The extended propagation equation of optical impulses in silica fibers

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Abstract. We considered the new equation of distribution of optical impulses in silica fibers. We also found its exact solitonic solution and analyzed extreme cases.

Introduction

The field of the optical impulse which is distributed in a single-mode optical fiber supporting the linear polarization condition has the form \cite{1}, \cite{3}, \cite{5}.

\[ E(r,t) = e^{-it} F(x,y) A(z,t) \exp[i(\beta_0 z - \omega_0 t)] \]  

where \( F(x,y) \) – Gaussian function, which has the form: \( \exp\left(-\frac{x^2+y^2}{w^2}\right) \) with radius of mode \( w \), \( A(z,t) \) - complex impulse envelope, \( \omega_0 \) – carrier frequency, \( \beta_0 = \omega_0 n(\omega_0)/c \) – central wave number.

Materials and Methods

In case of Kerr-type nonlinearity for envelope of optical impulse there is an equation

\[ i(\partial^2 A/\partial z^2 + \beta_1 \partial A/\partial t) + (1/2\beta_0) \left(\frac{\partial^2 A}{\partial z^2}\right)^2 - (\beta_0/2) \left(\frac{\partial^2 A}{\partial t^2}\right) + \gamma |A|^2 A = 0 \]  

here \( \beta_1 = 1/v_g \) - inverse quantity of group velocity, \( \beta_2 \) - variance of group velocity, \( \gamma \) - index of Kerr-type nonlinearity. The value \( \beta_2 \) is negative value in zone of silica transparency. Thus, equation (2) is elliptic equation. If we discard one of two derivatives in equation (2), then we get parabolic equation, which is reduced to form of Schrödinger equation.

Usually \cite{1-6} for silica fibers they use equation, which is received by rejection of second derivative with respect to coordinate. It is erroneous as it will be shown. Ordinarily, rejection of second derivative with respect to coordinate occurs at the initial stage of solution of wave equation in spectral representation \cite{1}, \cite{3}, \cite{5}, \cite{6}. At more detailed approach we can get equation (2), in which we can to estimate importance degree of second derivatives. Given that in area of the minimum losses for silica: \( \lambda \approx 1.55 \) \( \mu m \); \( \beta_2 = -20 \) (ps)\(^2\)/km; \( n = 1.45 \) and \( z = ct_1/n \) we can notice that coefficients of second derivatives with respect to coordinate and with respect to time \( t \) are differ for more than two orders. Thus, for silica waveguide it is better to use equation

\[ i(\partial^2 A/\partial z^2 + \beta_1 \partial A/\partial t) + (1/2\beta_2) \left(\frac{\partial^2 A}{\partial t^2}\right) + \gamma |A|^2 A = 0 \]  

but not equation...
\[ i(\partial A/\partial z + \beta_t A/\partial t) - (\beta_z/2)(\partial^2 A/\partial t^2) + \gamma A^2 A = 0 \]  
(4)

Fundamental soliton of equation (4) in laboratory reference system is given by
\[ A = \left(\sqrt{2h/\gamma} \exp[ibz]\sqrt{2\sqrt{b}(z-z_o-vt)/\sqrt{b_2}}\right) \]  
(5)

Here \( b \) - free argument and correction of central wave number, \( z_o \) - impulses coordinate at the initial time. Formula (5) is unsatisfactory; in view of velocity distribution of impulse is given by delta function. However, velocity of optical soliton [7], [8] is constant in the mean. The soliton solution of equation (3) in laboratory reference system is given by
\[ A = \left(\frac{2b(1+b/2\beta_o)}{\gamma c h^2\left(s\sqrt{2\beta_b b(1+b/2\beta_o)}(1-\beta_o\beta_b^2)\right)\right)^{1/2} \exp[ibz] \]  
(6)

were
\[ s(z,t) = z - z_o - vt, \]  
(7)
\[ v = v_g (1+b/\beta_o) \]  
(8)

is free of shortcoming because of arbitrary of parameter \( b \) and after all, we will show a similar solution of equation (2):
\[ A = \left(\frac{2b(1+b/2\beta_o)}{\gamma c h^2\left(s\sqrt{2\beta_b b(1+b/2\beta_o)}(1-\beta_o\beta_b^2)\right)\right)^{1/2} \exp[ibz] \]  
(9)

The dimensionless quantity \( \beta_o\beta_b^2 \leq 0.005 \). And being that it is under the radical sign, and then equation (9) coincides with the equation (6) with high degree of probability. This argues in favor of the equation (3) for silica fibers.

