our basic strategy is almost similar to theirs, but in fact the details and individual methods of our proofs are quite different from theirs. Although our Lipschitz bounds (Theorem [13 Proposition 2] are somewhat weaker than these sharp estimates by Beck and Mingione, our methods are rather elementary and do not appeal to the nonlinear potential theory [22] at all. The significant difference is that, compared with a key estimate obtained by a nonlinear iteration argument [4] Lemma 3.1], our iteration arguments in the proofs of Proposition [4 are rather classical and elementary. It should also be noted that another key estimate by Beck and Mingione lies in a Caccioppoli-type estimate [4 Lemma 4.5], and this is deduced from an weak formulation, which is almost similar to [4.9] in this paper. In the proof of [4 Lemma 4.5], they did fully use De Giorgi’s truncation but they did not use Moser’s iteration at all, whereas our key estimates in Proposition 2 are obtained by Moser’s iteration. They chose test functions which differ from those in our proof of Proposition 2 so that our arguments given in Section 4.1 are not needed. It is sure that they did both make use of approximation schemes and justify the convergence of approximated solutions, but their approaches concerning these are quite different from our direct and elementary ones given in Section 3 and Appendix.

1.5 Organization of the paper

We outline the contents of the paper.

Section 2 provides a proper definition of weak solutions to (1.2) in Definition 1. We also prove two properties of weak solutions, the minimizing property of weak solutions (Corollary 1) and the stability of weak solutions (Corollary 2). These two results are used later in Section 3 to complete the proof of main theorem.

Section 3 deals with approximation schemes. We introduce a parameter \( 0 < \epsilon \leq 1 \) and give suitable approximation schemes globally or locally. This approximation argument is inspired by DiBenedetto’s work in 1983 [10] and Krügel’s doctoral thesis in 2013 [24]. A justification for convergence is partially discussed by Krügel for some special cases. It is easy to modify arguments therein for general conditions. Results on convergence are used without proof in Section 3, and the precise proof of these is described in Lemma 4 in Appendix. In Section 3.1 via global approximation we prove Proposition 1 which states the converse of Corollary 1. In Section 3.2 we give a proof of Theorem 2] through local approximation, making use of Lemma 4.5 in Appendix, Corollary 1.2 in Section 2, and Proposition 2 in Section 3.5. Proposition 2 in Section 3.2 states a local a priori Lipschitz estimate for solutions to regularized equations, uniformly for an approximation parameter \( 0 < \epsilon \leq 1 \) and this plays an important role in the proof of Theorem 2. Proposition 2 will be proved in Section 4.2.

Section 4 establishes local a priori Lipschitz estimates for solutions to regularized equations, uniformly for \( 0 < \epsilon \leq 1 \). Section 4.1 presents some preliminaries for proofs of local a priori uniform Lipschitz estimates. In Section 4.2 we give a proof of Proposition 2] by Moser’s iteration. This proof is essentially a modification of that of [10 Proposition 3.3], and more general than that of [24 Lemma 4.9]. In Section 4.3 we also obtain another local a priori uniform Lipschitz estimate by De Giorgi’s truncation (Proposition 3). This is an adaptation of the proof of [21 Theorem 4.1, Method 1].

Appendix contains precise proofs of three lemmas (Lemma 3, 5), which are used throughout the paper.

2 Definition and basic properties of weak solutions

In Section 2 we define weak solutions to (1.2). A proper meaning of \( \nabla u/|\nabla u| \) is given in the sense of a subdifferential.
Definition 1. A pair \((u, Z)\) in \(W^{1,p}(\Omega) \times L^\infty(\Omega, \mathbb{R}^n)\) is called a weak solution to (1.2) when it satisfies
\[
\beta \int_\Omega (Z \mid \nabla \phi) \, dx + \int_\Omega \langle \nabla_Z E_p(\nabla u) \mid \nabla \phi \rangle \, dx = \int_\Omega f \phi \, dx
\]
(2.1)
for all \(\phi \in W^{1,p}_0(\Omega)\), and
\[
Z(x) \in \partial \Psi(\nabla u(x))
\]
(2.2)
for a.e. \(x \in \Omega\). Here \(\partial \Psi(\cdot) \subset \mathbb{R}^n\) denotes the subdifferential at \(\zeta_0 \in \mathbb{R}^n\) for the convex functional in \(\mathbb{R}^n\), \(\Psi(z) := |z|\).

\[i.e., \partial \Psi(\zeta_0) = \begin{cases} \frac{\zeta_0}{|\zeta_0|} \quad (\zeta_0 \neq 0), \\ \{w \in \mathbb{R}^n \mid |w| \leq 1\} \quad (\zeta_0 = 0). \end{cases}\]

For \(u \in W^{1,p}(\Omega)\), if there is \(Z \in L^\infty(\Omega, \mathbb{R}^n)\) such that \((u, Z)\) is a weak solution to (1.2), we simply say that \(u\) is a solution to (1.2) in weak sense.

Remark 1. To define a weak solution to (1.2), we may weaken the assumption \(n < q \leq \infty\). For example, if \(1 < p < n\), then equation (2.1) makes sense for \((p_0)' = \frac{np}{np - n + p} \leq q \leq \infty\), since the Sobolev embedding \(W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)\) holds true. We also note that if \(q > (p_0)'\), this embedding is compact. Similarly, for the proofs of Corollary 1.2, Lemma 1.4, and Proposition 1.1 it is possible to weaken the assumption \(n < q \leq \infty\). We omit this, however, since the assumption \(n < q \leq \infty\) is optimal for Lipschitz regularity. Throughout the paper we use the fact that, for a bounded Lipschitz domain \(V \subset \mathbb{R}^n\), continuous embeddings
\[W^{1,p}_0(V) \hookrightarrow L^q(V), \quad W^{1,p}(V) \hookrightarrow L^\infty(V)\]
hold true and they are compact if \(n < q \leq \infty\). See [1] Chapter 4 and 6] for the complete bibliography.

Remark 2. Local Hölder regularity of weak solutions to (1.2) can be easily obtained by perturbations from \(p\)-harmonic functions. More regularity property of vector field \(Z\) (for instance, Hölder regularity) is not discovered yet, which makes it difficult to obtain even local Lipschitz regularity for solutions to (1.1). We refer to [7] Section 2 and 3] as a related item.

Before showing basic properties for weak solutions to (1.2), here we state some elementary estimates on \(E_p, E_p^\varepsilon\). From (1.5), it is easy to get
\[E_p(0) = 0, \quad \nabla_x E_p(0) = 0.\]
(2.3)
Therefore we may take sufficiently small \(\varepsilon_0 \in (0, 1)\) such that
\[
\sup_{0 < \varepsilon \leq \varepsilon_0} |E_p^\varepsilon(0)| \leq 1 \quad \text{and} \quad \sup_{0 < \varepsilon \leq \varepsilon_0} |\nabla_x E_p^\varepsilon(0)| \leq 1.
\]
(2.4)
From (1.6)–(1.9) and (2.3), elementary calculation yields that
\[
\langle \nabla_x E_p(z_2) - \nabla_x E_p(z_1) \mid z_2 - z_1 \rangle \geq \begin{cases} c_1 \cdot C(p) |z_1 - z_2|^{p} & (p \geq 2), \\
c_1 |z_1 - z_2|^2 \left( |z_1|^2 + |z_2|^2 \right)^{p/2 - 1} & (1 < p < 2), \end{cases}
\]
(2.5)
\[
|\nabla_x E_p^\varepsilon(z_1) - \nabla_x E_p^\varepsilon(z_2)| \leq \begin{cases} c_2 \cdot C(p) \left( |z_1|^{p-2} + |z_2|^{p-2} \right) |z_1 - z_2| & (2 \leq p < \infty), \\
c_2 \cdot C(p) |z_1 - z_2|^{p-1} & (1 < p < 2), \end{cases}
\]
(2.6)
\[
|\nabla_x E_p(z_0)| \leq c_2 \cdot C(p) |z_0|^{p-1},
\]
(2.7)
\[
|\nabla_x E_p^\varepsilon(z_0) - \nabla_x E_p^\varepsilon(0)| \leq \begin{cases} c_2 \cdot C(p) \left( |z_0|^{p-1} \right) & (2 \leq p < \infty), \\
c_2 \cdot C(p) |z_0|^{p-1} & (1 < p < 2), \end{cases}
\]
(2.8)
\[
|E_p^\varepsilon(z_0) - E_p^\varepsilon(0)| \leq \begin{cases} C(c_2, p) \left( |z_0|^{p-1} + |\nabla_x E_p^\varepsilon(0)| |z_0| + |z_0|^p \right) & (2 \leq p < \infty), \\
C(c_2, p) \left( |\nabla_x E_p^\varepsilon(0)| |z_0| + |z_0|^p \right) & (1 < p < 2), \end{cases}
\]
(2.9)
\[ E_p^\epsilon(z_0) - E_p^\epsilon(0) \geq \langle \nabla_z E_p^\epsilon(z_0) - \nabla_z E_p^\epsilon(0) \rangle_{Z_0} \]
\[ \geq \begin{cases} c_1 \cdot C(p)|z_0|^p & (2 \leq p < \infty), \\ c_1 \left( \epsilon^2 + |z_0|^2 \right)^{p/2} - \epsilon^p & (1 < p < 2), \end{cases} \tag{2.10} \]

for all \( z_0, z_1, z_2 \in \mathbb{R}^n \) and \( 0 < \epsilon \leq 1 \). Here we omit the proof of (2.5)-(2.10). For details, see Lemma 4 in Appendix.

**Remark 3.** We can deduce an inequality of the type (1.5) from (1.8)-(1.9) and (2.9)-(2.10). Therefore we may assume (1.6)-(1.9) instead of (2.2).

As pointed out in Section 1.1, equation (1.2) derives from a minimizing problem of variational integral

\[ F_\Omega(u) := \beta \int_\Omega |\nabla u| \, dx + \int_\Omega E_p(\nabla u) \, dx - \int_\Omega f u \, dx \tag{2.11} \]

under a certain boundary condition. We first verify that a weak solution to (1.2) is a minimizer of the functional \( F_\Omega \) on a suitable function class.

**Corollary 1.** Let \( (u, Z) \in W^{1,p}(\Omega) \times L^\infty(\Omega, \mathbb{R}^n) \) be a weak solution to (1.2). Then we obtain \( F_\Omega(u) \leq F_\Omega(v) \) for all \( v \in u + W^{1,p}_0(\Omega) \). Here \( F_\Omega : W^{1,p}(\Omega) \rightarrow \mathbb{R} \) is defined as in (2.7). \( \square \)

**Proof.** We note that \( \partial E_p(z_0) = \{ \nabla_z E_p(z_0) \} \) for all \( z_0 \in \mathbb{R}^n \), since \( E_p \in C^1(\mathbb{R}^n) \) is convex. Combining this with (2.2), we have subgradient inequalities

\[ |\nabla v| - |\nabla u| \geq \langle Z \mid \nabla (v-u) \rangle, \quad E_p(\nabla v) - E_p(\nabla u) \geq \langle \nabla_z E_p(\nabla u) \mid \nabla (v-u) \rangle \quad \text{a.e. in } \Omega. \]

Testing \( \phi := v-u \in W^{1,p}_0(\Omega) \) in (2.11), we obtain

\[ 0 = \beta \int_\Omega \langle Z \mid \nabla (v-u) \rangle \, dx + \int_\Omega \langle \nabla_z E_p(\nabla u) \mid \nabla (v-u) \rangle \, dx - \int_\Omega f (v-u) \, dx \leq F_\Omega(v) - F_\Omega(u). \quad \square \]

We also mention the stability estimate of solutions, which is needed to complete the proof of Theorem 2.

**Corollary 2.** Let \( f_1, f_2 \in L^q(\Omega) (n < q \leq \infty) \). Assume that \( (u_1, Z_1), (u_2, Z_2) \in W^{1,p}(\Omega) \times L^\infty(\Omega, \mathbb{R}^n) \) satisfy

\[ -\beta \text{div}(\nabla u_j/|\nabla u_j|) - \text{div} \nabla_z E_p(\nabla u_j) \ni f_j \quad \text{in } \Omega \quad \text{for each } j \in \{1, 2\} \]

in weak sense. If \( u_1 - u_2 \in W^{1,p}_0(\Omega) \), then we obtain

\[ \| \nabla u_1 - \nabla u_2 \|_{L^p(\Omega)} \leq C(n, p, q, c_1, \Omega) \| f_1 - f_2 \|_{L^n(\Omega)}^{1/(p-1)} \tag{2.12} \]

for \( p \geq 2 \). For \( 1 < p < 2 \), instead of (2.12) we obtain

\[ \| \nabla u_1 - \nabla u_2 \|_{L^1(\Omega)} \leq C(n, p, q, p, c_1, c_2, \Omega) \left( 1 + \| \nabla u_2 \|_{L^p(\Omega)} + \| f_1 \|_{L^p(\Omega)}^{p'} \right) \| f_1 - f_2 \|_{L^p(\Omega)}^{1/2} \tag{2.13} \]

where \( p' := p/(p-1) \in (1, \infty) \) denotes the Hölder conjugate of \( p \).

