Phase Dependent Loss Analysis for RIS Systems

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Abstract—In this paper we focus on phase dependent loss (PDL), an important aspect of reconfigurable intelligent surfaces (RIS) where the signals reflected from the RIS elements are attenuated by varying amounts depending on the phase rotation provided by the element. To evaluate the effects of PDL, we analyse the SNR of a SIMO RIS-aided wireless link. We assume that the channel between the base station (BS) and RIS is a rank-1 LOS channel, while the user (UE)-BS and UE-RIS are correlated Rayleigh channels. The RIS design is optimal in the absence of PDL and maximizes the SNR in this scenario. Specifically, we derive an exact expression for the mean SNR in the presence of PDL. The attenuation function used for PDL was developed from a detailed circuit analysis of RIS elements. Leveraging the derived results, we analytically characterise the impact of PDL on the mean SNR. Numerical results are conducted to validate the derived expressions and verify the analysis.

I. INTRODUCTION

Research into reconfigurable intelligent surfaces (RISs) has shown that intelligently tuning the RIS phases can significantly improve performance in wireless systems. However, such works usually assume that reflections from the RIS elements experience a constant attenuation (CA). This is an oversimplification and assumes that the power of the reflected signal is independent of the phase shift at each RIS element. In this paper, we focus on the more general case [1], where the RIS phases affect the reflected signal strength, i.e., phase dependent loss (PDL). As an initial investigation, we focus on the effects of PDL on single user (SU) systems.

For SU systems, [2] derives an exact expression for the mean SNR where the user (UE) to RIS and RIS to base station (BS) channels experience Rayleigh fading and the direct channel between UE and BS is absent. [3] derives an exact expression for the optimal uplink (UL) mean SNR for systems where the UE-BS channel is rank-1 LOS and the UE-RIS and UE-BS channels are correlated Rayleigh. The LOS assumption in the RIS-BS channel has been considered and motivated in numerous works (e.g [4]). The authors in [3] leverage the mean SNR expression to provide insight on the impact of correlation on the mean SNR. In [5], the authors extend the exact mean SNR derivation in [3] to systems where the UE-BS and UE-RIS channels are correlated Ricean and derive a tight approximation to the mean rate. The authors again, leverage the mean SNR expression to provide insight on the impact of correlation and the Ricean K-factor on the mean SNR. However, the analysis in [2], [3], [5] assumes either perfect RIS reflection or reflections with CA.

In [1], a mathematical model is proposed for PDL. The attenuation function (loss function) is defined by three parameters; \( L_{\text{min}} \): the minimum amplitude of the loss function; \( \alpha \): the steepness of the loss function; \( \theta \): the shift of the loss function. These parameters are dependent on the circuit used to construct typical semiconductors for RIS reflective elements. Numerical results in [1] show that the model accurately matches the reflective response of a detailed circuit model for a semiconductor device used to construct typical RIS elements. Furthermore, characteristics of the circuit model resemble experimental results in the literature [1].

To the best of our knowledge, no analysis is available to characterise optimal system performance with PDL. Hence, the contributions of this paper are as follows:

- An exact mean SNR expression is derived for the optimal RIS phases using the PDL model in [1]. The optimal RIS design is based on the lossless case as there is no known optimal design in the presence of PDL.
- We evaluate the effects of PDL relative to a constant attenuation equal to the average PDL and show that CA results in a lower mean SNR.
- We analytically characterise the impact of the parameters in the loss function on the mean SNR.
- Numerical results validate the derived SNR expression and verify the effect of the loss function parameters \( L_{\text{min}}, \alpha, \theta \) on the mean SNR. We show that any impact caused by the loss function on the mean SNR becomes more pronounced as the size of the RIS increases. For typical parameter values, these effects are significant.

Notation: \( \mathbb{E}\{\cdot\} \) represents statistical expectation. \( \mathbb{R}\{\cdot\} \) is the Real operator. \( \|\cdot\|_2 \) denotes the \( \ell_2 \) norm. Upper and lower boldface letters represent matrices and vectors, respectively. \( \mathcal{CN}(\mu, Q) \) denotes a complex Gaussian distribution with mean \( \mu \) and covariance matrix \( Q \). \( U(a, b) \) denotes a uniform distribution on \( [a, b] \). The transpose, Hermitian transpose and complex conjugate operators are denoted as \( (\cdot)^T, (\cdot)^H, (\cdot)^* \), respectively. The trace and diagonal operators are denoted by \( \text{tr}\{\cdot\} \) and \( \text{diag}\{\cdot\} \), respectively. The angle of a vector \( x \) of length \( N \) is defined as \( \angle x = [\angle x_1, \ldots, \angle x_N]^T \) along with \( |x| = [|x_1|, \ldots, |x_N|]^T \). The exponential of a vector is defined as \( e^x = [e^{x_1}, \ldots, e^{x_N}]^T \). \( \otimes \) denotes the Kronecker product. \( B(z, w) \) denotes the beta function with parameters \( z, w \). \( 1_N \) denotes an \( N \times 1 \) vector with unit entries. The sinc function is defined by \( \text{sinc}(x) = \sin(\pi x) / (\pi x) \).

