NON-INNER AMENABILITY OF THE THOMPSON
GROUPS $T$ AND $V$

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Abstract. In this paper we prove that the Thompson groups $T$ and $V$
are not inner amenable. In particular, their group von Neumann alge-
bra do not have property $\Gamma$. Moreover, we prove that if the reduced
group $C^*$-algebra of $T$ is simple, then the Thompson group $F$ is non-
amenable. Furthermore, we give a few new equivalent characterizations
of amenability of $F$.

1. Introduction

The Thompson groups $F$, $T$ and $V$ were introduced by Richard Thomp-
son in 1965. They are countable discrete groups of piecewise linear bijections
of the half open interval $[0, 1)$ onto itself and satisfy $F \subseteq T \subseteq V$. We refer
to the paper [5] by Cannon, Floyd and Parry for a detailed introduction
to the subject. It is well-known that the Thompson groups $T$ and $V$ are
non-amenable, but it is a major open problem to decide whether or not
$F$ is amenable. All three groups are ICC, that is, all conjugacy classes of
non-trivial elements are infinite, and hence their von Neumann algebras $L_F$,
$L_T$ and $L_V$ are all factors of type $\text{II}_1$. Jolissaint [14] proved that $F$ is in-
ner amenable in the sense of Effros [9]. He later strengthened this result
by proving that $L_F$ is a McDuff factor; see [15]. In particular, this means
that $L_F$ has property $\Gamma$, and hence, by a result of Effros [9], $F$ is inner
amenable. The first and main result of this paper is that $T$ and $V$ are not
inner amenable, and hence, by the same result of Effros, $L_T$ and $L_V$ do not
have property $\Gamma$. Therefore, they are also not McDuff factors. The fact that
$F$ is inner amenable was proved in a different way by Ceccherini-Silberstein
and Scarabotti [6], a few years later.

Another result of this paper connects amenability of $F$ with simplicity
of the reduced group $C^*$-algebra of $T$. More precisely, we prove that if the
reduced group $C^*$-algebra $C^*_r(T)$ is simple, then $F$ is non-amenable. Shortly
after the results of this paper where announced by the first named author
at the Fields Institute in Toronto in March 2014, this result was re-obtained
by Breuillard, Kalantar, Kennedy and Ozawa [3], using completely differ-
ent methods. Very recently, this result was also re-proved by Le Boudec and
Matte Bon [19]. They also showed the converse of this statement. This result
is not the only connection between amenability of $F$ and simplicity of its reduced group $C^*$-algebra. Indeed, it has been well-known for some time that $F$ is amenable if and only if its reduced group $C^*$-algebra has a unique tracial state, and it has been a long-standing open problem set forth by de la Harpe (see, for example, [7]) to decide whether the reduced group $C^*$-algebra of a group is simple if and only if it has a unique tracial state. During the last few years tremendous progress has been made on the topic of $C^*$-simple groups (that is, groups whose reduced group $C^*$-algebra is simple) and groups with the unique trace property (that is, groups whose reduced group $C^*$-algebra has a unique tracial state). This new development began when Kalantar and Kennedy [16] gave new characterizations of $C^*$-simple groups in terms of boundary actions. Immediately after, Breuillard, Kalantar, Kennedy and Ozawa [3] proved several striking results, including that $C^*$-simplicity implies the unique trace property. Very recently, Le Boudec, [18], gave counterexamples to the remaining implication of the long-standing open problem of de la Harpe by providing examples of groups which are not $C^*$-simple, but have the unique trace property. This settled completely the relationship between $C^*$-simplicity and the unique trace property. Further examples of non-$C^*$-simple groups with the unique trace property were provided by Ivanov and Omland [13].

In addition to the results mentioned thus far, we investigate the $C^*$-algebras generated by the images of $F$, $T$ and $V$ via a representation discovered by Nekrashevych [20]. More precisely, we prove that these $C^*$-algebras are distinct, and that that one generated by the image of $V$ is the Cuntz algebra $O_2$. Furthermore, we give new characterizations of amenability of $F$ in terms of the size of certain explicit ideals in the reduced group $C^*$-algebras of $F$ and $T$, as well as in terms of whether certain convex hulls contain zero.

We end the introduction by giving the definition of the Thompson groups. The Thompson group $V$ is the set of all piecewise linear bijections of $[0,1)$ which are right continuous, have finitely many points of non-differentiability, all being dyadic rationals, and have a derivative which is a power of 2 at each point of differentiability. It is easy to see that these maps fix the set of dyadic rationals in $[0,1)$. The Thompson group $T$ consists of those elements of $V$ which define a homeomorphism of $[0,1)$, when identified with $\mathbb{R}/\mathbb{Z}$ in the standard way. Equivalently, $T$ is the set of elements in $V$ which have at most one point of discontinuity with respect to the usual topology on $[0,1)$. Finally, the Thompson group $F$ consists of those elements of $V$ which are, in fact, homeomorphisms of $[0,1)$, or, equivalently, the set of element $g \in T$ satisfying $g(0) = 0$. All three groups are finitely generated and also finitely presented in the following generators: $F$ is generated by $A$ and $B$; $T$ is generated by $A$, $B$ and $C$; and $V$ is generated by $A$, $B$, $C$ and $\pi_0$, where $A$, $B$, $C$ and $\pi_0$ are defined in [5]. Besides these standard generators, we will also consider the following element $D$ of $T$, given by $D(x) = x + \frac{2}{4}$, for $x \in [0,\frac{1}{4})$, and $D(x) = x - \frac{1}{4}$ for $x \in [\frac{1}{4},1)$. Note that $C^3 = D^4 = 1$. Moreover, the elements $C$ and $D$ also generate $T$ as a group, as it is easily seen that $A = D^2C^2$ and $B = C^2DA$. 


