Can one design a geometry engine?

On the (un)decidability of certain affine Euclidean geometries

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Abstract
We survey the status of decidability of the first order consequences in various axiomatizations of Hilbert-style Euclidean geometry. We draw attention to a widely overlooked result by Martin Ziegler from 1980, which proves Tarski’s conjecture on the undecidability of finitely axiomatizable theories of fields. We elaborate on how to use Ziegler’s theorem to show that the consequence relations for the first order theory of the Hilbert plane and the Euclidean plane are undecidable. As new results we add:

(A) The first order consequence relations for Wu’s orthogonal and metric geometries (Wen-Tsün Wu, 1984), and for the axiomatization of Origami geometry (J. Justin 1986, H. Huzita 1991) are undecidable.

It was already known that the universal theory of Hilbert planes and Wu’s orthogonal geometry is decidable. We show here using elementary model theoretic tools that

(B) the universal first order consequences of any geometric theory $T$ of Pappian planes which is consistent with the analytic geometry of the reals is decidable.

The techniques used were all known to experts in mathematical logic and geometry in the past but no detailed proofs are easily accessible for practitioners of symbolic computation or automated theorem proving.

Keywords Euclidean geometry · Automated theorem proving · Undecidability

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1 Introduction

"Truth cannot be demonstrated, only invented."

The bishop in Max Frisch’s play
Don Juan or the love of geometry, Act V.

Since the beginning of the computer era automated theorem proving in geometry remained a central topic and challenge for artificial intelligence. Already in the late 1950s, [28, 29], H. Gelernter presented a machine implementation of a theorem prover for Euclidean Geometry. Automated theorem proving in geometry is still a very active research topic both in AI1 and in mathematical logic, e.g. [59].

The very first idea for mechanizing theorem proving in Euclidean geometry came from the fact that till not long ago high-school students were rather proficient in proving theorems in planimetry using Euclidean style deductions. A modern treatment of Euclidean Geometry was initiated by D. Hilbert at the end of the 19th century [35], and a modern reevaluation of Euclidean Geometry can be found in [33]. Formalization in first order logic is thoroughly discussed in [1], and conceptual issues are discussed in [8, Part III].

On the high-school level one distinguishes between Analytic Geometry which is the geometry using coordinates ranging over the real numbers, and Synthetic Geometry which deals with points and lines with their incidence relation augmented by various other relations such as equidistance, orthogonality, betweenness, congruence of angles etc. A geometric statement is, in the most general case, a formula in second order logic SOL using these relations. However, it is more likely that for practical purposes full second order logic is rarely used. In fact, almost all the geometrical theorems2 proved in [35] are expressible by formulas of first order logic FOL with very few quantifier alternations, cf. also [6, 7]. The same applies to other Euclidean style systems of geometry, cf. [1, 46]. Instead, one uses statements expressed in a suitable fragment $F$ of second order logic, which can be full first order logic FOL (the Restricted Calculus in the terminology of [31]) or an even more restricted fragment, such as the universal $\forall$-formulas $\forall$, the existential $\exists$-formulas $\exists$, or $\forall\exists$-Horn formulas $\forall\exists$ of first order logic.

Many variants of Synthetic Euclidean Geometry are axiomatized in the language of first order logic by a finite set of axioms or axiom schemes $T \subseteq \text{FOL}$ if continuity requirements are discarded. It follows from the Completeness Theorem of first order logic that the first order consequences of $T$ are recursively (computably) enumerable. If full continuity axioms, which are not FOL-expressible, are added even the first order consequences of $T$ are not necessarily recursively enumerable.

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1The WEB-page GeoCoq, A formalization of the foundations of geometry in Coq, http://geocoq.github.io/GeoCoq/ is entirely dedicated to using the proof assistant Coq to question of geometry and lists ongoing research. The conference series International Conference on Automated Deduction in Geometry, ADG is a forum dedicated to the exchange of ideas and views, to the presentation of research results and progress, and to the demonstration of software tools on the intersection between geometry, computation, and automated reasoning. ADG organizes a workshop every two years. The previous editions were held in Nanning (China) 2018, Strasbourg (France) 2016, Coimbra (Portugal) 2014, Edinburgh (UK) 2012, Munich (Germany) 2010, Shanghai (China) 2008, Pontevedra (Spain) 2006, Gainesville (USA) 2004, Linz (Austria) 2002, Zurich (Switzerland) 2000, Bejing (China) 1998, and Toulouse (France) 1996. The proceedings of ADG 1996, ADG 1998, ADG 2000, ADG 2002, ADG 2004, ADG 2006, ADG 2010, ADG 2012, and ADG 2014 appeared as LNAI 1360, LNAI 1669, LNAI 2061, LNAI 2930, LNAI 3763, LNAI 4869, LNCS 6877, LNAI 7993, and LNAI 9201, respectively. The proceedings of ADG 2016 can be found at https://hal.imri.fr/hal-01334334.

2There are exceptions. Theorem 6 in Hilbert’s book [35] is really a countable family of first order statements, one for each finite set of points. This also applies to Theorem 9. Sometimes this is true for a particular theorem, but not for its proof. This occurs when theorem is first order definable, but depends on a second order axiom like the Archimedeian axiom (Theorems 34 and 35 for example).
A first order statement in the case of Analytic Geometry over the reals is a first order formula $\phi$ in the language of ordered fields and we ask whether $\phi$ is true in the ordered field of real numbers. By a celebrated theorem of A. Tarski announced in [67], and proven in [68], this question is mechanically decidable using quantifier elimination. However, the complexity of the decision procedure given by Tarski uses an exponential blowup for each elimination of a quantifier. This has been dramatically improved by G.E. Collins in 1975, reprinted in [22], giving a doubly exponential algorithm in the size of the input formula.

Further progress was and is slow. For a state of the art discussion, cf. [20, 22]. However, it is unlikely that a polynomial time algorithm exists for quantifier elimination over the ordered field of real numbers for existential formulas, because this would imply that in the computational model of Blum-Shub-Smale over the reals $\mathbb{R}$, [11], we would have $\mathbf{P}_\mathbb{R} = \mathbf{NP}_\mathbb{R}$, [52, 53], which is one of the open Millennium Problems, [23]. For formulas with unrestricted alternation of quantifiers a doubly exponential lower bound was given in [25]. Simply exponential upper bound for existential formulas were given by several authors. For a survey, see [20].

So what can a geometry engine for Euclidean Geometry try to achieve?

For a fixed fragment $\mathcal{F}$ of SOL in the language of Analytic or Synthetic Geometry we look at the following possibilities:

**Analytic Tarski Machine** ATM($\mathcal{F}$):

Input: A first order formula $\phi \in \mathcal{F}$ in the language of ordered fields.
Output: $\text{true}$ if $\phi$ is true in the ordered field of real numbers, and $\text{false}$ otherwise.

**Synthetic Tarski Machine** STM($\mathcal{F}$):

Input: A first order formula $\psi \in \mathcal{F}$ in a language of synthetic geometry.
Output I: a translation $\phi = \text{cart}(\psi)$ of $\psi$ into the language of analytic geometry.
Output II: $\text{true}$ if $\phi$ is true in the ordered field of real numbers, and $\text{false}$ otherwise.

**Geometric Theorem Generator** GTG($\mathcal{F}$):

Input: A recursive set of first order formulas $T \subseteq \text{FOL}$ (not necessarily in $\mathcal{F}$), in the language of some synthetic geometry.
Output: A non-terminating sequence $\phi_i : \in \mathbb{N}$ of formulas in $\mathcal{F}$ the language of the same synthetic geometry which are consequences of $T$.

**Geometric Theorem Checker** GTC($\mathcal{F}$):

Input: A recursive set of first order formulas $T$ and another formula $\phi \in \mathcal{F}$ in the language of some synthetic geometry.
Output: $\text{true}$ if $\phi$ is a consequence of $T$, and $\text{false}$ otherwise.

In the light of the complexity of quantifier elimination over the real numbers, [52, 53], designing *computationally feasible* Analytic or Synthetic Tarski Machines for various fragments $\mathcal{F}$ with the exception of $\mathcal{U}$ is a challenge both for Automated Theorem Proving (ATP) as well as for Symbolic Computation (SymbComp). Designing Geometric Theorem Generators GTG($\mathcal{F}$) is possible but seems pointless, because it will always output long sub-sequences of geometric theorems in which we are not interested.

In this paper we will concentrate on the challenge of designing Geometric Theorem Checkers GTC($\mathcal{F}$). This is possible only for very restricted fragments $\mathcal{F}$ of FOL, such as the universal formulas $\mathcal{U}$.
The main purpose of this paper is to bring negative results concerning Geometric Theorem Checkers GTC(\mathcal{G}) to a wider audience.

The negative results are based on a correspondence between sufficiently strong axiomatizations of Synthetic Euclidean Geometries and certain theories of fields consistent with the theory of the ordered field of real numbers.

A model of incidence geometry is an incidence structure which satisfies the axioms I-1, I-2 and I-3 from Section 3.2. An affine plane is a model of incidence geometry satisfying the Parallel Axiom (ParAx). An affine plane is Pappian if it additionally satisfies the axiom of Pappus (Pappus). In this paper an axiomatization of geometry \( T \) is sufficiently strong if all its models are affine planes.

Let \( \mathcal{F} \) be a field of characteristic 0. One can construct an Cartesian plane \( \Pi(\mathcal{F}) \) over \( \mathcal{F} \) which satisfies the Pappian axiom, and where all the lines are infinite. This construction turns out to be an example of a transduction as defined in Section 4.1. On the other side, if \( \Pi \) is an Pappian plane which has no finite lines then one can define inside \( \Pi \) its coordinate field \( \mathcal{F}(\Pi) \) which is of characteristic 0. Once one establishes that both constructions are first order definable one can say that the theory of ordered fields is bi-interpretable with the theory of Pappian planes. But the following theorem does not establish bi-interpretability with stating the required first order definability.

**Proposition 1.1** (F. Schur [60] and E. Artin [3]) (i) \( \mathcal{F} \) is a field of characteristic 0 iff \( \Pi(\mathcal{F}) \) is a Pappian plane with no finite lines.

(ii) \( \Pi \) is a Pappian plane with no finite lines iff \( \mathcal{F}(\Pi) \) is a field of characteristic 0.

(iii) The fields \( \mathcal{F} \) and \( \mathcal{F}(\Pi) \) are isomorphic.

(iv) The Pappian planes \( \Pi \) and \( \Pi(\mathcal{F}(\Pi)) \) are isomorphic as incidence structures.

A theory (set of formulas) \( T \subseteq \text{FOL}(\tau) \) is axiomatizable if the set of consequences of \( T \) is computably enumerable. \( T \) is decidable if the set of consequences of \( T \) is computable. \( T \) is undecidable if it is not decidable. \( T \) is complete if for every formula \( \phi \in \text{FOL}(\tau) \) without free variables either \( T \models \phi \) or \( T \models \neg \phi \). We note that if \( T \) is axiomatizable and complete, then \( T \) is decidable.

On the side of theories of fields we have several undecidability results:

**Proposition 1.2** (J. Robinson, 1949 [57]) (i) The theory of fields is undecidable. The same holds for fields of characteristic 0.

(ii) The theory of ordered fields is undecidable.

(iii) The theory of the field of rational numbers \( \langle \mathbb{Q}, +, \times \rangle \) is undecidable.

To show that the first order theory of affine geometry is undecidable we would like to use a classical tool from decidability theory, the details of which we explain in Section 4.

**Proposition 1.3** ([54], based on [69]) Let \( I \) be a first order translation scheme with associated transduction \( I^\ast \) which maps \( \tau \)-structures into \( \sigma \)-structures. Furthermore, let \( S \) be an undecidable first order theory over a relational vocabulary \( \sigma \) and let \( T \) be a theory over \( \tau \). Assume that \( I^\ast \) maps the models of \( T \) onto the models of \( S \), and that \( S \) is undecidable, then \( T \) is also undecidable.

The onto-condition needed for our purpose is usually stated in textbooks for mathematicians and logicians under various names. It is explicitly stated as the onto-condition in [26, 37]. In [61] it is called faithful, and in [51] it is called invertible. In this paper we use the formalism of translation schemes (with translations and transductions) to deal with the inter-
reducibility of theories. There are many variations of inter-reducibility between theories in the literature, see [17, 18, 27, 58, 65, 70, 71]. They all deal with an attempt to define the sameness of axiomatic theories and come under different names such as bi-interpretablity, mutual interpretability, faithful interpretability, synonymy, etc. The undecidability proofs we discuss in this paper can be modified for many of these notions of inter-reducibility. The notion of bi-interpretablity as defined in [27, 70, 71] stands out to be the most flexible tool. However, it is not the purpose of this paper to discuss all these variations.

