A notion of fine continuity for BV functions on metric spaces *

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Abstract

In the setting of a metric space equipped with a doubling measure supporting a Poincaré inequality, we show that BV functions are, in the sense of multiple limits, continuous with respect to a 1-fine topology, at almost every point with respect to the codimension 1 Hausdorff measure.

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1 Introduction

It is known, in the generality of a metric measure space \((X, d, \mu)\) equipped with a doubling measure \(\mu\) supporting a Poincaré inequality, that Newton-Sobolev functions \(u \in N^{1,p}(X)\) are \(p\)-quasicontinuous. This means that there exists an open set \(G \subset X\) of small \(p\)-capacity such that the restriction \(u|_{X \setminus G}\) is continuous, see e.g. [6] or [5]. From this, one can derive another result, which states that Newton-Sobolev functions are \(p\)-finely continuous at \(p\)-quasi every point, that is, almost every point with respect to the \(p\)-capacity, see [8] or [23] or [4, Theorem 11.40]. The concept of \(p\)-fine continuity means continuity with respect to a suitable topology, the \(p\)-fine topology, which is somewhat stronger than the metric topology. For previous results on fine topology and fine continuity in the Euclidean setting, see also e.g. [10, 16, 26].

In [24] it was shown that BV functions on metric spaces are \(1\)-quasicontinuous in the sense of multiple limits. In this paper we introduce a notion of \(1\)-fine topology, and show that BV functions are \(1\)-finely continuous (that is, continuous with respect to the \(1\)-fine topology) at \(1\)-quasi every point, again in the sense of multiple limits. This is given in Theorem 5.2. Instead of \(1\)-quasi every point, one may equivalently speak about \(\mathcal{H}\)-almost every point, where \(\mathcal{H}\) is the codimension 1 Hausdorff measure.

Our definition of the \(1\)-fine topology is based on a concept of \(1\)-thinness, which is analogous to a concept of \(p\)-fatness, with \(p > 1\), given in the metric setting in [9] and originally defined in [25]. Let us also note that the proofs for fine continuity given in [8] and [23] involve the theory of \(p\)-harmonic functions, for \(p > 1\). While some results on \(1\)-harmonic functions, known as functions of least gradient, have been derived in [14, 15, 19], we do not use this theory, relying on a geometric tool known as the boxing inequality instead.

2 Preliminaries

In this section we introduce the necessary definitions and assumptions.

In this paper, \((X, d, \mu)\) is a complete metric space equipped with a Borel regular outer measure \(\mu\) satisfying a doubling property, that is, there is a constant \(C_d \geq 1\) such that

\[
0 < \mu(B(x, 2r)) \leq C_d \mu(B(x, r)) < \infty
\]
for every ball $B = B(x, r)$ with center $x \in X$ and radius $r > 0$. We also assume that $X$ consists of at least two points.

In general, $C \geq 1$ will denote a constant whose particular value is not important for the purposes of this paper, and might differ between each occurrence. When we want to specify that a constant $C$ depends on the parameters $a, b, \ldots$, we write $C = C(a, b, \ldots)$. Unless otherwise specified, all constants only depend on the doubling constant $C_d$ and the constants $C_P, \lambda$ associated with the Poincaré inequality defined below.

A complete metric space with a doubling measure is proper, that is, closed and bounded subsets are compact. Since $X$ is proper, for any open set $\Omega \subset X$ we define $\text{Lip}_\text{loc}(\Omega)$ to be the space of functions that are Lipschitz in every $\Omega' \Subset \Omega$. Here $\Omega' \Subset \Omega$ means that $\Omega'$ is open and that $\overline{\Omega'}$ is a compact subset of $\Omega$. Other local spaces of functions are defined similarly.

For any set $A \subset X$ and $0 < R < \infty$, the restricted spherical Hausdorff content of codimension 1 is defined by

$$H_R(A) := \inf \left\{ \sum_{i=1}^{\infty} \frac{\mu(B(x_i, r_i))}{r_i} : A \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), r_i \leq R \right\}. \quad (2.1)$$

The codimension 1 Hausdorff measure of a set $A \subset X$ is given by

$$\mathcal{H}(A) := \lim_{R \to 0^+} \mathcal{H}_R(A).$$

The measure theoretic boundary $\partial^* E$ of a set $E \subset X$ is the set of points $x \in X$ at which both $E$ and its complement have positive upper density, i.e.

$$\limsup_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} > 0 \quad \text{and} \quad \limsup_{r \to 0^+} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} > 0.$$

The measure theoretic interior and exterior of $E$ are defined respectively by

$$I_E := \left\{ x \in X : \lim_{r \to 0^+} \frac{\mu(B(x, r) \setminus E)}{\mu(B(x, r))} = 0 \right\} \quad (2.2)$$

and

$$O_E := \left\{ x \in X : \lim_{r \to 0^+} \frac{\mu(B(x, r) \cap E)}{\mu(B(x, r))} = 0 \right\}. \quad (2.3)$$

A curve is a rectifiable continuous mapping from a compact interval into $X$. A nonnegative Borel function $g$ on $X$ is an upper gradient of an extended
real-valued function $u$ on $X$ if for all curves $\gamma$ on $X$, we have

$$|u(x) - u(y)| \leq \int_\gamma g \, ds,$$

where $x$ and $y$ are the end points of $\gamma$. We interpret $|u(x) - u(y)| = \infty$ whenever at least one of $|u(x)|$, $|u(y)|$ is infinite. Upper gradients were originally introduced in [17].

For $1 \leq p < \infty$, we consider the following norm

$$\|u\|_{N^{1,p}(X)} := \|u\|_{L^p(X)} + \inf \|g\|_{L^p(X)},$$

with the infimum taken over all upper gradients $g$ of $u$. The substitute for the Sobolev space $W^{1,p}(\mathbb{R}^n)$ in the metric setting is the Newton-Sobolev space

$$N^{1,p}(X) := \{u : \|u\|_{N^{1,p}(X)} < \infty\}.$$

For more on Newton-Sobolev spaces, we refer to [28, 5, 18].