II. SYSTEM MODEL

As shown in Fig. 1, we examine a RIS-aided single input multiple output (SIMO) system where a RIS with \( N \) reflective
elements is located close to a BS with \( M \) antennas such that a rank-1 LOS condition is achieved between the BS and RIS.

Fig. 1: System model (the red dashed line is the control link for the RIS phases).

A. Channel Model

Let \( h_d \in \mathbb{C}^{M \times 1} \), \( h_{ru} \in \mathbb{C}^{N \times 1} \), \( H_{br} \in \mathbb{C}^{M \times N} \) be the UE-BS, UE-RIS and RIS-BS channels, respectively. The diagonal matrix \( \Phi \in \mathbb{C}^{N \times N} \), where \( \Phi_{rr} = e^{j\theta_r} \) for \( r = 1, 2, \ldots, N \), contains the reflection coefficients for each RIS element. The global UL channel is thus represented by

\[
h = h_d + H_{br} \Phi L(\Phi) h_{ru},
\]

where \( L(\Phi) = \text{diag}(L(\phi_1), \ldots, L(\phi_N)) \), with the amplitude of the reflected signal at the \( n \text{-th} \) element is attenuated by the loss factor, \( L(\phi_n) \in [0, 1] \). Note that although the analysis in the paper is applicable to any loss function, we adopt the following practical loss model for RIS reflective elements based on detailed modeling of the RIS circuit elements in [1]

\[
L(\phi_n) = (1 - L_{\text{min}}) \left( \frac{\sin(\phi_n + \theta) + 1}{2} \right)^\alpha + L_{\text{min}}.
\]

The PDL model in (2) gives losses which are dependent on the RIS phases. The variables \( L_{\text{min}} \geq 0, \theta \geq 0 \) and \( \alpha \geq 0 \) are constants dependent on specific circuit implementations [1]. \( L_{\text{min}} \) controls the minimum amplitude of the loss function, \( \alpha \) controls the steepness of the loss function and \( \theta \) controls the mid-point position of the loss function. Note that perfect RIS phase reflection is achieved by setting \( L_{\text{min}} = 1 \) or \( \alpha = 0 \).

For \( h_d \) and \( h_{ru} \), we assume correlated Rayleigh channels:

\[
h_d = \sqrt{\beta_d} R_d^{1/2} u_d, \quad h_{ru} = \sqrt{\beta_u} R_u^{1/2} u_u,
\]

where \( \beta_d \) and \( \beta_u \) are the link gains, \( R_d \) and \( R_u \) are the correlation matrices for UE-BS and UE-RIS links respectively and \( u_d, u_u \sim \mathcal{CN}(0, I) \). The rank-1 LOS channel from RIS to BS has link gain \( \beta_{br} \) and is given by \( H_{br} = \sqrt{\beta_{br}} a_h a^H_b \), where \( a_h, a_b \) are topology specific steering vectors at the BS and RIS respectively. Note that our analysis hold for any steering vectors.

Note, that the correlation matrices, \( R_{ru} \) and \( R_d \), can represent any correlation model. For simulation purposes, we will use the well-known exponential decay model for correlation at the BS and adopt the sinc correlation model for correlation at the RIS [6, Eq. (11)]. Hence,

\[
(R_{ru})_{ik} = \text{sinc}(2d_{ik}) \text{sinc}(2d_{r}), \quad \rho_{ru} = \frac{d_{ik}}{d_{r}},
\]

\[
(R_d)_{ik} = \rho_{d}, \quad 0 \leq |\rho_{ru}| \leq 1, 0 \leq |\rho_{d}| \leq 1. \quad d_{ik} \text{ is the distance between the } i^\text{th} \text{ and } k^\text{th} \text{ antenna/element at the BS/RIS}. \quad d_b \text{ is the nearest-neighbour BS antenna separation measured in wavelength units.} \quad \rho_d \text{ and } \rho_{ru} \text{ are the nearest neighbour BS antenna and RIS element correlations, respectively.}
\]

B. Optimal RIS Matrix

Using (1), the received signal at the BS is,

\[
r = (h_d + H_{br} \Phi L(\Phi) h_{ru}) s + n \sim \mathcal{CN}(0, \sigma^2 I).
\]

For a SU system, matched filtering (MF) is optimal, with UL SNR, given by \( \text{SNR} = \frac{\| h_d \|^2}{\sigma^2} \). Thus, to maximize the SNR with lossless reflection \( L(\Phi) = I \), the optimal RIS matrix is given by [5, Eq. (4)],

\[
\Phi = \psi \text{diag}\{e^{j\phi_n}\} \text{diag}\{e^{-j\phi_{ru}}\},
\]

where \( \psi = \frac{a_h^H h_d}{|a_h^H h_d|} \). Thus, the UL SNR is

\[
\text{SNR} = \bar{\sigma}^2 \left( h_d^H h_d + 2 \Re\{ h_d^H H_{br} \Phi L(\Phi) h_{ru} \} \right)
\]

\[
+ h_{ru}^H L(\Phi) \Phi h_{ru}.
\]

In this paper, we assume that the optimal lossless design in (5) is used in the presence of phase dependent loss. This is reasonable as an optimal design in the presence of loss is unknown. Note that in obtaining (6), we set \( L(\Phi) = L(\Phi) \), since \( L(\Phi) \) is a positive real valued diagonal matrix.