Propositions 1.1 and 1.2 are not enough to prove that first order theory of affine geometry is undecidable. We have to verify all the conditions of Proposition 1.3.

In particular, we have to show:

(A) There is a first order translation scheme $RF_{field}$ such that for every Pappian plane $\Pi$ the structure $RF^*_{field}(\Pi)$ is a field.

(B) There is a first order translation scheme $PP_{\in}$ such that for every field $\mathcal{F}$ the structure $PP^*_{\in}(\Pi)$ is an Pappian plane.

(C) For every field $\mathcal{F}$ we have $RF^*_{field}(PP^*_{\in}(\mathcal{F})) \simeq \mathcal{F}$.

In the above statements the translation schemes are existentially quantified. They are given explicitly in Section 4. $RF^*_{field}$ and $PP^*_{\in}$ are the transductions induced by $RF_{field}$ and $PP_{\in}$ respectively, as defined in Section 4. Properties (A-C) are shown in detail in Section 5.5. While the existence of $PP_{\in}$ is rather straightforward, the existence of $RF_{field}$ with the necessary properties (B) and (C) requires the first order definability of the field of coordinates of affine planes. If $\Pi$ is a Hilbert plane or a Euclidean plane, the field of coordinates can be defined using segment arithmetic. This can be done using a first order translation scheme $FF_{field}$, which is somehow simpler than $RR_{field}$.

Only after having established (A) and (C) we can conclude:

**Theorem 1.1**

(i) The first order theory of Pappian planes undecidable.

(ii) The first order theory of affine geometry is undecidable.

(ii) follows from (i) because Pappian planes are obtained from affine planes by adding a finite number of axioms in the language of incidence geometry.

The ingredients for proving Theorem 1.1 were all implicitly available when Proposition 1.2 was published. I would also assume that Theorem 1.1 was known in Berkeley, but no detailed proof was written down. A. Tarski presented the result for projective planes at the 11th Meeting of the Association of Symbolic Logic already in 1949, [79]. An incomplete sketch of a proof Theorem 1.1 was published in 1961 by W. Rautenberg [55]. His more detailed proof of the projective case in [56] uses Proposition 1.2 and Lemma 4.4, but fails to note that something like Theorem 6.1 is needed to complete the argument. We discuss this in detail at the end of Section 4. It also seems that W. Szmielew planned to include a proof of Theorem 1.1 in her unfinished and posthumously published [66]. The only complete proof of Theorem 1.1 I could find in the literature appears in [16]. However, the arguments contain some fixable errors. One of the purposes of this paper is to give a conceptually clear account of what is needed to prove Theorem 1.1 by stating all the ingredients of the proof

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In [16, Section 7] the undecidability of affine spaces is stated (Corollary 7.38). It also states that the same applies for projective spaces, referring the reader to [78], but [78] does not mention projective spaces. It also discusses Proposition 1.1 in [16, Section 6], but fails to mention that Proposition 1.1 is needed to prove Theorem 1.1 (Theorem 7.37 in [16, Section 7]). Note that that in [16, Section 7] the numbering of theorems and corollaries is erroneously given as 1.37 and 1.38 instead of 7.37 and 7.38. It also attributes Proposition 1.2 erroneously to A. Tarski.
explicitly. They were surely known by most of the mathematicians and logicians interested in the logical aspects of geometry, but they were not written down in an accessible way so that graduate students of the 21st century in AI or automated deduction could reconstruct the proofs completely in a satisfactory manner.

To repeat this argument for other axiomatizations of extensions of affine geometry we need the following theorem of M. Ziegler:

**Theorem 1.2** (M. Ziegler, 1982 [12, 78])

(i) Let $T$ be a finite subset of the theory of the reals $\langle \mathbb{R}, +, \times \rangle$ and let $T^* = T \cup \{ n \neq 0, n \in \mathbb{N} \}$, where $n$ is shorthand for $\underbrace{1 + \ldots + 1}_n$. Both $T$ and $T^*$ are undecidable.

(ii) The same holds if $T$ is a finite subset of the theory of the complex numbers $\langle \mathbb{C}, +, \times \rangle$.

We paraphrase this theorem, following [64], by saying that the theory of real closed (algebraically closed) fields of characteristic 0 is finitely hereditarily undecidable.

Theorem 1.2 was conjectured by A. Tarski, but only proved in 1982 by M. Ziegler. Ziegler’s Theorem remained virtually unnoticed, having been published in German in the Festschrift in honor of Ernst Specker’s 60th birthday, published as a special issue of *L’Enseignement Mathématique*. In [62] the significance of the results of [78] is recognized. However, the book is written in German and is usually quoted for its presentation of Tarskian geometry. The discussion of Theorem 1.2 is buried there in the second part of the book dealing with metamathematical questions of geometry. This part of the book is difficult to absorb, both because of its pedantic style and its length. In short, the only reference to Theorem 1.2 within the framework of Automated Theorem Proving and Symbolic Computation is [13]. A very short and casual mention of Theorem 1.2 can also be found in [16].

The present paper gives a survey on the status of decidability of various axiomatizations of Hilbert-style Euclidean Geometry, including Wu’s metric geometry and the Origami geometry which are all undecidable, see Theorems 6.2, 6.4 and 6.6. None of these theorems are technically new. They all could have been proven with the tools used in the proof of Theorem 1.1 together with Ziegler’s Theorem 1.2. However, Theorem 6.2 is stated and proved only in [62], and Theorems 6.4 and 6.6 could not have been stated before the corresponding geometries were axiomatized. For Wu’s orthogonal geometry this would be 1984 respectively 1994, when the first translation from Chinese appeared [77], or 1986 [76]. For Origami geometry this would be at the earliest in 1989, [40], but rather in 2000 with [2].

The purpose of this paper is to discuss undecidability results in geometry addressing practitioners in Automated Theorem Proving, Artificial Intelligence, and Symbolic Computation. Although many variants of these results were stated and understood already in the early 1950s, I could not find references with detailed proofs which could be easily understood and reconstructed by graduate students of Logic in Computer Science. On the other hand the techniques described in this paper are well known in the mathematical logic community. Theorem 1.1 and some of its variations are given as an exercise in [37, Exercise 10 of Section 5.4]. Although the Theorems 6.4 and 6.6 are strictly speaking new, their proofs use the same techniques, together with Ziegler’s Theorem 1.2 from 1982.

We hope that our presentation of this material is sufficiently concise and transparent in showing the limitations of automating theorem proving in affine geometry. We restrict our discussion here to theories of affine Hilbert-style Euclidean geometries. However, the methods can be extended to Tarski-style Euclidean geometry, and to projective and hyperbolic

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4In [38] a proof was announced, which later was found containing in irreparable mistake, cf. [78].
geometries. A brief discussion of this is given in Section 8. However, this paper is not meant to be a complete survey of undecidability results in geometry.

1.1 An after-thought concerning computer-checkable proofs

As some authors misquote Proposition 1.3 by omitting the condition that $I^*$ has to map the models of $T$ onto the models of $S$, it would be interesting to know whether a proof-checking system would have helped in discovering the exact nature of the gap in the published incomplete proofs of Theorem 1.1. For work on defining the fields of coordinates for Hilbert planes within the framework of the proof assistant COQ the reader may consult [10].

1.2 Outline of the paper

In Section 2 we summarize what is known about the (un-)decidability of theories of fields. Theorems 1.2 and 2.2 show that the decidability of the theory of real closed fields and its elimination of quantifiers are very specific properties of this theory.

In Section 3 describe the geometrical theories which are the center of our discussion: Affine incidence geometry, Hilbert-style Euclidean geometry, Wu’s orthogonal geometry and Origami geometry.

In Section 4 we spell out the subtleties needed to derive undecidability of geometrical theories from the undecidability of corresponding theories of fields. Although the general idea is very intuitive, the argument given frequently in the literature tends to overlook that this reduction depends on deep theorems specific to geometry. Besides the one-one correspondence between geometrical theories and theories of fields one also needs the first order definability of the field of coordinates for affine incidence geometry. In Section 5 we do discuss the role the field of coordinates plays in the undecidability proofs. and show that it is first order definable. In Section 6 we finally give the complete proofs of undecidability of our geometrical theories, and in Section 7 we show that the consequences in the universal fragment $\forall \forall$ of these geometries are still decidable. In Section 8 we briefly discuss how to extend the methods described in this paper to other geometrical theories. In Section 9, finally, we summarize what we have achieved and propose some open problems.

2 Decidable and undecidable theories of fields

2.1 Background on fields

Let $\tau_{\text{field}}$ be the purely relational vocabulary consisting of a ternary relation $Add(x, y, z)$ for addition with $Add(x, y, z)$ holds if $x + y = z$, a ternary relation $Mult(x, y, z)$ for multiplication with $Mult(x, y, z)$ holds if $x \cdot y = z$, and two constants for the neutral elements 0 and 1. A field $\mathcal{F} = \langle A, Add_A, Mult_A, 0_A, 1_A \rangle$ is a $\tau_{\text{field}}$-structure satisfying the usual field axioms translated into the relational language augmented by the statements that $Add$ and $Mult$ are functions. Nevertheless, for the sake of readability, we write the relational formulas in the functional notation with $+$ and $\cdot$. Let $\tau_{\text{field}}$ be the purely relational vocabulary $\tau_{\text{field}} \cup \{ \leq \}$ where $\leq$ is a binary relation symbol. An ordered field $\mathcal{F} = \langle A, Add_A, Mult_A, 0_A, 1_A, \leq_A \rangle$ is a $\tau_{\text{field}}$-structure satisfying the usual axioms of ordered fields.

We sometimes also look at (ordered) fields as structures over a vocabulary containing function symbols. Let $\tau_{\text{field}}$ be the vocabularies with binary functions for addition
and multiplication, unary functions for negatives \(-x\) and inverses \(\frac{1}{x}\), and \(\tau_{\text{field}} = \tau_{\text{field}} \cup \{\leq\}\).

The difference between the relational and functional version lies in the notion of substructure. In the functional version substructures of (ordered) fields are (ordered) fields. Formulas in the functional version can be translated into formulas in the relational version but this requires the use of existential quantifiers.

Let \(B(x_1, \ldots, x_m, \bar{y})\) be a quantifier free formula with free variables \(x_1, \ldots, x_m, \bar{y}\). A formula \(\phi\) with free variables \(\bar{y}\) is universal if it is of the form

\[
\phi = \forall x_1, \ldots, \forall x_m B(x_1, \ldots, x_m, \bar{y}).
\]

A formula \(\psi\) is existential if it is of the form

\[
\psi = \exists x_1, \ldots, \exists x_m B(x_1, \ldots, x_m, \bar{y})
\]

Note that when translating a quantifier-free formula in \(\text{FOL}_{\text{field}}\) into the corresponding formula in \(\text{FOL}_{\text{field}}\) using the general algorithm of backward substitution, the result is not quantifier-free but in general a Boolean combination of existential formulas which is both equivalent to a \(\forall \exists\) and an \(\exists \forall\) formula. Translating a universal formula therefore results in an \(\forall \exists\)-formula and translating an existential formula results in an \(\exists \forall\)-formula. However, it is possible that, when translating a universal formula, the resulting formula is logically equivalent to a universal formula. Whether the formula obtained by the algorithm of backward substitution is logically equivalent to a simpler formula is undecidable in the general case.

Let \(\mathcal{F}\) be a field.

(i) For \(p\) a prime, \(\mathcal{F}\) is of characteristic \(p\) if \(1 + \ldots + 1 = 0\).

(ii) \(\mathcal{F}\) is of characteristic 0 if for all \(n \in \mathbb{N}\) we have that \(1 + \ldots + 1 \neq 0\).

(iii) \(\mathcal{F}\) is Pythagorean if every sum of two squares is a square,

\[
\forall x \forall y \forall z (x = y^2 + z^2 \rightarrow \exists u (u^2 = x)).
\]

(iv) \(\mathcal{F}\) is a Vieta field if every polynomial with coefficients in \(\mathcal{F}\) of degree at most 3 has a root in \(\mathcal{O}\).

