Cohomogeneity Two Nonsemisimple Isometric Actions

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Abstract
We describe the orbits of a cohomogeneity two Riemannian \(G\)-manifold \(M\) from topological point of view, under the conditions that \(G\) is nonsemisimple and \(M\) decomposes as a product of negatively curved Riemannian manifolds.

Keywords Riemannian manifold · Lie group · Isometry

Mathematics Subject Classification 53C30 · 57S25

1 Introduction

A manifold on which a group \(G\) acts is called a \(G\)-manifold. In this paper, we consider a complete Riemannian manifold \(M\) with the action of a closed and connected Lie subgroup \(G\) of isometries. Dimension of the orbit space is called the cohomogeneity of the action. Manifolds having actions of cohomogeneity zero are called homogeneous. A classic theorem about Riemannian manifolds of nonpositive curvature [22] states that a homogeneous Riemannian manifold \(M\) of nonpositive curvature is simply connected or it is diffeomorphic to a cylinder over a torus (i.e., it is diffeomorphic to \(\mathbb{R}^k \times T^s\), \(k + s = \dim M\)), and a theorem by S. Kobayashi [11] states that a homogeneous Riemannian manifold of negative curvature is simply connected. Therefore, it is diffeomorphic to \(\mathbb{R}^n\), \(n = \dim M\). There are many interesting theorems about topological properties of cohomogeneity one \(G\)-manifolds of nonpositive curvature [2,17,18,21].

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The authors of [21] studied cohomogeneity one G-manifolds of negative curvature. Among other results, they proved that if $M$ is a nonsimply connected negatively curved cohomogeneity one Riemannian G-manifold and dim$(M) \geq 3$, then either $M$ is diffeomorphic to $\mathbb{R}^k \times T^s$, $k + s = \text{dim} \ M$ or $\pi_1(M) = \mathbb{Z}$ and the principal orbits are covered by $S^{n-2} \times \mathbb{R}$, $n = \text{dim} \ M$, and $M^G/\Gamma$ is homeomorphic to one of the spaces $R$ and $[0, \infty)$. Also, topological properties of cohomogeneity one Riemannian manifolds of nonpositive curvature has been studied by many authors. But, classification of orbits and orbit spaces of cohomogeneity two Riemannian G-manifolds of nonpositive curvature is an open problem. This article follows previous papers [13–16], where we proved various results about topological properties of cohomogeneity two Riemannian G-manifolds. One of the main examples of Riemannian manifolds of nonpositive curvature is the product $M = M_1 \times \cdots \times M_k$ of Riemannian manifolds such that $M_i$, $1 \leq i \leq k$, has negative curvature. In the paper [15], we studied the orbits of cohomogeneity two G-manifolds of this kind, under the condition that $M^G \neq \emptyset$. In the present paper, we replace the condition $M^G \neq \emptyset$ by the condition that $G$ is nonsemisimple and there is no non-principal orbit of positive dimension. Among other results, we show that if $M$ is not simply connected then either it is homeomorphic to the product of a cohomogeneity one G-manifold with $\mathbb{R}$, or $\pi_1(M) = \mathbb{Z}^p$, $p \geq 1$. Our main result is Theorem 3.2.

The paper is organized as follows: We recall some definitions and prove some statements about Riemannian manifolds of nonpositive curvature in Preliminaries. In the section of results, first we mention a remark about the relations of the orbits of a G-manifold and the orbits of its universal covering manifold, which is important in the proof of our theorem. Then, we give our main theorem and its proof.

2 Preliminaries

In what follows, $M$ is a Riemannian manifold, $G$ is a closed and connected subgroup of the isometries of $M$, and $M^G = \{ x \in M : G(x) = x \}$. All geodesics of $M$ are considered to have unit speed.

A complete connected and simply connected Riemannian manifold of nonpositive curvature is called a Hadamard manifold. An isometry $\phi$ of a Hadamard manifold $M$ is called elliptic if it has fixed point, $\phi$ is called hyperbolic (parabolic) if the function $d_\phi^2 : M \to \mathbb{R}$ defined by $d_\phi^2(x) = d^2(x, \phi(x))$ has minimum point (has no minimum).

