Complementarity between signalling and local indeterminacy in quantum nonlocal correlations

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The correlations that violate the CHSH inequality are known to have complementary contributions from signaling and local indeterminacy. This complementarity is shown to represent a strengthening of Bell’s theorem, and can be used to certify randomness in a device-independent way, assuming neither the validity of quantum mechanics nor even no-signaling. We obtain general nonlocal resources that can simulate the statistics of the singlet state, encompassing existing results. We prove a conjecture due to Hall (2010) and Kar et al. (2011) on the complementarity for such resources.

PACS numbers: 03.65.Ud, 03.67.-a

I. INTRODUCTION

Quantum correlations are nonlocal in that they can violate Bell-type inequalities \[1,2\], which a local-realistic model cannot violate. A 1-bit signal \[3\] or a single use of separate Bell-type inequalities \[1, 2\], which a local-realistic model cannot violate. A 1-bit signal \[3\] or a single use of Bell’s theorem (or its variants) says that a bipartite correlation \(P(ab|xy)\) generated by local-realistic theories must satisfy the Clauser-Horne-Shimony-Holt (CHSH) inequality:

\[
\Lambda = E(0, 0) + E(0, 1) + E(1, 0) - E(1, 1) \leq 2, \quad (1)
\]

with \(a, b, x, y \in \{0, 1\}\). Here \(E(x, y) = P(a = b | xy) - P(a \neq b | xy)\). More generally, it applies to any bipartite correlation where outcomes \(a, b\) are assumed to be pre-determined, and \(x, y\) are freely chosen \[7\] and uncorrelated with the other party’s output.

A correlation \(P = P(a, b|x, y)\) is non-signaling if it satisfies:

\[
\begin{align*}
\sum_b P(a, b|x, y = 0) &= \sum_b P(a, b|x, y = 1) \equiv P(a|x), \\
\sum_a P(a, b|x = 0, y) &= \sum_a P(a, b|x = 1, y) \equiv P(b|y), \quad (2)
\end{align*}
\]

i.e., Alice knows nothing of Bob’s input, and vice versa. The amount of signal from Alice to Bob and Bob to Alice, respectively, can be quantified either statistically as \(S\) or entropically as \(H_S\), as follows:

\[
\begin{align*}
S_{A\to B} &= \sup_{x, x', y, b} |P(b|x, y) - P(b|x', y)|, \\
S_{B\to A} &= \sup_{y, y', a} |P(a|x, y) - P(a|x, y')|, \quad (3)
\end{align*}
\]

where \(P(a|x, y) = \sum_b P(a, b|x, y)\) and \(P(b|x, y) = \sum_a P(a, b|x, y)\). The signal

\[
S = \max\{S_{A\to B}, S_{B\to A}\} \quad (4)
\]

The entropic version of quantity of signal is

\[
H_S = \max_x \{\sup_y I(A : Y), \sup_y I(B : X)\}, \quad (5)
\]

where \(I(A : Y)\) denotes mutual information and \(A, B, X, Y\) are random variables representing \(a, b, x, y\).

The communication cost \(C\) of \(P\) is the minimum size of a classical message that must be exchanged between Alice and Bob in a classical protocol to reproduce \(P\). In general, this message must contain both the input and outcome information of the other party \[8\]. However, assuming that both parties have unrestricted access to shared randomness, and that measurement settings are chosen freely, the outcome information may be taken to be determined by the pre-shared randomness. Thus it suffices for the communication cost to be large enough to convey just the settings information. For the two-input two-outcome correlations considered here, this is just 1 bit. For example, the PR box is a non-signaling resource that satisfies the condition

\[
a \oplus b = x \cdot y, \quad (6)
\]
where the CHSH inequality to its algebraic maximum, going beyond the Tsirelson bound \( T \). It is described by the action

\[
P(a, b|x, y) = \begin{cases} \frac{1}{2} & \text{Eq. (6) holds} \\ 0 & \text{otherwise.} \end{cases}
\]

The indeterminacy of \( P \) can be quantified statistically as

\[
I \equiv \min_{x, y} \{ P(o|x, y), 1 - P(o|x, y) \},
\]

where \( o \) is the outcome on any one of the party’s side. If \( P \) is interpreted operationally, i.e., \( P \) is taken to be the correlation generated by measurements on a physical state, then it represents unpredictability \( \Pi \). If \( P \) is interpreted as a simulating resource or as an element of an underlying hidden-variable theory, then it represents indeterminacy \( \langle I \rangle \), a term which we also use generically here to describe a formal correlation \( P \). The information theoretic equivalent of \( I \) may be given by the measure

\[
H_I \equiv \sup_{x, y} H(O|x, y),
\]

where \( H(O|x, y) = -\sum_o p_o \log_2(p_o) \).

