EFFICIENT HIGH-ORDER IMPLICIT SOLVERS FOR THE
DYNAMIC OF THIN-WALLED BEAMS WITH OPEN CROSS
SECTION UNDER EXTERNAL ARBITRARY LOADINGS

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ABSTRACT. This paper aims to investigate, in large displacement and torsion context, the nonlinear dynamic behavior of thin-walled beams with open cross section subjected to various loadings by high-order implicit solvers. These homotopy transformations consist to modify the nonlinear discretized dynamic problem by introducing an arbitrary invertible pre-conditioner \([K^\ast]\) and an arbitrary path following parameter. The nonlinear strongly coupled equations of these structures are derived by using a 3D nonlinear dynamic model which accounts for large displacements and large torsion without any assumption on torsion angle amplitude. Coupling complex structural phenomena such that warping, bending-bending, and flexural-torsion are taken into account.

Two examples of great practical interest of nonlinear dynamic problems of various thin-walled beams with open section are presented to validate the

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efficiency and accuracy of high-order implicit solvers. The obtained results show that the proposed homotopy transformations reveal a few number of matrix triangulations. A comparison with Abaqus code is presented.

1. Introduction. The demand for engineering thin-walled structures with open cross section is continuously increasing on account of their ability to increase stability, functional requirements and reducing weight and cost in many fields of civil construction to mechanical and aerospace structures. During these last decades, many mechanical aspects have been taken into consideration in the design of these types of structures in order to improve their performance and extend operating life. One of important aspect of the design process is the dynamic response of these structures subjected to external dynamic loadings. Therefore, an accurate prediction of dynamic behavior corresponding to a given strength applied to these types of structures with open cross section is of fundamental importance in the design.

A realistic prediction must be pass obligatorily by either linear or nonlinear analyses according to the displacement and torsion sizes. When the displacement and torsion components of the structure are small, a linear modal analysis is generally used to accede to the structure dynamical characteristics and responses. But, as the displacement components and torsion became large a nonlinear dynamic analysis is necessary because the nonlinear dynamic effects are important and the structure show very complex structural dynamic behavior on account of coupling complex phenomena such that warping, bending-bending, and flexural-torsion.

The nonlinear dynamic of thin-walled beams with open cross section undergoing large displacements and large rotations is an interesting attractive research topic. The analysis of the nonlinear dynamic response of these types of thin-walled structures is a task frequently encountered in the engineering practice. However, the variability of the cross section makes this analysis more complex from a mathematical point of view.

During the last two decades an extensive research has been carried out on linear and nonlinear dynamic of thin-walled beams with constant open cross section undergoing large displacement and torsion subjected to various loading and different boundary conditions based on the total Lagrangian finite element formulations [7, 19] and the co-rotational formulations [6, 16]. Many finite element formulations and modelisations have been also proposed to analyze the linear and nonlinear vibration of these thin-walled structures [14, 20]. However, for thin-walled beams with open cross section, the research works which deal with nonlinear dynamics analysis has received less attention than thin-walled beams with constant open cross section. The research works are very limited concerning this topic [3, 27].

We deal in this paper with the nonlinear dynamic analysis of thin-walled beams with open cross section. The shortening effect, pre-buckling deformation, large torsion and flexural-torsional coupling and effect of load eccentricities are included in this nonlinear study. The objective is to investigate the influence of homotopies. Two other new efficient homotopies are considered in this paper, which are different from that adopted in [8]. The choice of these proposed homotopy transformations leads to strong reduction of stiffness matrix triangulations. This high-order implicit algorithm was applied successfully to solving instationary nonlinear problems [15, 28, 18] and to structural nonlinear dynamic problems [8, 9]. Two numerical examples of forced nonlinear dynamic problems of thin-walled beams with open cross section subjected to external arbitrary dynamic loads are analyzed to access the efficiency and the reliability of the developed high-order implicit algorithm by
considering new homotopy transformations. It is proven that the used implicit high-order algorithm is more reliable and less time consuming than the classical iterative methods and than the developed algorithm in [8].

The outline of this paper is as follows. Section 2 presents the considered 3D nonlinear dynamic model of thin-walled beams with open cross section and the derivation of the governing dynamic equations of motion of these thin-walled structures. The section 3 is devoted to the proposed numerical approach based on a high-order implicit algorithm. Its performance and comparisons with the Abaqus code and literature results are performed on two typical examples will be given in section 4. Finally the efficiency and the concluding remarks are drawing in section 5.

2. Nonlinear dynamic model for thin-walled beams with open cross section.

2.1. Statement of the problem and its nonlinear kinematic relations. Let us consider a 3D thin-walled beam element of length $L$ and open cross section of area $A(x)$ which occupies a domain of volume $\Omega$ bounded by the boundary $\partial \Omega$ as depicted in figure 1a. The beam is made of an homogenous, isotropic and elastic material with Young modulus $E$, shear modulus $\mu$ and mass density $\rho$. The used rectangular Cartesian co-ordinates system is $Gxyz$ of centre $G$ such that $Gy$ and $Gz$ are the transversal axes and $Gx$ is the initial longitudinal axis. The co-ordinates of shear centre $C$ located in $Gyz$ plane are $(y_c, z_c)$, those of a point $M$ on the variable section $A(x)$ are denoted by $y$, $z$ and $\omega$; with $\omega$ is the sectorial co-ordinate which characterizes the warping of the section at point $M$ for a nonuniform torsion (see figure 1b) [29]. In the framework of large displacements, large twist angle and small

![Figure 1. Thin-walled beam with open cross section, co-ordinates of the point M on the cross section contour](image)
deformations, the displacement components $u_M(x, y, z)$, $v_M(x, y, z)$, $w_M(x, y, z)$ of a point $M$ on the section contour are expressed by the following nonlinear relations \([8, 19, 21, 12]\):

$$
\{U_M\} = \begin{cases}
  u_M &= u(x) - y((1 + c)v' + w's) - z((1 + c)w' - v's) - \omega(y, z)\theta_x' \\
  v_M &= v(x) - (z - z_c)s + (y - y_c)c \\
  w_M &= w(x) + (y - y_c)s + (z - z_c)c
\end{cases}
$$

(1)

where $u(x)$ represents the axial displacement of $G$, the components $v(x)$ and $w(x)$ represent the displacements of shear point $C$ in $y$ and $z$ directions, $\theta_x$ is the torsion angle, $\omega$ represents the warping function and the two variables $c$ and $s$ are defined by the following trigonometric functions:

$$
c = \cos(\theta_x) - 1 \quad ; \quad s = \sin(\theta_x)
$$

(2)

The symbol $(\cdot)'$, in equation (1), denotes the derivation with respect to the coordinate $x$. Noting that the expressions (2) of trigonometric functions $c$ and $s$ are conserved in this model without any assumption on the torsion angle amplitude in both theoretical and numerical analysis. The equation (1) is strongly nonlinear incorporating the flexion and torsion coupling terms with the trigonometric functions $c$ and $s$. The Vlasov’s linear model [29] can be recovered from equation (1) by approximating the trigonometric functions $c$ and $s$ by 0 and $\theta_x$ respectively and using linear assumptions. The considered nonlinear Green-Lagrange strain tensor, taking into account the large displacements, membrane effect, bending, nonlinear warping effect, has the following components:

$$
\begin{cases}
  \epsilon_{xx} &= \epsilon - yk_z - zk_y - \omega\theta_x'' + \frac{1}{2}R^2\theta_x'^2 \\
  \epsilon_{xy} &= -\frac{1}{2}(z - z_c + \frac{\partial\omega}{\partial y})\theta_x' \\
  \epsilon_{xz} &= \frac{1}{2}(y - y_c - \frac{\partial\omega}{\partial x})\theta_x'
\end{cases}
$$

(3)

where $\epsilon$ is the membrane deformation component, $k_y$ and $k_z$ are beam curvatures about $y$ and $z$ axes and $R$ is the distance between the point $M$ and the shear centre $C$, expressed by:

$$
\begin{cases}
  \epsilon &= u' + \frac{1}{2}(v''^2 + w''^2) - \psi\theta_x' \\
  k_y &= (1 + c)w'' - sw'' \\
  k_z &= (1 + c)v'' - sw'' \\
  R &= \sqrt{(y - y_c)^2 + (z - z_c)^2}
\end{cases}
$$

(4)

with $\psi$ is the variable associated with membrane component given by:

$$
\psi = y_c((1 + c)w' - sv') - z_c((1 + c)v' + sw')
$$

(5)

