SWAN CONDUCTORS AND TORSION IN THE LOGARITHMIC DE RHAM COMPLEX

SİNAN ¨UNVER

1. Introduction

In the following $A$ is a discrete valuation ring with maximal ideal $m$, perfect residue field $k$, and field of fractions $K$, $S = \text{Spec} A$, with closed point $s$, and $X/S$ an arithmetic surface over $S$, i.e. an integral, regular scheme which is proper, flat and of relative dimension one over $S$. We also assume that the reduced special fiber $X_{s,\text{red}}$ is a strict normal crossings divisor in $X$, by this we mean that $X_{s,\text{red}}$ is a normal crossings divisor in $X$, and that the irreducible components of $X_{s,\text{red}}$ are regular schemes.

We endow $S$ with the log structure corresponding to the natural inclusion $\mathcal{O}_S - \{0\} \to \mathcal{O}_S$. Similarly, we endow $X$ with the log structure corresponding to the natural map $\mathcal{O}_X \cap j_* \mathcal{O}_{X_K} \to \mathcal{O}_X$, where $j : X_K \to X$ is the inclusion. Then the structure map from $X$ to $S$ becomes a map of fine log schemes. Let $\Omega_{X/S,\log}$ denote its logarithmic de Rham complex,

$\mathcal{O}_X \to \Omega_{X/S,\log}^1 \to \Omega_{X/S,\log}^2$.

Taking $A$-torsion in $\Omega_{X/S,\log}$, we obtain the complex $\Omega_{X/S,\log,\text{tors}}$, which is a complex supported on the special fiber of $X$. We will be mainly interested in the Euler characteristic $\chi(\Omega_{X/S,\log,\text{tors}})$ of $\Omega_{X/S,\log,\text{tors}}$, which will be defined as follows. If $K^\circ$ is a bounded complex of coherent sheaves on $X$ that is exact on the generic fiber of $X$, the hypercohomology groups $H^i(X, K^\circ)$ are modules of finite length over $A$, and we put

$$\chi(K^\circ) = \sum_i (-1)^i \text{length}_A(H^i(X, K^\circ)).$$

On the other hand, $X_K/K$ has a Swan conductor $\text{Sw}(X_K/K)$, defined as follows. Let $K'$ be the strict henselization of the completion of $K$ (with respect to its discrete valuation), and $\ell$ be a prime different from the characteristic $p$ of $k$. The action of the wild inertia group of $\text{Gal}(\overline{K}/K')$ on $H^i_{\text{et}}(X_{\overline{K}}, \mathbb{Q}_{\ell})$ factors through the wild inertia group of $\text{Gal}(L/K')$, for a finite Galois extension $L$ of $K'$, being the continuous action of a pro-$p$ group on a finite dimensional $\ell$-adic vector space. Let $\text{SW}_{L/K'}$ be the Swan module over $\mathbb{Z}_{\ell}$, which is the unique projective $\mathbb{Z}_{\ell}[\text{Gal}(L/K')]$ module having as
character the Swan character of $\text{Gal}(L/K')$. Then $\text{Sw}_{\mathcal{F}/K}(H^i_{\text{et}}(X_K, \mathbb{Q}_\ell))$ is defined to be
\[
\dim_{\mathbb{Q}_\ell} \text{Hom}_{\mathbb{Z}_\ell[G_L]}(\text{SW}_{L/K'}, H^i_{\text{et}}(X_{K'}, \mathbb{Q}_\ell)),
\]
where $G_L = \text{Gal}(L/K')$. This is independent of the choice of $L$ since for a finite Galois extension $L'$ of $K'$ containing $L$, we have
\[
\text{SW}_{L/K'} = \text{SW}_{L'/K'} \otimes_{\mathbb{Z}_L[G_L]} \mathbb{Z}_\ell[G_L].
\]
Finally we define
\[
\text{Sw}(X_K/K) = \sum_{i=0}^{2} (-1)^i \text{Sw}_{\mathcal{F}/K}(H^i_{\text{et}}(X_K, \mathbb{Q}_\ell)).
\]
Then we have the following theorem.

**Theorem 1.** With the notation above we have
\[
\chi(\Omega_{X/S, \text{log}, \text{tors}}) = -\text{Sw}(X_K/K).
\]
This can be viewed as the logarithmic version of Bloch’s theorem ([Bloch], Theorem 1). In fact we follow the method in [Bloch] closely, together with the recent work of Kato and Saito ([K-S]).

2. **Proof**

We need the Lemma 1.17 of [K-S] on the local structure of $X/S$.

**Lemma 1.** For each point $x \in X$ there is a Zariski open neighborhood $U$ of $x$ in $X$, a scheme $P$, etale over $\mathbb{A}^2_S = \text{Spec} A[t, s]$, and a closed immersion $i : U \to P$, such that $U$ is defined in $P$ by an equation of the form $\pi - ut^a s^b$, where $\pi$ is a uniformizer of $A$, $u$ is a unit in $P$, $t$ and $s$ form a system of parameters in $O_{X,x}$ when restricted to $X$, and $a, b$ are nonnegative integers.

