ASYMPTOTIC EXPANSIONS OF EXPONENTIALS OF DIGAMMA FUNCTION AND IDENTITY FOR BERNOULLI POLYNOMIALS

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Abstract. The asymptotic expansion of digamma function is a starting point for the derivation of approximants for harmonic sums or Euler-Mascheroni constant. It is usual to derive such approximations as values of logarithmic function, which leads to the expansion of the exponentials of digamma function. In this paper the asymptotic expansion of the function \( \exp(p\psi(x + t)) \) is derived and analyzed in details, especially for integer values of parameter \( p \). The behavior for integer values of \( p \) is proved and as a consequence a new identity for Bernoulli polynomials. The obtained formulas are used to improve know inequalities for Euler’s constant and harmonic numbers.

1. Introduction

Let \( \gamma \) denote Euler’s constant, \( H_n \) the harmonic number — the partial sum of harmonic series:
\[
H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n},
\]
and \( \psi \) digamma (psi) function.

Harmonic numbers, logarithm, Euler’s constant and digamma function are connected through well known relations, the main one is
\[
\psi(n + 1) = H_n - \gamma. \tag{1.1}
\]
Various approximations of digamma function are used in this relation and interpreted as approximation for the sequence \( (H_n) \) or \( \gamma \). In the recent paper [6] the constants \( a, b, c, d \) in the expression
\[
w_n = H_n - \ln\left(n + a + \frac{b}{x} + \frac{c}{n^2} + \frac{d}{n^3}\right) \tag{1.2}
\]
are calculated such that this sequence is a good approximation to \( \gamma \). The authors obtained four term expansion
\[
e^{\psi(x + 1)} = x + \frac{1}{2} + \frac{1}{24} + \frac{1}{48}x^2 + \frac{23}{5760}x^3 + O(x^{-4})
\]
which can be written conventionally as
\[
\psi(x + 1) = \log\left(x + \frac{1}{2} + \frac{1}{24} + \frac{1}{48}x^2 + \frac{23}{5760}x^3 + O(x^{-4})\right). \tag{1.3}
\]
The calculation was tedious with no clue how one can obtain general term. In fact, a question about generalization of this procedure is posed.

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This question is answered in the paper [11], but the algorithm proposed there is also complicated. It relies on the connection between logarithmic function and Bell polynomials, but the algorithm for calculation of this polynomials is not easy.

In this paper we shall explain a simple and useful algorithm which covers more general case of approximations of this type. We will not restrict ourself to finding a numerical sequence. Instead, there is a natural choice of at least one parameter which can better explain obtained expansions and related inequalities. So, we shall consider the expansion of the function \( e^{p\psi(x+t)} \), where \( p \) and \( t \) are arbitrary real numbers. The case when \( p \) is a natural number is especially important and curious, and it will be explained in details.

Replacing argument \( x + 1 \) by \( x + t \) leads to the better understanding of the involving expansions. The magnificent formula

\[
\psi(x + t) \sim \log x + \sum_{n=0}^{\infty} (-1)^n \frac{B_n(t)}{n} x^{-n}
\]

is a good example of the role of introducing a parameter in such expansions.

The following lemma about functional transformations of asymptotic series has his origin in Euler’s work. See e.g. [10] for the explanation in the case of Taylor series, and [7] for its use in the case of asymptotic series.

**Lemma 1.1.** Let \( s \) be a real number, \( a_0 = 1 \) and \( g(x) \) a function with asymptotic expansion

\[
g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}.
\]

Then for all real \( p \) it holds

\[
g(x)^p \sim \sum_{n=0}^{\infty} b_n(p) x^{-n},
\]

where

\[
b_0(p) = 1,
\]

\[
b_n(p) = \frac{1}{n} \sum_{k=1}^{n} [k(1 + p) - n]a_k b_{n-k}(p).
\] (1.4)

2. The main asymptotic approximations

The following theorem is proved in [7]:

