CONSTRUCTIONS OF CONTACT FORMS ON PRODUCTS AND PIECEWISE FIBERED MANIFOLDS

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Abstract. We study constructions of contact forms on closed manifolds. A notion of strong symplectic fold structure is defined and we prove that there is a contact form on $M \times X$ provided that $M$ admits such a structure and $X$ is contact. This result is extended to fibrations satisfying certain natural conditions. Some examples and applications are given.

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1. Introduction

In this paper we study constructions of contact forms on closed orientable manifolds. An intricate question of contact topology is whether any closed almost contact manifold admits a contact structure. It is solved positively only in dimensions three and five [19, 13, 7, 9]. However, even in low dimensions, this is usually very non-trivial to construct explicitly a contact form on a given almost contact manifold.

There are some obvious classes of almost contact manifolds. First of all, the product of an almost complex manifold $M$ (more generally, a stably almost complex manifold of even dimension) with a contact manifold $X$ is almost contact. It is known that $M \times X$ is contact if $M$ is an orientable surface and $X$ is contact (see [4] for the case of genus $> 0$ and [5] for $M = S^2$). Our aim and the principal motivation was to understand the case of $M$ closed and of arbitrary dimension.

There is a simple case when a contact form exists on $M \times X$. Let $(M, \omega)$ be an exact symplectic manifold (i.e., its symplectic form is exact, $\omega = d\beta$) and $\eta$ be a contact form on $X$. Then the product form $\beta + \eta$ is contact. Exact symplectic manifolds are necessarily open, so this cannot be applied directly to closed manifolds. However, if $M$ is compatibly decomposed into the sum of exact symplectic pieces, then there is a formula [12] which yields a contact form on $M \times S^1$. To be a bit more precise, $M$ is assumed to be a sum of exact symplectic cobordisms which meet at their convex ends and agree on their common boundaries. Thus $M$ is cut by a hypersurface and along it the symplectic forms of adjacent pieces yield a fold. We call such decomposition a strong symplectic fold of convex type (see Section 2 for the precise definition and comments on the formula).

Theorem 3.1 says that the product of a manifold with strong symplectic fold of convex type with a contact manifold is contact and it is the base for further construction and applications. The proof has two main ingredients: the Giroux - Mohsen [16, 17] theorem which states that any contact form can be deformed to a contact form given by an open

Formally we should write $p_M^* \beta + p_X^* \eta$, where $p_M, p_X$ are projections, but to simplify the notation we omit projections. For the same reason wedge signs are omitted in exterior products of forms.
book decomposition, and the heat flow deformation of a foliation to a contact form given by Altschuler and Wu [2]. Open books together with the Geiges - Stipsicz formula enable us to define a foliation and the heat flow applied to it gives a contact form.

Then we extend this theorem in two ways. First, we allow some bundles over the exact symplectic pieces. In particular, we prove that there exists a contact form on the total space of any bundle over a strong symplectic fold with contact fiber if some rather natural conditions are satisfied. For instance, this holds for any bundle over $S^{2n}$ if the structure group preserves the contact form of the fiber. Secondly, we show that in $M$ one can allow also concave folds, i.e., the fold is given by two concave ends of symplectic cobordisms (see Section 6).

We give a number of examples and applications. They include products $X \times S^{k_1} \times \ldots \times S^{k_r}$ provided that $X$ is a contact manifold and $k_1 + \ldots + k_r$ is even and $M \times S^{k_1} \times \ldots \times S^{k_r}$ if $M$ is a strong symplectic fold of contact type and $k_1 + \ldots + k_r$ is odd. We show also examples of homogenous spaces which are contact but have no invariant contact forms. Moreover, we show that some surgeries and blowing ups preserve contactness (cf. Proposition 5.5 and Example 7.5). We describe also a generalization of the open book construction of contact forms (see Section 8).

To give a sample of applications, consider the following fillability question. Any contact form $\lambda$ on $X$ yields the form $e^t \lambda$ on $X \times [0, 1]$ with symplectic exterior differential (called symplectification of $\lambda$). $(X, \lambda)$ is called fillable, if there exists a symplectic form on a manifold $W$ with $\partial W = X$ equal to $d(e^t \lambda)$ on a collar of the boundary. There are obstructions to fillability, in particular in dimension 3 no overtwisted form is fillable. However, there is an interesting and natural weaker question whether the product form $e^t \lambda + d\phi$, where $d\phi$ is the standard orientation form of $S^1$, extends to a contact form on $W \times S^1$. In Proposition 7.1 we construct such extensions from $X = S^{2n+1}$ to $D^{2n+2}$ for some forms on $S^{2n+1}$. If $n = 1$, one can use as $\lambda$ also some overtwisted forms. This shows that after multiplying with $S^1$ the obstruction to fillability disappears, at least for some classes of contact forms. This is a new proof of a result of Etnyre and Pancholi [10]. See Section 7 for details.

The constructions of contact forms on bundles over strong symplectic can be localized. This leads to a class of decompositions into fibered pieces which are still sufficient to get contactness. In Appendix A we give a preliminary version of this. We will study such notion together with its applications in a future paper. Appendix B contains sample computations in low dimensions performed using Mathematica.

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2. Preliminaries

We consider compact, smooth, orientable manifolds and we want to find constructions of smooth contact forms on a possibly large class of manifolds.

Geiges and Stipsicz [12] gave a formula which yields a contact form on products $M \times S^1$ for some closed $M$. Let us describe their construction in a slightly more general setup. We start with the definition of a structure which is crucial for our main theorem.
**Definition 2.1.** A strong symplectic fold structure of convex type on a compact manifold $M$ is a decomposition $M = W_- \cup W_+$, where $N = W_- \cap W_+$ is a hypersurface in $\text{Int} M$, together with exact symplectic forms $\omega_- = d\gamma_-$, $\omega_+ = d\gamma_+$ on respectively $W_-, W_+$, such that the forms satisfy the following convexity conditions on a tubular neighborhood $N \times [-1, 1]$ of $N$ and at $\partial M$:

1. $\gamma_- = e^t \lambda$ on $N \times [-1, 0] = N \times [-1, 1] \cap W_-$ and $\gamma_+(t) = e^{-t} \lambda$ on $N \times [0, 1] = N \times [-1; 1] \cap W_+$, where $t$ is the parameter of $[-1, 1]$ and $\lambda$ is a contact form on $N$.

2. the closure of every component of $M - N$ containing a component of $\partial M$ is an exact symplectic cobordism (by (1)), it is necessarily convex at the $N-$end, either convex or concave at the component of $\partial M$.

The hypersurface $N$ is called the fold locus. The product $N \times [a; b]$ endowed with the form $d(e^t \lambda)$ is called the symplectization of a contact form $\lambda$ on $M$. Hence, in the above definition we assume that on both sides of $N$ we have symplectizations of $\lambda$. If the Liouville vector field of $\omega_\pm$ is transverse to $N$, then one can deform the symplectic forms to symplectizations.

An obvious example is the double $W \cup (-W)$, where $W$ is a compact manifold with boundary and $W$ admits an exact symplectic form satisfying convexity condition (1) at $\partial W$. Note that a strong symplectic fold does not determine the orientation, since the orientations given by the symplectic forms on any two adjacent components of $M - N$ are opposite.

In our terminology we follow Ana da Silva [6]. She shows that on any closed stably almost complex manifold there exists a symplectic fold, i.e. a 2-form which is symplectic everywhere except for a hypersurface, where the form has fold singularities. A symplectic fold is globally defined and smooth. It is symplectic outside a hypersurface and gives opposite orientations on any two adjacent parts. However, in general the symplectic forms are not exact and the behavior along the singular hypersurface differs from what we require for strong symplectic folds. For instance, symplectic folds do not need give contact forms on the singular hypersurface.

**Theorem 2.2.** [12] If $M^{2m}$ admits a strong symplectic fold of convex type, then $M \times S^1$ is contact.

**Proof.** Let $d\phi$ denote the standard orientation form on $S^1$ and $p : M \times S^1 \to M$ be the projection. If $\omega_\pm = d\gamma_\pm$, then $p^*\gamma_\pm + d\phi$ are contact forms outside $N \times [-1, 1] \times S^1$.

Choose smooth functions $f, g : [-1, 1] \to \mathbb{R}$ such that:

1. $g$ is odd, equal to 1 near $t = -1$, equal to $-1$ near $t = 1$, and it is decreasing from $-1$ to 1,

2. $f$ is even, positive, equal to $e^{\pm t}$ near $\pm 1$ and increasing on $[-1, 0]$,

3. $f'g - g'f > 0$ on $[-1; 1]$.

Then the formula

$$\alpha = f\lambda + g \, d\phi$$

on $[-1, 1] \times N \times S^1$ yields a contact form on $N \times [-1, 1] \times S^1$ (with contact form $\lambda$ on $N$) which extends those defined above. In fact, it is not difficult to calculate:
\[ \alpha(d\alpha)^n = nf^{n-1}(f'g - fg')dt\lambda(d\lambda)^n d\theta > 0. \]

Geiges and Stipsicz apply this formula to show that for every closed orientable 4-manifold \( M \) the product \( M \times S^1 \) is contact. They use [3] where it is shown that any closed orientable 4-manifold admits a strong symplectic fold of convex type.

We want to use the above formula when the circle is replaced by a general contact manifold \( X \). For this purpose it is necessary to have a pair of contact forms on \( X \) defining opposite orientations and connected by a path of forms with controlled disruption of contactness. Moreover, one can see rather easily that the orientation change should be "one dimensional", for example given by changing the direction of a vector field transversal to the contact structure. To construct this we will use contact forms defined in terms of open book decompositions. So let us recall this construction.

