Explicit Implicit Function Theorem for All Fields

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Abstract
We give an explicit implicit function theorem for formal power series that is valid for all fields, which implies in particular Lagrange inversion formula and Flajolet-Soria coefficient extraction formula known for fields of characteristic 0.

Theorem 1. Let $K$ be an arbitrary field. If $P(X, Y) \in K[[X, Y]]$ and $f(X) \in K[[X]]$ are such that $f(0) = 0$, $P(X, f(X)) = f(X)$ and $P_Y(0, 0) = 0$. Then
\[
[X^n]f = \sum_{m \geq 1} [X^n Y^{m-1}] (1 - P_Y(X, Y)) P^m(X, Y).
\]
If the characteristic of $K$ is 0, we also have the following form
\[
[X^n]f = \sum_{m \geq 1} \frac{1}{m} [X^n Y^{m-1}] P^m(X, Y).
\]

Remark 1. The conditions $P(X, f(X)) = 0$ and $f(0) = 0$ imply that $P(0, 0) = 0$. As $P_Y(0, 0)$ is also 0, the sums in both expressions of $[X^n]f$ are finite.

Remark 2. When $P(X, Y) = X \phi(Y)$, where $\phi(X) \in K[[X]]$ and $\phi(0) \neq 0$, we obtain the Lagrange inversion formula
\[
[X^n]f = [Y^{n-1}] (\phi(X)^n - Y \phi'(X) \phi(X)^{n-1}).
\]
If the characteristic of $K$ is 0, we also have the following form
\[
[X^n]f = \frac{1}{n} [Y^{n-1}] \phi(Y)^n.
\]

Remark 3. When $P(X, Y)$ is a polynomial in $X$ and $Y$, we obtain a generalisation of Flajolet-Soria coefficient extraction formula [2].

Definition 1. Let $P(X, Y) \in K[[X, Y]]$, 
\[
P(X, Y) = \sum_{j=0}^{\infty} a_j(x) Y^j,
\]
where $a_j(x) \in K[[X]]$. We define
\[ P^{[m]}(X, Y) = \sum_{j=m}^{\infty} \binom{j}{m} a_j(x) Y^{j-m}. \]

for \( m \in \mathbb{N} \).

The motivation of the definition of \( P^{[m]} \) is to avoid the factorials in the denominators in the Taylor series, which do not make sense in positive characteristic. Once this obstacle is circumvented, the Taylor formula works as expected.

**Proposition 1.** Let \( K \) be an arbitrary field. Let \( P(X, Y) \in K[[X, Y]] \) and \( f(X) \in K[[X]] \) with \( f(0) = 0 \). Then

\[ P(X, Y) = \sum_{m=0}^{\infty} (Y - f(X))^m P^{[m]}(X, f(x)). \quad (*) \]

**Proof.** Let \( P(X, Y) = \sum_{j=0}^{\infty} a_j(X) Y^j \) with \( a_j(X) \in K[[X]] \) for \( j \in \mathbb{N} \). We prove that for all \( k \in \mathbb{N} \), the coefficient of \( Y^k \) in the left side and right side of (*) is equal. Indeed, we have

\[
\begin{align*}
[Y^k] & \left( \sum_{m=0}^{\infty} (Y - f(X))^m P^{[m]}(X, f(x)) \right) \\
& = \sum_{m=k}^{\infty} \binom{m}{k} (-f(X))^{m-k} \sum_{j=m}^{\infty} \binom{j}{m} a_j(x) f(X)^{j-m} \\
& = \sum_{j=k}^{\infty} a_j(X) f(X)^{j-k} \sum_{m=k}^{\infty} \binom{j}{m} \binom{m}{k} (-1)^{m-k} \\
& = \sum_{j=k}^{\infty} a_j(X) f(X)^{j-k} \sum_{m=k}^{\infty} \binom{j}{j-m-m-k-k} (-1)^{m-k} \\
& = a_k(X).
\end{align*}
\]

We have the last equality because \( \sum_{m=k}^{j} \binom{j}{j-m-m-k-k} (-1)^{m-k} = 1 \) if \( j = k \) and 0 if \( j > k \). This is because we have the multinomial expansion

\[
(a + b + c)^j = \sum_{k \leq m \leq j} \binom{j}{j-m-m-k-k} a^{j-m} b^{m-k} c^k.
\]

When we take \( a = 1, b = -1 \), the identity becomes

\[
c^j = \sum_{k=0}^{j} \left( \sum_{m=k}^{j} \binom{j}{j-m-m-k-k} (-1)^{m-k} \right) c^k.
\]

Seeing this as a polynomial identity in the variable \( c \) gives us the desired result. \( \Box \)
The following corollary is immediate.

