Negative heat capacity for a Klein-Gordon oscillator in non-commutative complex phase space

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Abstract

We obtain exact solutions to the two-dimensional Klein-Gordon oscillator in a non-commutative complex phase space to first order in the non-commutativity parameter. We derive the exact non-commutative energy levels and show that the energy levels split to $2m$ levels. We find that the non-commutativity plays the role of a magnetic field interacting automatically with the spin of a particle induced by the non-commutativity of complex phase space. The effect of the non-commutativity parameter on the thermal properties is discussed. It is found that the dependence of the heat capacity $C_V$ on the non-commutative parameter gives rise to a negative quantity. Phenomenologically, this effectively confirms the presence of the effects of self-gravitation induced by the non-commutativity of complex phase space.

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1 Introduction

There are many papers in the literature which are devoted to the study of various aspects of the Klein-Gordon oscillator in non-commutative (NC) space and NC phase space with the usual time coordinate \([1 - 3]\). However the extension of this study to the case of a two-dimensional (2D) NC complex space is limited \([4, 5]\). This topic is still very interesting since its phenomenological implications are important.

This paper is organized as follows. In Section 2, we discuss the Klein-Gordon oscillator in NC complex space. In Section 3, we study the Klein-Gordon oscillator in NC complex phase space. Then, the thermodynamic properties are studied in section 4. Finally, section 5 is devoted to a discussion.

2 2D Klein-Gordon oscillator in NC complex space

In a complex space, the NC complex coordinate operators \((\hat{z}, \hat{\bar{z}})\) and momentum operators \((\hat{p}_z, \hat{\bar{p}}_z)\) in 2D space are defined by \([4]\):

\[
\hat{z} = \hat{x} + i\hat{y} = z + i\theta p_z, \quad \hat{\bar{z}} = \hat{x} - i\hat{y} = \bar{z} - i\theta p_z, \quad \hat{p}_z = p_z, \quad \hat{\bar{p}}_z = p_{\bar{z}}. \quad (1)
\]

These operators satisfy the following commutation relations:

\[
[\hat{z}, \hat{\bar{z}}] = 2\theta, \quad [\hat{z}, \hat{p}_z] = [\hat{\bar{z}}, \hat{\bar{p}}_z] = 0, \quad [\hat{\bar{z}}, \hat{p}_z] = 2\hbar. \quad (3)
\]

Now, following ref. \([4]\), we review the Klein-Gordon oscillator in NC complex space. The Klein-Gordon oscillator in 2D complex space is defined by the following equation:

\[
(2p_{\bar{z}} + im\omega \hat{z}) (2p_z - im\omega \hat{\bar{z}}) \psi = (E^2 - m^2) \psi, \quad (4)
\]

which can be rewritten in commutative space as:

\[
(p_x^2 + p_y^2 + m^2 \omega^2 (x^2 + y^2) + 2m\omega L_z) \psi = (E^2 - m^2 + 2m\omega) \psi, \quad (5)
\]

with energy eigenvalues:

\[
E^2 = 2m\omega (n_x + n_y + m_\ell) + m^2. \quad (6)
\]

In the NC complex space the Klein-Gordon oscillator is described by the following equation:

\[
\left( \begin{array}{cc}
2p_z + im\omega \hat{z} & 0 \\
0 & 2p_{\bar{z}} - im\omega \hat{\bar{z}}
\end{array} \right) \psi = \left( \begin{array}{cc}
0 & (2p_z - im\omega \hat{z}) \\
(2p_{\bar{z}} + im\omega \hat{\bar{z}}) & 0
\end{array} \right) \psi = (E^2 - m^2) \psi. \quad (7)
\]
To solve this equation we use the NC complex coordinates:

$$\hat{z} = z + i\theta p_z,$$

$$\hat{\bar{z}} = \bar{z} - i\theta p_z,$$

$$\hat{p}_z = p_z,$$  \hspace{1cm} \hat{\bar{p}}_z = p_{\bar{z}}. \quad (10)$$

Inserting eqs. (8)-(10) into eq. (7), we have:

