Abstract

We consider a fundamental algorithmic question in spectral graph theory: Compute a spectral sparsifier of a random-walk matrix-polynomial

\[ L_\alpha(G) = D - \sum_{r=1}^{d} \alpha_r D \cdot (D^{-1}A)^r \]

where \( A \) is the adjacency matrix of a weighted, undirected graph, \( D \) is the diagonal matrix of weighted degrees, and \( \alpha = (\alpha_1, ..., \alpha_d) \) are nonnegative coefficients with \( \sum_{r=1}^{d} \alpha_r = 1 \). Recall that \( D^{-1}A \) is the transition matrix of random walks on the graph. In its linear form (when \( d = 1 \)), the matrix polynomial becomes \( D - A \), which is a Laplacian matrix, and hence this problem becomes the standard spectral sparsification problem, which enjoys nearly linear time solutions \([ST11, SS11]\). However, the sparsification of \( L_\alpha(G) \) appears to be algorithmically challenging as the matrix power \( (D^{-1}A)^r \) is defined by all paths of length \( r \), whose precise calculation would be prohibitively expensive (due to the cost of matrix multiplication and densification in the matrix powers).

In this paper, we develop the first nearly linear time algorithm for this sparsification problem: For any \( G \) with \( n \) vertices and \( m \) edges, \( d \) coefficients \( \alpha \), and \( \epsilon > 0 \), our algorithm runs in time \( O(d^2 \cdot m \cdot \log^2 n / \epsilon^2) \) to construct a Laplacian matrix \( \tilde{L} = D - \tilde{A} \) with \( O(n \log n / \epsilon^2) \) non-zeros such that

\[ \tilde{L} \approx_{\epsilon} L_\alpha(G) = D - \sum_{r=1}^{d} \alpha_r D \cdot (D^{-1}A)^r. \]

In the equation, \( \tilde{L} \approx_{\epsilon} L_\alpha(G) \) denotes that \( \tilde{L} \) and \( L_\alpha(G) \) are spectrally similar within a factor of \( 1 \pm \epsilon \) as defined in \([ST11]\).

Matrix polynomials arise in mathematical analysis of matrix functions as well as numerical solutions (such as Newton’s method) of matrix equations. Our work is particularly motivated by the algorithmic problems for speeding up the classic Newton’s method in applications such as computing the inverse square-root of the precision matrix of a Gaussian random field (in order to obtain \( i.i.d \) random samples of the graphic model), as well as computing the \( q^{th} \)-root transition (for \( q \geq 1 \)) in a time-reversible Markov model. The key algorithmic step for both

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applications is the construction of a spectral sparsifier of a constant degree \(^1\) random-walk matrix-polynomials introduced by Newton’s method. Our sparsification algorithm leads to a simpler and faster algorithm for these problems than the previous one [CCL+14] that circumvents the challenging problem of sparsifying high-degree random-walk matrix polynomials at the cost of slower convergences and complex approximation. Our algorithm can also be used to build efficient data structures for effective resistances for multi-step time-reversible Markov models, and we anticipate that it could be useful for other tasks in network analysis.

1 Introduction

Polynomials are used in many fields of mathematics and science for encoding equations that model various physical, biological and economical processes. In scientific computing and its underpinning numerical analysis, polynomials appear naturally in (truncated) Taylor series, the fast multipole method [GR87], and various numerical approximations. These computational methods are responsible for a large part of engineering and scientific simulations ranging from weather forecasting to earthquake modeling [SF73] to particle/galaxy simulation [GR87].

Like its scalar counterpart, matrix polynomials of the form, \(\sum_{i=0}^{d} c_i \cdot M^i\), arise in mathematical analysis of matrix functions and dynamical systems, as well as numerical solution of matrix equations. One class of matrices of particular importance in network analysis is the adjacency matrix of a weighted, undirected graph \(G\). We will denote these matrices using \(A\). If we use \(D\) to denote the diagonal matrix containing weighted degrees of vertices, \(D^{-1}A\) is the transition matrix of random walks on the graph. Powers of this matrix correspond to multiple steps of random walks on the graph, and are graphs themselves. However, they are usually dense, and are cost-prohibitive both in time and memory to construct when the input graph \(G\) is of large-scale. Our objective is to efficiently construct sparse approximations of this natural family of random-walk matrices.

1.1 Motivation from Gaussian Sampling and Fractional Markov Transition

A problem that motivates our study of random walk polynomials is the inverse square-root problem: find a linear operator \(C\) such that \(C^\top C\) is close to the inverse of the Laplacian matrix. Laplacian matrices are a subclass of SDDM matrices \(^2\) and have standard split-form representation of \(L = D - A\). When we apply an extension of the classical Newton’s method to this form we can reduce the problem to that of factoring \(D - (\frac{3}{4}D \cdot (D^{-1}A)^2 + \frac{1}{4}D \cdot (D^{-1}A)^3)\), which has smaller spectral radius, by using the matrix identity

\[
(D - A)^{-\frac{3}{2}} = \left( I + \frac{1}{2}D^{-1}A \right) \left( D - \left( \frac{3}{4}D \cdot (D^{-1}A)^2 + \frac{1}{4}D \cdot (D^{-1}A)^3 \right) \right)^{-\frac{3}{2}}. \tag{1}
\]

Finding inverse square-root factorizations is a key step in sampling from Gaussian graphical models: Given a graphical model of a Gaussian random field specified by its precision matrix \(\Lambda\) and potential vector \(h\), i.e., \(\Pr(x|\Lambda, h) \propto \exp(-\frac{1}{2}x^\top \Lambda x + h^\top x)\), efficiently generate \(i.i.d\) random samples from this multivariate Gaussian distributions [LW12]. If one can compute an efficient sparse representation of \(C = \Lambda^{-1/2}\), then one can convert \(i.i.d\) standard Gaussian random vector

\(^1\)In numerical algorithms where random-walk matrix-polynomials arise, the degree \(d\) of the polynomials is usually either a constant or bounded above by a polylogarithmic function in \(n\).

\(^2\)SDDM matrices are positive definite symmetric diagonal dominant matrices with non-positive off-diagonal elements.
z using \( x = Cz + \mu \) (where \( \mu = \Lambda^{-1}h \)) to i.i.d random vectors of a Gaussian random field that numerically approximates the one defined by \((\Lambda, h)\) [CCL+14]. Furthermore, if the precision matrix \( \Lambda = (\lambda_{i,j}) \) is symmetric diagonally dominant (SDD), i.e., for all \( i \), \( \lambda_{i,i} > \sum_{j \neq i} |\lambda_{i,j}| \), then one can reduce this factorization problem to the problem formulated by Equation (1) involving an SDDM matrix.

Then, in order to iteratively apply Equation (1) to build an efficient representation of the inverse square-root factor of \( D - A \), one needs to efficiently construct the second term in Equation (1),

\[
D - \left( \frac{3}{4}D \cdot (D^{-1}A)^2 + \frac{1}{4}D \cdot (D^{-1}A)^3 \right).
\]

The quadratic and cubic powers in this matrix can be very dense, making exact computations involving them expensive. Instead, we will directly compute an approximation of this matrix that still suffices for algorithmic purposes.