The isomorphism between $T$ and $V$

In order to prove that $T$ and $V$ are not inner amenable, we shall use that $T$ is isomorphic to a group of piecewise fractional linear transformations. This fact was originally proved by Thurston, as explained in [5, §7]. However, Thurston realized $T$ as Möbius transformations of the interval $[0,1]$, whereas we will use a slightly modified version of Thurston’s result, given by Imbert in [12, Theorem 1.1], where $T$ is realized as Möbius transformations of $\mathbb{R} \cup \{\infty\}$ instead. The construction is essentially the same, and we shall explain the difference in Remark 2.3 below. A description of the isomorphism is also given by Fossas in [10]. The second named author would like to thank Vlad Sergiescu for several fruitful discussions and his help in sorting out the origin of the particular version of Thurston’s result we are presenting here.

Recall that the group $\text{PSL}(2,\mathbb{Z}) = \text{SL}(2,\mathbb{Z})/\{\pm 1\}$ acts in a natural way on $\mathbb{R} \cup \{\infty\}$ via Möbius transformations, that is, for $x \in \mathbb{R} \cup \{\infty\}$,

$$g(x) = \frac{ax+b}{cx+d}, \quad \text{when} \quad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \{\pm 1\} \in \text{PSL}(2,\mathbb{Z}).$$

This action is faithful, in the sense that only the neutral element acts trivially. Recall also that the elements of $\text{PSL}(2,\mathbb{Z})$ act by homeomorphisms in an orientation-preserving way.

**Definition 2.1.** We denote by $\text{PPSL}(2,\mathbb{Z})$ the group of homeomorphisms of $\mathbb{R} \cup \{\infty\}$ which are piecewise in $\text{PSL}(2,\mathbb{Z})$, that is, piecewise of the form (1), and which have only finitely many breakpoints, all them being in $\mathbb{Q} \cup \{\infty\}$.

In the above definition, a breakpoint of an element $g \in \text{PPSL}(2,\mathbb{Z})$ should be understood as a point $x \in \mathbb{R} \cup \{\infty\}$ for which there is no open neighbourhood $U$ so that $g$ acts on $U$ as an element of $\text{PSL}(2,\mathbb{Z})$.

**Theorem 2.2** (Thurston). There exists a homeomorphism $\phi$ of $\mathbb{R} \cup \{\infty\}$ onto $\mathbb{R}/\mathbb{Z} = S^1$ such that the map $\Phi: \text{PPSL}(2,\mathbb{Z}) \rightarrow T$ given by

$$g \mapsto \phi \circ g \circ \phi^{-1}$$

is an isomorphism.

As mentioned, Thurston’s construction of the homeomorphism, see [5, §7], is different than the one in the theorem above, which is briefly explained in [12]. More details are given by Fossas in the proof of Theorem 2.2 in [10]. More precisely, one first defines an order preserving homeomorphism $\psi$ of $[-\infty, \infty]$ onto $[-\frac{1}{2}, \frac{1}{2}]$, and next obtains $\phi: \mathbb{R} \cup \{\infty\} \rightarrow \mathbb{R}/\mathbb{Z}$ from $\psi$ by identifying the endpoints of each of the two intervals, that is, $\phi(x) = \psi(x)+\mathbb{Z}$, for all $x \in \mathbb{R}$, and $\phi(\infty) = \frac{1}{2} + \mathbb{Z}$. The map $\psi$ is denoted by “?” in [10]. It is closely related to (but different from) the Minkowski question mark function, as explained in Remark 2.3 below. To construct $\psi$, one writes each element in $\mathbb{Q} \cup \{\pm \infty\}$ as a reduced fraction $\frac{p}{q}$ with $p,q \in \mathbb{Z}$ and $q \geq 0$, using the convention that $-\infty = -1$ and $\infty = \frac{1}{0}$.

Two reduced fractions $\frac{p}{q}$ and $\frac{r}{s}$ (possibly $\frac{-1}{0}$ and $\frac{1}{0}$) are called consecutive Farey numbers if $|ps -rq| = 1$. Given two consecutive Farey numbers one
defines
\[ \frac{p}{q} \oplus \frac{r}{s} = \frac{p + r}{q + s}. \]
Let now \( \psi \) be defined on \( \mathbb{Q} \cup \{ \pm \infty \} \) by letting \( \psi(-\infty) = -\frac{1}{2}, \psi(0) = 0, \psi(\infty) = \frac{1}{2}, \) and then recursively let \( \psi\left(\frac{p}{q} \oplus \frac{r}{s}\right) = \frac{1}{2}\left(\psi\left(\frac{p}{q}\right) + \psi\left(\frac{r}{s}\right)\right) \), whenever \( \frac{p}{q} \) and \( \frac{r}{s} \) are consecutive Farey numbers. As explained in [10], this map is well-defined, and it is a strictly increasing bijection of \( \mathbb{Q} \cup \{ \pm \infty \} \) onto \( \mathbb{Z}[\frac{1}{2}] \cap [-\frac{1}{2}, \frac{1}{2}] \). Therefore, it has a unique extension to a strictly increasing homeomorphism, also called \( \psi \), of \( [-\infty, \infty] \) onto \( [-\frac{1}{2}, \frac{1}{2}] \).

