Localization of spherical varieties

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Abstract. We prove some fundamental structural results for spherical varieties in arbitrary characteristic. In particular, we study Luna’s two types of localization and use it to analyze spherical roots, colors and their interrelation. At the end, we propose a preliminary definition of a \( p \)-spherical system.

1. Introduction

Let \( G \) be a connected reductive group defined over an algebraically closed ground field \( k \) of arbitrary characteristic \( p \). A normal \( G \)-variety \( X \) is called spherical if a Borel subgroup \( B \) of \( G \) has an open orbit in \( X \). In characteristic 0, there exists by now an extensive body of research on spherical varieties culminating in a complete classification (Luna-Vust [LV83], Luna [Lun01], Losev [Los09], Cupit-Foutou [CF09], and Bravi-Pezzini [BP11a, BP11b, BP11c]).

In positive characteristic, much less work has been done. Most papers dealing with spherical varieties in positive characteristic are restricted to particular examples (like flag or symmetric varieties) or other special classes of spherical varieties (like varieties obtained by reduction mod \( p \)).

The present paper is part of a program to develop a general theory of spherical varieties in arbitrary characteristic, leading possibly to a classification, as well. In this sense, the old paper [Kno91] on a characteristic free approach to spherical embeddings is already part of the program.

A crucial portion of Luna’s theory of spherical varieties depends on Akhiezer’s classification, [Akh83], of spherical varieties of rank 1. In the present paper we present the results which are independent of that classification. On the other hand, in the companion paper [Kno13] we determine all spherical varieties of rank 1 in arbitrary characteristic and present results whose proofs depend (so far) on it.

More precisely, in the present paper we recover most results of Luna’s paper [Lun97] on the “big cell”. We start with generalizing Luna’s fundamental relations for the colors of a spherical variety. At this point, we introduce additional data needed to describe a spherical variety in positive characteristic. These are certain \( p \)-powers \( q_{D,\alpha} \) where \( \alpha \) is a simple root and \( D \) is a color “moved by \( \alpha \)”. Our exposition of this part is different (and we think simpler) than Luna’s and seems to be new even in characteristic zero.

Next, we define the notion of spherical roots as properly normalized normal vectors to the valuation cone. Luna’s method of viewing them as weights attached to a wonderful variety does not generalize.
Next, we consider Luna’s construction from [Lun97] which is called localization at \( S \). Basically, it consists in analyzing the open Białynicki-Birula cell with respect to a dominant 1-parameter subgroup of \( G \). Our results are more general than Luna’s, even in characteristic zero, since Luna restricts his attention to wonderful varieties while we formulate everything for so-called toroidal varieties. From this, we derive Luna’s important result that the colors are, to a large extent, already determined by the spherical roots.

Then we consider a construction which is called localization at \( \Sigma \). This procedure amounts to analyzing \( G \)-orbits of a toroidal variety. Since this technique is mostly classical only the proof for the behavior of type-(a)-colors is new. Unfortunately, our results remain somewhat incomplete since it is unknown whether orbit closures in toroidal spherical embeddings are normal or not.

We use localization at \( \Sigma \) to prove the important non-positivity result Corollary 6.6. As opposed to Luna’s proof who uses Wasserman’s tables of rank-2-varieties [Was96] our proof is conceptual.

In the final section 7 we are making an attempt to generalize Luna’s notion of a spherical system to positive characteristic. This is a combinatorial structure describing the roots and the colors of a spherical variety. As additional data we propose the \( p \)-powers \( q_\alpha,D \) mentioned above and we hope that, at least for \( p \neq 2,3 \), these data are enough to describe a spherical variety. As for the axioms, we restrict ourselves to those which immediately generalize axioms in characteristic zero. Conditions which are only meaningful in positive characteristic (like bounding the denominators of the pairings \( \delta_D(\alpha) \)) are deferred to future work.

So additional axioms will have to be added on at a later stage.

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**Notation:** In the entire paper, the ground field \( k \) is algebraically closed. Its characteristic exponent is denoted by \( p \), i.e., \( p = 1 \) if \( \text{char } k = 0 \) and \( p = \text{char } k \), otherwise. The group \( G \) is connected reductive, \( B \subseteq G \) a Borel subgroup, and \( T \subseteq B \) a maximal torus. Let \( \Xi(T) = \Xi(B) \) be its character group. The set of simple roots with respect to \( B \) is denoted by \( S \subseteq \Xi(T) \).

A rational function \( f \) on \( X \) is \( B \)-semiinvariant if there is a character \( \chi_f \in \mathcal{X}(B) = \mathcal{X}(T) \) such that \( f(b^{-1}x) = \chi_f(b)f(x) \) for all \( b \in B \) and generic \( x \in X \). If \( X \) is spherical, the character \( \chi_f \) determines \( f \) up to a non-zero scalar. Let \( \Xi(X) \subseteq \Xi(T) \) be the set of characters of the form \( \chi_f \). It is a finitely generated abelian group whose rank is called the rank of \( X \). We also use \( \Xi_Q(X) := \Xi(X) \otimes \mathbb{Q} \) and \( \Xi_p(X) := \Xi(X) \otimes \mathbb{Z}_p \) with \( \mathbb{Z}_p := \mathbb{Z}[\frac{1}{p}] \).

### 2. Colors

Many properties of a spherical variety are determined by two sets of data and their interrelation: colors and valuations. We start with colors. Our results generalize those of [Lun97] in characteristic zero but the approach is different. We do not use compactifications but use the completeness of flag varieties instead.
Let $X$ be a spherical $G$-variety with group of characters $\Xi(X)$ and let
\[(2.1) \quad N_Q(X) = \text{Hom}(\Xi(X), \mathbb{Q}).\]

A color of $X$ is an irreducible divisor which is $B$- but not $G$-invariant. Every color $D$ produces an element $\delta_D \in N_Q(X)$ by
\[(2.2) \quad \delta_D(\chi_f) := v_D(f)\]
for all $B$-semiinvariants $f$. Here $v_D$ is the valuation of $k(X)$ attached to $D$. The color $D$ is, in general, not uniquely determined by $\delta_D$.

Since $X$ is spherical, we can choose a point $x_0 \in X$ such that $Bx_0$ is open and dense in $X$. Let $\Delta(X)$ be the set of colors of $X$. Since every color intersects the open $G$-orbit $Gx_0$ we have $\Delta(X) = \Delta(Gx_0) = \Delta(G/H)$ where $H = G_{x_0}$ is the isotropy subgroup scheme of $x_0$. We start by recalling a well-known formula for the number of colors.

2.1. Proposition. Let $G$ be a semisimple group and $H \subseteq G$ a spherical subgroup. Then
\[(2.3) \quad \#\Delta(G/H) = \text{rk} G/H + \text{rk} \Xi(H).\]

Proof. We compute the Picard group in two different ways. Set $X = G/H$. First, we have an exact sequence
\[(2.4) \quad \Xi(G) \to \text{Pic}^G X \to \text{Pic} X.\]
The left hand group is trivial since $G$ is semisimple. The cokernel of the right hand homomorphism is torsion by Sumihiro \[\text{Sum74}\]. On the other hand $\text{Pic}^G X = \Xi(H)$. Thus $\text{rk Pic} X = \text{rk} \Xi(H)$. Now let $X_0 = Bx_0 \subseteq X$ be the open $B$-orbit. Then the colors are the irreducible components of $X \setminus X_0$. Thus we have an exact sequence
\[(2.5) \quad k^\times = \mathcal{O}(X)^\times \to \mathcal{O}(X_0)^\times \to \mathbb{Z}^a \to \text{Pic} X \to \text{Pic} X_0 = 0\]
where $a = \#\Delta(X)$. By definition $\text{rk} \mathcal{O}(X_0)^\times/k^\times = \text{rk} X$. Thus $\text{rk Pic} X = a - \text{rk} X$. \[\Box\]

Given a simple root $\alpha \in S$, one can construct colors as follows: let $P_\alpha \subseteq G$ be the minimal parabolic subgroup corresponding to $\alpha$. Then $P_\alpha x_0$ is an open $B$-stable subset of $X$ which, according to [Kno95a, 3.2], decomposes into at most three $B$-orbits. One of them is the open $B$-orbit $Bx_0$; the others are of codimension 1 in $X$, hence their closures are colors. We say that these colors are moved by $\alpha$. Clearly, this just means that $P_\alpha D \neq D$. In particular, every color is moved by some (not necessarily unique) simple root.

A more precise description is as follows. Let $H_\alpha := (P_\alpha)x_0 = H \cap P_\alpha$ such that $P_\alpha x_0 = P_\alpha/H_\alpha$. Then the $B$-orbits in $P_\alpha x_0$ correspond to $H_\alpha^{\text{red}}$-orbits in $B \setminus P_\alpha \cong \mathbb{P}^1$. Let $\overline{H_\alpha}$ denote the image of $H_\alpha^{\text{red}}$ in $\text{Aut} \mathbb{P}^1 \cong \text{PGL}(2)$. Then, up to conjugation, there are four possibilities for $\overline{H_\alpha}$:

| Type of $\alpha$ | $\overline{H_\alpha}$ | colors |
|------------------|------------------------|--------|
| $(p)$            | $G_0$                  | —      |
| $(b)$            | $S_0U_0$               | $D$    |
| $(a)$            | $T_0$                  | $D, D'$|
| $(2a)$           | $N_0$                  | $D$    |

Here $G_0 = \text{PGL}(2)$. The subgroups $B_0$, $U_0$, and $T_0$ of $G_0$ are Borel subgroup, a maximal unipotent subgroup, and a maximal torus, respectively. Moreover, $S_0 \subseteq T_0$ and $N_0 =$
\(N_{G_0}(T_0)\). Thus, the set of simple roots decomposes as a disjoint union according to their type:

\[(7.2) \quad S = S^{(p)} \cup S^{(b)} \cup S^{(a)} \cup S^{(2a)}.\]

Observe that \(\alpha \in S^{(p)}\) if and only if the open \(B\)-orbit \(Bx_0\) is \(P_\alpha\)-invariant. Thus, \(S^{(p)}\) is the set of simple roots of the parabolic \(P_X\), the stabilizer of the open \(B\)-orbit.

Let \(D\) be a color moved by \(\alpha\). Then the morphism

\[(2.8) \quad \varphi_{D,\alpha} : P_\alpha \times B ^ D \to X\]

is generically finite. Its separable degree is \(1\), i.e., \(\varphi_{D,\alpha}\) is bijective if and only if \(\alpha\) is of type \((b)\) or \((a)\). It is \(2\) for \(\alpha\) of type \((2a)\). The inseparable degree of \(\varphi_{D,\alpha}\) will be denoted by \(q_{D,\alpha} \in p^{3\mathbb{N}}\).

