The Stress Tensor in Quenched Random Systems

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Abstract

The talk describes recent progress in understanding the behaviour of the stress tensor and its correlation functions at a critical point of a generic quenched random system. The topics covered include: (i) the stress tensor in random systems considered as deformed pure systems; (ii) correlators of the stress tensor at a random fixed point: expectations from the replica approach and \(c\)-theorem sum rules; (iii) partition function on a torus; (iv) how the stress tensor enters into correlation functions: subtleties with Kac operators.

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This talk is about the stress tensor in generic quenched random systems in which we expect the quenched averaged correlation functions to be those of some conformal field theory. Many of the results generalise to arbitrary dimension, but I shall take $d = 2$ for simplicity.

**Quenched random systems as deformed pure systems.**

Consider the quenched random system defined by the action

$$S = S_P + \sum_{i=1}^{N} \int h_i(r) \Phi_i(r) d^2r$$

$S_P$ is a non-random CFT; $\Phi_i(r)$ is a set of primary fields (assumed to be scalars – we can generalise to vector couplings). The $h_i(r)$ are quenched random variables, with $h_i(r) = 0$ and $h_i(r')h_j(r'') = \lambda_{ij}\delta(r' - r'')$. We take $N = 1$ for simplicity in this talk.

We are interested in the RG flow: $S_P \Rightarrow$ (random fixed point). The perturbation is not necessarily small: the idea is to see how objects in $S_P$ deform in the full theory. One of the tools will be replica group theory. There is an analogy with the use of group theory in atomic physics, where we can deduce the nature of the the splittings in the spectrum even when the couplings are relatively large.

Recall how Zamolodchikov considered deformed CFT in pure systems: the action is

$$S = S_0 - \lambda \int \Phi(r) d^2r$$

where $\lambda$ is a constant. The deformation of the $zz$ component of the stress tensor is, to first order in $\lambda$,

$$\delta T(z, \bar{z}) = \lambda \int_{|z' - z| > a} d^2z' T(z) \cdot \Phi(z', \bar{z}')$$

so that the conservation equation becomes

$$\partial_z T(z, \bar{z}) = \lambda \int d^2z' \delta(|z - z'|^2 - a^2)(z - z') \left( \frac{\Delta}{(z - z')^2} \Phi(z, \bar{z}) + \frac{1 - \Delta}{(z - z')} \partial_z \Phi(z, \bar{z}) + \cdots \right)$$

$$= -\partial_z \Theta$$

where

$$\Theta = -\pi \lambda (1 - \Delta) \Phi \quad (d = 2)$$

$$\propto -\lambda (d - x_\Phi) \Phi \quad \text{(general } d)$$

Note that no higher order terms in $\lambda$ arise, as long as no additional renormalisation is required. This is unnecessary if $x_\Phi < d$, but in general $\Theta \propto \beta(\lambda) \Phi_R$, so that $\Theta$ vanishes at the new IR fixed point.
Now do this for a random coupling $\lambda \to h(z, \bar{z})$:

$$\partial \bar{z} T = \int d^2 z' \delta(|z - z'|^2 - a^2)(z - z')h(z', \bar{z}') \left( \frac{\Delta}{(z - z')^2} \Phi(z, \bar{z}) + \frac{1 - \Delta}{(z - z')} \partial_z \Phi(z, \bar{z}) + \cdots \right)$$

but now $h(z', \bar{z}')$ is white noise. After some stochastic calculus, the result is

$$\partial \bar{z} T + \partial_z \Theta = K$$

where

$$\Theta(z, \bar{z}) = -\pi \left( \frac{1}{2} - \Delta_\Phi \right) h(z, \bar{z}) \Phi(z, \bar{z}) \quad (d = 2)$$

$$[\propto (d - \frac{d}{2} - x_\Phi) h \Phi] \quad (\text{general } d)$$

and

$$K = \frac{1}{2} \pi (h \partial_z \Phi - \Phi \partial_z h)$$

Note that $T$ and $\Theta$ are the components of the stress tensor for a given realisation of randomness, not the quenched average! The extra contribution $\frac{d}{2}$ comes from the white noise nature of $h(r)$. The derivative $\partial h$ does not make literal sense since $h(r)$ is a stochastic function: it is interpreted by integrating by parts in correlators. There is a similar equation relating $\mathcal{T}$ to the same $\Theta$: so that $T_{z\bar{z}} = T_{\bar{z}z} \propto \Theta$: the stress tensor is symmetric even in the random system, because local rotational symmetry is preserved (the results are slightly modified if the coupling is to a random vector).

