TRACES OF CM VALUES AND CYCLE INTEGRALS OF
POLYHARMONIC MAASS FORMS

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Abstract. As a generalization of harmonic weak Maass forms, we consider polyharmonic weak
Maass forms characterized by the repeating action of the $\xi$-operator. In this article, we construct a
basis for the space of such forms of half-integral weight. Furthermore, we give arithmetic formulas
for these Fourier coefficients in terms of traces of CM values and cycle integrals of polyharmonic
weak Maass forms.

1. Introduction

For a negative integer $d$, we denote by $Q_d$ the set of positive definite integral binary qua-
dratic forms of discriminant $d$. Under the usual right action of $\text{SL}_2(\mathbb{Z})$, it has finitely many classes
$Q_d/\text{SL}_2(\mathbb{Z})$. For each $Q \in Q_d$, the order of the stabilizer $w_Q := |\text{PSL}_2(\mathbb{Z})_Q|$ is equal to 3, 2, or
1 according as $Q$ is $\text{SL}_2(\mathbb{Z})$-equivalent to $X^2 + XY + Y^2$, $X^2 + Y^2$ up to a constant multiple, or
otherwise, respectively. Under these notations, the Kronecker-Hurwitz class number $H(d)$ is defined
by

$$H(d) := \sum_{Q \in Q_d/\text{SL}_2(\mathbb{Z})} \frac{1}{w_Q} \quad \text{for } d < 0,$$

and $H(0) = -1/12$ additionally. In 1975, Zagier [19] showed the generating function $\sum_{d \leq 0} H(d)q^{-d}$
is the holomorphic part of a certain harmonic Maass form of weight $3/2$ on $\Gamma_0(4)$. Here we put
$q := e^{2\pi iz}$ for $z \in \mathcal{H} := \{z = x + iy \mid y > 0\}$. He also considered the case replacing 1, the numerator
in the definition of $H(d)$, with the elliptic modular $j$-function, that is,

$$\text{Tr}_d(j - 744) := \sum_{Q \in Q_d/\text{SL}_2(\mathbb{Z})} \frac{j(\alpha_Q) - 744}{w_Q} \quad \text{for } d < 0,$$

with additional $\text{Tr}_0(j - 744) := 2$, $\text{Tr}_1(j - 744) := -1$. Here $\alpha_Q$ is the unique root of $Q(z, 1) = 0$ in
the upper half plane $\mathcal{H}$. Then his 2002 paper [20] asserts that this generating function $\sum_{d \leq 1} \text{Tr}_d(j - 744)q^{-d}$ is a weakly holomorphic modular form of weight $3/2$.

A simple question is whether one can give a similar result for another $\text{SL}_2(\mathbb{Z})$-invariant function $f$. For a negative integer $d$, let $\text{Tr}_d(f)$ be the modular trace function replacing $j(\alpha_Q) - 744$ with $f(\alpha_Q)$ in (1.1). Under this notation, the Kronecke-Hurwitz class number $H(d)$ can be written as

$H(d) = \text{Tr}_d(1)$. For example, we consider the function $-\log(y|\eta(z)|^4)$ appearing in Kronecker’s first
limit formula, where $\eta(z) := q^{1/24} \prod_{n=1}^\infty (1 - q^n)$ is the Dedekind eta function. This function is not

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holomorphic, but a sesquiharmonic Maass form of weight 0, according to Lagarias-Rhoades’ recent work [14]. Then the following is known.

**Theorem 1.1.** [2 Theorem 1.2] The generating function

\[ \sum_{d \leq 0} \text{Tr}_d(-\log(y|\eta(z)|^4))q^{-d} \]

is the holomorphic part of a polyharmonic Maass form of weight 3/2 and depth 2 on \( \Gamma_0(4) \). Here we need a suitable constant term \( \text{Tr}_0(-\log(y|\eta(z)|^4)) \).

This theorem can be generalized to any polyharmonic weak Maass forms of weight 0. Before that, we recall the definition of polyharmonic Maass forms. The integral weight case was introduced by Lagarias-Rhoades [14] in 2016. In this article, we are also concerned with the cases of half-integral weight. For a fixed \( k \in \frac{1}{2}Z \), let \( \Gamma = \text{SL}_2(\mathbb{Z}) \) if \( k \in \mathbb{Z} \), and \( \Gamma = \Gamma_0(4) \) if \( k \in \mathbb{Z} + 1/2 \). Then for any \( \gamma = [a \ b \ c \ d] \in \Gamma \), set

\[ j_k(\gamma, z) := \left\{ \begin{array}{ll} \sqrt{cz + d} & \text{if } k \in \mathbb{Z}, \\ (\frac{a}{d})\epsilon_d^{-1}\sqrt{cz + d} & \text{if } k \in \mathbb{Z} + 1/2, \end{array} \right. \]

where \( \sqrt{z} \) is the principal branch of the holomorphic square root, \( (\frac{a}{d}) \) is the Kronecker symbol, and \( \epsilon_d = 1 \) if \( d \equiv 1 \) (mod 4) and \( \epsilon_d = i \) if \( d \equiv 3 \) (mod 4). Then we define the weight \( k \) slash operator by \( (f|k\gamma)(z) = j_k(\gamma, z)^{-2k}f(\gamma z) \). For a fixed \( r \in \frac{1}{2}Z \), a complex-valued smooth function \( f \) on the upper half plane \( \mathcal{H} \) satisfying

1. \( (f|k\gamma)(z) = f(z) \) for any \( \gamma = [a \ b \ c \ d] \in \Gamma \),
2. \( f \) is annihilated by the repeating action of the \( \xi \)-operator \( \xi_k := 2iy^k\frac{\partial}{\partial z}, 2r \) times, that is,
\[
\xi_a \circ \cdots \circ \xi_k \circ \xi_{2-k} \circ \xi_k f(z) = 0, \quad (2r \text{ times}),
\]

where \( a = k \) if \( 2r \) is odd, and \( a = 2 - k \) if \( 2r \) is even,
3. there exists an \( \alpha \in \mathbb{R}_{>0} \) such that \( f(x + iy) = O(y^\alpha) \) as \( y \to \infty \), uniformly in \( x \in \mathbb{R} \),

is called a polyharmonic Maass form of weight \( k \) and depth \( r \). The space of all such forms is denoted by \( H^1_k \). In the particular case of \( r = 1 \), since the weight \( k \) hyperbolic Laplacian \( \Delta_k \) is expressed as \( \Delta_k = -\xi_2 \circ \xi_k \), the element of \( H^1_k \) is a harmonic Maass form, and \( H^1_{k/2} \) is the space of holomorphic modular forms. Moreover, we call them sesquiharmonic if \( r = 3/2 \). Further, we let \( H^r_{k,1} \) be the larger space of polyharmonic weak Maass forms defined by relaxing the third condition to \( f(x + iy) = O(e^{\alpha y}) \). Moreover in the half-integral weight case, we assume that the elements in \( H^r_{k,1} \) satisfy Kohnen’s plus-condition, that is, the \( n \)-th Fourier coefficient \( a(n, y) \) of \( f \in H^r_{k,1} \) in the form \( f(z) = \sum_{n \in \mathbb{Z}} a(n, y)e^{2\pi inx} \) vanishes unless \((-1)^{k-1/2}n \equiv 0, 1 \text{ (mod 4)} \). Then it can be easily checked that if \( f \) has polynomial (exponential) growth at \( i\infty \), then it also has at most polynomial (exponential) growth at other cusps 0, 1/2.

First we construct a basis for this space \( H^r_{k,1} \) as an analogue of the author’s earlier work [15] by using the Maass-Poincaré series of half-integral weight \( k \in \mathbb{Z} + 1/2 \),

\[ P_{k,m}(z, s) := \text{pr}_k^+(\sum_{\gamma \in \Gamma_0(4) \setminus \Gamma_0(4)} (\varphi_{k,m}|k\gamma)(z, s)), \quad \text{for } m \in \mathbb{Z}, \]
corresponding to
\[ \varphi_{k,m}(z, s) := \begin{cases} \Gamma(2s)^{-1}(4\pi|y|)^{-k/2} \lambda_{\mathrm{sgn}(m)} \frac{1}{2},s_{-1/2}(4\pi|y|)e^{2\pi i mx} & \text{if } m \neq 0, \\ y^{s-k/2} & \text{if } m = 0, \end{cases} \]

where \( \Pr_k \) is Kohnen’s projection operator \([13]\) and \( M_{\mu,\nu}(y) \) is \( M \)-Whittaker function. Moreover we consider its Laurent expansion
\[ P_{k,m}(z, s) = \begin{cases} \sum_{r \in \mathbb{Z}} F_{k,m,r}(z)(s + \frac{k}{2} - 1)^r & \text{if } k \leq 1/2, \\ \sum_{r \in \mathbb{Z}} G_{k,m,r}(z)(s - \frac{k}{2})^r & \text{if } k \geq 3/2. \end{cases} \]

Actually the coefficients satisfy \( F_{k,m,r}(z) = G_{k,m,r}(z) = 0 \) if \( r < 0 \) except for \( F_{1/2,n^2,-1}(z) \neq 0 \) with \( n \geq 0 \). In terms of these Laurent coefficients, we construct a basis for the space \( H^{r,1}_k \) of polyharmonic weak Maass forms of half-integral weight.