**Example.** Assume \(p > 3\) and let \(P \subseteq G\) be a subgroup scheme which contains \(^{-}B\), the Borel subgroup which is opposite to \(B\). In [Wen93, Wen94], Wenzel shows that such subgroup schemes are classified by functions \(f : S \to \mathbb{Z}_{\geq 0} \cup \{\infty\}\) where \(f(\alpha)\) can be defined as the supremum of the set of all \(n \in \mathbb{Z}_{\geq 0}\) such that \(P\) contains the \(n\)-th Frobenius kernel of \(P_\alpha\). See also [HL93] for a simplified account. Now let \(X = G/P\), a complete homogeneous \(G\)-variety. Then the following is easy to see: A simple root \(\alpha \in S\) is of type \((p)\) if and only if \(f(\alpha) = \infty\). All other simple roots are of type \((b)\) and they all move a different divisor \(D_\alpha\). Moreover, \(q_{D_\alpha,\alpha} = p^{f(\alpha)}\). In particular, this shows that in this example the numbers \(q_{D,\alpha}\) may be arbitrary \(p\)-powers.

To formulate the following permanence property we renormalize \(\delta_D\) as follows:

\[(2.9) \quad \delta^{(a)}_D := q_{D,\alpha} \delta_D.\]

**2.2. Lemma.** Let \(\pi : X_1 \to X_0\) be a finite surjective equivariant morphism between spherical \(G\)-varieties, let \(E\) be a color of \(X_1\) and let \(D = \pi(E)\) be its image in \(X_0\). Let, moreover, \(\alpha \in S\) be a simple root moving \(E\) (and \(D\)). Then \(\delta^{(a)}_D = \delta^{(a)}_E\).

**Proof.** We consider first the case that \(\alpha\) is of type \((a)\) or \((b)\) for \(X_0\). Then its type for \(X_1\) is the same. Moreover, both \(\varphi_{D,\alpha}\) and \(\varphi_{E,\alpha}\) are bijective and, as an equality of divisors, \(\pi^{-1}(D) = qE\) where \(q\) is some \(p\)-power. Thus, \(\delta_E = q\delta_D\). Now consider the diagram

\[(2.10) \quad \begin{array}{ccc}
P_\alpha \times B & \to & X_1 \\
\downarrow & & \downarrow \\
P_\alpha \times B ^ D & \xrightarrow{\varphi_{D,\alpha}} & X_0 \end{array}\]

It is cartesian, hence both horizontal arrows have the same (inseparable) degree, namely \(q_{D,\alpha}\). On the other hand, the top arrow has degree \(q_{E,\alpha}\). Hence

\[(2.11) \quad \delta^{(a)}_D = q_{D,\alpha} \delta_D = q_{E,\alpha} \delta_D = q_{E,\alpha} \delta_E = \delta^{(a)}_E.\]

Now assume that \(\alpha\) is of type \((2a)\) for \(X_0\). Then there are two cases. If \(\pi^{-1}(D)^{\text{red}}\) is irreducible then \(\alpha\) is of type \((2a)\) for \(X_0\), as well. Moreover, the degree of both horizontal arrows is now \(2q_{D,\alpha}\). From here one argues as above.

The second case is when \(\alpha\) of type \((a)\) for \(X_0\). Then \(\pi^{-1}(D)^{\text{red}} = E_1 \cup E_2\) has two components. As divisors, we have \(\pi^{-1}(D) = q_1 E_1 + q_2 E_2\). Thus \(\delta_{E_1} = q_1 \delta_D\) and \(\delta_{E_2} = q_2 \delta_D\). Moreover, as above, we get

\[(2.12) \quad 2\delta^{(a)}_D = 2q_{D,\alpha} \delta_D = q_1 q_{E_1,\alpha} \delta_D + q_2 q_{E_2,\alpha} \delta_D = \delta^{(a)}_{E_1} + \delta^{(a)}_{E_2}.\]
Now we claim that actually \( \delta_{E_1}^{(\alpha)} = \delta_{E_2}^{(\alpha)} \) which would prove our assertion.

To prove the claim, we may assume that \( X_0 = G/H_0 \) and \( X_1 = G/H_1 \) are homogeneous. Moreover, the cases proved above allow to replace \( H_0 \) and \( H_1 \) by \( H_0^{\text{red}} \) and \( H_1^{\text{red}} \), respectively. We can even replace \( H_1 \) be its connected component of unity since \( E \) cannot split any further (otherwise \( D \) would split into more than two components). Then \( H_1 \) is normal in \( H_0 \) and \( \pi \) is the quotient by the finite group \( \Gamma := H_0/H_1 \). Since \( \Gamma \) acts transitively on the connected components of the fibers of \( \pi \), there is an element \( g \in \Gamma \) which maps \( E_1 \) to \( E_2 \) which proves the claim. \( \square \)

For any simple root \( \alpha \) let \( \alpha^r \in N_\mathcal{Q}(X) \) be the restriction of \( \alpha^\vee \) to \( \mathcal{X}_\mathcal{Q}(X) \), i.e.,

\[
\alpha^r(\chi) = \langle \chi, \alpha^\vee \rangle \quad \text{for all } \chi \in \mathcal{X}_\mathcal{Q}(X)
\]

### 2.3. Proposition. Fix a simple root \( \alpha \in S \). Then the following relations hold:

| Type of \( \alpha \) | \( \delta_D^{(\alpha)} \) |
|-----------------|------------------|
| (p)             | \( \alpha^r = 0 \) |
| (b)             | \( \delta_D^{(\alpha)} = \alpha^r \) |
| (a)             | \( \delta_D^{(\alpha)} + \delta_D^{(\alpha^r)} = \alpha^r \) |
| (2a)            | \( \delta_D^{(\alpha)} = \frac{1}{2} \alpha^r \) |

**Proof.** Let \( X = G/H \) be homogeneous and put \( X_1 = G/H^{0, \text{red}} \). We claim that it suffices to prove the assertions for \( X_1 \). Indeed, if \( \alpha \) is of type \( (p) \) for \( X \) then the same holds for \( X_1 \) and the claim follows from \( \Xi_\mathcal{Q}(X) = \Xi_\mathcal{Q}(X_1) \). If \( \alpha \) is not of type \( (p) \) for \( X \) and \( X_1 \) then the claim follows immediately from Lemma 2.2 if \( \alpha \) has the same type for \( X \) and \( X_1 \). Otherwise, \( \alpha \) is of type \( (2a) \) for \( X \) (moving one color \( D \)) and of type \( (a) \) for \( X_1 \) (moving two colors \( E_1, E_2 \)). But then Lemma 2.2 implies

\[
\delta_D^{(\alpha)} = \frac{1}{2} (\delta_E^{(\alpha)} + \delta_E^{(\alpha)}) = \frac{1}{2} \alpha^r
\]

proving the claim.

Thus we may assume that \( H \) is connected and reduced. Then consider the following diagram

\[
\begin{array}{ccc}
X = G/H & \xleftarrow{p_1} & \xrightarrow{p_2} \ F \\
& \downarrow & \\
B \backslash G =: \ F
\end{array}
\]

Both morphisms \( p_1 \) and \( p_2 \) are smooth with connected fibers. Therefore, an irreducible \( B \)-stable divisors \( D \subset X \) corresponds to an irreducible \( H \)-stable divisor \( E \subset \ F \). Moreover, any \( B \)-semiinvariant rational function \( f \) on \( X \) corresponds to an \( H \)-invariant rational section \( s \) of the homogeneous line bundle \( \mathcal{L}_\chi \) (with \( \chi = \chi_f \)) on \( \mathcal{F} \). Furthermore, \( (D, f) \) is related to \( (E, s) \) by

\[
v_E(s) = v_D(f) = \delta_D(\chi).
\]

Now consider the \( \mathbb{P}^1 \)-fibration \( \pi : \mathcal{F} = B \backslash G \to \mathcal{F}_\alpha := P_\alpha \backslash G \). Moreover, let \( y \in \mathcal{F} \) be in the open \( H \)-orbit and let \( F \cong \mathbb{P}^1 \) be the fiber through \( y \).
Assume first that \( \alpha \) is of type \((p)\). Then the open \(B\)-orbit in \(X\) is \(P_\alpha\)-stable which translates into the open \(H\)-orbit in \(F\) being the preimage of an open set in \(P_\alpha \setminus G\). But then \(L_\chi\) is a pull-back from \(P_\alpha \setminus G\) which implies \(\langle \chi_f, \alpha^\vee \rangle = 0\).

Now assume that \(\alpha \in S^{(b)}\). Then \(E\) is the only \(H\)-invariant divisor mapping dominantly onto \(F_\alpha\). Moreover, since \(E \cap F\) consists of a single point, the map \(E \to \mathcal{F}_\alpha\) is generically bijective, hence purely inseparable. Its degree is \(q_{D,\alpha}\). Thus we get

\[
\langle \chi, \alpha^\vee \rangle = \deg L_\chi|_F = (s) \cdot F = v_E(s)E \cdot F = q_{D,\alpha}\delta_D(\chi) = \delta_D^{(\alpha)}(\chi)
\]

proving the assertion.

If \(\alpha \in S^{(a)}\) then there are two divisors \(E, E'\) mapping generically injectively to \(F_\alpha\) with degree \(q_{D,\alpha}\) and \(q_{D',\alpha}\), respectively. Then

\[
\langle \chi, \alpha^\vee \rangle = (s) \cdot F = (v_E(s)E + v_{E'}(s)E') \cdot F = q_{D,\alpha}\delta_D(\chi) + q_{D',\alpha}\delta_{D'}(\chi).
\]

Now we prove \(\alpha \in \Xi_p(X)\). By construction, there is an equivariant morphism \(P_\alpha x_0 \to \text{PGL}(2)/H\) with \(\hat{H}^{\text{red}} = T_0\). Thus the pull-back of any non-constant \(B_0\)-semi-invariant is \(\alpha\)-semi-invariant with character \(q_0\alpha\) for some \(p\)-power \(q_0\). The \(H_\alpha\)-linearization of \(L_{q_0\alpha}|_F\) factors through a \(\text{PGL}(2)\)-linearization. One reason is, e.g., that \(L_{-\alpha}\) is the relative canonical bundle of the fibration \(\pi\). This implies that \(s|_F\) has two zeroes of the same multiplicity on \(F\). Hence \(\delta_D^{(\alpha)}(\alpha) = \delta_D^{(\alpha)}(\alpha)\) and therefore both are equal to 1.

