The Casimir effect: medium and geometry

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Abstract

Theory of the Casimir effect is presented in several examples. Casimir–Polder-type formulas, Lifshitz theory and theory of the Casimir effect for two gratings separated by a vacuum slit are derived. Equations for the electromagnetic field in the presence of a medium and dispersion are discussed. The Casimir effect for systems with a layer of $2 + 1$ fermions is studied.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In the Casimir effect [1], one typically solves electromagnetic or scalar classical boundary problems and finds normal modes of the system. The summation over eigenfrequencies of normal modes of the system determines its ground state energy via a relation $E = \sum \hbar \omega_i/2$.

In this paper this relation for the ground state energy is applied to dielectrics or metals separated by a vacuum slit, which is the typical system in the Casimir effect. The interaction part of the Casimir energy of two or more separated bodies in a vacuum is always finite; this part determines Casimir forces measured in experiments.

Different geometries were studied in the theory of the Casimir effect. In most geometries, eigenfrequencies $\omega_i$ are not known explicitly; in this case, two main approaches exist to derive the Casimir energy. One commonly used approach in the Casimir effect is the zeta function technique [2, 3]. An alternative technique which proved to be efficient in the Casimir effect is the scattering approach; it is discussed in section 3 with an emphasis on flat and periodic geometries.

Typically, the Casimir interaction is considered local in space with a given frequency dispersion, thus allowing nonlocality of photon interaction in time direction. Spatial nonlocality of photon interaction arising due to fermions and existence of polarization diagrams in quantum electrodynamics make the issue technically more complicated. Spatial dispersion is particularly important in metals; it yields e.g. a Debye screening. Graphene is another example of the $2 + 1$
system where spatial dispersion cannot be neglected. A complete theory which would give a
description of $3+1$ systems with spatial dispersion in the Casimir effect is still absent.

A problem of spatial dispersion is closely related to the high-temperature behavior of the
Casimir force. The problem of the high-temperature asymptotics of the free energy between two
metals attracts the attention of theoreticians and experimentalists. Asymptotics of permittivity
at small frequencies is important at large separations between two metals or high temperatures.
Two models of metal permittivity, the plasma model and the Drude one, yield high-temperature
asymptotics of free energy differing by a factor of 2; see section 4.

Instead of using a selected model of permittivity, one can alternatively evaluate
components of the polarization operator in a medium and write equations of motions.
Components of the polarization operator can be evaluated from the first principles once
one knows properties of quasiparticles in the medium. This approach allows one to obtain
high-temperature behavior of the system from the first principles of quantum electrodynamics
and quantum field theory (see sections 4 and 5). Spatial dispersion of the polarization operator
in graphene is important for correct determination of high-temperature behavior; see section 5
for details.

A layout of the rest of this paper is the following. Section 2 is devoted to derivation of the
Casimir–Polder interaction [4] of an anisotropic atom with a flat surface (perfectly conducting
and dielectric surfaces are considered) on the basis of an approach developed in [5].

In section 3 the scattering theory approach to periodic systems is developed after
discussion of the Lifshitz formula for two parallel flat surfaces separated by a vacuum slit $\alpha$ [6].
The theory of the Casimir effect for periodic systems with frequency-dependent permittivity
was developed in [7]. A comparison of theory and experiments beyond applicability of the
proximity force approximation was performed for grating geometries in [7, 8].

In section 4 Maxwell equations in a medium with a polarization operator of fermions taken
into account explicitly are discussed. Free energy between two superconductors is analyzed at
large separations between superconductors. Importance of nonlocality and spatial dispersion
in the Casimir effect is emphasized.

In section 5 fermions in a flat $2+1$ layer are studied following ideas developed in section 4.
In particular, derivation of reflection coefficients from a layer with $2+1$ fermions is given
in section 5 in a novel way; diagrams corresponding to the Casimir–Polder energy and the
Lifshitz free energy for systems with a flat $2+1$ fermion layer are shown. Graphene is a
typical example of such a system [9], its properties in the Casimir effect have been studied by
different authors [10–16]. Finally, the high-temperature asymptotics of a graphene–parallel
ideal metal system is derived.

Additional references for further reading are given after the conclusions.

We use coordinates $x^1, x^2, x^3$ and $x, y, z$ interchangeably. The units $\hbar = c = k_B = 1$ are
used throughout this paper.

2. Casimir–Polder energy

We review the general formalism developed in [5] and then derive the Casimir–Polder energy
of an anisotropic atom above a perfectly conducting plane [4] and the energy of an anisotropic
atom above a flat dielectric.

The atom is modeled as a localized electric dipole at the point $(x^1, x^2, x^3) = (0, 0, \alpha)$,
which is described by the current $J_\mu(x)$:

$$J_\mu(x) = \sum_{i=1}^{3} d_i(t) \delta(x^1) \delta(x^2) \delta(x^3 - \alpha),$$

