More about One-Loop Effective Action of Open Superstring in $AdS_5 \times S^5$

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**Abstract:** We reconsider the calculation of the one-loop effective action for an open Green–Schwarz superstring in the $AdS_5 \times S^5$ background for a circular boundary loop. By an explicit computation of the ratio of relevant determinants, describing semi-classical fluctuations about the minimal surface in AdS and flat spaces, we show that it does not depend upon the AdS regularizing parameter $\epsilon$. The only dependence upon $\epsilon$ resides in the reparametrization path integral of the exponential of the classical boundary action. We analyze how the result depends on the choice of the boundary condition imposed on fluctuating fields and show that, despite the fact that the contribution of individual angular modes changes, the product over the modes remains unchanged.

**Keywords:** Wilson loop, AdS space, minimal surface, two-dimensional determinant, reparametrization path integral, Schwinger effect.
1. Introduction

The AdS/CFT correspondence states that the Wilson loop in $\mathcal{N} = 4 \, SU(N)$ super Yang-Mills is equal to the disk amplitude of an open IIB superstring with the Ramond-Ramond flux in the $AdS_5 \times S^5$ background \[1, 2\]. For a circular Wilson loop the supergravity approximation $\[3, 4\]$ to the disk amplitude indeed coincides with the limit of large ’t Hooft coupling $\lambda$ of the explicit result for the Wilson loop \[5\]. Also the $\lambda$-dependence of the pre-exponentials apparently coincides \[6\], while the comparison of the constant factors relies
on the one-loop computations \cite{7,8,9} of the effective action for superstring in $AdS_5 \times S^5$ that involves a nontrivial renormalization by subtracting the contribution from a reference contour (the straight line). These constant factors agree only up to a factor of 2, which is one of the motivations to repeat the computation by another method as is done in this Paper.

Circular Wilson loops also emerge in the study of the Schwinger process of pair production in a constant electric field, which is calculable in $\mathcal{N} = 4$ super Yang–Mills at large $\lambda$ via the AdS/CFT correspondence \cite{10}. It has recently been argued \cite{11} that at large coupling $\lambda$ there exists a critical value of the electric field like in string theory, contrary to what is the case at weak coupling. Using a representation of the string disk amplitude in AdS space through a path integral over reparametrizations of the boundary, it has recently been shown \cite{12} that quantum fluctuations about the minimal surface result in

$$W(\text{circle}) \propto e^{-2\pi\sqrt{\lambda R/\varepsilon}\sqrt{\lambda^{-3/4}}\left(\frac{R}{\varepsilon}\right)^{\nu/2}},$$

at the one-loop order. Here $R$ is the radius of the circle and $\varepsilon$ is a regularization parameter associated with moving the boundary of $AdS$ from $Z = 0$ to $Z = \varepsilon$, where $Z$ denotes the radial AdS coordinate.

The exponent in Eq. (1.1) is the classical action, i.e. the area of the minimal surface enclosed by a circle in the boundary \cite{3,4}. The induced metric is singular at the boundary and the regularization parameter $\varepsilon$ plays in the dual language of D-branes the role of the $U(1)$ boson mass \cite{4,5}

$$m = \frac{\sqrt{\lambda}}{2\pi\varepsilon},$$

associated with the breaking $U(N) \to U(1) \times U(N - 1)$. While it was shown \cite{4,13} that the $\varepsilon$-dependence of the classical action can be eliminated for (the dual of) the Wilson loop by a Legendre transformation, it plays a crucial role in the computations of the Schwinger effect of production of a pair of $U(1)$-bosons with masses $m$ given by Eq. (1.2). This is the reason why we shall concentrate in this Paper on the dependence of $W(\text{circle})$ upon $\varepsilon$.

The pre-exponential factor results from quantum fluctuations about the minimal surface. The factor $\lambda^{-3/4}$ was linked \cite{6} to the presence of three $SL(2,\mathbb{R})$ zero modes of the fluctuations about the classical solution. The $\varepsilon$-dependence of the pre-exponential factor displayed in Eq. (1.1) was obtained in Ref. \cite{12}, where it was argued that the value of $\nu$ is expected to be 3, conjectured again to be related with the number of the $SL(2,\mathbb{R})$ zero modes. Our goal in this Paper will be to reproduce this result, pursuing direct computations \cite{7,8,9} of the one-loop effective action, resulting from semi-classical quantum fluctuations of an open Green–Schwarz superstring in $AdS_5 \times S^5$ with the ends at the boundary circle.

In order to describe our results let us first recall what the analogous one-loop effective action looks like for an open bosonic string in flat space. In the Polyakov formulation \cite{14} the Liouville field $\varphi$, which emerges through the conformal (Weyl) factor in the metric tensor,

$$g_{ab} = e^{\varphi}\delta_{ab},$$

(1.3)
decouples in the bulk for the critical dimension $d = 26$, since the conformal anomaly is proportional to $d - 26$. However, its boundary value does not decouple for off-shell disk amplitudes even in $d = 26$ and results in a reparametrization path integral

$$Z_{\text{flat}} = \int Dt(s) \ e^{-KS_{\text{cl}}[t(s)]},$$

(1.4)

where $K$ is the string tension and $S_{\text{cl}}[t(s)]$ is a classical boundary action, which emerges after path-integrating over fluctuations of the open string with fixed ends, and whose explicit form depends on the choice of the coordinates parametrizing the string world sheet including its boundary. The path integral in Eq. (1.4) goes over the functions $t(s)$, reparametrizing the boundary, with non-negative derivative $dt(s)/ds \geq 0$, which is related to the boundary value of the Liouville field $\varphi_B$ as

$$\frac{dt(s)}{ds} = e^{\varphi_B/2}.$$  

(1.5)

While the necessity for reparametrizations of the boundary was emphasized long ago [15], only recently some progress has been achieved [16, 17] as to how to define the measure and actually compute the path integral in Eq. (1.4).

The disk amplitude for the Green–Schwarz string in $\text{AdS}_5 \times S^5$ with the circular boundary can be represented at one loop in a similar form

$$Z_{\text{AdS}} = \int Dt(s) \ e^{-\sqrt{\lambda}S_{\text{cl}}[t(s)]} Z^{(1)}_{\text{AdS}},$$

(1.6)

where $S_{\text{cl}}[t(s)]$ is explicitly constructed in Ref. [12] and $Z^{(1)}_{\text{AdS}}$ is the ratio of the determinants of second-order operators explicitly found in Ref. [7]. By computing this ratio of determinants we show in this Paper that $Z^{(1)}_{\text{AdS}}$ does not depend on the AdS regularizing parameter $\varepsilon$, so the $\varepsilon$-dependence of the one-loop effective action is entirely due to the reparametrization path integral in Eq. (1.4), reproducing the result of Ref. [12] displayed in Eq. (1.1).

This Paper is organized as follows. In Sect. 2 we briefly review the classical solution in AdS for the circular boundary. In Sect. 3 we describe the main results for the ratio of determinants, obtained in this Paper. In Sect. 4 we compute various ratios of 1D determinants and study their dependence on the choice of the boundary conditions. In Sect. 5 we concentrate on the $\varepsilon$-dependence of the ratios of relevant determinants. In Sect. 6 we compute the ratios of 2D determinants, multiplying the 1D determinants over angular modes, and derive the results listed in Sect. 3. We also compute the difference between the contributions from a circle and a straight line and show that it does not depend on $\varepsilon$. Sect. 7 briefly summarizes the results of this Paper. Appendices A to D are devoted to technical details of the calculations.

2. Preliminaries

The upper half-plane parametrization $z = x + iy \ (y > 0)$ used in Ref. [12] for constructing the boundary action in AdS space is not convenient for computing the ratio of determinants for circular geometry because the minimal value $y_{\text{min}}$, associated with the boundary,
depends on $x$ for the consistency with the boundary metric, as is pointed out there. For this reason the variables are not separated in a simple way. It is more convenient to conformally map the upper half-plane onto a unit disk and then the unit disk onto a strip $\sigma \in [0, \infty), \phi \in [0, 2\pi)$ as

$$\omega = e^{-\sigma + i\phi} = \frac{i - z}{1 + z}. \quad (2.1)$$

The boundary at $Z = Z_{\text{min}}$ now corresponds to

$$\sigma_{\text{min}} = \varepsilon \equiv \frac{1}{2} \ln \frac{R + Z_{\text{min}}}{R - Z_{\text{min}}}. \quad (2.2)$$

The spherical solution of Refs. [3, 4] for the embedding space coordinates $Y_{-1}, Y_0, Y_1, Y_2, Y_3, Y_4$, obeying

$$Y \cdot Y \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + Y_3^2 + Y_4^2 = -1, \quad (2.3)$$

reads

$$Y_1 + iY_2 = \frac{2\omega}{1 - \omega \bar{\omega}}, \quad (2.4a)$$

$$Y_{-1} = \frac{1 + \omega \bar{\omega}}{1 - \omega \bar{\omega}}, \quad (2.4b)$$

$$Y_4 = Y_0 = Y_3 = 0, \quad (2.4c)$$

or

$$Z \equiv \frac{R}{Y_{-1} - Y_4} = R \frac{1 - \omega \bar{\omega}}{1 + \omega \bar{\omega}}, \quad (2.5a)$$

$$X^1 + iX^2 \equiv Z(Y_1 + iY_2) = R \frac{2\omega}{1 + \omega \bar{\omega}}, \quad (2.5b)$$

on the Poincare patch, so the induced metric

$$d\ell^2 = \frac{d\omega d\bar{\omega}}{(1 - \omega \bar{\omega})^2}, \quad (2.6)$$

is the metric of the Lobachevsky plane for the Poincare disk. The solution (2.5) describes a sphere $(X^1)^2 + (X^2)^2 + Z^2 = R^2$ and corresponds to a circle of the radius $R$ in the boundary when $Z = 0$.