Finally, assume that \(\alpha \in S^{(2a)}\). Then there is one divisor \(E\) mapping generically \(2 : 1\) to \(F_\alpha\). The degree of inseparability of this map is \(q_{D,\alpha}\). Then \(E \cdot F = 2q_{D,\alpha}\) and therefore

\[
\langle \chi, \alpha^\vee \rangle = (s) \cdot F = v_E(s)E \cdot F = 2q_{D,\alpha}\delta_D(\chi) = 2\delta_D^{(\alpha)}(\chi),
\]

as claimed. \(\square\)

We note the following consequence:

**2.4. Corollary.** Let \(p \neq 2\) and let \(\alpha \in S\) be of type \((2a)\). Then \(\alpha \notin \Xi_p(X)\) and \(\langle \chi, \alpha^\vee \rangle\) is even for all \(\chi \in \Xi(X)\).

**Proof.** We keep the notation of the proof of Proposition 2.3. Let \(N_0 = \langle s_0 \rangle T_0\) and let \(n \in \mathbb{Z}\) with \(n\alpha \in \Xi(X)\). Then \(s_0\) acts on the \(T_0\)-invariant section of \(L_{q_0\alpha}|_F\) by multiplication with \((-1)^n\). Hence \(n\) is even and \(\alpha \notin \Xi_p(X)\). The rest follows directly from Proposition 2.3. \(\square\)

Now we analyze the case where a color is moved by more than one simple root.

**2.5. Lemma.** Let \(D\) be a color which is moved by two distinct simple roots \(\alpha_1\) and \(\alpha_2\). Then either \(\alpha_1, \alpha_2 \in S^{(b)}\) or \(\alpha_1, \alpha_2 \in S^{(a)}\). In the latter case let \(D'\) and \(D''\) be the second color moved by \(\alpha_1\) and \(\alpha_2\), respectively. Then \(D' \neq D''\).

**Proof.** Clearly, neither \(\alpha_1\) nor \(\alpha_2\) is of type \((p)\). Recall from [Kno95a, §2], that any \(B\)-orbit on \(X\) has a rank attached to it. Moreover, if \(\alpha \in S\) moves the color \(D\) then \(\text{rank } D = \text{rank } X\) if \(\alpha\) is of type \((b)\) and \(\text{rank } D = \text{rank } X - 1\) in case \(\alpha\) is of type \((a)\) or \((2a)\) ([Kno95a, §2 and Lemma 3.2]). This entails that \(\alpha_1, \alpha_2\) are either both of type \((b)\) or both of type \((a)\) or \((2a)\).

Suppose they are both of type \((2a)\). Then, since \(\alpha_1 \in \Xi_Q(X)\),

\[
0 < q_{D,\alpha_1}^{-1} = q_{D,\alpha_1}^{-1} \frac{1}{2} \alpha_1^\vee(\alpha_1) = \delta_D(\alpha_1) = q_{D,\alpha_2}^{-1} \frac{1}{2} \alpha_2^\vee(\alpha_1) \leq 0.
\]
Similarly, suppose \( \alpha_1 \) is of type \((a)\) and \( \alpha_2 \) is of type \((2a)\). Then
\[
(2.22) \quad 0 < q_{D,\alpha_1}^{-1} = \delta_D(\alpha_1) = q_{D,\alpha_2}^{-1} \frac{1}{2} \alpha_2^r(\alpha_1) \leq 0.
\]
This finishes the proof of the first part.

Now let both \( \alpha_1 \) and \( \alpha_2 \) be of type \((a)\) and suppose \( D' = D'' \). Then
\[
(2.23) \quad 0 < \frac{q_{D,\alpha_2}}{q_{D,\alpha_1}} + \frac{q_{D',\alpha_2}}{q_{D',\alpha_1}} = q_{D,\alpha_2} \delta_D(\alpha_1) + q_{D',\alpha_2} \delta_{D'}(\alpha_1) = \alpha_2^r(\alpha_1) \leq 0.
\]
\( \square \)

**Examples.** 1. Let \( G = SL(2) \times SL(2) \) and \( H = SL(2) \) embedded into \( G \) via \( id \times F_q \) where \( F_q \) is a Frobenius morphism. Then \( X := G/H \) has only one color \( D \). Moreover both simple roots \( \alpha_1, \alpha_2 \) are of type \((b)\) and \( D \) is moved by both of them. Furthermore, \( q_{D,\alpha_1} = 1 \) and \( q_{D,\alpha_2} = q \) which shows that \( q_{D,\alpha} \) may depend on \( \alpha \).

2. Let \( G = SL(3) \), let \( q \) be a \( p \)-power, and let \( H \) be the subgroup consisting of the matrices
\[
(2.24) \quad \begin{pmatrix} t^{q+2} & t^{-1} & t^{-2q-1} \\ t^{-q-2} & 1 & x^q \\ t^{-2q-1} & 1 & x^q \end{pmatrix} \quad \text{with } t \in G_m \text{ and } x, y \in G_a.
\]
Then both simple roots are of type \((a)\) and there are three colors \( D_0, D_1, D_2 \) where \( \alpha_i \) moves \( D_0 \) and \( D_1 \). Furthermore, \( q_{D_1,\alpha_1} = q_{D_0,\alpha_2} = q_{D_2,\alpha_2} = 1 \) while \( q_{D_0,\alpha_1} = q \). The values \( \delta_D(\alpha_i) \) are given by the following table:

\[
(2.25) \quad \begin{array}{ccc}
\alpha_1 & \delta_{D_0} & \delta_{D_1} \\
\qquad & q^{-1} & 1 \\
\alpha_2 & 1 & -q - 1 \\
\end{array}
\]

So indeed \( q \delta_{D_0} + \delta_{D_1} = \alpha_1^r \) and \( \delta_{D_0} + \delta_{D_2} = \alpha_2^r \).

Both examples show that \( G \) contains for \( p \geq 2 \) infinitely many conjugacy classes of self-normalizing spherical subgroups, a phenomenon which does not occur in characteristic zero.

**Remark.** The Lemma shows in particular, that one can assign unambiguously a type to any color. Thereby, one gets a decomposition
\[
(2.26) \quad \Delta(X) = \Delta^{(b)}(X) \cup \Delta^{(a)}(X) \cup \Delta^{(2a)}(X).
\]

3. **Spherical roots**

For a spherical variety \( X \), let \( \mathcal{V}(X) \) be the set of \( G \)-invariant \( \mathbb{Q} \)-valued valuations of the field \( k(X) \). The map
\[
(3.1) \quad \mathcal{V}(X) \to N_{\mathbb{Q}}(X) = \text{Hom}(\Xi(X), \mathbb{Q}) : v \mapsto (\chi_f \mapsto v(f))
\]
is injective ([Kno91, 1.8]). According to [Kno91, 5.3], it identifies \( \mathcal{V}(X) \) with a a finitely generated convex rational cone inside \( N_{\mathbb{Q}}(X) \) which contains the image of the antiodominant Weyl chamber under the projection \( \text{Hom}(\Xi(T), \mathbb{Q}) \to N_{\mathbb{Q}}(X) \). This can be phrased in terms of the dual cone \( \mathcal{V}(X)\hat{\vee} \) of \( \mathcal{V}(X) \): it is a finitely generated rational convex cone in \( \Xi_{\mathbb{Q}}(X) \) with \( \mathcal{V}(X) = (\mathcal{V}(X)\hat{\vee})\hat{\vee} \) and \( -\mathcal{V}(X)\hat{\vee} \subseteq \mathbb{Q}_{\geq 0} S \) where \( \mathbb{Q}_{\geq 0} S \) is the cone generated by the simple roots of \( G \). In particular, \( -\mathcal{V}(X)\hat{\vee} \) is a pointed cone. Thus, it has a canonical set of generators:
3.1. Definition. An element $\sigma \in \Xi(X)$ is called a spherical root of $X$ if
- $\mathbb{Q}_{\geq 0}\sigma$ is an extremal ray of $-\mathcal{V}(X)^\vee$ (thus $\sigma \in \mathbb{Q}S$) and
- $\sigma$ is a primitive element of $\mathbb{Z}S \cap \Xi_p(X)$.

The set of spherical roots is denoted by $\Sigma(X)$.

Clearly, each extremal ray of $-\mathcal{V}(X)^\vee$ contains a unique spherical root. Moreover, the spherical roots determine the valuation cone via

$\mathcal{V}(X) = \{ v \in N_\mathbb{Q}(X) | \sigma(v) \leq 0 \text{ for all } \sigma \in \Sigma(X) \}$, and are in bijection with faces of codimension 1 of $\mathcal{V}(X)$.

The normalization for a spherical root is chosen such that the following statement holds:

3.2. Lemma. Let $\varphi : X_1 \to X_2$ be a morphism of spherical varieties which is either purely inseparable or a quotient by a central subgroup scheme of $G$. Then $\Sigma(X_1) = \Sigma(X_2)$.

Proof. If $\varphi$ is purely inseparable then clearly $\mathcal{V}(X_1) = \mathcal{V}(X_2)$ and $\Xi_p(X_1) = \Xi_p(X_2)$. Hence $\Sigma(X_1) = \Sigma(X_2)$.

Now let $\varphi$ be the quotient by $A \subseteq Z(G)$ (which might be positive dimensional). Then

$\Xi_p(X_2) = \{ \chi \in \Xi_p(X_1) | \chi|_A = 1 \}$.

Since roots of $G$ are trivial on $A$, this implies

$\mathbb{Z}S \cap \Xi_p(X_2) = \mathbb{Z}S \cap \Xi_p(X_1)$.

Moreover, $N_\mathbb{Q}(X_2)$ is a quotient of $N_\mathbb{Q}(X_1)$ and $\mathcal{V}(X_1)$ is the preimage of $\mathcal{V}(X_2)$ (follows, e.g., from [Kno91, Thm. 6.1]). This implies $\Sigma(X_2) = \Sigma(X_1)$.

4. Localization at $S$

There are two types of constructions, called localization at $S$ and at $\Sigma$, respectively, which both allow to reduce a spherical variety to a simpler one. In this section we describe localization at $S$ which, in characteristic zero, was first introduced by Luna in [Lun97]. To this end, we first recall and prove some properties of the Bia\l\ynicki-Birula decomposition, [BB76], of a $\mathbb{G}_m$-variety.

