Directed unions of local quadratic transforms of regular local rings and pullbacks

Lorenzo Guerrieri, William Heinzer, Bruce Olberding and Matt Toeniskoetter

Abstract Let \( \{R_n, m_n\}_{n \geq 0} \) be an infinite sequence of regular local rings with \( R_{n+1} \) birationally dominating \( R_n \) and \( m_n R_{n+1} \) a principal ideal of \( R_{n+1} \) for each \( n \). We examine properties of the integrally closed local domain \( S = \bigcup_{n \geq 0} R_n \).

Keywords • regular local ring • local quadratic transform • valuation ring • pullback construction

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1 Introduction

Let \( R \) be a regular local ring with maximal ideal \( m = (x_1, \ldots, x_n)R \), where \( n = \text{dim } R \) is the Krull dimension of \( R \). Choose \( i \in \{1, \ldots, n\} \), and consider the overring \( R[\frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}] \) of \( R \). Choose any prime ideal \( P \) of \( R[\frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}] \) that contains \( m \). Then the ring \( R_1 := R[\frac{x_i}{x_i}, \ldots, \frac{x_n}{x_i}]_P \) is a local quadratic transform of \( R \), and \( R_1 \) is again a regular local ring and \( \text{dim } R_1 \leq n \). Iterating the process we obtain a sequence \( R = R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \) of regular local overrings of \( R \) such that for each \( i \), \( R_{i+1} \) is a local quadratic transform of \( R_i \). The sequence of positive integers \( \{\text{dim } R_i\}_{i \in \mathbb{N}} \)

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stabilizes, and \( \dim R_i = \dim R_{i+1} \) for all sufficiently large \( i \). If \( \dim R_i = 1 \), then necessarily \( R_i = R_{i+1} \), while if \( \dim R_i \geq 2 \), then \( R_i \not\subseteq R_{i+1} \).

The process of iterating local quadratic transforms of the same Krull dimension is the algebraic expression of following a closed point through a sequence of blow-ups of a nonsingular point of an algebraic variety, with each blow up occurring at a closed point in the fiber of the previous blow-up. This geometric process plays a central role in embedded resolution of singularities for curves on surfaces (see, for example, [3] and [8] Sections 3.4 and 3.5), as well as factorization of birational morphisms between nonsingular surfaces ([1] Theorem 3 and [32] Lemma, p. 538). These applications depend on properties of iterated sequences of local quadratic transforms of a two-dimensional regular local ring. For a two-dimensional regular local ring \( R \), Abhyankar ([1] Lemma 12] shows that the limit of this process of iterating local quadratic transforms \( R = R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots \) results in a valuation ring that birationally dominates \( R \); i.e., \( V = \bigcup_{i=0}^\infty R_i \) is a valuation ring with the same quotient field as \( R \) and the maximal ideal of \( V \) contains the maximal ideal of \( R \).

Moving beyond dimension two, examples due to David Shannon ([30] Examples 4.7 and 4.17) show that the union \( S = \bigcup_i R_i \) of an iterated sequence of local quadratic transforms of a regular local ring of Krull dimension > 2 need not be a valuation ring. The recent articles [21, 22] address the structure of such rings and how this structure encodes asymptotic properties of the sequence \( \{R_i\}_{i=0}^\infty \). We call \( S \) a quadratic Shannon extension of \( R \). In general, a quadratic Shannon extension need not be a valuation ring nor a Noetherian ring, although it is always an intersection of two such rings (see Theorem [2.2]).

The class of quadratic Shannon extensions separates naturally into two cases, the archimedean and non-archimedean cases. A quadratic Shannon extension \( S \) is non-archimedean if there is an element \( x \) in the maximal ideal of \( S \) such that \( \bigcap_{i>0} x^iS \neq 0 \). The class of non-archimedean quadratic Shannon extensions is analyzed in detail in [21 and 22]. We carry this analysis further in the present article by using techniques from multiplicative ideal theory to classify a non-archimedean quadratic Shannon extension as the pullback of a valuation ring of rational rank one along a homomorphism from a regular local ring onto its residue field. We present several variations of this classification in Lemma [4.3] and Theorems [4.8 and 5.1].

The pullback description leads in Theorem [4.5] to existence results for both archimedean and non-archimedean quadratic Shannon extensions contained in a localization of the base ring at a nonmaximal prime ideal. As another application, in Theorem [5.2] we use pullbacks to characterize the quadratic Shannon extensions \( S \) of regular local rings \( R \) such that \( R \) is essentially finitely generated over a field of characteristic 0 and \( S \) has a principal maximal ideal.

That non-archimedean quadratic Shannon extensions occur as pullbacks is also useful because of the extensive literature on transfer properties between the rings in a pullback square. In Section [6] we use the pullback classification along with structural results for archimedean quadratic Shannon extensions from [21] to show in Theorem [6.2] that a quadratic Shannon extension is coherent if and only if it is a valuation domain.
Our methods sometimes involve local quadratic transforms of Noetherian local domains that need not be regular local rings. To formalize these notions as well as those mentioned above, let \((R, m)\) be a Noetherian local domain and let \((V, m_V)\) be a valuation domain birationally dominating \(R\). Then \(mV = xV\) for some \(x \in m\). The ring \(R_1 = R[x/x^2]_{mV}^R[x/x^2]_{mV}\) is called a local quadratic transform (LQT) of \(R\) along \(V\). The ring \(R_1\) is a Noetherian local domain that dominates \(R\) with maximal ideal \(m_1 = m_V \cap R_1\). Since \(V\) birationally dominates \(R_1\), we may iterate this process to obtain an infinite sequence \(\{R_n\}_{n \geq 0}\) of LQTs of \(R_0 = R\) along \(V\). If \(R_n = V\) for some \(n\), then \(V\) is a DVR and the sequence stabilizes with \(R_m = V\) for all \(m \geq n\). Otherwise, \(\{R_n\}\) is an infinite strictly ascending sequence of Noetherian local domains.

If \(R\) is a regular local ring (RLR), it is well known that \(R_1\) is an RLR; cf. [28 Corollary 38.2]. Moreover, \(R = R_1\) if and only if \(\dim R \leq 1\). Assume that \(R\) is an RLR with \(\dim R \geq 2\) and \(V\) is minimal as a valuation overring of \(R\). Then \(\dim R_1 = \dim R\), and the process may be continued by defining \(R_2\) to be the LQT of \(R_1\) along \(V\). Continuing the procedure yields an infinite strictly ascending sequence \(\{R_n\}_{n \in \mathbb{N}}\) of RLRs all dominated by \(V\).

In general, our notation is as in Matsumura [26]. Thus a local ring need not be Noetherian. An element \(x\) in the maximal ideal \(m\) of a regular local ring \(R\) is said to be a regular parameter if \(x \notin m^2\). It then follows that the residue class ring \(R/\langle x\rangle\) is again a regular local ring. We refer to an extension ring \(B\) of an integral domain \(A\) as an overring of \(A\) if \(B\) is a subring of the quotient field of \(A\). If, in addition, \(A\) and \(B\) are local and the inclusion map \(A \rightarrow B\) is a local homomorphism, we say that \(B\) birationally dominates \(A\). We use UFD as an abbreviation for unique factorization domain, and DVR as an abbreviation for rank 1 discrete valuation ring. If \(P\) is a prime ideal of a ring \(A\), we denote by \(\kappa(P)\) the residue field \(A_P/P\) of \(A_P\).

## 2 Quadratic Shannon extensions

Let \((R, m)\) be a regular local ring with \(\dim R \geq 2\) and let \(F\) denote the quotient field of \(R\). David Shannon’s work in [30] on sequences of quadratic and monoidal transforms of regular local rings motivates our terminology quadratic Shannon extension in Definition 2.1.

**Definition 2.1.** Let \((R, m)\) be a regular local ring with \(\dim R \geq 2\). With \(R = R_0\), let \(\{R_n, m_n\}\) be an infinite sequence of RLRs, where \(\dim R_n \geq 2\) for each \(n\). If \(R_{n+1}\) is an LQT of \(R_n\) for each \(n\), then the ring \(S = \bigcup_{n \geq 0} R_n\) is called a quadratic Shannon extension of \(R\).

If \(\dim R = 2\), then the quadratic Shannon extensions of \(R\) are precisely the valuation rings that birationally dominate \(R\) and are minimal as a valuation overring of \(R\).

\[\text{1 In [21 and 22, the authors call } S \text{ a Shannon extension of } R. \text{ We have made a distinction here with monoidal transforms. Since } \dim R_n \geq 2, \text{ we have } R_n \subseteq R_{n+1} \text{ for each positive integer } n \text{ and } \bigcup_n R_n \text{ is an infinite ascending union.}\]
If \( \text{dim } R > 2 \), then, examples due to Shannon [30, Examples 4.7 and 4.17] show that there are quadratic Shannon extensions that are not valuation rings. Similarly, if \( \text{dim } R > 2 \), then there are valuations rings \( V \) that birationally dominate \( R \) with \( V \) minimal as a valuation overring of \( R \), but \( V \) is not a Shannon extension of \( R \). Indeed, if \( V \) has rank \( > 2 \), then \( V \) is not a quadratic Shannon extension of \( R \); see [14, Proposition 7].

These observations raise the question of the ideal-theoretic structure of a quadratic Shannon extension of a regular local ring \( R \) with \( \text{dim } R > 2 \), a question that was taken up in [21] and [22]. In this section we recall some of the results from [21] and [22] with special emphasis on non-archimedean quadratic Shannon extensions, a class of Shannon extensions that we classify in Sections 3 and 4.