An isometry $\phi$ is called axial if there is a geodesic $\gamma$ translated by $\phi$ (there exists a positive constant $c$ such that $\phi(\gamma(t)) = \gamma(t + c)$). $\gamma$ is called the axis of $\phi$.

If $\gamma$ is a geodesic in the Hadamard manifold $M$, we denote by $[\gamma]$ the collection of all geodesics which are asymptotic to $\gamma$. The collection of all asymptotic classes of the geodesics of $M$ is denoted by $M(\infty)$ and is called the ideal boundary of $M$ (see [8] for details). In fact, we can imagine $M \bigcup M(\infty)$ as a manifold with boundary such that $M$ is its interior and $M(\infty)$ is its boundary.

Remark 2.1 If $M_1$ and $M_2$ are Hadamard manifolds then their product $M = M_1 \times M_2$ is also a Hadamard manifold. Consider unit speed geodesics $\gamma_1$ and $\gamma_2$ in $M_1$ and $M_2$ and put $\gamma(t) = (\gamma_1(\sqrt{2}t), \gamma_2(\sqrt{2}t))$. Then, $\gamma$ is a unit speed geodesic in $M$. Let $\eta_1$ and

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\( \eta_2 \) be other unit speed geodesics in \( M_1 \) and \( M_2 \) and \( \eta \) be a geodesic in \( M \), defined in the similar way as \( \gamma \). If \( [\gamma_1] = [\eta_1] \) and \( [\gamma_2] = [\eta_2] \), then it is easy to show that \( [\gamma] = [\eta] \).

Consider the map \( \rho : M_1(\infty) \times M_2(\infty) \to M(\infty) \), defined by \( \rho([\gamma_1], [\gamma_2]) = [\gamma] \). Clearly, \( \rho \) is a one-to-one and well-defined map. Then, without loss of precision, we can denote \( [\gamma] \) by \(([\gamma_1], [\gamma_2])\). In this way, \( M_1(\infty) \times M_2(\infty) \) can be considered as a subset of \( M(\infty) \).

We call a point \( z \in M(\infty) \) a regular point at infinity, if \( z \) belongs to \( M_1(\infty) \times M_2(\infty) \) (that is \( z = (z_1, z_2), z_1 \in M_1(\infty), z_2 \in M_2(\infty) \)). A regular point in the infinity of the product of any finite number of Hadamard manifolds can be defined similarly.

**Remark 2.2** A Hadamard manifold \( M \) satisfies Axiom 1 (see [8]) if for any distinct points \( x, y \in M(\infty) \) there exists a geodesic \( \gamma \) joining \( x \) to \( y \).

**Lemma 2.3** Let \( \phi \) be a parabolic isometry on a Hadamard manifold \( M \).

1. If \( z \) is a fixed point at infinity for \( \phi \), then \( \phi \) leaves each horosphere \( S \) centered at \( z \) invariant (see [8] for definition of the horosphere).
2. If \( M \) has strictly negative curvature, then there is a unique fixed point \( z \in M(\infty) \) for \( \phi \).

**Proof** (1) is a direct consequence of [6, Lemma 3]. If \( M \) has strictly negative curvature then by [4, Lemma 9.10], \( M \) satisfies Axiom 1. Thus, by [8, Theorem 6.5], we get (2).

**Lemma 2.4** If \( H \) is a closed and connected solvable subgroup of the isometries of a Riemannian manifold \( M \) of strictly negative curvature, then one of the following statements is true:

1. \( M^H \neq \emptyset \).
2. \( H \) translates a unique geodesic.
3. There is a unique point in \( M(\infty) \) fixed by \( H \).

**Proof** By [6, Theorem 5], one of the following statements is true:

1. \( M^H \neq \emptyset \).
2. \( H \) translates a geodesic.
3. There is a point in \( M(\infty) \) fixed by \( H \).

Uniqueness of the geodesic in (2) comes from [4, Proposition 4.2]. If (1) and (2) are not true, then all elements of \( H \) are parabolic and uniqueness of the fixed point in (3) comes from Lemma 2.3(2).

