ON THE UNIQUENESS OF AF DIAGONALS IN REGULAR LIMIT ALGEBRAS

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Abstract. Necessary and sufficient conditions are obtained for the uniqueness of standard regular AF masas in regular limit algebras up to approximate inner unitary equivalence.

1. Introduction

A digraph algebra, or incidence algebra, is a subalgebra of a finite dimensional $C^*$-algebra which contains a maximal abelian self-adjoint algebra (masa). Such a masa is uniquely determined up to inner unitary equivalence and in this sense is intrinsic to the digraph algebra. By a regular limit algebra we mean either an algebraic limit, or operator algebra limit of a direct system $\{\phi_k : A_k \to A_{k+1} : k \in \mathbb{N}\}$ of digraph algebras with connecting maps $\phi_k$ that are regular. This regularity constraint requires that each $\phi_k$ is decomposable as a direct sum of elementary multiplicity one maps. For such a system a choice of matrix units provides a distinguished regular masa of the limit algebra and these masas (for a given system) are independent of this choice in the sense of being unique up to approximate inner unitary equivalence. See Proposition 3.3 of [8]. We refer to these masas as standard regular masas. An important problem for the classification theory of limit algebras is whether such a masa is intrinsic to the limit algebra itself, or, alternatively, depends on the particular system defining the algebra. More succinctly put, are two standard regular masas of a limit algebra conjugate by a star-extendible automorphism?

The significance of this uniqueness is that in exact parallel with the finite dimensional case (wherein the masa induces an intrinsic digraph or binary relation) it leads to the well-definedness of the spectrum, or semigroupoid, of the algebra and thence to complete isomorphism invariants.

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The uniqueness problem, which was first indicated explicitly in Power [18], is open for both algebraic and operator algebra limits. For self-adjoint limits, both algebraic and closed, it is now known that standard regular masas are unique, even to within approximate inner unitary equivalence. This fact is due to Kreiger [15] and a direct proof is given in [18]. However for non-selfadjoint algebras this stricter form of conjugacy may fail. This is shown in Donsig and Power [8] and in Power [19]. In the present article we resolve this strict uniqueness problem and obtain necessary and sufficient conditions for the uniqueness of standard regular masas up to approximate inner unitary equivalence.

A standard regular masa in an AF $C^*$-algebra, as indicated above, is in itself approximately finite and is regular in the usual sense that its normaliser generates the ambient AF $C^*$-algebra. We point out that although there is a converse to this in the algebraic limit case, there is no such converse for operator algebra limits. That is, a regular AF masa in an AF $C^*$-algebra need not be standard regular. We remark that the original terminology "regular", which has been borrowed for limit algebra embeddings, originates from Dixmier’s considerations [5] of masa types in von Neumann algebras.

The key notion that we need is that of functoriality for a family $\mathcal{F}$ of connecting homomorphisms between building block algebras, as introduced in Power [19]. This property ensures that if two algebras in $\text{Alglim}\mathcal{F}$ are star extendibly isomorphic then this isomorphism is induced by a commuting diagram in which all the maps belong to $\mathcal{F}$. Our main theorems here assert that regular masas are unique up to approximate inner equivalence for the algebras of the algebraic category $\text{Alglim}\mathcal{F}$ if $\mathcal{F}$ is functorial and algebras in the category $\text{Lim}\mathcal{F}$ if $\mathcal{F}$ is approximately functorial and a stability condition prevails. We also show that there is a converse implication in the case of certain saturated families. The usefulness of this connection lies in the fact that many families of maps are known to be functorial, as we indicate.

In the next section we consider standard regular masas in algebraic limits and in AF $C^*$-algebras. In section 3 we obtain the main theorem, for algebraic limits, that functoriality implies the approximate inner equivalence of masas. In sections 4 and 5 we give some applications to the classification of limit algebras both in terms of the spectrum invariant and the dimension module approach in Power [19]. In the final section we consider operator
algebra limits and the uniqueness of masas for approximately functorial families. Useful references for the sequel are Davidson [4], for C*-algebras, and Power [18] for limit algebras.

2. Standard regular AF masas

Let $\theta : A_1 \to A_2$ be a star-extendible homomorphism between digraph algebras, that is, an algebra homomorphism that coincides with a restriction of a C*-algebra map from $C^*(A_1)$ to $C^*(A_2)$. All maps are assumed henceforth to be star extendible unless otherwise stated. Let us say that $\theta$ has multiplicity one if, for every projection $p$ in $A_1$ with rank one it follows that $\theta(p)$ has rank less than or equal to one. A map $\phi : A_1 \to A_2$ is said to be regular, or 1-decomposable, if it admits a direct sum decomposition in terms of multiplicity one maps. If $C^*(A_1)$ is simple then such a decomposition is unique up to inner conjugacy. In general it is the multiplicity one decomposition with the maximum number of summands that is unique up to inner conjugacy.

Let $\{A_k, \phi_k\}$ be a direct system of digraph algebras with regular maps $\phi_k : A_k \to A_{k+1}$. By regularity it is possible to choose masas $C_k$ in $A_k$, successively, so that $\phi_k(N_{C_k}(A_k)) \subseteq N_{C_{k+1}}(A_{k+1})$. Here $N_C(A)$ denotes the partial isometry normaliser of $C$ in $A$, that is, the set of partial isometries $v$ for which $v^*Cv \subseteq C$ and $vCv^* \subseteq C$. Note that if matrix units are chosen for $A_k$ so that the diagonal matrix units span $C_k$, then $v \in N_{C_k}(A_k)$ if and only if $v$ is a partial isometry which is a linear combination of matrix units with unimodular coefficients. With the choice of masas understood we say that the map $\phi_k$ is a standard regular map.

The abelian algebra $C = \overrightarrow{\text{alg}} \lim(C_k, \alpha_k)$ is a maximal abelian self-adjoint subalgebra (masa) of the regular limit algebra $A = \overrightarrow{\text{alg}} \lim(A_k, \alpha_k)$. We refer to such a masa as a standard regular masa in $A$. Likewise, if $\mathfrak{A}$ is the closed limit algebra of the system then the closure $\mathfrak{C}$ of $C$ in $\mathfrak{A}$ is referred to as a standard regular masa.

In general a masa in an operator algebra is said to be regular if its partial isometry normaliser generates the algebra. The standard regular masas above are clearly regular in this sense. Also, for algebraic limits we have the following converse.

**Proposition 2.1.** Let $A$ be a regular algebraic limit of digraph algebras and let $C \subseteq A$ be a masa which is regular. Then $C$ is a standard regular masa.
Proof. It follows from the hypothesis that there is a sequence \( \{v_n\} \) of partial isometries in \( N_C(A) \) such that the algebra generated by \( \{v_n\} \) is equal to \( A \). Let \( E_k \) be the algebra generated by \( \{v_1, \ldots, v_k\} \) and let \( B_k \) be the \( C^* \)-algebra generated by \( \{v_1, \ldots, v_k\} \). Then \( B_k \) is a finite dimensional \( C^* \)-algebra. Moreover, it contains a masa which is spanned by the operators \( w^*w \) where \( w \) is any finite product of operators from the set \( \{v_1, \ldots, v_k, v_1^*, \ldots, v_k^*\} \).

