NONCOMMUTATIVE WEAK ORLICZ SPACES AND MARTINGALE INEQUALITIES

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Abstract. This paper is devoted to the study of noncommutative weak Orlicz spaces and martingale inequalities. Marcinkiewicz interpolation theorem is extended to include noncommutative weak Orlicz spaces as interpolation classes. In particular, we prove the weak type \( \Phi \)-moment Burkholder-Gundy inequality for noncommutative martingales through establishing a weak type \( \Phi \)-moment noncommutative Khintchine’s inequality for Rademacher’s random variables.

1. Introduction

Recently, the first two named authors proved an \( \Phi \)-moment Burkholder-Gundy inequality for noncommutative martingales in [5], i.e., the noncommutative analogue of the following inequality [7]: Let \( \Phi \) be an Orlicz function with \( 1 < p_\Phi \leq q_\Phi < \infty \). If \( f = (f_n)_{n \geq 1} \) is a \( L_\Phi \)-bounded martingale, then

\[
\int_{\Omega} \Phi\left(\left(\sum_{n=1}^{\infty} |df_n|^2\right)^{1/2}\right) dP \approx \sup_{n \geq 1} \int_{\Omega} \Phi(|f_n|) dP,
\]

where \( df = (df_n)_{n \geq 1} \) is the martingale difference of \( f \) and “ \( \approx \)” depends only on \( \Phi \). Notice that (1.1) is the well-known Burkholder-Gundy inequality for convex powers \( \Phi(t) = t^p \) (see [8]). In their remarkable paper [24], Pisier and Xu proved the noncommutative analogue of the Burkholder-Gundy inequality, which triggered a systematic research of noncommutative martingale inequalities. We refer to a recent book by Xu [27] for an up-to-date exposition of theory of noncommutative martingales. Evidently, the noncommutative \( \Phi \)-moment Burkholder-Gundy inequality implies those for \( L_\Phi \) norms, which were already known as particular cases of more general ones established by the first named author in [4].

In this paper, we continue this line of investigation. We will introduce noncommutative weak Orlicz spaces and prove the associated martingale inequalities. In particular, we will prove that noncommutative weak Orlicz spaces can be renormed as Banach spaces under a mild condition of \( \Phi \), and a weak type version of the \( \Phi \)-moment inequalities for noncommutative

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martingales obtained recently by the first two named authors [5]. To the best of our knowledge, this kind of weak type Φ-moment inequalities is new even in the commutative setting.

In [15], the authors prove the Burkholder-Gundy inequality for weak Orlicz spaces, using the arguments of stopping times and good-λ inequalities developed by Burkholder et al [6]. However, the concepts of stopping times and good-λ inequalities are, up to now, not well defined in the generic noncommutative setting (there are some works on this topic, see [3] and references therein). Instead, interpolation and noncommutative Khinchine inequalities play crucial roles in the proof of the noncommutative Burkholder-Gundy inequality mentioned above. Then, in order to prove the weak type Φ-moment Burkholder-Gundy inequality in the noncommutative setting, we need to prove the associated Khinchine type inequality. There are extensive works on various generalizations of the noncommutative Khinchine inequality in $L_p$-setting [16, 18], for instance, see [23] and references therein. Unfortunately, our weak type Φ-moment Khinchine inequality can not be obtained directly from ones established previously. By adapting natural and classical techniques in [16, 18, 19, 22], we obtain the required one. This is the key point of this paper.

The remainder of this paper is organized as follows. In Section 2 we present some preliminaries and notation on the noncommutative weak $L_p$ and Orlicz spaces. Noncommutative weak Orlicz spaces are presented in Section 3. In Section 4 we establish a Marcinkiewicz-type interpolation theorem for noncommutative weak Orlicz spaces and prove that noncommutative weak Orlicz spaces can be renormed as Banach spaces when Φ satisfies a mild condition. Finally, in Section 5, we will prove the weak type Φ-moment Burkholder-Gundy inequality for noncommutative martingales through establishing a weak type Φ-moment Khinchine’s inequality for Rademacher’s random variables. The style of proof follows mainly the arguments in [5].

In what follows, $C$ always denotes a constant, which may be different in different places. For two nonnegative (possibly infinite) quantities $X$ and $Y$, by $X \preceq Y$ we mean that there exists a constant $C > 0$ such that $X \leq CY$, and by $X \approx Y$ that $X \preceq Y$ and $Y \preceq X$.

2. Preliminaries

2.1. Noncommutative weak $L_p$ spaces. We use standard notation and notions from theory of noncommutative $L_p$-spaces. Our main references are [25] and [27] (see also [25] for more historical references). Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathbb{H}$ with a normal semifinite faithful trace $\tau$. For $0 < p < \infty$ let $L_p(\mathcal{M})$ denote the noncommutative $L_p$ space with respect to $(\mathcal{M}, \tau)$. As usual, we set $L_\infty(\mathcal{M}, \tau) = \mathcal{M}$ equipped with the operator norm. Also, let $L_0(\mathcal{M})$ denote the topological $*$-algebra of measurable operators with respect to $(\mathcal{M}, \tau)$.
For \( x \in L_0(\mathcal{M}) \) we define
\[
\lambda_s(x) = \tau(e_{s,\infty}(|x|)) \quad (s > 0)
\]
and
\[
\mu_t(x) = \inf \{ s > 0 : \lambda_s(x) \leq t \} \quad (t > 0),
\]
where \( e_{s,\infty}(|x|) = e_{(s,\infty)}(|x|) \) is the spectral projection of \( |x| \) associated with the interval \((s, \infty)\). The function \( s \mapsto \lambda_s(x) \) is called the distribution function of \( x \) and \( \mu_t(x) \) the generalized singular number of \( x \). It is easy to check that both functions \( \lambda_s(x) \) and \( \mu_t(x) \) are decreasing and continuous from the right on \((0, \infty)\).

For further information we refer the reader to [10].

For \( 0 < p < \infty \), we have the following Kolmogorov inequality
\[
(2.1) \quad \lambda_s(x) \leq \frac{\|x\|_p^p}{s^p}, \quad \forall s > 0,
\]
for any \( x \in L_p(\mathcal{M}) \). If \( x, y \) in \( L_0(\mathcal{M}) \), then
\[
(2.2) \quad \lambda_{2s}(x + y) \leq \lambda_{s/2}(x) + \lambda_{s/2}(y), \quad \forall s > 0.
\]
We will frequently use these two inequalities in the sequel.

For \( 0 < p < \infty \), the noncommutative weak \( L_p \) space \( L_w^p(\mathcal{M}) \) is defined as the space of all measurable operator \( x \) such that
\[
\|x\|_{L_w^p} := \sup_{t>0} t^{\frac{1}{p}} \mu_t(x) < \infty.
\]
Equipped with \( \|\cdot\|_{L_w^p} \), \( L_w^p(\mathcal{M}) \) is a quasi-Banach space. However, for \( p > 1 \) \( L_w^p(\mathcal{M}) \) can be renormed as a Banach space by
\[
x \mapsto \sup_{t>0} t^{-1+\frac{1}{p}} \int_0^t \mu_s(x)ds.
\]
On the other hand, the quasi-norm admits the following useful description
\[
(2.3) \quad \|x\|_{L_w^p} = \inf \{ c > 0 : t(\mu_t(x)/c)^p \leq 1, \forall t > 0 \}.
\]
Also, we have a description in terms of distribution function as following
\[
(2.4) \quad \|x\|_{L_w^p} = \sup_{s>0} s\lambda_s(x)^{\frac{1}{p}}.
\]

Recall that noncommutative weak \( L_p \) spaces can be presented through noncommutative Lorenz spaces, for details see Dodds et al [9] and Xu [26].