**Lemma 2.5** Let \( M = M_1 \times M_2 \times \cdots \times M_m \) such that for each \( i \), \( M_i \) is a Hadamard manifold of negative curvature, and suppose that \( \phi \in Iso(M) \) can be decomposed as the product \( \phi = \phi_1 \times \phi_2 \times \cdots \times \phi_m \) of non-elliptic isometries \( \phi_i \in Iso(M_i) \). If there is a geodesic \( \beta = (\beta_1, \ldots, \beta_m) \) such that \( \phi(\beta) = \beta \), then \( \beta \) is unique.

**Proof** Let \( \gamma = (\gamma_1, \ldots, \gamma_m) \) be another geodesic in \( M \) such that \( \phi(\gamma) = \gamma \). Then, for each \( i \), \( \phi_i(\gamma_i) = \gamma_i \) and \( \phi_i(\beta_i) = \beta_i \). We get from the uniqueness of the geodesic left invariant by non-elliptic isometry \( \phi_i \) [4, Proposition 4.2] that \( \gamma_i = \beta_i \). Thus, \( \beta = \gamma \).
**Remark 2.6** Let $G$ be a connected solvable Lie subgroup of the isometries of $M = M_1 \times M_2 \times \cdots \times M_m$ such that for each $i$, $M_i$ is simply connected with strictly negative curvature. Since $G$ is connected then each $g \in G$ can be decomposed as $g = g_1 \times \cdots \times g_m$, $g_i \in Iso(M_i)$ (see [10, Vol. 1, p. 240]). For each $i \in \{1, \ldots, m\}$, let $P_i : G \to Iso(M_i)$ be the map defined by $P_i(g_1, \ldots, g_i, \ldots, g_m) = g_i$. Put $G_i = P_i(G)$. Clearly, $G_i$ is a closed and connected solvable subgroup of the isometries of $M_i$, and we have:

$$G \subset G_1 \times \cdots \times G_m \quad (*)$$

**Lemma 2.7** Under the assumptions of Remark 2.6, there is a unique subset $\Omega$ of $M$ with the property that $G(\Omega) = \Omega$ and $\Omega$ is one of the following sets or a product of them.

1. $M^G$ or product of the fixed point set of some elements of $\{G_1, \ldots, G_m\}$.
2. Image of a geodesic.
3. A regular point at infinity of the product of some elements of the set $
   \{M_1, \ldots, M_m\}$.

**Proof** Since $M_i$ has strictly negative curvature, then by Remark 2.4, one of the following statements is true:

(I) For all $i$, $M_i^{G_i} \neq \emptyset$.

(II) For each $i$, there is a unique geodesic $\gamma_i$ in $M_i$ such that $G_i(\gamma_i) = \gamma_i$.

(III) There is a unique point $\xi_i \in M_i(\infty)$ for each $i$, such that $G_i(\xi_i) = \xi_i$.

(IV) $\{G_1, \ldots, G_m\}$ is a collection of the groups with properties (I) or (II) or (III).

In the case (I), $(M_1 \times \cdots \times M_m)^{(G_1 \times \cdots \times G_m)} \neq \emptyset$, then $M^{G} \neq \emptyset$. In the case (II), $\gamma = (\gamma_1, \ldots, \gamma_m)$ is a geodesic such that $(G_1 \times \cdots \times G_m)(\gamma) = \gamma$. Thus, $G(\gamma) = \gamma$ and by Lemma 2.5, $\gamma$ is unique. In a similar way if the case (III) is true, $\xi = (\xi_1, \ldots, \xi_m)$ will be the unique regular point in $M(\infty)$ fixed by $G$. Now, it is easy to show that if (IV) is true, then there is a unique set $\Omega$, which is a product of the sets similar to (I), (II), (III), such that $G(\Omega) = \Omega$. $\square$

### 3 Results

**Remark 3.1** [5]. If $M$ is a complete and connected Riemannian manifold and $G$ is a connected subgroup of $Iso(M)$, and if $\tilde{M}$ is the universal Riemannian covering manifold of $M$ with the covering map $\kappa : \tilde{M} \to M$, then there is a connected covering $\tilde{G}$ of $G$ with the covering map $\pi : \tilde{G} \to G$, such that $\tilde{G}$ acts isometrically on $\tilde{M}$ and

