The Descartes circles theorem and division by zero calculus

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Abstract. From the viewpoint of the division by zero \((0/0 = 1/0 = z/0 = 0)\) and the division by zero calculus, we will show that in the very beautiful theorem by Descartes on three touching circles is valid for lines and points for circles except for one case. However, for the exceptional case, we can obtain interesting results from the division by zero calculus.

1 Introduction

We recall the famous and beautiful theorem ([14]):

**Theorem (Descartes)** Let \(C_i (i = 1, 2, 3)\) be circles touching to each other of radii \(r_i\). If a circle \(C_4\) touches the three circles, then its radius \(r_4\) is given by

\[
\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}}. \tag{1}
\]

As well-known, circles and lines may be looked as the same ones in complex analysis, in the sense of stereographic projection and many reasons. Therefore, we will consider whether the theorem is valid for line cases and point cases for circles. Here, we will discuss this problem clearly from the division by zero viewpoint. The Descartes circle theorem is valid except for one case for lines and points for the three circles and for one exception case, we can obtain very interesting results, by the division by zero calculus.

2 The division by zero calculus

For any Laurent expansion around \(z = a\),

1
\[ f(z) = \sum_{n=-\infty}^{\infty} C_n(z-a)^n, \quad (2) \]

we obtain the identity, by the division by zero

\[ f(a) = C_0. \quad (3) \]

(Here, as convention, we consider as \( 0^0 = 1 \).)

For the correspondence (3) for the function \( f(z) \), we will call it the **division by zero calculus**. By considering the derivatives in (2), we can define any order derivatives of the function \( f \) at the singular point \( a \).

We have considered our mathematics around an isolated singular point for analytic functions, however, we do not consider mathematics at the singular point itself. At the isolated singular point, we consider our mathematics with the limiting concept, however, the limiting values to the singular point and the value at the singular point of the function are different. By the division by zero calculus, we can consider the values and differential coefficients at the singular point.

The division by zero \( (0/0 = 1/0 = z/0 = 0) \) is trivial and clear in the natural sense of the generalized division (fraction) against its mysterious and long history (see for example, [11]), since we know the Moore-Penrose generalized inverse for the elementary equation \( az = b \). Therefore, the division by zero calculus above and its applications are important. See the references [12, 2, 6, 13, 3, 4, 7, 9, 5, 10] for the details and the related topics. We regret that our common sense for the division by zero are still wrong; one typical comment for our division by zero results is given by some physician:

*Here is how I see the problem with prohibition on division by zero, which is the biggest scandal in modern mathematics as you rightly pointed out.*

However, in this paper we do not need any information and results in the division by zero, we need only the definition (3) of the division by zero calculus.

As stated already in [7], in general, for a circle with radius \( r \), its curvature is given by \( 1/r \) and by the division by zero, for the point circle, its curvature is zero. Meanwhile, for a line corresponding the case \( r = \infty \), its curvature is also zero, however, then we should consider the case as \( r = 0 \), not \( \infty \). For this reality and reasonable situation, look the paper. By this interpretation, we will show that the theorem is valid for line and point circle cases clearly. We would like to show that the division by zero \( (0/0 = 1/0 = z/0 = 0) \) is very natural also from the viewpoint of the Descartes circles theorem.
3 Results

We would like to consider all the cases for the Descartes theorem for lines and point circles, step by step.

3.1 One line and two circles case

We consider the case in which the circle $C_3$ is one of the external common tangents of the circles $C_1$ and $C_2$. This is a typical case in this paper. We assume $r_1 \geq r_2$. We now have $r_3 = 0$ in (1). Hence

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{0} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_1 \cdot 0} + \frac{1}{0 \cdot r_1}} = \frac{1}{r_1} + \frac{1}{r_2} \pm 2\sqrt{\frac{1}{r_1 r_2}}.$$

This implies

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

in the plus sign case. The circle $C_4$ is the incircle of the curvilinear triangle made by $C_1$, $C_2$ and $C_3$ (see Figure 1). In the minus sign case we have

$$\frac{1}{\sqrt{r_4}} = \frac{1}{\sqrt{r_2}} - \frac{1}{\sqrt{r_1}}.$$

In this case $C_2$ is the incircle of the curvilinear triangle made by the other three (see Figure 2).

