On the Waldspurger Formula and the Metaplectic Ramanujan
Conjecture over Number Fields

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Abstract. In this paper, by inputting the Bessel identities over the complex field in previous work of the authors, the Waldspurger formula of Baruch and Mao is extended from totally real fields to arbitrary number fields. This is applied to give a non-trivial bound towards the Ramanujan conjecture for automorphic forms on the metaplectic group $\tilde{SL}_2$ for the first time in the generality of arbitrary number fields.

1. Introduction

1.1. Backgrounds. In the work $[\text{Wal2}]$, Waldspurger proved his celebrated formula connecting the Fourier coefficients of a modular form of half integral weight to the twisted central $L$-values of a modular form of integral weight. The formula in $[\text{Wal2}]$ is over the rational field $\mathbb{Q}$.

In the work $[\text{BM3}]$, Baruch and Mao proved a very explicit Waldspurger-type formula for automorphic forms over totally real fields. Waldspurger’s formula was also extended by many other authors in various cases. An incomplete list (in the chronological order) includes the papers of Kohnen-Zagier $[\text{KZ, Koh}]$, Niwa $[\text{Niw}]$, Gross $[\text{Gro}]$, Shimura $[\text{Shi2}]$, Katok-Sarnak $[\text{KS}]$, Khuri-Makdisi $[\text{KM}]$, Kojima $[\text{Koj1}–\text{Koj4}]$, Prasanna $[\text{Pra}]$ and Altuğ-Tsimerman $[\text{AT}]$ (the function field case).

The Waldspurger-type formula of Baruch and Mao in $[\text{BM3}]$ may be regarded as the ultimate version of its kind—if there were no totally real restriction on the ground field—both by removing all hypotheses assumed by earlier authors as well as by making Waldspurger’s formula entirely explicit. For example, after the translation into the classical language (over the rational field $\mathbb{Q}$), the formula of Baruch and Mao is a generalization of the Kohnen-Zagier formula in $[\text{KZ}]$ for all fundamental discriminants $D$ without any restriction. It was later applied in $[\text{BM4}]$ and $[\text{HI}]$ to obtain the generalized Kohnen-Zagier formula for Maass forms and Hilbert modular forms respectively.

Applications of the Waldspurger-type formula in $[\text{BM3}]$ include:

2010 Mathematics Subject Classification. 11F37, 11F67, 11F70.

The first author is supported by the National Natural Science Foundation of China [Grant 11771131].
The equivalence of the Ramanujan conjecture for half integral weight forms with the Lindelöf hypothesis for twisted central $L$-values of integral weight forms, in the generality of totally real number fields;

(2). An efficient method of computing the central value of the twisted $L$-functions associated to the elliptic curve $X_0(11)$, by the aforementioned generalization of the Kohnen-Zagier formula.

A further application in the work of Cogdell, Piatetski-Shapiro and Sarnak [CPSS] (see also Blomer and Harcos [BH2]) is:

(3). A solution for the last open case of Hilbert’s eleventh problem on representing integers by positive definite integral ternary quadratic forms over totally real fields.

This is a consequence of the Ramanujan-Lindelöf equivalence in (1) combined with the subconvexity bound for twisted $L$-values for Hilbert modular forms over totally real fields in [CPSS] or [BH2]. See [Cog] for an account of their solution of Hilbert’s eleventh problem and related references.

1.2. Motivation. The Waldspurger formula of Baruch and Mao in [BM3] is a consequence of the local spectral theory from the viewpoint of Bessel identities over non-Archimedean fields and the real field in [BMI, BM2] and the global theory for a relative trace formula of Jacquet in [Jac]. These are incorporated nicely in the framework of Waldspurger [Wal1, Wal3] on the representation theoretic form of the Shimura correspondence [Shi1].

In Baruch and Mao’s project, the proof of the local Bessel identities is the foundational and technical part.

It has long been known that the Bessel functions for $GL_2(\mathbb{R})$ (and $\widehat{SL_2(\mathbb{R})}$), defined over $\mathbb{R} \setminus \{0\} = \mathbb{R}_+ \cup \mathbb{R}_-$, may be expressed in terms of classical Bessel functions on $\mathbb{R}_+$. The Bessel identities over $\mathbb{R}$ in [BM2] can be regarded as the representation theoretic interpretation of the classical exponential integral formulae of Weber and Hardy on the Fourier transform of Bessel functions on $\mathbb{R}_+$.

The Bessel functions for $PGL_2(\mathbb{C})$, expressed in terms of classical Bessel functions on $\mathbb{C} \setminus \{0\}$, were first discovered by Bruggemann and Motohashi in 2003 [BM6], and later by Lokvenec-Guleska [LG] in 2004 for $SL_2(\mathbb{C})$. The discovery was not long before the publication of Baruch and Mao’s work, and the complex analogue of the formulae of Weber and Hardy was not available at that time. There is however a remark in [BM3]:

"Our (Waldspurger-type) formula can be extended to all number fields once the local result in [BMI] and [BM2] is extended to the case of the complex field."

1. The first precise interpretation of classical Bessel functions in representation theory is due to Cogdell and Piatetski-Shapiro [CPS] (1990). The occurrence of Bessel functions in number theory however may be traced back to the formulae of Voronoi [Vor] (1904), Petersson [Pet] (1932) and Kuznetsov [Kuz] (1981). See [IK].

2. The spherical Bessel functions for $SL_2(\mathbb{C})$ actually appeared much earlier in [MW] (1990). The representation theory of Bessel functions for $GL_2(\mathbb{C})$ may be found in [BM5, Mot], [BBA] and [Qi3] Chapter 4."
Recently, tempted by its outcome indicated in this remark, the authors had a series of papers towards the Bessel identities over $\mathbb{C}$. It is now established in $[CQ]$ by the classical-like Weber-Hardy type exponential integral formula in $[Qi1, Qi2]$ for the Fourier transform of Bessel functions for $\text{PGL}_2(\mathbb{C})$ (see Remark 6.5 for more discussions). The present article is the final payment for these works—the Waldspurger formula over arbitrary number fields.

As an immediate application, we shall extend the Ramanujan-Lindelöf equivalence in (1) to arbitrary number fields. Moreover, combined with the $\text{GL}_2 \times \text{GL}_2$ subconvexity results over number fields in $[MY, \text{Wu, Mag}]$, we shall obtain the first nontrivial estimate towards the metaplectic Ramanujan conjecture for $\tilde{\text{SL}}_2$ over arbitrary number fields. Recall that this is a key step in the settlement of Hilbert’s eleventh problem in $[\text{CPSS}]$, but the ground field therein is only needed to be totally real thanks to the work of Siegel.

1.3. The formula of Waldspurger over arbitrary number fields. To derive our Waldspurger formula, we shall closely follow the approach of Baruch and Mao in $[BM]$.

It is a combination of two results, the basic Waldspurger’s formula and Waldspurger’s dichotomy result on theta correspondence.

Let $F$ be a number field and $\mathbb{A}$ be its ring of adèles. Given a nontrivial additive character $\psi$ on $\mathbb{A}/F$, we consider the global Shimura-Waldspurger correspondence $\Theta(\cdot, \psi)$ between automorphic representations $\pi$ of $\text{PGL}_2(\mathbb{A})$ and $\tilde{\pi}$ of $\tilde{\text{SL}}_2(\mathbb{A})$ $[\text{Wai1}]$.

The basic Waldspurger’s formula is the following simple statement.

**Theorem 1.1 (Restatement of Theorem 4.1).** Given $D \in F^\times$, define $\psi^D(x) = \psi(Dx)$, and let $\pi$ and $\tilde{\pi} = \Theta(\pi, \psi^D)$ correspond under the Shimura-Waldspurger correspondence with respect to $\psi^D$. We have

$$|d_\pi(S, \psi)|^2 L^S(\pi, 1/2) = |d_\pi(S, \psi^D)|^2. \tag{1.1}$$

In this theorem, $L^S(\pi, 1/2)$ is the central (partial) $L$-value for $\pi$, the constants $d_\pi(S, \psi)$ and $d_\pi(S, \psi^D)$ (see §2 for the definition) may be considered as the “leading” and the $D$-th Fourier coefficient of $\pi$ and $\tilde{\pi}$, respectively, where $S$ is a finite set of “bad” local places.

Simply speaking, the identity (1.1) follows from the comparison of two global distributions $I_\pi$ and $J_\pi$ (the relative trace formula of Jacquet) and the relations between the attached local distributions $I_{\pi, r}$ and $J_{\pi, r}$ (the local Bessel identities). See §5 and §6 for the details.

The Waldspurger formula for twisted central $L$-values is derived from the basic formula (1.1) simply by replacing $\pi$ by $\pi \otimes \chi_D$, where $\chi_D$ is the quadratic character associated to $D$. This leads to the consideration of $\Theta(\pi \otimes \chi_D, \psi^D)$ for varying $D \in F^\times$. Now the dichotomy result of Waldspurger in $[\text{Wai3}]$ gives an explicit classification of $\Theta(\pi \otimes \chi_D, \psi^D)$ according to a certain finite partition of $F^\times$. The identity (1.1) then yields a formula for $L^S(\pi \otimes \chi_D, 1/2)$ in terms of the $D$-th Fourier coefficient $d_\pi(S, \psi^D)$ of a fixed $\tilde{\pi}$, as long as $D$ lies in the subset of $F^\times$ in the partition attached to $\tilde{\pi}$. See §4 for the Waldspurger dichotomy and §4.2 for the precise statement of the Waldspurger formula for $L^S(\pi \otimes \chi_D, 1/2)$. 
Some remarks on the recent works of Lapid-Mao and Qiu are in order. Lapid and Mao have a sequence of papers [LM1]–[LM5], culminating in a Whittaker period formula for a cuspidal automorphic representation \( \pi \) of \( \tilde{Sp}_{2n} \) which is valid if \( F \) is totally real and \( \pi \) is a discrete series. In [Qiu], Qiu proved this formula for \( \tilde{SL}_2 \) over an arbitrary number field. Though in a different form, this may also be considered as a generalization of Waldspurger’s formula (the central value of \( L \)-functions for \( PGL_2 \) is now related to the Whittaker periods of automorphic forms on \( \tilde{SL}_2 \)) to all number fields. His approach is based on theta correspondence, and the most technical part is verifying the local identities in [Qiu, Lemma 4.2] by the isometry property of quadratic Fourier transform and some asymptotic estimates for the matrix coefficients and Whittaker functions of \( \pi \). Finally, we remark that, if our basic Waldspurger’s formula in Theorem 4.1 is assumed, Qiu’s Whittaker period formula for \( \tilde{SL}_2 \) may be deduced from that for \( PGL_2 \), while the latter is already known (see [LM3] §4 or [Qiu §2]).