1. Each deck transformation $\delta$ of the covering $\kappa : \tilde{M} \to M$ maps $\tilde{G}$-orbits onto $G$-orbits.
2. If $x \in M$ and $\tilde{x} \in \tilde{M}$ such that $\kappa(\tilde{x}) = x$ then $\kappa(\pi(\tilde{G}(\tilde{x}))) = G(x)$.
3. $\tilde{M}^{\tilde{G}} = \kappa^{-1}(M^G)$.
4. If $G$ is non-semisimple then $\tilde{G}$ is non-semisimple.
5. Deck transformation group, which we denote it by $\Delta$, centralizes $\tilde{G}$ (i.e., for each $\delta \in \Delta$ and $\tilde{g} \in \tilde{G}$, $\delta \tilde{g} = \tilde{g} \delta$).
Remark 3.2 Let $\tilde{M}$ be a Hadamard manifold and $S$ be a horosphere in $\tilde{M}$ related to the asymptotic class of geodesics $[\gamma]$ (i.e., all elements of $[\gamma]$ intersect $S$ perpendicularly). The function $f : \tilde{M} \to \mathbb{R}$, $f(p) = \lim_{t \to \infty} d(p, \gamma(t)) - t$, is called a Bussmann function. For each point $p \in \tilde{M}$, there is a point $\eta_S(p)$ in $S$ which is the unique point of $S$ nearest $p$, and the following map is a homeomorphism ([8, pp. 47, 58]):

$$\phi : \tilde{M} \to S \times \mathbb{R}, \quad \phi(p) = (\eta_S(p), f(p)).$$

Theorem 3.3 (see [15, Theorem 3.5]). Let $M^{n+2}$ be a nonsimply connected Riemannian manifold which is of cohomogeneity two under the action of a closed and connected subgroup $G$ of isometries, and suppose that $M$ can be decomposed as a product of Riemannian manifolds of negative curvature and $M^G \neq \emptyset$. Then,

(a) $M$ is diffeomorphic to $S^1 \times \mathbb{R}^{n+1}$ or $B \times \mathbb{R}^n$, where $B$ is the Moebius band.
(b) Each principal orbit is diffeomorphic to $S^n$.

As we mentioned in Introduction, classification of orbits of cohomogeneity two Riemannian manifolds of nonpositive curvature is an open problem and seems to be a difficult problem in general case. In the following theorem, we consider an important category of Riemannian manifolds of nonpositive curvature, containing products of negatively curved Riemannian manifolds. In direction of [15], we give a description of the manifold and its orbits under the condition that the acting group is non-semisimple.

Theorem 3.4 Let $M^{n+2}$ be a nonsimply connected Riemannian manifold such that it can be decomposed as a product of Riemannian manifolds of strictly negative curvature of dimension bigger than two, and let $G$ be a non-semisimple closed and connected subgroup of the isometries of $M$. If $M$ is of cohomogeneity two under the action of $G$, without non-principal orbits of positive dimension, then one of the following statements is true:

(a) $M$ is a parabolic manifold homeomorphic to $\frac{S}{\pi_1(M)} \times \mathbb{R}$. Where, $S$ is a horosphere in the universal Riemannian covering of $M$, and $\frac{S}{\pi_1(M)}$ is a cohomogeneity one $G$-manifold.
(b) $\pi_1(M) = \mathbb{Z}^p$ for some positive integer $p$, and all orbits are diffeomorphic to $\mathbb{R}^{n-p} \times T^p$.
(c) $M$ is diffeomorphic to $S^1 \times \mathbb{R}^{n+1}$ or $B \times \mathbb{R}^n$, where $B$ is the Moebius band, and each principal orbit is diffeomorphic to $S^n$.
(d) $\dim M = 3$ and $M$ has negative curvature. $\pi_1(M) = \mathbb{Z}$ and each orbit is diffeomorphic to $S^1$.

