Linear pencils on graphs and on real curves

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1 Introduction

Let $X$ be a general smooth irreducible complex curve of genus $g$ and assume $r$ and $d$ are nonnegative integers such that $\rho = g - (r + 1)(g - d + r) = 0$. From Brill-Noether Theory, it is known that $X$ has finitely many linear systems $g^r_d$ and that the natural associated 0-dimensional scheme $W^r_d$ parameterizing those linear systems is reduced and has $\lambda = g! \prod_{i=0}^{\rho} \frac{n^i}{(g-d+i)!}$ points.

In [4], a new method for considering Brill-Noether problems is introduced using linear systems on graphs. In particular, one obtains a metric graph $\Gamma$ of genus $g$ having exactly $\lambda$ linear systems $g^r_d$. In a degeneration from smooth curves of genus $g$ to a singular curve of genus $g$ with dual graph equal to $\Gamma$, the results from [2] imply that the generic curve $X$ has finitely many linear systems $g^r_d$. However, it is not clear whether all linear systems $g^r_d$ on $X$ have different specializations on $\Gamma$ (see [4, Conjecture 1.5]). In this note we solve this problem for the case $r = 1$.

In [5], the specialization from curves to graphs is studied for the situation of real curves. In particular, in that situation, the graph $\Gamma$ has a real structure compatible to the real structure on the curves and real linear systems on the curves specialize to real linear systems on the graph. Using suited lengths for the edges of the graph $\Gamma$ mentioned before, it is possible to introduce two real structures on $\Gamma$, including the trivial one. Using the previous results, we obtain the existence of real curves of genus $g$ having exactly $\lambda$ linear systems $g^1_d$ all of them being real (using the trivial structure) and having less than $\lambda$ real linear systems $g^1_d$ (using the non-trivial structure) in case $g \geq 6$. In this second case, the number $\lambda'$ of real linear systems $g^1_d$ can be computed and it turns out that $\lambda'/\lambda$ becomes 0 for $g \to \infty$.

The existence of real curves having $\lambda$ real linear systems $g^1_d$ also follows from degenerations to real rational cuspidal curves using [9] and it is proved using the theory of limit linear systems in [7]. In general, one has the following problem: determine all values $\lambda' \equiv \lambda \pmod{2}$ such that there exists a real curve $X$ having

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exactly $\lambda$ linear systems $g_4^1$ with exactly $\lambda'$ of them being real. The non-trivial real structure shows that $\lambda'$ can become much smaller than $\lambda$. It should be noted that from the result in [6], it follows that one cannot obtain the existence of real curves with $\lambda' \neq \lambda$ using degenerations to real cuspidal rational curves. In case $g$ is not of the type $2^n - 2$, then $\lambda$ is even and the smallest possible value of $\lambda'$ could be 0. Using the intersection of a general real cubic surface and a general real quadric containing no real lines, it is easy to find real curves of genus 4 having no real $g_1^3$. In [3], it is proved that there exist real curves of genus 8 having no real linear system $g_1^5$. At the moment, those seem to be the only known cases with $\lambda' = 0$.

2 Linear pencils on metric graphs

Let $G$ be a finite connected graph, possibly with multiple edges, but without loops. Denote by $V(G)$ the set of vertices of $G$ and by $E(G)$ the set of edges of $G$. Assume that a real positive number $\ell(e)$ is assigned to each edge $e$. If we identify each edge $e \in E(G)$ with a line segment $[0, \ell(e)]$, we get a compact connected metric space $\Gamma$, which is the metric graph associated to $G$. The genus $g$ of $\Gamma$ is defined as $|E(G)| - |V(G)| + 1$, i.e. the first Betti number of $G$, where $G$ is any underlying graph.

A divisor $D = n_1v_1 + \ldots + n_tv_t$ on $\Gamma$ is an element of the free abelian group $\text{Div}(\Gamma)$ on the points of the metric graph $\Gamma$. The degree of $D$ is the sum $n_1 + \ldots + n_t$ of the coefficients and $D$ is effective if each coefficient $n_i$ is nonnegative. Often, the terminology of chips is used when one deals with divisors on graphs, e.g. we say that the divisor $D$ has $k$ chips at $v$ if the coefficient of $v$ in $D$ is equal to $k$.

