On the classification of tight contact structures

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Abstract. Recently, there have been several breakthroughs in the classification of tight contact structures. We give an outline on how to exploit methods developed by Ko Honda and John Etnyre to obtain classification results for specific examples of small Seifert manifolds.

1. Introduction

After Eliashberg proved a classification for so-called overtwisted contact structures [4], work concentrated on the classification of tight contact structures, which turned out to be much more subtle and provide interesting relations to the topology of the underlying manifold. See [11] for an introduction to contact geometry and further references.

Until recently, the main tool to show that a contact structure on a manifold is tight is to show it is fillable. A contact structure is holomorphically fillable if it is the oriented boundary of a compact Stein 4-manifold. Gromov and Eliashberg showed that a fillable contact structure is tight [21, 5]. Moreover, fillability is preserved by Legendrian surgery [35, 6], thus providing a rich source of tight contact structures. Gompf’s extensive study on Legendrian surgery [19] enables one in particular to construct holomorphically fillable contact structures on many Seifert manifolds. Using Legendrian surgery and techniques from Seiberg-Witten-theory, Lisca and Matić [31] proved that for every integer \( n > 1 \) there exist at least \( n^2 \) tight contact structures on the Brieskorn homology spheres with reversed orientation, \( -\Sigma(2, 3, 6n-1) = M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{6n-1}) \). Later, they improved this lower bound to \( n - 1 \) in [32].

Contact structures induce a singular foliation on embedded surfaces and these are often easier to study than the contact structure itself. Motivated by work of Eliashberg and Gromov [9], Giroux introduced the notion of convex surfaces, i.e. surfaces whose characteristic foliation is cut transversely by a certain multicurve, called the dividing set [15]. This dividing set essentially determines the contact structure in a neighbourhood of the surface and is a convenient tool to study contact structures.

Exploiting this idea, Kanda gave a complete classification of tight contact structures on the 3-torus [27]; see also [16]. This led Honda to study so-called bypasses attached along convex surfaces, which provide a systematic tool for altering the dividing set of a convex surface. By splitting a contact 3-manifold along convex surfaces into simpler pieces and studying the possibilities of tight contact structures,
Honda gave a complete classification of tight contact structures on solid tori, toric annuli, Lens spaces in \([24]\), as well as torus bundles over the circle and circle bundles over closed Riemannian surfaces; see \([25]\). Many of these were independently obtained by Giroux \([16, 17, 18]\), and on some Lens spaces by Etnyre \([10]\).

Furthermore, Lisca proved in \([29]\) that the Poincaré homology sphere with reverse orientation \(-\Sigma(2,3,5)\) (this corresponds to the Seifert manifold \(M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})\) in the notation below) has no symplectically (weakly) semi-fillable contact structure, thus proving a conjecture of Gompf in \([19]\). Using the bypass technique in contact topology, Etnyre and Honda finally proved \([14]\) the nonexistence of a tight contact structure on \(M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})\), thereby providing the first example of a closed 3-manifold which admits no tight contact structure.

Lisca \([30]\) went further and proved (among other things) that the Seifert manifolds \(M(-\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\) and \(M(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})\) admit no (weakly) symplectically semi-fillable contact structure. From Lisca's examples, Etnyre and Honda proved that on the Seifert manifolds \(M(-\frac{1}{2}, \frac{1}{3}, \frac{1}{5})\) and \(M(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})\) there exist tight contact structures without symplectic fillings. These examples belong to the handful of Seifert manifolds which can be defined as torus bundles over the circle; see \([3]\) for fillability results on these manifolds.

Furthermore, the examples above are Seifert manifolds over the sphere \(S^2\) with three singular fibres. On ‘larger’ Seifert manifolds it is recently proven by Colin \([1]\) that every orientable Seifert manifold over a surface of genus \(g \geq 1\) has infinitely many non-isomorphic tight contact structures. Moreover, Colin \([2]\), see also Honda, Kazez, Matić \([26]\), proved that every closed irreducible orientable toroidal 3-manifold carries infinitely many contact structures.

Therefore the classification of tight contact structures on Seifert manifolds may provide interesting new insight to the topology of tight contact structures on 3-manifolds. In this work, we will demonstrate on two examples how to apply bypass techniques to obtain upper bounds on the number of tight contact structures on Seifert manifolds over the sphere with three singular fibres. In the examples below, tight contact structures are constructed using Legendrian surgery.

2. Basic contact geometry

A positive contact structure on an oriented 3-manifold \(M\) is a 2-plane field \(\xi = \ker \alpha \subset TM\), defined by a 1-form \(\alpha\) satisfying \(\alpha \wedge d\alpha > 0\). According to this definition, \(\xi\) is co-oriented by \(\alpha\) and oriented by \(d\alpha\) such that the orientation on \(M\) coincides with the orientation defined by \(\alpha \wedge d\alpha\).

2.1. Legendrian curves and twisting. A curve \(\gamma\) in a contact manifold \((M, \xi)\) everywhere tangent to \(\xi\) is called Legendrian. Throughout this paper, we assume curves to be closed, and we will refer to ‘arcs’ otherwise. Recall that every diffeomorphism between Legendrian curves extends to a contactomorphism of their neighbourhoods. A Legendrian curve \(\gamma\) in a contact manifold \((M, \xi)\) is endowed with a natural framing defined by a vector field along \(\gamma\) transverse to \(\xi\), called the contact framing. The twisting number \(t(\gamma, F)\) is defined as the number of right \(2\pi\) twists of the contact framing with respect to a preassigned framing \(F\) of \(\gamma\). In case \(\gamma\) is a Legendrian boundary component of an oriented surface \(S\), let \(F_S\) denote the framing of \(\gamma\) defined by \(S\). In this case we will write \(t(\gamma) := t(\gamma, F_S)\). When \(\gamma\) is null-homologous and \(S\) is a Seifert surface for \(\gamma\), the twisting number is called Thurston-Bennequin invariant and denoted by \(\text{tb}(\gamma)\).
We have another classical invariant for Legendrian knots, the rotation number \( r \): If \( \gamma \) is the Legendrian boundary of a Seifert surface \( S \), we define \( r(\gamma) \) as the number of revolution of its tangent \( \dot{\gamma} \) with respect to a trivialization of \( \xi|_S \). Note that, for any relative homology class \( \beta \in H_2(S, \gamma) \) and \( S \in \beta \) a representing surface, the rotation number of \( \gamma \) is independent of the choice of a trivialization but depends on \( \beta \), and reversing the orientation of \( \gamma \) reverses the sign of \( r \). We refer the reader to [11] for a detailed discussion of the following.

**Proposition 2.1 (Bennequin’s Inequality).** If \( \gamma \) is a Legendrian knot in a tight contact manifold \((M, \xi)\) and \( S \) a Seifert surface for \( \gamma \) with Euler characteristic \( \chi(S) \), then

\[
\text{tb}(\gamma) + |r(\gamma)| \leq -\chi(S).
\]

**2.2. Convex surface theory.** Assume \( S \) is a compact oriented surface embedded in a contact manifold \((M, \xi)\). The line field \( l_x = \xi_x \cap T_x S \) integrates to a singular foliation \( S_\xi \) of \( S \) called characteristic foliation. Recall that the singularities of \( S_\xi \) are exactly the points in \( S \) where the contact plane is tangent to \( S \). The characteristic foliation determines the contact structure in a tubular neighbourhood and one has a certain freedom to alter the characteristic foliation by perturbing the surface; see [11]. Generically, the amount of information needed to locally determine the contact structure can be reduced to a collection of curves on the surface \( S \).

A properly embedded orientable surface \( S \) in a contact manifold \((M, \xi)\) is called convex, if there exists a collection of curves \( \Gamma \) on \( S \) satisfying the following conditions:

1. \( S \setminus \Gamma = S^+ \cup S^- \)
2. \( \Gamma \) is transverse to the characteristic foliation \( S_\xi \) of \( S \)
3. There exists a vector field \( v \) and a volume form \( \theta \) on \( S \) such that the characteristic foliation is directed by \( v \), the flow of \( v \) expands \( \theta \) on \( S^+ \), contracts \( \theta \) on \( S^- \) and \( v \) points transversely out of \( S^+ \).

Recall that the existence of dividing curves \( \Gamma \) is equivalent to the existence of a contact vector field \( v \) transverse to the surface \( S \), determining the contact structure in a neighbourhood of the surface up to admissible isotopy, i.e. an isotopy \( \phi : S \times [0, 1] \to M \) such that \( \phi(S \times \{t\}) \) is transverse to \( v \) for all \( t \in [0, 1] \).

In [15], Giroux proved that every closed surface can be perturbed by a \( C^\infty \)-small isotopy to be convex. More generally, a compact surface with Legendrian boundary can be perturbed to be convex provided the twisting number of each boundary component is not positive. Moreover, the twisting number of a boundary component \( \partial S \) of a convex surface \( S \) determines the dividing set in a tubular neighbourhood of \( \partial S \). This follows from a relative version of Gray’s Theorem in dimension three; see Theorem 3.7 in [11]. We describe a standard tubular neighbourhood of a Legendrian boundary component \( \gamma \) as follows: After perturbing \( S \) we find a neighbourhood \( N \) of a boundary component \( \gamma \subset \partial S \) so that a collar neighbourhood \( A = N \cap S \) of \( \gamma \) in \( S \) has the form \( A = S^1 \times [0, 1] = (\mathbb{R}/\mathbb{Z}) \times [0, 1] \) with coordinates \((x, y)\) where \( \gamma = S^1 \times \{0\} \). In a neighbourhood \( A \times [-1, 1] \) of \( A \) with coordinates \((x, y, z)\) the contact 1-form is defined by \( \alpha = \sin(2\pi nx) \, dy + \cos(2\pi nx) \, dz \) for \( n = |r(\gamma)| \in \mathbb{Z}^+ \). Note that on this annulus the characteristic foliation consists of circles parallel to \( \gamma \), called Legendrian rulings and the dividing set consists of arcs.
transverse to the boundary, leading from one boundary component to another. Between two dividing arcs lies an arc of singularities, which we call Legendrian divides; see Fig. 1. If \( t(\gamma) = 0 \) then the contact structure is defined by \( \alpha = dz - ydx \).

In particular the twisting number of \( \gamma \) is related to the number of intersections of the dividing set \( \Gamma \) with \( \gamma \).

**Proposition 2.2.** Suppose \( S \) is a convex surface with Legendrian boundary in a contact manifold \((M, \xi)\) and \( \gamma \in \partial S \) is a boundary component of \( S \). Then

\[
(2.1) \quad t(\gamma) = -\frac{1}{2} \#(\gamma \cap \Gamma),
\]

where \( \#(\gamma \cap \Gamma) \) denotes the cardinality of the intersection \( \gamma \cap \Gamma \). Moreover, if \( \gamma \) is null-homologous and \( S \) a Seifert surface, then

\[
(2.2) \quad r(\gamma) = \chi(S^+) - \chi(S^-),
\]

where \( S^\pm \) is as in the definition of dividing set and \( \chi(S^\pm) \) denotes the Euler characteristic.

If \( \gamma \) is a Legendrian curve contained in a convex surface \( S \), i.e. not necessarily a boundary component, \( \gamma \) can be made to have a standard collar neighbourhood as depicted in Fig. 1 (where \( \gamma \) is a ruling curve in the interior); see [27]. Formula \( (2.1) \) is also valid in this case.

