The Chiral Soliton of the Nambu–Jona-Lasinio Model with Vector and Axial Vector Mesons †

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Abstract

The self-consistent chiral soliton of the Nambu–Jona-Lasinio model including the ω, ρ and a_1 (axial-) vector meson fields besides the chiral angle is investigated. The resulting energy spectrum of the one particle Dirac Hamiltonian is strongly distorted leading to a polarized Dirac sea which carries the complete baryon number. This supports Witten’s conjecture that baryons can be described as topological solitons. The exploration of the isoscalar mean squared radius of the nucleon exhibits that the repulsive character of the isoscalar vector field ω as well as the attractive features of the (axial-) vector mesons ρ and a_1 are reproduced in the Nambu–Jona-Lasinio model. The axial charge of the nucleon g_A comes out far too small. This can be understood as an artifact of the proper time regularization prescription.

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1. Introduction

In the last decade a description of baryons as chiral solitons proved to be quite successful. The soliton picture of baryons is based on large $N_C$ QCD considerations, $N_c$ being the number of colors. In the limit $N_c \to \infty$, QCD is equivalent to an effective theory of weakly interacting mesons [1]. Later Witten conjectured that in this effective meson theory baryons emerge as soliton solutions [2]. Furthermore, in the low-energy limit this meson theory is dominated by the pseudoscalar mesons, the would-be Goldstone bosons of spontaneously broken chiral symmetry described in form of a non-linear $\sigma$-model. In order to implement the chiral anomaly this non-linear $\sigma$-model has to be supplemented by the Wess-Zumino action. Introducing external gauge fields allows to extract the corresponding Noether currents. Especially, the baryon current arising from the Wess-Zumino term proves to be identical to the topological current thus supporting Skyrme’s original work [3].

Incorporating vector and axial vector mesons in the non-linear $\sigma$-model the Skyrme model arises in the limit of infinitely heavy vector and axialvector mesons. In addition, the inclusion of vector and axial vector mesons with their physical masses cures several deficiencies of the Skyrme model, as e.g. incomplete description of electromagnetic properties due to missing vector dominance [4] or wrong “high energy behavior” of $\pi-N$ phase shifts due to the higher order stabilization terms[5]. This demonstrates the important role of vector and axialvector mesons in the soliton description of baryons.

Although we have good confidence that the non-linear $\sigma$-model extended by vector mesons represents the low-energy approximation to the effective meson theory of QCD the actual effective theory is not explicitly known. Therefore Witten’s conjecture cannot be proven within QCD without using further approximations. On the other hand, the low-energy form of the effective meson theory is almost entirely determined by chiral symmetry. Therefore the soliton picture of baryons should not depend on the details of chiral QCD dynamics. This suggests to investigate whether Witten’s conjecture is fulfilled in simpler models for the quark flavor dynamics of QCD. For this purpose the Nambu–Jona-Lasinio (NJL) model is well suited. First of all, like QCD, it obeys chiral symmetry and, furthermore, it can be motivated as low-energy approximation to QCD[6]. Additionally, it can be bosonized by functional integral methods. The resulting effective meson theory is in satisfactory agreement with the low-energy meson data. In fact, its gradient expansion yields in leading order the linear $\sigma$-model and the Wess-Zumino action.

The bosonized NJL model shows that the topological current arises in leading order gradient expansion from the vacuum part of the baryon current. The vacuum is defined with all negative fermion states occupied and the positive energy states being empty. Therefore the topological current can describe a non-zero baryon charge only if the vacuum is charged requiring that the valence quarks are bound into the Dirac sea. Only then the chiral field can carry a non-trivial baryon number. Thus the key assumption underlying the soliton picture of baryons is that the valence quarks have joined the Dirac sea. Within the NJL model we can test whether this assumption is fulfilled.

In recent years the soliton of the NJL model has been extensively studied. First calculations were restricted to the chiral field[7, 8, 9]. Extending the model to also include the $\rho$ meson provides only minor changes to the chiral soliton[10]. In both calculations the energy eigenvalue corresponding to the valence quarks is positive. The physical picture changes drastically if the chiral partner of the $\rho$ meson, the $a_1$ axialvector meson, is added. Then the valence quarks become strongly bound and join the Dirac sea, i.e. the Skyrmion picture of baryons results[11]. However, in these calculations the isoscalar vector meson
ω was left out for technical reasons. The inclusion of the ω meson should even favor the Skyrmion picture since the ω introduces repulsion and hence increases the spatial size of the chiral field which in turn increases the binding of the quarks.

The inclusion of the ω meson leads to substantial technical complications because the ω field develops a non-zero time-like component in the chiral soliton. Due to the need for regularization the effective meson theory can be properly defined only via the continuation to Euclidean space. Thereby time-components of vector fields are continued to imaginary values, too. As a consequence the Euclidean action becomes complex and the evaluation of the NJL action for a soliton field leads to a non-Hermitean eigenvalue problem, see ref.[12] for a proper treatment. In this calculation only the ω meson and the chiral field have been included in a self-consistent soliton calculation. In the present paper we present the self-consistent soliton calculation in the NJL model with all low-lying two flavor vector and axial vector meson fields included.

2. Bosonization of the NJL Model

The starting point for the following considerations is the chirally invariant NJL model [13, 14]:

\[
\mathcal{L} = \bar{q}(i\slashed{D} - m^0)q + 2g_1 \sum_{i=0}^{N_f-1} \left( \left(\bar{q} \frac{\lambda^i}{2} q\right)^2 + \left(\bar{q} \frac{\lambda^i}{2} i\gamma_5 q\right)^2 \right) - 2g_2 \sum_{i=0}^{N_f-1} \left( \left(\bar{q} \frac{\lambda^i}{2} \gamma_\mu q\right)^2 + \left(\bar{q} \frac{\lambda^i}{2} \gamma_5 \gamma_\mu q\right)^2 \right),
\]

(2.1)

wherein \(q\) denotes the quark spinors and \(m^0\) the current quark mass matrix. Here we will work in the isospin limit, i.e. \(m^0_u = m^0_d = m^0\). The matrices \(\lambda^i/2\) are the generators of the flavor group \((\lambda^0 = \sqrt{2/N_f} 1)\). Furthermore we will restrict ourselves to two flavors \((N_f = 2)\) implying \(\lambda^i = \tau^i, i = 0, \ldots, 3\). The coupling constants \(g_1\) and \(g_2\) will be determined from mesonic properties, cf. e.g. refs. [14, 13, 16, 17] for the calculation of meson properties in the NJL model.

Applying standard functional integral bosonization techniques the model (2.1) can be rewritten in terms of composite meson fields [14]

\[
\mathcal{A} = \mathcal{A}_F + \mathcal{A}_m,
\]

\[
\mathcal{A}_F = \text{Tr} \log (i\slashed{D}) = \text{Tr} \log (i\slashed{D} + V + \gamma_5 A - (P_R \Sigma + P_L \Sigma^\dagger)),
\]

\[
\mathcal{A}_m = \int d^4x \left( -\frac{1}{4g_1} \text{tr}(\Sigma^\dagger \Sigma - m^0(\Sigma + \Sigma^\dagger)) + (m^0)^2 \right) - \frac{1}{4g_2} \text{tr}(V_\mu V^\mu + A_\mu A^\mu) \right).
\]

(2.2)

Here \(P_{R,L} = (1 \pm \gamma_5)/2\) are the projectors on right– and left–handed quark fields, respectively. \(V_\mu = \sum_{a=0}^3 V_{a,\tau^a}/2\) and \(A_\mu = \sum_{a=0}^3 A_{\mu,\tau^a}/2\) denote the vector and axial vector meson fields. \(V_\mu^\dagger\) and \(A_\mu^\dagger\) are real in Minkowski space. The complex field \(\Sigma\) describes the scalar and pseudoscalar meson fields, \(S_{ij} = S_{a,\tau^a}/2\) and \(P_{ij} = P_{a,\tau^a}/2\):

\[
\Sigma = S + iP = \xi_L^\dagger \Phi \xi_R,
\]

(2.3)
wherein we already introduced the angular decomposition of the complex field $\Sigma$ into a Hermitean field $\Phi$ and unitary fields $\xi_L$ and $\xi_R$ which are related to the chiral field by $U = \xi_L \xi_R$. The latter is conveniently expressed in terms of a chiral angle $\Theta$

$$U(x) = \exp (i\Theta(x)).$$

(2.4)

The quark determinant $A_F$ diverges and therefore requires regularization. As in a study of the mesonic sector of the NJL model[14] as well as in previous studies of the soliton sector[7, 8, 9, 10] we will use Schwinger’s proper time regularization[18] which introduces an $O(4)$-invariant cut-off $\Lambda$ after continuation to Euclidean space. For the regularization procedure it is necessary to consider the real and imaginary part of $A_F$ separately

$$A_F = A_R + A_I$$

$$A_R = \frac{1}{2} \text{Tr} \log (\bar{\psi}_E \psi_E)$$

$$A_I = \frac{1}{2} \text{Tr} \log ((\bar{\psi}_E)^{-1} \psi_E).$$

(2.5)

The real part $A_R$ diverges like $\log p^2$ for large momenta $p$ whereas the imaginary part $A_I$ does not contain any divergencies, i.e. it is finite without regularization. However, we believe that it has to be regularized also in order to have a consistent model. After all, the occurrence of the cutoff is a very crude way of mimicking the asymptotic freedom of QCD.

