Conserved Charges and Supersymmetry in Principal Chiral Models

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ABSTRACT

We report on investigations of local (and non-local) charges in bosonic and supersymmetric principal chiral models in 1+1 dimensions. In the bosonic PCM there is a classically conserved local charge for each symmetric invariant tensor of the underlying group. These all commute with the non-local Yangian charges. The algebra of the local charges amongst themselves is rather more subtle. We give a universal formula for infinite sets of mutually commuting local charges with spins equal to the exponents of the underlying classical algebra modulo its Coxeter number. Many of these results extend to the supersymmetric PCM, but with local conserved charges associated with antisymmetric invariants in the Lie algebra. We comment briefly on the quantum conservation of local charges in both the bosonic and super PCMs.

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1 Introduction

Integrable QFTs in two spacetime dimensions display many of the most important phenomena of higher-dimensional QFTs in a tractable setting. Non-linear sigma-models exhibit features such as asymptotic freedom, dynamical mass generation, and confinement; while within the class of Toda theories we find the famous sine-Gordon model—providing (together with the massive Thirring model) the prototype example of an exact equivalence in which perturbative and solitonic degrees of freedom are exchanged. Our understanding of such striking phenomena is strengthened by our ability to find exact S-matrices for many of these two dimensional theories. As well as a bootstrap principle and standard axioms like analyticity and unitarity, these S-matrices are highly constrained by the existence of ‘exotic’ conserved quantities, so that a multi-particle process must factorize into two-particle scatterings, and these in turn must obey the Yang-Baxter equation. It is the requirements of factorization and elasticity, above all else, which allows the S-matrix to be determined. The nature and properties of the ‘exotic’ charges which lie behind this can vary greatly from model to model, however.

We report here on results [1] for a particular class of non-linear sigma-models (and their supersymmetric extensions) with target manifold a compact Lie group: these are the principal chiral models or PCMs. In addition, much of what we do is naturally viewed against progress in understanding affine Toda field theories or ATFTs (though we shall not discuss them in any great detail beyond this introduction; see e.g. [2] for a review). ATFTs are also defined by Lie algebra data, this time a set of simple roots which specify the exponential interactions amongst a set of scalar fields. The two classes of models (PCMs and ATFTs) exemplify the very wide range of physical and mathematical behaviour encountered in integrable QFTs.

ATFTs can be treated in the classic perturbative manner, using Feynman diagrams to calculate order-by-order from the classical lagrangian after identifying the interactions among classical mass eigenstates. Such calculations confirm and complement the exact formulas for the S-matrices [3, 2]. The PCMs, in contrast, have quantum couplings which become very large at low energies, so that many quantum properties are impossible to extract analytically from the classical lagrangian. Under these circumstances the S-matrices [6, 7] must be checked by other means (see e.g. [8] and reference therein).

Another important difference concerns the nature of the conserved quantities responsible for factorization. ATFTs have infinitely many local conserved charges (i.e. integrals of local densities) which commute with one another. These are ‘exotic’ only in as much as they have higher-spins (and would therefore be forbidden in four dimensions by the Coleman-Mandula Theorem) with values running over the exponents of the underlying Lie algebra.
modulo its Coxeter number ([2] and references therein). The lowest exponent is always one, and the lowest-spin charge is thus energy-momentum. The conservation of these charges in particle fusions (i.e. three-point couplings in perturbation theory and certain S-matrix poles in exact scattering) can be characterised by an elegant geometrical construction known as Dorey’s rule [4, 2]. Each fusing is described by a triangle in the higher-dimensional space of roots, with the values of the individual charges being obtained by projecting this down onto a canonical set of planes through root space.

The PCMs have conserved charges which are much more ‘exotic’ from a conventional QFT standpoint: non-local quantities with an associated quantum group structure known as a Yangian [3, 10]. (ATFTs exhibit a related quantum group structure in the more complicated case when their coupling is imaginary, but we shall not dwell on this here—see [3] for a review.) These charges have non-integer or even indefinite spin, and, because they are the integrals of non-local densities, a non-trivial addition rule (coproduct) on asymptotic states. Their non-vanishing commutation relations may be regarded as a spacetime extension of internal Lie algebra symmetries.

Despite the profound differences between the local and non-local charges, they turn out to have a surprising commonality of features and consequences. One of these is the basic property of factorization of the S-matrix itself, which the two kinds of charges enforce in quite different ways. Another point of specific relevance to the work summarized here is the important result of Chari and Pressley [11]: that the Yangian quantum group fusing rule for particle multiplets in the PCM, and Dorey’s rule describing fusions for particles in ATFTs, are one and the same.

This naturally suggests some deeper underlying connections in integrable QFTs, and in particular it begs the questions of when local and non-local charges co-exist, and what their relationship might then be. In fact the existence of local conserved charges in PCMs has been known for some time, but they have received comparatively little attention. Our work [1] has led to a general construction of infinitely many local commuting charges in each PCM based on a classical algebra, with spins equal to the exponents modulo the Coxeter number, exactly as for ATFTs. We have also begun to investigate the analogous questions for the supersymmetric PCMs. These are known to have some novel features and are generally much less well-understood than their bosonic counterparts.
2 The classical bosonic PCM

2.1 The model in outline

The principal chiral model is defined by a field \( g(x) \) taking values in a compact Lie group \( G \) and governed by the lagrangian

\[
\mathcal{L} = \frac{1}{2} \text{Tr} \left( \partial_+ g^{-1} \partial_- g \right).
\] (2.1)