Let \( \bar{E}_k \) be the algebra generated by \( E_k \) and this masa. Then \( \bar{E}_k \) is a digraph algebra and the system \( \{\bar{E}_k, i_k\} \) associated with the inclusions \( i_k : \bar{E}_k \to \bar{E}_{k+1} \) is a regular system determining \( \tilde{A} \), and the masa \( C = \lim_{\to} (C_k, i_k) \) is a standard regular masa in \( A \).

It is well known that the regular masas in an AF \( C^* \)-algebra can be quite diverse and in particular may contain no proper projections. See Blackadar \([1]\) and also \([2]\). On the other hand we now show that even AF regular masas in an AF \( C^* \)-algebra may fail to be standard regular. To see this we need a proposition which follows from Theorem 4.7 in \([18]\).

**Proposition 2.2.** Let \( C \) be a standard regular (AF) masa in the AF \( C^* \)-algebra \( B \) and let \( E \) be a \( C^* \)-subalgebra of \( B \) containing \( C \). Then \( E \) is an AF \( C^* \)-algebra.

Recall that the \( 2^\infty \) Bunce-Deddens \( C^* \)-algebra \( E \) can be realised as the crossed product \( C^* \)-algebra \( C(S^1) \rtimes_\alpha G \) where \( G \) is the dyadic subgroup of the unit circle \( S^1 \) in the complex plane, acting by rotation. It is an important result of Kumjian \([16]\) (which is also conveniently presented in \([1]\)) that the flip automorphism \( \sigma : E \to E \) determined by

\[
\sigma(f)(z) = f(-z), \quad f \in C(S^1) \\
\sigma(u_g) = u_{g^{-1}}, \quad g \in G
\]

gives a symmetry for which the crossed product \( B_\sigma = E \rtimes_\sigma \mathbb{Z}_2 \) is an AF \( C^* \)-algebra.

We now note that \( E \) contains an AF regular masa, \( D \) say, which is also an AF regular masa for the AF \( C^* \)-algebra \( B_\sigma \). To identify \( D \) consider the subalgebra \( C(S^1) \rtimes_\alpha G_n \) for the subgroup \( G_n \) generated by \( g_n = e^{2\pi i/2^n} \). This is naturally isomorphic to \( M_{2^n}(C(S^1)) \) through the correspondence of the function \( f(z) = z \) in \( C(S^1) \) with

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & z \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0
\end{bmatrix}
\]
and the correspondence of the canonical unitary $u_n$ for $g_n$ with the scalar diagonal matrix

$$
\begin{bmatrix}
1 \\
\lambda_n \\
\lambda_n^2 \\
\vdots \\
\lambda_n^{2^n-1}
\end{bmatrix}
$$

where $\lambda_n = e^{2\pi i/2^n}$. (To make this explicit represent $C(S^1) \rtimes G$ on $L^2(S^1)$ by multiplication and rotation operators and consider the eigenspace decomposition of $u_n$.) Plainly $C(S^1) \rtimes \alpha G$ contains a natural subalgebra isomorphic to $M_{2^n}(\mathbb{C})$ in which the diagonal subalgebra, $D_n$ say, is the $C^*$-algebra generated by $u_n$. Observe now the elementary but significant fact that the action of $\sigma$ on $D_n$ is implemented by a permutation unitary. In particular the canonical unitary in the crossed product $E \rtimes_\sigma \mathbb{Z}_2$ normalises each $D_n$ and so normalises $D$. Since the unitary $f$ also normalises $D$ it follows now that $E \rtimes_\sigma \mathbb{Z}_2$ is generated by the normaliser of $D$.

**Theorem 2.3.** A regular AF masa in an AF $C^*$-algebra need not be standard regular.

**Proof.** We have $D \subseteq E \subseteq B$ where $B$ is the AF $C^*$-algebra $E \rtimes \mathbb{Z}_2$, $D$ is a regular AF masa in $B$ and $E$ is the Bunce-Deddens algebra. It follows from Proposition 2.2 that $D$ is not standard regular.

For other constructions generalising the example of Kunjian see Bratteli, Evans and Kishimolo [2].

It can be shown that the normalising element $f$ in our example above induces a homeomorphism of the spectrum of $D$ with a single fixed point. Normalisers of standard regular masas are known not to have this property and indeed fixed point sets are relatively open. A regular masa with this property is said to be a Cartan masa. (See Renault [20].) It seems to be an open problem, which is of interest in the $C^*$-algebra of Cantor dynamical systems (see [3] for example) whether AF Cartan masas are standard regular.

### 3. Functoriality and Approximate Inner Conjugacy

Let $\mathcal{F}$ be a family of star extendible homomorphisms between operator algebras. The following terminology was introduced in Power [19].
Definition 3.1. Let \( \{A_k, \phi_k\}, \{B_k, \theta_k\} \) be a pair of direct systems, where \( \phi_k, \theta_k \) belong to \( \mathcal{F} \) for all \( k \), and let \( \alpha_k : A_n \rightarrow B_m, \beta_k : B_n \rightarrow A_{m+1} \) be star extendible homomorphisms inducing a commuting diagram

\[
\begin{array}{ccc}
A_{n_1} & \xrightarrow{\alpha_1} & A_{n_2} & \xrightarrow{\alpha_2} & A_{n_3} \\
& \beta_1 \downarrow & & \beta_2 \downarrow & \\
B_{m_1} & \xrightarrow{\alpha_3} & B_{m_2} & \\
\end{array}
\]

Then \( \mathcal{F} \) is said to be functorial if for any such commuting diagram and any index \( k \) there are compositions

\[
\alpha_{r_k} \circ \beta_{r_k-1} \circ \ldots \circ \beta_k \circ \alpha_k,
\]

\[
\beta_{s_k} \circ \alpha_{s_k} \circ \ldots \circ \alpha_{k+1} \circ \beta_k,
\]

for some \( r_k, s_k \), which belong to \( \mathcal{F} \).

The significance of this property is that if the algebraic limit algebras of the systems \( \{A_k, \phi_k\}, \{B_k, \theta_k\} \) are star extendibly isomorphic then this isomorphism is necessarily an \( \mathcal{F} \)-isomorphism, that is, it is induced by maps from the family \( \mathcal{F} \). It follows that if \( G \) is a functor from a category of digraph algebras, with morphisms from \( \mathcal{F} \), to the category of abelian groups then (if \( \mathcal{F} \) is functorial) \( G \) extends to the category \( \text{Alglim}\mathcal{F} \) with star extendible morphisms. In particular one can define regular partial isometry homology groups (as defined in Davidson and Power \cite{DavidsonPower} and Power \cite{Power}) for \( \text{Lim}\mathcal{F} \) if \( \mathcal{F} \) is a functorial family of regular embeddings of digraph algebras.

We shall obtain the following.

**Theorem 3.2.** Let \( \mathcal{F} \) be a functorial family of regular star extendible homomorphisms between digraph algebras. Then for every algebra \( A \) in \( \text{Alglim}\mathcal{F} \) any two regular masas of \( A \) are conjugate by an approximately inner automorphism.

Also we obtain the following converse for a saturated family of connecting maps. A family \( \mathcal{F} \) of regular maps between digraph algebras is said to be *saturated* if it contains all regular maps \( A \rightarrow B \) for every pair \( A, B \) which are the domain and range algebras of a map in \( \mathcal{F} \).
Theorem 3.3. Let \( F \) be a saturated family of regular star extendible homomorphisms between digraph algebras and suppose that for every algebra \( A \) in \( \text{Alglim} F \) any two regular masas of \( A \) are conjugate by an approximately inner automorphism. Then \( F \) is functorial.

