SYMBOIC COMPUTATION IN HYPERBOLIC PROGRAMMING

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ABSTRACT. Hyperbolic programming is the problem of computing the infimum of a linear function when restricted to the hyperbolicity cone of a hyperbolic polynomial, a generalization of semidefinite programming. We propose an approach based on symbolic computation, relying on the multiplicity structure of the algebraic boundary of the cone, without the assumption of determinantal representability. This allows us to design exact algorithms able to certify the multiplicity of the solution and the optimal value of the linear function.

INTRODUCTION

Semidefinite programming (SDP) constitutes a popular class of convex optimization problems for which approximate solutions can be computed through a variety of numerical algorithms, the most efficient of which are based on primal-dual interior point methods. On the other hand, exact algorithms for general semidefinite programs have been developed only much more recently in the work of Henrion, Safey El Din and the first author in [9]. The optimizer in an SDP problem corresponds to a positive semidefinite real symmetric matrix. While symbolic algorithms obviously have a much higher complexity than numerical ones, finding exact solutions has many benefits, especially regarding certification of the solution. For instance, the rank of the optimizer, which is often a meaningful quantity, can be determined exactly by the algorithm in [9], and one can even optimize over all feasible points of bounded rank, which is a non-convex optimization problem [15].

In this paper, we consider analogous algorithmic questions in the more general framework of hyperbolic programming. We briefly summarize the underlying notions. A real homogeneous polynomial \( f \) in several variables \( x = (x_1, \ldots, x_n) \) is called hyperbolic with respect to a point \( e \in \mathbb{R}^n \) if \( f(e) \neq 0 \) and the polynomial \( f(te-a) \in \mathbb{R}[t] \) has only real zeros for every \( a \in \mathbb{R}^n \). The general determinant of symmetric matrices has this property with respect to the unit matrix \( e = I_d \), since \( \det(tI_d - A) \) is the classical characteristic polynomial of the real symmetric matrix \( A \). Hyperbolic polynomials can therefore be seen as generalized characteristic polynomials. If \( f \) is hyperbolic with respect to \( e \), the hypersurface defined by \( f \) bounds a convex cone containing \( e \), the hyperbolicity cone, generalizing the cone of positive semidefinite matrices in case of the determinant. The zeros of \( f(te-a) \) can be regarded as generalized eigenvalues of \( a \in \mathbb{R}^n \), and the multiplicity of the root \( t = 0 \) of \( f(te-a) \) as the corank of \( a \).

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A hyperbolic program is the convex optimization problem of minimizing a linear function over the hyperbolicity cone of a hyperbolic polynomial. Such cones have non-empty interior by construction (the interior will indeed contain the point \(e\)). Denote now by \(\mathcal{S}_d(\mathbb{R})\) the set of \(d \times d\) real symmetric matrices. Regular semidefinite programs (in which the feasible set has non-empty interior) correspond to the case in which \(f\) is the restriction of the determinant map \(\text{det}: \mathcal{S}_d(\mathbb{R}) \to \mathbb{R}\) to a linear subspace \(V \subset \mathcal{S}_d(\mathbb{R})\) containing a positive definite matrix. More precisely, for such \(V\), the polynomial \(f = \text{det}\big|_V\) is hyperbolic, and its hyperbolicity cone is the spectrahedron \(V \cap \mathcal{S}_d^+(\mathbb{R}) = \{M \in V : M \succeq 0\}\). If \(V\) does not contain positive definite matrices, \(V \cap \mathcal{S}_d^+(\mathbb{R})\) is still a spectrahedron, but \(f = \text{det}\big|_V\) is not hyperbolic.

Moreover, not every hyperbolic polynomial can be represented in this way (in fact, the set of representable polynomials is, in general, of strictly smaller dimension) which motivates the development of techniques that are independent of the determinantal representability of \(f\).

Hyperbolic programming can be solved numerically with interior point methods much like SDP [4, 8, 16]. One of the major challenges in hyperbolic programming, when compared to SDP, is the lack of an explicit duality theory, while SDP duality is always heavily exploited. The methods in [9] rely on the good properties of determinantal varieties, which provide an explicit non-singular lifting of the variety of symmetric matrices of bounded rank in a given subspace. The same is not available for hyperbolic programming. However, hyperbolicity of a real polynomial still imposes some strong conditions on the structure of the real part of the singular locus of the hypersurface.

Figure 1. An affine section of a hyperbolic quartic surface with four nodes.
Let us give an overview of our main results. Given a polynomial \( f \), hyperbolic with respect to \( e \in \mathbb{R}^n \), let \( \Lambda_+ \) be the hyperbolicity cone of \( f \) (Section 1) and let \( \Gamma_m \subset \mathbb{R}^n \) denote the set of points of multiplicity at least \( m \) (see Section 2). Furthermore, let \( L_e = \{ x \in \mathbb{R}^n : e^T x = 1 \} \) be the affine space orthogonal to the direction \( e \) (containing \( e \|_e \mathbb{R} \)). We show that if \( m \) equals the maximal multiplicity on \( \Lambda'_+ \), then \( \Lambda'_+ \) contains one of the real connected components of \( \Gamma'_m \), proving that \( \Gamma'_m \cap \Lambda_+ \) is the union of some components of \( \Gamma'_m \) (Proposition 6). Thus a point of maximal multiplicity (analogous to the minimal rank in SDP) can be found by sampling the connected components of \( \Gamma'_m \). Since this is an algebraic set (rather than just semialgebraic), this reduces to a standard problem in computational real algebraic geometry. Furthermore, we show that the more general convex hyperbolic programming problem over \( \Lambda'_+ \) is equivalent to computing local minimizers over the sets \( \Gamma'_m \) of the same linear function (Theorem 10). This can be carried out in practice using Lagrange multipliers, provided that the corresponding set of critical points has complex dimension 0. We use these results to design an exact algorithm for hyperbolic programming. Applying this to explicit examples yields interesting results that are discussed in the final part.

The structure of this paper is as follows. In Section 1, we summarize standard definitions and results about hyperbolicity cones. In Section 2, we describe the multiplicity structure of the algebraic boundary of \( \Lambda_+ \) and prove our main result on the maximal multiplicity. This is used to certify feasible multiplicities in the case where \( \Lambda_+ \) is the \( d \)-elliptope. In Section 3, we formalize the relationship between solutions to hyperbolic programming problems and the multiplicity loci \( \Gamma_m \). Our algorithm solving hyperbolic programming in exact arithmetic is implemented in Maple; we finally discuss the results of our tests.