Let $X$ be a complete normal $\mathbb{G}_m$-variety. Then for any $x \in X$, the limit

$$\pi(x) := \lim_{t \to 0} t \cdot x \in X$$

exists and is a $\mathbb{G}_m$-fixed point. Thus, letting $F$ to be the set of connected components of the fixed point set $X^{\mathbb{G}_m}$ we get a set-partition of $X$ by putting

$$X_Z := \{ x \in X | \pi(x) \in Z \}.$$ 

These are the Bia\l\ynicki-Birula cells which are indexed by $F$. Except when $X$ is smooth or projective they are, in general, not very well behaved. One cell is always good, though:

4.1. Proposition. Let $X$ be a complete normal $\mathbb{G}_m$-variety. Then there is a unique connected component $S$ of $X^{\mathbb{G}_m}$ (the source of $X$) such that $X_S$ is open. Moreover, the map $\pi_S := \pi_S : X_S \to S$ is affine and a categorical quotient by $\mathbb{G}_m$. In particular, the source $S$ is irreducible and normal.
4.2. Lemma. Let $X$ be as above. Then the general fibers of $\pi_S$ are irreducible and generically reduced.

Proof. Since $X_S$ is normal, the generic fiber of $\pi_S$ is geometrically unibranch ([EGA4, 6.15.6]). Since all irreducible components contain the $G_m$-fixed point, it follows that the generic fiber is geometrically irreducible. Thus, there is an open subset of $S$ over which all fibers are geometrically irreducible, as well ([Jou83, Thm. 4.10]).

The second property follows from the fact that $\pi_S$ is a categorical quotient. This entails that $k(S)$ is separable inside $k(X_S)$. Therefore, $\pi_S$ is smooth generically on $X_S$. □

There is a second well-behaved cell. For this, let $X^- := X$ but with the opposite $G_m$-action: $t \ast x := t^{-1} \cdot x$. Then the source $T$ of $X^-$ is called the sink of $X$. It is characterized by the fact that $\pi(x) \in T$ implies $x \in T$. Thus, $T = X_T$ is a Bialynicki-Birula cell by itself. Symmetrically, $S$ is the sink of $X^-$ and therefore $X^-_S = S$.

Now assume that $X$ is a $G$-variety for some connected reductive group $G$ and that the $G_m$-action is induced by a 1-parameter subgroup $\lambda : G_m \to G$. Then we put $X^\lambda := S$ and $X_\lambda := X_S$ and $\pi_\lambda = \pi_S$. Observe that $X^\lambda$ is not the entire fixed point set of $\lambda(G_m)$ but only a very special component.

Let $G^\lambda := C_G(\lambda(G_m))$ be the fixed point set under the conjugation action and let

\begin{equation}
G_\lambda := \{ g \in G \mid \pi_G(g) := \lim_{t \to 0^+} \lambda(t)g\lambda(t)^{-1} \text{ exists}\}.
\end{equation}

Then $G_\lambda$ is a parabolic subgroup with Levi complement $G^\lambda$ and the map $\pi_G : G_\lambda \to G^\lambda$ is the natural homomorphism with kernel $G^\lambda_\lambda$, the unipotent radical of $G_\lambda$. The following lemma is well-known, e.g., the proof given in [Lun97] carries over verbatim to positive characteristic.

4.3. Lemma. The open cell $X_\lambda$ is $G_\lambda$-invariant and $\pi_\lambda : X_\lambda \to X^\lambda$ is $G_\lambda$-equivariant where $G_\lambda$ acts on $X^\lambda$ via $\pi_G : G_\lambda \to G^\lambda$, i.e., $\pi_\lambda(gx) = \pi_G(g)\pi_\lambda(x)$ for all $g \in G_\lambda$ and $x \in X_\lambda$. Moreover, $X^\lambda$ consists of fixed points for the opposite unipotent radical $G^\lambda_\lambda$.

The next lemma provides a link between closed orbits in $X$ and closed orbits in $X^\lambda$:

4.4. Lemma. Let $X$ be a complete normal $G$-variety and let $Z \subseteq X^\lambda$ be a closed $G^\lambda$-orbit. Then $GZ \subseteq X$ is a closed $G$-orbit with $GZ \cap X^\lambda = Z$.

Proof. Since $Z$ is complete and homogeneous, it contains a unique fixed point $z$ for $\sim B^\lambda$, the Borel subgroup opposite to $B^\lambda$. Since $z$ is in the source, it is a $\sim G^\lambda_\lambda$-fixed point. So $z$ is fixed by $\sim B = \sim B^\lambda \sim G^\lambda_\lambda$ which implies that $GZ = GZ$ is a closed, hence closed $G$-orbit of $X$. Moreover, since $\lambda$ is dominant, we have $z \in (GZ)^\lambda$ and therefore $GZ \cap X^\lambda = (Gz)^\lambda = G^\lambda z = Z$. □

Recall that a complete spherical $G$-variety $X$ is called toroidal if no color of $X$ contains a $G$-orbit.
4.5. Proposition. Let $\lambda$ be a dominant 1-parameter subgroup and let $X$ be a complete toroidal spherical variety. Then $X^\lambda$ is a complete toroidal spherical $G^\lambda$-variety.

Proof. This is basically [Lun97, Prop. 1.4]. We recall the proof and check that it is characteristic free.

First of all, completeness of $X^\lambda$ is clear while irreducibility and normality were obtained in Proposition 4.1. Moreover, the dominance of $\lambda$ implies $B \subseteq G^\lambda$. Hence the open $B$-orbit of $X$ is contained in $X^\lambda$. Its image in $X^\lambda$ is an open $B^\lambda$-orbit. Thus, $X^\lambda$ is spherical, as well.

Now let $D \subseteq X^\lambda$ be a color containing the closed $G^\lambda$-orbit $Z$. Then $\pi_\lambda^{-1}(D)$ contains a unique component $E'$ which maps onto $D$ (by Lemma 4.2 and the fact that $D$ meets the open $G^\lambda$-orbit). Then $E$, the closure of $D$ in $X$, is a color which contains $Z$. Since $GZ$ is a closed $G$-orbit by Lemma 4.4 we have $GZ = \overline{BZ}$ and therefore $GZ \subseteq E$. It follows that $E$ is $G$-invariant since $X$ is toroidal. From this we get that $D = \pi_\lambda(E)$ is $G^\lambda$-invariant in contradiction to $D$ being a color.

A toroidal spherical variety $X$ determines a (pointed) fan $\mathcal{F} = \mathcal{F}(X)$ in $N_\mathbb{Q}(X)$ whose support is $\mathcal{V}(X)$. More precisely, for any $G$-orbit $Y \subseteq X$ the invariant valuations whose center in $X$ contain $Y$ form a cone $\mathcal{C}_Y \subseteq \mathcal{V}(X)$. Then $\mathcal{F}$ is the collection of cones of the form $\mathcal{C}_Y$.

The fan $\mathcal{F}$ is precisely the piece of information needed to reconstruct $X$ from its open $G$-orbit $X_0$. In fact, $X$ is the compactification of $X_0$ corresponding to the colored fan $(\mathcal{F}, \varnothing)$. See [Kno91] for more details.

Let $\lambda$ be a dominant 1-parameter subgroup and let $X$ be a complete toroidal spherical variety with associated fan $\mathcal{F}$. Then $\lambda$ induces, via restriction to $\Xi_\mathbb{Q}(X)$, an element $\lambda^* \in N_\mathbb{Q}(X)$. The dominance of $\lambda$ implies $-\lambda^* \in \mathcal{V}(X)$. Thus we can consider the fan

\[ \mathcal{F}^\lambda := \{ \mathcal{C} + \mathbb{Q}_{\geq 0}\lambda \mid -\lambda^* \in \mathcal{C} \in \mathcal{F} \} \]

One may think of $\mathcal{F}^\lambda$ as the restriction of $\mathcal{F}$ to a neighborhood of $-\lambda^*$. Visibly, this fan is not pointed since not all its members contain the line $\mathbb{Q}\lambda^*$. More precisely, let $\mathcal{C}(\lambda)$ be the unique cone in $\mathcal{F}$ such that $-\lambda^*$ is contained in its relative interior. Then $\mathcal{V}(\lambda) := \langle \mathcal{C}(\lambda) \rangle_\mathbb{Q} = \mathcal{C}(\lambda) + \mathbb{Q}_{\geq 0}\lambda^*$ is the unique element of $\mathcal{F}^\lambda$ which is a subspace. Thus,

\[ \mathcal{F}^\lambda := \{ \mathcal{C}/\mathcal{V}(\lambda) \mid \mathcal{C} \in \mathcal{F}^\lambda \} \]

is a pointed fan which lives in the vector space $N_\mathbb{Q}(X)/\mathcal{V}(\lambda)$ and is called the localization of $\mathcal{F}$ at $\lambda$.

4.6. Theorem. Let $\lambda$ and $X$ as in Proposition 4.5. Then $\Xi(X^\lambda) = \Xi(X) \cap V(\lambda)^\perp$, $N_\mathbb{Q}(X^\lambda) = N_\mathbb{Q}(X)/\mathcal{V}(\lambda)$, and $\mathcal{F}(X^\lambda) = \mathcal{F}(X)^\lambda$.

Proof. Let $z$ be an arbitrary $^\perp B$-fixed point in $X$. Then $z$ corresponds to the complete orbit $Gz$ and therefore to a cone $\mathcal{C}_{Gz}$ of maximal dimension in $\mathcal{F}(X)$. Let $P$ be the parabolic which is opposite to the reduced stabilizer $G^\text{red}_z$. Then the local structure theorem [Kno93, 1.2] asserts the existence of a normal affine $T$-variety $\overline{A}$ and a $T$-equivariant morphism $\varphi_0 : \overline{A} \to X$ such that the morphism

\[ \varphi : P_u \times \overline{A} \to X : (u, a) \mapsto u\varphi_0(a) \]
Lemma 4.4

Let $z \in X$ be as above. Then $\varphi^\lambda: P^\lambda_u \times \overline{A}^\lambda \rightarrow X^\lambda$ describes the local structure of $X^\lambda$ in a neighborhood of $z$.

