BISPECTRAL OPERATORS, DUAL ISOMONODROMIC DEFORMATIONS AND THE Riemann-Hilbert Dressing Method

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Abstract. A comparison is made between bispectral systems and dual isomonodromic deformation equations. A number of examples are given, showing how bispectral systems may be embedded into isomonodromic ones. Sufficiency conditions are given for the construction of rational solutions of isomonodromic deformation equations through the Riemann-Hilbert problem dressing method, and these are shown, in certain cases, to reduce to bispectral systems.

1. Bispectral Systems vs. Isomonodromic Deformations

We begin by listing a number of results characterizing bispectral systems and isomonodromic systems, respectively, with a view to comparison.

1.1. Bispectral Systems.

(1) In bispectral systems, we have a function $\psi(x, z)$ of the two variables $(x, z)$ that simultaneously satisfies a pair of eigenvalue equations

$$L\psi(x, z) = f(z)\psi(x, z)$$

$$\Lambda\psi(x, z) = \phi(x)\psi(x, z),$$

where $L(x)$ and $\Lambda(z)$ are differential operators in the indicated variables and $f(z)$, $\phi(x)$ are parametric families of eigenvalues depending on the other variable. Moreover [DG], for a suitable normalization, $L(x)$ takes the form

$$L(x) = \partial_x^r + a_r - 2\partial_x^{-2} + \cdots$$

where the coefficients $\{a_j(x)\}_{j=1}^{r-2}$ are rational in $x$ and the eigenvalues $f(z)$ depend polynomially on $z$, and similarly for the operator $\Lambda(z)$ and eigenvalues $\phi(x)$.

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(2) For $L$'s belonging to rank–1 bispectral algebras $[W]$, the eigenfunction $\psi$ may be viewed as the $t = (t_1 = x, 0, \ldots)$ value of a Baker-Akhiezer function $\psi_W(t, z)$ expressed, as usual, in terms of a $\tau$–function via the formula

\begin{equation}
\psi_W(t, z) = \frac{e^{-\sum_{m=1}^{\infty} t_m z^m} \tau_W(t)}{\tau_W(t)}.
\end{equation}

$[z] := \left( \frac{1}{z}, \frac{1}{2z}, \frac{1}{3z^3}, \cdots \right)$.

The $\tau$–function $\tau_W$ is rational, and has the form

\begin{equation}
\tau_W = \det(X + \sum_{j=1}^{\infty} j t_j (-Z)^{j-1}),
\end{equation}

where $(X, Z)$ is a pair of $n \times n$ matrices representing a point in Wilson’s complexified, prereduced Calogero–Moser phase space $C_n$. That is, their commutator is of the form

\begin{equation}
[X, Z] = -I + fg^T
\end{equation}

for a pair of vectors $f, g \in C^n$. The $t$–dependence of $\tau_W$ is thus given by flows generated by the commuting family of Hamiltonians

\begin{equation}
h_n := -\text{tr}(-Z)^n,
\end{equation}

which are obtained by applying the bispectral involution (see below) to the Calogero-Moser hierarchy. The Baker function $\psi_W$ at $t = (x, 0, \cdots)$ may be explicitly expressed in this case by the formula

\begin{equation}
\psi_W(x, z) = e^{xz} \det(I - (X + xI)^{-1}(Z + zI)^{-1}).
\end{equation}

(3) Solutions can be constructed from suitably defined “vacuum” solutions through the application of Darboux transformations [BHY1, BHY2, K, K].

(4) The Baker function $\psi_W$ and the $\tau$–function $\tau_W$ for rank–1 bispectral algebras correspond to an element $W$ of Wilson’s adelic Grassmannian $\text{Gr}^{ad}$ [W]. The existence of a bispectral pair $(L, \Lambda)$ follows from the fact that $\text{Gr}^{ad}$ admits a bispectral involution $b : \text{Gr}^{ad} \rightarrow \text{Gr}^{ad}$ defined by the commuting diagram of invertible maps

\begin{equation}
\begin{array}{ccc}
\cup_n C_n & \xrightarrow{b_W} & \cup_n C_n \\
W & \downarrow & W \\
\text{Gr}^{ad} & \xrightarrow{b} & \text{Gr}^{ad}
\end{array}
\end{equation}

where the map $W : \cup_n C_n \rightarrow \text{Gr}^{ad}$ is defined by formula (1.8) and $b_W : C_n \rightarrow C_n$ by

\begin{equation}
b_W : (X, Z) \mapsto (Z^t, X^t).
\end{equation}
1.2. Isomonodromic Deformations.