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1. Preliminaries

The following notation is used throughout. The ring of real polynomials in \( x = (x_1, \ldots, x_n) \) is denoted by \( \mathbb{R}[x] \), with its natural grading \( \mathbb{R}[x] = \bigoplus_d \mathbb{R}[x]_d \). The complex algebraic set defined by polynomials \( f = (f_1, \ldots, f_s) \) is denoted by \( Z(f) = \{ x \in \mathbb{C}^n : \forall i \ f_i(x) = 0 \} \), and its subset of real points by \( \mathbb{R}(f) = Z(f) \cap \mathbb{R}^n \). The closure of a set \( S \subset \mathbb{R}^n \) in the Euclidean topology (resp. Zariski topology) of \( \mathbb{R}^n \) is \( \overline{S} \) (resp. Zar(S)). Its Euclidean boundary is \( \partial S \).

Recall from the introduction that a homogeneous polynomial \( f \in \mathbb{R}[x] \) of degree \( d \) is called hyperbolic with respect to a point \( e \in \mathbb{R}^n \) if \( f(e) \neq 0 \) and \( f(te - a) \in \mathbb{R}[t] \) has only real roots for every \( a \in \mathbb{R}^n \). Up to rescaling \( f \), we may suppose that \( f(e) = 1 \), and we often say that \( f \) is just hyperbolic, without specifying the direction \( e \).

The polynomial

\[
t \mapsto \text{ch}_a(t; f, e) = f(te - a)
\]
is called the characteristic polynomial of \( a \in \mathbb{R}^n \) (with respect to \( f \) and \( e \)). This will be denoted simply by \( \text{ch}_a(t) \) when \( f, e \) are understood from the context. For \( a \in \mathbb{R}^n \), the ordered roots \( \lambda_1(a) \leq \lambda_2(a) \leq \cdots \leq \lambda_d(a) \) of \( \text{ch}_a \) are called the eigenvalues of \( a \). The set

\[
\Lambda_{++}(f, e) = \{ a \in \mathbb{R}^n : \lambda_1(a) > 0 \}
\]

is called the open hyperbolicity cone of \( f \) with respect to \( e \). This is an open, convex cone containing \( e \) \cite{5}. Note that there may exist several hyperbolicity cones associated to a given hyperbolic polynomial, and bounds on the number of such cones have been recently computed in \cite{12}; however, when \( f \) is irreducible, there is only one hyperbolicity cone, up to changing \( e \) with \(-e\) (see \cite{13}). We will simply denote \( \Lambda_{++}(f, e) \) by \( \Lambda_{++} \) when \( f, e \) are fixed. It is independent of the choice of \( e \) within \( \Lambda_{++} \), by the following basic result.

**Theorem 1** (Gårding \cite{5}, Renegar \cite{21}). The cone \( \Lambda_{++} \) is convex, open and coincides with the connected component of \( \mathbb{R}^n \setminus \mathbb{Z}_R(f) \) containing \( e \). Moreover, \( f \) is hyperbolic with respect to \( e' \) for all \( e' \in \Lambda_{++}(f, e) \), and \( \Lambda_{++}(f, e) = \Lambda_{++}(f, e') \).

The closure of \( \Lambda_{++} \) in the Euclidean topology is called the hyperbolicity cone. It is denoted by \( \Lambda_{+} \) and equals

\[
\Lambda_{+} = \overline{\Lambda_{++}} = \{ x \in \mathbb{R}^n : \lambda_1(x) \geq 0 \}.
\]

As mentioned in the introduction, we work over the affine section

\[
\Lambda'_+ = \Lambda_+ \cap L_e, \quad \text{where} \quad L_e = \{ x \in \mathbb{R}^n : e^T x = 1 \}.
\]

Note that the relative interior of \( \Lambda'_+ \) in \( L_e \) is not empty, since, for instance, it contains \( \frac{e}{\| e \|^2} \). We call the affine section \( \Lambda'_+ \) the standard section since it preserves the multiplicity structure of the cone \( \Lambda_+ \) (see Remark 5).

Since linear (LP) and semidefinite programming (SDP) are special instances of hyperbolic programming, we recall the description of their feasible cones in this setting. For LP, the polynomial splits into real linear factors, that is \( f = \ell_1 \cdots \ell_d \), with \( \ell_i \in \mathbb{R}[x]_1 \). For all \( e \in \mathbb{R}^n \) with \( \ell_i(e) > 0 \), \( i = 1, \ldots, d \), one has that

\[
\Lambda_+(f, e) = \{ x \in \mathbb{R}^n : \forall i \ \ell_i(x) \geq 0 \}
\]

is a polyhedron. For SDP, the hyperbolicity cone of \( f = \det(x_1 A_1 + \cdots + x_n A_n) \) in direction \( e \), with \( A_i \) real symmetric, and \( e_1 A_1 + \cdots + e_n A_n > 0 \), equals the cone of positive semidefinite matrices in the subspace spanned by \( A_1, \ldots, A_n \), that is, a spectrahedral cone.

2. **Multiplicities**

Let \( a \in \mathbb{R}^n \), and \( f \in \mathbb{R}[x] \) a hyperbolic polynomial (with respect to a fixed point \( e \in \mathbb{R}^n \)) of degree \( d \). The multiplicity of \( a \) is the multiplicity of \( t = 0 \) as an eigenvalue
Indeed, if Proof. The inclusion Lemma 2. on its algebraic closure. These will allow us to define explicit equations for algebraic lemmas give basic results on the multiplicity structure of the boundary of is independent of the hyperbolic direction of a. Thus if a \(a \in \Lambda_+\) for \(0 \leq m \leq d\), first considered in [21]. By [21, Prop.22], the function \(a \mapsto \mult(a)\) is independent of the hyperbolic direction \(e\) chosen within \(\Lambda_+^+(f, e)\). The following lemmas give basic results on the multiplicity structure of the boundary of \(\Lambda_+\) and on its algebraic closure. These will allow us to define explicit equations for algebraic relaxations of \(\Lambda_+\).

**Lemma 2.** For \(0 \leq m \leq d\), the set \(\partial^{\leq m} \Lambda_+\) is closed and \(\overline{\partial^{m} \Lambda_+} \subset \partial^{\leq m} \Lambda_+\).

