CAMPANATO ESTIMATES FOR THE GENERALIZED STOKES SYSTEM

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ABSTRACT. We study interior regularity of solutions of a generalized stationary Stokes problem in the plane. The main, elliptic part of the problem is given in the form \( \text{div}(A(Du)) \), where \( D \) is the symmetric part of the gradient. The model case is \( A(Du) = (\kappa + |Du|)^{p-2}Du \). We show optimal BMO and Campanato estimates for \( A(Du) \). Some corollaries for the generalized stationary Navier-Stokes system and for its evolutionary variant are also mentioned.

1. INTRODUCTION

Let \( \Omega \subset \mathbb{R}^2 \) be a domain. In this article we study properties of the local weak solution \( u \in W^{1,\phi}(\Omega) \) and \( \pi \in L^{\phi^*}(\Omega) \) of the generalized Stokes problem

\[
-\text{div}A(Du) + \nabla \pi = -\text{div}G \quad \text{in } \Omega, \\
\text{div}u = 0 \quad \text{in } \Omega
\]

for given \( G : \Omega \to \mathbb{R}^{2\times2}_{\text{sym}} \). Here \( u \) stands for the velocity of a fluid and \( \pi \) for its pressure. We do not need boundary condition, since our results are local. The model case is \( A(Q) = \nu(\kappa + |Q|)^{p-2}Q \) corresponding to power law fluids with \( \nu > 0, \kappa \geq 0, 1 < p < \infty \) and \( Q \) symmetric. But we also allow more general growth conditions, which include for example Carreau type fluids \( A(Q) = \mu_\infty Q + \nu(\kappa + |Q|)^{p-2}Q \) with \( \mu_\infty \geq 0 \) (see Subsection 2.2). In this article we are interested in the qualitative properties of \( A(Du) \) and \( \pi \) in terms of \( G \). The divergence form of the right-hand side is only for convenience of the formulation of the result, since every \( f \) can be written as \( -\text{div}G \) with \( G \) symmetric, see Remark 3.12.

System (1.1) originates in fluid mechanics. It is a simplified stationary variant of the system

\[
u_{t} - \text{div}A(Du) + |\nabla u|u + \nabla \pi = -\text{div}G, \quad \text{div}u = 0,
\]

where \( u \) stands for a velocity of a fluid and \( \pi \) for its pressure. The extra stress tensor \( A \) determines properties of the fluid and must be given by a constitutive law. If \( A(Q) = 2\nu Q \) with constant viscosity \( \nu > 0 \), then (1.2) is the famous Navier-Stokes system, which describes the flow of a Newtonian fluids. In the case of Non-Newtonian fluids however, the viscosity is not constant but may depend non-linearly on \( Du \). The power law fluids and the Carreau type fluids are such examples, which are widely used among engineers. For a more detailed discussion on the connection with mathematical modeling see e.g. [27, 28].

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\footnote{1}{We denote by \( Du \) the symmetric part of the gradient of \( u \), i.e. \( Du = \frac{(Vu + (Vu)^T)}{2} \).}
The existence theory for such type of fluids was initiated by Ladyzhenskaya [24, 25] and Lions [26].

The main result of the article are the following Campanato type estimates for the local weak solutions of (1.1).

**Theorem 1.1.** There is an $\alpha > 0$ such that for all $\beta \in [0, \alpha)$ a $C > 0$ exists such that for every ball $B$ with $2B \subset \Omega$

$$
\|A(Du)\|_{L^{1.2+\beta}(B)} + \|\pi\|_{L^{1.2+\beta}(B)} \leq C \left( \|G\|_{L^{1.2+\beta}(2B)} + R^{-\beta} \int_{2B} |A(Du) - (A(Du))_{2B}| dx \right).
$$

In particular, $G \in L^{1.2+\beta}(2B)$ implies $A(Du), \pi \in L^{1.2+\beta}(B)$.

The spaces $L^{1.2+\beta}(B)$ are the Campanato spaces, see Subsection 2.1. Our main theorem in particular includes the BMO-case (bounded mean oscillation), since BMO $= L^{1.2}$. Theorem 1.1 is a consequence of the refined BMO$_{\omega}$-estimates of Theorem 3.9, which also include the case VMO (vanishing mean oscillation). The upper bound $\alpha$ is given by the maximal (local) regularity of the homogeneous generalized Stokes system. Our estimates hold up to this regularity exponent. Due to the Campanato characterization of Hölder spaces $C^{\alpha,\alpha}$ our results can also be expressed in terms of Hölder spaces.

Our result is an extension of the results in [12] to the context of Non-Newtonian fluids. In [12] we studied the problem

$$
\begin{align*}
-\text{div}(A(\nabla u)) &= -\text{div}G \\
\Omega &\subset \mathbb{R}^n, n \geq 1, \text{ with similar growth conditions on } A \text{ but } Du \text{ replaced by the full gradient } \nabla u. \text{ The model case is } A(\nabla u) = |\nabla u|^{p-2}\nabla u \text{ with } 1 < p < \infty. \text{ In [12] we proved Theorem 1.1 for weak solutions of (1.3). The case } p \geq 2, \text{ has been studied first in [9].}
\end{align*}
$$

Theorem 1.1 is the limit case of the nonlinear Calderón-Zygmund theory, which was initiated by [17, 18]. Iwaniec proved that $G \in L^s$ with $s \geq p'$ implies $A(\nabla u) \in L^s$, where $p' = \frac{p}{p-1}$. See also [9] for related works. In the context of fluids the corresponding result was obtained in [11]. However, the limiting regularity of (1.1) for $G = 0$ restricts the transfer of integrability to the range $s \in [p', \frac{n}{n-2}p']$ for $n \geq 3$ and $s \in [p', \infty)$ for $n = 2$.

The reduced regularity for (1.1) with $G = 0$ is also the reason, why we can only treat the planar case $n = 2$ in this paper. The crucial ingredient for Theorem 1.1 are the decay estimates for the homogeneous case $G = 0$ in terms of the gradients. In this paper we are able to prove such decay estimates in the planar case $n = 2$, see Theorem 3.8. If such estimates can be proven for $n \geq 3$, then Theorem 1.1 would directly generalize to this situation. Unfortunately, this is an open problem, even in the absence of the pressure.

Theorem 1.1 can be used to improve the known regularity results for the stationary problem with convective term $|\nabla u|$, see Section 5, and for the instationary problem (1.2), see Section 5. The first $C^{1,\alpha}$-regularity for planar flows were obtained in the series of the articles [20, 21, 22] under various boundary conditions under the restriction $\kappa > 0$. See also [30, 1]. The stationary degenerate case $\kappa \geq 0$ was treated in [32] for $1 < p \leq 2$. To our knowledge the only result for $n \geq 2$ is the one obtained in [8] with $\kappa > 0$ and $1 < p \leq 2$ and small data and zero boundary values. Because of the zero boundary values (combined with the small data), we are not able to use this result for the higher regularity of the case $G = 0$.

