Gravitating dyons and the Lue-Weinberg bifurcation

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Gravitating t’Hooft-Polyakov magnetic monopoles can be constructed when coupling the Georgi-Glashow model to gravitation. For a given value of the Higgs boson mass, these gravitating solitons exist up to a critical value of the ratio of the vector meson mass to the Planck mass. The critical solution is characterized by a degenerate horizon of the metric. As pointed out recently by Lue and Weinberg, two types of critical solutions can occur, depending on the value of the Higgs boson mass. Here we investigate this transition for dyons and show that the Lue and Weinberg phenomenon is favored by the presence of the electric-charge degree of freedom.

I. INTRODUCTION

The Georgi-Glashow (GG) model consists of an SU(2) Yang-Mills field coupled to a (real) Higgs triplet. The self interation of the Higgs field by means of a Higgs potential leads to a spontaneous breakdown of the symmetry. One distinguished feature of this model is that it admits topological solitons: the celebrated t’Hooft-Polyakov monopoles [1,2].

By coupling the GG field theory to gravitation, one obtains the SU(2) Einstein-Georgi-Glashow model. Several years ago, it was shown that gravitating magnetic monopoles, as well as non-abelian black holes, exist in a certain domain of the space of the physical parameters of the model [3–6]. For instance for a fixed value of the Higgs boson mass, the gravitating monopoles exist up to a critical value \( \alpha_{\text{cr}} \) of the parameter \( \alpha \) (the ratio of the vector meson mass to the Planck mass). At the critical value \( \alpha_{\text{cr}} \), a critical solution with a degenerate horizon is reached. In particular, for small values of the Higgs boson mass, the critical solution where a horizon first appears corresponds to an extremal Reissner-Nordstrøm (RN) solution outside the horizon while it is non-singular inside.

Recently Lue and Weinberg reconsidered the equations of the self-gravitating magnetic monopoles and discovered an insofar not suspected phenomenon [6]. Indeed for large enough values of the Higgs boson mass, the critical solution is an extremal black hole with non-abelian hair and a mass less than the extremal RN value. An independent numerical analysis [7] of the equations has confirmed the results of Lue and Weinberg and reinforced the agreement between the numerical solutions and the algebraic conditions these solutions have to obey.

During the last months, there was a growing interest devoted to monopole and dyons, considered as classical solutions of the non-abelian Einstein-Born-Infeld-Higgs (EBIH) model [8–10]. This is motivated by the fact that the low energy effective action D-brane is related to a Born-Infeld model (see e.g. [11]). Because Born-Infeld models naturally contain a theta-term proportional to \( (F_{\mu\nu}\tilde{F}^{\mu\nu})^2 \), their nonlinearities are better tested by the dyons solutions which excite both magnetic and electric fields.

It is therefore natural to study if the gravitating dyons of the GG model are also sensitive to the Lue and Weinberg effect and to see if the degree of freedom conferring the classical lump its electric charge favors or attenuates this effect. In this report we extend the analysis of Ref. [6] to the case of spherically symmetric gravitating dyons. We show that the transition also occurs for dyons and that the phenomenon is enhanced by the presence of the new field describing the electric-charge degree of freedom.

II. THE EQUATIONS

We consider the SU(2) Einstein-Georgi-Glashow action [3–6]. In order to obtain static spherically symmetric globally regular solutions we employ Schwarzschild like coordinates for the metric

\[
ds^2 = -A^2 N dt^2 + N^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) ,
\]
and introduce, as usual, the mass function \( m(r) \) by means of

\[
N(r) = 1 - \frac{2m(r)}{r}.
\]

(2)

Then we use the standard Wu-Yang ansatz for the spatial components of the gauge field and the hedgehog ansatz for the Higgs fields (see e.g. [4,6])

\[
\vec{A}_r = 0, \quad \vec{A}_\theta = -\vec{e}_\phi \frac{1 - K(r)}{g}, \quad \vec{A}_\phi = \vec{e}_\theta \frac{1 - K(r)}{g} \sin \theta,
\]

(3)

and

\[
\vec{\phi} = \vec{e}_r H(r) v,
\]

(4)

with the standard unit vectors \( \vec{e}_r, \vec{e}_\theta \) and \( \vec{e}_\phi \) while \( v \) denotes the Higgs field’s vacuum expectation value. Since we want to describe dyon, we also employ the spherically symmetric ansatz for the time component of the gauge field [12–14].

\[
\vec{A}_0 = v J(r) \vec{e}_r.
\]

(5)

It is also convenient to introduce the dimensionless coordinate \( x \) and mass function \( \mu \),

\[
x = gvr, \quad \mu = gvm,
\]

(6)

as well as the dimensionless coupling constants \( \alpha, \beta \)

\[
\alpha^2 = 4\pi Ge^2, \quad \beta = \frac{M_H}{M_W},
\]

(7)

where \( G \) is Newton’s constant, \( M_H \) is the Higgs boson mass and \( M_W \) is the gauge boson mass.

