Spin correlation tensor for measurement of quantum entanglement in electron–electron scattering

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Abstract
We consider the problem of correct measurement of a quantum entanglement in the two-body electron–electron scattering. An expression is derived for a spin correlation tensor of a pure two-electron state. A geometric measure of a quantum entanglement as the distance between two forms of this tensor in entangled and separable cases is presented. Due to such definition, one does not need to look for the closest separable state to the analyzed state. We prove that introduced measure satisfies properties of a valid entanglement measure: nonnegativity, discriminance, normalization, non-growth under local operations and classical communication. This measure is calculated for a problem of electron–electron scattering. We prove that it does not depend on the azimuthal rotation angle of the second electron spin relative to the first electron spin before scattering. We specify how to find a spin correlation tensor and the related measure of a quantum entanglement in an experiment with electron–electron scattering. Finally, the introduced measure is extended to the mixed states.

Keywords: correlation tensor, electron polarization, entangled state, entanglement measure, electron–electron scattering

(Some figures may appear in colour only in the online journal)

1. Introduction
Quantum entanglement is of interest to modern physics, both from fundamental and applied points of view. The applied aspect of studying of an entanglement is related to its application in quantum information technology. The fundamental interest is related to the violation of the principle of locality in quantum mechanics. This principle was formulated originally in the form of the Einstein–Podolsky–Rosen paradox [1] and later as Bell’s theorem [2, 3]. The violation of the Bell’s inequalities, on which the theorem is based, was the first way of the identification of the quantum entanglement.

Now a few criteria are developed [4, 5] for the identification of quantum entanglement in a system. However, they do not give the quantitative information about it. Measures of a quantum entanglement serve for this purpose [5]. They have to satisfy a number of requirements [4]. For two-particle pure states, von Neumann entropy of reduced density operator satisfies all main requirements. To calculate entropy, it is necessary to find a density matrix of system. For measurement of a density matrix, the method of a quantum tomography is used [6]. However, a quantum tomography has not yet been performed for many problems. One of them is the two-body electron–electron scattering. Therefore, the search for a method of quantum entanglement measurement, which can be realized in the scattering experiment, is still desirable.

An approach based on the norm of a spin correlation tensor for the measurement of quantum entanglement of a system is well known [7, 8]. The advantage of this geometric measure of entanglement is that it can be measured experimentally. However, this approach is not physically obvious because the purely mathematical construction is used.

In this work, we will present a physically obvious geometric measure of a quantum entanglement based on a spin correlation tensor in electron–electron scattering. For that purpose we consider two-electron system in a state of...
coherent superposition of pairs of one-electron states [9]:
\[ |\psi\rangle = N (c_{++} |+\rangle |+\rangle + c_{+-} |+\rangle |-\rangle + c_{-+} |-\rangle |+\rangle + c_{--} |-\rangle |-\rangle ) \]
\[ + c_{-+} |-\rangle |+\rangle + c_{+-} |+\rangle |-\rangle ) \]
\[ + c_{--} |-\rangle |-\rangle ) \]  \hspace{1cm} (1)

In expression (1) \(|+\rangle\) and \(|-\rangle\) are orthonormal states of \(\alpha\)th electron with a spin ‘up’ and ‘down’ respectively concerning the allocated direction:
\[ a(+/) = I_{\alpha b} = a(-/-) \]
\[ a(+/) = 0 = a(-/-) \]
where \(I\) is the unit matrix. Here, and everywhere below, the indexes \(a, b \in \{1, 2\}\). Taking into account a normalization \(\langle \psi |\psi\rangle = 1\) for a state (1) we have
\[ N = (|c_{++}|^2 + |c_{+-}|^2 + |c_{-+}|^2 + |c_{--}|^2)^{-1/2} \]  \hspace{1cm} (3)

For the system in the state (1), we will obtain an expression for the spin correlation tensor. We will accept the norm of the density matrix for the description of quantum correlations in the two-electron system in state (1).

2. Spin correlation tensor

For the description of quantum correlations in the two-electron system the spin correlation tensor is used [9, 11]:
\[ T_{ij} := \langle \sigma_i |\sigma_j\rangle \]  \hspace{1cm} (4)

The mean value of the physical quantity \(A\) is written in terms of a wave function
\[ \langle A \rangle = \langle \psi |A|\psi\rangle \]  \hspace{1cm} (5)
or density matrix \(\rho := |\psi\rangle \langle \psi|\) in the form
\[ \langle A \rangle = Tr(\rho A), \]  \hspace{1cm} (6)
where \(Tr\) is a trace on pairs of one-electron states \(|\pm\rangle|\pm\rangle\) and \(|\pm\rangle|\mp\rangle\). Here and everywhere below the indexes \(i, j, k, l \in \{x, y, z\}\), with summation on the repeating indexes. We write the dimensionless projection of \(\alpha\)th electron spin on coordinate axis \(i\) as an operator:
\[ \sigma_{ai} := \begin{bmatrix} \sigma_{ai}^{11} & \sigma_{ai}^{12} \\ \sigma_{ai}^{21} & \sigma_{ai}^{22} \end{bmatrix} \]
\[ = \begin{bmatrix} |+\rangle & |-\rangle \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} |+\rangle \\ |-\rangle \end{bmatrix} \]  \hspace{1cm} (7)

where \(\sigma_{ai}^{mn}\) are elements of the Pauli matrices
\[ \sigma_i := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_i := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_i := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]  \hspace{1cm} (8)

2.1. Elements of tensor

Let us find elements of the tensor (4) taking into account the definition (5) for the system in state (1). From the expressions (1) and (7) taking into account the property (2) we have
\[ \langle \psi |\sigma_i N^{-1} \]  \hspace{1cm} (9)
\[ = (\sigma_i^{11} |+\rangle + \sigma_i^{12} |\rangle + \sigma_i^{21} |\rangle + \sigma_i^{22} |\rangle) \]  \hspace{1cm} (9)

\[ = (\sigma_i^{11} |+\rangle + \sigma_i^{12} |\rangle + \sigma_i^{21} |\rangle + \sigma_i^{22} |\rangle) \]  \hspace{1cm} (9)

Combining expressions (9) and (10) in the definition (4) we derive
\[ T_{ij} N^{-2} \]
\[ = (\sigma_i^{11} |+\rangle + \sigma_i^{12} |\rangle + \sigma_i^{21} |\rangle + \sigma_i^{22} |\rangle) \]  \hspace{1cm} (11)

Taking into account definitions of the Pauli matrices (8) from expression (11) we have for the rows of the tensor (4):
\[ T_{ij} N^{-2} \]
\[ = (\sigma_i^{11} |+\rangle + \sigma_i^{12} |\rangle + \sigma_i^{21} |\rangle + \sigma_i^{22} |\rangle) \]
\[ = (\sigma_i^{11} |+\rangle + \sigma_i^{12} |\rangle + \sigma_i^{21} |\rangle + \sigma_i^{22} |\rangle) \]