To each quadratic Shannon extension there is an associated collection of rank 1 discrete valuation rings. Let \( S = \bigcup_{i \geq 0} R_i \) be a quadratic Shannon extension of \( R = R_0 \). For each \( i \), let \( V_i \) be the DVR defined by the order function \( \text{ord}_{R_i} \), where for \( x \in R_i \), \( \text{ord}_{R_i}(x) = \sup \{ n | x \in m_i^n \} \) and \( \text{ord}_{R_i} \) is extended to the quotient field of \( R_i \) by defining \( \text{ord}_{R_i}(x/y) = \text{ord}_{R_i}(x) - \text{ord}_{R_i}(y) \) for all \( x, y \in R_i \) with \( y \neq 0 \). The family \( \{ V_i \}_{i=0}^{\infty} \) determines a unique set

\[
V = \bigcup_{n \geq 0} \bigcap_{i \geq n} V_i = \{ a \in F | \text{ord}_{R_i}(a) \geq 0 \text{ for } i \gg 0 \}.
\]

The set \( V \) consists of the elements in \( F \) that are in all but finitely many of the \( V_i \). In [21, Corollary 5.3], the authors prove that \( V \) is a valuation domain that birationally dominates \( S \), and call \( V \) the boundary valuation ring of the Shannon extension \( S \).

Theorem 2.2 records properties of a quadratic Shannon extension.

**Theorem 2.2.** [21, Theorems 4.1, 5.4 and 8.1] Let \( (S, m_S) \) be a quadratic Shannon extension of a regular local ring \( R \). Let \( T \) be the intersection of all the DVR overrings of \( R \) that properly contain \( S \), and let \( V \) be the boundary valuation ring of \( S \). Then:

1. \( \text{dim } S = 1 \) if and only if \( S \) is a rank 1 valuation ring.
2. \( S = V \cap T \).
3. There exists \( x \in m_S \) such that \( xS \) is \( m_S \)-primary, and \( T = S[1/x] \) for any such \( x \). It follows that the units of \( T \) are precisely the ratios of \( m_S \)-primary elements of \( S \) and \( \text{dim } T = \text{dim } S - 1 \).
4. \( T \) is a localization of \( R_i \) for \( i \gg 0 \). In particular, \( T \) is a Noetherian regular UFD.
5. \( T \) is the unique minimal proper Noetherian overring of \( S \).

In light of item 5 of Theorem 2.2, the ring \( T \) is called the Noetherian hull of \( S \).

The boundary valuation ring is given by a valuation from the nonzero elements of the quotient field of \( R \) to a totally ordered abelian group of rank at most 2 [22, Theorem 6.4 and Corollary 8.6]. In [22] the following two mappings on the quotient field of \( R \) are introduced as invariants of a quadratic Shannon extension. The first, \( e \), takes values in \( \mathbb{Z} \cup \{ \infty \} \), while the second, \( w \), takes values in \( \mathbb{R} \cup \{ -\infty, +\infty \} \). Both \( e \) and \( w \) are used in [22] to decompose the boundary valuation \( v \) of the quadratic
Shannon extension into a function that takes its values in $\mathbb{R} \oplus \mathbb{R}$ with the lexicographic ordering. The function $e$ is defined in terms of the transform $(aR_n)^{R_{n+i}}$ of a principal ideal $aR_n$ in $R_{n+i}$ for $i > n$; see [25] for the general definition of the (weak) transform of an ideal and [21] for more on the properties of the transform in our setting.

**Definition 2.3.** Let $S = \bigcup_{i \geq 0} R_i$ be a quadratic Shannon extension of a regular local ring $R$.

1. Let $a \in S$ be nonzero. Then $a \in R_n$ for some $n \geq 0$. Define
   $$e(a) = \lim_{i \to \infty} \text{ord}_{n+i}( (aR_n)^{R_{n+i}}).$$
   For $a, b$ nonzero elements in $S$, let $n \in \mathbb{N}$ be such that $a, b \in R_n$ and define $e(a) = e(a) - e(b)$. That $e$ is well defined is given by [22, Lemma 5.2].

2. Fix $x \in S$ such that $xS$ is primary for the maximal ideal of $S$, and define
   $$w : F \to \mathbb{R} \cup \{-\infty, +\infty\}$$
   by defining $w(0) = +\infty$, and for each $q \in F^*$,
   $$w(q) = \lim_{n \to \infty} \frac{\text{ord}_n(q)}{\text{ord}_n(x)}.$$

The structure of Shannon extensions naturally separate into those that are archimedean and those that are non-archimedean as in the following definition.

**Definition 2.4.** An integral domain $A$ is archimedean if $\bigcap_{n > 0} a^n A = 0$ for each nonunit $a \in A$.

An integral domain $A$ with $\text{dim} A \leq 1$ is archimedean.

Theorem 2.5, which characterizes quadratic Shannon extensions in several ways, shows that there is a prime ideal $Q$ of a non-archimedean quadratic Shannon extension $S$ such that $S/Q$ is a rational rank one valuation ring and $Q$ is a prime ideal of the Noetherian hull $T$ of $S$. In the next section this fact serves as the basis for the classification of non-archimedean quadratic Shannon extensions via pullbacks.

**Theorem 2.5.** Let $S = \bigcup_{n \geq 0} R_n$ be a quadratic Shannon extension of a regular local ring $R$ with quotient field $F$, and let $x$ be an element of $S$ that is primary for the maximal ideal $m_S$ of $S$ (see Theorem 2.2). Assume that $\text{dim} S \geq 2$. Let $Q = \bigcap_{n \geq 1} x^n S$, and let $T = S[1/x]$ be the Noetherian hull of $S$. Then the following are equivalent:

1. $S$ is non-archimedean.
2. $T = (Q : F Q)$.
3. $Q$ is a nonzero prime ideal of $S$.
4. Every nonmaximal prime ideal of $S$ is contained in $Q$.
5. $T$ is a regular local ring.
6. $\sum_{n=0}^{\infty} w(m_n) = \infty$, where $w$ is as in Definition 2.3 and, for each $n \geq 0$, $m_n$ is the maximal ideal of $R_n$. 


Moreover if (1)–(6) hold for $S$ and $Q$, then $T = S_Q, Q = QS_Q$ is a common ideal of $S$ and $T$, and $S/Q$ is a rational rank 1 valuation domain on the residue field $T/Q$ of $T$. In particular, $Q$ is the unique maximal ideal of $T$.

Proof. The equivalence of items 1 through 5 can be found in [22, Theorem 8.3]. That statement 1 is equivalent to 6 follows from [22, Theorem 6.1]. To prove the moreover statement, define $Q_{\infty} = \{a \in S \mid w(a) = +\infty\}$, where $w$ is as in Definition 2.3. By [22, Theorem 8.1], $Q_{\infty}$ is a prime ideal of $S$ and $T$, and by [22, Remark 8.2], $Q_{\infty}$ is the unique prime ideal of $S$ of dimension 1. Since also item 4 implies every nonmaximal prime ideal of $S$ is contained in $Q$, it follows that $Q = Q_{\infty}$. By item 5, $T = S[1/x]$ is a local ring. Since $xS$ is $m_S$-primary, we have that $T = S_Q$. Since $Q$ is an ideal of $T$, we conclude that $QS_Q = Q$ and $Q$ is the unique maximal ideal of $T$. By [22, Corollary 8.4], $S/Q$ is a valuation domain, and by [22, Theorem 8.5], $S/Q$ has rational rank 1. \[\square\]

We can further separate the case where $S$ is archimedean to whether or not $S$ is completely integrally closed. We recall the definition and result.

Definition 2.6. Let $A$ be an integral domain. An element $x$ in the field of fractions of $A$ is called almost integral over $A$ if $A[x]$ is contained in a principal fractional ideal of $A$. The ring $A$ is called completely integrally closed if it contains all of the almost integral elements over it.

Theorem 2.7. [21 Theorems 6.1, 6.2] Let $S$ be an archimedean quadratic Shannon extension. Then the function $w$ as in Definition 2.3 is a rank 1 nondiscrete valuation. Its valuation ring $W$ is the rank 1 valuation overring of $V$ and $W$ also dominates $S$. The following are equivalent:

(1) $S$ is completely integrally closed.
(2) The boundary valuation $V$ has rank 1; that is, $V = W$.

In Theorem 2.8 we recall from [22] the decomposition of the boundary valuation of a non-archimedean quadratic Shannon extension in terms of the functions $w$ and $e$ in Definition 2.3. For a decomposition of the boundary valuation in terms of $w$ and $e$ in the archimedean case, see [22, Theorem 6.4].

Theorem 2.8. [22 Theorem 8.5 and Corollary 8.6] Assume that $S$ is a non-archimedean quadratic Shannon extension of a regular local ring $R$ with quotient field $F$. Let $Q$ be as in Theorem 2.5 and let $e$ and $w$ be as in Definition 2.3. Then:

(1) $e$ is a rank 1 valuation on $F$ whose valuation ring $E$ contains $V$. If in addition $R/(Q \cap R)$ is a regular local ring, then $E$ is the order valuation ring of $T$.
(2) $w$ induces a rational rank 1 valuation $w'$ on the residue field $E/m_E$ of $E$. The valuation ring $W'$ defined by $w'$ extends the valuation ring $S/Q$, and the value group of $W'$ is the same as the value group of $S/Q$.
(3) $V$ is the valuation ring defined by the composite valuation of $e$ and $w'$.
Let \( z \in E \) such that \( m_E = zE \). Then \( V \) is defined by the valuation \( v \) given by

\[
v : F \setminus \{0\} \to \mathbb{Z} \oplus \mathbb{Q} : a \mapsto \left( \frac{e(a)}{e(z)}, \frac{w(a)e(z)}{w(z)e(a)} \right),
\]

where the direct sum is ordered lexicographically.

