TWO MATRIX WEIGHTED INEQUALITIES FOR COMMUTATORS WITH FRACTIONAL INTEGRAL OPERATORS

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Abstract. In this paper we prove two matrix weighted norm inequalities for the commutator of a fractional integral operator and multiplication by a matrix symbol. More precisely, we extend the recent results of the second author, Pott, and Treil on two matrix weighted norm inequalities for commutators of Calderon-Zygmund operators and multiplication by a matrix symbol to the fractional integral operator setting. In particular, we completely extend the fractional Bloom theory of Holmes, Rahm, and Spencer to the two matrix weighted setting with a matrix symbol.

1. Introduction

Let \( w \) be a weight on \( \mathbb{R}^d \) and let \( L^p(w) \) be the standard weighted Lebesgue space with respect to the norm

\[
\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^d} |f(x)|^p w(x) \, dx \right)^{\frac{1}{p}}.
\]

Furthermore, let \( A_{p,q} \) for \( p,q > 1 \) be the Muckenhoupt class of weights \( w \) satisfying

\[
\sup_{Q \subseteq \mathbb{R}^d} \left( \int_Q w(x) \, dx \right) \left( \int_Q w^{-\frac{q}{p}}(x) \, dx \right)^{\frac{p}{q}} < \infty
\]

where \( f_Q \) is the unweighted average over \( Q \) (which will also occasionally be denoted by \( m_Q \)). When \( p = q \) we write \( A_p := A_{p,p} \) as usual.

Given a weight \( \nu \), we say \( b \in \text{BMO}_\nu \) if

\[
\|b\|_{\text{BMO}_\nu} = \sup_{Q \subseteq \mathbb{R}^d \text{ is a cube}} \frac{1}{\nu(Q)} \int_Q |b(x) - m_Q b| \, dx < \infty
\]

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(where $\nu(Q) = \int_Q \nu$) so that clearly $\text{BMO} = \text{BMO}_\nu$ when $\nu \equiv 1$. Further, given a linear operator $T$, define the commutator $[M_b, T] = M_b T - T M_b$ with $M_b$ being multiplication by $b$. In the papers [HLW16, HLW17] the authors extended earlier work of S. Bloom [Blo85] and proved that if $u, v \in A_p$ and $T$ is any Calderón-Zygmund operator (CZO) then
\begin{equation}
\|[M_b, T]\|_{L^p(u) \to L^p(v)} \lesssim \|b\|_{\text{BMO}_\nu},
\end{equation}
where $\nu = (uv^{-1})^{\frac{1}{q}}$ and it was proved in [HLW17] that if $R_s$ is the $s^{th}$ Riesz transform then
\begin{equation}
\|b\|_{\text{BMO}_\nu} \lesssim \max_{1 \leq s \leq d} \|[M_b, R_s]\|_{L^p(u) \to L^p(v)}.
\end{equation}
Furthermore, let $I_\alpha$ be the fractional integral operator defined by the formula
\begin{equation}
I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-\alpha}} dy, \text{ for } 0 < \alpha < d.
\end{equation}
It was proved in [HRS16] that if $0 < \alpha < d$ and $\alpha/d + 1/q = 1/p$, if $u, v \in A_{p,q}$, and if $\nu = u^{\frac{1}{q}}v^{-\frac{1}{q}}$ then
\begin{equation}
\|[M_b, I_\alpha]\|_{L^p(u^{\frac{1}{q}}) \to L^p(v)} \approx \|b\|_{\text{BMO}(\nu)}.
\end{equation}
On the other hand, matrix weighted extensions and generalizations of (1.1) and (1.2) that surprisingly hold for two arbitrary matrix weights (and provided new results even in the scalar $p = 2$ setting of a single scalar weight) were proved in [IPT], and it is the purpose of this paper to extend the results of [IPT] to the fractional setting, providing matrix weighted extensions of (1.3) that hold for two arbitrary matrix weights. Note that for the rest of this paper we will assume that $0 < \alpha < d$ and $\alpha/d + 1/q = 1/p$.

In particular, for any linear operator $T$ acting on scalar valued functions on $\mathbb{R}^d$, we can canonically extend $T$ to act on $\mathbb{C}^n$ valued functions $\vec{f}$ by the formula $T\vec{f} := \sum_{j=1}^n \left(T \begin{pmatrix} f_j \end{pmatrix}_{\mathbb{C}^n} \right) \vec{e}_j$ where $\{\vec{e}_j\}$ is any orthonormal basis of $\mathbb{C}^n$ (and note that this is easily seen to be independent of the orthonormal basis chosen.) Let $W : \mathbb{R}^d \to M_{n \times n}$ be an $n \times n$ matrix weight (a positive definite a.e. $M_{n \times n}$ valued function on $\mathbb{R}^d$) and let $L^p(W)$ be the space of $\mathbb{C}^n$ valued functions $\vec{f}$ such that
\begin{equation}
\|\vec{f}\|_{L^p(W)} = \left(\int_{\mathbb{R}^d} |W^\frac{1}{2}(x)\vec{f}(x)|^p dx \right)^\frac{1}{p} < \infty.
\end{equation}
Furthermore, for $p, q > 1$ we will say that a matrix weight $W$ is a matrix $A_{p,q}$ weight (see [IM19]) if it satisfies
\[ \|W\|_{A_{p,q}} = \sup_{Q \subset \mathbb{R}^d} \left( \int_Q \left( \int_Q |W^{1/q}(x)W^{-1/q}(y)|^{p'} \, dy \right)^{q/p'} \, dx \right)^{1/q} < \infty \quad (1.4) \]
and when $p = q$ we say $W$ is a matrix $A_p$ weight (see [Rou03]).

Now for scalar weights $u$ and $v$, notice that by multiple uses of the $A_q$ property and Hölder’s inequality we have
\[ m_Q u \approx (m_Q u)^{1/q} (m_Q v)^{-1/q} \approx (m_Q u)^{1/q} (m_Q v)^{-1/q} \approx (m_Q u)^{1/2} (m_Q v)^{-1/2}. \]

Thus, $b \in \text{BMO}_\nu$ when $u$ and $v$ are $A_q$ weights if and only if
\[ \sup_{Q \subset \mathbb{R}^d} \left( \int_Q (m_Q v)^{1/q} (m_Q u)^{-1/q} |b(x) - m_Q b| \, dx \right)^{1/q} < \infty, \]
which is a condition that easily extends to the matrix weighted setting, noting that $A_{p,q} \subset A_q$ when $q > p$, since then $q' < p'$ and so Hölder’s inequality gives us
\[ \int_Q \left( \int_Q |W^{1/q}(x)W^{-1/q}(y)|^{q'} \, dy \right)^{q/q'} \, dx \leq \int_Q \left( \int_Q |W^{1/q}(x)W^{-1/q}(y)|^{p'} \, dy \right)^{p/p'} \, dx. \]

Namely, if $U, V$ are $n \times n$ matrix $A_{p,q}$ weights, then we define \( \text{BMO}^{p,q}_{U,V} \) to be the space of $n \times n$ locally integrable matrix functions $B$ where
\[ \|B\|_{\text{BMO}^{p,q}_{U,V}} = \sup_{Q \subset \mathbb{R}^d} \left( \int_Q \left( m_Q V^{1/q}(x)(B(x) - m_Q B)(m_Q U)^{-1/q} \right)^{1/q} \, dx \right) < \infty \]
so that $\|b\|_{\text{BMO}^{p,q}_{U,V}} \approx \|b\|_{\text{BMO}_\nu}$ if $U, V$ are scalar weights and $b$ is a scalar function. Note that the \( \text{BMO}^{p,q}_{U,V} \) condition is much more naturally defined in terms of reducing matrices, which will be discussed in Section 3.

