GLOBAL STRONG SOLUTION FOR THE INCOMPRESSIBLE FLOW OF LIQUID CRYSTALS WITH VACUUM IN DIMENSION TWO

XIAOLI LI*
College of Science, Beijing University of Posts and Telecommunications
Beijing 100876, China

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Abstract. This paper is devoted to the study of the initial-boundary value problem for density-dependent incompressible nematic liquid crystal flows with vacuum in a bounded smooth domain of \( \mathbb{R}^2 \). The system consists of the Navier-Stokes equations, describing the evolution of an incompressible viscous fluid, coupled with various kinematic transport equations for the molecular orientations. Assuming the initial data are sufficiently regular and satisfy a natural compatibility condition, the existence and uniqueness are established for the global strong solution if the initial data are small. We make use of a critical Sobolev inequality of logarithmic type to improve the regularity of the solution. Our result relaxes the assumption in all previous work that the initial density needs to be bounded away from zero.

1. Introduction. In this paper, we establish the well-posedness of the Ericksen-Leslie model of nematic liquid crystals formulated by Ericksen in [10, 11, 12] and Leslie [20] in the 1960’s. A simplified version of the Ericksen-Leslie model was introduced by Lin [22] and successfully analyzed by Lin-Liu [23–25] who used a modified Galerkin approach, and by Shkoller [31] who relied on a contraction mapping argument coupled with appropriate energy estimates. When the Oseen-Frank energy configuration functional reduces to the Dirichlet energy functional, the hydrodynamic equations of liquid crystals in \( \mathbb{R}^2 \) can be written as follows (see [21, 22, 26]):

\[
\begin{align*}
\rho_t + \nabla \cdot (\rho \mathbf{u}) &= 0, \\
(\rho \mathbf{u})_t + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P &= \mu \nabla \mathbf{u} - \lambda \nabla \cdot (\nabla \mathbf{d} \otimes \nabla \mathbf{d}), \\
\nabla \cdot \mathbf{u} &= 0, \\
d_t + \mathbf{u} \cdot \nabla \mathbf{d} &= \gamma (\Delta \mathbf{d} + |\nabla \mathbf{d}|^2 \mathbf{d}),
\end{align*}
\]

where \( \rho \in \mathbb{R} \) and \( \mathbf{u} \in \mathbb{R}^2 \) represent the density and the velocity field of the flow, \( \mathbf{d} \in \mathbb{S}^2 \) (the unit sphere in \( \mathbb{R}^3 \)) denotes the unit-vector field that represents the macroscopic molecular orientation of the liquid crystal material, \( P \in \mathbb{R} \) is the...
pressure (arising from the usual assumption of incompressibility \( \nabla \cdot \mathbf{u} = 0 \)), and they all depend on the spatial variable \( \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \) and the time variable \( t > 0 \). 

\[ \mu \nabla \mathbf{u} - P I_2 \] 

(2 \times 2 identity matrix) is the Cauchy stress tensor given by Stokes’ law and \( \mu \nabla \mathbf{u} \) stands for the fluid viscosity part of the stress tensor. The term \( \lambda \nabla \cdot (\nabla \mathbf{d} \circ \nabla \mathbf{d}) \) in the stress tensor represents the anisotropic feature of the system. The parameters \( \mu, \lambda, \gamma \) are positive constants that stand for viscosity, the competition between kinetic energy and potential energy, and microscopic elastic relaxation time or the Deborah number for the molecular orientation field, respectively. The symbol \( \nabla \mathbf{d} \circ \nabla \mathbf{d} \) denotes the \( 2 \times 2 \) matrix whose \((i, j)-\)th entry is \( \partial_i \mathbf{d} \cdot \partial_j \mathbf{d} \) for \( 1 \leq i, j \leq 2 \), and it is easy to see that \( \nabla \mathbf{d} \circ \nabla \mathbf{d} = (\nabla \mathbf{d})^T \nabla \mathbf{d} \), where \( (\nabla \mathbf{d})^T \) denotes the transpose of the matrix \( \nabla \mathbf{d} \).

Given the nature of liquid crystals, particularly the fact that these liquids are anisotropic, the occurrence of body couples and couple stresses in the Ericken-Leslie equations makes the theory somewhat unusual in continuum mechanics. Mathematically, it is the presence of couple stress that leads to the special coupling of the simplified system \( \{1\} \). The equations in \( \{1\} \) represent the conservation of mass, the conservation of momentum, the incompressibility of the fluid and the transported heat flow of harmonic maps into \( \mathbb{S}^2 \), respectively. The system was derived from the macroscopic continuum point of view to describe the time evolution of the materials under the influence of the coupling between the velocity field \( \mathbf{u}(\mathbf{x}, t) \), and the macroscopic description of the microscopic orientation configurations \( \mathbf{d}(\mathbf{x}, t) \) of (rod-like) liquid crystals. The system also exhibits the property of the anisotropy of the liquid crystal material which is exhibited in \( \{1d\} \) and its nonlinear coupling in \( \{1b\} \). The flow velocity does disturb the alignment of the molecules. More importantly, the converse is also true, that is, a change in the alignment will induce velocity. This velocity will in turn affect the time evolution of the director field. In this process, we cannot assume that the velocity field will remain small even when we start with zero velocity field.

As for the nonlinear constraint \( \mathbf{d} \in \mathbb{S}^2 \), i.e., \( |\mathbf{d}| = 1 \), one of the methods used to relax it is to consider a form of penalization, that is, not \( |\nabla \mathbf{d}|^2 \) in \( \{1d\} \), but the Ginzburg-Landau approximation \( \frac{1}{\varepsilon^2}(1-|\mathbf{d}|^2) \) for small \( \varepsilon \). For the homogeneous case (i.e. \( \rho \equiv 1 \)), Lin-Liu [21] proved local existence of the classical solutions the global existence of weak solutions

\[ (\mathbf{u}, \mathbf{d}) \in \big( L^2(0, T; H^1(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \big) \times \big( L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)) \big) \]

for all \( T \in (0, \infty) \) with large initial data under the assumption that \( \mathbf{u}_0 \in L^2(\Omega) \), \( \mathbf{d}_0 \in H^1(\Omega) \) with \( \mathbf{d}_0 |_{\partial \Omega} \in H^\frac{1}{2}(\partial \Omega) \) in the two-dimensional and three-dimensional cases. For any fixed \( \varepsilon \), the existence and uniqueness of global classical solution was also obtained if \( u_0 \in H^1(\Omega) \) and \( d_0 \in H^2(\Omega) \) either for dimension two or for dimension three when the fluid viscosity \( \mu \) is large enough. Taking the limit of \( \varepsilon \rightarrow 0 \) after discussing the cases for each \( \varepsilon \). This technique has been successfully used in a lot of other places (see [24, 25]). It also fits well with the general theory of Landau’s order parameter (see [20]). The partial regularity of the weak solution was investigated in [25] (and also [9, 28]), similar to the classical theorem by Caffarelli-Kohn-Nirenburgh [3] on the Navier-Stokes equations that asserts the one-dimensional parabolic Hausdorff measure of the singular set of any suitable weak solution is zero. For the density-dependent (nonhomogeneous) case, Liu [27] proved the global existence of weak solutions and classical solutions to the system of Smectic-A liquid crystals under the general condition of the initial density \( \rho_0 \) satisfying \( 0 < \alpha \leq \rho_0 \leq \beta \). The global existence of weak solutions in dimension
three was established by Liu-Zhang [29] if \( \rho_0 \in L^2(\Omega) \). Later Jiang-Tan [17] pointed out that the condition on initial density can be weaken to belong to \( L^\gamma(\Omega) \) for any \( \gamma \geq \frac{3}{2} \). With the Ginzburg-Landau penalty function, the global weak solution and large-time behavior to the compressible flow of liquid crystals were obtained in [33]. However, they cannot get the estimates uniformly for \( \varepsilon \), and therefore cannot take the limit \( \varepsilon \to 0 \).