**Theorem 1.2.** Let \( r \geq 1 \) be an integer. For \( k = \lambda_k + 1/2 \) with \( \lambda_k \in \mathbb{Z} \), we define an integer \( \ell_k \) by \( 2\lambda_k = 12\ell_k + k' \) where \( k' \in \{0, 4, 6, 8, 10, 14\} \). Moreover we put
\[ A_k := \begin{cases} 2\ell_k - (-1)^\lambda_k & \text{if } \ell_k \text{ is odd}, \\ 2\ell_k & \text{if otherwise}. \end{cases} \]

For each integer \( m \geq -A_k \) with \( (-1)^{\lambda_k - 1}m \equiv 0, 1(4) \), the unique weakly holomorphic modular form \( f_{k,m}(z) = q^{-m} + \sum_{n > A_k} a_k(m, n)q^n \) is given. Then \( \{f_{k,m}(z) | m \geq -A_k, (-1)^{\lambda_k - 1}m \equiv 0, 1(4)\} \) is a basis of \( H^{1/2}_{k,1} \). Further we have the decompositions \( H^{r,1}_k = H^{r,1}_{k,1} \oplus H^{r,1}_{k,2} \) and \( H^{r-1/2,1}_k = H^{r-1/2,1}_{k,1} \oplus H^{r-1/2,1}_{k,2} \), and they hold that \( H^{0,1}_k = \{0\} \) and

1. For \( k \leq 1/2, \)
   (a) \( \{F_{k,m,r-1}(z) | m > A_k, (-1)^{\lambda_k}m \equiv 0, 1(4)\} \) is a basis for \( H^{r,1}_{k,1} \).
   (b) \( \{F_{k,m,r-1}(z) | m \leq A_k, (-1)^{\lambda_k}m \equiv 0, 1(4)\} \) is a basis for \( H^{r,1}_{k,2} \).
2. For \( k = 1/2, \)
   (a) \( \{F_{1/2,m,r-1}(z) - 8\sqrt{\pi}d(m)F_{1/2,0,r-1}(z) | 0 < m \equiv 0, 1(4)\} \) is a basis for \( H^{r,1}_{1/2,1} \).
   (b) \( \{F_{1/2,m,r-1}(z) | 0 > m \equiv 0, 1(4)\} \cup \{F_{1/2,0,r-2}(z)\} \) is a basis for \( H^{r,1}_{1/2,2} \).
3. For \( k = 3/2, \)
   (a) \( \{G_{3/2,m,r}(z) | 0 < m \equiv 0, 3(4)\} \cup \{G_{3/2,0,r-1}(z)\} \) is a basis for \( H^{r,1}_{3/2} \).
   (b) \( \{G_{3/2,m,r-1}(z) - \frac{1}{\sqrt{\pi}d(m)}G_{3/2,0,r-1}(z) | 0 > m \equiv 0, 3(4)\} \) is a basis for \( H^{r-1/2,1}_{3/2} \).
4. For \( k \geq 5/2, \)
   (a) \( \{G_{k,m,r}(z) | m > A_k, (-1)^{\lambda_k}m \equiv 0, 1(4)\} \) is a basis for \( H^{r,1}_k \).
   (b) \( \{G_{k,m,r-1}(z) | m \leq A_k, (-1)^{\lambda_k}m \equiv 0, 1(4)\} \) is a basis for \( H^{r-1/2,1}_k \).
Here we put
\[
\tilde{F}_{k,m,r-1}(z) := F_{k,m,r-1}(z) + \sum_{A_k < n < 0 \atop (-1)^k n \equiv 0,1(4)} a_k(-m,n)F_{k,n,r-1}(z),
\]
\[
\tilde{G}_{k,m,r}(z) := m^{k-1}G_{k,m,r}(z) - \sum_{0 < n \leq A_k \atop (-1)^k n \equiv 0,1(4)} a_k(-n,m)n^{k-1}G_{k,n,r}(z),
\]
\[
\delta_{\Box}(m) := \begin{cases} 
1 & \text{if } m \text{ is a square number,} \\
0 & \text{otherwise.} 
\end{cases}
\]

In particular, we immediately obtain the following from this theorem.

**Corollary 1.3.** Let \( r \in \mathbb{Z}/2\mathbb{Z} \). The map \( \xi_k : H^r \rightarrow H^{r-1/2} \) is surjective.

**Remark.** Recently, Jeon-Kang-Kim \([12]\) constructed a basis for \( H^1_k \) for \( k \in \mathbb{Z}/2\mathbb{Z} \).

Next we investigate the Fourier coefficients of these polyharmonic weak Maass forms as another goal. We redefine the set \( Q_d \) for any \( d \in \mathbb{Z} \) by
\[
Q_d = \{Q(X,Y) = aX^2 + bXY + cY^2 \mid a, b, c \in \mathbb{Z}, b^2 - 4ac = d\}.
\]
In addition if \( d \) is not a square number, we split \( Q_d = Q_d^+ \sqcup Q_d^- \) into the subsets of forms with \( a > 0 \) and \( a < 0 \). For each fundamental discriminant \( D \) or \( D = 1 \), an integer \( d \) with \( dD < 0 \), and a \( \text{SL}_2(\mathbb{Z}) \)-invariant function \( f \), we define the twisted modular traces by
\[
\text{Tr}_{d,D}(f) := \sum_{Q \in Q_d^+/\text{SL}_2(\mathbb{Z})} \frac{\chi_D(Q)}{w_Q} f(\alpha_Q),
\]
where \( \chi_D \) is a genus character defined by
\[
\chi_D(Q) := \begin{cases} 
\left( \frac{D}{Q} \right), & (a, b, c, D) = 1 \text{ and } (r, D) = 1 \text{ where } Q \text{ represents } r, \\
0, & (a, b, c, D) > 1,
\end{cases}
\]
and well-defined as a character on \( Q_d/\text{SL}_2(\mathbb{Z}) \). On the other hand, for a non-square \( dD > 0 \), we define the twisted traces of cycle integrals by
\[
\text{Tr}_{d,D}(f) := \frac{1}{2\pi} \sum_{Q \in Q_d^+/\text{SL}_2(\mathbb{Z})} \chi_D(Q) \int_{\text{SL}_2(\mathbb{Z}) \setminus S_Q} f(z) \frac{\sqrt{dDdz}}{Q(z,1)},
\]
where \( S_Q \) is the geodesic characterized by the semi-circle \( a|z|^2 + bRe(z) + c = 0 \) oriented counterclockwise if \( a > 0 \) and clockwise if \( a < 0 \), and \( \text{SL}_2(\mathbb{Z}) \) is the stabilizer of \( Q \). Since the cycle integral is invariant under changing \( Q \) to \( -Q \) and it holds that \( \chi_D(-Q) = \text{sgn}(D)\chi_D(Q) \), we easily see that \( \text{Tr}_{d,D}(f) = 0 \) for \( d < 0 \) and \( D < 0 \).

To our purpose, we take any polyharmonic weak Maass form \( f \) of weight 0 as an input function. As shown in our earlier work \([15]\), the Laurent coefficients of the Niebur-Poincaré series \( G_m(z, s) \) span the space \( H^1_0 \) of polyharmonic weak Maass forms. Here this series is defined as the Poincaré series corresponding to
\[
\phi_m(z, s) := \begin{cases} 
2\pi|m|^{1/2}y^{1/2}I_{s-1/2}(2\pi|m|y)e^{2\pi imx} & \text{if } m \neq 0, \\
y^s & \text{if } m = 0,
\end{cases}
\]
where \( I_s(y) \) is the modified \( I \)-Bessel function. It was originally studied by Niebur \[17\]. Moreover we also want to consider the case of \( d < 0, D < 0 \) in some sense, so that we define the modified traces

\[
\tilde{\text{Tr}}_{d,D}(G_m(z,s)) := \left\{ \begin{array}{ll}
\frac{\Gamma(s)}{2^{s-1}\Gamma(s/2)^2} \sum_{Q \in \mathbb{Q}_d^*/\text{SL}_2(\mathbb{Z})_\infty} \chi_D(Q) \int_{S_Q} \phi_m(z,s) \sqrt{dDdz} \frac{Q(z,1)}{Q(z,1)}, & \text{if } dD < 0, \\
\text{if } dD > 0, dD \neq \square, 
\end{array} \right.
\]

according to Duke-Imamo glu-Tó th \[3\] Lemma 7, Proposition 5] and Jeon-Kang-Kim \[11\] (3.7)]. Note that for \( d > 0, D > 0 \) this is equal to \( 2^{1-s}\Gamma(s/2)^{-2}\Gamma(s)\pi\text{Tr}_{d,D}(G_m(z,s)) \). Combining the results of Duke-Imamo glu-Tó th \[3\] Lemma 4], Jeon-Kang-Kim \[11\] Theorem 4.4], \[11\] Theorem 4.1, 4.2], and Möbius inversion, we have

**Proposition 1.4.** Let \( k = \lambda k + 1/2 \). For any \( m \) satisfying \((-1)^{\lambda k}m \equiv 0, 1 \pmod{4} \), the Maass-Poincaré series \( P_{k,m}(z,s) \) has the Fourier expansion of the form

\[
P_{k,m}(z,s) = \varphi_{k,m}(z,s) + \sum_{(-1)^{\lambda k}n=0,1(4)} b_{k,m}(n,s)W_{\kappa,n}(y,s)e^{2\pi i n x},
\]

where

\[
W_{k,n}(y,s) := \left\{ \begin{array}{ll}
\Gamma(s + \text{sgn}(n)\frac{k}{2})^{-1}|n|^{k-1}(4\pi|n|y)^{-k/2}W_{\text{sgn}(n)}e_{\frac{k}{2}}(s-1/2)(4\pi|n|y) & \text{if } n \neq 0, \\
(4\pi)^{k-1}k^{1-s-k/2} & \text{if } n = 0,
\end{array} \right.
\]

and \( W_{\mu,\nu}(y) \) is \( W \)-Whittaker function. In particular, the symmetric property \( b_{k,m}(n,s) = b_{k,n}(m,s) \) holds. Moreover, for a non-zero integer \( d \) and a fundamental discriminant \( D \) or \( D = 1 \) such that \((-1)^{\lambda k}dD \) is not a square, the coefficients \( b_{k,m}(n,s) \) are given by

\[
b_{k,d}((-1)^{\lambda k}m^2D; \frac{s}{2} + \frac{1}{4}) = (-1)^{\lambda k + 1} \left\{ \begin{array}{ll}
\frac{m^2dD}{2^{k+1}} \sum_{n|m} \mu(n) \frac{D^n}{n} \tilde{\text{Tr}}_{(-1)^{\lambda k}d,D}(G_m/n(z,s)) & \text{if } m \neq 0, \\
2^{k+2}k^{2-k-1/4}|d|^{2k+4}k^{-1/4}D^{-k/2}L_D(s)^{-1} & \text{if } m = 0,
\end{array} \right.
\]

and for \( d = m = 0 \),

\[
b_{k,0}(0,s) = (-1)^{\lambda k + 1} \frac{\zeta(4s-2)}{\zeta(4s-1)}.
\]

where \( \mu(n) \) is the classical Möbius function, and \( L_D(s) := \sum_{n=1}^{\infty} \left( \frac{D}{n} \right) n^{-s} \) is the Dirichlet \( L \)-function.

