Spectrum of a composition operator with automorphic symbol
Robert F. Allen, Thong M. Le and Matthew A. Pons
Spectrum of a composition operator with automorphic symbol

Robert F. Allen, Thong M. Le and Matthew A. Pons
(Communicated by Stephan Garcia)

We give a complete characterization of the spectrum of composition operators, induced by an automorphism of the open unit disk, acting on a family of Banach spaces of analytic functions that includes the Bloch space and BMOA. We show that for parabolic and hyperbolic automorphisms the spectrum is the unit circle. For the case of elliptic automorphisms, the spectrum is either the unit circle or a finite cyclic subgroup of the unit circle.

1. Introduction

For an analytic self-map $\varphi$ of the open unit disk $\mathbb{D}$ and a Banach space $X$ of functions analytic on $\mathbb{D}$, we define the composition operator with symbol $\varphi$, denoted $C_{\varphi}$, by the rule $C_{\varphi}f = f \circ \varphi$ for all $f \in X$. The study of composition operators began formally with Nordgren’s paper [1968], where he explored properties of composition operators acting on the Hardy Hilbert space $H^2$. Since then the study has proved to be an active area of research, most likely due to the fact that the study of such operators lies at the intersection of complex function theory and operator theory.

The spectrum of $C_{\varphi}$ has been studied on many classical spaces of analytic functions, such as the Hardy spaces, Bergman spaces, weighted Hardy and Bergman spaces, Besov spaces, and the Dirichlet space. The interested reader is directed to [Cowen and MacCluer 1995] for general references.

The motivation for this paper was to determine the spectrum of a composition operator, induced by a disk automorphism, acting on the Bloch space. The Bloch space is the largest space of analytic functions on $\mathbb{D}$ that is Möbius invariant. This is one reason the Bloch space is a welcoming environment to study composition operators. The techniques developed apply to a larger class of spaces that includes the Bloch space.

The purpose of this paper is to determine the spectrum of $C_{\varphi}$ acting on a family of Banach spaces, where $\varphi$ is a disk automorphism. The spectrum will depend on

MSC2010: primary 47A10, 47B33; secondary 30H05.
Keywords: composition operator, spectrum, automorphism.
the fixed point classification of the automorphisms of \( \mathbb{D} \). This is a standard approach to the study of composition operators induced by automorphisms. We show the spectrum of \( C_\varphi \), acting on a particular family of Banach spaces, induced by a disk automorphism, must be a subset of the unit circle \( \partial \mathbb{D} \), and in some instances is the entire unit circle. Finally, we compare these results to particular examples of classical spaces.

2. Preliminaries

2A. Automorphisms. The automorphisms of the open unit disk \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \) are precisely the analytic bijections on \( \mathbb{D} \) which have the form

\[
\varphi(z) = \lambda \frac{a - z}{1 - \overline{a}z},
\]

where \( \lambda \) is a unimodular constant and \( a \) is a point in \( \mathbb{D} \). These automorphisms form a group under composition denoted by \( \text{Aut}(\mathbb{D}) \). Every element of \( \text{Aut}(\mathbb{D}) \) has two fixed points (counting multiplicity), and thus can be classified by the location of the fixed points:

- elliptic: one fixed point in \( \mathbb{D} \) and one in the complement of \( \overline{\mathbb{D}} \);
- parabolic: one fixed point on the unit circle \( \partial \mathbb{D} \) (of multiplicity 2);
- hyperbolic: two distinct fixed points on \( \partial \mathbb{D} \).

Two disk automorphisms \( \varphi \) and \( \psi \) are conformally equivalent if there exists a disk automorphism \( \tau \) for which \( \psi = \tau \circ \varphi \circ \tau^{-1} \). Many properties of automorphisms are preserved under conformal equivalence. The main advantage of conformal equivalence is in the placement of the fixed points. Every elliptic disk automorphism is conformally equivalent to one whose fixed point in \( \mathbb{D} \) is the origin.

Lemma 2.1. Let \( \varphi \) be an elliptic disk automorphism with fixed point \( a \) in \( \mathbb{D} \). Then \( \varphi \) is conformally equivalent to \( \psi(z) = \lambda z \) where \( \lambda = \varphi'(a) \).

Proof. Let \( \tau_a \) be the involution automorphism which interchanges 0 and \( a \), that is

\[
\tau_a(z) = \frac{a - z}{1 - \overline{a}z}.
\]

Define \( \psi = \tau_a \circ \varphi \circ \tau_a^{-1} \) on \( \mathbb{D} \). Since \( a \) is a fixed point of \( \varphi \), \( \psi \) fixes the origin, and is a rotation. So there is a unimodular constant \( \lambda \) such that \( \psi(z) = \lambda z \). To complete the proof, we will show \( \lambda = \varphi'(a) \). Observe \( \psi'(z) = \lambda \) for all \( z \in \mathbb{D} \). In particular

\[
\lambda = \varphi'(0) = \tau_a'(0) \varphi'(\tau_a(0)) \varphi'(\tau_a(0)) \tau_a'(0) = \varphi'(a) \tau_a'(a) \tau_a'(0) = \varphi'(a).
\]

Thus \( \varphi \) is conformally equivalent to the rotation \( \psi(z) = \varphi'(a)z \). \( \square \)
Every parabolic disk automorphism is conformally equivalent to one whose fixed point (of multiplicity 2) is 1. The following lemma is found as Exercise 2.3.5c of [Cowen and MacCluer 1995], and a complete proof can be found in [Pons 2007].

**Lemma 2.2** [Pons 2007, Lemma 4.1.2]. Let \( \varphi \) be a parabolic disk automorphism. Then \( \varphi \) is conformally equivalent to either

\[
\psi_1(z) = \frac{(1 + i)z - 1}{z + i - 1} \quad \text{or} \quad \psi_2(z) = \frac{(1 - i)z - 1}{z - i - 1}.
\]

Every hyperbolic disk automorphism is conformally equivalent to one whose fixed points in \( \partial \mathbb{D} \) are \( \pm 1 \).

**Lemma 2.3** [Nordgren 1968, Theorem 6]. Let \( \varphi \) be a hyperbolic disk automorphism. Then, for some \( r \in (0, 1) \), \( \varphi \) is conformally equivalent to

\[
\psi(z) = \frac{z + r}{1 + rz}.
\]

### 2B. The space of bounded analytic functions.

The set of analytic functions on \( \mathbb{D} \) is denoted by \( H(\mathbb{D}) \). The space of bounded analytic functions on \( \mathbb{D} \), denoted \( H^\infty = H^\infty(\mathbb{D}) \), is a Banach space under the norm

\[
\| f \|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.
\]

The bounded analytic functions on \( \mathbb{D} \) is a rich space containing many interesting types of functions, such as polynomials and Blaschke products. In addition, the disk algebra \( \mathcal{A}(\mathbb{D}) \), the set of analytic functions on \( \mathbb{D} \) continuous to \( \partial \mathbb{D} \), is a closed subspace of \( H^\infty \).

