Coherent States
of
Non-Linear Lie algebras:
Applications in Quantum Optics.

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Abstract

We present a general unified approach for finding the coherent states of polynomially deformed algebras such as the quadratic and Higgs algebras, which are relevant for various multiphoton processes in quantum optics. We give a general procedure to map these deformed algebras to appropriate Lie algebras. This is used, for the non compact cases, to obtain the annihilation operator coherent states, by finding the canonical conjugates of these operators. Generalized coherent states, in the Perelomov sense also follow from this construction. This allows us to explicitly construct coherent states associated with various quantum optical systems.

1 Introduction

Till recently, in quantum optics, only linear Lie Algebras have been used to give multiphoton coherent (including squeezed) states. However in other fields of physics, particularly string theories, both infinite dimensional algebras as well as non-linear "dynamical algebras" have been utilised for highly non-linear systems. Notable among these are the infinite dimensional Kac Moody and W Algebras and "q-algebras" and the finite dimensional Polynomial algebras such as the quadratic and "Higgs" algebras \cite{1, 2}. The quadratic algebra was discovered by Sklyanin \cite{3, 4}, in the context of statistical physics and field theory. The well-known Higgs algebra, a cubic algebra, was manifest in the study of the dynamical symmetries of the quantum oscillator and the Coulomb problem in a space of constant curvature\cite{5, 6}. These algebras have now found a place in quantum optics with the observation that quantum optical Hamiltonians describing multiphoton processes have symmetries which can be described by polynomially deformed SU(1,1) and SU(2) algebras \cite{7, 8}. In particular, as we shall see later, the trilinear boson Hamiltonian associated with various nonlinear processes in quantum optics, such as parametric amplification, frequency conversion, Raman and Brillouin scattering, and the interaction of two-level atoms with a single-mode radiation field, has the quadratically deformed SU(1,1) algebra as a dynamical symmetry. \cite{9}. Similarly the symmetry algebra for the quadratic boson
Hamiltonian for four photon processes is the Higgs algebra. By this we mean that the Hamiltonian can be written as \( H = aN_0 + bN_+ + cN_- \) with the N’s satisfying a quadratic algebra or a Higgs Algebra.

In this paper, we present a unified approach for finding the coherent states (CS) of these algebras \([10,11]\). Apart from its application to quantum optics, the method of construction presented here is quite general and will greatly facilitate the physical applications of these algebras to many quantum mechanical problems. The construction of the CS for the non-compact cases, is a two step procedure. First, we find the canonical conjugates of these operators. The CS, corresponding to the deformed algebras are then obtained by the action of the exponential of the respective conjugate operators on the vacuum \([12,13,14]\); this is in complete parallel to the harmonic oscillator case. Another CS, which in a sense to be made precise in the text, is dual to the first one, naturally follows from the above construction. We also provide a mapping between the deformed algebras and their undeformed counterparts. This connection is then utilized to find the CS in the Perelomov sense [11]. Apart from obtaining the known CS of the \( SU(1,1) \) algebra, we construct the CS for the quadratic, cubic and higher order polynomial algebras. Although our method is general, we will confine ourselves here to finding the CS of the deformed \( SU(1,1) \) and \( SU(2) \) algebras.

2 Polynomial Algebras as Symmetries of Multiphoton Hamiltonians.

A Polynomial deformation of a Lie algebra is defined in the following fashion in the Cartan-Weyl basis,

\[
\left[ H , E_{\pm} \right] = \pm E_{\pm} \ , \quad \left[ E_+ , E_- \right] = f(H) \ ,
\]

where \( f(H) \) is a polynomial function of H. The corresponding Casimir can be written in the form \([15]\),

\[
C = E_- E_+ + g(H) \ ,
\]

\[
= E_- E_+ + g(H - 1) .
\]

Here, \( f(H) = g(H) - g(H - 1) \). \( g(H) \) can be determined up to the addition of a constant. The eigenstates are characterized by the values of the Casimir operator and the Cartan subalgebra \( H \).

