WHEN CAN $l_p$-NORM OBJECTIVE FUNCTIONS BE MINIMIZED VIA GRAPH CUTS?

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Abstract. Techniques based on minimal graph cuts have become a standard tool for solving combinatorial optimization problems arising in image processing and computer vision applications. These techniques can be used to minimize objective functions written as the sum of a set of unary and pairwise terms, provided that the objective function is submodular. This can be interpreted as minimizing the $l_1$-norm of the vector containing all pairwise and unary terms. By raising each term to a power $p$, the same technique can also be used to minimize the $l_p$-norm of the vector. Unfortunately, the submodularity of an $l_1$-norm objective function does not guarantee the submodularity of the corresponding $l_p$-norm objective function. The contribution of this paper is to provide useful conditions under which an $l_p$-norm objective function is submodular for all $p \geq 1$, thereby identifying a large class of $l_p$-norm objective functions that can be minimized via minimal graph cuts.

1. Introduction

Many problems in image processing and computer vision can be formulated as minimization problems, where the objective function has the following form:

$$E(x) = \sum_{i \in V} \phi_i(x_i) + \sum_{i,j \in E} \phi_{ij}(x_i, x_j)$$  \hspace{1cm} (1)

where $G = (V, E)$ is an undirected graph and $x_i$ denotes the label of vertex $v \in V$ which must belong to a finite set of integers $\{0, 1 \ldots, K - 1\}$. We assume that both the unary terms $\phi_i(\cdot)$ and the pairwise terms $\phi_{ij}(\cdot, \cdot)$ are non-negative for all $i, j$. We seek a labeling $x = (x_1, \ldots, x_{|V|})$ for which $E(x)$ is minimal.

Finding a globally optimal solution to the labeling problem described above is NP-hard in the general case, but there are classes of objective functions for which efficient algorithms exist. Specifically, for the binary labeling problem, with $K = 2$, a globally optimal solution can be computed by solving a max-flow/min-cut problem on a suitably constructed graph, provided that all pairwise terms are submodular \cite{1, 4}. A pairwise term $\phi_{ij}$ is said to be submodular if

$$\phi_{ij}(0, 0) + \phi_{ij}(1, 1) \leq \phi_{ij}(0, 1) + \phi_{ij}(1, 0).$$  \hspace{1cm} (2)

For $K > 2$, the optimization problem can not be solved directly via graph cuts. The multi-label problem can, however, be reduced to a sequence of binary valued...
labeling problems using, e.g., the expansion move algorithm proposed by Boykov et al. [1]. The output of the expansion move algorithm is a labeling that is locally optimal in a strong sense, and that is guaranteed to lie within a multiplicative factor of the global minimum [1, 3]. With this in mind, we here restrict our attention to the binary label case, i.e., $K = 2$.

Since we assume all terms to be non-negative, minimizing $E$ can be seen as minimizing the $l_1$-norm of the vector containing all unary and pairwise terms. Here, we consider the generalization of this result to arbitrary $l_p$-norms, $p \geq 1$, and thus seek to minimize

$$
\left( \sum_{i \in V} \phi^p_i(x_i) + \sum_{i,j \in E} \phi^p_{ij}(x_i, x_j) \right)^{1/p},
$$

(3)

where $\phi^p_i(\cdot) = (\phi_i(\cdot))^p$ and $\phi^p_{ij}(\cdot, \cdot) = (\phi_{ij}(\cdot, \cdot))^p$. The value $p$ can be seen as a parameter controlling the balance between minimizing the overall cost versus minimizing the magnitude of the individual terms. For $p = 1$, the optimal labeling may contain arbitrarily large individual terms as long as the sum of the terms is small. As $p$ increases, a larger penalty is assigned to solutions containing large individual terms. In the limit as $p$ goes to infinity, a labeling that minimizes Eq. 3 is a strict minimizer in the sense of Levi and Zorin [4].

It is easily seen that minimizing Eq. 3 is equivalent to minimizing

$$
\sum_{i \in V} \phi^p_i(x_i) + \sum_{i,j \in E} \phi^p_{ij}(x_i, x_j),
$$

(4)

i.e., minimizing the sum of all unary and pairwise terms raised to the power $p$. Again, this labeling problem can be solved using minimal graph cuts, provided that all pairwise terms $\phi^p_{ij}$ are submodular. Unfortunately, submodularity of $\phi_{ij}$ does not in general imply submodularity of $\phi^p_{ij}$.

The contribution of this paper is to provide useful conditions under which $\phi^p_{ij}$ is submodular. Specifically, we show that if $\phi_{ij}$ is submodular and

$$
\max(\phi_{ij}(0, 0), \phi_{ij}(1, 1)) \leq \max(\phi_{ij}(1, 0), \phi_{ij}(0, 1)),
$$

(5)

then $\phi^p_{ij}$ is submodular for all $p \geq 1$.

### 2. Conditions for the Submodularity of $\phi^p$

This section presents our main result; conditions for the submodularity of $\phi^p$. We start by establishing a lemma that is central to the definition of this condition.