For the coordinates (2.1) the solution (2.5) and the metric (2.6) read

$$Z = R \tanh \sigma, \quad X^1 + iX^2 = \frac{Re^{i\phi}}{\cosh \sigma}, \quad X^0 = X^3 = 0, \quad (2.7)$$

and

$$d\ell^2 = \frac{1}{\sinh^2 \sigma} \left( d\sigma^2 + d\phi^2 \right). \quad (2.8)$$

\footnote{A subtlety is that to obtain a unit disk we conformally map the upper half-plane with infinity excluded. Then it has the Euler character one. If alternatively the upper half-plane is periodically identified along the real axis, it has topology of a cylinder and the Euler character zero.}
The solution (2.4) obeys the Euler–Lagrange equation

$$(-\Delta + 2)Y_i = 0, \quad \Delta = (1 - \omega \bar{\omega})^2 \frac{\partial^2}{\partial \omega \partial \bar{\omega}},$$  \hspace{1cm} \text{(2.9)}

or

$$\Delta = \sinh^2 \sigma \left( \frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial \phi^2} \right),$$  \hspace{1cm} \text{(2.10)}

for the coordinates (2.1). On the Poincare patch it takes the form

$$\partial^a \frac{1}{Z^2} \partial_a X^u = 0, \quad \text{(2.11a)}$$

$$\partial^a \frac{1}{Z^2} \partial_a Z + 2 \sqrt{g} = 0, \quad \text{(2.11b)}$$

where

$$\sqrt{g} = \frac{1}{\sinh^2 \sigma}.$$  \hspace{1cm} \text{(2.12)}

As usual, the $SL(2, \mathbb{R})$ transformation is an isometry of this spherical solution. The $SL(2, \mathbb{R})$ coordinate transformation of the upper half-plane reads

$$z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc = 1,$$  \hspace{1cm} \text{(2.13)}

with real $a, b, c$ and $d$ to preserve the boundary (= real axis). After the conformal mapping (2.1) it takes the form

$$\omega \rightarrow e^{i \alpha} \frac{\omega - \omega_0}{1 - \omega \bar{\omega}_0},$$  \hspace{1cm} \text{(2.14)}

with real $\alpha$ and complex $\omega_0$ ($\bar{\omega}_0$ denotes complex conjugation), which maps a unit disk onto itself.

Finally, we mention that for small $\sigma$ the solution (2.7) for a circle reproduces near the boundary the one for a straight line

$$X^1 = x, \quad X^2 = X^3 = X^4 = 0, \quad Z = y.$$  \hspace{1cm} \text{(2.15)}

For the former case the strip coordinates are more convenient, while for the latter case the upper half-plane coordinates are more convenient.

### 3. Results for the ratio of determinants

The ratio of one-loop determinants that enter Eq. (1.6) reads explicitly

$$Z_{\text{AdS}}^{(1)} = \frac{\det (-\Delta_{ij} + \delta_{ij})_{\text{ghost}}^{1/2} \det \left(-\tilde{\nabla}^2 + R^{(2)} / 4 + 1 \right)^{8/2}_{\text{Fermi}}}{\det (-\Delta_{ij} + \delta_{ij})_{\text{long.}}^{1/2} \det (-\Delta + 2)_{\text{Bose}}^{3/2} \det (-\Delta)_{\text{Bose}}^{5/2}}.$$  \hspace{1cm} \text{(3.1)}

Here $\Delta = \nabla^2 = g^{ab} \nabla_a \nabla_b$ stands for the Laplacian in general coordinates for an appropriate representation, and the masses (squared) equal 2 for transverse bosons, 1 for longitudinal bosons and 1 for fermions. They emerge because of the curvature of the embedding (AdS).
space. The operators in the ghost and longitudinal determinants are the same, but the ratio is generically not 1 because of different boundary conditions.

Our strategy to evaluate the $\varepsilon$-dependence of the ratio (3.4) is to utilize the fact that the analogous ratio of the determinants in flat space does obviously not depend on $\varepsilon$, so we assume that $Z^{(1)}_{\text{flat}} = 1$ as is expected from supersymmetry and calculate the ratio

$$
\frac{Z^{(1)}_{\text{AdS}}}{Z^{(1)}_{\text{flat}}} = \frac{\det (-\Delta)}{\det (-\Delta_{ij} + \delta_{ij})^{1/2}} \left[ \frac{\det \left( -\tilde{\nabla}^2 + R^{(2)}/4 + 1 \right)}{\det \left( -\tilde{\nabla}^2 + R^{(2)}/4 \right)} \right]^{8/2} \left[ \frac{\det (-\Delta)}{\det (-\Delta + 2)} \right]^{3/2},
$$

of massive to massless (associated with flat space) determinants, noting that the ghost determinants are the same. This is in contrast to the proposal [7] to evaluate the ratio of the $Z^{(1)}_{\text{AdS}}$'s for the circle and the straight line, where the $\varepsilon$-dependence, we are interested in, cancels out.

The massive determinants are not directly computable by the Seeley coefficients, widely used in the 1980’s for the massless case, because the variation with respect to the metric is not reduced to an anomaly. We employ instead the method advocated for this problem by Kruczenski and Tirziu [9], which is based on direct computation of 1D×angular determinants, applying the Gel’fand–Yaglom technique for the 1D determinants. This is possible because all determinants in Eq. (3.2) are to be calculated for the Dirichlet boundary condition.

The ratio (3.2) crucially simplifies if we compute it for a straight line rather than for a circle. This is legitimate since we are interested only in the $\varepsilon$-dependence of the ratio (3.2), which originates from the region of $\sigma \sim \varepsilon$ near the boundary, where the solution (2.7) for a circle can be substituted by the solution (2.15) for a straight line. For the bosonic and fermionic determinants this was explicitly demonstrated by computations in Ref. [9]. Of course this is not the case for $\varepsilon$-independent constants in the determinants, which are different for the circle and the straight line.

One may wonder if one can lose the $SL(2, \mathbb{R})$ zero modes when replacing the circle by the straight line? The answer is “no”, because the $SL(2, \mathbb{R})$ zero modes show up in the determinant of the ghost operator which is the same in AdS and flat space and therefore cancels in the ratio (3.2).

The Gel’fand–Yaglom technique expresses the ratio of 1D determinants through the (properly normalized) solutions of the equations

$$
\left( -\partial^2 + V_i(\sigma) \right) f_i(\sigma) = 0, \quad f_i(\varepsilon) = 0, \quad f_i'(\varepsilon) = 1 \quad i = 1, 2,
$$

as

$$
\frac{\det \left( -\partial^2 + V_1(\sigma) \right)}{\det \left( -\partial^2 + V_2(\sigma) \right)} = \frac{f_1(\infty)}{f_2(\infty)}.
$$

Applying this technique, we obtain the following results. The ratio of the massive to massless bosonic determinants is explicitly

$$
\prod_\omega \frac{\det^{3/2} \left( -\partial^2 + \omega^2 + 2/\sigma^2 \right)}{\det^{3/2} \left( -\partial^2 + \omega^2 \right)} = \prod_\omega \left( 1 + \frac{1}{\varepsilon \omega} \right)^3 = e^{\frac{3}{2} \left[ \ln(\Lambda \varepsilon) + 1 \right]}.
$$

(3.5)
The ratio of the massive to massless fermionic determinants is explicitly\(^2\)

\[
\prod_\omega \frac{\det^{4/2}(-\partial^2 + \omega^2 + 3/4\sigma^2 + \omega/\sigma) \det^{4/2}(-\partial^2 + \omega^2 + 3/4\sigma^2 - \omega/\sigma)}{\det^{4/2}(-\partial^2 + \omega^2 - 1/4\sigma^2 + \omega/\sigma) \det^{4/2}(-\partial^2 + \omega^2 - 1/4\sigma^2 - \omega/\sigma)}
= \prod_\omega \left(\frac{1 + \frac{1}{2\varepsilon\omega}}{-2\varepsilon\omega e^{-2\varepsilon\omega Ei(-2\varepsilon\omega)}}\right)^4 = e^{\frac{4}{5}[\ln(\Lambda\varepsilon) + \frac{1}{2} + C_1]}.
\]  

(3.6)

The ratio of the massive to massless longitudinal determinants is explicitly

\[
\prod_\omega \frac{\det^{1/2}(-\partial^2 + \omega^2 + 2/\sigma^2 + 2\omega/\sigma) \det^{1/2}(-\partial^2 + \omega^2 + 2/\sigma^2 - 2\omega/\sigma)}{\det(-\partial^2 + \omega^2)}
= \prod_\omega \left(1 + \frac{1}{\varepsilon\omega} + \frac{1}{2\varepsilon^2\omega^2}\right) = e^{\frac{1}{5}[\ln(\Lambda\varepsilon) + 1 + \pi/4 + 1/2 \ln 2]}.
\]  

(3.7)

Multiplying these three, we finally find

\[
\frac{Z_{\text{AdS}}^{(1)}}{Z_{\text{flat}}^{(1)}} = e^{(4-3-1)\frac{1}{5}\ln(\Lambda\varepsilon)+C_2/\varepsilon} = e^{C_2/\varepsilon}.
\]  

(3.8)

Our results differ from those of Ref. [9], where the ratio of the ghost to longitudinal determinants was assumed to be 1. For our results the \(\frac{1}{\varepsilon}\ln\varepsilon\) term coming from the bosonic and fermionic determinants is precisely canceled by the one coming from the ratio of the longitudinal determinants. The cancellation is as for the \(\frac{1}{\varepsilon}\ln\Lambda\) divergent parts:

\[
2 \times 1 \text{ (longitudinal)} + 3 \times 2 \text{ (transversal)} - 8 \times 1 \text{ (GS fermions)} = 0,
\]  

(3.9)

where the first figure in each term is the number of degrees of freedom and the second one is the proper mass squared. The remaining term is \(1/\varepsilon\), which does not spoil anything and is removable by the Legendre transformation like the classical singularity. It can be simply viewed as a renormalization of the \(U(1)\) boson mass (1.2).