In particular, a polynomial deformation of \( SL(2,R) \) is of the form \( N_0 = j_0 , N_+ = F(j_0,j)j_+ , N_- = F(j_0,j)j_- \) where the \( j_i \) are the ordinary \( SL(2,R) \) generators. \( [N_0,N_{\pm}] = \pm N_{\pm} \) and \( [N_+,N_-] = F(N_0) \) \([15,16]\). When \( F(N_0) \) is quadratic in \( N_0 \) the algebra is called quadratic algebra and if it is cubic in \( N_0 \) the "Higgs" algebra results.

When we consider Hamiltonians describing multiphoton processes it is not difficult to see that the symmetries of such Hamiltonians form non-linear Lie algebras. For example consider the triboson Hamiltonian,

\[
H = w_1 a_1^\dagger a_1 + w_2 a_2^\dagger a_2 + w_3 a_3^\dagger a_3 + \kappa a_1^\dagger a_2^\dagger a_3 + \kappa a_1^\dagger a_2 a_3
\]

Defining

\[
N_0 = (-a_1^\dagger a_1 + a_2^\dagger a_2 + a_3^\dagger a_3)/2
\]

\[
N_+ = a_1^\dagger a_2^\dagger a_3
\]

\[
N_- = a_1^\dagger a_2 a_3
\]

one can show that \( N_+ , N_- , N_0 \) satisfy a quadratic algebra.

Another example is that of the anharmonic oscillator with the Hamiltonian given by:

\[
H = \frac{1}{m} (a^\dagger a + \frac{1}{2}) + \frac{1}{n} (b^\dagger b + \frac{1}{2})
\]

The operators \( N_+ = a^m (b^\dagger)^n, N_- = b^n (a^\dagger)^n, N_0 = \frac{1}{m} (a^\dagger a + \frac{1}{2}) - \frac{1}{n} (b^\dagger b + \frac{1}{2}) \) commute with \( H \) and form a quadratic algebra for \( m=1, n=2 \) with:

\[
f(H) = -3N_0^2 + 2HN_0 + H^2 - 3/4.
\]
It can be brought to a more recognizable form after performing an invertible basis transformation:

\[ J_+ = \frac{1}{\sqrt{3}} N_+, \quad J_- = -\frac{1}{\sqrt{3}} N_-, \quad J_0 = N_0 - H/3, \]

where \( J_+, J_-, J_0 \) satisfy quadratic algebra with \( f(J_0) = J_0^2 + \left( \frac{1}{4} - \frac{4}{9} \right) H^2 \).

Thus the quadratic algebra represents the larger dynamical symmetry group of the anisotropic Harmonic oscillator.

For general multiphoton Hamiltonians:

\[ H = \sum_{i=0}^{1} w_i a_i^\dagger a_i + \kappa (a_0)^m (a_1^\dagger)^n + c.c \]  

we can define \( N_0, N_-, N_+ \) in an analogous way

\[ N_+ = a_0^m (a_1^\dagger)^n \]
\[ N_- = a_1^\dagger (a_0^\dagger)^m \text{ nonumber} \]
\[ N_0 = \frac{1}{m+n} (a_1^\dagger a_1 - a_0^\dagger a_0) \]

and show that we get n-dimensional Polynomial algebras as symmetry algebras for these Hamiltonians.

Similarly n-photon Dicke Models with Hamiltonians of the form:

\[ H = \sum \sigma_0 \epsilon + w_1 a_1^\dagger a_1 + \kappa \sum \sigma_-(i)(a_1^\dagger)^n \kappa \sum \sigma_+(i)(a_1)^n \]  

can be written as:

\[ H = aN_0 + bN_+ + cN_- \]

with

\[ N_0 = \sum \sigma_0 + a_1^\dagger a_1 \]
\[ N_- = \sum \sigma_-(i)(a_1^\dagger)^n \]
\[ N_+ = \sum \sigma_+(i)(a_1)^n \]

satisfying a polynomial Lie Algebra of order n.

