Matching Anonymized and Obfuscated Time Series to Users’ Profiles

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Abstract

Many popular applications use traces of user data to offer various services to their users; example applications include driver-assistance systems and smart home services. However, revealing user information to such applications puts users’ privacy at stake, as adversaries can infer sensitive private information about the users such as their behaviors, interests, and locations. Recent research shows that adversaries can compromise users’ privacy when they use such applications even when the traces of users’ information are protected by mechanisms like anonymization and obfuscation.

In this work, we derive the theoretical bounds on the privacy of users of these applications when standard protection mechanisms are deployed. We build on our recent study in the area of location privacy, in which we introduced formal notions of location privacy for anonymization-based location privacy-protection mechanisms. More specifically, we derive the fundamental limits of user privacy when both anonymization and obfuscation-based protection mechanisms are applied to users’ time series of data. We investigate the impact of such mechanisms on the tradeoff between privacy protection and user utility. In particular, we study achievability results for the case where the time-series of users are governed by an i.i.d. process. The converse results are proved both for the i.i.d. case as well as the more general Markov Chain model. We demonstrate that as the number of users in the network grows, the obfuscation-anonymization plane can be divided into two regions: in the first region, all users have perfect privacy; and, in the second region, no user has privacy.

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Index Terms

User-Data Driven Services (UDD), Privacy-Protection Mechanisms, Information Theoretic Privacy, Anonymization, Obfuscation.

I. INTRODUCTION

Various emerging systems and applications work by analyzing the data submitted by their users in order to serve them; we call such systems user-data driven (UDD) services. Examples of UDD services include smart cities, connected vehicles, smart homes, and connected healthcare devices, which have the promise of a significant societal impact. Unfortunately, the sheer volume of user data collected by these systems can compromise users’ privacy [2]. Even the use of standard privacy-protection mechanisms (specifically, anonymization of user identities and obfuscation of submitted data) does not guarantee users’ privacy: the adversaries are able to use powerful statistical inference techniques to learn sensitive private information of the users [2]–[8].

To illustrate the threat of privacy leakage, consider three popular UDD services: (1) Health care: Wearable monitors that constantly track user health variables can be invaluable in assessing individual health trends and responding to emergencies. However, such monitors produce long time-series of user data uniquely matched to the health characteristics of each user; (2) Smart homes: Emerging smart-home technologies such as fine-grained power measurement systems can help users and utility providers to address one of the key challenges of the twenty-first century: energy conservation. But the measurements of power by such devices can be mapped to users and reveal their lifestyle habits; and, (3) Connected vehicles: The location data provided by connected vehicles promises to greatly improve everyday lives by reducing congestion and traffic accidents. However, the matching of such location traces to prior behavior not only allows for user tracking, but also reveals a user’s habits. In summary, despite their potential impact on society and their emerging popularity, these services have one thing in common: their utility critically depends on their collection of user data, which puts users’ privacy at significant risk.

There are two main approaches to augment privacy in UDD services: identity perturbation (anonymization) [9]–[14], and location perturbation (obfuscation) [15]–[17]. In anonymization techniques, privacy is obtained by concealing the mapping between users and data, and the mapping is changed periodically to thwart statistical inference attacks that try to deanonymize the anonymized data traces by matching user data to known user profiles. In contrast, obfuscation
mechanisms aim at protecting privacy by perturbing user data, e.g., by adding noise to user locations.

The anonymization and obfuscation mechanisms improve user privacy at the cost of user utility. The anonymization mechanism works by frequently changing the pseudonym mappings of users; this improves privacy by reducing the length of time series that can be exploited by statistical analysis. However, this frequent change may also decrease the usability by concealing the temporal relation between a user’s locations, which may be critical in the utility of some systems, e.g., a dining recommendation system that makes suggestions based on the dining history of its users. On the other hand, obfuscation mechanisms work by adding noise to users’ collected data, e.g., location information. The added noise may degrade the utility of UDD applications. Thus, choosing the right level of privacy-protection mechanism is an important question, and understanding what levels of anonymization and obfuscation can provide theoretical guarantees of privacy is of interest.

We will consider the ability of an adversary to perform statistical analyses on time series and match those to descriptions of user behavior. In related work, Unnikrishnan [8] provides a comprehensive analysis of the asymptotic (in the length of the time series) optimal matching of time series to source distributions. However, there are several key differences between that analysis and the work here. First, Unnikrishnan [8] looks at the optimal matching tests, but does not consider any privacy metrics as considered in this paper, and a significant component of our study is demonstrating that mutual information converges to zero so we can conclude there is no privacy leakage (hence, “perfect privacy”). Second, the setting of [8] is different from us, as it does not consider: (a) obfuscation, which is one of the two major protection mechanisms; and (b) sources that are not independent and identically distributed (i.i.d.). Third, the setting of Unnikrishnan [8] assumes a fixed distribution on sources (i.e., classical inference), whereas we assume the existence of general (but possibly unknown) prior distributions for the sources (i.e., a Bayesian setting). Finally, we study the fundamental limits in terms of both the number of users and the number of observations, while Unnikrishnan [8] focuses on the case where the number of users are fixed, finite values.

We derive the fundamental limits of user privacy in UDD services in the presence of both anonymization and obfuscation protection mechanisms. We build on our previous work on formalizing privacy in location-based services [18]–[21], but we extensively expand that work here not just in application area but also user models and settings. In particular, our previous
work introduced the notion of perfect privacy for location-based services, and we derived the rate at which an anonymization mechanism should change the pseudonyms in order to achieve the defined perfect privacy. In this work, we expand the notion of perfect privacy to UDD services in general, and derive the conditions for it to hold when both anonymization and obfuscation-based protection mechanisms are employed.

We study both achievability and converse results. After discussing related work in detail in Section II and introducing the general framework in Section III, we consider an i.i.d. model extensively in Section IV and the first half of Section V. The i.i.d. model would apply directly to data that is sampled at a low rate. In addition, understanding the i.i.d. case can also be considered the first step toward understanding the more complicated case where there is dependency, as was done for anonymization-only LPPMs in [20] and will be done in Section V.C. In addition, dependency can only favor the adversary, so our extensive results for the i.i.d. case provide lower bounds on the achievable privacy in the case of dependency, which would be desirable if the dependency is poorly understood.

A. Summary of the Results

Given $n$, the total number of the users in a network, their degree of privacy depends on two parameters: (1) The number of observations $m = m(n)$ by the adversary per user for a fixed anonymization mapping (i.e., the number of observations before the pseudonyms are changed); and (2) the value of the noise added by the obfuscation technique (as defined in Section III, we quantify the obfuscation noise with a parameter $a_n$, where larger $a_n$ means a higher level of obfuscation). Intuitively, smaller $m(n)$ and larger $a_n$ result in stronger privacy, at the expense of lower utility for the users.

Our goal is to identify values of $a_n$ and $m(n)$ that satisfy perfect privacy in the asymptote of a large number of users ($n \to \infty$). We show that when the users’ data sets are governed by an i.i.d. process, the $m(n) - a_n$ plane can be divided into two areas. In the first area, all users have perfect privacy (as defined in Section III), and, in the second area, users have no privacy. Figure 1 shows the limits of privacy in the entire $m(n) - a_n$ plane. As the figure shows, in regions 1, 2, and 3, users have perfect privacy, while in region 4 users have no privacy.

For the case where the users’ data sets are governed by irreducible and aperiodic Markov chains with $r$ states and $|E|$ edges, we show that users will have no privacy if $m = cn^{2(\frac{4}{r-\tau}+\alpha)}$ and $a_n = c'n^{-\left(\frac{1}{r-\tau}+\beta\right)}$, for some constants $c > 0$, $c' > 0$, $\alpha > 0$, and $\beta > \frac{\alpha}{4}$. We also provide
Fig. 1: Limits of privacy in the entire $m(n) - a_n$ plane: in regions 1, 2, and 3, users have perfect privacy, and in region 4 users have no privacy.

some insights for the opposite direction (under which conditions users have perfect privacy) for the case of Markov chains.

II. RELATED WORK

Previously, mutual information has been used as a privacy metric in a number of settings, [22]–[29]. However, the framework and problem formulation for our setting (Internet of Things (IoT) privacy) is quite different from those encountered in previous work. More specifically, the IoT privacy problem we consider here is based on a large set of time-series data that belong to different users with different statistical patterns that has gone through a privacy-preserving mechanism, and the adversary is aiming at de-anonymizing and de-obfuscating the data.
Next, consider related works in smart-home/public environment Internet of Things (IoT) systems. The authors in [30] investigate privacy for IoT smart energy metering where the concern results from the fact that any privacy breach can reveal in-house activity. A ‘Dynamic Privacy Analyzer’ scheme is proposed as an attempt towards achieving a unique privacy metric that is derived from fundamental principles like robust statistics and information theory. To deal with concerns in different smart-home IoT devices such as lights, smoke-alarms, power switches, baby monitors, and weighing scales, which again makes it possible for the entities to snoop and intrude into the family’s activities, the authors in [31] suggest that device-level protections be augmented with network-level security solutions. This can be achieved through software defined networking technology when used to dynamically block/quarantine suspicious devices. In [32], the authors consider the privacy issues in public environments such as an IoT-enabled retail store. An IoT-enable retail store has the potential to provide rich, targeted information and services to users within the environment. However, to fully realize such potential, users must be willing to share certain data regarding their habits and preferences with the public IoT facility. To encourage users to share such information, the paper proposes a protocol to ensure the customers that their data will not be leaked to third parties.

As far as other applications of IoT is concerned, authors in [33] investigate privacy considerations for IoT applications on Smart Grids. In particular, the authors address three types of challenge domains: customer domain, information and communication domain, and the grid domain. In [34], the author tried to provide privacy for smart-metering systems with rechargeable batteries by obscuring the user’s power demand. In [35], the authors consider the privacy issue of wearable devices. These devices make it possible for health care services start a new phase in serving patients needs and monitor their health remotely. Finally, authors in [36], overview the privacy challenges in industrial IoT systems.

Among these applications, location privacy constitutes one of the most important aspects of IoT privacy. Location privacy preserving mechanisms (LPPMs) can be categorized into two main classes: identity perturbation LPPMs (anonymization) [9]–[14] and location perturbation LPPMs (obfuscation) [15]–[17].

There are various approaches suggested for identity perturbation LPPMs. Particularly, k-anonymity approaches help to keep each user’s identity indistinguishable within a group of \(k - 1\) other users [9], [16], [17], [37]–[44]. Another proposal changes users’ pseudonyms within areas called mix-zones [10], [45], [45]–[47]. There are also various approaches used by location
perturbation LPPMs. For instance, cloaking replaces each user’s location information with a larger region [16], [17], [39], [41], [48]–[63], while an alternative direction is to use dummy locations in the set of possible locations of users [64]–[69]. Some prior works combine techniques to achieve stronger location privacy. For instance, Freudiger et al. combine techniques from cryptography with mix-zones to improve location privacy [11].

Differential privacy, which aims at protecting queries on aggregated data, has also been applied to the problem of location privacy. Particularly, several identity perturbation [17], [70]–[73] and location perturbation [17], [74]–[77] LPPMs are based on differential privacy techniques. Dewri [78] combines k-anonymity and differential privacy to achieve higher location privacy. Alternatively, Andres et al. hide the exact location of a user in a region by adding Laplacian distributed noise to achieve a desired level of geo-indistinguishability [77]. In [79], it is proved that differential privacy arises out of maximizing entropy principle which is equal to minimizing information leakage and is measured using the mutual information notion. [80] tries to reduce the expected estimating loss by using differential privacy and constraining the privacy level.

Several studies aim at quantifying location privacy protection. Shokri et al. [13], [81] define the expected estimation error of the adversary as a metric to evaluate LPPM mechanisms. Ma et al. [12] use uncertainty about users’ location information to quantify user location privacy in vehicular networks. To defeat localization attacks and achieve privacy at the same time, Shokri et al. [15] proposed a method which finds optimal LPPM for an LS B given service quality constraints. Shokri et al. [15] design LPPM mechanisms that will defeat localization attacks. In [82] and [83], privacy leakage of location sharing and interdependent location privacy risks are quantified, respectively. A similar idea is proposed in [84] where the quantification model is based on the Bayes conditional risk. Yu et al. [85] combine two complementary notations, geo-indistinguishability and expected inference error, to achieve location privacy. The geo-indistinguishability is derived from differential privacy but cannot adequately protect users’ location against inference attacks of adversary by using prior information, and the expected inference error as privacy metric doesn’t consider posterior information obtained from pseudo-location. As a result, combining both notations gives users the opportunity to have location privacy. In addition, users are allowed to personalized error bound for different locations.

