An algebraic formula for the index of a 1-form on a real quotient singularity

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Abstract

Let a finite abelian group $G$ act (linearly) on the space $\mathbb{R}^n$ and thus on its complexification $\mathbb{C}^n$. Let $W$ be the real part of the quotient $\mathbb{C}^n/G$ (in general $W \neq \mathbb{R}^n/G$). We give an algebraic formula for the radial index of a 1-form on the real quotient $W$. It is shown that this index is equal to the signature of the restriction of the residue pairing to the $G$-invariant part $\Omega^G_\omega$ of $\Omega_\omega = \Omega^n_{\mathbb{R}^n,0}/\omega \wedge \Omega^{n-1}_{\mathbb{R}^n,0}$. For a $G$-invariant function $f$, one has the so-called quantum cohomology group defined in the quantum singularity theory (FJRW-theory). We show that, for a real function $f$, the signature of the residue pairing on the real part of the quantum cohomology group is equal to the orbifold index of the 1-form $df$ on the preimage $\pi^{-1}(W)$ of $W$ under the natural quotient map.

1 Introduction

For an analytic map $F : (\mathbb{R}^n,0) \to (\mathbb{R}^n,0)$ such that $F^{-1}_C(0) = 0$ ($F_C : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ is the complexification of $F$) one has the famous Eisenbud–Levine–Khimshiashvili algebraic formula for its local degree: [11, 19]. Let $F = (f_1, \ldots, f_n)$ and let $Q_F := E_{\mathbb{R}^n,0}/\langle f_1, \ldots, f_n \rangle$, where $E_{\mathbb{R}^n,0}$ is the ring of germs of analytic functions on $(\mathbb{R}^n,0)$. One has a natural residue pairing on $Q_F$.

In [11, 19] it is shown that the degree of $F$ is equal to the signature of the residue pairing. This can also be interpreted as a formula for the index of the singular point of the vector field $\sum f_i \frac{\partial}{\partial x_i}$ or of the 1-form $\omega = \sum f_i dx_i$. Moreover, the choice of a volume form permits to identify the algebra $Q_F$ (as a vector space) with the space $\Omega_\omega = \Omega^n_{\mathbb{R}^n,0}/\omega \wedge \Omega^{n-1}_{\mathbb{R}^n,0}$.

There exist notions of indices of vector fields and of 1-forms on singular varieties (see e.g. [2, 6]). One of them is the so-called radial index which

*Partially supported by DFG. The work of the second author (Sections 1, 2, 4, 6 and 9) was supported by the grant 16-11-10018 of the Russian Science Foundation. Keywords: group action, real quotient singularity, 1-form, index, signature formula. Mathematical Subject Classification – MSC2010: 14R20, 58K70, 57R18, 32S05.
is defined for a 1-form on an arbitrary singular variety. X. Gómez-Mont and P. Mardešić gave an analogue of the signature formula for the index of a vector field on an isolated (real) hypersurface singularity: [13, 14]. Some formulae expressing the index of a 1-form on a real isolated complete intersection singularity were given in [4, 5]. However, the pairing in [4] was defined in topological terms and the one in [5] was defined in non-local terms on a deformation of the singularity. Thus one can say that an algebraic signature formula for the index of a 1-form on a real singular variety (say on a hypersurface singularity) which can be considered as an analogue of those in [11, 19, 13, 14] does not exist. In the framework of attempts to find such a formula, there were defined (canonical) quadratic forms on analogues of the spaces $Q_F$ and $\Omega_\omega$ for 1-forms on isolated complete intersection singularities [7]. However, these quadratic forms appeared to be in general degenerate and relations of their signatures with indices remained unclear.

Equivariant (with respect to the action of a finite group $G$) versions of indices of $G$-invariant 1-forms were introduced in [8] as elements of the Burnside ring $A(G)$ of the group $G$. In particular, there was defined the equivariant radial index. There was also introduced the notion of an orbifold index of a $G$-invariant 1-form.

Let a finite abelian group $G$ act (linearly) on the space $\mathbb{R}^n$ and thus on its complexification $\mathbb{C}^n$. Let $W$ be the real part of the quotient $\mathbb{C}^n/G$. Note that in general $W \neq \mathbb{R}^n/G$. A (real) 1-form $\eta$ on $W$ defines a $G$-invariant (real) 1-form $\omega = \pi^* \eta$ on $\mathbb{C}^n$ ($\pi : \mathbb{C}^n \to \mathbb{C}^n/G$ is the quotient map). The radial index of the 1-form $\eta$ is equal to the reduction under the group homomorphism $r^{(0)} : A(G) \to \mathbb{Z}$ (see [3]) of the equivariant (radial) index of the $G$-invariant 1-form $\omega$ on the preimage of $W$. Here we give an algebraic formula for the indicated reduction of the equivariant index of a $G$-invariant 1-form on $\pi^{-1}(W)$ and thus an algebraic formula for the radial index of a 1-form on the real quotient $W$. It is shown that this index is equal to the signature of the restriction of the residue pairing to the $G$-invariant part $\Omega^G_\omega$ of $\Omega_\omega$.

For a germ $f$ of a quasihomogeneous function on $(\mathbb{C}^n, 0)$ with an isolated critical point at the origin invariant with respect to an appropriate action of a finite abelian group $G$, H. Fan, T. Jarvis, and Y. Ruan [12] defined the so-called quantum cohomology group. This group is considered as the main object of the quantum singularity theory (FJRW-theory). An analogue of this group can be defined for an arbitrary $G$-invariant function germ $f$. Let us denote it by $\mathcal{H}_{f,G}$. The vector space $\mathcal{H}_{f,G}$ is the direct sum of the spaces $(\Omega^C_{dfg})^G$ over the elements $g$ of the group $G$, where $f^g$ is the restriction of the function $f$ to the fixed point set of $g$, $(\Omega^C_{dfg})^G$ is the $G$-invariant part of the module $\Omega^C_{dfg}$. One has the residue pairing on each of the summands $(\Omega^C_{dfg})^G$ and thus on the space $\mathcal{H}_{f,G}$. If the function $f$ is real, one has a natural real part $\mathcal{H}^R_{f,G}$ of the space $\mathcal{H}_{f,G}$ and the residue pairing is real on it. We derive from the main result of the paper that the signature of the residue pairing on $\mathcal{H}^R_{f,G}$ is equal
to the orbifold index of the 1-form $df$ on $\pi^{-1}(W)$.

## 2 Equivariant index of a 1-form

The Burnside ring $A(G)$ of a finite group $G$ is the Grothendieck ring of finite $G$-sets, see, e.g., [20]. As an abelian group, $A(G)$ is generated by the classes $[G/H]$ for subgroups $H$ of the group $G$. For a topological space $X$ with a $G$-action, let $\chi(0)(X) := \chi(X/G)$ be the Euler characteristic of the quotient and let $\chi^{orb}(X, G)$ be the orbifold Euler characteristic of the $G$-space $X$ (see, e.g., [1, 17]). Applying $\chi(0)$ and $\chi^{orb}$ to finite $G$-sets one gets group homomorphisms $r^{(0)} : A(G) \to \mathbb{Z}$ and $r^{(1)} : A(G) \to \mathbb{Z}$. For an element $a = \sum_{H \subseteq G} a_{[G/H]} \in A(G)$ one has $r^{(0)}a = \sum_{H \subseteq G} a_{H}$. For an abelian group $G$, one has $r^{(1)}[G/H] = |H|$. One has a natural ring homomorphism $r : A(G) \to R_C(G)$ from the Burnside ring to the ring of (complex) representations of $G$ (sending a finite $G$-set to the vector space of functions on it).

