**Fundamental issues of quantum theory**
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**Abstract**

Though quantum theory is very successful in my respects, there are still many serious problems such as infrared divergences and ultraviolet divergences. It seems that perturbation theory and renormalization work well. When the quantum field theory is introduced to generalize quantum mechanics, there are two major changes or involutions, so to speak, of quantum theory. One is second quantization and the other is that a system of particles instead of single particle is considered. Most parts, if not all, of quantum theory are based on the Lagrangian. In QFT, it is obvious that second quantization relations imply that the constraints are introduced the calculus of variation and Lagrange’s multipliers must be adopted. In real world a set or collection of any objects, such as a group of men or a pile of stones, we can not expect they are equally likely. We can not assume that they have the same weight. Therefore, there must be a distribution which is associated with a system of particles. It means that a meaningful quantity is adjoined in the system of particles. It seems that these concepts, constraints and distribution are ignored in conventional approach. Further more, there are two versions of quantization relations, one is prior to the field equation and the other is posterior to the field equation. And it is very difficult to find the posterior one. If the posterior one is found, of course, these two versions of quantization must be the same one. Actually, it implies that there is recursive problem. In this paper, we will discuss these serious problems.

**Introduction**

We must notice that relations of the second quantization are constraints and a system of particles instead of single particle is treated in QFT. From the convention mathematical point of view, it must be considered in quantum field theory that two major concepts should be included. One is the method of the Lagrange’s multipliers and the other is probability distribution. Though it is very difficult job to construct a complete and perfect theory, we have started to study such two topics. It was pointed out by Hung-Ming Tsai et al in QTS3 conferences that the method of the Lagrange’s multipliers should be chosen to derive the field equation in gauge field theory. It was pointed out by Hung-Ming Tsai et al in PASCOS conferences that regular basis of field operators could resolve the soft infrared divergence of massless mediators, such as Maxwell field, in QFT. The probability amplitude which is introduced in quantum mechanics shall be discussed later. It was also pointed out by Hung-Ming Tsai in SUSY 06 that the probability density function of creators and annihilators in the Maxwell field operators are $1/E$ and it is not well-defined in $[0, \infty)$. Hence it will introduce both infrared and ultraviolet divergences naturally. The fuzzy relation and thermal factor are introduced in the same time to solve the ultraviolet divergence. In this paper, we are going to study and summarize these two topics in whole.
Consider a simple mathematical expression of quantum state vector $|\phi\rangle = c_+|+z\rangle + c_-|-z\rangle$. Here, $|c_+|^2$ and $|c_-|^2$ are interpreted as the probabilities that the electron in the state $|\phi\rangle$ will be found in the states $|+z\rangle$ and $|-z\rangle$ respectively. In order to applying this interpretation to the field operator in QED, the field operator $\Psi$ is rewritten as follows

$$\Psi = \sum_{\varepsilon=E=\pm} \sum_{p} \sum_{s} (u_{p}^{s} b_{p}^{s} + \bar{v}_{p}^{s} d_{p}^{s\dagger})$$

(1)

Hence

$$\bar{\Psi} = \sum_{\varepsilon=E=\pm} \sum_{p} \sum_{s} (u_{p}^{s} b_{p}^{s} + \bar{v}_{p}^{s} d_{p}^{s\dagger}).$$

(2)

Then the length of the vectors, $\bar{u}_{p}^{s} u_{p}^{s}$ and $\bar{v}_{p}^{s} v_{p}^{s}$, must be associated with the distribution $1/(1 + \exp((E - \mu)/k_{B}T))$ for fermions. Similarly, in the Maxwell field the associated the distribution is $1/(-1 + \exp((E - \mu)/k_{B}T))$. Since there are many concerns such as Lorentz invariance and Maxwell’s relations in classical electrodynamics, it is a nontrivial work to introduce these thermal factors, the distributions. In order to solve this problem, we adopt some results of the fuzzy theory. A simple example will give a hint that will lead to introduce the fuzzy theory into the quantum theory. The example is: A class of girls is introduced to a class of boys. Probably, some of them may get married. This probability behavior could be described by fuzzy theory. Therefore, fuzzy relations are introduced between the class of field operators and the vacuum state $|0\rangle$.