**Theorem 2.1.** The following asymptotic expansion is valid as \( x \to \infty \):

\[
\psi(x + t) \sim \log \left( \sum_{n=0}^{\infty} S_n(t) x^{-n+1} \right)
\] (2.1)

where \( S_0 = 1 \) and

\[
S_n(t) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} B_k(t) S_{n-k}(t), \quad n \geq 1.
\] (2.2)

The first few coefficients are:

\[
S_0 = 1,
\]

\[
S_1 = t - \frac{1}{2},
\]
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\[ S_2 = \frac{1}{24}, \]
\[ S_3 = -\frac{1}{24} t^2 + \frac{1}{18}, \]
\[ S_4 = \frac{1}{24} t^2 - \frac{1}{24} t + \frac{23}{5760}, \]
\[ S_5 = -\frac{1}{24} t^3 + \frac{1}{18} t^2 - \frac{23}{1220} t - \frac{17}{3840}, \]
\[ S_6 = \frac{1}{24} t^4 - \frac{1}{12} t^3 + \frac{23}{960} t^2 + \frac{17}{960} - \frac{10099}{2903040}. \]

The collapse of the degree in the polynomial \( S_2 \) is explained and various relations about polynomials \( (S_n) \) are proved. This theorem will be extended in our main theorem which covers more complex situation.

From this list of the coefficients one can write the following two expansions, for \( t = 0 \)
\[ \psi(x) \sim \log \left( x + \frac{1}{2} + \frac{1}{24} x - \frac{1}{48} x^2 + \frac{23}{5760} x^3 - \frac{17}{3840} x^4 - \frac{10099}{2903040} x^5 + \ldots \right). \] (2.3)

and for \( t = \frac{1}{2} \)
\[ \psi(x + \frac{1}{2}) \sim \log \left( x + \frac{1}{24} x - \frac{37}{5760} x^3 + \frac{10313}{2903040} x^5 - \frac{5509121}{1393459200} x^7 + \ldots \right) \] (2.4)

These expansions are used in the paper \([2]\) as approximants for harmonic sums, but the calculations of coefficients are made by other more tedious methods which cannot be improved to obtain general term of these sequences.

From (2.3) one can obtain the following expansions
\[ e^{2\psi(x)} \sim x^2 - x + \frac{1}{3} - \frac{1}{90} x^2 - \frac{1}{90} x^3 - \frac{1}{567} x^4 + \frac{43}{5670} x^5 + \ldots \]

and
\[ e^{4\psi(x)} \sim 2 x^3 - \frac{5 x^2}{3} - \frac{2 x}{3} + \frac{4}{455} + \frac{32}{2835} x^2 + \frac{32}{2835} x^3 + \ldots \]

The fact that in both expansions the term with \( x^{-1} \) is missing is not a coincidence. This is true for all even \( p \), but this is by no means obvious. We shall prove this in the next section.

Let us first derive asymptotic expansion of the function \( e^{p\psi(x+t)} \) where \( p \) is a real number. This expansion can be written in the form
\[ p\psi(x + t) \sim \log \left[ x^p \left( \sum_{n=0}^{\infty} S_n(t)x^{-n} \right)^p \right]. \]

Therefore, it is enough to apply algorithm given in Lemma \([11]\) to the calculated polynomials \( (S_n) \) to obtain the following:

**Theorem 2.2.** Let \( p \) be a real number. Function \( e^{p\psi(x+t)} \) has the following asymptotic expansion
\[ e^{p\psi(x+t)} \sim x^p \sum_{n=0}^{\infty} G_n(p,t)x^{-n} \] (2.5)

where \( (G_n) \) are defined by \( G_0 = 1 \) and
\[ G_n := \frac{1}{n} \sum_{k=1}^{n} [k(1 + p) - n] S_n(t) G_{n-k}. \] (2.6)
Let us denote in the sequel
\[ G(p, t, x) = x^p \sum_{n=0}^{\infty} G_n(p, t)x^{-n}. \] (2.7)

The polynomials \((G_n)\) are easy to calculate using any CAS as functions of both \(t\) and \(p\), but it is not easy to write it down. Polynomial \(S_n\) is of degree \(n - 2\) for \(n \geq 2\), but polynomials \(t \mapsto G_n(t, p)\) and \(p \mapsto G_n(t, p)\) will be, in general, of degree \(n\) — this is obvious from recurrent relation (2.6).