Let $(V, 0) \subset (\mathbb{R}^N, 0)$ be a germ of a (real) subanalytic space with an action of a finite group $G$ on it, and let $\omega$ be a $G$-invariant 1-form on $(V, 0)$ (that is, the restriction of a 1-form on $(\mathbb{R}^N, 0)$) with an isolated singular point at the origin. In [8] there was defined the equivariant (radial) index $\text{ind}^G(\omega; V, 0)$ as an element of the Burnside ring $A(G)$.

**Definition:** The orbifold index of the $G$-invariant 1-form $\omega$ is

$$\text{ind}^{orb}(\omega; V, 0) := r^{(1)}\text{ind}^G(\omega; V, 0).$$

Let $(V/G, 0)$ be the quotient of $V$ under the $G$-action and let $\eta$ be a 1-form on $(V/G, 0)$ with an isolated singular point at the origin.

**Proposition 1** One has

$$\text{ind}(\eta; V/G, 0) = r^{(0)}\text{ind}^G(\pi^*\eta; V, 0). \quad (1)$$

**Proof.** For a point $x \in V$, let $G_x := \{g \in G : gx = x\}$ be the isotropy subgroup of $x$, for a conjugacy class $[H] \in \text{Conjsub} G$ of subgroups of $G$, let $V([H]):=\{x \in V : G_x \in [H]\}$. The result [8, Proposition 4.9] says that

$$\text{ind}^G(\pi^*\eta; V, 0) = \sum_{[H] \in \text{Conjsub} G} \text{ind}(\eta; V^{([H])}/G, 0)[G/H],$$

where

$$\sum_{[H] \in \text{Conjsub} G} \text{ind}(\eta; V^{([H])}/G, 0) = \text{ind}(\eta; V/G, 0).$$

( the statement in [8, top of p. 290]).
Let $\mathbb{R}^n$ (and thus its complexification $\mathbb{C}^n$) be endowed with a linear action of the group $G$. For a $G$-invariant analytic germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated critical point at the origin, its Milnor fibre is

$$M_{f,\varepsilon} = f^{-1}(\varepsilon) \cap B_\delta^2,$$

where $0 < \|\varepsilon\| \ll \delta$ are small enough, $B_\delta^2$ is the ball of radius $\delta$ centred at the origin in $\mathbb{C}^n$. It has the homotopy type of a bouquet of $(n - 1)$-dimensional spheres.

Let the germ $f$ be real, that is, it takes real values on $(\mathbb{R}^n, 0) \subset (\mathbb{C}^n, 0)$. In this case one can define the real Milnor fibre (or rather fibres) of the germ $f$.

For $\varepsilon$ real (small enough) let

$$M_{f,\varepsilon}^\mathbb{R} = f^{-1}(\varepsilon) \cap B_\delta^2 \cap \mathbb{R}^n$$

be the real part of the Milnor fibre $M_{f,\varepsilon}$. It is a $C^\infty$-manifold of real dimension $(n - 1)$ with boundary. One can see that, as $C^\infty$-manifolds, the manifolds $M_{f,\varepsilon}$ are the same for positive $\varepsilon$ and also the same for negative $\varepsilon$. Thus there exist essentially two real Milnor fibres: $M_f^+$ and $M_f^-$. The Milnor fibres $M_f^\pm$ carry actions of the group $G$. One can show that the equivariant index

$$\text{ind}_G(df; \mathbb{R}^n, 0)$$

is equal to minus the reduced equivariant Euler characteristic $\chi_G(M_f^-) = \chi_G(M_f^-) - 1$ of the “negative” Milnor fibre: [8, Proposition 4.11]. The function $f$ induces an analytic function $\tilde{f}$ on $\mathbb{C}^n/G$ such that $f = \tilde{f} \circ \pi$.

Proposition 1 gives

$$\text{ind}(d\tilde{f}; \mathbb{R}^n/G, 0) = r^{(0)} \text{ind}_G(df; \mathbb{R}^n, 0).$$

(2)

3 Real quotient singularities

For a $G$-invariant analytic 1-form $\omega = \sum_{i=1}^n A_i(x)dx_i$ ($x := (x_1, \ldots, x_n)$) on $(\mathbb{R}^n, 0)$, let

$$\Omega_{\omega} := \Omega^n_{\mathbb{R}^n, 0}/\omega \wedge \Omega^{n-1}_{\mathbb{R}^n, 0}$$

and let

$$\Omega^C_{\omega} := \Omega^n_{\mathbb{C}^n, 0}/\omega \wedge \Omega^{n-1}_{\mathbb{C}^n, 0}.$$

The residue pairing

$$B^C_{\omega} : \Omega^C_{\omega} \otimes \Omega^C_{\omega} \to \mathbb{C}$$

is defined by

$$B^C_{\omega}(\zeta_1, \zeta_2) = \text{Res} \left[ \varphi_1(x)\varphi_2(x)dx \over A_1 \cdots A_n \right] = \frac{1}{(2\pi i)^n} \int \frac{\varphi_1(x)\varphi_2(x)}{A_1 \cdots A_n}dx,$$

where $dx := dx_1 \wedge \cdots \wedge dx_n$, $\zeta_i = \varphi_i(x)dx$ for $i = 1, 2$ and the integration is along the cycle given by the equations $\|A_k(x)\| = \delta_k$ with positive $\delta_k$ small.
enough. If the 1-form $\omega$ is real, the restriction of the pairing $B^G_\omega$ to $\Omega_\omega$ gives the (real) residue pairing

$$B_\omega : \Omega_\omega \otimes_R \Omega_\omega \to \mathbb{R}.$$  

Let $B^G_\omega : \Omega^G_\omega \otimes_R \Omega^G_\omega \to \mathbb{R}$ be its restriction to the $G$-invariant part $\Omega^G_\omega$ of $\Omega_\omega$. It is non-degenerate as well.

Note that the index of a complex valued 1-form $\omega$ on $(V,0)$ differs by the sign $(-1)^n$ from the index of its real part $\text{Re}\omega$ on $(V,0)$ [10, Remark 2.3] (see also [9]). Therefore, the (complex) equivariant index $\text{ind}^G(\omega; \mathbb{C}^n, 0)$ is $(-1)^n$ times the index $\text{ind}^G(\text{Re}\omega; \mathbb{C}^n, 0)$. It is possible to show that the image of the index $\text{ind}^G(\omega; \mathbb{C}^n, 0)$ under the map $r : A(G) \to R_C(G)$ is equal to the class $[\Omega^C_\omega]$ of the $G$-module $\Omega^C_\omega$: [10]. Therefore, for the $G$-invariant part $(\Omega^C_\omega)^G$ of $\Omega^C_\omega$, one has

$$\dim (\Omega^C_\omega)^G = r(0) \text{ind}^G(\omega; \mathbb{C}^n, 0).$$

Taking into account relations between dimensions of modules in the complex case and signatures of quadratic forms in the real case in the Eisenbud–Levine–Khimshiashvili theory, one might expect that

$$\text{sgn} B^G_\omega = r(0) \text{ind}^G(\omega; \mathbb{R}^n, 0).$$

However, in general this is not the case.

**Example 1** Let $\omega = df$, where $f(x) = x_1^2 + \cdots + x_n^2$ is a $\mathbb{Z}_2$-invariant function on $\mathbb{R}^n$ considered with the action $\sigma x = -x$ ($\sigma$ is the generator of $\mathbb{Z}_2$). One can see that $r(0) \text{ind}^G_\omega(\omega; \mathbb{R}^n, 0) = 1$ (since the 1-form $\omega$ is radial at the origin). The module $\Omega_\omega$ is generated by the 1-form $dx$ which is not $\mathbb{Z}_2$-invariant for $n$ odd. Therefore, in this case $\Omega^G_\omega = 0$ and $\text{sgn} B^G_\omega = 0$.

