Higher-order topological phases are characterized by protected states localized at the corners or hinges of the system. By applying time-periodic quenches to a two-dimensional lattice with balanced gain and loss, we obtain a rich variety of non-Hermitian Floquet second order topological insulating phases. Each of the phases is characterized by a pair of integer topological invariants, which predict the numbers of non-Hermitian Floquet corner modes at zero and \( \pi \) quasienergies. We establish the topological phase diagram of the model, and find a series of non-Hermiticity induced transitions between different Floquet second order topological phases. We further generalize the mean chiral displacement to two-dimensional non-Hermitian systems, and use it to extract the topological invariants of our model dynamically. This work thus extend the study of higher-order topological matter to more generic nonequilibrium settings, in which the interplay between Floquet engineering and non-Hermiticity yields fascinating new phases.

I. INTRODUCTION

Higher-order topological phases (HOTPs) have attracted great attention in recent years \([1,8]\). They are featured by localized states appearing at the boundaries of their domains. More precisely, an HOTP of order \( n \) (\( n > 1 \)) in spatial dimension \( d \) (\( \geq n \)) possesses topologically protected gapless states at its \( (d-n) \)-dimensional boundaries. Over the years, a rich variety of HOTPs have been found in insulating \([9,28]\), superconducting \([29,43]\) and semi-metallic \([44,49]\) systems, and further classified according to their protecting symmetries \([50,53]\). Experimentally, HOTPs have also been realized in solid-state \([54,58]\), photonic \([59,55]\), acoustic \([66,73]\) and electrical circuit \([74,77]\) platforms, triggering the interest over a wide range of research areas.

Recently, the study of HOTPs have been extended to nonequilibrium settings, in which time-periodic driving fields or gains and losses are applied to a given static system, leading to the discovery of Floquet HOTPs \([78-88]\) and non-Hermitian HOTPs \([89-96]\). The Floquet HOTPs are distinguished from their static cousins by their unique space-time symmetries, topological invariants, and anomalous Floquet corner or hinge states. On the other hand, the HOTPs in non-Hermitian systems are featured by non-Bloch topological invariants, hybrid higher-order skin modes and biorthogonal bulk-boundary correspondence. Yet, under more general conditions, a static system could subject to both time-dependent driving fields and non-Hermitian effects, and much less is known about the fate of HOTPs in such driven open systems. Moreover, the collaboration of drivings and dissipation may induce exotic non-Hermitian Floquet HOTPs that are absent in either closed Floquet systems or non-driven non-Hermitian systems, which certainly deserve careful investigations.

In this work, based on the coupled-wire construction of HOTPs \([78]\), we introduce a class of second order topological insulator (SOTI) model by coupling an array of one-dimensional (1D) topological insulators along a second spatial dimension with dimerized hoppings, as presented in Sec. II. Under the effects of time-periodic quenches and balanced onsite gains and losses, we find rich non-Hermitian Floquet SOTI phases in our system, which are protected by the sublattice and crystal symmetries. In Sec. III, we introduce a pair of integer topological invariants to characterize the found topological phases, and establish the topological phase diagram of our model. A series of topological phase transitions and non-Hermitian Floquet SOTI phases with large topological invariants are found by varying the amplitude of driving fields or the strength of gains and losses. Under the open boundary conditions (OBCs), many non-Hermitian Floquet zero and \( \pi \) modes emerge at the corners of the system, whose numbers are predicted by the bulk topological invariants, as shown in Sec. IV. In Sec. V, we propose a way to dynamically extract the topological invariants and detect the topological phase transitions of our system by measuring the mean chiral displacements of a wave packet. Finally, we summarize our results and discuss the possible experimental realizations of our model in Sec. VII.

II. MODEL AND Symmetry

In this section, we first introduce an SOTI model following the coupled-wire construction of static and Floquet SOTIs \([78]\). Our non-Hermitian Floquet SOTI system is then realized by applying time-periodic quenches and balanced onsite gains and losses to the static SOTI model.

We start with a prototypical tight-binding Hamiltonian \( H \), which describes particles hopping on a two-
FIG. 1. The schematic diagram of the lattice model described by Eq. (1). An array of SSH chains are stacked along the vertical (y) direction, coupled with each other by the hopping amplitudes $J_{10}$ and $J_{20}$, and also subject to an onsite potential bias $\pm 2\mu$.

dimensional (2D) square lattice,

$$
H = \sum_{i,j} [J + (-1)^i \delta](|i, j\rangle \langle i + 1, j| + \text{H.c.}) + \sum_{i,j} J_{10}|i, 2j\rangle \langle i, 2j + 1| + \text{H.c.} + \sum_{i,j} (-1)^j (iJ_{20}|i, j\rangle \langle i, j + 2| - \mu|i, j\rangle \langle i, j| + \text{H.c.}).
$$

Here $i$ ($j$) denotes the lattice site index along the $x$ ($y$) direction of the system. An illustration of the lattice model is presented in Fig. 1. Along the $x$-direction, $J - \delta$ ($J + \delta$) corresponds to the intrachain (intercell) hopping amplitude. Along the $y$-direction, $J_{10}$ and $J_{20}$ characterize the nearest- and next-nearest-neighbor hopping amplitudes, and $\mu$ denotes the strength of a staggered onsite potential. The system described by $H$ can thus be viewed as an array of tight-binding wires lying along the $y$-direction, with each of them being connected to its adjacent neighbors by Su-Schrieffer-Heeger (SSH)-type dimerized couplings. Such kind of "coupled-wire construction" has been demonstrated to be a powerful way of engineering both static and Floquet SOTIs in closed systems. Generally speaking, four zero-energy topological corner modes would appear in the system described by Eq. (1) if both the SSH-type couplings along the $x$-direction and the wires along the $y$-direction are set in topologically nontrivial regimes.

Taking periodic boundary conditions (PBCs) along both $x$, $y$ directions and performing Fourier transformations, we can express Eq. (1) in the momentum representation as $H = \sum_{k_x, k_y} |k_x, k_y\rangle \langle k_x, k_y|H(k_x, k_y)|k_x, k_y\rangle$, where the Hamiltonian matrix $H(k_x, k_y)$ has a Kronecker sum structure

$$
H(k_x, k_y) = H_x(k_x) \otimes \tau_0 + \sigma_0 \otimes H_y(k_y),
$$

with

$$
H_x(k_x) = [(J - \delta) + (J + \delta) \cos k_x] \sigma_x + (J + \delta) \sin k_x \sigma_y, \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad
$$

Here $k_x, k_y \in [\pi, \pi)$ are the quasienergies along $x$ and $y$ directions. $\sigma_0$ and $\tau_0$ are both $2 \times 2$ identity matrices, with $\tau_4 \equiv \sigma_0 \otimes \tau_0$. $\sigma_{x,y,z}$ and $\tau_{x,y,z}$ are Pauli matrices acting in the sublattice spaces in the $x$ and $y$ directions, respectively. It is well known that both $H_x(k_x)$ and $H_y(k_y)$ describe 1D topological insulators, which are characterized by integer winding numbers. When both the conditions $|J - \delta| < |J + \delta|$ and $|\mu| < |J_{20}|$ are satisfied, the 1D descendant systems $H_x(k_x)$ and $H_y(k_y)$ are both in topologically nontrivial phases at half-filling. In this case, according to the analysis in Ref. [75], the parent Hamiltonian $H$ describes an SOTI with four corner modes under the OBCs.