Giroux pointed out that for convex surfaces, the dividing set, not the particular characteristic foliation, essentially determines the contact structure in a neighbourhood. Namely:

**Theorem 2.3 (Giroux’s Flexibility Theorem, [15]).** Consider a surface \( S \), closed or compact with Legendrian boundary, in a contact manifold \((M, \xi)\). Assume \( \Gamma \) is a dividing set for the characteristic foliation \( S_\xi \) and \( F \) is another singular foliation on \( S \) divided by \( \Gamma \). Then there is an isotopy \( \phi : S \times [0,1] \to M \) of \( S \) such that \( \phi_0 = \text{id} \), \( (\phi_t(S))_\xi = \phi_t(F) \), \( (\phi_t(S))_\xi \) is divided by \( \Gamma \) for all \( t \in [0,1] \) and \( \Gamma \) is fixed.

On the other hand, on a convex surface in a tight contact manifold, two dividing sets of a characteristic foliation are isotopic. We will then, by slightly abusing language, refer to \( \Gamma \) as ‘the’ dividing set.
As a consequence of Giroux’s Flexibility Theorem, one can realize curves or arcs in a convex surface to be Legendrian:

**Theorem 2.4 (Legendrian Realization, [24]).** Consider a collection of disjoint properly embedded closed curves and arcs $C$ on a convex surface $S$, which satisfies the following properties:

(i) $C$ is transverse to the dividing set $\Gamma$ of $S$ and every arc in $C$ begins and ends on $\Gamma$,

(ii) every component of $S \setminus (\Gamma \cup C)$ has a boundary component which intersects $\Gamma$,

then there exists an isotopy $\phi : S \times [0, 1] \to M$ such that $\phi_0 = \text{id}$, $\phi_t(S)$ are all convex, $\phi_1(\Gamma) = \Gamma$ and $\phi_1(C)$ is Legendrian.

In a tight contact structure, the possibilities of dividing sets is rather restricted. Namely:

**Theorem 2.5 (Giroux’s criterion, [24]).** A convex surface (closed or compact with Legendrian boundary) $S$ other than the sphere $S^2$ has a tight neighbourhood if and only if no component of $S \setminus \Gamma_S$ bounds a disc. A convex sphere $S^2$ has a tight neighbourhood if and only if $\# \Gamma_{S^2} = 1$, i.e. if there exists exactly one dividing curve.

2.2.1. **Edge-rounding.** Next, we describe how to smooth out two convex surfaces intersecting transversely along a common Legendrian curve with a negative twisting number and moreover, to relate the dividing set of the two surfaces to the dividing set on the smoothed surface.

**Proposition 2.6 (Edge-Rounding Lemma, [24]).** Assume $S_1$ and $S_2$ are convex surfaces, with convex collar boundary, intersecting transversely inside a contact manifold $(M, \xi)$ along a common Legendrian boundary curve $\gamma$ with negative twisting number. Suppose that $S_1$ and $S_2$ are oriented such that smoothing the edge yields an oriented surface $S'$. Then the edge may be smoothed so that the dividing set of $S'$ is obtained from the dividing sets on $S_1$ and $S_2$, the dividing curves connect such that positive (negative) regions $S_1^\pm$ of $S_1$ connect to positive (negative) regions $S_2^\pm$ of $S_2$ as indicated in figure 4.

**Proof.** After possibly a small perturbation of a neighbourhood $N$ of $\gamma$ in $M$, we can consider the following situation: Consider $\mathbb{R}^2 \times (\mathbb{R}/\mathbb{Z})$ with coordinates $(x, y, z)$, and contact 1-form $\alpha = \sin(2\pi n z)dx + \cos(2\pi n z)dy$ for some $n \in \mathbb{Z}^>0$. Locally, a neighbourhood of $\gamma$ is contactomorphic to $N = \{x^2 + y^2 \leq \varepsilon\}$ and $\gamma$ is given by $x = y = 0$. The convex surfaces become $S_1 \cap N = \{x = 0, 0 \leq y < \varepsilon\}$ and $S_2 \cap N = \{y = 0, 0 \leq x < \varepsilon\}$ oriented by $\partial_x$ and $\partial_y$ respectively. The transverse vector field for $\{(x - \delta)^2 + (y - \delta)^2 = \delta^2\} \cap N$, $0 < \delta < \varepsilon$, is the inward-pointing radial vector $-\partial_x$ for the circle $\{(x - \delta)^2 + (y - \delta)^2 = \delta^2\}$. Take $S' = (S_1 \cup S_2 \setminus N_\delta) \cup \{(x - \delta)^2 + (y - \delta)^2 = \delta^2\} \cap N_\delta$, the dividing curve $z = \frac{1}{2m} - \frac{1}{4n}$ on $S_1$ then connects to the dividing curve $z = \frac{k}{2m} - \frac{1}{4n}$ on $S_2$, for $k = 0, \ldots, 2n - 1$; see Fig. 2.\[\square\]

2.2.2. **Bypasses and alteration of the dividing set.** By perturbing a surface, one can alter the characteristic foliation to assume certain normal forms. On convex surfaces, we want to alter directly the dividing set. A crucial tool for this is the use of bypasses, first exploited by Honda (see e.g. [24, 25]) with precursors in [27].
Figure 2. Rounding edges. (Shaded regions are positive, dashed lines are dividing curves and Legendrian divides are dotted.)

Figure 3. A bypass. (Dashed lines are dividing curves on $S$.)

**Definition 2.7.** Assume $S \subset M$ is a convex surface. A bypass for $S$ is an oriented embedded disc $D$ whose Legendrian boundary satisfies the following:

1. $\partial D$ is the union of two arcs $\gamma_1$, $\gamma_2$ which intersect at their endpoints.
2. $D$ intersects $S$ transversely along $\gamma_1$.
3. along $\gamma_1$, there are three elliptic tangencies in the characteristic foliation $D_\xi$, two of the same sign at the endpoints and one of different sign in the interior of $\gamma_1$.
4. along $\gamma_2$ there are at least three tangencies, all have the same sign but alternating indices.
5. there are no interior singular points of $D_\xi$.

See Fig. 3 for an illustration.

Observe that all singular points on $\partial D$ have the same sign except the one elliptic point in the interior of $\gamma_1$. We call this the sign of the bypass. The endpoints of $\gamma_1$ may be the same elliptic point, in this case we call $D$ a degenerate bypass.

We first explain how to find bypasses and then give a discussion regarding how bypasses are used to alter the dividing set of a convex surfaces. We discuss the
Assume $S$ is a convex surface with Legendrian boundary. After possibly perturbing $S$ we can further assume that all boundary tangencies are half-elliptic (see Lemma 3.2 in \cite{24}). If $t(\gamma) = -n \leq 0$ for $\gamma \subset \partial S$, then the dividing curves intersect $\gamma$ exactly $2n$ times. Suppose one of these dividing arcs is boundary-parallel, i.e. the arc cuts off a half-disc which has no further intersections with $\Gamma_S$. A nearby arc in the complement, parallel to this dividing arc, can be made Legendrian using the Realization Principle. After this, the arc bounds a bypass. Thus, we have as a general principle:

**Proposition 2.8.** Let $S$ be a compact surface having one Legendrian boundary with non-positive twisting number, other than $D^2$ with $t(\partial D^2) = -1$. After possibly a small perturbation we can assume that $S$ is convex and all singular points of $S_\xi$ on $\partial S$ are half-elliptic. Suppose further $\gamma$ is a boundary-parallel dividing curve. Then there exists a bypass which contains the half-disc cut off by $\gamma$.

In the sequel, we need bypass existence for two special surfaces: discs and annuli. We therefore consider these special cases as discussed in \cite{24}; see Fig. 4 for examples.

**Proposition 2.9.** Assume $D^2$ is a convex disc with Legendrian boundary lying inside a tight contact manifold and $t(\partial D^2) = -n < 0$. After possibly a small perturbation we can assume that all tangencies at the boundary are half-elliptic. Then $\Gamma_{D^2}$ consists of arcs which begin and end on $\partial D^2$. If $t(\partial D^2) < -1$, then there exists a bypass along $\partial D^2$.

**Proposition 2.10 (Imbalance Principle).** Assume $A = S^1 \times [0, 1]$ is a convex annulus with Legendrian boundary in a tight contact manifold. After possibly a small perturbation we can assume that all tangencies at the boundary are half-elliptic. If $t(S^1 \times \{0\}) < t(S^1 \times \{1\}) \leq 0$, then there exists a bypass along $S^1 \times \{0\}$.

Once we have a bypass for a convex surface, it can be used to manipulate the dividing set. The basic attachment process is described as follows:
Proposition 2.11 (Bypass attachment). Assume \( Q = [0, 1] \times [0, 1] \) is a convex square in a convex surface with three horizontal dividing arcs as in Fig. 5 (a). If there exists a bypass \( D \) for \( Q \), along a vertical Legendrian arc \( \delta \), we can isotope \( Q \) (fixing the boundary) by pushing \( Q \) across \( D \) such that the characteristic foliation has a dividing set as shown in Fig. 5 (b).

Proof. Because \( Q \) is convex, we can consider an \( I \)-invariant one-sided neighbourhood \( Q \times [0, \varepsilon] \), for some \( \varepsilon > 0 \), such that \( Q = Q \times \{\varepsilon\} \). Then, \( A' = \delta \times [0, \varepsilon] \) is a rectangle in standard form, (i.e. with horizontal linear characteristic foliation and parallel dividing arcs in \([0, \varepsilon]\)-direction), transverse to \( Q \). Then, \( A = A' \cup D \) is convex with piecewise smooth boundary. The endpoints of the arc \( \delta = A' \cap D \) are half-elliptic corners. In order to smooth the corners of \( A \), we convert half-elliptic points to full elliptic points. Finally, we apply the Pivot Lemma to smooth the corners; see [24]. Because \( A \) is convex, we can take an \( I \)-invariant neighbourhood \( N(A) = A \times [0, 1] \). The boundary components \( A_i = A \times \{i\} \), \( i = 0, 1 \) are copies of \( A \), i.e. have the same dividing set \( \Gamma \). Both \( A_0 \) and \( A_1 \) are oriented as boundary of \( N(A) \) and therefore corresponding regions of \( A_0 \setminus \Gamma \) and \( A_1 \setminus \Gamma \) have different signs. Now, using the Edge-Rounding Lemma we smooth out the four edges of \( N(A) \cup Q \times [0, \varepsilon] \) to obtain a surface \( Q' \) with dividing set as in Fig. 5 (b), which completes the proof. See Fig. 6 for an illustration.

In the sequel of this paper, we frequently encounter the situation where a bypass is attached along a torus. We first describe a standard normal form for a convex torus in a contact structure and explain then the consequences of the bypass attachment in this situation. On a convex torus \( T^2 \) in a tight contact manifold, we know by Giroux’s criterion (Theorem 2.5) that no dividing curve bounds a disc. Therefore, the dividing set \( \Gamma_{T^2} \) consists of \( 2n \) homotopic essential parallel dividing curves and the number \( n = \frac{1}{2} \# \Gamma_{T^2} \) is called the torus division number. Using some identification of \( T^2 \) with \( \mathbb{R}^2/\mathbb{Z}^2 \), the dividing curves have slope \( s \), called the boundary slope of the torus. Due to Giroux’s Flexibility Theorem we can deform the torus \( T^2 \) inside a neighbourhood of \( T^2 \subset M \), fixing the dividing set \( \Gamma_{T^2} \) so that the characteristic foliation \( T^2 \) consists of a 1-parameter family of closed curves, called Legendrian rulings, of the same slope \( r \), called ruling slope. Each component of \( T^2 \setminus \Gamma \) contains a line of singular points of slope \( s \), called Legendrian divide.

\footnote{1}{The Pivot Lemma allows to perturb a surface near an elliptic point such that any two transverse trajectories become smooth; see [8].}
convex torus in this (non-generic) form is said to be in standard form (see Fig. 7 for an illustration). An immediate consequence of Giroux’s Flexibility Theorem is the following:

**Proposition 2.12** (flexibility of Legendrian rulings, [24]). *Assume* $T^2$ *is a convex torus in standard form, and, using coordinates in* $\mathbb{R}^2/\mathbb{Z}^2$, *has boundary slope* $s$ *and ruling slope* $r$. *Then by a* $C^0$-small *perturbation near the Legendrian divides, we can modify the ruling slope from* $r \neq s$ *to any other* $r' \neq s$ (*∞ included*).

