Rationality of unitary $N = 2$ vertex operator superalgebras

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Abstract

In this note we prove that the vertex operator superalgebras associated to the unitary representations of the $N = 2$ superconformal algebra are rational.

Introduction

In this note we continue to study vertex operator superalgebras (SVOAs) associated to the representations of $N = 2$ superconformal algebras. In [A1] we presented the results which gave the complete classification of irreducible modules. We proved that SVOAs associated to the unitary $N = 2$ representations have finitely many irreducible modules. In non-unitary case the representation theory is more complicated (and perhaps, more interesting). The irreducible representations in non-unitary case have a nice description as a union of finitely many rational curves. We should mention that the results from [A1] were motivated by analysis of Eholzer and Gaberdiel from [EG], and from the series of papers [FST], [ST], [SS].

In the present paper we will make the first step in describing the complete category of modules for these SVOAs. Actually, we will prove that the category of finitely-generated modules for SVOAs associated to the unitary $N = 2$–representations is semisimple. This will imply that the corresponding SVOAs are rational. Our proof of rationality is based on the analysis of embedding diagrams of $N = 2$ Verma modules from [D] (see also [SS]), and the classification result from [A1].

We also believe that the main result of this note will be useful for the problem of the classification of rational SVOAs with a unitary central charges.
1 \( N = 2 \) superconformal vertex algebra

In this section we recall the results from [A1] on representation theory of SVOAs associated to the \( N = 2 \) superconformal algebra. We should also mention that the study of these SVOAs was initiated [EG].

\( N = 2 \) superconformal algebra \( \mathcal{A} \) is the infinite-dimensional Lie super algebra with basis \( L_n, T_n, G^\pm_r, C, \) \( n \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z} \) and (anti)commutation relations given by

\[
\begin{align*}
[L_m, L_n] &= (m - n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} \\
[L_m, G^\pm_r] &= \frac{1}{2}(m - r)G^\pm_{m+r} \\
[L_m, T_n] &= -nT_{n+m} \\
[T_m, T_n] &= \frac{C}{3}m\delta_{m+n,0} \\
[T_m, G^\pm_r] &= \pm G^\pm_{m+r} \\
\{G^+_r, G^-_s\} &= 2L_{r+s} + (r - s)T_{r+s} + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s,0} \\
[L_m, C] &= [T_n, C] = [G^\pm_r, C] = 0 \\
\{G^+_r, G^+_s\} = \{G^-_r, G^-_s\} &= 0
\end{align*}
\]

for all \( m, n \in \mathbb{Z}, r, s \in \frac{1}{2} + \mathbb{Z} \).

We denote the Verma module generated from a highest weight vector \( |h, q, c\rangle \) with \( L_0 \) eigenvalue \( h \), \( T_0 \) eigenvalue \( q \) and central charge \( c \) by \( M_{h,q,c} \). An element \( v \in M_{h,q,c} \) is called singular vector if

\[
L_n v = T_n v = G^\pm_r v = 0, \ n, r + \frac{1}{2} \in \mathbb{N},
\]

and \( v \) is an eigenvector of \( L_0 \) and \( T_0 \). Let \( J_{h,q,c} \) be the maximal \( U(\mathcal{A}) \)-submodule in \( M_{h,q,c} \). Then

\[
L_{h,q,c} = \frac{M_{h,q,c}}{J_{h,q,c}}
\]

is the irreducible highest weight module.

Now we will consider the Verma module \( M_{0,0,c} \). One easily sees that for every \( c \in \mathbb{C} \)

\[
G^{\pm}_{-\frac{1}{2}}|0, 0, c\rangle
\]
are the singular vectors in $M_{0,0,c}$. Set

$$V_c = \frac{M_{0,0,c}}{U(\mathcal{A})G^+_{\frac{-1}{2}}|0,0,c\rangle + U(\mathcal{A})G^-_{\frac{-1}{2}}|0,0,c\rangle}.$$ 

