Abstract

For a function field $K$ and fixed polynomial $F \in K[x]$ and varying $f \in F$ (under certain restrictions) we give a lower bound for the degree of the greatest prime divisor of $F(f)$ in terms of the height of $f$, establishing a strong result for the function field analogue of a classical problem in number theory.

1 Introduction

Let $F \in \mathbb{Z}[x]$ ($\mathbb{Z}$ denotes the ring of integers) be a fixed polynomial with integer coefficients, degree $\deg F \geq 2$ and with distinct roots (over the complex numbers). For $N \in \mathbb{Z}, N > 1$ denote by $\mathcal{P}(N)$ the largest prime factor of $N$. The problem of giving a lower bound for $\mathcal{P}(F(n))$ in terms of $n$ as $n \to \infty$ has been much studied. Pólya [9] proved that $\mathcal{P}(F(n)) \to \infty$ as $n \to \infty$ for the case $\deg F = 2$ and the general case can be deduced from Siegel’s theorem on the finiteness of integer points on curves with positive genus. Keates [3] proved a bound of the form

$$\mathcal{P}(F(n)) \gg_F \log \log n$$

(the implicit constant depending on $F$) for $\deg F = 2, 3$, after some special cases had been proved by Mahler [7], Nagell [8] and Schinzel [11]. Kotov [4], building on the work of Sprindzhuk [12], extended this result to $F$ of any degree $\geq 2$. It is conjectured that in fact

$$\mathcal{P}(F(n)) > (\deg F - 1 - \epsilon) \log n$$

for any fixed $F, \epsilon > 0$ and $n$ sufficiently large. This would follow from the conditional results of Granville [2] and Langevin [5, 6], which assume the ABC-conjecture.

We are concerned with the function field analogue of this problem. Let $p$ be a prime, $q$ its power and $\mathbb{F}_q$ the field with $q$ elements. Let $K$ be the function field of the curve $C_K$ defined over $\mathbb{F}_q$. By a curve we will always mean a smooth projective algebraic curve. For a function $f \in K^*$ we denote by $(f)$ its divisor which can be decomposed into its zero and polar components $(f) = (f)_0 - (f)_{\infty}$. 
The height of $f$ is defined to be $ht \ f = \deg(f)_0 = \deg(f)_\infty$. For a divisor $D$ on $K$ we denote by $\text{sup} \ D$ its support (the set of prime divisors appearing in $D$ with nonzero coefficient) and define

$$\delta(f) = \max_{P \in \text{sup}(f)} \deg P$$

(for $f$ not constant).

Now fix $p, q, K, 0 \neq F \in K[x]$. We are concerned with a lower bound for $\delta(F(f))$ in terms of $ht \ f$ as $ht \ f \to \infty$. For this problem we may assume without loss of generality that $F$ has no repeated irreducible factors in $K[x]$ (i.e. squarefree), otherwise just replace it with the product of its irreducible factors.

In the case of function fields it can happen that $\delta(F(f))$ stays bounded while $ht \ f \to \infty$. For example if $F \in \mathbb{F}_q[x]$ has constant coefficients, $t \in K$ and $f = t^k$, then $F(f) = F(t)^q$, so $\delta(F(f)) = \delta(F(t))$ while $ht \ f \to \infty$ as $k \to \infty$. Under certain restrictions on $F$ such pathologies do not occur and we will obtain a bound analogous to (2).

We denote by $\mathbb{F}_q \subset K$ the algebraic closures of $\mathbb{F}_q, K$ respectively. A polynomial $F \in K[x]$ is called separable if it has distinct roots over $\overline{K}$. Our main result is the following

**Theorem 1.** Let $q, K, 0 \neq F \in K[x]$ be fixed with $F$ squarefree (in $K[x]$). Assume that $F$ is either non-separable or has (at least) three distinct roots $a_1, a_2, a_3$ in $\overline{K}$ s.t.

$$\frac{a_1 - a_2}{a_1 - a_3} \notin \mathbb{F}_q.$$

Then there exists a constant $\lambda$ depending on $F$ s.t.

$$\delta(F(f)) > \log_q ht \ f - \lambda$$

for all $f \in K$ (for which $F(f)$ is not constant).