**Proof.** Test \( u_1 - u_2 \in W^{1,p}_0(\Omega) \) in each equation. Then we obtain

\[ \int_\Omega \langle Z_1 - Z_2 \mid \nabla (u_1 - u_2) \rangle \, dx + \int_\Omega \langle \nabla_z E_p(\nabla u_1) - \nabla_z E_p(\nabla u_2) \mid \nabla (u_1 - u_2) \rangle \, dx = \int_\Omega (f_1 - f_2)(u_1 - u_2) \, dx. \]

Since the subdifferential operator \( \partial \psi = \partial | \cdot | \) is monotone (see for instance (51)), we deduce that

\[ \langle Z_1 - Z_2 \mid \nabla (u_1 - u_2) \rangle \geq 0 \quad \text{a.e. in } \Omega \]

from (2.2). By (2.5), we obtain

\[ \langle \nabla_z E_p(\nabla u_1) - \nabla_z E_p(\nabla u_2) \mid \nabla (u_1 - u_2) \rangle \geq \begin{cases} c_1 \cdot C(p) |\nabla (u_1 - u_2)|^p & (p \geq 2), \\ c_1 (1 + |\nabla u_1|^2 + |\nabla u_2|^2)^{p/2} - |\nabla (u_1 - u_2)|^2 & (1 < p < 2), \end{cases} \quad \text{a.e. in } \Omega. \]
By the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{q'}(\Omega)$, we get for $p \geq 2$,
\[
c_1 \cdot C(p) \| \nabla u_1 - \nabla u_2 \|_{L^p(\Omega)}^p \leq \int_\Omega |f_1 - f_2| |u_1 - u_2| \, dx \leq C(n, p, q, \Omega) \| f_1 - f_2 \|_{L^{q}(\Omega)} \| \nabla u_1 - \nabla u_2 \|_{L^p(\Omega)}.
\]
From this we conclude (2.12). Similarly for $1 < p < 2$, we get
\[
\| \nabla u_1 - \nabla u_2 \|_{L^1(\Omega)} \leq \left( \int_\Omega |\nabla (u_1 - u_2)|^2 \left( 1 + |\nabla u_1|^2 + |\nabla u_2|^2 \right)^{p/2 - 1} \, dx \right)^{1/2} \left( \int_\Omega \left( 1 + |\nabla u_1|^2 + |\nabla u_2|^2 \right)^{1-p/2} \, dx \right)^{1/2}
\]
\[
\leq C(n, p, q, \Omega) \| f_1 - f_2 \|_{L^{q}(\Omega)}^{1/2} \| \nabla (u_1 - u_2) \|_{L^{p}(\Omega)}^{1/2} \left( \int_\Omega \left( 1 + |\nabla u_1|^{2-p} + |\nabla u_2|^{2-p} \right) \, dx \right)^{1/2}
\]
\[
\leq C(n, p, q, \Omega) \| f_1 - f_2 \|_{L^{q}(\Omega)} \left( 1 + \| \nabla u_1 \|_{L^p(\Omega)} + \| \nabla u_2 \|_{L^p(\Omega)} \right)
\]
by the Young inequality (see [19] Chapter 7.1, [25] Chapter 2.1 (3)). It suffices to show that
\[
\| \nabla u_1 \|_{L^p(\Omega)} \leq C(n, p, q, \beta, c_1, c_2, \Omega) \left( 1 + \| \nabla u_2 \|_{L^p(\Omega)} + \| f_1 \|_{L^{q}(\Omega)} \right)
\]
(2.14) to complete the proof of (2.13). By (1.15), the Young inequality, the Hölder inequality and the inequality
\[
\beta \int_\Omega |\nabla u_1| \, dx + \int_\Omega E_p(\nabla u_1) \, dx - \int_\Omega f_1 u_1 \, dx \leq \beta \int_\Omega |\nabla u_2| \, dx + \int_\Omega E_p(\nabla u_2) \, dx - \int_\Omega f_1 u_2 \, dx
\]
from Corollary (2.1) we get
\[
c_1 \| \nabla u_1 \|^p_{L^p(\Omega)} + \beta \| \nabla u_1 \|_{L^1(\Omega)} \leq c_2 \| \nabla u_2 \|^p_{L^p(\Omega)} + \beta \| \nabla u_2 \|_{L^1(\Omega)} + C(n, p, q, \Omega) \| f_1 \|_{L^q(\Omega)} \| \nabla (u_1 - u_2) \|_{L^p(\Omega)}
\]
\[
\leq C(\beta, c_2, \Omega) \left( 1 + \| \nabla u_2 \|^p_{L^p(\Omega)} \right) + C(n, p, q, \Omega) \| f_1 \|_{L^q(\Omega)} \| \nabla u_2 \|_{L^p(\Omega)}
\]
\[
+ C(n, p, q, \Omega) \| f_1 \|_{L^q(\Omega)} \| \nabla u_1 \|_{L^p(\Omega)}
\]
\[
\leq C(n, p, q, \beta, c_1, c_2, \Omega) \left( 1 + \| \nabla u_2 \|^p_{L^p(\Omega)} + \| f_1 \|_{L^q(\Omega)} \right) + C_1 \| \nabla u_1 \|^p_{L^p(\Omega)}.
\]
From this we conclude (2.14). \qed

## 3 Approximation schemes

For each $0 < \epsilon \leq 1$, we consider a weak solution $u^\epsilon$ to the equation
\[
- \nabla \cdot \nabla E^\epsilon (\nabla u^\epsilon) = f \in L^q(\Omega) \quad (n < q \leq \infty)
\]
in either $\Omega$ or Lipschitz subdomain $U \subset \Omega$. Here a family of strictly convex functions $\{E^\epsilon\}_{0 < \epsilon \leq 1} \subset C^\infty(\mathbb{R}^n)$ admits constants $0 < C_1 \leq C_2 < \infty$, independent of $0 < \epsilon \leq 1$, such that
\[
C_1 \left( \epsilon^2 + |z|^2 \right)^{p/2 - 1} \| \nabla z \|_{L^2(\Omega)} \leq \left( \nabla^2 E^\epsilon (z) \right) \| z \|_{L^2(\Omega)} \leq C_2 \left( \epsilon^2 + |z|^2 \right)^{p/2 - 1} \| z \|_{L^2(\Omega)} \| z \|_{L^p(\Omega)}
\]
(3.2) for all $z, \xi, \omega \in \mathbb{R}^n$ with $|z| \geq 1$. Especially in this paper, we consider
\[
\Psi^\epsilon (z) := \sqrt{\epsilon^2 + |z|^2}, \quad E^\epsilon (z) := \beta \Psi^\epsilon (z) + E^\epsilon_p (z) \quad \text{for } z \in \mathbb{R}^n,
\]
(3.3) where $E_p$ and $\{E^\epsilon_p\}_{0 < \epsilon \leq 1}$ satisfy (1.5)-(1.59). By direct calculation it is easy to check that eigenvalues of $\nabla^2 \Psi^\epsilon (z_0)$, the Hessian matrix of $\Psi^\epsilon$ at $z_0 \in \mathbb{R}^n$, are given by $\beta (\epsilon^2 + |z_0|^2)^{-1/2}$ and $\beta \epsilon^2 (\epsilon^2 + |z_0|^2)^{-3/2}$. Hence $E^\epsilon$ defined as in (3.4) satisfies (3.2)-(3.3) with $C_1 := c_1, C_2 := c_2 + \beta$.\]
Equation (3.3) derives from the Euler-Lagrange equation of the regularized variational integral

\[ F^\varepsilon_v(u) := \int_V E^\varepsilon(\nabla u) \, dx - \int_V f u \, dx, \]

where \( V = \Omega \) or \( V = U \subset \Omega \). We also define a functional \( F^\varepsilon_v : W^{1,p}(U) \to \mathbb{R} \) as in (2.11), replacing \( \Omega \) by \( U \).

**Lemma 1.** Functional \( F_v, F^\varepsilon_v (0 < \varepsilon \leq 1) \) are lower semi-continuous in \( W^{1,p}(V) \) with respect to the weak topology.

Lower semi-continuity of convex energy functionals with respect to the weak topology is generally discussed in [13] Chapter 8.2.2 (see also [15] Chapter 1.2, [20] Chapter 4.2 and 4.3). It is easy to prove Lemma 1 by making an adaptation of arguments therein. However, we give another simpler proof of Lemma 1 by showing that functionals \( F_v, F^\varepsilon_v \) are continuous with respect to the strong topology.

**Proof.** By [6] Corollary 3.9, we are reduced to showing that convex functionals \( F_v, F^\varepsilon_v (0 < \varepsilon \leq 1) \) are continuous in \( W^{1,p}(V) \) with respect to the strong topology. Fix \( 0 < \varepsilon \leq 1 \) and let \( \{v_n\}_{n=1}^{\infty} \subset W^{1,p}(V) \) satisfy \( v_n \to v (n \to \infty) \) in \( W^{1,p}(V) \) for some \( v \in W^{1,p}(V) \). Take any subsequence \( \{v_{n_k}\}_{k=1}^{\infty} \subset \{v_n\}_{n=1}^{\infty} \) and \( w \in L^p(V) \) such that

\[
\nabla v_{n_k} \to \nabla v (k \to \infty) \quad \text{a.e. in } V, \tag{3.5}
\]

\[
|v_{n_k}| \leq w \quad \text{a.e. in } V \text{ and for all } k \in \mathbb{N}, \tag{3.6}
\]

\[
v_{n_k} \to v (k \to \infty) \quad \text{in } L^p(V). \tag{3.7}
\]

By (3.5) and \( E^\varepsilon_p \in C^\infty(\mathbb{R}^n) \), we get

\[
E^\varepsilon(\nabla v_{n_k}) \to E^\varepsilon(\nabla v)(k \to \infty) \quad \text{a.e. in } V.
\]

By (2.9), (3.6) and the Young inequality, we can easily check that

\[
E^\varepsilon(\nabla v_{n_k}) \leq \left| E^\varepsilon_p(0) \right| + C(c_2, p) \left( e^{p-1} |\nabla v_{n_k}| + |\nabla E^\varepsilon_p(0)| |\nabla v_{n_k}| + |\nabla v_{n_k}|^p \right) \leq \left| E^\varepsilon_p(0) \right| + C(c_2, p) \left( e^p + |\nabla E^\varepsilon_p(0)|^p + w \right) \in L^1(V) \quad \text{a.e. in } V,
\]

uniformly for \( k \in \mathbb{N} \). From these we obtain

\[
\lim_{k \to \infty} F^\varepsilon_v(v_{n_k}) = \lim_{k \to \infty} \int_V E^\varepsilon(\nabla v_{n_k}) \, dx - \lim_{k \to \infty} \int_V f v_{n_k} \, dx = \int_V E^\varepsilon(\nabla v) \, dx - \int_V f v \, dx = F^\varepsilon_v(v)
\]

by Lebesgue’s dominated convergence theorem and (3.7). Hence it follows that \( F^\varepsilon_v(v_n) \to F^\varepsilon_v(v)(n \to \infty) \). This means that \( F^\varepsilon_v \) is strongly continuous in \( W^{1,p}(V) \). From (1.5) and \( E^\varepsilon_p \in C^1(\mathbb{R}^n) \), we similarly conclude that \( F_v \) is strongly continuous in \( W^{1,p}(V) \). \( \square \)

For each fixed \( u_0 \in W^{1,p}(V) \) and \( 0 < \varepsilon \leq 1 \), we can define

\[
u^\varepsilon := \arg \min \left\{ F^\varepsilon_v(v) \left| \ v \in u_0 + W^{1,p}_0(V) \right\right\} \in u_0 + W^{1,p}_0(V).
\]

Using the Young inequality, we can easily check that for all \( v \in u_0 + W^{1,p}_0(V) \),

\[
F^\varepsilon_v(v) \geq \int_V E^\varepsilon_p(0) \, dx + \int_V \langle \nabla E^\varepsilon_p(0), \nabla v \rangle \, dx + C(c_1, p) \int_V (|\nabla v|^p - 1) \, dx
\]

\[
- \|f\|_{L^\infty(V)} \|v - u_0\|_{L^\infty(V)} - \|f\|_{L^\infty(V)} \|u_0\|_{L^\infty(V)} \quad \text{(by (2.10) and the H"older inequality)}
\]

\[
\geq \frac{C(c_1, p)}{2} \|\nabla v\|^p_{L^p(V)} - \|f\|_{L^\infty(V)} \|v - u_0\|_{L^p(V)}
\]

\[
- C \left( n, p, q, c_1, V, E^\varepsilon_p(0), \nabla E^\varepsilon_p(0) \right) \left( 1 + \|f\|_{L^\infty(V)} \|u_0\|_{W^{1,p}(V)} \right)
\]

\[
\left( \text{by the Sobolev embedding } W^{1,p}(V), W^{1,p}_0(V) \hookrightarrow L^q(V) \right)
\]

\[
\geq \frac{C(c_1, p)}{4} \|\nabla v\|^p_{L^p(V)} - C \left( n, p, q, c_1, V, E^\varepsilon_p(0), \nabla E^\varepsilon_p(0) \right) \left( 1 + \|f\|_{L^\infty(V)} \|u_0\|_{W^{1,n}(V)} + \|f\|_{L^\infty(V)}^2 \right).
\]
Combining this with Lemma 4, we conclude that \( F_{\varepsilon} \) is coercive and weakly lower semi-continuous in \( u_0 + W^{1,p}_0(V) \). Hence the existence of a minimizer \( u^\varepsilon \in u_0 + W^{1,p}_0(V) \) is guaranteed by direct method (see for instance [13] Chapter 8.2.2, [16] Chapter I.3 and I.4, [20] Chapter 4.4). Uniqueness is clear by strict convexity of \( F_{\varepsilon} \) in \( u_0 + W^{1,p}(V) \), since \( E_{\varepsilon} \) is strictly convex. Similarly we can determine a unique function

\[
u := \arg \min \left\{ F_{\varepsilon}(v) \middle| v \in u_0 + W^{1,p}_0(V) \right\} \in u_0 + W^{1,p}_0(V)
\]

for each \( u_0 \in W^{1,p}(V) \). We note that it is easy to deduce that \( F_{\varepsilon} \) is coercive in \( u_0 + W^{1,p}_0 \) from (1.5). Lemma 4 in Appendix states that \( u^\varepsilon \to u \) in \( W^{1,p}(V) \) as \( \varepsilon \to 0 \), up to a subsequence. Results from Lemma 4 are used throughout Section 3.

### 3.1 Global approximation

From Corollary 4.1 if \( u \in W^{1,p} \) is a solution to (1.2) in weak sense, then \( u \) satisfies

\[
u = \arg \min \left\{ F_{\Omega}(v) \middle| v \in u + W^{1,p}_0(\Omega) \right\}.
\]

Proposition 4.1 states that the converse is true.

**Proposition 1.** Let \( f \in L^q(\Omega) \) \((n < q \leq \infty)\). Assume that \( u \in W^{1,p}(\Omega) \) satisfies (3.8). Then \( u \) is a solution to (1.2) in weak sense. That is, there exists \( Z \in L^\infty(\Omega, \mathbb{R}^n) \) such that \( (u, Z) \in W^{1,p}(\Omega) \times L^\infty(\Omega, \mathbb{R}^n) \) is a weak solution to (1.2).

