Fed+: A Unified Approach to Robust Personalized Federated Learning

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Abstract

We present a class of methods for robust, personalized federated learning, called Fed+, that unifies many federated learning algorithms. The principal advantage of this class of methods is to better accommodate the real-world characteristics found in federated training, such as the lack of IID data across parties, the need for robustness to outliers or stragglers, and the requirement to perform well on party-specific datasets. We achieve this through a problem formulation that allows the central server to employ robust ways of aggregating the local models while keeping the structure of local computation intact. Without making any statistical assumption on the degree of heterogeneity of local data across parties, we provide convergence guarantees for Fed+ for convex and non-convex loss functions and robust aggregation. The Fed+ theory is also equipped to handle heterogeneous computing environments including stragglers without additional assumptions; specifically, the convergence results cover the general setting where the number of local update steps across parties can vary. We demonstrate the benefits of Fed+ through extensive experiments across standard benchmark datasets as well as on a challenging real-world problem in financial portfolio management where the heterogeneity of party-level data can lead to training failure in standard federated learning approaches.

1 Introduction

Federated learning is a technique for training machine learning models without sharing data, introduced by McMahan et al. [2017] and Konečný et al. [2015, 2016], and steadily gaining momentum. Federated learning involves a possibly varying set of parties participating in a parallel training process through a centralized aggregator that has access only to the parties’ model parameters, or gradients, but not to the data itself. In federated learning, communication is minimized, as compared to parallel stochastic gradient descent (SGD), by parties performing a number of iterations locally before sending parameters to the aggregator. Federations tend to be diverse, leading to non-IID data across parties, and often include parties whose data can be considered to be outliers with respect to the others. Most algorithms, however, can trigger failure of the training process itself when parties are too heterogeneous in precisely the settings where federated learning could have the greatest benefit. Personalization of federated model training, when judiciously performed, is one means of avoiding such training failure. In addition, personalization of federated training allows for greater accuracy on
Fed+, pronounced FedPlus, is designed to address the issues of avoiding training failure, increasing robustness to outliers and stragglers, and improving performance on the applications of interest where party-level data distributions need not be similar across parties. Fed+ unifies many algorithms and offers provably-convergent personalization and robustness; this is achieved through a problem formulation that allows the central server to employ robust ways of aggregating the local models while keeping the structure of local computation intact. Fed+ does not make explicit assumptions on the distributions of the local data, which are assumed private to each party. Instead, we assume a global shared parameter space with locally computed loss functions. Like some personalized methods, Fed+ allows for data heterogeneity by relaxing the requirement that the parties must reach a full consensus. The Fed+ theory is also equipped to handle heterogeneous computing environments including stragglers without making additional assumptions; specifically, the convergence results cover the general setting where the number of local update steps across parties can vary.

To evaluate the performance of a federated learning aggregation method, it is important to assess it and the alternatives not only on common test sets but also on party-specific data, and on both benchmark problems and real-world applications. To this end, we examine a reinforcement learning-based financial portfolio management problem where federated learning allows portfolio managers to jointly train policies without revealing their private data. In addition, federated learning on heterogeneous data offers the benefits of multi-task learning in a privacy-protected manner, improving transferability of the models in the presence of non-stationarity, common in financial markets. We illustrate the training failure of federated reinforcement learning (RL) policies using standard algorithms, all of which are resolved by Fed+. We further illustrate Fed+ on the synthetic dataset created for FedProx by Li et al. [2020b] as well as datasets from the LEAF set of Caldas et al. [2018].

The contributions of this work are (i) the definition of a unified framework for robust, personalized federated learning, called Fed+; (ii) a convergence theory that covers the most important variants of the Fed+ algorithm, including convex and nonconvex loss functions, robust aggregation and stragglers; (iii) a comprehensive set of numerical experiments on both common and party-specific datasets, illustrating not only the benefit of Fed+ with respect to other algorithms, but also as compared to purely local training; (iv) a real-world application of federated learning that illustrates an example of training failure that occurs using standard federated aggregation methods.

## 2 Related Work

Li et al. [2020b] showed that FedAvg defined by McMahan et al. [2017] can converge to a point which is not a solution to the original problem and proposed to add a decreasing learning rate; with that they provide a theoretical convergence guarantee, even when the data is not IID, but the resulting algorithm is slow to converge. To handle non-IID data, Li et al. [2020a] introduced a regularization term in their FedProx algorithm. Li et al. [2020b]; Karimireddy et al. [2019] seek to explain the non-convergence of FedAvg while proposing new algorithms. Pathak and Wainwright [2020]; Charles and Konečný [2020]; Malinovsky et al. [2020] propose FedSplit and LocalUpdate, and Local Fixed Point, resp., and obtain tight bounds on the number of communication rounds required to achieve an ϵ accuracy. However, these algorithms all require the convergence of all parties to a common model.

Others have sought to increase robustness to corrupted updates and outliers. Pillutla et al. [2019] propose Robust Federated Aggregation (RFA) which replaces weighted arithmetic mean aggregation by an approximate geometric median. Yin et al. [2018] propose a Byzantine-robust distributed statistical learning algorithm based on the coordinate-wise median. Both RFA [Pillutla et al. [2019] and coordinate-wise median [Yin et al., 2018], involve training a single global model and neither is robust to non-IID data, leading in some cases to failure of the learning process.

Several recent works advocate, as we do, for a fully personalized approach whereby each client trains a local model while contributing to a global model. Mansour et al. [2020] propose clustering parties and solving an aggregate model within each cluster. While this would likely eliminate the training failure we observe in practice, it adds considerable overhead. Hanzely and Richtárik [2020] propose a local-global mixture method focused on reducing communication overhead for the smooth convex
setting. Deng et al. [2020] propose a method similar to our FedPlus+. T. Dinh et al. [2020] propose a procedure where each party optimizes its local loss and a (local version of) the global parameters. Hanzely et al. [2021] provide a unification of personalized federated aggregation methods in the smooth and convex setting. Li et al. [2021] propose a bilevel programming framework alternating between solving for the average solution and local solutions. The overall problem however is non-convex, even for convex party loss functions, and could be solved in two separate phases. Zhang et al. [2021] suggest personalizing the average aggregate solution as a set of weighted average aggregate solutions. The most important difference between Fed+ and the above methods is that only Fed+ allows for robust aggregation, both in the definition of the algorithm and in the convergence theory, handling the nonsmooth loss functions that result.

3 Illustration of Federated Learning Training Failure

Next, we illustrate the training failure that can occur in real-world federated learning settings on a federated reinforcement learning-based financial portfolio management problem. The key observation, see Figure [1a], is that replacing at each round the local party models with a common, aggregate model can lead to large spikes in model changes, triggering training failure for the federation as a whole. The figure shows the mean and standard deviation of the change in neural network parameter values before and after a federated learning aggregation step. FedAvg, RFA using the geometric median, coordinate-wise median, and FedProx are shown, as well as the no fusion case where each party trains independently on its own data, and the FedAvg+ version of Fed+. The standard methods all cause large spikes in the parameter change that do not occur without federated learning or with Fed+.

Such dramatic model change can lead to a collapse of the training process. The large spikes coincide precisely with training collapse, as shown in Figure [1c] (bottom four figures). Note that this example does not involve adversarial parties or party failure, as evidenced from the fact that single-party training (top curve) does not suffer failure. Rather, it shows a real-world problem where parties’ data are not drawn IID from a single dataset. It is conceivable that federated training failure may be a common occurrence in practice when forcing convergence to a common solution across parties.

A deeper understanding of the training failure can be gleaned from Figure [1c], which shows what occurs before and after an aggregation step and motivates the Fed+ approach. A local party update occurs in each subplot on the left side, at $\lambda = 0$. Values of $\lambda \in [0, 1]$ correspond to moving towards, but not reaching, the common, aggregate model. A right-hand side lower than the left-hand side means that a full step towards averaging (or using the median for) all parties, i.e., $\lambda = 1$, degrades local performance. Dashed lines represent the aggregate model at the previous round. Observe that local updates improve the performance from the previous aggregation indicated by the dashed lines. However, performance degrades after the subsequent aggregation, corresponding to the right-hand side of each subplot, where $\lambda = 1$. In fact, for FedAvg, RFA and FedProx, performance of the subsequent aggregation is worse than the previous value (dashed line).

4 The Fed+ Framework

Fed+ is designed to better handle federated learning in real-world settings including non-IID data across parties, parties having outlier data with respect to other parties, stragglers, in that updates are transmitted late, and an implicit requirement for the final trained model(s) to perform well on each party’s own datasets. To accomplish these goals, Fed+ takes a robust, personalized approach to federated learning, and, importantly, does not require all parties to converge to a single central point. Fed+ thus requires generalizing the objective of the federated learning training process, as follows.

