Multivariate Dynamical Sampling in $l^2(\mathbb{Z}^2)$ and Shift-Invariant Spaces Associated with Linear Canonical Transform *

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Abstract

In this paper, we investigate the multivariate dynamical sampling problem in $l^2(\mathbb{Z}^2)$ associated with the two-dimensional discrete time non-separable linear canonical transform (2D-DT-NS-LCT) and shift-invariant spaces associated with the two-dimensional non-separable linear canonical transform (2D-NS-LCT), respectively. Specifically, we derive a sufficient and necessary condition under which a sequence in $l^2(\mathbb{Z}^2)$ (or a function in a shift-invariant space) can be stably recovered from its dynamical sampling measurements associated with the 2D-DT-NS-LCT (or the 2D-NS-LCT). We also present a simple example to elucidate our main results.

Keywords: Multivariate dynamical sampling; linear canonical transform (LCT); shift-invariant spaces

1. Introduction

The linear canonical transform (LCT) is a general class of linear integral transformations with three free parameters, which includes many well-known

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linear transforms as its special cases, such as Fourier transform (FT), fractional FT, scaling operation, and Fresnel transform \[1-7\]. Therefore, there has been growing interest in studying the LCT and its properties pertaining to applications across the fields of signal processing and optics \[8-11\].

Signal sampling is a fundamental concept in digital signal processing, as it provides a bridge between continuous- and discrete-time signals. A variety of sampling theorems in the traditional FT domain have been generalized to the LCT domain in the broad sense that signals which are non-bandlimited in the FT domain may be bandlimited in the LCT domain \[12-18\]. In recent years, dynamical sampling has attracted empirical attention in the scientific community, which is a new way of signal sampling, compared with classical sampling techniques, and has potential applications to wireless sensor networks in the health, environment, and precision agriculture industries \[19-24\]. More specifically, the dynamical sampling refers to not only the signal \(f\) that is sampled but also its various states at different times \(\{t_1, t_2, \cdots, t_N\}\). It is to consider the problem of spatiotemporal sampling with which an initial state \(f\) of an evolution process \(f_t = A_t f\) is recovered from a combined set of coarse spatial samples \(\{f(X), f_{t_1}(X), \cdots, f_{t_N}(X)\}\) of \(f\) on \(X \subset \Omega\) at varying time levels \(\{t_1, t_2, \cdots, t_N\}\), where \(\Omega\) is the domain of \(f\). Most work of dynamical sampling aimed to derive some sufficient and necessary conditions on \(A\) and \(\Omega\) such that \(f\) can be stably recovered from its dynamical sampling measurements \(\{f(X), f_{t_1}(X), \cdots, f_{t_N}(X)\}\).

Of note, Aldroubi and his collaborators have intensively studied the dynamical sampling of signals in finite dimensional spaces \[25\], shift-invariant spaces \[26, 27\], and infinite dimensional spaces (e.g., \(l^2(Z)\)) \[28\]. Zhang et al \[29\] investigated the periodic nonuniform dynamical sampling in \(l^2(Z)\) and shift-invariant spaces associated with the FT. More recently, Liu et al \[30\] extended the dynamical sampling results associated with the FT to the special affine Fourier transform (also known as offset linear canonical transform) domain in a broader sense, i.e., signals are recovered from their dynamical sampling measurements associated with the special affine Fourier transform. However, all the mentioned dynamical sampling results above focus on one-dimensional signals. Motivated by applications in multi-dimensional signal systems, Zhang et al \[31\] considered the multivariate dynamical sampling in \(l^2(Z^d)\) and shift-invariant spaces associated with the \(d\)-dimensional FT.

To the best of our knowledge, there have been no studies on the multivariate dynamical sampling associated with the multi-dimensional LCT. In this paper, we therefore propose to investigate the multivariate dynamical sampling of signals in \(l^2(Z^2)\) associated with the two-dimensional discrete time non-separable LCT (2D-DT-NS-LCT, defined below) and shift-
invariant spaces associated with the two-dimensional non-separable LCT (2D-NS-LCT, defined below), respectively. We derive a sufficient and necessary condition under which a sequence in $l^2(\mathbb{Z}^2)$ (or a function in a shift-invariant space $V(\phi)$ that is generated by $\phi \in L^2(\mathbb{R}^2)$) can be stably reconstructed from its dynamical sampling measurements associated with the 2D-DT-NS-LCT (or the 2D-NS-LCT).