**Proof.** For each \( 0 < \varepsilon \leq 1 \), we set

\[ u^\varepsilon := \arg \min \left\{ F_{\Omega}(v) \middle| v \in u + W^{1,p}_0(\Omega) \right\} \in u + W^{1,p}_0(\Omega). \]

By Lemma 4 in Appendix, we have \( u^\varepsilon \to u \) in \( W^{1,p}(\Omega) \), up to a subsequence. We note that

\[ |\nabla \Psi^\varepsilon(\nabla u^\varepsilon)| = \frac{|\nabla u^\varepsilon|}{\sqrt{\varepsilon^2 + |\nabla u^\varepsilon|^2}} \leq 1 \quad \text{a.e. in } \Omega. \]

By [6] Corollary 3.30 and Theorem 4.9, again up to a subsequence, we may assume that

\[ |\nabla u^\varepsilon| \leq \nu \quad \text{a.e. in } \Omega \quad \text{and for all } 0 < \varepsilon \leq 1, \quad \nabla u^\varepsilon \to \nabla u \quad (\varepsilon \to 0) \quad \text{a.e. in } \Omega, \]

\[ \nabla \Psi^\varepsilon(\nabla u^\varepsilon) = \frac{\nabla u^\varepsilon}{\sqrt{\varepsilon^2 + |\nabla u^\varepsilon|^2}} \to Z \quad \text{in } L^\infty(\Omega, \mathbb{R}^n) \]

for some \( Z \in L^\infty(\Omega, \mathbb{R}^n) \), \( \nu \in L^p(\Omega) \). (3.10)–(3.11) imply that

\[ \|Z\|_{L^\infty(\Omega, \mathbb{R}^n)} \leq 1, \quad Z(x) = \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{if } \nabla u(x) \neq 0. \]

Hence \( Z \) satisfies (2.2). Consider the Euler-Lagrange equation of \( u^\varepsilon \), then we have

\[
\int_{\Omega} \langle \nabla \Psi^\varepsilon(\nabla u^\varepsilon) \rangle \nabla \phi \, dx + \int_{\Omega} \langle \nabla E_p^\varepsilon(\nabla u^\varepsilon) \rangle \nabla \phi \, dx = \int_{\Omega} f \phi \, dx
\]

for all \( \phi \in W^{1,p}_0(\Omega) \). We claim that

\[
\nabla \Psi^\varepsilon(\nabla u^\varepsilon) \to \nabla E_p(\nabla u) \quad \text{in } L^{p'}(\Omega, \mathbb{R}^n).
\]

From (1.9), (2.4), (2.6), (2.8) and the Arzelà-Ascoli theorem, we conclude that

\[
\nabla E_p^\varepsilon(\nabla u^\varepsilon) \to \nabla E_p(\nabla u) \quad (\varepsilon \to 0) \quad \text{compactly in } \mathbb{R}^n.
\]
Combining this result with (3.10), we get
\[ \nabla_z E_p^\varepsilon (\nabla u^\varepsilon) \to \nabla_z E_p (\nabla u) \quad \text{a.e. in } \Omega. \]

By (2.3), (2.7)-(2.8) and (3.9), we obtain
\[ |\nabla_z E_p^\varepsilon (\nabla u^\varepsilon) - \nabla_z E_p (\nabla u)| \leq |\nabla_z E_p^\varepsilon (0)| + |\nabla E_p (\nabla u)| + |\nabla_z E_p^\varepsilon (\nabla u^\varepsilon) - \nabla_z E_p^\varepsilon (0)| \]
\[ \leq 1 + c_2 \cdot C(p) |\nabla u|^{p-1} + c_2 \cdot C(p) \left( 1 + |\nabla u|^p \right) \]
\[ \leq C(c_2, p) \left( 1 + |\nabla u|^{p-1} + |\nabla u|^p \right) \in L^{p'}(\Omega) \quad \text{a.e. in } \Omega, \]
uniformly for \( 0 < \varepsilon \leq \varepsilon_0 \). Hence by Lebesgue’s dominated convergence theorem, we conclude (3.13). From (3.11)-(3.13), we easily verify that \( (u, Z) \in W^{1, p}(\Omega) \times L^\infty (\Omega, \mathbb{R}^n) \) satisfies (2.1) for all \( \phi \in W^{1, p}_0 (\Omega) \), and it completes the proof. \( \square \)

### 3.2 Local approximation and the proof of Theorem 2

Proposition 2 states a local a priori uniform Lipschitz estimate, which is proved in Section 4.2 later.

**Proposition 2.** Let \( f \in C^\infty(\Omega) \). Assume that \( u^\varepsilon \in C^\infty(U) \) is a classical solution to (3.7) in \( U \Subset \Omega \). Under the condition (3.2)-(3.3), we have

\[ \sup_{B_{\varepsilon R}} |\nabla u^\varepsilon| \leq \frac{C(n, p, q, C_1, C_2)}{(1 - \theta)^{n/p}} \left( 1 + \|f\|_{L^{n(1-\theta)}(B_{\varepsilon R})} + R^{-n/p} \|\nabla u^\varepsilon\|_{L^n(B_{\varepsilon R})} \right) \quad (3.14) \]

for any closed ball \( B_{\varepsilon R} \subset U \) with \( 0 < \varepsilon \leq 1 \), any \( 3 \leq n < q \leq \infty \) and \( 0 < \theta < 1 \). Even for \( n = 2 \), we have for each fixed \( 1 < \chi < 2^* = \infty \),

\[ \sup_{B_{\varepsilon R}} |\nabla u^\varepsilon| \leq \frac{C(n, p, \chi, C_1, C_2)}{(1 - \theta)^{2/(\chi-1)}} \left( 1 + \|f\|_{L^2(B_{\varepsilon R})}^{1/(2-\theta)} + R^{-2/p} \|\nabla u^\varepsilon\|_{L^2(B_{\varepsilon R})} \right) \quad (3.15) \]

instead of (3.14).

**Remark 4.** By interpolation and [16] Chapter V, Lemma 3.1], we easily obtain

\[ \sup_{B_{\varepsilon R}} |\nabla u^\varepsilon| \leq C(n, p, q, s, \theta, C_1, C_2) \left( 1 + \|f\|_{L^s(B_{\varepsilon R})}^{1/(s-\theta)} + R^{-n/s} \|\nabla u^\varepsilon\|_{L^s(B_{\varepsilon R})} \right) \quad (3.16) \]

for any closed ball \( B_{\varepsilon R} \subset U \) with \( 0 < \varepsilon \leq 1 \), any \( 2 \leq n < q \leq \infty \), \( 0 < \theta < 1 \) and \( 1 \leq s < p \).

In Subsection 4.3, we also show a local a priori uniform Lipschitz estimate in another way. This is weaker than (3.14)-(3.15) though.

For the proof of Theorem 2, we do not use a result of the strong convergence for global minimizers, given in Lemma 4. Instead, we use weaker results from Lemma 4 and a Fatou-type estimate proved in Lemma 5 in Appendix.

**Proof.** By the Hölder inequality, it suffices to consider the case \( n < q < \infty \). Fix \( \theta < \tau < 1 \) and \( B_{R \varepsilon} \subset U : B_R^\circ \Subset \Omega \). Here \( B_R^\circ \) denotes an open ball with its radius \( R \).

We first consider \( f \in C^\infty(\Omega) \). For each \( 0 < \varepsilon \leq 1 \) we set

\[ u^\varepsilon := \arg \min \left\{ F_u^\varepsilon(v) \mid v \in u + W^{1, p}(U) \right\}. \]

By (2.8), (2.10) and the inequalities

\[ \langle \nabla_z \Psi^\varepsilon(z_0) \rangle \geq 0, \quad |\nabla_z \Psi^\varepsilon(z_0)| \leq 1 \quad \text{for all } z_0 \in \mathbb{R}^n, \]

we can use results from [31] Chapter 7.1 and 7.4] to obtain \( u^\varepsilon \in L^\infty (U) \). Hence by [25] Chapter IV, Theorem 6.4, we conclude that \( u^\varepsilon \in C^\infty (U) \) and \( u^\varepsilon \) is a classical solution to (3.1) in \( U \) (see also [25] Chapter V, Theorem 6.1-6.3). We note

\[ u = \arg \min \left\{ F_U(v) \mid v \in u + W^{1, p}(U) \right\}. \]
by a similar argument given in the proof of Corollary [1]. By Lemma [2], we obtain $u^e \to u$ in $W^{1,p}(U)$ as $\epsilon \to 0$. Moreover we get (A.4).

We define

$$E_U(v) := \beta \|\nabla v\|_{L^1(U)} + \int_U E_p(\nabla v) \, dx.$$  

for each $v \in W^{1,p}(U)$. We easily check at once that

$$c_1^{1/p} \|\nabla v\|_{L^p(U)} \leq \left[ E_U(v) \right]^{1/p} \leq \left[ \beta \int_U |\nabla v| \, dx + c_2 \int_U |\nabla v|^p \, dx \right]^{1/p} \quad \text{(by (1.5) and $\beta > 0$)}$$

$$\leq \left[ (c_2 + 1) \int_U |\nabla v|^p \, dx + \int_U \beta' v^p \, dx \right]^{1/p} \quad \text{(by the Young inequality)}$$

$$\leq (c_2 + 1)^{1/p} \|\nabla v\|_{L^p(U)} + C(n, p) \beta^{p/2} R^{n/p} \quad \text{(by the Minkowski inequality)} \quad (3.17)$$

for all $v \in W^{1,p}(U)$. Also we have $E_U(u) = \liminf_{\epsilon \to 0} E_U'(u^\epsilon)$ from (A.4), since

$$E_U(u) - \int_U f u \, dx = F_U(u) = \liminf_{\epsilon \to 0} F_U'(u^\epsilon) = \liminf_{\epsilon \to 0} E_U'(u^\epsilon) - \int_U f u \, dx.$$  

Here we have used the compact embedding $W^{1,p}(U) \hookrightarrow L^q(U)$. Combining this fact with Proposition [2] (3.17), and Lemma [5], we obtain

$$\|\nabla u\|_{L^\infty(B_{R\epsilon})} \leq C(n, p, q, \beta, c_1, c_2, \theta, \tau) \left( 1 + \|f\|_{L^q(U)}^{1/(p-1)} + R^{-n/p} \liminf_{\epsilon \to 0} \|u^{\epsilon}\|_{L^p(B_{R\epsilon})} \right)$$

$$\leq C(n, p, q, \beta, c_1, c_2, \theta, \tau) \left( 1 + \|f\|_{L^q(U)}^{1/(p-1)} + R^{-n/p} \left[ E_U(u) \right]^{1/p} \right)$$

$$\leq C(n, p, q, \beta, c_1, c_2, \theta, \tau) \left( 1 + \|f\|_{L^q(U)}^{1/(p-1)} + R^{-n/p} \|\nabla u\|_{L^p(B_{R\epsilon})} \right).$$

Hence (1.11) holds true for $f \in C^\infty(\Omega)$.

We make a density argument to complete the proof. For $f \in L^q(\Omega)$ ($n < q < \infty$), fix a sequence $\{f_n\}_{n=1}^\infty \subset C^\infty(\Omega)$ such that $f_n \to f$ ($n \to \infty$) in $L^q(\Omega)$. We define for each $n \in \mathbb{N}$,

$$u_n := \arg \min \left\{ \beta \int_\Omega |\nabla v| \, dx + \int_\Omega E(p,\nabla v) \, dx - \int_\Omega f_n v \, dx \mid v \in u + W^{1,p}_0(\Omega) \right\} \in u + W^{1,p}(\Omega).$$

By Proposition [1] there exists a sequence $\{Z_n\}_{n=1}^\infty \subset L^\infty(\Omega, \mathbb{R}^n)$ such that $(u_n, Z_n) \in W^{1,p}(\Omega) \times L^\infty(\Omega, \mathbb{R}^n)$ is a weak solution to

$$-\beta \text{div}(\nabla u_n/|\nabla u_n|) - \text{div} \nabla_z E_p(\nabla u_n) \ni f_n \quad \text{in $\Omega$.}$$

From Corollary [2], we deduce that

$$\nabla u_n \rightharpoonup \nabla u \quad \text{in} \quad \begin{cases} L^p(\Omega) & (p \geq 2), \\ L^1(\Omega) & (1 < p < 2), \end{cases} \quad \text{as $n \to \infty$}$$

For $1 < p < 2$, the interpolation inequality $\|\nabla u_n - u_m\|_{L^p(U)} \leq \|\nabla(u_n - u_m)\|_{L^{1/p}(U)}^{1/p} \|\nabla u_n - u_m\|_{L^{1}(U)}$ and the estimate (3.16) imply that $\nabla u_n \rightharpoonup \nabla u (n \to \infty)$ in $L^p(U)$. Again by Lemma [5] we obtain

$$\|\nabla u\|_{L^\infty(B_{R\epsilon})} \leq C(n, p, q, \beta, c_1, c_2, \theta, \tau) \left( 1 + \liminf_{\epsilon \to 0} \left[ \|f_n\|_{L^q(U)}^{1/(p-1)} + R^{-n/p} \|\nabla u_n\|_{L^p(U)} \right] \right)$$

$$\leq C(n, p, q, \beta, c_1, c_2, \theta, \tau) \left( 1 + \|f\|_{L^q(U)}^{1/(p-1)} + R^{-n/p} \|\nabla u\|_{L^p(U)} \right).$$

This completes the proof of (1.11). \qed
4 Local a priori Lipschitz bounds

In Section 4 we prove local a priori uniform Lipschitz estimates for classical solutions to (3.1) with \( f \in C^\infty(\Omega) \).

Remark 5. For the special case (1.10), our result of local Lipschitz estimates (3.14) for solutions to (1.3) can be deduced as a special case of [4] Theorem 1.9 and 1.11. For (1.10) and (3.4), it is easily checked that

\[
c_1 \int_c^{(2 + \tau)^{p/2 - 1}} |z|^2 \, s \, ds \leq E'(z) \quad \text{for all } z \in \mathbb{R}^n \text{ with } |z| \geq 1,
\]

which is described as a coercive condition of the integrand \( E' \) (1.33)]. It is also noted that

\[
\frac{C_2 (e^2 + t^2)^{p/2 - 1}}{C_1 (e^2 + t^2)^{p/2 - 1}} \leq \frac{C_2(p) + \beta}{C_1(p)} < \infty \quad \text{for all } t \geq 1
\]

and thus the condition [4] (1.34)] holds true. Hence as a special case of [4] Theorem 1.11], we obtain

\[
c_1 \int_1 \left( e^2 + s^2 \right)^{p/2 - 1} \, s \, ds \leq C(n, p, \beta) \left( e^2 + s^2 \right)^{p/2 - 1}
\]

for each fixed closed ball \( B \subset \Omega \). By the Young inequality, we can easily check that

\[
(\text{Left Hand Side}) = \frac{c_1}{p} \left( e^2 + \|\nabla u^\epsilon\|_{L^p(B; \Omega)}^{p/2} \right)^{p/2} \quad \text{and that}
\]

\[
(\text{Right Hand Side}) \leq C(n, p, \beta) \left( 1 + R^{n-2} \|\nabla u^\epsilon\|_{L^p(B; \Omega)}^{p} + \|f\|_{L^p(B; \Omega)}^{p-1} + 1 \right). \]

Thus we obtain (1.13). For the case \( n = 2 \), more sophisticated estimate than (3.15) can similarly be concluded as a special case of [4] Theorem 1.11]. Their sharp results in [4] Theorem 1.9 and 1.11] cover a variety type of elliptic equations, and they can also be applied directly to (1.1). Moreover, it will work for general equations (1.2) or (3.1), as long as an integrand \( E_p \) or \( E_p^2 \) satisfies all the conditions described in [4] Section 1.3], including the coercive condition [4] (1.33)].