The following two families of functions will be used in the next section. To prove these functions are in \( H^\infty \), we take a geometric approach using conformal mappings of the plane. To this effect let \( \mathbb{H}_L \) and \( \mathbb{H}_R \) denote the open left and right half planes respectively, i.e., \( \mathbb{H}_L = \{ \text{Re } z < 0 \} \) and \( \mathbb{H}_R = \{ \text{Re } z > 0 \} \).

**Lemma 2.4.** For \( s \geq 0 \), the function

\[
f_s(z) = \exp \frac{s(z + 1)}{z - 1}
\]

is in \( H^\infty \).

**Proof.** If \( s = 0 \), then \( f_s \) is identically 1. So, \( f_s(z) \) is in \( H^\infty \). Now suppose \( s > 0 \). The function \( f_s \) is comprised of the functions

1. \( z \mapsto (z + 1)/(z - 1) \) (mapping \( \mathbb{D} \) onto \( \mathbb{H}_L \)),
2. \( z \mapsto sz \) (mapping \( \mathbb{H}_L \) onto \( \mathbb{H}_L \)),
3. \( z \mapsto e^z \) (mapping \( \mathbb{H}_L \) onto \( \mathbb{D} \setminus \{0\} \)).
So $f_s$ maps $\mathbb{D}$ into $\mathbb{D}$, as in Figure 1, and thus $f_s(z)$ is an element of $H^\infty$. □

**Lemma 2.5.** For real value $t$, the function

$$f_t(z) = \left( \frac{1+z}{1-z} \right)^{it}$$

is in $H^\infty$.

**Proof.** For $t = 0$, $f_t$ is identically 1, and thus is in $H^\infty$. Now suppose $t > 0$. We will rewrite the function $f_t$ as

$$f_t(z) = \exp \left( it \log \frac{1+z}{1-z} \right),$$

where $\log$ is the principle branch of the logarithm. Then $f_t$ is comprised of the functions

1. $z \mapsto (1+z)/(1-z)$ (mapping $\mathbb{D}$ onto $\mathbb{H}_r$),
2. $z \mapsto \log z$ (mapping $\mathbb{H}_r$ onto the horizontal strip $S_h = \{0 < \text{Im} z < 2\pi\}$),
3. $z \mapsto iz$ (mapping $S_h$ onto the vertical strip $S_v = \{-2\pi < \text{Re} z < 0\}$),
4. $z \mapsto e^z$ (mapping $S_v$ into $A(e^{-2\pi}, 1) = \{e^{-2\pi} < |z| < 1\}$).

So $f_t$ maps $\mathbb{D}$ into $A(e^{-2\pi}, 1) \subseteq \mathbb{D}$, as depicted in Figure 2. In the case of $t < 0$, the vertical strip $S_v$ becomes $\{0 < \text{Re} z < 2\pi\}$. The map $z \mapsto e^z$ takes $S_v$ into $A(1, e^{2\pi}) \subseteq e^{2\pi} \mathbb{D}$, as depicted in Figure 3. In either case, $f_t(z)$ is an element of $H^\infty$ since $\|f_t\|_\infty < e^{2\pi}$ for all $t \in \mathbb{R}$. □
These functions above, together with the monomials, play such a pivotal role in Section 3 that we denote the union of these functions by $\mathcal{F}$, i.e.,

$$\mathcal{F} = \{f_s : s \geq 0\} \cup \{f_t : t \in \mathbb{R}\} \cup \{z^k : k \in \mathbb{N}\}.$$  

**2C. Spectrum of $C_\varphi$.** In this section we collect useful results regarding the spectrum of operators on Banach spaces. For a bounded linear operator $T$ on a Banach space $X$, the spectrum of $T$ is given by

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}$$

where $I$ denotes the identity operator on $X$. The spectrum is a nonempty, closed subset of $\mathbb{C}$. The spectral radius of $T$ is given by

$$\rho(T) = \sup \{ |\lambda| : \lambda \in \sigma(T) \}.$$  

Due to the fact that the spectrum is closed, we have the spectrum of $T$ is contained in the closed disk centered at the origin of radius $\rho(T)$.

Determining the spectrum of a particular composition operator can be difficult depending on the symbol of the operator and the space on which it is acting. However, the difficulties can be avoided if the operator is similar to a “simpler” operator. Linear operators $S$ and $T$ (not necessarily bounded) on a Banach space $X$ are similar if there exists a bounded linear operator $U$ on $X$, having bounded inverse, such that $T = USU^{-1}$. If $S$ and $T$ are both bounded operators, then similarity preserves the spectrum.

**Theorem 2.6.** Let $S$ and $T$ be bounded operators on a Banach space $X$. If $S$ and $T$ are similar, then $\sigma(S) = \sigma(T)$.

**Proof.** Suppose $S$ and $T$ are similar operators on $X$. By definition, there exists an invertible, bounded operator $U$ such that $T = USU^{-1}$. Let $\lambda \in \mathbb{C}$ and observe that

$$T - \lambda I = USU^{-1} - \lambda I$$

$$= USU^{-1} - \lambda UU^{-1}$$

$$= USU^{-1} - U(\lambda I)U^{-1}$$

$$= U(S - \lambda I)U^{-1}.$$  


Thus, we have that $S - \lambda I$ is not invertible if and only if $T - \lambda I$ is not invertible. Therefore $\sigma(S) = \sigma(T)$. \hfill \square

3. Main results

In this section, we determine the spectrum of $C_\varphi$ for $\varphi$ a disk automorphism acting on a particular family of Banach spaces of analytic functions. The spaces we consider will be denoted by $\mathcal{X}$ and have the following properties:

(i) $\mathcal{X}$ contains $\mathcal{F}$,

(ii) for all $\varphi \in \text{Aut}(\mathbb{D})$, $C_\varphi$ is bounded on $\mathcal{X}$ and $\rho(C_\varphi) = 1$.

The set of automorphisms of $\mathbb{D}$, as seen previously, is a very nice subset of the analytic self-maps of $\mathbb{D}$. By property (ii), every composition operator induced by a disk automorphism is bounded on $\mathcal{X}$. In fact, every such composition operator is invertible. This result, that we prove below, can be viewed as a consequence of Theorem 1.6 of [Cowen and MacCluer 1995].

Proposition 3.1. Let $\varphi$ be a disk automorphism and $C_\varphi$ the induced composition operator on $\mathcal{X}$. Then $C_\varphi$ is invertible with inverse $C_\varphi^{-1} = C_{\varphi^{-1}}$.

Proof. Since $\varphi \in \text{Aut}(\mathbb{D})$, $\varphi$ is invertible, and $\varphi^{-1}$ is an automorphism. The composition operator $C_{\varphi^{-1}}$ is bounded by property (ii) and

\[
C_\varphi(C_{\varphi^{-1}}(f)) = C_\varphi(f \circ \varphi^{-1}) = f \circ \varphi^{-1} \circ \varphi = f,
\]

\[
C_{\varphi^{-1}}(C_\varphi(f)) = C_{\varphi^{-1}}(f \circ \varphi) = f \circ \varphi \circ \varphi^{-1} = f.
\]

Therefore, $C_\varphi$ is invertible with $C_\varphi^{-1} = C_{\varphi^{-1}}$. \hfill \square

Since the spectral radius of $C_\varphi$ on $\mathcal{X}$ is 1 for $\varphi \in \text{Aut}(\mathbb{D})$, we see that the search for the spectrum can be restricted to subsets of $\overline{\mathbb{D}}$. However, our search can be refined further to subsets of the unit circle.

