A SHORT PROOF OF THE BUCHSTABER–REES THEOREM

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To the memory of Robin Bullough

Abstract. We give a short proof of the Buchstaber–Rees theorem concerning symmetric powers. The proof is based on the notion of a formal characteristic function of a linear map of algebras.

1. Introduction

This paper is based on a talk given at the Robin K. Bullough Memorial Symposium in June 2009. Robin Bullough had been always interested in new algebraic and geometric structures arising from integrable systems theory (and mathematical physics in general) as well as in philosophical understanding of what ‘integrability’ is. We had had numerous conversations on that with him in the last ten years. The subject of this paper is related with both aspects. We use a method inspired by our studies in supermanifold geometry (namely, Berezinians and associated structures, ultimately rooted in quantum physics) to give a short direct proof of a theorem of Buchstaber and Rees concerning symmetric powers of algebras and spaces. Buchstaber and Rees’s notion of an “$n$-homomorphism”, motivated by the earlier studies of $n$-valued groups, cannot be separated from integrable systems in the broad sense. In recent years an understanding of ‘integrability’ of various objects has spread, according to which an ‘integrable’ case of any notion (a system of ODEs, a function, a manifold, ...) is a case somehow distinguished and discrete within the continuum of ‘generic’ (or ‘non-integrable’) cases. It is often related with some non-trivial algebraic identities. An approach to linear maps of algebras that we have put forward (see below), allows to isolate a hierarchy of ‘good’ classes of such maps, which may be also regarded as ‘integrable’. The notions of algebra homomorphisms and “$n$-homomorphisms” find their natural place in such a hierarchy based on the analysis of a formal characteristic function of a linear map of algebras that we introduced. We also see the next step of this hierarchy (the “$p|q$-homomorphisms”, which we hope to study further elsewhere).

In this paper we give a brief exposition of our general method together with its very concrete application, which is in the title.

A theory of “$n$-homomorphisms” (or “Frobenius $n$-homomorphisms”) of algebras was developed in a series of papers of V. M. Buchstaber and E. G. Rees (see particularly [Buchstaber & Rees 2002], [Buchstaber & Rees 2004], [Buchstaber & Rees 2008] and references therein). We recall the definition of an $n$-homomorphism in section 3 below. This notion originated in the studies of an analog for multi-valued groups of the Hopf algebra of

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functions on a group (namely, for an \( n \)-valued group, the coproduct in such an analog is an \( n \)-homomorphism) and was then identified with a structure discovered by Frobenius in his theory of higher group characters.

The main algebraic result of Buchstaber and Rees is the following fundamental theorem. For commutative associative algebras with unit \( A \) and \( B \) over real or complex numbers (the condition of the commutativity of \( A \) can be relaxed), under some technical assumptions on \( B \), there is a one-to-one correspondence between the algebra homomorphisms \( S^n A \to B \) and the \( n \)-homomorphisms \( A \to B \). Here \( S^n A \subset A^\otimes n \) is the symmetric power of \( A \) as a vector space with the algebra structure induced from \( A^\otimes n \). In particular, when \( A = C(X) \) for a compact Hausdorff space \( X \) and \( B = \mathbb{R} \), this gives the following extension of the classical theorem of [Gel’fand & Kolmogorov 1939]: for any \( n \), the symmetric power \( \text{Sym}^n X = X \times \ldots \times X / S_n \) is canonically embedded into the linear space \( C(X)^n \) so that the image of the embedding is the set of all \( n \)-homomorphisms \( C(X) \to \mathbb{R} \). The proof of this remarkable result in [Buchstaber & Rees 2002] is a tour de force of combinatorial ingenuity.

In [Khudaverdian & Voronov 2007a] we suggested the following construction. For an arbitrary linear map \( f \) of a commutative algebra \( A \) into a commutative algebra \( B \) we introduce a formal ‘characteristic function’ (a formal power series in \( z \))

\[
R_f(a, z) = \exp (f (\ln (1 + az)))
\]

and consider classes of maps \( f \) such that \( R_f(a, z) \) is a genuine function. A formal analysis of the behaviour of \( R_f(a, z) \) “at the infinity” leads to another crucial notion, of a ‘Berezinian’ on the algebra \( A \) associated with a linear map \( f: A \to B \) — shortly, an ‘\( f \)-Berezinian’. The study of the characteristic function and the \( f \)-Berezinian allows to single out certain classes of linear maps ‘with good properties’ among arbitrary linear maps of algebras, as follows.

Suppose the characteristic function \( R_f(a, z) \) is a linear function of the variable \( z \). This precisely characterizes those linear maps \( f: A \to B \) that are the ring homomorphisms.

