Total internal reflection tomography of small objects

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Abstract. The multiple signal classification (MUSIC) imaging method is applied to determine the locations of a collection of small anisotropic spherical scatterers in the framework of the total internal reflection tomography. Multiple scattering between scatterers is considered and the inverse scattering problem is nonlinear, which, however, is solved by the proposed fast analytical approach where no associated forward problem is iteratively evaluated. The paper also discusses the role of the polarization of incidence waves, the incidence angle, the separation of scatterers from the surface of the substrate, and the level of noise on the resolution of imaging.

1. Introduction
The resolution of far-field imaging system is limited by the fact the subwavelength information, which is encoded in the evanescent wave, is lost when the field is monitored far away from the object [1]. One of the approaches to improve resolution of imaging is near-field optics, such as near-field scanning optical microscopy (NSOM), total internal reflection microscopy (TIRM), and photon scanning tunneling microscopy (PSTM) [1,2]. However, there are certain limitations of the above mentioned modalities in the sense that only a two-dimensional image is produced although the sample may present a complicated three-dimensional structure [1, 3, 4]. To provide three-dimensional imaging capacity, it is desirable to solve the inverse scattering problem. In this paper, total internal reflection tomography (TIRT) is considered, where the sample is illuminated with different evanescent waves through a substrate in total internal reflection and then the permittivity of the sample is reconstructed from measurements of scattered fields at the far field.

For weak scattering samples, Born approximation can be applied and the solution to the TIRT can be obtained by constructing the pseudoinverse solution to a linearized inverse scattering problem [1, 4]. The pseudoinverse solution is analytical and non-iterative, but it is limited to only weakly scattering samples [5, 6]. For a sample that is not weakly scattering, one can resort to iterative numerical approaches. In [6], the TIRT is solved accurately by an iterative numerical approach, where the associated forward scattering problem is solved by the coupled dipole method (CDM) and an optimization problem is built up and solved by a conjugate gradient method. Many satisfying results are obtained by casting inverse scattering problems to nonlinear optimization problems and solving them through an iterative numerical approach, but its main drawback is the high computational cost.

We consider a special case in this paper where the sample is discrete scatterers that are much smaller than the wavelength. The small scatterers could be of any shape and their composing materials could be anisotropic. For this kind of special sample, the nonlinear inverse scattering
problem of TIRT is solved by an analytical and non-iterative method. The proposed method is fast in the sense that no associated forward problem is iteratively evaluated. In addition, there is no convergence problem involved. The positions of discrete scatterers are determined by the multiple signal classification (MUSIC) method [7–10]. After the positions are obtained, the scattering strengths of scatterers are determined by the least squares method. The orientations of electric dipoles induced in scatterers and their linear dependency are carefully analyzed under various polarizations of the illumination, so that the proposed MUSIC is able to correctly locate the scatterers. The paper also discusses the influences of propagating and evanescent waves, the substrate, the separation of scatterers from substrate surface, and the level of noise on the resolution of imaging. Numerical simulations show that the proposed algorithm produces subwavelength images under both noise-free and noisy scenarios.

2. Forward scattering problem

A collection of $M$ small scatterers are placed above a dielectric substrate of permittivity $\epsilon_s$ occupying the half space $z < 0$. The scatterers are illuminated by a total number of $N_t$ time-harmonic plane waves coming from the substrate. Some incidence wavevectors are in the $(x, z)$ plane and the rest are in the $(y, z)$ plane. Let $\theta_{inc}^i$ be the incidence angle of the $i$th illumination. The scattered field is measured at a total of $N_r$ positions, $\mathbf{r}_1^i, \mathbf{r}_2^i, \ldots, \mathbf{r}_{N_r}^i$, in the far field. The $M$ scatterers could be any shape, but for ease of presenting, only spherical scatterers are considered in this paper. The radii of scatterers, $a_1, a_2, \ldots, a_M$, are much smaller than the wavelength so that Rayleigh scattering occurs. The centers of the scatterers are located at $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M$. The spherical scatterers are made of anisotropic materials, and their permittivities are given by