In this work, we investigate whether the interplay between time-periodic drivings and dissipation effects can induce exotic non-Hermitian Floquet SOTI phases with multiple topological corner modes in the system described by $H$. To do so, we introduce balanced gain and loss to the staggered onsite potential $\mu$, i.e., by setting $\mu = u + iv$ with $u, v \in \mathbb{R}$. Furthermore, we apply piecewise time-periodic quenches to each of the wires along the $y$-direction, so that $H_y(k_y)$ becomes

$$
H_y(k_y, t) = \begin{cases} 
2J_1 \cos k_y \tau_x & \text{if } t \in [\ell T, \ell T + T/2) \\
2(\mu + J_2 \sin k_y) \tau_z & \text{if } t \in [\ell T + T/2, \ell T + T) 
\end{cases},
$$

where $t$ is time, $T$ is the driving period and $\ell \in \mathbb{Z}$ counts the number of driving periods. The form of $H_y(k_y)$ remains to be the same during the whole driving period. It is clear that with the driving fields, the hopping amplitude $J_1$ is only turned on in the first half of a driving period. In the second half of the period, the onsite potential $\mu$ and hopping amplitude $J_2$ are switched on. Since the parameters $\mu$, $J_1$ and $J_2$ only couple intracell degrees of freedom and nearest-neighbor unit cells, the periodic quenches of these parameters are expected to be achievable in recent cold atom [78] and photonic [99] experimental setups. Moreover, as will be made clear in the following sections, the choice of our quench protocol allows the system to close and reopen its spectral gaps alternatively at the quasienergies zero and $\pi$ with the change of system parameters. This has also been demonstrated before in the study of Hermitian Floquet SOTIs [78]. Our system could thus possess rich non-Hermitian Floquet SOTI phases, multiple topological phase transitions and many Floquet corner modes following the choice of our quench protocol.

With Eqs. (2) and (3), the time-dependent Hamil-
ton of our periodically quenched system can be expressed as \( H(k_x, k_y, t) = H_x(k_x) \otimes \tau_0 + \sigma_0 \otimes H_y(k_y, t) \).

The resulting Floquet operator, which generates the time evolution of the system over a complete driving period \( T \), is then given by 

\[
U(k_x, k_y) = \sum_{k_x, k_y} U(k_x, k_y) |k_x, k_y\rangle \langle k_x, k_y|,
\]

where

\[
U(k_x, k_y) = \mathcal{T} e^{-\frac{i}{\hbar} \int_0^T dt H(k_x, k_y, t)}.
\]

Here \( \mathcal{T} \) performs the time ordering, and we have set the unit of energy to be \( \hbar/T \), with \( \hbar = T = 1 \). Since the Hamiltonian of the system stays the same within the first and second halves of the driving period, the integration over time on the exponential of \( U(k_x, k_y) \) can be worked out analytically, leading to

\[
U(k_x, k_y) = e^{-\frac{i}{\hbar} H_x(k_x) \otimes \tau_0 + 2(\mu + J_2 \sin k_y) \sigma_0 \otimes \tau_x} \times e^{-\frac{i}{\hbar} H_x(k_x) \otimes \tau_0 + 2J_1 \cos k_y \sigma_0 \otimes \tau_x}.
\]

(6)

Noting that \( H_x(k_x) \otimes \tau_0 \) commutes with \( 2J_1 \cos k_y \sigma_0 \otimes \tau_x \) and \( 2(\mu + J_2 \sin k_y) \sigma_0 \otimes \tau_x \), the expression for \( U(k_x, k_y) \) can be further simplified to

\[
U(k_x, k_y) = e^{-iH_x(k_x) \otimes \tau_0 e^{-i(\mu + J_2 \sin k_y) \sigma_0 \otimes \tau_x} \times e^{-iJ_1 \cos k_y \sigma_0 \otimes \tau_x}}.
\]

(7)

Finally, expanding each term on the right hand side of \( U(k_x, k_y) \) into a Taylor series, and combining the relevant terms, we obtain

\[
e^{-iH_x(k_x) \otimes \tau_0} = \sum_{n=0}^{\infty} \frac{[-iH_x(k_x)]^n}{n!} \otimes \tau_0
\]

(8)

\[
e^{-i(\mu + J_2 \sin k_y) \sigma_0 \otimes \tau_x} = \sigma_0 \otimes \sum_{n=0}^{\infty} \frac{[-i(\mu + J_2 \sin k_y) \tau_x]^n}{n!}
\]

(9)

\[
e^{-iJ_1 \cos k_y \sigma_0 \otimes \tau_x} = \sigma_0 \otimes \sum_{n=0}^{\infty} \frac{(-iJ_1 \cos k_y \tau_x)^n}{n!}
\]

(10)

Plugging these three terms into the right hand side of Eq. (7), we arrive at

\[
U(k_x, k_y) = e^{-iH_x(k_x) \otimes \tau_0} e^{-i(\mu + J_2 \sin k_y) \sigma_0 \otimes \tau_x} e^{-iJ_1 \cos k_y \sigma_0 \otimes \tau_x}
\]

(11)

which gives the Floquet operator of our system at a fixed quasimomentum \((k_x, k_y)\). Without loss of generality, we choose to work within the topological flatband limit of \( H_x(k_x) \), which can be achieved by setting \( J = \Delta/2 \) [97]. Experimentally, such an SSH Hamiltonian can be realized in cold atom systems [100]. With these considerations, the Floquet operator of our system further simplifies to

\[
U(k_x, k_y) = e^{-iH_0(k_x)} e^{-ih_x(k_y) \tau_x} e^{-ih_x(k_y) \tau_x}.
\]

(12)

where

\[
H_0(k_x) = \Delta (\cos k_x \sigma_x + \sin k_x \sigma_y),
\]

(13)

\[
h_x(k_y) = J_1 \cos k_y,
\]

(14)

\[
h_y(k_y) = u + iv + J_2 \sin k_y.
\]

(15)

Note that \( U(k_x, k_y) \) is nonunitary due to the balanced gain and loss terms \( \pm iv \) in the staggered onsite potential \( \mu \tau_z \). In cold-atom systems, this non-Hermitian onsite potential maybe realized by kicking the atoms out of a trap by a resonant optical beam [101], or applying a radio-frequency pulse to excite atoms to an irrelevant state, in which an antitrap is further applied to induce the losses [102].