We call a separable squarefree polynomial $F \in K[x]$ exceptional if it fails the condition of Theorem 1. This is the function field analogue of the exceptional polynomials in $\mathbb{Z}[x]$ as defined in [4], to which the main method of [4] is not applicable but can be treated by other means (to obtain the bound (1)). Our notion of exceptional polynomial should not be confused with the notion of exceptional polynomials over $\mathbb{F}_q$ as defined in [1]. For exceptional polynomials we will obtain the following result:

**Theorem 2.** Let $F \in K[x]$ be a fixed separable polynomial.

i. The polynomial $F$ is exceptional if and only if there exist $s, t \in K, n \geq 0$ s.t. $F$ divides the polynomial

$$x^{q^n} - sx + t$$

(3)

and the latter polynomial is nonzero.
ii. If $F$ is exceptional then there is a sequence $f_k \in K$ s.t. $\mathfrak{ht}f_k \to \infty$ as $k \to \infty$ but $\delta(F(f_k))$ stays bounded.

iii. Assume that $F$ is exceptional and divides the nonzero polynomial (3) for some $s, t \in K, n \geq 0$. Let $\epsilon > 0$ be fixed. Then for $f \in K$ such that $sf - t$ is not a $p$-th power in $K$ and $\mathfrak{ht}f$ is sufficiently large (i.e. larger than some constant depending only on $F, \epsilon$) we have

$$\delta(F(f)) > \log_q \mathfrak{ht}f + \log_q(\deg F - 1) + \log_q(1 - 1/q) - \epsilon.$$  

(4)

Corollary 3. Let $F \in \mathbb{F}_q[x]$ be a fixed squarefree polynomial with constant coefficients. Then (4) holds (for any fixed $\epsilon > 0$) whenever $f \in K$ is not a $p$-th power in $K$ and $\mathfrak{ht}f$ is sufficiently large.

Proof. A squarefree polynomial with constant coefficients always divides a polynomial of the form $x^n - x$, so we can apply Theorem 2 with $s = 1, t = 0$. 

2 Preliminaries

For the proof of our results we will need the following proposition, which is an extension of the ABC-theorem for function fields.

Proposition 4. Let $K$ be the function field of the curve $C_K$ over $\mathbb{F}_q$ with genus $g_K$. Let $u \in K$ be a function which is not a $p$-th power in $K$ and $b_1, \ldots, b_m \in \mathbb{F}_q$. Then

$$\sum_{P \in \cup \sup(u - b_i)} \deg P \geq (m - 1)\mathfrak{ht}u - (2g_K - 2).$$

Proof. Consider the extension $\mathbb{F}_q(u) \subset K$. This is a separable geometric extension of function fields (because $u$ is not a $p$-th power in $K$) of degree $\mathfrak{ht}u$, so we may apply the Riemann-Hurwitz formula to obtain

$$2g_K - 2 \geq -2\mathfrak{ht}u + \sum_P (e_P - 1) \deg P,$$

where $e_P$ is the ramification index of the prime $P$ of $K$ in this extension (equality is obtained if all the $e_P$ are coprime to $p$, but we do not assume this). Restricting to the primes $P \in \cup_{i=1}^m \sup(u - b_i)$, which are exactly the primes lying over the primes $F_q(u)$ corresponding to the points $b_1, \ldots, b_m, \infty$ on $\mathbb{P}^1$ (considering $F_q(u)$ as the function field of $\mathbb{P}^1$) and using

$$\sum_{P \in \cup \sup(u - b_i)} e_P \deg P = [K : F_q(u)] \cdot \#\{b_1, \ldots, b_m, \infty\} = (m + 1)\mathfrak{ht}u$$

we obtain

$$2g_k - 2 + 2\mathfrak{ht}u \geq \sum_{P \in \cup \sup(u - b_i)} (e_P - 1) \deg P = (m + 1)\mathfrak{ht}u - \sum_{P \in \cup \sup(u - b_i)} \deg P.$$
Therefore
\[ \sum_{P \in \sup(u-b_i)} \deg P \geq (m-1)\text{ht }u - (2g_K - 2), \]
as required.

Taking \( m = 2, b_1 = 0, b_2 = 1 \) in the last proposition we obtain the ABC-theorem for function fields in the following form (see also [10, Theorem 7.17]):

**Proposition 5.** Let \( K \) be the function field of the curve \( C_K \) over \( \mathbb{F}_q \) with genus \( g_K \). Let \( u \in K \) be a function which is not a \( p \)-th power in \( K \). Then
\[ \sum_{P \in \sup(u) \cup \sup(u-1)} \deg P \geq \text{ht }u - (2g_K - 2). \]

## 3 Proof of Theorem 1

Let \( K \) be the function field of the curve \( C_K \) defined over \( \mathbb{F}_q \) and let \( F \in K[x] \) be a squarefree polynomial. We assume without loss of generality that \( F \) is monic (if \( c \) is the leading coefficient of \( F \) then \( \delta(F(f)) = \delta(F(f)/c) \) whenever \( \delta(F(f)) > \delta(c) \)).