Next we recall the definition and basic properties of functions of bounded variation on metric spaces, see [27]. See also e.g. [2, 11, 12, 29] for the classical theory in the Euclidean setting. For $u \in L^1_{\text{loc}}(X)$, we define the total variation of $u$ on $X$ to be

$$\|Du\|(X) := \inf \left\{ \liminf_{i \to \infty} \int_X g_{u_i} \, d\mu : u_i \in \text{Lip}_{\text{loc}}(X), u_i \to u \text{ in } L^1_{\text{loc}}(X) \right\},$$

where each $g_{u_i}$ is an upper gradient of $u_i$. The total variation is clearly lower semicontinuous with respect to convergence in $L^1_{\text{loc}}(X)$. We say that a function $u \in L^1(X)$ is of bounded variation, and denote $u \in \text{BV}(X)$, if $\|Du\|(X) < \infty$. By replacing $X$ with an open set $\Omega \subset X$ in the definition of the total variation, we can define $\|Du\|(\Omega)$. A $\mu$-measurable set $E \subset X$ is said to be of finite perimeter if $\|D\chi_E\|(X) < \infty$, where $\chi_E$ is the characteristic function of $E$. The perimeter of $E$ in $\Omega$ is also denoted by

$$P(E, \Omega) := \|D\chi_E\|(\Omega).$$

For any Borel sets $E_1, E_2 \subset X$ we have by [27, Proposition 4.7]

$$P(E_1 \cup E_2, X) \leq P(E_1, X) + P(E_2, X). \tag{2.4}$$

We will assume throughout that $X$ supports a $(1,1)$-Poincaré inequality, meaning that there exist constants $C_P > 0$ and $\lambda \geq 1$ such that for every ball
every locally integrable function $u$ on $X$, and every upper gradient $g$ of $u$, we have

$$\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq C pr \int_{B(x,\lambda r)} g \, d\mu,$$

where

$$u_{B(x,r)} := \int_{B(x,r)} u \, d\mu := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} u \, d\mu.$$

By applying the Poincaré inequality to approximating locally Lipschitz functions in the definition of the total variation, we get the following $(1,1)$-Poincaré inequality for BV functions. There exists a constant $C$ such that for every ball $B(x,r)$ and every $u \in L^1_{\text{loc}}(X)$, we have

$$\int_{B(x,r)} |u - u_{B(x,r)}| \, d\mu \leq Cr \frac{\|Du\|(B(x,\lambda r))}{\mu(B(x,\lambda r))}.$$  

For $\mu$-measurable sets $E \subset X$, the above can be written as

$$\min\{\mu(B(x,r) \cap E), \mu(B(x,r) \setminus E)\} \leq Cr P(E, B(x,\lambda r)).$$  

For $1 \leq p < \infty$, the $p$-capacity of a set $A \subset X$ is given by

$$\text{Cap}_p(A) := \inf \|u\|_{N^1,p(X)},$$

where the infimum is taken over all functions $u \in N^{1,p}(X)$ such that $u \geq 1$ in $A$. If a property holds for all points outside a set of $p$-capacity zero, we say that it holds for $p$-quasi every point, or $p$-quasieverywhere.

The relative $p$-capacity of a set $A \subset X$ with respect to an open set $\Omega \subset X$ is given by

$$\text{cap}_p(A, \Omega) := \inf \int_{\Omega} g_u^p \, d\mu,$$

where the infimum is taken over functions $u \in N^{1,p}(X)$ and upper gradients $g_u$ of $u$ such that $u \geq 1$ in $A$ and $u = 0$ in $X \setminus \Omega$. For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see e.g. [5].

The BV-capacity of a set $A \subset X$ is

$$\text{Cap}_{\text{BV}}(A) := \inf \|u\|_{\text{BV}(X)},$$

where the infimum is taken over all functions $u \in L^1_{\text{loc}}(X)$ and upper gradients $g_u$ of $u$ such that $u \geq 1$ in $A$ and $u = 0$ in $X \setminus \Omega$. For basic properties satisfied by capacities, such as monotonicity and countable subadditivity, see e.g. [5].
where the infimum is taken over functions \( u \in \text{BV}(X) \) that satisfy \( u \geq 1 \) in a neighborhood of \( A \). Note that we understand BV functions to be \( \mu \)-equivalence classes, whereas we understand Newton-Sobolev functions to be defined everywhere (even though \( \| \cdot \|_{N^1,p(X)} \) is then only a seminorm).

Given a set \( E \subset X \) of finite perimeter, for \( \mathcal{H} \)-almost every \( x \in \partial^* E \) we have

\[
\gamma \leq \liminf_{r \to 0^+} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \leq \limsup_{r \to 0^+} \frac{\mu(E \cap B(x,r))}{\mu(B(x,r))} \leq 1 - \gamma \tag{2.8}
\]

where \( \gamma \in (0, 1/2] \) only depends on the doubling constant and the constants in the Poincaré inequality, see [1, Theorem 5.4]. For an open set \( \Omega \subset X \) and a set \( E \subset X \) of finite perimeter, we know that

\[
P(E, \Omega) = \int_{\partial^* E \cap \Omega} \theta_E \, d\mathcal{H}, \tag{2.9}
\]

where \( \theta_E : X \to [\alpha, C_d] \) with \( \alpha = \alpha(C_d, C_P, \lambda) > 0 \), see [1, Theorem 5.3] and [3, Theorem 4.6].

