1. Introduction

In the past few years, classical solitonic solutions in string theory with higher-membrane structure have been actively investigated. These solutions are static multi-soliton solutions obeying a zero-force condition and saturating a Bogomol’nyi bound between mass and charge. In certain cases, exact solutions of bosonic and heterotic string theory may be constructed, each solution in principle corresponding to an exact conformal field theory (CFT) of the sigma-model. Although the solutions are initially conceived as perturbative expansions in the classical string parameter $\alpha'$ (the inverse string tension), the exact solutions acquire nonperturbative status. Being classical, the solitons are tree-level solutions in the (quantum) topological expansion of the string worldsheet, but are also nonperturbative in the loop parameter $e^{\phi_0}$. Therefore, full quantum string-loop extensions of these solutions await an understanding of nonperturbative string theory. Nevertheless, it is possible to use these solitons in nonperturbative calculations (such as vacuum-tunneling) since it is often the case that higher order corrections do not contribute to these effects.

In this work we confine ourselves to three classes of solitons, concentrating mainly on the construction of tree-level or exact static, multi-solitonic solutions in bosonic or heterotic string theory\textsuperscript{1}. We also briefly discuss the dynamics of these solitons\textsuperscript{2}. The three classes of solitons we consider are: instanton solutions, possessing instanton structure and local four-dimensional spherical symmetry, monopole solutions, possessing magnetic monopole features and local three-dimensional spherical symmetry, and “cosmic string” solutions, which represent extended states of fundamental strings and correspond to eight-dimensional instantons in $D = 10$. Both
the monopole and instanton solutions have “fivebrane” structure in $D = 10$ (i.e. they are $5 + 1$-dimensional objects in $9 + 1$-dimensional spacetime).

The motivation for the study of these solutions includes the following: to begin with, these solitons represent solutions of string theory in curved backgrounds, in contrast to the usual flat Minkowski background solutions. Another interest in these solutions stems from the application of known field-theoretic techniques (and their stringy analogs) in the physics of solitons and instantons to string theory. For example, the string instanton solutions, through vacuum tunneling computations, may lead to an understanding of the structure of the vacuum in string theory, much in the same manner as instantons are used in field theory. The string monopole solutions may appear in grand unification predictions of string theory, while the macroscopic string solitons may be used to represent extended string states. An especially noteworthy feature of the monopoles shown here is the cancellation between gauge and gravitational singularities in the action, a feature, which, if it survives quantization, promises to shed light on the nature of string theory as a finite theory of quantum gravity. A different point of view to these solutions is the study of the resultant string-inspired low-energy field theories, which may well capture the essential behaviour of these solitons (e.g. the singularity cancellation in the monopole solutions) without requiring an expansion in the full string theory. Finally, there is the open problem of proving the string/fivebrane duality conjecture, and its implications to low energy string theory, the singularity structure of string theory and its relation to the more familiar electric/magnetic duality conjecture of Montonen and Olive.

2. Bosonic and Heterotic Instantons

We first summarize the ’t Hooft ansatz for the Yang-Mills instanton, and then write down the tree-level bosonic axionic instanton solution. An exact bosonic solution, with corresponding CFT can be obtained for the special case of a linear dilaton wormhole. Finally, an exact multi-instanton solution of heterotic string theory is obtained, combining the Yang-Mills gauge solution with the bosonic axionic instanton.

