A DYNAMIC LOT SIZING MODEL WITH
PRODUCTION-OR-OUTSOURCING DECISION UNDER
MINIMUM PRODUCTION QUANTITIES

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ABSTRACT. In the real-world production process, the firms need to determine
the optimal production planning under minimum production quantity con-
straint in order to achieve economies of scale. However, the inventory cost will
hugely increase when there is a very large amount of production in a period and
also a large amount of total demands for the next few periods. This paper con-
siders a single-item dynamic lot sizing problem with production-or-outsourcing
decisions. In each period, the production level cannot be lower than a given
quantity in order to make full use of resources, but the outsourcing is unre-
stricted. The demands in a period can be backlogged. The production and
outsourcing costs are fixed-plus-linear, and the inventory and backlogging costs
are linear. We establish a mathematical programming model according to the
real problem in the firm. We explore some structural properties of the opti-
mal solution and use them to develop a dynamic programming algorithm to
solve the proposed problem. We further present a special case with stationary
production and outsourcing costs which can be solved with reduced computa-
tional complexities. In the end, we use three numerical instances to show how
to obtain the optimal solutions by using the dynamic programming algorithm.
Furthermore, we show that the policy of backlogging or outsourcing can reduce
the total cost.

1. Introduction. In the real world manufacturing industry, a production line al-
ways begins to produce resulting in high preparation (fixed) costs. Eventually,
the preparation costs are indeed fixed regardless of whether the firms produce one
product or multiple products. In this regard, the firms often set a lower bound for
the production in the production process to achieve economies of scale. In violent
market competition, make-to-order production has become one of the important op-
erations in the firm. Thus, the firms determine the production planning according
to the demands in current and future periods, because inventory cost will sharply
increase when the total demands are low and the lower bound is higher.

The minimum production quantities (lower bound for the production) for man-
ufacturing may be an economically induced restriction \([45]\). In lot-size production,
many small lots might not be cost effective, so that the minimum lot-size can be viewed as the break-even production amount [17]. Recently, applications of lot sizing models with minimum order quantities have been increasingly used in real world manufacturing, such as [21, 44, 26, 39]. Hence, the importance of considering minimum order quantities in the production process is obvious. The crucial problem for the firm is reducing the inventory cost when the total demands of current and future periods are lower. To deal with this problem, outsourcing, regarded as one of the most effective and popular tools, has been accepted to improve the production efficiency and reduce the operational costs [23, 50]. Furthermore, the dynamic lot sizing (DLS) model with different “strategies” is employed to decrease the total operational costs, for example, backlogging [1, 15].

To characterize the minimum production quantities and make-to-order in real life manufacturing, this paper aims to develop a dynamic programming algorithm (DP) to solve the dynamic lot sizing DLS problem with outsourcing and backlogging under minimum production quantities. We establish a single-item DLS model which includes the production-or-outsourcing and backlogging decisions in each period. Demands over \( T \) periods are deterministic, but are time-varying, and can be satisfied by production, outsourcing or inventory. The demands can also be backlogged. In each period, the production level cannot be lower than a given capacity and the outsourcing is unrestricted. The cost functions for production and outsourcing are fixed-plus-linear, and the inventory and backlogging cost function is linear. The problem is to determine the quantities to be produced and outsourced in each period so that all the demands are satisfied finally at a minimum cost. The proposed approach can guide the firm to make an accurate decision on production and outsourcing, i.e., when to produce (outsourcing) and how many to produce (outsourcing). The accurate decisions can help the firm reduce the total operational costs.

The DLS problem has received a significant amount of academic and practical attention since it was introduced by Wagner and Whitin [46]. The basic model is known as W-W model. There has been an abundance of literature on a variety of extensions of the W-W model since the seminal work of Wagner and Whitin [46]. The generalizations include DLS models with perishable inventory, learning in setup, time windows, remanufacturing, and production capacity limitation. A recent survey is summarized by Brahimi et al. [8] for lists of research papers on the DLS models.

As already stated, it is very common that the production level cannot be lower than a given quantity in order to achieve economies of scale in the real world. Göller and Voß [17] summarize the reasons for including minimum lot-sizes within production planning in practice. Anderson and Cheah [6] first consider a multi-item DLS problem where the production capacity has lower and upper limits. Constantino [13] solves the DLS problem with minimum production quantities by defining the strong valid inequalities. With the same method, Park and Klabjan [37] use the valid inequalities approach to solve the problem and improve the computational efficiency. Okhrin and Richter [29, 30] analyze the single-item DLS problem with minimum production quantities under linear production cost. They prove some necessary conditions and apply the DP technique to develop an exact algorithm to solve the problem in \( O(T^3) \) time. Hellion et al. [18, 19] study the single-item DLS problem with concave production and storage costs where the production capacity has lower and upper limits. They present an optimal algorithm in \( O(T^6) \) time.
Hellion et al. [20] continue studying the single-item capacitated DLS problem with concave production and storage costs, considering minimum production quantities and dynamic time windows. Based on some structural properties of the optimal solution, they present a DP algorithm, which is bounded in $O(T^5)$. For other research on DLS problems with minimum production quantities, see [21, 44, 26, 39]. All of the DLS models with minimum production quantities do not consider the outsourcing policy or backlogging. In practice, if the minimum production quantities are very large and the demands are satisfied by their own production for the firm, then the inventory cost will hugely increase. Therefore, the firm can use the outsourcing policy to satisfy the demands or let the demands be backlogged to decrease the inventory costs.