Their proofs of local Lipschitz bounds [4] Section 3 and 4], especially nonlinear iteration arguments in [4] Lemma 3.1, 4.7 and 4.8], contain pointwise nonlinear potential estimates. Our proofs of a priori local Lipschitz bounds in Section 4 however, does not require any potential estimate. Instead, we make iteration arguments which are rather classical and elementary.

4.1 Preliminaries for proofs of local a priori uniform Lipschitz estimates

Let \( u^\epsilon \in C^\infty(U) \) be a classical solution to (3.1) in \( U \) with \( f \in C^\infty(\Omega) \). For each fixed \( 0 < \epsilon \leq 1, k > 0 \), and \( i \in \{ 1, \ldots, n \} \), we define

\[
u_i := \frac{\partial u^\epsilon + k}{\partial u^\epsilon - k} \in W^{1,\infty}_\text{loc}(U).
\]

Here \( a_+ := \max\{a, 0\} \), \( a_- := \max\{-a, 0\} \). We also set

\[
w_k := k^2 + \sum_{i=1}^{n} u_i \in W^{1,\infty}_\text{loc}(U), \quad \hat{w}_k := k^2 + |\nabla u^\epsilon|^2 \in W^{1,\infty}_\text{loc}(U).
\]

Proposition 3.3] states a local a priori \( L^{\infty} - L^{p/2} \) estimate of \( \hat{w}_k \) \( (0 < \epsilon \leq 1) \) for classical solutions to

\[-\nabla \cdot E_p^\epsilon \nabla u^\epsilon = 0,
\]

where \( E_p^\epsilon \) satisfies (1.1) - (1.7]. The proof of Proposition is a modification of this one. The difference is that we should avoid an analysis for degenerate points. With this in mind, we defined the function \( w_k \in W^{1,\infty}_\text{loc}(U) \) \( k \geq 1 \) such that the support of \( \nabla w_k \) is contained in \( \{ x \in U \mid |\nabla u^\epsilon(x) > 1 \} \). We also need the compatibility of \( w_k \), \( \hat{w}_k \), and \( w_k \) \( 0 < \epsilon \leq 1 \) on a suitable set of regular points. In Section 4.1 we check this compatibility.

We first get the compatibility of \( w_k \) and \( \hat{w}_k \) for \( k \geq 1 \), which is described in [4].
Lemma 2. Let $\delta_1, \delta_2 > 0$. For each $t \in \mathbb{R}$, we define
\[
G_{\delta_1, \delta_2}(t) := \begin{cases} 
\delta_1^2 + (t - \delta_2)^2 & (t > \delta_2) \\
\delta_2^2 & (-\delta_2 \leq t \leq \delta_2) \\
\delta_2^2 + (t + \delta_2)^2 & (t < -\delta_2) 
\end{cases}
\]
Then there exists a constant $K = K(\delta_2/\delta_1) > 1$ such that
\[
G_{\delta_1, \delta_2} \leq \hat{G}_{\delta_1} \leq KG_{\delta_1, \delta_2} \quad \text{in } \mathbb{R}.
\]
Hence there exists a constant $C_0 = C_0(n) > 1$ such that
\[
\min \{ 1, C_0^{\sigma} \} \cdot \hat{w}_k^\sigma \leq \hat{w}_k^\sigma \leq \max \{ 1, C_0^{\sigma} \} \cdot w_k^\sigma \quad \text{in } U
\]
for all $\sigma \in \mathbb{R}$ and $k > 0$.

Proof. $0 < G_{\delta_1, \delta_2} \leq \hat{G}_{\delta_1}$ in $\mathbb{R}$ is clear by definition. It suffices to determine a constant $K = K(\delta_2/\delta_1) > 1$ such that $\hat{G}_{\delta_1} \leq KG_{\delta_1, \delta_2}$ in $\mathbb{R}$. We note that this is equivalent to
\[
\begin{align*}
\begin{cases} 
\delta_1^2 + t^2 \leq K\delta_1^2 & \text{for all } t \in [0, \delta_1), \\
\delta_2^2 + t^2 \leq K(\delta_2^2 + (t - \delta_2)^2) & \text{for all } t \in (\delta_2, \infty),
\end{cases}
\end{align*}
\]
since $G_{\delta_1, \delta_2}$ and $\hat{G}_{\delta_1}$ are symmetric. Solve two inequalities for $L > 1$,
\[
\begin{align*}
\begin{cases} 
0 \leq \inf \{ L\delta_1^2 - \delta_2^2 \} & \text{for all } 0 \leq t \leq \delta_2 \\
0 \leq \inf \{ L(\delta_1^2 + (t - \delta_2)^2) - \delta_1^2 - t^2 \} & \text{for all } t > \delta_2
\end{cases}
\end{align*}
\]
and then we obtain
\[
L \geq 1 + \frac{(\delta_2/\delta_1)^2}{2} \left( 1 + \sqrt{1 + 4(\delta_2/\delta_1)^2} \right) \equiv K(\delta_2/\delta_1).
\]
The constant $K = K(\delta_2/\delta_1) > 1$ determined as above satisfies (4.1).

Now we set $C_0(n) := K(\sqrt{n}) > 1$. We note
\[
w_k = \sum_{i=1}^{n} G_{k/\sqrt{n}, k}(\partial x_i u^\sigma) , \quad \hat{w}_k = \sum_{i=1}^{n} \hat{G}_{k/\sqrt{n}}(\partial x_i u^\sigma)
\]
by definition. Combining this fact with (4.1) implies that
\[
w_k \leq \hat{w}_k \leq C_0w_k \quad \text{in } U.
\]
\[4.2\] is an easy consequence of this result. \hfill \Box

It is easy to get
\[
\frac{1}{2} \left( k^2 + |z|^2 \right) \leq \epsilon^2 + |z|^2 \leq k^2 + |z|^2
\]
for all $0 < \epsilon \leq 1$ and $z \in \mathbb{R}^n$ with $|z| \geq k \geq 1$. This is clear by $\epsilon^2 \leq 1 \leq k^2$ and $k^2 + |z|^2 \leq 2|z|^2 \leq 2(\epsilon^2 + |z|^2)$. Hence we obtain for all $\sigma \in \mathbb{R}$ and $k \geq 1$,
\[
\min \{ 1, 2^{-\sigma} \} \cdot w_k^\sigma \leq \hat{w}_k^\sigma \leq \max \{ 1, 2^{-\sigma} \} \cdot w_k^\sigma \quad \text{in } \{ x \in U \mid |\nabla u^\sigma(x)| > k \}.
\]
\[4.3\] means the compatibility of $w_k(k \geq 1)$ and $w_\epsilon(0 < \epsilon \leq 1)$.

Throughout Section 4 we fix
\[
k := |f|^{1/(n-1)} + 1 \geq 1.
\]
and set a nonnegative function
\[
f_k := \frac{|f|^2}{w_k^{p-1}} \leq \left( \frac{|f|}{k^{p-1}} \right)^2.
\]
By the definition of $f_k$ and (4.3), it is obvious that
\[ \|f_k\|_{L^{n/(2\Lambda)}(B_R)} \leq 1. \] (4.5)
We also define two constants $0 < \lambda \leq \Lambda$ as
\[ \lambda := C_1 \min \{1, C_0(n)^{p/2-1}\} \min \{1, 2^{-1-p/2}\}, \quad \Lambda := C_2 \max \{1, C_0(n)^{p/2-1}\} \max \{1, 2^{-1-p/2}\}, \] (4.6)
which depend only on $n, p, C_1$ and $C_2$.

### 4.2 Moser’s iteration

By Moser’s iteration, we give a proof of Proposition 2.

**Proof.** Let $n \geq 3$. We divide the proof of (3.14) into 2 Steps.

**Step 1.** The aim of Step 1 is to prove the following Caccioppoli-type inequality.
\[ \int_{B_R} |\nabla (\eta v_\alpha)|^2 dx \leq C(n, p, q, \lambda, \Lambda) (1 + \alpha)^\beta \int_{B_R} v_\alpha^2 (|\nabla \eta|^2 + \eta^2) \, dx \] (4.7)
for any $\alpha \geq 0$ and $\eta \in C_1^\infty(B_R)$, where $v_\alpha := w_{k_0}^{(\alpha+p)/4} \in W_0^{1,\infty}(\Omega)$ and $\beta = \beta(n, q) \geq 2$ is a constant to be chosen later.

We prove (4.7) by a standard absorbing argument. For each fixed $i \in \{1, \ldots, n\}$, differentiate (3.1) with respect to $x_i$. Then using integration by parts, we have
\[ \int_{B_R} \left\{ \sum_{\beta=0}^d \nabla^\beta \left[ \left( \sum_{i=1}^n \nabla x_i u^\alpha \right) \nabla x_i \right] \eta \right\} \, dx = 0. \] (4.8)
for all $\phi \in W_0^{1,p}(B_R)$. We test $\phi := u_{i,k}w_{k}^{\alpha/2}\eta^2 \in W_0^{1,p}(B_R)$ in (4.8). We note that $\phi$ is supported in the superlevelset $\{x \in U \mid \nabla x_i u^\alpha(x) > k\}$, and hence we can replace $\nabla \nabla x_i u^\alpha$ by $\nabla u_{i,k}$. Summing over $i \in \{1, \ldots, n\}$, we obtain
\[ \int_{B_R} \left[ \sum_{i=1}^d \left( \nabla^2 \left[ \left( \sum_{i=1}^n \nabla x_i u^\alpha \right) \nabla x_i \right] \eta \right) \right] \, dx = 0. \] (4.9)

We set an integral
\[ J_\alpha := \int_{B_R} \eta^2 w_{k_0}^{(\alpha+p)/2-1} \sum_{i=1}^n |\nabla u_{i,k}|^2 \, dx + \frac{\alpha}{4} \int_{B_R} \eta^2 w_k^{(\alpha+p)/2-2} |\nabla w_k|^2 \, dx. \]
It is easy to obtain $I_1 \geq \lambda J_\alpha$ from (3.2), (4.2), (4.3) and (4.6). We estimate $|I_3|$ as following:
\[ |I_3| \leq C(n) \int_{B_R} |f| \left( \eta^2 w_k^{\alpha/2} \left( \sum_{i=1}^n |\nabla u_{i,k}|^2 \right)^{1/2} + \alpha \eta^2 w_k^{(\alpha-1)/2} |\nabla w_k| + w_k^{\alpha+1/2} |\eta||\nabla \eta| \right) \, dx \]
(by the Cauchy-Schwarz inequality)
\[ \leq \frac{\lambda}{2} J_\alpha + \frac{C(n)}{\lambda} (1 + \alpha) \int_{B_R} \eta^2 |f|^2 w_k^{(\alpha-p)/2+1} \, dx \]
\[ + C(n) \left( \frac{1}{1 + \alpha} \int_{B_R} w_k^{(\alpha+p)/2} |\nabla \eta|^2 \, dx + (1 + \alpha) \int_{B_R} \eta^2 |f|^2 w_k^{(\alpha-p)/2+1} \, dx \right) \]
(by the Young inequality)
\[ \leq \frac{\lambda}{2} J_\alpha + C(n, \lambda)(1 + \alpha) \left( \int_{B_R} v_\alpha^2 |\nabla \eta|^2 \, dx + \int_{B_R} f_k \eta^2 v_\alpha^2 \, dx \right). \] (by the definitions of $f_k, v_\alpha$)
From (4.9) we deduce that
\[ \frac{1}{2} J_\alpha \leq |I_2| + C(n, \lambda) \left( \int_{B_R} \hat{v}_\alpha^2 |\nabla \eta|^2 \, dx + (1 + \alpha) \int_{B_R} f_k \eta^2 v_\alpha^2 \, dx \right) . \]

The Cauchy-Schwarz inequality implies
\[ |\nabla w_k|^2 = 4 \sum_{j=1}^n \left( \sum_{i=1}^n u_{i,j} \partial_x u_{i,j} \right)^2 \leq 4 \sum_{j=1}^n \left( \sum_{i=1}^n u_{i,j}^2 \right)^2 \left( \sum_{i=1}^n |\partial_x u_{i,j}|^2 \right) \leq 4w_k \sum_{i=1}^n |\nabla u_{i,j}|^2 , \]

which leads to
\[ J_\alpha \geq \frac{1 + \alpha}{4} \int_{B_R} \eta^2 w_k^{(\alpha+p)/2-2} |\nabla w_k|^2 \, dx . \]

Combining this inequality with (3.3), (4.2)-(4.3), (4.6) and the Young inequality, we have
\[ |I_2| \leq \Lambda \int_{B_R} w_k^{(\alpha+p)/2-1} |\eta||\nabla w_k||\nabla \eta| \, dx \leq \frac{\Lambda(1 + \alpha)}{16} \int_{B_R} \eta^2 w_k^{(\alpha+p)/2-2} |\nabla w_k|^2 \, dx + \frac{4\Lambda^2}{\Lambda(1 + \alpha)} \int_{B_R} w_k^{(\alpha+p)/2} |\nabla \eta|^2 \, dx \leq \frac{1}{4} J_\alpha + \frac{4\Lambda^2}{\lambda} \int_{B_R} \eta^2 |\nabla \eta|^2 \, dx . \]

Therefore we obtain
\[ \int_{B_R} \eta^2 w_k^{(\alpha+p)/2-2} |\nabla w_k|^2 \, dx \leq C(n, \lambda, \Lambda) \left( \int_{B_R} \hat{v}_\alpha^2 |\nabla \eta|^2 \, dx + \int_{B_R} f_k \eta^2 v_\alpha^2 \, dx \right) . \]