Let $\psi : \Gamma \to \mathbb{R}$ be a continuous map such that $\psi$ is piecewise linear with finitely many pieces and with integer slopes on each edge of $\Gamma$. The divisor $\text{div}(\psi)$ of $\psi$ is equal to $\sum_{v \in \Gamma} \text{ord}_v(\psi)$, where $\text{ord}_v(\psi)$ is the sum of the incoming slopes of $\psi$ at $v \in \Gamma$. We say that two divisors $D$ and $D'$ are equivalent if and only if $D' - D = \text{div}(\psi)$ for some continuous map $\psi$ on $\Gamma$, and in this case, we write $D \sim D'$. The set $|D|$ of all effective divisors $D'$ equivalent with $D$ is the linear system corresponding to $D$. The rank $r(D)$ of $|D|$ or $D$ is defined as follows. If $|D| = \emptyset$, then $r(D) = -1$, otherwise $r(D) = r$ if and only if $|D - E| \neq \emptyset$ for each effective divisor $E$ on $\Gamma$ of degree $r$. For a more elaborate introduction to linear systems on metric graphs (e.g. the notion of the $v$-reduced divisor of $|D|$), we refer to [4].

Linear systems on metric graphs seem to obey analogous results as linear systems on algebraic curves, e.g. Riemann-Roch Theorem. In [2], Baker conjectures that also the following counterpart of the Brill-Noether Theorem holds and he proves the first part.

**Theorem 2.1** (Brill-Noether Theorem for metric graphs). Fix $g, r, d \geq 0$ and set $\rho = g - (r + 1)(g - d + r)$.
Proof. Since it is clear that \( |\ell| \) and we will prove this by induction on \( i \). Proposition 2.2. Let \( p \) be the divisor on \( \Gamma \) with one chip in \( v_i \), and such that in addition, the first and the last loop also contain one chip at the point \( v_0 \) and \( v_g \). Assume that \( \Gamma \) is the corresponding metric graph where the lengths \( \ell_i \) and \( m_i \) of the two edges from \( v_{i-1} \) to \( v_i \) are generic for each \( i \in \{1, \ldots, g\} \) (it suffices to take \( \ell_i = \ell \geq 2g-2 \) and \( m_i = 1 \) for all \( i \)). In \([4]\) is proved that such metric graphs are Brill-Noether general.

\[ \begin{array}{cccccc}
 v_0 & v_1 & v_2 & v_3 & v_4 & v_5 \\
 \end{array} \]

Figure 1: the graph \( G_5 \)

We will now focus on the case of linear pencils on \( \Gamma \), thus \( r = 1 \). If \( g \) is even and \( d = \frac{g}{2} + 1 \) (so \( \rho = 0 \)), it is proven in \([4]\) that there is a bijection between lattice paths \( p = (p_0, \ldots, p_g) \) in \( \mathbb{Z} \) satisfying \( p_0 = p_g = 1 \), \( p_i \geq 1 \) and \( p_i - p_{i-1} = \pm 1 \) for all \( i \in \{1, \ldots, g\} \) and linear pencils of degree \( d \) on \( \Gamma \) as follows. If \( p = (p_0, \ldots, p_g) \) is a path satisfying the conditions, let \( D_p \) be the divisor on \( \Gamma \) with one chip in \( v_0 \), one (extra) chip on the unique point \( w_i \) of the \( i \)th loop satisfying \( p_{i-1}v_i - 1 + w_i \sim p_i v_i \) if \( p_i - p_{i-1} = 1 \) and no (extra) chips on the \( i \)th loop if \( p_i - p_{i-1} = -1 \). Note that \( D_p \) is \( v_0 \)-reduced. The bijection maps \( p \) to the linear system \([D_p]^1\). So the number of linear pencils on \( \Gamma \) is equal to the Catalan number \( \lambda = \frac{1}{d} \binom{2d-2}{d-1} \).

Proposition 2.2. Let \( D_p \) be a divisor on \( \Gamma \) corresponding to a lattice path \( p = (p_0, \ldots, p_g) \). Then the linear system \([2D_p]^1\) has rank equal to two.

