POISSON STOCHASTIC PROCESS AND BASIC
SCHAUDER AND SOBOLEV ESTIMATES IN THE
THEORY OF PARABOLIC EQUATIONS

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Abstract. We show among other things how knowing Schauder or Sobolev-space estimates for the one-dimensional heat equation allows one to derive their multidimensional analogs for equations with coefficients depending only on time variable with the same constants as in the case of the one-dimensional heat equation. The method is quite general and is based on using the Poisson stochastic process. It also applies to equations involving non-local operators. It looks like no other method is available at this time and it is a very challenging problem to find a purely analytic approach to proving such results.

1. Introduction

In this paper we present a method allowing one, in particular, to obtain various estimates for the multidimensional second-order parabolic equations of main type with time dependent coefficients with the same constants as in the case of the one-dimensional heat equation, provided that the matrix of the second-order coefficients dominates the identity matrix.

The method is universal in the sense that it works in the same way for Hölder- or Sobolev-space estimates, for scalar equations and even for not necessarily parabolic systems. The main condition for it to work is that the equations should be commuting with space translations (more generally, should be commuting with a commutative group of affine mappings) and the estimates should be space-translation invariant as well.

We start with Section 2 and show our main idea on the example of deriving basic Schauder and Sobolev-space estimates for the heat equation in 2 space dimension from the similar estimates for the heat equation in 1 space dimension. Here we just use the Poisson process.

In Section 3 we show how the method works for multidimensional parabolic equations with measurable coefficients depending only on the time variable, provided that the matrix of the coefficients dominates the identity

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matrix. This time an integral of nonrandom functions against the Poisson process is involved.

As a corollary we obtain that for elliptic equations of main type with constant coefficients the constant in the estimate of the $C^\alpha$-semi-norm of the second-order directional derivatives of solutions through the $C^\alpha$-semi-norm of the free term is independent of the space dimension. The same is also noted for the $L_p$-estimates of the second-order directional derivatives of solutions through the $L_p$-norm of the free term.

In Section 4 we present our method in a more abstract form for evolution equations when the norms are not necessarily translation invariant, but invariant relative to a group of affine mappings of the space and the equations commute with that group. In Example 4.11 we show a result of application of our general theorem, Theorem 4.9, which allows us to obtain the Schauder estimates for a parabolic equation with space-dependent coefficients with the same constants as in the case of the 2 dimensional heat equation. In Example 4.13 we apply Theorem 4.9 to a hyperbolic system. In Example 4.14 we show an application of our results to the hyperbolic systems from §7.3.3 of Evans’s book [1].

Section 5 contains the proof of Theorem 4.4, which is used in Section 6 to prove Theorem 4.9. Finally, in Section 7 we present an extension of our method to treat non-local operators.

The origin of our ideas lies in the theory of stochastic partial differential equations (SPDEs) and can be found in the proof of Theorem 2.1 of [3]. This idea can be implemented quite formally without using the theory of SPDEs, see, for instance, [2] and [10], where still one needs to be familiar with the Itô stochastic integral with respect to the Wiener process albeit of nonrandom functions.

It turns out that replacing the Wiener process with the Poisson process in the original idea leads to much simpler SPDEs which, actually, are just usual equations with discontinuities in time at random well separated moments, dealing with which does not require any knowledge of stochastic integration. Turning to the Poisson processes has also an advantage that we can consider integro-differential equations (cf. Theorem 7.1).

At the same time we can easily recover the results obtained by using methods in [2] and [10]. The probabilistic reason (which is not used in the article) for that lies in the well-known central limit theorem according to which $(2\lambda)^{-1/2}(\pi_i^{\lambda,1} - \pi_i^{\lambda,2})$ tends in law to $w_t$ as $\lambda \downarrow 0$, where $\pi_i^{\lambda,j}$, $i = 1, 2$, are independent Poisson processes with intensity $\lambda$ and $w_t$ is a Wiener process.

In conclusion we note that the scope of applications of Theorems 4.4 and 4.9 is much wider than only the examples given in the article. For instance, one could consider integro-differential equations or higher order equations, or else the combinations of those. We plan to explore these possibilities in the near future.
In the whole article $T$ is a fixed number in $(0, \infty)$, $\mathbb{R}^d$ is a Euclidean space of points $x = (x^1, \ldots, x^d)$, $x^1, \ldots, x^d \in (-\infty, \infty)$, $S_1 := \{x \in \mathbb{R}^d : |x| = 1\}$ is the unit sphere, and the standard stipulation about the summation with respect to repeated indices is enforced. Also we use standard notation for derivatives, spaces, semi-norms, and norms which can be found in [6], [8], [9]. We only recall what Hölder functions spaces are. The space $C^{\alpha}(\mathbb{R}^d)$, $\alpha \in (0, 1)$, is the space of all real-valued functions $f$ on $\mathbb{R}^d$ for which the following norm

$$\|f\|_{C^{\alpha}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} (|f(x)| + |f|_{C^{\alpha}(\mathbb{R}^d)})$$

is finite, where

$$[f]_{C^{\alpha}(\mathbb{R}^d)} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

As usual, by $C^{2+\alpha}(\mathbb{R}^d)$ we mean the space of real-valued twice continuously differentiable functions $f$ on $\mathbb{R}^d$ having finite norm

$$\|f\|_{C^{2+\alpha}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} \left( |f(x)| + |Df(x)| + |D^2f(x)| \right) + |D^2f|_{C^{\alpha}(\mathbb{R}^d)},$$

where $Df$ is the gradient of $f$ and $D^2f$ is its Hessian.

2. ONE DIMENSIONAL HEAT EQUATION

Consider the problem of solving the equation

$$\partial_t u(t, x) = D^2 u(t, x) + f(t, x) \quad (2.1)$$

for $t \in (0, T)$, $x \in \mathbb{R}$, with zero initial condition, i.e., $u(0, \cdot) = 0$. To be more precise we treat the problem in the integral form:

$$u(t, x) = \int_0^t (D^2u(s, x) + f(s, x))ds, \quad t \in [0, T], \ x \in \mathbb{R}. \quad (2.2)$$

For a real-valued function $f(t, x)$, $t \in (0, T)$, $x \in \mathbb{R}^d$, write

$$f \in B_c((0, T), C^\infty_0(\mathbb{R}^d))$$

if $f$ is a Borel bounded function, such that $f(t, \cdot) \in C^\infty_0(\mathbb{R}^d)$ for any $t$, for any $n = 0, 1, \ldots$, the $C^n(\mathbb{R}^d)$-norms of $f(t, \cdot)$ are bounded on $(0, T)$, and the supports of $f(t, \cdot)$ belong to the same ball.

Fix $\alpha \in (0, 1)$ and $p \in (1, \infty)$. One knows (see, for instance, [6], [8], [9]) that if $f \in B_c((0, T), C^\infty_0(\mathbb{R}))$, then the above problem has a solution $u(t, x)$ having the following properties:

(a) it is continuous in $[0, T] \times \mathbb{R}$;
(b) $u(t, \cdot) \in C^{2+\alpha}(\mathbb{R})$, for any $t \in [0, T]$, and

$$\sup_{t \in [0, T]} \|u(t, \cdot)\|_{C^{2+\alpha}(\mathbb{R})} \leq N_0(T, \alpha) \sup_{t \in (0, T)} \|f(t, \cdot)\|_{C^\alpha(\mathbb{R})}, \quad (2.3)$$
where \( N_0(T, \alpha) \) is a (finite) constant depending only on \( T \) and \( \alpha \). There is only one solution with these properties and, furthermore,

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}} |u(t,x)| \leq T \sup_{(t,x) \in (0,T) \times \mathbb{R}} |f(t,x)|, \tag{2.4}
\]

\[
\sup_{t \in [0,T]} [D^2u(t,\cdot)]_{C^0(\mathbb{R})} \leq N_0(\alpha) \sup_{t \in (0,T)} [f(t,\cdot)]_{C^0(\mathbb{R})}, \tag{2.5}
\]

\[
\|D^2u\|_{L^p([0,T] \times \mathbb{R})}^p \leq N_p \|f\|_{L^p([0,T] \times \mathbb{R})}^p, \tag{2.6}
\]

where \( L^p \)-spaces are defined with respect to Lebesgue measure and \( N_0(\alpha), N_p \) are some constants.

Take a sequence \( \tau_1 = \tau_1(\omega), \tau_2 = \tau_2(\omega), \ldots \) of independent random variables defined on a probability space \((\Omega, \mathcal{F}, P)\) with common exponential distribution with parameter \( \lambda > 0 \), so that \( P(\tau_n > t) = e^{-\lambda t} \) for \( t \geq 0 \) and \( n = 1, 2, \ldots \). Define

\[
\sigma_0 = 0, \quad \sigma_n = \sum_{i=1}^{n} \tau_i, \quad n = 1, 2, \ldots, \quad \pi_t = \pi_t(\omega) = \sum_{n=1}^{\infty} I_{\sigma_n \leq t}
\]

(where \( I_{\sigma_n \leq t} \) denotes the indicator function of the event \( \{\sigma_n \leq t\} \)). We see that \( \pi_t \) is the number of consecutive sums of \( \tau_i \) which lie on \([0,t]\). The counting process \( \pi_t \) is known as a Poisson process with parameter \( \lambda \), for \( 0 \leq s \leq t < \infty \) and \( k = 0, 1, \ldots \) it holds that

\[
P(\pi_t - \pi_s = k) = \frac{[\lambda(t-s)]^k}{k!} e^{-\lambda(t-s)},
\]

and, moreover, \( \pi_t - \pi_s \) is independent of the trajectory \( \{\pi_r, r \in [0,s]\} \), which is to say that, for any positive integer \( K \) and \( s_1, \ldots, s_K \leq s \), the set of random variables

\[
\{I_{\sigma_n \leq s_k} (= I_{\pi_{s_k} \geq n}) : n = 1, 2, \ldots, \quad k = 1, 2, \ldots, K\}
\]

and \( \pi_t - \pi_s \) are independent. (That \( \pi_t \) introduced in this way possesses the above listed properties is often put under the rug. For the shortest check, we know, see Exercise 2.3.8 and the hint to it in [4]).

Then take a function \( f(t,x,y) \) of class \( B_c((0,T), C^\infty(\mathbb{R}^2)) \) and for each \( \omega \in \Omega \) and \( y \in \mathbb{R} \) solve the equation

\[
\partial_t u(t,x,y,\omega) = D^2_{x} u(t,x,y,\omega) + f(t,x,y-h\pi_t(\omega)) \tag{2.7}
\]

with zero initial data, where \( h \in \mathbb{R} \) is a parameter. As usual in probability theory in the sequel, more often than not, we do not indicate the dependence on \( \omega \). Moreover, we also drop the dependence on \( h \) in the sequel. By the above, there exists a unique solution \( u(t,x,y) \), depending on \( y \) and \( \omega \) as parameters, such that estimates (2.3), (2.4), (2.5), and (2.6) hold for each \( \omega \) and \( y \in \mathbb{R} \) if we replace \( u(t,x) \) and \( f(t,x) \) with \( u(t,x,y) \) and \( f(t,x,y-h\pi_t) \), respectively. Furthermore, since \( f \in B_c((0,T), C^\infty(\mathbb{R}^2)) \), \( u(t,x,y) \) is uniformly continuous with respect to \( y \) uniformly with respect to \( \omega, t, h \), and \( x \) (cf. the proof of Lemma 3.2).
By considering \( u(t, x, y + h\pi_t) \) on each interval \([\sigma_n, \sigma_{n+1})\) on which \( h\pi_t \) is constant, one easily derives that \( u(t, x, y + h\pi_t) \) satisfies

\[
\begin{align*}
    u(t, x, y + h\pi_t) &= \int_0^t \left[ D_x^2 u(s, x, y + h\pi_s) + f(s, x, y) \right] ds + \int_{[0, t]} g(s, x, y) d\pi_s \\
    &= \int_0^t \left[ D_x^2 u(s, x, y + h\pi_s) + f(s, x, y) \right] ds + \sum_{\sigma_n \leq t} g(\sigma_n, x, y),
\end{align*}
\]

where

\[
g(s, x, y) = u(s, x, y + h + h\pi_s) - u(s, x, y + h\pi_{s-})
\]

is the jump of the process \( u(t, x, y + h\pi_t) \) as a function of \( t \) at moment \( s \) if \( \pi_t \) has a jump at \( s \).

Here \( \pi_{s-} = \lim_{t \downarrow s} \pi_t \), \( s > 0 \). For instance, if \( t \in [\sigma_1, \sigma_2) \) we have

\[
\begin{align*}
    u(t, x, y + h\pi_t) &= \int_0^{\sigma_1} \left[ D_x^2 u(s, x, y + h\pi_s) + f(s, x, y) \right] ds \\
    &\quad + u(\sigma_1, x, y + h) - u(\sigma_1, x, y) + \int_{\sigma_1}^t \left[ D_x^2 u(s, x, y + h) + f(s, x, y) \right] ds.
\end{align*}
\]

The next result follows from the theory of stochastic integrals against \( \pi_t - \lambda t \) (see Exercise 2.7.8 in [4]). We provide a direct and self-contained proof although a more general situation will be encountered in Lemma 5.3 and treated in a more sophisticated way.

**Lemma 2.1.** For \( g \) introduced in (2.9) and \( t \leq T \) we have

\[
E\int_{[0, t]} g(s, x, y) d\pi_s = \lambda \int_0^t [v(s, x, y + h) - v(s, x, y)] ds,
\]

where

\[
v(t, x, y) := Eu(t, x, y + h\pi_t).
\]

**Proof.** First assume that \( t = 1 \). Fix \( x \) and \( y \) and set \( g(s) = g(s, x, y) \). The function \( g \) is bounded on \( \Omega \times (0, T) \) and \( \pi_{s-} \) is left-continuous with respect to \( s \). Therefore, if we define

\[
g_n(s) = g(k2^{-n}) = u(k2^{-n}, x + h + h\pi_{k2^{-n}-}) - u(k2^{-n}, x, y + h\pi_{k2^{-n}-})
\]

for \( s \in (k2^{-n}, (k+1)2^{-n}] \), \( k = 0, 1, \ldots \), then \( g_n(s) \to g(s) \) as \( n \to \infty \) for any \( s \in (0, t) \) \( \omega \) and \( \omega \), and

\[
\xi_n := \int_{[0, 1]} g_n(s) d\pi_s \to \int_{[0, 1]} g(s) d\pi_s =: \xi
\]

for any \( \omega \). By the dominated convergence theorem \( E\xi_n \to E\xi \).