The jump set of \( u \in \text{BV}(X) \) is the set

\[
S_u := \{ x \in X : u^-(x) < u^+(x) \},
\]

where \( u^-(x) \) and \( u^+(x) \) are the lower and upper approximate limits of \( u \) defined respectively by

\[
u^-(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x,r) \cap \{ u < t \})}{\mu(B(x,r))} = 0 \right\}
\]

and

\[
u^+(x) := \inf \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x,r) \cap \{ u > t \})}{\mu(B(x,r))} = 0 \right\}.
\]

In the Euclidean setting, results on the fine properties of BV functions can be formulated in terms of \( u^- \) and \( u^+ \), but in the metric setting, we need to consider a larger number of jump values. The reason for this is explained in Example 5.6. Thus we define the functions \( u^l, l = 1, \ldots, n := \lfloor 1/\gamma \rfloor \), as follows: \( u^1 := u^- \), \( u^n := u^+ \), and for \( l = 2, \ldots, n - 1 \) we define inductively

\[
u^l(x) := \sup \left\{ t \in \mathbb{R} : \lim_{r \to 0^+} \frac{\mu(B(x,r) \cap \{ u^{l-1}(x) + \delta < u < t \})}{\mu(B(x,r))} = 0 \quad \forall \delta > 0 \right\} \tag{2.10}
\]
provided $u^{l-1}(x) < u^{l}(x)$, and otherwise we set $u^{l}(x) = u^{r}(x)$. It can be shown that each $u^{l}$ is a Borel function, and $u^{\land} = u^{1} \leq \ldots \leq u^{n} = u^{r}$.

We have the following notion of quasicontinuity for BV functions.

**Theorem 2.1** ([24, Theorem 1.1]). Let $u \in \text{BV}(X)$ and let $\varepsilon > 0$. Then there exists an open set $G \subset X$ with $\text{Cap}_{1}(G) < \varepsilon$ such that if $y_{k} \to x$ with $y_{k}, x \in X \setminus G$, then

$$\min_{l_{2} \in \{1, \ldots, n\}} |u^{l_{1}}(y_{k}) - u^{l_{2}}(x)| \to 0$$

for each $l_{1} = 1, \ldots, n$.

**Remark 2.2.** The 1-capacity and Hausdorff contents are closely related: it follows from [13, Theorem 4.3, Theorem 5.1] that $\text{Cap}_{1}(A) = 0$ if and only if $\mathcal{H}(A) = 0$. Moreover, from [21, Lemma 3.4] it follows that $\text{Cap}_{1}(A) \leq 2C_{d}\mathcal{H}(A)$ for any set $A \subset X$. On the other hand, by combining [13, Theorem 4.3] and the proof of [13, Theorem 5.1], we obtain that

$$\mathcal{H}_{\varepsilon}(A) \leq C(C_{d}, C_{P}, \lambda, \varepsilon) \text{Cap}_{1}(A)$$

for any $A \subset X$ and $\varepsilon > 0$. Thus we can also control the size of the "exceptional set" $G$ in Theorem 2.1 and elsewhere by its $\mathcal{H}_{\varepsilon}$-measure, for arbitrarily small $\varepsilon > 0$.

### 3 Rigidity results for the 1-capacity

In order to prove our main result, Theorem 5.2, we need to be able to modify the "exceptional set" $G$ of Theorem 2.1 in a suitable way. In this section we show that sets can be enlarged in two different ways without increasing the 1-capacity significantly.

It is known that $\text{Cap}_{1}$ is an outer capacity, meaning that

$$\text{Cap}_{1}(G) = \inf\{\text{Cap}_{1}(U) : U \text{ is open and } U \supset G\}$$

for any $G \subset X$, see e.g. [5, Theorem 5.31]. The following result is in the same spirit as this fact.

**Lemma 3.1.** For any $G \subset X$, we can find an open set $U \supset G$ with $\text{Cap}_{1}(U) \leq C \text{Cap}_{1}(G)$ and $P(U, X) \leq C \text{Cap}_{1}(G)$. 

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Proof. We can assume that $\text{Cap}_1(G) < \infty$. According to Remark 2.2, we have $\mathcal{H}_{1/2}(G) \leq C \text{Cap}_1(G)$. Take a covering $\{B(x_i, r_i)\}_{i \in \mathbb{N}}$ of the set $G$ with $r_i \leq 1/2$ and

$$\sum_{i \in \mathbb{N}} \frac{\mu(B(x_i, r_i))}{r_i} \leq C \text{Cap}_1(G).$$

By [20, Lemma 6.2], for each $i \in \mathbb{N}$ there exists a radius $\tilde{r}_i \in [r_i, 2r_i]$ such that

$$P(B(x_i, \tilde{r}_i), X) \leq C_d \frac{\mu(B(x_i, \tilde{r}_i))}{\tilde{r}_i}.$$

By using the lower semicontinuity and subadditivity of perimeter, recall (2.4), we get

$$P\left(\bigcup_{i \in \mathbb{N}} B(x_i, \tilde{r}_i), X\right) \leq \sum_{i \in \mathbb{N}} P(B(x_i, \tilde{r}_i), X) \leq C_d \sum_{i \in \mathbb{N}} \frac{\mu(B(x_i, \tilde{r}_i))}{\tilde{r}_i} \leq C \text{Cap}_1(G).$$

So we can define $U := \bigcup_{i \in \mathbb{N}} B(x_i, \tilde{r}_i)$, with $U \supset G$, and then $P(U, X) \leq C \text{Cap}_1(G)$ and $\mathcal{H}_1(U) \leq C \text{Cap}_1(G)$, so that also $\text{Cap}_1(U) \leq C \text{Cap}_1(G)$ by Remark 2.2.

