Effect of the type I to type II Weyl semimetal topological transition on superconductivity

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The influence of recently discovered topological transition between type I and type II Weyl semimetals on superconductivity is considered. A set of Gorkov equations for weak superconductivity in Weyl semi-metal under topological phase transition is derived and solved. The critical temperature and superconducting gap both have spike in the point the transition point as function of the tilt parameter of the Dirac cone determined in turn by the material parameters like pressure. The spectrum of superconducting excitations is different in two phases: the sharp cone pinnacle is characteristic for a type I, while two parallel almost flat bands, are formed in type II. Spectral density is calculated on both sides of transition demonstrate different weight of the bands. The superconductivity thus can be used as a clear indicator for the topological transformation. Results are discussed in the light of recent experiments.

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I. INTRODUCTION

Effect of the Fermi surface topology on properties of a crystalline conductor was a subject of theoretical investigations over the year\cite{1}. Experimentally a continuous deformation of the Fermi surface (without changing the chemical nature of the crystal) can be experimentally achieved by external factors like pressure and electric field. Historically transitions associated with change of topology were called ”2.5 transition”\cite{2}. Transport in materials with low electron density is especially sensitive to modification of the Fermi surface\cite{3} while these transitions were predicted to make a great impact on the superconducting stat\cite{4}. Recently new class of such materials made the phenomenon important in a very different setting. A broad number of low electron density 2D and 3D Weyl semi-metals were discovered. Most of them are characterized by linear dispersion relation near the Fermi surface. Topology and even dimensionality of the small Fermi surface in these materials is linked to the Berry phases of their band structure\cite{5}. The first material of this class, graphene\cite{6}, exhibits the highest symmetry leading to linear ultra - relativistic spectrum, however most of the other materials are anisotropic. Examples include 2D Weyl semi-metals (WSM) silicene, germanene and borophene\cite{7,8} and 3D crystals\cite{9} Na$_3$Bi , and\cite{10} Cd$_3$As$_2$ and numerous layered organic compounds\cite{11,12}. Topological insulators (TI) like Bi$_2$Se$_3$ and others\cite{13} generally have Dirac cones on their surfaces of topological insulators.

It was realized recently that this variety of novel materials should be differentiated between the more isotropic ”type I’’ Weyl semimetal\cite{14,15} see Fig.1a, and highly anisotropic type II Weyl semi-metals in which the cone of the linear dispersion relation is tilted beyond a critical angle (Fig.1b). The newer type-II Weyl fermion materials, WTe$_2$ exhibits exotic phenomena such as angle dependent chiral anomalous\cite{16}. Discovery of a Weyl semimetal in TaAs offers the first Weyl fermion observed in nature and dramatically broadens the classification of topological phases. Other dichalcogenides like\cite{17} MoTe$_2$ and\cite{18} PtTe$_2$ were demonstrated to be type II Weyl semi - metals. In particular, the series Mo$_x$W$_{1-x}$Te$_2$ inversion-breaking, layered, tunable semimetals is already under study as a promising platform for new electronics and recently proposed to host Type II Weyl fermions\cite{19}. Some other materials, like layered organic compound $\alpha-(BEDT-TTF)_2$I$_3$, were long suspected\cite{20} to be a 2D type-II Dirac fermion. Several classes of materials were predicted via band structure calculations to undergo the I to II transition while doping or pressure is changed\cite{21}.

Theoretically physics of the transitions between the type I to type II Weyl semi-metals were considered in the context of superfluid phase\cite{22} A of He$_3$, layered organic materials in 2D\cite{23} and 3D Weyl semi-metals\cite{24}. The pressure modifies the spin orbit coupling that in turn determines the topology of the Fermi surface of these novel materials\cite{25}. Very recently the topological transition in Weyl semi - metal under pressure was observed\cite{26}.

This transition is an ideal example of the ”2.5 transition” mentioned above. Both the Fermi surface reconstruction and low electron density are present at the extreme in these materials. It is generally difficult to find good indicators for such a transition. Since superconductivity is especially affected by the ”2.5 transition”, it might serve as the indicator. Indeed some Weyl materials are known to be superconducting. Surface of the well known TI Bi$_2$Se$_3$ exhibits under certain conditions superconductivity of up to 8K, some organic materials like\cite{27} $\kappa-(BEDT-TTF)_2$X with $X = I_3$ or other anions\cite{28} are superconducting.

A detailed study of superconductivity in TI under hydrostatic pressure revealed a curious dependence of critical temperature of the superconducting transition on pressure\cite{29}. In intercalated $\text{Sr}_{0.065}\text{Bi}_2\text{Se}_3$ single crystals\cite{30} considered to be a pure material ambient weak superconductivity first is suppressed, but at high pressure of 6GPa reappears and reaches relatively high $T_c$ of 10K that persists till 80GPa. The increase is not gradual, but rather abrupt in the region of 15GPa\cite{31}. Superconductivity in similar TI compounds\cite{32,33,34} Bi$_2$Te$_3$ , Bi$_2$I$_3$ and 3D Weyl semi-metal\cite{35} HfTe$_5$ were also studied experimentally. The critical temperature $T_c$ in some of these systems shows a sharp maximum as a function of pressure. This contrasts with generally smooth dependence on pressure in other superconductors (not suspected to be Weyl materials) like a high $T_c$ cuprate\cite{36} YBCO. Various mechanisms of superconductivity in Dirac semi – metals and topological insulators turned superconductors have been considered theoretically\cite{37,38,39}. A theory predicted possibility of superconductivity in the type II Weyl semimetals was developed recently in the framework of Eliashiberg mode\cite{40}.

