RIGHT HOM-ALTERNATIVE ALGEBRAS

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Abstract. It is shown that every multiplicative right Hom-alternative algebra is both Hom-power associative and Hom-Jordan admissible. Multiplicative right Hom-alternative algebras admit Albert-type decompositions with respect to idempotents. Multiplication operators defined by idempotents in right Hom-alternative algebras are studied. Hom-versions of some well-known identities in right alternative algebras are proved.

1. Introduction

An algebra that satisfies
\[(xy)y = x(yy)\]
is called a right alternative algebra. If a right alternative algebra also satisfies the left alternative identity
\[(xx)y = x(xy),\]
then it is called an alternative algebra. For example, the 8-dimensional Cayley algebras are alternative algebras that are not associative \[1\]. Alternative algebras are closely related to other classes of non-associative algebras. In fact, alternative algebras are Jordan-admissible, Maltsev-admissible \[4, 10\], and power associative \[1, 2\]. Alternative algebras also satisfy the Moufang identities \[11\].

Generalizations of (left/right) alternative algebras, called (left/right) Hom-alternative algebras, were introduced by Makhlouf \[7\]. These (left/right) Hom-alternative algebras are defined by relaxing the defining identities in (left/right) alternative algebras by a linear self-map, called the twisting map. Construction results and examples of Hom-alternative algebras can be found in \[7, 16\]. Hom-type generalizations of classical algebras appeared in \[8\] in the form of Hom-Lie algebras, which were used to describe deformations of the Witt algebra and the Virasoro algebra. Hom-associative algebras were defined in \[8\] and further studied in \[9, 12, 13\]. For other Hom-type algebras, see \[8, 9, 14, 15, 16, 17\] and the references therein.

Many properties of alternative algebras have Hom-type generalizations. Indeed, the author proved in \[16\] that multiplicative Hom-alternative algebras are Hom-Jordan admissible and Hom-Maltsev admissible, and they satisfy Hom-versions of the Moufang identities. Moreover, in \[17\] the author showed that, as an immediate consequence of a much more general result, multiplicative right Hom-alternative algebras are Hom-power associative.

The purpose of this paper is to study right Hom-alternative algebras. In section 2 it is observed that the category of right Hom-alternative algebras is closed under twisting by self-weak morphisms (Theorem 2.6). A construction result in \[7\] of multiplicative right Hom-alternative algebras is then recovered as a special case (Corollary 2.8). In Example 2.9 an infinite family of distinct isomorphism
classes of multiplicative right Hom-alternative algebras are constructed. Each of these right Hom-alternative algebras is neither left Hom-alternative nor right alternative.

In section 3 it is shown with a short proof that every multiplicative right Hom-alternative algebra is Hom-power associative (Theorem 3.2). In section 4 it is shown that every multiplicative right Hom-alternative algebra is Hom-Jordan admissible (Theorem 4.3). Notice that in this result, left Hom-alternativity is not needed.

In section 5 the Hom-version of an idempotent is defined. A generalization of Albert’s decomposition is proved for multiplicative right Hom-alternative algebras (Proposition 5.5). In section 6 various multiplication operators on right Hom-alternative algebras induced by idempotents are studied.

In section 7 the Hom-versions of some well-known identities, including a Moufang identity, in right alternative algebras are proved.

2. Examples of right Hom-alternative algebras

The purposes of this section are to prove some construction results for right Hom-alternative algebras and to provide some examples of right Hom-alternative algebras that are neither left Hom-alternative nor right alternative.

2.1. Notations. Throughout the rest of this paper, we work over a fixed field \( k \) of characteristic 0. If \( V \) is a \( k \)-module and \( \mu: V \otimes^2 \to V \) is a bilinear map, then \( \mu^{op}: V \otimes^2 \to V \) denotes the opposite map, i.e., \( \mu^{op} = \mu \tau \), where \( \tau: V \otimes^2 \to V \otimes^2 \) interchanges the two variables. For \( x \) and \( y \) in \( V \), we sometimes write \( \mu(x, y) \) as \( xy \). For a linear self-map \( \alpha: V \to V \), denote by \( \alpha^n \) the \( n \)-fold composition of \( n \) copies of \( \alpha \), with \( \alpha^0 \equiv \text{Id} \).

We now provide some basic definitions regarding Hom-algebras.

**Definition 2.2.**

1. A Hom-module is a pair \( (A, \alpha) \) in which \( A \) is a \( k \)-module and \( \alpha: A \to A \) is a linear map, called the twisting map. A morphism \( f: (A, \alpha_A) \to (B, \alpha_B) \) of Hom-modules is a linear map of the underlying \( k \)-modules such that
   \[
   f\alpha_A = \alpha_B f.
   \]

2. A Hom-algebra is a triple \( (A, \mu, \alpha) \) in which \( (A, \alpha) \) is a Hom-module and \( \mu: A \otimes^2 \to A \) is a bilinear map, called the multiplication. A Hom-algebra \( (A, \mu, \alpha) \) and the corresponding Hom-module \( (A, \alpha) \) are often abbreviated to \( A \).

3. A Hom-algebra \( (A, \mu, \alpha) \) is multiplicative if \( \alpha \) is multiplicative with respect to \( \mu \), i.e.,
   \[
   \alpha\mu = \mu\alpha \otimes^2.
   \]

4. Let \( A \) and \( B \) be Hom-algebras. A weak morphism \( f: A \to B \) of Hom-algebras is a linear map \( f: A \to B \) such that
   \[
   f\mu_A = \mu_B f \otimes^2.
   \]
   A morphism \( f: A \to B \) is a weak morphism such that
   \[
   f\alpha_A = \alpha_B f.
   \]

From now on, an algebra \( (A, \mu) \) is also regarded as a Hom-algebra \( (A, \mu, \text{Id}) \) with identity twisting map.
Next we recall the Hom-type generalizations of (left/right) alternative and flexible algebras.

**Definition 2.3.** Let $(A, \mu, \alpha)$ be a Hom-algebra.

1. Define the **Hom-associator** as
   \[ a_A : A^{\otimes 3} \rightarrow A \] by
   \[ a_A = \mu(\mu \otimes \alpha - \alpha \otimes \mu). \] (2.3.1)

2. $A$ is called a **right Hom-alternative algebra** if
   \[ a_A(x, y, y) = 0 \] (2.3.2)
   for all $x, y \in A$. $A$ is called a **left Hom-alternative algebra** if
   \[ a_A(x, x, y) = 0 \]
   for all $x, y \in A$. $A$ is called a **Hom-alternative algebra** if it is both left Hom-alternative and right Hom-alternative.

3. $A$ is called a **Hom-flexible algebra** if
   \[ a_A(x, y, x) = 0 \]
   for all $x, y \in A$.