In proving our second rigidity result, we will use discrete convolutions of BV functions. By the doubling property of the measure $\mu$, given any scale $R > 0$ we can pick a covering of the space $X$ by balls $B(x_j, R)$, such that suitable dilated balls, say $B(x_j, 10\lambda R)$, have bounded overlap. More precisely, each $B(x_k, 10\lambda R)$ meets at most $C$ balls $B(x_j, 10\lambda R)$. Given such a covering, we can take a partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinate to it, such that $0 \leq \phi_j \leq 1$, each $\phi_j$ is a $C/R$-Lipschitz function, and $\text{supp}(\phi_j) \subset B(x_j, 2R)$ for each $j \in \mathbb{N}$ (see e.g. [7, Theorem 3.4]). Finally, we can define a discrete convolution $v$ of any $u \in \text{BV}(X)$ with respect to the covering by

$$v := \sum_{j=1}^\infty u_{B(x_j, 5R)} \phi_j.$$

We know that $v$ has an upper gradient

$$g = C \sum_{j=1}^\infty \chi_{B_j} \frac{\|Du\|(B(x_j, 10\lambda R))}{\mu(B(x_j, R))},$$

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see e.g. the proof of [22, Proposition 4.1], and so by the bounded overlap of the balls $B(x_j, 10\lambda R)$, we have $\| g \|_{L^1(Y)} \leq C \| Du \| (X)$. We also have $\| v \|_{BV(Y)} \leq \| v \|_{N^{1,1}(Y)}$ (since Lipschitz functions are dense in the class $N^{1,1}(Y)$, see e.g. [5, Theorem 5.1]), and thus

$$\| v \|_{BV(Y)} \leq C \| u \|_{BV(Y)}.$$  \hspace{1cm} (3.1)

If $u \in BV(Y)$ and each $v_i, i \in \mathbb{N}$, is a discrete convolution of $u$ at scale $1/i$, we know that for some $\bar{\gamma} = \bar{\gamma}(C_d, C_P, \lambda) \in (0, 1/2]$, 

$$(1 - \bar{\gamma})u^\wedge(y) + \bar{\gamma}u^\vee(y) \leq \liminf_{i \to \infty} v_i(y) \leq \limsup_{i \to \infty} v_i(y) \leq \bar{\gamma}u^\wedge(y) + (1 - \bar{\gamma})u^\vee(y)$$ \hspace{1cm} (3.2)

for $\mathcal{H}$-almost every $y \in X$, see [22, Proposition 4.1].

Recall the definitions of the 1-capacity and the BV-capacity from (2.6) and (2.7). By [13, Theorem 4.3] we know that

$$\text{Cap}_{BV}(A) \leq \text{Cap}_1(A) \leq C \text{Cap}_{BV}(A)$$ \hspace{1cm} (3.3)

for any $A \subset Y$.

Now we prove the following rigidity result for the 1-capacity. Recall from (2.2) the definition of the measure theoretic interior $I_G$ of a set $G$.

**Proposition 3.2.** Let $G \subset Y$ be an arbitrary set. Then

$$\text{Cap}_1(G \cup I_G \cup \partial^* G) \leq C \text{Cap}_1(G).$$

**Proof.** By (3.3) it is enough to prove this for $\text{Cap}_{BV}$ instead of $\text{Cap}_1$. We can assume that $\text{Cap}_{BV}(G) < \infty$. Fix $\varepsilon > 0$ and choose $u \in BV(Y)$ with $u \geq 0, u \geq 1$ in a neighborhood of $G$, and $\| u \|_{BV(Y)} \leq \text{Cap}_{BV}(G) + \varepsilon$. Let each $v_i \in \text{Lip}_{loc}(Y), i \in \mathbb{N}$, be a discrete convolution of $u$ at scale $1/i$, and let $N \subset Y$ be the set where (3.2) fails, so that $\mathcal{H}(N) = 0$. Thus we have (recall Remark 2.2)

$$\text{Cap}_{BV}(G \cup I_G \cup \partial^* G \setminus N) = \text{Cap}_{BV}(G \cup I_G \cup \partial^* G).$$

Clearly $u^\wedge \geq 1$ in $G \cup I_G$, and $u^\vee \geq 1$ in $\partial^* G$, so that

$$(1 - \bar{\gamma})u^\wedge(y) + \bar{\gamma}u^\vee(y) \geq \bar{\gamma}$$

Clearly $u^\wedge \geq 1$ in $G \cup I_G$, and $u^\vee \geq 1$ in $\partial^* G$, so that

$$(1 - \bar{\gamma})u^\wedge(y) + \bar{\gamma}u^\vee(y) \geq \bar{\gamma}$$
for every $y \in G \cup I_G \cup \partial^* G$, and so by (3.2),
\[
\liminf_{i \to \infty} v_i(y) \geq \frac{\tilde{\gamma}}{2} \quad \text{for every } y \in G \cup I_G \cup \partial^* G \setminus N.
\]
Define the sets
\[
G_i := \{x \in G \cup I_G \cup \partial^* G \setminus N : v_j(x) > \frac{\tilde{\gamma}}{2} \text{ for all } j \geq i\}, \quad i \in \mathbb{N}.
\]
Now we have $G_1 \subset G_2 \subset \ldots$ and $\bigcup_{i \in \mathbb{N}} G_i = G \cup I_G \cup \partial^* G \setminus N$. Since discrete convolutions are continuous, clearly $v_i > \frac{\tilde{\gamma}}{2}$ in a neighborhood of $G_i$ and so we can use $2v_i/\tilde{\gamma}$ to estimate the BV-capacity of $G_i$. Furthermore, by [13, Theorem 3.4] we know that the BV-capacity is continuous with respect to increasing sequences of sets, and so we get
\[
\text{Cap}_{BV}(G \cup I_G \cup \partial^* G) = \text{Cap}_{BV}(G \cup I_G \cup \partial^* G \setminus N) = \text{Cap}_{BV} \left( \bigcup_{i \in \mathbb{N}} G_i \right)
\]
\[
= \lim_{i \to \infty} \text{Cap}_{BV}(G_i) \leq \frac{2}{\tilde{\gamma}} \liminf_{i \to \infty} \|v_i\|_{BV(X)}
\]
by the choice of $u$. By letting $\varepsilon \to 0$, we get the result. \hfill \Box

4 The 1-fine topology

Our result on 1-fine continuity will be based on a concept of a fine topology on the space. Let us first consider some background concerning the case $1 < p < \infty$. The following definitions and facts are given in [8] and [5, Section 11.6]. A set $A \subset X$ is $p$-thin at $x \in X$ if
\[
\int_0^1 \left( \frac{\text{cap}_p(A \cap B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))} \right)^{1/(p-1)} \frac{dr}{r} < \infty.
\]
A set $U \subset X$ is $p$-finely open if $X \setminus U$ is $p$-thin at every $x \in U$. The collection of $p$-finely open sets is a topology on $X$, called the $p$-fine topology. Let $\overline{G}^p$ be the $p$-fine closure of $G \subset X$ (smallest $p$-finely closed set containing $G$). For an open set $\Omega \subset X$ with $\text{Cap}_p(X \setminus \Omega) > 0$ and $G \Subset \Omega$, we have
\[
\text{cap}_p(\overline{G}^p, \Omega) = \text{cap}_p(G, \Omega).
\]
A $p$-finely closed set is measure theoretically closed, as follows from [5, Corollary 11.25], and thus the measure theoretic closure $G \cup I_G \cup \partial^* G$ is a subset of $G^p$. Thus in the case $p > 1$, a stronger result than Proposition 3.2 holds.