(1) For these systems, we have an invertible $r \times r$ matrix–valued function $\Psi(x, z)$ that simultaneously satisfies a pair of equations

\[
\begin{align*}
\frac{\partial \Psi(x, z)}{\partial x} &= U(x, z)\Psi(x, z), \\
\frac{\partial \Psi(x, z)}{\partial z} &= V(x, z)\Psi(x, z),
\end{align*}
\]

where, in general, the $r \times r$ matrix–valued functions $U(x, z)$ and $V(x, z)$ are rational in $z$. The consistency condition

\[
\left[ \frac{\partial}{\partial z} - U, \frac{\partial}{\partial x} - V \right] = 0
\]

implies the invariance of the (generalized) monodromy data of the rational covariant derivative operator

\[
D_z := \frac{\partial}{\partial z} - V(x, z)
\]

under the deformations resulting from varying $x$. In general, there may be many deformation parameters $(t_1 = x, t_2, \cdots)$, each with an associated infinitesimal deformation operator

\[
D_j := \frac{\partial}{\partial t_j} - U_j(t_1, t_2, \cdots, z) \quad j = 1, 2, \ldots
\]

commuting with $D_z$ and amongst themselves, each determining a 1–parameter family of isomonodromic deformations. The structure of the corresponding deformation equations was analyzed in [JMU, JM].

(2) Isomonodromic systems of type (1.12a,b) have a Hamiltonian structure [JMU, H1, HTW] and are generated by certain spectral invariants of the matrix $V$. The logarithmic differential of the corresponding $\tau$–function is given by

\[
d(ln \tau) = \sum_j H_j dt_j,
\]

where $\{H_j\}_{j=1,2,\ldots}$ are the spectral invariant Hamiltonians generating the equations (1.12a,b).

(3) Solutions to eqs. (1.12a,b) can be constructed from “vacuum” solutions by application of the “dressing method” [NZMP, HI], based on the matrix Riemann–Hilbert problem. For suitably chosen vacuua and loop group elements, this gives rise to solutions having a rational dependence upon the deformation parameter.

(4) The isomonodromic $\tau$–functions correspond to certain special elements $W \in$ Gr of the general Segal–Wilson–Sato Grassmannian. Moreover, at least for the case where the element $V(x, z)$ has $n$ simple poles at finite $z$ and tends to a nonsingular finite value $X$ at $z = \infty$, there exists an equivalent
representation of the underlying dynamical equations as deformations of a second dual isomonodromic family of differential operators [H1]

\[
D_\lambda := \frac{\partial}{\partial \lambda} - \tilde{V}(x, \lambda),
\]

with \(\tilde{V}(x, \lambda)\) tending to a nonsingular finite limit \(Z^T\) at \(\lambda = \infty\). The corresponding dual matrix Baker function \(\tilde{\Psi}(x, \lambda)\) is a nonsingular \(n \times n\) matrix satisfying a similar system of equations

\[
\begin{align*}
\frac{\partial \tilde{\Psi}(x, \lambda)}{\partial x} &= \tilde{U}(x, \lambda) \tilde{\Psi}(x, \lambda) \quad (1.18a) \\
\frac{\partial \tilde{\Psi}(x, \lambda)}{\partial \lambda} &= \tilde{V}(x, \lambda) \tilde{\Psi}(x, \lambda) \quad (1.18b),
\end{align*}
\]

where the \(n \times n\) matrices \(\tilde{U}(x, \lambda)\) and \(\tilde{V}(x, \lambda)\) are rational in the second spectral parameter \(\lambda\). The dual system (1.18a,b) can be related to the original one (1.12a,b) by introducing an auxiliary symplectic vector space consisting of canonically conjugate pairs \((F, G)\) of \(n \times r\) rectangular matrices and applying a Hamiltonian quotienting procedure. The resulting matrices \(V\) and \(\tilde{V}\) in the definition of the operators \(D_z\) and \(D_\lambda\) respectively are given by the formulæ