We show in this paper that superconducting critical temperature and energy gap features is an efficient marker of the topological type I to type II transition under pressure. The $s$-wave pairing in a general Weyl semi-metal with tilted Dirac cones is considered in the framework of the boson mediated adiabatic regime. We calculate the critical temperature $T_c$ and the energy gap $\Delta$ at zero temperature for arbitrary tilt (controlled by pressure in certain materials) and find their sharp increase at topological transition point from type I to type II Weyl semi-metal.

II. TYPE I AND TYPE II WEYL SEMI - METAL WITH LOCAL PAIRING INTERACTION.

Weyl material typically possesses several sublattices. We exemplify the effect of the topological transition on superconductivity using the simplest possible model with just two sublattices denoted by $\alpha = 1, 2$. The band structure
FIG. 1. Spectrum of normal Weyl semimetal a. type I \((w/v = 0.5)\). b. Strongly tilted Dirac cone for the type II semimetals \((w/v = 1.2)\).

near the Fermi level of a 2D Weyl semi-metal is well captured by the following Hamiltonian,

\[
K = \int_r \psi_\alpha^L + (r) K_{\alpha\beta}^L \psi_\beta^L (r) + \psi_\alpha^R + (r) K_{\alpha\beta}^R \psi_\beta^R (r);
\]

\[
K_{\alpha\beta}^{L,R} = -i\hbar v \left( \nabla_x \sigma^x_{\alpha\beta} \mp \nabla_y \sigma^y_{\alpha\beta} \right) + (-i\hbar w \cdot \nabla - \mu) \delta_{\gamma\delta}.
\]

Here \(v\) is Fermi velocity, \(\mu\) - chemical potential, \(\sigma\) are Pauli matrices in the sublattice space and \(s\) is spin projection. Generally there are a number of pairs of points (Dirac cones) constituting the Fermi "surface" of such a material at chemical potential \(\mu = 0\). We restrict ourself to the case of just one left handed \((L)\) and one right handed \((R)\) Dirac points, typically but not always separated in the Brillouin zone. Generalization to several pairs is straightforward. The 2D velocity vector \(w\) defines the tilt of the cone, see dispersion relation of one of the cones in Fig.1. The graphene - like dispersion relation in Fig.1a for \(w = 0.5\) represents the type I Weyl semi-metal, while when the length of the tilt vector exceeds \(v\), Fig. 1b, the material becomes a type II Weyl semi - metal.

The effective electron-electron attraction due to either electron - phonon attraction and Coulomb repulsion (pseudopotential) or some unconventional pairing mechanism creates pairing. We assume that different valleys are paired independently and drop the valley indices (multiplying the density of states by \(2N_f\)). Further we assume the local singlet \(s\)-channel interaction Hamiltonian

\[
V = \frac{g^2}{2} \int dr \psi_{\alpha}^{\dagger} (r) \psi_{\beta}^{\dagger} (r) \psi_{\beta} (r) \psi_{\alpha} (r).
\]

As usual the interaction has a cutoff frequency \(\Omega\), so that it is active in an energy shell of width \(2\hbar \Omega\) around the Fermi level. For the phonon mechanism it is the Debye frequency.
FIG. 2. Spectrum of superconducting quasiparticles as function of momentum (in units of $\hbar \Omega/v$) for (a) Type I ($w/v = 0.5$) and (b) Type II ($w/v = 1.2$) Weyl semimetal for chemical potential $\mu = 3$ and energy gap $\Delta = 0.25$ (in units of phonon energy $\hbar \Omega$).

III. SPECTRUM OF EXCITATIONS IN THE SUPERCONDUCTING STATE.

Finite temperature properties of the condensate are described by the normal and the anomalous Matsubara Greens functions,

$$
G_{\alpha\beta}^{rs}(r_\tau, r'_\tau') = -\left\langle T_\tau \psi_{\alpha}^r(r_\tau) \psi_{\beta}^{s+}(r'_\tau') \right\rangle = \delta_{rs} g_{\alpha\beta}(r - r', \tau - \tau'); \\
F_{\alpha\beta}^{rs}(r_\tau, r'_\tau') = \left\langle T_\tau \psi_{\alpha}^r(r_\tau) \psi_{\beta}^{s}(r'_\tau') \right\rangle = -\varepsilon_{rs} f_{\alpha\beta}(r - r', \tau - \tau'); \\
F^{+rs}_{\alpha\beta}(r_\tau, r'_\tau') = \left\langle T_\tau \psi_{\alpha}^{s+}(r_\tau) \psi_{\beta}^{s+}(r'_\tau') \right\rangle = \varepsilon_{rs} f^{+}_{\alpha\beta}(r - r', \tau - \tau').
$$