The rest of this paper is organized as follows. In Section 2, we first introduce definitions of the 2D-NS-LCT, the 2D-DT-NS-LCT, and their related canonical convolution operators, and then obtain some important properties that will be utilized later. In Section 3 we consider the multivariate dynamical sampling in $l^2(\mathbb{Z}^2)$ (i.e., we derive a sufficient and necessary condition for sequences in $l^2(\mathbb{Z}^2)$ under which they can be recovered from their dynamical sampling measurements in a stable way) associated with the 2D-DT-NS-LCT, followed by the multivariate dynamical sampling in shift-invariant spaces associated with the 2D-NS-LCT in Section 4. In Section 5 we present an example to elucidate our main results, and conclude this paper in Section 6.

2. Preliminaries

The two-dimensional non-separable LCT (2D-NS-LCT) with parameter $\mathcal{M} \doteq [A, B; C, D]$ of a signal $f \in L^2(\mathbb{R}^2)$ is defined by

$$\left( L_{\mathcal{M}} f \right)(\xi) = \frac{1}{\sqrt{\det(iB)}} \int_{\mathbb{R}^2} f(u) e^{i\pi u^T B^{-1} A u - 2i\pi u^T B^{-1} \xi + i\pi \xi^T D B^{-1} \xi} du,$$  \hspace{1cm} (1)

where $u, \xi \in \mathbb{R}^2$ are two real column vectors, the superscript $(\cdot)^T$ denotes the transpose of a vector or matrix, $\det(\cdot)$ stands for the determinant of a matrix, and $A, B, C, D \in \mathbb{R}^{2 \times 2}$ are $2 \times 2$ real matrices with $B$ being non-singular. The matrix $\mathcal{M}$ is real and symplectic so that the following equations hold:

$$AB^T = BA^T, \quad CD^T = DC^T, \quad AD^T - BC^T = I,$$  \hspace{1cm} (2)

or

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I.$$  \hspace{1cm} (3)

Here, $I$ stands for the $2 \times 2$ identity matrix. From the perspective of group theory, the 2D-NS-LCT forms a symplectic group $Sp(4, \mathbb{R})$ with ten parameters. We refer the reader to [37, 38] for more details about the 2D-NS-LCT.

To begin with, we introduce a new canonical convolution operator associated with the 2D-NS-LCT. Let $\lambda_{\mathcal{M}}(t) = e^{i\pi t^T B^{-1} A t}$ be the chirp-modulation function, and define
\[ \overrightarrow{f}(t) := \lambda M(t)f(t) = e^{i\pi t^TB^{-1}At}(f(t)), \]
\[ \overleftarrow{f}(t) := \overline{\lambda M(t)f(t)} = e^{-i\pi t^TB^{-1}At}(f(t)), \]

where \( \overline{z} \) means the conjugate of \( z \).

**Definition 2.1.** Define the canonical convolution operator \( \ast_c \) of two functions \( f, g \in L^2(\mathbb{R}^2) \) associated with the 2D-NS-LCT as

\[ (f \ast_c g)(t) = \frac{\overline{\lambda M(t)}}{\sqrt{\det(iB)}}(\overrightarrow{f} \ast \overrightarrow{g})(t), \quad (4) \]

where \( \ast \) denotes the traditional convolution operator, i.e.,

\[ (f \ast g)(t) = \int_{\mathbb{R}^2} f(t - x)g(x)dx. \quad (5) \]

We then have the following lemma about the canonical convolution operator \( \ast_c \) of two functions in the 2D-NS-LCT domain.

**Lemma 2.2.** For two functions \( f, g \in L^2(\mathbb{R}^2) \), let \( h(t) = (f \ast_c g)(t) \). Then, the 2D-NS-LCT \( (LMh)(\xi) \) of \( h \) satisfies

\[ (LMh)(\xi) = \eta_M(\xi)(LMf)(\xi)(LMg)(\xi), \quad (6) \]

where \( \eta_M(\xi) = e^{i\pi \xi^TB^{-1}\xi} \).