Note that our result is optimal with respect to the regularity of $G$. All other planar results mentioned above need much stronger assumptions on the regularity of $G$. This is one of the advantages of the non-linear Calderón-Zygmund theory. This is the basis for our improved results in Section 4 and Section 5 for the system including the convective term.
It is based on the fact, that the convective term can be written as div(\(u \otimes u\)) using div\(u\) = 0 and therefore can be treated as a force term div\(G\).

2. Notation, Basic Definitions and Auxiliary Results

2.1. Notation. By \(B\) we will always denote a ball in \(\mathbb{R}^2\). We write \(B \subseteq \Omega\) if the closure of \(B\) is contained in \(\Omega\). Let \(|B|\) denote the volume of \(B\). Vector valued mappings are denoted by bold letters, e.g. \(u\), while single valued functions with regular letters, e.g. \(\eta\). For \(f \in L^1_{\text{loc}}(\mathbb{R}^2)\) we define component-wise

\[
\langle f \rangle_B = \frac{1}{|B|} \int_B f(x) \, dx,
\]

and \(M_B^p f = \int_B |f - \langle f \rangle_B| \, dx\), \(\langle M_B^2 f \rangle(x) = \sup_{B \ni x} M_B^2 f\).

The space BMO of function of bounded mean oscillations is defined via the seminorm (for \(\Omega\) open)

\[
\|f\|_{\text{BMO}(\Omega)} := \sup_{B \subseteq \Omega} \frac{1}{|B|} \int_B |f - \langle f \rangle_B| \, dx = \sup_{B \subseteq \Omega} M_B^2 f,
\]

saying that \(f \in \text{BMO}(\Omega)\), whenever its seminorm is bounded. Therefore \(f \in \text{BMO}(\mathbb{R}^2)\) if and only if \(M_B^2 f \in L^\infty(\mathbb{R}^2)\). We say that a function \(f \in \text{BMO}(\Omega)\) belongs to the subspace VMO\(_{\text{loc}}(\Omega)\) if \(\lim_{\varepsilon \to 0^+} \sup_{B \subseteq \Omega, |B| < \varepsilon} M_B^2 f = 0\). We need also the following refinements of BMO, see [31]. For a non-decreasing function \(\omega : (0, \infty) \to (0, \infty)\) we define

\[
M_{\omega, B}^2 f = \frac{1}{\omega(R)} \int_B |f - \langle f \rangle_B| \, dx,
\]

where \(R\) is the radius of \(B\). We define the seminorm

\[
\|f\|_{\text{BMO}_\omega(\Omega)} := \sup_{B \subseteq \Omega} M_{\omega, B}^2 f.
\]

The choice \(\omega(r) = 1\) gives the usual BMO seminorm, while \(\omega(r) = r^\alpha\) with \(0 < \alpha \leq 1\) induces the Campanato space \(C^{2, 2+\alpha}\). Its seminorm we denote \(\|u\|_{C^{2, 2+\alpha}(\Omega)}\).

By \(k B\), with \(k > 0\), we denote the ball with the same center and \(k\) times the radius. For functions \(f, g\) on \(\Omega\) we define \(\langle f, g \rangle := \int_\Omega f(x) g(x) \, dx\). Similarly also for mappings to \(\mathbb{R}^n\), \(n > 1\). We write \(f \sim g\) if and only if there exist constants \(c_0, c_1 > 0\), such that

\[
c_0 f \leq g \leq c_1 f,
\]

where we always indicate on what the constants may depend. Furthermore, we use \(c, C\) (no index) as generic constants, i.e. their values may change line to line but does not depend on the important quantities.

We say that a function \(\rho : [0, \infty) \to [0, \infty)\) is almost increasing if there is \(c > 0\) such that for all \(0 \leq s \leq t\) the inequality \(\rho(s) \leq c \rho(t)\) is valid. We say that \(\rho\) is almost decreasing if there is \(c > 0\) such that for all \(0 \leq s \leq t\) the inequality \(\rho(s) \geq c \rho(t)\) is valid. We say that \(\rho\) is almost monotone if it is almost increasing or almost decreasing.

For a mapping \(u : \Omega \to \mathbb{R}^2\) we define \(D u = (\nabla u + (\nabla u)^T)/2\), \(W u = (\nabla u - (\nabla u)^T)/2\) and \((|u| u)_{fj} = \sum_{i=1}^2 u_{ik} \partial_i u_{fj}\). In the parts of the article dealing with evolutionary problems we will assume that \(u : \Omega \times (0, T) \to \mathbb{R}^2\). In this case all operators \(\nabla, D, W\) and \(\text{div}\) are understood only with respect to the variable \(x \in \Omega\).

For \(P, Q \in \mathbb{R}^n\) with \(n \geq 1\) we define \(P : Q = \sum_{j=1}^n P_j Q_j\). The symbol \(\mathbb{R}^{2 \times 2}\) denotes the set of symmetric \(2 \times 2\) matrices. For a set \(M \subset \mathbb{R}^n\) we denote \(\chi_M\) as the characteristic function of the set \(M\), i.e. \(\chi(x) = 1\) if \(x \in M\) otherwise it is equal to zero. We write \(\mathbb{R}^{\geq 0} = [0, +\infty)\).
2.2. **N-functions and the extra stress tensor A.** The following definitions and results are standard in the context of N–function (see e. g. [29]).

**Definition 2.1.** A real function \( \varphi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0} \) is said to be an N-function if it satisfies the following conditions: There exists the derivative \( \varphi' \) of \( \varphi \). This derivative is right continuous, non-decreasing and satisfies \( \varphi'(0) = 0 \) and \( \varphi'(t) > 0 \) for \( t > 0 \). Especially, \( \varphi \) is convex.

**Definition 2.2.** We say that the N-function \( \varphi \) satisfies the \( \Delta_2 \)-condition, if there exists \( c_1 > 0 \) such that for all \( t \geq 0 \) it holds \( \varphi(2t) \leq c_1 \varphi(t) \). By \( \Delta_2(\varphi) \) we denote the smallest constant \( c_1 \). For a family \( \Phi \) of N-functions we define \( \Delta_2(\Phi) := \sup_{\varphi \in \Phi} \Delta_2(\varphi) \).

Let \( \varphi \) be an N-function. We state some of its basic properties. Since \( \varphi(t) \leq \varphi(2t) \) the \( \Delta_2 \)-condition is equivalent to \( \varphi(2t) \sim \varphi(t) \). The complementary function \( \varphi^\ast \) is given by

\[
\varphi^\ast(u) := \sup_{t \geq 0} (ut - \varphi(t)).
\]

It satisfies \( (\varphi^\ast)'(t) = (\varphi')^{-1}(t) \), where \( (\varphi')^{-1} \) is the right-continuous inverse of \( \varphi' \). Moreover, \( (\varphi^\ast)^\ast = \varphi \).