### 3 The relation of Shannon extensions to pullbacks

Let \( \alpha : A \to C \) be an extension of rings, and let \( B \) be a subring of \( C \). The subring \( D = \alpha^{-1}(B) \) of \( A \) is the pullback of \( B \) along \( \alpha : A \to C \).

\[
\begin{array}{ccc}
D & \\
\downarrow & \\
A & \\
\end{array} \xrightarrow{\alpha} \begin{array}{c} B \\
\downarrow \\
C
\end{array}
\]

Alternatively, \( D \) is the fiber product \( A \times_C B \) of \( \alpha \) and the inclusion map \( \iota : B \to C \); see, for example, [24, page 43].

The pullback construction has been extensively studied in multiplicative ideal theory, where it serves as a source of examples and generalizes the classical "\( D + M \)" construction. (For more on the latter construction, see [12].) We will be especially interested in the case in which \( A, B, C, D \) are domains, \( \alpha \) is a surjection and \( B \) has quotient field \( C \). In this case, following [11], we say the diagram above is of type \( \square^* \).

For a diagram of type \( \square^* \), the kernel of \( \alpha \) is a maximal ideal of \( A \) that is contained in \( D \). The quotient field \( C \) of \( B \) can be identified with the residue field of this maximal ideal. If \( A \) is local with \( \dim A \geq 1 \) and \( \dim B \geq 1 \), then \( A = DM \) is a localization of \( D \) and \( D \) is non-archimedean. These observations have a number of consequences for transfer properties between the ring \( D \) and the rings \( A \) and \( B \); see for example [9, 10, 11].

While pullback diagrams of type \( \square^* \) are often used to construct examples in non-Noetherian commutative ring theory, there are also instances where the pullback construction is used as a classification tool. A simple example is given by the observation that a local domain \( D \) has a principal maximal ideal if and only if \( D \) occurs in a pullback diagram of type \( \square^* \), where \( B \) is a DVR [23, Exercise 1.5, p. 7].

A second example is given by the fact that for nonnegative integers \( k < n \), a ring \( D \) is a valuation domain of rank \( n \) if and only if \( D \) occurs in pullback diagram of type \( \square^* \), where \( A \) is a valuation ring of rank \( n - k \) and \( B \) is a valuation ring of rank \( k \); see [9, Theorem 2.4]. Theorem 2.5 provides an instance of this decomposition in the present context. In the theorem, \( V \) is the pullback of \( E \) and \( W' \):

\[
\begin{array}{ccc}
V = \alpha^{-1}(W') & \xrightarrow{\alpha} W' \\
\downarrow & \downarrow \\
E & \xrightarrow{\alpha} E/m_E
\end{array}
\]
A third example classification via pullbacks of the form $\square^*$ is given by the classification of local rings of global dimension 2 by Greenberg [17, Corollary 3.7] and Vasoncelos [31]: A local ring $D$ has global dimension 2 if and only if $D$ satisfies one of the following:

(a) $D$ is a regular local ring of Krull dimension 2,
(b) $D$ is a valuation ring of global dimension 2, or
(c) $D$ has countably many principal prime ideals and $D$ occurs in a pullback diagram of type $\square^*$, where $A$ is a valuation ring of global dimension 1 or 2 and $B$ is a regular local ring of global dimension 2.

Motivated by these examples, we use the pullback construction in this and the next section to classify among the overrings of a regular local ring $R$ those that are non-archimedean quadratic Shannon extensions of $R$. We prove in Theorem 4.8 that these are precisely the overrings of $R$ that occur in pullback diagrams of type $\square^*$, where $A$ is a localization of an iterated quadratic transform $R_i$ of $R$ at a prime ideal $P$ and and $B$ is a rank 1 valuation overring of $R_i/P$ having a divergent multiplicity sequence. Thus a non-archimedean quadratic Shannon extension is determined by a rank 1 valuation ring and a regular local ring.

As a step towards this classification, in Theorem 3.1 we restate part of Theorem 2.5 as an assertion about how a non-archimedean quadratic Shannon extension can be decomposed using pullbacks. Much of the rest of this section and the next is devoted to a converse of this assertion, which is given in Theorems 4.8 and 5.1.

**Theorem 3.1.** Let $S$ be a non-archimedean quadratic Shannon extension. Then there is a prime ideal $P$ of $S$ such that $S_P$ is the Noetherian hull of $S$, $P = PS_P$ and $S/P$ is a rational rank 1 valuation ring. Theorem 3.1 follows from these observations. □

**Definition 3.2.** Let $R$ be a Noetherian local domain, let $\mathcal{V}$ be a rank 1 valuation ring dominating $R$ with corresponding valuation $\nu$, and let $\{(R_i, m_i)\}_{i=0}^{\infty}$ be the infinite sequence of LQTs along $\mathcal{V}$. Then the sequence $\{\nu(m_i)\}_{i=0}^{\infty}$ is the multiplicity sequence of $(R, \mathcal{V})$; see [15, Section 5]. We say the multiplicity sequence is divergent if $\sum_{i \geq 0} \nu(m_i) = \infty$.

**Remark 3.3.** Let $R$ be a regular local ring and let $\mathcal{V}$ be a rank 1 valuation ring birationally dominating $R$. If the multiplicity sequence of $(R, \mathcal{V})$ is divergent, then $\mathcal{V}$ is a DVR and $R_m = R_n$ for all $m \geq n$.

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2 If $R_n = R_{n+1}$ for some integer $n$, then $R_n = \mathcal{V}$ is a DVR and $R_m = R_n$ for all $m \geq n$. 

a quadratic Shannon extension of \( R \) [13]; Proposition 23] and \( \mathcal{V} \) has rational rank 1 [20, Proposition 7.3]. This is observed in [21, Corollary 3.9] in the case where \( \mathcal{V} \) is a DVR. In Proposition 3.4, we observe that \( \mathcal{V} \) is the union of the rings in the LQT sequence of \( R \) along \( \mathcal{V} \) for every Noetherian local domain \((R, m)\) birationally dominated by \( \mathcal{V} \).

**Proposition 3.4.** Let \((R, m)\) be a Noetherian local domain, let \( \mathcal{V} \) be a rank 1 valuation ring that birationally dominates \( R \), and let \( \{R_i\}_{i=0}^{\infty} \) be the infinite sequence of LQTs of \( R \) along \( \mathcal{V} \). If the multiplicity sequence of \((R, \mathcal{V})\) is divergent, then \( \mathcal{V} = \bigcup_{n \geq 0} R_n \). In particular, if \( \mathcal{V} \) is a DVR, then \( \mathcal{V} = \bigcup_{n \geq 0} R_n \).

**Proof.** Let \( \nu \) be a valuation for \( \mathcal{V} \) and let \( y \) be a nonzero element in \( \mathcal{V} \). Suppose we have an expression \( y = a_n/b_n \), where \( a_n, b_n \in R_n \). Since \( R_n \subseteq \mathcal{V} \), it follows that \( \nu(b_n) \geq 0 \). If \( \nu(b_n) = 0 \), then since \( \mathcal{V} \) dominates \( R_n \), we have \( 1/b_n \in R_n \) and \( y \in R_n \).

Assume otherwise, that is, \( \nu(b_n) > 0 \). Then \( b_n \in m_n \), and since \( \nu(a_n) \geq \nu(b_n) \), also \( a_n \in m_n \). Let \( x_n \in m_n \) be such that \( x_n R_{n+1} = m_n R_{n+1} \). Then \( a_n, b_n \in x_n R_{n+1} \), so the elements \( a_{n+1} = a_n/x \) and \( b_{n+1} = b_n/x \) are in \( R_{n+1} \). Thus we have the expression \( y = a_{n+1}/b_{n+1} \), where \( \nu(b_{n+1}) = \nu(b_n) - \nu(m_n) \).

Consider an expression \( y = a_0/b_0 \), where \( a_0, b_0 \in R_0 \). Then we iterate this process to obtain a sequence of expressions \( \{a_n/b_n\} \) of \( y \), with \( a_n, b_n \in R_n \), where this process halts at some \( n \geq 0 \) if \( \nu(b_n) = 0 \), implying \( y \in R_n \). Assume by way of contradiction that this sequence is infinite. For \( N \geq 0 \), it follows that \( \nu(b_0) = \nu(b_N) + \sum_{n=0}^{N-1} \nu(m_n) \). Then \( \nu(b_0) \geq \sum_{n=0}^{N} \nu(m_n) \) for any \( N \geq 0 \), so \( \nu(b_0) = \sum_{n=0}^{\infty} \nu(m_n) = \infty \), which contradicts \( \nu(b_0) < \infty \). This shows that the sequence \( \{a_n/b_n\} \) is finite and hence \( y \in \bigcup_{n \geq 0} R_n \). \( \square \)

**Remark 3.5.** Examples of \((R, \mathcal{V})\) with divergent multiplicity sequence such that \( \mathcal{V} \) is not a DVR are given in [20, Examples 7.11 and 7.12].