There is a global symmetry \( G_L \times G_R \) under which \( g \mapsto U_L g U_R^{-1} \). The current

\[
j_\pm = -g^{-1} \partial_\pm g
\] (2.2)

takes values in the Lie algebra \( \mathfrak{g} \) and corresponds to \( G_R \) transformations, while \( -gj_\pm g^{-1} \) corresponds to \( G_L \) transformations. The equations of motion for the PCM are

\[
\partial_- j_+ = -\partial_+ j_- = -\frac{1}{2} [j_+, j_-].
\] (2.3)

The energy-momentum tensor has components

\[
T_{\pm\pm} = -\frac{1}{2} \text{Tr}(j_\pm j_\pm), \quad T_{+-} = T_{-+} = 0, \quad \text{with} \quad \partial_- T_{++} = \partial_+ T_{--} = 0,
\] (2.4)

reflecting the classical conformal symmetry of the theory.

Every PCM has an important discrete symmetry \( \pi : g \mapsto g^{-1} \) which exchanges \( G_L \) and \( G_R \). Other discrete symmetries arise as outer automorphisms of \( G \) acting on the field \( g \). Thus we have a new symmetry \( \gamma : g \mapsto g^* \) when the defining representation of \( \mathfrak{g} \) is complex, while for \( \mathfrak{g} = \mathfrak{so}(2\ell) \) we also have \( \sigma : g \mapsto MgM^{-1} \) (where \( \det M = -1 \)) which exchanges the inequivalent spinor representations. Because higher derivatives of \( j_\pm \) do not transform in a simple fashion under these discrete symmetries, it is convenient to introduce

\[
j_{++} \equiv \partial_+ j_+, \quad j_{+++} \equiv \partial_+ j_{++} - \frac{1}{2} [j_+, j_{++}], \quad j_{++++} \equiv \partial_+ j_{+++} - \frac{1}{2} [j_+, j_{+++}], \quad \ldots
\]

(and similarly for the the minus components). It is then easy to show

\[
\pi : j_{+++} \mapsto -g j_{+++} g^{-1}, \quad \gamma : j_{+++} \mapsto (j_{+++})^* = -(j_{+++})^T
\] (2.5)

which will be useful later.

\footnote{Spacetime conventions: orthonormal and light-cone coordinates are related by \( x^\pm = \frac{1}{2}(t \pm x) \) and \( \partial_\pm = \partial_t \pm \partial_x \). Lie algebra conventions: We take \( \mathfrak{g} \) in its defining representation with anti-hermitian generators \( t^a \) obeying \( [t^a, t^b] = f^{abc} t^c \) and \( \text{Tr}(t^a t^b) = -\delta^{ab} \). For \( X \in \mathfrak{g} \) we write \( X = t^a X^a \) and \( X^a = -\text{Tr}(t^a X) \).}
The canonical structure of the classical PCM is defined by the Poisson brackets of the currents:

\[
\begin{align*}
\{j_+^a(x), j_+^b(y)\} &= f^{abc}(\frac{3}{2}j_+^c(x) - \frac{1}{2}j_-^c(x)) \delta(x-y) + 2\delta^{ab}\delta'(x-y) \\
\{j_+^a(x), j_-^b(y)\} &= \frac{1}{2} f^{abc}(j_+^c(x) + j_-^c(x)) \delta(x-y)
\end{align*}
\]

(2.6)

at equal time. These imply that the energy momentum tensor satisfies the classical, centre-less Virasoro algebra

\[
\{T_{++}(x), T_{++}(y)\} = 2\delta'(x-y)(T_{++}(x) + T_{++}(y))
\]

(2.7)

(and similarly for the minus components).

2.2 Local conserved charges

There are several categories of higher-spin, conserved, local charges in the PCM which have distinct characteristics.

- The conservation of the energy-momentum tensor (2.4) immediately implies

  \[
  \partial_-(T^m_{++}) = \partial_+(T^m_{--}) = 0.
  \]

  (2.8)

  Such conservation laws clearly hold in any classically conformally-invariant theory.

- In addition, we have

  \[
  \partial_- \text{Tr}(j^m_+) = \partial_+ \text{Tr}(j^m_-) = 0.
  \]

  (2.9)

  These are a consequence of (2.3); they depend on the detailed form of the PCM equations of motion, rather than on conformal invariance alone.

- The two previous categories may now be generalized as follows. Let \(d_{a_1a_2...a_m}\) be any totally symmetric invariant tensor, so that \(d_{c(a_1a_2...a_{m-1})b} = 0\). For each such tensor (or Casimir) there are conservation equations

  \[
  \partial_-(d_{a_1a_2...a_m}j^{a_1}_+j^{a_2}_+...j^{a_m}_+) = \partial_+(d_{a_1a_2...a_m}j^{-a_1}_-j^{-a_2}_-...j^{-a_m}_-) = 0.
  \]

  (2.10)

  The currents in (2.8) correspond to even-rank invariant tensors constructed from Kronecker deltas:

  \[
  d_{a_1a_2...a_m} = \delta_{a_1a_2}\delta_{a_3a_4}...\delta_{a_{2n-1}a_{2n}}
  \]

  (2.11)

  while those in (2.9) correspond to

  \[
  d_{a_1a_2...a_m} = \text{STr}(t^{a_1}t^{a_2}...t^{a_m})
  \]

  (2.12)
with ‘STr’ denoting the trace of a completely symmetrized product of matrices.