Therefore \(Z_{\text{AdS}}^{(1)}\) is equal to a constant that does not depend on \(\varepsilon\) after the Legendre transformation and is not essential in Eq. (1.6). Like in the flat space the Liouville field \(\varphi(x, y) \ (g_{ab} = e^{\varphi}\delta_{ab})\) decouples in the bulk, while its boundary value is related to the reparametrizing function \(t(s)\) as

\[
\frac{dt(s)}{ds} = e^{\varphi(s, \varepsilon)/2}.
\]  

(3.10)

We are thus left with the same boundary action \(S_{\text{cl}}[t(s)]\) as obtained in Ref. [12] (an extension of Douglas’ integral [18] to AdS space), which is to be substituted in the reparametrization path integral, reproducing the effective action displayed in Eq. (1.1).

In the rest of this Paper we present technicalities used for the derivation of this result.

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\(^2\)Here \(\text{Ei}(x)\) is the exponential integral.
4. Gel’fand-Yaglom meets Gel’fand-Dikii

As is already pointed out, the Gel’fand–Yaglom technique is applicable when the ratio of determinants is calculated for the Dirichlet boundary condition. In Ref. [9] it was imposed at $\sigma = \varepsilon$, as inherited from the regularization of the classical action. However, there is nothing special about this point for the classical solution (2.4) or (2.7), which obeys the boundary condition at $\sigma = 0$ rather than at $\sigma = \varepsilon$. One can therefore wonder if the results will change when we impose the Dirichlet boundary condition at $\sigma = 0$ rather than $\sigma = \varepsilon$ and consider $\varepsilon$ only as a parameter regularizing determinants for the given (Poincare) metric.

Using a more general technique, to be introduced shortly, we find that the answer to this question is that the 1D determinants will change only in a way that results in the same $\varepsilon$-dependence. In addition, we confirm the results obtained by the Gel’fand–Yaglom technique by this other method.

4.1 Determinants via diagonal resolvent

The ratio of the determinants of two Schrödinger operators of the form $-\partial^2 + \omega^2 + V(\sigma)$ can be related to the diagonal resolvent

$$R_\omega(\sigma, \sigma; V) \equiv \left\langle \sigma \left| \frac{1}{-\partial^2 + \omega^2 + V(\sigma)} \right| \sigma \right\rangle,$$

(4.1)

where $\omega^2$ is a spectral parameter and $\partial \equiv \partial / \partial \sigma$, as follows

$$R_\omega \equiv \frac{\det (-\partial^2 + \omega^2 + V(\sigma))}{\det (-\partial^2 + \omega^2)} = \exp \left[ \int d\omega^2 \int_\varepsilon^\infty d\sigma \left( R_\omega(\sigma, \sigma; V) - R_\omega(\sigma, \sigma; 0) \right) \right].$$

(4.2)

Here an overall constant is to be fixed by requiring that the ratio tends to 1 as $\omega \to \infty$, since the potential can then be disregarded. We have introduced $\varepsilon$ as a lower limit of the integral over $\sigma$, anticipating a divergence at small $\sigma$.

The diagonal resolvent $R_\omega(\sigma, \sigma; V)$ can be easily constructed via two solutions of the second-order equation

$$(-\partial^2 + \omega^2 + V(\sigma)) f_\pm(\sigma) = 0,$$

(4.3)

where $f_+(\sigma)$ vanishes as $\sigma \to \infty$ and $f_-(\sigma)$ obeys the boundary condition at the beginning of the interval. This can be either the Dirichlet or Neumann or mixed (Robin) boundary condition. The third case is needed for the ghost determinant [15, 19, 20]. The explicit formula is well known:

$$R_\omega(\sigma, \sigma; V) = \frac{f_+(\sigma)f_-(\sigma)}{f_+(\sigma)\partial f_-(\sigma) - f_-(\sigma)\partial f_+(\sigma)}.$$  

(4.4)

It is less known that this resolvent obeys the quadratic Gel’fand–Dikii equation

$$-2R_\omega\partial^2 R_\omega + (\partial R_\omega)^2 + 4(\omega^2 + V)R_\omega^2 = 1,$$

(4.5)

which may help to find it even when it is difficult to solve Eq. (4.3) explicitly. This method of computing the ratio of determinants is described in more detail in Ref. [21] for the case of fluctuations about an instanton for the double-well potential.
Finally, to compute the (logarithm of the) ratio of 2D determinants we have to sum over angular modes for a circular boundary:

$$\ln \frac{\det (-\partial_2^2 + V(\sigma))}{\det (-\partial_1^2)} = \sum_\omega \int d\omega^2 \int_{\epsilon}^{\infty} d\sigma \left( R_{\omega}(\sigma, \sigma; V) - R_{\omega}(\sigma, \sigma; 0) \right),$$

where $\omega$ runs over integers or half-integers for bosonic or fermionic determinants, respectively. For a straight line the sum over $\omega$ in Eq. (4.6) is to be replaced by an integral.

### 4.2 Bosonic determinant

For the bosonic determinant the potential is

$$V_b(\sigma) = \frac{2}{\sinh^2 \sigma}. \quad (4.7)$$

To make a connection with the results of Ref. [9], we begin with the case, where $f_-(\sigma)$ vanishes at $\sigma = \sigma_0$. We shall then set $\sigma_0$ either to $\epsilon$, reproducing the results of Ref. [9], or to 0, answering the question posed at the beginning of this Section.

The two solutions to Eq. (4.3) with the potential (4.7) are

$$f_+(\sigma) = (\coth \sigma + \omega) e^{-\omega \sigma}, \quad (4.8a)$$

$$f_-(\sigma) = (\coth \sigma - \omega) e^{\omega \sigma} - (\coth \sigma + \omega) e^{\omega(2\sigma_0-\sigma)} \frac{(\coth \sigma_0 - \omega)}{(\coth \sigma_0 + \omega)}, \quad (4.8b)$$

where $f_-(\sigma_0) = 0$ so that $R_{\omega}(\sigma, \sigma; V_b)$ obeys the boundary condition $R_{\omega}(\sigma_0, \sigma_0; V_b) = 0$. Then we have

$$R_{\omega}(\sigma, \sigma; V_b) = \frac{(\omega + \coth \sigma)}{2\omega^2 - 1} [\omega - \coth \sigma - e^{2\omega(\sigma_0-\sigma)}(\omega + \coth \sigma) \frac{(\omega - \coth \sigma_0)}{(\omega + \coth \sigma_0)}], \quad (4.9)$$

which satisfies Eq. (4.5).

Analogously in the free case of $V = 0$ we find

$$f_+(\sigma) = e^{\omega(\sigma_0-\sigma)}, \quad (4.10a)$$

$$f_-(\sigma) = \frac{\sinh \omega(\sigma_0 - \sigma)}{\omega}, \quad (4.10b)$$

so the free resolvent is

$$R_{\omega}(\sigma, \sigma; 0) = \frac{1 - e^{2\omega(\sigma_0-\sigma)}}{2\omega}, \quad (4.11)$$

reproducing the usual one when $\sigma_0 \to -\infty$. Equation (4.11) will be extensively used below, when evaluating the ratios of determinants.

These $f_-(\sigma)$’s shown in Eqs. (4.8b) and (4.10b) vanish at $\sigma = \sigma_0$, but the lower limit in the integral (4.2) which we denote again as $\epsilon$ could be greater then $\sigma_0$ ($\epsilon \geq \sigma_0$). In this case the solutions do not have zeros for $\sigma > \epsilon$, so $\epsilon$ plays simply the role of a regularization which is not directly related to the boundary condition. We can thus check to what extent the obtained results will be independent of $\sigma_0$, i.e. of the choice of the boundary condition.
For the ratio of the determinants (4.12) we obtain
\[
\mathcal{R}_\omega = \exp \left[ \frac{1}{2} (\coth \varepsilon - 1) \left( \text{Ei}[-2(\omega - 1)(\varepsilon - \sigma_0)] - \ln(\omega - 1) + \ln(\omega + 1) \right) \\
- \frac{1}{2} (\coth \varepsilon + 1) \text{Ei}[-2(\omega + 1)(\varepsilon - \sigma_0)] \\
- e^{2(\varepsilon - \sigma_0)\coth \sigma_0} \text{Ei}[-2(\omega + \coth \sigma_0)(\varepsilon - \sigma_0)] \frac{\sinh(\varepsilon - 2\sigma_0)}{\sinh \varepsilon} \right],
\]
(4.12)
where the exponential integral
\[
\text{Ei}(-x) \equiv - \int_x^\infty \frac{e^{-t}}{t} \, dt \to \ln x + \gamma E - x + O(x^2).
\]
(4.13)

A few comments concerning Eq. (4.12) are in order. When \(\sigma_0 \to \varepsilon\), it gives by the use of the asymptote (4.13)
\[
\mathcal{R}_\omega|_{\sigma_0=\varepsilon} = \frac{\omega + \coth \varepsilon}{\omega + 1},
\]  
(4.14)
reproducing the result of Ref. [9] obtained by the Gel’fand–Yaglom technique. For \(\omega \to +\infty\) the exponent in Eq. (4.12) vanishes, so the ratio of determinants tends to 1 as it should. Another interesting case is when \(\sigma_0 \sim \varepsilon\) but still \(\sigma_0 < \varepsilon\), say \(\sigma_0 = \varepsilon/2\). Then the term displayed in the third line of Eq. (4.12) can be disregarded. The same is true for \(\sigma_0 = 0\), when
\[
\mathcal{R}_\omega|_{\sigma_0=0} = \exp \left[ \frac{1}{2} (\coth \varepsilon - 1) \left( \text{Ei}[-2(\omega - 1)\varepsilon] - \ln(\omega - 1) + \ln(\omega + 1) \right) \\
- \frac{1}{2} (\coth \varepsilon + 1) \text{Ei}[-2(\omega + 1)\varepsilon] \right].
\]
(4.15)
As \(\omega \to +\infty\) this tends to 1, but behaves at large \(\omega\) smoother than (4.14). If alternatively \(\omega \ll 1/\varepsilon\) as \(\varepsilon \to 0\), we get
\[
\ln \mathcal{R}_\omega \xrightarrow{\varepsilon \to 0} - \ln(2\varepsilon) + \gamma E - 2 + \ln(\omega + 1)\]
(4.16)
This has the same \(\varepsilon\)-dependence as \(\varepsilon \to 0\) limit of the (logarithm of the) right-hand side of Eq. (4.14), but differs by a constant.