2.2. Noncommutative Orlicz spaces. Recall that noncommutative Orlicz spaces were respectively defined by Kunze [13] in an algebraic way (see also [2] for more general cases) and by Dodds et al [9] and by Xu [26] employing Banach space theory. The second approach based on the concept of Banach function spaces, among other properties, readily indicates similarities with the classical origins. We will take the second approach.

Let \( \Phi \) be an Orlicz function on \([0, \infty)\), i.e., a continuous increasing and convex function satisfying \( \Phi(0) = 0 \) and \( \lim_{t \to \infty} \Phi(t) = \infty \). Recall that \( \Phi \) is said to satisfy the \( \Delta_2 \)-condition if there is a constant \( C \) such that \( \Phi(2t) \leq C\Phi(t) \) for all \( t > 0 \). In this case, we denote by \( \Phi \in \Delta_2 \). It is easy to
check that $\Phi \in \triangle_2$ if and only if for any $a > 0$ there is a constant $C_a > 0$ such that $\Phi(at) \leq C_a \Phi(t)$ for all $t > 0$.

We will work with some standard indices associated to an Orlicz function. Given an Orlicz function $\Phi$. Since $\Phi$ is convex, $\Phi'(t)$ is defined for each $t > 0$ except for a countable set of points in which we take $\Phi'(t)$ as the derivative from the right. Then, we define

$$a_\Phi = \inf_{t > 0} \frac{t\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi = \sup_{t > 0} \frac{t\Phi'(t)}{\Phi(t)}.$$ 

(1) $1 \leq a_\Phi \leq b_\Phi \leq \infty$.

(2) The following characterizations of $a_\Phi$ and $b_\Phi$ hold:

$$a_\Phi = \sup \left\{ p > 0 : t^{-p}\Phi(t) \text{ is non-decreasing for all } t > 0 \right\};$$

$$b_\Phi = \inf \left\{ q > 0 : t^{-q}\Phi(t) \text{ is non-increasing for all } t > 0 \right\}.$$ 

(3) $\Phi \in \triangle_2$ if and only if $b_\Phi < \infty$.

See [20, 21] for more information on Orlicz functions and Orlicz spaces.

For an Orlicz function $\Phi$, the noncommutative Orlicz space $L_\Phi(\mathcal{M})$ is defined as the space of all measurable operators $x$ with respect to $(\mathcal{M}, \tau)$ such that

$$\tau\left(\Phi\left(\frac{|x|}{c}\right)\right) < \infty$$

for some $c > 0$. The space $L_\Phi(\mathcal{M})$, equipped with the norm

$$\|x\|_\Phi = \inf \left\{ c > 0 : \tau\left(\Phi\left(\frac{|x|}{c}\right)\right) < 1 \right\},$$

is a Banach space. If $\Phi(t) = t^p$ with $1 \leq p < \infty$ then $L_\Phi(\mathcal{M}) = L_p(\mathcal{M})$.

Noncommutative Orlicz spaces are symmetric spaces of measurable operators as defined in [9, 26].

3. Noncommutative weak Orlicz spaces

In the sequel, unless otherwise specified, we always denote by $\Phi$ an Orlicz function. Motivated by (2.3), we give the following definition

**Definition 3.1.** For an Orlicz function $\Phi$, define

$$L^w_\Phi(\mathcal{M}) = \{ x \in L_0(\mathcal{M}) : \exists c > 0 \text{ such that } \sup_{t > 0} t\Phi(\mu_t(x)/c) < \infty \},$$

equipped with

$$\|x\|_{L^w_\Phi} = \inf \{ c > 0 : \sup_{t > 0} t\Phi(\mu_t(x)/c) \leq 1, \forall t > 0 \}.$$ 

$L^w_\Phi(\mathcal{M})$ is said to be a noncommutative weak Orlicz space.

**Remark 3.1.** (1) It is easy to check that

$$\|x\|_{L^w_\Phi} = \inf \left\{ c > 0 : \frac{1}{\Phi^{-1}\left(\frac{1}{t}\right)} \mu_t(x)/c \leq 1, \forall t > 0 \right\}.$$ 

(2) For $0 < p < \infty$, if $\Phi(t) = t^p$ then $L^w_\Phi(\mathcal{M})$ is the noncommutative weak $L_p$-space as shown in (2.3).
Note that the noncommutative Orlicz space $L_{\Phi}(M)$ has the following description:

$$L_{\Phi}(M) = \left\{ x \in L_0(M) : \exists c > 0, \int_0^\infty \left[ t \Phi \left( \frac{\mu_t(x)}{c} \right) \right] \frac{dt}{t} \leq 1 \right\}$$

with the norm

$$\| x \|_{L_{\Phi}} = \inf \left\{ c > 0 : \int_0^\infty \left[ t \Phi \left( \frac{\mu_t(x)}{c} \right) \right] \frac{dt}{t} \leq 1 \right\}.$$

This shows that $L_{w\Phi}(M)$ has a close connection with $L_{\Phi}(M)$.

We have the following useful characterization of $L_{w\Phi}(M)$.

**Proposition 3.1.** Let $\Phi$ be an Orlicz function. For any $c > 0$ we have

$$\sup_{t > 0} t \Phi(\mu_t(x)/c) = \sup_{s > 0} \lambda_s(x) \Phi(s/c), \quad \forall x \in L_0(M).$$

Consequently,

$$L_{w\Phi}(M) = \left\{ x \in L_0(M) : \exists c > 0 \text{ such that } \sup_{s > 0} \lambda_s(x) \Phi(s/c) < \infty \right\},$$

and

$$\| x \|_{L_{w\Phi}} = \inf \left\{ c > 0 : \lambda_s(x) \Phi(s/c) \leq 1, \forall s > 0 \right\}.$$

**Proof.** Since $\lambda_s(x) = \lambda_{\mu(x)}(s)$, where $\lambda_{\mu(x)}$ is the distribution function of the function $t \to \mu_t(x)$ with respect the Lebesgue measure in $[0, \infty)$, it reduces to prove that

$$\sup_{t > 0} t \Phi(f^*(t)/c) = \sup_{s > 0} \lambda_f(s) \Phi(s/c),$$

for any nonnegative measurable function $f$ on $(0, \infty)$, where $\lambda_f$ is the distribution function of $f$ with respect to the Lebesgue measure on $[0, \infty)$ and $f^*$ is the rearrangement function of $f$ defined by

$$f^*(t) = \inf \left\{ s > 0 : \lambda_f(s) \leq t \right\}.$$

To this end, we consider a simple function $f = \sum_k a_k \chi_{A_k}$, where $a_k > 0$ and $A_k$ are measurable subsets of $[0, \infty)$ such that $|A_k| < \infty$ and $A_k \cap A_j = \emptyset$ whenever $k \neq j$. An immediate computation yields (3.2) holds for such a function. Since a nonnegative measurable function can be approximated almost everywhere by a sequence of nonnegative simple functions from below, a standard argument concludes (3.2) for any nonnegative measurable function. \qed

We collect some basic properties of noncommutative Orlicz spaces as follows.