### 3 Construction of Coherent States of Non-Linear Lie Algebras:Formalism.

Having seen that polynomially deformed Lie algebras occur in a large class of systems in quantum mechanics and quantum optics in particular, we now give the formalism for the construction of coherent states of these algebras. We do so by an ”Inonu-Wigner” type for construction of finding canonical conjugate of the annihilation operator and find the eigenstate of the annihilation operator by acting the exponential of the conjugate operator on the vacuum. This approach is then extended to the deformed algebras in a straightforward way. In the next section we shall show how this formalism can be used to explicitly construct the coherent states for application to multiphoton processes.

First introduce our method by first considering SU(1,1) for which the generators satisfy the commutation relations

\[ [K_+, K_-] = -2K_0, \quad [K_0, K_\pm] = \pm K_\pm. \]
Thus for this case one finds, \( f(K_0) = -2K_0 \) and \( g(K_0) = -K_0(K_0 + 1) \). The quadratic Casimir operator is given by 
\[
C = K_+ K_+ + g(K_0) = K_+ K_+ - K_0(K_0 + 1) .
\]
\( \tilde{K}_+ \), the canonical conjugate of \( K_- \), satisfies
\[
[K_-, \tilde{K}_+] = 1 ,
\]
can be written in the form,
\[
\tilde{K}_+ = K_+ F(C, K_0) .
\]
Eq.(11) then yields,
\[
F(C, K_0)K_- K_+ - F(C, K_0 - 1)K_+ K_- = 1 ;
\]
making use of the Casimir operator relation given earlier, one can solve for \( F(C, K_0) \) in the form,
\[
F(C, K_0) = \frac{K_0 + \alpha}{C + K_0(K_0 + 1)} .
\]
The constant, arbitrary, parameter \( \alpha \) in \( F \) can be determined by demanding that Eq.(11) is valid in the entire Hilbert space. For the purpose of clarity, we illustrate this point, with the one oscillator realization of the \( SU(1,1) \) generators.

The ground states defined by \( K_- \mid 0 > = \frac{1}{2} a^2 \mid 0 > = 0 \), are, \( | 0 > \) and \( | 1 > = a^\dagger \mid 0 > \), in terms of the oscillator Fock space. Making use of the results,
\[
K_0 \mid 0 > = \frac{1}{4} (2a^\dagger a + 1) \mid 0 > = \frac{1}{4} \mid 0 > ,
\]
and
\[
C \mid 0 > = \frac{3}{16} \mid 0 > ,
\]
we find that, \( [K_-, \tilde{K}_+] \mid 0 > = K_- \tilde{K}_+ \mid 0 > \), yields \( \alpha = \frac{3}{4} \). Similarly, for the other case
\[
[K_-, \tilde{K}_+] \mid 1 > = \mid 1 > ,
\]
leads to \( \alpha = \frac{1}{4} \). Hence, there are two disjoint sectors characterized by the \( \alpha \) values \( \frac{3}{4} \) and \( \frac{1}{4} \), respectively. These results match identically with the earlier known ones [13], once we rewrite \( F \) as,
\[
F(C, K_0) = \frac{K_0 + \alpha}{C + K_0(K_0 + 1)} ,
\]
\[
= \frac{K_0 + \alpha}{K_- K_+} .
\]
The unnormalized coherent state \( \mid \beta > \), which is the annihilation operator eigenstate, i.e, \( K_- \mid \beta > = \beta \mid \beta > \), is given in the vacuum sector by
\[
\mid \beta > = e^{\beta \tilde{K}_+^\dagger} \mid 0 > .
\]
Analogous construction holds in the other sector, where \( \alpha = \frac{1}{4} \). These states, which provide a realization of the Cat states[17], play a prominent role in quantum measurement theory. As has been noticed earlier [13], \( [K_-, \tilde{K}_+] = 1 \), also yields,
\[
[\tilde{K}_+^\dagger, K_+] = 1 .
\]
From this, one can find the eigenstate of \( \tilde{K}_+^\dagger \) operator, in the form,
\[
\mid \gamma > = e^{\gamma \tilde{K}_+} \mid 0 > .
\]
This CS, after proper normalization is the well-known Yuen state\cite{8}. Our construction can be easily generalized to various other realizations of the $SU(1,1)$ algebra.

