Efficient Algorithms for Maximum Link Scheduling in Distributed Computing Models with SINR Constraints

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Abstract—A fundamental problem in wireless networks is the maximum link scheduling (or maximum independent set) problem: given a set $L$ of links, compute the largest possible subset $L' \subseteq L$ of links that can be scheduled simultaneously without interference. This problem is particularly challenging in the physical interference model based on SINR constraints (referred to as the SINR model), which has gained a lot of interest in recent years. Constant factor approximation algorithms have been developed for this problem, but low complexity distributed algorithms that give the same approximation guarantee in the SINR model are not known. Distributed algorithms are especially challenging in this model, because of its non-locality.

In this paper, we develop a set of fast distributed algorithms in the SINR model, providing constant approximation for the maximum link scheduling problem under uniform power assignment. We find that different aspects of available technology, such as full/half-duplex communication, and non-adaptive/adaptive power control, have a significant impact on the performance of the algorithms; these issues have not been explored in the context of distributed algorithms in the SINR model before. Our algorithms’ running time is $O(g(L) \log^c m)$, where $c = 1, 2, 3$ for different problem instances, and $g(L)$ is the “link diversity” determined by the logarithmic scale of a communication link length. Since $g(L)$ is small and remains in a constant range in most cases, our algorithms serve as the first set of “sublinear” time distributed solution. The algorithms are randomized and crucially use physical carrier sensing in distributed communication steps.

I. INTRODUCTION

One of the most basic problems in wireless networks is the Maximum Link Scheduling problem (MAXLSP): given a set $L$ of links, compute the largest possible subset $L' \subseteq L$ of links that can be scheduled simultaneously without conflicts; this is also referred to as the one-shot scheduling [1] or max independent link set problem [2]. One of the main challenges for this problem is wireless interference, which limits the subsets of links that can transmit simultaneously. A commonly used model is based on “conflict graphs” [3], broadly referred to as graph-based interference models; examples of such models include: the unit disk graph model, the $k$-hop interference model, and the protocol model. MAXLSP is challenging in these models — the decision version of this problem is NP-Complete, but efficient constant factor approximation algorithms are known for many interference models [3], because of their inherent locality. However, graph based models are known to be inaccurate and an oversimplification of wireless interference. In recent years, a more realistic interference model based on SINR constraints (henceforth referred to as the SINR model) [1], [4] has gained a lot of interest: a set of links are feasible simultaneously if the signal to interference plus noise constraints are satisfied at all receivers (see Section II for the formal definition). This is much harder than graph based interference models because of the inherently non-local and non-linear nature of the model; only recently constant factor approximation algorithms have been developed in this model [2], [3], [6].

Since link scheduling is a common subroutine in many other problems, distributed algorithms with low complexity are crucial. A commonly studied model for distributed computing in wireless networks is the “Radio Broadcast Network (RBN)” model, in which the transmissions on two links conflict if the links interfere (in the corresponding graph-based model); variations have been studied of this model, depending on capabilities such as collision detection. Efficient distributed algorithms are known in the RBN model for MAXLSP, as well as other fundamental problems such as coloring and dominating set, e.g., [2]. A solution to MAXLSP computed in the RBN model might not be feasible in the SINR constraints (see, e.g., [1], [10]). Further, the distributed wireless communication mechanism can be quite different. In other words, a distributed algorithm in the RBN model cannot be implemented in general in the SINR-based model. Therefore, we need to rethink the design of distributed algorithms in the SINR model in a fundamentally new way.

In this paper, we focus on distributed algorithms for MAXLSP in the SINR model, which is defined in the following manner: at each time step of the algorithm, only those links for which the SINR constraints are satisfied at the receivers are successful. The goal of the algorithm is to end up with a feasible solution to MAXLSP, whose size is maximized. We have to rethink distributed algorithms in the SINR model for MAXLSP because of the fundamental differences between the graph-based and the SINR interference models. As mentioned earlier, even centralized algorithms for this
problem are much harder in the SINR model, than in the disk based interference model; recent work by [2, 5, 6, 11, 12] gives constant factor approximation algorithms for various instances of MAXLSP in the SINR model. The centralized algorithms of [2, 5, 6] are based on a greedy ordering of the links, which requires estimating the “affection,” (which, informally, is a measure of interference), at each stage (this is discussed formally later in Section III) — this is one of the challenges in distributed solutions to MAXLSP. We note that efficient time distributed algorithms for scheduling all the links (i.e., the coloring version) is already known [13]. Adapting them would immediately yield a distributed \( O(\log m) \)-approximation to MAXLSP, but it is not clear how to obtain a distributed \( O(1) \)-approximation. Further, an important aspect of MAXLSP is that the senders and receivers of all the links should know whether they have been chosen, since this is an important requirement in many networking applications; this seems to be difficult to ensure through random access based approaches.

A. Contributions

In this paper, we develop fast distributed constant factor approximation algorithms for MAXLSP, in which all nodes are constrained to use uniform power levels for transmission (we refer to this as MAXLSP\(^U\)), improving upon the results implied by [14, 15]. Our algorithms and the proofs build on ideas from [2, 5, 6] and [10], and one of the key technical contributions of our work is the notion of an “\( \omega_1, \omega_2 \)-ruling” (discussed below) and its distributed computation in the SINR model. Our results raise two new issues in the context of distributed algorithms in the SINR model — adaptive power control (i.e., the feature of using lower than the maximum power level, as needed), and full/half duplex communication (i.e., whether nodes can transmit and receive simultaneously). We find these features impact the performance of the algorithms quite a bit. We summarize some of the key aspects of the results and main challenges below.

(1) Performance and technology tradeoffs. In the case of "non-adaptive power control" (i.e., if all nodes are required to use fixed uniform power levels), we design a distributed algorithm that provably runs in \( O(g(L) \log^3 m) \) time and gives an \( O(1) \) approximation to the optimum solution for half duplex communication, and we improve the running time to \( O(g(L) \log^2 m) \) for the case of full duplex communication; here \( g(L) \) denotes the “link diversity”, which is the logarithm of the ratio of the largest to the smallest link length (this is defined formally in Section III). If nodes are capable of “adaptive power control” (i.e., they can vary power levels for scheduling, but not data transmission), we improve the running time of the above algorithm to \( O(g(L) \log^3 m) \) time for half duplex communication, and \( O(g(L) \log m) \) time for full duplex communication. Note that in the adaptive power control case, the algorithm uses varying power levels during its run, but the links which are selected finally use the fixed uniform power level for data transmission.

2) Key distributed subroutine. One of our key ideas is the parallelization of the link selection, which would have require sorting all links, processing a larger set of links simultaneously, and efficient filtering based on spatial and interference constraints in parallel. Moreover, it turns out that the usual notion of independence based on spatially separated nodes is inadequate because of the spatial separation of the sender and the receiver of a link: it is the senders which makes the distributed decision of transmission and the participation in the independent set, while the SINR model is receiver-oriented and it is hard for each sender of a candidate link to determine the interference caused by the chosen links at the corresponding receiver. One of the important steps of our algorithm involves the distributed construction of a “ruling” (a spatially-separated node cover, first introduced in [17]) which relates to the notion of independence and aids the solution to MIS and coloring problems in graph topologies [13, 19]. The extension of the notion of ruling and its computation in the SINR model is one of the important technical contributions of our paper. We believe this basic construct would be useful in other link and topology control problems.

(3) Sensing-based message-less distributed computing. We make crucial use of physical carrier sensing, and in solving MAXLSP\(^U\) we let the wireless nodes make distributed decisions purely based on the Received Signal Strength Indication (RSSI) measurement without the need of exchanging or decoding any messages. Given a threshold \( \text{Thres} \), a node is able to detect if the total sensed power strength is \( \geq \text{Thres} \). As discussed in [16, 17], this can be done using the RSSI measurement possible through the Clear Channel Assessment capability in the 802.11 standard. In this way, the protocol is much simplified such that the wireless nodes only need to control the physical layer to access the medium with a certain power or to sense the channel. Further, our algorithm uses constant size messages, and all the steps can be implemented within the model without additional capabilities or assumptions (as those made in [14]).

B. Key Challenges and Comparisons between Models

Comparison between interference models It is known that solutions to the link scheduling problems developed under the graph-based models can be inefficient, if not infeasible, under the SINR model. For instance, Le et al. [20] show that the longest-queue-first scheme may result in zero throughput under SINR constraints (unlike that in the graph based model) for the case of dynamic traffic. As for MAXLSP, it is easy to show that when all the transmitters have uniform transmission/interference ranges, an optimum solution developed under a graph-based model may turn out to be a solution whose size is a fraction of \( O((\frac{\text{d}}{\text{d}_{\text{tx}}})^2) \) of that of an optimum under the SINR model,
where $d_{\text{max}}$ is the length of the longest link and $d_{\text{tx}}$ is the uniform transmission range. This is because that given a set of links under the SINR model, as long as all the senders are separated by $cd_{\text{max}}$, where $c$ is some constant, all the links form an independent set. Since we are dealing with an arbitrary topology, $d_{\text{max}}$ may be small, leading to a much more conservative solution under the SINR model.

Comparison between distributed computing models In light of the huge amount of research on distributed algorithms in the RBN model for many problems, including MAXLSP (e.g., \cite{9, 21}), it is natural to ask if it might be possible to “reduce” the SINR model problem to the RBN model instead of developing new algorithmic techniques. Though it has not been rigorously proven, results from recent papers suggest this might not be feasible, or might only yield larger than constant factor gaps. For instance, Chafekar et al. \cite{10} discuss an instance where the solution in the “equivalent” RBN model could be significantly smaller than that in the SINR model; see also \cite{22}. Further, the RBN model does not allow for capabilities to determine the signal strength and make decisions based on that.