4.1 Problem Formulation

Consider a federation of $K$ parties with local loss functions $f_k : \mathbb{R}^d \to \mathbb{R}, k = 1, 2, \ldots, K$. The original FedAvg formulation [McMahan et al. 2017] involves training a central model $\tilde{w} \in \mathbb{R}^d$ by minimizing the average local loss over the $K$ parties:

$$
\min_{\tilde{w} \in \mathbb{R}^d, W \in \mathbb{R}^{d \times K}} \left[ F(W) := \frac{1}{K} \sum_{k=1}^{K} f_k(w_k) \right] \text{ subject to } w_k = \tilde{w}, k = 1, \ldots, K, \quad (1)
$$
(a) Change in weights before and after each aggregation round. Only Fed+ and local SGD ("no fusion") have no large spikes.

(b) Illustration of training collapse experienced using all standard methods except local SGD ("no fusion").

(c) Before and after federated model aggregation, along $\lambda \in [0, 1]$ between the local ($\lambda = 0$) and global ($\lambda = 1$) solutions.

Figure 1: Illustration of federated learning training failure on a real-world application when parties are forced to converge to a single, common aggregate solution.
where we use the notation \( W := (w_1, w_2, \ldots, w_K) \in \mathbb{R}^{d \times K} \) with \( w_k \in \mathbb{R}^d \) denoting the local model of party \( k \). In place of the hard equality constraints in (1), we advocate for a penalization-based approach and propose the following objective for the overall federated training process:

\[
\min_{W \in \mathbb{R}^{d \times K}} F_d(W) := \frac{1}{K} \sum_{k=1}^{K} \left[ f_k(w_k) + \mu B(w_k, A(W)) \right],
\]

(2)

where \( \mu > 0 \) is a user-chosen penalty constant, \( A \) is an aggregation function that outputs a central aggregate \( \hat{w} \in \mathbb{R}^d \) of \( w_1, \ldots, w_K \), and \( B(\cdot, \cdot) \) is a distance function that penalizes the deviation of a local model \( w_k \) from the central aggregate \( \hat{w} = A(W) \). Note that \( \hat{w} \) may be the mean, median, etc. When \( \mu = 0 \), problem (2) reduces to the non-federated setting where every party independently minimizes its local objective function. On the other hand, for \( \mu > 0 \) and \( A(W) = \frac{1}{K} \sum_k w_k \), if one sets \( B \) such that \( B(w, \hat{w}) = \infty \) if \( w \neq \hat{w} \) and \( B(w, \hat{w}) = 0 \) otherwise, then (2) is equivalent to problem (1). More generally, the distance function \( B \) may be the usual squared Euclidean distance, the metric induced by the \( \ell_p \) norm (denoted as \( \| \cdot \|_p \) with \( p \in [1, \infty] \), or other Bregman divergence measures such as

\[
B(w_k, \tilde{w}) = \frac{1}{2} \| w_k - \tilde{w} \|_Q^2,
\]

(3)

where \( Q \in \mathbb{R}^{d \times d} \) is a symmetric positive semi-definite matrix and \( \| w \|_Q := \sqrt{w^T Q w} \). By setting \( Q \) to be a diagonal matrix with non-negative entries, (3) can serve to weight each component’s contribution differently depending on the application and information available during model training.

### 4.2 Handling Robust Aggregation

We exploit the machinery of the function \( B \) to define a family of aggregation functions which includes the mean, geometric median, coordinate-wise median, etc. That is, the global model \( \hat{w} \) is computed by aggregating the current local models \( \{w_1, \ldots, w_K\} \) via

\[
\tilde{w} \leftarrow A(W) := \arg\min_{w \in \mathbb{R}^d} \frac{1}{K} \sum_{k=1}^{K} B(w_k, w).
\]

(4)

The mean aggregation function \( A \) is recovered by choosing \( B(w, w') = \| w - w' \|_2^2 \) in (4). The geometric median and coordinate-wise median aggregation are obtained by setting \( B(w, w') = \| w - w' \|_2 \) and \( B(w, w') = \| w - w' \|_1 \), resp. To unify the aggregation methods which include nonsmooth functions \( B \) in (2), we introduce the following general class of parameterized functions \( B \) where we choose a convex function \( \phi : \mathbb{R}^d \to [0, \infty] \) and a smoothness-robustness parameter \( \rho > 0 \):

\[
B(w_k, \tilde{w}) = \Phi_{\rho}(w_k - \tilde{w}), \quad \Phi_{\rho}(w) := \min_{w' \in \mathbb{R}^d} \left[ \phi(w') + \frac{1}{2\rho} \| w - w' \|_2^2 \right].
\]

(5)

We call the minimizer in the above problem the proximal operator of \( \phi \) and denote it as \( \text{prox}^\rho_\phi(w) \). Note that \( \Phi_{\rho} \) is a smooth function known as the Moreau envelope of \( \phi \). By choosing \( \phi \) to be the \( \ell_2 \) norm, we obtain a \((1/\rho)\)-smooth approximation of the geometric median aggregation as in [Pillutla et al., 2019]. Setting \( \phi \) to the \( \ell_1 \) norm gives a smooth approximation of coordinate-wise median aggregation. The usual mean aggregation is naturally recovered in both of these cases: (i) \( \phi(w) = \frac{1}{2} \| w \|_2^2 \), and (ii) \( \phi(w) = 0 \) if \( w = 0 \) and \( +\infty \) otherwise.

### 4.3 Personalization at the Local Parties

While problem (2) can be solved centrally, we are interested in the federated setting where at every round each active party \( k \) solves its own version of the problem, running \( E_k \) iterations of the following update, with learning rate \( \eta > 0 \):

\[
w_k \leftarrow \theta [w_k - \eta \nabla f_k(w_k)] + (1 - \theta) z_k, \quad \text{for } i = 1, \ldots, E_k.
\]

(6)

Note that personalization occurs via the party-specific regularization term \( z_k \) and the constant \( \theta \in (0, 1) \) controls the degree of regularization while training the local model using (6). In practice, the exact gradient \( \nabla f_k(w_k) \) in (6) is replaced by an unbiased random estimate. In standard methods, \( z_k \) is set to the current global model \( \tilde{w} \). Fed+ proposes however robust personalization as we shall see below.
4.4 Reformulation and Unification of the Objective

To obtain a unified framework that covers robustness and personalization, consider again the original FedAvg 1, expressed equivalently as:

$$
\min_{w, Z \in \mathbb{R}^{d \times K}, \tilde{w} \in \mathbb{R}^d} \frac{1}{K} \sum_{k=1}^K f_k(w_k) \quad \text{subject to} \quad w_k = z_k, \ z_k = \tilde{w}, \ k = 1, \ldots, K,
$$

(7)

where $Z := (z_1, \ldots, z_K)$ with $z_k \in \mathbb{R}^d$. While ADMM can be used to solve equality-constrained problems of the above form, we take a penalty-based approach, as in $Zhang et al. [2015]$, suitable for both convex and non-convex settings. Then, to handle heterogeneous data and computing environments across parties, we replace the equality constraints in (7) with penalty functions:

$$
\min_{w, Z \in \mathbb{R}^{d \times K}, \tilde{w} \in \mathbb{R}^d} H_{\mu, \alpha}(W, Z, \tilde{w}) := \frac{1}{K} \sum_{k=1}^K \left[ f_k(w_k) + \frac{\alpha}{2} \|w_k - z_k\|_2^2 + \mu \phi(z_k - \tilde{w}) \right],
$$

(8)

where $\alpha > 0$ is a user-chosen penalty constant and $\phi : \mathbb{R}^d \to [0, \infty]$ is a convex penalty function. The following proposition connects this new formulation (8) to our original objective (2).

