ALBERT ALGEBRAS AND CONSTRUCTION OF THE FINITE SIMPLE GROUPS $F_4(q)$, $E_6(q)$ AND $^2E_6(q)$ AND THEIR GENERIC COVERS

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Abstract. We give a uniform construction of the finite simple groups $E_6(q)$, $F_4(q)$ and $^2E_6(q)$, which does not require any special treatment for characteristics 2 or 3, and in particular avoids any mention of quadratic Jordan algebras. Although almost all the ingredients can already be found scattered through research papers spanning more than a century, a coherent, self-contained, account is hard to find in the literature.

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1. Introduction

The construction of the finite simple groups $E_6(q)$ and their triple covers (which exist whenever $q \equiv 1 \text{ mod } 3$) goes back over 100 years to the work of Dickson [9, 10]. This work has been, perhaps unjustly, somewhat neglected following Chevalley’s uniform construction in 1955 of what are now called Chevalley groups [6], which include five of the ten families of exceptional groups of Lie type, in particular $E_6(q)$. This is in spite of the fact that [6] constructs only the simple groups, and not their generic covers. Moreover, the representation is on the Lie algebra, which has dimension 78, as opposed to the smallest representation, which has dimension 27.

The other major breakthrough since Dickson is the discovery of the exceptional Jordan algebra (or Albert algebra), which (in the real case) was discovered by physicists in the 1930s as a by-product of an unsuccessful attempt to find an algebraic underpinning for quantum mechanics [10]. This 27-dimensional algebra consists of $3 \times 3$ Hermitian matrices over Cayley numbers, with multiplication $X \circ Y = \frac{1}{4}(XY + YX)$. Freudenthal [12] showed that $E_6$ is the stabiliser of the ‘determinant’, a certain cubic form defined on this space. Seligman showed that the automorphism group of a split Jordan algebra over any field $F$ is isomorphic to the Chevalley group $F_4(F)$. Jacobson [13, 14, 15] studied this construction of $F_4$ in detail and generalized the
construction of $E_6$ to arbitrary fields of characteristic not 2 or 3. By this stage it must have been implicit that the determinant is essentially the same as Dickson’s cubic form, although Jacobson does not refer to Dickson, and I have not found an explicit identification in the literature earlier than [17]. Moreover, fields of characteristic 2 and 3 are still problematic in the Jordan algebra context, although they were no obstacle to Dickson.

Chevalley and Schafer [7] showed that the algebra of derivations of the real Albert algebra is a Lie algebra of type $F_4$, and also showed how to extend this to $E_6$ by adjoining right-multiplications by matrices with trace 0. Corresponding descriptions of the groups of automorphisms, generated by maps $X \mapsto M^T M$ for certain $3 \times 3$ matrices $M$ over complex subfields of the Cayley numbers, are given by Jacobson [15], who attributes them to Freudenthal, in the revised Russian translation of [12]. See also [11], and the 1985 reprint of [12].

It was only in the late 1980s, when the maximal subgroup problem came to prominence, that there was renewed interest in Dickson’s work. Of particular note are Magaard’s unpublished thesis [17] on maximal subgroups of $F_4(q)$ in characteristic at least 5, and the series of papers by Aschbacher [1, 2, 3, 4, 5] on maximal subgroups of $E_6(q)$. In these papers, the 27-dimensional representation of the generic cover reveals much more structure than the 78-dimensional representation on the Lie algebra, and leads to strong restrictions on the shape of a maximal subgroup. However, the fact that Aschbacher apparently decided not to attempt to get a complete list of maximal subgroups means that there is still a need for a modern version of Dickson’s construction, to provide a starting point for investigation of this and other problems. It is our aim in this paper to develop this theory in a characteristic-free way, and in particular to remove the restriction to characteristic not 2 or 3. The main achievement is a relatively straightforward derivation of the group order, which is a notoriously difficult problem from the Lie-theoretic point of view.

2. The real exceptional Jordan algebra

First we recall the definition and basic properties of the real Albert algebra (or exceptional Jordan algebra), $J = \mathbb{J}_R$. It consists of $3 \times 3$ Hermitian matrices over the Cayley numbers (also known as octonions). We write

\[ (a, b, c \mid A, B, C) = \begin{pmatrix} a & C & B \\ C & b & A \\ B & A & c \end{pmatrix}. \]

(1)

The Jordan product $X \circ Y$ of two such matrices is $\frac{1}{2}(XY + YX)$, in terms of the ordinary matrix product $XY$. It can be readily checked that the algebra is closed under this multiplication. Moreover, $X \circ X = XX$, with the ordinary matrix product, so we shall write $X^2 = X \circ X$. Also, by commutativity we have

\[ (X \circ X) \circ X = X \circ (X \circ X), \]

so we write $X^3 = X \circ X \circ X$ (but note that we cannot write this as $XXX$, since it is not necessarily the case that $X(XX) = (XX)X$).

Now by explicit computation we can verify that any matrix $X = (a, b, c \mid A, B, C)$ in the exceptional Jordan algebra satisfies a form of the Cayley–Hamilton Theorem, specifically

\[ X^3 = \text{Tr}(X).X^2 + Q(X).X + \det(X).I \]

where the determinant $\det$ and the quadratic form $Q$ are defined by

\[ Q(X) = \frac{1}{2} (\text{Tr}(X^2) \quad - \quad \text{Tr}(X)^2) \]

\[ = A\overline{A} + B\overline{B} + C\overline{C} - ab - ac - bc \]

(2)

\[ \det(X) = abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + (AB)C + (BC)A. \]

(3)
It follows that any automorphism of the algebra preserves the trace \( \text{Tr}(X) \), the standard norm \( N(X) = \text{Tr}(X^2) \), and the determinant. Moreover, taking traces in the Cayley–Hamilton Theorem and re-arranging gives

\[
\text{det}(X) = \frac{1}{3} \text{Tr}(X^3) - \frac{1}{2} \text{Tr}(X^2) \text{Tr}(X) + \frac{1}{6} \text{Tr}(X)^3.
\]

Conversely, if the trace, the norm and the determinant are all preserved, then the multiplication can be recovered as follows. Polarizing the norm by

\[
2b(X, Y) = N(X + Y) - N(X) - N(Y)
\]

(5)
gives an inner product \( b \). Similarly, we may polarize the cubic form \( \text{Tr}(X^3) \) to obtain a symmetric trilinear form \( t \) given by

\[
24t(X, Y, Z) = \text{Tr}((X + Y + Z)^3) + \text{Tr}((X - Y - Z)^3) + \text{Tr}((Y - X - Z)^3) + \text{Tr}((Z - X - Y)^3)
\]

(6)
Now by explicit computation it can be checked that

\[
\text{Tr}((X \circ Y) \circ Z) = \text{Tr}(X \circ (Y \circ Z))
\]

(7)
and it then follows that

\[
t(X, Y, Z) = \text{Tr}((X \circ Y) \circ Z) = b(X \circ Y, Z).
\]

(8)
Since the norm \( N \) is positive-definite, \( b \) is non-singular, and therefore knowledge of the inner products \( b(X \circ Y, Z) \) as \( Z \) runs over a basis determines \( X \circ Y \) uniquely.

The (compact real form of the) Lie group \( F_4 \) may be defined as the automorphism group of the algebra \( \mathbb{J} \), although of course this was not the original definition. Similarly, a particular group of type \( E_6 \) is the group of linear maps which preserve the determinant. (This real form of \( E_6 \) is neither split nor compact.) We may alternatively define \( F_4 \) as the stabilizer of the identity matrix in \( E_6 \), since if the determinant is preserved, and the identity matrix is fixed, then the trace of \( X \) is \( t(I, I, X) \), and the norm of \( X \) is \( t(I, X, X) \), so these are also preserved.

If \( M \) is any \( 3 \times 3 \) matrix written over (any) complex subalgebra of the (real) octonions, then the operation \( X \mapsto \overline{M}^t XM \) makes sense, because each entry in \( \overline{M}^t XM \) is a sum of terms of the form \( m_1 x m_2 \), where \( m_1 \) and \( m_2 \) lie in this copy of the complex numbers, and so \( m_1(xm_2) = (m_1 x)m_2 \). It is clear by restricting to complex matrices \( X \) that such an operation can only preserve the determinant if \( |\det M| = 1 \). Conversely, we use the fact that any complex matrix of determinant \( \pm 1 \) is (plus or minus) a product of fundamental transvections, and check explicitly that the fundamental transvections preserve the determinant (see Lemma [3]). Thus any complex matrix of determinant \( \pm 1 \) preserves the determinant. On the other hand, if \( u \overline{u} = 1 \) but \( u \neq \pm 1 \), then it is easy to produce examples to show that \( \text{diag}(u, 1, 1) \) does not preserve the determinant. Hence the same is true for any matrix of determinant \( u \). Therefore a complex matrix \( M \) preserves the determinant if and only if \( \det M = \pm 1 \). Negating \( M \) if necessary, we may assume \( \det M = 1 \).