We now extend the above procedure to the quadratic algebra which is the relevant algebra in considering the coherent state of trilinear boson Hamiltonians\cite{23}. A example of such an algebra is the following:

$$[N_0, N_\pm] = \pm N_\pm \ , \ [N_+, N_-] = \pm 2N_0 + aN_0^2 \ .$$  \hspace{1cm} (25)$$

where the positive sign of $2N_0$ indicates a polynomially deformed $SU(2)$ and a negative sign indicates polynomially deformed $SU(1,1)$. In this case, $f_1(N_0) = \pm 2N_0 + aN_0^2 = g_1(N_0) - g_1(N_0 - 1)$, where,

$$g_1(N_0) = \pm N_0(N_0 + 1) + \frac{a}{3}N_0(N_0 + 1)(N_0 + \frac{1}{2}) \ .$$  \hspace{1cm} (26)$$

Representation theory of the quadratic algebra has been studied in the literature\cite{13}; it shows a rich structure depending on the values of ‘a’. In the non-compact case, i.e, for the values of ‘a’ such that the unitary irreducible representations (UIREP) are either bounded below or above, we can construct the canonical conjugate $\bar{N}_+$ of $N_-$ such that $[N_-, \bar{N}_+] = 1$. It is given by $\bar{N}_+ = N_+ F_1(C, N_0)$, with

$$F_1(C, N_0) = \frac{N_0 + \delta}{C - N_0(N_0 + 1) - \frac{a}{6}N_0(N_0 + 1)(N_0 + \frac{1}{2})} \ .$$  \hspace{1cm} (27)$$

As can be easily seen, in the case of the finite dimensional UIREP, $\bar{N}_+$ is not well defined since $F_1(C, N_0)$ diverges on the highest state. As mentioned earlier, the values of $\delta$ can be fixed by demanding that the relation, $[N_-, \bar{N}_+] = 1$, holds in the vacuum sector $|v >$ where, $|v >$ are annihilated by $N_-$. This gives $N_- \bar{N}_+ |v > = |v >$, which leads to $(N_0 + \delta) |v > = |v >$, the value of the Casimir operator, $C = N_- N_+ + aN_0$, can be easily calculated. Hence, the unnormalized coherent state $|\mu >; \ N_- | \mu > = \mu | \mu >$, is given by $e^{\mu \bar{N}_+} |v >$. The other coherent state originating from $[\bar{N}_+^\dagger, N_+] = 1$ is given by $|\nu > = e^{\nu \bar{N}_+} |v >$. This can be recognized as the (unnormalized) CS in the Perelomov sense. Depending on the UIREP being infinite or finite dimensional, this quadratic algebra can also be mapped in to $SU(1,1)$ and $SU(2)$ algebras, respectively; leaving aside the commutators not affected by this mapping, one gets,

$$[N_+, \bar{N}_+] = -2bN_0 \ ;$$  \hspace{1cm} (28)$$

where $b = 1$ corresponds to the $SU(1,1)$ and $b = -1$ gives the $SU(2)$ algebra. Explicitly,

$$\bar{N}_- = N_- G_1(C, N_0) \ ,$$  \hspace{1cm} (29)$$

and

$$G_1(C, N_0) = \frac{(N_0^2 - N_0)b + \epsilon}{C - g_1(N_0 - 1)} \ ,$$  \hspace{1cm} (30)$$

$\epsilon$ being an arbitrary constant. One can immediately construct CS in the Perelomov sense as $U |v >$, where $U = e^{\epsilon N_+ - \epsilon^* N_-}$. For the compact case, the CS are analogous to the spin and atomic coherent states\cite{19, 20}.