In a similar vein, according to [9, Definition 1.1] (which is based on [25]) a set $A \subset X$ is said to be $p$-fat at a point $x \in X$ if
\[
\limsup_{r \to 0^+} \frac{\text{cap}_p(A \cap B(x, r), B(x, 2r))}{\text{cap}_p(B(x, r), B(x, 2r))} > 0.
\]

By [5, Proposition 6.16] we know that for small $r > 0$ and $1 \leq p < \infty$, $\text{cap}_p(B(x, r), B(x, 2r))$ is comparable to $\mu(B(x, r))/r^p$. This motivates the following definition.

**Definition 4.1.** We say that $A \subset X$ is 1-thin at the point $x \in X$ if
\[
\lim_{r \to 0^+} \frac{\text{cap}_1(A \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0.
\]

We also say that a set $U \subset X$ is 1-finely open if $X \setminus U$ is 1-thin at every $x \in U$.

**Lemma 4.2.** The collection of 1-finely open sets is a topology on $X$ (called the 1-fine topology).

**Proof.** Let $\{U_i\}_{i \in I}$ be any collection of 1-finely open sets, and let $x \in \bigcup_{i \in I} U_i$. Then $x \in U_j$ for some $j \in I$. Thus
\[
\limsup_{r \to 0^+} r \frac{\text{cap}_1(B(x, r) \setminus \bigcup_{i \in I} U_i, B(x, 2r))}{\mu(B(x, r))} \\
\leq \limsup_{r \to 0^+} r \frac{\text{cap}_1(B(x, r) \setminus U_j, B(x, 2r))}{\mu(B(x, r))} = 0
\]
by the fact that $U_j$ is 1-finely open. Thus $\bigcup_{i \in I} U_i$ is a 1-finely open set. Next let $U_1, \ldots, U_k$ be 1-finely open sets, with $k \in \mathbb{N}$, and suppose $x \in \bigcap_{i=1}^k U_i$. 

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Then by the subadditivity of capacity

\[
\limsup_{r \to 0^+} r \frac{\text{cap}_1 \left( B(x, r) \setminus \bigcap_{i=1}^k U_i, B(x, 2r) \right) \mu(B(x, r))}{\mu(B(x, r))} \\
\leq \limsup_{r \to 0^+} \sum_{i=1}^k r \frac{\text{cap}_1 \left( B(x, r) \setminus U_i, B(x, 2r) \right) \mu(B(x, r))}{\mu(B(x, r))} \\
\leq \sum_{i=1}^k \limsup_{r \to 0^+} \frac{\text{cap}_1 \left( B(x, r) \setminus U_i, B(x, 2r) \right) \mu(B(x, r))}{\mu(B(x, r))} = 0
\]

by the fact that each \(U_i\) is 1-finely open. Thus \(\bigcap_{i=1}^k U_i\) is a 1-finely open set.

Let \(\overline{G}^1\) be the 1-fine closure of \(G \subset X\) (smallest 1-finely closed set containing \(G\)). In the case \(p > 1\), a crucial step in showing that Newton-Sobolev functions are \(p\)-finely continuous (i.e. continuous with respect to the \(p\)-fine topology) at \(p\)-quasi every point is showing that \(\text{cap}_p(\overline{G}^1, \Omega) = \text{cap}_p(G, \Omega)\), see the discussion earlier in this section.

**Open Problem.** Is it true that \(\text{Cap}_1(\overline{G}^1) = \text{Cap}_1(G)\) for every \(G \subset X\)?

For us it will be enough to have a weaker result that we prove in Proposition 4.4. Following [21], we first prove the following local version of the boxing inequality.

**Lemma 4.3.** Let \(x \in X\), let \(r > 0\), and let \(G \subset X\) be a \(\mu\)-measurable set with

\[
\frac{\mu(G \cap B(x, 2r))}{\mu(B(x, 2r))} \leq \frac{1}{2C_d^d}.
\]

Then \(\text{cap}_1(I_G \cap B(x, r), B(x, 2r)) \leq CP(G, B(x, 2r))\).

**Proof.** Fix \(y \in I_G \cap B(x, r)\). Since \(y \in I_G\), there exists \(s \in (0, r/32\lambda)\) such that

\[
\frac{\mu(G \cap B(y, s))}{\mu(B(y, s))} > \frac{1}{2}.
\]

On the other hand, for all \(t \in (r/32\lambda, r/16\lambda)\) we have \(B(x, 2r) \subset B(y, 128\lambda t)\) and then

\[
\frac{\mu(G \cap B(y, t))}{\mu(B(y, t))} \leq C_d \frac{\mu(G \cap B(x, 2r))}{\mu(B(x, 2r))} \leq \frac{1}{2}
\]

by the fact that each \(U_i\) is 1-finely open. Thus \(\bigcap_{i=1}^k U_i\) is a 1-finely open set.
by (4.1). Thus by repeatedly doubling the radius $s$, we eventually obtain a radius $t_y \in (0, r/16\lambda]$ such that

$$
\frac{1}{2C_d} < \frac{\mu(G \cap B(y, t_y))}{\mu(B(y, t_y))} \leq \frac{1}{2}.
$$