2.2. Governing equation of motion. To determine the governing nonlinear dynamic equation of motion, we use the principle of virtual work: $\delta(T + U - W_{ext}) = 0$, applied in its variational form, where $T$ is the work of acceleration forces, $U$ is the strain energy and $W_{ext}$ is the work of the applied external forces. The variation of the work of acceleration forces is given by:

$$
\delta T = \int_\Omega \rho(\ddot{u}_M\delta u_M + \ddot{v}_M\delta v_M + \ddot{w}_M\delta w_M)d\Omega
$$

(6)
where $\ddot{u}_M$, $\ddot{v}_M$ and $\ddot{w}_M$ are the acceleration vector components of point $M$. ($\dot{}$) indicates the second time derivative and $\delta(\cdot)$ denotes the virtual quantities. Using the equation (1) and taking into account for $\delta c = -s\delta \theta_x$ and $\delta s = (c + 1)\delta \theta_x$, the variations of virtual displacement components $\delta u_M$, $\delta v_M$ and $\delta w_M$ are expressed by:

\[
\begin{align*}
\delta u_M &= \dot{u} - y((1 + c)\dot{v}' + s\dot{w}' - ((1 + c)v' - sw')\dot{\theta}_x) \\
&- z((1 + c)\delta w' - s\delta v' - ((1 + c)v' + sw')\dot{\theta}_x) - \omega \ddot{\theta}_x \\
\delta v_M &= \dot{v} - ((1 + c)e_z + se_y)\delta \theta_x \\
\delta w_M &= \dot{w} - ((1 + c)e_y + se_z)\delta \theta_x
\end{align*}
\tag{7}
\]

where $e_y = y - y_c$ and $e_z = z - z_c$ are the eccentricities from the shear centre $C$. The components of the acceleration vector $\ddot{u}_M$, $\ddot{v}_M$ and $\ddot{w}_M$, obtained from equation (1), are:

\[
\begin{align*}
\ddot{u}_M &= \ddot{u} - y((1 + c)\ddot{v} + s\ddot{w} + 2((1 + c)\dot{v}' - sv')\dot{\theta}_x - ((1 + c)v' + sw')\dot{\theta}_x^2) \\
&+ ((1 + c)v' - sv')\dot{\theta}_x^2 z((1 + c)\ddot{w} - s\ddot{v} - 2((1 + c)\dot{v}' + sw')\dot{\theta}_x) \\
&- ((1 + c)v' + sw')\dot{\theta}_x^2 - (1 + c)v' + sw')\dot{\theta}_x - \omega \ddot{\theta}_x \\
\ddot{v}_M &= \ddot{v} - ((1 + c)e_z + se_y)\dot{\theta}_x - ((1 + c)e_z + se_y)\dot{\theta}_x^2 \\
\ddot{w}_M &= \ddot{w} + ((1 + c)e_y + se_z)\dot{\theta}_x - ((1 + c)e_y + se_z)\dot{\theta}_x^2
\end{align*}
\tag{8}
\]

Substituting equations (7) and (8) in equation (6), the variation of the work of acceleration forces, after the integration on elementary cross section area $dA$ writes as:

\[
\delta T = \rho \int_L A(x)(\ddot{u}\delta u + (\ddot{v} + (zc + czc + sy_c)\dot{\theta} + (yc + cy_c - sz_c)\dot{\theta}_x^2)\delta v) dx \\
+ \rho \int_L A(x)((\ddot{v} + (zc + czc + sy_c)\dot{\theta} + (yc + cy_c - sz_c)\dot{\theta}_x^2)\delta u dx \\
+ \rho \int_L A(x)((zc + czc + sy_c)\ddot{v} - (yc + cy_c - sz_c)\ddot{w})\delta \theta_x dx \\
+ \rho \int_L I_{\theta_x}(\ddot{\theta}_x)\delta \theta_x dx + \rho \int_L I_{\theta_x}dx + \rho \int_L I_{\theta_x}\ddot{\theta}_x dx \\
- \rho \int_L I_y((v' + c'u')Z_2\delta \theta_x dx + \rho \int_L I_y((1 + c)\delta w' - s\ddot{v}')Z_2dx \\
+ \rho \int_L I_{\omega}((1 + c)v' + s\ddot{w}')Z_1 dx + \rho \int_L I_{\omega} \ddot{\omega}_x \dot{\theta}_x dx
\tag{9}
\]

where $Z_1(x)$ and $Z_2(x)$ are defined by:

\[
\begin{align*}
Z_1 &= \ddot{v} + c\ddot{v}' + s\ddot{w}' + 2(\ddot{v}' + cw' - s\dot{w}')\dot{\theta}_x - (v' + cw' - sw')\dot{\theta}_x^2 \\
&+ (w' + cw' - sw')\dot{\theta}_x \\
Z_2 &= \ddot{w} + c\ddot{w}' - s\ddot{v}' - 2(\ddot{v}' + cw' + sw')\dot{\theta}_x - (w' + cw' - sw')\dot{\theta}_x^2 \\
&- (v' + cw' - sw')\dot{\theta}_x
\end{align*}
\tag{10}
\]

$I_y(x)$ and $I_z(x)$ are the second moments of area about $y$ and $z$ axes, $I_{\omega}(x)$ is the warping constant and $I_{\alpha}(x)$ is polar inertia moment about shear centre. These last
quantities are defined respectively by:
\[
\begin{align*}
I_y &= \int_A y^2 dA ; \\
I_z &= \int_A z^2 dA \\
I_\omega &= \int_A \omega^2 dA ; \\
I_0 &= \int_A \left( \frac{I_y + I_z}{A(x)} + y^2 + z^2 \right) dA
\end{align*}
\]  
(11)

The expression (9) can be written in terms of mass matrix \([M(\theta, \alpha)]\) and gyroscopic matrix \([C(\theta, \alpha)]\) in the following form:
\[
\delta T = \int_L \{\delta u\}[M(\theta, \alpha)]\{\ddot{u}\} dx + \int_L \{\delta u\}[C(\theta, \alpha)]\{\dot{u}\} dx
\]  
(12)

where \(\{u\}\) is defined by :
\[
\{u\} = \langle u, v, w, \theta, \dot{v}, \dot{w}, \dot{\theta} \rangle
\]  
(13)

The mass and gyroscopic matrices \([M(\theta, \alpha)]\) and \([C(\theta, \alpha)]\) can be decomposed in linear and nonlinear parts as follows:
\[
\begin{align*}
[M(\theta, \alpha)] &= [M_l] + [M_{nl}] \\
[C(\theta, \alpha)] &= [C_l] + [C_{nl}]
\end{align*}
\]  
(14)

with
\[
[M_l] = \rho \begin{bmatrix}
A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A & 0 & Az_c & 0 & 0 & 0 \\
0 & 0 & A & -Ay_c & 0 & 0 & 0 \\
0 & Az_c & -Ay_c & I_\omega & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_y & 0 \\
0 & 0 & 0 & 0 & 0 & I_\omega & 0
\end{bmatrix}
\]  
(15)

\[
[M_{nl}] = \rho \begin{bmatrix}
0 & 0 & 0 & \Lambda z & 0 & 0 & 0 \\
0 & 0 & 0 & -\Lambda y & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I_z & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2I_y & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  
(16)

\[
[C_l] = \rho \dot{\theta}_x \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & Ay_c & 0 & 0 & 0 \\
0 & 0 & 0 & Az_c & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  
(17)

\[
[C_{nl}] = \rho \dot{\theta}_x \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Lambda y & 0 & 0 & 0 \\
0 & 0 & 0 & \Lambda z & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  
(18)
where \( \Lambda_{vs} = s\Lambda' \), \( \Lambda_{ws} = s\Lambda'' \), \( \Lambda_{vc} = (c+1)\Lambda' \), \( \Lambda_{wc} = (c+1)\Lambda'' \), \( \Lambda_{vw} = \Lambda'\Lambda'' \); with \( \Lambda' = \nu' + c\nu' + sw' \), \( \Lambda'' = \nu'' + c\nu'' - sv' \), \( \Lambda_c = cz_c + sy_c \), \( \Lambda_y = cy_c - sz_c \). The variation \( \delta U \) of the strain energy is given by the following expression \([8, 12, 22, 23]\):

\[
\delta U = \int_{L} (N\delta \epsilon - M_y \delta k_y - M_z \delta k_y + M_{sv} \delta \theta_x + B\omega \delta \theta_x'' + \frac{1}{2} M_R \delta (\theta_x'')^2) \, dx
\]

where \( N \) is the axial force, \( M_y \) and \( M_z \) are the bending moments about the principal axes \( y \) and \( z \), \( B\omega \) is the bimoment acting on the cross section, \( M_{sv} \) is the St-Venant torsion moment (see figure 2). The higher order stress resultant \( M_R \) is the Wagners moment, responsible of nonlinear warping and playing an important role in stability and dynamical nonlinear behavior of the structure often neglected in literature.