Remark 2.3. Let \( ? \colon [0, 1] \rightarrow [0, 1] \) denote the Minkowski question mark function (see, for example, [22]). Then \( ? \) is constructed recursively in the same way as \( \psi \), but on the interval \([0, 1]\). However, one starts with the assignments
\[ ?(0) = 0 \quad \text{and} \quad ?(1) = 1, \]
while in the case of \( \psi \) (on the interval \([0, 1]\)) we have
\[ \psi(0) = 0 \quad \text{and} \quad \psi(1) = \frac{1}{4}. \]
Therefore \( \psi(x) = \frac{1}{4}?(x) \), for \( 0 \leq x \leq 1 \). Moreover, it is not hard to check that \( \psi \) satisfies the following symmetries \( \psi(-x) = -\psi(x) \) and \( \psi(x) = \frac{1}{2} - \psi(x) \), for \( x \in [0, \infty] \). Hence, \( \psi \) can be expressed in terms of the Minkowski question mark function by
\[ \psi(x) = \begin{cases} -\frac{1}{2} + \frac{1}{4}?(-\frac{1}{2}) & x \in [-\infty, -1] \\ -\frac{1}{4}?(-x) & x \in [-1, 0] \\ \frac{1}{4}?(x) & x \in [0, 1] \\ \frac{1}{2} - \frac{1}{4}?(\frac{1}{2}) & x \in [1, \infty] \end{cases}. \]

Thurston’s version of Theorem 2.2 states that \( T \) is isomorphic to the group of homeomorphisms of \([0, 1]\) with the endpoints identified, which are piecewise in \( \text{PSL}(2, \mathbb{Z}) \), have only finitely many breakpoints, all of which being in \( \mathbb{Q} \cap [0, 1] \). This isomorphism is realized by conjugating by the Minkowski question mark function \( ? \), instead of \( \psi \).

It is well-known (see [23, Example 1.5.3]) that \( \text{PSL}(2, \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}_2 \ast \mathbb{Z}_3 = \langle a, b \mid a^2 = b^3 = 1 \rangle \), with generators
\[ a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \{\pm 1\} \quad \text{and} \quad b = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \{\pm 1\}. \]
Note that \( \text{PSL}(2, \mathbb{Z}) \) is a subgroup of \( \text{PPSL}(2, \mathbb{Z}) \). Hence by [10, Remark 2.3], we have the following result.

Proposition 2.4. Let \( \Phi \colon \text{PPSL}(2, \mathbb{Z}) \rightarrow T \) be the isomorphism from Theorem 2.2, and let \( \Lambda \) denote the subgroup \( \Phi(\text{PSL}(2, \mathbb{Z})) \) of \( T \). Then
\[ D^2 = CA = \Phi(a) \quad \text{and} \quad C = \Phi(b) \]
are free generators of \( \Lambda \cong \mathbb{Z}_2 \ast \mathbb{Z}_3 \), of order 2 and 3, respectively.

We will also need the following well-known (and easy to prove) fact about the action of the group \( \text{PSL}(2, \mathbb{Z}) \) on \( \mathbb{R} \cup \{\infty\} \) by Möbius transformations.

Proposition 2.5. An element \( f \in \text{PSL}(2, \mathbb{Z}) \) is uniquely determined by its value on two distinct points in \( \mathbb{Q} \cup \{\infty\} \).
Remark 2.6. Note that we can translate the above result to one about $\Lambda$, by using the homeomorphism $\phi$. Indeed, $\phi$ restricts to a bijection from $Q \cup \{\infty\}$ to $[0, 1) \cap \mathbb{Z}[\frac{1}{2}]$, so an element in $\Lambda$ is uniquely determined by its value on two points in $[0, 1) \cap \mathbb{Z}[\frac{1}{2}]$.

3. Non-inner amenability of $T$ and $V$

In this section we prove that the Thompson groups $T$ and $V$ are not inner amenable. This answers a question that Ionut Chifan raised at a conference in Alba-Iulia in 2013. We start by recalling the fundamentals of inner amenability.

The reduced group $C^*$-algebra associated to a discrete group $G$, denoted by $C^*_r(G)$, is the $C^*$-algebra generated by the image of the left regular representation of $G$, $\lambda : G \to B(\ell^2(G))$, while the group von Neumann algebra $\mathcal{L}G$ associated to $G$, denoted by $\mathcal{L}G$, is the weak operator closure of $C^*_r(G)$ inside $B(\ell^2(G))$. The group von Neumann algebra is a type II$_1$-factor if and only if the group is ICC.

The von Neumann algebra $\mathcal{L}G$ of a discrete ICC group $G$ is said to have property $\Gamma$, if there exists a net $(u_i)_{i \in I}$ of unitaries in $\mathcal{L}G$ such that $\langle u_i \delta_x | \delta_e \rangle = 0$, for all $i \in I$, and $\lim_{i \in I} \|(u_i x - x u_i) \delta_e\| = 0$, for all $x \in \mathcal{L}G$.

Property $\Gamma$ can be defined for general finite factors, but we are only concerned here with the case where the von Neumann algebra is a group von Neumann algebra.

Recall that a discrete group $G$ is said to be inner amenable, if there exists a state $m$ on $\ell_\infty(G)$ such that $m(\delta_x) = 0$ and $m(f) = m(\alpha(g) f)$, for all $f \in \ell_\infty(G)$ and $g \in G$, where $\langle (\alpha(g) f)(h) = f(g^{-1} h g)$, for all $h \in G$. Inner amenability was introduced by Effros in 1975 in an attempt to characterize property $\Gamma$ for group von Neumann algebras in terms of a purely group theoretic property; see [9]. Therein he proved the following theorem.

Theorem 3.1 (Effros). For a discrete ICC group $G$, the following statements are equivalent:

(i) $G$ is inner amenable.

(ii) There exists a net $(\eta_i)_{i \in I}$ of unit vectors in $\ell^1(G)$ satisfying $\eta_i(e) = 0$ for all $i \in I$, and $\lim_{i \in I} \|\alpha(g) \eta_i - \eta_i\|_1 = 0$, for all $g \in G$.

(iii) There exists a net $(\xi_i)_{i \in I}$ of unit vectors in $\ell^2(G)$ satisfying $\xi_i(e) = 0$, for all $i \in I$, and $\lim_{i \in I} \|\alpha(g) \xi_i - \xi_i\|_2 = 0$, for all $g \in G$.