In this paper, we are interested in the existence and uniqueness of global strong solution \( (\rho, u, d, P) \) of the initial-boundary value problem of system (1) in a bounded smooth domain \( \Omega \subset \mathbb{R}^2 \) with the initial and boundary conditions:

\[
(\rho(x,0), u(x,0), d(x,0)) = (\rho_0(x), u_0(x), d_0(x)), \quad x \in \Omega, \tag{2}
\]

\[
(u(x,t), d(x,t)) = (0, d_0(x)), \quad (x,t) \in \partial \Omega \times (0, +\infty), \tag{3}
\]

with \( d_0 \in \mathbb{S}^2 \) being given with compatibility, \( \nabla \cdot u_0 = 0 \) in \( \Omega \) and \( d_0 \in C^1(\Omega) \) satisfying \( \nabla d_0 = 0 \) on the boundary \( \partial \Omega \) (see [15]). Note that the Dirichlet boundary condition for the velocity field implies non-slip on the boundary arising from viscous flows are known to stick to the kinematic boundary.

Compared with the Ginzburg-Landau approximation problem, \( |\nabla d|^2 \) in [14] brings us some new difficulties. Since the strong solutions of a harmonic map must be blowing up at finite time (see Chang-Ding-Ye [4] for the heat flow of harmonic maps), one cannot expect that there exists a global strong solution to system (1)-(3) with general initial data. For the homogeneous case of system (1)-(3), both the regularity and existence of global weak solutions were established by Lin-Lin-Wang [26]. More explicitly, they obtained both interior and boundary regularity theorem for such a flow under smallness conditions, and the existence of global weak solutions that are smooth away from at most finitely many singular times in any bounded smooth domain of \( \mathbb{R}^2 \). The reader is referred to Theorems 1.2, 1.3 in [26] for the details. Meanwhile, Hong [14] also showed the global existence of weak solution to this system in two dimensional space. Wang [32] established a global well-posedness theory for the incompressible liquid crystals for rough initial data, provided that \( \|u_0\|_{BMO^{-1}} + \|d_0\|_{BMO} \leq \varepsilon_0 \) for some \( \varepsilon_0 > 0 \). Assuming that the initial density \( \rho_0 \) has a positive bound from below and under smallness conditions on the initial data, Wen-Ding [30] got the global existence and uniqueness of the strong solution to (1)-(3) in Sobolev spaces in dimension two. For nonhomogeneous Navier-Stokes equations with initial density being bounded from above and below by some positive constants, Paicu-Zhang-Zhang [30] established the global well-posedness of solution when \( u_0 \in H^s(\mathbb{R}^2) \) for \( s > 0 \) in dimension two, or \( u_0 \in H^1(\mathbb{R}^3) \) satisfying \( \|u_0\|_{L^2} \|\nabla u_0\|_{L^2} \) being sufficiently small in dimension three. This result improves the work in [3], which required \( u_0 \in H^2(\mathbb{R}^d) \) for the existence and uniqueness of local solution, and a smallness condition prescribed on the fluctuation of the initial density for the global solution.

Throughout this paper, we denote \( L^q = L^q(\Omega) \), \( W^{k,q} = W^{k,q}(\Omega) \) and \( H^k = W^{k,2} \) and introduce

\[
H_0^1(\Omega) = \text{closure of } C_0^\infty(\Omega, \mathbb{R}^2) \text{ in the norm } \left( \int_\Omega |
abla v|^2 \, dx \right)^{\frac{1}{2}},
\]

\[
H = \text{closure of } C_0^\infty(\Omega, \mathbb{R}^2) \cap \{ v : \nabla \cdot v = 0 \} \text{ in } H_0^1(\Omega).
\]

We make the assumptions:

\[
0 \leq \rho_0 \in H^2, \quad u_0 \in H \cap (H^2)^2, \quad \nabla d_0 \in (H^2)^6. \tag{4}
\]
Observing the system (1) has the following scaling invariant property. If \( \rho(x,t), u(x,t), d(x,t) \) and \( P(x,t) \) solve (1), then, for each \( l > 0 \),
\[
\rho_l = \rho(lx,lt^2), \quad u_l = lu(lx,lt^2), \quad d_l = d(lx,lt^2) \quad \text{and} \quad P_l = l^2 P(lx,lt^2)
\]
also solve (1). One can encode this property by assigning a scaling dimension to each quantity as follows: each \( x_i \) has dimension 1, time variable \( t \) has dimension 2, \( \rho \) has dimension 0, \( u \) has dimension \(-1\), \( d \) has dimension 0, \( P \) has dimension \(-2\), \( \partial x_i \) has dimension \(-1\) and \( \partial_t \) has dimension \(-2\). This dimension analysis of equations enable us to build up various quantities, so called “energy”’s, associated with the solutions of (1) and is very important to the discussion of the partial regularity property of the solutions.

Note that \( \nabla \cdot (\nabla d \odot \nabla d) = \nabla \cdot ((\nabla d)^T \nabla d) = \nabla (|\nabla d|^2/2) + (\nabla d)^T \Delta d \), by using the fact \( \nabla \cdot u = 0 \) in \( \Omega \), we rewrite (1) as
\[
\rho_t + u \cdot \nabla \rho = 0, \tag{5a}
\]
\[
\rho u_t + \rho u \cdot \nabla u + \nabla P = \mu \Delta u - \lambda \nabla (|\nabla d|^2/2) - \lambda (\nabla d)^T \Delta d, \tag{5b}
\]
\[
\nabla \cdot u = 0, \tag{5c}
\]
\[
d_t + u \cdot \nabla d = \gamma (\Delta d + |\nabla d|^2 d). \tag{5d}
\]

The paper is written in the following way. In Section 2, we recall three useful lemmas and state our main result. In Section 3, we establish the basic energy law governing the system and employ the method that involves using higher order energy estimates to establish that the local strong solution does not blow up in finite time. In other words, the local strong solution can be extended to be a global one. And, the proof of the uniqueness property for the strong solutions is a standard procedure.

Throughout of the whole paper, sometimes, we make use of \( A \lesssim B \) in place of \( A \leq C_0 B \), where \( C_0 \) stands for a “harmless” constant whose exact meaning depends on the context, and \( A \approx B \) means that \( A \lesssim B \) and \( B \lesssim A \).