**Discussion 3.6.** Let \((R, m)\) be a Noetherian local domain, let \( \mathcal{V} \) be a rational rank 1 valuation ring that birationally dominates \( R \), and let \( \{R_i\}_{i=0}^{\infty} \) be the infinite sequence of LQTs of \( R \) along \( \mathcal{V} \). The divergence of the multiplicity sequence in Proposition 3.4 is a sufficient condition for \( \mathcal{V} = \bigcup_{n \geq 0} R_n \), but not a necessary condition; see Example 3.7. It would be interesting to understand more about conditions in order that the multiplicity sequence of \((R, \mathcal{V})\) is divergent. Example 3.7 illustrates that an archimedean Shannon extension \( S \) of a 3-dimensional regular local ring \( R \) may be birationally dominated by a rational rank 1 valuation ring \( V \), where \( S \subseteq V \). In this case by Proposition 3.4, the multiplicity sequence of \((R, V)\) must be convergent.

**Example 3.7.** Let \( x, y, z \) be indeterminates over a field \( k \). We first construct a rational rank 1 valuation ring \( V' \) on the field \( k(x, y) \). We do this by describing an infinite sequence \( \{ (R'_n, m'_n) \}_{n=0}^{\infty} \) of local quadratic transforms of \( R'_0 = k[x, y]_{(x, y)} \). To indicate properties of the sequence, we define a rational valued function \( v \) on specific generators of the \( m'_n \). The function \( v \) is to be additive on products. We set \( v(x) = v(y) = 1 \). This indicates that \( y/x \) is a unit in every valuation ring birationally dominating \( R'_1 \).

**Step 1.** Let \( R'_1 \) have maximal ideal \( m'_1 = (x_1, y_1) R_1 \), where \( x_1 = x, y_1 = (y/x) - 1 \). Define \( v(y_1) = 1/2 \).

Directed unions of local quadratic transforms of regular local rings and pullbacks
Remark 5.1(4). The local quadratic transform $z \rightarrow v$.

Step 1. Define $y_3 = (y_2/x_2) - 1$ and assign $v(y_3) = 1/4$. Then $x_3 = x_2$, $v(x_3) = 1/2$.

Step 2. The local quadratic transform $R'_2$ of $R'_1$ has maximal ideal $m'_2$ generated by $x_2 = x_1/y_1$, $y_2 = y_1$. We have $v(x_2) = 1/2$, $v(y_2) = 1/2$.

Step 3. Define $y_3 = (y_2/x_2) - 1$ and assign $v(y_3) = 1/4$. Then $x_3 = x_2$, $v(x_3) = 1/2$.

Step 4. The local quadratic transform $R'_3$ of $R'_2$ has maximal ideal $m'_4$ generated by $x_4 = x_3/y_3$, $y_4 = y_3$. Then $v(x_4) = v(y_4) = 1/4$.

Continuing this 2-step process yields an infinite directed union $(R'_n, m'_n)$ of local quadratic transforms of 2-dimensional RLRs. Let $V' = \bigcup_{n \geq 0} R'_n$. Then $V'$ is a valuation ring by [1, Theorem 12]. Let $v'$ be a valuation associated to $V'$ such that $v'(x) = 1$. Then $v'(y) = 1$ and $v'$ takes the same rational values on the generators of $m'_n$ as defined by $v$. Since there are infinitely many translations as described in Steps 2n+1 for each integer $n \geq 0$, it follows that $V'$ has rational rank 1, e.g., see [20, Remark 5.1(4)].

The multiplicity values of $\{R'_n, m'_n\}$ are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$, the sum of which converges to 3.

Define $V = V'(\frac{x}{x}, \frac{y}{y})$, the localization of the polynomial ring $V'[\frac{x}{x}, \frac{y}{y}]$ at the prime ideal $m'_V V'[\frac{x}{x}, \frac{y}{y}]$. One sometimes refers to $V$ as a Gaussian or trivial or Nagata extension of $V'$ to a valuation ring on the simple transcendental field extension generated by $\frac{x}{x}, \frac{y}{y}$ over $k(x, y)$. It follows that $V$ has the same value group as $V'$ and the residue field of $V$ is a simple transcendental extension of the residue field of $V'$ that is generated by the image of $\frac{x}{x}, \frac{y}{y}$ in $V/m_V$.

Let $v$ denote the associated valuation to $V$ such that $v(x) = 1$. It follows that $v(y) = 1$ and $v(z) = v(x^2y^2) = 4$. Let $R_0 = k[x, y, z]_{(x, y, z)}$. Then $R_0$ is birationally dominated by $V$. Let $\{(R_n, m_n)\}_{n \geq 0}$ be the sequence of local quadratic transforms of $R_0$ along $V$.

We describe the first few steps:

Step 1. $R_1$ has maximal ideal $m_1 = (x_1, y_1, z_1)R_1$, where $x_1 = x$, $y_1 = (y/x) - 1$, and $z_1 = z/x$. Also $v(y_1) = 1/2$.

Step 2. The local quadratic transform $R_2$ of $R_1$ along $V$ has maximal ideal $m_2$ generated by $x_2 = x_1/y_1$, $y_2 = y_1$ and $z_2 = z_1/y_1$. We have $v(x_2) = 1/2$, $v(y_2) = 1/2$ and $v(z_2) = 4 - 3/2 > 3/2$.

Step 3. The local quadratic transform $R_3$ of $R_2$ along $V$ has $y_3 = (y_2/x_2) - 1$, where $v(y_3) = 1/4$, and $x_3 = x_2$, $v(x_3) = 1/2$ and $v(z_3) > 1/2$.

Step 4. The local quadratic transform $R_4$ of $R_3$ along $V$ has maximal ideal $m_4$ generated by $x_4 = x_3/y_3$, $y_4 = y_3$ and $z_4 = z_3/y_3$.

The multiplicity values of the sequence $\{(R_n, m_n)\}_{n \geq 0}$ along $V$ are the same as that for $\{R'_n, m'_n\}$, namely $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$. Let $S = \bigcup_{n \geq 0} R_n$. Since $S$ is birationally dominated by the rank 1 valuation ring $V$, it follows that $S$ is an archimedean Shannon extension. Since we never divide in the $z$-direction, we have $S \subseteq R_z R$, and $S$ is not a valuation ring.
4 Quadratic Shannon extensions along a prime ideal

Let \( R \) be a Noetherian local domain and let \( \{ R_n \}_{n \geq 0} \) be an infinite sequence of LQTs of \( R = R_0 \). Using the terminology of Granja and Sanchez-Giralda [16] Definition 3 and Remark 4], for a prime ideal \( P \) of \( R \), we say the quadratic sequence \( \{ R_n \} \) is along \( R_P \) if \( \bigcup_{n \geq 0} R_n \subseteq R_P \).

Let \( P \) be a nonzero, nonmaximal prime ideal of a Noetherian local domain \((R, \mathfrak{m})\). Proposition 4.1 establishes a one-to-one correspondence between sequences \( \{ R_n \} \) of LQTs of \( R = R_0 \) along \( R_P \) and sequences \( \{ \overline{R_n} \} \) of LQTs of \( R_0 = R/P \).

**Proposition 4.1.** Let \( R \) be a Noetherian local domain and let \( P \) be a nonzero nonmaximal prime ideal of \( R \). Then there is a one-to-one correspondence between:

1. Infinite sequences \( \{ R_n \}_{n \geq 0} \) of LQTs of \( R_0 = R \) along \( R_P \).
2. Infinite sequences \( \{ \overline{R_n} \}_{n \geq 0} \) of LQTs of \( R_0 = R/P \).

Given such a sequence \( \{ R_n \}_{n \geq 0} \), the corresponding sequence is \( \{ R_n/(PR_P \cap R_n) \} \).

Denote \( S = \bigcup_{n \geq 0} R_n \) and \( \overline{S} = \bigcup_{n \geq 0} \overline{R_n} \), and let \( \overline{S} \) be the pullback of \( S \) with respect to the quotient map \( R_P \to \kappa(P) \) as in the following diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{\alpha^{-1}} & \overline{S} \\
\bigg\uparrow & & \bigg\uparrow \\
R_P & \xrightarrow{\alpha} & \kappa(P)
\end{array}
\]

Then \( \overline{S} = S + PR_P \) and \( \overline{S} \) is non-archimedean.

**Proof.** The correspondence follows from [18] Corollary II.7.15, p. 165]. The fact that \( \overline{S} = S + PR_P \) is a consequence of the fact that \( \overline{S} \) is a pullback of \( S \) and \( R_P \). That \( \overline{S} \) is non-archimedean is a consequence of the observation that for each \( x \in \mathfrak{m} \setminus P \), the fact that \( PR_P \subseteq \overline{S} \subseteq R_P \) implies \( PR_P \subseteq \chi^k \overline{S} \) for all \( k > 0 \).

**Lemma 4.2.** Assume notation as in Proposition 4.1. If \( \overline{S} \) is a rank 1 valuation ring and the multiplicity sequence of \( (R, \overline{S}) \) is divergent, then \( S = \overline{S} \).

**Proof.** Let \( v \) be a valuation for \( \overline{S} \) and assume that \( v \) takes values in \( \mathbb{R} \). Let \( f \in \overline{S} \).

We claim that \( f \in S \). Since \( \overline{S} = S + PR_P \), we may assume \( f \in PR_P \). Write \( f = \frac{g_n}{h_0} \), where \( g_0 \in P \) and \( h_0 \in R \setminus P \).