Moreover, if the group von Neumann algebra of $G$ has property $\Gamma$, then $G$ is inner amenable.

Effros conjectured that inner amenability of a discrete ICC group $G$ was, in fact, equivalent to property $\Gamma$ of $\mathcal{L}G$. In 2012, Vaes [24] provided a counterexample.

Recall that an action of a discrete group $G$ on a set $\mathcal{X}$ is a homomorphism from $G$ to the set of permutations of $\mathcal{X}$, and that such an action is said to be amenable if there exists a finitely additive probability measure on $\mathcal{X}$ which is invariant under the action. It straightforward to check that a discrete group $G$ is inner amenable if and only if the conjugation action of the group on $G \setminus \{e\}$ is amenable.
A key result for our proof of non-inner amenability of \( T \) and \( V \) is the following result due to Rosenblatt [21].

**Proposition 3.2** (Rosenblatt). Let \( G \) be a non-amenable discrete group acting on a set \( X \). If the stabilizer of each point is amenable, then the action of \( G \) is itself non-amenable.

From this we directly get the following sufficient condition for non-inner amenability.

**Corollary 3.3.** Let \( G \) be discrete a group. If \( G \) has a non-amenable subgroup \( H \) such that \( \{ g \in H : ghg^{-1} = h \} \) is amenable, for each \( h \in G \setminus \{ e \} \), then \( G \) is non-inner amenable.

Notice that, in the corollary above, any subgroup of \( G \) containing \( H \) will automatically fail to be inner amenable, as well.

**Theorem 3.4.** The Thompson groups \( T \) and \( V \) are not inner amenable.

*Proof.* Recall from Section 2 that \( \Lambda \) denotes the subgroup of \( T \) generated by \( C \) and \( D^2 \), and that conjugation by the map \( \phi \) from Theorem 2.2 restricts to an isomorphisms between \( \Lambda \) and \( PSL(2, \mathbb{Z}) \). Clearly \( \Lambda \) is not amenable, so, by Corollary 3.3, it suffices to prove that \( \{ g \in \Lambda : gf = fg \} \) is amenable, for all \( f \in V \setminus \{ e \} \). This will imply that both \( T \) and \( V \) are non-inner amenable.

We will consider separately the cases where \( f \in \Lambda \setminus \{ e \} \), \( f \in T \setminus \Lambda \) and \( f \in V \setminus T \), respectively. Fix \( f \in V \setminus \{ e \} \) and let us denote the subgroup \( \{ g \in \Lambda : gf = fg \} \) by \( H \).

First, suppose that \( f \in \Lambda \setminus \{ e \} \). Then \( H \) is the centralizer of \( f \) in \( \Lambda \), which is cyclic by Theorems 2.3.3 and 2.3.5 in [17]. In particular, it is amenable.

Now, suppose that \( f \in T \setminus \Lambda \). It is easy to see that, with \( \tilde{f} \) denoting \( \phi^{-1}f\phi \), we have \( \phi^{-1}H\phi = \{ h \in PSL(2, \mathbb{Z}) : h\tilde{f} = \tilde{f}h \} \). Let us show that this group is amenable, since then \( H \) will be amenable as well. Since \( f \notin \Lambda \), we know that \( \tilde{f} \notin PSL(2, \mathbb{Z}) \), and, in particular, \( \tilde{f} \) has at least two breakpoints. Let \( y_1, \ldots, y_n \) denote these breakpoints, and note that they all are in \( \mathbb{Q} \cup \{ \infty \} \). Suppose that \( h \in PSL(2, \mathbb{Z}) \) with \( h\tilde{f}h^{-1} = \tilde{f} \).

Clearly, the breakpoints of \( h\tilde{f}h^{-1} = h(y_1), \ldots, h(y_n) \), so since \( h\tilde{f}h^{-1} = \tilde{f} \), these must also be breakpoints of \( \phi^{-1}f\phi \). In other words, \( h \) permutes the breakpoints of \( \tilde{f} \). Now, because \( f \) has at least two breakpoints, it follows from Proposition 2.5 that \( h \) is uniquely determined by the corresponding permutation of \( y_1, \ldots, y_n \). Since there are only finitely many permutations of \( n \) elements, we deduce that \( \phi^{-1}H\psi \) is finite. In particular, \( H \) is also finite, and therefore amenable.

Last, suppose that \( f \in V \setminus T \). Then \( f \) is discontinuous, and since \( f \) is a bijection of \([0,1]\), it is easy to see that \( f \) must have at least two points of discontinuity. Since these points of discontinuity are all dyadic rationals and the elements of \( \Lambda \) are uniquely determined by their values of two dyadic rational numbers, by Remark 2.6, we can use the same argument as the one for \( T \setminus \Lambda \) to conclude that \( H \) is finite, since every element in \( \Lambda \) must permute points of discontinuity of \( f \). This proves that \( \{ g \in \Lambda : gf = fg \} \) is amenable, for all \( f \in V \setminus \{ e \} \). By Corollary 3.3, the conclusion follows.

The proof of the above theorem shows that, in fact, every subgroup of \( V \) containing \( \Lambda \) is non-inner amenable.
By the results of Effros (see Theorem 3.1), it follows that neither $L^T$ nor $L^V$ has property $\Gamma$, and hence they are not McDuff factors.

4. Simplicity of $C^*(T)$ implies non-amenability of $F$

In this section we prove that the Thompson group $F$ is non-amenable if the reduced group $C^*$-algebra of $T$ is simple. First, we recall the notion of weak containment of group representations.