Using definitions of the Pauli matrices (8) once again, from equalities (12)--(14) we obtain the spin correlation tensor for the two-electron system in state (1):
The tensor (15) has the following symmetry that the non-diagonal elements of the tensor $T_{12}$ and $T_{21}$, $T_{13}$ and $T_{31}$, $T_{23}$ and $T_{32}$ are connected with each other:

\[ c_{+-} \mapsto c_{-+}, \quad c_{-+} \mapsto c_{+-}. \quad (16) \]

2.2. Tensor in the absence of entanglement

In the absence of an entanglement, the spin correlation tensor (4) is equal to the tensor product of the electron polarization vectors (polarizations) [9, 11]:

\[ \tilde{T}_g \equiv \langle \sigma_1 \rangle \langle \sigma_2 \rangle = P_{\mu} P_\nu \quad (\Leftrightarrow \tilde{T} = P_1 \otimes P_2), \quad (17) \]

\[ P_{\mu} = \langle \sigma_\mu \rangle, \quad (18) \]

where $P_{\mu}$ is a polarization projection of \( \mu \)-th electron on axis $i$. Let us obtain expressions for polarizations.

According to definition (7) and a condition (2) for the projections of the electron polarizations in the system in state (1) we have

\[ P_{\mu} N^{-2} = \sigma_{11}^1 (|c_{++}|^2 + |c_{-+}|^2) + \sigma_{22}^1 (|\tilde{c}_{++}|^2 + |\tilde{c}_{-+}|^2) + \sigma_{12}^2 (|c_{+-}|^2 + |c_{-+}|^2). \]

\[ (19) \]

Taking into account definition of the Pauli matrices (8) we can simplify expressions (19) and (20):

\[ P_1 = N^2 \left[ \begin{array}{cc} 2 \text{Re}(c_{++} \tilde{c}_{++} + c_{-+} \tilde{c}_{-+}) & 2 \text{Im}(c_{++} \tilde{c}_{++} + c_{-+} \tilde{c}_{-+}) \\ 2 \text{Re}(c_{-+} \tilde{c}_{++} + c_{++} \tilde{c}_{-+}) & 2 \text{Im}(c_{-+} \tilde{c}_{++} + c_{++} \tilde{c}_{-+}) \end{array} \right], \quad (21) \]

\[ P_2 = N^2 \left[ \begin{array}{cc} 2 \text{Re}(c_{++} \tilde{c}_{++} + c_{-+} \tilde{c}_{-+}) & 2 \text{Im}(c_{++} \tilde{c}_{++} + c_{-+} \tilde{c}_{-+}) \\ 2 \text{Re}(c_{-+} \tilde{c}_{++} + c_{++} \tilde{c}_{-+}) & 2 \text{Im}(c_{-+} \tilde{c}_{++} + c_{++} \tilde{c}_{-+}) \end{array} \right]. \quad (22) \]

As well as non-diagonal elements of the spin correlation tensor (15), the electron polarizations (21) and (22) are connected with each other by substituting (16).

3. Tensor measure of quantum entanglement

As in the absence of entanglement the spin correlation tensor (4) is equal to the tensor product (17), we set the distance between them as the measure of entanglement in the system. Mathematically the distance between tensors is defined as the norm of their difference:

\[ E := ||T - \tilde{T}||. \quad (23) \]

As the norm we choose the scaled Euclidean norm:

\[ \forall A \in \mathbb{R}^{3 \times 3} \quad ||A|| = \sqrt{\text{tr}(AA^T)} / \sqrt{3} = \sqrt{\text{det}(A) / 3}. \quad (24) \]

Here and everywhere below $\text{tr}$ is a trace of the real $3 \times 3$ matrices.

The measure (23) has significant advantage in comparison with other geometric measures [5]. Due to its definition, we do not need to look for the closest separable state to the analyzed state $\rho$. We only calculate the distance between values of two tensor functions $T$ and $\tilde{T}$ for the same argument $\rho$.

The measure of the quantum entanglement is valid when it has the following properties [4, 5]:

1. nonnegativity, discriminance, normalization;
2. invariance under local unitary operations (LU);
3. non-growth under local operations and classical communication (LOCC).

Let us prove these properties for the measure (23). For this purpose we will use a definition of mean values in terms of a density matrix (6).

3.1. Nonnegativity, discriminance, normalization

**Proposition 1.** The measure (23) is nonnegative (nonnegativity):

\[ \forall \rho \quad E(\rho) \geq 0. \quad (25) \]

**Proof.** Property (25) is carried out for the measure (23) by definition of the norm. \( \square \)

**Proposition 2.** The criterion for separability of states is that the measure (23) is equal to zero (discriminance):

\[ \rho \text{ is separable} \iff E(\rho) = 0. \quad (26) \]

**Proof.** According to the proposition 1a in the work [12], taking into account that for a system of tenons the Bloch vector coincides with polarization by definition [7], we have

\[ \rho \text{ is separable} \iff T = P_1 \otimes P_2; \quad (27) \]

Property (26) for the measure (23) follows from the statement (27) and definition (17). \( \square \)

**Proposition 3.** The measure (23) for maximally entangled states is equal to one (normalization):

\[ \rho \text{ is maximally entangled} \implies E(\rho) = 1. \quad (28) \]

**Proof.** For two-particle system, maximally entangled states are Bell states [5]. In terms of function (1) one can write them as

\[ N = \frac{1}{\sqrt{2}}, \quad c_{++} = 0, \quad c_{-+} = +1, \quad c_{+1} = -1, \quad c_{-1} = 0; \]

\[ N = \frac{1}{\sqrt{2}}, \quad c_{++} = 0, \quad c_{-+} = +1, \quad c_{+1} = +1, \quad c_{-1} = 0; \]

\[ N = \frac{1}{\sqrt{2}}, \quad c_{++} = +1, \quad c_{-+} = 0, \quad c_{+1} = 0, \quad c_{-1} = -1; \]

\[ N = \frac{1}{\sqrt{2}}, \quad c_{++} = +1, \quad c_{-+} = 0, \quad c_{+1} = 0, \quad c_{-1} = +1; \quad (29) \]
for a singlet and three triplets respectively. From equalities (29), expressions (21), (22) and (15) follows that in all four conditions polarizations are equal to zero, and spin correlation tensors are diagonal matrices which elements are equal to ±1. Then taking into account definition (17) for the measure (23) the property (28) is carried out.

3.2. LU invariance

**Proposition 4.** The measure (23) is invariant under local unitary operations:

\[
E(\rho) = E(\rho'),
\]

\[
\rho' = (U_1^\dagger \otimes U_2^\dagger) \rho (U_1 \otimes U_2).
\]

Here \(U_a\) is the unitary operator acting on \(a\)th particle:

\[
U_a U_a^\dagger = U_a^\dagger U_a = L_a,
\]

where \(L_a\) is the unity operator acting on \(a\)th particle.