(1)
The condition of current conservation holds:
\[ \partial_t J^\mu = 0, \]
the expectation value of dipole moments is given by the formula [17]
\[ \langle T (d_j(t_1) d_k(t_2)) \rangle = -i \int_{-\infty}^{+\infty} \frac{\alpha_{j k}(\omega)}{2\pi} \ d\omega, \]
where \( \alpha_{j k}(\omega) \) for \( \omega > 0 \) coincides withatomic polarizability; a groundstate of the atom is represented by a bracket. We will use the following expression for interaction energy \( E \) of the atom with a system \( C \):
\[ E = \lim_{T_1 \to +\infty} \left( \ln \left[ \int \exp \left[ i S(A) + i \int J A \ d^3x \right] \right] \right) = \lim_{T_1 \to +\infty} -i \frac{\langle JD_C J \rangle}{2T_1}. \]
\( T_1 \) is a time interval; \( D_C \) is a propagator of photons in the presence of the system \( C \).
Consider the Feynman gauge of vector potentials. A free photon propagator in this gauge has the form
\[ D_{\mu \nu}^{(0)}(t, r) = -i \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} D_{\mu \nu}^{(0)}(\omega, r) e^{-i\omega t}, \]
where
\[ D_{\mu \nu}^{(0)}(\omega, r) = -g_{\mu \nu} e^{i\omega |r|}/(4\pi |r|) \]
and \( g_{\mu \nu} = (1, -1, -1, -1) \) (Heaviside–Lorentz units are used). The propagator satisfying perfectly conducting boundary conditions \( A_0|_{z=0} = A_x|_{z=0} = A_y|_{z=0} = 0 \) and \( \frac{\partial A}{\partial z}|_{z=0} = 0 \) on a plane located at \( z = 0 \) has the form
\[ D_{00}(\omega, r) = -\frac{e^{i\omega t}}{(4\pi |r|)} + \frac{e^{i\omega |r|}}{(4\pi r_1)} \]
\[ D_{ax}(\omega, r) = \frac{e^{i\omega t}}{(4\pi |r|)} - \frac{e^{i\omega |r|}}{(4\pi r_1)} \]
\[ D_{ay}(\omega, r) = \frac{e^{i\omega t}}{(4\pi |r|)} - \frac{e^{i\omega |r|}}{(4\pi r_1)} \]
\[ D_{az}(\omega, r) = \frac{e^{i\omega t}}{(4\pi |r|)} + \frac{e^{i\omega |r|}}{(4\pi r_1)}, \]
where \( r = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2} \) and \( r_1 = \sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2} \).
An image method yields opposite signs for Dirichlet boundary conditions and equal signs for Neumann boundary conditions in (7).
Performing integration over time, using (3), (4), (7) and subtracting the contribution of the free photon propagator (6) (to derive interaction energy of the atom with a plane at \( z = 0 \)), one obtains after the Wick rotation the Casimir–Polder result [4] for energy of the atom above a perfectly conducting plane:
\[ E = -\int_{0}^{+\infty} \frac{d\omega}{2\pi} \sum_{i=1}^{3} \alpha_{ji}(i\omega) \left[ -\omega^2 D_{ii} + \frac{\partial^2}{\partial x_i \partial x_i'} D_{0i} \right] \bigg|_{x'=x, y'=y, z'=z=0} \]
\[ = -\frac{1}{64\pi^2a^3} \int_{0}^{+\infty} d\omega \left( \alpha_{11}(i\omega)+\alpha_{22}(i\omega) \right) (4\omega^2 a^2 + 2\omega a + 1) e^{-2a\omega} \]
\[ + \frac{1}{64\pi^2a^3} \int_{0}^{+\infty} d\omega \left( \alpha_{33}(i\omega) \right) (2(2\omega a + 1) e^{-2a\omega}). \]
To derive the result for the atom above a flat dielectric surface, it is convenient to apply the Weyl formula [18]. It is instructive to solve the problem in the Feynman gauge as well. Components of the vector potential reflect from the surface in the following way:
\[ A_p \rightarrow rTE A_p \]
where $p$ is a transverse (perpendicular) direction and $l$ is a longitudinal direction to the vector $(k_x, k_y)$, $r_{\text{TE}}(\omega, k_x, k_y)$ is a reflection coefficient of the transverse electric (TE) wave and $r_{\text{TM}}(\omega, k_x, k_y)$ is a reflection coefficient of the transverse magnetic (TM) wave.

The free propagator has the same form (6). To make needed decomposition of waves, it is convenient to use the Weyl formula:

$$ \frac{e^{i\omega r}}{4\pi r} = \frac{i}{(2\pi)^2} \int \frac{d k_x d k_y}{2 \sqrt{\omega^2 - k_x^2 - k_y^2}} e^{i(k_x(x-x') + k_y(y-y') + \sqrt{\omega^2 - k_x^2 - k_y^2}(z-z'))} \tag{13} $$

valid for $z - z' > 0$. The atom is located at the point $(x, y, z = a)$; the surface of a dielectric is at $z = 0$. According to the Weyl formula, the propagator is decomposed into infinite number of plane waves having the wave vector $(k_x, k_y, \sqrt{\omega^2 - k_x^2 - k_y^2})$. A projection of $A_z$ perpendicular to the incidence plane is $A_z \cos(\theta) \exp(-ik_z'z') (\theta = \arctan(k_y/k_x))$. After reflection from the surface, it is transformed into $A_z r_{\text{TM}} \cos(\theta) \exp(ik_z'z')$. The projection of $A_z$ to the incidence plane and back to the $x$-axis yields the factor $A_z r_{\text{TE}} \cos^2(\theta) \exp(ik_z'z')$. The projection of $A_z$ to the incidence plane and back to the $y$-axis yields the factor $-A_z r_{\text{TM}} \sin^2(\theta) \exp(ik_z'z')$.

Thus after reflection from the surface one obtains

$$ A_z = \frac{i}{(2\pi)^2} \int_{-\infty}^{\infty} d k_x d k_y \frac{e^{i(k_x(x-x') + k_y(y-y') + \sqrt{\omega^2 - k_x^2 - k_y^2}(z-z'))}}{2 \sqrt{\omega^2 - k_x^2 - k_y^2}} (r_{\text{TE}} \cos^2(\theta) - r_{\text{TM}} \sin^2(\theta)). \tag{14} $$

Putting $x = x', y = y', z = z' = a$, it is convenient to perform integration over $\theta$ which yields $\pi$. Performing the Wick rotation and assuming reflection coefficients depend on $k = \sqrt{k_x^2 + k_y^2}$, one obtains the contribution of $D_{\text{xx}}$ to the Casimir–Polder energy:

$$ \frac{1}{(2\pi)^2} \int_0^{+\infty} d \omega \alpha_{\text{xx}}(i\omega) \int_0^{+\infty} d k \frac{r_{\text{TE}}(i\omega, k) - r_{\text{TM}}(i\omega, k)}{4\sqrt{\omega^2 + k^2}} e^{-2\omega\sqrt{\omega^2 + k^2}}. \tag{15} $$

Similar steps allow one to obtain the following resulting expression for the Casimir–Polder energy of an anisotropic atom above a dielectric:

$$ E = \frac{1}{(2\pi)^2} \int_0^{+\infty} d \omega \left( \alpha_{\text{xx}}(i\omega) + \alpha_{\text{yy}}(i\omega) \right) \right) 
\times \int_0^{+\infty} d k \sqrt{\omega^2 (r_{\text{TE}}(i\omega, k) - r_{\text{TM}}(i\omega, k)) - r_{\text{TM}}(i\omega, k)k^2} e^{-2\omega\sqrt{\omega^2 + k^2}} 
\times \int_0^{+\infty} d k \sqrt{\omega^2 (r_{\text{TE}}(i\omega, k) - r_{\text{TM}}(i\omega, k)) - r_{\text{TM}}(i\omega, k)k^2} e^{-2\omega\sqrt{\omega^2 + k^2}}. \tag{16} $$