**Definition 2.3.** An open book decomposition of \( X \) is given by

1. a codimension two submanifold \( B \subset X \) (called the binding),
2. a tubular neighborhood \( U \) of \( \partial B \) diffeomorphic to \( B \times D^2 \),
3. a fibration \( \pi : E = X - B \to S^1 \) with fiber \( P \) (called the page)

such that the monodromy of the fibration \( \pi \) is equal to the identity in \( P \cap U \) and \( \pi|U \) can be identified with the standard projection \( B \times (D^2 - \{0\}) \to S^1 \).

According to [21], one can associate a contact form (which we will call of open book type) with any open book decomposition satisfying the following conditions:

1. \( P \) is exact symplectic, i.e., \( P \) has 1-form \( \beta \) such that \( d\beta \) is symplectic on \( P \),
2. a tubular neighborhood \( U \) of \( \partial P \) is of convex type, which means that in a collar \( \partial P \times [0, \varepsilon) \) we have \( \beta = e^{-\varepsilon t} \nu \) with \( \nu \) contact on \( \partial P \),
3. the monodromy \( f : P \to P \) of \( \pi \) is exact, which means that \( f^* \beta - \beta = d\psi \) for some function \( \psi : P \to \mathbb{R} \).

Before we write a formula for such form, let us note that the main theorem of [16, 17] says that any contact form is homotopic (i.e., there exists a deformation through contact forms) to a form of open book type. It is not unique, but assuming that a contact form is of open book type does not restrict generality.

If \( f : P \to P \) is the monodromy of \( \pi \), we identify \( E \) with the quotient of \( P \times [0, 2\pi R] \) for some fixed \( R \), by the identification \( \Phi : (x, 0) \sim (f^{-1}(x), 2\pi R) \).

On \( P \times [0, 2\pi R] \) we put \( \eta_E = \beta + d\phi \) with

\[ \beta = \beta + u(\phi)d\psi \]

for some non-decreasing function \( u : [0; 2\pi R] \to [0; 1] \) so that for a small \( \varepsilon > 0 \)

\[ u(\phi) = \begin{cases} 
0 & \text{for } \phi \in [0; \varepsilon) \\
1 & \text{for } \phi \in (2\pi R - \varepsilon; 2\pi R].
\end{cases} \]

The form \( \beta \) descends to \( (P \times [0, 2\pi R])/\sim \) since
\[\Phi^*(\beta + d\psi + d\phi) = \beta + d\phi\] and \(\eta_E\) defines a smooth form on \(E\). Moreover, if dimension of \(P\) is \(2n\), then \(\eta_E(d\eta_E)^n = d\psi(d\beta)^n + n\beta(d\beta)^{n-1}u'(\phi)d\phi d\psi\). As \(d\beta^n > 0\) on \(P\) and for \(R\) big enough the derivative \(|u'(\phi)|\) can be made arbitrary small, \(\eta_E\) is contact.

**Remark 2.6.** As far as we know, such "enlarging the circle" trick has never been used before in this context. When we tried to apply the formulae we had been able to find in the literature, then we needed an additional assumption, essentially that the fibration \(E \to S^1\) was trivial. It was rather unexpected that the simple trick described above enabled us to solve this problem.

In the sequel we will use a deformation of such form to one having the opposite orientation of \(S^1\) in the fibration \(E \to S^1\). For this reason we have to consider the family of forms \(\eta_E = \beta + u d\psi(\phi) + l d\phi\) depending on \(l \in \mathbb{R}\). Now \(\Phi^*(\beta + d\psi + l \cdot d\phi) = \beta + l \cdot d\phi\), so \(\eta_E = \beta + l \cdot d\phi\) is well-defined on \((P \times [0, 2\pi R]) / \sim\) for any \(l \in \mathbb{R}\). We have

\[\eta_E(d\eta_E)^n = ld\phi \left((d\beta)^n - n\beta(d\beta)^{n-1}u'(\phi)d\psi\right).

(2.7)

Note that the formula implies that our choice of \(R\) does not depend on \(l\) and we get

**Proposition 2.8.** If \(R\) is large enough, then all forms in the family \(\eta_E\) are contact for \(l \neq 0\).

As the monodromy \(f\) is the identity near the boundary \(\partial P\), the form \(\bar{\beta} + l \cdot d\phi\) \((l \in \mathbb{R})\) is equal to \(\nu e' + l \cdot d\phi\) near the boundary of \(B \times D^2\) in polar coordinates \((r, \phi)\) on \(D^2\). Now we extend \(\bar{\beta} + l \cdot d\phi\) to \(B \times D^2\) by the formula

\[\alpha = h_1(r)\nu + l \cdot h_2(r)d\phi,\]

where

\[(2.9)\]

\[h_1(r) = \begin{cases} 2 & \text{for } r = 0 \\ e^{1-r} & \text{for } r \in [1; R], \end{cases}\]

is strictly decreasing with all derivatives at 0 vanishing,

\[(2.10)\]

\[h_2(r) = \begin{cases} r^2 & \text{near } r = 0 \\ 1 & \text{for } r \in [1; R] \end{cases}\]

and nondecreasing with \(h_1(r)h_2'(r) - h_1'(r)h_2(r) > 0\) (see the drawing below). As another simple calculation shows, the resulting form is contact on \(X\).
If \( l = \pm 1 \), for a suitable choice of \( u \) and \( R \) big enough, both forms \( \eta_{\pm} = \overline{\beta} \pm d\phi \) are contact. They determine opposite orientations and we use this pair of forms together with the family \( \eta_l, l \in \mathbb{R} \), in the sequel.

**Notation.** If \( \eta \) is one form of such a pair, then by \( \tilde{\eta} \) we denote the other one.

**Remark 2.11.** In some proofs in the sequel we use the following well-known fact: if \( \eta_1, \eta_2 \) are contact and homotopic on \( X \), then there is a topologically trivial symplectic cobordism \( M = X \times [0, 1] \) between \( (X, \eta_1) \) and \( (X, \eta_2) \). Let \( M \) be a compact manifold with boundary of contact type such that the resulting form on \( \partial M \) is \( \lambda \). If we have a homotopy from \( \lambda \) to \( \lambda' \), we can add a trivial cobordism to the boundary of \( M \) so that we get \( \lambda' \) on \( \partial M \). In particular, for a manifold with boundary of contact type, we can always assume that we have a contact form of open book type on \( \partial M \).

Our principal analytic tool is the heat flow deformation of a confoliation \([2]\). On a closed manifold \( Y^{2m+1} \) consider a confoliation, i.e. a 1-form \( \alpha \) satisfying the inequality \( \alpha \wedge (d\alpha)^m \geq 0 \). The points \( x \in Y \) where \( \alpha \wedge (d\alpha)^m > 0 \) are called contact (regular), the other (non-contact) points are called singular and the set of singular points will be denoted by \( \Sigma \). Altschuler and Wu show that under some assumptions, the heat flow can deform the confoliation to a contact form. To describe those assumptions we choose a Riemannian metric \( g \) on \( Y \) and consider the form \( \tau = \ast (\alpha \wedge (d\alpha)^{m-1}) \), where \( \ast \) denotes the Hodge star. Then at every point \( x \in Y \) we denote by \( \mathcal{D} \subset TY_x \) the orthogonal complement of \( \text{Null}(\tau)_x = \{ V \in T_x Y : \iota_V \tau = 0 \} \). At a contact point the subspace \( \mathcal{D} \) has dimension \( 2m \) and it is perpendicular to \( \text{Null}(\tau)_x \). At a point where rank of \( d\alpha \) on \( \ker \alpha \) is \( 2m-2 \), the dimension of \( \mathcal{D} \) is 2, and dim \( \mathcal{D} \) is zero at points where rank of \( d\alpha|\ker \alpha \) is less than 2m-2. A point \( x \) is called accessible if there is a smooth curve \( \sigma : [0, 1] \to Y \) such that \( z'(t) \in \mathcal{D} \) and is non-zero for all \( t \in [0, 1] \), \( z(0) = x \) and \( z(1) \) is a contact point. Thus we see that in the case when the rank of \( d\alpha|\ker \alpha \) is less than 2m-2 no singular point is accessible. Since we have to reduce the general case to that of corank at most 3, this is one of the main difficulties of our construction.

In the sequel we will use the following theorem.

**Theorem 2.12.** \([2]\) Suppose that \( Y \) is a closed manifold with a confoliation \( \alpha \). If every non-contact point of \( Y \) is accessible, then \( Y \) supports a contact form \( C^\infty \)-close to \( \alpha \).

### 3. Main theorem

Our main theorem is the following.

**Theorem 3.1.** If \((X^{2n+1}, \alpha)\) is a closed contact manifold and \( M^{2n} \) admits a strong symplectic fold of convex type, then \( X \times M \) is contact.

**Proof.** Consider the decomposition \( M = W_1 \cup (N \times [-1; 1]) \cup W_2 \) and the forms \( \omega_+ , \omega_-, \lambda \) given by the strong symplectic fold on \( M \). Here \( N = W_+ \cap W_- \) and \( N \times [-1, 1] \) is a tubular neighborhood of \( N \) with \( N \times [-1, 0] \subset W_- , N \times [0, 1] \subset W_+ , W_1 = W_- - N \times (-1, 0] , W_2 = W_+ - N \times [0, 1] , \omega_\pm = d\gamma_\pm \).

