Construction of Monodromy Matrix in the F - basis and scalar products in Spin chains.

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Abstract

We present in a simple terms the theory of the factorizing operator introduced recently by Maillet and Sanches de Santos for the spin- 1/2 chains. We obtain the explicit expressions for the matrix elements of the factorizing operator in terms of the elements of the Monodromy matrix. We use this results to derive the expression for the general scalar product for the quantum spin chain. We comment on the previous determination of the scalar product of Bethe eigenstate with an arbitrary dual state. We also establish the direct correspondence between the calculations of scalar products in the F - basis and the usual basis.

1. Introduction.

One of the most important open problems in the theory of quantum integrable models is the calculation of the correlation functions. In the framework of the Algebraic Bethe Ansatz method \[1\] the problem is the combinatorial complexity of calculations due to the structure of Bethe eigenstates. Only the very limited number of physical results for the correlation functions (or formfactors) of integrable models have been obtained from the first principles.

The concept of factorizing F - matrix was recently introduced by Maillet and Sanches de Santos \[2\] following the concept of Drinfeld’s twists in his theory of Quantum Groups. The existence of the factorizing matrix allows one by means of similarity transformation to make the elements of the Monodromy matrix totally symmetric with respect to an arbitrary permutation of indices 1, ..., N together with the corresponding inhomogeneity parameters \(\xi_1, \ldots, \xi_N\). The matrix \(F_{1\ldots N}\) is defined as follows. Consider any permutation \(\sigma \in S_N\). Then any component in the auxiliary space 0 of the monodromy matrix \(T_0(t)\) depending on the parameter \(t\) transforms as

\[
T_{0,\sigma 1\ldots \sigma N} = R_{\sigma 1\ldots \sigma N}(R_{\sigma 1\ldots \sigma N})^{-1},
\]

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where $R_{1\ldots N}^\sigma$ is the operator constructed from the elementary $S$-matrices. Then if the matrix $F_{1\ldots N}$ is defined according to

$$
(R_{1\ldots N}^\sigma)^{-1} = (F_{1\ldots N})^{-1}F_{\sigma_{1}\ldots\sigma_{N}},
$$

in the F- basis the monodromy matrix $T_{0}^{F} = FT_{0}F^{-1}$ is totally symmetric:

$$
T_{0,\sigma_{1}\ldots\sigma_{N}}^{F}(t) = T_{0,1\ldots N}^{F}(t), \quad \sigma \in S_{N}
$$

for arbitrary $t$. It was also realized that there exist the factorizing matrix $F$ which diagonalize the $A(D)$- component of the monodromy matrix. In this basis the operators $B$ and $C$ have quite a simple quasilocal form which recently made possible the direct computations for the important class of the (general) correlation functions [3]. In some respect this approach allows one to simplify the calculations in comparison with the general theory of scalar products [4], [5] and gives an alternative way to obtain many important results for the XXZ spin chain. Among them one can mention the derivation by Korepin the Gaudin formula for the norm of Bethe eigenstates [4] and the calculation of the scalar product of Bethe eigenstate with an arbitrary dual state [3], which leads straightforwardly to the determinant representation of the formfactors of basic (local) operators (see e.g. [7], [8] for the analogous calculations for the case of Bose-gas with $\delta$ - function interaction). Using the Algegraic Bethe Ansatz and the solution of the quantum inverse scattering problem the authors of ref.[9] obtained the multiple integral representation for the correlation functions found previously using the other methods (see for example [10] and references therein).

At the same time in the approach of ref.[2] the construction of the F - matrix itself and of the matrix elements of different operators in the F - basis is quite complicated. The construction involves the specially defined partial F - matrices and some steps (for example the proof that the factorizing matrix diagonalize the $A(D)$- operators) are based on a special recurrence procedures. Also the explicit expressions for the matrix elements of the operator $F$, which can be used in practical computations, was not presented. The generalization of the construction to the case of the other models is not straightforward.

The aim of the present paper is twofold. First, we present in a simple terms the theory of the factorizing operator introduced in ref.[2] and derive the new formulas for the matrix elements of the factorizing operator (Section 2). In comparison to the authors [2] we proceed in a different way. First of all we construct the matrix that diagonalizes the operator $A(t)$. We obtain the simple expression for its matrix elements and the matrix elements of the inverse operator in terms of the matrix elements of the products of the operators $B(t), C(t)$. These expressions have not been obtained previously. Then we find the expressions for the other operators $B, C$ in the new basis (Section 3). Afterwards we prove that the constructed operator is the factorizing operator as defined above. This procedure allows one to obtain
the simple expressions for the matrix elements of the factorizing operator which can be used in practical computations. The presented formalism could be useful for the other integrable models (for example the generalization to the case of XYZ- spin chain is straightforward).

Second, we apply the developed formalism to the calculation of the general scalar product in a way which is different from that of ref.[3] (Section 4). We obtain the general expression for the scalar product of ref.[4] in a direct (i.e. without any recurrence procedures) and a simple way. In some sense our calculation clarifies the mathematical structure underlying the derivation of ref’s [4], [5]. The key point in our derivation is the formulas for the matrix elements of the factorizing operator obtained in Section 2. Next, in Section 4, we supplement the derivation [3] of the determinant representation of the scalar product of Bethe eigenstate with an arbitrary dual state. We present here the new version of the proof given by Slavnov [4], which is based on the recurrence relations. We also establish the direct correspondence between the calculations of an arbitrary scalar products in the F - basis and the conventional basis in the Appendix C. The calculation of the sums for the scalar products presented here can serve as an independent proof of the formulas for the matrix elements of the Monodromy matrix in the F - basis.

2. Construction of the factorizing operator.

We consider in this paper the XXX or XXZ spin- 1/2 chains of finite length N. Before diagonalizing the operator $A$, let us fix the notations: the normalization of basic $S$ - matrix, the definition of monodromy matrix and write down the Bethe Ansatz equations. For the rational case (XXX- chain) the $S$- matrix has the form $S_{12}(t_1, t_2) = t_1 - t_2 + \eta P_{12}$, where $P_{12}$ is the permutation operator. In general (XXZ) case it can be written as

$$S_{12}(t_1, t_2) = \begin{pmatrix} a(t) & 0 & 0 & 0 \\ 0 & c(t) & b(t) & 0 \\ 0 & b(t) & c(t) & 0 \\ 0 & 0 & 0 & a(t) \end{pmatrix}^{(12)}, \quad t = t_1 - t_2.$$  

One can choose the normalization $a(t) = 1$ so that the functions $b(t)$ and $c(t)$ become:

$$\tilde{b}(t) = \phi(\eta)/\phi(t + \eta) \quad \tilde{c}(t) = \phi(t)/\phi(t + \eta),$$

where $\phi(t) = t$ for the isotropic (XXX) chain and $\phi(t) = \sinh(t)$ for the XXZ- chain. With this normalization the $S$-matrix satisfies the unitarity condition $S_{12}(t_1, t_2)S_{21}(t_2, t_1) = 1$. The monodromy matrix is defined as

$$T_0(t, \{\xi\}) = S_{10}(\xi_1, t)S_{20}(\xi_2, t)\ldots S_{N0}(\xi_N, t),$$

3
where $\xi_i$ are the inhomogeneity parameters. We define the operator entries in the auxiliary space (0) as follows:

$$\langle \beta | T_0 | \alpha \rangle = \begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix}_{\alpha \beta} ; \quad \alpha, \beta = (1, 2) = (\uparrow; \downarrow).$$

We denote throughout the paper $(\uparrow; \downarrow) = (1; 0)$ so that the pseudovacuum (quantum reference state) $|0\rangle = |\{00...0\}_N\rangle$. The triangle relation (Yang-Baxter equation) reads:

$$S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}, \quad R_{00'}T_0T_{0'} = T_{0'}T_0R_{00'}; \quad R_{00'} = P_{00'}S_{00'}.$$

The action of the operators on the pseudovacuum is: $A(t)|0\rangle = a(t)|0\rangle$ ($a(t) = \prod_\alpha c(\xi_\alpha - t)$), $D(t)|0\rangle = |0\rangle$, $C(t)|0\rangle = 0$. The Bethe Ansatz equations for the eigenstate of the Hamiltonian $\prod_{i=1}^M B(t_i)|0\rangle$ are

$$a(t_i) = \prod_{\alpha \neq i} \tilde{c}(t_\alpha - t_i)(\tilde{c}(t_i - t_\alpha))^{-1},$$

and the corresponding eigenvalue of the transfer - matrix $Z(t) = A(t) + D(t)$ is

$$\Lambda(t, \{t_\alpha\}) = a(t) \prod_{\alpha = 1}^M \tilde{c}^{-1}(t_\alpha - t) + \prod_{a=1}^M \tilde{c}^{-1}(t - t_\alpha),$$

where $t_\alpha$ are the solution of Bethe Ansatz equations.

