Algebras with non-periodic bounded modules

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Abstract

We study weakly symmetric special biserial algebra of infinite representation type. We show that usually some socle deformation of such an algebra has non-periodic bounded modules. The exceptions are precisely the algebras whose Brauer graph is a tree with no multiple edges. If the algebra has a non-periodic bounded module then its Hochschild cohomology cannot satisfy the finite generation property (Fg) introduced in [10].

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1 Introduction

Assume $\Lambda$ is a finite-dimensional selfinjective algebra over some field $K$. If $M$ is a finite-dimensional non-projective $\Lambda$-module, let

$$\ldots \rightarrow P_n \xrightarrow{d_n} P_{n-1} \rightarrow \ldots \rightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \rightarrow 0$$

be a minimal projective resolution of $M$. The module $M$ is called bounded if the dimensions of the projectives $P_n$ have a common upper bound, that is, $M$ has complexity one. The kernel of $d_n$ is the syzygy $\Omega^n(M)$; we say that the module $M$ is periodic if $\Omega^d(M) \cong M$ for some $d \geq 1$. A periodic module has complexity one but the converse need not hold. We call a module $M$ a criminal if it has complexity one but is not periodic. We would like to understand which algebras have criminals.

J. Alperin proved in [1] that the group algebra of a finite group does not have criminals when the coefficient field is algebraic over its prime field. On the other hand, R. Schulz discovered that there are four-dimensional selfinjective algebras which have criminals, see [17]. In the context of commutative algebra, there is a similar problem. Eisenbud proved in [7] that for complete intersections, if a finitely generated module has bounded Betti numbers then it is eventually periodic. He conjectured that this should be true for any commutative Noetherian local ring. However, counterexamples were constructed by Gasharov and Peeva [14].

Subsequently, a theory of support varieties was developed for modules of group algebras of finite groups. This is based on group cohomology and depends crucially on that the fact that it is Noetherian. It follows from this theory that a group algebra over an arbitrary field does not have criminals, so that Alperin’s theorem holds in general; for a proof see 2.24.4 in [3]. More recently, a support variety theory was
developed for modules of selfinjective algebras, based on Hochschild cohomology \[18, 10\]. This also requires suitable finite generation, namely the Hochschild cohomology $HH^\ast(\Lambda)$ should be Noetherian and the ext-algebra of $\Lambda$ should be finitely generated as a module over $HH^\ast(\Lambda)$. This condition is called (Fg) in [19], it is equivalent to (Fg1, 2) in [10]. Again, if (Fg) holds for $\Lambda$, so that $\Lambda$-modules have support varieties, then $\Lambda$ does not have criminals (see 5.3 in [10]).

The algebras studied by Schulz therefore do not satisfy (Fg). More generally, weakly symmetric algebra with radical cube zero were investigated in [11], [12]. The algebras in these papers which have criminals happen to be special biserial, therefore one may ask when a special biserial weakly symmetric algebra has criminals. Of course, if an algebra has a chance to have criminals it must have infinite representation type.

Here we study special biserial weakly symmetric $K$-algebras of infinite representation type, we assume $K$ is an algebraically closed field which contains non-roots of unity. An algebra is special biserial weakly symmetric if its basic algebra satisfies 2.1. Existence of criminals is invariant under Morita equivalence, and we will work throughout with basic algebras. We assume that the algebra is indecomposable, so that its quiver is connected. The algebras in 2.1 have socle relations involving scalar parameters, so that we have a family of algebras, which we write as $\Lambda_q$ where $q$ is the collection of the the socle scalars. Each algebra in a family is a socle deformation of the algebra $\Lambda_1$ for which all socle scalars are equal to 1.

Recall that if $\Lambda$ and $\Gamma$ are selfinjective, then $\Gamma$ is a socle deformation of $\Lambda$ if $\Gamma/soc(\Gamma)$ is isomorphic to $\Lambda/soc(\Lambda)$. For example when the field has characteristic 2 then the algebra studied by Schulz is a socle deformation of the group algebra of a Klein 4-group. There are similar socle deformations for group algebras of dihedral 2-groups, which are also special biserial and weakly symmetric.

Our main result answers when there is some choice for $q$ such that the algebra $\Lambda_q$ has criminals. The algebra $\Lambda_q$ has a Brauer graph $G_{\Lambda}$, which is independent of $q$, we define it in section 4. The Brauer graph generalises the Brauer tree for a block of a group algebra of finite type. For the algebras of infinite type, this graph is usually not a tree. We will prove the following:

**Theorem 1.1** Let $K$ be an algebraically closed field which contains some non-roots of unity. Assume $\Lambda_q$ is a family of indecomposable weakly symmetric special biserial $K$-algebras of infinite type. Then the following are equivalent:

(a) $\Lambda_q$ does not have criminals for any $q$.

(b) The Brauer graph $G_{\Lambda}$ is a tree with no multiple edges.
From the perspective of group representation theory, one might have expected that 'most' selfinjective algebras should satisfy (Fg) and have a support variety theory. Our Theorem suggests that this may not be the case.

A special biserial weakly symmetric algebra $\Lambda_q$ of finite type does not have criminals since it cannot have a non-periodic module, and its Brauer graph is a tree. As well, such an algebra is symmetric and is isomorphic to $\Lambda_1$. For such an algebra, there is a unique $\Omega$-orbit consisting of maximal uniserial modules and simple modules whose projective cover is uniserial. In the case of a block of a group algebra, this is due to J.A. Green [15], but it holds for arbitrary symmetric indecomposable Brauer tree algebras of finite type. Assume $\Lambda_q$ is weakly symmetric of infinite type whose Brauer graph is a tree. Then there is also a unique $\Omega$-orbit consisting of maximal uniserial modules generated by arrows and simple modules whose projective cover is uniserial, this follows from [16] and Section 4.1 in [6] when the algebra is symmetric.

Given the presentation of the algebra as in 2.1, it is easy to determine its Brauer graph. Our Theorem shows that the group algebra of a dihedral 2-group over characteristic 2 has a socle deformation with criminals, as well the group algebra of the alternating group $A_4$, or also other special biserial algebra occurring in algebraic Lie theory, see for example [13]. Algebras of dihedral type, as defined in [8], are special biserial and symmetric, hence they are examples for the algebras in the theorem. Of the ones which occur as blocks, only one has a Brauer graph which is a tree. The Hecke algebras of type $A$ which have tame representation type are also special biserial, there are two Morita equivalence classes, see [9]. Both have Brauer graphs which are trees.