We can assume that the contact form \( \alpha \) is of open book type with \( P , B \) denoting the page and the binding. We use the notation introduced in Section \([2, 3]\), function \( u \) (formula \([2, 5]\)), and \( h_1, h_2 \) (formulae...
We know \([2]\) that the necessary condition for accessibility is that \(\text{rank } d\tilde{\eta} \mid \ker \tilde{\eta} \geq 2(m+n-1)\). Unfortunately, on \(\Sigma = B \times \{0\} \times N \times \{0\}\) we have \(\text{rank } d\tilde{\eta} \mid \ker \tilde{\eta} < \)
2(m + n) - 2 since \( d\tilde{\eta}|T(X \times M)|_\Sigma = 2d\nu + d\lambda \) and \( \tilde{\eta}|T(X \times M)|_\Sigma = 2\nu + \lambda \). Thus the singular points are not accessible. In order to remedy this we change the conflation form making it asymmetric with respect to the decomposition \( W_1 \cup (N \times [-1; 1]) \cup W_2 \). Roughly speaking, we impose in this way some more transversality along the singular set. Define the form \( \eta \) on \( X \times M \) by the formula

\[
\eta = \begin{cases} 
\frac{1}{e} (\beta + d\phi + \gamma_\pm) & \text{on } B \times D^2 \times W_1 \\
\frac{1}{e^2} (h_1(r)\nu + f(t)\lambda + h_2(r)g(t)d\phi) & \text{on } B \times D^2 \times N \times [-1; 1] \\
e^{2\beta - d\phi + \gamma_+} & \text{on } B \times D^2 \times W_2.
\end{cases}
\]

In formula above \( k : [-1 - \varepsilon, 1 + \varepsilon] \to [e^{-1}; e] \) is a smooth, positive, non-decreasing function satisfying

\[
k(t) = \begin{cases} 
e^{-1} & \text{on } (-1 - \varepsilon; -1] \\
\varepsilon^t & \text{on } [-1 + \varepsilon; 1 - \varepsilon] \\
\varepsilon & \text{on } [1; 1 + \varepsilon)
\end{cases}
\]

with \( \varepsilon \) small enough.

Because \( k > 0 \) and \( \tilde{\eta} \) is contact on the complement of \( \Sigma \), hence \( \eta \) is also contact on \( X \times M - \Sigma \). By continuity, \( \eta(d\nu)^{m+n} \geq 0 \) on \( \Sigma \). Therefore we get again a smooth conflation with the same critical set \( \Sigma = B \times \{0\} \times N \times \{0\} \).

To apply \cite{2} we choose a Riemannian metric \( \langle \cdot, \cdot \rangle \) on \( X \times M \) such that near \( \Sigma \) submanifolds \( N, I = [-1, 1], B, D^2 \) are pairwise orthogonal. We will check that \( \eta \) satisfies the assumption of Theorem 2.12. Consider \( \tau = \ast (\eta(d\nu)^{m+n-1}) \) and \( D = \text{Null}(\tau)^\bot \). We will show that for every point \( (b, v) \in B \times \{0\} \times N \times \{0\}, \) the radial path \( z(r) = (b, (r, \phi), 0, v) \subset B \times D^2 \times N \times I \) (with \( z'(r) = \frac{\partial}{\partial r} \in TD^2 \) for \( r \in (0; R] \) and any fixed \( \phi \in [0, 2\pi] \)) satisfies \( z'(t) \in D \), hence every \( x \in \Sigma \) is accessible from a contact point. The proof is divided into two parts. We check first that we have \( D = TD^2 \) on \( \Sigma \) and then that \( z'(r) \in D \) for \( r \in (0; R] \).

**Lemma 3.5.** Under the assumptions above, \( D = TD^2 \) on \( \Sigma \).

**Proof.** By formula (3.4), \( \eta = e^\nu \tilde{\eta} \), hence

\[
d\eta = e^\nu dt\tilde{\eta} + e^\nu d\tilde{\eta} = 
e^\nu dt (h_1(r)\nu + f(t)\lambda + h_2(r)g(t)d\phi) + 
+ e^\nu (h_1(r)dr \nu + h_1(r)d\nu + f'(t)dt\lambda + f(t)d\lambda + h_2'(r)g(t)dr d\phi + h_2(r)g'(t)dtd\phi).
\]

Substituting \( t = r = 0 \) gives that \( \eta|T(X \times M)|_\Sigma = 2\nu + \lambda \) and \( d\eta|T(X \times M)|_\Sigma = 2d\nu + d\lambda + dt(2\nu + \lambda) \) with \( \Sigma = B \times \{0\} \times N \times \{0\} \). As \( dv^m = 0, d\lambda^n = 0 \) on \( \Sigma \), we easily calculate:

\[
\eta(d\eta)^{m+n} = (m + n - 1)(2d\nu + d\lambda)^{m+n-2}dt(2\nu + \lambda) = 
\eta(m + n - 1)2^{m-1}(dv)^{m-1}(d\lambda)^{n-1}dt(2\nu + \lambda) = 
=C dv^m d\lambda^n dt = C dvol_B dvol_N dt
\]

for some positive constant \( C \). Thus \( \ast (\eta(d\eta)^{m+n-1}) = \pm C dvol_{B^2} \) and \( D = TD^2 \). \( \square \)

The last lemma implies that \( z'(0) \in D \). However, it is not clear yet if \( z'(r) \in D \) beyond \( \Sigma \). So now we determine \( D \) for \( r > 0 \). The proof is an elementary but long computation, hence we skip some parts of it.
Since η is contact on X × M − Σ, by [2] we have that 2-form τ = * (η(ν))^{m+n-1} is of maximal rank (= 2(m + n)), and its nullity bundle Null(τ) is 1-dimensional. Thus D = Null(τ)⊥ is 2(m + n)-dimensional. For the remaining part of the proof, it is enough to check that Null(τ) is perpendicular to ∂τ on B × (D² − ∅) × N × ∅. By our choice of metric, ∂τ is perpendicular to Tr = B × S¹ r × N × I (with S¹ r = {p ∈ D² : |p| = r}, r > 0). Therefore once we show that for t = 0 the bundle null(τ) is tangent to Tr or, equivalently, that on Tr the form τ is degenerate (i.e., of rank < 2(n + m) = dim Tr), the proof of Theorem 3.1 is completed.

As in Lemma 3.5, substituting t = 0 in formula (3.6) gives \( \tilde{η}T(X × M)|S = h_1(r)ν + 2λ \) and \( d\tilde{η}T(X × M)|S = h'_1 dr ν + h_1 dv + dλ - h_2 dt dφ \) on \( S = B × D² × N × ∅ \). We obviously have \( (d\tilde{η})^{m+n-1} = (d\tilde{η}^r + d\tilde{η}^n)^{m+n-1} = (d\tilde{η})^{m+n-1} = (m + n - 1)(d\tilde{η})^{m+n-2} dt d\tilde{η} \) on S. Further, as \( dv^n = 0, dλ^n = 0 \) we get

\[
(d\tilde{η})^{m+n-1} = \left( \frac{m + n - 1}{n - 1} \right) (dλ)^{n-1}(h'_1 dr ν + h_1 dv - h_2 dt dφ)^m + \\
+ \left( \frac{m + n - 1}{n - 2} \right) (dλ)^{n-2}(h'_1 dr ν + h_1 dv - h_2 dt dφ)^m + \\
= (dλ)^{n-1}((dν)^{m-1}(D_1 dr ν + D_2 dt dφ) + D_3(dν)^{m-2} dr dt dφ) + \\
+ D_4(dλ)^{n-2}(dν)^{m-2} dr dt dφ
\]

for some functions \( D_i (i ∈ \{1, 2, 3, 4\}) \) of variable r. In a similar manner we calculate \( dt \tilde{η}(d\tilde{η})^{m+n-2} = dt \tilde{η}(h'_1 dr ν + h_1 dv + 2dλ - h_2 dt dφ)^{m+n-2} = \\
= dt \tilde{η}(h'_1 dr ν + h_1 dv + 2dλ)^{m+n-2} = \\
= dt \tilde{η} \left( \frac{m + n - 2}{n - 1} \right) ((h_1 dv)^{m-1}(2dλ)^{n-1} + (m + n - 2)(h_1 dv + 2dλ)^{m+n-3}h'_1 dr ν) .
\]

After arduous, but elementary computation, we get that

\[
η(d\tilde{η})^{m+n-1} = C_1 ν(dν)^{m-1} dr λ(dλ)^{n-1} + C_2 ν(dν)^{m-1} dφ(dλ)^{n-1} dt + \\
+ C_3(dν)^{m-1} dφ λ(dλ)^{n-1} dt + C_4 ν(dν)^{m-2} dr dφ λ(dλ)^{n-1} dt \\
+ C_5 ν(dν)^{m-1} dr dφ λ(dλ)^{n-2} dt + C_6 ν(dν)^{m-1} λ(dλ)^{n-1} dt
\]

for some functions \( C_i, i = 1, \ldots, 6 \) of variable r. Furthermore, \( ν = * (ν(dν)^{m-2}) \) in B and \( λ = * (λ(dλ)^{n-2}) \) in N both have maximal ranks equal to respectively 2m − 2 and 2n − 2. If we additionally set \( ν_1 = * ((dν)^{m-2}) \) in B and \( λ_1 = * ((dλ)^{m-2}) \) in N, then

\[
τ = * (η(d\tilde{η})^{m+n-1}) = E_1 dt dφ + E_2 λ_1 dr + E_3 dr ν_1 + E_4 ν + E_5 λ + E_6 dr dφ
\]

again for some functions \( E_i, i = 1, \ldots, 6 \) of variable r. The pullback of τ to \( T_r = B × S¹ r × N × I \) via the inclusion \( j : T_r ↪ M × X \) yields

\[
j^*τ = j^* ( * (η(d\tilde{η})^{m+n-1}) ) = E_1 dt dφ + E_4 ν + E_5 λ.
\]

The rank of this form is equal to \( 2(m-1)+(n-1)+2 = 2(m+n)−2 < 2(m+n) = \dim T_r \), hence τ ↦ Tr is degenerate on Tr. As we said earlier, this implies that Null(τ) is tangent to Tr, hence \( \frac{∂τ}{∂τ} ∈ \text{Null}(τ)⊥ = D \) for r > 0. This completes the proof.
4. CONTACT FORMS ON BUNDLES

In this section we discuss constructions of contact forms on bundles of two types:

1. exact bundles: bundles over a contact base with exact symplectic fiber and structure group of exact symplectomorphisms;
2. contact bundles: bundles over a strong symplectic fold of exact type with contact fiber and structure group of strict contactomorphisms.