We note that by direct calculation
\[ |\nabla v_\alpha|^2 = \frac{(\alpha + p)^2}{16} w_k^{(\alpha+p)/2-2} |\nabla w_k|^2 \leq C(p)(1 + \alpha)^2 w_k^{(\alpha+p)/2-2} |\nabla w_k|^2 . \]

From this it follows that
\[ \int_{B_R} \eta^2 |\nabla v_\alpha|^2 \, dx \leq C(n, p, \lambda, \Lambda)(1 + \alpha)^2 \left( \int_{B_R} \hat{v}_\alpha^2 |\nabla \eta|^2 \, dx + \int_{B_R} f_k \eta^2 v_\alpha^2 \, dx \right) , \]

and hence
\[ \int_{B_R} |\nabla (\eta v_\alpha)|^2 \, dx \leq C(n, p, \lambda, \Lambda)(1 + \alpha)^2 \left( \int_{B_R} \hat{v}_\alpha^2 |\nabla \eta|^2 \, dx + \int_{B_R} f_k \eta^2 v_\alpha^2 \, dx \right) . \quad (4.10) \]

For \( q = \infty \), it is easy to check from (4.5) and (4.10), that (4.7) is valid with \( \beta = 2 \). For \( 3 \leq n < q < \infty \), by the Hölder inequality and (4.5), we have
\[ \int_{B_R} f_k \eta^2 v_\alpha^2 \, dx \leq \| f_k \|_{L^{q/(\alpha n)}} \left( \int_{B_R} \left( \eta^2 v_\alpha^2 \right)^{\frac{2q}{q-2}} \, dx \right)^{1-2/q} \leq \| \eta v_\alpha \|^2_{L^{2/q}(B_R)} ; \]

Interpolation with \( L^2 \subset L^{2/q} \subset L^2 \) (Note that \( 2 < \frac{2q}{q-2} < \frac{2n}{n-2} = 2^* \) since \( 3 \leq n < q < \infty \)) and the Sobolev embedding \( W^{1,2}_0(B_R) \hookrightarrow L^2(B_R) \) imply that
\[ \| \eta v_\alpha \|^2_{L^{2/q}(B_R)} \leq \delta \int_{B_R} |\nabla (\eta v_\alpha)|^2 \, dx + C(n, q) \delta^{-\frac{2q}{q-2}} \int_{B_R} \eta^2 v_\alpha^2 \, dx \]

for any small number \( \delta > 0 \). Take \( \delta = c(1 + \alpha)^2 \) with \( c = c(n, p, q, \lambda, \Lambda) > 0 \) sufficiently small, then from (4.10) we obtain (4.7) with \( \beta := 2q/(q-n) \geq 2 \).

Step 2. From (4.7) we prove a local \( L^{\infty}-L^{p/2} \) estimate of \( w_k \) by Moser’s iteration.
Set \( \chi(n) := 2^* = n/(n-2) \in (1, \infty) \). We claim a reversed Hölder inequality
\[
\|w_k\|_{L^\frac{(\alpha+p)}{(\alpha+2^*)}(B_r)} \leq \left[ \int_{B_r} \left( (\eta(v_a))^2 \, dx \right)^{\frac{1}{\chi}} \right]^{\frac{1}{\chi}} \leq C(n) \int_{B_r} |\nabla (\eta v_a)|^2 \, dx 
\]
for all \( 0 < \rho < r \leq R \) and \( \alpha \geq 0 \). For any fixed \( 0 < \rho < r \leq R \), we take a cutoff function \( \eta \in C^\infty_c(B_r) \) such that
\[
0 \leq \eta \leq 1 \text{ in } B_r, \quad \eta = 1 \text{ in } B_{\rho}, \quad \text{and } |\nabla \eta| \leq \frac{2}{r-\rho} \text{ in } B_r.
\] (4.12)

Then we obtain
\[
\left( \int_{B_r} w_k^{(\alpha+p)/2} \, dx \right)^{1/\chi} \leq \left( \int_{B_r} (\eta v_a)^2 \, dx \right)^{1/\chi} \leq C(n) \int_{B_r} |\nabla (\eta v_a)|^2 \, dx
\]
(by the Sobolev embedding \( W_0^{1,2}(B_r) \hookrightarrow L^{2^*}(B_r) = L^{2^*}(B_r) \))
\[
\leq C(n, \rho, q, \lambda, \Lambda)(1 + \alpha)^\beta \int_{B_r} v_\alpha^2 \left( |\nabla \eta|^2 + |\eta|^2 \right) \, dx \quad \text{(by (4.11))}
\]
\[
\leq C(n, \rho, q, \lambda, \Lambda)(1 + \alpha)^\beta \left( \frac{1}{r-\rho} + 1 \right) \int_{B_r} v_\alpha^2 \, dx \quad \text{(by (4.12))}
\]
\[
\leq C(n, \rho, q, \lambda, \Lambda)(1 + \alpha)^\beta \int_{B_r} w_k^{(\alpha+p)/2} \, dx \quad \text{(note } 0 < r-\rho < 1),
\]
which implies (4.11).

For each \( N \in \mathbb{N} \cup \{0\} \), we define
\[
\alpha_N := p \left( \chi^N - 1 \right), \quad \gamma_N := \frac{p}{2} \chi^N, \quad \text{and } r_N := \left[ \theta + 2^{-N}(1 - \theta) \right] R.
\] (4.13)

We note \( \alpha_N + p = p \chi^N \) for each \( N \in \mathbb{N} \cup \{0\} \). Applying (4.11) with \( (\alpha, \rho, r) = (\alpha_N, r_{N+1}, r_N) \), we have for all \( N \in \mathbb{N} \cup \{0\} \)
\[
\|w_k\|_{L^{\gamma_N+1}(B_{r_{N+1}})} \leq C(n, \rho, q, \lambda, \Lambda) \left[ \frac{p \chi^N - p + 1}{2^{-N}(1 - \theta)R^2} \right]^{2(p \chi^N)} \|w_k\|_{L^{\gamma_N}(B_{r_N})}
\]
\[
\leq \left[ \left( \frac{C_1}{(1 - \theta)R^{4/p}} \right)^{\chi^N} \right] \|w_k\|_{L^{\gamma_N}(B_{r_N})}
\]
for some \( C_1 = C_1(n, \rho, q, \lambda, \Lambda) > 0 \). By iteration we can check that for each \( N \in \mathbb{N} \),
\[
\|w_k\|_{L^{\gamma_N}(B_{r_N})} \leq \|w_k\|_{L^{\gamma_N}(B_{r_{N-1}})}
\]
\[
\leq \left[ (1 - \theta)R^{\frac{4}{p} \sum_{j=1}^N \chi^j} \right] \left[ C_1 \sum_{j=0}^{\infty} \chi^j \right] \|w_k\|_{L^{\gamma_0}(B_0)} = C_1(n, \rho, q, \lambda, \Lambda) \frac{\|w_k\|_{L^{\gamma_0}(B_R)}}{(1 - \theta)R^{2^{N/p}}}
\]
Letting \( N \to \infty \), we obtain
\[
\|w_k\|_{L^{\infty}(B_{\theta R})} \leq C_1(n, \rho, q, \lambda, \Lambda) \frac{\|w_k\|_{L^{\gamma_0}(B_R)}}{(1 - \theta)R^{2^{N/p}}}
\]
Combining this result with (4.12) and the Minkowski inequality, we have
\[
\sup_{B_{\theta R}} |\nabla u^\ell| \leq C_0(n)^{1/2} \|w_k\|_{L^{\infty}(B_{\theta R})}^{1/2}
\]
\[
\leq C(n, \rho, q, \lambda, \Lambda) \left[ \frac{\|w_k\|_{L^{\gamma_0}(B_R)}}{(1 - \theta)R^{2^{N/p}}} \right] = C(n, \rho, q, \lambda, \Lambda) \left[ \frac{\|w_k\|_{L^{\gamma_0}(B_R)}}{(1 - \theta)R^{2^{N/p}}} \right] \left[ \int_{B_R} \left( k^2 + |\nabla u^\ell|^2 \right)^{p/2} \, dx \right]^{1/p}
\]
\[
\leq C(n, \rho, q, \lambda, \Lambda) \left( k + R^{-N/p} \|\nabla u^\ell\|_{L^p(B_R)} \right),
\]

Recall (4.4), and it completes the proof of (3.14).

We give a proof of (3.15) by making modifications of arguments given in Step 1 and 2. We note that (4.10) is valid even for \( n = 2 \) by the same computations, but that \( W^{1,2}_0(B_R) \hookrightarrow L^\infty(B_R) \) does not hold.

In Step 1, for \( 2 < q < \infty \), fix an arbitrary constant \( \frac{2q}{q-2} < k \leq \infty \), make an interpolation with \( L^2 \subset L^{k/2} \subset L^k \) and apply the Sobolev embedding \( W^{1,2}_0(B_R) \hookrightarrow L^k(B_R) \). Then from (4.10), we obtain (4.7) for some \( \beta = \beta(q, k) \geq 2 + \frac{1}{q-2} \). For \( q = \infty \), from (4.5) and (4.10), we can check that (4.7) is valid with \( \beta = 2 \), similarly as \( n \geq 3 \).

In Step 2, fix an arbitrary \( \chi \in (1, \infty) \) and apply the Sobolev embedding \( W^{1,2}_0(B_R) \hookrightarrow L^\chi(B_R) \). Then we obtain an alternative reversed Hölder inequality,

\[
\|w_k\|_{L^{(\alpha+p)/\gamma}(B_{rN})} \leq \left[ C(p, q, \chi, \Lambda, \frac{(p\chi^N - p + 1)^{\beta}}{2^{-N+1}(1-\theta)R^2}R^{2/k}\|w_k\|_{L^\chi(B_{rN})}} \right]^{2/(\alpha+p)}\|w_k\|_{L^{(\alpha+p)/\gamma}(B_{rN})}\
\]

instead of (4.11). Set \( \gamma, r_N \) as in (4.13), then we get

\[
\|w_k\|_{L^{\gamma N-\epsilon}(B_{rN+1})} \leq C(p, q, \chi, \Lambda, \frac{[(1-\theta)R]^{(1-1/p)\sum_{j=0}^\infty j^\beta}}{(1-\theta)^{4/p}(\chi)}\|w_k\|_{L^{\gamma N}(B_{rN})}^{2/(\alpha+p)}\|w_k\|_{L^{\gamma N}(B_{rN})}^{\alpha/(\alpha+p)}
\]

By iteration we can check that for each \( N \in \mathbb{N} \),

\[
\|w_k\|_{L^{\gamma N}(B_{rN}(\Omega))} \leq \|w_k\|_{L^{\gamma N}(B_{rN})} \leq \left[ (1-\theta)R \right]^{-\frac{\beta}{2}[1+\frac{\beta}{2}]\sum_{j=0}^\infty j^\beta} C(p, q, \chi, \Lambda, \frac{\|w_k\|_{L^{\gamma N}(B_{rN})}}{(1-\theta)R}^{1/p}(1-\theta)^{-\frac{4}{p}}\sum_{j=0}^\infty j^\beta)
\]

from which we conclude (3.15), similarly for \( n \geq 3 \).

\[ \square \]

Remark 6. Consider \( \beta = 0 \). Then (1.2) becomes

\[
\mathrm{div} \nabla u E_p(\nabla u) = f \quad \text{in} \quad \Omega.
\]

If \( u \in W^{1,p}(\Omega) \) is a weak solution to (4.14), then by making some modifications we conclude that

\[
\|\nabla u\|_{L^{\gamma N}(B_R)} \leq C(n, p, q, c_1, c_2, \theta) \left( \|f\|_{L^{1/(p-1)}(B_R)} + R^{-n/p} \|\nabla u\|_{L^{p}(B_R)} \right)
\]

for any fixed closed ball \( B_R \subset \Omega \) with its radius \( 0 < R \leq 1 \), any \( 2 \leq n < \infty \) and \( 0 < \theta < 1 \), instead of (1.11). Estimates of the type (4.15) can be seen in [4] Theorem 1.15 for solutions to uniformly elliptic systems. Compared with the proof of [4] Theorem 1.15, our proof of (4.15) is rather direct.

We give a sketch of the proof of (4.15).

\[ \text{Proof.} \]

We claim the following a priori estimate.

\[
\sup_{B_{r \delta}} |\nabla u^\epsilon| \leq C(n, p, q, c_1, c_2, \theta) \left( \delta + \|f\|_{L^{1/(p-1)}(B_R)} + R^{-n/p} \|\nabla u^\epsilon\|_{L^{p}(B_R)} \right)
\]

for any \( 0 < \delta < \delta \), any closed ball \( B_R \subset U \) with \( 0 < R \leq 1 \), any \( 3 \leq n < q \leq \infty \) and \( 0 < \theta < 1 \). Here \( u^\epsilon \in C^\infty(U) \) is a classical solution to

\[
-\mathrm{div} \nabla u^\epsilon E_p(\nabla u^\epsilon) = f \in C^\infty(\Omega) \quad \text{in} \quad U \subset \Omega.
\]
For each fixed $0 < \delta < 1$, we set

$$k := \delta + \|f\|_{L^p(B_R)}^{1/(p-1)} \geq \delta, \quad f_k := \frac{|f|^2}{w_k^{p-1}} \leq \left(\frac{|f|}{k^{p-1}}\right)^2.$$  

We note that (4.3) is valid for all $\sigma \in \mathbb{R}$ and $0 < \epsilon \leq \delta \leq k$. Using (4.6), (4.7), (4.8) and (4.9), we deduce (4.16), as in the proof of Proposition 2. Recalling all the proofs, we can easily check that Corollary 1 and Lemma 2 are valid even for $\beta = 0$. As in the proof of Theorem 2, we conclude from (4.15) that

$$\|\nabla u\|_{L^\infty(B_{2R})} \leq C(n, p, q, c_1, c_2, \theta) \left(\hat{\delta} + \|f\|_{L^p(B_R)}^{1/(p-1)} + R^{-n/p} \|\nabla u\|_{L^p(B_R)}\right)$$

for all $f \in L^q(\Omega)(n < q \leq \infty)$ and for any fixed $0 < \delta < 1$, from which we obtain (4.15). \hfill \square