### 4.3 Fermionic determinant

The fermionic potential reads [7, 8]
\[
V_{f\pm}(\sigma) = \frac{3}{4\sinh^2 \sigma} + \frac{1}{4} \pm \omega \coth \sigma,
\]
(4.17)
where the \(\pm\) sign refers to positive (negative) frequencies \(\omega\).

The treatment of the fermionic determinant follows that of the bosonic ones except for two major differences:
1) Anti-periodicity requires the angular modes to be \(e^{i r \phi}\) with half-integer \(r \in \mathbb{Z} + 1/2\).
2) The fermionic potential $V_f$ defined in (1.17) depends on $\omega$ itself. For this reason we have
\[
\mathcal{R}_{f \pm} = \frac{\det (-\partial^2 + \omega^2 + V_{f \pm}(\sigma))}{\det (-\partial^2 + (\omega \pm \frac{1}{2})^2)}
\]
\[
= \exp \left\{ \int_\varepsilon d\omega \int_\infty \sigma \left[ (2\omega \pm \coth \sigma) R_{\omega}(\sigma, \sigma; V_{f \pm}) - (2\omega \pm 1) R_{\omega \pm 1/2}(\sigma, \sigma; 0) \right] \right\},
\]
(4.18)
that generalizes Eq. (4.2) to the case of such an $\omega$-dependent potential. Also, the cases of positive and negative $\omega$ should now be treated separately, so the $\pm$ in Eq. (4.18) refer to positive or negative frequencies $r$.

For positive half-integer $\omega = r > 0$ the two solutions to Eq. (4.3) with the potential (1.17) are
\[
f_+ (\sigma) = \frac{e^{-\omega \sigma}}{\sqrt{\sinh \sigma}},
\]
(4.19a)
\[
f_- (\sigma) = \frac{e^{\omega (2\sigma_0 - \sigma)} (-\cosh \sigma + 2\omega \sinh \sigma)}{\sqrt{\sinh \sigma}} - \frac{e^{\omega (2\sigma_0 - \sigma)} (-\cosh \sigma_0 + 2\omega \sinh \sigma_0)}{\sqrt{\sinh \sigma}}.
\]
(4.19b)
Calculating the diagonal resolvent (4.4), substituting into Eq. (4.18) and integrating, we obtain for the ratio
\[
\ln \mathcal{R}_{f \pm} = \frac{1}{4} \left( \frac{\coth \varepsilon - 1}{\sinh \sigma} \right) \left\{ \mathrm{Ei} \left[ -(2\omega - 1)(\varepsilon - \sigma_0) \right] - \mathrm{Ei} \left[ -(2\omega + 1)(\varepsilon - \sigma_0) \right] - \ln(2\omega - 1) + \ln(2\omega + 1) \right\} + C,
\]
(4.20)
where $C$ is an integration constant that does not depend on $\omega$. It is given by
\[
C = \frac{1}{2} \ln \frac{\coth \varepsilon + 1}{2},
\]
(4.21)
as is derived in Appendix A using the semi-classical expansion.

For negative half-integer $r = -\omega < 0$ the two solutions to Eq. (4.3) with the potential (1.17) are
\[
f_+ (\sigma) = \frac{e^{-\omega \sigma} \cosh \sigma + 2\omega \sinh \sigma}{\sqrt{\sinh \sigma}},
\]
(4.22a)
\[
f_- (\sigma) = \frac{(-\coth \sigma + 2\omega \sinh \sigma)}{\sqrt{\sinh \sigma}} - \frac{e^{\omega (2\sigma_0 - \sigma)} (-\cosh \sigma_0 + 2\omega \sinh \sigma_0)}{\sqrt{\sinh \sigma}}.
\]
(4.22b)
The ratio (4.18) of the determinants reads
\[
\ln \mathcal{R}_{f -} = \frac{1}{4} \left( \frac{\coth \varepsilon - 1}{\sinh \sigma} \right) \left\{ \mathrm{Ei} \left[ -(2\omega - 1)(\varepsilon - \sigma_0) \right] - \ln(2\omega - 1) + \ln(2\omega + 1) \right\}
\]
\[
- \frac{1}{4} \left( \frac{\coth \varepsilon + 3}{\sinh \sigma} \right) \mathrm{Ei} \left[ -(2\omega + 1)(\varepsilon - \sigma_0) \right]
\]
\[
- \frac{1}{4} \frac{e^{2\sigma_0 - \sigma} \coth \sigma_0 \mathrm{Ei} \left[ -(2\omega + \coth \sigma_0)(\varepsilon - \sigma_0) \right]}{\sinh \varepsilon \sinh \sigma_0} 1 + \cosh \left[ 2(\varepsilon - \sigma_0) \right] - 2 \cosh (2\sigma_0)
\]
\[- C,
\]
(4.23)
with the same \(C\) as in Eq. (1.20) given by Eq. (1.21).

In the limit \(\sigma_0 \to \varepsilon\) we have
\[
\ln R_+|_{\sigma_0=\varepsilon} = C,
\]
and
\[
\ln R_-|_{\sigma_0=\varepsilon} = \ln(2\omega + \coth \varepsilon) - \ln(2\omega + 1) - C,
\]
reproducing the results of Ref. [9]. Alternatively, for \(\sigma_0 = 0\), we obtain
\[
\ln R_+|_{\sigma_0=0} = \frac{1}{4} \left( \coth \varepsilon - 1 \right) \left\{ \text{Ei}[-(2\omega - 1)\varepsilon] - \text{Ei}[-(2\omega + 1)\varepsilon] - \ln(2\omega - 1) + \ln(2\omega + 1) \right\} + C,
\]
and
\[
\ln R_-|_{\sigma_0=0} = \frac{1}{4} \left( \coth \varepsilon - 1 \right) \left\{ \text{Ei}[-(2\omega + 1)\varepsilon] - \ln(2\omega - 1) + \ln(2\omega + 1) \right\}
- \frac{1}{4} \left( \coth \varepsilon + 3 \right) \text{Ei}[-(2\omega + 1)\varepsilon] + \frac{\sinh \varepsilon}{2\varepsilon} e^{-2\omega \varepsilon} - C.
\]

For the product of these two ratios (one with positive and one with negative \(\omega\)) we obtain [cf. Eq. (4.12) for bosons]
\[
R_+ R_- = \exp \left[ \frac{1}{2} \left( \coth \varepsilon - 1 \right) \left\{ \text{Ei}[-(2\omega - 1)(\varepsilon - \sigma_0)] - \ln(2\omega - 1) + \ln(2\omega + 1) \right\}
- \frac{1}{2} (\coth \varepsilon + 1) \text{Ei}[-(2\omega + 1)(\varepsilon - \sigma_0)]
- \frac{1}{4} e^{(\varepsilon - \sigma_0)\coth \sigma_0} \text{Ei}[-(2\omega + \coth \sigma_0)(\varepsilon - \sigma_0)] \frac{1 + \text{cosh}[2(\varepsilon - \sigma_0)] - 2 \text{cosh}(2\sigma_0)}{\sinh \varepsilon \sinh \sigma_0} \right].
\]

If \(\sigma_0 \to \varepsilon\), we can use the expansion (4.13) to find
\[
R_+ R_-|_{\sigma_0=\varepsilon} = \frac{2\omega + \coth \varepsilon}{2\omega + 1},
\]
reproducing the result of Ref. [3] obtained by the other method. Alternatively, for \(\sigma_0 = 0\) the term displayed in the third line of Eq. (1.28) simplifies and we get
\[
R_+ R_-|_{\sigma_0=0} = \exp \left[ \frac{1}{2} (\coth \varepsilon - 1) \left\{ \text{Ei}[-(2\omega - 1)\varepsilon] - \ln(2\omega - 1) + \ln(2\omega + 1) \right\}
- \frac{1}{2} (\coth \varepsilon + 1) \text{Ei}[-(2\omega + 1)\varepsilon] + \frac{\sinh \varepsilon}{2\varepsilon} e^{-2\omega \varepsilon} \right].
\]

This is to be compared with Eq. (4.15) in the bosonic case.