**Proposition 3.2.** Let $\Phi$ be an Orlicz function.

1. If $\| x \|_{L_{w\Phi}} > 0$ then

$$\sup_{t > 0} t \Phi(\mu_t(x)/\| x \|_{L_{w\Phi}}) \leq 1 \quad \text{and} \quad \sup_{s > 0} \lambda_s(x) \Phi(s/\| x \|_{L_{w\Phi}}) \leq 1.$$
(2) \( \| x \|_{L^w_\Phi} \) is a quasi-norm on \( L^w_\Phi(\mathcal{M}) \). In particular,
\[
\| x + y \|_{L^w_\Phi} \leq 2(\| x \|_{L^w_\Phi} + \| y \|_{L^w_\Phi}), \quad \forall x, y \in L^w_\Phi(\mathcal{M}).
\]
(3) If \( \| x \|_{L^w_\Phi} \leq 1 \), then
\[
\sup_{t>0} t \Phi(\mu_t(x)) \leq \| x \|_{L^w_\Phi} \quad \text{and} \quad \sup_{s>0} \lambda_s(x) \Phi(s) \leq \| x \|_{L^w_\Phi}.
\]
(4) \( \| x \|_{L^w_\Phi} \leq \| x \|_{L_\Phi} \) for any \( x \in L_\Phi(\mathcal{M}) \). Consequently, \( L_\Phi(\mathcal{M}) \subset L^w_\Phi(\mathcal{M}) \).

**Proof.** (1) By the definition of \( \| x \|_{L^w_\Phi} \), there is a sequence \( \{ c_k \} \subset \mathbb{R}^+ \) such that \( c_k \downarrow \| x \|_{L^w_\Phi} \) and \( t \Phi(\mu_t(x)/c_k) \leq 1 \) for all \( t > 0 \). Since \( \Phi \) is continuous, taking \( k \to \infty \) we obtain the first inequality. The second inequality follows from (3.1) and the first one.

(2) If \( \| x \|_{L^w_\Phi} = 0 \), then there is a sequence \( \{ c_k \} \subset \mathbb{R}^+ \) such that \( c_k \downarrow 0 \) and \( t \Phi(\mu_t(x)/c_k) \leq 1, \forall t > 0 \). Since \( \Phi(t) \to \infty \) as \( t \to \infty \), it is concluded that \( \mu_t(x) = 0, \forall t > 0 \), which implies \( x = 0 \) from the fact that \( \lim_{t \to 0^+} \mu_t(x) = \| x \| \).

It is clear that \( \| a x \|_{L^w_\Phi} = |a| \| x \|_{L^w_\Phi} \). To prove the generalized triangle inequality, we let \( x, y \in L^w_\Phi(\mathcal{M}) \) and \( \| x \|_{L^w_\Phi} = a, \| y \|_{L^w_\Phi} = b \) with \( a, b > 0 \). By (1), we have
\[
t \Phi\left( \frac{\mu_t(x+y)}{2(a+b)} \right) \leq t \Phi\left( \frac{\mu_{t/2}(x) + \mu_{t/2}(y)}{2(a+b)} \right)
\leq \frac{t}{2} \Phi\left( \frac{\mu_{t/2}(x)}{a+b} \right) + \frac{t}{2} \Phi\left( \frac{\mu_{t/2}(y)}{a+b} \right)
\leq \frac{a}{a+b} \frac{t}{2} \Phi\left( \frac{\mu_{t/2}(x)}{a} \right) + \frac{b}{a+b} \frac{t}{2} \Phi\left( \frac{\mu_{t/2}(y)}{b} \right) \leq 1.
\]
Hence, \( \| x + y \|_{L^w_\Phi} \leq 2(a+b) = 2(\| x \|_{L^w_\Phi} + \| y \|_{L^w_\Phi}) \).

(3) If \( \| x \|_{L^w_\Phi} = 0 \), by (2) the inequality holds. Suppose \( \| x \|_{L^w_\Phi} = a \leq 1 \) and \( a \neq 0 \). By (1) we have that \( t \Phi(\mu_t(x)/a) \leq 1, \forall t > 0 \). From the convexity of \( \Phi \) and the fact \( \Phi(0) = 0 \), we have \( \Phi(at) \leq a \Phi(t), \forall t > 0 \), which implies that
\[
\frac{t}{a} \Phi(\mu_t(x)) \leq t \Phi(\mu_t(x)/a) \leq 1, \\forall t > 0.
\]
This gives the first inequality. The second inequality follows from (3.1) and the first one.

(4) Let \( x \in L_\Phi(\mathcal{M}), x \neq 0 \). Then, for any \( t > 0 \),
\[
t \Phi\left( \frac{\mu_t(x)}{\| x \|_{L_\Phi}} \right) \leq \int_0^t \Phi\left( \frac{\mu_s(x)}{\| x \|_{L_\Phi}} \right) ds \leq \int_0^\infty \Phi\left( \frac{\mu_s(x)}{\| x \|_{L_\Phi}} \right) ds \leq 1.
\]
Hence, \( \| x \|_{L^w_\Phi} \leq \| x \|_{L_\Phi} \) and \( L_\Phi(\mathcal{M}) \subset L^w_\Phi(\mathcal{M}) \). \( \square \)

Recall that for measurable operators \( x_n, x \) with respect to \( (\mathcal{M}, \tau) \), \( x_n \) converges to \( x \) in measure if and only if \( \lim_n \mu_t(x_n - x) = 0 \) for all \( t > 0 \). Then, we have
Proposition 3.3. Let $\Phi$ be an Orlicz function.

(1) If $\|x_n - x\|_{L^w_\Phi} \to 0$, then $x_n \to x$ in measure.
(2) $L^w_\Phi(M)$ is a quasi-Banach space.

Proof. (1) Suppose $\|x_n - x\|_{L^w_\Phi} \to 0$. Then there is a sequence $(c_n)$ of positive numbers with $\lim n c_n = 0$ such that

$$t \Phi \left( \frac{\mu(x_n - x)}{c_n} \right) \leq 1, \ \forall t > 0$$

for all $n$. Since $\Phi(t) \to \infty$ as $t \to \infty$, it is concluded that $\lim_n \mu(x_n - x) = 0$ for any $t > 0$. Hence, $x_n \to x$ in measure.

(2) By Proposition 3.2 (2), it suffices to prove that $L^w_\Phi(M)$ is complete. Suppose $x_n \in L^w_\Phi(M)$ such that $\lim_{m,n \to \infty} \|x_n - x_m\|_{L^w_\Phi} = 0$. Then, for any $1 > \varepsilon > 0$ there is an $n_0$ such that $\|x_n - x_m\|_{L^w_\Phi} < \varepsilon$ for all $n, m \geq n_0$. Since $L_0(M)$ is complete in the topology of the convergence in measure, by (1) there exists $x \in L_0(M)$ such that

$$\lim_{n \to \infty} \mu(x_n - x) = 0, \ \forall t > 0$$

Clearly,

$$x_n - x_m \to x_n - x \ \text{in measure}$$

as $m \to \infty$. By Proposition 3.2 (3), for any $n \geq n_0$ we have

$$t \Phi \left( \frac{\mu(x_n - x)}{\varepsilon} \right) \leq \lim_{m \to \infty} t \Phi \left( \frac{\mu(x_n - x_m)}{\varepsilon} \right) \leq \lim \inf_{m \to \infty} \|x_n - x_m\|_{L^w_\Phi} \leq 1,$$

for any $t > 0$. This yields $\|x_n - x\|_{L^w_\Phi} < \varepsilon$ and so $\lim_{n \to \infty} \|x_n - x\|_{L^w_\Phi} = 0$. Also, by (3.3) we obtain that $x \in L^w_\Phi(M)$. Hence, $L^w_\Phi(M)$ is complete. \qed

Remark 3.2. Clearly, $L^w_\Phi(M)$ is rearrangement invariant. Then, by Proposition 3.3 (2) we have that $L^w_\Phi(M)$ is a symmetric quasi-Banach space of measurable operators as defined in [26].