$$\epsilon_m = \frac{\bar{\epsilon}}{\bar{\epsilon} - k^2 \bar{M}} \cdot \mathrm{diag} \{ \epsilon_m^{(1)}, \epsilon_m^{(2)}, \epsilon_m^{(3)} \} \cdot \bar{\epsilon},$$

where $\epsilon_m^{(l)}$ is the permittivity element aligned to the $l$th principal axis of the $m$th scatterer, $l = 1, 2, 3$ and $m = 1, 2, \ldots, M$. We assume $\epsilon_m^{(l)} \neq \epsilon_0$ where $\epsilon_0$ is the permittivity of the background free space. The rotation matrix $\bar{\Omega}_m$ is determined by Euler angles [11].

When multiple scattering between scatterers is taken into account, the total incident electric field $\mathbf{E}_t^i(\mathbf{r}_k)$ upon the $k$th scatterer includes both the transmitted plane wave electric field $\mathbf{E}_t^i(\mathbf{r}_k)$ and the scattered fields from other scatterers. The total incident fields are governed by the Foldy-Lax equation [12, 13],

$$\tilde{\psi}_t^i = \tilde{\psi}_t^0 + \hat{\Phi} \cdot \bar{\Lambda} \cdot \tilde{\psi}_t^i,$$

where both $\tilde{\psi}_t^i$ and $\tilde{\psi}_t^0$ are $3M$-dimensional vectors, $\tilde{\psi}_t^i = \left[ \mathbf{E}_t^i(\mathbf{r}_1)^T, \mathbf{E}_t^i(\mathbf{r}_2)^T, \ldots, \mathbf{E}_t^i(\mathbf{r}_M)^T \right]^T$, $\tilde{\psi}_t^0 = \left[ \mathbf{E}_0^i(\mathbf{r}_1)^T, \mathbf{E}_0^i(\mathbf{r}_2)^T, \ldots, \mathbf{E}_0^i(\mathbf{r}_M)^T \right]^T$, where the superscript $T$ denotes the transpose. $\bar{\Lambda} = \mathrm{diag} \left[ \bar{\xi}_1, \bar{\xi}_2, \ldots, \bar{\xi}_M \right]$, where $\bar{\xi}_m$ is the scattering potential of the $m$th spherical scatterer whose expression can be found in [14]. $\hat{\Phi}$ is a $3M \times 3M$ matrix, consisting of $M \times M$ submatrices whose formulas in the $m$th row and $m'$th column $(m, m' = 1, 2, \ldots, M)$ are given by $i \omega \mu_0 \bar{\Theta}_t(\mathbf{r}_m, \mathbf{r}_{m'})$ for $m \neq m'$, and zero otherwise. Note that $\mu_0$ is the permeability of the background free space and $\bar{\Theta}_t(\mathbf{r}, \mathbf{r'}) = \bar{G}_0(\mathbf{r}, \mathbf{r'}) + \bar{G}_t(\mathbf{r}, \mathbf{r'})$ is the half space dyadic Green’s function with both $\mathbf{r}$ and $\mathbf{r'}$ in the upper medium $(z > 0$ and $z' > 0)$, where $\mathbf{r} = (x, y, z)$ and $\mathbf{r'} = (x', y', z')$. The total Green’s function includes free-space Green’s function $\bar{G}_0(\mathbf{r}, \mathbf{r'})$ and the reflection Green’s function $\bar{G}_t(\mathbf{r}, \mathbf{r'})$ due to the presence of the substrate [3].

Consider the electric field $\mathbf{E}(j, i)$ received at the $j$th measurement position $\mathbf{r}_j^i$ under the $i$th illumination with the incidence angle $\theta_{inc}^i$. The incidence plane wave is a linear composition of TE and TM waves. If the incidence wavevector is in the $(x, z)$ plane, the incidence wave from the substrate is given by

$$\mathbf{E}_{t,\text{sub}}^i(\mathbf{r}) = \alpha \mathbf{y} \exp(i k_0 \mathbf{k}_{\text{sub}} \cdot \mathbf{r}) + (1 - \alpha) \mathbf{y} \times \mathbf{k}_{\text{sub}} \exp(i k_0 \mathbf{k}_{\text{sub}} \cdot \mathbf{r})$$