Suppose we have an expression of the form \( f = \frac{g_n}{h_n} \), where \( g_n \in PR_P \cap R_n \) and \( h_n \in R_n \setminus PR_P \). Write \( m_n R_{n+1} = x R_{n+1} \) for some \( x \in m_n \). Since \( PR_P \cap R_n \subseteq m_n \), it follows that \( g_n = x g_{n+1} \) for \( g_{n+1} = \frac{g_n}{x} \in R_{n+1} \). Denote the image of \( h_n \in R_n \) in \( \mathbb{K} \) by \( \overline{h_n} \).

If \( v(\overline{h_n}) > 0 \) for some \( n_n \), then \( h_n = x h_{n+1} \) for \( h_{n+1} = \frac{h_n}{x} \in R_{n+1} \). Thus we have written \( f = \frac{g_{n+1}}{h_{n+1}} \), where \( g_{n+1} \in PR_P \cap R_{n+1} \) and \( h_{n+1} = \frac{h_n}{x} \in R_{n+1} \setminus PR_P \), such that \( v(\overline{f}) = v(\overline{h}) - v(\overline{m_n}) \).

Since \( \sum_{n \geq 0} v(\overline{m_n}) = \infty \) and \( v(\overline{h_n}) \) is finite, this process must halt with \( f = \frac{g_n}{h_n} \) as before such that \( v(\overline{h}) = 0 \). Since \( v(\overline{h_n}) = 0 \), \( h_n \) is a unit in \( \mathbb{F}_n \), so \( h_n \) is a unit in \( R_n \), and thus \( f \in R_n \).
Lemma 4.3. Let $P$ be a nonzero nonmaximal prime ideal of a regular local ring $R$. Let $\{R_n\}_{n \geq 0}$ be a sequence of LQTs of $R_0 = R$ along $R_P$ and let $\{\overline{R}_n\}$ be the induced sequence of LQTs of $\overline{R}_0 = R/P$ as in Proposition 4.1. Denote $S = \bigcup_{n \geq 0} R_n$ and $\overline{S} = \bigcup_{n \geq 0} \overline{R}_n$. Then the following are equivalent:

1. $S$ is the pullback of $\overline{S}$ along the surjective map $R_P \to \kappa(P)$.
2. The Noetherian hull of $S$ is $R_P$.
3. $\overline{S}$ is a rank 1 valuation ring and the multiplicity sequence of $(R, S)$ is divergent.

If these conditions hold, then $\overline{S}$ has rational rank 1.

Proof. (1) $\implies$ (2): As a pullback, the quadratic Shannon extension $S$ is non-archimedean (see the proof of Proposition 4.1). Let $x \in S$ be such that $xS$ is $m_S$-primary (see Theorem 2.2). By Theorem 2.5, the ideal $Q = \bigcap_{n \geq 0} x^n S$ is a nonzero prime ideal of $S$, every nonmaximal prime ideal of $S$ is contained in $Q$ and $T = S_Q$. Assumption (1) implies that $PR_P$ is a nonzero ideal of both $S$ and $R_P$. Hence $R_P$ is almost integral over $S$. We have $S \subseteq S_Q = T \subseteq R_P$, and $S_Q$ is an RLR and therefore completely integrally closed. It follows that $S_Q = R_P$ is the Noetherian hull of $S$.

(2) $\implies$ (3): Since the Noetherian hull $R_P$ of $S$ is local, Theorem 2.5 implies that $S$ is non-archimedean and $PR_P \subseteq S$. By Theorem 3.1, $\overline{S} = S/PR_P$ is a rational rank 1 valuation ring. The valuation $v$ associated to $\overline{S}$ is equal to the valuation $w'$ of Theorem 2.8. By item 2 of Theorem 2.8 and item 6 of Theorem 2.5, we have

$$\sum_{n=0}^{\infty} v(m_n) = \sum_{n=0}^{\infty} w'(m_n) = \infty.$$

(3) $\implies$ (1): This is proved in Lemma 4.2.

Remark 4.4. Let $P$ be a nonzero nonmaximal prime ideal of $R$ and let $S$ be a non-archimedean quadratic Shannon extension of $R$ with Noetherian hull $R_P$. The proof of Lemma 4.3 shows that the multiplicity sequence of $(R/P, S/PR_P)$ is given by $\{w(m_n)\}$, where $w$ is as in Definition 2.3.

With notation as in Lemma 4.3, examples where $\overline{S}$ is a rank 1 valuation ring that is not discrete are given in [20, Examples 7.11 and 7.12].

Theorem 4.5 (Existence of Shannon Extensions). Let $P$ be a nonzero nonmaximal prime ideal of a regular local ring $R$.

1. There exists a non-archimedean quadratic Shannon extension of $R$ with $R_P$ as its Noetherian hull.
2. If there exists an archimedean quadratic Shannon extension of $R$ contained in $R_P$, then $\dim R/P \geq 2$.

Proof. To prove item 1, we use a result of Chevalley that every Noetherian local domain is birationally dominated by a DVR [7]. Let $V$ be a DVR birationally dominating $R/P$. We apply Lemma 4.3 with this $R$ and $P$. Let $\{\overline{R}_n\}$ be the sequence of LQTs of $\overline{R}_0 = R/P$ along $V$. Let $S$ be the union of the corresponding sequence of LQTs of $R$ given by Proposition 4.1. Proposition 3.4 implies that $\overline{S} = V$ and
Lemma 4.3 implies that $S = \widetilde{S}$ is a non-archimedean Shannon extension with $R_P$ as its Noetherian hull.

For item 2, if $\dim R/P = 1$, then $\dim R_P = \dim R - 1$ since an RLR is catenary. If $S$ is an archimedean Shannon extension of $R$, then $\dim S \leq \dim R - 1$ by [21 Lemma 3.4 and Corollary 3.6]. Therefore $R_P$ does not contain the Noetherian hull of an archimedean Shannon extension of $R$ if $\dim R/P = 1$. □

Discussion 4.6. Let $P$ be a nonzero nonmaximal prime of a regular local ring $R$ such that $\dim R/P \geq 2$. We ask:

Question: Does there exists an archimedean quadratic Shannon extension of $R$ contained in $R_P$?

The question reduces to the case where $\dim R/P = 2$, for if $Q$ is a prime ideal of $R$ with $\dim R/Q \geq 2$, then there exists a prime ideal $P$ of $R$ such that $Q \subseteq P$ and $\dim R/P = 2$. Then $R_P \subseteq R_Q$. Hence a quadratic Shannon extension of $R$ contained in $R_P$ is contained in $R_Q$.

Assume that $P$ is a nonzero prime ideal of $R$ such that $\dim R/P = 2$. It is not difficult to see that the 2-dimensional Noetherian local domain $R_0 = R/P$ is birationally dominated by a rank 1 valuation domain $V$ of rational rank 2. Consider the infinite sequence of LQTs $\{R_n\}_{n \geq 0}$ of $R_0 = R/P$ along $V$ and let $\overline{S} = \bigcup_{n \geq 0} R_n$. Then $\overline{S}$ is birationally dominated by $V$. Each of the $R_n$ is a 2-dimensional Noetherian local domain and $\dim \overline{S}$ is either 1 or 2.

Let $\{R_n\}$ be the sequence of LQTs of $R$ given by Proposition 4.1 that corresponds to $\{R_n\}$, and let $S = \bigcup_n R_n$. Then $\dim R_n > 2$ for all $n$. Hence $S$ is a quadratic Shannon extension of $R$ and $S \subseteq R_P$. Let $p = PR_P \cap S$. Then $\overline{S}/p = \overline{S}$.

If $\dim \overline{S} = 1$ then there are no prime ideals of $S$ strictly between $p$ and $m_S$. Since $V$ has rational rank 2, the multiplicity sequence of $(\overline{R}, \overline{S})$ is convergent. Lemma 4.3 implies that the Noetherian hull of $S$ is not $R_P$. Hence if $\dim \overline{S} = 1$, there exists an archimedean quadratic Shannon extension $S$ of $R$ contained in $R_P$.

Theorem 4.8 implies the following:

Corollary 4.7. (Lipman [25 Lemma 1.21.1]) Let $P$ be a nonmaximal prime ideal of a regular local ring $R$. Then there exists a quadratic Shannon extension of $R$ contained in $R_P$.

In Theorem 4.8 we use Lemma 4.3 to characterize the overrings of a regular local ring $R$ that are Shannon extensions of $R$ with Noetherian hull $R_P$, where $P$ is a nonzero nonmaximal prime ideal of $R$. Note that by Theorem 2.5 such a Shannon extension is necessarily non-archimedean.

Theorem 4.8 (Shannon Extensions with Specified Local Noetherian Hull). Let $P$ be a nonzero nonmaximal prime ideal of a regular local ring $R$. The quadratic Shannon extensions of $R$ with Noetherian hull $R_P$ are precisely the rings $S$ such that $S$ is a pullback along the residue map $\alpha : R_P \to \kappa(P)$ of a rational rank 1 valuation ring birationally dominating $R/P$ whose multiplicity sequence is divergent.
Proof. If \( S \) is a quadratic Shannon extension with Noetherian hull \( R_P \), then by Lemma 4.3, \( S \) is a pullback along the map \( R_P \to \kappa(P) \) of a rational rank 1 valuation ring birationally dominating \( R/P \) whose multiplicity sequence is divergent.

Conversely, let \( S \) be such a pullback. Let \( \{R_n\}_{n \geq 0} \) denote the sequence of LQTs of \( R_0 = R/P \) along \( \mathcal{V} \) and let \( \{R_n\}_{n \geq 0} \) denote the induced sequence of LQTs of \( R_0 = R \) as in Proposition 4.1. Then Lemma 4.3 implies that \( S = \bigcup_{n \geq 0} R_n \), so \( S \) is a quadratic Shannon extension.