Proof. (following [K-S]) Since $X_{s,\text{red}}$ is a strict normal crossings divisor in $X$, there are at most two components of $X_{s,\text{red}}$ passing through $x$. If there is only one component we choose $t$ to be a local defining function for that component, and choose $s$ so that $\{t, s\}$ is a system of parameters at $x$. If there are two components we choose $t$ and $s$ to be the local defining functions for these two components. Now by considering the multiplicity of $X_s$ along the components of $X_{s,\text{red}}$ passing through $x$ we see that $X_s$ is defined by $t^a s^b = 0$ in a neighborhood of $x$ in $X$, for some nonnegative integers $a$ and $b$. Using $t$ and $s$ we get a map from an open neighborhood $U$ of $x$ to $\mathbb{A}^2$. Since $\{s, t\}$ is a system of parameters at $t$, and the residue field $k$, is perfect, by restricting $U$ if necessary, we may assume that this map is unramified. By restricting $U$ if necessary, we can factor this map as a closed immersion into a scheme $P$ followed by an etale map from $P$ to $\mathbb{A}^2_S$ ([EGA-4], Corollaire 18.4.7). Now since $\pi/(t^a s^b)$ is a unit in $U$ by restricting $U$ and $P$ if necessary, we can find a unit $u$ in $P$ such that $\pi - ut^a s^b$ vanishes on $U$. Since $U$ is a divisor in $P$, and $\pi - ut^a s^b$ vanishes on $U$ to see that $U$ is defined by $\pi - ut^a s^b$ it suffices to note that $\pi - ut^a s^b$ is not in $m^2_{P,x}$ (note that since $P$ is smooth over $S$, $\{\pi, t, s\}$ is a system of parameters for $O_{P,x}$).
Continuing with the notation of the lemma and denoting the conormal sheaves with $N$ we get an exact sequence,

$$0 \to N_{U/P} \to \Omega_{P/S}^1|_U \to \Omega_{U/S}^1 \to 0.$$  

In fact, this sequence is exact for any closed imbedding of $U$ into a scheme $P$ smooth over $S$. To see the injectivity of the map $N_{U/P} \to \Omega_{P/S}^1|_U$ we proceed as follows. If we denote the kernel of this map by $M$, we see that since $U_K/K$ is smooth, the map is injective over $U_K$, and hence $M|_{U_K} = 0$. On the other hand, since the imbedding $U \to P$ is regular $N_{U/P}$ is locally free. And therefore $M$, being a subsheaf of a locally free coherent sheaf on an integral scheme $U$, is zero. In the following we endow $P$ with the log structure associated to the inclusion $\mathcal{O}^* \to \mathcal{O}_P$, where $V = P - \{t^a s^b = 0\}$. We denote by $\Omega_{P/S,\log}^1$ the sheaf of log differentials where we endow $S$ with the trivial log structure. Then for the logarithmic differentials we get a similar exact sequence,

$$0 \to N_{U/P} \otimes_A \mathfrak{m}^{-1} \to \Omega_{P/S,\log}^1|_U \to \Omega_{U/S,\log}^1 \to 0,$$

where the map

$$\delta : N_{U/P} \otimes_A \mathfrak{m}^{-1} \to \Omega_{P/S,\log}^1|_U$$

is the map sending

$$(\pi - ut^a s^b) \otimes \pi^{-1} \text{ to } u^{-1}du + a.d\log t + b.d\log s.$$  

This resolution of $\Omega_{U/S,\log}^1$ gives a map

$$\alpha_U : \Omega_{U/S,\log}^1 \to \det \Omega_{U/S,\log}^1 \cong \text{Hom}(N_{U/P} \otimes_A \mathfrak{m}^{-1}, \Lambda^2 \Omega_{P/S,\log}^1|_U),$$

by the formula

$$\alpha_U(a)(b) = \tilde{a} \wedge \delta(b),$$

where $\tilde{a}$ is a section of $\Omega_{P/S,\log}^1$ that maps to $a$. Since any two resolutions of $\Omega_{U/S,\log}^1$ are homotopic and the maps $\alpha_U$ for different resolutions are compatible with the isomorphisms induced by the homotopies on $\det \Omega_{U/S,\log}^1$, we get a map

$$\alpha : \Omega_{X/S,\log}^1 \to \det \Omega_{X/S,\log}^1.$$  

Let $Z_U$ be the closed subscheme of $U$ defined by the section of the locally free sheaf $\Omega_{P/S,\log}^1$ corresponding to $\delta$. As above this does not depend on the imbedding, and hence defines a closed subscheme $Z$ of $X$. Note that $\Omega_{X/S,\log}^1$ is an invertible sheaf over $X - Z$, and hence $\alpha|_{X - Z}$ is an isomorphism. Let $\tilde{C}$ denote the complex

$$\alpha : \Omega_{X/S,\log}^1 \to \det \Omega_{X/S,\log}^1,$$

with $\Omega_{X/S,\log}^1$ in degree 1.