Finally, in order for \( M \) to preserve the identity element of the algebra, and hence to lie in \( F_4 \), it is necessary and sufficient to have the extra condition \( \overline{M}^t M = I \). It is shown in [11] that the compact real form of \( F_4 \) is generated by such elements.

3. Split octonions and the Dickson–Freudenthal determinant

Much the same constructions work over finite fields, except that there are obvious difficulties in characteristics 2 and 3 caused by dividing by 2 or 3. To overcome these difficulties we have to be careful to choose the most useful form of each definition from the various no-longer-equivalent versions.

For example, the usual ‘compact’ version of the octonions does not work in characteristic 2, so we use instead the ‘split’ version, which works over any field.
See for example [20 Section 4.3.3] for the equivalence of the two versions over finite fields of odd characteristic.

**Definition 1.** If $F$ is any field, the split octonion algebra over $F$ is an 8-dimensional vector space $\mathbb{O} = \mathbb{O}_F$ over $F$, with basis $\{e_i \mid i \in \pm I\}$, where $I = \{0, 1, \omega, \bar{\omega}\}$ and $\pm I = \{\pm0, \pm1, \pm\omega, \pm\bar{\omega}\}$, and bilinear multiplication given by

\[
\begin{align*}
(1) & \quad e_1 e_\omega = -e_\omega e_1 = e_{-\omega}; \\
(2) & \quad e_1 e_0 = e_{-0} e_1 = e_1; \\
(3) & \quad e_{-1} e_1 = -e_0 \quad \text{and} \quad e_0 e_0 = e_0;
\end{align*}
\]

and images under negating all suffices (including 0), and multiplying all suffices by $\omega$, where $\omega^2 = \bar{\omega}$ and $\omega \bar{\omega} = 1$. All other products of basis vectors are 0.

Thus $e_{\pm0}$ are orthogonal idempotents, and $e_0 + e_{-0} = 1$. This is essentially the same definition as given in (4.37) of [20], but with the basis vectors $x_1, \ldots, x_8$ in [20] corresponding respectively to $e_{-1}, e_0, \omega, e_0, e_{-0}, e_{-\omega}, e_{-\bar{\omega}}, e_1$. In this form of the octonions it no longer makes sense to talk about the ‘real part’ of $\sum_{i \in \pm I} \lambda_i e_i$ and we define instead the **trace** by

\[
\text{Tr}(\sum_{i \in \pm I} \lambda_i e_i) = \lambda_0 + \lambda_{-0}.
\]

Similarly the anti-automorphism $x \mapsto \bar{x}$ of the octonions now takes the form

\[
e_0 \leftrightarrow e_{-0}, e_i \mapsto -e_i (i \neq \pm0).
\]

Note that this anti-automorphism reverses the order of multiplication, in the sense that $\bar{xy} = y \bar{x}$, as is easily checked directly from the definition. Moreover, we see that $\text{Tr}(x) = x + \bar{x}$. It is easy to compute the norm $N(x) = x \bar{x}$ of an arbitrary element to be

\[
N(\sum_{i \in \pm I} \lambda_i e_i) = \sum_{i \in I} \lambda_i \lambda_{-i}.
\]

This norm can be polarized to obtain an inner product $B$ by

\[
B(x, y) = N(x + y) - N(x) - N(y).
\]

It is easy to see that $\mathbb{O}_F$ is non-commutative and non-associative, so that in general $x(yz) \neq (xy)z$. However, we do have the following.

**Lemma 1.** If $x, y, z \in \mathbb{O}_F$, then $\text{Tr}(x(yz)) = \text{Tr}((xy)z)$.

**Proof.** Since both sides are trilinear, it suffices to check on a basis. If $i + j + k \neq \pm0$, then

\[
\text{Tr}(e_i (e_j e_k)) = 0 = \text{Tr}((e_i e_j) e_k).
\]

Otherwise we show that in fact $e_i (e_j e_k) = (e_i e_j) e_k$, as follows. Using the symmetry we find there are just 8 cases to check, of which the following are representative sample:

\[
\begin{align*}
e_0 (e_1 e_{-1}) &= -e_0 e_{-0} = 0 = (e_0 e_1) e_{-1} \\
e_0 (e_{-1} e_1) &= -e_0 e_0 = -e_0 = e_{-1} e_1 = (e_0 e_{-1}) e_1 \\
e_1 (e_0 e_{-1}) &= e_1 e_{-1} = (e_1 e_0) e_{-1} \\
e_1 (e_\omega e_\bar{\omega}) &= e_1 e_{-1} = -e_{-0} = e_{-\omega} e_{\bar{\omega}} = (e_1 e_\omega) e_{\bar{\omega}}
\end{align*}
\]

Since $\text{Tr}(xy) = \text{Tr}(yx)$, it follows that $\text{Tr}(xyz)$ is independent of bracketing, and cyclic permutations of $x, y, z$. However, in general we have

\[
\text{Tr}(xyz) \neq \text{Tr}(xzy).
\]

It is also worth noting that the norm is multiplicative.

**Lemma 2.** If $x, y \in \mathbb{O}_F$, then $N(xy) = N(x) N(y)$.
Proof. We multiply the basis vectors on the left by an arbitrary element of \( \mathbb{O}_F \), say

\[
x = \sum_{i \in \pm 1} \lambda_i e_i,
\]

and obtain

\[
\begin{align*}
x e_0 &= \lambda_0 e_0 + \lambda_1 e_1 + \lambda_\omega e_\omega + \lambda_\bar{\omega} e_{\bar{\omega}} \\
x e_1 &= -\lambda_{-1} e_0 + \lambda_{-0} e_1 + \lambda_\bar{\omega} e_{\bar{\omega}} - \lambda_\omega e_\omega \\
x e_\omega &= -\lambda_{-\omega} e_0 - \lambda_0 e_1 + \lambda_\omega e_\omega - \lambda_{-1} e_{-\omega} \\
x e_{\bar{\omega}} &= -\lambda_{-\bar{\omega}} e_0 + \lambda_\omega e_{-\omega} - \lambda_{-1} e_\omega + \lambda_{-0} e_{\bar{\omega}}
\end{align*}
\]

and the corresponding equations with all subscripts negated, from which it is easy to see that \( N(x e_i) = 0 \), and the inner products of distinct basis vectors are all multiplied by \( N(x) = \sum_{i \in \pm 1} \lambda_i \lambda_{-i} \). Hence the result follows by linearity.

The final basic property of the split octonions is the Moufang law which comes in three equivalent versions.

**Lemma 3.** For all \( x, y, z, \in \mathbb{O} \), the following identities hold:

\[
\begin{align*}
x(yz)x &= (xy)(zx), \\
x(yzx) &= ((xy)z)y, \\
(xy)x &= x(y(xz)).
\end{align*}
\]

Proof. As we shall not need these identities in the rest of the paper, we merely sketch the proof of the first one. Let \( x = \sum_{i \in \pm 1} \lambda_i e_i \). By linearity we need only check the identity for \( y, z \) in the basis \( \{ e_i \mid i \in \pm 1 \} \), and by symmetry we may assume \( y = e_0 \) or \( e_1 \). We first compute the following:

\[
\begin{align*}
x e_0 &= \lambda_0 e_0 + \lambda_1 e_1 + \lambda_\omega e_\omega + \lambda_\bar{\omega} e_{\bar{\omega}} \\
e_0 x &= \lambda_0 e_0 + \lambda_{-1} e_{-1} + \lambda_{-\omega} e_{-\omega} + \lambda_{-\bar{\omega}} e_{-\bar{\omega}} \\
(xe_0)x = xe_0 x &= \lambda_0^2 e_0 - (\lambda_1 \lambda_{-1} + \lambda_\omega \lambda_{-\omega} + \lambda_\bar{\omega} \lambda_{-\bar{\omega}}) e_0 \\
+ &\lambda_0 (\lambda_1 e_1 + \lambda_\omega e_\omega + \lambda_\bar{\omega} e_{\bar{\omega}} + \lambda_{-1} e_{-1} + \lambda_{-\omega} e_{-\omega} + \lambda_{-\bar{\omega}} e_{-\bar{\omega}}) \\
x e_1 &= -\lambda_{-1} e_0 + \lambda_{-0} e_1 + \lambda_\bar{\omega} e_{\bar{\omega}} - \lambda_\omega e_\omega \\
e_1 x &= -\lambda_{-1} e_0 + \lambda_{-0} e_1 - \lambda_\bar{\omega} e_{\bar{\omega}} + \lambda_\omega e_\omega \\
(xe_1)x = xe_1 x &= \lambda_1^2 e_1 - (\lambda_0 \lambda_{-0} + \lambda_\omega \lambda_{-\omega} + \lambda_\bar{\omega} \lambda_{-\bar{\omega}}) e_1 \\
+ &\lambda_{-1} (\lambda_0 e_0 + \lambda_{-\omega} e_{-\omega} + \lambda_{-\bar{\omega}} e_{-\bar{\omega}} - \lambda_\omega e_\omega - \lambda_\bar{\omega} e_{\bar{\omega}})
\end{align*}
\]