**Proposition 1.** Problem (8) is a special case of (2), where the $A$ & $B$ functions are as defined in (4) & (5) respectively (with $\rho = \mu/\alpha$), and the following relation between the two optimization objectives holds:

$$
F_{\mu}(W) = \min_{Z \in \mathbb{R}^{d \times K}, \tilde{w} \in \mathbb{R}^d} H_{\mu, \alpha}(W, Z, \tilde{w}).
$$

(9)

**Proof.** Using $\rho = \mu/\alpha$, we have from (8):

$$
\min_{Z \in \mathbb{R}^{d \times K}} H_{\mu, \alpha}(W, Z, \tilde{w}) = \frac{1}{K} \sum_{k=1}^K \left[ f_k(w_k) + \mu \min_{\text{z}_k \in \mathbb{R}^d} \left\{ \frac{1}{2\rho} \|w_k - z_k\|_2^2 + \phi(z_k - \tilde{w}) \right\} \right]
$$

$$
= \frac{1}{K} \sum_{k=1}^K \left[ f_k(w_k) + \mu \min_{\text{v}_k \in \mathbb{R}^d} \left\{ \frac{1}{2\rho} \|(w_k - \tilde{w}) - v_k\|_2^2 + \phi(v_k) \right\} \right]
$$

$$
= \frac{1}{K} \sum_{k=1}^K \left[ f_k(w_k) + \mu \Phi_{\rho}(w_k - \tilde{w}) \right],
$$

$$
= \frac{1}{K} \sum_{k=1}^K \left[ f_k(w_k) + \mu \Phi_{\rho}(w_k - \tilde{w}) \right],
$$

(10)

where the 2nd equality is obtained by the change of variable $z_k \rightarrow \tilde{w} + v_k$ and the last two equalities are due to (5). Now, further minimizing (10) w.r.t. $w$ and using the definition (4), we arrive at (9). $\square$

Also, the Fed+ formulation (8) suggests a natural choice for the personalization function $R : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ which computes $z_k := R(\tilde{w}, w_k)$ as a (robust) combination of the current local and central model. To be precise, Fed+ proposes setting $R(\tilde{w}, w_k)$ by minimizing (8) w.r.t. $z_k$ while keeping $w_k$ and $\tilde{w}$ fixed. Thus, we have the following closed form update:

$$
z_k \leftarrow R(\tilde{w}, w_k) = \tilde{w} + \text{prox}_{\Phi_{\rho}}^{w_k}(w_k - \tilde{w}), \ \rho = \mu/\alpha.
$$

(11)

Next, we connect the local party’s update (10) to our new formulation (8).

**Proposition 2.** Let $\theta := \frac{1}{1+\alpha \eta}$. Then, the local update (10) (line 12 in Fed+ algorithm) is a gradient descent iteration with learning rate $\eta' := \frac{\eta}{1+\alpha \eta}$ applied to the following sub-problem:

$$
\min_{w_k \in \mathbb{R}^d} F_k(w_k; z_k, \tilde{w}) := f_k(w_k) + \frac{\alpha}{2} \|w_k - z_k\|_2^2 + \mu \phi(z_k - \tilde{w}),
$$

(12)

where $z_k$ & $\tilde{w}$ are kept fixed by setting them to $z_k^{t-1}$ and $\tilde{w}^{t-1}$ respectively.

6
Proof. The gradient descent iteration for the function \( F_k(\cdot; z_k^{-1}, \hat{w}^{t-1}) \) with stepsize \( \eta' := \frac{n}{1+\alpha\eta} \) is given by
\[
\begin{align*}
\hat{w}^t_k &\leftarrow \hat{w}^t_k - \eta' [\nabla f_k(w^t_k; z_k^{-1}) + \alpha (w^t_k - z_k^{-1})] \\
&= (1 - \alpha\eta') \left[ w^t_k - \frac{\eta'}{1 - \alpha\eta'} \nabla f_k(w^t_k) \right] + (\alpha\eta') z_k^{-1} \\
&= \left( \frac{1}{1 + \alpha\eta} \right) \left[ w^t_k - \eta \nabla f_k(w^t_k) \right] + \left( \frac{\alpha\eta}{1 + \alpha\eta} \right) z_k^{-1}.
\end{align*}
\]
Thus, we have the local update of the form \((\ref{eq:fedavg})\) (line \(12\) in Fed+ algorithm) where \( \theta := \frac{1}{1+\alpha\eta} \).

5 The Fed+ Algorithm

Fed+ is defined as a family of federated learning methods for solving problem \((\ref{eq:1})\) with \( B \) as in \((\ref{eq:3})\) and \( A \) as defined in \((\ref{eq:4})\). Fed+ is designed to allow for robust aggregation functions \( A \), where local copies of shared parameters are aggregated. Importantly, Fed+ does not require all parties to agree on a single common model. We argue that this offers the benefits of federation without the pitfall of training failure that can occur in real-world implementations of federated learning.

5.1 Fed+ Algorithm Description

The steps are provided in Algorithm 1. So as to encompass important special cases, Algorithm 1 introduces a number of parameters: \( \lambda \in [0, 1], \theta \in (0, 1], \) and \( R : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \). A main difference between our approach and other federated algorithms is that the parties do not set the aggregated central model as their starting point when performing the local update step \((\ref{eq:fedavg})\); i.e., parties need not set \( \lambda = 1 \) in line \((10)\) of Algorithm 1. Instead, Fed+ advocates initializing local models at each round with the value from the previous round (i.e. \( \lambda = 0 \)). This mitigates the dramatic changes in local models seen in Figure 1a.

**Algorithm 1** Fed+: Parties \( k = 1, \ldots, K \); aggregation fcn. \( A \); local iterations per round at party \( k \), \( E_k \); learning rate \( \eta \); and \( \theta \in (0, 1] \); \( \lambda \in [0, 1] \); and robust personalization fcn. \( R : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d \).

**Initialization:**
1: Each party \( k \) sends its initial local model \( \hat{w}^0_k \) to the Aggregator, which computes the central value \( \hat{w}^0 \leftarrow A(\mathcal{W}^0) \).

**Aggregator:**
2: for round \( t = 1, \ldots, T \) do
3: Sample parties to obtain \( S_t \subseteq \{1, \ldots, K\} \).
4: Send the current global model \( \hat{w}^{t-1} \) to each party \( k \in S_t \).
5: for each party \( k \in S_t \) in parallel do
6: \( \hat{w}^t_k \leftarrow \text{Local-Solve}(k, t, \hat{w}^{t-1}, \hat{w}^{t-1}_k) \). // Each party \( k \notin S_t \) sets \( \hat{w}^t_k \leftarrow \hat{w}^{t-1}_k \).
7: Party sends \( \hat{w}^t_k \) to Aggregator.
8: Compute the aggregated central model: \( \hat{w}^t \leftarrow A(\mathcal{W}^t) \).

**Local-Solve \((k, t, \hat{w}^{t-1}, \hat{w}^{t-1}_k)\):** Run on each active party \( k \in S_t \)
9: Compute a robust local model: \( z_k^{-1} := R(\hat{w}^{t-1}, \hat{w}^{t-1}_k) \).
10: Initialize the current local model: \( \hat{w}^t_k \leftarrow (1 - \lambda) \hat{w}^{t-1} + \lambda \hat{w}^{t-1} \).
11: for \( i = 1, \ldots, E_k \) do
12: \( \hat{w}^t_k \leftarrow \theta \hat{w}^t_k - \eta \nabla f_k(\hat{w}^t_k)\) \( + (1 - \theta) z_k^{-1} \).

5.2 Instantiations of Fed+

5.2.1 FedAvg+

A mean-aggregation based method with better training performance than FedAvg via personalization. Choose \( \phi(w) = \frac{1}{2} \|w\|^2 \). \( A \) is the mean, i.e., \( \hat{w}^t := \text{Mean}\{w^t_k : k \in S_t\} \) and \((\ref{eq:fedavg})\) takes the form: \( R(\hat{w}^t, w^t_k) = (1 - \lambda_k^t) w^t_k + \lambda_k^t \hat{w}^t \), where \( \lambda_k^t := \rho/(1 + \rho) \).
5.2.2 FedGeoMed+

A robust aggregation based method that offers stability in training in the presence of outliers/adversaries. Set $\phi(w) = \|w\|_2$. $A$ is a $\rho$-smoothed approximation of the Geometric Median, and \( R \) becomes:

$$R(\tilde{w}^t, w^t_k) = (1 - \lambda_k^t) w^t_k + \lambda_k^t \tilde{w}^t,$$

where $\lambda_k^t := \min \{1, \rho/\|w^t_k - \tilde{w}^t\|_2\}$.