**Figure 2.** Axial force \( N \), bending moments \( M_y \) and \( M_z \), bimoment \( B\omega \) and St-Venant torsion moment \( M_{sv} \)

The axial force \( N \), bending moments \( M_y \), \( M_z \), bimoment \( B\omega \), St-Venant torsion moment \( M_{sv} \), and the Wagners moment \( M_R \) are expressed by:

\[
\begin{align*}
N &= \int_A \sigma_{xx} \, dA = \int_A E\gamma_{xx} \, dA = EA\gamma + \frac{1}{2} EAI_0 \theta_x'^2 \\
M_y &= \int_A z\sigma_{xx} \, dA = \int_A zE\gamma_{xx} \, dA = -EI_y (k_y - \beta_y \theta_x'^2) \\
M_z &= \int_A y\sigma_{xx} \, dA = \int_A yE\gamma_{xx} \, dA = -EI_z (k_z - \beta_z \theta_x'^2) \\
M_{sv} &= 2 \int_A [\mu\gamma_{xz} (y - y_c - \frac{\partial \gamma}{\partial z}) - \mu\gamma_{xy} (z - z_c + \frac{\partial \gamma}{\partial y})] \, dA = \mu J \theta_x'' \\
B\omega &= -\int_A \omega\sigma_{xx} \, dA = -\int_A E\omega\gamma_{xx} \, dA = EI\omega (\theta_x'' - \beta_\omega \theta_x'^2) \\
M_R &= \int_A R^2 \sigma_{xx} \, dA = EAI_0 \gamma - 2EI_z \beta_y k_z - 2EI_y \beta_z k_y - 2EI\omega \beta_\omega \theta_x''
\end{align*}
\]  

where \( \beta_y \), \( \beta_z \) and \( \beta_\omega \) are Wagner’s coefficients. The expression (19) can be written in the following compact form:

\[
\delta U = \int_{L} \langle \delta \gamma \rangle \{ S \} \, dx
\]

where the stress vector \( \{ S \} \) is connected to the elastic behavior matrix \( [D(x)] \) of the thin-walled beam with open section cross through the constitutive law:

\[
\{ S \} = [D(x)] \{ \gamma \}
\]
with \( \{S\} \), \([D(x)]\) and \( \{\gamma\} \) are defined by:

\[
\{S\} = \begin{bmatrix} N & M_y & M_z & M_{sv} & B_\omega \end{bmatrix}
\]

\[
[D(x)] = E \begin{bmatrix} 0 & 0 & 0 & 0 & A I_0 \\ 0 & I_y & 0 & 0 & 0 & 0 & 2 I_y \beta_z \\ 0 & 0 & I_z & 0 & 0 & 0 & 0 & 2 I_z \beta_y \\ 0 & 0 & 0 & I_y & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
\{\gamma\} = \begin{bmatrix} \gamma \\ -k_y \\ -k_z \\ \theta_x \\ \theta_y \\ \theta_x \end{bmatrix}
\]

with \( J(x) \) is the St-Venant torsion given by:

\[
J(x) = \int_{A(x)} \left( (y - y_c - \frac{\partial \omega}{\partial y})^2 + (z - z_c - \frac{\partial \omega}{\partial y})^2 \right) dA
\]

(24)

Let us introduce the following vector:

\[
^t \{\theta\} = <u', v', w', \theta_x', v'', w'', \theta_x'' , \theta_x >
\]

(25)

Due to large torsion context, the strain vector \( \{\gamma\} \) is highly nonlinear. It can be written in the form:

\[
\{\gamma\} = \{\gamma_l\} + \{\gamma_{nl}(\{\theta\}, \{\theta\})\} + \{\gamma_{nl\alpha}(\{\theta\}, \{\alpha\})\}
\]

(26)

where \( \{\gamma_l\} \) is the linear part of \( \{\gamma\} \), \( \{\gamma_{nl}(\{\theta\}, \{\theta\})\} \) and \( \{\gamma_{nl\alpha}(\{\theta\}, \{\alpha\})\} \) are its nonlinear parts representing the quadratic and flexural-torsion coupling terms respectively and \( ^t \{\alpha\} \) denotes the vector:

\[
^t \{\alpha\} = <c, s, \psi>
\]

(27)

which takes into account of large rotation and twist coupling flexion, where:

\[
\psi = Q_c v' + R_c w' \quad \text{with} \quad Q_c = -y_c s - z_c (c + 1) \quad \text{and} \quad R_c = y_c (c + 1) - z_c s
\]

(28)

Introducing the vector \( \{\theta\} \) defined in (25), one can express the strain vector \( \{\gamma\} \) in terms of \( \{\theta\} \) by [22]:

\[
\{\gamma\} = \left( [H] + \frac{1}{2} [A(\{\theta\})] - [A_{\alpha}(\{\theta\}, \{\alpha\})] \right) \{\theta\}
\]

(29)

Using (29), the strain energy variation (21) and the constitutive law (22) are given by the matrix systems:

\[
\delta U = \int_L <\delta \theta > ([H] + [A(\{\theta\})] - [A_{\alpha}(\{\theta\}, \{\alpha\})])^t \{S\} dx
\]

(30)

\[
\{S\} = [D]([H] + \frac{1}{2} [A(\{\theta\})] - [A_{\alpha}(\{\theta\}))] \{\theta\}
\]
where \([H]\) is a constant matrix, \([A(\{\theta\})]\) and \([A_\alpha(\{\alpha\})]\) are nonlinear matrices. They are functions on the components of vectors \(\{\theta\}\) and \(\{\alpha\}\), \([A(\{\theta\},\{\alpha\})]\) is a matrix that results from the variation of \([A_\alpha(\{\alpha\})]\). All of these matrices are given in [22].

For arbitrary loadings, the external virtual work can be expressed, in cartesian co-ordinates, in terms of kinematic variables and resultant forces. Let us denote by \(\{f_j\} = <f_{xe}, f_{ye}, f_{ze}>\) the concentrated vector force applied at point \(M(x_j, y_j, z_j)\) of the cross section \(A_i\) (see figure 3a) and by \(\{F\} = <F_{xe}, F_{ye}, F_{ze}>\) the vector of distributed forces imposed on a part \(L_f\) of the beam (see figure 3c).

The variation \(\delta W_{ext}\) of the virtual work of external forces, proportional to the load parameter \(\lambda\), is expressed in terms of displacement variations of by:

\[
\delta W_{ext} = \lambda \int_{L_f} \int_{\partial A_i} (F_{xe} \delta u_M + F_{ye} \delta v_M + F_{ze} \delta w_M) ds dx + \lambda \sum_{N_c=1}^{N_c} \left( \sum_{n=1}^{n (f \{f_j\}) (i)} (f_{xe} \delta u_M + f_{ye} \delta v_M + f_{ze} \delta w_M) \right)
\]

(31)

where \(N_c\) in (31) is the number of loaded cross sections of the beam and \(n\) is the number of concentrated forces applied on the cross section \(A(x_i)\), which can be written as the sum of the work of the distributed forces and the work of the concentrated forces as:

\[
\delta W_{ext} = \delta W_d + \delta W_c
\]

(32)

Inserting the variation of kinematic relations (7) and taking into account for \(\delta c =

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Section beam under concentrated and distributed forces}
\end{figure}
Inserting (33) in (31) and replacing the 3D loadings by their statically equivalent forces on the mean fibre (see figures 3b and 3d), we obtain the following variations of works \( W_d \) and \( W_C \) of the distributed and concentrated forces respectively:

\[
\begin{align*}
\delta W_d & = \lambda \int_{L_f} (\hat{F}_{xz} \delta u + \hat{F}_{ye} \delta v + \hat{F}_{xe} \delta w + \hat{M}_{xc} \delta \theta_x + \hat{M}_{ze} \delta \theta_x + \hat{M}_{ye} \delta w' \\
& + \hat{B}_{w} \delta \theta_x') dx + \lambda \int_{L_f} (\hat{M}_{ze} \delta u' - \hat{M}_{ye} \delta v') \delta \theta_x dx \\
& + \lambda \int_{L_f} (\hat{M}_{ye} c + \hat{M}_{ze} s) \delta w' + (\hat{M}_{ze} c - \hat{M}_{ye} s) \delta v' + (\hat{M}_{xe} c) \\
& - (\hat{F}_{ye} \delta z + \hat{F}_{ye} \delta y) \delta \theta_x dx \\
& + \lambda \int_{L_f} (\hat{M}_{ze} (w' c - v' s) - \hat{M}_{ye} (v' c + w' s)) \delta \theta_x dx
\end{align*}
\]

\[
\begin{align*}
\delta W_C & = \lambda \sum_{i=1}^{N_i} (\hat{f}_{xz} \delta u + \hat{f}_{ye} \delta v + \hat{f}_{xe} \delta w + \hat{m}_{xe} \delta \theta_x + \hat{m}_{ze} \delta \theta_x + \hat{m}_{ye} \delta w' \\
& + \hat{b}_{w} \delta \theta_x' + (\hat{m}_{ze} \delta u' - \hat{m}_{ye} \delta v') \delta \theta_x + (\hat{m}_{xe} c - \hat{m}_{ye} s) \delta v' \\
& + (\hat{m}_{ye} c + \hat{m}_{ze} s) \delta w' + (\hat{m}_{xe} c - (\hat{f}_{xe} e_x + \hat{f}_{ye} e_y) s) \delta \theta_x \\
& + (\hat{m}_{ze} (w' c - v' s) - \hat{m}_{ye} (v' c + w' s)) \delta \theta_x)_{A(i)}
\end{align*}
\]