Finding an inverse square-root of an SDDM matrix is a special case of the following basic algorithmic problem in spectral graph theory and numerical analysis [CCL+14]:

Given an \( n \times n \) SDDM matrix \( M \), a non-zero integer \( q \), and an approximation parameter \( \epsilon \), compute an efficient sparse representation of an \( n \times n \) linear operator \( \tilde{C} \) such that

\[
M^{1/q} \approx_\epsilon \tilde{C}\tilde{C}^T
\]

where \( \approx_\epsilon \) is spectral similarity between linear operators which we will define at the start of Section 2.

The matrix \( q^{th} \)-root computation appears in several numerical applications and particularly in the analysis of Markov models [HL11]. For example, in his talk for Brain Davies’ 65 Birthday conference (2009), Nick Higham quoted an email that he received from a power company regarding the usage of an electricity network to illustrate the practical needs of taking the \( q^{th} \)-root of a Markov transition.

"I have an Excel spreadsheet containing the transition matrix of how a company’s [Standard & Poor’s] credit rating charges from on year to the next. I’d like to be working in eighths of a year, so the aim is to find the eighth root of the matrix."

In our case, note that when the graph is connected, \( D^{-1}A \) is the transition matrix of a reversible Markov chain [AF02], and the first order approximation of the \( q^{th} \)-root transition is \( I - (I - D^{-1}A)^{1/q} \). Extension of Newton’s method then leads to an iterative formula similar to Equation (1) for finding factorization of the \( q^{th} \) root. Thus, the key algorithmic task for obtaining a nearly linear time Newton(-like) algorithm for \( q^{th} \)-root factorizations is the efficient approximation of matrix polynomials akin to Equation (2).

### 1.2 Main Technical Contribution

We start with a definition that captures the matrix polynomials such like Equation (2) that arise in the application of Newton’s or Newton-like methods to graph Laplacians.

**Definition 1.1** (Random-Walk Matrix-Polynomials). Let \( A \) and \( D \) be the adjacency matrix and diagonal weighted degree matrix of a weighted, undirected graph \( G \) respectively. For a non-negative
vector $\mathbf{\alpha} = (\alpha_1, ..., \alpha_d)$ with $\sum_{r=1}^d \alpha_r = 1$, the matrix

$$L_\mathbf{\alpha}(G) = D - \sum_{r=1}^d \alpha_r D \cdot (D^{-1}A)^r$$

is a $d$-degree random-walk matrix-polynomial of $G$.

Random-walk matrix-polynomials naturally include the graph Laplacian $G$ as the linear case: when $d = 1$, the matrix polynomial becomes $L(G) = D - A$, which is the Laplacian matrix of $G$. In fact, the following proposition can be established by a simple induction, which we prove in Appendix A.

**Proposition 1.2** (Laplacian Preservation). For any weighted, undirected graph $G$ with adjacency matrix $A$ and diagonal matrix $D$, for every non-negative vector $\mathbf{\alpha} = (\alpha_1, ..., \alpha_d)$ such that with $\sum_{r=1}^d \alpha_r = 1$, the random-walk matrix-polynomial $L_\mathbf{\alpha}(G)$ remains a Laplacian matrix.

Consequently, applying spectral sparsification algorithms [ST11, SS11, BSS12] to $L_\mathbf{\alpha}(G)$ gives:

**Proposition 1.3** (Spectral Sparsifiers of Random-Walk Matrix Polynomials). For all $G$ and $\mathbf{\alpha}$ as in Proposition 1.2, for any $\epsilon > 0$, there exists a Laplacian matrix $\tilde{L} = D - \tilde{A}$ with $O(n \log n/\epsilon^2)$ non-zeros such that for all $x \in \mathbb{R}^n$

$$\left(1 - \epsilon\right) \cdot x^\top \tilde{L} x \leq x^\top \left(D - \sum_{r=1}^d \alpha_r D \cdot (D^{-1}A)^r\right) x \leq \left(1 + \epsilon\right) \cdot x^\top \tilde{L} x.$$  \hspace{1cm} (4)

The Laplacian matrix $\tilde{L}$ satisfying Equation (4) is called spectrally similar with approximation parameter $\epsilon$ to $L_\mathbf{\alpha}(G)$ [ST11]. The computation of a (nearly) linear size spectral sparsifier of a dense Laplacian matrix is a fundamental algorithmic problem in spectral graph theory that has been used in solving linear systems [ST14, KMP10] and combinatorial optimization [CKM+11]. The work of [SS11, ST11] showed that spectral sparsifier of $O(n \log n/\epsilon^2)$ non-zeros can be constructed in $O(m \log^2 n/\epsilon^2)$ time for any $n \times n$ Laplacian matrix $L$ with $m$ non-zeros. A recent technique of [PS14] can sparsify a degree 2 random-walk matrix-polynomial in nearly linear time.

In this paper, we give the first nearly linear time spectral-sparsification algorithm for all random-walk matrix-polynomials. Our sparsification algorithm is built on the following key mathematical observation that might be interesting on its own: One can obtain a sharp enough upper bound on the effective resistances of the high-order polynomial $D - D(D^{-1}A)^r$ from the combination of the linear $D - A$ and quadratic $D - AD^{-1}A$ polynomials.

This allows us to design an efficient path sampling algorithm that utilizes this mathematical observation to achieve the critical sparsification. We prove the following result which generalizes the works of [ST11, SS11, PS14].

**Theorem 1.4** (Random-Walk Polynomials Sparsification). For any weighted, undirected graph $G$ with $n$ vertices and $m$ non-zeros, for every non-negative vector $\mathbf{\alpha} = (\alpha_1, ..., \alpha_d)$ with $\sum_{r=1}^d \alpha_r = 1$, for any $\epsilon > 0$, we can construct in time $O(d^2 \cdot m \cdot \log^2 n/\epsilon^2)$ a spectral sparsifier with $O(n \log n/\epsilon^2)$ non-zeros and approximation parameter $\epsilon$ for the random-walk matrix-polynomial $L_\mathbf{\alpha}(G)$.

The total work of our sparsification algorithm depends quadratically in the degree of the polynomial, which could be expensive when $d = \Theta(n^c)$ for some constant $c > 0$. In Section 4 we present,
for even degrees \( d \), a more efficient algorithm to sparsify \( D - D(D^{-1}A)^d \). We will show that, for any positive integer \( r \), if we are given \( \tilde{A} \) such that
\[
D - \tilde{A} \approx D - D (D^{-1}A)^{2r},
\]
we can construct a sparse matrix \( \tilde{A}_x \) and a sparse matrix \( \tilde{A}_+ \) such that
\[
D - \tilde{A}_x \approx D - D (D^{-1}A)^{4r} \quad \text{and} \quad D - \tilde{A}_+ \approx D - D (D^{-1}A)^{2r+4}.
\]

Applying these two routines inductively gives an algorithm that, for any \( d \) divisible by 4, approximates the \( d \)-degree random-walk matrix-monomial in time polylogarithmic in \( d \). We can also extend this algorithm to handle all the even-degree monomials. Because when \( d = O(1/\epsilon) \), we can directly invoke Theorem 1.4 to sparsify \( D - D(D^{-1}A)^d \), and when \( d = \Omega(1/\epsilon) \), we know that \( D - D(D^{-1}A)^d \approx D - D(D^{-1}A)^{d+2} \), therefore any even-degree monomial can be replaced by a degree \( 4r \) monomial, while introducing a small error only.