By a representation of a group, we always mean a unitary representation, that is, a homomorphism from the given group to the unitary group of some Hilbert space. We denote the full group $C^*$-algebra of a discrete group $G$ by $C^*(G)$. If $\pi: G \to B(H)$ is a representation of $G$ on a Hilbert space $H$, then it extends uniquely to a representation of $C^*(G)$. We use the same symbol to denote this representation. If $\pi: G \to B(H)$ and $\rho: G \to B(K)$ are two representations of the group $G$, then we say that $\pi$ is weakly contained in $\rho$ if, for every $\xi \in H$ and $\varepsilon > 0$, there exist $\eta_1, \ldots, \eta_n \in K$ such that

$$\left| \langle \pi(g)\xi \mid \xi \rangle - \sum_{i=1}^n \langle \rho(g)\eta_i \mid \eta_i \rangle \right| < \varepsilon.$$ 

This is equivalent to the fact that there exists a $*$-homomorphism $\sigma$ from the $C^*$-algebra generated by $\rho(G)$ to the $C^*$-algebra generated by $\pi(G)$ such that $\pi = \sigma \circ \rho$. We need the following well-known result about weak containment.

**Proposition 4.1.** Let $G$ be a discrete group acting on a set $\mathcal{X}$. If all stabilizers of the action are amenable, then the representation of $G$ on $\ell^2(\mathcal{X})$ is weakly contained in the left regular representation of $G$.

For more about weak containment and connections to amenability see [1]. The following result is a well-known characterization of amenable actions. For a proof in the case where the action is left translation of the group on itself, the reader may consult [4, Theorem 2.6.8].

**Proposition 4.2.** Suppose that $G$ is a discrete group acting on a set $\mathcal{X}$, and let $\pi$ denote the induced representation on $\ell^2(\mathcal{X})$. Then the action of $G$ on $\mathcal{X}$ is non-amenable if and only if there exist $g_1, \ldots, g_n$ in $G$ so that

$$\left\| \frac{1}{n} \sum_{k=1}^n \pi(g_k) \right\| < 1.$$ 

In fact, if the action is non-amenable and $G$ is finitely generated, then $\left\| \frac{1}{n} \sum_{k=1}^n \pi(g_k) \right\| < 1$, for any set of elements $g_1, g_2, \ldots, g_n$ generating $G$.

In the following we will make use of the Cuntz algebra $O_2$. Recall that $O_2$ is the universal $C^*$-algebra generated by two isometries $s_1$ and $s_2$ with orthogonal range projections summing up to the identity. In other words, $O_2$ is the universal $C^*$-algebra generated by elements $s_1$ and $s_2$ satisfying the relations $s_1^*s_1 = 1$, $s_2^*s_2 = 1$ and $s_1s_2^* + s_2s_1^* = 1$. There is a canonical way to realize the Thompson groups as subgroups of the unitary group of $O_2$. This was discovered by Nekrashevych in [20], as kindly pointed out to us by Wojciech Szmyanski.
Let us describe the concrete model we will use for the Cuntz algebra $\mathcal{O}_2$. We think of it as a specific set of bounded operators on $\ell^2(\mathcal{X})$ where $\mathcal{X}$ denotes the set $\mathbb{Z}[\frac{1}{2}] \cap [0, 1)$. Define the operators $s_1$ and $s_2$ on $\ell^2(\mathcal{X})$ by
\[ s_1\delta_x = \delta_{x/2} \quad \text{and} \quad s_2\delta_x = \delta_{(1+x)/2}, \]
for all $x \in \mathcal{X}$. It is straightforward to check that $s_1$ and $s_2$ are isometries satisfying $s_1s_1^* + s_2s_2^* = 1$, so they generate a copy of the Cuntz algebra $\mathcal{O}_2$.

The groups $F$, $T$ and $V$ act by definition on the set $\mathcal{X}$, and we denote the induced representations on $\mathbb{B}(\ell^2(\mathcal{X}))$ by $\pi$. That is, $\pi(g)\delta_x = \delta_{g(x)}$, for all $x \in \mathcal{X}$ and $g \in V$. We use $\pi$ to denote this representation when restricted to $F$ and $T$, as well. As it so happens, the image of $\pi$ is contained in $\mathcal{O}_2$. In fact, one can check the following explicit identities:
\[ \pi(D) = s_2s_2s_1^*s_1^* + s_1s_1s_2^*s_1^* + s_1s_2s_1s_2^* + s_2s_1s_2s_2^*, \]
\[ \pi(C) = s_2s_2s_1^* + s_1s_1s_2^* + s_2s_1s_2s_2^*, \]
\[ \pi(D^2) = s_2s_1^* + s_1s_2^*, \]
\[ \pi(\pi_0) = s_2s_1^* + s_1s_2^* + s_2s_2s_2^*. \]

We denote the $C^*$-algebras generated by $\pi(F)$, $\pi(T)$ and $\pi(V)$ inside $\mathcal{O}_2$ by $C^*_\pi(F)$, $C^*_\pi(T)$ and $C^*_\pi(V)$, respectively.

**Proposition 4.3.** With the notation above, we have
\[ C^*_\pi(F) \subseteq C^*_\pi(T) \subseteq C^*_\pi(V) = \mathcal{O}_2. \]

**Proof.** By construction, $C^*_\pi(F) \subseteq C^*_\pi(T) \subseteq C^*_\pi(V) \subseteq \mathcal{O}_2$. We will prove that the first two inclusions are proper and that the last one is an equality.

It is easy to see that $C^*_\pi(F) \neq C^*_\pi(T)$, since $C\delta_0$ is a $C^*_\pi(T)$-invariant subspace which is not $C^*_\pi(T)$-invariant. Hence $C^*_\pi(F) \neq C^*_\pi(T)$.

