On a sequence of higher-order nonlinear diffusion-convection equations

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Abstract. Motivated by the existence of the thin film equations, fourth and sixth order, which are generalizations of the diffusion equation, we consider a sequence of higher order nonlinear diffusion-convection equations. We investigate this class of nonlinear equations from the point of view of Lie group analysis. In particular, we derive Lie symmetries, potential symmetries, nonclassical reductions and potential nonclassical reductions for a third and a fourth equation. The present work is a motivation for further investigation on this sequence of nonlinear partial differential equations. Certain general results of higher order equations of the sequence are given.

1. Introduction

The study of the nonlinear diffusion equations by means of symmetry methods started in by Ovsiannikovs [1] where he studied Lie symmetries of the equation

\[ u_t = [f(u)u_x]_x. \]

The member of the above class which appears in most applications has the form

\[ u_t = [u^n u_x]_x. \] (1)

Generalizations of the nonlinear diffusion equation (1) is the third order equation

\[ u_t = - [u^n u_{xx} + au^{n-1}u_x^2]_x, \] (2)

and the fourth order equation

\[ u_t = - [u^n u_{xxx} + au^{n-1}u_xu_{xx} + bu^{n-2}u_x^3]_x, \] (3)

where \( a \) and \( b \) are constants. A known case of the class (2) is the Harry-Dym equation \( u_t = 2[u^{-2}]_{xxx} \). The class (3) was introduced by King [2] and it is known as thin film equation. Equations (1) - (3) can be seen as a sequence of higher order diffusion-type equations. The corresponding fifth and sixth order equations of the sequence have been introduced in [2,3]. Symmetry properties (equivalence transformations, Lie symmetries, potential symmetries, nonclassical reductions, potential nonclassical reductions) have been presented in [4] and for the fifth and sixth order in [3].
In this spirit of the above works, we construct a sequence of higher order nonlinear diffusion-convection-type equations. The first term of the sequence is the well known diffusion-convection equation

\[ u_t = [u^n u_x]_x + u^m u_x \]  

(4)

which has considerable applications in Mathematical Physics and in particular in the study of porous media. Lie symmetries of (4) can be found in [5] and potential symmetries in [6]. The third order of the sequence has the form

\[ u_t = -[u^n u_{xxx} + a_1 u^{n-1} u_x^2 + b_1 u^m u_{xx}]_x, \]  

(5)

and the fourth order has the form

\[ u_t = -[u^n u_{xxxx} + a_1 u^{n-1} u_x u_{xx} + a_2 u^{n-2} u_x^3 + b_1 u^m u_{x} + b_2 u^{m-1} u_x^2]_x, \]  

(6)

where \( a_1, a_2, b_1 \) and \( b_2 \) are constants. In the first equation we take \( b_1 \neq 0 \) and in the second \((b_1, b_2) \neq (0, 0)\). We derive symmetry properties for equations (5) and (6) in the cases that they have nonlinear forms.

The method for finding Lie point symmetries is well-known and easy to apply. This is to search for generators

\[ \Gamma = \tau(t, x, u) \partial_t + \xi(t, x, u) \partial_x + \eta(t, x, u) \partial_u \]  

(7)

corresponding to the infinitesimal transformations

\( t' = t + \epsilon \tau(t, x, u), \quad x' = x + \epsilon \xi(t, x, u), \quad u' = u + \epsilon \eta(t, x, u) \)

to the first order of \( \epsilon \). Any such generators \( \Gamma \) satisfy the criterion of infinitesimal invariance, i.e. the action of the \( n \)th prolongation \( \Gamma^{(n)} \) of \( \Gamma \) on equation under consideration, where \( n \) is the order of equation, results in the conditions being an identity for all solutions of this equation.

Potential symmetries are nonlocal symmetries which can be obtained when the equation is written in conserved form. If we introduce potential variables \( v \) as a further unknown variable using a conservation law of the equation \( \Delta(x, u) \), we obtain a potential (auxiliary) system \( Z(x, u, v) \). Any Lie symmetry of \( Z(x, u, v) \) induces a symmetry for the \( \Delta(x, u) \). If at least one of the coefficients of the generator \( \Gamma \) which correspond to the independent variables \( x \) and dependent variable \( u \) depends explicitly on the potential variable \( v \), then the local symmetry of \( Z(x, u, v) \) induces a non-local (potential) symmetry of \( \Delta(x, u) \), otherwise the symmetry of \( Z(x, u, v) \) induces a Lie symmetry of \( \Delta(x, u) \). The method was first applied by Bluman and coauthors [7,8].