Suppose the characteristic function \( R_f(a, z) \) is a polynomial function of the variable \( z \). It can be shown that such a condition precisely characterizes the \( n \)-homomorphisms of algebras in the sense of Buchstaber and Rees (for some natural number \( n \)). Our construction gives a different approach to their theory. In particular, by using \( f \)-Berezinian (which is almost tautologically a multiplicative map), we obtain an effortless proof of the Buchstaber and Rees main theorem stated above. See Corollary 1 from Proposition 2 below, which is the crucial point of the proof; the way how it is obtained is the main advantage of our approach. It also leads to other substantial simplifications of the theory.

The next class of ‘good’ linear maps of algebras \( f: A \to B \) corresponds to the case when the characteristic function \( R_f(a, z) \) is rational. We call them the \( p|q \)-homomorphisms. Geometrically they correspond to some interesting generalization of symmetric powers. This is briefly discussed in the last section. See also in paper [Khudaverdian & Voronov 2007b].

Our approach is inspired by the previous work on invariants of supermatrices. In paper [Khudaverdian & Voronov 2005] we investigated the rational function

\[
R_A(z) = \text{Ber}(1 + Az)
\]

where \( A \) is an even operator on a superspace and \( \text{Ber} \) is its Berezinian (superdeterminant). It can be also written as \( R_A(z) = e^\text{str} \ln(1 + Az) \) where \( \text{str} \) stands for supertrace, due to the relation \( \text{Ber} e^X = e^{\text{str}X} \). Comparing the expansions of \( R_A(z) \) at zero and at infinity allowed us to establish non-trivial relations for the exterior powers of operators and spaces (in the
Grothendieck ring) and in particular obtain a new formula for the Berezinian as the ratio of certain polynomial invariants.

2. The formal characteristic function

Let $A$ and $B$ be two associative, commutative and unital algebras over $\mathbb{C}$ or $\mathbb{R}$. We shall study linear maps $f : A \to B$ that are not assumed to be algebra homomorphisms. For a fixed such map $f$, we say that an arbitrary function $\phi : A \to B$ is $f$-polynomial if its values $\phi(a)$, where $a \in A$, are given by a universal (i.e., independent of $a$) polynomial expression in $f(a), f(a^2), \ldots$. The ring of $f$-polynomial functions is naturally graded so that the degree of $f(a)$ is 1, the degree of $f(a^2)$ is 2, etc.

The characteristic function of a linear map of algebras $f$ is defined as the formal power series with coefficients in $B$

$$R_f(a, z) := \exp(f(\ln(1 + az))) = 1 + \psi_1(a)z + \psi_2(a)z^2 + \psi_3(a)z^3 + \ldots$$  \hspace{1cm} (1)

(see our paper [Kudraverdian & Voronov 2007]). It is a “function” of both $z$ and $a \in A$. The coefficients $\psi_k(a)$ of the series (1) are $f$-polynomial functions of $a$ of degree $k$, so that $\psi_k(\lambda a) = \lambda^k \psi_k(a)$. Indeed, by differentiating equation (1) w.r.t. $z$ we can see that $\psi_k(a)$ can be obtained by the Newton type recurrent formulae:

$$\begin{align*}
\psi_1(a) &= f(a), \\
\psi_{k+1}(a) &= \frac{1}{k+1} (f(a)\psi_k(a) - f(a^2)\psi_{k-1}(a) + f(a^3)\psi_{k-2}(a) - \ldots + (-1)^{k+1}f(a^{k+1})) .
\end{align*}$$

With respect to the linear map $f$, the characteristic function enjoys an obvious exponential property:

$$R_{f \cdot g}(a, z) = R_f(a, z)R_g(a, z) .$$

For a given linear map $f$, its characteristic function obeys the relations

$$R_f(a, z)R_f(a', z') = R_f(az + a'z' + aa'zz', 1) ,$$  \hspace{1cm} (2)

or

$$R_f(a, 1)R_f(b, 1) = R_f(c, 1) \quad \text{if} \quad 1 + c = (1 + a)(1 + b)$$  \hspace{1cm} (3)

(making sense as formal power series). This directly follows from the definition. We shall use these relations later.

