Massive dual gauge field and confinement in Minkowski space: Magnetic charge

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Gauge field configuration for a magnetic monopole and its dual configuration are studied in SU(2) gauge theory. We present a relation between the monopole field and its dual field. Since these fields can become massive, their massive Lagrangians are derived. In the dual case, an additional term appears. We show this term is necessary to produce a linear potential between a monopole charge and an antimonopole charge.

Subject Index B0,B3,B6
1 Introduction

Magnetic monopoles are considered to play an important role in quark confinement. Models based on the dual superconductor require monopoles and their condensation (see, e.g., [1]). In the Zwanziger’s formulation [2], two gauge fields, namely the usual gauge field and the dual gauge field, are used. To describe monopole condensation, a monopole field is introduced. Its vacuum expectation value (VEV) makes the dual gauge field massive. This massive field leads to a linear potential between a quark and an anti-quark.

In the extended QCD [3], a unit color vector $\hat{n}^A(x)$ in the internal space [4, 5] appears. Non-Abelian magnetic potential defined by $C^A_\mu = -(\hat{n} \times \partial_\mu \hat{n})^A/g$ describes the non-Abelian monopole [6]. In Ref. [7], we studied the SU(2) gauge theory in a nonlinear gauge, and derived the extended QCD with the massive magnetic potential $C^A_\mu$. In this paper, we study the relation between the Abelian magnetic potential and its dual potential. Based on this relation, we consider the confinement of magnetic charges.

In Sect. 2, we introduce the dual magnetic potential for magnetic monopoles. Next, by using the dual magnetic potential, we rewrite the Abelian part of the SU(2) Lagrangian with the monopole field. Massless magnetic potential is considered in Sect. 3, and massive magnetic potential is studied in Sect. 4. In Sect. 5, SU(2) gauge theory in the low energy region is studied. Because of the ghost condensation [8, 9] and the condensate $\langle A^+_\mu A^-_\mu \rangle$, the massive magnetic potential appears in this region [7]. Applying the result in Sect. 4, we obtain the low energy SU(2) Lagrangian with the dual magnetic potential. In Sect. 6, using the Lagrangian in Sect. 5, it is shown that the static potential between a magnetic monopole and an anti-monopole is a linear confining potential. Section 7 is devoted to summary. In Appendix A, notations are summarized. For both the massless case and the massive case, the monopole solutions and their dual solutions for a static magnetic charge are given in Appendix B. We also show that the relation between the Abelian magnetic potential and its dual potential, which is presented in the Sect. 2, is satisfied for the Dirac monopole. To obtain the Lagrangian in Sect. 5, the ghost condensation is necessary. In Ref. [10], we have shown that it happens in Euclidean space. In Appendix C, we show it in Minkowski space.

2 Dual magnetic potential

Let us consider a space-like gauge field $\tilde{C}_\mu$ which satisfies the equation of motion

$$\partial^\nu H_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} \partial_\nu \frac{n^\alpha}{n^\rho \partial_\rho} k^\beta, \quad H_{\mu\nu} = (\partial \wedge \tilde{C})_{\mu\nu}, \quad (2.1)$$
where \((\partial \wedge \tilde{C})_{\mu\nu} = \partial_\mu \tilde{C}_\nu - \partial_\nu \tilde{C}_\mu\), the space-like vector \(n^\alpha\) satisfies \(n_\alpha n^\alpha = -1\), and the magnetic current \(k^\beta\) satisfies \(\partial_\beta k^\beta = 0\). We call \(\tilde{C}_\mu\) magnetic potential. The dual field strength is defined by
\[
\mathcal{H}^{\mu\nu} = \frac{1}{2} \epsilon^{\nu\alpha\beta} H_{\alpha\beta} = \epsilon^{\nu\alpha\beta} \partial_\alpha \tilde{C}_\beta. \tag{2.2}
\]
Now we introduce a dual magnetic potential \(C_\mu\). Using the formula \(\epsilon^{\alpha\beta\rho\sigma} \epsilon_{\rho\sigma\mu\nu} = -2(\delta_\alpha^\mu \delta_\beta^\nu - \delta_\alpha^\nu \delta_\beta^\mu)\), we obtain
\[
H_{\mu\nu} = -H_{\rho\sigma} H^{\rho\sigma}. \tag{2.3}
\]
Therefore, if we define \(C_\mu\) as \(H_{\mu\nu} = (\partial \wedge C)_{\mu\nu}\), the kinetic term for \(C_\mu\) has the wrong sign \[4\]. We change this relation to
\[
H_{\mu\nu} = (\partial \wedge C)_{\mu\nu} + \Lambda_{\mu\nu}. \tag{2.4}
\]
If we impose the conditions
\[
\partial_\mu H_{\mu\nu} = \epsilon^{\nu\alpha\beta} \partial_\mu \partial_\alpha \tilde{C}_\beta = 0, \quad \partial_\nu H_{\mu\nu} = \epsilon^{\mu\alpha\beta} \partial_\nu \partial_\alpha \tilde{C}_\beta = 0, \tag{2.5}
\]
\(\Lambda^{\mu\nu}\) must satisfy
\[
\partial_\mu (\partial \wedge C)_{\mu\nu} + \partial_\mu \Lambda^{\mu\nu} = 0, \quad \partial_\nu (\partial \wedge C)_{\mu\nu} + \partial_\nu \Lambda^{\mu\nu} = 0. \tag{2.6}
\]
Eq. (2.6) holds if we choose
\[
\Lambda^{\mu\nu} = -\frac{n^\mu}{n^\rho \partial_\rho} \partial_\sigma (\partial \wedge C)^{\sigma\nu} + \frac{n^\nu}{n^\rho \partial_\rho} \partial_\sigma (\partial \wedge C)^{\sigma\mu}. \tag{2.7}
\]
Thus we obtain the relation
\[
\mathcal{H}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} \partial_\alpha \tilde{C}_\beta = (\partial \wedge C)_{\mu\nu} - \frac{n^\mu}{n^\rho \partial_\rho} \partial_\sigma (\partial \wedge C)^{\sigma\nu} + \frac{n^\nu}{n^\rho \partial_\rho} (\partial_\sigma (\partial \wedge C)^{\sigma\mu}. \tag{2.8}
\]
From Eq. (2.8), we can write \(H^{\mu\nu}\) as
\[
H^{\mu\nu} = (\partial \wedge \tilde{C})^{\mu\nu} = h^{\mu\nu}_1 + h^{\mu\nu}_2, \\
h^{\mu\nu}_1 = -\epsilon^{\mu\nu\alpha\beta} \partial_\alpha C_\beta, \quad h^{\mu\nu}_2 = \epsilon^{\mu\nu\alpha\beta} \frac{n_\alpha}{n^\rho \partial_\rho} \partial_\sigma (\partial \wedge C)^{\sigma\beta} = -\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \Lambda_{\alpha\beta}. \tag{2.9}
\]
This expression will be used in the following sections.