Of course, the result is known. The result was also well-known in Wasan geometry [16] with the Descartes circle theorem itself [17], [15].

3.2 Two lines and one circle case

In this case, the two lines have to be parallel, and so, this case is trivial, because then the two circles are congruent, by the division by zero $1/0 = 0$.
3.3 One point circle and two circles case

This case is another typical case for the theorem. Intuitively, for \( r_3 = 0 \), the circle \( C_3 \) is the common point of the circles \( C_1 \) and \( C_2 \). Then, there does not exist any touching circle of the three circles \( C_j; j = 1, 2, 3 \).

For the point circle \( C_3 \), we will consider it by limiting of circles touching the circles \( C_1 \) and \( C_2 \) to the common point. Then, we will examine the circles \( C_4 \) and the Descartes theorem.

Let us consider the circles \( C_1: (x + r_1)^2 + y^2 = r_1^2 \) and \( C_2: (x - r_2)^2 + y^2 = r_2^2 \) \((r_i > 0)\). We recall the parametric representation of any such circle \( C_3 \): H. Okumura and M. Watanabe gave a theorem in [3], for a real number \( z \), the point \((0, 2\sqrt{r_1r_2}/z)\) is denoted by \( V_z \), then

Theorem 7. The circle \( C_3 \) passing through \( V_{\pm1} \) for a real number \( z \neq \pm1 \) and touching the circles \( C_1 \) and \( C_2 \) can be represented by the equation

\[
\left( x - \frac{r_1 - r_2}{z^2 - 1} \right)^2 + \left( y - \frac{2z\sqrt{r_1r_2}}{z^2 - 1} \right)^2 = \left( \frac{r_1 + r_2}{z^2 - 1} \right)^2.
\]

By setting \( z = 1/w \), we will consider the case \( w = 0 \); that is, the case \( z = \infty \) in the classical sense; that is, the circle \( C_3 \) is reduced to the origin.

We look for the circle \( C_4 \) touching the three circles \( C_j; j = 1, 2, 3 \). We set

\[
C_4: (x - x_4)^2 + (y - y_4)^2 = r_4^2.
\]  

(4)

Then, from the touching property we obtain:

\[
x_4 = \frac{r_1r_2(r_1 - r_2)w^2}{D},
\]

\[
y_4 = \frac{2r_1r_2(\sqrt{r_1r_2} + (r_1 + r_2)w)}{D}
\]

and

\[
r_4 = \frac{r_1r_2(r_1 + r_2)w^2}{D},
\]

where

\[
D = r_1r_2 + 2\sqrt{r_1r_2}(r_1 + r_2)w + (r_1^2 + r_1r_2 + r_2^2)w^2.
\]

Notice that there are four sets of the solutions of \( x_4, y_4, r_4 \), but we consider only one set, because the other cases can be considered similarly.

By inserting these values to (1), we obtain

\[
f_0 + f_1w + f_2w^2 = 0,
\]
where

\[ f_0 = r_1r_2(x^2 + y^2), \]
\[ f_1 = 2\sqrt{r_1r_2((r_1 + r_2)(x^2 + y^2) - 2r_1r_2y)}, \]

and

\[ f_2 = (r_1^2 + r_1r_2 + r_2^2)(x^2 + y^2) + 2r_1r_2(r_2 - r_1)x - 4r_1r_2(r_1 + r_2)y + 4r_1^2r_2^2. \]