In [86], the authors propose a novel model for preserving location privacy in IoT in which they propose the so-called Enhanced Semantic Obfuscation Technique (ESOT). In [87], in the context of IoT based wireless sensor networks, the authors propose a flexible routing strategy,
called Multi-routing Random Walk, which protects the sensor’s location. Another approach has been proposed in [88].

The discussed studies demonstrate the growing importance of privacy. What is missing from the current literature is a solid theoretical framework for privacy that is general enough to encompass various privacy preserving methods in the literature. Such a framework will allow us to achieve provable privacy guarantees, obtain fundamental trade-offs between privacy and performance, and provide analytical tools to optimally achieve provable privacy. The closest works to this paper are [1], [18]–[21]. As mentioned, these works only consider anonymization and only achieveability. In this paper, by considering obfuscation and anonymization, we characterize the limits of privacy in the entire $m(n) - a_n$ plane by considering both achievability and converse results.

III. Framework

In this paper, we adopt a similar framework to that employed in [18]–[20]. The general set up is provided here, and the refinement to the precise models for this paper will be presented in the following sections. We assume a system with $n$ users with $X_u(k)$ denoting a sample of the data of user $u$ at time $k$, which would we like to protect from an interested adversary $\mathcal{A}$. We consider a strong adversary $\mathcal{A}$ that has complete statistical knowledge of the users’ data patterns based on the previous observations or other resources. In order to secure data privacy of users, both obfuscation and anonymization techniques are used as shown in Figure 2. In Figure 2, $Z_u(k)$ shows the (reported) sample of the data of user $u$ at time $k$ after applying obfuscation, and $Y_u(k)$ shows the (reported) sample of the data of user $u$ at time $k$ after applying anonymization. The adversary observes only $Y_u(k)$, $k = 1, 2, \cdots, m(n)$, where $m(n)$ is the number of observations of each user before the identities are permuted. The adversary then tries to estimate $X_u(k)$ by using those observations.

\[ X_u(k) \xrightarrow{\text{Obfuscation}} Z_u^{(n)}(k) \xrightarrow{\text{Anonymization}} Y_u^{(n)}(k) \]

Fig. 2: Applying obfuscation and anonymization techniques to users’ data samples.

Let $X_u^{(n)}$ be the $m(n) \times 1$ vector containing the sample of the data of user $u$, and $X^{(n)}$ be the
$m(n) \times n$ matrix with $u^{th}$ column equal to $X_u^{(n)}$;

\[
X_u^{(n)} = \begin{bmatrix}
X_u(1) \\
X_u(2) \\
\vdots \\
X_u(m)
\end{bmatrix}, \quad X^{(n)} = \begin{bmatrix}
X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}
\end{bmatrix}.
\]

**Data Samples Model:** We assume there are $r \geq 2$ possible values $(0, 1, \cdots, r - 1)$ for each sample of the users’ data. In the first part of the paper (perfect privacy analysis), we assume an i.i.d. model as motivated in Section I. At any time, $X_u(k)$ is equal to a value in \{0, 1, \cdots, r - 1\} according to a user-specific probability distribution. The collection of user distributions, which satisfy some mild regularity conditions discussed below, is known to the adversary $\mathcal{A}$, and he/she employs such to distinguish different users based on statistical matching of those user distributions to traces of user activity of length $m(n)$.

**Obfuscation Model:** The first step in obtaining privacy is to apply the obfuscation operation in order to perturb the users’ data samples. In this paper, we assume that each user has only limited knowledge of the characteristics of the overall population and thus we employ a simple distributed method in which the samples of the data of each user are reported with error with a certain probability, where that probability itself is generated randomly for each user. More precisely, let $Z_u^{(n)}$ be the vector which contains the obfuscated versions of user $u$’s data sample, and $Z^{(n)}$ is the collection of $Z_u^{(n)}$ for all users,

\[
Z_u^{(n)} = \begin{bmatrix}
Z_u^{(n)}(1) \\
Z_u^{(n)}(2) \\
\vdots \\
Z_u^{(n)}(m)
\end{bmatrix}, \quad Z^{(n)} = \begin{bmatrix}
Z_1^{(n)}, Z_2^{(n)}, \ldots, Z_n^{(n)}
\end{bmatrix}.
\]

To create a noisy version of data samples, for each user $u$ we independently generate a random variable $R_u^{(n)}$ that is uniformly distributed between 0 and $a_n$.\(^1\) The value of $R_u^{(n)}$ shows the probability that the user’s data sample are changed to a different data sample by obfuscation.

\(^1\)The uniform distribution assumption for $R_u^{(n)}$ is not necessary and the results can be extended to a general set of distributions when we employ $\frac{a_n}{2} = E[R_u^{(n)}]$. However, the assumption of a uniform random variable simplifies the presentation of the results significantly.
and \( a_n \) is termed the “noise level” of the system. For example, for the case of \( r = 2 \) where there are two states for users’ data pattern (state 0 and state 1), we can write

\[
Z^{(n)}_u(k) = \begin{cases} 
X_u(k), & \text{with probability } 1 - R^{(n)}_u \\
1 - X_u(k), & \text{with probability of } R^{(n)}_u
\end{cases}
\]

Note that the effect of the obfuscation is to alter the probability distribution function of each user across the \( r \) possibilities in a way that is unknown to the adversary, since it is independent of all past activity of the user, and hence the obfuscation inhibits user identification. For each user, \( R^{(n)}_u \) is generated once and is kept constant for the collection of samples of length \( m(n) \), thus providing a very low-weight obfuscation algorithm. We will discuss the extension to the case where \( R^{(n)}_u \) are regenerated independently over time in Section VI. There, we will also provide a discussion about obfuscation using continuous noise distributions (e.g., Gaussian noise).

**Anonymization Model:** Anonymization is modeled by a random permutation \( \Pi^{(n)} \) on the set of \( n \) users. The user \( u \) is assigned the pseudonym \( \Pi^{(n)}(u) \). For simplicity, we will employ \( \Pi(u) \) instead of \( \Pi^{(n)}(u) \) where the meaning is not ambiguous. \( Y^{(n)} \) is the anonymized version of \( Z^{(n)}_u \); thus

\[
Y^{(n)} = \text{Perm} \left( Z^{(n)}_1, Z^{(n)}_2, \ldots, Z^{(n)}_n ; \Pi \right) \\
= \left[ Z^{(n)}_{\Pi^{-1}(1)}, Z^{(n)}_{\Pi^{-1}(2)}, \ldots, Z^{(n)}_{\Pi^{-1}(n)} \right] \\
= \left[ Y^{(n)}_1, Y^{(n)}_2, \ldots, Y^{(n)}_n \right],
\]

where \( \text{Perm} \) is permutation operation with permutation function \( \Pi \). As a result, \( Y^{(n)}_u = Z^{(n)}_{\Pi^{-1}(u)} \) and \( Y^{(n)}_{\Pi(u)} = Z^{(n)}_u \).

**Adversary Model:** We protect against the strongest reasonable adversary. Through past observations or some other sources, the adversary is assumed to have complete statistical knowledge of the users’ patterns; in other words, he/she knows the probability distribution for each user on the set of data samples \( \{0, 1, \ldots, r-1\} \). The adversary also knows the value of \( a_n \) as it is a design parameter. However, the adversary does not know the realization of the random permutation \( \Pi \) or the realizations of random variables \( R^{(n)}_u \), as these are independent of the past behavior of the users.

In [20], perfect privacy is defined as follows:
Definition 1. User $u$ has perfect privacy at time $k$, if and only if
\[ \forall k \in \mathbb{N}, \lim_{n \to \infty} I(X_u(k); Y^{(n)}) = 0, \]
where $I(X;Y)$ denotes the mutual information between random variables (vectors) $X$ and $Y$.

In this paper, we also consider the situation in which there is no privacy:

Definition 2. User $u$ has no privacy at time $k$, if and only if there exists an algorithm for the adversary to estimate $X_u(k)$ perfectly as $n$ goes to infinity. In other words, as $n \to \infty$,
\[ \forall k \in \mathbb{N}, \quad P_e(1) \triangleq P\left(\widehat{X_u(k)} \neq X_u(k)\right) \to 0, \]
where $X_u(k)$ is the actual sample of the data of user $u$ at time $k$, $\widehat{X_u(k)}$ is the adversary’s estimated sample of the data of user $u$ at time $k$, and $P_e(u)$ is the error probability.

Notation: Throughout the paper, $X_n \overset{d}{\to} X$ denotes convergence in distribution. We use $P\left(X = x \mid Y = y\right)$ for the conditional probability of $X = x$ given $Y = y$. When we write $P\left(X = x \mid Y\right)$, we are referring to a random variable that is defined as a function of $Y$.

IV. Perfect Privacy Analysis: I.I.D. Case

A. Two-State Model

We first consider the two-state ($r = 2$) case which captures the salient aspects of the problem. For the two-state case, the sample of the data of user $u$ at any time is a Bernoulli random variable with parameter $p_u$, which is the probability of user $u$ being at location 1. Thus,
\[ X_u(k) \sim \text{Bernoulli}(p_u). \]

Per Section III, the parameters $p_u, u = 1, 2, \ldots, n$ are drawn independently from a continuous density function, $f_p(p_u)$, on the $(0, 1)$ interval. The density $f_p(p_u)$ might be unknown, so all it is assumed here is that such density exists. From the results of the paper, it will be evident that knowing or not knowing $f_p(p_u)$ does not change the results asymptotically. Further, we assume there are $\delta_1, \delta_2 > 0$ such that\footnote{The condition $\delta_1 < f_p(p_u) < \delta_2$ is not actually necessary for the results and can be relaxed; however, we keep it here to avoid unnecessary technicalities.}:
\[
\begin{cases} 
\delta_1 < f_p(p_u) < \delta_2, & p_u \in (0, 1), \\
 f_p(p_u) = 0, & p_u \notin (0, 1).
\end{cases}
\]
The adversary knows the values of \( p_u, u = 1, 2, \ldots, n \) and uses this knowledge to identify users. We will use capital letters (i.e., \( P_u \)) when we are referring to the random variable, and use lower case (i.e., \( p_u \)) to refer to the realization of \( P_u \).

In addition, since the user data \((X_u(k))\) are i.i.d. and have a Bernoulli distribution, the obfuscated data \((Z_u(k))\) are also i.i.d. with a Bernoulli distribution. Specifically,

\[
Z_u^{(n)}(k) \sim \text{Bernoulli} \left( Q_u^{(n)} \right),
\]

where

\[
Q_u^{(n)} = P_u (1 - R_u^{(n)}) + (1 - P_u) R_u^{(n)}
\]

\[
= P_u + (1 - 2P_u) R_u^{(n)},
\]

and recall that \( R_u^{(n)} \) is the probability that user \( u \)'s data sample is altered at any time. For convenience, define a vector where element \( Q_u^{(n)} \) is the probability that an obfuscated data sample of user \( u \) is equal to one:

\[
Q^{(n)} = \left[ Q_1^{(n)}, Q_2^{(n)}, \ldots, Q_n^{(n)} \right],
\]

and a vector containing the permutation of those probabilities after anonymization

\[
V^{(n)} = \text{Perm} \left( Q^{(n)}_1, Q^{(n)}_2, \ldots, Q^{(n)}_n; \Pi \right)
\]

\[
= \left[ Q_{\Pi^{-1}(1)}^{(n)}, Q_{\Pi^{-1}(1)}^{(n)}, \ldots, Q_{\Pi^{-1}(1)}^{(n)} \right]
\]

\[
= \left[ V_1^{(n)}, V_2^{(n)}, \ldots, V_n^{(n)} \right]
\]

where \( V_u^{(n)} = Q_{\Pi^{-1}(u)}^{(n)} \) and \( V^{(n)}_{\Pi(u)} = Q_u^{(n)} \). As a result, for \( u = 1, 2, \ldots, n \), the distribution of the data symbols for the user with pseudonym \( u \) is given by:

\[
Y_u^{(n)}(k) \sim \text{Bernoulli} \left( V_u^{(n)} \right) \sim \text{Bernoulli} \left( Q_{\Pi^{-1}(u)}^{(n)} \right).
\]

The following theorem states that if \( a_n \) is significantly larger than \( \frac{1}{n} \) in this two-state model, then all users have perfect location privacy independent of the value of \( m(n) \).