For any bounded complex $K$ of locally free coherent sheaves on $X$ which is exact outside a proper closed subscheme $Y$ we have a bivariant class

$$\text{ch}_Y^X(K) \text{ in } A(Y \to X)_Q$$
([Fulton], Chapter.18), which is the same for any other bounded complex of locally free coherent sheaves that is quasi-isomorphic to $K$ and exact outside $Y$. Therefore we can define $\chi^X(F)$ for any coherent sheaf $F$ supported on $Y$. If $Y$ is $X_s$ we will use the notation $\chi_s$ for $\chi^X$. Now by the Riemann-Roch theorem ([Fulton], Chapter.18; [Saito], Lemma.2.4) we have

$$\chi(\Omega^1_{X/S, \log, \text{tors}}) = \deg(\chi_s(\Omega^1_{X/S, \log, \text{tors}}) \cap \text{Td}(X/S)),$$

where $\text{Td}(X/S)$ is the Todd class of $X/S$. Since $C$ is exact outside $Z$, after choosing any resolution of $\Omega^1_{X/S, \log}$ we can define $\chi_s(C)$. Using the following lemma, we will work with $C$ instead of $\Omega^1_{X/S, \log, \text{tors}}$.

**Lemma 2.** With the notation above, we have

$$\chi_s(C) = \chi_s(\Omega^1_{X/S, \log, \text{tors}}).$$

**Proof.** Note that $\ker(\alpha) = \Omega^1_{X/S, \log, \text{tors}}$, since $\det\Omega^1_{X/S, \log}$ is an invertible sheaf and $\alpha$ is an isomorphism on the generic fiber. Therefore to finish the proof of the lemma we need to show that $\chi_s(\text{coker}(\alpha)) = \chi_s(\Omega^2_{X/S, \log, \text{tors}})$. First note that

$$\text{coker}(\alpha) \cong \det\Omega^1_{X/S, \log} \otimes \mathcal{O}_Z.$$

To see this we can work locally and choose an imbedding of $U$ as in Lemma.1. Let

$$u^{-1}du + a.d\log t + b.d\log s = (a + t.x)d\log t + (b + s.y)d\log s,$$

for some $x, y$ in $\mathcal{O}_U$. Then $Z$ is defined in $U$ by the ideal

$$(a + t.x, b + s.y), \text{ if } a \neq 0 \text{ and } b \neq 0, \text{ or by }$$

$$(a + t.x, y), \text{ if } a \neq 0 \text{ and } b = 0.$$

$\Omega^1_{X/S, \log}$ is generated by $dt$, $ds$, $d\log t$, (if $a \neq 0$), and $d\log s$ (if $b \neq 0$) subject to the relations

$$(a + t.x)d\log t + (b + s.y)d\log s = 0,$$

$$t.d\log t = dt, \text{ and } s.d\log s = ds.$$

Note that we may view $\det\Omega^1_{X/S, \log}$ as a subsheaf of $\Omega^1_{X_K/K, \log}$. If $a \neq 0$ and $b \neq 0$ $\det\Omega^1_{X/S, \log}$ is generated by

$$\frac{1}{b + s.y}d\log t, \text{ if } b + s.y \neq 0, \text{ or by }$$

$$\frac{1}{a + t.x}d\log s, \text{ if } a + t.x \neq 0.$$

Assume without loss of generality that $b + s.y$ is nonzero. Then the image of $\Omega^1_{X/S, \log}$ in $\det\Omega^1_{X/S, \log}$ is generated by

$$d\log t = (b + s.y)\frac{1}{b + s.y}d\log t, \text{ and } d\log s = (a + t.x)\frac{1}{b + s.y}d\log t.$$
Therefore the cokernel of $\alpha$ is $\det\Omega^1_{X/S,\log} \otimes \mathcal{O}_Z$. If $a \neq 0$ and $b = 0$ then $\det\Omega^1_{X/S,\log}$ is generated by
\[
\frac{1}{y} \, d\log t, \text{ if } y \neq 0, \text{ or by }
\frac{1}{a + t} \, d\log s, \text{ if } a + t \cdot x \neq 0,
\]
in $\Omega^1_{X/K,\log}$, and we similarly arrive at the conclusion.

Next, if $a \neq 0$ and $b \neq 0$, $\Omega^2_{X/S,\log}$ is generated by $d\log t \wedge d\log s$ with the relations
\[(a + t \cdot x) \, d\log t \wedge d\log s = 0, \text{ and } (b + s \cdot y) \, d\log t \wedge d\log s = 0.
\]
And if $a \neq 0$ and $b = 0$, $\Omega^2_{X/S,\log}$ is generated by $d\log t \wedge ds$ with the relations
\[(a + t \cdot x) \, d\log t \wedge ds = 0, \text{ and } y \, d\log t \wedge ds = 0.
\]

This shows that $\Omega^2_{X/S,\log}$ is an invertible sheaf on $Z$. And using again the local description we see that $\Omega^1_{X/S,\log}|Z$ is locally free of rank 2, and we have
\[
\Omega^2_{X/S,\log,tors} = \Omega^2_{X/S,\log} - \Lambda^2 \Omega^1_{X/S,\log}|Z.
\]