The efforts used in (34) and (35) are the forces \( (\hat{F}, \hat{f}) \), the bending and twisting moments \( (\hat{M}, \hat{m}) \) and Vlasov’s warping forces are denoted by \( (\hat{B}, \hat{b}) \). They are expressed by:

\[
\begin{align*}
\hat{F}_{xe} & = \int_{\partial A} F_{xs} ds \\
\hat{F}_{ye} & = \int_{\partial A} F_{ys} ds \\
\hat{F}_{xe} & = \int_{\partial A} F_{zs} ds \\
\hat{B}_{w} & = - \int_{\partial A} \omega F_{xs} ds
\end{align*}
\]

\[
\begin{align*}
\hat{f}_{xe} & = \sum_{j=1}^{n} f_{xej} \\
\hat{f}_{ye} & = \sum_{j=1}^{n} f_{yej} \\
\hat{f}_{xe} & = \sum_{j=1}^{n} f_{xej}
\end{align*}
\]

\[
\begin{align*}
\hat{m}_{xe} & = \sum_{j=1}^{n} (e_{yj} f_{xej} - e_{zj} f_{yej}) \\
\hat{m}_{ye} & = - \sum_{j=1}^{n} z_{j} f_{xej} \\
\hat{m}_{ze} & = - \sum_{j=1}^{n} y_{j} f_{xej}
\end{align*}
\]

\[
\hat{b}_{w} = - \sum_{j=1}^{n} \omega_{j} f_{xej}
\]
To have a matrix form, let us introduce the following vectors:

\[
\begin{aligned}
\{q\} &= <u, v, w, \theta> \\
\{\hat{F}_e\} &= <\hat{F}_{se}, \hat{F}_{ye}, \hat{F}_{ze}, \hat{M}_{xe}>
\end{aligned}
\]

\[
\begin{aligned}
\{\hat{M}_e\} &= <0, \hat{M}_{ze}, \hat{M}_{ye}, \hat{B}_{we}, 0, 0, 0, 0>
\end{aligned}
\]

\[
\begin{aligned}
\{\hat{f}_e\} &= <\hat{f}_{xe}, \hat{f}_{ye}, \hat{f}_{ze}, \hat{m}_{xe}>
\end{aligned}
\]

\[
\begin{aligned}
\{m_e\} &= <0, \hat{m}_{ze}, \hat{m}_{ye}, \hat{b}_{we}, 0, 0, 0, 0>
\end{aligned}
\]

The virtual work \(\delta W_{\text{ext}}\) (31) or (32) can be written in the following matrix form:

\[
\delta W_{\text{ext}} = \lambda \left( \int L_f (\delta q) \{\hat{F}_e\} + \delta \theta \{\hat{M}_e\} \right) dx
\]

\[
+ \int L_f <\delta \theta > \left( \left[ M_{x1}\right] \{\theta\} + \left[ M_{x2}\right] \{\alpha\} + \left[ M_{x3}\right] (\{\theta\}) \{\alpha\} \right) dx
\]

\[
+ \lambda \sum_{i=1}^{N_c} \left( \delta \theta \{\hat{f}_e\} + \delta \theta \{\hat{m}_e\} + <\delta \theta > \left( \left[ M_{x1}\right] \{\alpha\} + \left[ M_{x2}\right] (\{\theta\}) \{\alpha\} \right) A(i) \right)
\]

where \([M_{x1}],[M_{x2}]\) and \([M_{x3}(\{\theta\})]\) are respectively the rotation, flexion and torsion matrices which are functions of load eccentricities with respect to gravity and torsion centers. In the case of distributed forces, they are given by:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\hat{M}_{ye} & \hat{M}_{ze} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\(\left[ M_{x1}\right] = \)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\hat{M}_{ye} & \hat{M}_{ze} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\(\left[ M_{x2}\right] =\)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\hat{M}_{ye} & \hat{M}_{ze} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\([M_{x3}(\{\theta\})] = \)

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & (\hat{M}_{ye} v' + \hat{M}_{ze} w') & (\hat{M}_{ye} w' + \hat{M}_{ze} v') & 0
\end{bmatrix}
\]

Let us note that one can apply the external real forces away the middle line using the matrices \([M_{x1}],[M_{x2}]\) and \([M_{x3}(\{\theta\})]\) which represent the flexion and twist from gravity center \(G(y_G, z_G)\) and twist center \(C(y_c, z_c)\) that contribute in the stiffness matrix. Finally, using the above definitions and notations, the stationary condition
of the total potential energy gives the following system of the dynamic equations and material behavior:

\[
\begin{aligned}
&\int_{L} \delta u < [M((\theta), \{\alpha\})] [\ddot{u}] dx + \int_{L} \delta \theta > [C((\theta), \{\alpha\})] [\ddot{\theta}] dx \\
+ &\int_{L} < \delta \theta > ([H] + [A(\{\theta\})]) - [A_{o}(\{\alpha\})] - [\dot{A}(\{\theta\}, \{\alpha\})] [S] dx \\
- &\lambda \left( \int_{L} < \delta \theta > ([M_{x}]\{\theta\} + [M_{x}\{\alpha\}] + [M_{x3}\{\theta\}]) [\alpha] dx \right) \\
+ &\sum_{i=1}^{N_{e}} < \delta \theta > ([M_{x}]\{\theta\} + [M_{x}\{\alpha\}] + [M_{x3}\{\theta\}]) [\alpha]_{A(i)} \\
- &\lambda \left( \sum_{i=1}^{N_{e}} \left( < \delta q > \{\tilde{F}_{c}\} + < \delta \theta > \{\tilde{M}_{c}\} \right) dx \right) \\
+ &\sum_{i=1}^{N_{e}} \left( < \delta q > \{\tilde{F}_{c}\} + < \delta \theta > \{\tilde{M}_{c}\} \right)_{A(i)} = 0 \\
\{S\} &= [D][[H] + \frac{1}{2}[A(\{\theta\})] - [A_{o}(\{\alpha\})] [\theta]
\end{aligned}
\]

The highly nonlinear coupled system (41) governs the forced nonlinear dynamic state of thin-walled beams open cross section subjected to arbitrary external dynamic loads for large torsion and takes into account the flexural-torsion coupling and permits to study load eccentricity effects. Let us remind that any simplifying hypothesis on the torsion angle amplitude \( \theta \) have been made. The trigonometric functions \( c \) and \( s \) are incorporated in the vector \( \{\alpha\} \) as additional variables. The high nonlinearity comes from large torsion and flexural-torsion coupling through matrices \([A_{o}(\{\alpha\})]\) and \([\dot{A}(\{\theta\}, \{\alpha\})]\) in the stiffness rigidity matrix, mass matrix \([M(\{\theta\}, \{\alpha\})]\) and gyroscopic matrix \([C(\{\theta\}, \{\alpha\})]\). This system is highly nonlinear and all the equations (41) are fully coupled. The problem becomes more cumbersome in presence of eccentric loads. The adopted resolution strategy of this problem is described in what follows.