From this we already infer that $\Xi_\lambda(X^\lambda) = \Xi_\lambda(\overline{A}^\lambda) = V(\lambda)$. We claim that $\Xi(X^\lambda) = \Xi_\lambda(X^\lambda) \cap \Xi(X)$ which then proves the assertion $\Xi(X^\lambda) = \Xi(X) \cap V(\lambda)$. In fact, only “$\supseteq$” is an issue. To prove it let $\chi \in \Xi_\lambda(X^\lambda) \cap \Xi(X)$. Then there are $n \in \mathbb{Z}_{>0}$ and rational semiinvariants $f_\chi$ on $X$ and $f_{\pi_n}$ on $X^\lambda$ such that $f_\chi = \pi_n^* f_{\pi_n}$. Let $X' \subseteq X^\lambda$ be the open subset on which $f_{\pi_n}$ is regular. Then the normality of $X$ implies that $f_\chi$ is regular on $\pi_n^{-1}(X')$. Since $f_\chi$ is also $\lambda$-invariant we conclude that $f_\chi$ pushes down to a rational function on $X^\lambda$ which shows $\chi \in \Xi(X^\lambda)$, as claimed. The equality $N_\lambda(X^\lambda) = N_\lambda(X)/V(\lambda)$ follows immediately.

Finally, we compute the fan $\mathcal{F}(X^\lambda)$. Clearly, it suffices to determine its cones $C$ of maximal dimension corresponding to closed orbits. Lemma 4.4 and the discussion above show that the closed $G^\lambda$-orbits in $X^\lambda$ correspond precisely to those closed $G$-orbits $Gz$ in $X$ such that $-\lambda^\vee \in C_G$. In that case it is easy to check that the toroidal embedding $A^\lambda \rightarrow \overline{A}^\lambda$ corresponds to the cone $(C_G + \mathbb{Q}_{\geq 0}^\lambda)/V(\lambda)$. But these are precisely the cones of maximal dimension in $\mathcal{F}(X^\lambda)$ which shows $\mathcal{F}(X^\lambda) = \mathcal{F}(X^\lambda)$.

4.7. Corollary. Let $\lambda$ and $X$ be as above. Then

$$\Sigma(X^\lambda) = \Sigma(X) \cap V(\lambda) = \Sigma(X) \cap \lambda^\perp. \tag{4.9}$$

Proof. The valuation cone $\mathcal{V}(X^\lambda)$ equals the support of $\mathcal{F}(X^\lambda)$. Its codimension-1-faces are, by construction, by the construction of $\mathcal{V}(X)$ which contain $\mathcal{C}(\lambda)$. From $\Xi_\lambda(X^\lambda) = \Xi_\lambda(X) \cap V(\lambda)$ we get $\Sigma(X^\lambda) = \Sigma(X) \cap V(\lambda)$.

4.8. Proposition. Let $\lambda$ and $X$ be as above. Then $G X^\lambda = Y_0$ where $Y_0 \subseteq X$ is the $G$–orbit with $C_{Y_0} = \mathcal{C}(\lambda)$. Moreover, for any $G$-orbit $Y \subseteq X$ holds

$$\mathcal{C}(\lambda) \subseteq \mathcal{C}_Y \iff X^\lambda \cap Y \neq \emptyset \tag{4.10}$$

$$\mathcal{C}(\lambda) \supseteq \mathcal{C}_Y \iff X^\lambda \subseteq Y \tag{4.11}$$

Proof. First note that $X^\lambda$ is stable under $B = B^\lambda G^\mu_u$. Hence $G X^\lambda$ is closed, hence an orbit closure $Y_0$ in $X$. Choose a closed orbit $Gz$ in $Y_0$. The orbits of $X$ which contain
$Gz$ in their closure correspond precisely to the $T$-orbits in the slice $A$. It follows from (4.8) that this way $\overline{Y}_0$ corresponds to $A^\lambda$ which shows $C_{Y_0} = C(\lambda)$, as claimed.

The two equivalences follow easily: we have $X^\lambda \cap Y \neq \emptyset$ if and only if $Y \subseteq GX^\lambda = \overline{Y}_0$ if and only if $C_Y \supseteq C(\lambda)$ and $X^\lambda \subseteq Y$ if and only if $\overline{Y}_0 = GX^\lambda \subseteq Y$ if and only if $C(\lambda) \supseteq C_Y$. □

Next we compute the colors of $X^\lambda$. Let $D \subseteq X^\lambda$ be a color and let $D_0$ be its restriction to the open $G^\lambda$-orbit. Then $\pi^{-1}_\lambda(D_0)$ is irreducible by Lemma 4.2. Hence its closure $D^* \subset X$ is a $B$-stable prime divisor.

4.9. Proposition. Let $\lambda$ and $X$ be as above.

a) Let $\alpha \in S^\lambda := S(G^\lambda) = S \cap \lambda^\perp$ and let $D$ be a color of $X^\lambda$. Then

i) $\alpha$ has the same type for $X^\lambda$ as it has for $X$,

ii) $\delta_D$ is the restriction of $\delta_{D^*}$ to $\Xi_{\mathbb{Q}}(X^\lambda) \subseteq \Xi_{\mathbb{Q}}(X)$, and

iii) $D$ is moved by $\alpha$ if and only if $D^*$ is moved by $\alpha$. In that case $q_{D,\alpha} = q_{D^*,\alpha}$.

b) The map $D \mapsto D^*$ is a bijection between the set of colors of $X^\lambda$ and the set of colors of $X$ which are moved by some $\alpha \in S^\lambda$.

Proof. The first part of $\text{iii)}$ follows from $P_\alpha \subseteq G^\lambda$ and the equivariance of $\pi_\lambda$. Then $\text{b)}$ is an immediate consequence. For $\text{ii)}$ recall that $\pi^{-1}_\lambda(D_0)$ is even a reduced divisor by Lemma 4.2. Thus, $v_{D^*}(\pi^*_\lambda f) = v_D(f)$ for all $f \in k(X^\lambda)$. Moreover, there is a commutative diagram

\[
\begin{array}{ccc}
P_\alpha \times^B D^* & \xrightarrow{\varphi_{D^*,\alpha}} & X \\
P_\alpha \times^B \pi^{-1}_\lambda(D_0) & \xrightarrow{\varphi_{D^*,\alpha}} & X^\lambda \\
P_\alpha \times^B D_0 & \xrightarrow{\varphi_{D^*,\alpha}} & X^\lambda \\
P_\alpha \times^B D & \xrightarrow{\varphi_{D^*,\alpha}} & X^\lambda \\
\end{array}
\]

where the middle square is cartesian and the injections are open embeddings. It follows that $\varphi_{D,\alpha}^\lambda$ and $\varphi_{D^*,\alpha}^\lambda$ have the same inseparable degree, showing the second part of $\text{iii)}$.

Both morphisms also have the same separable degree (1 or 2). Thus, $\alpha$ is of type (2a) for $X^\lambda$ if and only if it is of type (2a) for $X$, proving part of $\text{i)}$. The other types (p), (b), and (a) are distinguished by the number of colors (0, 1, and 2, respectively) moved by $\alpha$.

Thus, the rest of $\text{i)}$ follows from $\text{iii)}$. □

5. The interrelation of roots and colors

In practice, the 1-parameter subgroup $\lambda$ has to be chosen diligently.

5.1. Lemma. Let $X$ be a complete toroidal spherical $G$-variety and $S' \subseteq S$. Then there is a 1-parameter subgroup $\lambda$ such that

a) $S(G^\lambda) = S'$ and
b) the connected center of $G^\lambda$ acts trivially on $X^\lambda$.

Proof. Let $F \subseteq N_\mathbb{Q}(T) := \text{Hom}(\Xi(T), \mathbb{Q})$ be the open face of the Weyl chamber defined by $\alpha = 0$ for all $\alpha \in S'$ and $\alpha > 0$ for $\alpha \in S \setminus S'$. Then $\lambda \in F$ is equivalent to a) (such $\lambda$ are called adapted to $S'$).

Now consider the projection $\pi : N_\mathbb{Q}(T) \rightarrow N_\mathbb{Q}(X)$. Since $\pi(F) \subseteq \mathcal{V}(X)$, the fan $\mathcal{F}$ associated to $X$ induces a complete fan $\mathcal{F}'$ on $\pi(F)$. Let $F^0 := F \setminus \bigcup \pi^{-1}(C)$ where $C$ runs through all $C \in \mathcal{F}$ with $\dim C < \dim \pi(F)$ which is obviously a dense open subset of $F$. We claim that $\lambda \in F^0$ ensures the second property b).

Indeed, $\pi(\lambda)$ lies by construction in the relative interior of a maximal dimensional cone $C'$ of $\mathcal{F}'$. This implies that $\pi(F) \subseteq V'$ where $V'$ is the span of $C'$. Now let $C \in \mathcal{F}$ be minimal with $C' \subseteq C$. Then $\pi(\lambda)$ is also in the relative interior of $C$. Thus, the subspace $V$ spanned by $C$ contains $\pi(F)$. Hence $(F)_Q \subseteq \pi^{-1}(V)$ which implies b). \hfill □

If the fan is changed one can do better.

5.2. Lemma. Let $X_0$ be a homogeneous spherical $G$-variety and $S' \subseteq S$. Then there is a 1-parameter subgroup $\lambda$ and a toroidal compactification $X$ of $X_0$ such that

a) $S(G^\lambda) = S'$ and

b) $\Xi_\mathbb{Q}(X^\lambda) = \langle \Sigma(X^\lambda) \rangle_Q$. 

Proof. Same construction as above but this time we choose $\mathcal{F}$ such that $\dim C$ is as large as possible, i.e., the dimension of the smallest face of $\mathcal{V}$ which contains $C'$. This means precisely b). \hfill □

Remark. A rather trivial application of the last lemma is when $S' = S$. Then $X^\lambda$ has the same roots and colors as $X_0$ but $\Xi_\mathbb{Q}(X^\lambda)$ is spanned by $\Sigma(X)$.

Following Luna [Lun97], an important application of this technique is:

5.3. Proposition. Let $\alpha \in S$ be a simple root.

If $p \neq 2$ then

a) $\alpha \in \Sigma(X)$ if and only if $\alpha$ is of type (a). Thus $S^{(\alpha)} = S \cap \Sigma(X)$.

b) $2\alpha \in \Sigma(X)$ if and only if $\alpha$ is of type (2a). Thus $S^{(2\alpha)} = S \cap \frac{1}{2} \Sigma(X)$.

If $p = 2$ then

c) $\alpha \in \Sigma(X)$ if and only if $\alpha$ is of type (a) or (2a). Thus $S^{(\alpha)} \cup S^{(2\alpha)} = S \cap \Sigma(X)$.

Proof. Without loss of generality we may replace $X$ by a toroidal compactification of its open $G$-orbit. The corresponding fan is denoted by $\mathcal{F}$. Choose $\lambda$ as in Lemma 5.1 with $S' := \{\alpha\}$. Then $G^\lambda$ acts on $X^\lambda$ only via a semisimple quotient $G_0$ of rank 1. Let $G_0/H_0$ be the open $G_0$-orbit in $X^\lambda$.