- Finally, the most general possibility is to take an arbitrary differential polynomial in the currents we have already discussed, *e.g.* \( \partial_+ \left( \text{Tr}(j_+^p) \partial_+ \text{Tr}(j_+^q) \right) = 0 \) follows immediately from (2.9). We shall not be directly concerned with such currents here.

These observations lead us to a more detailed consideration of invariant tensors. There are infinitely many invariant tensors for each algebra \( g \) but only \( \text{rank}(g) \) independent or primitive \( d \)-tensors and Casimirs (see *e.g.* [12]), whose degrees equal the exponents of \( g \) plus one. All other invariant tensors can be expressed as polynomials in these and the structure constants \( f_{abc} \). The primitive \( d \)-tensors for the classical algebras can all be chosen to be symmetrized traces, as in (2.12), with one exception. This exception is the Pfaffian invariant for \( so(2\ell) \), which has rank \( \ell \), and can be written

\[
d_{a_1...a_\ell} = \epsilon_{i_1j_1...i_\ell j_\ell} (t^{a_1})_{i_1j_1} \cdots (t^{a_\ell})_{i_\ell j_\ell}.
\]  

(2.13)

In the next section we will discuss the algebra of conserved charges arising from (2.10) for various choices of the invariant tensors \( d \). We denote these charges

\[
q_{\pm s} = d_{a_1a_2...a_m} \int_{-\infty}^{\infty} j_{\pm}^{a_1}(x) j_{\pm}^{a_2}(x) \cdots j_{\pm}^{a_m}(x) \, dx
\]

(2.14)

labelled by their spin, \( s = m - 1 \). It will be sufficient for many purposes to consider \( q_s \) with \( s > 0 \), and it is useful to introduce the notation

\[
\mathcal{J}_m = \text{Tr}(j_m^n), \quad \mathcal{P}_\ell = \epsilon_{i_1j_1...i_\ell j_\ell} (j_+)^n_{i_1j_1} \cdots (j_+)^n_{i_\ell j_\ell}
\]

(2.15)

for the currents corresponding to the tensors (2.12) and (2.13). Once again, sub-scripts denote the spin.

### 2.3 Non-local conserved charges

In addition to the local charges, there exist infinitely many conserved non-local charges in the bosonic PCM, generated by the obvious local charge

\[
Q^{(0)a} = \int_{-\infty}^{\infty} j_0^a(x) \, dx
\]

and the rather less obvious first non-local charge

\[
Q^{(1)a} = \int_{-\infty}^{\infty} j_1^a(x) \, dx - \frac{1}{2} f^{abc} \int_{-\infty}^{\infty} j_0^b(x) \int_{-\infty}^{x} j_0^c(y) \, dy \, dx.
\]

Under the Poisson bracket, these form a Yangian \( Y(g) \) [9, 10]. In fact there are two infinite sequences of such charges constructed from both \( j_\mu \) and \( -g j_\mu g^{-1} \), and so the model has a
charge algebra $Y_L(g) \times Y_R(g)$. (It can be checked that $Y_L$ and $Y_R$ commute.) These charges can be extracted from the monodromy matrix by a power series expansion in the spectral parameter, or via an iterative construction of their currents [10].

The non-local character of the Yangian charges means that they will not be additive on products of states in the quantum theory. Instead their action is given by the co-product rules $\Delta(Q^{(0)a}) = Q^{(0)a} \otimes 1 + 1 \otimes Q^{(0)a}$, which is essentially trivial, and $\Delta(Q^{(1)a}) = Q^{(1)a} \otimes 1 + 1 \otimes Q^{(1)a} + \frac{1}{2} f^{abc} Q^{(0)b} \otimes Q^{(0)c}$, which is non-trivial. These equations may also be interpreted classically as giving the values of the charges on widely-separated, localized configurations [9, 10].

It is natural to ask how the non-local charges behave with respect to the other conserved quantities in the PCM. It can be shown [1] that the non-local charges commute with the general local charge of type (2.14) in the classical theory: $\{q_s, Q^{(0)a}\} = \{q_s, Q^{(1)a}\} = 0$. The non-local charges are also classically Lorentz scalars, as can be checked by applying the boost operator $M$ and calculating $\{M, Q^{(0)a}\} = \{M, Q^{(1)a}\} = 0$. A subtle effect in the quantum theory is that the commutator of the non-local charges with the boost operator receives a correction at $O(\hbar^2)$, which is essential to the non-trivial structure of the S-matrices for the PCMs. Although the calculations have not been carried out, it seems unlikely to us that the commutators with the charges $q_s$ would receive similar modifications at the quantum level, since it is hard to see how this could be compatible with the Yangian-invariant S-matrices. Nevertheless, this is something which should be checked.

3 Commuting local charges in the bosonic PCM

3.1 Introductory comments and isolated examples

We have identified several sets of local conserved charges in any PCM, and an obvious question is whether these might be, or might contain, sets which mutually commute. At this stage one may be rather discouraged by the form of the current Poisson brackets (2.6). Both the lack of covariance and the presence of the $\delta'$ terms foreshadow potential complications with the charge algebra. Indeed, this was sufficient to prompt Faddeev and Reshetikin [13] to introduce and study an alternative classical limit of the quantum PCM that was more amenable to standard techniques like the classical inverse-scattering approach. We will persevere with (2.6), however, despite these ominous signs. We will see that it is possible not only to find sets of charges which mutually commute, but also to give a general definition: a universal formula valid for all the classical algebras.

