HOMOTOPY PERTURBATION METHOD AND CHEBYSHEV POLYNOMIALS FOR SOLVING A CLASS OF SINGULAR AND HYPERSINGULAR INTEGRAL EQUATIONS

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ABSTRACT. In this note, we review homotopy perturbation method (HPM), Discrete HPM, Chebyshev polynomials and its properties. Moreover, the convergences of HPM and error term of Chebyshev polynomials were discussed. Then, linear singular integral equations (SIEs) and hyper-singular integral equations (HSIEs) are solved by combining modified HPM together with Chebyshev polynomials. Convergences of the mixed method for the linear HSIEs are also obtained. Finally, illustrative examples and comparisons with different methods are presented.

1. Introduction. Integral equations occur naturally in many fields of science and engineering. A computational approach to solve integral equation is an essential work in scientific research. One of the most valuable mathematical tools in solving singular and hypersingular integral equations is the subject of singular and hypersingular integral operator in both pure and applied mathematics. A precise evaluation of singular integrals (SIs) and hypersingular integrals (HSIs) is only possible in rare cases. Therefore there is a need to enrich the approximate methods for evaluating it. By the implementation of various techniques, HSIs can be transformed into singular or weakly singular integrals whereas another group is based on the numerical computation of finite part integrals by various quadrature or cubature formulas. Singular integral equations (SIEs) and Hyper-singular integral equations (HSIEs) comes from a variety of mixed boundary value problems in mathematical physics such as solid mechanics (Chan et al. [6], electrodynamic (Davydov et al. [8], elasticity (Nik long and Eshkuvatov [35]), water wave scattering (Kanoria and Mandal [28]), radiation problems involving thin submerged plates (Parsons and Martin [38]) and fracture mechanics as well as aerodynamics (Lifanov [29] - [30]).

The numerical techniques can generally be applied to nonlinear problems in complicated computation domain. This is an obvious advantage of numerical methods over analytic ones that often handle nonlinear problems in simple domains. However, numerical methods give discontinuous points of a curve and thus it is often

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costly and time consuming to get a complete curve of results. Besides that numerical difficulties additionally appear if a nonlinear problem contains singularities or has multiple solutions. The numerical and analytic methods of nonlinear problems have their own advantages and limitations, and thus it is unnecessary for us to do one thing and neglect another. Generally, one delights in giving analytic solutions of nonlinear problems. There are some analytic techniques for nonlinear problems, such as perturbation techniques (see [7], [9], [36], [37] and literature cited there in for more details) that are well known and widely applied. By means of perturbation techniques, a lot of important properties and interesting phenomena of nonlinear problems have been revealed. One of the astonishing successes of perturbation techniques is the discovery of the ninth planet in the solar system, found in the vast sky at a predicted point. Perturbation techniques are essentially based on the existence of small or large parameters or variables called perturbation quantity. The existence of perturbation quantities is obviously a cornerstone of perturbation techniques.

HPM was proposed by Ji-Huan He in 1999 ([19]-[24]) and is the combination of two methods: the homotopy and the perturbation. The homotopy technique or continues mapping techniques, embeds a parameter \( p \) that typically ranges from zero to one \([0,1]\) and can be considered as small parameter. The changing process of \( p \) from zero to unity leads the perturbation \( H(v,p) \) varies from the initial solution to the original solution and in topology it is called deformation. HPM has been used for a wide range of problems; for finding the exact and approximate solutions of nonlinear ordinary differential equations (ODEs) (Ramos [39]), linear integral equations ([26], [43]), the integro-differential equations ([15], [10], [25]) and nonlinear integral equations ([16], [17], [18]). HPM has mainly two types of modifications ([17], [18]) and in the recent decades, modified homotopy perturbation method (MHPM) has been used to solve many types of integral equations including SIEs and HSIEs ([42],[13], [45], [31]).

Whenever HPM is applied to the integral or integro-differential equation (linear or nonlinear) of Fredholm or Volterra types many definite integrals need to be computed. In many cases the evaluation of integrals analytically is impossible, Due to such obstacle, Behiry et al. [5] introduced a discretized version of the ADM namely Discrete Adomian Decomposition method (DADM) for solving nonlinear Fredholm integral equations. In 2011, Allahviranloo and Ghanbari [3] have introduced Discrete homotopy analysis method (DHAM) and it arises when the quadrature rules are used to approximate the definite integrals which cannot be computed analytically. This method gives the numerical solution at nodes used in the quadrature rules. Similar technique can be used for HPM and can be named Discrete HPM.