Proof. Since it is clear that \([2D_p]^1\) has rank at least two, it suffices to prove that \([2D_p - 2v_0 - v_g]^1\) = 0. Let \( Q_i \) be the divisor attained from \( 2D_p - 2v_0 \) by moving as many chips as possible from the first \( i \) loops of \( \Gamma \) to \( v_i \), for each \( i \in \{0, \ldots, g\} \). Denote by \( q_i \) the number of chips of \( Q_i \) in \( v_i \). We claim that \( q_i = p_i - 1 \) for each \( i \) and we will prove this by induction on \( i \).

For \( i = 0 \), the divisor \( Q_0 = 2D_p - 2v_0 \) and thus \( q_0 = 0 = p_0 - 1 \) since \( D_p \) has exactly one chip in \( v_0 \). Now assume the claim holds for \( i \), hence \( q_i = p_i - 1 \). If \( p_{i+1} - p_i = -1 \), the divisor \( Q_{i+1} \) is attained from \( Q_i \) by moving the \( q_i \) chips in \( v_i \) as much as possible to \( v_{i+1} \). Since the edge lengths are general, only \( q_i - 1 = p_i - 2 = p_{i+1} - 1 \) chips end up in \( v_{i+1} \). If \( p_{i+1} - p_i = 1 \), the divisor \( Q_i \) has \( q_i \) chips in \( v_i \) and 2 chips at the point \( w_{i+1} \) of the \( (i+1) \)th loop satisfying \( p_i v_i + w_{i+1} \sim p_{i+1} v_{i+1} \), so only \( (q_i + 2) - 1 = p_i = p_{i+1} - 1 \) chips end up in \( v_{i+1} \). In both cases, we have \( q_{i+1} = p_{i+1} - 1 \).
The statement of the proposition now follows from the claim for $i = g$, since the $v_g$-reduced divisor $Q_g$ in $|2D - 2v_0|$ has $q_g = p_g - 1 = 0$ chips in $v_g$. □

Assume that the metric graph $\Gamma$ has integer edge lengths (thus $\ell_i, m_i \in \mathbb{Z}$). Let $G$ be the graph corresponding to $\Gamma$ with $\ell(e) = 1$ for each $e \in E(G)$, so $G$ is a refinement of $G_g$. Let $R$ be a complete discrete valuation ring with field of fractions $Q$ and algebraically closed residue field $k$. Then [2] Appendix B implies that there exists a regular arithmetic surface $X \to R$ whose generic fiber $X_Q$ is a smooth curve $X \to Q$ and whose special fiber $X_k$ has dual graph equal to $G$. Following [2] Section 2, the specialization map

$$\alpha : \text{Div}(X) \to \text{Div}(G)$$

sends effective (resp. principal) divisors to effective (resp. principal) divisors. Hereby, $\text{Div}(G)$ is the subgroup of $\text{Div}(\Gamma)$ consisting of divisors supported at $V(G)$. Let $\overline{Q}$ be the algebraic closure of $Q$ and algebraically closed residue field $k$. Then [2, Appendix B] implies that there exists a regular arithmetic surface $X \to R$ whose generic fiber $X_Q$ is a smooth curve $X \to Q$ and whose special fiber $X_k$ has dual graph equal to $G$. Following [2, Section 2], the specialization map

$$\tau_* : \text{Div}(X(\overline{Q})) \cong \text{Div}(X_{\overline{Q}}) \to \text{Div}(\Gamma)$$

satisfying

$$r(\tau_*(E)) \geq \dim(|E|)$$

for each $E \in \text{Div}(X_{\overline{Q}})$. This inequality implies that also $X_{\overline{Q}}$ is Brill-Noether general, but it is not clear whether each divisor on $\Gamma$ of rank $r$ and degree $d$ lifts to a divisor of rank $r$ and degree $d$ on $X_{\overline{Q}}$ (see [4, Conjecture 1.5]). Here we solve this problem for the case where $r = 1$ and $\rho = 0$.

**Theorem 2.3.** If $g$ is even and $d = \frac{g}{2} + 1$, each linear pencil $|D_p|$ on $\Gamma$ lifts to a unique linear pencil $g_1^d$ on $X_{\overline{Q}}$.