Then $V_c$ is a highest weight $\mathcal{A}$-module. Let $1$ denote the highest weight vector. Let $L_c = L_{0,0,c}$ be the corresponding simple module. Define the following four vectors in $V_c$:

$$\tau^\pm = G^\pm_{\frac{-1}{2}}1, \; j = T_{-1}1, \; \nu = L_{-2}1,$$

and set

$$G^+(z) = Y(\tau^+, z) = \sum_{n \in \mathbb{Z}} G^+_{n+\frac{1}{2}} z^{-n-2},$$
$$G^-(z) = Y(\tau^-, z) = \sum_{n \in \mathbb{Z}} G^-_{n+\frac{1}{2}} z^{-n-2},$$
$$L(z) = Y(\nu, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2},$$
$$T(z) = Y(j, z) = \sum_{n \in \mathbb{Z}} T_n z^{-n-1}.$$ (1.1)

It is easy to see that the fields $G^+(z), G^-(z), L(z), T(z)$ are mutually local and the theory of local fields (cf. [K], [Li]) implies the following result.

**Proposition 1.1** There is a unique extension of the fields (1.1) such that $V_c$ becomes vertex operator superalgebra (SVOA). Moreover, $L_c$ is a simple SVOA.

We are interested in the list of all irreducible $L_c$-modules. It is very easy to see that every irreducible $L_c$-module has to be irreducible highest weight $U(\mathcal{A})$-module. To verify this statement, it is enough to notice that the Zhu's algebra $A(L_c)$ is a certain quotient of the polynomial algebra $\mathbb{C}[x,y]$ (cf. [EG], [Z]).

We will now present the classification result from [A1].

**Definition 1.1** A rational number $m = t/u$ is called admissible if $u \in \mathbb{N}$, $t \in \mathbb{Z}$, $(t,u) = 1$ and $2u + t - 2 \geq 0$. 

For $m = \frac{t}{u}$ admissible set $c_m = \frac{3m}{m+2}$, $N = 2u + t - 2$ and

$$S^m = \{n - k(m + 2) \mid k, n \in \mathbb{Z}_+, n \leq N, k \leq u - 1\}.$$ 

Let

$$W^c_m = \left\{ \left( \frac{jk - \frac{1}{m+2}, \frac{j-k}{m+2}}{m+2} \right) \mid j, k \in \mathbb{N}_{\frac{1}{2}}, 0 < j, k, j + k \leq N + 1 \right\},$$

and if $m \notin \mathbb{N}$ let

$$D^c_m = \{ (h, q) \in \mathbb{C}^2 \mid q^2 + \frac{4h}{m+2} = \frac{r(r+2)}{(m+2)^2}, r \in S^m \setminus \mathbb{Z} \}.$$

Note that the set $W^c_m$ is finite and the set $D^c_m$ is union of finitely many rational curves.

**Theorem 1.1 [A1]** Assume that $m \in \mathbb{N}$ and $c = c_m$. Then the set

$$\{ L_{h,q,c} \mid (h, q) \in W^c_m \}$$

provides all $L_c$ irreducible modules for the SVOA $L_c$. So, irreducible $L_c$-modules are exactly all unitary modules for $N = 2$ superconformal algebra with the central charge $c$.

**Remark 1.1** Theorem [A1] shows that SVOA $L_{cm}$ for $m \in \mathbb{N}$ has exactly $\frac{(m+2)(m+1)}{2}$ non-isomorphic irreducible modules.

**Theorem 1.2 [A1]** Assume that $m \in \mathbb{Q}$ is admissible such that $m \notin \mathbb{N}$. Let $c = c_m$. Then the set

$$\{ L_{h,q,c} \mid (h, q) \in W^c_m \cup D^c_m \}$$

provides all $L_c$ irreducible modules for the SVOA $L_c$. 
2 Rationality of the SVOA $L_{c_m}$ for $m \in \mathbb{N}$

In this section we will present a proof of the rationality of the SVOA $L_{c_m}$. Theorem 1.1 gives that SVOA $L_{c_m}$ has finitely many irreducible modules. We will now finish the proof of rationality by proving that every finitely generated $L_{c_m}$–module is completely reducible. In almost all known cases the proof of rationality is based on the structure theory of certain categories of highest weight representations, for example the category $\mathcal{O}$ in the case of affine Lie algebras (cf. [DGK]). In order to modify the proof from affine case to superconformal algebras, one has to know the embedding diagrams for Verma modules. In the case of unitary representations of $\mathcal{N} = 2$ superconformal algebra, the correct embedding diagrams for Verma modules was done by M. Dorzäpf in [D]. He also proved that unitary $\mathcal{N} = 2$ Verma modules do not contain subsingular vectors, which implies that every submodule of a unitary Verma module is generated by singular vectors (cf. [D] and the corresponding references).