The main application of Lie symmetries is the construction of reduction operators that reduce the number of independent variables in a partial differential equation by one. In the case where the partial differential equation has two independent variables, the reduced equation is an ordinary differential equations. These reduction operators are obtained by solving the invariance surface condition

\[ \tau(t, x, u) u_t + \xi(t, x, u) u_x = \eta(t, x, u). \]  

(8)

Bluman and Cole [9] introduced the non-classical approach to derive reduction operators that cannot be obtained using the Lie method. In the method we require invariance of both equation under study and the invariance surface condition (8). A precise and rigorous definition of non-classical invariance was formulated in [10,11] as a generalization of the Lie definition of invariance. The necessary definitions, including those of equivalence of non-classical reduction operators, and relevant statements can be found in [12,13].

As we have stated earlier, when an equation is written in a conserved form, then it can be written as an auxiliary system with the introduction of a potential variable. Eliminating from
this system the original dependent variable we obtain the potential form of the original equation with the potential variable being the unknown function. Non-classical reduction for the potential equation induce potential non-classical reductions for the original equation.

In the following two sections we derive, Lie symmetries, potential symmetries, non-classical reductions and potential non-classical reductions for equations (5) and (6), respectively. We will omit most of the analysis which is required to obtain the desired results. Some results can be generalized to higher order equations of the sequence. Certain general results are given in the final section.

2. Analysis of equation (5)
2.1. Lie symmetries
We search for vector fields of the form (7) which generate one-parameter groups of point symmetry transformations of equation (5). These vector fields form the maximal Lie invariance algebra of this equation. Any such vector fields \( \mathcal{G} \) satisfy the criterion of infinitesimal invariance symmetry transformations of equation (5). These vector fields form the maximal Lie invariance algebra of this equation. Here we require that

\[
\mathcal{G} (3) \left\{ u_t + u^n u_{xxx} + (n + 2a_1)u^{n-1} u_x u_{xx} + a_1(n - 1)u^{n-2} u_x^3 + b_1 u^m u_{xx} + b_1 m u^{m-1} u_x^2 \right\} = 0
\]

(9)

identically, modulo equation (5). Since equation (5) is an evolution equation which is a polynomial in the pure derivatives of \( u \) with respect to \( x \), it can be shown that \( \tau = \tau(t) \) and \( \xi = \xi(t, x) [16] \). Using these simplifications, after elimination of \( u_t \), equation (9) becomes a multi-variable polynomial in the pure derivatives of \( u \) with respect to \( x \). Coefficients of these variables lead to the determining system which is a system of linear partial differential equations in the variables \( \tau(t) \), \( \xi(t, x) \) and \( \eta(t, x, u) \). The solution of the determining system, which depends on the values of the parameters \( n, m, a_1 \) and \( b_1 \), provides the desired Lie symmetries of equation (5). We omit further analysis and we state the results.

For arbitrary values of the parameters \( n, m, a_1 \) and \( b_1 \), equation (5) admits three Lie symmetries:

\[
\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = (2n - 3m)t \partial_t + (n - m)x \partial_x + u \partial_u.
\]

In the case where \( n = m = -3 \), \( a_1 = -2 \) and \( b_1 \) is arbitrary, equation (5) admits an additional fourth symmetry

\[
\Gamma_4 = e^{-b_1x}(\partial_x + b_1 u \partial_u).
\]

Lie symmetry analysis provides an approach to derive a wide class of solutions which are invariant with respect to different subgroups of the Lie symmetry group. These solutions invariant with respect to an one-parametric group of symmetries of a given partial differential equation can be obtained by solving an equation with independent variables reduced by one. Hence, in our case, since we study partial differential equations in two independent variables, the reduced equations are ordinary differential equations. The reduced equations are derived using ansatz which are obtained from one-dimensional subalgebras of the algebra of maximal symmetry. In order to find all possible invariant solutions, we need to construct an optimal system for such solutions. Ovsiannikov proved [15] that the optimal system of solutions consists of solutions invariant with respect to all proper inequivalent subalgebras of the symmetry algebra. For detail about how to construct optimal sets of subalgebras can be found in [14,15],