**Proof.** If $|E|$ would be a multiple $g_1^d$ on $X_{\overline{Q}}$, then it follows from the description of the tangent map to $W_1^d$ and the base point free pencil trick (see [1] Chapter IV, Section 4 and Chapter III, Section 3) that $\dim(|2E|) \geq 3$. This would imply that $|E|$ specializes to a linear pencil $|D_p|$ on $\Gamma$ satisfying

$$r(2D_p) = r(\tau_*(E)) \geq \dim(|2E|) \geq 3,$$

which is in contradiction with Proposition 2.2. So the scheme $W_1^d$ of $X_{\overline{Q}}$ consists of exactly $\lambda = \frac{1}{4}(2d-2)$ points.

Assume two different linear pencils $|E_1|$ and $|E_2|$ on $X_{\overline{Q}}$ specialize to the same linear pencil $|D_p|$ on $\Gamma$. Then $\dim(|E_1 + E_2|) \geq 3$ and $r(2D_p) = 2$, but on the other hand

$$r(2D_p) = r(\tau_*(E_1 + E_2)) \geq \dim(|E_1 + E_2|),$$

so we have a contradiction.

Since the graph $\Gamma$ has exactly $\lambda$ pencils $g_1^d$, it follows that $\tau_*$ induces a bijection between linear pencils $g_1^d$ on $X_{\overline{Q}}$ and on $\Gamma$. □
Assume $\ell_i = \ell$ and $m_i = 1$ for all $i \in \{1, \ldots, g\}$ and identify the two edges of $\Gamma$ between $v_{i-1}$ and $v_i$ with intervals $I_i = [0, 1]$ and $J_i = [0, \ell]$. In this case, there is a non-trivial involution $\sigma$ on $\Gamma$ sending $x \in I_i$ to $1-x \in I_{g-i}$ and $x \in J_i$ to $\ell - x \in J_{g-i}$. Note that the only fixed point of this involution is $v_g/2$.

**Proposition 2.4.** Let $p = (p_0, \ldots, p_g)$ be a lattice path corresponding to a divisor $D_p$ of rank one on $\Gamma$. Denote by $\sigma(p)$ the lattice path $(p_g, \ldots, p_0)$.

(i) $D_p \sim \sigma(D_{\sigma(p)})$.

(ii) The linear system $|D_p|$ is invariant under $\sigma$ if and only if $p = \sigma(p)$.

**Proof.** Define $f_i$ for all $i \in \{0, \ldots, g\}$ inductively by $f_0 = 0$ and

$$f_i = \begin{cases} f_{i-1} + p_{i-1} & \text{if } p_i - p_{i-1} = 1 \\ f_{i-1} + p_i & \text{if } p_i - p_{i-1} = -1 \end{cases}$$

Let $f : \Gamma \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} f_{i-1} + p_{i-1}x & \text{if } x \in I_i \\ f_{i-1} & \text{if } x \in [0, \ell - p_{i-1}] \subset J_i \\ f_{i-1} + x - (\ell - p_{i-1}) & \text{if } x \in [\ell - p_{i-1}, \ell] \subset J_i \end{cases}$$

if $p_i - p_{i-1} = 1$ and

$$f(x) = \begin{cases} f_{i-1} + p_i x & \text{if } x \in I_i \\ f_{i-1} + x & \text{if } x \in [0, p_i] \subset J_i \\ f_{i-1} + p_i & \text{if } x \in [p_i, \ell] \subset J_i \end{cases}$$

if $p_i - p_{i-1} = -1$. It is clear that $f$ is a piecewise linear with $f(v_i) = f_i$ and one can see that $\text{div}(f) = \sigma(D_{\sigma(p)}) - D$, hence $D_p \sim \sigma(D_{\sigma(p)})$.

If the linear system $|D_p|$ is invariant under $\sigma$, then $\sigma(D_{\sigma(p)}) \in |D_p|$ implies that $D_{\sigma(p)} \in |D_p|$. Since $D_p$ and $D_{\sigma(p)}$ are both $v_0$-reduced divisors in $|D_p|$, they must be equal, hence $p = \sigma(p)$.