(3)
where \( k_s = \omega (\mu_0 \epsilon_s)^{1/2} \), \( \mathbf{k}_{\text{sub}} = (\sin \theta_{\text{inc}} \hat{x} + \cos \theta_{\text{inc}} \hat{z}) \), and \( \alpha \) is a coefficient determining the portion of TE wave. Thus, the transmitted electric field, is given by

\[
\mathbf{E}_{\text{in}}(\mathbf{r}) = \left[ T_{\text{TE}} \mathbf{y} + T_{\text{TM}} (1 - \alpha) \mathbf{y} \times \mathbf{k}_t \right] \exp(ik_0 \mathbf{k}_t \cdot \mathbf{r})
\]

(4)

where \( T_{\text{TE}} \) and \( T_{\text{TM}} \) are transmission coefficients of TE and TM waves, respectively [3], and \( \mathbf{k}_t \) is the unit transmitted wavevector. If the incidence wavevector is in the \((y, z)\) plane, the transmitted electric field can be obtained similarly. The total incidence waves \( \psi_{\text{in}}^i(i) \) are obtained from (4) and (2),

\[
\bar{\psi}_{\text{in}}^i(i) = (\bar{I}_{3M} - \bar{\Phi} \cdot \bar{\Lambda})^{-1} \cdot \psi_{\text{in}}^i(i),
\]

(5)

where \( \bar{I}_{3M} \) is a 3M-dimensional identity matrix. Note that the symbol \((i)\) after \( \bar{\psi} \) denotes the \( i \)th illumination. The electric field \( \mathbf{E}(j, i) \) received at the \( j \)th measurement position \( \mathbf{r}_j \) is found to be

\[
\mathbf{E}(j, i) = \bar{G}(j) \cdot \bar{\Lambda} \cdot \bar{\psi}_{\text{in}}(i),
\]

(6)

where \( \bar{G}(j) = [\bar{G}_1(\mathbf{r}_j^1, \mathbf{r}_1), \bar{G}_1(\mathbf{r}_j^2, \mathbf{r}_2), \ldots, \bar{G}_1(\mathbf{r}_j^M, \mathbf{r}_M)] \) and the symbol \((j)\) after \( \bar{G} \) denotes the \( j \)th measurement. Since the measurement is carried out in the far field, \( \bar{G}_1(\mathbf{r}_j^s, \mathbf{r}_k) \) can be approximated by

\[
\bar{G}_E(\mathbf{r}, \mathbf{r}') = \frac{\exp(ik_0 \mathbf{r} \cdot \mathbf{r}')}{4\pi r} \left[ \exp(-ik_0 \mathbf{r} \cdot \mathbf{r}') (\bar{I}_3 - \bar{\rho} \bar{\rho}^*) + \exp(-ik_0 \mathbf{r} \cdot \mathbf{r}') (R_0 \mathbf{p}_E \bar{\rho} \mathbf{p}_E + R_M \mathbf{p}_M \bar{\rho} \mathbf{p}_M) \right],
\]

(7)

where the diacritical mark (‘ˀ’) denotes the flip of the sign of \( z \) in the expression, \( R_0 \) and \( R_M \) are the reflection coefficients of TE and TM waves [3], respectively, the vector \( \mathbf{p}_E \) is given by \(-\bar{\rho} \hat{x} + \bar{\rho} \hat{y} \), the vector \( \mathbf{p}_M \) is given by \(-\bar{\rho} \hat{x} - \bar{\rho} \hat{y} + \bar{\rho} \hat{z} \), \( \rho = (x^2 + y^2)^{1/2} \), and \( r = |\mathbf{r}| \).