For all \( \delta > 0 \) there exists \( c_\delta \) (only depending on \( \Delta_2(\varphi^\ast) \)) such that for all \( t, u \geq 0 \)

\[
(2.1) \quad tu \leq \delta \varphi(t) + c_\delta \varphi^\ast(u).
\]

This inequality is called Young’s inequality. For all \( t \geq 0 \)

\[
(2.2) \quad \frac{t}{2} \varphi'(\frac{t}{2}) \leq \varphi(t) \leq t \varphi'(t), \quad \varphi\left(\frac{\varphi^\ast(t)}{t}\right) \leq \varphi^\ast(t) \leq \varphi\left(\frac{2\varphi^\ast(t)}{t}\right).
\]

Therefore, uniformly in \( t \geq 0 \)

\[
(2.3) \quad \varphi(t) \sim \varphi'(t)t, \quad \varphi^\ast(\varphi'(t)) \sim \varphi(t),
\]

where the constants only depend on \( \Delta_2(\varphi, \varphi^\ast) \).

For an N-function \( \varphi \) with \( \Delta_2(\varphi) < \infty \), we denote by \( L^\varphi \) and \( W^{1, \varphi} \) the classical Orlicz and Sobolev-Orlicz spaces, i.e. \( u \in L^\varphi \) if and only if \( \int |u(x)| \varphi(dx) < \infty \) and \( u \in W^{1, \varphi} \) if and only if \( u, \nabla u \in L^\varphi \). By \( W^{1, \varphi}_0(\Omega) \) we denote the closure of \( C^\infty_0(\Omega) \) in \( W^{1, \varphi}(\Omega) \).

Throughout the paper we will assume that \( \varphi \) satisfies the following assumption.

**Assumption 2.3.** Let \( \varphi \) be an N-function with \( \varphi \in C^2((0, +\infty)) \cap C^1([0, +\infty)) \) such that \( \varphi'' \) is almost monotone on \((0, +\infty)\) and

\[
\varphi''(t) \sim t \varphi''(t)
\]

uniformly in \( t \geq 0 \). The constants hidden in \( \sim \) are called the characteristics of \( \varphi \).

We remark that if \( \varphi \) satisfies Assumption 2.3 below, then \( \Delta_2(\{\varphi, \varphi^\ast\}) \) \( < \infty \) will be automatically satisfied, where \( \Delta_2(\{\varphi, \varphi^\ast\}) \) depends only on the characteristics of \( \varphi \), see for example [2] for a proof. Most steps in our proof do not require that \( \varphi'' \) is almost monotone. It is only needed in Theorem 3.7 for the derivation of the decay estimates of Theorem 3.8.

Let us now state the assumptions on \( A \).

**Assumption 2.4.** Let \( \varphi \) hold Assumption 2.3. The vector field \( A : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2} \), \( A \in C^{0, 1}(\{\mathbb{R}^{2 \times 2} \setminus \{0\}) \cap C^0(\mathbb{R}^{2 \times 2}) \) satisfies the non-standard \( \varphi \)-growth condition, i.e. there are \( c, C > 0 \) such that for all \( P, Q \in \mathbb{R}^{2 \times 2} \) with \( P \neq 0 \)

\[
(2.4) \quad (A(P) - A(Q)) \cdot (P - Q) \geq c \varphi''(|P| + |Q|)|P - Q|^2,
\]

\[
|A(P) - A(Q)| \leq C \varphi''(|P| + |Q|)|P - Q|
\]
holds. We also require that $A(D)$ is symmetric for all $D \in \mathbb{R}^{2 \times 2}_{\text{sym}}$ and $A(0) = 0$.

Let us provide a few typical examples. If $\varphi$ satisfies Assumption 2.3. Then both $A(Q) := \varphi'(\|Q\|) \frac{Q}{\|Q\|}$ and $A(Q) := \varphi'(\|Q^{\text{sym}}\|) \frac{Q^{\text{sym}}}{\|Q^{\text{sym}}\|}$ satisfy Assumption 2.4. See [10] for a proof of this result. In this case, (1.1) is just the Euler-Lagrange equation of the local $W^{1,\varphi}_{\text{div}}$-minimizer of the energy $J(\mathbf{w}) := \int_{\Omega} \varphi(\|D\mathbf{w}\|) \, dx + \langle \mathbf{G}, \nabla \mathbf{w} \rangle$. Here $W^{1,\varphi}_{\text{div}}$ is the subspace of functions $\mathbf{w} \in W^{1,\varphi}$ with $\text{div} \mathbf{w} = 0$. The pressure acts as a Lagrange multiplier. This includes in particular the case of power law and Carreau type fluids:

(a) Power law fluids with $1 < p < \infty$, $\kappa \geq 0$ and $\nu > 0$

$$A(Q) = v(\kappa + |Q|)^{p-2}Q \quad \text{and} \quad \varphi(t) = \int_0^t v(\kappa + s)^{p-2} \, ds$$

or

$$A(Q) = v(\kappa^2 + |Q|^2)^{\frac{p-2}{2}}Q \quad \text{and} \quad \varphi(t) = \int_0^t v(\kappa^2 + s^2)^{\frac{p-2}{2}} \, ds.$$

(b) Carreau type fluids with $1 < p < \infty$, $\kappa, \mu_\omega \geq 0$ and $\nu > 0$

$$A(Q) = \mu_\omega Q + v(\kappa + |Q|)^{p-2}Q \quad \text{and} \quad \varphi(t) = \int_0^t \mu_\omega s + v(\kappa + s)^{p-2} \, ds.$$

(c) For $1 < p < \infty$, $\mu_\omega > 0$, and $\nu \geq 0$

$$A(Q) = \mu_\omega Q + v \text{arcsinh}(|Q|) \frac{Q}{|Q|} \quad \text{and} \quad \varphi(t) = \int_0^t \mu_\omega s + v \text{arcsinh}(s) \, ds.$$

We introduce the family of shifted N-functions $\{\varphi_a\}_{a \geq 0}$ by $\varphi'_a(t) := \varphi'(a + t)/(a + t)$. If $\varphi$ satisfies Assumption 2.3 then $\varphi''_a(t) \sim \varphi''(a + t)$ uniformly in $a, t \geq 0$. Moreover, $\Delta_2(\{\varphi_a, (\varphi_a)'\})_{a \geq 0} < \infty$ depending only on the characteristics of $\varphi$. Let use define $V : \mathbb{R}^{2 \times 2} \to \mathbb{R}^{2 \times 2}$

$$V(Q) = \sqrt{\varphi'(\|Q\|)\|Q\|} \frac{Q}{\|Q\|}.$$

In the special case of $A(Q) := \varphi'(\|Q\|) \frac{Q}{\|Q\|}$, the quantity $V(Q)$ is characterized by

$$|V(Q)|^2 = A(Q) \cdot Q \quad \text{and} \quad \frac{V(Q)}{|V(Q)|} = \frac{A(Q)}{|A(Q)|} = \frac{Q}{|Q|}.$$

The connection between $A$, $V$, and the shifted N-functions is best reflected in the following lemma, which is a summary of Lemmas 3, 21, and 26 of [10].