\textbf{Theorem 1.} For the above two-state model, if \( Z^{(n)} \) is the obfuscated version of \( X^{(n)} \), and \( Y^{(n)} \) is the anonymized version of \( Z^{(n)} \) as defined above, and

- \( m = m(n) \) is arbitrary;
- \( R_u^{(n)} \sim \text{Uniform}[0, a_n] \), where \( a_n \doteq c' n^{-(1-\beta)} \) for \( c' > 0 \) and \( \beta > 0 \);
then user 1 has perfect privacy. That is,

$$\forall k \in \mathbb{N}, \lim_{n \to \infty} I\left(X_1(k); Y^{(n)}\right) = 0.$$ 

**Intuition behind the Proof of Theorem 1:**

Since \(m(n)\) is arbitrary, the adversary is able to estimate very accurately (in the limit, perfectly) the distribution from which each data sequence \(Y_u^{(n)}, u = 1, 2, \cdots, n\) is drawn; that is, the adversary is able to accurately estimate the probability \(V_u^{(n)}, u = 1, 2, \cdots, n\). Clearly, if there were no obfuscation for each user \(u\), the adversary would then simply look for the \(i\) such that \(p_i\) is very close to \(V_i^{(n)}(k)\) and set \(\tilde{X}_i(k) = Y_u^{(n)}(k)\), resulting in no privacy for any user.

Hence, in order to make sure that the adversary obtains no information about \(X_1(k)\), the data of user 1, we need to establish that there are a large number of users, each whose probability \(p_u\) is such that, when obfuscated, it could have resulted in an obfuscated probability consistent with \(p_1\). Consider asking whether another probability \(p_2\) is sufficiently close enough to be confused with \(p_1\) after obfuscation; in particular, we will look for \(p_2\) such that, even if the adversary is given the obfuscated probabilities \(V^{(n)}_{\Pi(1)}\) and \(V^{(n)}_{\Pi(2)}\), he/she cannot associate these probabilities with \(p_1\) and \(p_2\). This requires that the distributions \(Q_1^{(n)}\) and \(Q_2^{(n)}\) of the obfuscated data of user 1 and user 2 have significant overlaps; we explore this next.

Recall that \(Q_i^{(n)} = P_i + (1 - 2P_i)R_i^{(n)}\), and \(R_i^{(n)} \sim Uniform[0, a_n]\). Thus, we know \(Q_i^{(n)}|P_i = p_i\) has a uniform distribution with length \((1 - 2p_i)a_n\). Specifically,

$$Q_i^{(n)}|P_i = p_i \sim Uniform(p_i, p_i + (1 - 2p_i)a_n).$$

Figure 3 shows the distribution of \(Q_i^{(n)}\) given \(P_i = p_i\).

Consider two cases: In the first case, the support of the distributions \(Q_1^{(n)}|P_1 = p_1\) and \(Q_2^{(n)}|P_2 = p_2\) are small relative to the difference between \(p_1\) and \(p_2\) (Figure 4); in this case, given the probabilities \(V^{(n)}_{\Pi(1)}\) and \(V^{(n)}_{\Pi(2)}\) of the anonymized data sequences, the adversary can associate those with \(p_1\) and \(p_2\) without error. In the second case, the support of the distributions \(Q_1^{(n)}|P_1 = p_1\)
and $Q_2(n)|P_2 = p_2$ is large relative to the difference between $p_1$ and $p_2$ (Figure 5), so it is difficult for the adversary to associate the probabilities $V^{(n)}_{\Pi(1)}$ and $V^{(n)}_{\Pi(2)}$ of the anonymized data sequences with $p_1$ and $p_2$. In particular, if $V^{(n)}_{\Pi(1)}$ and $V^{(n)}_{\Pi(2)}$ fall into the overlap of the support of $Q_1(n)$ and $Q_2(n)$, we will show the adversary can only guess randomly how to de-anonymize the data. Thus, if the ratio of the support of the distributions to $|p_1 - p_2|$ goes to infinity, the adversary’s posterior probability for each user converges to $\frac{1}{2}$, thus implying no information leakage on the user identities. More generally, if we can somehow guarantee that there will be a large set of users with $p_u$’s very close to $p_1$ compared to the support of $Q_1(n)|P_1 = p_1$, we will be able to obtain perfect privacy as demonstrated rigorously below.

Fig. 4: Case 1: The support of the distributions is small relative to the difference between $p_1$ and $p_2$.

Fig. 5: Case 2: The support of the distributions is large relative to the difference between $p_1$ and $p_2$.

Given this intuition, the formal proof proceeds as follows. Given $p_1$, we define a set $J^{(n)}$ of users whose parameter $p_u$ of their data distributions is sufficiently close to $p_1$ (Figure 5; case 2), so that it is likely that $Q_1(n)$ and $Q_u(n)$ cannot be readily associated with $p_1$ and $p_u$.

The purpose of Lemmas 1, 2, and 3 is to show that, from the adversary’s perspective, the users in the set $J^{(n)}$ are indistinguishable. More specifically, the goal is to show that the obfuscated data corresponding to each of these users could have been generated by any other users in $J^{(n)}$ in an equally likely manner. To show this, Lemma 1 employs the fact that, if the observed values of $N$
uniformly distributed random variable is within the intersection of their ranges, it is impossible to infer any information about the matching between the observed values and the distributions. That is, all possible \( N! \) matchings are equally likely. Lemmas 2 and 3 leverage Lemma 1 to show that even if the adversary is given a set that includes all of the pseudonyms of the users in set \( J^{(n)} \) (i.e., \( \Pi(J^{(n)}) = \{ \Pi^{-1}(u) \in J^{(n)} \} \)) he/she still will not be able to infer any information about the matching of each specific user in set \( J^{(n)} \) and his pseudonym. Then Lemma 5 uses the above fact to show that the mutual information between the data set of user 1 at time \( k \) and the observed data sets of the adversary converges to zero for large enough \( n \).

**Proof of Theorem 1:**

*Proof.* Note, per Lemma 6 of Appendix A, it is sufficient to establish the results on a sequence of sets with high probability. That is, we can condition on high-probability events.

Now, define the critical set \( J^{(n)} \) with size \( N^{(n)} = |J^{(n)}| \) as follows:

\[
J^{(n)} = \begin{cases} 
\{ i \in \{1, 2, \ldots, n \} : p_1 \leq P_i \leq p_1 + \epsilon_n; p_1 + \epsilon_n \leq Q_i^{(n)} \leq p_1 + (1 - 2p_1)a_n \}, & \text{for } 0 \leq p_1 < \frac{1}{2} \\
\{ i \in \{1, 2, \ldots, n \} : p_1 - \epsilon_n \leq P_i \leq p_1; p_1 + (1 - 2p_1)a_n \leq Q_i^{(n)} \leq p_1 - \epsilon_n \}, & \text{for } \frac{1}{2} \leq p_1 \leq 1
\end{cases}
\]

where \( \epsilon_n = \frac{1}{n^{1/\beta}} \), \( a_n = c'n^{-(1-\beta)} \) and \( \beta \) is defined in the statement of Theorem 1.

Note for large enough \( n \), if \( 0 \leq p_1 < \frac{1}{2} \), we have \( 0 \leq p_i < \frac{1}{2} \). As a result,

\[
Q_i^{(n)} \bigg| P_i = p_i \sim \text{Uniform}(p_i, p_i + (1 - 2p_1)a_n).
\]

We can prove that with high probability, \( 1 \in J^{(n)} \) for large enough \( n \), as follows. First

\[
Q_1^{(n)} \bigg| P_1 = p_1 \sim \text{Uniform}(p_1, p_1 + (1 - 2p_1)a_n).
\]

Now, according to Figure 7,

\[
P \left( 1 \in J^{(n)} \right) = 1 - \frac{\epsilon_n}{(1 - 2p_1)a_n} = 1 - \frac{1}{(1 - 2p_1)c'n^{2\beta}} \to 1.
\]

Now in the second step, we define the probability \( W_j^{(n)} \) for any \( j \in \Pi(J^{(n)}) = \{ \Pi(u) : u \in J^{(n)} \} \) as

\[
W_j^{(n)} = P \left( \Pi(1) = j \bigg| V^{(n)}, \Pi(J^{(n)}) \right).
\]

3The proof for the case \( \frac{1}{2} \leq p_1 \leq 1 \) is analogous and is omitted.
Fig. 6: Range of $P_i$ and $Q_i^{(n)}$ for elements of set $J^{(n)}$ and probability density function of $Q_i^{(n)} \mid P_i = p_i$.

Fig. 7: Range of $P_i$ and $Q_i^{(n)}$ for elements of set $J^{(n)}$ and probability density function of $Q_1^{(n)} \mid P_1 = p_1$.

$W_j^{(n)}$ is the conditional probability that $\Pi(1) = j$ after perfectly observing the values of the permuted version of obfuscated probabilities ($V^{(n)}$) and set including all of the pseudonyms of the users in set $J^{(n)}$. Since $V^{(n)}$ and $\Pi(J^{(n)})$ are random, $W_j^{(n)}$ is treated as a random variable. However, we will prove shortly that in fact $W_j^{(n)} = \frac{1}{N^{(n)}}$, for all $j \in \Pi(J^{(n)})$.

Note: Since, we are looking from the adversary’s point of view, the assumption is that all the values of $P_u$, $u \in \{1, 2, \cdots, n\}$ are known, so all of the probabilities are conditioned on the values of $P_1 = p_1, P_2 = p_2, \cdots, P_n = p_n$. Thus, to be accurate, we should write

$$W_j^{(n)} = P \left( \Pi(1) = j \mid V^{(n)}, \Pi(J^{(n)}), P_1, P_2, \cdots, P_n \right).$$

Nevertheless, for simplicity of notation, we often omit the conditioning on $P_1, P_2, \cdots, P_n$.

First, we need a lemma from elementary probability.

**Lemma 1.** Let $N$ be a positive integer, and let $a_1, a_2, \cdots, a_N$ and $b_1, b_2, \cdots, b_N$ be real numbers such that $a_i \leq b_i$ for all $i$. Assume that $X_1, X_2, \cdots, X_N$ are independent random variables such that

$$X_i \sim \text{Uniform}[a_i, b_i].$$
Let also $\gamma_1, \gamma_2, \cdots, \gamma_N$ be distinct real numbers such that
\[ \gamma_j \in \bigcap_{i=1}^{N} [a_i, b_i] \text{ for all } j \in \{1, 2, \ldots, N\} \]
Suppose that we know the event $E$ has occurred, meaning that the observed values of $X_i$’s is equal to the set of $\gamma_j$’s (but with unknown ordering), i.e.,
\[ E \equiv \{X_1, X_2, \cdots, X_N\} = \{\gamma_1, \gamma_2, \cdots, \gamma_N\}. \]
Then
\[ P(X_1 = \gamma_j | E) = \frac{1}{N}. \]

**Proof.** Lemma 1 is proved in Appendix B.
\[ \square \]

Using the above lemma, we can state our desired result for $W_j^{(n)}$.

**Lemma 2.** For all $j \in \Pi(J^{(n)})$, $W_j^{(n)} = \frac{1}{N^{(n)}}$.

**Proof.** We argue that the setting of this lemma is essentially equivalent to the assumptions in Lemma 1. First, remember that
\[ W_j^{(n)} = P\left(\Pi(1) = j \mid V^{(n)}, \Pi(J^{(n)})\right). \]
Note that $Q_u^{(n)} = P_u + (1 - 2P_u)R_u^{(n)}$, and since $R_u^{(n)}$ are uniformly distributed, $Q_u^{(n)}$ conditioned on $P_u$ are also uniformly distributed in the appropriate intervals. Moreover, since $V_u^{(n)} = Q_{\Pi^{-1}(u)}^{(n)}$, we conclude $V_u^{(n)}$ are also uniformly distributed. So looking at the definition of $W_j^{(n)}$, we can say the following: given the values of the uniformly distributed random variables $Q_u^{(n)}$, we would like to know which one of the values in $V^{(n)}$ is the actual value of $Q_1^{(n)} = V_{\Pi(1)}^{(n)}$, i.e., is $\Pi(1) = j$? This is equivalent to the setting of Lemma 1 as described further below.