Restricting the resolution of $\Omega^1_{X/S,\log}$ over $U$ to $Z_U$, we obtain
\[
0 \to L^1 i^* \Omega^1_{X/S,\log}|Z_U \to N_{U/P}|Z_U \otimes_A \mathfrak{m}^{-1} \to \Omega^1_{P/S,\log}|Z_U \to \Omega^1_{U/S,\log}|Z_U \to 0,
\]
where $i : Z \to X$ is the inclusion. Here the second and the fourth arrows are isomorphisms. In particular, we have
\[
L^1 i^* \Omega^1_{X/S,\log}|Z_U \cong N_{U/P}|Z_U \otimes_A \mathfrak{m}^{-1}.
\]
The exact sequence also shows that
\[
\Lambda^2 \Omega^1_{X/S,\log}|Z \cong \det\Omega^1_{X/S,\log}|Z \otimes L^1 i^* \Omega^1_{X/S,\log}.
\]

Using a filtration of $\mathcal{O}_Z$ with graded pieces supported on integral subschemes of $Z$, we see that to prove the lemma it is enough to show that $L^1 i^* \Omega^1_{X/S,\log}|T \cong \mathcal{O}_T$, for any integral curve $T$ in $Z$. For the rest of the proof we use the method of the proof of Proposition 3.1 in [Saito], in this very explicit (and easier) case. First note that if $k : U \to Q$ is a closed immersion with $Q$ smooth over $S$, then $k$ is a regular immersion. Since the inclusion $T_U \to U$ is also a regular immersion, we have an exact sequence of locally free sheaves on $T_U$,
\[
0 \to N_{U/Q}|T_U \to N_{T_U/Q} \to N_{T_U/U} \to 0.
\]
And similarly we have an exact sequence,
\[
0 \to N_{Q_s/Q}|T_U \to N_{T_U/Q} \to N_{T_U/Q_s} \to 0,
\]
in particular $N_{Q_s/Q}|T_U \to N_{T_U/Q}$ is injective. Furthermore for an immersion $U \to P$ as in Lemma 1 we claim that
\[
0 \to N_{P_s/P}|T_U \to N_{T_U/P} \to N_{T_U/U} \to 0
\]
is exact. To see this we only need to check that
\[
N_{P_s/P}|T_U = \ker(N_{T_U/P} \to N_{T_U/U}).
\]
Since this is a local question on $X$, by restricting $U$ and $P$ we will assume that $T_U$ is defined by $t$ on $U$. Denoting $\pi - ut^a s^b$ by $g$, we need to show that 

$$\pi/(\pi^2, \pi t) = \ker((t, g)/(t, g)^2 \rightarrow (t, g)/(t^2, g)).$$

Since by assumption $T$ is contained in $Z$, $X/S$ is not smooth along $T$, and so $a \geq 2$. Therefore $ut^a s^b \in (t^2)$, and $\pi \in (t^2, g)$. This shows that $\pi/(\pi^2, \pi t)$ is in the kernel. To see the converse we only need to note that,

$$g - \pi = ut^a s^b = 0 \text{ in } (t, g)/(t, g)^2,$$

since $ut^a s^b \in (t^2)$. This proves the claim. Tensoring the exact sequence with $m^{-1}$ and observing that 

$$N_{P_s/P}|T_U \otimes_A m^{-1} \cong \mathcal{O}_{T_U},$$

we obtain the exact sequence

$$0 \rightarrow \mathcal{O}_{T_U} \rightarrow N_{T_U/P} \otimes_A m^{-1} \rightarrow N_{T_U/U} \otimes_A m^{-1} \rightarrow 0.$$ 

On the other hand, using the isomorphism

$$L^1 i^* \Omega^1_{X/S, \log}|T_U \cong N_{U/P}|T_U \otimes_A m^{-1},$$

and the exact sequence

$$0 \rightarrow N_{U/P}|T_U \otimes_A m^{-1} \rightarrow N_{T_U/P} \otimes_A m^{-1} \rightarrow N_{T_U/U} \otimes_A m^{-1} \rightarrow 0,$$

we get an exact sequence

$$0 \rightarrow L^1 i^* \Omega^1_{X/S, \log}|T_U \rightarrow N_{T_U/P} \otimes_A m^{-1} \rightarrow N_{T_U/U} \otimes_A m^{-1} \rightarrow 0.$$ 

Therefore we see that

$$L^1 i^* \Omega^1_{X/S, \log}|T_U \cong \mathcal{O}_{T_U}$$

by viewing them both as the kernel of

$$N_{T_U/P} \otimes_A m^{-1} \rightarrow N_{T_U/U} \otimes_A m^{-1}.$$ 

If we take imbeddings of $U$ into $P$ and $P'$ as above, taking $Q = P \times P'$ we get the inclusions

$$L^1 i^* \Omega^1_{X/S, \log}|T_U \rightarrow N_{U/Q}|T_U \otimes_A m^{-1} \rightarrow N_{T_U/Q} \otimes_A m^{-1},$$

and

$$\mathcal{O}_{T_U} \cong N_{Q_s/Q}|T_U \otimes_A m^{-1} \rightarrow N_{T_U/Q} \otimes_A m^{-1}.$$ 