The list of its inequivalent subalgebras is

\[
\langle \Gamma_4 + \delta \Gamma_2 + \epsilon \Gamma_3 \rangle, \quad \langle \Gamma_4 + \epsilon \Gamma_3 \rangle, \quad \langle \Gamma_4 + \epsilon \Gamma_2 \rangle, \quad \langle \Gamma_3 \rangle
\]

\[
\langle \Gamma_3 + \epsilon \Gamma_1 \rangle n = \frac{3m}{2}, \quad \langle \Gamma_3 + \epsilon \Gamma_2 \rangle n = m, \quad \langle \Gamma_1 + \epsilon \Gamma_2 \rangle, \quad \langle \Gamma_2 \rangle.
\]
For each of the above subalgebra, we solve the corresponding invariant surface condition (8) to obtain the desired similarity reduction that reduces equation (5) into an ordinary differential equation. The results are tabulated in table 1.

**Table 1.** Subalgebras of the optimal system, similarity variables and similarity reductions.

| no. | Subalgebra | Similarity variable $\xi$ | Similarity reduction |
|-----|------------|----------------------------|----------------------|
| 1   | $\Gamma_4 + \delta \Gamma_2 + \epsilon \Gamma_3$ | $\ln(\delta e^{b_1x} + 1) - \frac{b_1\phi}{3e} \ln t$ | $u = e^{b_1x}(\delta e^{b_1x} + 1)^{\frac{1}{b_1}}\phi(\xi)$ |
| 2   | $\Gamma_4 + \epsilon \Gamma_3$ | $e^{b_1x} - \frac{b_1}{3e} \ln t$ | $u = e^{\frac{b_1}{3e}x} + b_1x \phi(\xi)$ |
| 3   | $\Gamma_4 + \epsilon \Gamma_2$ | $t$ | $u = e^{\frac{b_1}{3e}x} \phi(\xi)$ |
| 4a  | $\Gamma_3$, ($n \neq \frac{3m}{2}$) | $x t^{\frac{m-\frac{n}{2}}{m}}$ | $u = t^{\frac{3m-3n}{2m}} \phi(\xi)$ |
| 4b  | $\Gamma_3$, ($n = \frac{3m}{2}$) | $t$ | $u = e^{\frac{b_1}{3e}x} \phi(\xi)$ |
| 5   | $\Gamma_3 + \epsilon \Gamma_1$, ($n = \frac{3m}{2}$) | $x e^{-\frac{m}{2}}$ | $u = e^{\frac{b_1}{3e}x} \phi(\xi)$ |
| 6   | $\Gamma_3 + \epsilon \Gamma_2$, ($n = m$) | $x + \frac{c}{m} \ln t$ | $u = t^{-\frac{m}{2}} \phi(\xi)$ |
| 7   | $\Gamma_1 + \epsilon \Gamma_2$ | $x - ct$ | $u = \phi(\xi)$ |
| 8   | $\Gamma_2$ | $t$ | $u = \phi(\xi)$ |

Here $c$, $\delta$, $\epsilon$ are arbitrary constants and $\delta \epsilon \neq 0$. In the first three subalgebras $n = m = -3$.

Similarity reductions tabulated in table 1 maps equation (5) into an ordinary differential equation. For example, entry 4a leads to the reduced equation

$$\phi - (n - m)\xi \psi = (3m - 2n) \left( \phi^n \phi_{\xi \xi} + a_1 \phi^{n-1} \phi_{\xi}^2 + b_1 \phi^m \phi_{\xi} \right) \xi,$$

entry 5 to

$$m \xi \phi - 2 \phi = 2e \left( \phi^{3m \xi \xi} + a_1 \phi^{3m-2} \phi_{\xi}^2 + b_1 \phi^m \phi_{\xi} \right) \xi,$$

entry 6 to

$$\phi - c \phi = m \left( \phi^n \phi_{\xi \xi} + a_1 \phi^{n-1} \phi_{\xi}^2 + b_1 \phi^m \phi_{\xi} \right) \xi,$$

and entry 7, which is the traveling wave solution, reduces (5) to

$$c \phi = \left( \phi^n \phi_{\xi \xi} + a_1 \phi^{n-1} \phi_{\xi}^2 + b_1 \phi^m \phi_{\xi} \right) \xi.$$