The second equality in each line relies on homogeneity and unbroken invariance under spin rotations. The gap function in the $s$-wave channel is

$$
\Delta_{\alpha\gamma} = -\frac{g^2}{4} \varepsilon_{s1s2} \left\langle \psi_{\alpha}^{s+}(r_\tau) \psi_{\gamma}^{s2}(r_\tau) \right\rangle = \sigma^x_{\alpha\gamma} \Delta.
$$

In terms of Fourier transforms, $g_{\gamma\kappa}(r, \tau) = T \sum_{\omega p} \exp \left[ i (\omega_\tau + p \cdot r \cdot \tau) \right] g_{\gamma\kappa}(\omega, p)$, the Gorkov equations read (see Appendix A):

$$
(v_p \cdot \sigma_\beta + (i\omega + \mu - wp_x) \delta_{\gamma\beta}) g_{\beta\kappa}(\omega, p) + \Delta \sigma_\alpha^{s+} f^{+}_{\alpha\kappa}(\omega, p) = \delta_{\gamma\kappa}; \\
(v_p \cdot \sigma_\beta + (-i\omega + \mu - wp_x) \delta_{\gamma\beta}) f^{+}_{\beta\kappa}(\omega, p) - \Delta^* \sigma_\alpha^{s+} g_{\alpha\kappa}(\omega, p) = 0.
$$

The Matsubara frequency at temperature $T$ takes values $\omega_n = \pi T (2n + 1)$, and units are chosen such that $\hbar = 1$. We have chosen coordinates in such a way that the vector $w$ causing the tilt of the Dirac cone is oriented along the $x$ axis. In matrix form (in sub-lattice space) we obtain the solution (see Appendix A)

$$
\hat{g}(\omega, p) = \sigma^x \left( p \cdot \sigma^t + (-i\omega + \mu - wp_x) I \right) M^{-1}; \\
\hat{f}^{+}(\omega, p) = M^{-1} \Delta^*;
$$

where

$$
M = (v_p \cdot \sigma + (i\omega + \mu - wp_x) I) \sigma^x \left( v_p \cdot \sigma^t + (-i\omega + \mu - wp_x) I \right) + |\Delta|^2 \sigma^x,
$$
FIG. 3. Spectral density $A(p, E)$ for type I (a) ($w/v = 0.5$) and type II (b) ($w/v = 1.2$) superconducting semimetals. Here $p_x = 1, p_y = 0.1$ (green), 0.4 (red), 0.8 (blue) in units of $\hbar\Omega/v$. Here $\mu = 3, \Delta = 0.25$ (in the units of cut off phonon energy $\hbar\Omega$).

and $I$ is the identity matrix. The determinant of $M$,

$$\det M = -\left(\Delta^2 + \omega^2 + (vp - \mu + wp_x)^2\right)\left(\Delta^2 + \omega^2 + (-vp - \mu + wp_x)^2\right)$$

(8)

continued to physical energy, $\omega \to -iE$, gives the spectrum of quasiparticles

$$E^2 = \Delta^2 + (\pm vp - \mu + wp_x)^2,$$

(9)

depicted for type I and type II Weyl semimetals in Fig.2. The quasi-particle spectrum is very different. In a superconducting graphene-like material for $\mu >> \Delta$, the energy gap is minimal on the circle of radius $\mu/v$, see Fig.2a. Note that the sharp cone pinnacle in the excitations spectrum wedge is formed at the former cone location. For the type II Weyl semi-metal turned superconductor, Fig.2b, the low energy spectrum consists of two parallel flat bands, while the pinnacle disappears.

A. Spectral density.

The spectral density function

$$A(E, p) = -\frac{1}{\pi} \text{Im} \text{Tr} \hat{g}(-iE + \eta, p);$$

(10)

$$\text{Tr} \hat{g}(\omega, p) = \frac{2}{\det M} \left\{ \left(v^2p^2 - (\mu - wp_x - i\omega)^2\right)(\mu - wp_x + i\omega) - \Delta^2(\mu - wp_x - i\omega) \right\}$$

presented in Fig. 3, where different weight of the dispersion law branches accompanied Type I to Type II transition. Here $\eta$ is the "disorder" parameter, $\hat{g}(\omega, p)$ is defined by the Eq.6.

IV. CRITICAL TEMPERATURE AND THE ENERGY GAP.

The gap as function of the material parameters is determined by the equation

$$\Delta^* = \frac{\Delta^*}{2(2\pi)^3} T \sum_\omega \int dp \text{ Tr} [\sigma_x M^{-1}],$$

(11)
resulting in at zero temperature
\[
\frac{1}{g^2} = \frac{1}{(2\pi)^3} \int d\omega dp \frac{v^2 p^2 + \Delta^2 + (\mu - wp_x)^2 + \omega^2}{\left( \Delta^2 + (vp + wp_x - \mu)^2 + \omega^2 \right) \left( \Delta^2 + (vp - wp_x + \mu)^2 + \omega^2 \right)}.
\] (12)