1.4. The metaplectic Ramanujan conjecture. The first breakthrough in obtaining nontrivial bounds toward the Ramanujan conjecture for holomorphic modular forms of half-integral weight was achieved by Iwaniec in [Iwa1]. Later, Duke [Duk] got a similar bound for Maass forms. In their work, they apply a Petersson or Kuznetsov trace formula and then proceed to bound sums of Salié sums. This approach is conceptually direct, and it never goes through the Waldspurger formula.

Alternatively, the Ramanujan-Lindelöf equivalence in (1) established on the Waldspurger formula suggests that one may obtain nontrivial bounds toward the Ramanujan conjecture for cusp forms for \( \tilde{SL}_2 \) from subconvexity bounds for twisted \( L \)-functions for cusp forms of \( PGL_2 \). There is by now a great deal of machinery to deal with subconvexity problems, and so one can get even better bounds this way. Indeed, the bounds of Iwaniec and Duke were improved in [BHM, PY] and [BM4] for holomorphic modular forms and Maass forms of half-integral weight respectively. See also the aforementioned [CPSS] and [BH2] in the case of totally real fields.

The \( GL_2 \)-subconvexity problem is now completely solved by Michel and Venkatesh [MV] over arbitrary number fields. Later Han Wu [Wu] and Maga [Mag1, Mag2] proved by different methods the following subconvexity bound

\[
L(\pi \otimes \chi, 1/2) \ll C(\chi)^{\beta}
\]

for all \( \beta > 2\theta = \frac{3}{8} + \frac{1}{4}\theta \), where \( \pi \) is an automorphic representation of \( GL_2(\mathbb{A}) \), \( \chi \) is a Hecke character of \( \mathbb{A}^{\times} \) of (analytic) conductor \( C(\chi) \). The \( \theta \) is any exponent toward the

3The bounds in [BHM] and [BM4] may be further improved by the Burgess-type subconvexity bound in [BH1]. The results in [BH1] are valid for all Dirichlet characters. However, in our settings, we only need quadratic characters, and in this case the Weil-type subconvexity bound was known earlier in the work of Conrey and Iwaniec [CI]. A Weil-type bound for the holomorphic modular forms and Maass forms in the Kohnen space for \( \Gamma_0(4) \) may be deduced from [CI] (a small issue is that the conductor of the character is assumed to be odd in [CI]). In the same way, this is generalized by [PY] in the holomorphic case for arbitrary \( \Gamma_0(4r) \) with \( r \) odd and square-free. In view of the recent work [You], the generalization in the Maass case should come in the near future.
Ramanujan-Petersson conjecture for GL$_2(\mathbb{A})$. The best known value $\theta = \frac{7}{64}$ is due to Kim-Sarnak [Kim] over $\mathbb{Q}$, and to Blomer-Brumley [BB] over an arbitrary number field. Note that if $\theta = 0$ then we have the Burgess exponent $2\theta = \frac{3}{8}$.

Now in our settings (1.2) would imply
\[
L(\pi \otimes \chi_D, 1/2) \ll |D|_{\mathbb{A}, \infty}, \quad |D|_{\mathbb{A}, \infty} \to \infty,
\]
if $D$ is a square-free integer in $F^\times$ (see §4.2). Here $\mathbb{A}_{\infty}$ is the set of Archimedean places of $F$ and $|D|_\mathbb{A}_{\infty} = \prod_{v \in \mathbb{A}_{\infty}} |D|_v$. We shall obtain the following nontrivial bound on the Fourier coefficients of a cusp form for $\tilde{\text{SL}}_2(\mathbb{A})$. See §8 for the notations.

**Theorem 1.2.** Let $\tilde{\pi}$ be an irreducible cuspidal automorphic representation of $\tilde{\text{SL}}_2(\mathbb{A})$ in $\tilde{\mathbb{A}}_{\infty}$. Let $\tilde{\varphi}$ be a cusp form in the space of $\tilde{\pi}$, and $D$ be a square-free integer in $F^\times$. Then, for any $\alpha > \tilde{\theta} = \frac{3}{16} + \frac{\theta}{8}$, we have
\[
|d_2(\tilde{\varphi}, \mathbb{A}_{\infty}, \psi^D)| \ll |D|_{\mathbb{A}_{\infty}}^{\alpha - \frac{1}{2}}
\]
as $|D|_{\mathbb{A}_{\infty}} \to \infty$, where the implied constant depends only on $\tilde{\pi}, \tilde{\varphi}$ and $\alpha$. The $\theta$ is any exponent toward the Ramanujan-Petersson conjecture for GL$_2(\mathbb{A})$.

1.5. **Structure of the paper.** An outline of the paper is as follows. In §2 we introduce the two constants $d_2(S, \psi)$ and $d_2(S, \psi)$. In §3 we recollect Waldspurger’s results on theta correspondence [Wal1, Wal3]. In §4 we give the explicit statement of our Waldspurger formula. In §5 we review the relative trace formula of Jacquet in [Jac]. In §6 we review the local Bessel identities in [BM1, BM2, CQ]. In §7 we prove the Waldspurger formula. In §8 we prove the the Ramanujan-Lindelöf equivalence and establish a nontrivial bound toward the metaplectic Ramanujan conjecture.

**Notation.** Let $F$ be a number field, and $\mathbb{A}$ be its adèle ring. For a place $v$ of $F$, let $F_v$ be the corresponding local field, and $| \cdot |_v$ denote the normalized metric on $F_v$. When $v$ is a non-Archimedean place, let $O_v$ be the ring of integers in $F_v$, and $q_v$ be the order of the residue field.

For $D \in F_\times^\times$, let $\chi_D$ be the quadratic character of $\mathbb{A}_{\infty}/F_\times^\times$ associated to the field extension $F(\sqrt{D})$. Similarly, at a local place $v$, for $D \in F_v^{\times}$, we let $\chi_D$ be the quadratic character of $F_v^{\times}$ associated to the extension $F_v(\sqrt{D})$.

Let $\psi$ be a nontrivial additive character of $\mathbb{A}/F$, with $\psi = \otimes_v \psi_v$. For $D \in F_{\times}^{\times}$, define $\psi^D(x) = \psi(Dx)$.

**Groups.** Let $G = \text{GL}_2$, $S = \text{SL}_2$ and $\tilde{S} = \tilde{\text{SL}}_2$. Let $Z$ be the center of $G$, $B$ be the subgroup of $G$ consisting of upper triangular matrices, $B_{\tilde{S}}$ be the lift of $B \cap S$ in $\tilde{S}$. Note that $G/Z = \text{PGL}_2$. We shall use $1$ to denote the identity elements of these groups.

Let us now briefly recall the definition of the metaplectic group $\tilde{S}$. It consists of all pairs $(g, \epsilon)$, $g \in S$, $\epsilon \in \{\pm 1\}$, with the law of multiplication
\[
(g_1, \epsilon_1)(g_2, \epsilon_2) = (g_1g_2, \alpha(g_1, g_2)\epsilon_1\epsilon_2),
\]
where \( \alpha(g_1, g_2) \) is Kubota’s cocycle that we now describe. Set
\[
x(g) = \begin{cases} 
  c & \text{if } c \neq 0, \\
  d & \text{if } c = 0,
\end{cases}
\]
g = \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

On the other hand the Hilbert symbol \((a, b)\) is defined by \((a, b) = 1\) if \(a\) has the form \(a = x^2 - by^2\), \((a, b) = -1\) if not. Then
\[
\alpha(g_1, g_2) = \left( \frac{x(g_1 g_2)}{x(g_1)} \frac{x(g_1 g_2)}{x(g_2)} \right).
\]
The group \( \tilde{S} \) thus fits into the exact sequence
\[
1 \rightarrow \{ \pm 1 \} \rightarrow \tilde{S} \rightarrow S \rightarrow 1.
\]

Let
\[
n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \tilde{n}(x) = (n(x), 1), \quad w = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \tilde{w} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, 1),
\]
\[
t(a) = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix}, \quad \tilde{t}(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1), \quad z(c) = \begin{pmatrix} c & \alpha \\ 1 & c \end{pmatrix}.
\]

**Measures.** We fix the additive measure \( dx \) on \( F_v \) to be self dual for the character \( \psi_v \). Let the multiplicative measure be \( d^\times a = (1 - q_v^{-1})^{-1} \cdot |a|_v^{-1} da \), where \( q_v \) is the order of the residue field when \( v \) is non-Archimedean and \( q_v = \infty \) when \( v \) is Archimedean. Let \( dg = |a|_v d^\times c d^\times a dx dy \) be the measure on \( G(F_v) \) if we use the Bruhat coordinates \( g = (c)n(x) n(w) n(y) \) on \( G(F_v) \setminus B(F_v) \). The measure on \( Z(F_v) \) is \( dz(c) = d^\times c \), and we use the resulting quotient measure on \( G(F_v)/Z(F_v) \). For \( g \in \tilde{S}(F_v) \setminus \tilde{B}(F_v) \) with \( g = \tilde{n}(x) \tilde{h}(a) \tilde{n}(y) \), we define \( dg = |a|_v^2 d^\times a dx dy \) to be the measure on \( \tilde{S}(F_v) \).