By the relative isoperimetric inequality (2.5), this implies that

$$
\mu(B(y, t_y)) \leq C \mu(G \cap B(y, t_y)) \leq C t_y P(G, B(y, \lambda t_y)). \quad (4.2)
$$

Define the function

$$
w(z) := \max \left\{ 0, 1 - \frac{\text{dist}(z, B(y, 5\lambda t_y))}{5\lambda t_y} \right\}, \quad (4.3)
$$

so that $w = 1$ in $B(y, 5\lambda t_y)$ and $w = 0$ outside $B(y, 10\lambda t_y)$. Note that $w$ has an upper gradient $g := \frac{1}{5\lambda t_y} \chi_{B(y, 10\lambda t_y) \setminus B(y, 5\lambda t_y)}$. Then since $B(y, 10\lambda t_y) \subset B(x, 2r),

$$
\text{cap}_1(B(y, 5\lambda t_y), B(x, 2r)) \leq \int_{B(y, 10\lambda t_y)} g \, d\mu \leq \frac{\mu(B(y, 10\lambda t_y))}{5\lambda t_y}.
$$

Take a covering $\{B(y, \lambda t_y)\}_{y \in I_G \cap B(x, r)}$. By the 5-covering theorem, we can choose a countable disjoint collection $\{B(y_i, \lambda t_i)\}_{i \in \mathbb{N}}$ such that the balls $B(y_i, 5\lambda t_i)$ cover $I_G \cap B(x, r)$. Then we have by the countable subadditivity of capacity

$$
\text{cap}_1(I_G \cap B(x, r), B(x, 2r)) \leq \sum_{i \in \mathbb{N}} \text{cap}_1(B(y_i, 5\lambda t_i), B(x, 2r))
\leq \sum_{i \in \mathbb{N}} \frac{\mu(B(y_i, 10\lambda t_i))}{5\lambda t_i}
\leq C \sum_{i \in \mathbb{N}} \frac{\mu(B(y_i, t_i))}{t_i}
\leq \sum_{i \in \mathbb{N}} \frac{\mu(B(y_i, 5\lambda t_i))}{5\lambda t_i}
\leq \sum_{i \in \mathbb{N}} \mu(B(y_i, \lambda t_i))
\leq C \sum_{i \in \mathbb{N}} P(G, B(y_i, \lambda t_i))
\leq P(G, B(x, 2r)). \quad (4.2)
\]
It is easy to see that for any set $A \subset X$ and any ball $B(x, r)$,
\[
\text{cap}_1(A \cap B(x, r), B(x, 2r)) \leq C \mathcal{H}(A \cap B(x, r)).
\] (4.4)
This can be deduced by using suitable cutoff functions similar to those given in (4.3).

**Proposition 4.4.** For any $G \subset X$ we have $\text{Cap}_1(G^1) \leq C \text{Cap}_1(G)$.

**Proof.** We can assume that $\text{Cap}_1(G) < \infty$. First assume also that $G$ is open and that $P(G, X) < \infty$. By [4, Theorem 2.4.3] we know that if $\nu$ is a Radon measure on $X$, $t > 0$, and $A \subset X$ is a Borel set for which we have
\[
\limsup_{r \to 0^+} r \frac{\nu(B(x, r))}{\mu(B(x, r))} \geq t
\]
for all $x \in A$, then $\nu(A) \geq t \mathcal{H}(A)$. Since $G$ is of finite perimeter, we have $\mathcal{H}(\partial^* G) < \infty$ by (2.9). By using (4.4) and the above density result with $\nu = \mathcal{H}|_{\partial^* G}$, we get
\[
\limsup_{r \to 0^+} r \frac{\text{cap}_1(\partial^* G \cap B(x, r), B(x, 2r))}{\mu(B(x, 2r))} \leq C \limsup_{r \to 0^+} \frac{\mathcal{H}(\partial^* G \cap B(x, 2r))}{\mu(B(x, 2r))} = 0
\] (4.5)
for $\mathcal{H}$-almost every $x \in X \setminus \partial^* G$, that is, for every $x \in X \setminus (\partial^* G \cup N)$ with $\mathcal{H}(N) = 0$.

By Lemma 4.3 if $x \in X$ and $r > 0$ satisfy
\[
\frac{\mu(G \cap B(x, 2r))}{\mu(B(x, 2r))} \leq \frac{1}{2C^d_{\log_2(128 \lambda)}},
\]
then $\text{cap}_1(I_G \cap B(x, r), B(x, 2r)) \leq CP(G, B(x, 2r))$. Thus we get for all $x \in X \setminus (I_G \cup \partial^* G \cup N)$
\[
\limsup_{r \to 0^+} r \frac{\text{cap}_1(I_G \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} \leq C \limsup_{r \to 0^+} \frac{P(G, B(x, 2r))}{\mu(B(x, r))} \\
\leq C \limsup_{r \to 0^+} \frac{\mathcal{H}(\partial^* G \cap B(x, 2r))}{\mu(B(x, r))} \\
= 0.
\]
By combining this with (4.5), we have
\[
\limsup_{r \to 0^+} r \frac{\text{cap}_1((I_G \cup \partial^* G) \cap B(x, r), B(x, 2r))}{\mu(B(x, r))} = 0
\]
for all \( x \in X \setminus (I_G \cup \partial^* G \cup N) \). Since \( G \) is open, \( G \subset I_G \). Thus \( I_G \cup \partial^* G \cup N \supset G \) is a 1-finely closed set, so that \( \overline{G^1} \subset I_G \cup \partial^* G \cup N \). By Proposition 3.2 we have
\[
\text{Cap}_1(I_G \cup \partial^* G \cup N) = \text{Cap}_1(I_G \cup \partial^* G) \leq C \text{Cap}_1(G).
\]
Thus we have the result when \( G \) is open and of finite perimeter. In the general case, by Lemma 3.1 we can choose an open set \( U \supseteq G \) with \( \text{Cap}_1(U) \leq C \text{Cap}_1(G) \) and \( P(U, X) < \infty \). Thus we have
\[
\text{Cap}_1(G^1) \leq \text{Cap}_1(U^1) \leq C \text{Cap}_1(U) \leq C \text{Cap}_1(G).
\]

\( \square \)

5 Fine continuity

Since BV functions can have multiple jump values \( u^1, \ldots, u^n \) in their jump sets (recall the definition from (2.10)), we need to consider a notion of continuity for set-valued functions.