For the real part the proper time regularization consists in replacing the logarithm by a parameter integral

$$A_R \rightarrow -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \text{Tr} \exp \left(-s \bar{\psi}_E \psi_E \right),$$

(2.6)

which for $\Lambda \rightarrow \infty$ reproduces the logarithm up to an irrelevant constant. Since the operator $\bar{\psi}_E \psi_E$ is Hermitean and positive definite this integral is well defined. For the imaginary part the regularization procedure is equivalent,

$$A_I \rightarrow -\frac{1}{2} \int_{1/\Lambda^2}^{\infty} \frac{ds}{s} \text{Tr} \exp \left(-s (\bar{\psi}_E)^{-1} \psi_E \right),$$

(2.7)

however, in this case one has to be careful concerning the convergence of the integral, see section 3.

In order to determine the coupling constants $g_1$ and $g_2$ from the meson sector it is sufficient to only inspect $A_R$. Varying the regularized effective action with respect to the scalar and pseudoscalar fields yields the Dyson–Schwinger or gap equations

$$\langle S_{ij} \rangle = \delta_{ij} M$$

$$M = m^0 - 2g_1 \langle \bar{q}q \rangle$$

$$\langle \bar{q}q \rangle = -M^2 \frac{N_c}{4\pi^2} \Gamma(-1, M^2/\Lambda^2).$$

(2.8)

The quantity $M$ is the dynamically generated constituent quark mass and $\langle \bar{q}q \rangle$ the quark condensate. A non-vanishing quark condensate reflects spontaneous breaking of chiral symmetry.
Performing the derivative expansion of $A_R$ allows to read off the pion decay constant $f_\pi$ as the coefficient of the expression quadratic in the derivatives of the physical pion field $\pi = \sum_{a=1}^3 \pi^a \lambda^a$:

\[ A_R = \int d^4x \frac{f_\pi^2}{4} \text{tr} \partial_\mu \pi \partial^\mu \pi + \cdots. \]  

(2.9)

The inclusion of vector and axialvector mesons leads to pseudoscalar–axialvector meson mixing, especially $\pi - a_1$–mixing. This renormalizes the pion field and thus affects the pion decay constant which is then given by

\[ f_\pi^2 = \frac{6M^2}{g_V^2} \frac{1}{1 + 6M^2/m_\rho^2} \]  

(2.10)

where

\[ g_V = \left( \frac{1}{8\pi^2} \Gamma(0, \frac{M^2}{\Lambda^2}) \right)^{-1/2} \quad \text{and} \quad m_\rho^2 = \frac{g_V^2}{4g_2^2} \]  

(2.11)

are the universal vector coupling constant and the vector meson mass. As input quantities from experiment we use the pion decay constant $f_\pi = 93\text{MeV}$ and the $\rho$ meson mass $m_\rho = 770\text{MeV}$. For a given constituent quark mass $M$ we then determine the cut-off $\Lambda$ via eqn. (2.10) and subsequently $g_2$ via (2.11). It is important to note that the $\pi - a_1$–mixing increases $\Lambda$ significantly, e.g.

for $M = 350\text{MeV}$ we find $\Lambda = 1274\text{MeV}$ compared to $\Lambda = 649\text{MeV}$ when $\pi - a_1$–mixing is disregarded. Expanding $A_m$ up to second order in the pseudoscalar fields allows to express the current quark mass $m_0$ in terms of $g_1$ and the pion mass $m_\pi = 135\text{MeV}$: $m_0 = g_1 m_\pi^2 f_\pi^2/M$. Finally we employ the gap equation (2.8) to eliminate $g_1$ in terms of the constituent quark mass $M$ which from now on is considered as the only free parameter of the model.

### 3. The Energy Functional for Static Meson Fields

Next we will consider the energy functional of the static soliton in SU(2). After Wick rotation, i.e.

$x_0 = -ix_4$ and $V_0 = -iV_4$ the Euclidean Dirac operator corresponding to eqn. (2.2) is given by

\[ i\beta D_E = -\partial_\tau - h, \]

\[ h = \alpha \cdot p + iV_4 + i\gamma_5 A_4 + \alpha \cdot V + \gamma_5 \alpha \cdot A + \beta(P_R \Sigma + P_L \Sigma^\dagger) \]  

(3.1)

wherein $\tau$ denotes the Euclidean time. In Euclidean space $\tau$, $V_4$ and $A_4$ have to be considered as Hermitean quantities. This leads to a non–Hermitean Hamiltonian $h$ even for static configurations (i.e. $[\partial_\tau, h] = 0$) if non–vanishing time components of vector or axialvector meson fields are included.

We fix the scalar field at its vacuum value $\Phi = M\mathbf{1}$. This is a priori an unjustified approximation and indeed it has been shown that allowing $\phi$ to be space dependent leads to a collapse of the soliton in the case when only scalar and pseudoscalar fields are present. Employing, however, a well motivated generalization of the NJL model which includes an additional $\Phi^4$ potential in the mesonic action (2.2) yields a stable soliton with
the scalar field deviating only slightly from its vacuum value\cite{21,22}. Thus it is reasonable to keep the NJL model\cite{22,23} and restrict the scalar field to its vacuum value.

For the chiral field we impose the hedgehog \textit{ansatz}:

\[ U(x) = \exp \left(i \tau \cdot \hat{r} \Theta(r) \right). \]  

(3.2)

This configuration has vanishing ‘grand spin’ \[ G = l + \sigma/2 + \tau/2, \] \textit{i.e.} \[ [G, U] = 0. \] The only possible \textit{ansatz} for the isoscalar-vector field \( \omega \) with grand spin zero has vanishing spatial components\( (\omega_i = 0) \):

\[ V_{\mu}^0 = \omega_\mu = \omega(r) \delta_{\mu 4}. \]  

(3.3)

Parity invariance requires the isoscalar–axialvector meson field \( A_4 \) to vanish in the static limit. For the isovector- (axial) vector meson fields we use the spherically symmetric \textit{ansätze}

\[ \begin{align*}
V_a^4 & = 0, & V_i^a & = \epsilon^{akl} \hat{r}_k G(r), \\
A_a^4 & = 0, & A_i^a & = \hat{r}_i \hat{r}_a F(r) + \delta_{ia} H(r)
\end{align*} \]  

(3.4)

where the indices \( a, i \) and \( k \) run from 1 to 3. Note that \( V_i^a \) corresponds to the physical \( \rho \) meson while the physical axialvector meson \( a_1 \) is obtained from \( A_i^a \) and terms involving \( \partial_i (\hat{r}^a \Theta) \) after removing the pseudoscalar axialvector mixing. The Euclidean Dirac Hamiltonian now reads

\[ h = \alpha \cdot p + i \omega(r) + M \beta (\cos \Theta(r) + i \gamma_5 \tau \cdot \hat{r} \sin \Theta(r)) \]

\[ + \frac{1}{2} (\alpha \times \hat{r}) \cdot \tau G(r) + \frac{1}{2} (\sigma \cdot \tau) F(r) + \frac{1}{2} (\sigma \cdot \tau) H(r) \]  

(3.5)

which obviously is not Hermitean since \( \omega(r) \) is real giving rise to complex eigenvalues of \( h \). The matrix elements of the Hamiltonian \( (3.5) \) are evaluated in the quark spinor basis proposed in ref.\cite{23}. These spinors are characterized by their grand spin eigenvalue \( G \) and their parity transformation properties. Since \( (3.5) \) commutes with the grand spin operator and is parity invariant as well, the Hamiltonian \( (3.5) \) is diagonalized for each grand spin and parity eigenvalue separately. The momentum eigenvalues of the basis spinors are discretized by putting the system into a finite spherical box of radius \( D \) and demanding the upper components to vanish at the boundary for spinors with parity eigenvalue \( (1)_G \) and the lower components for spinors with parity eigenvalue \( (1)_{G-1} \). We list all matrix elements of operators appearing in \( (3.5) \) as well as the explicit form of the basis spinors in appendix A. The boundary conditions of ref.\cite{23} have the advantage that there are no spurious contributions to the equations of motion for the \( \omega \) field stemming from the fact that we only consider a finite basis in momentum space. The spurious contributions to the profile function \( G(r) \) of the \( \rho \) field appearing in the basis of ref.\cite{23} are well under control and may explicitly be eliminated. However, we will see later that there are finite size effects for the \( \omega \) meson profile originating from a large but finite \( D \).