**Theorem 1.5 (High Degree).** For any even integer \( d \), let \( L_{G_d} = D - D(D^{-1}A)^d \) be the \( d \)-step random walk matrix, For any \( \epsilon > 0 \), we can construct a graph Laplacian \( \tilde{L}_{\tilde{G}} \) with \( O(n \log n/\epsilon^2) \) nonzero entries, in total work \( O(m \cdot \log^3 n \cdot \log^5(d)/\epsilon^4) \), such that \( \tilde{L}_{\tilde{G}} \approx_{\epsilon} L_{G_d} \).

While we build our construction on the earlier work [ST11, SS11, PS14] for graph sparsification, we need to overcome some significant difficulties posed by high degree matrix polynomials, which appear be algorithmically challenging: The matrix \( (D^{-1}A)^r \) is defined by all paths of length \( r \), whose precise calculation would be prohibitively expensive due to the cost of matrix multiplication and densification in the matrix powers. Moreover, the algorithm of [PS14] relies an explicit clique-like representation of edges in the quadratic power and expanders, which is much more specialized.

### 1.3 Some Applications

Matrix polynomials are involved in numerical methods such as Newton’s or Newton-like methods, which have been widely used for finding solutions of matrix equations. Our sparsification algorithm can be immediately applied to speed up these numerical methods that involve SDDM matrices. For example, in the application of finding the inverse square root of an SDDM matrix \( D - A \), we can directly sparsify the cubic matrix polynomial given in Equation 2, and iteratively approximate the inverse-square root factor of \( D - A \) using Equation 1. This leads to a simpler and faster algorithm than the one presented in [CCL+14], which circumvents the challenging problem of sparsifying high-degree random-walk matrix polynomials at the cost of slower convergences and complex approximation. The simplicity of the new algorithm comes from the fact that we no longer need the Maclaurin series for conditioning the numerical iterations. The convergence analysis of the new algorithm follows the standard analysis of the Newton’s method, with careful adaptation to handle the approximation errors introduced by the spectral sparsification. The elimination of the Maclaurin series speeds up the previous algorithm by a factor of \( \log \log \kappa \), where \( \kappa \) is the relative condition number of \( D - A \).

In general, our sparsification algorithm can be used inside the Newton-like method for approximating the inverse \( q \)-th root of SDDM matrices to obtain a simpler and faster nearly linear time algorithm than the one presented in [CCL+14] for \( q \in \mathbb{Z}_+ \), with reduction formula as follows
\[
(I - X)^{-1/q} = \left( I + \frac{X}{2q} \right) \left[ \left( I + \frac{X}{2q} \right)^{2q} (I - X) \right]^{-1/q} \left( I + \frac{X}{2q} \right).
\]
Our mathematical and algorithmic advances enable the sparsification of the \((2q + 1)\)-degree polynomials in the middle, in turn speed up the previous algorithm by a factor of \(\log(\log(\kappa)/\epsilon)\).

By Proposition 1.2, the random-walk matrix polynomial \(L_\alpha(G) = D - \sum_{r=1}^{d} \alpha_r D \cdot (D^{-1}A)^r\) defines a weighted graph \(G_\alpha\) whose adjacency matrix is \(\sum_{r=1}^{d} \alpha_r D \cdot (D^{-1}A)^r\), and overlays \(d\) graphs induced by the multi-step random walks. While \(D - A\) and \(D - AD^{-1}A\) offers a good enough bound on the effective resistances of edges in \(G_\alpha\) for the purpose of path sampling, these estimates are relatively loose comparing with the standard approximation condition. Because spectral similarity implies effective-resistance similarity [ST11, SS11], our sparsification algorithm together with the construction of Spielman and Srivastava [SS11] provide an efficient data structure for effective resistances in \(G_\alpha\). After nearly linear preprocessing time, we can answer queries regarding approximate effective resistances in \(G_\alpha\) in logarithmic time.

Due to these connections to widely used tools in numerical and network analysis, we anticipate our nearly linear time sparsification algorithm could be useful for a variety of other tasks.

2 Background and Notation

We assume \(G = (V, E, w)\) is a weighted undirected graph with \(n = |V|\) vertices, \(m = |E|\) edges and edge weights \(w_e > 0\). Let \(A = (a_{i,j})\) denote the adjacency matrix of \(G\), i.e., \(a_{i,j} = w(i, j)\). We let \(D\) to denote the diagonal matrix containing weighted degrees of vertices. Note that \(D^{-1}A\) is the transition matrix of random walks on the graph and \(L_G = D - A\) is the Laplacian matrix of \(G\). It is well known that for any vector \(x = (x_1, ..., x_n)\),

\[
x^\top L_G x = \sum_{(u,v) \in E} (x_u - x_v)^2 w_{uv}
\]

We use \(G_r\) to denote the graph introduced by \(r\)-step random walks on \(G\). We have \(L_{G_2} = D - AD^{-1}A\), and \(L_{G_r} = D - D(D^{-1}A)^r\) in general.

In our analysis, we will make extensive use of spectral approximations based on the Loewner partial ordering of positive semidefinite matrices. Given two matrices \(X\) and \(Y\), we use \(Y \succeq X\) (or equivalently \(X \preceq Y\)) to denote that \(Y - X\) is positive semi-definite. Approximations using this ordering obey usual intuitions with approximations of positive scalars. We will also use a compressed, symmetric notation in situations where we have mirroring upper and lower bounds. We say \(X \approx_\epsilon Y\) when

\[
\exp(\epsilon) X \succeq Y \succeq \exp(-\epsilon) X,
\]

We use the following standard facts about this notion of approximation.

**Fact 2.1.** For positive semi-definite matrices \(X, Y, W\) and \(Z\),

1. if \(Y \approx_\epsilon Z\), then \(X + Y \approx_\epsilon X + Z\);
2. if \(X \approx_\epsilon Y\) and \(W \approx_\epsilon Z\), then \(X + W \approx_\epsilon Y + Z\);
3. if \(X \approx_{\epsilon_1} Y\) and \(Y \approx_{\epsilon_2} Z\), then \(X \approx_{\epsilon_1 + \epsilon_2} Z\);
4. if \(X\) and \(Y\) are positive definite matrices such that \(X \approx_\epsilon Y\), then \(X^{-1} \approx_\epsilon Y^{-1}\);
\( \tilde{G} = \text{GraphSampling}(G = \{V, E, w\}, \tau_e, M) \)

1. Initialize graph \( \tilde{G} = \{V, \emptyset\} \).
2. For \( i \) from 1 to \( M \):
   - Sample an edge \( e \) from \( E \) with \( p_e = \tau_e / (\sum_{e \in E} \tau_e) \). Add \( e \) to \( \tilde{G} \) with weight \( w(e) / (M \tau_e) \).
3. Return graph \( \tilde{G} \).

Figure 1: Pseudocode for Sampling by Effective Resistances

5. if \( X \approx \epsilon Y \) and \( V \) is a matrix, then \( V^\top X V \approx \epsilon V^\top Y V \).

