JOINT STATISTICS OF RANDOM WALK ON $Z^1$
AND ACCUMULATION OF VISITS

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Abstract. We obtain the joint distribution $P_N(X, K|Z)$ of the location $X$ of a one-dimensional symmetric next neighbor random walk on the integer lattice, and the number of times the walk has visited a specified site $Z$. This distribution has a simple form in terms of the one variable distribution $p_{N'}(X')$, where $N' = N - K$ and $X'$ is a function of $X$, $K$, and $Z$. The marginal distribution of $X$ and $K$ are obtained, as well as their diffusion scaling limits.

Key words. Symmetric random walks, walk on integer lattice, frequency of visits, walker visit number correlation

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1. Introduction. Random walks in random environments are models for an enormous number of biological, physical, — processes and when the walker alters the environment during its sojourn, the resulting phenomenology is many-faceted. This paper addresses a forerunner, arguably, of such a situation, in which we are content to obtain the distribution of the number of times $K$ the walker has visited or is visiting a specified lattice site $Z$ after its starting, but does not thereby alter the properties of that site. Extension to the reinforced random walk context is our implicit direction, but will not yet be attempted. However, we will try to employ tools that do extend to such more structured problems. Further, to keep our study as uncluttered as possible, we restrict our attention to the blatantly simplest case of a symmetric random walk on the one-dimensional integer lattice $Z^1$.

The specific question we will address is that of the correlation between the current location $X$ and the visit number $K$ at $Z$ since this will, in work soon to be reported, inform the question of the evolution of the pattern of visits. Our key result, Eq. (3.17), is that their joint distribution is expressible quite simply in terms of the elementary symmetric random walk, thereby having a readily visualized asymptotic form, but that this form has a somewhat unusual structure.

2. The symmetric Random Walk on $Z^1$. Let us review briefly [1, 2] the underlying problem of the statistics of a symmetric random walk making next neighbor moves from its initial location at the origin, as represented by the probability distribution.

\begin{equation}
    p_0(X) = \delta_{X,0},
\end{equation}

and ask for the $N^{th}$ step distribution $P_N(X)$, the probability that the walker has arrived at location $X$ after its $N^{th}$ random move. Clearly,

\begin{equation}
    p_{N+1}(X) = \frac{1}{2} (p_N(X - 1) + p_N(X + 1))
\end{equation}

\begin{equation}
    p_0(X) = \delta_{X,0},
\end{equation}
or in terms of the generating function

\begin{equation}
    \hat{p}(\lambda, X) = \sum_{N=0}^{\infty} \lambda^N p_N(x),
\end{equation}
we have instead
\begin{equation}
\dot{\hat{p}}(\lambda, X) - p_0(X) = \frac{\lambda}{2} (\dot{\hat{p}}(\lambda, X + 1) + \dot{\hat{p}}(\lambda, X - 1)).
\end{equation}

Eq. (2.4), with the convergence restriction \( \lim|X| \to \infty \hat{p}(\lambda, X) = 0 \), is solvable in standard fashion as
\begin{equation}
\hat{p}(\lambda, X) = \alpha^{-1} + \alpha \alpha^{\mid X \mid},
\end{equation}
where \( \alpha = \frac{1}{\lambda} - \sqrt{\frac{1}{\lambda^2} - 1} \) (from \( \lambda = 2\alpha/(1 + \alpha^2) \)).

On the other hand, the condition that the walker arrive at \( X \) in \( N \) steps after starting at 0 is equivalent to it having taken \( \frac{1}{2} \) \( (N + X) \) steps of +1 and \( \frac{1}{2} \) \( (N - X) \) steps of −1, thereby identifying it with a binomial distribution of move probability \( p = q = 1/2 \):
\begin{equation}
p_N(X) = \frac{1}{2^N} \left( \begin{array}{c} N \\ \frac{1}{2} (N + X) \end{array} \right),
\end{equation}
with the restriction of course that \( X \equiv N( \mod 2) \). The equivalence of (2.5) and (2.6) will be useful in the sequel: \( \hat{p} \) is useful for deconvolution in step number (and has convenient algebraic properties) whereas \( p \) is directly visualizable. In fact, if we set \( Y = \frac{1}{2} (N + X), 0 \leq Y \leq N \), then (2.6) takes the standard form \( 2^{-N} \binom{N}{Y} \), allowing us to immediately quote the \( N \)-asymptotic form \( (x \equiv X/N^{1/2}, \text{but } \Delta X = 2) \)
\begin{equation}
P(X) = \lim_{N \to \infty} N^{1/2} p_N \left( \frac{x}{\sqrt{N}} \right) = \frac{1}{\sqrt{2\pi}} e^{-x^2}.
\end{equation}