5. Straight-line limit (\(\varepsilon\)-dependence)

As is already pointed out, the case of a straight line can be obtained from the case of a circular boundary when distances from the boundary are of the order of \(\varepsilon\). This implies that we are dealing with the \(\omega \sim 1/\varepsilon\) limit. The formulas of this Section can be obtained as this limit of the proper formulas for a circle.
\section*{5.1 Massive bosons and fermions}

For the bosonic determinant with $\omega = |m|$ we obtain from Eq. (4.12)
\begin{equation}
\ln \frac{\det(-\partial^2 + \omega^2 + 2/\sigma^2)}{\det(-\partial^2 + \omega^2)} = \frac{1 - e^{-2\omega(\varepsilon - \sigma_0)}}{\omega \varepsilon} - \text{Ei}[-2\omega(\varepsilon - \sigma_0)] \\
- e^{2(\varepsilon - \sigma_0)/\sigma_0} \left(1 - \frac{2\sigma_0}{\varepsilon}\right) \text{Ei}[-2 \left(\omega + \sigma_0^{-1}\right) (\varepsilon - \sigma_0)]. \tag{5.1}
\end{equation}

For the fermionic determinants with positive $r = \omega$ we analogously obtain from Eq. (4.20)
\begin{equation}
\ln \frac{\det(-\partial^2 + \omega^2 + 3/(4\sigma^2) + \omega/\sigma)}{\det(-\partial^2 + \omega^2)} = \frac{1 - e^{-2\omega(\varepsilon - \sigma_0)}}{4\omega \varepsilon} + C. \tag{5.2}
\end{equation}

For the fermionic determinant with negative $r = -\omega$ we analogously obtain from Eq. (4.23)
\begin{equation}
\ln \frac{\det(-\partial^2 + \omega^2 + 3/(4\sigma^2) - \omega/\sigma)}{\det(-\partial^2 + \omega^2)} = \frac{1 - e^{-2\omega(\varepsilon - \sigma_0)}}{4\omega \varepsilon} - \text{Ei}[-2\omega(\varepsilon - \sigma_0)] \\
+ e^{(\varepsilon - \sigma_0)/\sigma_0} \left(-\frac{\varepsilon}{2\sigma_0} + 1 + \frac{\sigma_0}{2\varepsilon}\right) \text{Ei}[-(2\omega + \sigma_0^{-1}) (\varepsilon - \sigma_0)] - C. \tag{5.3}
\end{equation}

When $\sigma_0 = \varepsilon$ we get from Eqs. (5.1), (5.2) and (5.3)
\begin{align}
(5.1)|_{\sigma_0=\varepsilon} &= \ln(\omega + \varepsilon^{-1}) - \ln \omega, \tag{5.4a} \\
(5.2) + (5.3)|_{\sigma_0=\varepsilon} &= \ln(\omega + \varepsilon^{-1}/2) - \ln \omega, \tag{5.4b}
\end{align}
reproducing the results of Ref. [9].

Alternatively, for $\sigma_0 = 0$ we get from Eqs. (5.1), (5.2) and (5.3)
\begin{align}
(5.1)|_{\sigma_0=0} &= \frac{1 - e^{-2\omega\varepsilon}}{\omega \varepsilon} - \text{Ei}[-2\omega \varepsilon], \tag{5.5a} \\
(5.2) + (5.3)|_{\sigma_0=0} &= \frac{1 - e^{-2\omega\varepsilon}}{2\omega \varepsilon} - \text{Ei}[-2\omega \varepsilon] + \frac{1}{2} e^{-2\omega \varepsilon}, \tag{5.5b}
\end{align}
reproducing the limits of Eqs. (4.15), (4.30).

\section*{5.2 Longitudinal modes}

The two longitudinal modes have mass 1 and the corresponding ratio of determinants reads explicitly \cite{[7]}
\begin{equation}
\frac{\det(-\Delta_{ij} + \delta_{ij})^{1/2}}{\det(-\Delta)} = \prod_{\omega} \frac{\det^{1/2}(-\partial^2 + \omega^2 + 2/\sigma^2 + 2\omega/\sigma) \det^{1/2}(-\partial^2 + \omega^2 + 2/\sigma^2 - 2\omega/\sigma)}{\det(-\partial^2 + \omega^2)}, \tag{5.6}
\end{equation}
which can be treated similarly to the massive fermionic determinants in the previous Subsection.
The two independent solutions, obeying the Dirichlet boundary condition at \( \sigma = \sigma_0 \) and \( \sigma = \infty \), are

\[
f_+ (\sigma) = e^{-\omega \sigma},
\]

\[
f_- (\sigma) = e^{-\omega (\sigma_0 + \sigma)} \left[ \frac{e^{2\omega (1 + 2\sigma_0(\omega \sigma - 1))} - e^{2\omega_0(1 + 2\sigma_0(\sigma_0\omega - 1))}}{4\sigma_0\omega^3 \sigma} \right],
\]

(5.7a)

for \( \omega > 0 \) and

\[
f_+ (\sigma) = \frac{e^{-\omega (1 + 2\sigma_0(1 + \omega \sigma))}}{\omega \sigma},
\]

\[
f_- (\sigma) = e^{-\omega (\sigma_0 + \sigma)} \left[ \frac{e^{2\omega_0(1 + 2\sigma_0(1 + \sigma_0\omega))} - e^{2\omega_0(1 + 2\omega_0(1 + \omega \sigma))}}{4\sigma_0\omega^3 \sigma} \right],
\]

(5.7b)

for \( \omega < 0 \).

The solutions (5.7b) and (5.8b) are properly normalized to be used in the Gel’fand–Yaglom technique when \( \sigma_0 = \varepsilon \). We thus obtain

\[
(5.6) = \prod_{\omega} \left( 1 + \frac{1}{\varepsilon \omega} + \frac{1}{2\varepsilon^2 \omega^2} \right).
\]

(5.9)

Analogously, we can perform the computation of the ratio (5.6) by the Gel’fand–Dikii method using the formula

\[
\frac{\det (-\partial^2 + \omega^2 \pm V_{1\pm})}{\det (-\partial^2 + \omega^2)} = \exp \left\{ \int d\omega \int_\varepsilon^\infty d\sigma \left[ (2\omega \pm 2/\sigma) R_\omega (\sigma, \sigma; V_{1\pm}) - 2\omega R_\omega (\sigma, \sigma; 0) \right] \right\},
\]

(5.10)

with

\[
V_{1\pm} = \frac{2}{\sigma^2} \pm \frac{2\omega}{\sigma},
\]

(5.11)

which is an analog of Eq. (4.18), since the potential (5.11) is also \( \omega \)-dependent. For \( \sigma_0 = \varepsilon \) this reproduces Eq. (5.9).

For \( \sigma_0 = 0 \) we find from Eq. (5.10)

\[
\ln(5.6) = \int d\omega \left[ \frac{1}{\varepsilon \omega} + e^{-2\omega} \left( \frac{5}{4} \frac{1}{\varepsilon \omega} + \frac{1}{2} \varepsilon \omega \right) - \frac{1}{2} \text{Ei}(-2\varepsilon \omega) \right].
\]

(5.12)

### 5.3 Massless fermions

The last what remains to compute is the ratio of the determinants of massless fermions to bosons:

\[
\frac{\det (-\tilde{\nabla}^2 + R^{(2)}/4)}{\det (-\Delta)} = \prod_\omega \frac{\det (-\partial^2 + \omega^2 - 1/4\sigma^2 + \omega/\sigma) \det (-\partial^2 + \omega^2 - 1/4\sigma^2 - \omega/\sigma)}{\det^2 (-\partial^2 + \omega^2)},
\]

(5.13)

which enters the ratio (3.2).

The two independent solutions are

\[
f_+(\sigma) = e^{\omega \sigma} \sqrt{x} \text{Ei}(-2\omega \sigma),
\]

(5.14a)

\[
f_- (\sigma) = e^{\omega \sigma} \sqrt{x} \left( \text{Ei}(-2\omega \sigma) - \text{Ei}(-2\sigma_0) \right) \sqrt{\sigma_0} e^{\omega \sigma_0},
\]

(5.14b)
for \( \omega > 0 \) and

\[
\begin{align*}
  f_+ (\sigma) &= e^{-\omega \sigma} \sqrt{x}, \\
  f_- (\sigma) &= e^{-\omega \sigma} \sqrt{x} \text{Ei}(2\omega \sigma) - \text{Ei}(2\omega \sigma_0) \sqrt{\sigma_0} e^{-\omega \sigma_0},
\end{align*}
\]

(5.15a)

(5.15b)

for \( \omega < 0 \).

The solutions (5.14b) and (5.15b) are properly normalized to be used in the Gel'fand–Yaglom technique when \( \sigma_0 = \varepsilon \). We thus obtain

\[
\ln R = - \prod_\omega 2 e^{2\varepsilon \omega \varepsilon} \text{Ei}(-2\varepsilon \omega).
\]

(5.16)

Analogously, we can perform the computation of the ratio (5.13) by the Gel'fand–Dikii method using the straight line limit of Eq. (4.18). For \( \sigma_0 = \varepsilon \) this reproduces Eq. (5.16).

For \( \sigma_0 = 0 \) we obtain

\[
\ln R = \prod_\omega \frac{1}{2} \left[ (2 + e^{2\omega \varepsilon}) \text{Ei}(-2\omega \varepsilon) - e^{-2\omega \varepsilon} \text{Ei}(2\omega \varepsilon) \right].
\]

(5.17)

6. Multiplying over angular modes

We have described in the two previous Sections how to calculate 1D determinants. Our primary goal is to use the results to evaluate the ratio of 2D determinants of the type

\[
R = \frac{\det(-\Delta + \mu^2)}{\det(-\Delta)}.
\]

(6.1)

We deal with the case, where the Weyl factor of the metric in conformal coordinates \( \sigma \) and \( \phi \) depends only on one variable \( \sigma \), \( \sqrt{g} = V(\sigma) \), while the operator is diagonal with respect to the angular modes \( \exp[i\omega \phi] \). The ratio (6.1) is then the product over angular modes of the ratio of 1D determinants:

\[
\ln R = \sum_\omega \ln \left[ \frac{(-\partial^2 + \omega^2 + \mu^2 V(\sigma))}{(-\partial^2 + \omega^2)} \right].
\]

(6.2)

Here the sum over \( \omega \) runs over integers or half-integers in bosonic or fermionic determinants for the circular boundary, while \( \omega \in \mathbb{R} \) for its limiting case of the straight line, when

\[
\ln R = \int d\omega \ln \left[ \frac{\det(-\partial^2 + \omega^2 + \mu^2 V(\sigma))}{\det(-\partial^2 + \omega^2)} \right].
\]

(6.3)

Given the 1D determinants calculated above, it is possible to sum up the angular modes in Eq. (6.2) or integrate in Eq. (6.3). In particular, we can explicitly calculate the difference between the sum and the integral to show that it indeed does not depend on \( \varepsilon \) as was utilized in Sect. 3. This difference between the sum and the integral is computable by Plana’s summation formula

\[
\frac{1}{2} f(0) + \sum_{m=1}^{\infty} f(m) - \int_0^\infty d\omega f(\omega) = i \int_0^\infty dt \frac{f(it) - f(-it)}{e^{2\pi t} - 1}.
\]

(6.4)
which holds when \( f(z) \) is analytic for \( \text{Re} \ z \geq 0 \), in particular, on the imaginary axis.