Now assume $p = \sigma(p)$, hence $D_{\sigma(p)} \in |D_p|$. Let $E$ be an arbitrary divisor in $|D_p|$, so there exists a piecewise linear function $\psi : \Gamma \to \mathbb{R}$ such that $\text{div}(\psi) = E - D_p$. If we apply the involution $\sigma$ to this equation, we get that $\text{div}(\psi \circ \sigma) = \sigma(\text{div}(\psi)) = \sigma(E - D_p) = \sigma(E) - \sigma(D_p)$, hence $\sigma(E) \sim D_{\sigma(p)} \sim D_p$, so $\sigma(E) \in |D_p|$ and $|D_p|$ is invariant under the involution. \hfill \Box

This implies that the number $\lambda'$ of linear pencils on $\Gamma$ that are invariant under $\sigma$ is equal to the number of symmetric lattice paths. This number can be computed as follows. Using Andrè’s reflection method (see [8]), one can see that the number of symmetric lattice paths $p = (p_0, \ldots, p_g)$ with $p_g/2 = m$ (where $m \in \{1, \ldots, d\}$ with $m \equiv d \mod 2$) is equal to

$$m \left( \frac{d}{2} \right) \left( \frac{d-m}{2} \right) = \left( \frac{d-1}{2} \right) \left( \frac{d-m}{2} - 1 \right),$$

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hence $\lambda'$ is the sum of these numbers and equals the central binomial coefficient \( \binom{d-\frac{1}{2}}{d-\frac{1}{2}} \). In the following table, the values of $\lambda$ and $\lambda'$ are mentioned for small values of $d$. Note that the limit of the quotient $\lambda' / \lambda$ goes to zero if $d \to \infty$.

| $d$ | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|-----|----|----|----|----|----|----|----|----|----|
| $g$ | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 | 18 |
| $\lambda$ | 1  | 2  | 5  | 14 | 42 | 132| 429| 1430| 4862|
| $\lambda'$ | 1  | 2  | 3  | 6  | 10 | 20 | 35 | 70 | 126|

3 Linear pencils on real curves

In [3], it is proved that real linear systems on curves degenerate to real linear systems on graphs. We are going to explain that those arguments also imply that the degeneration respects complex conjugation of linear systems.

Let $R$ be a complete discrete valuation ring defined over $\mathbb{R}$ having residue field $\mathbb{R}$ and let $\mathcal{X} \to R$ be a regular arithmetic surface such that the generic fiber $X$ is smooth and geometrically irreducible. Let $Q$ be the quotient field of $R$, then $Q_C = Q \otimes_{\mathbb{R}} \mathbb{C}$ is a field (see [3 Lemma 5.1]). Let $R_C$ be the complete valuation ring extending $R$ in $Q_C$ and let $\mathcal{X}_C \to R_C$ be obtained by base change. We assume the special fiber $X_{0,C}$ (which is defined over $\mathbb{R}$) is strongly semistable. This means $X_{0,C}$ is reduced, all singular points of $X_{0,C}$ are nodes and its irreducible components are smooth. Associated to $X_{0,C}$, there is a dual graph $G$ and this dual graph has a real structure defined by the real structure on $X_{0,C}$.

Let $D$ be a divisor on the generic fiber $X_C$ and let $\overline{D}$ be the conjugated divisor. This means $\overline{D} = \sigma(D)$, where $\sigma$ is the involution on $X_C$ induced by the field extension $Q \subset Q_C$. Considering $X_C \subset \mathcal{X}_C$, one has closures $\text{cl}(D)$ and $\text{cl}(\overline{D})$ and one has $\text{cl}(\overline{D}) = \sigma(\text{cl}(D))$ (in the right hand side of this equality, the conjugation is induced by the field extension $\mathbb{R} \subset \mathbb{C}$ because $\mathcal{X}$ is defined over $\mathbb{R}$).