In terms of elements $x, y, z \in A$, we have
\[
   a_A(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz) \] (2.3.3)

When there is no danger of confusion, we will omit the subscript $A$ in the Hom-associator. When the twisting map $\alpha$ is the identity map, the above definitions reduce to the usual definitions of the associator, (left/right) alternative algebras, and flexible algebras.

It is well-known that alternative algebras are both Jordan-admissible and Maltsev-admissible \[4, 10\]. Moreover, alternative algebras satisfy the Moufang identities \[11\]. Generalizations of these results to multiplicative Hom-alternative algebras are proved in \[16\].

The following basic result gives the linearized form of the right Hom-alternative identity.

**Lemma 2.4** \[7\]. Let $(A, \mu, \alpha)$ be a Hom-algebra. Then the following statements are equivalent.

1. $A$ is right Hom-alternative.
2. $A$ satisfies
   \[ a_A(x, y, z) = -a_A(x, z, y) \] (2.4.1)
   for all $x, y, z \in A$.
3. $A$ satisfies
   \[ (xy)\alpha(z) + (xz)\alpha(y) = \alpha(x)(yz + zy) \] (2.4.2)
   for all $x, y, z \in A$.

**Proof.** The equivalence of the first two statements is part of \[7\] (Proposition 2.6). Indeed, starting from the right Hom-alternative identity \[2.3.2\], one replaces $y$ by $y + z$ to obtain \[2.4.1\]. Conversely, starting from \[2.4.1\], one sets $y = z$ to obtain the right Hom-alternative identity. The identity \[2.4.2\] is the expansion of \[2.4.1\] using \[2.3.3\]. \[\square\]

A Hom-alternative algebra is right Hom-alternative by definition. The following observation gives a necessary and sufficient condition under which the converse is true. This observation is a slight extension of \[7\] (Proposition 2.10) and is the Hom-version of \[2\] (Lemma 2).
Lemma 2.5. A right Hom-alternative algebra is Hom-alternative if and only if it is Hom-flexible.

Proof. Let \((A, \mu, \alpha)\) be a right Hom-alternative algebra. Setting \(y = x\) in (2.4.1), we have

\[ as(x, x, z) = -as(x, z, x) \]

for all \(x, z \in A\). Therefore, in a right Hom-alternative algebra, the left Hom-alternative identity is equivalent to the Hom-flexible identity.

Let us now observe that the category (with weak morphisms) of right Hom-alternative algebras is closed under twisting by self-weak morphisms.

Theorem 2.6. Let \((A, \mu, \alpha)\) be a right Hom-alternative algebra, and let \(\beta: A \to A\) be a weak morphism. Then the Hom-algebra

\[ A_\beta = (A, \mu_\beta = \beta \mu, \beta \alpha) \]  \hspace{1cm} (2.6.1)

is also a right Hom-alternative algebra. Moreover, if \(A\) is multiplicative and \(\beta\) is a morphism, then \(A_\beta\) is multiplicative.

Proof. In fact, given any Hom-algebra \(A\) and a weak morphism \(\beta: A \to A\), we have

\[ \beta^2 as_A = (\beta \mu)(\beta \mu \otimes \beta \alpha - \beta \alpha \otimes \beta \mu) \]

\[ = as_{A_\beta}. \]

This implies that, if \(A\) is right Hom-alternative, then so is \(A_\beta\). For the second assertion, assume that \(A\) is multiplicative and that \(\beta\) is a morphism. Then we have

\[ (\beta \alpha) \mu_\beta = \beta \alpha \beta \mu \]

\[ = \beta \mu \alpha \otimes \beta \otimes \beta \]

\[ = \mu_\beta (\alpha \beta) \otimes \beta \]

\[ = \mu_\beta (\beta \alpha) \otimes \beta. \]

This shows that \(A_\beta\) is multiplicative.

Two special cases of Theorem 2.6 follow. The following result says that each multiplicative right Hom-alternative algebra gives rise to a derived sequence of multiplicative right Hom-alternative algebras.

Corollary 2.7. Let \((A, \mu, \alpha)\) be a multiplicative right Hom-alternative algebra. Then

\[ A^n = (A, \mu^{(n)} = \alpha^n \mu, \alpha^{n+1}) \]

is a multiplicative right Hom-alternative algebra for each \(n \geq 0\).

Proof. The multiplicativity of \(A\) implies that \(\alpha^n: A \to A\) is a morphism. By Theorem 2.6, \(A_{\alpha^n} = A^n\) is a multiplicative right Hom-alternative algebra.

The following special case of Theorem 2.6 says that multiplicative right Hom-alternative algebras may arise from right alternative algebras and their morphisms. A twisting construction result of this form was first given by the author in [13] for G-Hom-associative algebras, which include Hom-associative and Hom-Lie algebras. The following adaptation to right Hom-alternative algebras appeared in [7] (Theorem 3.1).
Corollary 2.8. Let \((A, \mu)\) be a right alternative algebra and \(\beta : A \to A\) be an algebra morphism. Then
\[
A_\beta = (A, \mu_\beta = \beta \mu, \beta)
\]
is a multiplicative right Hom-alternative algebra.

Proof. This is the \(\alpha = \text{Id}\) case of Theorem 2.6. \(\square\)

Using Corollary 2.8 we now construct an infinite family of distinct isomorphism classes of multiplicative right Hom-alternative algebras that are neither left Hom-alternative nor right alternative.

Example 2.9. In \(\text{[3]}\) (p. 320-321) Albert constructed a five-dimensional right alternative algebra \((A, \mu)\) that is not left alternative. In terms of a basis \(\{e, u, v, w, z\}\) of \(A\), its multiplication \(\mu\) is given by
\[
e^2 = e, \quad eu = v, \quad ue = u, \quad ew = w - z, \quad ez = z = ze,
\]
where the unspecified products of the basis elements are all 0.