**Definition 2.1** SVOA $V$ is called **rational** if $V$ has finitely many irreducible modules, and if every finitely generated $V$–module is completely reducible.

In this section, let $m \in \mathbb{N}$, and $c = c_m = \frac{3m}{m+2}$. The results from [D] imply the following analysis.

For $(h, q) \in W^c$, we have that

$$h = \frac{jk - \frac{1}{2}}{m+2}, q = \frac{j - k}{m+2},$$

for

$$j, k \in \mathbb{N}_1, 0 < j, k, j + k \leq m + 1.$$

We are interested in the weights of all homogeneous singular vectors in Verma module $M_{h,q,c}$.

Theorem 4.A from [D] implies that the Verma module $M_{h,q,c}$ is reducible, and has infinitely many singular vectors

$$v_{h_1^i,q_1^i}, v_{h_2^i,q_2^i}, \ldots, v_{h_k^i,q_k^i}, \quad i \in \mathbb{N} \quad (2.1)$$

of the weights

$$h_1^i = h + i(m+2) - j - k, \quad q_1^i = q, \quad (2.2)$$

$$h_2^i = h + i(m+2) + j + k, \quad q_2^i = q, \quad (2.3)$$
\[h_3^i = h + k + (i - 1)(i(m + 2) + j + k), \quad q_3^i = q + 1, \quad (2.4)\]
\[h_4^i = h + k + (i + 1)(i(m + 2) - j - k), \quad q_4^i = q + 1, \quad (2.5)\]
\[h_5^i = h + j + (i - 1)(i(m + 2) + j + k), \quad q_5^i = q - 1, \quad (2.6)\]
\[h_6^i = h + j + (i + 1)(i(m + 2) - j - k), \quad q_6^i = q - 1. \quad (2.7)\]

Moreover, the maximal submodule \(M_{h/q,c}^1\) of the Verma module \(M_{h,q,c}\) is generated by three vectors:

\[v_{h_1^i,q_1^i}, v_{h_3^i,q_3^i}, v_{h_5^i,q_5^i}.\]

We have the following lemma.

**Lemma 2.1** Assume that \(v_{h',q'}\) is a singular vector of weight \((h',q')\) in the Verma module \(M_{h,q,c}\), where \(c = c_m, (h, q) \in W^c\). Then \((h', q') \notin W^c\).

**Proof.** Since the weights (2.4)-(2.7) provide the weights of all singular vectors in the Verma module \(M_{h,q,c}\), we have to check the statement of the Lemma for every singular vector in the list (2.1). For simplicity, we will only consider vectors \(v_{h_1^i,q_1^i}\). For other singular vectors, the considerations are completely analogous. So, we will prove that for every \(i \in \mathbb{N}\)

\[(h_1^i, q_1^i) \notin W^c.\]

First, we notice that

\[h_1^i = \frac{(j - i(m + 2))(k - i(m + 2)) - 1}{m + 2}, \quad q_1^i = \frac{j - k}{m + 2}.\]

Assume that \((h_1^i, q_1^i) \in W^c\). Then there are

\[\overline{j}, \overline{k} \in \mathbb{N}, 0 < \overline{j}, \overline{k}, \overline{j} + \overline{k} \leq m + 1,\]

such that

\[h_1^i = \frac{\overline{j} - \overline{k} - 1}{m + 2}, \quad q_1^i = \frac{\overline{j} - \overline{k}}{m + 2}. \quad (2.9)\]

But, the equation (2.5) implies that

\[\overline{j} = j - i(m + 2), \quad \overline{k} = k - i(m + 2)\]
or

\[ \bar{j} = -j + i(m + 2), \quad \bar{k} = -k + i(m + 2), \]

which in both cases contradicts the condition (2.8). \(\square\)

Lemma 2.1 and the fact that unitary Verma modules don’t contain sub-singular vectors imply the following proposition.