C. Organization

We discuss the network model and relevant definitions in Section I. We present the high-level distributed algorithm in Algorithm 1 with a constant approximation ratio in Section IV. We introduce and analyze the distributed algorithm to compute a ruling in Section V. In Section VI we show the detailed implementation for each step of the high-level Algorithm 1; we present a second method to implement Algorithm 1 in Section VII improving the running time by a logarithmic factor.

II. Related Work

There has been a lot of research on link scheduling and various related problems, because of their fundamental nature. Two broad versions of these problems are: scheduling the largest possible set of links from a given set (maximum independent set), and constructing the smallest schedule for all the links (minimum length schedule). These problems are well understood in graph based interference models and efficient approximation algorithms are known for many versions; see, e.g., \cite{3}. Distributed algorithms are also known for node and link scheduling (and many related problems) in the radio broadcast model \cite{7, 8, 9, 23, 24}. These algorithms are typically randomized and based on Luby’s algorithm \cite{23}, and run in synchronous polynomial time. There are varying assumptions on the kind of information and resources needed by individual nodes. For instance, \cite{24} require node degrees at each step (which might vary, as nodes become inactive). Moscibroda et al. \cite{27} develop algorithms that do not require the degree information, and run in $O(\log^2 n)$ time. In recent work, Afek et al. \cite{9} develop a distributed algorithm for the maximal independent set problem, which only requires the an estimate of the total number of nodes, but not degrees.

Link scheduling in the SINR model is considerably harder than in graph based models. Several papers developed $O(g(L))$-approximations for MAXLSP\(^U\), e.g., \cite{1, 4}, which have been improved to constant factor approximations by \cite{2, 5, 6} for uniform power assignments. Some of these papers use “capacity” \cite{5, 6} to refer to the maximum link scheduling; however, we prefer to avoid the term capacity in order to avoid confusion with the total throughput in a network, which has been traditionally referred to as the capacity (e.g., \cite{25}). Recently, Halldórsson and Mitra \cite{12} extend the $O(1)$ approx. ratio to a wide range of oblivious power assignments for both uni- and bi-directional links (including uniform, mean and linear power assignments). This has been improved by Kesselheim \cite{11}, who developed the first $O(1)$-algorithm for MAXLSP with power control and an thus an $O(\log m)$-algorithm for the minimum length schedule problem. Most of the results except those using uniform power assignments, assume unlimited power values; otherwise the results may degrade by a factor depending on the ratio of the maximum and minimum transmission power values.

Most of the above algorithms for scheduling in the SINR model are centralized and it is not clear how to implement them in a distributed manner efficiently. The closest results to ours are by Åseгрisson and Mitra \cite{14} and Dinitz \cite{15}, using game theoretic approaches; the former obtains a constant approx. ratio improving over the latter’s $O(\log m)$ approximation for MAXLSP\(^U\). Their running time can be much higher than ours, and they require additional assumptions (such as acknowledgements without any cost), which might be difficult to realize in the SINR model.

For the minimum length schedule problem (MINLSP) (where one seeks a shortest schedule to have all the links in $L$ transmit successfully) under a length-monotone sub-linear power assignment, Fanghanel et al. \cite{26} develop a distributed algorithm with an approximate ratio of $O(g(L))$ times a logarithmic factor. Recently, Kesselheim and Vöcking \cite{27} propose an $O(\log^2 m)$-approximate algorithm for any fixed length-monotone and sub-linear power assignment. The approx. ratio of that algorithm has been improved to $O(\log m)$ (matching the best performance of known centralized algorithms) by the analysis of Halldórsson and Mitra \cite{13}, who also prove that if all links uses the same randomized strategy, there exists a lower-bound of $\Omega(\log m)$ on the approx. ratio. However, it is not clear how to use these results for MINLSP to get a constant factor approximation for MAXLSP, in which the senders and receivers of all links know their status.

III. Preliminaries and Definitions

We let $V$ denote a set of tranceivers (henceforth, referred to as nodes) in the Euclidean plane. We assume $L$ is a
set of links with end-points in $V$, which form the set of communication requests for the maximum link scheduling problem at any given time, and $|L| = m$. Links are directed, and for link $l = (x(l), r(l))$, $x(l)$ and $r(l)$ denote the transmitter (or sender) and receiver respectively. For a link set $L'$, let $X(L')$ denote the set of senders of links in $L'$. Let $d(u, v)$ denote the Euclidean distance between nodes $u, v$. For link $l$, let $d(l) = d(x(l), r(l))$ denote its link length. For links $l, l'$, let $d(l', l) = d(x(l'), r(l))$. Let $d_{\min}$ and $d_{\max}$ denote the smallest and the largest transmission link lengths respectively. Let $B(v, d)$ denote the ball centered at node $v$ with a radius of $d$. Each transmitter $x(l)$ uses power $P(l)$ for transmission on link $l$; we assume commonly used path loss models [1], [4], in which the transmission on link $l$ is possible only if:

$$\frac{P(l)}{d^\alpha(l)} \geq \beta,$$

where $\alpha > 2$ is the “path-loss exponent”, $\beta > 1$ is the minimum SINR required for successful reception, $N$ is the background noise, and $\phi > 0$ is a constant (note that $\alpha, \beta, \phi$ and $N$ are all constants).

We partition the set of transmission links into non-overlapping link classes. We define link diversity $g(L) = \log_2 \frac{d_{\max}}{d_{\min}}$. Partition $L = \{L_i\}$, $i = 1, 2, \ldots, g(L)$, where each $L_i = \{l \mid 2^{i-1}d_{\min} \leq d(l) < 2^id_{\min}\}$ is the set of links of roughly similar lengths. Let $d_i = 2^i d_{\min}$, such that $d_i$ is an upperbound of link length of $L_i$; and $\forall i, \forall l \in L_i$, we define $d(l) = d_i$. In a distributed environment, nodes use their shared estimates of minimum and maximum possible link length to replace $d_{\min}$ and $d_{\max}$, as stated in the previous section. $g(L)$ in most cases $\leq 6 \log 10$ and remains a constant further, as discussed earlier, each link can compute which link class it belongs to. The reverse link of a link $l$, denoted by $l'$, is the same link with transmission direction inverted. For a link set $L$, We use $L'$ to denote the set of reverse links of $L'$.

**Wireless Interference.** We use physical interference model based on geometric SINR constraints (henceforth referred to as the SINR model), where a subset $L' \subseteq L$ of links can make successful transmission simultaneously if and only if the following condition holds for each link $l \in L'$:

$$\sum_{l' \in L' \setminus \{l\}} \frac{P(l')}{d^{\alpha}(l') \beta N} + N \geq \beta.$$  

Such a set $L'$ is said to be independent in the context.

**The Maximum Link Scheduling Problem (MAXLSP).**

Given a set of communication requests (links) $L$, the goal of the MAXLSP problem is to find a maximum independent subset of links that can be scheduled simultaneously in the SINR model. MAXLSP U is an instance of MAXLSP where links in a solution use a uniform power level for data transmission; note that this does not necessarily restrict scheduling to uniform power. In this paper, we use $OPT(L)$ to denote an optimal solution to the MAXLSP U, and thus $|OPT(L)|$ is the cardinality of the largest such independent set. As discussed earlier, computing $OPT(L)$ is NP-hard, and we focus on approximation algorithms. We say an algorithm gives a $C$-approximation factor if it constructs an independent link set $L' \subseteq L$ with $|L'| \geq |OPT(L)|/C$.

**Distributed Computing Model in the SINR-based Model.**

Traditionally, distributed algorithms for wireless networks have been studied in the radio broadcast model [7], [8], [29] and its variants. The SINR based computing model is relatively recent, and has not been studied that extensively. Therefore, we summarize the main aspects and assumptions underlying this model: (1) The network is synchronized and for simplicity we assume all slots have the same length. (2) All nodes have a common estimate of $m$, the number of links, within a polynomial factor; (3) For each link $l \in L$, $x(l)$ and $r(l)$ have an estimate of $d(l)$, but they do not need to know the coordinates or the direction in which the link is oriented; (4) All nodes share a common estimate of $d_{\min}$ and $d_{\max}$, the minimum and maximum possible link lengths; (5) We assume nodes have physical carrier sensing capability and can detect if the sensed signal exceeds a threshold. As discussed in [16], this can be done using the RSSI measurement possible through the Clear Channel Assessment capability in the 802.11 standard. Given a threshold $Thres$, we assume that a node is able to detect if the sensed power strength is $\geq Thres$.

**Sensed Power-strength and Affectance.** For ease of analysis based on links, we define affectance $[6]$ as that in [6], [12]: the affectance, caused by the sender of link $l'$ to the receiver of link $l$, is $A(l', l) = \frac{\beta}{1 - d^{\alpha}(l') \beta N}.$  

* The minimum link length is constrained by the device dimension, empirically at least 0.1 meter; the maximum link length depends on the type of the network, and is usually bounded by $10^5$ meters. For example, the Wi-Fi transmission range is below hundred meters and even long-distance Wi-Fi networks [28] have an experimental limit of hundred kilometers; in cellular networks the coverage is at most tens of kilometers; the transmission range in Bluetooth or 60GHz networks is smaller. This implies often $g(L) \leq \log 10^6$.