Note that in the case of isotropic polarizability of the atom in the $xy$ plane $\alpha_l(i\omega)$, expression (16) can be rewritten in the following form:

$$ E = \int_0^{+\infty} d \omega \int_0^{+\infty} d k \left( \Pi_{\text{I}}^{\text{xx}}(i\omega)D_{\text{I}}(i\omega, k) + \Pi_{\text{pp}}^{\text{xx}}(i\omega)D_{\text{pp}}(i\omega, k) + \Pi_{\text{zz}}^{\text{xx}}(i\omega)D_{\text{zz}}(i\omega, k) \right), \tag{17} $$
the positive imaginary frequency axis, the poles of coth zero-frequency Matsubara term). The contour above the real frequency axis is moved around zero frequency in the opposite direction (this semicircle integral is a contribution of the
written as
Matsubara frequencies
in every point of the momentum space
energy, one has to substitute
ω
φ(ω)
contour of integration in
Using the condition
Formula (17) can also be derived by changing the basis locally from the beginning using
f
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l
are zeroes and
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ω
and branch
Dii
(iω, k)
Dpp
(iω, k)
Dzz
(iω, k)
and
π
1
2
and free energy is rewritten in the form
nT
3. Free energy
Consider two bodies separated by a vacuum slit. To obtain the Casimir energy and free energy, one has to determine eigenfrequencies of normal modes of the electromagnetic field. Eigenfrequencies of normal modes can be summed up by making use of the argument principle [19, 20], which states

where


\begin{align}
D_{ll}(i\omega, k) &= \frac{1}{(2\pi)^2} \frac{-r_{TM}(i\omega, k)\sqrt{\omega^2 + k^2} e^{-2\omega \sqrt{\omega^2 + k^2}}}{2\omega^2}, \\
D_{pp}(i\omega, k) &= \frac{1}{(2\pi)^2} \frac{r_{TE}(i\omega, k) e^{-2\omega \sqrt{\omega^2 + k^2}}}{2\sqrt{\omega^2 + k^2}}, \\
D_{zz}(i\omega, k) &= \frac{1}{(2\pi)^2} \frac{-r_{TM}(i\omega, k)k^2 e^{-2\omega \sqrt{\omega^2 + k^2}}}{2\omega^2 \sqrt{\omega^2 + k^2}},
\end{align}

and \( \Pi_{ii}^{el}(i\omega) = \Pi_{pp}^{el}(i\omega) = \alpha_l(i\omega)\omega^2, \Pi_{zz}^{el}(i\omega) = \alpha_z(i\omega)\omega^2 \).

The equation for eigenfrequencies \( \omega_l \) of the corresponding problem of classical electrodynamics is \( f(\omega_l, \beta) = 0 \). For the Casimir energy, \( \phi(\omega) = \omega/2 \).

In the absence of dissipation (when eigenfrequencies of Maxwell equations are real), the contour of integration in \( \omega \) plane in (21) first passes around eigenfrequencies \( \omega_l \) and branch cuts on the positive real frequency axis as it is explained in detail, e.g. in [21]. To obtain free energy, one has to substitute \( \phi(\omega) = T \ln(2 \sinh(\omega/2T)) \) into (21). Then, free energy can be written as

\begin{align}
\mathcal{F} &= -\frac{T}{\pi} \int d\beta \int_0^{+\infty} d\omega \ln(2 \sinh(\omega/2T)) \text{Im} \frac{\partial}{\partial \omega} \ln f(\omega, \beta) \\
&= \frac{1}{2\pi} \int d\beta \int_0^{+\infty} d\omega \coth(\omega/2T) \text{Im} \ln f(\omega, \beta).
\end{align}

Using the condition \( f(-\omega^*, \beta) = f(\omega, \beta)^* \), one expands the contour of integration above the real frequency axis: \( \omega \in (-\infty + i\varepsilon \ldots + \infty + i\varepsilon) \) and adds to this contour a semicircle integral around zero frequency in the opposite direction (this semicircle integral is a contribution of the zero-frequency Matsubara term). The contour above the real frequency axis is moved around the positive imaginary frequency axis, the poles of \( \text{coth}(\omega/(2T)) \) (with residues \( 2T \)) yield Matsubara frequencies \( \omega_n = 2\pi n T \) and free energy is rewritten in the form

\begin{align}
\mathcal{F} &= T \int d\beta \sum_{n=0}^{\infty} \ln f(i\omega_n, \beta); \quad (\text{23})
\end{align}
prime means \( n = 0 \) term is taken with the coefficient 1/2. (Branch cuts on the real frequency axis disappear once one puts perfectly conducting plates at large separations. Scattering of electromagnetic waves between these perfectly conducting plates yields eigenfrequencies \( \omega \) located on positions of branch cuts. Eigenfrequencies \( \omega \) effectively transform into eigenfrequencies of scattering states which form branch cuts when perfectly conducting plates are moved to spatial infinity.) In the presence of dissipation, one can also use formulas (22) and (23); see a review [20].

Consider plane–plane geometry when two dielectric parallel slabs (slab 1: \( z < 0 \), slab 2: \( z > a \)) are separated by a vacuum slit (\( 0 < z < a \)) following [7]. In this case TE and TM modes are not coupled. The equation for TE eigenfrequencies is

\[
f(\omega, \beta = (k_x, k_y)) = 1 - r_{\text{TE down}}^{(1)}(k_x, k_y, \omega) r_{\text{TE up}}^{(2)}(k_x, k_y, \omega, a).
\]

(24)

Here \( r_{\text{TE down}}^{(1)}(k_x, k_y, \omega) \) is the reflection coefficient of a downward plane wave which reflects on a dielectric surface of slab 1 at \( z = 0 \), while \( r_{\text{TE up}}^{(2)}(k_x, k_y, \omega, L) \) is the reflection coefficient of an upward plane wave which reflects on a dielectric surface of slab 2 at \( z = a \). One can deduce from Maxwell equations that \( r_{\text{TE up}}^{(2)}(k_x, k_y, \omega, L) = r_{\text{TE down}}^{(2)}(k_x, k_y, \omega) \exp(2ik_xa) \) (\( r_{\text{TE down}}^{(2)}(k_x, k_y, \omega) \) is a reflection coefficient of a downward TE plane wave which reflects from dielectric slab 2 temporarily located at the position of slab 1, i.e. at \( z < 0 \)).

After substitution of (24) into formula (23), one obtains TE part contribution to free energy of two parallel plates. Contribution of TM part is obtained in full analogy. Free energy has the form

\[
\mathcal{F} = T \sum_{n=0}^{\infty} \int \frac{dk_x dk_y}{(2\pi)^2} \ln [(1 - e^{-2\pi \sqrt{\omega^2 + k_y^2}} r_{\text{TM}}^{(1)}(\omega, k_x, k_y))(1 - e^{-2\pi \sqrt{\omega^2 + k_y^2}} r_{\text{TM}}^{(2)}(\omega, k_x, k_y))],
\]

(25)

where \( \omega_n = 2\pi nT \) are Matsubara frequencies, the prime means the \( n = 0 \) term is taken with the coefficient 1/2, \( r_{\text{TM}}^{(1)} \equiv r_{\text{TE down}}^{(1)} \), \( r_{\text{TM}}^{(2)} \equiv r_{\text{TE down}}^{(2)} \), reflection coefficients are evaluated at Matsubara frequencies.