**Proof.** The inclusion \(\partial^{m} \Lambda_+ \subset \partial^{\geq m} \Lambda_+\) holds by definition. Hence we have to show that \(\partial^{\geq m} \Lambda_+\) is closed, which comes simply from the continuity of eigenvalues \(a \mapsto \lambda_i(a)\). Indeed, if \((a_i)_i \subset \partial^{\geq m} \Lambda_+\), then for all \(\epsilon > 0\)

\[
\lambda_1(a_\epsilon) = \ldots = \lambda_m(a_\epsilon) = 0 \leq \lambda_{m+1}(a_\epsilon) \leq \ldots \leq \lambda_d(a_\epsilon).
\]

Thus if \(a_\epsilon \xrightarrow{\epsilon \to 0^+} a\), then \(\lambda_1(a) = \ldots = \lambda_m(a) = 0 \leq \lambda_{m+1}(a) \leq \ldots \leq \lambda_d(a)\), hence \(\mult(a) \geq m\) as claimed. \(\square\)

Note that, typically, neither \(\partial^{m} \Lambda_+\) nor \(\partial^{\geq m} \Lambda_+\) are Zariski closed sets, since they are semialgebraic rather than algebraic. Often, in order to develop algebraic techniques, it is desirable to work with real algebraic sets. We therefore define the following:

\[
\Gamma_m = \{a \in \mathbb{R}^n : \mult(a) \geq m\}, \quad 0 \leq m \leq d.
\]

The sets \(\Gamma_m\) define a nested collection in \(\mathbb{R}^n\):

\[
\Gamma_d \subset \Gamma_{d-1} \subset \ldots \subset \Gamma_1 = \{a \in \mathbb{R}^n : f(a) = 0\} = Z_f = \Gamma_0 = \mathbb{R}^n.
\]

By Lemma 2, we have \(\overline{\partial^{m} \Lambda_+} \subset \Gamma_m \cap \Lambda_+\) (indeed, \(\Gamma_m \cap \Lambda_+ = \partial^{\geq m} \Lambda_+\)).

**Lemma 3.** For any \(0 \leq m \leq d\), the set \(\Gamma_m\) is real algebraic and satisfies

\[
Zar(\partial^m \Lambda_+) = Zar(\overline{\partial^m \Lambda_+}) \subset \Gamma_m.
\]

**Proof.** The equality \(Zar(S) = Zar(\overline{S})\) always holds, since the Zariski topology is coarser than the Euclidean topology. The inclusion \(Zar(\overline{\partial^m \Lambda_+}) \subset Zar(\Gamma_m)\) follows from \(\overline{\partial^m \Lambda_+} \subset \Gamma_m \cap \Lambda_+,\) proved in Lemma 2. Hence we only need to prove that \(\Gamma_m\) is real algebraic. Writing

\[
ch_\epsilon(t) = t^d + g_1(x)t^{d-1} + \ldots + g_{d-1}(x)t + g_d(x),
\]

we deduce that a point \(a \in \mathbb{R}^n\) lies in \(\Gamma_m\) if and only if \(t^m\) divides \(ch_\epsilon(t)\). This is the case if and only if all coefficients \(g_d = f(-x), g_{d-1}, \ldots, g_{d-m+1} \in \mathbb{R}[x]\) vanish at \(a\). \(\square\)
The defining equations for $\Gamma_m$ obtained in the proof can be made more explicit. Let us consider the modified characteristic polynomial $\text{ch}_a = f(te + a)$ at a point $a \in \mathbb{R}^n$, with respect to $f$ and $e$. Denote by $\sigma_i(y_1, \ldots, y_d) = \sum_{j_1 < \cdots < j_i} y_{j_1} \cdots y_{j_i}$, the $i$-th elementary symmetric polynomial on variables $y_1, \ldots, y_d$, then

$$\text{ch}_a = t^d + \sigma_1(\lambda(a)) t^{d-1} + \cdots + \sigma_{d-1}(\lambda(a)) t + f(a)$$

where $\lambda(a) = (\lambda_1(a), \ldots, \lambda_d(a))$ are the eigenvalues of $a$. Note that $f(x) = \sigma_d(\lambda(x)) = \lambda_1(x) \cdots \lambda_d(x)$, since $f(e) = 1$.

**Corollary 4.** For $0 \leq m \leq d$, we have $\Gamma_m = Z_{\mathbb{R}}(f, \sigma_{d-1}(\lambda), \ldots, \sigma_{d-m+1}(\lambda))$.

**Proof.** Indeed, in the proof of Lemma 3 we have shown that $\Gamma_m$ equals the real algebraic set $Z_{\mathbb{R}}(g_d, g_{d-1}, \ldots, g_{d-m+1})$, with $g_d := f(-x)$ and $g_i(x)$ is the coefficient of $t^{d-i}$ in $\text{ch}_x(t)$. Therefore, $g_i(-x) = \sigma_i(\lambda(x))$ for $i = 1, \ldots, d$ and hence

$$\Gamma_m = Z_{\mathbb{R}}(f(-x), \sigma_{d-1}(\lambda(-x)), \ldots, \sigma_{d-m+1}(\lambda(-x))).$$

The claim follows from the homogeneity of $f, \sigma_{d-1}(\lambda), \ldots, \sigma_{d-m+1}(\lambda)$. \hfill $\Box$

**Remark 5.** Suppose that $a \in \Lambda_+$ with $e^Ta \neq 0$. Then $x = \frac{e^Ta}{e^Ta} \in \Lambda'_+ \setminus \{0\}$ and $\text{mult}(x) = \text{mult}(a)$. We deduce that if $\Lambda_+ \cap \{x \in \mathbb{R}^n : e^T x = 0\} = \{0\}$, the standard section of a hyperbolicity cone is a base for $\Lambda_+$ (as a convex cone; see [1, Def. 8.3]) and has the same multiplicity structure as $\Lambda_+ \setminus \{0\}$.

We are particularly interested in computing the maximum multiplicity on the standard section of the cone $\Lambda_+$. For a hyperbolic polynomial $f \in \mathbb{R}[x]_d$, we define the integer

$$\max_{a \in \Lambda'_+} \text{mult}(a) = \max_{0 \leq t \leq d} \{t : \Gamma'_t \cap \Lambda_+ \neq \emptyset\}.$$

This is well defined since $\Lambda'_+ \neq \emptyset$ and $0 \leq \text{mult}(a) \leq d$ for all $a \in \mathbb{R}^n$. We now show the maximum multiplicity $m$ is attained on an entire real connected component of $\Gamma'_m$.

**Proposition 6.** Let $f \in \mathbb{R}[x]_d$ be hyperbolic with respect to $e \in \mathbb{R}^n$, and let $m = \max_{a \in \Lambda'_+} \text{mult}(a)$. For every (real) connected component $C$ of $\Gamma'_m$, with $C \cap \Lambda_+ \neq \emptyset$, we have

1. $C \subset \Lambda_+$
2. $C \cap \Gamma_{m+1} = \emptyset$.