Let \(\gamma, \delta, \epsilon \in \mathbb{k}\) be arbitrary scalars with \(\delta \neq 0, 1\). Consider the linear map \(\alpha_{\gamma, \delta, \epsilon} : A \to A\) given by
\[
\alpha(e) = e + \epsilon u + \epsilon v, \quad \alpha(u) = \delta u, \quad \alpha(v) = \delta v, \quad \alpha(w) = \gamma w, \quad \alpha(z) = \gamma z.
\]
We claim that \(\alpha\) is an algebra morphism on \(A\). Indeed, suppose \(x = \lambda_1 e + \lambda_2 u + \lambda_3 v + \lambda_4 w + \lambda_5 z, \ y = \theta_1 e + \theta_2 u + \theta_3 v + \theta_4 w + \theta_5 z\) are two arbitrary elements in \(A\) with all \(\lambda_i, \theta_j \in \mathbb{k}\). Then
\[
xy = \lambda_1 \theta_1 e + \lambda_2 \theta_1 u + \lambda_1 \theta_2 v + \lambda_1 \theta_4 w + (\lambda_4 \theta_5 - \lambda_3 \theta_4) z,
\]
\[
\alpha(x) = \lambda_1 e + (\lambda_1 \epsilon + \lambda_2 \delta) u + (\lambda_1 \epsilon + \lambda_3 \delta) v + \lambda_1 \gamma w + \lambda_5 \gamma z,
\]
and similarly for \(\alpha(y)\). A quick computation then shows that
\[
\alpha(xy) = \lambda_1 \theta_1 e + \theta_1 (\lambda_1 \epsilon + \lambda_2 \delta) u + \lambda_1 (\theta_1 \epsilon + \theta_2 \delta) v
+ \lambda_1 \theta_4 \gamma w + \gamma (\lambda_1 (\theta_5 - \theta_4) + \lambda_5 \gamma) z
= \alpha(x) \alpha(y).
\]
By Corollary 2.8 there is a multiplicative right Hom-alternative algebra
\[
A_\alpha = (A, \mu_\alpha = \alpha \mu, \alpha).
\]
We now prove the following statements.

(1) \(A_\alpha\) in \(\text{(2.9.2)}\) is not left Hom-alternative.
(2) \((A, \mu_\alpha)\) is not right alternative.
(3) If \(\alpha' = \alpha_{\gamma', \delta', \epsilon'} : A \to A\) corresponds to the scalars \(\gamma', \delta', \epsilon'\) such that \(\delta \not\in \{\delta', \gamma'\}\), then \(A_\alpha\) and \(A_{\alpha'}\) are not isomorphic as Hom-modules (and hence as Hom-algebras).

To see that \(A_\alpha\) is not left Hom-alternative, observe that
\[
as_{A_\alpha}(e, e, u) = \alpha^2 as_A(e, e, u)
= \alpha^2((ee)u - e(eu))
= \alpha^2(v)
= \delta^2 v,
\]
which is not 0 because \( \delta \neq 0 \). To see that \((A, \mu_\alpha)\) is not right alternative, observe that
\[
\mu_\alpha(\mu_\alpha(u, e), e) - \mu_\alpha(u, \mu_\alpha(e, e)) = \alpha \mu(\delta u, e) - \alpha \mu(u, e + \epsilon u + \epsilon v) = (\delta^2 - \delta)u,
\]
which is not 0 because \( \delta \neq 0, 1 \).

To prove the last statement, assume to the contrary that there is a Hom-module isomorphism \( f : A_\alpha \to A_{\alpha'} \). In particular, we have
\[
f(u) = \lambda_1 e + \lambda_2 u + \lambda_3 v + \lambda_4 w + \lambda_5 z
\]
for some scalars \( \lambda_i \), not all of which are 0. So we have
\[
\alpha' f(u) = \lambda_1 (e + \epsilon' u + \epsilon' v) + \delta'(\lambda_2 u + \lambda_3 v) + \gamma'(\lambda_4 w + \lambda_5 z).
\]
On the other hand, we have
\[
f\alpha(u) = f(\delta u) = \delta(\lambda_1 e + \lambda_2 u + \lambda_3 v + \lambda_4 w + \lambda_5 z).
\]
Since \( \alpha' f = f\alpha \) and \( \delta \neq 1 \), by comparing the coefficients of \( e \) in (2.9.3) and (2.9.4), we infer that
\[
\lambda_1 = 0.
\]
Using this in (2.9.3) and (2.9.4) and the assumption \( \delta \notin \{\delta', \gamma'\} \), we further infer that
\[
\lambda_i = 0
\]
for \( 2 \leq i \leq 5 \). We conclude that
\[
f(u) = 0,
\]
contradicting the assumption that \( f \) is a linear isomorphism. Therefore, \( A_\alpha \) and \( A_{\alpha'} \) are not isomorphic as Hom-modules.

\[\square\]

3. Hom-power associativity

The purpose of this section is to show that every multiplicative right Hom-alternative algebra is Hom-power associative. A more general result was proved in [17]. The point of the following proof is that, at least for multiplicative right Hom-alternative algebras, Hom-power associativity can be established by a short and direct proof.

Let us first recall the relevant definitions from [17].

**Definition 3.1.** Let \((A, \mu, \alpha)\) be a Hom-algebra, \( x \in A \), and \( n \) be a positive integer.

1. Define the **\( n \)th Hom-power** \( x^n \in A \) inductively by
   \[
x^1 = x, \quad x^n = x^{n-1}\alpha^{n-2}(x) \tag{3.1.1}
   \]
   for \( n \geq 2 \).
2. For positive integers \( i \) and \( j \), define
   \[
x^{i,j} = \alpha^{j-1}(x^i)\alpha^{i-1}(x^j). \tag{3.1.2}
   \]
3. We say that \( A \) is **\( n \)th Hom-power associative** if
   \[
x^n = x^{n-i,i} \tag{3.1.3}
   \]
   for all \( x \in A \) and \( i \in \{1, \ldots, n-1\} \).
4. We say that \( A \) is **Hom-power associative** if \( A \) is \( n \)th Hom-power associative for all \( n \geq 2 \).
Note that by definition

\[ x^n = x^{n-1,1} \]

for all \( n \geq 2 \).

If the twisting map \( \alpha \) is the identity map, then

\[ x^n = x^{n-1}x, \quad x^{i,j} = x^ix^j, \]

and \( n \)th Hom-power associativity reduces to

\[ x^n = x^{n-i}x^i \]

(3.1.4)

for all \( x \in A \) and \( i \in \{1, \ldots, n-1\} \). Therefore, Hom-powers and \((n)\) Hom-power associativity become Albert’s right powers and \((n)\) power associativity \cite{1, 2} if \( \alpha = Id \). Examples of and construction results for Hom-power associative algebras can be found in \cite{17}.

A well-known result of Albert \cite{1} says that (in characteristic 0, which is assumed throughout this paper) an algebra \((A, \mu)\) is power associative if and only if it is third and fourth power associative, i.e., the condition (3.1.4) holds for \( n = 3 \) and 4. Moreover, for (3.1.4) to hold for \( n = 3 \) and 4, it is necessary and sufficient that

\[ (xx)x = x(xx) \quad \text{and} \quad ((xx)x)x = (xx)(xx) \]

for all \( x \in A \). This result of Albert is remarkable because it shows that power associativity, which has infinitely many defining identities (namely, (3.1.4) for all \( n \)), is implied by just two identities. The Hom-versions of these statements are also true. More precisely, the author proved in \cite{17} that a multiplicative Hom-algebra \((A, \mu, \alpha)\) is Hom-power associative if and only if it is third and fourth Hom-power associative, which in turn is equivalent to

\[ x^2 \alpha(x) = \alpha(x)x^2 \quad \text{and} \quad x^4 = \alpha(x^2)\alpha(x^2) \]

(3.1.5)

for all \( x \in A \).