The Laplacian matrix is closely related to electrical flow [SS11, CKM+11]. For an edge with weight \( w(e) \), we view it as a resistor with resistance \( r(e) = 1 / w(e) \). Recall that the effective resistance between two vertices \( u \) and \( v \) \( R(u, v) \) is defined as the potential difference induced between them when a unit current is injected at one and extracted at the other. Let \( e_i \) denote the vector with 1 in the \( i \)-th entry and 0 everywhere else, the effective resistance \( R(u, v) \) equals to \( (e_u - e_v)^\top L^\dagger (e_u - e_v) \), where \( L^\dagger \) is the Moore-Penrose Pseudoinverse of \( L \). From this expression, we can see that effective resistance obeys triangle inequality. Also note that adding edges to a graph does not increase the effective resistance between any pair of nodes.

In [SS11], it was shown that oversampling the edges using upper bound on the effective resistance suffices for constructing spectral sparsifiers. The theoretical guarantees for this sampling process were strengthened in [KL13]. A pseudocode of this algorithm is given in Figure 1, and its guarantees can be stated as follows.

**Theorem 2.2.** Given a weighted undirected graph \( G \), and upper bound on its effective resistance \( Z(e) \geq R(e) \). For any approximation parameter \( \epsilon > 0 \), there exists \( M = O(\log n / \epsilon^2 \cdot (\sum_{e \in E} \tau_e)) \), with \( \tau_e = w(e) Z(e) \), such that with probability at least \( 1 - \frac{1}{n} \), \( \tilde{G} = \text{GraphSampling}(G, w, \tau_e, M) \) has at most \( M \) edges, and satisfies

\[
(1 - \epsilon) L_G \preceq L_{\tilde{G}} \preceq (1 + \epsilon) L_G.
\]

The equivalence between solving linear systems in such matrices and graph Laplacians, weakly-SDD matrices, and M-matrices are well known [ST04, DS08, KOSZ13, PS14].

### 3 Sparsification by Sampling Random Walks

We sparsify \( L_{G_r} = D - D(D^{-1}A)^r \) in two steps. In the first and critical step, we obtain an initial sparsifier with \( O(dm \log n / \epsilon^2) \) non-zeros for \( L_{G_r} \) using an upper bound estimate on the effective resistance of \( G_r \) obtained from \( L_G = D - A \) and \( L_{G_2} = D - AD^{-1}A \). In the second step, we apply the standard spectral sparsification algorithms to further reduce the number of nonzeros to \( O(n \log n / \epsilon^2) \).

In the first step sparsification, we bound the effective resistance on \( G_r \) using Lemma 3.1 (proof included in Appendix B), which allows us to use the resistance of any length-\( r \) path on \( \tilde{G} \) to upper
bound the effective resistance between its two endpoints on \( G_r \). Lemma 3.2 shows that we can sample by these estimates efficiently.

**Lemma 3.1 (Two-Step Supports).** For a graph Laplacian matrix \( L = D - A \), with diagonal matrix \( D \) and nonnegative off-diagonal \( A \), for all positive odd integer \( r \), we have

\[
\frac{1}{2}L_G \preceq L_{G_r} \preceq rL_G.
\]

(11)

and for all positive even integers \( r \) we have

\[
L_{G_2} \preceq L_{G_r} \preceq rL_{G_2}.
\]

(12)

For each length-\( r \) path \( p = (u_0 \ldots u_r) \) in \( G \), we have a corresponding edge in \( G_r \), with weight proportional to the chance of this particular path showing up in the random walk. We can view \( G_r \) as the union of these edges, i.e., \( L_{G_r}(u_0, u_r) = \sum_{p=(u_0,u_r)} w(p) \).

We bound the effective resistance on \( G_r \) in two different ways. If \( r \) is odd, \( G \) is a good approximation of \( G_r \), so we can obtain an upper bound using the resistance of a length-\( r \) (not necessarily simple) path on \( G \). If \( r \) is even, \( G_2 \) is a good approximation of \( G_r \), in this case, we get an upper bound by composing the effective resistance of 2-hop paths in different subgraphs of \( G_2 \).

**Lemma 3.2 (Upper Bound on Effective Resistance).** For a graph \( G \) with Laplacian matrix \( L = D - A \), let \( L_{G_r} = D - D(D^{-1}A)^r \) be its \( r \)-step random-walk matrix. Then, the effective resistance between two vertices \( u_0 \) and \( u_r \) on \( L_{G_r} \) is upper bounded by

\[
R_{G_r}(u_0, u_r) \leq \sum_{i=1}^{r} \frac{2}{A(u_{i-1}, u_i)},
\]

(13)

where \( (u_0 \ldots u_r) \) is a path in \( G \).

**Proof.** When \( r \) is a positive odd integer, by Lemma 3.1, we have that \( \frac{1}{2}L_G \preceq L_{G_r} \), which implies that for any edge \((u,v)\) in \( G \),

\[
R_{G_r}(u,v) \leq 2r_{G}(u,v) = \frac{2}{A(u,v)}.
\]

(14)

Because effective resistance satisfies triangular inequality, this concludes the proof for odd \( r \).

When \( r \) is even, by Lemma 3.1, we have that \( L_{G_2} \preceq L_{G_r} \). The effective resistance of \( D - AD^{-1}A \) is studied in [PS14] and restated in Appendix C. For the subgraph \( G_2(u) \) anchored at the vertex \( u \) (the subgraph formed by length-2 paths where the middle node is \( u \)), for any two of its neighbors \( v_1 \) and \( v_2 \), we have

\[
R_{G_r}(v_1,v_2) \leq R_{G_2(u)}(v_1,v_2) = \frac{1}{A(v_1,u)} + \frac{1}{A(u,v_2)}.
\]

(15)

Because we have the above upper bound for any 2-hop path \((v_1,u,v_2)\), by the triangular inequality of effective resistance, the lemma holds for even \( r \) as well.

Now we could sparsify \( G_r \) by sampling random walks according to the approximate effective resistance. The following mathematical identity will be crucial in our analysis.
Lemma 3.3 (A Random-Walk Identity). Given a graph $G$ with $m$ edges, consider the $r$-step random-walk graph $G_r$ and the corresponding Laplacian matrix $L_{G_r} = D - D(D^{-1}A)^r$. For a length-$r$ path $p = (u_0 \ldots u_r)$ on $G$, we have

$$w(p) = \frac{\prod_{i=1}^{r} A(u_{i-1}, u_i)}{\prod_{j=1}^{r-1} D(u_j, u_j)}.$$  

The summation of $w(p)Z(p)$ over all length-$r$ paths satisfies

$$\sum_p w(p) : Z(p) = 2rm. \quad (16)$$

Proof. We substitute the expression for $w(p)$ and $Z(p)$ in to the summation.