III. MEAN SNR

Here, we provide an exact result for the mean SNR, \( \mathbb{E}\{\text{SNR}\} \), building on the results in [3] for the mean SNR in a lossless scenario.

Theorem 1. The mean SNR is given by

\[
\mathbb{E}\{\text{SNR}\} = \bar{\sigma}^2 \left( \beta_d M + \sqrt{\beta_{br} \beta_d} \frac{\| R_d^{1/2} a_h \|}{\sqrt{\mu_1}} \frac{\pi}{2} \right)
\]

\[
+ \beta_{ru} \beta_d M (\mu_2 + F),
\]

where

\[
\mu_1 = \frac{4^\alpha (1 - L_{\text{min}})}{\pi} \left( \frac{2\alpha + 1}{2} \right)^2 + L_{\text{min}},
\]

\[
\mu_2 = \frac{2L_{\text{min}} (1 - L_{\text{min}})}{\pi} 4^\alpha \left( \frac{2\alpha + 1}{2} \right)^2 + L_{\text{min}},
\]

\[
F = \frac{1}{\pi} \sum_{r,s=1}^N \left( 1 - |\rho_{rs}|^2 \right)^2 2F_1 \left( \frac{3}{2}, \frac{3}{2}; 1; |\rho_{rs}|^2 \right) L_{rs},
\]

where

\[
L_{rs} = \mathbb{E}\{L(\phi_r)L(\phi_s)\}
\]

\[
= \int_0^{2\pi} \int_0^{2\pi} \left( L(s + \angle(\alpha_r) - 2\pi) L(t + \angle(\alpha_s) - 2\pi) \right)
\]

\[
\times g_{rs}(t-s) \, ds \, dt,
\]

\[
g_{rs}(x) = \frac{1 - |\rho_{rs}|^2}{4\pi^2} \left( \frac{1}{1 - v_{rs}(x)^2} - \frac{v_{rs}(x) \cos^{-1}(v_{rs}(x))}{(1 - v_{rs}(x)^2)^{3/2}} \right),
\]
$v_{rs}(x) = |\rho_{rs}| \cos(x - \angle(-\rho_{rs}))$, 

\(2F_1(\cdot)\) is the Gaussian hypergeometric function, $\rho_{rs} = (R_{ru})_{rs}$.

**Proof.** See App. A for the derivation of (7).

Note that $F$ is the only variable dependent on the correlations in $h_{ru}$ and also note that the variable $L_{rs}$ is a double integral of the loss function. In Sec. III-A and Sec. III-B, we derive exact results for special cases of $F$, $L_{rs}$ when $|\rho_{rs}| \in \{0, 1\}$. These correlation extremes provide useful benchmarks to evaluate the SNR trends.

**A. Special Case 1: Uncorrelated $h_{ru}$**

From (11), when $h_{ru}$ is uncorrelated then $\phi_r$ and $\phi_s$ are i.i.d for $r \neq s$. Hence,

$$E\{L(\phi_r)L(\phi_s)\} = (E\{L(\phi_r)\})^2 = \mu_r^2.$$  (14)

No correlation in $h_{ru}$ also implies that $\rho_{mn} = 0$ for all $m \neq n$. 

Using this result, (14) and [3, Eq. (10)], $F$ simplifies to

$$F_u = \frac{\mu_r^2 N(N-1)\pi}{4}.$$  (15)

Therefore, the mean SNR for an uncorrelated $h_{ru}$ channel is,

$$E\{SNR\} = \tilde{\tau}\left(\beta_d M + \sqrt{\beta_d\beta_r}\|R_{ru}^{-1/2}a_b\|\right) N\mu_r \frac{\pi}{2}$$

$$+ \beta_r\beta_r M(N\mu_2 + F_u).$$  (16)

Note that the mean SNR expression depends on PDL solely through the simple functions $\mu_1$ and $\mu_2$.