Consider the four-dimensional Euclidean action

$$ S = -\frac{1}{2g^2} \int d^4x \text{Tr} F_{\mu\nu} F^{\mu\nu}, \quad \mu, \nu = 1, 2, 3, 4. \tag{1} $$

For gauge group $SU(2)$, the fields may be written as $A_\mu = (g/2i)\sigma^a A_a^\mu$ and $F_{\mu\nu} = (g/2i)\sigma^a F^a_{\mu\nu}$ (where $\sigma^a$, $a = 1, 2, 3$ are the $2 \times 2$ Pauli matrices). A self-dual solution (but not the most general one) to the equation of motion of this action is given by the ’t Hooft ansatz

$$ A_\mu = i\overline{\Sigma}_{\mu\nu} \partial_\nu \ln f, \tag{2} $$

where $\overline{\Sigma}_{\mu\nu} = \overline{\eta}^{\mu\nu} (\sigma^i / 2)$ for $i = 1, 2, 3$, where

$$ \overline{\eta}^{\mu\nu} = -\overline{\eta}^{\nu\mu} = \epsilon^{\mu\nu}, \quad \mu, \nu = 1, 2, 3, $$

$$ = -\delta^{\mu\nu}, \quad \nu = 4 \tag{3} $$
and where $f^{-1} \Box f = 0$. The ansatz for the anti-self-dual solution is similar, with the $\delta$-term in Eq. (3) changing sign. To obtain a multi-instanton solution, one solves for $f$ in the four-dimensional space to obtain

$$f = 1 + \sum_{i=1}^{N} \frac{\rho_i^2}{|x - \vec{a}_i|^2},$$

(4)

where $\rho_i^2$ is the instanton scale size and $\vec{a}_i$ the location in four-space of the $i$th instanton.

It turns out that there is an analog to the Yang-Mills instanton in the gravitational sector of the string, namely the axionic instanton\(^7\). In its simplest form, this instanton appears as a solution for the massless fields of the bosonic string. The bosonic sigma model action can be written as

$$I = \frac{1}{4\pi \alpha'} \int d^2 x \left( \sqrt{\gamma} \gamma^{ab} \partial_a \varphi \partial_b \varphi + ie^{\alpha} \partial_a x^\mu \partial_b x^\nu B_{\mu\nu} + \alpha' \sqrt{\gamma} R(2) \phi \right),$$

(5)

where $g_{\mu\nu}$ is the sigma model metric, $\phi$ the dilaton and $B_{\mu\nu}$ the antisymmetric tensor, and where $\gamma_{ab}$ is the worldsheet metric and $R(2)$ the two-dimensional curvature. The classical equations of motion of the effective action of the massless fields is equivalent to Weyl invariance of $I$. A tree-level solution is given by any dilaton function satisfying $e^{-2\phi} e^{2\phi} = 0$ with

$$g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu} \quad \mu, \nu = 1, 2, 3, 4,$$

$$g_{ab} = \delta_{ab} \quad a, b = 5, \ldots, 26,$$

$$H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda\sigma} \partial^{\sigma} \phi \quad \mu, \nu, \lambda = 1, 2, 3, 4.$$  

(6)

In order to see the instanton structure of this solution, we define a generalized curvature $\hat{R}^i_{jkl}$ in terms of the standard curvature $R^i_{jkl}$ and $H_{\mu\nu\lambda}$:

$$\hat{R}^i_{jkl} = R^i_{jkl} + \frac{1}{2} \left( \nabla_l \nabla^i j_k - \nabla_k \nabla^i j_l \right) + \frac{1}{4} \left( H^{mk} j_k H^i_{lm} - H^m j_k H^i_{lm} \right).$$

(7)

One can also define $\hat{R}^i_{jkl}$ as the Riemann tensor generated by the generalized Christoffel symbols $\hat{\Gamma}^\mu_{\alpha\beta}$, where $\hat{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - (1/2)H^\mu_{\alpha\beta}$. The crucial observation for obtaining higher-loop and even exact solutions is the following. For any solution given by Eq. (6), we can express the generalized curvature in covariant form in terms of the dilaton field as

$$\hat{R}^i_{jkl} = \delta_{il} \nabla_k \nabla^i j_k - \delta_{ik} \nabla_l \nabla^i j_k + \delta_{jk} \nabla_i \nabla_l \phi - \delta_{jl} \nabla_k \nabla_i \phi \pm \epsilon_{ijkl} \nabla_l \nabla_m \phi \mp \epsilon_{ijkm} \nabla_l \nabla_m \phi,$$  