\[
\begin{align*}
V(x, z) &= X + F(Z - zI_r)^{-1}G^T \quad (1.19a) \\
\tilde{V}(x, z) &= Z^T + F^T(X^T - \lambda I_n)^{-1}G, \quad (1.19b),
\end{align*}
\]

and the corresponding spectral curves are birationally equivalent (with the rôles of the loop parameter \(\lambda\) and eigenvalue \(z\) interchanged). These are therefore related by the involutive duality map

\[
(X, Z, F, G, z, \lambda) \mapsto (Z^T, X^T, F^T, G^T, \lambda, z).
\]

### 1.3. A simple example

To further indicate the close correspondence between bispectral and isomonodromic systems, consider the following example of rank–1 bispectral systems (cf. [H2]). In the above notation, choose the matrices \(X\) and \(Z\) to be

\[
\begin{align*}
X &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.
\end{align*}
\]

The resulting bispectral Baker function is then

\[
\psi(x, a) = e^{xz} \left(1 - \frac{2}{xz} + \frac{2}{x^2z^2}\right),
\]

which satisfies the eigenvalue equations

\[
\begin{align*}
\left(\frac{\partial}{\partial z^3} - \frac{6}{z^2} \frac{\partial}{\partial z} + \frac{12}{z^3}\right) \psi &= z^3 \psi \quad (1.23a) \\
\left(\frac{\partial}{\partial x^3} - \frac{6}{x^2} \frac{\partial}{\partial x} + \frac{12}{x^3}\right) \psi &= x^3 \psi. \quad (1.23b)
\end{align*}
\]
In this case, we have two other single–valued solutions of the same system, due to the invariance under multiplication of either \( x \) or \( z \) by a cube root of 1. Let

\[
\Psi := \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 \\ \psi_{0,x} & \psi_{1,x} & \psi_{2,x} \\ \psi_{0,xx} & \psi_{1,xx} & \psi_{2,xx} \end{pmatrix}
\]

be the Wronskian matrix formed from the three solutions so obtained. Then \( \Psi \) satisfies

\[
\frac{\partial \Psi}{\partial x} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ z^3 - \frac{12}{x^2} & \frac{6}{xx} & 0 \end{pmatrix} \Psi = 0
\]

\[
\frac{\partial \Psi}{\partial z} - \begin{pmatrix} 0 & \frac{1}{z} & 0 \\ 0 & \frac{1}{z} & \frac{1}{2} \\ z^2 - \frac{12}{x^2} & \frac{6}{xx} & \frac{2}{z} \end{pmatrix} \Psi = 0,
\]

which shows that, viewing \( x \) as a deformation parameter, the monodromy of the rational covariant derivative operator entering in (1.25b) is independent of \( x \). Of course, the monodromy in this case is actually trivial, since the matrix Baker function \( \Psi \) is globally defined and single–valued. Moreover since the dependence on both the variables \( x \) and \( z \) is rational, the system could equally well be viewed as determining a 1-parameter family of rational covariant differential operators in the \( x \) variable, defined by eq. (1.25a), having trivial monodromy, with \( z \) viewed as the deformation parameter. A dual isomonodromic system may also be defined, just by introducing the dual matrix Baker function \( \tilde{\Psi} \) as the Wronskian matrix determined from \((\tilde{\psi}_0, \tilde{\psi}_1, \tilde{\psi}_2)\) through differentiation with respect to the \( z \)-variable,

\[
\tilde{\Psi} := \begin{pmatrix} \tilde{\psi}_0 & \tilde{\psi}_1 & \tilde{\psi}_2 \\ \tilde{\psi}_{0,z} & \tilde{\psi}_{1,z} & \tilde{\psi}_{2,z} \\ \tilde{\psi}_{0,zz} & \tilde{\psi}_{1,zz} & \tilde{\psi}_{2,zz} \end{pmatrix}
\]

\[
\tilde{\psi}_j(x, z) := \psi(x, \omega^j z), \quad \omega := e^{\frac{2\pi i}{3}}, \quad j = 0, 1, 2
\]

This then satisfies the system obtained from (1.25a,b) by interchanging the \( x \) and \( z \) variables.