For static configurations the eigenvalues of \( \partial_\tau \), \( i \Omega_n = i(2n + 1)\pi/T \), with \( n = 0, \pm 1, \pm 2, .. \) may be separated rendering the temporal part of the trace feasible.\footnote{The eigenfunctions of \( \partial_\tau \) assume anti-periodic boundary conditions in the Euclidean time interval \( T \). The \( \Omega_n \) are the analogues of the Matsubara frequencies with \( T \) figuring as inverse temperature.} Thus the eigenvalues \( \lambda_{n, \nu} \) of the operator \( \partial_\tau + h \) read:

\[ \lambda_{n, \nu} = -i \Omega_n + \epsilon_\nu = -i \Omega_n + \epsilon_\nu^R + i \epsilon_\nu^I. \]  

(3.6)
The fermion determinant is expressed in terms of the eigenvalues \( \lambda_{n,\nu} \):

\[
A_R = \frac{1}{2} \sum_{\nu,n} \log(\lambda_{n,\nu}\lambda_{n,\nu}^*) \quad \text{and} \quad A_I = \frac{1}{2} \sum_{\nu,n} \log\left(\frac{\lambda_{n,\nu}}{\lambda_{n,\nu}^*}\right). \tag{3.7}
\]

Using (3.6) the real part reads:

\[
A_R = \frac{1}{2} \sum_{\nu,n} \log((\Omega_n - \epsilon^I_\nu)^2 + (\epsilon^R_\nu)^2)
\]

\[
\rightarrow -\frac{1}{2} \sum_{\nu,n} \int_1^\infty \frac{d\tau}{\tau} \exp\left\{ -\frac{\tau}{\Lambda^2}((\Omega_n - \epsilon^I_\nu)^2 + (\epsilon^R_\nu)^2)\right\} \tag{3.8}
\]

according to the proper time regularization scheme (2.6). For large Euclidean time intervals \((T \to \infty)\) the temporal part of the trace may be performed

\[
A_R = -\frac{T}{2} \sum_{\nu} \int_{-\infty}^{\infty} \frac{dz}{2\pi} \int_1^\infty \frac{d\tau}{\tau} \exp\left\{ -\frac{\tau}{\Lambda^2}(z^2 + (\epsilon^R_\nu)^2)\right\} \tag{3.9}
\]

where we have shifted the integration variable \(z - \epsilon^I_\nu \rightarrow z\). For \(T \to \infty\) we may read off the Dirac sea contribution to the real part of the energy functional from \(A_R \rightarrow -TE^R_{\text{vac}}\):

\[
E^R_{\text{vac}} = \frac{N_C}{4\sqrt{\pi}} \sum_{\nu} |\epsilon^R_\nu| \Gamma\left( -\frac{1}{2}, (\epsilon^R_\nu/\Lambda)^2\right). \tag{3.10}
\]

For the imaginary part we obtain:

\[
A_I = \frac{1}{2} \left( \sum_{\nu} \sum_{n=-\infty}^{\infty} \log(\lambda_{\nu,n}) - \sum_{\nu} \sum_{n=-\infty}^{\infty} \log(\lambda_{\nu,n}^*) \right) = \frac{1}{2} \sum_{\nu} \sum_{n=-\infty}^{\infty} \log \frac{i\Omega_n - \epsilon^I_\nu}{i\Omega_n^* - \epsilon^I_\nu} \tag{3.11}
\]

where we have reversed the sign in the first sum over the integer variable \(n\). Next we express \(A_I\) in terms of a parameter integral:

\[
A_I = \frac{1}{2} \sum_{\nu} \sum_{n=-\infty}^{\infty} \int_{-1}^{1} d\lambda \frac{-i\epsilon^I_\nu}{i\Omega_n - \epsilon^R_\nu - i\lambda\epsilon^I_\nu}. \tag{3.12}
\]

In analogy to (3.10) we may carry out the temporal trace in the limit \(T \to \infty\):

\[
A_I = -\frac{i}{2} \sum_{\nu} \int_{-1}^{1} d\lambda T \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{\epsilon^I_\nu}{i(z - \lambda\epsilon^I_\nu)} \tag{3.13}
\]

Shifting the integration variable \(z - \lambda\epsilon^I_\nu \rightarrow z\) the integral over \(\lambda\) may be done

\[
A_I = -\frac{i}{2} \sum_{\nu} \epsilon^I_\nu \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{-2\epsilon^R_\nu}{z^2 + (\epsilon^R_\nu)^2}. \tag{3.14}
\]

\(A_I\) is regularized in proper time by expressing the integrand as a parameter integral:

\[
\frac{-1}{z^2 + (\epsilon^R_\nu)^2} \rightarrow \int_{1/\Lambda^2}^{\infty} d\tau \exp\{-\tau(z^2 + (\epsilon^R_\nu)^2)\} \tag{3.15}
\]
which obviously is finite for $\Lambda \to \infty$. Continuing the evaluation of $A_I$ in analogy to eqs. (3.8-3.10) we find for the contribution of the Dirac sea to the imaginary part of the Euclidean energy $E_{\text{vac}}^I$

$$E_{\text{vac}}^I = \frac{-N_C}{2} \sum \nu \epsilon^I_\nu \text{sign}(\epsilon_\nu^R) \begin{cases} 1, & A_I \text{ not regularized} \\ \mathcal{N}_\nu, & A_I \text{ regularized} \end{cases}$$

where

$$\mathcal{N}_\nu = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}, (\epsilon_\nu^R/\Lambda)^2\right)$$

are the vacuum “occupation numbers” in the proper time regularization scheme. The upper case, of course, corresponds to the limit $\Lambda \to \infty$. Obviously only the real part of the one-particle energy eigenvalue is relevant for the regularization of $A_I$. Eqn. (3.16) reveals that we have succeeded in finding a regularization scheme for $A_I$ that only involves quantities which are strictly positive definite. This is not evident from the definition of $A_I$ (2.5).

For soliton configurations with vanishing $\omega$ (i.e. $\epsilon_\nu = \epsilon^R_\nu$) there is no contribution from the imaginary part and eqn. (3.10) is the expression for the energy of the Dirac sea.

The total energy functional contains besides $E_{\text{vac}}^R$ and $E_{\text{vac}}^I$ also the valence quark energy

$$E_{\text{val}}^R = N_C \sum \nu \eta_\nu |\epsilon^R_\nu|, \quad E_{\text{val}}^I = N_C \sum \nu \eta_\nu \text{sign}(\epsilon^R_\nu) \epsilon_\nu^I$$

with $\eta_\mu = 0, 1$ being the occupation numbers of the valence quark and anti-quark states. Furthermore the meson energy is obtained by substituting the ansätze (3.2-3.4) into (2.2):

$$E_m = 4\pi \int dr r^2 \left( m_\pi^2 f_\pi^2 (1 - \cos \Theta(r)) + \frac{(m_\rho/\sqrt{2})^2 (G^2(r) + \frac{1}{2} F^2(r) + F(r) H(r) + \frac{3}{2} H^2(r) - 2\omega^2(r))}{G^2(r)} \right)$$

Note that we are working in the isospin limit which implies $m_\omega = m_\rho$. Continuing back to Minkowski space we find for the total energy functional:

$$E[\Theta, \omega, G, F, H] = E_{\text{val}}^R + E_{\text{val}}^I + E_{\text{vac}}^R + E_{\text{vac}}^I + E_m.$$  (3.20)

The equations of motion for the meson profiles are obtained by extremizing the static Minkowski energy (3.20). In a generic way we may write:

$$0 = \frac{\delta E}{\delta \phi} = \frac{\delta E_m}{\delta \phi} + \sum_{\kappa=R,I} \sum \mu \frac{\partial (E_{\text{val}}^R + E_{\text{val}}^I + E_{\text{vac}}^R + E_{\text{vac}}^I)}{\partial \epsilon^{\kappa}_\mu} \frac{\delta \epsilon^{\kappa}_\mu}{\delta \phi}$$

wherein $\phi$ denotes any of the meson profiles $\Theta, G, \omega, F$ or $H$. Since $h$ is not Hermitean (in Euclidean space) we have to distinguish between left and right eigenvectors of $h$. The corresponding eigenvalue equations read:

$$h |\Psi_\nu\rangle = \epsilon_\nu |\Psi_\nu\rangle \quad \langle \bar{\Psi}_\nu | h = \epsilon_\nu \langle \bar{\Psi}_\nu | \quad \text{i.e. } h^\dagger |\bar{\Psi}_\nu\rangle = \epsilon^{\dagger}_\nu |\bar{\Psi}_\nu\rangle.$$  (3.22)

The normalization condition is $\langle \bar{\Psi}_\mu | \Psi_\nu\rangle = \delta_{\mu\nu}$. In order to evaluate the derivatives $\delta \epsilon^{\kappa}_\mu /\delta \phi$ it is helpful to decompose the Hamiltonian operator (3.3) into Hermitean and anti-Hermitean parts

$$h = h_\Theta + i\omega$$  (3.23)
where \( h_\Theta \) includes all Hermitean terms of the Euclidean Dirac Hamiltonian \((3.5)\). Obviously both, \( h_\Theta \) and \( \omega \), are Hermitean implying \(|\tilde{\Psi}_\nu\rangle = |\Psi_\nu^*\rangle\). We may therefore extract the real and imaginary parts of the one particle energy eigenvalue

\[
\epsilon^R_\nu = \frac{1}{2}((\langle \Psi_\nu^*| h |\Psi_\nu\rangle + \langle \Psi_\nu| h^\dagger |\Psi_\nu^*\rangle)
= \langle \Psi_\nu^R| h_\Theta |\Psi_\nu^R\rangle - \langle \Psi_\nu^I| h_\Theta |\Psi_\nu^I\rangle - \langle \Psi_\nu^I| \omega |\Psi_\nu^R\rangle - \langle \Psi_\nu^R| \omega |\Psi_\nu^I\rangle,
\]

\[
\epsilon^I_\nu = \frac{1}{2}((\langle \Psi_\nu^*| h |\Psi_\nu\rangle - \langle \Psi_\nu| h^\dagger |\Psi_\nu^*\rangle)
= \langle \Psi_\nu^I| \omega |\Psi_\nu^I\rangle - \langle \Psi_\nu^I| \omega |\Psi_\nu^R\rangle + \langle \Psi_\nu^R| h_\Theta |\Psi_\nu^I\rangle + \langle \Psi_\nu^R| h_\Theta |\Psi_\nu^I\rangle)
\]