The method Lagrange’s multipliers

Generally, most optimization must be solved by the method Lagrange’s multipliers. The simple elementary example of calculus will show the result. Let $u = x^2 + 2x + 3y^4 - y + z^2 + z - 50$. In order to find the relative extreme values of $u$, it will yield the same result when the relations $xy = yx$, $yz = zy$ and $zx = xz$ are treated as constraints though the constraints are ignored in usual computations. But in functional calculus, we get new results when the imposed conditions or relations are treated as constraints.

The Hamiltonian operator and the Einstein’s energy equation

Based on the Lagrangian of a quantum system, the Hamilton’s operator can be obtained. The eigenvalue of Hamilton’s operator is the energy of the system. In gauge theory there are two kinds of fields. Each one is associated with a class of particles. Therefore, there must be more energy operators. In speaking of energy, the Einstein’s energy equation, $E^2 = p^2 + m^2$, plays an important role in beginning of developing the new field equation. Most free field equations, Klein-Gordon equation and Dirac equation, are obtained from the Einstein’s energy equation. Since there is only energy which is considered in quantum mechanics, the first quantization can give the energy operator. And hence the energy of the system is the eigenvalue of this energy operator. In gauge field theory there are at least two kinds of particles. Each particle has its energy equation. Clearly, there are more energy operators than before. Therefore, the field equations should be modified in the interaction field. It is the Lagrange’s multipliers that
make the field equations adjustable to fit the Einstein’s energy equation. The Einstein’s energy equation must be satisfied no matter whether the particles are free or not. In gauge field theory, the Dirac field equation deviates from free one. In the gauge field theory, if we apply the method Lagrange’s multipliers, then we are able to obtain the same field equation as free one. When energy is mentioned, it must be specified which energy, the energy of the system, the energy of electron or the energy of photon, is considered. The energy of the whole system must be derived from the Lagrangian no matter whether the quantization relations are treated as constraints or not. Now we quote some results of Hung-Ming Tsai in SUSY 06 to show how the method of Lagrange’s multipliers applying in the gauge field theory.

In order to study a simple gauge field theory, QED is taken as an example. The Lagrangian is not stated here, the quantization relations are defined by the differential equations.

Let

$$ S^M_{ij}(x - x', t - t') = \{\Psi_i(x, t), \overline{\Psi}_j(x', t')\}. $$

(3)

In order to solve the recursive problem, we must have some prior assumption. From the result of free field, we have the prior information as such

$$ (i\gamma^\mu \partial_\mu - m)_{\alpha\gamma} S^M_{\gamma\beta}(x - x', t - t') = 0, $$

with some initial condition

$$ S^M_{ij}(x - x', 0) = \delta_{ij} \delta(x - x'), $$

(4)

and

$$ \{\Psi_i(x, t), \Psi_j(x', t')\} = \{\overline{\Psi}_i(x, t), \overline{\Psi}_j(x', t')\} = 0. $$

(5)

By applying the method of Lagrange’s multiplier, the field equation can be obtained from the Lagrangian and the quantization relations

$$ i\gamma^\mu \partial_\mu \Psi - m \Psi + \Lambda \Psi = q\gamma^\mu A_\mu, $$

(7)

where \( \Lambda \) is a matrix of which the entries are the Lagrange’s multipliers. Now we are going to determine the entries of \( \Lambda \). Multiply both sides of the equation (7) by \( \overline{\Psi}(x', t') \) from the left, multiply both sides of the equation (7) by \( \overline{\Psi}(x', t') \) from the right and take the sum of them. We get

$$ (i\gamma^\mu \partial_\mu - m + \Lambda)_{\alpha\gamma} S^M_{\gamma\beta} = (q\gamma^\mu A_\mu)_{\alpha\gamma} S^M_{\gamma\beta}. $$

(8)

From (4) and (8), we get

$$ \Lambda_{\alpha\gamma} S^M_{\gamma\beta} = (q\gamma^\mu A_\mu)_{\alpha\gamma} S^M_{\gamma\beta}. $$