\[1\] We employ the metric \(g_{\mu\nu} = \text{diag}(1, -1, -1, -1)\), and the anti-symmetric \(\epsilon\) symbol with \(\epsilon^{0123} = 1\). The formulae related to \(\epsilon^{\mu\nu\alpha\beta}\) are summarized in Appendix A.
We note $H_{\mu\nu}$ is invariant under the transformations

$$\tilde{C}_\mu \to \tilde{C}_\mu + \partial_\mu \varepsilon, \quad C_\mu \to C_\mu + \partial_\mu \vartheta.$$ 

If we choose the gauges

$$n_\mu \tilde{C}^\mu = 0, \quad n_\mu C^\mu = 0,$$ 

Eq. (2.9) is solved as

$$\tilde{C}^\nu = \delta^\nu_{\alpha\beta} n_\mu \partial_\alpha n_\rho \partial_\rho C^\beta.$$ 

In Appendix B, as an example, $\tilde{C}_\mu$ and $C^\mu$ for a static magnetic charge $k^\beta \propto \delta^\beta_0 \delta(x) \delta(y) \delta(z)$ are presented. The magnetic potential $\tilde{C}_\mu$ is the Dirac monopole and its dual potential $C^\mu$ is the Coulomb potential. Eqs. (2.5) and (2.8) are fulfilled by these potentials, and the term $\Lambda^{\mu\nu}$ represents the Dirac string.

3 Abelian part of the SU(2) gauge theory

We consider the SU(2) gauge theory with structure constants $f^{ABC}$. Using the notations

$$F \cdot G = F^A G^A, \quad (F \times)^{AB} = f^{ACB} F^C, \quad (F \times G)^A = f^{ABC} F^B G^C, \quad A = 1, 2, 3,$$

the Lagrangian

$$L_{\text{inv}}(A) = -\frac{1}{4} G_{\mu\nu} \cdot G^{\mu\nu}, \quad G^A_{\mu\nu} = (\partial \wedge A^A)_{\mu\nu} + g(A_\mu \times A_\nu)^A$$

contains the Abelian part

$$L_{\text{Abel}}(A) = -\frac{1}{4} (\partial \wedge A^3)_{\mu\nu} (\partial \wedge A^3)^{\mu\nu}.$$ 

To incorporate the magnetic potential, we divide the gauge field $A^3_\mu$ into a classical part $b^A_\mu$ and a quantum part $a^A_\mu$ as

$$A^A_\mu = b^A_\mu + a^A_\mu, \quad b^A_\mu = \tilde{C} \delta^A_3.$$ 

For simplicity, we use the notation $(F + H)_{\mu\nu}(F + H)^{\mu\nu} = (F + H)^2$. Then, $L_{\text{Abel}}(A)$ becomes

$$L_{\text{Abel}}(b + a) = -\frac{1}{4} (F + H)^2, \quad F_{\mu\nu} = (\partial \wedge a^3)_{\mu\nu}, \quad H_{\mu\nu} = (\partial \wedge \tilde{C})_{\mu\nu}.$$ 

Next we introduce the magnetic current $k^\beta$. To reproduce the equation of motion (2.1), we consider the Lagrangian

$$L'_{\text{Abel}}(b + a) = -\frac{1}{4} (F + H + h_3)^2, \quad h_3^{\mu\nu} = -\epsilon^{\mu\nu\alpha\beta} \frac{n_\alpha}{n_\rho \partial_\rho} k_\beta.$$ 

\[2\] The field strength $H + h_3$ is the Zwanziger’s field strength $F = (\partial \wedge A) - (n \cdot \partial)^{-1} (n \wedge j_\beta)^d$ in Ref. [2].
Then, using Eq. (2.9), we rewrite Eq. (3.5) as

\[ \mathcal{L}'_{\text{Abel}}(b + a) = -\frac{1}{4}(F + h_2 + h_3)^2 - \frac{1}{4}h_{1\mu\nu}(h_1^{\mu\nu} + 2h_2^{\mu\nu} + 2h_3^{\mu\nu}), \]  

(3.6)

where \( \int dx F_{\mu\nu}h_1^{\mu\nu} = 0 \) has been used. As we stated in Eq. (2.3), the part

\[-\frac{1}{4}h_{1\mu\nu}h_1^{\mu\nu} = \frac{1}{4}(\partial \wedge C)^2 \]  

(3.7)

gives the kinetic term with the wrong sign. However, using the current conservation \( \partial_\mu k^\mu = 0 \), we obtain

\[-\frac{1}{4}h_{1\mu\nu}(2h_2^{\mu\nu} + 2h_3^{\mu\nu}) = -\frac{1}{2}(\partial \wedge C)^2 - C_\mu k^\mu. \]  

(3.8)

Thus we find the cross term \( 2h_1^{\mu\nu}h_2^{\mu\nu} \) changes the sign of the kinetic term for \( C_\mu \), \(^3\) and \(-\frac{1}{4}h_{1\mu\nu}(h_1^{\mu\nu} + 2h_2^{\mu\nu}) \) yields the correct kinetic term. Thus we obtain

\[ \mathcal{L}'_{\text{Abel}}(b + a) = -\frac{1}{4}(F + h_2 + h_3)^2 - \frac{1}{4}(\partial \wedge C)^2 - C_\mu k^\mu, \]  

(3.9)

Now we neglect the quantum part \( F_{\mu\nu} \). Then the classical solution \( C_\nu \) must satisfy the equation of motion

\[-\frac{1}{2}(h_2^{\mu\rho} + h_3^{\mu\rho}) \frac{\delta h_{2\mu\nu}}{\delta C^\nu} + \partial_\mu(\partial \wedge C)_{\mu\nu} - k_\nu = 0. \]  

(3.10)

However Eq. (3.10) is satisfied by

\[ \partial_\mu(\partial \wedge C)_{\mu\nu} - k_\nu = 0, \]  

(3.11)

because Eq. (3.11) leads to \( h_2^{\mu\nu} + h_3^{\mu\nu} = 0 \). If we insert \( h_2^{\mu\nu} + h_3^{\mu\nu} = 0 \) into the Lagrangian (3.9), the term \( h_2^{\mu\nu} \), which is related to the Dirac string \( \Lambda_{\alpha\beta} \), disappears.