The classical Poisson bracket algebra of local charges $q_{\pm s}$ ($s > 0$) of the general type
can be calculated from \((2.6)\). The terms involving \(\delta(x-y)\) (i.e. the ultra-local terms) always vanish by invariance of the \(d\)-tensors, leaving only contributions from the \(\delta'(x-y)\) terms (i.e. the non-ultra-local terms). It is clear from \((2.6)\) that these are absent too if we consider charges of opposite chiralities, implying
\[
\{q_s, q_{-r}\} = 0, \quad r, s > 0. \tag{3.1}
\]

For charges of the same chirality, however, the result is generally non-zero:
\[
\{q_s, q_r\} = (\text{const}) \int_{-\infty}^{\infty} dx d_{ca_1...a_s} d_{cb_1...b_r} j_{+}^{a_1} \cdots j_{+}^{a_s} \partial_x (j_{+}^{b_1} \cdots j_{+}^{b_r}). \tag{3.2}
\]

Note that this expression is anti-symmetric in \(s\) and \(r\), by integration by parts.

Our aim now is to find invariant tensors and conserved currents for which this expression vanishes, so that the charges commute. There are three circumstances in which this happens in a relatively simple way.

- For \(r = 1\) and \(d_{bc} = \delta_{bc}\), the integrand in \((3.2)\) is clearly a total derivative and the Poisson bracket vanishes. This simply means that all the local charges \((2.14)\) commute with energy and momentum: they are invariant under translations in space and time.

- If both currents are powers of the energy-momentum tensor \((2.8)\) then the integrand in \((3.2)\) can be written as a total derivative. This is actually a general feature of any classically conformally-invariant theory, whose energy-momentum tensor obeys the Virasoro algebra \((2.7)\). It is a simple consequence of this that the charges \(\int T_{++} dx\) commute with one another.

- The currents \(J_m\) defined in \((2.15)\) give rise to commuting charges \(\int J_m dx\) for \(g = so(\ell)\) or \(sp(\ell)\). For these currents we note that \((3.2)\) can be written
\[
\{q_s, q_r\} = (\text{const}) \int_{-\infty}^{\infty} dx \text{Tr}(t^c j_{+}^{a_1} \cdots j_{+}^{a_s}) \partial_x \text{Tr}(t^c j_{+}^{b_1} \cdots j_{+}^{b_r}). \tag{3.3}
\]
which can be simplified using the completeness condition \(X = -t^a \text{Tr}(t^a X)\) valid for any \(X\) in \(g\). For the orthogonal and symplectic algebras \(s\) and \(r\) are always odd (otherwise the currents vanish) and this implies that \((j_{+})^r\) and \((j_{+})^s\) also belong to \(g\). The completeness condition then implies that the integrand is proportional to \(\partial_x \text{Tr}(j_{+}^{r+s})\) and hence the charges commute.

The charges \(\int J_m dx\) have more complicated brackets for \(g = su(\ell)\). In this case the completeness condition holds only for traceless matrices and this property is of course

\(^4\) If \(X\) is in \(so(\ell)\) it is a real anti-symmetric matrix, and so if \(m\) is odd, \(X^m\) will also be real and anti-symmetric, and hence also in \(so(\ell)\). Similar arguments apply to \(g = sp(\ell)\).
spoiled by taking powers. The result is that (3.2) becomes
\[
\{q_s, q_r\} = (\text{const}) \int_{-\infty}^{\infty} dx \, \text{Tr}(j_s^+ \partial_x j_r^+)
\]
which is non-zero in general. Notice that the bracket nevertheless produces a conserved quantity which we recognize, namely a differential polynomial in the currents \( J_m \).

### 3.2 A general construction

We now investigate the possibility of more general sets of commuting charges in the PCM based on any classical group \( G \). One way to begin is to carry out some trial calculations for low-lying values of the spin. We can search systematically for polynomials \( K_{s+1}(J_m) \) which are homogeneous in the spin and which will give commuting charges
\[
q_s = \int K_{s+1} dx.
\]

For \( g = su(\ell) \) we find, after some laborious calculations, the following expressions:
\[
\begin{align*}
K_2 &= J_2 \\
K_3 &= J_3 \\
K_4 &= J_4 - \frac{3}{2\ell} J_2^2 \\
K_5 &= J_5 - \frac{10}{3\ell} J_3 J_2 \\
K_6 &= J_6 - \frac{5}{3\ell} J_3^2 - \frac{15}{4\ell} J_4 J_2 + \frac{25}{8\ell^2} J_2^3
\end{align*}
\]
(3.5)

These are the unique combinations (up to overall constants) for which the corresponding charges commute. For \( g = so(\ell) \) or \( sp(\ell) \) similar calculations reveal a family of currents with a single free parameter \( \alpha \). The first few examples are:
\[
\begin{align*}
K_2 &= J_2 \\
K_4 &= J_4 - \frac{3}{2}(3\alpha) J_2^2 \\
K_6 &= J_6 - \frac{3}{4}(5\alpha) J_4 J_2 + \frac{1}{8}(5\alpha)^2 J_2^3 \\
K_8 &= J_8 - \frac{2}{3}(7\alpha) J_6 J_2 - \frac{1}{4}(7\alpha) J_4^2 + \frac{1}{4}(7\alpha)^2 J_4 J_2^2 - \frac{1}{48}(7\alpha)^3 J_2^4
\end{align*}
\]
(3.6)

Notice that this one-parameter family interpolates the two simplest families we found previously for the orthogonal and symplectic algebras. When \( \alpha \to 0 \) we have \( K_{2m} \to J_{2m} \) and in the limit \( \alpha \to \infty \) we have (with a suitable rescaling) \( K_{2n} \to (J_2)^n \).