(3.24)

where we employed the decomposition \(|\Psi_\nu\rangle = |\Psi_\nu^R\rangle + i|\Psi_\nu^I\rangle\). Note also that \( \langle \Psi_\nu^*\rangle = \langle \Psi_\nu^R\rangle + i\langle \Psi_\nu^I\rangle\). We are now equipped with expressions for the real and imaginary parts of the energy eigenvalues \( \epsilon_\mu \) which are suitable to evaluate the derivatives with respect to the meson fields. For example we have:

\[
\frac{\delta \epsilon^I_\nu}{\delta \omega(r)} = r^2 \int \frac{d\Omega}{4\pi} \left( \langle r | \Psi_\nu^R \rangle \langle \Psi_\nu^R | r \rangle - \langle r | \Psi_\nu^I \rangle \langle \Psi_\nu^I | r \rangle \right).
\]

(3.25)

The expressions for the functional dependence of the energy eigenvalues may now be substituted into the equations of motion \((3.21)\). The individual equations for \( \Theta, \omega, G, F \) and \( H \) are displayed in appendix B where we also discuss the spurious contributions to the equation of motion for \( G \).

At this point it is indispensable to explain the differences between our approach and the treatment in ref. [24]. We would like to stress that in order to regularize the fermion determinant a continuation to Euclidean space has to be performed. In Euclidean space the regularized fermion determinant is well defined in terms of the complex energy eigenvalues \( \epsilon_\mu \) of the Euclidean Dirac Hamiltonian \((3.5)\). This final expression for the fermion determinant may then be continued back to Minkowski space yielding the energy functional \((3.20)\). Due to the non-linear structure of the regulator functions in \((3.10)\) and \((3.16)\) the operations “regularization” and “continuation” do not commute. However, this is just the procedure used by the authors of ref. [24]: The energy eigenvalues had firstly been continued back to Minkowski space and then the energy functional was regularized. In appendix C we present a detailed comparison of the two different treatments with the help of a toy model.

### 4. Numerical Results

In this section we present the numerical results characterizing the soliton solution. First we will explore the soliton containing all meson fields as discussed in the preceding sections. These results will then be compared to cases where some of the vector meson fields are switched off in order to examine the effects of various vector meson fields on the soliton.

The meson profiles of the soliton solution are determined by iteration. I.e. we start off with test profiles for \( \Theta, \omega, G, F \) and \( H \) to evaluate the energy eigenvalues \( \epsilon_\mu \) of the static Hamiltonian \((3.1)\). Subsequently the meson profiles undergo modifications according to the equations of motion \((3.21)\). In turn these modified profiles serve as input for the static Hamiltonian. This process is repeated until a convergent solution is obtained, i.e. a self-consistent meson field configuration is constructed.
Table 4.1. The soliton energy $E$ as well as its Dirac sea and mesonic contributions $E_{\text{vac}}$ and $E_{\text{m}}$ for different values of the constituent quark mass $M$. For the pion mass $m_\pi$ the physical value (135MeV) and the chiral limit ($m_\pi = 0$) are considered. Also shown is the energy of the ‘dived’ level ($\epsilon_{\text{val}}$).

| $M$ (MeV) | 300 | 350 | 400 |
|-----------|-----|-----|-----|
| $m_\pi$ (MeV) | 0  | 135 | 0  | 135 |
| $E$ (MeV) | 1117 | 1155 | 1024 | 1061 | 952 | 980 |
| $E_{\text{vac}}^R$ (MeV) | 690 | 690 | 569 | 572 | 504 | 503 |
| $E_{\text{vac}}^I$ (MeV) | 36 | 36 | 31 | 30 | 24 | 24 |
| $E_{\text{m}}$ (MeV) | 391 | 429 | 424 | 459 | 423 | 453 |
| $\epsilon_{\text{val}}^R/M$ | -0.11 | -0.11 | -0.44 | -0.42 | -0.60 | -0.61 |
| $\epsilon_{\text{val}}^I/M$ | 0.10 | 0.10 | 0.09 | 0.09 | 0.07 | 0.08 |

Table 4.2. Same as table (4.1) for $m_\pi = 0$, however the imaginary part is not regularized.

| $M$ (MeV) | 300 | 350 | 400 |
|-----------|-----|-----|-----|
| $E$ (MeV) | 1227 | 1210 | 1175 |
| $E_{\text{vac}}^R$ (MeV) | 669 | 563 | 460 |
| $E_{\text{vac}}^I$ (MeV) | 55 | 257 | 298 |
| $E_{\text{m}}$ (MeV) | 335 | 389 | 417 |
| $\epsilon_{\text{val}}^R/M$ | 0.02 | -0.28 | -0.47 |
| $\epsilon_{\text{val}}^I/M$ | 0.13 | 0.16 | 0.16 |

We have found stable self-consistent solutions for constituent quark masses in the range $300\text{MeV} \leq M \leq 400\text{MeV}$. In all these calculations the parameters have been kept at their physical values (cf. section 2). The existence of these stable solutions represents the main difference as compared to the results of ref. [24]. Those authors only find solutions for $m_\omega \geq 870\text{MeV}$.

In table (4.1) we display the energy $E$ of the self-consistent soliton solution. This table also contains the various contributors to $E$ as they appear in eqn.(3.20) for several values of the constituent mass $M$. Furthermore we compare to the results obtained in the chiral limit ($m_\pi = 0$).

The most striking result observed from table (4.1) is the fact that the real part of the energy eigenvalue associated with the valence quark state is negative! I.e. the valence quark has joined the Dirac sea and thus the baryon number is completely carried by the polarization of the Dirac sea. Thus the soliton of the NJL model supports the picture of baryons as topological solitons of meson fields. As a reminder we would like to mention that for $\epsilon_{\text{val}}^R < 0$ the valence quarks’ contribution to the energy is already contained in $E_{\text{vac}}^R$ and $E_{\text{vac}}^I$. Thus it must not explicitly be added in (3.18). We also observe from table (4.1) that if the imaginary part of the fermion determinant is regularized it only contributes a minor part to the total energy. However, we see from table (4.2) that this is no longer the case when the regularization of the imaginary part is abandoned. In that case the total energy $E$ is almost independent of the constituent quark mass $M$ while $E$ decreases by about 20% in case the imaginary part is regularized when $M$ is changed from 300MeV to 400MeV. For a non-regularized imaginary part the valence quarks appear to be slightly

*We do not exclude the existence of stable solutions for an even larger range of $M$.\)
Table 4.3. The soliton energy for various treatments of the NJL soliton. The meson fields in the first line denote the allowed meson profiles. All numbers are evaluated for a constituent quark mass $M = 350\text{MeV}$ and $m_\pi = 0$. If present, the imaginary part is regularized.

|                | $\pi$  | $\pi, \rho$ | $\pi, \omega$ | $\pi, \rho, a_1$ | $\pi, \omega, \rho, a_1$ |
|----------------|-------|-------------|---------------|------------------|-------------------------|
| $E$ (MeV)      | 1214  | 957         | 1310          | 1010             | 1024                    |
| $E_{vac}^R$ (MeV) | 561   | 655         | 669           | 615              | 569                     |
| $E_{vac}^I$ (MeV) | 0     | 0           | -56           | 0                | 31                      |
| $E_m$ (MeV)    | 0     | 155         | -42           | 395              | 424                     |
| $\epsilon_{\text{val}}^R/M$ | 0.62  | 0.14        | 0.50          | -0.13            | -0.44                   |
| $\epsilon_{\text{val}}^I/M$ | 0     | 0           | 0.24          | 0                | 0.09                    |

above the Dirac sea for $M=300\text{MeV}$. For all other cases the valence quarks’ energy remains negative, although less strongly bound in the Dirac sea than for a regularized imaginary part where the $\omega$ profile is less pronounced. Table (4.1) also reveals that the inclusion of a finite pion mass does not alter the results for the energy drastically. The total energy is increased by about 40MeV due to the additional term in $E_m$ (3.19).

Next we wish to investigate the role of the different meson fields for the NJL soliton. In order to do so we compare in table (4.3) the results obtained for the soliton energy in cases with different (axial-)vector mesons incorporated. Obviously, the inclusion of the $\omega$ field always increases the soliton energy while the $\rho$ and $a_1$ lower the energy. Though the latter result is anticipated the former is somewhat surprising since the meson profiles are obtained by extremizing the total energy. However, this increase is understood by taking into account that the $\omega$ field is proportional to the baryon number density which in turn is constrained by unit baryon number. We also observe from table (4.3) that the inclusion of any of the (axial-) vector mesons lowers the energy eigenvalue of the valence quark. The $a_1$ obviously effects the valence quark to join the Dirac sea. This important result was already obtained previously\[11\] and the main conclusion to be drawn out of our present calculation is the fact that this result is not spoiled by the $\omega$ meson which was not included in ref.\[11\]; on the contrary the $\omega$ meson gives an even stronger binding of the valence quark. This may be understood by noting that the $\omega$ is repulsive yielding a large spatial extension of the soliton. This, in due, causes the valence quark energy to drop.