It is easy to verify that the identities in (3.1.3) hold in every multiplicative right Hom-alternative algebra (\cite{17} Proposition 3.4). It follows that multiplicative right Hom-alternative algebras are Hom-power associative. We now give a more direct proof of this result, which is modeled after \cite{2} (Lemma 1), without relying on the results from \cite{17}.

**Theorem 3.2.** Every multiplicative right Hom-alternative algebra is Hom-power associative.

**Proof.** Let \((A, \mu, \alpha)\) be a multiplicative right Hom-alternative algebra. We prove the \( n \)th Hom-power associativity (3.1.3) of \( A \) by induction on \( n \geq 2 \). Pick an element \( x \) in \( A \). The case \( n = 2 \) of (3.1.3) is trivially true, since

\[ x^2 = xx = x^{1,1} \]

by definition.

Inductively suppose we already proved that \( A \) is \( k \)th Hom-power associative for all \( k \) in the range \( \{2, \ldots, n-1\} \) for some \( n \geq 3 \). We establish the \( n \)th Hom-power associativity (3.1.3) of \( A \) by induction on \( i \). Suppose \( i \in \{1, \ldots, n-2\} \). In (2.4.2) replace \((x, y, z)\) by

\[ \left( \alpha^{i-1}(x^{n-(i+1)}), \alpha^{n-i-2}(x^i), \alpha^{n-3}(x) \right) \quad (3.2.1) \]
Using the induction hypothesis, the left-hand side of (2.4.2) then becomes:
\[ x^{n-(i+1)i} \alpha^{-2}(x) + \left( \alpha^{i-1}(x^{n-(i+1)}) \alpha^{n-3}(x) \right) \alpha^{n-i-1}(x^i) \]
\[ = x^{n-1} \alpha^{-2}(x) + \alpha^{i-1} \left( x^{n-(i+1)} \alpha^{n-i-2}(x) \right) x^{n-i-1}(x^i) \]
\[ = x^{n} + \alpha^{i-1}(x^{n-i}) \alpha^{n-i-1}(x^i) \]
\[ = x^{n} + x^{n-i,i}. \] (3.2.2)

Likewise, by the induction hypothesis, the right-hand side of (2.4.2) becomes:
\[ \alpha(x^{n-(i+1)i}) \alpha^{n-i-2} \left( x^{i} \alpha^{i-1}(x) + \alpha^{i-1}(x)x^i \right) \]
\[ = \alpha^{i}(x^{n-(i+1)}) \alpha^{n-i-2} (x^{i+1} + x^{i+1}) \]
\[ = 2 \alpha^{i}(x^{n-(i+1)}) \alpha^{n-i-2}(x^{i+1}) \]
\[ = 2x^{n-(i+1),i+1}. \] (3.2.3)

Therefore, the special case of (2.4.2) with \((x, y, z)\) as in (3.2.1) says
\[ 2x^{n-(i+1),i+1} = x^{n} + x^{n-i,i} \] (3.2.4)
for \(i \in \{1, \ldots, n-2\}\). It follows from the definition \(x^{n} = x^{n-1,1}\) and (3.2.4) that
\[ x^{n} = x^{n-i,i} \]
for \(i \in \{1, \ldots, n-1\}\), so \(A\) is \(n\)th Hom-power associative. \(\square\)

Setting \(\alpha = \text{Id}\) in Theorem 3.2, we recover (3) (Lemma 1).

**Corollary 3.3.** Every right alternative algebra is power associative.

4. Hom-Jordan admissibility

The purpose of this section is to show that multiplicative right Hom-alternative algebras are Hom-Jordan admissible. It is well-known that alternative algebras are Jordan admissible. The Hom-version of this statement is also true. More precisely, the author proved in [16] that every multiplicative Hom-alternative algebra is Hom-Jordan admissible. Here we strengthen this result by removing the left Hom-alternativity assumption, while at the same time simplifying the argument.

Let us first recall some relevant definitions from [16].

**Definition 4.1.** Let \((A, \mu, \alpha)\) be a Hom-algebra.

(1) Define the **plus Hom-algebra**
\[ A^+ = (A, *, \alpha), \]
where \(* = (\mu + \mu^{\text{op}})/2\).

(2) \(A\) is called a **Hom-Jordan algebra** if \(\mu = \mu^{\text{op}}\) (commutativity) and the **Hom-Jordan identity**
\[ a s_A(x^2, \alpha(y), \alpha(x)) = 0 \]
is satisfied for all \(x, y \in A\), where \(s_A\) is the Hom-associator (2.3.1).

(3) \(A\) is called a **Hom-Jordan admissible algebra** if its plus Hom-algebra \(A^+\) is a Hom-Jordan algebra.
If $\alpha = Id$ then the above definitions reduce to the usual notions of plus algebras and Jordan (admissible) algebras.

The reader is cautioned that the above definition of a Hom-Jordan algebra is not the same as the one in [7]. A Hom-Jordan algebra in the sense of [7] is also a Hom-Jordan algebra in the sense of Definition 4.1, but the converse is not true.

Note that the Jordan product

$$x \ast y = \frac{1}{2}(\mu(x, y) + \mu(y, x)) = \frac{1}{2}(xy + yx) = y \ast x$$

is commutative and that

$$x \ast x = \mu(x, x) = x^2$$

for all $x \in A$.

**Lemma 4.2.** Let $(A, \mu, \alpha)$ be a Hom-algebra. Then the following statements are equivalent.

1. $A$ is Hom-Jordan admissible.
2. The condition

$$as_{A^+}(x^2, \alpha(y), \alpha(x)) = 0$$

holds for all $x, y \in A$.
3. The condition

$$(\alpha(x) \ast \alpha(y)) \ast \alpha(x^2) - \alpha^2(x) \ast (\alpha(y) \ast x^2) = 0$$

holds for all $x, y \in A$.

**Proof.** The equivalence of the first two statements follows from the commutativity of $\ast$ and the identity $x \ast x = x^2$. The equivalence of (4.2.1) and (4.2.2) follows by expanding the Hom-associator $as_{A^+}$ in (4.2.1) and using the commutativity of $\ast$. □

We are now ready for the main result of this section, which is the Hom-version of Theorem 2 in [3].