In Section 2, we define the algebras and summarise properties we need. In Section 3 we recall the definition of band modules, and we determine the $\Omega$-translates of certain band modules. Using this we prove the Theorem in Section 4.

It would be interesting to know whether the algebras in Theorem 1.1 for which the Brauer graph is a tree, always satisfy (Fg). More generally, one may ask whether any selfinjective algebra where modules have finite complexity, and which does not have criminals, must satisfy (Fg).
2 The algebras

Assume $\Lambda = KQ/I$ where $Q$ is a finite connected quiver and $I$ is an admissible ideal of the path algebra $KQ$, generated by a set of relations $\rho$.

**Definition 2.1** The algebra $\Lambda$ is special biserial and weakly symmetric if it satisfies the following:

1. Any vertex of $Q$ is either the source of two arrows and is the target of two arrows, or it is the source of one arrow and is the target of one arrow. We say that the vertex has valency two, or one, respectively.

2. If two different arrows $\alpha$ and $\beta$ start at vertex $i$, then for an arrow $\gamma$ ending at $i$, precisely one of the two paths $\gamma \alpha$ or $\gamma \beta$ is in $\rho$.

3. If two different arrows $\gamma$ and $\delta$ end at vertex $i$, then for an arrow $\alpha$ starting at $i$, precisely one of the two paths $\alpha \gamma$ or $\alpha \delta$ is in $\rho$.

4. For each vertex $i$ of $Q$ of valency two, there are different paths $C_i$ and $D_i$ of length $\geq 2$ starting and ending at $i$, and non-zero scalars $p_i, q_i \in K$ such that $p_iC_i + q_iD_i$ belongs to $\rho$.

5. For each vertex $i$ of $Q$ of valency one, there is a path $C_i$ of length $\geq 2$ such that $C_i \alpha$ belongs to $\rho$ where $\alpha$ is the arrow starting at $i$.

6. Any rotation of a path $C_i$ or $D_i$ in 4. or 5. is a path occurring in the relation of the form 4. or 5.

7. The set $\rho$ consists precisely of the relations described above.

The algebra depends on the parameters $p_i, q_i$, and we write $\Lambda = \Lambda_{q_i}$. We refer to relations 4. and 5. as 'socle relations'. The definition of a special biserial algebra is slightly more general, details may be found for example in [5]. If such an algebra has infinite representation type then there must be at least one vertex with valency two.

**Remark 2.2** (1) We identify as usual paths with the image in the algebra. So for example in $\Lambda$ we have $p_iC_i + q_iD_i = 0$ but $C_i$ and $D_i$ are non-zero. The element $C_i$ spans the socle of the indecomposable projective $e_i\Lambda$, and using this, one can write down a non-degenerate bilinear form on $\Lambda$ verifying that the algebra is indeed selfinjective. Hence it is weakly symmetric, noting that the simple quotient and the socle of $e_i\Lambda$ are isomorphic.
(2) Let $e_i$ be a vertex of $Q$. Then the indecomposable projective module $e_i\Lambda$ has a basis consisting of all proper initial subwords of $C_i$ and $D_i$ of positive length, together with $e_i$ and one of $C_i$ or $D_i$. In particular $\dim e_i\Lambda = |C_i| + |D_i|$ where we write $|\eta|$ for the number of arrows in $\eta$.

**Definition 2.3** Given a special biserial weakly symmetric algebra, there is an associated permutation $\sigma$ of the arrows of $Q$: For each arrow $\alpha$ of $Q$, define $\sigma(\alpha)$ to be the unique arrow such that $\alpha \cdot \sigma(\alpha)$ is non-zero in $\Lambda$.

We may write $\sigma$ as a product of disjoint cycles. Any monomial $C_i$ occurring in a socle relation is then the product $(\alpha_1 \alpha_2 \ldots \alpha_r)^m$ where $(\alpha_1 \alpha_2 \ldots \alpha_r)$ is a cycle of $\sigma$, and $m \geq 1$. Similarly $D_i = (\beta_1 \beta_2 \ldots \beta_s)^t$ where the product of the $\beta_j$ is taken over a cycle of $\sigma$. This cycle may or may not be the same as the cycle $(\alpha_1 \alpha_2 \ldots \alpha_r)$.

### 2.1 Candidates for criminals

The indecomposable non-projective $\Lambda$-modules are classified, they are ‘strings’ or ‘bands’. A description, and further details, may be found in [5] or [8]. A candidate to be a criminal must have complexity one.

1. If $M$ is a string module and is not of the form $\alpha\Lambda$ or $e\Lambda/\alpha\Lambda$ for an arrow $\alpha$ starting at vertex $e$, or is in the Auslander-Reiten component of such module, then $M$ has complexity $\geq 2$. One can see this for example by considering its Auslander-Reiten component translates $\tau^r M$ for $r \geq 2$. For a selfinjective algebra, the Auslander-Reiten translation $\tau$ is isomorphic to $\Omega^2 \circ \nu$ where $\nu$ is the Nakayama automorphism of the algebra, see for example [2], IV.3.7. As one can see from the construction of irreducible maps, the dimensions of $\tau^r(M)$ are unbounded for $r \geq 1$. Hence the dimensions of the modules $\Omega^{2r}(M)$ are also unbounded, which implies that $M$ has complexity $\geq 2$.

2. There are finitely many Auslander-Reiten components containing string modules of the form $\alpha\Lambda$ or $e\Lambda/\alpha\Lambda$. These string modules are permuted by $\Omega$ and hence are $\Omega$-periodic, since the set of arrows is finite. The action of $\Omega$ induces an equivalence of the stable module category, and it commutes with $\tau$, and it follows that all modules in these components must be periodic with respect to $\Omega$, and cannot be criminals.

3. Band modules have complexity one. They are parametrized by a band word $W$, a non-zero scalar $\lambda$, and a non-zero vector space $V$, we give details below. If $\lambda$ and $W$ are fixed, the corresponding band modules as $V$ varies, form one Auslander-Reiten component. Again, since $\Omega$ induces an equivalence of the stable module category and commutes with $\tau$, the component contains a criminal if and only if the band
module with $V = K$ is a criminal. Therefore we can focus on band modules where the space is $K$.