**Proof.** After transformation, the measure (23) takes the form:

\[
E(\rho') = ||T' - \tilde{T}'||,
\]

where tensors (4) and (17) can be written in terms of a density matrix \(\rho'\) (6). Then taking into account expression (31) and properties of a trace we have

\[
T'_{ij} = \text{Tr}(\rho' \sigma_{ij} \rho') = \text{Tr}(\rho (U_1 \otimes U_2) \sigma_{ij} (U_1^\dagger \otimes U_2^\dagger)) \Rightarrow
\]

\[
T_{ij} = \text{Tr}(\rho (U_1 \sigma_{ij} U_1^\dagger)(U_2 \sigma_{ij} U_2^\dagger)).
\]

The Pauli matrices form a basis in the space of Hermitian \(2 \times 2\) matrices with a zero trace. At the same time taking into account unitarity (32) \(\text{Tr}(U_a \sigma_a U_a^\dagger) = \text{Tr}(U_a^\dagger U_a \sigma_a) = \text{Tr}(\sigma_a) = 0\). Then in the expression (34), the quantity \(U_a \sigma_a U_a^\dagger\) can be expanded on the basis:

\[
U_a \sigma_a U_a^\dagger = Q_{a\alpha} \sigma_{\alpha},
\]

where \(Q_a\) is orthogonal \(3 \times 3\) matrix (see appendix A):

\[
Q_a Q_a^T = Q_a^T Q_a = I.
\]

At the same time the matrix \(Q_a\) can be taken out from under the trace \(T\).

From expression (34) and the expansion (35) we have:

\[
T^\prime_{ij} = \text{Tr}(\rho (Q_{a1} \sigma_{a1})(Q_{a2} \sigma_{a2})) = Q_{a1} Q_{a2} \text{Tr}(\rho \sigma_{a1} \sigma_{a2}) = Q_{a1} Q_{a2} T_{kl} \Rightarrow
\]

\[
T^\prime = Q_1 \otimes Q_2 \times T,
\]

where symbol \(\otimes\) means action of a matrix (on the left) on \(a\)th index of a tensor (on the right). Taking into account the expression (31), properties of a trace, the unitarity (32) and the expansion (35) we obtain:

\[
P^\prime_{\alpha\alpha} = \text{Tr}(\rho' \sigma_{\alpha\alpha}) = \text{Tr}(\rho(U_1 \otimes U_2) \sigma_{\alpha\alpha}(U_1^\dagger \otimes U_2^\dagger)) = \text{Tr}(\rho(U_0 \sigma_{\alpha\alpha} U_0^\dagger)) = Q_{b\alpha} P_{b\alpha}.
\]

Then for the transformed tensor product (17) we have

\[
T^\prime_{ij} = Q_{a1} P_{a1} Q_{a2} P_{a2} = Q_{a1} Q_{a2} T_{kl} \Rightarrow
\]

\[
\tilde{T}' = Q_1 \otimes Q_2 \times T.
\]

Taking into account expressions (37) and (38) the transformed measure (33) takes the form:

\[
E(\rho') = ||Q_1 \times Q_2 \times (T - \tilde{T})||.
\]

From expressions (39) and (24), properties of a trace and the orthogonality (36) we have

\[
3[E(\rho')]^2 = |\text{tr} \{[Q_1 \times Q_2 \times (T - T^\prime)]Q_1 \times Q_2 \times (T - T^\prime)\}|
\]

\[
= |\text{tr} \{[Q_1 Q_2 \times (T - T^\prime)]Q_2 \times (T - T^\prime)\}|
\]

\[
= |\text{tr} \{[Q_2 \times (T - T^\prime)]Q_2 \times (T - T^\prime)\}|
\]

\[
= |\text{tr} \{[T - T^\prime]Q_2^T Q_2 (T - T^\prime)\}|
\]

\[
= |\text{tr} \{[T - T^\prime](T - T^\prime)\} = 3||T - T^\prime||^2,
\]

from where the property (30) of the measure (23) follows.

3.3. Non-growth under measurements

**Proposition 5.** The measure (23) does not increase under measurements:

\[
E(\rho) \geq E(\rho'),
\]

where \(\rho\) is a density matrix of the original two-particle state, \(\rho'\) is a density matrix of the resulting two-particle mixed state. Without losing generality, we assume that the local measurements are *positive operator value measures* (POVMs) [5]. The local POVMs acting on a two-particle state generally have an appearance:

\[
\rho' = \sum_{mm'} (L_{m1} \otimes L_{m2}) \rho (L_{n1}^\dagger \otimes L_{n2}^\dagger),
\]

where \(\{L_{m}\}_n\) are linear, positive, keeping a trace operators having properties:

\[
\sum_n L_{m1} L_{m1}^\dagger = I, \quad [L_{m1}, L_{n1}^\dagger] = 0.
\]

**Proof.** After transformation, the measure (23) takes the form:

\[
E(\rho') = ||T' - \tilde{T}'||,
\]

where tensors (4) and (17) can be written in terms of a density matrix \(\rho'\) (6). Then taking into account expression (41) and
properties of a trace we have

\[
T'_\mathbf{i} = \text{Tr}(\rho' \sigma_1 \sigma_2) = \sum_{\mathbf{m}} \text{Tr}[\rho(L_{\mathbf{m}_1} \otimes L_{\mathbf{n}_2}) \sigma_1 \sigma_2 (L_{\mathbf{m}_1} \otimes L_{\mathbf{n}_2})] \\
= \sum_{\mathbf{m}} \text{Tr}(L_{\mathbf{m}_1} L_{\mathbf{n}_2} \sigma_1 \sigma_2) = \sum_{\mathbf{m}} \text{Tr}(\rho(L_{\mathbf{m}_1} L_{\mathbf{n}_2} \sigma_1 \sigma_2)) = \sum_{\mathbf{m}} \text{Tr}(\rho(L_{\mathbf{m}_1} \sigma_1 L_{\mathbf{n}_2} \sigma_2)) = : D_{\mathbf{m}_1} D_{\mathbf{n}_2} \\
\Rightarrow \\
T'_\mathbf{i} = \text{Tr}\left[\rho\left(\sum_{\mathbf{m}_1} L_{\mathbf{m}_1} \sigma_1 (L_{\mathbf{m}_1} \otimes L_{\mathbf{n}_2})\right)\left(\sum_{\mathbf{n}_2} L_{\mathbf{n}_2} \sigma_2 (L_{\mathbf{m}_1} \otimes L_{\mathbf{n}_2})\right)\right] \\
= \text{Tr}\left[\rho\left(\sum_{\mathbf{m}_1} L_{\mathbf{m}_1} \sigma_1 \sigma_2 (L_{\mathbf{m}_1} \otimes L_{\mathbf{n}_2})\right)\right] \\
= \sum_{\mathbf{m}_1} \text{Tr}(\rho(L_{\mathbf{m}_1} \sigma_1 \sigma_2)) = \sum_{\mathbf{m}_1} \text{Tr}(\rho(L_{\mathbf{m}_1} \sigma_1)) \text{Tr}(L_{\mathbf{m}_1} \sigma_2) \\
= D_{\mathbf{m}_1} D_{\mathbf{n}_2} = D_{\mathbf{m}_1} D_{\mathbf{n}_2} < 1. 
\tag{44}
\]

where \(D_{\mathbf{m}_1}\) is the real non-contraction 3 \times 3 matrix (see appendix A):

\[
D_{\mathbf{m}_1} D_{\mathbf{n}_2} = D_{\mathbf{m}_1}^T D_{\mathbf{n}_2} < 1.
\tag{45}
\]

At the same time the matrix \(D_{\mathbf{m}_1}\) can be taken out under the trace \(\text{Tr}\).