The examples we have just given are sufficient to suggest general definitions of infinite sets of currents which we can then prove yield commuting charges. For each of the classical...
algebras $su(\ell)$, $so(\ell)$, $sp(\ell)$, we introduce the generating functions $A(x, \lambda)$ and $F(x, \lambda)$ defined by

$$A(x, \lambda) = \exp F(x, \lambda) = \det(1 - \lambda j_+(x))$$

so that

$$F(x, \lambda) = \text{Tr} \log(1 - \lambda j_+(x)) = -\sum_{r=2}^{\infty} \frac{\lambda^r}{r} \mathcal{J}_r(x).$$

Observe that $A(x, \lambda)$ is a polynomial of degree at most $\ell$, with the coefficient of the term in $\lambda^\ell$ being $(-1)^\ell \det(j_+)$; on substituting the series expansion for $F$ into (3.7), we obtain non-trivial identities satisfied by the $\mathcal{J}_m$ as the coefficients of $\lambda^r$ must vanish for $r > \ell$ (for details see e.g. [12]).

We now define

$$\mathcal{K}_{s+1} = A(x, \lambda)^s \bigg|_{\lambda^{s+1}} = \exp \alpha s F(x, \lambda) \bigg|_{\lambda^{s+1}}$$

In extracting the coefficients indicated, we expand the generating function in ascending powers of $\lambda$. It can be shown from this definition that the charges defined commute when $\alpha = 1/h = 1/\ell$ for $su(\ell)$, or for $\alpha$ arbitrary for $so(\ell)$ and $sp(\ell)$ [1].

An important consequence of the formula (3.9) is that the spins of the charges which it defines repeat modulo the Coxeter number $h$. Consider first the case $g = su(\ell)$, with $\alpha = 1/h$. If $s/h$ is not an integer, then the expansion of $A(x, \lambda)^{s/h}$ is an infinite series, and the expression for $\mathcal{K}_{s+1}$ will be non-vanishing. If $s/h$ is an integer, however, then $A(x, \lambda)^{s/h}$ is a polynomial of degree $s$ in $\lambda$ and $\mathcal{K}_{s+1}$ vanishes according to the definition above. Thus for $g = su(\ell)$ the non-trivial charges have spins which run over all integers modulo the Coxeter number. For the algebras $g = so(\ell)$ or $sp(\ell)$ things work in a more trivial way, because $h$ is even, while the spins $s$ for the currents defined above are all the odd integers, so they certainly repeat modulo $h$.

So for each PCM based on a classical algebra we now have infinitely many commuting charges which come in sequences, each sequence being associated with an exponent of the algebra and with a corresponding primitive invariant tensor of type (2.12). But there is one primitive invariant tensor which is not of the type (2.12) and which has therefore been absent from our discussion so far. This is the Pfaffian in $d_\ell = so(2\ell)$. It is natural to expect that our results can be extended to include this last invariant, and this is indeed the case.

A direct computation with the first few examples listed in (3.6) shows that $\int \mathcal{K}_m \, dx$ commutes with the Pfaffian charge $\int \mathcal{P}_\ell \, dx$ provided we choose $\alpha = 1/h$, where $h = 2\ell - 2$ for $so(2\ell)$. But in fact the Pfaffian is just the first member of a series of conserved currents $\mathcal{P}_{\ell+a h}$ for integers $a \geq 0$ (where the subscript denotes the spin, as usual). It is rather remarkable that these currents can be defined from the same generating function $A(x, \lambda)$
we introduced above, and using the same formula (3.9), but with the coefficient to be extracted as part of an expansion in descending powers of \( \lambda \) rather than ascending powers of \( \lambda \). With this definition it can be shown that the charges \( \int K_m \, dx \) and \( \int P_{l+ah} \, dx \) all commute with one another for \( \alpha = 1/h \). Full details are given in [1].

In summary: in each PCM based on a classical algebra, we have found commuting local charges with spins equal to all the exponents modulo the Coxeter number. We have not investigated PCMs based on exceptional groups, but we have no reason to suspect they should behave any differently.

4 The classical supersymmetric PCM

4.1 The model in outline

The most efficient way to supersymmetrize the bosonic PCM is to introduce a superfield \( G(x, \theta) \) with values in \( \mathcal{G} \). The additional coordinates are real Grassmann numbers \( \theta^{\pm} \) with supercovariant derivatives \( D_{\pm} = \partial_{\theta^{\pm}} - i\theta^{\pm}\partial_{\pm} \) which we can use to write a manifestly supersymmetric lagrangian:\footnote{Spacetime conventions: Each index \( \pm \) signifies one unit of Lorentz spin on a bosonic object, but a 1/2-unit of spin on a fermionic object. Upper and lower indices denote opposite Lorentz weights.}

\[
\mathcal{L} = \text{Tr}(D_{+}G^{-1}D_{-}G) \tag{4.1}
\]

with \( \mathcal{G}_L \times \mathcal{G}_R \) symmetry. Corresponding to \( \mathcal{G}_R \) and \( \mathcal{G}_L \) we have the superspace currents

\[
J_{\pm} = -iG^{-1}D_{\pm}G \tag{4.2}
\]

and \( -GJ_{\pm}G^{-1} \) respectively. These take values in \( g \) (tensored with the appropriate underlying real Grassmann algebra) and they satisfy

\[
D_{+}J_{-} = D_{-}J_{+} = -\frac{i}{2}\{J_{+},J_{-}\}. \tag{4.3}
\]