**Proof.** First, note that (1) implies (2), because, by definition, $m$ is the maximal multiplicity of points in $\Lambda_+$.

To prove (1), let $C$ be a connected component of $\Gamma'_m$ intersecting $\Lambda_+$, and let $a \in C \cap \Lambda_+$. Note that $\text{mult}(a) = m$ by maximality of $m$. Suppose that there is $b \in C \setminus \Lambda_+$; hence $\text{mult}(b) \geq m$ and $\lambda_1(b) < 0$. Since $C$ is connected, there is a continuous path $\varphi : [0, 1] \to C$ with $\varphi(0) = a$ and $\varphi(1) = b$. For all $t \in [0, 1]$, $\text{mult}(\varphi(t)) \geq m$, and there exists $t_0 \in [0, 1]$ such that

1. $\varphi([0, t_0]) \subset \Lambda_+$ and
2. $\exists \delta > 0$ such that, $\forall 0 < \epsilon < \delta$, $\varphi(t_0 + \epsilon) \notin \Lambda_+$. 


Define \( a_0 = \varphi(t_0) \) and \( a_\epsilon = \varphi(t_0 + \epsilon) \), for \( 0 < \epsilon < \delta \), so that \( a_\epsilon \xrightarrow{\epsilon \to 0^+} a_0 \). Since \( a_\epsilon \not\in \Lambda_+ \), then \( \lambda_1(a_\epsilon) < 0 \) for all \( 0 < \epsilon < \delta \). More precisely, since \( \text{mult}(a_\epsilon) \geq m \), then for all \( \epsilon > 0 \) there exists \( i(\epsilon) \in \{1, \ldots, d\} \) such that

\[
\lambda_1(a_\epsilon) \leq \cdots \leq \lambda_{i(\epsilon)}(a_\epsilon) < 0
\]

\[
\lambda_{i(\epsilon)+1}(a_\epsilon) = \cdots = \lambda_{i(\epsilon)+m}(a_\epsilon) = 0
\]

\[
0 \leq \lambda_{i(\epsilon)+m+1}(a_\epsilon) \leq \cdots \leq \lambda_d(a_\epsilon).
\]

Passing to the limit for \( \epsilon \to 0^+ \), by the continuity of the eigenvalues and since \( a_0 \in \Lambda_+ \), we find that \( \lambda_1(a_0) = \cdots = \lambda_{m+1}(a_0) = 0 \), that is \( \text{mult}(a_0) \geq m + 1 \), which contradicts the fact that \( m \) is the maximum multiplicity. \( \square \)

Proposition 6 provides us with a way of computing the largest multiplicity on the standard section \( \Lambda'_+ \) of the hyperbolicity cone \( \Lambda_+ \), and of representing one point where this maximum value is attained. More precisely, consider the non-convex optimization problem

\[
\begin{align*}
\text{max} & \quad \text{mult}(x) \\
\text{s.t.} & \quad x \in \Lambda'_+(f,e)
\end{align*}
\]

By the hyperbolicity of \( f \), the interior of the feasible set \( \Lambda_+ \) is \( \Lambda_{++} \neq \emptyset \) (for instance, \( e \in \Lambda_{++} \)), hence this problem is feasible (and with non-empty interior). Hence there always exists \( a^* \in \Lambda'_+ \) such that \( \text{mult}(a^*) = m := \max_{a \in \Lambda'_+} \text{mult}(a) \). By Proposition 6, the whole connected component \( C^* \) of \( \Gamma'_m \) containing \( a^* \), is included in \( \Lambda_+ \).

First, by Remark 5, under the assumption that \( \Lambda_+ \) intersects \( \{x \in \mathbb{R}^n : e^T x = 0\} \) only in \( 0 \), we conclude that a solution of Problem (1) yields the maximum multiplicity over \( \Lambda_+ \setminus \{0\} \). We also conclude by applying Proposition 6, that Problem (1) can be solved by computing at least one point per connected component of the real algebraic sets \( \Gamma'_m, m = 1, \ldots, d-1 \). This is a central routine in computational algebraic geometry, for which exact algorithms have been designed, see e.g. [2, 20] or the monograph [3] with its references. By exact representation, we mean via a rational univariate representation [22]: this is a vector \( (q, q_0, q_1, \ldots, q_n) \in \mathbb{Q}[t]^{n+2} \) such that the set

\[
\left\{ \left( \frac{q_1(t)}{q_0(t)}, \ldots, \frac{q_n(t)}{q_0(t)} \right) : q(t) = 0 \right\}
\]

intersects every connected component of the given algebraic set. In (2), \( q \) and \( q_0 \) are coprime, therefore the set is well defined and finite. Its cardinality is bounded above by \( \deg q \).

The best arithmetic complexity bounds for computing representations as in (2) are essentially polynomial in the number of equations defining the algebraic set (which for \( \Gamma'_m \) is at most \( m+1 \), by Corollary 4) and in the maximum of their degree (bounded above by \( d \)) and singly exponential in the number of variables. These come out of the
so-called critical points method [3, Ch. 16], or the effective theory of polar varieties [24, 25].

In this paper we are not focusing on complexity results for hyperbolic programming, but our efforts are especially devoted to the design of algebraic methods for this problem. However, Proposition 6 and the mentioned results, according to the previous complexity analysis, give a singly exponential algorithm to represent a solution to Problem 1 in exact arithmetic, which is worth being highlighted.

We conclude this section with an example and comment on computational issues of Problem 1 for the case of the elliptope. This is the feasible set of the SDP-relaxation of the MAX-CUT combinatorial optimization problem [7].

Example 7 (Elliptope). Let \( d \in \mathbb{N} \) and \( n = \binom{d}{2} \), and let \( E_d \) be the \( d \)-elliptope. This is the spectrahedral cone of dimension \( n + 1 \) defined by the linear matrix inequality \( A(x) \succeq 0 \) with

\[
A(x) = \begin{bmatrix}
x_0 & x_{1,2} & \cdots & x_{1,d} \\
x_{1,2} & x_0 & \cdots & \vdots \\
\vdots & \ddots & \ddots & x_{d-1,d} \\
x_{1,d} & \cdots & x_{d-1,d} & x_0
\end{bmatrix}.
\]

That is \( E_d = \{ x = (x_0, x_{1,2}, \ldots, x_{d-1,d}) \in \mathbb{R}^{n+1} : A(x) \succeq 0 \} \). It is the linear section of codimension \( d - 1 \) of the cone of \( d \times d \) real symmetric positive semidefinite matrices whose main diagonal is constant. Every matrix in \( E_d \) is also called a correlation matrix (see [14] and references therein). Note that \( e = (1,0,\ldots,0) \) (corresponding to the identity matrix \( A(e) = \mathbb{I}_d \)) is in the interior of \( E_d \) and that \( L_e = \{ x \in \mathbb{R}^{n+1} : x_0 = 1/d \} \).