3. Space discretization and methodology of resolution.

3.1. Finite element discretization. The discrete form of the system (41) is obtained by recourse to the standard finite elements method approach [5, 10, 30, 4]. The thin-walled beam of slenderness \( L \) is meshed in many 3D beam elements with two nodes and seven degrees of freedom per node. For the shape functions, we adopt linear functions for axial displacement \( u \), and Hermite cubic functions for the variables \( v \), \( w \) and \( \theta \). Denoting by:

\[
\begin{gathered}
t_{i} \{r\} = < u, v, w, \theta_{x}, v', w', \theta_{x}' >_{i} \quad (i = 1, 2)
\end{gathered}
\]

the nodal displacements (degrees of freedom), the vectors \( \{q\}, \{\theta\} \) and their variations write in the form:

\[
\begin{aligned}
\{q\} &= [N]\{r\}_{e} \quad , \quad \{\delta q\} &= [N]\{\delta r\}_{e} \\
\{\theta\} &= [G]\{r\}_{e} \quad , \quad \{\delta \theta\} &= [G]\{\delta r\}_{e}
\end{aligned}
\]

where \([N]\) is the shape interpolation matrix and \([G]\) is a matrix which links the gradient vector \( \{\theta\} \) to nodal displacements. The substitution of equation (41) and (30) in eqs (38) leads, after assembly of elements, to
the following discretized form of the governing dynamic equation and constitutive law system:

\[
\begin{align*}
\sum_e \frac{1}{2} \int_{-1}^{1} \left( |\mathbf{F}_e| - M_e \right) \mathbf{v} d\xi + \\
\sum_e \frac{1}{2} \int_{-1}^{1} \left( |\mathbf{F}_e| - M_e \right) \mathbf{v} d\xi + \\
\sum_e \frac{1}{2} \int_{-1}^{1} \left( |\mathbf{F}_e| - M_e \right) \mathbf{v} d\xi \end{align*}
\]

(44)

In which the matrix \([\mathbf{N}G]\) that connects the displacement vector \(\{u\}\) to nodal displacements and \([B(\{\theta\}, \{\alpha\})]\) is defined by:

\[
B(\{\theta\}, \{\alpha\}) = [B_1] + [B_{nl}([\theta])] - [B_{nla}(([\theta], \{\alpha\}))] - [\hat{B}_{nla}((\theta), \{\alpha\})]
\]

(45)

The matrices \([B_1]\) and \([B_{nl}([\theta])]\) are familiar in nonlinear structural analysis. \([B_{nla}(([\theta], \{\alpha\}))]\) and \([\hat{B}_{nla}((\theta), \{\alpha\})]\) are additional geometric matrices resulting from large torsion and flexural-torsion coupling. All these matrices are defined in [22]. The problem (44) can be written in global coordinate system in the general form:

\[
\begin{align*}
\left\{ \begin{array}{l}
([\mathbf{M}_g] + [M_{nlg}([\theta_g], \{\alpha_g\})])[\mathbf{r}_g] + ([\mathbf{C}_{lg}([\theta_g])] + [\mathbf{C}_{nlg}([\theta_g], \{\alpha_g\})])\mathbf{\dot{r}}_g \\
+ [\mathbf{K}_g([\theta_g], \{\alpha_g\})][\mathbf{r}_g] - \lambda[M_{xg}([\theta_g])][\mathbf{r}_g] = \lambda[F_g]
\end{array} \right\}
\end{align*}
\]

(46)

with

\[
\begin{align*}
[M_{lg}] & = \sum_e \frac{1}{2} \int_{-1}^{1} |\mathbf{F}_e| [M_1] [\mathbf{N}G] d\xi \\
[M_{nlg}([\theta_g], \{\alpha_g\})] & = \sum_e \frac{1}{2} \int_{-1}^{1} |\mathbf{F}_e| [M_{nl}] [\mathbf{N}G] d\xi \\
[C_{lg}([\theta_g])] & = \sum_e \frac{1}{2} \int_{-1}^{1} |\mathbf{F}_e| [C_1] [\mathbf{N}G] d\xi \\
[C_{nlg}([\theta_g], \{\alpha_g\})] & = \sum_e \frac{1}{2} \int_{-1}^{1} |\mathbf{F}_e| [C_{nl}] [\mathbf{N}G] d\xi \\
[K_g([\theta_g], \{\alpha_g\})] & = \sum_e \frac{1}{2} \int_{-1}^{1} ([B(\{\theta\}, \{\alpha\})] + [D]([B] + [B_{nl}([\theta])]) - [B_{nla}(([\theta], \{\alpha\}))]) \mathbf{r}_g d\xi \\
[M_{xg}([\theta_g])] & = \sum_e \frac{1}{2} \int_{-1}^{1} |\mathbf{F}_e| ([M_{xg}] + [M_{xg}([\theta_g])]) [G] d\xi
\end{align*}
\]

(47)

where \([M_{lg}]\) and \([F_g]\) are the linear parts of mass and external forces, and \([M_{nlg}([\theta_g], \{\alpha_g\})], [C_{lg}([\theta_g])]\), \([C_{nlg}([\theta_g], \{\alpha_g\})]\) and \([M_{xg}([\theta_g])]\) are respectively the
coupling nonlinear parts of mass, the nonlinear quadratic part of gyroscopic dynamic matrix, the coupling nonlinear part of gyroscopic dynamic matrix and the quadratic part of external eccentric loads and $[K_g(\{\theta_g\}, \{\alpha_g\})]$ is the linear and nonlinear parts of stiffness and external forces. In order to simplify the problem (46), we neglect in what follows, the parts $[M_{nlg}(\{\theta_g\}, \{\alpha_g\})]$ and $[C_{nlg}(\{\theta_g\}, \{\theta_g\})]$. Then the problem (46) reduces to:

$$
[M_{lg}][\ddot{r}_g] + [C_{lg}(\{\theta_g\})][\dot{r}_g] + [K_g(\{\theta_g\}, \{\alpha_g\})][r_g] - \lambda [M_x(\{\theta_g\})][\alpha_g] = \lambda [F_g]
$$

(48)

The nonlinear system (48) is usually solved numerically by incremental iterative methods based on the Newmark direct integration [26] and Newton-Raphson method. These last is very consuming in terms of computational time since they require updating the different matrices at each time step or at each iteration.

The aim of this work is to propose three homotopy transformations coupled with high-order implicit solver for analysing the forced nonlinear vibration of thin-walled beams with open cross section subjected to arbitrary external dynamic load taking into account the influence of load eccentricities.

3.2. Solution strategy. The solution of the nonlinear discrete dynamic problem (48) is obtained by using efficient high-order implicit solvers derived by considering good homotopy transformations different from that used in [8]. These high-order implicit solvers are developed by combining the following five techniques: i) a time discretization by the Newmark scheme, ii) a change of variable, iii) appropriate homotopy transformations, iv) power series expansion technique and v) a continuation method. These techniques are detailed in next subsections.

3.2.1. Time discretization procedure. The time discretization of space discretized nonlinear dynamic problem (48) is performed by using unconditionally stable classical Newmark time scheme, widely used in dynamic problems [26]. Knowing the terms $\{r^n_g\}$ of (48), the velocities $\{\dot{r}^n_g\}$ and the acceleration $\{\ddot{r}^n_g\}$ at time $n\Delta t$; with $\Delta t$ is the time step, the corresponding quantities $\{r^{n+1}_g\}$, $\{\dot{r}^{n+1}_g\}$ and $\{\ddot{r}^{n+1}_g\}$ at time $(n+1)\Delta t$ are given by:

$$
\begin{align*}
\{\ddot{r}^{n+1}_g\} &= \{\dot{r}^n_g\} + \Delta t(\ddot{r}^n_g) + \frac{\Delta t^2}{2}((1 - 2\alpha)\{\ddot{r}^n_g\} + 2\alpha\{\ddot{r}^{n+1}_g\}) \\
\{\dot{r}^{n+1}_g\} &= \{\ddot{r}^n_g\} + \Delta t((1 - \beta)\{\ddot{r}^n_g\} + \beta\{\ddot{r}^{n+1}_g\})
\end{align*}
$$

(49)

where $\alpha$ and $\beta$ are the Newmark constants such that $2\alpha \leq \beta \leq \frac{1}{2}$ which determine both the accuracy and the stability of scheme. From equation (49), one can express the nodal acceleration $\{\ddot{r}^{n+1}_g\}$ at time $(n+1)\Delta t$ as follows:

$$
\{\ddot{r}^{n+1}_g\} = a_0(\{\ddot{r}^{n+1}_g\} - \{\ddot{r}^n_g\}) - a_1(\dot{r}^n_g) - a_2(\ddot{r}^n_g)
$$

(50)

where the coefficients $a_0$, $a_1$ and $a_2$ in (50) are defined by:

$$
a_0 = \frac{1}{\alpha \Delta t}, \quad a_1 = \frac{1}{\alpha \Delta t}, \quad a_2 = \frac{1}{2\alpha} - 1
$$

(51)