One can make the following formal transformation of the characteristic function of $f$ aimed at obtaining its ‘expansion near infinity’. More precisely: $R_f(a, z)$ is defined initially as a formal power series in $z$; it can be seen as the Taylor expansion at zero of some genuine function of the real or complex variable $z$ if such a function exists. Assume that it exists and keep the notation $R_f(a, z)$ for it. We have, by a formal transformation, that near the infinity in $z$,

$$R_f(a, z) = e^{f\ln(1+az)} = e^{f\ln(az(1+a^{-1}z-1))} = e^{f\ln(z)+f\ln a+f\ln(1+a^{-1}z-1)} = e^{\ln z}e^{f\ln a}e^{f\ln(1+a^{-1}z-1)} = z^{f(1)} \sum_{k \geq 0} e^{f\ln a} \psi_k(a^{-1}) z^{-k} = e^{f\ln a} z + e^{f\ln a} \psi_1(a^{-1}) z^{x-1} + e^{f\ln a} \psi_2(a^{-1}) z^{x-2} + \ldots$$
where we have denoted \( \chi = f(1) \). Here we assume whatever we may need for the calculation, e.g., that \( a^{-1} \) exists, and so on. Initially \( f(1) \in B \); an assumption that there is a Laurent expansion at infinity forces to conclude that \( \chi \) must be a number in \( \mathbb{Z} \). We also observe that the formal expression \( e^{f \ln z} \) arises as the coefficient of the leading term at infinity.

We are not using this heuristic argument in the next sections; however it may be helpful for understanding our approach. Instead of discussing how this formal calculation can be made rigorous, we shall go around it and apply arguments more specific for a particular case.

3. From characteristic function to \( n \)-homomorphisms

Suppose that the formal power series \( f \) terminates, i.e., \( R_f(a,z) \) is a polynomial function in \( z \) for all \( a \in A \), and that the degree of \( R_f(a,z) \) is bounded by some \( N \in \mathbb{N} \) independent of \( a \). We claim that in this case \( f(1) = n \) where \( n \) is a natural number and that \( R_f(a,z) \) is a polynomial of degree \( n \), i.e., the degree of \( R_f(a,z) \) is at most \( n \) for all \( a \) and is exactly \( n \) for some \( a \) (provided some technical assumption for the target algebra \( B \)).

Indeed, let \( \chi = f(1) \in B \). Consider \( R_f(1,z) = \exp[f(\ln(1+z))] = \exp[\chi \ln(1+z)] \). We show first that the element \( \chi \in B \) is a natural number. We have \( \exp[\chi \ln(1+z)] = (1+z)^\chi \) where \( (1+z)^\chi \) is considered is a formal power series:

\[
(1+z)^\chi = 1 + \chi z + \frac{\chi(\chi-1)}{2} + \cdots = \sum_{k=0}^{\infty} \frac{\chi(\chi-1)\cdots(\chi-k+1)}{k!} z^k.
\]

But \( R_f(a,z) \) is a polynomial of degree at most \( N \). Hence \( \chi(\chi-1)\cdots(\chi-k+1) = 0 \) for all \( k > N \). If in an algebra \( B \) the equation \( b(b-1)(b-2)\cdots(b-k) = 0 \) implies that \( b = j \) for some \( j = 0,1,\ldots,k \), the algebra \( B \) is called ‘connected’ (Buchstaber & Rees 2008). This is satisfied, for example, if \( B \) does not have divisors of zero. Provided such a condition for \( B \) holds, we conclude that \( \chi = n \) for some integer \( n \) between 1 and \( N \).

Now we show that the value \( n = f(1) \in \mathbb{N} \) gives the degree of the polynomial function \( R_f(a,z) \). For an arbitrary \( a \), we apply the above identity \( (1) \) to obtain

\[ R_f(a,z) = R_f(za,1) = R_f(z-1,1)R_f\left(\frac{1}{z} + a - 1,1\right) = z^n R_f\left(\frac{1}{z} + a - 1,1\right) \]  \hspace{1cm} (4)

(note that \( R_f(z-1,1) = e^{f(\ln z-1)} = e^{n \ln z} = z^n \)). More explicitly, the RHS of \( (1) \) has the form

\[ z^n \left[ 1 + f\left(\frac{1}{z} + a - 1\right) + \psi_2\left(\frac{1}{z} + a - 1\right) + \cdots + \psi_N\left(\frac{1}{z} + a - 1\right) \right]. \]  \hspace{1cm} (5)

Since the functions \( \psi_k(a) \) are \( f \)-polynomial of degrees \( k \), all terms in the bracket are inhomogeneous polynomials in \( z^{-1} \) of degrees 0, 1, \ldots, \( N \) respectively. We are given that \( R_f(a,z) \) is a polynomial in \( z \); it follows that \( N \leq n \). By multiplying in \( (5) \) through and comparing with the expansion in \( (1) \), we conclude that the degree of \( R_f(a,z) \) in \( z \) is at most \( n \),

\[ R_f(a,z) = 1 + \psi_1(a)z + \psi_2(a)z^2 + \psi_3(a)z^3 + \cdots + \psi_n(a)z^n, \]

for any \( a \). In particular, for \( a = 1 \), we have \( R_f(1,z) = (1+z)^n \) where the degree is exactly \( n \). This completes the proof of the claim above.