The electric fields \( \mathbf{E}(j, i) \) for all incidences and measurements can be written in a matrix of size \( 3N_r \times N_t \), referred to as the multistatic response (MSR) matrix,

\[
\bar{K} = \bar{R} \cdot \bar{\Lambda} \cdot (\bar{I}_{3M} - \bar{\Phi} \cdot \bar{\Lambda})^{-1} \cdot \bar{T},
\]

(8)

where \( \bar{R} = \left\{ [\bar{G}(1)]^T, [\bar{G}(2)]^T, \ldots, [\bar{G}(N_t)]^T \right\}^T \) is of size \( 3N_r \times 3M \) and \( \bar{T} = \left\{ \bar{\psi}_{\text{in}}(1)^T, \bar{\psi}_{\text{in}}(2)^T, \ldots, \bar{\psi}_{\text{in}}(N_t)^T \right\}^T \) is of size \( 3M \times N_t \). It is highlighted that the conditions \( 3M < N_t \) and \( 3M < 3N_r \) are assumed throughout the paper. In this case, the rank of \( \bar{K} \) is up to \( 3M \).

3. Inverse scattering problem

When incidence waves contain both TE and TM components, i.e., \( \alpha \neq 0 \) and \( \alpha \neq 1 \), three independent electric dipole components are induced in each scatterer. Eq.(8) shows that the range \( S_t \) of the MSR matrix \( \bar{K} \) is spanned by \( \mathbf{G}_x(\mathbf{r}_j), \mathbf{G}_y(\mathbf{r}_j) \) and \( \mathbf{G}_z(\mathbf{r}_j) \), that are the \([3(j - 1) + 1]^{\text{th}}, [3(j - 1) + 2]^{\text{th}}, \text{and } [3(j - 1) + 3]^{\text{th}}\) columns of matrix \( \bar{R} \), respectively, i.e., \( S_t = \text{span} \{ \mathbf{G}_x(\mathbf{r}_j), \mathbf{G}_y(\mathbf{r}_j), \mathbf{G}_z(\mathbf{r}_j); j = 1, 2, \ldots, M \} \). The singular value decomposition of the MSR matrix can be represented as \( \bar{K} \cdot \bar{v}_p = \sigma_p \bar{u}_p \) and \( \bar{K}^* \cdot \bar{u}_p = \sigma_p \bar{v}_p \), where superscript * denotes the Hermitian. Since the range \( S_n = \text{span} \{ \bar{u}_p, \sigma_p > 0 \} \) is orthogonal to the noise space \( S_n = \text{span} \{ \bar{u}_p, \sigma_p = 0 \} \) and the range can also be expressed as the span of \( \{ \mathbf{G}_x(\mathbf{r}_j), \mathbf{G}_y(\mathbf{r}_j), \mathbf{G}_z(\mathbf{r}_j); j = 1, 2, \ldots, M \} \), we have \( \bar{u}_p^* \bar{G}_l(\mathbf{r}_j) = 0 \) for \( \sigma_p = 0, j = 1, 2, \ldots, M, \) and \( l = x, y, z \). The MUSIC pseudo-spectrum is defined as

\[
\Phi(\mathbf{r}) = \log_{10} \left( \frac{1}{p_0} \sum_{\sigma_p = 0} \left| \bar{u}_p^* \bar{f}(\mathbf{r}) \right|^2 \right)^{-1/2},
\]

(9)

where \( p_0 \) is the number of vanishing singular values, the test function \( \bar{f}(\mathbf{r}) \) can be any linear combination of \( \mathbf{G}_x(\mathbf{r}), \mathbf{G}_y(\mathbf{r}), \) and \( \mathbf{G}_z(\mathbf{r}) \), and \( \bar{f}(\mathbf{r}) = \bar{f}(\mathbf{r}) / |\bar{f}(\mathbf{r})| \) is a normalized vector. The
pseudo-spectrum becomes infinite at positions of scatterers. In evaluating the pseudospectrum at each position of the investigated domain, the far field Green’s function is calculated. Since it is computationally easy to calculate the far field Green’s function that is shown in (7), MUSIC pseudospectrum can be obtained quickly.

If incidence waves contain only TE or TM components, dipoles in certain directions would not be induced and some dipoles become linearly dependent in some scenarios. In this case, the rank of the MSR matrix is less than 3M, and the test function \( \hat{f}(\mathbf{r}) \) should be a proper linear combination of \( \mathbf{G}_x(\mathbf{r}), \mathbf{G}_y(\mathbf{r}), \) and \( \mathbf{G}_z(\mathbf{r}) \) in order to locate positions of the scatterers.