Corollary 4.9. Let \( P \) be a prime ideal of the regular local ring \( R \) with \( \dim R/P = 1 \).

1. The quadratic Shannon extensions of \( R \) with Noetherian hull \( R_P \) are precisely the pullbacks along the residue map \( R_P \to \kappa(P) \) of the finitely many DVR overrings \( \mathcal{V}' \) of \( R_0 = R \) as in Proposition 4.1.

2. If \( R/P \) is a DVR, then \( R + PR_P \) is the unique quadratic Shannon extension of \( R \) with Noetherian hull \( R_P \).

Proof. The Krull-Akizuki Theorem [26, Theorem 11.7] implies that \( R/P \) has finitely many valuation overrings, each of which is a DVR. By Theorem 4.3, there is a one-to-one correspondence between these DVRs and the Shannon extensions of \( R \) with Noetherian hull \( R_P \). This proves item 1. If \( R/P \) is a DVR, then by item 1, the pullback \( R + PR_P \) of \( R/P \) along the map \( R_P \to \kappa(P) \) is the unique quadratic Shannon extension of \( R \) with Noetherian hull \( R_P \). This verifies item 2. \( \square \)

5 Classification of non-archimedean Shannon extensions

In Theorem 5.1 we classify the non-archimedean quadratic Shannon extensions \( S \) that occur as overrings of a given regular local ring \( R \). The classification is extrinsic to \( S \) in the sense that a prime ideal of an iterated quadratic transform of \( R \) is needed for the description of the overring \( S \) as a pullback. In Theorem 5.2 we give an intrinsic rather than extrinsic characterization of certain of the non-archimedean quadratic Shannon extensions with principal maximal ideal that occur in an algebraic function field of characteristic 0. In this case, we are able to characterize such rings in terms of pullbacks without the explicit requirement of a regular local “underring” of \( S \). This allows us to give an additional source of examples of non-archimedean quadratic Shannon extensions in Example 5.4.

Theorem 5.1 (Classification of non-archimedean Shannon extensions). Let \( R \) be a regular local ring with \( \dim R \geq 2 \), and let \( S \) be an overring of \( R \). Then \( S \) is a non-archimedean quadratic Shannon extension of \( R \) if and only if there is a ring \( \mathcal{V}' \), a nonnegative integer \( i \) and a prime ideal \( P \) of \( R_i \) such that...
(a) $\mathcal{V}$ is a rational rank 1 valuation ring of $\kappa(P)$ that contains the image of $R_i/P$ in $\kappa(P)$ and has divergent multiplicity sequence over this image, and

(b) $S$ is a pullback of $\mathcal{V}$ along the residue map $\alpha : (R_i)_P \to \kappa(P)$.

$$S = \alpha^{-1}(\mathcal{V}) \xrightarrow{} \mathcal{V}$$

$$\downarrow \quad \downarrow$$

$$(R_i)_P \xrightarrow{\alpha} \kappa(P)$$

**Proof.** Suppose $S$ is a non-archimedean quadratic Shannon extension, and let $\{R_i\}$ be the sequence of iterated QDTs such that $S = \bigsqcup R_i$. By Theorem 2.5, the Noetherian hull $T$ of $S$ is a local ring, and by Theorem 2.2, there is $i > 0$ and a prime ideal $P$ of $R_i$ such that $T = (R_i)_P$. Since $S$ is a non-archimedean quadratic Shannon extension of $R_i$, Theorem 4.8 implies there is a valuation ring $\mathcal{V}$ such that (a) and (b) hold for $i$, $P$, $S$ and $\mathcal{V}$.

Conversely, suppose there is a ring $\mathcal{V}$, a nonnegative integer $i$ and a prime ideal $P$ of $R_i$ that satisfy (a) and (b). By Theorem 4.8, $S$ is a quadratic Shannon extension of $R_i$ with Noetherian hull $(R_i)_P$. Thus $S$ is a quadratic Shannon extension of $R$ that is non-archimedean by Theorem 2.5. \hfill $\Box$

In contrast to Theorem 5.1, the pullback description in Theorem 5.2 is without reference to a specific regular local underring of $S$. Instead, the proof constructs one using resolution of singularities. Because our use of this technique is elementary, we frame our proof in terms of projective models rather than projective schemes. For more background on projective models, see [4, Sections 1.6 - 1.8] and [33, Chapter VI, §17]. Let $F$ be a field and let $k$ be a subfield of $F$. Let $t_0 = 1$ and assume that $t_1, \ldots, t_n$ are nonzero elements of $F$ such that $F = k(t_1, \ldots, t_n)$. For each $i \in \{0, 1, \ldots, n\}$, define $D_i = k[t_0/t_i, \ldots, t_n/t_i]$. The projective model of $F/k$ with respect to $t_0, \ldots, t_n$ is the collection of local rings given by

$$X = \{(D_i)_P : i \in \{0, 1, \ldots, n\}, P \in \text{Spec } (D_i)\}.$$

If $k$ has characteristic 0, then by resolution of singularities (see for example [8, Theorem 6.38, p. 100]) there is a projective model $Y$ of $F/k$ such that every regular local ring in $X$ is in $Y$, every local ring in $Y$ is a regular local ring, and every local ring in $X$ is dominated by a (necessarily regular) local ring in $Y$.

By a valuation ring of $F/k$ we mean a valuation ring $V$ with quotient field $F$ such that $k$ is a subring of $V$.

**Theorem 5.2.** Let $S$ be a local domain containing as a subring a field $k$ of characteristic 0. Assume that $\dim S \geq 2$ and that the quotient field $F$ of $S$ is a finitely generated extension of $k$. Then the following are equivalent:

1. $S$ has a principal maximal ideal and $S$ is a quadratic Shannon extension of a regular local ring $R$ that is essentially finitely generated over $k$.
2. There is a regular local overring $A$ of $S$ and a DVR $\mathcal{V}$ of $(A/\mathfrak{m}_A)/k$ such that
(a) \( \text{tr.deg}_k A/m_A + \dim A = \text{tr.deg}_k F, \) and
(b) \( S \) is the pullback of \( \mathcal{V} \) along the residue map \( \alpha : A \to A/m_A. \)

\[
\begin{array}{ccc}
S = \alpha^{-1}(\mathcal{V}) & \xrightarrow{\alpha} & \mathcal{V} \\
\downarrow & & \downarrow \\
A & \xrightarrow{\alpha} & A/m_A
\end{array}
\]

**Proof.** (1) \( \Rightarrow \) (2): Let \( x \in S \) be such that \( m_S = xS. \) By Theorem 2.2, \( S[1/x] \) is the Noetherian hull of \( S \) and \( S[1/x] \) is a regular ring. Since \( \dim S > 1, \) the ideal \( P = \bigcap_{i \geq 0} x^i S \) is a nonzero prime ideal of \( S \) (Exercise 1.5, p. 7). Hence \( S \) is non-archimedean. By Theorem 2.5, \( S_p \) is the Noetherian hull of \( S \) and hence \( S_p = S[1/x]. \) Let \( A = S_p \) and \( \mathcal{V} = S/P. \) By Theorem 3.3, \( S \) is a pullback of the DVR \( \mathcal{V} \) with respect to the map \( A \to A/m_A. \) By assumption, \( S \) is a quadratic Shannon extension of a regular local ring \( R \) that is essentially finitely generated over \( k. \) For sufficiently large \( i, \) we have \( A = S_p = (R_i)_{p \to R_i} \) by 21 Proposition 3.3. Since \( R_i \) is essentially finitely generated over \( R, \) and \( R \) is essentially finitely generated over \( k, \) we have that \( A \) is essentially finitely generated over \( k. \) By the Dimension Formula \[26 \] Theorem 15.6, p. 118,

\[
\text{tr.deg}_k A/m_A + \dim A = \text{tr.deg}_k F.
\]

This completes the proof that statement 1 implies statement 2.

(2) \( \Rightarrow \) (1): Let \( P = m_A. \) By item 2b, \( P \) is a prime ideal of \( S, \) \( A = S_P, \) \( P = PS_P \) and \( \mathcal{V} = S/P. \) Let \( x \in m_S \) be such that the image of \( x \) in the DVR \( S/P \) generates the maximal ideal. Since \( P = PS_P, \) we have \( P \subseteq xS. \) Consequently, \( m_S = xS, \) and so \( S \) has a principal maximal ideal.

To prove that \( S \) is a quadratic Shannon extension of a regular local ring that is essentially finitely generated over \( k, \) it suffices by Theorem 4.8 to prove:

(i) There is a subring \( R \) of \( S \) that is a regular local ring essentially finitely generated over \( k. \)

(ii) \( A \) is a localization of \( R \) at the prime ideal \( P \cap R. \)

(iii) \( \mathcal{V} \) is a valuation overring of \( (R + P)/P \) with divergent multiplicity sequence.