Performing integration on $\omega$ and momenta subject to restriction of being inside the energy shell of $\Omega$ around the Fermi level, see Fig. 4 one obtains in the adiabatic limit $\mu >> \Omega >> \Delta$, after integration over azimuthal angle (see Appendix B for details),

\[
\frac{1}{g^2} = \frac{\mu}{4\pi v^2} f(\kappa) \log \left[ \frac{2\Omega}{\Delta} \right].
\] (13)

Here $\kappa = w/v$ is the anisotropy parameter. The function $f$ is:

\[
f(\kappa) = \begin{cases} 
\frac{2}{(1-\kappa^2)^{1/2}} & \text{for } \kappa < 1 \\
\frac{2\kappa^2}{\pi(\kappa^2-1)^{1/2}} \left\{ 2\sqrt{1+\kappa} - 1 + \log \frac{2(\kappa^2-1)}{\kappa(1+\sqrt{1+\kappa})^2} \right\} & \text{for } \kappa > 1
\end{cases}
\] (14)

where $\delta = \pi a\Omega/v\hbar$ is the cutoff parameter, $a$ is the interatomic distance. (The effective density of states $f(\kappa)$ formally diverges at $\kappa = 1$. It’s result of linear dispersion relation in the model Weyl Hamiltonian. Indeed, the linear approximation of the dispersion relation in the Weyl semimetal is valid only for small neighborhood of the Dirac point. At higher energies the band spectrum is nonlinear and cut off by the band width $1/a$. Actual cutoff is not very important since the singularity is logarithmic).

The superconducting gap in this case therefore is (returning to physical units)

\[
\Delta = 2\hbar \Omega \exp \left[ -\frac{1}{\lambda f(\kappa)} \right].
\] (15)

where $\lambda = N(\mu) g^2$ is the conventional effective electron - electron dimensionless coupling constant and $\log [\gamma E] = 0.577$. Dependence is presented in Fig.5. The effective density of states (DOS) $D(E, \kappa) = N(E \to \mu) f(\kappa)$ in the
normal state for $N_f$ pairs of Dirac cones is $\frac{2N_f}{4\pi v^2}\mu f(\kappa)$. This DOS coincides with the spectral density given in Eq. (10) for $\Delta = 0$ integrated over momenta, $D(E, \kappa) = \int d\mathbf{p} A(E, \mathbf{p}, \kappa)$.

The critical temperature is defined from the gap equation, when the gap $\Delta$ vanishes:

$$\frac{1}{g^2} = \frac{T}{(2\pi)^2} \int d\mathbf{p} \sum_n \frac{\nu^2 p^2 + (\mu - wp_x)^2 + \omega_n^2}{[(wp_x + \mu)^2 + \omega_n^2][[(wp_x + \mu)^2 + \omega_n^2]}.$$ (16)

The sum over Matsubara frequencies can be done,

$$\frac{1}{g^2} = \frac{1}{4(2\pi)^2} \int d\mathbf{p} \left\{ \frac{\tanh |wp_x + \mu|}{|wp_x + \mu|} + \frac{\tanh |wp_x + \mu|}{|wp_x + \mu|} \right\}.$$(17)

In the adiabatic approximation, $\mu >> \Omega$, performing integration one obtains:

$$\frac{1}{g^2} \approx \frac{\mu f(\kappa)}{4\pi v^2} \left( \log \frac{\Omega}{2T_c} \tanh \left[ \frac{\Omega}{2T_c} \right] - \int_{E=0}^{\Omega} \frac{\log E}{\cosh^2 [E/2T_c]} \right) \equiv f(\kappa) R(T_c, \Omega, \mu).$$ (18)

Eq.(18) gives for critical temperature

$$T_c = 1.14\Omega \exp \left[ -\frac{1}{\lambda f(\kappa)} \right],$$ (19)

Fig.5 demonstrates that the critical temperature has a sharp spike at the transition point $w = v$. The ratio $\frac{2\Delta}{T_c}$ is still universal for any $w/v$.

To summarize, superconductivity in Weyl semi-metal with tilted Dirac cones that in the normal phase undergoes a topological ("2.5") transition from type I to type II semi-metal was theoretically considered. We calculated in the framework of the phonon mediated pairing model the spectrum of the superconducting quasi-particles, spectral density, critical temperature and the superconducting gap as function of the tilt parameter $w/v$. The critical temperature and the superconducting energy gap of Weyl semi-metal have a spike at the transition point $w/v = 1$. The quasi-particle spectrum in the superconducting state qualitatively discriminates between type I and type II see Fig.2. In particular the sharp cone pinnacle in the excitations spectrum wedge typical for a type I disappears, two parallel nearly flat bands, are formed. The spectral density $A(p, E)$ is also undergoing modification under type I to type II topological phase transition, see Fig.3.
V. DISCUSSION AND CONCLUSIONS.