Before characterizing the topological properties of our non-Hermitian Floquet system, we first analyze the symmetries that allow it to possess corner modes at zero and \( \pi \) quasienenergies under the OBCs. Following the established approach to the symmetry analysis of Floquet operators [103, 104], we first transform \( U(k_x, k_y) \) in Eq. (12) to a pair of symmetric time frames upon similarity transformations, yielding

\[
U_\alpha(k_x, k_y) = U_0(k_x) \otimes U_\alpha(k_y).
\]

(16)

Here \( \alpha = 1, 2 \) and

\[
U_0(k_x) = e^{-iH_0(k_x)},
\]

(17)

\[
U_1(k_y) = e^{-i\frac{h_x(k_y)}{2} \tau_x} e^{-i\frac{h_y(k_y)}{2} \tau_x} e^{-i\frac{h_y(k_y)}{2} \tau_x},
\]

(18)

\[
U_2(k_y) = e^{-i\frac{h_x(k_y)}{2} \tau_x} e^{-i\frac{h_y(k_y)}{2} \tau_x} e^{-i\frac{h_y(k_y)}{2} \tau_x}.
\]

(19)

It is clear that \( U(k_x, k_y) \), \( U_1(k_x, k_y) \) and \( U_2(k_x, k_y) \) are similar to one another, and therefore sharing the same complex Floquet quasienergy spectrum. Furthermore, both \( U_1(k_x, k_y) \) and \( U_2(k_x, k_y) \) possess the sublattice symmetry \( S = \sigma_z \otimes \tau_y \), i.e.,

\[
SU_\alpha(k_x, k_y)S = U_\alpha^{-1}(k_x, k_y).
\]

(20)

for \( \alpha = 1, 2 \), with \( S = S^\dagger \) and \( S^2 = \mathbb{1} \). Besides, we can also identify the diagonal (\( M_+ \)) and off-diagonal (\( M_- \)) spatial symmetries of \( U_\alpha \), i.e.,

\[
M_\pm U_\alpha(k_x, \pm k_y) M_\pm^{-1} = U_\alpha^{-1}(k_x, \pm k_y).
\]

(21)

The spatial symmetries \( M_\pm \) (which happen to be equal to \( S \) here) guarantee the zero- and \( \pi \)-quasienenergy Floquet topological modes, if presence, should appear at the four corners of the system under the OBCs, whereas the topological degeneracy of these non-Hermitian Floquet corner modes are protected by the sublattice symmetry \( S \) [28]. The sublattice symmetry \( S \) allows us to introduce a pair of integer winding numbers to characterize the topological phases of our system, as will be discussed in the following section.
III. TOPOLOGICAL INVARIANTS

With the relevant symmetries $S$ and $M_{\pm}$ being identified, we will now introduce the topological invariants of our system.

According to the topological classification of Floquet operators [103, 104], a Floquet system in one-dimension is characterized by integer winding numbers. This has been demonstrated for both Hermitian [105–108] and non-Hermitian [109–113] Floquet models. Since the Floquet operator $U(k_x, k_y)$ of our system in Eq. (12) has a Kronecker product structure, its topological invariants may be constructed from the winding numbers of descendant 1D models $U_0(k_x)$ and $U_\alpha(k_y)$ ($\alpha = 1, 2$) in the symmetric time frames. To do so, we first note that $U_0(k_x)$ is simply the evolution operator of a static SSH model over a period. Its topological winding number $w$ is therefore equal to 1 (0) in the topologically nontrivial (trivial) regime [97]. For the parameter choice in Eq. (13), we simply have $w = 1$. Furthermore, applying the Euler formula to Eqs. (18) and (19), we present the topological phase diagram of our non-Hermitian Floquet SOTI model in Eq. (30), we refer the reader to [97] for the details.

IV. TOPOLOGICAL PHASE DIAGRAM

In this section, based on the topological invariants introduced in Eq. (30), we present the topological phase diagram of our non-Hermitian Floquet SOTI model in typical situations.

From Eqs. (28) and (29), it is clear that $\nu_0 \neq 0$ ($\nu_\pi \neq 0$) in Eq. (30) if both $w$ and $w + 1/2$ or $w - 1/2$ are nonzero. As the parameters of the 1D descendant system in the $x$-direction in Eq. (13) has been set inside the topological nontrivial regime, we have the winding number $w = 1$ for $U_0(k_x)$. A topological phase transition in our system is then accompanied by the closing and reopening of a spectral gap of $U_\alpha(k_y)$ at the quasienergy zero or $\pi$ on the complex plane.

According to Eq. (25), the gapless condition of $U_\alpha(k_y)$ is determined by

$$\cos[\mathcal{E}(k_y)] = \cos[h_x(k_y)] \cos[h_z(k_y)] = \pm 1,$$

where the $+1$ ($-1$) on the right hand side of Eq. (31) corresponds to a gap closing at $\mathcal{E}(k_y) = 0$ [$\mathcal{E}(k_y) = \pi$]. With the help of Eqs. (14) and (15), Eq. (31) is equivalent to the following two equalities

$$\sin(u + J_2 \sin k_y) = 0, \quad \cos(J_1 \cos k_y) \cos(u + J_2 \sin k_y) \cosh v = \pm 1.$$ 

Combining them together, we can express the gapless condition of $U_\alpha(k_y)$ in Eq. (16) as

$$v = \pm \text{arccosh} \left\{ \frac{1}{\cos \left[ J_1 \sqrt{1 - (n\pi - u)^2/J_2^2} \right]} \right\},$$

where $n \in \mathbb{Z}$ and $|n\pi - u| < |J_2|$. Eq. (34) determines the boundaries between different topological phases in the parameter space, across which the system described...
by $U(k_x, k_y)$ in Eq. (12) is expected to change from one non-Hermitian Floquet SOTI phase to another.

In the following, we present the topological phase diagrams of our periodically quenched non-Hermitian lattice model Eq. (11) in three typical situations. In the first case, we show the phase diagram versus the real and imaginary parts of the onsite potential $\mu = u + iv$ in Fig. 2. The other system parameters are chosen as $J = \delta = \Delta/2 = \pi/40$, $J_1 = 0.5\pi$ and $J_2 = 5\pi$. The values of topological invariants $(\nu_0, \nu_\pi)$, obtained from Eqs. (28)-(30), are shown explicitly in Fig. 2 within each of the non-Hermitian Floquet SOTI phases. The black lines separating different phases (regions with different colors) in Fig. 2 are obtained from the gapless condition Eq. (34). From the phase diagram, we observe a series of topological phase transitions accompanied by quantized jumps of $\nu_0$ and/or $\nu_\pi$ by varying either the real or imaginary part of $\mu$. Therefore, the existence of balanced onsite gains and losses can indeed induce phase transitions and new types of non-Hermitian Floquet SOTIs in our system. Furthermore, we found a couple of SOTI phases characterized by large topological invariants $(\nu_0, \nu_\pi)$. Detailed numerical calculations suggest that the values of $(\nu_0, \nu_\pi)$ can be arbitrarily large with the increase of the hopping amplitude $J_2$. These SOTI phases originate from the interplay between the time-periodic driving fields and the onsite gains and losses. They are thus unique to non-Hermitian Floquet systems. Under the OBCs, a non-Hermitian Floquet SOTI phase with large invariants $(\nu_0, \nu_\pi)$ will also admit multiple quartets of topological corner modes at zero and $\pi$ quasienergies, as will be demonstrated in the next section.

In the second case, we present the topological phase diagram of our model versus the hopping amplitude $J_1$ and the imaginary part of onsite potential $\nu$ in Fig. 3. The other system parameters are fixed at $J = \delta = \Delta/2 = \pi/20$, $u = 0$, and $J_2 = 3\pi$. The values of topological invariants $\nu_0$ and $\nu_\pi$ for each of the phases are shown separately in the panels (a) and (b) of Fig. 3, respectively. Similar to the first case, we observe rich non-Hermitian Floquet SOTI phases and phase transitions at different values of $J_1$ and $\nu$. Moreover, around certain values of $J_1$ (e.g., $J_1 = 2.5\pi$), we find that by increasing the gain and loss strength $\nu$, the system can shift to topological phases with larger invariants, which could also support more quartets of corner modes under the OBCs. Such kinds of non-Hermiticity enhanced topological properties are usually unexpected in systems with losses. Therefore, it forms one of the defining features of our construction, with potential applications in preparing Floquet topological states and combating environmental effects in quantum information tasks.