Proof Following Remark 3.1, let $\tilde{M}$ be the universal Riemannian covering manifold of $M$ with the deck transformation group $\Delta$ and let $\tilde{G}$ be the corresponding connected covering of $G$ which acts isometrically and by cohomogeneity two on $\tilde{M}$. Since by assumptions of the theorem, $M$ can be decomposed as the product of Riemannian manifolds of strictly negative curvature, then $\tilde{M}$ can be decomposed as $\tilde{M} = \tilde{M}_1 \times \tilde{M}_2 \times \cdots \times \tilde{M}_m$ such that for all $i$, $\tilde{M}_i$ has strictly negative curvature, and each $\delta \in \Delta$ decomposes as $\delta = \delta_1 \times \delta_2 \times \cdots \times \delta_m$, $\delta_i \in \text{Iso}(\tilde{M}_i)$. If $M^G \neq \emptyset$, then we get
part (c) of the theorem from Theorem 3.3. Thus, in the rest of the proof we suppose that $M^G = \emptyset$ which implies $\tilde{M}^G = \emptyset$. Then, according to the assumptions of the theorem, all orbits must be regular and of dimension $n$. Since $G$ is non-semisimple, $\tilde{G}$ is non-semisimple. Let $H$ be a connected solvable normal subgroup of $\tilde{G}$. By Lemma 2.7, there exists a unique subset $\Omega$ of $\tilde{M}$ such that $H(\Omega) = \Omega$ and $\Omega$ is one of the following sets or product of them.

$A$: a totally geodesic submanifold of $\tilde{M}$ or a totally geodesic submanifold of the product of some elements of the set $\{M_1, \ldots, M_m\}$ (because, fixed point sets of connected subgroups of isometries are totally geodesic submanifolds).

$\gamma$: image of a geodesic of $\tilde{M}$ or a geodesic in the product of some elements of the set $\{M_1, \ldots, M_m\}$.

$\zeta$: a regular point at infinity of $\tilde{M}$ or a regular point at infinity for product of some elements of the set $\{M_1, \ldots, M_m\}$.

That is $\Omega \in \{\gamma, \zeta, A \times \gamma, A \times \zeta, \zeta \times \gamma, A \times \zeta \times \gamma\}$ (\*)

Since $H$ is normal in $\tilde{G}$, for each $g \in \tilde{G}$ we have:

$g^{-1} Hg = H \Rightarrow g^{-1} Hg(\Omega) = H(\Omega) \Rightarrow g^{-1} Hg(\Omega) = \Omega \Rightarrow H(g\Omega) = g\Omega.$

Now, from the uniqueness of $\Omega$ with the property $H(\Omega) = \Omega$, we get that $g\Omega = \Omega$, then

$\tilde{G}(\Omega) = \Omega$ (**)

Since the elements of $\Delta$ commute with the elements of $\tilde{G}$, for each $\delta \in \Delta$ we have $\tilde{G}(\delta\Omega) = \delta\tilde{G}(\Omega) = \delta\Omega$. Uniqueness of $\Omega$ implies that $\delta(\Omega) = \Omega$. Thus,

$\Delta(\Omega) = \Omega$ (***)

We consider now the seven cases of $\Omega$ in (*).

**Case 1.** $\Omega = \gamma$.

$\tilde{G}(\gamma) = \gamma$ and By (**), $\Delta(\gamma) = \gamma$. Since $\tilde{M}^G = \emptyset$, then $\gamma$ is a $\tilde{G}$-orbit. Since the cohomogeneity is two and all orbits are of dimension $n$, then $n = 1$. Since by assumptions of the theorem, for all $i$, $\dim M_i \geq 3$, then $\tilde{M} = M_1$. Thus, $M$ has strictly negative curvature. All $\tilde{G}$-orbits in $\tilde{M}$ are diffeomorphic to $\gamma$, then they are diffeomorphic to $R$. By Lemma 3.5 in chapter 12 of [7], $\Delta = Z$. Thus, each $G$-orbit in $M$ will be diffeomorphic to $R/Z = S^1$. This is part (d) of the theorem.