We discuss next an application of the results to recent measurement on superconductivity of a compound that have Weyl semi-metal signatures under pressure. In particular, in intercalated Sr_{0.063}Bi_{2}Se_{3} single crystal superconductivity first is suppressed, but at high pressure of 6GPa reappears and reaches relatively high Tc of 10K that persists till 80GPa. The increase is not gradual, but rather abrupt in the region of 15GPa. It was discovered recently that, while parameters of the system (the electron - phonon coupling, Debye frequency \Omega) between two structural phase transitions smoothly depend on a pressure, the tilt parameter is very sensitive to a stress. \cite{26–28} Our result might provide an explanation of a spike in dependence of Tc on pressure in single crystals observed in many superconducting semi-metals. \cite{25,28}

The experimental situation is not unambiguous. There are structural phase transitions that might cause an abrupt change of electron - phonon coupling. Efforts have been made to associate the changes of (\mu >> \Omega) in Weyl semi-metals has been addressed in detail by DasSarma’s group and in our earlier paper \cite{30}. The increase is not gradual, but rather abrupt in the region of 15GPa. The increase is not gradual, but rather abrupt in the region of 15,21

Problem of the applicability of the BCS approach and a related issue of Migdal’s theorem i.e. adiabatic limit (\mu >> \Omega) in Weyl semi-metals has been addressed in detail by DasSarma’s group and in our earlier paper. \cite{21} The energy gap in this systems does not exceed ten Kelvins, while \Omega is a hundred of Kelvins and chemical potential in thousands as in conventional optical phonon BCS. This justifies our BCS approach.

Although a very specific simple model of two dimensional single pair of Dirac cones was employed, it is expected that generalization to similar physical systems like the surfaces of 3D topological insulators and 3D Weyl semi - metals will lead to similar conclusions.

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VI. APPENDIX A: GORKOV EQUATIONS FOR A TWO SUBLATTICE SYSTEM.

In this Appendix Gorkov equations for a two sublattice system (index \alpha = 1, 2) in Matsubara representation \( t = -i \tau \) are briefly derived.

Equation of motion for creation and annihilation Matsubara operators read (\( h = 1 \))

\[
\frac{\partial \psi_\alpha^+}{\partial \tau} = [H, \psi_\alpha^+] ; \quad \frac{\partial \psi_\alpha}{\partial \tau} = [H, \psi_\alpha].
\]

Defining the Matsubara Green function as

\[
G_{\alpha \beta}^{ts}(X, X') = \delta^{ts} \delta_{\alpha \beta} (X, X') = - \langle T_\tau \left\{ \psi_\alpha^+ (X) \psi_\beta^+ (X') \right\} \rangle ; \quad (21)
\]

\[
F_{\alpha \beta}^{ts}(X, X') = - \varepsilon^{ts} f_{\alpha \beta} (X, X') = \langle T_\tau \left\{ \psi_\alpha (X) \psi_\beta (X') \right\} \rangle ; \quad (22)
\]

\[
F_{\alpha \beta}^{ts}(X, X') = \varepsilon^{ts} f_{\alpha \beta}^+ (X, X') = \langle T_\tau \left\{ \psi_\alpha^+ (X) \psi_\beta^+ (X') \right\} \rangle ,
\]

where \( X \) combines space and time and \( T_\tau \) is the imaginary time ordered product. \cite{32} The time derivatives using the equations of motion are written via the following commutators:

\[
\frac{\partial G_{\alpha \beta}^{ts}(X, X')}{\partial \tau} = - \langle T_\tau \left\{ [H, \psi_\beta^+ (X)] \psi_\alpha^+ (X') \right\} \rangle - \delta^{\tau \alpha} \delta^{\tau \beta} (X - X');
\]

\[
\frac{\partial F_{\alpha \beta}^{ts+}(X, X')}{\partial \tau} = \langle T_\tau \left\{ [H, \psi_\beta^+ (X)] \psi_\alpha^+ (X') \right\} \rangle .
\]

This leads to
\[
\frac{\partial G_{\gamma \kappa}^{xt} (X, X')}{\partial \tau} = - \left\langle T_{\tau} \left\{ - f_{\tau^\prime} \left\{ iv \sigma^i_{\gamma \beta} \delta^o_{\kappa \lambda} (r - r') - \mu' \delta^o_{\gamma \beta} (r - r') \right\} \psi^o_{\beta} (X') \psi^{i+}_{\kappa} (X') \right\} \right\rangle .
\]

(23)

\[
\frac{\partial F^{x+}_{\gamma \kappa} (X, X')}{\partial \tau} = \left\langle T_{\tau} \left\{ f_{\tau^\prime} \left\{ - iv \nabla_{\tau^\prime \lambda} \delta (r'' - r) \sigma^o_{\alpha \gamma} - \mu' \delta (r'' - r) \delta_{\alpha \gamma} \right\} \psi^{x}_{\gamma} (r'' M) \psi^{x+}_{\kappa} (X') \right\} \right\rangle .
\]

(24)

Here \( \mu' = \mu - wp_x \).

The correlators are calculated within the gaussian approximation below.