We will need a definition before we state our first result. We say that a linear operator $R$ acting on scalar functions is a fractional lower bound operator if for any $n \in \mathbb{N}$ and any $n \times n$ matrix weight $W$ we have
\[ \|W\|_{A_{p,q}}^{1/q} \lesssim \|T\|_{L^p(W^{1/q}) \to L^q(W)} \quad (1.5) \]
with the bound independent of $W$ (but not necessarily independent of $n$), and $\|T\|_{L^p(W^{1/q}) \to L^q(W)} < \infty$ if $W$ is a matrix $A_{p,q}$ weight.
Theorem 1.1. Let $T$ be any linear operator acting on scalar valued functions where its canonical $\mathbb{C}^n$ valued extension is bounded from $L^p(W^{\frac{d}{n}})$ to $L^q(W)$ for all $n \times n$ matrix $A_{p,q}$ weights $W$ and all $n \in \mathbb{N}$ with bound depending on $T,n,d,p,$ and $\|W\|_{A_{p,q}}$ (which is known to be true for fractional integral operators, see [IM19, Theorem 1.4].) If $U,V$ are $m \times m$ matrix $A_p$ weights and $B$ is an $m \times m$ locally integrable matrix function for some $m \in \mathbb{N},$ then

$$\|\left[MB,T\right]\|_{L^p(U^{\frac{d}{n}},W)\rightarrow L^q(V)} \lesssim \|B\|_{BMO_{V,U}^{p,q}}$$

(1.6)

with bounds depending on $T,m,d,p,$ $\|U\|_{A_{p,q}}$ and $\|V\|_{A_{p,q}}.$

Furthermore, for any fractional lower bound operator $T$ we have the lower bound estimate

$$\|B\|_{BMO_{V,U}^{p,q}} \lesssim \|\left[MB,T\right]\|_{L^p(U^{\frac{d}{n}},W)\rightarrow L^q(V)}$$

(1.7)

Like in [IPT], we will use matrix weighted arguments inspired by [GPTV04] in the next section to prove Theorem 1.1 in terms of a weighted BMO quantity $\|B\|_{\tilde{BMO}_{V,U}^{p,q}}$ that is equivalent to $\|B\|_{BMO_{V,U}^{p,q}}$ when $U$ and $V$ are matrix $A_{p,q}$ weights (see Theorem 3.1) but is much more natural for more arbitrary matrix weights $U$ and $V.$ More precisely, define

$$\|B\|_{\tilde{BMO}_{V,U}^{p,q}} = \sup_{Q \subseteq \mathbb{R}^d} \frac{1}{|Q|} \int_Q \left( \int_Q \left( V^{\frac{1}{q}}(x)(B(x) - B(y))U^{-\frac{1}{q}}(y) \right)^{p'} dy \right)^{\frac{1}{p'}} dx.$$

(1.8)

We will then give relatively short proofs of the following two results in Section 2.

Lemma 1.2. Let $T$ be any linear operator defined on scalar valued functions where its canonical $\mathbb{C}^n$ valued extension $T$ for any $n \in \mathbb{N}$ satisfies

$$\|T\|_{L^p(W^{\frac{d}{n}},W)\rightarrow L^q(W)} \leq \phi(\|W\|_{A_{p,q}})$$

for some positive increasing function $\phi$ (possibly depending on $T,d,n,p,q.$) If $U,V$ are $m \times m$ matrix $A_{p,q}$ weights and $B$ is a locally integrable $m \times m$ matrix valued function for some $m \in \mathbb{N},$ then

$$\|\left[MB,T\right]\|_{L^p(U^{\frac{d}{n}},W)\rightarrow L^q(V)} \leq \|B\|_{\tilde{BMO}_{V,U}^{p,q}} \phi \left( 3^{\frac{d}{q}} \left( \|U\|_{A_{p,q}} + \|V\|_{A_{p,q}} \right) + 1 \right)$$

Lemma 1.3. If $T$ is any fractional lower bound operator then for any $m \times m$ matrix $A_{p,q}$ weights $U,V$ and an $m \times m$ matrix symbol $B$ we
have
\[ \|B\|_{\widetilde{\text{BMO}}_{p,q}^V,U} \lesssim \|[M_B,T]\|_{L^p(U^{\frac{q}{p}}) \to L^q(V)} \]
where the bound depends possibly on \(n,p,d\) and \(T\) but is independent of \(U\) and \(V\).

As in [IPT], we will prove that the fractional integral operator is a fractional lower bound operator in Section 4 by utilizing the Schur multiplier/Wiener algebra ideas from [LT13], and thus recover (1.7). These arguments will in fact prove the following (see [IPT] for an analogous result with respect to the Riesz transforms). Here, for ease of notation, we set \(U' = U^{\frac{q'}{q}} - 1\) and \(V' = V^{\frac{q'}{q}} - 1\).

**Theorem 1.4.** Let \(U\) and \(V\) be any (not necessarily \(A_p\)) matrix weights. If \(B\) is any locally integrable \(m \times m\) matrix valued function then
\[ \max \left\{ \|B\|_{\widetilde{\text{BMO}}_{p,q}^V,U}, \|B\|_{\widetilde{\text{BMO}}_{q',p'}^{U',V'}} \right\} \lesssim \|[M_B,I_\alpha]\|_{L^p(U^{\frac{q}{p}}) \to L^q(V)}. \tag{1.9} \]

Note that the two quantities \(\|B\|_{\widetilde{\text{BMO}}_{p,q}^V,U}\) and \(\|B\|_{\widetilde{\text{BMO}}_{q',p'}^{U',V'}}\) are equivalent when \(U,V \in A_{p,q}\) (which will be proved in Section 4) and in general should be thought of as “dual” matrix weighted BMO quantities. Finally, we will show that an Orlicz “bumped” version of these conditions are sufficient for the general two matrix weighted boundedness of fractional integral operators. In particular, we will prove the following result in Section 5 (see [IPT] for an analogous result for Calderon-Zygmund operators)

**Proposition 1.5.** Let \(U\) and \(V\) be any \(m \times m\) matrix weights, and suppose that \(C\) and \(D\) are Young functions with \(\overline{\text{D}} \in B_{p,q}\) and \(\overline{\text{C}} \in B_{q'}\).

