INFORMATION DESIGN IN LARGE GAMES

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ABSTRACT. We define the notion of Bayes correlated Wardrop equilibrium for general nonatomic games with anonymous players and incomplete information. Bayes correlated Wardrop equilibria describe the set of equilibrium outcomes when a mediator, such as a traffic information system, provides information to the players. We relate this notion to Bayes Wardrop equilibrium. Then, we provide conditions—existence of a convex potential and complete information—under which mediation does not improve equilibrium outcomes. We then study full implementation and, finally, information design in anonymous games with a finite set of players, when the number of players tends to infinity.

1. Introduction

1.1. The problem. The time a morning commuter spends on the road to go from home to office depends on her choice and the choice of all other consumers. A good road choice depends on the information commuters have about the condition of the roads, the presence of accidents, and, more generally, the congestion on various parts on the network. Services like Waze or Google Map provide information about these aspects and such information can be used as a tool to reduce congestion. In a formal model, commuters play a game of incomplete information and their behavior leads to a equilibrium, which is in general inefficient. A suitable information design could lead to a more efficient equilibrium by inducing some correlation among the agents’ actions.
Inducing good equilibria by choosing an information structure is the focus of information design. One of the main features of traffic routing games is the presence of a huge number of players. The most common way to solve the computational issues that a large number of players create is to resort to nonatomic games, i.e., games where the action chosen by a single player is negligible. Actually these games are described by the flow of traffic that uses a specific route, rather than by the choice of every single player. The typical concept of equilibrium used in this context is the Wardrop equilibrium (WE).

In this paper we consider a large class of games, called anonymous nonatomic games, where the cost that a player incurs depends on her action and on the distribution of the remaining players’ actions; moreover, each player has a negligible weight. This class of games includes network traffic games and, more generally, congestion games as particular cases, but is much larger. The definition of WE generalizes directly to this larger class of games. We define a nonatomic version of correlated equilibrium (CE) and coarse correlated equilibrium (CCE) Their incomplete-information versions are at the core of the information design mechanism that we study in the paper. The big challenge of these models is to use mathematically sound concepts in a nonatomic environment.

1.2. Our contribution. In this paper we consider nonatomic cost minimization games and introduce the concepts of correlated Wardrop equilibrium and Bayes correlated Wardrop equilibrium, the analogs of correlated equilibrium (Aumann, 1974) and Bayes correlated equilibrium (Bergemann and Morris, 2016). We also introduce the weaker concept of coarse BCWE. We show that, for every information structure, a Bayes Wardrop equilibrium (BWE) outcome, defined by Wu et al. (2021) for a specific information structure, is a Bayes correlated Wardrop equilibrium (BCWE).

For complete information games with a convex potential—e.g., congestion games—we prove that every coarse correlated Wardrop equilibrium can be obtained as a mixture of WE and that all WE have the same cost. As a consequence, all coarse correlated Wardrop equilibrium have the same cost, as well. The implication of this is that in this case, there is no use for information design. We provide a condition on games with incomplete information under which the Bayes correlated Wardrop equilibrium which minimizes the social cost uses deterministic flows in each state.

We then study the problem of finding optimal information structures for a designer who has some objective function, such as minimizing the social cost. We show that an optimal BCWE exists and has finite support; moreover we bound the cardinality of this support. Then, we connect BCWE and BWE and we show that any BCWE with finite support and rational flows can be implemented by a BWE with a suitable information structure and obedient strategies. A weaker result holds in general in the sense that any BCWE can be weak*-approximated by a sequence of approximate BWE. We show that in games that admit a (strictly) convex potential in each state, for every information structure, BWE (flows) costs are unique.
Finally, we consider the relation between solution concepts for nonatomic games and finite games with a large number of players.

The approach of this paper is to keep the spirit of WE, i.e., all equilibrium concepts in nonatomic games are expressed through flows of players on different actions, rather than through actions of individual players. This approach avoids the use of the strong technical measurability assumptions that games with a continuum of players typically require.

1.3. Related literature. The reference paper for Bayesian persuasion with a single decision maker is Kamenica and Gentzkow (2011). Various papers have considered extensions to multiple decision makers, among them Arieli and Babichenko (2019); Bergemann and Morris (2016); Mathevet et al. (2020). The literature is surveyed in Bergemann and Morris (2019); Forges (2020); Kamenica (2019).

The concept of correlated equilibrium was introduced and studied by Aumann (1974, 1987). Properties of the set of correlated equilibria in both finite and infinite games were studied by Hart and Schmeidler (1989).

Congestion games with a finite number of players have been introduced by Rosenthal (1973). Potential games and their relation with congestion games have been studied by Monderer and Shapley (1996). Neyman (1997) proved that in games with finite number of players, convex strategy sets, and a convex potential, every correlated equilibrium is a mixture of Nash equilibria.

Wardrop (1952) studied a strategic model of traffic where each agent has a negligible weight and introduced the principle now known as Wardrop equilibrium: “At equilibrium, the delays along utilized paths are equal and no higher than those that would be experienced by an infinitesimal traffic element going through an unused route.” Beckmann et al. (1956) characterized Wardrop equilibria as the solution of a convex optimization program. The idea that equilibria of routing games are inefficient goes back to Pigou (1920). Measuring this inefficiency through the price of anarchy (Koutsoupias and Papadimitriou, 1999) has been the object of a huge literature in algorithmic game theory (see, e.g., Roughgarden, 2007; Roughgarden and Tardos, 2007).

Convergence of finite players potential games to nonatomic potential games has been studied by Sandholm (2001). The relation between Nash equilibria and Wardrop equilibria in congestion games has been studied by Haurie and Marcotte (1985) in the atomic splittable case and by Cominetti et al. (2021) in the atomic nonsplittable case. Similar results appear in the literature about mean-field games (see, e.g., Campi and Fischer, 2018, 2021; Lacker, 2020).

Some recent papers have studied the effect of public and private information in some classes of congestion games. Das et al. (2017) studied the problem of optimal information design in nonatomic routing games and focused of the case of i.i.d. signals. Tavafoghi and Teneketzis (2017, 2018) studied the design of information disclosure mechanism in routing games and showed that in general perfect disclosure is not optimal. Acemoglu et al. (2018) dealt with routing games where agents have different information sets about the available edges and can use only edges
in their information set. They considered an informational version of the well-known Braess paradox and provided conditions on the network for this informational paradox not to happen. Massicot and Langbort (2019) showed that, even for affine costs, the optimal information policy may be different from full information or no information. Liu and Whinston (2019) used the theory of Bayesian persuasion to propose a more efficient version of Waze that can approximate the social optimum in network routing games. Zhu and Savla (2020) considered nonatomic congestion games where the cost functions are state dependent and a subset of agents receives either a private or a public signal and the remaining agents stay uninformed; they studied the computational issue of finding the optimal information design. Castiglioni et al. (2020) considered the problem of computing optimal ex ante persuasive signaling schemes in atomic routing games. Wu et al. (2021) introduced the notion of Bayes Wardrop equilibrium (BWE) for a routing game with incomplete information when the information structure is given. In their model, each of a finite number of populations is endowed with a different signal. The authors determined the effect of population sizes on the relative value of information.

Ashlagi et al. (2008) introduced the mediation value—i.e., the ratio between the maximal welfare in a correlated equilibrium to the maximal welfare in a mixed-strategy equilibrium—and showed how this measure performs in congestion games. Diaz et al. (2009) showed that no mediator will improve the social welfare over the best Nash equilibrium in congestion games with malicious players. Roughgarden (2015) studied conditions under which bounds for the price of anarchy extend from pure to mixed, correlated, and coarse correlated equilibria.

The El Farol model goes back to Arthur (1994), who described a situation where a bar (El Farol, in Santa Fe) offers Irish music on Thursday nights. Being there is fun if there is some crowd, but starts getting unpleasant if the crowd is excessive. More recently it has been studied as a nonatomic game by Mitsche et al. (2013).

1.4. Organization of the paper. Section 2 introduces the model both under complete and incomplete information and relates the concepts of Bayes correlated equilibrium (BCE) and BCWE. Section 3 deals with potential games. Section 4 studies the design of optimal information mechanisms. Section 5 examines approximation of equilibria in large finite games with equilibria of nonatomic games. Appendix A describes the multi-population model. The remaining proofs are contained in Appendix B.