**Proposition 2.1** Let \( V_{h,q,c} \) be any highest weight \( U(A) \)-module such that the highest weight \( (h, q) \in W^c, c = c_m \). Assume that \( V_{h,q,c} \) is reducible. Then there is a highest weight \( U(A) \)-submodule \( N' \) with the highest weight \( (h', q') \notin W^c \).

Proposition 2.1 and Theorem 1.1 imply the following.

**Corollary 2.1** Assume that \( M \) is a highest weight \( U(A) \)-module such that \( M \) is a module for SVOA \( L_{c_m}, m \in \mathbb{N} \). Then \( M \) is irreducible.

**Remark 2.1** Proposition 2.1 and Corollary 2.1 can be also proved by using the results of A. M. Semikhatov and V. A. Sirota from [SS]. Their approach uses the relations between the embedding diagrams of \( N = 2 \)-modules and modules for affine Lie algebra \( \hat{sl}_2 \), and it can be applied on non-unitary highest weight \( N = 2 \)-modules. We hope to study the non-unitary case in our next publications (cf. [A3]).

Let \( \omega \) be the involutory anti-automorphism of \( A \), defined with \( c \mapsto c \) and

\[
L_m \mapsto L_{-m}, \quad T_m \mapsto T_{-m}, \quad G^+_{m - \frac{1}{2}} \mapsto G^-_{-m + \frac{1}{2}}, \quad G^-_{m - \frac{1}{2}} \mapsto G^+_{-m + \frac{1}{2}}
\]

for every \( m \in \mathbb{Z} \).
We will now prove that every finitely generated $L_c$-module is completely reducible. It is enough to consider $L_c$-modules with the following graduation:

\[ M = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}^+} M_m, \quad \dim M_m < \infty \]

$L_0$ acts semisimply on $M_m$.

We define $M^\omega = \bigoplus_{m \in \frac{1}{2}\mathbb{Z}^+} M^\omega_m$ with the following action $(af)(u) = f(\omega(a)u)$ for any $f \in M^\omega$, $a \in A$, $u \in M$. In the same way as in [DGK] one can see that $M \mapsto M^\omega$ is a contravariant functor and

\[ (M^\omega)^\omega \simeq M, \quad L^\omega_{h,q,c} \simeq L_{h,q,c} \] for any $(h, q) \in \mathbb{C}^2$.

**Theorem 2.1** For $m \in \mathbb{N}$ the SVOA $L_{cm}$ is rational.

We proved in Theorem 1.1 that SVOA $L_{cm}$ has finitely many irreducible modules. For a proof of rationality, it remains to prove that every finitely generated $L_{cm}$-module is completely reducible. To see this, it is enough to prove the following lemma (see also [DLM], [A2], [W] for proofs of rationality in some similar cases).

**Lemma 2.2** Let $c = c_m$ for $m \in \mathbb{N}$, and let $(h_1, q_1), (h_2, q_2) \in W^c$. Then every short exact sequence of $U(A)$-modules

\[ 0 \rightarrow L_{h_1, q_1, c} \xrightarrow{\iota} M \xrightarrow{\pi} L_{h_2, q_2, c} \rightarrow 0 \] (2.10)

splits.

**Proof.** Without loss of generality, we may assume that $h_2 \leq h_1$. Otherwise we can apply the functor $M \mapsto M^\omega$ to the short exact sequence to reverse it. Let $v_{h_2, q_2}$ be the highest weight vector of $L_{h_2, q_2, c}$. Pick a vector $v'_{h_2, q_2} \in M$ of weight $(h_2, q_2)$ such that $\pi(v'_{h_2, q_2}) = v_{h_2, q_2}$. We claim that $v'_{h_2, q_2}$ is a singular vector in $M$. Let $n \in \mathbb{N}$. Assume that $T_n v'_{h_2, q_2} \neq 0$. Then

\[ \pi(T_n v'_{h_2, q_2}) = T_n \pi(v'_{h_2, q_2}) = T_n v_{h_2, q_2} = 0. \]