**Theorem 4.3.** Every multiplicative right Hom-alternative algebra is Hom-Jordan admissible

**Proof.** Let $(A, \mu, \alpha)$ be a multiplicative right Hom-alternative algebra. By Lemma 4.2 it suffices to prove (4.2.3). Pick elements $x, y \in A$. First note that the right Hom-alternative identity (2.3.2) implies

$$0 = as(x, x, x) = x^2 \alpha(x) - \alpha(x)x^2.$$  (4.3.1)

Also, the special cases of (2.4.2) with $(x, y, z)$ replaced by $(x^2, \alpha(x), \alpha(y))$ and $(\alpha(x), \alpha(y), x^2)$ are

$$\alpha(x^2) (\alpha(x)\alpha(y) + \alpha(y)\alpha(x)) = (x^2\alpha(x)) \alpha^2(y) + (x^2\alpha(y)) \alpha^2(x)$$

and

$$\alpha^2(x) (\alpha(y)x^2 + x^2\alpha(y)) = (\alpha(x)\alpha(y)) \alpha(x^2) + (\alpha(x)x^2) \alpha^2(y),$$

respectively.
Using (4.3.1), (4.3.2), and (4.3.3), we now compute the left-hand side of (4.2.2) multiplied by 4:

\[
4 \left( (\alpha(x) * \alpha(y)) * \alpha(x^2) - 4\alpha^2(x) * (\alpha(y) * x^2) \right)
\]

\[
= (\alpha(x)\alpha(y) + \alpha(y)\alpha(x)) \alpha(x^2) + \alpha(x^2) (\alpha(x)\alpha(y) + \alpha(y)\alpha(x))
\]

\[- \alpha^2(x) (\alpha(y)x^2 + x^2\alpha(y)) - (\alpha(y)x^2 + x^2\alpha(y)) \alpha^2(x)
\]

\[
= (\alpha(x)\alpha(y)) \alpha(x^2) + (\alpha(y)\alpha(x)) \alpha(x^2) + (x^2\alpha(x)) \alpha^2(y) + (x^2\alpha(y)) \alpha^2(x)
\]

\[- (\alpha(x)\alpha(y)) \alpha(x^2) - (\alpha(y)x^2) \alpha^2(y) - (\alpha(y)x^2) \alpha^2(x) - (x^2\alpha(y)) \alpha^2(x)
\]

\[
= \alpha(yx) (\alpha(x)\alpha(x)) - ((yx)\alpha(x))\alpha^2(x)
\]

\[
= -as_A(yx, \alpha(x), \alpha(x)) = 0.
\]

This shows that (4.2.2) is satisfied, so \(A\) is Hom-Jordan admissible. \(\square\)

Setting \(\alpha = Id\) in Theorem 4.3, we recover \(\diamond\) (Theorem 2).

Corollary 4.4. Every right alternative algebra is Jordan admissible.

5. Albert-type decompositions

The purpose of this section is to study Albert-type decompositions for right Hom-alternative algebras with respect to idempotents.

Let us first define the Hom-version of an idempotent.

Definition 5.1. Let \((A, \mu, \alpha)\) be a Hom-algebra. An idempotent in \(A\) is an element \(e \in A\) that satisfies

\[
e^2 = e = \alpha(e),
\]

where \(e^2 = \mu(e, e)\).

In the above definition, if \(\alpha = Id\) then \(e \in A\) is an idempotent if and only if \(e^2 = e\). This, of course, is the usual definition of an idempotent in an algebra.

We first observe that the property of being an idempotent is preserved under the constructions in section 2. The following result is an immediate consequence of Definition 5.1.

Proposition 5.2. Let \((A, \mu, \alpha)\) be a Hom-algebra and \(e \in A\) be an idempotent.

(1) If \(\beta: A \rightarrow A\) is a linear map such that \(\beta(e) = e\), then \(e\) is an idempotent in \(A_{\beta} = (A, \mu_{\beta} = \beta\mu, \beta\alpha)\).

(2) The element \(e\) is an idempotent in \(A^n = (A, \mu^{(n)} = \alpha^n\mu, \alpha^{n+1})\)

for each \(n \geq 0\).

The following observation is the \(\alpha = Id\) special case of the first part of Proposition 5.2.
Corollary 5.3. Let \((A, \mu)\) be an algebra, \(e \in A\) be an idempotent, and \(\beta: A \to A\) be a linear map such that \(\beta(e) = e\). Then \(e\) is an idempotent in \(A_\beta = (A, \mu_\beta = \beta \mu, \beta)\).

Example 5.4. In the five-dimensional right alternative algebra \(A\) in Example 2.9, the basis element \(e\) is an idempotent. The map \(\alpha_{\gamma, \delta, \epsilon} = \alpha: A \to A\) in (2.9.1) fixes \(e\) if and only if \(\epsilon = 0\). Therefore, by Corollary 5.3 the element \(e\) is an idempotent in the multiplicative right Hom-alternative algebra \(A_\alpha\) in (2.9.2) if and only if \(\epsilon = 0\). □

Let \(e\) be an idempotent in a right alternative algebra \(A\). Then there is an **Albert decomposition**

\[
A = A_e(1) \oplus A_e(0),
\]

where

\[
A_e(i) = \{a \in A: ae = ia\}
\]

for \(i = 0, 1\). More precisely, every \(a \in A\) can be decomposed uniquely as

\[
a = ae + (a - ae)
\]

with \(ae \in A_e(1)\) and \((a - ae) \in A_e(0)\).

The following result is a Hom-type generalization of the Albert decomposition (5.4.1) for right Hom-alternative algebras.

**Proposition 5.5.** Let \((A, \mu, \alpha)\) be a multiplicative right Hom-alternative algebra in which \(\alpha\) is surjective and \(e \in A\) be an idempotent. Then there is a not-necessarily-direct sum

\[
A = A_e(\alpha) + A_e(0)
\]

of sub-Hom-modules, where

\[
A_e(i\alpha) = \{a \in A: ae = i\alpha(a)\}
\]

for \(i \in \{0, 1\}\). The sum (5.5.1) is a direct sum if \(\alpha\) is also injective.

**Proof.** First observe that, if \(\alpha\) is injective, then the intersection of \(A_e(\alpha)\) and \(A_e(0)\) is 0. Indeed, if \(x \in A\) belongs to both submodules, then

\[
\alpha(x) = xe = 0.
\]

The injectivity of \(\alpha\) implies that \(x = 0\).

To see that the submodule \(A_e(i\alpha)\) is a sub-Hom-module for \(i \in \{0, 1\}\), suppose \(a \in A_e(i\alpha)\). Then

\[
\alpha(a)e = \alpha(a)\alpha(e) = \alpha(ae) = \alpha(i\alpha(a)) = i\alpha(\alpha(a)).
\]

So \(\alpha(a)\) is also in \(A_e(i\alpha)\).