1.5. Notation. The cardinality of a finite set \( A \) is denoted by \(|A|\) and we use the notation

\[
\Delta_y(A) := \left\{ y : y_k \geq 0, \forall k \in A, \sum_{k \in A} y_k = y \right\}. \tag{1.1}
\]

We just write \( \Delta(A) \), when \( y = 1 \).

For \( n \in \mathbb{N} \), we write \([n] := \{1, \ldots, n\}\). For Borel distributions on \( \Delta(A) \), \( \mu_n \xrightarrow{w^*} \mu \) denotes weak* convergence of \( \mu_n \) to \( \mu \), i.e., \( \int f \, d\mu_n \rightarrow \int f \, d\mu \) for every continuous function \( f : \Delta(A) \rightarrow \mathbb{R} \).
Throughout the paper we consider cost games, i.e., games where players choose actions in order to minimize their cost.

2. **Anonymous nonatomic games**

2.1. **Complete information.** Throughout the paper, we consider a class of games that heuristically represent situations where the decision of a single player is negligible and the cost incurred by one player depends on the distribution of the other players’ actions, but not on the identity of who did what.

Consider a finite action set $\mathcal{A}$ and a mass of players, which, without loss of generality, is assumed to be 1 throughout. A flow vector $y = (y_a)_{a \in \mathcal{A}}$ indicates how the total mass of players splits over the different actions. The set $\mathcal{Y}$ of feasible flows is simply $\triangle(\mathcal{A})$.

An anonymous nonatomic game (ANG) $\Gamma = (\mathcal{A}, c)$ consists of a finite action set $\mathcal{A}$ and a profile of cost functions $c := (c_a)_{a \in \mathcal{A}}$, with $c_a : \mathcal{Y} \to \mathbb{R}$. In words, the cost $c_a$ of taking the action $a$ is a function of the flow $y$ over the action set $\mathcal{A}$.

**Definition 2.1.**

(a) A Wardrop equilibrium (WE) of the ANG $\Gamma$ is a flow vector $y$ such that, for every $a, b \in \mathcal{A}$, we have

$$y_a c_a(y) \leq y_a c_b(y). \quad (2.1)$$

(b) A correlated Wardrop equilibrium (CWE) of the ANG $\Gamma$ is a distribution $\mu$ over $\mathcal{Y}$ such that, for all $a, b \in \mathcal{A}$, we have

$$\int y_a c_a(y) \, d\mu(y) \leq \int y_a c_b(y) \, d\mu(y). \quad (2.2)$$

(c) A coarse correlated Wardrop equilibrium (CCWE) of the ANG $\Gamma$ is a distribution $\mu$ over $\mathcal{Y}$ such that, for all $a, b \in \mathcal{A}$, we have

$$\int \sum_{a \in \mathcal{A}} y_a c_a(y) \, d\mu(y) \leq \int c_b(y) \, d\mu(y). \quad (2.3)$$

The symbols $\text{WE}(\Gamma)$, $\text{CWE}(\Gamma)$, $\text{CCWE}(\Gamma)$ denote the sets of Wardrop equilibria, correlated Wardrop equilibria, and coarse correlated Wardrop equilibria of $\Gamma$, respectively.

The definition of WE is the usual one: only actions with the least cost receive positive flow. A CWE can be heuristically interpreted as follows. A mediator draws a flow vector $y$ according to $\mu$. For each action $a$, a fraction $y_a$ of the population is drawn at random to play action $a$. If we divide Eq. (2.2) by the total probability $\int y_a \, d\mu(y)$ of action $a$, then the left-hand-side is the expected payoff of playing $a$, conditionally of being recommended $a$. The right-hand-side is the expected payoff of playing $b$, conditionally of being recommended $a$. In a CWE, no player has a profitable deviation from any recommended action.

In a CCWE, no player has a profitable deviation that does not depend on the recommended action: the left-hand-side of Eq. (2.3) is the expected payoff of a player who always obeys the recommendation, while the right-hand-side is the expected payoff of a player who plays another action $b$ no matter what.
It is immediate to see that $\Delta(\text{WE}(\Gamma)) \subseteq \text{CWE}(\Gamma) \subseteq \text{CCWE}(\Gamma)$. The next example from Mitsche et al. (2013) illustrates the difference between CWE and WE. In the example, there exists a CWE in which the expected social cost is strictly smaller than in any WE, where the social cost of $y$ is defined as

$$ SC(y) := \sum_{a \in A} y_a c_a(y). \quad (2.4) $$

**Example 2.1 (El Farol).** There is a continuum of customers whose action set $A$ is $\{a, b\}$, which represent staying home and going to the bar, respectively. The cost functions are:

$$ c_a(y_a) = 1, \quad c_b(y_b) = \max\{2 - 4y_b, 4y_b - 2\}. \quad (2.5) $$

The game $\Gamma$ admits three WE: $(1, 0)$, $(3/4, 1/4)$, and $(1/4, 3/4)$, the social cost is equal to 1 for all of them. Take now $y^1 = (1, 0)$ and $y^2 = (1/2, 1/2)$. The distribution $\mu$ such that

$$ \mu(y^1) = \frac{1}{3}, \quad \mu(y^2) = \frac{2}{3} \quad (2.6) $$

is a CWE, since it satisfies the equilibrium constraints

$$ \frac{2}{3} = \frac{1}{3} y_a c_a(y^1) + \frac{2}{3} y_b c_b(y^2) \leq \frac{1}{3} y_a c_b(y^1) + \frac{2}{3} y_b c_b(y^2) = \frac{2}{3}, $$

$$ 0 = \frac{1}{3} y_b c_b(y^1) + \frac{2}{3} y_b c_b(y^2) \leq \frac{1}{3} y_a c_b(y^1) + \frac{2}{3} y_b c_b(y^2) = \frac{1}{3}. $$

This CWE is not a mixture of WE. Therefore $\Delta(\text{WE}(\Gamma)) \subseteq \text{CWE}(\Gamma)$. Moreover, the social cost of this CWE is $2/3$, which is lower than the social cost of each WE.

### 2.2. Incomplete information.

An anonymous nonatomic game with incomplete information (ANGII) $\Gamma = (A, \Theta, p, c)$ is given by an action set $A$, a finite state set $\Theta$, a full-support probability distribution over states $p \in \Delta(\Theta)$, and, for each $a \in A$, a continuous cost function $c_a : \mathcal{Y} \times \Theta \rightarrow \mathbb{R}$. A mapping $\mu : \Theta \rightarrow \Delta(\mathcal{Y})$ which associates a distribution over flows to each state, is called an outcome of the game. We denote by $\mathcal{M}$ the set of all outcomes.

We now define our solution concepts for games with incomplete information.

**Definition 2.2.** (a) A Bayes correlated Wardrop equilibrium (BCWE) of the ANGII $\Gamma$ is an outcome $\mu$ such that, for all $a, b \in A$, we have

$$ \sum_{\theta \in \Theta} p(\theta) \int y_a c_a(y, \theta) \, d\mu(y \mid \theta) \leq \sum_{\theta \in \Theta} p(\theta) \int y_a c_b(y, \theta) \, d\mu(y \mid \theta). \quad (2.7) $$

(b) A simple Bayes correlated Wardrop equilibrium (SBCWE) of the ANGII $\Gamma$ is a mapping $y(\cdot) : \Theta \rightarrow \mathcal{Y}$ such that, for all $a, b \in A$, we have

$$ \sum_{\theta \in \Theta} p(\theta) y_a c_a(y(\theta), \theta) \leq \sum_{\theta \in \Theta} p(\theta) y_a c_b(y(\theta), \theta). \quad (2.8) $$
(c) A coarse Bayes correlated Wardrop equilibrium (CBCWE) of the ANGII $\Gamma$ is an outcome $\mu$ such that, for all $b \in \mathcal{A}$, we have

$$\sum_{\theta \in \Theta} p(\theta) \int \sum_{a \in \mathcal{A}} y_{a\mu}(y, \theta) \, d\mu(y | \theta) \leq \sum_{\theta \in \Theta} p(\theta) \int \sum_{a \in \mathcal{A}} y_{a\mu}(y, \theta) \, d\mu(y | \theta).$$  

(2.9)

The symbols BCWE($\Gamma$), SBCWE($\Gamma$), CBCWE($\Gamma$) denote the sets of Bayes correlated Wardrop equilibria, simple Bayes correlated Wardrop equilibria, and coarse Bayes correlated Wardrop equilibria of $\Gamma$, respectively.

