Acceleration-extended Newton-Hooke symmetry and its
dynamical realization

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Abstract

Newton-Hooke group is the nonrelativistic limit of de Sitter (anti-de Sitter) group, which can be enlarged with transformations that describe constant acceleration. We consider a higher order Lagrangian that is quasi-invariant under the acceleration-extended Newton-Hooke symmetry, and obtain the Schrödinger equation quantizing the Hamiltonian corresponding to its first order form. We show that the Schrödinger equation is invariant under the acceleration-extended Newton-Hooke transformations. We also discuss briefly the exotic conformal Newton-Hooke symmetry in 2 + 1 dimension.

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I. INTRODUCTION

The Newton-Hooke (NH) symmetry can be obtained by taking the nonrelativistic limit of de Sitter (dS) or anti-de Sitter (AdS) symmetry.\(^1\) If we define \( \frac{c^2}{3} = \frac{1}{R^2} \), the parameter \( R \) has the dimension of time. When we take further contraction \( R \to \infty \), we will obtain the familiar Galileo symmetry and the corresponding standard Newton space-time. The NH symmetry and the corresponding NH space-time have interesting applications in nonrelativistic cosmology \(^3\) and even in String/M-theory \(^4, 5\), besides the significance in fundamental physics \(^1, 2, 6\).

Recently, acceleration-extended Galileo symmetry with central charges and its dynamical realization has been studied in Ref.\(^7\); these results have been generalized to the NH space-time, and it is shown that the acceleration-extended NH symmetry has one central charge in arbitrary dimension and three central charges in \( 2 + 1 \) dimension (the so-called exotic case) in Ref.\(^8\). In fact, the exotic central extension of NH symmetry, with or without acceleration extension, has been extensively studied in the literature \(^9, 10, 11\).

Following those results, in this paper, we focus on dynamical realization of the symmetry, i.e. the nonrelativistic Lagrangian on NH space-time which is quasi-invariant under the acceleration-extended NH symmetry, with one central charge. Furthermore, we obtain the first order form of the Lagrangian by introducing extra variables, and write down the Schödinger equation of the system under geometric quantization of the corresponding Hamiltonian. Naturally, we consider whether the Schödinger equation is invariant under the acceleration-extended NH transformations, and the answer is confirmative. In fact, since the (classical) acceleration-extended NH transformations act on the phase space (extended by extra variables) as nontrivial canonical transformations, the corresponding quantum ones are nontrivial unitary transformations on the Hilbert space. We also discuss briefly the exotic conformal NH algebra (in \( 2 + 1 \) dimension) and its dynamical realization.

We organize this paper as follows. In Section II we recall some known results on the NH algebra and add to it transformations describing constant acceleration. We also consider the central element \( \kappa \) there. Then, we construct the classical Lagrangian with higher order time derivatives which is quasi-invariant under the acceleration-extended Newton-Hooke

\(^1\) That is the so-called Newton-Hooke limit \(^1, 2\): the speed of light \( c \to \infty \), cosmological constant \( \Lambda \to 0 \), but \( c^2 \Lambda \) is kept fixed.
symmetry in Section III. In Section IV we obtain the extended phase space formulation from the first order form of the Lagrangian. We also derive the Schrödinger equation in standard way and show its invariance under the acceleration-extended (and also central extended) \( NH \) transformations. In the following section we discuss the exotic conformal \( NH \) algebra and its dynamics. The last section contains our conclusion.

II. ACCELERATION-EXTENDED NEWTON-HOOKE ALGEBRA WITH CENTRAL CHARGE

First, let us recall the \( NH \) algebra, which is the nonrelativistic limit of de Sitter algebra.\(^2\) The nonvanishing commutators are \((i, j, k, l = 1, 2, \cdots, D - 1)\)

\[
\begin{align*}
[J_{ij}, J_{kl}] &= \delta_{il}J_{jl} - \delta_{jl}J_{ik} + \delta_{jk}J_{il} - \delta_{il}J_{jk} \quad (1a) \\
[J_{ij}, A_k] &= \delta_{ik}A_j - \delta_{jk}A_i \quad (A_i = P_i, K_i) \quad (1b) \\
[H, K_i] &= P_i \quad (1c) \\
[H, P_i] &= \frac{K_i}{R^2} \quad (1d)
\end{align*}
\]

where \( J_{ij} \) are the generators of spatial rotation, \( H \) that of time translation, \( P_i \) those of spatial translation and \( K_i \) those of boost.