A reason for the difference between $\text{sgn} B^G_\omega$ and $r(0) \text{ind}^G(\omega; \mathbb{R}^n, 0)$ is the fact that a computation of $\text{sgn} B^G_\omega$ embraces singular points of a deformation of the 1-form $\omega$ which are not real, but become real after factorization by the group $G$. The latter means that the $G$-orbit of such a point is mapped into itself by the complex conjugation.

**Example 2** For the action of $\mathbb{Z}_2$ on $\mathbb{R}^n$ given by $\sigma x = -x$, the quotient $\mathbb{R}^n/\mathbb{Z}_2$ is the semialgebraic variety defined by

$$u^2_{ij} = u_{ii} \cdot u_{jj} \text{ for } 1 \leq i < j \leq n, \quad u_{ii} \geq 0 \text{ for } 1 \leq i \leq n,$$

(here $u_{ij} = x_ix_j$). Its Zariski closure is the cone $u^2_{ij} = u_{ii} \cdot u_{jj}$ (without the inequalities). It is the image under the quotient map $\mathbb{C}^n \to \mathbb{C}^n/\mathbb{Z}_2$ of the subset $\mathbb{R}^n \cup i\mathbb{R}^n \subset \mathbb{C}^n$ (a union of vector subspaces).
Let us describe the Zariski closure \( \overline{\mathbb{R}^n/G} \) of \( \mathbb{R}^n/G \), or rather its preimage under the quotient map \( \mathbb{C}^n \to \mathbb{C}^n/G \). It is shown in [18, Section 2], that, if the order of \( G \) is odd, then \( \mathbb{R}^n/G = \mathbb{R}^n/G \). Otherwise, for an element \( g \in G \) of even order, let \( \mathbb{R}^n_{g\pm} := \{ x \in \mathbb{R}^n \mid gx = \pm x \} \). The subspace \( \mathbb{R}^n_{g\pm} \) can also be described in the following way. Let \( \mathbb{R}^n = \bigoplus_\alpha \mathbb{R}^n_\alpha \) be the decomposition of the \( G \)-module \( \mathbb{R}^n \) into the submodules corresponding to the different irreducible representations \( \alpha \) of the group \( G \). Then \( \mathbb{R}^n_{g\pm} \) is the direct sum of the components \( \mathbb{R}^n_\alpha \) over representations \( \alpha \) such that \( \alpha(g) \) is the multiplication by \((\pm 1)\).

**Proposition 2** One has

\[
\pi^{-1}(\mathbb{R}^n/G) = \bigcup_{g \in G \text{ of even order}} (\mathbb{R}^n_{g+} \oplus i\mathbb{R}^n_{g-}).
\]

**Proof.** Let \( p \in \mathbb{C}^n \) be such that \( Gp = G\overline{p} \), that is, the complex conjugate \( \overline{p} \) of \( p \) satisfies \( \overline{p} = gp \) for a certain \( g \in G \) (in this case \( g \) is of even order). The vector \( u = \frac{1}{2}(p + \overline{p}) \) is real and satisfies the condition \( gu = u \), that is, \( u \in \mathbb{R}^n_{g+} \). The vector \( v = \frac{1}{2i}(p - \overline{p}) \) is also real and \( gv = -v \), that is, \( v \in \mathbb{R}^n_{g-} \). Therefore \( p = u + iv \in \mathbb{R}^n_{g+} \oplus i\mathbb{R}^n_{g-} \).

If \( \omega \) is a real analytic \( G \)-invariant 1-form on \( \mathbb{C}^n \) (that is, it is real on \( \mathbb{R}^n \)), it is real on \( \pi^{-1}(\mathbb{R}^n/G) \) as well. We are now ready to state the main result of the paper.

**Theorem 1** For a real analytic \( G \)-invariant 1-form \( \omega \) one has

\[
\text{sgn } B^G_\omega = r^{(0)}\text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0).
\]

In particular, for a \( G \)-invariant analytic function \( f \) on \( \mathbb{R}^n \) and for its push-forward \( \tilde{f} \) on \( \mathbb{C}^n/G \) one has

\[
\text{sgn } B^G_{df} = \text{ind}(df, \overline{\mathbb{R}^n/G}, 0).
\]

The proof will be given in Sections 4–6 and starts from the following statement.

**Proposition 3** There exists a real \( G \)-invariant deformation \( \tilde{\omega} \) of the 1-form \( \omega \) such that \( \tilde{\omega} \) has only non-degenerate singular points in \( \mathbb{C}^n \) (in a neighbourhood of the origin).

**Proof.** Let \( j : \mathbb{C}^n/G \to \mathbb{C}^N \) be a real embedding of the quotient into the affine space with the coordinates \( z_1, \ldots, z_N \). (It is given by \( z_j = \varphi_j(x) \), where \( \varphi_j \) are generators of the algebra of \( G \)-invariant functions on \( \mathbb{C}^n \).) For generic real \( \lambda_1, \ldots, \lambda_N \) the 1-form \( \tilde{\omega} = \omega + t \cdot (\sum_{j=1}^N \lambda_j dz_j) \) possesses the required property. \( \square \)
4 Laws of conservation of numbers

Let \( \tilde{\omega} \) be a real deformation of the 1-form \( \omega \) (not necessarily given by Proposition 3). The set \( \text{Sing}\tilde{\omega} \) of singular points of the 1-form \( \tilde{\omega} \) is a \( G \)-set. For a singular point \( p \in \text{Sing}\tilde{\omega} \) let \( G_p \) be its isotropy subgroup.

The equivariant index satisfies the following law of conservation of number (see [8, p. 295]).

**Proposition 4** One has

\[
\text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0) = \sum_{[p] \in \text{Sing}\tilde{\omega}/G} I_{G_p}^G(\text{ind}^{G_p}(\tilde{\omega}; \pi^{-1}(\mathbb{R}^n/G), p)),
\]

where the sum is over all the orbits of \( G \) on \( \text{Sing}\tilde{\omega} \), \( p \) is a representative of the orbit \([p]\), \( I_{G_p}^G \) is the induction map \( A(G_p) \to A(G) \) (sending \([G_p/H]\) to \([G/H]\)).

Since \( r(0)I_{G_p}^G = r(0) \), one has the following statement.

**Corollary 1** One has

\[
r(0)\text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0) = \sum_{[p] \in \text{Sing}\tilde{\omega}/G} r(0)I_{G_p}^G(\text{ind}^{G_p}(\tilde{\omega}; \pi^{-1}(\mathbb{R}^n/G), p)).
\]

One has a similar law of conservation of number for the signature of the residue pairing on the \( G \)-invariant part of \( \Omega_\omega \).

**Theorem 2** One has

\[
\text{sgn} B^G_\omega = \sum_{[p] \in \text{Sing}\tilde{\omega}/G} \text{sgn} B^{G_p}_{\tilde{\omega}, p},
\]

where the signature in a summand on the right hand side is computed at the point \( p \).