3. Joint Distribution. We now focus on a site \( Z > 0 \) together with the number of times, \( K \) it has been visited by the walker over the course of \( N \) steps. One can say that \( K \) is reacting passively to the passage of the walker, a situation that will be greatly expanded in future work, but we are at this state simply interested in
\begin{equation}
P_N(X, K | Z),
\end{equation}
the probability that after \( N \) steps from an initial
\begin{equation}
P_0(X, K | Z) = \delta_{K,0} \delta_{X,0}
\end{equation}
the walker is now at site \( X \), and site \( Z \) has been visited \( K \) times. The simplest way to deal with this situation is by constructing a generating function [3] over \( K \):
\begin{equation}
\tilde{P}_N(X, V | Z) = \sum_{K=0}^{\infty} V^K P_N(X, K | Z),
\end{equation}
where \( |V| \leq 1 \) allows convergence to be uniform in \( N \). The construction (3.3) is clearly equivalent to inserting a multiplicative weight \( V \) at each arrival at \( Z \), and so satisfies the weighted version of Eq. (2.2):
\begin{equation}
\tilde{P}_{N+1}(X, V | Z) = \frac{1}{2} (1 + (V - 1) \delta_{X,Z}) \left( \tilde{P}_N(X + 1, V | Z) + \tilde{P}_N(X - 1, V | Z) \right),
\end{equation}
\begin{equation}
\tilde{P}_0(X, V | Z) = \delta_{X,0}, \quad \text{and}
\end{equation}
\begin{equation}
\hat{\tilde{P}}(\lambda, X, V | Z) \equiv \sum_{N=0}^{\infty} \lambda^N \tilde{P}_N(X, V | Z).
\end{equation}
For further simplification, we again go over to the generating function over $N$, obtaining

$$\hat{P}(\lambda, X, V|Z) = \delta_{X,0} + \frac{\lambda}{2} (1 + (V - 1) \delta_{X,Z})$$

$$\left( \hat{P}(\lambda, X + 1, V|Z) + \hat{P}(\lambda, X - 1, V|Z) \right),$$

(3.5)

to be solved.

Eq. (3.5) is a bit more complex than (2.4), but yields to the same method of solution that was implicitly employed in obtaining (2.5) from (2.4). We set up a covering of $Z^1$ by three overlapping closed subspaces:

$$\begin{align*}
[I] & \quad X \in [-\infty, 0] \\
[II] & \quad X \in [0, Z] \\
[III] & \quad X \in [Z, \infty]
\end{align*}$$

(3.6)

and separate (3.5) into its actions on the corresponding open subspaces and their boundaries

$$\begin{align*}
X \to -\infty : & \quad \hat{P}(\lambda, X, V|Z) \to 0 \\
X \in (I) : & \quad \hat{P}(\lambda, X, V|Z) = \frac{\lambda}{2} \left( \hat{P}(\lambda, X + 1, V|Z) + \hat{P}(\lambda, X - 1, V|Z) \right) \\
X = 0 : & \quad \hat{P}(\lambda, 0, V|Z) = 1 + \\
& \quad \frac{\lambda}{2} \left( \hat{P}(\lambda, 1, V|Z) + \hat{P}(\lambda, -1, V|Z) \right) \\
X \in (II) : & \quad \hat{P}(\lambda, X, V|Z) = \frac{\lambda}{2} \left( \hat{P}(\lambda, X + 1, V|Z) + \hat{P}(\lambda, X - 1, V|Z) \right) \\
X = Z : & \quad \hat{P}(\lambda, X, V|Z) = \frac{\lambda V}{2} \left( \hat{P}(\lambda, Z + 1, V|Z) + \hat{P}(\lambda, Z - 1, V|Z) \right) \\
X \in (III) : & \quad \hat{P}(\lambda, X, V|Z) \\
& \quad X \to \infty : \quad \hat{P}(\lambda, X, V|Z) \to 0.
\end{align*}$$