Here are the first few polynomials \((G_n)\):

\[
G_0 = 1, \\
G_1 = \frac{1}{2}p(1 - 2t), \\
G_2 = \frac{1}{24}p(-2 + 3p + 12t - 12pt - 12t^2 + 12pt^2), \\
G_3 = \frac{1}{8}(-2 + p)p(-1 + 2t)(p + 4t - 4pt - t^2 + 4pt^2), \\
G_4 = \frac{1}{3520}p(15p^3 - 60p^2 + 20p + 48) - \frac{1}{48}(p^2(p - 2)(p - 3)t \\
+ \frac{1}{48}(p - 2)(p - 3)(3p - 2)t^2 - \frac{1}{12}p(p - 2)(p - 2)(p - 3)t^3 \\
+ \frac{1}{24}p(p - 1)(p - 2)(p - 3)t^4.
\]

In order to highlight some interesting details, we will choose some concrete values of one of these parameters. For example, \(p = 2\) and \(p = 3\) gives the following sequence:

\[
(p = 2) \\
G_0 = 1, \\
G_1 = 2t - 1, \\
G_2 = t^2 - t + \frac{1}{3}, \\
G_3 = 0, \\
G_4 = -\frac{1}{90}, \\
G_5 = \frac{2t - 1}{90}, \\
G_6 = -\frac{t^2}{30} + \frac{t}{30} - \frac{1}{56t}.
\]

\[
(p = 3) \\
G_0 = 1, \\
G_1 = 3t - \frac{3}{2}, \\
G_2 = 3t^2 - 3t + \frac{7}{8}, \\
G_3 = \frac{1}{16}(2t - 1)(8t^2 - 8t + 2), \\
G_4 = -\frac{9}{640}, \\
G_5 = \frac{9(2t - 1)}{1280}, \\
G_6 = \frac{201t^2}{35840} + \frac{9t}{640} = \frac{9}{640}.
\]

One can see the collapse of the polynomial \(G_{p+1}\). The similar thing can be noticed for each positive integer \(p\). For example, if \(p = 4\) then \(G_5 = 0\).

First, we shall deduce another, more natural recursion for polynomials \((G_n)\). It is based on the following result which is proved in [3] and [4, Theorem 3.1.].

Let \(F(x, s, t)\) be Wallis function, defined by

\[
F(x, s, t) = \frac{\Gamma(x + t)}{\Gamma(x + s)}
\]
Then, it holds

\[ F(x, s, t) \sim x^{t-s} \left[ \sum_{n=0}^{\infty} P_n(t, s)x^{-n} \right] \]

where \( P_0 = 1 \) and \( (P_n) \) is defined by

\[ P_n(t, s) = \frac{m}{n} \sum_{k=1}^{n} \frac{(-1)^{k+1}B_{k+1}(t) - B_{k+1}(s)}{k+1} \]

Now, let us choose \( m = p/(t-s) \). Taking the limit \( s \to t \), we obtain:

\[ \lim_{s \to t} F(x, s, t)^{p/(t-s)} = e^{p\psi(x+t)} \]

From the other side, using property of Bernoulli polynomials,

\[ p \lim_{s \to t} \frac{B_{k+1}(t) - B_{k+1}(s)}{(k+1)(t-s)} = pB_{k+1}'(t) = pB_k(t), \]

we obtain the following result.