Theorem 3.2. Let $\varphi$ be a disk automorphism and $C_\varphi$ the induced composition operator on $\mathcal{X}$. Then $\sigma(C_\varphi) \subseteq \partial \mathbb{D}$.

Proof. By property (ii) of $\mathcal{X}$, we have $\rho(C_\varphi) = 1$. So, $\sigma(C_\varphi) \subseteq \overline{\mathbb{D}}$. Since, by Proposition 3.1, $C_\varphi$ is invertible with the inverse $C_\varphi^{-1} = C_{\varphi^{-1}}$, then $0 \notin \sigma(C_\varphi)$. So, the function $f(z) = z^{-1}$ is analytic in some neighborhood of $\sigma(C_\varphi)$. By the Spectral Mapping Theorem (see Theorem 5.14 of [MacCluer 2009]), we have $\sigma(f \circ C_\varphi) = f(\sigma(C_\varphi))$, and so,

$$\sigma(C_{\varphi^{-1}}) = \sigma(C_\varphi^{-1}) = \sigma(C_\varphi)^{-1} = \{\lambda^{-1} : \lambda \in \sigma(C_\varphi)\}.$$ 

Since $\varphi^{-1} \in \text{Aut}(\mathbb{D})$, $\sigma(C_{\varphi^{-1}}) \subseteq \overline{\mathbb{D}}$. Thus for $\lambda \in \sigma(C_\varphi)$, both $\lambda$ and $\lambda^{-1}$ are in $\overline{\mathbb{D}}$. This implies $\lambda \in \partial \mathbb{D}$. So $\sigma(C_\varphi) \subseteq \partial \mathbb{D}$, as desired. \hfill \square
Since the disk automorphisms are classified into three categories, according to fixed points, we will treat each type of automorphism separately. However, the strategy to determine $\sigma(C_\psi)$ is the same. For a disk automorphism $\varphi$, we have shown $\varphi$ to be conformally equivalent to a particularly “nice” disk automorphism: in the elliptic case a disk automorphism that fixes 0, in the parabolic case a disk automorphism that fixes 1, and in the hyperbolic case a disk automorphism that fixes $\pm 1$. In the next result, we show that conformally equivalent automorphisms induce similar composition operators on $\mathcal{X}$. This result is not unique to the space $\mathcal{X}$, but is true for any space for which automorphisms induce bounded composition operators (see p. 250 of [Cowen and MacCluer 1995]).

**Proposition 3.3.** Let $\varphi$ and $\psi$ be conformally equivalent disk automorphisms. Then the induced composition operators $C_\varphi$ and $C_\psi$ on $\mathcal{X}$ are similar.

**Proof.** Suppose $\varphi$ and $\psi$ are conformally equivalent disk automorphisms. Then there exists a disk automorphism $\tau$ such that $\psi = \tau \circ \varphi \circ \tau^{-1}$. For $f \in \mathcal{X}$, observe

$$C_\psi f = f \circ (\tau \circ \varphi \circ \tau^{-1}) = ((f \circ \tau) \circ \varphi) \circ \tau^{-1} = (C_{\tau^{-1}} C_\varphi C_{\tau}) f.$$

Since $C_{\tau^{-1}}$ is bounded and invertible on $\mathcal{X}$ with $C_{\tau^{-1}}^{-1} = C_\tau$, then $C_\psi = C_{\tau^{-1}} C_\varphi C_{\tau}^{-1}$. Therefore $C_\varphi$ and $C_\psi$ are similar. \(\square\)

With Proposition 3.3 and Lemmas 2.1, 2.2, and 2.3, it suffices to determine the spectrum of composition operators induced by these “nice” disk automorphisms, since similarity of bounded operators preserves the spectrum.

**Theorem 3.4.** Let $\varphi$ be an elliptic disk automorphism with fixed point $a$ in $\mathbb{D}$. Then the spectrum of $C_\varphi$ acting on $\mathcal{X}$ is the closure of the positive powers of $\varphi'(a)$. Moreover, this closure is a finite subgroup of the unit circle if $\varphi'(a)^n = 1$ for some natural number $n$, and is the unit circle otherwise.

**Proof.** By Lemma 2.1, $\varphi$ is conformally equivalent to $\psi(z) = \lambda z$ where $\lambda = \varphi'(a)$. By Proposition 3.3, it suffices to show that $\sigma(C_\psi)$ is the closure of the positive powers of $\lambda$. Let $G = \langle \lambda \rangle = \{\lambda^k : k \in \mathbb{N}\}$, which is a subset of $\partial \mathbb{D}$ since $|\lambda| = 1$. For each $k \in \mathbb{N}$, the function $f_k(z) = z^k$ is in $\mathcal{X}$ by property (i), and we have $(C_\psi f_k)(z) = \lambda^k f_k(z)$. Thus $\lambda^k$ is an eigenvalue of $C_\psi$ corresponding to the eigenfunction $f_k$. So $G \subseteq \sigma(C_\psi)$, and since the spectrum is closed, we have $\overline{G} \subseteq \sigma(C_\psi) = \sigma(C_\varphi)$. If the order of $\lambda$ is infinite, then $G$ is dense in $\partial \mathbb{D}$, and so $\overline{G} = \partial \mathbb{D}$.

Now suppose $\lambda$ has order $m < \infty$. Then $G = \{\lambda^k : k = 1, \ldots, m\}$. So, $\overline{G} = G$. We now wish to show $\sigma(C_\psi) \subseteq \overline{G}$. Since $\sigma(C_\psi) \subseteq \partial \mathbb{D}$ by Theorem 3.2 it suffices to show that if $\mu \in \partial \mathbb{D} \setminus \overline{G}$ then $\mu \notin \sigma(C_\psi)$. Suppose $\mu \in \partial \mathbb{D} \setminus \overline{G}$.

Since $\mu \notin \overline{G}$, it clear that $\mu \notin G$ and $\mu^m \neq 1$. In order to show $\mu \notin \sigma(C_\psi)$, we will show that $C_\psi - \mu I$ is invertible by proving that for every $g \in \mathcal{X}$, there exists a unique $f \in \mathcal{X}$ such that $f \circ \psi - \mu f = g$. 