Reflection coefficients (Fresnel coefficients) for TM and electric flat waves approaching from vacuum the flat surface of a medium described by dielectric permittivity \( \varepsilon(\omega) \) are well known:

\[
\begin{align*}
|r_{\text{TM}}(\omega, k)|^2 &= \frac{\varepsilon(\omega) k_{z}^{(v)} - k_{z}^{(m)}}{\varepsilon(\omega) k_{z}^{(v)} + k_{z}^{(m)}},
\end{align*}
\]

(26)

\[
\begin{align*}
|r_{\text{TE}}(\omega, k)|^2 &= \frac{k_{z}^{(v)} - k_{z}^{(m)}}{k_{z}^{(v)} + k_{z}^{(m)}},
\end{align*}
\]

(27)

where

\[
\begin{align*}
k_{z}^{(v)} &= \sqrt{\omega^2 - k_x^2 - k_y^2},
\end{align*}
\]

(28)

\[
\begin{align*}
k_{z}^{(m)} &= \sqrt{\varepsilon(\omega) \omega^2 - k_x^2 - k_y^2}.
\end{align*}
\]

(29)

The Lifshitz result for two parallel semi-infinite dielectrics separated by a vacuum slit is obtained from (25) when Fresnel coefficients \( r_{\text{TM}}(i\omega, k), r_{\text{TE}}(i\omega, k) \) are substituted into (25) [6, 22].

The limit \( \varepsilon(\omega) \to +\infty \) corresponds to reflection coefficients of a perfectly conducting metal plane: \( r_{\text{TM}} = +1, r_{\text{TE}} = -1 \). In this case, one obtains from (25) energy of two parallel perfectly conducting plates at zero temperature, the result by Casimir [1]:

\[
E = \int \int \int \frac{d\omega dk_x dk_y}{(2\pi)^3} \ln (1 - e^{-2\pi \sqrt{\omega^2 + k_y^2}}) = -\frac{\pi^2}{720\omega_0^3}.
\]

(30)
Another limit is the high-temperature asymptotics \((4\pi Ta \gg 1)\) of the free energy for two parallel perfectly conducting plates. It can be obtained by evaluating contribution of the zero-frequency Matsubara term in \((25)\):

\[
\mathcal{F} \big|_{4\pi Ta \gg 1} = T \int \frac{dk_x dk_y}{(2\pi)^2} \ln(1 - e^{-2\alpha \sqrt{k_x^2 + k_y^2}}) = -\frac{T \zeta(3)}{8\pi a^2}.
\]

We will continue discussion of high-temperature behavior of free energy for models of metals with frequency dispersion of permittivity and for graphene systems in sections 4 and 5.

Consider now the system of two periodic gratings with a coinciding period \(d\) separated by a vacuum slit (figure 1) [7]. For this system, one has to consider reflection of downward and upward waves from a unit cell \(-\pi/d < k_x < \pi/d\). Imagine we remove the upper grating from the system. The reflection matrix of the downward wave is defined as \(R_{\text{down}}\).

The solution of Maxwell equations with a given \(m\) (\(m\) is an integer number determining the Brillouin zone of the downward wave with given \(\omega, k_x, k_y\)) for longitudinal components of the electromagnetic field outside the corrugated region \((z \geq h)\) may be written by making use of the Rayleigh expansion [23] for an incident monochromatic wave:

\[
E_y(x, z, m) = I_m^{(E)} \exp(ia_m x - i\beta_m z) + \sum_{n=-\infty}^{+\infty} R_{nm}^{(E)} \exp(i\alpha_n x + i\beta_n z), \tag{32}
\]

\[
B_y(x, z, m) = I_m^{(B)} \exp(i\alpha_m x - i\beta_m z) + \sum_{n=-\infty}^{+\infty} R_{nm}^{(B)} \exp(i\alpha_n x + i\beta_n z). \tag{33}
\]

\[
\alpha_n = k_x + 2\pi n/d, \quad \beta_n^2 = \omega^2 - k_y^2 - \alpha_n^2. \tag{34}
\]

This solution is valid outside any periodic structure in the \(x\) direction; in our notation, it is valid for \(z \geq h\). All other field components can be expressed in terms of longitudinal components \(E_y, B_y\) by standard formulas in waveguide theory. This can be done since the factor \(\exp(i k_y y)\) is conserved after reflection of the electromagnetic wave from a grating.

To construct the reflection matrix, one has to find Rayleigh expansions with the condition \(I_m^{(E)} = 0, I_m^{(B)} = 1\) and with the condition \(I_m^{(E)} = 1, I_m^{(B)} = 0\), \(m = -\infty \ldots + \infty\), \(m\) is an integer number. For actual calculations, one puts \(m = -J \ldots J\), where \(J\) is an upper and \(-J\) is...
Figure 2. A fictitious grating for which one evaluates $R_{2\downarrow}$.

Figure 3. The upper grating in figure 1 for which one evaluates $R_{2\uparrow}$; normal and lateral displacements from the fictitious grating shown in figure 2 are denoted by $L$ and $s$, respectively.

the lower limit in the sums (32), (33). In the case of reflection from a grating, the reflection matrix is given by

$$R_{1\downarrow}(k_x, k_y, \omega) = \begin{pmatrix} R^{(E)}_{n_{1\downarrow}q_{1}}(I^{(E)}_{m}) = \delta_{mq_{1}}, I^{(B)}_{m} = 0 \\ R^{(B)}_{n_{1\downarrow}q_{1}}(I^{(E)}_{m}) = \delta_{mq_{1}}, I^{(B)}_{m} = 0 \end{pmatrix}$$

(35)

Imagine now that we remove the lower grating from the system (see figure 3). We denote the reflection matrix of the upward wave as $R_{2\uparrow}$ then. Reflection matrices $R_{1\downarrow}, R_{2\uparrow}$ depend on wave vectors of incident waves, parameters of gratings and mutual location of the gratings. The equation for normal modes states

$$R_{1\downarrow}(k_x, k_y, \omega_{i})R_{2\uparrow}(k_x, k_y, \omega, L, s)\psi_{i} = \psi_{i},$$

(36)

where $\psi_{i}$ is an eigenvector describing the normal mode with a frequency $\omega_{i}$. Instead of equation (24), one obtains

$$f_{2}(\omega, \beta = (k_x, k_y)) = \det(I - R_{1\downarrow}(k_x, k_y, \omega)R_{2\uparrow}(k_x, k_y, \omega, L, s)).$$

(37)

For every $k_x, k_y$, the solution of $f_{2}(\omega_{i}) = 0$ yields possible eigenfrequencies $\omega_{i}$ of normal mode solutions of Maxwell equations.