We can easily check that Zagier’s generating functions of \( H(d) \) and \( \text{Tr}_{d}(j-744) \) are realized as \( G_{3/2,0,0}(z) \) and \( G_{3/2,-1,0}(z) = \frac{4}{\sqrt{\pi}}G_{3/2,0,0}(z) \) up to constant multiples, respectively. An important point of this proposition is, in the special case of \( m = 0 \), the coefficient \( b_{k,d}(0, \frac{s}{2} + \frac{1}{4}) \) on the left-hand side does not depend on the choice of \( D \), while the right-hand side depends on \( D \). For the example case of \( F_{1/2,0,0}(z) \), its \( d \)-th Fourier coefficient is calculated as follows.

**Corollary 1.5.** Let \( D \neq 1 \) be a fundamental discriminant. The \( d \)-th \((0 < d \neq \square \) and \( dD \neq \square \) \) Fourier coefficient of the holomorphic part of \( F_{1/2,0,0}(z) \) gives the equation

\[
\text{Tr}_{d,D}(-\log(y|\eta(z)|^4)) = \sqrt{|D|L_D(1)\text{Tr}_{d,1}(1)}.
\]
For $D < 0$, the left-hand side is equal to
\[ \sum_{Q \in \mathbb{Q}_{dD}/\text{SL}_2(\mathbb{Z})} -\chi_D(Q) \frac{\log(\text{Im}(\alpha_Q)|\eta(\alpha_Q)|^4)}{w_Q}, \]
and if $d$ is also a fundamental discriminant, the right-hand side is given by
\[ \sqrt{|D|} L_D(1) \frac{1}{2\pi} \sum_{Q \in \mathbb{Q}_d/\text{SL}_2(\mathbb{Z})} \int_{\text{SL}_2(\mathbb{Z})Q\backslash S_Q} \frac{\sqrt{ddz}}{Q(z,1)} = \sqrt{|D|} L_D(1) \frac{h(d)\log\varepsilon_d}{\pi}, \]
where $h(d)$ is the narrow class number of $\mathbb{Q}(\sqrt{d})$ and $\varepsilon_d$ is the smallest unit $> 1$ with positive norm. This equation was originally established by Kronecker [4, (41)]. (As a good reference, Duke-Imamoğu-Tóth [4] reviewed the history of Kronecker’s limit formula). For instance, $d = 5$ and $D = -4$, we have
\[ \log 1 \cdot 2 \cdot \pi = 2 \cdot \frac{\pi}{4} \cdot \log \frac{3 + \sqrt{5}}{2} \cdot \frac{1}{\pi} = \log \frac{1 + \sqrt{5}}{2}, \]
that is,
\[ \frac{\eta(-1+\sqrt{5}i)}{\eta(\sqrt{5}i)} |^4 = 1 + \sqrt{5}. \]
For a fundamental discriminant $d > 0$ and $D > 0$ with $dD \neq \square$, we have
\[ \sum_{Q \in \mathbb{Q}_{dD}/\text{SL}_2(\mathbb{Z})} \chi_D(Q) \int_{\text{SL}_2(\mathbb{Z})Q\backslash S_Q} -\log(y|\eta(z)|^4) \frac{\sqrt{ddz}}{Q(z,1)} = 2\sqrt{D} L_D(1) h(d)\log\varepsilon_d. \]
This formula was established by Siegel [4, (65)]. Moreover, Theorem 1.1, the case of $D = 1$ and $d < 0$, is given by combining Theorem 1.2 and Proposition 1.3, calculating the Fourier coefficients of the holomorphic part of $G_{3/2,0,1}(z)$. A generalization of Theorem 1.4 is obtained by taking some derivatives of $G_m(z,s)$ in $s$ and taking a limit as $s$ approaches 1. Therefore we see that the generating function of trace-values for polyharmonic weak Maass forms of weight 0 is the holomorphic part of a certain polyharmonic weak Maass form of weight 1/2 or 3/2. Furthermore, the Fourier coefficients for another half-integral weight are realized as the trace-values of shifted polyharmonic weak Maass forms (introduced in [1]).

The paper is organized as follows. First, in Section 2, we give basic properties of Whittaker functions. After that, we review some results on the Maass-Poincaré series, and explain our Proposition 1.4. Finally, through Section 4 to Section 6, we give a proof of Theorem 1.2 and in Section 7 we write down a proof of Corollary 1.5 and Theorem 1.1.

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2. The Whittaker Functions

In this section, we recall some basic properties of Whittaker functions based on [9, 9.22–9.23], [16] Chapter VII. For two parameters $\mu, \nu \in \mathbb{C}$, Whittaker functions $M_{\mu,\nu}(z)$ and $W_{\mu,\nu}(z)$ are the standard solutions to the Whittaker differential equation

(2.1) \[ \frac{d^2w}{dz^2} + \left( -\frac{1}{4} + \frac{\mu}{z} + \frac{1 - 4\nu^2}{4z^2} \right) w = 0. \]

If these parameters $\mu, \nu$ satisfy $\text{Re}(\nu \pm \mu + 1/2) > 0$ and $y > 0$, then Whittaker functions are represented by

\[ M_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}}e^{\frac{1}{2}y} \frac{\Gamma(1 + 2\nu)}{\Gamma(\nu + \mu + \frac{1}{2})\Gamma(\nu - \mu + \frac{1}{2})} \int_{0}^{1} t^{\nu+\mu-\frac{3}{2}}(1-t)^{\nu-\mu-\frac{1}{2}}e^{-yt}dt, \]

\[ W_{\mu,\nu}(y) = y^{\nu+\frac{1}{2}}e^{\frac{1}{2}y} \frac{1}{\Gamma(\nu - \mu + \frac{1}{2})} \int_{1}^{\infty} t^{\nu+\mu-\frac{1}{2}}(t-1)^{\nu-\mu-\frac{1}{2}}e^{-yt}dt. \]

Moreover, we define a modified version of Whittaker function $\mathcal{M}_{\mu,\nu}^{+}(z) := W_{-\mu,\nu}(ze^{\pi i})$ according to the paper [1]. This function $\mathcal{M}_{\mu,\nu}^{+}(z)$ is also a solution of (2.1), and always linearly independent to $W_{\mu,\nu}(z)$. Thus the Whittaker differential equation (2.1) has two linearly independent solutions $W_{\mu,\nu}(z)$ and $\mathcal{M}_{\mu,\nu}^{+}(z)$.

We now explain the Fourier expansion form of polyharmonic weak Maass forms. First we consider the integral depth case. Since any $f \in H_k^{r,1}$ satisfy the modular transformation law, we have $f(z + 1) = f(z)$, that is, $f(z)$ has the following Fourier expansion form

\[ f(z) = \sum_{n \in \mathbb{Z}} a(n, y)e^{2\pi i n z}. \]

By the second condition for the definition of polyharmonic Maass forms, it holds that $\Delta_k f(z) = 0$, where $\Delta_k$ is the hyperbolic Laplacian defined by

\[ \Delta_k := -y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) = -\xi_{2-k} \circ \xi_k. \]

Then each coefficient $a(n, y)$ satisfies a certain $2r$-order linear differential equation. For $k \neq 1$, Andersen-Lagarias-Rhoades [1] gave $2r$ linear independent solutions by

\[ u_{k,n}^{[j]}(y) := y^{-\frac{k}{2}} \frac{\partial^j}{\partial s^j} W_{\text{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}(4\pi |n|y) \bigg|_{s=\frac{k}{2}}, \]

\[ u_{k,n}^{[j]}(y) := y^{-\frac{k}{2}} \frac{\partial^j}{\partial s^j} \mathcal{M}_{\text{sgn}(n)\frac{k}{2}, s-\frac{1}{2}}^{+}(4\pi |n|y) \bigg|_{s=\frac{k}{2}}, \]

for $0 \leq j \leq r - 1$ if $n \neq 0$, and for $n = 0$,

\[ u_{k,0}^{[j]}(y) := \frac{\partial^j}{\partial s^j} y^{1-\frac{k}{2}-s} \bigg|_{s=\frac{k}{2}} = (-1)^j (\log y)^j y^{1-k}, \]

\[ u_{k,0}^{[j]}(y) := \frac{\partial^j}{\partial s^j} y^{s-\frac{1}{2}} \bigg|_{s=\frac{k}{2}} = (\log y)^j. \]
In the special case of $j = 0$, we can express these functions by simple functions,

$$u_{k,n}(y)e^{2\pi i nx} = y^{-\frac{k}{2}}W_{\text{sgn}(n)\frac{k}{2},\frac{k-1}{2}}(4\pi|n|y)e^{2\pi i nx}$$

$$= \begin{cases} (4\pi n)\frac{k}{2}q^n & \text{if } n > 0, \\ (4\pi|n|)\frac{k}{2}\Gamma(1-k, 4\pi|n|y)q^n & \text{if } n < 0, \end{cases}$$

$$u_{k,n}^{[0],+}(y)e^{2\pi i nx} = y^{-\frac{k}{2}}M_{\text{sgn}(n)\frac{k}{2},\frac{k-1}{2}}(4\pi|n|y)e^{2\pi i nx}$$

$$= (4\pi n)\frac{k}{2}q^n,$$  

where $\Gamma(s, y) := \int_y^\infty e^{-t}t^{s-1}dt$ is the incomplete Gamma function. Hence we have

**Proposition 2.1.** [I] Section 3] Let $f(z) \in H_k^{r,1}$ for $k \in \frac{1}{2}\mathbb{Z}$ and $r \in \mathbb{Z}_{>0}$. Then the Fourier-Whittaker expansion of $f(z)$ is given by

$$f(z) = \sum_{n=\infty}^{\infty} \sum_{j=0}^{r-1} \left( c_{n,j}^{-}\overline{u}_{k,n}^{[j],-}(y)e^{2\pi i nx} + c_{n,j}^{+}\overline{u}_{k,n}^{[j],+}(y)e^{2\pi i nx} \right),$$

where $c_{n,j}^{\pm} \in \mathbb{C}$. If $k \in \mathbb{Z} + 1/2$, then it is required to satisfy Kohnen’s plus-condition.