For completeness, we also present the phase diagram of our model versus the hopping amplitudes $J_1$ and $J_2$ in Fig. 4. It is clear that a series of topological phase transitions can be induced by varying both $J_1$ and $J_2$, yielding rich non-Hermitian Floquet SOTI phases. Furthermore, in certain ranges of $J_1$ (e.g., around $J_1 = 0.5\pi$),
FIG. 4. Topological phase diagram of the periodically quenched non-Hermitian lattice model (12) versus the hopping amplitudes \( J_1 \) and \( J_2 \). The other system parameters are \( \Delta = \pi/20, u = 0.2\pi \) and \( v = 1i \). The values of topological invariants \( \nu_0, \nu_\pi \) for each non-Hermitian Floquet SOTI phase with a uniform color are shown in panel (a) [(b)]. The black lines separating different phases are obtained from the gapless condition Eq. (35).

the magnitude of topological winding numbers \((\nu_0, \nu_\pi)\) tend to increase with \( J_2 \) monotonically. This observation again highlights the power of Floquet engineering in the realization of non-Hermitian SOTI phases with large topological invariants and multiple corner modes.

Note in passing that in the absence of the Floquet driving fields, our system could only possess a static non-Hermitian SOTI phase with winding number \( \nu_0 = 1 \), yielding at most four corner modes at zero energy under the OBCs. Thanks to the Floquet terms, the system could possess much richer SOTI phases with large topological winding numbers \((\nu_0, \nu_\pi)\), as presented by the phase diagrams. These phases are further subject to a \( \mathbb{Z} \times \mathbb{Z} \) topological characterization, and therefore totally different from the static SOTI phases. As will be demonstrated in the next section, the non-Hermitian Floquet SOTI phases also possess many corner modes at both zero and \( \pi \) quasienergies, with the \( \pi \) modes being unique to Floquet systems. Therefore, the Floquet term is essential in generating the rich topological features of our system.

To summarize, we find rich non-Hermitian Floquet SOTI phases with large topological invariants in our system. In the following two sections, we discuss two experimentally relevant signatures of the intriguing phases found in our system. We first present the Floquet spectrum and corner modes of our system under the OBCs, and establish the correspondence between the corner modes and the bulk topological invariants \((\nu_0, \nu_\pi)\). Next, we show how to extract the invariants \((\nu_0, \nu_\pi)\) from the nonunitary stroboscopic dynamics of easily prepared wave packets.

V. CORNER STATES AND BULK-CORNER CORRESPONDENCE

Under the OBCs, the Floquet operator of our periodically quenched lattice model Eq. (12) takes the form

\[
U = U_x \otimes U_y,
\]

where

\[
U_x = e^{-i \sum_{i,j} \frac{\Delta}{2} \left[(i,j)(i+1,j)+\text{H.c.}\right]}
\]

\[
U_y = e^{-i \sum_{i,j} \left[(i,j)(i,j+1)\right] - i \mu \sum_{i,j} J_1(i,j) + \text{H.c.}}
\]

The number of lattice sites along the \( x \) (\( y \)) direction is \( L_x = 2N_x \) (\( L_y = 2N_y \)), with \( N_x \) (\( N_y \)) being the number of unit cells. The Floquet quasienergy spectrum and corner modes of the model can be obtained by solving the eigenvalue equation \( U|\Psi\rangle = e^{-iE}|\Psi\rangle \), where \( E \) is the quasienergy and \( |\Psi\rangle \) is the corresponding Floquet right eigenvector. Note that due to the balanced gain and loss in the onsite potential \( \mu = u + iv \), the quasienergy \( E \) is in general a complex number. We define a quasienergy gap in this case as a point on the complex plane, which is avoided by all the bulk eigenstates for a given set of system parameters.

In the topological nontrivial regime, a 2D SOTI is featured by topologically protected zero energy modes around the corners of the lattice. In a non-Hermitian Floquet SOTI, there could be two types of topological corner modes, whose quasienergies are zero and \( \pi \). For the class of periodically quenched lattice model studied in this work, the physical origin of these corner modes can be directly inferred from the Kronecker product structure of Floquet operator in Eq. (35) and its underlying sublattice symmetry \( S \). As discussed in Sec. II, our system can be viewed as an array of 1D Floquet topological insulators (FTIs) lying along the \( y \)-direction, with each of them being connected to its adjacent neighbors by SSH-type dimerized couplings along the \( x \)-direction. The number of zero and \( \pi \) quasienergy edge modes of the 1D FTI is determined by its topological winding numbers \((w_0, w_\pi) = [(w_1 + w_2)/2, (w_1 - w_2)/2]\) following Eq. (28), whereas the number of zero-quasienergy edge modes of the 1D SSH chain is determined by its winding number \( w \). When the 1D FTIs and 1D SSH chains are coupled to from our 2D Floquet system, there are only two possibilities for the localized modes at zero and \( \pi \) quasienergies to appear. That is, if the 1D descendant systems \( U_x \) and \( U_y \) both possess zero quasienergy edge modes, they will couple to form a Floquet corner mode with quasienergy zero in the parent 2D system described by \( U \) in Eq. (35), and the total number of these zero-quasienergy corner modes
is determined by the invariant \( \nu_0 = w v_0 \). Similarly, if the 1D system \( U_x (U_y) \) possesses a zero (\( \pi \)) quasienergy edge mode, they will couple to form a Floquet corner mode with quasienergy \( \pi \) in the parent system \( U = U_x \otimes U_y \), and the total number of these \( \pi \) corner modes is determined by the invariant \( \nu_{\pi} = w v_{\pi} \). The zero- and \( \pi \)-corner modes are robust to perturbations, so long as the sublattice symmetry \( S \) is preserved. Moreover, the above analyses indicate that the number of non-Hermitian Floquet corner modes with quasienergy zero \( (\pi) \) in the 2D system is \( n_0 = n_{x0} n_{y0} \) \( (n_{\pi} = n_{x0} n_{y\pi}) \), where \( n_{x0} \) is the number of zero edge modes of \( U_x \) and \( n_{y0} \) \( (n_{y\pi}) \) is the number of zero \( (\pi) \) edge modes of \( U_y \). Combining these observations with the invariants \( (\nu_0, \nu_{\pi}) \) defined in Eq. (30), we could build the connection between the numbers of non-Hermitian Floquet corner modes \( (n_0, n_{\pi}) \) and the bulk topological numbers as

\[
 n_0 = 4|\nu_0|, \quad n_{\pi} = 4|\nu_{\pi}|. \tag{38}
\]

Eq. (38) establishes the bulk-corner correspondence of 2D chiral symmetric Floquet systems with the tensor product structure of Eq. (35), which also holds in the Hermitian limit \((\mu \in \mathbb{R})\) so long as the sublattice symmetry \( S \) is retained.

Note in passing that the system described by \( U_x \) in Eq. (36) is essentially static, and therefore could only possess edge modes at zero quasienergy. If a zero (zero or \( \pi \)) edge mode of the SSH chain (1D FTI) is coupled to a bulk mode of the 1D FTI (SSH chain), it may result in an edge state with a finite quasienergy in the 2D system. Such kinds of finite-quasienergy (i.e., \( E \neq 0, \pi \)) edge states are gapped, and their numbers will change with the system size as pointed out in Ref. [78] for Hermitian systems. They are thus trivial gapped edge states. Therefore, there are no edge states at \( E = 0, \pi \), and the topological numbers in Eq. (36) only counts the number of zero and \( \pi \) corner modes. The system is thus not a first-order topological system at \( E = 0, \pi \), although the topological numbers are well defined.