$$\left[ \left(1 + \frac{m\omega\theta}{2} \right)^2 (p_x^2 + p_y^2) + m^2\omega^2 (x^2 + y^2) - 2m\omega L_z - m^2\omega^2 \theta (L_z \pm 1) \right] \psi$$

$$= (E^2 - m^2 + 2m\omega) \psi. \quad (11)$$

The energy eigenvalues are given by:

$$E^2 = 2m\omega_{\varphi} (n_x + n_y + 1) + 2m\omega_{\theta} (m_\ell \pm 1) + m^2, \quad (12)$$

with $\omega_{\varphi} = \omega(1 - m\omega\theta/2)$. Such effects are similar to the normal Zeeman splitting of a particle with spin $1/2$ and thus the degeneracy of energy levels is completely removed. The oscillator is positioned in the four equivalent points $(z \uparrow, \bar{z} \uparrow, z \downarrow, \bar{z} \downarrow) \Leftrightarrow (z, \bar{z}, -z, -\bar{z})$. Therefore the eigenfunction $\psi(z, \bar{z})$ takes values in $C^4$, spin up, spin down, particle, antiparticle. This oscillator is described by two double-component spinors $[4, 6]$:

$$\psi_{n0} \begin{pmatrix} \psi_+^{n0} \\ \psi_-^{n0} \end{pmatrix}, \quad \text{and} \quad \psi_{0n} \begin{pmatrix} \psi_+^{0n} \\ \psi_-^{0n} \end{pmatrix},$$

where the sign $(\pm)$ signifies spin up or down, and the wave functions $\psi_{n0}$ and $\psi_{0n}$ have the following form:

$$\psi_{n0} (z, \bar{z}) = \sqrt{(m\omega)^{n+1} \pi n!} z^n \exp \left(-\frac{m\omega}{2} z \bar{z} \right), \quad (13a)$$

$$\psi_{0n} (z, \bar{z}) = \sqrt{(m\omega)^{n+1} \pi n!} \bar{z}^n \exp \left(-\frac{m\omega}{2} z \bar{z} \right). \quad (13b)$$

### 3 2D Klein-Gordon oscillator in NC complex phase space

In a NC complex phase space we replace the coordinate and momentum operators (relations (8-10)) by:

$$\hat{z} = z + i\theta p_z,$$

$$\hat{\bar{z}} = \bar{z} - i\theta p_z,$$

$$\hat{p}_z = p_z + i\bar{\theta} \bar{z},$$

$$\hat{\bar{p}}_z = p_{\bar{z}} - i\bar{\theta} z. \quad (17)$$
Inserting eqs. (14)-(17) into eq. (7), we have two equations:
\[
\left(1 - \frac{m\omega\theta}{2}\right)^2 \left(p_x^2 + p_y^2\right) + m^2\omega^2 \left(1 + \frac{\bar{\theta}}{2m\omega}\right)^2 (x^2 + y^2) + 2m\omega L_z - m^2\omega^3 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2}\right) (L_z + 1) \psi = (E^2 - m^2 + 2m\omega) \psi,
\]
and
\[
\left(1 - \frac{m\omega\theta}{2}\right)^2 \left(p_x^2 + p_y^2\right) + m^2\omega^2 \left(1 + \frac{\bar{\theta}}{2m\omega}\right)^2 (x^2 + y^2) + 2m\omega L_z - m^2\omega^3 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2}\right) (L_z - 1) \psi = (E^2 - m^2 + 2m\omega) \psi.
\]
The equations (18) and (19) are similar to the equation of motion for a fermion of spin 1/2 in a constant magnetic field. Under these conditions the equations (18) and (19) take the following form:
\[
\left(1 + \frac{m\omega\theta}{2}\right)^2 \left(p_x^2 + p_y^2\right) + m^2\omega^2 \left(1 + \frac{\bar{\theta}}{2m\omega}\right)^2 (x^2 + y^2) + 2m\omega L_z - m^2\omega^3 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2}\right) (L_z - 2s_z) \psi = (E^2 - m^2 + 2m\omega) \psi,
\]
with \(s_z = \pm 1/2\). The energy eigenvalues are given by:
\[
E^2 = 2m\Omega_\theta (n_x + n_y + 1) - 2m\omega (m_\ell + 1) - m^2\omega^2 \left(\theta + \frac{\bar{\theta}}{m^2\omega^2}\right) (m_\ell \pm 1) + m^2,
\]
where
\[
\Omega_\theta = \omega \left(1 + \frac{m\omega\theta}{2}\right) \left(1 + \frac{\bar{\theta}}{2m\omega}\right).
\]
We have thus shown that the non-commutativity effects are manifested in energy levels and thus the degeneracy of the levels is completely removed, so that they are split into \((2m_\ell)\) levels, similarly to the effects of a magnetic field interacting automatically with the spin of a particle.