Let us consider a linear map \( f : A \to B \) such that its characteristic function \( R_f(a,z) \) is a polynomial of degree \( n \) in \( z \), i.e., it is at most \( n \) for all \( a \) and it is exactly \( n \) for some \( a \). As we have found, the integer \( n \) is necessarily the value of \( f \) at \( 1 \in A \).
We shall show now that the class of such maps coincides with the class of $n$-homomorphisms introduced by Buchstaber and Rees. Let us recall their definition. For a given linear map $f: A \rightarrow B$, Buchstaber and Rees defined maps

$$\Phi_k: A \times \ldots \times A \rightarrow B,$$

for all $k = 1, 2, \ldots$, by a “Frobenius recursion formula”: $\Phi_1 = f$, and

$$\Phi_{k+1}(a_1, \ldots, a_{k+1}) = f(a_1)\Phi_k(a_2, \ldots, a_{k+1}) - \Phi_k(a_1a_2, \ldots, a_{k+1}) - \ldots - \Phi_k(a_2, \ldots, a_1a_{k+1}).$$

A linear map $f: A \rightarrow B$ is called a (Frobenius) $n$-homomorphism if $f(1) = n$ and $\Phi_k = 0$ for all $k \geq n + 1$ ([Buchstaber & Rees 1997, Buchstaber & Rees 2002]).

**Proposition 1.** The class of the linear maps $f: A \rightarrow B$ such that their characteristic functions $R_f(a, z)$ are polynomials of degree $n$ in $z$ coincides with the class of the Buchstaber–Rees Frobenius $n$-homomorphisms.

**Proof.** From the recursion formula, it is easy to show by induction that the multilinear maps $\Phi_k$ are symmetric. Therefore they are defined by the restrictions to the diagonal. Using induction again, one deduces that the functions $\varphi_k(a) = \Phi_k(a, \ldots, a)$ obey Newton-type recurrence relations similar to those satisfied by our functions $\psi_k(a)$ defined as the coefficients of the expansion (1). From here one can establish that $\Phi_k(a_1, \ldots, a_k)$ can be recovered from $\psi_k(a)$ by polarization (see remark below). Therefore the identical vanishing of the Frobenius maps $\Phi_k$ for all $k \geq n + 1$ is equivalent to the characteristic function $R_f(a, z)$ being a polynomial of degree $\leq n$. Suppose the characteristic function $R_f(a, z)$ is a polynomial of degree $n$, i.e., of degree $\leq n$ for all $a$ exactly $n$ for some $a$. Then $f(1) = \chi$ is an integer between 1 and $n$; if it is less than $n$, then the degree of $R_f(a, z)$ is less than $n$ for all $a$ as shown above. Hence $f(1) = n$. Conversely, if $f(1) = n$, then $R_f(1, z) = (1 + z)^n$ and $R_f(a, z)$ is a polynomial of degree $n$ as claimed. \hfill \Box

Therefore we can identify the $n$-homomorphisms as defined in [Buchstaber & Rees 1997, Buchstaber & Rees 2002] with the linear maps such that their characteristic functions are polynomials of degree $n$.

**Remark 1.** Here is a formula for the polarization of a homogeneous polynomial of degree $k$ (the restriction of a symmetric $k$-linear function to the diagonal):

$$\Phi_k(a_1, a_2, \ldots, a_k) = \sum_{r=1}^{k} \sum_{1 \leq i_1 < i_2 < \ldots < i_r \leq k} (-1)^{k+r} \psi_k(a_{i_1} + a_{i_2} + \cdots + a_{i_r}). \quad (6)$$

Here $\Phi_k(a_1, a_2, \ldots, a_k) = k! \psi_k(a)$. For example,

$$\Phi_2(a_1, a_2) = \psi_2(a_1 + a_2) - \psi_2(a_1) - \psi_2(a_2)$$

and

$$\Phi_3(a_1, a_2, a_3) = \psi_3(a_1 + a_2 + a_3) - \psi_3(a_1 + a_2) - \psi_3(a_1 + a_3) - \psi_3(a_2 + a_3) + \psi_3(a_1) + \psi_3(a_2) + \psi_3(a_3).$$