$\dagger$ Sometimes it shares the same definition with the term relative interference, e.g., in [2]; however, “relative interference” may refer to other forms, e.g., in [6].

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**Table I: Notation.**

| $G$  | network graph | $d(u, v)$ | dist. of $u$ and $v$ |
|------|---------------|-----------|---------------------|
| $V$  | set of nodes  | $L$       | set of links        |
| $n$  | #nodes        | $g(L)$    | link diversity      |
| $m$  | #links        | $OPT(L)$  | optimum instance    |
| $\alpha$ | path-loss exponent | $x(l)$ | sender of link $l$ |
| $\beta$ | SINR threshold | $r(l)$ | receiver of link $l$ |
| $N$  | background noise | $d(l)$ | length of link $l$ |
| $A$  | affectance    | $SP$      | sensed power        |
Likewise, we have affectance from a set \( L' \) of links, as \( A(L',l) = \sum_{l' \in L'} A(l',l) \). It can be verified that Inequality (3) is equivalent to \( A(L' \setminus \{ l \},l) \leq 1 \), signifying the success of data transmission on \( l \).

To simplify the analysis based on nodes, we define *sensed power-strength* \( SP(w,v) \), as the signal power that node \( v \) receives when only \( w \) is transmitting (which includes background noise); that is, \( SP(w,v) = P/d^a(w,v) + N \). Likewise, we have \( SP \) from a node set \( W \): \( SP(W,v) = \sum_{w \in W} P/d^a(w,v) + N \). Let \( Thres(d) = P/d^a + N \) be a function of distance \( d \), such that for any node \( v \), if any other node is transmitting in a range of \( d \), its sensed power will exceed \( Thres(d) \).

**Node Capabilities for Distributed Scheduling** (1) *Half/full Duplex Communication*: Wireless radios are generally considered *half duplex*, i.e., with a single radio they can either transmit or receive/sense but not both at the same time. *Full duplex* radios, which are becoming reality, enable wireless radios to perform transmission and reception/sensing simultaneously. (2) *Non-adaptive/adaptive Power*: Although links in a solution to MAXLSP employ a uniform power level for data transmission, they are usually capable of using adaptive power which vary across different power levels that may be *used for scheduling*. The capabilities can play a vital role in distributed computation.

### IV. Distributed Algorithm: Overview

In this section, we present the distributed algorithm for MAXLSP. Because the algorithm is quite complicated, we briefly summarize the sequential algorithm of [2], [5], [6] below, and then give a high-level description of the distributed algorithm and its analysis, without the implementation details of the individual steps in the SINR model. Section V describes the algorithm for computing a ruling in the full and half duplex models. The complete distributed implementation and other details are discussed in Sections VI and VII for the non-adaptive and adaptive power control settings, respectively.

#### A. The Centralized Algorithm

We discuss the centralized algorithm adapted from [2], [5], [6] for MAXLSP, which forms the basis for our distributed algorithm. The algorithm processes links in non-decreasing order of length. Let \( L \) be the initial set of links, and \( S \) the set of links already chosen (which is empty initially). Each iteration involves the following steps:

1. (1) picking the shortest link \( l \) in \( L \setminus S \) and removing \( l \) from \( L \),
2. (2) removing from \( L \) all the links in \( \{ l' \in L \setminus S : A(S,l') \geq c_0 \} \) where \( c_0 < 1 \) is a constant, i.e., all the links in \( L \setminus S \) that suffer from high interference caused by all chosen links in \( S \), and
3. (3) removing from \( L \) all the links in \( \{ l' \in L \setminus S : d(l',l) = c_1 d(l) \} \) where \( c_1 \) is a constant, i.e., all the nearby links of \( l \) in \( L \setminus S \).

The results of [2], [5], [6] show that: \( S \) is feasible (i.e., the SINR constraints are satisfied at every link), and \( |S| \) is within a constant factor of the optimum. Consider a link \( l \) that is added to \( S \) in iteration \( i \). The proof of feasibility of set \( S \) involves showing that for this link \( l \), the affectance due to the links added to \( S \) after iteration \( i \) is at most \( 1 - c_0 \), so that simultaneous transmission by all the links in \( S \) does not cause high interference for \( l \). The approximation factor involves the following two ideas: (1) for any link \( l \in S \), there can be at most \( O(1) \) links in \( OPT \) which are within distance \( c_0 d(l) \), and (2) in the set of links removed in step 2 due to the affectance from \( S \), there can be at most \( O(1) \) links in \( OPT \). We note that the second and third steps are reversed in [2], while [6] does not use the third step. However, we find it necessary for our distributed algorithm, which uses the natural approach of considering all the links in a given length class simultaneously (instead of sequentially). Our analysis builds on these ideas, and property (1) holds for our case without any changes. However, property (2) is more challenging to analyze, since many links are added in parallel. Another complication is that the distributed implementation has to be done from the senders’ perspective, so that the above steps become more involved.

#### B. Additional Definitions

![Figure 1](image-url)  
*Figure 1: Example of an \((\omega_1, \omega_2)\)-ruling: \( W = \{ v_i \} \cup \{ u_1 \} \) is the set of all dots (open and dark), while \( R_{\omega_1,\omega_2}(W) = \{ v_i \} \) is the set of all the open dots denotes a \((\omega_1, \omega_2)\)-ruling of \( W \). Note that all the nodes in \( W \) are \( \omega_2 \)-covered by \( R_{\omega_1,\omega_2}(W) \), while all the nodes in \( R_{\omega_1,\omega_2}(W) \) are \( \omega_1 \) away from each other.*

**Cover and Ruling** Let \( W, W' \) denote two node sets. We say a node \( u \) is \( \omega \)-covered by \( W \), if and only if \( \exists u' \in W' \) \( d(u,u') \leq \omega \); based on that, we say \( W \) is \( \omega \)-covered by \( W' \), or equivalently \( W' \) \( \omega \)-covers \( W \), if and only if every node in \( W \) is \( \omega \)-covered by \( W' \). An \((\omega_1, \omega_2)\)-ruling (where \( \omega_1 < \omega_2 \)) of \( W \), introduced in [17], is a node set denoted by \( R_{\omega_1,\omega_2}(W) \), such that...
algorithms and basic ideas in Section IV-A; now we discuss
an example to illustrate these notions.

The concept of ruling has a vital role in our algorithm: it is
considering Euclidean distance rather than graph distance. The

Here, we have adopted a generalized definition by consid-

We have provided a detailed discussion of the centralized
output parameter

...
the ball \( B(v, \omega_1) \) of any node \( v \in W_1 \); in the worst case, \( b_{max} \leq n \).

Algorithm 2: ConstructR\((\omega_1, \omega_2, W_1, W_2, b_{max})\): Distributed algorithm for computing an \((\omega_1, \omega_2)\)-ruling with full duplex radios.

```plaintext
input : \( \omega_1, \omega_2, W_1, W_2, b_{max} \)
output : \((\hat{R}, \hat{Z})\): an \((\omega_1, \omega_2)\)-ruling of \( W_1 \)

// Each \( v \in W_1 \cup W_2 \) does the following /*
1 for \( i_{\text{out}} = 0 \) to \( \log b_{max} + 1 \) do
2 for \( i_{\text{in}} = 1 \) to \( C_4 \log n \) do
3 if \( v \) is active then
4    /* Coordination Step (Lines 2-4): 1
5      slot */
6      \( U(v) \leftarrow 0 \);
7      if \( v \in W_1 \) then \( U(v) \) flips to 1 w/ prob. \( 1/b_{max} \):
8        if \( U(v) = 1 \) then
9          \( v \) transmits and senses, \( I(v) \leftarrow \) the power \( v \) receives in this slot;
10         if \( I(v) > \text{Thres}(\omega_1) \) then \( U(v) \leftarrow 0 \);
11      end /* Decision Step (Lines 5-8): 1
12      slot */
13      if \( U(v) = 1 \) then \( v \) transmits, \( v \) joins \( \hat{R} \);
14      /* inactive */
15    else
16      \( v \) senses, \( I(v) \leftarrow \) the power \( v \) receives in this slot;
17      if \( I(v) > \text{Thres}(\omega_1) \) then \( v \) joins \( \hat{Z} \);
18      /* inactive */
19    end
20 end
21 return \((\hat{R}, \hat{Z})\);
```

In this algorithm, we call an iteration of the outer loop (Line 1) a **phase**; we call an iteration of the inner loop (Line 2) a **round**, consisting of the **coordination step** (Lines 4 through 9) and the **decision step** (Lines 10 through 14). A node \( v \) is said to be **active** if \( v \) has not joined either \( \hat{R} \) or \( \hat{Z} \); otherwise, \( v \) becomes **inactive**.

In each round, the coordination step provides a probabilistic mechanism for active nodes to compete to get in the ruling (at Line 5). Lines 6 through 9 constitute a module to resolve the issue of sensing and transmitting at the same time, such that two nearby nodes do not both enter the ruling (i.e., Lemma \( V.3 \)). Next, during the decision step, a subset of active nodes decide to join \( \hat{R} \) or \( \hat{Z} \).

In each phase, there are \( C_4 \log n \) rounds, such that we can ensure a fraction of the node population have either joined \( \hat{R} \) or \( \hat{Z} \), and we expect the maximum number of active nodes in the nearby region of any active node to decrease by a half (proved in Lemma \( B.3 \) in Appendix \( B.A \)). After each phase, the probability for each active node to access the channel and compete doubles (at Line 5). After the total of \( \log b_{max} + 2 \) phases, we have Lemmas \( V.2 \), \( V.4 \), \( V.5 \) that lead to Theorem \( V.1 \).