Chebyshev polynomials are special group of polynomials whose properties and applications were discovered by the Russian mathematician Pafnuty Lvovich Chebyshev (He was born in Saint Petersburg, Russia, 1821-1894, graduated Moscow State University). Mason and Handscommb [34], collected authentic and highly regarded sources of Chebyshev polynomials. These polynomials play significant role in nearly every area of numerical analysis, including polynomial approximation, numerical integration, integral equations and spectral methods for partial differential equations.

Main aim of this note is to analysis homotopy perturbation method (HPM) and its applications in different problems of integral equations as well as Chebyshev Polynomials and its applications in SIEs and HSIEs.
2. Chebyshev Polynomials and its Applications. There are four kind of Chebyshev polynomials (Shahmorad and Ahdiaghdam [40])

\[ P_{\nu,n}(x) = \begin{cases} 
T_n(x) = \cos(n\theta), & x = \cos \theta, \ n = 0,1, \ldots \ \nu = 1, \\
U_n(x) = \frac{\sin((n+1)\theta)}{\sin \theta}, & x = \cos \theta, \ n = 0,1, \ldots, \ \nu = 2, \\
V_n(x) = \frac{\cos((n+1)\theta)}{\cos(\theta/2)}, & x = \cos \theta, \ n = 0,1, \ldots, \ \nu = 3, \\
W_n(x) = \frac{\sin((n+1/2)\theta)}{\sin(\theta/2)}, & x = \cos \theta, \ n = 0,1, \ldots, \ \nu = 4. 
\end{cases} \]  

(1)

The roots of Chebyshev polynomials are given by

\[ x_{\nu,k} = \begin{cases} 
\cos \left( \frac{(2k-1)\pi}{2(n+1)} \right), & k = 1, \ldots, n+1, \ \nu = 1, \\
\cos \left( \frac{k\pi}{n+2} \right), & k = 1, \ldots, n+1, \ \nu = 2, \\
\cos \left( \frac{(2k-1)\pi}{2n+3} \right), & k = 1, \ldots, n+1, \ \nu = 3, \\
\cos \left( \frac{2k\pi}{2n+3} \right), & k = 1, \ldots, n+1, \ \nu = 4. 
\end{cases} \]  

(2)

In Mason and Handscommb [34] shown that all 4 types of Chebyshev polynomials are orthogonal with corresponding weights.

Lemma 2.1. The Chebyshev polynomials satisfy the orthogonally conditions

\[ \int_{-1}^{1} \frac{\lambda_{\nu}(t)}{\sqrt{1-t^2}} P_{\nu,i}(t) P_{\nu,j}(t) dt = \begin{cases} 
0, & i \neq j, \ 
\pi, & i = j = 0, \ \nu = 1, \\
\pi/2, & i = j \neq 0, \ \nu = 1, \\
\pi/2, & i = j, \ \nu = 2, \\
\pi, & i = j \neq 0, \ \nu = \{3,4\}. 
\end{cases} \]  

(3)

where

\[ \lambda_{\nu}(t) = \begin{cases} 
1, & \nu = 1, \\
1-t^2, & \nu = 2, \\
1+t, & \nu = 3, \\
1-t, & \nu = 4. 
\end{cases} \]

All Chebyshev polynomials have degree of \( n \) and the sequence of Chebyshev polynomials are important in approximation theory. Roots of Chebyshev polynomials are called Chebyshev nodes and are used in polynomial interpolation. The resulting interpolation polynomial provides an approximation that is close to the polynomials of the best approximation to a continuous function under the maximum norm. Chebyshev polynomials form a special class of polynomials which suited for approximation of functions.

Let us consider mini-max approximation for the case of Chebyshev polynomials of the first kind. We say that \( 2^{1-n}T_n(x) \) is a monic polynomial, namely a polynomial with unit leading coefficient. The following two lemmas are of the alternation theorem.
Lemma 2.2. (Mason and Handscomb [34]): The minimax polynomial approximation of degree \( n - 1 \) to the function \( f(x) = x^n \) on \([-1, 1]\) is
\[
p_{n-1}(x) = x^n - 2^{1-n}T_n(x).
\]

Lemma 2.3. (Mason and Handscomb [34]): The polynomial \( \hat{T}_n(x) = 2^{1-n}T_n(x) \) is the mini-max approximation on \([-1, 1]\) to the zero function by a monic polynomial of degree \( n \) and
\[
\| \hat{T}_n \| = 2^{1-n}.
\]

Theorem 2.4. (Mason and Handscomb [34]): If \( f(x) \) is continuous and either of bounded variation or satisfying a Dini-Lipschitz condition on \([-1, 1]\), then its Chebyshev series expansion is uniformly convergent.