**Remark 7.** The significant difference between the proof of Proposition 2 and that of [24, Lemma 4.9] is that different test functions are chosen. For simplicity, let $f = \text{const.} = a$. In Krügel’s essential proof, we test $\phi := (u_1)_+ \cdot \left[w_1^+\right]^{\alpha/2} \eta^2 \in W_0^1,p(B_R)$ or $\phi := - (u_1)_- \cdot \left[w_1^-\right]^{\alpha/2} \eta^2 \in W_0^1,p(B_R)$ in (4.8). Here $a \geq 0$, and the functions $w_1^+, w_1^-$ are defined as

$$w_1^+ := 1 + \sum_{i=1}^n (u_1)_+^2 \in W_0^1,\text{loc}(U) \quad \text{and} \quad w_1^- := 1 + \sum_{i=1}^n (u_1)_-^2 \in W_0^1,\text{loc}(U).$$

From this, we make a similar absorbing argument for alternative integrals

$$J_a^\alpha := \int_{B_R} \eta^2 w_1^+ \left[w_1^\alpha\right]^{\alpha/2} \sum_{i=1}^n |\nabla (u_1)_+|^2 \, dx + \frac{\alpha}{4} \int_{B_R} \eta^2 \left[w_1^\alpha\right]^{\alpha/2} \left[|\nabla w_1|^2\right] \, dx.$$

Then, as in Step 1 of the proof of Proposition 2, we obtain

$$\int_{B_R} \eta^2 \left[w_1^\alpha\right]^{\alpha/2} \left[|\nabla w_1|^2\right] \, dx \leq C(a, n, p, c_1, c_2) \int_{B_R} \eta^2 \left[w_1^\alpha\right]^{\alpha/2} \left(|\nabla \eta|^2 + |\nabla \eta|^2\right) \, dx \leq C(a, n, p, c_1, c_2) \int_{B_R} \eta^2 \left[w_1^\alpha\right]^{\alpha/2} \left[|\nabla \eta|^2 + |\nabla \eta|^2\right] \, dx \quad (4.17)$$

from (4.2), (4.3) and (4.4). Here we note that (4.2) is not used. Krügel claims that

$$\int_{B_R} \eta^2 \left[w_1^\alpha\right]^{\alpha/2} \left[|\nabla w_1|^2\right] \, dx \leq C(a, n, p, c_1, c_2) \int_{B_R} \eta^2 \left[w_1^\alpha\right]^{\alpha/2} \left[|\nabla \eta|^2 + |\nabla \eta|^2\right] \, dx \quad (4.18)$$

by (4.17). From (4.18), we conclude that

$$\int_{B_R} |\nabla (\eta v_\alpha)|^2 \, dx \leq C(a, n, p, c_1, c_2)(1 + \alpha)^2 \int_{B_R} \eta^2 \left(|\nabla \eta|^2 + |\nabla \eta|^2\right) \, dx, \quad \text{where} \quad v_\alpha := w_1^{(\alpha/p)^4}$$

instead of (4.10), since we easily obtain

$$\int_{B_R} \eta^2 \left[w_1^\alpha\right]^{\alpha/2} \left[|\nabla w_1|^2\right] \, dx \leq C(a, n, p, c_1, c_2) \int_{B_R} \eta^2 \left[w_1^\alpha\right]^{\alpha/2} \left[|\nabla \eta|^2 + |\nabla \eta|^2\right] \, dx.$$  

The rest of Krügel’s proof is very similar to that of Proposition 2. Hence it suffices to prove (4.18) from (4.17). The problem is, however, that neither $w_1^+$ nor $w_1^-$ is compatible with $\tilde{w}_k$. That is, though it is clear that $w_1^+ \leq \tilde{w}_k$ in $U$, there does not exist $C = C(n) > 1$ such that $\tilde{w}_k \leq Cw_1^+$ in $U$. This makes it difficult to obtain (4.18) from (4.17), if $p/2 - 1 < 0$, i.e. $1 < p < 2$. We overcome this problem by taking other suitable test functions carefully.

### 4.3 De Giorgi’s truncation

By De Giorgi’s truncation, it is possible to obtain another local a priori uniform Lipschitz estimate, which is much rougher than the results in Proposition 2. More general and sophisticated estimate via De Giorgi’s truncation can be seen in the recent work by Beck and Mingione [4, Section 4.2], while our approach is rather classical and elementary.
Proposition 3. Let \( f \in C^\infty(\Omega) \). Assume that \( u^\infty \in C^\infty(U) \) is a classical solution to (3.7) in \( U \Subset \Omega \). Under the condition (3.2) and (3.3), we have

\[
\sup_{B_R} |\nabla u^\infty| \leq \frac{C(n, p, q, C_1, C_2)}{[(1-\theta)R]^q} \left( R^{n/p} \left[ 1 + \|f\|_{L^p(B_R)}^{1/p-1} \right] + \|\nabla u^\infty\|_{L^p(B_R)} \right)
\]

(4.19)

for any closed ball \( B_R \subset \Omega \) with \( 0 < R \leq 1 \), any \( 2 \leq n < q \leq \infty \) and \( 0 < \theta < 1 \). Here \( g = g(n, p, q) \geq n/p \) is a constant.

Proof. We divide the proof into 3 Steps.

Step 1. We set \( V_l := \left( \int_0^{w_k^l/p} - I \right)_+ \in W^{1,\infty}_{loc}(\Omega) \) and \( A(l, r) := \{ x \in B_r \mid w_k(x) > l^{2/p} \} \) for \( l \geq 0 \) and \( 0 < r \leq R \). The aim of Step 1 is to prove that there exists a constant \( \gamma = \gamma(n, q) \in (0, 2/n] \) such that

\[
\int_{A(l, r)} (\eta V_l)^2 \, dx \leq C(n, p, q, \lambda, A) \left( \mathcal{L}^n(A(l, r)) \right)^\gamma \int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + l^2 \mathcal{L}^n(A(l, r))^{1+\gamma}
\]

(4.20)

for all \( 0 < r < R, \eta \in C^1_c(B_r, [0, 1]) \) and \( l \geq l_0 := C_1 \mathcal{V}_0 \mathcal{L}^{1/2}(B_r) \). Here \( \mathcal{L}^n \) denotes \( n \)-dimensional Lebesgue measure, and \( C_1 = C(n, p, q, \lambda, A) > 0 \) is a constant which is chosen later.

We test \( \phi := u_{i,h} V_l \eta^2 \in W^{1,\infty}_{loc}(B_R) \) in (4.3). We note that all integrals range over the superlevelset \( \{ x \in U \mid \partial_x u^\infty(x) > k \} \cap A(l, r) \) and therefore we may replace \( \nabla \partial_x u^\infty, \nabla w_k \) by \( \nabla u_{i,h}, \nabla \left( w_k - l^{2/p} \right)_+ \) respectively. By summing over \( i \in \{1, \ldots, n\} \), we obtain

\[
\int_{A(l, r)} (\eta V_l)^2 \, dx \leq C(n, p, q, \lambda, A) \left( \mathcal{L}^n(A(l, r)) \right)^\gamma \int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + l^2 \mathcal{L}^n(A(l, r))^{1+\gamma}
\]

(4.21)

We set an integral

\[
J := \int_{A(l, r)} (\eta V_l)^2 \, dx \leq \frac{1}{2} \int_{A(l, r)} (\eta V_l)^2 \, dx + \frac{p}{4} \int_{A(l, r)} \eta^2 w_k^{p-2} \left| \nabla \left( w_k - l^{2/p} \right)_+ \right|^2 \, dx.
\]

(4.22)

\( I_1 \geq \lambda J \) is easily obtained from (3.2), (4.2) - (4.3) and (4.6). We note that

\[
w_k^{p/2} = V_l + l \text{ on } A(l, r), \text{ and hence } w_k^{p} \leq 2 \left( V_l^2 + l^2 \right) \text{ on } A(l, r).
\]

With this in mind, we obtain

\[
|I_3| \leq C(n) \int_{A(l, r)} |f| \left( \eta^2 V_l^2 \left( \sum_{i=1}^n |\nabla u_{i,h}|^2 \right) \right)^{1/2} + p \eta^2 w_k^{p-1} \left| \nabla \left( w_k - l^{2/p} \right)_+ \right| + V_l w_k^{1/2} \eta |\nabla \eta| \, dx
\]

(by the Cauchy-Schwarz inequality)

\[
\leq \frac{\lambda}{2} J + C(n) \left( \int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, r)} \eta^2 w_k |f|^2 \, dx \right)
\]

\[
+ \frac{C(n)}{2} \left( \int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, r)} \eta^2 w_k |f|^2 \, dx \right)
\]

(by the Young inequality)

\[
\leq \frac{\lambda}{2} J + C(n, p, \lambda) \left( \int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, r)} \eta^2 f_k \left( w_k^p + V_l w_k^p \right) / f_k \, dx \right)
\]

(by the definition of \( f_k \))

\[
\leq \frac{\lambda}{2} J + C(n, p, \lambda) \left( \int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, r)} \eta^2 f_k \left( V_l^2 + l V_l + l^2 \right) \, dx \right).
\]

20
From this and (4.21), we deduce that

$$J \leq |I_2| + C(n, p, A)\left(\int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, r)} \eta^2 f_k \left(V_l^2 + |V_l|^2 + f_k^2\right) \, dx\right).$$

By dropping the first term in (4.22), we have

$$J \geq \frac{p}{4} \int_{A(l, r)} \eta^2 |w_k|^{p-2} \left|\nabla \left(w_k - f_k^2/\lambda\right)\right|^2 \, dx = \frac{1}{p} \int_{A(l, r)} \eta^2 |\nabla V_l|^2 \, dx.$$

By (3.3), (4.2)-(4.3), (4.6) and the Young inequality, we obtain

$$|I_2| \leq \Lambda \int_{A(l, r)} \eta |\nabla \eta| |V_l|^2 |\nabla |^{p/2-1} \left|\nabla \left(w_k - f_k^2/\lambda\right)\right|^2 \, dx = \frac{2\Lambda}{p} \int_{A(l, r)} \eta |\nabla \eta| |\nabla V_l|^2 \, dx.$$

From these it follows that

$$\int_{A(l, r)} \eta^2 |\nabla V_l|^2 \, dx \leq C(n, p, A, \lambda)\left(\int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, r)} \eta^2 f_k \left(V_l^2 + |V_l|^2 + f_k^2\right) \, dx\right),$$

and hence

$$\int_{A(l, r)} |\nabla (\eta V_l)|^2 \, dx \leq C(n, p, A, \lambda)\left(\int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + \int_{A(l, r)} \eta^2 f_k \left(V_l^2 + |V_l|^2 + f_k^2\right) \, dx\right).$$

(4.23)

From (4.23) we verify that (4.20) is valid. We first consider $n \geq 3$. By $0 \leq \eta \leq 1$, (4.5), the Hölder inequality and the Sobolev embedding $W_0^{1,2}(B_r) \hookrightarrow L^2(B_r)$, we obtain

$$\int_{A(l, r)} \eta^2 f_k V_l^2 \, dx \leq \|f_k\|_{L^{n/(2n-1)}(B_r)} \mathcal{L}^n(A(l, r))^{-2+1/q} \leq \mathcal{L}^n(A(l, r))^{-2+1/q},$$

$$\int_{A(l, r)} \eta^2 f_k V_l^2 \, dx \leq \|f_k\|_{L^{2/(n-2)}(B_r)} \left(\int_{B_r} |\nabla (\eta V_l)|^2 \right)^{1/2} \mathcal{L}^n(A(l, r))^{-2+1/q} \leq C(n) \mathcal{L}^n(A(l, r))^{-2+1/q} \int_{A(l, r)} |\nabla (\eta V_l)|^2 \, dx,$$

$$I \int_{A(l, r)} \eta^2 f_k V_l \, dx \leq I \|f_k\|_{L^{1/(2n-1)}(B_r)} C(n) \left(\int_{B_r} |\nabla (\eta V_l)|^2 \right)^{1/2} \mathcal{L}^n(A(l, r))^{-1+1/(2n-2)} \leq \delta \int_{A(l, r)} |\nabla (\eta V_l)|^2 \, dx + \frac{C(n)}{\delta} \mathcal{L}^n(A(l, r))^{1+2+4/q}$$

for any $\delta > 0$. Take $\delta = \delta(n, p, \lambda, \alpha) > 0$ sufficiently small, and assume that

$$\mathcal{L}^n(A(l, r)) \leq c(n, p, q, \lambda, \alpha)$$

(4.24)

for some sufficiently small constant $0 < c < 1$. Then from (4.23), we have

$$\int_{A(l, r)} |\nabla (\eta V_l)|^2 \, dx \leq C(n, p, \lambda, \alpha) \left(\int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + I^2 \mathcal{L}^n(A(l, r))^{-2+1/q}\right).$$

We can choose $C_*$ such that the assumption (4.24) holds true for all $l \geq l_0 = C_* \|V_0\|_{L^2(B_{R_1})}$; since by the Hölder inequality we obtain

$$\mathcal{L}^n(A(l, r)) \leq \frac{1}{7} \int_{A(l, r)} w_k^{p/2} \, dx \leq \frac{\mathcal{L}^n(A(l, r))^{1/2}}{l} \|V_0\|_{L^2(B_{R_1})}$$

and hence $\mathcal{L}^n(A(l, r)) \leq \left(\frac{\|V_0\|_{L^2(B_{R_1})}}{l}\right)^2.$
Again by the Hölder inequality and the Sobolev embedding $W^{1,2}_0(B_r) \hookrightarrow L^2(B_r)$, we get

$$
\int_{A(l, r)} (\eta V_l)^2 \, dx \leq C(n) \mathcal{L}^n(A(l, r))^{2/n} \int_{A(l, r)} |\nabla (\eta V_l)|^2 \, dx
\leq C(n, \rho, \Lambda) \left( \mathcal{L}^n(A(l, r))^{2/n} \int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + l^2 \mathcal{L}^n(A(l, r))^{1+2/n-2/q} \right).
$$

for all $l \geq l_0$. We note (4.24), and hence conclude that $\gamma(n, q) := 2/n - 2/q > 0$ satisfies (4.20) for $n \geq 3$. For $n = 2$, fix $2 < \kappa < \infty$ and use the Sobolev embedding $W^{1,2}_0(B_r) \hookrightarrow L^{\kappa}(B_r)$. Then, by a similar argument we realize that (4.20) is valid for some $\gamma = \gamma(q, \kappa) \in (0, 1 - 2/q)$.