Since the order of $\lambda$ is $m$, we have

$$\psi^{(m)}(z) = (\psi \circ \cdots \circ \psi)^{(m)}(z) = \lambda^m z = z.$$  

By repeated composition with $\psi$, we obtain the system of linear equations:

\[
\begin{align*}
    f \circ \psi - \mu f &= g \\
    f \circ \psi^{(2)} - \mu (f \circ \psi) &= g \circ \psi \\
    & \vdots \\
    f - \mu (f \circ \psi^{(m-1)}) &= g \circ \psi^{(m-1)}.
\end{align*}
\]

This system of linear equations can be expressed as the matrix equation $A\vec{x} = \vec{b}$ where

\[
A = \begin{bmatrix}
    -\mu & 1 & 0 & 0 & \cdots & 0 \\
    0 & -\mu & 1 & 0 & \cdots & 0 \\
    \vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
    \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \vdots & \cdots & \cdots & \cdots & 1 \\
    1 & 0 & \cdots & \cdots & 0 & -\mu
\end{bmatrix}
\quad \text{and} \quad 
\vec{x} = \begin{bmatrix}
    f \\
    f \circ \psi \\
    \vdots \\
    f \circ \psi^{(m-2)} \\
    f \circ \psi^{(m-1)}
\end{bmatrix}, \quad \text{and} \quad \vec{b} = \begin{bmatrix}
    g \\
    g \circ \psi \\
    \vdots \\
    g \circ \psi^{(m-2)} \\
    g \circ \psi^{(m-1)}
\end{bmatrix}.
\]

The determinant of $A$ is $(-1)^m(\mu^m - 1)$, which is not zero since $\mu \notin G$. Thus there is a unique solution for $\vec{x}$. It gives us the unique solution $f$, which is a finite linear combination of function in $\mathcal{X}$ of the form $g \circ \psi^{(j-1)}$ for $j = 1, \ldots, m$, and thus $f$ is in $\mathcal{X}$. It follows that $C_\psi - \mu I$ is invertible. So, $\mu \notin \sigma(C_\psi)$. Therefore, $\sigma(C_\psi) = \sigma(C_\psi) \subseteq \overline{G}$. \hfill \qed

**Theorem 3.5.** Let $\varphi$ be a parabolic disk automorphism. Then the spectrum of $C_\varphi$ acting on $\mathcal{X}$ is the unit circle.

**Proof.** From Lemma 2.2, $\varphi$ is conformally equivalent to either

\[
\psi_1(z) = \frac{(1 + i)z - 1}{z + i - 1} \quad \text{or} \quad \psi_2(z) = \frac{(1 - i)z - 1}{z - i - 1}.
\]

By Theorems 3.2 and 3.3 it suffices to show that $\partial \mathbb{D}$ is a subset of $\sigma(C_{\psi_1})$ and $\sigma(C_{\psi_2})$.

First suppose $\varphi$ is conformally equivalent to $\psi_1$. Consider the function

$$f_\varphi(z) = \exp s(z + 1) \frac{s(z + 1)}{z - 1}$$
for $s \geq 0$. By property (i), $f_s$ is in $\mathcal{X}$. Observe

$$(C_{\psi_1} f_s)(z) = f_s(\psi_1(z)) = f_s \left( \frac{(1 + i)z - 1}{z + i - 1} \right)$$

$$= \exp \left( \frac{(1 + i)z - 1}{z + i - 1} + 1 \right) = \exp \frac{s((1 + i)z - 1 + z + i - 1)}{(1 + i)z - 1 - z - i + 1}$$

$$= \exp \frac{s(2 + i)z + i - 2)}{i(z - 1)} = \exp \frac{s((1 - 2i)z + 1 + 2i)}{z - 1}$$

$$= \exp \left( \frac{s(z + 1)}{z - 1} - 2is \right) = e^{i(-2s)} f_s(z).$$

So, $f_s$ is an eigenfunction of $C_{\psi_1}$ for $s \geq 0$. Then, $\partial \mathbb{D} = \{ e^{i(-2s)} : s \geq 0 \}$ is a subset of $\sigma(C_{\psi_1})$. If $\varphi$ is conformally equivalent to $\psi_2$, then by a similar calculation, we have

$$(C_{\psi_2} f_s)(z) = e^{2is} f_s(z),$$

and so $\partial \mathbb{D} = \{ e^{2is} : s \geq 0 \}$ is a subset of $\sigma(C_{\psi_2})$. Therefore, $\sigma(C_{\varphi}) = \partial \mathbb{D}$, as desired.

Theorem 3.6. Let $\varphi$ be a hyperbolic disk automorphism. Then the spectrum of $C_{\varphi}$ acting on $\mathcal{X}$ is the unit circle.

Proof. From Lemma 2.3, $\varphi$ is conformally equivalent to $\psi(z) = (z + r)/(1 + rz)$ for some $r \in (0, 1)$. By Theorems 3.2 and 3.3 it suffices to show that $\partial \mathbb{D} \subseteq \sigma(C_{\psi})$. Consider the function

$$f_t(z) = \left( \frac{1 + z}{1 - z} \right)^{it}$$

for $t \in \mathbb{R}$. By property (i), $f_t$ is in $\mathcal{X}$. Observe

$$(C_{\psi} f_t)(z) = f_t(\psi(z)) = f_t \left( \frac{z + r}{1 + rz} \right)$$

$$= \left( \frac{1 + \frac{z + r}{1 + rz}}{1 - \frac{z + r}{1 + rz}} \right)^{it} = \left( \frac{1 + rz + z + r}{1 + rz - z - r} \right)^{it}$$

$$= \left( \frac{(r + 1)z + (r + 1)}{(r - 1)z - (r - 1)} \right)^{it}$$

$$= \left( \frac{r + 1}{r - 1} \right)^{it} f_t(z).$$

So, $f_t$ is an eigenfunction of $C_{\psi}$ for $t$ real. Then $\partial \mathbb{D} = \{ ((r + 1)/(r - 1))^{it} : 0 < r < 1, t \in \mathbb{R} \}$ is a subset of $\sigma(C_{\psi}) = \sigma(C_{\varphi})$. Therefore $\sigma(C_{\varphi}) = \partial \mathbb{D}$, as desired.
4. Examples and comparisons

In this section we first consider examples of spaces that satisfy the properties of $\mathcal{X}$. For these spaces, our results characterize the spectrum of composition operators induced by disk automorphisms. Lastly, we consider spaces that do not satisfy the properties of $\mathcal{X}$ but for which the spectrum of composition operators induced by automorphisms is known. We will compare the spectra for those spaces with the characterization for $\mathcal{X}$.

4A. Examples. First, we will discuss examples of spaces that satisfy the properties of $\mathcal{X}$.

4A1. Bounded analytic functions. The property (i) of $\mathcal{X}$ is satisfied by $H^\infty$ by Lemmas 2.4 and 2.5. In fact, on $H^\infty$, any analytic self-map of $\mathbb{D}$ induces a bounded composition operator $C_\phi$ such that $\|C_\phi\| = 1$. Equality is achieved since $H^\infty$ contains the constant function 1. The spectral radius formula (see Theorem 5.15 of [MacCluer 2009]) then implies that $\rho(C_\phi) = 1$. Thus, property (ii) is satisfied. Thus $H^\infty$ belongs to the family of Banach spaces of analytic functions $\mathcal{X}$.

4A2. Bloch space. The Bloch space on $\mathbb{D}$, denoted $B = B(\mathbb{D})$, is the space of analytic functions on $\mathbb{D}$ such that $\beta(f) = \sup_{z \in \mathbb{D}} (1 - |z|^2) |f'(z)| < \infty$. The quantity $\beta(f)$ is a seminorm, called the Bloch seminorm. The Bloch space is a Banach space under the norm

$$\|f\|_B = |f(0)| + \beta_f.$$  

It is well-known that $B$ is a Banach space of analytic functions that contains $H^\infty$, and thus satisfies property (i) of $\mathcal{X}$. In fact, every analytic self-map of $\mathbb{D}$ induces a bounded composition operator on $B$ (see [Arazy et al. 1985, p. 126]). Donaway, in his Ph.D. thesis, proved the spectral radius of every composition operator induced by an analytic function on $\mathbb{D}$, and in particular the disk automorphisms, is 1, by [Donaway 1999, Corollary 3.9]. So the Bloch space satisfies all the properties of $\mathcal{X}$.