The repulsive character of the $\omega$ field may also be observed directly from radial behavior of the chiral angle, $\Theta(r)$. In fig. (4.1) we display $\Theta(r)$ for various treatments of the NJL soliton. $\Theta(r)$ develops the largest tail in the case when the $\omega$ meson field is the only one added to the chiral field\[12\]. The axialvector field provides a significant attraction resulting in a slope for the chiral field which is larger than in the case when $\Theta(r)$ is the only field being present. The inclusion of the $\omega$ meson on top of the isovector mesons $\rho$ and $a_1$ alters the chiral angle only slightly. A similar behavior can be observed for the $\rho$ meson profile $G(r)$ (cf. fig. (4.2)) as well as the axialvector meson profiles $F(r)$ and $H(r)$ (cf. fig. (4.3)). On the other hand the inclusion of the isovector mesons on top of the $\pi - \omega$ system significantly reduces the strength of the $\omega$ meson profile as may be seen in fig. (4.4).

Of course, in the present form the soliton carries neither good spin nor isospin quantum numbers, hence, it does not describe physical baryon states. Following the work of Adkins,\[\dagger\]

\[\dagger\]We do, however, not consider a system containing the $\rho$ and $\omega$ mesons besides the pseudoscalar fields.
Fig. 4.1. The chiral angle $\Theta(r)$ as a function of the radial distance $r$. The solid line corresponds to the case when all vector mesons are included; the dashed line to the $\pi - \omega$ system; the short dashed to the $\pi - \rho$ system; the long dashed to the $\pi - \rho - a_1$ system and the dashed dotted to the case when the chiral angle is the only field present.

Fig. 4.2. The vector meson profile $G(r)$ as a function of the radial distance $r$. Solid line: all vector meson fields are present; short dashed denotes the $\pi - \rho$ system and the long dashed to the $\pi - \rho - a_1$ system.

Nappi and Witten[25] the soliton is projected onto baryon quantum numbers by a cranking procedure which introduces time dependent collective coordinates for the zero modes $R(t)$. Explicitly one imposes the ansatz:

$$\xi(x, t) = R(t) \xi(x) R^\dagger(t) \quad R(t) \in SU(2). \quad (4.1)$$

and similarly for the isovector fields $\rho$ and $a_1$. Although this approach has been successfully applied to the SU(2) NJL model of pseudoscalar fields only[26] and even been generalized to SU(3)[27] this is not the whole story in the presence of vector mesons. The collective rotation $R(t)$ excites additional vector meson components as e.g. the space components of $\omega$ and the time components of $\rho$ which are absent in the static case[4]. Since the investigation of these excitation is beyond the scope of this paper we will not discuss the resulting baryon spectrum, especially the nucleon-$\Delta$ mass splitting. Nevertheless there are a few baryon properties which dependent only on the static fields and may thus be evaluated at the present stage; the most prominent of which is the axial charge of the nucleon, $g_A$.

In the NJL model the current field identities hold[14]. Therefore the axial current $J_{5\mu}$ is directly proportional to the axial field:

$$J_{5\mu} = -\frac{1}{4g_2} A_{\mu}^i \quad (4.2)$$

wherein the superscript denotes the isospin component. Noting that $g_A$ is obtained as the matrix element of $J_{5\mu}$ at zero momentum transfer we immediately obtain

$$g_A = -\frac{2\pi}{g_2} \int dr r^2 \left[ H(r) + \frac{1}{3} F(r) \right] \langle D_{33} \rangle_p. \quad (4.3)$$

$\langle D_{33} \rangle_p = 1/3$ refers to the matrix element of $\text{tr}(R\tau_3 R^\dagger \tau_3)$ between proton states. It is important to mention that eqn. (4.3) represents an exact result which is not subject to

Fig. 4.3. The axialvector meson profile $F(r)$ and $H(r)$ as functions of the radial distance $r$. $H(r)$ is non-vanishing at the origin. The case when all vector mesons are present is denote by the solid and dashed lines. The $\pi - \rho - a_1$ system is represented by the long dashed and dashed dotted lines.
Fig. 4.4. The vector meson profile $\omega(r)$ as a function of the radial distance $r$. The solid line denotes the case when all vector mesons are present while the dashed line represents the $\pi - \omega$ system.

Table 4.4. The axial charge $g_A$ of the nucleon in the various treatments of the soliton in the NJL model. The last line corresponds to the case when the imaginary part is not regularized.

| $M$ (MeV) | 300 | 350 | 400 |
|-----------|-----|-----|-----|
| $\pi$     | —   | 0.78| 0.73|
| $\pi, \omega$ | —   | 0.98| 1.03|
| $\pi, \rho, a_1$ | 0.31| 0.27| 0.13|
| $\pi, \omega, \rho, a_1$ | 0.34| 0.25| 0.23|
| $\pi, \omega, \rho, a_1$ | 0.54| 0.39| 0.28|

renormalization due to $\pi - a_1$ mixing. Making use of the equation of motion for the axialvector profiles $H(r)$ and $F(r)$ (see appendix B) we may reexpress $g_A$ as a mode sum over quark spinors:

$$g_A = -\frac{N_C}{3} \sum_{\mu} \left\{ \langle \psi^R_\mu | \sigma_3 | \psi^I_\mu \rangle - \langle \psi^I_\mu | \sigma_3 | \psi^R_\mu \rangle + \langle \psi^I_\mu | \sigma_3 | \psi^I_\mu \rangle + \langle \psi^R_\mu | \sigma_3 | \psi^R_\mu \rangle \right\} \eta_\mu$$

The regulator functions $f_{R,I}$ are listed in appendix B (cf. eqns. (B.3,B.4)). The mode sum (4.4) has the advantage that it may be employed in models without axialvector mesons.

Unfortunately our numerical results which are listed in table (4.4) are somewhat discouraging since they are well below the numerical value $g_A = 1.25$. This is especially pronounced in case the valence quark energy is negative. In order to understand the origin of this shortcoming let us consider the case when the isoscalar field $\omega$ is absent. Then eqn. (4.4) reduces to

$$g_A = -\frac{N_C}{3} \eta_{val} \langle val | \sigma_3 | val \rangle + \frac{N_C}{6} \sum_{\mu} \text{sign}(\epsilon_\mu) \text{erfc}(|\epsilon_\mu|/\Lambda) \langle \mu | \sigma_3 | \mu \rangle$$

which is identical to the expression derived in ref.[28]. Eqn. (4.5) reveals that once the valence quark energy has become negative its contribution to $g_A$ is strongly suppressed by the proper time regularization. This suggests that a regularization prescription which does not affect low-lying states as strongly as the proper time scheme would be highly desirable in order to describe $g_A$ correctly. This consideration is supported by a simple modification of eqn. (4.5). In the sum over all eigenstates we replace the complementary error function by a sharp cut-off function. Then the contributions from the low-lying states are not affected by the regularization procedure. Choosing, e.g., a constituent mass of 300MeV the prediction for $g_A$ increases drastically to 1.04 in the $\pi - \rho - a_1$ system. Of course, this exploration does not represent a consistent calculation but merely demonstrates that $g_A$ strongly depends on the regularization description. If we consider the $\pi - \omega$ system the strong repulsion of the $\omega$ field transfers to an increased prediction.
Table 4.5. The isoscalar radius $\langle r^2_{I=0}\rangle^{1/2}$ of the nucleon in various treatments of the NJL model. The radii are given in fm.

| $\bar{M}$ (MeV) | 300 | 350 | 400 |
|-----------------|-----|-----|-----|
| $\pi$           | —   | 0.89| 0.76|
| $\pi,\rho,a_1$  | 0.55| 0.55| 0.56|
| $\pi,\omega$    | —   | 2.06| 1.77|
| $\pi,\omega,\rho,a_1$ | 1.39| 1.40| 1.29|

for $g_A$. This is also obvious from fig.(4.1) since in the absense of axialvector fields $g_A$ may also be obtained from the size of the “pion tail” [25].

We may also investigate the isoscalar radius of the nucleon without explicitly performing the collective quantization. Again due to the current field identity the isoscalar charge density is proportional to the $\omega$ profile yielding the isoscalar radius:

$$
\langle r^2_{I=0}\rangle = N^{-1} \frac{4m^2}{NCG^2V} \int d^3r \, r^2 \, \omega(r). \tag{4.6}
$$

The normalization factor

$$
N = \frac{4m^2}{NCG^2V} \int d^3r \, \omega(r) \tag{4.7}
$$

is one if the imaginary part of the determinant remains unregularized.