**Lemma 2.5.** Let $A$ satisfy Assumption 2.4. Then

$$(A(P) - A(Q)) \cdot (P - Q) \sim |V(P) - V(Q)|^2$$

$$\sim \varphi_Q(||P - Q||)$$

uniformly in $P, Q \in \mathbb{R}^{2 \times 2}$. Moreover,

$$A(Q) \cdot Q \sim |V(Q)|^2 \sim \varphi(|Q|),$$

and

$$|A(P) - A(Q)| \sim \left(\varphi_Q\right)'(||P - Q||),$$

uniformly in $P, Q \in \mathbb{R}^{2 \times 2}$. 

As a further consequence of Assumption 2.3 there exists $1 < p \leq q < \infty$ and $K_1 > 0$ such that
\begin{equation}
\phi(st) \leq K_1 \max\{s^p, s^q\}\phi(t)
\end{equation}
for all $s, t \geq 0$. The exponents $p$ and $q$ are called the lower and upper index of $\phi$, respectively. We say that $\phi$ is of type $T(p, q, K_1)$ if it satisfies (2.6), where we allow $1 \leq p \leq q < \infty$ in this definition.

The following two lemmas show an important invariance in terms of shifts.

**Lemma 2.6** (Lemma 22, [13]). Let $\phi$ hold Assumption 2.3. Then $(\phi|_{P})^s(t) \sim (\phi^s)_{|A(P)}(t)$ holds uniformly in $t \geq 0$ and $P \in \mathbb{R}^{2\times 2}$. The implicit constants depend on $p$, $q$ and $K_1$ only.

We define
\begin{equation}
\bar{p} := \min\{p, 2\} \text{ and } \bar{q} := \max\{q, 2\}.
\end{equation}

**Lemma 2.7.** Let $\phi$ be of type $T(p, q, K_1)$ and $P \in \mathbb{R}^{2\times 2}$, then there is a $K$ depending on $K_1, p, q$ such that $\phi|_{P}$ is of type $T(p, \bar{p}, K)$ and $(\phi|_{P})^s$ and $(\phi^s)_{|A(P)}$ are of type $T(\bar{q}^s, \bar{q}^s, K)$.

The proof can be found in [12, Lemma 2.3]. Finally, we can deduce from Lemma 2.7 the following versions of Young’s inequality. For all $\delta \in (0, 1]$ and all $t, s \geq 0$ it holds
\begin{equation}
ts \leq K \delta^{1-\bar{q}} \phi(t) + \delta \phi^s(s),
\end{equation}
\begin{equation}
ts \leq \bar{\delta} \phi(t) + K^{p-1} \delta^{1-p} \phi^s(s).
\end{equation}

3. PROOF OF THE MAIN THEOREM

Let $u, \pi$ be the local weak solution of (1.1), in the sense that $u \in W^{1,\varphi}_{\text{div}}(\Omega)$, $\pi \in L^{\varphi^*}(\Omega)$, and
\begin{equation}
\forall \xi \in W^{1,\varphi} \left(\Omega\right) : \left< A(D\xi) + G\xi, \nabla \pi \right> - \left< \nabla u, \nabla \pi \right> = \left< G, D\xi \right>,
\end{equation}
where we used that $A(Du)$ and $G$ are symmetric. To omit the pressure, we will use divergence free test function, i.e.
\begin{equation}
\forall \xi \in W^{1,\varphi}_{0,\text{div}} \left(\Omega\right) : \left< A(D\xi), D\xi \right> = \left< G, D\xi \right>.
\end{equation}
The method of the proof of Theorem 1.1 was developed in [12] for elliptic systems with the main part depending on full gradient of solutions. It is based on a reverse Hölder inequality, an approximation by the problem with zero right hand side and a decay estimate for this approximation. These three properties are discussed in the subsequent subsections. Note that the restriction to the planar case and $\varphi''$ almost monotone is only needed for the decay estimate of Subsection 3.3. The first two subsections are valid independently of these extra assumptions.

3.1. **Reverse Hölder inequality.** In this section we show the reverse Hölder estimate for solutions of (1.1). To prove the result we need a Sobolev-Poincaré inequality in the Orlicz setting from [10, Lemma 7].

**Theorem 3.1** (Sobolev-Poincaré). Let $\psi$ be an $N$-function such that $\psi$ and $\psi^*$ satisfy the $\Delta_2$-condition. Then there exists $0 < \theta < 1$ and $c > 0$, which depend only on $\Delta_2(\{\psi, \psi^*\})$ such that $\psi^\theta$ is almost convex\(^2\) and the following holds. For every ball $B \subset \mathbb{R}^n$ with radius $R$
and every \( \mathbf{v} \in W^{1, \psi}(B) \) holds

\[
\int_B \psi\left( \frac{|\mathbf{v} - \langle \mathbf{v} \rangle_B|}{R} \right) d\mathbf{x} \leq c \left( \int_B \psi^\theta(|\nabla \mathbf{v}|) d\mathbf{x} \right)^{\frac{1}{\theta}}.
\]

**Remark 3.2.** It is not possible to replace full gradient on the right hand side with the symmetric one only. Consider \( \mathbf{v} = (x_2, -x_1) \) on the unit ball.

We also need the following version of the Korn’s inequality for Orlicz spaces, which is a minor modification of the one in [14, Theorem 6.13]. See [5] for sharp conditions for Korn’s inequality on Orlicz spaces.

**Lemma 3.3.** Let \( B \subset \mathbb{R}^n \) be a ball. Let \( \psi \) be an \( N \)-function such that \( \psi \) and \( \psi^* \) satisfy the \( \Delta_2 \)-condition (for example let \( \psi \) satisfy Assumption 2.3). Then for all \( \mathbf{v} \in W^{1, \psi}(B) \) with \( \langle \mathbf{Wv} \rangle_B = 0 \) the inequality

\[
\int_B \psi(|\nabla \mathbf{v}|) d\mathbf{x} \leq C \int_B \psi(|\mathbf{Dv}|) d\mathbf{x}
\]

holds. The constant \( C > 0 \) depends only on \( \Delta_2(\{\psi, \psi^*\}) < \infty \).