Using this we see that the isomorphism $L^1 i^* \Omega^1_{X/S, \log}|T_U \cong \mathcal{O}_{T_U}$ does not depend on the choice of the local imbedding satisfying Lemma.1. Therefore we obtain

$$L^1 i^* \Omega^1_{X/S, \log}|T \cong \mathcal{O}_T.$$ 

This finishes the proof of the lemma. □

Using Lemma.2 we see that

$$\chi(\Omega^1_{X/S, \log, \text{tors}}) = \deg(\text{ch}_s(\Omega^1_{X/S, \log, \text{tors}}) \cap \text{Td}(X/S)) = \deg(\text{ch}_s(C) \cap \text{Td}(X/S)).$$

Let

$$F : 0 \rightarrow F_m \rightarrow \cdots \rightarrow F_0 \rightarrow 0$$
be a complex of locally free coherent sheaves on \(X\), which is exact outside a proper closed set \(Y\). This exact sequence on \(X - Y\) gives a canonical trivialization over \(X - Y\), of the line bundle

\[
\det F = \bigotimes_{0 \leq i \leq m} (\det F_i)^{\otimes (-1)^i}.
\]

This gives a rational section \(s\) of \(\det F\). Denote the image of the divisor of \(s\) in the Chow group \(\mathbb{A}_s Y\) of \(Y\), by \(\gamma\). We will need the following lemma.

**Lemma 3.** We have the equality

\[
\text{ch}^X_{Y, 1}(F) \cap [X] = \gamma, \text{ in } \mathbb{A}_s Y.
\]

**Proof.** Let \(f_i = \text{rank} F_i\), \(F_{-1} = 0\), and

\[
G_i = \text{Grass}_{f_i}(F_i \oplus F_{i-1}), \text{ the Grassmannian of } f_i \text{ planes in } F_i \oplus F_{i-1},
\]

for \(0 \leq i \leq m\). Let \(\xi_i\) be the tautological subbundle, of rank \(f_i\), of \(F_i \oplus F_{i-1}\) on \(G_i\). Let

\[
G = G_m \times \cdots \times G_0, \text{ with the projections } p_i : G \to G_i, \text{ and } \pi : G \to X.
\]

Let

\[
\xi = \sum_{i=0}^m (-1)^i p_i^* \xi_i, \text{ and } \det \xi = \bigotimes_{0 \leq i \leq m} (\det p_i^* \xi_i)^{\otimes (-1)^i}.
\]

Furthermore we denote the kernel of \(d_i : F_i \to F_{i-1}\) by \(K_i\), of rank \(k_i\), and \(H_i = \text{Grass}_{k_i}(F_i)\). Finally, let \(W\) be the closure of \(\varphi(X \times \mathbb{A}^1)\) in \(G \times \mathbb{P}^1\), where \(\varphi : X \times \mathbb{A}^1 \to G \times \mathbb{A}^1\) is the map sending \((x, \lambda)\) to

\[
(\text{Graph}(\lambda \cdot d_{m_i}(x)), \cdots, \text{Graph}(\lambda \cdot d_0(x)), \lambda).
\]

Over \(\varphi((X - Y) \times \mathbb{A}^1)\), \(\det \xi\) has a natural trivialization since

\[
\det p_i^* \xi_i \cong \det \pi^* K_i \otimes \det \pi^* K_{i-1} \text{ over } \varphi((X - Y) \times \mathbb{A}^1).
\]

This trivialization gives a divisor, say \(D\), on \(\varphi(X \times \mathbb{A}^1)\). \(D\) is supported on \(G_Y \times \mathbb{A}^1\).

\[
\pi_*([D_0]) = \gamma \text{ in } \mathbb{A}_s Y
\]

Let \(t \in \mathbb{P}^1 - \{0, \infty\}\). As

\[
\pi_*([D_0]) = \pi_*([D_t]),
\]

we will be done if we can show that \(\pi_*([D_t])\) is equal to \(\text{ch}^X_{Y, 1}(F) \cap [X]\). For this, by definition, we need to show that

\[
\pi_*([D_t]) = \pi_*([\text{ch}_1(\xi) \cap [T]]),
\]

where \([T] = [W_\infty] - [\tilde{X}]\), and \(\tilde{X}\) is the irreducible component of \(W_\infty\) which projects birationally onto \(X\). Note that over \(X - Y\), \(\varphi\) can be extended to a function

\[
\varphi : (X - Y) \times \mathbb{P}^1 \to G \times \mathbb{P}^1
\]

as follows. For \((x, [\lambda_0, \lambda_1]) \in (X - Y) \times \mathbb{P}^1\), and \(0 \leq i \leq m\), let \(\varphi_i(x, [\lambda_0, \lambda_1])\) denote the point of \(G_i\) over \(x\) that corresponds to