(4) We also note that a special biserial algebra has infinite representation type if and only if there are band modules. This is proved in [20], Theorem 1 and Lemma 2 (with the terminology of primitive $V$-sequences, for translating terminology, see [21]).

3 Band modules

We are looking for criminals, and therefore we focus on band modules. We start by describing the parameter set. It is convenient to identify a vertex $i$ of $Q$ with the corresponding idempotent $e_i$ of the path algebra $KQ$.

**Definition 3.1** Let $e_0, \ldots, e_m$ and $f_0, \ldots, f_m$ be vertices in $Q$ of valency two, and let $e_{m+1} = e_0$. A band word $W$ is a sequence $(a_i, b_i)_{i=0}^m$ where the $a_i$ and $b_i$ are paths in $Q$ between vertices of valency two, where $a_i : e_i \mapsto f_i$ for $0 \leq i \leq m$, and $b_i : e_i \mapsto f_{i-1}$ for $1 \leq i \leq m + 1$, such that $a_i$ and $b_{i-1}$ are proper initial subpaths of the $C_i$ and $D_i$. Moreover, the sequence $(a_i, b_i)$ must be minimal with these properties.

That is, there is no shorter sequence $(\tilde{a}_i, \tilde{b}_i)$ with the same properties such that $(a_i, b_i)$ is the concatenation of copies of $(\tilde{a}_i, \tilde{b}_i)$.

The band word $W$ may be described by a quiver:

$$e_0 \xrightarrow{a_0} f_0 \leftarrow e_1 \xrightarrow{a_1} f_1 \leftarrow \cdots f_m \xleftarrow{b_m} e_{m+1} = e_0$$

Note that we do not specify the the names of the arrows occurring in the paths $a_i, b_i$, since we will not need these. For details, we refer to [5] or [8].

For example, if all vertices of the quiver have valency two, then there is such band word where all the $a_i$ and $b_i$ are arrows. In this case, the minimality condition holds precisely if all the $e_i$ are distinct, equivalently if all the $f_i$ are distinct.

**Definition 3.2** The band module $M(\lambda)$ associated to the band word $W$ as in 3.1 and a vector space $V$, labelled by a parameter $0 \neq \lambda \in K$, is defined as follows:

1. For each vertex along the paths $a_i$ and $b_i$, except the for the start vertex of $b_m$, we take a copy of $V$. We identify the space at the start of $b_m$ with the space at the start of $a_0$.

2. The first arrow of $a_0$ acts by multiplication with an indecomposable Jordan block matrix with eigenvalue $\lambda$. 


3. All other arrows occurring in the paths \(a_i, b_i\) act as identity.

We will only take \(V = K\), then the first arrow of \(a_0\) is multiplication by \(\lambda\). The module has dimension \(\sum_{i=0}^{m} |a_i| + |b_i|\) where \(|\eta|\) is the number of arrows in the path \(\eta\). It is indecomposable, and \(M(\lambda) \cong M(\mu)\) only if \(\lambda = \mu\).

**Remark 3.3** The arrow which acts by a non-identity scalar need not be the first arrow of \(a_0\). There are variations which give isomorphic modules, details are discussed in [5], or [21].

**Example 3.4** To illustrate the shorthand notation, let \(m = 0\) and \(a_0 = \alpha_1\alpha_2\alpha_3\), and \(b_0 = \beta\), then the word written in detail is

\[
e_0 \xrightarrow{\alpha_3} \cdot \xrightarrow{\alpha_3} \cdot \xrightarrow{\alpha_3} f_0 \leftarrow e_0.
\]

The module \(M(\lambda)\) as defined in 3.2 associated to this word and \(V = K\), is four-dimensional.

We fix the word \(W\), and the module \(M(\lambda)\), and we will now determine \(\Omega\)-translates for \(M(\lambda)\). Note that \(\Omega^2(M(\lambda))\) will be a band module defined by the same word \(W\), and therefore we only need to calculate two steps. This requires using the socle relations for the vertices \(f_t\) and \(e_t\) occurring in the word \(W\). We fix now the notation for these, so that we can keep track over the paths \(a_t\) and \(b_t\).

**Notation 3.5** We write the socle relation relation starting and ending at the vertex \(f_t\) in the form

\[
(\theta_t) \quad p_t(A_t a_t) + q_t(B_t b_t) = 0,
\]

where \(A_t, B_t\) are paths, and \(p_t, q_t\) are non-zero scalars. Here

\[
A_t : f_t \rightarrow e_t, \quad B_t : f_t \rightarrow e_{t-1}
\]

(taking indices modulo \(m + 1\)). Similarly we write the socle relations starting and ending at vertex \(e_t\) in the form

\[
(\theta'_t) \quad p'_t(a_t A_t) + q'_t(b_{t-1} B_{t-1}) = 0,
\]

where \(p'_t\) and \(q'_t\) are non-zero scalars.
Remark 3.6 Note that with this notation, we have $a_t B_t = 0 = b_t A_t$ for each $t$. For example, let $\alpha$ be the last arrow of $a_t$. Since $a_t A_t$ is non-zero, we know that $A_t$ must start with $\sigma(\alpha)$ where $\sigma$ is the permutation defined in 2.3. The first arrow of $B_t$ is the other arrow starting at $f_t$, say this is $\beta$, and by condition 2. of Definition 2.1 we have $\alpha \beta = 0$ and hence $a_t B_t = 0$. Similarly we have $A_t b_{t-1} = 0 = B_t a_{t+1}$.

Proposition 3.7 Let $v := \prod_{i=0}^{m} (q_i/p_1) \prod_{i=0}^{m} (p'_i/q'_i)$. Then $\Omega^2(M(\lambda))$ is isomorphic to $M(v\lambda)$.

Remark 3.8 Hence we have that $\Omega^{2r}(M(\lambda)) \cong M(v^r\lambda)$ for $r \geq 1$. This shows directly that $M(\lambda)$ has a bounded projective resolution, that is, its complexity is one. If $v = 1$ then $M(\lambda)$ has $\Omega$-period at most two, and this occurs when the algebra is symmetric. As well $v$ might be some root of unity but $v \neq 1$. If so then $M(\lambda)$ is still periodic but it can have a larger period. Then we see that $\Omega^2(M(\lambda))$ is not isomorphic to $\tau(M(\lambda))$ and we deduce that the Nakayama automorphism is non-trivial. Our main interest here is in algebras for which $v$ is not a root of unity. Note that the parameter $v$ depends only on the band word $W$ but not on $\lambda$. We say that ’$v$ is the parameter for $W$’.