Finally combining with the cusp condition. Corollary A.3 in [I] asserts that $u_{k,n}^{[j],+}(y)$ grows exponentially as $y \to \infty$, while $u_{k,n}^{[j],-}(y)$ decays exponentially as $y \to \infty$ for $n \neq 0$. By the growth condition at $i\infty$, the Fourier coefficients $c_{n,j}^{\pm} = 0$ for almost all indices $(n, j)$. If all coefficients $c_{n,j}^{+} = 0$ for $n \neq 0$, then $f \in H_k^{r}$ strictly. In order to consider the case of half-integral depth, we recall the following lemma.

**Lemma 2.2.** [15] Lemma 2.2] Under the above notations, we have

$$\xi_k(u_{k,n}^{[j],-}(y)e^{2\pi i nx})$$

$$= \begin{cases} j(1-k)u_{2-k,-n}^{[j-1],-}(y)e^{-2\pi i nx} - j(j-1)u_{2-k,-n}^{[j-2],-}(y)e^{-2\pi i nx} & \text{if } n > 0, \\ -u_{2-k,-n}^{[j],-}(y)e^{-2\pi i nx} & \text{if } n < 0, \end{cases}$$

$$\xi_k(u_{k,n}^{[j],+}(y)e^{2\pi i nx})$$

$$= \begin{cases} -u_{2-k,-n}^{[j],+}(y)e^{-2\pi i nx} & \text{if } n > 0, \\ j(1-k)u_{2-k,-n}^{[j-1],+}(y)e^{-2\pi i nx} - j(j-1)u_{2-k,-n}^{[j-2],+}(y)e^{-2\pi i nx} & \text{if } n < 0, \end{cases}$$

$$\xi_k(u_{k,0}^{[j],-}(y)) = (-1)^j\left( j u_{2-k,0}^{[j-1],+}(y) + (1-k)u_{2-k,0}^{[j],+}(y) \right),$$

$$\xi_k(u_{k,0}^{[j],+}(y)) = (-1)^{j-1}j u_{2-k,0}^{[j-1],-}(y),$$

where we put $u_{k,n}^{[j],\pm}(y) = 0$ for any $j < 0$.

By virtue of this lemma, we see that a function $f \in H_k^{r,1}$ with integer $r$ is strictly in $H_k^{r-1/2,1}$ if and only if $c_{n,r-1} = 0$ for all $n \leq 0$ and $c_{n,r-1}^{+} = 0$ for all $n > 0$. 
3. The Fourier expansion of the Maass-Poincaré series

From now, we explain analytic and algebraic properties of the Fourier coefficients of the Maass-Poincaré series based on [3, 10, 11].

3.1. Analytic aspect. For \( k \in \frac{1}{2} \mathbb{Z} \) and integer \( m \in \mathbb{Z} \), let

\[
\phi_{k,m}(z, s) := \begin{cases} 
\Gamma(2s)^{-1}(4\pi|m|y)^{-k/2}M_{\text{sgn}(m)\frac{s-1}{2}}(4\pi|m|y)e^{2\pi imx} & \text{if } m \neq 0, \\
y^{s-k/2} & \text{if } m = 0,
\end{cases}
\]

and define the corresponding Poincaré series by

\[
P_{k,m}(z, s) := \sum_{\gamma \in \Gamma_{\infty}\backslash \Gamma} (\phi_{k,m}|_{k}\gamma)(z, s),
\]

where \( \Gamma := \text{SL}_2(\mathbb{Z}) \) if \( k \in \mathbb{Z} \) and \( \Gamma := \Gamma_0(4) \) if \( k \in \mathbb{Z} + 1/2 \). This series is called the Maass-Poincaré series, and converges absolutely and uniformly on compact subset in \( \Re(s) > 1 \). Moreover we set

\[
P_{k,m}(z, s) := \begin{cases} 
P_{k,m}(z, s) & \text{if } k \in \mathbb{Z}, \\
pr_{k}^+(P_{k,m}(z, s)) & \text{if } k \in \mathbb{Z} + 1/2,
\end{cases}
\]

where \( pr_k^+ \) is Kohnen’s projection operator introduced by Kohnen [13]. For \( k = \lambda_k + 1/2 \), this operator is given by

\[
pr_k^+(g) := (-1)^{\frac{\lambda_k+1}{2}}\frac{1}{2\sqrt{2}}\left( \sum_{\nu \pmod{4}} (g|_{k}A)|_{k}B_{\nu} \right) + \frac{1}{2}g
\]

where

\[
A := \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}, \quad B_{\nu} := \begin{bmatrix} 1 & 0 \\ 4\nu & 1 \end{bmatrix},
\]

and the slash operator for (\( \gamma, \phi(z) \)) is defined by \( g|_{k}(\gamma, \phi(z)) := \phi(z)^{-2k}g(\gamma z) \). We can easily check that this \( P_{k,m}(z, s) \) is an eigenfunction of the hyperbolic Laplacian,

\[
\Delta_k P_{k,m}(z, s) = (s - \frac{k}{2})(1 - \frac{k}{2} - s)P_{k,m}(z, s).
\]

Furthermore, it can be meromorphically continued in \( s \) to \( \Re(s) > 1/2 \) except for possibly finitely many simple poles at points of the discrete spectrum of \( \Delta_k \). (See [7, Section 3]). It is known that the Fourier expansion of \( P_{k,m}(z, s) \) can be expressed in terms of the Kloosterman sums and Bessel functions.

**Proposition 3.1.** [10, Theorem 3.2] If \( k \) and \( m \) are integers and \( \Re(s) > 1 \), then the Poincaré series \( P_{k,m}(z, s) \) has the Fourier expansion

\[
P_{k,m}(z, s) = \phi_{k,m}(z) + \sum_{n \in \mathbb{Z}} c_{k,m}(n, s)W_{k,n}(y, s)e^{2\pi i nx},
\]
where the coefficients \(c_{k,m}(n,s)\) are given by

\[
2\pi i^{-k} \sum_{c > 0} \frac{K(m,n,c)}{c} \times \begin{cases}
|mn|^{\frac{1-k}{2}} J_{2s-1} \left( \frac{4\pi \sqrt{|mn|}}{c} \right) & \text{if } mn > 0, \\
|mn|^{\frac{1-k}{2}} I_{2s-1} \left( \frac{4\pi \sqrt{|mn|}}{c} \right) & \text{if } mn < 0, \\
2^{k-1} \pi^{s+k/2-1} |m+n|^{s-k/2} e^{1-2s} & \text{if } mn = 0, m + n \neq 0, \\
2^{2k-2} \pi^{k-1} \Gamma(2s)(2c)^{1-2s} & \text{if } m = n = 0.
\end{cases}
\]

Here \(I_s(y)\) and \(J_s(y)\) are Bessel-functions, and we put

\[
K(m,n,c) := \sum_{d(c)^* \text{ ad } \equiv 1(c)} e \left( \frac{am + dn}{c} \right),
\]

called the Kloosterman sum.

**Proposition 3.2.** \([10, \text{Theorem 4.4}]\) Let \(k = \lambda_k + 1/2\). For any \(m\) and \(s\) satisfying \((-1)^k m \equiv 0, 1 \, (\text{mod } 4)\) and \(\text{Re}(s) > 1\), the function \(P_{k,m}(z,s) = pr_k^+ (\mathcal{R}, z, s)\) has the Fourier expansion

\[
P_{k,m}(z,s) = \varphi_{k,m}(z,s) + \sum_{(-1)^k n \equiv 0,1(4)} b_{k,m}(n,s) W_{k,n}(y,s) e^{2\pi i n x},
\]

where the coefficients \(b_{k,m}(n,s)\) are given by

\[
2\pi i^{-k} \sum_{c > 0} \left( \frac{4}{c} \right) \frac{\tilde{K}_k(m,n,4c)}{4c} \times \begin{cases}
|mn|^{\frac{1-k}{2}} J_{2s-1} \left( \frac{4\pi \sqrt{|mn|}}{4c} \right) & \text{if } mn > 0, \\
|mn|^{\frac{1-k}{2}} I_{2s-1} \left( \frac{4\pi \sqrt{|mn|}}{4c} \right) & \text{if } mn < 0, \\
2^{k-1} \pi^{s+k/2-1} |m+n|^{s-k/2} e^{1-2s} & \text{if } mn = 0, m + n \neq 0, \\
2^{2k-2} \pi^{k-1} \Gamma(2s)(8c)^{1-2s} & \text{if } m = n = 0.
\end{cases}
\]

Here we put

\[
\tilde{K}_k(m,n,c) := \sum_{d(c)^* \text{ ad } \equiv 1(c)} \left( \frac{c}{d} \right) e^{2\pi i \left( \frac{am + dn}{c} \right)},
\]

called the generalized Kloosterman sum.