Note that since $1 \in J^{(n)}$, $\Pi(1) \in \Pi(J^{(n)})$. Therefore, when searching for the value of $\Pi(1)$, it is sufficient to look inside the set $\Pi(J^{(n)})$. Therefore, instead of looking among all the values of $V_u^{(n)}$, it is sufficient to look at $V_u^{(n)}$ for $u \in \Pi(J^{(n)})$. Let’s show these values by $V_{\Pi}^{(n)} = \{v_1, v_2, \cdots, v_{N^{(n)}}\}$, so
\[ W_j^{(n)} = P\left(\Pi(1) = j \mid V_{\Pi}^{(n)}, \Pi(J^{(n)})\right). \]
Thus we have the following scenario: $Q_i^{(n)}$, $i \in J^{(n)}$ are independent random variables, and
\[ Q_i^{(n)} | P_i = p_i \sim Uniform[p_i, p_i + (1 - 2p_i)a_n]. \]
Also \( v_1, v_2, \ldots, v_{N(n)} \) are the observed values of \( Q_i^{(n)} \) with unknown ordering (unknown mapping \( \Pi \)). We also know from the definition of set \( J^{(n)} \) that

\[
P_i \leq p_1 + \epsilon_n \leq Q_i^{(n)},
\]

\[
Q_i^{(n)} \leq p_1 (1 - 2a_n) + an \leq P_i (1 - 2a_n) + an,
\]

so we can conclude

\[
v_j \in \bigcap_{i=1}^{N(n)} [p_i, p_i + (1 - 2p_i)a_n] \quad \text{for all } j \in \{1, 2, \ldots, N^{(n)}\}.
\]

We know the event \( E \) has occurred, meaning that the observed values of \( Q_i^{(n)} \)'s is equal to the set of \( v_j \)'s (but with unknown ordering), i.e.,

\[
E \equiv \{Q_i^{(n)}, i \in J^{(n)}\} = \{v_1, v_2, \ldots, v_{N(n)}\}.
\]

Then according to Lemma 1,

\[
P(Q_i^{(n)} = v_j | E, P_1, P_2, \ldots, P_n) = \frac{1}{N^{(n)}}.
\]

Note that there is a subtle difference between this lemma and Lemma 1 as here \( N^{(n)} \) is a random variable while \( N \) is a fixed number in Lemma 1. Nevertheless, since the assertion holds for every fixed \( N \), it also holds for the case where \( N \) is treated as a random variable. Now, note that

\[
P(Q_1^{(n)} = v_j | E, P_1, P_2, \ldots, P_n)
= P(\Pi(1) = j | E, P_1, P_2, \ldots, P_n)
\]

\[
= P(\Pi(1) = j | V_\Pi^{(n)}, \Pi(J^{(n)}), P_1, P_2, \ldots, P_n)
\]

\[
= W_j^{(n)}.
\]

Thus, we can conclude

\[
W_j^{(n)} = \frac{1}{N^{(n)}}.
\]

\( \square \)

In third step, we define \( \overline{W}_j^{(n)} \) for any \( j \in \Pi(J^{(n)}) \) as

\[
\overline{W}_j^{(n)} = P(\Pi(1) = j | Y^{(n)}, \Pi(J^{(n)})).
\]

\( \overline{W}_j^{(n)} \) is the conditional probability that \( \Pi(1) = j \) after observing the values of anonymized version of obfuscated samples of users’ data \( (Y^{(n)}) \) and aggregate set including all the pseudonyms
of the users in set \(J^{(n)}\) (i.e., \(\Pi(J^{(n)}) \triangleq \{\Pi^{-1}(u) \in J^{(n)}\}\). Since \(Y^{(n)}\) and \(\Pi(J^{(n)})\) are random, \(\tilde{W}_j^{(n)}\) is treated as a random variable. Now, in following lemma, we will prove \(\tilde{W}_j^{(n)} = \frac{1}{N^{(n)}}\), for all \(j \in \Pi(J^{(n)})\) by using Lemma 3.

Note in the following lemma, we want to show even if the adversary is given a set including all of the pseudonyms of the users in set \(J^{(n)}\), he/she cannot match each specific user in set \(J^{(n)}\) and his pseudonym.

**Lemma 3.** For all \(j \in \Pi(J^{(n)})\), \(\tilde{W}_j^{(n)} = \frac{1}{N^{(n)}}\).

**Proof.** First, note that

\[
\tilde{W}_j^{(n)} = \sum_{\forall v} P\left(\Pi(1) = j \mid Y^{(n)}, \Pi(J^{(n)}), V^{(n)} = v\right) P\left(V^{(n)} = v \mid Y^{(n)}, \Pi(J^{(n)})\right)
\]

Also, we note that given \(V^{(n)}\), \(\Pi(J^{(n)})\) and \(Y^{(n)}\) are independent. Intuitively, this is because when observing \(Y^{(n)}\), any information regarding \(\Pi(J^{(n)})\) is leaked through estimating \(V^{(n)}\). This can be rigorously proved similar to the proof of Lemma 1 in [20]. We can state this fact as

\[
P\left(Y^{(n)}(k) \mid Y^{(n)} = v_u, \Pi(J^{(n)})\right) = P\left(V^{(n)}(k) \mid V^{(n)} = v_u\right) = v_u.
\]

The right and left hand side are given by \(Bernoulli(v_u)\) distributions.

As a result,

\[
\tilde{W}_j^{(n)} = \sum_{\forall v} P\left(\Pi(1) = j \mid \Pi(J^{(n)}), V^{(n)} = v\right) P\left(V^{(n)} = v \mid Y^{(n)}, \Pi(J^{(n)})\right)
\]

Note \(W_j^{(n)} = P\left(\Pi(1) = j \mid \Pi(J^{(n)}), V^{(n)}\right)\), so

\[
\tilde{W}_j^{(n)} = \sum_{\forall v} W_j^{(n)} P\left(V^{(n)} = v \mid Y^{(n)}, \Pi(J^{(n)})\right) = \frac{1}{N^{(n)}} \sum_{\forall v} P\left(V^{(n)} = v \mid Y^{(n)}, \Pi(J^{(n)})\right) = \frac{1}{N^{(n)}}.
\]

□

To show that no information is leaked, we need to show that the size of set \(J^{(n)}\) goes to infinity. This is established in Lemma 4.
Lemma 4. If \(N(n) \triangleq |J(n)|\), then \(N(n) \to \infty\) with high probability as \(n \to \infty\). More specifically, there exists \(\lambda > 0\) such that
\[
P\left(N(n) > \frac{\lambda}{2} n^2\right) \to 1.
\]

Proof. Lemma 4 is proved in Appendix C. \(\square\)

In the final step, we define \(\tilde{W}_{j}^{(n)}\) for any \(j \in \Pi(J(n))\) as
\[
\tilde{W}_{j}^{(n)} = P\left(X_1(k) = 1 \mid Y^{(n)}, \Pi(J(n))\right).
\]

\(\tilde{W}_{j}^{(n)}\) is the conditional probability that \(X_1(k) = 1\) after observing the values of anonymized version of obfuscated samples of users’ data \((Y^{(n)})\) and aggregate set including all the pseudonyms of the users in set \(J(n)\) \((\Pi(J(n)))\). \(\tilde{W}_{j}^{(n)}\) is treated as a random variable because \(Y^{(n)}\) and \(\Pi(J(n))\) are random. Now, in following lemma, we will prove \(\tilde{W}_{j}^{(n)}\) converges in distribution to \(\rho_1\).

Note that this is the probability from the adversary’s point of view. That is, given that the adversary has observed \(Y^{(n)}\) as well as the extra information \(\Pi(J(n))\), what can he/she infer about \(X_1(k)\)?

Lemma 5. For all \(j \in \Pi(J(n))\), \(\tilde{W}_{j}^{(n)} \overset{d}{\to} \rho_1\).

Proof. We know
\[
\tilde{W}_{j}^{(n)} = \sum_{j \in \Pi(J(n))} P\left(X_1(k) = 1 \mid \Pi(1) = j, Y^{(n)}, \Pi(J(n))\right) P\left(\Pi(1) = j \mid Y^{(n)}, \Pi(J(n))\right),
\]
and according to definition of \(\tilde{W}_{j}^{(n)} = \left(\Pi(1) = j \mid Y^{(n)}, \Pi(J(n))\right)\), we have
\[
\tilde{W}_{j}^{(n)} = \sum_{j \in \Pi(J(n))} P\left(X_1(k) = 1 \mid \Pi(1) = j, Y^{(n)}, \Pi(J(n))\right) \tilde{W}_{j}^{(n)}
\]
\[
= \frac{1}{N(n)} \sum_{j \in \Pi(J(n))} P\left(X_1(k) = 1 \mid \Pi(1) = j, Y^{(n)}, \Pi(J(n))\right).
\]

We now claim that
\[
P\left(X_1(k) = 1 \mid \Pi(1) = j, Y^{(n)}, \Pi(J(n))\right) = \rho_1 + o(1).
\]

The reasoning goes as follows. Given \(\Pi(1) = j\) and knowing \(Y^{(n)}\), we know that
\[
Y_{\Pi(1)}^{(n)}(k) = Z_1^{(n)}(k) = \begin{cases} X_1^{(n)}(k), & \text{with probability of } 1 - R_1^{(n)}, \\ 1 - X_1^{(n)}(k), & \text{with probability of } R_1^{(n)}. \end{cases}
\]
Thus, given \( Y_j^{(n)}(k) = 1 \), then according to Bayes’ rule,
\[
P \left( X_1(k) = 1 \mid \Pi(1) = j, Y^{(n)}, \Pi(J^{(n)}) \right) = \left( 1 - R_1^{(n)} \right) \frac{P(X_1(k) = 1)}{P(Y_{\Pi(1)}^{(n)}(k) = 1)}
\]
\[
= \left( 1 - R_1^{(n)} \right) \frac{P_1}{p_1(1 - R_1^{(n)}) + (1 - p_1)R_1^{(n)}}
\]
\[
= 1 - o(1),
\]
and similarly, given \( Y_j^{(n)}(k) = 0 \),
\[
P \left( X_1(k) = 1 \mid \Pi(1) = j, Y^{(n)}, \Pi(J^{(n)}) \right) = R_1^{(n)} \frac{P(X_1(k) = 1)}{P(Y_{\Pi(1)}^{(n)}(k) = 0)}
\]
\[
= R_1^{(n)} \frac{P_1}{p_1(1 - R_1^{(n)}) + (1 - p_1)R_1^{(n)}}
\]
\[
= o(1).
\]

Note that by the independence assumption, the above probabilities do not depend on the other values of \( Y_u^{(n)}(k) \) (as we are conditioning on \( \Pi(1) = j \)). Thus, we can write
\[
\overline{W}_j^{(n)} = \frac{1}{N^{(n)}} \sum_{j \in \Pi(J^{(n)})} P \left( X_1(k) = 1 \mid \Pi(1) = j, Y^{(n)}, \Pi(J^{(n)}) \right)
\]
\[
= \frac{1}{N^{(n)}} \sum_{j \in \Pi(J^{(n)}, Y_j^{(n)}(k) = 1)} (1 - o(1)) + \frac{1}{N^{(n)}} \sum_{j \in \Pi(J^{(n)}, Y_j^{(n)}(k) = 0)} o(1).
\]

First, note that since \( \left| \left\{ j \in \Pi(J^{(n)}), Y_j^{(n)}(k) = 0 \right\} \right| \leq N^{(n)} \), the second term above converges to zero. Thus,
\[
\overline{W}_j^{(n)} \rightarrow \frac{\left| \left\{ j \in \Pi(J^{(n)}), Y_j^{(n)}(k) = 1 \right\} \right|}{N^{(n)}}.
\]
Since for all \( j \in \Pi(J^{(n)}), Y_j^{(n)}(k) \sim Bernoulli (p_1 + o(1)) \), by a simple application of Chebyshev’s inequality, we can conclude \( \overline{W}_j^{(n)} \rightarrow p_1 \). Appendix D provides the detail. \( \square \)

As a result,
\[
X_1(k) | Y^{(n)}, \Pi(J^{(n)}) \rightarrow Bernoulli(p_1),
\]
thus,
\[
H \left( X_1(k) \mid Y^{(n)}, \Pi(J^{(n)}) \right) \rightarrow H (X_1(k))
\]
We know conditioning reduces entropy, so
\[
H \left( X_1(k) \left| Y^{(n)}, \Pi^{(n)} \right. \right) \leq H \left( X_1(k) \left| Y^{(n)} \right. \right);
\]
and as a result,
\[
\lim_{n \to \infty} H \left( X_1(k) \right) - H \left( X_1(k) \left| Y^{(n)} \right. \right) \leq 0
\]
\[
\lim_{n \to \infty} I \left( X_1(k); Y^{(n)} \right) \leq 0.
\]
By knowing that \( I \left( X_1(k); Y^{(n)} \right) \) cannot take any negative value, we can conclude that
\[
I \left( X_1(k); Y^{(n)} \right) \to 0.
\]

B. Extension to \( r \)-States

Now, assume users’ data samples can have \( r \) possibilities \((0, 1, \cdots, r - 1)\), and \( p_u(i) \) shows the probability of user \( u \) having data sample \( i \). We define the vector \( p_u \) and the matrix \( P \) as
\[
p_u = \begin{bmatrix} p_u(1) \\ p_u(2) \\ \vdots \\ p_u(r-1) \end{bmatrix}, \quad P = [p_1, p_2, \cdots, p_n].
\]
We assume \( p_u(i) \)’s are drawn independently from some continuous density function, \( f_P(p_u) \), which has support on a subset of the \((0,1)^{r-1}\) hypercube (Note that the \( p_u(i) \)’s sum to one, so one of them can be considered as the dependent value and the dimension is \( r - 1 \)). In particular, define the range of distribution as
\[
R_P = \{(x_1, \cdots, x_{r-1}) \in (0,1)^{r-1} : x_i > 0, \quad x_1 + x_2 + \cdots + x_{r-1} < 1 \}.
\]
Figure 8 shows the range \( R_P \) for the case where \( r = 3 \).