The question naturally arises: is it possible in general to embed bispectral systems into isomonodromic ones, such that the matrix Riemann-Hilbert problem used in the dressing method construction of solutions to the latter systems reduces to the Darboux transformations used in constructing the former? In the following sections, this idea will be further developed. A set of criteria is given, within the Riemann–Hilbert problem setting, giving rise to isomonodromic systems derived from suitably defined “vacuum” solutions. Imposing further restrictions on the data entering the Riemann–Hilbert problem, we deduce sufficient conditions for the solution to depend rationally on the deformation parameter. Although a set of sufficient conditions for these systems to reduce to bispectral ones is not yet determined, we are able to illustrate via examples and elementary calculations that rank–1 bispectral systems can be recovered from this approach.
2. ISOMONODROMIC SYSTEMS AND THE RIEMANN–HILBERT PROBLEM

2.1. The Dressing Method for Isomonodromic Systems. In the matrix Riemann–Hilbert (RH) problem approach to zero–curvature equations depending upon a spectral parameter \( z \), one usually starts with a particular \( r \times r \) matrix function \( \Psi_0(t,z) \), viewed as the “vacuum” solution, depending on a number of commuting flow parameters \( \mathbf{t} = (t_1, t_2, \cdots) \) and chooses, along some suitably defined, closed contour \( \Gamma \) in the \( z \)–plane, with interior region \( \Gamma^+ \) and exterior \( \Gamma^- \), a smooth, nonsingular \( r \times r \) matrix–valued function \( H(z) \) of the spectral parameter. (In the group theoretical formulation, \( H \) is interpreted as a loop group element.) The Zakharov–Shabat dressing method \([NZMP]\) then consists of solving the associated matrix RH problem for the conjugated matrix

\[
\chi^{-1}(t,z)\chi_{-}(t,z) = \Psi_0(t,z)H(z)\Psi_0^{-1}(t,z),
\]

where \( \chi_{\pm} \) are holomorphic functions of \( z \) in the regions \( \Gamma_{\pm} \) which, in the regular case, are nonsingular \( r \times r \) matrices in their domain of holomorphicity (with a suitable normalization condition at some point, usually \( z = \infty \), to guarantee uniqueness of the solution). The “dressed” matrix Baker function \( \Psi(x,z) \) is then obtained by applying \( \chi_{\pm} \) as a gauge transformation to the vacuum solution

\[
\Psi = \chi_{+}\Psi_0H_{+} = \chi_{-}\Psi_0H_{-}^{-1},
\]

where the nonsingular \( r \times r \) matrices \( H_{\pm}(z) \) are \( t \)–independent and chosen in any way that respects the factorization

\[
H(z) = H_{+}(z)H_{-}(z).
\]

These just serve to determine a basis for the fundamental solution and may, in particular, be chosen either so that eq. (2.3) itself is a solution to the RH problem for the fixed element \( H \) or, alternatively, that one of the two factors \( H_{\pm} \) is equal to \( H \) itself, and the other is the identity matrix.

If the vacuum matrix Baker function satisfies the Zakharov–Shabat (ZS) system

\[
\frac{\partial\Psi_0}{\partial t_j} = U_{j,0}(t,z)\Psi_0,
\]

where the \( U_{j}^{(0)} \)'s are \( r \times r \) matrix–valued functions depending rationally on the spectral parameter \( z \), with a given pole structure and asymptotic limit as \( z \to \infty \), the dressed Baker function \( \Psi(t,z) \) will satisfy a system having the same pole and asymptotic structure in \( z \)

\[
\frac{\partial\Psi}{\partial t_j} = U_{j}(t,z)\Psi_0,
\]

where the \( U_{j} \)'s are obtained by applying the same gauge transformation to the vacuum covariant derivative operators as that appearing in eq. (2.2). It follows that the compatibility conditions for (2.4), the zero–curvature conditions, are satisfied identically in \( z \).
To adapt this procedure to isomonodromic deformation equations, we must assume that, in addition to the usual vacuum ZS equations (2.4), \( \Psi_0 \) also satisfies a compatible differential equation with respect to the spectral parameter