### 4 Massive Abelian part of the SU(2) gauge theory

In the previous paper \(^2\), we have shown that there appears the mass terms

\[ \mathcal{L}_m = \frac{m^2}{2} \left( 2a_\mu^3 \tilde{C}^\mu + \tilde{C}_\mu \tilde{C}^\mu \right), \]  

(4.1)

where the mass squared \( m^2 \) is defined in Eq. (5.5), and the derivation of \( \mathcal{L}_m \) is explained briefly in Sect. 5. Using Eq. (2.11) and integration by parts, the first term becomes

\[ \int dx m^2 a_\mu^3 \tilde{C}^\mu = -\frac{1}{2} \int dx F_{\mu\nu}h_4^{\mu\nu}, \quad h_4^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} \frac{n_\alpha}{n_\rho \partial_\rho} m^2 C_\beta, \]  

(4.2)

\(^3\) Similar result is found in the study of the interaction energy of two magnetic monopoles. The cross term of the Coulomb part and the string part changes the sign of the energy of the Coulomb interaction \(^1\).
and the second term becomes

$$\int dx \frac{m^2}{2} \tilde{C}_\mu \tilde{C}^\mu = \int dx \frac{m^2}{2} \left[ C_\mu C^\mu + \frac{n_\beta}{n^\rho \partial_\rho} C_\mu (\Box \delta_\mu - \partial_\mu \partial_\nu) \frac{n_\beta}{n^\sigma \partial_\sigma} C^\nu \right], \quad (4.3)$$

where Eq.(A3) and the gauge condition $n_\nu C^\nu = 0$ have been used.

If we apply Eq.(A2) and $n_\nu C^\nu = 0$, Eq.(4.3) is rewritten as

$$\int dx \left( \frac{m^2}{2} C_\mu C^\mu - \frac{1}{4} h^{\mu\nu}_2 h_{4\mu\nu} \right). \quad (4.4)$$

Now, combining the kinetic term (3.5) with the mass term (4.1), we consider the Lagrangian

$$L_{mAbel}(b + a) = L'_{Abel} + L_m. \quad (4.5)$$

Applying Eqs.(3.9), (4.2) and (4.4), it becomes

$$L_{mAbel}(b + a) = -\frac{1}{4} (F + h_2 + h_3 + h_4)^2 - \frac{1}{4} (\partial \wedge C)^2 + \frac{m^2}{2} C_\mu C^\mu - C_\mu k^\mu + \Omega, \quad (4.6)$$

where Eq.(4.6) is satisfied by the equation of motion

$$-\frac{1}{2} \left( h^\mu_2 + h^\mu_3 + h^\mu_4 \right) \frac{\delta h^\mu_2}{\delta C^\nu} + \left( \Box + m^2 \right) C^\nu - \partial_\nu \partial_\mu C^\mu - k_\nu = 0. \quad (4.7)$$

Eq.(4.6) is satisfied by the equation of motion

$$\left[ (\Box + m^2) \delta_\nu^\mu - \partial_\nu \partial_\mu \right] C^\nu - k^\mu = 0, \quad (4.7)$$

because Eq.(4.7) leads to $h^{\mu\nu}_2 + h^{\mu\nu}_3 + h^{\mu\nu}_4 = 0$. For a static magnetic charge, a solution of Eq.(4.7) is presented in Appendix B.

We note, using Eq.(A2), $\Omega$ is rewritten as

$$\Omega = C^\mu \frac{m^2}{2} \frac{n_\alpha n^\alpha}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n^\sigma} \right) \left[ \partial_\lambda (\partial \wedge C)^{\lambda\nu} + m^2 C^\nu - 2 k^\nu \right]. \quad (4.8)$$
SU(2) gauge theory in the low energy region

5.1 Derivation of the massive magnetic potential

In this subsection, we review the derivation of the Lagrangian with the massive magnetic potential \[ \text{[7]} \]. Let us consider the Lagrangian

\[
\mathcal{L}(b + a) = \mathcal{L}_{\text{inv}}(b + a) + \mathcal{L}_\varphi(a, b),
\]

where \( \mathcal{L}_{\text{inv}}(A) \) is defined in Eq.\([3, 4]\), and \( A_\mu^A = b_\mu^A + a_\mu^A \). In the background covariant gauge, a gauge-fixing part is chosen as

\[
\mathcal{L}_\varphi(a, b) = \frac{\alpha_1}{2} B \cdot B + B \cdot [D_\mu(b)a^\mu + \varphi - w] + \frac{i\epsilon}{2} [D_\mu(b)D^\mu(b + a) + g\varphi \times c] - \frac{\varphi \cdot \varphi}{2\alpha_2},
\]

where \( D_\mu(A) = \partial_\mu + gA_\mu \times \), \( \varphi^A \) is a field in the adjoint representation, and \( w^A \) is a constant. If \( \varphi^A \) is integrated out, Eq.\([5,2]\) gives

\[
\mathcal{L}_{\text{NL}} = B \cdot D_\mu(b)a^\mu + i\epsilon \cdot [D_\mu(b)D^\mu(b + a)c] + \frac{\alpha_1}{2} B \cdot B + \frac{\alpha_2}{2} B \cdot \bar{B} - B \cdot w,
\]

where \( \alpha_1 \) and \( \alpha_2 \) are gauge parameters, and \( \bar{B} = -B + ig\bar{c} \times c \). Namely Eq.\([5,1]\) gives the Lagrangian in the nonlinear gauge \([12]\) with the constant \( w^A \). Although \( \varphi^A \) is the auxiliary field which represents \( \alpha_2 \bar{B}^A \), because of the quartic ghost interaction \( \frac{\alpha_2}{2} \bar{B} \cdot \bar{B} \), it acquires the VEV \( \varphi^A = \varphi_0 \delta^A_3 \) under the energy scale \( \mu_0 = \Lambda e^{-4\pi^2/(\alpha_2 g^2)} \) \([13]\), where \( \Lambda \) is a momentum cut-off. \(^4\) Substituting \( \varphi^A(x) = \varphi_0 \delta^A_3 + \tilde{\varphi}^A(x) \) into Eq.\([5,2]\), we obtain