We can make the following central extension in arbitrary dimension \( D \):

\[
[P_i, K_j] = -im\delta_{ij} \quad (2)
\]

as in the Galilei case, where \( m \) describes the nonrelativistic mass appearing in the \( NH \) quantum mechanics.\(^2\)

We can also add to \( NH \) algebra generators \( F_i \) describing acceleration transformations to enlarge it to the so-called \( \hat{NH} \) algebra. It is known that the constant acceleration transformations in NH space-time can be obtained by some appropriate combination of spatial translations and special conformal transformations.\(^1\) Written in terms of the so-called static coordinates in NH space-time, the acceleration transformation is

\[
x_i' = x_i + 2R^2(\cosh \frac{t}{R} - 1)b_i \quad (3)
\]

\(^2\) The anti-de Sitter case can be dealt with in parallel, without any difficult.
which is the same as that in the Galileo case, when $R \to \infty$. Combining it with the standard NH transformation $[2, 4]$, we can have the full $\hat{N}H$ transformation as follows:

$$
\begin{align*}
    x'_i &= v_i R \sinh \frac{t}{R} + a_i \cosh \frac{t}{R} + 2R^2(\cosh \frac{t}{R} - 1)b_i + O_i^j x_j \\
    t' &= t + a
\end{align*}
$$

(4)

where $O_i^j$ is the transformation matrix generated by spatial rotation $J_{ij}$; $v_i$ is a “velocity” corresponding to boost $K_i$; $a_i$ and $b_i$ are spatial translations generated by momentum operators $P_i$ and “accelerations” generated by acceleration operators $F_i$, respectively. Of course, in the space-time realization of the NH (or $\hat{N}H$) symmetry, there is no room for the central parameter $m$, i.e. we must have $[P_i, K_j] = 0$.

The generators of transformation (4) can be represented by tangent vector fields on the NH space-time. By definition, the generators of spatial translations are described by the vector fields

$$
P_i = \left. \frac{\partial x'_j}{\partial a_i} \right|_{a=b=v=0, O=1} = \cosh \frac{t}{R} \partial x_i
$$

(5)

Similarly, we can write down the vector fields of time translation $H$, boost $K_i$, spatial rotation $J_{ij}$ as well as acceleration generators $F_i$ as follows:

$$
\begin{align*}
    H &= \frac{\partial}{\partial t}, \\
    K_i &= R \sinh \frac{t}{R} \cdot \frac{\partial}{\partial x_i} \\
    J_{ij} &= x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \\
    F_i &= 2R^2 \left(\cosh \frac{t}{R} - 1\right) \cdot \frac{\partial}{\partial x_i}
\end{align*}
$$

(6)

One can check that the above form of generators satisfy the relations (1a-1d) and

$$
[J_{ij}, F_k] = \delta_{ik} F_j - \delta_{jk} F_i, \quad [H, F_i] = 2K_i
$$

(7)

The other commutators of $F_i$ are vanishing. In fact, without considering the space-time realization, one can show that the Jacobi identity for generators $(H, P_i, F_j)$ implies $m = 0$ if $[K_i, F_j] = 0$. Just relaxing this condition, in arbitrary space-time dimension $D$, we can introduce at least one central element $\kappa$ that appears as (see [8])

$$
[K_i, F_j] = 2\kappa \delta_{ij}
$$

(8)

Using the Jacobi identities, we can obtain

$$
[P_i, K_j] = -\frac{\kappa}{R^2} \delta_{ij}
$$

(9)

with other commutators kept unchanged.
III. LAGRANGIAN WITH ACCELERATION-EXTENDED NEWTON-HOOKE (QUASI-)SYMMETRY IN ARBITRARY DIMENSION $D$