(13)

In particular, we deduce by linearity that \( (xy)x = x(yx) \) for all \( x, y \in \mathbb{O} \). (Similar calculations show that \( x(xy) = (xy)x \) and \( (yx)x = (yx)x \).) We now have to calculate the left-hand side of the identity in the following cases, and check equality with the right-hand side, which is either zero or given above:

\[
\begin{align*}
y = e_0, z = e_0, & \quad yz = e_0 \\
y = e_0, z = e_{-0}, & \quad yz = 0 \\
y = e_0, z = e_1, & \quad yz = 0 \\
y = e_{-0}, z = e_1, & \quad yz = e_1 \\
y = e_1, z = e_1, & \quad yz = 0 \\
y = e_1, z = e_{-\omega}, & \quad yz = 0 \\
y = e_{-1}, z = e_1, & \quad yz = e_0 \\
y = e_{-\omega}, z = e_{-\omega}, & \quad yz = e_1
\end{align*}
\]

(14)

These calculations are left to the reader.

**Definition 2.** Let \( \mathbb{J} = \mathbb{J}_F \) be the set of \( 3 \times 3 \) Hermitian matrices with entries in \( \mathbb{O}_F \), that is matrices

\[
X = (a, b, c \mid A, B, C) = \begin{pmatrix} a & C & \bar{B} \\ C & b & A \\ \bar{B} & A & c \end{pmatrix}
\]
with \( a = \overline{a}, \ b = \overline{b}, \ c = \overline{c}. \) The trace of \( X \) is \( \text{Tr}(X) = a + b + c, \) the norm of \( X \) is
\[
Q(X) = A\overline{A} + B\overline{B} + C\overline{C} - ab - ac - bc
\]
and the Dickson–Freudenthal determinant of \( X \) is
\[
\det(X) = abc - aA\overline{A} - bB\overline{B} - cC\overline{C} + \text{Tr}(ABC).
\]

(The definition of the determinant in [20], in (4.130) and elsewhere, is wrong.) Notice that we are not defining a Jordan product on \( J, \) so \( J \) is not a Jordan algebra.

We show next that the Dickson–Freudenthal determinant as defined here is equivalent to Dickson’s original cubic form \([9]\) in 27 variables. First define 27 variables \( a, \ b, \ c, \ A_i, \ B_i, \ C_i, \) where \( A = \sum_{i \in \pm 1} A_i e_i \) and similarly for \( B_i \) and \( C_i. \) Then we calculate the determinant as
\[
\det(X) = abc - \sum_{i \in I} (aA_i A_{-i} + bB_i B_{-i} + cC_i C_{-i})
\]
\[
+ \sum_{i+j+k = \pm 0} (\text{Tr}(e_i e_j e_k)) A_i B_j C_k,
\]
where the coefficients \( \text{Tr}(e_i e_j e_k) \) of the 32 terms in the last sum are all \( \pm 1. \) Further calculation gives
\[
\text{Tr}(e_i e_j e_k) = +1
\]
when \( (i, j, k) \) is a cyclic rotation of a multiple of \( (0, 0, 0) \) or \( (1, \bar{\omega}, \omega), \) and
\[
\text{Tr}(e_i e_j e_k) = -1
\]
for cyclic rotations of multiples of \( (1, \omega, \bar{\omega}) \) or \( (1, 0, -1). \)

Dickson’s 27 variables were called \( x_i, y_j \) and \( z_{ij} = -z_{ji}, \) where \( i, j \in \{1, 2, 3, 4, 5, 6\}, \) and the cubic form is
\[
\sum_{i,j} x_i y_j z_{ij} + \sum z_{ijkl} z_{mn}
\]
where the second sum is over all partitions \( \{i, j\}, \{k, l\}, \{m, n\} \) of \( \{1, 2, 3, 4, 5, 6\}, \) ordered so that \( iklmn \) is an even permutation of 123456.

To translate between the two cubic forms, let \( a = z_{13}, \ b = z_{26}, \ c = z_{45}, \) and the other 24 variables as follows.
\[
\begin{array}{cccc|ccc}
\hline
i & A_i & B_i & C_i & A_{-i} & B_{-i} & C_{-i} \\
\hline
0 & z_{25} & z_{43} & z_{16} & z_{46} & z_{15} & z_{23} \\
1 & y_3 & y_6 & y_5 & x_1 & x_2 & x_4 \\
\omega & x_3 & x_6 & x_5 & -y_1 & -y_2 & -y_4 \\
\bar{\omega} & z_{56} & z_{35} & z_{63} & z_{42} & z_{14} & z_{21} \\
\hline
\end{array}
\]

Observe that the symmetry \( (a, b, c)(A, B, C) \) corresponds to \( (1, 2, 4)(3, 6, 5), \) so that we only need to check 17 of the 45 terms. The (easy) calculations are omitted—in fact the determinant is exactly the negative of Dickson’s cubic form.

Hence we may interpret the determinant of the split Jordan algebra as a cubic form over any field, and then follow Dickson and define \( SE_0(q) \) for any \( q \) to be the group of \( \mathbb{F}_q \)-linear maps which preserve this cubic form over \( \mathbb{F}_q. \) Similarly, we may define \( F_4(q) \) to be the subgroup of \( SE_0(q) \) consisting of those maps which fix the identity element. Notice in particular that we now have a definition of \( F_4(q) \) in characteristic 2 which completely avoids the need for introducing the ‘quadratic Jordan algebras’ of McCrimmon \([13].\)
4. Some elements of $E_6(q)$

In this section we write down some elements of $SE_6(q)$, which we shall later show are enough to generate the whole group. All these elements will be encoded as $3 \times 3$ matrices $M$, written over some commutative subring of $\mathbb{O}$, and acting on $X \in J$ via $X \mapsto \overline{M}^\top XM$. In fact, most of the proofs in this section also work for arbitrary octonion algebras over arbitrary fields.

First observe that the coordinate permutations, generated by

\[(a, b, c \mid A, B, C) \mapsto (c, a, b \mid C, A, B)\]

\[(a, b, c \mid A, B, C) \mapsto (a, c, b \mid A, C, B)\]

(21)

preserve the determinant. These are encoded respectively by the matrices

\[
\begin{pmatrix}
  0 & 1 & 0 \\
  0 & 0 & 1 \\
  1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & 1 & 0
\end{pmatrix}.
\]

Lemma 4. Let

\[
M_x = \begin{pmatrix}
  1 & x & 0 \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{pmatrix},
\]

for any $x \in \mathbb{O}$. If $X = (a, b, c \mid A, B, C)$ then

\[
\overline{M_x}^\top XM_x = (a, ax + b + (\overline{x}C + \overline{C}x), c \mid A + \overline{A}, B, B, ax + C),
\]

and $\det(\overline{M_x}^\top XM_x) = \det(X)$.

Proof. The calculation of $\overline{M_x}^\top XM_x$ is an easy exercise. The individual terms of the determinant are as follows:

\[
\begin{align*}
  abc & \mapsto abc + a^2cx + ac(\overline{C}x + C) \\
  -aA A & \mapsto -a\overline{A} - a(Ax) - a\overline{x}B \overline{B} \\
  -bBB & \mapsto -b\overline{B}B - ax\overline{A}B - (\overline{x}C + \overline{C}x)B \overline{B} \\
  -cCC & \mapsto -c\overline{C} - ac(\overline{x}C + \overline{C}x) - a^2cx \\
  (AB)C & \mapsto (AB)C + (\overline{B})\overline{C}C + ax\overline{A}B + a(AB)x \\
  \overline{C}.(\overline{B}.A) & \mapsto \overline{C}.(\overline{B}.A) + ax\overline{A}B + (B\overline{B})C + a\overline{x}B \overline{B}
\end{align*}
\]

(22)

and it is easy to see that all the terms on the right-hand side cancel out, except those in $\det(a, b, c \mid A, B, C)$.

Now if two matrices $M$ and $N$ both lie in $SE_6(q)$, and are both written over the same 2-dimensional subring of the octonions, then there is sufficient associativity to show that the action of $M$ followed by the action of $N$ is the same as the action of $MN$, that is

\[
\overline{(MN)}^\top X (MN) = \overline{N}^\top (\overline{M}^\top XM)N.
\]

In other words, we can multiply together the generators of $SE_6(q)$ as long as the entries stay within the same 2-dimensional subring.

In this way we obtain 48 root groups by putting $x = \lambda e_i$ (for arbitrary $\lambda \in F$ and fixed $i$) in one of the six off-diagonal positions.