$$\sum_{p=(u_0 \ldots u_r)} w(p)Z(p) = \sum_p \left( \sum_{i=1}^{r} \frac{2}{A(u_{i-1}, u_i)} \right) \left( \frac{\prod_{j=1}^{r} A(u_{j-1}, u_{j})}{\prod_{j=1}^{r-1} D(u_j, u_j)} \right) \quad (17)$$

$$= 2 \sum_p \sum_{i=1}^{r} \left( \frac{\prod_{j=1}^{i-1} A(u_{j-1}, u_{j}) \prod_{j=1}^{r-1} A(u_{j}, u_{j+1})}{\prod_{j=1}^{r-1} D(u_j, u_j)} \right) \quad (18)$$

$$= 2 \sum\sum_{e \in G} \left( \sum_{p \text{ with } (u_{i-1}, u_i) = e} \frac{\prod_{j=1}^{i-1} A(u_{j-1}, u_{j}) \prod_{j=1}^{r-1} A(u_{j}, u_{j+1})}{\prod_{j=1}^{r-1} D(u_j, u_j)} \right) \quad (19)$$

$$= 2 \sum\sum_{e \in G} \left( \sum_{p \text{ with } (u_{i-1}, u_i) = e} \frac{\prod_{j=1}^{i-1} A(u_{j-1}, u_{j}) \prod_{j=1}^{r-1} A(u_{j}, u_{j+1})}{\prod_{j=1}^{r-1} D(u_j, u_j)} \right) \quad (20)$$

$$= 2 \sum\sum_{e \in G} \sum_{i=1}^{r-1} \frac{\prod_{j=1}^{i-1} A(u_{j-1}, u_{j}) \prod_{j=1}^{r-1} A(u_{j}, u_{j+1})}{\prod_{j=1}^{r-1} D(u_j, u_j)} \quad (21)$$

$$= 2mr. \quad (22)$$

From Equation 18 to 19, instead of enumerating all paths, we first fix an edge $e$ to be the $i$-th edge on the path, and then extend from both ends of $e$. From Equation 20 to 21, we sum over indices iteratively from $u_0$ to $u_{i-1}$, and from $u_i$ to $u_r$. Because $D(u, u) = \sum_v A(u, v)$, this summation over all possible paths anchored at $(u_{i-1}, u_i)$ equals to 1. \hfill \square

Now we show that we can perform Step (2) in GraphSampling efficiently. We take samples in the same way we cancel the terms in the previous proof. Recall that sampling an edge from $G_r$ corresponds to sampling a path of length $r$ in $G$.

Sample a path $p$ from $G$ with probability proportional to $\tau_p = w(p)Z(p)$:

a. Pick an integer $k \in [1 : r]$ and an edge $e \in G$, both uniformly at random.

b. Perform $(k-1)$-step random walk from one end of $e$.

c. Perform $(r-k)$-step random walk from the other end of $e$.

d. Keep track of $w(p)$ during the process, and finally add a fraction of this edge to our sparsifier.
Lemma 3.4. There exists an algorithm for the Step (2) in GraphSampling, such that after preprocessing with work $O(n)$, it can draw an edge $e$ as in Step (2) with work $O(r \cdot \log n)$.

Proof. The task is to draw a sample $p = (u_0 \ldots u_r)$ from the multivariate distribution $D$

$$Pr(u_0 \ldots u_r) = \frac{1}{2rm} \left( \sum_{i=1}^{r} \frac{2}{A(u_{i-1}, u_i)} \right) \cdot \left( \prod_{i=1}^{r} A(u_{i-1}, u_i) \prod_{j=1}^{r} D(u_i, u_i) \right). \tag{23}$$

For any fixed $k \in [1 : r]$ and $e \in G$, we can rewrite the distribution as

$$Pr(u_0 \ldots u_r) = Pr((u_{k-1}, u_k) = e) \cdot Pr(u_0 \ldots u_{k-2}, u_{k+1} \ldots u_r | (u_{k-1}, u_k) = e)$$
$$= Pr((u_{k-1}, u_k) = e) \cdot Pr(u_0 \ldots u_{k-2} | u_{k-1}) \cdot Pr(u_{r} \ldots u_{k+1} | u_{k})$$
$$= Pr((u_{k-1}, u_k) = e) \cdot \prod_{i=k-1}^{r} Pr(u_{i-1} | u_i) \cdot \prod_{i=k+1}^{r} Pr(u_i | u_{i-1}) \tag{24}$$

Note that $Pr((u_{k-1}, u_k) = e) = \frac{1}{m}$, and $Pr(u_{i-1} | u_i) = A(u_i, u_{i-1}) / D(u_i, u_i)$. The three terms in Equation 24 corresponds to Step (a)-(c) in the sampling algorithm stated above. With linear preprocessing time, we can draw an uniform random edge in time $O(\log n)$, and we can also simulate two random walks with total length $r$ in time $O(r \log n)$, so Step (2) in GraphSampling can be done within $O(r \log n)$ time. 

Combining this with spectral sparsifiers gives our algorithm for efficiently sparsifying low-degree polynomials.

Proof of Theorem 1.4. First we show on how to sparsify a degree-$d$ monomial $L_{G_d} = D - D \cdot (D^{-1}A)^d$. We use the sampling algorithm described in Theorem 2.2, together with upper bounds on effective resistance of $G_d$ obtained from $D - A$ and $D - AD^{-1}A$. The total number of samples requires is $O(dm \log n / \epsilon^2)$. We use Lemma 3.4 to draw a single edge from $G_r$, where we sample $d$-step random walks on $D - A$, so the total running time is $O(d^2 m \log^2 n / \epsilon^2)$. Now that we have a spectral sparsifier with $O(dm \log n / \epsilon^2)$ edges, we can sparsify one more time to reduce the number of edges to $O(n \log n / \epsilon^2)$ by [ST11, SS11] in time $O(dm \log^2 n / \epsilon^2)$.

To sparsify a random-walk matrix-polynomial $L_{\alpha}(G)$, we sparsify all the even/odd terms together, so the upper bound of effective resistance in Lemma 3.2 still holds. To sample an edge, we first decide the length $r$ of the path, according to the probability distribution $Pr(\text{length} = r | r \text{ is odd/even}) \propto \alpha_r$. 

4 Sparsification of Higher Degree Matrix-Monomials

We now put together the components of the algorithm for sparsifying higher degree monomials. Given a positive even integer $2r$ and $\tilde{A}$ such that $D - \tilde{A} \approx_\epsilon D - D(D^{-1}A)^{2r}$, we will show that we can efficiently compute

1. $\tilde{A}_x$ such that $D - \tilde{A}_x \approx_\epsilon D - AD^{-1}A \approx_\epsilon L_{G_{4r}}$ in Subsection 4.1, and

2. $\tilde{A}_+ \approx_\epsilon D - (AD^{-1})^2 \tilde{A}(D^{-1}A)^2 \approx_\epsilon L_{G_{2r+4}}$ in Subsection 4.2.

Putting these together then gives our main theorem about sparsifying high degree monomials from Theorem 1.5.
4.1 Constructing $\tilde{A}_x$

We construct $\tilde{A}_x$ by sparsifying $D - \tilde{A}D^{-1}\tilde{A}$ with the algorithm described in Section 3. The remaining is to show that spectral approximation holds under squaring, i.e.,

$$D - \tilde{A} \approx \epsilon' D - D(D^{-1}A)^{2r}$$

(25)

$$\Rightarrow D - \tilde{AD}^{-1}\tilde{A} \approx \epsilon' D - D(D^{-1}A)^{4r},$$

(26)

which directly follows Lemma 4.3 and Lemma 4.4. The result is summarized in the following lemma.