For the rest, let us first prove that $C^*_\pi(V) = \mathcal{O}_2$, and afterwards that $C^*_\pi(T) \neq \mathcal{O}_2$. Our strategy for showing that $C^*_\pi(V) = \mathcal{O}_2$ is to prove that $C^*_\pi(V)$ contains $s_1$ and $s_2$. In fact, since $s_2 = \pi(D^2)s_1$, it suffices to show that it contains $s_1$. To ease notation, let us denote $\ell^2(\mathbb{Z}[\frac{1}{2}] \cap [0, \frac{1}{2}))$ and $\ell^2(\mathbb{Z}[\frac{1}{2}] \cap [\frac{1}{2}, 1))$ by $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Consider the subgroup $V_0$ of $V$ consisting of elements $g \in V$ satisfying $g(x) = x$, for all $x \in [0, \frac{1}{2})$. Note that, for $g \in V_0$, both $\mathcal{H}_1$ and $\mathcal{H}_2$ are invariant subspaces for $\pi(g)$, so that we can write $\pi(g) = \pi(g)|_{\mathcal{H}_1} \oplus \pi(g)|_{\mathcal{H}_2}$. It is easy to see using Theorem 3.2 that the action of $V_0$ on $[\frac{1}{2}, 1)$ is non-amenable. Thus, by Proposition 4.2, there exist elements $g_1, \ldots, g_n$ in $V_0$, such that $\|\frac{1}{n} \sum_{k=1}^n \pi(g_k)|_{\mathcal{H}_1}\| < 1$. Since $\frac{1}{n} \sum_{k=1}^n \pi(g_k)|_{\mathcal{H}_1} = 1_{\mathcal{H}_1}$, we deduce that $(\frac{1}{n} \sum_{k=1}^n \pi(g_k))^m$ converges in norm, as $m \to \infty$, to the projection of $\ell^2(\mathcal{X})$ onto $\mathcal{H}_1$, that is, to $s_1s_1^*$. Hence $s_1s_1^*$, and therefore also $s_2s_2^* = 1 - s_1s_1^*$, belong to $C^*_\pi(V)$. It is straightforward to check that
\[ s_1 = \pi(A)s_1s_1^* + \pi(D^2A^{-1})s_2s_2^*. \]
This implies that $s_1 \in C^*_\pi(V)$, and we conclude that $C^*_\pi(V) = \mathcal{O}_2$.

Last, let us prove that $C^*_\pi(T) \neq \mathcal{O}_2$. We do this by exhibiting two different states $\phi_0$ and $\phi_1$ on $\mathcal{O}_2$ which agree on $C^*_\pi(T)$. Let $\phi_0$ denote the vector state given by $\delta_0$. It is well-known that elements of the form $s_1s_2 \cdots s_n s_1^* s_2^* \cdots s_n^*$ span a dense subspace of $\mathcal{O}_2$. Moreover, it is easy to check that $\phi_0(s_1^*(s_1^* s_1^*)^m) = 1$, for all $n, m \geq 0$, while $\phi_0$ is zero on the rest
of these elements. Let \( \phi_1 \) denote the composition of \( \phi_0 \) with the automorphism of \( \mathcal{O}_2 \) that interchanges \( s_1 \) and \( s_2 \). Then, for all \( n, m \geq 0 \), \( \phi_1 \) satisfies 
\[
\phi_1(s_1 \cdot \ldots \cdot s_n s_1^* \cdot \ldots \cdot s_m^*) = 0 \quad \text{unless } i_1 = \ldots = i_n = 2 \quad \text{and } j_1 = \ldots = j_m = 2,
\]
in which case \( \phi_1(s_2^2(s_2^m)) = 1 \). It follows easily from Nekrashevych’s description of the representation \( \pi \) (see [20, Section 9]) that for all \( g \in V \), either \( \phi_k(\pi(g)) = 0 \) or \( \phi_k(\pi(g)) = 1 \), where \( k = 0, 1 \). Similarly, it is easily seen that, for \( g \in V \), \( \phi_0(\pi(g)) = 1 \) if and only if \( \lim_{x \to 0} g(x) = 0 \), and that \( \phi_1(\pi(g)) = 1 \) if and only if \( \lim_{x \to 1} g(x) = 1 \). In particular, we see that \( \phi_k(\pi(g)) = 1_F(g) \), for all \( g \in T \) and \( k = 0, 1 \). Thus \( \phi_0 \) and \( \phi_1 \) agree on \( \pi(T) \), so clearly they also agree on \( C^*_\pi(T) \). It follows that \( C^*_\pi(T) \neq \mathcal{O}_2 \), as the two states clearly do not agree on \( \mathcal{O}_2 \). 

It turns out that the representation \( \pi \) is closely connected to amenability of the Thompson group \( F \), as we show in the following.

**Proposition 4.4.** The Thompson group \( F \) is amenable if and only if \( \pi \) is contained in the left regular representation of \( T \). This is further equivalent to the fact that \( \pi \) is contained in the left regular representation of \( F \).

**Proof.** Clearly, \( \pi|_F \) is weakly contained in the left regular representation of \( T \) if \( \pi \) is weakly contained in the left regular representation of \( T \). Suppose now that \( F \) is amenable. Since \( T \) acts transitively on \( X \), all the stabilizers of the action are isomorphic. It follows from Proposition 4.1 that \( \pi \) is weakly contained in the left regular representation of \( T \), as \( F \) is the stabilizer of 0.

Suppose that \( \pi \) is weakly contained in the left regular representation of \( F \). Let \( p \) denote the projection onto \( \mathbb{C} \delta_0 \), which is a \( \pi(F) \)-invariant subspace of \( \ell^2(X) \). Then \( p \pi |_F \) is the trivial representation of \( F \). This is contained in \( \pi \), so by transitivity of weak containment, we deduce that the trivial representation of \( F \) is weakly contained in the left regular representation of \( F \). By [1, Theorem G.3.2], this is equivalent to amenability of \( F \).