Suppose that the reflection matrix $R_{2\downarrow}$ for a reflection from the fictitious imaginary grating located as in figure 2 is known in coordinates $(x, z)$. Performing a change of coordinates $z = -z_{1} + L, x = x_{1} - s$ $(s < d)$ in (32) and (33), it is possible to obtain a matrix $R_{2\downarrow}$ for reflection of upward waves from a grating with the same profile turned upside down, displaced from the lower grating by $\Delta x = s$, $\Delta z = L$ (see figure 3). It follows that

$$R_{2\uparrow}(k_x, k_y, \omega, L, s) = Q^*(s)K(k_x, k_y, \omega, L)R_{2\downarrow}(k_x, k_y, \omega)K(k_x, k_y, \omega, L)Q(s).$$

(38)
where \( R_{\text{down}}(k_x, k_y, i\omega) \) is a reflection matrix of downward waves from the grating in the system of coordinates \((x, z)\) depicted in figure 2. Here \( K(k_x, k_y, i\omega, L) \) is a diagonal \( 2(2J+1) \) matrix of the form

\[
K(k_x, k_y, i\omega, L) = \begin{pmatrix} G_1 & 0 \\ 0 & G_1 \end{pmatrix},
\]

(39)

with matrix elements \( e^{-L\sqrt{i\omega^2 + k_y^2 + k_x^2}} \) on a main diagonal of a matrix \( G_1 \). Due to exponential factors in (39), all resulting expressions are finite. The lateral translation \( 2(2J+1) \) diagonal matrix \( Q(s) \) is defined as follows:

\[
Q(s) = \begin{pmatrix} G_2 & 0 \\ 0 & G_2 \end{pmatrix},
\]

(40)

with matrix elements \( e^{2\pi i s/d} \) on a main diagonal of the matrix \( G_2 \).

The summation over eigenfrequencies is performed by making use of formula (21), which yields the Casimir energy of two parallel gratings on a unit surface:

\[
E = \frac{1}{(2\pi)^3} \int_0^{+\infty} d\omega \int_{-\infty}^{+\infty} dk_y \int_{-\pi}^{\pi} dk_x \ln \det (I - R_{\text{down}}(k_x, k_y, i\omega)R_{\text{up}}(k_x, k_y, i\omega, L, \varphi)).
\]

(41)

Here \( \varphi = 2\pi s/d \), and \( s \) is a lateral displacement of two gratings. This is an exact expression valid at zero temperature for two arbitrary parallel gratings with coinciding periods \( d \) separated by a vacuum slit.

Free energy on a unit surface \( \mathcal{F} \) in the system of two gratings can be written as follows:

\[
\mathcal{F}(L, \varphi) = \frac{T}{\pi^2} \sum_{n=0}^{+\infty} \int_0^{+\infty} dk_x \int_0^{\pi/d} dk_s \ln \det (I - R_{\text{down}}(k_x, k_y, i\omega_n)R_{\text{up}}(k_x, k_y, i\omega_n, L, \varphi)).
\]

(42)

Here \( \omega_n = 2\pi n T \) is a Matsubara frequency. The \( n = 0 \) term is multiplied by \( 1/2 \). Formula (42) is valid for arbitrary profile and arbitrary dielectric permittivity of each grating.

4. Polarization

Interaction of the electromagnetic potential \( A_\mu \) with a current \( J \) given by (1) and (2) yields a formalism with frequency-dependent dielectric permittivity \( \varepsilon(\omega) \). This interaction is local and gauge invariant. Polarization bubbles make interaction of photons in a medium essentially nonlocal. Nonlocality in time is reflected in the frequency-dependent permittivity \( \varepsilon(\omega) \). More general nonlocality leads to spatial dispersion, which by now led to intensive discussions [24–30].

It is instructive to write equations of motion in the presence of a medium with a polarization operator taken into account:

\[
\partial_\nu F^{\nu\mu} + \Pi^{\mu\nu} A_\nu = -j^\mu,
\]

(43)

\( j^\mu \) is an external current, and \( \Pi^{\mu\nu} \) is a polarization operator.

In momentum space, equations of motion can be written as follows:

\[
\text{div } \vec{E} + \frac{\Pi^{0n} E_n}{i\omega} = j^0(\omega, \vec{k})
\]

(44)

\[-\omega^2 E^m + (\text{rotrot } \vec{E})^m + \Pi^{mn} E^n = i\omega j^m(\omega, \vec{k}).
\]

(45)

Here \( \text{div } \vec{E} \equiv ik^s E^s \), (rotrot \( \vec{E} \))^m = \((k^2 \delta^{mn} - k^m k^n)E^n \).
The components of the 3 + 1 polarization operator can be evaluated using techniques of quantum electrodynamics in condensed matter. A general structure of the polarization operator resulting from gauge invariance in the presence of a medium is the following [31] (there is no Lorentz invariance in the presence of a medium, only rotation symmetry is conserved):

$$\pi^{0n}(\omega, \vec{k}, T) = \frac{\omega k^n \pi^{00}(\omega, \vec{k}, T)}{k^2}$$  \hspace{1cm} (46)

$$\pi^{mn}(\omega, \vec{k}, T) = \frac{\omega^2 k^m k^n \pi^{00}(\omega, \vec{k}, T)}{k^4} + \left( \delta^{mn} - \frac{k^m k^n}{k^2} \right) \pi^{00}(\omega, \vec{k}, T).$$  \hspace{1cm} (47)

Due to these properties, equation (44) can be rewritten as

$$\text{(div} \, \vec{E}) \left( 1 - \frac{\pi^{00}(\omega, \vec{k})}{k^2} \right) = j^0(\omega, \vec{k}).$$  \hspace{1cm} (48)

In this approach the longitudinal dielectric permittivity is naturally defined as $\varepsilon_l(\omega, \vec{k}) = 1 - \pi^{00}(\omega, \vec{k})/k^2$, which coincides with random phase approximation (RPA) result for the longitudinal dielectric permittivity [32].