**Corollary 1.** Let $K$ be an arbitrary field. Let $Q(X,Y) \in K[[X,Y]]$ and $f(X) \in K[[X]]$ be such that $f(0) = 0$ and $Q(X,f(X)) = 0$. Then there exists $R(X,Y) \in K[[X,Y]]$ such that $Q(X,Y) = (Y - f(X))R(X,Y)$.

**Definition 2.** For the formal power series in $K((X_1, \ldots, X_m))$

$$P(X_1, X_2, \ldots, X_m) = \sum_{n_i > -\mu} a_{n_1 n_2 \ldots n_m} X_1^{n_1} X_2^{n_2} \ldots X_m^{n_m}$$

its (principal) diagonal $\mathcal{D}f(t)$ is defined as the element in $\kappa((T))$

$$\mathcal{D}P(T) = \sum a_{n_1 \ldots n} T^n.$$

The following proposition is a generalization of Proposition 2 from [1], the only difference in the proof is in the first step where we use Corollary 1 to factorize $Q(X,Y)$.

**Proposition 2.** Let $K$ be an arbitrary field. Let $Q(X,Y) \in K[[X,Y]]$ and $f(X) \in K[[X]]$ be such that $f(0) = 0$, $Q(X,f(X)) = 0$ and $Q'_Y(0,0) \neq 0$, then

$$f(X) = \mathcal{D} \left( Y^2 \frac{Q'_Y(XY,Y)}{Q(X,Y)} \right).$$

**Proof.** Using Corollary 1 we can write $Q(X,Y) = (Y - f(X))R(X,Y)$ with $R(X,Y) \in K[[X,Y]]$. We have $R(0,0) \neq 0$ because $f(0) = 0$ and $Q'_Y \neq 0$. Then

$$\frac{1}{Q(X,Y)} Q'_Y(X,Y) = \frac{1}{Y - f(X)} + \frac{R'_Y(X,Y)}{R(X,Y)}.$$

Replacing $X$ by $XY$ and multiplying by $Y^2$ we get

$$\mathcal{D} \left( Y^2 \frac{Q'_Y(XY,Y)}{Q(X,Y)} \right) = \mathcal{D} \left( \frac{Y^2}{Y - f(X)} \right) + \mathcal{D} \left( Y^2 \frac{R'_Y(XY,Y)}{R(X,Y)} \right). \quad (†)$$

For the first term on the right side of (†) we have

$$\mathcal{D} \left( \frac{Y^2}{Y - f(X)} \right) = \mathcal{D} \left( \frac{Y}{1 - \frac{1}{Y} f(X)} \right) = \mathcal{D} \left( \sum_{n=0}^{\infty} Y^{-n+1} f(X)^n \right) = \mathcal{D} \left( f(X) \right) = f(X).$$

For the second term, as $R(0,0) \neq 0$, $\frac{R'_Y(XY,Y)}{R(X,Y)}$ is a power series in $XY$ and $Y$, so when we multiply this by $Y^2$ there is no diagonal term.

$\square$
Proof of Theorem. Let the power series $Q(X, Y)$ be defined as $Q(X, Y) = P(X, Y) - Y$, then $Q_Y(0, 0) = P_Y(0, 0) - 1 \neq 0$, and $Q(X, f(X)) = P(X, f(X)) - f(X) = 0$. According to Proposition

$$f = \mathcal{D}\{Y^2Q_Y(XY, Y)/Q(XY, Y)\}$$
$$= \mathcal{D}\{Y^2(P'_Y(XY, Y) - 1)/(P(XY, Y) - Y)\}$$
$$= \mathcal{D}\{Y(1 - P'_Y(XY, Y))/(1 - \frac{P(XY, Y)}{Y})\}$$
$$= \mathcal{D}\{Y(1 - P'_Y(XY, Y))(1 + \sum_{m \geq 1} (\frac{P(XY, Y)}{Y})^m)\}.$$

We have the last equality due to the fact that $P_Y(0, 0) = 0$, $\frac{P(XY, Y)}{Y}$ has no constant term, and therefore $1/(1 - \frac{P(XY, Y)}{Y}) = 1 + \sum_{m \geq 1} (\frac{P(XY, Y)}{Y})^m$.

As in each term of $Y(1 - P'_Y(XY, Y))$ the power of $Y$ is larger than that of $X$, it cannot contribute to the diagonal. Therefore,

$$f_n = [X^nY^m]Y(1 - P'_Y(XY, Y))(1 + \sum_{m \geq 1} (\frac{P(XY, Y)}{Y})^m)$$
$$= [X^nY^m]Y(1 - P'_Y(XY, Y))(\sum_{m \geq 1} (\frac{P(XY, Y)}{Y})^m)$$
$$= \sum_{m \geq 1} [X^nY^{m-1}](1 - P'_Y(X, Y))P(X, Y)^m.$$

\[\square\]

References

[1] H. Furstenberg, “Algebraic functions over finite fields”, J. Algebra 7, 271–277 (1967).

[2] M. Soria, Thèse d’habilitation (1990), LRI, Orsay.