Then
\[ \|[M_B,I_\alpha]\|_{L^p(U^{\frac{q}{p}}) \to L^q(V)} \lesssim \min \{\kappa_1, \kappa_2\} \]
where
\[ \kappa_1 = \sup_Q \|V^{\frac{q}{p}}(x)(B(x) - B(y))U^{-\frac{1}{2}}(y)\|_{C_x,Q} \|D_y,Q} \]
\[ \kappa_2 = \sup_Q \|V^{\frac{q'}{p}}(x)(B(x) - B(y))U^{-\frac{1}{2}}(y)\|_{D_y,Q} \|C_x,Q} \]

We refer the reader to Section 5.2 in [CUIM18] for the standard Orlicz space related definitions used in the statement of Proposition 1.5.

It is important to emphasize that Theorem 1.4 and Proposition 1.5 are new, even in the scalar setting of a single weight.
2. Intermediate fractional upper and lower bounds

As stated in the introduction, we will give short proofs of Lemma 1.2 and Lemma 1.3 in this section, beginning with Lemma 1.2.

2.1. Proof of Lemma 1.2. Define the $2 \times 2$ block matrix-valued function $Φ = Φ_{U,V,B} : \mathbb{R}^d \rightarrow M_{2\times2}(\mathbb{C})$ by

$$Φ(x) := \begin{pmatrix} V^{\frac{1}{\theta}}(x) & 0 \\ 0 & U^{\frac{1}{\theta}}(x) \end{pmatrix} \begin{pmatrix} I & B(x) \\ 0 & I \end{pmatrix} = \begin{pmatrix} V^{\frac{1}{\theta}}(x) & V^{\frac{1}{\theta}}(x)B(x) \\ 0 & U^{\frac{1}{\theta}}(x) \end{pmatrix},$$

so that for a.e. $x \in \mathbb{R}^d$,

$$Φ^{-1}(y) = \begin{pmatrix} V^{-\frac{1}{\theta}}(y) & -B(y)U^{-\frac{1}{\theta}}(y) \\ 0 & U^{-\frac{1}{\theta}}(y) \end{pmatrix}.$$  

Thus, we have

$$Φ^TΦ^{-1} = \begin{pmatrix} V^{\frac{1}{\theta}}TV^{-\frac{1}{\theta}} & V^{\frac{1}{\theta}}[M_B,T]U^{-\frac{1}{\theta}} \\ 0 & U^{\frac{1}{\theta}}TU^{-\frac{1}{\theta}} \end{pmatrix}.$$  

Note that $W := (Φ^*Φ)^{\frac{\theta}{2}}$ is a matrix weight and, by polar decomposition, there exists a unitary a.e. matrix function $Ψ$ such that $Φ(x) = Ψ(x)W^{\frac{1}{\theta}}(x)$. This gives us that

$$||T||_{L^p(W^{\frac{1}{\theta}})}_{L^q(Ψ)} = ||W^{\frac{1}{\theta}}TW^{-\frac{1}{\theta}}||_{L^p \rightarrow L^q} = ||Φ^TΦ^{-1}||_{L^p \rightarrow L^q} \approx \max \{ ||V^{\frac{1}{\theta}}TV^{-\frac{1}{\theta}}||_{L^p \rightarrow L^q}, ||V^{\frac{1}{\theta}}[M_B,T]U^{-\frac{1}{\theta}}||_{L^p \rightarrow L^q}, ||U^{\frac{1}{\theta}}TU^{-\frac{1}{\theta}}||_{L^p \rightarrow L^q} \} \geq ||V^{\frac{1}{\theta}}[M_B,T]U^{-\frac{1}{\theta}}||_{L^p \rightarrow L^q} = ||[M_B,T]||_{L^p(Ψ^\frac{1}{\theta}) \rightarrow L^q(Ψ)}$$

Using the assumption in Lemma 1.2 that $||T||_{L^p(W^{\frac{1}{\theta}})} \lesssim φ(||W||_{A_{p,q}})$

we get that

$$||[M_B,T]||_{L^p(Ψ^\frac{1}{\theta}) \rightarrow L^q(Ψ)} \lesssim φ(||W||_{A_{p,q}}). \tag{2.1}$$

Unravelling the $A_{p,q}$ condition for $W$, we obtain

$$||W||_{A_{p,q}} = \sup_Q \int_Q \left( \int_Q ||W^{\frac{1}{\theta}}(x)W^{-\frac{1}{\theta}}(y)||^p \, dy \right)^{\frac{1}{p}} \, dx$$

$$= \sup_Q \int_Q \left( \int_Q ||Φ(x)Φ^{-1}(y)||^p \, dy \right)^{\frac{1}{p}} \, dx$$
\[ \leq 3^{\frac{q}{p'}} \left( \|U\|_{A_{p,q}} + \|V\|_{A_{p,q}} + \sup_Q \int_Q \left( \int_Q \left\| V^\frac{1}{q}(x) (B(x) - B(y)) U^\frac{1}{q'}(y) \right\| dy \right)^{\frac{p'}{p}} dx \right) \]
\[ = 3^{\frac{q}{p'}} \left( \|U\|_{A_{p,q}} + \|V\|_{A_{p,q}} + \|B\|^{\frac{q}{p'}-p,q}_{BMO_{V,U}} \right) \]
and thus
\[ \|[M_B, I_\alpha]||_{L^p(U^\frac{q}{p'}) \rightarrow L^q(V)} \lesssim \phi \left( 3^{\frac{q}{p'}} \left( \|U\|_{A_{p,q}} + \|V\|_{A_{p,q}} + \|B\|^{\frac{q}{p'}-p,q}_{BMO_{V,U}} \right) \right). \]

Re-scaling with \( B \) replaced by \( B \|B\|^{-1}_{BMO_{V,U}} \) now completes the proof.

2.2. **Proof of Lemma 1.3.** We now prove Lemma 1.3. Let \( W \) and \( \Phi \) be defined as in the previous subsection, so that
\[ \left( \|U\|_{A_{p,q}} + \|V\|_{A_{p,q}} + \|B\|^{\frac{q}{p'}-p,q}_{BMO_{V,U}} \right)^{\frac{q}{q'}} \approx \|W\|^{\frac{q}{q'}}_{A_{p,q}} \lesssim \|T\|_{L^p(W^\frac{1}{q'}) \rightarrow L^q(W)} \]
\[ \lesssim \|[M_B, T]||_{L^p(U^\frac{q}{q'}) \rightarrow L^q(V)} + \|T\|_{L^p(U^\frac{q}{q'}) \rightarrow L^q(U)} + \|T\|_{L^p(V^\frac{q}{q'}) \rightarrow L^q(V)}. \]

Clearly we may assume that \( \|[M_B, T]||_{L^p(U^\frac{q}{q'}) \rightarrow L^q(V)} < \infty \) and so by assumption all quantities above are finite. Re-scaling \( B \mapsto rB \) for \( r > 0 \), dividing by \( r \), and taking \( r \to \infty \), we obtain
\[ \|B\|_{BMO_{V,U}} \lesssim \|[M_B, I_\alpha]||_{L^p(U^\frac{q}{q'}) \rightarrow L^q(V)}. \]
which is the desired lower bound.