For the determinants in Eqs. (4.14), (4.29) with \( \sigma_0 = \varepsilon \) we have

\[
f(m) = \ln(m + a),
\]

and, using the formula (4.552) from Ref. [22], we obtain for the right-hand side of Eq. (6.4)

\[
-2 \int_0^\infty \frac{dt}{e^{2\pi t} - 1} = -\ln \Gamma(1 + a) + \left( a + \frac{1}{2} \right) \ln a - a + \frac{1}{2} \ln(2\pi).
\]

This results in the identity

\[
\frac{1}{2} \ln a + \sum_{m=1}^\infty \ln(m + a) - \int_0^\infty d\omega \ln(\omega + a)
\]

\[
= -\ln \Gamma(1 + a) + \left( a + \frac{1}{2} \right) \ln a - a + \frac{1}{2} \ln(2\pi) \text{ as } a \to \infty.
\]

Therefore, the integral exactly equals the sum as \( a \sim 1/\varepsilon \to \infty \) and no constant emerges in the difference.

An analog of Eq. (6.7) for \( \sigma_0 = 0 \) looks quite similar when \( a \ll 1/\varepsilon \to \infty \)

\[
\frac{1}{2} \text{Ei}(-2a\varepsilon) + \sum_{m=1}^\infty \text{Ei}[-2(m + a)\varepsilon] - \int_0^\infty d\omega \text{Ei}[-2(\omega + a)\varepsilon]
\]

\[
\approx -\ln \Gamma(1 + a) + \left( a + \frac{1}{2} \right) \ln a - a + \frac{1}{2} \ln(2\pi), \text{ for } a \ll 1/\varepsilon.
\]

This formula is derived in Appendix B. Again the difference between the sum and the integral does not depend on \( \varepsilon \).

6.1 Straight line

It is also seen from the above formulas that the difference between the sum and the integral is UV finite. We should be careful, however, at this point because, as is already pointed out in footnote [1], the upper half-plane bounded by the straight line with periodically identified ends has the Euler character zero rather than one as for the disk. The logarithmic UV divergence in the ratio of determinants is proportional to the (exponential of the) difference of the Euler characters [15]. This is demonstrated by explicit calculations of this Subsection. Modulo this subtlety both the dependence on a UV cutoff and the \( \varepsilon \)-dependence of the ratio of determinants for the circle can be evaluated from the integral, associated with the ratio of determinants for a straight line. In the ratio of determinants (3.2) the UV divergence cancels, since they are computed for the surfaces with the same Euler character.

Using the UV regularization \( \Lambda \) by cutting off high frequencies at \( \omega = \Lambda \), which is described in Appendix C, we obtain as \( \Lambda \to \infty \) the following results.

**Massive bosons.** We get from Eq. (5.4a) for \( \sigma_0 = \varepsilon \)

\[
\int_0^\Lambda d\omega \ln \left( 1 + \frac{1}{\omega \varepsilon} \right) = \frac{1}{\varepsilon} \left( \ln(\Lambda \varepsilon) + 1 \right),
\]

and from Eq. (5.5a) for \( \sigma_0 = 0 \)

\[
\int_0^\Lambda d\omega \left[ \frac{1 - e^{-2\omega \varepsilon}}{\omega \varepsilon} - \text{Ei}(-2\omega \varepsilon) \right] = \frac{1}{\varepsilon} \left( \ln(2\Lambda \varepsilon) + \gamma_E + \frac{1}{2} \right).
\]
Massive fermions. We get from Eq. (5.4b) for $\sigma_0 = \varepsilon$

$$\int_0^\Lambda d\omega \ln \left(1 + \frac{1}{2\omega\varepsilon}\right) = \frac{1}{2\varepsilon} \left(\ln(2\Lambda\varepsilon) + 1\right), \quad (6.11)$$

and from Eq. (5.5b) for $\sigma_0 = 0$

$$\int_0^\Lambda d\omega \left[\frac{1 - e^{-2\omega\varepsilon}}{2\omega\varepsilon} - \text{Ei}(-2\omega\varepsilon) + \frac{1}{2} e^{-2\omega\varepsilon}\right] = \frac{1}{2\varepsilon} \left(\ln(2\Lambda\varepsilon) + \gamma_E + \frac{3}{2}\right) . \quad (6.12)$$

Longitudinal modes. The logarithm of the right-hand side of Eq. (5.9) for $\sigma_0 = \varepsilon$ equals

$$\int_0^\Lambda d\omega \ln \left(1 + \frac{1}{\varepsilon \omega} + \frac{1}{2\varepsilon \omega^2}\right) = \frac{1}{\varepsilon} \left(\ln(\Lambda\varepsilon) + 1 + \frac{\pi}{4} + \frac{1}{2} \ln 2\right). \quad (6.13)$$

For $\sigma_0 = 0$ we analogously find

$$\ln (5.12) = \frac{1}{\varepsilon} \left(\ln(2\Lambda\varepsilon) + \gamma_E + \frac{7}{4}\right). \quad (6.14)$$

Massless fermions. The logarithm of the right-hand side of Eq. (5.16) for $\sigma_0 = \varepsilon$ equals

$$\int_0^\Lambda d\omega \ln (-2 e^{2\omega\varepsilon} \varepsilon \text{Ei}(-2\varepsilon\omega)) = \frac{1}{\varepsilon} \left(\frac{1}{2} \ln(\Lambda\varepsilon) - C_1\right), \quad C_1 = 0.438934 , \quad (6.15)$$

where the constant $C_1$ is found numerically. For the constant $C_2$ in Eq. (3.8) this gives

$$C_2 = \frac{3}{2} \ln 2 - 2 - \frac{\pi}{4} + 4C_1 \approx 0.01 . \quad (6.16)$$

Because of the occurred cancellation we cannot exclude that $C_2$ is actually zero. A more precise analysis is required at this point.

For $\sigma_0 = 0$ the logarithm of Eq. (5.17) equals

$$\ln (5.17) = -\frac{1}{2\varepsilon} \left(\ln(2\Lambda\varepsilon) + \gamma_E + 1\right). \quad (6.17)$$

For the constant $C_2$ in Eq. (3.8) this gives

$$C_2 = \frac{7}{4} \quad (6.18)$$

We see from the explicit computations of this Subsection that in the ratios of determinants the $\frac{1}{\varepsilon} \ln \varepsilon$ terms are the same for $\sigma_0 = \varepsilon$ and $\sigma_0 = 0$, while the $1/\varepsilon$ terms change.
6.2 Circle

**Massive bosons.** We obtain from Eq. (4.14) for $\sigma_0 = \varepsilon$

$$\frac{1}{2} \ln \frac{1}{\varepsilon} + \sum_{m=1}^{\Lambda} \left[ \ln \left( m + \frac{1}{\varepsilon} \right) - \ln(m+1) \right] = \left( \frac{1}{\varepsilon} - 1 \right) \ln \Lambda + \frac{1}{\varepsilon} \ln \varepsilon + \frac{1}{\varepsilon} - \frac{1}{2} \ln(2\pi), \quad (6.19)$$

and from Eq. (4.15) for $\sigma_0 = 0$

$$-\frac{1}{2} \left[ \ln(2\varepsilon) + \gamma_E - 2 \right] + \sum_{m=1}^{\Lambda} \left( \frac{1}{\varepsilon} - 1 \right) \left\{ \text{Ei}[-2(m-1)\varepsilon] - \text{Ei}[-2(m+1)\varepsilon] - \ln(m-1) + \ln(m+1) \right\} - \sum_{m=1}^{\infty} \text{Ei}[-2m\varepsilon] = \left( \frac{1}{\varepsilon} - 1 \right) \ln \Lambda + \frac{1}{\varepsilon} \left( \ln(2\varepsilon) + \gamma_E + \frac{3}{2} \right) - \frac{1}{2} \ln(2\pi). \quad (6.20)$$

**Massive fermions.** Substituting $\omega = m - 1/2$, we get from Eq. (4.29) for $\sigma_0 = \varepsilon$

$$\sum_{m=1}^{\Lambda} \left[ \ln \left( m + \frac{1}{2\varepsilon} - \frac{1}{2} \right) - \ln m \right] = \frac{1}{2} \left( \frac{1}{\varepsilon} - 1 \right) \ln \Lambda + \frac{1}{2\varepsilon} \ln(2\varepsilon) + \frac{1}{2\varepsilon} - \frac{1}{2} \ln(2\pi), \quad (6.21)$$

and from Eq. (4.30) for $\sigma_0 = 0$

$$\sum_{m=1}^{\Lambda} \left( \frac{1}{\varepsilon} - 1 \right) \left\{ \text{Ei}[-2(m-1)\varepsilon] - \text{Ei}[-2m\varepsilon] - \ln(m-1) + \ln(m) \right\} - \sum_{m=1}^{\infty} \text{Ei}[-2m\varepsilon] + \frac{1}{2} \sum_{m=1}^{\infty} e^{-(2m-1)\varepsilon} = \frac{1}{2} \left( \frac{1}{\varepsilon} - 1 \right) \ln \Lambda + \frac{1}{2\varepsilon} \left( \ln(2\varepsilon) + \gamma_E + 3 \right) - \frac{1}{2} \ln(2\pi). \quad (6.22)$$

6.3 Circle minus straight line

Subtracting the contributions of the circle and the straight line, we obtain for the differences:

$$\text{(6.19)} - \text{(6.9)} = - \ln \Lambda - \frac{1}{2} \ln(2\pi), \quad (6.23a)$$

$$\text{(6.20)} - \text{(6.10)} = - \ln \Lambda - \frac{1}{2} \ln(2\pi), \quad (6.23b)$$

$$\text{(6.21)} - \text{(6.11)} = - \frac{1}{2} \ln \Lambda - \frac{1}{2} \ln(2\pi), \quad (6.23c)$$

$$\text{(6.22)} - \text{(6.12)} = - \frac{1}{2} \ln \Lambda - \frac{1}{2} \ln(2\pi), \quad (6.23d)$$

which coincide for $\sigma_0 = \varepsilon$ and $\sigma_0 = 0$ both for bosons and fermions. In Appendix D we reproduce this computation, using an extension of the $\zeta$-function regularization.