(v) \(\mathcal{F}\) is formally real if 0 cannot be written as a sum of nonzero squares, i.e., for all \(n \in \mathbb{N}\) we have

\[
\forall x_1, \ldots, x_n \left( \sum_{i=1}^{n} x_i^2 = 0 \rightarrow \bigwedge_{i=1}^{n} (x_i^2 = 0) \right)
\]

(vi) \(\mathcal{F}\) is algebraically closed if every non-constant polynomial with coefficients in \(\mathcal{F}\) has a root in \(\mathcal{F}\). We denote by \(\text{ACF}_0\) the first order sentences of fields describing an algebraically closed field of characteristic 0.

An ordered field \(\mathcal{O}\) is a field \(\mathcal{F}\) with an additional binary relation \(\leq\) which is compatible with the arithmetic relations of \(\mathcal{F}\). An ordered field is always of characteristic 0.

Let \(\mathcal{O}\) be an ordered field.

(i) \(\mathcal{O}\) is Euclidean if every positive element has a square root,

\[
\forall x (x \geq 0 \rightarrow \exists y (y^2 = x)).
\]

(ii) An ordered field is Pythagorean (Vieta, formally real) if it is an ordered field and as a field is Pythagorean (Vieta, formally real).
An ordered field $\mathcal{O}$ is real closed if every positive element in $\mathcal{O}$ has a square root,

$$\forall x \exists y (x \geq 0 \rightarrow y^2 = x)$$

and every polynomial of odd degree with coefficients in $\mathcal{O}$ has a root in $\mathcal{O}$.

$$\forall x_0, \ldots, x_{2n+1} (x_{2n+1} \neq 0) \rightarrow \exists y \sum_{i=0}^{2n+1} x_i y^i = 0$$

We denote by $RCF$ the first order sentences of ordered fields describing a real closed field.

### 2.2 Undecidable theories of fields

We now are ready to apply J. Robinson’s and M. Ziegler’s Theorem (Theorems 1.2 and 1.2) in order to show the following:

**Theorem 2.1** Let $T$ one of the first order theories over the vocabulary of (ordered) fields listed below. Then the set of first order consequences of $T$ is undecidable (not computable but computably enumerable).

(i) The theory of fields and of ordered fields (J. Robinson).
(ii) The theory of Pythagorean fields and ordered Pythagorean fields (M. Ziegler).
(iii) The theory of Vieta fields and ordered Vieta fields (M. Ziegler).
(iv) The theory of Pythagorean fields and ordered Pythagorean fields of characteristic 0 (M. Ziegler).
(v) The theory of ordered Euclidean fields.

**Proof** First we note that each of these theories has the field of (ordered) real numbers as a model. Furthermore each of them is either finite, or of the form

$$T^* = T \cup \{ n \neq 0, n \in \mathbb{N} \}$$

with $T$ finite. Hence we can apply Theorem 1.2.

### 2.3 Decidable theories of fields

In order to prove decidability of the theory Tarskian Geometry A. Tarski (and A. Seidenberg independently) proved the following:

**Proposition 2.1** (A. Tarski and A. Seidenberg [9]) The first order theory $RCF \subseteq FOL_{f-field}$ is recursively axiomatized, complete and admits elimination of quantifiers, and therefore is decidable.

**Proposition 2.2** (A. Tarski [68]) The first order theory $ACF_0 \subseteq FOL_{f-field}$ is recursively axiomatized, complete and admits elimination of quantifiers, and therefore is decidable.

**Remark 2.1** To prove decidability one has to prove additionally in both Propositions 2.1 and 2.2 that equality and inequality (and comparison by $\leq$) of constant terms of $\tau_{f-field}$
(τ_f-ofield) is decidable. We also note that quantifier elimination is not possible if the theories are expressed in FOL_ofield respectively FOL_field.

However, even in FOL_f_ofield respectively FOL_f_field the method of quantifier elimination cannot be used for other theories compatible with the theories RCF or ACF_0.

**Theorem 2.2** ([47]) Assume \( T \subseteq \text{FOL}_f_{-\text{field}} \) (\( T \subseteq \text{FOL}_f_{-\text{ofield}} \)) is a theory of (ordered fields) which has the complex (real) numbers as a model, and \( T \) admits elimination of quantifiers, then \( T \) is equivalent to ACF_0 (RCF).

**Problem 2.1** Is there a decidable theory \( T \) of ordered fields (requiring infinitely many axioms) which has no real closure?

Inside the field of real numbers there exists a minimal Pythagorean \( P \) (Euclidean \( E \), Vieta \( V \)) field, which is the intersection of all Pythagorean (Euclidean, Vieta) subfields in \( \mathbb{R} \). The theory of the minimal field of characteristic 0, the field \( \mathbb{Q} \) of the rationals \( \mathbb{Q} \) is undecidable by Proposition 1.2(iii).

**Problem 2.2** Are the complete theories of (ordered) fields of \( P \), \( E \) or \( O \) undecidable?

Theorem 1.2 holds not only for finite subtheories of real or algebraically closed fields, of characteristic 0, but also for finite characteristic and for certain formally \( p \)-adic fields. In [64], many more infinitely axiomatizable theories of fields are shown to be finitely hereditarily undecidable.

### 2.4 The universal consequences of a theory of fields

Our next observation concerns the universal consequences of a theory of fields.

The following lemma is a special case of Tarski’s Theorem for universal formulas proven in every textbook on model theory, e.g., [37]. It is stated for the vocabulary \( \tau_f-\text{field} \) to make sure that every substructure is closed under the arithmetic operations.

**Lemma 2.1** Let \( \mathbb{F} \) be a field and \( \mathbb{F}_0 \) be a subfield. Let \( \theta \in \text{FOL}(\tau_f-\text{field}) \) be a universal formula with parameters from \( \mathbb{F}_0 \), and \( \mathbb{F} \models \theta \). Then \( \mathbb{F}_0 \models \theta \).

The same also holds for ordered fields.

**Proposition 2.3**

1. For every set \( F \subset \text{FOL}_f_{-\text{field}} \) such that all its models are fields of characteristic 0, and \( F \) is consistent with ACF_0, and for every universal \( \theta \in \text{FOL}_f_{-\text{field}} \) we have:

\[
F \models \theta \text{ iff } \text{ACF}_0 \models \theta.
\]

Hence the universal consequences of \( F \) are decidable.

2. For every set \( F_o \subset \text{FOL}_f_{-\text{ofield}} \) such that all its models are ordered fields, \( F_o \) is consistent with RCF, and for every universal \( \theta \in \text{FOL}_f_{-\text{ofield}} \) we have:

\[
F_o \models \theta \text{ iff } \text{RCF} \models \theta.
\]

Hence the universal consequences of \( F_o \) are decidable.

**Proof** (i): As ACF_0 is complete and \( F \) is consistent with ACF_0 we have that ACF_0 \( \models F \). Let \( F \models \theta \). Then also ACF_0 \( \models \theta \).
Conversely, assume $ACF_0 \models \theta$. Now we use that $\theta$ is universal. By Lemma 2.1, $\theta$ holds in every subfield $\mathcal{F}$ of an algebraically closed fields of characteristic 0. By a classical theorem of Algebra, [63], every field of characteristic 0 has an algebraically closed extension which satisfies $ACF_0$. Hence $T \models \theta$.

(ii): The proof is similar, using real closures instead.

In [77] a special case of the decidability in Proposition 2.3(i) is proved, where the decision procedure is given using Hilbert’s Nullstellensatz and Gröbner bases, rather than via quantifier elimination. We discuss this further in Section 7. This makes the decision procedure seemingly less complicated than in the case of the decidability in Proposition 7.1(ii). A comparison of the complexity of the two cases may be found in [41].

**Problem 2.3** For which theories of fields $F$ is the set of existential formulas $\phi \in E$ derivable from $T$ decidable?

The answer is positive for $ACF_0$ and $RCF$ by Propositions 2.1 and 2.2. In spite of recent results by J. Koengismann [42, 43] on decidability of theories of fields, Problem 2.3 is open for the field of rational numbers $\mathbb{Q}$.

**Problem 2.4** Is the existential theory of the field $(\mathbb{Q}, +, \times, 0, 1)$ decidable?

### 3 Axioms of geometry: Hilbert, Wu and Huzita-Justin

In this section we first discuss various vocabularies used when axiomatizing Hilbert-style affine geometries in First Order Logic, and then we collect some of Hilbert’s (and Hilbert-style) axioms of geometry which we need in the sequel, and which are all true when one considers the analytic geometry of the plane with real coordinates.

### 3.1 The vocabularies of geometry

Models of plane geometry are called *planes*. These models differ in their basic relations. The universe is always two-sorted, consisting of Points and Lines and the most basic relation is *incidence* $\in$ with $p \in \ell$ to be interpreted as a point $p$ is coincident with a line $\ell$. Other relations are

- **Equidistant**: $\text{Eq}(p_1, p_2, p'_1, p'_2)$ to be interpreted as two pairs of points $p_1, p_2$ and $p'_1, p'_2$ have the same distance.
- **Orthogonality**: $\text{Or}(\ell_1, \ell_2)$ to be interpreted as two lines are orthogonal (perpendicular)/
- **Equiangular**: $\text{An}(p_1, p_2, p_3, p'_1, p'_2, p'_3)$ to be interpreted as two triples of points define the same angle.
- **Betweenness**: $\text{Be}(p_1, p_2, p_3)$ to be interpreted as three distinct points are on the same line and $p_2$ is between $p_1$ and $p_3$.
- **P-equidistant**: $\text{Peq}(\ell_1, p, \ell_2)$ to be interpreted as the point $p$ has the same distance from two lines $\ell_1$ and $\ell_2$.
- **L-equidistant**: $\text{Leq}(p_1, \ell, p_2)$ to be interpreted as the two points $p_1$ and $p_2$ have the same distance from the line $\ell$. 
- **Symmetric Line**: $\text{SymLine}(p_1, \ell, p_2)$ to be interpreted as the two points $p_1$ and $p_2$ are symmetric with respect to the line $\ell$.
We define now the following vocabularies:

\( \tau \): The vocabulary of incidence geometry, which uses incidence alone, possibly extended with a few symbols for specific constants.

\( \tau_{\text{hilbert}} \): The vocabulary of Hilbertian style geometry: Incidence, Betweenness, Equidistance and Equiangularity [33].

\( \tau_{\text{wu}} \): The vocabulary of Wu’s Orthogonal geometry: Incidence, Equidistance, Orthogonality [77, Chapter 2].

\( \tau_{\text{origami}} \): The vocabulary used to describe Origami constructions: Incidence, Symmetric Line, L-equidistant, Orthogonality [30]

\( \tau_{\text{o–origami}} \): An alternative version for describing Origami constructions. Incidence, Orthogonality, hence \( \tau_{\text{o–origami}} = \tau_{\text{wu}} \).

We note that all these vocabularies contain the symbol \( \in \) for the incidence relation.

### 3.2 Incidence geometries

**Axioms using only the incidence relation**

(I-1): For any two distinct points \( A, B \) there is a unique line \( l \) with \( A \in l \) and \( B \in l \).

(I-2): Every line contains at least two distinct points.

(I-3): There exists three distinct points \( A, B, C \) such that no line \( l \) contains all of them.

They can be formulated in FOL using the incidence relation only.

**Parallel axiom** We define: \( \text{Par}(l_1, l_2) \) or \( l_1 \parallel l_2 \) if \( l_1 \) and \( l_2 \) have no point in common. We use in this paper Playfair’s version of the Parallel Axiom, because it can be formulated using the incidence relation alone.

\( \text{Par}(l_1, l_2) \) can be formulated in FOL using the incidence relation only, hence also the Parallel Axiom.

**Pappus’ axiom**

(Pappus): Given two lines \( l, l' \) and points \( A, B, C \in l \) and \( A', B', C' \in l' \) such that \( AC' \parallel A'C \) and \( BC' \parallel B'C \). Then also \( AB' \parallel A'B' \).

**Axioms of infinity** We give here a version of the axiom of infinity which depends on the parallel axiom (ParAx), but has the advantage that all the points the existence of which is asserted can be constructed by geometrical constructions.