3. **Proof of Theorem 1.1**

We now prove Theorem 1.1 (assuming that \( I_\alpha \) is a fractional lower bound operator, which will be proved in the next section) by proving that \( \|B\|_{BMO_{V,U}} \approx \|B\|^{\frac{q}{p'}-p,q}_{BMO_{V,U}} \) when \( U, V \) are matrix \( A_{p,q} \) weights (see Theorem 3.1). To do this we need the concept of a reducing matrix. In particular, for any norm \( \rho \) on \( \mathbb{C}^n \) there exists a positive definite \( n \times n \) matrix \( A \) where for any \( \vec{e} \in \mathbb{C}^n \) we have
\[ n^{-1} |A\vec{e}| \leq \rho(\vec{e}) \leq |A\vec{e}| \]
(see [NT96, Lemma 11.4]).
In particular, for any matrix weight \( U \) and measurable \( 0 < |E| < \infty \) there exists \( n \times n \) matrices \( U_E, U'_E \) where for any \( \vec{e} \in \mathbb{C}^n \) we have
\[
|U_E \vec{e}| \approx \left( \int_E |U_{\frac{1}{q}}(x)\vec{e}|^q \, dx \right)^{\frac{1}{q}} , \quad |U'_E \vec{e}| \approx \left( \int_E |U_{-\frac{1}{q}}(x)\vec{e}|^{p'} \, dx \right)^{\frac{1}{p'}} .
\]

Similarly for a matrix weight \( V \) we will use the notation \( V_E \) and \( V'_E \) for these reducing matrices. Using reducing matrices in conjunction with elementary linear algebra, it is easy to see that for a matrix weight \( U \) we have
\[
\|U\|^{\frac{q}{p},q}_{A_{p,q}} \approx \sup_Q \left\| U_Q U'_Q \right\| = \sup_Q \left\| U'_Q U_Q \right\| \approx \sup_Q \left( \int_Q \left( \int_Q \|W_{\frac{1}{q}}(x)W_{-\frac{1}{q}}(y)\|^q \, dx \right)^{\frac{p'}{q}} \, dy \right)^{\frac{1}{p'}}
\]
and similarly an easy application of Hölder’s inequality gives us that
\[
\left| \langle \vec{e}, \vec{f} \rangle_{\mathbb{C}^n} \right| \leq |U_Q \vec{e}| |U'_Q \vec{f}|
\]
for \( \vec{e}, \vec{f} \in \mathbb{C}^n \), which clearly implies that
\[
\|U_Q^{-1}(U'_Q)^{-1}\| \leq 1 . \tag{3.1}
\]

We now prove the following matrix weighted John-Nirenberg type theorem, which should be thought of as a fractional generalization of the matrix weighted John-Nirenberg theorem from \([\text{IPT}]\).

**Theorem 3.1.** If \( U, V \) are two \( m \times m \) matrix weights such that \( U, V \in A_{p,q} \) and \( B \) is an \( m \times m \) locally integrable matrix function, then the following are equivalent (where the suprema is taken over all cubes \( Q \)).

\( 1 \) \[
\sup_Q \int_Q \|V_Q(B(x) - m_Q B)U_Q^{-1}\| \, dx
\]
\( 2 \) \[
\sup_Q \left( \int_Q \left\| V_{\frac{1}{q}}(x)(B(x) - m_Q B)U_Q^{-1}\right\|^q \, dx \right)^{\frac{1}{q}}
\]
\( 3 \) \[
\sup_Q \left( \int_Q \left\| U_{-\frac{1}{q}}(x)(B^*(x) - m_Q B^*)\left(V_Q^{-1}\right)^{-1}\right\|^{p'} \, dx \right)^{\frac{1}{p'}}
\]
\( 4 \) \[
\sup_Q \left( \int_Q \left( \int_Q \left\| V_{\frac{1}{q}}(x)(B(x) - B(y))U_{-\frac{1}{q}}(y)\right\|^{p'} \, dy \right)^{\frac{q}{p'}} \, dx \right)^{\frac{1}{q}}
\]
\( 5 \) \[
\sup_Q \left( \int_Q \left( \int_Q \left\| V_{\frac{1}{q}}(x)(B(x) - B(y))U_{-\frac{1}{q}}(y)\right\|^q \, dy \right)^{\frac{p'}{q}} \, dx \right)^{\frac{1}{p'}}
\]
We need to discuss some duality properties of matrix \( \mathcal{A}_{p,q} \) weights. To better keep track of the exponents and matrix weights that correspond to a reducing matrix we temporarily use the notation \( \mathcal{V}_Q(W,q) \), \( \mathcal{V}_Q(W,p,q) \) to denote reducing matrices where

\[
|\mathcal{V}_Q(W,q)\mathcal{e}| \approx \left( \int_Q \left| W^{\frac{q}{q'}}(x)\mathcal{e} \right|^q dx \right)^{\frac{1}{q}}, \quad |\mathcal{V}_Q'(W,p,q)\mathcal{e}| \approx \left( \int_Q \left| W^{-\frac{1}{q'}}(x)\mathcal{e} \right|^{p'} dx \right)^{\frac{1}{p'}}
\]

so that

\[
\|W\|_{\mathcal{A}_{p,q}} \approx \sup_Q \|\mathcal{V}_Q(W,q)\mathcal{V}_Q'(W,p,q)\|^{p'}, \quad \|W\|_{\mathcal{A}_q} \approx \sup_Q \|\mathcal{V}_Q(W,q)\mathcal{V}_Q'(W,q,q)\|^{q}.
\]

Moreover

\[
|\mathcal{V}_Q(W^{-\frac{q}{q'}},p')\mathcal{e}| \approx \left( \int_Q \left| W^{-\frac{1}{q'}}(x)\mathcal{e} \right|^{p'} dx \right)^{\frac{1}{p'}} \approx \mathcal{V}_Q'(W,p,q)\mathcal{e} \quad (3.2)
\]

and similarly

\[
|\mathcal{V}_Q'(W^{-\frac{q'}{q}},q',p')\mathcal{e}| \approx \left( \int_Q \left| W^{\frac{q}{q'}}(x)\mathcal{e} \right|^q dx \right)^{\frac{1}{q}} \approx \mathcal{V}_Q(W,q)\mathcal{e}, \quad (3.3)
\]

which (as observed in [IM19]) means that \( W \in \mathcal{A}_{p,q} \) if and only if \( W^{-\frac{q}{q'}} \in \mathcal{A}_{q',p'} \). Note that with this notation we have

\[
\mathcal{V}_Q = \mathcal{V}_Q(V,q), \quad \mathcal{V}_Q' = \mathcal{V}_Q'(V,p,q)
\]

and a similar statement holds for \( U \).