We remark two symmetric properties for the coefficients \(b_{k,m}(n,s)\). First, it follows immediately from the above explicit formula that \(b_{k,m}(n,s) = b_{k,n}(m,s)\). The generalized Kloosterman sum satisfies that

\[
\tilde{K}_{3/2}(m,n,c) = -i \tilde{K}_{1/2}(-m,-n,c),
\]

\[
\tilde{K}_{k+2}(m,n,c) = \tilde{K}_k(m,n,c).
\]

Then it is convenient to put

\[
K^+(m,n,4c) := (1 - i) \left( 1 + \left( \frac{4}{c} \right) \right) \tilde{K}_{1/2}(m,n,4c),
\]

and we have
Lemma 3.3. Let $k = \lambda_k + 1/2$. Then
\[
\sqrt{2}i^{-k} \left(1 + \left(\frac{1}{c}\right)^k\right) K_k(m, n, 4c) = \begin{cases} 
(-1)^{\frac{\lambda_k}{2}} K^+(m, n, 4c), & \text{if } \lambda_k : \text{even,} \\
(-1)^{\frac{\lambda_k+1}{2}} K^+(-m, -n, 4c), & \text{if } \lambda_k : \text{odd,}
\end{cases}
\]
and
\[
b_{k,m}(n, s) = (-1)^{\frac{\lambda_k+1}{2}} b_{1/2, (-1)^{\lambda_k} m}((-1)^{\lambda_k} n, s) \times \begin{cases} 
|m n|^{1-2k} \pi^{\frac{2k-1}{2}} |m+n|^{-\frac{2k-1}{4}} & \text{if } m \neq 0, n \neq 0, \\
2^{2k-1} \pi^{k-k-\frac{1}{2}} & \text{if } mn = 0, m+n \neq 0, \\
2^{2k-1} & \text{if } m = n = 0.
\end{cases}
\]

Consequently, our goal is shifted to investigate the coefficients $b_{1/2, m}(n, s)$. This coefficient was studied by Duke-Imamoğlu-Tóth [3].

3.2. Algebraic aspect. For an integer $m$, we consider the Niebur-Poincaré series $G_m(z, s)$ defined by
\[
G_m(z, s) := \sum_{\gamma \in \text{SL}_2(\mathbb{Z})} (\phi_m | 0 \gamma)(z, s),
\]
where $\phi_m(z, s)$ is defined by
\[
\phi_m(z, s) := \begin{cases} 
2\pi|m|^{1/2} y^{1/2} I_{\frac{s}{2}}(2\pi|m| y)e^{2\pi i m z} & \text{if } m \neq 0, \\
y^s & \text{if } m = 0,
\end{cases}
\]
as before. Here there is one remark about the relation to the Maass-Poincaré series. Since it holds that [9, (9.235)]
\[
M_{0, \nu}(z) = 2^{2\nu} \Gamma(\nu + 1) z^{1/2} I_{\nu}(\frac{z}{2}),
\]
for $m \neq 0$, we have
\[
\varphi_{0,m}(z, s) = \Gamma(2s)^{-1} 2^{2s-1} \Gamma(s + 1/2) \pi^{-1/2} \phi_m(z, s).
\]
Moreover by the Legendre duplication formula
\[
\Gamma(2s) = \frac{2^{2s-1} \Gamma(s) \Gamma(s + 1/2)}{\pi^{1/2}},
\]
thus we have
\[
\varphi_{0,m}(z, s) = \Gamma(s)^{-1} \phi_m(z, s) \quad \text{for } m \neq 0.
\]
From now, we consider the modified traces explained in Section [1]
\[
\tilde{\text{Tr}}_{d,D}(G_m(z, s)) := \begin{cases} 
\sum_{Q \in \mathcal{Q}_{dD}^+/\text{SL}_2(\mathbb{Z})} \chi_D(Q) G_m(\alpha_Q, s), & \text{if } dD < 0, \\
\frac{\Gamma(s)}{2^{s-1} \Gamma(s/2)^2} \sum_{Q \in \mathcal{Q}_{dD}^+/\text{SL}_2(\mathbb{Z})} \chi_D(Q) \int_{S_Q} \phi_m(z, s) \sqrt{dD} dz Q(z, 1), & \text{if } dD > 0, dD \neq \Box,
\end{cases}
\]
for a fundamental discriminant $D$ or $D = 1$ and a non-zero integer $d$ such that $dD$ is not a square. By the proof of [3, Proposition 5] and [11] Theorem 4.1, 4.2, for Re$(s) > 1$ and $m \neq 0$, we have

$$
\widetilde{\text{Tr}}_{d,D}(G_m(z,s)) = \pi |2m|^{1/2}|dD|^{1/4} \sum_{n|m} \left( \frac{D}{n} \right) n^{-1/2} \sum_{c>0} K^+(d, \frac{m^2D}{n^2}, 4c) \frac{4c}{4c} 
$$

$$
\times \left\{ \begin{array}{ll}
I_{s-\frac{1}{2}} \left( \frac{\pi}{c} \sqrt{\frac{m^2D}{n^2}} |dD| \right), & \text{if } dD < 0, \\
J_{s-\frac{1}{2}} \left( \frac{\pi}{c} \sqrt{\frac{m^2D}{n^2}} |dD| \right), & \text{if } dD > 0, dD \neq 0
\end{array} \right.
$$

while for $m = 0$ we have

$$
\widetilde{\text{Tr}}_{d,D}(G_0(z,s)) = 2^{s-1}|dD|^{s/2}L_D(s) \sum_{c>0} \frac{K^+(d, 0, 4c)}{(4c)^{s+1/2}},
$$

where $L_D(s) := \sum_{n=1}^{\infty} \left( \frac{D}{n} \right) n^{-s}$ is the Dirichlet $L$-function. Comparing this formula with Proposition 3.2 we can see that

$$
\widetilde{\text{Tr}}_{d,D}(G_m(z,s)) = \sum_{n|m} \left( \frac{D}{n} \right) b_{1,d} \left( \frac{m^2D}{n^2}, \frac{s}{2} + \frac{1}{4} \right), \quad \text{if } m \neq 0,
$$

$$
2^{s-1-\frac{1}{2}} \frac{\pi}{c} |D|^{s} L_D(s) b_{0,d} \left( 0, \frac{s}{2} + \frac{1}{4} \right), \quad \text{if } m = 0.
$$

Furthermore, by using the Möbius inversion, this can be written in the form

$$
b_{1/2,d}(m^2D, \frac{s}{2} + \frac{1}{4}) = \sum_{n|m} \mu(n) \left( \frac{D}{n} \right) \widetilde{\text{Tr}}_{d,D}(G_{m/n}(z,s)), \quad \text{if } m \neq 0,
$$

$$
2^{1-s} \frac{\pi}{c} |D|^{s} L_D(s) \widetilde{\text{Tr}}_{d,D}(G_0(z,s)), \quad \text{if } m = 0.
$$

Finally, we combine this form and Lemma 3.3. For a non-zero integer $d$ and a fundamental discriminant $D$ or $D = 1$ such that $(-1)^{\lambda_k} dD$ is not a square, the coefficients $b_{k,m}(n,s)$ are given by

$$
b_{k,d}((-1)^{\lambda_k} m^2D, \frac{s}{2} + \frac{1}{4}) = (-1)^{\frac{\lambda_k+1}{2}}
$$

$$
\times \left\{ \begin{array}{ll}
|m^2D|^{-\frac{2k-1}{4}} \sum_{n|m} \mu(n) \left( \frac{D}{n} \right) \widetilde{\text{Tr}}_{d,D}((-1)^{\lambda_k} G_{m/n}(z,s)) & \text{if } m \neq 0, \\
2^{k+s} \frac{\pi}{c} |D|^{2k-1} L_D(s) \widetilde{\text{Tr}}_{d,D}((-1)^{\lambda_k} G_0(z,s)) & \text{if } m = 0.
\end{array} \right.
$$

As a remark, for $d = 0$ and $m \neq 0$, we see that $b_{1/2,0}(n,s) = b_{1/2,n}(0,s)$. For the remaining case of $d = m = 0$ is given by

$$
b_{1/2,0}(0,s) = \frac{\pi^{1/2} 2^{5/2 - 6s} \Gamma(2s)}{\zeta(4s - 2) \zeta(4s - 1)}.
$$

(see [3, (2.24)]). This concludes the proof of Proposition 4.4.

4. Polyharmonic Maass forms, Lagarias-Rhoades

In this section, we give a basis for the space $H_k^r$ as a half-integral weight analogue of Lagarias-Rhoades’ work [14]. Let $k \in \mathbb{Z} + 1/2$. We recall that $F_{k,m,r}(z)$ and $G_{k,m,r}(z)$ are defined as

$$
P_{k,m}(z,s) = \begin{cases} 
\sum_{r \in \mathbb{Z}} F_{k,m,r}(z)(s + \frac{r}{2} - 1)^r & \text{if } k \leq 1/2, \\
\sum_{r \in \mathbb{Z}} G_{k,m,r}(z)(s - \frac{r}{2})^r & \text{if } k \geq 3/2.
\end{cases}
$$
Here we remark that the coefficients $F_{k,m,r}(z)$ and $G_{k,m,r}(z)$ vanish if $r < 0$ except for $F_{1/2,n^2,-1}(z) \neq 0$ with $n \geq 0$. Then we see that these coefficients satisfy the following recurrence relations, (see [13 Section 5]). For $m \neq 0$,

$$\xi_k F_{k,m,r}(z) = (4\pi|m|)^{1-k}\left\{ (1-k)G_{2-k,-m,r}(z) + G_{2-k,-m,r-1}(z) \right\},$$

$$\xi_k G_{k,m,r}(z) = (4\pi|m|)^{1-k}F_{2-k,-m,r-1}(z),$$

and for $m = 0$,

$$\xi_k F_{k,0,r}(z) = (1-k)G_{2-k,0,r}(z) + G_{2-k,0,r-1}(z),$$

$$\xi_k G_{k,0,r}(z) = F_{2-k,0,r-1}(z).$$

Our main goal of this section is that we show the following theorem.

**Theorem 4.1.** Let $r \geq 1$ be an integer, and $k = \lambda_k + 1/2$. Then

1. For $k \leq -1/2$, $H^{1/2}_k = \{0\}$ and \{\(F_{k,0,0}(z), \ldots, F_{k,0,r-1}(z)\)\} is a basis for $H^{r+1/2}_k$.
2. For $k = 1/2$, \(F_{1/2,0,-1}(z), \ldots, F_{1/2,0,r-2}(z)\) is a basis for $H^{r-1/2}_1$.
3. For $k = 3/2$, \(H^{3/2}_k = \{0\}\) and \{\(G_{3/2,0,0}(z), \ldots, G_{3/2,0,r-1}\)\} is a basis for $H^{r+1/2}_{3/2}$.
4. For $k \geq 5/2$, we have \(H^r_k = H_{3/2}^r = E^r_k + S_k\), where $S_k$ consists of holomorphic cusp forms on $\Gamma_0(4)$ and $E_k^r$ is spanned by \{\(G_{k,0,0}(z), \ldots, G_{k,0,r-1}\)\}.