Next, observe that

\[
E\xi_n = \sum_{k=0}^{2^n-1} Eg(k2^{-n})(\pi_{(k+1)2^{-n}} - \pi_{k2^{-n}}).
\]
Here, owing to the way \( g \) was constructed, \( g(k2^{-n}) \) is uniquely defined once we know the values of the random variables \( I_{\sigma_i \leq t} \) for all \( i = 1, 2, \ldots, \) and all \( t \leq k2^{-n} \), and, as we have said, the increments of \( \pi_s \) after time \( k2^{-n} \) are independent of those random variables. Hence, the expectations of the products on the right in (2.11) are equal to the products of expectations, and since \( E\pi_t = \lambda t \), we conclude, that

\[
E\xi_n(t) = \lambda E \sum_{k=0}^{2^n-1} g(k2^{-n})2^{-n} = \lambda E \int_0^1 y_n(s) \, ds
\]

\[
\rightarrow \lambda E \int_0^1 g(s) \, ds = \lambda \int_0^1 Eg(s) \, ds.
\]

Since, for any \( s > 0 \), we have \( \pi_s = \pi_{s-} \) (a.s.), it holds that

\[
Eg(s) = v(s, x, y + h) - v(s, x, y).
\]

We have thus proved the lemma if \( t = 1 \). If it is not, one should just replace above \( k2^{-n} \) and \( (k+1)2^{-n} \) with \( tk2^{-n} \) and \( t(k+1)2^{-n} \). This proves the lemma.

By taking expectations of both sides of (2.8) we now obtain the existence part in the following result.

**Lemma 2.2.** Let \( f \in B_c((0, T), C_0^\infty(\mathbb{R}^2)) \), \( h \in \mathbb{R} \) and \( \lambda > 0 \). Then there exists a unique continuous function \( v(t, x, y) \), \( t \in [0, T] \), \( x, y \in \mathbb{R} \), satisfying the equation

\[
\partial_t v(t, x, y) = D_x^2 v(t, x, y) + \lambda [v(t, x, y + h) - v(t, x, y)] + f(t, x, y) \quad (2.12)
\]

for \( t \in (0, T) \), \( x, y \in \mathbb{R} \), with zero initial condition and such that \( v(t, \cdot, y) \in C^{2+\alpha}(\mathbb{R}) \) for any \( t \in (0, T) \), \( y \in \mathbb{R} \) and

\[
\sup_{(t, y) \in [0, T] \times \mathbb{R}} \|v(t, \cdot, y)\|_{C^{2+\alpha}(\mathbb{R})} \leq N_0(T, \alpha) \sup_{(t, y) \in (0, T) \times \mathbb{R}} \|f(t, \cdot, y)\|_{C^\alpha(\mathbb{R})}.
\]

(2.13)

Furthermore,

\[
\sup_{(t, z) \in [0, T] \times \mathbb{R}^2} |v(t, z)| \leq T \sup_{(t, z) \in (0, T) \times \mathbb{R}^2} |f(t, z)|,
\]

\[
\sup_{(t, y) \in [0, T] \times \mathbb{R}} \|D_x^2 v(t, \cdot, y)\|_{C^\alpha(\mathbb{R})} \leq N_0(\alpha) \sup_{(t, y) \in (0, T) \times \mathbb{R}} \|f(t, \cdot, y)\|_{C^\alpha(\mathbb{R})},
\]

(2.14)

\[
\|D_x^2 v\|_{L_p((0, T) \times \mathbb{R}^2)} \leq N_p \|f\|_{L_p((0, T) \times \mathbb{R}^2)}
\]

(where \( N_0(T, \alpha) \), \( N_0(\alpha) \) and \( N_p \) are the same as in (2.3), (2.5) and (2.6)).

Proof. Uniqueness follows from (2.4) if \( \lambda T \leq 1/4 \) and extends beyond \( 1/(4\lambda) \) by steps of size \( 1/(4\lambda) \).

All claimed estimates, apart from the last one, are obtained in the same manner following the example:

\[
\sup_{y \in \mathbb{R}} \|D_x^2 v(t, \cdot, y)\|_{C^\alpha(\mathbb{R})} \leq \sup_{y \in \mathbb{R}} E\|D_x^2 v(t, \cdot, y + h\pi_t)\|_{C^\alpha(\mathbb{R})},
\]
where, for any $t \leq T$ and $\omega$,
\[
\sup_{y \in \mathbb{R}} [D^2_x u(t, \cdot, y + h\pi t)]_{C^\alpha(\mathbb{R})} = \sup_{y \in \mathbb{R}} [D^2_x u(t, \cdot, y)]_{C^\alpha(\mathbb{R})}
\leq N_0(\alpha) \sup_{y \in \mathbb{R}, s < t} [f(s, \cdot, y - h\pi s)]_{C^\alpha(\mathbb{R})} \leq N_0(\alpha) \sup_{y \in \mathbb{R}, s < T} [f(s, \cdot, y)]_{C^\alpha(\mathbb{R})},
\]
which leads to (2.14).

The last $L_p$-estimate is obtained by replacing the above sups with integrals:
\[
\int_0^T \int_{\mathbb{R}^2} |D^2_x v(t, x, y)|^p \, dy \, dx \, dt \leq E \int_0^T \int_{\mathbb{R}^2} |D^2_x u(t, x, y + h\pi t)|^p \, dy \, dx \, dt
\leq E \int_0^T \int_{\mathbb{R}^2} |f(t, x, y - h\pi t)|^p \, dy \, dx \, dt = N_p E \int_0^T \int_{\mathbb{R}^2} |f(t, x, y)|^p \, dy \, dx \, dt.
\]
The lemma is proved.

We succeeded in adding in the right-hand side of (2.1) the first-order difference without changing constants in our estimates.

In our next step, we do with (2.12) almost the same thing as with (2.1) adding another finite difference. Namely, we introduce $v(t, x, y)$ depending also on $\omega$ as a unique solution of
\[
\partial_t v(t, x, y) = D^2_x v(t, x, y) + \lambda [v(t, x, y + h) - v(t, x, y)] + f(t, x, y + h\pi t)
\]
with zero initial condition. Then by just repeating the above computations, we see that
\[
w(t, x, y) := Ev(t, x, y - h\pi t)
\]
satisfies
\[
\partial_t w(t, x, y) = D^2_x w(t, x, y) + \lambda [w(t, x, y + h) - 2w(t, x, y) + w(t, x, y - h)] + f(t, x, y)
\]
(2.15) and admits the same estimates as in Lemma 2.2.

Then we take $\lambda = h^{-2}$ in (2.15) and let $h \downarrow 0$. With some extra work, to be presented later (see the proof of Lemma 3.2), one can show that the solutions $w = w_h$ of (2.15) with $\lambda = h^{-2}$ converge to a function $v(t, x, y)$, that is infinitely differentiable with respect to $(x, y)$ for any $t$ with any derivative bounded on $[0, T] \times \mathbb{R}^2$, is continuous in $[0, T] \times \mathbb{R}^2$, equals zero for $t = 0$, satisfies
\[
\partial_t v(t, x, y) = \Delta v(t, x, y) + f(t, x, y)
\]
in $(0, T) \times \mathbb{R}^2$ and for which all the estimates in Lemma 2.2 hold true with the same constants.

One knows that bounded continuous in $[0, T] \times \mathbb{R}^2$ solution of (2.16) having continuous second-order derivatives with respect to $(x, y)$ and vanishing at $t = 0$ are unique, and we conclude that, for any such solution the estimates in Lemma 2.2 hold true.
Take a unit vector \( l_1 \in \mathbb{R}^2 \) and a unit vector \( l_2 \in \mathbb{R}^2 \) orthogonal to \( l_1 \). Let \( S \) be an orthogonal transformation of \( \mathbb{R}^2 \) such that \( Se_i = l_i, \ i = 1, 2 \), where \( e_1, e_2 \) is the standard basis in \( \mathbb{R}^2 \), and set \( f(t, xe_1 + ye_2) = f(t, x, y), \ v(t, xe_1 + ye_2) = v(t, x, y) \),

\[
S(x, y) = xl_1 + yl_2, \quad g(t, x, y) = f(t, S(x, y)), \quad w(t, x, y) = v(t, S(x, y)).
\]

Since the Laplacian is rotation invariant, we have

\[
\partial_t w(t, x, y) = \Delta w(t, x, y) + g(t, x, y)
\]

and, since \( g \) is as good as \( f \), we conclude by defining

\[
K = \sup_{(t, y) \in (0, T) \times \mathbb{R}} \sup_{x_1, x_2 \in \mathbb{R}, x_1 \neq x_2} \frac{|g(t, x_1, y) - g(t, x_2, y)|}{|x_1 - x_2|^\alpha}
\]

that

\[
\sup_{(t, y) \in [0, T] \times \mathbb{R}} \sup_{x_1 \neq x_2} \frac{|D^2_x w(t, x_1, y) - D^2_x w(t, x_2, y)|}{|x_1 - x_2|^\alpha} \leq N_0(\alpha)K. \tag{2.17}
\]

Observe that, as is easy to see,

\[
D^2_x w(t, x, y) = (D^2_{l_1} v)(t, S(x, y)) = (D^2_{l_2} v)(t, x_l + yl_2),
\]

where

\[
D^2_{l_i} = l^i l^j D_{ij} \quad \text{and} \quad D_{ij} = \partial / \partial x^i, \quad D_{ij} = D_i D_j.
\]

Therefore, the left-hand side of (2.17) equals

\[
\sup_{(t, y) \in [0, T] \times \mathbb{R}} \sup_{x, \mu, \nu \in \mathbb{R}, \mu \neq \nu} \frac{|D^2_{l_1} v(t, \mu x_1 + x l_1 + y l_2) - D^2_{l_1} v(t, \nu x_1 + x l_1 + y l_2)|}{|\mu - \nu|^{\alpha}}
\]

\[
= \sup_{(t, z) \in [0, T] \times \mathbb{R}^2} \sup_{\mu \neq \nu} \frac{|D^2_{l_1} v(t, \mu x_1 + z) - D^2_{l_1} v(t, \nu x_1 + z)|}{|\mu - \nu|^{\alpha}}.
\]

Similarly the right-hand side of (2.17) is transformed and we get that for any (actually, only one) bounded continuous in \( [0, T] \times \mathbb{R}^2 \) solution \( v \) of (2.16) having continuous second-order derivatives with respect to \( (x, y) \) and vanishing at \( t = 0 \) and any unit vector \( l \in \mathbb{R}^2 \)

\[
\sup_{(t, z) \in [0, T] \times \mathbb{R}^2} \sup_{\mu \neq \nu} \frac{|D^2_{l_1} v(t, \mu z + z) - D^2_{l_1} v(t, \nu z + z)|}{|\mu - \nu|^{\alpha}} \leq N_0(\alpha) \sup_{(t, z) \in (0, T) \times \mathbb{R}^2} \sup_{\mu \neq \nu} \frac{|f(t, \mu z + z) - f(t, \nu z + z)|}{|\mu - \nu|^{\alpha}}. \tag{2.18}
\]

Also, since the Jacobian of the above \( S(x, y) \) equals one, for any unit vector \( l \in \mathbb{R}^2 \)

\[
\int_0^T \int_{\mathbb{R}^2} |D^2_{l_1} v(t, z)|^p \, dz \, dt \leq N_p \int_0^T \int_{\mathbb{R}^2} |f(t, z)|^p \, dz \, dt. \tag{2.19}
\]
3. Multidimensional second-order parabolic equations

**Theorem 3.1.** Let \( a(t) = (a^{ij}(t)) \) be a \( d \times d \) symmetric matrix-valued Borel measurable function on \((0, T)\) such that
\[
a^{ij}(t)\lambda^i \lambda^j \geq |\lambda|^2
\]
for all \( t \in (0, T) \) and \( \lambda \in \mathbb{R}^d \) and
\[
\int_0^T \text{tr} a(t) \, dt < \infty.
\]
Then for any \( f \in C_c([0, T] \times \mathbb{R}^d) \) there exists a unique continuous in \([0, T] \times \mathbb{R}^d\) solution \( u(t, x) \) of the equation
\[
\partial_t u(t, x) = a^{ij}(t)D_{ij}u(t, x) + f(t, x)
\]
for all \( t \in (0, T) \) and such that, for any \( t \in [0, T] \), \( u(t, \cdot) \in C^{2+\alpha}([0, T] \times \mathbb{R}^d) \) and, for any \( i, j = 1, \ldots, d \) and unit vector \( l \in \mathbb{R}^d \), we have
\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |u(t, x)| \leq T \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |f(t, x)|,
\]
\[
\sup_{t \in [0, T]} [D_{ij}u(t, \cdot)]_{C^\alpha(\mathbb{R}^d)} \leq N'(\alpha)N_0(\alpha) \sup_{t \in [0, T]} [f(t, \cdot)]_{C^\alpha(\mathbb{R})},
\]
\[
\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} [D^2_{ij}u(t, x + l \cdot)]_{C^\alpha(\mathbb{R})} \leq N_0(\alpha) \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} |f(t, x + l \cdot)|_{C^\alpha(\mathbb{R})},
\]
\[
\|D^2_{ij}u\|_{L^p((0, T) \times \mathbb{R}^d)}^p \leq N_p\|f\|_{L_p((0, T) \times \mathbb{R}^d)}^p,
\]
where \( N_0(\alpha), N_p \) are the constants from Section 2 (see (2.5) and (2.6)) and \( N'(\alpha) \) is a constant specified in Lemma 3.3.

We see, in particular, that the \( L^1 \)-norms of \( a^{ij}(t) \) do not influence the constants in the estimates.

**Lemma 3.2.** The assertions of Theorem 3.1, apart from (3.5), hold true if \( a^{ij} = \delta^{ij} \).

**Proof.** We proceed by induction on \( d \). Assume that the lemma is true for a particular \( d \) and repeat the construction in Lemma 2.2 treating \( x \) there as a point in \( \mathbb{R}^d \) and replacing \( D_x^2 \) with the Laplacian \( \Delta_x \) in \( \mathbb{R}^d \). Then, under the assumption that we are given \( f(t, x, y), \ t \in (0, T), x \in \mathbb{R}^d, y \in \mathbb{R} \), which is of class \( B_c((0, T), C^\infty_c(\mathbb{R}^{d+1})) \), we arrive at the conclusion that, for any \( h > 0 \), the equation
\[
\partial_t u_h(t, x, y) = \Delta_x u_h(t, x, y) + f(t, x, y)
\]
\[
+h^{-2}[u_h(t, x, y + h) - 2u_h(t, x, y) + u_h(t, x, y - h)],
\]
where \( t \in (0, T), x \in \mathbb{R}^d, y \in \mathbb{R} \), with zero initial condition has a unique continuous in \([0, T] \times \mathbb{R}^{d+1}\) solution \( u_h(t, x, y) = u_h(t, z) \), where \( z = (x, y) \), such that
\[
\sup_{(t, z) \in [0, T] \times \mathbb{R}^{d+1}} |u_h(t, z)| \leq T \sup_{(t, z) \in (0, T) \times \mathbb{R}^{d+1}} |f(t, z)|.
\]
compact set of \([0, T] \times \mathbb{R}^{d+1}\) of \(u(x, y)\) with respect to \((x, y)\), where \(e_1\) is the first basis vector and \(\delta > 0\).