III. INTERPLAY OF SIGNALING AND INDETERMINACY IN NONLOCAL CORRELATIONS

A correlation \( P \) generated by two-input, two-output bipartite measurements on a physical state, or which can be used as a resource to reproduce such correlations, can be decomposed as a convex combination of deterministic correlations or ‘boxes’ (for which \( P(a, b|x, y) = 0 \) or \( 1 \) that are 1-bit strategies, having the form \( P(a, b|x, y) = \delta_{f(x,y)}^{a} \delta_{g(y)}^{b} \)) or \( P(a, b|x, y) = \delta_{f(x)}^{a} \delta_{g(y)}^{b} \) (with \( C = 1 \)) or 0-bit strategies, having the form \( P(a, b|x, y) = \delta_{f(x)}^{a} \delta_{g(y)}^{b} \) (\( C = 0 \) \( \Pi \)).

We may uniformly average some pairs of the above signaling boxes to create non-signaling correlations. For example, a uniform average of \( P_{1}^{+} = P(ab|x,y) = \delta_{0}^{a} \delta_{x}^{b} \) and \( P_{1}^{-} = P(ab|x,y) = \delta_{1}^{a} \delta_{x}^{b} \), results in the PR box \( \Pi \). We call pairs like \( P_{1}^{+} \) as signaling pairs, with \( P_{1}^{+} \) as the signaling complements of \( P_{1}^{-} \). By averaging signal complements non-uniformly, we obtain resources of intermediate signaling. A complete listing of the deterministic signaling correlations that satisfy the PR box condition \( \Pi \) are given in Table I. The no-signaling polytope has 8 nonlocal vertices, corresponding to the PR boxes, characterized by the three bits \( \mu_{1}, \mu_{2}, \mu_{3} \), which define the general PR box relation \( a \oplus b = x \cdot y \oplus \mu_{1} x \oplus \mu_{2} y \oplus \mu_{3} \).

### Theorem 1

For correlation \( P \) in Eq. (\( \Pi \))

\[
S + 2I \geq C.
\]

### Proof sketch

We first consider simulating \( P \) that is simulable using strategies in the scope (PR box) \( \mu_{j} = 0 \). We do not require individual signal pairs to be balanced.

| Input | \( S_{1}^{+} \) | \( S_{1}^{-} \) | \( S_{2}^{+} \) | \( S_{2}^{-} \) | \( S_{3}^{+} \) | \( S_{3}^{-} \) | \( S_{4}^{+} \) | \( S_{4}^{-} \) | \( S_{5}^{+} \) | \( S_{5}^{-} \) | \( S_{6}^{+} \) | \( S_{6}^{-} \) | \( S_{7}^{+} \) | \( S_{7}^{-} \) | \( S_{8}^{+} \) | \( S_{8}^{-} \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| 00    | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 |
| 01    | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 |
| 10    | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 | 00 | 11 |
| 11    | 01 | 10 | 01 | 10 | 01 | 10 | 01 | 10 | 01 | 10 | 01 | 10 | 01 | 10 | 01 | 10 |
From Table I, it is seen that Bob receives a signal from Alice setting $y = 0$, when the strategies are $S_{y=0}^A, S_{y=1}^B, S_{x=0}^B$, and $S_{x=1}^B$. The probabilities of these strategies thus determine the signal $S_{v=0}^{A \rightarrow B}$ in the resources. Thus, by imbalancing this and the other 4 signal complements and denoting the signal in each case by $s_k$, we have

\[
S_{y=0}^{A \rightarrow B} = (p_1^0 + p_2^0 + p_3^0 + p_4^0) - (p_1^0 + p_2^0 + p_3^0 + p_4^0) \equiv s_1
\]