\[
\mathcal{L}_\varphi(a, b) = \frac{\alpha_1}{2} B \cdot B + B \cdot [D_\mu(b)a^\mu + \tilde{\varphi}] + \frac{i\epsilon}{2} [D_\mu(b)D^\mu(b + a) + v \times g\tilde{\varphi} \times c] - \frac{(v + g\tilde{\varphi}) \cdot (v + g\tilde{\varphi})}{2\alpha_2 g^2},
\]

where \( v = g\varphi_0 \), and to preserve the BRS symmetry, \( w^A \) is chosen as \( w^A = \varphi_0 \delta^A_3 \) \([14, 15]\). Since \( \varphi_0 \delta^A_3 \) selects the unbroken U(1) direction, we incorporate the Abelian monopole as \( b_\mu^A = \tilde{C}_\mu \delta^A_3 \) \([16]\).

When \( v \neq 0 \), it is known that ghost loops bring about tachyonic gluon masses \([10, 17]\). In Ref.\([7]\), we have shown that the ghost determinant \( \det[D_\mu(b)D^\mu(b + a) + v \times] \) yields the tachyonic mass terms

\[
- \frac{m^2}{2}[a_\mu^+ a_-^\mu + a_\mu^3 a_\mu^3], \quad m^2 = \frac{g^2 v}{32\pi},
\]

where \( a_\mu^+ a_-^\mu = a_\mu^a a^a_\mu / 2 \) \( (a = 1, 2) \). We note, contrary to the quantum part \( a_\mu^A, b_\mu^A \) does not have tachyonic mass.

\(^4\) In Ref.\([10]\), the ghost condensation \( \varphi_0 \neq 0 \) is shown in Euclidean space. In Appendix C, we explain that this phenomenon happens in Minkowski space as well.
To avoid the tachyonic masses, we introduced the source term $M^2a^+a^{-\mu}$ into the Lagrangian, and constructed the effective potential for $\Phi = \langle a^+_\mu a^{-\mu} \rangle$. However, at the lowest order, we can obtain the effective potential by the following simple procedure. First add the tachyonic mass terms (5.5) and the source term to the Lagrangian. Next replace $a^+_\mu a^{-\mu}$ to $\Phi + a^+_\mu a^{-\mu}$. Thus the terms which contain $\Phi$, $m^2$, or $M^2$ are
\begin{align*}
-V(\Phi) &= -g^2\Phi + \frac{m^2}{2} - M^2 \right) a^+_\mu a^{-\mu} - \left(g^2\Phi + \frac{m^2}{2}\right) a^+_\mu a^{-\mu} - g^2\Phi[2b^3a^3\mu + b^3b^3\mu], \quad (5.6) \\
V(\Phi) &= \frac{1}{2}(g^2\Phi^2 + m^2\Phi).
\end{align*}

The interaction term
\begin{equation*}
-g^2\left[ (b + a)_{\mu} \times (b + a)_{\nu} \right] \cdot [(b + a)^{\mu} \times (b + a)^{\nu}]
\end{equation*}
in $L_{\text{inv}}(b + a)$ becomes
\begin{equation*}
-\frac{g^2}{2}(a^+_\mu a^{-\mu})^2 - g^2(a^+_\mu a^{-\mu})[(b^3_{\nu} + a^3_{\nu})(b^{3\nu} + a^{3\nu})] + \frac{g^2}{2}(a^+_{\mu}a^{-\nu})(a^-_{\nu}a^{-\nu}) + g^2[a^+_{\mu}(b^3_{\nu} + a^3_{\nu})][a^-_{\nu}(b^3_{\nu} + a^3_{\nu})]. \quad (5.7)
\end{equation*}
The factors $g^2\Phi^2$ and $g^2\Phi$ in Eq.(5.6) come from the first and the second terms in Eq.(5.7). From the minimum of $V(\Phi)$, we have the VEV
\begin{equation*}
\Phi = -\frac{m^2}{2g^2}, \quad (5.8)
\end{equation*}
and Eq.(5.6) becomes
\begin{equation*}
M^2a^+_\mu a^{\mu} + \frac{m^2}{2}[2a^3_{\mu}\tilde{C}^\mu + \tilde{C}_\mu\tilde{C}^\mu]. \quad (5.9)
\end{equation*}
If we write a tree Lagrangian for $a^+_\mu$ as $a^+_\mu(\Delta^{\mu\nu} + M^2g^{\mu\nu})a^-_{\nu}$, $M^2$ is determined by the equation
\begin{equation*}
-\frac{m^2}{2g^2} = i\langle x|\text{tr} \left( \Delta + M^2 \right)^{-1}|x \rangle.
\end{equation*}
Namely, although the component $a^3_{\mu}$ is massless, the components $a^\pm_{\mu}$ have mass $M$. The $m^2$ part in Eq.(5.9) is $\mathcal{L}_m$ in Eq.(1.1).

We summarize the result. After the ghost condensation $v = g\varphi_0 \neq 0$, we can introduce the magnetic potential $\tilde{C}_\mu$ and the VEV $\Phi = \langle a^+_\mu a^{-\mu} \rangle = -m^2/(2g^2)$. The Lagrangian $L_{\text{inv}}$ gives
\begin{equation*}
\tilde{L}_{\text{inv}} = -\frac{1}{4}(F + H)^2 + \frac{m^2}{2}[2a^3_{\mu}\tilde{C}^\mu + \tilde{C}_\mu\tilde{C}^\mu] + M^2a^+_\mu a^{-\mu} - \frac{g}{2}(F_{\mu\nu} + H_{\mu\nu})(a^\mu \times a^\nu)^3 - \frac{g^2}{4}(a^\mu \times a^\nu)^3(a^\mu \times a^\nu)^3 - \frac{1}{4}(\hat{D}_\mu a^\nu - \hat{D}_\nu a^\mu)(\hat{D}^\mu a^\nu - \hat{D}^\nu a^\mu)a, \quad (5.10)
\end{equation*}
where $(\hat{D}_\mu a^\nu)^a = (\partial_\mu a^\nu + gA^3_{\mu} \times a^\nu)^a$ with $a = 1, 2$. 
5.2 Lagrangian with the magnetic potential $\tilde{C}_\mu$