The interpretation of BCWE is similar to the interpretation of CWE: for each state $\theta \in \Theta$, a mediator who knows the state draws a flow vector $y$ at random according to the distribution $\mu(\cdot | \theta)$ and then, for each $a \in \mathcal{A}$, recommends a random fraction $y_a$ of players to play $a$. The outcome is a BCWE if there is no incentive to deviate from the mediator’s recommendation. A SBCWE is a BCWE where in each state, the distribution $\mu(\cdot | \theta)$ is a degenerate distribution $\delta_\mu$. In a CBCWE, no player can profitably deviate from obedience to a fixed action that does not depend on the recommendation.

2.3. Bayes Wardrop equilibria and Bayes correlated Wardrop equilibria.

Wu et al. (2021) introduced the notion of Bayes Wardrop equilibrium (BWE) for an incomplete information routing game with a given information structure. Their definition of BWE is easily adapted beyond routing games to our more general model.

An information structure $(y, \mathcal{T}, \pi)$ for the ANGII $\Gamma = (\mathcal{A}, \Theta, p, c)$ is given by: a finite set $\mathcal{K}$ of populations, where the size of population $k$ is $y^k \geq 0$ with $\sum_{k \in \mathcal{K}} y^k = 1$; for each population $k$, a finite type set $\mathcal{T}^k$; and a mapping $\pi : \Theta \rightarrow \Delta(\mathcal{T})$, where $\mathcal{T} := \times_{k \in \mathcal{K}} \mathcal{T}^k$.

For each state $\theta \in \Theta$, the type profile $\tau \in \mathcal{T}$ is drawn with probability $\pi(\tau | \theta)$ and the entire population $k$ observes the type $\tau^k \in \mathcal{T}^k$. For $k \in \mathcal{K}$, consider the map $y^k(\cdot) : \mathcal{T}^k \rightarrow \Delta^k(\mathcal{A})$ and let $y^k_a(\tau^k)$ denote the flow of players who observed type $\tau^k$ and chose action $a$. Notice that $\sum_{a \in \mathcal{A}} y^k_a(\tau^k) = y^k$. Define

$$y_a(\tau) = \sum_{k \in \mathcal{K}} y^k_a(\tau^k) \quad \text{and} \quad y(\tau) = (y_a(\tau))_{a \in \mathcal{A}}.$$

(2.10)

A BWE is a profile $y(\cdot) = (y^k(\cdot))_{k \in \mathcal{K}}$ such that, for all $k \in \mathcal{K}$, for all $\tau^k \in \mathcal{T}^k$, and for all $a, b \in \mathcal{A}$, if $y^k_a(\tau^k) > 0$, then

$$\sum_{\tau^{-k} \in \mathcal{T}^{-k}} \sum_{\theta \in \Theta} p(\theta) \pi(\tau^k, \tau^{-k} | \theta)c_a(y(\tau), \theta) \leq \sum_{\tau^{-k} \in \mathcal{T}^{-k}} \sum_{\theta \in \Theta} p(\theta) \pi(\tau^k, \tau^{-k} | \theta)c_b(y(\tau), \theta).$$

(2.11)

That is, for each population $k$ and type $\tau^k$, a positive mass chooses action $a$ only if it has the least expected cost, conditional on $\tau^k$.

A mapping $\mu : \Theta \rightarrow \Delta(\mathcal{Y})$ is a BWE outcome of $\Gamma = (\mathcal{A}, \Theta, p, c)$ with information structure $(y, \mathcal{T}, \pi)$ if there exists a BWE $y(\cdot)$ such that for every $\theta \in \Theta$ and
for every $\omega \in \mathcal{Y}$,
\[
\mu(\omega \mid \theta) = \sum_{\tau: y(\tau) = \omega} \pi(\tau \mid \theta).
\]  
(2.12)

**Proposition 2.3.** Given an ANGII, for every information structure, a BWE outcome is a BCWE.

**Proof.** Let $y(\cdot)$ be a BWE. Multiplying by $y_a^k(\tau^k)$, Eq. (2.11) gives $\forall k, \forall \tau^k, \forall a, b,$
\[
\sum_{\theta \in \Theta} p(\theta) \sum_{\tau^{-k} \in \mathcal{T}^{-k}} y_a^k(\tau^k) \pi(\tau^k, \tau^{-k} \mid \theta) c_a(y(\tau), \theta)
\leq \sum_{\theta \in \Theta} p(\theta) \sum_{\tau^{-k} \in \mathcal{T}^{-k}} y_a^k(\tau^k) \pi(\tau^k, \tau^{-k} \mid \theta) c_b(y(\tau), \theta).
\]

Summing over $k$ and then over $\tau^k$ we get:
\[
\forall a, b, \sum_{\theta} p(\theta) \sum_{\tau} \pi(\tau \mid \theta) y_a(\tau) c_a(y(\tau), \theta) \leq \sum_{\theta} p(\theta) \sum_{\tau} \pi(\tau \mid \theta) y_a(\tau) c_b(y(\tau), \theta).
\]
Letting $\mu(y \mid \theta) = \sum_{\tau: y(\tau) = y} \pi(\tau \mid \theta)$ we conclude that $\mu$ is a BCWE. $\square$ $\square$

### 3. Potential Games

We now study the interesting subclass of potential games.

**Definition 3.1.** An ANG $\Gamma = (\mathcal{A}, c)$ is a potential game if there exists $\Phi: \mathcal{Y} \rightarrow \mathbb{R}$ such that, for every $a \in \mathcal{A}$ and $y \in \mathcal{Y}$, we have
\[
\frac{\partial \Phi(y)}{\partial y_a} = c_a(y).
\]  
(3.1)

The function $\Phi$ is called the potential of $\Gamma$.

An ANGII $\Gamma = (\mathcal{A}, \Theta, p, c)$ is a potential game if for every $\theta \in \Theta$, there exists a function $\Phi_\theta: \mathcal{Y} \rightarrow \mathbb{R}$ such that, for every $a \in \mathcal{A}$ and $y \in \mathcal{Y}$, we have
\[
\frac{\partial \Phi_\theta(y)}{\partial y_a} = c_a(y, \theta).
\]  
(3.2)

**Example 3.1.** Among the most extensively studied anonymous nonatomic potential games, are congestion games\(^1\) (see, e.g., Roughgarden, 2007). One of the main applications of congestion games is traffic routing as is reflected in the terminology, yet the model is more general. There is a finite set of resources $\mathcal{E}$, a set of actions $\mathcal{A} \subseteq 2^\mathcal{E}$ and for each resource $e$, a continuous nondecreasing cost function $c_e: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. For $a \in \mathcal{A}$, the symbol $y_a$ denotes the flow of players who choose action $a$. For each resource $e$, the load on resource $e$ is $x_e := \sum_{a \in e} y_a$. The symbols $y := (y_a)_{a \in \mathcal{A}}$ and $x := (x_e)_{e \in \mathcal{E}}$ denote the flow and load profiles, respectively. The cost of using resource $e$ is $c_e(x_e)$ and the cost of choosing action $a$ is obtained additively: $c_a(y) = \sum_{e \in e} c_e(x_e)$ with a common abuse of notation. In routing traffic, models there is an underlying oriented graph with origin and

\(^1\)Other examples of anonymous nonatomic potential games include Cournot competition, random matching games and all games with only two actions (see, for example, Sandholm, 2001).
destination nodes, the resources are edges of the graph and the actions are the paths from the origin to the destination.\(^2\)

Congestion games are potential games and the potential is given by

\[
\Phi(y) = \sum_{e \in E} \int_0^{\sum_{a \in e} y_a} c_e(u) \, du. \tag{3.3}
\]

Since cost function \(c_e\) of resources are nondecreasing, this function is convex. It is well known that all minimizers of this function have the same cost profiles \((c_e(x_e))_{e \in E}\). In other words, Wardrop equilibria of a congestion game are not necessarily unique, but all have the same costs.