Assume first $p \neq 2$. Then $\alpha \in S^{(\alpha)}(X)$ iif $\alpha \in S^{(\alpha)}(X^\lambda)$ iif $H_0^{\text{red}} \sim T_0$ iif $\alpha \in \Sigma(X^\lambda)$ iif $\alpha \in \Sigma(X)$. This proves a).

The argument for b) is analogous with $T_0$ replaced by $N(T_0)$. Finally, for $p = 2$ we argue with $\Sigma(G_0/T_0) = \Sigma(G_0/N(T_0)) = \{\alpha\}$. \hfill □
Remarks. 1. The proposition shows that in case $p \neq 2$ the type of the simple roots can be recovered from $S^{(p)}$ and $\Sigma(X)$ as follows:

\begin{equation}
\alpha \in S \text{ is of type } \begin{cases} 
(p) & \text{if } \alpha \in S^{(p)} \\
(a) & \text{if } \alpha \in \Sigma(X) \\
(2a) & \text{if } 2\alpha \in \Sigma(X) \\
(b) & \text{otherwise}
\end{cases}
\end{equation}

This way, all colors can be recovered but some might appear multiple times (see Lemma 2.5). For colors of type $(b)$, that behavior is controlled by $\Sigma(X)$, as well (see the following Proposition 5.4).

2. For $p = 2$ and $\alpha \in S^{(2a)}$ it is tempting to define the corresponding spherical root to be $2\alpha$ instead of $\alpha$. This would make parts $a)$ and $b)$ of Proposition 5.3 work uniformly in all characteristics. We opted against this procedure. The main reason is that otherwise spherical roots would not be roots (possibly not simple) of some root system. Example: For $p = 2$ and $X = \text{SL}(3)/\text{SO}(3)$ the two roots are $\alpha_1$ and $\alpha_1 + \alpha_2$ (see [Sch11]) which are visibly contained in an $A_2$-root system. On the other hand, the root $\alpha_1$ is of type $(2a)$ and the set $\{2\alpha_1, \alpha_1 + \alpha_2\}$ is not part of any root system.

5.4. Proposition.

For two distinct simple roots $\alpha_1, \alpha_2 \in S^{(b)}$ are equivalent:

$a)$ $\alpha_1$ and $\alpha_2$ move the same color $D$.

$b)$ $\alpha_1$ and $\alpha_2$ are orthogonal to each other and $q_1\alpha_1 + q_2\alpha_2 \in \Sigma(X)$ for two $p$-powers $q_1, q_2$ (one of which is necessarily equal to 1).

In this case

\begin{equation}
q_1^{-1}\alpha_1^r = q_2^{-1}\alpha_2^r.
\end{equation}

Moreover, $D$ is not moved by any other simple root.

Proof. Again replace $X$ by a toroidal compactification and choose $\lambda$ as in Lemma 5.1 with $S' = \{\alpha_1, \alpha_2\}$. Now $G^\lambda$ acts on $X^\lambda$ via a semisimple group $G_0$ of rank 2. The simple roots of $G_0$ are $\alpha_1$ and $\alpha_2$, and both are of type $(b)$ with respect to $X^\lambda$.

Assume first $b)$. Then $G_0$ is of type $A_1A_1$. An easy inspection of its subgroups shows that $X^\lambda$ has a spherical root of the given form if and only if its open orbit is isogenous to $\text{SL}(2) \times \text{SL}(2)/(F_{q_1} \times F_{q_2})\text{SL}(2)$ (with $F_q$=Frobenius morphism of $\text{SL}(2)$). In that case one checks that $a)$ holds for $X^\lambda$ and therefore for $X$.

Conversely assume $a)$. Then $X^\lambda$ has precisely one color $D$ which is moved by both simple roots. Thus, (2.14) implies that a relation like (5.2) holds. In particular, the rank of $X^\lambda$ is 1.

Now one could use the classification of spherical varieties of rank $\leq 1$ in [Kno13] and conclude that $G_0$ is of type $A_1A_1$ having a root of the given form. A selfcontained argument goes as follows. We may assume that $X = G/H$ is homogeneous where $H$ is reduced and connected. The color and the halfline $\nu(X)$ lie opposite to each other. By [Kno91, Thm. 6.7] the variety $X$ is affine, thus $H$ is reductive. Since there is only
one color, formula (2.3) implies that $H$ is semisimple. Moreover, the dimension formula [Kno91, Thm. 6.6] shows that $\dim H = 3, 4, 5, 7$ for $G = A_1A_1, A_2, B_2, G_2,$ respectively. This shows that $G$ is isogenous to $SL(2) \times SL(2)$ and that $H \cong SL(2)$ is embedded diagonally using the Frobenius morphisms. The assertion $b)$ follows.

Formula (5.2) follows immediately from (2.14). Finally, assume $\alpha_3 \in S$ moves $D,$ as well. Then $\alpha_3 \in S^{(b)}$ (Lemma 2.5). Thus, by the above, $\alpha_3$ would be orthogonal to $\alpha_1$ and $\alpha_2.$ Moreover, $q_1^2\alpha_1 + q_3\alpha_3 \in \Sigma(X)$ for some $p$-powers $q_1, q_3.$ Now equation (5.2) implies the contradiction
\[(5.3) \quad 2q_1^2q_1^{-1} = \langle q_1^2\alpha_1 + q_3\alpha_3, q_1^{-1}\alpha_1^\vee \rangle = \langle q_1^2\alpha_1 + q_3\alpha_3, q_2^{-1}\alpha_2^\vee \rangle = 0.\]

\[\Box\]

6. Localization at $\Sigma$

Localization at $S$ allows one to pass from $S$ to a subset. There is a second, older, kind of localization which does the same thing with $\Sigma(X).$ Geometrically, it simply corresponds to looking at an orbit in a carefully chosen toroidal embedding. The next result summarizes what was already known about localization at $\Sigma$:

6.1. Proposition. Let $X$ be a toroidal spherical variety and let $Y \subseteq X$ be an orbit. Put $V := \langle C \rangle^\perp \subseteq \Xi_{\mathbb{Q}}(X).$ Then
\[\begin{align*}
a) \ &\Xi_p(Y) = \Xi_p(X) \cap V. \\
b) \ &\Sigma(Y) = \Sigma(X) \cap V. \\
c) \ &S^{(p)}(Y) = S^{(p)}(X).
\end{align*}\]

Proof. Part $a)$ follows e.g. from [Kno91, Thm. 1.3]. Moreover, $\mathcal{V}(Y) = (\mathcal{V}(X) + V)/V$ where $V = \langle C \rangle_{\mathbb{Q}}$ (follows from [Kno91, Satz 7.4]) which implies $b).$ Part $c)$ follows for example from the fact that all closed orbits in any toroidal compactification of $X$ are of the form $G/Q$ with $Q^{\text{red}} = \mathbb{P}$ where $P$ is the parabolic corresponding to $S^{(p)}.$ \[\Box\]

If $p \neq 2,$ then the remark after Proposition 5.3 allows now to determine the type of a simple root for $Y.$

\begin{align*}
6.1) \quad &S^{(p)}(Y) = S^{(p)}(X) \\
6.2) \quad &S^{(a)}(Y) = S^{(a)}(X) \cap V \\
6.3) \quad &S^{(2a)}(Y) = S^{(2a)}(X) \cap V \\
6.4) \quad &S^{(b)}(Y) = S^{(b)}(X) \cup (S^{(a)}(X) \setminus V) \cup (S^{(2a)}(X) \setminus V)
\end{align*}

For $p = 2$ equations (6.2) and (6.3) have to be replaced by the weaker equality
\[(6.5) \quad S^{(a)}(Y) \cup S^{(2a)}(Y) = (S^{(a)}(X) \cap V) \cup (S^{(2a)}(X) \cap V)
\]

The next lemma shows (in particular) that moreover
\[(6.6) \quad S^{(2a)}(Y) \subseteq S^{(2a)}(X) \cap V\]

6.2. Lemma. Let $X$ and $Y$ be as above and let $\alpha \in S \cap V$ be of type $(a)$ for $X.$ Then $\alpha$ is also of type $(a)$ for $Y.$ Moreover, let $D$ be a color of $X$ which is moved by $\alpha.$ Then $E = (D \cap Y)^{\text{red}}$ is a color of $Y$ which is moved by $\alpha$ and there is a $p$-power $q$ such that $\delta_E$ is the restriction of $q\delta_D$ to $\Xi_p(Y).$
Proof.} We plan to use localization at \( S \) but face the problem that \( C_Y \) may not meet \( N_Q(X) \), the image of the antidominant Weyl chamber in \( N_Q(X) \). To bypass this problem, we go one dimension up: the group \( \mathcal{C} := G \times G_m \) acts on \( \mathcal{X}^0 := X \times G_m \). Then \( N_Q(\mathcal{X}^0) = N_Q(X) \oplus \mathbb{Q} \) and \( \mathcal{V}(\mathcal{X}^0) = \mathcal{V}(X) \times \mathbb{Q} \). Now choose any \( v_0 \) in the relative interior of \( N^-(X) \cap \{ \alpha = 0 \} \subset \mathcal{V}(X) \) and put \( \mathcal{C} := (C \times 0) + \mathbb{Q}_{\geq 0} (v_0, 1) \). Choose any fan \( \mathcal{F} \) whose support is \( \mathcal{V}(\mathcal{X}^0) \) and which contains \( \mathcal{C} \). This gives rise to a toroidal \( \mathcal{C} \)-variety \( \mathcal{X} \). Moreover, \( \mathcal{X} \) contains \( Y \times G_m \) since \( C_Y \) is a face of \( \mathcal{C} \). Its closure is denoted by \( \overline{Y} \).