It remains to show that the sum \(A_e(\alpha) + A_e(0)\) is all of \(A\). Pick an arbitrary element \(b \in A\). The surjectivity of \(\alpha\) implies that \(b = a\alpha(a)\) for some \(a \in A\). Consider the decomposition

\[
b = ae + (b - ae).
\]
We claim that $ae \in A_e(\alpha)$ and that $(b - ae) \in A_e(0)$. Indeed, we have

$$(ae)e = (ae)\alpha(e)$$
$$= \alpha(a)(ee)$$
$$= \alpha(a)\alpha(e)$$
$$= \alpha(a).$$

This shows that $ae \in A_e(\alpha)$. Likewise, we have

$$(b - ae)e = be - (ae)\alpha(e)$$
$$= \alpha(a)e - \alpha(a)(ee)$$
$$= 0,$$

which shows that $(b - ae) \in A_e(0)$. \hfill \Box

In particular, if $\alpha = Id$ in Proposition 5.5, then $A_e(Id) = A_e(1)$ and the decomposition $A_e(Id) \oplus A_e(0)$ becomes the Albert decomposition (5.4.1).

6. Multiplication operators induced by idempotents

In this section, we study multiplication operators in right Hom-alternative algebras, especially those induced by idempotents. Let us first fix some notations.

**Definition 6.1.** Let $(A, \mu, \alpha)$ be a Hom-algebra and $x$ be an element in $A$. Define $L_x, R_x : A \to A$ to be the operators of left and right multiplication by $x$, acting from the right, i.e.,

$$aL_x = xa$$
$$aR_x = ax$$

for all $a \in A$.

The convention of $L_x$ and $R_x$ acting from the right is often used in the literature, e.g., [2, 3]. We also allow the twisting map $\alpha : A \to A$ to act from the right, i.e.,

$$a\alpha = \alpha(a)$$

for $a \in A$. The operators $\alpha, L_x, R_x$ are all in Hom($A, A$), the module of linear operators on $A$ acting from the right. Since composition of linear operators is associative, Hom($A, A$) is a Lie algebra under the commutator bracket,

$$[f, g] = fg - gf$$

for $f, g \in$ Hom($A, A$).

**Lemma 6.2.** Let $(A, \mu, \alpha)$ be a right Hom-alternative algebra. Then

$$R_x R_{\alpha(x)} = \alpha R_x$$
$$L_y L_{\alpha(y)} - \alpha L_{xy} = L_x R_{\alpha(y)} - R_y L_{\alpha(x)}$$

for all $x, y \in A$.

**Proof.** The desired identities (6.2.1) and (6.2.2) are the operator forms of the right Hom-alternative identity (2.3.2) and the linearized right Hom-alternative identity (2.4.2), respectively. \hfill \Box
We now restrict our attention to the multiplication operators induced by idempotents.

**Proposition 6.3.** Let \((A, \mu, \alpha)\) be a right Hom-alternative algebra and \(e \in A\) be an idempotent. Then
\[
R_e^{n+1} = \alpha^n R_e
\] (6.3.1)
for all \(n \geq 0\), and
\[
L_e^2 - \alpha L_e = [L_e, R_e].
\] (6.3.2)
If, in addition, \(A\) is multiplicative, then
\[
[\alpha, L_e] = 0 = [\alpha, R_e]
\] (6.3.3)

**Proof.** Write \(R = R_e\). The condition (6.3.1) is clearly true for \(n = 0\). Inductively, suppose (6.3.1) is true for a particular \(n\). Then we have
\[
R^{n+2} = R^{n+1} R
= \alpha^n R R \quad \text{(by induction hypothesis)}
= \alpha^n \alpha R,
\]
where the last equality follows from (6.2.1) because \(\alpha(e) = e = e^2\). Therefore, (6.3.1) is true for all \(n\).

The equality (6.3.2) is the special case of (6.2.2) with \(x = y = e\).

For the last assertion, the multiplicativity of \(A\) implies
\[
L_x \alpha = \alpha L_x(x) \quad \text{and} \quad R_x \alpha = \alpha R_x(x)
\]
for all \(x \in A\). Restricting to the special case \(x = e\), we obtain (6.3.3). \(\square\)

A linear self-map \(f: V \to V\) on a module \(V\) is said to be an **idempotent** if \(f^2 = f\). We now introduce the Hom-version of this concept.

**Definition 6.4.** Let \((A, \mu, \alpha)\) be a Hom-algebra, \(f: A \to A\) be a linear map acting from the right, and \(n\) be an integer. We say that \(f\) is an \(\alpha^n\)-idempotent if
\[
f^2 = \alpha^n f
\]
in Hom\((A, A)\).

In particular, (6.3.1) with \(n = 1\) says
\[
R_e^2 = \alpha R_e.
\] (6.4.1)
In other words, in a right Hom-alternative algebra, the right multiplication operator \(R_e\) is an \(\alpha\)-idempotent. On the other hand, (6.3.2) tells us that the left multiplication operator \(L_e\) is not an \(\alpha\)-idempotent. The following result, which is the Hom-version of [3] (Lemma 4), says that \(L_e\) is not too far from being an \(\alpha\)-idempotent.

**Theorem 6.5.** Let \((A, \mu, \alpha)\) be a multiplicative right Hom-alternative algebra and \(e \in A\) be an idempotent. Then
\[
(L_e^2 - \alpha L_e)^2 = 0 = ([L_e, R_e])^2.
\]
Proof. By (6.3.2) it suffices to prove

$$(L^2_e - \alpha L_e)^2 = 0.$$  \hspace{1em} (6.5.1)

Let us write $L = L_e$ and $R = R_e$. First note that

$$(L^2 - \alpha L)^2 = L^4 - 2\alpha L^3 + \alpha^2 L^2 \quad (\text{by } (6.3.3))$$

$$= (L^3 - \alpha L^2)L - \alpha(L^3 - \alpha L^2). \quad (\text{6.5.1})$$

To show that $(L^2 - \alpha L)^2 = 0$, we next compute $(L^3 - \alpha L^2)$ in two ways. On the one hand, we have

$$L^3 - \alpha L^2 = L(L^2 - \alpha L) \quad (\text{by } (6.3.3))$$

$$= L^2 R - LRL \quad (\text{by } (6.3.2))$$

$$= (\alpha L + LR - RL)R - LRL \quad (\text{by } (6.3.2)) \quad (6.5.2)$$

On the other hand, we have

$$L^3 - \alpha L^2 = (L^2 - \alpha L)L$$

$$= LRL - RL^2 \quad (\text{by } (6.3.2))$$

$$= LRL - R(\alpha L + LR - RL) \quad (\text{by } (6.3.2)) \quad (6.5.3)$$

Comparing (6.5.2) and (6.5.3), we obtain

$$LRL = \alpha LR. \quad (6.5.4)$$

Using (6.5.4) in (6.5.3), we obtain

$$L^3 - \alpha L^2 = \alpha LR - RLR. \quad (6.5.5)$$

Finally, we have

$$(L^3 - \alpha L^2)L = \alpha LRL - RLRL \quad (\text{by } (6.5.5))$$

$$= \alpha^2 LR - \alpha RLR \quad (\text{by } (6.5.4) \text{ and } (6.3.3)) \quad (6.5.6)$$

Comparing (6.5.1) and (6.5.6), we conclude that $(L^2 - \alpha L)^2 = 0$. \hfill \Box

The next result shows that, although $L_e$ is not $\alpha$-idempotent, there is an operator of degree 3 in $L_e$ that is $\alpha^4$-idempotent. This is the Hom-version of an observation in [3] (section 6).