**Proposition 3.2.** If an ANG \(\Gamma\) has a convex potential, then

\[
\text{CCWE}(\Gamma) = \text{CWE}(\Gamma) = \Delta(\text{WE}(\Gamma)). \tag{3.4}
\]

In addition, all WE (and therefore all CWE and CCWE) have the same costs profiles: all actions with positive flow have the same cost in all equilibria.

Proposition 3.2 shows that, for ANGs with a convex potential, distributions over WE exhaust all CWE. Thus to minimize any expected cost function over the set of CWE, it is enough to focus on WE. We now examine the case of ANGIIs and ask whether it is enough to focus on SBCWE, the analog of WE.

The next example shows that Proposition 3.2 does not extend to incomplete information. That is, in games with a convex potential but incomplete information, the convex hull of the set of simple BCWE outcomes might be strictly included in the set of BCWE.

**Example 3.2.** There are two states \(\Theta = \{0, 1\}\) with uniform prior distribution \(p(\theta) = 1/2\) and two actions \(A = \{a, b\}\). The cost functions are \(c_a(y_a, \theta) = 3 - 3\theta\) and \(c_b(y_b, \theta) = 1\). The following outcome is a BCWE:

\[
\mu((1, 0) \mid \theta = 1) = 1 \quad \text{and} \quad \mu((1, 0) \mid \theta = 0) = \mu((0, 1) \mid \theta = 0) = 1/2.
\]

This BCWE is not a convex combination of SBCWE since \(y(\cdot)\) such that \(y(1) = y(0) = (1, 0)\) is not a SBCWE, yet it must have probability 1/2 under \(\mu\).

However, for a particular class of games with binary actions and for minimizing the social cost over the set of BCWE, the next proposition shows that it is enough to focus on SBCWE.

**Proposition 3.3.** Consider an ANGII \(\Gamma\) with two actions \(A = \{a, b\}\) and such that for every \(a \in A\) and \(\theta \in \Theta\), the function \(y \mapsto y_a c_a(y, \theta)\) is convex and the function \(y \mapsto c_a(y, \theta)\) is concave. Then for every BCWE of \(\Gamma\), there exists a SBCWE whose expected social cost is weakly smaller in each state \(\theta\).

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\(^2\)In this description, we assume that all players have the same origin and destination. However, the model and our results extend to multiple populations with different origins and destinations, see Appendix A.
implies that Wu et al. do not imply the existence of a convex potential. This is not a corollary of Proposition 3.2.

Zhu and Savla (2021) proposed Proposition 3.3.

In this section, we consider the problem of minimizing the expected social cost,

$$\min_\mu \sum_{\theta} p(\theta) \int \sum_{a \in A} y_a c_a(y, \theta) \mu(dy | \theta),$$

s.t. $$\forall a, b \sum_{\theta} p(\theta) \int y_a c_a(y, \theta) \mu(dy | \theta) \leq \sum_{\theta} p(\theta) \int y_a c_b(y, \theta) \mu(dy | \theta).$$

Since there are two actions, $$y_b = 1 - y_a$$, the incentive constraint where $$a$$ is the recommended action and $$b$$ is the deviation is

$$\sum_{\theta} p(\theta) \int y_a c_a(y, \theta) \mu(dy | \theta) \leq \sum_{\theta} p(\theta) \int (1 - y_b) c_b(y, \theta) \mu(dy | \theta),$$

which can be written as

$$\sum_{\theta} p(\theta) \int y_a c_a(y, \theta) \mu(dy | \theta) + \sum_{\theta} p(\theta) \int y_b c_b(y, \theta) \mu(dy | \theta) \leq \sum_{\theta} p(\theta) \int c_b(y, \theta) \mu(dy | \theta). \quad (3.5)$$

For each action $$a$$ and $$\theta$$, define $$\bar{y}_a(\theta) = \int y_a d\mu(\theta)$$ as the expected flow. Since the mappings $$y \mapsto y_a c_a(y, \theta)$$ are convex and $$y \mapsto c_a(y, \theta)$$ are concave,

$$\sum_{\theta} p(\theta) \int y_a c_a(y, \theta) \mu(dy | \theta) \geq \sum_{\theta} p(\theta) \bar{y}_a(\theta) c_a(\bar{y}(\theta), \theta),$$

and

$$\sum_{\theta} p(\theta) \int c_b(y, \theta) \mu(dy | \theta) \leq \sum_{\theta} p(\theta) c_b(\bar{y}(\theta), \theta).$$

Therefore, Eq. (3.5) implies that $$\theta \mapsto \bar{y}(\theta)$$ is a SBCWE with a weakly smaller social cost. \(\square\)

Remark 3.1. Notice that when the state space is a singleton, i.e., the game is of complete information, Proposition 3.3 is not a corollary of Proposition 3.2, since the assumptions of Proposition 3.3 do not imply the existence of a convex potential.

A special case of Proposition 3.3 is in Tavafoghi and Teneketzis (2018); Zhu and Savla (2020) with binary actions and affine cost functions.

4. Designing optimal information structures

In this section, we consider the problem of a designer who looks for the best BCWE with respect to some optimization criterion, e.g., minimizing the social cost.

First, we show that the optimal BCWE exists and has finite support, and we bound the size of this support. Then, we prove that for any BCWE $$\mu$$ with finite support and rational flows, there exists an information structure such that the induced game has a BWE (in the sense of Wu et al., 2021) that induces the outcome $$\mu$$. Furthermore, as in the revelation principle for finite incomplete information
games (Myerson, 1982), the information structure we construct is a symmetric, direct system of action recommendations, and equilibrium strategies are obedient.

Finally, we show that BCWE can be fully implemented in games with a strictly convex potential in the sense that, for every BCWE \( \mu \), there exists an information structure such that the unique BWE outcome is the BCWE \( \mu \). If the potential is just convex, then equilibrium costs can be fully implemented: all equilibria have the same cost.

Let \( c^D : \mathcal{Y} \times \Theta \to \mathbb{R} \) be the designer’s continuous cost function. The designer’s goal is to minimize her expected cost over BCWE(\( \Gamma \)):

\[
\min_{\mu \in \mathcal{M}} \sum_{\theta \in \Theta} p(\theta) \int c^D(y, \theta) \, d\mu(y \mid \theta)
\]

s.t. \( \sum_{\theta \in \Theta} p(\theta) \int y_a c_a(y, \theta) \, d\mu(y \mid \theta) \leq \sum_{\theta \in \Theta} p(\theta) \int y_a c_b(y, \theta) \, d\mu(y \mid \theta) \),

\( \forall a, b \in \mathcal{A} \).

**Proposition 4.1.** The program (P) admits a solution \( \mu \) with finite support, whose cardinality is at most \( |\Theta|(|\mathcal{A}|^2 + 1) \).

**Proof.** First, we remark that a BCWE \( \mu \) which belongs to \( \Delta(\mathcal{Y})^\Theta \) can equivalently represented by \( \tilde{\mu} \in \Delta(\mathcal{Y})^\Theta \). That is, the mediator draws from \( \tilde{\mu} \) a vector of flows \( \tilde{y} = (y(\theta))_{\theta \in \Theta} \in \mathcal{Y}^\Theta \), where \( \tilde{\mu} \) is defined by \( d\tilde{\mu}(\tilde{y}) = \times_{\theta} d\mu(y(\theta) \mid \theta) \).

Program (P) can be written as

\[
\min_{\tilde{\mu}} \int \sum_{\theta} p(\theta) c^D(y(\theta), \theta) \, d\tilde{\mu}(\tilde{y}),
\]

s.t. \( \forall a, b \int \sum_{\theta} p(\theta) y_a(\theta)c_a(y(\theta), \theta) \, d\tilde{\mu}(\tilde{y}) \leq \int \sum_{\theta} p(\theta) y_a(\theta)c_b(y(\theta), \theta) \, d\tilde{\mu}(\tilde{y}) \).