In order to construct the operator $\hat{O} = \hat{O}_{1...N}$ which diagonalizes the operator $A(t)$ ($\hat{O}^{-1}A\hat{O} = \text{diag}(A)$), let us first construct the eigenfunctions of the operator $A$. One can do it in two different ways. First, note that $A$ - is a triangular matrix in a sense that it makes particles (the spin-up - coordinates) move to the right on the lattice $1...N$. Thus its eigenvalues are coincide with its diagonal matrix elements and therefore are characterized by the set of integers $(n_1,...n_M)$ - the spin-up positions. Let us denote this eigenfunctions by $|\phi(n_1,...n_M)\rangle$. Clearly, $|\{00..0\}_N-M11..1\rangle_M$ is an eigenstate of $A(t)$. Therefore, considering the permutation

$$S_{10}...S_{N0} \rightarrow S_{10}...S_{N0}S_{n_M0}...S_{n_10} \quad (n_1 < n_2 < ... < n_M),$$

we realize that

$$|\phi(n_1,...n_M)\rangle = T_{n_1}T_{n_2}...T_{n_M}|n_1, n_2, ... n_M\rangle \quad (2)$$

where

$$T_n = S_{n+1,n}S_{n+2,n}...S_{Nn},$$

is an eigenstate of the operator $A(t)$. Note that if we modify the given permutation interchanging $n_i \leftrightarrow n_j$ the operator in the right-hand side of (2) modifies but the state $|\phi(\{n\})\rangle$ remains the same. For example, for $M = 2$: $T_{n_2}T_{n_1}' = T_{n_1}T_{n_2}S_{n_2n_1}^{-1}$, where $n_1 < n_2$ and the
where the matrix (in the same sense as the operator diagonal (see (5)). Now we immediately get the simple formula for the matrix elements of the elements of this operator are equal to and the operator \(A\) identity (dual states: and construct the inverse operator. To find this operator we have to consider the following prime means the absence of the term \(S_{n_1n_2}\) in \(T_{n_1}\). Since \(S_{n_1n_2}|n_1, n_2\rangle = |n_1, n_2\rangle\) the state remains the same.

The second way - is to consider the state

\[
|\phi(n_1, \ldots n_M)\rangle = B(\xi_{n_1})B(\xi_{n_2})\ldots B(\xi_{n_M})|0\rangle, \quad (n_i \neq n_j).
\]

Using the fundamental commutation relation:

\[
A(t)B(q) = \frac{1}{c(q-t)}B(q)A(t) - \frac{b(q-t)}{c(q-t)}B(t)A(q),
\]

and the fact that for the pseudovacuum state \(A(\xi)|0\rangle = 0\), we find again that \(|\phi(\{n\})\rangle\) \((\{n\} = \{n_1, \ldots n_M\})\) is an eigenstate of \(A(t)\) with the eigenvalue \(A_{\{n\}\{n\}}(t) = \prod_{\alpha \neq n_k} c(\xi_{\alpha} - t)\), and the operator \(A(t)\) in the new basis has the following form:

\[
A^F(t) = \prod_{\alpha=1}^N \left( \begin{array}{c} \overline{c}(\xi_{\alpha} - t), \quad \alpha \neq n_k \\ 1, \quad \alpha = n_k \end{array} \right)
\]

One can see that the states \(|\phi(\{n\})\rangle\) (8) and (9) are coincide. This can be seen using the identity

\[
B(\xi_n) = T_n\langle 0|S_{10} \ldots S_{n-1,0}P_{n0}|1\rangle
\]

and ordering the (commuting) operators \(B(\xi_{n_1})\) in eq.(8) according to the prescription \(n_1 < n_2 < \ldots < n_M\), so that the second operator in the last formula simply creates the particle (spin- up) at the site \(n\) with the amplitude equal to unity.

Now we can introduce the diagonalizing operator \(\hat{O} = \hat{O}_{1\ldots N}\) (we will see later that it is also the factorizing operator). Let us define the operator \(\hat{O}\) such that

\[
|\phi(\{n\})\rangle = \hat{O}(\{n\})|\{n\}\rangle = \hat{O}|\{n\}\rangle,
\]

where \(\hat{O}(\{n\})\) is given by the equation (8). Clearly the operator \(A^F(t) = \hat{O}^{-1}A(t)\hat{O}\) - is diagonal (see (9)). Now we immediately get the simple formula for the matrix elements of the factorizing operator:

\[
\hat{O}_{\{m\}\{n\}} = \langle \{m\}|B(\xi_{n_1})B(\xi_{n_2})\ldots B(\xi_{n_M})|0\rangle,
\]

where \(\{m\} = \{m_1, \ldots m_M\}\). From this expression (see (9)) one see that \(\hat{O}\) is the triangular matrix (in the same sense as the operator \(A((t))\)). Let us show that the operator \(\hat{O}\) is invertible and construct the inverse operator. To find this operator we have to consider the following dual states:

\[
\langle \tilde{\phi}(n_1, \ldots n_M)| = \langle 0|C(\xi_{n_1})C(\xi_{n_2})\ldots C(\xi_{n_M})\rangle.
\]

Define the operator \(\hat{O}\) analogously to the previous case as \(\langle \tilde{\phi}(\{n\})| = \langle \{n\}|\hat{O}\). The matrix elements of this operator are equal to