**Proof.** The possibility to deform the 1-form \( \tilde{\omega} \) to a one with only non-degenerate singular points permits us to assume that already \( \tilde{\omega} \) has this property. It is known that the bilinear form \( B_\omega \) is the limit (when the deformation parameter tends to zero) of the following bilinear form \( B_\tilde{\omega} \) on the space \( L_{\text{Sing}\tilde{\omega}} \) of \( n \)-forms of the form \( \varphi(x)dx \) where \( dx = dx_1 \wedge \cdots \wedge dx_n \) and \( \varphi \) is a real function on \( \text{Sing}\tilde{\omega} \). (A function \( \varphi \) on \( \text{Sing}\tilde{\omega} \) is called real if \( \varphi(z) = \varphi(-z) \) for all \( z \in \text{Sing}\tilde{\omega} \).)

For two real functions \( \varphi \) and \( \psi \), the value of \( B_\omega \) is defined by

\[
B_\omega(\varphi(x)dx, \psi(x)dx) = \sum_{z \in \text{Sing}\tilde{\omega}} \frac{\varphi(z)\psi(z)}{\mathcal{J}_\omega(z)},
\]

where \( \mathcal{J}_\omega \) is the Jacobian of the 1-form \( \tilde{\omega} \): if \( \tilde{\omega} = \sum_{i=1}^n A_i(x)dx_i \), then \( \mathcal{J}_\omega(x) = \det \left( \frac{\partial A_i(x)}{\partial x_j} \right) \).

7
Consider the action of the complex conjugation on \( \text{Sing}\tilde{\omega} \). It commutes with the action of \( G \). Let \( \mathcal{G} \) be the group generated by \( G \) and by the complex conjugation (the direct product of \( G \) and \( \mathbb{Z}_2 \)). The set \( \text{Sing}\tilde{\omega} \) is the disjoint union of the orbits \( \mathcal{G}p \) for representatives of the classes \([p] \in \text{Sing}\tilde{\omega}/\mathcal{G}\). Each orbit \( \mathcal{G}p \) either coincides with \( Gp \) (if \( \overline{p} \in \mathcal{G}p \)) or is the disjoint union of \( Gp \) and \( G\overline{p} \) (if \( \overline{p} \not\in \mathcal{G}p \)). The bilinear form \( B_{\overline{\omega}} \) is the direct sum of its restrictions to the corresponding subspaces \( L_{\mathcal{G}p} \subset L_{\text{Sing}\tilde{\omega}} \) for \([p] \in \text{Sing}\tilde{\omega}/\mathcal{G}\) and the action of \( G \) preserves these spaces. Let \( \det : G \to \mathbb{Z}_2 \) be the natural (determinant) homomorphism. (The action of an element \( g \in G \) belongs to \( \text{SL}(n; \mathbb{R}) \) if and only if \( \det g = 1 \).) Let \( K \) be the kernel of \( \det \).

Let \( \mathcal{G}p = Gp \sqcup \mathcal{G}\overline{p} \). If the isotropy subgroup \( G_p \) of the point \( p \) is not contained in \( K \), then \( L_{\mathcal{G}p}^G = 0 \) and thus the signature of the corresponding summand is equal to zero. If \( G_p \subset K \), the space \( L_{\mathcal{G}p}^G \) is two-dimensional and is generated by the forms \( \varphi_{\text{Re}}d\mathbf{x} \) and \( \varphi_{\text{Im}}d\mathbf{x} \) where the function \( \varphi_{\text{Re}} \) has values \( \pm 1 \) on the points of \( Gp \) (so that \( \varphi_{\text{Re}}(p) = 1 \), \( \varphi_{\text{Re}}(gz) = (\det g)\varphi_{\text{Re}}(z) \), \( \varphi_{\text{Re}}(\overline{z}) = \varphi_{\text{Re}}(z) \)) and the function \( \varphi_{\text{Im}} \) has values \( \pm i \) (so that \( \varphi_{\text{Im}}(p) = i \), \( \varphi_{\text{Im}}(gz) = (\det g)\varphi_{\text{Im}}(\overline{z}) \), \( \varphi_{\text{Im}}(\overline{z}) = -\varphi_{\text{Im}}(z) \)). Equation 5 gives the following matrix of the pairing \( B_{\overline{\omega}} \) on these elements

\[
\begin{pmatrix}
  m(J^{-1} + \overline{J}^{-1}) & m(J^{-1} - \overline{J}^{-1})i \\
  (J^{-1} - \overline{J}^{-1})i & m(J^{-1} + \overline{J}^{-1})
\end{pmatrix},
\]

where \( m \) is the number of elements in the orbit \( Gp \), \( J := J(p) \). The signature of this matrix is equal to zero and therefore pairs of complex conjugate \( G \)-orbits do not contribute to \( \text{sgn} B_{\overline{\omega}}^G \).

Let \( \mathcal{G} = Gp \). If \( G_p \not\subset K \), then \( L_{\mathcal{G}p}^G = 0 \) as above and thus it gives zero impact to \( \text{sgn} B_{\overline{\omega}}^G \). The isotropy group \( G_p \) acts non-trivially on \( 1d\mathbf{x} \) and therefore \( \text{sgn} B_{\overline{\omega}}^{G_p} = 0 \).

Assume now that \( \mathcal{G} = Gp \) and \( G_p \subset K \). Let \( \overline{p} = g_0p \), \( g_0 \in G_p \), that is, \( p \in i\mathbb{R}^{n_0-} \oplus \mathbb{R}^{n_0+} \) in terms of Proposition 2. The space \( \mathbb{R}^{n_0-} \) is even-dimensional if \( \det g_0 = 1 \) and is odd-dimensional if \( \det g_0 = -1 \). The space \( L_{\mathcal{G}p}^G \) is one-dimensional. If \( \det g_0 = 1 \) (or \( \det g_0 = -1 \)), then the space \( L_{\mathcal{G}p}^G \) is generated by the \( n \)-form \( \varphi d\mathbf{x} \), where the (real) function \( \varphi \) on \( G_p \) is such that \( \varphi(gz) = (\det g)\varphi(z) \) for \( g \in G \) and it has values \( \pm 1 \) (or \( \pm i \)) on the points of \( Gp \) so that \( \varphi(p) = 1 \), \( \varphi(\overline{p}) = 1 \) (or \( \varphi(p) = i \), \( \varphi(\overline{p}) = -i \) respectively). If \( \det g_0 = 1 \), then the value of the quadratic form \( B_{\overline{\omega}} \) on \( \varphi \) is equal to \( \frac{m}{J(p)} \), where \( m = |G_p| \). Otherwise (if \( \det g_0 = -1 \)) it is equal to \( -\frac{m}{J(p)} \). Real coordinates on \( \pi^{-1}(\mathbb{R}^n/G) \) at the point \( p \) are \( ix_1, \ldots, ix_k, x_{k+1}, \ldots, x_n \), where \( x_1, \ldots, x_k \) are the coordinates on \( \mathbb{R}^{n_0-} \) and \( x_{k+1}, \ldots, x_n \) are the coordinates on \( \mathbb{R}^{n_0+} \). The value of the Jacobian of \( \tilde{\omega} \) in these coordinates differs from the value of the Jacobian in the coordinates \( x_1, \ldots, x_n \) by the sign \((-1)^k \). Thus the value of the quadratic form \( B_{\overline{\omega},p} \) on the \((G_p\)-invariant!) \( n \)-form \( d\mathbf{x} \) is equal to \( (-1)^k \frac{1}{J(p)} \).

Thus in this case the impact of \( L_{\mathcal{G}p}^G \) to the signature of \( B_{\overline{\omega}} \) also coincides with the signature of \( B_{\overline{\omega},p} \). \( \square \)
5 Reduction to one-dimensional representations

Due to Theorem 2, the statement of Theorem 1 can be verified for $G$-invariant 1-forms which are non-degenerate at the origin. Deforming the 1-form in the class of non-degenerate (and $G$-invariant) ones, we can assume that the 1-form $\omega$ has the shape

$$\omega = \sum_{i,j=1}^{n} \ell_{ij} x_j dx_i.$$  
(We shall call 1-forms of this type linear ones.)