Nowadays, outsourcing has become a popular way to improve the efficiency of inventory systems and reduce the operational costs in practice. Make-or-buy (production-or-outsourcing) is one of key problems faced by the purchasing departments of plenty of large manufacturing companies [23]. Chu and Chu [11, 12] address a real-life single-item DLS problem in a refinery for crude oil procurement. They consider two policies under bounded inventory, with one policy. In the first the demands in a period can be backlogged, and in the other the demands in a period can be outsourced. They develop a DP algorithm to solve the problem in polynomial time under different cost functions. Chu et al. [10] address a production planning problem arising in the manufacture of luxury goods. They model the problem as a single item DLS problem with backlogging, outsourcing and inventory capacity. The production cost function is fixed-plus-linear, and the holding, backlogging and outsourcing cost functions are assumed to be linear. They show that the problem can be solved in $O(T^4 \log T)$. Zhong et al. [50] consider a DLS problem with inventory bounds and outsourcing. They develop DP algorithms to solve the problem under different cost functions. From the modeling point of view, the DLS model with linear outsourcing cost function shares an important similarity with the DLS model with stockout (or lost sales). The DLS models with stockout (or lost sales) have been extensively studied [1, 3, 4, 7, 15, 27, 42, 43, 47]. In these studies, the stockout cost function is linear. We assume that the outsourcing cost function is fixed-plus-linear, thus, these models with stockout can be regarded as a special case for our outsourcing model.

There are two main goals in production operation, minimizing total cost and maximizing service level. While the two goals are always contradictory, it is the high service level that meets the demands timely and efficiently, while maximizing the meeting of demands will increase the inventory level and then will increase the holding cost, subsequently increasing the total cost. Given this, the demands may not always be satisfied timely and usually may be backlogged in reality. Since there are plenty of studies on the DLS models with backlogging [14, 16, 21, 22, 25, 49], we do not offer any discussion on this area.

The early research of the DLS models with minimum production quantities mainly focuses on the design of effective algorithms to obtain optimal solution, but these existing methods only include the production decision without considering the production outsourcing and backlogging. Although there are many improvements for the DLS problem with minimum production quantities in the early work, they neglect some realistic factors for the operation in the firm. Therefore, we study a DLS model with minimum production quantities by considering outsourcing and backlogging strategies. When minimum production quantities or the fixed costs of
production are larger and the demands are relatively smaller, the firms can adopt
the policy of outsourcing to satisfy the demands in order to decrease the inventory
costs. In addition, the firm can adopt a backlogging policy to decrease the inven-
tory costs, that is the demands in a period will be satisfied in one future period.
Of course, the combination of outsourcing and backlogging policies can reduce the
total operation costs hugely. Furthermore, the combination policies can also bal-
ace the operation costs and service level. To summarize, in the case of minimum
production quantities, the combination of outsourcing and backlogging policies are
more realistic in real life manufacturing and can provide more accurate decision
supports for production operations.

The remainder of this paper is organized as follows. We establish the DLS
model with outsourcing and backlogging under the minimum production quantities
constraint in Section 2. In Section 3, we explore several structural properties and
use them to develop a forward algorithm to solve the problem. A special case is
discussed with stationary production and outsourcing costs in Section 4. We use
three numerical instances to show how to obtain the optimal solutions by using the
dynamic programming algorithm and give managerial insights in Section 5. Section
6 concludes the paper with ideas for future research.

2. Model formulation. Let us consider a planning horizon of $T$ periods. De-
note $d_t$ as the known and time-varying demands for a single product in period $t$,
$1 \leq t \leq T$. At each period $t$, the demands $d_t$ will be satisfied by production and/or
by outsourcing at the beginning of period $t$, and/or by inventory available at the
end of period $t - 1$. For $1 \leq t \leq T$, let $X_t(O_t)$ be the production (outsourcing)
amount in period $t$. The production amount cannot be lower than the minimum
production quantities $X_{\min}$ and the outsourcing amount is unrestricted. Without
loss of generality, the production (outsourcing) is assumed to instantaneously oc-
cur at the beginning of each production (outsourcing) point. Period $t$ is called a
production point (period) if $X_t > 0$. Similarly, period $t$ is called an outsourcing
point (period) if $O_t > 0$. Let $C_t^P(X_t)$ be the cost to produce $X_t$ units and $C_t^O(O_t)$
be the cost to outsource $O_t$ units. The production (outsourcing) cost function is
fixed-plus-liner, that is

$$C_t^P(X_t) = \begin{cases} K_t + c_t X_t & \text{if } X_t > 0 \\ 0 & \text{if } X_t = 0 \end{cases}$$ (1)

and

$$C_t^O(O_t) = \begin{cases} k_t + o_t O_t & \text{if } O_t > 0 \\ 0 & \text{if } O_t = 0 \end{cases}.$$ (2)

The cost parameters $K_t$, $c_t$, $k_t$ and $o_t$, together with $h_t$ and $b_t$, are summarized in
Table 1.

According to the study of Lee and Zipkin [23], the unit outsourcing cost is larger
than the unit production cost and the fixed outsourcing cost is smaller than the
fixed production cost. We follow this assumption, that is $o_t > c_t$ and $K_t > k_t$,
$1 \leq t \leq T$.