4 Thermodynamic properties of the 2D Klein-Gordon oscillator in NC complex phase space

The thermodynamic functions associated with the NC complex oscillator are also of interest. The eigenvalues of the 2D Klein-Gordon oscillator in NC complex phase space are [7]:
\[
E^\pm = \pm m\sqrt{\lambda_{\theta\bar{\theta}} + \gamma_{\theta\bar{\theta}} n}, \quad n = 0, 1, 2, \ldots,
\]

3
where
\[
\lambda_{\theta\bar{\theta}} = 1 + \gamma_{\theta\bar{\theta}} - \frac{2\omega}{m}(\ell + 1) - \omega^2 \left( \theta + \frac{\bar{\theta}}{m^2\omega^2} \right)(\ell \pm 1), \quad \ell = 0, 1, \ldots, \tag{24}
\]
and
\[
\gamma_{\theta\bar{\theta}} = 2 \frac{\Omega_{\theta}}{m}. \tag{25}
\]

We concentrate, firstly, on the calculation of the partition function \( Z(\beta, \theta, \bar{\theta}) \), defined as:
\[
Z(\beta, \theta, \bar{\theta}) = \sum_{n,s} \exp \left[ -\beta (E_{n,s} - E_{0,s}) \right], \tag{26}
\]
where \( \beta = 1/k_B T \) is the Boltzmann factor, and \( E_0 \) is the background energy. Therefore we have the single-oscillator partition function with \( \ell = 0 \), from eq. (26):
\[
Z(\beta, \theta, \bar{\theta}) = \sum_{n=0}^{\infty} \exp \left[ -\beta m \left( \sqrt{1 + \gamma_{\theta\bar{\theta}} n} - 1 \right) \right] + \sum_{n=0}^{\infty} \exp \left[ -\beta m \left( \sqrt{\lambda_{\theta\bar{\theta}} + \gamma_{\theta\bar{\theta}} n} - \sqrt{\lambda_{\theta\bar{\theta}}} \right) \right], \tag{27}
\]
where
\[
\lambda_{\theta\bar{\theta}} = 1 + 2\omega^2 \left( \theta + \frac{\bar{\theta}}{m^2\omega^2} \right). \tag{28}
\]

On the other hand, the Euler-Maclaurin formula [8] is:
\[
\sum_{x=0}^{\infty} f(x) = \frac{1}{2} f(0) + \int_{0}^{\infty} f(x) dx - \sum_{p=1}^{\infty} \frac{1}{(2p)!} B_{2p} f^{(2p-1)}(0), \tag{29}
\]
where
\[
B_{2n} = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{p=1}^{\infty} p^{-2n}, \tag{30}
\]
are the Bernoulli numbers. Using the Euler-Maclaurin formula (eq. (29)), and after a simple calculation, the partition function in eq. (27) can be written as:
\[
Z(\beta, \theta, \bar{\theta}) = 1 + \frac{2}{\gamma_{\theta\bar{\theta}} \beta^2} \left[ (1 + \beta) + \left( 1 + \beta \sqrt{\lambda_{\theta\bar{\theta}}} \right) \right] + \frac{B_2}{2} \left( e^{\beta f_1^{(1)}} + e^{\beta \sqrt{\lambda_{\theta\bar{\theta}}} f_2^{(1)}} \right) + \frac{B_4}{24} \left( e^{\beta f_1^{(3)}} + e^{\beta \sqrt{\lambda_{\theta\bar{\theta}}} f_2^{(3)}} \right) + \ldots, \tag{31}
\]
where
\[
f_1^{(1)} = -\frac{\gamma_{\theta\bar{\theta}} \beta m}{2} e^{-\beta}, \tag{32a}
\]
\[
f_2^{(1)} = -\frac{\gamma_{\theta\bar{\theta}} \beta m}{2 \sqrt{\lambda_{\theta\bar{\theta}}}} e^{-\beta \sqrt{\lambda_{\theta\bar{\theta}}}}, \tag{32b}
\]
and