Assume that

$$\mathbb{R}^n = \bigoplus_{\alpha} \mathbb{R}^n_{\alpha}$$  
(6)

is the decomposition of the $G$-module $\mathbb{R}^n$ into parts corresponding to different irreducible representations $\alpha$ of the group $G$. (Each representation $\alpha$ is either one- or two-dimensional.)

Proposition 5 The 1-form $\omega$ is the direct sum of 1-forms $\omega_{\alpha}$ on $\mathbb{R}^n_{\alpha}$.

Proof. Assume that $\sum x_i^2$ is a $G$-invariant quadratic form on $\mathbb{R}^n$. Then the mapping $\mathbb{R}^n \to \mathbb{R}^n$ defined by

$$(x_1, \ldots, x_n) \mapsto (\sum \ell_{1j} x_j, \ldots, \sum \ell_{nj} x_j)$$

is a $G$-invariant operator. According to Schur’s lemma, this operator is the direct sum of operators on the subspaces $\mathbb{R}^n_{\alpha}$. □

Proposition 6 The space of $G$-invariant non-degenerate linear 1-forms on the subspace $\mathbb{R}^n_{\alpha}$ is connected if $\dim \alpha = 2$ and has two components if $\dim \alpha = 1$ and $\mathbb{R}^n_{\alpha} \neq 0$.

Proof. We shall identify linear 1-forms with operators as in Proposition 5. If the representation $\alpha$ is one-dimensional, then all operators $\mathbb{R}^n_{\alpha} \to \mathbb{R}^n_{\alpha}$ are $G$-equivariant. Thus the space of non-degenerate $G$-invariant linear 1-forms on $\mathbb{R}^n_{\alpha}$ can be identified with $\text{GL}(s, \mathbb{R})$, where $s = \dim \mathbb{R}^n_{\alpha}$ and it has two connected components if $s \neq 0$. The complexification of a two-dimensional representation $\alpha$ is the sum of two one-dimensional complex representations $\beta$ and $\overline{\beta}$. A $G$-equivariant operator from the space of the representation $\alpha$ to itself splits into operators from $\beta$ to itself (given by the multiplication by $z_1$) and from $\overline{\beta}$ to itself (given by the multiplication by $z_2$). The fact that the operator is real yields that $z_2 = \overline{z_1}$. Therefore the space of operators from $\alpha$ to itself can be identified with $\mathbb{C}$ and the space of non-degenerate $G$-invariant 1-forms on $\mathbb{R}^n_{\alpha}$ can be identified with $\text{GL}(s, \mathbb{C})$ ($s = \dim \mathbb{R}^n_{\alpha}/2$) which is connected. □

Propositions 5 and 6 imply that it is sufficient to verify the statement of Theorem 1 for a 1-form $\omega$ such that its restriction to $\mathbb{R}^n_{\alpha}$ is $d(\sum x_i^2)$ if $\alpha$ is
two-dimensional and is $d(\sum \pm x_i^2)$ if $\alpha$ is one-dimensional. (In the latter case one can assume that the number of minus signs is $\leq 1$.)

Let, as above, $\omega = \bigoplus \omega_\alpha$, where $\omega_\alpha$ is of type $d(\sum \pm x_i^2)$ (with only plus signs for two-dimensional representations). Let $\omega'$ be the direct sum of the 1-forms $\omega_\alpha$ with one-dimensional $\alpha$ (defined on the direct sum $(\mathbb{R}^n)'$ of the subspaces $\mathbb{R}^n_\alpha$ with $\dim \alpha = 1$).

**Proposition 7** One has

$$\text{sgn } B^G_{\omega'} = \text{sgn } B^G_\omega, \quad r(0) \text{ind}^G(\omega'; \pi^{-1}((\mathbb{R}^n)'/G), 0) = r(0) \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0). \quad (7)$$

**Proof.** Equation (7) is obvious.

To show (8), we shall prove a somewhat stronger statement:

$$\text{ind}^G(\omega'; \pi^{-1}((\mathbb{R}^n)'/G), 0) = \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0).$$

The complement $\pi^{-1}(\mathbb{R}^n/G) \setminus \pi^{-1}((\mathbb{R}^n)'/G)$ is the disjoint union of $(G$-invariant) strata of the form

$$\Xi = \prod_{\text{some } \beta: \dim \beta = 2} (\mathbb{R}^n_\beta)^* \times \prod_{\text{some } \beta: \dim \beta = 2} (i\mathbb{R}^n_\beta)^* \times \prod_{\text{some } \alpha: \dim \alpha = 1} (\mathbb{R}^n_\alpha)^* \times \prod_{\text{some } \alpha: \dim \alpha = 1} (i\mathbb{R}^n_\alpha)^*,$$

where $(\mathbb{R}^n_\cdot)^* := \mathbb{R}^n_\cdot \setminus \{0\}$.

We have the equality

$$\text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0) = \text{ind}^G(\omega'; \pi^{-1}((\mathbb{R}^n)'/G), 0) + \sum_{\Xi} \text{ind}^G(\omega; \Xi, 0).$$

We shall prove that $\text{ind}^G(\omega; \Xi, 0) = 0$ for each $\Xi$. Assume that this is already proven for strata of lower dimensions. The equivariant index $\text{ind}^G(\omega; \Xi, 0)$ on the closure

$$\Xi' = \prod_{\text{some } \beta: \dim \beta = 2} \mathbb{R}^n_\beta \times \prod_{\text{some } \beta: \dim \beta = 2} i\mathbb{R}^n_\beta \times \prod_{\text{some } \alpha: \dim \alpha = 1} \mathbb{R}^n_\alpha \times \prod_{\text{some } \alpha: \dim \alpha = 1} i\mathbb{R}^n_\alpha$$

of the stratum $\Xi$ satisfies the equality

$$\text{ind}^G(\omega; \Xi, 0) = \text{ind}^G(\omega; \Xi', 0) + \sum_{\Sigma \subseteq \Xi} \text{ind}^G(\omega; \Sigma, 0) + \text{ind}^G(\omega; \Xi, 0), \quad (9)$$

where

$$\Xi' = \prod_{\text{some } \alpha: \dim \alpha = 1} \mathbb{R}^n_\alpha \times \prod_{\text{some } \alpha: \dim \alpha = 1} i\mathbb{R}^n_\alpha$$

and the sum is over all strata of lower dimensions inside $\Xi$. This sum is assumed to be equal to zero. The latter summand is equal to $k[G/G_\Xi]$, where
$k$ is an integer and $G_\Xi$ is the isotropy group of each point of $\Xi$. The natural homomorphism $A(G) \to \mathbb{Z}$ sends the class of a $G$-set to its number of elements. The number of elements of $\text{ind}^G(\omega; \Xi, 0)$ or of $\text{ind}^G(\omega; \Xi', 0)$ is the usual (integer valued) index of $\omega$ on $\Xi$ or on $\Xi'$ respectively. These two (usual) indices coincide since the 1-form $\omega$ on $\prod \mathbb{R}^n_\beta \times \prod i\mathbb{R}^n_\beta$ is the differential of a quadratic function with even numbers of plus and minus signs. Therefore $k = 0$ and $\text{ind}^G(\omega; \Xi, 0) = 0$. \hfill \square

Proposition 7 permits to verify the statement of Theorem 1 for the space $\mathbb{R}^n$ with only one-dimensional irreducible representations of $G$. We shall show that one can assume in addition that $\mathbb{R}^n$ does not contain the trivial representation $1$. Let

$$(\mathbb{R}^n)^\prime := \bigoplus_{\dim \alpha = 1} \mathbb{R}^n_\alpha$$

and let $\omega'' := \omega|_{(\mathbb{R}^n)^\prime}$.