To compute $\tilde{w}^t$ from $\{w^t_k : k \in S_t\}$ the aggregator runs the following two step iterative procedure initialized with $\tilde{w} = w_{mean} := \text{Mean}\{w^t_k : k \in S_t\}$ until $\tilde{w}$ converges:

$$v_k \leftarrow \max \{0, 1 - (\rho/\|w^t_k - \tilde{w}\|_2)\} (w^t_k - \tilde{w}, \forall k \in S_t,$$

$$\tilde{w} \leftarrow w_{mean} - \text{Mean}\{v_k : k \in S_t\}.\)

5.2.3 FedCoMed+

FedCoMed+ offers the benefits of robust aggregation via the median with added flexibility in allowing different parameters for each coordinate. Choose $\phi(w) = \|w\|_1$. $A$ is a $\rho$-smoothed approximation of the Coordinate-wise Median and \( R \) takes the following form:

$$R(\tilde{w}^t, w^t_k) = (1 - \Lambda_k^t) w^t_k + \Lambda_k^t \tilde{w}^t,$$

where $\Lambda_k^t(i, i) := \min \{1, \rho/\|w^t_k(i) - \tilde{w}^t(i)\|\}$, $i = 1, \ldots, d$.

To compute $\tilde{w}^t$ from $\{w^t_k : k \in S_t\}$ the aggregator starts with $\tilde{w} = w_{mean} := \text{Mean}\{w^t_k : k \in S_t\}$ and runs the following two step iterative procedure until $\tilde{w}$ converges:

$$v_k \leftarrow \max \{0, w^t_k - \tilde{w}^t - \rho \text{sign}(w^t_k - \tilde{w}^t)\}, \forall k \in S_t,$$

$$\tilde{w} \leftarrow w_{mean} - \text{Mean}\{v_k : k \in S_t\}.\)

5.3 Hybridization via the Unified Fed+ Framework with Layer-specific $\phi$

The unification of aggregation methods through a single formulation allows for seamlessly combining different methods of aggregation and personalization applied to different layers in training deep neural networks. For example, initial layers may use FedAvg+, while final layers may benefit from FedCoMed+ and greater personalization via $\rho$.

To this end, let $w = (w_{[1]}, \ldots, w_{[L]})$, where $w_{[l]}$ denotes the weights of the $l$-th layer. In this case, we define

$$\phi(w) := \sum_{l=1}^{L} \phi_l(w_{[l]}), \quad (14)$$

where $\phi_l$, $l = 1, \ldots, L$, can be chosen independently giving rise to different methods of robust personalization for each layer.

5.4 Deriving Existing Algorithms from Fed+

Many federated learning methods fit into the Fed+ framework and can be obtained by setting the parameters in Algorithm 1 appropriately.

5.4.1 Local (Stochastic) Gradient Descent without Federation

Let $\lambda = 0$, $\theta = 1$. Then the resulting algorithm is party-specific local SGD.

5.4.2 FedAvg

The classic and non-personalized FedAvg method can be obtained by setting $\lambda = 1$, $\theta = 1$, $E_k = E$ for all $k$, $\tilde{w}^t = \text{Mean}\{w^t_k : k \in S_t\}$.

5.4.3 RFA

The non-personalized robust federated averaging approach, which uses geometric median as the aggregation function, is obtained when we set $\lambda = 1$, $\theta = 1$. $\tilde{w}^t = \text{Geometric Median}\{w^t_k : k \in S_t\}$. 


5.4.4 Coordinate-wise median

A non-personalized coordinate-wise median based aggregation is obtained when we set \( \lambda = 1, \theta = 1 \), and \( \tilde{w}^t = \text{CoordinateWiseMedian}\{w^t_k : k \in S_t\} \).

5.4.5 FedProx

FedProx is obtained when one sets \( \lambda = 1, \theta = \frac{1}{1+\mu\eta} \), \( \mathcal{R}(\tilde{w}^t, w^t_k) = \tilde{w}^t = \text{Mean}\{w^t_k : k \in S_t\} \).

6 Fixed Points of Fed+

Here, we characterize the fixed points of Fed+ algorithm for solving problem \((9)\) under Assumption \(1\). Let \( \mathcal{R} \) be as in \((11)\) and \((\mathbf{W}^*, \mathbf{Z}^*, \tilde{w}^*) \) be a fixed point of Fed+. Then, the following conditions are satisfied:

\[
\frac{1}{K} \sum_{k=1}^{K} \tilde{f}_k(z_k^t) = 0, \quad w^t_k = z_k^t - \frac{1}{\alpha} \nabla \tilde{f}_k(z_k^t), \quad z_k^t = \tilde{w}^* + \text{prox}_\phi(w^*_k - \tilde{w}^*), \quad k = 1, \ldots, K. \tag{17}
\]

**Proof.** We start with the following observations about Fed+: \( \forall t \geq 1 \),

\[
w^t_k = z_k^{t-1} - \frac{1}{\alpha} \nabla \tilde{f}_k(z_k^{t-1}), \quad k = 1, \ldots, K, \tag{18}
\]

\[
\tilde{w}^t = \frac{1}{K} \sum_{k=1}^{K} w^t_k - \frac{1}{K} \sum_{k=1}^{K} \text{prox}_\phi(w^t_k - \tilde{w}^t), \tag{19}
\]

\[
z_k^t = \tilde{w}^t + \text{prox}_\phi(w^t_k - \tilde{w}^t), \quad k = 1, \ldots, K, \tag{20}
\]

where the 1st equation is a direct consequence of \((15)\), the 2nd one comes from \((34)\) and the last one is by choice of \( \mathcal{R} \). Therefore, for a fixed point, the 2nd & 3rd equations in \((17)\) obviously hold. Now, for a fixed point, we also have the following from \((19)\):

\[
\tilde{w}^* = \frac{1}{K} \sum_{k=1}^{K} w^*_k - \frac{1}{K} \sum_{k=1}^{K} \text{prox}_\phi(w^*_k - \tilde{w}^*). \tag{21}
\]

Replacing the first \( w^*_k \) in \((21)\) with \( z_k^* - \frac{1}{\alpha} \nabla \tilde{f}_k(z_k^*) \) and subsequently \( z_k^* \) by \( \tilde{w}^* + \text{prox}_\phi(w^*_k - \tilde{w}^*) \), we get

\[
\tilde{w}^* = \frac{1}{K} \sum_{k=1}^{K} \left[ z_k^* - \frac{1}{\alpha} \nabla \tilde{f}_k(z_k^*) \right] - \frac{1}{K} \sum_{k=1}^{K} \text{prox}_\phi(w^*_k - \tilde{w}^*) \\
= \frac{1}{K} \sum_{k=1}^{K} \left[ \tilde{w}^* + \text{prox}_\phi(w^*_k - \tilde{w}^*) \right] - \frac{1}{\alpha K} \sum_{k=1}^{K} \nabla \tilde{f}_k(z_k^*) - \frac{1}{K} \sum_{k=1}^{K} \text{prox}_\phi(w^*_k - \tilde{w}^*) \\
= \tilde{w}^* - \frac{1}{\alpha K} \sum_{k=1}^{K} \nabla \tilde{f}_k(z_k^*). \tag{22}
\]

This, we have the first equation in \((17)\). \(\square\)
With the help of above Theorem, we now analyze two extreme choices of $R$ in part (a) & (b) of the following Corollary:

**Corollary 1.** Consider the Fed+ algorithm under Assumption \[1\] Let $(W^*, Z^*, \bar{w}^*)$ be a fixed point of Fed+. Then, the following are true:

(a) If Fed+ sets $R(\bar{w}, w_k) = w_k$, then

$$w_k^* \in \arg\min_w f_k(w), \ k = 1, \ldots, K.$$  \hspace{1cm} (23)

(b) If Fed+ uses $R(\bar{w}, w_k) = \bar{w}$ along with $A(W) = \frac{1}{K} \sum_k w_k$, then

$$\frac{1}{K} \sum_{k=1}^K \tilde{f}_k(\bar{w})^* = 0, \ w_k^* = \bar{w}^* - \frac{1}{\alpha} \nabla \tilde{f}_k(\bar{w})^*, \ k = 1, \ldots, K.$$  \hspace{1cm} (24)

(c) If Fed+ employs $R(\bar{w}, w_k) = (1 - \gamma)w_k + \gamma \bar{w}$, $\gamma \in (0, 1)$ with $A(W) = \frac{1}{K} \sum_k w_k$, then

$$w_k^* = \bar{w}^* - \frac{1}{\alpha \gamma} \nabla \tilde{f}_k((1 - \gamma)w_k^* + \gamma \bar{w}^*), \ k = 1, \ldots, K, \text{ where } \bar{w}^* = \frac{1}{K} \sum_{k=1}^K w_k^*.$$  \hspace{1cm} (25)