**Proof of Theorem 3.1.** Recall from the introduction that \( \mathcal{A}_{p,q} \subseteq \mathcal{A}_q \). Thus, we have from [IPT, Corollary 4.7] that (1) \( \iff \) (2) \( \iff \) (6).

Moreover, since \( U, V \in \mathcal{A}_{p,q} \) if and only if \( U^{-\frac{q'}{q}}, V^{-\frac{q'}{q}} \in \mathcal{A}_{q',p'} \), we have that \( U^{-\frac{q'}{q}}, V^{-\frac{q'}{q}} \in \mathcal{A}_{p',q} \). The fact that \( V \in \mathcal{A}_{p,q} \) tells us that (by the \( \mathcal{A}_{p,q} \) property and (3.1))

\[
(1) = \sup_Q \int_Q \|\mathcal{U}_Q^{-1}(B^* - mQB^*)\mathcal{V}_Q\| \, dx \approx \sup_Q \int_Q \|\mathcal{U}_Q'(B^* - mQB^*)(\mathcal{V}_Q')^{-1}\| \, dx
\]
which by (3.2) is nothing but (1) with respect to matrix weights $V^{-\frac{p}{q}}, U^{-\frac{p}{q}}$, the symbol $B^*$, and the exponent $p'$, and thus (1) equivalent to (2) with respect to $V^{-\frac{p}{q}}, U^{-\frac{p}{q}}, B^*$, and $p'$, which (again by (3.2)) is nothing but (3). This tells us that (3) $\iff$ (1) $\iff$ (2) $\iff$ (6).

Furthermore, (6) $\leq$ (5) by an easy application of Hölder’s inequality, since $p'/q' > 1$. Thus, if we can show that (5) $\lesssim$ (2) + (3) then we will have (3) $\iff$ (1) $\iff$ (2) $\iff$ (6) $\iff$ (5). To that end,

\[
\left(\int_Q \left( \int_Q \left| V_{\frac{1}{q'}}(x) (B(x) - B(y)) U^{-\frac{1}{q'}}(y) \right|^q \frac{dx}{dy} \right)^{\frac{p'}{q'}} dy \right)^{\frac{1}{p'}} \lesssim \left( \int_Q \left( \int_Q \left| V_{\frac{1}{q'}}(x) (B(x) - m_Q B) U^{-\frac{1}{q'}}(y) \right|^q \frac{dx}{dy} \right)^{\frac{p'}{q'}} dy \right)^{\frac{1}{p'}} + \left( \int_Q \left( \int_Q \left| V_{\frac{1}{q'}}(x) (B(y) - m_Q B) U^{-\frac{1}{q'}}(y) \right|^q \frac{dx}{dy} \right)^{\frac{p'}{q'}} dy \right)^{\frac{1}{p'}} = (A) + (B).
\]

Using the matrix $A_{p,q}$ property we get

\[
(A) \leq \left( \int_Q \left( \int_Q \left| V_{\frac{1}{q'}}(x) (B(x) - m_Q B) U_{Q}^{-1} \right|^q \left| U_Q U^{-\frac{1}{q'}}(y) \right|^q \frac{dx}{dy} \right)^{\frac{p'}{q'}} dy \right)^{\frac{1}{p'}} = \left( \int_Q \left( \int_Q \left| V_{\frac{1}{q'}}(x) (B(x) - m_Q B) U_{Q}^{-1} \right|^q \frac{dx}{dy} \right)^{\frac{p'}{q'}} \left| U_Q U^{-\frac{1}{q'}}(y) \right|^p dy \right)^{\frac{1}{p'}} \lesssim \|U\|_{A_{p,q}}^{\frac{1}{p}} (2).
\]

and likewise

\[
(B) \leq \left( \int_Q \left( \int_Q \left| V_{\frac{1}{q'}}(x) V_Q \right|^q \left| (V_Q)^{-1}(B(y) - m_Q B) U^{-\frac{1}{q'}}(y) \right|^q \frac{dx}{dy} \right)^{\frac{p'}{q'}} dy \right)^{\frac{1}{p'}} = \left( \int_Q \left( \int_Q \left| V_{\frac{1}{q'}}(x) V_Q \right|^q \frac{dx}{dy} \right)^{\frac{p'}{q'}} \left| (V_Q)^{-1}(B(y) - m_Q B) U^{-\frac{1}{q'}}(y) \right|^p dy \right)^{\frac{1}{p'}} \lesssim \|V\|_{A_{p,q}}^{\frac{1}{p}} (3).
\]
Finally, as was observed already, (1) is equivalent to (1) with respect to $V^{-\frac{p}{q}}_\pi, U^{-\frac{p}{q}}_\pi, B^*$, and $p'$, which is equivalent to (5) with respect $V^{-\frac{p}{q}}_\pi, U^{-\frac{p}{q}}_\pi, B^*$, and $p'$, which is easily be seen to be nothing but (4).

\[\square\]

4. Commutator lower bound: proof of Theorem 1.4

In this section we will prove Theorem 1.4 and in the process prove that $I_\alpha$ is a fractional lower bound operator (which will complete the proof of Theorem 1.1). As stated in the introduction, this will be done by modifying Wiener algebra arguments from by [IPT, LT13]. Let $W$ be a matrix weight and suppose that $\tilde{f} \in L^p \cap L^p \left(W^{\frac{p}{q}}_\pi\right)$ and $\tilde{g} \in L^q \cap L^{q'} \left(W^{-\frac{q'}{q}}_\pi\right)$. Let $E \subset \mathbb{R}^d$ be measurable. For any $t \in \mathbb{R}^d$, define

$$k_{\alpha,t}(x, y) := e^{-2\pi i t \cdot x} k_\alpha(x, y) e^{2\pi i t \cdot y}$$

where $k_\alpha(x, y) = |x - y|^{\alpha-d}$. We then have from Hölder’s inequality that

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{E \times E}(x, y) k_{\alpha,t}(x, y) \left\langle \tilde{f}(y), \tilde{g}(x) \right\rangle_{C^n} dy \, dx \right|$$

$$= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \chi_{E \times E}(x, y) k_{\alpha,t}(x, y) \left\langle W^{\frac{1}{q}}_\pi(x) \tilde{f}(y), W^{-\frac{1}{q}}_\pi(x) \tilde{g}(x) \right\rangle_{C^n} dy \, dx \right|$$

$$\leq ||\chi_E I_\alpha \chi_E||_{L^p \left(W^{\frac{p}{q}}_\pi\right) \rightarrow L^q(W)} ||\tilde{f}||_{L^p \left(W^{\frac{p}{q}}_\pi\right)} ||\tilde{g}||_{L^{q'} \left(W^{-\frac{q'}{q}}_\pi\right)}$$

Thus, if $\psi = \hat{\rho}$ for $\rho \in L^1(\mathbb{R}^d)$ (where $\hat{\cdot}$ denotes Fourier transform) then,