In both cases our results require some further properties of the bundles. For exact bundles of type \(1\) we will need the following property. Let \(E\) with fiber \(F\) hypersurface \(H\) such that the map \(A: H \times F \to H \times F\) is obtained by gluing the product pieces along \(H\) with \(A\). The definition applies also in the case when \(B\) is the circle, then as the hypersurface one can take a single point.

Given an exact symplectic manifold \((M, \omega = d\beta)\), denote by \(Ex(M, \beta)\) the group of exact symplectomorphisms and by \(Ex(M, \partial M, \beta)\) the group of exact symplectomorphisms equal to the identity near the boundary.

**Proposition 4.1.** If \(\pi: E \to B\) is a bundle with compact contact base \((B, \mu)\), compact exact symplectic fiber \((F, \omega = d\beta)\), the structure group contained in the group of exact symplectomorphisms \(Ex(F, \beta)\) and defined on a hypersurface \(H \subset B\) if its restriction to \(B - H\) is trivial and there is a map \(a: H \to G\) such that the map \(A: H \times F \to H \times F\) is smooth and the bundle is obtained by gluing the product pieces along \(H\) with \(A\). The definition applies also in the case when \(B - H\) is connected. If \(B\) is the circle, then as the hypersurface one can take a single point.

A contact form on bundles

**Proposition 4.2.** Let \((W, \omega_0 = d\beta)\) be a compact exact symplectic manifold, \(\pi: E \to W\) a bundle over \(W\) with a closed contact fiber \((X, \eta_0)\). If the structure group of the bundle...
is contained in the group $\text{Cont}(X, \eta_0)$ of diffeomorphisms preserving the contact form $\eta_0$ (strict contactomorphisms), then $E$ admits a contact form. If the bundle is trivial over $\partial X$ and $\beta = e^t \lambda$ in a collar $U$ of $\partial W$, where $\lambda$ is a contact form on $\partial W$, then one may require the form to be the product form $K e^t \lambda + \eta_0$ in a collar of $U \times X$, where $K$ is a large enough positive constant.

Proof. We will use the symplectization of the fiber and the well-known Thurston construction of symplectic forms on bundles. Let $\{U_s\}_{s \in S}$ be an open cover of $W$ with local trivializations $\Psi_s : \pi^{-1}(U_s) \cong U_s \times X$. If $\{f_s\}_{s \in S}$ is the partition of unity subordinated to $\{U_s\}_{s \in S}$, then we define a symplectic form $\omega = d(K \pi^* \beta + e^t(\sum_{s \in S} f_s \Psi_s^* \eta_0))$ on $E \times [-\varepsilon, \varepsilon]$ for some $K$ big enough and $\varepsilon > 0$. Let $R$ be the Reeb vector field of $\eta_0$. Its interior products with $\eta_0, d\eta_0$ are $\iota_R \eta_0 \equiv 1, \iota_R d\eta_0 \equiv 0$. Since $\eta_0$ is preserved by the structure group of the bundle, there is a horizontal vector field $\tilde{R}$ on $E$ such that its pushforward by $\Psi_s$ is equal to $R$ for any $s \in S$. This implies that $\tilde{R}$ is the Reeb field of $\Psi_s^* \eta_0 | \pi^{-1}(w)$ for any $s$ and $w \in W$. Thus, if $\eta = \sum_{s \in S} f_s \Psi_s^* \eta_0$, then we have $\iota_{\tilde{R}} d\eta \equiv 1, \iota_{\tilde{R}} \eta \equiv 0$. Therefore for the Liouville vector field $L$ of $\omega$ we have $\iota_L \omega = K \pi^* \beta + e^t \eta$. If we additionally apply $\iota_R$ to the last equation, we get $-\iota_{L, R} \omega = -\iota_{\tilde{R}} (e^t d\eta + e^t d\eta) = e^t \iota_L d\eta = e^t$. This implies that $L$ is transversal to $E$, hence $E \cong E \times \{0\} \subset E \times [-\varepsilon, \varepsilon]$ is contact. The additional convexity property of $\partial X$ follows from the fact that one can take $U$ as a trivialization chart (i.e. $U \in \{U_s\}_{s \in S}$).

Consider now contact bundles over a strong symplectic fold. We will prove a generalization of Theorem 3.1 in this case.

Let $(X, \eta)$ be a closed contact manifold and let $(W_{\pm}, N, \lambda_{\pm})$ be a strong symplectic fold of convex type on $M$. Consider a bundle $E \to M$ with fiber $X$ and let $E_{\pm} \to W_{\pm}$ denote its restrictions to $W_{\pm}$. We assume that the bundle is trivial over the fold locus $N = \partial W_- \cap \partial W_+, E_- \cong X$ with respect to $(X, \eta)$, $E_+ \cong X$ with respect to $(X, \eta')$. We also assume that there exists a contact form $\eta_0$ of open book type on $X$ such that $\eta$ is homotopic to $\eta_0$ and $\eta'$ to $\tilde{\eta}_0$.

Theorem 4.3. If $E$ is the total space of contact fibration over a strong symplectic fold of exact type and satisfies the above assumptions, then there exists a contact form on $E$.

For $E_+$ we apply Proposition 4.2. Over the collar $N \times [-1, 1]$ the bundles are product, thus the arguments used in the product case work. To be more precise, we start from the contact forms on $E_+$ given by the contactness of those bundles. Since the bundles are trivial over $N \times [-1, 1]$, we can use the homotopies $\eta \sim \eta_0, \eta' \sim \tilde{\eta}_0$ to get the form $\beta + \eta_0$ over $N \times [-\varepsilon, 0]$ and $\beta + \tilde{\eta}_0$ over $N \times [0, \varepsilon]$ for some $\varepsilon > 0$. Having established this, we can apply the same arguments which were used to prove Theorem 3.1.

Corollary 4.4. If one of the fibrations $E_+ \to W_{\pm}$ is trivial, then $E$ admits a contact form. In particular, this holds for any contact fibration over a sphere with its standard strong symplectic fold $S^{2n} = D^{2n}_- \cup D^{2n}_+$.

5. Some applications

Results of the previous sections give a constructive way to show that some manifolds are contact. We present now a series of examples. First of all we discuss the problem of
existence of strong symplectic folds on manifolds which in general seems to be a difficult question.

Let us recall that $W$ is the trace of a (single) surgery of index $k + 1$ on $M^{2n+1}$ if $W$ is obtained by attaching a handle of index $k + 1$ on $M \times [0, 1]$. It means that $W$ is diffeomorphic to $M \times [0, 1] \cup (D^{k+1} \times D^{2n-k+1})$, where $f : S^k \times D^{2n-k+1} \to M \times \{1\}$ is the attaching map of the handle. In particular, $\partial W = M \cup (-M')$, where $M' = (M - f(S^k \times D^{2n-k+1})) \cup (D^{k+1} \times S^{2n-k})$ is the result of the surgery on $M$.

The following classical result of Eliashberg [8] (cf. also [22] and Ch. 6 of [14]) is the basic tool to construct some examples.

**Theorem 5.1.** Let $\lambda$ be a contact form on a $(2n + 1)$-dimensional manifold $M$ and let $W$ be the trace of a surgery on $M$ of index $k + 1$ with $1 \leq k \leq n$ and $n > 1$. If the almost complex structure on $M \times [0, 1]$ determined by $\lambda$ extends to $W$, then there exists an exact symplectic form $\omega$ on $W$ such that $\omega$ is the symplectization of $\lambda$ near $M \times \{0\}$ as well as the symplectization of a contact form in a collar of $M'$. In particular, $M'$ admits a contact form. Furthermore, if $V$ is a compact connected almost complex $(2n + 2)$-dimensional manifold ($n > 1$) and $V$ admits a Morse function maximal on $\partial V$ such that indices of all critical points are less or equal to $n + 1$, then $V$ admits a symplectic structure with convex boundary (the boundary is of contact type). A Morse function with the required properties exists if and only if $V$ has the homotopy type of a CW-complex of dimension at most $n + 1$.

Let us call any manifold $V$ having the above properties of Weinstein type. Thus the double of a manifold of Weinstein type admits a strong symplectic fold.

**Remark 5.2.** The contact surgery in dimension 4 requires some additional assumption on framings of the attaching spheres of 2-handles, see [18] [8] or [14], Ch. 6.3, 6.4.

We can give now examples of whole families of contact manifolds.

**Proposition 5.3.** The following manifolds admit contact structures:

1. $S^{k_1} \times ... \times S^{k_r}$ if $k_1 + ... + k_r$ is odd;
2. $S^{k_1} \times ... \times S^{k_r} \times X$ if $X$ is a closed contact manifold and $k_1 + ... + k_r$ is even;
3. $M \times S^{k_1} \times ... \times S^{k_r}$, if $M$ is a closed manifold with a strong symplectic fold and $k_1 + ... + k_r$ is odd;
4. $M \times X$, if $M$ is a closed orientable 4-manifold and $X$ is contact;
5. $\Sigma \times X$, where $X$ is contact and $\Sigma$ is a closed oriented surface

**Proof.** Both $D^{2k}$ and $D^{2k+1} \times S^{2l+1}$ with $k \geq l$ are Weinstein manifolds, thus taking the doubles we see that $S^{2k}$ and $S^{2k+1} \times S^{2l+1}$ admit strong symplectic folds with any $k, l$. Therefore the first three cases follow by induction. To get [14] one has to use existence of strong symplectic folds on closed orientable 4-manifolds [3]. In the last statement it is enough to notice that any orientable surface has a strong symplectic fold. This statement was first proved in [4] for $\Sigma$ of genus $g > 0$ and for $\Sigma = S^2$ in [5].