**Replica formulation.**

How does all this appear within the replica formulation?

$$\mathcal{Z}^n = \int \mathcal{D}h e^{-\lambda (1/2) \int \Phi} \text{Tr} e^{-\sum_a S_{P,a} + \int h \sum_a \Phi \partial \Phi}$$

$$= \text{Tr} \int \mathcal{D}h e^{-\lambda (1/2) \int \Phi} e^{-\sum_a S_{P,a} + \frac{1}{2} \lambda \int \sum_{a \neq b} \Phi_a \Phi_b}$$

which has the form of a translationally invariant perturbed CFT.

The replicated theory has a stress tensor $\mathcal{T}$ which is a deformation of $\sum_a T_a$, so

$$\partial \bar{z} \mathcal{T} + \partial_z \vartheta = 0$$

where $\vartheta = -\frac{1}{2} \lambda \pi (1 - 2\Delta) \sum_{a \neq b} \Phi_a \Phi_b$. Note that neither $\mathcal{T}$ nor $\vartheta$ are the components of the true stress tensor, discussed in the previous section.

At the new fixed point, $\vartheta = 0$, and

$$\langle \mathcal{T}(z) \mathcal{T}(0) \rangle = c(n)/2z^4$$
where, by the \( c \)-theorem sum rule,
\[
c(n) - nc_P = -(12/\pi) \int r^2 \langle \vartheta(r) \vartheta(0) \rangle_c d^2 r
\]
This has the following interpretation:
\[
\langle TT \rangle = \sum_{a,b} \langle T_a T_b \rangle = n \langle T_1 T_1 \rangle + n(n-1) \langle T_1 T_2 \rangle
\]
\[
\sim n \left( \langle TT \rangle - \langle T \rangle \langle T \rangle \right)
\]
so that, at the random fixed point
\[
\langle TT \rangle_c = \frac{c_{\text{eff}}}{2z^4}
\]
where \( c_{\text{eff}} = c'(0) \), and
\[
\delta c_{\text{eff}} = -3\pi \lambda^2 (1 - 2\Delta)^2 \lim_{n \to 0} (1/n) \sum_{a \neq b \neq c \neq d} \int r^2 \langle \Phi_a(r) \Phi_b(r) \Phi_c(0) \Phi_d(0) \rangle_c d^2 r
\]
\[
= -3\pi(1 - 2\Delta)^2 \int r^2 h(r) h(0) \langle \Phi(r) \Phi(0) \rangle_c d^2 r
\]
\[
= -\frac{3\pi(1 - 2\Delta)^2}{\text{area}} \int r^2 h(r_1) h(r_2) \langle \Phi(r_1) \Phi(r_2) \rangle_c d^2 r_1 d^2 r_2
\]
The second expression follows from undoing the replacement
\( h \to \lambda \sum_a \Phi_a \) in the gaussian integration. The last line expresses the fact that the quenched average is unnecessary if instead we average over the whole system: this version of the \( c \)-theorem sum rule thus applies to a \textit{given realisation} of the randomness. Note there is no obvious positivity: we expect that \( h \Phi > 0 \), but the above involves \( h(\Phi - \langle \Phi \rangle) \).

However, there are in addition \( n - 1 \) other independent components of the deformed stress tensor:
\[
\mathcal{T} = \sum_a T_a
\]
\[
\tilde{T}_a = T_a - (1/n) \mathcal{T}
\]
where \( \sum_a \tilde{T}_a = 0 \). These combinations are chosen to transform according to irreducible representations of \( S_n \), so that they should deform into conformal fields at the new fixed point, with well-defined scaling dimensions \((2 + \delta(n), \delta(n))\). It may be checked in perturbation theory that \( \delta \neq 0 \).

In the undeformed theory,
\[
\langle \tilde{T}_a \tilde{T}_b \rangle = \left( \delta_{ab} - \frac{1}{n} \right) \frac{c}{2z^4}
\]
so we choose, at the new fixed point,

\[
\langle \tilde{T}_a \tilde{T}_b \rangle = \left( \delta_{ab} - \frac{1}{n} \right) \frac{c(n)}{2n} \frac{1}{z^4 (zz)_{2\delta(n)}}
\]

Then

\[
\langle T \rangle \langle T \rangle = \lim_{n \to 0} \langle T_1 T_2 \rangle
\]

\[
= \lim_{n \to 0} \langle (\tilde{T}_1 + (1/n) T)(\tilde{T}_2 + (1/n) T) \rangle
\]

\[
= \lim_{n \to 0} \frac{c'(0)}{2z^4} \left( -\frac{1}{n} (zz)^{-2\delta(n)} + \frac{1}{n} \right)
\]

\[
= \frac{\tilde{c}_{\text{eff}}}{2z^4} \ln(z\bar{z})
\]

where

\[
\tilde{c}_{\text{eff}} = 2c'(0)\delta'(0)
\]

Now \( \tilde{T}_a \) is not conserved: in fact

\[
\partial_z \tilde{T}_a + \partial_{\bar{z}} \tilde{\vartheta}_a = K_a
\]

where

\[
\tilde{\vartheta}_a = -\pi \lambda \left( \frac{1}{2} - \Delta \right) \Phi_a \sum_{c \neq a} \Phi_c
\]

\[
K_a = \frac{1}{2} \pi \lambda \sum_{b \neq a} \left( \Phi_a \partial_z \Phi_b - \Phi_b \partial_z \Phi_a \right)
\]

This is equivalent to the previous equation for a fixed \( h(r) \) by the substitution \( \lambda \sum_b \Phi_b \to h(r) \). From the above one can derive a sum rule for \( \delta \tilde{c}_{\text{eff}} \) in terms of suitably averaged correlators of \( \Phi \) (but once again there is no positivity).