Let us show that any soliton solution of equation (2) is some function of hyperbolic cosine, regardless of the type of nonlinearity. Let us to reduce this equation (2) for complex envelope of optical impulse to form of equation for optical intensity \( I = I(s(z,t)) \) by substitution
\[ A(z,t) = \sqrt{2\gamma} \exp{ibz} \]  
(10)

were \( b \) – free argument and correction of central wave number \( \beta_o \). This parameter is always expressed in terms of the peak intensity or the peak value of intensity.

Substitution of (10) into (2) gives us (after separation of real part from imaginary part of obtained solution) these two equations for optical intensity of envelope:
\[ (1+b/\beta_o)(\partial^2 I/\partial z^2) + (b_g^2)/(v/\beta_o)(\partial^2 I/\partial t^2) = 0 \]  
(11)
\[ (1/\beta_o)(\partial^2 I/\partial z^2) - (\partial^2 I/\partial t^2) - \beta_o(\partial^2 I/\partial t^2) = \left[b + (b^2/2\beta_o) - \Delta\beta(I)\right] I^2. \]  
(12)

Thus, we obtain two equations for one unknown function. Equation (11) is linear and homogeneous equation. This allows us to write its general solution, which is an arbitrary differentiable function \( I = I(s(z,t)) \), were
\[ s(z,t) = z - z_o - vt, \]  
(13)
\[ v = v_g (1+b/\beta_o). \]  
(14)

Here we take into account that \( \beta_o = 1/v_g \). In other words equation (11) defines an argument of the unknown function. In that case the form of the function remains arbitrary. Thus, the required function is a progressive wave with unknown wave profile and constant velocity (14). The wave profile describe by equation (12), which can be reduced as ordinary differential equation
\[ \left[b + (b^2/2\beta_o) - \Delta\beta(I)\right] I^2. \]  
(15)

For ease of calculation let us proceed in (15) to the dimensionless independent variable \( \tau \):
\[ \tau = s\left(b(1+b/2\beta_o)/(1-\beta_o\beta_o^2)\right)^{1/2} \]  
(16)

As a result, equation (11) is of the form
\[ 2\tau I - I^2 = 4I^2 - 8\beta I^2 \]  
(17)

were
\[ \gamma = \beta_\alpha (h(b + 2\beta_\alpha)) \]  
\[ (18) \]

The derivative with respect to \( \tau \) is designated as a point. As is easy to check that equation (17) is equivalent to the normal system of two Hamiltonian first-order equations:

\[ I = \partial H / \partial P_1 \]  
\[ (19) \]
\[ \dot{P}_1 = -\partial H / \partial I \]  
\[ (20) \]

with Hamiltonian function

\[ H = (P_1^2 - 1)I + 2\gamma \beta(I) \]  
\[ (21) \]

were \( B(I) \) – antiderivative for function \( \Delta \beta(I) \), namely

\[ B(I) = \int \Delta \beta(I) dI \]  
\[ (22) \]

It is well known that the Hamiltonian system (19), (20) also describes one-dimensional point particle with coordinate \( I \). Therefore, equation (17) is also the Euler-Lagrange equation for this one-dimensional mechanical particles. In classical dynamics of particles dimensionless variable \( \tau \), defined by equation (16), is used as time. Thus, the Hamiltonian system (19), (20), obtained for the intensity envelope of the optical impulse, has a mechanical sense, if the \( I \) is understood not as a field function, but as a coordinate of the point. Since the Hamilton function (21) does not depend explicitly on time, and then the mechanical energy of this point is stored.

The solutions \( I(\tau) \), which asymptotically tends to zero at infinity, exist (if they certainly exist) with zero mechanical energy of point. This means that the envelope of the optical impulse in this case is a localized function. Such localized solutions of the equation (17) are of interest first of all.