2.2. Potential symmetries

Equation (5) is written in conserved form and hence, by introducing a potential variable $v$, it can be written as a system,

$$v_x = u, \quad v_t = -u^n u_{xx} - a_1 u^{n-1} u_x^2 - b_1 u^m u_x.$$

We search for vector fields of the form

$$\Delta = \tau(t, x, u, v) \partial_t + \xi(t, x, u, v) \partial_x + \eta(t, x, u, v) \partial_u + \theta(t, x, u, v) \partial_v,$$

which generate one-parameter groups of point symmetry transformations of the system (20).

Any such symmetry induces a potential symmetry of the original equation (5) in the case where $(\tau_v, \xi_v, \eta_v) \neq (0, 0, 0)$. Otherwise, it projects into a point symmetry of the original equation.
We obtain potential symmetries in two cases. If \( n = 0, \ m = 1, \ a_1 = -1 \) and \( b_1 \) arbitrary, then system (20) admits the Lie symmetry
\[
\Delta_1 = e^{-b_1 v}(b_1 u \partial_u - \partial_v)
\]
which induces a potential symmetry for equation (5) and the rest of the symmetries project into Lie point symmetries for equation (5). If \( n = -3, \ m = -2, \ a_1 = -3 \) and \( b_1 \) arbitrary, we derive the following potential symmetries
\[
\Delta_1 = 9t \partial_t - b_1 vx \partial_x + u(b_1 ux + b_1 v + 3) \partial_x + (3v - 2b_1^2 t) \partial_v
\]
\[
\Delta_2 = \phi(t, v) \partial_x - \phi_v u^2 \partial_u,
\]
where \( \phi(t, v) \) satisfies the linear partial differential equation
\[
\phi_t + \phi_{vvv} + b_1 \phi_{vv} = 0.
\]
The infinite dimensional symmetry \( \Delta_2 \) suggests that in this case, system (20) can be linearized. In fact the mapping
\[
t' = t, \ x' = v, \ u' = \frac{1}{u}, \ v' = x
\]
connects the linear system
\[
v'_x = u', \ v'_t + u'_x + b_1 u'_x = 0
\]
and the nonlinear system
\[
v_x = u, \ v_t + u^{-3} u_{xx} - 3u^{-4} u_x^2 + b_1 u^{-2} u_x = 0.
\]
From the above mapping, we construct the one-to-one contact transformation
\[
dt' = dt, \ dx' = udx - \left[(u^{-3} u_x)_x + b_1 u^{-2} u_x\right] dt, \ u' = \frac{1}{u}
\]
which maps the linear partial differential equation
\[
u'_t + u'_{x't} + b_1 u'_{x't} = 0
\]
into the nonlinear equation
\[
u_t + \left[u^{-3} u_{xx} - 3u^{-4} u_x^{-2} + b_1 u_x^{-2}\right] x = 0.
\]

2.3. Non-classical reductions
We find reduction operators that cannot be obtained by Lie’s classical method. We require invariance of equation (5) in conjunction with its invariant surface condition (8), under the infinitesimal transformations generated by (7). The nonclassical reduction operators can be of regular (\( \tau = 1 \)) and of singular types (\( \tau = 0, \ \xi = 1 \)). The problem of finding singular reduction operators reduces to solving an initial partial differential equation and therefore this case is very difficult to be solved. It is known as the “no-go” case and is often omitted from consideration. See more about “no-go” case in [17–22]. Even in the case of regular nonclassical reduction operators the problem is still difficult. This is due to the fact that the system of determining equations for finding the coefficients of nonclassical reduction operators is a system of nonlinear partial differential equations.