Let $I_t$ be the inventory held in period $t$, which excludes the quantities used to
satisfy $d_t$ at the beginning of period $t$. Inventory cost is incurred on inventory
carried from one period to the next period. Note that a negative $I_t$ denotes the
quantities of backlogged demand in period $t$. We assume that the inventory cost
function is linear, that is

\[ H_t(I_t) = \begin{cases} h_t I_t & \text{if } I_t > 0 \\ b_t I_t & \text{if } I_t < 0 \\ 0 & \text{if } I_t = 0 \end{cases} \]  

(3)

Let \( B_t \) be the units of backlogging in period \( t \), that is \( B_t = -I_t \) if \( I_t < 0 \).

**Table 1. The basic notation**

| \( K_t \) | fixed (setup) costs for production in period \( t \), \( t = 1, 2, \ldots, T \); |
| \( c_t \) | per unit production cost in period \( t \), \( t = 1, 2, \ldots, T \); |
| \( k_t \) | fixed (setup) costs for outsourcing in period \( t \), \( t = 1, 2, \ldots, T \); |
| \( o_t \) | per unit outsourcing cost in period \( t \), \( t = 1, 2, \ldots, T \); |
| \( h_t \) | per unit holding cost in period \( t \), \( t = 1, 2, \ldots, T \); |
| \( b_t \) | per unit backlogging cost in period \( t \), \( t = 1, 2, \ldots, T \); |

Define a binary variable \( \delta(x) \),

\[ \delta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases} \]  

(4)

Based on the definitions and assumptions, our problem with outsourcing and minimum production quantities can be formulated as the following mathematical program:

\[
\text{Min } \sum_{t=1}^{T} (K_t \delta(X_t) + k_t \delta(O_t) + c_t X_t + o_t O_t + h_t I_t + b_t B_t) \\
\text{Subject to:}
\]

\[
I_{t-1} + X_t - I_t + O_t - B_{t-1} - B_t = d_t \quad (1 \leq t \leq T) \\
X_t, I_t, O_t, B_t \geq 0 \quad (1 \leq t \leq T) \\
X_t \geq X_{\text{min}} \quad (1 \leq t \leq T) \\
I_0 = B_0 = B_T = 0
\]

(5)

(6)

(7)

(8)

(9)

The objective of our problem aims to minimize the total production, outsourcing and inventory cost. Constraint (6) represents the well-known material balance equation. Constraint (7) is the nonnegative constraint. Constraint (8) means that the minimum production quantities are satisfied. Without loss of generality, we assume zero inventory at the beginning of period 1 and no backlogging at the beginning of period 1 and at the end of period \( T \) in Constraint (9).

Period \( t \) is called a regeneration point (period) if \( 0 \leq I_t < d_t \).

3. **Some properties and forward algorithm.** In this section, we state several important properties of the optimal solution under time-varying production and outsourcing costs. Then, they will be used to develop a forward algorithm to solve the problem.

A property similar to the following has been proven in Zangwill [49], hence its proof is omitted.

**Property 1.** There exists an optimal solution such that:

(A) \( I_t B_t = 0 \) for each \( t, 1 \leq t \leq T \).

(B) \( X_t B_t = 0 \) and \( O_t B_t = 0 \) for each \( t, 1 \leq t \leq T \).
Part (A) is the important orthogonal condition. Part (B) says that if we produce or outsource in a period \( t \), then there is no backlogging in that period.

**Property 2.** [6]. There is an optimal solution such that:

(A) \( I_{t-1}X_t(X_p - X^\text{min}) = 0 \) for each \( t \) and \( p \) satisfying \( 1 \leq p < t \leq T \), where \( X_p \geq X^\text{min} \) and \( X_i = 0 \) for each \( p < i < t \).

(B) If \( X_p > X^\text{min} \) and \( I_p - 1 > 0 \), then \( X_p = \sum_{i=p}^{t} d_i - I_{p-1} \) for some \( p \leq t \leq T \).

Part (A) of the property generalizes the Zero-Inventory-Property. If we perform a production when there are some on-hand inventories in a period \( t \), then the previous production must be the minimum production quantities. Part (B) says that if we produce more than the minimum production quantities in a period \( p \), then the sum of the production and the inventory before that period must equal exactly to the quantities to satisfy the demands up to the end of some future period \( t \).

**Property 3.** There exists an optimal solution such that:

(A) \( X_tO_t = 0 \) for each \( t, 1 \leq t \leq T \).

(B) For one production point \( t' \) and one outsourcing point \( t'' \), \( 1 \leq t' < t'' \leq T \), if there is no production point and no outsourcing point between \( t' \) and \( t'' \), then

\[
O_{t''} = \sum_{l=t''}^{t_\ast} d_l - I_{t''-1} \quad \text{for} \quad t'' \leq t_\ast \leq T.
\]

(C) For two consecutive outsourcing points \( t' \) and \( t'' \), \( 1 \leq t' < t'' \leq T \), if there is no production point between \( t' \) and \( t'' \), then \( O_{t''} = \sum_{l=t''}^{t_\ast} d_l - I_{t''} \) for some \( t'' \leq t_\ast \leq T \).

**Proof of (A).** Since \( a_t > c_t \) and \( K_t > k_t \), \( C_t^P(X_t) \) and \( C_t^O(O_t) \) have a unique intersection point (see Fig. 1). Assume the production (or outsourcing) \( x \) in period \( t \) is to satisfy the demands from period \( t' \) to \( t'' \), \( t' \leq t \leq t'' \), if \( x < x_\ast \), that is if the cumulative demands are smaller, then there is no production in period \( t \). If \( x \geq x_\ast \), that is if the cumulative demands are bigger, then there is no outsourcing in period \( t \).