**Proof.** To prove (a), we apply Theorem [1] with $\phi = 0$. This choice of $\phi$ leads to $z_k = R(\bar{w}, w_k) = w_k$ from [11]. Therefore, Fed+ boils to applying the proximal point algorithm $w_k^* = \text{prox}_{\frac{1}{\alpha} \nabla \tilde{f}_k}^*(w_k^*)$, $t = 1, \ldots, T$, at each local party $k = 1, \ldots, K$. Therefore, we obtain the result [23] as $w_k^* = \text{prox}_{\frac{1}{\alpha} \nabla \tilde{f}_k}^*(w_k^*)$. Alternatively, setting $z_k^* = w_k^*$ in [17], we get

$$w_k^* = w_k^* - \frac{1}{\alpha} \nabla \tilde{f}_k(w_k^*) \implies w_k^* \in \arg\min_w f_k(w).$$  \hspace{1cm} (26)

Next, we prove part (b) by applying Theorem [1] with the following choice of $\phi$: $\phi(w) = 0$ iff $w = 0$ and $+\infty$ otherwise. This particular $\phi$ corresponds to the choice $z_k = R(\bar{w}, w_k) = \bar{w}$ from [11]. Also, the aggregation function $A$ becomes the mean as from [4] and [5] we get:

$$A(W) = \arg\min_w \frac{1}{K} \sum_{k=1}^K \Phi_{\mu}(w_k - \bar{w}), \text{ where } \Phi_{\mu}(w) = \frac{1}{2\mu} \|w\|^2.$$  

Now, putting $z_k^* = \tilde{w}^*$ in [17], we arrive at [24]. Finally, we show part (c) by setting $\phi(w) = \frac{1}{2} \|w\|^2$, $w \in \mathbb{R}^d$ in Theorem [1]. Let the constant $\mu$ (or $\rho$) be set in such a way that $(\mu/\alpha) = \rho = \gamma/(1 - \gamma)$. Then, (11) becomes $z_k = R(w, w_k) = (1 - \gamma)w_k + \gamma \bar{w}$. Also, like in part (b), the aggregation function $A$ becomes the mean here as well. Now, we complete the proof by using $z_k^* = (1 - \gamma)w_k^* + \gamma \bar{w}^*$ in [17]:

$$w_k^* = (1 - \gamma)w_k^* + \gamma \bar{w}^* - \frac{1}{\alpha} \nabla \tilde{f}_k(z_k^*) \implies w_k^* = \bar{w}^* - \frac{1}{\alpha \gamma} \nabla \tilde{f}_k(z_k^*).$$

\[ \square \]

Note that part (b) of the Corollary recovers the fixed point result of FedProx given in [Pathak and Wainwright, 2020]. In the next Proposition, we characterize the fixed points of Fed+ for the general case where $R$ and $A$ need not be defined through a common $\phi$ function.

**Proposition 3.** Consider the Fed+ algorithm with an arbitrary aggregation function $A$ and the following personalization function: $z_k = R(\bar{w}, w_k) = (1 - \gamma_k)w_k + \gamma_k \bar{w}$, $\gamma_k \in (0, 1)$, $k = 1, \ldots, K$. Then, under Assumption \[1\] the following holds for any fixed point $(W^*, Z^*, \bar{w}^*)$ of Fed+:

$$w_k^* = \bar{w}^* - \frac{1}{\alpha \gamma_k} \nabla \tilde{f}_k((1 - \gamma_k)w_k^* + \gamma_k \bar{w}^*), \ k = 1, \ldots, K, \text{ where } \bar{w}^* = A(w_1^*, \ldots, w_K^*).$$  \hspace{1cm} (27)

Moreover, if the aggregation function $A$ (such as Mean, Geometric Median, Coordinate-wise Median) in Fed+ satisfy the following translation & sign invariance property

$$\forall w, w_1, \ldots, w_K \in \mathbb{R}^d, \ A(w - w_1, \ldots, w - w_K) = w - A(w_1, \ldots, w_K),$$  \hspace{1cm} (28)

then the following holds:

$$A\left(\frac{1}{\alpha \gamma_1} \nabla \tilde{f}_1(z_1^*), \ldots, \frac{1}{\alpha \gamma_K} \nabla \tilde{f}_K(z_K^*)\right) = 0, \text{ where } z_k^* = (1 - \gamma_k)w_k^* + \gamma_k \bar{w}^*, \ k = 1, \ldots, K.$$  \hspace{1cm} (29)
Proof. Similar to the proof of Theorem we have the following from Fed+ algorithm: \( \forall t \geq 1 \),

\[
\begin{align*}
\mathbf{w}_k^t &= z_k^{t-1} - \frac{1}{\alpha} \nabla f_k(z_k^{t-1}), \quad k = 1, \ldots, K, \\
\tilde{\mathbf{w}}_k^t &= A(\mathbf{w}_1^t, \ldots, \mathbf{w}_K^t), \\
\mathbf{z}_k^t &= (1 - \gamma_k)\mathbf{w}_k^t + \gamma_k\tilde{\mathbf{w}}_k^t, \quad k = 1, \ldots, K.
\end{align*}
\]

For a fixed point, we thus have:

\[
\begin{align*}
\mathbf{w}_k^* &= z_k^* - \frac{1}{\alpha} \nabla f_k(z_k^*), \quad k = 1, \ldots, K, \\
\tilde{\mathbf{w}}^* &= A(\mathbf{w}_1^*, \ldots, \mathbf{w}_K^*), \\
\mathbf{z}_k^* &= (1 - \gamma_k)\mathbf{w}_k^* + \gamma_k\tilde{\mathbf{w}}^*, \quad k = 1, \ldots, K.
\end{align*}
\]

Now, replacing the first \( z_k^* \) in (30) by (32), we arrive at

\[
\mathbf{w}_k^* = \tilde{\mathbf{w}}^* - \frac{1}{\alpha \gamma_k} \nabla f_k(z_k^*), \quad k = 1, \ldots, K.
\]

Thus, we have (27). Further, using (33) in (31) and utilizing the property (28) gives us (29).

\( \square \)

7 Convergence Analysis of Fed+

Here, we analyze the convergence properties and fixed points of the Fed+ algorithm. The parameters \( \alpha > 0, \rho > 0, \) and \( \eta > 0 \) are tunable unless specified otherwise. For the rest of this section, we will use the following setting for the parameters in Algorithm 1. (i) \( \phi : \mathbb{R}^d \to [0, \infty] \) is any convex function with easy to compute proximal operator, (ii) the function \( R \) is set as in (11), (iii) the initialization parameter \( \lambda \) in line 10 is set to 0, (iv) set \( \theta := \frac{1}{\eta + \alpha} \) in line 12 and (v) compute the aggregation step in line 8 via (4), where \( B \) is given by (5). To implement this aggregation step \( \tilde{\mathbf{w}} \leftarrow A(\mathbf{w}_1, \ldots, \mathbf{w}_K) \) for a general choice of \( \phi \), we propose the following iterative procedure initialized with \( \tilde{\mathbf{w}} = \mathbf{w}_{mean} := \text{Mean}\{\mathbf{w}_1, \ldots, \mathbf{w}_K\} \):

\[
\tilde{\mathbf{w}} \leftarrow \mathbf{w}_{mean} - \text{Mean}\{\text{prox}_\phi(\mathbf{w}_1 - \tilde{\mathbf{w}}), \ldots, \text{prox}_\phi(\mathbf{w}_K - \tilde{\mathbf{w}})\}.
\]

(34)

The above setting immediately gives rise to the following property:

\[
(\mathbf{z}_1^t, \ldots, \mathbf{z}_K^t, \tilde{\mathbf{w}}^t) = \underset{\mathbf{z} \in \mathbb{R}^{d \times K}, \tilde{\mathbf{w}} \in \mathbb{R}^d}{\text{argmin}} H_{\mu, \alpha}(\mathbf{W}^t, \mathbf{Z}, \tilde{\mathbf{w}}), \quad t = 1, 2, \ldots
\]

(35)

Now, to analyze Fed+, we make the following smoothness assumption:

**Assumption 2.** For each \( k = 1, 2, \ldots, K \), \( f_k : \mathbb{R}^d \to \mathbb{R} \) is differentiable and the gradient \( \nabla f_k \) is Lipschitz continuous with constant \( L_f \), i.e., \( \| \nabla f_k(w) - \nabla f_k(w') \|_2 \leq L_f \| w - w' \|_2, \forall w, w' \in \mathbb{R}^d \).

Let us first recall the following well-know descent lemma [Beck, 2015] for functions with Lipschitz continuous gradient.