The fact that the differences do not depend upon whether $\sigma_0 = \varepsilon$ or $\sigma_0 = 0$ seems to be quite nontrivial since the contributions of individual modes with a certain $m$ do change. Their product over $m$ gives, however, the same both for the bosonic and fermionic determinants. The physical implication of this fact is that only the distances far away from the boundary ($\gg \varepsilon$) contribute to the differences.
7. Conclusion and Outlook

The main result of this Paper is based on an evaluation of the ratio of 2D determinants, that emerge in the one-loop effective action of the open Green–Schwarz superstring in $AdS_5 \times S^5$ background, for a circular boundary. We have concentrated on the dependence of the ratio on the parameter $\varepsilon$, regularizing the near-boundary singularity in AdS, and have shown that it does not involve a term $\frac{1}{\varepsilon} \ln \varepsilon$ in the exponent, coming from the bosonic and fermionic determinants, since this term cancels against the one coming from the longitudinal determinant. We differ at this point from the results of Ref. [9], where this term remained and a special procedure of dealing with it was implemented. The only possible dependence of the ratio upon $\varepsilon$ is like the exponential of $1/\varepsilon$ which is similar to that of the classical action and is removable by a Legendre transformation.

The remaining $\varepsilon$-dependence resides in the reparametrization path integral of the exponential of the classical boundary action in AdS space, which is explicitly constructed in Ref. [12]. It is a counterpart of the one in flat space, known as Douglas’ integral [18], whose minimization with respect to functions, reparametrizing the boundary, or equivalently boundary metrics, reconstructs a minimal surface. The cancellation of determinants, mentioned in the previous paragraph, is like decoupling of the conformal factor (or the Liouville field) in the bulk, which happens in the critical dimension $d = 10$ for the Green–Schwarz superstring.

If the cancellation of the bulk determinants continues to higher loops, say because of supersymmetry, then the reparametrization path integral itself may be equivalent to the exact result [3] for the circle. This interesting possibility deserves further study.

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A. Semi-classical correction at large $\omega$

We can compute the fermionic determinant at large $\omega = r$ by a semi-classical expansion in $1/\omega$. For $\omega \gg 1/\varepsilon$ we can find the diagonal resolvent by iteratively solving the Gel’fand–Dikii equation (1.3):

$$R_{\omega}(\sigma, \sigma; V) = \frac{1}{2\omega} - \frac{V_{f\pm}(\sigma)}{4\omega^3} + \mathcal{O}(\omega^{-5}),$$  \hspace{1cm} (A.1)

where $V_{f\pm}$ is given by Eq. (1.17) with $\omega$ substituted by $r$. The meaning of this procedure is that we include this factor in the definition of the potential, performing the expansion in the inverse spectral parameter $\omega$ for an arbitrary potential.
The $1/\omega^3$ term in Eq. (A.1) coincides with the first Gel’fand–Dikii differential polynomial. Integrating over $\sigma$ and $\omega$, we obtain for the logarithm of the ratio
\[
\int_r^\infty d\omega \frac{1}{2\omega^2} \int_\epsilon^\infty d\sigma \, V_{f\pm}(\sigma) = \pm \frac{1}{2} \int_\epsilon^\infty d\sigma \, (\coth \sigma - 1) = \frac{1}{2} \left[ \varepsilon - \ln(2\sinh \varepsilon) \right] \xrightarrow{\varepsilon \to 0} \pm \frac{1}{2} \ln(2\varepsilon). \quad (A.2)
\]
It does not vanish as $r \to \infty$ because $V_{f\pm} \propto \rho$ itself.

We are now in a position to determine the constant $C$ introduced in Eq. (4.20). Noting that $\ln R_{f\pm} \to C$ from Eq. (4.20), we deduce that
\[
C = \frac{1}{2} \ln \coth \varepsilon + \frac{1}{2} \varepsilon \xrightarrow{\varepsilon \to 0} - \frac{1}{2} \ln(2\varepsilon). \quad (A.3)
\]
This reproduces the result of Ref. [9].

**B. Summing over angular modes for $\sigma_0 = 0$**

The sum over angular modes, i.e. over $m$ in the bosonic case or $r = m + 1/2$ in the fermionic case, involves for $\sigma_0 = 0$ an exponential of the sum
\[
\sum_{m=1}^\infty \text{Ei} \left[ -2(m + a)\varepsilon \right] = -\int_{2\varepsilon}^\infty \frac{dt}{t^2} e^{-at}, \quad (B.1)
\]
which is obviously convergent for $a > -1$. Notice that this produces something different from what results from first expanding in $\varepsilon$ and then summing over $m$. The summation is hence not interchangeable with the $\varepsilon \to 0$ limit.

For small $\varepsilon$ and $a \sim 1$ we can evaluate the integral in Eq. (B.1) as
\[
\int_{2\varepsilon}^\infty \frac{dt}{t^2} e^{-at} \approx \frac{1}{2\varepsilon} + \left( a + \frac{1}{2} \right) \left[ \ln(2\varepsilon) + \gamma_E \right] + \ln(1+a) - \frac{1}{2} \ln(2\pi) - \sum_{n=2}^\infty B_n(-a) \frac{(2\varepsilon)^{n-1}}{(n-1)n!}, \quad (B.2)
\]
where
\[
B_0(-a) = 1, \quad B_1(-a) = -a - \frac{1}{2}, \quad B_2(-a) = a^2 + a + \frac{1}{6}, \quad (B.3)
\]
are the Bernoulli polynomials.

In Eq. (B.2) $1/\varepsilon$ remarkably emerges as a regularized contribution of high modes. If we substituted each term in the sum (B.1) by its $\varepsilon \to 0$ asymptote and used the $\zeta$-function regularization, we would not get this term. This happens because the $m \to \infty$ and $\varepsilon \to 0$ limits do not commute.

The integral which is the companion of the sum (B.1) reads
\[
- \int_0^\infty d\omega \, \text{Ei} \left[ -2(\omega + a)\varepsilon \right] \approx \int_{2\varepsilon}^\infty \frac{dt}{t^2} e^{-at} \approx \frac{1}{2\varepsilon} + a \left[ \ln(2a\varepsilon) + \gamma_E - 1 \right] - a^2 \varepsilon + \mathcal{O}(\varepsilon^2), \quad (B.4)
\]
for $a \ll 1/\varepsilon$. We can calculate them separately because both the sum and the integral are convergent. The difference of the two reproduces Eq. (6.8).
C. Regularization by “proper frequency”

The usual proper time regularization of functional determinants is defined as
\[
-\text{tr} \ln(-\Delta + V)_{\text{reg}} = \int dx \int_{\Lambda-2}^{\infty} \frac{dt}{t} \langle x | e^{t(\Delta - V)} | x \rangle = \int dx \int_{1}^{\infty} \frac{d\tau}{\tau} \langle x | e^{\tau(\Delta - V)/\Lambda^2} | x \rangle.
\]

We use instead the regularization consisting in cutting off very high frequencies:
\[
-\text{tr} \ln(-\Delta + V)_{\text{reg}} = \int dx \int_{0}^{\Lambda^2} d\omega^2 R_\omega(x, x; V)
\]
\[
= \int dx \int_{0}^{\Lambda^2} d\omega^2 \int_{0}^{\infty} dt \langle x | e^{t(\Delta - \omega^2 - V)} | x \rangle
\]
\[
= \int dx \int_{0}^{\infty} \frac{dt}{t} \left(1 - e^{-t\Lambda^2}\right) \langle x | e^{t(\Delta - V)} | x \rangle
\]
\[
= \int dx \int_{0}^{\infty} \frac{d\tau}{\tau} \left(1 - e^{-\tau}\right) \langle x | e^{\tau(\Delta - V)/\Lambda^2} | x \rangle.
\]

The two look similar, the difference shows up only for eigenvalues of the order of the cutoff.