4.1. **Weight 1/2 case.** This section is based on the work of Duke-Imamoğlu-Tóth [3]. They showed

$$\lim_{s \to \frac{3}{4}} \left( P_{1/2,0}(z, s) - \frac{3\pi}{s - \frac{3}{4}} \theta(z) \right) = 3\hat{Z}_+(z),$$

where $\theta(z) := \sum_{r \in \mathbb{Z}} q^{r^2}$ and $\hat{Z}_+(z)$ was given by [3 Theorem 4]

$$\hat{Z}_+(z) = \sum_{d > 0} \operatorname{Tr}_{d,1}(1) q^d + \text{(non-holomorphic part)}.$$ 

This formula is an analogue of the Kronecker limit formula, and one example that the generating function of trace-values is realized as the holomorphic part of a polyharmonic weak Maass form. Consider the Laurent expansion

$$P_{1/2,0}(z, s) = \sum_{r=-1}^{\infty} F_{1/2,0,r}(z)(s - \frac{3}{4})^r.$$ 

Then by the important property

$$\Delta^{r+2}_{1/2} F_{1/2,0,r}(z) = \left( s - \frac{1}{4} \right) \left( \frac{3}{4} - s \right) F_{1/2,0}(z, s),$$

we see that $\Delta^{r+2}_{1/2} F_{1/2,0,r}(z) = 0$. Furthermore from Proposition [3-2] for Re($s$) > 1 we have

$$P_{1/2,0}(z, s) = y^{s-1/4} + \sum_{n=0,1(4)} b_{1/2,0}(n, s) W_{1/2,0}(y, s) e^{2\pi inx}.$$ 

By the analytic continuation of $P_{1/2,0}(z, s)$ to $s = \frac{3}{4}$, we can obtain the Fourier expansion form of the function $F_{1/2,0,r}(z)$ from this Fourier expansion. This does not have exponentially growing
terms, thus we see that each coefficient $F_{1/2,0,r}(z)$ is a polyharmonic Maass form of weight $1/2$ and depth $r + 2$ (actually depth $r + 3/2$). Now we show the following lemma.

**Lemma 4.2.** The set $\{F_{1/2,0,-1}(z), \ldots, F_{1/2,0,r-2}(z)\}$ is a basis for $H_{1/2}^r = H_{1/2}^{r-1/2}$.

**Proof.** This proof is based on [10, Proof of Proposition 10.1]. It is known that the space $H_{1/2}^1 = M_{1/2}$ has dimension 1 and is spanned by theta function $\theta(z)$. For every $f(z) \in H_{1/2}^1$, by Proposition 2.1, it has a Fourier expansion of the form

$$f(z) = \sum_{n \equiv 0,1(4)} c_n u_{1/2,n}^+[y] e^{2\pi inx} + c_0^+.$$ 

By the action of $\xi_{1/2}$, we have

$$\xi_{1/2} f(z) = \frac{1}{2} c_0^- u_{3/2,0}^+(y) - \sum_{0 < n \equiv 0,3(4)} c_n^- u_{3/2,n}(y) e^{2\pi inx}$$

$$= \frac{1}{2} c_0^- - \sum_{0 < n \equiv 0,3(4)} c_n^- (4\pi n)^3/4 q^n,$$

that is, $\xi_{1/2} f(z)$ is a holomorphic modular form of weight $3/2$. However there is no non-zero such form, thus we obtain $f$ has depth $1/2$ actually. Similarly we can show $H_{1/2}^r = H_{1/2}^{r-1/2}$ by induction. Next we show $\dim H_{1/2}^{r+1} \leq \dim H_{1/2}^r + 1$. We assume that $\dim H_{1/2}^r = m$ and $\dim H_{1/2}^{r+1} \geq m + 2$. Then we can take $m + 2$ linearly independent functions $f_0(z) = \theta(z), f_1(z), \ldots f_{m+1}(z) \in H_{1/2}^{r+1}$. On the other hand, for $1 \leq i \leq m + 1$, it holds that $0 \neq \Delta_{1/2} f_i(z) \in H_{1/2}^r$. By our assumption of $\dim H_{1/2}^r = m$, there are some constants $\alpha_i \in \mathbb{C}$ such that $\sum_{i=1}^{m+1} \alpha_i \Delta_{1/2} f_i(z) = 0$. Thus $g(z) := \sum_{i=1}^{m+1} \alpha_i f_i(z)$ satisfies $\Delta_{1/2} g(z) = 0$, that is, $g(z) = \theta(z)$ up to a constant multiple. This contradicts our assumption.$\square$

4.2. **Weight 3/2 case.** This section is based on the work of Jeon-Kang-Kim [10]. As we mentioned before, there is no holomorphic modular form of weight $3/2$. They showed

$$P_{3/2,0}(z, \frac{3}{4}) = G_{3/2,0,0}(z) = -12E_{3/2}(z),$$

where $E_{3/2}(z) \in H_{3/2}^1$ is Zagier’s Eisenstein series of weight $3/2$ given by

$$E_{3/2}(z) = \sum_{d \leq 0} H(d) q^{-d} + \frac{1}{16\pi \sqrt{y}} \sum_{n \in \mathbb{Z}} \beta(4\pi n^2 y) q^{-n^2}.$$ 

Here $\beta(s) = \int_1^\infty t^{-3/2} e^{-st} dt$. From Proposition 3.2 we have

$$P_{3/2,0}(z,s) = y^{s-3/4} + \sum_{n \equiv 0,3(4)} b_{3/2,0}(n,s) W_{3/2,n}(y,s) e^{2\pi inx}.$$ 

In the same way as the weight 1/2 case, we can show the following lemma.

**Lemma 4.3.** The set $\{G_{3/2,0,0}(z), \ldots, G_{3/2,0,r-1}(z)\}$ is a basis for $H_{3/2}^r = H_{3/2}^{r+1/2}$. 
4.3. Weight \( k \leq -1/2 \) or \( 5/2 \leq k \) cases. For \( k \geq 5/2 \), the function
\[
P_{k,0}(z, s) = \text{pr}_k^+(\mathcal{P}_{k,0}(z, s))
\]
\[
= \text{pr}_k^+ \left( \sum_{\gamma \in \Gamma_0(4) \setminus \Gamma(4)} y^{s-k/2} |_{k} \gamma \right)
\]
is known as Cohen’s Eisenstein series. Then we see that \( H_{1/2}^1 k \) is spanned by \( G_{k,0,0}(z) = P_{k,0}(z, k/2) \) and holomorphic cusp forms \( S_k \). As for the case of \( k \leq -1/2 \), there is no holomorphic modular form of weight \( k \). The remaining proof is the same as above.

Remark. For any cusp form \( f \in S_k \), a preimage \( F \) such that \( \xi_{2-k} F = f \) is not in \( H_{1/2}^1 k \) but in \( H_{1/2}^1 \). (see [14, Section 6.3]).

5. Duke-Jenkins basis

Duke-Jenkins [5, 6] constructed a standard basis for the space \( H_{1/2}^1 k \) of weakly holomorphic modular forms. For \( k = \lambda_k + 1/2 \), we define an integer \( \ell_k \) by \( 2\lambda_k = 12\ell_k + k' \) where \( k' \in \{0, 4, 6, 8, 10, 14\} \). Moreover we put
\[
A_k := \begin{cases} 
2\ell_k - (-1)^{\lambda_k} \text{ if } \ell_k \text{ is odd,} \\
2\ell_k \text{ if otherwise}
\end{cases}
\]
For each integer \( m \geq -A_k \) satisfying \( (-1)^{\lambda_k - 1} m \equiv 0, 1(4) \), there exists the unique weakly holomorphic modular form \( f_{k,m}(z) \) with Fourier expansion of the form
\[
f_{k,m}(z) = q^{-m} + \sum_{n > A_k} a_k(m, n) q^n.
\]
These coefficients satisfy the duality
\[
a_k(m, n) = -a_{2-k}(n, m).
\]
As for \( k \geq 2 \), it is known that the functions \( G_{k,m,0}(z) \) with \( m > 0 \) span the space \( S_k \) of holomorphic cusp forms. More precisely, Rhoades [18] showed the following lemma.

Lemma 5.1. [18, Theorem 1.21] Let \( k \in \frac{1}{2} \mathbb{Z} \) with \( k \geq 2 \) and \( \mathcal{I} \) be a finite set of positive integers. Then
\[
\sum_{m \in \mathcal{I}} \alpha_m G_{k,m,0}(z) \equiv 0
\]
if and only if there exists a weakly holomorphic modular form of weight \( 2 - k \) with principal part at \( \infty \) equal to
\[
\sum_{m \in \mathcal{I}} \frac{\alpha_m}{m^{k-1}} q^{-m}.
\]

From now on, we reconstruct the Duke-Jenkins basis in terms of \( F_{k,m,r}(z) \) and \( G_{k,m,r}(z) \). First, let \( k = \lambda_k + 1/2 \leq -1/2 \) and \( m > 0 \). Since \( P_{k,m}(z, s) \) converges in \( \text{Re}(s) > 1 \), we immediately see
that

\[
F_{k,-m,0}(z) = P_{k,-m}(z, 1 - k/2) = \varphi_{k,-m}(z, 1 - k/2) + \sum_{(1)^k \mathbb{n} \equiv 0, 1(4)} b_{k,-m}(n, 1 - k/2) W_{k,n}(y, 1 - k/2) e^{2\pi inx}
\]

\[
= q^{-m} - \frac{\Gamma(1 - k, 4\pi my)}{\Gamma(1 - k)} q^{-m} + \sum_{(1)^k \mathbb{n} \equiv 0, 1(4)} b_{k,-m}(n, 1 - k/2) W_{k,n}(y, 1 - k/2) e^{2\pi inx}.
\]

Comparing with the Duke-Jenkins basis \(f_{k,m}(z) = q^{-m} + \sum_{n > A_k} a_k(m, n) q^n\), for \(k \leq -1/2\) and \(m \geq -A_k > 0\), we see that

\[
f_{k,m}(z) - \left\{ F_{k,-m,0}(z) + \sum_{(1)^k \mathbb{n} \equiv 0, 1(4)} a_k(m, n) F_{k,n,0}(z) \right\}
\]

is a harmonic function and bounded on the upper half plane \(\mathbb{H}\). Thus this difference is a constant, that is, equal to 0.