It should be stressed that the choice of additive measure does not matter for the statement of our theorems.

**The Weil constant.** Define the Weil constant \( \gamma(a, \psi_v^P) \) over \( F_v \) to satisfy
\[
\int \Phi^{2D}(x) \psi_v^P(ax^2) dx = |a|_v^{-1/2} \gamma(a, \psi^P) \int \Phi(x) \psi_v^P(-a^{-1}x^2) dx,
\]
where \( \Phi \) is a Schwartz-Bruhat function on \( F_v \) and
\[
\tilde{\Phi}(x) = \int \Phi(y) \psi_v^P(xy) dy.
\]

**Representations.** We use \( \pi \) to denote an irreducible cuspidal representation of \( G(F_v) \) with trivial central character, and use \( \tilde{\pi} \) to denote an irreducible cuspidal representation of \( \tilde{S}(F_v) \). We have \( \pi = \otimes_v \pi_v \) and \( \tilde{\pi} = \otimes_v \tilde{\pi}_v \) as the restricted tensor products of representations over local fields. Let \( V_\pi, V_{\tilde{\pi}}, V_{\pi_v}, V_{\tilde{\pi}_v} \) denote the underlying spaces of \( \pi, \tilde{\pi}, \pi_v \) and \( \tilde{\pi}_v \) respectively. We use \( \| \varphi \| \) to denote the norm of a vector \( \varphi \) in \( V \); namely, let \( \| \varphi \| = (\varphi, \varphi)^{1/2} \) if \( (., .) \) is the Hermitian form on a space \( V \).

Let the \( L \)-function \( L(\pi, s) \) and the \( \epsilon \)-factor \( \epsilon(\pi, s) \) be defined as in [11]. We have \( L(\pi, s) = \prod L(\pi_v, s) \) and \( \epsilon(\pi, s) = \prod \epsilon(\pi_v, s, \psi_v) \); however \( \epsilon(\pi_v, 1/2) = \epsilon(\pi_v, 1/2, \psi_v) \) is independent on \( \psi_v \).
We use $S$ to denote a finite set of local places. As usual, let $S_{\infty}$ denote the set of all Archimedean places. Let $| |_S$ be the product of the local metrics over $S$. Let $L^S(\pi, s) = \prod_{v \notin S} L(\pi_v, s)$ be the partial $L$-function above the complement of $S$.

We say that $S$ contains all bad places if it contains all $v$ that are Archimedean or have even residue characteristic. Then it is known that the covering $\tilde{\text{SL}}_2(F_v)$ splits over $\text{SL}_2(O_v)$ if $v \notin S$. Let $V^S_\pi$ or $V^S_S$ be the subspace of vectors in $V_\pi$ or $V_S$ which are invariant under $\prod_{v \notin S} \text{GL}_2(O_v)$ or $\prod_{v \notin S} \text{SL}_2(O_v)$ respectively.

### 2. Definition of Two Constants

In this section, we define the two constants $d_\varepsilon(S, \psi)$ and $d_\varepsilon(S, \psi)$ that occur in Theorem [1.1. For $D \in F^\times$, $d_\varepsilon(S, \psi^D)$ and $d_\varepsilon(S, \psi^D)$ may be regarded as the $D$-th Fourier coefficients for $\pi$ and $\pi$ respectively.

#### 2.1. Definition of $d_\varepsilon(S, \psi)$.

2.1.1. **Whittaker model on $G$.** Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Let $V_\pi \subset L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$ be the underlying space of $\pi$. The global $\psi$-Whittaker model of $\pi$ consists of all the Whittaker functions $W^\psi_\varepsilon$ defined by

\[ W^\psi_\varepsilon(g) = W^\psi_\varepsilon(g) = \int_{\mathbb{A}/F} \varphi(n(x)g)\psi(-x)dx, \quad \varphi \in V_\pi. \]

For any given local component $\pi_v$, we shall fix a nontrivial $\psi_v$-Whittaker functional $L_v$, satisfying

\[ L_v(\pi_v(n(x))v) = \psi_v(x)L_v(v), \quad v \in V_\pi. \]

Let $S$ be a finite set of places that contains all bad places along with places $v$ where $\pi_v$ is not unramified. For $v \notin S$, let $\varphi_{0,v}$ be the unique vector in $V_\pi$ fixed under the action of $G(O_v)$ such that $L_v(\varphi_{0,v}) = 1$. Subsequently, we shall always assume that a pure tensor $\varphi = \otimes_v \varphi_v$ in $V^S_\pi$ has local component $\varphi_v = \varphi_{0,v}$ outside $S$.

We note that

\[ L(\varphi) = W^\psi_\varepsilon(1) \]

gives rise to a $\psi$-Whittaker functional on $V_\pi$. From the uniqueness of the local Whittaker functional, $L$ can be expressed as a product of $L_v$. Precisely, there is a constant $c_1(\pi, S, \psi, \{L_v\})$ such that

\[ W^\psi_\varepsilon(1) = c_1(\pi, S, \psi, \{L_v\}) \prod_{v \in S} L_v(\varphi_v) \]

for any pure tensor $\varphi \in V^S_\pi$.

2.1.2. **Hermitian forms for $G$.** The space $V_\pi$ has the Hermitian form

\[ (\varphi, \varphi') = \int_{Z(\mathbb{A})G(F)\backslash G(\mathbb{A})} \varphi(g)\overline{\varphi'(g)}dg. \]
Given the nontrivial Whittaker functional $L_v$ on $\pi_v$, we can define a $G(F_v)$-invariant Hermitian form on $V_{\pi_v}$ by

$$
(v, v') = \left. \int_{F_v} L_v(\pi_v(t(a))v)L_v(\pi_v(t(a))v') \frac{da}{|a_v|} \right|_{v, v' \in V_{\pi_v}}.
$$

(2.4)

From the uniqueness of $G_v$-invariant Hermitian forms, we infer that there is a constant $c_2(\pi, S, \psi, \{L_v\}) > 0$ such that

$$
\|\varphi\| = c_2(\pi, S, \psi, \{L_v\}) \prod_{v \in S} \|\varphi_v\|,
$$

(2.5)

for any tensor $\varphi \in V^S_{\pi}$.

2.1.3. The constant $d_\pi(S, \psi)$. We define

$$
d_\pi(S, \psi) = |c_1(\pi, S, \psi, \{L_v\})/c_2(\pi, S, \psi, \{L_v\})|.
$$

(2.6)

It is easy to verify that $d_\pi(S, \psi)$ is well defined in the sense that it is independent on the choice of the linear forms $L_v$. Also $d_\pi(S, \psi)$ is independent on the the choice of the additive measure on $F_v$. It follows from (2.2) and (2.5) that

$$
d_\pi(S, \psi) = \frac{|W_\psi(1)|}{|\varphi|} \prod_{v \in S} \frac{\|\varphi_v\|}{|L_v(\varphi_v)|}
$$

(2.7)

for any tensor $\varphi \in V^S_{\pi}$ such that $L_v(\varphi_v) \neq 0$ for $v \in S$. We let $e(\varphi_v, \psi_v)$ denote the square of the local factor occurring in (2.7), namely,

$$
e(\varphi_v, \psi_v) = \|\varphi_v\|^2/|L_v(\varphi_v)|^2.
$$

(2.8)

It is clear that the quotient on the right is independent on the choice of $L_v$.

2.2. Definition of $d_\pi(S, \psi)$.

2.2.1. Whittaker model on $\tilde{S}$. We proceed in the same way as in §2.1.1. Let $\tilde{\pi}$ be an irreducible cuspidal automorphic representation of $\tilde{S}$. For $\tilde{\varphi} \in V_{\tilde{\pi}}$, define

$$
\tilde{W}_{\tilde{\varphi}}(g) = \tilde{W}_{\tilde{\varphi}}^\varphi(g) = \int_{A_v/F_v} \tilde{\varphi}(\tilde{n}(x)g)\varphi(-x)dx.
$$

(2.9)

Later, we shall use the abbreviation $\tilde{W}_D^\varphi(g) = \tilde{W}_{\tilde{\varphi}}^{\varphi}D(g)$ for $D \in F^\times$.

Let us assume at the moment that $\tilde{\pi}$ has a nontrivial $\psi$-Whittaker model. Then locally each $\tilde{\pi}_v$ has a nontrivial $\psi_v$-Whittaker model, unique up to a scalar multiple. We shall fix the corresponding $\psi_v$-Whittaker functional $\tilde{L}_v$, satisfying

$$
\tilde{L}_v(\tilde{\pi}_v(\tilde{n}(x))\tilde{\nu}) = \psi_v(x)\tilde{L}_v(\tilde{\nu}), \quad \tilde{\nu} \in V_{\tilde{\pi}_v}.
$$

Let $\tilde{S}$ be a finite set of places that contains all bad places along with places $v$ where $\tilde{\pi}_v$ is not unramified. For $v \notin \tilde{S}$, let $\tilde{\varphi}_{0,v}$ be the unique vector in $V_{\tilde{\pi}_v}$ fixed under the action of $\text{SL}_2(O_v)$ such that $\tilde{L}_v(\tilde{\varphi}_{0,v}) = 1$. It will be understood that a pure tensor $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v$ in $V^S_{\tilde{\pi}}$ has local component $\tilde{\varphi}_v = \tilde{\varphi}_{0,v}$ outside $\tilde{S}$.