\begin{align}
    f_1^{(3)} &= \left[ \frac{-3\beta m (\gamma_{\theta\theta})^3}{8} - \frac{3\beta^2 m^2 (\gamma_{\theta\theta})^3}{8} - \frac{3\beta^3 m^3 (\gamma_{\theta\theta})^3}{8} \right] e^{-\beta}, \\
    f_2^{(1)} &= \left[ \frac{-3\beta m (\gamma_{\theta\theta})^3}{8 (\lambda_{\theta\theta})^{5/2}} - \frac{3\beta^2 m^2 (\gamma_{\theta\theta})^3}{8 (\lambda_{\theta\theta})^2} - \frac{3\beta^3 m^3 (\gamma_{\theta\theta})^3}{8 (\lambda_{\theta\theta})^{3/2}} \right] e^{-\beta \sqrt{\lambda_{\theta\theta}}},
\end{align}

with \( B_2 = 1/6 \) and \( B_4 = -1/30 \). By replacing Eqs. (32)–(33) into Eq. (31), we obtain the partition function in NC complex phase space as:

\begin{align}
    Z(\beta, \theta, \tilde{\theta}) &= 1 + \frac{2}{\gamma_{\theta\theta}} \left( 1 + \sqrt{\lambda_{\theta\theta}} \right) \beta^{-1} + \frac{4}{\gamma_{\theta\theta}} \beta^{-2} + \\
    &\quad + \frac{\gamma_{\theta\theta}}{24} \left[ 1 + \frac{1}{\sqrt{\lambda_{\theta\theta}}} - \frac{(\gamma_{\theta\theta})^2}{80} \left( 1 + \frac{1}{(\lambda_{\theta\theta})^{5/2}} \right) \right] \beta^+ + \\
    &\quad - \frac{(\gamma_{\theta\theta})^3}{1920} \left( 1 + \frac{1}{(\lambda_{\theta\theta})^{3/2}} \right) \beta^2 - \frac{(\gamma_{\theta\theta})^3}{1920} \left( 1 + \frac{1}{(\lambda_{\theta\theta})^{3/2}} \right) \beta^3. \tag{34}
\end{align}

Hence there is a characteristic temperature which divides the temperature range into two regions: \( \beta \gg \beta_0 = 1/mc^2 \) for very low temperatures, and \( \beta \ll \beta_0 \) for very high temperatures. In this context, we derive the thermodynamic properties of our system, such as the total energy, entropy, free energy and specific heat, which are given by:

\begin{align}
    \langle E(\beta, \theta, \tilde{\theta}) \rangle &= -\frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \tilde{\theta}), \quad C_V = -k_B \beta^2 \frac{\partial \langle E(\beta, \theta, \tilde{\theta}) \rangle}{\partial \beta}, \tag{35a} \\
    F &= -\frac{1}{\beta} \ln Z(\beta, \theta, \tilde{\theta}), \quad S = -\frac{1}{T} F - k_B \frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \tilde{\theta}). \tag{35b}
\end{align}