**Theorem 2.3.** For any real \( p \neq 0 \) function \( e^{p\psi(x+t)} \) has the asymptotic expansion

\[ G_n = \frac{p}{n} \sum_{k=1}^{n} (-1)^{k+1}B_k(t)G_{n-k}. \]  

**3. Integer values of \( p \)**

Now we can prove the main results.

**Theorem 3.1.** Polynomials \( (G_n) \) satisfy the following identity

\[ G_n(p, s + t) = \sum_{k=0}^{n} \binom{p-n+k}{k} G_{n-k}(p, s)t^k. \]  

1) If \( p \) is not positive integer, then \( G_n(p, t) \) is polynomial of degree \( n \) in both variables.

   Let us suppose that \( p \) is positive integer. Then

2) For \( n \leq p \), \( t \to G_n \) is polynomial of degree \( n \).

3) For \( n \geq p + 1 \), \( t \to G_n \) is polynomial of degree \( \leq n - p - 1 \).

**Proof.** Let \( p, t \) and \( s \) be arbitrary real numbers. The expansion for the function \( G(p, s + t, x) \) can be written in two different ways:

\[ G(p, s + t, x) = \sum_{n=0}^{\infty} G_n(p, s + t)x^{-n+p} \]

\[ = \sum_{n=0}^{\infty} G_n(p, s)(x + t)^{-n+p} \]

\[ = \sum_{n=0}^{\infty} G_n(p, s) \sum_{k=0}^{\infty} \binom{-n+p}{k} t^k x^{-n+p-k} \]

\[ = \sum_{n=0}^{\infty} \left[ \sum_{k=0}^{n} \binom{p-n+k}{k} G_{n-k}(p, s)t^k \right] x^{-n+p} \]
which proves (3.1).

The assertions 1) and 2) are obvious from this explicit formula. Let us prove 3). Suppose \( n \geq p + 1 \). Then, from (3.1) we can write

\[
G_n(p, t) = \sum_{k=0}^{n} (-1)^k \binom{n - p + 1}{k} G_{n-k}(p, 0) t^k.
\]

If \( n = p + 1 \), the sum is reduced to the first term:

\[
G_{p+1}(p, t) = G_{p+1}(p, 0),
\]

so, polynomial \( G_{p+1}(p, t) \) is of degree 0, or it is equal to zero identically. For \( n > p + 1 \),

\[
G_n(p, t) = \sum_{k=0}^{n-p-1} (-1)^k \binom{n - p + 1}{k} G_{n-k}(p, 0) t^k
\]

and this is polynomial of degree at most \( n - p - 1 \). This degree can be reduced if \( G_{p+1}(p, 0) = 0 \). \( \square \)

It will be shown that for each even \( p \), it holds \( G_{p+1}(0) = 0 \), so, the asymptotic expansion of \( e^{p\psi(x)} \) does not contain the member with power \( x^{-1} \). This requires another treatment.

**Theorem 3.2.** If \( p \) is an even natural number, then \( G_{p+1} \equiv 0 \) and \( t \mapsto G_{p+2+k} \) is polynomial of degree \( k \).

**Proof.** Let us treat \( p \) as a real-valued variable. Then the polynomial \( p \mapsto G_n(p, t) \) is of degree \( n \). Let us denote its coefficients by:

\[
G_n(p, t) = \sum_{k=0}^{n} G_{n,k}(p)t^n.
\]

Since for the function \( G(p, t, x) = e^{p\psi(x+t)} \) it holds \( \frac{\partial G}{\partial x} = \frac{\partial G}{\partial t} \), we have

\[
\sum_{n=1}^{\infty} \frac{\partial G_n}{\partial t} x^{-n+p} = \sum_{n=1}^{\infty} G_{n-1}(-n + p + 1)x^{-n+p},
\]

therefore

\[
\frac{\partial G_n(p, t)}{\partial t} = (p + 1 - n)G_{n-1}(p, t).
\]

Now, from the (3.2) it follows

\[
G_{n,k} = \frac{p + 1 - n}{k} G_{n-1,k-1}.
\]