**B. Special Case 2: Perfect Correlation in $h_{ru}$**

With perfect correlation in $h_{ru}$, $|\rho_{rs}| = 1$ for $r, s = 1, \ldots, N$. Hence, from [3, Eq. (13)], $F$ can be rewritten as $F = \sum_{r=1}^{N} \sum_{s=1}^{N} L_{rs}$. Under perfect correlation, we can exactly compute $E\{L(\phi_r)L(\phi_s)\}$. Following App. A, we can express the $i^{th}$ RIS phase as $\phi_i = \angle a^h_{sd} h_{sd} + \angle(a_j)_i - \angle h_{ru,i}$. Hence,

$$E\{L(\phi_r)L(\phi_s)\} =$$

$$E\left\{L(\angle a^h_{sd} h_{sd} + \angle(a_j)_r - \angle h_{ru,r})\times\right.$$

$$\left.L(\angle a^h_{sd} h_{sd} + \angle(a_j)_s - \angle h_{ru,s})\right\}$$

$$= E\left\{L(w + \angle(a_j)_r)L(w + \angle(a_j)_s)\right\}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} L(w + \angle(a_j)_r)L(w + \angle(a_j)_s) dw.$$  (16)

Using Result 2 in App. B, the solution to the above integral is

$$F_c = N(N-1) \left(\frac{A_1 A_2 2^{a_1+1}}{\pi} B(\frac{2a_1+1}{2}, \frac{2a_1+1}{2}) + A_2^2\right)$$

$$+ \sum_{r=1}^{N} \sum_{s \neq r}^{N} \frac{A_2^2 2^{2a_1-1}}{\pi \sin(2\pi\alpha)} [\sin(2\pi\alpha) - 1 - \cos(2\pi\alpha)] \left[\Re\{\mathcal{I}\} \quad \Im\{\mathcal{I}\}\right]$$

with

$$\mathcal{I} = 2\pi(\gamma^2 - 1)\alpha F_1\left(-2\alpha, 2\alpha + 1; 1; \frac{1 - \gamma_1}{2}\right),$$  (17)

where $\gamma_1 = \gamma / \sqrt{\gamma^2 - 1}$ and $\gamma = \cos\left(\frac{\angle(a_j)_s - \angle(a_j)_r}{2}\right)$. $A_1 = 1 - L_{\min}$ and $A_2 = L_{\min}$. Therefore, the mean SNR for a fully correlated channel is,

$$E\{SNR\} = \tilde{\tau}\left(\beta_d M + \sqrt{\beta_d\beta_r}\|R_{ru}^{-1/2}a_b\|\right) N\mu_1 \frac{\pi}{2}$$

$$+ \beta_r\beta_r M(N\mu_2 + F_c).$$  (19)

**IV. IMPACT OF LOSS FUNCTION ON THE MEAN SNR**

In this section, we explore the impact of the circuit-dependent parameters $L_{\min}, \alpha, \theta$ on the mean SNR. These parameters only impact the variables $\mu_1, \mu_2, L_{rs}$ in the mean SNR expression (7). While the broad impact of $L_{\min}, \alpha, \theta$ is intuitive from the loss function (2), in this section we present analysis to support and quantify these effects.

**A. Phase Shift of the PDL Function: $\theta$**

The parameter, $\theta$, which controls the midpoint position of the loss function does not affect the mean SNR as $E\{L(\phi_r)\}$ and $E\{L(\phi_r)L(\phi_s)\}$ are averaged over an entire $2\pi$ period. Therefore, the mean SNR is independent of $\theta$.

**B. Steepness of the PDL Function: $\alpha$**

The parameter $\alpha$ only affects the beta functions in $\mu_1$ and $L_{rs}$. From [7, Eq. (8.384.4)], we have

$$B(x, x) = 2^{x - 2x} B(1/2, x),$$  (20)

which is a useful result as it appears in both $\mu_1$ and $L_{rs}$. Firstly, note that the series representation of $B$ (20) given in [7, Eq. (8.382.3)] shows that $B(1/2, x)$ decreases in value as $x \to \infty$. Therefore, $B\left(\frac{2a_1+1}{2}, \frac{2a_1+1}{2}\right)$ and $B\left(\frac{2a_1+1}{2}, \frac{4a_2+1}{2}\right)$ are monotonically decreasing functions in $\alpha$ since $\alpha \geq 0$. Hence, from (8)-(9), $\mu_1$ and $\mu_2$ benefit from having small $\alpha$. In terms of $L_{rs}$, note that $L_{rs}$ is a double integral over positive functions as $L(\phi_n) \in (0, 1)$ and $L_{rs}$ is a positive function for all $v_{rs}(x)$ (see. Result 3 in App. B). Therefore, $L_{rs}$ benefits from having small $\alpha$ since (2) increases as $\alpha$ decreases. In summary, the mean SNR benefits from having a small $\alpha$ parameter.

**C. Minimum Amplitude of the PDL Function: $L_{\min}$**

Let $c_1 = 4^{\alpha/2} \sqrt{\frac{2a_1+1}{2}, \frac{2a_1+1}{2}}$ and $c_2 = 16^\alpha \sqrt{\frac{2a_1+1}{2}, \frac{4a_2+1}{2}}$, then $\mu_1$ can be rewritten as

$$\mu_1 = c_1 + L_{\min}(1 - c_1),$$  (21)

and $L(\cdot) \in [0, 1]$ implies that $L_{\min} \in [0, 1]$. Using the results in Result 4 of App. B, we can infer that for $\alpha \geq 0$, $1 \geq c_2 - 2c_1$ and $c_1 \geq c_2$ so $\mu_2$ also increases with $L_{\min} \in [0, 1]$. In terms of $L_{rs}$, note that $L_{rs}$ is a double integral over positive functions as $L(\phi_n) \in (0, 1)$ and $L_{rs}$ is a positive function for all $v_{rs}(x)$ (see. Result 3 in App. B). Therefore, $L_{rs}$ benefits from having large $L_{\min}$ since (2) increases in value as $L_{\min}$ increases. In summary, the mean SNR benefits from high values of $L_{\min}$.
V. Results

