1. Introduction

Studies of excitation, ionization, and capture processes for ion-atom collision are developed over the last 25 years [1–10]. These processes are significantly more challenging, both from experimental and theoretical points of view. One process that is mainly suitable to study the few-body dynamics is ionization because of involving three unbound particles in the final state. The ionization process of atoms by charged particles is an important challenge problem for atomic physics, astrophysics, and radiation physics [11–13]. Precise determinations of the various forms of differential cross-sections such as: triple, double, and single with varied kinematics conditions offer interesting problems to applied mathematics [6]. Investigations of the problem of ionization of excited hydrogen atoms by charged particles are interested in astrophysical, and fusion plasmas research [11, 13].

For the ionization process that contains an excited atom, the radial distance between the electron and the nucleus has a much longer tail than for the ground states. Thus; interaction involving higher angular momentum becomes very important for excited atoms by ion impact. The 2p-hydrogen ionization differential cross sections reveal some preferential ejection direction due to the specific forms of their electron distribution [14]. The hydrogen atom is both the simplest atom and also the most omnipresent in the Universe and forms the prototype target for collision studies due to its structure being known analytically for its high precision.

The collision system is well represented as two-body at relative high collision energy and therefore the Born-model gives fair results especially at forward scattering angles [15]. At small scattering angles, both of the first- and the second-Born approximations give reasonable results. But, at backward angles, the result of the FBA is weak, which means that the projectile-target interaction plays an important role [2]. The interaction part between the projectile and the target nucleus is the key element of the elastic scattering explanation. So, it does not contribute to the FBA. The influence of the projectile charge on the outcomes of the collision processes becomes weaker as the incident energy increased [3].

The second-Born approximation could be expected to give a good representation of the ionization processes at relatively high energies, especially for forward scattering angles. This is because the condition for the validity of the Born-model is \( v_e / v \ll 1 \) (where \( v_e \) is the mean velocity of the bound electron in its initial state and \( v \) is the velocity of the incident projectile) and varies as \( 1/n \). It is expected that the SBA becomes more successful as the principal quantum number \( n \) of the initial state increases [16].

In studying the triple differential cross-sections of ion-atom collisions at intermediate collision energies, there are four independent parameters, the two polar angles \( \theta_1 \) and \( \theta_2 \), the azimuthal angle \( \phi \), and the ejected...
energy $E_e$. Many attempts have been made to determine the TDCSs in the coplanar geometry plane [17, 18] and in both the scattering and the perpendicular planes [19]. For the electron ejection outside the scattering plane, the comparison between experiment and theory is much less favorable. On the perpendicular plane, surprising peak structures are observed but could not be fully explained by theory [20].

In the present work, the ionization of H (2p) bombard by proton and antiproton is investigated. Both of the triple and single differential cross-sections are calculated. The calculations are performed for the magnetic sub-levels of the 2p-state. The effect of the projectile electric charge sign is examined on the differential cross-sections. These differential cross sections in electron or projectile variables are also major important as they are very important to the collision dynamic. In section 2, the formalism of the problem is outlined. The obtained results are illustrated and discussed in section 3. Finally, the work is concluded in section 4. Atomic units are used throughout calculations.

2. Formalism

In the ionization process, the Born approximation uses a scattering wave function approximated by the product of the target atom’s wave function and a plane wave for the incident projectile folded between the projectile-electron potential and a final state wave function which is the product of a scattered ion plane wave and a Coulomb wave for the ionized electron. The ionization of atomic hydrogen allows us to test accurately this electron potential and a Coulomb wave for the ionized electron. The ionization of atomic hydrogen allows us to test accurately this approximation because the eigenstates of the target are known analytically. The triple differential cross section measures the momentum distribution of the ionized electron with eject energy and a scattering angle of the projectile is defined as [21]:

$$\frac{d^3\sigma}{d\Omega_f d\Omega_e dE_e} = \frac{\mu k_f k_e}{(2\pi)^3} |T^{\text{Born}}_{fi}|^2.$$

Where $k_i (k_f)$ are the initial (final) momenta of the projectile, $k_e$ and $E_e = k_e^2/2$ are the momentum and energy of the ejected electron, and $\mu$ is the reduced mass of the colliding particles. Also, $d\Omega_i (d\Omega_f)$ denote to the element of solid angle for the ejected electron (scattered projectile), whereas the energy interval of the ejected electron is represented by $dE_e$.