The Cubic algebra, which is also popularly known as the Higgs algebra in the literature, manifested in the study of the degeneracy structure of eigenvalue problems in a curved space\cite{6}. The generators satisfy,

$$[M_0, M_\pm] = \pm M_\pm \ , \ [M_+, M_-] = 2cM_0 + 4hM_0^3 \ ,$$  \hspace{1cm} (31)$$

where, $f_2(M_0) = 2cM_0 + 4hM_0^3 = g_2(M_0) - g_2(M_0 - 1)$, and

$$g_2(M_0) = cM_0(M_0 + 1) + hM_0^2(M_0 + 1)^2 \ .$$  \hspace{1cm} (32)$$
Analysis of its representation theory yields a variety of UIREP's, both finite and infinite dimensional, depending on the values of the parameters $c$ and $h$ [21]. Physically, $h$ represents the curvature of the manifold. In the non-compact case the canonical conjugate is given by,

$$
\hat{M}_+ = M_+ F_2(C, M_0) \tag{33}
$$

where,

$$
F_2(C, M_0) = \frac{M_0 + \zeta}{C - cM_0(M_0 + 1) - hM_0^2(M_0 + 1)^2} \tag{34}
$$

As before, the annihilation operator eigenstate is given by

$$
| \rho >_i = e^{\rho\hat{M}_+} | p >_i \tag{35}
$$

where, $| p >_i$ are the states annihilated by $M_-$. Like the previous cases, the dual algebra yields another coherent state. This algebra can also be mapped in to $SU(1,1)$ and $SU(2)$ algebras, as has been done for the quadratic case:

$$
[M_+, \hat{M}_-] = -2dM_0 \tag{36}
$$

where, $d = 1$ and $d = -1$ correspond to the $SU(1,1)$ and $SU(2)$ algebras respectively. Here,

$$
\hat{M}_- = M_- G_2(C, M_0) \tag{37}
$$

where,

$$
G_2(C, M_0) = \frac{(M_0^2 - M_0)d + \sigma}{C - g_2(M_0 - 1)} \tag{38}
$$

$\sigma$ being a constant. The coherent state in the Perelomov sense is then $U|v >$, where, $U = e^{\phi M_+ - \phi^* M_-}$. We would like to point out that, earlier the generators of the deformed algebra have been written in terms of the undeformed ones [15]. However, in our approach the undeformed $SU(1,1)$ and $SU(2)$ generators are constructed from the deformed generators [16].

### 4 Explicit construction of the Coherent States for Physical Application.

We now construct the state explicitly for purposes of application. First we show that this method, indeed, gives us well known $SU(1,1)$ Barut-Giradello (pair coherent) states for $SU(1,1)$ in the familiar form [22]. The action on Hilbert Space of the generators is given in the original Barut Giradello representation by:

$$
K_0 | \phi > = (\phi + m) | \phi, m > \tag{39}
$$

$$
K_+ | \phi, m > = \frac{1}{\sqrt{2}} \sqrt{(m+1)(-2\phi + m)} | \phi, m + 1 > \tag{40}
$$

$$
K_- | \phi, m > = \frac{1}{\sqrt{2}} \sqrt{m(-2\phi + m + 1)} | \phi, m - 1 > \tag{41}
$$

The Coherent state $| \alpha >$ is given by

$$
| \alpha > = e^{\alpha K_+} | \phi, 0 > \tag{42}
$$

where $[K_-, \tilde{K}_+] = 1$ and $\tilde{K}_+ = K_+ F(C, K_0)$

$$
F(C, K_0) = \frac{1}{C - g(K_0)} = \frac{1}{C + \frac{1}{2}K_0(K_0 + 1)} \tag{43}
$$
Here operators on eigenfunctions of suitable basis transformations gives the symmetry algebra of most trilinear Hamiltonians \[23\] \[24\]. The action of the