(3.7)

Since the general solution of

$$a < X < b : \quad f(X) = \frac{\lambda}{2} (f(X + 1) + f(X - 1))$$

(3.8)

is given by

$$a \leq X \leq b : \quad f(X) = A \alpha^{-X} + B \alpha^{X}$$

where \( \alpha = \frac{1}{\lambda} - \left( \frac{1}{\lambda^2} - 1 \right)^{1/2} \),

(3.9)
we have from (3.7)

\[ X \in [I] : \hat{P}_I (\lambda, X, V|Z) = A \alpha^{-X} \]
\[ X \in [II] : \hat{P}_{II} (\lambda, X, V|Z) = B \alpha^{X} + C \alpha^{-X} \]
\[ X \in [III] : \hat{P}_{III} (\lambda, X, V|Z) = D \alpha^{X} \]

where \(A, B, C, D\) are functions of \(\lambda, V\), and \(Z, \alpha\) of \(\lambda\) alone. The uniqueness of \(\hat{P}\) at \(X = 0\) and \(X = Z\) then implies two equalities from (3.10), two from (3.7):

\[ A = B + C \]
\[ D \alpha^{Z} = B \alpha^{Z} + C \alpha^{-Z} \]
\[ A = 1 + \frac{\lambda}{2} (B \alpha + C \alpha^{-1} + \alpha \alpha) \]
\[ D \alpha^{Z} = \frac{\lambda V}{2} (B \alpha^{Z-1} + C \alpha^{1-Z} + D \alpha^{Z+1}) \]

immediately solved using \(\lambda/2 = \alpha/(1 + \alpha^2)\) as

\[ A = \frac{1 + \alpha^2}{1 - \alpha^2} (1 - \alpha^{2Z}) + \frac{1 + \alpha^2}{1 + \alpha^2 - 2 \alpha^2 V} V \alpha^{2Z} \]
\[ B = \frac{1 + \alpha^2}{1 - \alpha^2}, \quad C = -\frac{1 + \alpha^2}{1 - \alpha^2} \alpha^{2Z} + \frac{1 + \alpha^2}{1 + 2^Z - 2 \alpha^2 V} V \alpha^{2Z}, \]
\[ D = \frac{1 + \alpha^2}{1 + \alpha^2 - 2 \alpha^2 V} V. \]

Substituting into (3.10) and using a unifying notation we conclude that

\[ \hat{P} (\lambda, X, V|Z) = \frac{1 + \alpha^2}{1 - \alpha^2} \left( \alpha^{X} - \alpha^{-Z} \alpha^{X-Z} \right) + \frac{1 + \alpha^2}{1 + \alpha^2 - 2 \alpha^2 V} V \alpha^{2Z} \alpha^{X-Z}. \]

Eq. (3.13) can also be obtained by strictly combinatorial reasoning, but this approach does not extend to reinforced walks.

Extracting \(P_N (X, K|Z)\) from (3.13) means taking the coefficient of \(\lambda^N V^K\), which is far from obvious. But it can be done indirectly. We first extract the coefficient of \(V^K\):

\[ \hat{P} (\lambda, X, K|Z) \equiv \text{coef } V^K \text{ in } \hat{P} (\lambda, X, V|Z) \]
\[ = \frac{1 + \alpha^2}{1 - \alpha^2} \left( \alpha^{X} - \alpha^{-Z} \alpha^{X-Z} \right) \delta_{K,0} \]
\[ + \left( \frac{2 \alpha^2}{1 + \alpha^2} \right)^{K-1} \alpha^{Z} \alpha^{X-Z} \left( 1 - \delta_{K,0} \right), \]

and then observe that

\[ \text{coef } (\lambda^N) \text{ in } \frac{1 + \alpha^2}{1 - \alpha^2} \alpha^{Y|} = p_N (Y) \]
whereas

\[
\text{coef } (\lambda^N) \left( \frac{2\alpha^2}{1 + \alpha^2} \right)^{K-1} \alpha^{|Y|} = \text{coef } (\lambda^N)(\alpha\lambda)^{K-1} \alpha^{|Y|}
\]

\[
= \text{coef } (\lambda^{N+1-K}) \alpha^{|Y|+K-1} = \text{coef } (\lambda^{N+1-K}) \frac{1 - \alpha^2}{1 + \alpha^2} \frac{1 + \alpha^2}{1 - \alpha^2} \alpha^{|Y|+K-1}
\]