(9)

Setting \( t \) to be \( t' \) and taking the three dimensional integration of \( x' \), we get

$$ \Lambda_{\alpha\beta} = (q\gamma^\mu A_\mu)_{\alpha\beta}. $$

(10)
since \( S^M_{\alpha\beta}(x-x',0) = \delta_{\alpha\beta}\delta(x-x') \). Therefore, the field equations are

\[
(i\gamma^\mu \partial_\mu - m)\Psi = 0, \tag{11}
\]

and

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2} \right)A^\mu = q\gamma^\mu \Psi. \tag{12}
\]

Clearly, the energy of the electron which is obtained from the Dirac field equation does not deviate from the Einstein’s energy equation. We stress it again that the energy operator of the whole system must be derived from the Lagrangian, the Hamiltonian. When this operator apply to the vacuum state \(|0\rangle\), the energy of the energy of the system is obtained. It is obvious the constraints, the quantization relation must be obeyed during the computation process. It is of no doubt that the result is the same as before, if the thermal factor is ignored. But the whole process is more consistency in the sense of mathematics.

In order to describe the natural phenomenon, the distributions are introduced in the relations between the field operators and the vacuum state \(|0\rangle\). When a non-constant polynomial of the whole field operators \( \Psi \) and \( \bar{\Psi} \) operates on the vacuum state \(|0\rangle\), there exists a fuzzy relation between the set of operators, such as creators, annihilators and their products in this polynomial, and the vacuum state \(|0\rangle\). Hence this operation yields a distribution which is inserted in the conventional operation \( \Psi |0\rangle \).

**Fuzzy Theory and the Distribution**

According to the spin of the particles, there are two distributions which are associated with the particles. Let the \( T_f \) be defined

\[
T_f = \sqrt{1/(1 + \exp((E-\mu)/k_B T))}\sqrt{m/E}. \tag{13}
\]

When the field \( \Psi \) is spin 1/2, from the fuzzy relation, the operation \( \Psi |0\rangle \) shall yield the outcome result

\[
\Psi |0\rangle = \sum_p \sum_s \frac{N}{(2\pi)^{3/2}} T_f(b_p^s |0\rangle u^s(p) \exp \varphi + d_p^{s\dagger} |0\rangle v^s(p) \exp(-\varphi)).
\]

where \( \varphi = i(p \cdot x - Et) \), \( E = \sqrt{m^2 + |p|^2} \) and \( N \) is a factor for normalization.

**The Parameter of System with Distribution**

When Hung-Ming Tsai, Po-Yu Tsai and Lu-Hsing Tsai was organizing the paper for the Proceedings of the 3rd International Symposium on Quantum Theory and Symmetries \textbf{QTS3}, Cincinnati, Ohio, 10-14 September 2003, they discovered the distribution of a system of particles must be considered. Therefore, the behavior of the system depends on some parameters such as chemical potential and temperature. Therefore, most measured quantities must be temperature dependent in QFT. In order to introduce these parameters, the fuzzy relation is adopted. Due to the fuzzy relations between the field operators and the vacuum state \(|0\rangle\), the thermal factors \( \alpha \sqrt{1/(1 + \exp((E-\mu)/k_B T))} \) is taken for fermions and \( \beta \sqrt{1/(-1 + \exp((E-\mu)/k_B T))} \) is taken for bosons. Usually, \( \alpha = 1 \) for Dirac field, \( \beta = 1/\omega_k \) for massless Maxwell field. The complete
work are shown in the 14th International Conference on Supersymmetry and the Unification of Fundamental Interactions SUSY ’06, University of California, Irvine, California, June 12–17 2006. This is their major contributions in quantum field theory since the ultraviolet divergence and infrared divergence shall be removed by fuzzy relation. It seems that the work is completed when quantization relations, the Lagrange’s multipliers, fuzzy relation and thermal factors are all put together. Actually, there some problems must be studied since there is nothing to do with the uncertainty principle in theory relativity and hence there is no uncertainty in the energy equation. It will be all right in sense of uncertainty principle if the life time of particles, electron, proton etc., are almost infinite, that is, $\Delta t = \infty$. Hence that $\Delta E = 0$ is possible.