With these ansatz and definitions, the classical equations of the model reduce to the following system of five coupled radial equations: [3,4]

\[
\mu' = \alpha^2 \left( \frac{x^2 J^2}{2A^2} + \frac{J^2 K^2}{A^2 N^2} + NK'^2 + \frac{1}{2}N x^2 H'^2 \\
+ \frac{(K^2 - 1)^2}{2x^2} + H^2 K^2 + \frac{\beta^2}{8} x^2 (H^2 - 1)^2 \right),
\]

(8)

and

\[
A' = \alpha^2 x \left( \frac{2J^2 K^2}{A^2 N^2 x^2} + \frac{2K^2}{x^2} + H'^2 \right) A,
\]

(9)

for the functions parametrizing the geometry (the prime indicates the derivative with respect to \( x \)); for the matter functions we obtain the equations

\[
(ANK')' = AK \left( \frac{K^2 - 1}{x^2} + H^2 - \frac{J^2}{A^2 N} \right),
\]

(10)

and

\[
\left( \frac{x^2 J'}{A} \right)' = \frac{2JK^2}{AN},
\]

(11)
\[ (x^2 A N H')' = AH \left( 2K^2 + \frac{\beta^2}{2} x^2(H^2 - 1) \right). \] (12)

The equations of motion depend on two physical parameters \( \alpha, \beta \). This form of the parameters are used in [3] but several alternative notations were used in previous papers. In [3] they use \( a, b \) with

\[ a = 2\alpha^2 \quad , \quad b = \frac{1}{4} \beta^2 = \frac{1}{2} \tilde{\beta}^2, \] (13)

and in passing we redefined by \( \tilde{\beta} \) the parameter labelled \( \beta \) in [13,14].

In solving the equations, \( \alpha, \beta \) are imposed by hand; the third physical parameter to be fixed by hand is the electric charge of the dyon, noted \( Q \). It is encoded [12,15] into the asymptotic behaviour of the function \( J(x) \)

\[ J(x)|_{x \to \infty} = J_\infty - Q x + o(\frac{1}{x^2}) \] (14)

So that \( Q \) is imposed as a boundary condition of the function \( x^2 J' \) at \( x = \infty \).

The regularity of the solution at the origin, the finiteness of the mass and the requirement that the metric (1) approaches the Minkowski metric for \( x \to \infty \) lead to the following set of boundary conditions [14]

\[ \mu(0) = 0 \quad , \quad K(0) = 1 \quad , \quad H(0) = 0 \quad , \quad J(0) = 0 \] (15)

\[ A(\infty) = 1 \quad , \quad K(\infty) = 0 \quad , \quad H(\infty) = 1 \quad , \quad (x^2 J')|_{x \to \infty} = Q . \] (16)

which fully specify the problem.

In absence of gravity (i.e. \( a = 0, N = A = 1 \)) the first two equations are trivial and the solutions are the dyons of Julia and Zee [12]. They were studied numerically in some details recently [15]. In particular, it was found that for \( b = 0 \) dyons exist for an arbitrary value of \( Q \) while for \( b \neq 0 \) the dyons exist only to a maximal value \( Q_{cr} \) (see Fig. 8 of [15]); for too high values of \( Q \) the value \( J_\infty \) becomes larger than one and, as seen from Eq.(10), \( K \) becomes oscillating. Within the range of \( \beta \) that we will explore in this paper (gravitating) dyons exist for \( Q \leq 0.7 \).

The equations (8)-(12) were studied in details in [13,14] for the case \( \beta = 0 \). In particular, it was shown that the non-Abelian gravitating dyon bifurcates at \( \alpha = \alpha_{cr} \) into an extremal RN solution with

\[ N(x) = \left( x - \alpha \sqrt{1 + Q^2} \right)^2 \quad , \quad A(x) = 1 \] (17)

\[ K(x) = 0 \quad , \quad H(x) = 1 \quad , \quad J(x) = Q\left( \frac{1}{\alpha \sqrt{1 + Q^2}} - \frac{1}{x} \right) \] (18)

defined on \( x \in [\alpha \sqrt{1 + Q^2}, \infty] \). In the next section we study the dyons solution for \( b > 0 \).