The relation between homogeneous polynomial functions and symmetric multilinear functions is standard, but explicit formulas are not easy to find in the literature.
Remark 2. It may be noted that the ‘ghost’ of our characteristic function \( R_f(a, z) \) (to which we came motivated by the study of Berezinians in [Khudaverdian & Voronov 2005]) did appear in relation with \( n \)-homomorphisms but was never recognized. We can see these instances in hindsight. The initial definition of an “\( n \)-ring homomorphism” in [Buchstaber & Rees 1996] (with a bit different normalization from that adopted later) used a certain monic polynomial \( p(a, t) = t^n - \ldots \) of degree \( n \). That definition was quickly abandoned starting from [Buchstaber & Rees 1997] in favour of the Frobenius recursion. In hindsight one can relate the Buchstaber and Rees polynomial \( p(a, t) \) with our characteristic function when it is a polynomial of a given degree \( n \), by the formula \( p(a, t) = t^n R_f(a, -\frac{1}{t}) \). In [Buchstaber & Rees 2002, Corollary 2.12], the maps \( \Phi_k(a, \ldots, a) \) for an \( n \)-homomorphism were assembled into a certain generating function, which was then re-written in the exponential form. The resulting power series can be identified with our characteristic function. That series was used only in the proof of their Theorem 2.9 about the sum of \( n \) - and \( m \)-homomorphisms, while the central Theorem 2.8 concerning the relation of the Frobenius \( n \)-homomorphisms with the algebra homomorphisms of the symmetric powers, was obtained in [Buchstaber & Rees 2002] by a long combinatorial argument.

4. Berezinian and a proof of the key statement

Let \( f: A \to B \) be an arbitrary linear map of algebras. We define formally the \( f \)-Berezinian on \( A \) as a map \( \text{ber}_f: A \to B \) by the formula
\[
\text{ber}_f(a) := \exp f(\ln a) = R_f(a - 1, 1)
\]
when it makes sense. (Compare the Liouville formulas for matrices, \( \det X = \exp \text{tr} \ln X \) and \( \text{Ber} X = \exp \text{str} \ln X \), with the ordinary and super determinants, respectively.) Note that the \( f \)-Berezinian \( \text{ber}_f(a) = e^{f \ln a} \) appeared in the heuristic calculation in section 2 above as the leading coefficient of the characteristic function \( R_f(a, z) \) “at infinity”.

**Theorem 1.** The function \( \text{ber}_f \) is multiplicative:
\[
\text{ber}_f(a_1 a_2) = \text{ber}_f(a_1) \text{ber}_f(a_2)
\]
(whenever both sides make sense).

**Proof.** By the definition:
\[
\exp f(\ln(a_1 a_2)) = \exp f(\ln a_1 + \ln a_2) = \exp (f \ln a_1 + f \ln a_2) = \exp f(\ln a_1) \exp f(\ln a_2).
\]

This holds even if the algebra \( B \) is non-commutative, but \( f \) is a ‘trace’, i.e., \( f(a_1 a_2) = f(a_2 a_1) \).

Let \( f \) be an \( n \)-homomorphism. Then \( \text{ber}_f(a) = \exp f(\ln a) \) is an \( f \)-polynomial function of \( a \) with values in \( B \):
\[
\text{ber}_f(a) = \exp f(\ln(1 + (a - 1))) = R_f(a - 1, 1) = 1 + f(a - 1) + \psi_2(a - 1) + \cdots + \psi_n(a - 1),
\]
and is well-defined for all \( a \).

**Proposition 2.** For an \( n \)-homomorphism,
\[
\text{ber}_f(a) = \psi_n(a).
\]
Proof. Consider the equality
\[ 1 + \psi_1(a)z + \psi_2(a)z^2 + \psi_3(a)z^3 + \cdots + \psi_n(a)z^n = z^n \left[ 1 + f \left( \frac{1}{z} + a - 1 \right) + \psi_2 \left( \frac{1}{z} + a - 1 \right) + \cdots + \psi_n \left( \frac{1}{z} + a - 1 \right) \right] \]
(compare formulas (4), (5); we have legitimately set \( N = n \)). Collecting all the terms of degree \( n \) in \( z \), we arrive at the identity
\[ \psi_n(a) = 1 + f(a - 1) + \psi_2(a - 1) + \cdots + \psi_n(a - 1) , \]
where the RHS is by the definition \( \text{ber}_f(a) \).

This proposition is the crucial step in our proof of the Buchstaber–Rees theorem.

Corollary 1. For an \( n \)-homomorphism \( f \), the function \( \psi_n(a) \) is multiplicative in \( a \).

This multiplicativity of the function \( \psi_n(a) \) for \( n \)-homomorphisms is the central fact in the Buchstaber–Rees theorem. Establishing this fact was the main difficulty of the proof in \[\text{Buchstaber \\& Rees 2002}\], where it was deduced by complicated combinatorial arguments. In our approach this fact comes about almost without effort.