The following are two examples for illustrating noncommutative weak Orlicz spaces.

Example 3.1. Let $\Phi(t) = t^a \ln(1 + t^b)$ with $a > 1$ and $b > 0$. It is easy to check that $\Phi$ is an Orlicz function and $p_\Phi = a$ and $q_\Phi = a + b$. Then, $L^w_\Phi$ can not be coincide with any $L^p_\Phi$.

Example 3.2. Let $\Phi(t) = t^p(1 + c \sin(p \ln t))$ with $p > 1/(1 - 2c)$ and $0 < c < 1/2$. Then, $\Phi$ is an Orlicz function and $p_\Phi = q_\Phi = p$. It is clear that $\Phi$ is equivalent to $t^p$ and hence $L^w_\Phi = L^p_\Phi$.

Let $a = (a_n)$ be a finite sequence in $L^w_\Phi(M)$, we define

$$\|a\|_{L^w_\Phi(M, \ell^2_\Phi)} = \left\| \left( \sum_n |a_n|^2 \right)^{1/2} \right\|_{L^w_\Phi} \ \text{and} \ \|a\|_{L^w_\Phi(M, \ell^2_\Phi)} = \left\| \left( \sum_{n \geq 0} |a_n^*|^2 \right)^{1/2} \right\|_{L^w_\Phi},$$

respectively. Then, we have
Proposition 3.4. \( \| \|_{L^w_\Phi(\mathcal{M},\ell^2_C)} \) and \( \| \|_{L^w_\Phi(\mathcal{M},\ell^2_R)} \) are two quasi-norms on the family of all finite sequences in \( L^w_\Phi(\mathcal{M}) \).

Proof. To see this, let us consider the von Neumann algebra tensor product \( \mathcal{M} \otimes \mathcal{B}(\ell^2) \) with the product trace \( \tau \otimes \text{tr} \), where \( \mathcal{B}(\ell^2) \) is the algebra of all bounded operators on \( \ell^2 \) with the usual trace \( \text{tr} \). \( \tau \otimes \text{tr} \) is a semifinite normal faithful trace. The associated noncommutative weak Orlicz space is denoted by \( L^w_\Phi(\mathcal{M} \otimes \mathcal{B}(\ell^2)) \). Now, any finite sequence \( a = (a_n)_{n \geq 0} \) in \( L^w_\Phi(\mathcal{M}) \) can be regarded as an element in \( L^w_\Phi(\mathcal{M} \otimes \mathcal{B}(\ell^2)) \) via the following map

\[
a \mapsto T(a) = \begin{pmatrix}
a_0 & 0 & \cdots \\
a_1 & 0 & \cdots \\
\vdots & \vdots & \ddots 
\end{pmatrix},
\]

that is, the matrix of \( T(a) \) has all vanishing entries except those in the first column which are the \( a_n \)'s. Such a matrix is called a column matrix, and the closure in \( L^w_\Phi(\mathcal{M} \otimes \mathcal{B}(\ell^2)) \) of all column matrices is called the column subspace of \( L^w_\Phi(\mathcal{M} \otimes \mathcal{B}(\ell^2)) \). Since

\[
\|a\|_{L^w_\Phi(\mathcal{M},\ell^2_C)} = \|T(a)\|_{L^w_\Phi(\mathcal{M} \otimes \mathcal{B}(\ell^2))} = \|T(a)\|_{L^w_\Phi(\mathcal{M} \otimes \mathcal{B}(\ell^2))},
\]

then \( \|\|_{L^w_\Phi(\mathcal{M},\ell^2_C)} \) defines a quasi-norm on the family of all finite sequences of \( L^w_\Phi(\mathcal{M}) \). Similarly, we can show that \( \|\|_{L^w_\Phi(\mathcal{M},\ell^2_R)} \) defines a quasi-norm on the family of all finite sequences of \( L^w_\Phi(\mathcal{M}) \). \( \square \)

We define \( L^w_\Phi(\mathcal{M},\ell^2_C) \) (resp. \( L^w_\Phi(\mathcal{M},\ell^2_R) \)) to be the completion of all finite sequences in \( L^w_\Phi(\mathcal{M}) \) under the norm \( \|\|_{L^w_\Phi(\mathcal{M},\ell^2_C)} \) (resp. \( \|\|_{L^w_\Phi(\mathcal{M},\ell^2_R)} \)). It is clear that a sequence \( a = (a_n)_{n \geq 0} \) in \( L^w_\Phi(\mathcal{M}) \) belongs to \( L^w_\Phi(\mathcal{M},\ell^2_C) \) (resp. \( L^w_\Phi(\mathcal{M},\ell^2_R) \)) if and only if

\[
\|a\|_{L^w_\Phi(\mathcal{M},\ell^2_C)} := \sup_{n \geq 0} \left( \sum_{k=0}^{n} |a_k|^2 \right)^{\frac{1}{2}} \| \Phi \| < \infty
\]

(resp. \( \|a\|_{L^w_\Phi(\mathcal{M},\ell^2_R)} := \sup_{n \geq 0} \left( \sum_{k=0}^{n} |a_k|^2 \right)^{\frac{1}{2}} \| \Phi \| < \infty \)).

\( L^w_\Phi(\mathcal{M},\ell^2_C) \) and \( L^w_\Phi(\mathcal{M},\ell^2_R) \) are evidently quasi-Banach spaces, but we will see in Sect. 4 that they can be renormed as Banach spaces provided \( \Phi \) satisfies a mild condition.

4. INTERPOLATION

The main result of this section is a Marcinkiewicz type interpolation theorem for noncommutative weak Orlicz spaces. We first introduce the following definition.

Definition 4.1. Let \( \mathcal{M} \) (resp. \( \mathcal{N} \)) be a von Neumann algebra with a normal semifinite faithful trace \( \tau \) (resp. \( \nu \)). A map \( T : L_0(\mathcal{M}) \rightarrow L_0(\mathcal{N}) \) is said to be quasilinear if
Then, by the real interpolation of noncommutative 
\[ L_p(\mathcal{M}) \]
for \( i = T \) with equivalent quasi-norms. Since 
1.6.11 of [27]), we have 
\[ x, y \]
Recall that for any 
\[ T(x + y) \leq K(u^*|Tx|u + v^*|Ty|v). \]
In addition, if \( K = 1 \) we call \( T \) a sublinear operator.