Just as in the bosonic case, there are discrete symmetries \( \pi : G \mapsto G^{-1} \) and \( \gamma : G \mapsto G^* \) (and also \( \sigma : G \mapsto MGM^{-1} \) for \( g = \text{so}(2\ell) \)). We find

\[
\pi : J_{+++} \mapsto -GJ_{+++}G^{-1}, \quad \gamma : J_{+++} \mapsto (J_{+++})^* = -(J_{+++})^T
\]

where we have again introduced convenient quantities

\[
J_{++} \equiv D_{+}J_{+} + \frac{i}{2}\{J_{+},J_{+}\}, \quad J_{++} \equiv -iD_{+}J_{++} + \frac{1}{2}[J_{+},J_{++}], \quad \ldots \tag{4.4}
\]
which have simple behaviour. Note that the brackets appearing above are graded, and that the factors of $i$ ensure that $J_{++}$ is always a combination of anti-hermitian Lie algebra generators with real (possibly Grassmann algebra-valued) coefficients.

To reveal the component ($x$-space) content of the super PCM we can expand

$$G(x, \theta) = g(x)(1 + i\theta^+ \psi_+(x) + i\theta^- \psi_-(x) + i\theta^+ \theta^- \sigma(x)).$$

The fermions $\psi_\pm(x)$ take values in $\mathfrak{g}$ and are the superpartners of the group-valued fields $g(x)$. The field $\sigma(x)$ is auxiliary and can be eliminated from the action algebraically to produce four-fermion interaction terms. The corresponding expansion of the superspace currents is

$$J_\pm(x, \theta) = \psi_\pm(x) + \theta^\pm j_\pm(x) + \ldots$$

where $j_\pm = -g^{-1} \partial_\pm g - i\psi^2_\pm$ (4.5)

and the precise form of the higher components of the currents will not be needed. The equations of motion (4.3) imply that the bosonic current is conserved, $\partial_- j_+ + \partial_+ j_- = 0$, although it does not obey (2.3). The remaining consequences of (4.3) are equations of motion for the fermions.

The classical super PCM is superconformally invariant, with the non-vanishing components of the super energy-momentum tensor obeying

$$D_- \text{Tr}(J_+ J_{++}) = D_+ \text{Tr}(J_- J_{--}) = 0.$$

When expanded in components this contains conservation equation for both the supersymmetry current and the conventional (bosonic) energy momentum tensor.

4.2 Conserved charges

The supersymmetric PCM contains infinitely many local and non-local conserved quantities, some of which resemble their bosonic counterparts, others arising in conjunction with novel features. It has been known for a long time [14] that the Yangian charges generalize to the supersymmetric theory with no significant modification of their properties. In particular, they commute with supersymmetry, so there is no enhancement of Yangian symmetry. We shall therefore concentrate on the local charges.

The simplest local conserved currents in the bosonic PCM are powers of the energy-momentum tensor (2.8). A super energy-momentum tensor is a fermionic quantity, however, so we cannot take powers of it to obtain new conservation laws in quite the same way. Let us therefore turn directly to the generalizations of (2.9) and (2.10).
• The conservation laws (2.9) in the bosonic PCM can be generalized to the supersymmetric PCM in two different ways. First, we have

\[ D_- \text{Tr}(J_{+}^{2n+1}) = 0 \]  

which is odd under the discrete symmetry \( \pi \). The power of \( J_+ \) must be an odd integer, otherwise the expression would vanish identically, by Fermi statistics. Second, we have

\[ D_- \text{Tr}(J_{+}^{2n-1} J_{++}) = 0 \]  

which is even under \( \pi \). The power of \( J_+ \) must again be odd, this time to prevent the expression being a total \( D_+ \) derivative and hence giving a trivial conservation equation. Both (4.6) and (4.7) follow directly from the superspace equations of motion.

• As in the bosonic case, we can re-express and generalize these conservation equations by writing them in terms of invariant tensors. The equation (4.6) becomes

\[ D_- (\Omega_{a_1 a_2 \ldots a_{2n+1}} J_{+}^{a_1} J_{+}^{a_2} \ldots J_{+}^{a_{2n+1}}) = 0 \]  

where the odd-rank invariant tensor

\[ \Omega_{a_1 a_2 \ldots a_{2n+1}} = f_{[a_1 a_2} b_1 \ldots b_{n} d^{b_1 \ldots b_n}]_{a_{2n+1}]} \]  

is totally anti-symmetric. In a similar fashion, the second kind of conservation equation (4.7) becomes

\[ D_- (\Lambda_{a_1 a_2 \ldots a_{2n}} J_{+}^{a_1} \ldots J_{+}^{a_{2n-1}} J_{++}^{a_{2n}}) = 0 \]  

where now the relevant invariant tensor is even-rank,

\[ \Lambda_{a_1 a_2 \ldots a_{2n-1} a_{2n}} = f_{[a_1 a_2} b_1 \ldots b_{n-1} d^{b_1 \ldots b_{n-1}] a_{2n]} a_{2n+1} \]  

It has a more complicated structure in that it is antisymmetric only on its first \( 2n-1 \) indices.

It seems natural that in a theory which contains fermionic currents we should find conservation laws involving antisymmetric invariant tensors. There can clearly only be finitely many of these. We note that both \( \Omega \) and \( \Lambda \) are defined above in terms of some symmetric invariant tensor \( d \). They are non-vanishing when \( d \) is one of the finite number of primitive symmetric tensors which we mentioned previously (see e.g. [12]).