A function \( g(x) \) satisfies Dini-Lipschitz condition if
\[
|g(\theta + \delta) - g(\theta)| \leq w(\delta),
\]
where \( w(\delta) \) is a modulus of continuity with \( w(\delta)\log(\delta) \to 0 \) as \( \delta \to 0 \).

Theorem 2.5. (Mason and Handscomb [34]): If the function \( f(x) \) has \( m + 1 \) continuous derivatives on \([-1, 1]\), then
\[
|f(x) - S^T_n(x)| \leq O(n^{-m}),
\]
for all \( x \in [-1, 1] \), where
\[
S^T_n(x) = \sum_{k=0}^{n} c_k T_k(x).
\]

Theorem 2.6. (Mason and Handscomb [34]) (Lagrange interpolation): Given a function \( f \) that is defined at \( n+1 \) points \( a = x_0 < x_1 < \ldots < x_n = b \) in \([a, b]\) there exists a unique polynomial of degree smaller than or equal to \( n \) such that
\[
P_n(x_i) = f(x_i), \quad i = 0, \ldots, n
\]
This polynomial is given by
\[
P_n(x) = \sum_{i=0}^{n} f(x_i) L_i(x),
\]
where \( L_i(x) \) is defined by
\[
L_i(x) = \frac{\pi_{n+1}(x)}{(x - x_i) \pi_{n+1}(x_i)} = \frac{\prod_{j=0, j\neq i}^{n} (x - x_j)}{\prod_{j=0, j\neq i}^{n} (x_i - x_j)}, \quad \pi_{n+1}(x) = \prod_{j=0}^{n} (x - x_j).
\]
Additionally, if \( f \) is continuous on \([a, b]\) and \( n+1 \) times differentiable in \((a, b)\), then for any \( x \in [a, b]\) there exists a value \( \xi_x \in (a, b) \), depending on \( x \), such that
\[
R_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \pi_{n+1}(x).
\]
By considering the estimation for Lagrange interpolation error (Theorem 2.6) and mini-max property of \( \hat{T}_n(x) \) (Lemma 2.3) the following theorem can be proved.

Theorem 2.7. (Mason and Handscomb [34]): Under condition Theorem 2.2 and mini-max property of \( \hat{T}_n(x) \) the following error bound is true
\[
|R_n(x)| = \frac{|f^{(n+1)}(\xi_x)|}{(n+1)!} \| \hat{T}_n(x) \| \leq \frac{1}{2^n (n+1)!} \| f^{(n+1)} \|.
\]
Chebyshev polynomials are used not only in polynomial approximations and in evaluation of singular and hypersingular integral operators of the form

\[ I_\alpha(T_n, m, r) = \int_{-1}^{1} \frac{T_n(s) (1 - s^2)^{m-1/2}}{(s - r)^\alpha} ds, \quad |r| < 1, \]

\[ I_\alpha(U_n, m, r) = \int_{-1}^{1} \frac{U_n(s) (1 - s^2)^{m-1/2}}{(s - r)^\alpha} ds, \quad |r| < 1, \]

where \( T_n(s) \) and \( U_n(s) \) are the Chebyshev polynomials of the first and second kinds, respectively.

In 2003, Chan et al. [6], found the exact solution of the Eq. (5) for the cases \( \alpha = \{1, 2, 3, 4\} \) and \( m = \{0, 1, 2, 3\} \). Moreover, a systematic approaches for evaluating these integrals when \( \alpha > 4 \) and \( m > 3 \) is provided. The integrals are also evaluated for \( |r| > 1 \) in order to calculate the stress intensity factors. Chebyshev polynomials are widely used in many areas of numerical analysis for instance approximate solution of a system of singular integral equations (Shahmorad and Ahdiaghdam [40]), numerical solution of nonlinear Volterra integral equations (Maleknejad et al. [32]), nonlinear boundary value problems (Shaban et al. [41]), Fredhol-Folterra type integro-differential equation (general and system respectively) (Daschigol [11] - [12]), singular and hypersingular integral equations (Eshkuvatov et al. [14], and Abdulkawi [1]) and so on.

3. The Homotopy Perturbation Method (HPM). To illustrate the basic ideas of the homotopy perturbation method, Ji-Huan He [19] consider the following nonlinear differential equation

\[ A(\bar{U}) = f(\bar{r}), \quad \bar{r} \in \Omega, \]

with boundary conditions

\[ B\left(\bar{U}, \frac{\partial \bar{U}}{\partial n}\right) = 0, \quad \bar{r} \in \Gamma, \]

where \( A \) is a differential operator, \( B \) is a boundary operator, \( f(\bar{r}) \) is a known analytic function, and \( \Gamma \) is the boundary of the domain \( \Omega \).