\[
S_{y=1}^{A \rightarrow B} = (p_1^1 + p_2^1 + p_3^1 + p_4^1) - (p_1^1 + p_2^1 + p_3^1 + p_4^1) \equiv s_2
\]

\[
S_{x=0}^{B \rightarrow A} = (p_1^0 + p_2^1 + p_3^0 + p_4^1) - (p_1^0 + p_2^1 + p_3^0 + p_4^1) \equiv s_3
\]

\[
S_{x=1}^{B \rightarrow A} = (p_1^2 + p_2^1 + p_3^0 + p_4^0) - (p_1^2 + p_2^1 + p_3^0 + p_4^0) \equiv s_4.
\]

Therefore, $\sum_{j=1}^4 (p_j^0 - p_j^1) = s_1 + s_2 + s_3 + s_4$. Since $P$ in (13) has non-vanishing probability only in 1-way strategies, and thus its communication cost is $C = \sum_{j=1}^4 (p_j^0 + p_j^1)$, it follows that

\[
\sum_{j=1}^4 p_j^0 = \frac{C + s_1 + s_2 + s_3 + s_4}{2}
\]

\[
\sum_{j=1}^4 p_j^1 = \frac{C - s_1 - s_2 - s_3 - s_4}{2}.
\]

From Table I we have $P(00|00) \geq \sum_{j=1}^4 p_j^0$ and $P(11|00) \geq \sum_{j=1}^4 p_j^1$, so that

\[
P(00|00) \geq \frac{C + s_1 + s_2 + s_3 + s_4}{2}
\]

\[
P(11|00) \geq \frac{C - s_1 - s_2 - s_3 - s_4}{2}.
\]

The inequalities above follow from the fact that $P(00|00)$ etc. may have contributions also from the local strategies. (If $C = 1$, we would have equalities here.) By the same method we have all the remaining conditional probabilities

\[
P(00|01) \geq \frac{C + (s_1 + s_2 - s_3 + s_4)}{2}
\]

\[
P(11|01) \geq \frac{C - (s_1 + s_2 - s_3 + s_4)}{2}
\]

\[
P(00|10) \geq \frac{C + (-s_1 + s_2 + s_3 + s_4)}{2}
\]

\[
P(11|10) \geq \frac{C - (-s_1 + s_2 + s_3 + s_4)}{2}
\]

\[
P(01|11) \geq \frac{C + (-s_1 + s_2 + s_3 - s_4)}{2}
\]

\[
P(10|11) \geq \frac{C - (-s_1 + s_2 + s_3 - s_4)}{2}.
\]

Let us consider case $[A]$ $s_1 \leq s_2 \leq s_3 \leq s_4$ and $s_1 + s_4 > s_2 + s_3$. From definition (8), we have

\[
I \geq \frac{C + (-s_1 + s_2 + s_3 - s_4)}{2},
\]

and Ineq. (11) using assumption [A1]. If we consider the case $[A2]$ $s_1 + s_4 < s_2 + s_3$, then

\[
I \geq \frac{C - (-s_1 + s_2 + s_3 - s_4)}{2}
\]

from which, once again, Eq. (11) follows, using condition [A2]. Repeating the above exercise for all other cases, Eq. (11) is seen to hold in a similar fashion. Since each scope (i.e., PR box) can be converted to another using reversible local operations (12), the result holds true for any mixture of the scopes. 

For an arbitrary nonlocal correlation $P$, our result (11) implies

\[
S + 2I > 0.
\]

Eq. (18) can be interpreted as an operational version of Bell’s inequality, derived under the assumptions of signal-locality ($S = 0$) and predictability ($I = 0$) (10). Our result Eq. (11) is then seen to represent a strengthening of Bell’s theorem, Eq. (15).