**Theorem V.1 (Correctness).** Algorithm \( 2 \) terminates in \( O(\log n \log b_{max}) \) time. By the end of the algorithm: (1) \( \hat{R} \) forms an \((\omega_1, \omega_2)\)-ruling of \( W_1 \), w.h.p. Lemmas \( V.2 \), \( V.4 \), \( V.5 \) show that \( \hat{Z} \) complements \( \hat{R} \) in \( W_1 \) and partially in \( W_2 \) with desired properties, w.h.p. To help the reading flow and due to the page limit, we defer much of the technical content — the proof of Lemma \( V.2 \) (which involves Lemmas \( B.1 \), \( B.2 \), and \( B.3 \)) and the proof of Lemma \( V.5 \) to Appendix \( B \).

**Lemma V.2 (Completion).** By the end of the algorithm, all nodes in \( W_1 \) have joined either \( \hat{R} \) or \( \hat{Z} \), i.e., all nodes in \( W_1 \) become inactive, w.h.p.

Lemma \( V.2 \) implies that \( \hat{Z} \cap W_1 = W_1 \setminus \hat{R} \). We say a node \( v \in \hat{R} \) is “good,” if and only if \( d(v, v') \geq \omega_1, \forall v' \in \hat{R} \) and \( v' \neq v \). In Algorithm 2, when a node enters \( \hat{R} \), it makes sure that there are no other ones entering \( \hat{R} \) within a range of \( \omega_1 \), and it deactivates all the active nodes in the same range. Therefore, we have the following Lemmas \( V.3 \) and \( V.4 \).

**Lemma V.3 (Quality of \( \hat{R} \)).** All nodes in \( \hat{R} \) are good, with probability of 1.

**Lemma V.4 (Quality of \( \hat{Z} \): Part 1).** \( \hat{Z} \) contains all the nodes in \( W_1 \cup W_2 \setminus \hat{R} \) that are \( \omega_2 \)-covered by \( \hat{R} \), with probability of 1.

**Lemma V.5 (Quality of \( \hat{Z} \): Part 2).** Further, suppose all nodes in \( \hat{R} \) are good, then all nodes in \( \hat{Z} \) are \( \omega_2 \)-covered by \( \hat{R} \), \( \forall \omega_2 \geq \frac{36}{\alpha_2 - 2} \log \omega_1 \).

**Half Duplex Communication.** Now, we assume that nodes are in the half duplex mode, so that they cannot perform transmission and reception/sensing at the same time. In Algorithm 2, Lines 6 through 9 make use of the full duplex capability, such that Lemma \( V.3 \) is true. To account for the case of half duplex communication, if we replace the one-slot deterministic full duplex mechanism (Lines 6 through 9) with a randomized \( O(\log n) \)-time loop — illustrated by the following lines of pseudo code — we have Lemma \( V.6 \) for half duplex communication as the counterpart of Lemma \( V.3 \) for full duplex. The cost incurred includes (1) the increase in the total running time to obtain an \((\omega_1, \omega_2)\)-ruling by \( O(\log n) \), and (2) a weakened statement in Lemma \( V.6 \).
In replacement of Lines 6 through 9 in Algorithm 2 for using half duplex radios.

1 for $j = 1$ to $C_5 \log n$ do /* resolving half duplex communication */
   /* in each slot */
   if $U(v) = 1$ then $v$ transmits with prob. 1/2;
   if $v$ does not transmit then
      $v$ senses, $I(v) \leftarrow$ the power $v$ receives in this slot;
      if $I(v) > \text{Thres}(\omega_1)$ then $U(v) \leftarrow 0$;
   /* stops */
2 end

Lemma V.6 (Quality of $\hat{R}$: Half Duplex Mode). All nodes in $R$ are good, w.h.p.

Since Lemmas V.2, V.4 and V.5 remain valid, we obtain the following theorem for the half duplex case.

Theorem V.7 (Half Duplex). There exists a modified version of ConstructR$(\omega_1, \omega_2, W_1, W_2, b_{\text{max}})$ for the half duplex case, such that it finishes in $O((\log^2 m \log b_{\text{max}}))$ time and by the end of the algorithm: (1) $\hat{R}$ forms an $(\omega_1, \omega_2)$-ruling of $W_1$, and (2) $\hat{Z} \cap W_1 = W_1 \setminus \hat{R}$ and $\{v \in W_2 : v$ is $\omega_1$-covered by $\hat{R}\} \subseteq \hat{Z} \cap W_2 \subseteq \{v \in W_2 : v$ is $\omega_2$-covered by $\hat{R}\}$, w.h.p.

VI. DISTRIBUTED IMPLEMENTATION WITH NON-ADAPTIVE UNIFORM TRANSMISSION POWER FOR SCHEDULING

Putting everything together, we present in this section the distributed implementation of Algorithm 1 when restricted to using one uniform power level for scheduling.

Theorem VI.1 (Performance). Our distributed implementation of Algorithm 1 with non-adaptive uniform transmission power has the following properties:
(1) in half duplex mode, it terminates in $O(g(L) \log^3 m)$ time,
(2) in full duplex mode, it terminates in $O(g(L) \log^2 m)$ time, and
(3) in both modes, it produces a constant-approximate solution to MAXLSP$^\phi$.

For the $i$th phase of Algorithm 1, we present the distributed implementation that works even when there is only one fixed power level available. We assign $\gamma_2$ a constant value $\geq (36 + 1)\gamma_1^2 - 2\gamma_1$, and let $\omega_1 = \gamma_1 d_m$, and $\omega_2 = \gamma_2 d_i$. The distributed implementation goes as follows.

Distributed Implementation: 1st Step: With Algorithm 1 we run CheckA$(J_i, \hat{P}_i, \cup_{j<i} S_j)$ to implement the 1st step.

Algorithm 4: CheckA$(Y, Y', S)$: Distributed algorithm for checking affectance.

input : Link sets $Y, Y', S$
output: $Y^a = \{x(l) : l \in Y, A(S, l) \leq \psi(1 - (1/\phi^{1/\alpha})^{1/\alpha})\}$,
        $Y^b = \{x(l) : l \in Y', A(S, l) \leq \psi(1 - (1/\phi^{1/\alpha})^{1/\alpha})\},$
        $Y^a \cap \hat{Y} = \{x(l) : A(S, l) > \psi(1 - (1/\phi^{1/\alpha})^{1/\alpha})\}$
        $Y^b \cap \hat{Y} = \{x(l) : A(S, l) > \psi(1 - (1/\phi^{1/\alpha})^{1/\alpha})\}$
/* in i time slot: */
1 if $l \in S$ then $x(l)$ transmits;
2 else if $l \in Y \cup Y'$ then
   3 $x(l)$ senses, $SP(X(S), x(l)) \leftarrow$ the power $x(l)$ receives;
   4 if $SP(X(S), x(l)) \leq \text{Thres}(\beta \psi(1 - (1/\phi^{1/\alpha})^{1/\alpha}))$ then
      5 if $l \in Y$ then $x(l)$ joins $Y^a$; if $l \in Y'$ then $x(l)$ joins $Y^b$;
   6 else
      7 $x(l)$ joins $\hat{Y}^a$; if $l \in Y'$ then $x(l)$ joins $\hat{Y}^b$;
end
9 return $Y^a, Y^b, \hat{Y}^a, \hat{Y}^b$.

for phase $i$ in Algorithm 1 ∀$l \in J_i \cup \hat{J}_i$, on Line 3 of Algorithm 4 we get $SP(X(S), x(l)) = \sum_{l' \in \cup_{j<i} S_j} P_{d^a(T)} + N$. Then, since

$$SP(X(S), x(l)) \leq \text{Thres}(\beta \psi(1 - (1/\phi^{1/\alpha})^{1/\alpha})),$$

is equivalent to

$$A(\cup_{j<i} S_j, l) = \frac{\beta}{1 - d^a(l)} \sum_{l' \in \cup_{j<i} S_j} d^a(l'),$$

the sets of links whose sender nodes are in $Y^a, Y^b, \hat{Y}^a, \hat{Y}^b$ correspond to $J^a_i, J^b_i, \hat{J}^a_i, \hat{J}^b_i$ in Algorithm 1 respectively.

Distributed Implementation: 2nd Step: Recall that for a link set $L'$, $X(L')$ is the set of all sender nodes. To implement the 2nd step for phase $i$ in Algorithm 1 we feed $b_{\text{max}} = m$ to Algorithm 2 and run ConstructR$(\omega_1, \omega_2, X(J^a_i), X(J^b_i), m)$. Thus, we obtain an $(\omega_1, \omega_2)$-ruling $\hat{R}$ of $X(J^a_i)$ and $\hat{Z}$ that complements $\hat{R}$ in...
O(\log^3 m) time for half duplex and O(\log^2 m) time for full duplex. Then, the sets of links whose sender nodes are in \( R, Z \) respectively correspond to \( X(J^*_1), X(J^*_2) \) in Algorithm \( \Box \).

**Distributed Implementation: 3rd Step:** The 3rd step of Algorithm \( \Box \) means all the links in class \( L_i \) and those longer links removed in the 1st step exit Algorithm \( \Box \). Because our algorithm is sender based, the corresponding links will quit upon the decision of their sender nodes in the 1st and the 2nd steps.