We require several lemmas and the following additional terminology. A projection or partial isometry in a digraph algebra is said to be standard with respect to a given masa, if it belongs to the normaliser of that masa. Also we write \( \text{rk}(X) \) for the rank of an operator \( X \) in a finite dimensional C*-algebra. The proof of the first lemma is routine.

Lemma 3.4. Let \( \{P_1, \ldots, P_s\}, \{Q_1, \ldots, Q_s\} \) be sets of pairwise orthogonal standard projections in a digraph algebra \( A \), and let \( E_1, \ldots, E_m \) be the minimal projections of \( A \cap A^* \). If \( \text{rk}(E_i P_j E_i) = \text{rk}(E_i Q_j E_i) \) for all \( i, j \), then there exists a standard unitary \( U \in A \cap A^* \) (which is in fact of permutation type) such that \( U^* P_j U = Q_j \) for each \( 1 \leq j \leq s \).

The next lemma shows that if two standard regular embeddings are inner equivalent then they are also inner equivalent by a standard unitary.

Lemma 3.5. Let \((A_1, C_1), (A_2, C_2)\) be digraph algebras with specified masas and let \( \phi_i : A_1 \to A_2, i = 1, 2 \) be standard regular embeddings for which there exists a unitary \( U \in A_2 \) such that \( U^* \phi_1(\cdot)U = \phi_2(\cdot) \). Then there exists a unitary \( V \in \text{NC}_2(A_2) \) such that \( V^* \phi_1(\cdot)V = \phi_2(\cdot) \).

Proof. Since \( \phi \) and \( \psi \) are standard regular embeddings we can decompose each into a maximal direct sum of multiplicity one maps,

\[
\phi = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_s, \quad \psi = \phi_1 \oplus \phi_2 \oplus \ldots \oplus \phi_s.
\]

We first note that since \( \phi \) and \( \psi \) are unitarily equivalent, there is indeed the same number of summands in each decomposition, as written. (This number is simply the number of edges in the Bratteli diagram for the C*-algebra extension map.) Let \( Q_i = \phi_i(1), P_i = \psi_i(1), q_i^l = \phi_i(e_{l,i}) \) and \( p_i^l = \psi_i(e_{l,i}) \). Clearly, each \( q_i^l \) and \( p_i^l \) is a rank one projection in \( C_2 \) and \( Q_i = \sum_{l=1}^s q_i^l, P_i = \sum_{l=1}^s p_i^l \). Now, for each \( 1 \leq i \leq s \), the map \( U^* \phi_i(\cdot)U \) is a multiplicity one summand of \( \phi_2 \) and by the uniqueness of the decomposition, is unitarily equivalent to \( \psi_{k_i} \) for some \( 1 \leq k_i \leq s \). We note that since \( U \) is block diagonal, \( \text{rk}(E_r q_i^l E_r) = \text{rk}(E_r p_i^{k_i} E_r) \) for each block projection \( E_r \) in \( A_2 \). Thus by Lemma 3.5, we can find a unitary \( V_i \in \text{NC}_2(A_2) \) such that \( V_1^* q_i^l V_1 = p_i^{k_i} \) for all \( i, l \). Thus the maps \( A \phi_i \) and \( \psi_{k_i} \) agree on the diagonal
matrix units. Since these maps have multiplicity one and are star extendible it follows that
they are conjugate by a unitary in $C_2$. Thus, $\phi^{i}_{1}$ and $\phi^{k}_{2}$ are conjugate by a unitary in
$N_{C_2}(A_2)$ and the desired conclusion follows on combining these equivalences.

We can now obtain a key regularising lemma.

**Lemma 3.6.** Let $(A_1, C_1), (A_2, C_2), (A_3, C_3)$ be digraph algebras with specified masas and
suppose that the following diagram commutes:

$$
\begin{array}{ccc}
(A_1, C_1) & \xrightarrow{\theta} & (A_3, C_3) \\
\phi_1 \downarrow & & \phi_2 \downarrow \\
(A_2, C_2) & &
\end{array}
$$

where $\phi_1$ and $\theta$ are standard regular and $\phi_2$ is regular. Then there exists a unitary $U$ in
the commutant of $\theta_1(A_1)$ in $A_3$ such that $\text{Ad}U \circ \phi_2$ is standard regular.

**Proof.** Since $\phi_2 : A_2 \to A_3$ is regular, there exists a unitary $U_1 \in A_3$ such that $\text{Ad}U_1 \circ \phi_2 : A_2 \to A_3$ is standard regular. Thus the maps $\theta : A_1 \to A_3$ and $\text{Ad}U_1 \circ \phi_2 \circ \phi_1 : A_1 \to A_3$
are unitarily equivalent standard regular maps. By Lemma 3.5 we can find a unitary $U_2 \in N_{C_3}(A_3)$ such that $\text{Ad}U_2 \circ \text{Ad}U_1 \circ \phi_2 \circ \phi_1 = \theta_1$. Set $U = U_1 U_2$ and the map $\text{Ad}U \circ \phi_2$ is standard regular with $\text{Ad}U \circ \phi_2 \circ \phi_1 = \theta$, completing the proof.

We now obtain the uniqueness of masas in the following slightly stronger formulation of
Theorem 3.2.

**Theorem 3.7.** Let $\mathcal{F}$ be a functorial family of regular star extendible embeddings between
digraph algebras and let $A, B$ be algebras of $\text{Alglim}\mathcal{F}$ with standard regular masas $C, D$
respectively. If $\psi : A \to B$ is a star extendible isomorphism then $\psi$ is approximately inner
equivalent to an isomorphism $\hat{\psi} : A \to B$ with $\hat{\psi}(C) = D$.

**Proof.** Let $(A, C) = \text{alg lim}((A_k, C_k), \phi_k), (B, D) = \text{alg lim}((B_k, D_k), \theta_k)$. By functoriality,
we can find subsystems $(A_{n_k}, C_{n_k}), (B_{m_k}, D_{m_k})$ and maps $\alpha_k, \beta_k$ such that the following
diagram commutes:
where the maps $\hat{\phi}_i$ and $\hat{\theta}_i$ are compositions of the given embeddings and the $\alpha_i, \beta_i$ are regular, but not necessarily standard regular, for each $i$.

We construct sequences of unitaries, $\hat{V}_i, \hat{U}_i$ with the following properties:

1. $\hat{V}_i \in B_{m_i}, \hat{U}_i \in A_{n_i+1}, i = 1, 2, \ldots$.
2. the maps $\hat{\alpha}_i = (\text{Ad}\hat{V}_i \circ \alpha_i): A_n \rightarrow B_m$, $\hat{\beta}_i = (\text{Ad}\hat{U}_i \circ \beta_i): B_m \rightarrow A_{n+1}$ are standard regular,
3. $\hat{\beta}_i \circ \hat{\alpha}_i = \hat{\phi}_i$, $\hat{\alpha}_{i+1} \circ \hat{\beta}_i = \hat{\theta}_i$ for all $i$.