From expression (44) and the expansion (45) we have

\[
T'_\mathbf{i} = \text{Tr}[\rho(D_{\mathbf{m}_1} \sigma_1)(D_{\mathbf{n}_2} \sigma_2)] \\
= D_{\mathbf{m}_1} D_{\mathbf{n}_2} \text{Tr}(\rho \sigma_1 \sigma_2) = D_{\mathbf{m}_1} D_{\mathbf{n}_2} T_{\mathbf{i}} \\
\Rightarrow \\
T'_\mathbf{i} = D_{\mathbf{m}_1} D_{\mathbf{n}_2} T_{\mathbf{i}} \\
= D_{\mathbf{m}_1} D_{\mathbf{n}_2} T_{\mathbf{i}}. 
\tag{47}
\]

Taking into account the expression (41), properties of a trace, properties (42) and the expansion (45) we obtain:

\[
P'_\mathbf{i} = \text{Tr}(\rho' \sigma_1) \\
= \sum_{\mathbf{m}_1} \text{Tr}[\rho(L_{\mathbf{m}_1} \otimes L_{\mathbf{n}_2}) \sigma_1 (L_{\mathbf{m}_1} \otimes L_{\mathbf{n}_2})] \\
= \sum_{\mathbf{m}_1} \text{Tr}[\rho(L_{\mathbf{m}_1} \sigma_1 L_{\mathbf{m}_1} \otimes L_{\mathbf{n}_2} L_{\mathbf{n}_2})] \\
= \text{Tr}\left[\rho\left(\sum_{\mathbf{m}_1} L_{\mathbf{m}_1} \sigma_1 L_{\mathbf{m}_1}\right)\right] = \text{Tr}(\rho L_{\mathbf{m}_1} \sigma_1 L_{\mathbf{m}_1}) \\
= \text{Tr}(\rho L_{\mathbf{m}_1} \sigma_1) \text{Tr}(L_{\mathbf{m}_1}) = \text{Tr}(\rho L_{\mathbf{m}_1} L_{\mathbf{m}_1}) \\
= D_{\mathbf{m}_1} D_{\mathbf{m}_1}. 
\tag{48}
\]

One also can show that \(P'_\mathbf{j} = D_{\mathbf{m}_1} P_{\mathbf{j}}\). Then for the transformed tensor product (17) we have

\[
\tilde{T}'_\mathbf{i} = D_{\mathbf{m}_1} P_{\mathbf{j}} D_{\mathbf{m}_2} P_{\mathbf{j}} = D_{\mathbf{m}_1} D_{\mathbf{m}_2} T_{\mathbf{i}} \\
\Rightarrow \\
\tilde{T}' = D_{\mathbf{m}_1} \otimes D_{\mathbf{m}_2} \otimes (T - \tilde{T}). 
\tag{49}
\]

Taking into account expressions (47) and (48) the transformed measure (43) takes the form:

\[
E(\rho') = ||D_{\mathbf{m}_1} D_{\mathbf{m}_2} (T - \tilde{T})||. 
\tag{49}
\]

From expressions (49) and (24), properties of a trace and the property (46) we have

\[
3E(\rho') \tilde{T}' = \text{tr}\{[D_{\mathbf{m}_1} D_{\mathbf{m}_2} (T - \tilde{T})][D_{\mathbf{m}_1} D_{\mathbf{m}_2} (T - \tilde{T})]\} \\
\leq \text{tr}\{[D_{\mathbf{m}_1} D_{\mathbf{m}_2} (T - \tilde{T})][D_{\mathbf{m}_1} D_{\mathbf{m}_2} (T - \tilde{T})]\} \\
\leq \text{tr}\{[D_{\mathbf{m}_1} D_{\mathbf{m}_2} (T - \tilde{T})][D_{\mathbf{m}_1} D_{\mathbf{m}_2} (T - \tilde{T})]\} \\
= \text{tr}(T - \tilde{T})D_{\mathbf{m}_1} D_{\mathbf{m}_2} D_{\mathbf{m}_1} D_{\mathbf{m}_2} (T - \tilde{T}) \\
\leq \text{tr}(T - \tilde{T}) (T - \tilde{T}) = 3||T - \tilde{T}||^2, 
\tag{49}
\]

from where the property (40) of the measure (23) follows. 

\section{4. Non-growth under LOCC}

\subsection{Proposition 6.}

The measure (23) does not increase under local operations and classical communication \(\Phi_{\text{LOCC}}\):

\[
E(\rho) \geq E(\Phi_{\text{LOCC}}(\rho)). 
\tag{50}
\]

\textbf{Proof.} LOCC can be decomposed into four basic kinds of operations [13].

I. Appending an ancillary system not entangled to the state of the original system. It is obvious that appending cannot change the tensor (4) and polarizations (18). Therefore the measure (23) is invariant under appending.

II. Performing a unitary transformation. The measure (23) is invariant under the unitary transformations (30).

III. Performing measurements. The measure (23) does not increase under the measurements (40).

IV. Throwing away (tracing out) part of the system. It is obvious that after this operation in two-particle system the entanglement is equal to zero.

As for the measure (23) all four requirements are fulfilled, for it property (50) is true.

Thus, according to the properties proved in this section, the measure of quantum entanglement (23) is valid.