If a bypass is attached along some Legendrian ruling on $T^2$, we can push the torus across the bypass, using the bypass attachment (Proposition 2.11), which yields a new torus with different boundary conditions. If the torus division number $n$ of $T^2$ is greater than one, this will yield a torus with division number $n - 1$. In the case $n = 1$ attaching a bypass does not change the torus division number but the boundary slope of the torus; see [24]. In order to describe how the new boundary conditions are obtained from the old, we first recall the Farey tessellation of the...
hyperbolic disc: Consider the hyperbolic unit disc \( \mathbb{H} = \{(x, y) : x^2 + y^2 \leq 1\} \). We label the point \((1, 0)\) as \(0 = \frac{0}{1}\), the point \((-1, 0)\) as \(\infty = \frac{1}{0}\) and join the two points by an arc. Now label inductively points on \(S^1 = \partial \mathbb{H}\) as follows (for \(y > 0\)): assume we have labelled two points \(\infty \geq \frac{p}{q}, \frac{p'}{q'} \geq 0\) (where numerators and denominators are relatively prime). Label the point half way between \(\frac{p}{q}\) and \(\frac{p'}{q'}\) along the shorter arc on \(S^1\) by \(\frac{p + p'}{q + q'}\). Connect two points \(\frac{p}{q}\) and \(\frac{p'}{q'}\) by an arc if the corresponding shortest integral vectors form an integral basis of \(\mathbb{Z}^2\); see Fig. 8.

**Theorem 2.13** (Honda, [24]). Assume \(T\) is a convex torus in standard form with \(\#\Gamma = 2\) and boundary slope \(s = s(T)\). If a bypass \(D\) is attached to \(T\) along a Legendrian ruling curve of slope \(r \neq s\), then the resulting convex torus \(T'\) will have \(\#\Gamma_{T'} = 2\) and boundary slope \(s'\) which is obtained as follows: take the arc \([r, s] \subset \partial \mathbb{H}\) obtained by starting from \(r\) and moving counterclockwise until we hit \(s\). On this arc, \(s'\) is the point which is closest to \(r\) and has an edge from \(s'\) to \(s\).

### 2.3. Tight contact structures on basic blocks.

A key principle in the classification of tight contact structures on 3-manifolds is to cut along convex surfaces to obtain simpler pieces on which the classification is known. In this subsection, we review the basic properties of tight contact structures on various simple pieces referred to as basic blocks.

#### 2.3.1. The 3-ball

The following key Theorem was proven by Eliashberg in [7]:

**Theorem 2.14.** Assume there exists a contact structure \(\xi\) on a neighbourhood of \(\partial B^3\) such that \(\partial B^3\) is convex and \(\#\Gamma_{\partial B^3} = 1\). Then there exists a unique extension of \(\xi\) to a tight contact structure on \(B^3\), up to an isotopy relative to \(\partial B^3\).

#### 2.3.2. The solid torus \(S^1 \times D^2\)

Assume \(\gamma \subset M\) is a Legendrian curve with a negative twisting number \(t(\gamma) = n\) with respect to some fixed framing. The
standard tubular neighbourhood $N(\gamma)$ of $\gamma$ is defined as solid torus $S^1 \times D^2$ with coordinates $(z, (x, y))$ and contact 1-form $\alpha = \sin(2\pi nz)dx + \cos(2\pi nz)dy$ and $\gamma = \{(z, (x, y)) : x = y = 0\}$. With respect to the fixed framing of $\gamma$, we may identify $\partial N(\gamma) = \mathbb{R}^2 / \mathbb{Z}^2$ such that the meridian is $(1, 0)^T$ and the longitude (fixed by the framing) is $(0, 1)^T$. Then the boundary slope is $s(\partial N(\gamma)) = \frac{1}{n}$.

In standard neighbourhoods of Legendrian curves the model standard tubular neighbourhood provides a unique tight contact structure. This fact was used extensively by Kanda [27], and proved (in a slightly different form) by Makar-Limanov in [33]; we refer to Theorem 6.7 in [11].

**Proposition 2.15.** There exists a unique tight contact structure on $S^1 \times D^2$ with a fixed convex boundary with $\#\Gamma_{\partial(S^1 \times D^2)} = 2$ and slope $s(\partial(S^1 \times D^2)) = \frac{1}{n}$, where $n$ is a negative integer. With the possibility of modifying the characteristic foliation on the boundary using the Flexibility Theorem (Proposition 2.12), the tight contact structure is isotopic to the standard neighbourhood of a Legendrian curve with twisting number $n$.

Decreasing the twisting number of a Legendrian curve is feasible, as commonly understood, by adding a ‘zigzag’ in the front projection; see [11]. Increasing the twisting number is not an easy task, but possible in the presence of bypasses.

**Proposition 2.16 (Twist Number Lemma, [24]).** Consider a Legendrian curve $\gamma$ in a contact manifold $(M, \xi)$ with twisting number $n$ relative to a fixed framing and $N$ a standard tubular neighbourhood of $\gamma$. If there exists a bypass attached to a Legendrian ruling curve of $\partial N$ of slope $r$ and $\frac{1}{r} \geq n + 1$, then there exists a Legendrian curve with twisting number $n + 1$ isotopic to $\gamma$. Notice that this isotopy cannot be a Legendrian isotopy because the twisting number changes.

Suppose we have given a solid torus $S^1 \times D^2$ and an oriented identification of the boundary torus $\partial(S^1 \times D^2)$ with $\mathbb{R}^2 / \mathbb{Z}^2$ so that $(1, 0)^T$ corresponds to the meridian and $(0, 1)^T$ corresponds to a longitude. Assuming the boundary of $S^1 \times D^2$ is a torus in standard form with torus division number one, the number of tight contact structures are determined by the boundary slope, i.e. the slope of the dividing curves. More precisely:

**Theorem 2.17 (Theorem 2.3, [24]).** Let $S^1 \times D^2$ be a solid torus with convex boundary $T^2$ in standard form. If $\#\Gamma_{T^2} = 2$ and the boundary slope $s(T^2) = -\frac{p}{q}$, $p \geq q > 1$, $(p, q) = 1$ and continued fraction expansion

$$-\frac{p}{q} = r_0 - \frac{1}{r_1 - \frac{1}{r_2 - \cdots - \frac{1}{r_k}}} \quad \text{with all } r_i < -1,$$

then there are exactly $|r_0 + 1)(r_1 + 1)\ldots(r_{k-1} + 1)r_k|$ tight contact structures on $S^1 \times D^2$ up to isotopy fixing the boundary.

**Proposition 2.18 (Lemma 3.16, [12]).** Assume $S^1 \times D^2$ has convex boundary with boundary slope $s < 0$. Then we can find a convex torus parallel to the boundary $\partial(S^1 \times D^2)$ with any boundary slope in $[s, 0)$.

2.3.3. **Toric annuli $T^2 \times [0, 1]$.** Assume a toric annulus $T^2 \times I$ is given in coordinates $(x, y, z) \in \mathbb{R}^2 / \mathbb{Z}^2 \times [0, 1]$. A standard tight contact structure is given by $\alpha = \sin(\frac{\pi}{2}z)dx + \cos(\frac{\pi}{2}z)dy$. Note that the boundary $\partial(T^2 \times I) = T_0 - T_1$ consists (after a perturbation) of convex tori with boundary slope 0 and $\infty$ respectively. It
is not hard to see that the tori \( T^2 \times \{ z \} \) are linearly foliated and the boundary slopes decrease as \( z \) increases. More generally, one obtains models for tight contact structures on toric annuli with different slopes on the boundary \( T_0, T_1 \) by changing the chosen interval \( I \) on the \( z \)-axis.

**Proposition 2.19** *(Proposition 4.16, [24]).* Assume a toric annulus \( T^2 \times I \) has convex boundary in standard form and the boundary slope on \( T_i = T^2 \times \{ i \} \) is \( s_i \), \( i = 0, 1 \) respectively. Then we can find convex tori parallel to \( T_0 \) with any boundary slope \( s \) in \( [s_1, s_0] \) (if \( s_0 < s_1 \) this means \( [s_1, \infty) \cup [-\infty, s_0] \)).

On the other hand, consider a tight contact structure \( \xi \) on a toric annulus \( T^2 \times I \) with convex boundary and boundary slope \( s_i = s(T_i) \), \( i = 0, 1 \). We say \( \xi \) is *minimally twisting* (in the \( I \)-direction) if every convex torus parallel to the boundary has slope \( s \in [s_1, s_0] \).

We will outline the classification of tight contact structures in thickened tori \( T^2 \times I \). For a detailed description, we refer to Honda [24]. To state the Theorems, we first recall the notion of the relative Euler class. Consider a complex line bundle \( \xi \) on a 3-manifold \( M \) with boundary \( \partial M \). Assume \( \xi|_{\partial M} \) has a nowhere vanishing section \( s \). We may define the *relative Euler class* \( \varepsilon(\xi, s) \in H^2(M, \partial M) \) as the obstruction to extending \( s \) to the whole manifold. It is related to the Euler class by the following exact sequence:

\[
H^1(\partial M) \xrightarrow{\Delta} H^2(M, \partial M) \rightarrow H^2(M) \rightarrow H^2(\partial M) \rightarrow 0
\]

The following two Lemmas are useful for the calculation of the relative Euler class of contact structures. The proofs are found in Section 4.2 of [24].

**Lemma 2.20.** Let \((M, \xi)\) be a contact manifold with convex boundary, and \( s \) a fixed section of \( \xi|_{\partial M} \).

1. If \( \Sigma \subset M \) is a closed convex surface with positive (resp. negative) region \( R_+ \) (resp. \( R_- \)) divided by \( \Gamma_\Sigma \), then \( \langle \varepsilon(\xi), \Sigma \rangle = \langle \varepsilon(\xi, s), \Sigma \rangle = \chi(R_+) - \chi(R_-) \).

2. If \( \Sigma \subset M \) is a compact convex surface with Legendrian boundary on \( \partial M \) and regions \( R_+ \) and \( R_- \), and \( s \) is homotopic to \( s' \) which coincides with \( \gamma \) for every oriented connected component \( \gamma \) of \( \partial \Sigma \), then \( \langle \varepsilon(\xi, s), \Sigma \rangle = \chi(R_+) - \chi(R_-) \).

**Lemma 2.21.** Let \((M, \xi)\) be a tight contact manifold with convex boundary consisting of tori. Then the relative Euler class \( \langle \varepsilon(\xi, s), \Sigma \rangle \) is independent of the slope of the Legendrian rulings, where \( s \) is a nonzero section of \( \xi \) tangent to the ruling curves.

In the following we use the Euler class only in its relative form. Thus, when \( \partial M \) is a union of convex tori in standard form, we will write \( \varepsilon(\xi) \) instead of \( \varepsilon(\xi, s) \) with an abuse of notations if the section \( s \) comes from a vector field tangent to the Legendrian rulings.

We say that two rational slopes in \( \mathbb{Q} \cup \{ \infty \} \) are *consecutive* if they are joined by an edge in the Farey tessellation. The very basic building blocks for contact structures are the minimally twisting tight contact structures on \( T^2 \times I \) whose boundary slopes \( s_0 \) and \( s_1 \) are consecutive. Such contact structures are called *basic slices*. We have the following classification result for basic slices.
Theorem 2.22 (\cite{24}, Section 4.3). Given consecutive \( s_0 \) and \( s_1 \), there are, up to isotopy fixed on the boundary, two minimally twisting tight contact structures on \( T^2 \times I \) with boundary in standard form, \( \# \Gamma_{t_i} = 2 \) for \( i = 0, 1 \), and boundary slopes \( s_0 \) and \( s_1 \). The two contact structures are distinguished by their relative Euler class, and both can be contact-embedded in a tight contact structure on \( T^3 \).