\[
\{(v_i, v_{i-1}) \in F_i(x) \oplus K_{i-1}(x) : \lambda_0 v_{i-1} = \lambda_1 v_i \} \subseteq F_i(x) \oplus F_{i-1}(x).
\]
Then
\[ \varphi(x, [\lambda_0, \lambda_1]) = (\varphi_m(x, [\lambda_0, \lambda_1]), \ldots, \varphi_0(x, [\lambda_0, \lambda_1])). \]
And \( \tilde{X} \) is the closure of \( \varphi((X - Y) \times \{\infty\}) \) in \( G \times \{\infty\} \). Now as
\[
\det q_i^* \xi_i \cong \det \pi^* K_i \otimes \det \pi^* K_{i-1} \text{ over } \varphi((X - Y) \times (\mathbb{P}^1 - \{0\})),
\]
det\( \xi \) has a natural trivialization over \( \varphi((X - Y) \times (\mathbb{P}^1 - \{0\})) \), which gives a divisor, say \( D' \), on \( W - W_0 \), supported on \( G_Y \times (\mathbb{P}^1 - \{0\}) \). If \( t \in \mathbb{P}^1 - \{0, \infty\} \) then \( [D_t] = [D'_t] \) in \( A_sW \), being divisors associated to the same line bundle \( \text{det} \xi|_{W_t} \). Noting that \( [D'_t] = [D'_{\infty}] \), we only need to show that
\[
\pi_*(D'_{\infty}) = \pi_*(D'.T) \text{ in } A_s Y,
\]
or that
\[
\pi_*(D'_{\infty}, \tilde{X}) = 0 \text{ in } A_s Y.
\]
If
\[ \Psi : X - Y \to H \]
is the map that sends \( x \) to \( (K_m(x), \ldots, K_0(x)) \), and
\[ t : H \to G \]
is the map that sends \( (V_m, \ldots, V_0) \) to \( (V_m \oplus V_{m-1}, \ldots, V_0) \), then
\[
\tilde{X} = t(\Psi(X - Y)).
\]
However, as \( t^*(D'_{\infty}) \) is the divisor corresponding to a section of
\[
\otimes_{0 \leq i \leq m} \det q_i^* \xi_i \cong \otimes_{0 \leq i \leq m} (\det q_i^* \xi_i \otimes \det q_{i-1}^* \xi_{i-1}) \otimes (\text{det} \xi)^{-1} \cong \mathcal{O}_H,
\]
that is nonzero on \( H_{X - Y} \), where \( \xi_i \) is the tautological sub bundle of \( H_i \), and
\[ q_i : H \to H_i \]
is the projection, we see that
\[
\pi_*(D'_{\infty}, \tilde{X}) = 0.
\]
This finishes the proof of the lemma. \( \square \)

Let \( 0 \to E_m \to \cdots \to E_1 \to \Omega^1_{X/S, \log} \to 0 \) be a resolution of \( \Omega^1_{X/S, \log} \) by locally free sheaves of finite rank. Now consider the complex
\[ E : 0 \to E_m \to \cdots \to E_1 \to \det \Omega^1_{X/S, \log} \to 0, \]
where the map \( E_1 \to \det \Omega^1_{X/S, \log} \) is the composition of the differential
\[ d_1 : E_1 \to \Omega^1_{X/S, \log}, \]
and the canonical map
\[ \alpha : \Omega^1_{X/S, \log} \to \det \Omega^1_{X/S, \log}. \]
Applying Lemma 3 to \( E \) we obtain
\[ \chi(\Omega^1_{X/S, \log, \text{tors}}) = \deg(\text{ch}_s(C) \cap \text{Td}(X/S)) \]
\[ = \deg(\text{ch}_s(E) \cap \text{Td}(X/S)) = \deg(\text{ch}_{s, 2}(E) \cap [X]). \]
We will need the following lemma.