The finite $\widehat{NH}$ transformation (4) gives the infinitesimal transformation as follows:

$$\delta x_i = (\delta O)_{ij} x_j + \cosh \left( \frac{1}{R} \delta a_i + R \sinh \left( \frac{1}{R} \delta v_i + 2R^2 (\cosh \left( \frac{1}{R} \right) - 1) \delta b_i \right) \right)$$

$$\delta t = \delta a$$  \hspace{1cm} (10)

where $\delta a_i$, $\delta b_i$, $\delta a$ and $(\delta O)_{ij}$ are infinitesimal parameters corresponding to the finite ones in (4).

Then, we should look for a Lagrangian which is quasi-invariant\(^3\) under transformations (10) and which, in the limit $R \to \infty$, goes back to the Lagrangian with higher order time derivatives in Galileo case (see [7]), namely:

$$L|_{R\to\infty} = \frac{\kappa}{2} \ddot{x}_i^2$$  \hspace{1cm} (11)

where an over dot means $\frac{d}{dt}$ as usual.

The extension of this Lagrangian to the NH case can be obtained through the following substitution [8]:

$$\ddot{x}_i^2 \to \ddot{x}_i^2 + \frac{1}{R^2} \dot{x}_i^2$$  \hspace{1cm} (12)

Then, we can obtain the Lagrangian in NH case:

$$L = \frac{\kappa}{2} \left( \ddot{x}_i^2 + \frac{1}{R^2} \dot{x}_i^2 \right)$$  \hspace{1cm} (13)

It is easy to check that the Lagrangian (13) is quasi-invariant under transformations (10). Thus we get a (classical) dynamical model, whose Lagrangian with higher order time derivatives has the $\widehat{NH}$ algebra as its (quasi-)symmetry algebra.

\(^3\) That, following Ref.[7], means that the Lagrangian is invariant up to a boundary term of the form $\frac{d}{dt} f(x_i, \dot{x}_i)$, in comparison with the invariant one with usual $\frac{d}{dt} f(x_i)$.
IV. FIRST ORDER LAGRANGIAN, HAMILTONIAN FORMALISM AND THE EXTENDED PHASE SPACE

A. first order Lagrangian, Hamiltonian formalism

Introducing $y_i = \dot{x}_i$ as independent coordinates, one can put the Lagrangian (13) into its first order form as follows:

$$L = p_i (\dot{x}_i - y_i) + q_i \dot{y}_i - \frac{1}{2\kappa} q_i^2 + \frac{\kappa}{2R^2} y_i^2$$  \hspace{1cm} (14)

which can also be derived from the Lagrangian in Ref. [11]. Using the Faddev-Jackiw procedure [13, 14] one obtains the following nonvanishing Poisson brackets:

$$\{x_i, p_j\} = \delta_{ij}, \quad \{y_i, q_j\} = \delta_{ij}$$  \hspace{1cm} (15)

The Hamiltonian which follows from (14) has the form

$$H = p_i y_i + \frac{1}{2\kappa} q_i^2 - \frac{\kappa}{2R^2} y_i^2$$  \hspace{1cm} (16)

The equations of motion which can be obtained from Euler-Lagrange equations or Hamilton equations with (15-16), besides the equations $y_i = \dot{x}_i$, are

$$\dot{p}_i = 0, \quad \dot{y}_i = \frac{q_i}{\kappa}, \quad p_i + \dot{q}_i - \frac{\kappa}{R^2} y_i = 0$$  \hspace{1cm} (17)