Remark 3. The apparatus of characteristic functions allows to obtain easily many other facts. For example, if \( f \) and \( g \) are \( n \)- and \( m \)-homomorphisms \( A \to B \), respectively, then the exponential property of characteristic functions immediately implies that \( f + g \) is an \((n+m)\)-homomorphism, since its characteristic function is the product of polynomials of degrees \( \leq n \) and \( \leq m \). If \( g \) is an \( m \)-homomorphism \( A \to B \) and \( f \) is an \( n \)-homomorphism \( B \to C \), then \( R_{fg}(a,z) = e^{fg \ln(1 + az)} = e^{f \ln R_g(a,z)} = \text{ber}_f R_g(a,z) \). Since we know that \( R_g(a,z) \) is a polynomial in \( z \) of degree at most \( m \), and the \( f \)-Berezinian \( \text{ber}_f b \) is a polynomial in \( b \in B \) of degree \( n \), we conclude that \( R_{fg}(a,z) \) has degree at most \( nm \) in \( z \), therefore \( f \circ g \) is an \( nm \)-homomorphism. (The first statement was established in \[\text{Buchstaber \\& Rees 2002}\] by a similar argument, see Remark 2, while the second statement was obtained in paper \[\text{Buchstaber \\& Rees 2008}\] in a much harder way as a corollary of the main theorem.)

5. A completion of the proof

Let us formulate the main theorem of Buchstaber and Rees.

Theorem 2 (\[\text{Buchstaber \\& Rees 2002}\]). There is a one-to-one correspondence between the \( n \)-homomorphisms \( A \to B \) and the algebra homomorphisms \( S^n A \to B \). The algebra homomorphism \( F : S^n A \to B \) corresponding to an \( n \)-homomorphism \( f : A \to B \) is given by the formula
\[ F(a_1, \ldots, a_n) = \frac{1}{n!} \Phi_n(a_1, \ldots, a_n) , \]
where in the left-hand side a linear map from \( S^n A \) is written as a symmetric multilinear function.

Here \( \Phi_n(a_1, \ldots, a_n) \) is the top non-vanishing term of the Frobenius recursion for \( f \).

Basing on the key result established in the previous section (Corollary 1), we can complete the proof of this theorem as follows.
Let $\text{Alg}^n(A, B)$ be the set of all $n$-homomorphisms from an algebra $A$ to an algebra $B$. We shall construct two mutually inverse maps between the spaces $\text{Alg}^n(A, B)$ and $\text{Alg}^1(S^n A, B)$, thus establishing their one-to-one correspondence:

$$\text{Alg}^1(S^n A, B) \xrightarrow{\alpha} \text{Alg}^n(A, B).$$

To every algebra homomorphism $F \in \text{Alg}^1(S^n A, B)$ we shall assign an $n$-homomorphism $\alpha(F) \in \text{Alg}^n(A, B)$, and to every $n$-homomorphism $f \in \text{Alg}^n(A, B)$ we shall assign an algebra homomorphism $\beta(f) \in \text{Alg}^1(S^n A, B)$, in the following way.

It is convenient to introduce an $n \times n$ matrix $L(a)$ with entries in the algebra $A^{\otimes n} = A \otimes \ldots \otimes A$, where

$$L(a) = \text{diag} \left[ a \otimes 1 \otimes \cdots \otimes 1, 1 \otimes a \otimes 1 \otimes \cdots \otimes 1, \ldots, 1 \otimes 1 \otimes \cdots \otimes 1 \otimes a \right].$$

The map $a \mapsto L(a)$ is a matrix representation $A \to \text{Mat}(n, A^{\otimes n})$. Consider an equation:

$$F(\text{tr} \ln(1 + L(a)z)) = R_f(a, z). \tag{8}$$

The coefficients of the determinant in the left-hand side take values $a$ priori in the algebra $A^{\otimes n}$, but they actually belong to the subalgebra $S^n A$. Let (8) hold identically in $z$. We shall show that, for a given $F$, equation (8) uniquely defines $f$ so that we can set $\alpha(F) := f$, and conversely, for a given $f$, equation (8) uniquely defines $F$ so that we can set $\beta(f) := F$. (Then the maps $\alpha$ and $\beta$ will automatically be mutually inverse.)

Suppose $F \in \text{Alg}^1(S^n A, B)$ is given. Then, by comparing the linear terms in (8), we see that $f$ should be given by the formula

$$f(a) = F(\text{tr} L(a)). \tag{9}$$

We have $\text{tr} L(a) = a \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes 1 \otimes \cdots \otimes a \in S^n A$.

**Remark 4.** The element $a \otimes 1 \otimes \cdots \otimes 1 + \ldots + 1 \otimes 1 \otimes \cdots \otimes a$ appears in Buchstaber & Rees (2008) where it is denoted $\Delta(a)$.