In equation (45) the part $\Pi^{mn}E^m$ can be expanded in powers of $k^2$. The leading term of this expansion is

$$\Pi^{mn}E^m|_{\vec{k}| \to 0} = -\frac{k^m(\vec{k} \vec{E})}{k^2} \left( \varepsilon_l(\omega, \vec{k}, T)|_{\vec{k}| \to 0} - 1 \right) \omega^2 - \left( E^m - \frac{k^m(\vec{k} \vec{E})}{k^2} \right)$$

$$\times (\varepsilon_l(\omega, \vec{k}, T)|_{\vec{k}| \to 0} - 1) \omega^2,$$  \hspace{1cm} (49)

where

$$\varepsilon_l(\omega, \vec{k}, T) \equiv 1 - \left( \frac{\pi^{00}(\omega, \vec{k}, T)}{k^2} \right),$$  \hspace{1cm} (50)

$$\varepsilon_t(\omega, \vec{k}, T) \equiv 1 - \frac{\pi^{tr}(\omega, \vec{k}, T)}{\omega^2}$$  \hspace{1cm} (51)

are frequency-, momentum- and temperature-dependent longitudinal and transverse dielectric permittivities.

In the limit of zero temperature $T \to 0$, the longitudinal and the transverse dielectric permittivities coincide for $|\vec{k}| \to 0$, $\varepsilon_l(\omega, \vec{k}, T)|_{\vec{k}| \to 0, T \to 0} = \varepsilon_t(\omega, \vec{k}, T)|_{\vec{k}| \to 0, T \to 0} \equiv \varepsilon(\omega)$, and the term (49) results in

$$\Pi^{mn}E^m|_{\vec{k}| \to 0, T \to 0} = -\left( \varepsilon(\omega) - 1 \right) E^m.$$  \hspace{1cm} (52)

The term (52) is a standard one in the theory of a medium without spatial dispersion.

Neglecting dependence on $\vec{k}$ in the polarization operator, we obtain the standard equation for propagation of waves in a dielectric medium with frequency-dependent dielectric permittivity $\varepsilon(\omega)$:

$$-\omega^2 \varepsilon(\omega)E^m + \text{(rot} \, \vec{E})^m = i\omega j^m(\omega, \vec{k}).$$  \hspace{1cm} (53)

The equation in a medium has the same form (53) in the coordinate space, where the operation rot in (53) is a standard one in the coordinate space.

At $T = 0$, one obtains $\lim_{|\vec{k}| \to 0} \Pi^{00}(\omega, \vec{k})/|\vec{k}|^2 = \omega_p^2/\omega^2$ already for non-relativistic electron gas, $\omega_p^2 = 4\pi m e^2/m$, $\omega_p$ is plasma frequency, $m$ is electron mass and $e$ is charge density. So the general scheme outlined in this part of the paper naturally yields
Lindhard RPA result for \(\varepsilon_{l}(\omega, \vec{k})\) [32] and the plasma model of dielectric permittivity in the limit \(|\vec{k}| \to 0\) of the Lindhard formula: \(\varepsilon_{l}(\omega, \vec{k})|_{|\vec{k}| \to 0} = \varepsilon(\omega) = 1 - \omega_{p}^{2}/\omega^{2}\).

Several comments must be made about the plasma–Drude high-temperature problem in the Casimir effect. Permittivity in metals at small frequencies is a Drude one \((\varepsilon_{\text{Drude}}(\omega) = 1 - \omega_{p}^{2}/\omega(\omega + i\gamma))\) because the current is proportional to the electric field \(\vec{E}\), and as a result the pole of the first order is required in the permittivity. Optical data for metal permittivities \(\varepsilon(\omega)\) at small frequencies are in good agreement with a Drude model of the permittivity [33].

Due to this, it is challenging to explain results of measurements of the Casimir force between two metals at \(T = 300\) K, which were found to be in agreement with a plasma model of the permittivity [34].

The main difference of plasma–Drude results follows from the zero-frequency TE Matsubara term in free energy. The zero-frequency Matsubara term determines the asymptotics of the free energy at large separations \(4\pi Ta \gg 1\). The Drude model predicts \(r_{\text{TE}}(i\omega = 0, k) = 0 (\lim_{\omega \to 0} \varepsilon(i\omega)\omega^{2} = 0)\) and \(r_{\text{TM}}(i\omega = 0, k) = 1\), while the plasma model predicts \(r_{\text{TE}}(i\omega = 0, k) \neq 0 (\lim_{\omega \to 0} \varepsilon(i\omega)\omega^{2} = \omega_{p}^{2})\) and \(r_{\text{TM}}(i\omega = 0, k) = 1\). At large separations between two metals, the zero-frequency Matsubara term yields the leading contribution to free energy, and the asymptotical result for a plasma model of permittivity is two times larger than the result for a Drude model of permittivity.

In superconductors, the current is proportional to vector potential \(A_{\mu}\) (or, better to say, it is proportional to \(A_{\mu} - \partial_{\mu}\phi\), where \(\phi\) is the phase of the wavefunction of Cooper pairs in the ground state of a superconductor). To study the limit of large separations, one needs the zero-frequency Matsubara term. Suppose \(\Pi_{\mu}(\omega = 0, \vec{k}, T) = m_{0}^{2}(T) + \gamma(T)\vec{k}^{2} + O(\vec{k}^{4})\). In this case, we obtain

\[
r_{\text{TM}}(i\omega = 0, k) = 1, \quad r_{\text{TE}}(i\omega = 0, k) = k_{z}^{(e2)} - k_{z}^{(m2)} \over k_{z}^{(e2)} + k_{z}^{(m2)},
\]

where

\[
k_{z}^{(e2)} = \sqrt{k_{x}^{2} + k_{y}^{2}}, \quad k_{z}^{(m2)} = \sqrt{m_{0}^{2}(T)/(1 + \gamma(T)) + k_{x}^{2} + k_{y}^{2}}.
\]

Note these expressions are valid for small \(k\) only; nevertheless for a nonzero \(m_{0}(T)\), such behavior of \(\Pi_{\mu}(\omega = 0, \vec{k}, T)\) for small \(k\) leads to the ideal metal asymptotics of free energy at separations \(4\pi Ta \gg 1\): \(\mathcal{F} \sim -T \zeta(3)/(8\pi a^{2})\).