We take the infinitesimal $\hat{N}H$ transformations of the variables as

$$\delta t = \delta a$$

$$\delta x_i = (\delta O)_i^j x_j + \cosh \frac{t}{R} \delta a_i + R \sinh \frac{t}{R} \delta v_i + 2R^2 (\cosh \frac{t}{R} - 1) \delta b_i + y_i \delta a$$

$$\delta y_i = (\delta O)_i^j y_j + \frac{1}{R} \sinh \frac{t}{R} \delta a_i + \cosh \frac{t}{R} \delta v_i + 2R \sinh \frac{t}{R} \delta b_i + q_i \delta a$$

$$\delta q_i = (\delta O)_i^j q_j + \frac{\kappa}{R^2} \cosh \frac{t}{R} \delta a_i + \frac{\kappa}{R} \sinh \frac{t}{R} \delta v_i + 2\kappa \cosh \frac{t}{R} \delta b_i + p_i \delta a + \frac{\kappa}{R^2} y_i \delta a$$

$$\delta p_i = (\delta O)_i^j p_j$$  \hspace{1cm} (18)

We should note that these transformations leave the Lagrangian (14) invariant. Indeed, performing the transformations (18), one obtains

$$\delta L = \frac{d}{dt} \left( \frac{\kappa}{R^2} y_i \cosh \frac{t}{R} \delta a_i + \frac{\kappa}{R} y_i \sinh \frac{t}{R} \delta v_i + 2\kappa y_i \cosh \frac{t}{R} \delta b_i - y_i p_i \delta a + \frac{\kappa}{R^2} y_i^2 \delta a \right)$$  \hspace{1cm} (19)
which is a total derivative. We can also check that the infinitesimal transformations in phase space are canonical transformations. In fact, the transformed canonical coordinates and canonical momenta are

\[
x'_i = x_i + (\delta O)_i^j x_j + \cosh \frac{t}{R} \delta a_i + R \sinh \frac{t}{R} \delta v_i + 2R^2 (\cosh \frac{t}{R} - 1) \delta b_i + y_i \delta a
\]

\[
y'_i = y_i + (\delta O)_i^j y_j + \frac{1}{R} \sinh \frac{t}{R} \delta a_i + \cosh \frac{t}{R} \delta v_i + 2R \sinh \frac{t}{R} \delta b_i + \frac{q_i}{\kappa} \delta a
\]

\[
q'_i = q_i + (\delta O)_i^j q_j + \frac{\kappa}{R^2} \cosh \frac{t}{R} \delta a_i + \frac{\kappa}{R} \sinh \frac{t}{R} \delta v_i + 2\kappa \cosh \frac{t}{R} \delta b_i - p_i \delta a + \frac{\kappa}{R^2} y_i \delta a
\]

\[
p'_i = p_i + (\delta O)_i^j p_j
\]

and it is easy to show that the Poisson brackets of canonical variables, up to first order of the infinitesimal parameters, are

\[
\{x'_i, p'_j\} = \delta_{ij}, \quad \{y'_i, q'_j\} = \delta_{ij}
\]

with the others vanishing.

We can also get the Noether charges by the Noether theorem corresponding to the Lagrangian (14). When the time \(t\) is fixed, the \(\widehat{NH}\) transformations become

\[
\delta x_i = (\delta O)_i^j x_j + \cosh \frac{t}{R} \delta a_i + R \sinh \frac{t}{R} \delta v_i + 2R^2 (\cosh \frac{t}{R} - 1) \delta b_i
\]

\[
\delta y_i = (\delta O)_i^j y_j + \frac{1}{R} \sinh \frac{t}{R} \delta a_i + \cosh \frac{t}{R} \delta v_i + 2R \sinh \frac{t}{R} \delta b_i
\]

\[
\delta q_i = (\delta O)_i^j q_j + \frac{\kappa}{R^2} \cosh \frac{t}{R} \delta a_i + \frac{\kappa}{R} \sinh \frac{t}{R} \delta v_i + 2\kappa \cosh \frac{t}{R} \delta b_i
\]

\[
\delta p_i = (\delta O)_i^j p_j
\]

Using the Noether theorem we can obtain the Noether charges as follow:

\[
P_i = p_i \cosh \frac{t}{R} + q_i \frac{1}{R} \sinh \frac{t}{R} \frac{1}{R^2} y_i \cosh \frac{t}{R}
\]

\[
K_i = p_i R \sinh \frac{t}{R} + q_i \cosh \frac{t}{R} - \frac{\kappa}{R} y_i \sinh \frac{t}{R}
\]

\[
F_i = 2p_i R^2 (\cosh \frac{t}{R} - 1) + 2q_i R \sinh \frac{t}{R} - 2\kappa y_i \cosh \frac{t}{R}
\]

It is easy to see that the Noether charges become the same as that in the acceleration-extended Galileo case, when \(R \to \infty\).