By using the division by zero calculus for \( w = 0 \), we obtain, for the first, for \( w = 0 \), the second by setting \( w = 0 \) after dividing by \( w \) and for the third case, by setting \( w = 0 \) after dividing by \( w^2 \),

\[ x^2 + y^2 = 0 \quad (5) \]
\[ (r_1 + r_2)(x^2 + y^2) - 2r_1r_2y = 0 \quad (6) \]
\[ (r_1^2 + r_1r_2 + r_2^2)(x^2 + y^2) + 2r_1r_2(r_2 - r_1)x - 4r_1r_2(r_1 + r_2)y + 4r_1^2r_2^2 = 0. \quad (7) \]

Note that (6) is the red circle in Figure 3 and its radius is

\[ \frac{r_1r_2}{r_1 + r_2}, \quad (8) \]

and (7) is the green circle in Figure 3 whose radius is

\[ \frac{r_1r_2(r_1 + r_2)}{r_1^2 + r_1r_2 + r_2^2}. \quad (9) \]

When the circle \( C_3 \) is reduced to the origin, of course, the inscribed circle \( C_4 \) is reduced to the origin, then the Descartes theorem is not valid. However, by the division by zero calculus, then the origin of \( C_4 \) is changed suddenly for the cases (5), (6) and (7), and for the circle (6), the Descartes theorem is valid for \( r_3 = 0 \), surprisingly.
Indeed, in (1) we set $\xi = \sqrt{r_3}$, then (1) is as follows:

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{\xi^2} \pm \frac{2}{\xi^2} \sqrt{\frac{\xi^2}{r_1 r_2} + \left( \frac{1}{r_1} + \frac{1}{r_2} \right)}.$$

and so, by the division by zero calculus at $\xi = 0$, we have

$$\frac{1}{r_4} = \frac{1}{r_1} \pm \frac{1}{r_2},$$

which is (8). Note, in particular, that the division by zero calculus may be applied in many ways and so, for the results obtained should be examined some meanings. This circle (6) may be looked a circle touching the origin and two circles $C_1$ and $C_2$, because by the division by zero calculus

$$\tan \frac{\pi}{2} = 0,$$

that is a popular property (see [4]).

Meanwhile, the circle (7) touches the circles $C_1$, $C_2$ and the beautiful circle with center $(r_2 - r_1, 0)$ with radius $r_1 + r_2$. The each of the areas surrounded by the last circle and the circles $C_1$, $C_2$ is called an arbelos, and the circle (5) is the famous Bankoff circle of the arbelos.

For $r_3 = -(r_1 + r_2)$, from the Descartes identity (1), $r_4$ equals (9). That is, when we consider that the circle $C_3$ is changed to the circle with center $(r_2 - r_1, 0)$ with radius $r_1 + r_2$, the Descartes identity holds. Here, the minus sign shows that the circles $C_1$ and $C_2$ touch $C_3$ internally from the inside of $C_3$.

### 3.4 Two point circles and one circle case

This case is trivial, because the externally touching circle is coincident with one circle.

### 3.5 Three points case and three lines case

In these cases we have $r_j = 0, j = 1, 2, 3$ and the formula (1) shows that $r_4 = 0$. This statement is trivial in the general sense.

As the solution of the simplest equation

$$ax = b,$$  \hspace{1cm} (10)

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to
solve the equation (10) is impossible. We saw for different parallel lines or
different parallel planes, their common points are the origin in [4]. Certainly
they have the common point of the point at infinity and the point at infinity
is represented by zero. However, we can understand also that they have no
common points, because the point at infinity is an ideal point. The zero will
represent some impossibility.

In the Descartes theorem, three lines and three points cases, we can un-
derstand that the touching circle does not exist, or it is the point and so the
Descartes theorem is valid.

4 Conclusion

By the division by zero calculus, we were able to give a general theorem of
Descartes for containing lines and point circles for circles, simply. At the
same time, we showed that the division by zero calculus is natural for the
famous Descartes circles theorem clearly.

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