\section{4. Tensor measure of quantum entanglement in a scattering problem}

\subsection{Electron–electron scattering problem}

Let us calculate measure of quantum entanglement (23) in a problem of Coulomb electron–electron scattering [10]. In this case, in expression (1) we set (see appendix B)

\[
\begin{align*}
\cos(\Omega/2) & = \varphi, \\
\cos(\Omega/2) & = \varphi, \\
\cos(\Omega/2) & = \varphi, \\
\cos(\Omega/2) & = \varphi, \\
\cos(\Omega/2) & = \varphi.
\end{align*}
\tag{51}
\]

In expressions (51)–(53) \(\Omega\) and \(\varphi\) are polar (relative to the axis \(z\)) and azimuthal (relative to the axis \(x\)) rotation angles of the 2nd electron polarization before scattering (polarization of the 1st electron is oriented in the \(z\)-direction), \(\psi_s\) is symmetric wave function, \(\psi_a\) is anti-symmetric wave function, \(f\) is the scattering amplitude in the center of mass of the interacting electrons, \(\theta\) is scattering angle in the center of mass frame, \(\alpha = 1/\nu_{\text{rel}}\) is the dimensionless factor, \(\nu_{\text{rel}}\) is the relative electron velocity in atomic units.

\[
f(\theta) \sim \csc(\theta/2)^2 \exp[-i\alpha \ln(1 - \cos \theta)]. 
\tag{53}
\]
4.2. \( \phi \) independence of entanglement measure

**Proposition 7.** The measure (23) in the problem (1), (51)–(53) does not depend on azimuthal angle:

\[
E = \text{const}(\phi).
\]

**Proof.** Let us segregate the sum in the measure (23) into three blocks:

\[
3E^2 = T_{ij}T_{ij} - 2T_{ij}T_{ij} + T_{ij}T_{ij}.
\]

At the same time everywhere below we will consider that according to expressions (3) and (51) \( N = \text{const}(\phi). \)

Let us consider the block \( T_{ij}T_{ij} \). According to expressions (15) and (51) elements \( T_{ij}T_{ij} \), \( T_{ij}T_{ij} \), \( T_{ij}T_{ij} \), and \( T_{ij}T_{ij} \) do not depend on \( \phi \), and also one can see that

\[
T_{ij}T_{ij} = |N^2c_++\bar{c}_+|^2 = \text{const}(\phi),
\]

\[
T_{ij}T_{ij} = |N^2c_-\bar{c}_+|^2 = \text{const}(\phi).
\]

Consequently, the block \( T_{ij}T_{ij} \) does not depend on \( \phi \):

\[
T_{ij}T_{ij} = \text{const}(\phi).
\]

Let us consider the block \( T_{ij}T_{ij} \). From expressions (15), (21) and (22) taking into account equalities (51) we have

\[
T_{ij}T_{ij} = N^2(c_++\bar{c}_+)^2P_{ij}^2,
\]

\[
T_{ij}T_{ij} = N^2(c_-\bar{c}_+)^2P_{ij}^2.
\]

Substituting in these expressions \( P_{ij} \) and \( P_{ij} \) (22), taking into account equalities (51) we see that

\[
T_{ij}T_{ij} + T_{ij}T_{ij} + T_{ij}T_{ij} + T_{ij}T_{ij} = N^8 |c_+\bar{c}_+|^2 = \text{const}(\phi).
\]

Consequently, the block \( T_{ij}T_{ij} \) does not depend on \( \phi \):

\[
T_{ij}T_{ij} = \text{const}(\phi).
\]

Let us consider the block \( T_{ij}T_{ij} \). From expressions (17), (21), (22) and (51) we have

\[
T_{ij}T_{ij} = (P_{ij}^2 + P_{ij}^2)P_{ij}^2 + P_{ij}^2
\]

\[
= |N^2c_+\bar{c}_+|^2P_{ij}^2 = \text{const}(\phi),
\]

\[
T_{ij}T_{ij} = P_{ij}^2|N^2c_+\bar{c}_+|^2 = \text{const}(\phi),
\]

\[
T_{ij}T_{ij} = P_{ij}^2|N^2c_+\bar{c}_+|^2 = \text{const}(\phi).
\]

Consequently, the block \( T_{ij}T_{ij} \) does not depend on \( \phi \):

\[
T_{ij}T_{ij} = \text{const}(\phi).
\]

Thus, property (54) for measure (23) follows from the equalities (55)–(58). 

4.3. Numerical calculation

Let us plot the graph of quantum entanglement measure (23) taking into account equalities (24), (15), (17), (21), (22), (51)–(53) (figures 1 and 2). Calculations confirmed that it does not depend on the azimuthal angle \( \phi \). Therefore, without losing the generality, we set \( \phi = 0 \) in the following figures. From them we see that the entanglement reaches a maximum at \( \theta = \pi/2 \), where it is equal to one. The entanglement is equal to zero at \( \theta = 0, \pi \). One can observe a peak broadening about a point \( \theta = \pi/2 \) with increase of angle \( \Omega \). These results completely conform and supplement the ones obtained in article [10] where the entropy of one of the electrons of the correlated electron pair was the measure of the entanglement.
The spin correlation tensor \( \rho_{ij} \) and the electron polarizations (18) can be found on the basis of the dimensionless projections of electron spins to coordinate axes measured in experiment. For this purpose, it is necessary to measure the spin correlation tensor and the electron polarizations. In this section they act as phenomenological quantities. One can measure them even for a system, which has no microscopic model, and give the quantitative assessment to a quantum entanglement in it.

The spin correlation tensor (4) and the electron polarizations (18) can be found on the basis of the dimensionless projections of electron spins to coordinate axes measured in experiment. After each act of scattering both electrons are detected separately by two analyzers. Analyzers register the spin projections after \( n \)th act of scattering \( \sigma_{i1}^{(m)} \) and \( \sigma_{j2}^{(m)} \) on the axes chosen for them \( i \) and \( j \) respectively. Spin projections take on values \( \pm 1 \) that corresponds to eigenvalues of operators (7).

In experiment, we measure projections of electron spins at various combinations of the axes chosen for analyzers. Data of the experiment are registered in the form of 3 \( \times \) 3 matrix \( \mathbb{T} = [\mathbb{T}_{ij}] \). Each element of the matrix is the table of two columns and \( M_{ij} \) rows of data:

\[
\mathbb{T}_{ij} = [(\sigma_{i1}^{(m)}, \sigma_{j2}^{(m)}); \sigma_{i1}^{(m)}, \sigma_{j2}^{(m)} \in \{+1, -1\}, m = 1, \ldots, M_{ij}].
\]

The table (59) corresponds to the set of measurements of electron spins projections on axes \( i \) and \( j \). It allows to find the corresponding element of spin correlation tensor (4) on the basis of the products \( \sigma_{i1}^{(m)} \sigma_{j2}^{(m)} \). As well as spin projections, their products take on values \( \pm 1 \) that corresponds to the eigenvalues of the operator \( \sigma_{i1} \sigma_{j2} \) (see appendix C).

When calculating projections of electron polarizations (18) we use data (59). Minimum errors of these quantities can be expected if all available data for the corresponding spin projections are used:

\[
S_{ij} = |\bigcup_{l} \mathbb{T}_{ij}(l), 1 \rangle, \text{card}(S_{ij}) = N_{ij} = \Sigma_{j} M_{ij},
\]

\[
S_{j2} = |\bigcup_{l} \mathbb{T}_{j2}(l), 2 \rangle, \text{card}(S_{j2}) = N_{j2} = \Sigma_{j} M_{ij}.
\]

The array \( S_{ij} \) is the first column of the table, which is obtained by combining all elements of the matrix \( \mathbb{T} \) in the row \( i \). The array \( S_{j2} \) is the second column of the table, which is obtained by combining all elements of the matrix \( \mathbb{T} \) in the column \( j \).