As already indicated in the previous section our numerical results contain some spurious finite size contributions. These finite size effects are manifested in a contribution to the $\omega$ meson profile which is proportional to $D^{-3}$. Especially we find that our numerical solution for $\omega(r)$ accquires a finite value at the edge of the box: $\omega(r = D) \sim D^{-3}$. From eqn. (4.6) it may be observed that then $\langle r^2_{I=0}\rangle$ would diverge as $D \to \infty$. We eliminate this spurious contribution by enforcing $\omega(r)$ to vanish at the boundary. This is accomplished by including a Lagrange multiplier $\lambda$ in the energy functional (3.20):

$$
E[\Theta,\omega,G,F,H] \longrightarrow E[\Theta,\omega,G,F,H] + \lambda \int d^3r \, f_\epsilon(r) \omega^2(r) \tag{4.8}
$$

wherein $f_\epsilon(r)$ is a positive radial function which vanishes everywhere except within a small vicinity $\epsilon$ of $r = D$. By iteration of the modified equations of motions $\lambda$ is adjusted such that the additional term vanishes. Of course, there is some arbitrariness in choosing $f_\epsilon(r)$. We demand the change of the total energy with $\lambda$ included to deviate from the case $\lambda = 0$ by less than 1MeV. In the region of physical interest ($r \leq 2\text{fm}$) $\omega(r)$ remains almost unaltered and we obtain the desired effect that $\langle r^2_{I=0}\rangle$ stays finite as $D \to \infty$ and assumes a constant value. In case the $\omega$ field is not present we use the corresponding unregularized mode sum (B.6) as input in eqn. (4.6). The resulting data are displayed in table (4.5). The repulsive effect of the isoscalar vector field $\omega$ is obvious and is not completely compensated by the attraction provided by the isovector fields $\rho$ and $a_1$. I.e. the prediction still overestimates the experimental value $\langle r^2_{I=0}\rangle^{1/2} \approx 0.8\text{fm}.$

5. Conclusions
We have found stable soliton solutions in the NJL model with all low-lying vector and axialvector meson fields included. The isoscalar vector meson $\omega$ is excited only if the imaginary part of the Euclidean action is taken into account. It is essential to note that the necessary regularization can only be performed in Euclidean space. This implies that the energy functional can be defined properly only if one continues to Euclidean space, regularizes and then continues back to Minkowski space. This is quite distinct from the procedure of ref. [24] where the one-particle energy eigenvalues had firstly been continued to Minkowski space and the resulting energy functional was “regularized”. As a matter of fact, their results are quite different from ours. Once they incorporate the $\omega$ meson they do not even find stable soliton solutions for the physical value of the $\omega$ mass.

For reasonable values of the constituent quark mass we find that the energy eigenvalue corresponding to the valence quark state is negative. Therefore the baryon number is carried by the asymmetry of the vacuum. Thus the NJL model supports Witten’s conjecture, i.e. the Skyrmion picture of the baryon. In this respect we remark that the $\omega$ meson is even decreasing the valence quark level despite its repulsive nature. This can be understood by noting that the $\omega$ meson broadens the other meson profile functions thereby increasing the binding energy for the valence quark state. However, if the imaginary part is not regularized this effect is inverted.

The calculation of the isoscalar mean square radius of the nucleon is plagued by a numerical deficiency stemming from the finite spatial extension where we iterate the equations of motion. We have introduced a Lagrange multiplier to keep the result for the isoscalar radius finite. The prediction for the isoscalar mean square radius of the nucleon overestimates the experimental value by about 40-50%. Nevertheless we have seen the repulsion provided by the isoscalar vector field as well as the attraction due to the isovector (axial-) vector fields.

The axial coupling $g_A$ comes out much too small. We have demonstrated that this behavior is due to a deficiency of the proper time regularization which attaches “weight factors” smaller than unity already to levels with quite a small energy. We therefore conclude that a regularization which does not effect low energy levels is highly desirable.

In order to calculate other observables one has to project the soliton solution on good spin and flavor states as was done for the NJL soliton with pseudoscalars only in refs. [26, 28, 30]. However, including the vector and axialvector mesons this is a quite involved task since components which vanish on the classical level but get excited by the collective rotation (4.1) have to be taken into account. On the other hand, we have seen that the NJL soliton is close to a Skyrmion. Thus it makes sense to use the self-consistent meson profiles in an action obtained by the gradient expansion of the fermion determinant. A first step towards this direction is to calculate static properties in this approximation and compare those with the exact results.

Finally we would like to remark that problems arising from the regularization can be overcome only if one takes a renormalizable quark model as starting point. E.g. one way were to use a bilocal effective quark interaction leading to bilocal meson fields [31]. Whether such models allow for soliton solutions and whether these solutions support Witten’s conjecture is, of course, an interesting question which deserves further studies.

Appendix A: Matrix elements of the static Hamiltonian

For our numerical calculations we have used as basis the orthonormal eigenstates of
the free Dirac Hamiltonian

\[ h_0 = \alpha p + \beta M. \] (A.1)

Since the static Hamiltonian commutes with the grand spin operator \( G = l + \sigma/2 + \tau/2 \), our basis spinors are characterized by the grand spin quantum number \( G \) in addition to the total angular momentum \( j = G \pm 1/2 \) and the orbital angular momentum \( l = G, G \pm 1 \). The latter also determines the parity eigenvalue of the spinor, \( \pi = (-1)^l \). The grand spin eigenstates:

\[ |GMjl\rangle = \sum_{m,j,m_l} C_{j,m_j,1/2M-m_j}^{jM} C_{m_l,1/2m_l-m_l}^{m_l} |l m_l\rangle | \frac{1}{2} m_j - m_l\rangle | \frac{1}{2} M - m_j\rangle_I \] (A.2)

are two component spinors in both spin(\( S \))- and isospin(\( I \))-space. The momentum, and therefore the energy, is discretized by putting the system in a spherical box of radius \( D \) and requiring appropriate boundary conditions. For the calculations reported in this paper we used the boundary conditions of ref. [23*]. The discrete momenta \( q_n^G \) are then given by requiring

\[ j_G(q_n^G D) = 0. \] (A.3)

where \( j_G \) are the spherical Bessel functions. The energy eigenvalues corresponding to the momenta \( q_n \) in (A.3) are \( E_n^G = \pm \sqrt{(q_n^G)^2 + m^2} \). Now the Dirac spinors are easily constructed:

(i) \( j = G + \frac{1}{2}; \ l = G \):

\[ |G + + n\rangle = N_{G+1n}^G \left( \begin{array}{c} iw_n^{G+} j_G(q_n^G r) \\ j_G(q_n^G r) \end{array} \right) |GM(G+\frac{1}{2})G\rangle |G M (G+\frac{1}{2}) (G+1)\rangle \] (A.4)

(ii) \( j = G - \frac{1}{2}; \ l = G \):

\[ |G - + n\rangle = N_{G+1n}^G \left( \begin{array}{c} iw_n^{G+} j_G(q_n^G r) \\ -j_G(q_n^G r) \end{array} \right) |GM(G-\frac{1}{2})G\rangle |G M (G-\frac{1}{2}) (G-1)\rangle \] (A.5)

(iii) \( j = G + \frac{1}{2}; \ l = G + 1 \):

\[ |G + - n\rangle = N_{G+1n}^G \left( \begin{array}{c} iw_n^{G+} j_{G+1}(q_n^G r) \\ -j_{G+1}(q_n^G r) \end{array} \right) |GM(G+\frac{1}{2})G\rangle |G M (G+\frac{1}{2}) (G+1)\rangle \] (A.6)

(iv) \( j = G - \frac{1}{2}; \ l = G - 1 \):

\[ |G - - n\rangle = N_{G+1n}^G \left( \begin{array}{c} iw_n^{G+} j_{G-1}(q_n^G r) \\ j_{G-1}(q_n^G r) \end{array} \right) |GM(G-\frac{1}{2})G\rangle |G M (G-\frac{1}{2}) (G-1)\rangle \] (A.7)

*For a discussion of a different type of boundary condition see ref. [27].
where we used the following abbreviations

\[ N_{Gn}^{-L} = D^{-\frac{D}{2}} |j_G(q_n^L D)|^{-1} \]
\[ w_n^{L+} = \sqrt{1 + \frac{m}{E_n^L}} \]
\[ w_n^{L-} = \text{sign} (E_n^L) \sqrt{1 - \frac{m}{E_n^L}}. \] (A.8)