This definition of sublinear operators in the noncommutative setting is
due to Q.Xu and first appeared in Ying Hu’s thesis [11] (see also [12]).
Recall that for any \( x, y \in L_0(\mathcal{N}) \) there exist two partial isometrics \( u, v \in \mathcal{N} \) such that
\[
|\alpha x + \beta y| \leq u^*|\alpha x + \beta y|u + v^*|\alpha x + \beta y|v,
\]
(see [1]) and then a linear operator is sublinear. We recall that a quasilinear
operator \( T : L_0(\mathcal{M}) \rightarrow L_0(\mathcal{N}) \) with 0 < \( p \leq q \leq \infty \),
\[
\| Tx \|_L^q \leq C \| x \|_L^p, \quad \forall x \in L_p(\mathcal{M}).
\]

The classical Marcinkiewicz interpolation theorem has been extended to
include Orlicz spaces as interpolation classes by A.Zygmund, A.P.Calderón,
S.Koizumi, I.B.Simonenko, W.Riordan, H.P.Heinig and A.Torchinsky (for
references see [21]). The following result is a noncommutative analogue of
the Marcinkiewicz type interpolation theorem for weak Orlicz spaces.

**Theorem 4.1.** Let \( \mathcal{M} \) (resp. \( \mathcal{N} \)) be a von Neumann algebra with a normal
semifinite faithful trace \( \tau \) (resp. \( \nu \)). Suppose 0 < \( p_0 < p_1 \leq \infty \). Let \( T : L_0(\mathcal{M}) \rightarrow L_0(\mathcal{N}) \) be a quasilinear operator and simultaneously of weak type
\((p_i, p_i)\) for \( i = 0 \) and \( i = 1 \). If \( \Phi \) is an Orlicz function with \( p_0 < a_\Phi \leq b_\Phi < p_1 \), then there exists a constant \( C > 0 \) such that
\[
\sup_{t>0} t\Phi[\mu_t(Tx)] \leq C \sup_{t>0} t\Phi[\mu_t(x)]
\]
for all \( x \in L^w_\Phi(\mathcal{M}) \). Consequently,
\[
\| Tx \|_{L^w_\Phi(\mathcal{N})} \lesssim \| x \|_{L^w_\Phi(\mathcal{M})}, \quad \forall x \in L^w_\Phi(\mathcal{M}).
\]

**Proof.** We chose \( \theta_1, \theta_2, r_0, r_1 \) such that
\[
p_0 < r_0 < a_\Phi < b_\Phi < r_1 < p_1
\]
and
\[
0 < \theta_1, \theta_2 < 1, \quad \frac{1}{r_k} = \frac{(1-\theta_k)}{p_0} + \frac{\theta_k}{p_1}, \quad k = 0, 1.
\]
Then, by the real interpolation of noncommutative \( L_p \) spaces (cf., Corollary
1.6.11 of [27]), we have
\[
(L_{p_0}(\mathcal{M}), L_{p_1}(\mathcal{M}))_{\theta_k,q} = L_{r_k,q}(\mathcal{M}), \quad k = 0, 1,
\]
with equivalent quasi-norms. Since \( T \) is simultaneously of weak type \((p_i, p_i)\)
for \( i = 0 \) and \( i = 1 \), we obtain that
\[
\| Tx \|_{L^w_{p_0}} \leq A_0 \| x \|_{L^w_{p_0}}, \quad \forall x \in L^w_{p_0}(\mathcal{M}),
\]
and

\[(4.5) \quad \|Tx\|_{L^p_\alpha} \leq A_1 \|x\|_{L^p_\alpha}, \quad \forall x \in L^w_{p_1}(\mathcal{M}),\]

where \(A_0, A_1\) are both constants which depend only on \(p_0, p_1\), and the weak type \((p_i, p_i)\) norms of \(T\) for \(i = 0\) and \(i = 1\).

Now, take \(x \in L^w_{p_0}(\mathcal{M})\). For any \(\alpha > 0\) let \(x = x_0^\alpha + x_1^\alpha\), where \(x_0^\alpha = xe_{(\alpha, \infty)}(|x|)\). Since \(t^{-\alpha} \Phi(t)\) is an increasing function in \((0, \infty)\), by Proposition 3.2 (1) and (4.4) we have

\[
\lambda_\alpha(Tx_0^\alpha) \leq \alpha^{-\tau_0} \|Tx_0^\alpha\|_{L^w_\alpha}^{\tau_0} \\
\leq \alpha^{-\tau_0} A_0^{\tau_0} \|x_0^\alpha\|_{L^w_\alpha}^{\tau_0} \\
= \alpha^{-\tau_0} A_0^{\tau_0} \sup_{t > 0} t^{\tau_0} \lambda_t(x_0^\alpha) \\
\leq A_0^{\tau_0} \sup_{t > \alpha} \left(\frac{t}{\alpha}\right)^{\tau_0} \lambda_t(x) \\
\leq A_0^{\tau_0} \sup_{t > \alpha} \frac{\Phi(t)}{\Phi(\alpha)} \lambda_t(x) \\
\leq A_0^{\tau_0} \frac{\alpha^{\tau_0}}{\Phi(\alpha)} \sup_{t > 0} \Phi(t) \lambda_t(x).
\]

Also, since \(t^{-\alpha} \Phi(t)\) is a decreasing function in \((0, \infty)\), by Proposition 3.2 (1) and (4.5) we obtain similarly

\[
\lambda_\alpha(Tx_1^\alpha) \leq \frac{A_1^{\tau_1}}{\Phi(\alpha)} \sup_{t > 0} \Phi(t) \lambda_t(x).
\]

On the other hand, by the sublinearity of \(T\) and the basic properties of the distribution function \(\lambda(|x|)\), such as \(\lambda(a^*a) = \lambda(aa^*)\) and \(\lambda_{\alpha + \beta}(x + y) \leq \lambda_\alpha(x) + \lambda_\beta(y)\) for any \(x, y \geq 0\), we have that

\[
\lambda_{2K\alpha}(Tx) \leq \nu(E_{(2K\alpha, \infty)}[K(u^*|Tx_0^\alpha|u + v^*|Tx_1^\alpha|v)]) \\
\leq \lambda_\alpha(u^*|Tx_0^\alpha|u) + \lambda_\alpha(v^*|Tx_1^\alpha|v) \\
\leq \lambda_\alpha(|Tx_0^\alpha|) + \lambda_\alpha(|Tx_1^\alpha|),
\]

where the first and third inequalities use the fact that \(0 \leq a \leq b\) implies \(E_{(\alpha, \infty)}(a)\) is equivalent to a subprojection of \(E_{(\alpha, \infty)}(b)\) (e.g., [10]). Then, by (4.6) we have

\[
\lambda_{2K\alpha}(Tx) \leq \frac{A_0^{\tau_0}}{\Phi(2K\alpha)} \sup_{t > 0} \Phi(t) \lambda_t(x) + \frac{A_1^{\tau_1}}{\Phi(2K\alpha)} \sup_{t > 0} \Phi(t) \lambda_t(x) \\
= \frac{C}{\Phi(2K\alpha)} \sup_{t > 0} \Phi(t) \lambda_t(x).
\]

By Proposition 3.1 we obtain the desired inequality (4.2). \(\square\)

**Remark 4.1.** We set

\[
L_p(\mathcal{N})_{\text{Her}} = \{x \in L_p(\mathcal{N}) : x^* = x\}.
\]
If $T$ is simultaneously of weak types $L_{p_i}(\mathcal{M})_{\text{Her}} \rightarrow L_{p_i}(\mathcal{N})_{\text{Her}}$ for $i = 0$ and $i = 1$, then the conclusion of Theorem 4.1 holds for any hermitian operator $x \in L_{\Phi}(\mathcal{M})$. The proof is the same as above and omitted.

We have the following corollaries.