Then, owing to (3.9) and the fact that any derivative of any order of \(f\) is in \(B_\delta((0, T), C_0^\infty(\mathbb{R}^{d+1}))\), we conclude that any finite-difference of any order of \(u_h\) is bounded on \(\mathbb{R}^{d+1}\) uniformly with respect to \(t\) and \(h\). It follows that \(u_h\) is infinitely differentiable with respect to \((x, y)\) and any derivative of any order is bounded on \([0, T] \times \mathbb{R}^{d+1}\). Then equation (3.8) itself (always considered in the integral form as (2.2)) shows that these derivatives are Lipschitz continuous in \(t\). Thus, the family \(u_h\) is equi-Lipschitz in each compact set of \([0, T] \times \mathbb{R}^{d+1}\) and the same holds for any derivative with respect to \((x, y)\) of \(u_h\).

Now by the Arzelà-Ascoli theorem there is a sequence \(u_{h_n}\), \(h_n \downarrow 0\), which converges uniformly on any set \([0, T] \times \{|(x, y)| \leq R\}, R \in (0, \infty)\), along with any derivative with respect to \((x, y)\) of \(u_{h_n}\) and \(\partial_t u_{h_n}\).

Writing (3.8) in the integral form as (2.2) and passing to the limit as \(n \to \infty\), we conclude that there exists a continuous function \(u(t, x, y)\) in \([0, T] \times \mathbb{R}^{d+1}\), which is infinitely differentiable with respect to \((x, y)\) with any derivative bounded on \([0, T] \times \mathbb{R}^{d+1}\); moreover, the equation

\[
\partial_t u(t, x, y) = \Delta_{x,y} u(t, x, y) + f(t, x, y)
\]

holds in \((0, T) \times \mathbb{R}^{d+1}\) and estimates (3.9), (3.10), and (3.11) are valid with \(u\) in place of \(u_h\).

Uniqueness of such solutions is a simple consequence of the maximum principle. The invariance of the Laplacian in \(\mathbb{R}^{d+1}\) under rotations shows that estimates (3.4), (3.6), and (3.7) are true with \(\mathbb{R}^{d+1}\) in place of \(\mathbb{R}^d\) for any unit vector \(l \in \mathbb{R}^{d+1}\) (cf. (2.18) and (2.19)). The lemma is proved. \(\square\)

The following lemma shows that (3.5) follows from (3.6).

**Lemma 3.3.** Let \(u \in C^{2+\alpha}(\mathbb{R}^d)\) be such that, for any unit vector \(l \in \mathbb{R}^d\), we have

\[
\sup_{x \in \mathbb{R}^d} |D^2 u(x + l \cdot)|_{C^\alpha(\mathbb{R})} \leq 1.
\]

Then there exists a constant \(N'(\alpha)\) such that for any \(i, j = 1, \ldots, d\) we have

\[
M := |D_{ij} u|_{C^\alpha(\mathbb{R}^d)} \leq N'(\alpha).
\]
Next, take \( x = \theta \) where in \( B \) a quadratic polynomial \( p \) and observe, that if \( r \) then by the mean-value theorem for any unit vector \( l \in \mathbb{R}^d \) and \( t \geq 0 \)
\[
|u(x + tl) - T_{x_0}(x_0 + tl)| = (1/2)t^2|D^2u(x_0 + \theta l) - D^2u(x_0)| \leq (1/2)t^{2+\alpha},
\]
where \( \theta \in (0, t) \). It follows that for any \( r \in (0, \infty) \) and \( x_0 \in \mathbb{R}^d \) there exists a quadratic polynomial \( p(x) \) such that
\[
|u(x) - p(x)| \leq (1/2)r^{2+\alpha}
\]
in \( B_r(x_0) = \{ x : |x - x_0| < r \} \).

Observe that by the mean-value theorem, for \( h > 0 \),
\[
|D_{ij}u(x) - \delta_{h,i}\delta_{h,j}u(x)| \leq M(2h)^\alpha.
\]
Next, take \( x_1, x_2 \in \mathbb{R}^d \), choose \( h = \varepsilon|x_1 - x_2| \), where \( \varepsilon \) is such that
\[
(2\varepsilon)^\alpha = 1/4,
\]
and observe, that if \( r = |x_1 - x_2| + 2h \), then all six points \( x_k, x_k + he_i, x_k + he_i + he_j, k = 1, 2 \), can be encompassed by a ball of radius \( r \) (centered at \( x_1 \)).

By the above, for an appropriate quadratic polynomial \( p \) (we use the fact that \( \delta_{h,i}\delta_{h,j}p \) is constant since it is a second-order difference of a quadratic polynomial)
\[
|D_{ij}u(x_1) - D_{ij}u(x_2)| \leq (1/2)M|x_1 - x_2|\alpha
\]
\[
+|\delta_{h,i}\delta_{h,j}(u - p)(x_1) - \delta_{h,i}\delta_{h,j}(u - p)(x_2)|,
\]
where the last term is less than
\[
|\delta_{h,i}\delta_{h,j}(u - p)(x_1)| + |\delta_{h,i}\delta_{h,j}(u - p)(x_2)|
\]
\[
\leq 3r^{2+\alpha}h^{-2} = 3(1 + 2\varepsilon)^{2+\alpha}\varepsilon^{-2}|x_1 - x_2|\alpha.
\]
The arbitrariness of \( x_1 \) and \( x_2 \) now yields the desired result with
\[
N(\alpha) = 6(1 + 2\varepsilon)^{2+\alpha}\varepsilon^{-2}.
\]
The lemma is proved. \( \square \)

In the sequel, given a unit vector \( l \in \mathbb{R}^d \), we denote by \( ll^* \) the \( d \times d \) matrix with entries \( l^Tl \).

**Lemma 3.4.** Let the assertions of Theorem 3.1 be true for a given \( a(t) \) satisfying the assumptions of the theorem and such that it is continuous. Let \( \nu(t) \) be a real-valued continuous function on \( [0, T] \) and \( l \in \mathbb{R}^d \) be a unit vector. Then the assertions of Theorem 3.1 hold true for \( a(t) + \nu^2(t)ll^* \), as well, with the same constants in the estimates (hence the constants are independent of \( \nu(t) \) and \( l \)).

**Proof.** Introduce
\[
b_t = l \int_{[0,t]} \nu(s) \, d\pi_s \quad (= l \sum_{\sigma_n \leq t} \nu(\sigma_n) = l \sum_{s \leq t} \nu(s)(\pi_s - \pi_{s-})).
\]
Observe that for $0 \leq s \leq t < \infty$

$$E(b_t - b_s) = \lambda \int_s^t \nu(r) \, dr$$  \hspace{1cm} (3.12)

(which is easily proved if $\nu$ is piece-wise constant, and then extended to continuous $\nu$ by standard arguments, cf. the proof of Lemma 2.1).

Then take a function $f(t, x)$ of class $B_c((0, T), C_0^\infty(\mathbb{R}^d))$ and for each $\omega$ solve the equation

$$\partial_t u(t, x) = a^{ij}(t)D_{ij}u(t, x) + f(t, x - hb_t)$$

with zero initial data, where $h \in \mathbb{R}$ is a parameter. In the sequel we drop the dependence on $h$. By assumption, there exists a unique continuous in $[0, T] \times \mathbb{R}^d$ solution $u(t, x)$ depending on $\omega$ as parameter such that estimates (3.4), (3.5), (3.6), and (3.7) hold for each $\omega$ if we replace $f(t, x)$ with $f(t, x - hb_t)$ (which, by the way, does not affect the right-hand sides of these estimates). Furthermore, since $f \in B_c((0, T), C_0^\infty(\mathbb{R}^d))$, $u(t, x)$ is uniformly continuous with respect to $x$ uniformly with respect to $\omega, t$, and $h$ (cf. the proof of Lemma 3.2).

By considering $u(t, x)$ on each interval $[\sigma_n, \sigma_{n+1})$ on which $\pi_t$, and hence $b_t$, are constant, one easily derives that $u(t, x + hb_t)$ satisfies

$$u(t, x + hb_t) = \int_0^t [a^{ij}(s)D_{ij}u(s, x + hb_s) + f(s, x)] \, ds + \int_{(0,t]} g(s, x) \, d\pi_s,$$

where

$$g(s, x) := u(s, x + hl\nu(s) + hb_{s-}) - u(s, x + hb_{s-}).$$

By introducing

$$g_n(s, x) = u(k2^{-n}, x + hl\nu(k2^{-n} + hb_{k2^{-n}-}) - u(k2^{-n}, x + hb_{k2^{-n}-})$$

for $s \in (k2^{-n}, (k + 1)2^{-n}]$, $k = 0, 1, \ldots$, using the continuity of $\nu(t)$ and (3.12), and repeating the proof of Lemma 2.1, we arrive at the conclusion that

$$E \int_{(0,t]} g(s, x) \, d\pi_s = \lambda \int_0^t [v(s, x + h\nu(s)) - v(s, x)] \, ds,$$

where

$$v(t, x) = Eu(t, x + hb_t).$$

Then (3.13) yields

$$\partial_t v(t, x) = a^{ij}(t)D_{ij}v(t, x) + \lambda[v(t, x + h\nu(t)) - v(t, x)] + f(t, x).$$

As in Section 2, $v$ is a unique solution of this equation for which all estimates claimed in the theorem hold true.

After that we solve

$$\partial_t w(t, x) = a^{ij}(t)D_{ij}w(t, x) + \lambda[w(t, x + h\nu(t)) - w(t, x)] + f(t, x + hb_t)$$
and repeat the end of Section 2 to conclude that for each \( h > 0 \) there exists a continuous function \( u_h(t, x) \) on \([0, T] \times \mathbb{R}^d\), which is a unique solution of
\[
\partial_t u_h(t, x) = a^{ij}(t)D_{ij}u_h(t, x) + f(t, x) + h^{-2}[u_h(t, x + hl\nu(t)) - 2u_h(t, x) + u_h(t, x - hl\nu(t))]
\]
in \((0, T) \times \mathbb{R}^d\) with zero initial condition and for which all estimates claimed in the theorem hold true.

As in the proof of Lemma 3.2, a subsequence \( u_{h_n} \) converges to the function we are after. The lemma is proved. □

**Proof of Theorem 3.1.** Uniqueness is easily derived from the maximum principle. (Just in case, if the reader sees any obstacle in the fact that \( a^{ij} \) may be unbounded, have in mind that a trivial time change (i.e., \( u(t, x) = v(\int_0^t \text{tr} a(s)ds, x) \)) reduces the general situation to the one with \( \text{tr} a(t) \equiv 1 \). Actually, after the time change the new matrix may degenerate, but this is not an obstacle for the maximum principle for parabolic equations to hold, see, for instance, Theorem 4.1 of [7]. Also see Corollary 3.6 there.) To prove the existence of solutions, by having in mind a simple passage to the limit (we say more about this in Theorem 4.5 and its proof in Section 6 or send the reader to the end of the present proof) and approximating \( a(t) \) by \( a_n(t) = a(t)I_{\text{tr} a(t) \leq n} + (\delta^{ij})I_{\text{tr} a(t) > n} \), we may assume that \( a(t) \) is bounded. By the same token we may assume that there exists a constant \( \varepsilon > 0 \) such that
\[
a^{ij}(t)\lambda_i\lambda_j \geq (1 + 2\varepsilon)|\lambda|^2
\]
for all \( t \in (0, T) \) and \( \lambda \in \mathbb{R}^d \).

Then for the matrix \( \hat{a}(t) = (\hat{a}^{ij}(t)) = (a^{ij}(t) - \delta^{ij}) \) we have
\[
\hat{a}^{ij}(t)\lambda_i\lambda_j \geq 2\varepsilon|\lambda|^2,
\]
for all \( t \in (0, T) \) and \( \lambda \in \mathbb{R}^d \). By assumption \( \text{tr} \hat{a}(t) \) is also bounded, so that \( a(t) \) takes values in a closed subset \( \Gamma \) of the set \( S(M) \) of symmetric \( d \times d \)-matrices \( a \) such that
\[
a^{ij}(t)\lambda_i\lambda_j > \varepsilon|\lambda|^2, \quad \lambda \neq 0, \quad \text{tr} a < M.
\]

One knows that there exist \( n \in \{1, 2, \ldots\} \), vectors \( l_1, \ldots, l_n \in \mathbb{R}^d \), and real-analytic real-valued functions \( \nu_1(a), \ldots, \nu_n(a) \) on \( S(M) \), such that for \( a \in \Gamma \) it holds that
\[
a = \sum_{k=1}^n \nu_k^2(a)l_k l_k^\ast
\]
(for instance, see Section 1 in [5]). In particular,
\[
a(t) = (\delta^{ij}) + \sum_{k=1}^n \nu_k^2(t)l_k l_k^\ast,
\]
where \( \nu_k(t) = \nu_k(\hat{a}(t)) \). The functions \( \nu_k(t) \) are continuous if \( a(t) \) is continuous, and, therefore, by using Lemma 3.2 and an obvious induction on the number of terms in (3.15) along with Lemma 3.4 we conclude that the
theorem holds true under the additional assumptions that \(a(t)\) is continuous and (3.14) holds.

To abandon the continuity assumption, we find uniformly bounded smooth \(a_n(t), \ n = 1, 2, \ldots,\) satisfying (3.1), such that \(a_n(t) \to a(t)\) as \(n \to \infty\) for almost all \(t\).

We extend \(a\) to the whole \(\mathbb{R}\) by setting \(a(t) = a(T/2),\) if \(t \geq T\) or \(t \leq 0\). Then we consider standard mollifiers \((\rho_n) \subset C^\infty_0(\mathbb{R})\) and introduce the matrices \(a_n(t) = (a_{ij}^n(t)),\)

\[ a_{ij}^n(t) = (a_{ij}^n * \rho_n)(t), \quad t \in \mathbb{R}. \]

It is clear that each \(a_n(t)\) is symmetric and non-negative and depends continuously on \(t\); moreover

\[ \sup_{t \in \mathbb{R}} \text{tr} a_n(t) \leq \sup_{t \in (0,T)} \text{tr} a(t) \]

and

\[ a_{ij}^n(t)\lambda^i\lambda^j \geq |\lambda|^2, \quad t \in \mathbb{R}, \lambda \in \mathbb{R}^d. \]

Let us consider solutions \(u_n\) of

\[ u_n(t, x) = \int_0^t a_{ij}^n(s) D_{ij} u_n(s, x) ds + \int_0^t f(s, x) ds, \quad (3.16) \]

the ones obtained according to the first part of the proof.

We can use estimates (3.4), (3.5), and (3.6) with \(u\) replaced by \(u_n\). Moreover, using also (3.16) we deduce that the family \(u_n\) is equi-Lipschitz in each compact set of \([0, T] \times \mathbb{R}^d\); the same holds for any derivative with respect to \(x\) of \(u_n\).

By the Arzelà-Ascoli theorem there is a subsequence which we still denote by \(u_n\) which converges uniformly on any set \([0, T] \times \{ |x| \leq R \}, \ R \in (0, \infty),\) along with any derivative with respect to \(x\) of \(u_n\).