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi \left( \frac{x - y}{\epsilon} \right) \chi_{E \times E}(x, y) k_\alpha(x, y) \left\langle \tilde{f}(y), \tilde{g}(x) \right\rangle_{C^n} dy \, dx \right|$$

$$= \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i \frac{x - y}{\epsilon}} \left( \int_{\mathbb{R}^d} \rho(\epsilon t) e^{-2\pi i t \cdot (x - y)} dt \right) \chi_{E \times E}(x, y) k_\alpha(x, y) \left\langle \tilde{f}(y), \tilde{g}(x) \right\rangle_{C^n} dy \, dx \right|$$

$$= ||\chi_E I_\alpha \chi_E||_{L^p \left(W^{\frac{p}{q}}_\pi\right) \rightarrow L^q(W)} ||\tilde{f}||_{L^p \left(W^{\frac{p}{q}}_\pi\right)} ||\tilde{g}||_{L^{q'} \left(W^{-\frac{q'}{q}}_\pi\right)} ||\rho||_{L^1(\mathbb{R}^d)},$$

(4.1)

which means that (4.1) holds if $\psi$ is in the Wiener algebra $W_0(\mathbb{R}^d) := \{\hat{\rho} : \rho \in L^1(\mathbb{R}^d)\}$. To prove Theorem 1.4 we will use (4.1) with $\psi(x) = |x|^{d-\alpha} \phi(x)$ where $\phi \in C^\infty_c(\mathbb{R}^d)$. While it is likely known that such a function lies in $W_0(\mathbb{R}^d)$, a precise reference seems difficult to find and
thus we will prove it (using a simple idea from [DT83]) for the sake of completeness.

**Proposition 4.1.** Let $0 < \alpha < d$. If $\phi \in C_c^\infty(\mathbb{R}^d)$, then $|.|^{d-\alpha} \phi \in W_0(\mathbb{R}^d)$.

**Proof.** Let $F(x) = |x|^{d-\alpha}$ and $\mathcal{F}(x) = |x|^{d-\alpha} \phi(x)$. If $\beta \in \{0, 1\}^d$, then by an easy induction we have

$$D^\beta F(x) = \Phi_\beta(x)|x|^{d-\alpha-2|\beta|}$$

where $\Phi_\beta(x)$ is the sum of monomials of degree $|\beta|$ in $d$ variables, which means that

$$|D^\beta \mathcal{F}(x)| \lesssim |x|^{d-\alpha-|\beta|}$$

(4.2)

Now let $1 < \delta < \min\{1 + \frac{d-\alpha}{\alpha}, 2\}$. Using Hölder’s inequality, a standard integration by parts argument, and the Hausdorff-Young inequality, we then have by an argument identical to the one used in [IPT, Lemma 3.2] that

$$\left\| \mathcal{F} \right\|_{L^1(\mathbb{R}^d)} \lesssim \left( \int_{\mathbb{R}^d} |D^\beta \mathcal{F}(x)|^\delta \, dx \right)^{\frac{1}{\delta}} < \infty$$

which is finite by (4.2), since

$$\delta|\beta|-(d-\alpha) < \left(1 + \frac{d-\alpha}{\alpha}\right)[d-(d-\alpha)] = \left(1 + \frac{d-\alpha}{\alpha}\right)\alpha = d.$$

Fourier inversion now immediately completes the proof. \qed

Applying (4.1) with $\psi(x) = |x|^{d-\alpha} \phi(x)$ where $\phi \in C_c^\infty(\mathbb{R}^d)$, we obtain the inequality

$$\left| \int_E \int_E \frac{1}{e^{d-\alpha} \phi} \left( \frac{x-y}{\epsilon} \right) \left\langle \tilde{f}(y), \tilde{g}(x) \right\rangle \, dy \, dx \right| \leq \|x E I_{\alpha} x E\|_{L^p(W_{\frac{d}{n}})} \|\tilde{f}\|_{L^p(W_{\frac{d}{n}})} \|\tilde{g}\|_{L^p(W_{\frac{d}{n}})^{\prime}} \|\rho\|_{L^1(\mathbb{R}^d)}.$$  

(4.3)

We proceed by citing a uniform boundedness result which was (implicitly) proved in [IM19, Proposition 3.1].

**Proposition 4.2.** Let $E$ be measurable with $0 < |E| < \infty$ and define the fractional averaging operator by $A_E: L^p(W_{\frac{d}{n}}) \rightarrow L^q(W)$ by

$$A_E \tilde{f} := \frac{\chi_E}{|E|^{1-\frac{q}{n}}} \int_E \tilde{f}(x) \, dx,$$
then
\[
||\mathcal{W}_E^* \mathcal{W}_E|| \approx ||A_E||_{L^p\left(W^{\frac{d}{q}}\right)} \rightarrow_{L^q(W)} \cdot
\]  
(4.4)

We can now state and prove the main technical lemma of this section, which immediately proves that $I_\alpha$ is a fractional lower bound operator.

Lemma 4.3. If $B$ is a ball and $E \subset B$ with $|E| > 0$, then
\[
||\mathcal{W}_E^* \mathcal{W}_E|| \lesssim \left[\frac{|E|}{|B|}\right]^{1-\frac{\alpha}{d}} ||\chi_E I_\alpha \chi_E||_{L^p\left(W^{\frac{d}{q}}\right)} \rightarrow_{L^q(W)} \cdot
\]

Proof. Suppose that $\vec{f} \in L^p \cap L^p\left(W^{\frac{d}{q}}\right)$ and $\vec{g} \in L^q' \cap L^q'\left(W^{-\frac{d}{q}}\right)$

Clearly we have
\[
\left[\frac{|E|}{|B|}\right]^{1-\frac{\alpha}{d}} \left\langle A_E \vec{f}, \vec{g} \right\rangle_{L^2(\mathbb{R}^d)} = \frac{1}{|B|^{1-\frac{\alpha}{d}}} \int_E \int_E \left\langle \vec{f}(y), \vec{g}(x) \right\rangle_{C^n} dy dx.
\]

Let $B$ have radius $\epsilon$ and pick $\phi \in C^\infty_c(\mathbb{R}^d)$ such that $\phi = 1$ on the open ball $B(0,2)$. If $x, y \in B$ then $|x - y| \leq 2\epsilon$ and therefore (4.3) gives us that

\[
\left| \frac{1}{|B|^{1-\frac{\alpha}{d}}} \int_E \int_E \left\langle \vec{f}(y), \vec{g}(x) \right\rangle_{C^n} dy dx \right| \lesssim \left| \frac{1}{(\epsilon^d)^{1-\frac{\alpha}{d}}} \int_E \int_E \left\langle \vec{f}(y), \vec{g}(x) \right\rangle_{C^n} dy dx \right|
\]

\[
\lesssim ||\chi_E I_\alpha \chi_E||_{L^p\left(W^{\frac{d}{q}}\right)} \rightarrow_{L^q(W)} \left\| \vec{f} \right\|_{L^p\left(W^{\frac{d}{q}}\right)} \left\| \vec{g} \right\|_{L^q'\left(W^{-\frac{d}{q}}\right)}
\]

which means
\[
\left\langle A_E \vec{f}, \vec{g} \right\rangle_{L^2(\mathbb{R}^d)} \lesssim \left[\frac{|E|}{|B|}\right]^{1-\frac{\alpha}{d}} ||\chi_E I_\alpha \chi_E||_{L^p\left(W^{\frac{d}{q}}\right)} \rightarrow_{L^q(W)} \left\| \vec{f} \right\|_{L^p\left(W^{\frac{d}{q}}\right)} \left\| \vec{g} \right\|_{L^q'\left(W^{-\frac{d}{q}}\right)}.
\]

(4.5)

Duality, the density of $L^p \cap L^p\left(W^{\frac{d}{q}}\right)$ in $L^p\left(W^{\frac{d}{q}}\right)$ and $L^q \cap L^q'\left(W^{-\frac{d}{q}}\right)$ in $L^q'\left(W^{-\frac{d}{q}}\right)$ (see [CUMR16, Proposition 3.7]), and (4.4) now completes the proof.