**Lemma 1.** Let $a, b, c, d, p \in \mathbb{R}$, with $p > 1$ and $a, b, c, d \geq 0$. If $a + b \leq c + d$ and $\max(a, b) \leq \max(c, d)$ then $a^p + b^p \leq c^p + d^p$.

**Proof.** Showing that $a^p + b^p \leq c^p + d^p$ is equivalent to showing that $a^p + b^p - c^p - d^p \leq 0$. We assume, without loss of generality, that $a \geq b$ and $c \geq d$ so that $\max(a, b) = a$ and $\max(c, d) = c$.

\footnote{As a counterexample, consider the two-label pairwise term $\phi$ given by $\phi(0, 0) = 3$, $\phi(1, 1) = 0$, and $\phi(0, 1) = \phi(1, 0) = 2$. It is easily verified that $\phi$ is submodular, while $\phi^2$ is not.}
If \( b < d \) then \( b^p < d^p \) and, since also \( a^p \leq c^p \), the lemma trivially holds. For the remainder of the proof, we will therefore assume that \( b \geq d \). It then holds that \( c \geq a \geq b \geq d \).

If \( c = 0 \), then also \( a = b = c = 0 \) and the lemma holds. For the remainder of the proof, we will therefore assume that \( c > 0 \).

If \( a + b < c + d \), then \( c + d - a - b > 0 \) and so \( a < a + (c + d - a - b) = c + d - b \).

Let \( A = c + d - b \). Since \( d - b \leq 0 \), it holds that \( c \geq A \). Thus the numbers \( A, b, c, d \) satisfy the conditions given in the lemma: \( A + b = c + d \) and \( \max(A, b) \leq \max(c, d) \). Since \( a^p + b^p \leq A^p + b^p \) it follows that if the lemma holds for \( A, b, c, d \) then it also holds for \( a, b, c, d \). For the remainder of the proof, we will therefore assume that \( a + b = c + d \). It follows that \( b = c + d - a \) and so

\[
a^p + b^p - c^p - d^p = a^p + (c + d - a)^p - c^p - d^p. \tag{6}
\]

From the condition \( a \geq b \), it follows that \( (c + d)/2 \leq a \leq c \). Let

\[
f(x) = x^p + (c + d - x)^p \tag{7}
\]

be a function defined on the domain \( x \in [(c + d)/2, c] \). We have

\[
f'(x) = px^{p-1} - p(c + d - x)^{p-1}. \tag{8}
\]

and

\[
f''(x) = (p-1)px^{p-2} + (p-1)p(c + d - x)^{p-2}. \tag{9}
\]

Setting \( f'(x) = 0 \) yields

\[
px^{p-1} - p(c + d - x)^{p-1} = 0 \Leftrightarrow px^{p-1} = p(c + d - x)^{p-1} \tag{10}
\]

\[
\Leftrightarrow x = c + d - x \tag{11}
\]

\[
\Leftrightarrow x = (c + d)/2. \tag{12}
\]

The function \( f(x) \) thus has a single stationary point at \( x = (c + d)/2 \) which coincides with the lower bound of the function domain. Since

\[
f''((c + d)/2) = 2(p-1)p((c + d)/2)^{p-2} > 0 \tag{13}
\]

this stationary point is a local minimum. Therefore, the maximum of \( f(x) \) is attained at the upper bound of the domain \( x = c \), and so \( f(x) \leq f(c) = c^p + d^p \) on its domain.

Returning to Eq. \([6]\) we now have

\[
a^p + b^p - c^p - d^p = a^p + (c + d - a)^p - c^p - d^p \tag{14}
\]

\[
= f(a) - c^p - d^p \tag{15}
\]

\[
\leq c^p + d^p - c^p - d^p \tag{16}
\]

\[
= 0. \tag{17}
\]

This concludes the proof.

\[\square\]

**Theorem 1.** Let \( \phi \) be a submodular pairwise term. If \( \max(\phi(0,0),\phi(1,1)) \leq \max(\phi(1,0),\phi(0,1)) \), then \( \phi^p \) is also submodular, for any real \( p \geq 1 \).
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Proof. Taking $a = \phi(0,0)$, $b = \phi(1,1)$, $c = \phi(1,0)$ and $d = \phi(0,1)$, the theorem follows directly from Lemma[1]. □

3. Conclusions

We have presented a condition under which a pairwise term $\phi^p$ is submodular for all $p \geq 1$, thereby identifying a large class of $l_p$-norm objective functions that can be minimized via minimal graph cuts. The conditions derived here are easy to verify for a given set of pairwise terms, and thus make it easier to apply minimal graph cuts for solving labeling problems with $l_p$-norm objective functions, without having to explicitly prove the submodularity of the pairwise terms for each specific $p$.

It should be noted that even when there are non-submodular pairwise terms, graph cut techniques may still be used to find approximate solutions [2]. Nevertheless, submodularity remains an important property for determining the feasibility of optimizing labeling problems via minimal graph cuts.

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