2. Useful lemmas and the main results. The following result is quite standard in dimension two (as a matter of fact, it is a straightforward generalization of the one presented in [19], also see Lemma 2.4 in [31] and Ladyzenskaya [18]):

**Lemma 2.1.** If \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \), then the following inequality is true for every function \( f \in H^1 \):
\[
\|f\|_{L^4} \leq C(\|f\|_{L^1} + \|\nabla f\|_{L^2} \|f\|_{L^2}^2),
\]
where the constant \( C \) depends only on \( \Omega \). Especially, if \( f \big|_{\partial \Omega} = 0 \), then
\[
\|f\|_{L^4} \leq 2 \|\nabla f\|_{L^2} \|f\|_{L^2}^2.
\]

Next, let us recall the classical regularity theory for Stokes equations (see [13] for its proof):

**Lemma 2.2.** Assume that \((u,P) \in H \times H^1\) is a weak solution of the stationary Stokes problem
\[
\begin{cases}
-\Delta u + \nabla P = F, \quad \text{in} \ \Omega, \\
\nabla \cdot u = 0, \quad \text{in} \ \Omega, \\
u \big|_{\partial \Omega} = 0,
\end{cases}
\]
and \( F \in (L^q)^2 \), \( 1 < q < \infty \), then it holds that
\[
\|u\|_{W^{2,q}} \leq C(\|F\|_{L^q} + \|u\|_{H^1}),
\]
where the constant \( C \) depends on \( \Omega \) and \( q \). Moreover, if \( F \in (H^1)^2 \), then
\[
\|u\|_{H^3} \leq C(\|F\|_{H^1} + \|u\|_{H^1}),
\]
where the constant \( C \) depends only on \( \Omega \).

Finally, we state a critical Sobolev inequality of logarithmic type, which is originally due to Brezis-Wainger [2]. The reader is referred to Section 2 in [16] for the proof.

**Lemma 2.3.** Assume \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^2 \) and \( f \in L^2(s,t;H^1) \cap L^2(s,t;W^{1,q}) \) with some \( q > 2 \) and \( 0 \leq s < t \leq \infty \). Then it holds that
\[
\|f\|_{L^2(s,t;L^\infty)} \leq C \left( 1 + \|f\|_{L^2(s,t;H^1)} \right)^{\frac{1}{q} \left( \ln^+ \|f\|_{L^2(s,t;W^{1,q})} \right)^{\frac{1}{2}}},
\]
where the constant \( C \) depends only on \( \Omega \) and \( q \), and is independent of \( s,t \).

Our main result about the global existence of a unique strong solution for the initial boundary value problem reads:

**Theorem 2.4.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain. Assume the initial data \((\rho_0, u_0, d_0)\) satisfies the regularity [4] and the compatibility condition
\[
-\mu \Delta u_0 + \nabla p_0 + \lambda \nabla : (\nabla d_0 \otimes \nabla d_0) = \rho_0^\frac{1}{2} g, \quad \forall \, x \in \Omega
\]
with some \((p_0, g) \in H^1 \times (L^2)^2\). Furthermore, suppose that there exists a sufficiently small constant \( \delta > 0 \), such that
\[
\int_{\Omega} \left( \frac{1}{\rho_0^\delta} |u_0|^2 + |\nabla d_0|^2 \right) \, dx \leq \delta,
\]
then system (1)-(3) has a unique global strong solution \((\rho, u, d, P)\), with
\[
0 \leq \rho \in C([0, \infty); H^2), \quad \rho_t \in L^\infty_{\text{loc}}(0, \infty; L^2),
\]
\[
u \in C([0, \infty); H \cap (H^2)^2), \quad u_t \in L^2_{\text{loc}}(0, \infty; H), \quad \sqrt{\rho} u_t \in L^\infty_{\text{loc}}(0, \infty; (L^2)^2),
\]
\[
\nabla d \in C([0, \infty); (H^1_0)^6 \cap (H^2)^6), \quad \nabla d_t \in L^2_{\text{loc}}(0, \infty; (H^1_0)^6) \cap L^\infty_{\text{loc}}(0, \infty; (L^2)^6),
\]
\[P \in C([0, \infty); H^1) \cap L^2_{\text{loc}}(0, \infty; H^2) \quad \text{and} \quad |d| = 1 \, a.e. \, (x, t) \in \Omega \times [0, \infty).
\]

**Remark 1.** We would like to point out if \( d \) is a constant map, then [1] reduces to the nonhomogeneous Navier-Stokes equations. A straightforward application of Theorem 2.4 is that there exists a unique global strong solution enjoying the above regularity property for the 2D density-dependent Navier-Stokes system with vacuum, which erases the assumption that the initial density is strictly positive (see [H]).
3. Proof of Theorem 2.4. Note that the local existence of a unique strong solution with vacuum to \((1)-(3)\) in a bounded domain of \(\mathbb{R}^2\) can be established following the procedure of Wen-Ding [34] or Choe-Kim [5]. Therefore, this paper is devoted to showing global estimates for the density, velocity and the molecule orientational direction vector.

We give some \textit{a priori} estimates globally in time to get Theorem 2.4. The method here is to find a new energy inequality involving the following quantities:

\[
A(T) = \sup_{0 \leq t \leq T} \left( \|\rho(t)\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \|\nabla d(t)\|_{L^2}^2 \right) + \|\sqrt{\rho}u\|_{L^\infty(0,T;L^2)}^2 \\
+ \int_0^T \left( \|u(t)\|_{H^3}^2 + \|\nabla d(t)\|_{H^3}^2 \right) dt + \int_0^T \left( \|u_t(t)\|_{H^1}^2 + \|\nabla d_t(t)\|_{H^1}^2 \right) dt.
\]

Suppose \(0 < T^* < \infty\) is the maximum time for the existence of strong solution to \((1)-(3)\). In other words, \((\rho, u, d, P)\) is a strong solution to \((1)-(3)\), in \(\Omega \times (0, T]\) for any \(0 < T < T^*\), but not a strong solution in \(\Omega \times (0, T^*]\). The main aim is to show that under the assumption of Theorem 2.4, there is a bound \(B > 0\) depending only on the initial data and \(T^*\) such that

\[
\sup_{0 \leq t \leq T^*} A(T) \leq B. \tag{6}
\]

We can then show without much difficulty that \(T^*\) is not the maximum time provided (6) holds, i.e., we can extend the strong solution beyond \(T^*\), which is the desired contradiction. With the regularities of the solution we get, the proof of the uniqueness by the energy estimate of the difference between two different solutions is straightforward (see Step 4 in [34]).

Now we outline the proof of Theorem 2.4 into several steps:

\textbf{Step 1. \((L^\infty\) bound for \(\rho\)).} As (5a) is a transport equation for \(\rho\), it follows easily from Proposition 3.1 in [7] that

\[
\|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}, \quad \forall t \in (0, T^*). \tag{7}
\]

\textbf{Step 2. \textit{(Basic energy inequality).}} We derive the energy inequality: multiplying (1b) by \(u\), integrating over \(\Omega\), then multiplying (1d) by \((\Delta d + \nabla d^2 d)\) and integrating over \(\Omega\) and finally, by adding two results above and using (1a), noticing that

\[
\int_{\Omega} \nabla P \cdot u \, dx = \int_{\Omega} \nabla \left( \frac{|\nabla d|^2}{2} \right) \cdot u \, dx = 0,
\]

that, by taking advantage of \(|d| = 1\),

\[
(d_t + u \cdot \nabla d) \cdot |\nabla d|^2 d = \frac{1}{2} \left( |\nabla d|^2 d_t^2 + u \cdot \nabla |d|^2 |\nabla d|^2 \right) = 0,
\]

and that, as \(d_t = 0\) on \(\partial \Omega\),

\[
\int_{\Omega} d_t \cdot \Delta d \, dx = -\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla d|^2 \, dx,
\]

we get the identity

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} (\rho|u|^2 + \lambda |\nabla d|^2) \, dx + \int_{\Omega} (\mu|\nabla u|^2 + \lambda\gamma|\Delta d + |\nabla d|^2 d|^2) \, dx = 0,
\]
for all \( t \in (0, T^*) \), which implies

\[
\int_{\Omega} (\rho|u|^2 + \lambda|\nabla d|^2) \, dx + 2 \int_0^T \int_{\Omega} (\mu|\nabla u|^2 + \lambda \gamma |\Delta d|^2 + |\nabla d|^2|d|^2) \, dx \, dt
\]

\[
= \int_{\Omega} (\rho_0|u_0|^2 + \lambda|\nabla d_0|^2) \, dx,
\]

(8)

for a.a. \( t \in (0, T) \), \( 0 < T < T^* \).