After the spheres are located by the MUSIC method, their polarization tensors can also be determined. When multiple scattering effect is considered, the inverse problem of retrieving polarization tensors is nonlinear. Despite the nonlinearity, there are non-iterative analytical algorithms for the problem [13,15,16]. The following two-step least squares method is a modified version of the method proposed in [13].

Let \( \mathbf{J} = \tilde{\mathbf{A}} \cdot (\tilde{\mathbf{T}}_{3M} - \tilde{\Phi} \cdot \tilde{\Lambda})^{-1} \cdot \tilde{T} \) and treat it as an unknown in (8), whose least squares solution is given by \( \mathbf{J} = \tilde{\mathbf{T}}^1 \cdot \tilde{K} \), where \( \dagger \) denotes pseudoinverse of a matrix [17]. On the other hand, \( \mathbf{J} \) can be rewritten as
\[
\mathbf{J} = \tilde{\mathbf{A}} \cdot (\tilde{T} + \tilde{\Phi} \cdot \tilde{J}).
\]

Since the matrix \( \tilde{\mathbf{A}} \) consists of diagonal 3 \times 3 sub-matrices, the \( i \)-th sub-matrices \( \tilde{\mathbf{A}}_i \) is obtained by the least squares method,
\[
\tilde{\mathbf{A}}_i = \tilde{\mathbf{J}}_i \cdot \left[ (\tilde{T} + \tilde{\Phi} \cdot \tilde{J})^T \right]_i^T \quad i = 1, 2, \ldots, M.
\]

where \( \tilde{\mathbf{J}}_i \) denotes the corresponding \( i \)-th 3-row sub-matrices of \( \mathbf{J} \).

It is stressed that (11) yields exact solution when there is no noise in the measurement of MSR matrix \( \tilde{K} \) and the positions of scatterers are correctly estimated by MUSIC pseudo-spectrum.

4. Numerical examples

In numerical simulations, the permittivity of the substrate \( \varepsilon_s \) is equal to 2.25\( \varepsilon_0 \). There are 33 plane waves coming from the substrate: 17 plane waves are in the \((x, z)\) plane with incidence angles evenly distributed in the range \((-90^\circ, 90^\circ)\), and 16 plane waves are in the \((y, z)\) plane with only the normal incidence wave absent compared with the incidence waves in the \((x, z)\) plane. The scattered field is measured in the far field on a 10-by-10 grid with increment 2\( \lambda \). The grid is in the \( z = 5\lambda \) plane and its center is directly above the origin.

4.1. Influence of the polarization of incidence waves

Consider two spheres located at \( \mathbf{r}_1 = (0.05\lambda, 0, 0.1\lambda) \) and \( \mathbf{r}_2 = (-0.05\lambda, 0, 0.1\lambda) \), respectively. The radii of both spheres are \( \lambda/30 \) and both of their permittivities are 10\( \varepsilon_0 \tilde{\varepsilon}_3 \). The influence of the polarization of the incidence wave on MUSIC imaging will be investigated. Noise-free synthetic data generated from (8) are used in the following numerical examples to test the influence of the polarization.

First, consider TM incidence, i.e., \( \alpha \) in (3) is equal to 0. Green’s functions corresponding to test electric dipoles with orientations \( \mathbf{x}, \mathbf{y}, \mathbf{z}, \) and \( \frac{1}{\sqrt{3}}(\mathbf{x} + \mathbf{y} + \mathbf{z}) \) are used to produce MUSIC pseudospectrum, respectively, and the imaging results are shown in Fig. 1. The pseudospectra generated from \( \mathbf{x} \) and \( \mathbf{z} \) oriented dipoles correctly locate the two small spheres (Fig. 1(a) and 1(c)), whereas it fails to do so in the other two cases. This is because the induced \( \mathbf{y} \) oriented dipoles in the spheres are always dependent. For the TM incidence wave with wavevector \( \mathbf{k}_0^{\text{ub}} \) in the \((x, z)\) plane, the induced dipoles are also oriented in the \((x, z)\) plane. Similarly, incidence wavevectors in the \((y, z)\) plane induce dipoles oriented in the \((y, z)\) plane. Therefore, only incidences with wavevectors in the \((y, z)\) plane play the role when the test dipole is \( \mathbf{y} \) oriented. However, the positions of the two spheres differ only in the \( x \) component. Therefore,
the incident electric fields onto the two spheres are equal, so as the induced electric dipoles. Mathematically, the Green’s function corresponding to $\hat{y}$ oriented dipoles at the sphere, $\vec{G}_y(r_1)$ or $\vec{G}_y(r_2)$, individually is not in the range of $\vec{K}$. Instead, the Green’s function corresponding to an array of $\hat{y}$ oriented dipoles, $\vec{G}_y(r_1) + \vec{G}_y(r_2)$, is in the range of $\vec{K}$. Since the dependency of $\hat{y}$ induced dipoles is due to the positions of spheres, a slight perturbation of the position of one of the spheres may lead to correct imaging result. Fig. 3 shows the result for the same case as Fig. 1(b) except that the $y$ coordinate of the first sphere is slightly shifted to 0.001$\lambda$.