**Lemma 4.1.** Let the graph Laplacian $L_G = D - A$ and $L_{\tilde{G}_x} = D - \tilde{A}$ for $r \in \mathbb{Z}_+$, such that (1) $D - \tilde{A} \approx \epsilon' D - D(D^{-1}A)^{2r}$, and (2) $\tilde{A}$ contains $m$ nonzero entries, then for any $\epsilon > 0$, we can construct in total work $O(m \log^3 n/\epsilon)$, a graph Laplacian $L_{\tilde{G}_x} = D - \tilde{A}_x$ with $\tilde{A}_x$ containing at most $O(n \log^2 n/\epsilon^2)$ nonzero entries, such that

$$D - \tilde{A}_x \approx \epsilon' \epsilon D - D(D^{-1}A)^{4r}.$$ 

(27)

First, we will start with the fact that $D - AD^{-1}A$ is the Schur complement of the matrix

$$\begin{bmatrix} D & -A \\ -A & D \end{bmatrix},$$

onto the second half of the vertices.

This is to allow the use of the following Lemma regarding Schur complement:

**Fact 4.2** (Lemma B.1. from [MP13]). Suppose $M$ and $\tilde{M}$ are positive semi-definite matrices satisfying $M \approx \epsilon \tilde{M}$, then their Schur complements on the same set of vertices also satisfy $M_{\text{schur}} \approx \epsilon \tilde{M}_{\text{schur}}.$

We also need the following facts about even random walks.

**Lemma 4.3.** If $0 \nless A$ and

$$(1 - \epsilon) (D - A) \nless D - \tilde{A} \nless (1 + \epsilon) (D - A),$$

(28)

then

$$(1 - \epsilon) (D + A) \nless D + \tilde{A} \nless (1 + \epsilon) (D + A).$$

(29)

**Proof.** Rearranging the leftmost condition in Equation 28 gives

$$\tilde{A} \nless \epsilon D + (1 - \epsilon) A$$

(30)

Adding $D$ to both sides then gives

$$D + \tilde{A} \nless (1 + \epsilon) D + (1 - \epsilon) A.$$ 

(31)

Combining with $0 \nless A$ gives

$$D + \tilde{A} \nless (1 + \epsilon) (D + A).$$

(32)

Similarly, we can prove

$$(1 - \epsilon) (D + A) \nless D + \tilde{A}.$$ 

(33)

$\square$
Lemma 4.4. If $D - A \approx \epsilon$, $D - \tilde{A}$ and $D + A \approx \epsilon$, $D + \tilde{A}$, then

$$D - AD^{-1}A \approx D - \tilde{AD}^{-1}\tilde{A}. \quad (34)$$

Proof. Because of Lemma 4.2, it suffices to show that

$$\begin{bmatrix} D & -A \\ -A & D \end{bmatrix} \approx \epsilon \begin{bmatrix} D & -\tilde{A} \\ -\tilde{A} & D \end{bmatrix}. \quad (35)$$

Consider a test vector

$$\begin{bmatrix} x \\ y \end{bmatrix},$$

we have

$$\begin{bmatrix} x^\top & y^\top \end{bmatrix} \begin{bmatrix} D & -A \\ -A & D \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \left[ (x + y)^\top (D - A)(x + y)^\top + (x - y)^\top (D + A)(x - y)^\top \right]. \quad (36)$$

which leads to Equation 35, since we have $D - A \approx \epsilon D - \tilde{A}$ and $D + A \approx \epsilon D + \tilde{A}$.

4.2 Constructing $\tilde{A}_+$

We can then extend this squaring routine by composing its walk with smaller random walks on each side. That is, we sparsify $D - (AD^{-1})^2 \tilde{A} (D^{-1}A)^2$ with the support of $D - AD^{-1}A$ and $D - \tilde{A}$. First, we need to prove that the symmetric composition preserve the spectral approximation.

Lemma 4.5. If $D - A \approx \epsilon D - \tilde{A}$, then for any symmetric $\tilde{A}$ such that $0 \preceq \tilde{A} \preceq D$, we have

$$D - \tilde{AD}^{-1}AD^{-1}\tilde{A} \approx D - \tilde{AD}^{-1}\tilde{AD}^{-1}\tilde{A}. \quad (37)$$

Proof. Since $\tilde{AD}^{-1} = (D^{-1}\tilde{A})^\top$, we can compose the given identity with $D^{-1}\tilde{A}$ using Fact 2.1.5 to give:

$$\left( \tilde{AD}^{-1} \right)(D - A)(D^{-1}\tilde{A}) \approx \epsilon \left( \tilde{AD}^{-1} \right)(D - \tilde{A})(D^{-1}\tilde{A}),$$

or if expanded:

$$\tilde{AD}^{-1}\tilde{A} - \tilde{AD}^{-1}AD^{-1}\tilde{A} \approx \epsilon \tilde{AD}^{-1}\tilde{A} - \tilde{AD}^{-1}\tilde{AD}^{-1}\tilde{A}$$

Also, since $D - \tilde{A} \succ 0$, we have

$$D - \tilde{AD}^{-1}\tilde{A} \succ 0,$$

which allows us to write an approximation of the form

$$D - \tilde{AD}^{-1}\tilde{A} \approx \epsilon D - \tilde{AD}^{-1}\tilde{A},$$

and add it to the approximation above using Fact 2.1.2. This turns the $\tilde{AD}^{-1}\tilde{A}$ terms into $D$ terms, and completes the proof.

Now we prove that, with the support of $D - AD^{-1}A$ and $D - \tilde{A}$, there exists an efficient algorithms to conduct the edge sampling, which leads to the efficient sparsification.
Lemma 4.6. For graph Laplacian $L_G = D - A$ and $L_{\tilde{G}_r} = D - \tilde{A}$ for $r \in \mathbb{Z}_+$, such that (1) $D - \tilde{A} \approx_{1/\epsilon} D - D (D^{-1} A)^{2r}$, and (2) $\tilde{A}$ and $A$ each contains at most $m$ nonzero entries, then for any $\epsilon > 0$, we can construct in total work $O(m \log^3 n / \epsilon^4)$, a graph Laplacian $L_{\tilde{G}_+} = D - \tilde{A}_+$ with $\tilde{A}_+$ containing at most $O(n \log n / \epsilon^2)$ nonzero entries, such that

$$D - \tilde{A}_+ \approx_{1/\epsilon'} D - D (D^{-1} A)^{2r+4}.$$ (37)

Proof. By Lemma 4.5, we have

$$D - (AD^{-1})^2 \tilde{A} (D^{-1} A)^2 \approx_{1/\epsilon} D - (AD^{-1})^2 D (D^{-1} A)^{2r} (D^{-1} A)^2 = D - D (D^{-1} A)^{2r+4}.\hspace{1cm} (38)$$

Next step is to support $D - (AD^{-1})^2 \tilde{A} (D^{-1} A)^2$ with $D - \tilde{A}$ and $D - AD^{-1} A$.

First, we have

$$\exp(-\epsilon) (D - AD^{-1} A) \approx \exp(-\epsilon) \left( D - D (D^{-1} A)^{2r+4} \right) \approx D - (AD^{-1})^2 \tilde{A} (D^{-1} A)^2.\hspace{1cm} (39)$$

We can also show that

$$\exp(-2\epsilon) \left( D - \tilde{A} \right) \approx \exp(-\epsilon) \left( D - D (D^{-1} A)^{2r} \right) \approx \exp(-\epsilon) \left( D - D (D^{-1} A)^{2r+4} \right) \approx D - (AD^{-1})^2 \tilde{A} (D^{-1} A)^2.\hspace{1cm} (40)$$

To sample $(AD^{-1})^2 \tilde{A} (D^{-1} A)^2$ efficiently, let’s consider the length-5 path

$$p = (u_0, u_1, u_2, u_3, u_4, u_5),$$

where $(u_0, u_1), (u_1, u_2), (u_3, u_4), (u_4, u_5)$ are edges in $A$ and $(u_2, u_3)$ in $\tilde{A}$.