This implies

\[ T_n v'_{h_2, q_2} = \iota(u) \] (2.11)

for some nonzero $u \in L_{h_1, q_1, c}$, since the short sequence is exact. Comparing the weights of both sides of equation (2.11), we have $h_2 - n = h_1 + \alpha$ for
nonzero $\alpha \in \frac{1}{2}\mathbb{Z}_+$. It follows that $h_2 = h_1 + \alpha + n > h_1$, which is a contradiction. In the same way the assumptions $G_{n-\frac{1}{2}}^+ v_{h_2,q_2}^l \neq 0$, $L_n v_{h_2,q_2}^l \neq 0$ for $n > 0$ lead to contradiction.

Denote by $M'$ the highest weight submodule of $M$ generated by the singular vector $v_{h_2,q_2}^l$. Assume that $M'$ is reducible. Then Lemma 2.1 implies that $M'$ has a highest weight submodule $N' \neq 0$ with the highest weight $(h',q') \notin W_{c_m}$. Then irreducibility of $L_{h_2,q_2,c}$ implies that $N' \subseteq \text{Ker}(\pi)$, which contradicts the fact that short sequence is exact. In this way we have proved that $M'$ is irreducible. This implies that $M \cong M' \oplus L_{h_1,q_1,c}$, and we have proved that the sequence (2.10) is exact. \(\square\)

Let us conclude this section by giving one characterization of the category of $L_{c_m}$-modules. We know from the embedding structure that the maximal submodule of the Verma module $M_{0,0,c}$ is generated three vectors. Two such vectors are $G_{-\frac{1}{2}}^+ |0,0,c\rangle$, and the third vector has relatively charge zero and level $m+1$. In particular, we have that the maximal submodule of $U(A)$-module $V_{c_m}$ is generated by one singular vector $v_{\text{sing}}$ of level $m+1$. Moreover, the results from [ST], [FST] imply that the maximal submodule of $V_{c_m}$ is generated by the vector $G_{-m-\frac{3}{2}}^- \cdots G_{-\frac{3}{2}}^- 1$ (or equiv. $G_{-m-\frac{3}{2}}^+ \cdots G_{-\frac{3}{2}}^+ 1$), and that the singular vector is given by the following formulae:

$$v_{\text{sing}} = G_{\frac{1}{2}}^+ \cdots G_{m+\frac{1}{2}}^+ G_{-m-\frac{3}{2}}^- \cdots G_{-\frac{3}{2}}^- 1.$$

The previous analysis and the vertex operator superalgebra structure on $V_{c_m}$ and $L_{c_m}$ imply the following lemma.

**Lemma 2.3** Let $M$ be a $U(A)$-module of central charge $c_m$. Then the following statements are equivalent:

(i) $M$ is a $L_{c_m}$-module,

(ii) $Y(G_{-m-\frac{3}{2}}^- \cdots G_{-\frac{3}{2}}^- 1, z)M = 0$, (or equiv. $Y(G_{-m-\frac{3}{2}}^+ \cdots G_{-\frac{3}{2}}^+ 1, z)M = 0$),

(iii) $: \partial^{(m)} G^- (z) \cdots G^- (z) : M = 0$

(or equiv. $: \partial^{(m)} G^+ (z) \cdots G^+ (z) : M = 0$).

Now Lemma 2.3 and Theorem 2.1 imply the following.
Corollary 2.2 Assume that $M$ is a finitely generated $U(A)$–module of the central charge $c_m$ such that: $\partial^{(m)}G^-(z)\cdots G^-(z): M = 0$. Then $M$ is a direct sum of unitary representations with the central charge $c_m$.

Remark 2.2 The statement of the Corollary 2.2 doesn’t need vertex algebra language, but our proof uses vertex algebras. This statement was also known to physicist, and it was used in [FS] for some cohomology calculation.

Remark 2.3 Let $e, f, h$ be the generators of $sl_2$. It is well-known that the integrable $\hat{sl}_2$–modules of level $m$ are characterized by the field-relation $e(z)^{m+1} = 0$, and the Corollary 2.2 gives $N = 2$ interpretation of this fact. It will be also interesting to translate some other properties of the representation theory of $\hat{sl}_2$ in $N = 2$ language.

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