**Proof of (B).** Suppose there is an outsourcing point \( t'' \) from period \( t_\ast \) to \( T \). Assume that an optimal solution exists with \( O_{t''} = \sum_{l=t''}^{t_\ast} d_l - I_{t''-1} + \varepsilon \) and \( O_{t''} = \sum_{l=t_\ast+1}^{T} d_l - \varepsilon \),

\[
0 < \varepsilon < d_{t_\ast+1}, \quad \text{if} \quad a_{t''} + \sum_{l=t''}^{t_\ast} h_l \geq a_{t''} + \sum_{l=t_\ast+1}^{t''-1} b_l, \quad \text{then reduce} \quad O_{t''} \quad \text{from} \quad \sum_{l=t''}^{t_\ast} d_l - I_{t''-1} + \varepsilon
\]

\[
to \sum_{l=t''}^{t_\ast} d_l - I_{t''-1} \quad \text{and increase} \quad O_{t''} \quad \text{to} \quad O_{t''} + \varepsilon, \quad \text{this alteration does not incur any additional fixed costs and saves the cost} \quad (a_{t''} + \sum_{l=t''}^{t_\ast} h_l - a_{t''} - \sum_{l=t_\ast+1}^{t''-1} b_l)\varepsilon.
\]

If \( a_{t''} + \sum_{l=t''}^{t_\ast} h_l < a_{t''} + \sum_{l=t_\ast+1}^{t''-1} b_l, \quad \text{then reduce} \quad O_{t''} \quad \text{from} \quad \sum_{l=t_\ast+1}^{T} d_l - \varepsilon \quad \text{to} \quad \sum_{l=t_\ast+2}^{T} d_l \quad \text{and increase} \quad O_{t''} \quad \text{to} \quad O_{t''} = \sum_{l=t''}^{T} d_l - I_{t''-1}, \quad \text{this alteration does not incur any additional fixed costs and saves the cost} \quad (a_{t''} + \sum_{l=t''}^{t_\ast} b_l - a_{t''} - \sum_{l=t''-1}^{t_\ast+1} h_l)\varepsilon.
\]

If there is a production point \( t'' \) from period \( t_\ast \) to \( T \), the proof is similar and we omit the details.
Proof of (C). The proof is similar to (B), we have omitted the details due to space limitations.

From Property 2 and 3, we have $I_T < X_{\min}$ in an optimal solution. This means that the inventories at the end of the final period are always lower than the minimum production quantities.

From Property 2 and 3, we can also draw an interesting conclusion. An optimal solution exists such that: the demands in a period are satisfied by production from at most two production points or are satisfied by the combination of production and outsourcing from one production point and one outsourcing point or are satisfied by outsourcing from one outsourcing point.

\[ \text{Cost} = k_t + o_t x \]

\[ K_t + c_t x \]

\[ X_{\min} \]

\[ x^* \]

\[ \text{Quantity} x \]

**Figure 1.** The sketch for production and outsourcing decisions

Based on the above properties, the problem can be solved efficiently by a DP algorithm, let $F(t)$ be the optimal cost to solve the $t$-period problem and $I_t$ be the inventory amount produced in period $i$ and held in period $t$. The other notations are assigned as follows:

(i) $F(i, j, t)$ is the optimal cost to supply the demands from period $j$ to period $t$, where $i$ is the largest indexed production point and there is no outsourcing point from period $i$ to period $t$, $1 \leq i \leq j \leq t$;

(ii) $F(i, j, j_1, j_2, ..., g_n, J_n, t)$ is the optimal cost to supply the demands from period $j$ to period $j_1' - 1$ by production and supply the demands from period $j_1'$ to period $t$ by outsourcing, where $i$ is the largest indexed production point, and $j_1$ is the first outsourcing point and $g_n$ is largest indexed production point from period $i$ to period $t$, and $g_n$ is largest indexed production point before $j_n$, $1 \leq i \leq j \leq j_1 \leq j_1' \leq g_2 \leq j_2 \leq ... \leq g_n \leq j_n \leq t$;

(iii) $F(i, j, j_1, ..., g_n, J_n, t)$ is the optimal cost to supply the demands from period $j$ to period $j_1' - 1$ by production and supply the demands from period $j_1'$ to period $t$ by outsourcing, where $i$ is the largest indexed production point, and $j_1$ is the first outsourcing point and $j_n$ is largest indexed production point from period $i$ to period $t$, and $g_1$ is the largest indexed regeneration point before $j_1$, and $g_n$ is largest indexed production point before $j_n$, $1 \leq i \leq j \leq j_1 \leq ... \leq g_n \leq j_n \leq t$;

(iv) $F(e, i, t)$ is the optimal cost to supply the demands from period $e$ to period $t$, where $i$ is the largest indexed production point, and $e$ is the largest indexed regeneration point before period $i$, and there is no outsourcing point from period $i$ to period $t$, $0 \leq e \leq i$, $1 \leq i \leq t$;
(v) \( F(e, i, j_1, j_1', g_2, j_2, ..., g_n, j_n, t) \) is the optimal cost to supply the demands from period \( e \) to period \( j_1' \) by production and supply the demands from period \( j_1 \) to period \( t \) by outsourcing, where \( i \) is the largest indexed production point, and \( e \) is the largest indexed regeneration point before period \( i \), and \( j_1 \) is the first outsourcing point and \( j_n \) is largest indexed production point from period \( i \) to period \( t \), and \( g_n \) is largest indexed production point before period \( j_n \), \( 0 \leq e \leq i \), \( 1 \leq i \leq j_1' \leq g_2 \leq j_2 \leq ... \leq g_n \leq j_n \leq t \);