Firstly, since $\alpha_1: A_{n_1} \rightarrow B_{m_1}$ is regular, there exists a unitary $\hat{V}_1 \in B_{m_1}$ such that $\text{Ad}\hat{V}_1 \circ \alpha_1$ is standard regular. Let $\hat{\alpha}_1 = \text{Ad}\hat{V}_1 \circ \alpha_1$. Now suppose we have constructed unitaries $\hat{V}_1, \ldots, \hat{V}_k$, $\hat{U}_1, \ldots, \hat{U}_{k-1}$ with the above properties. We now construct $\hat{U}_k$ and $\hat{V}_{k+1}$. Set $\beta_k = \text{Ad}(\beta_k(V_k^*)) \circ \beta_k$. Since $V_k^* \in B_{m_k}$ is unitary, $\beta_k(V_k^*) \in A_{n_{k+1}}$ is unitary (or may be extended to one if $\beta_k$ is non unital) and for each $a \in A_{n_k}$,

$$\beta_k'(\hat{\alpha}_k(a)) = \beta_k(V_k)\beta(V_k^*\hat{\alpha}_k(a)V_k)\beta_k(V_k^*) = \beta_k(V_kV_k^*\hat{\alpha}_k(a)V_kV_k^*) = \beta_k(\hat{\alpha}_k(a)) = \hat{\phi}_k(a).$$

Since $\beta_k'$ is regular and the maps $\hat{\alpha}_k$ and $\hat{\phi}_k$ are standard regular, by Lemma 3.5, we can find a unitary $U_k \in A_{n_{k+1}}$ such that $\text{Ad}(U_k) \circ \beta_k'$ is standard regular and $\text{Ad}(U_k) \circ \beta_k' \circ \hat{\alpha}_k = \hat{\phi}_k$. Now set $\hat{U}_k = \beta_k(V_k^*)U_k$, and we obtain a unitary with the required properties. Similarly, set $\alpha_{k+1}' = \text{Ad}(\alpha_{k+1}(U_k^*)) \circ \alpha_{k+1}$ and repeat the above argument to find a unitary $V_{k+1} \in B_{m_{k+1}}$ such that $\text{Ad}(V_{k+1}) \circ \alpha_{k+1}'$ is standard regular and $\text{Ad}(V_{k+1}) \circ \alpha_{k+1}' \circ \hat{\beta}_k = \hat{\theta}_k$. Now let $\hat{V}_{k+1} = \alpha_{k+1}(U_k^*)V_{k+1}$. The required map $\hat{\psi}$ is determined by the new commuting diagram that is $\hat{\psi} = \lim_k \hat{\alpha}_k$ with inverse, $\lim_k \hat{\beta}_k$. It is clear from the construction of $\hat{\psi}$. That it is a star extendible isomorhism which is approximately inner equivalent to $\psi$ and that $\hat{\psi}(C) = D$. 

\hfill \Box
We now prove Theorem 3.3.

**Proof.** As before let \((A, C), (B, D)\) be algebras in \(\text{Alglim}\, \mathcal{F}\) with standard regular masas arising from direct systems \(\{\phi_k : (A_k, C_k) \to (A_{k+1}, C_{k+1})\}\), \(\{\theta_k : (B_k, D_k) \to (B_{k+1}, D_{k+1})\}\) respectively. Let \(\Psi : A \to B\) be a star extendible isomorphism. Then \(\Psi\) is induced by a commuting diagram:

\[
\begin{array}{ccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 \\
\downarrow & & \downarrow & & \downarrow \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3
\end{array}
\]

(after relabelling subsystems for notational simplicity) in which the horizontal maps belong to \(\mathcal{F}\) and the crossover maps are merely star extendible. We wish to construct subsequences \(\{n_k\}, \{m_k\}\) such that the crossover maps in the induced diagram:

\[
\begin{array}{ccc}
A_{n_1} & \longrightarrow & A_{n_2} & \longrightarrow & A_{n_3} \\
\downarrow & & \downarrow & & \downarrow \\
B_{m_1} & \longrightarrow & B_{m_2} & \longrightarrow & B_{m_3}
\end{array}
\]

all belong to \(\mathcal{F}\). Let \(\psi : A \to B\) be the isomorphism corresponding to the first diagram above. We have that \(D' = \Psi(C)\) is a standard regular masa in \(B\) and by the uniqueness hypothesis there exists a sequence of unitaries, \(\{U_k\}\), with \(U_k \in B_k\) for all \(k\) which determines an automorphism \(\Phi\) with \(D' = \Psi(D)\).

Since \(\Psi(C) = D'\) it follows that \(\Psi(N_C(A)) = N_{D'}(B)\) and we can find sequences \(\{n_k\}, \{m_k\}\) and corresponding crossover maps (as in the diagram of Definition 3.1) such that

\[
\alpha_k(N_{C_{n_k}}(A_{n_k})) \subseteq N_{U_{m_k}^*D_{m_k}U_{m_k}}(B_{m_k}),
\]

\[
\beta_k(N_{U_{m_k}^*D_{m_k}U_{m_k}}(B_{m_k})) \subseteq N_{C_{n_{k+1}}}(A_{n_{k+1}})
\]

for each \(k\), where the maps \(\alpha_k\) and \(\beta_k\) are compositions of the given embeddings. Since \(U_{m_k} \in B_{m_k}\) and \(U_{m_k}^*D_{m_k}U_{m_k}\) is a masa in \(D_{m_k}\) it follows that \(\alpha_k\) and \(\beta_k\) are regular for all \(k\). Since \(\mathcal{F}\) is saturated these maps belong to \(\mathcal{F}\), as required.

\[\square\]

4. **Functoriality and the Spectrum**

Let \(\mathcal{F}_{\text{nest}}^{\text{reg}}\) be the family of regular embeddings between finite dimensional nest algebras and let \(\mathcal{F}_{T_r}^{\text{reg}}\) be the subfamily of (star extendible) maps between \(T_r\)-algebras, the nest algebras
with \( r \times r \) block upper triangular structure. Also let \( \mathcal{F}^{reg}_H \) be the family of regular maps between \( H \)-algebras, where an \( H \)-algebra is a digraph algebra whose reduced digraph is \( H \).

The family \( \mathcal{F}^{reg}_{T_2} \) is functorial for the trivial reason that all star extendible maps between \( T_2 \)-algebras are regular (see Heffernan [12]). The functoriality of \( \mathcal{F}^{reg}_{T_3} \) is shown in Power [13] and we expect the families \( \mathcal{F}^{reg}_{T_r} \), for \( r \geq 4 \), and \( \mathcal{F}^{nest}_{nest} \) also to be functorial.

Let us, for convenience, say that \( H \) is functorial if \( \mathcal{F}^{reg}_H \) is functorial. From Donsig and Power [8] and Power [19], it is known that various bipartite graphs, including the 4-cycle graph \( D_4 \), are not functorial. It would be very interesting to determine precisely which digraphs \( H \) are functorial since non-functoriality of necessity occurs in a subtle algebraic way. A deeper understanding of this is necessary to determine invariants for the approximate inner conjugacy classes of standard regular masas.