(3.16)

\[
= \text{coef } (\lambda^{N+1-K}) \frac{\lambda}{2\alpha} \frac{1 + \alpha^2}{1 - \alpha^2} (\alpha^{|Y|+K-1} - \alpha^{|Y|+K+1})
\]

\[
= \text{coef } (\lambda^{N-K}) \frac{1}{2} \frac{1 + \alpha^2}{1 - \alpha^2} (\alpha^{|Y|+K-2} - \alpha^{|Y|+K})
\]

\[
= \frac{1}{2} (p_{N-K} (|Y| + K - 2) - p_{N-K} (|Y| + K))
\]

It follows from (3.14), (3.15), (3.16) that

(3.17)

\[
P_N (X, K|Z) = (p_N (X) - p_N (Z + |X - Z|)) \delta_{K,0}
\]

\[+ \frac{1}{2} (p_{N-K} (|X - Z| + Z + K - 2) - p_{N-K} (|X - Z| + Z + K)) (1 - \delta_{K,0}).
\]

4. The X and K Marginals. If we want just \(P_N(X|Z)\), it is necessary to sum \(P_N(X, K|Z)\) over \(K\), which is equivalent to setting \(V = 1\) in \(P_N(X, V|Z)\). A somewhat simpler path is to first set \(V = 1\) in \(\hat{P}(\lambda, X, V|Z)\) of (3.13), obtaining

(4.1)

\[
\hat{P}(\lambda, X, 1|Z) = \frac{1 + \alpha^2}{1 - \alpha^2} \alpha^{X|},
\]

and then convert from \(\lambda\) to \(N\):

(4.2)

\[
P_N (X|Z) = \hat{P}_N (X, 1|Z) = p_N (X) = \left(\frac{1}{2}\right)^N \left(\frac{N}{2(N + X)}\right)\]

if \(N - X \equiv 0 \pmod{2}\),

the input symmetric random walk, as must be the case

The \(K\)-marginal

(4.3)

\[
\mathcal{P}_N (K|Z) = \sum_{X=-\infty}^{\infty} P_N (X, K|Z)
\]

is almost as simple, if less familiar. Here, we only need to proceed directly: substitute (3.17) into (4.3) and carry out the summation over \(X\). In terms of the basic cumulant

(4.4)

\[
F_N (X) = \sum_{Y=-\infty}^{X} p_N (Y)
\]

this is readily evaluated as

(4.5)

\[
\mathcal{P}_N(K|Z) = F_N (Z - 1) \delta_{K,0}
\]

\[+ \frac{1}{2} [p_{N-K} (Z + K - 2) + 2p_{N-K} (Z + K - 1) + p_{N-K} (Z + K)] (1 - \delta_{K,0}).
\]
or

\( P_N(K|Z) = F_N(Z-1) \delta_{K,0} + (1 - \delta_{K,0}) \begin{cases} 1/2^{N-K} \binom{N-K}{(N+Z-1)/2} & \text{if } N - Z \equiv 1 \pmod{2} \\ 1/2^{N+1-K} \binom{N+1-K}{(N+Z)/2} & \text{if } N - Z \equiv 0 \pmod{2} \end{cases} \)

Eqn. (3.17), (4.2) and (4.5) solve the combinatorial problem of determining the two-variable and two one-variable marginal distributions of the random variables \( X \) and \( K \), but they don’t present us with an immediate picture of what is going on. That is traditionally done by finding the \( N \)-dependence of various means and covariances, by looking for the distribution resulting for asymptotic \( N \), or by both. We adopt the third strategy, which can be carried out in either order of its two components. To start, we may compute the low order moments. Since these require only those of the basic symmetric random walk, the results

\[
E_N(X|Z) = \sum_{X=-N}^{N} X P_N(X|Z) = 0
\]