The transition matrix element of the Born approximation, which takes into account the coupling between the initial and final states of the target atom, is defined as [22]:

$$T^{\text{Born}}_{fi} = \left\langle \Psi_{k_i}^{-}\left(\bar{r}\right) e^{ik \bar{r}} \middle| V + \frac{1}{E - H_0 + i\eta} \middle| \Psi_{k_f}\left(\bar{r}\right) e^{ik \bar{r}} \right\rangle.$$

Where $\Psi_{k_i}$ is the initial eigenstate and $\Psi_{k_f}\left(\bar{r}\right) e^{ik \bar{r}}$ is the final continuum eigenstate of the atomic hydrogen and $\bar{r}$ ($\bar{R}$) is the position vector of the electron of the atom (the projectile) relative to the atomic nucleus. The operator $(E - H_0 + i\eta)^{-1}$ can be viewed as a propagator and the Born-series for transition matrix can therefore be pictured as a multiple transition series in which the projectile interacts repeatedly with the potential $V$ and propagates freely between two interactions (in the limit $\eta \to 0^+).$ The interaction potential between a projectile with charge $Z_p$ and the hydrogen atom is given by:

$$V(\bar{r}, \bar{R}) = Z_p \left[ \frac{1}{\bar{R}} - \frac{1}{|\bar{R} - \bar{r}|} \right].$$

The eigenfunctions of the bound states of the hydrogen atom can be given as [22]:

$$\Psi_{n\ell m} = \sum_{s=0}^{n-\ell-1} B_{s}^{nl} r^{s} e^{-\hat{\delta} r} Y_{s}^{m}(\hat{r}),$$

where $\delta = \frac{1}{n^2} Y_{s}^{m}(\hat{r})$ are the Spherical Harmonics, and

$$B_{s}^{nl} = (-1)^{l} \sqrt{\frac{(n + 1)!}{(n - l - 1)!}} \frac{2^{l+1} \theta(n + 1 - l)}{(n - l - s - 1)!s!(2l + s + 1)!} \frac{2^{s+l+1}}{n^{n+l+1}}.$$

The continuum wave function is given as [22]

$$\Psi_{k_f}^{-}\left(\bar{r}\right) = N(k_e) e^{\frac{\hat{x}_e \cdot \bar{r}}{k_e}} F_1(-i/k_e, 1; -i(k_e r + \hat{x}_e \bar{r})),$$

where $1F_1(-i/k_e, 1; -i(k_e r + \hat{x}_e \bar{r})$ is the confluent-hypergeometric function, and

$$N(k_e) = e^{\frac{k_e^2}{2}} \Gamma(1 + i/k_e).$$
The first-Born term \( T^{B1} = \langle \psi_{k^{-}}(\vec{r}) e^{ik_{f} \cdot \vec{r}} | \varphi_{nlm}(\vec{r}) e^{ik_{i} \cdot \vec{r}} \rangle \) can be written as [5]:

\[
T^{B1} = -\frac{4\pi Z_{p}}{q^{2}} F_{nlm}(\vec{q}),
\]

with the momentum transferred to the electron from the projectile motion and \( F_{nlm}(\vec{q}) = \langle \psi_{k^{-}}(\vec{r}) | e^{ik_{f} \cdot \vec{r}} | \varphi_{nlm}(\vec{r}) \rangle \) is a form factor. Also, the second Born term \( T^{B2} = \langle \psi_{k^{-}}(\vec{r}) e^{ik_{f} \cdot \vec{r}} | V - \frac{1}{\vec{r}_{f} + \vec{r}_{i}} V \ | \varphi_{nlm}(\vec{r}) e^{ik_{i} \cdot \vec{r}} \rangle \) is given in a simplified version of the second Born approximation [7], in which the closure relation is employed to perform the summation over the intermediate states. Hence, \( T^{B2} \) can be given in the form:

\[
T^{B2} = \frac{4\mu Z_{p}^{2}}{\pi} \lim_{q \to 0} \int \frac{F_{nlm}(q_{f}) + F_{nlm}(q_{i}) - 2F_{nlm}(q_{i})}{q_{f}^{2} - q_{i}^{2} - i\eta} d\vec{k},
\]