### VII. Distributed Implementation with Adaptive Transmission Power for Scheduling

In this section, suppose we have multiple power levels at our disposal on each node\( \dagger \). We present how this aids the distributed implementation of Algorithm \( \Box \).

**Theorem VII.1 (Performance).** Our distributed implementation of Algorithm \( \Box \) with adaptive transmission power has the following properties:

1. In half duplex mode, it terminates in \( O(g(L) \log^2 m) \) time.
2. In full duplex mode, it terminates in \( O(g(L) \log m) \) time, and
3. In both modes, it produces a constant-approximate solution to MAXLSP\( \dagger \).

Again, note that these adaptive power levels are only for scheduling in the control phase; for data transmission in the resulting independent set \( S \) we still use one uniform power level. Specifically, we require that (1) nodes have access to a set of \( \Theta(g(L)) \) power levels; and (2) for each \( i \in [1, g(L)] \), there exists a power level \( P_i \) to use such that \( (\frac{d_i}{\gamma_3})^{1/\alpha} = \gamma_3 d_i \), where \( \gamma_3 \) is a constant.

We present a second method to implement the 2nd step of each phase in Algorithm \( \Box \) reducing the running time by one logarithmic factor, by (1) performing a preprocessing to reduce \( b_{\max} \) to some constant \( C_0 \) in \( O(\log m) \) time, (2) running Algorithm \( \Box \) with the constant \( C_0 \) in \( O(\log^2 m) \) time with half duplex radios and \( O(\log m) \) time with full duplex radios, and (3) performing a postprocessing to obtain the sets of links required as a result of 2nd step of Algorithm \( \Box \) in one slot.

We introduce a new constant \( \gamma_4 \geq (36 \alpha - 1)^{1/2} \alpha - 2 \gamma_1 \), and assign \( \gamma_2 \) a constant value \( \geq \gamma_3 + \gamma_4 \). For the \( i \)th phase of Algorithm \( \Box \) let \( \omega_1 = \gamma_1 d_i, \omega_2 = \gamma_2 d_i, \omega_3 = \gamma_3 d_i \), and \( \omega_4 = \gamma_4 d_i \).

We reuse the implementation for the 1st and the 3rd steps from the previous version. We implement the 2nd step of each phase in Algorithm \( \Box \) with the following three substeps.

#### A. Preprocessing: Constant Density Dominating Set

Scheideler, Richa, and Santi in [16] propose a distributed protocol to construct a constant density dominating set of nodes under uniform power assignment within \( O(\log m) \) slots. They define \( \text{Dom}(W, P_i) \) as a dominating set of a node set \( W \) with transmission power of \( P_i \) on each node, such that \( W \) is \( d_i \)-covered \( \text{Dom}(W, P_i) \), where \( d_i \) is the transmission range under \( P_i \). Then, by “constant density”, they mean that \( \text{Dom}(W, P_i) \) is a \( O(1) \)-approximation of the minimum dominating set of \( W \), such that within the transmission range \( d_i \) of each node in \( W \) there are at most a constant number \( C_0 \) of nodes chosen by \( \text{Dom}(W, P_i) \).

At phase \( i \) of Algorithm \( \Box \) after the 1st step of checking affectance, we execute the protocol on the node set \( X(J^*_i) \) with power \( P_i \) which corresponds to a transmission range of \( \omega_3 \), and thus obtain a constant density dominating set \( \text{Dom}(X(J^*_i), P_i) \) out of \( X(J^*_i) \). \( \text{Dom}(X(J^*_i), P_i) \) has the following properties:

1. \( \text{Dom}(X(J^*_i), P_i) \in X(J^*_i) \);
2. \( \text{dominating set} \): all the node in \( X(J^*_i) \) \( \omega_3 \)-covered by \( \text{Dom}(X(J^*_i), P_i); \) and
3. \( \text{constant density} \): \( \forall v \in X(J^*_i), 1 \leq |B(v, \omega_3) \cap \text{Dom}(X(J^*_i), P_i)| \leq C_9 \), where \( C_9 \) is a constant.

#### B. Construction of Ruling \( X(J^*_i) \)

\( \text{ConstructR}(\omega_1, \omega_4, \text{Dom}(X(J^*_i), P_i), X(J^*_i), C_9) \) produces \( \hat{R} \) as an \( \langle \omega_1, \omega_4 \rangle \)-ruling of \( \text{Dom}(X(J^*_i), P_i) \), and \( \hat{Z} \) such that

1. \( \hat{Z} \cap X(J^*_i) = \text{Dom}(X(J^*_i), P_i) \setminus \hat{R}; \)
2. \( \hat{Z} \cap X(J^*_i) \supseteq \{ v \in X(J^*_i) : v \text{ is } \omega_1 \text{-covered by } \hat{R} \}; \) and
3. \( \hat{Z} \cap X(J^*_i) \subseteq \{ v \in X(J^*_i) : v \text{ is } \omega_4 \text{-covered by } \hat{R} \}, \) i.e., \( \hat{Z} \) is \( \omega_4 \)-covered by \( \hat{R} \).

We argue that \( \hat{R} \) is an \( \langle \omega_1, \omega_2 \rangle \)-ruling of \( X(J^*_i) \) due to the following two properties: (1) \( \hat{R} \subseteq X(J^*_i) \) and any two nodes in \( \hat{R} \) are \( \omega_1 \)-separated, and (2) \( X(J^*_i) \) is \( \omega_2 \)-covered by \( \hat{R} \). Property (2) can be deduced from the facts below: (1) \( X(J^*_i) \) is \( \omega_3 \)-covered by \( \text{Dom}(X(J^*_i), P_i) \) due to the preprocessing step, (2) \( \text{Dom}(X(J^*_i), P_i) \) is \( \omega_4 \)-covered by \( \hat{R} \) as a result of \( \text{ConstructR}(\omega_1, \omega_4, \text{Dom}(X(J^*_i), P_i), X(J^*_i), C_9) \), and (3) \( \omega_4 + \omega_3 \leq \omega_2 \) by our construction. Therefore, \( \hat{R} \) corresponds to \( X(J^*_i) \) in the 2nd step of Algorithm \( \Box \).

#### C. Postprocessing: Accounting for \( X(J^*_i) \)

\( \text{Construct} \hat{Z}' \triangleq \hat{Z} \cup \left(X(J^*_i) \setminus \hat{R}\right); \) the following is true:

1. \( \hat{Z}' \cap X(J^*_i) = X(J^*_i) \setminus \hat{R}; \)
2. \( \hat{Z}' \cap X(J^*_i) \supseteq \{ v \in X(J^*_i) : v \text{ is } \omega_1 \text{-covered by } \hat{R} \}; \) and
3. \( \hat{Z}' \cap X(J^*_i) \subseteq \{ v \in X(J^*_i) : v \text{ is } \omega_2 \text{-covered by } \hat{R} \}, \) i.e., \( \hat{Z}' \) is \( \omega_2 \)-covered by \( \hat{R} \).
Therefore, \( \hat{Z}' \) corresponds to \( X(J_i^o \mapsto \cdot) \) in the 2nd step of Algorithm \( \Box \).

VIII. Conclusion

In this paper, we present the first set of fast distributed algorithms in the SINR model with a constant factor approximation guarantee for MaxLSP\(^U \). We extensively study the problem by accounting for the cases of half/full duplex and non-adaptive/adaptive power availability for scheduling. The non-local nature of this model and the asymmetry between senders and receivers makes this model very challenging to study. Our algorithm is randomized and crucially relies on physical carrier sensing for the distributed communication steps, without any additional assumptions. Our main technique of distributed computation of a ruling is likely to be useful in the design of other distributed algorithms in the SINR model.

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APPENDIX A

APPENDIX TO SECTION IV

A. Proof of Lemma [IV.1]

The statement of Lemma [IV.1] is equivalent to that \( \forall l \in S, A(S_i \setminus \{i\}, l) \leq 1 \). Let \( l \) be an arbitrary link in \( S \), and w.l.o.g., we assume \( l \in S_i \), and thus \( l \in L_i \). In each phase \( j < i \), because \( J^j_i \cup J^{j+}_i \cup J^{j-}_i \geq J_j \), all the links in \( \cup_{j<i} L_j \) have been removed from \( J \) at the end of phase \( j \).

Due to the 2nd step, \( A(\cup_{j<i} S_j, l) \leq \psi(1-(\frac{\phi}{\beta(1+\phi)})^{1/\alpha}) \).

First, we show that \( A(\cup_{j<i} S_j, l) \leq \psi \).