**Case 2.** $\Omega = \zeta$.

Let $[\gamma]$ be the asymptotic class of the geodesics related to $\zeta$. We have $\tilde{G}(\zeta) = \zeta$ and by (**), $\Delta(\zeta) = \zeta$ (i.e, $\tilde{G}([\gamma]) = [\gamma]$ and $\Delta([\gamma]) = [\gamma]$). First, suppose that there is an axial element $\delta \in \Delta$ and let $\lambda$ be the unique geodesic such that $\delta\lambda = \lambda$ (uniqueness of $\lambda$ comes from Lemma 2.5). If $g \in \tilde{G}$ then $\delta(g\lambda) = g\delta\lambda = g\lambda$. Thus, we get from $\tilde{G}$.
the uniqueness of $\lambda$ that $g\lambda = \lambda$. Then, $\lambda$ is a $\tilde{G}$-orbit, and we get part (d) of the theorem as like as Case 1.

Now, suppose that all elements of $\Delta$ are non-axial. Non-identity elements of $\Delta$ are without fixed points, then they must be parabolic and $M$ will be a parabolic manifold. By Lemma 2.3, for each $\delta \in \Delta$ and each horosphere $S$ related to the asymptotic class $[\gamma]$, $\delta S = S$. Fix a horosphere $S$ related to $[\gamma]$. Put $M_1 = \frac{S}{\Delta}$ and let $\eta_\delta$ and $f$ be the maps defined in Remark 3.2. The homeomorphism $\phi : M \rightarrow S \times \mathbb{R}$ mentioned in Remark 3.2, induces a homeomorphism $\phi_1 : \frac{M}{\Delta} = M \rightarrow \frac{S}{\Delta} \times \mathbb{R}$, such that $\phi_1(x) = (\kappa \eta_\delta(x), f(x))$, $\bar{x} \in \kappa^{-1}(x)$.

If there is a $g \in \tilde{G}$ which is axial and $\lambda$ is its unique axis, then we get from the fact that the elements of $\Delta$ and $g$ commute that for all $\delta \in \Delta$, $\delta(\lambda)$ is also an axis for $g$. Since the axis is unique then $\delta(\lambda) = \lambda$. Now, if $g' \in \tilde{G}$, then again we get from the uniqueness of the axis $\lambda$ for $\delta$ that $g'\lambda = \lambda$, thus $\tilde{G}(\lambda) = \lambda$, and we get part (d) of the theorem as like as case 1.

Now, suppose that all elements of $\tilde{G}$ are non-axial. If for some $g \in \tilde{G}$ and $x \in \tilde{M}$, $gx = x$, then for the geodesic $\lambda$ in $[\gamma]$ which passes from $x$, we have $g\lambda = \lambda$, and $g$ must be axial which is contradiction. Thus, we can assume that the elements of $\tilde{G}$ are parabolic and by Lemma 2.3, $\tilde{G}(S) = S$. Thus, $S$ is a cohomogeneity one $\tilde{G}$-manifold and $\frac{S}{\Delta}$ is a cohomogeneity one $G$-manifold. This is part (a) of the theorem.

**Case 3.** $\Omega = A$.

Similar to the previous cases, we have $\tilde{G}(A) = A$ and $\Delta(A) = A$. Put $A_1 = \kappa(A)$. $A$ is a nontrivial totally geodesic submanifold of $\tilde{M}$, thus $A_1$ is a totally geodesic submanifold of $M$. Since $\Delta(A) = A$, then $\Delta$ is equal to deck transformation group of $A_1$, and $\pi_1(A_1) = \Delta = \pi_1(M)$. Since all orbits are of dimension $n$, we have for all $x \in A$:

\[
n = \dim \tilde{G}(x) \leq \dim A < \dim \tilde{M} = n + 2 \Rightarrow \dim A = n \quad \text{or} \quad n + 1
\]

Now, we consider $\dim A = n$, $\dim A = n + 1$, separately.

**I** $\dim A = n$.

In this case, $A$ is a $\tilde{G}$-orbit and $A_1$ must be a $G$-orbit of nonpositive curvature. Thus, we get part (b) of the theorem (Because, a homogeneous Riemannian manifold of nonpositive curvature is diffeomorphic to product of a torus and a euclidean space [22]).