3.1 The case $m = 0$

We prove the Proposition first for a band word with $m = 0$, this needs slightly different (and less) notation. In this case we have paths $a, b : e \mapsto f$ and the socle relations at $e$ and $f$ are of the form

$$p(Aa) + q(Bb) = 0, \quad p'(aA) + q'(bB) = 0$$

where $p, q, p'$ and $q'$ are non-zero scalars.

Fix some non-zero $\lambda \in K$, we want to construct the band module $M(\lambda)$ as a submodule of $f\Lambda$. That is, we look for an element $w \in f\Lambda$ such that $wa = \lambda \cdot wb$ and such that this is a non-zero element in the socle of $f\Lambda$, ie, it is a non-zero scalar multiple of $Aa$.

Definition 3.9 Let $w := \lambda A - \frac{q}{p} B \in f\Lambda$. Since $Ba = 0$ and $Ab = 0$ (see 3.5), we have

$$wa = \lambda Aa = -\lambda(q/p) Bb \quad \text{and} \quad wb = -(q/p) Bb$$
and hence $wa = \lambda wb$, and this is non-zero in the socle. Let $a = \alpha_1\alpha_2\ldots\alpha_r$ where the $\alpha_i$ are arrows, and let $b = \beta_1\ldots\beta_s$ for arrows $\beta_j$. We may write down a basis for $w\Lambda$, where each basis vector spans the 1-dimensional space at a vertex of the quiver described in 3.1, showing that $w\Lambda$ is of the form as in 3.2. Namely, take the basis
\[ w, \lambda^{-1}w\alpha_1, \lambda^{-1}w\alpha_1\alpha_2, \ldots, \lambda^{-1}wa, \ w\beta_1, \ w\beta_1\beta_2, \ w\beta_1\beta_2\ldots\beta_{s-1}. \]
Hence $w\Lambda$ is isomorphic to $M(\lambda)$.

3.1.1 The module $\Omega(M(\lambda))$.

We find $\Omega(M(\lambda))$, this can be identified with the kernel of the homomorphism
\[ \psi : e\Lambda \rightarrow w\Lambda, \ \psi(x) := wx. \]
We see that
\[ w(a - \lambda b) = \lambda Aa + (q/p)\lambda Bb = 0. \]
Hence if $\zeta := a - \lambda b \in e\Lambda$, then $\zeta\Lambda$ is a submodule of $\Omega(M(\lambda))$. We compare dimensions; the dimension of $\zeta\Lambda$ is $|A| + |B|$. As well the dimension of $w\Lambda$ is $|a| + |b|$ and hence the sum of the dimension is equal to the dimension of $e\Lambda$. It follows that $\zeta\Lambda = \Omega(M(\lambda))$.

3.1.2 The module $\Omega^2(M(\lambda))$.

We identify $\Omega^2(M(\lambda)) \cong \Omega(\zeta\Lambda)$ with the kernel of the map $\psi^+ : f\Lambda \rightarrow \zeta\Lambda$ given by left multiplication with $\zeta$. Let $w^+ := \lambda Aa - (q'/p')B$, then
\[ (a - \lambda b)w^+ = \lambda Aa + \lambda(q'/p')bB = 0. \]
As before, we compare dimensions and deduce that $w^+\Lambda = \Omega^2(M(\lambda))$.
We identify $w^+\Lambda$. First we have $w^+a = \lambda Aa$ and $w^+b = -(q'/p')Bb = Aa$ and hence
\[ w^+a = \lambda(v(w^+b)). \]
As well, this is a non-zero element in the socle of $f\Lambda$. Hence $\Omega^2(M(\lambda)) \cong M(v\lambda)$ where $v = (q/p)(p'/q')$, as stated in the Proposition.

Example 3.10 (1) Let $\Lambda$ be the local algebra with generators $x, y$ and relations
\[ x^2 = 0 = y^2, \ p(yx)^2 + q(xy)^2 = 0. \]
where $p, q$ are non-zero scalars.
We have the band word $W$ given by $a = x$ and $b = y$. The relevant socle relations are then
\[ p(Ax) + q(By) = 0 \quad \text{and} \quad p'(xA) + q'(yB) = 0 \]
with $A = yxy$ and $B = xyx$ and $p' = q, q' = p$. Therefore we have that $v = (q/p)^2$.
If $q/p$ is not a root of unity then the modules $M(\lambda)$ are criminals for the algebra with these parameters.
If $\text{char}(K) = 2$, then the algebra with $q = p = 1$ is isomorphic to the group algebra of the dihedral group of order 8.

(2) There is family of commutative special biserial local algebras with generators $x, y$, and relations
\[ xy = 0 = yx, \quad p(x^r) + q(y^s) = 0 \]
for $r, s \geq 2$ and $p, q$ non-zero scalars. We have again the band word $W$ given by $a = x$ and $b = y$. Writing down the socle relations in this case, we see that the parameter $v$ is equal to 1 in this case.

(3) Let $\Lambda$ be the algebra with quiver
\[ \begin{tikzpicture}
\node (0) at (0,0) {$\alpha$};
\node (1) at (1,0) {$1$};
\node (2) at (0,-1) {$0$};
\node (3) at (1,-1) {$\gamma$};
\path[->] (0) edge (2)
(2) edge (3)
(3) edge (1);
\end{tikzpicture} \]
and relations
\[ pa^2 + q(\beta \gamma)^s = 0, \quad \alpha \beta = 0, \gamma \alpha = 0. \]
where $p, q \neq 0$. Take the words $a = \alpha : e_0 \to e_1$ and $b = \beta \gamma : e_0 \to e_1$, then $(a, b)$ is a band word. In this case $A = \alpha$ and $B = (\beta \gamma)^{s-1}$. The two socle relations we need in this case are identical since $aA = Aa$ and $bB = Bb$. The parameter $v$ is equal to 1. When $s = 2$, this algebra occurs as a tame Hecke algebras, see \[9\].

3.2 The case $m \geq 1$

We take a band word $W$ as described in 3.1 with $m \geq 1$, and we will construct $M(\lambda)$ by specifying generators, as a submodule of $\oplus_{t=0}^{m-1} f_t \Lambda$. 