Indeed, more is true. If we apply the matrices $M_x$ and $M_y$ in turn to $X$ we obtain

\[
(a, b + ax + ay \overline{B} + (\overline{x}C + \overline{C}x) + (\overline{y}(ax + C) + (ax + C)y), c \mid A + \overline{A}, B, B, ax + ay),
\]

(23)
which is the same as the image of $X$ under the action of $M_x y$. Thus the matrices $M_x$ generate an elementary abelian group of order $q^8$. Similarly, if we follow $M_x$ by

\[
\begin{pmatrix}
1 & 0 & y \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

we obtain

\[
(a, b + ax \bar{x} + \bar{x}C + \bar{x}x, c + ag \bar{y} + By + \bar{y}B | A + \bar{x}B + \bar{x}y, B + a \bar{y}, C + ax)
\]

so in fact we obtain an elementary abelian group of order $q^{16}$ in this way.

More elements may be obtained by the following computations in one of the $2 \times 2$ blocks. If $u \in \mathbb{O}$ is invertible, then we have

\[
\begin{pmatrix}
1 & u - 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & u^{-1} - 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-u & 1
\end{pmatrix} = \begin{pmatrix}
u & 0 \\
0 & u^{-1}
\end{pmatrix}.
\]

Hence the group contains the diagonal matrices

\[
M = \text{diag}(u, \bar{u}, 1) = \begin{pmatrix}
u & 0 \\
0 & u^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

where $u \in \mathbb{O}$ satisfies $u \bar{u} = 1$, which acts on $J$ as

\[(a, b, c | A, B, C) \mapsto (a, b, c | uA, Bu, uC \bar{u}).\]

By using the Moufang law one can show directly that these matrices preserve the determinant, though of course this follows from the calculations already done. Since we have $(uA)(Bu) = u(AB)u$, repeated use of the identities $\text{Tr}(AB) = \text{Tr}(BA)$ and $\text{Tr}(A(BC)) = \text{Tr}((AB)C)$ implies that

\[
\text{Tr}((uA)(Bu)(\bar{C}u)) = \text{Tr}(ABC).
\]

The other terms in the determinant are easy to deal with.

Next we analyse the group generated by these diagonal matrices. Consider the action on $C$, that is the map $C \mapsto \bar{C}u$. Since reflection in 1 is the map $x \mapsto -\bar{x}$, reflection in $u$ is the map $y \mapsto -u\bar{y}u$, and the given action is the composition of these two maps. As $u$ ranges over all octonions of norm 1, therefore, the action generated is that of $\Omega_8^+(q)$. Indeed, by using all of the diagonal matrices we can get a similar result for reflections in vectors $u$ of arbitrary norm, and hence get an action of $SO_8^+(q)$. The kernel of this action is given by $u \in \mathbb{F}_q$, and thus we have an action of a group of shape $C_{q-1} SO_8^+(q)$ on $J$.

Now extend this to the action on the 10-space of matrices of the form

\[(a, b, 0 | 0, 0, C).
\]

The elements $M_x$ and their transposes extend the action $SO_8^+(q)$ to $SO_{10}^+(q)$, preserving the norm $C \bar{C} - ab$. Again we have a kernel of order $q-1$, giving a group of shape $C_{q-1} SO_{10}^+(q)$.

5. The White Points

In order to calculate the group order we count the ‘rank 1’ matrices, otherwise known as the ‘white’ vectors. In order to obtain a construction which works also in characteristics 2 and 3, we define these purely in terms of the determinant.

**Definition 3.** For a fixed non-zero $W \in J$, the expression $\text{det}(W + X)$ is a cubic form in the variables of $X$, and has a cubic term $\text{det}(X)$, a quadratic term, a linear term, and a constant term $\text{det}(W)$.

1. If the linear term is identically zero, then $W$ is called white.
2. If the constant term $\text{det}(W)$ is non-zero, then $W$ is called black.
3. Otherwise, $W$ is called grey.
A white/grey/black point is a 1-dimensional subspace spanned by a white/grey/black vector.

By analogy with ordinary $3 \times 3$ matrices, we may think of white, grey and black matrices as having rank $1, 2, 3$ respectively. For example, $(1, 0, 0 | 0, 0, 0)$ is white because

$$\det(1 + a, b, c | A, B, C) = bc - A\overline{A} + \det(a, b, c | A, B, C)$$

has zero linear term. Similarly, $(1, 1, 1 | 0, 0, 0)$ is black because it has determinant 1. Finally, $(1, 1, 0 | 0, 0, 0)$ is grey because

$$\det(1 + a, 1 + b, c | A, B, C) = c + (a + b)c - A\overline{A} - B\overline{B} + \det(a, b, c | A, B, C)$$

has zero constant term but non-zero linear term. The terms white, grey and black were introduced by Cohen and Cooperstein [8]. Jacobson [15] uses the equivalent terms rank 1, rank 2 and rank 3. Aschbacher [1] calls them respectively singular, brilliant non-singular, and dark.

**Lemma 5.** A non-zero element $(a, b, c | A, B, C)$ of $\mathbb{F}$ is white if and only if one of the following holds:

1. at least one of the diagonal entries (say $c$) is non-zero, and $(a, b, c | A, B, C)$ is of the form $c_\top v$, where $v = (x, y, 1) = (B/c, A/c, 1)$, or
2. $a = b = c = 0$, $A\overline{A} = B\overline{B} = C\overline{C} = 0$ and $AB = BC = CA = 0$.

**Proof.** Suppose that $W = (a, b, c | A, B, C)$ is white, and let $X = (p, q, r | P, Q, R)$, so that the terms in $\det(W + X)$ which are linear in $p, q, r, P, Q, R$ are

$$bcp + acq + abr - A\overline{A}p - B\overline{B}q - C\overline{C}r - a(P\overline{A} + \overline{A}P) - b(Q\overline{B} + \overline{B}Q) - c(R\overline{C} + \overline{C}R) + \operatorname{Tr}(PBC + QCA + RAB)$$

(26)

This can be re-written as

$$(bc - A\overline{A})p + (ac - B\overline{B})q + (ab - C\overline{C})r + \operatorname{Tr}((BC - a\overline{A})P + (CA - b\overline{B})Q + (AB - c\overline{C})R)$$

(27)

For this to be identically zero, it is necessary and sufficient that the following equations be satisfied:

$$bc = A\overline{A},$$
$$ac = B\overline{B},$$
$$ab = C\overline{C},$$
$$BC = a\overline{A},$$
$$CA = b\overline{B},$$
$$AB = c\overline{C}.$$  

(28)

Now if any of $a, b, c$ is non-zero, say $c \neq 0$, we have

$$b = A\overline{A}/c$$
$$a = B\overline{B}/c$$
$$\overline{c} = AB/c,$$

(29)

and hence

$$\begin{pmatrix} a & C & \overline{B} \\ \overline{c} & b & A \\ B & \overline{A} & c \end{pmatrix} = \frac{1}{c} \begin{pmatrix} B \\ A \\ c \end{pmatrix} \cdot \begin{pmatrix} B & A & c \end{pmatrix}.$$  

(30)

On the other hand, if $a = b = c = 0$, then the equations reduce to

$$A\overline{A} = B\overline{B} = C\overline{C} = AB = BC = CA = 0.$$

$\square$
Theorem 1. The number of white vectors is $(q^9 - 1)(q^8 + q^4 + 1)$.

Proof. First suppose Lemma 5(i) holds, and disjoin cases according to how many of $a, b, c$ are non-zero. If all three of $a, b, c$ are non-zero, then there are $q - 1$ choices for each of $a, b, c$, and $q^7 - q^3$ choices for each of $x, y$, making

$$(q - 1)^3(q^7 - q^3)^2$$

such vectors in all. If just two of them are non-zero, say $b$ and $c$, then there are $q^7 - q^3$ choices for $y$ and $q^7 + q^4 - q^3$ choices for $x$ (any isotropic octonion, or 0), making

$$3(q - 1)^2(q^7 - q^3)(q^7 + q^4 - q^3)$$

in all. If just one of them is non-zero, then there are $q^7 + q^4 - q^3$ choices for each of $x, y$, making

$$3(q - 1)(q^7 + q^4 - q^3)^2$$

in all.

In the second case of Lemma 5 we disjoin cases according to how many of $A, B, C$ are non-zero. If all three of $A, B, C$ are non-zero, then there are

$$(q^3 - 1)(q^3 + 1) = q^7 + q^4 - q^3 - 1$$

choices for $A$, and the condition $AB = 0$ leaves $q^4 - 1$ choices for $B$. The conditions $BC = 0$ and $CA = 0$ leave $q^5 - 1$ choices for $C$, making

$$(q^4 - 1)^2(q^6 - 1)$$

in total. Similarly, if just two of $A, B, C$ are non-zero, there are

$$3(q^4 - 1)^2(q^3 + 1)$$

choices; and if just one is non-zero, there are $3(q^4 - 1)(q^3 + 1)$ choices. Adding together these six expressions gives the total $(q^9 - 1)(q^8 + q^4 + 1)$ as claimed. \qed

For clarity, let us define $\mathcal{G} = SE_6(q)$, that is the group of $\mathbb{F}_q$-linear maps which preserve the determinant, and define $G$ to be the group generated by the matrices $M_x$, their transposes and images under permutations of the three coordinates. We have shown that $G \leq \mathcal{G}$. It is our aim to show that $G = \mathcal{G}$, and deduce the order of the group from this.