Take (9) as the definition of $f$. Evidently, it is a linear map $A \to B$. Calculate its characteristic function. We have

$$R_f(a, z) = e^{f \ln(1+az)} = e^{F(\text{tr} L(\ln(1+az)))} = e^{F(\text{tr} \ln(1+L(a)z))} = F(e^{\text{tr} \ln(1+L(a)z)}) = F(\text{det}(1+L(a)z))$$

(note that both $L$ and $F$ can be swapped with functions, as we did in the calculation above, because they are algebra homomorphisms). So equation (8) is indeed satisfied. In particular, the characteristic function of $f$ is a polynomial of degree $n$. Hence $f$ is an $n$-homomorphism $A \to B$. We have constructed the desired map $\alpha: \text{Alg}^1(S^n A, B) \to \text{Alg}^n(A, B)$.

Suppose now $f \in \text{Alg}^n(A, B)$ is given. We wish to define $F$ from (8). We need to show the existence of a linear map $F: S^n A \to B$ and that it is an algebra homomorphism. Assume that a linear map $F$ satisfying (8) exists. We shall deduce its uniqueness. By developing the left-hand side of (8), we see that this equation specifies $F$ on all the elements of $S^n A$ of the form $\text{tr} \wedge^k L(a) \in S^n A$, including $\det L(a)$. In particular, we should have

$$F(\det L(a)) = \text{ber}_f(a). \tag{10}$$
Take $[10]$ as the definition of $F$ on such elements. Note that the elements of the form $\det L(a) = a \otimes \ldots \otimes a \in S^n A$ linearly span $S^n A$, so if a linear map $F: S^n A \to B$ satisfying $[10]$ exists, this formula determines it uniquely. Moreover, by replacing $a$ by $1 + az$ in $[10]$, we see that equation $[30]$ will be automatically satisfied. For the existence, observe that $\text{ber}_f(a) = \psi_n(a)$ and $\psi_n$ is the restriction on the diagonal of the symmetric multilinear map $\frac{1}{n!} \Phi: A \times \ldots \times A \to B$, which corresponds to a linear map $S^n A \to B$. It is the map $F$ we are looking for. (We recover the Buchstaber–Rees formula $[17]$.) It remains to check that so obtained $F$ is indeed an algebra homomorphism. This immediately follows from Corollary $[11]$ which tells that $F$ is multiplicative on the elements $a \otimes \ldots \otimes a$, where it is $\psi_n(a)$, because the elements $a \otimes \ldots \otimes a$ span the whole algebra $S^n A$. So we have arrived at the desired map $\beta: \text{Alg}^p(A,B) \to \text{Alg}^q(S^n A,B)$, and the maps $\alpha$ and $\beta$ are mutually inverse by the construction.

This concludes the proof.

6. Rational characteristic functions and $p|q$-homo- morphisms

Our notion of characteristic function readily suggests the next step in the investigation of linear maps of algebras with good properties (after the $n$-homo- morphisms). We shall consider it now briefly.

Suppose the characteristic function $R_f(a, z)$ of a linear map $f: A \to B$ is not a polynomial in $z$, but a rational function that can be written as the ratio of polynomials of degrees $p$ and $q$. We call such a linear map a $p|q$-homo- morphism. One can deduce that for a $p|q$-homo- morphism the value $f(1)$ must be an integer: $\chi = f(1) = p - q$.

Examples. The negative $-f$ of a ring homomorphism $f$ is a $0|1$-homo- morphism. The difference $f_{(p)} - f_{(q)}$ of a $p$-homo- morphism $f_{(p)}$ and a $q$-homo- morphism $f_{(q)}$ is a $p|q$-homo- morphism. In particular, a linear combination of algebra homomorphisms of the form $\sum n_i f_i$, where $n_i \in \mathbb{Z}$, is a $p|q$-homo- morphism with $\chi = \sum n_i$, $p = \sum_{n_i > 0} n_i$, and $q = -\sum_{n_i < 0} n_i$. (This follows from the exponential property of the characteristic function.)

The geometric meaning of $p|q$-homo- morphisms is related with a certain generalization of the notion of symmetric powers.

Consider a topological space $X$. We define its $p|q$-th symmetric power $\text{Sym}^{p|q}(X)$ as the identification space of $X^{p+q} = X^p \times X^q$ with respect to the action of the group $S_p \times S_q$ together with the relations

$$(x_1, \ldots, x_{p-1}, y, x_{p+1}, \ldots, x_{p+q-1}, y) \sim (x_1, \ldots, x_{p-1}, z, x_{p+1}, \ldots, x_{p+q-1}, z).$$

The algebraic analog of $\text{Sym}^{p|q}(X)$ is the $p|q$-th symmetric power of a commutative associative algebra with unit $A$, which we denote $S^{p|q} A$. We define the algebra $S^{p|q} A$ as the subalgebra $\mu^{-1} (S^{p-1} A \otimes S^{q-1} A)$ in $S^p A \otimes S^q A$, where $\mu: S^p A \otimes S^q A \to S^{p-1} A \otimes S^{q-1} A \otimes A$ is the multiplication of the last arguments.