Second, consider TE incidence, i.e., $\alpha$ in (3) is equal to 1. Fig. 2 shows the MUSIC pseudospectra generated from the Green’s functions corresponding to the four aforementioned dipoles. Only the pseudospectrum generated from $\hat{y}$ oriented dipole correctly locates the spheres (Fig. 2(b)). Whereas the result shown in Fig. 2(a) is attributed to the dependency of the $\hat{x}$ oriented induced dipoles, similar to the cases of Fig. 1(b) and 1(d), the results shown in Fig. 2(c) and 2(d) are due to the fact that there is no $z$ component in the induced dipoles under TE incidences.

Last, consider a mixture of TE and TM incidences, i.e., $0 < \alpha < 1$. Fig. 4 shows the MUSIC pseudospectrum for $\alpha = 0.3$. Since all $\hat{x}$, $\hat{y}$, and $\hat{z}$ oriented dipoles are induced within each sphere and each component of the dipoles in one sphere is linearly independent on the same component in the other sphere, MUSIC pseudospectrum corresponding to every test dipole is able to locate the spheres.

In summary, it is obvious to conclude that the MUSIC imaging using a combination of TE and TM incidence works robustly regardless of the orientation of test dipole.
which is smaller than the poor resolution of type B incidence is due to small magnitude of the transverse wavevector, $k_{\perp}$.

Fig. 5(b) shows that type B incidence fails to locate spheres at both $20 \log_{10}$ measured MSR levels. The noise level is defined to be the signal-to-noise ratio in dB that is respectively.

(8), and then white Gaussian noise is added to the imaging. To include noise effect, noise-free and TM incidences. To investigate the influence of evanescent and propagating waves on imaging, different types of incidence are chosen. The 33 incidence waves introduced in the beginning of Section 4, which contain only evanescent waves, are defined to be incidence type A. The incidence waves with 33 wavevectors evenly distributed in the range $(\psi, 0)$, $(0, 0)$, $(\psi, 0)$, and $(0, -\psi/6, 0)$, respectively.

Since the permittivity of the substrate is $2.25\varepsilon_0$, the critical angle ($\theta_c$) is approximately $42^\circ$. To investigate the influence of evanescent and propagating waves on imaging, different types of incident waves are chosen. The 33 incidence waves introduced in the beginning of Section 4, evenly distributed in the range $(-90^\circ, 90^\circ)$ and containing both evanescent and propagating waves, are referred to as incidence type A. The incidence waves with 33 wavevectors evenly distributed in the range $\theta^{inc}_i \in (-42^\circ, 42^\circ)$, which contain only propagating waves, are defined to be incidence type B. The incidence waves with 33 wavevectors evenly distributed in the range $\theta^{inc}_i \in (-90^\circ, -42^\circ) \cup (42^\circ, 90^\circ)$, which contain only evanescent waves, are defined to be incidence type C. In order to reduce the effect of unbalanced aperture and at the same time keep both propagating and evanescent wave incidences, incidence type D is defined to be the incidence waves with the same total number of incidence wavevectors evenly distributed in the range $\theta^{inc}_i \in (-66^\circ, -21^\circ) \cup (21^\circ, 66^\circ)$.