(8)

It easily follows that

$$\hat{R}^i_{jkl} = \pm \frac{1}{2} \epsilon_{kl} \epsilon_{mn} \hat{R}^i_{jmn}.$$  

(9)

So the instanton appears in the gravitational sector of the string in the (anti) self-duality of the generalized curvature. A tree-level multi-instanton solution is therefore given by Eq. (6) with the dilaton given by

$$e^{2\phi} = C + \sum_{i=1}^{N} \frac{Q_i}{|x - \vec{a}_i|^2},$$

(10)
where \( Q_i \) is the charge and \( \vec{a}_i \) the location in the four-space \((1234)\) (the transverse space) of the \( i \)th instanton. In the spherically symmetric case \( e^{2\phi} = Q/r^2 \), we can explicitly solve the higher order equations of motion by rescaling the dilaton and fixing the metric and antisymmetric tensor in lowest order form. For example, the two-loop dilaton is given by \( e^{2\phi} = Q/r^{2(1-\frac{2}{3})} \). To get an exact solution in this linear dilaton case, we notice that the sigma-model action can be decomposed according to 

\[
I = I_1 + I_3,
\]

where for \( u = 1/r \)

\[
I_1 = \frac{1}{4\pi\alpha'} \int d^2x \left( Q(\partial u)^2 + \alpha' R^{(2)} \phi \right)
\]

is the action for a Feigin-Fuchs Coulomb gas, a one-dimensional CFT with central charge given by \( c_1 = 1 + 6\alpha'(\partial\phi)^2 \) and \( I_3 \) is the Wess–Zumino–Witten action on an \( SU(2) \) group manifold with central charge

\[
c_3 = \frac{3k}{k+2} \simeq 3 - \frac{6}{k} + \frac{12}{k^2} + \ldots
\]

where \( k = Q/\alpha' \), the level of the WZW model, is an integer, from the quantization condition on the Wess-Zumino term. Thus \( Q \) is not arbitrary, but is quantized in units of \( \alpha' \).

We use this splitting to obtain exact expressions for the fields by fixing the metric and antisymmetric tensor field in their lowest order form and rescaling the dilaton to all orders in \( \alpha' \). The resulting expression for the dilaton is

\[
e^{2\phi} = \frac{Q}{r^{1+\frac{2}{k}}}
\]

for arbitrary \( \alpha' \).

We now turn to the heterotic solution. The tree-level supersymmetric vacuum equations for the heterotic string are given by

\[
\delta \psi_M = (\nabla_M - \frac{1}{4} H_{MAB} \Gamma^{AB}) \epsilon = 0,
\]

\[
\delta \lambda = (\Gamma^A \partial_A \phi - \frac{1}{2} H_{AMC} \Gamma^{ABC}) \epsilon = 0,
\]

\[
\delta \chi = F_{AB} \Gamma^{AB} \epsilon = 0,
\]

where \( \psi_M, \lambda \) and \( \chi \) are the gravitino, dilatino and gaugino fields. The Bianchi identity is given by

\[
dH = \alpha' \left( \text{tr} R \wedge R - \frac{1}{30} \text{Tr} F \wedge F \right).
\]

The \((9 + 1)-\)dimensional Majorana-Weyl fermions decompose down to chiral spinors according to \( SO(9,1) \supset SO(5,1) \otimes SO(4) \) for the \( M^{9,1} \rightarrow M^{5,1} \times M^4 \) decomposition. Let \( \mu, \nu, \lambda, \sigma = 1, 2, 3, 4 \) and \( a, b = 0, 5, 6, 7, 8, 9 \). Then the ansatz

\[
g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu},
\]

\[
g_{ab} = \eta_{ab},
\]

\[
H_{\mu\nu\lambda} = \pm \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \phi
\]