(vi) \( F(e, i, g_1, j_1, ..., g_n, j_n, t) \) is the optimal cost using production to supply the demands from period \( e \) to period \( g_1 - 1 \) and using outsourcing to supply the demands from period \( g_1 \) to period \( t \), where \( i \) is the largest indexed production point, and \( e \) is the largest indexed regeneration point before period \( i \), and \( j_1 \) is the first outsourcing point and \( j_n \) is the largest indexed production point from period \( i \) to period \( t \), and \( g_1 \) is the largest indexed regeneration point before \( j_1 \), and \( g_n \) is the largest indexed regeneration point before \( j_n \), \( 0 \leq e \leq i \), \( 1 \leq i \leq g_1 \leq j_1 \leq ... \leq g_n \leq j_n \leq t \);

(vii) \( F(g_n, j_n, t) \) is the optimal cost using production to supply the demands from period \( g_n \) to period \( t \) if there is no production point for the \( t \)-period problem, \( j_n \) is the largest indexed production point and \( g_n \) is the largest indexed regeneration point before \( j_n \), \( 0 \leq g_n \leq j_n \), \( 1 \leq j_n \leq t \).

From the above definitions the DP recursion is as follows:

\[
F(t) = \min \left\{ \begin{array}{l}
F(j - 1) + F(i, j, t); \\
F(j - 1) + F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t);
\end{array} \right. \]

\[
F(t) = \min \left\{ \begin{array}{l}
F(j - 1) + F(i, j, t); \\
F(j - 1) + F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t);
\end{array} \right. \]

\[
F(t) = \min \left\{ \begin{array}{l}
F(j - 1) + F(i, j, t); \\
F(j - 1) + F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t);
\end{array} \right. \]

\[
F(t) = \min \left\{ \begin{array}{l}
F(j - 1) + F(i, j, t); \\
F(j - 1) + F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t);
\end{array} \right. \]

\[
F(t) = \min \left\{ \begin{array}{l}
F(j - 1) + F(i, j, t); \\
F(j - 1) + F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t);
\end{array} \right. \]

\[
F(t) = \min \left\{ \begin{array}{l}
F(j - 1) + F(i, j, t); \\
F(j - 1) + F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t);
\end{array} \right. \]

\[
F(t) = \min \left\{ \begin{array}{l}
F(j - 1) + F(i, j, t); \\
F(j - 1) + F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t);
\end{array} \right. \]

We will consider seven cases according to whether there is a production point or not and if the inventories at the beginning of the production (and outsourcing) point are smaller than the demands in that period. For the cases \( F(i, j, t), F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t), F(i, j, g_1, j_1, ..., g_n, j_n, t), \) the inventories at the end of period \( i - 1 \) (at the beginning of period \( i \)) are not smaller than the demands in period \( i \). For the case \( F(e, i, t), F(e, i, j_1, j_1', g_2, j_2, ..., g_n, j_n, t), F(e, i, g_1, j_1, ..., g_n, j_n, t), \) the inventories at the end of period \( i - 1 \) (at the beginning of period \( i \)) are smaller than the demands in period \( i \). For the cases \( F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t) \) and \( F(e, i, j_1, j_1', g_2, j_2, ..., g_n, j_n, t) \), the inventories at the end of period \( j - 1 \) are not smaller than the demands in period \( j \). Similarly, for the cases \( F(i, j, g_1, j_1, ..., g_n, j_n, t) \) and \( F(e, i, g_1, j_1, ..., g_n, j_n, t) \), the inventories at the end of period \( j - 1 \) are smaller than the demands in period \( j \). For the case \( F(g_n, j_n, t) \), there is no production point, that is all the demands from period \( 1 \) to period \( t \) are satisfied by outsourcing. In the following, we will explain the calculations for \( F(i, j, t), F(i, j, j_1, j_1', g_2, j_2, ..., g_n, j_n, t), F(i, j, g_1, j_1, ..., g_n, j_n, t), F(e, i, t), F(e, i, j_1, j_1', g_2, j_2, ..., g_n, j_n, t), F(e, i, g_1, j_1, ..., g_n, j_n, t) \) and \( F(g_n, j_n, t) \).

For the case \( F(i, j, t) \), assume that period \( p \) is the largest production point before period \( i \) and that there is no outsourcing point, then \( I_{p,j-1} < d_j \). If period \( p \) is the largest outsourcing point before period \( i \) and there is no production point, then \( I_{p,j-1} = 0 \). Therefore, the cost for satisfying the demands from period \( j \) to \( t \) is the sum of production costs and inventory costs. The sketch for the computation of
shown in Fig. 2. Thus,

\[ F(i, j, t) = K_i + c_i \min \{ \sum_{l=j}^{t} d_l - I_{p,j-1} - X_{\text{max}}^{\text{min}}, X_{\text{min}}^{\text{min}} \} \]

\[ + \sum_{l=j}^{j-1} h_l( \min \{ \sum_{t} d_l - I_{p,j-1} - X_{\text{max}}^{\text{min}}, X_{\text{min}}^{\text{min}} \} ) \]

\[ + \sum_{l=j}^{j-1} h_l( \min \{ \sum_{t} d_l - I_{p,j-1} - X_{\text{max}}^{\text{min}}, X_{\text{min}}^{\text{min}} \} - \sum_{k=j}^{l} d_k ) \]

(11)

The sketch for Figure 2.