Therefore,

\[
G_{n,k} = \binom{p - n + k}{k} G_{n-k,0}.
\]
Let us arrange coefficients in the following table:

\[
\begin{array}{cccc}
G_{0,0} & G_{1,1} & \quad & \\
G_{1,0} & G_{2,2} & \quad & \\
G_{2,0} & G_{3,3} & \quad & \\
G_{3,0} & & & \\
\vdots & & & \\
G_{n-1,0} & G_{n-1,1} & G_{n-1,2} & \cdots \quad G_{n-1,n-1} \\
G_{n,0} & G_{n,1} & \quad & G_{n,n-1} \\
G_n & 0 & & G_n,n
\end{array}
\]

If some of the coefficients \(G_{n,k}\) in this table are equal to zero for some particular value of \(p\), then the same is true for all coefficients \(G_{n+1,k+1}, G_{n+2,k+2}, \ldots\).

To finish proof of the theorem, we should prove that

\[G_{p+1,k} = 0, \quad \text{for all } k\]

if \(p\) is an even number.

First, note that for \(k \geq 1\) we read from (3.3) that \(G_{n,k}\) is divisible by \((p - n + 1) \cdots (p - n + k)\). Therefore, for each integer value of \(p\), all coefficients \(G_{p+1,k}\), \(k \geq 1\) are equal to zero. Then, of course, \(G_{p+1}(p, t) \equiv G_{p+1}(p, 0)\), but, also, the degree of all subsequent polynomials are reduced by \(p\).

To end this proof, we need the following

**Lemma 3.3.** It holds

\[G_n(p, 1) = (-1)^n G_n(p, 0).\]  

(3.4)

In other words,

\[
\sum_{k=0}^{n} G_{n,k}(p, 0) = (-1)^n G_{n,0}(p, 0)
\]

From here, if \(p\) is an even number

\[G_{p+1,0} = -\frac{1}{2} \sum_{k=1}^{n} G_{p+1,k}(p + 1, 0) = 0.
\]

and the theorem is proved.

Finally, let us prove previous lemma using induction. We have, using (2.8)

\[G_0(p, 1) = 1 = G_0(p, 0),\]

\[G_1(p, 1) = B_1(1)G_0(p, 1) = -B_1(0)G_0(p, 0) = -G_1(p, 0).\]

Let us suppose that

\[G_k(p, 1) = (-1)^k G_k(p, 0)\]

is satisfied for all \(k = 0, 1, \ldots , n - 1\). Then

\[G_n(p, 1) = \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} B_k(1) G_{n-k}(p, 1)\]

\[= \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} (-1)^k B_k(0)(-1)^{n-k} G_k(p, 0)\]

\[= (-1)^n \frac{1}{n} \sum_{k=1}^{n} (-1)^{k+1} B_k(0) G_{n-k}(p, 0)\]

\[= (-1)^n G_n(p, 0).\]
4. Fixed values of \( t \)

Interesting value is \( t = 1 \), wherefrom one obtains

\[
\begin{align*}
G_0 &= 1, \\
G_1 &= \frac{p}{2}, \\
G_2 &= \frac{1}{24}p(3p - 2), \\
G_3 &= \frac{1}{48}p^2(p - 2), \\
G_4 &= \frac{1}{5760}p(15p^3 - 60p^2 + 20p + 48), \\
G_5 &= \frac{1}{11520}p^2(p - 4)(3p^2 - 8p - 12).
\end{align*}
\]

For \( t = \frac{1}{2} \) this sequence is reduced to even members, i.e. \( G_{2n+1} = 0 \) and:

\[
\begin{align*}
G_0 &= 1, \\
G_2 &= \frac{p}{24}, \\
G_4 &= \frac{p(5p - 42)}{5760}, \\
G_6 &= \frac{p(35p^2 - 882p + 11160)}{2903040}.
\end{align*}
\]

Here \( p \mapsto G_{2n}(p, t) \) is a polynomial of degree only \( p \), which shows that the interplay between \( p \) and \( t \) is not so obvious.