**Definition 5.1.** Let \( \mathcal{U} \) be a topology on \( X \). We say that the function \( y \mapsto \{u^1(y), \ldots, u^n(y)\} \) is *upper hemicontinuous* with respect to \( \mathcal{U} \) at the point \( x \) if for every \( \varepsilon > 0 \), there exists \( U \in \mathcal{U} \) with \( x \in U \) such that
\[
\min_{l_2 \in \{1, \ldots, n\}} |u^{l_1}(y) - u^{l_2}(x)| < \varepsilon
\]
for each \( l_1 = 1, \ldots, n \) and all \( y \in U \).

Now we can prove the main result of this paper.

**Theorem 5.2.** Let \( u \in \text{BV}(X) \). Then the function \( y \mapsto \{u^1(y), \ldots, u^n(y)\} \) is 1-finely upper hemicontinuous, i.e. upper hemicontinuous with respect to the 1-fine topology, at \( \mathcal{H} \)-almost every \( x \in X \).
Proof. Take sets $G_i \subset X$ with $\text{Cap}_1(G_i) < 1/i$, $i \in \mathbb{N}$, as given by our quasicontinuity-type result, Theorem 2.1. Then also

$$\text{Cap}_1(G_i^1) < C/i$$

by Proposition 4.4. For 1-quasi every and thus for $\mathcal{H}$-almost every $x \in X$, we have $x \notin \bigcap_{i \in \mathbb{N}} G_i^1$. Fix such $x$, so that $x \notin G_j^1$ for some $j \in \mathbb{N}$, and fix $\varepsilon > 0$. Theorem 2.1 gives a radius $r > 0$ such that

$$\min_{l_2 \in \{1, \ldots, n\}} |u^{l_1}(y) - u^{l_2}(x)| < \varepsilon$$

for each $l_1 = 1, \ldots, n$ and all $y \in B(x, r) \setminus G_j$, in particular for all $y \in B(x, r) \setminus G_j^1$. But $B(x, r) \setminus G_j^1$ is a 1-finely open set containing $x$. Thus we have the result.

Corollary 5.3. Let $u \in N^{1,1}(X)$. Then $u$ is 1-finely continuous at 1-quasi every $x \in X$.

Recall that we understand functions in the class $N^{1,1}(X)$ to be defined everywhere, unlike BV functions that are defined only up to sets of $\mu$-measure zero.

Proof. Since Lipschitz functions are dense in $N^{1,1}(X)$, see [6] or [5, Theorem 5.1], we have $N^{1,1}(X) \subset \text{BV}(X)$, so that Theorem 5.2 applies to $u$. By [21, Theorem 4.1, Remark 4.2], there exists $N \subset X$ with $\text{Cap}_1(N) = \mathcal{H}(N) = 0$ such that every $x \in X \setminus N$ is a Lebesgue point, that is,

$$\lim_{r \to 0^+} \int_{B(x, r)} |u - u(x)| \, d\mu = 0.$$

Thus $u(x) = u^1(x) = \ldots = u^n(x)$ for every such $x$. Assume that the function $y \mapsto \{u^1(y), \ldots, u^n(y)\}$ is 1-finely upper semicontinuous at $x \in X \setminus N$, which is true for $\mathcal{H}$-almost every and thus 1-quasi every point $x \in X$. Let $\varepsilon > 0$. By Theorem 5.2 there exists a 1-finely open set $U \ni x$ such that

$$\min_{l_2 \in \{1, \ldots, n\}} |u^{l_1}(y) - u^{l_2}(x)| < \varepsilon$$

for each $l_1 = 1, \ldots, n$ and all $y \in U$. Then $U \setminus N$ is a 1-finely open set containing $x$, and $|u(y) - u(x)| < \varepsilon$ for all $y \in U$. \qed
Now consider the following. We know (see e.g. [29, Remark 5.9.2]) that if \( u \in L^1(X) \) has a Lebesgue point at \( x \in X \), i.e.

\[
\lim_{r \to 0^+} \int_{B(x, r)} |u - u^\vee(x)| \, d\mu = 0,
\]

then there exists a set \( A_x \ni x \) with density 1 at \( x \), such that \( u^\vee|_{A_x} \) is continuous (instead of \( u^\vee \) we could consider some other pointwise representative). Similarly, by using the analogs of Lebesgue's differentiation theorem for BV functions, see [24, Theorem 5.3], we obtain the following.

**Proposition 5.4.** Let \( u \in BV(X) \). Then for \( H \)-almost every \( x \in X \) there exists a set \( A_x \ni x \) with density 1 at \( x \) such that if \( y_k \to x \) with \( y_k \in A_x \), then

\[
\min_{l_2 \in \{1, \ldots, n\}} |u^{l_1}(y_k) - u^{l_2}(x)| \to 0
\]

for each \( l_1 = 1, \ldots, n \).

From Theorem 5.2 we get the following strengthening of this result.

**Proposition 5.5.** Let \( u \in BV(X) \). Then for \( H \)-almost every \( x \in X \) there exists a set \( A_x \ni x \) with

\[
\lim_{r \to 0^+} \frac{\mathcal{H}_1(B(x, r) \setminus A_x)}{\mu(B(x, r))} = 0
\]

such that if \( y_k \to x \) with \( y_k \in A_x \), then

\[
\min_{l_2 \in \{1, \ldots, n\}} |u^{l_1}(y_k) - u^{l_2}(x)| \to 0
\]

for each \( l_1 = 1, \ldots, n \).