Passing to the limit as \(n \to \infty\) in (3.16) we conclude that there exists a continuous function \(u(t, x)\) in \([0, T] \times \mathbb{R}^d\), which is infinitely differentiable with respect to \(x\) with any derivative bounded on \([0, T] \times \mathbb{R}^d\). Such function \(u\) is a solution to (3.3). Moreover estimates (3.5), (3.6) and (3.7) hold for \(u\).

\[ \square \]

**Corollary 3.5.** Let \(u \in C^\infty_0(\mathbb{R}^d)\) and assume that \(a(t)\) in Theorem 3.1 is independent of \(t\), i.e., \(a(t) = a\). Set

\[ f = a_{ij} D_{ij} u. \]

Then for all \(i, j = 1, \ldots, d\) and unit vector \(l \in \mathbb{R}^d\) we have

\[ [D_{ij} u]_{C^0(\mathbb{R}^d)} \leq N'(\alpha) N_0(\alpha) [f]_{C^\alpha(\mathbb{R}^d)}, \]

\[ \|D_l^2 u\|_{L_p(\mathbb{R}^d)}^p \leq N_p \|f\|_{L_p(\mathbb{R}^d)}^p. \]
Proof. Let $T > 0$. The function $u(t,x) := u(x)t/T$ is a unique bounded solution of
\[
\partial_t u(t,x) = a^{ij} D_{ij} u(t,x) + g_T(t,x)
\]
with zero initial condition, where $g_T(t,x) = u(x)/T - f(x)t/T$. By Theorem 3.1
\[
[D_{ij} u]_{C^0(\mathbb{R}^d)} \leq N'(\alpha) N_0(\alpha) \left( [f]_{C^0(\mathbb{R}^d)} + (1/T)[u]_{C^0(\mathbb{R}^d)} \right),
\]
\[
\int_{\mathbb{R}^d} |D^2_T u(x)|^p \, dx \int_0^T (t/T)^p \, dt \leq N_p \int_0^T \int_{\mathbb{R}^d} |u(x)/T - (t/T)f(x)|^p \, dxdt,
\]
\[
\int_{\mathbb{R}^d} |D^2_T u(x)|^p \, dx \leq (p + 1) N_p/T \int_0^T \int_{\mathbb{R}^d} [u(x)/T + (t/T)f(x)]^p \, dxdt \leq (p + 1) N_p \int_0^1 \int_{\mathbb{R}^d} |u(x)/T + sf(x)|^p \, dxds,
\]
and our assertions follow after letting $T \to \infty$. □

Remark 3.6. For fixed $T \in (0,\infty)$ denote by $N_p(d)$ the least constant $N$ such that
\[
\|D^2_T u\|_{L_p((0,T) \times \mathbb{R}^d)} \leq N \|f\|_{L_p((0,T) \times \mathbb{R}^d)}
\]
for any unit vector $l \in \mathbb{R}^d$, $f \in B_c((0,T), C_0^\infty(\mathbb{R}^d))$, and any bounded continuous in $[0,T] \times \mathbb{R}^d$ solution $u$ of the equation
\[
\partial_t u = \Delta u + f \tag{3.17}
\]
in $(0,T) \times \mathbb{R}^d$ with zero initial condition. It turns out that
\[
N_p(d) = N_p(1).
\]
Indeed, by Theorem 3.1, $N_p(d) \leq N_p(1)$. On the other hand, let
\[
\mathcal{W}_p^{1,2}(0,T \times \mathbb{R}^d) = \{ u \in \mathcal{W}_p^{1,2}([0,T] \times \mathbb{R}^d)) \cap C([0,T], L_p(\mathbb{R}^d)) : u(0,\cdot) = 0 \}
\]
(for the definition of $\mathcal{W}_p^{1,2}([0,T] \times \mathbb{R}^d)$, see for instance, page 153 in [6]).

We know that the operator $\partial_t - \Delta$ maps $\mathcal{W}_p^{1,2}([0,T] \times \mathbb{R}^d)$ onto $L_p((0,T) \times \mathbb{R}^d)$ in a one-to-one way and has a bounded inverse. Furthermore, the set $B_c((0,T), C_0^\infty(\mathbb{R}^d))$ is dense in $L_p((0,T) \times \mathbb{R}^d)$. It follows that $N_p(d)$ is the least constant $N$ such that for any $u \in \mathcal{W}_p^{1,2}([0,T] \times \mathbb{R}^d)$ and unit vector $l \in \mathbb{R}^d$ we have
\[
\|D^2_T u\|_{L_p((0,T) \times \mathbb{R}^d)} \leq N \|\partial_t u - \Delta u\|_{L_p((0,T) \times \mathbb{R}^d)}.
\]

Now, let $u(t,x)$ be a function of class $\mathcal{W}_p^{1,2}([0,T] \times \mathbb{R}^d)$ and let $\zeta(x') = \zeta(x^2,\ldots,x^d)$ be a nonzero function of class $C_0^\infty(\mathbb{R}^{d-1})$. Introduce $u_n(t,x) = u(t,x^1)\zeta(x'/n)$. By definition, $(D_{11} = \partial^2/\partial x^1)$
\[
\int_0^T \int_{\mathbb{R}} |D_{11} u(t,x)|^p \, dx dt \int_{\mathbb{R}^{d-1}} \zeta^p(y/n) \, dy
\]

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\[
N_p(d) \int_0^T \int_{R^d} |\zeta(x'/n)| [\partial_t u(t, x^1) - D_{11} u(t, x^1)]
- n^{-2} u(t, x^1) (\Delta \zeta)(x'/n) |^p \, dxdt,
\]
\[
\int_0^T \int_{R} |D_{11} u(t, x)|^p \, dxdt \int_{R^{d-1}} \zeta^p(y) \, dy
\leq N_p(d) \int_0^T \int_{R^d} |\zeta(x')| [\partial_t u(t, x^1) - D_{11} u(t, x^1)]
- n^{-2} u(t, x^1) (\Delta \zeta)(x') |^p \, dxdt.
\]
By letting \( n \to \infty \) we get
\[
\int_0^T \int_{R} |D^2 u(t, x)|^p \, dxdt \leq N_p(d) \int_0^T \int_{R} |\partial_t u(t, x) - D^2 u(t, x)|^p \, dxdt
\]
and, since this is true for any element \( u \) of \( W^{1,2}_p([0,T] \times R^d) \), we have \( N_p(d) \geq N_p(1) \).

4. General setting. Main results

Let \( W \) be a set consisting of real-valued (Borel) measurable functions \( u = u_t = u_t(x) \) on \([0, T] \times R^d \). In Sections 2 and 3 we only considered bounded solutions. Therefore, we assume that the elements of \( W \) are bounded and even uniformly bounded as required in Assumption 4.1 (i) below.

Let \( \mathcal{G} \) be a commutative group of affine volume-preserving transformations of \( R^d \). If \( g, h \in \mathcal{G} \) by \( gh \) we mean the composition of the two transformations.

Remark 4.1. We draw the reader’s attention to the fact that, since each \( g \in \mathcal{G} \) is measure-preserving, its Jacobian equals one.

As usual, if \( f(x) \) is a function on \( R^d \) and \( g \in \mathcal{G} \), we define \( (gf)(x) = f(gx) \), where \( gx \) is the image of \( x \) under mapping \( g \).

By \( B((0, T), \mathcal{G}) \) we denote the set of bounded measurable \( \mathcal{G} \)-valued functions on \((0, T)\).

Fix a constant \( K \in [0, \infty) \).

Assumption 4.1. (i) For any \( u \in W \) we have
\[
\sup_{(t,x) \in [0,T] \times R^d} |u_t(x)| \leq K.
\]
(ii) (Convexity of \( W \).) If \((\Omega, \mathcal{F}, P)\) is a probability space and \( u(\omega) = u_t(\omega, x) \) is an \( \mathcal{F} \times B([0,T] \times R^d) \)-measurable function such that \( u(\omega) \in W \) for any \( \omega \), then the function \( E[u_t(x)] \) belongs to \( W \) (by \( E \) we indicate the expectation with respect to \( P \) and by \( B([0,T] \times R^d) \) we mean the Borel \( \sigma \)-field on \([0,T] \times R^d) \).
(iii) ("Shift" invariance of \( W \).) For \( u \in W \) and any bounded measurable \( \mathcal{G} \)-valued function \( g_t \) given on \([0,T] \), the function \( u_t(g_t x) \) is in \( W \).
Let \( L := \{ L_t, t \in (0, T) \} \), be a family of linear operators
\[
L_t : C^\infty_0(\mathbb{R}^d) \to B(\mathbb{R}^d)
\]
\((B(\mathbb{R}^d))\) denotes the space of real-valued bounded and Borel functions defined on \(\mathbb{R}^d\) and take and fix
\[
f \in B((0, T) \times \mathbb{R}^d), \quad u_0 \in B(\mathbb{R}^d), \quad (4.1)
\]
where \(B((0, T) \times \mathbb{R}^d)\) is the set of Borel bounded functions on \((0, T) \times \mathbb{R}^d\).

**Assumption 4.2.** The couple \((L, f)\) is \(W\)-regular in the following sense.

(i) \((G \text{ and } L \text{ commute.})\) For any \(t \in (0, T)\) and \(g \in G\), we have \(gL_t = L_t g\).

(ii) For any \(\zeta \in C_0^\infty(\mathbb{R}^d)\), \(L_t \zeta(x) := (L_t \zeta)(x)\) is measurable with respect to \((t, x)\) and
\[
\int_{[0,T] \times \mathbb{R}^d} |L_t \zeta(x)| dt dx < \infty.
\]

(iii) There is a mapping \(B((0, T), G) \to W\) mapping every bounded measurable \(G\)-valued functions \(h = h_t, t \in (0, T)\), into \(u[h] \in W\) such that \(u = u[h]\) satisfies the equation
\[
u_t(x) = u_0(x) + \int_0^t [L_r^* u_r(x) + (h_r f_r)(x)] dr, \quad t \in [0, T], \quad x \in \mathbb{R}^d, \quad (4.2)
\]
in the sense specified below (see \((4.4)\)).

(iv) For any \(h', h'' \in B((0, T), G)\) and \((t, x) \in [0, T] \times \mathbb{R}^d\), we have
\[
|u_t[h'](x) - u_t[h''](x)| \leq K \int_0^t \sup_{y \in \mathbb{R}^d} |f_r(h'_r y) - f_r(h''_r y)| dr. \quad (4.3)
\]

**Remark 4.2.** Assumption 4.2 (iv) implies that, for any \(h \in B((0, T), G), x \in \mathbb{R}^d\), and \(t \leq s \leq T\) we have
\[
u_t[h](x) = u_t[h \wedge s](x).
\]
Indeed, it is enough to use \((4.3)\) with \(h' = h\) and \(h'' = h \wedge s\).

We say that \(u \in W\) satisfies \((4.2)\) if, for any \(\zeta \in C_0^\infty(\mathbb{R}^d)\) and \(t \in [0, T]\),
\[
(u_t, \zeta) := \int_{\mathbb{R}^d} u_t(x) \zeta(x) dx = (u_0, \zeta) + \int_0^t (u_s, L_s \zeta) ds + \int_0^t (h_s f_s, \zeta) ds. \quad (4.4)
\]

**Remark 4.3.** In light of Assumptions 4.1 (i), 4.2 (ii), and \((4.1)\), the right-hand side of \((4.4)\) makes sense for any \(u \in W\) and defines a continuous function of \(t\). Therefore, for any \(h \in B((0, T), G)\) and \(\zeta \in C_0^\infty(\mathbb{R}^d)\), the function \((u_t[h], \zeta)\) is continuous on \([0, T]\).

**Theorem 4.4.** Suppose that \(W, G, K, L, u_0,\) and \(f\), described above, satisfy Assumptions 4.1 and 4.2. Then, for any \(g^{(1)}, ..., g^{(n)} \in B((0, T), G)\) and \(\lambda_1, ..., \lambda_n \geq 0\), the couple, consisting of the family of operators \(\hat{L}_t\), such that
\[
\hat{L}_t^* = L_t^* + \sum_{i=1}^n \lambda_i (g^{(i)}_t - 1), \quad (4.5)
\]
where 1 stands for the operation of multiplying by one, and $f$, is $W$-regular.

This theorem is proved in Section 5.

To state our second general result we need one more assumption on $W$.

**Assumption 4.3.** For any sequence $u^k \in W$ and a bounded function $u = u_t(x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$, such that

$$
\int_{\mathbb{R}^d} u^k(x) \zeta(x) \, dx \to \int_{\mathbb{R}^d} u_t(x) \zeta(x) \, dx
$$

for any $t \in [0, T]$ and $\zeta \in C^\infty_0(\mathbb{R}^d)$, there exists $w \in W$ such that $w_t = u_t$ (a.e.) on $\mathbb{R}^d$ for any $t \in [0, T]$.

The main consequence of Assumption 4.3 is the following technical result.

**Theorem 4.5.** Suppose that Assumptions 4.1 (i) and 4.3 are satisfied. Let $\{L^k_t, t \in (0, T)\}$, $k = 0, 1, \ldots$, be a sequence of families of linear operators mapping $C^\infty_0(\mathbb{R}^d)$ into $B(\mathbb{R}^d)$ subject to the following conditions:

a) For each $k$, Assumption 4.2 (ii) is satisfied with $L^k_t$ in place of $L_t$;

b) For any $\zeta \in C^\infty_0(\mathbb{R}^d)$, we have

$$
\lim_{k \to \infty} \int_{(0,T) \times \mathbb{R}^d} |(L^k_t - L_t)\zeta(x)| \, dt \, dx = 0;
$$

c) For each $k = 1, 2, \ldots$, there exists $u^k \in W$ such that for any $\zeta \in C^\infty_0(\mathbb{R}^d)$ and $t \in [0, T]$,

$$
(u^k_t, \zeta) = (u_0, \zeta) + \int_0^t (u^k_s, L^k_s \zeta) \, ds + \int_0^t (f_s, \zeta) \, ds. \tag{4.6}
$$

Then there exists $u^0 \in W$ for which (4.6) holds with 0 in place of $k$ for any $\zeta \in C^\infty_0(\mathbb{R}^d)$ and $t \in [0, T]$.

This theorem is proved in Section 6. Theorem 4.5 allows us to improve the result of Theorem 4.4 under slightly heavier assumptions. (The conjecture is that, actually, Assumption 4.3 is not necessary in Theorem 4.6.)

**Theorem 4.6.** Suppose that $W$, $\mathcal{G}$, $K$, $L$, $u_0$, and $f$, described above, satisfy Assumptions 4.1, 4.2, and 4.3. Let $g^{(1)}, \ldots, g^{(n)} \in B((0, T), \mathcal{G})$ and $\lambda_1(t), \ldots, \lambda_n(t)$ be nonnegative bounded measurable functions. Then for any $h \in B((0, T), \mathcal{G})$ there exists $u \in W$ such that (4.4) holds for any $\zeta \in C^\infty_0(\mathbb{R}^d)$ and $t \in [0, T]$ with

$$
L_s + \sum_{i=1}^n \lambda_i(s)(g^{(i)}_s - 1),
$$

in place of $L_s$.