### 4.1 Results and discussions

For very low temperatures the partition function in eq. (34) can be written as:

\begin{align}
    Z(\beta, \theta, \tilde{\theta}) &\simeq \frac{\gamma_{\theta\theta}}{24} \left[ 1 + \frac{1}{\sqrt{\lambda_{\theta\theta}}} - \frac{(\gamma_{\theta\theta})^2}{80} \left( 1 + \frac{1}{(\lambda_{\theta\theta})^{5/2}} \right) \right] \beta^+ + \\
    &\quad - \frac{(\gamma_{\theta\theta})^3}{1920} \left( 1 + \frac{1}{(\lambda_{\theta\theta})^{3/2}} \right) \beta^2 - \frac{(\gamma_{\theta\theta})^3}{1920} \left( 1 + \frac{1}{(\lambda_{\theta\theta})^{3/2}} \right) \beta^3. \tag{36}
\end{align}

The mean energy \( \langle E(\beta, \theta, \tilde{\theta}) \rangle \) of the systems is

\begin{align}
    \langle E(\beta, \theta, \tilde{\theta}) \rangle &= -\frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \tilde{\theta}) \sim 0, \tag{37}
\end{align}

and the specific heat \( C_V \) of the systems is:

\begin{align}
    C_V &= -k_B \beta^2 \frac{\partial \langle E(\beta, \theta, \tilde{\theta}) \rangle}{\partial \beta} \sim -3k_B, \tag{38}
\end{align}

5
which is clearly a negative quantity. In other words this means that the Klein-
Gordon oscillator in NC complex phase space is similar to a self-gravitating
system that is discussed in the framework of a non-extensive kinetic theory (see
[9] and references therein).

The free energy $F$ is given by:

$$F = -\frac{1}{\beta} \ln Z(\beta, \theta, \bar{\theta}) \sim 0,$$

and the entropy of the systems is:

$$S = -\frac{1}{T} F - k_B \frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \bar{\theta}) \sim 0.$$

For very high temperatures the partition function in eq. (34) can be written
as:

$$Z(\beta, \theta, \bar{\theta}) \simeq 1 + \frac{2}{\gamma \theta} \left(1 + \sqrt{\lambda \theta}\right) \beta^{-1} + \frac{4}{\gamma a \bar{\theta}} \beta^{-2},$$

and the mean energy $\langle E(\beta, \theta, \bar{\theta}) \rangle$, and the specific heat $C_V$ of the systems are
given by:

$$\langle E(\beta, \theta, \bar{\theta}) \rangle = -\frac{\partial}{\partial \beta} \ln Z(\beta, \theta, \bar{\theta}) \sim 2 \beta^{-1},$$

$$C_V = \frac{\partial \langle E(\beta, \theta, \bar{\theta}) \rangle}{\partial T} \sim 2k_B.$$

Note that these results are similar to those of the Dirac oscillator under a mag-
netic field in a NC space [5].

We shown in figures [1, 2 and 3] comparisons of the partition function $Z$ as a
function of $\beta$, the thermodynamic function $E/mc^2$ of the Klein Gordon Bosons
as a function of $\tau$ and the heat capacity $C_V/k_B$ of Klein-Gordon Bosons as a
function of $\tau$, for different values of $\theta$ and $\bar{\theta}$.

![Figure 1: Comparison of the partition function $Z$ as a function of $\beta$ for different values of $\theta$ and $\bar{\theta}$.](image)
5 Conclusions

In this paper we started from a Klein-Gordon oscillator in a NC complex phase space. Using the Moyal product method, we derived the deformed Klein-Gordon oscillator and showed that it similar to the Klein-Gordon equation for a particle with spin 1/2 in a uniform magnetic field in NC phase space [2]. We solved this equation exactly and found that the NC energy levels split into $2m_\ell$ levels. Thus the system without spin in a NC complex coordinate space has an added advantage that the spin effect is automatically manifested. The statistical quantities of the 2D Klein-Gordon oscillator in a NC complex phase space were investigated and the effect of the NC parameter on thermal properties was discussed. It was found that the dependence of the specific heat $C_V$ on the NC parameter gives rise to a negative quantity. Phenomenologically, this effectively confirms the presence the effects of self-gravitation at this level.
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