For the same \( X \), \( F^*_{\alpha \beta} = - \left\langle T_{\tau} \psi_{\beta}^{*} \psi_{\alpha}^{*} \right\rangle \rightarrow F^{x+}_{\alpha \beta} = F_{\beta \alpha} \). In gaussian approximation and using the spin symmetry one obtains and using definition \( \Delta_{\alpha \gamma} = - \frac{g^2}{4} e^{s_{x \tau}} \left\langle \psi^{\alpha} \psi^{\gamma} (X) \right\rangle \). Thus one obtains the first Gorkov equation:

\[
- \frac{\partial}{\partial \tau} g_{\gamma \kappa} (X, X') - iv \sigma^i_{\gamma \beta} \nabla^i_{\tau} g_{\beta \kappa} (X, X') + \mu' g_{\beta \kappa} (X, X') + \Delta_{\alpha \gamma} f^{x+}_{\alpha \kappa} (X, X') = \delta^{\kappa \gamma} \delta (X - X')
\]

(25)

Similarly the second equation takes a form:

\[
\frac{\partial}{\partial \tau} f^{x+}_{\gamma \kappa} (X, X') - iv \sigma^i_{\gamma \beta} \nabla^i_{\tau} f^{x+}_{\beta \kappa} (X, X') + \mu' f^{x+}_{\beta \kappa} (X, X') - \Delta_{\alpha \gamma} g_{\alpha \kappa} (X, X') = 0
\]

(26)

For uniform system, using the Fourier transform, \( g_{\gamma \kappa} (X) = T \sum_{\omega p} \exp \left\{ i \left( -\omega \tau + pr \right) \right\} g_{\gamma \kappa} (\omega, p) \), with Matsubara frequencies, \( \omega_n = 2\pi T (n + 1/2) \), one obtains

\[
(v p \sigma^i_{\gamma \beta} + (i \omega + \mu - wp_x) \delta_{\gamma \beta}) g_{\beta \kappa} (\omega, p) + \Delta_{\alpha \gamma} f^{x+}_{\alpha \kappa} (\omega, p) = \delta^{\kappa \gamma};
\]

(27)

\[
(v p \sigma^i_{\beta \gamma} + (-i \omega + \mu - wp_x) \delta_{\beta \gamma}) f^{x+}_{\beta \kappa} (\omega, p) - \Delta_{\alpha \gamma} g_{\alpha \kappa} (\omega, p) = 0.
\]

(28)

The singlet Ansatz

\[
\Delta_{\alpha \gamma} = \sigma^i_{\alpha \gamma} \Delta
\]

(29)

simplifies the equations:

\[
(v p \sigma^i_{\gamma \beta} + (i \omega + \mu - wp_x) \delta_{\gamma \beta}) g_{\beta \kappa} (\omega, p) + \Delta_{\alpha \gamma} f^{x+}_{\alpha \kappa} (\omega, p) = \delta^{\kappa \gamma};
\]

(30)

\[
(v p \sigma^i_{\beta \gamma} + (-i \omega + \mu - wp_x) \delta_{\beta \gamma}) f^{x+}_{\beta \kappa} (\omega, p) - \Delta_{\alpha \gamma} g_{\alpha \kappa} (\omega, p) = 0.
\]

(31)

Casting this in the matrix form,

\[
(v p \sigma^i + i \omega + \mu - wp_x) \hat{g} + \Delta \sigma^i \hat{f}^+ = I;
\]

(32)

\[
(v p \sigma^{iz} - i \omega + \mu - wp_x) \hat{f}^+ - \Delta^* \sigma^i \hat{g} = 0
\]

(33)

it is easily solved. Substituting \( g \) from the second equation,

\[
\sigma^i (v p \sigma^{iz} + (-i \omega + \mu - wp_x) I) \hat{f}^+ = \Delta^* \hat{g},
\]

(34)

one explicitly obtains the anomalous correlator

\[
\hat{f}^+ = M^{-1} \Delta^*.
\]

(35)
The correlator determining the excitation spectrum consequently is

\[ \hat{g} = \sigma^x \left( p^i \sigma^i + (-i\omega + \mu - wp_x) I \right) M^{-1} \Delta^*. \]

The dispersion relations in the text are defined by zeroes of the determinant,

\[ det [M] = - \left( \Delta^2 + (vp - \mu + wp_x)^2 + \omega^2 \right) \left( \Delta^2 + (vp + \mu - wp_x)^2 + \omega^2 \right). \tag{37} \]

In this section, the details of the calculation of the superconducting gap at zero temperature and of the critical temperature are given.

**VII. APPENDIX B. ZERO TEMPERATURE GAP.**

At zero temperatures the gap is determined by the self consistency equation:

\[ \Delta^* = - \frac{\Delta^*}{2(2\pi)^3} \int d\omega dp \text{Tr} [\sigma_x M^{-1}] \tag{38} \]

At zero temperature the summation over Matsubara frequencies was replaced by integration. The resulting integral,

\[ \frac{1}{g^2} = \frac{1}{(2\pi)^3} \int d\omega dp \frac{vp^2 + \Delta^2 + (\mu - wp_x)^2 + \omega^2}{(\Delta^2 + (vp + wp_x - \mu)^2 + \omega^2)(\Delta^2 + (vp - wp_x + \mu)^2 + \omega^2)}, \tag{39} \]

is first performed over \( \omega \):

\[ \frac{1}{g^2} = \frac{1}{4(2\pi)^2} \int_{\theta=0}^{2\pi} d\theta \int p dp \left\{ \frac{1}{(\Delta^2 + (\epsilon - \mu)^2)^{1/2}} + \frac{1}{(\Delta^2 + (\epsilon + \mu)^2)^{1/2}} - \frac{1}{(\Delta^2 + (\epsilon - \mu)^2)^{1/2} + (\Delta^2 + (\epsilon + \mu)^2)^{1/2}} \right\} \tag{40} \]

The resulting expression is already given in polar coordinates and in addition the energy as function of the polar angle

\[ \epsilon (p, \theta) = vp + wp_x = vp (1 + \kappa \cos \theta), \tag{41} \]

was used. Here \( \kappa = \frac{w}{v} \).