**Uncertainty Principle and the Mediators**

Based on the uncertainty principle, Yukawa predicted not only existence of the pie mesons but also the mass of these mediators. Using the method of the Lagrange’s multipliers, we obtain the field equations in QED are

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0, \quad (14)$$

and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial t^2}\right)A^\mu = q\bar{\Psi}\gamma^\mu \Psi. \quad (15)$$

We find the equation (14), the Dirac field equation, is different from the Einstein’s energy equation, $E^2 = p^2 + m^2$, while the equation (15), the Maxwell equation, deviates from the Einstein’s energy equation, $E^2 = p^2 + m^2$. We argue that this term, $q\bar{\Psi}\gamma^\mu \Psi$, in the right hand side of equation (15) is the balance term of the Einstein’s energy equation, $E^2 = p^2 + m^2$. Since life time of mediators is not infinite, there must be some uncertainty in this equation.

**Conclusion and Discussion**

To construct a consistent theory is one thing, to verify this theory is another thing. It takes a long time to verify the theory by experimental results. Though the string theory is nice theory, there is no experimental result to verify it so far. Hung-Ming Tsai, Po-Yu Tsai and Lu-Hsing Tsai have predicted that most meaningful quantities derived in QFT such as energy levels of hydrogen atoms, gyromagnetic ratio of electron and muon etc. might be dependent on the temperature $T$ if the temperature is well defined. It also takes long time to verify whether it is right or not.

In the Dirac field the thermal factor $\alpha\sqrt{1/(1 + \exp((E - \mu)/k_B T))}$, $\alpha$ is a constant, $\alpha = 1$, while in the Maxwell equation the thermal factor $\beta\sqrt{1/(-1 + \exp((E - \mu)/k_B T))}$, $\beta$ is momentum $k$ and hence energy $E$ dependent, $\beta = 1/\omega_k$. If we force that $\beta$ is a constant, $\beta = 1$, then the fractional operator might be introduced. Let the operator $D_t$ be defined $D_t = \partial/\partial t$. Let $D_t^{1/2}D_t^{1/2} = D_t$. Obviously, we are going to adopt the fractional functional calculus since $D_t^{1/2}$ will be introduced into QFT.

Let the Maxwell field be $A^\mu$. Let $A_{1/2}^\mu = D_t^{1/2}A^\mu$. Ignoring the gauge fixed, radiation gauge or Lorentz gauge, $A_{1/2}^\mu$ satisfy the Maxwell field equa-
tion, the wave equation. Roughly speaking, in gauge field theory, $A^\mu$ can be replaced by $A^\mu_{1/2}$, since the invariance of local gauge transformation implies there must be some adjoint field $A^\mu$ which, the symbols in sense of mathematical operation, collaborates with gauge transformation. The new field satisfies the wave equation. Clearly, the adjoint field (symbol) is not unique. The new field $A^\mu_{1/2}$ is chosen instead of $A^\mu$, then there will be no infrared divergence if $\beta$ in $\beta \sqrt{1/(-1 + \exp((E - \mu)/k_B T))}$ is a constant, $\beta = 1$. Since some problems, such as the covariance in the Lorentz transformation and homogenous dimension of each term in the Lagrangian and the field equation must be solved, some new matrix must be introduced in the Lagrangian. This is not a simple work. Fractional calculus is very popular in the applied mathematics. It takes time to develop the new theory of QFT by fractional functional calculus.

In this paper, we try to make quantum field theory more consistent in the senses of mathematics and physics. It is not an occasional case to introduce the thermal factor in constrained QFT because most distributions of particles are derived by optimizing an objective function with constraints. These constraints are: the total energy of the system of particles is equal to a finite constant and the total number of the system of particles is equal to a finite constant. Finally, the temperature and chemical potential of the system are obtained directly or indirectly from the Lagrange’s multipliers of these constraints.

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