Given two lines \( l_1, l_2 \) which are not parallel, we denote by \( C = l_1 \times l_2 \) the unique point \( C \) such that \( C \in l_1 \) and \( C \in l_2 \). We denote by \( l_1 = \text{par}(l, C) \) the unique line \( l_1 \) such that \( C \in l_1 \) and \( l_1 \parallel l \). If \( l \) is defined by two points \( A \) and \( B \) we also write \( \text{par}(AB, C) \) for \( \text{par}(l, C) \).

\( \text{InfLines} \): Given distinct \( A, B, C \) and \( l \) with \( A \in l, B, C \notin l \) we define \( A_1 = \text{par}(AB, C) \times l \), and inductively, \( A_{n+1} = \text{par}(A_nB, C) \times l \). Then all the \( A_i \) are distinct.

Note that (InfLines) is not first order definable but is an axiom scheme which consists of an infinite set of first order formulas with infinitely many new constant symbols for the points \( A_i \), and the incidence relation. It is stronger than just saying there infinitely many distinct points. It says that there are no lines which have only finitely many points.
The latter has a simpler form consisting of the formulas $\text{Dist}_n$ a where $\text{Dist}_n$ says that for every line $l$ there are $n$ distinct points incident with $l$. But $\text{Dist}_n$ only says the points $A_n$ exist without giving a construction. $\text{Dist}_n$ does not depend on the parallel axiom. However in the presence of (ParAx) we will always use (InfLines).

(De-1): If $AA', BB', CC'$ intersect in one point or are all parallel, and $AB \parallel A'B'$ and $AC \parallel A'C'$ then $BC \parallel B'C'$.

(De-2): If $AB \parallel A'B'$, $AC \parallel A'C'$ and $BC \parallel B'C'$ then $AA', BB', CC'$ are all parallel.

The two Desargues axioms are first order definable using the incidence relation only.

**Affine plane:** Let $\tau \subseteq \tau$ be a vocabulary of geometry. A $\tau$-structure $\Pi$ is an (infinite) affine plane if it satisfies (I-1, I-2, I-3 and the parallel axiom (ParAx) and (InfLines). We denote the set of these axioms by $T_{\text{affine}}$.

**Pappian plane:** $\Pi$ is a Pappian plane if additionally it satisfies the Axiom of Pappus (Pappus). We denote the set of these axioms by $T_{\text{pappus}}$.

In the literature the definition of affine planes vary. Sometimes the parallel axiom is included, and sometimes not. We always include the parallel axiom, unless indicated explicitly otherwise.

### 3.3 Hilbert style geometries

**Axioms of betweenness**

(B-1): If $Be(A, B, C)$ then also $Be(C, B, A)$ and $A, B, C$ are distinct and there is $l$ with $A, B, C \in l$.

(B-2): For every distinct $A \neq B$ there is $C$ with $Be(A, B, C)$.

(B-3): For each distinct $A, B, C \in l$ exactly one point of the points $A, B, C$ is between the two others.

(B-4): (Pasch) Assume the points $A, B, C$ and $l$ in general position, i.e. the three points are not on one line, none of the points is on $l$. Let $D$ be the point at which $l$ and the line $AB$ intersect. If $Be(A, D, B)$ there is $D' \in l$ with $Be(A, D', C)$ or $Be(B, D', C)$.

The axioms of betweenness are all first order expressible in the language with incidence relation and the betweenness relation.

**Congruence axioms: Equidistance** We write for $Eq(A, B, C, D)$ the usual $AB \cong CD$.

(C-0): $AB \cong AB \cong BA$.

(C-1): Given $A, B, C, C', l$ with $C, C' \in l$ there is a unique $D \in l$ with $AB \cong CD$ and $Be(C, C', D)$ or $Be(C, D, C')$.

(C-2): If $AB \cong CD$ and $AB \cong EF$ then $CD \cong EF$.

(C-3): (Addition) Given $A, B, C, D, E, F$ with $Be(A, B, C)$ and $Be(D, E, F)$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

Note that (C-1) and (C-3) use the betweenness relation $Be$. Hence they are first order definable using the incidence, betweenness and equidistance relation.

**Congruence axioms: equiangularity** We denote by $\overrightarrow{AB}$ the directed ray from $A$ to $B$, and by $\angle(ABC)$ the angle between $\overrightarrow{AB}$ and $\overrightarrow{BC}$. For the congruence of angles $An(A, B, C, A', B', C')$ we write $\angle(ABC) \cong \angle(A'B'C')$.
(C-4): Given rays $AB$, $AC$ and $DF$ there is a unique ray $DE$ with $\angle(BAC) \cong \angle(EDF)$ on a given side of the ray $DF$.

(C-5): Congruence of angles is an equivalence relation.

(C-6): (Side-Angle-Side) Given two triangles $ABC$ and $A'B'C'$ with $AB \cong A'B'$, $AC \cong A'C'$ and $\angle BAC \cong \angle B'A'C'$ then $BC \cong B'C'$, $\angle ABC \cong \angle A'B'C'$ and $\angle ACB \cong \angle A'C'B'$.

**Axiom E:** Let $A$ be a point and $BC$ be a line segment. A circle $\Gamma(A, BC)$ is the lieu of all points $U$ such that $AU \cong BC$. A point $D$ is inside the circle $\Gamma(A, BC)$ if there is $U$ with $AU \cong BC$ and $Be(A, D, U)$. A point $D$ is outside the circle $\Gamma(A, BC)$ if there is $U$ with $AU \cong BC$ and $Be(A, U, D)$.

(AxE): Given two circles $\Gamma, \Delta$ such that $\Gamma$ contains at least one point inside, and one point outside $\Delta$, then $\Gamma \cap \Delta \neq \emptyset$.

**Hilbert plane:** Let $\tau$ with $\tau_{hilbert} \subseteq \tau$ be a vocabulary of geometry. A $\tau$-structure $\Pi$ is an (infinite) Hilbert plane if it satisfies (I-1, I-2, I-3), (B-1, B-2, B-3, B-4) and (C-1, C-2, C-3, C-4, C-5, C-6).

We denote the set of these axioms by $T_{hilbert}$

**P-Hilbert plane:** $\Pi$ is a P-Hilbert plane if it additionally satisfies (ParAx).

We denote the set of these axioms by $T_{p-hilbert}$

**Euclidean plane:** $\Pi$ is a Euclidean plane if it is a P-Hilbert plane which also satisfies Axiom E.

We denote the set of these axioms by $T_{euclid}$

### 3.4 Axioms of orthogonal geometry

**Congruence axioms: Orthogonality** We denote by $l_1 \perp l_2$ the orthogonality of two lines $Or(l_1, l_2)$. We call a line $l$ isotropic if $l \perp l$. Note that our definitions do not exclude this.

(O-1): $l_1 \perp l_2$ iff $l_2 \perp l_1$.

(O-2): Given $O$ and $l_1$, there exists exactly one line $l_2$ with $l_1 \perp l_2$ and $O \in l_2$.

(O-3): $l_1 \perp l_2$ and $l_1 \perp l_3$ then $l_2 \parallel l_3$.

(O-4): For every $O$ there is an $l$ with $O \in l$ and $l \perp l$.

(O-5): The three heights of a triangle intersect in one point.

Note that the axioms O-1,..., O-5 implicitly define orthogonality.

**Axiom of Symmetric Axis and Transposition**

(AxSymAx): Any two intersecting non-isotropic lines have a symmetric axis.

(AxTrans): Let $l$, $l'$ be two non-isotropic lines with $A, O, B \in l$, $AO \cong OB$ and $O' \in l'$ there are exactly two points $A', B' \in l'$ such that $AB \cong A'B' \cong B'A'$ and $A'O' \cong O'B'$.

The two axioms are equivalent in geometries satisfying the Incidence, Parallel, Desargues and Orthogonality axioms together with the axiom of infinity.

**Orthogonal Wu plane:** Let $\tau$ with $\tau_{wu} \subseteq \tau$ be a vocabulary of geometry. A $\tau$-structure $\Pi$ is an orthogonal Wu plane if it satisfies (I-1, I-2, I-3), (O-1, O-2, O-3, O-4, O-5), the axiom of infinity (InfLines), (ParAx), and the two axioms of Desargues (D-1) and (D-2).

We denote the set of these axioms by $T_{o-wu}$
Metric Wu plane: \( \Pi \) is a metric Wu plane if it satisfies additionally the axiom of symmetric axis (AxSymAx) or, equivalently, the axiom of transposition (AxTrans). We denote the set of these axioms by \( T_{m-wu} \).

The axiomatization of orthogonal is due to W. Wu [74, 76, 77], see also [49].

3.5 The origami axioms

A line which is obtained by folding the paper is called a fold. The axioms (H-1) to (H-6) are known as Huzita’s axioms. Axiom (H-7) was discovered by K. Hatori. Jacques Justin and Robert J. Lang also found axiom (H-7), [75]. The axioms (H-1)-(H-7) only express closure under folding operations, and do not define a geometry. To make it into an axiomatization of geometry we have to assume that these operations are performed on an affine plane.

We follow here [30]. The original axioms and their expression as first order formulas in the vocabulary \( \tau_{origami} \) are as follows:

\[ (H-1): \quad \forall P_1, P_2 \exists ! l(P_1 \in l \land P_2 \in l) \]
\[ (H-2): \quad \forall P_1, P_2 \exists ! lSymLine(P_1, l, P_2) \]
\[ (H-3): \quad \forall l_1, l_2 \exists k \forall P (P \in k \rightarrow Peq(l_1, P, l_2)) \]
\[ (H-4): \quad \forall P, l \exists ! k \forall P (P \in k \land \exists l(SymLine(P_1, l_3, Q_1)) \land (\exists Q_2 SymLine(P_2, l_3, Q_2) \land Q_2 \in l_2)) \]
\[ (H-5): \quad \forall P_1, P_2 l_1 \exists ! l \forall P (P \in l \land \exists l_2(SymLine(P_1, l_2, P_2)) \land P_2 \in l_1)) \]
\[ (H-6): \quad \forall P_1, P_2 l_1, l_2 \exists l_3 ((\exists Q_1 SymLine(P_1, l_3, Q_1) \land Q_1 \in l_1)) \land (\exists Q_2 SymLine(P_2, l_3, Q_2) \land Q_2 \in l_2)) \]
\[ (H-7): \quad \forall P, l_2, l_3 \exists (Or(l_2, l_3) \land (\exists Q_2 SymLine(P, l_3, Q) \land Q \in l_1)) \]

Affine Origami plane: Let \( \tau \) with \( \tau_{origami} \subseteq \tau \) be a vocabulary of geometry. A \( \tau \)-structure \( \Pi \) is an affine Origami plane if it satisfies (I-1, I-2, I-3), the axiom of infinity (InfLines), (ParAx) and the Huzita-Hatori axioms (H-1) - (H-7).

We denote the set of these axioms by \( T_{a-origami} \).

**Proposition 3.1** The relations SymLine and Peq are first order definable using Eq and Or with existential formulas over \( \tau_{field} \). Hence the axioms (H-1)-(H-7) are first order definable in FOL(\( \tau_{wu} \)).
Proof (i) $\text{SymLine}(P_1, \ell, P_2)$ iff there is a point $Q \in \ell$ such that $\text{Or}((P_1, Q), \ell)$, $\text{Or}((P_2, Q), \ell)$ and $\text{Eq}(P_1, Q, P_2, Q)$.

(ii) $\text{Peq}(\ell_1, P, \ell_2)$ iff there exist points $Q_1, Q_2$ such that $\text{Or}((P, Q_1), \ell_1)$, $\text{Or}((P, Q_2), \ell_2)$, $\text{Eq}(P, Q_1)$ and $\text{Eq}(P, Q_2)$.

\[ \square \]

4 Proving undecidability of geometrical theories

In this section we spell out how one can apply J. Robinson’s Proposition 1.2 or M. Ziegler’s Theorem (Theorem 1.2) to prove undecidability of geometric theories.

4.1 Translation schemes

We first introduce the formalism of translation schemes, transductions and translation. In [69] this was first used, but not spelled out in detail. Our approach follows [44, Section 2]. To keep it notationally simple we explain on an example. Let $\tau$ be a vocabulary consisting of one binary relation symbol $R$, $\sigma$ be a vocabulary consisting of one ternary relation symbol $S$. In general, if $\tau$ and $\sigma$ are purely relational vocabularies, the definition can be extended in a straightforward way. If the vocabularies contain function symbols (and constants) one has to be a bit more careful when extending the definitions below. However, for our purpose here, this is not needed.