\[
\frac{\partial \Psi_0}{\partial z} = V_0(t, z) \Psi_0,
\]

where the \( r \times r \) matrix-valued function \( V_0 \) depends polynomially on \( z \). This implies that the (generalized) monodromy data of the vacuum rational covariant derivative operator

\[
D^0_z := \frac{\partial}{\partial z} - V_0(t, z)
\]

is invariant under the deformations generated by varying the parameters \( \{t_j\}_{j=1,2,...} \). In order to guarantee that the resulting “dressed” operator

\[
D_z := \frac{\partial}{\partial z} - V(t, z),
\]

obtained through application of the gauge transformation (2.2) to \( D^0_z \), should also be rational in the spectral parameter \( z \), further restrictions must be put on the choice of \( H(z) \). A sufficient set of conditions for \( H(z) \) is given in the following.

**Proposition 2.1.** If \( H(z) \) satisfies a differential equation of the form

\[
\frac{dH(z)}{dz} = r(z)H(z) + Q(z),
\]

where \( r(z) \) is a rational matrix-valued function and \( Q(z) \) is a distribution with support at a finite number of points, and the splitting (2.3) is chosen such that

\[
H_+(z) := H(z), \quad H_-(z) = I,
\]

the resulting operator \( D_z \) will depend rationally on \( z \), and its (generalized) monodromy will be preserved under deformations in the parameters \( \{t_j\}_{j=1,2,...} \) determined compatibly by the equations (2.5). If \( Q(z) \) vanishes, then \( V(t, z) \) can be expressed

\[
V(t, z) = V_-(t, z) + V_+(t, z),
\]

where \( V_+ \) is a polynomial in \( z \), of the same degree as \( V_0 \), and \( V_- \) is rational in \( z \), with the same pole support as \( r(z) \), and vanishing at \( z = \infty \).

The proof of this proposition is quite elementary and will not be detailed here. However, the sense of the derivative appearing in eq. (2.8) needs to be clarified since, a priori, \( H(z) \) is only defined along the contour \( \Gamma \). What is meant here is that we assume that \( H(z) \) may be extended to a neighborhood of \( \Gamma \), though not necessarily as a holomorphic function. If it can be extended holomorphically, terms like \( Q(z) \) are absent. In particular, \( H(z) \) could be piecewise constant along \( \Gamma \), and \( Q(z) \) could have its support just on \( \Gamma \) itself. This case will not be the one of interest to us in the following, but it does play an important role in the solution.
of certain classes of isomonodromic deformation equations through the Riemann-Hilbert problem method (cf. [HI]).

In the following, we shall restrict ourselves to the case when \( Q(z) \equiv 0 \), and seek to impose suitable restrictions on the vacuum solution \( \Psi_0 \) and the element \( H \) in order that the resulting matrices \( (U_j(t, z), V(t, z)) \) appearing in eqs. (2.5) and (2.8) depend rationally, not only on the spectral parameter \( z \), but also on the deformation parameters \( \{t_j\} \). We begin in the next subsection by giving two examples that lead to such rational solutions, and which may be reduced to particular cases of Wilson’s rank-1 bispectral systems.

### 2.2. Two Examples Reducing to Rank-1 Bispectral Systems.

The following examples show how an appropriately chosen pair \((\Psi_0, H)\) leads, through solution of the Riemann-Hilbert problem, to isomonodromic systems that reduce to bispectral ones. We restrict ourselves to \( 3 \times 3 \) matrices, with \( \chi_- \) rational, and the pole support of both \( \chi_- \) and \( H \) at \( z = 0 \). To obtain a correspondence between the scalar and matrix systems, we restrict ourselves to the subgroup of the full loop group \( \tilde{\text{SL}}(3) \) consisting of elements satisfying the invariance condition

\[
TH(\omega z)T^{-1} = H(z)
\]

where

\[
\omega := e^{2\pi i/3}, \quad T := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},
\]

with similar restrictions on \( \Psi_0, V_0 \) and \( U_0 \). Define the matrix

\[
\Sigma := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

In both the following examples we choose the vacuum Baker function as

\[
\Psi_0(x, z) := e^{xz\Sigma},
\]

so that

\[
U_0 := z\Sigma \quad V_0 := x\Sigma.
\]