(16)
with constant chiral spinors $\epsilon_\pm$ solves the supersymmetry equations with zero background fermi fields provided the YM gauge field satisfies the instanton (anti)self-duality condition

$$F_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu}^{\lambda\sigma} F_{\lambda\sigma}. \quad (17)$$

A perturbative solution representing a supersymmetric fivebrane was first derived by Strominger$^9$. In the absence of a gauge sector, the multi-fivebrane solution is identical to the tree-level type II supersymmetric fivebrane solution of Duff and Lu$^{10}$, derived in terms of the dual seven-form formulation of supergravity, with $K = e^{-2\phi} H = dA$, where $A$ is the antisymmetric six-form associated with a fivebrane. An exact solution is obtained as follows. Define a generalized connection by

$$\Omega_{\pm M} = \omega_{M}^{AB} \pm H_{AB} \quad (18)$$

embedded in an SU(2) subgroup of the gauge group, and equate it to the gauge connection $A_\mu$ so that $dH = 0$ and the corresponding curvature $R(\Omega_{\pm})$ cancels against the Yang-Mills field strength $F$. As in the bosonic case, for $e^{-2\phi} e^{2\phi} = 0$ with the above ansatz, the curvature of the generalized connection can be written in covariant form in terms of the dilaton as in Eq. (8) from which it follows that both $F$ and $R$ are (anti)self-dual. This solution becomes exact since $A_\mu = \Omega_{\pm \mu}$ implies that all the higher order corrections vanish. The self-dual solution for the gauge connection is then given by the 't Hooft ansatz. An interesting feature of the heterotic solution is that it combines a YM instanton structure in the gauge sector with an axionic instanton structure in the gravity sector. In addition, the heterotic solution has finite action.

Note that the single instanton solution in the heterotic case carries through to higher order without correction to the dilaton. This seems to contradict the bosonic solution by suggesting that the expansion for the central charge $c_3$ terminates at one loop. This contradiction is resolved by noting that for an $N = 4$ worldsheet supersymmetric solution$^{13}$ the bosonic contribution to the central charge is given by

$$c_3 = \frac{3k'}{k' + 2}, \quad (19)$$

where $k' = k - 2$. This reduces to

$$c_3 = 3 - \frac{6}{k} = 3 - \frac{6\alpha'}{Q}, \quad (20)$$

which indeed terminates at one loop order. The exactness of the splitting then requires that $c_1$ not get any corrections from $(\partial \Phi)^2$ so that $c_1 + c_3 = 4$ is exact for the tree-level value of the dilaton.$^7,^{13}$

3. Heterotic Multimonopole

We now turn to a solution with monopole-like structure. We begin with a simple modification of the 't Hooft ansatz which leads to a multimonopole solution.
in field theory, not in the BPS limit. In analogy with the previous section, we obtain an exact heterotic multimonopole solution\(^\text{14}\).

Going back to the ’t Hooft ansatz (Eq. (1)-(3)), if we single out a direction in the transverse four-space (say \(x_4\)) and assume all fields are independent of this coordinate. Then the solution for \(f\) satisfying \(f^{-1}
abla f = 0\) can be written as

\[
f = 1 + \sum_{i=1}^{N} \frac{m_i}{|\vec{x} - \vec{a}_i|},
\]

where \(m_i\) is the charge and \(\vec{a}_i\) the location in the three-space (123) of the \(i\)th monopole. If we make the identification \(\Phi \equiv A_4\) then the Lagrangian density for the above ansatz can be rewritten as

\[
F_{\mu\nu}^a F_{\mu\nu}^a = F_{ij}^a F_{ij}^a + 2 F_{k4}^a F_{k4}^a = F_{ij}^a F_{ij}^a + 2 D_\mu \Phi^a D^\mu \Phi^a, \quad (22)
\]

which has the same form as the Lagrangian density for YM + massless scalar field in three dimensions. We now go to 3 + 1 dimensions with the Lagrangian density (signature \((-+++))

\[
L = -\frac{1}{4} G_{\mu\nu}^a G^{\mu\nu a} - \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a. \quad (23)
\]