We want to interpret a $\sigma$ structure on $k$-tuples of elements of a $\tau$-structure.

A $\tau - \sigma$-translation scheme $\Phi = (\varphi, \varphi_S)$ consists of a $\tau$-formula $\varphi(\bar{x})$ with $k$ free variables and a formula $\varphi_S$ with $3k$ free variables. $\Phi$ is quantifier-free if all its translation formulas are quantifier-free.

Let $A = \langle A, R^A \rangle$ be a $\tau$-structure. We define a $\sigma$-structure $\Phi^*(\mathcal{A}) = \langle B, S^B \rangle$ as follows: The universe is given by

\[ B = \{ \bar{a} \in A^k : \mathcal{A} \models \varphi(\bar{a}) \} \]

and

\[ S^B = \{ \bar{b} \in A^{k \times 3} : \mathcal{A} \models \varphi_S(\bar{b}) \} \]

$\Phi^*$ is called a transduction.\(^5\)

Let $\theta$ be a $\sigma$-formula. We define a $\tau$-formula $\Phi^\varphi(\theta)$ inductively by substituting occurrences of $S(\bar{b})$ by their definition via $\varphi_S$ where the free variables are suitable named. $\Phi^\varphi$ is called a translation.

The fundamental property of translation schemes, transductions and translation is the following:

**Proposition 4.1** (Fundamental Property of Translation Schemes) Let $\Phi$ be a $\tau - \sigma$-translation scheme, and $\theta$ be a $\sigma$-formula, hence $\Phi^\varphi(\theta)$ is a $\tau$-formula.

\[ \mathcal{A} \models \Phi^\varphi(\theta) \text{ iff } \Phi^*(\mathcal{A}) \models \theta \]

If $\theta$ has free variables, the assignment have to be chosen accordingly. Furthermore, if $\Phi$ is quantifier-free, and $\theta$ is a universal formula, $\Phi^\varphi(\theta)$ is also universal.

\(^5\)This terminology was put forward in the many papers by B. Courcelle, cf. [21].
In order to use translation schemes to prove decidability and undecidability of theories we need two lemmas.

**Lemma 4.1** Let $\Phi$ be a $\tau - \sigma$-translation scheme.

(i) Let $\mathcal{A}$ be a $\tau$-structure. If the complete first order theory $T_0$ of $\mathcal{A}$ is decidable, so is the complete first order theory $T_1$ of $\Phi^*(\mathcal{A})$.

(ii) There is a $\tau$-structure $\mathcal{A}$ such that the complete first order theory $T_1$ of $\Phi^*(\mathcal{A})$ is decidable, but the complete first order theory $T_0$ of $\mathcal{A}$ is undecidable.

(iii) If however, $\Phi^\sharp$ is onto, i.e., for every $\phi \in \text{FOL}(\tau)$ there is a formula $\theta \in \text{FOL}(\sigma)$ with $\Phi^\sharp(\theta)$ logically equivalent to $\phi$, then the converse of (i) also holds.

(iv) Let $T \subseteq \text{FOL}(\tau)$ be a decidable theory and $T' \subseteq \text{FOL}(\sigma)$ and $\Phi^*$ be such that $\Phi^*|_{\text{Mod}(T)} : \text{Mod}(T) \to \text{Mod}(T')$ be onto. Then $T'$ is decidable.

**Proof** (i): This follows from Proposition 4.1. $\mathcal{A} \models \Phi^\sharp(\theta)$ iff $\Phi^*(\mathcal{A}) \models \theta$, hence, $\Phi^\sharp(\theta) \in T_1$ iff $\theta \in T_0$. As $T_0$ is decidable, we can decide whether $\Phi^\sharp(\theta) \in T_0$, and also, whether $\theta \in T_1$.

(ii) Let $\mathcal{A} = \langle \mathbb{N}, +_N, \times_N \rangle$ where addition and multiplication are ternary relations. $T_0(\mathcal{A})$ is undecidable by Gödel’s Theorem.

Now let $\Phi^*(\mathcal{A})$ be $\langle \mathbb{N}, +_A, \times_A \rangle$ where $+_A = +_N$ but $\times_A = +_N$. $\Phi^*(\mathcal{A})$ is like Pressburger Arithmetic, but has two names ($+_A$ and $\times_A$) for the same addition. Hence the complete theory of $\Phi^*(\mathcal{A})$ is decidable.

(iii): If we assume $T_0$ to be decidable, we can only decide whether $\phi \in T_1$ for $\phi$ of the form $\phi = \Phi^\sharp(\theta)$.

(iv): Let $\theta \in \text{FOL}(\sigma)$. We want to check whether $T' \models \theta$.

Let $\mathcal{B} \models T'$. As $\Phi^*$ is onto, there is $\mathcal{A} \models T$ and $\Phi^*(\mathcal{A}) = \mathcal{B}$.

Now we have, using Proposition 4.1

$\mathcal{B} \models \neg \theta$ iff $\mathcal{A} \models \Phi^\sharp(\neg \theta)$ iff $T' \not\models \theta$ iff $T \not\models \Phi^\sharp(\theta)$

But by assumption $T$ is decidable, hence $T'$ is decidable.

**Remark 4.1** The condition that $\Phi^\sharp$, resp. $\Phi^*$ have to be onto is often overlooked in the literature.\(^6\)

We shall need one more observation:

**Lemma 4.2** Let $T \subseteq \text{FOL}(\tau)$ and $\phi \in \text{FOL}(\tau)$. Assume $T$ is decidable. Then $T \cup \{\phi\}$ is also decidable.

**Proof** This follows from the semantic version of the Deduction Theorem of First Order Logic:

$T \cup \{\phi\} \models \theta$ iff $T \models (\phi \rightarrow \theta)$

\(^6\)Theorems 1.36 and 1.37 as stated in [16] are only true when one notices that their Theorems 1.20 and 1.21 imply that the particular transductions used in Theorems 1.36 and 1.37 are indeed onto. However, this is not stated there.
4.2 Interpretability

A theory $T \subseteq \text{FOL}(\tau)$ is \textit{finitely axiomatizable} if there is a finite $T'$ which is axiomatizable and has the same set of consequences as $T$. $T$ is \textit{essentially undecidable} if no theory $T' \subseteq \text{FOL}(\tau)$ extending $T$ is decidable. $T$ is \textit{completely undecidable} if there is a finite subtheory $T' \subseteq T$ which is essentially undecidable.

Let $S \in \text{FOL}(\sigma)$ and $T \in \text{FOL}(\tau)$ be two theories over disjoint vocabularies. $S$ is \textit{interpretable} in $T$, if there exists a first order translation scheme $\Phi$ such that

$$\Phi^*(T) \models S.$$ 

$S$ is \textit{weakly interpretable} in $T$, if there exists a theory $T'$ over the same vocabulary as $T$, and a translation scheme $\Phi$ such that

$$\Phi^*(T') \models S.$$

**Lemma 4.3** ([15, Statement (e3) on page 602]7) Assume $S$ is a theory which is

(i) finitely axiomatizable,
(ii) essentially undecidable, and
(iii) weakly interpretable in a theory $T$ using a translation scheme $\Phi$.

Then $T$, and every subtheory of $T$, is undecidable.

Moreover, there is a theory $T'$ with $T \subseteq T'$ and with the same vocabulary as $T$, which is essentially undecidable.

Let $M$ be a class of $\tau$-structures closed under isomorphisms. A $\tau - \sigma$-translation scheme $\Phi$ is \textit{invertible on $M$} if there exists a $\sigma - \tau$-translation scheme $\Psi$ such that for all $\mathcal{A} \in M$

$$\Psi^*(\Phi^*(\mathcal{A})) \simeq \mathcal{A}$$

and for all $\mathcal{B} \in \Phi^*(M)$

$$\Phi^*(\Psi^*(\mathcal{B})) \simeq \mathcal{B}.$$ 

Clearly, if $\Phi$ is invertible on $M$, $\Phi^*|_M : M \rightarrow \Phi^*(M)$ is onto.

**Lemma 4.4** Let $\mathcal{A}$ be a $\sigma$-structure and $\mathcal{A}'$ be a $\tau$-structure, and let $\Phi$ be a $\tau - \sigma$-translation scheme such that $\Phi^*(\mathcal{A}') = \mathcal{A}$. Let $S$ be the complete theory of $\mathcal{A}$. Assume $S$ is undecidable. Let $T \subseteq \text{FOL}(\tau)$ with $\mathcal{A}' \models T$. and assume that $\Phi$ is invertible on $M = \{\mathcal{A} : \mathcal{A} \models T\}$. Then

(i) $S$ is weakly interpretable in $T$, and
(ii) $T$ is undecidable.

**Proof** (ii) follows from (i) and Lemma 4.3.

To see (i) we use that $\mathcal{A}' \models T$ and use as $T'$ the complete theory of $\mathcal{A}'$. Now the invertibility of $\Phi^*$ allows us to complete the argument. 

In [56] Lemma 4.4 is stated without the invertibility assumption as the Interpretations-theorem. In the particular application in [56], $S$ is the complete theory of the field of rational numbers, which is undecidable by Proposition 1.2. The translation scheme $\Phi$ is vaguely

---

7This is the earliest reference I could find for this formulation.
sketched as \( PP \), and its inverse is not defined at all. We will show in the next section that both \( PP \) and \( RR \) are first order definable. Theorem 6.1 implies that both \( PP \) and \( RR \) are invertible. This allows us to complete the gap in [56] in the proof of Theorem 1.1. However, Theorem 6.1 only appears explicitly in [19] and in [66] and were not available in 1962.

5 The role of coordinates

5.1 Analytic geometry over fields of characteristic 0

Given a field \( \mathcal{F} \) or an ordered field \( \mathcal{O} \) we define the following relations in \( \mathcal{F} (\mathcal{O}) \), where elements \( P = (x, y) \) are called points and \( \ell = (a, b, c) = \{(x, y) : ax + by = c\} \) are called lines. Similarly, we write \( P_i = (x_i, y_i) \) and \( \ell_i = (a_i, b_i, c_i) \). In \( \tau_{\text{field}} \) points are defined using a quantifier-free formula and lines are defined using an existential formula. In \( \tau_{\text{field}} \) both are defined using a quantifier-free formula.

Incidence: \( P \in \ell \iff ax + by + c = 0 \). In \( \tau_{\text{field}} \) this is a quantifier-free formula.

Equidistance: \( Eq(P_1, P_2, P_3, P_4) \iff (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_3 - x_4)^2 + (y_3 - y_4)^2 \). In \( \tau_{\text{field}} \) this is a quantifier-free formula.

Orthogonality: \( Or(\ell_1, \ell_2) \) (or \( \ell_1 \perp \ell_2 \)) iff \( a_1 a_2 + b_1 b_2 = 0 \). In \( \tau_{\text{field}} \) this is a quantifier-free formula.

For equiangularity we have to work a bit more. Let \( \ell = (a, b, c) \) be a line. The slope of \( \ell \) is defined as \( sl(\ell) = \frac{a}{b} \). Now let \( \ell_1, \ell_2 \) be two lines intersecting at the point \( p \), let and \( k \) a line with \( Or(k, \ell_1) \) intersecting \( \ell_i \) at \( Q_i \) (\( i = 1, 2 \)). The angle \( \angle(Q_1, P, Q_2) \) is an acute angle. For acute angles we define

\[
\tan(Q_1, P, Q_2) = \left| \frac{sl(\ell_1) - sl(\ell_2)}{1 + sl(\ell_1)sl(\ell_2)} \right|.
\]

We now give a quantifier-free definition of equiangularity in right triangles.

Rectangular: \( \text{rectangular}(P_1, P_2, P_3) \iff Or((P_1, P_2), (P_1, P_3)) \).

Equiangular: Assume we have two rectangular triangles \( P_1 P_2 P_3 \) and \( Q_1 Q_2 Q_3 \) with \( \text{rectangular}(P_1, P_2, P_3) \) and \( \text{rectangular}(Q_1, Q_2, Q_3) \) we define \( \text{An}(P_1, P_2, P_3, Q_1, Q_2, Q_3) \) iff \( \tan(P_1, P_2, P_3) = \tan(Q_1, Q_2, Q_3) \).

In \( \tau_{\text{field}} \) this is a quantifier-free formula.

If the field is an ordered field we define additionally:

Colinear: \( \text{Col}(P_1, P_2, P_3) \iff \exists \ell \left( \bigwedge_{i=1}^{3} P_i \in \ell \right) \).