**Step 3. (Estimates for \( \|(\sqrt{\rho}u_t, \nabla d_t)\|_{L^2(0,T;L^2)} \) and \( \|(\nabla u, \Delta d)\|_{L^\infty(0,T;L^2)} \)).**

Multiplying (5b) by \( u_t \) and integrating over \( \Omega \), it yields

\[
\frac{\mu}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} \rho|u_t|^2 \, dx = -\int_{\Omega} \rho u \cdot \nabla u \cdot u_t \, dx - \lambda \int (\nabla d)^T \Delta d \cdot u_t \, dx.
\]

(9)

Let us estimate the terms on right-hand of (9). By virtue of Hölder’s inequality and Young’s inequality, we have

\[
|\int_{\Omega} \rho u \cdot \nabla u \cdot u_t \, dx| \leq \frac{1}{2} \|\sqrt{\rho}u_t\|_{L^2}^2 + C\|u\|_{L^\infty} \|\nabla u\|_{L^2}^2,
\]

where (7) is used, and

\[
-\int (\nabla d)^T \Delta d \cdot u_t \, dx = -\frac{d}{dt} \int (\nabla d)^T \Delta d \cdot u \, dx + \int (\nabla d_t)^T \Delta d \cdot u \, dx
\]

\[
= -\int (\nabla d_t \cdot \nabla) (\nabla d) \cdot u \, dx - \int (\nabla d)^T \nabla d_t : \nabla u \, dx
\]

\[
\leq -\frac{d}{dt} \int (\nabla d)^T \Delta d \cdot u \, dx + C\|\Delta d\|_{L^2} \|u\|_{L^\infty}^2
\]

\[
+ C\|\nabla d\|_{L^\infty} \|\nabla u\|_{L^2} + \frac{1}{2}\|\nabla d_t\|_{L^2}^2,
\]

where

\[
\nabla d_t \cdot \nabla = \begin{pmatrix} \nabla d_{1t} \cdot \nabla \\ \nabla d_{2t} \cdot \nabla \\ \nabla d_{3t} \cdot \nabla \end{pmatrix} = \begin{pmatrix} \sum \partial_i d_{1t} \partial_i \\ \sum \partial_i d_{2t} \partial_i \\ \sum \partial_i d_{3t} \partial_i \end{pmatrix}, \quad d = (d_1, d_2, d_3)^T.
\]

Here we have used \( u |_{\partial \Omega} = 0 \), and the fact that \( \|\Delta d\|_{L^2} \approx \|\nabla^2 d\|_{L^2} \) given \( \nabla d_0 = 0 \) on the boundary \( \partial \Omega \).

Finally, we conclude

\[
\frac{\mu}{2} \frac{d}{dt} \int_{\Omega} |\nabla u|^2 \, dx + \frac{\lambda}{2} \frac{d}{dt} \int (\nabla d_t)^T \Delta d_t \cdot u_t \, dx + \frac{1}{2} \int_{\Omega} \rho |u_t|^2 \, dx
\]

\[
\leq C\|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2 + C\lambda \|u\|_{L^\infty} \|\Delta d\|_{L^2}^2 + C\lambda \|\nabla u\|_{L^2} \|\nabla d\|_{L^\infty} + \frac{\lambda}{2} \|\nabla d_t\|_{L^2}^2.
\]

(10)

Similarly, multiplying \( \nabla d_t \) by \( \Delta d_t \), integrating over \( \Omega \), using Hölder’s inequality and Young’s inequality, we obtain, bearing in mind the facts that \(|d| = 1\), \( u = 0 \) and
∇d₀ = 0 on ∂Ω, that
\[ γ \frac{d}{dt} \int_{Ω} |Δd|^2 \, dx + \int_{Ω} |∇d|^2 \, dx \]
\[ = \int_{Ω} u \cdot ∇d \cdot Δd \, dx - γ \int_{Ω} |∇d|^2 d \cdot Δd \, dx \]
\[ ≤ \frac{1}{2} |∇d|_L^2 + C (\|u\|_{L^∞} \|Δd\|_{L^2}^2 + \|∇d\|_{L^∞}^2 (\|∇u\|_{L^2}^2 + γ (\|Δd\|_{L^2}^2 + \|Δd\|_{L^2}^2))) \cdot \]

Next, taking advantage of Lemma 2.1, we have
\[ \frac{d}{dt} \int_{Ω} |Δd|^2 \, dx + \int_{Ω} |∇d|^2 \, dx \]
\[ ≤ (\|u\|_{L^∞} + (\|∇d\|_{L^2}^2 + 1)\|∇d\|_{L^∞}^2) \|Δd\|_{L^2}^2 + \|∇u\|_{L^2}^2 \|∇d\|_{L^∞}^2. \]

Setting
\[ E(t) = \int_{Ω} (\rho|u|^2 + λ|∇d|^2) \, dx, \quad E_0 = \int_{Ω} (\rho_0|u_0|^2 + λ|∇d_0|^2) \, dx, \]
then, 8 implies that for any \(0 < t \leq T\),
\[ E(t) + \int_{0}^{t} \int_{Ω} (\rho|u|^2 + λ|∇d|^2) \, dx \, dt \leq E_0. \] (12)

Multiplying (11) by \(\tilde{C}E_0 + λ\), integrating the result with respect to time, we have, for every \(0 \leq s < T < T^∗\),
\[ (\tilde{C}E_0 + λ) \int_{Ω} |Δd(T)|^2 \, dx + (\tilde{C}E_0 + λ) \int_{0}^{T} \int_{Ω} |∇d|^2 \, dx \, dt \]
\[ ≤ (\tilde{C}E_0 + λ) \int_{Ω} |Δd(s)|^2 \, dx \]
\[ + C_0(\tilde{C}E_0 + λ) \int_{0}^{T} (\|u\|_{L^∞}^2 + (\|∇d\|_{L^2}^2 + 1)\|∇d\|_{L^∞}^2)\|Δd\|_{L^2}^2 + \|∇u\|_{L^2}^2 \|∇d\|_{L^∞}^2 \, dt, \] (13)

where \(C_0\) is the constant included in (11).