**Proposition 6.** Assume that there exist three distinct roots \( a_1, a_2, a_3 \in \overline{K} \) of \( F \) s.t.
\[ \tau = \frac{a_1 - a_2}{a_1 - a_3} \notin \mathbb{F}_q. \]
Then the assertion of Theorem 1 holds for \( F \).

**Proof.** Let \( L \) be the splitting field of \( F \) over \( K \), \( C_L \) its underlying curve with genus \( g_L \). Since \( \tau \notin \mathbb{F}_q \), for some \( k \) the element \( \tau \in L \) is not a \( p^k \)-th power in \( L \). Take any \( f \in K \). Denote
\[ u = \frac{f - a_2}{a_1 - a_2}, v = \frac{f - a_3}{a_1 - a_3}. \]
It is not possible that both \( u \) and \( v \) are \( p^k \)-th powers in \( L \) because then so would be \( \tau = v/u \) which we assumed is not the case. Assume (by symmetry) that \( u \) is not a \( p^k \)-th power and let \( l \leq k \) be the largest integer s.t. \( u \) is a \( p^l \)-th power in \( L \). Applying Proposition 5 to the function \( u^{1/p^l} \in L \) (which is not a \( p \)-th power) we obtain
\[ \sum_{P \in \sup(u) \cup \sup(u-1)} \deg P \geq p^{-l}u - (2g_L - 2). \]

Note that
\[ u - 1 = \frac{f - a_1}{a_1 - a_2}. \]
Let $a_1, \ldots, a_{\deg F} \in L$ be the other roots of $F$, so that $F(x) = \prod_{i=1}^{\deg F} (x - a_i)$. We have

$$F(f) = \prod_{i=1}^{\deg F} (f - a_i) = u(u - 1)(a_1 - a_2)^2 \prod_{i=3}^{\deg F} (f - a_i). \quad (6)$$

Denote

$$M = \max_{1 \leq i, j \leq \deg F} \max_{P \in \sup(a_i - a_j)} \deg P.$$

Let $P$ be a prime divisor of $L$ with $\deg P > M$. Then $P$ is a pole of $f - a_i$ for one $i$ iff it is a pole of each $f - a_j, 1 \leq j \leq \deg F$. Also $P \in \sup(u)$ iff $P \in \sup(f - a_2)$ and $P \in \sup(u - 1)$ iff $P \in \sup(f - a_1)$. We see that if $P \in \sup(u)$ then it cannot cancel out in the product on the right hand side of (6) and so $P \in \sup(F(f))_L$. The same holds if $P \in \sup(u - 1)$.

We will denote by $O(1)$ quantities which are bounded by a constant depending only on $F$. We will use the notation $P \in \PDiv(L)$ to mean that $P$ is a prime divisor of $L$ and similarly with $K$. We have

$$\sum_{P \in \PDiv(L) : \deg P \leq M} \deg P = O(1)$$

and so using (5) we obtain

$$\sum_{P \in \sup(F(f))_L} \deg P \geq \sum_{P \in \sup(u), \sup(u - 1)} \deg P - O(1) \geq p^{-1}\ht_{L}u - O(1). \quad (7)$$

For any prime divisor $Q$ of $K$ we have

$$\sum_{P \in F_{Q^{\nu}} \over P \over Q} \deg P \leq \frac{[L : K]}{\nu} \deg Q,$$

where $F_{Q^{\nu}}$ is the field of constants of $L$ (equality occurs if $Q$ is unramified). Therefore

$$\sum_{Q \in \sup(F(f))_K} \deg Q \geq \frac{\nu p^{-1}}{[L : K]} \ht_{L}u - O(1).$$