The solution of Hamiltonian equations (19), (20), and therefore equation (17) is easily obtained using the theory of canonical transformations. As follows from Jacobi-Poincare theorem, if there exists a twice-differentiable function, \( S(I, p, \tau) \) such that \( \partial^2 S / \partial \tau^2 \neq 0 \), then transformation \( (I, P_1) \leftrightarrow (q, p) \) is canonical equation and generated by function \( P_1 \) with new Hamilton function \( \overline{H} \):

\[ P_1 = \partial S / \partial I; \quad q = \partial S / \partial p \]  
\[ (23) \]
\[ \overline{H}(q, p, \tau) = H(I(q, p, \tau), P_1(q, p, \tau), \tau) + \left( \partial S / \partial \tau \right) I(q, p, \tau), p, \tau) \]  
\[ (24) \]

In the theory of canonical transformations for two-dimensional phase space, we can take as a basis the four types of generating functions, which depend on one new and one old variables. For further purposes, the most appropriate is a function that depends on the old impulse and the new coordinate: \( F = R_2(P_1, q, \tau) \). Then the explicit form of the transformation can be found from equations

\[ I = \partial F / \partial P_1; \quad p = \partial F / \partial q, \]  
\[ (25) \]

and new Hamiltonian function

\[ \overline{H}(q, p, \tau) = H(I(q, p, \tau), P_1(q, p, \tau), \tau) - \left( \partial F / \partial \tau \right) P_1(q, p, \tau), q, \tau) \]  
\[ (26) \]

Let us consider that the 1-st component in Hamiltonian is quiescent Hamiltonian and the 2-nd is disturbance, due to nonlinear response of the medium. Let's move from dynamic system \( \{I, P_1, H\} \) to system \( \{q, p, \overline{H}\} \), with interaction by using the generating function \( F(P_1, q, \tau) \), such that in new variables the new Hamilton function would be defined by the disturbance only. This can be done by using the generating function

\[ F = q\text{Arth}P_1 - q \tau \]  
\[ (27) \]

which is the complete integral of the unperturbed Hamilton-Jacobi equation: \( \partial F / \partial \tau = \left( P_1^2 - 1 \right)F / \partial P_1 \). From equations (21) follows the explicit form of the transformations:

\[ I = qch^2(p + \tau); \quad P_1 = \tau \text{th}(p + \tau), \]  
\[ (28) \]

and from equation (22) follows the explicit form of the new Hamilton function:

\[ \overline{H} = \gamma B(qch^2(p + \tau)), \]  
\[ (29) \]

new Hamilton equations
\[ \dot{q} = \partial H / \partial \dot{p}; \quad \dot{p} = - \partial H / \partial \dot{q} \]  

(30)
determine the dynamics of the new canonical variables \( q \) and \( p \):

\[ \dot{q} = 4\gamma ch(p + \tau)sh(p + \tau)\Delta \beta(qch^2(p + \tau)), \quad \dot{p} = - 2\gamma h^2(p + \tau)\Delta \beta(qch^2(p + \tau)). \]  

(31)

As the understanding of the interaction Hamiltonian (29) depends on time in hand \( (p + \tau) \) the first integral of the system can be easily guessed

\[ 2\gamma B(qch^2(p + \tau)) - q = E = 0 \]  

(32)

Truly, taking derivative with respect to time of this expression and taking into account (29) and (30) we will get the identity. Mechanical energy \( E \) is taken as zero (it is not the energy of the optical impulse, it is just an arbitrary constant, which in the mechanical interpretation makes sense of the mechanical energy of one-dimensional particles). To simplify the solution of the dynamic problem in the understanding of the interaction divide the first equation (31) for the second. The result is

\[ \dot{q} / q = - 2\gamma pth(p + \tau) \]  

(33)

The form of the last equation does not depend on the type of nonlinear response function, and is determined only by the canonical transformation (28), which, in turn, is determined by the unperturbed Hamiltonian \( H_o = \{ p^2 - 1 \} H \). This unperturbed Hamiltonian is defined in the end of the linear part of equation (2). If one first integral of the system will be known, will reduce the system order by one, therefore, to one equation. As such equation it is necessary to take equation (33). As a second equation to solve the problem, you should take the first integral (32), from which it follows that \( q \) is some function of the hyperbolic cosine of the argument \( (p + \tau) \). Substitution of this function into (33) gives the time dependence \( p(\tau) \), and that completes the solution to the problem.

So, for example, for Kerr nonlinearity \( B(\lambda) = 4\pi \lambda^2 \): \( p = - 2\tau \), \( q = 1/2\gamma h^2(\tau) \) and it is an exact solution.

**Conclusion**

Thus, it is shown that the problem of finding localized solutions of the extended nonlinear equation the propagation of optical pulses in fibers (2) for silica fibers is reduced to a simpler equation of parabolic type (3). It differs significantly from generally accepted in the literature on fiber optics equation (4). Solutions of all three equations are in the laboratory frame of reference.

Besides, the problem of finding localized solutions of equation (2) is reduced by using the theory of Hamiltonian systems to two equations (32) and (33). The first of equations is algebraic.

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