It is a straightforward exercise to show that $G$ acts transitively on the set of white points. On the other hand, it is obvious from the definition that $\mathcal{G}$ also preserves this set. Hence it is sufficient to show that the stabilizer of a white point in $\mathcal{G}$ is equal to the stabilizer of a white point in $G$.

Theorem 2. The stabilizer in $G$ of a white point is at least a group of shape $q^{16}.C_{q-1}.SO_{10}^+(q)$, where in characteristic 2, we interpret $SO_{10}^+(q)$ as meaning $\Omega_{10}^+(q)$.

Proof. We consider the stabilizer of the white point spanned by $(1, 0, 0 | 0, 0, 0)$. This is invariant under an elementary abelian group of order $q^{16}$ generated by elements of the shape

$$
\begin{pmatrix}
1 & 0 & 0 \\
x & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
y & 0 & 1
\end{pmatrix}.
$$

Similarly it is invariant under the action of all diagonal matrices and

$$
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & x \\
0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & y & 1
\end{pmatrix},
$$

which together generate $C_{q-1}.SO_{10}^+(q)$. Hence we easily see a subgroup of $G$ of shape $q^{16}.C_{q-1}.SO_{10}^+(q)$ fixing this white point. \qed
Next we show that the suborbits are the same in both groups.

**Lemma 6.** Given any white point $W$, 

1. there are exactly $q(q^3+1)(q^8-1)/(q-1)$ white points $X$ such that all points in $\langle W, X \rangle$ are white.
2. there are exactly $q^6(q^4+1)(q^5-1)/(q-1)$ white points $Y$ such that $\langle W, Y \rangle$ contains only two white points.

Moreover, the stabilizer in $G$ of $W$ acts transitively on the points $X$, and transitively on the points $Y$. Hence the permutation actions of $G$ and of $\bar{G}$ on the white points each have rank 3, with the given suborbit lengths.

**Proof.** We may assume that $W$ is spanned by $(1, 0, 0 | 0, 0, 0)$.

1. Hence in the first part we are counting the remaining points spanned by a vector of shape $(a, 0, 0 | 0, B, C)$. As in the proof of Theorem 1, the conditions on $B$ and $C$ result in the number of solutions for $B$ and $C$ being 

   $$(q^4-1)^2(q^3+1) + 2(q^4-1)(q^3+1) = (q^8-1)(q^3+1).$$

   Dividing by $q-1$ for the scalars, and multiplying by $q$ for the choice of $a$, gives us the result.

2. Obviously all white points not already counted have the second property.

   The number of them is easily computed.

Transitivity is immediate using the action of the group $q^{16}.C_{q-1}.SO^+_{10}(q)$ already exhibited. \thickmuskip=2mu \medmuskip=2mu \normalmuskip=2mu \rightskip=0pt \leftskip=0pt \relax \hfill \Box

**Theorem 3.** The stabilizer in $\bar{G}$ of a white point is at most a group of shape $q^{16}.C_{q-1}.SO^+_{10}(q)$, where again we interpret $SO^+_{10}(q)$ as meaning $\Omega^+_{10}(q)$ in characteristic 2.

**Proof.** We again consider the stabilizer of the white point spanned by 

$$(1, 0, 0 | 0, 0, 0).$$

First note that this stabilizer fixes the 17-space of matrices of the form 

$$(a, 0, 0 | 0, B, C).$$

Hence it acts on the 10-dimensional quotient space. Now the trilinear form obtained by polarizing the determinant induces a bilinear form on this quotient, by substituting the original white vector as the first variable. This bilinear form is invariant up to scalar multiplication, and therefore the action of the point stabilizer on the 10-dimensional quotient can be no bigger than already given. In particular, any element of the kernel of this action maps $(0, 1, 0 | 0, 0, 0)$ to a matrix of the form $(0, 1, 0 | 0, 0, C)$, and maps $(0, 0, 1 | 0, 0, 0)$ to $(0, 0, 1 | 0, B, 0)$.

But we already have a group of order $q^{16}$ permuting these pairs of matrices regularly, so we may assume that the two white points spanned by $(0, 1, 0 | 0, 0, 0)$ and $(0, 0, 1 | 0, 0, 0)$ are fixed. Now the white points which are adjacent to both of these span the 8-space $\{(0, 0, 0 | A, 0, 0)\}$, so this 8-space is fixed. Similarly the 8-spaces $\{(0, 0, 0 | 0, B, 0)\}$ and $\{(0, 0, 0 | 0, 0, C)\}$. As the white points are just the isotropic vectors in these 8-spaces, the action on any one of them can be no more than the orthogonal group already exhibited.

Hence we may assume that our element of the kernel acts trivially on one: say on the $(0, 0, 0 | A, 0, 0)$. Now we have a large number of pairs of non-adjacent white points which are fixed, and for every one of these pairs, the 8-space of white points which are adjacent to both is also fixed. This is enough to show that the kernel of the action is no bigger than the group already exhibited. \thickmuskip=2mu \medmuskip=2mu \normalmuskip=2mu \rightskip=0pt \leftskip=0pt \relax \hfill \Box

As a consequence, we now have:
Corollary 1.

\[ |SE_6(q)| = q^{36}(q^{12} - 1)(q^9 - 1)(q^8 - 1)(q^6 - 1)(q^5 - 1)(q^2 - 1). \]

Define \( E_6(q) \) to be the quotient of \( SE_6(q) \) by any scalars it contains. Note that a scalar \( \lambda \) is in \( SE_6(q) \) if and only if \( \det(\lambda X) = \det(X) \) for all \( X \), that is if and only if \( \lambda^3 = 1 \). Hence \( SE_6(q) \) is a triple cover of \( E_6(q) \) if \( q \equiv 1 \mod 3 \), and \( SE_6(q) \cong E_6(q) \) otherwise. To prove that \( E_6(q) \) is simple, we use Iwasawa’s Lemma:

Lemma 7. If \( G \) is a perfect, primitive permutation group, and the point stabiliser has a normal abelian subgroup whose \( G \)-conjugates generate \( G \), then \( G \) is simple.

Now consider the action of \( E_6(q) \) on the white points. This action is obviously primitive and faithful. Now \( SE_6(q) \) is generated by the conjugates of \( M_x \), which lies in an abelian normal subgroup of the stabilizer \( q^{16} C_{q-1} SO^{+}_{10}(q) \) of a point. In particular, \( M_x \) lies in the derived group, so the group is perfect. Hence, by Iwasawa’s Lemma, \( E_6(q) \) is simple.

6. Building the Building

The classification of white vectors above allows us to classify the subspaces which consist entirely of white vectors.

Theorem 4. If \( W \) is a subspace of \( \mathbb{J} \) consisting entirely of white vectors (and 0), then \( W \) is taken by an element of \( SE_6(q) \) to one of the following:

\[
W_1 = \langle (1,0,0 | 0,0,0) \rangle \\
W_2 = \langle W_1, (0,0,0 | 0,e,-1) \rangle \\
W_3 = \langle W_2, (0,0,0 | 0,e,\omega,0) \rangle \\
W_4 = \langle W_3, (0,0,0 | 0,e,\omega,0) \rangle \\
W_5 = \langle W_4, (0,0,0 | 0,0,0,0) \rangle \\
W_6 = \langle W_5, (0,0,0 | 0,0,0,0) \rangle \\
\]

\[(31)\]

Proof. First observe that there is a 6-space consisting entirely of white vectors, spanned by

\[
(1,0,0 | 0,0,0), (0,0,0 | 0,0,0), (0,0,0 | 0,e,-1), (0,0,0 | 0,B,0),
\]

where \( B \in \langle e,-1,e,\omega,e,\omega,e,\omega,e,\omega,e,\omega,e,\omega,e\rangle \), which is obviously maximal. Moreover, the root elements already given act as transvections on this 6-space \( W_6 \), and generate a group which acts on it as \( SL_6(q) \). Therefore it suffices to prove that every pure white subspace is contained in an image under the group of \( W_5 \) or \( W_6 \).