Example. For the algebra of polynomials in one variable $A = \mathbb{C}[x]$, it can be shown that the algebra $S^{p|q} A$ is the algebra of all polynomial invariants of $p|q$ by $p|q$ matrices. In more detail, consider even matrices $p|q$ by $p|q$ where the entries are regarded as indeterminates of appropriate parity (over complex numbers). The general linear supergroup $GL(p|q)$ acts on such matrices by conjugation. ‘Polynomial invariants’ of $p|q$ by $p|q$ matrices are the
polynomial functions of the matrix entries invariant under such action. Denote their algebra
by \( I_{pq} \). The problem is to describe the algebra \( I_{pq} \) in terms of functions of the \( p + q \)
eigenvalues \( \lambda_i, \mu_a \) similar to the classical case \( q = 0 \). The restriction of a polynomial invariant
of \( pq \) by \( pq \) matrices to the diagonal matrices is clearly a \( S_p \times S_q \)-invariant polynomial (or an
element of \( S_p^A \otimes S_q^A \) where \( A = \mathbb{C}[x] \)). Such a polynomial \( f(\lambda_1, \ldots, \lambda_p, \mu_1, \ldots, \mu_q) \)
separately symmetric in \( \lambda_i \) and \( \mu_a \) extends to an element of \( I_{pq} \) if and only if it satisfies
an extra condition which is equivalent to \( f \in S^{pq}_A \). (Here there is a great difference with
the rational invariants, for which no extra condition arises.) This is a non-trivial theorem
that can be traced to \cite{Berezin1977} (see \cite{Berezin1987}, p. 294; see also \cite{Sergeev1982}. See
discussion in \cite{KhudaverdianVoronov2005}.

(The algebra described in the example above and its deformations have recently become
important in integrable systems, see, e.g., \cite{SergeevVeselov2004}.)

There is a relation between the algebra homomorphisms \( S^{pq}_A \rightarrow B \) and \( pq \)-homomorphisms
\( A \rightarrow B \). To each homomorphism \( S^{pq}_A \rightarrow B \) canonically corresponds a \( pq \)-homomorphism
\( A \rightarrow B \).

**Example.** For the algebra of functions on a topological space \( X \), the element \( x = [x_1, \ldots, x_{p+q}] \in
\operatorname{Sym}^{pq}(X) \) defines a \( pq \)-homomorphism \( \operatorname{ev}_x : C(X) \rightarrow \mathbb{R} \) by the formula
\[
a \mapsto a(x_1) + \ldots + a(x_p) - \ldots - a(x_{p+q}).
\]
This gives a natural map from \( \operatorname{Sym}^{pq}(X) \) to the dual space \( A^* \) of the algebra \( A = C(X) \),
which generalizes the Gelfand–Kolmogorov map \( X \rightarrow A^* \) and the Buchstaber–Rees map
\( \operatorname{Sym}^n(X) \rightarrow A^* \).

By using formulas from \cite{KhudaverdianVoronov2005}, the condition that \( f : A \rightarrow B \) is
a \( pq \)-homomorphism can be expressed by the algebraic equations
\[
\begin{align*}
f(1) = p - q \quad \text{and} \quad & \begin{bmatrix}
\psi_k(f,a) & \ldots & \psi_{k+q}(f,a) \\
\vdots & \ddots & \vdots \\
\psi_{k+q}(f,a) & \ldots & \psi_{k+2q}(f,a)
\end{bmatrix} = 0 
\end{align*}
\tag{11}
\]
for all \( k \geq p - q + 1 \) and all \( a \in A \), where \( \psi_k(f,a) = \psi_k(a) \) are the coefficients in the expansion
of the characteristic function \( f \). The determinants arising in (11) are the well-known Hankel
determinants. The condition that they identically vanish replaces the condition \( \psi_k(a) = 0 \)
for all \( k \geq n + 1 \) for an \( n \)-homomorphism.

We see that the image of \( \operatorname{Sym}^{pq}(X) \) in \( A^* \), where \( A = C(X) \), under the map \( x \mapsto f = \operatorname{ev}_x \),
satisfies equations (11). The system (11) can be regarded as an infinite system of polynomial
equations for the ‘coordinates’ \( f(a) \) of a point \( f \) in the infinite-dimensional vector space \( A^* \).

A conjectured statement is that the solutions of equations (11) give precisely the image
of \( \operatorname{Sym}^{pq}(X) \) in \( A^* \). This would be an exact analog of the Gelfand–Kolmogorov and
Buchstaber–Rees theorems. The corresponding algebraic statement should be a one-to-one
 correspondence between the \( pq \)-homomorphisms \( A \rightarrow B \) and the algebra homomorphisms
\( S^{pq}_A \rightarrow B \) extending the correspondence given by our formula (8).

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