\[
\hat{O}_{\{m\}\{n\}} = \langle 0|C(\xi_{m_1})C(\xi_{m_2})\ldots C(\xi_{m_M})|\{n\}\rangle.
\]
Let us calculate the following scalar product
\[
\langle \hat{\phi}(\{m\})|\phi(\{n\}) \rangle = \langle \{m\} | \hat{O} \hat{O} | \{n\} \rangle = \langle 0 | C(\xi_{m_1})C(\xi_{m_2})...C(\xi_{m_M})B(\xi_{n_1})B(\xi_{n_2})...B(\xi_{n_M}) | 0 \rangle.
\]This scalar product can be calculated using another well known relation relation between the elements of the monodromy matrix following from the Yang-Baxter equation:
\[
[B(q), C(t)] = \hat{b}(t - q) \frac{D(q)A(t) - D(t)A(q)}{\hat{c}(t - q)},
\]
and using again that for any site \(i\), \(A(\xi_i)|0\rangle = 0\). Moving the operators \(A\) and \(D\) to the right, repeating consequently the relation (4) and the analogous relation for \(D\) (which differs from eq.(4) by the interchange \(t \leftrightarrow q\) in the coefficients between the products of the operators) and using the equations
\[
C(\xi_n)B(\xi_n)|0\rangle = \left( \prod_{\alpha \neq n} \hat{c}(\xi_{\alpha} - \xi_n) \right) |0\rangle,
\]
we obtain that the matrix \(\hat{f}\) is diagonal:
\[
\langle \hat{\phi}(\{m\})|\phi(\{n\}) \rangle = \langle \{m\} | \hat{O} \hat{O} | \{n\} \rangle = \langle \{m\} | \hat{f} | \{n\} \rangle = \left( \prod_i \delta_{n_i,m_i} \right) f(\{n\}).
\]
The corresponding diagonal matrix elements can be found either using the procedure mentioned above or, which is the simplest way, using the representation (6) (and the same for the operator \(C(t)\)). In fact, taking into account the triangularity of \(\hat{O}\) and \(\hat{\phi}\) we obtain:
\[
\langle \{n\} | \hat{f} | \{n\} \rangle = \sum_{\{m\}} \hat{O}_{\{n\}\{m\}} \hat{O}_{\{m\}\{n\}} = \hat{O}_{\{n\}\{n\}} = \hat{O}_{\{n\}\{n\}}.
\]
The diagonal matrix elements can be easily calculated using the representation (6) and the formula (6). We get:
\[
f(n_1,...n_M) = \prod_k \left( \prod_{\alpha \neq n_k,n_j} \hat{c}(\xi_{\alpha} - \xi_{n_k}) \right).
\]
Thus we obtained the inverse matrix \(\hat{O}^{-1}\):
\[
\hat{O} \hat{O} = \hat{f}, \quad \hat{O}^{-1} = \hat{f}^{-1} \hat{O}.
\]
Before proceeding with evaluation of the matrix elements of the other operators in the new basis, let us mention some usefull properties of the operator \(\hat{O}\), and prove that that, in fact, it is the factorizing operator in a sense of the definition (1): \(\hat{O} = F^{-1}\). First, \(\hat{O}\) and \(\hat{O}^{-1}\) are the triangular matrices (upper triangular as \(A(t)\)). Second, the pseudovacuum state is an
eigenstate of $\hat{O}$ ($\hat{O}^{-1}$) with the eigenvalue equal to unity. In general we have the following equations for arbitrary number of particles $n$:

$$\hat{O}|\{00..0\}_{N-n}\{11..1\}_n\rangle = 1 |\{00..0\}_{N-n}\{11..1\}_n\rangle$$

$$\langle\{11..1\}_n|00..0\rangle_{N-n} \hat{O} = 1 \langle\{11..1\}_n|00..0\rangle_{N-n}|,$$

and the same formulas for the inverse operators. From this formulas one can already suspect that $\hat{O}$ is the factorizing operator. Indeed for the particular permutation (4) $\sigma(\{n\})$ the factorizing condition is represented as $\hat{O}(\hat{O}^{\sigma(\{n\})})^{-1} = \hat{O}(\{n\})$, where $\hat{O}(\{n\}) = T_{n_1}..T_{n_M}$ and is fulfilled at least for the state $|\{n\}\rangle$ since due to the last formulas $\hat{O}^{\sigma(\{n\})^{-1}}|\{n\}\rangle = |\{n\}\rangle$. The rigorous proof goes as follows. We construct the operator that acting on the state $|\{n\}\rangle$ produces the state $\hat{O}(\{n\})|\{n\}\rangle$. It is easy to see that the operator that fulfills the above requirement is:

$$\hat{O} = \hat{F}_1\hat{F}_2\ldots\hat{F}_N, \quad \hat{F}_i = (1 - \hat{n}_i) + T_i\hat{n}_i,$$

where $\hat{n}_i$ is the operator of the number of particles (spin up) at the given site $i$. The operators $\hat{F}_i$ entering (13) do not commute and their ordering in eq.(13) is important. To prove the factorizing property of this operator it is sufficient to consider only one particular permutation, say the the permutation $(i, i+1)$ since all the others can be obtained as a superposition of these ones for different $i$. We will show in the Appendix A that

$$\hat{O} = S_{i+1,i} \hat{O}^{(i,i+1)};$$

(14)

Evidently, in contrast to (3), for any transmutation $\sigma \in S_N$ we will obtain only one operator $R^{\sigma}_{1..N}$ on the left of the operator $\hat{O}$. Thus it is proved that $\hat{O}$ is the factorizing operator in a sense of eq.(3).

3. Construction of the matrix elements.

In this section we calculate the matrix elements of $B(t)$ and $C(t)$ - operators in the F-basis: $B^F(t) = FB(t)F^{-1} = \hat{O}^{-1}B(t)\hat{O}$ (and the same for $C(t)$). The general scheme to perform the calculations is to use the formalism developed in the previous section, which leads to the following chain of equations:

$$B(t)|\phi(\{n\})\rangle = B(t)\hat{O}|\{n\}\rangle = \hat{O}B^F(t)|\{n\}\rangle = \sum_x \phi(x, t, \{n\})\hat{O}|\{n\}, x\rangle,$$

where $|\{n\}, x\rangle$ is the state corresponding to the new set of coordinates with an extra spin-up at the site $x$. Thus acting by the operator $B(t)$ on an eigenstate (3)

$$B(t)B(\xi_1)...B(\xi_{n_M})|0\rangle = \sum_x (B(\xi_x)B(\xi_1)...B(\xi_{n_M})|0\rangle) \phi(x, t, \{n\}),$$

(15)
we see that \( \phi(x, t, \{n\}) \) is exactly the matrix element in the new basis. To get the single term in the sum in eq. (15), we act by the operator \( A(\xi) \) \((x \neq n_k)\) at both sides of this equation. Using again the property \( A(\xi)|0\rangle = 0 \) and eq. (11) we get for the left-hand side of (15)

\[
A(\xi_x)B(t)B(\xi_{n_1})...B(\xi_{n_M})|0\rangle = -\left(\tilde{b}(t - \xi_x)/\tilde{c}(t - \xi_x)\right) B(\xi_x)A(t)B(\xi_{n_1})...B(\xi_{n_M})|0\rangle,
\]

while for the right-hand side we get the single term with \( B(\xi_x) \), which can be easily evaluated using again eq. (11) and the formula

\[
A(\xi_x)B(\xi_x)|0\rangle = \left(\prod_{\alpha \neq x} \tilde{c}(\xi_\alpha - \xi_x)\right) B(\xi_x)|0\rangle,
\]

which can be proved by direct computations. After the cancellation of similar terms at both sides of eq. (15) we get

\[
\phi(x, t, \{n\}) = -\left(\tilde{b}(t - \xi_x)/\tilde{c}(t - \xi_x)\right) \prod_{\alpha \neq n_k} \tilde{c}(\xi_\alpha - t) \left(\prod_{\alpha \neq n_k} \tilde{c}(\xi_\alpha - \xi_x)\right)^{-1}.
\]

Then, using the equality

\[
-\frac{\tilde{b}(t - \xi_x)}{\tilde{c}(t - \xi_x)} \tilde{c}(\xi_x - t) = \tilde{b}(\xi_x - t),
\]

we finally obtain in the operator form:

\[
B^F(t) = \sum_x \sigma_x^+ \tilde{b}(\xi_x - t) \prod_{\alpha \neq x} \begin{cases} \tilde{c}(\xi_\alpha - t)(\tilde{c}(\xi_\alpha - \xi_x))^{-1}, & \alpha \neq n_k \\ 1, & \alpha = n_k \end{cases}.
\] (16)

With this expression one can prove the equation \( \prod_i B^F(\xi_{n_i})|0\rangle = |\{n\}\rangle \) which is consistent with the formulas of the previous section.