**Corollary 6.6.** Let $(A, \mu, \alpha)$ be a multiplicative right Hom-alternative algebra, $e \in A$ be an idempotent, and $L = L_e$. Then the linear operator

$$T = 3\alpha^2 L^2 - 2\alpha L^3 \quad (6.6.1)$$

satisfies

$$T^{n+1} = \alpha^{4n} T \quad (6.6.2)$$

for all $n \geq 0$. In particular, $T$ is $\alpha^4$-idempotent, i.e.,

$$T^2 = \alpha^4 T.$$
Proof. The condition (6.6.2) is trivially true when \( n = 0 \). When \( n = 1 \), we compute using (6.3.3) as follows:

\[
T^2 = 4\alpha^2 L^6 - 12\alpha^3 L^5 + 9\alpha^4 L^4
= (4\alpha^2 L^2 - 4\alpha^3 L - 3\alpha^4)(L^4 - 2\alpha L^3 + \alpha^2 L^2) + \alpha^4 T
= (4\alpha^2 L^2 - 4\alpha^3 L - 3\alpha^4)(L^2 - \alpha L)^2 + \alpha^4 T
= \alpha^4 T.
\]

The last equality holds because \((L^2 - \alpha L)^2 = 0\) by Theorem 6.5. Inductively, suppose (6.6.2) is true for a particular \( n \). Then we have:

\[
T^{n+2} = T^{n+1} T
= \alpha^{4n} \alpha^4 T \quad \text{(by induction hypothesis)}
= \alpha^{4(n+1)} T.
\]

This finishes the induction and proves (6.6.2) for all \( n \). \( \square \)

In the context of Lemma 6.3, the operators \( L_e \) and \( R_e \) do not commute, as can be seen from (6.3.2). Our next result says that the \( \alpha^4 \)-idempotent operator \( T \) (6.6.1) does commute with \( R_e \). This result is the Hom-version of [3] (Lemma 7).

**Corollary 6.7.** Let \((A, \mu, \alpha)\) be a multiplicative right Hom-alternative algebra, \( e \in A \) be an idempotent, \( L = L_e \), and \( R = R_e \). Then

\[
[T, R] = 0,
\]

where \( T = 3\alpha^2 L^2 - 2\alpha L^3 \) is the operator in Corollary 6.6.

**Proof.** First we claim that

\[
k(L^{k+1} - \alpha L^k) = [L^k, R] \quad (6.7.1)
\]

for all \( k \geq 1 \). The case \( k = 1 \) is true by (6.3.2). Inductively, suppose (6.7.1) is true for a particular \( k \). Then we have:

\[
k(L^{k+2} - \alpha L^{k+1}) = kL(L^{k+1} - \alpha L^k) \quad \text{(by (6.3.3))}
= L[L^k, R] \quad \text{(by induction hypothesis)}
= L^{k+1} R - LRL^k
= L^{k+1} R - (L^2 - \alpha L + RL)L^k \quad \text{(by (6.3.2))}
= [L^{k+1}, R] - (L^{k+2} - \alpha L^{k+1}).
\]

This finishes the induction step and proves (6.7.1) for all \( k \).

Using (6.3.3) and (6.7.1), we compute as follows:

\[
[T, R] = (3\alpha^2 L^2 - 2\alpha L^3) R - R(3\alpha^2 L^2 - 2\alpha L^3)
= 3\alpha^2 [L^2, R] - 2\alpha [L^3, R]
= 3\alpha^2 (2)(L^3 - \alpha L^2) - 2\alpha (3)(L^4 - \alpha L^3)
= -6\alpha (L^2 - \alpha L)^2.
\]

Since \((L^2 - \alpha L)^2 = 0\) by Theorem 6.3, we conclude that \([T, R] = 0\). \( \square \)
7. Identities in right Hom-alternative algebras

The purpose of this section is to prove the following Hom-type generalizations of some well-known and frequently-used identities in right alternative algebras from [6].

**Theorem 7.1.** Let \((A, \mu, \alpha)\) be a multiplicative right Hom-alternative algebra. Then the following identities hold for all \(w, x, y, z \in A\).

\[
\begin{align*}
\text{(7.1.1a)} & \quad \alpha(x, \alpha(y), yz) = \alpha(x, y, z)\alpha^2(y). \\
\text{(7.1.1b)} & \quad \alpha(x, \alpha(w), yz) + \alpha(x, \alpha(y), w) = \alpha(x, w, z)\alpha^2(y) + \alpha(x, y, z)\alpha^2(w). \\
\text{(7.1.1c)} & \quad \alpha(wx, \alpha(y), \alpha(z)) + \alpha(\alpha(w), \alpha(x), [y, z]) = \alpha^2(w)\alpha(x, y, z) + \alpha(w, y, z)\alpha^2(x). \\
\text{(7.1.1d)} & \quad \alpha(\alpha(x), y^2, \alpha(z)) = \alpha(x, \alpha(y), yz + yz). \\
\text{(7.1.1e)} & \quad (\alpha(x)\alpha(y))\alpha^2(y) = \alpha^2(x)(yz)\alpha(y). \\
\text{(7.1.1f)} & \quad \alpha(x, y, z)\alpha^2(y)\alpha^3(z) = \alpha(\alpha(x, y, z)\alpha(yz)).
\end{align*}
\]

Setting \(\alpha = \text{Id}\) in Theorem 7.1, we obtain the corresponding identities in right alternative algebras, all of which can be found in [6]. In (7.1.1e) the bracket \([,]\) is the commutator bracket of \(\mu\), i.e., \([,]\) = \(\mu - \mu^{op}\). The identity (7.1.1d) is one of the Hom-Moufang identities in [16], in which (7.1.1d) was established with the additional assumption of left Hom-associativity.

To simplify the presentation, we adopt the following notations.

**Definition 7.2.** Let \((A, \mu, \alpha)\) be a Hom-algebra, and let \(w, x, y, z \in A\). Define

\[
\begin{align*}
(x, y, z) &= \alpha_A(x, y, z), \quad x' = \alpha(x), \quad x'' = \alpha^2(x), \quad x''' = \alpha^3(x).
\end{align*}
\]

Define the function \(f: A^\otimes 4 \to A\) by

\[
f(w, x, y, z) = (wx, y', z') - (w', xy, z') + (w', x', yz) - w''(x, y, z) - (w, x, y)z''
\]

for \(w, x, y, z \in A\).