Any Lie group of odd dimension is obviously almost contact. However, no general construction of contact forms on compact Lie group is known. It can be proved that except for rank 1 there is no G-invariant contact forms on G. The product $S^3 \times S^3 \times S^3$ is an example of simply connected Lie group which admits a contact form but no
$G$–invariant contact form. Some examples of contact forms on quotient spaces $G/H$ which do not admit $G$–invariant contact forms can be obtained from Theorem 4.3. For instance, the following is true.

**Proposition 5.4.** For any even $n$, the homogenous space $SO(n + 3)/SO(n)$ is contact, but admits no $SO(n + 3)$–invariant contact form.

**Proof.** The space $SO(n + 2)/SO(n)$ has a $SO(n + 2)$–invariant contact form given by the circle fibration $SO(n + 2)/SO(n) \to SO(n + 2)/(SO(n) \times SO(2))$ with symplectic base. Moreover, the space $SO(n + 3)/SO(n)$ has no $SO(n + 3)$-invariant contact form. Both statements follow from Alekseevski’s description of contact homogeneous spaces [1]. Consider now the bundle $SO(n + 3)/SO(n) \to SO(n + 3)/SO(n + 2)$ with fiber $SO(n + 2)/SO(n)$. If $n$ is even, then on the base $SO(n + 3)/SO(n + 2) = S^{n+2}$ we have the obvious strong symplectic fold. The structure group of the bundle is $SO(n + 2)$, thus the assumptions of Theorem 4.3 are satisfied. In fact, one can use Corollary 4.4 to show that there exists a contact form on the total space of the bundle. □

Another example of this type is the space $SO(2k + 1)/SU(k)$ of ”special unitary twistors” on $S^{2k}$ which is fibered over $SO(2k+1)/SO(2k) = S^{2k}$ with fiber $SO(2k)/SU(k)$.

We will describe now examples of a modification which can be performed on a manifold with a strong symplectic fold. Assume that $M^{2m}$ admits a strong symplectic fold $W_- \cup W_+$ with the fold locus $N$. We say that a surgery on a sphere $S^{k-1} \subset M$ is *symmetric of index* $k$, if it is performed using an embedding $\phi : S^{k-1} \times D^{2m-k+1} \to M$ such that $\phi = \phi_0 \times id_{D^1}$, where $\phi_0 : S^{k-1} \times D^{2m-k} \to N$ is an embedding and $D^1$ corresponds to the transversal disk of a tubular neighborhood of $N$.

**Proposition 5.5.** Consider a manifold $M$ of dimension $2m > 4$ with a strong symplectic fold of convex type. If $M'$ is obtained from $M$ by a symmetric surgery of index $k \leq m$ such that the stable almost complex structure of $M$ extends to $M'$, then $M'$ has a strong symplectic fold structure.

**Proof.** We have $M' = (M - \phi(S^{k-1} \times D^{2m-k+1})) \cup (D^k \times S^{2m-k})$. Decompose $S^{2m-k}$ into the sum of two disks $D_- \cup D_+$ such that the decomposition corresponds to cutting the sphere by $N$. Then we obtain a decomposition $M' = W'_- \cup W'_+$ such that $W'_\pm = W_\pm \cup (D^k \times D_\pm)$. Thus both parts are given by attaching handles $D^k \times D_\pm$ of index $k$ to respectively $W_-, W_+$. Because $k \leq m$ and by assumption the almost complex structures on $W_-, W_+$ extend to these handles, given symplectic forms extend to $W_\pm$. □

**Corollary 5.6.** If $M^{2m}$ admits a strong symplectic fold $(k + n = 2m)$, then so does the connected sum $M\#(S^k \times S^n)$.

**Proof.** The proposition can be applied, since connected sum with $S^k \times S^n$ is obtained by the surgery on a trivially embedded sphere $S^{k-1}$ (or on $S^{n-1}$) and thus we can assume that $k \leq n$. □

6. Concave folds and strong symplectic folds of general type

Till now we considered decompositions of a manifold $M$ into the sum of two exact symplectic cobordisms $W_1$ and $W_2$ having the same contact boundary $N$ at their convex ends. If $M$ is closed, the symplectic cobordisms cannot have concave ends, thus they
should be symplectic fillings of the contact form on the fold locus. Our present purpose is to extend this construction to the case where \( M \) is decomposed into several pieces, each being a symplectic cobordism having possibly concave ends as well.

In case when \( W_1, W_2 \) meet in such a way that one of the ends is concave and one is convex (and the contact forms at the boundary are equal), one can apply standard gluing of two symplectic cobordisms, which assembles two symplectic cobordisms into one, simplifying the decomposition. Thus the substantial cases are when two convex ends or two concave ends meet.

Consider now the case of concave ends of two symplectic cobordisms meeting at \((N, \lambda)\). We assume that in a collar neighborhood \( N \times [-1, 1] \) of \( N \) we have the form \( e^{-t}\lambda \) for \( t \in [-1; 0] \) and \( e^{t}\hat{\lambda} \) for \( t \in [0; 1] \). Note that the orientations given by the forms on the two sides coincide, unlike the case of convex folds.

We explain now how to use Theorem 3.1 to obtain a contact form on the product of the sum of such two cobordisms by a contact manifold \( X \).

**Lemma 6.1.** Suppose that \((X, \alpha)\) and \((N, \lambda)\) are closed contact manifolds. Then there exists a contact form on \( X \times N \times [-1, 1] \) equal to \( \alpha + e^{-t}\lambda \) near \( X \times N \times \{-1\} \) and to \( \alpha + e^{t}\hat{\lambda} \) near \( X \times N \times \{1\} \).

**Proof.** We apply Theorem 3.1 after switching the role of \( X \) and \( N \).

For this purpose we define positive functions

\[
g_1(t) = \begin{cases} 1 & \text{near } t = -1 \\ e^t & \text{near } t = -\frac{1}{2} \end{cases},
\]

on \([-1, -\frac{1}{2}]\) and

\[
g_2(t) = \begin{cases} e^{-t} & \text{near } t = \frac{1}{2} \\ 1 & \text{near } t = 1 \end{cases}.
\]

on \([\frac{1}{2}, 1]\).

The contact form \( g_1(t)(\alpha + e^{-t}\lambda) \) on \( X \times N \times [-1, -\frac{1}{2}] \) extends to \( e^t\alpha + \lambda \) on \( X \times N \times [-\frac{1}{2}, 0] \).

Similarly, the contact form \( g_2(t)(\alpha + e^{t}\hat{\lambda}) \) on \( X \times N \times [\frac{1}{2}, 1] \) extends to \( e^{-t}\alpha + \hat{\lambda} \) on \( X \times N \times [0, \frac{1}{2}] \). Thus we can apply Theorem 3.1 to construct the form with required properties.

\( \Box \)

In Lemma 6.1 in order to get a contact structure on the product of this manifold by a contact one, we need a contact form \( \lambda \) on one end of \( N \times [-1, 1] \) and \( \hat{\lambda} \) on the other, while on the contact factor the form does not depend on \( t \in [-1, 1] \). This is too restrictive for applications and we will show that the construction of contact forms is possible also if the pair of forms is \((\lambda + \alpha)\) and \((\lambda + \hat{\alpha})\) on the two sides.

**Lemma 6.2.** Suppose that \((X, \alpha)\) and \((N, \lambda)\) are closed contact manifolds. Then there exist two contact forms on \( X \times N \times [-1, 1] \), both equal to \( e^{-t}\lambda + \alpha \) near \( N \times \{-1\} \) and one equal to \( e^{t}\hat{\lambda} + \alpha \), the other to \( e^{t}\lambda + \hat{\alpha} \) near \( N \times \{1\} \).

**Proof.** In case of the pair \((e^{-t}\lambda + \alpha, e^{t}\hat{\lambda} + \alpha)\) we apply Lemma 6.1. If we have the pair \((e^{-t}\lambda + \alpha, e^{t}\lambda + \hat{\alpha})\) we use Theorem 3.1 and Lemma 6.1 to \( X \times M \times [-1, 1] \) divided into 4 parts:
(1) $X \times N \times [-1, -\frac{1}{2}]$ with the form $e^{-t}\lambda + \alpha$,
(2) $X \times N \times [-\frac{1}{2}, 0]$ with the form $e^t\lambda + \alpha$,
(3) $X \times N \times [0, \frac{1}{2}]$ with the form $e^{-t}\lambda + \hat{\alpha}$,
(4) $X \times N \times [\frac{1}{2}, 1]$ with the form $e^t\lambda + \hat{\alpha}$.

The forms are defined such that crossing the convex fold at 0 corresponds to passing from $(\alpha, \lambda)$ to $(\hat{\alpha}, \lambda)$ and crossing concave folds at $\pm \frac{1}{2}$ is the swap between $(\alpha, \lambda)$, $(\alpha, \hat{\lambda})$ and back. In all cases one of the previously described constructions works. Thus we get a contact form on $X \times N$.

Now we are in position to extend the notion of strong symplectic fold to allow concave folds.

Consider a closed hypersurface $N \subset Int M$ and denote by $W_i, i=1,\ldots,k$ the connected components of $M - N$ compactified by adding adjacent components of $N$. Hence $W_i$ is just the closure of a component of $M - N$. Let $N = \bigcup_s N_s$ denote the decomposition of $N$ into the sum of connected components.