**B. Schrödinger equation of the system and its invariance**

We describe the quantization of classical system by the method of geometric quantization here. It is known that the classical system is described by the Poisson algebra of
observables defined on the phase space of the system. If we quantized the classical system, we can obtain the corresponding quantum states on the Hilbert space. The relation of Poisson brackets of the classical observables and commutators of Hermitian operators is:

\[ [\hat{f}_1, \hat{f}_2] = i\hbar \{f_1, f_2\} \]  \hspace{1cm} (24)

where \( f_1, f_2 \) are the classical observables and \( \hat{f}_1, \hat{f}_2 \) their quantum counterparts. Here, corresponding to the brackets (15), the commutators are:

\[ [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{y}_i, \hat{q}_j] = i\hbar \delta_{ij} \]  \hspace{1cm} (25)

For a classical system whose degrees of freedom is \( n \), generalized coordinates are \( Q_i \) and generalized momenta are \( P_i \) \( (i = 1, \cdots, n) \). The symplectic 2-form on the phase space \( M \) is

\[ \omega = dP_i \wedge dQ_i \]  \hspace{1cm} (26)

Now, in our case, the symplectic 2-form is

\[ \omega = dp_i \wedge dx_i + dq_i \wedge dy_i \]  \hspace{1cm} (27)

Corresponding to every observable as a function \( f \) on the phase space \( M \), we can define a vector field \( X_f \) satisfying

\[ i_{X_f} \omega + df = 0 \]  \hspace{1cm} (28)

We should pre-quantize the observables first, i.e. give the Hermitian operator \( \hat{f} \) corresponding to every observable \( f \), satisfying (24). The Hermitian operator can be given by the following equation (16):

\[ \hat{f} = -i\hbar (X_f - \frac{i}{\hbar} i_{X_f} \theta) + f \]  \hspace{1cm} (29)

where \( \theta \) is the symplectic potential, i.e. the 1-form satisfying \( \omega = d\theta \). In our case, the symplectic potential can be simply taken as

\[ \theta = p_i dx^i + q_i dy^i \]  \hspace{1cm} (30)

Then we should polarize the Hermitian operators. Because our case is relatively simple, the Hermitian operators obtained after polarization are just the Hermitian operators on the Hilbert space, corresponding to the classical observables.
Thus we can obtain the Hermitian operators on the Hilbert space corresponding to the observables in NH space-time in this way. The operators are:

\[
\hat{x}_i = x_i, \quad \hat{p}_i = -i\hbar \frac{\partial}{\partial x_i}, \quad \hat{y}_i = y_i, \quad \hat{q}_i = -i\hbar \frac{\partial}{\partial y_i}
\]

(31)

So the Schrödinger equation of the system can be obtained as

\[
(-\frac{\hbar^2}{2\kappa} \frac{\partial^2}{\partial y_i^2} - i\hbar y_i \frac{\partial}{\partial x_i} - \frac{\kappa}{2R^2} y_i^2)\varphi = i\hbar \frac{\partial}{\partial t} \varphi
\]

(32)