Figure 1. The triple differential cross sections for ionization of H atoms in 2p₀-state (a-panel), 2p₁-state (b-panel) and 2p-state (c-panel) at 30 keV proton; solid curves (antiproton; dot curves) impact, for ejected energy 5 eV into the scattering plane (\( \varphi_{e} = 0^\circ \)) at scattering angle 1 mrad.

where \( \vec{k} \) is a parameter represents the average wave number. The expression (9) suggests that the momentum transferred to the electron from the projectile motion takes place during two steps with \( \vec{q}_{f} = \vec{k}_{f} - \vec{k} \) followed by \( \vec{q}_{i} = \vec{k}_{i} - \vec{k}_{f} \) momentum transfers.

The form factor \( F_{nlm}(\vec{q}) \) in equations (8) and (9), corresponding to an initial state \( \varphi_{nlm}(\vec{r}) \) and the final state \( \psi_{k^{-}}(\vec{r}) \) is obtained in the form [5]:

\[
\int \psi_{k^{-}}(\vec{r}) e^{-ik_{i} \cdot \vec{r}} | V - \frac{1}{\vec{r}_{f} + \vec{r}_{i}} V \ | \psi_{nlm}(\vec{r}) e^{ik_{f} \cdot \vec{r}} d\vec{r}.
\]
Figure 2. The same as figure 1 but into the perpendicular plane ($\phi_i = 90^\circ$).

$$F_{\text{fin}}(\hat{q}) = \sum_{i=0}^{\ell \pm 1} B_i^\ell N^\ell(k,\phi_i) \int r^{\ell+1} e^{-i\vec{q} \cdot \vec{r}} e^{-i\vec{k} \cdot \vec{r}} Y_{\ell}^m(\vec{r})$$

$$\int F_{i}(i/k_e, 1, i(k_e + \vec{k}_e, \vec{r}) d\vec{r},$$

which can be obtained as:

$$F_{\text{fin}}(\hat{q}) = \frac{4\pi}{\ell^2} \sum_{i=0}^{\ell \pm 1} \sum_{\ell'^{\prime}=0}^{\ell} \sum_{\ell''=0}^{\ell} D_{\ell'^{\prime} \ell''}^{\ell + 1} B_i^\ell C_{\ell'^{\prime} \ell''}$$

$$\frac{\partial^{\ell+1}}{\partial \alpha^{\ell+1}} [A^{\ell'-1} Z^\mu (1 - Z)^{-\ell'-i/k_e}].$$

Where

$$A = q^2 + k_e^2 + (\delta - \alpha)^2 - 2q_k e,$$

$$Z = \frac{2k_e^2 - 2\vec{q} \cdot \vec{k}_e + 2i\alpha \phi_e (\delta - \alpha)}{A},$$

$$C_{\ell'^{\prime} \ell''} = \frac{1}{\ell!} (2\ell)! \exp \left[ \frac{\pi}{2k_e} \right] \frac{\Gamma(\ell + 1) \Gamma(\nu - \ell') \Gamma(\nu + \ell - \ell') i/k_e + 1}{\Gamma(-\ell') \Gamma(\nu + \ell - \ell' + 1)},$$

and

$$D_{\ell'^{\prime} \ell''}^{\ell + 1} = (k_e)^{\ell - \ell'} q^{\ell''} \sqrt{\frac{4\pi (2\ell' + 1)!}{(2\ell' + 1)! (2\ell' - 2\ell'' + 1)!}} (\ell'' m'; \ell - \ell' m - m'| \ell m) Y_{\ell'}^m(q) Y_{-\ell' - m'}(-k_e).$$

Here $(l_i, m_i; l - l_i, m - m_i|l, m)$ are the Clebsch–Gordan coefficients.
The form factor corresponding to the differential cross-section for proton (antiproton)-hydrogen atom being initially in the $2P_0$- and $2P_{+1}$-substates can be obtained, respectively, as:

$$
F_{210}(\bar{q}) = i \frac{16\pi}{\sqrt{3}} 6^{3/2} \exp \left[ \frac{\pi}{2k_e} \Gamma(1 - i/k_e) \frac{(1 - Z)^{2-\frac{i}{k_e}}}{A^3} \right] \left\{ k_e Y_0^0(k_e) (1 - i/k_e) [2\delta(1 - Z)^2 + (1 - Z)(1 + \delta Z/k_e)] + q Y_1^0(q) [2\delta(1 - Z) + (\delta Z - ik_e)(1 + i/k_e)] - q Y_1^1(q) (1 - i/k_e) [2\delta Z(1 - Z) + (1 - Z)(\delta Z - ik_e) + Z(1 + i/k_e)((\delta Z - ik_e))] \right\},
$$