4A3. Analytic functions of bounded mean oscillation. The space of analytic functions on $\mathbb{D}$ with bounded mean oscillation on $\partial \mathbb{D}$, denoted BMOA, is defined to be the set of functions in $H^2(\mathbb{D})$ such that

$$\|f\|_* = \sup_{z \in \mathbb{D}} \|f \circ \tau_a - f(z)\|_{H^2} < \infty,$$

where $H^2$ is defined in Section 4B1. The space BMOA is a Banach space under the norm

$$\|f\|_{\text{BMOA}} = |f(0)| + \|f\|_*.$$  

It is well-known that BMOA is a Banach space of analytic functions, a subspace of the Bloch space, and contains $H^\infty$ as a subspace since $\|f\|_{\text{BMOA}} \leq 3\|f\|_\infty$.  


Thus property (i) is satisfied by BMOA. The following result shows property (ii) is satisfied by BMOA also.

**Theorem 4.1.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$. Then $C_\varphi$ acting on BMOA is bounded and $\rho(C_\varphi) = 1$.

**Proof.** As a result of the Littlewood subordination principle (see Theorem 1.7 of [Duren 1970]), every analytic self-map $\varphi$ of $\mathbb{D}$ induces a bounded composition operator on BMOA.

To compute the spectral radius of $C_\varphi$ acting on BMOA, we first estimate the norm. By Corollary 2.2 of [Laitila 2009] there is a constant $M > 0$, independent of $\varphi$, such that

$$\|C_\varphi\| \leq M \left( \sup_{a \in \mathbb{D}} \|\tau_{\varphi(a)} \circ \varphi \circ \tau_a\|_{H^2} + \log \frac{2}{1 - |\varphi(0)|^2} \right). \quad (4-1)$$

Since the function $\tau_{\varphi(a)} \circ \varphi \circ \tau_a$ is a composition of self-maps of the disk, the first term on the right is bounded above by 1. Also,

$$\frac{1}{1 - |\varphi(0)|^2} \leq \frac{1 + |\varphi(0)|}{1 - |\varphi(0)|} \leq \frac{2}{1 - |\varphi(0)|}$$

and hence

$$\log \left( \frac{2}{1 - |\varphi(0)|^2} \right) \leq \log \left( \frac{4}{1 - |\varphi(0)|} \right) \leq 2 \log 2 - \log(1 - |\varphi(0)|).$$

Applying these estimates to Equation (4-1), we have

$$\|C_\varphi\| \leq M(1 + 2 \log 2) - M \log(1 - |\varphi(0)|).$$

This immediately implies that

$$\|C_{\varphi_n}\| \leq M(1 + 2 \log 2) - M \log(1 - |\varphi_n(0)|)$$

and it follows that $\rho(C_\varphi) = 1$ for all bounded composition operators acting on BMOA by Theorem 3.7 of [Donaway 1999]. \[\square\]

Thus BMOA satisfies all the properties of $\mathcal{X}$.

**4B. Comparisons.** We now investigate spaces that do not satisfy the properties of $\mathcal{X}$. We compare the spectrum of induced composition operators on these spaces with those on $\mathcal{X}$.

**4B1. Hardy spaces.** For $1 \leq p < \infty$, the Hardy space, denoted $H^p = H^p(\mathbb{D})$, is the space of analytic functions on $\mathbb{D}$ such that

$$\|f\|_{H^p} = \sup_{0 < r < 1} \int_\mathbb{D} |f(re^{i\theta})|^p \frac{d\theta}{2\pi} < \infty.$$
Under this norm, the Hardy spaces are Banach spaces and for $p = 2$ it is a Hilbert space.

It is well known that $H^p$ is a Banach space of analytic functions that contains $H^\infty$ as a subspace. For the cases of an elliptic or parabolic automorphism $\varphi$, it is the case that $\rho(C_\varphi) = 1$ and the spectrum of $C_\varphi$ on $H^p$ is the same as for $C_\varphi$ acting on $X$ (see Theorem 3.9 of [Cowen and MacCluer 1995]). However, it is not the case that the spectral radius is 1 for every composition operator induced by an automorphism. In fact, if $\varphi$ is hyperbolic, then $\rho(C_\varphi) = \frac{\max\{1, \varphi'(a)^{-1/p}\}}{\min\{1, \varphi'(a)^{s}\}}$ where $a$ is the Denjoy–Wolff point of $\varphi$ (see Theorem 3.9 of [Cowen and MacCluer 1995]). In this situation, $\varphi'(a) < 1$ thus making $\rho(C_\varphi) > 1$. In turn, the spectrum is the annulus $\frac{1}{\varphi'(a)^{1/p}} \leq |z| \leq \frac{1}{\varphi'(a)^{-1/p}}$ (see Theorem 4.9 of [Hyvärinen et al. 2013]).

4B2. Weighted Bergman spaces. For $1 \leq p < \infty$ and $\alpha > -1$, the standard weighted Bergman space, denoted $A^p_\alpha = A^p_\alpha(\mathbb{D})$, is the space of analytic functions on $\mathbb{D}$ such that

$$
\|f\|_{A^p_\alpha}^p = \int_{\mathbb{D}} (1 - |z|^2)^\alpha |f(z)|^p \, dA(z) < \infty,
$$

where $dA(z)$ is the normalized Lebesgue area measure on $\mathbb{D}$. The weighted Bergman spaces are Banach spaces under the norm $\|\cdot\|_{A^p_\alpha}$.

It is well known that $A^p_\alpha$ is a Banach space of analytic functions that contains $H^\infty$ as a subspace. For the cases of an elliptic or parabolic automorphism $\varphi$, it is the case that $\rho(C_\varphi) = 1$ and the spectrum of $C_\varphi$ on $A^p_\alpha$ is the same as for $C_\varphi$ acting on $X$ (see Lemma 4.2 and Theorem 4.14 of [Hyvärinen et al. 2013]). However, as was the case for the Hardy spaces, it is not the case that the spectral radius is 1 for every composition operator induced by an automorphism. In fact, if $\varphi$ is hyperbolic, then

$$
\rho(C_\varphi) = \max\left\{\frac{1}{\varphi'(a)^s}, \frac{1}{\varphi'(b)^s}\right\}
$$

where $s = (\alpha + 2)/p$, $a$ is the Denjoy–Wolff point and $b$ is the other fixed point of $\varphi$ (see Theorem 4.6 of [Hyvärinen et al. 2013]). In turn, the spectrum contains the annulus

$$
\min\left\{\frac{1}{\varphi'(a)^s}, \frac{1}{\varphi'(b)^s}\right\} \leq |z| \leq \max\left\{\frac{1}{\varphi'(a)^s}, \frac{1}{\varphi'(b)^s}\right\}
$$

(see Corollary 4.7 of [Hyvärinen et al. 2013]).