For the operator \( C(t) \) proceeding in a similar way and using the relation (11), we get

\[
C(t)B(\xi_{n_1})...B(\xi_{n_M})|0\rangle = B(\xi_{n_1})C(t)B(\xi_{n_2})...B(\xi_{n_M})|0\rangle
\]

\[
-\left(\tilde{b}(t - \xi_{n_1})/\tilde{c}(t - \xi_{n_1})\right) D(\xi_{n_1})A(t)B(\xi_{n_2})...B(\xi_{n_M})|0\rangle.
\] (17)

Using the symmetry between \( n_1, ... n_M \), we can concentrate on the term which describes the flipping of the spin on the \( n_1 \)-site and consider only the second term in the right-hand side of the equation (17). Commuting \( A \) and \( D \) operators to the right in (17) and denoting \( \xi_{n_1} = \xi_x \) we obtain similarly to the previous case

\[
C^F(t) = \sum_x \sigma_x^- \tilde{b}(\xi_x - t) \prod_{\alpha \neq x} \begin{cases} \tilde{c}(\xi_\alpha - t), & \alpha \neq n_k \\ (\tilde{c}(\xi_x - \xi_\alpha))^{-1}, & \alpha = n_k \end{cases}.
\] (18)
The operators (16) and (18) are quasilocal i.e. they describe the flipping of the spin on a single site with the amplitude depending on the positions of spins on all the other sites of the chain. The operator $D_F(t)$ can be found using either the same method or the quantum determinant relation and has a (quasi)bilocal form.

4. Calculation of the scalar products.

In this section we use the developed formalism to obtain the expressions for the correlation functions for the spin chains. We use the factorizing operator and, in particular, the expression for its matrix elements (7) to obtain the expression for the general correlation function (scalar product)

$$S_M(\{\lambda\}, \{t\}) = \langle 0|C(\lambda_1)C(\lambda_2)\ldots C(\lambda_M)B(t_1)B(t_2)\ldots B(t_M)|0\rangle,$$

where $\{\lambda\}$ and $\{t\}$ are two arbitrary sets of parameters (not necessarily satisfying the Bethe Ansatz equations). The correlation function can be represented in the following form

$$\langle 0|Tr_{0,1,\ldots,0_{2M}}(\sigma^+_0\ldots\sigma^+_{0_M}\sigma^-_{0_{M+1}}\ldots\sigma^-_{0_{2M}}T_0\ldots T_{0_{2M}})|0\rangle$$

The auxiliary spaces $0_1,\ldots,0_{2M}$ can be considered as a lattice consisting of $2M$ sites with the corresponding spectral parameters $\lambda_1,\ldots,\lambda_M, t_1,\ldots, t_M$. Rearranging the basic $S$-matrices entering the product of the monodromy matrices we arrive at the operator

$$\tilde{T}_1\ldots\tilde{T}_N,$$

where the new monodromy matrices act in the auxiliary space instead of the original quantum space:

$$\tilde{T}_n = S_{n0_1}S_{n0_2}\ldots S_{n0_{2M}}.$$

Obviously using this matrices the correlator can be represented as the following matrix element in the new quantum space $0_1,\ldots,0_{2M}$:

$$\langle\{00..0\}_M\{11..1\}_M|\tilde{A}(\xi_1)\ldots\tilde{A}(\xi_N)|\{11..1\}_M\{00..0\}_M\rangle$$

(we use the symmetry $A(t) \leftrightarrow D(t), 0 \leftrightarrow 1$ here). Transforming the operators $\tilde{A}(\xi_i)$ to the F-basis we find at $\xi_i = 0$:

$$S_M = \sum_{\{n\}}\langle\{00..0\}_M\{11..1\}_M|\hat{O}|\{n\}\rangle\langle\{n\}|\hat{A}(0)\rangle^N\langle\{n\}|\hat{O}^{-1}|\{11..1\}_M\{00..0\}_M\rangle.$$

The sum is over the states labeled by the positions of $M$ particles $\{n\} = n_1\ldots n_M$ on a lattice consisting of $2M$ sites with the inhomogeneity parameters $\lambda_1,\ldots,\lambda_M, t_1,\ldots, t_M$. We use $\hat{O}^{-1} = \hat{f}^{-1}\hat{O}$ and the representations (10), (12) for the matrix elements in the last formula. Let
us denote by $\mu_1, \ldots, \mu_M$ the parameters (from the set $\{\{\lambda\}, \{t\}\}$) corresponding to the sites $n_1 \ldots n_M$, and by $\nu_1, \ldots, \nu_M$ the rest of the parameters so that $\{\lambda\} \cup \{t\} = \{\mu\} \cup \{\nu\}$. The first matrix element in the formula for $S_M$ is well known (see for example [4], [11]):

$$
\langle \{00..0\}_M | B(\mu_1) \ldots B(\mu_M) | 0 \rangle = \langle \{11..1\}_M | B'(\mu_1) \ldots B'(\mu_M) | 0 \rangle,
$$

(19)

where $B'(\mu)$ are the same operators defined on the lattice consisting of $M$ sites with the inhomogeneity parameters $t_1 \ldots t_M$. The second matrix element can be reduced to the same expression with the parameters $\lambda_1 \ldots \lambda_M$ (using the symmetry $C \leftrightarrow B, 0 \leftrightarrow 1$). Then using the formula for the matrix elements of the operator $\hat{f}^{-1}$ (see eq.[12]) we finally obtain the expression:

$$
S_M(\{\lambda\}, \{t\}) = \sum_{n_1, \ldots, n_M} \left( \prod_j a(\nu_j) \right) \Phi_M(t, \mu) \Phi_M(\lambda, \mu) \prod_{i,j} \frac{1}{c(\mu_i - \nu_j)},
$$

(20)

where we denoted by $\Phi_M(\xi, t)$ the functions in the right hand side of eq.[19]. The determinant representation of this function is well known (see for example [4], [7]). We present the basic properties of this function used in the calculations in the Appendix B. The functions $a(\nu)$ in eq.[20] are exactly the functions defined above $a(\nu) = \prod_\alpha c(\xi_\alpha - \nu)$ while in the rest of this formula due to the definition of the matrices $\tilde{T}_i$ one should interchange the arguments in the functions $\tilde{c}^{-1}(\nu_i - \nu_j)$ which is taken into account in eq.[20] (or one could make the replacement $\eta \to -\eta$). Using the properties of this functions one can represent the general formula [24] in a different way:

$$
\sum_{m=0}^M \sum_{k,n} \left( \prod(a(\lambda_n)a(t_k)) \right) \Phi_m(t_k, \lambda_\beta) \Phi_{M-m}(\lambda_n, t_\alpha) \prod_{i,j} \frac{1}{c(\lambda_\beta - \lambda_n)} \frac{1}{c(t_\alpha - t_k)} \frac{1}{c(t_\alpha - \lambda_n)} \frac{1}{c(\lambda_\beta - t_k)},
$$

where the sum is over the two sets $k_1, \ldots k_m$ and $n_1, \ldots n_{M-m}$. We used the following simplified notations in this formula. We devided the set $\{t\}$ into two subsets $\{t\} = \{t_k\} \cup \{t_\alpha\}$ where $\{t_k\} = (t_{k_1}, \ldots, t_{k_m}) \in \{\nu\}$, $\{t_\alpha\} \in \{\mu\}$ and analogously $\{\lambda\} = \{\lambda_n\} \cup \{\lambda_\beta\}$, $\{\lambda_n\} = (\lambda_{n_1}, \ldots, \lambda_{n_{M-m}}) \in \{\nu\}$, $\{\lambda_\beta\} \in \{\mu\}$. The products in the last formula are over the indices labeling the elements of the corresponding sets. One can further rewrite this formula using the expressions for the functions $\Phi_m$ to get the formula which leads to the determinant representation for $S_M$ after the special dual fields are introduced [3].