(12)

and

$$
F_{21+1}(\bar{q}) = i \frac{16\pi}{\sqrt{3}} 6^{3/2} \exp \left[ \frac{\pi}{2k_e} \Gamma(1 - i/k_e) \frac{(1 - Z)^{2-\frac{i}{k_e}}}{A^3} \right] \left\{ k_e Y_1^1(k_e) (1 - i/k_e) [2\delta(1 - Z)^2 + (1 - Z)(1 + \delta Z/k_e)] + q Y_1^0(q) [2\delta(1 - Z) + (\delta Z - ik_e)(1 + i/k_e)] - q Y_1^1(q) (1 - i/k_e) [2\delta Z(1 - Z) + (1 - Z)(\delta Z - ik_e) + Z(1 + i/k_e)((\delta Z - ik_e))] \right\},
$$

(13)

The present work is devoted to studying the structure of the TDCS for ionization of hydrogen atoms initially in the $2p$-state by proton (antiproton) impact. It is shown that how certain structure is calculated in the FBA of the ionization process. Structures of the SBA arise only from a correlation sequence of two binary collisions. Structures of TDCS arising from the same physical effect can assume very different shapes depending upon the charge and mass of the projectile. The triply differential cross-section in equation (1) can be written as:

**Figure 3.** The same as figure 1 but for scattering angle 0.5 deg.
in which the TDCS includes terms proportional to $Z^2_p$ or $Z^3_p$ only.

By integrating the TDCS over $\Theta$ and $\Phi$, the single differential cross-section, which describes the shared energy between the ejected electron and the scattered ion can be calculated as:

$$\frac{d^3\sigma}{d\Omega_t d\Omega_e dE_e} = \frac{\mu}{(2\pi)^3} k_j k_e \left| T^{R_i}_j \right|^2 \left( 1 + \text{Re} \frac{T^{R_i}_j}{T^{R_i}_j} \right),$$

where $T^{R_i}_j$ is the angular differential cross-section of emission electron with angular momentum $m = 0$ and $m = +1$, respectively.

3. Results and discussions

In this section, the closure second-Born approximation is applied to calculate the triple differential cross sections (TDCSs) for ionization of hydrogen atoms in the 2p-state by proton (solid-curves) and antiproton (dot-curves) impact as functions of the ejected electron angle $0^\circ \leq \theta_e \leq 360^\circ$. The parameter $\bar{k}$ is taken as

$$\bar{k} = \sqrt{k_i^2 + 2\mu I}$$

where $I$ is the ionization potential of the target atom. The calculations are performed at

![Figure 4. The same as figure 3 but into the perpendicular plane ($\phi = 90^\circ$).](image-url)
impact energy 30 keV for 5 and 50 eV ejected electron energies at scattering angles 1 mrad and 0.5 deg. The computed TDCSs for the 2p_{-1} state are identical to those computed for the 2p_{+1} state in both the scattering ($\varphi_e = 0^\circ$) and the perpendicular ($\varphi_e = 90^\circ$) planes. The triple differential cross sections for the 2p state are computed via equation (16) that is defined as the sum TDCS of emission electron with angular momentum $m = 0$ and $m = +1$.

Figure 1 presents the second-Born TDCSs for a scattering of proton (solid-curved) and antiproton (dot-curves) by hydrogen atom initially in the states 2p\(_0\) (a-panel), 2p\(_{+1}\) (b-panel), and 2p\(_c\) (c-panel) at impact energy 30 for 5 eV ejected energy and 1 mrad scattering angle; where the electron is emitted into the scattering plane. It is noticed that the results in the binary region $0^\circ \leq \theta_e \leq 180^\circ$ are flat, while they have maximum peaks around $\theta_e = 285^\circ$ in the recoil region $180^\circ \leq \theta_e \leq 360^\circ$. The contribution of calculations due to the 2p\(_{+1}\) substate to the 2p-calculations is much greater than that due to the 2p\(_0\) substate. It is observed that the projectile electric charge has a negligible effect on the cross-sections in the binary region, while the antiproton calculations are much greater than the proton ones in the recoil region, especially around the peak.