We have already shown that there is a unique orbit of the group on pure white 1-spaces and 2-spaces, so we may take the latter to be spanned by \( (1,0,0 | 0,0,0) \) and \( (0,0,0 | 0,e,-1,0) \). Now all vectors which together with \( (1,0,0 | 0,0,0) \) span a pure white 2-space are of the form \( (a,0,0 | 0,B,C) \). Therefore our space contains white vectors of shape \( (0,0,0 | 0,B,C) \), where \( B \) lies in some totally isotropic subspace of the octonions, which, using the action of the orthogonal group, may be taken to be one of

\[
\langle e,-1,e,\omega,e,\omega,e\rangle, \langle e,-1,e,\omega,e,\omega,e\rangle, \langle e,-1,e,\omega,e,\omega,e\rangle, \langle e,-1,e,\omega,e,\omega,e\rangle.
\]

Then \( C \) lies in the corresponding annihilator \( \langle e,-1,e,\omega \rangle \) (in the first case) or \( \langle e,-1 \rangle \) (in the second and last cases) or 0 (in the third case). All of these contain at least a 4-space in common with \( W_6 \), and by transitivity on these 4-spaces, we see that there is just one more orbit on maximal white subspaces, with representative the 5-space spanned by \( (1,0,0 | 0,0,0) \) and \( (0,0,0 | 0,B,0) \) with \( B \in \langle e,-1,e,\omega,e,\omega,e \rangle \). Since any pure white 4-space is contained in a unique pure white 6-space, the result follows. \( \square \)
By adjoining appropriate root groups to the subgroup of \(2^2.P\Omega_6^+(q).S_3\) which fixes \(W_i\) or \(W'_i\), it is easy to obtain generators for the stabilizers. For \(i = 1, 2, 3, 5, 6\), these turn out to be five of the six maximal parabolic subgroups. The other maximal parabolic subgroup fixes the 10-space \(W_{10}\) defined by adjoining to \(W_5\) the 5-space spanned by \((0, 1, 0 \mid 0, B, 0)\) with \(B \in (e_{-0}, e_{-\bar{\omega}}, e_{-\bar{\bar{\omega}}}, e_1)\). In other words,
\[
W_{10} = \{(a, 0, 0 \mid 0, B, C)\}.
\]
Notice that \(W_{10}\) has a quadratic form defined on it, which is invariant up to scalar multiplication. With respect to this form, the white points are isotropic, while the non-isotropic points are grey.

7. Duality and the subgroup \(F_4(q)\)

There is a second, ‘dual’, action of \(SE_6(q)\) on the set \(J\) of \(3 \times 3\) octonion Hermitian matrices, whereby a matrix \(M\) acts as
\[
M : X \mapsto M^{-1}X(\overline{M^\top})^{-1}.
\]
To see that this is an action, we need to show that any relation between the original actions of \(M\) by
\[
M : X \mapsto \overline{M^\top}XM
\]
also holds for \((\overline{M^\top})^{-1}\). But by symmetry, any word in the original (right-)actions of the \(M\) corresponds to the reverse word in the (left-)actions of the corresponding \(\overline{M^\top}\). In particular, given any relator satisfied by the actions of matrices \(M_i\), the reverse relator is satisfied by the corresponding \(\overline{M_i^\top}\). Hence the original relator is satisfied by \((\overline{M_i^\top})^{-1}\).

This implies that the map \(M \mapsto (\overline{M^\top})^{-1}\) on the given generators of \(SE_6(q)\) induces an automorphism of \(SE_6(q)\). It is easy to see that it is not inner, so we shall call it duality.

Now if \(M\) is a generator of \(SE_6(q)\) fixed by this duality automorphism, then \(M = (\overline{M^\top})^{-1}\), so \(\overline{M^\top}IM = I\). In other words, \(M\) fixes the identity element of \(J\), so \(M\) lies in \(F_4(q)\).

For example, the diagonal elements \(\text{diag}(u, \overline{\omega}, 1)\) with \(u\overline{\omega} = 1\) satisfy this condition. So do the elements
\[
\begin{pmatrix}
1 & x & 0 \\
-x & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]
provided \(x\overline{x} = 0\). These elements are in \(SE_6(q)\) because
\[
\begin{pmatrix}
1 & x \\
-x & 1
\end{pmatrix}
= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix}.
\]
Hence by putting \(x = \lambda e_i\) we obtain 8 root groups, becoming 24 when we allow coordinate permutations as well. (The other 24 root groups lie in the subgroup generated by the diagonal matrices.) The case \(x = \lambda e_0\) realises precisely the short root element displayed in (4.105) of [20].

As noted in [20], the normalizer of a maximal torus can be found inside a subgroup of shape \(2^2.P\Omega_8^+(q).S_3\). If we take the diagonal elements \(\text{diag}(u, \overline{\omega}, 1)\) and \(\text{diag}(1, u, \overline{\omega})\) with \(u = \lambda e_{-1} + \lambda^{-1}e_i\), and adjoin the coordinate permutations, then we obtain the normalizer of a maximal split torus.

The long root elements also lie in \(2^2.P\Omega_8^+(q)\). For example we may take the product of the three group elements given by
\[
\begin{align*}
\text{diag}(1 + e_{-1}, 1 - e_{-1}, 1), \\
\text{diag}(1 - \lambda e_{-1}, 1 + \lambda e_{-1}, 1), \\
\text{diag}(1 - e_{-1} + \lambda e_{\bar{\omega}}, 1 + e_{-1} - \lambda e_{\bar{\bar{\omega}}}, 1)
\end{align*}
\]
(32)
to give the long root element displayed in (4.104) of \[20\].

One way to compute the order of $F_4(q)$ is to count the primitive idempotents. The official definition in terms of the Jordan algebra is that they are idempotents $X$ (in the sense that $X \circ X = X$) with trace 1. However, this definition does not necessarily work in characteristic 2 or 3, and they may alternatively be defined as white vectors with trace 1, so that it is not necessary to treat these characteristics differently.

**Definition 4.** An element of $\mathbb{J}$ is called a primitive idempotent if it is a white vector with trace 1.

A straightforward calculation shows that there are precisely

$$q^8(q^4 + q^4 + 1)$$

primitive idempotents, and

$$(q^8 - 1)(q^8 + q^4 + 1) = (q^{12} - 1)(q^4 + 1)$$

white vectors of trace 0. More precisely, the trace can be non-zero only in the first three of the six cases in the proof of Theorem \[1\]. The number of choices of $a, b, c$ which give trace 1 is $q^2 - 3q + 3$ if all are non-zero, and $3(q - 2)$ if two are non-zero, and 3 if just one is non-zero. Hence the total number of primitive idempotents is

$$(q^2 - 3q + 3)(q^7 - q^3)^2 + 3(q - 2)(q^7 - q^3)(q^7 + q^4 - q^3) + 3(q^7 + q^4 - q^3)^2$$

which simplifies to $q^8(q^4 + q^4 + 1)$. Subtracting $q - 1$ times this from the total number of white vectors gives the number with trace 0.

It is now clear that $F_4(q)$ acts transitively on the primitive idempotents. To calculate the group order we need only calculate the order of the stabilizer of one of the primitive idempotents. We already know that the stabilizer in $SE_6(q)$ of a white point is a group of shape $q^{16}C_{q - 1}.SO_{10}^+(q)$, so we just need to calculate the subgroup of this which preserves the identity element of $\mathbb{J}$. In the case of a trace 1 white point, such as $(1, 0, 0 | 0, 0, 0)$, this is easily seen to be a subgroup $\text{Spin}_8(q)$.

In particular the formula for the group order is, independently of the characteristic,

$$|F_4(q)| = q^{24}(q^{12} - 1)(q^8 - 1)(q^6 - 1)(q^2 - 1).$$

Note also that the stabilizer of a white point of trace 0 is a group of shape

$$q^{7+8}C_{q - 1}.SO_7(q),$$

which is one of the maximal parabolic subgroups of $F_4(q)$. To prove simplicity of $F_4(q)$, we can apply Iwasawa’s Lemma to the action on the white points of trace 0.

Details are given in Section 4.8.7 of \[20\].

8. THE COMPACT REAL FORM OF $E_6$

We constructed the finite groups $E_6(q)$ by analogy with the split real form of $E_6$, which is defined in terms of the exceptional Jordan algebra over the split octonions. In a similar way, we shall construct the finite groups $^2E_6(q)$ by analogy with the compact real form of $E_6$. But we have seen that in order to describe the latter, it is not sufficient just to replace the split octonions by the compact octonions. Instead, we must first extend the scalars from $\mathbb{R}$ to $\mathbb{C}$ to obtain the complexification $SE_6(\mathbb{C})$. Then we compactify by decreeing that a certain Hermitian form be invariant. Now, just as in the construction of the unitary groups, there is some choice as to which Hermitian form to use. It is not obvious a priori which (if any) is the ‘best’.