Integrating (10) with respect to time, for every \(0 \leq s < T < T^∗\), we have
\[ \frac{μ}{2} \int_{Ω} |∇u(T)|^2 \, dx + λ \int_{Ω} ((∇d) \cdot u)(T) \, dx + \frac{1}{2} \int_{s}^{T} \int_{Ω} ρ|u|^2 \, dx \, dt \]
\[ ≤ \frac{μ}{2} \int_{Ω} |∇u(s)|^2 \, dx + λ \int_{Ω} ((∇d) \cdot u)(s) \, dx \]
\[ + \int_{s}^{T} (C\|u\|_{L^∞}^2 \|∇u\|_{L^2}^2 + C\lambda\|u\|_{L^∞}^2 \|Δd\|_{L^2}^2 + C\lambda\|∇u\|_{L^2}^2 \|∇d\|_{L^∞}^2 + \frac{λ}{2} \|∇d\|_{L^2}^2) \, dt. \] (14)

Noticing that, using integration by parts, one can easily get
\[ \int_{Ω} \sum_{i,j} ∂_i d_j d_j u_i \, dx \]
\[ = - \int_{Ω} \sum_{i,j,k} ∂_i d_j ∂_j d_k u_i \, dx - \int_{Ω} \sum_{i,j,k} ∂_k d_j ∂_j d_k u_i \, dx \]
\[ = - \int_{Ω} \sum_{i,j,k} ∂_i d_j ∂_j d_k u_i \, dx - \int_{Ω} \sum_{i,j,k} ∂_i (\frac{|∂_kd_j|^2}{2}) u_i \, dx \]
which means, by using Lemma 2.1 that

$$\left| \int (\nabla d)^T \Delta d \cdot u \, dx \right| \leq \frac{\mu}{4\lambda} \|\nabla u\|_{L^2}^2 + \tilde{C} \|\nabla d\|_{L^2}^2 \|\Delta d\|_{L^2}^2. \tag{16}$$

Finally, taking advantage of the estimates (13), (14) and (16), we conclude that, for every $0 \leq s < T < T^*$,

$$\begin{align*}
\frac{\mu}{4} \|\nabla u(T)\|_{L^2}^2 + (\tilde{C}E_0 + \lambda) \|\Delta d(T)\|_{L^2}^2 + \int_s^T \left( \frac{1}{2} \|\sqrt{\rho} u\|_{L^2}^2 + (\tilde{C}E_0 + \lambda \tilde{C} \|\nabla d(s)\|_{L^2}^2) \right) dt \\
\leq \frac{3\mu}{4} \|\nabla u(s)\|_{L^2}^2 + (\tilde{C}E_0 + \lambda \tilde{C} \|\nabla d(s)\|_{L^2}^2) \|\Delta d(s)\|_{L^2}^2 + \lambda \tilde{C} \|\nabla d(T)\|_{L^2}^2 \|\Delta d(T)\|_{L^2}^2 \\
+ \int_s^T \left( (\tilde{C}E_0 + \lambda \tilde{C}) 0 \|\nabla d\|_{L^2}^2 \right) dt \\
+ \int_s^T \left( \lambda \tilde{C} \|\nabla u\|_{L^\infty}^2 + (\tilde{C}E_0 + \lambda \tilde{C} \|\nabla d\|_{L^2}^2) \right) \|\Delta d\|_{L^2}^2 dt,
\end{align*}$$

where $C_0$ is the constant included in (11).

Recalling the energy balance (8) that $\sup_t \|\nabla d\|_{L^2}^2 \leq \frac{1}{\lambda} E_0$, one can easily from inequality (17) to get

$$\begin{align*}
\|\nabla u(T)\|_{L^2}^2 + \|\Delta d(T)\|_{L^2}^2 + \int_s^T (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) dt \\
\lesssim (\|\nabla u(s)\|_{L^2}^2 + \|\Delta d(s)\|_{L^2}^2) \exp \left\{ C \int_s^T (\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) dt \right\} + 1, \tag{18}
\end{align*}$$

here $C$ depends on $\tilde{C}, C_0, E_0, \lambda$ and the regularity parameters.

Denote

$$\Psi(t) = e + \sup_{0 \leq \tau \leq t} (\|u(\tau)\|_{H^1_0}^2 + \|\nabla d(\tau)\|_{H^1_0}^2) + \int_0^t (\|\sqrt{\rho} u\|_{L^2}^2 + \|\nabla d\|_{L^2}^2) \, dr,$$

by the scaling argument used in Section 1, we can see the new energy $\|\nabla u(t)\|_{L^2}^2 + \|\Delta d(t)\|_{L^2}^2$ contained in $\Psi(t)$ is of higher order than $E(t)$. Higher order estimates of the density, velocity and the molecule orientational can be done in a standard way once $\|u, \nabla d\|_{H^1}$ is uniformly bounded with respect to time. Now, for every $0 \leq s < T < T^*$, one can easily from (12) and (18) to get

$$\Psi(T) \lesssim \Psi(s) \exp \left\{ C \int_s^T (\|u\|_{L^\infty}^2 + \|\nabla d\|_{L^\infty}^2) \, dt \right\}. \tag{19}$$

To continue, we need to get estimates on the norms $\|u\|_{L^2(s,T;L^\infty)}$ and $\|\nabla d\|_{L^2(s,T;L^\infty)}$. From Lemma 2.3, we have

$$\begin{align*}
\|u\|_{L^2(s,T;L^\infty)}^2 + \|\nabla d\|_{L^2(s,T;L^\infty)}^2 \\
\lesssim 1 + (\|u\|_{L^2(s,T;H^1)}^2) \left( \ln^+ \|u\|_{L^2(s,T;W^{1,4})} + \ln^+ \|\nabla d\|_{L^2(s,T;W^{1,4})} \right), \tag{20}
\end{align*}$$

Applying the operator $\nabla$ to (5d), we get

$$\gamma \Delta (\nabla d) = \nabla d_t + \nabla (u \cdot \nabla d) - \gamma (\nabla |d|^2) \cdot d. \tag{21}$$
It follows from the classical regularity theory for Stokes equations, as indicated by Lemma 2.2 and for elliptic equations that
\[
\|u\|_{W^{2,4}} \lesssim \|u\|_{H_0^1} + \|\nabla u - \nabla(\frac{\nabla d}{2}) - (\nabla d)\Delta d\|_{L^4},
\]
and
\[
\|\nabla d\|_{W^{2,4}} \lesssim \|\nabla d\|_{H_0^1} + \|\nabla d_i + \nabla(u \cdot \nabla d) - \nabla(\nabla d^2 d)\|_{L^4}.
\]
By virtue of the Sobolev imbedding
\[
W^{2,4}(\Omega) \hookrightarrow W^{1,4}(\Omega)
\]
for \(n = 2\), the Hölder inequality and (7), we have
\[
\|u\|_{L^2(s,T;W^{1,4})} \lesssim \|u\|_{L^2(s,T;H_0^1)} + \|\sqrt{\rho}u\|_{L^2(s,T;L^2)} + \|\nabla d\|_{L^2(s,T;H_0^1)}\|\Delta d\|_{L^\infty(s,T;L^2)}
\]
and
\[
\|\nabla d\|_{L^2(s,T;W^{1,4})} \lesssim \|\nabla d\|_{L^2(s,T;H_0^1)} + \|\nabla d_i\|_{L^2(s,T;L^2)} + \|\nabla u\|_{L^\infty(s,T;L^2)}\|\nabla d\|_{L^2(s,T;H_0^1)}
\]
\[
+ \|u\|_{L^2(s,T;H_0^1)}\|\Delta d\|_{L^\infty(s,T;L^2)}
\]
\[
+ \|\nabla d\|_{L^2(s,T;H_0^1)}(1 + \|\nabla d\|_{L^\infty(s,T;L^2)})\|\Delta d\|_{L^\infty(s,T;L^2)}.
\]
Here we have used Lemma 2.1 to estimate \(\|\nabla d\|_{L^\infty(s,T;L^4)}\).