Define the constant $\tilde{c}_1(\tilde{\pi}, S, \psi, \{\tilde{L}_v\})$ such that, for all pure tensors $\tilde{\varphi}$ in $V^S_{\tilde{\pi}}$,

$$
\tilde{W}_{\tilde{\varphi}}(1) = \tilde{c}_1(\tilde{\pi}, S, \psi, \{\tilde{L}_v\}) \prod_{v \in \tilde{S}} \tilde{L}_v(\tilde{\varphi}_v).
$$

(2.10)
2.2.2. Hermitian forms for $\tilde{S}$. The space $V_2$ has the Hermitian form

\begin{equation}
(\tilde{\varphi}, \tilde{\varphi}') = \int_{S(F)\tilde{S}(A)} \tilde{\varphi}(g)\overline{\tilde{\varphi}'(g)}dg;
\end{equation}

here $S(F)$ splits in $\tilde{S}(A)$ so that we may regard $S(F)$ as a discrete subgroup of $\tilde{S}(A)$. The local Hermitian form on $V_2$, may be defined similar to (2.4), though the definition is more complicated. According to [BM1 §9] and [BM2 §15, 16], there is a choice of $\psi^\delta_L$-Whittaker functionals $\tilde{L}^\delta_\psi$ (could be trivial) on $V_2$, for $\delta$ in a collection of representatives of $F^\times/F^\times 2$ containing 1 ($\tilde{L}^1_\psi = \tilde{L}_\psi$), such that

\begin{equation}
(\tilde{\varphi}, \tilde{\varphi}') = \sum_{\delta} \frac{|2|_v}{2} \int_{F^\times_\psi} \tilde{L}^\delta_\psi(\tilde{\chi}_v(a)\tilde{\varphi})\overline{\tilde{L}^\delta_\psi(\tilde{\chi}_v(a)\tilde{\varphi}')}, \quad \tilde{\varphi}, \tilde{\varphi}' \in V_2,
\end{equation}

is an $\tilde{S}(F_\psi)$-invariant Hermitian form. Define the constant $\tilde{c}_2(\tilde{\chi}, S, \psi, \{\tilde{L}_\psi\})$ such that

\begin{equation}
\|	ilde{\varphi}\| = \tilde{c}_2(\tilde{\chi}, S, \psi, \{\tilde{L}_\psi\}) \prod_{v \in S} \|	ilde{\varphi}_v\|
\end{equation}

for all pure tensors $\tilde{\varphi}$ in $V_2^S$.

2.2.3. The constant $d_{\tilde{\chi}}(S, \psi)$. We define

\begin{equation}
d_{\tilde{\chi}}(S, \psi) = |\tilde{c}_1(\tilde{\chi}, S, \psi, \{\tilde{L}_\psi\})/\tilde{c}_2(\tilde{\chi}, S, \psi, \{\tilde{L}_\psi\})|.
\end{equation}

This definition is independent on the choice of the linear forms $\tilde{L}_\psi$ and the additive measure on $F_\psi$. Additionally, we set $d_{\tilde{\chi}}(S, \psi) = 0$ when $\tilde{\chi}$ does not have a nontrivial $\psi$-Whittaker model. We have

\begin{equation}
d_{\tilde{\chi}}(S, \psi) = \frac{|\tilde{W}_{\tilde{\chi}}(1)|}{\|	ilde{\varphi}\|} \prod_{v \in S} \frac{\|	ilde{\varphi}_v\|}{|\tilde{L}_v(\tilde{\varphi}_v)|}
\end{equation}

for any pure tensor $\tilde{\varphi} \in V_2^S$ such that $\tilde{L}_v(\tilde{\varphi}_v) \neq 0$ for $v \in S$. We let $e(\tilde{\varphi}_v, \psi_v)$ denote the square of the local factor in (2.15), namely,

\begin{equation}
e(\tilde{\varphi}_v, \psi_v) = \|	ilde{\varphi}_v\|^2/|\tilde{L}_v(\tilde{\varphi}_v)|^2.
\end{equation}

Note that the quotient on the right is independent on the choice of $\tilde{L}_v$. When working with $\psi_v^D$, we shall denote by $\tilde{L}_v^D$ the $\psi_v^D$-Whittaker functional and by $\|	ilde{\varphi}_v\|_D$ the local Hermitian norm of $\tilde{\varphi}_v$ constructed from $\tilde{L}_v^D$.

3. Results on the Theta Correspondence à la Waldspurger

In this section, we describe Waldspurger’s beautiful results in [Wal1] [Wal3] on the theta correspondence, both local and global, between PGL$_2$ and SL$_2$.

Given $\vartheta = \otimes_v \psi_v$, let $\vartheta(\cdot, \psi_v)$ and $\Theta(\cdot, \psi)$ be the local and global theta correspondence, respectively, as defined in [Wal1] [Wal3].

3.1. Local theory of Waldspurger. Let $\pi_v$ be an irreducible unitary representation of PGL$_2(F_v)$. For $D \in F^\times_\psi$, define

\[ e(D, \pi_v) = \chi_D(-1)e(\pi_v, 1/2)/e(\pi_v \otimes \psi_D, 1/2), \]
which takes value in \{±1\}. Set
\[ F^\pm_v(\pi_v) = \{ D \in F^\times_v : \varepsilon(D, \pi_v) = \pm 1 \}, \]
and we have a partition \( F^\times_v = F^+_v(\pi_v) \cup F^-_v(\pi_v). \)

**Theorem 3.1 (Waldspurger).** Let \( P_{0,v} \) be the set of equivalence classes of discrete series of \( \text{PGL}_2(F_v) \).

1. When \( \pi_v \notin P_{0,v}, F^\times_v(\pi_v) = F^+_v \) and \( \Theta(\pi_v \otimes \chi_D, \psi^D) = \Theta(\pi_v, \psi_v) \) for all \( D \in F^\times_v \).
2. When \( \pi_v \in P_{0,v}, \) there are two distinct representations \( \hat{\pi}^+_v = \Theta(\pi_v, \psi_v) \) and \( \hat{\pi}^-_v \) of \( \hat{\text{SL}}_2(F_v) \), such that
   \[ \Theta(\pi_v \otimes \chi_D, \psi^D_v) = \begin{cases} \hat{\pi}^+_v & \text{if } D \in F^+_v(\pi_v), \\ \hat{\pi}^-_v & \text{if } D \in F^-_v(\pi_v). \end{cases} \]
3. \( \Theta(\pi_v \otimes \chi_D, \psi^D_v) = \hat{\pi}^\pm_v \) if and only if \( \hat{\pi}^\pm_v \) has a nontrivial \( \psi^D \)-Whittaker model.

### 3.2. Global theory of Waldspurger

Let \( \hat{\text{A}}_{00} \) be the set of irreducible cuspidal automorphic representations of \( \hat{\text{SL}}_2(\hat{\text{A}}) \) that are orthogonal to the theta series attached to one dimensional quadratic forms. These are precisely those representations that admit the Shimura lift. For \( \hat{\pi}_1 \) and \( \hat{\pi}_2 \) in \( \hat{\text{A}}_{00} \), we say they are near equivalent, denoted by \( \hat{\pi}_1 \sim \hat{\pi}_2 \), if \( \hat{\pi}_1,\pi_v \sim \hat{\pi}_2,\pi_v \) for almost all places \( v \). Use \( \hat{\text{A}}_{00} \) to denote the quotient of \( \hat{\text{A}}_{00} \) by this relation.

Let \( \hat{\text{A}}_{01} \) be the set of irreducible cuspidal automorphic representations of \( \text{PGL}_2(\hat{\text{A}}) \) such that \( L(\pi \otimes \chi_D, 1/2) \neq 0 \) for some \( D \in F^\times \).

Let \( \Sigma = \Sigma(\pi) \) be the set of places \( v \) such that \( \pi_v \in P_{0,v} \). Given \( D \in F^\times \), let \( \varepsilon(D, \pi) = (\varepsilon(D, \pi_v))_{v \in \Sigma} \). Note that \( \varepsilon(D, \pi) \in \{±1\}^{[\Sigma]} \). We have
\[ \varepsilon(\pi \otimes \chi_D, 1/2) = \varepsilon(\pi, 1/2) \prod_{v \in \Sigma} \varepsilon(D, \pi_v). \]

We shall use \( \varepsilon = (\varepsilon_v)_{v \in \Sigma} \) to denote an element in \( \{±1\}^{[\Sigma]} \). Given such an \( \varepsilon \), define
\[ F^\varepsilon(\pi) = \{ D \in F^\times : \varepsilon(D, \pi) = \varepsilon \}. \]
Then we get a partition
\[ F^\times = \bigcup_{\varepsilon \in \{±1\}^{[\Sigma]}} F^\varepsilon(\pi). \]

**Theorem 3.2 (Waldspurger).** Let notations be as above.

1. \( \Theta(\pi, \psi) \cong \otimes_v \Theta(\pi_v, \psi_v) \) if \( \Theta(\pi, \psi) \neq 0 \); \( \Theta(\hat{\pi}, \psi) = \otimes_v \Theta(\hat{\pi}_v, \psi_v) \) if \( \Theta(\hat{\pi}, \psi) \neq 0 \).
2. \( \Theta(\pi, \psi) \neq 0 \) if and only if \( L(\pi, 1/2) \neq 0 \). \( \Theta(\pi, \psi) \neq 0 \) if and only if \( \hat{\pi} \) has a nontrivial \( \psi \)-Whittaker model.
3. For \( \hat{\pi} \in \hat{\text{A}}_{00} \), there is a unique \( \pi \) associated to \( \hat{\pi} \), such that \( \Theta(\hat{\pi}, \psi^D) \otimes \chi_D = \pi \) whenever \( \Theta(\hat{\pi}, \psi^D) \neq 0 \). Denote this association by \( \pi = S_{\psi}(\hat{\pi}) \). The map \( S_{\psi} \) then defines a bijection between \( \hat{\text{A}}_{00} \) and \( \text{A}_{01} \).
4. Let \( \pi = S_{\psi}(\hat{\pi}) \). There is a bijection between the near equivalence class of \( \hat{\pi} \) and the subset of \( \{±1\}^{[\Sigma]} \) comprising those \( \varepsilon \) such that \( \prod_{v \in \Sigma} \varepsilon_v = \varepsilon(\pi, 1/2) \); denote by \( \hat{\pi}^\varepsilon \) the representation corresponding to \( \varepsilon \). For convenience, set \( \hat{\pi}^\varepsilon = 0 \) if \( \prod_{v \in \Sigma} \varepsilon_v \neq \varepsilon(\pi, 1/2) \).
Let $\pi = S\phi(\tilde{\pi})$, then the near equivalence class of $\tilde{\pi}$ consists of all the nonzero $\Theta(\pi \otimes \chi_D, \psi^D)$. Precisely, for $D \in F^\times(\pi)$,

$$\Theta(\pi \otimes \chi_D, \psi^D) = \begin{cases} \tilde{\pi}^\cdot, & \text{if } L(\pi \otimes \chi_D, 1/2) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

4. The Formula of Waldspurger

We are now ready to state our main theorems. Their proofs will be provided in §7 after the relative trace formula and local Bessel identities are introduced.