Now choose any \( v \in \mathcal{C}_0 \) small enough such that \( v + v_0 \) is still in the relative interior of \( N^-(X) \cap \{ \alpha = 0 \} \) and choose \( a \in \mathbb{Z}_{>0} \) such that \( v_1 := a(v + v_0, 1) \) is the image of a 1-parameter subgroup \( \lambda \) of \( \mathcal{C} \). Then, by construction, \( S^\lambda = \{ \alpha \} \) and \( \mathcal{G}^\lambda \) is of semisimple rank 1. Moreover, \( \mathcal{X}^{\lambda} \) is contained in \( \overline{Y} \) by \textit{Proposition 4.8}. Thus, we get a diagram

\[
\begin{array}{ccc}
\tilde{Y}^\lambda & \longrightarrow & \overline{Y}^\lambda \longrightarrow \mathcal{X}^\lambda \\
| & & | \\
\tilde{\pi}_\lambda & \longrightarrow & \pi_\lambda \\
\end{array}
\]

where \( \tilde{Y} \) is the normalization of \( \overline{Y} \) and where the vertical arrows represent Bialynicki-Birula contractions on the open cell. By \textit{Proposition 4.9}, the type of \( \alpha \) on \( \mathcal{X}^{\lambda} \) is \((a)\). This means that the open \( \mathcal{G}^{\lambda} \)-orbit in \( \mathcal{X}^{\lambda} \) is of the form \( \mathcal{G}^{\lambda} / H_0 \) where \( H_0^{\text{red}} \) is diagonalizable.

Now we argue that \( \nu \) is purely inseparable. Indeed, the open orbit in \( \tilde{Y}^\lambda \) is \( \mathcal{G}^{\lambda} / H_1 \) with \( H_1^{\text{red}} \subset \subset H_0^{\text{red}} \subset T \). This already shows that \( \alpha \) is of type \((a)\) for \( \tilde{Y} \), hence for \( \overline{Y} \) and \( Y \). Since both \( H_1^{\text{red}} \) and \( H_0^{\text{red}} \) are linearly reductive abelian groups we have

\[
[H_0^{\text{red}} : H_1^{\text{red}}] = [\Xi_p(\tilde{Y}^{\lambda}) : \Xi_p(\mathcal{X}^{\lambda})].
\]

On the other hand

\[
\Xi_p(\tilde{Y}^{\lambda}) = \Xi_p(\tilde{Y}) = \Xi_p(\overline{Y}) = \Xi_p(\mathcal{X}) \cap \overline{V} = \Xi_p(\mathcal{X}^{\lambda})
\]

where \( \overline{V} := \langle \mathcal{C} \rangle \). This shows that \( \nu \) is generically injective and therefore purely inseparable.

The color \( D \) gives rise to the color \( D \times G_m \) of \( \mathcal{X} \). Its image \( D_0 \) in \( \mathcal{X}^{\lambda} \) is a color of \( \mathcal{X}^{\lambda} \). Now \( E_0 = \nu^{-1}(D_0)^{\text{red}} \) is a color of \( \tilde{Y}^{\lambda} \). Clearly \( \nu^{-1}(D_0) = qE_0 \) for some \( p \)-power \( q \). Finally, the closure of \( (\tilde{\pi}_\lambda)^{-1}(E_0) \) is a color of \( \overline{Y} \) which is of the form \( E \times G_m \) where \( E = (D \cap Y)^{\text{red}} \).

Recall \( V = \langle \mathcal{C} \rangle_Q \). Then

\[
\overline{V} = \{(\chi, -v_0(\chi) \mid \chi \in V) \simeq V
\]

and \( \Xi_Q(\mathcal{X}) = \Xi_Q(X) \oplus \mathbb{Q}, \Xi_Q(\overline{Y}) = V \oplus \mathbb{Q}, \) and \( \Xi_Q(\mathcal{X}^{\lambda}) \). Thus for \( \chi \in \Xi_Q(X) \):

\[
\delta_D(\chi) = \delta_{D \times G_m}(\chi, 0) = \delta_{D \times G_m}(\chi, -v_0(\chi)) = \delta_{D_0}(\chi, -v_0(\chi))
\]

and similarly \( \delta_D(\chi) = \delta_{E}(\chi, -v_0(\chi)) \) for all \( \chi \in V \). Thus for \( \chi \in V \):

\[
\delta_D(\chi) = \delta_{D_0}(\chi, -v_0(\chi)) = q\delta_{E_0}(\chi, -v_0(\chi)) = q\delta_{E}(\chi).
\]

\[\square\]
Localized at $\Sigma$ is, a priori, not possible for all subsets of $\Sigma(X)$. Therefore, we define:

**6.3. Definition.** A subset $\Sigma'$ of $\Sigma(X)$ is called a set of neighbors if there is $v \in \mathcal{V}(X)$ such that

$$
\Sigma' = \{ \sigma \in \Sigma(X) \mid v(\sigma) = 0 \}.
$$

Equivalently, $\Sigma'$ is a set of neighbors if $Q_{\geq 0} \Sigma'$ is a face of $Q_{\geq 0} \Sigma(X)$. Two spherical roots $\alpha$ and $\beta$ are called neighbors if they are distinct and if $\{\alpha, \beta\}$ is a set of neighbors.

Clearly, if $\Sigma(X)$ is linearly independent then all subsets are sets of neighbors. This is always the case if $p \neq 2$ (see [Bri90] for char $k = 0$ and [Kno13, Corollary 4.8] for the general case). For $p = 2$ and $X = SL(4)/SO(4)$, Schalke has shown (unpublished) that $\Sigma(X) = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3\}$. Since $\alpha_1 + (\alpha_2 + \alpha_3) = (\alpha_1 + \alpha_2) + \alpha_3$, the pairs $(\alpha_1, \alpha_2 + \alpha_3)$ and $(\alpha_1 + \alpha_2, \alpha_3)$ are not neighbors. In fact, $\mathcal{V}(X)$ is the cone over a quadrangle and the given pairs correspond to opposite faces.

**6.4. Lemma.** If $\alpha, \beta \in \Sigma(X)$ are multiples of simple roots then they are neighbors.

**Proof.** The set $Q_{\geq 0} \alpha + Q_{\geq 0} \beta$ is already a face of $Q_{\geq 0} S$ and therefore also for the smaller cone $Q_{\geq 0} \Sigma(X)$.

The following statement can be used to exclude certain configurations of colors and roots (see [Kno13, Proof of Thm. 4.5]).

**6.5. Proposition.** Let $D \in \Delta(X)^{(a)}$ be moved by $\alpha \in S^{(a)}$ and let $\sigma \in \Sigma(X)$ be a neighbor of $\alpha$ with $\delta_D(\sigma) > 0$. Then $\sigma \in S^{(a)}$ and $D$ is moved by $\sigma$, as well.

**Proof.** We first reduce to the case that $X$ is of rank 2 with $\Sigma = \{\alpha, \sigma\}$. Because $\alpha$ and $\sigma$ are neighbors one can choose a pointed cone $\mathcal{C}$ inside $\mathcal{V}(X) \cap \{\alpha = \sigma = 0\}$ which is of codimension 2 in $N_0(X)$. Let $\overline{X}$ be the simple embedding corresponding to $\mathcal{C}$ and let $Y$ be its closed orbit. Then **Proposition 6.1** implies $\Sigma(Y) = \{\alpha, \sigma\}$. Moreover, using the remark after **Lemma 5.2** there is a spherical variety $Y'$ with $\Sigma(Y') = \{\alpha, \sigma\}$ and $\mathrm{rk} Y' = 2$. Moreover, **Lemma 6.2** implies that $Y'$ still has a color $E$ moved by $\alpha$ with $\delta_E(\sigma) = q_0 \delta_D(\sigma) > 0$. Let us assume that we can prove that $\sigma \in S^{(a)}(Y')$ and that $E$ is moved by $\sigma$. Clearly, $D$ is moved by $\sigma$ in $X$, as well. Moreover, since $\alpha \in \Sigma(X)$, either $\alpha \in S^{(a)}(X)$ or $p = 2$ and $\alpha \in S^{(2a)}(X)$. But the latter case cannot happen since then $D$ could not be moved by any other simple root (**Lemma 2.5**). So $\sigma \in S^{(a)}(X)$, which finishes the reduction step.

From now on we assume $\mathrm{rk} X = 2$ and, without loss of generality, that $X = G/H$ where $H$ is reduced. Since, $\delta_D(\sigma) > 0$ by assumption and $\delta_D(\alpha) = q_{D,\alpha} > 0$, the cone generated by $\mathcal{V}(X)$ and $\delta_D$ is the entire space $N := N_0(X)$. From that we get a morphism $X = G/H \rightarrow G/P$ with rank $G/P = \dim N/N = 0$ ([Kno91, 4.6])\(^1\). Hence $P$ is a parabolic subgroup with an identification $\Delta(G/P) = \Delta(G/H) \setminus \{D\}$. We may choose $P$ in such a way that it is opposite to $B$.

\(^1\)The idea for this construction is due to Guido Pezzini.
Every $\beta \in S \setminus S^{(p)}$ moves at least one color and $\alpha$ moves even two. Assume first that these colors are not all different. Then, according to Lemma 2.5 and Proposition 5.4 there are two possibilities:

a) $\sigma = \alpha_1 + q\alpha_2$ with $\alpha_1, \alpha_2 \in S$ orthogonal. But then $\alpha_1$ and $\alpha_2$ would move the same color in $G/P$ which is impossible.

b) $S^{(a)}$ contains another element $\beta$ besides $\alpha$. But then $\beta \in \Sigma(X)$ (Proposition 5.3), hence $\sigma = \beta \in S^{(a)}$. Moreover, there is a color $D'$ moved by both $\alpha$ and $\sigma$. We claim that $D' = D$. Suppose not. Then $D$ and $D'$ are the two colors moved by $\alpha$ and Proposition 2.3 implies the contradiction

$$
(6.14) \quad \delta_D^{(a)}(\sigma) = \langle \sigma, \alpha^\vee \rangle - \delta_D^{(a)}(\sigma) = \langle \sigma, \alpha^\vee \rangle - \frac{q_{D',\alpha}}{q_{D',\sigma}} \delta_D^{(a)}(\sigma) = \langle \sigma, \alpha^\vee \rangle - \frac{q_{D',\alpha}}{q_{D',\sigma}} < 0.
$$

Thus, we are exactly in the asserted situation, i.e., $\sigma \in S^{(a)}$ moving $D$.

So, assume from now on that the colors moved by all the $\beta \in S \setminus S^{(p)}$ are different. Then

$$
(6.15) \quad \#\Delta(G/P) \geq \#S \setminus S^{(p)}.
$$

Consider a toroidal completion $\overline{X}$ of $X$. Then the morphism $X \to G/P$ extends to $\overline{X}$ ([Kno91, 4.1]). Every closed orbit in $\overline{X}$ is isogenous to $G/\sim P_X$ where $P_X$ is the parabolic attached to $S^{(p)}$. Hence $P_X \subseteq P_{\text{red}}$. Thus (6.15) implies $P = P_X$.