4B3. Weighted Banach spaces. For $0 < p < \infty$, the standard weighted Banach space on $\mathbb{D}$, denoted $H^\infty_p = H^\infty_p(\mathbb{D})$, is the space of analytic functions on $\mathbb{D}$ such that

$$
\|f\|_{H^\infty_p} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^p |f(z)| < \infty,
$$

The weighted Banach spaces are, not surprisingly, Banach spaces under $\|\cdot\|_{H^\infty_p}$. 
It is well known that $H^\infty_p$ is a Banach space of analytic functions that contain $H^\infty$ as a subspace. For the cases of an elliptic or parabolic automorphism $\varphi$, it is the case that $\rho(C_\varphi) = 1$ and the spectrum of $C_\varphi$ on $H^\infty_p$ is the same as for $C_\varphi$ acting on $X$ (see Lemma 4.2 and Theorem 4.14 of [Hyvärinen et al. 2013]). However, as was the case for the Hardy spaces, it is not the case that the spectral radius is 1 for every composition operator induced by an automorphism. In fact, if $\varphi$ is hyperbolic, then

$$\rho(C_\varphi) = \max \left\{ \frac{1}{\varphi'(a)^p}, \frac{1}{\varphi'(b)^p} \right\}$$

where $a$ is the Denjoy–Wolff point and $b$ is the other fixed point of $\varphi$. In turn, the spectrum contains the annulus

$$\min \left\{ \frac{1}{\varphi'(a)^p}, \frac{1}{\varphi'(b)^p} \right\} \leq |z| \leq \max \left\{ \frac{1}{\varphi'(a)^p}, \frac{1}{\varphi'(b)^p} \right\}$$

(see Theorem 4.6 and Corollary 4.7 of [Hyvärinen et al. 2013]).

4B4. Dirichlet space. The Dirichlet space on $\mathbb{D}$, denoted $D$, is the space of analytic functions on $\mathbb{D}$ such that

$$\int_\mathbb{D} |f'(z)|^2 \, dA(z) < \infty$$

where $dA$ denotes the normalized Lebesgue area measure on $\mathbb{D}$. Under the norm

$$\|f\|_D^2 = |f(0)|^2 + \int_\mathbb{D} |f'(z)|^2 \, dA(z)$$

the Dirichlet space has a Hilbert space structure. Although not every analytic self-map of $\mathbb{D}$ induce bounded composition operators on $D$, univalent maps, and thus the automorphisms, of $\mathbb{D}$ do.

Independently, Donaway [1999, Corollary 3.11] and Martín and Vukotić [2005, Theorem 7] showed that composition operators on $D$ induced by univalent self-maps of $\mathbb{D}$, and thus the automorphisms, have spectral radius 1. However, by direct calculation, one can see that the functions in $F$ are not contained in the Dirichlet space; for the case of $f_s$ this is shown in [Pons 2010] (see p. 455). Despite $D$ not satisfying all the properties of $X$, the spectrum of automorphism induced composition operators on $D$ are precisely the same as those on $X$.

To overcome the lack of eigenfunctions, the authors in [Higdon 1997] and [Gallardo-Gutiérrez and Montes-Rodríguez 2003] used two new approaches. In [Higdon 1997], the author produces approximate eigenfunctions and in [Gallardo-Gutiérrez and Montes-Rodríguez 2003] unitary similarity is the key tool.

Remark 4.2. For all of the spaces discussed in Sections 4A and 4B (and those discussed in the next section), the spectrum of $C_\varphi$ when $\varphi$ is elliptic will be the
same as that for $C_\psi$ acting on $\mathcal{X}$. This is due to the fact that the eigenfunctions are the monomials, which are contained in all of these spaces.

5. Open questions

We end this paper with open questions which were inspired while developing the examples and comparisons in Sections 4A and 4B.

5A. The little Bloch space. While the Bloch space contains the polynomials, they are not dense in $B$. The closure of the polynomials with respect to $\|\cdot\|_B$ is called the little Bloch space, denoted $B_0 = B_0(\mathbb{D})$. More formally, the little Bloch space consists of the functions $f \in B$ such that

$$\lim_{|z| \to 1} \left| 1 - |z|^2 \right| |f'(z)| = 0.$$ 

From Theorem 12 of [Arazy et al. 1985], bounded composition operators on $B_0$ are induced exactly by functions in $B_0$, which include the automorphisms. Donaway also proved the spectral radius of every bounded composition operator on $B_0$ is 1. Thus property (ii) is satisfied by $B_0$. However, the following result shows that $\mathcal{F}$ is not contained in $B_0$, and thus property (i) of $\mathcal{X}$ is not satisfied.

Theorem 5.1. The functions $f_s$ and $f_t$, for $s > 0$ and $t \neq 0$, are not contained in the little Bloch space.

Proof. Consider the function

$$f_t(z) = \exp \left( it \log \frac{1+z}{1-z} \right).$$

We show that this function is not in $B_0$ for $t \in \mathbb{R} \setminus \{0\}$. Taking the derivative,

$$f_t'(z) = f_t(z) \left( it \frac{1-z}{1+z} \right) = f_t(z) \frac{2it}{(1-z)(1+z)}.$$

For $t > 0$, $|f_t(z)| \geq e^{-2\pi}$ and, for $t < 0$, $|f_t(z)| \geq 1$. In either case, there is a constant $C > 0$ such that $|f_t(z)| \geq C$ for all $t \in \mathbb{R} \setminus \{0\}$ and all $z \in \mathbb{D}$. Hence

$$|f_t'(z)| \geq \frac{2C|t|}{|z-1||z+1|}.$$ 

To show that $f_t \not\in B_0$, we need to show that

$$\lim_{|z| \to 1} \left( 1 - |z|^2 \right) |f_t'(z)| \neq 0.$$ 

To see this, first observe that

$$\lim_{|z| \to 1} \left( 1 - |z|^2 \right) |f_t'(z)| \geq \lim_{|z| \to 1} \left( 1 - |z|^2 \right) \frac{2C|t|}{|z-1||z+1|}.$$
by our estimate from above. If we now take a radial path to 1, that is, we set \( z = r \) and let \( r \uparrow 1 \), we have

\[
\lim_{r \to 1^-} (1 - r^2) \frac{2C|t|}{(1 - r)(1 + r)} = 2C|t| > 0
\]

when \( t \neq 0 \). Thus

\[
\lim_{|z| \to 1} (1 - |z|^2) \frac{2Ct}{|z - 1||z + 1|} \neq 0
\]

for \( t \neq 0 \), and hence \( f_t \) is not in \( \mathcal{B}_0 \).