In a general renormalisation scheme we find that

\[
\tilde{\vartheta}_a = -\frac{1}{2} \pi \left( \vartheta(\lambda) + \delta(n) \right) \sum_{c \neq a} (\Phi_a \Phi_c)_R - (1/n) \vartheta
\]

so that \( \tilde{\vartheta}_a = O(n) \) at the random fixed point. One should, however, be cautious in setting it to zero at \( n = 0 \) in correlation functions, because of factors of \( 1/n \).

**The torus partition function.**

For a general conformal field theory, the torus partition function encodes its operator content. In the replicated theory for general \( n \) we expect therefore

\[
\mathcal{Z}^n = (q\bar{q})^{-c(n)/24} \left( 1 + q^2 + (n-1)q^{2+\delta(n)} + q^2 + \cdots \right)
\]
Now this should equal 1 when \( n = 0 \). It is clear to see how the \( O(q^2) \) and \( O(\bar{q}^2) \) terms cancel, but all the descendants of these must do so in addition! It is easy to see that this requires postulating the existence of new Virasoro primaries at each level, whose scaling dimensions coincide with those of the descendents of more relevant operators at \( n = 0 \). This suggests that there is massive degeneracy of Virasoro primaries as \( n \to 0 \), suggesting that there is an underlying extended algebra in all such theories, possibly supersymmetry, even when it is not apparent.

It is interesting to compute the quenched free energy \( \ln Z = \left(\frac{\partial}{\partial n}\right)|_{n=0} Z = -(c_{\text{eff}}/24) \ln(q\bar{q}) - \delta'(0)(q^2 + \bar{q}^2) \ln(q\bar{q}) + \cdots \)

where \( \delta'(0) = \tilde{c}_{\text{eff}}/2c_{\text{eff}} \). We see the appearance of logarithms, and also the second effective central charge \( \tilde{c}_{\text{eff}} \). There are still many unresolved questions, including how modular invariance works, and how to characterise boundary states.

### Operator product expansions.

Let us begin by describing the so-called "\( c \to 0 \) catastrophe". For any primary operator \( \phi \) in any CFT, its OPE with itself takes the form

\[
\phi(z, \bar{z}) \cdot \phi(0, 0) = \frac{a_\phi}{z^{2\Delta} \bar{z}^{2\bar{\Delta}}} \left( 1 + \frac{2\Delta}{c} z^2 T + \cdots + \frac{4\Delta \bar{\Delta}}{c^2} z^2 \bar{z}^2 (T \bar{T}) + \cdots \right)
\]

so, in the 4-point function,

\[
\langle \phi\phi\phi\phi \rangle \propto a_\phi^2 (1 + (2\Delta/c)^2 (c/2)\eta^2 + \cdots + O(1/c^4) c^2 (\eta\bar{\eta})^2 + \cdots)
\]

where \( \eta \) is the cross-ratio. There is an obvious problem as \( c \to 0 \). There are three possible resolutions:

1. other operators in \( \cdots \) cancel the divergence;
2. \( a_\phi \to 0 \) as \( c \to 0 \);
3. \((\Delta, \bar{\Delta}) \to (0, 0) \) as \( c \to 0 \).

Let us see what happens in the replica approach. Set \( \Phi = \sum_a \Phi_a, \tilde{\Phi}_a = \Phi_a - (1/n)\Phi \). These are chosen to transform according to irreducible representations of the permutation group of the replicas. In the pure theory, the OPEs are schematically

\[
\tilde{\Phi}_a \cdot \tilde{\Phi}_a = (1 - 1/n)(z\bar{z})^{-4\Delta} \left( 1 + \frac{2\Delta}{cn} z^2 T + \frac{2\Delta}{c} z^2 \bar{T}_a + \cdots \right)
\]

\[
\Phi \cdot \Phi = n(z\bar{z})^{-4\Delta} \left( 1 + \frac{2\Delta}{cn} z^2 T + \frac{2\Delta^2}{(cn)^2} (z\bar{z})^2 T \bar{T} + \frac{2\Delta^2}{c^2} (z\bar{z})^2 \sum_a \bar{T}_a T_a + \cdots \right)
\]
which deform into