**Theorem 4.5.** If \( C^*_\pi(T) \) is simple, then \( F \) is non-amenable.

**Proof.** Suppose that \( F \) is amenable, and let us then prove that \( C^*_\pi(T) \) is not simple. By Proposition 4.4, \( \pi \) is weakly contained in the left regular representation of \( T \), so there exists a \( \ast \)-homomorphism \( \sigma \) from \( C^*_\pi(T) \) to \( C^*_\pi(T) \) such that \( \pi = \sigma \circ \lambda \). Our goal is to show that the kernel of \( \sigma \) is a non-trivial ideal in \( C^*_\pi(T) \). Since \( \sigma \) is obviously not the zero map, we only need to show that its kernel contains a non-trivial element. The \( \ast \)-homomorphism \( \lambda \) is injective on the complex group algebra \( \mathbb{C}T \), so it suffices to find an element \( x \neq 0 \) in \( \mathbb{C}T \) such that \( \pi(x) = 0 \).

Consider the elements \( a \) and \( b \) of \( T \) (and their product) defined by

\[
\begin{align*}
a &= CDC, \\
b &= D^2 CDCD^2, \\
ab &= ba
\end{align*}
\]
Obviously, $a$ and $b$ commute, and it is easy to see that the values $(a(x), b(x))$ and $(ab(x), x)$ agree, up to a permutation, for all $x \in [0, 1)$. From this we deduce that $\pi(a + b)\delta_x = \pi(ab + e)\delta_x$, for all $x \in \mathcal{X}$. In particular, $\pi(a + b - ab - e) = 0$, so that the kernel of $\sigma$ is non-trivial. Thus $C^*_r(T)$ is not simple.

As mentioned in the introduction, Le Boudec and Matte Bon [19] recently proved the converse implication to the theorem above. They also proved therein that $C^*_r(V)$ is simple, and that $C^*_r(F)$ is simple if $C^*_r(T)$ is simple. The converse of this latter implication was already proved by Breuillard, Kalantar, Kennedy and Ozawa [3].

It is still unknown whether the reduced group $C^*$-algebras of $T$ is simple, but it is, however, known that it has a unique tracial state. This was proved by Dudko and Medynets [8].

**Remark 4.6.** We believe that is is a well-known fact that $F$ is non-amenable if and only if its reduced group $C^*$-algebra has a unique tracial state, although we have not been able to find an explicit reference for this. It follows, for example, from the fact that $C^*_r(F)$ has at most two extremal tracial states, as showed by Dudko and Medynets [8]. A different proof of this is based on the new characterization of groups whose reduced group $C^*$-algebra has a unique tracial state given by Breuillard, Kalantar, Kennedy and Ozawa. In the following we briefly indicate the argument.

It was shown in the aforementioned paper that $C^*_r(F)$ has a unique tracial state if and only if the amenable radical of $F$ is trivial. However, it is a well-known fact (see [5, Theorem 4.3]) that every non-trivial normal subgroup of $F$ contains a copy of $G$. Hence the amenable radical is trivial if and only if $F$ is non-amenable.

We end this section by giving a few new equivalent characterizations of amenability of $F$. We will make use of the following well-known result, for which a proof can be found in [11, Lemma 4.1] (in the case where the action is left translation of the group on itself).

**Lemma 4.7.** Let $G$ be a discrete group acting on a set $\mathcal{X}$, and let $\sigma$ be the corresponding representation on $l^2(\mathcal{X})$. Then $\|x + y\| \geq \|x\|$, for $x, y \in \mathbb{R}_+G$.

By scrutinizing the proof of Theorem 4.5 one sees that, with the notation therein, the Thompson group $F$ is non-amenable if the closed two-sided ideal generated by the element $1 + \lambda(ab) - \lambda(a) - \lambda(b)$ inside $C^*_r(T)$ is the whole of $C^*_r(T)$. This element is not unique with this property. Indeed, the same holds true for $\lambda(x)$, where $x$ is an element of the complex group algebra $\mathbb{C}T$ such that $\pi(x) = 0$. Bleak and Juschenko [2] established a partial converse to Theorem 4.5 by proving that such $x$ exists if the Thompson group $F$ is non-amenable. The equivalence of the first two statements in the following proposition is a strengthening of their result, in the sense that we exhibit a concrete such element $x$.

**Proposition 4.8.** With $a$ and $b$ as above, the following are equivalent:

1. The Thompson group $F$ is non-amenable.
2. The closed two-sided ideal generated by $1 + \lambda(ab) - \lambda(a) - \lambda(b)$ in $C^*_r(T)$ is all of $C^*_r(T)$. 

(3) The closed convex hull of \( \{ \lambda(hah^{-1}) + \lambda(hbh^{-1}) : h \in T \} \) contains 0.

(4) The closed two-sided ideal generated by \( 1 + \lambda(ab) - \lambda(a) - \lambda(b) \) in \( C^*_r(F) \) is all of \( C^*_r(F) \).

(5) The closed convex hull of \( \{ \lambda(hah^{-1}) + \lambda(hbh^{-1}) : h \in F \} \) contains 0.

Proof. We prove that (1) implies (4) and (5), that (4) implies (2), that (5) implies (3), that (2) implies (1), and that (3) implies (1).

Some of these implications are straightforward. That (4) implies (2) follows from the fact that the inclusion \( C^*_r(F) \subseteq C^*_r(T) \) is unital. That is, if (4) holds, then the closed two-sided ideal in \( C^*_r(T) \) generated by \( 1 + \lambda(ab) - \lambda(a) - \lambda(b) \) contains \( C^*_r(F) \), and, in particular, the unit. That (5) implies (3) follows directly from the fact that the inclusion \( C^*_r(F) \subseteq C^*_r(T) \) is isometric.