We present numerical results to verify the analysis in Sec. IV. We do not consider cell-wide averaging as the focus is on the SNR distribution over fast fading. Hence, we present results for fixed link gains for which the geometric model for the deployment of the UE, BS and RIS is adopted from [8] and shown in Fig. 2. In particular, since the RIS-BS link is

\[ \beta_{br} = d_{br}^{-2} \]

where \( d_{br} = 51 \text{m} \). For the other channels, we use the distance-dependent path loss model,

\[ \beta_{di} = C_d d_{di}^{-\alpha_d}, \quad \beta_{ri} = C_d d_{ri}^{-\alpha_d}, \quad (23) \]

where \( C_d = -30 \text{ dB} \) is the path loss at a reference distance of 1m, \( d_{ri} = 21.0238 \text{m} \) and \( d_{ri} = 30.167 \text{m} \) is the UE-RIS and UE-BS separation distances respectively, \( \alpha_d = 2.8 \) and \( \alpha_d = 3.5 \) are the path loss exponents for the BS and RIS size scenarios, respectively. These values give the path gains of \( \beta_{di} = -81.7077 \text{ dB} \) and \( \beta_{ri} = -67.0360 \text{ dB} \). Distances \( d_{ri} \) and \( d_{ri} \) were computed using elementary trigonometry where \( d = 30 \text{m} \) and \( d = 1 \text{m} \). The power of the transmitted signal is \( E_s = 1 \) and the noise power is \( \sigma^2 = -65 \text{ dBm} \).

For simulation purposes, we use the VURA model outlined in [9], but in the \( y \) - \( z \) plane with equal spacing in both dimensions at both the RIS and BS. The \( y \) and \( z \) components of the steering vector at the BS are \( a_{b,y} \) and \( a_{b,z} \) which are given by

\[
\begin{cases}
1, e^{i2\pi d_{ri} \sin(\theta_A) \sin(\omega_A)} & \text{and} \\
1, e^{i2\pi d_{ri} \cos(\theta_A) \cos(\omega_A)} & \text{and}
\end{cases}
\]

respectively. Similarly at the RIS, \( a_{r,y} \) and \( a_{r,z} \) are defined by

\[
\begin{cases}
1, e^{i2\pi d_{ri} \sin(\theta_B) \sin(\omega_B)} & \text{and} \\
1, e^{i2\pi d_{ri} \cos(\theta_B) \cos(\omega_B)} & \text{and}
\end{cases}
\]

respectively, where \( M = M_y M_z, N = N_y N_z \) with \( M_y, M_z \) being the number of antenna columns and rows at the BS and \( N_y, N_z \) being the number of columns and rows of RIS elements. \( d_{ri} = 0.5 \) and \( d_{ri} \) are BS/RIS element spacings in wavelength units. Note that the value of \( d_{ri} \) is set to satisfy a particular correlation level \( \sin^2(2d_{ri}) = \rho_{ri} \) as per (4). The steering vectors at the BS and RIS are then given by

\[ a_{b} = a_{b,y} \otimes a_{b,z}, \quad a_{r} = a_{r,y} \otimes a_{r,z}, \quad (24) \]

respectively. \( \theta_A \) and \( \omega_A \) are elevation/azimuth angles of arrival (AOAs) at the BS and \( \theta_B, \omega_B \) are the corresponding angles of departure (AODs) at the RIS. The elevation/azimuth angles are selected based on the following geometry representing a range of \( \text{LOS}\_br \) links with less elevation variation than azimuth variation: \( \theta_D \sim U[70^\circ, 90^\circ], \omega_D \sim U[-30^\circ, 30^\circ], \theta_A = 180^\circ - \theta_D, \omega_A = \omega_D - 30^\circ, 30^\circ \). For all results in this paper we use a single sample from this range of angles given by \( \theta_D = 77.1^\circ, \omega_D = 19.95^\circ, \theta_A = 109.9^\circ, \omega_A = -29.9^\circ \).

