Improving the Quality of Random Number Generators by Applying a Simple Ratio Transformation

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Abstract

It is well-known that the quality of random number generators can often be improved by combining several generators, e.g. by summing or subtracting their results. In this paper we investigate the ratio of two random number generators as an alternative approach: the smaller of two input random numbers is divided by the larger, resulting in a rational number from $[0, 1]$.

We investigate theoretical properties of this approach and show that it yields a good approximation to the ideal uniform distribution. To evaluate the empirical properties we use the well-known test suite TestU01. We apply the ratio transformation to moderately bad generators, i.e. those that failed up to 40% of the tests from the test battery Crush of TestU01. We show that more than half of them turn into very good generators that pass all tests of Crush and BigCrush from TestU01 when the ratio transformation is applied. In particular, generators based on linear operations seem to benefit from the ratio, as this breaks up some of the unwanted regularities in the input sequences.

Thus the additional effort to produce a second random number and to calculate the ratio allows to increase the quality of available random number generators.

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1. Introduction

Stochastic simulation is an important tool to study the behavior of complex stochastic systems that cannot be mathematically analyzed, as it is often the case e.g. in network models for traffic, communication or production. The basis of these simulation tools is a (pseudo-) random input provided by a random number generator.

A random number generator (RNG for short) is an algorithm that produces sequences of numbers that, viewed as an observation of a random experiment, can be modeled mathematically by a sequence of independent, identically distributed random variables (i.i.d. rvs). As basis for most simulations, these rvs should have the uniform distribution $U(0, 1)$ on the interval $[0, 1]$.

Of course, a deterministic algorithm can only approximate this mathematical model. Its deviance from the model is used to measure the quality of the generator. A lot of investigations have been made into that direction for different types of generators proposed over time, see e.g. [1] or [2] for a survey. In particular, a large number of empirical tests have been developed to assess the quality of RNGs.

In this paper we are going to show that one can transform many simple, moderately good generators into statistically excellent ones using the ratio of their output. We show this with the test suite TestU01 from [3], which has become a standard for RNG testing.

A general framework for RNGs producing numbers in the interval $[0, 1]$ is described in [4]. It consists of a finite set $S$ of internal states, a function $f : S \to S$ describing the recursion, a seed state $s_0$ and an evaluation function $g : S \to [0, 1]$. Starting with the seed state $s_0$, a sequence of states $(s_i)_{i \geq 0}$ is constructed using the recursion $s_{i+1} := f(s_i), i \geq 0$. The random numbers returned are $u_i := g(s_i), i \geq 0$.

Well-known examples for RNGs are the linear congruential generators (LCG) of order 1. Here, $f(s) := (as + c) \mod M$ for some constants $a, c, M \in \mathbb{N}$ and $S := \{0, 1, \ldots, M - 1\}$. The integers $s$ that serve as internal states are turned into output values from $[0, 1]$ by the evaluation function $g(s) := s/M$.  

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random number generation, combination of random generators, ratio of uniform random variables, empirical testing of random generators
In many popular generators the internal state consists of one or more integers from a bounded set $\mathcal{N}_M := \{0, 1, \ldots, M - 1\}$. The evaluation step then selects one of these integers and divides it by $M$ as in the example above. This is the case e.g. in LCGs of higher order, in many so-called Lagged Fibonacci generators and more complex combinations of RNGs as given in [1], [5], [6] or [7]. This way of producing random numbers from $[0, 1]$ using a division by a constant $M$ will be referred to in the sequel as the direct approach.

In this paper, we assume that we are given a RNG that internally produces integers $x_i \in \mathcal{N}_M$ as above, but the final evaluation step is a more complex transformation that uses two consecutive values $x_{2i}, x_{2i+1} \in \mathcal{N}_M$ and returns

$$u_i := \frac{\min\{x_{2i}, x_{2i+1}\}}{\max\{x_{2i}, x_{2i+1}\}}$$

where the cases $x_{2i} \cdot x_{2i+1} = 0$ and $x_{2i} = x_{2i+1}$ will be considered in detail in Subsection 2.1. We shall call (1) the ratio transformation and $x_1, x_2, \ldots$ its input sequence from the base RNG. To our knowledge, this type of ratio transformation was first used in [8].

To make the comparison between the ratio transformation and the direct approach fairer, we will consider an extension of this approach that also uses two random numbers and returns

$$w_i := \frac{x_{2i}}{M} + \frac{x_{2i+1}}{M^2}.$$  

We call this the direct-2 approach. Note that our ratio transformation is computationally slightly more complex than direct-2.

We start the paper with a theoretical analysis of the ratio transformation. We show that the cumulative distribution function (cdf) of the ratio transformation of two independent discrete random variables $X_1, X_2$, that are both uniformly distributed over $\mathcal{N}_M$, approximates the continuous uniform distribution on $[0, 1]$. This approximation, though not as close as by the direct approaches, yields values that are much less regularly distributed than those from the direct approaches. We believe that this is the reason for the superior empirical behaviour.

The empirical quality of the ratio transformation is tested using the standard test batteries CRUSH and BigCrush from [2]. Typically, simple RNGs fail many of these tests with the direct approach. However, if the ratio transformation is applied, the number of tests failed is reduced considerably. In
particular, we tested the RNGs from [2] that failed up to 40% of the 144 tests of Crush. When the ratio transformation is applied to their output, about half of them passed all tests of Crush, whereas an application of the direct-2 approach could hardly improve their performance. This shows that the ratio transformation is able to turn simple, moderately good RNGs into excellent ones. First observations in that direction were reported in [8].