Next we consider the case of \(k = \lambda_k + 1/2 \geq 5/2\). By Lemma \([3]\) and an easy remark

\[
A_k + A_{2-k} = \begin{cases} 
-1 & \text{if } k = \lambda_k + 1/2 \text{ with } \lambda_k \equiv \ell_k(2), \\
-3 & \text{if } k = \lambda_k + 1/2 \text{ with } \lambda_k \not\equiv \ell_k(2),
\end{cases}
\]

we see that \(\{G_{k,m,0}(z) \mid 0 < m \leq A_k\}\) is a basis for \(S_k\). As for \(k \geq 5/2\) and \(-m < 0\),

\[
G_{k,-m,0}(z) = P_{k,-m}(z, k/2)
\]

\[
= \varphi_{k,-m}(z, k/2) + \sum_{(1)^k \mathbb{n} \equiv 0, 1(4)} b_{k,-m}(n, k/2) W_{k,n}(y, k/2) e^{2\pi inx}
\]

\[
= \frac{1}{\Gamma(k)} \left\{ q^{-m} + \sum_{(1)^k \mathbb{n} \equiv 0, 1(4), n > 0} n^{-1} b_{k,-m}(n, k/2) q^n \right\}.
\]

Similarly we have that

\[
f_{k,m}(z) - \Gamma(k) G_{k,-m,0}(z)
\]

is a holomorphic cusp form for weight \(k \geq 5/2\). For \(m = 0\), we have

\[
G_{k,0,0}(z) = P_{k,0}(z, k/2)
\]

\[
= 1 + \sum_{(1)^k \mathbb{n} \equiv 0, 1(4), n > 0} b_{k,0}(n, k/2) \frac{n^{-1}}{(k-1)!} q^n,
\]

that is, \(f_{k,0}(z) - G_{k,0,0}(z)\) is a holomorphic cusp form.

Finally, we consider the cases of \(k = 1/2\) and \(3/2\) separately. In these cases, the coefficients \(b_{k,m}(n, s)\) has a possible pole at \(s = 3/4\). For \(k = 1/2\), Duke-Imamoğlu-Tóth showed

**Lemma 5.2.** \([3]\) Let \(m \equiv 0, 1(4)\).

1. If \(m = 0\), then \(F_{1/2,0,-1}(z) = \frac{3}{4\pi} \theta(z) \in M_{1/2}^1\).
(2) If \(m < 0\), then \(F_{1/2, m, 0}(z) \in M_{3/2}'\).

In the same way, for \(m > 0\) we see that
\[
f_{1/2, m}(z) = F_{1/2, -m, 0}(z),
\]
and \(f_{1/2, 0}(z) = \theta(z) = \frac{4\pi}{3} F_{1/2, 0, -1}(z)\). As for \(k = 3/2\), Jeon-Kang-Kim showed

**Lemma 5.3.** [10] Proposition 5.1] Let \(m \equiv 0, 3(4)\).

1. If \(m > 0\), then \(G_{3/2, m, 0}(z) = 0\).
2. If \(m = 0\), then \(G_{3/2, 0, 0}(z) = -12E_{3/2}(z) \in H_{3/2}'\).
3. If \(m < 0\) and \(-m\) is not a square, then \(G_{3/2, m, 0}(z) \in M_{3/2}'\).
4. If \(m < 0\) and \(-m\) is a non-zero square, then
\[
G_{3/2, m, 0}(z) - \frac{4}{\sqrt{\pi}} G_{3/2, 0, 0}(z) \in M_{3/2}'.
\]

More precisely, for \(m < 0\) but \(-m \not\equiv \square\),
\[
G_{3/2, m, 0}(z) = \frac{2}{\sqrt{\pi}} \left( q^m + \sum_{0 < n \equiv 3(4)} b_{3/2, m}(n, 3/4)n^{1/2}q^n \right),
\]
and for \(-m = \square\),
\[
G_{3/2, m, 0}(z) - \frac{4}{\sqrt{\pi}} G_{3/2, 0, 0}(z) = \frac{2}{\sqrt{\pi}} \left( q^m - 2 + O(q) \right).
\]

In conclusion, we obtain the following proposition.

**Proposition 5.4.** For \(k = \lambda_k + 1/2\) with \(\lambda_k \in \mathbb{Z}\), we define an integer \(\ell_k\) by \(2\lambda_k = 12\ell_k + k'\) where \(k' \in \{0, 4, 6, 8, 10, 14\}\). Moreover we put
\[
A_k := \begin{cases} 2\ell_k - (-1)^{\lambda_k} & \text{if } \ell_k \text{ is odd,} \\ 2\ell_k & \text{if otherwise.} \end{cases}
\]

For each integer \(m \geq -A_k\) with \((-1)^{\lambda_k}m \equiv 0, 1(4)\), the unique weakly holomorphic modular form \(f_{k, m}(z) = q^{-m} + \sum_{n > A_k} a_k(m, n)q^n\) is expressed in terms of the functions \(F_{k, m, r}(z)\), \(G_{k, m, r}(z)\) as follows.

1. For \(k \leq -1/2\),
\[
f_{k, m}(z) = F_{k, -m, 0}(z) + \sum_{A_k < n < 0} a_k(m, n)F_{k, n, 0}(z).
\]

2. For \(k = 1/2\) and \(m > 0\), \(f_{1/2, m}(z) = F_{1/2, -m, 0}(z)\), and \(f_{1/2, 0}(z) = \frac{4\pi}{3} F_{1/2, 0, -1}(z) = \theta(z)\).

3. For \(k = 3/2\),
   a. If \(m\) is not a square, then \(f_{3/2, 2m}(z) = \frac{\sqrt{\pi}}{2} G_{3/2, -m, 0}(z)\).
   b. If \(m\) is a non-zero square, then
\[
f_{3/2, m}(z) = \frac{\sqrt{\pi}}{2} \left( G_{3/2, -m, 0}(z) - \frac{4}{\sqrt{\pi}} G_{3/2, 0, 0}(z) \right).
\]
(4) For \( k \geq 5/2 \), the set \( \{ G_{k,m,0}(z) \mid 0 < m \leq A_k \} \) is a basis for the space \( S_k \) of holomorphic cusp forms.
(a) For \( m > 0 \), \( f_{k,m}(z) - \Gamma(k)G_{k,-m,0}(z) \) is a holomorphic cusp form.
(b) For \( m = 0 \), \( f_{k,0}(z) - G_{k,0,0}(z) \) is a holomorphic cusp form.
(c) For \( m < 0 \), \( f_{k,m}(z) \) is a holomorphic cusp form.

Remark. For \( k \geq 5/2 \), we can also express the Duke-Jenkins basis in terms of our functions \( G_{k,m,r}(z) \) explicitly. For example, Jeon-Kang-Kim [12] gave such expression by using the Petersson inner product.

6. Proof of Theorem 1.2

First, we consider the case of \( k \leq -1/2 \). By Proposition 3.2 and the relation
\[
M_{\mu,\nu}(y) = \frac{\Gamma(1+2\nu)}{\Gamma(\nu-\mu+\frac{1}{2})} e^{\pi i \mu} M_{\mu,\nu}(y) + \frac{\Gamma(1+2\nu)}{\Gamma(\nu+\mu+\frac{1}{2})} e^{-\pi i (\nu-\mu+\frac{1}{2})} W_{\mu,\nu}(y),
\]
for \( m \neq 0 \), we have the following Fourier expansion of the form
\[
F_{k,m,r}(z) = \frac{1}{r!} \frac{\partial^r}{\partial s^r} P_{k,m}(z,s) \bigg|_{s=-k/2} = \sum_{j=0}^r c_{m,j}^{(r)} u_{k,m}^{[j]}(y) e^{2\pi i mx} + \sum_{\lambda k,n \equiv 0,1(4)} \sum_{j=0}^r c_{n,j}^{(r)} u_{k,n}^{[j]}(y) e^{2\pi i nx}.
\]

Since the exponentially growing terms come from \( u_{k,m}^{[j]}(y) \), we see that the set \( \{ F_{k,m,r-1}(z) \mid (1)^k m \equiv 0,1(4) \} \) spans \( H_{k}^{r,1} \oplus H_{k}^{r-1/2,1} \) by Proposition 2.1 and Theorem 4.1. We recall that \( H_{k}^{r,1} = H_{k}^{r-1/2,1} \oplus H_{k}^{r,1} \). Here we put
\[
\tilde{F}_{k,m,r-1}(z) := F_{k,m,r-1}(z) + \sum_{A_k < n < 0} \sum_{\lambda k,n \equiv 0,1(4)} a_k(-m,n) F_{k,n,r-1}(z).
\]

Then we have
\[
\xi_k \circ \Delta_k^{-1} F_{k,m,r-1}(z) = -(4\pi |m|)^{1-k}(k-1)^r G_{2-k,-m,0}(z), \quad \text{for} \ m > A_k, \ (\Longleftrightarrow m \geq -A_{2-k}),
\]
\[
\Delta_k^{-1} \tilde{F}_{k,-m,r-1}(z) = (k-1)^{-r-1} \left( F_{k,-m,0}(z) + \sum_{A_k < n < 0} a_k(m,n) F_{k,n,0}(z) \right)
\]
\[
= (k-1)^{-r-1} f_{k,m}(z), \quad \text{for} \ m \geq -A_k.
\]