Now we can verify that the equation (32) is invariant under infinitesimal transformations (18). First, we get the Hermitian operators corresponding to the transformed observables, again using the geometric quantization:

\[
\hat{x}_i' = x_i + (\delta O)_i^j x_j + \cosh \frac{t}{R} \delta a_i + R \sinh \frac{t}{R} \delta v_i + 2R^2 (\cosh \frac{t}{R} - 1) \delta b_i = x'_i
\]

\[
\hat{y}_i' = y_i + (\delta O)_i^j y_j + \frac{1}{R} \sinh \frac{t}{R} \delta a_i + \cosh \frac{t}{R} \delta v_i + 2R \sinh \frac{t}{R} \delta b_i - i\hbar \frac{\delta a_i}{\kappa} \frac{\partial}{\partial y_i}
\]

\[
\hat{p}_i' = -i\hbar (\frac{\partial}{\partial x_i} + (\delta O)_i^j \frac{\partial}{\partial x_j})
\]

\[
\hat{q}_i' = -i\hbar (\frac{\partial}{\partial y_i} + (\delta O)_i^j \frac{\partial}{\partial y_j}) - i\hbar \delta a_i \frac{\partial}{\partial x_i} + \frac{\kappa}{R^2} \cosh \frac{t}{R} \delta a_i + \frac{\kappa}{R} \sinh \frac{t}{R} \delta v_i + 2\kappa \cosh \frac{t}{R} \delta b_i + \frac{\kappa}{R^2} \delta a_i
\]

(33)

It is easy to check that the above transformed operators still satisfy the standard commutators

\[
[\hat{x}_i', \hat{p}_j'] = i\hbar \delta_{ij}, \quad [\hat{y}_i', \hat{q}_j'] = i\hbar \delta_{ij}
\]

(34)

as already indicated by equations (21) and (24). As is well known, in this case the transformations act on the Hilbert space as similarity transformations. Under certain Hermitian condition, these transformations become unitary ones.

Then, for simplicity, we consider rotations, spatial translations, time translation, accelerations and boosts one by one. First, it is obvious that equation (32) is invariant under rotations if the wave function \( \varphi(x, y, t) \) is invariant. Second, the spatial translations are

\[
\hat{x}_i' = x_i + \cosh \frac{t}{R} \delta a_i
\]

\[
\hat{y}_i' = y_i + \frac{1}{R} \sinh \frac{t}{R} \delta a_i
\]

\[
\hat{p}_i' = -i\hbar \frac{\partial}{\partial x_i}
\]

\[
\hat{q}_i' = -i\hbar \frac{\partial}{\partial y_i} + \frac{\kappa}{R^2} \cosh \frac{t}{R} \delta a_i
\]

\[
t' = t
\]

(35)
From the chain rule of derivatives, we can obtain the transformed time derivative:

\[
\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \frac{\delta a_i}{R^2} \sinh \frac{t}{R} \frac{\partial}{\partial x_i} + \frac{\partial}{\partial y} \frac{\delta a_i}{R^2} \cosh \frac{t}{R} \frac{\partial}{\partial y_i} + \frac{\partial}{\partial q} \frac{\delta a_i}{R^2} \kappa \sinh \frac{t}{R} \frac{\partial}{\partial q_i} \tag{36}
\]

If we substitute equations (35) and (36) into equations (16) and (32) with every quantity replaced by the corresponding primed one, omitting the higher order terms and because of \( \frac{\partial}{\partial q_i'} = \frac{1}{i \hbar} y_i' \), we will obtain the following equation:

\[
(-\frac{\hbar^2}{2\kappa} \frac{\partial}{\partial y_i} - i \hbar y_i \frac{\partial}{\partial x_i} - \frac{\kappa}{2R^2} y_i^2) \varphi' = i \hbar \frac{\partial}{\partial t} \varphi' \tag{37}
\]

It is now obvious that equation (32) is invariant under spatial translations, if the wave function \( \varphi(x, y, t) \) is itself invariant. Similarly, we can also show that the Schrödinger equation is invariant under time translation, boosts and acceleration transformations.