For \( \ell = (a, b, c) \) and \( P_i = (x_i, y_i) \) we can write this as

\[
\exists a, b, c \left( \bigwedge_{i=1}^{3} ax_i + bxy_i + c = 0 \right)
\]

which is equivalent to

\[
\det \begin{pmatrix}
x_1 & y_1 & 1 \\
x_2 & y_2 & 1 \\
x_3 & y_3 & 1 
\end{pmatrix} = 0
\]

which in \( \tau_{\text{field}} \) is a quantifier-free formula.
Betweenness: \( \text{Be}(P_1, P_2, P_3) \) iff

\[
\text{Col}(P_1, P_2, P_3) \land \\
[(x_1 \leq x_2 \leq x_3) \land (y_1 \leq y_2 \leq y_3)) \lor ((x_3 \leq x_2 \leq x_1) \land (y_3 \leq y_2 \leq y_1)) \lor \\
((x_1 \leq x_2 \leq x_3) \land (y_3 \leq y_2 \leq y_1)) \lor ((x_3 \leq x_2 \leq x_1) \land (y_1 \leq y_2 \leq y_3))]
\]

which in \( \tau_f \) of field is a quantifier-free formula.

**Definition 5.1** Given a field with universe \( A \), let \( \text{Points}(\mathcal{F}) = A^2 \) and \( \text{Lines}(\mathcal{F}) = A^3 \). For a field \( \mathcal{F} \), respectively an ordered field \( \mathcal{O} \), we define

(i) \( \Pi_\in(\mathcal{F}) \) to be the two sorted structure

\[
\langle \text{Points}(\mathcal{F}), \text{Lines}(\mathcal{F}); \in_\mathcal{F} \rangle.
\]

The quantifier-free first order translation scheme \( PP_\in = (\text{Points}, \text{Lines}, \in) \) satisfies \( PP_\in^\ast(\mathcal{F}) = \Pi_\in(\mathcal{F}) \).

(ii) \( \Pi_{wu}(\mathcal{F}) \) to be the two sorted structure

\[
\langle \text{Points}(\mathcal{F}), \text{Lines}(\mathcal{F}); \in_\mathcal{F}, \text{Eq}_\mathcal{F}, \text{Or}_\mathcal{F} \rangle.
\]

The quantifier-free first order translation scheme \( PP_{wu} = (\text{Lines}, \in, \text{Eq}, \text{Or}) \) satisfies \( PP_{wu}^\ast(\mathcal{F}) = \Pi_{wu}(\mathcal{F}) \).

(iii) \( \Pi_{hilbert}(\mathcal{O}) \) to be the two sorted structure

\[
\langle \text{Points}(\mathcal{O}), \text{Lines}(\mathcal{O}); \in_\mathcal{O}, \text{Eq}_\mathcal{O}, \text{An}(\mathcal{O}), \text{Be}_\mathcal{O} \rangle.
\]

The quantifier-free first order translation scheme \( PP_{hilbert} = (\text{Lines}, \in, \text{Eq}, \text{An}, \text{Be}) \) satisfies \( PP_{hilbert}^\ast(\mathcal{F}) = \Pi_{hilbert}(\mathcal{F}) \).

This gives us:

**Proposition 5.1** The translation schemes \( PP_\in \), \( PP_{wu} \) and \( PP_{hilbert} \) are quantifier-free first order translation schemes.

5.2 Properties of \( PP_\in \) and \( PP_{wu} \)

We summarize now the properties needed of these translation schemes and their induced transductions and translations. Here, and in the next section we call these properties the correctness of the translation schemes, because they state that they behave as needed to prove undecidability results.

**Theorem 5.1** (Correctness of \( PP_\in \) and \( PP_{wu} \)) Let \( PP_\in^\ast \) and \( PP_{wu}^\ast \) be the transductions induced by the translation schemes \( PP_{wu} \) and \( PP_{wu} \) respectively.

(i) ([33, 14.1]) If \( \mathcal{F} \) is a field, then \( PP_\in^\ast(\mathcal{F}) \) satisfies the incidence axioms \((I_1)-(I_3)\), the Parallel Axiom and the Pappus Axiom.

(ii) ([33, 14.4]) If \( \mathcal{F} \) is a field of characteristic 0, then \( PP_\in^\ast(\mathcal{F}) \) satisfies additionally the Axioms of Infinity, i.e., is an infinite Pappian plane.

(iii) ([77]) If \( \mathcal{F} \) is a Pythagorean field of characteristic 0, then \( PP_{wu}^\ast(\mathcal{F}) \) satisfies \((I-1)-(I-3), (O-1)-(O-5)\), the Parallel Axiom, the Axiom of Infinity, the Axiom of Desargues and the Axiom of Symmetric Axis, which are axioms of a metric Wu plane.

(iv) ([2]) If \( \mathcal{F} \) is a Vieta field, then \( PP_{wu}^\ast(\mathcal{F}) \) satisfies the Huzita-Hatori axioms \((H-1)-(H-7)\).
Can one design a geometry engine? On the (un)decidability of certain...

Theorem 5.2 (Correctness of $P_{hilbert}$) (i) ([33, 17.3]) If $\mathcal{O}$ is an ordered Pythagorean field, then $PP_{hilbert}^*(\mathcal{O})$ satisfies (I-1) - (I-3), (B-1) - (B-4) (C-1) -(C-6) and the Parallel Axiom, which are axioms of a Hilbert Plane which satisfies the parallel axiom.

(ii) ([33, 17.3]) If $\mathcal{O}$ is an ordered Euclidean field, then $PP_{hilbert}^*(\mathcal{O})$ is a Hilbert Plane which satisfies the parallel axiom and Axiom E.

5.3 Introducing coordinates

We have seen in the last section how to get models of geometry using coordinates in a field. Now we want to find a way to define a skew field or a field of coordinates from a model $\Pi$ of geometry. We say that we want to coordinatize $\Pi$. This problem has a long tradition and was solved already in the 19th century. The inverse situation, namely defining a geometry using a field of coordinates, was already the subject of Descartes’ treaty La Géométrie, published in 1637, [24].

There are two accepted ways of finding a field of coordinates in a Pappian plane: If we have the notion of equidistance and betweenness available, we can define an arithmetic of line segments. This is discussed in detail in [33, Chapter 18]. In the absence of betweenness and congruence, but in the presence of the Parallel Axiom, one can use Pappus’ Axiom to define the arithmetic operations even in a Pappus plane. This was first done by K.G.C. von Staudt [72, 73], a student of C.F. Gauss, before D. Hilbert’s [35]. The first modern treatments of defining a field of coordinates for affine and projective planes was given by M. Hall [32] and possibly by E. Artin [4] which as only published in 1965 in E. Artin’s collected papers. M. Hall’s version can easily be seen to be first order definable.

Definition 5.2 Let $\tau$ a vocabulary for geometry, $T \subseteq \text{FOL}(\tau)$ a set of axioms of geometry, $T_f$ be a set of axioms for fields in $\tau_{fields}$ or $\tau_{ofields}$. We say that the models of $T$ have a first order coordinatization in fields satisfying $T_f$ if there exists a first order translation scheme $CC_{field}$ such that

(a) for every $\Pi$ which satisfies $T$ the structure $CC_{field}^*(\Pi) (CC_{a-field}^*(\Pi))$ is a field which satisfies $T$;
(b) for every field $\mathcal{F}$ which satisfies $T_f$, the $\tau$-structure $PP_{\tau}(\mathcal{F})$ satisfies $T$;
(c) For every field $\mathcal{F}$ which satisfies $T_f$ we have $CC_{field}(PP_{\tau}(\mathcal{F})) \simeq \mathcal{F}$;
(d) For every $\tau$-structure $\Pi$ which satisfies $T$ we have $PP_{\tau}(CC_{field}(\Pi)) \simeq \Pi$.

We have formulated the definition in terms of the relational vocabularies for fields to make the use of translation schemes simple. As we deal here with full first order logic, there is no loss of generality.

In order to deduce undecidability of geometric theories using undecidability of theories of fields we will need the following:

Theorem 5.3 (Segment Arithmetics) Every P-Hilbert plane has a first order coordinatization $FF_{field}$ (via segment arithmetic).

---

8In the case of Pappian planes we always get a field.
9In the case of hyperbolic geometry there is different way: Hilbert’s arithmetic of ends, cf. [33, Chapter 41].
10Modern in the sense of van der Waerden’s Modern Algebra, in the spirit of axiomatic mathematics.
Theorem 5.4 (Planar Ternary Rings) Every infinite Pappus plane without finite lines has a first order coordinatization \( \mathbb{R} \mathbb{R}_{\text{field}} \) (via planar ternary rings).

We will define \( \mathbb{F}_{\text{field}} \) (Lemma 5.1) and \( \mathbb{R} \mathbb{R}_{\text{field}} \) (Lemma 5.5) in the next subsections and show that \( \mathbb{F}_{\text{field}} \) and \( \mathbb{R} \mathbb{R}_{\text{field}} \) are indeed FOL-definable.

5.4 Segment arithmetic

Given a Hilbert plane \( \Pi \) which satisfies the Parallel Axiom, we now want to show that one can interpret in \( \Pi \) an ordered field of coordinates \( \mathcal{F}_{\text{hilbert}}(\Pi) \). Note that orthogonality \( Or(\ell_1, \ell_2) \) of lines is definable in every Hilbert plane using equiangularity. We follow essentially [33, Chapter 4].

Fix a line segment \( 1 = [A_0, A_1] \) given by two points \( A_0, A_1 \).

We first define the commutative semiring \( \mathcal{S}_{\text{hilbert}}(\Pi) \) as follows:

Positive elements: Equivalence classes \([P_1, P_2]\) of pairs of points \( P_1, P_2 \) with \( Eq(P_1, P_2) \).

Zero element: The equivalence class \([A_0, A_0]\).

Unit element: The equivalence class \([A_0, A_1]\).

Positive addition: Choose three points \( P_1, P_2, P_3 \) such that \( Be(P_1, P_2, P_3) \). Then we put \([P_1, P_2] + [P_2, P_3] = [P_1, P_3]\). If \( P_1, P_2, P_3 \) or not colinear, we always can choose \( P_1', P_2', P_3' \) with \( Be(P_1', P_2', P_3') \) such that \([P_1, P_2] = [P_1', P_2']\) and \([P_2, P_3] = [P_2', P_3']\).

Positive multiplication: Let \( P_0, P_1, P_2, P_3, P_4 \) be points such that \( Be(P_0, P_1, P_2) \) and \( Be(P_0, P_3, P_4) \) and the lines \((P_1, P_3)\) and \((P_2, P_4)\) are parallel and the lines \((P_1, P_2)\) and \((P_3, P_4)\) are orthogonal, and \([P_0, P_3] = [A_0, A_1] = 1\) is the unit length. Then we put for \( a = [P_0, P_1] \) and \( b = [P_0, P_4] \) the product \( ab = [P_0, P_2] \).

One easily verifies now:

Proposition 5.2 The arithmetic operations defined as above are definable in FOL in the vocabulary \( \tau_{\text{hilbert}} \).

Proposition 5.3 In any Hilbert plane (even without the Parallel Axiom) addition of line segments as defined above is well-defined, commutative, associative. Furthermore, if \( a, b \) are two line segments, then one of the following holds:

(i) \( a = b \),

(ii) There is a line segment \( c \) such that \( a + c = b \),

(iii) There is a line segment \( d \) such that \( a = b + d \).

Proposition 5.4 In any Hilbert plane \( \Pi \) with the Parallel Axiom multiplication of line segments as defined above is well-defined, associative, and has \( 1 \) as its neutral element. Furthermore, for all line segments \( a, b, c \) we have

(i) \( a(b + c) = ab + ac \)

(ii) There is a unique \( d \), such that \( ad = 1 \),

(iii) If \( \Pi \) is also Pappian, then multiplication is also commutative.

Lemma 5.1 In every semiring satisfying Propositions 5.3 and 5.4 we can define an ordered field. In fact this field is definable using the first order translation scheme \( \mathbb{F}_{\text{field}} \) and \( \mathbb{F}_{\text{ord}} \) built using the formulas which define the equivalence classes of positive elements, zero element, unit element, positive addition and positive multiplication.
Proof We use the standard construction the same way as one constructs the ordered field of rational numbers \( \mathbb{Q} \) from the ordered semiring of the natural numbers \( \mathbb{N} \).