It follows that the above multimonopole ansatz is a static solution with \(A_0^a = 0\) and all time derivatives vanish (it is straightforward to verify the equations of motion). The solution in 3 + 1 dimensions has the form

\[
\Phi^a = \pm \frac{1}{g} \delta^{ai} \partial_i \omega,
\]

\[
A_\mu^a = \epsilon^{akj} \partial_j \omega, \quad (24)
\]

where \(\omega \equiv \ln f\). This solution represents a multimonopole configuration with sources at \(\vec{a}_i = 1, 2\ldots N\). A simple observation of far field and near field behaviour shows that this solution does not arise in the Prasad-Sommerfield limit. In particular, the fields are singular near the sources and vanish as \(r \to \infty\). This solution can be thought of as a multi-line source instanton solution, each monopole being interpreted as an “instanton string”.

The topological charge of each source is easily computed (\(\hat{\Phi}^a \equiv \Phi^a/|\Phi|\)) to be

\[
Q = \int d^3 x \kappa_0 = \frac{1}{8\pi} \int d^3 x \epsilon_{ijk} \epsilon^{abc} \partial_i \hat{\Phi}^a \partial_j \hat{\Phi}^b \partial_k \hat{\Phi}^c = 1. \quad (25)
\]

The magnetic charge of each source is then given by \(m_i = Q/g = 1/g\). It is also straightforward to show that the Bogomol’yi bound

\[
G_{ij}^a = \epsilon_{ijk} D_k \Phi^a \quad (26)
\]

is saturated by this solution. Finally, it is easy to show that the magnetic field \(B_i = \frac{1}{2} \epsilon_{ijk} G_{jk}^a\) (where \(G_{\mu\nu}^a \equiv \hat{\Phi}^a F_{\mu\nu}^a - (1/g) \epsilon^{abc} \hat{\Phi}^a D_\mu \hat{\Phi}^b D_\nu \hat{\Phi}^c\) is the gauge-invariant electromagnetic field tensor defined by ’t Hooft) has the the far field limit behaviour of a multimonopole configuration:

\[
B(\vec{x}) \to \sum_{i=1}^{N} \frac{m_i (\vec{x} - \vec{a}_i)}{|\vec{x} - \vec{a}_i|^3}, \quad \text{as} \quad r \to \infty. \quad (27)
\]
As usual, the existence of this static multimonopole solution owes to the cancellation of the gauge and Higgs forces of exchange—the “zero-force” condition.

We have presented all the monopole properties of this solution. Unfortunately, this solution as it stands has divergent action near each source, and this singularity cannot be simply removed by a unitary gauge transformation. This can be seen for a single source by noting that as \( r \to 0 \), \( A_k \to \frac{1}{2} (U^{-1} \partial_k U) \), where \( U \) is a unitary \( 2 \times 2 \) matrix. The expression in parentheses represents a pure gauge, and there is no way to get around the \( 1/2 \) factor in attempting to “gauge away” the singularity. The field theory solution is therefore not very interesting physically.

In analogy with the previous section, we can write down a monopole-like solution in bosonic string theory and obtain higher-order corrections in the spherically symmetric case by rescaling the dilaton. However, a CFT description is elusive in this case as there is no corresponding natural splitting of the sigma-model action. We turn instead to the heterotic multimonopole solution. The derivation of this solution closely parallels that of the heterotic multi-instanton, but in this case, the solution possesses three-dimensional (rather than four-dimensional) spherical symmetry near each source. The reduction is effected by singling out a direction in the transverse space. An exact solution is now given by

\[
\begin{align*}
  g_{\mu\nu} &= e^{2\phi} \delta_{\mu\nu}, \quad g_{ab} = \eta_{ab}, \\
  H_{\mu\nu\lambda} &= \pm \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma \phi, \\
  e^{2\phi} &= e^{2\phi_0} f, \\
  A_\mu &= i \sum_{\mu} \partial_\mu \ln f,
\end{align*}
\]