Similar to the work in Ref. [4], we search for nonclassical operators for equation (5). It turns out that equation (5) admits a nonclassical reduction operator only in the case where
\( n = m = a_1 = -\frac{1}{2} \) and \( b_1 \) is an arbitrary constant. We find that the nonclassical operator has the form

\[ \Gamma = \partial_t + \phi(x)\sqrt{u}\partial_u, \quad (11) \]

where \( \phi(x) \) is a solution of the ordinary differential equation

\[ \frac{d^3\phi}{dx^3} + b_1\frac{d^2\phi}{dx^2} + \frac{1}{2}\phi^2 = 0. \quad (12) \]

Solving the corresponding characteristic system, we deduce that operator (11) produces the mapping

\[ u = \left[ \frac{1}{2}\phi(x)t + F(x) \right]^2 \quad (13) \]

that reduces (5) into the ordinary differential equation

\[ \frac{d^3F}{dx^3} + b_1\frac{d^2F}{dx^2} + \frac{1}{2}\phi F = 0. \quad (14) \]

The reduction mapping (13) can be interpreted as an ansatz with the two new unknown functions \( \phi(x) \) and \( F(x) \) which reduces (5) into the system of two ordinary differential equations (12) and (14).

### 2.4 Non-classical potential reductions

The idea is to search for nonclassical operators for the potential system (20). Equivalently, we can search for such operators for the potential form of (5), namely,

\[ v_t + v^n_x v_{xxx} + a_1 v^{-1}_x v_{xx}^2 + b_1 v_x^n v_{xx} = 0 \quad (15) \]

which can be obtained by eliminating the variable \( u \) in the potential system (20). It appears that it is easier to search for nonclassical operators for equation (15) instead of the potential system (20).

Here we derive two members of (15) that admit a nonclassical operator. In particular, equation

\[ v_t + v^{-1}_x v_{xxx} + b_1 v^{-1}_x v_{xx} = 0 \quad (16) \]

admits the operator

\[ \Delta_1 = \partial_t + c\partial_x + \phi(x + ct)\partial_v \quad (17) \]

and equation

\[ v_t + v^{-2}_x v_{xxx} - 3v^{-3}_x v_{xx}^2 + b_1 v^{-1}_x v_{xx} = 0 \quad (18) \]

admits the operator

\[ \Delta_2 = \partial_t + \phi(v + ct)\partial_x + c\partial_v. \quad (19) \]

In both operators, the function \( \phi(\omega) \) is a solution of the nonlinear third order ordinary differential equation

\[ \frac{d^3\phi}{d\omega^3} + b_1\frac{d^2\phi}{d\omega^2} + \phi\frac{d\phi}{d\omega} = 0. \]

Reduction (17) produces the travelling wave solution

\[ v(t, x) = \frac{1}{2c}\Phi(x + ct) + \Psi(x - ct), \]

where \( \Phi(x) \) and \( \Psi(x) \) are arbitrary functions.
where \( \Phi(\omega) \) and \( \Psi(\zeta) \) are, respectively, solutions of the nonlinear third order ordinary differential equations

\[
\frac{d^3\Phi}{d\omega^3} + b_1 \frac{d^2\Phi}{d\omega^2} + \frac{1}{2} \left( \frac{d\Phi}{d\omega} \right)^2 = 0, \quad \frac{d^3\Psi}{d\zeta^3} + b_1 \frac{d^2\Psi}{d\zeta^2} - c \left( \frac{d\Psi}{d\zeta} \right)^2 = 0.
\]

We point out that the two nonlinear equations (16) and (18) are connected by the pure hodograph transformation

\[ x \mapsto v, \quad u \mapsto x. \]

3. Analysis of equation (6)

3.1. Lie symmetries

For arbitrary values of the parameters \( n, m, a_1, a_2, b_1 \) and \( b_2 \), equation (6) admits three Lie symmetries:

\[ \Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = (3n - 4m)t\partial_t + (n - m)x\partial_x + u\partial_u. \]

In the case where \( n = m = -4, a_1 = -7, a_2 = 8, b_2 = -3b_1 \) and \( b_1 \) is arbitrary, equation (6) admits an additional fourth symmetry

\[ \Gamma_4 = e^{-b_1x}(\partial_x + b_1u\partial_u). \]

Similar to the previous section, we can construct the optimal system of subalgebras that lead to all Lie possible reductions of equation (6) to ordinary differential equations.