Let us derive the scalar product of the Bethe eigenstate with an arbitrary dual state along the lines of ref.[3] starting from the formula (20). The set of the parameters $\{t\}$ obey the
Bethe equations so for each \( a(t_i) \) in the sum (20) one can substitute the function

\[
f(t_i) = \prod_{\alpha \neq i} \frac{\tilde{c}(t_i - t_{\alpha})}{c(t_i - t_{\alpha})} = \prod_{\alpha \neq i} \frac{\phi(t_{\alpha} - t_i - \eta)}{\phi(t_{\alpha} - t_i + \eta)}.
\]

Then the idea is that one can calculate the sum in (21) for an arbitrary smooth function \( a(\lambda) \), which behaves at least as a constant at infinity, used for the terms \( a(\lambda_j) \) (not necessarily equal to \( a(\lambda) = \prod_\alpha \tilde{c}(\xi_\alpha - \lambda) \)). In that case the sum has the simple poles in the variables \( \{t\} \) and \( \{\lambda\} \) at the points \( t_i = \lambda_j \) and don’t have any other poles, for example the poles at \( \lambda_i = \lambda_j \), which exist if, according to [3] one considers \( a(\lambda_j) \) as an independent variables. These poles was not considered in [3] (actually only the behaviour of the function in the parameters \( t_i \) was considered - obviously there is no poles at \( t_i = t_j \) if the Bethe Ansatz equations are taken into account). One can use the symmetry of \( S_M \) in \( \{t\} \) and \( \{\lambda\} \) and single out the residue at \( \lambda_1 \to t_1 \) (which is contained in the term with \( \lambda_1 \in \{\nu\}, t_1 \in \{\mu\} \) and vice versa):

\[
S_M (\{\lambda\}, \{t\}, a(\nu_j)) |_{\lambda_1 \to t_1} \to \eta \frac{a(\lambda_1) - f(t_1)}{t_1 - \lambda_1} \prod_{\alpha \neq 1} \frac{1}{\tilde{c}(t_{\alpha} - t_1)} \frac{1}{\tilde{c}(\lambda_\alpha - t_1)} S_{M-1} (\{\lambda\}', \{t\}', a'(\nu_j)),
\]

where \( \{\lambda\}', \{t\}' \) do not contain \( \lambda_1, t_1 \) and

\[
a'(\nu_j) = a(\nu_j) \frac{\tilde{c}(\nu_j - t_1)}{\tilde{c}(t_1 - \nu_j)}.
\]

The functions \( f(t_i) \) entering the sum \( S_{M-1} \) are also modified so that the terms with \( t_1 \) are absent in their definition. Also the behaviour of \( S_M \sim 1/\lambda_1 \) at \( \lambda_1 \) going to infinity should be taken into account. It can be easily proved that the same recurrence relation at \( \lambda_1 \to t_1 \) is obeyed by the following expression:

\[
S_M (\{\lambda\}, \{t\}) = \frac{1}{\prod_{i<j} (t_i - t_j) \prod_{i<j} (\lambda_i - \lambda_j)} \det_{ij} (M_{ij}(t, \lambda))
\]

\[
M_{ij}(t, \lambda) = \frac{\eta}{(t_i - \lambda_j)} \left( a(\lambda_j) \prod_{\alpha \neq i} (t_\alpha - \lambda_j + \eta) - \prod_{\alpha \neq i} (t_\alpha - \lambda_j - \eta) \right),
\]

where at the end of calculations one should take the actual form of the function \( a(\lambda) \). So we proved that for an arbitrary function \( a(\lambda) \) the residues of the only existing poles at \( t_i = \lambda_j \) satisfy the same recursion relation in both formulas. Since the function (in each variable) is completely determined by the positions of (all) poles and the corresponding residues (along with the behaviour at the infinity) the two functions (22) and (20) should coincide for an arbitrary function \( a(\lambda) \). To complete the proof, one should check the formula (22) for \( M = 1, 2 \). Equivalently, one can check the coefficients corresponding to the terms \( \prod_{i=1}^M a(\lambda_i) \) and \( \prod_{i=1}^M a(t_i) = 1 \) in eq.(22) which are respectively

\[
\Phi_M(\lambda, t) \prod_{i,j} \frac{1}{\tilde{c}(t_i - \lambda_j)}, \quad \Phi_M(\lambda, t) \prod_{i,j} \frac{1}{\tilde{c}(\lambda_i - t_j)}.
\]
in agreement with (20). The formula (22) is written for the rational case. The generalization to the case of XXZ chain is obvious. This formula can also be represented through the Jacobian as:

\[
S_M(\{\lambda\}, \{t\}) = (-1)^M \frac{\prod_{i,j}(t_i - \lambda_j)}{\prod_{i < j}(t_i - t_j) \prod_{j < i}(\lambda_i - \lambda_j)} \det_{ij} \left( \frac{\partial}{\partial t_i} \Lambda(\lambda_j; \{t_\alpha\}) \right).
\]

Here the sign \((-1)^M\) can be absorbed into the product \(\prod_{i \neq j}(\lambda_i - t_j) = (-1)^M \prod_{i \neq j}(t_i - \lambda_j)\). The orthogonality of two different Bethe eigenstates was shown in ref. [8]. From this expression taking the limit \(\lambda_i \to t_i\) one can easily obtain the formula for the norm of the Bethe eigenstate:

\[
N_M(t) = \eta^M \frac{\prod_{i \neq j}(t_i - t_j + \eta)}{\prod_{i \neq j}(t_i - t_j)} \det_{ij} \left[ -\frac{\partial}{\partial t_j} \ln \left( \frac{a(t_i)}{f(t_i)} \right) \right],
\]

where \(f(t_i) = \prod_{\alpha \neq i} \tilde{c}(t_\alpha - t_i)/\tilde{c}(t_i - t_\alpha)\). Note that the matrix \(N_{ij} = -\partial/\partial t_j \ln (a(t_i)/f(t_i))\) can also be represented in the form:

\[
N_{ij} = \frac{2\eta}{(t_{ij} + \eta)(t_{ij} - \eta)}, \quad i \neq j,
\]

\[
N_{ii} = -\left. \frac{\partial}{\partial t_i} \ln (a(t_i)) \right| - \sum_{\alpha \neq i} \frac{2\eta}{(t_{\alpha i} + \eta)(t_{\alpha i} - \eta)},
\]

where \(t_{ij} = t_i - t_j\). The first term in \(N_{ii}\) should be understood as \(-(\ln(a(t)))'\) \((t = t_i)\). To calculate the correlators in the continuum limit it can be useful to represent the matrix \(N\) as a product of two matrices as

\[
N_{ij} = \sum_{\alpha} \left( \delta_{i\alpha} + (1 - \delta_{i\alpha}) N_{\alpha \alpha} (N_{\alpha \alpha})^{-1} \right) (\delta_{\alpha j} N_{jj}),
\]

where \(N_{ii} \sim R(t_i)\) - the density of the variables \(t_i\) in the continuum limit. The determinant expressions for the formfactors can be rather straightforwardly obtained from the above formulas using the explicit solution of quantum inverse scattering problem presented in ref. [3].

In conclusion, for the calculation of correlation functions using the formfactors, it seems necessary to develop the method to evaluate the dependence of the formfactors of local operators \(\sigma_i^{\pm, z}\) on the quantum numbers (quasiparticle numbers) characterizing the ground and the excited states, the problem which was not solved by now. The physical results have been obtained only for the simplest formfactors of the operator \(\sigma^z\) for the massless [3] and massive [12] (Baxter’s formula for spontaneous magnetization) regimes of the XXZ chain.