We propose two distinct tests on Example 7. The results that we present have been obtained on a desktop PC, with CPU architecture with the following characteristics: Intel(R) Xeon(R) CPU E5-4620 0 @ 2.20GHz. For the sake of reproducibility, the corresponding Maple scripts are made available on the webpage of the first author\(^1\).

Test 8. Since we work in the affine space \( L_e \) defined in Example 7, we put \( x_0 = 1/d \). We recall that \( \det A \) is hyperbolic with respect to \( e = (1,0,\ldots,0) \). The authors of [14] proved that the vertices of \( E_d \) (defined as those boundary points whose normal cone is full-dimensional) are characterized as all the rank one matrices in \( A(x) \) with entries in \( \pm 1 \). These are exactly the connected components (in this case, isolated real points) of \( \Gamma_{d-1} \), that is \( \Gamma_{d-1} \subset E_d \), and each rank one matrix in \( \Gamma_{d-1} \) maximizes the multiplicity on \( E_d \setminus \{0\} \): indeed, 0 is the unique positive semidefinite matrix of trace 0, hence the assumption in Remark 5 is satisfied. If \( M \) is one of these rank one matrices, then \( (1/d)M \in E_d \cap L_e \) maximizes the multiplicity on \( E_d \cap L_e \).

These points can be computed efficiently in practice. We make use of the Maple library SPECTRA [10], which is targeted to computing low rank solutions of linear matrix inequalities. As explained in [10], the command \texttt{SolveLMI(A,[all],[1])} (when

\(^1\)www.unilim.fr/pages_perso/simone.naldi/software.html
called in a Maple worksheet where \textsc{spectra} has been previously loaded, and where the variable \( A \) is instantiated to \( A(x) \), with \( x_0 = 1/d \) computes all components with highest multiplicity, namely isolated matrices of rank 1.

To give an idea of performances, we are able to solve our problem for \( d \leq 5 \) in less than half second, or for \( d = 8 \) (corresponding to an ellipope of dimension 21, and computing \( 2^7 = 128 \) solutions) in around 2.5 minutes.

A second test is performed for the ellipope, without exploiting the spectrahedrality of \( \mathcal{E}_d \), but just relying on the hyperbolicity of \( \det A \).

\textit{Test 9.} We generate the sets \( \Gamma_m \), \( m = 1, \ldots, d \), as the zero locus of the polynomials defined in Corollary 4. For every \( m \), we sample the real connected components of \( \Gamma'_m \) and check how many of the solutions lie in \( \mathcal{E}_d \), using the Maple library \textsc{raglib} [23]. The function \textsc{PointsPerComponents} allows us to sample the real connected components of the sets \( \Gamma'_m \), \( m = 1, \ldots, d \), computing rational parametrizations as in (2). The goal is to sample many points on the boundary of \( \mathcal{E}_d \), possibly with different multiplicities. Note that, \textit{a priori}, Proposition 6 guarantees that this method yields at least one feasible point for the maximum multiplicity in \( \mathcal{E}_d \cap L_e \).

The results are summarized in Table 1. For a fixed \( d \) and for a given multiplicity \( m = 1, \ldots, d - 1 \), the number of sample points computed by \textsc{raglib} is given in the third column, those lying on \( \mathcal{E}_d \cap L_e \) in the fourth, and those of the expected multiplicity in the fifth column; then, we report on the average time on 1000 tries on our standard desktop PC.

We also note that one computes points with multiplicity which can be larger than the expected one (that is, matrices whose rank is smaller than expected). For example, for \( d = 3 \) and \( m = 1 \), the four points on \( \Gamma'_1 \cap \mathcal{E}_d \) actually belong to \( \Gamma_2 \subset \Gamma_1 \), and correspond to those solutions computed in the subsequent step \( m = 2 \). Moreover, in contrast with \textsc{spectra} in Test 8, one can even sample multiplicities smaller than the maximal one, as for the case of the 4-ellipope. Already for the 5-ellipope, however, the computation becomes quite prohibitive, which is coherent with the exponential arithmetic complexity of the algorithms implemented in \textsc{raglib}.

In these tests, we have seen that points of maximum multiplicity can be computed efficiently in practice using LMI exact solvers in the case of hyperbolic polynomials.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|}
\hline
\( d \) & \( m \) & \# samples & \# feasible & mult = \( m \) & CPU time \\
\hline
2 & 1 & 2 & 2 & 2 & 0.07 s \\
3 & 1 & 10 & 4 & 0 & 0.7 s \\
3 & 2 & 4 & 4 & 4 & 5.1 s \\
4 & 1 & 36 & 10 & 6 & 26 s \\
4 & 2 & 36 & 36 & 30 & 24 s \\
4 & 3 & 8 & 8 & 8 & 34 s \\
\hline
\end{tabular}
\caption{Sample points on the \( d \)-ellipope}
\end{table}
with determinantal representation. On the other hand, larger multiplicities can be computed on general hyperbolicity cones $\Lambda_+(f,e)$ by sampling the loci $\Gamma'_m$, but with clear limitations in terms of the degree of $f$.

3. HYPERBOLIC PROGRAMMING

Hyperbolic programming is a convex optimization problem specified as follows. We are given a homogeneous polynomial $f \in \mathbb{R}[x]$ of degree $d$, with $x = (x_1, \ldots, x_n)$, hyperbolic with respect to $e \in \mathbb{R}^n$, and a linear map $\ell : \mathbb{R}^n \to \mathbb{R}$.

The hyperbolic program associated to data $(f, e, \ell)$ is

\[
\ell^* = \inf_{x} \ell(x) \quad \text{s.t. } x \in \Lambda'_+(f, e).
\]

Assumption. We assume, without loss of generality, that in Problem (3) $\ell(x)$ and $e^T x$ are independent linear forms. Indeed, if $\ell(x) = \lambda e^T x$ for some $\lambda \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$, then $\ell(x)$ is constant and equal to $\lambda$ over the feasible set $\Lambda'_+(f, e)$, and the hyperbolic program is trivial.