Next consider the function

\[
f_s(z) = \exp \frac{s(z + 1)}{z - 1}.
\]

We will show that this function is not in \( \mathcal{B}_0 \) for \( s > 0 \). First observe that

\[
f'_s(z) = \left( \exp \frac{s(z + 1)}{z - 1} \right) \frac{-2s}{(z - 1)^2}
\]

and thus we aim to show that

\[
\lim_{|z| \to 1} (1 - |z|^2) |f'_s(z)| = \lim_{|z| \to 1} (1 - |z|^2) \exp \left( \frac{s(z + 1)}{z - 1} \right) \left| \frac{2s}{|1 - z|^2} \right| \neq 0.
\]

Fix \( x_0 < 0 \) and consider the sequence \( \{z_n\} \) defined by

\[
z_n = \frac{x_0 + in + 1}{x_0 + in - 1}.
\]

Since \( x_0 < 0 \), this sequence is contained in the unit disk and \( \{z_n\} \to 1 \) as \( n \to \infty \). To obtain our conclusion, we show

\[
\lim_{n \to \infty} (1 - |z_n|^2) \exp \frac{s(z_n + 1)}{z_n - 1} \left| \frac{2s}{|1 - z_n|^2} \right| \neq 0.
\]

First observe that the map \( \psi(z) = (z + 1)/(z - 1) \) is its own inverse and hence \( \psi(z_n) = x_0 + in \) for each \( n \in \mathbb{N} \). Thus

\[
\left| \exp \frac{s(z_n + 1)}{z_n - 1} \right| = \left| \exp(sx_0 + isn) \right| = e^{sx_0} > 0.
\]

Substituting,

\[
\lim_{n \to \infty} (1 - |z_n|^2) \exp \frac{s(z_n + 1)}{z_n - 1} \left| \frac{2s}{|1 - z_n|^2} \right| = \lim_{n \to \infty} e^{sx_0} (1 - |z_n|^2) \left| \frac{2s}{|1 - z_n|^2} \right|.
\]

Next,

\[
1 - |z_n|^2 = \frac{-4x_0}{(x_0 - 1)^2 + n^2}
\]
Thus
\[
\lim_{n \to \infty} e^{sx_0} \left( 1 - |z_n|^2 \right) = \lim_{n \to \infty} e^{sx_0} \left( \frac{-4x_0}{(x_0 - 1)^2 + n^2} \right) = \lim_{n \to \infty} (-2sx_0)e^{sx_0} > 0
\]
and hence \( f_s \) is not in \( B_0 \) for \( s > 0 \). \( \square \)

For the little Bloch space, we leave the reader with the following question.

**Question 1.** For \( \varphi \) a parabolic or hyperbolic automorphism, what is the spectrum of \( C_\varphi \) on the little Bloch space?

**5B. Analytic functions of vanishing mean oscillation.** Like the Bloch space, the polynomials are contained in BMOA, but they are not dense in BMOA. We denote by VMOA the closure of the polynomials in \( \| \cdot \|_{BMOA} \). VMOA is the space of analytic functions with vanishing mean oscillation on \( \partial D \), formally defined as the functions \( f \in BMOA \) such that
\[
\lim_{|a| \to 1} \| f \circ \tau_a - f(a) \|_{H^2} = 0.
\]

By Corollary 4.2 of [Laitila 2009], \( C_\varphi \) is bounded on VMOA if and only if \( \varphi \in VMOA \). So every automorphism induces a bounded composition operator on VMOA. By the same argument as in Section 4A3, the spectral radius of \( C_\varphi \) induced by a disk automorphism is 1. Thus property (ii) of \( \mathcal{X} \) is satisfied. Since VMOA is a subspace of the little Bloch space (see [Gallardo-Gutiérrez et al. 2013]), it follows that VMOA does not satisfy property (i), a corollary of Theorem 5.1.

**Corollary 5.2.** The functions \( f_s \) and \( f_t \), for \( s > 0 \) and \( t \neq 0 \), are not contained in VMOA.

For VMOA, we leave the reader with the following question.

**Question 2.** For \( \varphi \) parabolic or hyperbolic automorphism, what is the spectrum of \( C_\varphi \) on VMOA?

**Acknowledgements**

The authors would like to thank the referees for their careful review of the manuscript and their many suggestions, especially to generalize the results to a more general family of Banach spaces. The work of the second author was conducted while an undergraduate student at the University of Wisconsin-La Crosse and funded by the College of Science and Health Dean’s Distinguished Fellowship.
References

[Arazy et al. 1985] J. Arazy, S. D. Fisher, and J. Peetre, “Möbius invariant function spaces”, J. Reine Angew. Math. 363 (1985), 110–145. MR 814017 Zbl 0566.30042

[Cowen and MacCluer 1995] C. C. Cowen and B. D. MacCluer, Composition operators on spaces of analytic functions, Studies in Advanced Mathematics 20, CRC Press, Boca Raton, FL, 1995. MR 1397026 Zbl 0873.47017

[Donaway 1999] R. L. Donaway, Norm and essential norm estimates of composition operators on Besov-type spaces, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–University of Virginia. MR 2698630

[Duren 1970] P. L. Duren, Theory of $H^p$ spaces, Pure and Applied Mathematics Vol. 38, Academic Press, New York, 1970. MR 0268655 Zbl 0215.20203

[Gallardo-Gutiérrez and Montes-Rodríguez 2003] E. A. Gallardo-Gutiérrez and A. Montes-Rodríguez, “Adjoints of linear fractional composition operators on the Dirichlet space”, Math. Ann. 327:1 (2003), 117–134. MR 2005124 Zbl 1048.47016

[Gallardo-Gutiérrez et al. 2013] E. A. Gallardo-Gutiérrez, M. J. González, F. Pérez-González, C. Pommerenke, and J. Rättyä, “Locally univalent functions, VMOA and the Dirichlet space”, Proc. Lond. Math. Soc. (3) 106:3 (2013), 565–588. MR 3048550 Zbl 1276.30026

[Higdon 1997] W. M. Higdon, Composition operators of the Dirichlet space, Ph.D. thesis, Michigan State University, 1997.

[Hyvärinen et al. 2013] O. Hyvärinen, M. Lindström, I. Nieminen, and E. Saukko, “Spectra of weighted composition operators with automorphic symbols”, J. Funct. Anal. 265:8 (2013), 1749–1777. MR 3079234 Zbl 1325.47054

[Laitila 2009] J. Laitila, “Weighted composition operators on BMOA”, Comput. Methods Funct. Theory 9:1 (2009), 27–46. MR 2478261 Zbl 1163.47018

[MacCluer 2009] B. D. MacCluer, Elementary functional analysis, Graduate Texts in Mathematics 253, Springer, New York, 2009. MR 2462971 Zbl 1170.46002

[Martín and Vukotic 2005] M. J. Martín and D. Vukotic, “Norms and spectral radii of composition operators acting on the Dirichlet space”, J. Math. Anal. Appl. 304:1 (2005), 22–32. MR 2124646 Zbl 1071.47028

[Nordgren 1968] E. A. Nordgren, “Composition operators”, Canad. J. Math. 20 (1968), 442–449. MR 0223914 Zbl 0161.34703

[Pons 2007] M. A. Pons, Composition Operators on Besov and Dirichlet Type Spaces, Ph.D. thesis, University of Virginia, 2007.