The edge corresponds to $p$ has weight $w(p)$ as

$$w(p) = \frac{A(u_0, u_1) A(u_1, u_2) \tilde{A}(u_2, u_3) A(u_3, u_4) A(u_4, u_5)}{D(u_0, u_1) D(u_1, u_2) D(u_2, u_3) D(u_3, u_4)}.\hspace{1cm} (42)$$

And it has upper bound on effective resistance $Z(p)$ as

$$Z(p) = \frac{\exp(\epsilon)}{A(u_0, u_1)} + \frac{\exp(\epsilon)}{A(u_1, u_2)} + \frac{\exp(2\epsilon)}{A(u_2, u_3)} + \frac{\exp(\epsilon)}{A(u_3, u_4)} + \frac{\exp(\epsilon)}{A(u_4, u_5)}.$$ (43)

The sampling algorithm is similar to that described in Lemma 3.4, with the middle edge random-walk replaced with the random walk on graph $\tilde{A}$. Also, the edge index in the path is not uniform among 1, 2, . . . , 5 as in Lemma 3.4, i.e., it is now proportional to number of edges in the corresponding random-walk and the additional $\exp(\epsilon)$ terms occurred from the chain of spectral approximation. However, the total work for path sampling remains the same for $\epsilon = O(1)$. After the initial sparsification by path sampling, can we further sparsify the graph with spectral sparsifiers. \qed
4.3 Combining $A_+$ and $A_\times$

Combining these two components allows us to prove our main result on sparsifying high degree monomials.

Proof of Theorem 1.5. When $d$ is divisible by 4, we start with $\tilde{A}$ such that $D - \tilde{A} \approx \epsilon' D - A D^{-1} A$, and invoke Lemma 4.1 and Lemma 4.6 $k = O(\log d)$ times in total, to reach an approximation for $D - D(D^{-1} A)^d$. Moreover, if we invoke Lemma 4.1 and Lemma 4.6 with approximation parameter $\epsilon/(2k)$, and apply [ST11] with $\epsilon/2$ on the final output to obtain $\tilde{G}$, the total work is bounded by $O(m \log^3 n \log^5 d)/\epsilon^4$, and $\tilde{G}$ satisfies $L_{\tilde{G}} \approx L_G$.

When $d$ is equal to $4r + 2$ for some integer $r$, we first check if $d \leq 4/\epsilon$, in which case we can directly use Theorem 1.4 to sparsify $L_G$. When $d > 4/\epsilon$, we can produce a sparsifier for degree $4r$ monomial with error $\epsilon/2$, $L_{\tilde{G}} \approx \epsilon/2 L_{G_{4r}}$ and use it directly. This is because

$$(1 - \lambda^{4r}) \leq (1 - \lambda^{4r+2}) \leq (1 + \frac{1}{2r})(1 - \lambda^{4r}) \leq (1 + \frac{\epsilon}{2})(1 - \lambda^{4r}), \quad \forall \lambda \in (-1, 1) \text{ and integer } r. \quad (44)$$

So in this situation, we know that $L_{G_{4r}} \approx \epsilon/2 L_{G_d}$. Combining the two spectral approximation using Fact 2.1.3 gives $L_{\tilde{G}} \approx L_{G_d}$. □

Note that $G_d$ with even $d$ is usually enough the study the long term effect of the random-walk\(^3\). However, it is an intriguing question both mathematically and algorithmically if one can sparsify $G_d$ for all $d$ with work polylogarithmic in $d$.

5 Extension to SDDM Matrices

Our path sampling algorithm from Section 3 can be generalized to a SDDM matrix with splitting $D - A$. The idea is to split out the extra diagonal entries to reduce it back to the Laplacian case. Of course, the $D^{-1}$ in the middle of the monomial is changed, however, it only decreases the true effective resistance so the upper bound in Lemma 3.2 still holds without change. The main difference is that we need to put back the extra diagonal entries, which is done by multiplying an all 1 vector through $D - D(D^{-1} A)^r$.

The follow Lemma can be proved similar to Lemma 1.2.

**Lemma 5.1** (SDDM Preservation). If $M = D - A$ is an SDDM matrix with diagonal matrix $D$ and nonnegative off-diagonal $A$, for any nonnegative $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\sum_{r=1}^d \alpha_r = 1$, $M_\alpha = D - \sum_{r=1}^d \alpha_r D(D^{-1} A)^r$ is also an SDDM matrix.

Our algorithm is a direction modification of the algorithm from Section 3. To analyze it, we need a variant of Lemma 3.2 that bounds errors w.r.t. the matrix by which we measure effective resistances. We use the following statement from [Pen13].

**Lemma 5.2** (Lemma B.0.1. from [Pen13]). Let $A = \sum_i y_i^T y_e$ and $B$ be $n \times n$ positive semi-definite matrices such that the image space of $A$ is contained in the image space of $B$, and $\tau$ be a set of estimates such that $\tau_e \geq y_e^T B^1 y_e \quad \forall e$.

\(^3\)The obvious exception is that when the random-walk is periodic
Then for any error $\epsilon$ and any failure probability $\delta = n^{-d}$, there exists a constant $c_\delta$ such that if we construct $\hat{A}$ using the sampling process from Figure 1, with probability at least $1 - \delta = 1 - n^{-d}$, $\hat{A}$ satisfies:

$$\hat{A} - \epsilon(A + B) \preceq \hat{A} \preceq A + \epsilon(A + B).$$

**Theorem 5.3.** Let $M = D - A$ be an SDDM matrix with diagonal $D$, nonnegative off-diagonal $A$ with $m$ nonzero entries, for any nonnegative $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $\sum_{r=1}^d \alpha_r = 1$, we can define $M_\alpha = D - \sum_{r=1}^d \alpha_r(D^{-1}A)^r$. For any approximation parameter $\epsilon > 0$, we can construct an SDDM matrix $\tilde{M}$ with $O(n \log n/\epsilon^2)$ nonzero entries, in time $O(m \cdot \log^2 n \cdot d^2/\epsilon^2)$, such that $\tilde{M} \approx \epsilon M_\alpha$.

**Proof.** We look at each monomial separately. First, by Lemma 5.1, $M_r$ is an SDDM matrix. It can be decomposed as the sum of two matrices, a Laplacian matrix $L_r = D_r - D(D^{-1}A)^r$, and the remaining diagonal $D_{\text{extra}}$. As in the Laplacian case, a length-$r$ paths in $D - A$ corresponds to an edge in $L_r$. We apply Lemma 5.2 to $M_r$ and $L_r = \sum_{e \in P} e_y e^\top$, where $P$ is the set of all length-$r$ paths in $D - A$, and $e_y$ is the column of the incidence matrix associated with $e$.