D. Summing using the Lerch function

An alternative to the regularization by cutting off high frequencies from Appendix C is the “supersymmetric summation” used in [9], where the following series appears
\[
\sum_{m=1}^{\infty} e^{-\mu m} \left[ \ln(m + a) - \ln(m + b) \right] = -\int_{0}^{\infty} \frac{dt}{t} \left( e^{-at} - e^{-bt} \right).
\]

We have used here the integral representation of the Lerch transcendent
\[
\sum_{m=1}^{\infty} \frac{z^m}{(m + a)^s} = \frac{z}{\Gamma(s)} \int_{0}^{\infty} \frac{dt}{t} t^{s-1} \frac{e^{-at}}{e^t - z}.
\]

To prove Eq. (D.1), we expand
\[
\frac{1}{e^{t+\mu} - 1} = \sum_{m=1}^{\infty} e^{-m(t+\mu)},
\]
and use
\[
-\int_{0}^{\infty} \frac{dt}{t} \left[ e^{-(a+m)t} - e^{-(b+m)t} \right] = \ln(m + a) - \ln(m + b).
\]

To compute the integral in Eq. (D.1), we rewrite it as
\[
-\int_{0}^{\infty} \frac{dt}{t} \left( e^{-at} - e^{-bt} \right) = -\int_{0}^{\infty} \frac{dt}{t} \left( e^{-at} - e^{-bt} \right) \left( \frac{1}{e^{t+\mu} - 1} - \frac{1}{t + \mu} + \frac{1}{t + \mu} \right).
\]

We have
\[
-\int_{0}^{\infty} \frac{dt}{t} \left( e^{-at} - e^{-bt} \right) = \frac{1}{\mu} \ln \frac{a}{b} - \frac{1}{\mu} \left[ e^{\mu a} \text{Ei}(a\mu) - e^{b\mu} \text{Ei}(b\mu) \right]
\]
\[
= \left( a - b \right) \left( \ln \frac{1}{\mu} - \gamma_E + 1 \right) - a \ln a + b \ln b + O(\mu),
\]
For small $\mu$ we obtain for the integral on the right-hand side of Eq. (D.1):

$$\langle D.1 \rangle = (a - b) \left( \ln \frac{1}{\mu} - \gamma_E \right) - \ln \Gamma(1 + a) + \ln \Gamma(1 + b) + O(\mu),$$  \hspace{1cm} (D.8)

which coincides with the formula

$$\sum_{m=1}^{\Lambda} [\ln (m + a) - \ln (m + b)] = (a - b) \ln \Lambda - \ln \Gamma(1 + a) + \ln \Gamma(1 + b),$$  \hspace{1cm} (D.9)

obtained by using Eq. (6.7) with the regularization of Appendix [C], if

$$\Lambda = \frac{1}{\mu} e^{-\gamma_E}.$$  \hspace{1cm} (D.10)

For the bosonic determinants we obtain: circle for $\sigma_0 = \varepsilon$

$$\frac{1}{2} \ln \frac{1}{\varepsilon} + \sum_{m=1}^{\infty} e^{-\mu m} \left[ \ln \left( m + \frac{1}{\varepsilon} \right) - \ln (m + 1) \right] = \left( \frac{1}{\varepsilon} - 1 \right) \ln \Lambda + \frac{1}{\varepsilon} \ln \varepsilon + \frac{1}{\varepsilon} - \frac{1}{2} \ln (2\pi),$$  \hspace{1cm} (D.11)

and circle for $\sigma_0 = 0$

$$- \frac{1}{2} [\ln (2\varepsilon) + \gamma_E - 2] + \sum_{m=1}^{\infty} \frac{1}{2} \left( \frac{1}{\varepsilon} - 1 \right) \left\{ \text{Ei}[ -2(m-1)\varepsilon ] - \text{Ei}[ -2(m+1)\varepsilon ] \right\} - \sum_{m=1}^{\infty} \text{Ei}[ -2(m+1)\varepsilon ]
$$

$$= \left( \frac{1}{\varepsilon} - 1 \right) \ln \Lambda + \frac{1}{\varepsilon} \left( \ln (2\varepsilon) + \gamma_E + \frac{1}{2} \right) - \frac{1}{2} \ln (2\pi),$$  \hspace{1cm} (D.12)

straight line for $\sigma_0 = \varepsilon$

$$\int_{0}^{\infty} d\omega \ e^{-\mu \omega} \ln \left( 1 + \frac{1}{\omega \varepsilon} \right) = \frac{1}{\varepsilon} \left( \ln (\Lambda \varepsilon) + 1 \right),$$  \hspace{1cm} (D.13)

and straight line for $\sigma_0 = 0$

$$\int_{0}^{\infty} d\omega \ e^{-\mu \omega} \left[ \frac{1 - e^{-2\omega \varepsilon}}{\omega \varepsilon} - \text{Ei}(2\omega \varepsilon) \right] = \frac{1}{\varepsilon} \left( \ln (2\Lambda \varepsilon) + \gamma_E + \frac{1}{2} \right).$$  \hspace{1cm} (D.14)

For the difference of the circle and the straight line we find

$$\langle D.11 \rangle - \langle D.13 \rangle = - \ln \Lambda - \frac{1}{2} \ln (2\pi),$$  \hspace{1cm} (D.15a)

$$\langle D.12 \rangle - \langle D.14 \rangle = - \ln \Lambda - \frac{1}{2} \ln (2\pi),$$  \hspace{1cm} (D.15b)
which again coincide. The divergent term in both cases is owing to the difference in the Euler characters (see footnote 1).

Fermions: circle for \( \sigma_0 = 0 \) with positive \( r = m - 1/2 \) \((m \geq 1)\)

\[
\sum_{m=1}^{\infty} e^{-\mu r} \ln R_f^+ \bigg|_{\sigma_0=0} = \sum_{m=1}^{\infty} e^{-\mu(m-1/2)} \frac{1}{4} (\coth \varepsilon - 1) \left\{ \text{Ei} [-2(m-1)\varepsilon] - \text{Ei} [-2m\varepsilon] \right. \\
- \ln(m-1) + \ln m \left. \right\} + \sum_{m=1}^{\infty} e^{-\mu(m-1/2)} C \\
= \frac{1}{4} (\coth \varepsilon - 1) \left[ \ln(2\varepsilon) + \gamma_E + \ln \frac{1}{\mu} \right] + C \frac{1}{\mu},
\]  
(D.16)

where \( C \) is given by Eq. (A.3), and with negative \( r = -m - 1/2 \) \((m \geq 0)\)

\[
\sum_{m=0}^{\infty} e^{-\mu |r|} \ln R_f^- \bigg|_{\sigma_0=0} = \sum_{m=0}^{\infty} e^{-\mu(m+1/2)} \left\{ \frac{1}{4} (\coth \varepsilon - 1) \left[ \text{Ei} [-2(m+1)\varepsilon] - \ln m + \ln(m+1) \right] \\
- \frac{1}{4} (\coth \varepsilon + 3) \text{Ei} [-2(m+1)\varepsilon] + \frac{1}{2\varepsilon} e^{-(2m+1)\varepsilon} - C \right\} \\
= \frac{1}{4} (\coth \varepsilon - 1) \left[ \ln(2\varepsilon) + \gamma_E + \ln \frac{1}{\mu} \right] + \frac{3}{4\varepsilon} + \frac{1}{2} \ln(2\varepsilon) + \gamma_E \\
- \frac{1}{2} \ln(2\pi) - C \frac{1}{\mu},
\]  
(D.17)

For the sum we find

\[
(D.16) + (D.17) = \frac{1}{2} (\coth \varepsilon - 1) \ln \Lambda + \frac{1}{2\varepsilon} \left[ \ln(2\varepsilon) + \gamma_E + \frac{3}{2} \right] - \frac{1}{2} \ln(2\pi).
\]  
(D.18)

The contribution of the straight line \((\sigma_0 = 0)\) is

\[
\int_{0}^{\infty} d\omega \ e^{-\mu \omega} \left[ \frac{1 - e^{-2\omega \varepsilon}}{2\omega \varepsilon} - \text{Ei} [-2\omega \varepsilon] + \frac{1}{2} e^{-2\omega \varepsilon} \right] = \frac{1}{2\varepsilon} \left[ \ln \Lambda + \ln(2\varepsilon) + \gamma_E + \frac{3}{2} \right],
\]  
(D.19)

so that we find for the difference

\[
(D.18) - (D.19) = -\frac{1}{2} \ln \Lambda - \frac{1}{2} \ln(2\pi).
\]  
(D.20)

Fermions with an alternative “bosonic” regularization: circle for \( \sigma_0 = 0 \) with positive \( r = m - 1/2 \) \((m \geq 1)\)

\[
\sum_{m=1}^{\infty} e^{-\mu r} \ln R_f^+ \bigg|_{\sigma_0=0} = \sum_{m=1}^{\infty} e^{-\mu m} \frac{1}{4} (\coth \varepsilon - 1) \left\{ \text{Ei} [-2(m-1)\varepsilon] - \text{Ei} [-2m\varepsilon] \right. \\
- \ln(m-1) + \ln m \left. \right\} + \sum_{m=1}^{\infty} e^{-\mu(m-1/2)} C \\
= \frac{1}{4} (\coth \varepsilon - 1) \left[ \ln(2\varepsilon) + \gamma_E + \ln \frac{1}{\mu} \right] + C \left[ \frac{1}{\mu} - \frac{1}{2} \right].
\]  
(D.21)
and with negative $r = -m - 1/2$ ($m \geq 0$)

$$
\sum_{m=0}^{\infty} e^{-\mu|r+1/2|} \ln R_f \bigg|_{\sigma=0} = \sum_{m=0}^{\infty} e^{-\mu m} \left\{ \frac{1}{4} \left( \coth \varepsilon - 1 \right) \left[ \text{Ei}(-2m\varepsilon) - \ln m + \ln(m+1) \right] - \frac{1}{4} \left( \coth \varepsilon + 3 \right) \text{Ei}[-2(m+1)\varepsilon] + \frac{1}{2\varepsilon} e^{-(2m+1)\varepsilon} - C \right\} \\
= \frac{1}{4} \left( \coth \varepsilon - 1 \right) \left[ \ln(2\varepsilon) + \gamma_E + \ln \frac{1}{\mu} + \frac{3}{4\varepsilon} + \frac{1}{2} \left( \ln(2\varepsilon) + \gamma_E \right) \right] - \frac{1}{2} \ln(2\pi) - C \left( \frac{1}{\mu} + \frac{1}{2} \right) .
\tag{D.22}
$$

For the sum we find

$$
(D.16) + (D.17) = \frac{1}{2} \left( \coth \varepsilon - 1 \right) \ln \Lambda + \frac{1}{2\varepsilon} \left[ \ln(2\varepsilon) + \gamma_E + \frac{3}{2} \right] - \frac{1}{2} \ln(2\pi) - C ,
\tag{D.23}
$$

and

$$
(D.23) - (D.19) = -\frac{1}{2} \ln \Lambda - \frac{1}{2} \ln(2\pi) - C ,
\tag{D.24}
$$

which differs from (D.20).

We have thus reproduced in this Appendix the results of Sect. [1].

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