For any link \( l' \in \cup_{j<i} S_j \),

\[
A(l', \frac{\tilde{T}}{l}) = \frac{\beta}{1 - \frac{d^\alpha(l', l)}{P/(SN)}} d^\alpha(l', l) \leq A(\cup_{j<i} S_j, \frac{\tilde{T}}{l}) < 1.
\]

Hence,

\[
d(x(l'), x(l)) = d(l', \frac{\tilde{T}}{l}) \geq (\frac{\beta d^\alpha(l', l)}{1 - \frac{d^\alpha(l', l)}{P/(SN)}})^{1/\alpha} \geq \frac{\beta}{(1+\phi)^1/\alpha} \frac{d(l', l)}{d^\alpha(l', l)}\]

implies

\[
d(l', \frac{\tilde{T}}{l}) = \frac{d(x(l'), s(l))}{d(x(l'), x(l))} \geq d(x(l'), x(l)) - d(l', l) \geq 1 - (\frac{\phi}{\beta(1+\phi)})^{1/\alpha}.
\]

By referring to the definition of affectance and

\[
A(\cup_{j<i} S_j, l) = \sum_{l' \in \cup_{j<i} S_j} \frac{\beta}{1 - \frac{d^\alpha(l', l)}{P/(SN)}} d^\alpha(l', l) \quad \text{and}
\]

\[
A(\cup_{j<i} S_j, l) \leq \sum_{l' \in \cup_{j<i} S_j} \frac{\beta}{1 - \frac{d^\alpha(l', l)}{P/(SN)}} d^\alpha(l', l)
\]

we obtain

\[
A(\cup_{j<i} S_j, l) \leq \left(1 - (\frac{\phi}{\beta(1+\phi)})^{1/\alpha}\right)^{\alpha} A(\cup_{j<i} S_j, l) = \psi.
\]

Next, it suffices to show that

\[
A(\cup_{j \geq i} S_j \setminus \{i\}, l) \leq 1 - \psi.
\]

At phase \( i \), \( \omega_i = \gamma_i d_i \); the 1st step in the phase ensures that when link \( l \) is added to \( S_i \), any link \( l' \in \cup_{j \geq i} L_j \) with \( d(x(l'), x(l)) < \omega_i \) will not get in \( S_i \). Therefore, for the set \( \cup_{j \geq i} S_j \setminus \{i\} \), we have: (1) all the nodes in \( X(\cup_{j \geq i} S_j \setminus \{i\}) \) have a mutual distance of at least \( \omega_i \); (2) the distance from any node in \( X(\cup_{j \geq i} S_j \setminus \{i\}) \) to \( r(l) \) is at least \( \omega_i - d(l); \) and (3) \( \omega_i - d(l) > \omega_i/2 > 0 \). According to Proposition [A.1] by using \( \gamma_1 = \frac{\beta}{1-\phi} \frac{(\alpha-1)\alpha}{\beta(1+\phi)} \),

\[
SP \left( X(\cup_{j \geq i} S_j \setminus \{i\}), r(l) \right) \leq \frac{36(\alpha-1)}{\alpha-2} \frac{P}{(\omega_1 - d(l))^{1/\alpha}} + N < \frac{12(\alpha-1)}{\alpha-2} \frac{P}{(\omega_2 - d(l))^{1/\alpha}} + N.
\]

It is easy to verify \( A(\cup_{j \geq i} S_j \setminus \{i\}, l) < 1 - \psi \).

**Proposition A.1.** \( \forall V' \in V \) and \( \forall v \notin V' \), if (1) all the nodes in \( V' \) are at least \( \rho_1 \) away from each other; (2) the distance between \( v \) and any node in \( V' \) is at least \( \rho_2 \), (3) \( \rho_2 > \rho_1/2 \), then \( SP(V', v) < \frac{36(\alpha-1)}{\alpha-2} \frac{P}{\rho_2^{1/\alpha}} + N \).

**Proof:** We bound the sensed power strength by partitioning the plane into concentric rings all centered at \( v \), each of width \( \rho_2 \), via a similar technique to that in [4].

Let \( Ring(i) \) denote the \( i \)th ring (where \( i = 1, 2, \ldots \)), which contains every node \( v' \) that satisfies \( i \rho_2 \leq d(v', v) < (i+1) \rho_2 \); let \( V'(i) \) denote the subset of nodes in \( V' \) that fall in \( Ring(i) \). We notice the following facts: (1) For any two nodes \( v, v' \in V'(i) \), two disk centered at \( v, v' \) respectively with a radius of \( \rho_1/2 \) are non-overlapping. (2) For any node \( v \in V'(i) \), such a disk is fully contained in an extended ring \( Ring'(i) \) of \( Ring(i) \), with an extra width of \( \rho_1/2 \) at each side of \( Ring(i) \). The area (denoted by \( D \)) of each of such disks is \( D = \pi(\rho_1/2)^2 \). The area (denoted by \( D(i) \)) of \( Ring(i) \) is

\[
D(i) = \pi[(i+1)\rho_2 + \rho_1/2]^2 - (i\rho_2 - \rho_1/2]^2] \leq 3\pi(2i+1)^2.
\]

Using \( |V'(i)| \leq D(i)/D \leq 12(2i+1)\rho_2^2/\rho_2^2 \), we obtain

\[
SI(V', w) \leq \sum_{i=1}^{\infty} |V'(i)| \frac{P}{i^\alpha \rho_2^2} + N
\]

\[
\leq \sum_{i=1}^{\infty} \frac{12(2i+1)\rho_2^2 P}{i^\alpha} + N
\]

\[
\leq \frac{36(\alpha-1)}{\alpha-2} \frac{P}{\rho_2^2} + N.
\]

**B. Proof of Theorem IV.2**

For a node \( v \), we define \( B(v, d) \) as the ball centered at \( v \) with a radius of \( d \). With a parameter \( \gamma > 1 \), we then define a link set \( B^\gamma_v(l) \), such that for a link \( l \in L_i \), \( B^\gamma_v(l) \) contains all and only the links in \( \{l_j : j \geq i\} \), with their senders in the ball \( B(x(l), \gamma d_i) \); in other words, \( B^\gamma_v(l) \) contains the links with similar or longer lengths, whose senders are \( (\gamma d_i) \)-covered by \( x(l) \). For a set \( L' \subseteq L \), \( B^\gamma_v(L') \) is defined as \( \cup_{l \in L'} B^\gamma_v(l) \).

**Lemma A.2 (Spatial Constraint).** \( \forall \gamma > 1, \forall L' \subseteq L, \forall l \in L', |OPT(B^\gamma_v(l) \cap L')| \leq C_1(\gamma), \) where \( C_1(\gamma) = \frac{(2\gamma+1)^\gamma}{\beta} \).

**Proof:** Let \( k \) be an arbitrary link in \( OPT(B^\gamma_v(l) \cap L') \). For any link \( k' \in OPT(B^\gamma_v(l) \cap L'), d(k', k) \leq d(x(k'), x(l)) + d(x(l), x(k')) + d(k') \leq (2\gamma + 1) d(k) \).
Therefore,
\[
1 \geq A(\text{OPT}(B_{l_1}^2(l) \cap L')) \setminus \{k\}, k) \\
= \beta \sum_{k' \in \text{OPT}(B_{l_1}^2(l) \cap L') \setminus \{k\}} \frac{d'^{(k)}}{d'^{(k')}} \\
\geq \frac{\beta}{1 - \frac{d'^{(k)}}{\beta^3(3N)}} \\
\geq \beta(2\gamma + 1)^{-\alpha}|OPT(B_{l_1}^2(l) \cap L')|.
\]
It follows that
\(|OPT(B_{l_1}^2(l) \cap L')| \leq (2\gamma + 1)^\alpha/\beta = C_3(\gamma).\)

**Lemma A.3** (Affection Constraint). \(\forall \psi^* > 0\) and \(\forall L', L'' \subseteq L, \text{ if } L' \cap L'' = \emptyset \text{ and } A(L', \overline{l}) > \psi^*\) for any link \(l \in L''\), then \(|OPT(L'')| \leq C_2(\psi^*)|L'|\), where \(C_2(\psi^*) = \frac{(2(2\beta)^{1/\alpha} - 1)}{(2(2\beta)^{1/\alpha} - 1)^{\alpha/\beta}}/\psi^* + 1\).

**Proof:** If \(|OPT(L'')| > 0\), we can express it as \(|OPT(L'')| = b|L'| + g\), such that \(b\) and \(g\) are non-negative integers and \(1 \leq g \leq |L'|\). We create \(|L'|\) bins, each of which has a capacity of \(b\) links; we pack the links in \(OPT(L'')\) to the bins via a first-fit sweep through the links in \(L'\):

1. We order the links in \(L'\) arbitrarily; let \(l_j\) denote the \(j\)th link in \(L'\), and let \(Bin_j\) denote the \(j\)th bin. Let set \(L^* = OPT(L'')\) initially; then, the sweep proceeds in \(|L'|\) rounds.
2. In the \(i\)th round (where \(i = 1, 2, \ldots, |L'|\)), we pick \(b\) links in \(L^*\) whose senders are the nearest \(b\) nodes to the sender \(x(l_i)\) of the \(i\)th link in \(L'\), and we remove those \(b\) links from \(L^*\) and put them into \(Bin_i\).

The completion of the above packing means that in \(OPT(L''),\) we have \(b|L'|\) links “near”, and \(g \in [1, |L'|]\) links “far” from the senders of links in \(L'\). If \(b \geq \frac{(2(2\beta)^{1/\alpha} - 1)^{\alpha}}{(2(2\beta)^{1/\alpha} - 1)^{\alpha/\beta}}\), we are done. Therefore, for the rest of the proof we assume that \(b > \frac{(2(2\beta)^{1/\alpha} - 1)^{\alpha}}{(2(2\beta)^{1/\alpha} - 1)^{\alpha/\beta}}\), and we show that \(b \leq C_2(\psi^*) - 1 = \frac{(2(2\beta)^{1/\alpha} - 1)^{\alpha}}{(2(2\beta)^{1/\alpha} - 1)^{\alpha/\beta}}/\psi^*\) in this case.