With these notations, if \(A\) is multiplicative, then

\[
(xy)' = x'y', \quad (x, y, z)' = (x', y', z')
\]

The right Hom-alternative identity (2.3.2) and its linearized form (2.4.1) now say

\[
(x, y, y) = 0 \quad \text{and} \quad (x, y, z) = -(x, z, y),
\]

respectively.

We need the following preliminary result.

**Lemma 7.3.** Let \((A, \mu, \alpha)\) be a multiplicative Hom-algebra. Then

\[
f(w, x, y, z) = 0
\]

for all \(w, x, y, z \in A\).

**Proof.** Simply expand the five Hom-associators in \(f\) (7.2.1), and observe that the resulting ten terms add to 0. \(\square\)
The $\alpha = Id$ special case of (7.3.1) is known as the Teichmüller identity, which holds in any algebra. We refer to (7.3.1) as the Hom-Teichmüller identity.

In the rest of this section, $(A, \mu, \alpha)$ denotes a multiplicative right Hom-alternative algebra, and $w, x, y, z$ denote arbitrary elements in $A$. We now prove the identities in Theorem 7.1.

**Proof of (7.1.1a).** We need to prove
\[
(x', y', yz) = (x, y, z) y''. \tag{7.3.2}
\]
Using (2.3.2), (2.4.1), and the Hom-Teichmüller identity, we have:
\[
0 = f(x, y, z) - f(x, z, y) + f(x, y, z)
\]
\[
= (xy, y', z') - (x', yy, z') + (x', y', yz) - x''(y, y, z) - (x, y, y)z''
\]
\[
- (xz, y', y') + (x, zy, y') - (x', z, yy) + x''(z, y, y) + (x, z, y)y''
\]
\[
+ (xy, z', y') - (x', y, y') + (x', y', zy) - x''(y, z, y) - (x, y, z)y''
\]
\[
= 2(x', y', yz) - 2(x, y, z)y''.
\]
Dividing by 2 in the above computation yields (7.3.3).

**Proof of (7.1.1b).** We need to prove
\[
g(x, w, y, z) \overset{\text{def}}{=} (x', w', yz) + (x', y', wz) - (x, w, z)y'' - (x, w, z)y'' = 0. \tag{7.3.3}
\]
Linearize the identity (7.3.2) by replacing $y$ by $y + w$. The result is exactly (7.3.3).

**Proof of (7.1.1c).** We need to prove
\[
h(w, x, y, z) \overset{\text{def}}{=} (wx, y', z') + (w', x', [y, z]) - w''(x, y, z) - (w, y, z)x'' = 0. \tag{7.3.4}
\]
Using (2.4.1), the Hom-Teichmüller identity, and (7.3.1), we have:
\[
0 = f(w, x, y, z) - g(w, x, y, z)
\]
\[
= (wx, y', z') - (w', xy, z') + (w', x', yz) - w''(x, y, z) - (w, x, y)z''
\]
\[
- (w', z', xy) - (w', x', zy) + (w, z, y)x'' + (w, x, y)z''
\]
\[
= (wx, y', z') + (w', x', yz) - w''(x, y, z) - (w', x', zy) - (w, y, z)x''
\]
\[
= h(w, x, y, z).
\]
This proves (7.3.4).

**Proof of (7.1.1d).** By (2.4.1), the identity (7.1.1d) is equivalent to
\[
(x', z', y') = (x', yz + zy'). \tag{7.3.5}
\]
Using (2.3.2), (2.4.1), and the Hom-Teichmüller identity, we have:

\[ 0 = f(x, z, y, y) \]
\[ = (xz, y', y') - (x', zy, y') + (x', z', yy) - x''(z, y, y) - (x, z, y)y'' \]
\[ = -(x', zy, y') + (x', z', y^2) + (x, y, z)y'' \]
\[ = -(x', zy, y') + (x', z', y^2) + (x', y', yz) \quad \text{(by (7.3.3))} \]
\[ = -(x', zy, y') + (x', z', y^2) - (x', yz, y'). \]

This proves (7.3.5). \( \square \)

**Proof of (7.1.1e).** We need to prove

\[ ((xy)z')y'' = x''((yz)y'). \]  \hfill (7.3.6)

We compute as follows:

\[ ((xy)z')y'' = (x, y, z)y'' + (x'(yz))y'' \quad \text{(by (2.3.1))} \]
\[ = (x', y', yz) + (x'(yz))y'' \quad \text{(by (7.3.3))} \]
\[ = -(x', yz, y') + (x'(yz))y'' \quad \text{(by (2.4.1))} \]
\[ = -(x'(yz))y'' + x''((yz)y') + (x'(yz))y'' \quad \text{(by (2.3.1))} \]
\[ = x''((yz)y'). \]

This proves the Hom-Moufang identity (7.1.1e). \( \square \)

**Proof of (7.1.1f).** By interchanging \( y \) and \( z \), the desired identity (7.1.1f) is equivalent to

\[ ((x, z, y)z')y'' = (x, z, y)'(yz)'y''. \]

Using the linearized right Hom-alternative identity (2.4.1) in the previous line, it follows that (7.1.1f) is equivalent to

\[ ((x, y, z)z')y'' = (x, y, z)'(yz)'y''. \]  \hfill (7.3.7)

To prove (7.3.7), first observe that:

\[ (x'', y'', y'(zz)) = (x', y', zz)y'' \quad \text{(by (7.3.2))} \]
\[ = (x', yz, z')y'' + (x', zy, z')y'' \quad \text{(by (7.3.5))} \]
\[ = (x', yz, z')y'' - (x', z', yz)y'' \quad \text{(by (2.4.1))} \]
\[ = (x', yz, z')y'' - (x, y, z)y'' \quad \text{(by (7.3.2))} \]
\[ = (x', yz, z')y'' + ((x, y, z)z')y''' \quad \text{(by (2.4.1))}. \]

Now using (7.3.3) and (2.3.2) we have:

\[ 0 = g(x', y', yz, z') \]
\[ = (x'', y', (yz)z') + (x'', (yz)'y'z') - (x', y', z')(yz)'y''' - (x', yz, z')y''' \]
\[ = (x'', y', (yz)z') + (x'', y'z', y'z') - (x, y, z)'(yz)'y''' - (x', yz, z')y''' \]
\[ = (x'', y', (yz)z') - (x, y, z)'(yz)'y''' - (x', yz, z')y''' \]
\[ = ((x, y, z)z')y''' - (x, y, z)'(yz)'y''' \quad \text{(by (7.3.8))}. \]

This proves the identity (7.3.7). \( \square \)
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