4.1. The basic Waldspurger’s formula.

THEOREM 4.1. Let $\pi$ be an irreducible cuspidal automorphic representation of $\text{GL}_2(\mathbb{A})$, trivial on the center, with $L(\pi, 1/2) \neq 0$. Let $D \in F^\times$. Set $\tilde{\pi}_D = \Theta(\pi, \psi^D)$. Suppose that $S$ is a finite set of places of $F$ which contains all bad places, all places $v$ where $\psi_v$ or $\psi^D_v$ is ramified, along with all places where $\pi_v$ or $\tilde{\pi}_D,v$ is not ramified. We have

$$|d_\pi(S, \psi)|^2 L^S(\pi, 1/2) = |d_{\tilde{\pi}_0}(S, \psi^D)|^2. \tag{4.1}$$

Recall that $L^S(\pi, s)$ is the partial $L$-function $\prod_{v \notin S} L(\pi_v, s)$ and that $d_\pi(S, \psi)$ and $d_{\tilde{\pi}_0}(S, \psi^D)$ are the two constants defined as in §2.

In view of (2.7) and (2.15), we may reformulate (4.1) more explicitly as follows.

COROLLARY 4.2. Let $\varphi \in V^\otimes_\pi$ and $\varphi^D \in V^\otimes_{\tilde{\pi}_D}$ be pure tensors such that $L_v(\varphi_v) \neq 0$ and $\tilde{L}_v(\varphi_v^D) \neq 0$ for all $v \in S$. Let the local constants $e(\varphi_v, \psi_v)$ and $e(\varphi_v, \psi^D_v)$ be defined as in (2.8) and (2.16) respectively. Then

$$\frac{|\tilde{W}_\varphi(1)|^2}{|\varphi|^2} = \frac{|W_\varphi(1)|^2 L(\pi, 1/2)}{|\varphi|^2} \prod_{v \notin S} \frac{e(\varphi_v, \psi)}{e(\varphi_v, \psi^D_v) L(\pi_v, 1/2)}. \tag{4.2}$$

4.2. The Waldspurger formula for twisted central $L$-values. Applying Theorem 4.1 to $\pi \otimes \chi_D$, we get the Waldspurger formula for $L(\pi \otimes \chi_D, 1/2)$.

For $D \in F^\times$, we say, with slight abuse of terminology, that $D$ is a square-free integer if $|q_v|^{\epsilon_v^{-1}} \leq |D|_v \leq 1$ for all non-Archimedean places $v$; clearly, one has $|D|_v = 1$ or $q_v^{-1}$ if the place $v$ is odd. Note that the discriminant of any quadratic extension of $F$ is a square-free integer.

According to Theorem 3.2 there is a bijection $S_\phi$ between $\tilde{\mathcal{A}}_{00}$ and $\mathcal{A}_{0,\ell}$. For $\pi \in \mathcal{A}_{0,\ell}$, let $\{\tilde{\pi}^\epsilon : \prod_{v \in \Sigma} \epsilon_v = \epsilon(\pi, 1/2)\}$ be the packet $S_{\tilde{\pi}}^{-1}(\pi)$ of representations in $\tilde{\mathcal{A}}_{00}$ corresponding to $\pi$. Here $\epsilon = (\epsilon_v)_{v \in \Sigma} \in \{1, -1\}^{[\Sigma]}$, with $\Sigma = \Sigma(\pi)$ defined as in §3.2. For $D \in F^\times(\pi)$, we have $\Theta(\pi \otimes \chi_D, \psi^D) = \tilde{\pi}^\epsilon$ or $0$ according as $L(\pi \otimes \chi_D, 1/2) = 0$ or not.

THEOREM 4.3. Let $\pi$ and $\tilde{\pi}^\epsilon$ be as above. Let $S$ be a finite set of places containing all bad places along with the places $v$ where $\pi_v$ or $\psi_v$ is not unramified. Then for any square-free integer $D \in F^\times$, if $D \in F^\epsilon(\pi)$, then $d_{\tilde{\pi}^\epsilon}(S, \psi^D) = 0$ whenever $\epsilon \neq \epsilon_0$, but, for this $\epsilon_0$, we have

$$|d_{\tilde{\pi}^\epsilon}(S, \psi^D)|^2 = |d_\pi(S, \psi)|^2 L^S(\pi \otimes \chi_D, 1/2)/|D|_S. \tag{4.3}$$
More explicitly, for pure tensors \( \varphi = \otimes_v \varphi_v \in V^S_a \) and \( \tilde{\varphi} = \otimes_v \tilde{\varphi}_v \in V^S_a \), with \( \varphi_v = \varphi_{0,v} \) and \( \tilde{\varphi}_v = \tilde{\varphi}_{0,v} \) for all \( v \neq S \), we have

\[
(4.4) \quad \frac{[\hat{W}_S^D(1)]^2}{\|\varphi\|^2} = \frac{[W_\varphi(1)]^2 L(\pi \otimes \chi_D, 1/2)}{\|\varphi\|^2} \prod_{v \in S} e(\varphi_v, \psi_v) L(\pi_v \otimes \chi_D, 1/2)|D_v|.
\]

5. Review of a Relative Trace Formula of Jacquet

In this section, we give a brief review of the relative trace formula of Jacquet in [Jac]. Note that he acknowledges the inspiration from Iwaniec’s work [Iwa1, Iwa2].

5.1. Definition of the global distributions \( I(f, \psi) \) and \( J(f', \psi^D) \). Let \( C_c^\infty(Z(\mathbb{A}) \backslash G(\mathbb{A})) \) denote the space of smooth compactly supported functions on \( Z(\mathbb{A}) \backslash G(\mathbb{A}) \). For \( f \in C_c^\infty(Z(\mathbb{A}) \backslash G(\mathbb{A})) \) define a kernel function

\[
K(x, y; f) = \sum_{\xi \in Z(F) \backslash G(F)} f(x^{-1} \xi y).
\]

Define the distribution \( I(f, \psi) \) to be

\[
(5.1) \quad I(f, \psi) = \int_{\mathbb{A}^\times / F^\times} \int_{\mathbb{A}/F} K(t(a), n(x); f) \psi(x) dx d^\times a.
\]

Let \( C_c^\infty(\mathcal{S}(\mathbb{A})) \) denote the space of genuine smooth compactly supported functions on \( \mathcal{S}(\mathbb{A}) \). For \( f' \in C_c^\infty(\mathcal{S}(\mathbb{A})) \) define

\[
K(x, y; f') = \sum_{\xi \in S(F)} f'(x^{-1} \xi y).
\]

Define the distribution \( J(f', \psi^D) \) to be

\[
(5.2) \quad J(f', \psi^D) = \int_{\mathbb{A}/F} \int_{\mathbb{A}^\times / F} K(\tilde{n}(x), \tilde{n}(y); f') \psi^D(\tilde{x} + y) dx dy.
\]

The relative trace formula is an identity between the distributions \( I(f; \psi) \) and \( J(f'; \psi^D) \).

5.2. Comparison of orbital integrals. The relative trace formula identity follows from comparing the orbital integrals on the geometric side. See [Jac §3.1, 4, 6.1, 6.2, 7].

Proposition 5.1 (Jacquet). Suppose that \( f \) and \( f' \) are products of local functions \( f_v \in C_c^\infty(Z(F_v) \backslash G(F_v)) \) and \( f'_v \in C_c^\infty(\mathcal{S}(F_v)) \), that is, \( f = \otimes_v f_v \) and \( f' = \otimes_v f'_v \). We have

\[
I(f, \psi) = \prod S^+_{\varphi_v}(f_v) + \prod S^-_{\varphi_v}(f_v) + \sum_{\xi \in F^\times} \prod O_{\varphi_v}(n(\xi) w; f_v),
\]

\[
J(f', \psi^D) = \prod S^+_{\varphi_v}(f'_v) + \prod S^-_{\varphi_v}(f'_v) + \sum_{\xi \in F^\times} \prod O_{\varphi_v}(\tilde{w}\tilde{n}(\xi) \tilde{w}; f'_v),
\]

in which

\[
O_{\varphi_v}(g; f_v) = \int_{F_v} \int_{F_v} f_v(t(a) n(x)) \psi_v(x) dx d^\times a,
\]

\[
O_{\varphi_v}(g; f'_v) = \int_{F_v} \int_{F_v} f'_v(n(x) \tilde{n}(y)) \psi^D_v(x + y) dx dy,
\]