V. EXOTIC CONFORMAL NEWTON-HOOKE ALGEBRA AND ITS DYNAMICAL REALIZATION IN \( D = 2 + 1 \) DIMENSION

A. exotic conformal Newton-Hooke algebra

To obtain the conformal NH algebra, one has to add to the acceleration-extended NH algebra (1a-1d, 7) another two generators, dilatation \( D \) and expansion \( K \), which together with the Hamiltonian form a subalgebra:

\[
[D, H] = -H - \frac{1}{2R^2} K, \quad [K, H] = -2D, \quad [D, K] = K \tag{38}
\]

The generators \( D \) and \( K \) are scalars:

\[
[D, J] = [K, J] = 0 \tag{39}
\]

Here, \( J \) is the rotation generator defined as \( J_{ij} = \varepsilon_{ij} J \) in \( D = 2 + 1 \), with \( \varepsilon_{ij} \) the standard antisymmetric tensor: \( \varepsilon_{12} = -\varepsilon_{21} = 1 \). \( D \) and \( K \) also satisfy

\[
[D, P_i] = -P_i + \frac{1}{2R^2} F_i, \quad [D, K_i] = 0, \quad [D, F_i] = F_i \tag{40}
\]

\[
[K, P_i] = -2K_i, \quad [K, K_i] = -F_i, \quad [K, F_i] = 0. \tag{41}
\]
One can show that the realization of the above Lie algebra on the $D = 2 + 1$ nonrelativistic space-time is given in terms of differential operators as

\[ H = \partial_t, \quad P_i = -\cosh \frac{t}{R} \partial_i, \quad K_i = -\sinh \frac{t}{R} \partial_i \]

\[ F_i = -2R^2(\cosh \frac{t}{R} - 1)\partial_i, \quad D = R \sinh \frac{t}{R} \partial_t + \cosh \frac{t}{R} x_i \partial_i \]

\[ K = 2R^2(\cosh \frac{t}{R} - 1)\partial_t + 2R \sinh \frac{t}{R} x_i \partial_i, \quad J = -\varepsilon_{ij} x_i \partial_j \]

(42)

It is easy to see that the conformal NH algebra goes back to conformal Galileo algebra (see [17]) in the limit of $R \to \infty$.

It is easy to work out the spacetime infinitesimal transformations generated by the conformal NH algebra:

\[ \delta x_i = -\cosh \frac{t}{R} \delta a_i - \sinh \frac{t}{R} \delta v_i - 2R^2(\cosh \frac{t}{R} - 1)\delta b_i + \cosh \frac{t}{R} x_i \delta d + 2R \sinh \frac{t}{R} x_i \delta e + \delta \varepsilon_{ij} x_j \]

\[ \delta t = \delta a + R \sinh \frac{t}{R} \delta d + 2R(\cosh \frac{t}{R} - 1)\delta e \]

(43)

where $\delta d$ is the infinitesimal parameter for the dilatation $D$, and $\delta e$ that for the expansion $K$.

One can extend the conformal NH algebra by introducing an exotic central element $\Theta$ in $(2+1)$ dimension. This element is introduced into the Lie bracket for two NH boosts [18]:

\[ [K_i, K_j] = \Theta \varepsilon_{ij} \]

(44)

As a consequence the Lie-bracket $[P_i, K_j]$ becomes also nonvanishing:

\[ [P_i, K_j] = -2\Theta \varepsilon_{ij}. \]

(45)

The conformal NH algebra with the modified relations (44,45) will be called exotic conformal Newton-Hooke algebra hereafter.

B. the dynamical model of the conformal Newton-Hooke symmetry

As in the case of the acceleration-extended NH algebra (see [18]), we should also find a Lagrangian which is quasi-invariant under the conformal NH transformation (43) and reduces to that in the Galilei case in the limit of $R \to \infty$, i.e. [19]

\[ L|_{R \to \infty} = -\frac{\Theta}{2} \varepsilon_{ij} \dot{x}_i \ddot{x}_j \]

(46)
Again, we can take the following substitution

\[ \varepsilon_{ij} \dot{x}_i \ddot{x}_i \rightarrow \varepsilon_{ij} \dot{x}_i \ddot{x}_i + \frac{1}{R^2} x_i \dot{x}_i \]