Step 2. The aim of Step 2 is to prove that

$$
\int_{A(l_0+l_0, \rho R)} V_{l_0+0}^2 \, dx = 0
$$

for $L_0 = C_* \|V_0\|_{L^2(B_R)}$. Here $C_* = C_*(n, p, q, \Lambda, \rho, \theta, R) > 0$ is a constant which is chosen later. For any fixed $0 < \rho < r \leq R$, take a cutoff function $\eta \in C_c^1(B_r, [0, 1])$ as in (4.12). We note that for any $L > l \geq l_0$,

$$
\mathcal{L}^n(A(L, r)) = \mathcal{L}^n \left( \left\{ x \in B(r) \mid w^{p/2}_k - l > L - l \right\} \right) \leq \frac{1}{(L-l)^2} \int_{A(l, r)} V_l^2 \, dx,
$$

and

$$
\int_{A(L, r)} V_l^2 \, dx \leq \int_{A(l, r)} V_l^2 \, dx \quad \text{(since } A(L, r) \subset A(l, r), V_L \leq V_l \text{ in } \Omega).
$$

Hence by (4.20), we obtain

$$
\int_{A(L, r)} V_l^2 \, dx \leq \int_{A(l, r)} (\eta V_l)^2 \, dx
\leq C(n, \rho, q, \Lambda, l) \left( \int_{A(l, r)} V_l^2 |\nabla \eta|^2 \, dx + L^2 \mathcal{L}^n(A(L, r)) \right) \mathcal{L}^n(A(L, r))^{\gamma}
\leq C(n, \rho, q, \Lambda, l) \left( \frac{1}{(L-l)^2} + \frac{L^2}{(L-l)^2} \right) \frac{1}{(L-l)^2} \left( \int_{A(l, r)} V_l^2 \, dx \right)^{1+\gamma}
$$

for any $L > l \geq l_0$ and $0 < \rho < r \leq R$. Now we use an iteration argument. For each $N \in \mathbb{N} \cup \{0\}$, set

$$
l_N := l_0 + L_0 \left( 1 - 2^{-N} \right), \quad r_N := \left\{ \theta + 2^{-N}(1 - \theta) \right\} R, \quad \text{and } a_N := \|V_{l_N}\|_{L^2(B_{r_N})}.
$$

By (4.25), we get

$$
a_{N+1} \leq C(n, \rho, q, \Lambda, l) \left[ \frac{2^{N+1}}{(1-\theta) R} + 2^{N+1} \right] \frac{2^{N} \gamma (N+1)}{L_0} a_N \leq \frac{C_0(n, p, q, \lambda, \Lambda)}{(1-\theta) R} l_0^{-\gamma} 2^{(1+\gamma)N} a_N^{1+\gamma}
$$

for any $N \in \mathbb{N} \cup \{0\}$. Set

$$
L_0 := \left[ \frac{C_0}{(1-\theta) R} \right]^{1/\gamma} 2^{N/2^{1-\gamma}} \|V_0\|_{L^2(B_R)} \geq C_* \|V_0\|_{L^2(B_R)}.
$$

Then we obtain

$$
a_0 = \|V_{l_0}\|_{L^2(B_R)} \leq \|V_0\|_{L^2(B_R)} \geq \left[ \frac{C_0}{(1-\theta) R} l_0^{-\gamma} \right]^{-1/\gamma} \left( 2^{1+\gamma} \right)^{-1/2}
$$

and hence $a_N \to 0$ as $N \to \infty$, by (23) Chapter 2, Lemma 4.7. From this we have

$$
0 \leq \int_{A(l_0+l_0, \theta R)} V_{l_0+0}^2 \, dx \leq \liminf_{N \to \infty} \int_{A(l_N, r_N)} V_{l_N}^2 \, dx = \liminf_{N \to \infty} a_N^{1+\gamma} = 0.
$$
which implies that
\[ \|V_0\|_{L^\infty(B_R)} = \|w_k\|_{L^\infty(B_R)}^{p/2} \leq (C_0 + C_n^*) \|V_0\|_{L^2(B_R)}. \] (4.26)

**Step 3.** Set \( g := 2/(py) \geq n/p \). We make an interpolation argument to prove
\[ \|V_0\|_{L^\infty(B_R)} \leq C(n, p, q, \Lambda) \frac{\|V_0\|_{L^1(B_R)}}{(1 - \theta R)^{p \gamma}}. \] (4.27)

We note that \( \|V_0\|_{L^2(B_R)} \leq \|V_0\|_{L^1(B_R)}^{1/2} \|V_0\|_{L^2(B_R)}^{1/2} \). By (4.26) and the Young inequality, we obtain
\[ \|V_0\|_{L^\infty(B_R)} \leq \left( C_0 + \frac{C_0}{1 - \theta R} \right)^{1/p} \|V_0\|_{L^2(B_R)} \leq \frac{C(n, p, q, \Lambda)}{(1 - \theta R)^{p \gamma}} \|V_0\|_{L^1(B_R)}^{1/2} \|V_0\|_{L^2(B_R)}^{1/2} \]
\[ \leq \frac{1}{2} \|V_0\|_{L^\infty(B_R)} + C(n, p, q, \Lambda) \frac{\|V_0\|_{L^1(B_R)}}{(1 - \theta R)^{p \gamma}}. \]

Hence (4.27) follows from [16, Chapter V, Lemma 3.1]. By (4.2), (4.27) and the Minkowski inequality, we obtain
\[ \sup_{B_R} |\nabla u^\varepsilon| \leq C(n)^{1/2} \|V_0\|_{L^1(B_R)}^{1/p} \]
\[ \leq \frac{C(n, p, q, \Lambda)}{(1 - \theta R)^{p \gamma}} \|V_0\|_{L^1(B_R)}^{1/2} \left( \int_{B_R} \left( k^2 + |\nabla u^\varepsilon|^2 \right)^{p/2} dx \right)^{1/p} \]
\[ \leq \frac{C(n, p, q, \Lambda)}{(1 - \theta R)^{p \gamma}} \left( R^{n/p} k + \|\nabla u^\varepsilon\|_{L^p(B_R)} \right). \]

Recall (4.14), and it completes the proof of (4.19). \( \square \)

**Remark 8.** If \( n \geq 3 \) and \( q = \infty \), we may take \( g = n/p \) and therefore (4.14) is obtained.

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## A Elementary proofs of three lemmas

In Appendix, we give precise proofs of three lemmas for completeness. Most of the proofs are elementary in the sense that we just use standard tools of calculus, measure theory, convex analysis, functional analysis and real analysis.

### A.1 Vector inequalities

Vector inequalities (2.3)–(2.9) are used throughout Section 2.4.

**Lemma 3.** Let \( F_p, E^\varepsilon_p \) satisfy (1.3)–(1.9) and (2.3). Then we obtain inequalities (2.5)–(2.10).

For the special case (1.10) and \( \varepsilon = 1 \), proofs of (2.5) are given [29, Section 12], and [29, Lemma 13.3 and 30.1]. Here we give a generalized proof of inequalities (2.5)–(2.10) via smooth approximation.

**Proof.** By (1.9), the proof of (2.5) is completed by showing that
\[ \langle \nabla_z E^\varepsilon_p(z_2) - \nabla_z E^\varepsilon_p(z_1), z_2 - z_1 \rangle \geq \left\{ \begin{array}{ll} c_1 \cdot C(p) |z_1 - z_2|^p & (2 \leq p < \infty), \\ c_1 |z_1 - z_2|^2 (\varepsilon^2 + |z_1|^2 + |z_2|^2)^{p/2-1} & (1 < p < 2), \end{array} \right. \]
for all \( z_1, z_2 \in \mathbb{R}^n, 0 < \varepsilon \leq 1 \). \hfill (A.1)
For each fixed \(0 < \epsilon < 1\) and \(z_1, z_2 \in \mathbb{R}^n\), we have
\[
\langle \nabla_z E^\epsilon_p(z_2) - \nabla_z E^\epsilon_p(z_1) \mid z_2 - z_1 \rangle = \int_0^1 \langle \nabla^2_z E^\epsilon_p(tz_2 + (1-t)z_1) \mid z_2 - z_1 \rangle \, dt \\
\geq c_1 |z_1 - z_2|^2 \int_0^1 \left( \epsilon^2 + |t z_2 + (1-t)z_1|^2 \right)^{p/2 - 1} \, dt.
\]

Here we have used (A.10), (A.1) for \(1 < p < 2\) is easily obtained by a simple inequality
\[
\left( \epsilon^2 + |tz_2 + (1-t)z_1|^2 \right)^{p/2 - 1} \geq \left( \epsilon^2 + |z_1|^2 + |z_2|^2 \right)^{p/2 - 1} \quad \text{for } 0 \leq t \leq 1.
\]

Even for \(2 \leq p < \infty\), we get (A.1) as following.
\[
|z_1 - z_2|^2 \int_0^1 \left( \epsilon^2 + |tz_2 + (1-t)z_1|^2 \right)^{p/2 - 1} \, dt \geq \int_0^1 |tz_2 + (1-t)z_1|^{p-2} \, dt \geq C(p)|z_1 - z_2|^p.
\]

For the last inequality, see the proof of [11, Chapter I, Lemma 4.4].

We note that (1.14) implies that for each fixed \(z_0 \in \mathbb{R}^n\), eigenvalues of the positive real symmetric matrix \(\nabla^2_z E^\epsilon_p(z_0)\) are all no greater than \(c_2 \left( \epsilon^2 + |z_0|^2 \right)^{p/2 - 1}\). Hence for all \(z_1, z_2 \in \mathbb{R}^n, 0 < \epsilon \leq 1\),
\[
|\nabla_z E^\epsilon_p(z_1) - \nabla_z E^\epsilon_p(z_2)| = \int_0^1 \nabla^2_z E^\epsilon_p(tz_1 + (1-t)z_2) \cdot (z_1 - z_2) \, dt \leq c_2 |z_1 - z_2| \int_0^1 \left( \epsilon^2 + |z_1|^2 + |z_2|^2 \right)^{p/2 - 1} \, t \, dt.
\]

For \(p \geq 2\), we easily obtain (2.6) by using a simple inequality
\[
\left( \epsilon^2 + |tz_2 + (1-t)z_1|^2 \right)^{p/2 - 1} \leq C(p) \left( \epsilon^{p-2} + |z_1|^{p-2} + |z_2|^{p-2} \right) \quad \text{for all } 0 \leq t \leq 1.
\]

For \(1 < p < 2\), we get (2.6) as following,
\[
|z_1 - z_2|^2 \int_0^1 \left( \epsilon^2 + |tz_2 + (1-t)z_1|^2 \right)^{p/2 - 1} \, t \, dt \leq |z_1 - z_2| \int_0^1 |z_2 + t(z_1 - z_2)|^{p-2} \, dt \leq C(p)|z_1 - z_2|^{p-1}.
\]

For the last inequality, see the proof of [11, Chapter I, Lemma 4.4]. Let \((z_1, z_2) = (z_0, 0)\). Then we obtain (2.8) for \(1 < p < 2\) as following.
\[
|\nabla_z E^\epsilon_p(z_0) - \nabla_z E^\epsilon_p(0)| \leq c_2 \cdot C(p) \left( \epsilon^{p-2} |z_0| + |z_0|^{p-1} \right) \leq c_2 \cdot C(p) \left( \epsilon^{p-1} + |z_0|^{p-1} \right).
\]

Here we have used the Young inequality. The proof of (2.8) for \(2 \leq p < \infty\) is similar. Letting \(\epsilon \to 0\), we conclude (2.9) from (2.4) and (2.8).

(2.9) is easily deduced from (2.8). For \(2 \leq p < \infty\), we calculate
\[
|E^\epsilon_p(z_0) - E^\epsilon_p(0)| = \left| \int_0^1 \langle \nabla_z E^\epsilon_p(tz_0) \mid tz_0 \rangle \, dt \right| \\
\leq |\nabla E^\epsilon_p(0)| |z_0| \int_0^1 \langle \nabla E^\epsilon_p(tz_0) - \nabla E^\epsilon_p(0) \mid tz_0 \rangle \, dt \\
\leq \frac{1}{2} |\nabla E^\epsilon_p(0)| |z_0| + c_2 \cdot C(p) \int_0^1 \left( \epsilon^{p-1} |t| + t^p |z_0|^p \right) \, dt \\
\leq C(c_2, p) \left( \epsilon^{p-1} |z_0| + |\nabla E^\epsilon_p(0)| |z_0| + |z_0|^p \right) \quad \text{for all } z \in \mathbb{R}^n.
\]

The proof of (2.9) for \(1 < p < 2\) is similar.

For the proof of (2.10), we note that \(\partial E^\epsilon_p(z_0) = \{ \nabla E^\epsilon_p(z_0) \}\) for all \(z_0 \in \mathbb{R}^n\), since \(E^\epsilon_p \in C^\infty\) is convex. Hence by the subgradient inequality, we obtain
\[
E^\epsilon_p(z_0) - E^\epsilon_p(0) - \langle \nabla E^\epsilon_p(0) \mid z_0 \rangle \geq \langle \nabla E^\epsilon_p(z_0) - \nabla E^\epsilon_p(0) \mid z_0 \rangle
\]
for all \(z_0 \in \mathbb{R}^n\). We can also easily check at once that
\[
|z_0|^2 \left( \epsilon^2 + |z_0|^2 \right)^{p/2 - 1} \left( \epsilon^2 + |z_0|^2 \right)^{p/2 - 1} - \epsilon^2 \left( \epsilon^2 + |z_0|^2 \right)^{p/2 - 1} \geq \left( \epsilon^2 + |z_0|^2 \right)^{p/2 - 1} - \epsilon^p
\]
for all \(0 < \epsilon < 1, z_0 \in \mathbb{R}^n\) and \(1 < p < 2\). Combining these inequalities with (A.1), we conclude (2.10).
A.2 A justification for convergence of minimizers

Lemma 4 is used in Section 3 to justify that a sequence of local or global minimizers \( \{u^\varepsilon\}_{0 < \varepsilon \leq 1} \) converges to a minimizer \( u \).