**Lemma 4.** We have \( \text{ch}_{s, 2}(E) \cap [X] = c_{s, 2}(\Omega^1_{X/S, \log}) \cap [X] \) in \( (A_sX_s)_\mathbb{Q} \).
Proof. First of all note that we have \( \text{ch}_{s,1}(E) \cap [X] = 0 \) by Lemma.1, hence
\[
\text{ch}_{s,2}(E) \cap [X] = \frac{c_{s,1}^2(E) \cap [X]}{2} - c_{s,2}(E) \cap [X] = -c_{s,2}(E) \cap [X].
\]
Let \( E' \) denote the complex \( 0 \to E_m \to \cdots \to E_1 \to 0 \), where we put \( E_1 \) in degree 0. We use the notation in Lemma.3, where the objects with \( ' \) refer to those corresponding to \( E' \).
We have
\[
\text{ch}_{s,2}(E) \cap [X] = -c_{s,2}(E) \cap [X] = \pi_*(-c_2(\sum_{i=0}^{m} (-1)^i [p_i^* \xi_i]) \cap [T]),
\]
and
\[
= \pi_*(-c_2(\sum_{i=1}^{m} (-1)^i [p_i^* \xi_i]) - c_1(\sum_{i=1}^{m} (-1)^i [p_i^* \xi_i]) \cdot c_1(p_0^* \xi_0)
- c_2(p_0^* \xi_0) \cap [T]).
\]
Since \( \xi_0 \) is a line bundle this is equal to
\[
\pi_*(-c_2(\sum_{i=1}^{m} (-1)^i [p_i^* \xi_i]) - c_1(\sum_{i=1}^{m} (-1)^i [p_i^* \xi_i]) \cdot c_1(p_0^* \xi_0) \cap [T])
= \pi_*(-c_2(\sum_{i=1}^{m} (-1)^i [p_i^* \xi_i] + c_1^2(p_0^* \xi_0)) \cap [T])
- \pi_*(c_1(\sum_{i=1}^{m} (-1)^i [p_i^* \xi_i]) \cdot c_1(p_0^* \xi_0) \cap [T])
= \pi_*((-c_2(\sum_{i=1}^{m} (-1)^i [p_i^* \xi_i]) + c_1^2(p_0^* \xi_0)) \cap [T])
- (c_{s,1}(E) \cap [X]) \cdot c_1(\xi_0) \quad \text{(since \( p_0 = \pi \)).}
\]
Lemma.3 shows that this is equal to
\[
\pi_*((-c_2(\sum_{i=1}^{m} (-1)^i [p_i^* \xi_i]) + c_1^2(p_0^* \xi_0)) \cap [T])
= \pi_*((-c_1^2(\sum_{i=1}^{m} (-1)^i+1 [p_i^* \xi_i]) + c_2(\sum_{i=1}^{m} (-1)^i+1 [p_i^* \xi_i])
+ c_1^2(p_0^* \xi_0)) \cap [T]).
\]
If \( f : G \to G' \) is the projection, note that we have \( f^* p_i^* \xi_i' \cong p_i^* \xi_i \), for \( 2 \leq i \leq m \). Therefore the last expression is equal to
\[
\pi_*((-c_1^2(\sum_{i=1}^{m} (-1)^i+1 [p_i^* \xi_i]) + c_2(\sum_{i=1}^{m} (-1)^i+1 [f^* p_i^* \xi_i'])
+ c_1(\sum_{i=1}^{m} (-1)^i+1 [f^* p_i^* \xi_i']) \cdot c_1([p_i^* \xi_i] - [f^* p_i^* \xi_i'])
+ c_2([p_i^* \xi_i] - [f^* p_i^* \xi_i']) + c_1^2(p_0^* \xi_0)) \cap [T]).
\]
This is equal to

\[ \pi_*(c_1^2(p_0^*\xi_0) + c_1([p_0^*\xi_0] - [p_1^*\xi_1] + [f^*p_1^*\xi_1]) \cdot c_1([p_1^*\xi_1] - [f^*p_1^*\xi_1]) + c_2([p_1^*\xi_1] - [f^*p_1^*\xi_1]) + c_2^2(p_0^*\xi_0) \cap [T]) + c_{s,2}(\Omega_{X/S,\log}^1) \cap [X] ] \]

Now, since \( p_1^*\xi_1 \) is the tautological subbundle of \( \pi^*E_1 \oplus \pi^*\det\Omega_{X/S,\log}^1 \) on \( G \), we have an exact sequence \( 0 \to p_1^*\xi_1 \to \pi^*E_1 \oplus \pi^*\det\Omega_{X/S,\log}^1 \to Q \to 0 \), with \( Q \) a line bundle. Therefore we obtain

\[ c_2([p_1^*\xi_1] - [\pi^*E_1]) = c_2([\pi^*\det\Omega_{X/S,\log}^1] - [Q]) = c_2(\pi^*\det\Omega_{X/S,\log}^1) - c_1([\pi^*\det\Omega_{X/S,\log}^1] - [Q]) \cdot c_1(Q) - c_2(Q). \]

Since \( \xi_0 \) and \( Q \) are line bundles

\[ c_1(Q) = c_1([\pi^*E_1] + [\pi^*\det\Omega_{X/S,\log}^1] - [p_1^*\xi_1]), \]

and

\[ c_2([p_1^*\xi_1] - [\pi^*E_1]) = -c_1([p_1^*\xi_1] - [\pi^*E_1]) \cdot c_1([\pi^*E_1] + [\pi^*\det\Omega_{X/S,\log}^1] - [p_1^*\xi_1]). \]

Combining this with the expression for \( \text{ch}_{s,2}(E) \cap [X] \) above we obtain

\[ \text{ch}_{s,2}(E) \cap [X] = c_{s,2}(\Omega_{X/S,\log}^1) \cap [X]. \]

Using Lemma 4 we obtain

\[ \chi(\Omega_{X/S,\log,\text{tors}}) = \deg(\text{ch}_{s,2}(E) \cap [X]) = \deg(c_{s,2}(\Omega_{X/S,\log}^1) \cap [X]). \]

Now the logarithmic version of Bloch’s conductor formula ([K-S], Theorem 1.15) says that

\[ \deg(c_{s,2}(\Omega_{X/S,\log}^1) \cap [X]) = -\text{Sw}(X_K/K). \]

Combining this with the above we obtain

\[ \chi(\Omega_{X/S,\log,\text{tors}}) = -\text{Sw}(X_K/K). \]

We now give a consequence of the proof of Theorem 1. In the following if \( C \) is a 0-dimensional subscheme of \( X \), and \( [G] \) and \( [H] \) are curves in \( X \), we denote by \( \deg[C] \) the degree of \( C \) with respect to \( k \), and by \( [G] \cdot [H] \) the intersection number of the curves \( G \) and \( H \). Let \( D \subseteq X \) be a curve supported on the special fiber \( X_s \), \( L \) a line bundle on \( X \), and \( K \) and \( E \) divisors on \( X \) such that \( \mathcal{O}(K) \cong \det\Omega_{X/S}^1 \), the dualizing sheaf of \( X/S \), and \( \mathcal{O}(E) \cong L \). Then we have the following lemma.