Proof of Theorem 1.4. We assume that $||[M_B, I_\alpha]||_{L^p\left(U^{\frac{d}{q}}\right)} \rightarrow_{L^q(V)} < \infty$, or the lower bound holds trivially. Let $B$ be a ball and, for each $M > 0,$
define
\[ E_M := \{ x \in B : \max \{ ||U(x)||, ||U^{-1}(x)||, ||V(x)||, ||V^{-1}(x)|| \} < M \} . \] (4.6)

By continuity of Lebesgue measure, there exists \( M \) such that \( 2|E_M| > |B| \). Further, define
\[ ||W||_{A_{p,q}(E_M)} := \int_{E_M} \left( \int_{E_M} \left| W^{\frac{1}{q}}(x) W^{-\frac{1}{q}}(y) \right|^{p'} \, dy \right)^{\frac{q}{p'}} \, dx \] (4.7)
and
\[ ||B||_{\overline{BMO}^{p,q}_{V,U}(E_M)} := \left( \int_{E_M} \left( \int_{E_M} \left| V^{\frac{1}{q}}(x)(B(x) - B(y)) U^{-\frac{1}{q}}(y) \right|^{p'} \, dy \right)^{\frac{q}{p'}} \, dx \right)^{\frac{1}{q}} . \] (4.8)

Let \( W \) and \( \Phi \) be defined as in Subsection 2.1. Using ideas from that subsection, we have
\[
||W||_{A_{p,q}(E_M)} = \int_{E_M} \left( \int_{E_M} \left| W^{\frac{1}{q}}(x) W^{-\frac{1}{q}}(y) \right|^{p'} \, dy \right)^{\frac{q}{p'}} \, dx = \int_{E_M} \left( \int_{E_M} \left| \Phi(x) \Phi^{-1}(y) \right|^{p'} \, dy \right)^{\frac{q}{p'}} \, dx
\approx ||U||_{A_{p,q}(E_M)} + ||V||_{A_{p,q}(E_M)} + ||B||_{\overline{BMO}^{p,q}_{V,U}(E_M)},
\]
so that
\[
\left( ||U||_{A_{p,q}(E_M)} + ||V||_{A_{p,q}(E_M)} + ||B||_{\overline{BMO}^{p,q}_{V,U}(E_M)} \right)^{\frac{1}{q}} \approx ||W||_{A_{p,q}(E_M)} \approx ||W_{E_M} W_{E_M}||
\leq \left[ \frac{||B||}{|E_M|} \right]^{1 - \frac{q}{p'}} \left| \chi_{E_M} I\alpha \chi_{E_M} \right|_{L^p \left( W^{\frac{1}{q}}(x) \right) \rightarrow L^q(W)}
\leq ||[M_B, I\alpha]||_{L^p \left( W^{\frac{1}{q}}(x) \right) \rightarrow L^q(V)} + \left| \chi_{E_M} I\alpha \chi_{E_M} \right|_{L^p \left( W^{\frac{1}{q}}(x) \right) \rightarrow L^q(U)} + \left| \chi_{E_M} I\alpha \chi_{E_M} \right|_{L^p \left( V^{\frac{1}{q}}(y) \right) \rightarrow L^q(V)} .
\]

By assumption, all quantities above are finite so we can rescale with the replacement \( B \mapsto rB \), for \( r > 0 \). Upon dividing by \( r \) and taking \( r \rightarrow \infty \), we obtain
\[
||B||_{\overline{BMO}^{p,q}_{V,U}(E_M)} \lesssim ||[M_B, I\alpha]||_{L^p \left( W^{\frac{1}{q}}(x) \right) \rightarrow L^q(V)} .
\]

Applying Fatou’s lemma with \( M \rightarrow \infty \) and taking the supremum over all balls \( B \) in \( \mathbb{R}^d \), we obtain
\[
||B||_{\overline{BMO}^{p,q}_{V,U}} \lesssim ||[M_B, I\alpha]||_{L^p \left( W^{\frac{1}{q}} \right) \rightarrow L^q(V)} ,
\]
and a slight modification to the arguments above proves that
\[ \|B\|_{BMO_u,v'} \lesssim \| [M_B, I_\alpha] \|_{L^p(U^{\frac{1}{q}})} \to L^q(V) \]
which proves Theorem 1.4.

5. Proof of Theorem 1.5

Our proof is a combination and modification of the arguments in [IPT, CUIM18, Li06]. For additional information on Orlicz spaces, see e.g., [BL12].

Proposition 5.1. There exists \(2^d\) dyadic grids \(D_t, t \in \{0, 1\}^d\) where
\[
\left\langle V^{\frac{1}{q}} [M_B, I_\alpha] U^{\frac{1}{q}} f, g \right\rangle_{L^2} \lesssim \sum_{t \in \{0, 1\}^d} \sum_{Q \in D_t} \frac{1}{|Q|^{\frac{1}{q}}} \int_Q \int_Q \left\langle V^{\frac{1}{q}} (x)(B(x) - B(y)) U^{\frac{1}{q}}(y) f(y), g(x) \right\rangle_{C^n} |dx\,dy|
\]

Proof. Noting that \(V^{\frac{1}{q}} [M_B, I_\alpha] U^{\frac{1}{q}}\) is an integral operator with kernel \(V^{\frac{1}{q}}(x)(B(x) - B(y)) U^{\frac{1}{q}}(y)|x - y|^{1-q}\), the proof is almost identical to the proof of [IM19, Lemma 3.8].