This gives us the first order translation schemes \( FF_{field} \) and \( FF_{ofield} \).

**Theorem 5.5** (Correctness of \( FF_{field} \) and \( FF_{ofield} \)) Let \( \Pi \) be a Hilbert Plane which satisfies the Parallel Axiom.

(i) \( FF^*_{field}(\Pi) \) is a field of characteristic 0 which can be uniquely ordered to be an ordered field \( F_{\Pi} \).

(ii) Let \( F_{\Pi} = FF^*_{field}(\Pi) \) be the ordered field of segment arithmetic in \( \Pi \). Then \( F \) is Pythagorean and \( PP^*_\text{hilbert}(F) \) is isomorphic to \( \Pi \).

(iii) An ordered field \( O \) is Pythagorean iff \( PP^*_\text{hilbert}(O) \) is a Hilbert Plane which satisfies the Parallel Axiom.

(iv) \( \Pi \) is a Euclidean plane iff \( FF^*_{field}(\Pi) \) is a Euclidean field.

(v) \( F \) is a Euclidean fields iff \( PP^*_\text{hilbert}(F) \) is a Euclidean plane.

Proof A proof may be found in [33, Theorems 20.7, 21.1 and 21.2].

5.5 Planar ternary rings

In order to use the undecidability of the theory of fields, we have to find a first order transduction \( RR_e \) which turns any Pappian plane \( \Pi \) into a field \( RR_e(\Pi) \) without using the betweenness relation \( Be \). Fortunately, this can be done using Planar Ternary Rings, which were introduced by M. Hall in [32]. M. Hall credits [36, 73] for the original idea. A good exposition can be found in [19, 66]. We follow here almost verbatim [39], which is a particularly nice exposition of [32]. The study of ternary rings was developed to study non-Desarguesian planes. The results specialize to Pappian planes which is what we use in our context.

Let \( \Pi \) be an affine plane satisfying (I-1)- (I-3) (ParAx), with two distinguished lines \( \ell_0, m_0 \) in \( \Pi \). Let \( O \) be the point of intersection of \( \ell_0 \) and \( m_0 \).

**Lemma 5.2** There is a formula \( bij(x, y, d) \in FOL_e \) which, for every line \( \delta \) different from \( \ell_0 \) and \( m_0 \) such that \( O \in \delta \), defines a bijection between \( \ell_0 \) and \( m_0 \).

Proof Let \( x \in \ell_0 \) and \( z(x) \) be the intersection with \( \delta \) of the line \( m_1 \) parallel to \( m_0 \) containing \( x \). Let \( y(x) \in m_0 \) be the intersection of the line \( \ell_1 \) parallel to \( \ell_0 \) containing \( z(x) \). Clearly \( f_\delta : \ell_0 \rightarrow m_0 \) given by \( f_\delta(x) = y(x) \) is a bijection and is FOL definable by a formula \( bij(x, y, \delta) \).

We will define a structure \( RR_\Pi \) with universe a set \( K \) (which we take to be \( \ell_0 \)). Thinking of \( \ell_0 \) and \( m_0 \) as axes of a coordinate system we can identify the points of \( \Pi \) with pairs of points in \( K^2 \). The projection of a point \( P \) onto \( \ell_0 \) is defined by the point \( x \in \ell_0 \) which is the intersection of the line \( m_1 \) parallel to \( m_0 \) with \( P \in m_1 \). The projection of a point \( P \) onto \( m_0 \) is defined analogously. The point 0 has coordinates \((0, 0)\). Furthermore, we fix an arbitrary point \( 1 \in \ell_0 \) different from 0 which has coordinates \((1, 0)\).

Next we define the **slope** of a line \( \ell \) in \( \Pi \) to be an element \( sl(\ell) \in K \cup \{\infty\} \) If \( \ell \) is parallel to \( \ell_0 \) its slope is 0 and it is called a horizontal line. If \( \ell \) is parallel to \( m_0 \) its slope is \( \infty \) and it
is called a vertical line. For \( \ell \) not vertical, let \( \ell_1 \) be the line parallel to \( \ell \) and passing through 0. Let \((1, a)\) be the coordinates of the intersection of \( \ell_1 \) with the line vertical line \( \ell_2 \) passing through \((1, 0)\). Then the slope \( sl(\ell) = a \).

This shows:

**Lemma 5.3** There is a first order formula \( \text{slope}(\ell, a, \delta) \in \text{FOL}_e \) which expresses \( sl(\ell) = a \) with respect to the auxiliary line \( \delta \).

**Lemma 5.4**

(i) Two lines \( \ell, \ell_1 \) have the same slope, \( sl(\ell) = sl(\ell_1) \) iff they are parallel.

(ii) For the line \( \delta \) we have \( sl(\delta) = 1 \) (because \((1, 1) \in \delta\)).

We now define a ternary operation \( T : K \rightarrow K \). We think of \( T(a, x, b) = (ax + y) \) as the result of multiplying \( a \) with \( x \) and then adding \( b \). But we yet have to define multiplication and addition.

Let \( a, b, x \in K \). Let \( \ell \) be the unique line with \( sl(\ell) = a \neq \infty \) intersecting the line \( m_0 \) at the point \( P_1 = (0, b) \). Let \( \ell_1 = \{(x, z) \in K^2 : z \in K \} \). For every \( x \in K \) the line \( \ell \) intersects \( \ell_1 \) at a unique point, say \( P_2 = (x, y) \). We set \( T(a, x, b) = y \).

**Lemma 5.5** There is a formula \( Ter(a, x, b, y, \delta) \in \text{FOL}_e \) which expresses that \((a, x, b) = y \) with respect to the auxiliary line \( \delta \).

**Lemma 5.6** The ternary operation \( T(a, x, b) \) has the following properties and interpretations:

**T-1:**

\[
T(1, x, 0) = T(x, 1, 0) = x
\]

\( T(1, x, 0) = x \) means that the auxiliary line \( d = \{(x, x) \in K^2 : x \in K \} \) is a line with \( sl(d) = 1 \).

\( T(x, 1, 0) = x \) means that the slope of the line \( \ell \) passing through \((0, 0)\) and \((1, x)\) is given by \( sl(\ell) = x \). This is the true interpretation of the slope in analytic geometry.

**T-2:**

\[
T(a, 0, b) = T(0, a, b) = b
\]

The equation \( T(a, 0, b) = b \) means that the line \( \ell \) defined by \( T(a, x, b) = y \) intersects \( m_0 \) at \((0, b)\) (which is the meaning of \( ax + b \) in analytic geometry).

The equation \( T(0, a, b) = b \) means that the horizontal line \( \ell_1 \) passing through \((0, b)\) consists of the points \((a, b) \in K^2 : a \in K \).

**T-3:**

For all \( a, x, y \in K \) there is a unique \( b \in K \) such that \( T(a, x, b) = y \)

This means that for every slope \( s \) different from \( \infty \) there is a unique line \( \ell \) with \( sl(\ell) = s \) passing through \((x, y)\).

**T-4:**

For every \( a, a', b, b' \in K \) and \( a \neq a' \) the equation \( T(a, x, b) = T(a', x, b') \) has a unique solution \( x \in K \).

This means that two lines \( \ell_1 \) and \( \ell_2 \) with different slopes not equal to \( \infty \) intersect at a unique point \( P \).

**T-5:**

For every \( x, y, x', y' \in K \) and \( x \neq x' \) there is a unique pair \( a, b \in K \) such that \( T(a, x, b) = y \) and \( T(a', x', b) = y' \).

This means that any two points \( P_1, P_2 \) not on the same vertical line are contained in a unique line \( \ell \) with slope different from \( \infty \).

A structure \( \langle K, T_K \rangle \) with a ternary operation \( T_K \) satisfying (T-1)-(T-5) is called a planar ternary ring PTR. If the PTR \( \langle K, T_K \rangle \) arises from a Pappian plane we define addition by \( add_T(a, b, c) \) by \( T(a, 1, b) = c \) and multiplication by \( mult_T(a, x, c) \) by \( T(a, x, 0) = c \).
We define now the translation schemes \( RR_{ptr} = (\text{line, } Ter) \) and \( RF_{field} = (\text{line, } addT, \text{ multT}) \): The transduction \( RR^*_{ptr} \) maps incidence planes \( \Pi \) into structures with universe \( K \) defined by the formula \( \text{line}(x, \ell_0) \) given by \( x \in \ell_0 \) and the ternary function \( Ter \). The transduction \( RF^*_{ptr} \) maps Pappian planes \( \Pi \) into structures with universe \( K \) defined by the \( \text{line} \) and with two binary operations \( add_T, \text{ mult}_T \).

With these definitions we get:

\[ \textbf{Theorem 5.6 (Correctness of } RR_{ptr} \text{ and } RF_{field}) \] Let \( \Pi \) be plane satisfying the incidence axioms I-1, I-2, and I-3 from Section 3 with distinguished lines \( \ell, m, d \) and points \( O = (0, 0) \) and \( I = (1, 0) \).

(i) \( RR^*_{ptr}(\Pi) \) is a planar ternary ring.

(ii) \( \Pi \) is a (infinite) Pappian plane iff \( RF^*_{field}(\Pi) \) is a field (of characteristic 0).

A detailed proof may be found in [19, 66].

\[ \textbf{6 Undecidable geometries} \]

\[ \textbf{6.1 Incidence geometries} \]

First we look \( \tau_e \)-structures, i.e., at models of the incidence relation alone. To prove undecidability, the correctness of the translation scheme \( RR_{field} \), Theorem 5.6, is not enough. We still have to show that \( RR^*_{field} \) is onto as a transduction from Pappus planes to fields.

\[ \textbf{Theorem 6.1} \] ([66, Section 4.5]) (i) If \( \Pi \) is a Pappus plane there is a field \( \mathcal{F}_\Pi \) such that \( PP^*_{\epsilon}(\mathcal{F}_\Pi) \) is isomorphic to \( \Pi \).

(ii) If additionally \( \Pi \) satisfies (InfLines), \( \mathcal{F}_\Pi \) is a field of characteristic 0.

In fact, \( \mathcal{F}_\Pi \) can be chosen to be \( RR^*_{field}(\Pi) \) from Theorem 5.4.

\[ \textbf{Corollary 6.1} \] \( RR^*_{field} \) is onto as a transduction from Pappus planes to fields.

We now can use Proposition 1.2(i) to prove Theorem 1.1, which states that the theory \( T_{pappus} \) of Pappus planes is undecidable.

\[ \textbf{Proof of Theorem 1.1} \] Assume, for contradiction, that \( T_{pappus} \) is decidable. By Corollary 6.1, \( RR^*_{field} \) is onto, hence the theory of fields is decidable, which contradicts Proposition 1.2.

\[ \textbf{Proof} \] (Alternative proof of Theorem 1.1) We could also use Proposition 1.2(iii) to prove Theorem 1.1. Let \( Q \) be the field of rational numbers. In this case we use Lemma 4.4 with \( \mathcal{A} = PP^*(Q) \) and \( \mathcal{A}' = RR^*(\mathcal{A}) \). \( RR^*(\mathcal{A}) \simeq Q \), by Theorem 6.1. So \( S \) is the complete theory of \( Q \), which is undecidable by Proposition 1.2(iii), hence \( T_{pappus} \) is undecidable.

This alternative proof does not work for Hilbert planes and Euclidean planes, because we do not know whether the complete theories of the fields \( P \) and \( E \) are undecidable.
6.2 Hilbert planes and Euclidean planes

Similarly, the correctness of the translation scheme $FF^*_\text{field}$, Theorem 5.5, is not enough. We need one more lemma.\textsuperscript{11}

**Lemma 6.1** Let $\mathcal{F}$ be a Pythagorean field and $\Pi_\mathcal{F} = PP^*_\text{hilbert}(\mathcal{F})$. Then $FF^*_\text{field}(\Pi_\mathcal{F})$ is isomorphic to $\mathcal{F}$.