From these result, we see that the functions \( F_{k,m,r-1}(z) \) and \( \tilde{F}_{k,m,r-1}(z) \) consist bases for \( H_{k}^{r,1} \) and \( H_{k}^{r-1/2,1} \), respectively. As for \( k \geq 5/2 \) and \( m > A_k \), similarly we have
\[
\xi_k \circ \Delta_k^{-1} G_{k,m,r}(z) = (4\pi)^{1-k}(1-k)^{-r-1} \left( F_{2-k,-m,0}(z) + \sum_{A_{2-k} < n < 0} a_{2-k}(m,n) F_{2-k,n,0}(z) \right)
\]
\[
= (4\pi)^{1-k}(1-k)^{-r-1} f_{2-k,m}(z).
\]
For the remaining cases $k = 1/2$ and $3/2$, we see that

\[
\xi_{1/2} \circ \Delta_{1/2}^{r-1} \left( F_{1/2,m,r-1}(z) - 8\sqrt{m} \delta(m) F_{1/2,0,r-1}(z) \right)
\]

\[
= -\left(-\frac{1}{2}\right)^r (4\pi m)^{1/2} \left( G_{3/2,-m,0}(z) - \frac{4}{\sqrt{\pi}} \delta(m) G_{3/2,0,0}(z) \right).
\]

\[
\Delta_{3/2}^{r-1} \left( G_{3/2,m,r-1}(z) - \frac{4}{\sqrt{\pi}} \delta(m) G_{3/2,0,0}(z) \right)
\]

\[
= \left(-\frac{1}{2}\right)^{r-1} \left( G_{3/2,m,0}(z) - \frac{4}{\sqrt{\pi}} \delta(m) G_{3/2,0,0}(z) \right).
\]

By a similar argument as [14, Section 6], we conclude this proof.

7. Examples: Corollary 1.5 and Theorem 1.1

Finally, as examples, we compute the Fourier coefficients of the holomorphic parts of $F_{1/2,0,0}(z)$ and $G_{3/2,0,1}(z)$. Throughout this section, we assume that a positive integer $d$ and the product $dD$ are not square numbers. By Theorem 1.2, the function $F_{1/2,0,0}(z)$ is a polyharmonic Maass form of weight $1/2$ and depth $3/2$. By (4.1), we have

\[
F_{1/2,0,0}(z) = \frac{\partial}{\partial s} \left( (s - \frac{3}{4}) P_{1/2,0}(z, s) \right) \Big|_{s=3/4}.
\]

By Proposition 1.4, the d-th Fourier coefficient of its holomorphic part is given by

\[
\frac{\partial}{\partial s} \frac{b_{1/2,d}(0,s)}{\Gamma(s - 3/4)} \Big|_{s=\frac{3}{4}} d^{-1/2} = \frac{\partial}{\partial s} \frac{2^{3/2 - 2s} \pi^{s+1/4} |D|^{-s + 1/4} \tilde{\text{Tr}}_{d,D}(G_0(z, 2s - 1/2))}{\Gamma(s - 3/4)} \Big|_{s=\frac{3}{4}}.
\]

Here it is known that

\[
L_D(2s - 1/2)^{-1} = \begin{cases} 
2(s - 3/4) + O((s - 3/4)^2) & \text{if } D = 1, \\
L_D(1)^{-1} + O(s - 3/4) & \text{if } D \neq 1,
\end{cases}
\]

\[
G_0(z, 2s - 1/2) = \frac{\pi}{2\zeta(4s - 1)} \left( \frac{1}{2(s - 3/4)} + 2\gamma - \log 4 - \log(|\eta(z)|^4) + O(s - 3/4) \right),
\]

\[
\Gamma(s - 3/4)^{-1} = (s - 3/4) + \gamma(s - 3/4)^2 + O((s - 3/4)^3),
\]

and if $D \neq 1$, then it holds that $\text{Tr}_{d,D}(1) = 0$. From these properties, we immediately see that if $D = 1$,

\[
\frac{\partial}{\partial s} \frac{b_{1/2,d}(0,s)}{\Gamma(s - 3/4)} \Big|_{s=\frac{3}{4}} d^{-1/2} = \frac{\partial}{\partial s} \frac{2^{3-4s} \pi^{s+5/4} \Gamma(2s - 1/2) \text{Tr}_{d,1}(G_0(z, 2s - 1/2))}{\Gamma(s - 1/2)^2} \frac{\Gamma(2s - 1/2) \text{Tr}_{d,1}(G_0(z, 2s - 1/2))}{\Gamma(s - 3/4)} \Big|_{s=\frac{3}{4}}
\]

\[
= \frac{\pi^2}{d^{1/2} \Gamma(1/2)^2} \frac{\pi}{2\zeta(2)} \text{Tr}_{d,1}(1)
\]

\[
= \frac{3}{\sqrt{d}} \text{Tr}_{d,1}(1),
\]
while if \( D < 0 \), then
\[
\frac{\partial}{\partial s} b_{1/2,d}(0,s) \Big|_{s=\frac{d}{4}} d^{-1/2} = \frac{\pi |D|^{-1/2}}{d^{1/2}} \frac{\pi}{2\zeta(2)L_D(1)} \text{Tr}_{d,D}(-\log(\eta(z)\vert^4))
\]
\[
= \frac{3}{\sqrt{d|D|L_D(1)}} \text{Tr}_{d,D}(-\log(\eta(z)\vert^4)),
\]
and if \( D > 1 \), then
\[
\frac{\partial}{\partial s} b_{1/2,d}(0,s) \Big|_{s=\frac{d}{4}} d^{-1/2} = \frac{\partial}{\partial s} \frac{2^{3-4s} s^{+5/4} D^{-s+1/4}}{d^{1/2}L_D(2s-1/2)} \frac{\Gamma(2s-1/2)}{\Gamma(s-1/4)^2} \frac{\text{Tr}_{d,D}(G_0(z,2s-1/2))}{\Gamma(s-3/4)} \Big|_{s=\frac{d}{4}}
\]
\[
= \frac{\pi^2 D^{-1/2}}{d^{1/2}L_D(1)} \frac{\pi}{2\zeta(2)L_D(1)} \text{Tr}_{d,D}(-\log(\eta(z)\vert^4))
\]
\[
= \frac{3}{\sqrt{d|D|L_D(1)}} \text{Tr}_{d,D}(-\log(\eta(z)\vert^4)).
\]

Comparing the \( d \)-th coefficient, we have Corollary 1.5.

As another application, we next consider Theorem 1.1. Similar as above, we compute the holomorphic part of \( G_{3/2,0,1}(z) \) in the case of \( D = 1 \). By the definition, we have
\[
G_{3/2,0,1}(z) = \frac{\partial}{\partial s} P_{3/2,0}(z,s) \Big|_{s=3/4}.
\]

By Proposition 1.4 its holomorphic part is given by
\[
\sum_{0 < d \equiv 0,3(4)} d^{1/2} \frac{\partial}{\partial s} b_{3/2,0}(d,s) \Big|_{s=\frac{d}{3}} \frac{q^d}{\zeta(2s-1/2)\Gamma(s+3/4)} \frac{\zeta(2s-1/2)\Gamma(s+3/4)}{s=\frac{d}{3}} \frac{q^d}{\zeta(2s-1/2)\Gamma(s+3/4)}
\]
\[
= - \sum_{0 < d \equiv 0,3(4)} \frac{\partial}{\partial s} \frac{25/2-2s+s^{3/4}\zeta(2s-1/2)\Gamma(s+3/4)}{\zeta(2s-1/2)\Gamma(s+3/4)} \frac{q^d}{\zeta(2s-1/2)\Gamma(s+3/4)}
\]
\[
\frac{25/2-2s+s^{3/4}}{\zeta(2s-1/2)\Gamma(s+3/4)} = 8\pi(s-3/4) - 8\pi \left( 2\gamma + \log 4 - \log \pi + \psi(3/2) \right) (s-3/4)^2 + O((s-3/4)^3),
\]
\[
G_0(z,2s-1/2) = \frac{\pi}{2\zeta(4s-1)} \left( \frac{1}{2\pi(s-3/4)} + 2\gamma - \log 4 - \log(\eta(z)\vert^4) + O(s-3/4) \right)
\]
\[
= \frac{3}{2\pi(s-3/4)} + \frac{3}{\pi} \left( 2\gamma - \log 4 - \log(\eta(z)\vert^4) - \frac{12\zeta'(2)}{\pi^2} \right) + O(s-3/4).
\]

Then we get the Fourier expansion of the form
\[
- \sum_{0 < d \equiv 0,3(4)} \frac{\partial}{\partial s} \frac{25/2-2s+s^{3/4}\zeta(2s-1/2)\Gamma(s+3/4)}{\zeta(2s-1/2)\Gamma(s+3/4)} \frac{q^d}{\zeta(2s-1/2)\Gamma(s+3/4)} \Big|_{s=\frac{d}{3}}
\]
\[
= \sum_{0 < d \equiv 0,3(4)} \left[ -24 Tr_{-d,1}(-\log(\eta(z)\vert^4)) + 12 \left( 2 - 3\gamma + \log \frac{16}{\pi} + \frac{24\zeta'(2)}{\pi^2} \right) Tr_{-d,1}(1) \right] q^d.
\]
Here we use the formula for the digamma function \( \psi(3/2) = -\gamma - \log 4 + 2 \). Recall that the generating function

\[
1 - 12 \sum_{0 < d \equiv 0, 3(4)} \text{Tr}_{d,1}(1) q^d
\]

is the holomorphic part of \( G_{3/2,0,0}(z) \). Therefore the function

\[
G_{3/2,0,1}(z) + \left(2 - 3\gamma + \log \frac{16}{\pi} + \frac{24\zeta'(2)}{\pi^2}\right) G_{3/2,0,0}(z)
\]

is also polyharmonic Maass form of weight 3/2 and depth 2, and its holomorphic part is given by

\[
\left(2 - 3\gamma + \log \frac{16}{\pi} + \frac{24\zeta'(2)}{\pi^2}\right) - 24 \sum_{0 < d \equiv 0, 3(4)} \text{Tr}_{d,1}(-\log (y|\eta(z)|^4)) q^d.
\]

This concludes the proof of Theorem 1.1 and gives the suitable constant coefficient.

**References**

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