Define, for \( a, b \in \mathcal{A} \),

\[
\tilde{z}_{a,b}(\tilde{y}) = \sum_{\theta} p(\theta) y_a(\theta)c_b(y(\theta), \theta), \quad (4.1)
\]

\[
\tilde{z}(\tilde{y}) = (\tilde{z}_{a,b}(\tilde{y}))_{(a,b) \in \mathcal{A} \times \mathcal{A}} \in \mathbb{R}^{\mathcal{A} \times \mathcal{A}}, \quad (4.2)
\]

\[
c^D(\tilde{y}) = \sum_{\theta} p(\theta) c^D(y(\theta), \theta), \quad (4.3)
\]

\[
\mathcal{Z} = \{(z, c) \in \mathbb{R}^{\mathcal{A} \times \mathcal{A}} \times \mathbb{R} : \exists \tilde{y} \in \mathcal{Y}^\Theta \text{ s.t. } (z, c) = (\tilde{z}(\tilde{y}), c^D(\tilde{y}))\}. \quad (4.4)
\]

The objective function and the constraints are linear with respect to \((\tilde{z}(\tilde{y}), c^D(\tilde{y}))\) so Program (P) is equivalent to

\[
\min \{ c \text{ s.t. } (z, c) \in \text{co}(\mathcal{Z}) \text{ and } \forall a, b \in \mathcal{A}, z_{a,a} \leq z_{a,b} \}.
\]

From Carathéodory’s theorem, any point in \(\text{co}(\mathcal{Z})\) can be obtained as a convex combination of \(|\mathcal{A} \times \mathcal{A}| + 2 \) points and as \(\mathcal{Z}\) is connected, this can be reduced to \(|\mathcal{A} \times \mathcal{A}| + 1 \) points (Fenchel, 1929). Since any \(\tilde{y}\) induces at most \(|\Theta|\) different flows, the social cost of any BCWE can be obtained with a BCWE that randomizes over a set of flows with cardinality at most \(|\Theta|(|\mathcal{A}|^2 + 1)\). \(\square\) \(\square\)
Having established the existence of a finite support optimal BCWE, the next step is to show that it can be implemented as a BWE of the original incomplete information game with an appropriate information structure. First, we consider the case where every flow in the support of \( \mu \) is a vector with rational components.

**Proposition 4.2.** If \( \mu \) is a BCWE with finite support and rational flows, then there exists an information structure \((y, T, \pi)\) and a BWE with outcome \( \mu \).

In addition, the information structure is direct: for every \( k \in K \), \( T^k = A \); and the BWE is obedient: the whole population \( k \) plays a upon observing the type \( \tau^k = a \).

**Proposition 4.2** follows from the following lemma.

**Lemma 4.3.** Let \( \mu \) be a BCWE with finite support. Then there exists a sequence \( \varepsilon_k \searrow 0 \), a sequence of information structures and associated \( \varepsilon_k \)-BWE flow \( \mu_k \), such that \( \mu_k \Rightarrow \mu \). Furthermore, if all flows in the support of \( \mu \) are rational, then we can choose \( \varepsilon_k = 0 \) and \( \mu_k = \mu \).

Proof. Let \( \mu \) be a BCWE with finite support \( \mathcal{Y}^\ast \). Then, for all \( a, b \in A \),

\[
\sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^\ast} p(\theta)\mu(y \mid \theta)y_a c_a(y, \theta) \leq \sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^\ast} p(\theta)\mu(y \mid \theta)y_a c_b(y, \theta). \tag{4.5}
\]

We use vectors with rational components to approximate flows in \( \mathcal{Y}^\ast \). For every \( y \in \mathcal{Y}^\ast \), \( a \in A \), and \( k \in \mathbb{N} \), there exist \( N^k(y_a) \in \mathbb{N} \) and \( \delta^k > 0 \) such that

\[
\sum_{a \in A} N^k(y_a) = k \quad \text{and} \quad \left| \frac{N^k(y_a)}{k} - y_a \right| \leq \delta^k, \tag{4.6}
\]

and \( \delta^k \searrow 0 \). Let \( y^k \) denote the flow \( (y^k_a)_{a \in A} \) with \( y^k_a = N^k(y_a)/k \).

We construct an information structure as follows. There are \( k \) population of players, each population having the same size \( y^k = 1/k \). The set of types is \( T^k = A \) for each population \( k \). Conditional on state \( \theta \), the mediator draws \( y \in \mathcal{Y}^\ast \) with probability \( \mu(y \mid \theta) \), then recommends action \( a \) to a subset of populations of size \( N^k(y_a) \), so that, conditionally on \( y \) and \( \theta \), the probability that population \( k \) is recommended \( a \) is \( N^k(y_a)/k = y^k_a \). The expected cost of playing \( b \) conditional of being recommended \( a \) (multiplied by the total probability of \( a \)) is

\[
\sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^\ast} p(\theta)\mu(y \mid \theta)y^k_a c_b(y^k, \theta). \tag{4.7}
\]

The mapping \( y \mapsto y^k_a c_b(y, \theta) \) is uniformly continuous on \( \mathcal{Y} \) for each pair of actions \( (a, b) \), thus, there exists a common modulus of continuity \( \omega(\cdot) \) with \( \lim_{\delta \searrow 0} \omega(\delta) = 0 \) such that, \( \forall \theta \in \Theta, \forall (a, b) \in A \times A, \forall (y, z) \in \mathcal{Y} \times \mathcal{Y} \),

\[
|y^k_a c_b(y, \theta) - z^k_a c_b(z, \theta)| \leq \omega(\max_b|y_b - z_b|). \tag{4.8}
\]

It follows from Eqs. (4.5), (4.6) and (4.8) that

\[
\sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^\ast} p(\theta)\mu(y \mid \theta)y^k_a c_a(y^k, \theta) \leq \sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^\ast} p(\theta)\mu(y \mid \theta)y^k_a c_b(y^k, \theta) + 2\omega(\delta^k).
\]
We have thus constructed an $\varepsilon^k$-BWE with $\varepsilon^k = 2\omega(\delta^k)$. The induced outcome weak$^\ast$ converges to $\mu$ since for any continuous $f : \mathcal{Y} \times \Theta \to \mathbb{R}$,

$$\sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^\kappa} \mu(y | \theta) f(y^k, \theta) \xrightarrow{k \to \infty} \sum_{\theta \in \Theta} \sum_{y \in \mathcal{Y}^\kappa} \mu(y | \theta) f(y, \theta).$$

If flows in $\mathcal{Y}^\kappa$ have rational coefficients with common denominator $k$, then we can choose $y^k = y$ and we have a 0-BWE.

The combination of Propositions 2.3 and 4.2 can be interpreted as a revelation principle for anonymous nonatomic games with incomplete information: whatever the information structure, a BWE induces a BCWE, and for every BCWE there exists a direct recommendation system that implements this BCWE with obedient equilibrium strategies. This revelation principle also applies to complete information games: if a complete information game is extended with some information structure, then a BWE of the extended game induces a CWE; conversely, for every CWE of a complete information game, there exists a direct recommendation system that implements this CWE with obedient equilibrium strategies.$^3$

In general, when a BCWE induces irrational flows or an infinite number of possible flows, we cannot implement a BWE with a finite information structure as defined in Wu et al. (2021). Nevertheless, the next proposition provides an approximate implementation result.

**Proposition 4.4.** Let $\mu$ be a BCWE of the ANGII $\Gamma$. Then there exist a sequence $\varepsilon^k \searrow 0$, a sequence of information structures, and corresponding $\varepsilon^k$-BWE outcomes $\mu^k$ such that $\mu^k \xrightarrow{w} \mu$.

Proposition 4.4 follows from Lemma 4.3 and Lemma B.2, which shows that the set of BCWE with finite support is dense in the set of BCWE.

**4.1. Full implementation.** Full implementation refers to the implementation of a BCWE in every BWE. As in Morris et al. (2020), full implementation guarantees that, under the chosen information structure, all equilibria induce the same outcome. First, we show that in games with convex potential, the BWE costs are unique and that in games with a strictly convex potential, the BWE itself is unique.

This applies to congestion games since they have a convex potential. Moreover, if the cost function $c_e$ of each resource $e \in \mathcal{E}$ is strictly increasing and if for every action $a \in \mathcal{A}$ there exists a resource $e \in a$ such that $e \notin b$, for every $b \neq a$, then the congestion game has a strictly convex potential. This holds, for instance, in parallel networks with strictly increasing costs.

**Proposition 4.5.** Consider a potential ANGII. If the potential is convex in each state, then for every information structure there exists a unique BWE cost profile.