**Proposition 8** If $\omega_1 = d(\sum x_i^2)$, then one has

$$\text{sgn} B_{\omega''}^G = \text{sgn} B_{\omega}^G,$$

$$r^{(0)} \text{ind}^G(\omega''; \pi^{-1}((\mathbb{R}^n)^\prime/G), 0) = r^{(0)} \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0).$$

**Proof.** Again (10) is obvious. The equation

$$\text{ind}^G(\omega''; \pi^{-1}((\mathbb{R}^n)^\prime/G), 0) = \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0)$$

follows from the fact that $\pi^{-1}((\mathbb{R}^n)^\prime/G) = \pi^{-1}((\mathbb{R}^n)^\prime/G) \times \mathbb{R}^1$ and the 1-form $\omega_1$ on $\mathbb{R}^1$ is radial. \hfill \square

**Proposition 9** Let $\omega_1 = d(-x_1^2 + \sum_{i>1} x_i^2)$ (that is, it is the differential of a quadratic function with one minus sign), let $\omega'_1 := d(\sum_{i>1} x_i^2)$ and let $\omega' := \omega'_1 \oplus \bigoplus_{\alpha \neq 1} \omega_\alpha$. Then one has

$$\text{sgn} B_{\omega'}^G = -\text{sgn} B_{\omega}^G,$$

$$r^{(0)} \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0) = -r^{(0)} \text{ind}^G(\omega'; \pi^{-1}(\mathbb{R}^n/G), 0),$$

and therefore

$$\text{sgn} B_{\omega}^G = r^{(0)} \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0).$$

**Proof.** Equation (12) is obvious. To prove that

$$\text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0) = -\text{ind}^G(\omega'; \pi^{-1}(\mathbb{R}^n/G), 0),$$

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let us consider the family $\omega(t)$ of 1-forms defined by

$$\omega(t) = d(x_1^3 - tx_1 + \sum_{i>1} x_i^2) \oplus \bigoplus_{\alpha \neq 1} \omega_{\alpha}.$$  

For $t > 0$, the 1-form $\omega_t$ has no singular points on $\pi^{-1}(\mathbb{R}^n/G)$. For $t < 0$ it has one (non-degenerate) singular point of type $\omega$ and one of type $\omega'$. In both cases the equivariant indices of the singular points sum up to the equivariant index of $\omega(0)$.

6 The case of one-dimensional non-trivial representations

Propositions 7, 8, and 9 permit to verify the statement of Theorem 1 for the space $\mathbb{R}^n$ with only non-trivial one-dimensional representations and

$$\omega = d(\sum \pm x_i^2).$$  

(14)

The first idea would be to compute both sides of Equation (3) for the forms of the described type and to compare the results. The left hand side (the signature) is easily computed: it is equal to 0 if the representation of $G$ is not in $\text{SL}(n; \mathbb{R})$ and to $\pm 1 = \text{ind}(\omega, \mathbb{R}^n, 0)$ if the representation is inside $\text{SL}(n; \mathbb{R})$. The problem is that the computation of the right hand side (namely of $\text{ind}_G$: we do not know how it is possible to compute $r(0)\text{ind}_G$ directly) is not complicated for a particular case, however it leads to rather involved combinatorial relations for certain functions on the set of subgroups which we could not understand.

The difficulty to compute $\text{ind}_G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0)$ is related to the fact that the function $f$ such that $\omega = df$ takes both positive and negative values on $\pi^{-1}(\mathbb{R}^n/G)$. If $\omega = df$ where $f(x) > 0$ for $x \in \pi^{-1}(\mathbb{R}^n/G) \setminus \{0\}$, then $\text{ind}_G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0) = 1$.

**Proposition 10** Let $\omega_0 := d(\sum x_i^4)$. Then one has

$$\text{sgn} B_{\omega_0}^G = r(0)\text{ind}_G(\omega_0; \pi^{-1}(\mathbb{R}^n/G), 0) = 1.$$  

**Proof.** The equation

$$\text{ind}_G(\omega_0; \pi^{-1}(\mathbb{R}^n/G), 0) = 1$$  

follows from the fact that the function $\sum x_i^4$ is positive on $\pi^{-1}(\mathbb{R}^n/G) \setminus \{0\}$ (and therefore $\omega_0$ is radial on $\pi^{-1}(\mathbb{R}^n/G)$).

A basis of the space $\Omega_{\omega_0}$ consists of the $n$-forms $\frac{x^k}{k!}d\mathbf{x}$, where $\frac{x^k}{k!} := x_1^{k_1} \cdots x_n^{k_n}$ with $0 \leq k_1 \leq 2$. The Jacobian $J_{\omega_0}$ of $\omega_0$ is $2^n \prod_{i=1}^n x_i^2$. It is known that the
pairing $B_{\omega_0}$ (at least up to an automorphism of $\Omega_{\omega_0}$ defined by the multiplication by an invertible function) can be computed in the following way. Let $\ell$ be a $G$-invariant linear function on $\Omega_{\omega_0}$ such that $\ell(J_{\omega_0}) > 0$. Then

$$B_{\omega_0}(\phi d\mathbf{x}, \psi d\mathbf{x}) = \ell(\phi, \psi).$$

Let us take the function $\ell$ to be equal to zero on all the elements of the basis except $x^2 d\mathbf{x}$ ($\mathbf{x} = (2, \ldots, 2)$) where it is equal to 1. We have to consider the pairing on the subspace generated by $G$-invariant basis (monomial) $n$-forms. The $n$-form $x^k d\mathbf{x}$ ($\mathbf{x} = (1, \ldots, 1)$) is $G$-invariant and one has $B_{\omega_0}(x^k d\mathbf{x}, \mathbf{x} d\mathbf{x}) = 1$. If a basis element $x^k d\mathbf{x}$, $k \neq 1$, is $G$-invariant, then the element $x^{2-k} d\mathbf{x}$ is $G$-invariant as well and we have

$$B_{\omega_0}(x^k d\mathbf{x}, x^{2-k} d\mathbf{x}) = B_{\omega_0}(x^k d\mathbf{x}, x^{2-k} d\mathbf{x}) = 0, \quad B_{\omega_0}(x^k d\mathbf{x}, x^{2-k} d\mathbf{x}) = 1.$$

Therefore $\text{sgn} B_{\omega_0}^G = 1$. □

Now we can finish the proof of Theorem 1. If $G = \{e\}$ (the trivial group) Theorem 1 is just the Eisenbud–Levine–Khimshiashvili theorem. Assume that Theorem 1 is proved for all groups of order smaller than the order of $G$. Let $\omega$ be as in (14).