The paper is organized as follows: in Section 2 we investigate theoretical properties of (1) under the assumption that the inputs are from ideal RNGs. In Section 3 we report on empirical tests with the Crush/BigCrush test battery from [2] with different types of base RNGs, each time comparing the direct approaches with the ratio transformation. Finally, we give a short conclusion.

2. The Mathematical Model

2.1. The Cumulative Distribution Function of the Ratio Transformation

In this Section we investigate the mathematical model of the ratio (1) where we replace the input sequence by random variables with a uniform distribution. We prove that the ratio transformation preserves uniformity at least approximately.

For completeness, we first give a simple theorem which states that the ratio of two continuous $U(0,1)$-distributed independent random variables is again $U(0,1)$-distributed. Then, we will study the more complicated situation where the input is from a discrete uniform distribution.

Note that for a continuous random variable $U$ with distribution $U(0,1)$, its cdf is $F_U(t) = P(U \leq t) = t$ and its density $f_U(t) = 1$ for $t \in [0,1]$.

**Theorem 1.** Let $U_1, U_2$ be two independent and identically distributed random variables with distribution $U(0,1)$ and define the ratio

$$Z := \frac{\min\{U_1, U_2\}}{\max\{U_1, U_2\}}$$  \hspace{1cm} (2)

where we put $0/0 = 0$. Then $Z$ has distribution $U(0,1)$, too.

**Proof.** As $P(U_1 \cdot U_2 = 0) = P(U_1 = U_2) = 0$ we may exclude these two cases and obtain

$$P(Z \leq t) = P\left(\frac{\min\{U_1, U_2\}}{\max\{U_1, U_2\}} \leq t\right) = 2P(0 < U_1 \leq tU_2)$$
\[ = 2 \int_0^1 P(U_1 \leq tu) \, du = 2 \int_0^1 tu \, du = 2t/2 = t, \]

for all \( t \in [0,1] \). \( \square \)

Now we assume that the input is from two \textit{i.i.d.} random variables \( X_1, X_2 \) with the \textit{discrete} uniform distribution \( U(N_M) \) on \( N_M = \{0, \ldots, M - 1\} \), i.e.

\[ P(X_1 = k, X_2 = l) = \frac{1}{M^2} \quad \text{for all } k, l \in N_M. \] (3)

Then, we have \( P(X_1 \cdot X_2 = 0) \approx 2/M \) and \( P(X_1 = X_2 > 0) \approx 1/M \). A straightforward extension of the ratio (1) would choose 0 as return value in case \( X_1 \cdot X_2 = 0 \) and return value 1 if \( X_1 = X_2 > 0 \), see [8]. As it is a common practice with RNGs to completely avoid 0, 1 as return values, we introduce two replacement values \( \varepsilon_0 \) and \( \varepsilon_1 \) with \( 0 < \varepsilon_0 < 1 - \varepsilon_1 < 1 \), and let them appear with almost equal small probability (\( \approx 1.5/M \)). We therefore define

\[ \varepsilon_0 := \frac{M - 1 + \lfloor M/2 \rfloor}{2M^2} \]
\[ \varepsilon_1 := \frac{2M - 1 - \lfloor M/2 \rfloor}{2M^2} \] (4)

where \( \lfloor a \rfloor \) is the largest integer less or equal \( a \in \mathbb{R} \). Note that \( 0 < \varepsilon_0, \varepsilon_1 < 1/(M - 1) \) and that for large \( M \), \( \varepsilon_0 \approx \varepsilon_1 \approx 0.75/M \). The motivation for this particular choice will become clear from Theorem 3 below.

In the sequel we will use the following ratio transformation of two inputs \( x_1, x_2 \in N_M \)

\[ h(x_1, x_2) := \begin{cases} 
\varepsilon_0 & \text{if } x_1 = 0 < x_2 \text{ or } 0 \leq x_1 = x_2 \leq \lfloor M/2 \rfloor - 1, \\
\min\{x_1, x_2\} & \text{if } x_1 \cdot x_2 > 0 \text{ and } x_1 \neq x_2, \\
\max\{x_1, x_2\} & \text{if } x_2 = 0 < x_1 \text{ or } x_1 = x_2 \geq \lfloor M/2 \rfloor, \\
1 - \varepsilon_1 & \text{if } x_1 = x_2 > 0 < x_2, \\
1 & \text{if } x_1 = x_2 = 0 \text{ or } x_1 = x_2 > 0 \geq \lfloor M/2 \rfloor,
\end{cases} \] (5)

i.e. we split the cases \( x_1 \cdot x_2 = 0 \) and \( x_1 = x_2 \) more or less evenly between the two values \( \varepsilon_0 \) and \( 1 - \varepsilon_1 \). The next Theorem gives the cdf of the discrete random variable \( h(X_1, X_2) \). We show in Theorem 3 that this is a close approximation of the cdf of \( U(0,1) \).
Theorem 2. Let $X_1, X_2$ be i.i.d. $U(N_M)$-distributed random variables and $Y = h(X_1, X_2)$. Then the cdf of $Y$ is given by

$$P(Y \leq t) = \begin{cases} 
0 & \text{if } 0 \leq t < \varepsilon_0 \\
2\varepsilon_0 & \text{if } \varepsilon_0 \leq t < \frac{1}{M-1} \\
2\varepsilon_0 + \frac{2}{M^2} \sum_{k=1}^{M-1} \lfloor tk \rfloor & \text{if } \frac{1}{M-1} \leq t < 1 - \varepsilon_1 \\
1 & \text{if } 1 - \varepsilon_1 \leq t \leq 1 
\end{cases}$$