First notice that, since there is only one isomorphism type of complex octonion algebra, we can take either the basis $\{e_i | i \in \pm I\}$ defined above for the ‘split’ real octonions, or the basis $\{1 = i_\infty, i_0, i_1, \ldots, i_6\}$ for the ‘compact’ real octonions, or
Cayley numbers. We can switch between the two by a base change such as the following:

\[
\begin{align*}
1 & = e_0 + e_{-0}, \\
i_1 & = e_\omega + e_{-\omega}, \\
i_2 & = e_{\bar{\omega}} + e_{-\bar{\omega}}, \\
i_4 & = e_{-1} + e_1, \\
j i_3 & = e_0 - e_{-0}, \\
j i_0 & = e_\omega - e_{-\omega}, \\
j i_5 & = e_{\bar{\omega}} - e_{-\bar{\omega}}, \\
j i_6 & = e_{-1} - e_1,
\end{align*}
\]

(33)

where \(j\) denotes \(\sqrt{-1}\) in the scalar copy of \(\mathbb{C}\).

Now there is an obvious Hermitian form \(h\) on the complex octonions obtained by defining the basis \(\{e_i\}\) to be orthonormal. There is another obvious Hermitian form \(h_2\) defined by saying that the basis \(\{i_t\}\) is orthonormal. We show next that \(h = 2h_2\).

Let us write \(x'\) for the complex conjugate \(a - bj\) of \(x = a + bj\). Now there are two different ways we might want to extend \(\cdot\) to the octonions. Given an octonion

\[A = \sum_t \alpha_t i_t = \sum_i \beta_i e_i\]

we define

\[
\begin{align*}
A^* & = \sum_t \alpha'_t i_t \\
A' & = \sum_i \beta'_i e_i
\end{align*}
\]

(34)

Since \(e'_t = e_{-1}\) we have

\[A^* = \sum_i \beta'_i e_{-1}\]

and since \(i_t^* = i_t\) for \(t = \infty, 1, 2, 4\) and \(i_t^* = -i_t\) otherwise, we have

\[A' = \sum_{t=\infty,1,2,4} \alpha'_t i_t - \sum_{t=0,3,5,6} \alpha'_t i_t.\]

Then we can compute

\[h_2(A) = \sum_t \alpha_t \alpha'_t = (A\overline{A} + A^*\overline{A})/2 = (\overline{A}A^* + \overline{A} A)/2\]

and therefore

\[h(A) = \sum_i \beta_i \beta'_i = \overline{A} A + A^* \overline{A} = \overline{A} A^* + \overline{A} A.\]

So now define an Hermitian form \(H\) on \(J_\mathbb{C}\) by

\[H(a, b, c \mid A, B, C) = a a' + b b' + c c' + h(A) + h(B) + h(C).\]

Thus \((1, 0, 0 \mid 0, 0, 0), (0, 0, 0 \mid e_i, 0, 0)\) and rotations of these form an orthonormal basis. Then the subgroup of \(SE_6(\mathbb{C})\) which preserves this Hermitian form is in fact the compact real form of \(SE_6\). If we prefer to use the basis \(\{i_t\}\) for the octonions, then we have

\[H(a, b, c \mid A, B, C) = a a' + b b' + c c' + h_2(A) + h_2(B) + h_2(C) + h_2(\overline{A}) + h_2(\overline{B}) + h_2(\overline{C}).\]

This shows that \(H\) is the ‘natural’ Hermitian form induced on \(J_\mathbb{C}\) by the ‘natural’ Hermitian form \(h_2\) on the complex octonions.

We now extend \(\cdot\) also to \(J_\mathbb{C}\) by defining

\[
X^* = (a', b', c' \mid A^*, B^*, C^*)
\]
Lemma 8. If $x, y, z \in \mathbb{C}$, then

1. $x(yx) = (yx)x = (x\overline{x} z)\overline{y}$;
2. $\text{Tr}((xy)(z\overline{y})) = x\overline{x} \text{Tr}(yz)$.

Proof. 

1. $x(yx - \text{Tr}(yx)) = -x(y\overline{x}) = -x(x\overline{y}) = -(x\overline{x})\overline{y}$.
2. $\text{Tr}((xy)(z\overline{y})) = \text{Tr}((z\overline{x})xy) = \text{Tr}((z\overline{x})xy) = x\overline{x} \text{Tr}(yz)$.

Now we are mapping by a matrix of the form

$$\begin{pmatrix} x & y & 0 \\ -\overline{y} & x' & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $x$ is real and $y$ is a multiple of $i_t$ for some $t$. By explicit computation we see that the action of this matrix on $X = (a, b, c \mid A, B, C)$ is given by

$$a \mapsto ax^2 - x\text{Tr}(C\overline{y}) + by^*\overline{y},$$
$$b \mapsto b(x')^2 + x'\text{Tr}(C\overline{y}) + ay\overline{y},$$
$$c \mapsto c,$$
$$A \mapsto x^*A + \overline{y}B,$$
$$B \mapsto Bx - \overline{A}\overline{y},$$
$$C \mapsto xx^*C + ax\overline{y} - bx^*y^* - y^*\overline{C}y.$$ (36)

Then we can compute the new value of $H$ term by term as follows. First consider the terms in $A$ and $B$.

$$\text{Tr}(AA^*) \mapsto xx'\text{Tr}(A\overline{A}^*) + x\text{Tr}(A^*B\overline{y}) + x'\text{Tr}(AB^*y^*)$$

$$\text{Tr}(BB^*) \mapsto xx'\text{Tr}(B\overline{B}^*) - x\text{Tr}(B\overline{y}A^*) - x^*\text{Tr}(B^*y^*A)$$

and by applying the lemma and using the fact that $xx^* + y\overline{y} = 1$ we see that the sum of these two terms is preserved. Now $cc'$ is fixed, and the other terms are as follows. (Note that all calculations are in the (complex) quaternion subalgebra generated by $C$ and $y$, so we can use associativity.)

$$aa' \mapsto x^2x'^2aa' + x^2y\overline{y}ab' - x^2x'\text{Tr}(C\overline{y})a$$

$$+x^2y\overline{y}a'b + y\overline{y}b'y' - y^*x'\text{Tr}(C\overline{y})b$$

$$-xx'\text{Tr}(C\overline{y})a' - y\overline{y}x\text{Tr}(C\overline{y})b' + xx'\text{Tr}(C\overline{y})\text{Tr}(C'\overline{y})$$

(37)
natural basis to be orthonormal for the Hermitian form. He does not mention the over the field of order given matrices lie in the compact real form of $E$ which is the product of complex conjugation with the ordinary duality map whose centralizer is $F$.

Using the lemma, we have

$$\text{Tr}(C^*) \quad \Rightarrow \quad \text{Tr}(x' y^*) = (x y^*)^* = x^* y^*$$

Adding these together, and collecting like terms we find the coefficient of $aa'$ is

$$(x')^2 + y y^*$ $(xx' + \overline{yy}) = (x')^2 + \overline{yy} (xx')$$

and similarly, so is the coefficient of $bb'$. Next, the coefficient of $ab'$ is

$$2 x y y^* - \text{Tr}(x^2 y y^*) = 0,$$

while the coefficient of $a$ is

$$-x x^2 \text{Tr}(C^*) y + x y y^* \text{Tr}(C^*) = \text{Tr}(x y y^* C^*)$$

$$= x^2 x' \text{Tr}(C^* - y C^*) + x y y^* \text{Tr}(C^* - C^*)$$

and the coefficient of $b$ is

$$-y y^* x' \text{Tr}(C^*) - x x'^2 \text{Tr}(C^*) + x^2 \text{Tr}(y^* C^*) = 0.$$}

The remaining terms are as follows:

$$(x')^2 \text{Tr}(C^*) + \text{Tr}(y^* C y y^*) + \text{Tr}(C^*) y + \text{Tr}(y^* y C^*) - 2 \text{Tr}(C^* C y^*)$$

Using the lemma, we have

$$\text{Tr}(C^*) \quad \Rightarrow \quad \text{Tr}(y y^*) \text{Tr}(C^*) = \text{Tr}(y y^*) \text{Tr}(C^*)$$

and hence this expression reduces to $\text{Tr}(C^*)$. This concludes the proof that the given matrices lie in the compact real form of $E_6$.

It is not hard to see that the given generators $M$ satisfy the relation

$$M \in \mathbb{M} = I,$$

which may also be expressed by saying that they centralize the twisted duality map induced by

$$M \mapsto (M^\top)^{-1},$$

which is the product of complex conjugation with the ordinary duality map whose centralizer is $F_4$.

9. Aschbacher’s construction of $E_6(q)$.

Aschbacher defines $E_6(q)$ just in terms of Dickson’s cubic form, by defining the natural basis to be orthonormal for the Hermitian form. He does not mention the Jordan algebra or octonions at all. Of course this is equivalent to taking the space $J$ over the field of order $q^2$, and defining the Hermitian form so that the vectors $(1, 0, 0 \mid 0, 0, 0), (0, 0, 0 \mid e_i, 0, 0)$ and rotations form an orthonormal basis.