**Lemma 4.** Let \( E_p, \{E_p^\varepsilon\}_{0 < \varepsilon \leq 1} \) satisfy (1.3)–(1.5). For bounded domain \( V \subset \mathbb{R}^n \) with Lipschitz boundary, assume that \( u \in W^{1,p}(V) \) satisfies

\[
\begin{align*}
F(v) &= \inf \left\{ F(v') \left| v' \in u + W^{1,p}_0(V) \right. \right\}. 
\end{align*}
\]

For each \( 0 < \varepsilon \leq 1 \), we define

\[
\begin{align*}
u^\varepsilon &= \inf \left\{ F(v') \left| v' \in u + W^{1,p}_0(V) \right. \right\} \in u + W^{1,p}_0(V). 
\end{align*}
\]

Then \( u^\varepsilon \rightharpoonup u(\varepsilon \to 0) \) in \( W^{1,p}(V) \) and

\[
\lim_{\varepsilon \to 0} F(v^\varepsilon) = F(u) = \lim_{\varepsilon \to 0} F(u^\varepsilon) = F(u).
\]

Moreover, up to a subsequence we obtain \( u^\varepsilon \to u \) in \( W^{1,p}(V) \).

In [24, Theorem 3.3], Krügel, inspired by the proof of [27, Theorem 6.1], discussed weak or strong convergence of minimizers, for the special case where \( E_p \) and \( E_p^\varepsilon \) are sphere symmetric and \( f = \text{const} \). For the reader’s convenience, we give a proof of Lemma 4 by generalizing Krügel’s idea.

**Proof.** We first note that

\[
F(v) \leq F(v^\varepsilon) \to F(v) \quad \text{as } \varepsilon \to 0
\]

for each fixed \( v \in W^{1,p}(V) \). (A.5) is clear by (1.3), (2.4), (2.9) and Lebesgue’s dominated convergence theorem.

We prove that \( u^\varepsilon \rightharpoonup u \) in \( W^{1,p}(V) \) and (A.4). For each \( 0 < \varepsilon \leq 1 \), we note \( u^\varepsilon - u \in W^{1,p}_0(V) \). By the Poincaré inequality, we get

\[
\|u^\varepsilon\|_{L^p(V)} \leq \|u\|_{L^p(V)} + C(n, p, V)\|\nabla u - \nabla v\|_{L^p(V)} \leq C(n, p, V) \left( \|u\|_{W^{1,p}(V)} + \|\nabla u^\varepsilon\|_{L^p(V)} \right).
\]

By (1.8) and \( F^\varepsilon(u^\varepsilon) \leq F^\varepsilon(u) \) from (A.3), we get

\[
\|\nabla u^\varepsilon\|_{L^p(V)}^{p} \leq \frac{1}{c_1} \int_V E_p(\nabla u^\varepsilon) \, dx \leq \frac{1}{c_1} \beta \int_V (|\varepsilon|^2 + |\nabla u^\varepsilon|^2) \, dx + \int_V E_p^\varepsilon(\nabla u^\varepsilon) \, dx
\]

\[
\leq \frac{1}{c_1} \beta \int_V (|\varepsilon|^2 + |\nabla u|^2) \, dx + \int_V E_p(\nabla u) \, dx + \int_V f(u^\varepsilon - u) \, dx
\]

\[
\leq C(n, p, \beta, c_1, c_2, V) \left( 1 + \int_V (1 + |\nabla u|^2)^{p/2} \, dx + \|f\|_{L^{q}(V)} \|\nabla(u^\varepsilon - u)\|_{L^p(V)} \right)
\]

\[
\leq \frac{\|\nabla u^\varepsilon\|_{L^p(V)}^p}{2} + C(n, p, \beta, c_1, c_2, V) \left( 1 + \left( \|f\|_{L^{q}(V)}^p + 1 \right) \|\nabla u\|_{L^p(V)}^p \right).
\]

Here we have used the Sobolev embedding \( W^{1,p}_0(V) \hookrightarrow L^{q}(V) \) and the Young inequality. Hence \( \{u^\varepsilon\}_{0 < \varepsilon \leq 1} \subset u + W^{1,p}_0(V) \) is bounded. Assume that \( u^{\varepsilon N} \rightharpoonup v \in u + W^{1,p}_0(V) \) for some sequence \( \{\varepsilon N\}_{N=1}^\infty \subset (0, 1) \) such that \( \varepsilon N \to 0 \) as \( N \to \infty \). We note that

\[
F(v) \leq F(v^\varepsilon) \leq F^\varepsilon(u^\varepsilon) \leq F^\varepsilon(u) \to F(u) \quad \text{as } \varepsilon \to 0
\]

(A.6)

by (A.2)–(A.3) and (A.5). By Lemma 4 we have

\[
F(v) \leq F(v^\varepsilon) \leq \liminf_{N \to \infty} F(v^{\varepsilon N}) = F(u),
\]

which implies \( v = u \). Hence we obtain \( u_\varepsilon \rightharpoonup u(\varepsilon \to 0) \) in \( W^{1,p}(V) \). Again by (A.6), we conclude (A.4).
By [6] Proposition 3.32 and the compact embedding \( W^{1,p}(V) \hookrightarrow L^p(V) \), we are reduced to showing that
\[
\limsup_{\epsilon \to 0} \| \nabla u^\epsilon \|_{L_p(V)} \leq \| \nabla u \|_{L_p(V)} \quad \text{up to a subsequence}
\]
to complete the proof. By (1.6), we can check that a smooth functional \( \hat{E}^\epsilon(z) := E^\epsilon(z) - C_\epsilon (\epsilon^2 + |z|^2)^{p/2} \) is convex in \( \mathbb{R}^n \) for sufficiently small \( C_\epsilon = C_\epsilon(\epsilon^1, p) > 0 \). By (1.8), \( \hat{E}(z) := \lim_{\epsilon \to 0} \hat{E}^\epsilon(z) = \beta |z| + E_p(z) - C_\epsilon |z|^p \) is also convex in \( \mathbb{R}^n \). We note that
\[
C_\epsilon \left( \int_V \left( \epsilon^2 + |\nabla u|^2 \right)^{p/2} \, dx - \int_V \left( \epsilon^2 + |\nabla u^\epsilon|^2 \right)^{p/2} \, dx \right) = \int_V \left( \hat{E}(\nabla u^\epsilon) - \hat{E}(\nabla u) \right) \, dx + \int_V f(u - u^\epsilon) \, dx + \left[ F_p^\epsilon(u) - F_p^\epsilon(u^\epsilon) \right]
\]
by the definitions of \( \hat{E}^\epsilon \) and \( F_p^\epsilon \). For \( z_0 \in \mathbb{R} \), we define
\[
h(z_0) := \nabla \hat{E}_p(z_0) + \left( \beta - pC_\epsilon |z|^{-1} \right) \text{sgn}(z_0) \in \mathbb{R}^n, \quad \text{where} \quad \text{sgn}(z_0) := \begin{cases} \frac{z_0}{|z_0|} & (z_0 \neq 0), \\ 0 & (z_0 = 0). \end{cases}
\]
It is easy to check that \( h(z_0) \in \partial \hat{E}(z_0) \) for all \( z_0 \in \mathbb{R}^n \). Moreover, we can check that for \( z_0 \in \mathbb{R}^n \) and \( 0 < \epsilon \leq 1 \),
\[
\nabla \hat{E}^\epsilon(z_0) = \nabla \hat{E}_p(z_0) + \frac{\beta |z_0| \text{sgn}(z_0)}{\sqrt{\epsilon^2 + |z|^2}} - pC_\epsilon \left( \epsilon^2 + |z|^2 \right)^{p/2 - 1} |z_0| \text{sgn}(z_0) \to h(z_0)(\epsilon \to 0). \quad (A.7)
\]
Since \( \partial \hat{E}^\epsilon(z_0) = \{ \nabla \hat{E}^\epsilon(z_0) \} \) for all \( z_0 \in \mathbb{R} \), we have
\[
\hat{E}^\epsilon(\nabla u^\epsilon) - \hat{E}(\nabla u) \geq \langle \nabla \hat{E}^\epsilon(\nabla u) - \nabla \hat{E}(\nabla u) \rangle = \langle \nabla \hat{E}^\epsilon(\nabla u) - h(\nabla u) \rangle \langle \nabla (u^\epsilon - u) \rangle + \langle h(\nabla u) \rangle \langle \nabla (u^\epsilon - u) \rangle \quad \text{a.e. in} \ V.
\]

Hence we obtain
\[
\begin{aligned}
C_\epsilon \int \left( \epsilon^2 + |\nabla u^\epsilon|^2 \right)^{p/2} \, dx &\geq C \int |\nabla u^\epsilon|^p \, dx + \int \left( f(u - u^\epsilon) \right) \, dx + \left[ F_p^\epsilon(u) - F_p^\epsilon(u^\epsilon) \right] \\
&= I_1(\epsilon) + I_2(\epsilon) \\
&= I_1(\epsilon) + I_2(\epsilon) + \int \langle h(\nabla u) \rangle \langle \nabla (u^\epsilon - u) \rangle \, dx.
\end{aligned}
\]

We claim that, up to a subsequence,
\[
\lim_{\epsilon \to 0} I_k(\epsilon) = 0 \quad \text{for all} \ k \in \{1, 2, 3, 4\}. \quad (A.9)
\]

Once we apply the compact embedding \( W^{1,p}(V) \hookrightarrow L^q(V) \), \( I_2(\epsilon) \to 0(\epsilon \to 0) \) up to a subsequence is clear by the compact embedding \( W^{1,p}(V) \hookrightarrow L^q(V) \). \( I_2(\epsilon) \to 0(\epsilon \to 0) \) follows from (A.4). \( h(\nabla u) \in L^p(V, \mathbb{R}^n) \) is clear by (2.7). Since \( \nabla u^\epsilon \to \nabla u(e \to 0) \) in \( L^p(V, \mathbb{R}^n) \), we obtain \( I_3(\epsilon) \to 0(\epsilon \to 0) \). From (A.7), it is clear that \( \nabla \hat{E}^\epsilon(\nabla u) \to h(\nabla u)(e \to 0) \) a.e. in \( V \). By (2.4) and (2.7), we get
\[
\begin{aligned}
\| \nabla \hat{E}^\epsilon(\nabla u) - h(\nabla u) \|_V &\leq C(p, C_\epsilon \left( |\nabla \hat{E}_p(\nabla u) - \nabla \hat{E}_p(0)|^p + |\nabla \hat{E}_p(0)|^p + \left( 1 + |\nabla u|^2 \right)^{p/2} \right) + \beta \| u \|
\end{aligned}
\]

uniformly for \( 0 < \epsilon \leq 6_0 \). From these, we conclude that \( \nabla \hat{E}^\epsilon(\nabla u) \to h(\nabla u) \) in \( L^p(V, \mathbb{R}^n) \) by Lebesgue’s dominated convergence theorem. We note that
\[
\sup_{0 < \epsilon \leq 1} \| \nabla u^\epsilon - \nabla u \|_{L_p(V)} < \infty.
\]
since we have already checked that \( \{ u^\epsilon \}_{0 < \epsilon \leq 1} \subset u + W^{1,p}_0(V) \) is bounded in \( W^{1,p}(V) \). Hence by the Hölder inequality, we deduce that \( I_1(\epsilon) \to 0(\epsilon \to 0) \). From (A.8) and (A.9), by letting \( \epsilon \to 0 \) we obtain
\[
C \limsup_{\epsilon \to 0} \| \nabla u^\epsilon \|^p_{L_p(V)} = C \limsup_{\epsilon \to 0} \int_V |\nabla u^\epsilon|^p \, dx = C \limsup_{\epsilon \to 0} \int_V \left( \epsilon^2 + |\nabla u|^2 \right)^{p/2} \, dx = C \| \nabla u \|^p_{L_p(V)}
\]
up to a subsequence. Here we have used Lebesgue’s dominated convergence theorem for the last equality. This completes the proof. \( \Box \)
A.3 A Fatou-type estimate

Lemma 5 is used in the proof of main theorem in Section 3.2.

**Lemma 5.** Let \((E, \|\cdot\|_E)\) be a Banach space and \(X \subset \mathbb{R}^n\) be a \(\mathcal{L}^n\)-measurable set. Suppose that sequences \(\{u_N\}_{N=1}^{\infty} \subset L^p(X, E) (1 \leq p < \infty)\), \(\{C_N\}_{N=1}^{\infty} \subset [0, \infty)\) satisfy

\[
\text{ess sup}_{x \in X} \|u_N(x)\|_E \leq C_N \quad \text{for all } N, \quad (A.10)
\]

\[
u_N \rightharpoonup u \quad \text{in } L^p(X, E) \quad \text{as } N \to \infty \quad (A.11)
\]

for some \(u \in L^p(X, E)\). Then we have

\[
\text{ess sup}_{x \in X} \|u(x)\|_E \leq \liminf_{N \to \infty} C_N. \quad (A.12)
\]

**Proof.** We may assume that \(C_\infty := \liminf_{N \to \infty} C_N < \infty\), since otherwise (A.12) is clear. Since \(\mathcal{L}^n\) is \(\sigma\)-finite, it suffices to show that \(\mathcal{L}^n(X_{\epsilon, r}) = 0\) for all \(\epsilon, r > 0\), where

\[
X_{\epsilon, r} := \{x \in X \mid |x| \leq r \text{ and } \|u(x)\|_E > C_\infty + \epsilon\} \subset X
\]

is a \(\mathcal{L}^n\)-measurable set. By (A.11), it is clear that

\[
u_N \rightharpoonup u \quad \text{in } L^p(Y, E) (N \to \infty)
\]

for any fixed \(\mathcal{L}^n\)-measurable set \(Y \subset X\). Hence for each fixed \(\epsilon, r > 0\), we obtain

\[
(C_\infty + \epsilon)(\mathcal{L}^n(X_{\epsilon, r}))^{1/p} \leq \left(\int_{X_{\epsilon, r}} \|u(x)\|_E^p dx\right)^{1/p} \quad \text{(by the definition of } X_{\epsilon, r})
\]

\[
\leq \liminf_{N \to \infty} \left(\int_{X_{\epsilon, r}} \|u_N(x)\|_E^p dx\right)^{1/p} \quad \text{(since the norm map is weakly lower semi-continuous)}
\]

\[
\leq (\mathcal{L}^n(X_{\epsilon, r}))^{1/p} \liminf_{N \to \infty} C_N = C_\infty (\mathcal{L}^n(X_{\epsilon, r}))^{1/p} \quad \text{(by (A.10))}
\]

Since \(\mathcal{L}^n(X_{\epsilon, r}) \leq \mathcal{L}^n(\{x \in \mathbb{R}^n \mid |x| \leq r\}) < \infty\), this implies \(\mathcal{L}^n(X_{\epsilon, r}) = 0\), which completes the proof of (A.12). \(\square\)

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