When $r$ is an odd integer, we have

$$y_e (M_r)^{-1} y_e \leq 2y_e (D - A)^{-1} y_e,$$

and when $r$ is an even integer, we have

$$y_e (M_r)^{-1} y_e \leq 2y_e (D - AD^{-1}A)^{-1} y_e.$$  

Let $e$ denote the edge corresponds to the length-$r$ path $(u_0, \ldots, u_r)$, the weight of $e$ is

$$w(e) = w(u_0 \ldots u_r) = y_e^\top y_e = \prod_{i=1}^r A(u_{i-1}, u_i) \leq \prod_{i=1}^r A(u_{i-1}, u_i) / \prod_{i=1}^r D_g(u_i, u_i).$$

where $D_g(u, v) = \sum_{v \neq u} A(u, v)$, so we have the same upper bound as the Laplacian case, and we can sample random walks in the exact same distribution. By Lemma 5.2 there exists $M = O(r \cdot m \cdot \log n/\epsilon^2)$ such that with probability at least $1 - \frac{1}{n}$, the sampled graph $\tilde{G} = \text{GRAPHSAMPLING}(G_r, \tau_\epsilon, M)$ satisfies

$$M_r - \frac{1}{2} \epsilon(L_r + M_r) \preceq L_{\tilde{G}} + D_{\text{extra}} \preceq M_r + \frac{1}{2} \epsilon(L_r + M_r).$$

Now if we set $\tilde{M} = L_{\tilde{G}} + D_{\text{extra}}$, we will have

$$(1 - \epsilon)M_r \preceq \tilde{M} \preceq (1 + \epsilon)M_r.$$ 

Note that $D_{\text{extra}}$ can be computed efficiently by computing $\text{diag}(M, 1)$ via matrix-vector multiplications. 

**6 Remarks**

We gave nearly-linear time algorithms for generating sparse approximations of several classes of random-walk matrix-polynomials. As our study of this problem is motivated by the low degree case such as for speeding up numerical methods for solving matrix equations, our results only gives part
of the picture for the high degree case: we are only able to sparsify even-degree monomials, and this routine calls spectral sparsifiers with error of $\epsilon = 1/\log d$ at each step. Obtaining better algorithms for approximating the structures of long-range random-walk matrices is an intriguing mathematical and algorithmic question, partially due to the non-commutativity of matrix products. Extending our algorithm to any $d$, and allowing for higher error tolerances at each step are natural directions for future work. Furthermore, we conjecture that any degree $n$ random-walk matrix-polynomial can be sparsified in nearly-linear time.

Our algorithms for the low degree case is based on path sampling. This routine has analogs in widely used combinatorial network analysis routines such as distance estimation [KTF09] and subgraph counting [JG05]. We believe further investigating this connection will lead to improved algorithms, as well as models that better explain the effectiveness of many existing ones.

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A Laplacian Preservation

Proposition 1.2 (Laplacian Preservation). For any weighted, undirected graph $G$ with adjacency matrix $A$ and diagonal matrix $D$, for every non-negative vector $\alpha = (\alpha_1, ..., \alpha_d)$ such that with $\sum_{r=1}^{d} \alpha_r = 1$, the random-walk matrix-polynomial $L_\alpha(G)$ remains a Laplacian matrix.
Proof. First note that $L_\alpha(G) = \sum_{r=1}^{d} \alpha_r D(D^{-1}A)^r$ is symmetric and has non-positive off-diagonals, so to prove that $L_\alpha(G)$ is also a Laplacian matrix, we only need to show the off-diagonals sum to the diagonal. Fix an integer $r$ and a row index $i$, we study the $i$-th row sum $S_r$ of $D(D^{-1}A)^r$.

For $r = 1$, we have that the row sum $S_1$ of $i$-th row of $A$ gives $S_1 = \sum_j A_{i,j} = D_{i,i}$. We show that the row sum $S_{r+1}$ can be reduce to $S_r$ as follows,

$$S_{r+1} = \sum_k \left( (D(D^{-1}A)^r)_{i,k} \cdot D^{-1}_{k,k} \cdot \sum_j A_{k,j} \right) = \sum_k (D(D^{-1}A)^r)_{i,k} = S_r \quad (50)$$

By induction, we have that $S_n = \cdots = S_1 = D_{i,i}$. Thus, the $i$-th row sum of $L_\alpha(G)$

$$\sum_j (L_\alpha(G))_{i,j} = \sum_{r=1}^{t} \alpha_r S_r = D_{i,i}. \quad (51)$$

Therefore, $L_\alpha(G)$ is a Laplacian matrix. □

B Support from Linear and Quadratic Terms

Lemma 3.1 (Two-Step Supports). For a graph Laplacian matrix $L = D - A$, with diagonal matrix $D$ and nonnegative off-diagonal $A$, for all positive odd integer $r$, we have

$$\frac{1}{2} L_G \preceq L_G r \preceq L_G. \quad (11)$$

and for all positive even integers $r$ we have

$$L_G 2 \preceq L_G r \preceq \frac{r}{2} L_G. \quad (12)$$

Proof. Let $X = D^{-\frac{1}{2}} A D^{-\frac{1}{2}}$, for any integer $r$, the statements are equivalent to

$$\frac{1}{2} (I - X) \preceq I - X^{2r+1} \preceq (2r + 1) (I - X) \quad (52)$$

$$I - X^2 \preceq I - X^{2r} \preceq r (I - X^2). \quad (53)$$

Because $X$ can be diagonalized by unitary matrix $U$ as $\Lambda = UXU^\top$, where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\lambda_i \in [-1, 1]$ for all $i$. Therefore we can reduce the inequalities to the scalar case, and we conclude the proof with the following inequalities:

$$\frac{1}{2}(1 - \lambda) \leq 1 - \lambda^{2r+1} \leq (2r + 1) (1 - \lambda), \quad \forall \lambda \in (-1, 1) \text{ and odd integer } r;$$

$$1 - \lambda^2 \leq 1 - \lambda^{2r} \leq r (1 - \lambda^2), \quad \forall \lambda \in (-1, 1) \text{ and even integer } r. \quad (54)$$

□
C Effective Resistance on Rank One Graph

Proposition C.1 (Claim 6.3. from [PS14]). Given a graph of size \( n \) with the Laplacian matrix \( L = D - \frac{1}{d}aa^\top \), where \( D_{i,i} = (a_i s)/d \) with \( s = \sum_{i=1}^{n} a_i \). The effective resistance for edge \((i, j)\) is

\[
\frac{d}{s} \left( \frac{1}{a_i} + \frac{1}{a_j} \right).
\]

Proof. Let \( e_i \) denote the vector where the \( i \)-th entry is 1, and 0 everywhere else. We have

\[
dL \left( \frac{e_i - e_j}{a_i - a_j} \right) = (s - a_i)e_i - \sum_{k \neq i} a_k e_k - (s - a_j)e_j + \sum_{k \neq j} a_k e_k = s(e_i - e_j). \tag{56}
\]

Therefore

\[
(e_i - e_j)^\top L^\top (e_i - e_j) = \frac{d}{s} (e_i - e_j)^\top \left( \frac{e_i}{a_i} - \frac{e_j}{a_j} \right) = \frac{d}{s} \left( \frac{1}{a_i} + \frac{1}{a_j} \right).
\]

\[\square\]