Assuming that $\ht_{L}f > \ht_{L}a_1$ we see that $\ht_{L}f = \ht_{L}u + O(1)$ and using $\ht_{L}f = \frac{[L : K]}{\nu} \ht_{K}f$ we obtain

$$\sum_{Q \in \sup(F(f))_K} \deg Q \geq p^{-1}\ht_{K}f - O(1). \quad (8)$$

Let $d = \delta(F(f))$ be the degree of the largest prime divisor of $K$ appearing in the support of $(F(f))_K$. By the prime number theorem for function fields (see [10, Theorem 5.12]) for every natural number $e$ we have

$$\sum_{Q \in \PDiv(K) : \deg Q = e} \deg Q = q^e(1 + o(1)),$$

and so
with the $o(1)$ term tending to zero as $e \to \infty$, so

$$\sum_{q \in C_K \text{ deg } q \leq d} \deg Q \leq (1 + o(1)) \left( \sum_{e=1}^{d} q^e \right) = (1 + o(1))(1 - 1/q)^{-1}q^d.$$ 

By (8) we obtain

$$(1 + o(1))(1 - 1/q)^{-1}q^d \geq p^{-1}\text{ht}_KF - O(1) \quad (9)$$

and taking logarithms this becomes

$$\delta(F(f)) = d \geq \log_q\text{ht}_KF - O(1)$$

as required. \qed

**Proposition 7.** Assume that $F$ is non-separable. Then the assertion of Theorem 1 holds for $F$.

**Proof.** Since $F$ is squarefree and non-separable it has a non-separable irreducible factor $F_1 \in K[x]$. It must be of the form $F_1(x) = G(x^p)$ with $G \in K[x] \setminus F_q[x]$ monic. Of course $G$ is also irreducible over $K$. Let $L$ be the maximal separable extension of $K$ contained in the splitting field of $G$ over $K$. Over $L$ we have a factorization of the form

$$G(x) = \prod_{i=1}^{m} \left(x^{p^r} - \alpha_i\right), \alpha_i \in L \quad (10)$$

for some $r \geq 0$, with the $\alpha_i$ distinct. If all the $\alpha_i$ are $p$-th powers in $L$ then the coefficients of $G$ are $p$-th powers in $L$ and therefore also in $K$ (because the extension $K \subset L$ is separable), so $F_1(x) = G(x^p)$ is a $p$-th power of a polynomial in $K[x]$, which is impossible because $F_1$ divides $F$ and $F$ is squarefree. Therefore we may assume one and therefore all the $\alpha_i$ (since they are conjugate over $K$) are not $p$-th power in $L$.

Take some $f \in K$ and denote $u = \alpha_1^{-1}f^{p^{r+1}}$. Since $\alpha_1$ is not a $p$-th power in $L$, neither is $u$. We have

$$u - 1 = \alpha_1^{-1}\left(f^{p^{r+1}} - \alpha_1\right).$$

We may apply Proposition 5 to $u$ to obtain

$$\sum_{P \in \text{sup}(u) \cup \text{sup}(u-1)} \deg P > \text{ht}_Lu - O(1) = p^{r+1}\text{ht}_Lf - O(1).$$

However

$$\sum_{P \in \text{sup}(u)} \deg P \leq \sum_{P \in \text{sup}(f)_L} \deg P + \sum_{P \in \text{sup}(\alpha_1)} \deg P = \text{ht}_Lf + O(1),$$

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so
\[
\sum_{P \in \text{sup}(u-1)} \deg P > (p^{r+1} - 1) \text{ht}_L f - O(1). \tag{11}
\]

By (10) we have
\[
F_1(f) = m \prod_{i=1}^{m} (f^{p^{r+1}} - \alpha_i).
\]

As in the proof of Case 1 we see that a prime divisor \( P \) of \( L \) of sufficiently large degree (depending only on \( F \)) occurring in \( \text{sup}(u-1)_L \) must also occur in \( \text{sup}(F_1(f))_L \). Using (11) and arguing in the same way as in the proof of Proposition 7 we obtain
\[
\delta(F_1(f)) > \log_q \text{ht}_L f - O(1),
\]

Denote \( H = F/F_1 \in K[x] \). Let \( Q \) be a prime divisor of \( K \). There exists a constant \( N \) depending only on \( F \) s.t. if \( \deg Q > N \) then \( Q \) is a pole of either \( F_1(f), H(f) \) iff it is a pole of \( f \) (we just take \( N \) to be the maximum of the degrees of all the poles of the coefficients of \( F_1, H \)). For such \( Q \) if \( Q \in \text{sup}(F_1(f))_K \) then also \( Q \in \text{sup}(F(f))_K \) (zeroes and poles cannot cancel out by those of \( H(f) \)). Therefore
\[
\delta(F(f)) > \delta(F_1(f)) - O(1) > \log_q \text{ht}_L - O(1),
\]
as required. \( \square \)

Now Theorem 1 follows by combining Propositions 6 and 7.