while \( S^+_{\varphi_v}(f_v) \) and \( S^+_{\varphi_v}(f'_v) \) are certain singular orbital integrals which are determined by the asymptotics of \( O_{\varphi_v}(n(a) w; f_v) \) and \( O_{\varphi_v}(\tilde{w}\tilde{n}(\xi) \tilde{w}; f'_v) \) at the boundary \( a = 0 \) respectively.
For an irreducible cuspidal representation $\pi$ of $G(\mathbb{A}_F)$, trivial on the central, define
\begin{equation}
I_{\pi}(f, \psi) = \sum_{\varphi} Z(\pi(f) \varphi) W_{\varphi}(1),
\end{equation}
with $\{\varphi\}$ an orthonormal basis of $V_{\pi}$; here for $\varphi \in V_{\pi}$
\begin{align*}
\pi(f) \varphi &= \int_{Z(\mathbb{A})/G(\mathbb{A})} f(g) \pi(g) \varphi \, dg, \\
W_{\varphi}(1) &= \int_{K/F^\times} \varphi(n(x)) \psi(-x) \, dx, \\
Z(\varphi) &= \int_{K^x/F^x} \varphi(t(a)) \, da.
\end{align*}
For an irreducible cuspidal representation $\tilde{\pi}$ of $\tilde{S}(\mathbb{A})$ define
\begin{equation}
J_{\tilde{\pi}}(f', \psi^D) = \sum_{\varphi} W_{\tilde{\pi}(f') \varphi}(1) \tilde{W}^D_{\varphi}(1),
\end{equation}
with $\{\varphi\}$ an orthonormal basis of $V_{\tilde{\pi}}$; here for $\varphi \in V_{\tilde{\pi}}$
\begin{equation}
\tilde{\pi}(f') \varphi = \int_{\tilde{S}(\mathbb{A})} f'(g) \tilde{\pi}(g) \varphi \, dg.
\end{equation}

Moreover, Jacquet established the following result on the matching between Hecke functions on $Z(F_v) \backslash G(F_v)$ and $\tilde{S}(F_v)$ at almost all places ([Jac §2, 5, 8]).

**Proposition 5.3 (Jacquet).** Suppose that $F_v$ is a local non-Archimedean field, of odd residual characteristic, and that $\psi_v$ and $\psi_v^D$ are both unramified. Then there is a canonical choice of isomorphism between the Hecke algebras
\begin{equation}
H(Z(F_v) \backslash G(F_v)) \sim H(\tilde{S}(F_v))
\end{equation}
such that Hecke functions $f_v$ and $f_v'$ match if they correspond under the isomorphism (5.3).

Combining Propositions 5.1, 5.2 and 5.3, we have the following theorem.

**Theorem 5.4 (Jacquet).** Fix a finite set of places $S$ that contains all the bad places and the places where $\psi$ or $\psi^D$ is ramified. Let $f$ and $f'$ be as in Proposition 5.1. Assume that the local functions $f_v$ and $f_v'$ match as in Proposition 5.2 for each $v$ and that $f_v \in H(Z(F_v) \backslash G(F_v))$ and $f_v' \in H(\tilde{S}(F_v))$ correspond to one another via the isomorphism (5.3) in Proposition 5.3 for each $v \notin S$. Then
\begin{equation}
I(f, \psi) = J(f', \psi^D).
\end{equation}
The distributions $I_\tau(f, \psi)$ and $J_\tau(f', \psi^D)$ are the contributions from $\pi$ and $\tilde{\pi}$ to $I(f, \psi)$ and $J(f', \psi^D)$ in their spectral decompositions, respectively.

Recall that, if $\pi_\circ$ is unramified, there is a vector $\varphi_{o,e}$ that is fixed under the action of $G(O_v)$. For each Hecke function $f_\circ$ in $H(Z(F_o) \backslash G(F_o))$ there is a constant $\hat{f}_\circ(\pi_\circ)$ such that

$$\pi_\circ(f_\circ) \varphi_{o,e} = \hat{f}_\circ(\pi_\circ) \varphi_{o,e}. \quad (5.5)$$

Similarly, if $\tilde{\pi}_e$ is unramified, let $\tilde{\varphi}_{o,e}$ be a vector that is fixed under $S(O_e)$, then for each $f'_e$ in $H(\tilde{S}(F_e))$ there is a constant $\tilde{\hat{f}}_e(\tilde{\pi}_e)$ with

$$\tilde{\pi}_e(f'_e) \tilde{\varphi}_{o,e} = \tilde{\hat{f}}_e(\tilde{\pi}_e) \tilde{\varphi}_{o,e}. \quad (5.6)$$

**Theorem 5.5.** Let $\pi$ be a cuspidal representation of $G(\mathbb{A})$ with trivial central character such that the distribution $I_r(f, \psi)$ is nontrivial.

1. There is a unique cuspidal representation $\tilde{\pi}$ of $\tilde{S}$ such that if $f$ and $f'$ match as in Theorem 5.4 then

$$I_\tau(f, \psi) = J_\tau(f', \psi^D). \quad (5.7)$$

2. Suppose that $S$ satisfies the condition in Theorem 5.4 and contains all the places where $\pi_\circ$ or $\tilde{\pi}_e$ is not unramified. For $v \notin S$, if the Hecke functions $f_\circ$ and $f'_e$ correspond to each other under the isomorphism in Proposition 5.3, then

$$\hat{f}_\circ(\pi_\circ) = \tilde{\hat{f}}_e(\tilde{\pi}_e). \quad (5.8)$$

3. $\tilde{\pi} = \Theta(\pi, \psi^D)$.

**Remark 5.6.** The distribution $I_r(f, \psi)$ is nontrivial if and only if $L(\pi, 1/2) \neq 0$.

In this theorem, (1) and (2) are due to Jacquet, while (3) is proven in [BM3] by combining the work of Jacquet and the theory of Waldspurger (Theorem 3.1 and 3.2).

**6. Review of the Local Bessel Distributions and Their Bessel Identities**

In this section, we review the local Bessel identities established in [BM1], [BM2] and [CQ] which complement the global identity (5.7) in Theorem 5.5. We shall retain the notations and assumptions as in Theorem 5.5.

**6.1. The local relative Bessel distribution $I_{\pi, v}(f_v, \psi_v)$.** As in §2.1, we fix a choice of the local $\psi_v$-Whittaker functional $L_v$ on $V_\pi$. For each $v \in S$, fix an orthonormal basis $\{\varphi_{i,v}\}$ of $V_\pi$. For $v \notin S$, let $\varphi_{o,e}$ be the normalized spherical vector as in §2.1. In view of (2.5), after rescaled by the factor $c_2(\pi, S, \psi_v, \{L_v\})^{-1}$, the tensor products formed by these $\varphi_{i,v}$ and $\varphi_{o,e}$ give rise to an orthonormal basis for $V_\pi^S$, say, denoted by $\{\varphi_{i(S)}\}$. With our choice of $f$, the operator $\pi(f)$ preserves $V_\pi^S$ but annihilates its orthogonal complement in $V_\pi$. Then the expression in (5.4) becomes

$$I_\pi(f, \psi) = \sum_{\varphi_{i(S)}} Z(\pi(f)\varphi_{i(S)}) W_{\varphi_{i(S)}}(1). \quad (6.1)$$

Using the Hecke theory for $GL_2$, it is easy to verify the following lemma.
Lemma 6.1. For a pure tensor \( \varphi \in V_{\pi}^S \), we have
\[
Z(\varphi) = c_1(\pi, S, \psi, \{L_\nu\})L(\pi, 1/2) \prod_{v \in S} P_v(\varphi_v),
\]
with
\[
P_v(\varphi_v) = \frac{1}{L(\pi_v, s)} \int_{F_v^\times} L_v(\pi_v(t(a))\varphi_v) |a_v|^{s-1/2} d^\times a |_{s=1/2}.
\]

We now define the local Bessel distribution as in \([BM1, BM2, CQ]\).
\[
I_v(f_v, \psi_v) = \sum_{\varphi_v} P_v(\pi_v(f_v)\varphi_v)L_v(\varphi_v).
\]

Note that this definition is independent on the choice of \( L_v \) at the beginning.

The following proposition is readily established on (2.2), (2.6) and (6.1)-(6.4).

Proposition 6.2. Let notations be as above. We have
\[
I_v(f, \psi) = L(\pi, 1/2)|d_\pi(S, \psi)|^2 \prod_{v \in S} I_v(f_v, \psi_v) \prod_{v \notin S} \hat{f}_v(\pi_v).
\]

6.2. The local Bessel distribution \( J_{\tilde{\pi}}(f'_v, \psi'_v) \). We can apply the same argument to the distribution \( J_{\tilde{\pi}}(f'_v, \psi'_v) \). The corresponding local Bessel distribution is defined by
\[
J_{\tilde{\pi}}(f'_v, \psi'_v) = \sum_{\varphi'_v} \tilde{L}_v(\tilde{\pi}_v(f'_v)\varphi'_v)\tilde{L}_v(\varphi'_v).
\]
where the sum is over a orthonormal basis \( \{\tilde{\varphi}'_v\} \) of \( V_{\tilde{\pi}} \). Again this definition is independent on the local \( \psi'_v \)-Whittaker functional \( \tilde{L}_v(\cdot) \) that we choose.

Proposition 6.3. Let notations be as above. We have
\[
J_v(f', \psi) = |d_\pi(S, \psi)|^2 \prod_{v \in S} J_v(f'_v, \psi'_v) \prod_{v \notin S} \hat{f}_v(\pi_v).
\]

6.3. The local Bessel identity.

Theorem 6.4. Let \( \pi_v \) be an irreducible unitary representation of \( G(F_v) \) with trivial central character. Let \( D \in F_v^\times \). Put \( \tilde{\pi}_v = \theta(\pi_v, \psi'_v) \). Suppose that \( f_v \in C_c^\infty(\hat{Z}(F_v) \setminus G(F_v)) \) and \( f'_v \in C_c^\infty(\hat{S}(F_v)) \) match as in Proposition 5.2. We have
\[
J_{\tilde{\pi}}(f'_v, \psi'_v) = |2D_v|e(\pi_v, 1/2)L(\pi_v, 1/2)I_v(f_v, \psi_v),
\]
in which \( J_{\tilde{\pi}}(f'_v, \psi'_v) \) and \( I_v(f_v, \psi_v) \) are defined as in (6.3) and (6.6) respectively.