**Proof.** From [14, Theorem 6.13] we know that

\[
\int_B \psi(|\nabla \mathbf{v} - \langle \nabla \mathbf{v} \rangle_B|) d\mathbf{x} \leq C \int_B \psi(|\mathbf{Dv} - \langle \mathbf{Dv} \rangle_B|) d\mathbf{x}.
\]

Using \( \langle \mathbf{Wv} \rangle_B = 0 \) we have \( \nabla \mathbf{v} = (\nabla \mathbf{v} - \langle \nabla \mathbf{v} \rangle_B) + \langle \mathbf{Dv} \rangle_B \). Thus, by triangle inequality and (3.4) we get

\[
\int_B \psi(|\nabla \mathbf{v}|) d\mathbf{x} \leq c \int_B \psi(|\mathbf{Dv} - \langle \mathbf{Dv} \rangle_B|) d\mathbf{x} + c \int_B \psi(|\langle \mathbf{Dv} \rangle_B|) d\mathbf{x},
\]

where we also used \( \Delta_2(\psi) < \infty \). Now, the claim follows by triangle inequality and Jensen’s inequality. \( \square \)

As in [12] we need a reverse Hölder estimate for the oscillation of the gradients. Additional difficulties arise due to the symmetric gradient and the hidden pressure (so that the test functions must be divergence free).

**Lemma 3.4.** Let \( \mathbf{u} \) be a local weak solution of (1.1) and \( B \) be a ball satisfying \( 2B \subset \Omega \). There exists \( \Theta \in (0, 1) \) and \( c > 0 \) only depending on the characteristics of \( \Phi \), such that for all \( \mathbf{P}, \mathbf{G}_0 \in \mathbb{R}^{2 \times 2} \),

\[
\int_B \left| \mathbf{V}(\mathbf{Du}) - \mathbf{V}(\mathbf{P}) \right|^2 d\mathbf{x} \leq c \left( \int_{2B} \left| \mathbf{V}(\mathbf{Du}) - \mathbf{V}(\mathbf{P}) \right|^{2\Theta} d\mathbf{x} \right)^{\frac{1}{\Theta}} + c \int_{2B} (\Phi^*_{|A|\mathbf{P}}(|\mathbf{G} - \mathbf{G}_0|) d\mathbf{x}
\]

holds. The constant \( c > 0 \) depends only on the characteristics of \( \Phi \in T(p, q, K) \) and the constants in Assumption 2.4.

**Proof.** Let \( \eta \in C_0^\infty(2B) \) with \( \chi_B \leq \eta \leq \chi_{3B/2} \) and \( |\nabla \eta| \leq c/R \), where \( R \) is the radius of \( B \). We define \( \psi = \eta^\Phi(\mathbf{u} - \mathbf{z}) \), where \( \mathbf{z} \) is a linear function such that \( \langle \mathbf{u} - \mathbf{z} \rangle_{2B} = 0 \), \( \mathbf{Dz} = \mathbf{P} \), and \( \mathbf{Wz} = \langle \mathbf{Wu} \rangle_{2B} \). We cannot use \( \psi \) as test function in the pressure free formulation (3.2), since its divergence does not vanish. Therefore, we correct \( \psi \) by help of the Bogovskiî
operator Bog from [3]. In particular, \( w = \text{Bog}(\text{div} \psi) \) is a special solution of the auxiliary problem

\[
\begin{align*}
\text{div} w &= \text{div} \psi \quad \text{in } \frac{3}{2} B \\
w &= 0 \quad \text{in } \partial (\frac{3}{2} B).
\end{align*}
\]

We extend \( w \) by zero outside of \( \frac{3}{2} B \). It has been shown in [14, Theorem 6.6] that \( \nabla w \) can be estimated in terms of \( \text{div} \psi \) in any suitable Orlicz spaces. In our case we use the following estimate in terms of \( \varphi_{\mathcal{P}} \).

\[
\int_{2B} \varphi_{\mathcal{P}}(|Dw|) \, dx \leq C \int_{2B} \varphi_{\mathcal{P}}(|\text{div} \psi|) \, dx.
\]

The constant \( C > 0 \) depends only on the characteristics of \( \varphi \).

Using \( \text{div} u = 0 \), we have

\[
\text{div} \psi = \nabla (\eta^7)(u - z) + \eta^7 \text{div}(u - z) = \nabla \eta^{-1} \nabla (u - z) - \nabla \eta^7 \text{tr} P.
\]

This implies

\[
\int_{2B} \varphi_{\mathcal{P}}(|Dw|) \, dx \leq C \int_{2B} \varphi_{\mathcal{P}} \left( \frac{|u - z|}{R} \right) \, dx + C \int_{2B} \varphi_{\mathcal{P}}(|\text{tr} P|) \, dx.
\]  

We define \( \xi := \psi - w = \eta^7(u - z) - w \), then \( \text{div} \xi = 0 \), which ensures that \( \xi \) is a valid test function for (1.1). We get

\[
\langle A(Du) - A(P), \eta^7(Du - P) \rangle = \langle G - G_0, \eta^7(Du - P) \rangle + \langle G - G_0, (u - z) \otimes_{\text{sym}} \nabla (\eta^7) \rangle
\]

\[
- \langle A(Du) - A(P), (u - z) \otimes_{\text{sym}} \nabla (\eta^7) \rangle
\]

\[
- \langle G - G_0, Dw \rangle + \langle A(Du) - A(P), Dw \rangle.
\]

The symbol \( \otimes_{\text{sym}} \) denotes the symmetric part of \( \otimes \), i.e., \( (f \otimes_{\text{sym}} g)_{ij} := (f_{ij} + f_{ji})/2 \) for \( f, g \in \mathbb{R}^2 \). We divide (3.6) by \( |2B| \) and estimate the two sides. Concerning the left hand side we find by Lemma 2.5

\[
|2B|^{-1} \langle A(Du) - A(P), \eta^7(Du - P) \rangle \sim \int_{2B} \eta^7 |V(Du) - V(P)|^2 \, dx =: (I).
\]

We estimate the right hand side of (3.6) by Young’s inequality (2.8) for \( \varphi_{\mathcal{P}} \) with \( \delta \in (0, 1) \) using also \( (\varphi_{\mathcal{P}})^* \sim (\varphi^*)_{|A(P)|} \) (see Lemma 2.6).

\[
(I) \leq c \delta \int_{2B} (\varphi^*)_{|A(P)|}(|G - G_0|) \, dx + \delta \int_{2B} \eta^7 \varphi_{\mathcal{P}}(|Du - P|) \, dx
\]

\[
+ c \delta \int_{2B} \varphi_{\mathcal{P}} \left( \frac{|u - z|}{R} \right) \, dx + c \delta \int_{2B} \varphi_{\mathcal{P}}(|Dw|) \, dx
\]

\[
+ \delta \int_{2B} \eta^7 \text{tr} \varphi^*_{|A(P)|}(|A(Du) - A(P)|) \, dx
\]

\[
= : (II) + (III) + (IV) + (V) + (VI).
\]
Now we use Lemma 2.5 to estimate \((III + (VI) \leq \delta c(I))\), so these terms can be absorbed. Moreover, by (3.5)

\[
(IV) + (V) \leq c(IV) + c \int_{2B} \varphi_P(|\text{tr}\, P|) \, dx.
\]