### III. GRAVITATING DYONS

In order to make the paper self consistent, we first briefly recall how the magnetic monopole solutions approach critical solutions, when the vector boson mass, i.e. \( a \), is varied, while the ratio of the Higgs boson mass to the vector boson mass, i.e. \( b \), is kept fixed. Recently, Lue and Weinberg [3] realized that there are two regimes of \( b \), each with its own type of critical solution.

In the first regime \( b \) is small (\( b < 25 \)), and the metric function \( N(r) \) of the monopole solutions possesses a single minimum. As the critical solution is approached, i.e. as \( a \to a_{cr} \), the minimum of the function \( N(r) \) decreases until it reaches zero at \( r = r_0 \). The limiting solution corresponds to an extremal RN black hole solution with horizon radius \( r_h = r_0 \) and unit magnetic charge for \( r \geq r_0 \). Consequently, also the mass of the limiting solution
and NA-types. Technically, it would involve the diagonalisation of a 3*3 matrix (rather than a 2*2) and leads to $Q_{cr} = 0$ (i.e. for $\alpha = \alpha_{cr}$ the transition point (figure is that the line $a_{cr}$ increases, the NA-type appears for lower values of $\alpha$). The critical solution thus possesses an extremal horizon at $r_{cr} < r_0$, and corresponds to an extremal black hole with non-abelian hair and a mass less than the extremal RN value. Consequently, we refer to this second limiting approach as NA-type (non-abelian-type) behaviour.

To summarize: the non-Abelian gravitating monopole exist on a portion of the $(a,b)$ plane limited by a curve $a_{cr}(b)$. Somewhere on this curve (at $(a_{cr} = 3/2, b_{cr} \approx 26.7)$) there is a critical point separating the RN-type and NA-type of ending of the solution.

By solving Eqs.(8) to (13) we were able to show that the same phenomenon occurs for dyon solutions. Before to present the details of the two regimes, we discuss the solutions in the case $Q = 1/2$, $b = 50$ which is generic of the NA-type (the corresponding critical $\alpha$ is $\alpha_{cr} \approx 0.83175$).

The profiles of the functions $A, K, H$ at the approach of the critical value $\alpha_{cr}$ was described at length in [6] and obey a similar pattern in the case of dyon. Therefore we present here the profiles of the functions $N$ and $J$.

We found it instructive to superpose on our figures the profiles of the functions

$$\frac{J}{A}, \quad B \equiv \frac{J}{A\sqrt{N}}.$$  \hfill (19)

The ratio $J/A$ naturally appears when one eliminates the function $A$ (by using (13)) from the other equations. The special combination $B$ becomes a constant when the solution approaches an extremal RN black holes (17),(18). Moreover it takes the value

$$B_{RN} = \frac{Q}{\alpha \sqrt{1 + Q^2}}$$  \hfill (20)

this can be checked from the numerical solutions.

So on Figs.1-4, the functions $N, J, J/A, B$ are represented as functions of the variable $y$

$$y \equiv \frac{x}{\alpha \sqrt{1 + Q^2}}.$$  \hfill (21)

It is such that $y_0 = 1$ (with an obvious notation we also define $y_s$ as the value of the interior minimum of N). Figure 1 is drawn for $\alpha = 0.75$, i.e. much before $\alpha_{cr}$. The function $N$ still present a unique minimum around $y = 1$. Figs. 2, 3 are obtained for $\alpha = 0.8315$. Now $N$ has developed a second minimum at $y \approx 0.80$. We observe that $J$ (and then also $J/A$ and $B$) deviates only little from zero for $y \in [0, 0.8]$. A little before $y = 0.80$ the function $B$ increases very quickly to attain the value (20) around $y = 1$. We also notice that $J/A$ develops a ripple between the two minima on $N$. This general tendency of the functions is confirmed by Fig. 4 drawn very close to the critical $\alpha$. The function $B$ here increases very steeply (in fact quasi linearly) from zero at $y_s \approx 0.795$ to the value (20) at $y = 1$.

By solving the equations for several values of $a,b$ and $Q$ we obtain a strong numerical evidence that the two behaviours found in [6] persist in the case of dyons. This statement is illustrated by Figs. 5,6 where the ratio $y_s = r_s/r_0$ characterizing the NA-type of behaviour is plotted respectively as a function $Q$ (for a few fixed values of $b$) and as a function of of $b$ (for $Q = 0$ and $Q = 1/2$) [14]. The two figures indicates in particular that, when $Q$ increases, the NA-type appears for lower values of $b$.

Corresponding to Fig. 6, the critical value $a_{cr}$ is also plotted in Fig. 7. The remarkable fact indicated by this figure is that the line $a_{cr}(b)$ in the $(a,b)$ plane is independent of $Q$ (this was also observed in [14] for $b = 0$). Only the transition point $(a_{cr}, b_{cr})$ is moving with $Q$ on the line.