We start with listing a few helpful relations involving the two-component grand spin eigenstates (A.2) only:

\[ \langle GMj_1|\tau \cdot \hat{\mathbf{r}}|GMj_2l_2 \rangle = \begin{cases} 
\frac{1}{2G+1} & \text{for } j_1 = j_2 = G + \frac{1}{2}, |l_2 - l_1| = 1 \\
-\frac{1}{2G+1} & \text{for } j_1 = j_2 = G - \frac{1}{2}, |l_2 - l_1| = 1 \\
-2\sqrt{G(G+1)} & \text{for } j_1 = G \pm \frac{1}{2}, j_2 = G \mp \frac{1}{2}, |l_2 - l_1| = 1 \\
0 & \text{otherwise}
\end{cases} \] (A.9)

\[ \langle GMj_1| (\sigma \times \hat{\mathbf{r}}) \tau |GMj_2l_2 \rangle = \begin{cases} 
\frac{1}{2G+1} & \text{for } j_1 = j_2 = G + \frac{1}{2}, l_1 = l_2 \\
\frac{1}{2G+1} & \text{for } j_1 = j_2 = G - \frac{1}{2}, l_1 = l_2 \\
2\sqrt{G(G+1)} & \text{for } j_1 = G \pm \frac{1}{2}, j_2 = G \mp \frac{1}{2}, l_1 = l_2 = G \\
2\sqrt{G(G+1)} & \text{for } j_1 = G \pm \frac{1}{2}, j_2 = G \mp \frac{1}{2}, |l_2 - l_1| = 2 \\
0 & \text{otherwise}
\end{cases} \] (A.10)

and

\[ \langle GMj_1| (\sigma \cdot \tau) |GMj_2l_2 \rangle = \begin{cases} 
1 & \text{for } j_1 = j_2 = G \pm \frac{1}{2}, l_1 = l_2 = G \pm 1 \\
-\frac{2G-1}{2G+1} & \text{for } j_1 = j_2 = G - \frac{1}{2}, l_1 = l_2 = G \\
-\frac{2G-3}{2G+1} & \text{for } j_1 = j_2 = G + \frac{1}{2}, l_1 = l_2 = G \\
4\sqrt{G(G+1)} & \text{for } j_1 = G \pm \frac{1}{2}, j_2 = G \mp \frac{1}{2}, l_1 = l_2 = G \\
0 & \text{otherwise}
\end{cases} \] (A.11)

and

\[ (\sigma \cdot \tau + 1)^2 = 4 \] (A.13)
\[ [(\sigma \cdot \hat{\mathbf{r}})(\tau \cdot \hat{\mathbf{r}})]^2 = 1 \] (A.14)
\[ [\sigma \cdot \tau - (\sigma \cdot \hat{\mathbf{r}})(\tau \cdot \hat{\mathbf{r}})]^2 = 2 - 2(\sigma \cdot \hat{\mathbf{r}})(\tau \cdot \hat{\mathbf{r}}). \] (A.15)
In the following we shall display all matrix elements which are needed for the static Hamiltonian.

We introduce some additional abbreviations:

\[ a'^{\nu \nu}_{nm} = u^G_{nm} w^G_{m} j_l(q^G_{n} r) j_\nu(q^G_{m} r) \]
\[ b'^{\nu \nu}_{nm} = u^G_{nm} w^G_{m} j_l(q^G_{n} r) j_\nu(q^G_{m} r) \]
\[ c'^{\nu \nu}_{nm} = u^G_{nm} w^G_{m} j_l(q^G_{n} r) j_\nu(q^G_{m} r) \]
\[ N_{nm} = \text{det} \left( j_{G+1}(q^G_{n} D) \right) \]

(A.16)

1. \( \hat{\mathcal{O}} = \beta f(r) \)

\[ \langle G^+ n | \hat{\mathcal{O}} | G^+ m \rangle = N_{nm} \int_0^1 dxx f(Dx) \left( a^{G^+G}_{nm} - b^{G^+G}_{nm} \right) \]
\[ \langle G^- n | \hat{\mathcal{O}} | G^- m \rangle = N_{nm} \int_0^1 dxx f(Dx) \left( a^{G^-G}_{nm} - b^{G^-G}_{nm} \right) \]
\[ \langle G^+ n | \hat{\mathcal{O}} | G^- m \rangle = N_{nm} \int_0^1 dxx f(Dx) \left( a^{G^+G}_{nm} + b^{G^+G}_{nm} \right) \]
\[ \langle G^- n | \hat{\mathcal{O}} | G^+ m \rangle = N_{nm} \int_0^1 dxx f(Dx) \left( a^{G^-G}_{nm} + b^{G^-G}_{nm} \right) \]

2. \( \hat{\mathcal{O}} = i \mathbf{\tau} \cdot \hat{\mathbf{r}} \beta \gamma_5 f(r) \)

\[ \langle G^+ n | \hat{\mathcal{O}} | G^+ m \rangle = N_{nm} \int_0^1 dxx f(Dx) \frac{1}{2G+1} \left( c^{G^+G}_{nm} + c^{G+G}_{nm} \right) \]
\[ \langle G^- n | \hat{\mathcal{O}} | G^- m \rangle = N_{nm} \int_0^1 dxx f(Dx) \frac{1}{2G+1} \left( c^{G^-G}_{nm} + c^{G-G}_{nm} \right) \]
\[ \langle G^+ n | \hat{\mathcal{O}} | G^- m \rangle = N_{nm} \int_0^1 dxx f(Dx) \frac{2\sqrt{G(G+1)}}{2G+1} \left( c^{G^+G}_{nm} - c^{G+G}_{nm} \right) \]
\[ \langle G^- n | \hat{\mathcal{O}} | G^+ m \rangle = N_{nm} \int_0^1 dxx f(Dx) \frac{2\sqrt{G(G+1)}}{2G+1} \left( c^{G^-G}_{nm} - c^{G-G}_{nm} \right) \]
\[ \langle G^+ n | \hat{\mathcal{O}} | G^- m \rangle = N_{nm} \int_0^1 dxx f(Dx) \frac{2\sqrt{G(G+1)}}{2G+1} \left( c^{G^+G}_{nm} + c^{G+G}_{nm} \right) \]
\[ \langle G^- n | \hat{\mathcal{O}} | G^+ m \rangle = N_{nm} \int_0^1 dxx f(Dx) \frac{2\sqrt{G(G+1)}}{2G+1} \left( c^{G^-G}_{nm} + c^{G-G}_{nm} \right) \]

3. \( \hat{\mathcal{O}} = (\mathbf{\alpha} \times \hat{\mathbf{r}}) \cdot \mathbf{\tau} f(r) \)

\( ^1\)We employ the standard Dirac representation for the \( \gamma \) matrices.
\[
\langle G + n | \hat{O} | G + m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{2(G+1)}{2G+1} \left( c_{nm}^{G-1G} + c_{nm}^{G+1G} \right)
\]
\[
\langle G - n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{-2G}{2G+1} \left( c_{nm}^{G-1G} + c_{nm}^{G+1G} \right)
\]
\[
\langle G + n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{4\sqrt{G(G+1)}}{2G+1} \left( c_{nm}^{G-1G} - c_{nm}^{G+1G} \right)
\]
\[
\langle G - n | \hat{O} | G + m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{4\sqrt{G(G+1)}}{2G+1} \left( c_{nm}^{G-1G} - c_{nm}^{G+1G} \right)
\]

4. \( \hat{O} = (\sigma \cdot \hat{r})(\tau \cdot \hat{r})f(r) \)

\[
\langle G + n | \hat{O} | G + m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{-1}{2G+1} \left( a_{nm}^{G+1G} + b_{nm}^{G+1G} \right)
\]
\[
\langle G - n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{1}{2G+1} \left( a_{nm}^{G+1G} + b_{nm}^{G+1G} \right)
\]
\[
\langle G + n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{2\sqrt{G(G+1)}}{2G+1} \left( a_{nm}^{G+1G} - b_{nm}^{G+1G} \right)
\]
\[
\langle G - n | \hat{O} | G + m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{2\sqrt{G(G+1)}}{2G+1} \left( a_{nm}^{G+1G} - b_{nm}^{G+1G} \right)
\]
\[
\langle G + n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{1}{2G+1} \left( a_{nm}^{G-1G} + b_{nm}^{G-1G} \right)
\]
\[
\langle G - n | \hat{O} | G + m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \frac{1}{2G+1} \left( a_{nm}^{G-1G} + b_{nm}^{G-1G} \right)
\]

5. \( \hat{O} = (\sigma \cdot \tau)f(r) \)

\[
\langle G + n | \hat{O} | G + m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \left( -\frac{2G+3}{2G+1} a_{nm}^{G+1G} + b_{nm}^{G+1G} \right)
\]
\[
\langle G - n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \left( -\frac{2G-1}{2G+1} a_{nm}^{G-1G} + b_{nm}^{G-1G} \right)
\]
\[
\langle G + n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \left( a_{nm}^{G+1G} + 4\sqrt{G(G+1)} a_{nm}^{G+1G} \right)
\]
\[
\langle G - n | \hat{O} | G + m \rangle = N_{nm} \int_0^1 dx^2 f(Dx) \left( a_{nm}^{G+1G} + 4\sqrt{G(G+1)} a_{nm}^{G+1G} \right)
\]
\[ \langle G - n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx x^2 f(Dx) \left( a_{nm}^{G+1G+1} - \frac{2G+3}{2G+1} b_{nm}^{GG} \right) \]
\[ \langle G + - n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx x^2 f(Dx) \left( a_{nm}^{G-1G-1} - \frac{2G-1}{2G+1} b_{nm}^{GG} \right) \]
\[ \langle G + - n | \hat{O} | G - m \rangle = N_{nm} \int_0^1 dx x^2 f(Dx) \left( -\frac{4\sqrt{G(G+1)}}{2G+1} b_{nm}^{GG} \right) \]
\[ \langle G - n | \hat{O} | G + m \rangle = N_{nm} \int_0^1 dx x^2 f(Dx) \left( -\frac{4\sqrt{G(G+1)}}{2G+1} b_{nm}^{GG} \right) \]

Appendix B: Equations of motion for the meson profiles

In this appendix the equations of motion (3.21) are displayed in detail. The functional derivative of the one particle energies with respect to the fields yields expressions involving the eigenfunctions (3.22). On the other hand, the derivative of the total energy with respect to the one particle energies results in regularization functions for the vacuum part and the occupation number for the valence part. As the eigenfunctions (3.22) occur only in certain combinations in the equations of motion it is convenient to define quark density matrices.