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**Appendix A.**

12
We give in this appendix the proof that the operator (13) introduced in the text is the factorizing operator:
\[ \hat{O} = \hat{F}_1\hat{F}_2\ldots\hat{F}_N, \quad \hat{F}_i = (1 - \hat{n}_i) + T_i\hat{n}_i, \] (23)
As it was explained in section 2 it is sufficient to prove the property \( \hat{O} = S_{i+1,i}^{(i,i+1)} \hat{O} \) for the one particular permutation \((i, i + 1)\). Consider the component \( \hat{F}_i\hat{F}_{i+1} \) in eq. (23). Substituting in this term \( T_i = S_{i+1,i} T_i^{(i+2\ldots N)} \), we obtain:
\[ \hat{F}_i\hat{F}_{i+1} = (1 - \hat{n}_i)(1 - \hat{n}_{i+1}) + S_{i+1,i} T_i^{(i+2\ldots N)} T_i^{(i+2\ldots N)} \hat{n}_i \hat{n}_{i+1} + \]
\[ S_{i+1,i} T_i^{(i+2\ldots N)} \hat{n}_i(1 - \hat{n}_{i+1}) + T_i^{(i+2\ldots N)} \hat{n}_{i+1}(1 - \hat{n}_i). \] (24)
Moving \( S_{i+1,i} \) to the right in the second term of eq. (24) using the Yang-Baxter equation, using the unitarity of the \( S \)-matrix and taking into account that \( S_{i,i+1}\hat{n}_i\hat{n}_{i+1} = \hat{n}_i\hat{n}_{i+1} \), we rewrite (24) in the following form:
\[ S_{i+1,i} (1 - \hat{n}_i)(1 - \hat{n}_{i+1}) + S_{i+1,i} T_i^{(i+2\ldots N)} T_i^{(i+2\ldots N)} \hat{n}_i \hat{n}_{i+1} + \]
\[ S_{i+1,i} T_i^{(i+2\ldots N)} \hat{n}_i(1 - \hat{n}_{i+1}) + T_i^{(i+2\ldots N)} \hat{n}_{i+1}(1 - \hat{n}_i) \] (25)
Comparing eq. (24) and eq. (23), we see that the expression in the brackets in (25) coincide with the right-hand side of (24) up to the interchange of the indices \( i \leftrightarrow i + 1 \) (together with the corresponding parameters \( \xi_i, \xi_{i+1} \)). In the expression (23) the indices \( i, i + 1 \) are contained only on the left from the operator \( \hat{F}_i\hat{F}_{i+1} \) in the operators \( T_n \) for \( n < i \):
\[ T_n = S_{n+1,n} \ldots S_{i,n} S_{i+1,n} \ldots S_{N,n}. \]
Thus moving the matrix \( S_{i+1,i} \) in (25) to the left and using the Yang-Baxter equation, we interchange the indices \( i, i + 1 \) in the whole expression for the operator \( \hat{O} \) (23) and arrive exactly to the formula (14).

Appendix B.

In this appendix we derive the expression for the function \( \Phi_M(\xi, t) \) - the partition function of a six-vertex model with domain wall boundary conditions, and point out its properties used in the text. For simplicity we consider the rational case. We define this function using the operators \( B(t) \) corresponding to the monodromy matrix defined according to \( \hat{T}_0(t) = S_{01}S_{02} \ldots S_{0M} \) with the inhomogeneity parameters \( \xi_1, \ldots, \xi_M \) (that corresponds to its definition in the text):
\[ \Phi_M(\{\xi\}, \{t\}) = \langle \{111..1\}_M | B(t_1)B(t_2) \ldots B(t_M) | 0 \rangle. \]
It is just the same function that appears in the formula (20). The function is determined entirely by the following properties.

1) At $M = 1$, $\Phi_1 = \eta/(t - \xi + \eta)$.
2) $\Phi_M(\{\xi\}, \{t\})$ is a symmetric function of the variables $\{t\}$ and separately $\{\lambda\}$.
3) If one chooses the initial (non-unitary) normalization of the $S$-matrix (and the corresponding $B$ operators), $\Phi_M$ is a polynomial of degree $M - 1$ in each of the variables $t_i, \lambda_i$. The polynomial in $t_i$ is completely fixed by its values at $M$ different points (for example at the points $\xi_j$).
4) At $t_1 = \xi_1$ the function reduces to $\Phi_{M-1}(\{\xi_i\}_{i \neq 1}, \{t_i\}_{i \neq 1})$ and the same for all the other pairs of variables. This can be easily seen using the symmetry property, by placing the corresponding variable $\xi_i$ to the end of the chain (at M-th site). Alternatively, one can use the representation of $B$-operators in the F-basis (14).

These properties determine the function $\Phi_M$ unambiguously (see eq.(21) in the text). One should mention that it does not have poles at $t_i = t_j$ and $\xi_i = \xi_j$ since at these values the two lines or columns in the determinant coincide.

One can also determine the function $\Phi_M(\xi, t)$ directly using the operators in the F-basis. This was done in ref.[3]. Let us present here this proof with some technical modifications which could be important for the other problems of this type and will be used in the Appendix C.

Consider the determinant

$$\det_{ij}(M_{ij}) = \det_{ij} \left( \frac{1}{(t_i - \lambda_j)(t_i - \lambda_j + \eta)} \right).$$

First, we modify the first line of the matrix $M_{1j}$ adding a linear combination of the other lines so that the determinant $\det M$ is unchanged:

$$M'_{1j} = M_{1j} + \sum_{x \neq 1} C_{x} M_{xj}$$

where the coefficients are chosen to be

$$C_{x} = -\prod_{\beta \neq j} \frac{(t_x - \lambda_\beta + \eta)}{(t_1 - \lambda_\beta + \eta)} \prod_{\alpha \neq 1, x} \frac{(t_1 - t_\alpha)}{(t_x - t_\alpha)}.$$

It is necessary to calculate the following sum:

$$\sum_{x \neq 1} C_{x} M_{xj} = -\prod_{\alpha \neq 1} \frac{(t_1 - t_\alpha)}{(t_1 - \lambda_j + \eta)} \sum_{x \neq 1} f(t_x) \prod_{\alpha \neq 1, x} \left( \frac{1}{t_x - t_\alpha} \right) \frac{1}{(t_1 - t_x)} \frac{1}{(t_x - \lambda_j)},$$

(26)

where

$$f(t_x) = \prod_{\beta \neq j} \frac{(t_x - \lambda_\beta + \eta)}{(t_1 - \lambda_\beta + \eta)}.$$
To calculate the sum in eq. (26) consider the integral in the complex plane over the circle of large radius which is equal to zero due to the behaviour of the integrand at infinity:

\[
\oint dz \frac{f(z)}{(z-t_1)(z-\lambda_j) \prod_{\alpha \neq 1}(z-t_\alpha)} = 0,
\]

where

\[
f(z) = \prod_{\beta \neq j} \frac{(z-\lambda_\beta + \eta)}{(t_1-\lambda_\beta + \eta)}.
\]

The sum of the residues at \( z = t_\alpha \) is equal to the sum in (26), so it is easily calculated to be

\[
f(t_1) \frac{1}{(t_1-\lambda_j)(t_1-\lambda_j + \eta)} \prod_{\alpha \neq 1} (t_1-t_\alpha) \prod_{\beta \neq j} (\lambda_j - \lambda_\beta + \eta),
\]

Thus combining all terms we obtain the following expression for the new elements of the first line:

\[
M'_{1j} = \frac{1}{(t_1-\lambda_j)(t_1-\lambda_j + \eta)} \prod_{\alpha \neq 1} (t_1-t_\alpha) \frac{(\lambda_j - \lambda_\beta + \eta)}{(t_1-\lambda_\beta + \eta)} \prod_{\beta \neq j} (\lambda_\beta - \lambda_j + \eta).
\]

The same formula could be obtained for the first column of the matrix \( M_{ij} \). Note also that taking the coefficients \( C_x \) in a different way we could obtained the other expressions of this type. For example, taking the coefficients

\[
M'_i = M_i + \sum_{x \neq 1} C_x M_{ix}, \quad C_x = -\prod_{\alpha} (\lambda_x - t_\alpha) \prod_{\beta \neq 1,x} (\lambda_x - \lambda_\beta),
\]

we obtain the new expression of the first column of the matrix \( M' \) in the form:

\[
M'_i = \frac{1}{(t_i-\lambda_1)(t_i-\lambda_1 + \eta)} \prod_{\alpha \neq i} (t_i-t_\alpha + \eta) \frac{(\lambda_1 - \lambda_\beta)}{(\lambda_1 - t_\beta)} \prod_{\beta \neq 1} (t_i-\lambda_\beta + \eta),
\]

which is different from (27). Note also an interesting property of the function \( \Phi_M(\xi, t) \), which can be seen from these formulas. Namely the function \( \Phi_M \) is equal to zero at the points \( \xi_i = \xi_j \pm \eta \). We will use this property in the Appendix C.