Let \( v_0 \) and \( v_1 \) be shortest integer vectors representing the slopes \( s_0 \) and \( s_1 \) of a basic slice, such that \((v_1 - v_0)\) is a positively oriented basis. The possible relative Euler classes of the basic slices are the Poincaré duals of the homology classes represented by \( \pm (v_1 - v_0) \). For this reason, in what follows we will refer to the isotopy class of a basic slice as its sign.

The basic slices are basic in the sense that any minimally twisting tight contact structure on \( T^2 \times I \) can be decomposed into basic slices, as explained in the following Theorem:

Theorem 2.23 (Lemma 4.12, \cite{24}). Given a minimally twisting, tight contact structure on \( T^2 \times I \) with boundary slopes \( s_0 \) and \( s_1 \), we can find a partition \( 0 = t_0 < \ldots < t_k = 1 \) of \([0, 1]\) such that \( T^2 \times \{t_i\} \) is a convex torus in standard form with slope \( s_{t_i} \) for any \( i = 0, \ldots, k \). These slopes form a counterclockwise sequence in the arc \([s_1, s_0]\) with the property that \( s_{t_i} \) and \( s_{t_{i+1}} \) are consecutive and there is no edge in the Farey tessellation joining \( s_{t_i} \) and \( s_{t_{i+2}} \).

Moreover such a basic slices decomposition is minimal in the sense that any other basic slices decomposition of the given tight contact structure on \( T^2 \times I \) is a further decomposition of this one.

Note that the intermediate slopes of the basic slices decomposition depend only on \( s_0 \) and \( s_1 \) and are independent of the isotopy class of the contact structure.

Conversely, given a decomposition into basic slices, we have the following gluing Theorem, which is a particular case of the more general Theorem 4.24 in \cite{24}.

Theorem 2.24. Let \( s_1 < s_0 \) be rational slopes. Then every choice of signs for the basic slices in the basic slices decomposition associated to \( s_0 \) and \( s_1 \) realizes a minimally twisting, tight contact structure on \( T^2 \times I \) with slopes \( s_0 \) and \( s_1 \).

We observe that, unlike basic slices, in general these tight contact structures cannot be contact-embedded into a tight contact structure on \( T^3 \). This is possible if and only if the relative Euler class is \( \pm (v_1 - v_0) \).

We may ask when two different choices of signs for the basic slices give the same contact structure. We say the basic slices in \( T^2 \times [t_j, t_l] \) form a continued fraction block if there is a slope \( r \) such that there is an edge in the Farey tessellation joining \( r \) and \( t_i \) for all \( j \leq i \leq l \). To understand the origin of the name, see \cite{24}, where this concept appeared for the first time in this context. The importance of this notion comes from the fact that the sign of basic slices belonging to the same continued fraction block can be shuffled without affecting the isotopy type of the contact structure on \( T \times I \). This is a nontrivial result whose proof can be found in \cite{24}.

In this paper we will use the property of continued fraction blocks only in the following case. Let \( T^2 \times I \) carry a minimally twisting tight contact structure with boundary slopes \( s_0 = -\frac{1}{n} \), for \( n > 0 \), and \( s_1 = \infty \). Then all the basic slices of its decomposition belong to the same continued fraction block, and therefore their signs can be shuffled.
As a result of the classification of basic slices, the basic slices decomposition, and this last fact about continued fraction blocks, we can derive the following classification Theorem for tight contact structures on $T^2 \times I$:

**Theorem 2.25** (Proposition 4.22, [24]). The minimally twisting tight contact structures on $T^2 \times I$ with standard boundary, $\#\Gamma_{T_1} = 2$ and boundary slopes $s_0$ and $s_1$ are distinguished up to isotopy fixed on the boundary by their relative Euler class.

Actually, in this Theorem Honda proves more than what he states: in fact he shows that, after normalising the boundary slopes to $s_0 = -1$ and $s_1 < -1$, the tight contact structures are distinguished by the value their relative Euler class takes at a horizontal annulus with Legendrian boundary. He proves this fact by showing that each basic slice in the decomposition gives a contribution to the value of the relative Euler class which is bigger than the sum of the contributions of the basic slices belonging to all the preceding continued fraction blocks. The same arguments prove the following Corollary.

**Corollary 2.26.** The minimally twisting tight contact structures on $T^2 \times I$ with standard boundary, $\#\Gamma_{T_1} = 2$ and boundary slopes $s_0 = 0$ and $s_1 < -1$ are distinguished up to isotopy fixed on the boundary by the value their relative Euler class takes at a horizontal annulus with Legendrian boundary.

The number of minimally twisting tight contact structures on $T^2 \times I$ with fixed dividing set on the boundary is finite, and is expressed as a function of the continued fraction representation of $s_1$, after normalising $s_0$ to $-1$ by a change of coordinates, as in the case of solid tori in Theorem 2.17.

We mention also the case where the boundary slopes are equal. Then, either any convex intermediate torus $T \subset T^2 \times I$ has the same slope as the boundary tori, or there is a convex torus of slope $s$ for any $s \in \mathbb{Q}$. This is a consequence of Proposition 2.19.

The tight contact structures on $T^2 \times I$ with boundary slopes $s_0 = s_1$ such that all the intermediate convex tori have the same slope are called non-rotative.

### 3. Legendrian $(-1)$ surgery

A very useful method to construct contact structures on a manifold is given by Legendrian surgery. Together with the fact that, in particular, Legendrian $(-1)$ surgery preserves fillability one can construct tight contact structures. Suppose $\gamma$ is a Legendrian curve in a contact manifold $(M, \xi)$ and we fix a framing $F$ of $\gamma$ such that the twisting number is zero $\tau(\gamma, F) = 0$; for example take $F$ to be the contact framing of $\gamma$. We then find a standard neighbourhood $N(\gamma) = S^1 \times D^2$ with convex boundary so that the dividing set $\partial N(\gamma)$ consists of two parallel curves. Take an oriented identification $-\partial(M \setminus N(\gamma)) = \partial N(\gamma) \cong \mathbb{R}^2/\mathbb{Z}^2$ so that $(1, 0)^T$ corresponds to the meridian and $(0, 1)^T$ to the longitude given by a dividing curve. Thus the boundary slope $s(\partial N(\gamma))$ is infinite and the meridian has slope zero. Let $M' = (M \setminus N(\gamma)) \cup_f N(\gamma)$, where $f : \partial N(\gamma) \to -\partial(M \setminus N(\gamma))$ is a diffeomorphism corresponding to

\[
\begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix} \in SL_2(\mathbb{Z}).
\]

Topologically this corresponds to a $(-1)$ Dehn surgery along $\gamma$ with respect to the chosen framing, and the contact structure can be glued together, after possibly
adjusting the characteristic foliation, since the dividing sets on \(-\partial(M \setminus N(\gamma))\) and \(f(\partial N(\gamma))\) are isotopic. More generally one can define Legendrian \((r)\) surgery, for \(r \in \mathbb{Q}\), as described in \cite{3}.

Legendrian \((-1)\) surgery corresponds to a handle body construction in the sense of \cite{19,35} and thus preserves fillability. Recall that a contact manifold \((M, \xi)\) is called holomorphically fillable if it is the oriented boundary of a compact Stein surface. For example the standard tight contact structure on the three-sphere \(\partial V\) is holomorphically fillable. It is a remarkable result of Gromov \cite{21} and Eliashberg \cite{5} that fillable contact structures are tight. Furthermore:

\textbf{Theorem 3.1 (Eliashberg, \cite{6}). If \((M', \xi')\) is obtained from a holomorphically fillable contact manifold \((M, \xi)\) by Legendrian \((-1)\) surgery as described above, then \((M', \xi')\) is holomorphically fillable.}

In fact both Theorem remain true for any notion of fillability (see \cite{13} for a survey), but Legendrian \((-1)\) surgery does not preserve tightness in case of contact manifolds with boundary, as pointed out in \cite{23}.

### 4. Applications

In this section, we will show how to obtain the classification of tight contact structures in two specific cases: the Brieskorn homology spheres \(\pm \Sigma(2, 3, 11)\). Both manifolds are Seifert fibred spaces over the sphere with three singular fibres.

Consider a Seifert manifold \(M\) with three singular fibres over \(S^2\). \(M\) is described by Seifert invariants \((\frac{\alpha_1}{\alpha_2}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3})\), we refer to \cite{22} for an introduction.

Assume \(V_i\) are solid tori \(S^1 \times D^2\) with core curves \(F_i, i = 1, 2, 3\). We identify \(\partial V_i\) with \(\mathbb{R}^2/\mathbb{Z}^2\) by choosing \((1, 0)^T\) as the meridional direction and \((0, 1)^T\) as a longitudinal direction. Furthermore consider \(S^1 \times \Sigma\), where \(\Sigma\) is a three-punctured sphere, i.e. a pair of pants. We identify each boundary component of \(-\partial(S^1 \times \Sigma)\) with \(\mathbb{R}^2/\mathbb{Z}^2\) by setting \((0, 1)^T\) as the direction of the \(S^1\)-fibre and \((1, 0)^T\) as the direction given by \(-\partial((pt) \times \Sigma)\). Then we obtain the Seifert manifold \(M(\frac{\alpha_1}{\alpha_2}, \frac{\beta_2}{\alpha_2}, \frac{\beta_3}{\alpha_3})\) by attaching the solid tori \(V_i\) to \(S^1 \times \Sigma\), where the attaching maps \(A_i : \partial V_i \to -\partial((S^1 \times \Sigma)\) are given by

\[
A_i = \begin{bmatrix}
\alpha_i & \gamma_i \\
-\beta_i & \delta_i
\end{bmatrix} \in SL(2, \mathbb{Z})
\]

\textbf{Remark 4.1.} Note that we often refer to the same surface by different names. For example \(\partial V_i\) and \(\partial(M \setminus V_i)\) denote the same torus, but the identification with \(\mathbb{R}^2/\mathbb{Z}^2\) is different.

\textbf{Remark 4.2.} Unless stated otherwise, properly embedded surfaces in a contact manifold are understood to be convex, if possible.

Note that the three singular fibres \(F_i\) may be isotoped to be Legendrian so that their twisting numbers \(n_i\) are particularly negative. Recall that a standard neighbourhood \(V_i\) of \(F_i\) with convex boundary has boundary slope \(\frac{\alpha_i}{n_i}\). Furthermore we may assume that the ruling slope on \(-\partial(M \setminus V_i)\) is infinite, thereby using the flexibility of Legendrian rulings (Proposition \cite{2,12}). Starting with this initial configuration, we do the following:
In the first step, we try to maximise the twisting numbers of the singular fibres. For this, consider a vertical annulus $A = S^1 \times I$ with Legendrian boundary along ruling curves of two different tori $V_i$, $V_j$. If the Imbalance Principle forces a bypass on $A$ we may apply the Twist Number Lemma (Proposition 2.16) to increase one of the twisting numbers $n_i$ or $n_j$. Repeating this process, two different situations might occur.

(a) Either there exists a bypass on $A$ however we cannot apply the Twist Number Lemma. In this case, we can thicken the tori by attaching the bypass. In the cases below this yields an infinite boundary slope on $-\partial(M \setminus V_i)$. Consider a vertical annulus from a Legendrian divide to the other two tori, we can thicken all three tori so that $s(-\partial(M \setminus V_i)) = \infty$, $i = 1, 2, 3$.