4 Proof of Theorem 2

Let \( F \in K[x] \) be a separable polynomial of degree \( m = \deg F, a_1, ..., a_m \in \overline{K} \) the roots of \( F \).

4.1 Proof of Theorem 2(i)

Let \( F \) be exceptional. We want to show that it must divide a nonzero polynomial of the form
\[
x^n - sx + t, s, t \in K, n \geq 0, \tag{12}
\]
as asserted in Theorem 2(ii). Let \( L \) be the splitting field of \( F \) over \( K \) and \( \mathbb{F}_q^v \) the field of constants in \( L \). Since \( F \) is exceptional, for all distinct \( 1 \leq i, j, k \leq m \) we have \((a_i - a_j)/(a_i - a_k) \in \mathbb{F}_q^v \). Equivalently, there exist \( \alpha, \beta \in L \) and \( b_i \in \mathbb{F}_q^v, 1 \leq i \leq m \) s.t. \( a_i = \alpha b_i + \beta \). Consider the polynomial
\[
G(x) = \prod_{b \in \mathbb{F}_q^v} (x - ab - \beta) = x^{q^v} - \alpha^{q^v-1} x + \alpha^{q^v-1} \beta - \beta^{q^v} \in L[x] \tag{13}
\]
(to see that this identity holds just substitute \( ab + \beta \) into the RHS to see that it is a root for every \( b \in \mathbb{F}_q^v \)). If \( G \in K[x] \) then \( F \) divides \( G \) which has the form
If $G \not\in K[x]$ then there exists an automorphism $\sigma$ of $L$ over $K$ s.t. $G^\sigma \neq G$ ($G^\sigma$ is obtained from $G$ by applying $\sigma$ to each coefficient). The polynomial $F$ divides $G$ and since $F \in K[x]$ it also divides $G^\sigma$. Therefore $F$ divides $G - G^\sigma$. But from (13) we see that $G - G^\sigma$ is linear, so $F$ must be linear and already has the form (12). This concludes the proof of one implication of Theorem 2(i).

To prove the other implication assume that $F$ divides $G(x) = x^{q^n} - sx + t$ for some $n \geq 1, s, t \in K$ (if $F$ is linear it is obviously exceptional, so we may assume $n \geq 1$). There exist $\alpha, \beta \in \overline{K}$ s.t.

$$\alpha^{q^n} = s, \alpha^{q^n-1}\beta - \beta^{q^n} = t.$$  

The roots of $G$ over $\overline{K}$ are precisely $\alpha b + \beta, b \in \mathbb{F}_{q^n}$, so the roots of $F$ have the form $a_i = \alpha b_i + \beta, b_i \in \mathbb{F}_{q^n}$ and $F$ is exceptional. This concludes the proof of Theorem 2(i).

### 4.2 Proof of Theorem 2(ii)

Suppose $F$ is exceptional and therefore divides the nonzero polynomial

$$G(x) = x^{q^n} - sx + t, s, t \in K.$$  

Since the assertion of Theorem 2(ii) is trivial for $F$ linear we assume that $n \geq 1$. If $s = 0$ then $G$ has only one root over $\overline{K}$ and $F$ cannot be separable unless it is linear. Hence we assume that $s \neq 0$. Choose some $f_0 \in K$ with a pole $P$ of degree $\deg P > \delta(s), \delta(t)$ and define recursively

$$f_{k+1} = \frac{f_k^{q^n} + t}{s}, k \geq 0.$$  

The prime divisor $\deg P$ is a pole of multiplicity $q^{kn}$ of $f_{k+1}$, therefore $\text{ht} f_{k+1} \to \infty$ as $k \to \infty$. Now observe that

$$G(f_{k+1}) = f_{k+1}^{q^n} - sf_{k+1} + t = \frac{f_k^{q^{2n}} + t^{q^n}}{s^{q^n}} - f_k^{q^n} = \frac{\left(f_k^{q^n} - sf_k + t\right)^{q^n}}{s^{q^n}} = \left(G(f_k)\right)^{q^n},$$

so

$$\delta(G(f_{k+1})) \leq \max(\delta(G(f_k)), s)$$

and therefore $\delta(F(f_k))$ stays bounded as $k \to \infty$.