It is also natural to restrict to subclasses of regular embeddings. Thus, it is known from [8] that the family of so called rigid embeddings between \( 2n \)-cycle algebras, for \( n \geq 3 \), is functorial. Also one of the key results in Hopenwasser and Power [14] is that the subfamilies \( \mathcal{F}^{loc}_{nest} \) and \( \mathcal{F}^{oc}_{nest} \) of \( \mathcal{F}^{nest}_{nest} \), consisting of certain locally order conserving embeddings and order conserving embeddings, respectively, are functorial. (Order conserving maps are more general than the order preserving maps previously considered in [17], [8].)

We now recall the definition of the spectrum \( R(A, \mathcal{C}) \) associated with a (closed) regular limit algebra \( A \) and a standard regular masa \( \mathcal{C} \).

Let \( \{A_k, \phi_k\} \) be a regular direct system as before and let \( \{e^k_{i,j}\} \) be a matrix unit system for \( A_k \) such that for fixed \( k \) the projections \( \{e^k_{i,i}\} \) span \( C_k \) and \( \phi_k \) maps each matrix unit \( e^k_{i,j} \) to a sum of matrix units in \( \{e^{k+1}_{i,j}\} \). For each point \( x \) in the Gelfand space \( M(\mathcal{C}) \) there is a unique sequence

\[ q^1_x \geq q^2_x \geq \ldots \]

where \( q^k_x \) is the projection in \( \{e^k_{i,i}\}_i \) with \( x(q^k_x) = 1 \). Conversely for each sequence of decreasing minimal projections as above, there is an associated point \( x \) in \( M(\mathcal{C}) \) which is the continuous extension of the functional \( x_0 \) such that \( x_0(e^k_{i,i}) = 1 \) if \( e^k_{i,i} = q^k_x \), and \( x_0(e^k_{i,i}) = 0 \) otherwise.
The topology on $M(C)$ is generated by the compact clopen sets which are the supports of the Gelfand transforms of the projections $\{e^k_{ii}\}$. It follows that a basic clopen neighbourhood of $x$ is

$$\{y : q^k_y = q^k_x \text{ for } k = 1, \ldots, k_o\}.$$ 

Define $E^k_{ij} \subset M(C) \times M(C)$ to be the analogous set

$$\{(x, y) : q^k_x = e^k_{ii}, q^k_y = e^k_{jj}, q^l_x = e^k_{ij}q^l_y(e^k_{ij})^*, l = k, k+1, \ldots\},$$

and define the binary relation $R = R(\{e^k_{ij}\})$ to be the union $\bigcup E^k_{ij}$. The sets $E^k_{ij}$ can be viewed more intuitively as the graphs of a partial homeomorphism of $M(C)$ induced by the $e^k_{ij}$. The topological binary relation $R(A, C)$ (and $R(A, C)$) are defined to be this binary relation with the topology generated by the sets $E^k_{ij}$, as clopen sets. In the triangular case, for which there is a unique masa, this is also called the spectrum of the algebras.

It is an elementary fact that a partial isometry in $A$ which normalises $C$ is of the form $cw$ with $c$ a unitary in $C$ and $w$ a sum of matrix units. There is a corresponding fact for closed limits given by Lemma 5.5 of [9], which is important for the general theory. It follows from this that $R(A, C)$ is in fact independent of the choice of matrix unit system chosen for the pair $A, C$. We can now obtain the following.

**Theorem 4.1.** Let $F$ be a functorial family of regular star extendible maps between digraph algebras. Then for each algebra $A$ in $\text{Alglim} F$ the spectrum $R(A)$ is well defined and is a complete invariant for star extendible isomorphism.

**Proof.** Theorem 3.2 and the discussion above shows that the spectrum is a well defined invariant. The completeness of the invariant follows from Theorem 7.5 of [9] simplified to the algebraic case.

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5. **Functoriality and dimension module invariants.**

Recently the second author has introduced general dimension module invariants for limit algebras, both algebraic and closed, associated with certain families $F$ of connecting embeddings. We shall review these constructions and then show that the functoriality of $F$ is the essential requirement for the well-definedness of the dimension module in the case of certain algebraic limits.
In this section we widen the class of building block algebras to unital subalgebras of finite dimensional C*-algebras, which we refer to as finitely acting operator algebras. The conditions of the following definition are natural requirements for a family of maps between building block algebras.

**Definition 5.1.** Let $F$ be a family of star extendible homomorphisms between finitely acting operator algebras. Then $F$ is said to be an algebraically closed family if the following conditions hold.

(i) $F$ is closed under unitary conjugation; if $\phi: A_1 \to A_2$ is in $F$, and $u$ is a unitary element of $A_2$ (and hence of $A_2 \cap A_2^*$) then $Adu \circ \phi$ is in $F$.

(ii) $F$ is closed under compositions; if $\phi: A_1 \to A_2$ and if $\psi: A_2 \to A_3$ are in $F$ then so too is $\psi \circ \phi$.

(iii) $F$ is matricially stable; if $\phi: A_1 \to A_2$ is in $F$ then the map $\phi: A_1 \otimes M_n \to A_2 \otimes M_n$ is in $F$.

(iv) $F$ is sum closed; if $\phi, \psi: A_1 \to A_2$ are in $F$ then the map $\phi \oplus \psi$ with domain $A_1$ and range $A_2 \otimes M_2$ belongs to $F$.

The set of all regular unital star extendible maps $T_r \otimes M_n \to T_r \otimes M_m$, for $m, n \in \mathbb{N}$, is an instance of a unital algebraically closed family, that is, one for which all the maps are unital. On the other hand it is natural to define a matricially closed algebraically closed family as one which contains all the multiplicity one (star extendible) maps $A_1 \to A_1 \otimes M_n$, for all $n$, and for all algebras $A_1$ which are the domain or range algebra of maps in $F$.

As a final point of terminology we associate with the algebraically closed family $F$ the associated algebraically closed family $\widetilde{F}$ of maps

$$\psi: E_1 \oplus \cdots \oplus E_k \to F_1 \oplus \cdots \oplus F_l,$$

which are of the form $\Sigma \oplus \psi_{ij}$ where each map $\psi_{ij}: E_i \to F_j$ belongs to $F$.

Suppose now that $E$ is a fixed finitely acting operator algebra. To each star extendible map $\phi: E \otimes M_n \to E \otimes M_m$, for $n, m \in \mathbb{Z}$ there is an associated map $\tilde{\phi}$ in

$$\text{Hom}(E \otimes K_0, E \otimes K_0),$$
the set of star extendible endomorphisms of the stable algebra of $E$. Here we write $K_0$ for the (unclosed) stable algebra $M_\infty(C)$ of $C$. The set of these maps is closed under compositions and closed under the sum operation, as in (iv) above. It follows that the set of unitary equivalence classes $[\tilde{\phi}]$, which we indicate as

$$Hom_u(E \otimes K_0, E \otimes K_0),$$

has the structure of a semiring with additive unit, namely the class of the zero map. (Unitaries here are taken, as usual, in the unitisation of $E \otimes K_0$.) Write $F_E$ for the set of all maps $\phi$ as above. If $F \subseteq F_E$ is an algebraically closed family then we write $V_F$ for the subsemiring of $Hom_u(E \otimes K_0, E \otimes K_0)$ determined by the classes $[\tilde{\phi}]$ for $\phi$ in $F$.

For some illustrative examples we note the following. These and others are discussed more fully in [19].