(6)

Proof. The smallest nonzero value that $\min\{X_1, X_2\}$ may attain is $\frac{1}{M-1}$ and similarly, $\frac{M-2}{M-1} = 1 - \frac{1}{M-1}$ is the largest value smaller than 1. We have $0 < \varepsilon_0 < \frac{1}{M-1}$ and $\frac{M-2}{M-1} < 1 - \varepsilon_1$. Therefore, using (3)

$$P(Y \leq \varepsilon_0) = P(Y = \varepsilon_0) = P(X_1 = 0 < X_2 \text{ or } 0 \leq X_1 = X_2 \leq \lfloor M/2 \rfloor - 1)$$

$$= P(X_1 = 0 < X_2) + P(0 \leq X_1 = X_2 \leq \lfloor M/2 \rfloor - 1)$$

$$= \sum_{k=1}^{M-1} P(X_1 = 0, X_2 = k) + \sum_{k=0}^{\lfloor M/2 \rfloor - 1} P(X_1 = X_2 = k)$$

$$= \frac{M - 1 + \lfloor M/2 \rfloor}{M^2} = 2\varepsilon_0.$$  

(7)

Similarly, one may show

$$P(Y = 1 - \varepsilon_1) = 2\varepsilon_1.$$  

(8)

For $t \in \left[\frac{1}{M-1}, \frac{M-2}{M-1}\right]$ we have, using the symmetry of the joint distribution of $X_1, X_2$,

$$P(Y \leq t) = P(Y = \varepsilon_0) + P(Y \leq t, X_1 \cdot X_2 > 0, X_1 \neq X_2)$$

$$= 2\varepsilon_0 + P(\frac{\min\{X_1, X_2\}}{\max\{X_1, X_2\}} \leq t, \ X_1 \cdot X_2 > 0)$$

$$= 2\varepsilon_0 + 2 \cdot P(X_1 \leq tX_2, \ X_1 > 0)$$

$$= 2\varepsilon_0 + 2 \sum_{k=1}^{M-1} P(X_2 = k)P(X_1 \leq kt) = 2\varepsilon_0 + \frac{2}{M^2} \sum_{k=1}^{M-1} \lfloor tk \rfloor,$$

here $t < 1$ implies $X_1 \neq X_2$, which proves the Theorem. \qed
2.2. The Deviation of $Y$ from the Uniform Distribution

We want to measure the quality of the ratio transformation $Y$ by the maximal deviation of its cdf $F_Y(t) := P(Y \leq t)$ as given in Theorem 2 from the cdf of a $U(0,1)$-distributed random variable $U$ with $F_U(t) = t, t \in [0,1]$. This difference

$$
\Delta_Y := \sup_{t \in [0,1]} |F_Y(t) - t|
$$

is also called Kolmogoroff-Smirnov-distance (KS-distance).

![Figure 1: The cdf of the ratio transformation for $M = 10$. The diagonal corresponds to the cdf of $U(0,1)$.](image)

Theorem 3 shows, that the distribution of the ratio transformation $Y$ approaches the true $U(0,1)$ distribution in KS-distance as $M \to \infty$. Moreover, the largest deviation of $F_Y$ from $F_U$ takes place at the two artificial extreme points $\varepsilon_0, 1 - \varepsilon_1$.

**Theorem 3.** Let $Y$ and $\Delta_Y$ be defined as above. Then

$$
\Delta_Y \leq \max\{\varepsilon_0, \varepsilon_1\} \approx \frac{3}{4} \frac{1}{M}.
$$

**Proof.** First note that for the KS-distance between an increasing step function $F$ with jump points $t_i, i = 1, \ldots, L$, and the continuous cdf $F_U(t) = t$ on $[0,1]$, it is sufficient to check the distances at the jump points, more precisely with $t_0 := 0$

$$
\sup_{t \in [0,1]} |F(t) - t| = \max_{i=1,\ldots,L} \{|F(t_i) - t_i|, |F(t_{i-1}) - t_i|\}. \tag{9}
$$
Determining $\Delta_Y$ turns out to be quite involved and we will only sketch the most important steps here. First note that $F_Y$ has jumps at $\varepsilon_0, 1 - \varepsilon_1$ and the possible values of the ratio expression $k/l \in D$ where

$$D := \left\{ \frac{k}{l} \mid k, l \in \mathbb{N}, 1 \leq k < l \text{ and } k, l \text{ coprime} \right\}. \tag{10}$$

From (7) and (8) we see that the maximal deviation between $F_Y$ and $F_U$ outside of $D$ is $\max\{\varepsilon_0, \varepsilon_1\}$. For the proof that $\max\{\varepsilon_0, \varepsilon_1\}$ is also the bound for $\Delta_Y$, it only remains to show that

$$\max \left\{ \left| F_Y(\varepsilon_0) - \frac{1}{M-1} \right|, \sup_{t \in \left[\frac{1}{M-1}, \frac{M-2}{M-1}\right]} |F_Y(t) - t| \right\} \leq \max\{\varepsilon_0, \varepsilon_1\}. \tag{11}$$

Note that here the first term has to be included to check the jump of the cdf from the left at the minimal value $\frac{1}{M-1}$ of $D \subset \left[\frac{1}{M-1}, \frac{M-2}{M-1}\right]$. The long and technical proof of (11) is sketched in the Appendix.