As follows from equation (45), results (54) and (55) correspond to zero-frequency reflection coefficients for the electromagnetic wave reflecting from a superconductor (if the Drude model of the permittivity is valid for the normal component of the two-fluid model of a superconductor; see also [35, 36]). Indeed, the photon field becomes massive with a mass \(m_{0}(T)\) as follows from equation (45); the condition \(\Pi_{\mu}(\omega = 0, \vec{k})|_{|\vec{k}| \to 0} = m_{0}^{2}(T)\) is the principal condition characterizing superconductivity. Equation (45) and the property \(i\omega_{B} = \text{rot}\vec{E}\) yield the equation for a magnetic field with a mass \(m_{0}(T)\); the Meissner effect immediately follows from the equation for a magnetic field. Zero-frequency reflection coefficients (54), (55) coincide with zero-frequency reflection coefficients for the plasma model of permittivity if one puts formally \(m_{0}^{2}(T)/(1 + \gamma(T)) = \omega_{p}^{2}\). In fact, if one considers a plasma model description of metals at large separations (or high temperatures), one should try to explain at the same time why superconductivity is not present in the same system.

In my opinion, one of the principal questions in the Casimir effect is to understand the conditions under which it is possible to use the approximation (52) and the resulting
equation (53) in the Casimir effect, i.e. when it is possible to neglect spatial dispersion. Spatial dispersion (i.e. dependence of $\Pi^{\mu\nu}$ on the frequency $\omega$ and the wave vector $\vec{k}$) should be taken into account in metals and it is important in metals indeed; it yields Debye screening, which immediately follows from $\varepsilon_l(\omega = 0, \vec{k})$. Of course, spatial dispersion makes the transition from momentum space to coordinate space complicated from the mathematical point of view. However, it is worth studying the issue since the solution of the problem of high-temperature behavior of the Casimir force between two metals can hardly be obtained without a detailed study of spatial dispersion and properties of the polarization operator at finite temperature.

5. Fermions in a layer

In this section we find reflection coefficients for TE and TM modes reflecting from the $2+1$ layer with fermions at $z = 0$. High-temperature results for a graphene–ideal metal system are discussed.

Equations

$$\partial_\mu F^{\mu\nu} + \delta(z)\Pi^{\nu\rho}A_\rho = 0$$

lead to conditions

$$\partial_z A_{ml}|_{z=+0} - \partial_z A_{ml}|_{z=-0} = \Pi_{ml}A^l|_{z=0}.$$  

(59)

Let us consider the condition

$$\partial_0 A^0 + \partial_\ell A^\ell + \partial_p A^p = 0.$$  

(60)

Here, a direction of the wave vector in the $xy$ plane is along the coordinate $\ell$, and $p$ is a transverse direction. In fact, the condition (60) is quite convenient for a description of TE and TM modes of the propagating electromagnetic wave.

For a nonzero $A_p$, the condition $\partial_p A_p = 0$ and conditions $A_0 = A_z = A_0 = 0$ describe propagation of the TE electromagnetic wave (the electric field is parallel to the surface $z = 0$) since $E_p \sim A_p$.

For the TE wave, we have

$$A_p = e^{ikl}e^{ikz} + r_{TE} e^{ikl} e^{-ikz} \quad \text{for} \quad z < 0$$  

(61)

$$A_0 = e^{ikz} \varepsilon^{kl}_{TE} \quad \text{for} \quad z > 0$$  

(62)

and

$$\Pi_{ml} A^m = -\Pi_{pp}(\omega, k)A_p.$$  

(63)

Here $k_0 = \omega^2 - k^2$, $k_0 = k_0^2 + k_0^2$. From continuity of potentials at $z = 0$, one obtains $1 + r_{TE} = r_{TE}$. Now one substitutes (61) and (62) into (59) and uses (63) to obtain

$$r_{TE}(\omega, k) = \frac{\Pi_{pp}(\omega, k)}{2\sqrt{\omega^2 + k^2} - \Pi_{pp}(\omega, k)} = \frac{\Pi_{pp}(\omega, k)}{2\sqrt{\omega^2 + k^2}} \left(1 - \frac{\Pi_{pp}(\omega, k)}{2\sqrt{\omega^2 + k^2}}\right)^{-1}. $$  

(64)

Conditions $A_p = A_z = 0$, $k_0 A_0 = k A_1$ describe the TM wave. This choice of vector potentials describes TM wave since $E_z \sim \partial_z A_0$ or $B_p \sim \partial_z A_1$. For $A_0$, we have

$$A_0 = e^{ikl}e^{ikz} + r_{A0} e^{ikl} e^{-ikz} \quad \text{for} \quad z < 0$$  

(65)

$$A_0 = e^{ikz} \varepsilon^{kl} I_{k_0} \quad \text{for} \quad z > 0.$$  

(66)
Two conditions follow from gauge invariance:

\[ \omega \Pi_{00}(\omega, k) - k \Pi_{0i}(\omega, k) = 0, \]
\[ \omega \Pi_{ij}(\omega, k) - k \Pi_{il}(\omega, k) = 0, \]

which yield

\[ \omega^2 \Pi_{00}(\omega, k) = k^2 \Pi_{il}(\omega, k). \]  

(67)

Thus, one obtains

\[ \Pi_{00}A^0 + \Pi_{0i}A^i = -\frac{\omega^2 - k^2}{\omega^2} \Pi_{il}A_0 = -\frac{k^2}{\omega^2} \Pi_{il}A_0, \]

(68)

and from (59) the condition

\[ 2i k_c (t_{A_0} + r_{A_0} - 1) = -\frac{k^2}{\omega^2} \Pi_{il}A_0 (1 + r_{A_0}) \]

(69)

follows. After use of \( r_{TM}(i\omega, k) = -r_{A_0}(i\omega, k) \) and the Wick rotation, one obtains

\[ r_{TM}(i\omega, k) = -\frac{\sqrt{\omega^2 + k^2}}{2\omega^2} \Pi_{il}(i\omega, k) \left( 1 - \frac{\sqrt{\omega^2 + k^2}}{2\omega^2} \Pi_{il}(i\omega, k) \right)^{-1}. \]

(70)

It is interesting to see how these results can be interpreted in terms of diagrams. Consider first 2 + 1 fermions interacting with an atom.

In the gauge \( A_0 = 0 \), the longitudinal part of the free photon propagator has the form (\( i = 1, 2 \)):

\[ D_{ij}^L(i\omega, k, z) = \frac{k_i k_j \sqrt{\omega^2 + k^2} e^{-|z|\sqrt{\omega^2 + k^2}}}{2\omega^2}, \]

(71)

the transverse part of the free photon propagator has the form:

\[ D_{ij}^T(i\omega, k, z) = \left( \delta_{ij} - \frac{k_i k_j}{k^2} \right) \frac{e^{-|z|\sqrt{\omega^2 + k^2}}}{2\sqrt{\omega^2 + k^2}}. \]

(72)

Consider the TE part of formula (17). Substituting (64) into (17) and using (72), we obtain

\[ \Pi_{pp}^aD_{pp} = \Pi_{pp}^a D_{pp}^T(2a) \cdot r_{TE} \]
\[ = \Pi_{pp}^aD_{pp}^T(a)(\Pi_{pp} + \Pi_{pp}D_{pp}^T(0)\Pi_{pp} + \cdots)D_{pp}^T(a). \]

(73)

Expression (73) has a clear diagrammatic representation (one of the terms is shown in figure 4). The sum of the terms in round parentheses in (73) is the sum of RPA diagrams in the two-dimensional layer (note that photon propagators do not depend on \( z \) there). Taking
the trace of (73), one obtains the TE contribution to the Casimir–Polder energy of interaction of the atom and 2 + 1 fermion layer.