[Pons 2010] M. A. Pons, “The spectrum of a composition operator and Calderón’s complex interpolation”, pp. 451–467 in Topics in operator theory. Volume 1. Operators, matrices and analytic functions, Oper. Theory Adv. Appl. 202, Birkhäuser, Basel, 2010. MR 2723292 Zbl 1217.47050

Received: 2015-06-24 Revised: 2015-08-26 Accepted: 2015-09-07

callen@uwlax.edu Department of Mathematics and Statistics, University of Wisconsin-La Crosse, La Crosse, WI 54601, United States

thmle@ucdavis.edu Department of Computer Science, University of California, Davis, Davis, CA 95616, United States

mapons@noctrl.edu Department of Mathematics, North Central College, Naperville, IL 60540, United States

mathematical sciences publishers
INVOLVE YOUR STUDENTS IN RESEARCH

*Involve* showcases and encourages high-quality mathematical research involving students from all academic levels. The editorial board consists of mathematical scientists committed to nurturing student participation in research. Bridging the gap between the extremes of purely undergraduate research journals and mainstream research journals, *Involve* provides a venue to mathematicians wishing to encourage the creative involvement of students.

**MANAGING EDITOR**

Kenneth S. Berenhaut  
Wake Forest University, USA

**BOARD OF EDITORS**

| Name                  | Institution                                      |
|-----------------------|--------------------------------------------------|
| Colin Adams           | Williams College, USA                             |
| John V. Baxley        | Wake Forest University, NC, USA                   |
| Arthur T. Benjamin    | Harvey Mudd College, USA                          |
| Martin Bohner         | Missouri U of Science and Technology, USA         |
| Nigel Boston          | University of Wisconsin, USA                      |
| Amarjit S. Budhiraja  | U of North Carolina, Chapel Hill, USA             |
| Pietro Cerone         | La Trobe University, Australia                    |
| Scott Chapman         | Sam Houston State University, USA                 |
| Joshua N. Cooper      | University of South Carolina, USA                 |
| Jem N. Corcoran       | University of Colorado, USA                       |
| Toka Diagana          | Howard University, USA                             |
| Michael Dorff         | Brigham Young University, USA                     |
| Sever S. Dragomir     | Victoria University, Australia                    |
| Behrouz Emamizadeh    | The Petroleum Institute, UAE                      |
| Joel Foisy            | SUNY Potsdam, USA                                 |
| Errin W. Fulp         | Wake Forest University, USA                       |
| Joseph Gallian        | University of Minnesota Duluth, USA               |
| Stephan R. Garcia     | Pomona College, USA                               |
| Anant Godbole         | East Tennessee State University, USA              |
| Ron Gould             | Emory University, USA                              |
| Andrew Granville      | Université Montréal, Canada                       |
| Jerrold Griggs        | University of South Carolina, USA                 |
| Sat Gupta             | U of North Carolina, Greensboro, USA              |
| Jim Haglund           | University of Pennsylvania, USA                   |
| Johnny Henderson      | Baylor University, USA                            |
| Jim Hoste             | Pitt College, USA                                 |
| Natalia Hrinchenko    | Prairie View A&M University, USA                  |
| Glenn H. Hurbert      | Arizona State University,USA                      |
| Charles R. Johnson    | College of William and Mary, USA                  |
| K. B. Kulasekera      | Clemson University, USA                           |
| Gerry Ladas           | University of Rhode Island, USA                   |
| Suzanne Lenhart       | University of Tennessee, USA                      |
| Chi-Kwong Li          | College of William and Mary, USA                  |
| Robert B. Lund        | Clemson University, USA                           |
| Gaven J. Martin       | Massey University, New Zealand                    |
| Mary Meyer            | Colorado State University, USA                    |
| Emil Minchev          | Ruse, Bulgaria                                    |
| Frank Morgan          | Williams College, USA                             |
| Mohammad Sal Moslehiān| Ferdowsi University of Mashhad, Iran              |
| Zuhair Nashed         | University of Central Florida, USA                |
| Ken Ono               | Emory University, USA                             |
| Y.-F. S. Pétermann    | Université de Genève, Switzerland                 |
| Robert J. Plemmons    | Wake Forest University, USA                       |
| Carl B. Pomerance     | Dartmouth College, USA                            |
| Vadim Ponomarenko     | San Diego State University, USA                   |
| Bjorn Poonen          | UC Berkeley, USA                                  |
| Yuval Peres           | Microsoft Research, USA                           |
| Joseph O’Rourke       | Smith College, USA                                |
| Jem N. Corcoran       | Emory University, USA                              |
| Robert W. Robinson    | University of Georgia, USA                        |
| James Propp           | U Mass Lowell, USA                                |
| Joseph H. Przytycki   | George Washington University, USA                 |
| Richard Rebarber      | University of Nebraska, USA                       |
| Filip Saidak          | U of North Carolina, Greensboro, USA              |
| James A. Sellers      | Penn State University, USA                        |
| Andrew J. Sierge      | Honorary Editor                                   |
| Ann Trenk             | Wellesley College, USA                            |
| Ravi Vakil            | Stanford University, USA                          |
| Antonio Vecchio       | Consiglio Nazionale delle Ricerche, Italy         |
| Ram U. Verma          | University of Toledo, USA                         |
| John C. Wierman       | Johns Hopkins University, USA                     |
| Michael E. Zieve      | University of Michigan, USA                       |

**PRODUCTION**

Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2016 is US $160/year for the electronic version, and $215/year (+$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

**PUBLISHED BY**

[mathematical sciences publishers](http://msp.org/)

© 2016 Mathematical Sciences Publishers
An iterative strategy for Lights Out on Petersen graphs
BRUCE TORRENCE AND ROBERT TORRENCE 721

A family of elliptic curves of rank \( \geq 4 \)
FARZALI IZADI AND KAMRAN NABARDI 733

Splitting techniques and Betti numbers of secant powers
REZA AKHTAR, BRITTANY BURNS, HALEY DOHRMANN, HANNAH HOGANSON, OLA SOBIESKA AND ZEROTTI WOODS 737

Convergence of sequences of polygons
ERIC HINTIKKA AND XINGPING SUN 751

On the Cheremak–Delgado lattices of split metacyclic \( p \)-groups
ERIN BRUSH, JILL DIETZ, KENDRA JOHNSON-TESCH AND BRIANNE POWER 765

The left greedy Lie algebra basis and star graphs
BENJAMIN WALTER AND AMINREZA SHIRI 783

Note on superpatterns
DANIEL GRAY AND HUA WANG 797

Lifting representations of finite reductive groups: a character relation
JEFFREY D. ADLER, MICHAEL CASSEL, JOSHUA M. LANSKY, EMMA MORGAN AND YIFEI ZHAO 805

Spectrum of a composition operator with automorphic symbol
ROBERT F. ALLEN, THONG M. LE AND MATTHEW A. PONS 813

On nonabelian representations of twist knots
JAMES C. DEAN AND ANH T. TRAN 831

Envelope curves and equidistant sets
MARK HUIBREGTSE AND ADAM WINCHELL 839

New examples of Brunnian theta graphs
B YOUNGWOOK JANG, ANNA KRONAEUR, PRATAP LUITEL, DANIEL MEDICI, SCOTT A. TAYLOR AND ALEXANDER ZUPAN 857

Some nonsimple modules for centralizer algebras of the symmetric group
CRAIG DODGE, HARALD ELLERS, YUKIHIKE NAKADA AND KELLY POHLAND 877

Acknowledgement 899