For the case \( F(i, j, j_1, j'_1, g_2, j_2, \ldots, g_n, j_n, t) \), periods \( g_2-1, \ldots, g_n-1 \) are all regeneration points. The cost for satisfying the demands from periods \( j \) to \( t \) is the sum of production, inventory, backlogging and outsourcing costs. The sketch for the computation of \( F(i, j, j_1, j'_1, g_2, j_2, \ldots, g_n, j_n, t) \) is shown in Fig. 3. Thus,

\[ F(i, j, j_1, j'_1, g_2, j_2, \ldots, g_n, j_n, t) \]

\[ = K_i + c_i \min \{ \sum_{l=j}^{j'_1-1} d_l - I_{p,j-1} - X_{\text{max}}^{\text{min}}, X_{\text{min}}^{\text{min}} \} \]

\[ + \sum_{l=j}^{j_1-1} h_l( \min \{ \sum_{t} d_l - I_{p,j-1} - X_{\text{max}}^{\text{min}}, X_{\text{min}}^{\text{min}} \} ) \]

\[ + \sum_{l=j}^{j_1-1} h_l( \min \{ \sum_{t} d_l - I_{p,j-1} - X_{\text{max}}^{\text{min}}, X_{\text{min}}^{\text{min}} \} - \sum_{k=j}^{l} d_k ) \]

\[ + k_{j_1} + k_{j_2} + \ldots + k_{j_n} + o_{j_1}( \sum_{l=j'_1}^{j_1} d_l - I_{i,j'_1-1} ) + o_{j_2} \sum_{l=g_2}^{g_n-1} d_l \]

\[ + \ldots + o_{j_n} \sum_{l=g_{n-1}}^{j'_1} d_l + \sum_{l=g_2}^{j_2-1} \sum_{l=g_2}^{l} b_l d_k + \ldots + \sum_{l=g_n}^{j_n-1} \sum_{l=g_n}^{l} b_l d_k \]

\[ + \sum_{l=j_1}^{j_1-1} h_l( \sum_{k=j'_1}^{j_1} d_l - I_{i,j'_1-1} ) + \sum_{l=j_1}^{j_2-1} \sum_{l=j_1}^{l+1} h_l d_k + \sum_{l=j_2}^{j_3-1} \sum_{l=j_2}^{l+1} h_l d_k \]

\[ + \ldots + \sum_{l=j_n}^{l-1} \sum_{l=j_n}^{l+1} h_l d_k \]
For the case $F(i, j, g_1, j_1, ..., g_n, j_n, t)$, periods $g_1 - 1, g_2 - 1, ..., g_n - 1$ are all regeneration points. The cost for satisfying the demands from periods $j$ to $t$ is the sum of production, inventory, backlogging and outsourcing costs. The sketch for the computation of $F(i, j, g_1, j_1, ..., g_n, j_n, t)$ is shown in Fig. 4. Thus,

$$F(i, j, g_1, j_1, ..., g_n, j_n, t) = K_i + c_i \min \{ \sum_{l=j}^{j_1-1} d_l - I_{p,j-1}, X_{\min}^1, X_{\min}^{e-1} \}$$

$$+ \sum_{l=j}^{j_1-1} h_l(\min \{ \sum_{l=j}^{j_1-1} d_l - I_{p,j-1}, X_{\min}^1, X_{\min}^{e-1} \})$$

$$+ \sum_{l=j}^{j_1-1} h_l(\min \{ \sum_{l=j}^{j_1-1} d_l - I_{p,j-1}, X_{\min}^1, X_{\min}^{e-1} \} - \sum_{l=k=j}^t d_k)$$

$$+ k_{j_1} + k_{j_2} + ... + k_{j_n} + o_{j_1}(\sum_{l=g_1}^{j_1-1} d_l - I_{g_1-1}) + o_{j_2} \sum_{l=g_2}^{g_3-1} d_l$$

$$+ ... + o_{j_n} \sum_{l=g_{n-1}}^{t} d_l + \sum_{l=g_1}^{j_1-1} b_l(d_{g_1} - I_{g_1-1}) + \sum_{l=g_1+1}^{j_1-1} \sum_{l=k=g_1+1}^t b_l d_k$$

$$+ \sum_{l=g_2}^{j_2-1} \sum_{k=g_2}^{j_1-1} b_l d_k + ... + \sum_{l=g_n}^{j_2-1} \sum_{k=g_n}^{j_1-1} b_l d_k + \sum_{l=j_1}^{j_1-1} h_l(\sum_{l=k=j_1}^t d_l - I_{j_1-1})$$

$$+ \sum_{l=j_1}^{j_1-1} \sum_{k=j_1+1}^{j_1-1} h_l d_k + \sum_{l=j_2}^{j_2-1} \sum_{k=j_2+1}^{j_1-1} h_l d_k + ... + \sum_{l=j_n}^{j_n-1} \sum_{k=j_n+1}^{j_1-1} h_l d_k$$

For the case $F(e, i, t)$, period $e - 1$ is a regeneration point. The cost for satisfying the demands from periods $e$ to $t$ is the sum of production, inventory and backlogging costs. The sketch for the computation of $F(e, i, t)$ is shown in Fig. 5. Thus,

$$F(e, i, t) = K_i + c_i \min \{ \sum_{l=e}^{t} d_l - I_{e-1}, X_{\min}^1, X_{\min}^{e-1} \} + \sum_{l=e}^{t} b_l(d_e - I_{e-1})$$
\[ F(i, j, j_1', ..., j_n, j_t, t) = K_i + c_i \min \{ \max \{ \sum_{l=e}^{t} d_l - I_{e-1}, X_{e-1}^{\min} \}, X_{e-1}^{\min} \} + \sum_{l=e}^{i-1} b_l (d_e - I_{e-1}) \]

\[ + \sum_{l=e+1}^{i-1} \sum_{k=e+1}^{l} b_l d_k + \sum_{l=e}^{t} h_l (\min \{ \max \{ \sum_{l=e}^{t} d_l - I_{e-1}, X_{e-1}^{\min} \}, X_{e-1}^{\min} \} - \sum_{k=e}^{l} d_k) \]