1. For $E = C, V_{F_E}$ is the semiring $\mathbb{Z}_+$.

2. For $F = F_{reg}^{T_r}, V_F$ is a semigroup semiring $\mathbb{Z}_+[S_r]$, where $S_r$ is the semigroup of order-preserving maps from $\{1, \ldots, r\}$ to $\{1, \ldots, r\}$.

3. For the operator algebra $E \subseteq M_2$ consisting of the matrices

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

the semiring $V_{F_E}$ is isomorphic to

$$\mathbb{Z}_+[S^1] \oplus \mathbb{Z}_+$$

where the elements $\theta \oplus 0$ (with $\theta \in S^1$) and $0 \oplus 1$ correspond, respectively, to the classes of the maps

$$\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \to \begin{bmatrix} a & b\theta \\ a \end{bmatrix},$$

$$\begin{bmatrix} a & b \\ a \end{bmatrix} \to \begin{bmatrix} a & b & 0 & 0 \\ 0 & a & 0 & 0 \\ a & b \\ 0 & a \end{bmatrix}.$$

4. For $F_{T_3}$ the semiring $V_{F_{T_3}}$ is uncountable.
5. Let $E$ be the operator algebra in $M_3 \oplus M_2 \oplus M_2$ given by the set of matrices

$$\begin{bmatrix} a & x & z \\ b & y & c \end{bmatrix} \oplus \begin{bmatrix} a \\ b \end{bmatrix} \oplus \begin{bmatrix} b & y & c \end{bmatrix}.$$  

Although $E$ is isometrically isomorphic to $T_3$ it is not star extendibly isomorphic. Moreover, in contrast with $T_3$, the semiring $V_{F_E}$ is finitely generated. In fact $V_{F_E}$ is a finite semigroup semiring.

For the remainder of the section we assume that $F$ is an algebraically closed subfamily of $F_E$. Following the next definition we define the dimension module $V_F(A)$ with reference to a specific presentation $A = \text{alg lim}(A_k, \phi_k)$. The issue we shall address is to determine condition on $F$ under which the right $V_F$-module $V_F(A)$ is independent of the choice of presentation.

Let $A \in \text{Alglim F}$ with presentation $\text{alg lim}(A_k, \phi_k)$ with each $\phi_k \in F$. Each algebra $A_k$ is a direct sum of $E$-algebras, that is, an algebra of the form

$$B = E \otimes M_{n_1} \oplus \cdots \oplus E \otimes M_{n_r}.$$  

Define $V_F(A_k)$ to be the monoid of inner unitary equivalence classes of star extendible homomorphisms $\psi : E \to B \otimes K_0$, where the partial embeddings belong to $F$. Then $V_F(A_k)$ is naturally a direct sum of $r$ copies of $V_F$ with the natural right $V_F$-module structure.

For each $k$ we have the induced $V_F$-module homomorphism

$$\hat{\phi}_k : V_F(A_k) \to V_F(A_{k+1})$$

given by

$$\hat{\phi}_k([\psi]) = [\phi_k \circ \psi].$$

Plainly $\hat{\phi}$ respects the right $V_F$-action, which is to say that for $[\theta]$ in $V_F$,

$$\hat{\phi}_k([\psi][\theta]) = (\hat{\phi}_k([\psi]))[\theta].$$

**Definition 5.2.** Let $F \subseteq F_E$ be an algebraically closed family. The dimension module of the direct system $\{A_k, \phi_k\}$, for the family $F$, is the right $V_F$-module

$$V_F(\{A_k, \phi_k\}) = \lim_{\text{alg lim}}(V_F(A_k), \hat{\phi}_k).$$
The direct limit is taken in the category of additive abelian semigroups and endowed with the induced right $V_F$-action. We tentatively define $V_F(A)$ to be the $V_F$-module $V_F(\{A_k, \phi_k\})$.

Note that if $E = C$ then $V_F = Z_+$ and $V_F(A)$ is naturally isomorphic to the positive cone $K_0(A)_+$. Define the scale $\Sigma_F(A_k)$ of $V_F(A_k)$ as the subset of classes $[\psi]$ for maps $\psi: E \to A_k$ where $A_k$ is identified with $A_k \otimes \mathbb{C}p$ for some rank one projection $p$. Also, define the scale $\Sigma_F(A)$ of $V_F(A)$ to be the union of the images of the scales $\Sigma_F(A_k)$ in $V_F(A)$. Writing $G_F(A)$ for the enveloping group of $V_F(A)$ we obtain the scaled ordered abelian group

$$(G_F(A), V_F(A), \Sigma_F(A))$$

endowed with the $V_F$-module structure.

In [19] it was shown that $V_F$ has cancellation and the following classification was obtained. This generalises the scaled $K_0$-group classification (as in Elliot [9]) for the case $E = C$ of ultramatricial algebras.

**Theorem 5.3.** Let $E \subseteq M_n$ be an operator algebra, let $\mathcal{F} \subseteq \mathcal{F}_E$ be an algebraically closed family of maps and let $A, A'$ belong to $\text{Alglim}(\tilde{\mathcal{F}})$. If $\Gamma$ is a $V_F$-module isomorphism from $V_F(A)$ to $V_F(A')$ then $A \otimes K_0$ and $A' \otimes K_0$ are star extendibly isomorphic. If, moreover, $\Gamma$ gives a bijection from $\Sigma_F(A)$ to $\Sigma_F(A')$ then $A$ and $A'$ are star extendibly isomorphic.

If $\mathcal{F}$ is functorial then the converse of these assertions hold and the $V_F$-module $V_F(-)$ is a complete invariant for stable star extendible isomorphism, whilst the pair $(V_F(-), \Sigma_F(-))$ is a complete invariant for star extendible isomorphism.

The first part of the theorem is proven by showing that $\Gamma$ can be lifted to a star extendible isomorphism $\Phi: A \to A'$. The lifting is not, of course, unique, although liftings are unique up to approximate inner equivalence.

We now obtain a new involvement of functoriality in connection with the (tentative) dimension module invariant $V_F(A)$. The following definition of functoriality for $V_F(-)$ is a natural formulation of the well-definedness of $V_F(-)$ for the category $\text{Alglim} \mathcal{F}$. The theorem below shows that the functoriality of $\mathcal{F}$ is necessary for this well-definedness.
**Definition 5.4.** The dimension module $V_F(\{A_k, \phi_k\})$ is said to be functorial for the category $\text{Alglim} \mathcal{F}$ if for any star extendible isomorphism $\Phi : A \to B$ between limit algebras $A = \text{alg lim}(A_k, \phi_k), B = \text{alg lim}(B_k, \theta_k)$ in $\text{Alglim}\mathcal{F}$ there is a $V_F$-module isomorphism $\Gamma : V_F(\{A_k, \phi_k\}) \to V_F(\{B_k, \theta_k\})$ with lifting equal to $\phi$.

**Theorem 5.5.** $V_F(-)$ is functorial for $\text{Alglim} \mathcal{F}$ if and only if $\mathcal{F}$ is a functorial family.

**Proof.** That $V_F(-)$ is functorial if $\mathcal{F}$ is functorial is immediate. Suppose then that $V_F(-)$ is functorial and let $\Phi : A \to B$ be a star extendible isomorphism. In particular $\Phi$ is induced by a commuting diagram of maps $\alpha_k, \beta_k$ as in the statement of Definition 3.1.