**Proposition 11** Under the assumptions above, one has

$$\text{sgn} B_{\omega}^G = r^{(0)} \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0).$$

**Proof.** Let us consider the family $\omega(t) = \omega_0 + t\omega$. For $t > 0$ the 1-form $\omega(t)$ has a non-degenerate singular point at the origin of type $\omega$. Theorem 2 gives

$$\text{sgn} B_{\omega_0}^G = \text{sgn} B_{\omega}^G + \sum_{p \neq 0} \text{sgn} B_{\omega(t), p}^G$$

and

$$r^{(0)} \text{ind}^G(\omega_0; \pi^{-1}(\mathbb{R}^n/G), 0) = r^{(0)} \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0) + \sum_{p \neq 0} r^{(0)} \text{ind}^G_{\omega(t)}(\omega_0; \pi^{-1}(\mathbb{R}^n/G), p).$$

By Proposition 10 the left hand sides of (15) and (16) are equal to each other. By the assumption

$$\text{sgn} B_{\omega(t), p}^G = r^{(0)} \text{ind}^G_{\omega(t)}(\omega_0; \pi^{-1}(\mathbb{R}^n/G), 0).$$

Therefore

$$\text{sgn} B_{\omega}^G = r^{(0)} \text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0).$$

□
7 Quantum cohomology group and pairings

Let \((\mathbb{C}^n,0)\) be endowed with an action of a finite abelian group \(G\). Without loss of generality we can assume that the action is linear, that is, it is induced by a representation of \(G\). Let \(f : (\mathbb{C}^n,0) \to (\mathbb{C},0)\) be a germ of a \(G\)-invariant holomorphic function with an isolated critical point at the origin. For \(g \in G\), let \((\mathbb{C}^n)^g\) be the fixed point set (a vector subspace) \(\{x \in \mathbb{C}^n \mid gx = x\}\) and let \(n_g\) be the dimension of \((\mathbb{C}^n)^g\). The restriction of \(f\) to \((\mathbb{C}^n)^g\) will be denoted by \(f^g\). The germ \(f^g : ((\mathbb{C}^n)^g,0) \to (\mathbb{C},0)\) has an isolated critical point at the origin. Let \(M_{f^g} := M_{f^g,\varepsilon}\) (\(\varepsilon\) small enough) be the Milnor fibre of the germ \(f^g\). The group \(G\) acts on \(M_{f^g}\) and thus on its homology and cohomology groups.  

Definition: (cf. \[12\]) The quantum cohomology group of the pair \((f,G)\) is 

\[
\mathcal{H}_{f,G} = \bigoplus_{g \in G} \mathcal{H}_g,
\]

where \(\mathcal{H}_g := H^{n_g-1}(M_{f^g}; \mathbb{C})^G = H^{n_g-1}(M_{f^g}/G; \mathbb{C})\) is the \(G\)-invariant part of the vanishing cohomology group \(H^{n_g-1}(M_{f^g}; \mathbb{C})\) of the Milnor fibre of \(f^g\). If \(n_g = 1\), this means the cohomology group \(\tilde{H}^0(M_{f^g}; \mathbb{C})\) reduced modulo a point. If \(n_g = 0\), one assumes \(H^{-1}(M_{f^g}; \mathbb{C})\) to be one dimensional with the trivial action of \(G\). (This means that the “critical point” of the function of zero variables is considered as a non-degenerate one and thus has Milnor number equal to one.)

Remark 1 In \[12\] the space \(\mathcal{H}_g\) is defined as 

\[
\mathcal{H}_g := H^{n_g}(B^{2n}_\delta, M_{f^g}; \mathbb{C})^G.
\]

However this space is canonically isomorphic to \(H^{n_g-1}(M_{f^g}; \mathbb{C})^G\) with the conventions for \(n_g = 0, 1\) in the definition above.

Let \(\Omega^C_{f^g} := \Omega^{n_g}_{\mathbb{C}^{n_g},0}/df^g \wedge \Omega^{n_g-1}_{\mathbb{C}^{n_g},0}\). One has a canonical isomorphism between \(\Omega^C_{f^g}\) and \(H^{n_g-1}(M_{f^g}; \mathbb{C})\) (for \(n_g = 0, 1\) as well). (A differential \(n_g\)-form \(\eta\) on \((\mathbb{C}^{n_g},0)\) gives a well-defined \((n_g - 1)\)-form \(\eta/d\delta^g\) on \(M_{f^g}\) which corresponds to an element of \(H^{n_g-1}(M_{f^g}; \mathbb{C})\).) This isomorphism respects the action of the group \(G\) on both spaces and thus defines an isomorphism between \((\Omega^C_{f^g})^G\) and \(H^{n_g-1}(M_{f^g}; \mathbb{C})^G\). One also has the residue pairing \(B^C_{f^g} := B^C_{dg^g}\) on the space \(\Omega^C_{f^g}\).

Assume that the function \(f\) is quasihomogeneous, that is, there exist positive integers (the weights) \(w_1, \ldots, w_n\) and \(d\) (the quasidegree) such that

\[
f(\lambda^{w_1}x_1, \ldots, \lambda^{w_n}x_n) = \lambda^d f(x_1, \ldots, x_n).
\]

The classical monodromy transformation of \(f\) is a map \(\phi_f\) from the Milnor fibre \(M_f = M_{f,\varepsilon}\) to itself induced by rotating the value \(\varepsilon\) around the the
origin in \( \mathbb{C} \) counterclockwise. For a quasihomogeneous \( f \), the monodromy transformation can be chosen as the restriction to the Milnor fibre of the (linear) transformation

\[
J_f(x_1, \ldots, x_n) = (\exp(2\pi i \cdot w_1/d)x_1, \ldots, \exp(2\pi i \cdot w_n/d)x_n).
\]

The transformation \( J := J_f \) preserves the function \( f \). Assume that \( J \in G \). (This is the condition on the (abelian) group \( G \) to be admissible: see [12, Proposition 2.3.5].) In this situation one also has the following pairing on the space \( H^{n-1}(M_f; \mathbb{Z}) \cong H^n(\mathbb{C}^n, M_f; \mathbb{Z}) \). Let \( M_f = M_{f+} := f^{-1}(\varepsilon) \) and \( M_{f-} := f^{-1}(\varepsilon) \). One has a well-defined pairing \( \langle \circ \rangle : H_n(\mathbb{C}^n, M_{f+}; \mathbb{Z}) \otimes H_n(\mathbb{C}^n, M_{f-}; \mathbb{Z}) \to \mathbb{Z} \) defined by the "intersection number". Let \( I = \sqrt{J} \) be the linear transformation

\[
I(x_1, \ldots, x_n) = (\exp(\pi i \cdot w_1/d)x_1, \ldots, \exp(\pi i \cdot w_n/d)x_n).
\]

The transformation \( I \) sends \( M_{f+} \) to \( M_{f-} \) and thus defines a map

\[
I_* : H_n(\mathbb{C}^n, M_{f+}; \mathbb{Z}) \to H_n(\mathbb{C}^n, M_{f-}; \mathbb{Z}).
\]

For \( A, B \in H_n(\mathbb{C}^n, M_f; \mathbb{Z}) \) let

\[
\langle A, B \rangle := \langle A \circ I_* B \rangle.
\]

This is a (non-degenerate) pairing (neither symmetric, nor skew-symmetric) on \( H_n(\mathbb{C}^n, M_f; \mathbb{Z}) \). It defines a pairing on the dual space \( H^n(\mathbb{C}^n, M_f; \mathbb{Z}) \) and thus on its subspace \( H^n(\mathbb{C}^n, M_f; \mathbb{Z})^G \cong H^{n-1}(M_f; \mathbb{Z})^G \).

In [12] this pairing is identified with the residue pairing on \( (\Omega^G_{\mathbb{C}})^G \cong H^{n-1}(M_f; \mathbb{C})^G \). However this seems not to be the case. The residue pairing is symmetric, whereas the pairing \( \langle \ , \ \rangle \) on \( H^{n-1}(M_f; \mathbb{C})^G \) described above is either symmetric or skew-symmetric depending on the number of variables. Indeed,

\[
\langle A, B \rangle := \langle A \circ I_* B \rangle = \langle I_* A \circ I_*^2 B \rangle = \langle I_* A \circ J_* B \rangle.
\]

If the (relative) cycle \( B \) is \( J_* \)-invariant (this is the case when \( J \in G \)), then one has

\[
\langle A, B \rangle = \langle I_* A \circ B \rangle = (-1)^n\langle B \circ I_* A \rangle = (-1)^n\langle B, A \rangle.
\]

Thus the pairing \( \langle \ , \ \rangle \) is symmetric if \( n \) is even and skew-symmetric if \( n \) is odd.