(4.7)

\[
E_N(X^2|Z) = \sum_{X=-N}^{N} X^2 P_N(X|Z) \equiv N
\]

are immediate. The corresponding \( K \)-moments are tedious to compute by hand (we have done so), but are eased by the use of Mathematica and we find

\[
E_N(K|Z) = \sum_{K=1}^{N} K P_N(K|Z)
\]

\[
= \begin{cases} \sum_{K=1}^{N} \frac{K}{2^{N-K}} \binom{N-K}{(N+Z-1)/2} & \text{if } N - Z \equiv 1 \pmod{2} \\ \sum_{K=1}^{N} \frac{K}{2^{N+1-K}} \binom{N+1-K}{(N+Z)/2} & \text{if } N - Z \equiv 0 \pmod{2} \end{cases}
\]

(4.8)

\[
= 2^{1-N} \binom{N-1}{(N+Z-1)/2} 2F1 \left( 2, \frac{1}{2} \left( Z + 1 - N \right), 1 - N; 2 \right) - 2(N+1) \binom{N-1}{(N+Z-1)/2} 2F1 \left( N+2, \frac{1}{2} \left( Z + 1 + N \right), 1 + N, 2 \right) & \text{if } N - Z \equiv 1 \pmod{2},
\]

\[
= 2^{1-N} \binom{N}{(N+Z)/2} 2F1 \left( 2, \frac{1}{2} \left( Z - N \right) , -N; Z \right) - 2(N+2) \binom{N-1}{(N+Z)/2} 2F1 \left( N+3, \frac{1}{2} \left( Z + N + 2 \right), N + 2; 2 \right) & \text{if } N - Z \equiv 10 \pmod{2},
\]

in terms of Hypergeometric function \( 2F1 \). Similarly, \( E_N(K^2|Z) \) involves \( 3F2 \).

5. Asymptotics and Conclusion. Although the explicit combinatorial results (3.17), (4.7) do not contribute to ready visualization, they do tell us about the nature of large \( N \) asymptotics which, to put in a more positive light, can be regarded as conversion to a Brownian motion context. The simplification afforded by this limit has as well the enormous advantage of bringing out qualitative aspects that may be concealed e.g., by analysis of various moments. Let us see what this strategy reveals in our particular case.

In diffusion scaling, the transformation

\[
X = N^{1/2} x, \quad K = N^{1/2} k
\]

(5.1)
is called for. Since \( \Delta k = 1/N^{1/2} \), but \( X \equiv N \pmod 2 \) \( \Delta x = 2/N^{1/2} \), the probability in \((X, K)\) space is then transformed as \( N \to \infty \) to a probability density in \((x, k)\) space. In addition, the change of scale in \( X \)-space tells us that we should also scale \( Z \) as

\[
(5.2) \quad Z = N^{1/2} z.
\]

The probability density in \((x, k)\) space then becomes

\[
(5.3) \quad P(x, k|z) = \lim_{N \to \infty} \frac{N}{2} P_N \left( N^{1/2} x, N^{1/2} k | N^{1/2} z \right).
\]

We can now apply (2.7) to (3.17) and use \( \delta_{K,0} \to N^{-1/2} \delta(k) \) with the convention that \( \int_{-\infty}^{\infty} f(k) \delta(k) \, dk = f(0) \) to obtain without difficulty

\[
(5.4) \quad P(x, k|z) = \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{1}{2} x^2} - e^{-\frac{1}{2} (z+|x-z|)^2} \right) \delta(k) + \frac{1}{2} \frac{1}{\sqrt{2\pi}} (k + z + |x - z|) e^{-\frac{1}{2} (k+z+|x-z|)^2},
\]

divided of course into three regions, as was (3.6). The density maxima are shown (bold lines) in the accompanying figure.

Now the \( x \)-marginal is of course that of the basic symmetric Brownian walk, but the \( k \)-marginal is available from either (4.6) or (5.4),

\[
(5.5) \quad P(k|z) = (2/\pi)^{1/2} e^{-\frac{1}{2} (k+z)^2} + C(z) \delta(k)
\]

where \( C(z) \) assures normalization.

It is clear that the first few joint moments, as in (4.7), say very little about the structure of the joint distribution, that the asymptotic form is both simpler and more informative. (Note the \( k + z \) dependence of (5.5)) We will take advantage of this in work soon to be reported, in which the development of the full pattern of visits is in question.

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