The fact that the curve corresponding to $Q = 1/2$ overtakes the value $a_{cr} = 3/2$ is not in contradiction with [6] (i.e. for $Q = 0$). Expanding the dyon equations around a double zero of the function $N(r)$ (i.e. $r = r_*$ or $r = r_0$) is the same way as in [6] would lead in principle to a $Q$-depending critical value of $a$ limiting the RN and NA-types. Technically, it would involve the diagonalisation of a 3*3 matrix (rather than a 2*2) and leads to
a more involved algebra. However, owing that $Q$ is related to the asymptotic decay of the function $J$ it is very unlikely that an expansion of the solution around $r_*$ (or $r_0$) could lead to an explicit expression for $a_{cr}(Q)$ and therefore we refrain from performing such an expansion.

Clearly, Figs. 6, 7 report data which is obtained from a numerical analysis of the equation and probably the NA-type solution (with $Q = 0$ or $Q \neq 0$) exists for still slightly lower values of $b$. These solutions cannot be constructed because of the limitation in accuracy of our numerical routine. In order to support this statement we report in Fig. 8 the value $N_{min}$ of the (RN-type) minimum of $N$ when the internal minimum first appears, this as a function of $b$. As argued in [3] we believe that the transition between RN and NA-types of behaviour occurs just when this curve crosses zero. An extrapolation of the curves gives $btr \approx 26.7$ for $Q = 0$ and $btr \approx 18.7$ for $Q = 1/2$.

IV. CONCLUSION

The Einstein-Georgi-Glashow model constitutes a good theoretical laboratory for testing the properties of gravitational solitons. The solutions are particularly rich and obey non-trivial patterns of bifurcations of several types [3][4][5]. Here we have analyzed numerically the equations for the spherically symmetric gravitating dyons in this model, focusing our attention on the intermediate region of the Higgs boson mass parameter.

Our results gives strong evidence of two different scenarii of ending of the non-abelian dyons at some critical value $a = a_{cr}(b)$, confirming the recent result of [6]. Three points deserve to be further stressed. (i) The domain of existence of gravitating dyons in the (a,b) plane seems to be independant of $Q$. (ii) The transition bewtween the RN and NA regime occurs at a $Q$-depending value $(a_{cr}, b_{cr})$ of the parameters. For monopole ($Q = 0$) we have $a_{cr} = 3/2$, $b_{cr} \approx 26.7$. Dyons have $a_{cr} > 3/2$ and the precise value depends (rather indirectly) on the charge $Q$. (iii) At non-zero values of $b$ the dyons solutions exist up to a maximal value $Q_{cr}(b)$, limiting the domain of investigation of the equations.

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[16] The data of Figs. 6,7,8 for $Q = 0$ is taken from [3].
**Figure captions**

Fig. 1.
The profiles of the functions $N, J, J/A, B$ as functions of $y$ for $Q = 1/2$, $b = 50$ and $\alpha = 0.75$.

Fig. 2.
The profiles of the functions $N, J, J/A, B$ as functions of $y$ for $Q = 1/2$, $b = 50$ and $\alpha = 0.8315$.

Fig. 3.
Enlarged view of Fig. 2 in the region $Y = 0.9$ where the two minima of $N$ occur.

Fig. 4.
The profiles of the functions $N, J, J/A, B$ as functions of $y$ for $Q = 1/2$, $b = 50$ and $\alpha = 0.83175$.

Fig. 5
The ratio $y^* = r^*/r_0$ is presented as a function of $Q$ in the NA-type regime for $\beta = 8$ ($b = 32$) and for $\beta = 6.5$ ($b = 21.125$).

Fig. 6
The ratio $y^* = r^*/r_0$ is presented as a function of $b$ in the NA-type regime for monopole solutions and for dyon solutions with charge $Q = 1/2$.

Fig. 7
The critical value $a_{cr} - 1$ is presented as a function of $b$ in the NA-type regime for monopole solutions and for dyon solutions with charge $Q = 1/2$.

Fig. 8
The value of the function $N(r)$ at the first minimum where the second minimum first appears is presented as a function of the value of $b$, for monopole solutions and for dyon solutions with charge $Q = 1/2$. 
Figure 1

$\alpha = 0.75$
Figure 2

\[ \alpha = 0.8315 \]
Figure 3

$\alpha = 0.8315$
Figure 4

\[ \alpha = 0.83175 \]
Figure 5

- $\beta = 8.0$
- $\beta = 6.5$
Figure 6
Figure 7

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure7}
\end{figure}
Figure 8

$Q = 0.5$

$Q = 0.0$