Let $Y = G/H_1 \subset \overline{X}$ be the rank-1-orbit corresponding to the halfline $\mathcal{V}(X) \cap \{\sigma = 0\}$. Then $\Sigma(Y) = \{\sigma\}$. Because of $P = P_X$, the fiber $P_X/H_1^{\text{red}}$ is one-dimensional and therefore isomorphic to $P^1$, $G_m$, or $A^1$. The first case is impossible since $H_1$ is not parabolic. The second case is excluded since $H_1$ is not horospherical. Thus $P_X/H_1^{\text{red}} \cong A^1$.

This means in particular that (a conjugate of) $H_1$ contains the maximal torus $T$ of $G$ and that $H_1$ contains all root subgroups $U_\gamma$ which are contained in $P_X$ except for one, which is denoted by $\gamma$, and which lies in $P_X^\gamma$. The $U_\beta$ corresponding to $\beta \in S$ generate the maximal unipotent subgroup of $G$. This implies $\gamma \in S$. Moreover $U_\gamma$ is a 1-dimensional module for the Levi part of $P_X$. This shows that $H_1^{\text{red}}$ is, in fact, induced from $\text{PGL}(2)/G_m$ (on induction in arbitrary characteristic see [Kno13, §2, in particular Lemma 2.1]). Hence $\sigma \in S^{(a)}$. But in that case (6.15) would be a strict inequality which is not true because of $P = P_X$. \hfill $\square$

Proposition 6.5 can be used to give bounds for $\delta_D(\sigma)$:

6.6. Corollary. With $\alpha$ and $\sigma$ as above, assume that $\sigma \notin S$ or that $\sigma \in S$ but does not move either color moved by $\alpha$. Then

$$
(6.16) \quad q_{D,\alpha}^{-1} \langle \sigma, \alpha^\vee \rangle \leq \delta_D(\sigma) \leq 0.
$$

Proof. The righthand inequality follows directly from Proposition 6.5. For the lighthand inequality apply Proposition 6.5 to the other color $D'$ moved by $\alpha$ and observe that

$\delta_D^{(a)}(\sigma) = \alpha^\vee - \delta_D^{(a)}(\sigma)$ (Proposition 2.3). \hfill $\square$

In positive characteristic, these bounds are less valuable since we didn’t derive a bound on the denominator of $\delta_D(\sigma)$. Such a bound exists (in terms of the $q_{D,\alpha}$’s) and will be included in a future paper. Then (6.16) leaves only finitely many possibilities for $\delta_D(\sigma)$. 18
7. The $p$-spherical system

We summarize what we have proved so far in terms of a combinatorial structure which generalizes Luna’s spherical systems. But first, we need some more terminology:

7.1. Definition. Let $G$ be a connected reductive group.

a) An element $\sigma \in \Xi_G(T)$ is called a spherical root for $G$ if there is a spherical $G$-variety $X$ such that $\sigma$ is a spherical root for $X$. The set of spherical roots for $G$ is denoted by $\Sigma(G)$.

b) A spherical root $\sigma \in \Sigma(G)$ is compatible to a subset $S(p) \subseteq S$ if there is a spherical $G$-variety $X$ with $\sigma \in \Sigma(X)$ and $S(p) = S(p)(X)$.

Remarks. 1. Proposition 6.1 shows that in the definition above one may assume without loss of generality $\text{rk} \ X = 1$.

2. Spherical varieties of rank 1 have been classified by Akhiezer [Akh83] in characteristic zero and [Kno13] in general. In particular, for every $G$ there is a complete description of $\Sigma(G)$ (see [Kno13, §2 and §7]).

3. One result of that classification is that $\Sigma(G)$ is infinite unless $\text{char} \ k = 0$ or $G$ is simple of rank $\leq 2$.

For the following recall that $\mathbb{Z}_p = \mathbb{Z}[\frac{1}{p}]$ and $\Xi_p := \Xi \otimes \mathbb{Z}_p$ for any abelian group $\Xi$.

7.2. Definition. Let $p \neq 2$. Then a $p$-spherical system for $G$ consists of

- a subgroup $\Xi \subseteq \Xi(T)$,
- a subset $\Sigma \subseteq \Xi_p \cap \Sigma(G)$,
- a subset $S(p) \subseteq S$,
- a finite set $\Delta^{(a)}$,
- a map $\delta : \Delta^{(a)} \rightarrow \text{Hom}(\Xi, \mathbb{Z}) : D \mapsto \delta_D$, and
- a map $S \setminus (S(p) \cup S^{(a)}) \rightarrow p^N : \alpha \mapsto q_\alpha$ where $S^{(a)} := S \cap \Sigma$.

Of course, these data are subject to some conditions. Here, we list only those which are straightforward generalizations of Luna’s axioms. It is safe to say that more axioms have to be imposed which deal specifically with issues of positive characteristic. We keep the notation that $\alpha^r$ denotes the restriction of $\alpha^v$ to $\Xi$.

A1 All $\sigma \in \Sigma$ are primitive vectors of $\mathbb{Z}S \cap \Xi_p$.

A2 $\alpha^r = 0$ for all $\alpha \in S(p)$.

A3 Every $\sigma \in \Sigma$ is compatible to $S(p)$.

A4 For all $D \in \Delta^{(a)}$ and $\sigma \in \Sigma \setminus S^{(a)}$ holds $\delta_D(\sigma) \leq 0$.

A5 For every $\alpha \in S^{(a)}$ there are exactly two $D \in \Delta^{(a)}$ with $\delta_D(\alpha) > 0$. Conversely, for every $D \in \Delta^{(a)}$ there is at least one $\alpha \in S^{(a)}$ with $\delta_D(\alpha) > 0$.

A6 For $\alpha \in S^{(a)}$ let $D^+ \neq D^- \in \Delta^{(a)}$ with $\delta_{D^+}(\alpha) > 0$. Then $q_{\alpha,D^+} := \delta_{D^+}(\alpha)^{-1} \in p^N$ and $q_{\alpha,D^+} - q_{\alpha,D^-} = \alpha^r$. 

19
A7 Let $\alpha \in S$ with $2\alpha \in \Sigma$. Then $\alpha \not\in \Xi_p$ and \( \frac{1}{2}\alpha^r(\Xi_p) \subseteq \mathbb{Z}_p \). Moreover, $\alpha^r(\sigma) \leq 0$ for all $\sigma \in \Sigma \setminus \{2\alpha\}$.

A8 Let $q\alpha_1 + \alpha_2 \in \Sigma$ with $\alpha_1 \perp \alpha_2$. Then $q^{-1}\alpha_1^r = \alpha_2^r$ and $q^{-1}q_{\alpha_1} = q_{\alpha_2}$.

The point is, of course, that for $p \neq 2$ every homogeneous spherical variety $X$ gives rise to a $p$-spherical system. More specifically we put
\[
(7.1) \quad \Xi := \Xi(X), \quad \Sigma := \Sigma(X), \quad S^{(p)} := S^{(p)}(X), \quad \Delta^{(a)} := \Delta^{(a)}(X), \quad \delta_D := \delta_D^X.
\]
The only new constituents are the $p$-powers. For $\alpha \in S \setminus (S^{(p)} \cup S^{(a)}) = S^{(b)} \cup S^{(2a)}$ we define $q_\alpha$ as $q_{\alpha,D}$ from Proposition 2.3 where $D$ is the unique color moved by $\alpha$.

Now we verify all axioms.

A1 Holds by definition of $\Sigma(X)$.

A2 See Proposition 2.3.

A3 Follows from the definition of “compatibility”.

A4 This is Corollary 6.6 in conjunction with [Kno13, Corollary 4.8] which implies that for $p \neq 2$ any two spherical roots are neighbors.

A5 The first part follows also from Corollary 6.6 and Proposition 2.3. The second part holds by definition of $\Delta^{(a)}(X)$.

A6 This follows from Proposition 2.3.

A7 The first part is Proposition 5.3(b) and Corollary 6.4. The second follows from [Kno13, Theorem 4.5].

A8 This is Proposition 5.4 and Proposition 2.3.

Remarks. 1. In characteristic 0, Luna (see [Lun01, 5.1]) used Wasserman’s tables [Was96] of spherical rank-2-varieties to verify the axioms. So our approach is more conceptual in that it uses only the classification of rank-1- but not of rank-2-varieties².

2. The case $p = 2$ requires some modifications. To distinguish simple roots of type $(a)$ and $(2a)$ we redefine $S^{(a)}$ as
\[
(7.2) \quad S^{(a)} := \{ \alpha \in S \cap \Sigma \mid \delta_D(\alpha) > 0 \text{ for some } D \in \Delta^{(a)} \}
\]
This works indeed for spherical systems coming from spherical varieties: Suppose there is $\alpha \in S^{(2a)}(X)$ and $D \in \Delta^{(a)}$ with $\delta_D(\alpha) > 0$. Then $D$ is moved by some $\beta \in S^{(a)}$. Since $\alpha$ and $\beta$ are neighbors (Lemma 6.4) we get a contradiction to Proposition 6.5.

With this change all axioms hold for $p = 2$ except for one: in A4 one has to require that $\sigma$ and $\alpha$ are neighbors. Observe that A7 is vacuously satisfied.

3. It is a natural question whether spherical varieties are classified by their $p$-spherical system. In characteristic zero, the answer is “yes” according to work by Luna [Lun01], Losev [Los09], Cupit-Foutou [CF09], and Bravi-Pezzini [BP11a, BP11b, BP11c]. For $p \neq 2$ or 3, it might be possible that the $p$-spherical system determines the variety uniquely. For example, all complete homogeneous varieties are classified by $p$-spherical systems with $\Xi = 0$ (see the example before Lemma 2.2). Furthermore, the author convinced himself

²Note, that the tables in [Was96] are slightly incomplete: of the series $G = \text{Sp}(2n) \times \text{Sp}(2)$, $H = \text{Sp}(2n-2) \times \text{Sp}(2)$ with $n \geq 2$ only the first case, $n = 2$, is stated. I would like to thank Guido Pezzini for pointing that out to me.
that this also holds for spherical varieties of rank 1. If $p = 2$ or $p = 3$ then uniqueness does not even hold for complete homogeneous varieties (see [Wen94, Prop. 4]) due to exceptional isogenies. If $p = 2$ then uniqueness is wrong already for $G = SL(2)$ as then $G$ contains non-standard horospherical subgroup schemes (see [Kno95b]).

4. The above list of axioms A1–A8 is definitely only preliminary. Even in the rank-1-case they do not suffice. For example, there is no axiom bounding the lattice $\Xi$ from below. We plan to return to this problem in the future.

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