As the KS-distance $\Delta_Y$ is determined by the probability of the two additional points $\varepsilon_0, 1 - \varepsilon_1$, $\Delta_Y$ could be further lowered by replacing $\varepsilon_0$ with $\varepsilon'_0, \varepsilon''_0 \in (0, \frac{1}{M-1})$, each with half the probability of $\varepsilon_0$ and therefore smaller jumps of $F_Y$. Similarly, one could replace $1 - \varepsilon_1$. Then $\Delta_Y$ would be dominated by the leftmost term of inequality (15) of Theorem 6 in the Appendix, which is again bounded by $(7/4 - 2\sqrt{2})/M + 2/M^2 = 0.6715/M + 2/M^2$ as is shown in the Appendix. Since this is not a real improvement, we stick to the simpler form of the ratio transformation as given in definition (5).

Next we want to compare the maximal deviation $\Delta_Y$ with the deviation under the two direct approaches. To be more formal, let $X_1, X_2$ be i.i.d., $U(\mathbb{N}_M)$-distributed rvs. Then

$$V := \frac{X_1}{M} + \frac{1}{2M} \quad \text{and} \quad W := \frac{X_1}{M} + \frac{X_2}{M^2} + \frac{1}{2M^2}, \tag{12}$$

where we use a slight shift of the results in both cases again to avoid the value 0. Then $V$ corresponds to the simple direct method and $W$ to direct-2 approach. A first impression of the cdfs of these variables is given in Figure 2. We define

$$\Delta_V := \sup_{t \in [0,1]} |P(V \leq t) - t|, \quad \Delta_W := \sup_{t \in [0,1]} |P(W \leq t) - t|. \tag{13}$$
As would be expected from Figure 2, at least $\Delta_W$ is much smaller than $\Delta_Y$. The next Theorem gives simple bounds on $\Delta_V, \Delta_W$.

**Theorem 4.** With $V, W$ and $\Delta_V, \Delta_W$ as above we have

$$\Delta_V \leq \frac{1}{2M} \quad \text{and} \quad \Delta_W \leq \frac{1}{2M^2}$$

**Proof.** The simple proof is omitted

Theorem 3 and 4 show that, with respect to the KS-distance, both direct approaches approximate the $U(0,1)$ closer than the ratio transformation as is also obvious from Figure 1 and 2. Nevertheless, these figure also give a clue why the ratio transformation performs better in practice: its structure is far more irregular than that of the direct approaches.

### 2.3. The Set of Possible Values in $[0,1]$}

The next Theorem gives the exact number of values in $\mathcal{D}$ as defined in (10). Here we use the *Euler totient function* $\phi(k)$, that gives the number of integers $m, 1 \leq m \leq k$, that are relatively prime to $k$.

**Theorem 5.** Assume that $M > 2$, then the number of values that the ratio transformation $Y$ may attain is

$$N(M) := \sum_{k=2}^{M-1} \phi(k) + 2.$$
Proof. We put \( D_l := \{ \frac{k}{m} \mid 0 < k < m < l \} \), then \( D = D_M \). We prove

\[
|D_l| = \sum_{k=2}^{l-1} \phi(k) \quad (14)
\]

for \( l > 2 \), then the assertion follows as \( Y \) takes on values in \( D \cup \{ \varepsilon_0, 1 - \varepsilon_1 \} \) only. We have \( D_3 = \{ 1/2 \} \), hence the assertion follows for \( l = 3 \) as \( \phi(2) = 1 \).

Now assume that (14) holds for some \( l \geq 3 \). Then all values from \( D_l \) must be contained in \( D_{l+1} \). The additional values in \( D_{l+1} \) must all have the form \( k/l \) where \( 1 \leq k < l \). In order that these have not yet appeared in \( D_l \), \( k \) and \( l \) must be relatively prime, hence there are exactly \( \phi(l) \) additional values and

\[
|D_{l+1}| = |D_l| + \phi(l) = \sum_{k=2}^{l-1} \phi(k) + \phi(l) = \sum_{k=2}^{l} \phi(k).
\]

According to [9], for large \( M \), \( N(M) \) can be approximated by

\[
\frac{3(M-1)^2}{\pi^2} + O \left( (M-1) \log(M-1)^{2/3} (\log \log(M-1))^{4/3} \right)
\]

which is much more than the \( M \) different values possible under the direct approach though it is less than the \( M^2 \) different values from the direct-2 approach.

2.4. Further Theoretical Properties

Nevertheless, the number of different values actually produced by a RNG is limited by the length of its period. The direct approach obviously keeps the period length \( T \) of its base RNG. The period length of pairs \((x_{2i}, x_{2i+1})\) is \( T_1 := T/2 \) if \( T \) is even and \( T_1 := 2T \) if \( T \) is odd. So the period length \( T_0 \) of the ratio transformation should be a divisor of \( T_1 \). Presently, we cannot give any further results on \( T_0 \).