Let \(l\) denote a “far” link in \(OPT(L'')\) that is out of any bins. We have for any link \(k \in Bin_i, d(x(k), x(l)) < d(x(l), x(l_i))\), implying that \(d(x(l), x(l_i)) < 2d(x(l), x(l_i))\) due to triangle inequality. Since \(Bin_i \cup \{l\} \subseteq OPT(L')\), we have
\[
1 \geq A(Bin_i, l) = \beta \sum_{k \in Bin_i} \frac{d'^{(k)}}{d'^{(k)}} \\
\geq \frac{\beta b}{1 - \frac{d'^{(l)}}{\beta^3(3N)}} \\
\geq \beta b \left(\frac{2d(x(l), x(l_i))}{d(l) + 1}\right)^\alpha.
\]
That leads to that \(d(l) \leq \frac{2(2\beta)^{1/\alpha} - 1}{\beta(1 + \phi)}d(x(l), x(l_i)).\) Then for any link \(k \in Bin_i,\)
\[
d(k, l) = d(x(k), r(l)) \\
\leq d(x(k), x(l_i)) + d(x(l_i), x(l)) + d(x(l), r(l)) \\
\leq \frac{2(2\beta)^{1/\alpha} - 1}{\beta(1 + \phi)}d(x(l_i), x(l)).
\]
Since \(\bigcup Bin_i \cup \{l\} \subseteq OPT(L'')\), we have
\[
1 \geq A(\bigcup_{i=1}^{L'} Bin_i, l) = \beta \sum_{k \in bin_i} \frac{d'^{(l)}}{d'^{(k)}} \\
\geq \frac{\beta b}{1 - \frac{d'^{(l)}}{\beta^3(3N)}} \\
\geq \frac{b}{(2(2\beta)^{1/\alpha} - 1)^{\alpha}} A(L', \overline{l})
\]
The last inequality above holds because
\[
A(L', \overline{l}) = \beta \sum_{l' \in L'} \frac{d'^{(l')}}{d'^{(l)}} \geq \psi^*.
\]
Therefore, \(b \leq \frac{(2(2\beta)^{1/\alpha} - 1)^{\alpha}}{(2(2\beta)^{1/\alpha} - 1)^{\alpha/\beta}}/\psi^*\) and \(|OPT(L'')| \leq (b + 1)|L'| \leq C_2(\psi^*)|L'|.\)

We define \(J^a = \bigcup_{i=1}^{L'} J^a_i, J^b = \bigcup_{j=1}^{J^a} J^b_j, J^c = \bigcup_{j=1}^{J^b} J^c_j, J^d = \bigcup_{j=1}^{J^c} J^d_j\) which contains all the links removed in the 1st step in Algorithm\[\text{I}\] due to the affection constraints. At each phase \(i, X(J^i)\) is an \((\omega_1, \omega_2, \omega_3)\)-ruling of \(X(J^i)\), and all the nodes in \(X(J^i)\) are \(\omega_2\)-covered by \(X(J^i)\); we choose all the links in \(J^i\) to add to \(S\), discard all the links in \(J^d\) and \(J^a\) (for failing affection check), and also discard all the links in \(J^c\) (because of their proximity to the chosen links).

We have
\[
L = \bigcup_{i=1}^{L'} J^a_i \cup J^a_i \cup J^b_i \cup J^d_i \\
\subseteq \bigcup_{i=1}^{L'} J^a_i \cup J^b_i \cup J^d_i \cup B_{\gamma_1}^\infty(J^b_i)
\]
Due to Lemma A.3 \(|OPT(J^a) \cup J^b| \leq C_2(\psi(1 - \frac{\alpha}{\beta(1 + \phi)^{1/\alpha}}))|S|\); due to Lemma A.2 \(|OPT(B_{\gamma_1}^\infty(S))| \leq \sum_{l \in S} |OPT(B_{\gamma_1}^\infty(l))| \leq C_1(\gamma_2)|S|\).

Therefore,
\[
|OPT(L)| \leq |OPT(J^a) \cup J^b| + |OPT(B_{\gamma_1}^\infty(S))| \\
\leq (C_1(\gamma_2) + C_2(\psi(1 - \frac{\alpha}{\beta(1 + \phi)^{1/\alpha}})))|S|.
\]
APPENDIX B
APPENDIX TO SECTION Vi

Recall that we call an iteration of the outer loop (Line 1) a phase of the algorithm, and an iteration of the inner loop (Line 2) a round; recall that \( B(v, d) \) denotes the ball centered at \( v \) with a radius of \( d \). Let \( A_{t_1}^{W_1}(v, d) \) denote the set of active nodes in set \( W_1 \) that fall in the ball \( B(v, d) \) at time point \( t \); we will explicitly point \( t \) out whenever we use \( A_{t_1}^{W_1}(v, d) \).

A. Proof of Lemma Vi.2

The following definitions are only involved in Lemmas B.1 and B.2. Let \( \eta \) be a constant \( > \left( \frac{96 \alpha - 1}{2} \right)^{-1/\alpha} \). In one round (which corresponds to one iteration of the inner loop) of Algorithm 2 let \( U \in W_1 \) be the set of nodes with \( U(\cdot) = 1 \) at line 5 in the coordination step. We say an active node \( v \in W_1 \) is “lucky” in a round, with \( t_0 \) being the time that the round starts, if and only if

1. \( v \in U \);
2. \( U \cap A_{t_0}^{W_1}(v, \eta \omega_1) = \{ v \} \), i.e., \( v \) has no nearby active nodes in \( U \);
3. \( SP(U \setminus A_{t_0}^{W_1}(v, \eta \omega_1), v) < Thres(\omega_1) \), i.e., total power received from faraway active nodes is small.

In a round if \( v \) gets lucky, \( U(v) \) will remain 1 till the end of that round, and thus will elect to be included in \( \hat{R} \) and will cause all nodes in \( A_{t_0}^{W_1}(v, \omega_1) \) to get into \( Z \).

**Lemma B.1.** In a round \( i_{\text{out}} \) of phase \( i_{\text{out}} \), with \( t_1 \) being the time that the round starts, suppose that for each active node \( u \in W_1 \), \( |A_{t_1}^{W_1}(u, \omega_1)| \leq 2^{(\log b_{\max} - \log \omega_1 + 1)} \) at the beginning of the round, then the probability for an arbitrary active node in \( W_1 \) to be lucky in the round is at least

\[
2^{-((\log b_{\max} - \log \omega_1 + 1) + 3 + 2(\eta + 1)^2)}.
\]

**Proof:** For a round at phase \( i_{\text{out}} \), we prove the statement in the following four steps.

1. First, for any active node \( v \in W_1 \), the probability for \( v \) to be in \( U \) is \( \text{Prob}(U(v) = 1) = 2^{-((\log b_{\max} - \log \omega_1 + 2))} \).
2. Second, for any active node \( v \in W_1 \), due to the packing property we upper-bound the size of \( A_{t_1}^{W_1}(v, \omega_1) \) as

\[
|A_{t_1}^{W_1}(v, \omega_1)| \leq \frac{(\pi \omega_1 + \omega_1/2)^2}{\pi (\omega_1/2)^2} \max_v |A_{t_1}^{W_1}(v', \omega_1)| \leq (2\eta + 1)^2 2^{(\log b_{\max} - \log \omega_1 + 1)}.
\]

Then, because \( \log b_{\max} + 1 \geq i_{\text{out}} \), \( 2^{(\log b_{\max} - \log \omega_1 + 2)} \geq 2 \), the probability for all nodes other than \( v \) in \( A_{t_1}^{W_1}(v, \omega_1) \) to not appear in \( U \) (i.e., to remain silent) is

\[
\text{Prob}(A_{t_1}^{W_1}(v, \omega_1) \cap U \setminus \{ v \} = \emptyset) \geq \prod_{u \in A_{t_1}^{W_1}(v, \omega_1)} (1 - \text{Prob}(U(u) = 1)) \geq \left( 1 - 2^{-((\log b_{\max} - \log \omega_1 + 2))} \right) |A_{t_1}^{W_1}(v, \omega_1)| \geq (1 - 2^{-((\log b_{\max} - \log \omega_1 + 2))}) \geq (1/4)^{(2\eta + 1)^2/2} = 2^{-2(\eta + 1)^2},
\]

3. Third, for any active node \( v \in W_1 \), we lower-bound the probability that \( v \)'s received power \( SP(U \setminus A_{t_1}^{W_1}(v, \omega_1), v) \) from outside of the ball \( B(v, \omega_1) \) is “low” — i.e., below \( \text{Thres}(\omega_1) \) — by (i) partitioning the plane into concentric rings via a similar technique to that in [4], [30], and (ii) referring to an \( (\eta \omega_1, \omega_1) \)-ruling of \( \text{Ring}(\omega_1) \). Then by noticing that (i) \( \text{Ring}(\omega_1) \subseteq \bigcup_{v' \in R(h)} A_{t_1}^{W_1}(v', \omega_1) \), and (ii) for any two nodes \( v', u' \in R(h), d(v', u') > \eta \omega_1 \), we have

\[
\mathbb{E}\{ U \cap \text{Ring}(\omega_1) \} \geq \sum_{v' \in \text{Ring}(\omega_1)} \sum_{u' \in A_{t_1}^{W_1}(v', \omega_1)} \mathbb{E}\{ U(u') \} \geq \sum_{v' \in R(h)} \sum_{u' \in A_{t_1}^{W_1}(v', \omega_1)} \frac{b_{\max}}{2^{\omega_1 - 2}} \leq 2 |R(h)|.
\]

To bound the cardinality of \( R(h) \), we use the following facts: (i) for any two nodes \( v', u' \in R(h) \), two disk centered at \( v', u' \) respectively with a radius of \( \eta \omega_1/2 \) are non-overlapping; and (ii) For any node \( v' \in \text{Ring}(\omega_1) \), such a disk is fully contained in an extended ring \( \text{Ring}'(\omega_1) \) of \( \text{Ring}(h) \), with an extra width of \( \eta \omega_1/2 \) at each side of \( \text{Ring}(h) \). Then, by referring to the ratio between the areas of \( \text{Ring}(\omega_1) \) and a disk, we have \( |R(h)| \leq 8(2h + 1) \); Inequality (4) yields \( \mathbb{E}\{ U \cap \text{Ring}(\omega_1) \} \leq 2^4(2h + 1) \).