Now one need to get the estimate for \(\|\Delta d\|_{L^2}\) at first. In fact, Since
\[
\|\Delta d\|^2 \leq 2\|\Delta d + |\nabla d|^2 d|^2 + |\nabla d|^4,
\]
then
\[
\|\Delta d\|_{L^2}^2 \leq 2\|\Delta d + |\nabla d|^2 d|^2 + |\nabla d|_{L^2}^4,
\]
\[
\lesssim \|\Delta d + |\nabla d|^2 d|^2 + |\Delta d|^2_{L^2} + |\nabla d|_{L^2}^2.
\]
Recalling the energy balance (8) and bearing in mind that \(\sup_t \|\nabla d\|_{L^2}^2 \leq \frac{1}{\lambda} E_0\), we have
\[
\|\Delta d\|_{L^2}^2 \lesssim \|\Delta d + |\nabla d|^2 d|^2 + \frac{1}{\lambda} E_0 \|\Delta d\|_{L^2}^2.
\]
If there exists a constant \(\delta > 0\) such that \(\frac{1}{\lambda} E_0 \leq \delta\) and \(\tilde{C}_0 \delta < 1\), where \(\tilde{C}_0\) is the constant included in (24), then
\[
\|\Delta d\|_{L^2}^2 \lesssim \|\Delta d + |\nabla d|^2 d|^2\|_{L^2}.
\]
Making use of (22), (23) and (8), we have, from (20), for every \(0 \leq s < T < T^*\),
\[
\|u\|_{L^2(s,T;L^\infty)} + \|\nabla d\|_{L^2(s,T;L^\infty)} \leq C_1 \left(1 + (\|u\|_{L^2(s,T;H^1)} + \|\nabla d\|_{L^2(s,T;H^1)}) \ln(C(E_0, T^*)\Psi(T))\right).
\]
Substituting (25) into (19), one has
\[
\Psi(T) \lesssim \Psi(s) (C(E_0, T^*)\Psi(T)^2)^{C_1}(\|u\|_{L^2(s,T;H^1)} + \|\nabla d\|_{L^2(s,T;H^1)}).
\]
Using the energy balance (8) and choosing \(s\) close enough to \(T^*\) such that
\[
\lim_{T \to T^*} C_1(\|u\|_{L^2(s,T;H^1)} + \|\nabla d\|_{L^2(s,T;H^1)}) \leq \frac{1}{4},
\]
it follows from (26) that $\Psi(T)$ bounded, and which implies
\[
\sup_{0<T<T^*} \left\{ \| (u(T), \nabla d(T)) \|_{H^1}^2 + \int_0^T \| (\sqrt{\rho}u_t, \nabla d_t) \|_{L^2}^2 dt \right\} < \infty. \tag{27}
\]

**Step 4. (Estimates for $\| (u, \nabla d) \|_{L^2(0,T;H^2)}$ and $\| (u, \nabla d) \|_{L^1(0,T;L^\infty)}$).** Having (27) at hand, assume that
\[
\sup_{0<T<T^*} \left\{ \| (u(T), \nabla d(T)) \|_{H^1}^2 + \int_0^T \| (\sqrt{\rho}u_t, \nabla d_t) \|_{L^2}^2 dt \right\} \leq C_2.
\]

With the help of Lemmas 2.1, 2.2 and the regularity theory for elliptic equations, we have
\[
\| u \|_{H^2} \lesssim \| u \|_{H^1} + \| -\rho u_t - \rho \rho \nabla - \nabla (\frac{|\nabla d|^2}{2} - (\nabla d)^T \Delta d) \|_{L^2} \\
\lesssim \| u \|_{H^1} + \| \sqrt{\rho}u_t \|_{L^2} + \| u \|_{L^\infty} \| \nabla u \|_{L^2} + \| \nabla d \|_{L^\infty} \| \Delta d \|_{L^2},
\]
and
\[
\| \nabla d \|_{H^2} \lesssim \| \nabla d \|_{H^1} + \| \nabla d_t + \nabla (u \cdot \nabla d) - \nabla (|\nabla d|^2 d) \|_{L^2} \\
\lesssim \| \nabla d \|_{H^1} + \| \nabla d_t \|_{L^2} + \| u \|_{L^\infty} \| \Delta d \|_{L^2} + \| \nabla u \|_{L^2} \| \Delta d \|_{L^\infty} \\
+ \| \nabla d \|_{L^\infty} (1 + \| \nabla d \|_{L^2}) \| \Delta d \|_{L^2}.
\]

Now, taking (8) into account and using Gagliardo-Nirenberg inequality
\[
\| u \|_{L^\infty} \lesssim \| u \|_{L^2}^{\frac{1}{2}} \| u \|_{H^2}^{\frac{1}{2}}, \quad \| \nabla d \|_{L^\infty} \lesssim \| \nabla d \|_{L^2}^{\frac{1}{2}} \| \nabla d \|_{H^2}^{\frac{1}{2}}, \tag{28}
\]
we obtain
\[
\| u \|_{H^2} + \| \nabla d \|_{H^2} \\
\lesssim \| \sqrt{\rho}u_t \|_{L^2} + \| \nabla d_t \|_{L^2} + (\| u \|_{L^\infty} + \| \nabla d \|_{L^\infty} + 1)(\| \nabla d \|_{L^2} + \| \Delta d \|_{L^2}) \\
\lesssim \| \sqrt{\rho}u_t \|_{L^2} + \| \nabla d_t \|_{L^2} \\
+ \left( (\| u \|_{H^2} + \| \nabla d \|_{H^2})^2 (\| u \|_{L^2} + \| \nabla d \|_{L^2})^2 + 1 \right) \left( \| \nabla u \|_{L^2} + \| \Delta d \|_{L^2} \right) \\
\lesssim \| u \|_{L^2} + \| \nabla d \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla d \|_{L^2} \| \nabla u \|_{L^2} + \| \Delta d \|_{L^2} \\
+ \| \sqrt{\rho}u_t \|_{L^2} + \| \nabla d_t \|_{L^2} + \| \nabla u \|_{L^2} + \| \Delta d \|_{L^2}. \]

By solving the inequality, it yields
\[
\| u \|_{H^2} + \| \nabla d \|_{H^2} \lesssim \| \sqrt{\rho}u_t \|_{L^2} + \| \nabla d_t \|_{L^2} + \| \nabla u \|_{L^2} + \| \Delta d \|_{L^2} + (\| \nabla u \|_{L^2} + \| \Delta d \|_{L^2})^3,
\]
and then, there exists a constant $C_3$ depending only on $C_2$, such that
\[
\sup_{0<T<T^*} \left\{ \| u \|_{L^2(0,T;H^2)} + \| \nabla d \|_{L^2(0,T;H^2)} \right\} \leq C_3. \tag{29}
\]

Furthermore, (28) together with (29) give rise to
\[
\sup_{0<T<T^*} \left\{ \| u \|_{L^4(0,T;L^\infty)} + \| \nabla d \|_{L^4(0,T;L^\infty)} \right\} \leq C_4, \tag{30}
\]
where, similar to $C_3$, the constant $C_4$ depends only on $C_2$.