Remark 6.5. The proof of this theorem is quite technical. The identity is established in \([BM1, BM2] \) and \([CQ]\) when \( v \) is non-Archimedean, real and complex, respectively. In the former two papers of Baruch and Mao, the complexity lies mostly in the representation theory aspect, although the real case is analytic in nature. In the complex case, however, the representation theory is the simplest—\( \tilde{SL}_2(\mathbb{C}) \) is just the trivial double cover of \( SL_2(\mathbb{C}) \)—while the difficulty is to prove the complex analogue of the exponential integral formulae of Weber and Hardy—we now have to integrate twice, radially and angularly. The proof is done in two papers of the second author \([Qi1, Qi2]\), and it involves a number of classical
formulae for special functions (Bessel and Kummer), quite a lot of combinatorial identities (the combinatorial theory of Bessel functions therein should be of some independent interest), the method of stationary phase and a little bit of differential equation theory.

7. Proof of Theorem 4.1 and Theorem 4.3

7.1. Proof of Theorem 4.1 Let \( \pi \) and \( \tilde{\pi} = \Theta(\pi, \psi^D) \) be as in the theorem. Let \( f = \otimes f_v \) and \( f' = \otimes f'_v \) match as in Theorem 5.4. By Theorem 5.3(1, 3), we have
\[
I_\pi(f, \psi) = J_\tilde{\pi}(f', \psi^D).
\]
By Proposition 6.2 and 6.3, this equality may be explicitly written as
\[
L(\pi, 1/2)|d_\pi(S, \psi)|^2 \prod_{v \in \Sigma} I_{\pi_v}(f_v, \psi_v) \prod_{v \notin \Sigma} \hat{f}_v(\pi_v) = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in \Sigma} J_{\tilde{\pi_v}}(f'_v, \psi^D_v) \prod_{v \notin \Sigma} \hat{f}'_v(\tilde{\pi}_v).
\]
It then follows from Theorem 5.5(2) and 6.4 that
\[
L(\pi, 1/2)|d_\pi(S, \psi)|^2 = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in \Sigma} |2D_v| \epsilon(\pi_v, 1/2)L(\pi_v, 1/2).
\]
As \( |2D_v| = 1 \) for \( v \notin \Sigma \), we get \( \prod_{v \notin \Sigma} |2D_v| = 1 \). As \( \epsilon(\pi_v, 1/2) = 1 \) for \( v \notin \Sigma \) and \( \epsilon(\pi, 1/2) = 1 \), we get \( \prod_{v \notin \Sigma} \epsilon(\pi_v, 1/2) = 1 \). Thus follows the identity in the theorem.

7.2. Proof of Theorem 4.3 Let \( \pi \) and \( \tilde{\pi}^e \) be as in the case. Assume \( D \in F^{\infty}(\pi) \).
We first consider the case \( \epsilon_0 \neq \epsilon \). Then \( \epsilon(D, \pi_v) = \epsilon_0, \epsilon \neq \epsilon_0 \) for some \( v \in \Sigma \).
Thus \( \theta(\epsilon_0 \otimes \chi_D, \psi^D_v) \neq \tilde{\pi}^e \). By Theorem 5.1, \( \tilde{\pi}^e \) does not have a nontrivial \( \psi^D \)-Whittaker functional, which then implies that \( \tilde{\pi}^e \) does not have a nontrivial \( \psi^D \)-Whittaker model. Hence \( d_{\tilde{\pi}^e}(S, \psi^D) = 0 \) by definition.
Let \( S_D \) be a finite set of places such that \( |D| = 1 \) if \( v \notin S_D \). Put \( S_1 = S \cup S_D \), \( S_2 = S_1 \setminus S \).
We now consider \( \Theta(\pi \otimes \chi_D, \psi^D) \). First of all, by Theorem 3.2(5), \( \Theta(\pi \otimes \chi_D, \psi^D) \) is equal to either \( \tilde{\pi}^e \) or 0.
Suppose \( \Theta(\pi \otimes \chi_D, \psi^D) = 0 \). Then \( L(\pi \otimes \chi_D, 1/2) = 0 \). By Theorem 3.2(2, 3), \( \tilde{\pi}^e \) does not have a \( \psi^D \)-Whittaker model; otherwise we would have \( \Theta(\tilde{\pi}^e, \psi^D) \otimes \chi_D = \pi \) and therefore \( \tilde{\pi}^e = \Theta(\pi \otimes \chi_D, \psi^D) \). Hence \( d_{\tilde{\pi}^e}(S, \psi^D) = 0 \). Thus the identity 4.3 holds in this case.
Next we consider the case \( \Theta(\pi \otimes \chi_D, \psi^D) = \tilde{\pi}^e \). For \( v \notin S \), \( \pi \otimes \chi_D \) is unramified at \( v \). Moreover, as \( \psi_v \) is unramified, the representation \( \theta(\pi_v \otimes \chi_D, \psi_v) \) is also unramified. We apply Theorem 4.1 and get
\[
|d_{\pi \otimes \chi_D}(S_1, \psi_v)|^2 L^{S_1}(\pi \otimes \chi_D, 1/2) = |d_{\pi_v}(S_1, \psi^D_v)|^2.
\]
According to Lemma 7.1 in [BM3], we have \( d_{\pi \otimes \chi_D}(S_1, \psi) = d_\pi(S_1, \psi) \) and hence
\[
|d_\pi(S_1, \psi_v)|^2 L^{S_1}(\pi \otimes \chi_D, 1/2) = |d_{\pi_v}(S_1, \psi^D_v)|^2.
\]
Take vectors \( \varphi = \otimes \varphi_v \in V_\pi \) and \( \tilde{\varphi} = \otimes \tilde{\varphi}_v \in V_{\tilde{\pi}^e} \), with \( \varphi_v = \varphi_{\alpha, v} \) and \( \tilde{\varphi}_v = \tilde{\varphi}_{\alpha, v} \) for \( v \notin S \), then by (2.7) and (2.15) we have
\[
|d_\pi(S_1, \psi_v)|^2 = |d_\pi(S_1, \psi)|^2 \prod_{v \in S_2} e(\varphi_{\alpha, v}, \psi_v)
\]
and
and
\[|d_{\hat{\varphi}^n}(S,\psi^D)|^2 = |d_{\hat{\varphi}^n}(S,\psi)|^2 \prod_{\nu \in \mathbb{S}_2} e(\varphi_{\nu,\nu},\psi_{\nu}^D),\]
where the constants \(e(\varphi_{\nu,\nu},\psi_{\nu}), e(\varphi_{\nu,\nu},\psi_{\nu}^D)\) are defined by (2.8) and (2.16), respectively. It then follows that
\[(7.1) \quad |d_{\pi}(S,\psi)|^2 L^S(\pi \otimes \chi_D, 1/2) = |d_{\hat{\varphi}^n}(S,\psi^D)|^2 \prod_{\nu \in \mathbb{S}_2} \frac{e(\varphi_{\nu,\nu},\psi_{\nu}^D)}{e(\varphi_{\nu,\nu},\psi_{\nu})}.\]

For \(\nu \in \mathbb{S}_2\), \(\pi_\nu\) is unramified and unitary, so \(\hat{\pi}_\nu^o = \theta(\pi_\nu \otimes \chi_D, \psi_{\nu}^D) = \theta(\pi_\nu, \psi_{\nu})\) by Theorem 3.1(1). At this point, we invoke Proposition 8.1 and 8.2 in [BM3] as follows.

**Proposition.** Let \(\nu\) be a non-Archimedean place, with odd residue characteristic. Suppose that \(\psi_\nu\) is unramified and that \(|D|_\nu = 1\) or \(q_\nu^{-1}\). Let \(\pi_\nu\) be an unramified unitary representation of \(G(F_\nu)\). Let \(\varphi_{\nu,\nu}\) and \(\hat{\varphi}_{\nu,\nu}\) be the unramified vectors of \(\pi_\nu\) and \(\hat{\pi}_\nu = \theta(\pi_\nu, \psi_{\nu})\) defined as in (8.2) respectively. Then
\[(7.2) \quad \frac{e(\varphi_{\nu,\nu},\psi_{\nu}^D)}{e(\varphi_{\nu,\nu},\psi_{\nu})} = \frac{1}{|D|_\nu L(\pi_\nu \otimes \chi_D, 1/2)}.\]

As \(D\) is square-free, we may apply the formula (7.2) for \(\nu \in \mathbb{S}_2\). Since \(S_1 = S \cup \mathbb{S}_2\), it follows from (7.1) and (7.2) that
\[(7.3) \quad |d_{\pi}(S,\psi)|^2 L^S(\pi \otimes \chi_D, 1/2) = |d_{\hat{\varphi}^n}(S,\psi^D)|^2 / |D|_{S_2}.
\]
Finally note that \(|D|_\nu = 1\) for \(\nu \notin S_1\) and hence \(|D|_{S_2} = 1\). So \(1 / |D|_{S_2} = |D|_S\). We get (4.3) and the theorem.

8. Metaplectic Ramanujan Conjecture and Lindelöf Hypothesis

In this final section, we discuss the connection between the metaplectic Ramanujan conjecture and the Lindelöf hypothesis.