Therefore, \( v \)'s received power from outside of the ball
Lemma B.2. In a round $i_{in}$ of phase $i_{out}$, with $t_1$ being the time that the round starts, suppose that for each active node $u \in W_1$, $|A_{t_1}^{W_1}(u, \omega_1)| \leq 2^{\log b_{max} - i_{out} + 1}$; then, for an arbitrary active node $v \in W_1$ with $|A_{t_1}^{W_1}(v, \omega_1)| \geq 2^{\log b_{max} - i_{out}}$, the probability that $v$ becomes inactive by the end of the round is at least a constant $C_0$, where $0 < C_0 = 2^{-3(3 + (2n + 1)^2)} < 1$.

Proof: In a round of phase $i_{out}$, a sufficient condition for $v$ to become inactive by the end of the round is that either $v$ or any node in $A_{t_1}^{W_1}(v, \omega_1)$ enters $\tilde{R}$, such that $v$ either enters $\tilde{R}$ or $\tilde{Z}$ and exits the algorithm. Further, that either $v$ or any node in $A_{t_1}^{W_1}(v, \omega_1)$ gets lucky satisfies this condition. Therefore, the probability for $v$ to become inactive in the round is at least

$$\sum_{v' \in A_{t_1}^{W_1}(v, \omega_1)} \text{Prob}(v' \text{ is lucky}) \geq |A_{t_1}^{W_1}(v, \omega_1)| \cdot 2^{-\left(\log b_{max} - i_{out} + 3 + (2n + 1)^2\right)} \geq 2^{-3(3 + (2n + 1)^2)}.$$

Lemma B.3. Let $E_{i_{out}}$ denote the event that at the end of the phase $i_{out}$, $|A_{t_1}^{W_1}(u, \omega_1)| \leq \frac{b_{max}}{2^{\alpha}}$ for every active node $u \in W_1$, where $t_1$ is the time that the last round in $i_{out}$ ends, $\forall i_{out}$, $\text{Prob}(E_{i_{out}}) \geq 1 - i_{out}/nC_7$, for some positive constant $C_7$.

Proof by induction: At the end of the phase $i_{out} = 0$, this is trivial. Suppose that for $i_{out} = i \geq 1$, the statement is true; we show that it still holds for $i_{out} = i + 1$. At the beginning of phase $i + 1$, if we already have event $E_{i+1}$, we are done; otherwise, let $v$ denote an active node such that $|A_{t_2}^{W_1}(v, \omega_1)| \geq \frac{b_{max}}{2^{\alpha}}$, where $t_2$ is the time that the first round in $i + 1$ starts. We show $\text{Prob}(E_{i+1}) \geq 1 - \frac{i + 1}{nC_7}$, as below:

1. We choose a constant $C_4 \geq \frac{C_3 + 1}{\log n}$ for the inner loop at Line 2 of Algorithm 2.

2. Under the induction assumption for phase $i$, we have: for any round $i_{in}$ during phase $i + 1$, with $t_3$ being the time that this round starts, if $|A_{t_1}^{W_1}(v, \omega_1)| \leq \frac{b_{max}}{2^{\alpha}}$, we are done; otherwise, as long as $\frac{b_{max} - 1}{b_{max}} \geq |A_{t_3}^{W_1}(v, \omega_1)| \geq \frac{b_{max}}{2^{\alpha}}$, the probability for $v$ to turn inactive during the round is at least $C_6$ according to Lemma B.2. Then, the probability for $v$ to become inactive by the end of phase $i + 1$ (consisting of $C_4 \log n$ rounds) is at least $1 - (1 - C_6)^{C_4 \log n} \geq 1 - \frac{1}{nC_7}$.

3. By considering both the conditional probability and the fact that there are at most $n$ such nodes as $v$,

$$\text{Prob}(E_{i+1}) \geq \text{Prob}(E_{i+1} \mid E_i) \text{Prob}(E_i) \geq (1 - \frac{n}{nC_7})(1 - \frac{i + 1}{nC_7}) \geq 1 - \frac{i + 1}{nC_7}.$$

At the end of phase $i_{out} = \log b_{max} + 1$, with $t_4$ being the time that the last round in phase $i_{out}$ ends, we have that with probability of at least $1 - \frac{1}{nC_7}$, for every active node $u \in W_1$, $|A_{t_1}^{W_1}(u, \omega_1)| \leq \frac{b_{max} - 1}{b_{max}}$, which means that all the nodes in $W_1$ have joined either $\tilde{R}$ or $\tilde{Z}$, with probability of at least $1 - \frac{1}{nC_7}$, concluding the proof of Lemma B.3.

B. Proof of Lemma B.5

Proof by contradiction. Suppose there exists a node $v \in \tilde{Z}$ that is not $\omega_2$-covered by $\tilde{R}$. Therefore, all the nodes in $\tilde{R}$ are outside of the ball $B(v, \omega_2)$. We calculate $S\tilde{P}(\tilde{R}, v)$ and derive the conflict. We notice the following three facts:

1. All the nodes in $\tilde{R}$ have a mutual distance of at least $\omega_1$;
2. The distance from any sender node in $\tilde{R}$ to $v$ is $> \omega_2$; and
3. $\omega_2 > \omega_1/2 > 0$. Due to Proposition A.1,

$$S\tilde{P}(\tilde{R}, v) \leq 2^{n \frac{\alpha - 2}{\omega_1^2 \omega_2^2} - 2} \sum_{i=0}^{\alpha - 2} \frac{P}{\alpha - 2 - \omega_1^2 \omega_2^2} \leq 2^{n \frac{\alpha - 2}{\omega_1^2 \omega_2^2} - 2} \text{Thres}(\omega_1) \leq \text{Thres}(\omega_1).$$

According to the condition for $v$ to enter $\tilde{Z}$, we must have $S\tilde{P}(\tilde{R}, v) > \text{Thres}(\omega_1)$, which lies the contradiction.

C. Proof of Lemma B.6

Suppose there is “bad” node $v \in \tilde{R}$ such that there exists a node $v' \in \tilde{R}$ and $d(v, v') < \omega_1$. We call such a $v'$ a “bad” partner of $v$. The only possible situation for $v$ to have a “bad” partner $v'$ is that $v$ and $v'$ enter $\tilde{R}$ in the same round; otherwise, one of them should have been “pushed” into $\tilde{Z}$.
during the decision step of a round when the other enters \( \hat{R} \).

Let \( u \) denote an arbitrary node in \( \hat{R} \). The necessary and sufficient condition for \( u \) to be bad (or to have a bad partner) is that, in the coordination step of the round when \( u \) enters \( \hat{R} \), there exists at least one active node \( u' \in W_1 \) such that, (1) \( d(u, u') < \omega_1 \), (2) \( u \) and \( u' \) are in \( U \), and (3) \( u \) and \( u' \) made the same random binary decisions all through the \( C_5 \log m \) slots of the coordination step.

W.l.o.g., we assume \( u \) enters \( \hat{R} \) at round \( i_{in} \) in phase \( i_{out} \). Let \( t_1 \) denote the time that round \( i_{in} \) in phase \( i_{out} \) begins. The probability that \( u \) is bad equals the probability for at least one other active node \( u' \in A_{W_1}^W(u, \omega_1) \) (i.e., \( d(v, u') < \omega_1 \)) to enter \( \hat{R} \) at round \( i_{in} \) in phase \( i_{out} \).

\[
\text{Prob}(u \text{ is bad}) \leq \sum_{u' \in A_{W_1}^W(u, \omega_1)} \text{Prob}(U(u') = 1) \frac{1}{2C_5 \log n}
\]

\[
\leq \frac{1}{n C_5} \frac{|A_{W_1}^W(u, \omega_1)|}{2 \log b_{max} - i_{out} + 2}
\]

\[
\leq \frac{1}{n C_5} \frac{\max_{w \in W_1} |A_{W_1}^W(w, \omega_1)|}{2 \log b_{max} - i_{out} + 2}
\]

Further, due to Lemma [B.3]

\[
\text{Prob}(\max_{w \in W_1} |A_{W_1}^W(w, \omega_1)| \leq 2 \log b_{max} - i_{out} + 1) \geq 1 - \frac{i_{out} - 1}{n C_7} \geq 1 - \frac{1}{n C_7 - 1}.
\]

Therefore,

\[
\text{Prob}(u \text{ is good}) \geq (1 - \frac{1}{n C_5})(1 - \frac{1}{n C_7 - 1})
\]

\[
\geq 1 - \frac{2}{n \min\{C_5, C_7 - 1\}}.
\]

Finally, since \( \hat{R} \) contains at most \( n \) nodes, the probability that there are no bad nodes in \( \hat{R} \) is at least \( 1 - \frac{1}{n \min\{C_5, C_7 - 1\}} \).