In the previous paper [11], to remove the string, we performed the singular gauge transformation [16]. Then the Lagrangian with the massive non-Abelian magnetic potential was obtained. However, since we want to use the dual magnetic potential in this paper, we introduce the magnetic current $k^\beta$ as in Eq. (3.3). Namely replacing $H^{\mu\nu}$ with $H^{\mu\nu} + h^{\mu\nu}_3$, Eq. (5.10) gives

$$\tilde{\mathcal{L}}'_\text{inv} = \mathcal{L}'\text{Abel} + \mathcal{L}_m + M^2a_\mu^+a^-\mu - \frac{g}{2}(a_\mu \times a_\nu)^3(F^{\mu\nu} + H^{\mu\nu} + h^{\mu\nu}_3)$$

$$- \frac{g^2}{4}(a_\mu \times a_\nu)^3(a^\mu \times a^\nu)^3 - \frac{1}{4}({\tilde{D}}_\mu a_\nu - {\tilde{D}}_\nu a_\mu)^a({\tilde{D}}^\mu a^\nu - {\tilde{D}}^\nu a^\mu)^a,$$

$$\mathcal{L}'\text{Abel} + \mathcal{L}_m = -\frac{1}{4}(F + H + h_3)^2 + \frac{m^2}{2} \left(2a_\mu^3\tilde{C}^\mu + \tilde{C}_\mu C^\mu\right).$$

The classical solution $\tilde{C}_\mu$ satisfies the equation of motion

$$\partial_\mu(\partial \wedge \tilde{C})^{\mu\nu} + m^2\tilde{C}^{\mu\nu} - \epsilon^{\nu\alpha\mu\beta}\frac{n_\alpha n_\beta}{n^\rho n_\rho} k_\beta = 0. \quad (5.11)$$

If we use it, the linear terms on $a_\mu^3$ disappear, and $\tilde{\mathcal{L}}'_\text{inv}$ becomes

$$\tilde{\mathcal{L}}'_\text{inv} = -\frac{1}{4}F^2 - \frac{1}{4}(\partial \wedge \tilde{C} + h_3)^2 + \frac{m^2}{2}\tilde{C}_\mu \tilde{C}^\mu + M^2a_\mu^+a^-\mu - \frac{g}{2}(a_\mu \times a_\nu)^3(F^{\mu\nu} + H^{\mu\nu} + h^{\mu\nu}_3)$$

$$- \frac{g^2}{4}(a_\mu \times a_\nu)^3(a^\mu \times a^\nu)^3 - \frac{1}{4}({\tilde{D}}_\mu a_\nu - {\tilde{D}}_\nu a_\mu)^a({\tilde{D}}^\mu a^\nu - {\tilde{D}}^\nu a^\mu)^a. \quad (5.12)$$

This is the low energy effective Lagrangian with the magnetic potential $\tilde{C}_\mu$. For a static magnetic charge, a solution $\tilde{C}_\mu$ which satisfies Eq. (5.11) is presented in Appendix B.

5.3 Lagrangian with the dual magnetic potential $C_\mu$

In Sect. 4, using the dual potential $C_\mu$, we have shown that the Lagrangian $\mathcal{L}_{\text{mAbel}} = \mathcal{L}'\text{Abel} + \mathcal{L}_m$ is written as in Eq. (4.3). Since $C_\mu$ satisfies Eq. (4.7), the equality $h^{\mu\nu}_2 + h^{\mu\nu}_3 + h^{\mu\nu}_4 = 0$ holds. We apply this equality to $\mathcal{L}_{\text{mAbel}}$ and $(a_\mu \times a_\nu)^3(H^{\mu\nu} + h^{\mu\nu}_3)$. Thus Eq. (5.12) is rewritten as

$$\tilde{\mathcal{L}}'_\text{inv} = -\frac{1}{4}F^2 - \frac{1}{4}(\partial \wedge C)^2 + \frac{m^2}{2}C_\mu C^\mu - C_\mu k^\mu + \Omega$$

$$- \frac{1}{4}({\tilde{D}}_\mu a_\nu - {\tilde{D}}_\nu a_\mu)^a({\tilde{D}}^\mu a^\nu - {\tilde{D}}^\nu a^\mu)^a + M^2a_\mu^+a^-\mu$$

$$- \frac{g}{2}(a_\mu \times a_\nu)^3(F^{\mu\nu} + h^{\mu\nu}_1 - h^{\mu\nu}_4) - \frac{g^2}{4}(a_\mu \times a_\nu)^3(a^\mu \times a^\nu)(5.13)$$
where, using Eq. (2.11), \((\hat{D}_\mu a_\nu)^a\) becomes
\[
(\hat{D}_\mu a_\nu)^a = \partial_\mu a_\nu^a + gf^{a3b} \left( a_\mu^a + \epsilon_{\mu\alpha\beta}\frac{n^\alpha}{n^\rho} \frac{\partial_\rho C^\lambda}{n^\sigma} \right) a_\nu^b.
\]

Eq. (5.13) is the low energy effective Lagrangian with the dual magnetic potential \(C_\mu\).