3.2.2. Change of variable. Let us introduce the following change of variable:

$$
\{r^{n+1}_g\} = \{r^n_g\} + \{\Delta r_g\}
$$

(52)

where the increment $\{\Delta r_g\}$ is the new unknown. The insertion of expressions (50), (51) and (52) in equation (48) leads to the form of space and time discretized
dynamic equation, expressed in terms $n$ time step $\Delta t$: 

$$
\begin{align*}
(K/I_g) + [K_{nlg}((\theta_g^n), (\alpha_g^n))] + a_0[M/I_g] + a_0\beta\Delta t((1 - a_1\beta\Delta t)[C_{qq}((\dot{r}_g^n)]) \\
+ (\Delta t(1 - \beta) - a_2\beta\Delta t)[C_{qq}((\dot{r}_g^n)))](\Delta r_g) \\
+ a_0\beta\Delta t[C_{qq}((\Delta r_g))][(1 - a_1\beta\Delta t)(\dot{r}_g^n) + (\Delta t(1 - \beta) - a_2\beta\Delta t)(\dot{r}_g^n)] \\
+ ([K_{nlg}((\Delta \theta_g), (\alpha_g^n)] + [K_{nlg}((\Delta \theta_g), (\Delta \alpha_g)))](\dot{r}_g^n) \\
- \lambda([M_x((\Delta \theta_g))]\{\alpha_g^n\} + ((1 - a_1\beta\Delta t)C_{qq}([\dot{r}_g^n])) \\
+ (\Delta t(1 - \beta) - a_2\beta\Delta t)[C_{qq}([\dot{r}_g^n])][(1 - a_1\beta\Delta t)(\dot{r}_g^n) + (\Delta t\beta\Delta t)(\dot{r}_g^n)] + \\
(\Delta t(1 - \beta) - a_2\beta\Delta t)(\dot{r}_g^n) + ([K/I_g] + [K_{nlg}((\theta_g^n), (\alpha_g^n)))](\dot{r}_g^n) - \\
(\Delta t(1 - \beta) - a_2\beta\Delta t)(\dot{r}_g^n) + ([K/I_g] + [K_{nlg}((\theta_g^n), (\alpha_g^n)))](\dot{r}_g^n) - \\
\lambda([M_x((\theta_g^n))]\{\alpha_g^n\}) + [F_g] - [M_I](a_1(\dot{r}_g^n) + a_2(\dot{r}_g^n)) \\
+ ((a_0\beta\Delta t)^2[C_{qq}((\Delta r_g)))] + [K_{nlg}((\Delta \theta_g), (\alpha_{g})](\Delta r_g) \\
+ [K_{nlg}((\theta_g^n), (\Delta \alpha_g))]\{\alpha_g^n\} - \\
\lambda([M_x((\Delta \theta_g))]\{\alpha_g^n\} + [M_x((\theta_g^n))]\{\Delta \alpha_g\}) = 0
\end{align*}
$$

(53)

The equation (53) can be written in a form of a linear part $[K_T]\{\Delta r_g\}$ and a nonlinear part $\{F_g^{nl}\}$ given by:

$$
[K_T]\{\Delta r_g\} + \{F_g^{nl}\} = \{SM_g\}
$$

(54)

where $[K_T]$ is the stiffness matrix including the stiffness, mass, gyroscopic terms and the Newmark coefficients, $\{F_g^{nl}\}$ represents a nonlinear vector and $\{SM_g\}$ is right hand-side. To solve (54) with a reduced computational effort and to decrease the computational cost in matrix inversions, a robust and efficient numerical strategy is needed.

In the next, three high-order implicit solvers are proposed and compared. they are obtained by introducing some homotopy transformations.

3.2.3. The homotopy transformations. The homotopy technique is one of the important techniques widely used to approximate, in mathematical physics, the solutions of nonlinear static or dynamic problems. This technique was originally proposed by [11] and used after by [2]. It consists to modify the time discretized problem (54) by introducing an adimensional artificial parameter “a” and a pre-conditioner arbitrary matrix $[K^*]$ in this problem. This pre-conditioner arbitrary matrix is homogeneous to the tangent stiffness matrix. Let us note that there exist many manners to perform this modification according to choice of the homotopy transformation. Hence, an appropriate choice of the homotopy transformation is of a great importance because it leads to strong reduction of computational effort and cost. To solve the time discretized problem (54), we present, in this section, three different kinds of homotopy transformations which will be compared in order to select the better in terms of computational cost.
**First homotopy transformation.** In order to overcome the inversion of the tangent stiffness matrix at each time step, we adopt an homotopy technique used in the previous work of the same authors [8]. Its consists to introduce an arbitrary invertible matrix and apply an homotopy transformation which acts both on quadratic term and on the right hand side as follows [8, 15, 28, 9, 13]:

$$\[ K^* \{ \Delta r_g \} + a([K_T] - [K^*]) \{ \Delta r_g \} + a\{ F^{nl}_g \} = a\{ SM_g \} \] \quad (55)$$

This homotopy transformation allows to transform continuously the system from the original, when $a = 1$, to the easier system to solve with $a = 0$. This algorithm is denoted by $Alg_1$.

**Perturbation technique.** To solve the nonlinear problem (55), the perturbation method is used and “$a$” is considered as a perturbation parameter [25, 24]. The unknowns $\Delta r_g$ are sought in the form of an integro-power series with respect to homotopy parameter “$a$” starting from order 1 and truncated at order $p$ as follows:

$$\Delta r_g = a\Delta r_{g1} + ... + a^p\Delta r_{gp} \quad (56)$$

Inserting the series representation (56) in the homotopy system (55) and after equaling like power of “$a$”, we get at each order, the following set of sequence recurrent linear systems:

Order $p = 1$:

$$[K^*]\{ \Delta r_{g1} \} = \{ SM_g \} \quad (57)$$

Order $p \geq 2$:

$$[K^*]\{ \Delta r_{gp} \} = ([K^*] - [K_T])\{ \Delta r_{g(p-1)} \} - \{ F^{nl}_{g(p-1)} \} \quad (58)$$

Each linear system ((57)-(58)) is solved numerically. Only the right-hand sides vectors are computed for each of these linear systems.

**Validity range.** The validity range of series representation (56) is $[0, a_{max}]$, where the maximal homotopy parameter $a_{max}$ is estimated by the criterion which is expressed in terms of the truncation order of a defined tolerance parameter $\varepsilon$ and right hand sides of linear equations verified by the terms of series (56), given by the following relationship:

$$a_{max} = \left( \frac{\varepsilon}{\| F^{nl}_{g(p+1)} \|} \right)^{1/p+1} \quad (59)$$

The parameter $a_{max}$ which depends of time allows us to determine the maximum time $t_{max}$ of validity range of series which is defined by the following inequality $a_{max} \geq 1$. This maximum time will be considered later as an initial condition for the next step of the continuation procedure. When this inequality becomes false, we increase the truncation order of development in Taylor series.

**Second homotopy transformation.** This second homotopy transformation acts only on the right hand side as follows:

$$[K^*]\{ \Delta r_g \} + a([K_T] - [K^*])\{ \Delta r_g \} + \{ F^{nl}_g \} = a\{ SM_g \} \quad (60)$$

This algorithm is denoted by $Alg_2$. 
b.1 Perturbation technique. Introducing the polynomial representation (56) in the homotopy problem system (60), and equating like power of \( \text{“} a \text{”} \), a set of following linear problems is obtained at each order:

Order \( p = 1 \):
\[
[K^*]\{\Delta r_{g1}\} = \{SM_g\}
\]

Order \( p \geq 2 \):
\[
[K^*]\{\Delta r_{gp}\} = ([K^*] - [KT])\{\Delta r_{g(p-1)}\} - \{F_{gp}^n\}
\]

c- Third homotopy transformation. This last homotopy transformation acts only on quadratic term as follows:
\[
[K^*]\{\Delta r_g\} + a([KT] - [K^*])\{\Delta r_g\} + a\{F_{gp}^n\} = \{SM_g\}
\]

This algorithm is denoted by Alg3.

c.1 Perturbation technique. In contrary to previous cases, let us search the unknown \( \Delta r_g \) of the homotopy problem (63) in the from of the following series expansion with respect to homotopy parameter “\( a \)” truncated at order \( p \):
\[
\{\Delta r_g\} = \{\Delta r_{g0}\} + a\{\Delta r_{g1}\} + ... + a^p\{\Delta r_{gp}\}
\]

Substituting the series representation (64) in (63) and equating like power of “\( a \)” , we get, at each order, the following linear problems:

Order \( p = 0 \):
\[
[K^*]\{\Delta r_{g0}\} = \{SM_g\}
\]

Order \( p \geq 1 \):
\[
[K^*]\{\Delta r_{gp}\} = ([K^*] - [KT])\{\Delta r_{g(p-1)}\} - \{F_{p}^n\}
\]

these three homotopy transformations (55), (60) and (63) differ principally by the way to perturb the nonlinear quadratic term and/or the right hand side. The aim of this work is to investigate the homotopy transformation effect on the computational cost for accede to nonlinear dynamic response of thin-walled beams with open cross section. The developed high-order implicit solvers described previously, will be applied and tested, in the next, on two examples of nonlinear dynamic problem of thin-walled beams with open cross section under dynamic loadings.