**IV. CERTIFIED RANDOMNESS**

Randomness, while very important in modern science and industry for simulations, is nevertheless an elusive concept (13). Given a purported source of randomness, it is difficult to ascertain its random nature without characterizing the detailed structure and mechanism behind it. Randomness certified by Bell’s theorem provides a way out of this difficulty (14, 15). If Bell’s inequality is violated by the observed correlation $P$ between two distant parties, Alice and Bob, whose measurements are spacelike-separated, then as signaling is fundamentally disallowed, Eq. (11) implies that there is an irreducible randomness in $P$, irrespective of a detailed characterization of the devices used. Thus a bound on randomness obtained by a Bell test is device-independent. Our above results can be used to generalize this idea in two ways: one is that quantum mechanics is not assumed, and, further nor is no-signaling.
It is known that \( C \geq \frac{\Lambda(P)}{2} - 1 \) [11]. Substituting this in Eq. (11), we find:

\[
I \geq \frac{\Lambda(P)}{4} - \frac{1 + S}{2},
\]

(19)
as the amount of randomness certified by a Bell test in the presence of signaling. Intuitively, the greater the signal, the larger the classical explanation for a Bell’s inequality violation [16], and hence lower the certifiable randomness.

Rewriting Eq. (19), we obtain a version of the relaxed Bell’s inequality

\[
\Lambda(P) - 2 \leq 2S + 4I,
\]

(20)
where the amount of CHSH inequality violation (in the l.h.s) is bounded by the signaling and indeterminacy in the correlation (cf. a similar result in Ref. [7]).

V. COMPLEMENTARITY BETWEEN SIGNALING AND INDETERMINACY IN SIMULATING SINGLET STATISTICS

If the correlation \( P \) is used as a resource to simulate the correlations in a physical theory, then Eq. (11) represents the complementarity for the simulating resources. Now, modelled as a mixture of local and nonlocal strategies, correlations representing a singlet have no local contribution [17]. Thus, consider as a resource the general signaling, nonlocal box obtained by the convex combination of the 1-bit strategies of Table 4

\[
P = \sum_{j=1}^{4}(p_{j}^{+}S_{j}^{+} + p_{j}^{-}S_{j}^{-}),
\]

(21)
where \( \sum_{j=1}^{4}(p_{j}^{+} + p_{j}^{-}) = 1 \). The protocol of Toner and Bacon [3] corresponds to the case of setting all \( p_{j}^{+} \) in Eq. (21) to 0 except one (say, \( p_{1}^{+} = 1 \)). The PR box simulation of Cerf et al. [4] corresponds to the case of setting all \( p_{j}^{+} \) in Eq. (21) to 0 except those belonging to one signaling pair, which are both equally weighted (say, \( p_{1}^{+} = p_{1}^{-} = \frac{1}{2} \)). The more general simulation presented by Kar et al. [6] corresponds to the case of setting all \( p_{j}^{+} \) in Eq. (21) to 0 except those belonging to one pair, which are not required to be equally weighted (say, \( p_{1}^{+} + p_{1}^{-} = 1 \) and \( p_{j}^{+} \neq p_{j}^{-} \)). In our notation, all these nonlocal resources belong to the same signaling pair. Our result follows straightforwardly from the observation that the simulation protocols of Refs. [4, 6] work even when \( P \) is generalized as in Eq. (21) with unrestricted signal domain in the same PR scope, essentially because each of the underlying deterministic strategies considered satisfies the condition [11].

A general resource of the type (21) drawn from any other, fixed scope (a different triple of values \( \mu_{j} \)) would also do, since the different PR boxes are mutually transformable through reversible local relabelling.

For completeness, we give the full protocol that simulates the singlet state correlation using resource \( P \) and pre-shared randomness \( \theta_{1} \) and \( \theta_{2} \), which are independently and uniformly distributed directional vectors. Alice (Bob) is given vector \( \hat{x} (\hat{y}) \) and outputs binary number \( x (y) \) taking value 0 or 1. To simulate singlet statistics, they must satisfy:

\[
x \oplus y|x, y = \frac{1 + \hat{x} \cdot \hat{y}}{2}.
\]

(22)
where the overline indicates the expectation value. To this end, Alice computes \( x = \text{sgn}(\hat{x} \cdot \hat{\theta}_{1}) \oplus \text{sgn}(\hat{x} \cdot \hat{\theta}_{2}) \), which she inputs into the resource \( P \). Here \( \text{sgn}(z) = 0 \) (1) if \( z < 0 \) (\( z \geq 0 \)). Using output \( a \) from the resource, Alice obtains:

\[
x = a \oplus \text{sgn}(\hat{x} \cdot \hat{\theta}_{1}).
\]