In Fig. 3, we verify the mean SNR expression in (7) for varying values of \( N, \theta = 0.2, \rho_{di} = 0.7, \rho_{ri} \in \{0, 0.95, 1\} \) and the typical parameter values, \( L_{min} = 0.2, \alpha = 1.6, \) given in [1]. For the special cases of \( \rho_{ri} = 0 \) and \( \rho_{ri} = 1 \), we use (15) and (17) to compute \( F \), respectively. For all correlation and RIS size scenarios, the analytical mean SNR agrees with simulations. Notice that even with PDL, the mean SNR grows as \( O(N^2) \), identical to the growth of the mean SNR without PDL [10]. Also shown is the benchmark case of CA, where each element, \( L(\phi_i) \), of the loss matrix is replaced by \( \mu_1 \). As can be seen, CA reduces the mean SNR relative to variable attenuation. To explain this, we note that this property is also observed for standard maximal ratio combining (MRC). Consider MRC for an \( M \times 1 \) channel, \( h \). If the \( i \)-th channel coefficient is attenuated by \( a_i \), then the SNR is proportional to \( \sum_{i=1}^{M} a_i^2 |h_i|^2 \). Since \( \sum_{i=1}^{M} a_i^2 \geq M \bar{a}^2 \) it follows that the mean SNR is reduced by CA. The SNR gap between PDL and CA is increased by larger \( N \) and \( \rho_{ri} \) and the relative change is over 20% for \( N = 100, \rho_{ri} = 1 \).
Fig. 4: Simulated and analytical mean SNR for $L_{\text{min}} = \{0.2, 0.5, 0.95, 1\}$, $\theta = \{0.2, 0.42\}$, $N = \{16, 64\}$.

SNR is steeper for $N = 64$ compared to $N = 16$. Hence, as the number of RIS elements increases, the initial drop in mean SNR is more pronounced. Also, note that when $N = 16$, the separation gap between the mean SNR curves for the three $L_{\text{min}}$ values is smaller than those in the case of $N = 64$. Therefore, as the number of RIS elements increases, altering $L_{\text{min}}$ has a greater effect on the mean SNR. From Fig. 4, we see that the SNR for the typical parameters, $L_{\text{min}} = 0.2$, $\alpha = 1.6$, and $N = 64$ is approximately 30% of the SNR without PDL, which is a significant reduction. Hence we can expect a significant reduction in mean SNR for practical RIS systems.

VI. CONCLUSION

We derive an exact expression for the mean SNR where the RIS elements experience PDL. Specifically, the amplitude of the reflections from the RIS element are dependent on the optimal RIS phases which maximize the SNR in the absence of PDL. The attenuation function used for PDL is dependent on three parameters which control the minimum amplitude, steepness and shift of the attenuation function. We analytically characterise the impact of PDL on the mean SNR, offering insight into how PDL impacts the mean SNR performance. The analysis shows that the mean SNR only depends on the minimum amplitude and the steepness parameters. Having a larger minimum amplitude increases the mean SNR and having a steeper attenuation function decreases the mean SNR. This effect is enhanced when the number of RIS elements increases.

APPENDIX A
DERIVATION OF MEAN SNR

For ease of notation, we define the three terms in the SNR expression (6) by $\text{SNR} \triangleq \bar{r} (S_1 + S_2 + S_3)$. We then compute $E\{\text{SNR}\}$ by considering each term in the expression.

**Term 1:** Using [5, Eq. (52)] and (3), we have

\[ E\{S_1\} = \beta_d M. \tag{25} \]

**Term 2:** Substituting the optimal RIS matrix (5) and the channels $\mathbf{h}_{d}^{\text{opt}}, \mathbf{h}_{\text{br}}, \mathbf{h}_{\text{ru}}$ from Sec. II-A into $S_2$,

\[ E\{S_2\} = 2\sqrt{\beta_{\text{br}}} \times \mathbb{R}\{E\{\mathbf{h}_{d}^{H} a_{d} a_{d}^{H} \psi \text{diag}\{a_{d}\} \text{diag}\{e^{-j\theta_{\text{ru}}}\} L(\Phi) \mathbf{h}_{\text{ru}}\}\} \]

\[ = 2\sqrt{\beta_{\text{br}}} \mathbb{R}\{E\{a_{d}^{H} \mathbf{h}_{d}^{H} 1_{N}^{T} L(\Phi) \text{diag}\{e^{-j\theta_{\text{ru}}}\} \mathbf{h}_{\text{ru}}\}\} \]

\[ = 2\sqrt{\beta_{\text{br}}} \mathbb{R}\{E\{a_{d}^{H} \mathbf{h}_{d}^{H} 1_{N}^{T} L(\Phi) \mathbf{h}_{\text{ru}}\}\}. \tag{26} \]

The matrix $L(\Phi)$ depends on $e^{j\alpha_{\text{e}} h_{d}^{H} a_{d}^{H}}$ and $e^{-j\theta_{\text{ru}}}$. Hence,

\[ E\{S_2\} = 2\sqrt{\beta_{\text{br}}} \sum_{r=1}^{N} \mathbb{E}\{a_{d}^{H} \mathbf{h}_{d}^{H}\} \mathbb{E}\{L(\phi_{r})\} \mathbb{E}\{|h_{\text{ru},r}|\}, \]

which is obtained by realising that $L(\phi_{r})$ is independent of both $a_{d}^{H} \mathbf{h}_{d}$ and $|h_{\text{ru},r}|$. Noting that $\phi_{r} = \angle a_{d}^{H} \mathbf{h}_{d} + \angle(a_{\alpha})$, $-\angle h_{\text{ru},r} \sim U[0, 2\pi]$, it follows that

\[ E\{L(\phi_{r})\} = \frac{1}{2\pi} \int_{0}^{2\pi} L(x) \, dx. \tag{27} \]