Now write $G(x) = F(x)H(x), H \in K[x]$. Let $P$ be a prime divisor not appearing in the supports of the coefficients of $F,H$. Then for any $f \in K^\times, P$ is a pole of $F(f)$ iff it is a pole of $f$ and of $H(f)$. We see that sufficiently large (depending only on the coefficients of $F,H$) prime divisor cannot be canceled out when we multiply $F(f)$ by $H(f)$, so if $\delta(F(f))$ is sufficiently large we have $\delta(G(f)) \geq \delta(F(f)), \text{so} \delta(F(f_k))$ is also bounded as $k \to \infty$. This concludes the proof of Theorem 2(ii).
4.3 Proof of Theorem 2(iii)

Assume that $F$ is exceptional. As in the proof of Theorem 1 we will also assume without loss of generality that $F$ is monic. If $F$ is linear the assertion of Theorem 2(iii) is obvious, so we assume $\deg F = m \geq 2$. Assume that $F$ divides

$$G(x) = x^q^n - sx + t, s, t \in K, n \geq 1.$$ 

We fix one such $G$ once and for all, so $s, t, n$ are also fixed. We have $s \neq 0$, otherwise $F$ would be linear. It follows that $G$ is separable because its derivative is $-s \in K^\times$. Let $\alpha, \beta \in \overline{K}$ be such that

$$\alpha q^n = t, \quad \alpha q^n - 1 = \beta.$$ 

Then the roots of $G$ are $\alpha b + \beta, \quad b \in \mathbb{F}_{q^n}$, and the roots of $F$ have the form $a_i = \alpha b_i + \beta, b_i \in \mathbb{F}_{q^n}, 1 \leq i \leq m$. Let $L$ be the splitting field of $G$ over $K \mathbb{F}_{q^n}$ (the composite of the fields $K, \mathbb{F}_{q^n}$). It is a separable extension of $K$ since $G$ is separable. Since $\deg G \geq 2$ and $s \neq 0$, $G$ has at least two distinct roots $\alpha b + \beta, \alpha b' + \beta$ from which it follows that $\alpha, \beta \in L$.

Now let $f \in K$ be such that $sf - t$ is not a $p$-th power in $K$. Denote $u = (f - \beta)/\alpha \in L$. We claim that $u$ is not a $p$-th power in $L$. Suppose to the contrary that $u$ is a $p$-th power. Then so is $u - b$ for any $b \in \mathbb{F}_{q^n}$. Now

$$G(f) = \prod_{b \in \mathbb{F}_{q^n}} (f - \alpha b - \beta) = \alpha q^n \prod_{b \in \mathbb{F}_{q^n}} (u - b),$$

so $G(f)$ is a $p$-th power in $L$ and therefore also in $K$, because $K \subset L$ is a separable extension. But $G(f) = f^q^n - sf + t$, so $sf - t$ is also a $p$-th power, a contradiction. Therefore $u$ is not a $p$-th power.

Now we apply Proposition 4 to the field $L$, function $u$ and constants $b_1, ..., b_m$. We obtain the inequality

$$\sum_{P \in \cup_{i=1}^{m} \sup(u - b_i)_L} \deg P > (m - 1)\text{ht}_L u - O(1),$$

where $O(1)$ stands for a quantity bounded by a constant depending only on $F$ and $G$ (the latter was fixed for a given exceptional $F$). Since

$$F(f) = \prod_{i=1}^{m} (f - \alpha b_i - \beta) = \alpha^m \prod_{i=1}^{m} (u - b_i)$$

we see that a prime divisor $P$ of $L$ with $\deg P > \delta(\alpha)$ and appearing in $\cup_{i=1}^{m} \sup(u - b_i)_L$ must also appear in $\sup(F(f))_L$ (note that $u - b_i$ have the same poles for $1 \leq i \leq m$), so

$$\sum_{P \in \sup(F(f))_L} \deg P > (m - 1)\text{ht}_L u - O(1). \quad (14)$$
Now denoting $d = \delta(F(f))$ and arguing as in the proof of Proposition 6 (where we deduced (9) from (7)) we deduce from (14) that

$$(1 + o(1))(1 - 1/q)^{-1}q^d > (m - 1)ht_K f - O(1),$$

$(o(1)$ is a quantity tending to 0 as $ht_K f \to \infty$ for fixed $F, G$) from which it follows that

$$d > \log_q ht_K f + \log_q (m - 1) + \log_q (1 - 1/q) - o(1),$$

which is exactly the assertion of Theorem 2(iii).

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