**Theorem 4.1.** For \( t = \frac{1}{2} \), polynomials \( G_{2n+1} \) vanishes, \( G_{2n} \) has the degree \( n \) and can be calculated from

\[
G_{2n} = -\frac{p}{2n} \sum_{k=1}^{n} (1 - 2^{-2k})B_{2k}G_{2n-2k}.
\]  

**Proof.** We will use the expression (2.8). It is enough to use the property of Bernoulli polynomials

\[
B_k\left(\frac{1}{2}\right) = (1 - 2^{-2k})B_k.
\]

Hence, for all odd \( k \) we have \( B_k = 0 \), so, using (2.8)

\[
G_n = \frac{p}{n} \sum_{k=1}^{n} B_k\left(\frac{1}{2}\right)G_{n-k}
= \frac{p}{n} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^{2k+1}(1 - 2^{2k})B_{2k}G_{n-2k}
\]

Hence, for odd \( n \) there is no summands on the right, and for even \( n \) one obtains (4.1). From there, it is evident that \( G_{2n} \) is polynomial of degree \( n \). \qed
5. An identity for Bernoulli polynomials

Polynomials \((G_n)\) have explicit formula through Bernoulli polynomials, which is not friendly for its calculations. It is similar to the expression from [11]:

**Theorem 5.1.** The following explicit formula for coefficients \(G_n\) is valid:

\[
(-1)^n G_n(p, t) = \sum_{r=1}^{n} \frac{(-p)^r}{r!} \sum_{k_1 + \ldots + k_r = n} \frac{B_{k_1}(t) \cdots B_{k_r}(t)}{k_1 \cdots k_r}. \tag{5.1}
\]

**Proof.** We have

\[
G(p, t, x) = x^p \exp \left( -p \sum_{k=0}^{\infty} \frac{B_k(1-t)}{k} x^{-k} \right)
\]

\[
= x^p \left( 1 + \sum_{r=1}^{\infty} \frac{(-p)^r}{r!} \left[ \sum_{k=0}^{\infty} \frac{B_k(1-t)}{k} x^{-k} \right]^r \right)
\]

Hence

\[
G_n(p, t) = \sum_{r=1}^{n} \frac{(-p)^r}{r!} \sum_{k_1 + \ldots + k_r = n} \frac{B_{k_1}(1-t) \cdots B_{k_r}(1-t)}{k_1 \cdots k_r}
\]

and from here one obtains (5.1). \(\square\)

From the derivation of this expression it is obvious that indexes \(k_1, k_2, \ldots, k_r\) are not ordered, so, the combinations which lead to the same monomial should be counted carefully.

Taking into account the result from Theorem 3.2 that left side vanishes for \(n = p + 1\), where \(p\) is even, we can write the following identity for Bernoulli polynomials.

**Corollary 5.2.** For each natural number \(n\), we have:

\[
\sum_{r=1}^{2n+1} \frac{(-2n)^r}{r!} \sum_{k_1 + \ldots + k_r = 2n+1} \frac{B_{k_1}(t) \cdots B_{k_r}(t)}{k_1 \cdots k_r} = 0. \tag{5.2}
\]

For example, for \(n = 1\) and \(n = 2\) the following identities are fullfilled:

\[
\begin{align*}
\frac{2}{3} B_3(t) + 2B_1(t)B_2(t) - \frac{4}{3} B_1(t)^3 &= 0, \\
-\frac{4}{3} B_5(t) + 4B_1(t)B_4(t) + \frac{8}{3} B_2(t)B_3(t) - \frac{22}{3} B_1(t)^2 B_3(t) & \\
-8B_1(t)B_2(t)^2 + \frac{64}{3} B_1(t)^3 B_2(t) - \frac{128}{15} B_1(t)^5 &= 0.
\end{align*}
\]

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