Note that (27) is a generic calculation for any loss function. For the loss function given by (2),

\[ E\{L(\phi_{r})\} = \frac{1 - L_{\text{min}}}{2} \int_{0}^{2\pi} (1 + \sin(x + \theta))^\alpha \, dx + L_{\text{min}} \]

\[ = \frac{4^\alpha (1 - L_{\text{min}})}{\pi} B\left(\frac{2\alpha + 1}{2}, \frac{2\alpha + 1}{2}\right) + L_{\text{min}} \]

\[ \triangleq \mu_1, \tag{28} \]

where (a) uses Result 1 in App. B to evaluate the integral. To complete the solution for $E\{S_2\}$, we need to compute $E\{a_{d}^{H} \mathbf{h}_{d}^{H}\}$ and $\sum_{r=1}^{N} \mathbb{E}\{|h_{\text{ru},r}|\}$ which can be computed exactly using [3, Eq. (22)]. Hence.

\[ E\{S_2\} = \sqrt{\beta_{\text{br}} E_{\text{br}} E_{\text{ru}}} |R_{1/2}^{1/2} a_{b}| N \mu_1 \frac{\pi}{2}. \tag{29} \]

**Term 3:** Substituting the optimal RIS matrix, (5), and the channels $\mathbf{h}_{\text{br}}, \mathbf{h}_{\text{ru}}$ from Sec. II-A into $S_3$,

\[ E\{S_3\} = \beta_{\text{br}} M \mathbb{E}\{h_{\text{br}}^{H} L(\Phi) \mathbf{h}_{\text{ru}}^{H} a_{d} a_{d}^{H} \} \mathbb{E}\{e^{j\theta_{\text{ru}}}\} \mathbb{E}\{\text{diag}\{e^{-j\theta_{\text{ru}}}\}\} \mathbb{E}\{\text{diag}\{a_{\alpha}^{H}\}\} a_{\alpha} \]

\[ \times \mathbb{E}\{a_{d}^{H} a_{d} a_{d}^{H} \} \mathbb{E}\{e^{-j\theta_{\text{ru}}}\} \mathbb{E}\{L(\Phi)\} \mathbb{E}\{h_{\text{ru}}\} \]

\[ = \beta_{\text{br}} M \sum_{r=1}^{N} \sum_{s=1}^{N} \mathbb{E}\{h_{\text{ru},r}^{*} L(\phi_{r}) e^{j\theta_{\text{ru},r}} e^{-j\theta_{\text{ru},s}} L(\phi_{s}) h_{\text{ru},s}\} \]

\[ = \beta_{\text{br}} M \left( \sum_{r=1}^{N} \mathbb{E}\{|h_{\text{ru},r}|^2\} \mathbb{E}\{L^2(\phi_{r})\} \right) \]

\[ + \sum_{r=1}^{N} \sum_{s=1}^{N} \sum_{r \neq s} \mathbb{E}\{|h_{\text{ru},r}| \, |h_{\text{ru},s}| \} \mathbb{E}\{L(\phi_{r}) L(\phi_{s})\}. \tag{30} \]

The first term in (30) requires $\mathbb{E}\{|h_{\text{ru},r}|^2\} = \beta_{\text{ru}}$. To obtain $\mathbb{E}\{L^2(\phi_{r})\}$, we expand the square of (2),

\[ L^2(\phi_{r}) = (1 - L_{\text{min}})^2 2^{-2\alpha} \sin^2(\phi_{r} + \theta) + 1 + L_{\text{min}}^2 \]

\[ + 2^{-\alpha - 1} L_{\text{min}} (1 - L_{\text{min}}) (\sin(\phi_{r} + \theta) + 1)^\alpha \]

\[ \triangleq L_1 + L_2 + L_3. \tag{31} \]
The mean of the first term is
\[ \mathbb{E} \{ L_1 \} = \frac{(1 - L_{\text{min}})^2}{2\pi a^4} \int_0^{2\pi} (1 + \sin(x + \theta))^{2\alpha} \, dx \]
\[ \approx \frac{(1 - L_{\text{min}})^2}{\pi} 16^n \beta \left( \frac{4\alpha + 1}{2}, \frac{4\alpha + 1}{2} \right), \quad (32) \]
where, in (a), Result 1 in App. B is used to evaluate the integral. The mean of the second term is simply \( \mathbb{E} \{ L_2 \} = L_{\text{min}} \). The mean of the third term is,
\[ \mathbb{E} \{ L_3 \} = \frac{(1 - L_{\text{min}})^2}{\pi} B \left( \frac{2\alpha + 1}{2}, \frac{2\alpha + 1}{2} \right), \quad (33) \]
where, in (a), Result 1 in App. B is used to evaluate the integral. Summing the three expectations, we have
\[ \mathbb{E} \{ L^2(\phi_r) \} = \frac{(1 - L_{\text{min}})^2}{\pi} B \left( \frac{2\alpha + 1}{2}, \frac{2\alpha + 1}{2} \right) + L_{\text{min}}^2 + \frac{(1 - L_{\text{min}})^2}{\pi} B \left( \frac{4\alpha + 1}{2}, \frac{4\alpha + 1}{2} \right) \]
\[ \triangleq \mu_2. \quad (34) \]