Definition 3.11 Define elements in the direct sum $\oplus_{t=0}^{m} f_t \Lambda$:

\[
\begin{align*}
v_0 & := (c_0 A_0, 0, \ldots, d_m B_m) \\
v_1 & := (d_0 B_0, c_1 A_1, 0, \ldots, 0) \\
v_2 & := (0, d_1 B_1, c_2 A_2, 0, \ldots) \\
\ldots \\
v_m & := (0, \ldots, 0, d_{m-1} B_{m-1}, c_m A_m)
\end{align*}
\]

where the $c_i$ and the $d_i$ are non-zero scalars.

With this, we have

\[
\begin{align*}
v_0 a_0 & = (c_0 (A_0 a_0), 0, \ldots, 0) \\
v_1 b_0 & = (d_0 (B_0 b_0), 0, \ldots, 0) \\
v_1 a_1 & = (0, c_1 (A_1 a_1), 0, \ldots) \\
v_2 b_1 & = (0, d_1 (B_1 b_1), 0, \ldots) \\
\ldots \\
v_m a_m & = (0, \ldots, 0, c_m (A_m a_m)) \\
v_0 b_m & = (0, 0, \ldots, d_m (B_m b_m)).
\end{align*}
\]

For any choice of scalars $c_i$ and $d_i$, the elements $v_i$ generate a submodule of $\oplus_{t=0}^{m} f_t \Lambda$, and its dimension depends only on the length of the $a_i$ and $b_i$ and one finds that the dimension is equal to $\dim M(\lambda)$. We can see from the parameters when it is isomorphic to $M(\lambda)$.

Lemma 3.12 The submodule of $\oplus_{t=0}^{m} f_t \Lambda$ generated by $v_0, v_1, \ldots, v_m$ is isomorphic to $M(\lambda)$ if and only if

\[
\lambda p_0 d_0 + q_0 c_0 = 0 \quad \text{and} \quad p_t d_t + q_t c_t = 0
\]

for $1 \leq t \leq m$.

Proof We need $v_0 a_0 = \lambda v_1 b_0$, that is

\[
c_0 (A_0 a_0) = \lambda d_0 (B_0 b_0)
\]

We have $(B_0 b_0) = -(p_0/q_0)(A_0 a_0)$ and $A_0 a_0 \neq 0$. Substituting this gives the first equation. Similarly we need $v_1 a_1 = v_0 b_1$, that is

\[
c_1 (A_1 a_1) = d_1 (B_1 b_1)
\]
Using $B_1b_1 = -(p_1/q_1)(A_1a_1)$ gives the second equation. Similarly the other equations follow. Conversely, if all these identities hold then $\sum_{t=0}^m v_t \Lambda \cong M(\lambda)$. □

We continue with the notation

$$c_0 = \lambda p_0, \quad c_i = p_i, \quad d_j = -q_j \quad (1 \leq i \leq m, \, 0 \leq j \leq m).$$

We construct the first two steps of a minimal projective resolution.

### 3.2.1 The module $\Omega(M(\lambda))$

Let $\Psi : P_0 = \bigoplus_{t=0}^m e_t \Lambda \rightarrow \bigoplus_{t=0}^m f_t \Lambda$ be the map given by left multiplication with the matrix

$$
\begin{pmatrix}
\lambda p_0 A_0 & -q_0 B_0 & 0 & \ldots & 0 & 0 \\
0 & p_1 A_1 & -q_1 B_1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & p_{m-1} A_{m-1} & -q_{m-1} B_{m-1} \\
-\lambda p_m B_m & 0 & \ldots & 0 & 0 & p_m A_m
\end{pmatrix}.
$$

Then $\Psi$ takes the standard generators of $P_0$ to $v_0, v_2, \ldots, v_m$, and hence the image of $\Psi$ is $M(\lambda)$. We know that $\Omega(M(\lambda))$ has minimal projective cover of the form

$$P_1 = \bigoplus_{t=0}^m f_t \Lambda \xrightarrow{\Psi_1} P_0$$

and $\Psi_1$ is given by left multiplication with a matrix of the form

$$
\begin{pmatrix}
r_0 a_0 & 0 & 0 & \ldots & 0 & s_m b_m \\
s_0 b_0 & r_1 a_1 & 0 & \ldots & 0 & 0 \\
0 & s_1 b_1 & r_2 a_2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & s_{m-1} b_{m-1} & r_m a_m
\end{pmatrix}.
$$

Here the $r_t$ and $s_t$ are non-zero scalars. By comparing dimensions, we see that $\text{Im}(\Psi_1) = \Omega(M(\lambda))$ if and only if the product of the matrices $\Psi \Psi_1$ is zero.

The matrix $\Psi \Psi_1$ is diagonal, with diagonal entries

$$
\begin{align*}
\lambda p_0 r_0 (A_0 a_0) + (-q_0) s_0 (B_0 b_0), \\
p_1 r_1 (A_1 a_1) + (-q_1) s_1 (B_1 b_1), \\
\vdots \\
p_t r_t (A_t a_t) + (-q_t) s_t (B_t b_t)
\end{align*}
$$

for $t \leq m$. Substitute $(-q_t) B_t b_t = p_t (A_t a_t)$ and cancel. It follows that:
Lemma 3.13 We have \( \text{Im}(\Psi_1) = \text{Ker}(\Psi) \) if and only if \( \lambda r_0 + s_0 = 0 \) and \( r_t + s_t = 0 \) \((1 \leq t \leq m)\).

We assume this now, and we identify the image of \( \Psi_1 \) with \( \Omega(M(\lambda)) \).

3.2.2 The module \( \Omega^2(M(\lambda)) \).

Let \( P_2 = P_0 = \bigoplus_{t=0}^m e_t A \), and define \( \Psi_2 : P_2 \to P_1 \) to be the map given by left multiplication with a matrix of the same form as that of \( \Psi \), that is

\[
\begin{pmatrix}
  c_0^+ A_0 & d_0^+ B_0 & 0 & \ldots & 0 \\
  0 & c_1^+ A_1 & d_1^+ B_1 & 0 & \ldots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  d_m^+ B_m & 0 & \ldots & 0 & c_m^+ A_m
\end{pmatrix}.
\]

Here the \( c_i^+ \) and the \( d_i^+ \) are again non-zero scalars. We may apply Lemma 3.12 again to identify the image of \( \Psi_2 \). That is, \( \text{Im}(\Psi_2) = M(\mu) \) where \( \mu \) is determined by the identities

\[
\mu \cdot p_0^+ d_0^+ + q_0^+ c_0^+ = 0 \quad \text{and} \quad p_t^+ d_t^+ + q_t^+ c_t^+ = 0 \quad (1 \leq t \leq m).
\]

That is

\[
(*) \quad c_0^+ = -\mu(p_0^+/q_0^+)d_0^+, \quad \text{and} \quad c_t^+ = -(p_t^+/q_t^+)d_t^+.
\]

for \( 1 \leq t \leq m \).