Now the derivation of the formula (27) is straightforward. Using the operators \( B(t) \) in the \( F \)-basis and acting on the vacuum state by one of the commuting operators, say the operator \( B^F(t_1) \) first, we get the formula

\[
\Phi_M(\xi, t) = \sum_{i=1}^{M} f_{\xi_i}(t_1) F(\xi, t), \quad f_{\xi_i}(t_1) = \tilde{b}(t_1 - \xi_i) \prod_{\alpha \neq i} \tilde{c}(t_1 - \xi_\alpha) \frac{\tilde{c}(\xi_i - \xi_\alpha)}{c(\xi_i - \xi_\alpha)}
\]

(we change the arguments in comparison with eq. (13) due to our definition of the function \( \Phi_M \)), where the function \( F(\xi, t) \) corresponds to the action of the other operators \( B^F(t_j) \). Each
term in this sum corresponds to the occupied site \( i \), so due to the form of the operators \( B^F \) the action of the other operators

\[
B^F(t_M) \ldots B^F(t_3)B^F(t_2)
\]
on this state does not contain the variable \( \xi_i \). Thus we single out the dependence of the function on the variables \( \xi_i \) and \( t_1 \), and the function \( F(\xi, t) \) is equal to

\[
F(\xi, t) = \Phi_{M-1}(\{\xi_\beta\}_{\beta \neq i}, \{t_\alpha\}_{\alpha \neq 1}),
\]
and does not contain these variables. Apart from the factor in front of the determinant that procedure corresponds to the first line development of the determinant in (21). One can check that starting from the last formula we get exactly the first line development of the determinant with the matrix elements \( M'_{1i} \) instead of \( M_{1i} \). In fact, assuming that the function \( \Phi_{M-1} \) is given by the formula (21) we obtain from the last formula:

\[
\Phi_M(\xi, t) = \frac{\prod_{i,j}(t_i - \xi_j)}{\prod_{i<j}(t_i - t_j) \prod_{j<i}(\xi_i - \xi_j)} \sum_i (-1)^i \det(M_{\alpha\beta})^{(M-1)}_{\alpha \neq 1, \beta \neq i}(\xi, t)
\]

\[
\left\{ \frac{\eta}{(t_1 - \xi_i)(t_1 - \xi_i + \eta)} \prod_{\alpha \neq 1} (t_1 - t_\alpha) \prod_{\beta \neq i} (\xi_i - \xi_\beta + \eta) \right\}.
\]
Comparing this formula with (27) we see that the last term in the brackets is exactly equal to \( M'_{1i} \) and we obtain the exactly the determinant of the matrix \( M' \). Since we have shown that the two determinants are equal this procedure gives us the recurrence relation which proves the formula (21).

Appendix C.

In this appendix we will establish the direct correspondence between the calculations of the correlators in the F- basis and the previous calculations [4] based on the commutation relations (or, alternatively using the method of the present paper). Namely, starting from the expression of the scalar product in the F- basis we obtain the general formula (20) for an arbitrary parameters \( \{t\}, \{\lambda\} \). To obtain the expression of the scalar product in the F- basis we insert the complete set of states labeled by the coordinates \( \{x\} = (x_1, \ldots x_M) \) as follows:

\[
S_M = \sum_{\{x\}} \langle 0|C^F(\lambda_1) \ldots C^F(\lambda_M)\{x\}\rangle \langle \{x\}|B^F(t_1) \ldots B^F(t_M)|0\rangle.
\]
One can use the expressions for \( B^F, C^F \) from Section 3 to find the expression for \( S_M \) (which is possible due to the quasilocal structure of these operators). However the simplest way is to
reduce the matrix elements to the scalar products of the usual form:

\[
S_M(\{\lambda\}, \{t\}) = \sum_{x_1, \ldots, x_M} \left( \prod_{i,j, \alpha_i \neq x_j} \frac{1}{\hat{c}(\xi_{\alpha_i} - \xi_j)} \right) \left( 0 | C(\lambda_1) \ldots C(\lambda_M) B(\xi_{x_1}) \ldots B(\xi_{x_M}) | 0 \right)
\]

\[
= \sum_{\{x\}} \prod_{\{\xi\}} \frac{1}{\hat{c}(\xi_{\alpha} - \xi_x)} \left( \Phi_M(\lambda, \xi_x) \prod_{\{\lambda\}} \frac{1}{\hat{c}(\xi_\lambda - \lambda)} \right) \left( \Phi_M(t, \xi_x) \prod_{\{t\}} \frac{1}{\hat{c}(\xi_x - t)} \right),
\]

where we have introduced the sets of parameters \(\{\xi\} = \{\xi_x\} \cup \{\xi_\alpha\}\) with \(\{\xi_x\} = (\xi_{x_1} \ldots \xi_{x_M})\) and the products are over all elements of the corresponding sets of parameters \((\{\xi_x\}, \{\xi_\alpha\}, \{t\}, \{\lambda\})\). The function given by the product of two brackets in eq.28 has only simple poles at \(\xi_{x_i} = t_j\) and \(\xi_{x_i} = \lambda_j\). We remind that the function \(\Phi_M(\lambda, \xi_x) \prod(\xi_x - \lambda + \eta)\) has no poles in \(\xi_{x_i}\). The behaviour of this function at the infinity is \(\sim 1/\xi_{x_i}^2\) for each \(\xi_{x_i}\). To take the sum we apply the same method as in the Appendix B. The problem is that now we have to sum over \(M\) coordinates \(x_1, \ldots, x_M\) with the condition \(x_i \neq x_j\). First, as an example, let us consider the following simplified sum which is quiet similar to that in eq.28:

\[
S = \sum_{x_1, \ldots, x_M} \left( \prod_{i,j, \alpha_i \neq x_j} \frac{1}{\xi_{\alpha_i} - \xi_j} \prod_{i,j} \frac{1}{\xi_{x_i} - \lambda_j} \right).
\]

To calculate this sum let us multiply both the denominator and the numerator by \(\prod_{i \neq j}(\xi_{x_i} - \xi_{x_j})\). Then we obtain the following sum:

\[
\sum_{x_1, \ldots, x_M} \prod_{i \neq j}(\xi_{x_i} - \xi_{x_j}) \prod_{\alpha \neq x_1}(\xi_\alpha - \xi_{x_1}) \ldots \prod_{\alpha \neq x_M}(\xi_\alpha - \xi_{x_M}) \prod_{i,j}(\xi_{x_i} - \lambda_j),
\]

where the sum is still over the configurations with \(x_i \neq x_j\). However one can relax this condition since in the denominator we have the function which is not equal to zero at \(x_i = x_j\), while the numerator is equal to zero at \(x_i = x_j\). Then the sum is represented by the product of the sums of the type (28) (with some degree of \(\xi_{x_i}\) in the numerator) which can be easily calculated by the same method. In fact, by means of the polynomial

\[
\prod_{i \neq j}(\xi_i - \xi_j) = \sum_{k_1, \ldots, k_M} C_{k_1, \ldots, k_M} \xi_{k_1}^{k_1} \ldots \xi_{k_M}^{k_M},
\]
(k_i \in \mathbb{Z}^+)$ the sum is represented as

$$S = \sum_{k_1, \ldots, k_M} C_{k_1, \ldots, k_M} \left( \sum_{x_1} \frac{\xi_i^{k_1}}{f(\xi_{x_1})} \right) \cdots \left( \sum_{x_M} \frac{\xi_i^{k_M}}{f(\xi_{x_M})} \right),$$

where

$$f(\xi_{x_i}) = \prod_{\alpha \neq x_i} (\xi_\alpha - \xi_{x_i}) \prod_{j=1}^M (\lambda_j - \xi_{x_i}).$$