Let $\Gamma : V_F(A) \to V_F(B)$ be a $V_F(-)$-module isomorphism with lifting equal to $\Phi$. First (irrespective of this lifting) we construct a natural map $\hat{\psi} : V_F(A_1) \to V_F(B_{m_1})$ so that the following diagram commutes.

$$
\begin{array}{ccc}
V_F(A_1) & \longrightarrow & V_F(\{A_k, \phi_k\}) \\
\downarrow \hat{\psi} & & \downarrow \Gamma \\
V_F(B_{m_1}) & \longrightarrow & V_F(\{B_k, \theta_k\})
\end{array}
$$

Let $\eta : E \to E$ be the identity map, with class $[\eta]$ in $V_F$. Let $[\eta]_1$ denote the corresponding class in $V_F(A_1)$ with image $[\eta]_\infty$ in $V_F(\{A_k, \phi_k\})$. Then $\Gamma([\eta]_\infty) = g$ for some element $g$ in $V_F(\{A_k, \theta_k\})$ which in turn is the image of a class $[\psi]$ in $V_F(B_{m_1})$, for some $m_1$, where $\psi$ belongs to $\mathcal{F}$. Since the natural maps

$$
V_F(A_1) \to V_F(\{A_k, \phi_k\}) \to V_F(\{B_k, \theta_k\})
$$

$$
V_F(B_{m_1}) \to V_F(\{B_k, \theta_k\})
$$

are $V_F$-module maps and $[\eta]$ is a generator for $V_F(A_1)$ as a $V_F$-module, the desired diagram follows.

Since $\Gamma$ is induced by $\Phi$, by hypothesis, we may increase $m_1$, if necessary, so that $\hat{\psi}$ agrees with the induced map $\gamma : V_F(A_1) \to V_F(B_{m_1})$ given by $\gamma([\theta]) = [\phi \circ \theta]$ where $\phi : A_1 \to B_{m_1}$ is the restriction of $\Phi$ in $A_1$ and $m_1$ is suitably large. Thus in particular, $[\phi] = [\phi \circ \eta] = \gamma([\eta])$ is a class in $V_F(B_{m_1})$ which is to say, since $\mathcal{F}$ is closed under unitary equivalence, that $\phi$
belongs to $\mathcal{F}$. Note that $\phi$ has the form

$$\alpha_r \circ \alpha_{r-1} \circ \cdots \circ \alpha_1.$$ 

Since $\Gamma^{-1}$ is induced by $\Phi^{-1}$ we deduce that $\mathcal{F}$ is functorial.

6. Approximate Functoriality and Closed Limits

We now turn our attention to the operator algebras determined by the closed limits of regular systems and obtain standard regular masa uniqueness for algebras determined by an approximately functorial family, at least in the case of pertubationally stable digraph algebras. Furthermore, it has been a folk lore conjecture that, as in the self adjoint case, two (closed) regular limit algebras are (isometrically) isomorphic if and only if their corresponding algebraic limits are isomorphic. An isomorphism between algebraic limit algebras extends by continuity to an isomorphism between the closures of the algebras. However to construct a map between algebraic limits given a map between the closures is extremely problematic and the issue remains open in general. We provide a partial solution to this problem and prove that the conjecture holds for algebras built from certain approximately functorial families.

The following two key lemmas will enable us to bring to bear the results of Section 3. We remark that elementary examples show that close star extendible emmbeddings need not be inner unitarily equivalent.

**Lemma 6.1.** Let $A_1$ and $A_2$ be digraph algebras. Then there is a constant $c$ such that if $\phi_i : A_1 \rightarrow A_2$ for $i = 1, 2$ are regular maps and $\|\phi_1 - \phi_2\| < c$, where $c = c(A_1)$ depends only on the reduced digraph of $A_1$, then there exists a unitary $U \in A_2$ such that $\phi_2(\cdot) = Ad(U) \circ \phi_1(\cdot)$.

Proof. The lemma follows readily from the special (triangular) case in which $A_1$ is the digraph algebra $A(G)$ with $G$ a connected reduced digraph. Let $p_1, \ldots, p_r$ be the (rank one) atomic projections of $A_1 \cap A_1^*$ and let $P_1, \ldots, P_s$ denote the block projections in $A_2$. Note that if $\alpha : A_1 \rightarrow A_2$ is a multiplicity one star extendible embedding, then there is an associated index map $\pi : \{1, \ldots, r\} \rightarrow \{1, \ldots, s\}$ such that $P_{\pi(i)} \alpha(p_i) \neq 0$, for $i = 1, \ldots, r$. Moreover it is elementary to prove that $\pi$ is a complete inner conjugacy invariant for $\alpha$.
We next note the following simple algebraic criterion that a regular map \( \phi : A_1 \to A_2 \) should possess a multiplicity one summand inner conjugate to \( \alpha \) in its decomposition.

Let \( v_1, v_2, \ldots, v_n \) be a sequence of matrix units taken from \( A_1 \) or \( A_1^* \) with the following two properties.

1. The product \( v_1 v_2 \ldots v_n \) is a non-zero projection.
2. Each \( p_i \) is the initial or final projection of at least one \( v_j \).

Then, writing \( \hat{\alpha}(p_i) \) for the block projection \( P_{\pi(i)} \) and denoting the star extension of \( \alpha \) by \( \alpha \) also, we have

\[
0 \neq \alpha(v_1 v_2 \ldots v_n) = \alpha(v_1) \hat{\alpha}(v_1^* v_1) \alpha(v_2) \hat{\alpha}(v_2^* v_2) \ldots \alpha(v_n) \hat{\alpha}(v_n^* v_n).
\]

On the other hand, if \( \beta : A_1 \to A_2 \) is a multiplicity one embedding which is not conjugate to \( \alpha \) then

\[
0 = \beta(v_1) \hat{\alpha}(v_1^* v_1) \beta(v_2) \hat{\alpha}(v_2^* v_2) \ldots \beta(v_n) \hat{\alpha}(v_n^* v_n).
\]

Moreover, if \( \phi : A_1 \to A_2 \) is a direct sum of multiplicity one embeddings then the product

\[
\phi(v_1) \hat{\alpha}(v_1^* v_1) \phi(v_2) \hat{\alpha}(v_2^* v_2) \ldots \phi(v_n) \hat{\alpha}(v_n^* v_n)
\]

splits as a corresponding direct sum of products, one for each summand, and so we conclude that this “test product” is non-zero precisely when \( \phi \) has a summand conjugate to \( \alpha \). Now, if \( \|\phi_1 - \phi_2\| \leq \frac{1}{n+1} \) then, for each \( \alpha \), the test product for \( \phi_1 \) is at most distance \( \frac{n}{n+1} \) from the test product for \( \phi_2 \). Since test products have norm zero or one, the statement will follow by a simple induction argument on the number of summands in the decomposition for \( \phi_1 \).

The next lemma is a stable version of Lemma 3.6.