The quark scalar density matrix \( \rho(x, y) \) may be decomposed into contributions due to valence and sea quarks:

\[
\rho(x, y) = \rho_R^{val} + \rho_I^{val} + \rho_R^{vac} + \rho_I^{vac},
\]
\[
\rho_R^{val}(x, y) = \sum_\nu \left\{ \psi_R^\nu(x) \bar{\psi}_R^\nu(y) - \psi_I^\nu(x) \bar{\psi}_I^\nu(y) \right\} \eta_\nu,
\]
\[
\rho_I^{val}(x, y) = \sum_\nu \left\{ \psi_I^\nu(x) \bar{\psi}_I^\nu(y) + \psi_R^\nu(x) \bar{\psi}_R^\nu(y) \right\} \eta_\nu,
\]
\[
\rho_R^{vac}(x, y) = \sum_\nu \left\{ \psi_R^\nu(x) \bar{\psi}_R^\nu(y) - \psi_I^\nu(x) \bar{\psi}_I^\nu(y) \right\} f_R(\epsilon_\nu/\Lambda),
\]
\[
\rho_I^{vac}(x, y) = \sum_\nu \left\{ \psi_I^\nu(x) \bar{\psi}_I^\nu(y) + \psi_R^\nu(x) \bar{\psi}_R^\nu(y) \right\} f_I(\epsilon_\nu/\Lambda). \tag{B.1}
\]

Similarly, the quark number (or baryon number) density matrix \( b(x, y) \) is written as

\[
b(x, y) = b_R^{val} + b_I^{val} + b_R^{vac} + b_I^{vac},
\]
\[
b_R^{val}(x, y) = \sum_\nu \left\{ \psi_R^\nu(x) \psi_R^\nu(y) - \psi_I^\nu(x) \psi_I^\nu(y) \right\} \eta_\nu,
\]
\[
b_I^{val}(x, y) = -\sum_\nu \left\{ \psi_I^\nu(x) \psi_I^\nu(y) + \psi_R^\nu(x) \psi_R^\nu(y) \right\} \eta_\nu,
\]
\[
b_R^{vac}(x, y) = \sum_\nu \left\{ \psi_R^\nu(x) \psi_R^\nu(y) - \psi_I^\nu(x) \psi_I^\nu(y) \right\} f_R(\epsilon_\nu/\Lambda),
\]
\[
b_I^{vac}(x, y) = -\sum_\nu \left\{ \psi_I^\nu(x) \psi_I^\nu(y) + \psi_R^\nu(x) \psi_R^\nu(y) \right\} f_I(\epsilon_\nu/\Lambda). \tag{B.2}
\]

Note that the baryon density matrix \( b(x, y) \) differs from the scalar density matrix \( \rho(x, y) \) not only by the additional factor \( \gamma_0 \) but also by an exchange of the regulator functions \( f_R \) and \( f_I \) which are given as the derivatives of the energy functional with respect to the energy eigenvalue \( \epsilon_\nu \):

\[
f_R(\epsilon_\nu/\Lambda) = \begin{cases} 
-\frac{1}{2} \text{sign}(\epsilon_\nu) \mathcal{N}_\nu, & A_I \text{ not regularized}, \\
-\frac{1}{2} \text{sign}(\epsilon_\nu) \mathcal{N}_\nu + \frac{1}{\sqrt{\pi}} (\epsilon_\nu/\Lambda) \exp(- (\epsilon_\nu/\Lambda)^2), & A_I \text{ regularized} 
\end{cases} \tag{B.3}
\]
\[ f_I(\epsilon^R / \Lambda) = \begin{cases} 
\frac{1}{2} \text{sign}(\epsilon^R) & \mathcal{A}_I \text{ not regularized} \\
\mathcal{N}_\nu & \mathcal{A}_I \text{ regularized} 
\end{cases} \]  

with the vacuum occupation number \( \mathcal{N}_\nu \) being defined in (3.17). With these definitions the equations of motion (3.21) for the meson profiles \( \Theta, \omega, G, F \) and \( H \) read

\[
\sin \Theta(r) = \frac{M}{m_\pi^2 f_\pi^2} N_c \text{tr} \int \frac{d\Omega}{4\pi} \left( \sin \Theta(r) - i\gamma_5 \tau \cdot \hat{r} \cos \Theta(r) \right) \rho(x, \bar{x}), 
\]

\[
\omega(r) = \frac{g_3^2}{4m_\rho^2} N_c \text{tr} \int \frac{d\Omega}{4\pi} b(x, \bar{x}),
\]

\[
G(r) = -\frac{g_3^2}{4m_\rho^2} N_c \text{tr} \int \frac{d\Omega}{4\pi} \left( (\gamma \times \hat{r}) \cdot \tau \right) \rho(x, \bar{x}),
\]

\[
F(r) = -\frac{g_3^2}{4m_\rho^2} N_c \text{tr} \int \frac{d\Omega}{4\pi} \beta \left( 3(\sigma \cdot \hat{r})(\tau \cdot \hat{r}) - (\sigma \cdot \tau) \right) \rho(x, \bar{x}),
\]

\[
H(r) = \frac{g_3^2}{4m_\rho^2} N_c \text{tr} \int \frac{d\Omega}{4\pi} \beta \left( (\sigma \cdot \hat{r})(\tau \cdot \hat{r}) - (\sigma \cdot \tau) \right) \rho(x, \bar{x}).
\]

The traces are over Dirac and isospin indices only. Note that only the \( \omega \) meson profile is given by a trace over the baryon number density. All other meson profiles have as “source” the scalar quark density. Without regularization the difference between both densities would have only been a factor \( \gamma_0 = \beta \). However, the regularization makes the relation between the two densities, and therefore the relation between the \( \omega \) and the other meson profiles, highly non-linear.

For the basis of ref. [23] the RHS of eqn. (B.7) does not vanish for the vacuum configuration yielding some spurious contributions to \( G(r) \). These may, however, be eliminated explicitly by subtracting the corresponding value of the RHS of eqn. (B.7) at each step of the iteration procedure. These spurious contributions are absent when the boundary conditions proposed in ref. [27] are employed at the expense of a non-vanishing RHS of eqn. (B.6) for the vacuum configuration. This then would give similar spurious contributions to \( \omega(r) \). Thus we conclude that for the problem at hand both sets of boundary conditions are equally well suited.

### Appendix C: Continuation to Euclidean space in a toy model

In this appendix we discuss the differences of our approach [12] and the treatment in [24] in the continuation prescription to Euclidean space with the help of a 2 \( \times \) 2 dimensional toy model.

The non-Hermitian structure of our Hamiltonian (3.5) is already represented in the simple model:

\[
h = p \begin{pmatrix} 
0 & -i \\
i & 0
\end{pmatrix} + i \omega_4 \begin{pmatrix} 
1 & 1 \\
1 & 1
\end{pmatrix}.
\]

\[\text{This equality is not the case when } e.g. \text{ matrix elements of } SU(3) \text{ generators are considered.}\]

\[\text{We thank the anonymous referee of our earlier paper [2] for bringing this toy model to our attention.}\]
Comparing the structure of $h$ with a Dirac Hamiltonian we would like to identify the variable $p$ with the “momentum”. Note that $\omega_4$ is real. The eigenvalues are

$$\epsilon_{1,2} = i\omega_4 \pm \sqrt{p^2 - \omega_4^2}. \quad \text{(C.2)}$$

Then the energy functional is given by (ignoring the subtraction of the vacuum part, $\omega_4 = 0$):

$$E^R_{\text{vac}} = \frac{N_C}{2\sqrt{\pi}} \sqrt{p^2 - \omega_4^2} \Gamma(-\frac{1}{2}, (p^2 - \omega_4^2)/\Lambda^2) \quad E^I_{\text{vac}} = 0 \quad \text{\text{(C.3)}}$$

for the case that $|p| > |\omega_4|$. Otherwise we have from eqns. (3.9) and (3.16)‡:

$$E^R_{\text{vac}} = \frac{N_C}{4\sqrt{\pi}} \Lambda \quad \text{and} \quad E^I_{\text{vac}} = 0 \quad \text{\text{(C.4)}}$$

\textit{i.e.} a constant energy functional and thus no contribution to equation of motion. Contrary to this, using the prescription of ref.[24] yields a dynamical contribution of this mode:

$$E^R_{\text{vac}} = \frac{N_C}{2\sqrt{\pi}} \sqrt{p^2 + \omega_0^2} \Gamma(-\frac{1}{2}, (p^2 + \omega_0^2)/\Lambda^2) \quad E^I_{\text{vac}} = 0 \quad \text{\text{(C.5)}}$$

For large “momenta” $p$ eqn. (C.5) obviously is the continuation of eqn. (C.3) and in both prescriptions the contribution of states with large $p$ is suppressed via the regularization. However, we see that for $|p| < |\omega_4|$ the two approaches are no longer identical. Especially, we see that in our prescription (C.4) the states with small $p$ are not effected by the cut-off $\Lambda$ in contrast to ref.[24]. This, of course, provides a physical motivation to use our approach. It is also obvious from (C.3) that for $|p| < |\omega_4|$ there is still a contribution to the equation of motion in the formulation of ref.[24] in contradiction to the formulation of the determinant in Euclidean space.

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‡Consistency of eqns. (3.14) and (3.16) enforces to define sign(0) = 0.
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