Proof. For real variable $r$ and integer $k \geq 1$ set $\kappa_k(r) = [kr]/k$, where $[r]$ stands for the integer part of $r$. Note that $|\kappa_k(r) - r| \leq 1/k$, for any $r \in \mathbb{R}$.
Set
\[ J^0_t = \sum_{i=1}^{n} \lambda_i(t)(g^{(i)}_t - 1), \quad J^k_t = \sum_{i=1}^{n} \kappa_k(\lambda_i(t))(g^{(i)}_t - 1). \]

Observe that, for an integer \( N \), which is larger than all \( \lambda_i(t) \), we have
\[ J^k_t = \sum_{i=1}^{n} \sum_{j=1}^{Nk} (i/k) I_{\{\kappa_k(\lambda_i(t)) = j/k\}} (g^{(i)}_t - 1) = \sum_{i=1}^{n} \sum_{j=1}^{Nk} (i/k)(g^{(ijk)}_t - 1), \]
where \( g^{(ijk)}_t = g^{(i)}_t \) if \( \kappa_k(\lambda_i(t)) = j/k \) and \( g^{(ijk)}_t = 1 \) otherwise.

It follows by Theorem 4.4 that for any \( k \geq 1 \) there exists \( u^k \in W \) such that (4.4) holds for any \( \zeta \in C^\infty_0(\mathbb{R}^d) \) and \( t \in [0, T] \) with \( L_s + J^k_s \) in place of \( L_s \).

Furthermore, for any \( \phi \in C^\infty_0(\mathbb{R}^d) \)
\[ \int_{(0,T)\times\mathbb{R}^d} |(J^0_s - J^k_s)\phi(x)| \, dt \, dx \]
\[ \leq \sum_{i=1}^{n} \int_{(0,T)\times\mathbb{R}^d} |\lambda_i(t) - \kappa_k(\lambda_i(t))| |(g^{(i)}_t - 1)\phi(x)| \, dt \, dx \]
\[ \leq (2T/k) \int_{\mathbb{R}^d} |\phi(x)| \, dx \]
(recall that the Jacobian of \( g^{(i)}_t \) is one) which tends to zero as \( k \to \infty \). An application of Theorem 4.5 finishes the proof of the present theorem. \( \Box \)

Next, let \( \mathfrak{N} \) be a subset of the space of affine transformations of \( \mathbb{R}^d \) and suppose that \( \mathcal{G} \) in the beginning of the section is given as
\[ \mathcal{G} = \{ e^{t\nu} : t \in \mathbb{R}, \nu \in \mathfrak{N} \}, \] (4.7)
where by \( e^{t\nu} \) we mean a transformation \( g(t) \) defined as a unique solution of the equation
\[ g(t) = 1 + \int_0^t \nu g(s) \, ds. \] (4.8)

Also for any \( \nu \in \mathfrak{N} \) we introduce a mapping \( \nu^0 \) by the formula
\[ \nu^0 x = \nu x - \nu 0. \]

Notice that the \( \nu^0 \)'s are linear mappings, which we identify with matrices in a usual way. Of course, we keep the assumption that \( \mathcal{G} \) is a commutative group of volume-preserving transformations.

Note in passing that, in case \( \mathcal{G} \) is given by (4.7), the volume-preserving assumption is satisfied if and only if \( \text{tr} \nu^0 = 0 \) for any \( \nu \in \mathfrak{N} \). Interestingly enough, this “if and only if” statement will never be used in the future.

With any \( \nu \in \mathfrak{N} \) we associate an operator \( M_\nu \) acting on functions \( \phi : \mathbb{R}^d \to \mathbb{R} \) by the formula
\[ M_\nu \phi(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^d} \left[ \phi(e^{\varepsilon \nu} x) - 2\phi(x) + \phi(e^{-\varepsilon \nu} x) \right], \ x \in \mathbb{R}^d, \]
whenever the limit on the right exists for all $x$.

Observe that if $\phi$ is twice continuously differentiable, then

$$M_\nu \phi(x) = \frac{d^2}{(d\varepsilon)^2} \phi(e^{\varepsilon \nu} x) |_{\varepsilon=0} = \frac{d}{d\varepsilon} \left\{ [D_i \phi](e^{\varepsilon \nu} x)(\nu e^{\varepsilon \nu} x)^i \right\} |_{\varepsilon=0}$$

$$= \frac{d}{d\varepsilon} \left\{ [D_i \phi](e^{\varepsilon \nu} x)(\nu^0 e^{\varepsilon \nu} x)^i \right\} |_{\varepsilon=0} + \frac{d}{d\varepsilon} \left\{ [D_i \phi](e^{\varepsilon \nu} x) \right\} |_{\varepsilon=0}$$

$$= (\nu^0 x)^i (\nu x)^j D_{ij} \phi(x) + (\nu^0 \nu x)^i D_i \phi(x) + (\nu x)^j D_{ij} \phi(x)$$

$$= (\nu x)^i (\nu x)^j D_{ij} \phi(x) + (\nu^2 x - \nu x)^i D_i \phi(x).$$

**Example 4.7.** Let $\nu = \nu_1$ by $\nu x = \nu_1 x$ on $\mathbb{R}^d$. Then (4.8) becomes

$$g(t)x = x + \int_0^t \nu g(s)x \, ds = x + \int_0^t I \, ds = x + tl.$$

Observe that in this example, for smooth $\phi$, we have $M_\nu \phi(x) = D_t^2 \phi(x)$. Thus, if $\mathcal{N} = \{ \nu_1 : l \in \mathbb{R}^d, ||l|| = 1 \}$, then $G$ is the set of shifts of $\mathbb{R}^d$ and $G$ is a commutative group. Just in case, observe that, for such $\mathcal{N}, \nu_1 \nu_2 = \nu_2 \nu_1$ unless $l_1 = l_2$ although $e^{t \nu_1} e^{t \nu_2} = e^{t \nu_2} e^{t \nu_1}$ always.

**Example 4.8.** Let $\nu x = Qx$, where $Q$ is a skew-symmetric $d \times d$-matrix. Then $g(t)x = e^{lt} x = (\exp[tQ])x$, where $\exp[tQ]$ is an orthogonal matrix. In this example, for smooth $\phi$,

$$M_\nu \phi(x) = (Qx)^i (Qx)^j D_{ij} \phi(x) + (Q^2 x)^i D_i \phi(x).$$

**Theorem 4.9.** Suppose that $W, G, K, L, u_0$ and $f$ satisfy Assumptions 4.1 and 4.2 with $G$ from (4.7) and suppose that $W$ also satisfies Assumption 4.3. Then, for any $\mu^{(1)}, \ldots, \mu^{(n)} \in B((0,T), \mathcal{N})$ equation (4.2) with

$$L_t^* + \sum_{i=1}^n M_{\mu^{(i)}}$$

in place of $L_t^*$ has a solution in $W$.

We prove this theorem in Section 6.

**Remark 4.10.** We concentrate on the case of scalar equations (4.2) only to slightly simplify the presentation. The results similar to Theorems 4.4, 4.6, and 4.9 also hold for systems, when $u_t(x)$ are vector- rather than real-valued functions. The reader will easily adjust our proofs to the case of systems.

**Example 4.11.** Let $d = 2$, $\alpha \in (0,1)$, and $L_t = \Delta$. We know that for any $f \in B_c((0,T), C_0^\infty (\mathbb{R}^2))$

$$u_t(x) = \int_0^t \Delta u_s(x) + f_s(x) \, ds, \quad t \leq T, x \in \mathbb{R}^2,$$

the equation

$$u_t(x) = \int_0^t \int_0^T [\Delta u_s(x(t,s)) + f(t,s)] \, ds \, dt, \quad t \leq T, x \in \mathbb{R}^2,$$
has a unique continuous solution such that
\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}^2} |u_t(x)| + \sup_{t \in [0,T]} \int_{\mathbb{R}^2} |u_t(x)| \, dx 
\leq N_0 \left[ \int_0^T \int_{\mathbb{R}^2} |f_t(x)| \, dx \, dt + \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} |f_t(x)| \right],
\]
(4.11)
\[
\sup_{t \in [0,T]} [D^2_t u_t]_{C^0(\mathbb{R}^2)} \leq N_0 \sup_{t \in [0,T]} [f_t]_{C^0(\mathbb{R}^2)}
\]
(4.12)
for any \( l \in S_1 = \{ |x| = 1 \} \), where \( N_0 \) and \( N_\alpha \) are some constants.

We claim that, if (4.9) holds, the equation
\[
u_t(x) = \int_0^t [\Delta u_s(x) + Mu_s(x) + f_s(x)] \, ds,
\]
where
\[
M\phi(x) = (x^2)^2 D_{11}\phi(x) - 2x^1x^2 D_{12}\phi(x) + (x^1)^2 D_{22}\phi(x) - x^1 D_1\phi(x) - x^2 D_2\phi(x),
\]
has a continuous solution, which satisfies estimates (4.11) and (4.12) (with the same \( N_0 \) and \( N_\alpha \)).

With the goal of applying Theorem 4.9, fix \( f \) as in (4.9) and denote by \( A_0 \) and \( A_\alpha \) the right-hand sides of (4.11) and (4.12), respectively. Then introduce
\[
W = \{ u \in B([0,T] \times \mathbb{R}^2) : u_t \in C^{2+\alpha}(\mathbb{R}^d), \ t \in [0,T], \ \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} |u_t(x)| + \sup_{t \in [0,T]} \int_{\mathbb{R}^2} |u_t(x)| \, dx \leq A_0, \ \sup_{t \in [0,T]} [D^2_t u_t]_{C^0(\mathbb{R}^2)} \leq A_\alpha \ \forall l \in S_1 \},
\]
and let \( \mathfrak{H} = \{ tQ : t \in \mathbb{R} \} \), where \( Q = (Q_{ij}) \) is a 2 \times 2-matrix, \( Q_{ii} = 0 \), \( Q_{12} = 1 \), \( Q_{21} = -1 \), \( i = 1,2 \). Note that since \( Q \) is skew-symmetric, \( G = \{ e^{tQ} : t \in \mathbb{R} \} \) is the group of rotations of \( \mathbb{R}^2 \) about the origin.

In light of Example 4.8 and Theorem 4.9, to prove our claim, it suffices to check that Assumptions 4.1, 4.2, and 4.3 are satisfied for the above \( W \) and \( \mathfrak{H}, u_0 = 0 \) and \( \Delta \) in place of \( L_t \).

Assumption 4.1 (i) is obviously satisfied. Assumption 4.1 (ii) is satisfied since, for instance,
\[
\sup_{t \in [0,T]} [D^2_t E u_t]_{C^0(\mathbb{R}^2)} \leq \sup_{t \in [0,T]} E [D^2_t u_t]_{C^0(\mathbb{R}^2)} \leq A_\alpha.
\]
Moreover, using that \( |g x| = |x|, \ g \in G \), we deduce that for any bounded measurable \( G \)-valued function \( g_t \) given on \([0,T]\)
\[
[D^2_t (u_t(g_t \cdot)))]_{C^0(\mathbb{R}^2)} = [(D^2_{g_t} u_t)(g_t \cdot))]_{C^0(\mathbb{R}^2)} = [D^2_{g_t} u_t]_{C^0(\mathbb{R}^2)} \leq A_\alpha.
\]
By adding to this that
\[
\int_{\mathbb{R}^2} |u_t(g_t x)| \, dx = \int_{\mathbb{R}^2} |u_t(x)| \, dx
\]
since \( \det g_t = 1 \), we conclude that the function \( u_t(g_t x) \) is in \( W \) and Assumption 4.1 is satisfied.

Assumption 4.2 (ii) is obviously satisfied and requirement (i) is satisfied since the Laplacian is rotation invariant. As long as Assumption 4.2 (iii) is concerned, observe that, for any \( h \in B((0, T), \mathcal{G}) \), we have \( h_t f_t = f_t(h_t) \in B_c((0, T), C_0^\infty(\mathbb{R}^2)) \), so that equation (4.10) with \( h_s f_s \) in place of \( f_s \) has a unique continuous solution and estimates (4.11) and (4.12) are valid with \( h_t f_t \) in place of \( f_t \). As is seen from the above arguments, this replacement does not change the right-hand sides of (4.11) and (4.12), which implies that Assumption 4.2 (iii) is satisfied. That Assumption 4.2 (iv) is satisfied is a simple consequence of the maximum principle.

To check Assumption 4.3, we consider a sequence \( u^k \) which converges in the specified weak sense to a function \( u \) defined on \([0, T] \times \mathbb{R}^2\). We fix \( t \in [0, T] \). Possibly passing to a subsequence and using the Arzelà-Ascoli theorem, we find that there exists \( w_t \in C^{2+\alpha}(\mathbb{R}^2) \) such that, along the subsequence, \( u^k_t \), \( D_i u^k_t \), and \( D_{ij} u^k_t \) converge to \( w_t \), \( D_t w_t \), and \( D_{ij} w_t \), respectively, uniformly on each compact subset of \( \mathbb{R}^2 \). In principle it could happen that along a different subsequence \( u^{k'}_t \), \( D_i u^{k'}_t \), and \( D_{ij} u^{k'}_t \) converge to \( w'_t \), \( D_t w'_t \), and \( D_{ij} w'_t \) uniformly on each compact subset of \( \mathbb{R}^2 \) and \( w_t \neq w'_t \). However, along both subsequences

\[
\int_{\mathbb{R}^d} u^k_t \zeta \, dx \to \int_{\mathbb{R}^d} u_t \zeta \, dx
\]

for any \( \zeta \in C_0^\infty(\mathbb{R}^d) \). It follows that

\[
w_t = w'_t = u_t
\]

in \( \mathbb{R}^d \) almost everywhere, and, since \( w_t \) and \( w'_t \) are continuous, \( w_t = w'_t \) everywhere.

Thus, for each \( t \in [0, T] \), the sequences \( u^k_t \), \( D_i u^k_t \), and \( D_{ij} u^k_t \) converge to \( w_t \), \( D_t w_t \), and \( D_{ij} w_t \), respectively, uniformly on each compact subset of \( \mathbb{R}^2 \) as \( k \to \infty \). Since \( u^k_t(x) \) are Borel measurable as functions of \((t, x)\), so is \( w_t(x) \). The fact that \( w \) satisfies the inequalities entering the definition of \( W \) is obvious. This proves our claim.

Remark 4.12. In Theorem 4.9 we could consider more general operators like

\[
L_t^* + \sum_{i=1}^n M_{\mu^{(i)}} + \sum_{j=1}^m F_{\nu^{(j)}},
\]

(4.13)

where \( F_{\nu^{(j)}} \) are first-order operators defined by

\[
F_{\nu}(\phi)(x) = |D_t \phi(x) (\nu x)|^\delta.
\]

The conclusion of Theorem 4.9 remains true since the substitution \( v_t(x) = u_t(g^{(1)}(t) \cdots g^{(m)}(t) x) \), where \( g^{(i)}(t) = e^{\nu^{(i)}_t} \), converts the equation for \( v_t(x) \) not containing \( F \)'s into an equation for \( u_t(x) \) with the additional first-order
terms. Of course, the free term will change. But it will satisfy the same estimates as before the above change of variables.