At rather large numbers of measurements \( \{M_{ij}\}_{ij} \) from tables (59) and arrays (60) we have for spin correlation tensor (4) and projections of electron polarizations (18)

\[
\langle \sigma_{i1} \sigma_{j2} \rangle = \frac{1}{M_{ij}} \sum_{m} (1 \leq m \leq M_{ij}) \sigma_{i1}^{(m)} \sigma_{j2}^{(m)},
\]

\[
\langle \sigma_{i1} \rangle = \frac{1}{N_{i}} \sum_{n} (1 \leq n \leq N_{i}) \sigma_{i1}^{(n)},
\]

respectively. In expressions (61) and (62) Iverson notation is used [14]; brackets with the statement are equal to one if it is true, and are equal to zero if it is false. Taking into account definition (17) these expressions allow finding the measure (23). Thus, the quantum entanglement in the two-electron system can be measured in an experiment on electron–electron scattering.

The suggested model of experiment relates to quantum tomography. It allows finding the density matrix using the spin correlation tensor [9]. Therefore, it is also possible to find von Neumann entropy of reduced density operator [4] and compare two measures of quantum entanglement in the same experiment.

6. Tensor measure of quantum entanglement for mixed states

The measure of quantum entanglement (23) can be extended to the mixed states via the use of the convex roof (or hull) construction [15]. Such approach is as follows.

A density matrix of two-electron system acts in the composite Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), where \( \mathcal{H}_n \) is the Hilbert space of \( n \)th electron states. Let us consider pure-state decomposition of the mixed state density matrix:

\[
\rho = \sum_{n} p_n \rho_n, \quad [p_n \geq 0, \rho_n \text{ is pure}]_n, \quad \sum_{n} p_n = 1.
\]

We define the quantum entanglement measure of the mixed state \( \rho \) as the average entanglement measure (23) of the pure states of the decomposition (63), minimized over all possible decompositions \( \{p_n, \rho_n\}_n \) [15]:

\[
E_{CR}(\rho) := \min_{\{p_n, \rho_n\}_n} \sum_{n} p_n E(\rho_n).
\]
Let us prove the properties of the valid measure of quantum entanglement (see section 3) for the extension \((64)\).

**Proposition 8.** The measure \((64)\) is nonnegative (nonnegativity):
\[
\forall \rho \quad E_{CR}(\rho) \geq 0.
\]  

**Proof.** Property \((65)\) is carried out for the measure \((64)\) due to property \((25)\). □

**Proposition 9.** The necessary condition for separability of the mixed states is that the measure \((64)\) is equal to zero (weak discrimination):
\[
\rho \text{ is separable } \Rightarrow E_{CR}(\rho) = 0.
\]

**Proof.** A mixed state \(\rho\) is called separable, when it can be represented as a convex sum of product states [5]:
\[
\rho \text{ is separable: } \Leftrightarrow \rho = \sum_{s=1}^{M} q_s \tau_{s1} \otimes \tau_{s2},
\]
where density matrices \(\{\tau_{si}\}_{s=1}^{M}\) act in the Hilbert space \(\mathcal{H}_a\). Without losing generality, we assume that \(\{\tau_{si}\}_{s=1}^{M}\) are the density matrices of pure states. Then taking into account definition \((67)\) and statement \((26)\) for the measure \((64)\) the property \((66)\) is carried out: \(E_{CR}(\rho) = \min_{(R,s)} \sum_{s=1}^{M} q_s E(\tau_{s1} \otimes \tau_{s2}) = 0\). □

**Proposition 10.** The measure \((64)\) for maximally entangled states is equal to one (normalization):
\[
\rho \text{ is maximally entangled } \Rightarrow E_{CR}(\rho) = 1.
\]

**Proof.** From definition \((63)\) and normalization \((28)\), we can see that for the measure \((64)\) the maximally entangled states are the Bell states and arbitrary convex sums of them (i.e. mixed states). From here, for the measure \((64)\) the property \((68)\) follows. □

Finally, for the measure \((64)\), LU invariance, non-growth under measurements and non-growth under LOCC follow directly from the statements \((30)\), \((40)\) and \((50)\) respectively. Thus, according to the properties proved in this section, the measure of quantum entanglement \((64)\) is valid.

7. **Conclusion**

In this work, we considered the problem of correct measurement of a quantum entanglement in the two-body electron–electron scattering. An expression is derived for a spin correlation tensor in case of a pure two-electron state. On its basis, we proposed geometric measure of a quantum entanglement in a system of two particles. It is the distance between two forms of this tensor: in the entangled and separable cases, that makes this measure physically obvious. Also due to such definition, one does not need to look for the closest separable state to the analyzed state. As distance between tensors, we used the scaled Euclidean norm of their difference. It enabled the proof of the properties of a measure confirming its validity: nonnegativity, discrimination, normalization, non-growth under local operations and classical communication.

We calculated the measure of quantum entanglement suggested in this paper for a problem of Coulomb electron–electron scattering. We revealed numerically and proved analytically that it does not depend on the azimuthal rotation angle of the second electron spin relative to the first electron spin before scattering. We also suggested a procedure of measurement of a spin correlation tensor in the electron–electron scattering. It allows finding a measure of a quantum entanglement in the experiment even in the absence of a microscopic model of the studied system. Finally, we extended the introduced measure to the mixed states.

Thus, the significant features of the tensor measure of a quantum entanglement proposed in this work are:

- physical obviousness;
- validity;
- measurability in experiment;
- extendability to mixed states.

The prospect of further development of this approach is generalization to the multi-electron states. The suggested experimental procedure sets a direction for the correct measurement of quantum entanglement in electron–electron scattering.

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**Appendix A. Properties of matrices \(Q_a\) and \(D_a\)**

In this appendix, we will prove properties of matrices \(Q_a\) \((35)\) and \(D_a\) \((45)\) which we used in section 3 at the proof of properties of the measure \((23)\) (propositions 4 and 5). We will carry out proofs on the basis of expression for products of the Pauli matrices:
\[
\sigma_{ia} \sigma_{ju} = \imath \epsilon_{ijk} \sigma_{iu} + I_j I_k,
\]  

(A1)

where \(\epsilon_{ijk}\) is Levi–Civita symbol. Having picked up a trace from expression \((A1)\) on pairs of one-electron states \(|\pm\rangle_1|\pm\rangle_2\) and \(|\pm\rangle_1 |\mp\rangle_2\), we have
\[
\text{Tr}(\sigma_{ia} \sigma_{ju}) = I_j \text{Tr}(I_a).
\]  

(A2)

Everywhere below we use expression \((A2)\) and properties of the trace. For brevity we also use references to formulas without their names, symbols of an implication \(\Rightarrow\) and biconditional \(\Leftrightarrow\).
A.1. Orthogonality of matrix $Q_a$