The ratio transformation may also be applied to base RNGs that produce numbers from \([0, 1]\) (instead of \( N_M \)) as is supported by Theorem [1]. In case the base RNG uses the direct method, the denominator \( M \) of the two consecutive random numbers cancels out in the ratio transformation and Theorem [2] applies.
Note that we could save the additional call of the RNG for the ratio transformation, if we would use overlapping pairs \((x_i, x_{i+1})\) in [1], such that for each step we need to invoke the base RNG only once. As could be expected, this leads to strongly correlated results and is therefore not investigated further in this paper. If, however, the input values to the ratio come from two different RNGs, one may look at [1] as a non-linear combination of RNGs that showed an excellent quality in first empirical tests.

The theoretical investigation presented in this Section show that the ratio transformation might be a candidate to produce good random numbers, but it is not clear why this could be better than the direct approaches. This will become evident from the empirical results of the next Section.

3. Experimental study

3.1. The Test Set-up

The theoretical results of the Section 2 assumed that the input sequence was from an ideal generator. We shall now investigate the empirical behavior of the ratio transformation when the input sequence is from RNGs used in practice, in particular from fast and simple generators that may not be very good on their own. We shall show that for many classical generators the sequences from the ratio transformation are much better than the sequences obtained from the same RNG with the two direct approaches.

We use the test batteries CRUSH and BigCRUSH from the test suite TestU01 of L’Ecuyer and Simard (version 1.2.3 from [3]), described in [2]. In that paper, 92 widely used or well known RNGs from different publications or software packages are described and tested. The \(H_0\) hypothesis in these tests is “the random numbers \(u_1, u_2, \ldots\) are observations from \(U(0, 1)\)” and the software reports the right \(p\)-values for each of the tests. A test is failed if the \(p\)-value lies outside the interval \([10^{-10}, 1 - 10^{-10}]\) and results are considered ‘suspicious’ if the \(p\)-value lies in \([10^{-10}, 10^{-4}] \cup [1 - 10^{-4}, 1 - 10^{-10}]\) as specified in [2]. There are 144 tests applied in one run of CRUSH, some of which are based on identical statistical procedures but with different parameters.

We looked at those 57 RNGs used in TestU01 that failed at least one but not more than 40% of the 144 tests used in CRUSH. For each of these generators we list in Table 1 the original results from TestU01, the results from an application of our ratio transformation and the results from an application of direct-2 as in (13) under CRUSH. If there were no failures or suspicious results in CRUSH, we further applied the BigCRUSH.
We reuse here the names of the generators from [2], e.g. ‘LCG(M, a, c)’
denotes a linear congruential generator with modulus M, multiplicator a
and additive constant c, for the exact definition of the other generators we refer
to [2].

| Generator                                      | Crush | BigCrush | time for 10^8 |
|------------------------------------------------|-------|----------|---------------|
| | Dir | Dir2 | Rat | Dir2 | Rat | Dir | Rat |
| LCG(2^{46}, 5^{11}, 0) | 38(2) | 38(2) | 0 | 0 | 0.35 | 1.62 |
| LCG(2^{48}, 25214903917, 11) | 21(1) | 21(5) | 0 | 0 | 0.35 | 1.58 |
| Java.util.Random | 9(3) | 11(2) | 0 | 0 | 0.52 | 1.88 |
| LCG(2^{48}, 5^{19}, 0) | 21(2) | 27(5) | 0 | 0 | 0.35 | 1.59 |
| LCG(2^{48}, 33952834046453, 0) | 24(5) | 29(5) | 0 | 0 | 0.38 | 1.54 |
| LCG(2^{48}, 4485709377909, 0) | 24(5) | 33(7) | 0 | 0 | 0.36 | 1.58 |
| LCG(2^{59}, 13^{13}, 0) | 10(1) | 12(2) | 0 | 0 | 0.36 | 1.75 |
| LCG(2^{63}, 5^{19}, 1) | 5 | 6 | 0 | 0 | 0.37 | 1.62 |
| LCG(2^{63}, 9219741426499971445, 1) | 5(1) | 7(2) | (1) | | | |
| LCG(2^{31} − 1, 16807, 0) | 42(9) | 40(5) | 12(5) | 42(9) | 40(5) | 12(5) | |
| LCG(2^{31} − 1, 397204094, 0) | 38(4) | 43(3) | 15(1) | | | | |
| LCG(2^{31} − 1, 742938285, 0) | 42(5) | 40(2) | 12(6) | 42(5) | 40(2) | 12(6) | |
| LCG(2^{31} − 1, 950706376, 0) | 42(4) | 43(1) | 17(1) | | | | |
| LCG(10^{12} − 11, 427419669081, 0) | 22(2) | 19(5) | 0 | 0 | 1.2 | 3.2 | |
| LCG(2^{61} − 1, 2^{30} − 2^{19}, 0) | 1(4) | 7(1) | 0 | 0 | 2.07 | 4.94 | |
| Wichmann-Hill | 12(3) | 9(3) | 0 | 0 | 4.16 | 9.51 | |
| CombLec88 | 1 | 2 | 0 | 0 | 0.73 | 2.31 | |
| Knuth(38) | 1(1) | 1(3) | 0 | 0 | 0.86 | 3.27 | |
| DengLin(2^{31} − 1, 2, 46338) | 11(1) | 1(2) | 0 | 0 | 0.147 | 4.46 | |
| DengLin(2^{31} − 1, 1, 22993) | 2 | 0 | 0 | 0 | 0.147 | 4.47 | |
| LFnib(2^{31}, 55, 24, +) | 9 | 1 | 1(3) | | | | |
| LFnib(2^{31}, 55, 24, −) | 11 | 1 | 1(3) | | | | |
| ran3 | 11 | 1 | 1(1) | | | | |
| LFnib(2^{48}, 607, 273, +) | 2 | 1 | 0 | | | | |
| Unix-random-64 | 57(6) | 51(8) | 35(2) | | | | |
| Unix-random-128 | 13 | 1 | 6(1) | | | | |
| Unix-random-256 | 8 | 0 | 15(1) | | | | |
| Knuth-ran_array2 | 3 | 3 | 0 | | | | |
| SWB(2^{24}, 10, 24) | 30 | 9(3) | 16(2) | 30 | 9(3) | 16(2) | |
| SWB(2^{24}, 10, 24)[24, 48] | 6(1) | 0 | 0 | | | | |
| SWB(2^{32} − 5, 22, 43) | 8 | 2 | 4(2) | | | | |
| SWB(2^{31}, 8, 48) | 8(2) | 14(3) | 9(1) | | | | |
| Mathematica-SWB | 15(3) | 28(2) | 16 | | | | |
| SWB(2^{32}, 222, 237) | 2 | 0 | 1(1) | | | | |
| GFSR(250, 103) | 8 | 77(3) | 2 | | | | |
| GFSR(521, 32) | 7 | 77(2) | 1(1) | | | | |
| GFSR(607, 273) | 8 | 77(1) | (1) | | | | |
| Ziff98 | 6 | 74(4) | 0 | 0 | 0.38 | 1.83 | |
| T800 | 8(4) | 14(4) | 5(5) | | | | |
| TT800 | 12(4) | 8(1) | 0 | 0 | 0.53 | 2.13 | |
| MT19937 | 2 | 2 | 0 | 0 | 0.9 | 3.12 | |
| WELL1024a | 4 | 4 | 0 | 0 | 0.63 | 2.61 | |
| WELL19937a | 2(1) | 2 | 0 | 0 | 0.52 | 2.65 | |
| LFSR113 | 6 | 6 | 0 | 0 | 0.56 | 2.16 | |