4. Numerical results and discussions. We test in the following the efficiency and accuracy of the high-order implicit solvers for analyzing the nonlinear dynamic of thin-walled beams with open cross section subjected to various dynamic loading and different boundary conditions. Two examples are examined to demonstrate the performance of these numerical approaches. In these numerical applications, we have chosen the pre-conditioner equal to the tangent stiffness matrix \([KT] \) evaluated at each end of the continuation step.

4.1. Nonlinear dynamic of an U-mono-symmetrical thin-walled beam with open cross section under discrete loading. We consider a clamped-free mono-symmetrical beam with \( U \) cross section of length \( L = 9m \) subjected to concentrated dynamical loading \( F_z(t) \) applied at the section \( B \) (see figure 4a). The temporal evolution of dynamical loading \( F_z(t) \) is given in the figure 4b. All numerical computations are carried out by the following material and geometrical data which are similar to those used in [8, 17]. The material data are \( E = 210GPa, \nu = 0.33, \rho = 7850kg/m^3 \) and the geometrical data of the sections are reported in figures 4a and 5 where the section varies according to the relationship \( h(x) = -\frac{0.15}{L}x + 0.3. \)
The dynamical loading is applied at point $O$ (see figure 5b) with the eccentricities $e_y = 0.065m$ and $e_z = 0.15m$ with respect to centroid of the end cross section where $y_c = -0.06m$, $z_c = 0m$.

Figure 4. External dynamical loading and its time evolution applied on the U-mono-symmetrical thin-walled beam with open cross section

Figure 5. Geometrical characteristics of sections $A$ and $B$

The nonlinear dynamical analysis is performed in time range $[0, 4s]$ with a time step $\Delta t = 10^{-3}s$. To compare the results of high-order implicit solvers to those of the Abaqus code [1], we take as a truncation order $p = 15$ and a tolerance parameter $\epsilon = 10^{-6}$. The domain of the considered beam is discretized in 40 3D beam elements [12] for high-order implicit solvers and in 4800 $C3D20R$ solid elements for the Abaqus code.

The time evolutions of vector components $(u(L,t), v(L,t), w(L,t), \theta_x(L,t))$ at node 41 (point $B$) compared to those given by the Abaqus code are depicted in figure 6a-d. These results show a good agreement between the results obtained by high-order implicit solvers and those of Abaqus code. Comparing the appearances of obtained curves with those given in the works of [8, 17] concerning thin-walled beams with open constant cross section, we observe that there is a change in the vibrational behavior when the cross section is variable. The full response curves of figure 6 is obtained by high-order implicit solvers using three inversions of the matrix $[K^*]$. Whereas, the obtaining of same response curves by the Abaqus code requires 8965 iterations. Subsequently, we will study the influence of some parameters on the effectiveness of high-order implicit solvers.
a- Effect of the time step. In this paragraph, we study the influence of time step on the parameters of high-order implicit solvers (optimal truncation order: optimal order, Residue: Log|Res|). In the table 1, we remark that the optimal order increases when the time step increases for a tolerance parameter fixed to \( \epsilon = 10^{-5} \). From these results, we notice that the solver Alg3 is better than the others because its optimal order is the smallest.

| Solvers | Alg1 | Alg2 | Alg3 |
|---------|------|------|------|
| \( \Delta t \) | Optimal order | Log|Res| | Optimal order | Log|Res| | Optimal order | Log|Res| |
| \( 10^{-3} \) | 10 | -3.73 | 9 | -3.71 | 8 | -3.71 |
| \( 2 \times 10^{-3} \) | 12 | -3.72 | 10 | -3.71 | 9 | -3.70 |
| \( 3 \times 10^{-3} \) | 13 | -3.69 | 11 | -3.67 | 10 | -3.65 |

Table 1. Comparison between three solvers Alg1, Alg2 and Alg3: Influence of time step

b- Effect of truncation order \( p \). Here, we study the influence of the truncation order on the number of matrix inversions \([K^*]\) denoted by IM, the number of right hand sides denoted by RHS and the computation time denoted by CPU for the three studied solvers with a time step \( \Delta t = 10^{-3} \) and a tolerance parameter \( \epsilon = 10^{-5} \). In Table 2, we notice that the three solvers diverge for the truncation order \( p = 7 \),

Figure 6. Response curves obtained by the high-order implicit solver Alg3 and by Abaqus code. Time evolution of displacement components \((u(L,t), v(L,t), w(L,t), \theta_x(L,t))\)
the solver $\text{Alg}_3$ starts to operate from the truncation order $p = 8$, the solver $\text{Alg}_2$ from $p = 9$ and the solver $\text{Alg}_1$ from $p = 10$. For the truncation order $p = 10$, for example, the solver $\text{Alg}_3$ requires less matrix inversions and less the time CPU than the others. We also notice that the number of matrix inversions $IM$, the number of right hand sides $RHS$ and the CPU time increase with the truncation order $p$.

| Solver | $\text{Alg}_1$ | $\text{Alg}_2$ | $\text{Alg}_3$ |
|--------|----------------|----------------|----------------|
| $p$    | $a_{\text{max}} < 1$ | $a_{\text{max}} < 1$ | $a_{\text{max}} < 1$ |
| 7      | $a_{\text{max}} < 1$ | 2810 36000 4252 | 2711 36000 4102 |
| 8      | 2850 40000 4900 | 2600 40000 4470 | 2480 40000 4262 |
| 9      | 630 60000 12376 | 612 60000 12023 | 520 60000 10210 |
| 20     | 320 80000 25896 | 309 80000 25000 | 280 80000 22640 |

Table 2. Comparison between three solvers $\text{Alg}_1$, $\text{Alg}_2$ and $\text{Alg}_3$: Effect of truncation order

4.2. Nonlinear dynamic of a cantilever bi-symmetrical beam with steel I section under eccentric loading. In this example, we analyze the nonlinear dynamic behavior of a cantilever bi-symmetrical beam with steel I section subjected to an eccentric trapezoidal load $F_z(t)$ applied at end $B$. The considered data are the same used in [27]; the Young’s modulus $E = 210GPa$, the shear modulus $\mu = 8.0769 \times 10^7 kPa$, the mass density $\rho = 7850 kg/m^3$, the length $L = 8m$, the flange thickness $t_f = 0.03 m$, the web thickness $t_w = 0.012 m$, the section varies according to the relationship $h(x) = (2t_f + 0.5(1.5 - x/L))$ (see figure 8a) and a constant width $b = 0.30 m$ (see figure 7a). The beam clamped-free is subjected at free end to an eccentric trapezoidal load $F_z(t)$ plotted in the figure 7b. This beam is discretized in 40 3D elements for high-order implicit solvers and in 1020 3D solid elements $C3D20R$ for the Abaqus code. The nonlinear dynamic analysis is made in the time range $[0, 0.2s]$ with a time step $\Delta t = 0.001s$. For the resolution, we take a truncation order $p = 6$ and a tolerance $\epsilon = 10^{-3}$ for high-order implicit solvers. In this example, the load $F_z(t)$ is applied at point $O$ with eccentricities $e_y = -0.03m$ and $e_z = 0.155m$ with respect to centroid of the cross section where $y_c = 0m$ and $z_c = 0m$ (see figure 8b).
The time evolution curves of components \((u(L,t), v(L,t), w(L,t), \theta_x(L,t))\) are reported in figure 9, compared to result computed by Abaqus code and those obtained by Spountzakis et al. [27]. These response curves show a very good agreement with those obtained by the Abaqus code and by Spountzakis et al..
In table 3, we study the influence of time step on the truncation order and on the residue. We remark that the optimal order increases with the time step. The solver Alg₃ is always better than the others because its optimal order is less than the those of others. In the following table 4, we study the influence of the truncation order

| Solver | Alg₁ | Alg₂ | Alg₃ |
|--------|------|------|------|
| Δt     | Optimal order | Log|Res| Optimal order | Log|Res| Optimal order | Log|Res|
| 10⁻³   | 6    | -5.23| 4    | -5.2  | 3    | -5.13 |
| 210⁻³  | 12   | -5.10| 9    | -4.80 | 7    | -4.62 |
| 310⁻³  | 14   | -4.91| 11   | -4.79 | 8    | -4.60 |
| 410⁻³  | 15   | -4.88| 12   | -4.70 | 10   | -4.55 |