3.2. Potential symmetries

The potential system

\[
v_x = u, \quad v_t = -u^n u_{xxx} - a_1 u^{n-1} u_x u_{xx} - a_2 u^{n-2} u_x^3 - b_1 u^m u_{xx} - b_2 u^{m-1} u_x^2 \]

(20)

of equation (6) admits Lie symmetries that induce potential symmetries for (6) in four cases. In particular, we find the following results:

(i) \( (n, m, a_1, a_2, b_1, b_2) = (-4, -3, -10, 15, b_1, -3b_1) : \Delta = \phi(t, v)\partial_x - \phi_v u^2 \partial_u \) where \( \phi(t, v) \) is a solution of the linear equation \( \phi_t + \phi_{vvv} + b_1 \phi_{vuv} = 0 \).

(ii) \( (n, m, a_1, a_2, b_1, b_2) = (-4, -4, -10, 15, 0, b_2) : \Delta = v\partial_x - u^2 \partial_u \)

(iii) \( (n, m, a_1, a_2, b_1, b_2) = (0, 1, -3, 2, b_1, 0) : \Delta = e^{-b_1v}(b_1u\partial_u - \partial_v) \)

(iv) \( (n, m, a_1, a_2, b_1, b_2) = (0, 0, -4, 3, b_1, -3b_1) : \Delta = 2uv\partial_u + v^2 \partial_v \)

The infinite dimensional symmetry in the first case leads to the mapping

\[ t' = t, \quad x' = v, \quad u' = \frac{1}{u}, \quad v' = x \]

which connects the linear system

\[
v'_{x'} = u', \quad v'_{x'} + u'_{x'x'} + b_1 u'_{x'x'} = 0
\]

and the nonlinear system

\[
v_x = u, \quad v_t = -u^{-4} u_{xxx} + 10u^{-5} u_x u_{xx} - 15u^{-6} u_x^3 - b_1 u^{-3} u_{xx} + 3b_1 u^{-4} u_x^2.
\]

The above mapping, in turn, leads to one-to-one contact transformation that connects the corresponding equation (6) and the linear equation \( u_t + u_{xxx} + b_1 u_{xxx} = 0 \).
3.3. Non-classical reductions
Equation \( u_t = - \left[ u^{-\frac{1}{2}} u_{xxx} - \frac{3}{2} u^{-\frac{3}{2}} u_x u_{xx} + \frac{3}{4} u^{-\frac{5}{2}} u_x^2 + b_1 u^{-\frac{1}{2}} u_{xx} - \frac{1}{2} b_1 u^{-\frac{3}{2}} u_x^2 \right]_x \)
admits the nonclassical operator (11) that produces the ansatz (13), where the functions \( \phi(x) \) and \( F(x) \) are solutions of the system of ordinary differential equations
\[
\frac{d^4 \phi}{dx^4} + b_1 \frac{d^3 \phi}{dx^3} + \frac{1}{2} \phi^2 = 0, \quad \frac{d^4 F}{dx^4} + b_1 \frac{d^3 F}{dx^3} + \frac{1}{2} \phi F = 0.
\]

3.4. Non-classical potential reductions
The potential equation
\[
v_t + v_x^{-1} v_{xxxx} + b_1 v_x^{-1} v_{xxx} = 0 \tag{21}
\]
admits the nonclassical operator (17) and the potential equation
\[
v_t + v_x^{-3} v_{xxxx} - 10 v_x^{-4} v_{xxx} v_{xx} + 15 v_x^{-5} v_x^3 + b_1 v_x^{-2} v_{xxx} - 3 b_1 v_x^{-3} v_{xx}^2 = 0 \tag{22}
\]
admits the nonclassical operator (19), where \( \phi(\omega) \) is a solution of the ordinary differential equation
\[
\frac{d^4 \phi}{d\omega^4} + b_1 \frac{d^3 \phi}{d\omega^3} + \phi \frac{d\phi}{d\omega} = 0.
\]
Equations (21) and (22) are connected by the hodograph transformation \( x \mapsto v, \, v \mapsto x \).