Let $\overline{Q}$ be the algebraic closure of $Q$ and let $\overline{Q}'$ be the real closure. We obtain a field extension $\overline{Q}' \subset \overline{Q}$ of degree 2. On $\overline{X} = X \times_{\overline{Q}} \overline{Q}'$, it defines an involution $\sigma$ and if $D$ is a divisor on $\overline{X}$, then we write $\overline{D}$ to denote $\sigma(D)$. The divisor $D + \overline{D}$ is defined over $\overline{Q}'$. In particular, there exists a finite extension $Q \subset K$ with $K \subset \overline{Q}'$ such that $D + \overline{D}$ is defined over $K$ and $D$ is defined over $K \otimes_{\mathbb{R}} \mathbb{C} = K_C$. Let $R_K$ (resp. $R_{K,C}$) be the extension of $R$ in $K$ (resp. $K_C$) and consider the base extensions $\mathcal{X} \times_R R_K$ and $\mathcal{X} \times_R R_{K,C}$. It is proved in [3] that there exists a family $\mathcal{X}_K \to R_K$ defined over $\mathbb{R}$ such that by base change we obtain a desingularization $\mathcal{X}_{K,C} \to R_{K,C}$ of $\mathcal{X} \times_R R_{K,C}$; the special fiber $X_{0,K,C}$ is strongly semistable and its graph $G_K$ with the associated real structure has a natural weighted structure such that the associated metric graph is equal to $\Gamma$ with its real structure. The associated divisor on $G_K$ defines a divisor $\tau_*(D)$ on $\Gamma$ and one has $\tau_*(\overline{D}) = \overline{\tau_*(D)}$ on $\Gamma$. In case $D = \overline{D}$, these arguments are explained in full detail in [3] showing that $\tau_*(D)$ is a real divisor on $\Gamma$ in that case.
We consider the metric graph $\Gamma$ used in the proof of Proposition 2.4. It has two real structures: the trivial one and the one defined by $\sigma$. From [5, Proposition 5.9], it follows that both real structures can be obtained from a degeneration with $X_{0,C}$ a totally degenerated curve. This means that all components of $X_{0,C}$ have genus 0, in particular $g(X_{0,C}) = g(\overline{X}) = g(\Gamma) = g$. Now assume such a degeneration is chosen.

**Proposition 3.1.** The specialization induces a bijection between the set of real linear systems $g^1_d$ on $X$ and the set of real linear systems $g^1_d$ on $\Gamma$.

**Proof.** Theorem 2.3 implies that the specialization map induces a bijection between the sets of all linear systems $g^1_d$. From [5], it follows that under the specialization a real $g^1_d$ on $X$ corresponds to a real $g^1_d$ on $\Gamma$. In case $g$ is a non-real $g^1_d$ on $X$, then $g$ and $\overline{g}$ are two different linear systems on $X$, hence $\tau_*(g) \neq \tau_*(\overline{g})$. Using the equality $\tau_*(\overline{g}) = \tau_*(g)$ mentioned above, we get $\tau_*(g) \neq \tau_*(g)$, hence $\tau_*(g)$ is non-real on $\Gamma$.

Using Proposition 2.4 in the case the real structure on $\Gamma$ is induced by $\sigma$ and using a general specialization of the generic fiber of $X \to R$ to a complex curve $X$ of genus $g$ defined over $\mathbb{R}$, we obtain the following.

**Theorem 3.2.** There exist smooth complex curves $X$ defined over $\mathbb{R}$ of genus $g = 2d - 2$ having exactly $\lambda$ linear systems $g^1_d$ such that all of them (resp. exactly $\lambda'$ of them) are real.

**Remark 3.3.** In case the real structure on $\Gamma$ is induced by $\sigma$, we can say a little bit more about the structure of the real curve. We can choose $X_{0,C}$ such that the component $C_{g/2}$ corresponding to $v_{g/2}$ has a non-empty real locus or an empty real locus. The other components are obtained from pairs of disjoint complex conjugated copies of $\mathbb{P}^1$. In the first case, the deformation is a smooth real curve of genus $g$ having exactly one connected component in the real locus. In the second case, we obtain a real curve with an empty real locus. In the case of real curves with empty real locus, there exist two types of linear systems invariant under complex conjugation distinguished by the fact whether or not they contain a real divisor. Unfortunately, using our methods, we are not able to distinguish between those possibilities.

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