Since the objective function of Problem (3) is linear and the feasible set is convex, when the infimum is attained at $x^*$, then $x^* \in \partial(\Lambda'_+) \subset \partial \Lambda_+$. The boundary $\partial \Lambda_+$ is defined, locally, by the coefficients of the modified characteristic polynomial $\text{ch}_{-x}(t)$, as in Corollary 4. We now formalize this relationship between solutions to Problem (3) and multiplicity loci.

A local minimizer of a continuous function $\ell : \mathbb{R}^n \to \mathbb{R}$ on a set $S \subset \mathbb{R}^n$ is a point $x^* \in S$ such that there exists an open set $U \subset \mathbb{R}^n$ with $x^* \in U$ and $\ell(x^*) \leq \ell(x)$ for all $x \in U \cap S$. We can reduce Problem (3) to the computation of local minimizers on the multiplicity loci $\Gamma'_m$ as follows.

Theorem 10. Let $f, e, \ell$ be the data defining Problem (3). Let $x^* \in \Lambda'_+(f, e)$ with $\ell(x^*) = \ell^*$, and let $m^* = \text{mult}(x^*)$. Then $x^*$ is a local minimizer of $f$ on $\Gamma'_m$.

Proof. Since we are minimizing a linear function over a non-empty convex set, the minimizer (if it exists) belongs to the boundary of the feasible set, hence $m^* > 0$.

Next, we denote by $C^*$ the connected component of $\Gamma'_m$ containing $x^*$. First, we suppose that $C^* \not\subset \Lambda_+$ and that $W \cap \mathcal{L}_c \cap (Z_R(f) \setminus \Lambda_+) \neq \emptyset$ holds for every open set $W \subset \mathbb{R}^n$ containing $x^*$. We show that this situation cannot occur. Indeed, since $C^* \not\subset \Lambda_+$, we easily deduce $W \cap (\Gamma'_m \setminus \Lambda'_+) \neq \emptyset$ for all $W$ as above. Let $B_k$ be the open ball with center $x^*$ and radius $1/k$, for $k \in \mathbb{N} \setminus \{0\}$. For all such $k$, we can choose $x(k) \in B_k \cap (\Gamma'_m \setminus \Lambda'_+)$, yielding a sequence $x(k) \xrightarrow{k \to \infty} x^*$. We deduce that $\lambda_1(x(k)) < 0$ holds for all $k$, and hence $\lambda_{m^*+1}(x(k)) \leq 0$ (because at least $m^*$ eigenvalues of $x(k)$ must vanish). Passing to the limit, we find $\lambda_{m^*+1}(x^*) = 0$; since
$x^* \in \Lambda'_+ $, we get $\lambda_j(x^*) = 0$ for $j = 1, \ldots, m^* + 1$, which implies $\text{mult}(x^*) \geq m^* + 1$, which is a contradiction.

Now, two cases remain to be analyzed:

Case A: $C^* \subset \Lambda_+$. Thus $\ell(x^*) \leq \ell(x)$ for all $x \in C^*$, that is $x^*$ is a (global) minimizer of $\ell$ on $C^*$, hence a local minimizer of $\ell$ on $\Gamma_{m^*}$.

Case B: There is an open set $W \subset \mathbb{R}^n$ with $x^* \in W$ and $W \cap (Z_R(f) \setminus \Lambda_+) \cap L_e = \emptyset$. In other words, $W$ meets $Z_R(f)$ only at feasible points. Hence $x^*$ minimizes $\ell$ on $W \cap \Gamma_{m^*}$, hence it is a local minimizer of $\ell$ on $\Gamma_{m^*}$. \hfill $\square$

We now give a formal description of an algorithm for Problem (3). The idea is to represent local minimizers of the map $\ell$ on the set $\Gamma_m$ via first-order conditions involving additional Lagrange multipliers. Let $f_1, \ldots, f_m$ be the polynomials defining $\Gamma_m$ (see Corollary 4). A local minimizer $x^*$ of $\ell$ on $\Gamma_m$ is encoded by the following system of equations:

\begin{equation}
\begin{aligned}
f_1 &= 0, \quad \ldots, \quad f_m = 0, \quad e^T x = 1 \\
z_1 \nabla f_1 + \cdots + z_m \nabla f_m + z_{m+1} e &= \nabla \ell
\end{aligned}
\end{equation}

which means that there exists $z^* \in \mathbb{R}^{m+1}$ such that $(x^*, z^*)$ satisfies system (4). When the number of singular points of the complex set $\{x \in \mathbb{C}^n : \forall i = 1, f_i(x) = 0, e^T x = 1\}$ is finite, which turns out to be often satisfied, the solutions of system (4) consist of these singular points and the smooth minimizers of $\ell$. Note that if $\ell(x)$ and $e^T x$ are dependent linear forms, then $e$ and $\nabla \ell$ are multiples, and hence system (4) has infinitely many solutions (all feasible points are critical). Our assumption that $\ell, e^T x$ are independent excludes this pathological situation.

We suppose now that we are given a routine $\text{RP}$ that, when given as input a zero-dimensional ideal $I \subset \mathbb{R}[x, z]$, returns the rational parametrization (2) for the finite set $Z(f \cap \mathbb{R}[x]) \subset \mathbb{C}^n$. Algorithms to compute such parametrizations have appeared e.g. in [22, 6, 11]. The routine $\text{LAG}$ is supposed to build the system (4) from data $f, e, \ell, m$. The formal description of our algorithm is as follows:

Algorithm 1 SolveHP

1. procedure SolveHP($f, e, \ell$)  
2. \hspace{1cm} $L \leftarrow \{\}$  
3. \hspace{1cm} for $m = 1, \ldots, d - 1$ do  
4. \hspace{2cm} $L \leftarrow L \cup \text{RP}(\text{LAG}(f, e, \ell, m))$  
5. \hspace{1cm} return $L$

The output of SolveHP is a set of rational parametrizations. The union of their solutions contains the solution to Problem (3), according to Theorem 10.

3.1. Examples. We have implemented Algorithm 1 in Maple and used it to recover exact information on examples from the literature.
Example 11 (Quartic symmetroids). We consider the list of nodal quartic symmetroids given in [19]. The authors of [19] associate to every transversal quartic spectrahedron \( S = \{ x = (x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : A(x) \succeq 0 \} \), where \( A(x) = x_0 A_0 + x_1 A_1 + x_2 A_2 + x_3 A_3 \), for some \( A_i \in S_4(\mathbb{R}) \), a couple \((\rho, \sigma)\) of nonnegative integers, where \( \rho \) corresponds to the number of nodes (quadratic singularities) of the real projective hypersurface \( \{ x \in \mathbb{P}^3(\mathbb{R}) : \det A(x) = 0 \} \), and \( \sigma \leq \rho \) is the number of nodes lying on \( \partial S \). We denote the quartic of type \((\rho, \sigma)\) in the list in [19] by \( S_{\rho, \sigma} \).