\[|\alpha| = \sum_n \frac{\alpha^n}{n!} (K_+ F(C, K_0))^n |\phi, 0| = \sum_n \frac{\alpha^n}{n!} (K_+)^n F(C, K_0) \ldots F(C, K_0 + n - 1) |\phi, 0| \]

substituting the values of \(F\) we get:

\[|\alpha| = \sum_n \frac{\alpha^n}{n!} \frac{1}{(-\phi + \frac{1}{2}) \ldots (-\phi + \frac{n-1}{2})} (K_+)^n |\phi, 0| = \sum_n \frac{\alpha^n}{n!} (2)^n \frac{\Gamma(-2\phi)}{\Gamma(-2\phi + n)} \sqrt{n!(-2\phi + n-1)!} |\phi, n|\]

\[= C \sqrt{\Gamma(-2\phi)} \sum_n \frac{(\sqrt{2}\alpha)^n}{\sqrt{n!(n+1)!}} \frac{1}{\Gamma(2\phi + n)} |\phi, n|\]

Which is precisely the well-known state of Barut and Giradello up to the normalization coefficient \(C\).

First we consider a quadratic algebra and then show the general explicit construction. For the quadratic case we take an illustrative algebra relevant to the trilinear boson cases described in section 2. A typical algebra satisfied by the generators is given:

\[\left[ N_+ , N_- \right] = N_+ [N_0 , N_-] - N_- [N_+ , N_-] = -3N_0^2 + 4\epsilon N_0 - \epsilon^2 \] where \(\epsilon\) is a constant and various values of \(\epsilon\) and suitable basis transformations gives the symmetry algebra of most trilinear Hamiltonians \(23\) \(24\). The action of the operators on eigenfunctions of \(N_0\) is given by:

\[N_0 |n >= (n + \frac{1}{2}) |n >\]

\[N_+ |n > = \sqrt{(n + \frac{3}{2} - \epsilon)(n + 1)(n + \frac{1}{2} - \epsilon)} |n + 1 >\]

\[N_- |n > = \sqrt{(n - \frac{1}{2} - \epsilon)n(n + \frac{1}{2} - \epsilon)} |n - 1 >\]

Here \(g(N_0) = -(N_0 - \epsilon)(N_0 + \frac{1}{2})(N_0 + 1 - \epsilon)\) and \(g(N_0) - g(N_0 - 1) = 3N_0^2 - 4\epsilon N_0 + \epsilon^2\)

From our construction the CS therefore is:

\[|\alpha| = e^{\alpha N_+} |0 >= \sum_n \frac{\alpha^n}{n!} (N_+)^n |0 >\]

Thus:

\[|\alpha| = \sum_n \frac{\alpha^n}{n!} (N_+ F(N_0, C))^n |0 >\]

Constructing the \(F's\) from \(g(N_0)\) we get:

\[|\alpha| = \sum_n \frac{\alpha^n}{n!} (N_+)^n F(N_0) F(N_0 + 1) \ldots F(N_0 + n - 1) |0 >\]

\[= \sum_n \frac{\alpha^n}{n!} (N_+)^n \frac{N_0 + \delta}{(N_0 - \epsilon)(N_0 + \frac{1}{2})(N_0 + 1 - \epsilon)} \ldots \frac{N_0 + n - 1 + \delta}{(N_0 + n - 1 - \epsilon)(N_0 + n - \frac{1}{2})(N_0 + n - \epsilon)} |0 >\]

\[= \sum_n \frac{\alpha^n}{n!} \frac{(-\frac{1}{2} - \epsilon)!(\frac{1}{2} - \epsilon)!}{(n - \frac{1}{2} - \epsilon)!} \frac{1}{\delta!} (N_+)^n |0 >\]

\[= C \sqrt{\Gamma(\frac{1}{2} - \epsilon)} \frac{3}{2} \sqrt{\Gamma(\frac{1}{2} - \epsilon)} \sum_n \frac{\alpha^n}{\sqrt{n!(n + 1)! (n + \frac{3}{2} - \epsilon)}} |n >\]

(51)
$C$ is the normalization coefficient, which can be easily determined.