$$\tilde{\Phi}_a \cdot \tilde{\Phi}_a = (1 - 1/n)(z\bar{z})^{-4\Delta_\Phi} \left( 1 + \frac{2\Delta_\Phi}{c(n)}z^2\mathcal{T} + \text{const. } z^2(\delta_0^c(n))_{\tilde{T}_a} + \cdots \right)$$

$$\Phi \cdot \Phi = n(z\bar{z})^{-4\Delta_\Phi} \left( 1 + \frac{2\Delta_\Phi}{c(n)}z^2\mathcal{T} + \frac{2\Delta_\Phi^2}{c(n)^2}(z\bar{z})^2\mathcal{T}\mathcal{T} + \text{const. } (z\bar{z})^{2+\delta_2(n)}\mathcal{M} + \cdots \right)$$

where $\mathcal{M}$ is a new primary operator with dimensions $(2 + \delta, 2 + \delta_2(n))$. $\tilde{\Phi}$ and $\Phi$ resolve the “$c \to 0$ catastrophe” according to schemes 1 and 2 respectively. Their 4-point functions have the form

$$\langle \tilde{\Phi}_a \tilde{\Phi}_a \tilde{\Phi}_a \tilde{\Phi}_a \rangle \sim 1 + \delta'(0)\eta^2\ln(\eta\bar{\eta}) + \cdots$$

$$\langle \Phi \Phi \Phi \Phi \rangle \sim n \left( \eta^2 + \cdots + \delta'_2(0)(\eta\bar{\eta})^2\ln(\eta\bar{\eta}) + \cdots \right)$$

Note that the connected correlators of $\Phi$ all vanish proportional to $c$ as $n \to 0$.

$\Phi_a \equiv \tilde{\Phi}_a + (1/n)\Phi$ and $\Phi$ are an example of a logarithmic pair: at $c = 0$

$$\langle \Phi_a(z, \bar{z})\Phi_a(0, 0) \rangle \sim (z\bar{z})^{-4\Delta} \ln(z\bar{z})$$

$$\langle \Phi_a(z, \bar{z})\Phi(0, 0) \rangle \sim (z\bar{z})^{-4\Delta}$$

$$\langle \Phi(z, \bar{z})\Phi(0, 0) \rangle = 0$$

It turns out that Kac operators are always examples of the second solution to the $c = 0$ catastrophe:

- Def.: a Kac operator $\phi$ has scaling dimensions at some fixed position in the Kac table for a range of $c$ including 0.

Now only other Kac operators can appear in the OPE $\phi \cdot \phi$: this excludes a companion of $\mathcal{T}$, which would have dimension $(2 + \delta, \delta)$, which does not appear in the Kac table. Hence we must have resolution 2: $a_\phi \to 0$ as $c \to 0$. (But note that $\mathcal{M}$ with dimensions $(2 + \delta_2, 2 + \delta_2)$ does exist, giving rise to $(\eta\bar{\eta})^2\ln(\eta\bar{\eta})$ terms in the 4-point function. Explicit calculations confirm this.) If we choose $a_\phi \propto c^p$, one can show that the $2N$-point connected correlator goes like $c^{N(p-1)+1}$, so it is natural to take $p = 1$. This is exactly what happens in physical examples of percolation ($((Q \to 1)$-Potts model) or self-avoiding walks (O($n \to 0$) model), where Kac operators enter into physical quantities only through derivatives wrt $c$ of correlators. This suggests that Kac operators in such $c = 0$ theories are always the partner of a (non-Kac) logarithmic operator. In the above examples these other operators may be identified away from $c = 0$. 


Summary

- The stress tensor in a general quenched random system, with a given distribution of impurities, satisfies
  \[ \partial_z T + \partial_z \Theta = K \]
  with explicit expressions for \( \Theta \) and \( K \).

- at a random fixed point,
  \[
  \langle TT \rangle_c = \frac{c_{\text{eff}}}{2z^4} \\
  \langle TT \rangle = \left( \frac{\tilde{c}_{\text{eff}}}{2z^4} \right) \ln(z\bar{z})
  \]

- there are sum rules for the change in \( c_{\text{eff}} \) and \( \tilde{c}_{\text{eff}} \) along a RG trajectory between 2 fixed points, in terms of physically measurable correlators.

- there must be a massive degeneracy of operators at \( c = 0 \). This suggests an extended symmetry, but the candidates \( \tilde{T} \) for its generators are not holomorphic fields!

- some operators solve the “\( c \to 0 \) catastrophe” by having connected correlators which are all \( O(c) \) – this is true of all Kac operators – but the physics is in the \( O(c) \) term and is therefore invisible in the theory at \( c = 0 \). This suggests that approaches to taking the quenched average which work exactly at \( c = 0 \), such as supersymmetry, cannot expose all the physics.

References

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