From any \( 3 \times 3 \) matrix Baker function \( \Psi \) satisfying the invariance condition (2.12), we obtain three scalar Baker functions by summing along the rows; i.e., by taking the components of the vector obtained by applying \( \Psi \) to the vector with all entries equal to 1.

\[
\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} := \Psi \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.
\]
Example 2.2.1. Choose

\( H(z) := \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & z \\ z^{-2} & 0 & 0 \end{pmatrix} \)

as the loop group element. The solution to the Riemann-Hilbert problem is then given by

\( \chi_- = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{x} & 1 & 0 \\ \frac{1}{x^2} & -\frac{2}{x^2} & 1 \end{pmatrix} \),

which is rational in \( z \). The corresponding isomonodromic system (1.12a,b), (1.13) is then defined by

\( U(x, z) = \begin{pmatrix} \frac{1}{x} & z & 0 \\ 0 & \frac{1}{x} & z \\ z & 0 & -\frac{2}{x} \end{pmatrix} \), \quad V(x, z) = \begin{pmatrix} \frac{1}{x} & x & 0 \\ 0 & \frac{1}{x} & x \\ x & 0 & -\frac{2}{x} \end{pmatrix} \).

Summing over the rows of \( \Psi \) gives

\( \psi_1 = e^{xz}, \quad \psi_2 = e^{xz} \left( 1 - \frac{1}{xz} \right), \quad \psi_3 = e^{xz} \left( 1 - \frac{2}{x^2} + \frac{2}{x^2 z^2} \right) \).

These are all rank-1 bispectral wave functions which are eigenfunctions, respectively, of the following pairs of operators

\( L_1 = \frac{\partial}{\partial x}, \quad \Lambda_1 = \frac{\partial}{\partial z}, \)

\( L_2 = \frac{\partial^2}{\partial x^2} - \frac{2}{x^2}, \quad \Lambda_2 = \frac{\partial^2}{\partial z^2} - \frac{2}{z^2}, \)

\( L_3 = \frac{\partial^3}{\partial x^3} - \frac{6}{x^2} \frac{\partial}{\partial x} + \frac{12}{x^3}, \quad \Lambda_3 = \frac{\partial^3}{\partial z^3} - \frac{6}{z^2} \frac{\partial}{\partial z} + \frac{12}{z^3}. \)

In terms of Wilson’s correspondence, \( \psi_1 \) is just the vacuum Baker function, \( \psi_2 \) corresponds to the point in the 1–particle Calogero-Moser phase space with \((X, Z) = (0, 0)\), and \( \psi_3 \) is just the example of section 1.3, corresponding to the point in the 2–particle phase space with matrices \((X, Z)\) given in (1.21). Note, however, that the correspondence between the isomonodromic and bispectral systems given here is not the same as the one given in section (1.3).

Example 2.2.2. Now choose

\( H(z) := \begin{pmatrix} 0 & 0 & z^2 \\ z^{-1} & 0 & 0 \\ 0 & z^{-1} & 0 \end{pmatrix} \).
The solution to the Riemann-Hilbert problem in this case gives
\[(2.24) \quad \chi_-=\begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{xz} & 1 & 0 \\ 0 & -\frac{1}{xz} & 1 \end{pmatrix},\]
and the corresponding isomonodromic system (1.12a,b), (1.13) is defined by
\[(2.25) \quad U(x,z) = \begin{pmatrix} \frac{2}{z} & x & 0 \\ 0 & -\frac{1}{z} & x \\ \frac{z}{x} & 0 & -\frac{1}{z} \end{pmatrix}, \quad V(x,z) = \begin{pmatrix} \frac{2}{z} & x & 0 \\ 0 & -\frac{1}{z} & x \\ \frac{z}{x} & 0 & -\frac{1}{z} \end{pmatrix}.\]