As in [Li06], we will make use of the following well known fact about Orlicz spaces

Proposition 5.2. For an increasing, convex function \(\Phi\), if
\[
\|f\|_{\Phi,Q} = \inf \{\lambda > 0 : \int_Q \Phi \left( \frac{|f(y)|}{\lambda} \right) dy \leq 1\} \quad \text{and} \quad \|f\|_{\Phi,Q}^* = \sup_{s > 0} \left\{ s \int_Q \Phi \left( \frac{|f(y)|}{s} \right) dy \right\}
\]
then \(\|f\|_{\Phi,Q} \leq \|f\|_{\Phi,Q}^* \leq 2 \|f\|_{\Phi,Q}

Proof of Proposition 1.5. Clearly it is enough to prove for a dyadic grid \(D\) that
\[
\sum_{Q \in D} \left| Q \right|^{\frac{1}{q}} \int_Q \int_Q \left\langle V^{\frac{1}{q}} (x)(B(x) - B(y)) U^{\frac{1}{q}}(y) f(y), g(x) \right\rangle_{C^n} |dx\,dy| \lesssim \min \{\kappa_1, \kappa_2\} \|f\|_{L^p} \|g\|_{L^{p'}}.
\]

Also, by Fatou’s lemma, it is enough to assume that \(f, g\) are bounded with compact support. For that matter, the generalized Hölder inequality gives
\[
\sum_{Q \in D} \left| Q \right|^{\frac{1}{q}} \int_Q \int_Q \left\langle V^{\frac{1}{q}} (x)(B(x) - B(y)) U^{\frac{1}{q}}(y) f(y), g(x) \right\rangle_{C^n} |dx\,dy|
\]
$$\leq \min\{\kappa_1, \kappa_2\} \sum_{Q \in \mathcal{D}} |Q|^{1 + \frac{\alpha}{d}} \|\tilde{f}\|_{\mathcal{D}, Q} \|\tilde{g}\|_{\mathcal{C}, Q}$$

Fix $a > 2^{d+1}$ and define the collection of cubes

$$Q^k = \{Q \in \mathcal{D} : a^k < \|\tilde{f}\|_{\mathcal{D}, Q} \leq a^{k+1}\},$$

and let $S^k$ be the disjoint collection of $Q \in \mathcal{D}$ that are maximal with respect to the inequality $\|\tilde{f}\|_{\mathcal{D}, Q} > a^k$. Note that since $\tilde{f}$ is bounded with compact support, such maximal cubes are guaranteed to exist. Set $S = \bigcup_k S^k$. We now continue the above estimate:

$$\sum_k \sum_{Q \in Q^k} |Q|^{1 + \frac{\alpha}{d}} \|\tilde{f}\|_{\mathcal{D}, Q} \|\tilde{g}\|_{\mathcal{C}, Q} \leq \sum_k a^{k+1} \sum_{Q \in Q^k} |Q|^{1 + \frac{\alpha}{d}} \|\tilde{g}\|_{\mathcal{C}, Q}$$

$$= \sum_k a^{k+1} \sum_{P \in S^k} \sum_{Q \in Q^k} |Q|^{1 + \frac{\alpha}{d}} \|\tilde{g}\|_{\mathcal{C}, Q}.$$  (5.1)

We now estimate the inner sum. Let $\ell(P) = 2^{-m_0}$, so

$$\sum_{Q \subset P} |Q|^{1 + \frac{\alpha}{d}} \|\tilde{g}\|_{\mathcal{C}, Q} = \sum_{m = m_0}^{\infty} 2^{-m} \sum_{\ell(Q) = 2^{-m}} \sum_{Q \subset P} |Q| \|\tilde{g}\|_{\mathcal{C}, Q}.$$  (5.2)

However, Proposition 5.2 gives us that

$$\sum_{\ell(Q) = 2^{-m}} \sum_{Q \subset P} |Q| \|\tilde{g}\|_{\mathcal{C}, Q} \leq \sum_{\ell(Q) = 2^{-m}} |Q| \inf_{s > 0} \left\{ s + s \int_Q C \left( \frac{|\tilde{f}(y)|}{s} \right) dy \right\}$$

$$\leq \inf_{s > 0} \left\{ \sum_{\ell(Q) = 2^{-m}} |Q| s + s \int_Q C \left( \frac{|\tilde{f}(y)|}{s} \right) dy \right\}$$

$$= \inf_{s > 0} \left\{ |P| s + s \int_P C \left( \frac{|\tilde{f}(y)|}{s} \right) dy \right\}$$

$$= |P| \inf_{s > 0} \left\{ s + s \int_P C \left( \frac{|\tilde{f}(y)|}{s} \right) dy \right\} \leq 2 |P| \|\tilde{g}\|_{\mathcal{C}, P}.$$

Plugging this into (5.2) we get
\[
\sum_{m=m_0}^{\infty} 2^{-ma} \sum_{Q \subset P} |Q| \|\vec{g}\|_{C,Q} \lesssim 2^a P \|\vec{g}\|_{\overline{C},P} \sum_{m=m_0}^{\infty} 2^{-ma} \lesssim |P|^{1+\frac{d}{q}} \|\vec{g}\|_{\overline{C},P}
\]

and combining this with (5.1) gives

\[
\sum_{k} a^{k+1} \sum_{P \in S^k} \sum_{Q \subset P} |Q|^{1-\frac{d}{q}} \|\vec{g}\|_{C,Q} \lesssim \sum_{k} \sum_{P \in S^k} |P|^{1+\frac{d}{q}} \|f\|_{\mathcal{D},P} \|\vec{g}\|_{\overline{C},P}
\]

\[
= \sum_{P \in S} |P| \left( |P|^{\frac{d}{q}} \|f\|_{\mathcal{D},P} \right) \left( \|\vec{g}\|_{\overline{C},P} \right) \leq \sum_{P \in S} |P| \inf_{x \in P} M_{\mathcal{D}}^a f(x) M_{\overline{C}} \vec{g}(x)
\]

where as usual, \(M_{\mathcal{D}}^a f(x) = \sup_{D \ni x} |P|^{\frac{d}{q}} \|f\|_{\mathcal{D},P}\) and when \(\alpha = 0\) we omit the superscript.

For each \(P \in S\), define

\[
E_P = P \setminus \bigcup_{P' \subseteq P} P'.
\]

Then by Proposition A.1 in [CUMP11], since \(f, \vec{g}\) are bounded with compact support, we have that the sets \(E_P\) are pairwise disjoint and \(|E_P| \geq \frac{1}{2} |P|\). Thus,

\[
\sum_{P \in S} |P| \inf_{x \in P} M_{\mathcal{D}}^a f(x) M_{\overline{C}} \vec{g}(x) \leq 2 \sum_{P \in S} |E_P| \inf_{x \in P} M_{\mathcal{D}}^a f(x) M_{\overline{C}} \vec{g}(x)
\]

\[
\leq 2 \sum_{P \in S} \int_{E_P} M_{\mathcal{D}}^a f(x) M_{\overline{C}} \vec{g}(x) \, dx \leq 2 \int_{\mathbb{R}^d} M_{\mathcal{D}}^a f(x) M_{\overline{C}} \vec{g}(x) \, dx
\]

\[
\leq 2 \|M_{\mathcal{D}}^a f\|_{L^q} \|M_{\overline{C}} \vec{g}\|_{L^q} \lesssim \|f\|_{L^p} \|\vec{g}\|_{L^p}.
\]

\[
\square
\]

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