Figure 2 presents the corresponding results to those in figure 1 but the electron is emitted into the perpendicular plane. It is noticed that the TDCSs are symmetric about $\theta_e = 180^\circ$ at which the results due to the 2p\(_{+1}\) initial state have minimum peaks, while the results due to the 2p\(_0\) initial state show maximum peaks at which the peak due to antiproton interaction is much higher than that due to the proton one. However, the 2p\(_0\) cross-sections show three maximum and two minimum peaks. On the other hand, the 2p\(_{+1}\) cross-sections show two maximum and one minimum peaks. The curves illustrate that the proton-impact process gives cross sections greater than those in the antiproton process; the situation is exchanged in the region $120^\circ \leq \theta_e \leq 240^\circ$, for the results due to the 2p\(_0\) initial state. However, the effect of the projectile electric charge on the cross-sections due to the 2p\(_0\) initial state is much greater than it on those due to the 2p\(_{+1}\) initial state which has a major effect on the 2p

![Figure 5](image_url)
initial state results. It is easy to see that the effect of the contribution of calculations due to the $2p_{+1}$ substate to the $2p$ calculations is much greater than that due to the $2p_0$ substate.

Figures 3 and 4 display the second-Born TDCSs for proton (antiproton) $-\text{H}(2p_0,2p_{+1},2p)$ collisional system at 30 keV incident energy, $0.5^\circ$ scattering angle, and 5 eV ejected energy into the scattering and perpendicular planes, respectively. In figure 3, the curves show a maximum recoil peak around $\theta_e = 320^\circ$ for $2p_0$ substate and three recoil peaks with different heights for $2p_{+1}$ substate. The curves in figure 4 show that the TDCSs are symmetric about $\theta_e = 180^\circ$ and two sharp peaks around $\theta_e = 80^\circ, 280^\circ$. The results in figures 3 and 4 illustrate a negligible effect of the sign of the projectile electric charge on the calculations and the contribution of calculations due to the $2p_{+1}$ initial substate to the $2p$ initial state calculations is much greater than that due to the $2p_0$ initial substate.

The second-Born TDCSs of $\text{H}(2p_0,2p_{+1},2p)$ by 30 keV proton and antiproton impact are illustrated in figures 5 and 6 for 50 eV ejected electrons into the scattering and perpendicular planes, respectively. The scattering angle is taken to be 1 mrad. The curves in figure 5 show flat TDCSs in the binary region and sharp recoil peaks around $\theta_e = 280^\circ$ and an additional small peak for the results due to the $2p_0$ initial state around $\theta_e = 340^\circ$. In figure 6 the TDCSs, for ejection of the electron into the perpendicular plane, are flat except for sharp binary peaks around $\theta_e = 20^\circ$. The curves in figures 5 and 6 show that the effect of the sign of the projectile electric charge on the calculations can be neglected except in the case of the $2p_0$ initial state at $\theta_e \geq 310^\circ$.

Single differential cross-sections (SDCSs) are obtained from the TDCSs by numerical integrations as indicated in section 2. If the integration is performed over $d\Omega_i$ and $d\Omega_f$, one obtains the SDCS as a function of $E_e$. Figure 7 represents the single differential cross-sections for ionization of H($2p$) atom by 30 keV proton impact. The SDCSs are calculated by the first-Born approximation. It is noticed that the SDCSs are steeply decreasing, with small humps, as increasing of the ejection energy.

![Figure 6. The same as figure 5 but into the perpendicular plane ($\phi_i = 90^\circ$).](image)
4. Conclusions

I have investigated the ionization triple differential cross-sections of $\text{H}(2\text{p}), 2\text{p}_0, 2\text{p}_1$ atoms caused by 30 keV proton and antiproton impact. It is found that the TDCSs are significantly affected by the initial substate of the target atom. The contribution of results due to the $2\text{p}_1$ initial substate to the $2\text{p}$ initial state results is much greater than that due to the $2\text{p}_0$ initial substate. The influence of the sign of the projectile electric charge plays an important role in ionization at small scattering angles and small ejection energies.

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