4. Final remarks
In the spirit of generalizing the nonlinear diffusion equation to higher order equations (the fourth and sixth order known as thin film equations), we present a sequence of equations which has as first term the nonlinear diffusion convection equation (4). In the present work, we have considered the second and third terms of the sequence which are respectively, the third order nonlinear equation (5) and the fourth order (6). The next two terms of the sequence is the nonlinear fifth order equation
\[
u_t = - \left[ u^n u_{xxxx} + a_1 u^{n-1} u_x u_{xxx} + a_2 u^{n-1} u_x^2 u_{xx} + a_3 u^{n-2} u_x^2 u_{xx} + a_4 u^{n-3} u_x^4 \right]_x
\]
and the sixth order equation
\[
u_t = - \left[ u^n u_{xxxx} + a_1 u^{n-1} u_x u_{xxx} + a_2 u^{n-1} u_x^2 u_{xx} + a_3 u^{n-2} u_x^2 u_{xx} + a_4 u^{n-2} u_x u_{xx}^2 \\
+ a_5 u^{n-3} u_x^3 u_{xx} + a_6 u^{n-4} u_x^5 + b_1 u^m u_{xxxx} + b_2 u^{m-1} u_x u_{xxx} + b_3 u^{m-1} u_x^2 \right]_x.
\] (23)

The corresponding equations with \( b_i = 0 \), have been studied in [3]. Similar to the analysis presented for equations (5) and (6), we can derive the corresponding results for equations (23) and (24) or for the corresponding higher order equations. For example, for arbitrary values of the parameters, the p-th order equation, term of the sequence, admits three Lie symmetries
\[
\Gamma_1 = \partial_t, \quad \Gamma_2 = \partial_x, \quad \Gamma_3 = [(p-1)n - pm]t \partial_t + (n - m)x \partial_x + u \partial_u.
\]
An additional Lie symmetry exists for specific values of the parameters $m$, $n$, $a_i$ and $b_i$. In the case of fifth order these values are

$$(n, m, a_1, a_2, a_3, a_4, b_1, b_2, b_3) = (-5, -5, -11, -7, 59, -48, b_1, -10b_1, 15b_1)$$

and for the sixth order are

$$(n, m, a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5) = (-6, -6, -16, -25, 125, 160, -605, 384, b_1, -15b_1, -10b_1, 105b_1, -105b_1).$$

We note that equation of the class (5) that admits nonclassical reduction can be written in the form

$$\frac{\partial u}{\partial t} = -2 \left( \frac{\partial^3}{\partial x^3} \sqrt{u} + b_1 \frac{\partial^2}{\partial x^2} \sqrt{u} \right)$$

and the corresponding equation from the class (6) has the form

$$\frac{\partial u}{\partial t} = -2 \left( \frac{\partial^4}{\partial x^4} \sqrt{u} + b_1 \frac{\partial^3}{\partial x^3} \sqrt{u} \right).$$

Based on these observations, we present the following result: The nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = -2 \left( \frac{\partial^p}{\partial x^p} \sqrt{u} + b_1 \frac{\partial^{p-1}}{\partial x^{p-1}} \sqrt{u} \right), \quad p \geq 2$$

has a solution of the form

$$u(x, t) = \left[ \frac{1}{2} \phi(x) t + F(x) \right]^2$$

where the functions $\phi(x)$ and $F(x)$ are solutions of the system of ordinary differential equations

$$\frac{d^p \phi}{dx^p} + b_1 \frac{d^{p-1} \phi}{dx^{p-1}} + \frac{1}{2} \phi^2 = 0, \quad \frac{d^p F}{dx^p} + b_1 \frac{d^{p-1} F}{dx^{p-1}} + \frac{1}{2} \phi F = 0.$$

Similarly, the results on non-classical potential reductions can be generalized. The p-th order nonlinear equation

$$\frac{\partial v}{\partial t} + \frac{1}{2} \frac{\partial^p v}{\partial x^p} + b_1 \frac{1}{2} \frac{\partial^{p-1} v}{\partial x^{p-1}} = 0$$

admits the nonclassical operator (17), where $\phi(\omega)$ is a solution of the ordinary differential equation

$$\frac{d^p \phi}{d\omega^p} + b_1 \frac{d^{p-1} \phi}{d\omega^{p-1}} + \phi \frac{d \phi}{d\omega} = 0.$$

The equivalent equation which connected to the above equation by the pure hodograph transformation admits the nonclassical operator (19).

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