Since \(P\) is constant, \(\text{tr}\, P = \text{div}\, z\) and \(\text{div}\, u = 0\), we can estimate

\[
(3.7) \quad \int_{2B} \varphi_P(|\text{tr}\, P|) \, dx = \left( \int_{2B} (\varphi_P)^{\theta}(|\text{div}\, (u - z)|) \, dx \right)^{\frac{1}{\theta}} \leq \left( \int_{2B} (\varphi_P)^{\theta}(|Du - Dz|) \, dx \right)^{\frac{1}{\theta}}.
\]

It remains to estimate (IV). We use Sobolev-Poincaré inequality of Theorem 3.1 with \(\psi = \varphi_P\) such that \((\varphi_P)^{\theta}\) is almost convex and

\[
(IV) = c \int_{2B} \varphi_P \left( \frac{|u - z|}{R} \right) \, dx \leq c \left( \int_{2B} \varphi_P(|\nabla u - \nabla z|) \, dx \right)^{\frac{1}{\theta}}
\]

with \(\theta \in (0, 1)\). The constants and \(\theta\) are independent of \(|P|\), since the \(A_2(\{\varphi_\theta\}_{\theta \geq 0})\) is bounded in terms of the characteristics of \(\varphi\).

As \(\langle W(u - z) \rangle_{2B} = 0\) we find by Korn’s inequality (Lemma 3.3) with \(\psi = \varphi_P^\theta\) (almost convex) and \(Dz = P\) that

\[
(IV) \leq c \left( \int_{2B} \varphi_P^\theta(|Du - Dz|) \, dx \right)^{\frac{1}{\theta}}.
\]

The above estimates and Lemma 2.5 show that

\[
(IV) + (V) \leq c \left( \int_{2B} \varphi_P^\theta(|Du - Dz|) \, dx \right)^{\frac{1}{\theta}} \leq c \left( \int_{2B} |V(Du) - V(P)|^{2\theta} \, dx \right)^{\frac{1}{\theta}}.
\]

The lemma is proved.

Lemma 3.4 allows to obtain exactly as in [12] the next corollary, compare with [12, Corollary 3.5].

**Corollary 3.5.** Let the assumptions of Lemma 3.4 be satisfied. Then for all \(P \in \mathbb{R}^{2 \times 2}_{\text{sym}}\)

\[
\int_B |V(Du) - V(P)|^2 \, dx \leq c (\varphi^\star)_{|A|} \left( \int_{2B} |A(Du) - A(P)| \, dx \right) + c (\varphi^\star)_{|A|} ||G||_{\text{BMO}(2B)}
\]

The constants only depend on the characteristics of \(\varphi\) and the constants in Assumption 2.4.

3.2. **Approximation property.** Let \(u\) be a local weak solution of (1.1) and \(B\) be a ball satisfying \(2B \subset \Omega\). We consider a solution \(h, \rho\) of the homogeneous problem

\[
\begin{align*}
-\text{div}\, A(Dh) + \nabla \rho &= 0 \quad \text{in } \Omega, \\
\text{div}\, u &= 0 \quad \text{in } \Omega, \\
h &= u \quad \text{on } \partial \Omega.
\end{align*}
\]

(3.8)

The next lemma estimates the natural distance between \(u\) and its approximation \(h\).
Lemma 3.6. For every \( \delta > 0 \) there exists \( c_\delta \geq 1 \) such that

\[
\int_B |V(Du) - V(Dh)|^2 \, dx \leq \delta (\varphi^+)(|A(Du)|_{2B}) \left( \int_{2B} |A(Du) - \langle A(Du) \rangle_{2B}| \, dx \right)
\]

\[\begin{align*}
&+ c_\delta (\varphi^+)(|A(Du)|_{2B})(\|G\|_{BMO(2B)})
\end{align*}\]

holds. The constants depend only on the characteristics of \( \varphi \) and the constants in Assumption 2.4.

Proof. The estimate is obtained by testing the difference of the equations for \( u \) and \( h \) by \( u - h \). The proof is exactly as in [12, Lemma 4.2]. One just needs to replace the gradient by the symmetric gradient. \( \square \)

3.3. Decay estimate. In this section we derive decay estimates for our approximation \( h \).

The main ingredient is the following theorem which can be found in [11, Theorem 3.6]. It is valid in any dimension but needs \( \varphi'' \) to be almost monotone. This is the only place in the paper, where we need this assumption on \( \varphi'' \).

Theorem 3.7. Let \( \varphi'' \) be almost monotone. If \( h \) is a weak solution of (3.8), then there is an \( r > 2 \) such that for every ball \( Q \subset B \) with radius \( R > 0 \)

\[
R^2 \left( \int_{\frac{1}{2}Q} |\nabla V(Dh)|^r \, dx \right)^{\frac{2}{r}} \leq C \int_Q |V(Dh) - \langle V(Dh) \rangle_Q|^2 \, dx.
\]

The constants \( C \) and \( r \) depend only on the characteristics of \( \varphi \) and the constants in Assumption 2.4.

The regularity \( V \in W^{1,r} \) with \( r > 2 \) ensures in two space dimensions that \( V \) is Hölder continuous. This is the reason, why our estimates can only be applied to planar flows. It is an open question if \( V(Vu) \) is Hölder continuous in higher dimensions.

This provides the following decay estimates in the plane:

Theorem 3.8. There exists \( \gamma > 0 \) such that for every \( \lambda \in (0, 1] \)

\[
\int_{\lambda B} |V(Dh) - \langle V(Dh) \rangle_{\lambda B}|^2 \, dx \leq C\lambda^{2\gamma} \int_B |V(Dh) - \langle V(Dh) \rangle_B|^2 \, dx.
\]

The constant \( C \) and \( \gamma \) depend only on the characteristics of \( \varphi \) and the constants in Assumption 2.4.

Proof. The result is clear if \( \lambda \geq \frac{1}{2} \), so we can assume \( \lambda \in (0, \frac{1}{2}) \). Let \( R \) denote the radius of \( B \). We compute by Poincaré inequality on \( \lambda B \), Jensen’s inequality with \( r > 2 \), enlarging the domain of integration and Theorem 3.7

\[
\int_{\lambda B} |V(Dh) - \langle V(Dh) \rangle_{\lambda B}|^2 \, dx \leq C(\lambda R)^2 \int_{\lambda B} |\nabla V(Dh)|^2 \, dx
\]

\[
\leq C(\lambda R)^2 \left( \int_{\lambda B} |\nabla V(Dh)|^r \, dx \right)^{\frac{2}{r}} \leq CR^2 \lambda^{2(1-\frac{2}{r})} \left( \int_{\frac{1}{2}B} |\nabla V(Dh)|^r \, dx \right)^{\frac{2}{r}}
\]

\[
\leq C\lambda^{2(1-\frac{2}{r})} \int_B |V(Dh) - \langle V(Dh) \rangle_B|^2 \, dx.
\]

As \( r > 2 \) the proof is completed. \( \square \)
3.4. Proof of the main theorem. Theorem 1.1 is a corollary of the following more general theorem.