To demonstrate the topological phase transitions and bulk-corner correspondence of our system, we present the Floquet spectrum of \( U \) in Eq. (35) under the OBCs for two typical examples. In order to show the evolution of spectral gaps with the system parameters in a more transparent manner, we introduce a pair of spectral gap functions, defined as

\[
 G_0 = \frac{1}{\pi} \sqrt{(\text{Re}E)^2 + (\text{Im}E)^2}, \tag{39}
\]

\[
 G_\pi = \frac{1}{\pi} \sqrt{|(\text{Re}E - \pi)^2 + (\text{Im}E)^2|}. \tag{40}
\]

It is clear that when the system becomes gapless at the quasienergy \( E = 0 \) \((E = \pm \pi)\), we will have \( G_0 = 0 \) \((G_{\pi} = 0)\). In Fig. 5(a), we show the evolutions of \( G_0 \) (red circles) and \( G_{\pi} \) (blue lines) versus the real part \( u \) of the onsite potential. The other system parameters are chosen as \( \Delta = \pi/20 \), \( J_1 = 0.5\pi \), \( J_2 = 5\pi \) and \( v = 0.5 \). The numbers of non-Hermitian Floquet zero and \( \pi \) corner modes \( (n_0, n_{\pi}) \) are denoted explicitly in each panel. The ticks \( u_i \) \( (i = 1, 2) \) and \( v_i \) \( (i = 1, 2, 3, 4, 5) \) along the horizontal axis are the bulk phase transition points deduced from Eq. (34).

![FIG. 5. Spectral gap functions \( G_0 \) (red circles) and \( G_{\pi} \) (blue lines) versus the real and imaginary parts of onsite potential \( \mu = u + iv \). The system parameters are \( \Delta = \pi/20 \), \( J_1 = 0.5\pi \), \( J_2 = 5\pi \) and \( v = 0.5 \) \((u = 0.5\pi)\) for panel (a) [(b)]. The numbers of non-Hermitian Floquet topological corner modes at zero \((G_0 = 0)\) and \( \pi \) \((G_{\pi} = 0)\) quasienergies \( (n_0, n_{\pi}) \) are denoted explicitly in each panel. The ticks \( u_i \) \( (i = 1, 2) \) and \( v_i \) \( (i = 1, 2, 3, 4, 5) \) along the horizontal axis are the bulk phase transition points deduced from Eq. (34).](image-url)
fixed at $\Delta = \pi/20$, $J_1 = 2.5\pi$, $J_2 = 3\pi$ and $u = 0$. It is clear that with the increase of $v$, the numbers of corner modes $\langle n_0, n_x \rangle$ changes from (4, 0) to (12, 0) across $v_1$ and from (12, 0) to (12, 8) across $v_2$, coinciding with the bulk-corner relation Eq. (35). Such an enhancement of topological signatures in deeper non-Hermitian regimes is intriguing, which might be used to design new topological state preparation schemes and achieve quantum information tasks in open systems. To see the numbers and profiles of the Floquet corner modes more explicitly, we show the first twenty states of the system at $v = 2$ in Fig. 6(b), with the other system parameters chosen to be the same as in Fig. 6(a). The twelve (eight) non-Hermitian Floquet corner modes at the quasienergy zero ($\pi$) are denoted by red circles (blue dots), whose probability distributions are shown in Figs. 7(a)-(c) [Figs. 7(d)-(e)]. Here we plotted the distributions of right eigenvectors of $U$ in Eq. (35) in the lattice representation, and similar results can be obtained from the left eigenvectors. We see that the zero and $\pi$ modes are indeed well-localized around the four corners of the 2D lattice, which are protected by the sublattice symmetry $\mathcal{S} = \sigma_z \otimes \tau_y$ introduced in Sec. III.

For completeness, in Fig. 8 we present the gap functions with respect to the quasimomentum $k_x$ ($k_y$) by taking the PBC (OBC) along $x$-direction and OBC (PBC) along $y$-direction of the lattice. For all the three cases considered in Fig. 8, the systems are set in non-Hermitian Floquet SOTI phases, and the gap functions $G_0$ and $G_\pi$ are found to be gapped at $G_0 = G_\pi = 0$. This means that in the complex Floquet spectrum of the system, all possible 1D edge states are gapped at the quasienergies zero and $\pi$, as expected for SOTI phases.

VI. DYNAMICAL CHARACTERIZATION OF THE TOPOLOGICAL PHASES

The mean chiral displacement (MCD) is first introduced as the time-averaged chiral displacement $\langle \mathcal{C} \rangle$ of an initially localized wave packet in a 1D lattice within the symmetry classes AIII and BDI [114]. Later, it is generalized to Floquet systems [106, 115], non-Hermitian systems [110, 111], interacting systems [116], systems in other symmetry classes [108] and higher physical dimensions [78]. Experimentally, the MCD has been measured in cold atom [117] and photonic [118] setups. In this section, we extend the definition of MCD to 2D non-Hermitian Floquet systems with sublattice symmetry, and demonstrate how to extract the topological invariants of our model dynamically from the MCDs.

For a 2D lattice model with the sublattice symmetry $\mathcal{S}$, we define its chiral displacement operator as $\mathcal{C} = \hat{r} \otimes \mathcal{S}$, with $\hat{r}$ being the unit cell position operator. The chiral displacement of a wave packet $\rho_0$ in the symmetric time frame $\alpha$ is then given by

$$C_\alpha(t) = \text{Tr} \left( \rho_0 \hat{U}^\dagger \alpha U^\alpha \right),$$

where $\alpha = 1, 2$, $t$ counts the number of driving periods, and the trace $\text{Tr}(\cdots)$ is taken over all degrees of freedom of the system. To build the connection between $C_\alpha$ and the topological invariants $\langle v_0, v_x \rangle$ of the system in the most straightforward manner, we prepare the initial state $\rho_0$ in the central unit cell $(m, n) = (0, 0)$ of the lattice,
with all the four sublattices being uniformly filled, i.e., \( \rho_0 = |0, 0, 0, 0 \rangle \otimes \mathbb{I}/4 \). For our periodically quenched lattice model Eq. (12), \( U_\alpha \) is given by the inverse Fourier transform of Eq. (16), and \( \tilde{U}_\alpha \) is defined such that if \( |\Psi\rangle \) is a right eigenvector of \( U_\alpha \) with eigenvalue \( e^{-iE} \), it is the left eigenvector of \( \tilde{U}_\alpha \) with the same eigenvalue.