We require that \( \text{Im}(\Psi_2) = \text{Ker}(\Psi_1) \). By comparing dimension, this is again equivalent with \( \Psi_1 \Psi_2 = 0 \). The matrix \( \Psi_1 \Psi_2 \) is diagonal, with diagonal entries

\[
c_0^+ r_0(a_0 A_0) + d_m^+ s_m(b_m B_m) \\
c_1^+ r_1(a_1 A_1) + d_0^+ s_0(b_0 B_0) \\
\quad \vdots \\
c_t^+ r_t(a_t A_t) + d_{t-1}^+ s_{t-1}(b_{t-1} B_{t-1})
\]

(for \( 1 \leq t \leq m \)).

We substitute \( b_{t-1} B_{t-1} = -(p_t^+/q_t^+)a_t A_t \), so we require that

\[
c_0^+ r_0 q_m' - d_m^+ s_m p_0' = 0 \\
c_t^+ r_t q_{t-1}' - d_{t-1}^+ s_{t-1} p_t' = 0
\]

(for \( 1 \leq t \leq m \)).
We know from Lemma 3.13 that $\lambda r_0 + s_0 = 0$ and $r_l + s_l = 0$ for $1 \leq l \leq m$. We may take $s_t = -1$ for all $t$, and then $r_0 = \lambda^{-1}$ and $r_l = 1$ for $1 \leq l \leq m$. With this, we get $\Psi_1 \Psi_2 = 0$ if and only if

\[
(\star\star) \quad \lambda^{-1} c_0^+ = -d_m^+ (p_0'/q_0') \quad \text{and} \quad c_l^+ = -d_{l-1}^+ (p_l'/q_0')
\]

(for $1 \leq l \leq m$).

**The proof of Proposition 3.2 for $m \geq 1$.**

We take the product of all identities in (**), and get

\[
(\star\star\star) \quad \lambda^{-1} \prod_{t=0}^{m} c_t^+ = (-1)^{m+1} \prod_{t=0}^{m} d_t^+ \cdot \prod_{t=0}^{m} p_t' \prod_{t=0}^{m} q_t'
\]

We also take the product over all identities in (*) and get

\[
\prod_{t=0}^{m} c_t^+ = \mu \cdot (-1)^{m+1} \prod_{t=0}^{m} d_t^+ \cdot \prod_{t=0}^{m} p_t \prod_{t=0}^{m} q_t
\]

 Substitute this into (***) and cancel, and we get $\mu = \lambda v$ where $v$ is the number in the statement of Proposition 3.2. This proves that

\[
\Omega^2 (M(\lambda)) \cong M(\lambda v)
\]

\[
\square
\]

4 The proof of the Theorem

Let $\Lambda_q$ be special biserial weakly symmetric, and let $\sigma$ be the permutation of the arrows such that $\alpha \cdot \sigma(\alpha)$ is non-zero in the algebra. Write $\sigma$ as a product of disjoint cycles.

We define the Brauer graph of $\Lambda_q$ as follows. It is the undirected graph whose vertices are the cycles of $\sigma$. Let $\sigma_1$ and $\sigma_2$ be two cycles of $\sigma$, then the edges between $\sigma_1$ and $\sigma_2$ are labelled by the crossings of $\sigma_1$ and $\sigma_2$. These are the vertices $i$ of $Q$ such that both $\sigma_1$ and $\sigma_2$ pass through $i$ (counted with multiplicities). There is a cyclic ordering of the edges adjacent to a given vertex $\sigma_i$ of the graph; the successor of edge $e$ is edge $f$ if $f$ comes next after $e$ along the path in $Q$ given by $\sigma_i$. This graph is connected, and is independent of $q$, we denote it by $G_\Lambda$.

Note that the edges of $G_\Lambda$ only see the vertices of $Q$ of valency two; we do not need details about vertices with valency one. This means that this graph is slightly
different from the usual definition of a Brauer graph, where vertices of $Q$ with valency
one are also recorded, the corresponding edges $e$ of the graph have the property that
one of the adjacent vertices is adjacent only to this edge $e$. Hence our graph is a
tree if and only if the usual Brauer graph is a tree.
Note also that once we know that the graph $G_\Lambda$ has no multiple edges and no cycles
then it must be a tree. As well, we do not need to go into details about the cyclic
ordering around a vertex.

**Example 4.1** (1) Let $\Lambda_q$ be a ‘Double Nakayama algebra’ with $n$ vertices for $n \geq 2$,
where $\sigma$ is a product of disjoint 2-cycles. That is, $\Lambda_q = KQ/I$ where $Q$ is the quiver

and we label the vertices by $\mathbb{Z}_n$ and the arrows are $a_i : i \mapsto i + 1$ and $b_i : i + 1 \mapsto i$.
The ideal $I$ is generated by $a_{i+1}a_i$, $b_ib_{i+1}$ and

$$p_i(a_ib_i)^{r_i} + q_i(b_{i-1}a_{i-1})^{r_i-1}$$

(for $i \in \mathbb{Z}_n$, where $r_i \geq 1$). Then the Brauer graph has $n$ vertices and is a cycle.
When $r_i = 1$ for all $i$ so that the radical has cube zero, some socle deformation does
not satisfy (Fg), by [11]. By Theorem 1.1 this holds for arbitrary $r_i \geq 1$. One can
show that for an arbitrary special biserial weakly symmetric algebra with the above
quiver, the Brauer graph is a cycle.