The second term in (30) requires \( \mathbb{E} \{ |h_{ru,r}|^2 \} \text{ and } \mathbb{E} \{ L(\phi_r) | \phi_r \} \). Using [11, Eq. (11)] we have
\[ \mathbb{E} \{ |h_{ru,r}|^2 \} = \pi \left( 1 - |\rho_{ik}|^2 \right) F_1 \left( \frac{3}{2}, \frac{3}{2}; 1; |\rho_{ik}|^2 \right), \quad (35) \]
where \( F_1(\cdot) \) is the Gaussian hypergeometric function and \( \rho_{ij} = (R_{ru,i})^2 \).

The final expectation required is \( \mathbb{E} \{ L(\phi_r) | \phi_r \} \). Let \( x = \angle h_{ru,r}, y = \angle h_{ru,s} \), then the joint density of phases \( x, y \) is given by [12, Eq. (3.12)],
\[ f_{X,Y}(x,y) = \frac{1 - |\rho_{rs}|^2}{8\pi^2} \frac{\partial^2}{\partial x^2} (\cos^{-1}(\lambda))^2 \]
\[ = \frac{1 - |\rho_{rs}|^2}{4\pi^2} \frac{\partial}{\partial \lambda} \left( \cos^{-1}(\lambda) (1 - \lambda^2)^{-1/2} \right) \]
\[ = \frac{1 - |\rho_{rs}|^2}{4\pi^2} \left( \frac{1}{1 - \lambda^2} - \frac{\lambda \cos^{-1}(\lambda)}{(1 - \lambda^2)^{3/2}} \right) \]
\[ \triangleq g(x, y), \quad (36) \]
with
\[ \lambda = |\rho_{rs}| \cos(x - y - \angle(-\rho_{rs})). \quad (37) \]
Recall that each optimal RIS phase is \( \phi_r = \angle a_i^T h_d + \angle(a_i)_v - \angle h_{ru,r} \). To obtain the joint density of \( \phi_r, \phi_s \) defined by \( f_{r,s}(x, y) \), let \( Z = \angle a_i^T h_d, a = \angle(a_i)_v, \) and \( b = \angle(a_i)_s \). Then conditioned on \( Z = z \), we have the conditional PDF
\[ f_{r,s|z}(u, v | z) = f_{X,Y}(z + a - u, z + b - v) \]
\[ = g(u - v + a - b), \]
where the domain of \( u \) and \( v \) is \( z + a - 2\pi \leq u \leq z + a \) and \( z + b - 2\pi \leq v \leq z + v \) respectively. This gives,
\[ \mathbb{E} \{ L(\phi_r) | \phi_s \} = \mathbb{E} \{ L(\phi_r) | \phi_s, Z \} \]
\[ = \int_0^{2\pi} \frac{1}{2\pi} \int_z^{z+b} \int_{z+a}^{z+a-2\pi} L(u)(u) \, du \, dv \, dz \times g(v - u + a - b) \, du \, dv \, dz. \]
Let \( s = u + 2\pi - a \) and \( t = v + 2\pi - b \), then
\[ \mathbb{E} \{ L(\phi_r)| \phi_s \} \]
\[ = \int_0^{2\pi} \frac{1}{2\pi} \int_z^{z+2\pi} \int_z^{z+2\pi} L(s + a - 2\pi)(t + b - 2\pi) \times \frac{1}{2\pi} \int_z^{z+b-2\pi} \int_{z+a-2\pi}^{z+a} L(u)(u) \, du \, dv \, dz \times g(v - u + a - b) \, du \, dv \, dz. \]
Therefore, the second term of (30) is given by
\[ F = \sum_{r=1}^{N} \sum_{s=1}^{N} \pi f_{r,s} \left( 1 - |\rho_{rs}|^2 \right) \left( 3/2, 3/2; 1; |\rho_{rs}|^2 \right) \]
\[ \times \int_0^{2\pi} \int_0^{2\pi} L(s + a - 2\pi)(t + b - 2\pi) g(t - s) \, ds \, dt. \]

Combining (25), (29) and (40), completes the derivation.

**APPENDIX B**

**FOUR SUPPORTING RESULTS**

For reasons of space, proof of the following results can be found in [13]. Derivations of Result 1: \( f_0^{2\pi} (1 + \sin(x + a))^b \, dx \) and Result 2: \( f_0^{2\pi} X^a \, dx \) are in [13, App. B, App. C] respectively. Result 3: \( \eta_{ru} \) is a positive function is shown in [13, App. D]. Result 4: \( c_1 = c_2 = 1 \) when \( \alpha = 0 \) and decrease with \( \alpha \) so that their limits are zero as \( \alpha \to \infty \) is in [13, App. E].

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