In the following, we will relate the long-time average of \( C_\alpha(t) \) to the topological invariants of 2D non-Hermitian Floquet operators with the structure of Eq. (16) and the sublattice symmetry \( S \). Note that in the Hermitian limit, we simply have \( \tilde{U}_\alpha = \alpha \), and our derivations below will also hold. Taking the trace in Eq. (41) explicitly and inserting the identities in the lattice and momentum representations, we find

\[
C_\alpha(t) = \frac{1}{4} \sum_{k_x, k_y, k_x', k_y'} \sum_{m, n} e^{i(k_x, m + k_y, n)} N_{\alpha}^{k_x, k_y} S_{\alpha}^{k_x', k_y'} \left( \tilde{U}_\alpha^{ik} \right) \left( \tilde{U}_\alpha^{ik'} \right)
\]

where the trace \( \text{tr} \left[ \cdots \right] \) in the second line is only taken over the sublattice degrees of freedom. Using the Fourier expansion \( m, n = \frac{1}{N_x N_y} \sum_{k_x, k_y} e^{i(k_x, m + k_y, n)} |k_x, k_y, k_x', k_y' \rangle \langle k_x, k_y | m, n \rangle \)

Eq. (42) can be simplified to

\[
C_\alpha(t) = \frac{1}{4} \sum_{k_x, k_y, k_x', k_y'} \sum_{m, n} e^{i(k_x, m + k_y, n)} N_{\alpha}^{k_x, k_y} S_{\alpha}^{k_x', k_y'} \left( \tilde{U}_\alpha^{ik} \right) \left( \tilde{U}_\alpha^{ik'} \right)
\]

With the help of summation formulas \( \sum_{m} m e^{i(k_x, k_y - k_x', k_y')} = iN_x \delta_{k_x, k_x'} \delta_{k_y, k_y'} \) and \( \sum_{n} n e^{i(k_x - k_x', k_y - k_y')} = iN_y \delta_{k_y, k_y'} \), we further obtain

\[
C_\alpha(t) = \frac{1}{4} \sum_{k_x, k_y, k_x', k_y'} \sum_{m, n} \left( i\partial_{k_x} \delta_{k_x, k_x'} \right) \left( i\partial_{k_y} \delta_{k_y, k_y'} \right) \left( \tilde{U}_\alpha^{ik} \right) \left( \tilde{U}_\alpha^{ik'} \right)
\]

Finally, taking the continuous limit \( N_j \to \infty \), we have \( N_j \delta_{k_x, k_x'} \to \delta(k_x - k_x') \) and \( \sum_{k_x, k_y} \to \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \)

For our periodically quenched lattice model, the expression of chiral displacement \( C_\alpha(t) \) can be further simplified. Noting the tensor product structure of Floquet operator \( U_\alpha \) in Eq. (16) and the expression of sublattice symmetry operator \( S = \sigma_x \otimes \tau_y \), we can write \( C_\alpha(t) \) as a product of chiral displacements in the descendant 1D systems as \( C_\alpha(t) = C_x(t) C_{\alpha y}(t) \), where

\[
C_x(t) = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \left( \tilde{U}_\alpha^{ik} \right) \left( \tilde{U}_\alpha^{ik'} \right)
\]

\[
C_{\alpha y}(t) = \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \left( \tilde{U}_\alpha^{ik} \right) \left( \tilde{U}_\alpha^{ik'} \right)
\]

Summing up the chiral displacements \( C_\alpha(t) \) over different numbers \( t \) of the driving period and taking the long-time average, we obtain the MCD of our system in the \( \alpha \)'s time frame as

\[
\overline{C}_\alpha = \lim_{t \to \infty} \frac{1}{t} \sum_{t'} C_x(t') \times \int_{-\pi}^{\pi} \frac{dk_x}{2\pi} \int_{-\pi}^{\pi} \frac{dk_y}{2\pi} \left( \tilde{U}_\alpha^{ik} \right) \left( \tilde{U}_\alpha^{ik'} \right)
\]

where we have inserted a normalization factor \( \text{tr} \left[ \tilde{U}_\alpha^{ik} \right] \) to compensate for the changing norm of the state during the nonunitary evolution. Note that the same expression for \( \overline{C}_\alpha \) can be derived if the dynamics is expressed in the biorthogonal basis [120].
In previous studies, it has been shown that under the limit $\lim_{N_\text{c} \to \infty} \frac{1}{N_\text{c}} \sum_{i=1}^{N_\text{c}} C_z(t')$ is averaged to $w/2$ \cite{100}. and the second line in Eq. (48) converges to $w_\alpha/2$ \cite{110}. Putting together, we would obtain

$$\bar{C}_\alpha = \frac{w w_\alpha}{4} = \frac{\nu_\alpha}{4}$$

(49)

for $\alpha = 1, 2$ according to Eq. (29). Therefore, with the help of Eq. (30), we establish the connection between the topological invariants $(\nu_0, \nu_\pi)$ and the MCDs as

$$\nu_0 = 2(\bar{C}_1 + \bar{C}_2) \equiv 2C_0,$$  

(50)

$$\nu_\pi = 2(\bar{C}_1 - \bar{C}_2) \equiv 2C_\pi.$$  

(51)

These relations have been derived before for Hermitian Floquet SOTIs \cite{78}. Upon appropriate modifications, we find that they also hold in non-Hermitian Floquet systems with the sublattice symmetry $S$. Experimentally, by measuring the MCDs $(\bar{C}_1, \bar{C}_2)$ of the dynamics over a long time-duration, we would be able to extract the topological invariants $(\nu_0, \nu_\pi)$ for the class of non-Hermitian Floquet SOTI models studied in this work.

To be concrete, we present a typical example of the recombined MCDs $(C_0, C_\pi)$ obtained numerically from Eq. (48) for our periodically quenched lattice model Eq. (12) in Fig. 9. The system parameters are chosen to be $\Delta = \pi/20$, $J_1 = 0.5\pi$, $J_2 = 5\pi$, $u = 0.25\pi$, and the dynamics is averaged over $M = 100$ driving periods. From Fig. 9, we see clearly that the value of $C_0$ or $C_\pi$ gets a quantized jump every time when the imaginary part $\nu$ of the onsite potential reaches a topological phase transition point $v_i$ ($i = 1, ..., 10$), as predicted by Eq. (34). Furthermore, between each pair of adjacent transition points, the values of $(2C_0, 2C_\pi)$ remain quantized, equaling to the topological invariants $(\nu_0, \nu_\pi)$ of the corresponding non-Hermitian Floquet SOTI phase as shown in Fig. 9. Putting together, we verified the correctness of the relations in Eqs. (50) and (51) between the bulk topological invariants and MCDs of non-Hermitian Floquet SOTIs with sublattice symmetry. In the meantime, these results demonstrate the usefulness of MCDs in characterizing and detecting topological phases and phase transitions in 2D non-Hermitian Floquet systems. Numerically, we observe good quantizations of $(2C_0, 2C_\pi)$ for an average over as few as $M = 15$ driving periods, which should be well within reach under current experimental conditions.

In experiments, the MCD could be detected in both cold atom and photonic systems. In a photonic setup, the MCD could be obtained from the the quantum walk of twisted photons by measuring the Zak phase \cite{114}, or from the chiral intensity distribution of structured light \cite{116}. In a cold atom setup, the MCD can be obtained from the time-of-flight images at different time steps of the evolution of a wave packet, which is initially prepared at central unit cell of the lattice and then subjected to periodically switched lattice parameters \cite{117,118}. Since our system can be viewed as the Kronecker sum of two 1D systems, and the non-Hermitian term can be engineered in both cold atom and photonic systems, we expect that the MCDs we introduced could be detectable in both cold atom and photonic setups.