**Proof.** In the proof of Theorem 5.2 we showed that for \( H \)-almost every \( x \in X \), there exists a 1-finely open set \( U \ni x \) such that if \( y_k \to x \) with \( y_k \in U \), then

\[
\min_{l_2 \in \{1, \ldots, n\}} |u^{l_1}(y_k) - u^{l_2}(x)| \to 0
\]

for all \( l_1 = 1, \ldots, n \). By using first Remark 2.2 and then [5, Proposition 6.16], we get for small \( r > 0 \)

\[
\mathcal{H}_1(B(x, r) \setminus U) \leq C \operatorname{Cap}_1(B(x, r) \setminus U) \leq C \operatorname{cap}_1(B(x, r) \setminus U, B(x, 2r)).
\]

Thus we can take \( A_x = U \).
Roughly speaking, if Proposition 5.4 says that the complement of $A_x$ cannot have significant “volume” close to $x$, Proposition 5.5 says that it cannot have significant “surface area” either.

The reason for considering more than two jump values is explained in the following example, which is essentially from [24, Example 5.1].

**Example 5.6.** Consider the one-dimensional space

$$X := \{ x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0 \text{ or } x_2 = 0 \}$$

consisting of the two coordinate axes. Equip this space with the Euclidean metric inherited from $\mathbb{R}^2$, and the 1-dimensional Hausdorff measure. This measure is doubling and supports a $(1, 1)$-Poincaré inequality. Moreover, we can take $\gamma = 1/4$ in (2.8), and then the number of jump values defined in (2.10) is $n = 1/\gamma = 4$. Let

$$u := \chi_{\{x_1 > 0\}} + 2\chi_{\{x_2 > 0\}} + 3\chi_{\{x_1 < 0\}} + 4\chi_{\{x_2 < 0\}}.$$

For brevity, denote the origin $(0, 0)$ by $0$. Now $S_u = \{0\}$ with $\mathcal{H}(\{0\}) = 2$, and $(u^1(0), u^2(0), u^3(0), u^4(0)) = (1, 2, 3, 4)$. The function

$$x \mapsto \{u^1(x), u^2(x), u^3(x), u^4(x)\}$$

is easily seen to be upper hemicontinuous everywhere (even with respect to the metric topology), but this would not be the case if we considered fewer than 4 jump values.

The following very simple example demonstrates that we cannot in general have 1-fine upper hemicontinuity at every point.

**Example 5.7.** Let $X = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ with the Euclidean distance and the 2-dimensional Lebesgue measure $\mathcal{L}^2$. Since we now have $\gamma = 1/2$ in (2.8), see e.g. [2] Theorem 3.59], we only consider the two jump values $u^\wedge$ and $u^\vee$ of a given BV function $u$, recall (2.10). Denoting $x = (x_1, x_2)$, let

$$u(x_1, x_2) = \begin{cases} 
1, & x_1 < 0, \\
0, & x_1 \geq 0, x_2 \leq 0, \\
2, & x_1 \geq 0, x_2 > 0.
\end{cases}$$
Clearly \( u \in \text{BV}(X) \). For brevity, denote the origin \((0,0)\) by 0. Take \( \varepsilon = 1 \). If \( x \mapsto (u^\wedge(x), u^\vee(x)) \) were 1-finely continuous at the origin, there would exist a 1-finely open set \( U \ni 0 \) with
\[
\min\{|u^\wedge(y) - u^\wedge(0)|, |u^\wedge(y) - u^\vee(0)|\} < 1
\]
for all \( y \in U \), so necessarily \( u^\wedge(y) = 0 \) or \( u^\wedge(y) = 2 \) at these points. However, \( U \) must necessarily intersect \( \{y_1 < 0\} \), since
\[
\liminf_{r \to 0^+} r \cap 1_{\{y_1 < 0\} \cap B(0,r)}(B(0,2r)) > 0.
\]
On the other hand, \( u^\wedge(y) = 1 \) for all \( y \in \{y_1 < 0\} \). Thus 1-fine upper hemicontinuity fails at the origin. However, it does hold at every other point, and \( \mathcal{H}(0) = 0 \).

Note that if \( E \subset X \) and \( u = \chi_E \), then \( x \in I_E \) means that \( u^\wedge(x) = u^\vee(x) = 1 \), \( x \in O_E \) means that \( u^\wedge(x) = u^\vee(x) = 0 \), and \( x \in \partial^* E \) means that \( u^\wedge(x) = 0 \) and \( u^\vee(x) = 1 \).

The following example concerning the enlarged rationals illustrates the need to consider upper hemicontinuity with respect to the 1-fine topology instead of the metric topology.

**Example 5.8.** Consider the Euclidean space \( \mathbb{R}^2 \). Let \( \{q_i\}_{i \in \mathbb{N}} \) be an enumeration of \( \mathbb{Q} \times \mathbb{Q} \subset \mathbb{R}^2 \), and define
\[
E := \bigcup_{i \in \mathbb{N}} B(q_i, 2^{-i}).
\]
Clearly \( L^2(E) \leq \pi \). By the lower semicontinuity and subadditivity of perimeter, see [2,4], we can estimate
\[
P(E, \mathbb{R}^2) \leq \sum_{i=1}^{\infty} P(B(q_i, 2^{-i}), \mathbb{R}^2) \leq 2\pi \sum_{i=1}^{\infty} 2^{-i},
\]
so that \( P(E, \mathbb{R}^2) < \infty \), and then also \( \mathcal{H}(\partial^* E) < \infty \). However, \( \partial E = \mathbb{R}^2 \setminus E \). Thus, denoting \( u := \chi_E \), for every \( x \in O_E \) there exists a sequence \( y_k \to x \) with \( y_k \in E \subset I_E \) such that
\[
u^\wedge(y_k) = u^\vee(y_k) = 1 \neq 0 = u^\wedge(x) = u^\vee(x).
\]
Thus at almost every point \( x \in \mathbb{R}^2 \setminus E \), the function \( y \mapsto \{u^\wedge(y), u^\vee(y)\} \) fails to be upper hemicontinuous with respect to the metric topology.
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