Generally speaking the operator \( A \) can be divided into two parts \( L \) and \( N \), where \( L \) is a linear, and \( N \) is a nonlinear operator. Then Eq. (6), can be rewritten as follows

\[ L(\bar{U}) + N(\bar{U}) = f(\bar{r}). \]

We construct a homotopy \( \bar{V}(\bar{r}, p) : \Omega \times [0, 1] \rightarrow \mathbb{R} \), which satisfies

\[ H(\bar{V}, p) = (1 - p) \left[ L(\bar{V}) - L(\bar{U}_0) \right] + p \left[ A(\bar{V}) - f(\bar{r}) \right] = 0, \quad p \in [0, 1], \quad \bar{r} \in \Omega \]

or equivalently

\[ H(\bar{V}, p) = L(\bar{V}) - L(\bar{U}_0) + pL(\bar{U}_0) + p \left[ N(\bar{V}) - f(\bar{r}) \right] = 0, \]

where \( p \in [0, 1] \) is an embedding parameter and \( \bar{U}_0 \) is an initial approximation of Eq. (6) which satisfies the boundary conditions (7). Clearly, from Eq. (9) we have

\[ H(\bar{V}, 0) = L(\bar{V}) - L(\bar{U}_0) = 0, \]

\[ H(\bar{V}, 1) = A(\bar{V}) - f(\bar{r}) = 0. \]

\[ \frac{\partial \bar{U}}{\partial n} = 0, \quad \bar{r} \in \Gamma. \]
The changing process of \(\bar{V}(\bar{r},p)\) changing from zero to unity is just that of \(\bar{U}_0(\bar{r})\) to \(\bar{U}(\bar{r})\). In topology, this is called deformation, and \(L(\bar{V}) - L(\bar{U}_0)\), \(A(\bar{V}) - f(\bar{r})\) are called homotopic. If embedding parameter \(p, (0 \leq p \leq 1)\) is considered as a small parameter, then we can naturally assume that the solution of Eq (10) can be given as a power series in \(p\) for \(\bar{V}\), i.e.,

\[
\bar{V} = \bar{V}_0 + p\bar{V}_1 + p^2\bar{V}_2 + \ldots = \sum_{m=0}^{\infty} p^m\bar{V}_m. \tag{11}
\]

Setting \(p = 1\) results in the approximate solution of Eq. (6):

\[
U = \lim_{p \to 1} \bar{V} = \bar{V}_0 + \bar{V}_1 + \bar{V}_2 + \ldots = \sum_{m=0}^{\infty} \bar{V}_m.
\]

The coupling of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated the limitations of the traditional perturbation methods. On the other hand, this technique can have full advantage of the traditional perturbation techniques. The series (11) is convergent for most cases, however the convergent rate depends on the non-linear operator of \(A(\bar{U})\), the following opinions are suggested by Ji-Huan He [19]:

1. The second derivative of \(N(\bar{U})\) with respect to \(\bar{U}\) must be small because the parameter \(p\) may be relatively large, i.e., \(p \to 1\).
2. The norm of \(L^{-1}\frac{\partial N}{\partial \bar{V}}\) must be smaller than one, so that the series (11) converges.

In 2009, Biazar and Aminikhah [4] continue the study of convergence of HPM and establish the following theorem

**Theorem 3.1. (Sufficient Condition of Convergence):** Suppose that \(X\) and \(Y\) are Banach spaces and \(N : X \to Y\) is a contractive nonlinear mapping, that is

\[
\forall \bar{w}, \bar{w}^* \in X : \|N(\bar{w}) - N(\bar{w}^*)\| \leq \gamma \|\bar{w} - \bar{w}^*\|, \ 0 < \gamma < 1.
\]

Then according to Banach’s fixed point theorem \(N\) has a unique fixed point, that is \(N(\bar{u}) = \bar{u}\). Assume that the sequence generated by homotopy perturbation method can be written as

\[
\bar{W}_n = N(\bar{W}_{n-1}), \ \bar{W}_{n-1} = \sum_{i=0}^{n-1} \bar{w}_i, \ n = 1, 2, \ldots
\]

and suppose that \(\bar{W}_0 = \bar{w}_0 \in B_r(\bar{w})\) where \(B_r(\bar{w}) = \{\bar{w}^* \in X : \|\bar{w}^* - \bar{w}\| < r\}\), then we have

\[
(i) \ W_n \in B_r(\bar{w}), \ (ii) \lim_{n \to \infty} \bar{W}_n = \bar{w}.
\]