(14)
The cost for satisfying the demands from periods \( e \) to \( t \) is the sum of production, inventory, backlogging and outsourcing costs. The sketch for the computation of \( F(e, i, j_1, \ldots, j_n, t) \) is shown in Fig. 7. Thus, \( F(e, i, j_1, \ldots, j_n, t) \)

\[
= K_i + c_i \min \left\{ \max \left\{ \sum_{l=e}^{t} d_l - I_{e-1} \right\}, X_{e}^{\text{max}} \right\} + \sum_{l=e}^{t} b_l (d_l - I_{e-1}) \\
+ \sum_{l=e+1}^{t} b_l d_k + \sum_{l=i}^{j_1-1} h_l (\min \left\{ \max \left\{ \sum_{l=e}^{t} d_l - I_{e-1} \right\}, X_{e}^{\text{max}} \right\} - \sum_{k=i}^{j_1-1} d_k) \\
+ k_{j_1} + k_{j_2} + \ldots + k_{j_n} + o_{j_1} \left( \sum_{l=g_1}^{j_1-1} d_l - I_{g_1-1} \right) + o_{j_2} \sum_{l=g_2}^{j_2-1} d_l + \ldots + o_{j_n} \sum_{l=g_n}^{j_n-1} d_l \\
+ \sum_{l=g_1}^{j_1-1} \sum_{k=g_1+1}^{j_1-1} b_l d_k + \sum_{l=g_2}^{j_2-1} \sum_{k=g_2}^{j_2-1} b_l d_k + \ldots + \sum_{l=g_n}^{j_n-1} \sum_{k=g_n}^{j_n-1} b_l d_k \\
+ \sum_{l=j_1}^{j_1-1} h_l (\sum_{l=1}^{j_1-1} d_l - I_{j_1-1}) + \sum_{l=j_2}^{j_2-1} \sum_{k=j_2+1}^{j_2-1} h_l d_k + \ldots + \sum_{l=j_n}^{j_n-1} \sum_{k=j_n+1}^{j_n-1} h_l d_k \\
\sum_{l=g_1}^{j_1-1} h_l d_k + \sum_{l=g_2}^{j_2-1} h_l d_k + \sum_{l=g_n}^{j_n-1} h_l d_k
\]

(16)

For the case \( F(g_n, j_n, t) \), periods \( g_1 - 1, g_2 - 1, \ldots, g_n - 1 \) are all regeneration points. The cost for satisfying the demands from periods \( g_n \) to \( t \) is the sum of outsourcing, inventory and backlogging costs. The sketch for the computation of \( F(g_n, j_n, t) \) is shown in Fig. 8. Thus, \( F(g_n, j_n, t) \)

\[
= k_{j_n} + o_{j_n} \sum_{l=g_n}^{j_n-1} d_l + \sum_{l=g_n}^{j_n-1} \sum_{k=g_n}^{j_n-1} b_l d_k + \sum_{l=j_n}^{t} \sum_{k=1}^{j_n} h_l d_k
\]

(17)
4. A special case. In this section, we consider a special case for stationary production and outsourcing cost, that is $C^P_t(X_t) = C^P(X_t)$ and $C^O_t(O_t) = C^O(X_t)$. The following properties are specified in this special case.

Property 4. An optimal solution exists such that:

(A) If $I_{t-1} \geq d_t$ for each $1 \leq t \leq T$, then $X_t = 0$ and $O_t = 0$.

(B) If $X_t > 0$ or $O_t > 0$ for each $1 \leq t \leq T$, then $X_t = X^{\text{min}}$ or $X_t = \sum_{l=t}^{t^*} d_l - I_{t-1}$, or $O_t = \sum_{l=t}^{t^*} d_l - I_{t-1}$ for some $t \leq t^* \leq T$.

Proof of (A). If $I_{t-1} \geq d_t$ is an optimal solution for $P(t)$, a cheaper feasible production (or outsourcing) plan can also be constructed as follows, we consider two cases:
If $I_{t-1} \geq \sum_{u=t}^{T} d_u$, then reduce the production (or outsourcing) in period $t$ to zero, this alteration saves the production (or outsourcing) costs in period $t$; (2) if $\sum_{u=t}^{t^*} d_u \leq I_{t-1} \leq \sum_{u=t}^{t^*} d_u, \ t+1 \leq t^* \leq T$, then transfer the production (or outsourcing) amount from period $t$ to $t^*$, this alteration saves the inventory costs and does not incur any additional costs.

Proof of (B). As the proof is similar to Property 3(B), we have omitted the details.

As for the stationary production and outsourcing cost, the problem can be solved efficiently by a DP algorithm:

$$F(t) = \min \begin{cases} F(e-1) + F(e, i, t); \\ F(e-1) + F(e, i, g_1, j_1, ..., g_n, j_n, t); \\ F(g_n - 1) + F(g_n, j_n, t) \end{cases}$$

(18)

where $F(e, i, t)$, $F(e, i, g_1, j_1, ..., g_n, j_n, t)$ and $F(g_n, j_n, t)$ have been computed according to the general cost structure, we will not repeat them.

