On Symmetry Properties of Quaternionic Analogs of Julia Sets

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By means of theory group analysis, some algebraic and geometrical properties of quaternion analogs of Julia sets are investigated. We argue that symmetries, intrinsic to quaternions, give rise to the class of identical Julia sets, which does not exist in complex number case. In the case of quadratic quaternionic mapping $X_{k+1} = X_k^2 + C$ these symmetries mean, that the shape of fractal Julia set is completely defined by just two numbers, $C_0$ and $|C|$. Moreover, for given $C_0$ the vector part of the Julia set may be obtained by rotation of a two-dimensional Julia subset of arbitrary plane, comprising $C$, around the axis $n = C/|C|$.

I. INTRODUCTION

Beautiful and unexpected fractal properties of Julia-Fatou and Mandelbrot sets, spawned by a simple iteration rule

$$z_{k+1} = z_k^2 + C,$$

with $z$ and control parameter $C$ being complex numbers, have been intensively studied by many authors [1, 2, 3]. Iterations of arbitrary starting point $z_0$ in accordance with (1) ultimately result in the confinement of all $z_k, k \geq N_{\text{max}} >> 1$ in a finite region, a basin of attraction, of some $z_i$. These $z_i$ are called attractors; they are completely defined by control parameter $C$.

A key concept of fractal set research is the special case of $z = \infty$ attractor. In fact, each $C$ of (1) classifies all points of the complex plane as belonging to either runaway subset or prisoners one. If $z_0$ belongs to runaway subset, then (1) leads to $z = \infty$. In the other case, ultimate cycles of $z_k$ will reside in a basin of some finite attractor.

A fascinating boundary of the basin of attraction of $z = \infty$ is called Julia set. All

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information about shapes and topologies of these sets is encoded in \( C \). These sets are mostly fractals.

Prisoners of

\[
C_{k+1} = C_k^2 + C_k
\]

(i.e. of quadratic mapping \( \mathbb{I} \) with starting point \( z_0 = 0 \)) form the famous Mandelbrot set. It classifies all Julia sets as either connected or unconnected ones. The latter are known as Cantor dust or Fatou sets.

There were many attempts to apply fractal sets to various fields of physics (see \([4, 5]\)). It was clear from the very beginning, that many attractive features of these sets were due to remarkable properties of algebra of complex numbers. This suggests to generalize the quadratic mapping \( \mathbb{I} \) to different algebras and to study their behavior.

In paper \([6]\) such a generalization was undertaken for double numbers, which differ from complex ones just in definition of the imaginary unit \( \varepsilon \), i.e. \( \varepsilon^2 = 1 \). Replacement of complex numbers by double numbers actually means a transition from the Euclidean plane to the pseudoeuclidean one, the latter being of special interest for physics, in particular for relativistic kinematics. Another example of validity of double numbers give two-dimensional relativistic models, which are prevalent in modern field theories. They also may be successfully described in terms of double numbers \([\mathbb{I}]\).

Generally speaking, all hypercomplex algebras, i.e. algebra of double and dual numbers, of quaternions, biquaternions and of octanions, have proved to be very convenient in numerous physical applications. It may be explained by their close relations with geometries of Euclidean and pseudoeuclidean spaces and with spaces of constant curvature \([\mathbb{I}]\).

In this paper we focus on quaternion algebra. Really, the 'quaternion language' seems to be especially natural for physics of our four-dimensional space-time. It has been already successfully exploited in describing of rigid body motion \([\mathbb{I}]\), in searching for instanton solutions of Yang-Mills equations \([\mathbb{I}]\), in the problem of supersymmetric oscillator \([\mathbb{I}]\) and many others.