4. **Modified Homotopy Perturbation Method (MHPM).** To improve the efficiency of the HPM, a few modifications have been made by many researches. For instance, Javidi and Golbabai [42] added the accelerating parameter named \(M\) to the perturbation equation for obtaining the approximate solution for nonlinear Fredholm integral equation of the second kind. This extra parameter \(M\) is defined such a way that second iteration \(U_2 = 0\). For the expention of the nonlinear term they have used Adomian polynomials. Ghorbani and Saberi-Nadjafi [26] added a series of parameter and selective functions to HPM to find the semi-analytical
solutions of nonlinear Fredholm and nonlinear Volterra integral equations of the
second kind. Let us describe the modification of HPM more in details.
Consider the nonlinear Volterra integral equation of the form
\[ y(x) = g(x) + \int_a^x k(x, t) [y(t)]^r \, dt, \quad a \leq x, t \leq b. \]  
(12)
To explain the HPM, let us rewrite Eq. (12) as
\[ L(u) = u(x) - g(x) - \int_a^x k(x, t) [u(t)]^r \, dt = 0, \]  
(13)
with the solution \( u(x) = y(x) \) and define homotopy \( H(u, p) \) as follows
\[ H(u, p) = (1 - p) F(u) + pL(u) = 0, \quad p \in [0, 1], \]  
(14)
where \( F(u) \) is a functional operator with solution, say \( u_0 \), which can be obtained
easily.
When embedding parameter \( p \) monotonically increase from zero to one then
the trivial problem \( F(u) = 0 \) is continuously deformed to the original problem
\( L(u) = 0 \), i.e.
\[ H(u, 0) = F(u), \quad H(u, 1) = L(u). \]
Searching the solution of Eq. (14) in the series form with embedding parameter
\[ u = \sum_{n=0}^{\infty} p^n u_n \]  
(15)
and substituting (15) into (14) and equating the terms with identical powers of \( p \),
we obtain
\[ u_0 (x) = g(x), \]
\[ u_{n+1} (x) = \int_a^x k(x, t) H_n (t) \, dt, \quad n = 0, 1, 2, ..., \]  
(16)
where the \( H_n \)'s are so called He’s polynomial [17], which can be calculated as follows
\[ H_n (u_0, u_1, ..., u_n) = \frac{1}{n!} \frac{\partial^n}{\partial p^n} \left[ \left( \sum_{k=0}^{n} p^k u_k \right) \right]_{p=0}, \quad n = 0, 1, 2, ... \]
This is standard convex HPM which leads to a rapid convergence and sometimes
gives exact results.
For the modified HPM we do the following changes. Let
\[ F(u) = u(x) - \sum_{m=0}^{N} \alpha_m v_m (x) \]
and homotopy perturbation is chosen as
\[ H(u, p) = u(x) - pg(x) + (p - 1) \sum_{m=0}^{N} \alpha_m v_m (x) \]
\[ -p \int_a^x k(x, t) [u(t)]^r \, dt = 0, \]  
(17)
where $\alpha = [\alpha_m]_{m=0}^{N}$ are called the accelerating components of the parameter and $v = [v_m]_{m=0}^{N}$ are selective functions. It is easy to verify that the solutions

$$H(u_0, 0) = F(u_0) = u_0 - \sum_{m=0}^{N} \alpha_m v_m(x),$$

and $H(u, 1) = L(u)$ also satisfies Eq. (12).

By using modified HPM (17) they (Ghorbani and Saberi-Nadjafi [18]) were able to solve many nonlinear integral equations which have more than one solutions. In 2009, Zaid [44], has introduced efficient modification of the homotopy perturbation method that will facilitate the calculations. Then, a comparative study between the new modification and the homotopy perturbation method were carried out. Numerical examples are investigated to show the features of the technique. The modified method accelerates the rapid convergence of the series solution and reduces the size of work. Let us describe the new modifications a bit more in detail.

Let us rewrite the Eq. (12) in the form

$$L(v) + N(v) - g(x) = 0, \quad x \in \Omega,$$

where $L$ is a linear operator and $N$ is nonlinear operator.

In the new modification, the modified form of the HPM can be established based on the assumption that the function $g(x)$ in Eq. (12) can be divided into two parts, namely $g_0(x)$ and $g_1(x)$

$$g(x) = g_0(x) + g_1(x),$$

and construct homotopy $v(x, p) : \Omega \times [a, b] \to \mathbb{R}$ in the form

$$H(v, p) = (1 - p) [L(v) - L(u_0)] + p [L(v) + N(v) - g_1(x)] = g_0(x).$$

If the function $g(x)$ can be replaced by a series of infinite components, then it can be expressed in Taylor series

$$g(x) = \sum_{n=0}^{\infty} g_n(x).$$

Under this assumption they have constructed the homotopy function $v(x, p) : \Omega \times [a, b] \to \mathbb{R}$ which satisfies

$$H(v, p) = (1 - p) [L(v) - L(u_0)] + p [L(v) + N(v)] = \sum_{n=0}^{\infty} p^n g_n(x).$$

Obviously, from Eqs. (19) and (21), we have

$$H(v, 0) = L(v) - L(u_0) = g_0,$$

$$H(v, 1) = L(v) + N(v) - g(x) = 0.$$

The changing process of $p$ from zero to unity is just that of $v(x, p)$ from $u_0(x)$ to $u(x)$. In topology, this called deformation, $L(v) - L(u_0)$ and $L(v) + N(v) - g(x)$ are homo-topic as the effectiveness of MHPM, the author has shown many examples.