The goal is to perform a random analysis on the solutions of SDP instances over these sets. While this is similar to [18, Table 2], our exact viewpoint can certify the multiplicity at a given solution; indeed, once the representation (2) is computed, isolating the real roots of \( q \) allows to compute the signs of the coefficients of the characteristic polynomial \( \det(tI - A(x)) \) exactly, hence to decide feasibility and multiplicity. The same is not possible with standard SDP solvers. We draw random linear forms

| \( S_{\rho, \sigma} \) | \( m^* = 1 \) | \( m^* = 2 \) | \( S_{\rho, \sigma} \) | \( m^* = 1 \) | \( m^* = 2 \) |
|-------------------|---------|---------|-------------------|---------|---------|
| \( S_{2,2} \)      | 98%     | 2%      | \( S_{4,0} \)      | 100%    | 0%      |
| \( S_{4,4} \)      | 62%     | 38%     | \( S_{6,2} \)      | 32%     | 68%     |
| \( S_{6,6} \)      | 58%     | 42%     | \( S_{8,4} \)      | 17%     | 83%     |
| \( S_{8,8} \)      | 22%     | 78%     | \( S_{10,6} \)     | 7%      | 93%     |
| \( S_{10,10} \)    | 75%     | 25%     | \( S_{6,0} \)      | 100%    | 0%      |
| \( S_{2,0} \)      | 100%    | 0%      | \( S_{8,2} \)      | 15%     | 85%     |
| \( S_{4,2} \)      | 36%     | 64%     | \( S_{10,4} \)     | 14%     | 86%     |
| \( S_{6,4} \)      | 46%     | 54%     | \( S_{8,0} \)      | 100%    | 0%      |
| \( S_{8,6} \)      | 63%     | 37%     | \( S_{10,2} \)     | 18%     | 82%     |
| \( S_{10,8} \)     | 86%     | 14%     | \( S_{10,0} \)     | 100%    | 0%      |

Table 2. Multiplicities on random quartic symmetroids

\( \ell \in \mathbb{Q}[x_0, x_1, x_2, x_3]_1 \) with coefficients uniformly distributed in \( \mathbb{Z} \cap [-100, 100] \), and we compute the solution in (3) with \( f = \det A \) and \( e = I_4 \). Note that in this case the standard section of the hyperbolicity cone is given by \( 1 = \langle I_4, A \rangle = \text{Trace}(A) \), hence we restrict the homogeneous pencil \( A(x) \) to the affine space of matrices with trace 1.

The multiplicity of a solution \( x^* \in S_{\rho, \sigma} \) in this case corresponds to the corank of \( A(x^*) \). In Table 2 we report on the percentage for the multiplicity at a minimizer on 1000 tries. There are only two possible multiplicities, that is 1 and 2. Feasible points with multiplicity 2 correspond to the singularities of the determinant lying on \( S_{\rho, \sigma} \).

We finally generated other representatives of the classes. We observe that percentages can change; indeed these depend not only on the topology of the symmetroid, but also on how the singularities on \( \partial S_{\rho, \sigma} \) are exposed. We believe that this approach can be useful to solve similar classification problems of larger size. ■
Algorithm 1 still works without the assumption that $f$ has a determinantal representation. We test our algorithm on one such example.

**Example 12.** Let $A(x)$ be a $5 \times 5$ homogeneous symmetric linear matrix in 4 variables $x_0, x_1, x_2, x_3$, with $A(e) > 0$ for some $e \in \mathbb{R}^4$. Let $f = \det A$. Then the directional derivative of $f$ in direction $e$, that is the polynomial

$$D_e^{(1)}(f) = \sum_{i=0}^n e_i \frac{\partial f}{\partial x_i},$$

is hyperbolic with respect to $e$ (hence the same is true for the $k$–th derivative $D_e^{(k)}(f)$, $1 \leq k \leq 5$, by induction), but in general does not admit a determinantal representation. For example, let 

$$A(x) = \begin{bmatrix}
  x_0 + x_3 & 2x_1 + 2x_3 & x_1 + 3x_3 & x_2 & x_2 + 3x_3 \\
  2x_1 + 2x_3 & x_0 + 4x_1 + 3x_3 & x_1 + x_2 + 6x_3 & x_1 + x_2 - 2x_3 & x_1 + x_2 + 4x_3 \\
  x_1 + 3x_3 & x_1 - x_2 + 6x_3 & x_0 + x_1 + 8x_3 & -x_1 - x_2 - 3x_3 & -x_1 - x_2 + 6x_3 \\
  x_2 & x_1 + x_2 - 2x_3 & -x_1 - x_2 - 3x_3 & x_0 + x_2 + x_3 & x_1 + 2x_2 - x_3 \\
  x_2 + 3x_3 & x_1 + x_2 + 4x_3 & -x_1 - x_2 + 6x_3 & x_1 + 2x_2 - x_3 & x_0 + x_2 + 4x_3
\end{bmatrix}.$$ 

Then $f = \det A$ is hyperbolic with respect to $e = (1, 0, 0, 0)$ (corresponding to the identity matrix $I_5$), and the quintic real hypersurface \(\{x \in \mathbb{R}^4 : f(x) = 0\}\) has four singularities. As in the previous example, we cut the hyperbolicity cone with the condition $\text{Trace}(A(x)) = 1$ defining the affine space $L_e$. The derivative $D_e^{(1)}(f)$ defines a singular quartic, still with four nodes and hyperbolic with respect to $e$ (see Figure 1), which is not representable as a determinant of a symmetric pencil. Let $\Lambda_+(D_e^{(1)}(f), e)$ be its hyperbolicity cone.