Continued on next page
Table 1: No. of tests failed in Crush and BigCrush. The results are for those 57 RNGs from [2], which failed at least one and maximally 40% of the Crush tests in their original form. Generators that are much improved by the ratio transformation are marked with a ★.

Table 1 – continued from previous page

| Generator         | Crush | BigCrush | time for $10^8$ |    |    |
|-------------------|-------|----------|----------------|----|----|
|                   | Dir   | Dir2     | Rat            |    |    |
| LFSR258           | 6     | 6        | 0              | 0  | 0.68 |
| ★                 |       |          |                |    |     |
| Marsa-xor64 (13, 7, 17) | 8(1) | 8        | 0              | 0  | 0.41 |
| ★                 |       |          |                |    |     |
| ↓ Matlab-rand     | 5     | 1        | 6(2)           | 0  | 0.42 |
| ★                 |       |          |                | 1  |     |
| SuperDuper-73     | 3     | 5        | 0              | 0  | 0.38 |
| GiftMultiCarry    | 25(3) | 22(5)    | 0              | (1)|     |
| ★                 | 40(4) | 33(4)    | 6              |    |     |
| R-MultiCarry      | 1     | 1        | 0              | 0  | 0.42 |
| Marsa-xor64 (13, 7, 17) | 8(1) | 8        | 0              | 0  | 0.41 |
| ★                 |       |          |                |    |     |
| ICG($2^{31} - 1, 1, 1$) | 6   | 5(2)     | 4              |    |     |
| ★                 |       |          |                |    |     |
| ICG($2^{31} - 1, 22211, 11926380$) | 5   | 5        | 0              | 5(6)|     |
| ICG($2^{31} - 1, 1, 1$) | 6   | 5(2)     | 4              |    | 1.82|
| ★                 |       |          |                | 5(6)|     |
| EICG($2^{31} - 1, 1288490188, 1$) | 6   | 6(1)     | 4              |    | 1.82|
| ★                 |       |          |                | 5(6)|     |
| SNWeyl            | 56(12)| 65(11)   | 19(7)          | 1  |     |
| Coveyou-64        | 1     | 1        |                |    |     |

In Table 1 columns 3-7 show the number of failed or suspicious tests under different transformations and in the two test batteries. The number of suspicious result is given in ‘()’. ‘Dir’ refers to the original RNG, ‘Dir2’ to the direct-2 approach and ‘Rat’ to the results from ratio transformation. The first column marks the success of the ratio transformation: 26 of the 57 RNGs (≈ 45.6%) became excellent after an application of the ratio transformation, they passed all tests in Crush and BigCrush and are marked with a ★. Another 9 RNGs marked with ○ have a few suspicious results in Crush or pass all tests in Crush and fail a few in BigCrush. Those without a mark are RNGs that are improved by the ratio transformation but still fail with some tests in Crush. For four RNGs only, results were degraded by an application of the ratio transformation, these are marked with a ↓.

Note that even the well-known ‘Mersenne-Twister’ MT19937 and its derivatives WELL1024a and WELL19937a could benefit from the ratio transformation.