(b) There exists no bypass on $A$. In this case a tubular neighbourhood of $V_i \cup V_j \cup A$ is a piecewise smooth torus with exactly four edges. Rounding the edges using the Edge-Rounding Lemma (Proposition 2.6), we obtain a torus with boundary slope $s$, which can be thought of as the boundary of a neighbourhood of the third singular fibre $F_k$. In case $s < s(\partial(M \setminus V_k))$ we can eventually increase the twisting number of $F_k$.

In either case, this process ends in a configuration with fixed boundary conditions on the basic blocks $V_i$, $i = 1, 2, 3$ and $M \setminus \{V_1 \cup V_2 \cup V_3\}$. Combinations of tight contact structures on the basic blocks give a possible tight contact structure on $M$. Since there are finitely many tight contact structures on each basic piece, we obtain an upper bound on the number of tight contact structures on $M$.

(2) In the second step we try to further analyse combinations of tight contact structures on the basic blocks. Observe that if we were able to find further bypasses to thicken one $V_i$ such that $V_i$ contains a neighbourhood $V_i'$ of $F_i$ so that the boundary slope $s(\partial V_i')$ is zero, we would find an overtwisted disc as meridional disc with boundary a Legendrian divide, and thus reduce the number of potentially tight contact structures on $M$.

In the examples below, we are able to find further bypasses and eventually find an overtwisted disc in case there exists a thickening so that $-\partial(M \setminus V_i)$ has infinite boundary slope.

(3) We finally construct tight contact structures by Legendrian surgery to show that the upper bound is sharp.

**Remark 4.3.** Note that the strategy of constructing tight contact structures by Legendrian surgery yields fillable contact structures. Hence this strategy is not successful in every case; see [13] and section 5.

### 4.1. The case $\Sigma(2,3,11)$

In this subsection, $M$ denotes the Brieskorn homology sphere $\Sigma(2,3,11)$. This corresponds to the manifold $M(\frac{1}{2}, -\frac{1}{3}, -\frac{1}{11})$ in the notation above.

**Theorem 4.4.** On the Seifert manifold $\Sigma(2,3,11)$ there exist, up to isotopy, exactly two tight contact structures, which are both holomorphically fillable.

The attaching maps are given by

$$A_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 11 & -6 \\ 2 & -1 \end{bmatrix}.$$
Assume the singular fibres $F_i$ are (simultaneously) isotoped to Legendrian and further isotoped such that their twisting numbers $n_i$ are particularly negative. The standard neighbourhood of $F_i$ is denoted by $V_i$ and the slope of the dividing curves on $\partial V_i$ is $\frac{1}{n_i}$. Because $A_1 \cdot (n_1, 1)^T = (2n_1 + 1, -n_1)^T$, $A_2 \cdot (n_2, 1)^T = (3n_2 - 1, n_2)^T$ and $A_3 \cdot (n_3, 1)^T = (11n_3 - 6, 2n_3 - 1)^T$, we calculate the boundary slopes on $-\partial(M \setminus V_i)$ ($i = 1, 2, 3$) to be $\frac{-n_1}{2n_1 + 1}$, $\frac{-n_3}{3n_3 - 2}$ and $\frac{-3n_2 - 1}{11n_3 - 6}$, respectively.

4.1.1. Increasing twisting numbers of singular fibres. We try to increase the twisting numbers of the singular fibres as far as possible. As described above, we start by assuming the twisting numbers $n_i < 0$ are particularly negative.

**Lemma 4.5.** We can increase the twisting numbers $n_i$ of the singular fibres $F_i$, $i = 1, 2, 3$, up to $n_1 = -1$, $n_2 = n_3 = 0$.

**Proof.** Using the flexibility of Legendrian rulings, we modify the Legendrian rulings on each $\partial(M \setminus V_i)$ to have infinite slope. Consider a vertical annulus $S^1 \times I$ from $\partial(M \setminus V_1)$ to $\partial(M \setminus V_2)$ such that the boundary consists of Legendrian ruling curves on the tori. Observe that the boundary of this annulus intersects the dividing curves on $\partial(M \setminus V_i)$ exactly $2(2n_1 + 1)$ and $2(3n_2 - 1)$ times respectively.

If $2n_1 + 1 \neq 3n_2 - 1$, then, due to the Imbalance Principle (Proposition 2.10), there exists a bypass along a Legendrian ruling curve either on $\partial(M \setminus V_1)$ or $\partial(M \setminus V_2)$. The Legendrian rulings on $\partial V_1$ have slope $-2$ and we can apply the Twist Number Lemma (Proposition 2.16) to increase the twisting number of a singular fibre by one as long as $n_1 < -1$. A similar argument shows that we can use the Twist Number Lemma to increase $n_2$ as long as $n_2 < 0$.

Assume $2n_1 + 1 = 3n_2 - 1$ and there exists no bypass on a vertical annulus $A = S^1 \times I$ between $\partial(M \setminus V_1)$ and $\partial(M \setminus V_2)$. We cut along the tori connected by $A$ and round the corners using the Edge-Rounding Lemma: For this, observe that a neighbourhood of $M \setminus (V_1 \cup V_2 \cup S^1 \times I)$ is a piecewise smooth solid torus with four edges. Using the Edge-Rounding Lemma (Proposition 2.16), each rounding changes the slope by an amount $-\frac{1}{2n_1 + 1}$. Because there are four edges to round, we get on the boundary $\partial(M \setminus V_1 \cup V_2 \cup S^1 \times I)$ the slope

$$\frac{-n_1}{2n_1 + 1} + \frac{2(n_1 + 1)}{2n_1 + 1} + \frac{-1}{2n_1 + 1} = \frac{1}{3} \frac{n_1 + 1}{2n_1 + 1}.$$

Note that we identified this torus with $\mathbb{R}^2/\mathbb{Z}^2$ in the same way as $\partial(M \setminus V_3)$. Since $A_3 \cdot (6n_1 + 3, n_1 + 1)^T = (3, -n_1 + 5)^T$, this corresponds to slope $s = -\frac{1}{3} n_1 + \frac{2}{3}$ when measured using $\partial V_3$. Now $s > 1$ for $n_1 < 0$ and we find a standard neighbourhood $V_3$ of $F_3$ with infinite boundary slope, corresponding to $n_3 = 0$. We remark that the boundary slope becomes $\frac{1}{6}$, when measured with respect to $-\partial(M \setminus V_3)$.

Next, to increase the twisting number $n_2$, take a vertical annulus $S^1 \times I$ from a Legendrian ruling on $\partial(M \setminus V_2)$ to a Legendrian ruling on $\partial(M \setminus V_3)$. Observe that if $n_2 < -2$, we have $|3n_2 - 1| > 6$ and thus there exists a bypass on the $V_2$ side along a vertical Legendrian ruling, which allows us to increase $n_2$ up to $-1$ by the Twist Number Lemma. A similar argument shows that we can increase $n_1$ up to $-2$.

Now the slopes on $-\partial(M \setminus V_i)$ are $-\frac{2}{3}, \frac{1}{3}$ and $\frac{1}{6}$ respectively. Taking, once more, a vertical annulus between $V_1$ and $V_2$, we find a bypass due to the Imbalance Principle and are finally able to increase $n_2$ to 0 and $n_1$ to $-1$. \qed
We have now arrived at $n_1 = -1$, $n_2 = n_3 = 0$. Note that the boundary slopes on $-\partial(M \setminus V_i)$ are $-1$, $0$ and $\frac{1}{6}$ respectively. Take again a vertical annulus between $V_1$ and $V_2$. There are two possibilities: Either there exists a bypass along both boundary components or not. If there is a bypass, the cutting and rounding construction yields a torus of infinite slope. We use vertical annuli from a Legendrian divide of this torus to thicken each $V_i$ to $V'_i$ s.t. $-\partial(M \setminus V'_i)$ has infinite boundary slope. In case there is no bypass, we perform a cutting and rounding construction on $V_1$ and $V_2$ as in the proof of Lemma 4.5 to obtain a further thickening of $V_3$ to $V_3'$ such that $-\partial(M \setminus V'_3)$ has boundary slope $0$. We have shown that there are two possibilities, distinguished by whether or not there exists a thickening of all $V_i$ such that the boundary slope with respect to $-\partial(M \setminus V_i)$ is infinite for $i = 1, 2, 3$. We will primarily be concerned with

4.1.2. The case when a thickening to infinite slope exists. We will now show that all possible tight contact structures arising in this case are overtwisted. We do this by patching together meridional discs of two solid tori thus obtaining a surface with boundary on the third torus and relate its dividing set to the dividing curves given on the discs. This may produce a bypass which allows a further thickening, i.e. increasing the twisting and eventually becoming overtwisted. In order to do this patching we have to examine the possible tight contact structures on the complement of the singular fibres $S^1 \times \Sigma \cong M \setminus (\cup_i V'_i)$. Consider $\Sigma = \{1\} \times \Sigma$. Each boundary component of $\Sigma$ intersects the dividing set on the corresponding torus twice and therefore contains exactly two half-elliptic points. The following two Lemmata are proven by Etnyre and Honda in [14]. We enclose the proofs for the reader’s convenience.

**Lemma 4.6.** The dividing set on $\Sigma$ consists of arcs, each connecting two different boundary components.

**Proof.** Assume there is a boundary-parallel dividing arc as shown, for example, in cases (A) and (B) of Fig. 9. This implies the existence of a bypass along some $\partial(M \setminus V'_i)$. Attaching this bypass yields a thickening $V''_i$ of $V'_i$ with slope $0$. Take a vertical annulus from a Legendrian dividing curve on $\partial(M \setminus V''_j)$ ($j \neq i$) to $\partial(M \setminus V''_i)$. We find a bypass on this annulus producing a further thickening to $V'''_i$ such that $\partial(M \setminus V'''_i)$ has infinite boundary slope. Therefore, by Proposition 2.18 we find a neighbourhood $V_i$ of $F_i$ so that the boundary slope of $\partial V_i$ is zero. A meridional disc in $V_i$ whose boundary is a Legendrian divide on $\partial V_i$ is an overtwisted disc. Possible configurations of the dividing set on $\Sigma$ without boundary-parallel arcs are as shown in Fig. 9 (C), up to twisting as shown in Fig. 9 (D).

**Lemma 4.7.** There exists a unique tight contact structure on $S^1 \times \Sigma$, up to isotopy moving the boundary, where the configuration of the dividing set on $\Sigma$ is given as in Lemma 4.6.

**Proof.** We cut along $\Sigma$ and round the edges using the Edge-Rounding Lemma thus obtaining a solid two-handlebody. We can arrange the dividing set on the boundary so that two meridional discs intersect the dividing set exactly twice; see Fig. 10. Cutting along these two discs we obtain a three-ball. Since there is a unique tight contact structure on the three-ball (Theorem 2.14) and the dividing curves on the surface we cut along are determined by the initial data, we must have a unique tight contact structure on $S^1 \times \Sigma$. □
Using the flexibility of Legendrian rulings, we can choose the slopes on the $V'_i$ to be 0, this is possible since the boundary slopes, measured on $\partial V'_i$ are $-2$, $3$, and $\frac{11}{6}$, respectively. Take a meridional disc $D_i$ in $V'_i$ such that $\partial D_i$ consists of a Legendrian ruling curve. On $-\partial(M \setminus V'_i)$, these discs have slopes $-\frac{1}{2}$, $\frac{1}{3}$ and $\frac{2}{11}$ respectively.

Since $\#(\partial D_i \cap \Gamma_{\partial(M \setminus V'_i)}) = 4$, $6$ and $22$, we obtain $tb(\partial D_i) = -2$, $-3$ and $-11$ and the possible constellations of dividing curves on $D_i$, distinguished by their relative Euler number, i.e. the rotation number of $\partial D_i$ according to $r(\partial D_i) = \chi(D^+_i) - \chi(D^-_i)$.