In 2016, Eshkuvatov et al. [13] were able successfully implemented the modified HPM together with Chebyshev polynomials for the following HSIEs of the first kind.

$$\frac{1}{\pi} \int_{-1}^{1} \frac{K(x, t)}{(x-t)^2} \varphi(t) \, dt + \int_{-1}^{1} L_1(x, t) \varphi(t) \, dt = f(x), \quad -1 < x < 1,$$
where \( K(x,t) \) is constant on the diagonal of the region

\[
D = \{ (s,t) \in \mathbb{R}^2 \mid -1 \leq x, t \leq 1 \}
\]

i.e. kernel has the form

\[
K(x,t) = c_0 + (t-x)(Q(x) + Q_1(x,t)).
\]

To find bounded solution of Eq. (22) we search solution in the form \( \varphi(x) = \sqrt{1-x^2}u(t) \), and rewrite the Eq. (22) as follows

\[
Hu + Cu + Lu = f,
\]

(24)

where

\[
Hu = \frac{c_0}{\pi} \int_{-1}^{1} \sqrt{1-t^2}u(t) \, dt,
\]

\[
Cu = \frac{Q(x)}{\pi} \int_{-1}^{1} \sqrt{1-t^2}u(t) \, dt,
\]

\[
Lu = \frac{1}{\pi} \int_{-1}^{1} \sqrt{1-t^2}L(x,t)u(t) \, dt.
\]

For the convergence of the HPM let us rewrite the Eq. (24) in the form

\[
Hu + \lambda(C + L)u = f,
\]

(25)

and search solution of (25) in the series form

\[
v(x) = \sum_{k=0}^{\infty} p^k v_k(x).
\]

(26)

Standard HPM for Eq. (25) has the scheme

\[
v_0 = H^{-1}(u_0),
\]

\[
v_1 = H^{-1}(f - u_0 - \lambda(C + L)v_0)
\]

\[
v_k = H^{-1}(-\lambda(C + L)v_{k-1}), \quad k \geq 2.
\]

(27)

For the scheme of MHPM, let us rewrite (24) as equivalent form

\[
\tilde{S}u + \tilde{L}u = f,
\]

(28)

where

\[
S = H + C + L, \quad \tilde{S} = H + C + L_n, \quad \tilde{L} = L - L_n,
\]

with

\[
\|S - \tilde{S}\| \xrightarrow{n \to \infty} 0, \quad \|\tilde{L}\| = \|L - L_n\| \xrightarrow{n \to \infty} 0.
\]

Computing scheme for the MHPM is

\[
\tilde{S}v_0 = \sum_{j=0}^{m} \alpha_j g_j(x),
\]

\[
\tilde{S}v_1 = -\tilde{L}(v_0) + f - \sum_{j=0}^{m} \alpha_j g_j(x),
\]

\[
\tilde{S}v_k = -\tilde{L}(v_{k-1}), \quad k \geq 2.
\]

(29)

So, under these assumption we have proved the following theorems
Remark 1. Theorems 4.1 shows the fact that the HPM is convergent for all \( x \in \mathbb{R} \) if \( |\lambda| (C + L) \) is a continuous function, then the series \( \sum_{n=0}^{\infty} a_n \) is norm convergent to the exact solution \( u(x) \) on the interval \([-1, 1]\) for each \( p \in [-1, 1] \).

Theorem 4.2. (Eshkuvatov et al. [13]) Let \( K(x, t) \) be defined by (23) and \( Q_1(x, t), L(x, t) \in C(D) \) and \( f \in C[-1, 1] \) be continuous functions. In addition, let \( v_1(x) \) be generated by (27) and the following inequality

\[ |\lambda| (C + L) \leq c_0, \]

holds and initial guess \( u_0(t) \) is chosen as a continuous function, then the series \( \sum_{n=0}^{\infty} a_n \) is norm convergent to the exact solution \( u(t) \) on the interval \([-1, 1]\) for each \( p \in [-1, 1] \).

\[ \gamma = \|S^{-1}\| \|L\| < 1, \]

holds and selective functions \( g_j(x) \), \( j = 0, 1, \ldots, n \) are chosen as Chebyshev polynomials and initial guess \( u_0(t) \) is chosen as a linear combination of selective function, then the series \( \sum_{n=0}^{\infty} a_n \) is norm convergent to the exact solution \( u(t) \) on the interval \([-1, 1]\) for each \( p \in [-1, 1] \).