VII. SUMMARY AND DISCUSSION

In this work, we found rich non-Hermitian Floquet SOTI phases in periodically quenched 2D lattices with balanced gain and loss. Each of the phases is characterized by a pair of integer topological invariants $\nu_0$ and $\nu_\pi$, which allow us to establish the topological phase diagram of the model. We further observed multiple non-Hermitian Floquet SOTI phases with large topological invariants and various gain or loss-induced topological phase transitions. Under the OBCs, the invariants $\nu_0$ and $\nu_\pi$ predict the numbers of protected Floquet corner modes at the quasienergies zero and $\pi$. Thanks to the interplay between the periodic drivings and non-Hermitian effects, we found a series of non-Hermitian Floquet SOTI phases with many zero and $\pi$ corner modes, which might be useful in topological state preparations, detections and quantum information technologies. Finally, we introduced a generalized version of the mean chiral displacement, which could capture the topological invariants of our system through the wave packet dynamics.

Before discussing the experimental realization of our model and possible future directions, the essential role played by the non-Hermitian term in our system deserve.
to be emphasized. First, a series of topological phase transitions can be induced by varying the non-Hermitian term as reflected in the phase diagrams, and rich non-Hermitian Floquet SOTI phases could emerge after these transitions. These new phases could persist only when the system is subject to both the driving fields and the non-Hermitian effects. Therefore, they are unique to non-Hermitian Floquet systems, different from any phases that may appear in the system if the non-Hermitian term is switched off. Second, in the phase diagram with respect to $J_1$ and $\nu$, we observe that with the increase of the non-Hermitian term $\nu$ around $J_1 = 2.5\pi$, the system can undergo a transition from a non-Hermitian Floquet SOTI phase with winding numbers $(\nu_0, \nu_1) = (1, 0)$ to another phase with $(\nu_0, \nu_1) = (3, -2)$. This means that the resulting phase could carry larger topological invariants and more topological corner modes when the gains and losses become stronger, which clearly runs counter to the belief that the non-Hermitian term is usually destructive for topological phases. The underlying physics behind this intriguing observation is again the interplay between the losses and driving fields, for which the non-Hermitian term is necessary. Putting together, the SOTI phases discovered in our system are different from those in static systems, in the sense that the former and later are characterized by distinct topological invariants and phase transitions. They are also different from SOTI phases in Hermitian Floquet systems, as the non-Hermitian term could create new phase transitions and SOTI phases with even larger topological invariants compared with the Hermitian counterparts. Our work thus extend the study of SOTIs to physical settings with both driving terms and more topological corner modes when the gains and losses, and unraveled the richness of non-Hermitian Floquet SOTI phases that can appear in such situations.

A candidate setup in which the bulk Floquet operator of our system might be realizable is the nitrogen-vacancy-center in diamond [107, 121]. By applying a universal dilation scheme, an arbitrary non-Hermitian model with a finite number of bands can in principle be mapped to a Hermitian Hamiltonian in an enlarged Hilbert space [121]. The non-Hermitian Floquet band structure and dynamics of our system can then be studied with the help of the dilated Hamiltonian and its resulting unitary evolution, in which the periodic driving can also be implemented [107]. In the definition of our model, we have set the driving period $T = 1$, leading to a dimensionless driving frequency $\omega = 2\pi$. The other system parameters used in the phase diagrams are either smaller then or comparable to $\omega$. According to the setups introduced in [107, 121], we expect that the choices of system parameters in our model should be within reach under current or near-term experimental conditions. Another possible setup that could be used to realize our model is the cold atom system. In cold atom systems, there are mature technologies to realize topological bands in different physical dimensions [122, 123]. An SOTI might then be realized by loading ultracold atoms into the orbital angular momentum states of an optical lattice [124]. The non-Hermitian term in our system might be engineered by staggered onsite atom losses. To obtain such losses, one could introduce resonant couplings between the ground and excited states of atoms, which realizes the effective loss for the ground state and also controls the staggered loss [91]. The staggered loss is further equivalent to the staggered gain-loss configuration in our system up to a constant. Finally, the periodic quenches can be achieved by stepwise Raman-induced couplings [113]. In the cold atom setup realized by Ref. [113], the magnitude of the hopping rate is $\hbar\Omega$, where the Raman-coupling rate $\Omega \sim 2\pi \times 2.3$kHz. The driving period realized in the experiment is around 0.22ms, which corresponds to a driving frequency $\omega \sim 2\pi \times 4.5$kHz. From these experimental data, it is clear that the realized Floquet hopping amplitude $\hbar\Omega$ and the energy scale of driving photon $\hbar\omega$ are comparable. On the other hand, the driving frequency of our model is $\omega = 2\pi$ in dimensionless units, and the Floquet hopping terms $J_1$ and $J_2$ are set within the range of $(0, 3\pi)$ for most of our numerical examples. Therefore, referring to the experiment performed in Ref. [113], the system parameters involved in our numerical simulations are expected to be reasonable, as their magnitudes are either smaller than or comparable to the (dimensionless) driving frequency $\omega = 2\pi$. Putting together, we expect that our model should be realizable in cold atom systems as well in the context of current or near-term experimental technologies.

In future work, it would be interesting to generalize our strategies to the engineering of non-Hermitian Floquet HOTPs in other symmetries classes and higher spatial dimensions. For example, due to the sublattice symmetry $\mathcal{S}$, the spatial symmetries $\mathcal{M}_\pm$ and the configuration of hopping amplitudes $J_{1,2}$, the corner modes of our model are expected to appear at the four corners of a lattice with a square-shaped boundary. In systems with honeycomb or kagome lattice structures, the SOTI phases would be protected by a different set of crystal and rotational symmetries, and the corner modes might be observable under a triangular-shaped boundary [52]. Finding non-Hermitian Floquet SOTI phases in such kinds of lattices would be an interesting topic for further study. Moreover, in superconducting systems, the interplay between driving and non-Hermitian effects may also induce multiple quartets of Floquet Majorana corner modes, which are potentially useful in realizing certain topological quantum computing tasks [79, 88].