**Lemma 1.** Let \( f : \mathbb{R}^d \to \mathbb{R} \) be continuously differentiable and \( \nabla f \) be Lipschitz continuous with constant \( L > 0 \). Then, the following holds:

\[
f \left( \mathbf{w} - \frac{1}{L} \nabla f(\mathbf{w}) \right) \leq f(\mathbf{w}) - \frac{1}{2L} \| \nabla f(\mathbf{w}) \|_2^2, \forall \mathbf{w} \in \mathbb{R}^d.
\]

(36)

**Proposition 4.** Under Assumption 2 and the stepsize choice \( \eta = 1/L_f \), the following holds for Fed+:

\[
F_k(\mathbf{w}_k^t; z_{k}^{t-1}, \tilde{\mathbf{w}}^{t-1}) \leq F_k(\mathbf{w}_k^{t-1}; z_{k}^{t-1}, \tilde{\mathbf{w}}^{t-1}) - \frac{1}{2(L_f + \alpha)} \| \nabla F_k(\mathbf{w}_k^{t-1}; z_{k}^{t-1}, \tilde{\mathbf{w}}^{t-1}) \|_2^2,
\]

(37)

where \( F_k \) is defined in (12) and the gradient is w.r.t. \( \mathbf{w}_k \).

**Proof.** From Proposition 2 we know that the local update (6) (line 12 in Fed+ algorithm) is a gradient descent iteration with learning rate \( \eta' = \eta(1 - \alpha) = \frac{1}{L_f + \alpha} \) applied to the function \( F_k(\cdot; z_{k}^{t-1}, \tilde{\mathbf{w}}^{t-1}) \). Clearly, \( \nabla F_k(\cdot; z_{k}^{t-1}, \tilde{\mathbf{w}}^{t-1}) \) is Lipschitz continuous with constant \( L = (L_f + \alpha) \). Therefore,
applying the above Lemma, we have the following after one gradient descent iteration (starting with \(w^t_k\)) at the Local-Solve subroutine: \(\forall k \in S_t,\)
\[
F_k(w^t_k; z^{t-1}_k, \tilde{w}^{t-1}) \leq F_k(w^{t-1}_k; z^{t-1}_k, \tilde{w}^{t-1}) - \frac{1}{2L}||\nabla F_k(w^{t-1}_k; z^{t-1}_k, \tilde{w}^{t-1})||^2_2. \tag{38}
\]
Now, note the fact that \(F_k(w^t_k; z^{t-1}_k, \tilde{w}^{t-1})\) remains non-increasing after each gradient descent step. This completes the proof. \(\square\)

Combining the relation (35) with (37) we derive the following convergence result for Fed+:

**Theorem 2.** Assume that \(H_{\mu,\alpha}\) in (8) is bounded from below, parties are sampled with equal probability. Then, under Assumption 2 and the stepsize choice \(\eta = 1/L_f\), the following holds for Fed+:
\[
\lim_{t \to \infty} \mathbb{E} \left[ \sum_{k=1}^{K} \|\nabla F_k(w^{t-1}_k; z^{t-1}_k, \tilde{w}^{t-1})\|_2^2 \right] = 0, \tag{39}\]
where the expectation is with respect to the random subsets \(S_t, \ t \geq 1.\)
Moreover, the federated objective \(F_{\mu}(W^t)\) monotonically decreases with round \(t\) and converges to a value \(\bar{F}_{\mu} \geq \min_{W^0} F_{\mu}(W^0).\) Additionally, if the \(f_k\)'s are convex, all parties are active in every round, and the level set \(\mathcal{S} = \{ (W, Z, \tilde{w}) : H_{\mu,\alpha}(W, Z, \tilde{w}) \leq H_{\mu,\alpha}(W^0, Z^0, \tilde{w}^0) \}\) is compact, then \(\lim_{t \to \infty} F_{\mu}(W^t) = \min_{W} F_{\mu}(W)\) and the rate of convergence is \(O(1/t)\).

**Proof.** We start the proof with following observations from the Proposition 1 and the definition (11):
\[
\bar{w}^t = \arg\min_{w \in \mathbb{R}^d} \left[ \min_{Z \in \mathbb{R}^{d \times K}} H_{\mu,\alpha}(W^t, Z, \tilde{w}) \right],
\]
\[
(z_1^t, \ldots, z_K^t) = \arg\min_{Z \in \mathbb{R}^{d \times K}} H_{\mu,\alpha}(W^t, Z, \tilde{w}^t), \ t = 1, 2, \ldots
\]
Combining the above, we have the following:
\[
(z_1^t, \ldots, z_K^t, \bar{w}^t) = \arg\min_{Z \in \mathbb{R}^{d \times K}, \bar{w} \in \mathbb{R}^d} H_{\mu,\alpha}(W^t, Z, \bar{w}), \ t = 1, 2, \ldots \tag{40}
\]
This implies
\[
H_{\mu,\alpha}(W^t, Z_i^t, \tilde{w}^t) \leq H_{\mu,\alpha}(W^t, Z_i^{t-1}, \tilde{w}^{t-1}), \ t = 1, 2, \ldots \tag{41}
\]
Before moving further, we introduce the following notation \(F_k^t(w) := F_k(w; z_k^{t-1}, \tilde{w}^{t-1}).\) Now, we have the following from Proposition 1
\[
F_k^t(w_k^t) \leq F_k^t(w_k^{t-1}) - \frac{1}{2L}||\nabla F_k^t(w_k^{t-1})||^2_2, \ \forall k \in S_t \tag{42}
\]
where \(L := (L_f + \alpha).\) Moreover, \(w_k^t = w_k^{t-1}\) for all \(k \not\in S_t\) implies that
\[
F_k^t(w_k^t) \leq F_k^t(w_k^{t-1}), \ \forall k \not\in S_t \tag{43}
\]
Summing (42) and (43), we get: \(\forall t = 1, 2, \ldots,\)
\[
H_{\mu,\alpha}(W^t, Z_i^{t-1}, \tilde{w}^{t-1}) \leq H_{\mu,\alpha}(W^{t-1}, Z_i^{t-1}, \tilde{w}^{t-1}) - \frac{1}{2KL} \sum_{k \in S_t} ||\nabla F_k^t(w_k^{t-1})||^2_2, \tag{44}
\]
We can also express (44) in expectation form:
\[
\mathbb{E}[H_{\mu,\alpha}(W^t, Z_i^{t-1}, \tilde{w}^{t-1})] \leq H_{\mu,\alpha}(W^{t-1}, Z_i^{t-1}, \tilde{w}^{t-1}) - \frac{p}{2KL} \sum_{k=1}^{K} ||\nabla F_k^t(w_k^{t-1})||^2_2, \tag{45}
\]
where the expectation is w.r.t the random subset \(S_t\) and \(p \in (0, 1]\) is the probability of \(k \in S_t.\) Taking, expectations w.r.t \(S_1, S_2, \ldots, S_t\) (i.e. all the randomness), we get: \(\forall t = 1, 2, \ldots,\)
\[
\mathbb{E}[H_{\mu,\alpha}(W^t, Z_i^{t-1}, \tilde{w}^{t-1})] \leq \mathbb{E}[H_{\mu,\alpha}(W^{t-1}, Z_i^{t-1}, \tilde{w}^{t-1})] - \frac{p}{2KL} \sum_{k=1}^{K} \mathbb{E}||\nabla F_k^t(w_k^{t-1})||^2_2. \tag{46}
\]
Combining (46) and (41), we have: \( \forall t = 1, 2, \ldots, \)
\[
\mathbb{E}[H_{\mu, \alpha}(W^t, Z^t, \tilde{w}^t)] \leq \mathbb{E}[H_{\mu, \alpha}(W^{t-1}, Z^{t-1}, \tilde{w}^{t-1})] - \frac{p}{2KL} \sum_{k=1}^{K} \mathbb{E}[\|\nabla F_k^t(w_{k}^{t-1})\|_2^2]. \tag{47}
\]

Summing over all \( t \) and using the fact \( H_{\mu, \alpha} \) is bounded below, we arrive at (39).