Remark 1. Theorems 4.1 shows the fact that the HPM is convergent for all \( x \in (-1, 1) \) if \( |\lambda| (C + L) \leq c_0 \) holds i.e. artificial parameter \( \lambda \) should be chosen as much as small. From Theorems 4.2 we can see MHPM is convergent for any \( x \in (-1, 1) \).

In 2016, Zulkarnain et al. [45] consider HSIEs of the second kind

\[ \varphi(x) - \frac{1}{\pi} \int_{-1}^{1} K(x, t) \varphi(t) dt - \frac{1}{\pi} \int_{-1}^{1} L_1(x, t) \varphi(t) dt = f(x), \quad -1 < x < 1, \quad (30) \]

where \( \varphi(x) \) is the unknown function of \( x \) to be determined, \( K(x, t) \) and \( L_1(x, t) \) are the square integrable kernels on \( D = \{(s, t) \in \mathbb{R}^2 \} : -1 \leq s, t \leq 1 \}. Assume that \( K(x, t) \) is constant on the diagonal of the region (see Eq. (23)). To solve Eq. (30) mixed method (MHPM and Chebyshev polynomial approximation) are used and as an effectiveness of the method few examples are shown. Look at Examples 5.4 and 5.5.

5. Numerical Examples. In this section we consider five examples.

Example 5.1 (Mandal and Bhattacharya [33]). Consider HSIE (22) of the form

\[ \frac{1}{\pi} \int_{-1}^{1} \frac{\varphi(t) dt}{(x-t)^2} = 1, \quad -1 < x < 1. \quad (31) \]

The exact solution of Eq. (31) is \( \varphi(x) = \sqrt{1-x^2} \) and \( c_0 = 1, f(x) = 1 \). In Eshkuvatov et al. [13] it is shown that both HPM and MHPM give identical solution. In Table 1 comparisons of three methods are given.

Example 5.2 (Abdulkawi et al. [2]). Consider HSIE of the form

\[ \frac{1}{\pi} \int_{-1}^{1} \frac{\varphi(t) dt}{(x-t)^2} + \frac{1}{\pi} \int_{-1}^{1} (\sin x) t^4 \varphi(t) dt = -5(16x^4 - 12x^2 + 1) - \frac{\sin x}{32}, \quad -1 < x < 1, \quad (32) \]

with exact solution \( \varphi(x) = \sqrt{1-x^2} (16x^4 - 12x^2 + 1) \).

For this example condition of Theorem 4.1 is not hold therefore we do comparison between modified HPM and method given in Abdulkawi et al. [2].
Table 1. Comparisons with other methods

| $x$  | Error term in [45] | Error of HPM | Error of MHPM in [28] |
|------|------------------|--------------|-----------------------|
| -1   | 0                | 0            | 0                     |
| -0.5 | $13 \times 10^{-17}$ | 0            | 0                     |
| 0    | 0                | 0            | 0                     |
| 0.5  | $7.8 \times 10^{-18}$ | 0            | 0                     |
| 1    | 0                | 0            | 0                     |

Table 2. Comparisons with other methods

| $x$  | Error term in [45] | Error of MHPM [28] |
|------|------------------|--------------------|
| -1.0 | 0.0              | 0                  |
| -0.8 | $2.1 \times 10^{-10}$ | 0                  |
| -0.4 | $3.4 \times 10^{-9}$ | 0                  |
| 0.0  | $2.0 \times 10^{-9}$ | 0                  |
| 0.4  | $3.3 \times 10^{-9}$ | 0                  |
| 0.8  | $2.7 \times 10^{-9}$ | 0                  |
| 1.0  | 0.0              | 0                  |

Example 5.3  (Eshkuvatov et al. [13]): Let us consider HSIEs of the form

$\frac{1}{\pi} \int_{-1}^{1} \frac{2 + tx(t-x)}{(x-t)^2} \varphi(t) dt + \frac{1}{\pi} \int_{-1}^{1} \left( \frac{1}{t+2} + \frac{1}{x+2} \right) \varphi(t) dt = f(x), \quad (33)$

where

$f(x) = -\frac{20\sqrt{3}}{x^2 + 2} - \frac{10x^2}{x+2} \left( 2 - \sqrt{3} + x \right) + 10 \left( 2 - \sqrt{3} \right) x$

$+ \frac{10}{3} \left( 2\sqrt{3} - 3 \right) + \frac{10\left( 2 - \sqrt{3} \right)}{x+2}.$

Exact solution is $\varphi(x) = \sqrt{1-x^2} \frac{10}{x^2 + 2}$.