In the adiabatic (BCS) limit, when \( \mu >> \Omega >> \Delta \), and the integration is limited to the shell shown in Fig. 4 in both phases. Changing the integration variable to \( \epsilon \), one obtains:

\[ \frac{1}{g^2} = \frac{1}{4(2\pi)^2} \int_0^{2\pi} d\theta \text{sign} (1 + \kappa \cos \theta) \int_{\epsilon=-\Omega}^{\Omega} \frac{d\epsilon}{(\epsilon + \mu)^2} \left( \Delta^2 + \epsilon^2 \right)^{1/2} = \tag{42} \]

\[ = \frac{\mu}{4(2\pi)^2} f (\kappa) \int_{\epsilon=-\Omega}^{\Omega} \left( \Delta^2 + \epsilon^2 \right)^{1/2} = \tag{43} \]

\[ = \frac{\mu}{4\pi^2} f (\kappa) \log \left[ \frac{\Omega + \sqrt{\Delta^2 + \Omega^2}}{\Delta} \right], \tag{44} \]

Here the integration over azimuthal angle \( \theta \) was obtained by replacing variables \( \theta \) by \( x = v/w + \cos \theta \). In the case \( w/v < 1 \) (type I semimetal phase), \( x > 0 \) and integral over the azimuthal angle reads

\[ f (\kappa) = \frac{1}{\pi} \int_\theta \frac{1}{(1 + \kappa \cos \theta)^2} = \frac{2}{(1 - \kappa^2)^{3/2}}. \tag{45} \]
In the case $w > v$ (type II semimetal phase) the integral might be modified as

$$ f \left( \frac{w}{v} \right) = -\frac{1}{\pi} \int_{v/w}^{0} dx \frac{\text{sign}(x)}{\sqrt{1 - (x - v/w)^2} x^2} = $$

$$ = \frac{1}{\pi} \int_{v/w - 1}^{0} dx \frac{1}{\sqrt{1 - (x - v/w)^2} x^2} - \frac{1}{\pi} \int_{0}^{v/w + 1} dx \frac{1}{\sqrt{1 - (x - v/w)^2} x^2} \tag{47} $$

Represented Eq. (46) in equivalent form

$$ \lim_{\delta \to 0} \frac{1}{\pi} \int_{x=\delta}^{-v/w+1} dx \left( \frac{1}{\sqrt{1 - (x + 1/w)^2}} - \frac{1}{\sqrt{1 - (x - 1/w)^2}} \right) - $$

$$ - \frac{1}{\pi} \int_{-v/w+1}^{v/w+1} dx \frac{1}{\sqrt{1 - (x - v/w)^2} x^2} \tag{48} $$

and performing the integration in (48), we obtain

$$ f(\kappa) = \frac{2\kappa^2}{\pi(\kappa^2 - 1)^{3/2}} \left\{ 2\sqrt{1 + \kappa} - 1 + \log \frac{2(\kappa^2 - 1)}{\kappa(1 + \sqrt{1 + \kappa})^2} \right\} \tag{50a} $$

from which the Eq. (18) of the paper follows. Here $\delta = \pi a\Omega/\hbar$ is the cutoff parameter.

VIII. APPENDIX C. CRITICAL TEMPERATURE.

The critical temperature is defined from the second Gorkov equation with vanishing $\Delta$:

$$ \frac{1}{g^2} = \frac{T}{(2\pi)^2} \int dp \sum_n \frac{v^2 p^2 + (\mu - wp_x)^2 + \omega_n^2}{(vp + wp_x - \mu)^2 + \omega_n^2} \left[ (vp - wp_x + \mu)^2 + \omega_n^2 \right] \tag{51} $$

Performing summation over $\omega_n$, one obtains,

$$ \frac{1}{g^2} = \frac{1}{4(2\pi)^2} \int_{0}^{2\pi} dp \int_{0}^{\pi} \left\{ \tanh \left( \frac{2T}{\Omega} \right) \frac{\epsilon}{|p(v + w\cos \theta)|} \right\} \tag{52} $$

In the adiabatic approximation, $\mu >> \Omega$, we get as in the previous case

$$ \frac{1}{g^2} = \frac{\mu}{8\pi^2} f(\kappa) \left\{ 2 \left( \frac{\log \frac{\Omega}{2T_c}}{2T_c} - \int_{0}^{\Omega} \frac{dz}{\cosh^2 \left( \frac{z}{2T_c} \right)} \right) + \frac{\Omega}{\mu} \right\} \tag{53} $$

In this limit Eq.(53) reads for critical temperature

$$ T_c = \frac{\pi}{4\gamma_E} \Omega \exp \left[ -\frac{4\pi v^2}{\mu g^2 f(\kappa)} \right], \tag{54} $$

where $\log [\gamma_E] = 0.577.$

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I. M. Lifshitz, M. Ya. Azbel, M. I. Kaganov, "Electron Theory of Metals", Springer, 2012.