6 Magnetic charge confinement

6.1 The use of the Lagrangian (5.12)

The classical part of the Lagrangian (5.12) is
\[
-\frac{1}{4}(\partial \wedge \tilde{C} + h_3)^2 + \frac{m^2}{2} \tilde{C}^\mu \tilde{C}_\mu = \frac{1}{2} \tilde{C}^\mu (D_m^{-1})_{\mu\nu} \tilde{C}^\nu - \tilde{C}^\mu K_\mu
- \frac{1}{2} k_\mu \frac{n_\alpha n_\beta}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n_\sigma} \right) k^\nu,
\]
where
\[
(D_m^{-1})_{\mu\nu} = g_{\mu\nu}(\Box + m^2) - \partial_\mu \partial_\nu, \quad K_\mu = \epsilon_{\mu\alpha\beta} \frac{n_\alpha}{n^\rho} \frac{\partial_\rho k^\beta}{n^\sigma}.
\]
and the last term comes from \(-h_3^2/4\). The equation (6.1) can be written as
\[
\frac{1}{2} \left( \tilde{C}^\mu - K_\sigma D_{\mu\sigma}^\mu \right) (D_m^{-1})_{\mu\nu} \left( \tilde{C}^\nu - D_{\beta\nu}^\beta K_\beta \right) = \frac{1}{2} K_\mu D_{\mu\nu}^\mu K_\nu - \frac{1}{2} k_\mu \frac{n_\alpha n_\beta}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n_\sigma} \right) k^\nu,
\]
where
\[
D_{\mu\nu}^\mu = \frac{g_{\mu\nu}}{\Box + m^2} + \frac{\partial_\mu \partial_\nu}{m^2(\Box + m^2)}.
\]
If we apply Eq. (5.11), the first term in Eq. (6.3) vanishes. Using Eq. (A3), \(\partial_\nu K_\nu = 0\) and the current conservation \(\partial_\mu k^\mu = 0\), we find the second term in Eq. (6.3) becomes
\[
-\frac{1}{2} K_\mu D_{\mu\nu}^\nu K_\nu = -\frac{1}{2} K_\mu \frac{1}{\Box + m^2} K_\mu
- \frac{1}{2} k_\mu \frac{1}{\Box + m^2} k^\nu + \frac{1}{2} k_\nu \frac{n_\beta}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n_\sigma} \right) k^\nu.
\]
As \(\Box = \Box + m^2 - m^2\), Eq. (6.5) is rewritten as
\[
-\frac{1}{2} K_\mu D_{\mu\nu}^\nu K_\nu = -\frac{1}{2} k_\mu \frac{1}{\Box + m^2} k^\mu - \frac{1}{2} k_\mu \frac{m^2}{\Box + m^2} \frac{n_\beta}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n_\sigma} \right) k^\nu
+ \frac{1}{2} k_\nu \frac{n_\beta}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n_\sigma} \right) k^\nu.
\]
The last term cancels out the third term in Eq. (6.3). Thus we obtain the magnetic current-current correlation

\[ L_{kk} = -\frac{1}{2} k^\mu D^\mu_m K_\nu - \frac{1}{2} k^\mu \frac{n_\beta n_\beta}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n_\sigma} \right) k^\nu \]

\[ = -\frac{1}{2} k^\mu \Box + m^2 k^\mu - \frac{1}{2} k^\mu m^2 \frac{n_\beta n_\beta}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n_\sigma} \right) k^\nu. \]  \hspace{1cm} (6.7)

If we replace the magnetic current \( k_\mu \) with the color electric current \( j_\mu \), Eq. (6.7) becomes the electric current-current correlation, which was derived in the framework of the dual Ginzburg-Landau model [18–20]. So, by replacing electric charges with magnetic charges, we can apply the results in these references. For a static magnetic monopole-antimonopole pair, the current is chosen as

\[ k^\mu(x) = Q_m g^{\mu 0} \{ \delta(x - a) - \delta(x - b) \}, \]  \hspace{1cm} (6.8)

where the magnetic charge is \( Q_m \), and the position of the monopole (antimonopole) is \( a \) \( (b) \). We write \( r = a - b \), \( r = |r| \) and \( n^\mu = (0, n) \). The vector \( n \) is chosen as \( n \parallel r \). Then the correlation (6.7) gives the monopole-antimonopole potential [18–20]

\[ V_m(r) = V_Y(r) + V_L(r), \quad V_Y(r) = \frac{-Q_m^2 e^{-mr}}{4\pi} \frac{1}{r}, \]

\[ V_L(r) = \sigma r + O(e^{-mr}), \quad \sigma = \frac{Q_m^2 m^2}{8\pi} \ln \left( \frac{m^2 + m_\chi^2}{m^2} \right), \]  \hspace{1cm} (6.9)

where \( m_\chi \) is the ultraviolet cutoff for the \( p_T \), which is the momentum component perpendicular to \( r \). Thus the magnetic monopoles are confined by the linear potential \( V_L(r) \).

We comment on the scale \( m_\chi \). In the usual dual superconductor model, \( m_\chi \) is the scale that the QCD-monopole condensation vanishes [20]. In our model, since the ghost condensation happens at the scale \( \mu_0 = \Lambda e^{-4\pi^2/(\alpha g^2)} \) and it yields the mass for \( \tilde{C}_\mu \), \( m_\chi \) is the scale \( \mu_0 \). As we showed in Ref. [13], \( \mu_0 \) coincides with the QCD scale parameter \( \Lambda_{QCD} \).

---

5 If we choose \( n \) as \( n \parallel r \), we obtain \( V_L(r) \propto r_n K_0(m r_T) \) [18, 19], where \( r = r_n + r_T \) with \( n \perp r_T \), and \( r_T = |r_T| \). Since the modified Bessel function \( K_0(m r_T) \) is positive and decreases rapidly as \( r_T \) increases, the configuration with \( r_T \ll 1/m \) and \( r_n \approx r \) contributes to \( V_L \). The meaning of this configuration will be discussed in the next paper.
6.2 The use of the Lagrangian (5.13)

Next we study the classical part of Eq.(5.13), i.e.,
\[-\frac{1}{4}(\partial \wedge C)^2 + \frac{m^2}{2}C^\mu C_\mu - C_\mu k^\mu + \Omega.\] (6.10)

Using \(\Omega\) in Eq.(4.8), Eq.(6.10) is rewritten as
\[\frac{1}{2}C^\mu \left\{ g_{\mu\nu} + m^2 \frac{n_\alpha n^\alpha}{n^\rho n_\rho}  \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n^\sigma} \right) \right\} \left[ (D_m^{-1})^{\nu\beta} C_\beta - 2k^\nu \right].\] (6.11)

From the equation of motion (4.7), \(C_\mu = (D_m)^{\mu\beta} k_\beta\) is derived. By substituting it into Eq.(6.11), we find Eq.(6.11) coincides with Eq.(6.7).

We note the second term in Eq.(6.7), which yields the linear potential \(V_L\), comes from the \(h_{3\mu\nu}h_4^{\mu\nu}/4\) term in \(\Omega\).