*Proof* For every $a \in \mathcal{F}$ with $a \geq 0$ there is a segment in $\Pi_\mathcal{F}$ of the form $[(0,0), (a,0)]$ on the line $y = 0$, and every segment $[(0,0), (b,0)]$ on the line $y = 0$ corresponds to an element $b \in \mathcal{F}$. We conclude that there is an isomorphism $f$ of ordered fields between $\mathcal{F}$ and the segments on the line $y = 0$. Similarly, there is an isomorphism of ordered fields $g$ between the segments on the line $y = 0$ and the field $FF^*_\text{field}(\Pi_\mathcal{F})$. Composing the two isomorphisms gives the required isomorphism between $\mathcal{F}$ and $FF^*_\text{field}(\Pi_\mathcal{F})$. \qed

**Corollary 6.2** (i) $FF^*_\text{field}$ is onto as a transduction from Hilbert planes to ordered Pythagorean fields.

(ii) $FF^*_\text{field}$ is onto as a transduction from Euclidean planes to ordered Euclidean fields.

Using Ziegler’s Theorem 1.2, Lemma 4.1, Theorem 5.5 and Corollary 6.2 we conclude:

**Theorem 6.2** (i) The set of first order sentences true in all Hilbert planes which satisfy the Parallel Axiom is undecidable.

(ii) The set of first order sentences true in all Hilbert planes is undecidable.

(iii) The set of first order sentences true in all Euclidean planes is undecidable.

6.3 Wu’s geometry

We discuss here two systems of orthogonal geometry\textsuperscript{12}, Wu’s orthogonal geometry $T_{o-wu}$ and Wu’s metric geometry $T_{m-wu}$. Recall that in Wu’s orthogonal geometry we have as basic relation incidence $A \in l$ and Orthogonality $Or(l_1, l_2)$, but neither betweenness nor equidistance. We also require that models of Wu’s geometry are infinite Pappian planes without finite lines.

**Theorem 6.3** (i) Let $\mathcal{F}$ be a Pythagorean field of characteristic 0. Then $PP_{wu}(\mathcal{F})$ is a metric Wu plane.

(ii) Conversely, let $\Pi$ be a metric Wu plane then $RF^*_\text{field}(\Pi)$ is a Pythagorean field of characteristic 0.

(iii) Furthermore, $RF^*_\text{field}(PP_{wu}(\mathcal{F}))$ is isomorphic to $\mathcal{F}$ and $PP_{wu}(RF^*_\text{field}(\Pi))$ is isomorphic to $\Pi$.

\textsuperscript{11}In [16] it is overlooked that Lemma 6.1 is needed in order to apply Lemma 4.1. In [62] it is used properly but not explicitly stated.

\textsuperscript{12}Here an orthogonal geometry is an axiomatization of geometry where orthogonality is one of the basic relations to be characterized axiomatically. In [3] an orthogonal geometry is a vector space equipped with a quadratic form.
Corollary 6.3 \( RF_{field}^* \) maps metric Wu planes onto Pythagorean fields.

Theorem 6.4 The consequence problem for (Wu-metric) is undecidable. The consequence problem for (Wu-orthogonal) is undecidable.

Proof (i): Use Ziegler’s Theorem 1.2 for \( AFC_0 \), Lemma 4.1, Theorem 5.6 and Corollary 6.3.

(ii): We observe that (Wu-metric) is obtained from (Wu-orthogonal) by adding one more axiom. This gives that if (Wu-orthogonal) were decidable, so would (Wu-metric) be decidable, which contradicts (i).

6.4 Origami geometry

In Origami Geometry we have also points and lines, the incidence relation \( A \in l \), the orthogonality relation \( Or(l_1, l_2) \), a relation \( SymP(A, l, B) \) and a relation \( d(A, l_1, l_2) \).

The intended interpretation of \( SymP(A, l, B) \) states that \( A \) and \( B \) are symmetric with respect to \( l \), i.e., \( l \) is perpendicular to the line \( AB \) and intersects \( AB \) at a point \( C \) such that \( Eq(A, C) \) and \( Eq(C, B) \).

The intended interpretation of \( d(A, l_1, l_2) \) states that the point \( A \) has the same perpendicular distance from \( l_1 \) and \( l_2 \).

Clearly, \( SymP(A, l, B) \) and \( d(A, l_1, l_2) \) are definable in a Hilbert plane using \( \in \), \( Eq \), \( Or \).

An Origami Plane is an infinite Pappian plane without finite lines which satisfies additionally the axioms (H-1) to (H-7).

Theorem 6.5 ([2]) If \( \Pi \) is an Origami plane, then \( RF_{field}^*(\Pi) \) is a Vieta field. Conversely, for every Vieta field \( \mathcal{F} \) the structure \( PP_{origami}^*(\mathcal{F}) \) is an Origami field.

Corollary 6.4 \( RR_{field}^* \) maps Origami planes onto Vieta fields.

Using Ziegler’s Theorem 1.2, Lemma 4.1 and 4.2 we can now apply Theorem 6.5 and Corollary 6.4 to conclude:

Theorem 6.6 (i) The consequence problem for Origami Planes is undecidable.

(ii) The consequence problem for the Huzita axioms (H-1)-(H-7) is undecidable.

7 Decidability for fragments of first order logic

High-school geometry is based on diagrams representing configurations in Euclidean geometry. The use of diagrams is discussed from a logical point of view in [1, 46]. Problems in high-school geometry are usually of the form

Given a configuration between points \( \bar{p} \) (and lines) described by a quantifier-free formula \( \phi(\bar{p}) \) show that these points also satisfy a quantifier-free formula \( \psi(\bar{p}) \)

\[ \sigma(\bar{p}) : \forall \bar{p}(\phi(\bar{p}) \rightarrow \psi(\bar{p})) \]

A typical example would be:
Of the three altitudes $\ell_1, \ell_2, \ell_3$ of a triangle $P_1 P_2 P_2$ which intersect pairwise at the points $P_{1,2}$, $P_{1,3}$, $P_{2,3}$, show that $P_{1,1} = P_{1,2} = P_{1,3}$.

The formula $\sigma$ is a universal Horn formula in $\Omega \cap \Omega = \Omega \cap \Omega$.

In the literature the following was observed:

**Proposition 7.1** ([41]) The universal consequences of $T_{m-wu}$ are decidable.

([48]) The universal consequences of $T_{p-hilbert}$ and $T_{euclid}$ are decidable.

Using a simple model theoretic argument we can generalize Proposition 7.1 to Theorem 7.1 below.

We need some preparation. The reader can easily verify the following:

**Proposition 7.2** The formulas in the definition of the translation scheme

$$PP = \langle \phi_{\text{points}}, \phi_{\text{lines}}, \phi_{\varepsilon}, \phi_{Eg}, \phi_{Or}, \phi_{An}, \phi_{Be} \rangle$$

can be written in the vocabulary $\tau_{o,+,*}$ as quantifier-free formulas

$$QP = \langle \psi_{\text{points}}, \psi_{\text{lines}}, \psi_{\varepsilon}, \psi_{Eg}, \psi_{Or}, \psi_{An}, \psi_{Be} \rangle$$

In particular if $\theta \in FOL_{hilbert}$ is a universal formula, then $\hat{\theta} = QP^\sharp(\theta) \in FOL(\tau_{o,+,*})$ is a universal formula.

The general form of Proposition 7.1 can now be stated as follows:

**Theorem 7.1** $T \subseteq FOL(\tau_{hilbert})$ be a geometrical theory such that

(i) $T \models T_{pappus}$.

(ii) The set of formulas

$$FT = \{ \phi \in FOL(\tau_{o,+,*}) : \phi = PP^\sharp(\psi) \land \psi \in T \}$$

is consistent with $RCF$.

(iii) For every $\Pi$ with $\Pi \models T$ $PP^* (RR^*(\Pi))$ is isomorphic to $\Pi$.

Then the universal consequences of $T$ are decidable.

The analogous statement also holds for $T \subseteq FOL(\tau_{wu})$ and $T \subseteq FOL(\tau_{origami})$ where $RCF$ is replaced by $ACF_0$.

**Proof of Theorem 7.1** Let $T$ be as required and $\theta$ be universal. We want to show that $T \models \theta$ iff $RFC \models QP^\sharp(\theta)$. The latter can be decided using the Theorems 2.1.

We have: $T \models \theta$ iff for every plane $\Pi$ with $\Pi \models T$ also $\Pi \models \theta$.

By Theorem 6.1 $\Pi$ is isomorphic $PP^* (RR^*(\Pi))$ and also to $QP^* (RR^*(\Pi))$. By Theorem 4.1 $\Pi \models \theta$ iff $(RR^*(\Pi)) \models QP^\sharp(\theta)$. By the definition of $QP$ the formula $QP^\sharp(\theta)$ is universal. Therefore $(RR^*(\Pi)) \models QP^\sharp(\theta)$ iff $RCF \models QP^\sharp(\theta)$ by Lemma 2.3.

In the case of fields rather than ordered fields, we show that $T \models \theta$ iff $ACF_0 \models QP^\sharp(\theta)$ which can be decided using Theorem 2.2.

Proposition 7.1 now follows easily using Theorems 2.1 and 2.2.

We also get:
Corollary 7.1 The universal consequences of $T_{\text{origami}}$ formulated as formulas in FOL ($\tau_{wu}$) are decidable.

Proof This follows from the characterization of the fields corresponding to $T_{\text{origami}}$ as the Vieta fields (Theorem 6.5).

More decidability for the universal consequences can be obtained from axiomatizations of geometrical constructions using more than just ruler and compass, cf. [50].

8 Beyond Hilbert-style affine Euclidean geometries

The present paper does not intend to be a complete catalog of decidable and undecidable axiomatizations of geometry. Its main purpose was to isolate the ingredients needed to pass from the undecidability of a theory of fields to the undecidability of a corresponding geometry.

In each of the following cases, undecidability results are known. However, providing all the details for a proof for some of the cases would still be a serious project for today’s graduate students in AI and mechanized theorem proving, even if all this is well known for mathematical logicians interested in geometry.

Projective, elliptic and hyperbolic geometry: The undecidability of projective, elliptic and hyperbolic geometry is stated in [56] with a sketch of a proof. In the case of hyperbolic geometry the field of coordinates is obtained using Hilbert’s arithmetic of ends, as described in [33, Chapter 41]. Starting with a field one build the Poincaré model of hyperbolic geometry, cf. [33, Chapter 39]. Using the arithmetic of ends one can recover a field of coordinates from a hyperbolic plane. But one has to verify that this field of coordinates is isomorphic to the original field (and vice versa). This seems to be true, but I have not seen in the literature a proof of this, and checking the details goes beyond the scope of this paper. It remains also to be worked out in detail which axiomatizations of projective, elliptic or hyperbolic geometry can be proved undecidable using Ziegler’s theorem.

Tarski’s axiomatization of Euclidean geometry: Hilbert’s and Tarski’s axiomatizations of Euclidean geometry differ in the choice of vocabulary, [5]. For a conceptual discussion, see [7]. It is known that Hilbert planes are bi-interpretable with Tarski’s axioms A1-A9, and Hilbert planes with the parallel axioms are bi-interpretable with Tarski’s axioms A1-A10, [14]. This implies that the first order consequences of A1-A10 (and hence of A1-A9) are undecidable.

Geometry in higher dimensions: Our discussion so far concerned plane geometry. The undecidability results should remain true, but one has to check the details which depend on the choice of basic relations and sorts and the axiomatization of higher dimensional geometry. A detailed discussion of higher dimensional geometries also goes beyond the scope of this paper.

9 Conclusions

We have discussed the decidability of the consequence problem for various axiomatizations of Euclidean geometry. The purpose of the paper was to make the metamathematical methods discussed in [62] and in [16] more accessible to the research communities of symbolic
computation and automated theorem proving. In particular, we wanted to draw attention to Ziegler’s Theorem 1.2, and spell out in detail what is needed to draw its consequences for geometrical theories. We have also listed some open problems concerning the decidability of theories of fields if restricted to fragments of first order logic such as $\mathcal{U}$, $\mathcal{E}$, $\mathcal{F}$.

In writing this expository paper we also included new applications of these methods to Wu’s orthogonal geometry and to the geometry of paper folding Origami. These results, both undecidability of first order consequences and decidability of universal consequences, can be easily extended to theories of geometric constructions going beyond ruler and compass or paper folding, cf. [33, 50].

From a complexity point of view, we see that the consequence problem for first order formulas is either undecidable or, in the case of Tarski’s decidability results, prohibitively difficult. We have also shown that in the cases discussed, the consequence problem for universally quantified formulas is decidable, possibly in non-deterministic polynomial time. What is left open, and remains a challenge for future research, is the decidability question for existential and $\forall\exists$-Horn formulas $\mathcal{E}$ and $\mathcal{F}$.

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