5. Numerical instance. Example 1. Let the demands for the first 10 periods be (8, 6, 9, 29, 5, 6, 5, 15, 5, 6). Let the minimum order quantity $S$ be 30. The cost parameters are as follows: $K_t = 90, c_t = 4, h_t = 2, b_t = +\infty, C^O_t(O_t) = +\infty, 1 \leq t \leq 10$.

The optimal solution to the problem with no backlogging and outsourcing is to produce in periods 1, 4 and 8. The production in period 1 is used to satisfy the demands from period 1 to 3 and part of the demands in period 4. A part of the demands in period 4 and the demands from period 5 to 7 are satisfied by production in period 4. The demands from period 8 to 10 are satisfied by production in period 8. Let $X^*_t$ denote the optimal production quantities in period $t$. The detailed results are shown in Table 2.

| t | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|----|
| $d_t$ | 8 | 6 | 9 | 29 | 5 | 6 | 5 | 15 | 5 | 6 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $X^*_t$ | 30 | 30 | 0 | 0 | 30 | 30 | 0 | 30 | 0 | 30 |
| $F(t)$ | 254 | 286 | 300 | 526 | 532 | 556 | 606 | 786 | 856 | 874 |

Example 2. Let the backlogging costs be $b_t = 2, 1 \leq t \leq 10$. Let other parameters be the same as in Example 1.
The optimal solution to the problem with backlogging is to produce in periods 4 and 8. The production in period 4 is used to satisfy the demands from period 1 to 5. The production in period 8 is used to satisfy the demands from period 6 to 10. The detailed results are shown in Table 3.

| Table 3. Summary of Computations of Example 2 |
|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $d_t$ | 8 | 6 | 9 | 29 | 5 | 6 | 5 | 15 | 5 | 6 |
| $X^*_t$ | 30 | 0 | 30 | 0 | 0 | 30 | 0 | 0 | 0 | 52 |
| $X^*_t$ | 0 | 0 | 0 | 63 | 0 | 0 | 0 | 68 | 0 | 0 |
| $X^*_t$ | 0 | 0 | 0 | 0 | 57 | 0 | 0 | 0 | 30 | 0 |
| $X^*_t$ | 0 | 0 | 0 | 57 | 0 | 0 | 0 | 31 | 0 | 0 |
| $X^*_t$ | 0 | 0 | 0 | 0 | 0 | 63 | 0 | 0 | 0 | 30 |
| $F(t)$ | 254 | 258 | 268 | 388 | 418 | 466 | 546 | 682 | 688 | 748 |

Example 3. Let outsourcing costs be $k_t = 50$ and $o_t = 6$. Let other parameters be the same as in Example 1.

The optimal solution to the problem with outsourcing is to produce in period 4 and outsource in periods 1 and 8. The outsourcing in period 1 is used to satisfy the demands from period 1 to 3. The production in period 4 is used to satisfy the demands from period 4 to 7. The outsourcing in period 8 is used to satisfy the demands from period 8 to 10. The detailed results are shown in Table 4.

| Table 4. Summary of Computations of Example 3 |
|---|---|---|---|---|---|---|---|---|---|---|
| $t$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $d_t$ | 8 | 6 | 9 | 29 | 5 | 6 | 5 | 15 | 5 | 6 |
| $X^*_t, O^*_t$ | 0, 8 | 0, 14 | 0, 0 | 0, 23 | 0, 0 | 0, 0 | 0, 0 | 0, 30 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 23 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $X^*_t, O^*_t$ | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 | 0, 0 |
| $F(t)$ | 98 | 146 | 236 | 448 | 472 | 532 | 582 | 722 | 762 | 822 |

The optimal cost for Example 2 (backlogging is permitted) is 748, which is a 14% reduction as compared with the optimal cost for Example 1 (no backlogging and outsourcing). The optimal cost for Example 2 (outsourcing is permitted) is a 6% reduction as compared with the optimal costs for Example 1. Based on the results
from Example 1, 2 and 3, we can argue that the combination policies of backlogging and outsourcing can dramatically reduce the total cost.

6. Conclusions and future research directions. Considering the production outsourcing and backlogging is realistic factors and important for real life manufacturing systems. This paper presents a significant variant of the traditional dynamic lot sizing problem with a minimum production quantities constraint.

We consider the problem where (i) production below a given level is not allowed, (ii) outsourcing is allowed and unrestricted. (iii) Backlogging is permitted. A mathematical programming model is established according to the real problem in the firm. We explore some structural properties of the optimal solution and use them to develop a dynamic programming algorithm to solve the proposed problem. In addition, we further present a special case with stationary production and outsourcing costs which can be solved with reduced computational complexities. In the end, three numerical instances are used to show how to obtain the optimal solutions by using the dynamic programming algorithm. Furthermore, our results show that the policy of backlogging or outsourcing can reduce the total cost. The combination of two policies can provide more accurate decision supports for production operations.

There are some interesting research directions worth pursuing and we have detailed them, as follows. First, the storage capacity is unlimited in our model, while the products that need temperature control may have a limited storage capacity. It would be of interest to solve the problem under limited storage capacity. Second, the demand is an exogenous variable, which is independent of product price. In situations when the demands depend on the product price, we can merge operations management and marketing [38, 48]. Third, the vast majority of companies can benefit from economies of scope in producing multiple products. Another possible direction for future research on the subject is to incorporate multi-item, even though the analytical complexity will obviously increase. Fourth, learning in setup is an important phenomenon in practice [9], the inclusion of learning in setup can also provide opportunities for further analysis.

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