Let us prove that (2) implies (1), and that (3) implies (1), or, equivalently, that the negation of (1) implies the negation of (2), as well as the negation of (3). Assume that \( F \) is amenable. Then by Proposition 4.4, \( \pi \) is weakly contained in the left regular representation of \( T \), that is, there exists a *-homomorphism \( \sigma \) so that \( \pi = \sigma \circ \lambda \). As explained in the proof of Theorem 4.5, \( 1 + \lambda(ab) - \lambda(a) - \lambda(b) \) is in the kernel of \( \sigma \), which is a non-trivial ideal. Hence the negation of (2) holds. Since \( \sigma \) is a contraction, to prove the negation of (3) it suffices to show that \( \|x\| \geq 1 \), for every \( x \in \text{conv} \{ \pi(hah^{-1}) + \pi(hbh^{-1}) : h \in T \} \). As explained in the proof of Theorem 4.5, that \( \pi(a) + \pi(b) = 1 + \pi(ab) \). Hence

\[
\text{conv} \{ \pi(hah^{-1}) + \pi(hbh^{-1}) : h \in T \} = 1 + \text{conv} \{ \pi(hbh^{-1}) : h \in T \}.
\]

If \( x \) is a finite convex combination of elements of the form \( habh^{-1} \) with \( h \in T \), then \( x \in \mathbb{R}_+T \). By Lemma 4.7, we conclude that \( \|1 + x\| \geq \|1\| = 1 \). Continuity then ensures that \( \|1 + x\| \geq 1 \), for every \( x \in \text{conv} \{ \pi(hah^{-1}) + \pi(hbh^{-1}) : h \in T \} \). Hence the negation of (3) holds.

We are left to prove that (1) implies (4) and (5), so assume that \( F \) is non-amenable. To prove (4), it suffices to show that the closed two-sided ideal in \( C^*_r(F) \) generated by \( 1 + \lambda(ab) - \lambda(a) - \lambda(b) \) contains \( 1 \). As it clearly contains the set

\[
\text{conv} \left\{ \lambda(h)(1 + \lambda(ab) - \lambda(a) - \lambda(b))\lambda(h)^*: h \in F \right\}
\]

\[
= 1 + \text{conv} \left\{ \lambda(habh^{-1}) - \lambda(hah^{-1}) - \lambda(hbh^{-1}) : h \in F \right\},
\]

it suffices to show that the closed convex hull on the right hand side contains 0. Let \( \varepsilon > 0 \) be given. As mentioned in Remark 4.6, \( F \) has the unique trace property. Let \( F_1 \) and \( F_2 \) denote the subgroups of \( F \) consisting of elements \( f \in F \) so that \( f(x) = x \), for all \( x \in [0, \frac{1}{2}] \), and \( f(x) = x \), for all \( x \in [\frac{1}{2}, 1] \), respectively. Note that \( b \in F_1 \) and \( a \in F_2 \). Since \( F_1 \) and \( F_2 \) are both isomorphic to \( F \), they both have the unique trace property. It follows from [11, Corollary 4.4] that there exist positive real numbers \( s_1, \ldots, s_n, t_1, \ldots, t_m \) with \( \sum_{k=1}^n s_k = \sum_{k=1}^m t_k = 1 \), and \( g_1, \ldots, g_n \in F_1 \) and \( h_1, \ldots, h_m \in F_2 \) so that

\[
\left\| \sum_{k=1}^n s_k \lambda(g_k b g_k^{-1}) \right\| < \varepsilon \quad \text{and} \quad \left\| \sum_{k=1}^m t_k \lambda(h_k a h_k^{-1}) \right\| < \varepsilon.
\]

Let us, for simplicity of notation, denote the elements \( \sum_{k=1}^m t_k \lambda(h_k a h_k^{-1}) \) and \( \sum_{k=1}^n s_k \lambda(g_k b g_k^{-1}) \) by \( a \) and \( b \), respectively. As the elements of the
subgroups $F_1$ and $F_2$ commute, it is straightforward to check that
\[
\sum_{i=1}^n \sum_{j=1}^m s_i t_j \lambda(g_i h_j) (\lambda(a) + \lambda(b)) \lambda(g_i h_j)^* = \tilde{a} + \tilde{b}.
\]
Since $\sum_{i=1}^n \sum_{j=1}^m s_i t_j = 1$, the left hand side belongs to the convex hull of $\{\lambda(h a h^{-1}) + \lambda(h b h^{-1}) : h \in F\}$, and since $\|\tilde{a} + \tilde{b}\| < 2\varepsilon$, we conclude that the convex hull of $\{\lambda(h a h^{-1}) + \lambda(h b h^{-1}) : h \in F\}$ contains elements of arbitrarily small norm. Thus (5) holds.

Using similar calculations, it is straightforward to check that
\[
\sum_{i=1}^n \sum_{j=1}^m s_i t_j \lambda(g_i h_j) (\lambda(ab) - \lambda(a) - \lambda(b)) \lambda(g_i h_j)^* = \tilde{a} \tilde{b} - \tilde{a} - \tilde{b}.
\]
Again, the left hand side belongs to the convex hull of $\{\lambda(h a b h^{-1}) - \lambda(h a h^{-1}) - \lambda(h b h^{-1}) : h \in F\}$.

Since $\|\tilde{a} \tilde{b} - \tilde{a} - \tilde{b}\| < \varepsilon^2 - 2\varepsilon$, we conclude that the closure of this convex hull contains 0, and as mentioned above, this means that the ideal generated by $1 + \lambda(ab) - \lambda(a) - \lambda(b)$ in $C^*_r(F)$ is the whole of $C^*_r(F)$. This proves that (1) implies (4) and (5), which concludes the proof. \qed

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