(28)

where in this case \( f \) is given by Eq. (21). If we again identify the scalar field as \( \Phi \equiv A_4 \), then the gauge and scalar fields may be simply written in terms of the dilaton as

\[
\begin{align*}
  \Phi^a &= -\frac{2}{g} \delta^a_i \partial_i \phi, \\
  A^a_k &= -\frac{2}{g} \epsilon^{akj} \partial_j \phi
\end{align*}
\]

(29)

for the self-dual solution. For the anti-self-dual solution, the scalar field simply changes sign. Here \( g \) is the YM coupling constant of the heterotic string. Note that \( \phi_0 \) drops out in Eq. (29).

The above solution (with the gravitational fields obtained directly from Eqs. (21) and (28)) represents an exact multimonopole solution of heterotic string theory and has the same structure in the four-dimensional transverse space as the above multimonopole solution of the YM + scalar field action. If we identify the (123) subspace of the transverse space as the space part of the four-dimensional spacetime (with some appropriate toroidal compactification) and take the timelike direction as the usual \( X^0 \), then the monopole properties of the field theory solution carry over directly into the string solution. Note that the metric line-element resembles that of the Kaluza-Klein monopole\(^{15} \). The string monopole solution is
stable as a result of its saturation of a Bogomol’nyi bound between ADM mass and charge\textsuperscript{16}.

The string action contains a term $-\alpha' F^2$ which also diverges as in the field theory solution. However, this divergence is precisely cancelled by the term $\alpha' R_\pm^2(\Omega_\pm)$ in the $O(\alpha')$ action. This result follows from the exactness condition $A_\mu = \Omega_{\pm\mu}$ which leads to $dH = 0$ and the vanishing of all higher order corrections in $\alpha'$. Another way of seeing this is to consider the higher order corrections to the bosonic action\textsuperscript{12}. All such terms contain the tensor $T_{MN\Phi Q}$, a generalized curvature incorporating both $R(\Omega_\pm)$ and $F$. The ansatz is constructed precisely so that this tensor vanishes identically\textsuperscript{7,14}. The action thus reduces to its finite lowest order form and can be calculated directly for a multi-source solution from the expressions for the massless fields in the gravity sector.

The divergences in the gravitational sector in heterotic string theory thus serve to cancel the divergences stemming from the field theory solution. This solution thus provides an interesting example of how this type of cancellation can occur in string theory, and supports the promise of string theory as a finite theory of quantum gravity. Another point of interest is that the string solution represents a supersymmetric multimonopole solution coupled to gravity, whose zero-force condition in the gravity sector (cancellation of the attractive gravitational force and repulsive antisymmetric field force) arises as a direct result of the zero-force condition in the gauge sector (cancellation of gauge and Higgs forces of exchange) once the gauge connection and generalized connection are identified.

4. String Solitons

In recent work\textsuperscript{17}, Dabholkar \textit{et al}. presented a low-energy analysis of macroscopic superstrings and discovered several interesting analogies between macroscopic superstrings and solitons in supersymmetric field theories. The main result of this work centers on the existence of exact multi-string solutions of the low-energy supergravity super-Yang-Mills equations of motion. In addition, Dabholkar \textit{et al}. find a Bogomolnyi bound for the energy per unit length which is saturated by these solutions, just as the Bogomolnyi bound is saturated by magnetic monopole solutions in ordinary Yang-Mills field theory. The solution may be outlined as follows. The action for the massless spacetime fields (graviton, axion and dilaton) in the presence of a source string can be written as

$$ S = \frac{1}{2\kappa^2} \int d^Dx \sqrt{g} \left( R - \frac{1}{2}(\partial \phi)^2 - \frac{1}{12} e^{-2\alpha\phi} H^2 \right) + S_\sigma, \quad (30) $$