The associated bispectral Baker functions obtained by summing over the rows of \(\Psi\) are
\[(2.26) \quad \psi_1 = e^{xz}, \quad \psi_2 = e^{xz}\left(1 - \frac{2}{xz}\right), \quad \psi_3 = e^{xz}\left(1 - \frac{1}{xz}\right),\]
so \(\psi_1\) and \(\psi_3\) coincide with \(\psi_1\) and \(\psi_3\), respectively, of the previous example, while \(\psi_2\) here is a bispectral eigenfunction of the pair of operators
\[(2.27) \quad L = \frac{\partial^3}{\partial x^3} - \frac{6}{x^2} \frac{\partial}{\partial x}, \quad \Lambda = \frac{\partial^3}{\partial z^3} - \frac{6}{z^2} \frac{\partial}{\partial z}.\]

This case corresponds to the point in the Calogero-Moser phase space with matrices
\[(2.28) \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.\]

In both these examples, \(\chi_-\) is rational, and its only pole, at \(z = 0\), is coincident with that of \(H_z H^{-1}\). In the last subsection, some preliminary results will be given showing how isomonodromic systems may more generally be obtained, such that the dependence on the deformation parameter is also rational, but with an arbitrary number of poles in \(r(z)\). The structure of the solutions will be seen to closely resemble that of bispectral systems.

### 2.3. Rational Isomonodromic Systems.

We now make some more restrictive assumptions about the solution to the Riemann-Hilbert problem, which are sufficient to assure that the matrices \(U_j(t,z)\) and \(V(t,z)\) are rational functions of the deformation parameters \((x = t_1, t_2 \cdots)\). In the following, we assume that the contour \(\Gamma\) is sufficiently large that all poles of \(r(z)\) are contained in the interior region \(\Gamma_+\), and choose the normalization of \(\chi_-(z)\) to be such that
\[(2.29) \quad \chi_-(\infty) = 1.\]

We must of course first require that the matrices \(U_0,j(t,z), V_0(t,z)\) entering in the vacuum equations (2.5), (2.6) should themselves depend rationally on the deformation parameters. Now also assume that the solution to the Riemann-Hilbert problem is such \(\chi_-\) is rational in \(z\). It follows that its pole support is contained in that of \(r(z)\). We now add the additional requirement that the difference
\[(2.30) \quad \chi_+(z)\Psi_0(z) r(z) \Psi_0^{-1}(z) \chi_+^{-1}(z) = r(z)\]
be holomorphic in \(\Gamma_+\). This means, for example, when \(r(z)\) has only first order poles at the points \(z = z_1, \cdots z_n\), with residue matrices \(\{r_i\}_{i=1}^n\), that the matrix \(\chi_+(z_i)\Psi_0(z_i)\) must commute with \(r_i\), for each \(z = z_i\). We then have the following result
Proposition 2.2. Under the above assumptions, (i.e., that $\chi_-, U_{0,j}$ and $V_0$ are all rational in the deformation parameters, and $\chi_+(z)\Psi_0(z)$ is such that the expression (2.30) is holomorphic in $z$ inside $\Gamma_+$), the matrices $U_j(t, z)$, $V(t, z)$ obtained through the dressing transformation are also rational in the parameters $t = (t_1, t_2 \cdots)$.

Again, the proof is elementary and will not be detailed here. However, to show the close similarity with bispectral systems, at least for the case of rank–1 bispectral algebras, we will consider the detailed structure of the resulting Baker matrix $\Psi$ in the case when $r(z)$ has only simple poles and $V_0(x, z)$ is chosen as independent of $z$. (Here $x = t_1$ and the other deformation parameters are chosen to vanish.) This means that the vacuum Baker matrix $\Psi_0$, as in the scalar case, is a linear exponential in $z$, with matrix exponent $zV_0(x)$. (These latter restrictions are imposed only for the sake of simplicity and comparison with the bispectral case; the following formulae can easily be modified to cover the general case with higher order poles and polynomial dependence on $z$.) For this case, $r(z)$ and $\chi_-(x, z)$ have the general form

\begin{align}
(2.31a) & \quad r(z) = \sum_{j=1}^{n} \frac{r_j}{z - z_j}, \\
(2.31b) & \quad \chi_-(x, z) = I + \sum_{j=1}^{n} \frac{Q_j}{z - z_j},
\end{align}

where the residue matrices \( \{r_j\} \) are constants and the \( Q_j \)'s are rational functions of \( x \), whose explicit form will be determined under the assumptions of Proposition 2.2. The relation (2.29) in this case implies the equalities

\begin{equation}
(2.32) \quad \chi_+(z_i)\Psi_0(z_i)r_i\Psi_0^{-1}(z_i)\chi_+^{-1}(z_i) = r_i, \quad i = 1, \cdots n.
\end{equation}