**Lemma 5.** We have the equality

\[ \chi(i_*(L|_D)) = [E] \cdot [D] - \frac{1}{2} \chi(\omega_{D/k}), \]

where \( i : D \to X \) is the inclusion.
Proof. By the Riemann-Roch theorem we have
\[ \chi(i_*(L|_D)) = \deg(ch^X_D(L|_D) \cap \text{Td}(X/S)). \]
By applying the Riemann-Roch theorem to the closed immersion \( i : D \to X \) we obtain
\[
ch^X_D(i_*(L|_D)) = i_*(ch^D_D(L|_D) \cdot \text{Td}(N_{D/X})^{-1})
\]
\[
= (1 + c_1(L|_D)) \cdot (1 + \frac{1}{2} c_1(\mathcal{O}(-D)|_D)).
\]
Combining this with the above we obtain
\[ \chi(i_*(L|_D)) = [E] \cdot [D] - \frac{([K] + [D]) \cdot [D]}{2}. \]
Finally using the adjunction formula for \( D \to X \) we obtain the expression in the statement of the lemma.
\[ \Box \]
Let \( Z_1 \) and \( Z_2 \) denote the closed subschemes of \( Z \) consisting of the components of \( Z \) which have codimension 1 and codimension 2 in \( X \) respectively. With this notation, we have the following corollary.

**Corollary 1.** We have the following equality
\[ \text{Sw}(X_K/K) = \chi(\Omega^1_X/K, \log, \text{tors}) + (2[Z_1] - [Z_1, \text{red}]) \cdot [Z_1] - \frac{1}{2} \chi(\omega_{Z_1/k}) - \deg[Z_2]. \]

Proof. Using Theorem 1 we see that we only need to prove the equality
\[ \chi(\Omega^2_X/K, \log) = ([Z_1, \text{red}] - 2[Z_1]) \cdot [Z_1] + \frac{1}{2} \chi(\omega_{Z_1/k}) + \deg[Z_2]. \]
The proof of Lemma 2 shows that
\[ \chi(\Omega^2_X/K, \log) = \chi(\det\Omega^1_X/K, \log|_Z). \]
Then we have
\[ \chi(\Omega^2_X/K, \log) = \chi(\det\Omega^1_X/K, \log|_Z) + \deg[Z_2]. \]
Using the lemma above we obtain that
\[ \chi(\Omega^2_X/K, \log) = [K_{\log}] \cdot [Z_1] - \frac{1}{2} \chi(\omega_{Z_1/k}) + \deg[Z_2], \]
where \( K_{\log} \) is a divisor on \( X \) such that \( \mathcal{O}(K_{\log}) \cong \det\Omega^1_X/K, \log \). Now we also have
\[ [K_{\log}] = [K] - [Z_1] + [Z_1, \text{red}]. \]
To see this we only need to look at the multiplicities at codimension 1 points. Using the notation in the proof of Lemma 2, viewing \( \det\Omega^1_X/K \) as a subsheaf of
\[ \Omega^1_{X_K/K} \cong \Omega^1_{X_K/K, \log}, \]
it is generated by
\[ \frac{1}{t^{a-1}y} \text{dlog}t, \text{ if } y \neq 0, \text{ or by } \]
\[ \frac{1}{t^{a-1}(a + t.x)} \text{d}s, \text{ if } a + t.x \neq 0, \]
in a Zariski neighborhood of a point with \( a \neq 0 \) and \( b = 0 \). Using the similar description of \( \det \Omega^1_{X/S,\log} \) in the proof of Lemma 2, we arrive at the formula as claimed above. Using this and the adjunction formula we obtain that

\[
\chi(\Omega^2_{X/S,\log}) = ([Z,\text{red}] - 2[Z_1]) \cdot [Z_1] + \frac{1}{2} \chi(\omega_{Z_1/k}) + \deg[Z_2].
\]

\[\Box\]

References

[Bloch] Bloch, S.: De Rham Cohomology and Conductors of Curves. Duke Math. Journal 54.2 (1987), 295-308.

[Fulton] Fulton, W.: Intersection theory. Springer-Verlag, Berlin, 1984.

[K-S] Kato, K., Saito, T.: Conductor Formula of Bloch. Preprint, (2001).

[Saito] Saito, T.: Self-Intersection 0-Cycles and Coherent Sheaves on Arithmetic Schemes. Duke Math. Journal 57.2 (1988), 555-578.