To get a better idea of the meaning of the above superspace conservation equations it is instructive to expand them in component fields, using (4.5). On doing this we find that (4.8) produces fermionic and bosonic conserved currents

\[ \Omega_{a_1 a_2 \ldots a_{2n+1}} \psi_{a_1} \psi_{a_2} \ldots \psi_{a_{2n+1}} j_{a_2n+1}^+ \]  

\[ \Omega_{a_1 a_2 \ldots a_{2n+1}} \psi_{a_1} \psi_{a_2} \ldots \psi_{a_{2n+1}} j_{a_2n+1}^+ \]  

(4.12)
The fermionic and bosonic currents resulting from (4.10) are more complicated. They can be written, up to terms proportional to the expressions in (4.12), as

\[ d_{a_1a_2a_3\ldots a_{n+1}}j_{+}^a\psi^a_+F^q_+\ldots F_{+}^{a_{n+1}}, \quad d_{a_1a_2a_3\ldots a_{n+1}}(n_j^a j_{+}^a + i\psi^a_+ \partial_+ \psi^a_+)F^q_+\ldots F_{+}^{a_{n+1}} \]

where we have introduced the bosonic quantity \( F^a_+ = if^a_{bc}\psi^b_+\psi^c_+ \). Notice that in either family of conservation laws, the fermionic and bosonic currents have spins \( n+\frac{1}{2} \) and \( n+1 \) respectively, and so the corresponding conserved charges have spins \( n-\frac{1}{2} \) and \( n \) respectively. The \( d \) tensors being primitive then implies that the values of \( n \) are precisely the exponents of the algebra.

The Poisson bracket structure of the super PCM and the resulting algebra of its local currents is significantly more complicated than in the bosonic case. For this reason we shall not attempt to give a detailed discussion here. One can derive results for the families (4.6) and (4.7) which are similar in many respects to those we have described for the bosonic PCM. The charges can be shown to have simple brackets amongst themselves, including many which vanish. Finally, all of these charges commute with the Yangian. We intend to give a full account of these results in a forthcoming paper.

5 Remarks on quantum conserved charges

The character of the bosonic and super PCMs changes dramatically on quantization. The (super)conformal invariance of the classical theories is broken, and the dimensionless classical coupling (which we have suppressed throughout) is replaced by a mass-scale. The theories are strongly coupled in the infra-red so that quantum computations from the classical action are usually formidable to say the least.

While the non-local charges have been successfully studied at the quantum level, the situation for the local charges is more complicated, and only indirect results are presently available. To find some indication of whether the classical charges we have been studying are also present in the quantum theory, we can use the method of Goldschmidt and Witten, summarized as follows.

5.1 Goldschmidt-Witten counting

Suppose we have linearly-independent conservation equations \( \partial_- j_i = 0 \) or \( D_- J_i = 0 \) (in the supersymmetric case) with \( i = 1, \ldots, n \) and that these have a common prescribed

\[ \text{The Goldschmidt-Witten method in superspace was discussed in } [16] \text{ and applied to the supersymmetric } O(N) \text{ sigma-model.} \]
behaviour under all symmetries of the theory. The only quantum modifications which can appear on the right-hand sides of these equations are operators with the same mass dimension and the same behaviour under continuous and discrete symmetries. Let \( A_i \) with \( i = 1, \ldots, p \) be a linearly-independent set of such operators. We can also enumerate the linearly-independent total-derivative terms \( B_i \) with \( i = 1, \ldots, q \) which have the same symmetry properties. Since each of the \( B_i \)s is expressible as a combination of \( A_i \)s we must have \( q \leq p \). Now if \( n - p + q > 0 \), then there are at least this many combinations of the classical conservation equations which survive in the quantum theory, because this is the number of linearly-independent combinations for which the right-hand side is guaranteed to be a (super)spacetime divergence.

We can now apply these arguments to the bosonic and supersymmetric PCMs, specializing to \( G = SU(\ell) \) for simplicity. It is important to consider the behaviour of each current under both the continuous symmetries and the discrete symmetries \( \pi \) and \( \gamma \). Starting with the bosonic model, it so happens that all the currents we list below have the same behaviour under \( \pi \) and \( \gamma \), and so we describe them simply as even or odd.

- **Spin-2**: \( \text{Tr}(j^2_+) \), even; this is the energy-momentum tensor. There is one anomaly \( A_1 = \text{Tr}(j_+ j^+ j^+ j^+) \) and one derivative \( B_1 = \partial_+ \text{Tr}(j_+ j^+) \) with \( A_1 = B_1 \). The conservation law therefore survives quantum-mechanically, as we expect, but its modification reflects the non-vanishing of the trace of the quantum energy-momentum tensor, corresponding to the breaking of conformal symmetry.

- **Spin-3**: \( \text{Tr}(j^3_+) \), odd. There is one anomaly \( A_1 = \text{Tr}(j_+ j^+ j^+ j^+ j^+) \) and one derivative \( B_1 = \partial_+ \text{Tr}(j_+ j^+ j^+ j^+) \) with \( A_1 = B_1 \); the conservation again survives quantization.