**Example 4.13.** As mentioned in Remark 4.10 results similar to Theorems 4.4, 4.6 and 4.9 also hold for systems. Without going into too much detail, we just give an example of the following hyperbolic system in $\mathbb{R}^2$:

$$\partial_t u_t(x) = v_t(x), \quad \partial_t v_t(x) = D_{11} w_t(x)$$

on $[0, T] \times \mathbb{R}^2$ with initial condition $w_0(x) = \zeta(x^1)\eta(x^2)$, $v_0(x) = \zeta'(x^1)\eta(x^2)$, where $\zeta, \eta \in C_0^\infty(\mathbb{R})$ are fixed function (of one variable and $\zeta'$ is the derivative of $\zeta$). Assume that $\zeta, \eta \geq 0$. Of course, $x^2$ enters system (4.14) only as parameter.

Take $\mathcal{R}$ and $\mathcal{G}$ from Example 4.7 and define $W$ as the collection of Borel $\mathbb{R}^2$-valued functions $u_t(x) = (\psi_t(x), \phi_t(x))$ on $[0, T] \times \mathbb{R}^2$ such that

$$\psi \geq 0 \text{ in } [0, T] \times \mathbb{R}^2 \text{ (a.e.),}$$

$$\int_{[0, T] \times \mathbb{R}^2} \psi_t(x) \, dx \, dt \leq T \int_{\mathbb{R}^2} w_0(x) \, dx,$$

$$\int_{[0, T] \times \mathbb{R}^2} |\phi_t(x)| \, dx \, dt \leq T \int_{\mathbb{R}^2} |v_0(x)| \, dx.$$

Of course, given an $\mathbb{R}^2$-valued function $(\psi(x), \phi(x))$ and $g \in \mathcal{G}$, we define $g(\psi(x), \phi(x)) = (\psi(gx), \phi(gx))$. Then, obviously, Assumption 4.1 is satisfied. Also observe that since by definition $g(\psi(x), \phi(x)) = (\psi(gx), \phi(gx))$, the operator $M_{v_t}$ from Example 4.7 will act on vector-valued functions by the formula $M_{v_t}(\psi(x), \phi(x)) = (D_t^1 \psi(x), D_t^2 \phi(x))$ if $\psi$ and $\phi$ are smooth enough.

Next, we define $L_t$ to be a $2 \times 2$ matrix whose entries are operators: $L_t^{11} = L_t^{22} = 0$, $L_t^{12} = D_{11}$, and $L_t^{21}$ is a unit operator. Finally, set $f \equiv 0$.

Then system (4.14) in the integral form becomes (4.2) and, for any bounded measurable $\mathcal{G}$-valued functions $h = h_t$, $t \in (0, T)$, it has a solution

$$u_t(x) = (\zeta(x^1 + t) \eta(x^2), \zeta'(x^1 + t) \eta(x^2))$$

(4.15)

(independent of $h$). This shows that Assumption 4.2 is also satisfied. Assumption 4.3 is easily verified as well, and by a vector-valued counterpart of Theorem 4.9 we obtain that the parabolic system

$$\partial_t w_t(x) = v_t(x) + \Delta w_t(x), \quad \partial_t v_t(x) = D_{11} w_t(x) + \Delta v_t(x),$$

$t \in [0, T]$, $x \in \mathbb{R}^2$, with initial data $w_0(x) = \zeta(x^1)\eta(x^2)$ and $v_0(x) = \zeta'(x^1)\eta(x^2)$ has a solution (in the sense explained after Assumption 4.2) belonging to $W$.

In particular, for this solution $w_t(x) \geq 0$ (a.e.). Actually, this result comes as no surprise since $(w_t, v_t) = T_t u_t$, where $u_t$ is defined in (4.15) and $T_t$ is the heat semigroup acting on $\mathbb{R}^2$-valued functions. We just wanted to show that our main results are applicable to systems of equations and not only in what concerns a priori estimates for scalar equations.
Example 4.14. Consider the following hyperbolic system taken from §7.3.3 of [1]
\[ \partial_t u_r^t(x) + B_j^r D_j u_k^t(x) = g_r^t(x) \] (4.16)
where \( r = 1, \ldots, m \), in \((0, T) \times \mathbb{R}^d\) with zero initial condition, where the \( m \times m \) constant matrices \( B_j := (B_j^r) \), \( j = 1, \ldots, d \), are such that for any \( \xi \in \mathbb{R}^d \), the matrix \( \xi^j B_j \) has \( m \) real eigenvalues. Assume that \( g_t(x) = (g_r^t(x)) \) is a \( \mathbb{R}^m \)-valued measurable functions such that
\[ \int_0^T \| g_t \|^2_{H^s(\mathbb{R}^d; \mathbb{R}^m)} \, dt = A < \infty, \]
where \( s > m + d/2 \) and \( H^s(\mathbb{R}^d; \mathbb{R}^m) = W^s_2(\mathbb{R}^d; \mathbb{R}^m) \) are the usual fractional Sobolev spaces of \( \mathbb{R}^m \)-valued functions (see their definitions, for instance, in §5.8.4 of [1]). By closely following the proof of Theorem 5 in §7.3.3 of [1] (given there for \( g = 0 \) but with nonzero initial value) one arrives at the conclusion that (4.16) with zero initial condition has a unique solution in \( \text{class } W \) which consists of measurable functions \( u_t(x) \) on \([0, T] \times \mathbb{R}^d\), such that \( u_t \in C^{0,1}_{\text{Lipschitz}}(\mathbb{R}^d; \mathbb{R}^m) \) for any \( t \in [0, T] \) and \( \| u_t \|_{L^2([0, T] \times \mathbb{R}^d; \mathbb{R}^m)} + \sup_{t \in [0, T]} \| u_t \|_{C^{0,1}_{\text{Lipschitz}}(\mathbb{R}^d; \mathbb{R}^m)} \leq N'A, \) (4.17)
where \( N' \) is a constant independent of \( g \). As in Example 4.11 one checks that the assumptions of Theorem 4.9 are satisfied with obvious matrix-valued first-order differential operator and \( \mathcal{G} \) being the group of translations.

Now take a bounded measurable \( d \times d \)-matrix valued function \( a = a_t \) which is symmetric and nonnegative for any \( t \in [0, T] \). Define \( \sigma_t = a_t^{1/2} \).

One knows that \( \sigma_t \) is also measurable and if \( \sigma_t^{(i)} \) is the \( i \)th column of \( \sigma(t) \), \( i = 1, \ldots, d \), then for smooth \( \phi = \phi(x) \)
\[ a_t^{ij} D_{ij} \phi = \sum_{i=1}^d D_{\sigma_t^{(i)}}^2 \phi \]
(cf. Example 4.7). Therefore, by Theorem 4.9 system (4.16) with the additional terms on the right-hand side \( a_t^{ij} D_{ij} u_r^t(x) \) has a solution of class \( W \). In particular, estimate (4.17) holds for the solution of the new system with the same right-hand side. Observe that the system is of unknown type, because no nondegeneracy assumption is imposed on \( a_t \).

It is worth mentioning that the fact that estimate (4.17) holds for the new system with a constant \( N' \) independent of \( a \) can also be obtained by closely following the proof of Theorem 5 in §7.3.3 of [1].

Remark 4.15. It could be that in each of the above examples one can prove our assertions by examining the classical proofs. However, the whole point is that under some easily verified conditions we have a unified method of adding new term into the equations without caring much as of why an how the sets \( W \) were proved to be appropriate in any particular problem.
Just in case, we recall that all equations are understood in a weak sense as in (4.4).

5. Proof of Theorem 4.4

We need some preparations. Again take independent and identically exponentially distributed with parameter \( \lambda > 0 \) random variables \( \tau_1, \tau_2, ... \) defined on a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and construct \( \pi_t \) as in Section 2. For \( t \geq 0 \) introduce \( \mathcal{F}_t \) as the smallest \( \sigma \)-fields in \( \Omega \) containing all sets of the form \( \{ \omega : \pi_s(\omega) = k \} \), \( s \leq t, k = 0, 1, ... \). Since, for \( t > s \), \( \pi_t - \pi_s \) is independent of \( \pi_r, r \leq s, \pi_t - \pi_s \) and \( \mathcal{F}_s \) are independent.

Also take \( g \in B((0, T), \mathcal{G}) \), extend it to \( [0, \infty) \) by setting \( g_0 = 1 \) and \( g_t = 1 \) for \( t \geq T \), where 1 is the operator of multiplying by 1, and define \( h_t = h_t(\omega) \in \mathcal{G} \) for \( t \geq 0 \) and \( \omega \in \Omega \) by

\[
h_t = g_{\sigma_n} h_{\sigma_n} - \quad \text{for} \quad t \in [\sigma_n, \sigma_{n+1}), \quad (5.1)
\]

\( n = 0, 1, ..., \) where \( \sigma_0 = 0 = : 0 \) and \( h_0 x := x, x \in \mathbb{R}^d \). In other terms,

\[
h_t = \prod_{n \leq \tau_t} g_{\sigma_n} = \prod_{n \leq \tau_t} g_{\sigma_n \land t}.
\]

Observe that the random variables \( \sigma_n \land t \) are \( \mathcal{F}_t \)-measurable because, for constant \( c \geq 0 \), the set \( \{ \omega : \sigma_n(\omega) \land t \leq c \} \) coincides with \( \Omega \) if \( c \geq t \), and if \( c \in [0, t) \), this set is \( \{ \omega : \sigma_n(\omega) \leq c \} = \{ \omega : \pi_c(\omega) \geq n \} \in \mathcal{F}_c \subset \mathcal{F}_t \). Since \( g_t \) is measurable, \( g_{\sigma_n \land t} \) is \( \mathcal{F}_t \)-measurable. It follows that \( h_t \) is \( \mathcal{F}_t \)-measurable for each \( t \), or, in other words, the process \( h_t \) is \( \mathcal{F}_t \)-adapted.

The construction of the stochastic process \( h \) with values in \( \mathcal{G} \) is inspired by the one of the simpler process \( b_t \) used in the proof of Lemma 3.4.

Also note that the number of jumps of \( \pi_t \) on \( [0, T] \) is finite and, therefore, \( h_t(\omega) \) is bounded on \( [0, T] \) for any \( \omega \).

Before the next result recall that the notation \( u_t[h] \) is introduced in Assumption 4.2, and \( (u_t, \zeta) \) in (4.4) and, according to what is said in the beginning of Section 4, \( gx \) is the image of \( x \) under mapping \( g \in \mathcal{G} \).

**Lemma 5.1.** Let \( h \) be introduced by (5.1) and let \( \tilde{h} \in B((0, T), \mathcal{G}) \). Then

(i) For any \( \zeta \in C_0^\infty(\mathbb{R}^d) \), the process \( \eta_t := (u_t[h(\omega)\tilde{h}], \zeta), t \in [0, T] \) is continuous and \( \mathcal{F}_t \)-adapted.

(ii) For any nonrandom bounded measurable \( \mathcal{G} \)-valued function \( \beta_t, t \in (0, T] \), the function

\[
u_t[h(\omega)\tilde{h}](h_{t-1}(\omega)\beta_t x)
\]

is \( \mathcal{F}_T \times B([0, T] \times \mathbb{R}^d) \)-measurable and belongs to \( W \) for any \( \omega \).

Proof. We will see that the assertions of the lemma hold true no matter which \( f \), satisfying (4.1), is taken in (4.2) in construction of \( u_t[h] \). Therefore, by replacing \( f \) in (4.2) with \( f' = \tilde{h} f \) we reduce the general situation to the one where \( \tilde{h} \equiv 1 \), which we assume henceforth.

(i). The continuity of \( \eta_t \) follows from Remark 4.3. To investigate its measurability properties, we need the separable Banach space \( L_1((0, T), \mathcal{G}) \).
of measurable and integrable \( \mathcal{G} \)-valued functions on \([0,T]\). Notice that any element \( \alpha \in \mathcal{G} \) is an affine transformation and \( \alpha x \) has a unique representation as \( a_\alpha x + b_\alpha \), where \( a_\alpha \) is a linear mapping and \( b_\alpha \) is a vector. The norms of \( a_\alpha \) and \( b_\alpha \) are well defined and we make the space, say \( \Lambda \), of affine transformation of \( \mathbb{R}^d \) a linear normed space by setting

\[
|\alpha' - \alpha''| = |a_{\alpha'} - a_{\alpha''}| + |b_{\alpha'} - b_{\alpha''}|
\]

After that we introduce the norm in the linear space \( L_1((0,T), \mathcal{G}) \) by setting

\[
||\alpha||_{L_1((0,T), \mathcal{G})} = \int_0^T |\alpha_t| \, dt.
\]

As any \( L_1 \)-space relative to Lebesgue measure of functions on \((0,T)\) with values in finite-dimensional spaces, the space \( L_1((0,T), \mathcal{G}) \) is Polish.

Next, we take continuous \( \Lambda \)-valued functions \( \phi^m(\alpha) \), \( m = 1,2,... \), on \( \Lambda \) each of which is bounded and such that \( \phi^m(\alpha) = \alpha \) for \( |\alpha| \leq m \).

Observe that, if \( \alpha^n \in L_1((0,T), \mathcal{G}) \), \( n = 0,1,... \), are such that \( \alpha^n \to \alpha^0 \) in \( L_1((0,T), \mathcal{G}) \) as \( n \to \infty \), then, for any fixed \( m = 1,2,... \), \( \alpha^{mn} := \phi^m(\alpha^n) \in B((0,T), \mathcal{G}) \), so that \( u_t[\alpha^{mn}](x) \) are well defined. We claim that in this situation \( u_t[\alpha^{mn}](x) \to u_t[\alpha^0](x) \) uniformly on \([0,T] \times \mathbb{R}^d \) as \( n \to \infty \).

To prove this claim, thanks to Assumption 4.2 (iv), it suffices to show that, for any fixed \( m \),

\[
I_n := \int_0^T \sup_{y \in \mathbb{R}^d} |f_r(\alpha^{mn}_r y) - f_r(\alpha^{m0}_r y)| \, dr \to 0 \quad (5.3)
\]
as \( n \to \infty \). As usual, it suffices to prove (5.3) assuming that \( \alpha^n_t \to \alpha^0_t \) for almost any \( t \). For such \( t \) and any \( y \) we have \( f_t(\alpha^{mn}_t y) - f_t(\alpha^{m0}_t y) \to 0 \) by continuity. Furthermore the functions \( f_t(\alpha^{mn}_t y) - f_t(\alpha^{m0}_t y) \) are supported in the same ball and are uniformly continuous (\( t \) and \( m \) are fixed). Therefore, the convergence \( f_t(\alpha^{mn}_t y) - f_t(\alpha^{m0}_t y) \to 0 \) is uniform on \( \mathbb{R}^d \), and this implies (5.3) by the dominated convergence theorem.