The two Lemmata above imply that the tight contact structure on $S^1 \times \Sigma$ is contactomorphic to an $I$-invariant contact structure on a $T^2 \times I$ with a neighbourhood of a vertical Legendrian curve of zero twisting removed. View the $T^2 \times I$
Figure 11. Possible constellations of dividing curves on the punctured torus $P$, determined by the ‘signs’ of the bypasses on the meridional discs. (Dividing curves are dashed lines.)

(minus $S^1 \times D^2$) from the above Lemma as the region between $\partial V'_1$ and $\partial V'_2$ (minus $\partial V'_3$) i.e. assume $T_0 = \partial V'_2$ and $T_1 = -\partial V'_1$. We write $T_t = T^2 \times \{t\}$, $t \in [0,1]$.

Now pick three copies of meridional discs $D_1$ in $V_1$ and two copies of meridional discs $D_2$ in $V_2$. Due to the $I$-invariance of $\xi$, we have a 1-parameter family of positive regions $(T_t)^+ = (T^2)^+ \times \{t\}$. Consider $D_1$ and $D_2$ such that $(D_1 \cap T_0) = \partial V'_1$ and $(D_1 \cap T_1) = \delta \times \{1\}$, where $\delta$ is a union of Legendrian arcs on $(T^2)^+$. Now $P = D_1 \cup D_2 \cup \delta \times [0,1]$ is a punctured torus. After smoothing the corners using the Pivot-Lemma, $P$ has smooth boundary $\partial P \subset \partial V'_3$ with slope $-\frac{1}{2} + \frac{1}{1} = -\frac{1}{6}$, measured using $\partial(M \setminus V'_3)$.

Case (1) $D_1$ and $D_2$ have bypasses of the same sign. Then the dividing set on $P$ contains a boundary-parallel curve, i.e. there exists a bypass on $P$. We can think of this bypass as attached on $-\partial(M \setminus V'_3)$, along a ruling curve of slope $\frac{1}{6}$. Attaching this bypass (Theorem 2.13) yields a thickening of $V'_3$ to $V''_3$ such that $-\partial(M \setminus V''_3)$ has boundary slope 1. Repeating the argument of Lemma 4.5 shows that we can increase the twisting numbers $n_1$ and $n_2$ to $-1$ and 0 respectively. Thus we can thicken $V''_3$ further to $V'''_3$ such that the boundary slope of $-\partial(M \setminus V'''_3)$ is 0. Recall that we started in a stage where the boundary slope of $-\partial(M \setminus V'_3)$ is zero. We assumed to find a thickening that this slope becomes infinite and showed that we are then able to thicken further to obtain slope zero again. Thus we find a neighbourhood $V$ of $F_3$ so that $s(\partial V) = 0$, and a meridional disc in $V$ bounding a Legendrian divide is an overtwisted disc.

Case (2) $D_1$ and $D_2$ have different sign. Then there is a bypass on $D_3$ of the same sign as a bypass on $D_1$ or $D_2$. Assume $D_1$ and $D_3$ contain bypasses of the same sign. A similar argument as in Case (1) shows that patching eleven copies of $D_1$ and two copies of $D_3$ yields a surface whose boundary is contained in $\partial(M \setminus V'_2)$ with slope $-\frac{1}{2} + \frac{2}{11} = -\frac{7}{22}$. A bypass on each $D_1$ and $D_3$ joins to a bypass on the patched surface, if both have the same sign. Thus we find a bypass along $V'_2$ and its
attaching yields a thickening to $V_3''$ such that the boundary slope of $-\partial(M \setminus V_3'')$ is 1. Repeating the argument of Lemma 4.5 shows that we can again arrange $n_1 = -1$ and $n_3 = 0$. A cutting and rounding construction gives a further thickening of $V_1'$ to $V_3'$ such that $-\partial(M \setminus V_3')$ has boundary slope $-1$ and hence contains an overtwisted disc.

Similarly, if $D_2$ and $D_3$ contain a bypass of the same sign, we patch together eleven copies of $D_2$ and three copies of $D_3$ to obtain a surface whose boundary is contained in $\partial(M \setminus V_1')$ with slope $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. We find a bypass and its attaching yields a thickening of $V_1'$ to $V_3''$ such that $-\partial(M \setminus V_3'')$ has boundary slope 0. Make $n_2$ and $n_3$ again particularly negative and the same argument as in Lemma 4.5 shows that we can increase both $n_2$ and $n_3$ to 0. Then, cutting and rounding along a vertical annulus between $V_1$ and $V_2$ gives neighbourhood $V_3''$ of $F_3$ with boundary slope 1 when measured using $-\partial(M \setminus V_3'')$. Since the boundary slope on $-\partial(M \setminus V_3'')$ is zero, we find by Proposition 2.18 a neighbourhood $V$ of $F_1$ so that $-\partial(M \setminus V)$ has infinite boundary slope. Take a vertical annulus $A = S^1 \times I$ from a Legendrian divide on $\partial(M \setminus V)$ to $\partial(M \setminus V_3'')$. There exists a bypass on $A$ whose attachment yields a further thickening of $V_3''$ to $V_3'''$ where $\partial(M \setminus V_3''')$ has slope zero. Therefore, $V_3'''$ contains, by Proposition 2.18 a neighbourhood $V$ of $F_3$ so that the boundary slope of $-\partial(M \setminus V)$ is $\frac{1}{1}$. A meridional disc in $V$ with boundary a Legendrian divide on $\partial V$ is overtwisted. Hence we have eliminated all possibilities in case there exists a thickening of the $V_i$ such that $s(-\partial(M \setminus V_i))$ is infinite.

4.1.3. The case when no thickening exists. We are left now with the case when there exists no thickening of the standard neighbourhoods so that the boundary slopes of the complements is infinite. We have the following conditions: for the first singular fibre $F_1$ we obtained twisting number $n_1 = -1$, hence a standard neighbourhood $V_1$ has boundary slope $-1$. Measured using $\partial(M \setminus V_1)$, the boundary slope is 1, because $A_1 \cdot (-1, 1)^T = (-1, 1)^T$. For the second singular fibre $F_2$ we obtained twisting number $n_2 = 0$ and hence a standard neighbourhood of $F_2$ has infinite boundary slope, which corresponds to slope 0, when measured using $\partial(M \setminus V_2)$. Lastly, for the third singular fibre, the twisting number is $n_3 = 0$ and the slope on $-\partial(M \setminus V_3)$ is $\frac{1}{6}$. A cutting and rounding construction along a vertical annulus between $V_1$ and $V_2$ yields a further thickening of $V_3$ such that $-\partial(M \setminus V_3)$ has boundary slope 0.

In the first and second solid torus $V_1$ and $V_2$ there exists exactly one tight contact structure as standard neighbourhood of Legendrian fibres. Because $A_3^{-1} \cdot (1, 0)^T = (-1, -2)^T$ we find two tight contact structures on $V_3$.

The remaining block is $\Sigma \times S^1 = M \setminus (V_1 \cup V_2 \cup V_3)$, where we can arrange the boundary components of the pair of pants $\Sigma$ to be Legendrian along the boundary components of $\Sigma \times S^1$. With this boundary conditions there exists exactly one tight contact structure on this block. This is due to the following Lemma, as part of Lemma 5.1. in [25].

**Lemma 4.8.** If for $S^1 \times \Sigma$, where $\Sigma$ is a pair of pants, we have on the boundary tori $\partial(S^1 \times \Sigma) = T_1 + T_2 + T_3$ slopes 1, 0, 0 respectively, then there exists exactly one tight contact structure on $S^1 \times \Sigma$ with no vertical Legendrian curve.

Thus there are at most two tight contact structures on $M$. In the next section, we use Legendrian surgery to see that there are two Stein fillable contact structures on $M$.
4.1.4. Construction of a tight contact structure. We will describe now how to establish a tight contact structure on $M$ by Legendrian surgery. The Seifert manifold $M(\frac{1}{2}, -\frac{1}{3}, -\frac{2}{11})$ has a surgery description as the left hand side of Fig. 12. By performing one (-1)-Rolfsen twist on the (3) and ($\frac{11}{2}$) fibre, we obtain the surgery description as shown on the right hand side; see [20]. Observe that we have the continued fraction expansions $-\frac{3}{2} = [-2, -2]$ and $-\frac{11}{9} = [-2, -2, -2, -2, -3]$, thus the surgery description as at the bottom in Fig. 12. Because the surgery coefficients are $-2$ or $-3$ we may conclude that there are exactly two Legendrian realizations of this link, where each component of the link will have $tb = -1$ and hence $r = 0$, except one with $tb = -2$ and $r = \pm 1$. Note that the difference in the rotation number in a component of the two Legendrian realizations implies that the Chern classes of the corresponding Stein surfaces are different, thus that the contact structures on the boundary are non isotopic; see Theorem 4.20 in [24] and [31, 32] for details.

4.2. The case $-\Sigma(2, 3, 11)$. In this subsection $M$ will denote the Seifert manifold over $S^2$ with three singular fibres with invariants $(-\frac{1}{2}, -\frac{1}{3}, -\frac{2}{11})$, corresponding to the Brieskorn homology sphere $-\Sigma(2, 3, 11)$.

Theorem 4.9. On the Seifert manifold $-\Sigma(2, 3, 11)$ there exists, up to isotopy, exactly one tight contact structure, which is holomorphically fillable.

Assume $V_i$ are tubular neighbourhoods of the singular fibres $F_i$, $i = 1, 2, 3$, and identify $M \setminus \cup V_i$ with $\Sigma \times S^1$, where $\Sigma$ is a pair of pants. We identify $\partial V_i$ and $-\partial(M \setminus V_i)$ with $\mathbb{R}^2/\mathbb{Z}^2$ as in the previous example. The gluing maps $A_i : \partial V_i \to -\partial(M \setminus V_i)$ are given by

$$A_1 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 11 & 6 \\ -2 & -1 \end{bmatrix}.$$
4.2.1. Increasing the twisting number of the singular fibres. We begin by increasing the twisting number of the singular fibres as far as possible in a similar way as in the previous example. We start by assuming the singular fibres $F_i$ are simultaneously isotoped to Legendrian curves with twisting numbers $n_i$ very negative. The slopes of $\partial V_i$ are $-\frac{1}{n_i}$, while the slopes of $-\partial(M \setminus V_i)$ are $\frac{n_1}{2n_1-1}$, $\frac{n_2}{3n_2+1}$, and $-\frac{2n_1+1}{11n_3+6}$ respectively.

**Lemma 4.10.** We can increase the twisting numbers $n_1$ and $n_2$ up to $-2$, and the twisting number $n_3$ up to $-1$.

**Proof.** Using Proposition 2.12, we modify the Legendrian rulings on each $-\partial(M \setminus V_i)$ to have infinite slope. Consider a convex annulus $A$ whose boundary consists of Legendrian rulings of $-\partial(M \setminus V_1)$ and $-\partial(M \setminus V_2)$. If $2n_1 - 1 \neq 3n_2 + 1$ the Imbalance Principle provides a bypass along a Legendrian ruling either in $\partial(M \setminus V_1)$ or in $\partial(M \setminus V_2)$. Using such a bypass we can apply the Twist Number Lemma to increase the twisting number $n_i$ of a singular fibre by one as long as $n_1 < 0$ and $n_2 < -1$.