Consider formula (25) for two parallel layers with fermions separated by a distance $a$. One obtains a diagrammatic representation of free energy expanding the logarithm. The factor $1/n$ is a standard factor arising in the representation of a thermodynamic potential in terms of closed loop diagrams with $n$ equivalent clusters connected by $n$ photon lines. The cluster is shown in figure 5; typical diagrams contributing to the Lifshitz free energy are shown in figure 6. Recent discussions of related multiple scattering techniques can be found in [46, 47].

Graphene is a typical 2 + 1 system that can be studied along the lines described in this part. Quasiparticles in graphene [9] obey a linear dispersion law $\omega = v_F k$ ($v_F \approx c/300$ is the Fermi velocity, $c$ is the speed of light) at energies less than 2 eV. There are $N = 4$ species of fermions in graphene. The polarization operator for 2 + 1 fermions at finite temperature was found in [10].

Define $\text{tr} \, \Pi \equiv \Pi_{mm}^0$. Due to property (67) and

$$\text{tr} \, \Pi(i\omega, k) = \Pi_{00}(i\omega, k) \frac{\omega^2 + k^2}{k^2} - \Pi_{pp}(i\omega, k),$$

one can rewrite reflection coefficients (64) and (70) in the form given in [10]:

$$r_{\text{TM}}(\omega, k) = \frac{k_z \Pi_{00}}{k_z \Pi_{00} + 2ik^2}, \quad r_{\text{TE}}(\omega, k) = -\frac{k_z^2 \Pi_{00} + k^2 \text{tr} \, \Pi}{k_z^2 \Pi_{00} + k^2 (\text{tr} \, \Pi - 2ik_z)}.$$
For $k \to 0$ one obtains for the zero mass gap and zero chemical potential ($\alpha$ is the coupling constant):

$$\Pi^{00}(i\omega = 0, k) = \frac{4\alpha N T \ln 2}{v_F^2} + \frac{\alpha N k^2}{12T} + \cdots,$$

$$\text{tr} \, \Pi(i\omega = 0, k) - \Pi^{00}(i\omega = 0, k) = \frac{\alpha N v_F^2 k^2}{6T} + \cdots.$$ 

For the ideal metal $r_{TM} = 1$, $r_{TE} = -1$ for all frequencies and wave vectors. Zero Matsubara TM and TE terms yield the following high-temperature behavior of free energy (25) in the graphene–ideal metal system:

$$F_{0TM} = -\frac{T \zeta(3)}{16\pi a^2} + \cdots, \quad (76)$$

$$F_{0TE} = -\frac{\alpha N v_F^2}{192\pi a^3} + \cdots. \quad (77)$$

Here

$$-\frac{T \zeta(3)}{16\pi a^2} \equiv F_{\text{Drude}}|_{T \to \infty} = \frac{1}{2} F_{\text{id}}|_{T \to \infty}$$

is the high-temperature asymptotics of the metal–metal system with a Drude model of permittivity used [37–45], which is equal to one half of the high-temperature asymptotics in the metal–metal system with ideal boundary conditions (31) or the plasma model of permittivity used [34]. The zero-frequency TE Matsubara term is suppressed by a factor $\alpha N v_F^2$ and additional power of $1/(Ta)$. 

The high-temperature asymptotics of free energy in a graphene–metal system coincides with the Drude high-temperature asymptotics of the metal–metal system. Detailed analysis [10] shows that the high-temperature behavior in the graphene–metal system takes place already at separations of the order of 100 nm at temperature $T = 300$ K, which makes systems with graphene very promising for experimental studies of the finite temperature Casimir effect. 

The example of this section shows that nonlocality arising due to interaction of photons via a polarization operator plays a crucial role in the high-temperature asymptotics of the system. Vacuum effects arising due to fermions are essentially nonlocal and lead to spatial dispersion. One of the most challenging problems in field theory is to unify nonlocality in momentum space with boundary problems in coordinate space. The Casimir effect is a natural area for further developments in this direction.

6. Conclusions

The Casimir effect is an area of research where one applies and develops methods of quantum field theory in the presence of a medium. Throughout this paper I discuss several techniques which proved to be efficient in the theory of the Casimir effect.

The Casimir–Polder effect is studied in the Feynman gauge of vector potentials in section 2; the Casimir–Polder energy of an anisotropic atom above a flat surface with boundary conditions of the ideal conductor and above a dielectric surface is found. The scattering approach is an efficient method in the theory of the Casimir effect; it is introduced in section 3 and applied to flat and periodic geometries separated by a vacuum slit. 

Properties of the medium determine important characteristics of Casimir systems such as the high-temperature asymptotics of free energy once one knows the dielectric permittivity of the medium. An alternative approach is to consider properties of the medium through evaluation of components of the polarization operator; in the current paper, this idea is
developed in sections 4 and 5. In section 5 exceptional finite temperature properties of systems with graphene are discussed.

Further reading. A discussion of interaction in the Casimir effect is given in the paper [48]. In the review [20] applications of the argument principle to flat geometries are considered; dissipation and dispersion are discussed. Papers [21, 49] yield a detailed derivation of the Lifshitz result for the plasma model of permittivity. The finite temperature Casimir effect is reviewed in [50]; see also the review [51]. Examples of scattering theory applied to spherical and cylindrical geometries can be found in papers [52–55] and in chapter 10 of the book [56]. The Casimir repulsion due to the Chern–Simons term was studied in [57, 58]; scattering of electromagnetic waves on a plane with the Chern–Simons term was considered in [59]. The Casimir–Polder interaction and macroscopic QED are reviewed in [60–62]; the scattering approach to the Casimir–Polder effect is discussed in [63]. Recent advances in microstructured geometries and the Casimir effect are summarized in a review [64]. Techniques for evaluation of functional determinants are discussed in [65]; methods of heat kernel and zeta function are presented in reviews [66, 67] and books [2, 3, 68, 69] in detail. One can find examples of the zeta function technique in the Casimir effect problems in papers [70–75].

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