**Step 5. (Estimates for $\|\sqrt{\rho} u_t, \nabla d_t\|_{L^\infty((0,T);L^2)}$ and $\|\nabla(u, \Delta d_t)\|_{L^2((0,T);L^2)}$).**

Differentiating (5b) with respect to time, taking inner product with $u_t$ and integrating by parts, bearing in mind (1a) and (1c), we derive

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho |u_t|^2 \, dx + \mu \int_\Omega |\nabla u_t|^2 \, dx
$$

$$
= -2 \int_\Omega \rho u_t \cdot \nabla u_t \cdot u_t \, dx - \int_\Omega \rho u_t \cdot \nabla(u \cdot \nabla u_t) \, dx - \int_\Omega \rho u_t \cdot \nabla u \cdot u_t \, dx \tag{31}
$$

$$
- \lambda \int_\Omega ((\nabla d)^T \Delta d)_t \cdot u_t \, dx.
$$

Now we will estimate the right-hand side of (31) term by term, using Hölder’s inequality and Cauchy’s inequality, we get

$$
-2 \int_\Omega \rho u_t \cdot \nabla u_t \cdot u_t \, dx \lesssim \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u_t\|_{L^\infty} \|\nabla u_t\|_{L^2} \leq \frac{\mu}{8} \|\nabla u_t\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \|\sqrt{\rho} u_t\|_{L^2}^2,
$$

$$
- \int_\Omega \rho u_t \cdot \nabla(u \cdot \nabla u_t) \, dx
$$

$$
\lesssim \int_\Omega |\rho u_t| |u| |\nabla u_t| \, dx + \int_\Omega |\rho u_t| |u|^2 |\nabla u| \, dx + \int_\Omega |\rho u|^2 |\nabla u_t| \, dx
$$

$$
\lesssim \|u\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} + \|\sqrt{\rho} u_t\|_{L^2} \|u\|_{L^\infty} \|\nabla u\|_{L^2}
$$

$$
+ \|u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2}
$$

$$
\leq \|u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2} \|\nabla u_t\|_{L^2} + \|u\|_{L^\infty} \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|u\|_{L^2}^2
$$

$$
+ \frac{\mu}{8} \|\nabla u_t\|_{L^2}^2 + C \|\sqrt{\rho} u_t\|_{L^2}^2 \|u\|_{L^2}^2,
$$

and

$$
- \lambda \int_\Omega ((\nabla d)^T \Delta d)_t \cdot u_t \, dx
$$

$$
= -\lambda \int_\Omega (\nabla d)^T \Delta d \cdot u_t \, dx + \lambda \int_\Omega (\nabla d_t \cdot \nabla d)^T (\nabla d) \, u_t \, dx
$$

$$
- \lambda \int_\Omega (\nabla d)^T \nabla d_t \cdot \nabla u_t \, dx
$$

$$
\lesssim \|\nabla d_t\|_{L^4} \|\Delta d\|_{L^2} \|u_t\|_{L^4} + \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} \|\nabla u_t\|_{L^2}
$$

$$
\leq \frac{\mu}{8} \|\nabla u_t\|_{L^2}^2 + \frac{\gamma}{4} \|\Delta d_t\|_{L^2}^2 + C \|\nabla d_t\|_{L^2}^2 \|\Delta d\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2 \|\nabla d_t\|_{L^2}^2.
$$

Putting all the estimates above together and combining with (31), it yields to

$$
\frac{d}{dt} \int_\Omega \rho |u_t|^2 \, dx + \mu \int_\Omega |\nabla u_t|^2 \, dx
$$

$$
\leq \frac{\gamma}{4} \|\Delta d_t\|_{L^2}^2 + C \|u\|_{L^\infty}^2 + \|u\|_{L^2}^2 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|\nabla d\|_{L^\infty}^2
$$

$$
+ \|\Delta d\|_{L^2}^2 \|\nabla d_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^2 + C \|u\|_{L^\infty}^2 \|\nabla u\|_{L^2}^2. \tag{32}
$$
Similarly, as for (34), we simply take \( t \)-derivative, multiply the result by \( \Delta d_t \), integrate over \( \Omega \) and use integration by parts to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla d_t|^2 \, dx + \gamma \int_{\Omega} |\Delta d_t|^2 \, dx \\
= \int_{\Omega} (\mathbf{u} \cdot \nabla d) \cdot \Delta d_t \, dx + \int_{\Omega} (\mathbf{u} \cdot \nabla d_t) \cdot \Delta d_t \, dx - \gamma \int_{\Omega} |\nabla d|^2 \, d_t \cdot \Delta d_t \, dx \\
- 2\gamma \int_{\Omega} (\nabla d : \nabla d_t) d_t \cdot \Delta d_t \, dx
\]
\[
\lesssim \|\nabla u_t\|_{L^2} \|\nabla d_t\|_{L^2} \|\nabla d\|_{L^\infty} + \|u_t\|_{L^4} \|\nabla d_t\|_{L^4} \|\Delta d\|_{L^2} + \|u\|_{L^\infty} \|\nabla d_t\|_{L^2} \|\Delta d_t\|_{L^2} \\
+ \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} \|\Delta d_t\|_{L^2} \\
\lesssim \frac{\mu}{4} \|\nabla u_t\|_{L^2}^2 + \gamma \|\Delta d_t\|_{L^2}^2 + C \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} \|\Delta d_t\|_{L^2} \\
+ C \|\nabla d\|_{L^\infty} \|\nabla d_t\|_{L^2} \|\Delta d_t\|_{L^2}.
\]
\[
(33)
\]
Now, we have, by (32) and (33),
\[
\frac{d}{dt} \int_{\Omega} (\rho|u_t|^2 + |\nabla d_t|^2) \, dx + \int_{\Omega} (|\nabla u_t|^2 + |\Delta d_t|^2) \, dx \\
\lesssim (\|u\|_{L^\infty} + \|u\|_{H^2} + \|\nabla d\|_{L^\infty} + |\Delta d|_{L^2} + \|\nabla d\|_{L^2} \|\Delta d\|_{L^2}^2) + \|\nabla u_t\|_{L^2}^2 + \|u_t\|_{H^2}^2 + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2}^2.
\]
In accordance with (27), (29), (30), then it comes directly from Gronwall’s inequality that
\[
\sup_{0 < T < T^*} \left\{ \|\sqrt{\rho} u_t(T), \nabla d_t(T)\|_{L^2} + \int_0^T \|\nabla u_t, \Delta d_t\|_{L^2}^2 \, dt \right\} < \infty. \tag{34}
\]
With (34), if the inequality (25) is reconsidered, one has
\[
\sup_{0 < T < T^*} \{ \|u\|_{H^2} + \|\nabla d\|_{H^2} \} < \infty. \tag{35}
\]
**Step 6. (Estimates for \( \|\nabla \rho\|_{L^\infty(0,T;H^1)} \) and \( \|\nabla d\|_{L^2(0,T;H^2)} \)).** First, we prove \( u \) enjoys higher regularity in space. By using Lemmas 2.1, 2.2 and (7), one has
\[
\|u\|_{W^{2,4}} \lesssim \|u\|_{H^1} + \|\nabla u\|_{L^2} + \|\nabla d\|_{L^\infty} \|\nabla^2 d\|_{L^4} \\
\lesssim \|u\|_{H^1} + \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla d\|_{L^4} \|\nabla^2 d\|_{L^4} \\
\lesssim \|u\|_{H^1} + \|\nabla u_t\|_{L^2} + \|\nabla u\|_{L^\infty} \|\nabla u\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla d\|_{L^2} \|\nabla^2 d\|_{L^2},
\]
which implies
\[
\|u\|_{L^2(0,T;W^{2,4})} \lesssim \|\nabla u_t\|_{L^2(0,T;L^2)} \|\nabla u\|_{L^\infty(0,T;L^2)} \|\nabla^2 u\|_{L^2(0,T;L^2)} \|\nabla^2 d\|_{L^2(0,T;L^2)}.
\]
In accordance with (27), (29), (30), (34) and (35), we conclude
\[ \sup_{0 < T < T^*} \{ \| u \|_{L^2(0,T;W^{2,4})} \} < \infty. \]