Since \( F \) is a finitely generated field extension of \( k \) and \( A \) (as a localization of \( S \)) has quotient field \( F, \) there is a finitely generated \( k \)-subalgebra \( D \) of \( A \) such that the quotient field of \( D \) is \( F. \) By item 2a, \( A/P \) has finite transcendence degree over \( k. \) Let \( a_1, \ldots, a_n \) be elements of \( A \) whose images in \( A/P \) form a transcendence basis for \( A/P \) over \( k. \) Replacing \( D \) with \( D[a_1, \ldots, a_n], \) and defining \( p = P \cap D, \) we may assume that \( A/P \) is algebraic over \( k(p) = D/pD. \) In fact, since the normalization of an affine \( k \)-domain is again an affine \( k \)-domain, we may assume also that \( D \) is an integrally closed finitely generated \( k \)-subalgebra of \( A \) with quotient field \( F. \) Since \( D \) is a finitely generated \( k \)-algebra, \( D \) is universally catenary. By the Dimension Formula \[26 \] Theorem 15.6, p. 118, we have

\[
\dim D/p + \text{tr.deg}_k k(p) = \text{tr.deg}_k F.
\]
Therefore, item 2a implies
\[ \dim D_p + \text{tr.deg}_k \kappa(p) = \dim A + \text{tr.deg}_k A/P. \]
Since \( A/P \) is algebraic over \( \kappa(p) \), we conclude that \( \dim D_p = \dim A \).

The normal ring \( A \) birationally dominates the excellent normal ring \( D_p \), so \( A \) is essentially finitely generated over \( D_p \) [19 Theorem 1]. Therefore \( A \) is essentially finitely generated over \( k \).

Since \( A \) is essentially finitely generated over \( k \), the local ring \( A \) is in a projective model \( X \) of \( F/k \). As discussed before the theorem, resolution of singularities implies that there exists a projective model \( Y \) of \( F/k \) such that every regular local ring in \( X \) is in \( Y \), every local ring in \( Y \) is a regular local ring, and every local ring in \( X \) is dominated by a local ring in \( Y \).

Since \( A \) is a regular local ring in \( X \), \( A \) is a local ring in the projective model \( Y \). Let \( x_0, \ldots, x_n \in F \) be nonzero elements such that with \( D_i := k[x_0/x_i, \ldots, x_n/x_i] \) for each \( i \in \{0, 1, \ldots, n\} \), we have
\[ Y = \bigcup_{i=0}^{n} \{(D_i)_Q : Q \in \text{Spec} \,(D_i)\}. \]

Since \( S \) has quotient field \( F \), we may assume that \( x_0, \ldots, x_n \in S \). Since \( A \) is in \( Y \), there is \( i \in \{0, 1, \ldots, n\} \) such that \( A = (D_i)_{P_i} \).

By item 2b, \( \mathcal{V}' = S/P \) is a valuation ring with quotient field \( A/P \). For \( a \in A \), let \( \overline{a} \) denote the image of \( a \) in the field \( A/P \). Since \( S/P \) is a valuation ring of \( A/P \), there exists \( j \in \{0, 1, \ldots, n\} \) such that
\[ (\{x_k/x_i\}_{k=0}^{n})(S/P) = (x_j/x_i)(S/P). \]

Notice that \( x_j/x_i = 1 \notin P \). Hence at least one of the \( x_k/x_i \notin P \), and Equation (1) implies \( x_j/x_i \notin P \). Since \( A = S_P \) and \( P = PS_P \), every fractional ideal of \( S \) contained in \( A \) is comparable to \( P \) with respect to set inclusion. Therefore \( P \subseteq (x_j/x_i)S \). This and Equation (1) imply that
\[ (x_0/x_i, \ldots, x_n/x_i)S = (x_j/x_i)S. \]

Multiplying both sides of Equation (2) by \( x_i/x_j \) we obtain
\[ D_j = k[x_0/x_j, \ldots, x_n/x_j] \subseteq S. \]

Let \( R = (D_j)_{m \subseteq D_j} \). Since \( Y \) is a nonsingular model, \( R \) is a regular local ring with \( R \subseteq S \subseteq A \).

We observe next that \( A = R_{P \cap R} \). Since \( R \subseteq A \), we have that \( A \) dominates the local ring \( A' \) := \( R_{P \cap R} \). The local ring \( A' \) is a member of the projective model \( Y \), and every valuation ring dominating the local ring \( A \) in \( Y \) dominates also the local ring \( A' \) in \( Y \). Since \( Y \) is a projective model of \( F/k \), the Valuative Criterion for Properness [18 Theorem II.4.7, p. 101] implies no two distinct local rings in \( Y \) are dominated by the same valuation ring. Therefore, \( A = A' \), so that \( A = R_{P \cap R} \).
Finally, observe that since \( V = S/P \) is a DVR overring of \((R + P)/P\), the multiplicity sequence of \( S/P \) over \((R + P)/P\) is divergent. By Theorem 4.8, \( S \) is a quadratic Shannon extension of \( R \) with Noetherian hull \( A = R_{P^{\infty}} \). By Theorem 2.5, \( S \) is non-archimeean, so the proof is complete. \( \Box \)

As an application of Theorem 5.2, we describe for a finitely generated field extension \( F/k \) of characteristic 0 the valuation rings with principal maximal ideal that arise as quadratic Shannon extensions of regular local rings that are essentially finitely generated over \( k \), i.e., the valuation rings on the Zariski-Riemann surface of \( F/k \) that arise from desingularization followed by infinitely many successive quadratic transforms of projective models. Recall that a valuation ring \( V \) of \( F/k \) is a divisorial valuation ring if \( \text{tr.deg}_k V/m_V = \text{tr.deg}_k F - 1 \).

Such a valuation ring is necessarily a DVR (apply, e.g., \( \text{[1, Theorem 1]} \)).

**Corollary 5.3.** Let \( F/k \) be a finitely generated field extension where \( k \) has characteristic 0, and let \( S \) be a valuation ring of \( F/k \) with principal maximal.

1. Suppose rank \( S = 1 \). Then there is a sequence \( \{R_i\} \) (possibly finite) of LQTs of a regular local ring \( R \) essentially finitely type over \( k \) such that \( S = \bigcup_i R_i \). This sequence is finite if and only if \( S \) is a divisorial valuation ring.

2. Suppose rank \( S > 1 \). Then \( S \) is a quadratic Shannon extension of a regular local ring essentially finitely generated over \( k \) if and only if \( S \) has rank 2 and is contained in a divisorial valuation ring of \( F/k \).

**Proof.** For item 1, assume rank \( S = 1 \). By resolution of singularities, there is a nonsingular projective model \( X \) of \( F/k \) with function field \( F \). Let \( R \) be the regular local ring in \( X \) that is dominated by \( S \). Let \( \{R_i\} \) be the sequence of LQTs of \( R \) along \( S \). If \( \{R_i\} \) is finite, then \( \dim R_i = 1 \) for some \( i \), so that \( R_i \) is a DVR. Since \( S \) is a DVR between \( R_i \) and its quotient field, we have \( R_i = S \). Otherwise, if \( \{R_i\} \) is infinite, then Proposition 3.4 implies \( S = \bigcup_i R_i \) since \( S \) is a DVR. That the sequence is finite if and only if \( S \) is a divisorial valuation ring follows from \( \text{[1, Proposition 4]} \).

For item 2, suppose rank \( S > 1 \). Assume first that \( S \) is a Shannon extension of a regular local ring essentially finitely generated over \( k \). By \( \text{[21, Theorem 8.1]} \), \( \dim S = 2 \). By Theorem 5.2, \( S \) is a contained in a regular local ring \( A \subseteq F \) such that \( A/m_A \) is the quotient field of a proper homomorphic image of \( S \) and

\[
\text{tr.deg}_k A/m_A + \dim A = \text{tr.deg}_k F. \tag{3}
\]

We claim \( A \) is a divisorial valuation ring of \( F/k \). Since \( A/m_A \) is the quotient field of a proper homomorphic image of \( S \), it follows that

\[
\text{tr.deg}_k A/m_A < \text{tr.deg}_k F. \tag{4}
\]

From equations 3 and 4, we conclude that \( \dim A \geq 1 \). As an overring of the valuation ring \( S \), \( A \) is also a valuation ring. Since \( A \) is a regular local ring that is not a field, it follows that \( A \) is a DVR. Thus \( \dim A = 1 \) and equation 4 implies that
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\[ \text{tr.deg}_k A/m_A = \text{trdeg}_k F - 1, \]

which proves that \( A \) is a divisorial valuation ring.

Conversely, suppose rank \( S = 2 \) and \( S \) is a quadratic Shannon extension of a regular local ring that is essentially finitely generated over \( k \). Theorem 5.2 and rank \( S = 2 \) imply \( S \) is contained in a regular local ring \( A \) with \( \dim A = 1 \) and

\[ \text{tr.deg}_k A/m_A + 1 = \text{trdeg}_k F. \]

Thus \( A \) is a divisorial valuation ring.

Finally, suppose rank \( S = 2 \) and \( S \) is contained in a divisorial valuation ring \( A \) of \( F/k \). Since \( S \) is a valuation ring of rank 2 with principal maximal ideal it follows that \( m_A \subseteq S \) and \( S/m_A \) is DVR. Since \( A \) is a divisorial valuation ring, we have

\[ \text{tr.deg}_k A/m_A + \dim A = \text{trdeg}_k F. \]

As a DVR, \( A \) is a regular local ring, so Theorem 5.2 implies \( S \) is a quadratic Shannon extension of a regular local ring that is essentially finitely generated over \( k \). \qed

Example 5.4. Let \( k \) be a field of characteristic 0, let \( x_1, \ldots, x_n, y_1, \ldots, y_m \) be algebraically independent over \( k \), and let

\[ A = k(x_1, \ldots, x_n)[y_1, \ldots, y_m]. \]

Let \( \alpha : A \to k(x_1, \ldots, x_n) \) be the canonical residue map. Then for every DVR \( V \) of \( k(x_1, \ldots, x_n)/k \), the ring \( S = \alpha^{-1}(V) \) is by Theorem 5.2 a quadratic Shannon extension of a regular local ring that is essentially finitely generated over \( k \). As in the proof that statement 2 implies statement 1 of Theorem 5.2, the Noetherian hull of \( S \) is \( A \).