Optimizing generic linear functions over $\Lambda_+(D_e^{(1)}(f), e)$ yields solutions of multiplicity one (smooth boundary points) for 64% of the time, and solutions corresponding to singular points (of multiplicity 2) for 36% of the time, on average. An example of multiplicity two is any multiple of the vector with coordinates

$$x_0 = \frac{1}{2}, \quad x_1 = 0, \quad x_2 = \frac{1}{2}, \quad x_3 = 0$$

which in this case are rational numbers. A smooth point on the boundary of $\Lambda_+(D_e^{(1)}(f), e)$ (multiplicity one), whose coordinates are given as elements of certified rational intervals, with 10 significant decimal digits, is:

$$\begin{align*}
x_0 &\in [1.69735298372573029285, 21216912291574321063] \\
&\approx 1.437713900 \\
x_1 &\in [-29707767148024593931, -147573952589676412928] \\
&\approx -0.2013076605 \\
x_2 &\in [-18765770800641154993, -18765770800640685591] \\
&\approx -0.2543236116 \\
x_3 &\in [21153099539285965043, 661034354527671111] \\
&\approx 0.01791737208.
\end{align*}$$

In our last example, we show how our algorithm can certify lower bounds of Renegar’s method, which uses derivative cones for hyperbolic programming.

**Example 13** (Nie, Parrilo, Sturmfels [17]; Saunderson, Parrilo [26]). We consider the semidefinite representation of the 3-ellipse, as computed in [17]. Given $n$ points
of derivative relaxations and look at the sequence of optimal values. \( \Lambda \) of the direction of the base points could contain one or more singularity, and if this happens, these coincide with some has the semidefinite representation. Consider the relaxations of \( E \) and also the optimal value of the linear function on the solution. In order to measure the error when considering Renegar relaxations for solving hyperbolicity programs, we consider the relaxations of \( E_{\ell} \) and also the multiplicity, and the optimal value of the linear function on the solution. In order to measure the error when considering derivative relaxations, which decreases when considering derivative relaxations, which implies the following fact: the higher the derivative relaxation order is, the faster the exact representation, and hence the lower bound, can be computed. We

\[
\begin{align*}
\text{Table 3, } k \text{ denotes the order of derivation of } f, \text{ and } x^*, m^*, f(x^*) \text{ denote the minimizer, its multiplicity, and the optimal value of } \ell \text{ on the given derivative cone } \Lambda_+(D_{\ell}^{(k)}(f), e), \text{ respectively (here } D_{\ell}^{(k)}(f) \text{ denotes the } k-\text{th derivative of } f \text{ in direction } e). \text{ Moreover, we report in the fifth and sixth column, the degree of the polynomial } q(t) \text{ in the rational representation which is computed by our algorithm (cf. (2)) and the degree of the coordinates of } x^* \text{ (as algebraic numbers over } \mathbb{Q}). \text{ We first remark that the value in the fifth column decreases when considering derivative relaxations, which implies the following fact: the higher the derivative relaxation order is, the}
\end{align*}
\]

\[
\begin{align*}
f = 9x^8 - 72x^7z + 36x^6y^2 - 96x^6yz - 1564x^5y^2z^2 - 216x^5y^2z + 960x^5yz^2 + 9912x^5z^3 + \\
+ 54x^4y^4 - 288x^4y^3z - 4748x^4y^2z^2 + 12256x^4yz^3 + 70782x^4z^4 - 216x^3y^4z + \\
+ 1920x^3y^3z^2 + 17424x^3y^2z^3 - 71040x^3yz^4 - 262296x^3z^5 + 36x^2y^6 - 288x^2y^6z - \\
- 4804x^2y^4z^2 + 27712x^2y^3z^3 + 137228x^2y^2z^4 - 564384x^2yz^5 - 616140x^2z^6 - \\
- 72xy^6 + 960xy^5z^2 + 7512xy^4z^3 - 76416xy^3z^4 - 389688xy^2z^5 + 1372608xyz^6 + \\
+ 1610280xz^7 + 2880y^3z^5 - 58014yz^5 + 283572y^3z^5 - 349728z^5 - \\
- 457380y^2z^6 + 1723680yz^7 + 893025z^8.
\end{align*}
\]

restricted to the plane \( z = 1 \). Let \( A = A(x, y, z) \) be the linear matrix representation of \( E_3 \) given in [26, Example 1], and hence \( f = \det A \). The corresponding 3-ellipse \( E_3 \) has the semidefinite representation \( \{(x, y) \in \mathbb{R}^2 : A(x, y, 1) \succeq 0\} \). The boundary \( \partial E_n \) could contain one or more singularity, and if this happens, these coincide with some of the base points \( p_1, \ldots, p_k \). This is the case for the 3-ellipse we consider, which contains the point \((3, 0)\). The polynomial \( f \) is hyperbolic with respect to \( e = (1, 1, 1) \), hence \( L_e \) is given by the equation \( x + y + z = 1 \).

Our exact algorithm for hyperbolic programs, as we have already remarked, is able to certify rational intervals containing the coordinates of a solution, its multiplicity, and also the optimal value of the linear function on the solution. In order to measure the error when considering derivative relaxations for solving hyperbolicity programs, we consider the relaxations of \( E_3 \), namely the hyperbolicity cones of the derivatives of \( f \) in the direction \( e = (1, 1, 1) \). The infimum of the linear function \( \ell(x, y, z) = x + 2y + 3z + 4 \) on \( E_3 \cap L_e \) is attained at the unique point of multiplicity 2, that is at \((3/4, 0, 1/4)\) (projectively equivalent to \((3, 0, 1)\)). We optimize the same linear function over the derivative relaxations and look at the sequence of optimal values.

In Table 3, \( k \) denotes the order of derivation of \( f \), and \( x^*, m^*, f(x^*) \) denote the minimizer, its multiplicity, and the optimal value of \( \ell \) on the given derivative cone \( \Lambda_+(D_{\ell}^{(k)}(f), e) \), respectively (here \( D_{\ell}^{(k)}(f) \) denotes the \( k \)-th derivative of \( f \) in direction \( e \)). Moreover, we report in the fifth and sixth column, the degree of the polynomial \( q(t) \) in the rational representation which is computed by our algorithm (cf. (2)) and the degree of the coordinates of \( x^* \) (as algebraic numbers over \( \mathbb{Q} \)). We first remark that the value in the fifth column decreases when considering derivative relaxations, which implies the following fact: the higher the derivative relaxation order is, the faster the exact representation, and hence the lower bound, can be computed. We
also remark that for \( k = 0, 1, 2 \) the value in the sixth column is lower. This is because the polynomial \( q \) is reducible, and for \( k = 0 \) (resp \( k = 1, 2 \)) has a linear (resp. degree 30, degree 26) factor which corresponds to the minimum polynomial of the extensions \( \mathbb{Q}[x_i^k] \) over \( \mathbb{Q} \). Indeed, in these cases the variety of critical points factors over \( \mathbb{Q} \).

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