Solution: Standard HPM is not suitable for solving this equation as it is not satisfies the conditions in Theorem 4.1. For the modified HPM, we choose the selective functions $g_j(x) = \phi_j(x)$, $j = 0, 1, \ldots, m$, where $\phi_j(x) = \sqrt{\frac{2}{\pi}} T_j(x)$ is the orthonormal Chebyshev polynomials of the first kind. Numerical results is given in TABLE 3.

Example 5.4: (Zulkarnain et al. [45]): Consider HSIE (30) of the form

$\varphi(x) - \frac{1}{\pi} \int_{-1}^{1} \frac{2/5 + x(2t + t^4) + 2x^2(1 - t^3) + x^3t^2}{(x-t)^2} \varphi(t) dt$

$- \frac{1}{2\pi} \int_{-1}^{1} (3x + t) \varphi(t) dt = f(x), \quad (34)$

where $f(x) = 24x^4 + \sqrt{1-x^2} (12x^2 + 4x - 3)$.
Table 3. Error terms for different value of $n$

| $x$    | Exact Solution | Error MHPM for $m=n=6$ | Error MHPM for $m=n=10$ |
|--------|----------------|------------------------|------------------------|
| -0.9999 | 0.14140368029  | 1.0851729 $10^{-4}$    | 1.4040446 $10^{-10}$   |
| -0.901  | 3.94739842327  | 3.5501594 $10^{-4}$    | 1.8203460 $10^{-9}$    |
| -0.436  | 5.75413468725  | 3.0648319 $10^{-4}$    | 2.6221447 $10^{-8}$    |
| -0.015  | 5.03721659280  | 8.6784464 $10^{-5}$    | 1.7385004 $10^{-8}$    |
| 0.015   | 4.96222081226  | 1.9992451 $10^{-5}$    | 1.8524990 $10^{-8}$    |
| 0.436   | 3.69436233615  | 3.3916634 $10^{-1}$    | 1.9700974 $10^{-8}$    |
| 0.901   | 1.4954122584   | 1.2745747 $10^{-4}$    | 1.3403888 $10^{-8}$    |
| 0.9999  | 0.04714084491  | 3.3327100 $10^{-5}$    | 3.5400390 $10^{-10}$   |

The exact solution of Eq. (34) is $\varphi(x) = \sqrt{1-x^2}(12x^2 + 4x - 3)$. In Zulkarnain et al. [29] it is shown that MHPM gives identical solution.

Example 5.5: (Zulkarnain et al. [45]): Consider HSIE of the second kind in the form

$$\varphi(x) - 2\pi \sqrt{1-x^2} + \frac{\pi}{2} \sqrt{1-x^2} \int_{-1}^{1} \frac{\varphi(t)}{(x-t)^2} dt,$$  \hspace{1cm} (35)

with exact solution $\varphi(x) = \sqrt{1-x^2} \frac{5\pi}{\pi + 2}$.

In Zulkarnain et al. [45], by applying the MHPM with selective function $g_i(x) = t^j$, $j = 0, 1, ..., m$ it is found that the approximate solution is

$$\varphi_n(x) = \sqrt{1-x^2} \frac{2\pi^{3/2}}{\pi + 2}.$$

Thus, maximum error between exact and approximate solution is attained at $x = 0$ i.e.

$$\|(\varphi(x) - \varphi_n(x))\| = \frac{1}{\pi + 2} \left| 5\pi - 2\pi^{3/2} \sqrt{2} \right| \equiv 0.008099931.$$

However, whenever selective functions $g_i(x)$ are chosen as Chebyshev polynomials $g_i(x) = \phi_j(x) = \sqrt{2} T_j(x)$, $j = 0, 1, ..., m$ then approximate solution coincides with the exact solution.

6. Conclusion. In this work, the standard and modified HPM together with Chebyshev polynomials are used to find the approximate solution of the first and second kind of HSIEs. The approximate solutions obtained by the HPM are compared with exact solutions. The theoretical aspect supported by the numerical examples. It is shown that the modified HPM gives better approximation than the standard HPM. Examples shows that the modified HPM ables to handle the problem that can not be solved by standard HPM. Modified HPM is effective and reliable method for solving HSIE of the first and second kind. It can be concluded that the modified He’s homotopy perturbation method together with Chebyshev polynomials is very powerful and efficient technique in finding exact and approximate solutions for wide classes of SIEs and HSIEs.
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