On the other hand, combining (43) and (41), we get: \( \forall t = 1, 2, \ldots, \)
\[
H_{\mu, \alpha}(W^t, Z^t, \tilde{w}^t) \leq H_{\mu, \alpha}(W^{t-1}, Z^{t-1}, \tilde{w}^{t-1}) - \frac{1}{2KL} \sum_{k \in S_t} \|\nabla F_k^t(w_{k}^{t-1})\|_2^2. \tag{48}
\]

Now, from (40) and (9) we see that \( F_\mu(W^t) = H_{\mu, \alpha}(W^t, Z^t, \tilde{w}^t) \). Thus, from (48) we have that \( \{F_\mu(W^t)\}_{t=0} \) is monotonically non-decreasing; therefore, also converges to some real value say \( F_\mu \) because \( H_{\mu, \alpha} \) is bounded below. The rest of the proof, when \( f_k \)'s are convex, follows from Theorem 3.7 in Beck [2015] as (48) and (40) together suggest that Fed+ is basically an (approximate) alternating minimization approach for solving (9).

\[\square\]

8 Experiments

8.1 Standard Federated Learning Datasets

We curated a diverse set of synthetic and non-synthetic datasets from [Li et al., 2020a] and the LEAF set of [Caldas et al., 2018]. Data is randomly split for each local party into an 80% training set and a 20% testing set. For all experiments, the number of local iterations per round \( E = 20 \), the number of selected parties per round \( K = 10 \), and the batch size is 10. The neural network models for all datasets are the same as those of [Li et al., 2020a]. Learning rates are 0.01, 0.03, 0.003 and 0.3 for synthetic, MNIST and FEMNIST and Sent140 datasets, respectively. The experiments used a fixed regularization parameter \( \alpha = 0.01 \) for each party’s Local-Solve and the parameter \( \rho \) is set to 1000, 10 and 10 for FedAvg+, FedGeoMed+ and FedCoMed+ methods, respectively. On the Sent140 dataset, we found that initializing the local model to a mixture model (i.e. setting \( \lambda = 0.001 \) instead of the default \( \lambda = 0 \)) at the beginning of every Local-Solve subroutine for each party gives the best performance. We simulate the federated learning setup (1 aggregator \( K \) parties) on a commodity hardware machine with 16 Intel® Xeon® E5-2690 v4 CPU and 2 NVIDIA® Tesla P100 PCIe GPU.

To generate non-identical synthetic data, we follow a similar setup to that of [Li et al., 2020a], additionally imposing heterogeneity among parties. In particular, for each party \( k \), we generate samples \( (X_k, Y_k) \) according to the model \( y = \arg \max(\text{softmax}(Wx + b)) \), \( x \in \mathbb{R}^{60}, W \in \mathbb{R}^{10 \times 60}, b \in \mathbb{R}^{10}. \) We model \( W_k \sim N(uk, 1), b_k \sim N(0, \zeta), x_k \sim N(v_k, \Sigma) \), where the covariance matrix \( \Sigma \) is diagonal with \( \Sigma_{j,j} = j^{-1.2} \). Each element in the mean vector \( u_k \) is drawn from \( N(B_k, 1), B_k \sim N(0, \beta) \). Therefore, \( \beta \) controls how much the local models differ from each other and \( \zeta \) controls how much the local models differ from that of other parties. In order to better characterize statistical heterogeneity and study its effect on convergence, we choose \( \zeta = 1000 \) and \( \beta = 10 \). There are \( K = 30 \) parties in total and the number of samples on each party follows a power law. Hyperparameters are the same as those of [Li et al., 2020a] and use their reported best \( \mu \) for FedProx.

First, we consider a convex classification problem using MNIST [LeCun et al., 1998] with multinomial logistic regression. To impose statistical heterogeneity, we distribute the data among \( K = 1000 \) parties such that each party has samples of only one digit and the number of samples per party follows a power law. The input is a flattened 784-dimensional (28 \( \times \) 28) image, and the output is a class label between 0 and 9. Second, we study the more complex 62-class Federated Extended MNIST [Cohen et al., 2017; Caldas et al., 2018] (FEMNIST) of [Li et al., 2020a]. Heterogeneous data partitions are generated by subsampling 10 lower case characters (‘a’-‘j’) from EMNIST and distributing only 5 classes to each party, with \( K = 200 \) parties in total. The input is a flattened 784-dimensional (28 \( \times \) 28) image, and the output is a class label between 0 and 9. Lastly, to address non-convex settings, we consider sentiment analysis on tweets from Sentiment140 [Go et al., 2009] (Sent140) with a two layer LSTM binary classifier containing 256 hidden units with pretrained 300D GloVe embedding [Pennington et al., 2014]. Each twitter account corresponds to a party
with $K = 772$ in total. The model takes as input a sequence of 25 characters, embeds each into a 300-dimensional space using Glove and outputs one character per training sample after 2 LSTM layers and a densely-connected layer. We consider the highly heterogeneous setting where there are 90% stragglers; see [Li et al., 2020a] for details.

\[ w_i \cdot \eta_i = \sum_{t=0}^{T} w_{i,t} = 1 \text{ for all } t. \]

The initial value of $w_0$ is $(1, 0, \ldots, 0)^T$. At the end of time $t$, the weights evolve according to $w_t' = (y_t \circ w_{t-1})/(y_t \cdot w_{t-1})$, where $\circ$ is the element-wise division. Define $w_{t-1}$ as the portfolio weight vector at the beginning of time $t$ where its $i^{th}$ element $w_{i,t-1}$ represents the proportion of asset $i$ in the portfolio after capital reallocation and $\sum_{t=0}^{T} w_{i,t} = 1$ for all $i$. The portfolio is initialized with $w_0 = (1, 0, \ldots, 0)^T$. At the end of the round, the weights evolve according to $w_t' = (y_t \circ w_{t-1})/(y_t \cdot w_{t-1})$, where $\circ$ is the element-wise division. The reallocation from $w_t'$ to $w_t$ is gotten by selling and buying relevant assets. Paying all fees, this reallocation shrinks the portfolio value by $\rho_t$. The normalized close price matrix at $t$ is $Y_t = [v_{t-1} \circ v_{t} \circ v_{t-2} \circ \cdots v_{t-1} \circ v_t | 1]$ where $1 = (1, 1, \ldots, 1)^T$ and $t$ is the time embedding. The

![Figure 2: Performance of FedAvg+, FedGeoMed+ and FedCoMed+ is superior to that of the baselines.](image)

In Figure 2, we report the test performance of the baseline algorithms FedAvg, FedProx, RFA, coordinate-wise median and Fed+. The baseline robust algorithms perform the worst on these non-IID data sets. Fed+ often speeds up the learning convergence, as shown in Figure 2 and improves performance on these datasets by 28.72%, 6.24%, 11.32% and 13.89%, resp. In particular, the best Fed+ algorithm can improve the most competitive implementation of the baseline FedProx on these four datasets by 9.90% on average. FedAvg+ achieves similar performance to FedGeoMed+ on MNIST and FEMNIST, but but fails to outperform the robust variants of Fed+, FedGeoMed+ and FedCoMed+, on the synthetic and Sent140 datasets, highlighting the benefit of robust statistics.

We also evaluate the impact of increasing the number of parties in training on test accuracy. On the synthetic dataset, average test accuracy improves from 70.22% to 90.73% to 98.03% when the number of parties participating in training goes from $K = 3$ to $K = 15$ to $K = 30$. The average is taken over FedAvg+, FedGeoMed+ and FedCoMed+. On MNIST, average accuracies of Fed+ are 69.80%, 81.34%, and 83.36% when the number of parties in training goes from $K = 100$ to $K = 500$ to $K = 1000$, resp. On FEMNIST, average accuracies of Fed+ are 25.16%, 68.71%, and 78.66% when the number of parties in training goes from $K = 20$ to $K = 100$ to $K = 200$, resp. On the Sent140 dataset, average accuracies of Fed+ are 57.13%, 60.77%, and 65.43% when the number of parties goes from $K = 77$ to $K = 386$ to $K = 772$, resp. This shows the benefit of using Fed+ increases as the number of parties increases.

### 8.2 Financial Portfolio Management Problem

Financial portfolio management is the process of sequentially allocating wealth to a collection of assets in consecutive trading periods [Markowitz, 1959; Haugen and Haugen, 2001]. Assume the financial market is sufficiently liquid such that any transactions can be executed immediately with minimal market impact.