Then, we assume there are \( \delta_1, \delta_2 > 0 \) such that:
\[
\begin{cases}
\delta_1 < f_P(p_u) < \delta_2, & p_u \in R_P. \\
f_P(p_u) = 0, & p_u \notin R_P.
\end{cases}
\]
The obfuscation is similar to the two-state case. Specifically, for \( j \in \{0, 1, \cdots, r - 1\} \) we can write
Fig. 8: $R_P$ for case $r = 3$.

$$P(Z_u^{(n)}(k) = j | X_u(k) = i) = \begin{cases} 1 - R_u^{(n)}, & \text{for } j = i. \\ R_u^{(n)/r - 1}, & \text{for } j \neq i. \end{cases}$$

**Theorem 2.** For the above $r$-state model, if $Z^{(n)}$ is the obfuscated version of $X^{(n)}$, and $Y^{(n)}$ is the anonymized version of $Z^{(n)}$ as defined previously, and

- $m = m(n)$ is arbitrary;
- $R_u^{(n)} \sim Uniform[0, a_n]$, where $a_n \equiv c'n^{-\left(\frac{1}{r-1} - \beta\right)}$ for $c' > 0$ and $\beta > 0$;

then user 1 has perfect privacy. That is,

$$\forall k \in \mathbb{N}, \quad \lim_{n \to \infty} I\left(X_1(k); Y^{(n)}\right) = 0.$$

The proof of Theorem 2 is similar to the proof of Theorem 1. The major difference is that instead of the random variables $P_{iu} Q_u^{(n)}, V_u^{(n)}$, we need to consider the random vectors $P_u, Q_u^{(n)}, V_u^{(n)}$. Similarly, for user $u$, we define the vector $Q_u^{(n)}$ as

$$Q_u^{(n)} = \begin{bmatrix} Q_u^{(n)}(1) \\ Q_u^{(n)}(2) \\ \vdots \\ Q_u^{(n)}(r - 1) \end{bmatrix}.$$
Note that in this case
\[
Q_u^{(n)}(l) = P_u(l)\left(1 - R_u^{(n)}(l)\right) + \left(1 - P_u(l)\right)\frac{R_u^{(n)}}{r - 1}
\]
\[
= P_u + \left(1 - rP_u\right)\frac{R_u^{(n)}}{r - 1},
\]

We also need to define the critical set \( J^{(n)} \). To do so, we define for \( l = 0, 1, \ldots, r - 1 \) the set \( J_l^{(n)} \) as follows. If \( 0 \leq p_1(l) < \frac{1}{r} \), then
\[
J_l^{(n)} = \left\{ i \in \{1, 2, \ldots, n\} : p_1(l) \leq p_i(l) \leq p_1(l) + \epsilon_n; p_1(l) + \epsilon_n \leq Q_i^{(n)}(l) \leq p_1(l) + (1 - r p_1(l))\frac{a_n}{r - 1} \right\},
\]
where \( \epsilon_n \triangleq \frac{1}{n^{r-1} - \beta} \) and \( \beta \) is defined in the statement of Theorem 2, i.e., \( a_n = c'n^{-\left(\frac{1}{r-1} - \beta\right)} \). If \( \frac{1}{r} \leq p_1(l) \leq 1 \), then
\[
J_l^{(n)} = \left\{ i \in \{1, 2, \ldots, n\} : p_1(l) - \epsilon_n \leq p_i(l) \leq p_1(l); p_1(l) + (1 - r p_1(l))\frac{a_n}{r - 1} \leq Q_i^{(n)}(l) \leq p_1(l) - \epsilon_n \right\}.
\]
We will then define the critical set \( J^{(n)} \) as follows:
\[
J^{(n)} = \bigcap_{l=0}^{r-1} J_l^{(n)}.
\]
Using these, we can repeat the same arguments in the proof of Theorem 1 to complete the proof.

V. Converse Results: No Privacy Region

In this section, we prove that if the number of observations by the adversary is larger than its critical value and the noise level is less than its critical value, then the adversary can find an algorithm to successfully estimate users’ data samples with arbitrarily small error probability. Combined with the results of the previous section, this implies that asymptotically (as \( n \to \infty \)), privacy can be achieved if and only if at least one of the two techniques (obfuscation or anonymization) are used above their thresholds. This statement needs a clarification as follows:

Looking at the results of [20], we notice that anonymization alone can provide perfect privacy if \( m(n) \) is below its threshold. On the other hand, the threshold for obfuscation requires some anonymization: In particular, the identities of the users must be permuted once to prevent the adversary from readily identifying the users.
Fig. 9: $p_1$, sets $B^{(n)}$ and $C^{(n)}$ for case $r = 2$.

**A. Two-State Model**

Let us start with the simplest case: i.i.d. two-state model. The data sample of users at any time is a Bernoulli random variable with parameter $p_u$.

As before, we assume that $p_u$’s are drawn independently from some continuous density function, $f_P(p_u)$, on the $(0, 1)$ interval. Specifically, there are $\delta_1, \delta_2 > 0$ such that:

$$\begin{cases} 
    \delta_1 < f_P(p_u) < \delta_2, & p_u \in (0, 1), \\
    f_P(p_u) = 0, & p_u \notin (0, 1).
\end{cases}$$

**Theorem 3.** For the above two-state mode, if $Z^{(n)}$ is the obfuscated version of $X^{(n)}$, and $Y^{(n)}$ is the anonymized version of $Z^{(n)}$ as defined, and

- $m = cn^{2+\alpha}$ for $c > 0$ and $\alpha > 0$;
- $R_u^{(n)} \sim \text{Uniform}[0, a_n]$, where $a_n \equiv c' n^{-(1+\beta)}$ for $c' > 0$ and $\beta > \frac{\alpha}{4}$;

user 1 has no privacy as $n$ goes to infinity. In other words, there exists an algorithm for the adversary to produce the estimation of $X_1(k)$ such that for large enough $n$,

$$\forall k \in \mathbb{N}, \quad P_e(1) \equiv P\left(\overline{X_1(k)} \neq X_1(k)\right) \to 0.$$ 

Since this is a converse result, we give an explicit detector at the adversary and show that it can be used by the adversary to recover the true data pattern of user 1.

**Proof.** The adversary first inverts the anonymization mapping $\Pi$ to obtain $Z_1(k)$, and then estimates the value of $X_1(k)$ from that. To invert the anonymization, the adversary calculates the empirical probability that each string is in state 1 and then assigns the string with the empirical probability closest to $p_1$ to user 1.

Formally, for $u = 1, 2, \cdots, n$, the adversary computes $Y_u^{(n)}$, the empirical probability of user $u$ being in state 1, as follows

$$\overline{Y_u^{(n)}} = \frac{Y_u^{(n)}(1) + Y_u^{(n)}(2) + \cdots + Y_u^{(n)}(m)}{m}.$$
Thus,
\[ \overline{Y}_{\Pi(u)}^{(n)} = \frac{Z_u^{(n)}(1) + Z_u^{(n)}(2) + \cdots + Z_u^{(n)}(m)}{m}. \]

As shown in Figure 9, define
\[ B^{(n)} = \{ x \in (0, 1); p_1 - \Delta_n < x < p_1 + \Delta_n \}, \]
where \( \Delta_n = \frac{1}{n^{1+\alpha}} \) and \( \alpha \) is defined in the statement of Theorem 3. We claim that for \( m = c n^{2+\alpha} \), \( a_n = c' n^{-(1+\beta)} \), and large enough \( n \),
\[
1) \quad P \left( \overline{Y}_{\Pi(1)}^{(n)} \in B^{(n)} \right) \to 1.
\]
\[
2) \quad P \left( \bigcup_{j=2}^{n} \left( \overline{Y}_{\Pi(j)}^{(n)} \in B^{(n)} \right) \right) \to 0.
\]
Thus, the adversary can identify \( \Pi(1) \) by examining \( \overline{Y}_{u}^{(n)} \)'s and assigning the one in \( B^{(n)} \) to user 1.

First, we show that as \( n \) goes to infinity,
\[ P \left( \overline{Y}_{\Pi(1)}^{(n)} \in B^{(n)} \right) \to 1. \]

We can write
\[
P \left( \overline{Y}_{\Pi(1)}^{(n)} \in B^{(n)} \right) = P \left( \frac{\sum_{i=1}^{m} Z_1^{(n)}(i)}{m} \in B^{(n)} \right) \]
\[
= P \left( p_1 - \Delta_n < \frac{\sum_{i=1}^{m} Z_1^{(n)}(i)}{m} < p_1 + \Delta_n \right) \]
\[
= P \left( m p_1 - m \Delta_n - m Q_1^{(n)} < \sum_{i=1}^{m} Z_1^{(n)}(i) - m Q_1^{(n)} < m p_1 + m \Delta_n - m Q_1^{(n)} \right). \]

Note that for any \( u \in \{1, 2, \cdots, n\} \), we have
\[
|p_u - Q_u^{(n)}| = |1 - 2p_u R_u^{(n)}| \leq R_u^{(n)} \leq a_n.
\]
So we can conclude
\[
P\left(\frac{Y^{(n)}}{\Pi(1)} \in B^{(n)}\right) = P\left(mp_1 - m\Delta_n - mQ_1^{(n)} < \sum_{i=1}^{m} Z_i^{(n)}(i) - mQ_1^{(n)} < mp_1 + m\Delta_n - mQ_1^{(n)}\right)
\geq P\left(-m\Delta_n + ma_n < \sum_{i=1}^{m} Z_i^{(n)}(i) - mQ_1^{(n)} < -ma_n + m\Delta_n\right)
= P\left(\sum_{i=1}^{m} Z_i^{(n)}(i) - mQ_1^{(n)} \bigg| < m(\Delta_n - a_n)\right).
\]
Since \(a_n \to 0\), for \(p_1 \in (0, 1)\) and large enough \(n\), we can say \(p_1 + a_n < 2p_1\). According to Chernoff bound,
\[
P\left(\left| \sum_{i=1}^{m} Z_i^{(n)}(i) - mQ_1^{(n)} \right| < m(\Delta_n - a_n)\right) \geq 1 - 2e^{-\frac{m(\Delta_n - a_n)^2}{3m(\Pi)'^2}}
\geq 1 - 2e^{-\frac{1}{4p_1+an}(n^{2+\alpha})(\frac{1}{n^{1+\alpha}} \frac{a'}{a})^2}
\geq 1 - 2e^{-\frac{\alpha''}{6p_1}n^{\frac{a}{2}}} \to 1.
\]
As a result, as \(n\) becomes large,
\[
P\left(\frac{Y^{(n)}}{\Pi(1)} \in B^{(n)}\right) \to 1.
\]
Now, we need to show that as \(n\) goes to infinity,
\[
P\left(\bigcup_{j=2}^{n} \frac{Y^{(m)}}{\Pi(j)} \in B^{(n)}\right) \to 0.
\]
First, we define
\[
C^{(n)} = \{x \in (0, 1); p_1 - 2\Delta_n < x < p_1 + 2\Delta_n\},
\]
and claim as \(n\) goes to infinity,
\[
P\left(\bigcup_{j=2}^{n} \left(P_j \in C^{(n)}\right)\right) \to 0.
\]
Note
\[
4\Delta_n \delta_1 < P\left(P_j \in C^{(n)}\right) < 4\Delta_n \delta_2.
\]
and according to the union bound, for large enough $n$,

$$
P\left(\bigcup_{j=2}^{n} \left( P_j \in C^{(n)} \right) \right) \leq \sum_{j=2}^{n} P\left( P_j \in C^{(n)} \right) \leq 4n\Delta_n \delta_2 = 4n\frac{1}{n^{1+\frac{2}{5}}} \delta_2 = 4n^{-\frac{3}{5}} \delta_2 \rightarrow 0.
$$

As a result, we can conclude that all $p_j$'s are outside of $C^{(n)}$ for $j \in \{2, 3, \cdots, n\}$ with high probability.