**Definition 6.3.** A strong symplectic fold on a compact manifold $M$ is given by:

1. a decomposition $\{W_i\}_{i \in I}$ of compact codimension 0 submanifolds, $M = \bigcup_i W_i$, obtained by cutting $M$ by a hypersurface $N \subset Int M$;
2. a family of contact forms $(N_s, \lambda_s)$;
3. exact symplectic forms $\omega_i = d\beta_i$ on $W_i$ such that each $\omega_i$ yields a symplectic cobordism structure on $W_i$ with some convex ends and some concave ends and each pair $\omega_i, \omega_j$ satisfies one of the following compatibility condition for every connected component $N_s$ of $N$ with $N_s \subset W_i \cap W_j$:
   a. $\beta_i = e^t\lambda_s$ in $N_s \times [-1, 0] \subset W_i$ and $\beta_j = e^{-t}\lambda_s$ in $N_s \times [0, 1] \subset W_j$ where $t$ is the parameter of $[-1,1]$ (convex fold: a convex end of $W_i$ meets a convex end of $W_j$ at $N_s$);
   b. $\beta_i = e^{-t}\lambda_s$ on $N_s \times [-1, 0] \subset W_i$ and $\beta_j = e^t\lambda_s$ on $N_s \times [0, 1] \subset W_j$, where $t$ is the parameter of $[-1,1]$ (concave fold: a concave end of $W_i$ meets a concave end of $W_j$);

As before, the hypersurface $N$ is called the fold locus. We assumed that every piece $W_i$ is a symplectic cobordism, hence the forms $\omega_i$ are either convex or concave along any component of the boundary of $M$.

From the discussion of this section we obtain the following extension of Theorem 3.1.

**Theorem 6.4.** If $X$ is a closed contact manifold and $M$ admits a strong symplectic fold, then $X \times M$ is contact.
7. Some further applications

To illustrate usefulness of concave folds consider the question of fillability of contact manifolds (by a symplectic one). It is well-known that no overtwisted contact form $\lambda$ on a compact 3-manifold $M$ is fillable, i.e., there is no compact manifold with boundary of contact type (convex boundary) having overtwisted contact form on the boundary. Constructions based on fibrations, for instance the open book technique, lead to the following question. Is there a similar obstruction to fill up by a compact contact manifold the product of an overtwisted 3-manifold by $S^1$? In other words, we ask if the form $e^{-t}\lambda + d\theta$ on $M \times [0, \varepsilon) \times S^1$ can be extended to a contact form on a compact manifold $W$ such that $\partial W = M \times \{0\} \times S^1$ ($d\theta$ denotes the standard form on $S^1$). Below we show examples that fillability in this sense is possible.

Given two connected contact manifolds $(X, \alpha)$, $(X', \alpha')$ oriented compatibly with contact structures, one can perform 1-surgery such that the resulting manifold is the connected sum $X \# X'$. Then by the contact surgery (Theorem 5.1) we get a contact form on the connected sum. Since we need some choices to perform such operation, the result is not defined uniquely, but its homotopy class is already unique. By slight abuse of language we denote the contact form obtained in this way by $\alpha \# \alpha'$.

**Proposition 7.1.** If $n > 0$ and $\lambda$ is any contact form on $S^{2n+1}$, then the form $e^{-t}(\lambda \# \lambda) + d\theta$ on a collar of the boundary $(S^{2n+1} \# S^{2n+1}) \times (0, \varepsilon) \times S^1 \subset D^{2n+2} \times S^1$ extends to a contact form on $D^{2n+2} \times S^1$.

**Proof.** Consider the symplectizations $e^{-t}\lambda$ on $S^{2n+1} \times [-1, 0]$ and $e^{t}\lambda'$ on $S^{2n+1} \times [0, 1]$. Gluing these manifolds along $S^{2n+1} \times \{0\}$ we get a manifold with a concave fold $S^{2n+1} \times \{0\}$ and boundary $S^{2n+1} \cup \{-S^{2n+1}\}$. We can perform contact 1-surgery by adding a 1-handle to the boundary which makes the boundary connected and diffeomorphic to $S^{2n+1}$. The manifold $W$ obtained by the surgery is diffeomorphic to $S^{2n+1} \times S^1 - D^{2n+2}$. Using Theorem 5.1 for this handle we get a strong symplectic fold on $W$ extending the symplectizations and with the boundary $(S^{2n+1}, \lambda \# \lambda)$ of contact type (note that we still have the fold $S^{2n+1} \times \{0\}$ in the interior of $W$). By Lemma 6.1 there is a contact form $\eta$ on $W \times S^1$ equal to $\lambda \# \lambda + d\theta$ on the boundary. Denote by $S \subset Int W$ the circle given as the sum of intervals $x_0 \times [-1, 1] \subset S^{2n+1} \times [-1, 1]$ and $y_0 \times [-1, 1]$ in the handle, where $y_0 \times \{\pm 1\}$ are attached to $x_0 \times \{\pm 1\}$ by the attaching map of the handle. The (topological) surgery of index 2 on $W$ with the attaching circle $S$ and the standard framing of the normal bundle yields the disk $D^{2n+2}$. Moreover, the standard almost complex structure on $W$ extends to the 2-handle. To finish the proof we have to show that the surgery applied to $\eta$ yields another contact form on its result. $D^{2n+2} \times S^1$ is obtained from $W \times S^1$ by the 2-surgery multiplied by $S^1$. The product of a 2-handle by $S^1$ decomposes into two handles on $W \times S^1$, one of index 2 on $W \times S^1$ and one of index 3 attached to the result of the first surgery. Since the manifold $W \times S^1$ is of dimension at least 5 and the given almost contact structure is compatible with the surgeries, we get a contact form on $D^{2n+2} \times S^1$. Finally, the surgeries are done in the interiors of manifolds in each step of the construction, hence they preserve the form we have obtained previously in a neighborhood of the boundary sphere.
Thus we get the following corollary that was first proved in [10].

**Corollary 7.2.** There exists an overtwisted contact form $\lambda$ on $S^3$ such that the form $e^{-t}\lambda + d\theta$ extends from a collar $S^3 \times S^1 \times [0, \varepsilon)$ to a contact form on $D^4 \times S^1$.

This property can be applied to prove the following special case of results proved in [9,7].

**Proposition 7.3.** If $M^5$ is closed almost contact and admits an open book decomposition with trivial monodromy, then it is contact.

**Proof.** Let $P$ denote the page of the open book. The almost contact structure of $M$ gives a stably almost complex structure on $P$. For an open manifold stably almost complex structure determines an almost complex structure. It follows from basic facts of the Morse - Smale theory that there exists a Morse function $f : P \to [0, 4]$ with one minimum ($= 0$), constant and maximal ($= 4$) on $\partial P$. This function has critical points only of indices $q = 0, 1, 2, 3$ and such that the value of $f$ at a critical point of index $q$ is $q$. Denote $W_i = f^{-1}([i - \frac{1}{2}, i + \frac{1}{2}])$, $i = 0, 1, 2, 3, 4$. Then $W_0$ is diffeomorphic to $D^4$, $W_i$ contains only critical points of indices $i$ and $W_4 = \partial P \times [\frac{3}{4}, 4]$. Let $\lambda$ be an overtwisted contact form on $S^3$ such that $e^t \lambda + d\theta$ extends to a contact form on $D^4 \times S^1$. Since $P$ is almost complex, then by the contact surgery Theorem 5.1 we extend the form $e^t \lambda$ to 1-handles of $W_1$. This makes $W_1$ a symplectic cobordism with concave end $f^{-1}(\frac{1}{2})$ and convex end $f^{-1}(\frac{3}{2})$. Since the surgeries can be performed far from overtwisted disks, the contact form on the latter can be assumed again overtwisted. On an overtwisted 3-manifold one can perform contact surgery on every framing, so this holds for $W_2$. In the same manner we make $W_3$ a symplectic cobordism with concave end $f^{-1}(\frac{1}{2})$ and convex end $f^{-1}(\frac{3}{2})$. Namely, we use the (unique up to homotopy) overtwisted form $\mu$ representing the almost contact structure of $f^{-1}(\frac{3}{2})$. In this way we get symplectic structures which agree with the almost complex structure of $P$. Since the homotopy class of an overtwisted form is determined by the homotopy class of the contact distribution, the contact forms on $f^{-1}(\frac{3}{2})$ obtained from $W_2$ and $W_3$ are homotopic, hence by Remark 2.11 can be assumed equal. Finally, on $W_4$ we put symplectization of the form $\hat{\mu}$, where $\mu$ is the form used in $W_3$. In this way we get a strong symplectic fold on $W_1 \cup W_2 \cup W_3 \cup W_4$ with fold locus $f^{-1}(\frac{1}{2}) \cup f^{-1}(\frac{3}{2})$, where the fold at $f^{-1}(\frac{1}{2})$ is convex and at $f^{-1}(\frac{3}{2})$ is concave. Therefore, by Theorem 3.1 and Lemma 6.1 we have a contact form on the product with $S^1$. Since the form on $f^{-1}(\frac{1}{2}) \times S^1$ extends to $D^4 \times S^1$, we get also a contact form on $P \times S^1$. By the construction, in a collar of $\partial P \times S^1$ this form is the product of a convex form on $W_4$ by the standard form on $S^1$ at $\partial P$. It can be extended to $\partial P \times D^2 \subset M$ exactly as it is done in the case of the open book construction. This completes the proof.

Let us illustrate Theorem 6.1 by the following examples.