**Corollary 4.1.** Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a von Neumann algebra with a normal semifinite faithful trace $\tau$ (resp. $\nu$). Suppose $0 < p_0 < p_1 \leq \infty$. Let $T : L_0(\mathcal{M}) \rightarrow L_0(\mathcal{N})$ be a quasilinear operator and simultaneously of strong type $(p_i, p_i)$ for $i = 0$ and $i = 1$, i.e.,

$$\|Tx\|_{L_{p_0}} \lesssim \|x\|_{L_{p_0}}, \quad \forall x \in L_{p_0}(\mathcal{M}),$$

and

$$\|Tx\|_{L_{p_1}} \lesssim \|x\|_{L_{p_1}}, \quad \forall x \in L_{p_1}(\mathcal{M}).$$

Let $\Phi$ be an Orlicz function with $p_0 < a_\Phi \leq b_\Phi < p_1$. Then, the conclusion of Theorem 4.1 holds.

**Proof.** If $T$ is of strong type $(p, p)$, by the Kolmogorov inequality (2.1) we immediately conclude that $T$ is of weak type $(p, p)$.

An appeal to Theorem 4.1 yields the result. \hfill \Box

**Corollary 4.2.** Let $\Phi$ be an Orlicz function with $1 < a_\Phi \leq b_\Phi < \infty$. Then

$$\|x\|_{L_w^\Phi} \approx \inf \left\{ c > 0 : t_\Phi \left( \frac{1}{t} \int_0^t \mu_s(x) ds / c \right) \leq 1, \forall t > 0 \right\}.$$

Consequently, $L_w^\Phi(\mathcal{M})$ can be renormed as a Banach space.

**Proof.** Since $\mu_t(x)$ is decreasing in $t \in (0, \infty)$, we immediately get

$$\|x\|_{L_w^\Phi} \leq \inf \left\{ c > 0 : t_\Phi \left( \frac{1}{t} \int_0^t \mu_s(x) ds / c \right) \leq 1, \forall t > 0 \right\}.$$

Conversely, let $1 < p \leq \infty$. Define $S : f(t) \mapsto \frac{1}{t} \int_0^t |f(s)| ds$ for $f \in L_p(0, \infty)$. Then, by the classical Hardy-Littlewood inequality there exists a constant $A_p > 0$ such that

$$\|Sf\|_p \leq C_p \|f\|_p, \quad \forall f \in L_p(0, \infty).$$

Consequently,

$$\|Tx\|_p \leq A_p \|x\|_p, \quad \forall x \in L_p(\mathcal{M}),$$

where

$$Tx := \frac{1}{t} \int_0^t \mu_s(x) ds, \quad x \in L_0(\mathcal{M}).$$

Since $T$ is sublinear, by Corollary 4.1 we obtain the reverse inequality and hence (4.7) holds. \hfill \Box

**Corollary 4.3.** Let $\Phi$ be an Orlicz function with $1 < a_\Phi \leq b_\Phi < \infty$. Let $p_\Phi^w$ and $q_\Phi^w$ be respectively the lower and upper Boyd indices of $L_w^\Phi(\mathcal{M})$. Then,

$$a_\Phi \leq p_\Phi^w \leq q_\Phi^w \leq b_\Phi.$$
Proof. Let $1 \leq p < a_\Phi \leq b_\Phi < q < \infty$. Suppose $T$ is a linear operator defined on $L_{p,1}[0,\infty) + L_{q,1}[0,\infty)$, which is simultaneously of weak type $(p,p)$ and weak type $(q,q)$ in the sense of [14]. Take $p_0, q_0$ such that $p < p_0 < a_\Phi \leq b_\Phi < q_0 < q$. Then by Theorem 2.b.11 in [14], we have that $T$ is simultaneously of strong type $(p_0,p_0)$ and strong type $(q_0,q_0)$. Using Corollary 4.1, we get $p < p_0^w \leq q_0^w < q$. This completes the proof. □

5. MARTINGALE INEQUALITIES

In this section, we will prove the weak type $\Phi$-moment versions of martingale transformations, Stein’s inequalities, Khintchine’s inequalities for Rademacher’s random variables, and Burkholder-Gundy martingale inequalities in the noncommutative setting. We mainly follows the arguments in [5] using Theorem 4.1 and Corollary 4.1.

In the sequel, without otherwise specified, we always denote by $\mathcal{M}$ a finite von Neumann algebra with a normalized normal faithful trace $\tau$. Let $(\mathcal{M}_n)_{n \geq 0}$ be an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ such that $\bigcup_{n \geq 0} \mathcal{M}_n$ generates $\mathcal{M}$ (in the $w^*$-topology). $(\mathcal{M}_n)_{n \geq 0}$ is called a filtration of $\mathcal{M}$. The restriction of $\tau$ to $\mathcal{M}_n$ is still denoted by $\tau$. Let $\mathcal{E}_n = \mathcal{E}(\cdot|\mathcal{M}_n)$ be the conditional expectation of $\mathcal{M}$ with respect to $\mathcal{M}_n$.

A non-commutative $L^w_\Phi$-martingale with respect to $(\mathcal{M}_n)_{n \geq 0}$ is a sequence $x = (x_n)_{n \geq 0}$ such that $x_n \in L^w_\Phi(\mathcal{M}_n)$ and

$$\mathcal{E}_n(x_{n+1}) = x_n$$

for any $n \geq 0$. Let $\|x\|_{L^w_\Phi} = \sup_{n \geq 0} \|x_n\|_{L^w_\Phi}$. If $\|x\|_{L^w_\Phi} < \infty$, then $x$ is said to be a bounded $L^w_\Phi$-martingale.

For convenience, we denote the weak type $\Phi$-moment of $x$ by

$$\|x\|_{\Phi_w(\mathcal{M})} := \sup_{t>0} t\Phi(\mu_t(x)), \quad x \in L_0(\mathcal{M}).$$

We write $\|x\|_{\Phi_w} = \|x\|_{\Phi_w(\mathcal{M})}$ in short when no confusion occurs.

Let $\alpha = (\alpha_n) \subset \mathbb{C}$ be a sequence. Recall that a map $T_\alpha$ on the family of martingale difference sequences defined by $T_\alpha(dx) = (\alpha_n dx_n)$ is called the martingale transform of symbol $\alpha$. It is clear that $(\alpha_n dx_n)$ is indeed a martingale difference sequence. The corresponding martingale is $T_\alpha(x) = \sum_n \alpha_n dx_n$.

**Theorem 5.1.** Let $\alpha = (\alpha_n) \subset \mathbb{C}$ be a bounded sequence and $T_\alpha$ the associated martingale transform. Let $\Phi$ be a Orlicz function such that $1 < a_\Phi \leq b_\Phi < \infty$. Then, for all bounded $L^w_\Phi$-martingales $x = (x_n)$, we have

$$\|T_\alpha x\|_{\Phi_w} \lesssim \|x\|_{\Phi_w},$$

where $\lesssim$ depends only on $\Phi$ and $\sup_n |\alpha_n|$. Consequently,

$$\|x\|_{\Phi_w} \approx \|\sum_n \varepsilon_n dx_n\|_{\Phi_w}, \quad \forall \varepsilon_n = \pm 1,$$

for any bounded $L^w_\Phi$-martingales $x = (x_n)$, where “$\approx$” depends only on $\Phi$. 
Theorem 5.2. Let $\Phi$ be an Orlicz function with $1 < a_\Phi \leq b_\Phi < \infty$. Then,

$$\left\| \left( \sum_n |E_n(a_n)|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w} \lesssim \left\| \left( \sum_n |a_n|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w}$$

(5.3)

for any finite sequence $(a_n)$ in $L^w_\Phi(M)$. Similarly, we have

$$\left\| \left( \sum_n |E_n(a_n^*)|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w} \lesssim \left\| \left( \sum_n |a_n|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w}$$

(5.4)

for any finite sequence $(a_n)$ in $L^w_\Phi(M)$.