with the source terms contained in the sigma model action $S_\sigma$ given by

$$ S_\sigma = -\frac{\mu}{2} \int d^2\sigma (\sqrt{\gamma} \gamma^{mn} \partial_m X^\mu \partial_n X^\nu g_{\mu\nu} e^{\alpha\phi} + \epsilon^{mn} \partial_m X^\mu \partial_n X^\nu B_{\mu\nu}), \quad (31) $$

with $\alpha = \sqrt{2/(D-2)}$ and $\gamma_{mn}$ a worldsheet metric to be determined. The sigma model action $S_\sigma$ describes the coupling of the string to the metric, antisymmetric tensor
field and dilaton. The first part of the action $S$ above represents the effective action for the massless fields in the spacetime frame and whose equations of motion are equivalent to conformal invariance of the underlying sigma model. The combined action thus generates the equations of motion satisfied by the massless fields in the presence of a macroscopic string source. The static solution to the equations of motion is given by

$$
\begin{align*}
\phi &= \alpha E(r) \\
B &= -2D-2E(r)
\end{align*}
$$

where $x^1$ is the direction along the string, $r = \sqrt{\vec{x} \cdot \vec{x}}$ and

$$
e^{-E(r)} = \begin{cases} 
1 + \frac{M}{18G_{\mu\nu} \ln(r)} & D > 4 \\
1 - \frac{M}{12G_{\mu\nu} \ln(r)} & D = 4
\end{cases}
$$

for a single static string source. The solution can be generalized to an arbitrary number of static string sources by linear superposition of solutions of the $(D-2)$-dimensional Laplace’s equation. A similar solution incorporating the eight-dimensional instanton of Grossman, Kephart and Stasheff in the gauge sector was derived by Duff and Lu18.

The force exerted on a test string moving in given background fields is obtained from the sigma model equation of motion

$$
\nabla_m (\gamma^{mn} \nabla_n \gamma^{\mu}) = -\Gamma^\mu_{\nu\rho} \partial_m \gamma^{\nu} \partial_n \gamma^{\rho} + \frac{1}{2} H^\mu_{\nu\rho} \partial_m \gamma^{\nu} \partial_n \gamma^{\rho} e^{\gamma^{mn}},
$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols calculated from the sigma model metric $G_{\mu\nu} = \gamma_{\mu\nu} e^{\alpha\phi}$. We make the usual distinction between the sigma model metric and the Einstein metric; spacetime indices are raised and lowered by contraction with $G_{\mu\nu}$; worldsheet indices are denoted by $m$ and $n$. Consider a stationary test string in the background of a source string located at the origin. Assume further that both strings run along the $x^1$ direction and have the same orientation. We use conformal gauge for the test string and get $X^0 = \tau$, $X^1 = \sigma$, $\gamma_{mn} = \text{diag}(-1, +1)$ and $\epsilon^{01} = +1$. The transverse force then vanishes

$$
\frac{d^2}{d\tau^2} X^i = -2\Gamma^i_{00} + H^i_{00} = 0.
$$

Note that if the test string and source string were oppositely oriented then the second term would appear with a negative sign and there would be a net attractive force. Also note that the no-force condition depends only on the general ansatz and not on the precise form of the solution.

The zero-force condition arises from the cancellation of long-range forces of exchange of the massless fields of the string (the graviton, axion and dilaton) and can be seen explicitly from Eq. (34). This is a perfect analog to the zero-force
condition of Manton for magnetic monopoles, which requires that the attractive scalar exchange force precisely cancel the repulsive vector exchange force when the Bogomolnyi bound is attained. Dabholkar et al. show that a similar Bogomolnyi bound is satisfied by their string soliton solutions, further strengthening the analogy with the monopoles. In the next section, however, we shall see that in contrast to BPS monopoles, the string solitons also obey a zero dynamical force condition.