**Example 7.4.** If $M$ is a $S^1$-bundle over $X \times N$ with contact $(X, \lambda), (N, \lambda')$, then $M$ is contact (in particular, $X \times S^1 \times S^1$ is). Begin with the trivial bundle. Write $N \times S^1 = N \times [0, \frac{1}{2}] \cup N \times [\frac{1}{2}, \frac{1}{2}] \cup N \times [\frac{1}{2}, \frac{3}{2}] \cup N \times [\frac{3}{2}, 1]$, where $N \times \{0\}$ and $N \times \{1\}$ are identified. On these four parts put $e^t \lambda, e^{-t+\frac{1}{2}} \lambda$, $e^{-t-\frac{1}{2}} \lambda'$, $e^{-t+1} \lambda'$, respectively. This gives a strong symplectic fold structure on $N \times S^1$. Now take products with $(X, \lambda)$ for $N \times [0, \frac{1}{2}], N \times [\frac{3}{2}, 1]$ and with $(X, \hat{\lambda})$ for $N \times [\frac{1}{2}, \frac{1}{2}], N \times [\frac{1}{2}, \frac{3}{2}]$. So we have the following sequence of forms:
By Theorem 6.4, there exists a contact form on $X \times N \times S^1$. By [15], this extends to any circle bundle over $X \times N$.

**Example 7.5.** Consider a closed contact manifold $M$ of dimension 5 and a homotopically trivial circle $S$ embedded in $M$. Then the manifold $M'$ obtained from $M$ by the blow-up along $S$ is contact.

**Proof.** We can deform the given contact form on $M$ to one given by an open book with the binding $B$, the page $P$ and the fibration $E \to S^1$. Then $S$ can be deformed to a section of the fibration, say to a circle given by a point near $\partial P$, where the fibration is product. A tubular neighborhood of such $S$ is the product of a small disk $D^4 \subset Int P$ by $S$. On $\mathbb{C}P^2$ there is a strong symplectic fold $W_- \cup W_+$ of convex type by [3]. Cutting another small (Darboux) disks $D^4$ in $W_-$ and identifying boundary spheres of $D^4$, $D^4$ we get the connected sum $P \# \mathbb{C}P^2$ and a strong symplectic fold on it (with concave fold at the connected sum sphere). Consider the following decomposition of $M$:

$B \times D^2$, the product neighborhood of the binding, $(U - D^4) \times S^1$, where $U$ is a collar of $\partial P$, $(W_- - D^4) \times S^1, W_+ \times S^1$ and the fibration over $S^1$ with fiber $P - U$ given by the open book structure. On the fibration we have a contact form. By the assumption, this form is product near the boundary. Other pieces are products, thus we can apply Theorem 6.4 to get a contact form on $M$.

**Remark 7.6.** Note that $P \# \mathbb{C}P$ does not admit any exact symplectic form with contact type boundary, so the example cannot be obtained by modification of the open book. It was explained to us by András Stipsicz that this property follows from the fact that any spherical homology 2-class in a closed 4-manifold with self-intersection number $-1$ is represented by a symplectic submanifold. The same argument, combined with a result of McDuff [20] shows that there is no strong symplectic fold of convex type on $\mathbb{C}P^2 - Int D^4$.

**8. A generalization of the open book construction**

We give now a generalization of the open book construction allowing bindings of codimensions greater that 2. This is a decomposition of a manifold into two pieces. This decomposition is much more symmetric than the open book.

Consider two compact manifolds $X,Y$ with non-empty boundaries, of dimensions $2n, 2m$ respectively. Assume that they are endowed with exact symplectic forms $\omega_X = d\beta_X$, $\omega_Y = d\beta_Y$, both with convex type boundaries. Let $\beta_X = e^{\mu} \theta_X$, $\beta_Y = e^{-\mu} \theta_Y$ in
collars $\partial X \times (-1,0], \partial Y \times [0,1)$ of boundaries, where $\mu_{\partial X}, \mu_{\partial Y}$ are some contact forms. In this notation $s \in [-1,1]$ and both boundaries correspond to $s = 0$. Let $E$ be the total space of a bundle over $\partial Y$ with fiber $X$ defined on a hypersurface $H_{\partial Y} \subset \partial Y$ and with the structure group $\text{Ex}(X, \partial X, \beta_X)$ of exact symplectomorphisms of $X$ equal to the identity near the boundary. Similarly, assume that the bundle $F \to \partial X$ with fiber $Y$ is defined on a hypersurface $H_{\partial X} \subset \partial X$, and its structure group is $\text{Ex}(Y, \partial Y, \beta_Y)$. The assumptions on structure groups imply that $\partial E = \partial X \times \partial Y = \partial F$.

**Proposition 8.1.** Under the above assumptions, $E \cup_{\partial X \times \partial Y} F$ is contact.

**Proof.** Consider $\tilde{X} = X \cup \partial X \times [0, \log R_X]$ obtained from $X$ by adding a long collar, with $\beta_X = e^{s} \mu_{\partial X}$ for $s \in [-1, \log R_X]$. In this way the contact form on the boundary is multiplied by the constant $R_X$. Analogously, $Y$ is enlarged to $\tilde{Y} = Y \cup \partial Y \times [-\log R_Y, 0]$ with $\beta_Y = e^{-s} \mu_{\partial Y}$ for $s \in [-\log R_Y, 1]$ (we assume $R_X, R_Y \geq 1$). Let $\tilde{E}$ denote the obvious extension of $E$ to a bundle with fiber $\tilde{X}$, and similarly $\tilde{F}$ the extension of $F$. Proposition 4.1 gives a contact form on $\tilde{X}$, and our claim follows. Thus we can choose $\psi$ as a contact form on $\tilde{X}$, $\mu_{\partial X}$ for $s \in [-1, \log R_X]$,

$$\eta_E d\eta_E^{n+1} = e^{n} \mu_{\partial X} d\mu_{\partial X}^{n-1} \left( D_1 R_Y^{m} \mu_{\partial Y} d\mu_{\partial Y}^{m-1} ds + D_2 R^{m-1} u' \mu_{\partial Y} d\mu_{\partial Y}^{m-2} ds dt d\tilde{\psi} + D_3 R^{-2} u' \mu_{\partial Y}^{m-1} dt d\tilde{\psi} + D_4 R^{m-1} \mu_{\partial Y}^{m-1} ds d\tilde{\psi} \right),$$

where $D_i, i = 1, 2, 3, 4$ are again constants depending only on $m, n$.

It follows from this formula that the choice of $R_Y$ is independent of the extension by the long collar and our claim follows. Thus we can choose $R = R_X = R_Y$ such that there are contact forms on $\tilde{E}$ and $\tilde{F}$ that restrict to $R(\mu_{\partial X} + \mu_{\partial Y})$ on $\partial E = \partial F = \partial X \times \partial Y$. Let $K = \log R$. After the change the parameter in $[-K, 1]$ replacing $s$ with $s + 2K$, the form $\eta_F$ on $\partial X \times \partial Y \times [K, 1 + 2K]$ becomes $R_{\mu_{\partial X}} + e^{s+K} \mu_{\partial Y}$.

Let $\psi : [K, 1 + 2K] \to \mathbb{R}$ be a positive smooth function such that $\psi = e^{s-K}$ near $s = K$ and $\psi = 1$ near $1 + 2K$, regarded as a function on $F$ (we simply extend it from the collar $\partial F \times [0,1]$ to whole $F$). Then $\psi \eta_F$ is contact and it smoothly agrees with $\eta_E$ along $\partial E = \partial F = \partial X \times \partial Y$. Thus we get a smooth contact form on $E \cup F$.

9. **Concluding remarks**

We do not know any example of closed stably almost complex manifold which admits no strong symplectic folds. On the other hand, it is anything but obvious if any symplectic manifold has a strong symplectic fold. In particular, it would be interesting to decide whether complex projective spaces admit strong symplectic folds.

The standard Morse - Smale theory shows that for any closed manifold $M^{2m}$ one can find a decomposition $M = W_+ \cup N \cup W_-$, where $N = \partial W_+ = \partial W_- = W_+ \cap W_-$ with both $W_+, W_-$ having the homotopy type of complexes of dimension at most $m$. If $M$ is stably almost complex, then $W_\pm$ are almost complex and we have exact symplectic forms on both parts by contact surgery. The resulting contact forms $\lambda_-, \lambda_+$ on $N$ define
homotopic almost contact structures on \( N \), but the question whether they are homotopic (as contact forms) is apparently difficult. If they do, we would get a strong symplectic fold of convex type on \( M \). Quite possibly, the general type of strong symplectic folds can be useful in this problem, as the arguments used for Example 7.3 and Proposition 7.3 indicate.

10. Appendix A: Contact piecewise fibered structures

In this appendix we describe a preliminary version of a structure generalizing all cases we considered till now and still sufficient to provide a contact form on a manifold endowed with such structure. This is obtained by localization, requiring that each piece of such decomposition is one of described previously with appropriate compatibility conditions along intersections assumed.

Let \( Y \) be a compact orientable manifold. Given a hypersurface \( H \subset Int Y \), let \( \{ Y_i \} \) denote the collection of connected components of \( Y - H \) compactified by adding components of \( H \) contained in the closure of \( Y_i \). Our basic assumption is that each \( Y_i \) is a fibration of one of the following two types:

1. a contact fibration with a closed contact fiber \((X_i, \alpha_i)\) over an exact symplectic cobordism \((W_i, d\mu_i)\) trivial in a neighborhood of \( \partial W_i \), or
2. the fibration over a closed contact manifold \((X_i, \alpha_i)\), defined on a hypersurface in \( X_i \), such that the fiber is an exact symplectic cobordism \((W_i, d\mu_i)\) and the structure group is the group \( Ex(W_i, \partial W_i, \mu_i) \) of exact symplectomorphisms equal to the identity in a collar of \( \partial W_i \).