If $2n_1 - 1 = 3n_2 + 1$, and there exist no bypasses on $A$, we get stuck in this operation. Suppose we are in this case: we cut along $A$ and round the edges, obtaining a torus with slope $\frac{2n_1+1}{11n_3+6}$ isotopic to $\partial(M \setminus V_3)$. This slope corresponds to $-\frac{1}{3}n_2 - 2$ in $\partial V_3$, and is non-negative when $n_2 \leq -4$, therefore we can find a standard neighbourhood $V_3$ of $F_3$ with infinite boundary slope. This boundary slope becomes $-\frac{1}{6}$ if measured with respect to $-\partial(M \setminus V_3)$. To further increase $n_2$ take an annulus between $\partial(M \setminus V_2)$ and $\partial(M \setminus V_3)$. If $n_2 < -2$, we have $|3n_2 + 1| > 6$ and thus there exists a bypass attached to $\partial(M \setminus V_2)$ which allows us to increase $n_2$ by one so that we can start again. In this way we can increase $n_1$ and $n_2$ up to $-2$. When $n_1 = n_2 = -2$ the boundary slopes are $\frac{2}{6}$ on $-\partial(M \setminus V_1)$ and $-\frac{2}{6}$ on $-\partial(M \setminus V_2)$. By the Imbalance Principle, a convex annulus between $\partial(M \setminus V_2)$ and $\partial(M \setminus V_3)$ produces a bypass attached to $\partial(M \setminus V_3)$ as long as $5 < |11n_3 + 6|$, thus we can use the Twist Number Lemma to increase $n_3$ up to $-1$.

**Lemma 4.11.** Let us suppose $n_1 = n_2 = -2$, $n_3 = -1$ and $A$ is a convex vertical annulus whose boundary consists of Legendrian rulings of $\partial(M \setminus V_1)$ and $\partial(M \setminus V_2)$. If $A$ has a boundary-parallel dividing curve, then the twisting numbers can be increased up to $n_1 = 0$, $n_2 = n_3 = -1$ and moreover there is a regular fibre with twisting number zero.

**Proof.** If there is a boundary-parallel dividing curve, then $A$ carries a bypass on each side after perturbing its characteristic foliation. Using these bypasses, we can further increase $n_1$ and $n_2$ up to $-1$. By the Imbalance Principle, we can find one more bypass in an annulus between $\partial(M \setminus V_1)$ and $\partial(M \setminus V_2)$ on the side of $\partial(M \setminus V_1)$. This bypass increases the twisting number $n_1$ up to $0$. The slope of $-\partial(M \setminus V_1)$ is $0$, and the slope of $-\partial(M \setminus V_2)$ is $-\frac{1}{3}$, therefore two possibilities for an annulus $A$ between $\partial(M \setminus V_1)$ and $\partial(M \setminus V_2)$ are given: either $A$ carries a bypass for $\partial(M \setminus V_1)$, or not. If such a bypass exists, then all the boundary slopes can be made infinite, and we can decrease the twisting $n_3$ to $-1$. If there is no such bypass, cutting along $A$ and rounding edges yields a torus with slope $0$, which is $-2$ when measured in $\partial V_3$. In $V_3$ we find a convex torus with slope $-\frac{1}{6}$, which corresponds to infinite slope in $-\partial(M \setminus V_3)$. □
4.2.2. The case when a thickening to infinite slope exists. In this subsection we will show that there are no tight contact structures on $-\Sigma(2, 3, 11)$ with a regular fibre with twisting number zero. We suppose such a fibre exists and argue by contradiction.

Let $V_i$ be a standard neighbourhood of $F_i$. Then $M \setminus V_i$ is diffeomorphic to $\Sigma_0 \times S^1$ for a pair of pants $\Sigma_0$, and has boundary slopes $0, -\frac{1}{2}$, and $-\frac{1}{5}$. We use vertical annuli from a regular fibre with twisting number zero to thicken each $V_i$ to $V'_i$ such that $-\partial(M \setminus V'_i)$ has infinite slope. We can find another pair of pants $\Sigma_1 \subset \Sigma_0$ such that $\Sigma_1 \times S^1$ is diffeomorphic to $M \setminus V'_i$. The arguments in Lemma 4.6 apply to show that the dividing set of $\Sigma_1$ looks like in Figure 9 (C).

Note that $(\Sigma_0 \setminus \Sigma_1) \times S^1$ is the disjoint union of three thickened tori $T_i \times I$ such that $T_1 \times \{0\} = -\partial(M \setminus V_1')$ and $T_i \times \{1\} = -\partial(M \setminus V_i')$ for $i = 1, 2, 3$; see Fig. 13. $T_1 \times I$ is a basic slice, $T_2 \times I$ is a union of two basic slices $T_2 \times [0, \frac{1}{2}]$ with slopes $-\frac{1}{2}$ and $-1$, and $T_2 \times \left[\frac{1}{2}, 1\right]$ with slopes $-1$ and $\infty$, and $T_3 \times I$ is a union of five basic slices $T_3 \times \left[\frac{i}{5}, \frac{i+1}{5}\right]$ for $i = 0, \ldots, 4$ with slopes $-\frac{1}{5-i}$ and $-\frac{1}{4-i}$. We observe that the basic slices which compose the tight contact structures on the $T_i \times I$ belong to the same continued fraction block.

The contact structures of the previous Lemma are described by the three numbers $p_i$ of positive basic slices in $T_i \times I$, for $i = 1, 2, 3$.

Our strategy to find overtwisted discs is cutting (a suitable sub-manifold of) $M \setminus V_i \cup V_j$, for some $i$ and $j$, along a vertical annulus $A$ and rounding the edges of the cut open manifold to obtain a neighbourhood of the third singular fibre with boundary slope zero. Slope zero on $V_i$ corresponds to $\frac{1}{2}$ on $-\partial(M \setminus V_1)$, to $-\frac{1}{3}$ on $-\partial(M \setminus V_2)$, and to $-\frac{1}{4}$ on $-\partial(M \setminus V_3)$. We will call these slopes critical slopes.

Remark 4.12. This technique differs from the ‘disc patching’ used in section 4.1.2 of the previous example. We remark that both techniques work for each example and encourage the reader to perform both proofs with the technique not used.

The following Lemma allows us to control the dividing set of such an annulus $A$.
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Lemma 4.13. Let $\Sigma$ be a pair of pants and $\xi$ a tight contact structure on $\Sigma \times S^1$. Suppose that the boundary $-\partial(\Sigma \times S^1) = T_1 \cup T_2 \cup T_3$ consists of tori in standard form with $\# T_i = 2$ for $i = 0, 1, 2$, and slopes $s(T_0) = \frac{p_0}{q}$, $s(T_1) = \infty$, $s(T_2) = \frac{p_2}{q}$. Suppose also that there exists a pair of pants $\Sigma^* \subset \Sigma$ such that $\Sigma \times S^1$ decomposes as $\Sigma \times S^1 = \Sigma^* \times S^1 \cup (0 \times I) \cup (T_2 \times I)$, where $\xi|_{\Sigma^* \times S^1}$ is the tight contact structure with infinite boundary slopes described in Lemma 4.7, and $\xi|_{T_1 \times I}$ is minimally twisting for $i = 0, 2$.

Suppose that one of the following holds:

1. $s(T_0) = s(T_1) = -\frac{1}{q}$ and $\xi|_{T_0 \times I}$ is isotopic to $\xi|_{T_2 \times I}$.
2. $s(T_2) < 0$ and $\xi|_{T_1 \times I}$, for $i = 0, 2$, decomposes into basic slices of the same sign (i.e. with relative Euler class $\pm (q, p_i - 1)$).

Then there exists a convex annulus with Legendrian boundary consisting of vertical Legendrian rulings of $T_0$ and $T_2$ without boundary parallel dividing curves.

Proof. Let $\xi'$ be obtained in the following way from a tight contact structure on $T^2 \times I$ isotopic to $\xi|_{T_0 \times I}$. Remove a standard neighbourhood $U''$ of a vertical Legendrian ruling of a standard torus parallel to $T^2 \times \{0\}$ and contained in its invariant neighbourhood, then thicken $U''$ to $U'$ with infinite boundary slope attaching the bypasses coming from the annulus between a vertical Legendrian ruling of $U''$ and a Legendrian divide of $T^2 \times \{1\}$. By proposition 2.19, there is a solid torus $U$ between $U''$ and $U'$ with boundary slope $\frac{p_2}{q}$ because $\frac{p_2}{q} \in [-\frac{1}{q}, -\infty)$. In case (1) this operation is not necessary and we can simply take $U = U''$. In a similar way we can find a collar $C$ of $T^2 \times \{0\}$ in $T^2 \times I \setminus U$ with boundary slopes $\frac{p_i}{q}$ and $\infty$.

We identify $T^2 \times I \setminus U$ to $\Sigma \times S^1$ so that $T^2 \times \{0\}$ corresponds to $T_0$, $T^2 \times \{1\}$ corresponds to $-T_1$, $\partial U$ corresponds to $T_2$, $C$ corresponds to $T_0 \times I$, and $U'' \setminus U$ corresponds to $T_2 \times I$. A convex annulus with Legendrian boundary $A \subset (\Sigma \times S^1, \xi')$ between $T_0 = T^2 \times \{0\}$ and $T_2 = \partial U$ contained in the invariant neighbourhood of $T^2 \times \{0\}$ has no boundary parallel dividing arcs. To prove the Lemma we only need to show that $\xi$ and $\xi'$ are isotopic.

The dividing set of $\Sigma^* \subset (\Sigma \times S^1, \xi)$ cannot contain a boundary parallel dividing arc, otherwise there would be a convex torus with slope 0 in $\Sigma' \times S^1$ (see the proof of Lemma 4.10). By construction, $(\Sigma \times S^1, \xi)$ contact embeds in $T^2 \times S^1$ with a minimally twisting tight contact structure. A convex torus with slope 0 around $T_2$ would give an overtwisted disc in $T^2 \times I$, while between $T_0 \times \{1\}$ and $T_1$, both with infinite boundary slope, would contradict minimally twisting. This proves that $\xi|_{\Sigma \times S^1}$ and $\xi'|_{\Sigma \times S^1}$ induce the same dividing set on $\Sigma$, therefore they are isotopic by Lemma 4.7.

Take two vertical annuli with Legendrian boundary $A_i$, for $i = 0, 2$, between $T_i$ and $T_1$, we have

$$\langle e(\xi'), A_0 \rangle = \langle e(\xi'), A_2 \rangle$$

because $[A_0] = [A_2] + [A]$ in $H_2(\Sigma \times S^1, \partial(\Sigma \times S^1))$ and $\langle e(\xi'), A \rangle = 0$ by Lemma 2.20. Decompose $A_i = B_i \cup B'_i$, such that $B_i = A_i \cap T_i \times I$, and $B'_i = A_i \cap \Sigma \times S^1$ for $i = 0, 2$. We can suppose that all these annuli have Legendrian boundary. Since

$$\langle e(\xi'|_{\Sigma \times S^1}), B'_0 \rangle = \langle e(\xi'|_{\Sigma \times S^1}), B'_2 \rangle = 0$$

we get

$$\langle e(\xi'|_{T_0 \times I}), B_0 \rangle = \langle e(\xi'|_{T_2 \times I}), B_2 \rangle$$. 
Corollary 2.26 after a change of coordinates, we conclude \( 1 \) has slope \( T \) same as \( p \), with the same signs as the two basic slices in between the rotations.

\[
\langle e(\xi|T_i\times I), B_i \rangle = \langle e(\xi|T_i\times I), B_i \rangle
\]

for \( i = 0, 2 \) because \( \langle e(\xi|T_2\times I), B_2 \rangle = \langle e(\xi|T_0\times I), B_0 \rangle = \langle e(\xi|T_0\times I), B_0 \rangle \). Applying Corollary 2.26 after a change of coordinates, we conclude \( \xi|T_i\times I \) is isotopic to \( \xi'|T_i\times I \) for \( i = 0, 2 \), and this ends the proof of the Lemma.