Now, applying the operator \( \nabla \) to (5a), multiplying the result by \( \nabla \rho \) and integrating over \( \Omega \), we obtain
\[ \frac{d}{dt} \int_{\Omega} |\nabla \rho|^2 \, dx \lesssim \int_{\Omega} |\nabla u| |\nabla \rho|^2 \, dx \lesssim \| \nabla u \| \| \nabla \rho \|^2_{L^2}. \]

Similarly, higher order estimate for \( \rho \) can be obtained:
\[ \frac{d}{dt} \int_{\Omega} |\nabla^2 \rho|^2 \, dx \lesssim \int_{\Omega} (|\nabla u| |\nabla^2 \rho|^2 + |\nabla^2 u| |\nabla \rho| |\nabla^2 \rho|) \, dx \lesssim \| \nabla u \|_{L^\infty} \| \nabla^2 \rho \|^2_{L^2} + \| \nabla^2 u \|_{L^1} \| \nabla \rho \|_{L^1} \| \nabla^2 \rho \|_{L^2}. \]

By virtue of the Sobolev embeddings
\[ W^{1,4} \hookrightarrow L^\infty, \quad H^1 \hookrightarrow L^4 \]
and Gronwall's inequality, we have
\[ \frac{d}{dt} \int_{\Omega} |\nabla \rho|^2 \, dx \lesssim \| \nabla u \|_{W^{1,4}} \| \nabla \rho \|^2_{L^2}, \quad \frac{d}{dt} \int_{\Omega} |\nabla^2 \rho|^2 \, dx \lesssim \| \nabla u \|_{W^{1,4}} \| \nabla \rho \|^2_{H^1}, \]
and
\[ \| \nabla \rho(T) \|^2_{H^1} \lesssim \| \nabla \rho_0 \|^2_{H^1} \exp \left( \int_0^T \| \nabla u \|_{W^{1,4}} dt \right). \]

Therefore, it follows from (36) and (7) that
\[ \sup_{0 < T < T^*} \{ \| \rho \|_{L^\infty(0,T;H^2)} \} < \infty. \]

Similarly, by using Hölder's inequality,
\[ \| u \|_{H^3} \lesssim \| u \|_{H^1} + \| -\rho u_t - \rho u \cdot \nabla u - \nabla (|\nabla d|^2 / 2) \Delta d \|_{H^1} \]
\[ \lesssim \| u \|_{H^1} + \| u \|_{H^1} + \| \nabla \rho \|_{L^4} \| u \|_{L^4} + \| u \|_{L^\infty} \| \nabla u \|_{H^1} + \| \nabla u \|_{L^2}^2 \]
\[ + \| \nabla \rho \|_{L^4} \| u \|_{L^\infty} \| \nabla u \|_{L^1} + \| \nabla d \|_{L^\infty} \| \nabla d \|_{H^2} + \| \nabla^2 d \|_{H^1}^2, \]
which implies that
\[ \| u \|_{L^2(0,T;H^3)} \lesssim \| u \|_{L^2(0,T;H^1)} + \| \nabla u \|_{L^2(0,T;L^2)} + \| \rho \|_{L^\infty(0,T;H^2)} \| \nabla u \|_{L^2(0,T;L^2)} + \| u \|_{L^\infty(0,T;L^\infty)} \| \nabla u \|_{L^2(0,T;H^1)} + \| \nabla u \|_{L^2(0,T;H^2)} \| \nabla d \|_{L^\infty(0,T;H^1)}, \]
and that
\[ \sup_{0 < T < T^*} \{ \| u \|_{L^2(0,T;H^3)} \} < \infty. \]

Repeating the same procedure to equation (21), we have
\[ \| \nabla d \|_{H^3} \lesssim \| \nabla d \|_{H^1} + \| \nabla d_t + \nabla (u \cdot \nabla d) - \nabla (|\nabla d|^2 d) \|_{H^1} \]
\[ \lesssim \| \nabla d \|_{H^1} + \| \nabla d_t \|_{H^1} + \| \nabla u \|_{L^4} \| \nabla d \|_{L^4} + \| \nabla^2 u \|_{L^4} \| \nabla d \|_{L^4} + \| \nabla u \|_{L^4} \| \nabla^2 d \|_{L^4} + \| u \|_{L^\infty} \| \nabla d \|_{L^4}^2 + \| \nabla^2 d \|_{L^4} \| \nabla d \|_{L^\infty} + \| \nabla^2 d \|_{L^4}^2, \]
\[ + \| \nabla d \|_{L^4} \| \nabla^2 d \|_{L^4} + \| \nabla d \|_{L^\infty} \| \nabla^3 d \|_{L^2}. \]
which gives rise to
\[ \| \nabla d \|_{L^2(0,T;H^3)} \lesssim \| \nabla d \|_{L^2(0,T;H^3)} + \| \Delta d \|_{L^2(0,T;L^2)} + \| \nabla u \|_{L^2(0,T;H^3)} \| \nabla d \|_{L^\infty(0,T;H^1)} 
\]
\[ + \| u \|_{L^2(0,T;H^3)} \| \nabla d \|_{L^\infty(0,T;H^1)} + \| \nabla u \|_{L^2(0,T;H^3)} \| \nabla d \|_{L^\infty(0,T;H^2)} 
\]
\[ + \| \nabla d \|_{L^2(0,T;H^2)} \| \nabla d \|_{L^\infty(0,T;H^2)} + \| \nabla d \|_{L^2(0,T;H^2)} \| \nabla d \|_{L^\infty(0,T;H^2)} 
\]
\[ + \| \nabla d \|_{L^\infty(0,T;H^1)} \| \nabla d \|_{L^1(0,T;H^2)}, \]
and therefore
\[ \sup_{0 < T < T^*} \left\{ \| \nabla d \|_{L^2(0,T;H^3)} \right\} < \infty. \tag{39} \]

Now, it follows from (37)-(39) that
\[ \sup_{0 < T < T^*} \left\{ \| \rho \|_{L^\infty(0,T;H^2)} + \int_0^T \| (u, \nabla d) \|_{H^3}^2 dt \right\} < \infty. \]

Collecting all the estimates obtained in **Step 1-Step 6**, we have proved that (6) holds. The proof of Theorem 2.4 is completed.

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E-mail address: xlli@bupt.edu.cn