**Proposition A1.** $Q_a$ is orthogonal matrix:

$$Q_a Q_a^T = Q_a^T Q_a = I. \quad (A3)$$

**Proof.** (A2), (32) and (35) \implies

$$\begin{aligned}
(Q_a Q_a^T)_{hl} \text{Tr}(L_h) &= Q_{l_h} Q_{j_h}^T L_h \text{Tr}(L_h) \\
&= Q_{l_h} Q_{j_h} \text{Tr}(\sigma_{a,l,j}) = \text{Tr}((Q_{l_h} Q_{j_h}) (Q_{j_h}^T \sigma_{a,j})) \\
&= \text{Tr}(U_{l_h} U_{l_h}^T L_h U_{j_h} U_{j_h}^T) = \text{Tr}(\sigma_{a,l,j}) \\
&= L_{l_h} \text{Tr}(L_h) \implies (Q_a Q_a^T)_{hl} = L_{l_h} \implies (A3). \quad \blacksquare
\end{aligned}$$

A.2. Expression for matrix $D_a^T$

**Proposition A2.** In terms of operators $[L_{na}]$ the matrix $D_a^T$ is written as

$$D_a^T \sigma_{ja} = \sum_n L_{na} \sigma_{na} L_{na}^+. \quad (A4)$$

**Proof.** (A2) and (45) \implies

$$\begin{aligned}
(D_a D_a^T)_{hl} \text{Tr}(L_h) &= D_{l_h} D_{j_l} L_h \text{Tr}(L_h) \\
&= D_{l_h} D_{j_l} \text{Tr}(\sigma_{a,l,j}) = D_{l_h} \text{Tr}(\sigma_{a,l,j} D_{j_l} \sigma_{a,j}) \\
&= D_{l_h} \text{Tr}(\sigma_{a} \sum_n L_{na}^+ \sigma_{na} L_{na}) \\
&= D_{l_h} \sum_n \text{Tr}(L_{na} \sigma_{na} L_{na}^+) \\
&= D_{l_h} \text{Tr}(X_{j_l} \sigma_{a,j} \sigma_{a,j}) = D_{l_h} X_{j_l} \text{Tr}(\sigma_{ja} \sigma_{ja}) \\
&= (D_a X_a)_{hl} \text{Tr}(L_h) \implies (A4). \quad \blacksquare
\end{aligned}$$

A.3. Commutator of matrices $D_a$ and $D_a^T$

**Proposition A3.** Matrices $D_a$ and $D_a^T$ commute:

$$[D_a, D_a^T] = 0. \quad (A5)$$

**Proof.** (A2), (A4), (42) and (45) \implies

$$\begin{aligned}
(D_a^T D_a)_{hl} \text{Tr}(L_h) &= D_{l_h}^T D_{j_l} L_h \text{Tr}(L_h) \\
&= D_{l_h}^T D_{j_l} \text{Tr}(\sigma_{a,l,j}) = D_{l_h}^T \text{Tr}(\sigma_{a,l,j} D_{j_l} \sigma_{a,j}) \\
&= D_{l_h}^T \sum_n \text{Tr}(L_{ma} \sigma_{ma} L_{ma}^+ \sigma_{ma} L_{ma}) \\
&= D_{l_h}^T \sum_n \text{Tr}(L_{ma}^+ \sigma_{ma} L_{ma} \sigma_{ma} L_{ma}^+) \\
&= D_{l_h}^T \text{Tr}(D_{l_h}^T \sigma_{ja} D_{j_l} \sigma_{ja}) = D_{l_h} D_{j_l} \text{Tr}(\sigma_{ja} \sigma_{ja}) \\
&= (D_a D_a^T)_{hl} \text{Tr}(L_h) \implies (A5). \quad \blacksquare
\end{aligned}$$

A.4. The contractive property of matrix $D_a$

**Proposition A4.** In case of POVMs matrix $D_a$ is contractive matrix:

$$D_a^T D_a \leq I \iff \forall b \in \mathbb{R}^3 \quad b^T D_a^T D_a b \leq b^T b. \quad (A6)$$

**Proof.** (A2), (A4), (42) and (45) \implies

$$\begin{aligned}
\left\{ \begin{array}{ll}
b^T D_a^T D_a b \text{Tr}(L_h) &= b_a D_{l_h} D_{j_l} b_i L_h \text{Tr}(L_h) = b_a D_{l_h} D_{j_l} b_j \text{Tr}(\sigma_{a,l,j}) \\
&= b_a b_i \text{Tr}[(D_{l_h}^T \sigma_{a,l,j})(D_{j_l} \sigma_{a,j})] \\
&= b_a b_i \sum_m \text{Tr}(L_{ma}^+ \sigma_{ma} L_{ma} \sigma_{ma} L_{ma}^+) \\
&= b_a b_i \sum_m \text{Tr}(L_{ma}^+ \sigma_{ma} L_{ma} \sigma_{ma} L_{ma}^+) + \text{Tr}(L_{ma}^+ \sigma_{ma} L_{ma} \sigma_{ma} L_{ma}^+) \\
&= b_a b_i \sum_m \text{Tr}[(L_{ma}^+ \sigma_{ma} L_{ma}) (L_{ma}^+ \sigma_{ma} L_{ma})] + \text{Tr}(L_{ma}^+ \sigma_{ma} L_{ma} b_i b_i L_{ma}^+) \\
&= \frac{1}{2} \sum_m \text{Tr}[L_{ma}^+ \sigma_{ma} L_{ma} b_i b_i L_{ma}^+], \\
A := b_a \sigma_{ma} L_{ma}^+ L_{ma}, \quad B := L_{ma}^+ L_{ma} \sigma_{ma} b_i, \\
0 \leq \text{Tr}[(A - B)(A - B)^T] \implies \text{Tr}(AB^T) + \text{Tr}(BA^T) \leq \text{Tr}(AA^T) + \text{Tr}(BB^T)
\end{array} \right\} \quad \Rightarrow
\end{aligned}$$

$$\begin{aligned}
\left\{ \begin{array}{ll}
b^T D_a^T D_a b \text{Tr}(L_h) &\leq \frac{1}{2} \sum_m \text{Tr}[(b_a \sigma_{ma} L_{ma}^+ L_{ma}) (b_i \sigma_{ma} L_{ma}^+ L_{ma})] + \text{Tr}(L_{ma}^+ \sigma_{ma} b_i b_i L_{ma}^+) \\
&= \frac{1}{2} \sum_m [b_a b_i \text{Tr}(\sigma_{ma} L_{ma}^+ L_{ma} \sigma_{ma} L_{ma}) + b_i b_a \text{Tr}(L_{ma}^+ \sigma_{ma} L_{ma} \sigma_{ma} L_{ma})] \\
&= \frac{1}{2} \sum_m [b_a b_i \text{Tr}(\sigma_{ma} L_{ma}^+ L_{ma} \sigma_{ma} L_{ma}) + b_i b_a \text{Tr}(L_{ma}^+ \sigma_{ma} L_{ma} \sigma_{ma} L_{ma})] \\
&= \frac{1}{2} [b_a b_i \text{Tr}(\sigma_{ma} L_{ma}^+ L_{ma} \sigma_{ma} L_{ma}) + b_i b_a \text{Tr}(L_{ma}^+ \sigma_{ma} L_{ma} \sigma_{ma} L_{ma})] \\
&= \frac{1}{2} [b_a b_i \text{Tr}(\sigma_{ma}) + b_i b_a \text{Tr}(\sigma_{ma} L_{ma}^+ L_{ma} \sigma_{ma} L_{ma})] \\
&= \frac{1}{2} [b_a b_i \text{Tr}(L_{ma}) + b_i b_a \text{Tr}(L_{ma})] = \frac{1}{2} [b_a b_i \text{Tr}(L_{ma}) + b_i b_a \text{Tr}(L_{ma})] \implies (A6). \quad \blacksquare
\end{array} \right\}
\end{aligned}$$
Appendix B. Construction of wave function for electron–electron scattering problem