(47)

to get the dynamical model of the conformal NH symmetry. So we obtain the higher order Lagrangian of the conformal NH symmetry as follows

\[ L = -\frac{\Theta}{2} \varepsilon_{ij} \dot{x}_i \ddot{x}_i + \frac{1}{R^2} x_i \dot{x}_i \]

(48)

We can show that the Lagrangian is quasi-invariant under NH conformal transformations (43). Indeed, performing the Lagrangian (48) one obtains

\[ \delta L = \frac{d}{dt} \left( \frac{\Theta}{2} \varepsilon_{ij} (\dot{x}_j \frac{1}{R} \sinh \frac{t}{R} + \frac{1}{R^2} x_j \cosh \frac{t}{R}) \delta a_i + \frac{\Theta}{2} \varepsilon_{ij} (\dot{x}_j \frac{1}{R} \cosh \frac{t}{R} + \frac{1}{R^2} x_j \sinh \frac{t}{R}) \delta v_i \right) + \frac{\Theta}{2} \varepsilon_{ij} (\dot{x}_j R \sinh \frac{t}{R} - x_j \cosh \frac{t}{R} - x_j) \delta b_i - \frac{\Theta}{2} \varepsilon_{ij} \dot{x}_j \ddot{x}_i \frac{1}{R} \sinh \frac{t}{R} \delta d - \Theta \varepsilon_{ij} \dot{x}_j \ddot{x}_i \cosh \frac{t}{R} \delta e \] 

(49)

Introducing \( y_i = \dot{x}_i \) as an independent coordinate the Lagrangian (48) can be put into the following first order form

\[ L = P_i (\dot{x}_i - y_i) - \frac{\Theta}{2} \varepsilon_{ij} (y_i \dot{y}_j + \frac{1}{R^2} x_i y_i) \]

(50)

It can be checked that the first order form of the Lagrangian (50) becomes that in Galileo case in the limit of \( R \rightarrow \infty \), see [19]. Then we can obtain the Hamiltonian following from the Lagrangian (50):

\[ H = P_i y_i + \frac{\Theta}{2R^2} \varepsilon_{ij} x_i y_j \]

(51)

One can also obtain the following Poisson brackets from (50) due to the Faddeev-Jackiw procedure [13, 14]:

\[ \{ x_i, P_j \} = \delta_{ij}, \quad \{ y_i, y_j \} = \frac{\varepsilon_{ij}}{\Theta} \]

(52)

The equations of motion, which can be obtained from the Euler-Lagrange equations or the Hamilton equations, are:

\[ y_i = \dot{x}_i, \quad \dot{y}_i - \frac{\varepsilon_{ij}}{\Theta} P_j + \frac{1}{2R^2} \varepsilon_{ij} \dot{x}_j = 0, \quad \dot{P}_i + \frac{\Theta}{2R^2} \varepsilon_{ij} y_j = 0 \]

(53)
VI. CONCLUSION

Following Ref. [8], we mainly discuss the acceleration-extended Newton-Hooke algebra and its dynamical realization in arbitrary dimension in this paper. We first recall the NH algebra, and then extend it by adding acceleration generators to it. The central charges are also introduced. We present the Lagrangian with higher order time derivatives that is quasi-invariant under the \( \hat{NH} \) transformations. After that, we obtain the quantities on the phase space using the first order form of the Lagrangian, and show that the \( \hat{NH} \) transformations on the phase space are canonical transformations. The Noether charges corresponding to the \( \hat{NH} \) transformations are obtained from the Noether theorem. The Schrödinger equation is obtained in standard way, and by geometric quantization we also show that it is invariant under the \( \hat{NH} \) transformations, which now act on the corresponding Hilbert space as unitary transformations. Finally, we discuss the exotic conformal NH algebra and its dynamical realization briefly.

It is easy to see that all the results in this paper go back to those in the Galileo case in the limit of \( R \to \infty \). On the other hand, although we give only the explicit results on the NH space-time contracted from the dS one, all the discussions in our paper can be as well applied to the AdS case. The quantum dynamical realization of the exotic conformal NH symmetry is left for future work. Further investigations on some possible physical implications of these results are even more interesting.

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