- **Spin-4**: currents \( \text{Tr}(j^4_+) \) and \( (\text{Tr}(j^2_+))^2 \) are both even under each of the discrete symmetries. The anomalies and derivatives with these symmetries are

\[
A_1 = \text{Tr}(j_+ j^+ j^+ j^+) \quad B_1 = \partial_+ \text{Tr}(j_+ j^+ j^+ j^+)
\]
\[
A_2 = \text{Tr}(j_+ j^+ j^+ j^+) \quad B_2 = \partial_+ \left( \text{Tr}(j_+ j^+ j^+ j^+) \text{Tr}(j^2_+) \right)
\]
\[
A_3 = \text{Tr}(j_+ j^+ j^+ j^+) \quad B_3 = \partial_+ \text{Tr}(j^3_+)
\]
\[
A_4 = \text{Tr}(j^2_+ \{j_+ j^+ j^+ j^+ \}) \quad B_4 = \partial_\ \text{Tr}(j^2_+)
\]
\[
A_5 = \text{Tr}(j_+ j^+ j^+ j^+ j^+)
\]

Since \( n = 2, p = 5, q = 4 \), we conclude that there is at least one linear combination of the currents which is conserved in the quantum theory.

- For higher values of the spin, the Goldschmidt-Witten method is inconclusive. For instance, we find for spin-5 (odd) that \( n = 2, p = 8, q = 6 \); while for spin-6 (even) we have \( n = 5, p = 25, q = 18 \).
Turning now to the supersymmetric $SU(\ell)$ PCM, we find the following results:

- **Spin-3/2**: $\text{Tr}(J^+_+ J_{++})$, even under both $\pi$ and $\gamma$; this is the super-energy-momentum tensor. There is one anomaly $A_1 = \text{Tr}(J_- J_{+++})$ and one derivative $B_1 = D_+ \text{Tr}(J_- J_{++})$ with $A_1 = -B_1$. Supersymmetry and translation invariance therefore survive in the quantum theory, as expected.

- **Spin-3/2**: $\text{Tr}(J^3_+ J_{++})$, odd under $\pi$, even under $\gamma$. There is one anomaly $A_1 = \text{Tr}(J_- [J_+, J_{++}])$ and one derivative $B_1 = D_+ \text{Tr}(J_- J^2_+)$ with $A_1 = B_1$. This current too survives.

- **Spin-5/2**: $\text{Tr}(J^3_+ J_{++})$, even under $\pi$, odd under $\gamma$. The lists of anomalies and derivatives with these symmetries are
  
  \begin{align*}
  A_1 &= \text{Tr}(J_- \{ J^2_+, J_{+++} \}) \\
  A_2 &= \text{Tr}(J_- J_+ J_{+++}) \\
  A_3 &= \text{Tr}(\{ J_-, J_+ \} J^2_+) \\
  B_1 &= D_+ \text{Tr}(J_+ J_- J_{++}) \\
  B_2 &= D_+ \text{Tr}([J_+, J_{++}] J_+) \\
  B_3 &= D_+ \text{Tr}(J_+ J^4_+) \\
  B_4 &= D_- (J_+ J^2_{++})
  \end{align*}

  Since the former out-number the latter we do not necessarily have quantum conservation.

- **Spin-5/2**: $\text{Tr}(J^5_+)$, odd under both $\pi$ and $\gamma$. This time we find
  
  \begin{align*}
  A_1 &= \text{Tr}(J_- \{ J^2_+, J_{+++} \}) \\
  A_2 &= \text{Tr}(J_- \{ J^3_+, J_{+++} \}) \\
  A_3 &= \text{Tr}(\{ J_-, J^3_+ \} J_{++}) \\
  A_4 &= \text{Tr}(J_+ \{ J_+, J_- \} J_+ J_{++}) \\
  B_1 &= D_+ \text{Tr}(J_+ J^2_{++}) \\
  B_2 &= D_+ \text{Tr}(J_- [J_+, J_{++}] J_+) \\
  B_3 &= D_+ \text{Tr}(J_- J^4_{++}) \\
  B_4 &= D_- (J_+ J^2_{++})
  \end{align*}

  and so this conservation law survives quantization.

- For higher values of the spin the results are inconclusive, just as in the bosonic case. As illustrations, for spin-7/2 (odd/even) we have $n = 2$, $p = 28$, $q = 20$; while for spin-7/2 (even/even) we find $n = 2$, $p = 27$, $q = 20$.

### 5.2 Implications of quantum conservation laws

The counting arguments described above are sometimes sufficient to demonstrate the existence of a quantum conserved charge, but they are by no means necessary. The fact that they fail in most instances should certainly not be interpreted as meaning that the classical equations in question do not generalize, but merely that these arguments are insufficient to settle the matter one way or the other. Moreover, it is believed that the existence of just one additional conserved charge of higher-spin—which the counting establishes for both the bosonic and super $SU(\ell)$ PCMs—is sufficient to guarantee integrability and factorization.
of the $S$-matrix. Since this in turn implies infinitely many more conserved quantities, it would be somewhat surprising if, one charge being conserved, the others were not.

The survival of particular local charges which are even or odd under some discrete symmetry (which we can call generically ‘parity’) can have important implications for the spectrum. Indeed, the contrasts between the local charges in the bosonic and supersymmetric cases are reflected in the multiplets which are required for the construction of consistent $S$-matrices [6, 7]. In the bosonic case, odd-parity charges appear only in conjunction with complex representations of $g$, and it is only such multiplets which form parity doublets. In the supersymmetric case, the odd-parity family of currents (4.8) is always present, matching nicely the assumptions of [1], where particle multiplets also appear in parity doublets.

Finally, we return to the main theme of our introductory remarks. The quantum conservation of a full set of local charges with spins equal to the exponents would provide a natural explanation of the occurrence of Dorey’s rule in the fusions of the PCM $S$-matrices. This is immediate for the simply-laced cases, but there are additional subtleties for the non-simply-laced theories, as discussed in [1]. Their resolution is an interesting topic for future work.

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