---

$^3$For instance, in Example 2.1, the cost minimizing CWE $\mu(1, 0) = \frac{1}{2}$ and $\mu(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ is implemented as a BWE with the following information structure: there are two populations of size $\gamma_1 = \gamma_2 = \frac{1}{2}$, and $\pi(a, a) = (a, b) = \pi(b, a) = \frac{1}{2}$, and the incentive compatibility conditions of the CWE are equivalent to the obedience constraints.
If the potential is strictly convex in each state, then for every information structure there exists a unique BWE.

From Propositions 4.2 and 4.5, we have the following full implementation result for games with (strictly) convex potential.

**Corollary 4.6.** Consider an ANGII with convex potential and let µ be a BCWE with finite support and rational flows. There exists an information structure \((y, T, \pi)\) such that the ex-ante social cost of every BWE of the extended game is the same as the ex-ante social cost of µ. If the potential is strictly convex, then the BWE is unique and implements the BCWE µ. The implementation can be achieved with \(T^k = A\) and obedient equilibrium strategies.

5. CONVERGENCE OF ATOMIC TO NONATOMIC GAMES

Given an ANGII \(\Gamma = (A, \Theta, p, c)\), we define \(n\)-player games with incomplete information \(\Gamma_n = ([n], A, \Theta, p, (c_i)_{i \in [n]})\). Each player has the same action set \(A\). Every action profile \(a \in A^n\) induces a flow vector \(y(a) = (y_a)_{a \in A}\) defined by

\[
y_a = \frac{1}{n} \sum_{i \in N} 1_{\{a_i = a\}}.
\]

The cost function \(C_i: A^n \rightarrow \mathbb{R}\) of player \(i\) is given by

\[
C_i(a) = c_{a_i}(y(a)).
\]

The game \(\Gamma_n\) is anonymous: the cost of a player depends on her action and on the distribution of all players’ strategies. We now recall the definition of Bayes correlated equilibrium for finite games (Bergemann and Morris, 2016), adapted to anonymous games.

**Definition 5.1.** A Bayes correlated equilibrium (BCE) of an \(n\)-player game is a mapping \(\beta^n: \Theta \rightarrow \Delta(A^n)\) such that for all \(i \in [n]\) and for all \(a, b \in A\), we have

\[
\sum_{a_{-i} \in A_{-i}} \sum_{\theta \in \Theta} p(\theta) \beta^n(a, a_{-i} | \theta) c_a(y(a, a_{-i}), \theta) \leq \sum_{a_{-i} \in A_{-i}} \sum_{\theta \in \Theta} p(\theta) \beta^n(a, a_{-i} | \theta) c_a(y(b, a_{-i}), \theta).
\]

A Bayes correlated equilibrium outcome is a mapping \(\mu^n: \Theta \rightarrow \Delta(Y)\) induced by a BCE \(\beta^n:\)

\[
\mu^n(y | \theta) = \sum_{a: y(a) = y} \beta^n(a | \theta).
\]

In words, there is mediator who knows the state, draws an action profile \(a = (a_i, a_{-i})\), and recommends action \(a_i\) to player \(i\). Eq. (5.5) is the BCE constraint stating that player \(i\) who is recommended \(a_i = a\), prefers to play \(a\) over any other action \(b\). Bergemann and Morris (2016) show that the set of BCE outcomes is the set of all Bayes-Nash equilibrium outcomes, for all possible information structures over the state set \(\Theta\).
The next result shows that the BCE outcomes converge to the BCWE of the nonatomic game $\Gamma$ as the number of players tends to infinity.

**Proposition 5.2.** Let $\mu^n$ be a BCE outcome of the game $\Gamma_n$. Then any weak$^*$ accumulation point $\mu$ of the sequence $\mu^n$, is a BCWE of the ANGII $\Gamma$.

Proposition 5.2 also applies to games with complete information, where a BCE is simply a CE (Aumann, 1974). Since the set of CWE of games with complete information and convex potential coincides with the set distributions over WE (Proposition 3.2) we get the following corollary:

**Corollary 5.3.** Let the ANG $\Gamma$ have a convex potential and let $\mu^n$ be a correlated equilibrium outcome of $\Gamma_n$. Then any weak$^*$ accumulation point $\mu$ of the sequence $\mu^n$, is an element of $\triangle(\text{WE}(\Gamma))$.

The next result complements Proposition 5.2 by showing that any BCWE is a limit of approximate BCE of the games with $n$ players.

**Definition 5.4.** Given $\varepsilon > 0$, an $\varepsilon$-BCE of $\Gamma_n$ is a map $\beta^n : \Theta \rightarrow \Delta(A^n)$ such that for all $i \in [n]$ and for all $a, b \in A$, we have

$$
\sum_{a_{-i} \in A_i} \sum_{\theta \in \Theta} p(\theta) \beta^n(a, a_{-i} | \theta) c_a(y(a, a_{-i}), \theta) \\
\leq \sum_{a_{-i} \in A_i} \sum_{\theta \in \Theta} p(\theta) \beta^n(a, a_{-i} | \theta) c_a(y(b, a_{-i}), \theta) - \varepsilon.
$$

(5.5)

The induced outcome is an $\varepsilon$-BCE outcome.

**Proposition 5.5.** Let $\mu$ be a BCWE of the ANGII $\Gamma$. There exists a sequence $\varepsilon_n \searrow 0$ and a sequence of $\varepsilon_n$-BCE outcomes $\mu^n$ of $\Gamma_n$ such that $\mu^n \xrightarrow{w^*} \mu$.

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Appendix A. Multiple Populations

To simplify the exposition, we have considered symmetric players in the main text. Yet, the analysis and most of the results extend to heterogenous players. Precisely, assume that there is a finite set of populations \( K \). Each player in population \( k \in K \) has a finite set of actions \( \mathcal{A}^k \). The mass of each population is normalized to 1. Let \( \mathcal{Y}^k = \Delta(\mathcal{A}^k) \) be the set of flows over \( \mathcal{A}^k \) and \( \mathcal{Y} = \times_k \mathcal{Y}^k \). A flow profile is denoted by \( y = (y^k)_{k \in K} \). For each \( (k, a) \) with \( a \in \mathcal{A}^k \), there is a continuous cost function \( c^k_a : \Theta \times \mathcal{Y} \rightarrow \mathbb{R} \). The cost of an individual in population \( k \) who chooses action \( a \) is \( c^k_a(\theta, y) \).

For each state \( \theta \), a Wardrop equilibrium of the game with complete information at \( \theta \) is a flow profile \( y = (y^k)_{k \in K} \) such that for all \( k \), and all \( a, b \in \mathcal{A}^k \)

\[
y^k_{a|\theta|y} \leq y^k_{b|\theta|y}.
\]

In other words, \( \forall k, a, y^k_a > 0 \implies c^k_a(\theta, y) = \min_b c^k_b(\theta, y) \).

The definition of Bayes correlated Wardrop equilibrium extends as follows. The other equilibrium notions are defined analogously.

Definition A.1. A Bayes correlated Wardrop equilibrium of multi-population game is a mapping \( \mu : \Theta \rightarrow \Delta(\mathcal{Y}) \) such that for every \( k \in K \) and \( a, b \in \mathcal{A}^k \):

\[
\sum_{\theta} p(\theta) \int y^k_{a|\theta|y} c^k_a(\theta, y) \, d\mu(y|\theta) \leq \sum_{\theta} p(\theta) \int y^k_{b|\theta|y} c^k_b(\theta, y) \, d\mu(y|\theta).
\]

The multi-population game is a potential game if for every \( \theta \) there is a function \( \Phi_\theta : \mathcal{Y} \rightarrow \mathbb{R} \) such that \( \frac{\partial \Phi_\theta}{\partial y^k_b} = c^k_a(\theta, y) \). As in a single population complete information game, if \( \Phi \) is convex, then the set of WE is the set of minimizers of \( \Phi \), for each population all actions with positive flow have the same cost across all WE, and we have \( \text{CCWE} = \text{CWE} = \Delta(\text{WE}) \) (see the Proof of Proposition 3.2 in Appendix B, which is done for general multi-population games). The convergence results of
Section 5 also extend, and they are proved in Appendix B for multi-population games.\footnote{The implementation results of BCWE as BWE are not extended to multi-population games because a BWE in Wu et al. (2021) is defined only for homogeneous action spaces and cost functions.}

Appendix B. Proofs

Proof of Proposition 3.2. We do the proof for general multi-population games. We first introduce the following lemma:

Lemma B.1. If $\Phi$ is convex, then the set of WE is the set of minimizers of $\Phi$.