8.1. Statement of conjectures. Let \(\tilde{\pi} \in \tilde{A}_{000}\). Suppose that \(S\) is a finite set of places of \(F\) containing all bad places along with places \(v\) where \(\pi_v\) is not unramified. For \(S_0 \subset S\) and \(\hat{\varphi} \in V^D_{\tilde{\pi}}\), we define
\[(8.1) \quad d_{\tilde{\pi}}(\hat{\varphi}, S_0, \psi^D) = \frac{\|\hat{\varphi}\| D}{\|\hat{\varphi}\|} \prod_{\nu \in S_0} \frac{\|\tilde{\varphi}_\nu\| D}{\|\tilde{\varphi}_\nu\| D}.
\]
This constant is well defined and independent on the choice of the \(\psi^D\)-Whittaker functional \(\hat{L}_D\). Moreover, in view of (2.15), we also have
\[(8.2) \quad d_{\tilde{\pi}}(\hat{\varphi}, S_0, \psi^D) = d_{\tilde{\pi}}(S,\psi^D) \prod_{\nu \in S \setminus S_0} \frac{\|\tilde{\varphi}_\nu\| D}{\|\tilde{\varphi}_\nu\| D}.
\]
Recall that \(S_\infty\) is the set of Archimedean places of \(F\) and that \(|D|_{S_\infty} = \prod_{\nu \in S_\infty} |D|_\nu\) for \(D \in F^\times\).

We can now state the metaplectic Ramanujan conjecture as follows.
Conjecture 8.1 (Metaplectic Ramanujan conjecture). Let $\tilde{\pi}$ be an irreducible cuspidal automorphic representation of $\tilde{\text{SL}}_2(\mathbb{A})$ in $\tilde{\Delta}_{00}$. Let $\tilde{\varphi}$ be a cusp form in $V_{\tilde{\varphi}}$. Let $D$ be a square-free integer in $F^\times$. For all $\alpha > 0$, we have

$$\left| d_2(\tilde{\varphi}, S_{\infty}, \psi^D) \right| \ll |D|_{S_{\infty}}^{-\frac{\alpha}{2}}$$

as $|D|_{S_{\infty}} \to \infty$, where the implied constant depends only on $\tilde{\pi}, \tilde{\varphi}$ and $\alpha$. Equivalently, for all $\alpha > 0$, we have

$$\left| \tilde{W}^D_{\varphi}(1) \right| \prod_{v \in S_{\infty}} e(\tilde{\varphi}_v, \psi^D)^{\frac{\alpha}{2}} \ll |D|_{S_{\infty}}^{-\frac{\alpha}{2}}$$

as $|D|_{S_{\infty}} \to \infty$.

In the classical language, the constant $d_2(\varphi, S_{\infty}, \psi^D)$ is indeed a renormalized $D$-th Fourier coefficient of the cusp form $\varphi$. Details may be found for example in [Wal2 III], [BM3 §9] and [BM4 §2].

Next we state a special case of the Lindelöf hypothesis, which is a conjecture on the bound of central value of twisted $L$-functions for $\text{PGL}_2$.

Conjecture 8.2 (Lindelöf hypothesis). Let $\pi$ be an irreducible cuspidal automorphic representation of $\text{PGL}_2(\mathbb{A})$. Let $D$ be a square-free integer in $F^\times$. For all $\beta > 0$, we have

$$\left| L^{S_{\infty}}(\pi \otimes \chi_D, 1/2) \right| \ll |D|_{S_{\infty}}^\beta$$

as $|D|_{S_{\infty}} \to \infty$, where the implied constant depends only on $\pi$ and $\beta$.

8.2. The Ramanujan-Lindelöf equivalence.

Theorem 8.3. The inequality (8.3) holds for $\alpha > 0$ (and for all $\tilde{\pi}$ and $\tilde{\varphi}$ as in Conjecture 8.1) if and only if the inequality (8.5) holds for $\beta = 2\alpha > 0$ (and for all $\pi$ as in Conjecture 8.2). In particular, Conjecture 8.1 is equivalent to Conjecture 8.2.

Proof. Let $\pi \in A_{0,1}$ and $\tilde{\pi} \in \tilde{\Delta}_{00}$ be given such that $\pi = S_{\varphi}(\tilde{\pi})$. Restrict $D$ to range in the set of square-free integers in $F^\times$ with $\Theta(\pi \otimes \chi_D, \psi^D) = \tilde{\pi}$ so that $d_2(S, \psi^D)$ and $L(\pi \otimes \chi_D, 1/2)$ are nonzero and that the identity (4.3) in Theorem 4.3 is nontrivial. We then rewrite the identity (4.3) as follows,

$$\left| d_2(\tilde{\varphi}, S_{\infty}, \psi^D) \right|^2 |D|_{S_{\infty}} \prod_{v \in S \setminus S_{\infty}} C_v(D, \pi_v, \tilde{\varphi}_v) = \left| d_2(S, \psi) \right|^2 L^{S_{\infty}}(\pi \otimes \chi_D, 1/2),$$

with

$$C_v(D, \pi_v, \tilde{\varphi}_v) = \left| D_v \right| L(\pi_v \otimes \chi_D, 1/2) \cdot \frac{\|\tilde{\varphi}_v\|_{D_v}^2}{\|\tilde{\varphi}_v\|_{D_v}^2}.$$

Here we have used the identity (8.2).

For each $v \in S \setminus S_{\infty}$, since $D$ is square-free, it is clear that $|D_v|$ and $L(\pi_v \otimes \chi_D, 1/2)$ take only finitely many possible nonzero values, depending on $\pi_v$ (in the latter case). It is also proven in [BM3 Lemma 7.2] that there are only finitely many possible nonzero values of $\|\tilde{\varphi}_v\|_{D_v}^2/\|\tilde{\varphi}_v\|_{D_v}^2$ when $D$ varying in the set of square-free integers. Combining these, we have

$$C_v(D, \pi_v, \tilde{\varphi}_v) \approx_{\pi_v, \tilde{\varphi}_v} 1.$$
The asymptotic notation here means that there are positive constants $A = A(\pi, \varphi_0)$ and $B = B(\pi, \varphi_0)$, depending only on $\pi_0$ and $\varphi_0$, such that $B < |C_\nu(D, \pi, \varphi)| < A$.

Now we turn to the proof of our theorem.

Assume first that (8.3) holds for $\alpha > 0$. Given $\pi$, let $S$ be a fixed finite set of places so that $d_\pi^r \in S^{-1}(\pi)$, fix a cusp form $\varphi_0 \in V_{\pi}$. Applying (8.3) for these $\pi^r$ and $\varphi_0$, along with (8.6)-(8.8), we get
\[ L^S \left( \pi \otimes \chi_D, 1/2 \right) \ll \left| \frac{d_\pi(\varphi, S, \psi_D)}{D_{S,\infty}^\alpha} \right| \leq |D|_{S,\infty}^{\frac{\alpha}{2}} |D|_{S,\infty} \to \infty, \]
for all square-free $D$ in $F^\ell(\pi)$ (note that if $D \in F^\ell(\pi)$ but $\Theta(\pi \otimes \chi_D, \psi_D) = 0$ then $L(\pi \otimes \chi_D, 1/2) = 0$ and the inequality above trivially). This proves (8.5) with $\beta = 2\alpha$. Note that the implied constant in each step depends (ultimately) only on $\pi$ and $\alpha$.

The reverse direction may be proven similarly. Assume that (8.5) holds for $\beta = 2\alpha > 0$. Given $\tilde{s}$, let $\pi = S(\tilde{s})$ and let $S$ be fixed as before. Assume $D$ is such that $\Theta(\pi \otimes \chi_D, \psi_D) = \tilde{s}$, as otherwise $d_\pi(\varphi, S, \psi_D) = 0$ and hence $d_\pi(\varphi, S, \psi_D) = 0$. Applying (8.3) for $\pi$, along with (8.6)-(8.8), we get
\[ |d_\pi(\varphi, S, \psi_D)| \leq |L^S \left( \pi \otimes \chi_D, 1/2 \right)|D_{S,\infty}^{-\frac{1}{2}} \ll |D|_{S,\infty}^{\frac{\alpha}{2}}, |D|_{S,\infty} \to \infty, \]
as desired. The implied constant depends only on $\tilde{s}, \varphi$ and $\alpha$.

Q.E.D.

8.3. Nontrivial bound toward the metaplectic Ramanujan conjecture. In view of (1.2) and (1.3), the inequality (8.5) holds for any $\beta > 2\bar{\theta} = \frac{3}{8} + \frac{1}{4} \theta$. As a consequence of Theorem 8.3 we have the following theorem.

Theorem 8.4. Let $\theta$ be any exponent toward the Ramanujan-Petersson conjecture for $GL_2(\mathbb{A})$. Then the bound in (8.3) holds for any $\alpha > \bar{\theta} = \frac{3}{16} + \frac{1}{8} \theta$.

Note that $\theta = 0$ is the Ramanujan-Petersson conjecture while the Kim-Sarnak $\theta = \frac{7}{64}$ is the best known exponent.

It is suggested in [Iwa1] that the bound in (8.3) would hold trivially for any $\alpha > \frac{1}{4}$ while (8.5) with $\beta > \frac{1}{2}$ is the convexity bound. The exponent in Theorem 8.4 was obtained over the rational field $\mathbb{Q}$ in [BH] and [BM] for holomorphic modular forms and Maass forms of half-integral weight respectively. Moreover, the Burgess-type exponent $\tilde{\theta} = \frac{3}{16}$ may be deduced from [BH] and the Weyl-type $\bar{\theta} = \frac{1}{6}$ was achieved in [PY] in the holomorphic case. Our exponent also matches that in [BH] in the totally real field case. In comparison, the exponent of Iwaniec and Duke is $\bar{\theta} = \frac{9}{16}$.

Acknowledgements. We are grateful to Jim Cogdell for helpful commentaries and discussions on this work. We thank Gergely Harcos and the referee for bringing to our attention the Burgess and Weyl-type subconvexity bounds and the work of Qiu on the Whittaker period formula.

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