II. QUATERNIONS

Let us first briefly remind the fundamentals of quaternion algebra. The quaternion \( X \) is a set of 4 real numbers \( x_0, x_1, x_2, x_3 \) with 3 imaginary units \( i, j, k \). The definition and
commutation properties are as follows:

\[ X = x_0 + i x_1 + j x_2 + k x_3, \quad (3) \]

where

\[
\begin{align*}
  i^2 &= j^2 = k^2 = -1; \\
  ij &= -ji = k; \\
  ki &= -ik = j; \\
  jk &= -kj = i;
\end{align*}
\]

Hence *quaternions* are non-commutative with respect to multiplication, so that

\[ X_1 \cdot X_2 \neq X_2 \cdot X_1. \quad (5) \]

In some applications a representation of *quaternion* (3) as a pair of complex numbers,

\[ X = (x_0 + i x_1) + (x_2 + i x_3)j, \quad (6) \]

may turn to be useful. The traditional notation of arbitrary quaternion (3) as a combination of scalar \( x_0 \) and three-dimensional vector \( x \) parts,

\[ X = x_0 + x, \quad (7) \]

may be very convenient in many problems. In terms of (7), the operation of quaternion conjugation is simply

\[ \bar{X} = x_0 - x. \]

The product of two arbitrary elements of the quaternionic algebra (i.e. of two four-dimensional vectors) in the traditional notations reads:

\[ X X' = x_0 x'_0 - (x \cdot x') + x_0 x' + x'_0 x + [xx']. \quad (8) \]

Consequently, the square of any quaternion \( X \) looks like

\[ X^2 = XX = x_0^2 - |x|^2 + 2x_0 x. \quad (9) \]

It should be noted, that operation of division by nonzero element (\( q \neq 0 \)) is well defined for *quaternions*.

The set of quaternions forms a group with respect to the operations of addition and multiplication.
III. QUATERNIONIC ANALOGS OF FRACTAL SETS

The above-mentioned algebraic properties of quaternions permit to introduce a quaternionic analog of the original complex number Julia-Fatou algorithm (1). Namely,

$$X_{k+1} = X_k^2 + C,$$  \hspace{1cm}(10)

with $X$ and control parameter $C$ being quaternions. This iteration rule maps the four-dimensional Euclidean space on itself. Direct computer experiments justify, that (10) classifies again all quaternions with respect to $X = \infty$ as belonging to either prisoner or runaway subsets. The border of these subsets is a quaternionic analog of Julia set; it is fully defined by coefficients of mapping rule, e.g. $C$ in case of (10). However, in contrast to complex and double numbers, these fractal sets are four-dimensional.

The first treatment of this problem was given by A. Norton in Ref. [12]. The author have discussed some aspects of quadratic mappings

$$X_{k+1} = AX_k^2 + B$$ \hspace{1cm}(11)

with $A, B$ and $X$ being quaternions. It was noted, that the general case the problem was very difficult both for discussion and for representation of results.

To simplify the problem, the consideration of Ref. [12] was restricted to a special case of coefficients $A$ and $B$ belonging to the complex number subalgebra of quaternions. The author argued, that this choice allowed to reduce the dimensions of the resulting Julia sets. However, no good arguments were proposed to support this statement.

A particular attention of Ref. [12] had been paid to the analysis of topological properties of quaternionic Julia sets. It was noted, that, due to noncommutativity of quaternions, there actually exist three different non-equivalent generalizations of (1). Besides (11), one is also to discuss

$$X_{k+1} = X_kAX_k + B,$$ \hspace{1cm}(12)
$$X_{k+1} = X_k^2A + B.$$

The main goal of our research is to provide an instrument for establishing of common features and differences of fractal sets realized by algorithms (10)–(12) from algebraic, theoretical group point of view.
Let us consider transformations connected with multiplications and additions of quaternions as transformations of the group of motion of four-dimensional Euclidean space \( \mathbb{R}^4 \). It is easy to check that (10)–(12) are invariant under the following transformations:

\[
X'_k = QX_k\bar{Q}, \quad A' = QA\bar{Q}, \quad B' = QB\bar{Q}, \quad C' = QC\bar{Q},
\]

where quaternion \( Q \) satisfies the condition

\[
Q\bar{Q} = 1,
\]

i.e. \( \bar{Q} = Q^{-1} \).

Note, that due to commutativity of complex and double numbers the analogous transformations, \( q = e^{i\varphi} \), mean just trivial rotation by angle \( \varphi \) of the whole plane.

Transformations (13) are inner automorphisms of the \textit{division ring} of quaternions. They are isomorphic to the transformations of the group of three-dimensional rotations \( \text{SO}(3, \mathbb{R}) \). Note, that \( x_0 \) and \( |x| \) stay invariant under such transformations.