The MUSIC pseudospectra under the four different types of incidences are shown in Fig. 5. In the numerical example, 30dB white Gaussian noise is added, the incidence waves are a mixture of TE and TM modes with $\alpha = 0.3$, and the test dipole direction is chosen to be $\frac{\hat{x} + \hat{y} + \hat{z}}{\sqrt{3}}$. Fig. 5(b) shows that type B incidence fails to locate spheres at both $z = 0.1\lambda$ and $z = 0.9\lambda$ levels. The poor resolution of type B incidence is due to small magnitude of the transverse wavevector, which is smaller than $k_0$. In comparison, the magnitude of the transverse wavevector is up to
1.5k0 for the incidence type C. In Fig. 5(c), type C incidence produces much better imaging results at level of z = 0.1λ, where the sphere at (−0.1λ, −0.1λ, 0.1λ) is almost identified. The two spheres at level z = 0.9λ are not detected. This is because evanescent waves decay away from the substrate surface and the induced dipoles at level z = 0.9λ are relatively weak. The MUSIC pseudospectra of both incidences type A and type D correctly locate the spheres at level z = 0.1λ, as shown Fig. 5(a) and 5(d). However, at level z = 0.9λ, the presence of weak evanescent waves only slightly improves the imaging result compared with Fig. 5(b), and due to the noise, spheres at this level cannot be clearly identified.

Compared with propagating wave incidences, evanescent wave incidences have better resolution ability but weaker robustness in presence of noise. Fig. 6 shows the MUSIC imaging result when 50dB white Gaussian noise is added, with other conditions the same as those in Fig. 5. The four spheres at both z = 0.1λ and z = 0.9λ levels are correctly located when the incidences contain both propagating and evanescent waves, as shown in Fig. 6(a) and Fig. 6(d). In comparison, the imaging ability is weaker if only propagating or evanescent waves are contained in the incidence.

After the positions of spheres are obtained, their polarization tensors can be obtained by the non-iterative inversion method introduced in Section 3. The accuracy of the estimation of polarization tensors is quantified by using a percent error that is defined as
\[ E = \left( \sum_{i=1}^{M} |\vec{\Lambda}_i - \vec{\Lambda}_i'|^2 \right)^{1/2} \cdot \left( \sum_{i=1}^{M} |\vec{\Lambda}_i|^2 \right)^{-1/2} \times 100\%, \quad (12) \]

where \( \vec{\Lambda}_i' \) is the retrieved polarization tensor. For the incidence type A with 50dB noise, the normalized percent error is \( E = 0.024\% \).

### 4.3. Influence of the substrate

Due to the presence of the substrate, the magnitude of transverse wavevector is larger than that of free space \( (k_0) \) when the incidence angle is larger than the critical angle. If the substrate is absent, all of the aforementioned four types of incidences contain only propagating waves with magnitudes of transverse wavevectors smaller than \( k_0 \). Numerical simulations are carried out under the same condition as Fig. 5 except that the substrate is replaced by free space. For brevity, the results are not shown in this paper. The results indeed show that the imaging results under incidence type A and D are not as good as those shown in Fig. 5.

### 5. Conclusion

This paper deals with the inverse scattering problem of determining the locations and polarization tensors of a collection of small anisotropic spherical scatterers in the framework of the total internal reflection tomography. Multiple scattering effect is accounted and the inverse scattering problem is nonlinear. MUSIC imaging algorithm is proposed to determine the locations of the spheres and least squares method is proposed to retrieve their polarization tensors. Both methods are analytical approaches and there is no iterative evaluation of the associated forward scattering problem. The MUSIC algorithm is found to be more effective under the incidence with a mixture of TE and TM polarization. The influences of the noise, the substrate, the incidence angle, and the distance of scatterers to the substrate on imaging are also investigated, and it is observed that evanescent waves, characterized with large transverse wavevectors, help to improve the resolution, whereas they are vulnerable in presence of noise, especially when the scatterers are far from the surface of the substrate. The proposed method provides a new tool in the application of sub-wavelength imaging of small objects in the framework of total internal reflection tomography.

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