The following is the weak type $\Phi$-moment version of noncommutative Kintchine's inequalities for Rademacher's sequences.

Theorem 5.3. Let $\Phi$ be an Orlicz function and $\{\varepsilon_k\}$ a Rademacher's sequence on a probability space $(\Omega, P)$.

1. If $1 < a_\Phi \leq b_\Phi < 2$, then for any finite sequence $\{x_k\}$ in $L^w_\Phi(M)$

$$\left\| \sum_k \varepsilon_k x_k \right\|_{\Phi_w(L^\infty(\Omega) \otimes M)} \approx \inf \left\{ \left\| \left( \sum_k |y_k|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w(M)} + \left\| \left( \sum_k |z_k|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w(M)} \right\}$$

where the infimum runs over all decompositions $x_k = y_k + z_k$ with $y_k, z_k \in L^w_\Phi(M)$ and "$\approx$" depends only on $\Phi$.

2. If $2 < a_\Phi \leq b_\Phi < \infty$, then for any finite sequence $\{x_k\}$ in $L^w_\Phi(M)$,

$$\left\| \sum_k \varepsilon_k x_k \right\|_{\Phi_w(L^\infty(\Omega) \otimes M)} \approx \left\| \left( \sum_k |x_k|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w(M)} + \left\| \left( \sum_k |x_k^*|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w(M)}$$

(5.6)

where "$\approx$" depends only on $\Phi$.
Proof. By the argument in [5], we need only to prove the lower estimate of (5.5). By the analogue argument in [18], we are reduced to show for any finite sequence \( \{x_k\} \) in \( L^w_\Phi(M) \),

\[
\inf \left\{ \left\| \left( \sum_{k=0}^n |y_k|^2 \right)^{\frac{2}{\Phi}} \right\|_{\Phi_w} + \left\| \left( \sum_{k=0}^n |z_k|^2 \right)^{\frac{2}{\Phi}} \right\|_{\Phi_w} \right\}
\lesssim \left\| \sum_{k=0}^n x_k z^k \right\|_{\Phi_w(L^\infty(T) \otimes M)},
\]

(5.7)

where the infimum runs over all decomposition \( x_k = y_k + z_k \) with \( y_k \) and \( z_k \) in \( L^w_\Phi(M) \).

To this end, we consider \( N = L^\infty(T) \otimes M \) equipped with the tensor product trace \( \nu = \int \otimes \tau \) and \( A = H^\infty(T) \otimes M \). Then, \( A \) is a finite maximal subdiagonal algebras of \( N \) with respect to \( E = \int \otimes I_M : N \to M \) (e.g., see [25]). Since \( L^1(N) = L^1(T, L^1(M)) \) we can define Fourier coefficients for any \( f \in L^1(N) \) by

\[
\hat{f}(n) = \frac{1}{2\pi} \int_T f(z) \bar{z}^n dm(z), \quad \forall n \in \mathbb{Z},
\]

where \( dm \) is the normalized Lebesgue measure on \( T \). It is easy to check that \( A = \{ f \in N : \hat{f}(n) = 0, \forall n < 0 \} \).

For any \( n \in \mathbb{Z} \) we define \( F_n \) the linear mapping such that \( F_n(f) = \hat{f}(n) \) for any \( L^1(N) \). Then \( F_n \) is both a contract from \( L^1(N) \) into \( L^1(M) \) and from \( N \) into \( M \). Hence, for an Orlicz function \( \Phi \) with \( 1 < a_\Phi \leq b_\Phi < \infty \), by Corollary 4.1 we have

\[
\| \hat{f}(n) \|_{\Phi_w} \lesssim \| f \|_{\Phi_w}, \quad \forall f \in L^w_\Phi(N),
\]

(5.8)

for any \( n \in \mathbb{Z} \).

Lemma 5.1. Let \( \Phi \) be an Orlicz function with \( 1 < a_\Phi \leq b_\Phi < \infty \). For any finite sequence \( \{f_k\} \) in \( L^w_\Phi(N) \) and any \( n \in \mathbb{Z} \), we have

\[
\left\| \sum_k |\hat{f}_k(n)|^2 \right\|_{\Phi_w(\mathcal{M})} \lesssim \left\| \sum_k |f_k|^2 \right\|_{\Phi_w(N)}.
\]

Proof. Let \( 1 \leq k \leq K \). Applying (5.8) on \( M_K(\mathcal{M}) \) instead of \( \mathcal{M} \) with

\[
f = \sum_{k=1}^K E_k \otimes f_k = \begin{pmatrix} f_1 & 0 & \ldots & 0 \\ f_2 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ f_K & 0 & \ldots & 0 \end{pmatrix}_{K \times K}
\]

yields the required inequality. \( \square \)
For an Orlicz function $\Phi$, we denote by $H^w_\Phi(A)$ the completion of $A$ under the quasinorm $\| \cdot \|_{L^w_\Phi(N)}$. If $\Phi$ is an Orlicz function with $1 < a_\Phi \leq b_\Phi < \infty$, then by Corollary 4.3 we have $L^w_\Phi(N) \subset L^1(N)$ and

$$H^w_\Phi(A) = \{ f \in L^w_\Phi(N) : \hat{f}(n) = 0, \forall n < 0 \}.$$

In this case,

$$H^1(A) \cap L^w_\Phi(N) = H^w_\Phi(A).$$

**Lemma 5.2.** Let $\Phi$ be an Orlicz function with $1 < a_\Phi \leq b_\Phi < \infty$. Let $\Phi(2)(t) = \Phi(t^2)$. Then, for any $f \in H^w_\Phi(A)$ and $\varepsilon > 0$, there exist two functions $g, h \in H^w_{\Phi(2)}(A)$ such that $f = gh$ with

$$\max \left\{ \| |g|^2 \|_{\Phi_u(N)}, \| |h|^2 \|_{\Phi_u(N)} \right\} \lesssim \| |f| \|_{\Phi_u(N)} + \varepsilon.$$

**Proof.** By slightly modifying the proof of Lemma 4.1 in [5] we can prove this lemma and omit the details. □

**Lemma 5.3.** Let $\Phi$ be an Orlicz function with $2 < a_\Phi \leq b_\Phi < \infty$. Let $\{ I_n = (3^n, 3^n) : n \in \mathbb{N} \}$ and $\triangle_n$ the Fourier multiplier by the indicator function $\chi_{I_n}$, i.e.

$$\triangle_n(f)(z) = \sum_{k \in I_n} \hat{f}(k)z^k$$

for any trigonometric polynomial $f$ with coefficients in $L^w_\Phi(M)$. Then,

$$\left\| \left( \sum_{n} \triangle_n(f)^* \triangle_n(f) \right)^{\frac{1}{2}} \right\|_{\Phi_u(N)} \lesssim \| |f| \|_{\Phi_u(N)},$$

for any $f \in H^w_\Phi(N)$.