**Theorem 4.14.** \( M(-\frac{1}{2}, \frac{1}{2}, \frac{2}{5}) \) carries no tight contact structure with a Legendrian regular fibre with twisting number zero.

**Proof.** For any choice of the number \( p_i \) of positive basic slices in \( T_i \times I \), for \( i = 0, 1, 2 \), we will find a convex torus in \( M \setminus \bigcup V_i \) with critical slope. Most of the possible tight contact structures fall into one of the following cases.

**Case (1)** We work between \( V_2 \) and \( V_3 \). If in \( T_3 \times I \) there are two basic slices with the same signs as the two basic slices in \( T_2 \times I \), we can arrange them so that \( T_3 \times [\frac{5}{6}, 1] \) is isotopic to \( T_2 \times I \). The manifold \( M \setminus (V_1' \cup V_2 \cup V_3 \cup (T_3 \times [0, \frac{5}{6}])) \) has boundary slopes \( \infty, -\frac{1}{2}, -\frac{3}{5} \), and by Lemma 4.13 we can find a convex annulus \( A \) between \( T_2 \times \{0\} \) and \( T_3 \times \{\frac{5}{6}\} \) whose dividing curves go from a component of the boundary to the other one. See Fig. 14. After cutting along \( A \) and rounding the edges, we obtain a torus with slope \(-\frac{1}{2}\) isotopic to \( \partial(M \setminus V_1) \). Because the critical slope for \( -\partial(M \setminus V_1) \) is \( \frac{1}{2} \), \( M \) is overtwisted. This case excludes all the candidate tight contact structures except for the ones with \( p_2 = 0, p_3 = 4, 5, p_2 = 1, p_3 = 0, 5, \) or \( p_2 = 2, p_3 = 0, 1 \).

**Case (2)** We work between \( V_1 \) and \( V_3 \). Suppose that all the three basic slices in \( T_3 \times [\frac{2}{3}, 1] \) have the same sign as \( T_1 \times I \). We decrease the twisting number \( n_3 \) to \(-1\) and take a standard neighbourhood \( V_1'' \) so that \( -\partial(M \setminus V_1'') = T_1 \times \{-1\} \) has slope \( \frac{1}{3} \). We can choose the sign of the basic slice \( T_1 \times [-1, 0] \) so that it is the same as \( T_1 \times [0, 1] \). To show this fact, embed \( V_1 \) in the standard \( S^3 \), and perform stabilisation there, see [12]. We add \(+1\) or \(-1\) to the rotation number, according to the sign of the stabilisation, and it turns out that the relative Euler class of the contact structure on \( V_1 \setminus V_2 \) calculated on a vertical annulus is the difference between the rotations.

The manifold \( M \setminus (V_1'' \cup V_2' \cup V_3 \cup (T_3 \times [0, \frac{2}{3}])) \) has boundary slopes \( \frac{1}{3}, -\frac{1}{3}, \infty \), and by Lemma 4.13 we can find a convex annulus \( A \) between \( T_1 \times \{-1\} \) and
Figure 15. Case 2. The tori without indication of the slope are meant to have infinite slope.

$T_3 \times \{\frac{2}{3}\}$ as in figure 16 without boundary-parallel arcs, and by cutting along $A$ and rounding the edges we obtain a torus with slope $\frac{1}{3}$ isotopic to $\partial (M \setminus V_2)$. This gives an overtwisted disk because $-\frac{1}{3}$ is the critical slope for $-\partial (M \setminus V_2)$. This case excludes the contact structures with $p_1 = 0$ and $p_3 \leq 2$, or the contact structures with $p_1 = 1$ and $p_3 \geq 3$.

**Case (3)** Now we work between $V_1$ and $V_2$. Suppose that the basic slices which compose $T_2 \times I$ have the same sign as $T_1 \times I$. We decrease the twisting number of the singular fibres $F_1$ and $F_2$ to $-2$ and take standard neighbourhoods $V_i''$, for $i = 1, 2$, so that $-\partial (M \setminus V_i'') = T_1 \times \{-1\}$ has slope $\frac{2}{3}$, $-\partial (M \setminus V_2'') = T_2 \times \{-1\}$ has slope $-\frac{2}{3}$, and all the basic slices in $T_1 \times [-1, 1]$ and $T_2 \times [-1, 1]$ have the same sign. The manifold $M \setminus (V_1'' \cup V_2'' \cup V_3')$ has boundary slopes $\frac{2}{3}, -\frac{2}{3}, \infty$, therefore by Lemma 4.13 we can find a convex annulus $A \subset M \setminus V_3 \cup (T_3 \times I)$ between $T_1 \times \{-1\}$ and $T_2 \times \{-1\}$ without boundary-parallel dividing curves. See figure 16. Then, after cutting along $A$ and rounding the edges, we find a torus $V_3''$ isotopic to $-\partial (M \setminus V_3)$ with slope $\frac{1}{3}$. The thickened torus between $-\partial (M \setminus V_3)$ and $-\partial (M \setminus V_3'')$ has both boundary slopes $-\frac{1}{3}$, and contains a torus with infinite slope, hence, by the classification Theorem for thickened tori, it contains an intermediate convex torus for each slope. In particular, it contains a convex torus with slope $-\frac{2}{3}$, which is the critical slope for $-\partial (M \setminus V_3)$, therefore it gives an overtwisted disk around $F_3$. This case excludes the contact structures with $p_1 = 0$, $p_2 = 0$ and $p_3 = 1$, $p_2 = 2$.

We can exclude the remaining candidate tight contact structures with $p_1 = 1$, $p_2 = 1$, $p_3 = 0$, or $p_1 = 0$, $p_2 = 1$, $p_3 = 5$ in the following way. We go back to $V_2$ and $V_3$ and arrange the basic slices in $T_2 \times I$ such that $T_2 \times \{\frac{1}{3}, 1\}$ has the same sign as the basic slices in $T_3 \times I$ and consider an annulus $A$ between $T_3 \times \{\frac{1}{3}\}$ and $T_3 \times \{\frac{2}{3}\}$ without boundary parallel dividing curves. Cutting along $A$ and rounding the edges we obtain a torus with slope $-1$ isotopic to $-\partial (M \setminus V_1)$. This torus has slope $1$ in the basis of $\partial V_1$, which corresponds to increasing the twisting number of the singular fibre $F_1$ up to $n_1 = 1$. Now we can decrease $n_1$ again choosing the sign of $T_1 \times I$, showing that the contact structures with $p_1 = 1$, $p_2 = 1$, $p_3 = 0$, or $p_1 = 0$, $p_2 = 1$, $p_3 = 5$ are isotopic to the contact structures with $p_1 = 0$, $p_2 = 1$, $p_3 = 0$, or $p_1 = 1$, $p_2 = 1$, $p_3 = 5$ respectively, which have already been shown to be overtwisted in case (2).
4.2.3. The case when no thickening to infinite slope exists. It remains to analyse only the case when the convex annulus $A$ between $\partial(M \setminus V_1)$ and $\partial(M \setminus V_2)$ of Lemma 4.11 carries no bypasses. Since $-\partial(M \setminus V_1)$ has slope $\frac{2}{5}$ and $-\partial(M \setminus V_2)$ has slope $-\frac{2}{5}$, the dividing set of $A$ consists of 10 dividing arcs going from one side of the annulus to the other. We say that an arc in $A$ is horizontal if its algebraic intersection with $\Sigma \subset M \cup V_i \cong \Sigma \times S^1$ is zero.

**Proposition 4.15.** The manifold $M(-\frac{1}{2}, \frac{1}{3}, \frac{2}{11})$ carries at most one tight contact structure.

**Proof.** Let $\phi_t$ be an isotopy of $A$ such that $\phi_0 = id$ and $\phi_1(\Gamma_A)$ is a collection of horizontal arcs, and extend it to an isotopy on the whole $M$. We can consider $\phi_1^*\xi$ instead of $\xi$ and suppose without loss of generality that the dividing arcs of $A$ are horizontal.

Cutting $M \setminus V_1 \cup V_2$ along $A$ and rounding the edges yields a solid torus with boundary slope $-\frac{1}{5}$, calculated with respect to the basis of $-\partial(M \setminus V_3)$, which corresponds to slope $-\frac{1}{11}$ in the basis of $\partial V_3$. By Proposition 4.11, there exists only one tight contact structure up to isotopy on $V_3$ with this boundary condition, thus the candidate tight contact structure on $M(-\frac{1}{2}, \frac{1}{3}, \frac{2}{11})$ is unique up to isotopy. □

4.2.4. Construction of the tight contact structure. We will use Kirby calculus to show that this manifold can be represented as Legendrian surgery on a link in $S^3$. This will prove that $M$ has at least one holomorphically fillable, and therefore tight, contact structure.

The structure of Seifert fibration of the manifold $M(-\frac{1}{2}, \frac{1}{3}, \frac{2}{11})$ gives the surgery presentation shown by (a) in Fig. 17. By a slam-dunk on the $(2)$ component of the link, we obtain the link (b). Next, we perform a Rolfsen twist on the $(-\frac{1}{2})$ component to obtain the link (c). After another Rolfsen twist around the $(-1)$ component, we obtain diagram (d) in Fig. 17 and finally, after one inverse slam-dunk, we obtain diagram (e). This link can be made Legendrian with the Thurston-Bennequin invariant of each component one more than the surgery coefficient, see (f) for a Legendrian realization. This proves that $M(-\frac{1}{2}, \frac{1}{3}, \frac{2}{11})$ carries at least one tight contact structure, and therefore ends the classification.
5. What we can do, and what we can't do

In this section we give a quick overview of similar results on the classification of tight contact structures on Seifert manifolds we are able to do with the techniques exposed in this paper, and point out some problems we have met. In addition to the Poincaré homology sphere with reversed orientation studied by Etnyre and Honda in [14] and the two examples treated in this paper, we are able to give a classification in several other cases. Among them there are the manifold $M(\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2})$, where the existence of a unique tight contact structure has been proven by S. Schönenberger; see [34], the Poincaré homology sphere $M(\frac{1}{3}, -\frac{5}{3}, -\frac{1}{3})$, and the manifold $M(-\frac{1}{2}, -\frac{3}{4}, -\frac{1}{3})$. Moreover P. Ghiggini in his Ph.D. thesis is working towards the classification of tight contact structures on Seifert manifolds over the torus.

We would like to point out that in the examples computed in this paper, like in most of the other results mentioned above, the tight contact structures one obtains happen to be holomorphically fillable. In fact, the presence of possibly tight contact structures for which no Stein filling is known is a major source of difficulties in achieving a complete classification. The problems one has to face when dealing with non-holomorphically fillable contact structures are twofold.

The first problem is the proof of tightness, in fact we have very few techniques to do that, the main ones being holomorphic or symplectic fillability and the gluing techniques developed by Colin [2] and Honda [23]. In particular, the gluing Theorems require the presence of incompressible surfaces, which do not exist in manifolds whose fundamental group is finite, such as small Seifert manifolds. An example is $M(-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{3})$, which has been proved by Lisca to be not fillable in any sense, and where only an upper bound for the number of tight contact structures is known.
The second problem is distinguishing the tight contact structures. For Stein fillable contact structures this is made easy by a result from Seiberg-Witten theory due to Lisca and Matić \cite{31} and Kronheimer and Mrowka \cite{28}, which gives necessary conditions for holomorphically fillable contact structures to be isotopic. Another way to distinguish contact structures is through their homotopy classification as 2-planes fields given by Gompf \cite{19} in terms of algebraic topological invariants. An example of what happens outside the range of applicability of both methods is given by the family of manifolds $M(-\frac{1}{2}, \frac{1}{3}, \frac{2}{n-1})$ for $n > 3$, where we are able to give both an upper and a lower bound on the number of tight contact structures, but the problem of determining whether some of them are isotopic or not remains open.

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