5. Dynamics

We now summarize some recent results on the dynamics of these solitons. While the static force vanishes as a result of the cancellation of long-range forces of exchange, the force between two moving solitons is in general nonvanishing and depends on the velocities of the solitons. The most complete answer would be given by a full time-dependent solution of the equations of motion of the above action for the case of an arbitrary number of sources moving with arbitrary transverse velocities. These equations, however, are much more difficult to solve for moving sources than for a static configuration. Even a two-soliton solution is in general quite intractable for this class of actions.

We first examine the scattering of these solitons using the above test-string approach. This entails solving the constraint equation for the worldsheet metric obtained by varying the worldsheet Lagrangian \( \mathcal{L} \). The resultant solution for the worldsheet metric along with the static solution for the spacetime metric, antisymmetric tensor field and dilaton from the static ansatz for a single source string are then substituted into the Lagrangian, whose equations yield the dynamics of the test string in the source string background. A flat kinetic Lagrangian is obtained, suggesting trivial scattering (i.e. a zero dynamical force) in this limit.

We then address the scattering problem from a string-theoretic point of view. The winding configuration described by \( X(\sigma, \tau) \) describes a soliton string state. It is therefore a natural choice for us to compare the dynamics of these states with the above string solitons in order to determine whether we can identify these solitons with infinitely long fundamental strings. Accordingly, we study the scattering of the winding states in the limit of large winding radius. We find that the Veneziano amplitude obtained also indicates trivial scattering in this limit, providing evidence for the identification of the string solitons with infinitely long macroscopic fundamental strings.

Finally, we turn to soliton-soliton scattering. In the low-velocity limit, multi-soliton solutions trace out geodesics in the static solution manifold, with distance defined by the Manton metric on moduli space manifold. In the absence of a full time-dependent solution to the equations of motion, these geodesics represent a good approximation to the low-energy dynamics of the solitons. For BPS monopoles, the Manton procedure was implemented by Atiyah and Hitchin. Computing the Manton metric on moduli space for the scattering of the soliton string solutions in \( D = 4 \) (we expect that the same result will hold for arbitrary \( D \geq 4 \)) we find that the metric is flat to lowest nontrivial order in the string tension. This result
implies trivial scattering of the string solitons and is consistent with the above two calculations, and thus provides even more compelling evidence for the identification of the string soliton with the underlying fundamental string.

For the instanton and monopole fivebranes\textsuperscript{20}, both the test-fivebrane limit and the metric on moduli space also yield a zero dynamical force condition. Since a fundamental theory of fivebranes has not yet been constructed, there is no corresponding Veneziano amplitude computation with which to compare. For the instantons, it is sufficient to demonstrate Ricci flatness of the Manton metric to obtain trivial scattering while for the monopoles a flat metric can be explicitly computed. The zero dynamical force can be seen as a direct consequence of the exactness condition of equating the gauge and spin connections\textsuperscript{16}.

6. Future Directions

While all the solutions we discussed are classical, one can still conceive of situations in which quantum corrections to the instantons, for example, drop out in nonperturbative computations (such as for vacuum tunnelling). To this end, a vertex operator representation of the instantons would be highly desirable. The most interesting feature of the heterotic monopole solution is the cancellation between gauge and gravitational singularities. If this is an intrinsically “stringy” feature, then it presumably occurs in a larger context within string theory, in which case the full quantum string loop extension of this solution promises to shed light on the nature of string theory as a finite theory of quantum gravity. If this cancellation is of a more accidental nature, then it would pay to concentrate more on the corresponding low-energy field theory, whose quantization is presumably far simpler. In either case, it would be interesting to see whether the singularity cancellation occurs in a quantized solution, or in the context of blackhole type solutions. We have compelling dynamical evidence for the identification of the string solitons with macroscopic fundamental strings, but an exact heterotic solution seems most natural in the context of the conjectured dual fundamental theory of fivebranes. While the construction of such a theory remains elusive, there is so far solid evidence to support the duality conjecture (see M. J. Duff’s contribution to this volume).

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