Table 3. Comparison between three solvers Alg₁, Alg₂ and Alg₃:
Influence of time step on the number of matrix inversions IM, number of right hand sides RHS and the CPU time. From this table, we notice that the solver Alg₃ starts to operate for the truncation order \( p = 3 \) contrary to the solvers Alg₂ and Alg₁ begin to operate respectively for truncation orders \( p = 4 \) and \( p = 6 \). We remark also that the solver Alg₃ is always better than the others.

| Solver | Alg₁ | Alg₂ | Alg₃ |
|--------|------|------|------|
| \( p \) | \( a_{\text{max}} < 1 \) | \( a_{\text{max}} < 1 \) | \( a_{\text{max}} < 1 \) |
| 2      | \( a_{\text{max}} < 1 \) | \( a_{\text{max}} < 1 \) | \( a_{\text{max}} < 1 \) |
| 3      | \( a_{\text{max}} < 1 \) | \( a_{\text{max}} < 1 \) | \( 12 \) | \( 600 \) | \( 26 \) |
| 4      | \( a_{\text{max}} < 1 \) | \( 13 \) | \( 800 \) | \( 42 \) | \( 11 \) | \( 800 \) | \( 30 \) |
| 5      | \( a_{\text{max}} < 1 \) | \( 8 \) | \( 1000 \) | \( 46 \) | \( 6 \) | \( 1000 \) | \( 34 \) |
| 6      | \( 15 \) | \( 1200 \) | \( 102 \) | \( 7 \) | \( 1200 \) | \( 54 \) | \( 5 \) | \( 1200 \) | \( 36 \) |
| 7      | \( 4 \) | \( 1400 \) | \( 126 \) | \( 2 \) | \( 1400 \) | \( 65 \) | \( 1 \) | \( 1400 \) | \( 40 \) |

Table 4. Comparison between three solvers Alg₁, Alg₂ and Alg₃:
Effect of truncation order

5. Conclusion. In this paper, high-order implicit solvers for solving the nonlinear dynamics problem of thin-walled beams with open cross section under external dynamic loadings are investigated. These high-order implicit solver are obtained using an appropriate and suitable homotopy transformations. The formulation of this problem is based on a 3D nonlinear dynamic model taking into account large displacement and large torsion. No assumptions on the amplitude of the torsion angle are considered in the constitutive law and in the derivation of the nonlinear dynamic governing equations. These solvers are developed by associating four techniques: 1) the space and time discretization procedures, 2) a change of variable, 3) an appropriate homotopy transformation and 4) a perturbation technique followed of continuation process. The space and time discretizations are performed respectively by the finite elements and by the classical implicit Newmark scheme. A 3D beam elements having two nodes with seven degrees of freedom are adopted in mesh process. The efficiency and accuracy of the high-order implicit solvers are tested on typical examples of nonlinear dynamic of thin-walled beams with open section under external dynamic loadings. The obtained results are in good agreement with those
computed by the Abaqus code and with those available in literature. This comparison confirms the robustness, accuracy and efficiency of the high-order implicit solvers.

REFERENCES

[1] Abaqus, Version 6.11 Documentation, Dassault Systemes Simulia Corp, Providence, RI, USA, 2011.
[2] E. L. Allgower and K. Georg, Numerical Continuation Methods: An Introduction, Springer series in Computational Mathematics, 1990.
[3] R. D. Ambrosini, J. D. Riera and R. F. Danesi, Dynamic analysis of thin-walled and variable open section beams with shear flexibility, International Journal for Numerical Methods in Engineering, 38 (1995), 2867–2885.
[4] K. J. Bathe, Finite Elements Procedures, Prentice-Hall, New Jersey, 1996.
[5] J. L. Batoz and G. Dhatt, Modélisation des structures par éléments finis, Hermès, Paris, 1990.
[6] K. Behdinan, M. C. Stylianou and T. Tabarrok, Co-rotational dynamic analysis of flexible beams, Computer Methods in Applied Mechanics and Engineering, 154 (1998), 151–161.
[7] P. Betsch and P. Steinmann, Constrained dynamics of geometrically exact beams, Computational Mechanics, 31 (2003), 49–59.
[8] O. Bourihan, B. Braikat, M. Jamal, F. Mohri and N. Damil, Dynamic analysis of a thin-walled beam with open cross section subjected to dynamic loads using a high-order implicit algorithm, Engineering Structures, 120 (2016), 133–146.
[9] S. Boutmir, B. Braikat, M. Jamal, N. Damil, B. Cochelin and M. Potier-Ferry, Des solveurs implicites d’ordre supérieurs pour les problèmes de dynamique non linéaire des structures, Revue Européenne des Élémens Finis, 13 (2004), 449–460.
[10] M. A. Crisfield, Nonlinear Finite Elements Analysis of Solids and Structures, John Willey and Sons, 1991.
[11] E. Dale Martin, A technique for accelerating iterative convergence in numerical integration with application in transonic aerodynamics, Lectures notes in Physics, 47 (1976), 123–139.
[12] A. Ed-dinari, H. Mottaqui, B. Braikat, M. Jamal, F. Mohri and N. Damil, Large torsion analysis of thin-walled open sections beams by the asymptotic numerical method, Engineering Structures, 81 (2014), 240–255.
[13] Y. Guevel, G. Girault and J. M. Cadou, Numerical comparisons of high-order nonlinear solvers for the transient Navier-Stokes equations based on homotopy and perturbation techniques, Journal of Computational and Applied Mathematics, 289 (2015), 356–370.
[14] D. Haijuan, Nonlinear free vibration analysis of asymmetric thin-walled circularly curved beams with open section, Thin-Walled Structures, 46 (2008), 107–112.
[15] M. Jamal, B. Braikat, S. Boutmir, N. Damil and M. Potier-Ferry, A high order implicit algorithm for solving instationary nonlinear problems, Computational Mechanics, 28 (2002), 375–380.
[16] T. N. Le, J. M. Battini and M. Hjiaj, Efficient formulation for dynamics of corotational 2D beams, Computational Mechanics, 48 (2011), 153–161.
[17] T. N. Le, J. M. Battini and M. Hjiaj, Corotational formulation for nonlinear dynamics of beams with arbitrary thin-walled open cross-sections, Computer and Structures, 134 (2014), 112–127.
[18] S. Mesmoudi, A. Timesli, B. Braikat, H. Lahmam and Z. Harrouni, A 2D mechanical–thermal coupled model to simulate material mixing observed in friction stir welding process, Engineering with Computers, (2017), 1–11.
[19] F. Mohri, N. Damil and M. Potier Ferry, Large torsion finite element model for thin-walled beams, Computers and Structures, 86 (2008), 671–683.
[20] F. Mohri, L. Azzar and M. Potier-Ferry, Vibration analysis of buckled thin-walled beams with open sections, Journal of Sound and Vibration, 275 (2004), 434–446.
[21] F. Mohri, N. Damil and M. Potier-Ferry, Linear and nonlinear stability analyses of thin-walled beams with nonsymmetric sections, Thin-Walled Structures, 48 (2010), 299–315.
[22] F. Mohri, N. Damil and M. Potier-Ferry, Large torsion finite element model for thin-walled beams, Computers and Structures, 86 (2008), 671–683.
[23] F. Mohri, A. Ed-dinari and N. Damil, A beam finite element for nonlinear analysis of thin-walled elements, Thin Walled Structures, 46 (2008), 981–990.
[24] H. Mottaqui, B. Braikat and N. Damil, Discussion about parameterization in the asymptotic numerical method: Application to nonlinear elastic shells, *Computer Methods in Applied Mechanics and Engineering*, 199 (2010), 1701–1709.

[25] H. Mottaqui, B. Braikat and N. Damil, Local parameterization and the asymptotic numerical method, *Mathematical Modelling of Natural Phenomena*, 5 (2010), 16–22.

[26] N. Newmark, A method of computation for structural dynamics, *Journal of the Engineering Mechanics Division, Proceeding of ASCE*, (1959), 67–94.

[27] E. J. Sapountzakis and I. C. Dikaros, Nonlinear flexural-torsional dynamic analysis of beams of variable doubly symmetric cross section-application to wind turbine towers, *Nonlinear Dynamics*, 73 (2013), 199–227.

[28] A. Timesli, B. Braikat, H. Lahmam and H. Zahrouni, A new algorithm based on moving least square method to simulate material mixing in friction stir welding, *Engineering Analysis with Boundary Elements*, 50 (2015), 372–380.

[29] V. Z. Vlasov, Thin walled elastic beams, Eyrolles, French translation: Pièces longues en voiles minces, Paris, 1965.

[30] O. C. Zienkiewicz and R. Taylor, *The Finite Element Method, Solid and Fluid Mechanics and Non-linearity*, Book Company, 1987.

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