It then follows from the dressing method that the polynomial (in $z$) and rational parts $V_+$, $V_-$, respectively, of the matrix $V(x, z)$ entering in Proposition 2.1 are just

\begin{equation}
(2.33) \quad V_+(x, z) = V_0(x), \quad V_-(x, z) = r(z).
\end{equation}

Thus, under the given assumptions, the residue matrices in the polar part of $V(x, z)$ are constants, and the polynomial part is the same as for the vacuum solution. The dressing transformation gives

\begin{equation}
(2.34) \quad (V_0(x) + r(z))\chi_-(x, z) = \chi_-,z(x, z) + \chi_-(x, z)V_0(x).
\end{equation}

Substituting the expression (2.31a,b) for $\chi_-(x, z)$ and $r(z)$ and equating the polar parts gives the relations

\begin{align}
(2.35a) & \quad \left(\text{ad}_{V_0} - \sum_{j=1}^{n} \frac{r_j}{z_i - z_j}\right)Q_i + \sum_{j=1}^{n} \frac{r_i}{z_i - z_j}Q_j = -r_j \\
(2.35b) & \quad r_jQ_j = -Q_j, \quad j = 1 \cdots n.
\end{align}
To express the solution of this linear algebraic system it is convenient to introduce the \( nr \times r \) matrices formed from a column of \( r \times r \) blocks consisting of the residue matrices

\[
R := \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}, \quad Q := \begin{pmatrix} Q_1 \\ \vdots \\ Q_n \end{pmatrix},
\]

as well as the \( nr \times nr \) matrix \( X \), consisting of \( r \times r \) blocks \( \{X_{ij}\}_{i,j=1,\ldots,n} \) given by

\[
X_{ii} := -\sum_{j=1}^{n} \frac{r_j}{z_i - z_j} \tag{2.37a}
\]

\[
X_{ij} := \frac{r_j}{z_i - z_j}, \quad i \neq j. \tag{2.37b}
\]

Eq. (2.35a) may then be written more concisely as

\[
\left( \text{ad}V_0 \otimes I + X \right)Q = -R. \tag{2.38}
\]

Solving, and substituting into the expression (2.31b) for \( \chi_- \) and (2.2) for \( \Psi \) gives

\[
\Psi(x,z) = \left( I + E^T (Z - zI)^{-1} (X + \text{ad}V_0 \otimes I)^{-1} R \right), \tag{2.39}
\]

where \( E \) is the \( nr \times r \) matrix consisting of a column of \( n \) blocks, each of which is the \( r \times r \) identity matrix

\[
E = \begin{pmatrix} I \\ \vdots \\ I \end{pmatrix}, \tag{2.40}
\]

while \( Z \) is the \( np \times np \) block–diagonal matrix whose \( r \times r \) diagonal blocks are just

\[
Z = -\text{diag} (z_1, z_2, \cdots, z_n) \otimes I. \tag{2.41}
\]

The commutator of the matrices \( X \) and \( Z \) is given by

\[
[X, Z] = -R + ER^T, \tag{2.42}
\]

where \( R \) is the \( nr \times nr \) block diagonal matrix with \( j \)’th \( r \times r \) diagonal block equal to \( R_j \).

We see that the rational \( x \)–dependence contained in formula (2.39) comes entirely from the factor \( (X + \text{ad}V_0 \otimes I)^{-1} \). Comparison with the formulae in [W] for the Baker functions for rank–1 bispectral algebras shows the close resemblance to eq. (2.39), while eq. (2.42) is the analogue of the relation (1.5) satisfied by the associated Calogero–Moser matrices.
Although the above scheme does not guarantee reduction to scalar bispectral systems, the examples of the previous subsection show that they do so at least in particular cases. Also, although the conditions of Proposition (2,2) can be used constructively when looking for isomonodromic systems involving rational dependence on the deformation parameters, it would be more satisfying to find sufficient conditions for rationality, and for bispectrality, purely in terms of the input data consisting of the vacuum solution $\Psi_0$ and the group element $H$.

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