Now, we claim that given all $p_j$'s are outside of $C^{(n)}$, $P\left( \overline{Y_{\Pi(j)}} \in B^{(n)} \right)$ is small. Remember that for any $u \in \{1, 2, \cdots, n\}$, we have $\left| p_u - Q_u^{(n)} \right| \leq a_n$.

Now, noting the definitions of the sets $B^{(n)}$ and $C^{(n)}$, we can write

$$
P\left( \overline{Y^{(n)}_{\Pi(j)}} \right) \leq P\left( \left| \overline{Y^{(n)}_{\Pi(j)}} - Q_j \right| > (\Delta_n - a_n) \right).
$$

According to the Chernoff bound,

$$
P\left( \left| \sum_{i=1}^{m} Z^{(n)}_j(i) - mQ^{(n)}_j \right| > m(\Delta_n - a_n) \right) \leq 2e^{-\frac{m(\Delta_n - a_n)^2}{3Q_j^{(n)}}} \leq 2e^{-\frac{1}{3p_1 + 2\alpha} q_n^2 (\frac{1}{n^{1+\frac{2}{5}}} - \frac{\alpha'}{n^{1+\frac{2}{5}}})^2} \leq 2e^{-\frac{\alpha''}{6p_1 n^{2/5}}}. \tag{1}
$$

We also know that as $n$ goes to infinity, $2e^{-\frac{\alpha''}{6p_1 n^{2/5}}}$ goes to zero. As a result,

$$
P\left( \overline{Y^{(n)}_{\Pi(j)}} \in B^{(n)} \right) \rightarrow 0.
$$

Now, by using union bound, we have

$$
P\left( \bigcup_{j=2}^{n} \left( \overline{Y^{(m)}_{\Pi(j)}} \in B^{(n)} \right) \right) \leq \sum_{j=2}^{n} P\left( \overline{Y^{(m)}_{\Pi(j)}} \in B^{(n)} \right) \leq n \left( 2e^{-\frac{\alpha''}{6p_1 n^{2/5}}} \right),
$$
and thus, as $n$ goes to infinity,

$$P \left( \bigcup_{j=2}^{n} \left( \overline{Y}_{\Pi(j)}^{(m)} \in B^{(n)} \right) \right) \to 0.$$ 

So the adversary can successfully recover $Z_1^{(n)}(k)$. Since $Z_1^{(n)}(k) = X_1(k)$ with probability $1 - R_u^{(n)} = 1 - o(1)$, the adversary can recover $X_1(k)$ with vanishing error probability.  

\[ \square \]

**B. Extension to $r$-States**

Now, assume users’ data samples can have $r$ possibilities $(0, 1, \ldots, r - 1)$, and $p_u(i)$ shows the probability of user $u$ having data sample $i$. We define the vector $p_u$ as

$$p_u = \begin{bmatrix} p_u(1) \\ p_u(2) \\ \vdots \\ p_u(r-1) \end{bmatrix}, \text{ for each } u \in \{1, 2, \ldots, n\}.$$  

We also assume $p_u$’s are drawn independently from some continuous density function, $f_{P}(p_u)$, which has support on a subset of the $(0, 1)^{r-1}$ hypercube. In particular, define the range of distribution as

$$R_{P} = \{ (x_1, \ldots, x_{r-1}) \in (0, 1)^{r-1} : x_i > 0, x_1 + x_2 + \cdots + x_{r-1} < 1 \}.$$ 

Then, we assume there are $\delta_1, \delta_2 > 0$ such that:

$$\begin{align*}
\delta_1 < f_{P}(p_u) < \delta_2, & \quad p_u \in R_{P}. \\
f_{P}(p_u) = 0, & \quad p_u \notin R_{P}.
\end{align*}$$

**Theorem 4.** For the above $r$-states mode, if $Z^{(n)}$ is the obfuscated version of $X^{(n)}$, and $Y^{(n)}$ is the anonymized version of $Z^{(n)}$ as defined, and

- $m = cn^{\frac{2}{r} + \alpha}$ for $c > 0$ and $0 < \alpha < 1$;
- $R_u^{(n)} \sim \text{Uniform}[0, a_n]$, where $a_n \triangleq c' n^{-\frac{1}{r} + \beta}$ for $c' > 0$ and $\beta > \frac{\alpha}{4}$,

then user 1 has no privacy as $n$ goes to infinity. In other words, there exists an algorithm for the adversary to produce the estimation of $\overline{X}_1(k)$ such that for large enough $n$,

$$\forall k \in \mathbb{N}, \quad P_e(1) \triangleq P \left( \overline{X}_1(k) \neq X_1(k) \right) \to 0.$$
Fig. 10: \( p_1 \), sets \( B'(n) \) and \( C'(n) \) in \( R_P \) for case \( r = 3 \).

The proof of Theorem 4 is similar to the proof of Theorem 3, so we just provide the general idea. We similarly define the empirical probabilities \( Y_j^{(m)} \). The difference is that now for each \( j \in \{1, 2, \ldots, n\} \), \( Y_j^{(m)} \) is a vector of size \( r - 1 \). In other words,

\[
Y_j^{(m)} = \left[ Y_j^{(m)}(1), Y_j^{(m)}(2), \ldots, Y_j^{(m)}(r - 1) \right],
\]

where \( Y_j^{(m)}(i) \) is the empirical probability that the user with pseudonym \( j \) has data sample \( i \).

Define sets \( B'(n) \) and \( C'(n) \) as

\[
B'(n) \triangleq \{(x_1, \ldots, x_{r-1}) \in R_P : p_1(i) - \Delta'_n < x_i < p_1(i) + \Delta'_n, \quad i = 1, \ldots, r - 1\},
\]

\[
C'(n) \triangleq \{(x_1, \ldots, x_{r-1}) \in R_P : p_1(i) - 2\Delta'_n < x_i < p_1(i) + 2\Delta'_n, \quad i = 1, \ldots, r - 1\},
\]

where \( \Delta'_n = \frac{1}{n^{1+\frac{1}{2}}} \). Figure 10 shows \( p_1 \) and the sets \( B'(n) \) and \( C'(n) \) for the case \( r = 3 \).

We claim for \( m = cn^{\frac{5}{2} + \alpha} \) and as \( n \) become large:

1) \( P \left( \bigcup_{i=1}^{r-1} \left[ Y_j^{(m)}(i) \in B'(n) \right] \right) \rightarrow 1 \)

2) \( P \left( \bigcup_{j=2}^{n} \left[ Y_j^{(m)}(i) \in B'(n) \right] \right) \rightarrow 0 \)

This can be shown similar to the above proof that we provided for the two-state case. Thus, the adversary can de-anonymize the data and then recover \( X_1(k) \) with vanishing error probability as in the two-state model.
C. Markov Chain Model

So far, we have assumed users’ data sample can have \( r \) possibilities \((0, 1, \cdots, r-1)\) and users’ patterns are i.i.d. Here, we model users’ pattern by using Markov chains, to capture the dependency of the users’ pattern over time. Again, we assume there are \( r \) possibilities (the number of states in the Markov chains). Let \( E \) be the set of edges. More specifically, \((i, j) \in E\) if there exists an edge from \( i \) to \( j \) with probability \( p(i, j) > 0 \). What distinguishes different users is their transition probabilities \( p_u(i, j) \) (the probability that user \( u \) jumps from state \( i \) to state \( j \)). The adversary knows the transition probabilities of all users. The model for obfuscation and anonymization is exactly the same as before.

We show that the adversary will be able to estimate the data samples of the users with low error probability if \( m(n) \) and \( a_n \) are in the appropriate range. The key idea is that the adversary can focus on a subset of transition probabilities that are sufficient for recovering the entire transition probability matrix. By estimating those from the observed data and matching with the known transition probabilities of the users, the adversary will be able to first de-anonymize the data, and then estimate the locations of users. In particular, note that for each state \( i \), we must have

\[
\sum_{j=1}^{r} p_u(i, j) = 1, \quad \text{for each } u \in \{1, 2, \cdots, n\}
\]

so the Markov chain of user \( u \) is completely determined by a subset of size \( d = |E| - r \) of transition probabilities. Let’s write this subset in a new vector as follows:

\[
p_u = \begin{bmatrix}
p_u(1) \\
p_u(2) \\
\vdots \\
p_u(|E| - r)
\end{bmatrix}, \quad \text{for each } u \in \{1, 2, \cdots, n\}.
\]

We also consider \( p_u \)'s are drawn independently from some continuous density function, \( f_P(p_u) \), which has support on a subset of the \((0, 1)^{|E| - r}\) hypercube. Let \( R_p \subset \mathbb{R}^d \) be the range of acceptable values for \( P_u \). As before, we assume there are \( \delta_1, \delta_2 > 0 \), such that:

\[
\begin{cases}
\delta_1 < f_P(p) < \delta_2, & p \in R_p. \\
f_P(p) = 0, & p \notin R_p.
\end{cases}
\]

Using the above observations, we can show the following theorem.
Theorem 5. For an irreducible, aperiodic Markov chain with \( r \) states and \(|E|\) edges as defined above, if \( Z^{(n)} \) is the obfuscated version of \( X^{(n)} \), and \( Y^{(n)} \) is the anonymized version of \( Z^{(n)} \), and

- \( m = cn^\frac{2}{|E|-r} + \alpha \) for \( c > 0 \) and \( \alpha > 0; \)
- \( R_i^{(n)} \sim \text{Uniform}[0, a_n] \), where \( a_n \doteq c' n^{-\frac{|E|-\beta}{4}} \) for \( c' > 0 \) and \( \beta > \frac{\alpha}{4} \); then the adversary can successfully identify the pattern of user 1 as \( n \) goes to infinity. In other words, for large enough \( n \),

\[
\forall k \in \mathbb{N}, \quad P_e(1) \doteq P \left( X_1(k) \neq \widehat{X}_1(k) \right) \to 0.
\]

The proof has a lot of similarity to the i.i.d. case, so we provide a sketch, mainly focusing on the differences. We argue as follows. If the total number of observations per user is \( m = m(n) \), then define \( M_i(u) \) to be the total number of visits by user \( u \) to state \( i \), for \( i = 0, 1, \ldots, r - 1 \). Since the Markov Chain is irreducible and aperiodic, and \( m(n) \to \infty \), all \( \frac{M_i(u)}{m(n)} \) converge to their stationary values. Now conditioned on \( M_i(u) = m_i(u) \), the transitions from state \( i \) to state \( j \) for user \( u \) follow a multinomial distribution with probabilities \( p_u(i, j) \).