In this appendix, we will derive the expressions for coefficients in wave function (51), which we used in the section 4 considering the problem of Coulomb electron–electron scattering [10].

Let us construct the wave function of the system of two electrons scattered on each other. Taking into account the Pauli exclusion principle, we write it in the antisymmetric form

\[ |\psi\rangle = N (\psi_1 |x\rangle + \psi_2 |x\rangle), \]  

(B1)

where \(N\) is normalization factor, \(\psi_1\) and \(\psi_2\) are symmetric and antisymmetric wave functions (depend on coordinates of electrons), \(x\) and \(\bar{x}\) are symmetric and antisymmetric spin wave functions. We write the functions \(\psi_1\) and \(\psi_2\) in the form (52), leaving in the scattering amplitudes [16] only parts, which include dependence on the scattering angle \(\theta\) (53).

The functions \(x\) and \(\bar{x}\) we will find as follows. Before scattering, the spin of the first electron is oriented in the \(z\)-direction (the direction ‘up’) and that of the second one is rotated by a polar angle \(|\Omega|\) relative to the \(z\)-direction and by an azimuth angle \(\varphi\), relative to the \(x\)-axis. Then, before the symmetrization, the spin wave function is written as

\[ |\chi_{12}\rangle = |+\rangle U_{12} |+\rangle, \]

(B2)

\[ U_{12} = \cos(|\Omega|/2) - i \sin(|\Omega|/2) \sigma_{12}, \]

(B3)

\[ \sigma_{12} = -\sin(\varphi) \sigma_2 + \cos(\varphi) \sigma_1, \]

(B4)

where \(U_{12}\) is the rotation matrix of the second electron spin. (B4), (7), (8) and (2), \(\Rightarrow\)

\[ \sigma_{12} |+\rangle_2 = ie^{i\varphi} |-\rangle_2. \]

(B5)

(B2), (B3) and (B5) \(\Rightarrow\)

\[ |\chi_{12}\rangle = |+\rangle U_{12} |+\rangle, \]

\[ U_{12} = \cos(|\Omega|/2) + i \sin(|\Omega|/2) \sigma_{12}, \]

(B7)

\[ |\chi_{12}\rangle = |\chi_\Omega\rangle + |\chi_\bar{\Omega}\rangle, \]

(B8)

\[ |\chi_\Omega\rangle = |\chi_0\rangle \langle \chi_0| \chi_{12}\rangle, \]

\[ |\chi_\bar{\Omega}\rangle = \sum_{n=1}^{3} |\chi_n\rangle \langle \chi_n| \chi_{12}\rangle. \]

(B9)

The only antisymmetric state of the Bell states is singlet:

\[ |\chi_\Omega\rangle = \frac{1}{2} e^{i\varphi} \sin(|\Omega|/2) (|+\rangle_1 |-\rangle_2 - |-\rangle_1 |+\rangle_2). \]

(B10)

(B7), (B6) and (B10) \(\Rightarrow\)

\[ |\chi_\bar{\Omega}\rangle = \frac{1}{2} e^{i\varphi} \sin(|\Omega|/2) (|+\rangle_1 |+\rangle_2 + |-\rangle_1 |-\rangle_2 - |+\rangle_1 |+\rangle_2). \]

(B11)

(1), (B1), (B10) and (B11) \(\Rightarrow\) (51).

Appendix C. Eigenvalues of operator \(\sigma_i \sigma_j\)

In this appendix, we will find eigenvalues of the operator \(\sigma_i \sigma_j\) which were used in the section 5 at the description of procedure of experiment for measurement of spin correlation tensor (4).

Proposition C1. The operator \(\sigma_i \sigma_j\) has twice degenerate eigenvalues which are equal to \(\pm 1\).

Proof. (7) \(\Rightarrow\)

\[ \sigma_{i} \sigma_{j} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]

\[ \sigma_{i} \sigma_{j} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

\[ \sigma_{i} \sigma_{j} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \]

\[ \sigma_{i} \sigma_{j} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \]

\[ \sigma_{i} \sigma_{j} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Let us solve a problem on eigenvalues for the obtained \(4 \times 4\) matrix:

\[ \left[ \begin{array}{cc} \sigma_{i} \sigma_{j}^{1} & \sigma_{i} \sigma_{j}^{2} \\ \sigma_{i} \sigma_{j}^{12} & \sigma_{i} \sigma_{j}^{12} \end{array} \right] \left[ \begin{array}{c} u \\ v \end{array} \right] = \lambda \left[ \begin{array}{c} u \\ v \end{array} \right]. \]

(C1)

For the solution of the equation on eigenvalues, we use properties of determinant:

\[ \det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \det(D - CA^{-1}B), \]

\[ \det(cA) = c^n \det(A) \quad (A \in \mathbb{C}^{n \times n}). \]

(C2)

(C1), (8) and (C2) \(\Rightarrow\)

\[ j = x \Rightarrow 0 = \begin{pmatrix} -\lambda & \sigma_i & -\lambda \\ \sigma_i & -\lambda \end{pmatrix} \]

\[ = \det(-\lambda) \det(-\lambda + \sigma_i \lambda^2) \Rightarrow \ldots \]

\[ j = y \Rightarrow 0 = \begin{pmatrix} -\lambda & -i\sigma_i \\ i\sigma_i & -\lambda \end{pmatrix} \]

\[ = \det(-\lambda) \det(-\lambda + i\sigma_i \lambda^2) \Rightarrow \ldots \]

\[ j = z \Rightarrow 0 = \begin{pmatrix} -\lambda & \sigma_i & O \\ \sigma_i & -\lambda \end{pmatrix} \]

\[ = \det(-\lambda) \det(-\sigma_i - \lambda) \Rightarrow \ldots \]

\[ \lambda_{1,2} = \pm 1, \quad \lambda_{3,4} = -1. \]

Here \(O\) is zero matrix, on the repeating index with underlining, there is no summing up. Thus, the operator \(\sigma_i \sigma_j\) has twice degenerate eigenvalues which are equal to \(\pm 1\).
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