**Proof.** The proof can be done as similar to the one of Lemma 4.2 in [5] by using Corollary 4.1 and the details are omitted. □

Now, we are ready to prove (5.7). Indeed, the proof can be obtained by using Lemmas 5.1, 5.2 and 5.3 as similar to the one of Theorem 4.1 in [5]. We omit the details. □

**Remark 5.1.** (1) Note that Khintchine’s inequality is valid for $L_1$-norm in both commutative and noncommutative settings (cf., [18]). We could conjecture that the right condition in Theorem 5.3 (1) should be $b_\Phi < 2$ without the additional restriction one $1 < a_\Phi$. However, our argument seems to be inefficient in this case. We need new ideas to approach it.

(2) Evidently, the weak type $\Phi$-moment Khintchine inequalities in Theorem 5.3 imply those for $L^w_\Phi$ norms, which, by Corollary 4.2, can be considered as particular cases of more general ones in [17] and then in [19, 22].

Now, we are in a position to state and prove the weak type $\Phi$-moment version of noncommutative Burkholder-Gundy martingale inequalities.
Theorem 5.4. Let $\mathcal{M}$ be a finite von Neumann algebra with a normalized normal faithful trace $\tau$ and $(\mathcal{M}_n)_{n \geq 0}$ an increasing filtration of subalgebras of $\mathcal{M}$. Let $\Phi$ be an Orlicz function and $x = \{x_n\}_{n \geq 0}$ a noncommutative $L^w_{\Phi}$-martingale with respect to $(\mathcal{M}_n)_{n \geq 0}$.

(1) If $1 < a_\Phi \leq b_\Phi < 2$, then

$$\|x\|_{\Phi_w} \approx \inf \left\{ \left\| \left( \sum_{n=0}^{\infty} |dy_n|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w} + \left\| \left( \sum_{n=0}^{\infty} |dz_n|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w} \right\}$$

where the infimum runs over all decomposition $x_n = y_n + z_n$ with $\{dy_n\}$ in $L^w_{\Phi}(\mathcal{M}, l^2_C)$ and $\{dz_n\}$ in $L^w_{\Phi}(\mathcal{M}, l^2_R)$ and $\approx$ depends only on $\Phi$.

(2) If $2 < a_\Phi \leq b_\Phi < \infty$, then

$$\|x\|_{\Phi_w} \approx \left\| \left( \sum_{n=0}^{\infty} |dx_n|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w} + \left\| \left( \sum_{n=0}^{\infty} |dx_n^*|^2 \right)^{\frac{1}{2}} \right\|_{\Phi_w},$$

where $\approx$ depends only on $\Phi$.

Proof. The proof is similar to the one of Theorem 5.1 in [5] through using Theorem 5.3 and the details are omitted. \qed

Remark 5.2. All inequalities above are left open for $1 < a_\Phi \leq 2 \leq b_\Phi < \infty$. At the time of this writing, we do not see how to formulate a meaningful statement for this case. However, our argument works well in the commutative case for all cases $1 < a_\Phi \leq b_\Phi < \infty$.

References

[1] C.A.Akemann, J.Anderson and G.K.Pedersen, Triangle inequalities in operator algebras, Linear Multilinear Algebra 11 (1982), 167-178.

[2] M.H.A.Al-Rashed and B.Zegarlinski, Noncommutative Orlicz spaces associated to a state, Studia Math. 180 (2007), 199-209.

[3] S.Attal and A.Coquio, Quantum stopping times and quasi-left continuity, Ann.Inst.H.Poincaré Probab.Statist. 40 (2004), 497-512.

[4] T.N.Bekjan, $\Phi$-inequalities of non-commutative martingales, Rocky Mountain J. Math. 36 (2006), 401-412.

[5] T.N.Bekjan and Z.Chen, Interpolation and $\Phi$-moment inequalities of noncommutative martingales, Probab. Theory Relat. Fields, DOI 10.1007/s00440-010-0319-2, online first, in press.

[6] D.L.Burkholder, Distribution function inequalities for martingales, Ann.Probab. 1(1) (1973), 19-42.

[7] D.L.Burkholder, B.Davis and R.Gundy, Integral inequalities for convex functions and operators on martingales, Proc. 6th Berkeley Symp. II: 223-240, 1972.

[8] D.L.Burkholder and R.Gundy, Extrapolation and interpolation of quasi-linear operators on martingales, Acta Math. 124 (1970), 249-304.

[9] P.G.Dodds, T.K.Dodds, and B.de Pagter, Noncommutative Banach function spaces, Math.Z. 201 (1989), 583-587.

[10] T.Fack and H.Kosaki, Generalized $s$-numbers of $\tau$-measure operators, Pacific. J. Math. 123 (1986), 269-300.

[11] Y.Hu, Théorèmes ergodiques et théorèmes d’extrapolation non commutatifs, Thesis, Université de Franche-Comté, 2007.
Noncommutative weak Orlicz spaces

[12] Y.Hu, Noncommutative extrapolation theorems and applications, Illinois J.Math. 53(2) (2009), 463-482.
[13] W.Kunze, Noncommutative Orlicz spaces and generalised Arens algebras, Math. Nachr. 147 (1990), 123-138.
[14] J.Lindenstrauss and L.Tzafriri, Classical Banach space II, Springer-Verlag, Berlin, 1979.
[15] P.Liu, Y.Hou, and M.Wang, Weak Orlicz space and its applications to martingale theory, Sci. China Math. 53 (4) (2010), 905-916.
[16] F.Lust-Piquard, Inégalites de Khintchine dans $c_p$ ($1 < p < \infty$), C. R. Acad. Sci. Paris 303 (1986), 289-292.
[17] F.Lust-Piquard, A Grothendieck factorization theorem on 2-convex Schatten spaces, Isreal J. Math. 79 (1992), 331-365.
[18] F.Lust-Piquard and G.Pisier, Noncommutative Khintchine and Paley inequalities, Arkiv für Mat. 29 (1991), 241-260.
[19] F.Lust-Piquard and Q.Xu, The little Grothendieck theorem and Khintchine inequalities for symmetric spaces of measurable operators, J.Funct.Anal. 244 (2007), 488-503.
[20] L.Maligranda, Indices and interpolation, Dissert. Math. 234 (1985), Polska Akademia Nauk, Inst. Mat.
[21] L.Maligranda, Orlicz Spaces and Interpolation, Seminars in Mathematics, Departamento de Matemática, Universidade Estadual de Campinas, Brasil, 1989.
[22] C.Le Merdy and F.Sukochev, Rademacher averages on noncommutative symmetric spaces, J.Funct.Anal. 255 (2008), 3329-3355.
[23] G.Pisier, Remarks on the non-commutative Khintchine inequalities for $0 < p < 2$, J. Funct. Anal. 256 (2009), 4128-4161.
[24] G.Pisier and Q.Xu, Non-commutative martingale inequalities, Commun.Math.Phys. 189 (1997), 667-698.
[25] G.Pisier and Q.Xu, Noncommutative $L^p$-spaces, in: Handbook of the Geometry of Banach Spaces, vol.2, Ed. W.B.Johnson and J.Lindenstrauss, 1459-1517, North-Holland, Amsterdam, 2003.
[26] Q.Xu, Analytic functions with values in lattices and symmetric spaces of measurable operators, Math. Proc. Cambridge Phil. Soc. 109 (1991), 541-563.
[27] Q.Xu, Noncommutative $L_p$-Spaces and Martingale Inequalities, book manuscript, 2008.

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