7 Summary and comment

We studied the low energy effective SU(2) gauge theory in Minkowski space. In the low energy region, the ghost condensation \(g\varphi_0 \neq 0\) happens, and the SU(2) symmetry breaks down to the U(1) symmetry. We introduced the Abelian magnetic potential \(\tilde{C}_\mu\) as a classical solution, and presented the relation between \(\tilde{C}_\mu\) and its dual potential \(C_\mu\) in Minkowski space. It was shown that the term \(h_{2\mu}^{\mu\nu} = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} \Lambda_{\alpha\beta}\), which is the Dirac string essentially, plays an important role to derive the correct Lagrangian for \(C_\mu\).

When \(g\varphi_0 \neq 0\), the quantum parts of the gauge field acquire the tachyonic masses. These tachyonic masses are removed by the condensate \(\langle A_\mu^+ A^{-\mu} \rangle = \frac{1}{2}((A_\mu^1 A_\mu^1 + A_\mu^2 A_\mu^2))\). At the same time, this condensate makes classical parts of the gauge field massive. Thus the magnetic potential \(\tilde{C}_\mu\) and its dual potential \(C_\mu\) become massive. The effective low energy Lagrangian with \(\tilde{C}_\mu\) is presented in Eq.(5.12), and that with \(C_\mu\) is Eq.(5.13).

If there are static magnetic charges \(Q_m\) and \(-Q_m\), the classical field \(\tilde{C}_\mu\) connects them. The static potential between them is \(V_Y(r) + V_L(r)\), where \(V_Y\) is the Yukawa potential and \(V_L\) is the linear potential. Namely the linear confining potential appears. If we use the dual potential \(C_\mu\), the Lagrangian is not Eq. (5.12) but Eq.(5.13). However, because of the term \(\Omega\) in Eq.(4.8), the same static potential is obtained.

Usually the Dirac string is considered to be unphysical. We cannot detect it. In fact, in the massless case, the equation of motion (3.11) for \(C_\mu\) leads to \(h_2^{\mu\nu} + h_3^{\mu\nu} = 0\), and \(h_2^{\mu\nu}\) disappears from the Lagrangian (3.9). However, \(h_2^{\mu\nu}\) is necessary to produce the correct kinetic term for the dual gauge field. Namely theoretical consistency requires the string term.
When the field \( C_\mu \) becomes massive, this situation changes a little. The equation of motion (4.7) for \( C_\mu \) leads to
\[
\rho_\mu^\nu + \rho_3^\mu^\nu + \rho_4^\mu^\nu = 0,
\]
and \( \rho_2^\mu^\nu \) can be removed from the Lagrangian (4.5). However there is the remnant \( \rho_3^\mu^\nu \rho_4^\mu^\nu / 4 \) in \( \Omega \). This term is the origin of the linear potential.

A The \( \epsilon \) symbol and notation

In this paper, we employ the metric \( g_{\mu\nu} = \text{diag}(1, -1, -1, -1) \). The antisymmetric \( \epsilon \) symbol defined by
\[
\epsilon^{0123} = 1
\]
satisfies the formulae
\[
\epsilon^{\alpha\beta\rho\sigma} \epsilon_{\alpha\lambda\mu\nu} = -2(\delta_\mu^\alpha \delta_\nu^\beta - \delta_\nu^\alpha \delta_\mu^\beta), \quad (A1)
\]
From Eq.(A1), the following relations are obtained:
\[
\epsilon_{\mu\nu\alpha\beta} n^\alpha J^\beta \rho_p = 2J_\mu \frac{n_\beta n_\gamma}{(n^\rho \partial_\rho)^2} \left( g_{\mu\nu} - \frac{n_\mu n_\nu}{n_\sigma n_\sigma} \right) K^\nu, \quad (A2)
\]
\[
\epsilon_{\mu\nu\alpha\beta} n^\nu \partial_\alpha J^\beta \rho_p = J_\mu \frac{1}{(n^\rho \partial_\rho)^2} \left\{ (n^\gamma \partial_\gamma)^2 g_{\mu\nu} - n_\sigma n_\sigma \Box g_{\mu\nu} + n_\sigma n_\sigma \partial_\mu \partial_\nu + n_\mu n_\nu \Box - n_\sigma \partial_\sigma \partial_\mu n_\nu - n_\sigma \partial_\sigma n_\mu n_\nu \right\} K^\nu. \quad (A3)
\]
For simplicity, we use the notations
\[
(\partial \wedge C)^\mu^\nu = \partial^\mu C^\nu - \partial^\nu C^\mu, \quad \partial_\sigma (\partial \wedge C)^{\sigma\nu} = \Box C^\nu - \partial^\nu \partial_\sigma C^\sigma \quad (A4)
\]
and
\[
H^2 = H_{\mu\nu} H^{\mu\nu},
\]
where \( H_{\mu\nu} \) is an antisymmetric tensor.

B Monopole solutions and dual solutions

In this appendix, we present monopole solutions and dual solutions for a static magnetic charge
\[
k^\beta = \delta_0^\beta 4\pi N \frac{g}{y} \delta(x)\delta(y)\delta(z), \quad (N = \pm 1, \pm 2, \cdots). \quad (B1)
\]
The term \( \Lambda^{\mu\nu} \) defined in Eq.[2.7] is discussed.
B.1 The massless case

Choosing the Dirac string on the negative z-axis, it is shown that the monopole solution

\[ \bar{C}_\mu = \frac{N}{g} (\cos \theta - 1) \partial_\mu \phi = \frac{N}{g} \frac{z - r}{r \rho^2} (0, -y, x, 0) \]  

(B2)

satisfies the equation

\[ \partial_\mu (\partial \wedge \bar{C})^{\mu \nu} - \epsilon^{\nu \alpha \beta} n_\alpha \partial_\mu k_\beta = 0, \]  

(B3)

where \((r, \theta, \phi)\) are spherical coordinates, \(\rho = \sqrt{x^2 + y^2}\), and the space-like vector is chosen as \(n^\rho = \delta^\rho_3\). The corresponding dual potential satisfies the equation

\[ \partial_\mu (\partial \wedge C)^{\mu \nu} - k^\nu = 0, \]  

(B4)

and the solution is

\[ C^\mu = \frac{N}{g} \frac{1}{r} \delta^\mu_0. \]  

(B5)

Now, using \(\bar{C}^\mu\), we calculate \(H^{\mu \nu}\). Substituting Eq.(B2) into Eq.(2.2), we find

\[ H^{0j} = -\frac{N}{g} \frac{x^j}{r^3}, \quad (j = 1, 2), \quad H^{03} = -\frac{N}{g} \frac{x^3}{r^3} - \frac{4\pi N}{g} \theta(-z) \delta(x) \delta(y), \]  

(B6)

where \((\partial^2_x + \partial^2_y) \ln \rho = 2\pi \delta(x) \delta(y)\) has been used. As \(\nabla^2 (1/r) = -4\pi \delta(r)\), it is easy to show \(\partial_\nu H^{\mu \nu} = \partial_j H^{0j} = 0\). Namely Eq.(2.5) is satisfied.