(23)
Bob computes the quantity \( y = \text{sgn}(\hat{y} \cdot \hat{\theta}_{+}) \oplus \text{sgn}(\hat{y} \cdot \hat{\theta}_{-}) \), where \( \hat{\theta}_{\pm} = \hat{\theta}_{1} \pm \hat{\theta}_{2} \), which input into \( P \), produces output \( b \). Bob uses this to compute:

\[
y = b \oplus \text{sgn}(\hat{y} \cdot \hat{\theta}_{+}) \oplus 1.
\]

(24)
This yields

\[
x \oplus y = a \oplus b \oplus \text{sgn}(\hat{x} \cdot \hat{\theta}_{1}) \oplus \text{sgn}(\hat{y} \cdot \hat{\theta}_{+}) \oplus 1
\]

\[
= \sum_{j}(P_{j}^{+} \oplus P_{j}^{-})xy \oplus \text{sgn}(\hat{x} \cdot \hat{\theta}_{1}) \oplus \text{sgn}(\hat{y} \cdot \hat{\theta}_{+}) \oplus 1
\]

\[
= xy \oplus \text{sgn}(\hat{x} \cdot \hat{\theta}_{1}) \oplus \text{sgn}(\hat{y} \cdot \hat{\theta}_{+}) \oplus 1,
\]

(25)
from which Eq. (22) follows using the method of Ref. [4].

Now, 1 bit is sufficient to simulate the singlet, since the general resource (21) has a communication cost of 1 bit. That this is also necessary [17] follows from the optimality of the Toner-Bacon protocol. Accordingly, we set \( C = 1 \) in Eq. (11), obtaining the complementary relation

\[
S + 2I \geq 1
\]

(26)
for signal and indeterminacy contributions in correlations in singlet statistics. This was conjectured by Hall [7]. If we consider a non-signaling model of quantum mechanics, we set \( S = 0 \) in Eq. (26), so that \( I = \frac{1}{2} \). Thus, 1 bit of randomness can be certified using singlets (cf. [11]).

To obtain the entropic version of the above, we note that entropic indeterminacy is, using Eq. (22), just

\[
H_{I} \equiv -I \log_{2}(I) - (1 - I) \log_{2}(1 - I)
\]

(27)
For a model with signal \( S \) from Alice to Bob, there is a setting of Bob such that the probability of an outcome, \( p \), shifts to \( p + S \), when Alice toggles her input. Thus, the entropic signal is given by \( H_{S} = H(p + \frac{1}{2}) - \frac{1}{2}H(p) - \frac{1}{2}H(p + S) \), from which it follows, by optimizing over \( p \), that

\[
H_{S} \geq 1 - H \left( \frac{1 - S}{2} \right).
\]

(28)
From Eqs. (27) and (28), we have
\[ H_S + H_I \geq 1, \quad \text{(29)} \]
conjectured by Hall [7] and Kar et al. [6].

VI. DISCUSSIONS

The complementarity of contributions from signaling and local indeterminacy to nonlocal correlations was derived, and shown to represent a strengthening of Bell’s theorem. Our result, which applies to arbitrary degrees of violation of Bell’s inequality, was used to verify a conjecture about the complementarity in the resources required to simulate singlet statistics. Finally we obtain a bound on the randomness that can be certified by non-locality even in the presence of signaling.

The complementarity (26) unifies a number of results on the simulation of singlet statistics. Leggett [18] and Gröblacher et al. [19] proposed non-signaling models with local bias, which were shown to be incapable of reproducing singlet statistics. Local bias is equivalent in our terminology to \( I < \frac{1}{2} \), and since \( S = 0 \) here, such models fail to satisfy Ineq. (26). Thus complementarity explains why such models fail to simulate singlet statistics. It also provides an alternative proof of the result obtained by Branciard et al. [20], that any non-signaling model of singlet statistics must have unbiased marginals (\( I = \frac{1}{2} \)).

Acknowledgments

SA acknowledges support through the INSPIRE fellowship [IF120025] by the Department of Science and Technology, Govt. of India and Manipal University graduate programme. RS acknowledges support from the DST project SR/S2/LOP-02/2012.

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