(2) Let $\Lambda_q$ be an algebra whose quiver is of type $\tilde{Z}$ (with the notation of [11]), and
where $\sigma$ is a product of 2-cycles together with 1-cycles for the two loops. That is,
$\Lambda_q = KQ/I$ where $Q$ is the quiver

where $I$ is generated by the following relations (we assume $n > 0$).

$$ca_0, \quad b_0c, \quad a_{n-1}d, \quad db_{n-1}, \quad a_ia_{i+1}, \quad b_ib_{i-1}$$

$$p_0c^2 + q_0(a_0b_0)^{r_0}, \quad p_i(b_{i-1}a_{i-1})^{r_i-1} + q_i(a_ib_i)^{r_i}, \quad p_n(b_{n-1}a_{n-1})^{r_n-1} + q_nd^2$$
where \( 1 \leq i \leq n - 1 \), and \( r_i \geq 1 \). The coefficients \( p_0, \ldots, p_n \) and \( q_0, \ldots, q_n \) are non-zero scalars. Then the Brauer graph of \( \Lambda_q \) is a line. When \( r_i = 1 \) for all \( i \), it was shown in [11] that one can modify the presentation and have all scalar parameters equal \( \pm 1 \). By the Lemma below, this holds for arbitrary \( r_i \). When \( r_i = 1 \) for all \( i \), the result of [11] shows that the algebra satisfies (Fg). It would be interesting to know whether it is always the case.

(3) Let \( \Lambda_q \) be the local algebra as in 3.10(1). Then \( \sigma = (x, y) \) and hence the Brauer graph has one vertex with a double edge. Hence by the Theorem, for some \( q \), the algebra has criminals.

(4) Let \( \Lambda_q \) be the commutative local algebra as in 3.10(2). Then \( \sigma = (x)(y) \), the product of two cycles each of length one, and the Brauer graph has two vertices and one edge between them. Of course we can see here directly that if we rescale generators, then the scalar parameters in the socle relations can be changed to 1 (or anything non-zero).

Remark 4.2 We assume \( \Lambda \) has infinite type, then the Brauer graph cannot be just one vertex: If so then the permutation \( \sigma \) would be one cycle with no self-crossings. For such an algebra, all vertices have valency one and it is a Nakayama algebra, of finite type.

We first prove the implication (b) \( \Rightarrow \) (a) of Theorem 1.1. When the Brauer graph of the algebra is a tree, one can always rescale the arrows and achieve that all scalar parameters are equal to 1 (see also the example (2) above). Namely we have the following.

Lemma 4.3 Assume \( \Lambda_q \) is a weakly symmetric and special biserial algebra whose Brauer graph \( G_\Lambda \) is a tree. Then \( \Lambda_q \) is isomorphic to \( \Lambda_1 \), the algebra where all parameters are equal to 1. The algebra \( \Lambda_q \) does not have criminals.

Proof We will show that by rescaling some arrows one can achieve that all socle parameters become 1. Note that rescaling arrows does not change the zero relations of length two.

We fix a cycle \( \sigma_0 \) of \( \sigma \) which has only one neighbour in \( G_\Lambda \). For any vertex \( \sigma_i \) of the Brauer graph, there is a unique path in \( G_\Lambda \) of shortest length from \( \sigma_0 \) to \( \sigma_i \). Define the ‘distance’ \( d(\sigma_i) \) to be the number of edges of this path. Note also that if \( d(\sigma_i) > 0 \) then \( \sigma_i \) has unique neighbour \( \sigma_t \) with \( d(\sigma_t) = d(\sigma_i) - 1 \).
We prove the Lemma by induction on the distance. If \( d = 0 \) then the cycle is \( \sigma_0 \), and we keep its arrows as they are.

For the inductive hypothesis, assume that for all cycles with \( d(\sigma_t) < d \), the arrows in it have been scaled so that the relevant socle relations have scalars equal to 1. Now take \( \sigma_t \) such that \( d(\sigma_t) = d + 1 \). Then there is a unique \( \sigma_t \) joined in the Brauer graph to \( \sigma_t \) such that \( d(\sigma_t) = d \). Let \( j \) be the edge between \( \sigma_t \) and \( \sigma_i \) in the Brauer graph. That is, \( j \) is a vertex in \( Q \) of valency two. Consider the socle relation at \( j \),

\[
p_j C_j + q_j D_j = 0
\]

Say the arrows in \( C_j \) are the arrows of \( \sigma_t \), so that \( C_j = (\alpha_1 \alpha_2 \ldots \alpha_r)^{m_t} \) where \( \sigma_t = (\alpha_1 \alpha_2 \ldots \alpha_r) \) and \( m_t \geq 1 \). Then \( D_j = (\beta_1 \beta_2 \ldots \beta_s)^{m_i} \) where \( \sigma_i = (\beta_1 \beta_2 \ldots \beta_s) \) and \( m_i \geq 1 \). We replace a single arrow in \( \sigma_i \), namely \( \beta_1 \), by \( \beta_1' := c \beta_1 \) where \( c \) is a root of \( x^{m_i} - (q_i/p_i) \). Then \( \beta_1' \) is an arrow, and replacing \( \beta_1 \) by \( \beta_1' \) does not affect zero relations of length two. The monomial \( D'_j := (\beta_1' \beta_2 \ldots \beta_s)^{m_i} \) is an element in the socle. By the choice of \( c \) it follows that

\[
p_j(C_j + D'_j) = 0 \quad \text{and hence } C_j + D'_j = 0.
\]

Since we have not changed any of the \( \alpha_u \)'s, the relations fixed earlier are not altered.

We may take all parameters equal to 1, and then by Proposition 3.7, the module \( M(\lambda) \) for any band word is periodic. By the discussion in Section 2.1, the algebra \( \Lambda_1 \) does not have any criminals at all. \( \Box \)

We observe that we could equally well have signs 1 and \(-1\). With this one can show that the algebra is symmetric if its Brauer graph is a tree.

For the implication \((a) \Rightarrow (b)\) of Theorem 1.1, we start with the following. We will use band words, as defined in 3.1, which involve vertices of valency two. When the word \( W \) is fixed, we write \( \mathcal{E} = \{e_0, \ldots, e_m\} \) and \( \mathcal{F} = \{f_0, \ldots, f_m\} \) for the sets of these vertices (which depend on \( W \)).

**Lemma 4.4** Assume \( \Lambda_\mathbf{q} \) has a band word \( W \) for which \( \mathcal{E} \neq \mathcal{F} \) and where the \( f_i \) are distinct and the \( e_i \) are distinct. Then for some choice of \( \mathbf{q} \) the algebra has criminals.