**Theorem 3.9.** Let $B \subset \mathbb{R}$ be a ball. Let $u$, $\pi$ be a local weak solution of (1.1) on $2B$, with $\varphi$ and $A$ satisfying Assumption 2.4. Let $\omega : (0, \infty) \to (0, +\infty)$ be non-decreasing such that for some $\beta \in (0, \frac{2}{n})$ the function $\omega(r)^{-\beta}$ is almost decreasing, where $\gamma$ is defined in Theorem 3.8 and $\overline{p}$ in (2.7). Then

$$\|\pi\|_{\text{BMO}(\omega(B))} + \|A(Du)\|_{\text{BMO}(\omega(B))} \leq c M_{\omega,2B}^4(A(Du)) + c \|G\|_{\text{BMO}(\omega(2B))}.$$  

The constants depend only on the characteristics of $\varphi$ and the constants in Assumption 2.4.

**Proof.** The proof of the estimate of $A(Du)$ follows line by line the proof of [12, Theorem 5.3]. It is based on Corollary 3.5, Lemma 3.6 and Theorem 3.8. We will therefore omit the proof here and restrict ourselves to the additional estimates for the pressure.

We define $H = A(Du) - G$. It holds $H \in \text{BMO}(B) \subset \text{BMO}(B)$. We fix a ball $Q \subset B$. Then equation (1.1) implies that

$$\forall \xi \in W^{1,2}_0(\Omega) : \langle \pi - \langle \pi \rangle_Q, \text{div} \xi \rangle = \langle H - \langle H \rangle_Q, \nabla \xi \rangle.$$  

Let $\xi \in W^{1,2}_0(Q)$ be the solution of the auxiliary problem

$$\text{div} \xi = \pi - \langle \pi \rangle_Q \quad \text{in } Q, \quad \xi = 0 \quad \text{on } \partial Q.$$  

The existence of such a solution is ensured by the Bogovskii operator [4] and we have $\|\nabla \xi\|_{L^2(Q)} \leq C \|\pi - \langle \pi \rangle_Q\|_{L^2(Q)}$. The constant $C > 0$ is independent of $Q$. Inserting such $\xi$ into (3.10) we get

$$\|\pi - \langle \pi \rangle_Q\|_{L^2(Q)} = \|\pi - \langle \pi \rangle_Q, \text{div} \xi\rangle = \langle H - \langle H \rangle_Q, \nabla \xi \rangle.$$  

This and $\|\nabla \xi\|_{L^2(Q)} \leq C \|\pi - \langle \pi \rangle_Q\|_{L^2(Q)}$ implies $\|\pi - \langle \pi \rangle_Q\|_{L^2(Q)} \leq C \|H - \langle H \rangle_Q\|_{L^2(Q)}$. We find by Jensen’s inequality

$$(M_Q^0)^2 \leq \int_Q |\pi - \langle \pi \rangle_Q|^2 dx \leq C \int_Q |H - \langle H \rangle_Q|^2 dx \leq C \|H\|_{\text{BMO}(Q)}^2.$$  

In the last inequality we used the John-Nirenberg estimate, [15, Corollary 6.12]. It follows that $\pi \in \text{BMO}(B)$ and $\|\pi\|_{\text{BMO}(Q)} \leq C \|H\|_{\text{BMO}(Q)}$. This implies that

$$M_{\omega,Q}^r(\pi) \leq C \frac{1}{\omega(R_Q)} \|H\|_{\text{BMO}(Q)} \leq C \|H\|_{\text{BMO}(B)}$$  

using the monotonicity of $\omega$. Since $Q$ is arbitrary, we have $\|\pi\|_{\text{BMO}(\omega(B))} \leq \|H\|_{\text{BMO}(\omega(B))}$. Now $H = A(Du) - G$ and the estimate for $A(Du)$ conclude the proof. \qed

The choice $\omega(t) = 1$ in Theorem 3.9 gives the BMO estimate. However, the choice $\omega(t) = t^2$, $\beta \in (0, 2\gamma/\overline{p})$ Theorem 3.9 gives the estimates in Campanato space $\mathcal{L}^{1,2+\beta}$, compare [12, Corollary 5.5]. This is just Theorem 1.1.

**Remark 3.10.** It is possible to transfer the Hölder continuity of $A(Du)$ to $Du$ and $\nabla u$. Let us discuss the case of power-law and Carreau type fluids. This follows from the fact that $A^{-1} \in C^{\kappa,\sigma}_{\text{loc}}$ for some $\sigma > 0$. If $\kappa = 0$, then $\sigma = \min \{1, p'/2 - 1\}$. If $\kappa > 0$, then $\sigma = 1$. Now, $A(Du) \in C^{0,\beta}$ implies $Du \in C^{0,\beta,\sigma}$. Due to Korn’s inequality we get $\nabla u \in C^{0,\beta,\sigma}$ as well.
Remark 3.11. Note that if $G \in \text{VMO}(2B)$ in Theorem 3.9 we get that $A(Du) \in \text{VMO}(B)$. Indeed, since $G \in \text{VMO}(2B)$ there exists a nondecreasing function $\tilde{\omega} : (0, \infty) \to (0, \infty)$ with $\lim_{r \to 0} \tilde{\omega}(r) = 0$, such that $\|G\|_{\text{BMO}(B_r)} \leq \tilde{\omega}(r)$, for all $B_r \subset 2B$. Defining $\omega(r) = \min\{\tilde{\omega}(r), r^{\frac{n}{p}}\}$ we obtain by Theorem 3.9 the $\text{BMO}_\omega$-estimate for $A(Du)$ and $\pi$, which imply that both are in VMO (compare [12, Corollary 5.4]).

Remark 3.12. Let us now assume that the right hand side of (1.1) is not given in divergence form $-\text{div}G$ with $G$ symmetric, but rather as $f \in L^2$ with $s \geq 2$.

Let $w \in W^{2,s}(2B) \cap W^{1,1}(2B)$ and $\sigma \in W^{1,2}(2B)$ with $\langle \sigma \rangle_{2B} = 0$ be the unique solution of the Stokes problem $-\text{div}Dw + \nabla \sigma = f$ and $\text{div}w = 0$ in $2B$ with $w = 0$ on $\partial(2B)$. Then $f = -\text{div}G$ for $G := Dw - \sigma \text{Id}$ and $G$ is symmetric. If $s = 2$, then $G \in W^{1,2}(2B) \to \text{VMO}(2B)$.