We now give an outline of the method of explicit construction of coherent states for general multiphoton processes for which the generators satisfy the algebra \[ [N_0, N_+] = N_+ \quad [N_0 N_-] = -N_- \quad [N_+ N_-] = g(N_0) - g(N_0 - 1) \]

The action on eigenstates of $N_0$ is given by

\[
N_0 | j, m > = j + m | j, m > \\
N_+ | j, m > = \sqrt{C(j) - g(j + m)} | j, m + 1 > \\
N_- | j, m > = \sqrt{C(j) - g(j + m - 1)} | j, m - 1 >
\]

where $C(j) = g(j - 1)$

Coherent state is given by

\[
| \alpha > = e^{\alpha N_+} | j, 0 > \\
= \sum_n \frac{\alpha^n (N_+)^n}{n!} \frac{N_0 + \delta}{g(j - 1) - g(N_0)} \cdots \frac{N_0 + n - 1 + \delta}{g(j - 1) - g(N_0 + n - 1)} | j, 0 > \\
= C \sum_n \frac{\alpha^n}{\sqrt{g(j - 1) - g(j)} \cdots (g(j - 1) - g(j + n - 1))} | j, n >
\]

A discussion of coherent states is incomplete without showing that these states do give a resolution of the identity and that they are over complete. From the resolution of the identity we have:

\[
\int d\sigma(\alpha^*, \alpha) |\alpha\rangle \langle \alpha| = 1
\]

Within the polar decomposition ansatz

\[
d\sigma(\alpha^*, \alpha) = r \sigma(r) d\theta dr
\]

with $r = |\alpha|$ and a yet unknown positive density $\sigma$ which provides the measure. For the general case we have:

\[
2\pi \int_0^\infty dr \sigma(r) r^{2n+1} = C(g(j - 1) - g(j)) \cdots (g(j - 1) - g(j + n - 1))
\]

For the various cases the substitution of the explicit value of $g(j)$ then reduces the expression on the R.H.S to a rational function of Gamma Functions and the measure $\sigma$ can be found by an inverse Mellin transform. For example for the SU(1,1) $g(j) = \frac{1}{2}(j)(j + 1)$ case the R.H.S. becomes in the BG representation

\[
2\pi \int_0^\infty dr \sigma(r) r^{2n+1} = C \frac{\Gamma(n + 1)\Gamma(-2\phi + n)}{\Gamma(-2\phi)}
\]

where $C$ is a numerical constant and from the inverse Mellin transform we get $\sigma(r) = Cr^{-2\phi + 1}K_{\frac{1}{2} + \phi}(2r)$

For the quadratic case the R.H.S. becomes

\[
\int_0^\infty dr \sigma(r) r^{2n+1} = \Gamma(n + 1) \frac{\Gamma(\frac{1}{2} - \varepsilon + n)\Gamma(\frac{1}{2} + \varepsilon + n)}{\Gamma(\frac{1}{2} - \varepsilon)\Gamma(\frac{1}{2} + \varepsilon)}
\]

and $\sigma(r)$ can be determined to be a confluent hypergeometric function from the inverse Mellin transformation formula:

\[
\int_0^\infty r^{b-1}\Phi(a, c, -r) = \frac{\Gamma(b)\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)}
\]

For the general case the measure will be a Meijer’s G-function.
5 Conclusion

To conclude, we have found a general method for constructing the coherent states for various polynomially deformed algebras for quantum optical systems whose dynamics is governed by multilinear boson Hamiltonians. Since the method is algebraic and relies on the group structure of Lie algebras, the precise nature of the non-classical behaviour of these CS can be easily inferred from our construction. It will be of particular interest to see the time development of the system and the physical role of the deformation parameters. Since many of these algebras are related to quantum mechanical problems with non-quadratic, non-linear Hamiltonians, a detailed study of the properties of the CS associated with these non-linear and deformed algebras is of physical relevance [26, 27]. This is the subject of our current and future work [28].

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