Proof. This result follows from Sandholm (2001, Proposition 3.1), who proves that the set of WE is the set of Karush–Kuhn–Tucker (KKT) points of the minimization problem $\min\{\Phi(y) : y \in \mathcal{Y}\}$. Since $\Phi$ is convex, the KKT points are the minimizers of $\Phi$. \hfill $\square$

Then clearly $\Delta(\arg \min \Phi) = \arg \min_{\mu \in \Delta(\mathcal{Y})} E_{\mu} \Phi \subseteq \text{CCWE}$. Conversely, take a CCWE $\mu$ and suppose that there is $z \in \Delta(\mathcal{Y})$ such that $\Phi(z) < \int \Phi(y) \, d\mu(y)$. Using the fact that $\Phi(z) - \Phi(y) \geq \nabla \Phi(y)(z - y)$ and integrating, we obtain

$$\int \nabla \Phi(y)(z - y) \, d\mu(y) \leq \Phi(z) - \int \Phi(y) \, d\mu(y) < 0.$$ 

We have

$$\nabla \Phi(y)(z - y) = \sum_{k \in K} \sum_{a \in A^k} (z_a^k - y_a^k) c_a^k(y),$$

so

$$\sum_{k \in K} \sum_{a \in A^k} \int y_a^k c_a^k(y) \, d\mu(y) > \sum_{k \in K} \sum_{a \in A^k} z_a^k \int c_a^k(y) \, d\mu(y) \geq \sum_{k \in K} \min_{a \in A^k} \int c_a^k(y) \, d\mu(y).$$

Thus, there exists $k \in K$ such that

$$\sum_{a \in A^k} \int y_a^k c_a^k(y) \, d\mu(y) > \sum_{a \in A^k} \int c_a^k(y) \, d\mu(y) \geq \min_{a \in A^k} \int c_a^k(y) \, d\mu(y).$$

contradicting the assumption that $\mu$ is a CCWE.

Next, we show that all WE have the same costs. We know from Lemma B.1 that the WE minimize the potential, i.e., they are the solutions of the following convex optimization problem:

$$\min \left( \Phi(y) : \forall k, \sum_a y_a^k = 1, \forall a \in A, k \in K, y_a^k \geq 0 \right).$$

The Lagrangian of this problem is

$$L(y, \lambda) = \Phi(y) - \sum_{k \in K} \lambda^k \left( \sum_{a \in A} y_a^k \right) - \sum_{a \in A} \sum_{k \in K} \lambda_{a,k} y_a.$$
with $\lambda := ((\lambda^k)_{k \in K^*}, (\lambda_{a,k})_{a \in A^*, k \in K^*})$. From the KKT theorem, $\hat{y}$ is a solution if and only if there exists $\hat{\lambda}$ such that $(\hat{y}, \hat{\lambda})$ satisfies the KKT conditions: For all $a \in A, k \in K$, \[
abla_{y_a} \Phi(\hat{y}) = \hat{\lambda}^k + \lambda_{a,k}^k; \quad \lambda_{a,k}^k y_a^k = 0; \quad \lambda_{a,k}^k \geq 0; \quad \sum_{a \in A} y_a^k = 1.
\] This is if and only if for all $(y, \lambda)$, \[\mathcal{L}(\hat{y}, \hat{\lambda}) \leq \mathcal{L}(\hat{y}, \hat{\lambda}) \leq \mathcal{L}(y, \lambda),\] (see Rockafellar, 1970, Theorem 28.3, page 281). From the exchange property, if $(\hat{y}, \hat{\lambda})$ and $(\hat{\lambda}, \hat{\lambda})$ are such saddle points, then $(\hat{y}, \hat{\lambda})$ and $(\hat{\lambda}, \hat{\lambda})$ are also saddle points.

Consider the relative interior of the set $\text{WE}$. For each $k \in K$ there exist subsets of actions $B^k \subseteq A^k$ such that $x_{k \in K} B^k$ is the support of all flows in this relative interior. To see this, notice that for each pair of $\text{WE}$ $y, z$ such that $y$ in the relative interior and each $t \in (0, 1), ty + (1 - t)z$ is also in the relative interior. Therefore, for each $a, k$, whenever $y^k_a > 0$ for some $\text{WE}$, this must be true for all points in the relative interior of $\text{WE}$.

Consider now two points $\hat{y}, \hat{y}$ in the relative interior of $\text{WE}$. For every $a^k \in B^k$, we have $\hat{y}_a > 0, \hat{y}_a > 0$ and, \[\nabla_{y_a} \Phi(\hat{y}) = \hat{\lambda}^k = \lambda^k = \nabla_{y_a} \Phi(\hat{y}).\] Thus, for every $k \in K$ and $a \in B^k$, $c^k_a(\hat{y}) = c^k_a(\hat{y})$ and all points in the relative interior of $\text{WE}$ have the same costs. By continuity, all points in $\text{WE}$ have the same costs (notice that the support can only shrink when approaching the boundary).

Proof of Proposition 4.2. Directly from Lemma 4.3. □
Proof of Proposition 4.4. We introduce the following lemma.

Lemma B.2. The set of $\text{BCWE}$ with finite support is dense in the set of $\text{BCWE}$.

Proof. Suppose that there exists a $\text{BCWE}$ $\mu$ which does not belong to the closure $C$ of the set of $\text{BCWE}$ with finite support. This set $C \subseteq (\Delta(\mathcal{Y}))^{\Theta}$ is convex and closed. From the separation theorem, for each $\theta$, there exists a continuous function $f(\cdot, \theta) : \mathcal{Y} \rightarrow \mathbb{R}$ such that
\[\sum_{\theta} \int f(y, \theta) \, d\mu(y \mid \theta) < \inf \left\{ \sum_{\theta} \int f(y, \theta) \, d\nu(y \mid \theta) : \nu \in C \right\}\] which contradicts Proposition 4.1. □

Then, Proposition 4.4 directly follows from Lemmas 4.3 and B.2. □

Proof of Proposition 4.5. Consider a game with incomplete information and an information structure $\sum_k \gamma^k = 1, \mathcal{T} = x_{k \in K} T^k$ and $\pi : \Theta \rightarrow \Delta(\mathcal{T})$. Consider the
Proposition 3.2

Equilibrium flows are unique for any information structure.

A WE of this auxiliary game is a BWE for this information structure, with \( \hat{\gamma}_a^{k,r} \) for every \( a \in A \), \( r \in T^k \), \( k \in K \). Suppose that there is a potential \( \Phi_\theta \) in each state \( \theta \),

\[
\Phi_\pi (\hat{\gamma}) = \sum_{\theta,T} \Phi(\pi | \theta) \Phi_\theta \left( \left( \sum_{k=1}^K \hat{\gamma}_a^{k,r} \right)_{a \in A} \right),
\]

is a potential for the auxiliary game: \( \partial \Phi_\pi (\hat{\gamma}) / \partial \gamma_a^{k,r} = c_a^{k,r}(\hat{\gamma}) \). If \( \Phi_\theta \) is convex for each \( \theta \), then so is \( \Phi_\pi \) and by Proposition 3.2 the equilibrium cost profile is unique for any information structure. If \( \Phi_\theta \) is strictly convex for each \( \theta \), then so is \( \Phi_\pi \) and by Lemma B.1 equilibrium flows are unique for any information structure. □

Proof of Proposition 5.2. We prove more generally the result for multi-population games and general weights. Consider a game \( \Gamma_n \) with \( n = \sum_k n^k \) players where there are \( n^k \) in population \( k \). Each player \( i \) in population \( k \) has a weight \( w_i^k \) with \( \sum_{i=1}^{n^k} w_i^k = 1 \) for each \( k \). We show that for every sequence of weights \( \{w^k(n)\} \) such that \( \max_k w_i^k(n) \to 0 \) and sequence \( \{\mu^n\} \) of BCE outcomes of the \( n \)-player games, any weak-* accumulation point \( \mu \) of \( \{\mu^n\} \) is a BCWE. The result of the proposition is obtained in the particular case of a single population with weight \( w_i^k = 1/n \) for every player.