Hence, \( u_t[\phi^m(\alpha)](x) \) is continuous with respect to \( \alpha \in L_1((0,T), \mathcal{G}) \) uniformly with respect to \((t,x)\).

Next, coming back to \( h(\omega) \) observe that for any \( \alpha \in L_1((0,T), \mathcal{G}) \) the random function

\[
\rho(\alpha, h) := \int_0^T |\alpha_t - h_t| \, dt = \sum_{n \leq \pi_T} \int_{\sigma_n \wedge T}^{\sigma_{n+1} \wedge T} |\alpha_t - \prod_{i \leq n} g_{\sigma_i \wedge T}| \, dt
\]
is \( \mathcal{F}_T \)-measurable. Therefore, we have

\[
\{ \omega : \rho(\alpha, h(\omega)) \leq \rho \} \in \mathcal{F}_T
\]
for any \( \alpha \in L_1((0,T), \mathcal{G}) \) and \( \rho > 0 \). Since \( L_1((0,T), \mathcal{G}) \) is a Polish space, we get that \( h(\omega) \) is an \( \mathcal{F}_T \)-measurable \( L_1((0,T), \mathcal{G}) \)-valued function.

Now we conclude that, since \( u_t[\phi^m(\alpha)](x) \) is continuous in \( \alpha \) and \( h(\omega) \) is \( \mathcal{F}_T \)-measurable, \( u_t[\phi^m(h(\omega))](x) \) is \( \mathcal{F}_T \)-measurable. By observing that \( h_0(\omega) \) is bounded for each \( \omega \) by definition, we conclude that \( u_t[\phi^m(h(\omega))](x) \to \)
Remark 5.2. If $\beta_t$ is a predictable $\mathcal{G}$-valued process, $\zeta \in C_0^\infty(\mathbb{R}^d)$, $h$ is taken from (5.1), and $\hat{h} \in B((0,T), \mathcal{G})$, then $(u_t[h(\omega)\hat{h}], \zeta(\beta_t(\omega)\cdot))$ is predictable. This follows from the fact that $(u_t[h(\omega)\hat{h}], \zeta(\beta_t(\omega)\cdot))$ is predictable for any $\beta \in \mathcal{G}$ and is continuous with respect to $\beta$ so that it is jointly measurable with respect to $(\omega, t, \beta)$.

We are going to use the following.
Lemma 5.3. Let $\xi_t$ be a predictable process such that
\[ E \int_0^t |\xi_s| \, ds < \infty. \]
Then
\[ E \int_{[0,t]} \xi_s \, d\pi_s = \lambda E \int_0^t \xi_s \, ds. \] (5.4)

This lemma follows from Theorem 16 and the comments after it in Section III.5 on page 118 of [11]. Since going through the material before that theorem can be somewhat painful for inexperienced reader we give a short proof.

First of all we note that it suffices to concentrate on bounded processes $\xi_t$. This follows from the monotone convergence theorem by a routine argument. In that case the lemma is just Exercise 2.7.8 of [4] and its solution, given below, is outlined in the hint to this exercise.

One fixes $t > 0$ and introduces two measures on $\mathcal{F} \times \mathcal{B}(0,t]$.
\[ \mu(B) = E \int_{[0,t]} I_B(\omega, r) \, d\pi_r, \quad \nu(B) = \lambda E \int_{[0,t]} I_B(\omega, r) \, dr. \]

When $B = A \times (a, b]$, with $A \in \mathcal{F}_a$, and $a < b \leq t$, we have $\mu(B) = EI_A(\pi_b - \pi_a) = \lambda P(A)(b-a) = \nu(B)$ because $I_A$ and $\pi_b - \pi_a$ are independent. The equality $\mu(B) = \nu(B)$ for $B = A \times (a, b]$ is also easily verified for other dispositions of $a, b, t$. Thus, $\mu = \nu$ on such sets $B$. Since the collection of such $B$ is a $\Pi$-system (see the definition of $\Pi$-system in [4]), by a very general fact from measure theory (see Lemma 2.3.18 in [4]) $\mu(B) = \nu(B)$ on the smallest $\sigma$-field containing all such $B$, that is, on $\mathcal{P}$.

We thus have proved (5.4) if $\xi_s(\omega)$ is the indicator of a predictable set. The same equality is true if $\xi_s(\omega)$ is a finite linear combination of the indicators of predictable sets with nonrandom coefficients. Since bounded measurable functions admit uniform approximations by finite linear combinations of the indicators of measurable sets, (5.4) holds for all bounded predictable processes and the lemma is proved. \hfill \Box

Remark 5.4. The reader may feel uncomfortable encountering the above measure-theoretic arguments which we easily avoided in Sections 2 and 3. Unfortunately, these arguments are necessary in the general theory. To see this, observe that
\[ \int_{[0,t]} \pi_s \, d\pi_s = \sum_{n=0}^{\pi_t} n = (1/2)\pi_t(\pi_t + 1), \quad E \int_{[0,t]} \pi_s \, d\pi_s = \lambda t + \lambda^2 t^2 / 2. \]

At the same time
\[ E \int_0^t \pi_s \, ds = \int_0^t E\pi_s \, ds = \lambda \int_0^t s \, ds = \lambda t^2 / 2, \]
and (5.4) does not hold for $\xi_s = \pi_s$. 
However,
\[ \int_{[0,t]} \pi_{s-} d\pi_s = \int_{[0,t]} [\pi_s - 1] d\pi_s = (1/2)\pi_t (\pi_t - 1), \quad E \int_{[0,t]} \pi_{s-} d\pi_s = \lambda t^2 / 2. \]
By the way, one of consequences of these calculations and Lemma 5.3 is that the process \( \pi_t \) is not predictable, although \( \pi_{t-} \) is.

**Proof of Theorem 4.4.** Obvious induction on \( n \) allows us to concentrate on \( n = 1 \) and assume that \( \lambda = \lambda_1 > 0 \). Next, the requirements (i) and (ii) of Assumption 4.2 are obviously satisfied for the operators whose formal adjoints are defined in (4.5). To check the remaining requirements, take \( g \in B((0,T), G) \), take \( h \) and \( \zeta \) as in Lemma 5.1, take any \( \hat{h} \in B((0,T), G) \), and consider the process
\[ \xi_t = (u_t[h\hat{h}], \zeta(h_t\cdot)), \]
where and below we drop the argument \( \omega \) as usual. This process is well-defined since changing variables (recall Remark 4.1) we get
\[ |(u_t[h(\omega)\hat{h}], \zeta(h_t\cdot))| \leq \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |u_t[h(\omega)\hat{h}](x)| \int_{\mathbb{R}^d} |\zeta(y)|dy. \]
By the same reason the processes \( \xi_{t,r} = (u_t[h\hat{h}], \zeta(h_r\cdot)) \) are well defined for \( t, r \in [0,T] \). In addition, for any fixed \( r \in [0,T] \), viewing \( \zeta(h_r x) \) just as another \( C_0^\infty(\mathbb{R}^d) \)-function, for \( t \in [\sigma_n, \sigma_{n+1}] \) we obtain
\[ \xi_{t,r} = \xi_{\sigma_n,r} + \int_{\sigma_n}^t (u_s[h\hat{h}], L_s \zeta(h_r\cdot)) ds + \int_{\sigma_n}^t (h_s \hat{h}_s f_s, \zeta(h_r\cdot)) ds. \]
We substitute here \( r = \sigma_n \) and observe that for \( t \in [\sigma_n, \sigma_{n+1}] \) the function \( h_t \) does not change and equals \( h_{\sigma_n} \) and
\[ \xi_{t,\sigma_n} = (u_t[h\hat{h}], \zeta(h_{\sigma_n}\cdot)) = (u_{\sigma_n}[h\hat{h}], \zeta(h_{\cdot}\cdot)) = \xi_{\sigma_n}. \]
Then we conclude that similarly to (2.12), for \( t \in [\sigma_n, \sigma_{n+1}] \),
\[ \xi_t = \xi_{\sigma_n} + \int_{\sigma_n}^t (u_s[h\hat{h}], L_s \zeta(h_s\cdot)) ds + \int_{\sigma_n}^t (h_s \hat{h}_s f_s, \zeta(h_s\cdot)) ds. \]
At time \( t = \sigma_{n+1} \) the process \( h_t \) jumps from \( h_{\sigma_{n+1}-} \) to \( h_{\sigma_{n+1}} = g_{\sigma_{n+1}} h_{\sigma_{n+1}-} \), so that
\[ \xi_{\sigma_{n+1}-} = \xi_{\sigma_n} + \int_{\sigma_n}^{\sigma_{n+1}} (u_s[h\hat{h}], L_s \zeta(h_s\cdot)) ds \]
\[ + \int_{\sigma_n}^{\sigma_{n+1}} (h_s \hat{h}_s f_s, \zeta(h_s\cdot)) ds, \]
\[ \xi_{\sigma_{n+1}} = \xi_{\sigma_{n+1}-} + [(u_{\sigma_{n+1}}[h\hat{h}], \zeta(g_{\sigma_{n+1}} h_{\sigma_{n+1}-} \cdot)) - \xi_{\sigma_{n+1}-}]. \]
It follows easily that for (each \( \omega \) and) \( t \in [0,T] \) we have
\[ (u_t[h\hat{h}], \zeta(h_t\cdot)) = (u_0, \zeta) + \int_0^t (u_s[h\hat{h}], L_s \zeta(h_s\cdot)) ds. \]
\[
\int_0^t (\hat{h}_s \hat{f}_s, \zeta(h_{s-})) ds \\
+ \int_{(0,t]} [(u_s[\hat{h}], \zeta(g_h h_{s-})) - \xi_{s-}] d\pi_s. 
\]

(5.5)

The above formulas show that \( \xi_{t-} \) is a well-defined left-continuous process, which is \( \mathcal{F}_t \)-adapted since \( \xi_t \) is such (cf.
Lemma 5.1). We observe also that, by Remark 5.2 and the
fact that \( h_{t-} \) is left-continuous, \( \mathcal{F}_t \)-adapted, and hence predictable process, the last
integrand is predictable.

Of course, we want to take expectations of both sides of (5.5) and use
Lemma 5.3. Introduce
\[
v_t(x) = u_t[h(\omega)\hat{h}](h_t^{-1}(\omega)x).
\]

By Lemma 5.1 we have \( v \in W \) for any \( \omega \). In particular, \( |v_t(x)| \leq K \).
Hence, changing variables (see Remark 4.1) we find
\[
E \int_0^t |\xi_{s-}| ds = E \int_0^t |\xi_s| ds = E \int_0^t |(v_s, \zeta)| ds \leq KT \int_{\mathbb{R}^d} |\zeta(y)| dy < \infty.
\]

Similarly,
\[
E \int_0^t (|u_s[\hat{h}]|, |\zeta(g_h h_{s-})|) ds \leq E \int_{[0,T] \times \mathbb{R}^d} |v_s(g_h^{-1} h_{s-} x)| |\zeta(x)| dx ds
\]
\[
\leq KT \int_{\mathbb{R}^d} |\zeta(y)| dy < \infty.
\]

Dealing with other terms on the right in (5.5) presents no problem either, and, after taking the expectations of both sides and using Fubini’s theorem, we obtain
\[
E(u_t[h\hat{h}](h_t^{-1}x), \zeta) = (u_0, \zeta) + \int_0^t (Ev_s, L_s \zeta) ds + \int_0^t (\hat{h}_s f_s, \zeta) ds
\]
\[
+ \lambda \int_0^t (Ev_s(g_h^{-1} - Ev_s, \zeta) ds. 
\]

(5.6)

Since \( P(t \in \{\sigma_1, \sigma_2, \ldots\}) = 0 \), we have \( E(u_t[h\hat{h}](h_t^{-1}x), \zeta) = E(v_t, \zeta) \).
Furthermore,
\[
E(|v_t|, |\zeta|) \leq K \int_{\mathbb{R}^d} |\zeta(y)| dy < \infty,
\]

which allows us to use Fubini’s theorem and conclude that \( E(v_t, \zeta) = (Ev_t, \zeta) \).
This and (5.6) show that the function
\[
w_t[\hat{h}](x) = w_t(x) := E_{v_t}(x) = E_{u_t[h\hat{h}](h_t^{-1}x)},
\]

which belongs to \( W \) by Lemma 5.1 and by assumption, satisfies
\[
w_t(x) = u_0(x) + \int_0^t [L^*_r w_r(x) + \lambda(g_r^{-1} - 1) w_r(x) + \hat{h}_r f_r(x)] dr.
\]
This equation coincides with
\[ w_t(x) = u_0(x) + \int_0^t [\hat{L}_r w_r(x) + \hat{h}_r f_r(x)] \, dr \]
if we take \( n = 1 \) and \( g^{(1)}_t = g^{-1}_t \), which is as arbitrary as a member of \( B((0, T), G) \) could be. Hence Assumption 4.2 (iii) is satisfied. Finally, if \( h', h'' \in B((0, T), G) \), then due to our assumptions
\[
|w_t[h'](x) - w_t[h''](x)| = |Eu_t[hh'](h^{-1}_t x) - Eu_t[hh''](h^{-1}_t x)| \leq \sup_{x, \omega} |u_t[hh'](x) - u_t[hh''](x)|
\]
\[
\leq K \sup_{\omega, x} \int_0^t \sup_{y \in \mathbb{R}^d} |f_r(h'_r h_r y) - f_r(h''_r h_r y)| \, dr = K \int_0^t \sup_{y \in \mathbb{R}^d} |f_r(h'_r y) - f_r(h''_r y)| \, dr,
\]
which shows that Assumption 4.2 (iv) is satisfied as well and proves the theorem.

6. Proof of Theorems 4.5 and 4.9

**Proof of Theorem 4.5.** By Assumption 4.1 (i) all \( u^k_t(x) \), \( k \geq 1 \), are uniformly bounded on \( [0, T] \times \mathbb{R}^d \). Then there exists a subsequence still denoted by \( u^k \) and a bounded (Borel) function \( u \) on \( [0, T] \times \mathbb{R}^d \) such that for any \( \zeta \in L_1([0, T] \times \mathbb{R}^d) \) we have
\[
\int_{[0,T] \times \mathbb{R}^d} u^k_t(x) \zeta_t(x) \, dx \, dt \to \int_{[0,T] \times \mathbb{R}^d} u_t(x) \zeta_t(x) \, dx \, dt.
\]
Next, take \( \zeta \in C_{0}^\infty(\mathbb{R}^d) \) and write that by definition
\[
(u^k_t, \zeta) = (u_0, \zeta) + \int_0^t \left[ (u^k_s, L^0_s \zeta) + (f_s, \zeta) \right] \, ds + F^k_t, \quad (6.1)
\]
where
\[
F^k_t = \int_0^t (u^k_s, (L^k_s - L^0_s) \zeta) \, ds.
\]
Let us fix \( t \in (0, T] \). In light of Assumption 4.1 (i) and requirement b) in the theorem we have that \( F^n_t \to 0 \) as \( n \to \infty \).