Next, to calculate \(H^{\mu \nu}\), we use the dual potential \(C^\mu\). From Eqs.(2.8) and (B5), \(H^{\mu \nu}\) becomes

\[ H^{0j} = -\partial_j C^0 = -\frac{N}{g} \frac{x^j}{r^3}, \quad (j = 1, 2), \quad H^{03} = -\frac{N}{g} \frac{x^3}{r^3} + \Lambda^{03}. \]  

(B7)

We follow the Zwanziger’s definition [2]

\[ \frac{1}{\partial_z} f(x, y, z) = a \int_0^\infty f(x, y, z - s) ds - (1 - a) \int_0^\infty f(x, y, z + s) ds, \]  

(B8)

and, to put the Dirac string on the negative z-axis, we set \(a = 0\). This choice gives

\[ \frac{1}{\partial_z} \delta(z) = -\theta(-z), \]  

and we find

\[ \Lambda^{03} = \frac{1}{\partial_z} (\Box C^0) = -\frac{4\pi N}{g} \theta(-z) \delta(x) \delta(y). \]  

Thus \(\Lambda^{\mu \nu}\) term represents the Dirac string part, and Eq.(B7) coincides with Eq. (B6).
B.2 The massive case

In this case, $\tilde{C}^\mu$ fulfills the equation

$$\partial^\mu(\partial \wedge \tilde{C})^{\mu\nu} + m^2 \tilde{C}^\nu - \epsilon^{\nu\alpha\beta\gamma\rho\delta} n_\alpha \partial_\mu k_\beta = 0.$$  \hspace{1cm} (B9)

By modifying Eq.(B2), we find Eq.(B9) is satisfied by the solution

$$\tilde{C}_\mu = \frac{N g \rho - r}{g r \rho^2} e^{-mr}(0, -y, x, 0).$$  \hspace{1cm} (B10)

In the same way, the dual potential

$$\tilde{C}_\mu = \frac{N e^{-mr}}{g r} \delta^\mu_0$$  \hspace{1cm} (B11)

defeals the equation

$$\partial^\mu(\partial \wedge \tilde{C})^{\mu\nu} + m^2 \tilde{C}^\nu - k^\nu = 0.$$  \hspace{1cm} (B12)

C Ghost condensation in Minkowski space

At the one-loop level, the Lagrangian (5.4) yields the effective potential for $v = g \phi_0$ as

$$V_M(v) = \frac{v^2}{2 \alpha_2 g^2} + i V_{gh}(v), \quad V_{gh}(v) = \int \frac{d^4 p}{(2\pi)^4} \ln[(-p^2)^2 + v^2].$$  \hspace{1cm} (C1)

In Ref. [10], we calculated $V_{gh}$ directly, and showed Eq.(C1) becomes

$$V_M(v) = \frac{v^2}{2 \alpha_2 g^2} - i \frac{v^2}{32 \pi}. \hspace{1cm} (C2)$$

Therefore the condition $V'_M(v) = 0$ gives $v = 0$.

However, when $v = 0$, the integrand $\ln[(-p^2)^2 + v^2]$ diverges at $p^2 = 0$, and the calculation in Ref. [10] is inapplicable. So we should replace $p^2$ with $p^2 + i\epsilon$ as usual, and set $\epsilon \to 0$ after the $p$-integration. Thus we consider

$$i V_{gh}(v) = i \int \frac{d^4 p}{(2\pi)^4} \ln[(-p^2 - i\epsilon - iv)(-p^2 - i\epsilon + iv)]. \hspace{1cm} (C3)$$

When $\epsilon < v$, if we take the limit $\epsilon \to 0$, the result (C2) is obtained. In the $p^0$ plane, since the pole ${|p|^2 - i\epsilon + iv}^{1/2}$ (${-|p|^2 - i\epsilon + iv}^{1/2}$) is in the first quadrant (the third quadrant), the usual Wick rotation is inapplicable. On the other hand, when $\epsilon > v$, the poles ${|p|^2 - i\epsilon \pm iv}^{1/2}$ are in the fourth quadrant, and $-{|p|^2 - i\epsilon \mp iv}^{1/2}$ are in the second quadrant. Then
we can apply the usual Wick rotation, which is performed by the replacement $p^0 \to -ip^4$ and $\int dp^0 \to i \int dp^4$. After that, we can take the limit $\epsilon \to 0$, and we find

$$V_M(v) = \frac{v^2}{2\alpha_2 g^2} - \int\frac{d^4 p_E}{(2\pi)^4} \ln [(p_E^2 - iv)(p_E^2 + iv)], \quad (C4)$$

where $(p_E)_\mu = (p, p^4)$ is the Euclidean four-momentum. This is the usual Euclidean potential, and its minimum is at $v \neq 0$ \[13\]. Namely the ghost condensation happens in Minkowski space as well.

We make a comment. The one-loop diagrams in Fig.C1 lead to the series

$$\int\frac{d^4 p}{(2\pi)^4 i} \ln (-p^2 - i\epsilon)^2 + \sum_{n=1}^{\infty} \frac{-1}{n} \int\frac{d^4 p}{(2\pi)^4 i} \left\{ -\frac{v^2}{(-p^2 - i\epsilon)^2} \right\}^n. \quad (C5)$$

Under the condition

$$\left| \frac{v^2}{(p^2 + i\epsilon)^2} \right| < 1, \quad (C6)$$

this series converges as

$$-i \int\frac{d^4 p}{(2\pi)^4} \ln (-p^2 - i\epsilon)^2 - i \int\frac{d^4 p}{(2\pi)^4} \ln \left[ 1 + \frac{v^2}{(-p^2 - i\epsilon)^2} \right]. \quad (C7)$$

This expression gives the potential $V_{gh}$. For an arbitrary value of $p^2$, the condition (C6) is satisfied if $\epsilon > v$. Namely, the condition $\epsilon > v$ is required for convergence.

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