**Lemma 6.2.** Let \( (A_1, C_1), (A_2, C_2), (A_3, C_3) \) be digraph algebras with chosen masas and let \( \theta_1 : A_1 \to A_3 \) and \( \phi_1 : A_1 \to A_2 \) be standard regular maps. If \( \phi_2 : A_2 \to A_3 \) is a (merely) regular map and \( \|\phi_2 \circ \phi_1 - \theta_1\| < c \), then there exists a unitary \( U \in A_3 \) such that \( \text{Ad}(U) \circ \phi_2 \) is standard regular and \( \text{Ad}(U) \circ \phi_2 \circ \phi_1 = \theta_1 \).
Proof Since $\|\phi_2 \circ \phi_1 - \theta_1\| < c$, it follows from Lemma 6.1 that the maps $\phi_2 \circ \phi_1$ and $\theta_1$ are inner conjugate. Let $V \in A_3$ be a unitary such that $\text{Ad}(V) \circ \phi_2 \circ \phi_1 = \theta_1$ and replace $\phi_2$ by the regular map $\text{Ad}(V) \circ \phi_2$ and we can apply Lemma 3.6.

We say that a diagram

\[
\begin{array}{cccccc}
A_1 & \overset{\phi_1}{\longrightarrow} & A_2 & \overset{\phi_2}{\longrightarrow} & A_3 & \overset{\phi_3}{\longrightarrow} \\
\downarrow{\mu_1} & \downarrow{\nu_1} & \downarrow{\mu_2} & \downarrow{n\nu_2} & \downarrow{\mu_3} & \\
B_1 & \overset{\psi_1}{\longrightarrow} & B_2 & \overset{\psi_2}{\longrightarrow} & & \\
\end{array}
\]

is approximately commuting if

\[
\Sigma \| \phi_k - \nu_k \circ \mu_k \| + \Sigma \| \psi_k - \nu_{k+1} \circ \mu_k \| < \infty.
\]

It is well known and easy to verify that such a diagram determines a star extendible isomorphism between the two limit algebras determined by the horizontal maps. The converse of this certainly holds if the building block algebras form a perturbationally stable family in the sense of the following definition from Power [19]. This property was established for $T_r$-algebras in Haworth [11].

**Definition 6.3.** The family $\mathcal{E}$ has the stability property, or is perturbationally stable, if for each algebra $A_1$ in $\mathcal{E}$ and $\epsilon > 0$ there is a $\delta > 0$ such that to each algebra $A_2$ in $\mathcal{E}$ and star-extendible embedding

\[
\phi : A_1 \to C^*(A_2), \quad \text{with} \quad \phi(A_1) \subseteq \delta A_2
\]

there is a star-extendible embedding $\psi : A_1 \to A_2$ with $\| \phi - \psi \| \leq \epsilon$.

**Definition 6.4.** Let $\mathcal{F}$ be a family of maps between digraph algebras and let

\[
\begin{array}{cccccc}
A_1 & \overset{\phi_1}{\longrightarrow} & A_2 & \overset{\phi_2}{\longrightarrow} & A_3 & \overset{\phi_3}{\longrightarrow} \\
\downarrow{\mu_1} & \downarrow{\nu_1} & \downarrow{\mu_2} & \downarrow{n\nu_2} & \downarrow{\mu_3} & \\
B_1 & \overset{\psi_1}{\longrightarrow} & B_2 & \overset{\psi_2}{\longrightarrow} & & \\
\end{array}
\]

be an approximately commuting diagram in which all the connecting (horizontal) maps belong to $\mathcal{F}$. Then $\mathcal{F}$ is said to be approximately functorial if for all such diagrams there exist compositions

\[
\alpha_k : A_{n_k} \to A_{n_k-1} \to B_{m_k}
\]

\[
\beta_k : B_{m_k} \to B_{m_k+1-1} \to A_{n_{k+1}}
\]

for $k = 1, 2, \ldots$, such that $\text{dist}(\alpha_k, \mathcal{F}) \to 0$ and $\text{dist}(\beta_k, \mathcal{F}) \to 0$ as $k \to \infty$. 
Theorem 6.5. Let $F$ be an approximately functorial family of regular maps between digraph algebras from a pertubationally stable family. Then we have the following:

1. Any two standard regular masas of an algebra in $\text{Lim} F$ are conjugate by an approximately inner automorphism.
2. Whenever $\Phi : A \to B$ is an isomorphism between algebras in $\text{Lim} F$, there exists an isomorphism $\Psi_0 : A_0 \to B_0$ between the corresponding algebraic limits for the systems giving rise to $A$ and $B$. Moreover, the extension of $\Psi_0$ to the closed algebras gives rise to a map $\Psi : A \to B$ which is approximately inner equivalent to $\Phi$.

Proof It will be enough to prove the second assertion. Let $A$ and $B$ be isomorphic algebras in $\text{Lim} F$ arising from the regular systems $\{(A_k, C_k), \phi_k\}$ and $\{(B_k, D_k), \theta_k\}$ respectively. Further, let $A_0$ and $B_0$ denote the algebraic limits of these systems. Let $\Phi : A \to B$ denote the given isomorphism which, by perturbational stability, is determined by the approximately commuting diagram

By approximate functoriality we can find an approximately commuting sub-diagram (which we re-label for notational convenience)

where the crossover maps belong to $F$. We may further assume that

$$\|\beta_i \circ \alpha_i - \phi_i\| < c(A_i),$$

$$\|\alpha_{i+1} \circ \beta_i - \theta_i\| < c(B_i)$$

for all $i$.

We can now repeat the argument of the proof of Theorem 3.2 except we use the more powerful Lemma 6.2 to construct the sequences of unitaries, $\hat{V}_i, \hat{U}_i$. 

We have thus created an exactly commuting diagram:

\[
\begin{array}{ccc}
A_1 & \xleftarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 & \xrightarrow{\phi_3} \\
\downarrow{\alpha_1} & \swarrow{\beta_1} & \downarrow{\alpha_2} & \swarrow{\beta_2} & \downarrow{\alpha_3} & \\
B_1 & \xleftarrow{\theta_1} & B_2 & \xrightarrow{\theta_2} & \\
\end{array}
\]

and so the algebraic limits, \( A_0 \) and \( B_0 \) of the systems are isomorphic. If \( \Psi_0 \) is the isomorphism corresponding to the above diagram and \( \Psi \) its extension by continuity to the closed limits then, by construction, \( \Psi \) is approximately inner equivalent to \( \Phi \).

**Theorem 6.6.** Let \( \mathcal{F} \) be a saturated family of regular embeddings between digraph algebras coming from a perturbationally stable family. Suppose that for every algebra in \( \text{Lim}\mathcal{F} \) any two standard regular masas are conjugate by an approximately inner automorphism. Then \( \mathcal{F} \) is approximately functorial.

The proof is entirely analogous to the proof of Theorem 3.3.

It has been shown in Haworth [10] that the family of regular embeddings between \( T_3 \)-algebras is approximately functorial. This a result of some depth; in contrast to maps between \( T_2 \)-algebras, star extendible algebra homomorphisms between \( T_3 \)-algebras need not be regular. Combining this with the results above, and the stability result of [11], we obtain the theorems below. We conjecture that standard regular masas are similarly unique in the \( T_r \) case also.

**Theorem 6.7.** Any pair of standard regular masas in an operator algebra limit of \( T_3 \)-algebras are conjugate by an approximately inner automorphism.

**Theorem 6.8.** Two algebras of \( \text{Alglim}^{\text{reg}}_{T_3} \) are star extendibly isomorphic if and only if their norm closures are isomorphic.

**Theorem 6.9.** The spectrum is a well defined complete isometric isomorphism invariant for the operator algebras of \( \text{Lim}^{\text{reg}}_{T_3} \).

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