**Proof** Suppose, say, \( f_i \notin \mathcal{E} \). The socle relation at \( f_i \) which we denoted by \( (\theta_i) \) contributes the factor \( q_i/p_i \) to the parameter \( v \) of the word \( W \). The relation \( (\theta_i) \) does not occur elsewhere since the \( f_j \) are distinct and \( f_i \) is not in \( \mathcal{E} \). Take \( q \in K \) which is not a root of unity, and take \( q_i := q \) and set all other parameters for socle relations equal to 1. Then \( v = q \) and hence a module \( M(\lambda) \) with this word is a criminal for \( \Lambda_\mathbf{q} \).
Proposition 4.5 Assume we have a band word \( W \) where \( \mathcal{E} = \mathcal{F} \), of size \( m+1 \), and let \( \pi \) be the permutation of \( m+1 \) with \( e_{\pi(i)} = f_i \). Then the following are equivalent:

1. The permutation \( \sigma \) takes the last arrow of \( a_i \) to the first arrow of \( a_{\pi(i)} \) for all \( i \);
2. \( v = 1 \) where \( v \) is the parameter for \( W \).

**Proof** Fix a vertex \( i \), the socle relations for \( f_i \) and for \( e_{\pi(i)} \) are equal. Recall we have written them as

\[
p_i(A_i a_i) + q_i(B_i b_i) = 0, \quad p'_{\pi(i)}(a_{\pi(i)} A_{\pi(i)}) + q'_{\pi(i)}(b_{\pi(i)} - b_{\pi(i)} - 1 B_{\pi(i)} - 1) = 0.
\]

For the moment, view \( p_i, q_i \) and \( q'_{\pi(i)}, p'_{\pi(i)} \) as indeterminates. The contribution of these to the parameter \( v \) is

\[
\frac{q_i}{p_i} \cdot \frac{p'_{\pi(i)}}{q'_{\pi(i)}}.
\]

Hence \( v = 1 \) if and only if for all \( i \) we have \( q_i = q'_{\pi(i)} \) and \( p_i = p'_{\pi(i)} \). This holds if and only if for all \( i \) we have \( A_i a_i = a_{\pi(i)} A_{\pi(i)} \), (or equivalently \( B_i b_i = b_{\pi(i)} - 1 B_{\pi(i)} - 1 \)).

Recall that a rotation of the path \( A_i a_i \) is a non-zero element in the algebra. Therefore if \( \alpha \) is the last arrow of \( a_i \) then \( \sigma(\alpha) \) must be the first arrow of \( A_i \).

Since \( A_i a_i \) is either \( a_{\pi(i)} A_{\pi(i)} \) or \( b_{\pi(i)} - 1 B_{\pi(i)} - 1 \), the arrow \( \sigma(\alpha) \) is the first arrow of precisely one of \( a_{\pi(i)} \) or \( b_{\pi(i)} - 1 \).

Hence \( A_i a_i = a_{\pi(i)} A_{\pi(i)} \) if and only if \( \sigma(\alpha) \) is the first arrow of \( a_{\pi(i)} \).

\[ \square \]

**Proof of (a) ⇒ (b) of Theorem 1.1.**

Assume that for any \( \mathbf{q} \) the algebra \( \Lambda_\mathbf{q} \) does not have criminals. We must show that the Brauer graph \( G_\Lambda \) is a tree with no multiple edges.

Take a band word \( W \) as in 3.1 in which all paths \( a_i \) and \( b_i \) have minimal length, that is, all vertices along these paths other than \( e_i \) and \( f_i \) (if any) have valency one. Such \( W \) must exist, we refer to the \( a_i \) and \( b_i \) as minimal paths in this proof.

1. We claim that \( \mathcal{E} = \mathcal{F} \). Note that by the minimality condition in 3.1 the vertices \( e_0, \ldots, e_m \) are pairwise distinct, similarly the \( f_0, \ldots, f_m \) are pairwise distinct. Hence the claim follows from Lemma [4].

2. We claim that \( \mathcal{E} \) is the set of all vertices of valency two of \( \mathcal{Q} \): Suppose not. Since \( \mathcal{Q} \) is connected, there must be a minimal path \( \gamma \) say starting or ending at some vertex of valency two \( e \) which is not in \( \mathcal{E} \), and ending or starting at some vertex \( e_i \in \mathcal{E} \). But there are only two minimal paths starting at \( e_i \) and two minimal paths ending at \( e_i \), and \( \gamma \) must then be one of the \( a_i \) or \( b_i \), a contradiction.

3. We claim that the set of arrows occurring in the paths \( \{a_0, a_1, \ldots, a_m\} \) is invariant under the permutation \( \sigma \):
If $\alpha$ is an arrow ending at a vertex $j$ of valency one then clearly $\sigma(\alpha)$ is the arrow starting at $j$. So we only need to know that $\sigma$ takes the last arrow of some $a_i$ to the the first arrow of some $a_l$. We have no criminals, therefore $v = 1$ and the claim holds by Proposition 4.5.

Then as well, $\sigma$ leaves the set of arrows invariant occurring in any of the paths $\{b_0, b_1, \ldots, b_m\}$.

This means that we can colour the cycles of $\sigma$ by the two colours $a$ and $b$, and this gives a colouring for the vertices of $G_A$.

(4) By considering the word $W$ we see that that for each vertex $i$ of valency two, there is one $a$-cycle and one $b$-cycle passing through $i$. Hence the Brauer graph does not have edges between two cycles of the same colour, and there is no edge in $G_A$ starting and ending at the same cycle of $\sigma$.

(5) The graph $G_A$ does not have multiple edges: Suppose $\sigma_1, \sigma_2$ are cycles which pass through vertices $e \neq f$. Then we can find a band word $W$ of the form

$$e \xrightarrow{a_0} f \xleftarrow{b_0} e$$

Namely, say $\sigma_1$ is an $a$-cycle, then take for $a_0$ the shortest path consisting of arrows in $\sigma_1$ from $e$ to $f$, and take $b_0$ similarly. Then the parameter $v$ for this word is $\neq 1$, by the Lemma.

So far, we have proved that the Brauer graph has a colouring of vertices with colours $a$ and $b$ where the colours alternate, and it does not have multiple edges.

(6) The graph $G_A$ does not have a cycle:

If there is a cycle in the graph, then this cycle must have an even number of vertices, since the vertices of $G_A$ are coloured by two alternating colours. If we take part of each cycle in the appropriate direction, then we get a band word with $\mathcal{E}$ and $\mathcal{F}$ disjoint, since the number of vertices is even. By Lemma 4.4, this gives rise to a criminal. We have now proved that $G_A$ is a connected graph with no multiple edges, and without cycles, and hence $G_A$ must be a tree. □

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