Given the above, the setting is now very similar to the i.i.d. case. Each user is uniquely characterized by a vector \( p_u \) of size \(|E| - r \). We define the sets \( B''^{(n)} \) and \( C''^{(n)} \) as

\[
B''^{(n)} \doteq \{(x_1, \ldots, x_d) \in R^d : p_1(i) - \Delta''_n < x_i < p_1(i) + \Delta''_n, i = 1, \ldots, d\},
\]

\[
C''^{(n)} \doteq \{(x_1, \ldots, x_d) \in R^d : p_1(i) - 2\Delta''_n < x_i < p_1(i) + 2\Delta''_n, i = 1, \ldots, d\},
\]

where \( \Delta''_n = \frac{1}{n^{|E|-r+\frac{3}{4}}} \), and \( d = |E| - r \). Then, we can show that for the stated values of \( m(n) \) and \( a_n \), as \( n \) becomes large:

1) \( P \left( \bigcup_{i=1}^{n} \bigcup_{j=1}^{n} \left( Y^{(m)}_{\Pi(j)} \in B''^{(n)} \right) \right) \to 0. \)

2) \( P \left( \bigcup_{i=1}^{n} \bigcup_{j=2}^{n} \left( Y^{(m)}_{\Pi(j)} \in B''^{(n)} \right) \right) \to 0. \)

which means that the adversary can estimate the pattern of user 1 with vanishing error probability. The proof is very similar to the proof of the i.i.d. case; however, there are two differences that need to be addressed:

First, the probability of observing an erroneous observation is not exactly given by \( R_i^{(n)} \). In fact, a transition is distorted if at least one of its nodes is distorted. So if the actual transition is from state \( i \) to state \( j \), then the probability of an erroneous observation is

\[
R_i^{(n)} + R_j^{(n)} - 2R_i^{(n)}R_j^{(n)}.
\]
Nevertheless, here the order only matters and the above expression is still in the order of \( a_m = O\left(n^\left(\frac{1}{|r| + \beta}\right)\right)\).

The second difference is more subtle. As opposed to the i.i.d. case, the error probabilities are not completely independent. In particular, if \( X_u(k) \) is reported in error, then both the transition to that state and from that state are reported in error. This means that there is a dependency between errors of adjacent transitions. We can address this issue in the following way:

The adversary makes his decision only based on a subset of the observations. More specifically, the adversary looks at only odd-numbered transitions: First, third, fifth, etc., and ignores the even-numbered transitions. This way, the number of observations is effectively reduced from \( m \) to \( \frac{m}{2} \) which again does not impact the order of the result (remember that the Markov chain is aperiodic). However, the adversary has now access to observations with independent errors.

VI. PERFECT PRIVACY ANALYSIS: MARKOV CHAIN MODELS

So far, we have provided both achievability and converse results for the i.i.d. case. However, we have only provided the converse results for the Markov chain case. Here, we investigate achievability for Markov chain models. It turns out that for this case, the assumed obfuscation technique is not sufficient to achieve a reasonable level of privacy. Loosely speaking, we can state that if the adversary can make enough observations, then he can break the anonymity. The culprit is the fact that the observed sequence by the adversary is no longer a Markov chain; namely it can be modeled by a hidden Markov chain. This allows the adversary to successfully estimate the obfuscation random variable \( R_u^{(n)} \) as well as the \( p_u(i,j) \) values for each sequence, hence successfully de-anonymize the sequences.

More specifically, as we will see below, there is a fundamental difference between the i.i.d. case and the Markov chain case. In the i.i.d. case, if the noise level is beyond a relatively small threshold, the adversary will be clueless in de-anonymizing the data, let alone in recovering the actual values of the data sets for users, no matter how large \( m \) is. On the other hand, in the Markov chain case, if \( m = m(n) \) is large enough, then the adversary can easily de-anonymize the data. To better illustrate this, let’s consider a very simple example.

**Example 1.** Consider the simple scenario where there are only two states and users’ data samples change between the two states according to the Markov chain shown in Figure 11. What distinguishes the users is their different values of \( p \). Now suppose we use the same obfuscation
method as before. That is, to create a noisy version of data samples, for each user \( u \) we generate the random variable \( R_u(n) \) that is the probability that the data sample of the user is changed to a different data sample by obfuscation. Specifically,

\[
Z_u^{(n)}(k) = \begin{cases} 
X_u(k), & \text{with probability } 1 - R_u(n), \\
1 - X_u(k), & \text{with probability } R_u(n).
\end{cases}
\]

Fig. 11: A state transition diagram.

To analyze this problem, we can construct the underlying Markov chain as follows. Each state in this Markov chain is identified by two values: the real state of the user, and the observed value by the adversary. In particular, we can write

\[
(\text{Real value, Observed value}) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}.
\]

Figure 12 shows the state transition diagram of this new Markov chain.

Fig. 12: The state transition diagram of the new Markov chain.
We know

\[ \pi_{00} = \pi_0 (1 - R) = \frac{p}{1 + p} (1 - R). \]
\[ \pi_{01} = \pi_0 R = \frac{p}{1 + p} R. \]
\[ \pi_{10} = \pi_1 R = \frac{1}{1 + p} R. \]
\[ \pi_{11} = \pi_1 (1 - R) = \frac{1}{1 + p} (1 - R). \]

The observed process by the adversary is not a Markov chain; nevertheless we can define limiting probabilities. In particular, let \( \theta_0 \) be the limiting probability of observing a zero. That is, we have

\[ \frac{M_0}{m} \sim \theta_0, \quad \text{as } n \to \infty, \]

where \( m \) is the total number of observations by the adversary, and \( M_0 \) is the number of 0’s observed. Then,

\[ \theta_0 = \pi_{00} + \pi_{10} = \frac{(1 - R)p + R}{1 + p}. \]

Also, let \( \theta_1 \) be the limiting probability of observing a one, so

\[ \theta_1 = \pi_{01} + \pi_{11} = \frac{pR + (1 - R)}{1 + p} = 1 - \theta_0. \]

Now by estimating \( \theta_0 \), the adversary obtains

\[ \hat{\theta}_0 = \frac{(1 - R)p + R}{1 + p}. \quad (1) \]

Note that if the number of observations by the adversary can be arbitrarily large, the adversary can obtain an arbitrarily accurate estimate of \( \theta_0 \). The adversary can obtain another equation easily. For example, let \( \theta_{01} \) be the limiting portion of transitions from state 0 to 1 in the observed chain by the adversary. We can write

\[ \theta_{01} = P \{(00 \to 01), (00 \to 11), (10 \to 01), (10 \to 11)\} \]

\[ = \pi_{00}(1 - R) + \pi_{10}PR + \pi_{10}(1 - p)(1 - R). \]

As a result,

\[ \hat{\theta}_{01} = \frac{p(1 - R)^2 + R (PR(1 - R)(1 - p))}{1 + p}. \quad (2) \]

Again, if the number of observations can be arbitrarily large, the adversary can obtain an arbitrarily accurate estimate of \( \theta_{01} \). By solving the Equations 3 and 4, the adversary can successfully recover \( R \) and \( p \); thus, he/she can successfully de-anonymize users’ locations.
VII. DISCUSSION

A. Markov Chain Model

As opposed to the i.i.d. case, we see from Section III that if we do not limit \( m = m(n) \), the assumed obfuscation method will not be sufficient to achieve perfect privacy. There are a few natural questions here. First, for a given noise level, what would be the maximum \( m(n) \) that could guarantee perfect privacy in this model? The more interesting question is, how can we possibly modify the obfuscation technique to make it more suitable for the Markov chain model? A natural solution seems to be re-generating the obfuscation random variables \( R_u^{(n)} \) periodically. This will keep the adversary from easily estimating them by observing a long sequence of data at a small increase in complexity. In fact, this will make the obfuscation much more robust to modeling uncertainties and errors. It is worth noting, however, that this change would not affect the other results in the paper. That is, even if the obfuscation random variables are re-generated frequently, it is relatively easy to check that all the previous theorems in the paper remain valid. However, the increase in robustness to modeling errors will definitely be a big advantage. Thus, the question is how often should the random variables \( R_u^{(n)} \) be re-generated to strike a good balance between complexity and privacy? These are all interesting questions for future research.

B. Obfuscating the Sample of Users’ Data Using Continuous Noise

Here, we argue that for the setting of this paper, continuous noise such as Gaussian noise is not a good option to obfuscate the sample of users’ data with the finite alphabet when we want to achieve perfect privacy. For better understanding, let’s consider a simple example.

**Example 2.** Consider the simple scenario where the users’ data sets are governed by i.i.d. model and the number of possible values for each sample of the users’ data \( r \) is equal to 2 (two-state model). Note the data sequence for user \( u \) is a Bernoulli random variable with parameter \( p_u \).

Here we assume that the actual sample of the data of user \( u \) at time \( k \) (\( X_u(k) \)) is obfuscated using a noise with Gaussian distribution \( (S_u^{(n)}(k)) \), and \( Z_u^{(n)}(k) \) is the obfuscated version of \( X_u(k) \). That is, we can write

\[
Z_u^{(n)}(k) = X_u(k) + S_u^{(n)}(k), \quad S_u^{(n)}(k) \sim N\left(0, R_u^{(n)}\right)
\]

where \( R_u^{(n)} \) is chosen from some distributions. For simplicity, we can consider \( R_u^{(n)} \sim N\left(0, a_n^2\right) \) where \( a_n \) is the noise level.
We also apply anonymization technique in order to anonymize $Z_u^{(n)}(k)$, and as before $Y_u^{(n)}(k)$ is the reported sample of the data of user $u$ at time $k$ after applying anonymization technique. According to Section III, anonymization is modeled by a random permutation $\Pi^{(n)}(u)$ on the set of $n$ users.

Now the question is as follows: Is it possible to achieve perfect privacy independent of the number of adversary’s observation ($m(n)$) while using this continuous noise ($S_u^{(n)}(k)$) to obfuscate the sample of users’ data?

Note that, the density function of the reported sample of the data of user $u$ after applying obfuscation is

$$f_{Z_u^{(n)}}(z) = p_u f_{S_u^{(n)}(k)}(z - 1) + (1 - p_u) f_{S_u^{(n)}(k)}(z)$$

$$= p_u \frac{1}{\sqrt{2\pi R_u^{(n)}}} e^{-\frac{(z-1)^2}{2R_u^{(n)}}} + (1 - p_u) \frac{1}{\sqrt{2\pi R_u^{(n)}}} e^{-\frac{z^2}{2R_u^{(n)}}}.$$  

In this case, when the number of adversary’s observation of the sample of users’ data is large, the adversary can estimate the values of $P_u$ and $R_u^{(n)}$ for each user with an arbitrarily small error probability. As a result, the adversary can de-anonymize the data and then recover $X_u(k)$. The conclusion here is that continuous noise distribution gives too much information to adversary when used for obfuscation of finite alphabet data. A method to remedy this issue is to regenerate the random variables $R_u^{(n)}$ frequently (similar to our previous discussion for Markov chains). Understanding the optimal frequency and perfect analysis in this case is an interesting future research direction.

C. Relation to Differential Privacy

Differential privacy is mainly used when there is a statistical database of users’ sensitive information and the goal is to protect an individual’s data while publishing aggregate information about the database [17], [70]–[73]. The goal of differential privacy is publishing aggregate queries with low sensitivity. It means the effect of changes in a single individual on the outcome of the aggregated information is negligible.

In [89] three different approaches for differential privacy are presented. The one that best matches to our setting is stated as

$$\frac{P(X_1(k) = x_1 | Y^{(n)})}{P(X_1(k) = x_2 | Y^{(n)})} \leq e^{\epsilon} \frac{P(X_1(k) = x_1)}{P(X_1(k) = x_2)},$$
where $Y^{(n)}$ is the set of reported data sets. It means $Y^{(n)}$ has limited effect on the probabilities assigned by the attacker. In differential privacy, user 1 has strongest differential privacy when $\epsilon_r = 0$.

In Lemma 5, we proved that if user 1 has perfect privacy, this implies that asymptotically (for large enough $n$)

$$P\left(X_1(k) = x_1 \mid Y^{(n)}\right) \rightarrow P\left(X_1(k) = x_1\right).$$

(3)

$$P\left(X_1(k) = x_2 \mid Y^{(n)}\right) \rightarrow P\left(X_1(k) = x_2\right).$$

(4)

As a result, by using (3) and (4) we can conclude that if we satisfy perfect privacy condition in this paper, we can also satisfy differential privacy with $\epsilon_r = 0$, i.e., the strongest case of differential privacy.

**VIII. CONCLUSIONS**

In this paper, we considered both obfuscation and anonymization techniques to achieve privacy. We characterized the limits of privacy in the entire $m(n) - a_n$ plane for the i.i.d. case. The privacy level of the users depend on both $m(n)$ (number of observations per user by the adversary for a fixed anonymization mapping) and $a_n$ (noise level). That is, larger $m(n)$ and smaller $a_n$ indicate weaker privacy. In this paper, we obtained the exact values of the thresholds for $m(n)$ and $a_n$. We showed that if $m(n)$ is fewer than $O\left(n^{\frac{2}{\alpha-1}}\right)$, or $a_n$ is bigger than $\Omega\left(n^{-\frac{1}{\alpha-1}}\right)$, users have perfect privacy. On the other hand, if none of the above conditions are satisfied, users have no privacy. For the case, where the users’ patterns are modeled by Markov chains, we obtained a no-privacy region in the $m(n) - a_n$ plane.