If this is satisfied, then every component of \( H \) is the product of \( X_i \) by a component of the boundary of the symplectic cobordism \( W_i \). Let us denote by \( N_{is}, s = 1, \ldots, l_s \) components of \( \partial W_i \) and by \( \lambda_{is} \) the contact form induced on \( N_{is} \) by \( \mu_i \) (which is either convex or concave at \( N_{is} \)).

If \( N_{is} = N_{jr} \) is a connected component of the intersection \( Y_i \cap Y_j \cap H \), then we assume that one of the following conditions is satisfied:

1. \( N_{is} \) is a convex end of \( W_j \), \( N_{jr} \) is a convex end of \( W_j \) and \( X_i = X_j \);
2. \( N_{is} \) is a concave end of \( W_j \), \( N_{jr} \) is a concave end of \( W_j \) and \( X_i = X_j \);
3. \( N_{is} \) is a convex end of \( W_j \), \( N_{jr} \) is a convex end of \( W_j \), \( N_{is} = X_j \), \( N_{jr} = X_i \);
4. \( N_{is} \) is a concave end of \( W_j \), \( N_{jr} \) is a concave end of \( W_j \), \( N_{is} = X_j \), \( N_{jr} = X_i \);
5. \( N_{is} \) is a concave end of \( W_i \), \( N_{jr} \) is a convex end of \( W_j \) and \( X_i = X_j \);
6. \( N_{is} \) is a convex end of \( W_i \), \( N_{jr} \) is a concave end of \( W_j \) and \( X_i = X_j \).

Finally, we assume compatibility of the forms on the adjacent ends of \( Y_i \)'s. In all the cases above we require one the following conditions, according to the list above:

1. \( \lambda_{is} = \lambda_{jr} \) and \( \alpha_i = \alpha_j \) or \( \lambda_{is} = \hat{\lambda}_{jr} \) and \( \alpha_i = \alpha_j \);
2. \( \lambda_{is} = \lambda_{jr} \) and \( \alpha_i = \tilde{\alpha}_j \) or \( \lambda_{is} = \tilde{\lambda}_{jr} \) and \( \alpha_i = \alpha_j \);
3. \( \lambda_{is} = \alpha_j \) and \( \alpha_i = \lambda_{jr} \);
4. \( \lambda_{is} = \alpha_j \) and \( \alpha_i = \tilde{\lambda}_{jr} \);
5. \( \lambda_{is} = \tilde{\lambda}_{jr} \) and \( \alpha_i = \alpha_j \);
6. \( \lambda_{is} = \lambda_{jr} \) and \( \alpha_i = \alpha_j \).

If \( \partial Y_i \) contains a connected component of \( \partial Y_j \), then in a collar of that component we have the product of \( X_i \) and an end of \( W_i \) (either convex or concave).
Remark 10.1. We allow a component of $H$ to be the boundary of two different ends of one $Y_i$ (when $i = j$ in the list above). In particular, it is possible that $Y - H$ is connected.

One can explain our assumptions by saying that the fold locus $H$ divides the manifold $M$ into a number of fibrations carrying contact fibered structure with both fibrations and forms product near any component of $H$. Under our compatibility conditions we can apply either Theorem 4.3 or Lemma 6.2.

Definition 10.2. A decomposition of $M$ satisfying the assumptions above is called a contact piecewise fibered structure on $M$.

Theorem 10.3. If $M$ admits a contact piecewise fibered structure, then $M$ is contact.

Sketch of the proof. Consider a component $Y_i$ of the decomposition. As we explained in Sections 2 and 3, it admits a contact form equal to $\lambda_{ij} + p^i \alpha_i$, or to $p^i \lambda_{ij} + \alpha_i$, in a collar of the $j$-th component of $\partial Y_i$, depending on the type of the fiber on $Y_i$. Furthermore, $\varepsilon = \pm 1$ depending on convex/concave type of the fold. Under the compatibility conditions we use Theorem 4.3 or Lemma 6.2 to extend those forms through $H$ and we get a global contact form on $M$.

Remark 10.4. One can allow that instead of equalities in the compatibility conditions one assumes equality up to homotopy, for instance up to the multiplication by a constant. This can be always reduced to the equality case by extending the adjacent end (which is $\varepsilon \lambda_{ij}$, $t \in [0, 1]$) from $[0, 1]$ to $[0, R]$ for $R$ appropriately chosen and applying the trick of Lemma 6.2.

11. Appendix B: Computations in Mathematica

We present here some of the calculations which led us to the proof of Theorem 3.1. The result was first checked using Mathematica’s package ”Differential forms” (Frank Zizza and Ulrich Jentschura [11]) in low dimensions. Namely, for $t_1 = 0$ (for technical reasons we slightly change notation to adapt it for our purposes) and around a point $(b, d, n, 0) \in B \times D^2 \times N \times I$ we take coordinate system in which $\beta = d[z1] + x1d[y1]$, $\lambda = d[z2] + x2d[y2]$. Further, on disk $D^2$ we take coordinate system $(x, y)$. In these coordinates we set $h_1 = 2 - (x^2 + y^2)^2$ and $h_2 = x^2 + y^2$ (hence in the formula below $h_1$ is equal to $2 - r^4$ near $r = 0$ so that it is of class $C^3$). Then the following expressions are equal respectively to $\eta$ and $d\eta$ :

\begin{align*}
\text{eta1} &:= (2 - (x^2 + y^2)^2)(d[z1] + x1d[y1]) + (d[z2] + x2d[y2]) \\
\text{deta1} &:= (-4x^3 - 4xy^2)d[x] \wedge d[z1] + (-4x^3x_1 - 4x_1y^2)d[x] \wedge d[y1] + \\
&(-4x^2y - 4y^3)d[y] \wedge d[z1] + (-4x^2x_1y - 4x_1y^3)x1d[y] \wedge d[y1] + \\
&(2 - x^4 - 2x^2y^2 - y^4)d[t1] \wedge d[z1] + (2 - x^4 - 2x^2y^2 - y^4)d[x1] \wedge d[y1] + \\
&(2x_1 - x_1x_1 - 2x_1x_1y^2 - x_1y^4)d[t1] \wedge d[y1] + d[x2] \wedge d[y2] + \\
&d[t1] \wedge d[z2] + x2d[t1] \wedge d[y2] - xd[t1] \wedge d[y] + yd[t1] \wedge d[x] \\
\end{align*}

Now $\tau = \lambda (\eta \wedge (d\eta)^3)$ can be computed in two steps: first we calculate

ExteriorProduct[eta1, deta1, deta1, deta1]

and later
where the percent sign refers to τ where the percent sign refers to τ

Then \( \tau = * (\eta \wedge (d\eta)^3) \) is given by

\[
(24(x^2 + y^2)^2) \ dx_1 \wedge dy_1 + (-96x(-1 + x)xy(x^2 + y^2)^2) \ dt_1 \wedge dx_1 + \\
(-24x(x^2 + y^2)(x^2 + x_1y^2)) \ dx_1 \wedge dx_2 + (6x(-2 + x^2 + 2x_2y^2 + y^4)) \ dx \wedge dz_1 + \\
(+6y(-2 + x^2 + 2x_2y^2 + y^4)) \ dy \wedge dz_1 + (-24(x^2 + y^2)^2(-2 + x^4 + 2x_2y^2 + y^4)) \ dx_2 \wedge dy_2 + \\
(24x(x^2 + y^2)^2(-2 + x^4 + 2x_2y^2 + y^4)) \ dx_2 \wedge dz_2 + (6x(-2 + x^4 + 2x_2y^2 + y^4)^2) \ dx \wedge dz_2 + \\
(-24(x^2 + y^2)(-2 + x^4 + 2x_2y^2 + y^4)) \ dy \wedge dz_2 + (24(x^2 + y^2)(-2 + x^4 + 2x_2y^2 + y^4)) \ dt_1 \wedge dy_1 + \\
(-24y(x^2 + y^2)(-2 + x^4 + 2x_2y^2 + y^4)) \ dt_1 \wedge dx + \\
(-24(-1 + x)xy(x^2 + y^2)(-2 + x^4 + 2x_2y^2 + y^4)) \ dx_1 \wedge dz_2
\]

and \( \tau^4 \) is equal to

\[
(-1990656(x^2 + y^2)^6(-2 + x^4 + 2x_2y^2 + y^4)^3) \ dt_1 \wedge dx \wedge dx_1 \wedge dx_2 \wedge dy \wedge dy_1 \wedge dy_2 \wedge dz_1 + \\
(-1990656x_2(x^2 + y^2)^6(-2 + x^4 + 2x_2y^2 + y^4)^3) \ dt_1 \wedge dx \wedge dx_1 \wedge dx_2 \wedge dy \wedge dy_1 \wedge dz_1 \wedge dz_2 + \\
(-1990656x_1(x^2 + y^2)^6(-2 + x^4 + 2x_2y^2 + y^4)^4) \ dt_1 \wedge dx \wedge dx_1 \wedge dx_2 \wedge dy \wedge dy_2 \wedge dz_1 \wedge dz_2 + \\
(-1990656(x^2 + y^2)^6(-2 + x^4 + 2x_2y^2 + y^4)^4) \ dt_1 \wedge dx \wedge dx_1 \wedge dx_2 \wedge dy \wedge dy_1 \wedge dy_2 \wedge dz_2,
\]

As \( \iota_R d\text{vol}_{g_R} = \tau^4 \) for some \( R \in \text{lin} \{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \} \), hence the Reeb field \( R_\tau \) is equal to \( R \) because \( \iota_R \tau^4 = \iota_R \text{vol}_{g_R} = 0 \). The field \( R_\tau \) is obviously perpendicular to \( \frac{\partial}{\partial \tau} \) (away from the degenerate set \( \Sigma \)).

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