Each of the sums over $x_i$ can be rewritten as in the Appendix B:

$$\sum_{x_1} \frac{\xi_i^{k_1}}{f(\xi_{x_1})} = \sum_{x_1} \frac{\lambda_i^{k_1}}{f(\lambda_{x_1})}, \quad \tilde{f}(\lambda_{x_1}) = \prod_{\alpha} (\xi_\alpha - \lambda_{x_1}) \prod_{j \neq x_1} (\lambda_j - \lambda_{x_1}).$$

Then we obtain the product of the other sums of this type and the procedure can be repeated in the opposite direction. Thus we obtain the following simple result for this sum: $S = \prod_{\alpha=1}^N \prod_{i=1}^M (\xi_\alpha - \lambda_i)^{-1}$. Now using the same method one can calculate directly the sum (28).

Schematically, for each of the variables $\xi_{x_i}$ one can take either the pole at $t_i$ or the pole at $\lambda_j$. Taking the pole at $t_i$ we get $1/a(t)$ which cancels the corresponding factor in front of the sum in (28). Using the notations of the formula (20), that means that the corresponding $t_i$ belongs to the set $\{\mu\}$. We should also obtain the factor $\tilde{c}^{-1}(\mu - \nu)$ arriving to the formula (20). In fact, using the same method as before we represent this sum as:

$$\sum_{x_1, \ldots, x_M; x_i \neq x_j} \prod_{j=1}^M \left( \frac{1}{\prod_{\alpha \neq x_j} \tilde{c}(\xi_\alpha - \xi_{x_j}) \prod_{i=1}^M \tilde{c}(\xi_{x_i} - t_i) \prod_{i \neq j} \tilde{c}(\xi_{x_i} - \xi_{x_j})} \right) \prod_{i \neq j} (\xi_{x_i} - \xi_{x_j}) \prod_{i \neq j} (\xi_{x_i} - \xi_{x_j} + \eta)^{-1} \Phi_M(t, \xi_x) \Phi_M(\lambda, \xi_x)$$

The function in the second line of this formula $F(\xi_x, t, \lambda)$ - is the function of $3M$ variables. The function $F$ has the following properties. 1) $F$ is equal to zero at $x_i = x_j$, so we can relax the condition $x_i \neq x_j$ and sum over all $x_i$ independently. 2) The function $F(\xi_x, t, \lambda)$ does not have poles at the points $\xi_{x_i} = \xi_{x_j} \pm \eta$ since the function $\Phi_M(t, \xi_x)$ is equal to zero at these points. This can be seen from the representation given by the last equation of the Appendix B. The only poles contained in the function $F$ are at $\xi_{x_i} = t_j - \eta$ and $\xi_{x_i} = \lambda_j - \eta$. 3) The behaviour of $F$ at $|\xi_{x_i}| \to \infty$ is $\sim 1/\xi_{x_i}^2$. We can take the sum consequently over $x_1, \ldots, x_M$. Taking the sum over $x_1$ we obtain residues at some $t_i$ or $\lambda_i$ (which corresponds to the set $\{\mu\}$). One can see that the residues corresponding to the poles contained in the function $F$ are equal to zero due to the terms of the first line of the last formula. Then we repeat this procedure for $x_2, \ldots, x_M$. Note that the pole we take for $x_2$ should not coincide with the pole we took at the previous stage (for $x_1$). One could formulate the same procedure in another, equivalent way. Consider the function

$$F'(\xi_x, t, \lambda) = F(\xi_x, t, \lambda) \prod_{i} (\xi_x - t + \eta) \prod_{i} (\xi_x - \lambda + \eta) =$$

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\[
\prod_{i \neq j} c(\xi_{x_i} - \xi_{x_j}) \Phi_M(t, \xi_x) \Phi_M(\lambda, \xi_x) \prod_{i,j} (\xi_{x_i} - t_j + \eta) \prod_{i,j} (\xi_{x_i} - \lambda_j + \eta)
\]

This function does not have poles at all and therefore is a polynomial. Then the sum is represented as the product of independent sums (as for our simplified example). Thus, we finally obtain the result:

\[
\sum_{n_1 \ldots n_M} \prod_{i=1}^M \frac{1}{a(\mu_i)} \prod_{i,j} \frac{1}{c(\mu_i - \nu_j)} \Phi_M(t, \mu) \Phi_M(\lambda, \mu),
\]

where the notations are the same as in the equation (20). Thus the equation (20) is reproduced. Note that one could start from the equation (20) and derive the equation (28) using the same method substituting \( \prod_i a(\nu_i) = \prod_i a(t_i) / \prod_i a(\mu_i) \) and taking the sum over \( \mu_1, \ldots, \mu_M \) with \( a(\mu) = \prod_{\alpha=1}^N c(\xi_{\alpha} - \mu) \) (taking the residues corresponding to the poles at the points \( \mu_i = \xi_{x_i} \)).

In conclusion, let us present two equivalent ways to rewrite the expression (20) for the case when the parameters \( \{t\} \) satisfy the Bethe Ansatz equations. Expressing the functions \( a(t_k) \) and using the explicit form of the function \( \Phi_M \) we get:

\[
S_M(\lambda, t) = \frac{1}{\prod_{i<j}(t_i - t_j)} \prod_{j<i}(\lambda_i - \lambda_j) \sum_{k,n} (-1)^{P_k} (-1)^{P_n} \prod (a(\lambda_n))
\]

\[
\det(t_k, \lambda_\beta) \det(\lambda_n, t_\alpha) \prod(\lambda_\beta - \lambda_n + \eta)(t_k - t_\alpha + \eta)(t_\alpha - \lambda_n + \eta)(\lambda_\beta - t_k + \eta),
\]

where again the products is over all elements of the corresponding sets of parameters. Here the sign factors \( P_k, P_n \) depend on the sets of the coordinates \( k_1, \ldots, k_m, n_1, \ldots, n_{M-m} \) in order to obtain the factors in front of the sum from the terms \( \prod(t_k - t_\alpha) \) and \( \prod(\lambda_\beta - \lambda_n) \). We also used the simplified notations for the determinants (21). One can further rewrite this formula as follows:

\[
S_M(\lambda, t) = \frac{1}{\prod_{i<j}(t_i - t_j)} \prod_{j<i}(\lambda_i - \lambda_j) \sum_{k,n} (-1)^{P_k} (-1)^{P_n} (-1)^{\lambda M_m} \prod (a(\lambda_n))
\]

\[
\prod(t - \lambda_n + \eta) \prod(t - \lambda_\beta + \eta) \det(t_k, \lambda_\beta) \det(\lambda_n, t_\alpha) \prod \left( \frac{t_k - t_\alpha + \eta}{t_k - \lambda_n + \eta} \right) \prod \left( \frac{\lambda_\beta - \lambda_n + \eta}{\lambda_\beta - t_\alpha + \eta} \right),
\]

where we denote for example \( \prod(t - \lambda_n + \eta) = \prod \prod_{i=1}^M (t_i - \lambda_n + \eta) \). The other notations are the same as before. In this formula the poles at \( t_i = \lambda_j \) are contained in the determinants. If the two last products are dropped out this formula becomes the result of the decomposition of the determinant in eq. (22). Then one can see explicitly that the residues in the poles in the formulas (20) and (22) satisfy the same recurrence relations.
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