For fields of odd characteristic, we can just mimic everything we did for the compact real form. However, in characteristic 2 we have the usual problem that the octonions are not spanned by the vectors $i$; it is necessary therefore to change basis to the $e_i$ before reducing modulo 2. With this small change, we obtain generators for all the finite groups $E_6(q)$ in all characteristics.
10. Another real form of $E_6$, and an alternative construction of $^2E_6(q)$.

There is another Hermitian form one might want to use on the octonions, namely $h_1$ defined by

$$h_1(A) = \sum_i \beta_i \beta_i' = A\overline{A} + A'\overline{A} = \overline{AA'} + \overline{A}A'.$$

This induces the Hermitian form $H_1$ on the Albert algebra, where

$$H_1(a, b, c | A, B, C) = aa' + bb' + cc' + h_1(A) + h_1(B) + h_1(C).$$

This Hermitian form is not positive definite, so it defines a non-compact real form of $E_6$, in fact the form called $E_6(2)$. But on reducing modulo $p$, we again obtain the finite groups $^2E_6(q)$. For certain purposes, this basis seems to be more useful than the one Aschbacher uses. Since this construction does not (so far as I am aware) appear explicitly in the literature, we give some more details here.

The group $^2E_6(q)$ is usually defined as the subgroup of $E_6(q^2)$ consisting of those elements which commute with the automorphism which is the product of the automorphism given above with the field automorphism $x \mapsto x^q$ on all coefficients.

To generate $^2E_6(q)$, therefore, we first need to take the Albert algebra over $F_q$. Let $J = J_F$, where $F = F_q^2$. Denote by $'$ the automorphism of $O_F$ induced by the Frobenius automorphism $\lambda \mapsto \lambda^q$ of $F$ of order 2, that is, if $x = \sum_{i \in \pm 1} \lambda_i e_i$ then $x' = \sum_{i \in \pm 1} \lambda_i^q e_i$. Then there is a twisted duality map $*$ on $J$ defined by

$$X^* = X'^T = \overline{X}.$$

This induces the above-mentioned automorphism of the group, which acts on the generators $M$ by $M \mapsto (M'^T)^{-1}$. For $M$ to centralize this automorphism, therefore, we must have $\overline{M}^T M = I$.

There is a notion of twisted Jordan algebra, in which there is a new product $*$ defined in terms of the ordinary Jordan product $X \circ Y$ by

$$X * Y = (X \circ Y)'$$

Here we shall define the group in a slightly different way, as hinted above. Let $H_1$ be the Hermitian form defined on $J$ by

$$H_1(a, b, c | A, B, C) = aa' + bb' + cc' + \text{Tr}(A\overline{A} + B\overline{B} + C\overline{C}).$$

Then the simply-connected group $^2SE_6(q)$ is the subgroup of $SE_6(q^2)$ which preserves $H_1$. As long as the characteristic is not 2, this Hermitian form may be described in terms of the Jordan algebra as $H(X) = \text{Tr}(X \circ X')$.

In order to produce generators for $^2SE_6(q)$, we consider matrices $M$ which satisfy $M^1 M = I$, where $M^1$ is defined by applying the field automorphism $x \mapsto x^q$ to every coefficient in $M^T$. For example, if $x = \lambda e_i$ for some $i \in \pm 1$, then the matrix

$$N_x = \begin{pmatrix} 1 & x & 0 \\ \overline{\overline{\lambda}^q} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is such a matrix. Since $e_i \overline{\overline{\lambda}^q} = 0$ we have $x \overline{\overline{\lambda}^q} = 0$, so

$$\begin{pmatrix} 1 & x \\ -\overline{\overline{\lambda}^q} & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\overline{\overline{\lambda}^q} & 1 \end{pmatrix}$$

and therefore the given matrix $N_x$ lies in $SE_6(q^2)$. To check that it preserves $H_1$, we first prove a small lemma.

Lemma 9. If $x\overline{\overline{\lambda}^q} = 0$, and $y, z \in O_F$, then

1. $x(yz) = x\text{Tr}(yz)$;
2. $\text{Tr}((xy)(z\overline{\overline{\lambda}^q})) = 0$. 


Proof. (1) $x(yx - \text{Tr}(yx)) = x(\overline{yx}) = x(\overline{x}) = (x\overline{x}) = 0$.
(2) $\text{Tr}((xy)(zx)) = \text{Tr}((zx)(xy)) = \text{Tr}(((z\overline{x})y) = \text{Tr}((z\overline{x}x))y) = 0.$

Now by explicit computation we see that $N_x$ maps $X = (a, b, c \mid A, B, C)$ to

$(a - \text{Tr}(C\overline{x}), b + \text{Tr}(C\overline{x}), c \mid A + \overline{B}, B - \overline{A}, C + ax - bx' - x'C).$

Then we can compute the new value of $H_1$ term by term as follows:

\[
\begin{align*}
aa' &\mapsto aa' - a\text{Tr}(C\overline{x}) - a'\text{Tr}(C\overline{x}) + \text{Tr}(C\overline{x})\text{Tr}(C'\overline{x}) \\
bb' &\mapsto bb' + b\text{Tr}(C'\overline{x}) + b'\text{Tr}(C\overline{x}) + \text{Tr}(Cx)\text{Tr}(C'\overline{x}) \\
c'C' &\mapsto cc'
\end{align*}
\]

\[
\begin{align*}
\text{Tr}(A\overline{A}') &\mapsto \text{Tr}(A\overline{A}') + \text{Tr}(A\overline{B}) + \text{Tr}(AB\overline{x'}) + \text{Tr}((\overline{B})(B'x')) \\
\text{Tr}(B\overline{B}) &\mapsto \text{Tr}(B\overline{B}) - \text{Tr}(A\overline{B}) - \text{Tr}(BxA') + \text{Tr}((\overline{A})(x'A')) \\
\text{Tr}(C\overline{C}) &\mapsto \text{Tr}(C\overline{C}) + \text{Tr}(a'C\overline{x}) + axC - b'C\overline{x} - bx'C
\end{align*}
\]

(43)

and using the lemma we see that all cross terms cancel out, as required.

We also see that $F_4(q)$ is a subgroup of $2^2SE_6(q)$, because on the $F_4$-subspace $J_{F_4}$ all elements of $F_4(q)$ preserve the standard norm, which is just the restriction of $H_1$. We may now take the same generators for $F_4(q)$ as before, consisting of certain matrices $M$ which are fixed by the field automorphism.

Then adjoin to $F_4(q)$ the matrix

\[
M = \begin{pmatrix} x & 0 & 0 \\ 0 & x^q & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where $x \in F_{q^2} \setminus F_q$ satisfies $x^{1+q} = 1$. More generally, take matrices

\[
M = \begin{pmatrix} a & b & 0 \\ -b^q & a^q & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

where $a^{1+q} + b^{1+q} = 1$.

The extra root elements are given by matrices like

\[
\begin{pmatrix} 1 & \lambda e_0 & 0 \\ -\lambda^q e_{-0} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & \lambda e_i & 0 \\ \lambda^q e_i & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ for } i = \pm 1, \pm \omega, \pm \overline{\omega}.
\]

With a certain amount of calculation it is now possible to show that this group has exactly three orbits on the white points for $E_6(q^2)$. The lengths of these orbits are as follows:

(1) $(q^9 + 1)(q^{12} - 1)(q^5 + 1)/(q^2 - 1)$,
(2) $(q^4 + 1)(q^9 + 1)(q^{12} - 1)(q^{12} - 1)/(q^2 - 1)$ and
(3) $q^{16}(q^8 + q^4 + 1)(q^6 + 1)/(q + 1)$.

Of these, the first two are isotropic with respect to $H_1$, while the last is non-isotropic. Now we know the stabilizer in $SE_6(q^2)$ has shape $q^{32}.\text{Spin}^+_1(q^2).C_{q^2-1}$ and it is now not too difficult to see what the stabilizers in $2^2SE_6(q)$ must be. A point in the last orbit has a stabilizer of shape $\text{Spin}^-_{10}(q).C_{q+1}$, from which we deduce the order of $2^2SE_6(q)$, that is,

\[
|2^2SE_6(q)| = q^{36}(q^{12} - 1)(q^9 + 1)(q^{12} - 1)(q^6 - 1)(q^6 + 1)/(q^2 - 1).
\]

The three orbits are distinguished as follows. Any white vector $v$ determines a 17-space, which is the radical of the quadratic form determined by $v$, and hence determines the radical of $H_1$ on this 17-space. If $v$ belongs to this last space, then $v$ is of type (1), which Aschbacher calls emerald; and in fact the radical of $H_1$ on
the 17-space is just $\langle v \rangle$. Otherwise, if $H_1(v) = 0$, then $v$ is of type (2). Finally, $v$ is of type (3) if $H_1(v) \neq 0$.

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