ACKNOWLEDGEMENT

L. Z. is supported by the National Natural Science Foundation of China (Grant No. 11905211), the China Postdoctoral Science Foundation (Grant No. 2019M662444), the Fundamental Research Funds for the Central Universities (Grant No. 841912008), the Young Talents Project at Ocean University of China.
[1] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Science 357, 61 (2017).
[2] W. A. Benalcazar, B. A. Bernevig, and T. L. Hughes, Phys. Rev. B 96, 245115 (2017).
[3] Z. Song, Z. Fang, and C. Fang, Phys. Rev. Lett. 119, 246402 (2017).
[4] J. Langbehn, Y. Peng, L. Trifunovic, F. von Oppen, and P. W. Brouwer, Phys. Rev. Lett. 119, 246401 (2017).
[5] K. Hashimoto, X. Wu, and T. Kimura, Phys. Rev. B 95, 165443 (2017).
[6] F. Schindler, A. M. Cook, M. G. Vergniory, Z. Wang, S. S. P. Parkin, B. A. Bernevig, and T. Neupert, Sci. Adv. 4, eaat0346 (2018).
[7] F. Liu and K. Wakabayashi, Phys. Rev. Lett. 118, 076803 (2017).
[8] R.-J. Slager, L. Rademaker, J. Zaanen, and L. Balents, Phys. Rev. B 98, 201114(R) (2018).
[9] S. Franca, J. van den Brink, and I. C. Fulga, Phys. Rev. B 98, 011114(R) (2018).
[10] M. Ezawa, Phys. Rev. Lett. 121, 116801 (2018).
[11] E. Khalaf, Phys. Rev. B 97, 205136 (2018).
[12] F. Liu, H.-Y. Deng, and K. Wakabayashi, Phys. Rev. Lett. 122, 086804 (2019).
[13] L. Trifunovic and P. W. Brouwer, Phys. Rev. X 9, 011012 (2019).
[14] F. K. Kunst, G. van Miert, and E. J. Bergholtz, Phys. Rev. B 97, 241405(R) (2018).
[15] K. Kudo, T. Yoshida, and Y. Hatsugai, Phys. Rev. Lett. 123, 196402 (2019).
[16] T. J. Tugel, V. Chua, and T. L. Hughes, Phys. Rev. B 100, 115126 (2019).
[17] F. Zangeneh-Nejad and R. Fleury, Phys. Rev. Lett. 123, 053902 (2019).
[18] O. Pozo, C. Repellin, and A. G. Grushin, Phys. Rev. Lett. 123, 247401 (2019).
[19] M. J. Park, Y. Kim, G. Y. Cho, and S. Lee, Phys. Rev. Lett. 123, 216803 (2019).
[20] Y. Hwang, J. Ahn, and B.-J. Yang, Phys. Rev. B 100, 205126 (2019).
[21] H. Araki, Phys. Rev. Research 2, 012009(R) (2020).
[22] L. Li, M. Umer, and J. Gong, Phys. Rev. B 98, 205422 (2018).
[23] R. Chen, C.-Z. Chen, J.-H. Gao, B. Zhou, and D.-H. Xu, Phys. Rev. Lett. 124, 036803 (2020).
[24] H. Li and K. Sun, Phys. Rev. Lett. 124, 036401 (2020).
[25] Y. Xu, Z. Song, Z. Wang, H. Weng, and X. Dai, Phys. Rev. Lett. 122, 256402 (2019).
[26] R. Queiroz and A. Stern, Phys. Rev. Lett. 123, 036802 (2019).
[27] R. Kozlovsky, A. Graf, D. Kochan, K. Richter, and C. Gorini, Phys. Rev. Lett. 124, 126804 (2020).
[28] R. Queiroz, I. C. Fulga, N. Avraham, H. Beidenkopf, and J. Cano, Phys. Rev. Lett. 123, 266802 (2019).
[29] Z. Yan, Phys. Rev. Lett. 123, 177001 (2019).
[30] Z. Yan, F. Song, and Z. Wang, Phys. Rev. Lett. 121, 096803 (2018).
[31] R.-X. Zhang, W. S. Cole, X. Wu, and S. Das Sarma, Phys. Rev. Lett. 123, 167001 (2019).
[32] C. Zeng, T. D. Stanescu, C. Zhang, V. V. Scaramela, and S. Tewari, Phys. Rev. Lett. 123, 060402 (2019).
[33] M. Geier, L. Trifunovic, M. Hoskam, and P. W. Brouwer, Phys. Rev. B 97, 205135 (2018).
[34] X. Zhu, Phys. Rev. Lett. 122, 236401 (2019).
[35] S. A. A. Ghorashi, X. Hu, T. L. Hughes, and E. Rossi, Phys. Rev. B 100, 020509(R) (2019).
[36] D. Varjas, A. Lau, K. Pöyhönen, A. R. Akhmerov, Phys. Rev. Lett. 123, 196401 (2019).
[37] S.-B. Zhang and B. Trauzettel, Phys. Rev. Research 2, 012018(R) (2020).
[38] S. Franca, D. V. Efremov, and I. C. Fulga, Phys. Rev. B 100, 075145 (2019).
[39] Q. Wang, C.-C. Liu, Y.-M. Lu, and F. Zhang, Phys. Rev. Lett. 121, 186801 (2018).
[40] C.-H. Hsu, P. Stano, J. Klinovaja, and D. Loss, Phys. Rev. Lett. 121, 196801 (2018).
[41] Y. Volpe, D. Loss, and J. Klinovaja, Phys. Rev. Lett. 122, 126402 (2019).
[42] R.-X. Zhang, W. S. Cole, and S. Das Sarma, Phys. Rev. Lett. 122, 187001 (2019).
[43] X.-H. Pan, K.-J. Yang, L. Chen, G. Xu, C.-X. Liu, and X. Liu, Phys. Rev. Lett. 123, 156801 (2019).
[44] M. Lin and T. L. Hughes, Phys. Rev. B 98, 241103(R) (2018).
[45] D. Calugaru, V. Juricic, and B. Roy, Phys. Rev. B 99, 041301(R) (2019).
[46] B. Roy, Phys. Rev. Research 1, 032048(R) (2019).
[47] M. Ezawa, Phys. Rev. Lett. 120, 026801 (2018).
[48] B. J. Wieder, Z. Wang, J. Cano, X. Dai, L. M. Schoop, B. Bradlyn, and B. A. Bernevig, Nat. Commun. 11, 627 (2020).
[49] Z. Wang, B. J. Wieder, J. Li, B. Yan, and B. A. Bernevig, Phys. Rev. Lett. 123, 186401 (2019).
[50] E. Khalaf, H. C. Po, A. Vishwanath, and H. Watanabe, Phys. Rev. X 8, 031070 (2018).
[51] E. Cornfeld and A. Chapman, Phys. Rev. B 99, 075105 (2019).
[52] L. Trifunovic and P. W. Brouwer, arXiv:2003.01144.
[53] J. Kruthoff, J. de Boer, J. van Wezel, C. L. Kane, and R.-J. Slager, Phys. Rev. X 7, 041069 (2017).
[54] F. Schindler, Z. Wang, M. G. Vergniory, A. M. Cook, A. Murani, S. Sengupta, A. Yu. Kasumov, R. Debloch, S. Jeon, I. Drozdov, H. Bouchiat, S. Guéron, A. Yazdani, B. A. Bernevig, and T. Neupert, Nat. Phys. 14, 918-924 (2018).
[55] S. N. Kempkes, M. R. Slot, J. J. van den Broeke, P. Capiod, W. A. Benalcazar, D. Vanmaekelbergh, D. Bercioux, I. Swart, and C. Morais Smith, Nature Materials 18, 1292-1297 (2019).
[56] R.-X. Zhang, F. Wu, and S. Das Sarma, Phys. Rev. Lett. 124, 136407 (2020).
[57] Y. Yang, Z. Jia, Y. Wu, Z.-H. Hang, H. Jiang, and X. C. Xie, Science Bulletin 65, 531 (2020).
[118] D. Xie, T.-S. Deng, T. Xiao, W. Gou, T. Chen, W. Yi, and B. Yan, Phys. Rev. Lett. 124, 050502 (2020).

[119] The formula $\text{tr}(A \otimes B) = \text{tr}(A)\text{tr}(B)$ for the tensor product of operators $A$ and $B$ has been used to arrive at this relation.

[120] D. C. Brody, J. Phys. A: Math. Theor. 47, 035305 (2014).

[121] Y. Wu, W. Liu, J. Geng, X. Song, X. Ye, C.-K. Duan, X. Rong, and J. Du, Science 364, 878-880 (2019).

[122] N. R. Cooper, J. Dalibard, and I. B. Spielman, Rev. Mod. Phys. 91, 015005 (2019).

[123] D.-W. Zhang, Y.-Q. Zhu, Y. X. Zhao, H. Yan, and S.-L. Zhu, Advances in Physics 67, 253-402 (2019).

[124] G. Pelegrí, A. M. Marques, V. Ahufinger, J. Mompart, and R. G. Dias, Phys. Rev. B 100, 205109 (2019).