Thus, we actually deal with a class of equivalent iteration rules

\[
X'_{k+1} = A'(X'_k)^2 + B',
\]

which are bound with (11) by relations (13–14). Applying these transformations to the mapping (11), we obtain equivalent rules

\[
X_{k+1}' = (X'_k)^2 + C'.
\]

To illustrate this symmetry manifestation, let us show, that there is a freedom in orientation of vector \( C' \) (remember, that \( C'_0 = C_0, |C'| = |C| \)). These transformations are analogous to the plane transformations of the Lorentz group, found in Ref. [14]. For vector parts of quaternions \( C \) and \( C' \) the plane transformation quaternion \( Q \) and its conjugate \( \bar{Q} \) are:

\[
Q = \frac{C + C'}{\sqrt{-\left(C + C'\right)^2}} \frac{C}{\sqrt{-C^2}},
\]

\[
\bar{Q} = \frac{C}{\sqrt{-C^2}} \frac{\bar{C} + \bar{C}'}{\sqrt{-\left(C + C'\right)^2}} = \frac{C + C'}{\sqrt{-\left(C + C'\right)^2}} \frac{\bar{C}'}{\sqrt{-C'^2}}.
\]

It may be convenient to rotate vector \( C \) so that to orient vector \( C' \) along, say, vector \( i \):

\[
C' = C_0 + |C| i.
\]
If all starting points $X_0$ of (11) lie in this complex plane, the following points $X_k$ will lie in this plane too. Thus, one obtains a complex number Julia set, which is a subset of the full quaternionic one.

The most important example of the profit of (13) in studies of quaternionic Julia set symmetries is given by transformations, that leave control parameter $C$ intact, i.e.

$$C' = QC\bar{Q} = C, \quad Q\bar{Q} = 1. \quad (20)$$

It is straightforward to check that the proper quaternion is (see Ref. [8]):

$$Q = \frac{1 + n \tan \frac{\varphi}{2}}{\sqrt{1 + \tan^2 \frac{\varphi}{2}}} = \cos \frac{\varphi}{2} + n \sin \frac{\varphi}{2}, \quad (21)$$

where $n = C/|C|$.

These transformations form the $O(2)$ group. They are similar to the gauge symmetries of field theories. One can consider the resulting Julia set as a projective space and bundle manifold. An invariance of fractal sets under transformation (21) means that projection of this sets in the three dimensional subspace are axial symmetric.

These symmetries does not change the zero quaternion component. They do exist for each $C_O$, but corresponding two-dimensional Julia subsets are different even for the same $|C|$. It fact, quaternionic symmetries mean, that one needs just two numbers, $C_O$ and $|C|$, to describe all possible shapes of quaternionic Julia sets.

Besides continuous symmetries, quadratic mappings of all hypercomplex algebras also possess a special obvious invariance under discrete reflections,

$$X \leftrightarrow -X. \quad (22)$$

This invariance is due to two-valued property of $X^2$. Really, if some point $X_k$ belongs to a Julia set, then the same is true for $-X_k$. This symmetry allows to save the computer time at calculations of fractal pictures.

It may be interesting to mention, that the special choice of parameters $A$ and $B$ in Ref. [12] as complex numbers was not obligatory. Symmetries (13) imply, that it would suffice just to orient vector parts of both quaternions in the same direction. The resulting Julia sets would be obtained by rotation of arbitrary plane Julia subset around the axis oriented along the direction of their vector parts. The transverse cross-sections of these Julia sets consist of concentric circles.
IV. CONCLUSIONS

Quaternion algebra provides non-trivial generalizations of usual \textit{Julia} sets. Although these sets are much more complicated, the intrinsic quaternionic symmetries allow to simplify the problem. It turns out that in case of quadratic mapping (\textup{(10)}) all essentially different quaternionic analogs of \textit{Julia} sets may be enumerated by just two numbers, \(C_0\) and \(|C|\). Due to \(O(2)\) symmetry, the three-dimensional part of quaternionic \textit{Julia} sets may be restored by rotation of some arbitrary two-dimensional (e.g. complex number) \textit{Julia} subset around the axis \(n = C/|C|\), that lies in this plane.

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