Following [Jiang et al., 2017], let $t$ denote the index of asset trading days and $v_{i,t} = \{1, \ldots, m\}$ the closing price of the $i^{th}$ asset at time $t$, where $m$ is the number of assets in a given asset universe. The price vector $v_t$ consists of the closing prices of all $m$ assets. An additional dimension (the first dimension indexed by 0) in $v_t$, $v_{0,t}$, denotes the cash price at time $t$. We normalize all temporal variations in $v_t$ with respect to cash so $v_{0,t}$ is constant for all $t$. Define the price relative vector at time $t$ as $y_t = v_{t+1} \circ v_t = (1, v_{1,t+1}/v_{1,t}, \ldots, v_{m,t+1}/v_{m,t})^\top$ where $\circ$ denotes element-wise multiplication. Define $w_{t-1}$ as the portfolio weight vector at the beginning of time $t$ where its $i^{th}$ element $w_{i,t-1}$ represents the proportion of asset $i$ in the portfolio after capital reallocation. Define $\rho_t$ as the buy/sell fee; let $\rho_{t-1}$ denote the portfolio value at the beginning of $t$ and $\rho_t'$ at the end, so $\rho_t = \beta_t \rho_t'$. The normalized close price matrix at $t$ is $Y_t = [v_{t-1} \circ v_t \circ v_{t-2} \circ \cdots v_{t-1} \circ v_t | 1]$ where $1 = (1, 1, \ldots, 1)^\top$ and $t$ is the time embedding.
financial portfolio management is formulated as a reinforcement learning (RL) problem where, at time step \( t \), the agent observes the state \( s_t \triangleq (Y_t, w_{t-1}) \), takes an action (portfolio weights) \( a_t = w_t \) and receives an immediate reward \( r(s_t, a_t) \triangleq \ln(\rho_{t}/\rho_{t-1}) = \ln(\beta \cdot y_t \cdot w_{t-1}) \). The objective of RL agent is to maximize an objective function over policies \( \mu_\theta \) parameterized by \( \theta \): 
\[
\max_{\mu_\theta} J(\mu_\theta) = \max_{\mu_\theta} \sum_{t=1}^{T} \ln (\beta \cdot y_t \cdot w_{t-1})/T.
\]

Given \( K \) parties, each with its own trading asset universe and data \( I_k \) for \( k \in \{1, 2, \ldots, K\} \). The number of assets in each universe \( I_k \) is the same. Each party’s task is a Markov decision processes \((S_k, A, \mathcal{P}_k, r, \tau, \gamma)\) where \( \tau \) is the decision horizon, \( \gamma \in (0, 1) \) is the discount factor, and the action space \( A \) and the reward function \( r \) are common to all parties. The state space \( S_k \) depends on each party’s trading asset universe \( I_k \) and the transition kernel density \( \mathcal{P}_k \) should be inferred from the underlying stock price dynamics of \( I_k \). Each party is maximizing its objective and updating its RL agent using the deterministic policy gradient [Silver et al. 2014] through the federation.

The \( K \) agent models follow the Ensemble of Identical Independent Evaluators (EIIE) topology with the online stochastic batch learning of [Jiang et al. 2017], the latter of which samples mini-batches consecutively in time to train the EIIE networks. We choose the discount factor \( \gamma = 0.99 \), time embedding \( l = 30 \) and use a batch size of 50 and a learning rate of \( 5 \times 10^{-5} \) for all experiments. Other experimental details including the choice of time embedding and the RNN implementation of the EIIE for agent models are the same as those of [Jiang et al. 2017]. We conduct our experiments on 10 virtual machines where each machine has 32 Intel® Xeon® Gold 6130 CPUs and 128 GB RAM and can support up to 5 party processes.

The experiments are performed on 50 assets from the S&P500 technology sector\(^2\). Each party \( k \) constructs its own asset universe \( I_k \) by randomly choosing 9 assets and pre-trains a private data augmentation method. We use geometric Brownian motion, variable-order Markov model and generative adversarial network, on the assets of each in \( I_k \). S&P price data from 2006–2018 is used for training data augmentation and the RL policies; 2019 price data is used for testing.

The number of global rounds \( T = 800 \) and local RL iterations is equal over all parties and set to \( E = 50 \). We use a fixed regularization parameter \( \alpha = 0.01 \) for each party’s Local-Solve. In practice, we find that initializing the local model to a mixture model (i.e. using a small positive value of \( \lambda \) instead of the default \( \lambda = 0 \)) at the beginning of every Local-Solve subroutine for each party yields good performance. Such a mixture model is computed using a convex combination of each party’s local model and latest global model with weight \( \lambda = 0.001 \). In addition, we keep \( \lambda^t_k \) (also the diagonal entries for the diagonal matrix \( \Lambda^t_k \) ) constant for all \( t \). Results are based on training over a sufficiently wide grid of fixed \( \lambda^t_k \) (typically 10-13 values on a multiplicative grid of resolution \( 10^{-1} \) or \( 10^{-3} \)). Results are for the best fixed \( \lambda^t_k \) selected individually for each experiment.

Consider now that \( K = 50 \) parties (portfolio managers), each with an RL agent performing portfolio management on its own asset universe. Each party shares its RL model updates through federated learning and receives an updated model that can improve the performance on its own asset universe. Each portfolio manager generates one year of synthetic time-series data using their own data generation method for each of their assets every 1000 RL local training iterations and appended to the last day’s real closing prices of the assets. This combined synthetic–real data, which is strictly confidential to each party, is used to train its own RL agent. All parties participate in every round of federated training. Two metrics measure performance: (i) the annualized geometric average return, and (ii) to take into account risk and volatility, the Sharpe ratio, \( \mathbb{E}[X] / \sqrt{\text{var}[X]} \), where \( X \) is the random return, i.e., the additional return per unit increase in risk.

Average performance is shown in Table 1. Average portfolio values in the testing period were shown already in Figure 5A. The Fed+ algorithms significantly outperform the original FedAvg, RFA, coordinate-wise median and FedProx algorithms. Average learning curves across baseline methods were shown in Figure 1B the top curve being the average of the 50 single-party training. In spite of having no adversarial parties, training failure occurs with all of the standard federated learning algorithms. As seen in Figure 5A, no training failure occurs with the Fed+ algorithms, rather, Fed+ stabilizes each party’s learning process. Figure 5A and Table 1 show that Fed+ significantly

\(^2\)We use the following 50 assets from the S&P500 technology sector: AAPL, ADBE, ADI, ADP, ADS, AKAM, AMD, APH, ATVI, AVGO, CHTR, CMCSA, CRM, CSCO, CTL, CTSH, CTXS, DIS, DISH, DXC, FB, FFIV, FISV, GLW, GOOG, IBM, INTC, INTU, IPG, IT, JNPR, KLAC, LRCX, MA, MCHP, MSFT, MSI, NFLX, NTAP, OMC, PAYX, QCOM, SNPS, STX, T, TEL, VZ, WDC, WU, XRX.
Table 1: Average performance of different methods over 50 parties.

| Method             | Ann. return | Sharpe | Method             | Ann. return | Sharpe |
|--------------------|-------------|--------|--------------------|-------------|--------|
| FedAvg             | 0.99%       | 0.11   | FedAvg+            | 32.93%      | 1.67   |
| RFA                | 4.90%       | 0.30   | FedGeoMed+         | 32.23%      | 1.64   |
| Coordinate-wise med.| 21.71%      | 1.17   | FedCoMed+          | 32.47%      | 1.65   |
| FedProx            | -1.46%      | -0.02  | –                  | –           | –      |

(a) Average portfolio values during the test period. Fed+ is superior to the baselines by a large margin.

(b) Average learning performance of Fed+. Compared with the training failure of Figure 1b using the baseline algorithms, there is no training failure with Fed+.

Figure 3: Results on the financial portfolio management problem.

outperforms FedAvg, RFA, coordinate-wise median and FedProx, improving the annualized return by 26.01% and the Sharpe ratio by 1.26 on average. For FedAvg+, we investigate how the risk-return performance improves as more parties share their local tasks. In particular, compared with single-party training (no-fusion), the maximum improvement in annualized return are 4.06%, 12.57% and 35.50%, and in Sharpe ratio are 0.19, 0.63 and 1.52 when $K = 5, 25$ and 50, showing the benefit of Fed+ as the number of parties increases.

9 Conclusion

Fed+ was designed to better handle the heterogeneity inherent in federated settings: the lack of IID data, the need for robustness to outliers and stragglers, and the requirement to perform well on
party-specific data. The Fed+ class of methods unifies numerous algorithms through a formulation that allows for robust ways of aggregating the local models whilst keeping the structure of local computation intact. We provide convergence guarantees for Fed+ for convex and non-convex loss functions, robust aggregation, and for the case of stragglers. We illustrate a real-world application from financial markets where forcing diverse parties to solve for a common solution across all parties leads to failure of the training process. Negative societal impacts of federated learning in general can be mitigated by requiring users to opt-in to any federated training process. In particular, personalization of the models can reduce the negative impacts that may occur from averaging models for a population, including bias. One limitation of this work is the joint treatment of personalization and robustness, which could in the future be decoupled. Another future work of interest is to explore further NN-layer-specific aggregation functions as made possible through the Fed+ formulation.

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