The last two columns in Table 1 give the runtime in seconds needed to produce $10^8$ random numbers of the original generator (column ‘Dir’) and of its ratio transformation (column ‘Rat’) based on the implementations in TestU01. The ratio transformation is 3 to 4 times slower than the original
RNG. Note that this can be improved with better implementations and, in particular, if the base RNG is integrated into the ratio transform instead of being called as an external program. Then e.g. the ratio transformation for LCG(2^{63}, 5^{19}, 1) needs only 0.9 seconds (instead of 1.62 in Table 1) and for LCG(2^{59}, 13^{13}, 0) it becomes 0.82 seconds (instead of 1.75). All times were measured on a multi-core 64 bit i7-processor with 2.2 GHz under the Ubuntu operating system.

While the ratio transformation improved many results justifying the additional effort of the second random number (and the division operation), this is not the case with the competitor, the direct-2 approach in (13). In many cases results are not improved by an application of the direct-2 approach (see column ‘Dir2’ in Table 1). Only for 4 RNGs the direct-2 approach passed all tests in both CRUSH and BIGCRUSH, and for another one all tests in CRUSH were passed. In 17 cases, the results became even worse after an application of the direct-2 approach. This may in part be due to the structure of the tests that mainly rely on the first 32 Bits of the numbers produced and these do not change when direct-2 is applied.

4. Summary and conclusion

In this paper we investigated the impact of a simple ratio transformation on the quality of RNGs, in particular in comparison with the direct approach that is used by most congruential generators. The theoretical properties showed a less regular, but somewhat coarser behaviour than the direct approach. The statistical tests, however, demonstrated the strength of the ratio: it breaks up the linear regularities of its base RNG and turns many mediocre RNGs into excellent ones that pass all tests of BIGCRUSH.

Including the ratio transformation in random number generators would increase their running time but would also give much better results in many cases.

5. Appendix: Maximal KS-Distance of the Ratio Transformation

To complete the proof of Theorem 3 b), namely \( \Delta_Y \leq \max\{\varepsilon_0, \varepsilon_1\} \), it remains to show that the maximal distance apart from the jumps at \( \varepsilon_0, 1 - \varepsilon_1 \) is also bounded by \( \max\{\varepsilon_0, \varepsilon_1\} \).
We number the elements of the set \( D \) as defined in (10) as \( D = \{t_1, t_2, \ldots, t_{L-1}\} \) where
\[
t_1 = \frac{1}{M - 1} < t_2 < \cdots < t_{L-1} = \frac{M - 2}{M - 1}
\]
and set \( t_0 := \varepsilon_0, t_L := 1 - \varepsilon_1 \) and \( t_{-1} := 0 \). Then we know from (9) that
\[
\Delta_Y = \sup_{t \in [0,1]} |F_Y(t) - t| = \max_{i=0, \ldots, L} \left\{ |F_Y(t_i) - t_i|, |F_Y(t_{i-1}) - t_i| \right\}
\]

**Theorem 6.**

a) \[
\max_{i=2, \ldots, L-1} \left\{ |F_Y(t_i) - t_i|, |F_Y(t_{i-1}) - t_i| \right\} \leq \frac{7/4 - \sqrt{2}}{M} + \frac{2}{M^2} \leq \max\{\varepsilon_0, \varepsilon_1\} \tag{15}
\]
b) \[
\max \left\{ |F_Y(t_1) - t_1|, |F_Y(t_0) - t_1| \right\} = |F_Y(t_1) - t_1| \leq \varepsilon_0 \tag{16}
\]

**Proof.** Part a)

1. From Theorem 2 we have
\[
F_Y(t) - t = 2\varepsilon_0 + \frac{2}{M^2} \sum_{k=1}^{M-1} |tk| - t = 2\varepsilon_0 - \frac{2}{M^2} \left( \frac{M^2}{2} t - \sum_{k=1}^{M-1} |tk| \right).
\]

For \( 1 \leq i, j \leq L - 1 \), i.e. for \( t_i, t_j \in D \) define
\[
\Delta(i, j) := \frac{M^2}{2} t_i - \sum_{k=1}^{M-1} t_j k = \sum_{k=1}^{M-1} (kt_i - [t_j k]) + \frac{M}{2} t_i.
\]

Then
\[
\max_{i=2, \ldots, L-1} \left\{ |F_Y(t_i) - t_i|, |F_Y(t_{i-1}) - t_i| \right\} \tag{17}
\]

\[
= \max_{i=2, \ldots, L-1} \left\{ |2\varepsilon_0 - \frac{2}{M^2} \Delta(i, i)|, |2\varepsilon_0 - \frac{2}{M^2} \Delta(i, i - 1)| \right\}
\]

\[
= \max \left\{ 2\varepsilon_0 - \frac{2}{M^2} \min_{i=2, \ldots, L-1} \{\Delta(i, i), \Delta(i, i - 1)\}, \right. \left. \frac{2}{M^2} \max_{i=2, \ldots, L-1} \{\Delta(i, i), \Delta(i, i - 1)\} - 2\varepsilon_0 \right\}.
\]
2. Now let \( t_i = \frac{m}{n} \) and \( T = T(n) := \lfloor \frac{M-1}{n} \rfloor \), then we have for any \( 1 \leq k \leq M-1 \)

\[
[t_{i-1}k] = \begin{cases} 
[t_{i}k] - 1 & \text{if } k = nj \text{ for some } 1 \leq j \leq T \\
[t_{i}k] & \text{if } k \neq nj \text{ for all } 1 \leq j \leq T \end{cases}
\]  \hspace{1cm} (18)