If $s > 2$, then $G \in W^{1,s}(2B) \to L^{1,2+\left(1-\frac{2}{s}\right)}(2B) = C^{0,1-\frac{2}{s}}(2B)$. In particular, Theorem 3.9 is applicable and for all $s \geq 2$ we get the result.

The case $s = 2$ is obviously the limiting one in this setting. In the case of the $p$-Laplacian, i.e. no symmetric gradient and no pressure, it has been proven in [7, 16] that $f \in L^{p,1}(\mathbb{R}^n)$ (Lorentz space; subspace of $L^p$) implies $A(\text{Vu}) \in L^{p,1}$. It is an interesting open problem, if this also holds for the system with pressure and symmetric gradients (at least in the plane). Note that our results imply in this situation $A(Du), \pi \in \text{VMO}$ for $n = 2$.

4. An application to the stationary Navier-Stokes problem

In this section we present an application of the previous results to the generalized Navier-Stokes problem. We assume that $u \in W^{1,\phi}(\Omega), \text{div}u = 0$ and $\pi \in L^{p}(\Omega)$ are local weak solutions of the generalized Navier-Stokes problem, in the sense that

$$\forall \xi \in W^{1,\phi}_0(\Omega): \langle A(Du), D\xi \rangle - \langle \pi, \text{div}\xi \rangle = \langle G + u \otimes u, D\xi \rangle$$

for a given mapping $G : \Omega \to \mathbb{R}^{2 \times 2}_{\text{sym}}$.

In order to handle the convective term we need the condition

$$\liminf_{s \to +\infty} \frac{\phi(s)}{s^r} > 0 \quad \text{for some } r > \frac{3}{p}.$$ 

We have the following result

Theorem 4.1. Let $\varphi$ and $A$ satisfy Assumption 2.4 and (4.2). Let $u$ be a local weak solution of (4.1) on $\Omega$. Let $\beta \in (0, \frac{\alpha}{p-\alpha})$ ($\alpha$ is defined in Theorem 3.8 and $\varphi$ in Lemma 2.7). If $B$ is a ball with $2B \subset \Omega$ and $G \in L^{1,2+\beta}(2B)$, then $A(Du), \pi \in L^{1,2+\beta}(B)$.

Proof. According to [11, Remark 5.3] we get that $Du \in L^q(3B/2)$ for all $q > 1$. Consequently by the Korn inequality and the Sobolev embedding we get that $\pi \otimes u \in L^{1,1+\beta}(3B/2)$. Applying Theorem 1.1 we get the result. \hfill $\Box$

Exactly as in Remark 3.10 it is possible to transfer the Hölder continuity of $A(Du)$ to $Du$ and $\nabla u$.

Remark 4.2. A similar result has been proved also in [21], provided $\kappa > 0$, by a completely different method, which requires the stronger assumption $\text{div}G \in L^q(2B)$ for some $q > 2$.

The same result was also proved in [32] for power law fluids with $p \in (3/2, 2]$ and $\kappa \geq 0$, again under the stronger assumption $\text{div}G \in L^q(2B)$ for some $q > 2$. 

By our method we reprove these known results and improve them by weakening the assumption on the data of the problem.

5. An Application to the Parabolic Stokes Problem

Now we apply the previous results to the evolutionary variant of the problem (1.1). We set $T > 0$ and $I = (0, T)$, $\Omega_T = \Omega \times I$ and assume that $u \in L^\infty(I, L^2(\Omega))$ with $Du \in L^q(\Omega_T)$ is a local weak solution of the problem

\[
\partial_t u - \text{div}(A(Du)) + \nabla \pi = f \quad \text{in} \; \Omega_T,
\]

\[
\text{div} u = 0 \quad \text{in} \; \Omega_T.
\]

(5.1)

If the system of equations (5.1) is complemented by a suitable boundary and initial condition and if the data of the problem are sufficiently smooth it is possible to show existence of a solution that moreover satisfies

\[
\partial_t u \in L^\infty(I, L^2(\Omega)),
\]

(5.2)

see for example [23, 19, 6]. If we know such regularity of $\partial_t u$ and $f$ is smooth, it is easy to reconstruct the pressure $\pi$ in such a way that $\pi \in L^q(\Omega_T)$ with some $q > 1$ and

\[
\forall \xi \in C^\infty_0(\Omega_T) : \int_0^T -\langle \partial_t u, \xi \rangle + \langle A(Du) - \pi I, \nabla \xi \rangle dt = \int_0^T \langle f, \xi \rangle dt.
\]

(5.3)

The constant $q$ is determined by the requirement $A(Du) \in L^q(\Omega_T)$.

Applying the results from the previous sections of this article we obtain the next simple corollary.

**Corollary 5.1.** Let $A$ and $\varphi$ satisfy Assumption 2.4. Let $u \in L^\infty(I, L^2(\Omega))$ with $Du \in L^q(\Omega_T)$ and $\text{div} u = 0$ in $\Omega_T$ solve the problem (5.1) and satisfy (5.2). Let $B$ be a ball with $2B \subset \Omega$ and $f \in L^\infty(I, L^2(\Omega))$. Then $A(Du), \pi \in L^\infty(I, \text{VMO}(B))$.

**Proof.** The result is immediate consequence of $\partial_t u \in L^\infty(I, L^2(\Omega))$ and Remark 3.12. \qed

**Remark 5.2.** Certainly, we can obtain a similar result for the problem (5.1) with convection, as soon as $u \otimes u \in L^\infty(I, \text{VMO}(\Omega))$. This follows for example from the fact that $V(Du) \in W^{1,2}(I, L^2(\Omega)) \cap L^2(I, W^{1,2}(\Omega))$. Such kind of regularity is obtained, if it is possible to test with $\partial_t^2 u$ and $\Delta u$.

In [23] a method was developed to construct regular solutions of (5.1). The essential assumption was that the growth of $A$ is sufficiently fast. It was necessary to assume that

\[
\liminf_{s \to +\infty} \frac{\varphi(s)}{s^r} > 0 \quad \text{for some } r > \frac{4}{3}.
\]

(5.4)

This assumption was not due to the presence of the convective term in the analysis of [23]. It was necessary to overcome problems connected with the anisotropy of the evolutionary problem (5.1). The previous corollary is a first step to improve these results. If it is possible to show $\partial_t u \in L^\infty(I, L^r(\Omega))$ for some $r > 2$. Then for $f \in L^\infty(I, L^r(\Omega))$, we find by Remark 3.12 that $A(Du) \in L^\infty(I, C^{0,\tilde{\beta}}(\Omega))$ for $\tilde{\beta} \in (0, 1 - \frac{2}{r}) \cap (0, \frac{4}{3r})$. This implies (locally) bounded gradients $Vu$. So far the results of this paper are of local nature. An extension of this technique up to the boundary would imply globally bounded gradients $Vu$ and we could reconstruct the result of [23] for the generalized Stokes problem without the restriction (5.4).
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