There are two consequences of this fact. First, the right-hand sides of (6.1) converge as \( k \to \infty \) to
\[
(u_0, \zeta) + \int_0^t \left[ (u_s, L^0_s \zeta) + (f_s, \zeta) \right] \, ds.
\]
Secondly, the left-hand sides of (6.1) also converge for any \( \zeta \in C_0^\infty(\mathbb{R}^d) \) to the limit, say \( \phi_t(\zeta) \), which is a generalized function. Since
\[
|\phi_t(\zeta)| \leq K \int_{\mathbb{R}^d} |\zeta(x)| \, dx,
\]
\( \phi_t \) can be extended to a linear continuous functional on \( L^1(\mathbb{R}^d) \) and so there exists a (bounded) function \( u = u_t(x) \), \((t, x) \in [0, T] \times \mathbb{R}^d \), such that
\[
\phi_t(\zeta) = \int_{\mathbb{R}^d} u_t(x) \zeta(x) \, dx, \quad \int_{\mathbb{R}^d} u_t^k(x) \zeta(x) \, dx \to \int_{\mathbb{R}^d} u_t(x) \zeta(x) \, dx
\]
for any \( t \in [0, T] \) and \( \zeta \in C_0^\infty(\mathbb{R}^d) \).

Another way to get the same result is to fix \( R > 0 \), take the ball \( B \) of radius \( R \) centered at the origin, and take a subsequence \( u_t^k \) such that \( u_t^k I_B \) converges weakly in \( L_2(B) \) to a function \( u_B^t \). Then, obviously,
\[
\int_{\mathbb{R}^d} u_B^t(x) \zeta(x) \, dx = \phi_t(\zeta)
\]
for any \( \zeta \in C_0^\infty(B) \). This holds for any weakly convergent subsequence of \( u_t^k I_B \), and shows that the weak limit is always the same. Hence, the whole sequence \( u_t^k I_B \) converges weakly in \( L_2(B) \) to \( u_B^t \). Of course, (6.3) implies that, for balls \( B' \subset B'' \), \( u_B^{t'} = u_B^{t''} \) on \( B' \) and this allows us to define \( u_t \) on \( \mathbb{R}^d \) for which (6.2) hold for any \( \zeta \in C_0^\infty(\mathbb{R}^d) \).

By Assumption 4.3 there exists \( w \in W \) such that \( w_t = u_t \) (a.e.) on \( \mathbb{R}^d \) for any \( t \in [0, T] \). It follows that
\[
\int_{\mathbb{R}^d} u_t^k(x) \zeta(x) \, dx \to \int_{\mathbb{R}^d} u_t(x) \zeta(x) \, dx
\]
for any \( t \in [0, T] \) and \( \zeta \in C_0^\infty(\mathbb{R}^d) \). Hence, for any \( t \in [0, T] \) and \( \zeta \in C_0^\infty(\mathbb{R}^d) \)
\[
(w_t, \zeta) = (u_0, \zeta) + \int_0^t \left[ (u_s, L_0^0 \zeta) + (h_s f_s, \zeta) \right] ds. \tag{6.5}
\]

Next, note that, for any smooth function \( \eta_t(x) \) with compact support in \((0, T) \times \mathbb{R}^d \), on the one hand, by definition of \( u_t(x) \)
\[
\lim_{k \to \infty} \int_{(0, T) \times \mathbb{R}^d} u_t^k(x) \eta_t(x) \, dx \, dt = \int_{(0, T) \times \mathbb{R}^d} u_t(x) \eta_t(x) \, dx \, dt.
\]
On the other hand, owing to (6.4), by the dominated convergence theorem,
\[
\lim_{k \to \infty} \int_0^T dt \int_{\mathbb{R}^d} u_t^k(x) \eta_t(x) \, dx = \int_0^T dt \int_{\mathbb{R}^d} w_t(x) \eta_t(x) \, dx.
\]
It follows that \( u_t(x) = w_t(x) \) (a.e.) in \((0, T) \times \mathbb{R}^d \) and we can replace \( u_s \) with \( w_s \) in (6.5) without violating this equality. This proves the theorem. \( \square \)

We build our proof of Theorem 4.9 entirely on Theorems 4.4 and 4.5 thus avoiding using probability theory. We need the following. Recall that if \( \nu \in \mathfrak{N} \), we set \( \nu^0 \nu = \nu \sigma - \nu \sigma_0 \), and \( \nu^0 \) is a linear mapping.

**Lemma 6.1.** Let \( \nu \in \mathfrak{N} \) and let \( g \) be the solution of (4.8). Then, for any \( \zeta \in C_0^\infty(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \),
\[
\zeta(g^{-1}(t)x) = \zeta(x) - (\nu x)^i D_i \zeta(x) t + (1/2) \left[ (\nu x)^i (\nu x)^j D_{ij} \zeta(x) + (\nu^0 \nu x)^i D_i \zeta(x) \right] t^2 + o(t^2)
\]
as \( t \downarrow 0 \).
Proof. For any \( y \in \mathbb{R}^d \) we have \( \dot{g}(t)y = \nu g(t)y + \nu 0 \). The solution of this equation which equals \( y \) at \( t = 0 \) is
\[
g(t)y = e^{\nu t}y + \int_0^t e^{\nu t_s} \nu 0 \, ds.
\]
It follows that
\[
g^{-1}(t)x = e^{-\nu t}x - \int_0^t e^{\nu (s-t)} \nu 0 \, ds
\]
and the results follow by Taylor’s formula. The lemma is proved. \( \square \)

**Proof of Theorem 4.9.** Take \( \mu^{(1)}, ..., \mu^{(n)} \in B((0,T),\mathfrak{N}) \) and for \( k = 1, 2, ..., \) set
\[
L^k_t = L_t + \sum_{r=1}^n M_t^{(r)k}.
\]
where
\[
M_t^{(r)k} \phi(x) := k^2[\phi(e^{\mu^{(r)}_{s} / k} x) - 2 \phi(x) + \phi(e^{-\mu^{(r)}_{s} / k} x)].
\]
Observe that \( M_t^{(r)k} \) are formally self-adjoint, so that
\[
L^*_t = L^*_t + \sum_{r=1}^n M_t^{(r)k}
\]
and by Theorem 4.4, for any \( k \geq 1 \), and \( h \in B((0,T),\mathcal{G}) \) there exists \( u^k \in W \) satisfying
\[
(u^k_t, \zeta) = (u^0_0, \zeta) + \int_0^t (u^k_s, L^k_s \zeta) \, ds + \int_0^t (h_s f_s, \zeta) \, ds. \tag{6.6}
\]
for any \( \zeta \in C_0^\infty(\mathbb{R}^d) \) and \( t \in [0,T] \).

Then define
\[
L^0_t = L_t + \sum_{r=1}^n M_t^{(r)}.
\]
Observe that owing to the boundedness of the \( \mu^{(r)} \)'s, it follows easily from the arguments in the proof of Lemma 6.1 that there is a ball \( B \) such that \( M_t^{(r)k} \zeta = 0 \) outside \( B \) for all \( k \) and \( s \in (0,T) \) and
\[
M_t^{(r)k} \zeta(x) \rightarrow M_t^{(r)}(x) \zeta(x)
\]
as \( k \rightarrow \infty \) uniformly with respect to \( s \in (0,T) \) and \( x \in B \). It now follows by Theorem 4.5 that there exists \( u^0 \in W \) for which (6.6) holds with 0 in place of \( k \) for any \( \zeta \in C_0^\infty(\mathbb{R}^d) \) and \( t \in [0,T] \). This is exactly what we need because simple manipulations show that, for \( \nu \in \mathfrak{N} \),
\[
M_{\nu} \zeta(x) = (\nu x)^i D_i \left[ (\nu x)^j D_j \zeta(x) \right],
\]
so that the operators \( M_{\nu} \) are formally self-adjoint, and this proves the theorem. \( \square \)
7. Possible Extensions to Non-local Operators

Assumption 7.1. We are given a family \( \{ \nu_t(A), t \in (0, T) \} \) of measures on Borel subsets of \( \mathbb{R}^d \) such that

(i) \( \nu_t(\{0\}) = 0 \) for any \( t \in (0, T) \),

(ii) \( \nu_t(A) \) is a (Borel) measurable function of \( t \in (0, T) \),

(iii) we have

\[
\int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu_t(dx) < \infty \quad \forall t \in (0, T), \quad \int_{(0,T) \times \mathbb{R}^d} (1 \wedge |x|^2) \nu_t(dx)dt < \infty.
\]

Assumption 7.2. We are given \( W, G, K, L, u_0, f \) as in Theorem 4.9 with \( G \) being the group of translations.

Introduce

\[
L_t^0 = L_t + J_{\nu_t}, \tag{7.7}
\]

where, for \( \phi \in C_0^\infty(\mathbb{R}^d) \) and measure \( \nu \),

\[
J_{\nu_t}\phi(x) = \int_{\mathbb{R}^d} \left[ \phi(x + y) - \phi(x) - y^i D_i \phi(x) I_{|y| \leq 1}(y) \right] \nu(dy).
\]

As a side observation recall that if \( \nu \) is a measure on \( \mathbb{R}^d \setminus \{0\} \) such that

\[
\int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(dx) < \infty,
\]

the operator \( J_{\nu_t} \) is known in probability theory as the generator of a unique in law Lévy process associated to \( \nu \) (this process is without Gaussian part; see [12] and [4]).

One knows (and we will see this again in the proof of Theorem 7.1) that, owing to Assumption 7.1, \( J_{\nu_t}\phi(x) \) is well defined for any \( \phi \in C_0^\infty(\mathbb{R}^d) \).

Standard measure theoretic arguments show that \( J_{\nu_t}\phi(x) \) is a measurable function of \((t, x)\) for any \( \phi \in C_0^\infty(\mathbb{R}^d) \).

Theorem 7.1. Under the above assumptions for any \( h \in B((0,T),G) \) there exists \( u \in W \) such that (4.4) holds for any \( \zeta \in C_0^\infty(\mathbb{R}^d) \) and \( t \in [0, T] \) with \( L_t^0 \) in place of \( L_t \).

Proof. Notice that by Taylor’s formula for any \( \phi \in C_0^\infty(\mathbb{R}^d) \), if \( |y| \leq 1 \), then

\[
|\phi(x + y) - \phi(x) - y^i D_i \phi(x)| \leq |y|^2 N \sup_{|z| \leq 1} |D^2 \phi(x + z)|,
\]

where \( N \) is a constant. Below by \( N \) we denote generic constants which may change from one occurrence to another. It follows that

\[
|J_{\nu_t}\phi(x)| \leq N \sup_{|z| \leq 1} |D^2 \phi(x + z)| \int_{|y| \leq 1} |y|^2 \nu_t(dy) + \int_{|y| \geq 1} (|\phi(x + y)| + |\phi(x)|) \nu_t(dy),
\]

and owing to Assumption 7.1 (iii) and Fubini’s theorem we see that Assumption 4.2 (ii) is satisfied with \( L_t^0 \) in place of \( L_t \).
Furthermore, for Borel sets $A \subset \mathbb{R}^d$, define $\nu_k^t(A) = \nu_t(A \cap B_k)$, where $B_k = \{|x| \leq k\}$. Then the above manipulations show that, for $k \geq 1$

\[
\delta^k := \int_{(0,T) \times \mathbb{R}^d} |(J_{\nu_t} - J_{\nu_k^t})\phi(x)| \, dt \, dx
\]

\[
\leq \int_{(0,T) \times \mathbb{R}^d} \int_{|y| \geq k} (|\phi(x + y)| + |\phi(x)|) \, \nu_t(dy) \, dt \, dx
\]

\[
\leq N \int_{(0,T)} \nu_t(B_k^c) \, dt,
\]

which tends to zero as $k \to \infty$ by the dominated convergence theorem (see (iii) in Assumption 7.1). It follows that, if we introduce $L_k^t$ by (7.7) with $J_{\nu_k^t}$ in place of $J_{\nu_t}$, then condition b) of Theorem 4.5 is fulfilled. Of course, condition a) is fulfilled as well by the above. Now thanks to Theorem 4.5 to prove our theorem, it suffices to prove it with $\nu_k^t$ in place of $\nu_t$.

Hence, below we assume that $\nu_t(B^c) = 0$, where $B$ is a ball (independent of $t$). We can play the same trick with small jumps. Set this time $\nu_k^t(A) = \nu_t(A \cap B_{1/k}^c)$ (of course, this $\nu_k^t$ is different from the above one, but it is convenient to forget the above $\nu_k^t$ and introduce $\delta^k$ by the same formula with the new $\nu_k^t$). Then

\[
\delta^k \leq N \int_{(0,T) \times \mathbb{R}^d} \sup_{|z| \leq 1} \left| D^2 \phi(x + z) \right| \int_{|y| \leq 1/k} |y|^2 \, \nu_t(dy) \, dt \, dx
\]

\[
\leq N \int_{(0,T)} \int_{|y| \leq 1/k} |y|^2 \, \nu_t(dy) \, dt,
\]

which again tends to zero as $k \to \infty$ by the dominated convergence theorem.

Since this measures $\nu_k^t$ are finite, we now see that we may concentrate on the case in which $\nu_t$ are finite measures with support in a ball $B$ independent of $t$. One more simplification is achieved by introducing

\[
\nu_t^k = \nu_t 1_{\{|\nu_t(\mathbb{R}^d)\leq k\}},
\]

in which case

\[
\delta^k = \int_{(0,T) \times \mathbb{R}^d} |J_{\nu_t^k} \phi(x)| 1_{\{|\nu_t(\mathbb{R}^d)\geq k\}} \, dt \, dx \to 0
\]

by the dominated convergence theorem.

Thus, we need only consider the case in which $t \mapsto \nu_t(\mathbb{R}^d)$ is bounded and $\nu_t$ have support in a ball $B = \{|x| \leq R\}$.

In that case

\[
J_{\nu_t^k} \phi(x) = \int_{B} [\phi(x + y) - \phi(x)] \, \nu_t(dy) + b_t \cdot D\phi(x),
\]

where

\[
b_t = \int_{|y| \leq 1} y \, \nu_t(dy).
\]
For $y \in \mathbb{R}^d$ set $\kappa_k(y) = (\kappa_k(y^1), \ldots, \kappa_k(y^d))$ and introduce

$$J^k_{\nu_t} \phi(x) = \int_B \left[ \phi(x + \kappa_k(y)) - \phi(x) \right] \nu_t(dy) + k[\phi(x + b_t/k) - \phi(x)].$$

As is (very) easy to see

$$\delta^k = \int_{(0,T) \times \mathbb{R}^d} |(J_{\nu_t} - J^k_{\nu_t}) \phi(x)| \, dt \, dx \to 0$$
as $k \to \infty$ and to finish the proof it only remains to refer to Theorem 4.6 after observing that

$$J^k_{\nu_t} \phi(x) = \sum_{z \in (1/k)\mathbb{Z}^d \atop |z| \leq R+1} [\phi(x+z) - \phi(x)] \nu_t(\{y : \kappa_k(y) = z\}) + k[\phi(x + b_t/k) - \phi(x)],$$

where the sum contains only finite number of terms. The theorem is proved.

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