Future research in this area needs to characterize the exact privacy/no-privacy regions for the Markov models. It is also important to consider different ways to obfuscate users’ data sets, and study the utility-privacy trade-offs for different kinds of obfuscation techniques.

**APPENDIX A**

**Lemma 6 and its Proof**

Here, we state that we can condition on high-probability events.

**Lemma 6.** Let $p \in (0, 1)$, and $X \sim Bernoulli(p)$ be defined on a probability space $(\Omega, \mathcal{F}, P)$. Consider $B_1, B_2, \cdots$ be a sequence of events defined on the same probability space such that
\( P(B_n) \to 1 \) as \( n \) goes to infinity. Also, let \( Y^{(1)}, Y^{(2)}, \ldots \) be a sequence of random vectors (matrices) in the same probability space, then:

\[
I(X; Y^{(n)}) \to 0 \quad \text{iff} \quad I(X; Y^{(n)}|B_n) \to 0.
\]

**Proof.** First, we prove that as \( n \) becomes large,

\[
H(X|B_n) - H(X) \to 0. \tag{5}
\]

Note that as \( n \) goes to infinity,

\[
P(X = 1) = P \left( X = 1 \bigg| B_n \right) P(B_n) + P \left( X = 1 \bigg| \overline{B_n} \right) P(\overline{B_n})
\]

\[
= P \left( X = 1 \bigg| B_n \right),
\]

thus, \( X \bigg| B_n \overset{d}{\to} X \), and as \( n \) goes to infinity,

\[
H(X|B_n) - H(X) \to 0.
\]

Similarly as \( n \) becomes large,

\[
P \left( X = 1 \bigg| Y^{(n)} = y \right) \to P \left( X = 1 \bigg| Y^{(n)} = y, B_n \right),
\]

and

\[
H \left( X|Y^{(n)} = y, B_n \right) - H \left( X|Y^{(n)} = y \right) \to 0. \tag{6}
\]

Remembering that

\[
I \left( X; Y^{(n)} \right) = H(X) - H(X|Y^{(n)}), \tag{7}
\]

and using (5), (6), and (7), we can conclude that as \( n \) goes to infinity,

\[
I \left( X; Y^{(n)}|B_n \right) - I \left( X, Y^{(n)} \right) \to 0.
\]

As a result, for large enough \( n \),

\[
I \left( X; Y^{(n)} \right) \to 0 \iff I \left( X; Y^{(n)}|B_n \right) \to 0.
\]

\( \square \)
APPENDIX B

PROOF OF LEMMA 1

Here, we provide a formal proof for Lemma 1 which we restate as follows.

Let $N$ be a positive integer, and let $a_1, a_2, \cdots, a_N$ and $b_1, b_2, \cdots, b_N$ be real numbers such that $a_i \leq b_i$ for all $i$. Assume that $X_1, X_2, \cdots, X_N$ are $N$ independent random variables such that

$$X_i \sim Uniform[a_i, b_i].$$

Let also $\gamma_1, \gamma_2, \cdots, \gamma_N$ be real numbers such that

$$\gamma_j \in \bigcap_{i=1}^{N} [a_i, b_i] \quad \text{for all } j \in \{1, 2, \ldots, N\}.$$

Suppose that we know the event $E$ has occurred, meaning that the observed values of $X_i$’s is equal to the set of $\gamma_j$’s (but with unknown ordering), i.e.,

$$E \equiv \{X_1, X_2, \ldots, X_N\} = \{\gamma_1, \gamma_2, \ldots, \gamma_N\}.$$

Then

$$P(X_1 = \gamma_j | E) = \frac{1}{N}.$$

Proof. Define the sets $\mathfrak{P}$ and $\mathfrak{P}_j$ as follows.

$\mathfrak{P}$ = The set of all permutations $\Pi$ on $\{1, 2, \ldots, N\}$.

$\mathfrak{P}_j$ = The set of all permutations $\Pi$ on $\{1, 2, \ldots, N\}$ such that $\Pi(1) = j$.

So, we have $|\mathfrak{P}| = N!$ and $|\mathfrak{P}_j| = (N - 1)!$. Then

$$P(X_1 = \alpha_j | E) = \frac{\sum_{\pi \in \mathfrak{P}_j} f_{X_1, X_2, \ldots, X_N}(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \cdots, \gamma_{\pi(N)})}{\sum_{\pi \in \mathfrak{P}} f_{X_1, X_2, \ldots, X_N}(\gamma_{\pi(1)}, \gamma_{\pi(2)}, \cdots, \gamma_{\pi(N)})}$$

$$= \frac{(N - 1)! \prod_{i=1}^{N} \frac{1}{b_i-a_i}}{N! \prod_{i=1}^{N} \frac{1}{b_i-a_i}}$$

$$= \frac{1}{N}.$$
APPENDIX C

PROOF OF LEMMA 4

Here, we provide a formal proof for Lemma 4 which we restate as follows. The following lemma confirms that the number of elements in $J^{(n)}$ goes to infinity as $n$ becomes large.

If $N^{(n)} \triangleq |J^{(n)}|$, then $N^{(n)} \to \infty$ with high probability as $n \to \infty$. More specifically, there exists $\lambda > 0$ such that

$$P \left( N^{(n)} > \frac{\lambda}{2} n^\beta \right) \to 1.$$ 

Proof. We prove the lemma for $0 \leq p_1 < \frac{1}{2}$. A similar proof can be used for $p_1 \geq \frac{1}{2}$.

Define the events $A, B$ as

$$A \equiv p_1 \leq P_i \leq p_1 + \epsilon_n$$

$$B \equiv p_1 + \epsilon_n \leq Q_i^{(n)} \leq p_1 + (1 - 2p_1)a_n.$$ 

Then, for $i \in \{1, 2, \ldots, n\}$ and $0 \leq p_1 < \frac{1}{2}$:

$$P \left( i \in J^{(n)} \right) = P (A \cap B)$$

$$= P(A) P \left( B \mid A \right).$$

So, given $p_1 \in (0, 1)$ and the assumption $0 < \delta_1 < f_p < \delta_2$, for $n$ large enough, we have

$$P(A) = \int_{p_1}^{p_1 + \epsilon_n} f_p(p) dp,$$

so we can conclude that

$$\epsilon_n \delta_1 < P(A) < \epsilon_n \delta_2.$$ 

We can find a $\delta$ such that $\delta_1 < \delta < \delta_2$ and

$$P(A) = \epsilon_n \delta.$$ (8)

We know

$$Q_i^{(n)} \bigg| P_i = p_i \sim Uniform \left( p_i, p_i + (1 - 2p_i)a_n \right),$$

so according to Figures 6, for $p_1 \leq p_i \leq p_1 + \epsilon_n$,

$$P(B \mid P_i = p_i) = \frac{p_1 + (1 - 2p_1)a_n - p_1 - \epsilon_n}{p_i + (1 - 2p_1)a_n - p_i}$$

$$= \frac{p_1 + (1 - 2p_1)a_n - \epsilon_n}{1 - 2p_1)a_n}$$

$$\geq \frac{1 - 2p_1)a_n - \epsilon_n}{1 - 2p_1)a_n}$$

$$= 1 - \frac{\epsilon_n}{1 - 2p_1)a_n},$$
which implies
\[ P(B \mid A) \geq 1 - \frac{\epsilon_n}{(1 - 2p_1)a_n}. \] (9)

Using (8) and (9), we can conclude
\[ P\left( i \in J^{(n)} \right) \geq \epsilon_n \delta \left( 1 - \frac{\epsilon_n}{(1 - 2p_1)a_n} \right). \]

Then, we can say that \( N^{(n)} \) has a binomial distribution with expected value of \( N^{(n)} \) greater than \( n \epsilon_n \delta \left( 1 - \frac{\epsilon_n}{(1 - 2p_1)a_n} \right) \), and by substituting \( \epsilon_n \) and \( a_n \), we get
\[ E\left[ N^{(n)} \right] \geq \delta \left( n^{\frac{\beta}{2}} - \frac{1}{c'(1 - 2p_1)} \right) \geq \lambda n^{\frac{\beta}{2}}. \]

Now by using the above followed by Chebyshev’s inequality, as \( n \) goes to infinity,
\[ P\left( N^{(n)} \leq \frac{\lambda}{2} n^{\frac{\beta}{2}} \right) \leq P\left( N^{(n)} \leq \frac{E\left[ N^{(n)} \right]}{2} \right) \]
\[ \leq P\left( \left| N^{(n)} - E\left[ N^{(n)} \right] \right| \geq \frac{E\left[ N^{(n)} \right]}{2} \right) \]
\[ \leq \frac{4 \text{Var}\left( N^{(n)} \right)}{(E\left[ N^{(n)} \right])^2}. \]

Since \( N^{(n)} \) is binomial, we also have
\[ \text{Var}\left( N^{(n)} \right) \leq E\left[ N^{(n)} \right], \]
so we can conclude for large enough \( n \),
\[ P\left( N^{(n)} \leq \frac{\lambda}{2} n^{\frac{\beta}{2}} \right) \leq \frac{4}{E\left[ N^{(n)} \right]} \]
\[ \leq \frac{4}{\lambda n^{\frac{\beta}{2}}} \rightarrow 0. \]

Therefore,
\[ P\left( N^{(n)} > \frac{\lambda}{2} n^{\frac{\beta}{2}} \right) \rightarrow 1 \]
As a result, \( N^{(n)} \rightarrow \infty \) with high probability for large enough \( n \). \( \square \)
Let \( p_1 \in (0, 1) \), and let \( N^{(n)}(n) \) be a random variable as above, i.e., \( N^{(n)} \to \infty \) as \( n \to \infty \). Consider the sequence of independent random variables \( Y_i^{(n)} \sim \text{Bernoulli}(p_i^{(n)}) \) for \( i = 1, 2, \cdots, N^{(n)}(n) \) such that

1) For all \( n \) and all \( i \in \{1, 2, \cdots, N^{(n)}(n)\} \), \( |p_i^{(n)} - p_1| \leq \zeta_n \).

2) \( \lim_{n \to \infty} \zeta_n = 0 \).

Define

\[
\bar{Y}^{(n)} = \frac{1}{N^{(n)}} \sum_{i=1}^{N^{(n)}} Y_i^{(n)},
\]

then \( \bar{Y}^{(n)} \overset{d}{\to} p_1 \).

**Proof.** Note

\[
E[\bar{Y}^{(n)}] = \frac{1}{N^{(n)}} \sum_{i=1}^{N^{(n)}} p_i^{(n)}
\leq \frac{1}{N^{(n)}} \sum_{i=1}^{N^{(n)}} (p_1 + \zeta_n)
= \frac{1}{N^{(n)}} \cdot N^{(n)}(p_1 + \zeta_n)
= p_1 + \zeta_n.
\]

Similarly we can prove \( E[\bar{Y}^{(n)}] \geq p_1 - \zeta_n \). Since as \( n \) becomes large, \( \zeta_n \to 0 \) and \( p_1 \in (0, 1) \), we can conclude

\[
\lim_{n \to \infty} E[\bar{Y}^{(n)}] = p_1. \tag{10}
\]

Also,

\[
\text{Var}\left(\bar{Y}^{(n)}\right) = \frac{1}{(N^{(n)})^2} \sum_{i=1}^{N^{(n)}} p_i (1 - p_i)
\leq \frac{1}{(N^{(n)})^2} \sum_{i=1}^{N^{(n)}} (p_1 + \zeta_n)(1 - p_1 + \zeta_n)
= \frac{1}{(N^{(n)})^2} \cdot N^{(n)}(p_1 + \zeta_n)(1 - p_1 + \zeta_n)
= \frac{1}{N^{(n)}} (p_1 + \zeta_n)(1 - p_1 + \zeta_n).
\]
Thus,

\[
\lim_{n \to \infty} \text{Var}(\bar{Y}^{(n)}) = 0 \tag{11}
\]

By using (10), (11), and Chebyshev’s inequality, we can conclude

\[
\bar{Y}^{(n)} \overset{d}{\to} p_1
\]

\[\square\]

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