### 8 Orbifold index and quantum cohomology group

Let \( (V, 0) \subset \mathbb{C}^N \) be a germ of a complex analytic \( n \)-dimensional variety with an action of a finite group \( G \) on it. In Section 2 we defined the orbifold index \( \text{ind}^\text{orb}(\omega; X, 0) \) for a \( G \)-invariant 1-form on \((X, 0)\), where \((X, 0)\) is a germ of a (real) subanalytic space and \( \omega \) has an isolated singular point at the origin. It can be defined in the same way in the case of a complex valued 1-form on \((V, 0)\) with an isolated singular point at the origin.

From [16, Theorem 2] one can derive the following result.
Theorem 3 One has
\[ \text{ind}(\omega; \mathbb{C}^n/G, 0) = \dim(\Omega^C_{\omega})^G. \]

Proof. This follows from \cite[Theorem 2]{16} since the \( G \)-invariant part corresponds to the trivial representation. \( \square \)

Let \( (\mathbb{C}^n, 0) \) be endowed with a linear action of a finite abelian group \( G \) and let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a germ of a \( G \)-invariant holomorphic function with an isolated critical point at the origin as in Sect. 7. Let \( \bar{f} \) be the corresponding function on \( (\mathbb{C}^n/G, 0) \) \( (f = \bar{f} \circ \pi, \text{ where } \pi \text{ is the quotient map } \mathbb{C}^n \to \mathbb{C}^n/G) \) and let
\[ Q^C_{f,G} := \bigoplus_{g \in G} (\Omega^C_{\bar{f}g})^G. \]

Definition: The orbifold dimension of \( Q^C_{f,G} \) is
\[ \dim^{\text{orb}} Q^C_{f,G} := \sum_{g \in G} (-1)^n \dim(\Omega^C_{\bar{f}g})^G. \]

Theorem 4 One has
\[ \dim^{\text{orb}} Q^C_{f,G} = (-1)^n \text{ind}^{\text{orb}}(df; \mathbb{C}^n, 0) = -\chi^\text{orb}(M_f), \]
where \( \chi^\text{orb}(M_f) \) is the reduced orbifold Euler characteristic of the Milnor fibre of \( f \).

Proof. One has
\[ \text{ind}^{\text{orb}}(\text{Re } df; \mathbb{C}^n, 0) = \sum_{g \in G} \text{ind}(\text{Re } d\bar{f}^g; (\mathbb{C}^n)^g/G, 0). \] \hfill (17)

For the index of the complex 1-form \( df \) this implies (taking into account that the index of a complex valued 1-form differs by the sign \((-1)^n\) from the index of its real part):
\[ (-1)^n \text{ind}^{\text{orb}}(df; \mathbb{C}^n, 0) = \sum_{g \in G} (-1)^n \text{ind}(d\bar{f}^g; (\mathbb{C}^n)^g/G, 0). \]

By Theorem 8 and the definition of the orbifold dimension of \( Q^C_{f,G} \), we get the first equality. The second equality follows from the equality
\[ \text{ind}^G(df; \mathbb{C}^n, 0) = (-1)^{n-1} \chi^G(M_f) \]
(see \cite[p. 297]{8}) by applying the homomorphism \( r^{(1)} \) to both sides of the equation. \( \square \)
Now let $f$ be real and let

$$Q_{f,G} := \bigoplus_{g \in G} \Omega^G_{f,g}, \quad B_{df} := \bigoplus_{g \in G} B^G_{df,g}.$$ 

Then Theorem 1 implies the following corollary.

**Corollary 2** One has

$$\text{sgn } B_{df} = \text{ind}^\text{orb}(df; \pi^{-1}(\mathbb{R}^n/G), 0).$$

From [8, Proposition 4.11] one can derive that

$$\text{ind}^\text{orb}(df; \pi^{-1}(\mathbb{R}^n/G), 0) = -\chi^\text{orb}(M_{f_{\mathbb{R}}}, G),$$

where $f_{\mathbb{R}}$ is the restriction of $f$ to $\pi^{-1}(\mathbb{R}^n/G)$ (a real valued function), $M_{f_{\mathbb{R}}}$ is the “negative” Milnor fibre $f_{\mathbb{R}}^{-1}(-\varepsilon) \cap B_\delta$ of the function $f_{\mathbb{R}}$ ($\varepsilon > 0$ is small enough).

## 9 G-signature and the equivariant index

If $\omega$ is a real analytic $G$-invariant 1-form on $(\mathbb{C}^n, 0)$ with an isolated singular point at the origin (in $\mathbb{C}^n$), the pairing $B_{\omega}$ on $\Omega_\omega = \Omega^m_{\mathbb{R}^n, 0}/\omega \wedge \Omega^{n-1}_{\mathbb{R}^n, 0}$ is also $G$-invariant. In this situation one has a notion of its $G$-signature $\text{sgn}^G B_{\omega}$ as an element of the ring $R_{\mathbb{R}}(G)$ of (real) representations of the group $G$: see, e. g., [15, 3]. One can show that if the order of the group $G$ is odd, the reduction $r_{\mathbb{R}}\text{ind}^G(\omega; \mathbb{R}^n), 0$ of the equivariant index of the 1-form $\omega$ under the natural map $r_{\mathbb{R}} : A(G) \to R_{\mathbb{R}}(G)$ is equal to the $G$-signature $\text{sgn}^G B_{\omega}$ of the quadratic form $B_{\omega}$. This is essentially proved in [15, 3]. (In these papers the statement is formulated in terms of a $G$-degree of a map.) Together with Theorem 1 this permits to conjecture that, for a finite abelian group $G$, the $G$-signature $\text{sgn}^G B_{\omega}$ might be equal to the reduction of the equivariant index $\text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0)$. This is not the case.

**Example 3** Let the group $G = \mathbb{Z}_2$ act on $\mathbb{R}^2$ by $\sigma(x, y) = (-x, -y)$ and let $\omega = df$, where $f(x, y) = x^2 - y^2$. One has $\text{sgn}^G B_{\omega} = -1$. The preimage $\pi^{-1}(\mathbb{R}^n/G)$ is $\mathbb{R}^2 \cup i\mathbb{R}^2$. The equivariant index $\text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0)$ is equal to $1 + \text{ind}^G(\omega; \mathbb{R}^2 \setminus \{0\}, 0) + \text{ind}^G(\omega; i\mathbb{R}^2 \setminus \{0\}, 0)$, where $\text{ind}^G(\omega; \mathbb{R}^2 \setminus \{0\}, 0) = c_1[\mathbb{Z}_2/(e)], \text{ind}^G(\omega; i\mathbb{R}^2 \setminus \{0\}, 0) = c_2[\mathbb{Z}_2/(e)]$, $1 + 2c_1 = \text{ind}(\omega; \mathbb{R}^2, 0) = -1, 1 + 2c_2 = \text{ind}(\omega; i\mathbb{R}^2, 0) = -1$. Thus $\text{ind}^G(\omega; \pi^{-1}(\mathbb{R}^n/G), 0) = 1 - 2(1 + \sigma) = -1 - 2\sigma$, where $\sigma$ is the non-trivial one-dimensional representation of the group $\mathbb{Z}_2$.  

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