**II** $\dim A = n + 1$.

$A_1$ is a cohomogeneity one $G$-manifold of non-positive curvature, without singular orbits. Consider the following two cases separately:

(II-1) For all $\delta \in \Delta$, $d_\delta^2$ has no minimum point.

(II-2) There is a $\delta \in \Delta$ such that $d_\delta^2$ has minimum point.

(II-1): In this case by [17, Lemma 3.2], $\Delta$ maps each orbit of $A$ onto itself. Thus by a similar way in the proof of [17, Lemma 3.6], all orbits of $A_1$ are diffeomorphic to $T^p \times \mathbb{R}^{n-p}$ for some nonnegative integer $p$, thus we get part (b) of the theorem.

(II-2): By [4, Proposition 4.2], the minimum point set of $d_\delta^2$ is equal to the image of all geodesics translated by $\delta$. But by Lemma 2.5, there is at most one geodesic translated by $\delta$. Let $\gamma$ be the unique geodesic such that $\delta(\gamma) = \gamma$. Since the elements of $\tilde{G}$ and
Δ commute, then we get from the uniqueness of γ that \( \widetilde{G}(\gamma) = \gamma \), and we get part (d) of the theorem in the similar way as Case 1.

**Case 4.** \( \Omega \in \{A \times \gamma, A \times \xi, \xi \times \gamma, A \times \xi \times \gamma\} \).

By dimensional reasons, this cases can not occur. We give the proof for \( \Omega = A \times \gamma \), other cases are similar. We can assume (after a possible rearrangement of the indices) that

\[
\widetilde{M} = M_1 \times M_2 \times \ldots M_k \times M_{k+1} \times M_{k+2} \times \cdots \times M_m
\]

\[
A \subset M_1 \times M_2 \times \ldots M_k, \quad \gamma \subset M_{k+1} \times M_{k+2} \times \cdots \times M_m
\]

By \((*)\), \( \widetilde{G}(A \times \gamma) = A \times \gamma \). Thus, \( A \times \gamma \) is a union of \( \widetilde{G} \)-orbits. Since \( A \) and \( \gamma \) are nontrivial then the codimension of \( A \) in \( M_1 \times M_2 \times \ldots M_k \) is at least 1, and because for all \( i \), \( \dim M_i \geq 3 \), then the codimension of \( \gamma \) in \( M_{k+1} \times M_{k+2} \times \cdots \times M_m \) is at least 2. Thus, the codimension of \( A \times \gamma \) in \( \widetilde{M} \) will be at least 3. This is contradiction (because, \( A \times \gamma \) is union of orbits which have codimension two in \( \widetilde{M} \)).

**Remark 3.5** In Theorem 3.4, decomposability of \( M \) to the product of negatively curved manifolds can be replaced by the weaker condition of decomposability of the universal covering manifold \( \widetilde{M} \) to negatively curved manifolds and decomposability of \( \Delta \) (see [17], definition of UND-manifolds and examples).

**Remark 3.6** In case (a) of Theorem 3.4, \( M \) is homeomorphic to the product of a cohomogeneity one manifold \( S_{\Delta}^1 \) with \( \mathbb{R} \). By proof of the theorem, in this case, there is no singular orbit. Since the orbit space of cohomogeneity one manifolds with no singular orbit are homeomorphic to \( S^1 \) or \( R \), then the orbit space of \( M \) under the action of \( G \) will be homeomorphic to \( S^1 \times R \) or \( R^2 \). Study of the orbits in this case reduces to the study of the orbits of cohomogeneity one actions on horospheres. In the special case when \( M \) has constant negative curvature (decomposition of \( M \) has one factor of constant negative curvature), the horospheres of \( \widetilde{M} \) are isometric to \( R^{n+1} \), and from the known results about cohomogeneity one actions on flat Riemannian manifolds, orbits of \( M \) are diffeomorphic to \( \mathbb{R}^k \times T^{n-k} \), for some positive integer \( k \).

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