I. M. Lifshitz, Zh. Eksp. Teor. Fiz. 38 1569 (1960), (Sov. Phys. JETP 11 1130 (1960)).

Ya.M. Blanter, M.I. Kaganov, A.V. Pantsulaya, A.A. Varlamov, Physics Reports, 245, 159 (1994).

C. N. Louis and K. Iyakutti, Phys. Rev. B 67, 094509 (2003).

G. E. Volovik, "Lifshitz transitions via the type-II Dirac and type-II Weyl points", arXiv:1604.00849v5 (2016); Y. Xu, F. Zhang, and C. Zhang, Phys. Rev. Lett., 115, 265304 (2015).

K. Novoselov, A. K. Geim, S. Morozov, D. Jiang, M. Katsnelson, I. Grigorieva, S. Dubonos, and A. Firsov, Nature 438, 197 (2005); M.I. Katsnelson, "Graphene. Carbon in Two Dimensions", Cambridge University Press, 2012.

B. Feng, Jin Zhang, Qing Zhong, W. Li, S. Li, H. Li, P. Cheng, S. Meng, L. Chen & K. Wu, Nature Chemistry 8, 563 (2016); M. E. Dvila, L. Xian, S. Cahangirov, A. Rubio and G. Le Lay, New J. Phys., 16, 095002 (2014); P.-H. Shih, Y.-H. Chiu, J.-Y. Wu, F.-L. Shyu, M.-F. Lin, Coulomb excitations of monolayer germainene, arXiv:1604.07275v4 (2016).

M. Neupane, S.-Y. Xu, R. Sankar, et al., Nat. Commun. 5 3786 (2014); Z. K. Liu, et al., Nat. Mater. 13, 677 (2014).

Z. Wang, X.-F. Zhou, X. Zhang, Q. Zhu, H. Dong, M. Zhao, and A. R. Oganov, Nanot. Lett. 15, 6182 (2015); L. He, Y. Jia, S. Zhang, X. Hong, C. Jin & S. Li, Quantum Materials 1 16014 (2016).

V. Z. Kresin and W. A. Little, ed., “Organic Superconductivity”, Springer Science+Business Media, 1990; V. Vulcanescu, S. Wanka, D. Beckmann, and J. Wosnitza, E. Balthes and D. Schweitzer, W. Strunz and H. J. Keller, Phys. Rev. B 23, 4952 (2005).

S. Katayama, A. Kobayashi, Y. Suzumura, J. Phys. Soc. Japan 11 27 (1952).

F. Sun and J. Ye, "Type I and Type II Weyl fermions, Topological depletions and sub-leading scalings across topological phase transitions", arXiv:1610.03171v1 (2016).

G. E. Volovik, "Lifshitz transitions via the type-II Dirac and type-II Weyl points", arXiv:1604.00849v5 (2016); Y. Xu, F. Zhang, and C. Zhang, Phys. Rev. Lett., 115, 265304 (2015).
28 P. P. Kong, et al, J. Phys. Condens. Matter 25, 362204 (2013).
29 Y. Qi, W. Shi, P. G. Naumov, N. Kumar, W. Schnelle, O. Barkalov, C. Shekhar, H. Borrmann, C. Felser, B. Yan, and S. A. Medvedev, Phys. Rev. B 94, 054517 (2016).
30 P. L. Alireza et al, "Accessing the Full Superconducting Dome in Pristine YBa$_2$Cu$_3$O$_{6+x}$ Under Pressure", arxiv: 1610.09790v1 (2016).
31 S. Das Sarma and Q. Li, Phys. Rev. B 88, 081404(R) (2013); P.M.R. Brydon, S. Das Sarma, H.-Y. Hui, and J. D. Sau, Phys. Rev. B 90, 184512 (2014); D.P. Li, B. Rosenstein, B. Ya. Shapiro, and I. Shapiro, Phys. Rev. B 90, 054517 (2014).
32 M. Alidoust, K. Halterman, and A. A. Zyuzin, "Superconductivity in Type-II Weyl Metals", arXiv:1612.05003v1 (2016).
33 L. Fu and E. Berg, Phys. Rev. Lett. 105, 097001 (2010).
34 J.-L. Zhang et al. Front. Phys., 7, 193 (2012).
35 A. A. Abrikosov, L. P. Gor’kov, I. E. Dzyaloshinskii, "Quantum field theoretical methods in statistical physics", Pergamon Press, New York (1965).
36 R. L. Stillwell, Z. Jenei, S. T. Weir, Y. K. Vohra, and J. R. Jeffries, Phys. Rev. B 93, 094511 (2016); D. VanGennep et al, "Pressure-induced superconductivity in the giant Rashba system BiTeI", arXiv:1610.08038v1 (2016).
37 A. A. Abrikosov, L. P. Gor’kov, I. E. Dzyaloshinskii, "Quantum field theoretical methods in statistical physics", Pergamon Press, New York (1965).