To prove (18), note first that, as \( m,n \) are coprime, \( [t_{i}k] = t_{i}k \) holds iff \( k = nj \) for some \( j \in \mathbb{N} \). If this is the case then \( [t_{i-1}k] \leq t_{i-1}k < t_{i}k = [t_{i}k] \) and therefore \( [t_{i-1}k] \leq [t_{i}k] - 1 \). On the other hand

\[
t_i > t_i - \frac{1}{k} = \frac{m}{n} - \frac{1}{nj} = \frac{mj - 1}{nj} \in \mathcal{D} \cup \{0\},
\]

hence \( t_i - \frac{1}{k} \leq t_{i-1} \) and

\[
[t_{i-1}k] \geq \lfloor (t_i - \frac{1}{k})k \rfloor = [t_{i}k] - 1.
\]

For the second case of (18), we have \( [t_{i}k] < t_{i}k \) as \( k \neq nj \). Assume \( t_{i-1}k < [t_{i}k] \), then

\[
[t_{i-1}k] \leq t_{i-1}k < [t_{i}k] < t_{i}k \quad \text{and} \quad t_{i-1} < \frac{[t_{i}k]}{k} < t_{i}.
\]

This contradicts the fact that \( t_{i-1} \) and \( t_{i} \) are consecutive elements in \( \mathcal{D} \). Hence (18) is proved.

3. Using (18), we have

\[
\Delta(i, i - 1) = \sum_{k=1}^{M-1} (kt_i - [t_{i-1}k]) + \frac{M}{2} t_i
\]

\[
= \sum_{k=1}^{M-1} (kt_i - [t_{i}k]) + T(n) + \frac{M}{2} t_i = \Delta(i, i) + T(n)
\]

as there are exactly \( T(n) = \lfloor \frac{M-1}{n} \rfloor \) values of \( 1 \leq k \leq M - 1 \) that have \( k = nj \) for some \( j \in \mathbb{N} \).

4. As \( m, n \) in \( t_i = \frac{m}{n} \) are coprime, we have for the central part in \( \Delta(i, i) \)

\[
\sum_{k=1}^{M-1} (kt_i - [t_{i}k]) = \sum_{k=1}^{M-1} \left( \frac{km}{n} - \left\lfloor \frac{km}{n} \right\rfloor \right) = \frac{1}{n} \sum_{k=1}^{M-1} (mk \mod n)
\]
\[
\begin{align*}
&= \frac{1}{n} \left( \sum_{\nu=0}^{T-1} \sum_{l=0}^{n-1} (\nu + l) m \mod n \right) + \sum_{l=0}^{d} \left( (Tn + l) m \mod n \right) \\
&= \frac{1}{n} \left( T \frac{n(n-1)}{2} + \sum_{l=0}^{d} (lm \mod n) \right)
\end{align*}
\]

where \( T = \lfloor \frac{M-1}{n} \rfloor \) as before and \( d = (M - 1) \mod n \). Here, the last sum may be bounded as follows:

\[
\sum_{l=1}^{d} l \leq \sum_{l=0}^{d} (lm \mod n) \leq \sum_{l=n-d}^{n-1} l.
\]

5. Thus we obtain a lower bound for \( \Delta(i, i) \) as

\[
\Delta(i, i) \geq \frac{1}{n} \left( T \frac{n(n-1)}{2} + \sum_{l=0}^{d} i \right) + \frac{Mm}{2n} \geq \frac{1}{2} \left( n(T^2 + T) + \frac{M^2}{n} \right) - MT
\]

\[
\geq M(\sqrt{T^2 + T} - T) \geq M(\sqrt{2} - 1) =: \Delta_{\text{lower}}
\]

where we used \( 2\sqrt{xy} \leq x + y \) for \( x, y \geq 0 \) and \( T \geq 1 \).

6. In a similar way we obtain

\[
\Delta(i, i - 1) = \Delta(i, i) + T \leq -\frac{1}{2} \left( n(T^2 + T) + \frac{M^2}{n} \right) + 3/2M + TM - 1
\]

\[
\leq M \left( 3/2 + T - \sqrt{T^2 + T} \right) - 1
\]

\[
\leq M(5/2 - \sqrt{2}) =: \Delta_{\text{upper}}
\]

7. Inserting these last two bounds into (17) we obtain

\[
\max_{i=2, \ldots, L-1} \left\{ |F_Y(t_i) - t_i|, |F_Y(t_{i-1}) - t_i| \right\}
\]

\[
\leq \max \left\{ 2\varepsilon_0 - \frac{2}{M^2} \Delta_{\text{lower}}, \frac{2}{M^2} \Delta_{\text{upper}} - 2\varepsilon_0 \right\}
\]

\[
< 2 \left( \frac{7/4 - \sqrt{2}}{M^2} \right) M + 1
\]

\[
\leq \frac{7/2 - 2\sqrt{2}}{M} + \frac{2}{M^2}
\]

\[
= 0.6715729 \frac{1}{M} + \frac{2}{M^2}
\]

which proves part a) of the Theorem.

Part b) follows by evaluating \( F_Y(t_1) \) with \( t_1 = \frac{1}{M-1} \).

\[\square\]
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