Conversely, suppose \( S \) is a \( k \)-subalgebra of \( F \) with principal maximal ideal such that \( S \) is a quadratic Shannon extension of a regular local ring that is essentially finitely generated over \( k \) and \( S \) has Noetherian hull \( A \). As in the proof that statement 1 implies statement 2 of Theorem 5.2, there is a DVR \( V \) of \( k(x_1, \ldots, x_n)/k \) such that \( S = \alpha^{-1}(V) \).

It follows that there is a one-to-one correspondence between the DVRs of \( k(x_1, \ldots, x_n)/k \) and the quadratic Shannon extensions \( S \) of regular local rings that are essentially finitely generated over \( k \), have Noetherian hull \( A \), and have a principal maximal ideal.

Theorem 5.2 concerns quadratic Shannon extensions of regular local rings that are essentially finitely generated over \( k \). Example 5.3 is a quadratic Shannon extension of a regular local ring \( R \) in a function field for which \( R \) is not essentially finitely generated over \( k \).

Example 5.5. Let \( F = k(x, y, z) \), where \( k \) is a field and \( x, y, z \) are algebraically independent over \( k \). Let \( \tau \in k[[x]] \) be a formal power series in \( x \) such that \( x \) and \( \tau \) are algebraically independent over \( k \). Set \( y = \tau \) and define \( V = k[[x]] \cap k(x, y) \). Then \( V \) is a DVR on the field \( k(x, y) \) with maximal ideal \( xV \) and residue field \( V/xV = k \). Let \( V(z) = V[z]_{xV[z]} \). Then \( V(z) \) is a DVR on the field \( F \) with residue field \( k(z) \), and \( V(z) \)
is not essentially finitely generated over \( k \). Let \( R = V[[z, x]]V[[z]] \). Notice that \( R \) is a 2-dim RLR. The pullback diagram of type \( \square^1 \)

\[
\begin{array}{ccc}
S = \alpha^{-1}(k[[z, x]]V[[z]]) & \longrightarrow & k[[z, x]]V[[z]] \\
\downarrow & & \downarrow \\
V(z) & \longrightarrow & k(z)
\end{array}
\]

defines a rank 2 valuation domain \( S \) on \( F \) that is by Theorem 5.1 a quadratic Shannon extension of \( R \). For each positive integer \( n \), define \( R_n = R[[z, x]]V[[z]] \). Then \( S = \bigcup_{n \geq 1} R_n \).

6 Quadratic Shannon extensions and GCD domains

As an application of the pullback description of non-archimedean quadratic Shannon extensions given in Section 4, we show in Theorem 6.2 that a quadratic Shannon extension \( S \) is coherent, a GCD domain or a finite conductor domain if and only if \( S \) is a valuation domain. We extend this fact to all quadratic Shannon extensions \( S \), regardless of whether \( S \) is archimedean, by applying structural results for archimedean quadratic Shannon extensions from [21].

**Definition 6.1.** Following McAdam in [27], an integral domain \( D \) is a finite conductor domain if for elements \( a, b \) in the field of fractions of \( D \), the \( D \)-module \( aD \cap bD \) is finitely generated. A ring is said to be coherent if every finitely generated ideal is finitely presented. Chase [6, Theorem 2.2] proves that an integral domain \( D \) is coherent if and only if the intersection of two finitely generated ideals of \( D \) is finitely generated. Thus a coherent domain is a finite conductor domain. An integral domain \( D \) is a GCD domain if for all \( a, b \in D \), \( aD \cap bD \) is a principal ideal of \( D \) [12, page 76 and Theorem 16.2, p.174]. It is clear from the definitions that a GCD domain is a finite conductor domain.

Examples of GCD domains and finite conductor domains that are not coherent are given by Glaz in [13, Example 4.4 and Example 5.2] and by Olberding and Saydam in [29, Prop. 3.7]. Every Noetherian integral domain is coherent, and a Noetherian domain \( D \) is a GCD domain if and only if it is a UFD. Noetherian domains that are not UFDs are examples of coherent domains that are not GCD domains.

**Theorem 6.2.** Let \( S \) be a quadratic Shannon extension of a regular local ring. The following are equivalent:

1. \( S \) is coherent.
2. \( S \) is a GCD domain.
3. \( S \) is a finite conductor domain.
4. \( S \) is a valuation domain.
Proof. It is true in general that if \( S \) is a valuation domain, then \( S \) satisfies each of the first 3 items. As noted above, if \( S \) is coherent or a GCD domain, then \( S \) is a finite conductor domain. To complete the proof of Theorem 6.2 it suffices to show that if \( S \) is not a valuation domain, then \( S \) is not a finite conductor domain. Specifically, we assume \( S \) is not a valuation domain and we consider three cases. In each case, we find a pair of principal fractional ideals of \( S \) that is not finitely generated.

Case 1: \( S \) is non-archimedean. By Theorem 2.2 there is a unique dimension 1 prime ideal \( Q \) of \( S \), \( QT = Q \) and \( S \) is the Noetherian hull of \( S \). If \( \dim S = 1 \), then, as a regular local ring, \( S \) is a DVR; this, along with the fact that \( Q = QT \) implies \( S \) is a DVR, contrary to the assumption that \( S \) is not a valuation domain. Therefore \( \dim S \geq 2 \), and there exist elements \( f, g \in Q \) that have no common factors in the UFD \( S \). Consider \( I = fS \cap gS \), let \( x \in m_S \) such that \( \sqrt{xS} = m_S \) (see Theorem 2.2), and let \( a \in I \). Since \( a \in fS \cap gS = fgS \), we can write \( a = fg \) for some \( y \in S \).

Now \( gy \in QS = Q \) and \( fy \in QS = Q \). So \( \frac{gy}{x} \in S \) and \( \frac{fy}{x} \in S \). Thus \( \frac{x}{y} = \frac{g}{a} \in I \). This shows that \( I = I \) and so \( m_S I = I \). Since \( I \neq \{0\} \), Nakayama’s Lemma implies that \( I \) is not finitely generated.

Case 2: \( S \) is archimedean, but not completely integrally closed. By Theorem 2.2 \( \dim S \geq 2 \). We claim that \( m_S \) is not finitely generated as an ideal of \( S \). Since \( \dim S \geq 1 \), if \( m_S \) is a principal ideal, then \( \cap m_S \) is a nonzero prime ideal of \( S \), a contradiction to the assumption that \( S \) is archimedean. Thus \( m_S \) is not principal. By [21 Proposition 3.5], this implies \( m_S^2 = m_S \). From Nakayama’s Lemma it follows that \( m_S \) is not finitely generated. Since \( S \) is not completely integrally closed, there is an almost integral element \( \theta \) over \( S \) that is not in \( S \). By [21 Corollary 6.6], \( m_S = \theta^{-1}S \cap S \).

Case 3: \( S \) is archimedean and completely integrally closed. By Theorem 2.2 \( \dim S \geq 2 \). By Theorems 2.2 and 2.7, \( S = T \cap W \), where \( W \) is the rank 1 nondiscrete valuation ring with associated valuation \( w(\cdot) \) as in Definition 2.3 and \( T \) is a UFD that is a localization of \( S \). Since \( \sum_{n \geq 0} w(m_n) < \infty \) by Theorem 2.3 and since \( m_n S \) is principal and generated by a unit of \( T \) for \( n \gg 0 \), the \( w \)-values of units of \( T \) generate a non-discrete subgroup of \( \mathbb{R} \).

Since \( S \) is archimedean, Theorem 2.4 implies \( T \) is a non-local UFD. Therefore there exist elements \( f, g \in S \) that have no common factors in \( T \). As in Case 1, we consider \( I = fS \cap gS \). Since \( S = T \cap W \), it follows that

\[
I = (fT \cap gT) \cap (fW \cap gW) = fT \cap gT \cap \{a \in W \mid w(a) \geq \max\{w(f), w(g)\}\}.
\]

Assume without loss of generality that \( w(f) \geq w(g) \).

For \( a \in I \), write \( a = (\frac{a}{x}) f \) in \( S \) and consider \( w(a) \). Since \( \frac{a}{x} \) is divisible by \( g \) in \( T \), it is a non-unit in \( T \), and thus it is a non-unit in \( S \). Since \( W \) dominates \( S \), it follows that \( w(\frac{a}{x}) > 0 \) and thus \( w(a) > w(f) \).

We claim that \( m_SI = I \). Since the \( w \)-values of the units of \( T \) generate a non-discrete subgroup of \( \mathbb{R} \), for any \( \varepsilon > 0 \), there exists an unit \( x \) in \( T \) with \( 0 < w(x) < \varepsilon \). Then for \( a \in I \) and for some \( x \) with \( 0 < w(x) < w(a) - w(f) \), we have \( \frac{a}{x} \in I \) and
thus \( a \in m_S I \). Since \( m_S I = I \) and \( I \neq (0) \), Nakayama’s Lemma implies that \( I \) is not finitely generated.

In every case, we have constructed a pair of principal fractional ideals of \( S \) whose intersection is not finitely generated. We conclude that if \( S \) is not a valuation domain, then \( S \) is not a finite conductor domain. \( \square \)

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