Given an action profile \( a = (a^k)_k \), the induced flow \( y(a) = (y^k(a^k))_k \) is such that for each \( k \) and \( a \in A^k \), \( y_a^k(a^k) = \sum_i w_i^k(n) \Pi \{a_i^k = a \} \). A Bayes correlated equilibrium of the \( n \)-player game is \( \beta^n : \Theta \to \Delta(\prod_k A^k) \) such that for all \( k \), \( i = 1, \ldots, n^k \), \( \forall a, b, b^k \in A^k \),

\[
\sum_{\theta,a,i} p(\theta) \beta^n(a, a_{-i}, a^{-k} | \theta) c_a(y^k(a^k, a_{-i}), y^{-k}(a^{-k}), \theta) \leq \sum_{\theta,a,i} p(\theta) \beta^n(a, a_{-i}, a^{-k} | \theta) c_a(y^k(b^k, a_{-i}), y^{-k}(a^{-k}), \theta)
\]

Equivalently, this rewrites as follows

\[
E \left[ \Pi \{a_i^k = a \} c_a^k(y(a), \theta) \right] \leq E \left[ \Pi \{a_i^k = a \} c_a^k(y^k(a^k) + w_i^k(n) (\Pi_b - \Pi_a), y^{-k}(a^{-k}), \theta) \right],
\]

where \( \Pi_a \) is the flow such that \( y_a = 1 \). Multiply by \( w_i^k(n) \) and sum over \( i = 1, \ldots, n^k \) to get

\[
E \left[ \sum_i w_i^k(n) \Pi \{a_i^k = a \} c_a^k(y(a), \theta) \right]
\]
\[
\leq \mathbb{E} \left[ \sum_i w_i^k (n) \mathbb{1} \{ a_i^k = a \} c_a^k (y^k (a) \mathbb{1} (a - 1) a, y^{\mathbb{1} (a - 1)} (\theta) \} \right].
\]

The LHS is
\[
\sum_{\theta, y} \mathbb{P} (\theta, y (a) = y) \mathbb{E} \left[ \sum_i w_i^k (n) \mathbb{1} \{ a_i^k = a \} c_a^k (y (a), \theta) \mid y (a) = y, \theta \right]
\]
\[
= \sum_{\theta, y} \mathbb{P} (\theta, y (a) = y) \mathbb{E} \left[ \sum_i w_i^k (n) \mathbb{1} \{ a_i^k = a \} c_a^k (y (a), \theta) \mid y (a) = y \right] c_a^k (y, \theta)
\]
\[
= \sum_{\theta, y} \mathbb{P} (\theta, y (a) = y) y_a^k \mathbb{E} (y, \theta)
\]
\[
= \sum_{\theta} p (\theta) \int y_a^k \mathbb{E} (y, \theta) d\mu^* (y \mid \theta).
\]

where \( \mu^* (\cdot \mid \theta) \in \Delta (\mathcal{Y}) \) is the (marginal) distribution of the flow \( y (a) \) induced by \( \beta^0 (\cdot \mid \theta) \). Consider a sequence \( \{ \beta^m \} \) of BCE of the \( n \)-player game such that \( \max_i w_i^m \rightarrow 0 \) and that for each \( \theta, \mu^0 (\cdot \mid \theta) \) converges weak* to some \( \mu (\cdot \mid \theta) \) in \( \mathcal{Y} \) (take a subsequence if needed). The LHS tends to \( \sum_{\theta} p (\theta) \int y_a^k \mathbb{E} (y, \theta) d\mu (y \mid \theta) \).

The RHS is:
\[
\text{RHS} = \mathbb{E} \left[ \sum_i w_i^k (n) \mathbb{1} \{ a_i^k = a \} c_a^k (y (a), \theta) \right] +
\]
\[
\sum_{\theta} \mathbb{P} (\theta) \int \left( c_a^k (y (a), \theta) \right) d\mu^* (y \mid \theta)
\]
\[
= \sum_{\theta} p (\theta) \int y_a^k \mathbb{E} (y, \theta) d\mu^* (y \mid \theta) + \text{Error term}.
\]

The first term tends to \( \sum_{\theta} p (\theta) \int y_a^k \mathbb{E} (y, \theta) d\mu (y \mid \theta) \). For the second term, note that the finite family of functions \( \{ y \mapsto c_a^k (y, \theta) \} \) is uniformly equicontinuous on the compact \( \mathcal{Y} \). That is, \( \forall \varepsilon > 0, \exists \delta > 0, \text{s.t.} \forall w, \delta, \theta, a, b, \forall y \in \mathcal{Y}, |c_a^k (y + w (\delta - \delta), \theta) - c_b (y, \theta) | \leq \varepsilon \). Thus for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that \( \forall n \geq \delta, \forall k, i, w_i^k (n) \leq \delta, \) and thus \( |\text{Error term}| \leq \varepsilon \) for all \( n \geq \delta \). The RHS thus tends to \( \sum_{\theta} p (\theta) \int y_a^k \mathbb{E} (y, \theta) d\mu (y \mid \theta) \), which proves that \( \mu \) is a BCWE. \( \square \)

**Proof of Proposition 5.5.** From Lemma B.2 it suffices the prove the result for a BCWE with finite support. The construction is the same as in the proof of Lemma 4.3. The main difference is that a population with size \( 1/n \) is now a single player whose deviation changes the flow by \( 1/n \). Let \( \mu \) be a BCWE with finite support:

\[
\forall a, b, \sum_{\theta, y} p (\theta) \mu (y \mid \theta) y_a c_a (y, \theta) \leq \sum_{\theta, y} p (\theta) \mu (y \mid \theta) y_a c_b (y, \theta).
\]

(B.1)
Let $\mathcal{Y}^* = \bigcup_\theta \text{supp } \mu(\cdot \mid \theta)$. Approximating the numbers $y_a$, for $a \in A$ and $y \in \mathcal{Y}^*$ by sequences of rationals, for every integer $n$, one can find integers $(N^n_a(y))_{a \in A, y \in \mathcal{Y}^*}$ such that for all $y \in \mathcal{Y}^*$, $\sum_a N^n_a(y) = n$ and $|N^n_a(y) - y_a| \leq \delta_n$ for all $y \in \mathcal{Y}^*$, $a \in A$ with $\lim_n \delta_n = 0$. Denote by $y_n$ the flow $\left(\frac{N^n_a(y)}{n}\right)_{a \in A}$.

Consider the $n$-player game where each player $i$ has weight $1/n$ and construct $\beta^n : \Theta \to \Delta(A^n)$ as follows. Conditional on state $\theta$, the mediator draws $y \in \mathcal{Y}^*$ with probability $\mu(y \mid \theta)$, then recommends action $a$ to a subset of players of cardinality $N^n_a(y)$ chosen uniformly. Conditional on $\theta, y$, the probability that player $i$ is recommended $a$ is $\frac{N^n_a(y)}{n}$. The (non-normalized) expected cost of a player who is recommended $a$ and plays $b$ is

$$\sum_{\theta, y} p(\theta) \mu(y \mid \theta) \frac{N^n_a(y)}{n} c_b(y_n + \frac{1}{n}(1_b - 1_a), \theta).$$

We have,

$$\left| \frac{N^n_a(y)}{n} c_b(y_n + \frac{1}{n}(1_b - 1_a), \theta) - y_a c_b(y_n, \theta) \right| \leq \frac{N^n_a(y)}{n} \left| c_b(y_n + \frac{1}{n}(1_b - 1_a), \theta) - c_b(y_n, \theta) \right| + \frac{N^n_a(y)}{n} \left| c_b(y_n, \theta) - y_a c_b(y_n, \theta) \right|,$$

thus,

$$\left| \frac{N^n_a(y)}{n} c_b(y_n + \frac{1}{n}(1_b - 1_a), \theta) - y_a c_b(y_n, \theta) \right| \leq \omega \left( \frac{1}{n} \right) + \omega(\delta_n) := \frac{\epsilon_n}{2},$$

where $\omega(\cdot)$ is a modulus of continuity common to all mappings $y \mapsto c_a(y, \theta), y \mapsto y_a c_b(y, \theta)$, $(a, b) \in A \times A$. It follows from Eq. (B.1) that any unilateral deviation cannot profit by more that $\epsilon_n$. \qed