**Proof.** According to (1), (4) and (5), we have

\[
(LMH)(\xi) = \frac{1}{\sqrt{\det(iB)}} \int_{\mathbb{R}^2} h(u)e^{i\pi u^TB^{-1}Au - 2i\pi u^TB^{-1}\xi + i\pi \xi^TDB^{-1}\xi} du
\]

\[
= \frac{1}{\sqrt{\det(iB)}} \int_{\mathbb{R}^2} \overline{\lambda M(u)} \left( \int_{\mathbb{R}^2} \overrightarrow{f}(u - x)\overrightarrow{g}(x)dx \right)
\times e^{i\pi u^TB^{-1}Au - 2i\pi u^TB^{-1}\xi + i\pi \xi^TDB^{-1}\xi} du
\]

\[
= \frac{1}{\det(iB)} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(u - x)e^{i\pi(u-x)^TB^{-1}A(u-x)}g(x)e^{i\pi x^TB^{-1}Ax}
\times e^{-2i\pi u^TB^{-1}\xi + i\pi \xi^TDB^{-1}\xi} dudx
\]

\[
= \frac{\eta_M(\xi)}{\sqrt{\det(iB)}} \int_{\mathbb{R}^2} f(u - x)e^{i\pi(u-x)^TB^{-1}A(u-x)-2i\pi(u-x)^TB^{-1}\xi + i\pi \xi^TDB^{-1}\xi} du
\]
\[
\times \frac{1}{\sqrt{\det (iB)}} \int_{\mathbb{R}^2} g(x) e^{i\pi x^T B^{-1} A x - 2i\pi x^T B^{-1} \xi + i\pi T DB^{-1} \xi} dx
\]
\[
= \overline{\nu_M} (\xi) (L_M f)(\xi) (L_M g)(\xi),
\]
which completes the proof. \(\square\)

We next give definitions of the two-dimensional discrete time non-separable LCT (2D-DT-NS-LCT) of a sequence in \(l^2(\mathbb{Z}^2)\), the canonical convolution of a sequence in \(l^2(\mathbb{Z}^2)\) and a function in \(L^2(\mathbb{R}^2)\), and periodic functions, respectively.

**Definition 2.3.** Let \(s = s(k)\) be a sequence in \(l^2(\mathbb{Z}^2)\), i.e., \(\sum_{k \in \mathbb{Z}^2} |s(k)|^2 < +\infty\). The 2D-DT-NS-LCT of \(s\) is defined by
\[
(L_M s)(\xi) = \frac{1}{\sqrt{\det (iB)}} \sum_{k \in \mathbb{Z}^2} s(k) e^{i\pi k^T B^{-1} A k - 2i\pi k^T B^{-1} \xi + i\pi T DB^{-1} \xi}.
\] (7)

**Definition 2.4.** The canonical convolution operator \(*_{sd}\) of a sequence \(s \in l^2(\mathbb{Z}^2)\) and a function \(\phi \in L^2(\mathbb{R}^2)\) is defined as
\[
h(t) = (s *_{sd} \phi)(t) = \frac{\overline{\lambda_M}(t)}{\sqrt{\det (iB)}} \sum_{k \in \mathbb{Z}^2} \lambda_M(k) s(k) \lambda_M(t - k) \phi(t - k).
\] (8)

**Definition 2.5.** A function \(\phi(t)\) is said to be periodic with periodicity matrix \(M \in \mathbb{R}^{2 \times 2}\), if
\[
\phi(t + Mn) = \phi(t), \quad \text{for all } t \in \mathbb{R}^2 \text{ and any } n \in \mathbb{Z}^2,
\] (9)
where \(M\) is non-singular.

In the following, we derive a canonical convolution theorem of a sequence in \(l^2(\mathbb{Z}^2)\) and a function in \(L^2(\mathbb{R}^2)\).

**Lemma 2.6.** For \(s \in l^2(\mathbb{Z}^2), \phi \in L^2(\mathbb{R}^2)\), let \(h(t) = (s *_{sd} \phi)(t)\). Then,
\[
(L_M h)(\xi) = \overline{\nu_M}(\xi) (L_M s)(\xi) (L_M \phi)(\xi).
\] (10)
Moreover, \(|(L_M s)(\xi)|\) is periodic with periodicity matrix \(B\).

**Proof.** According to (1), (7) and (8), we have
\[
(L_M h)(\xi)
= \frac{1}{\sqrt{\det (iB)}} \int_{\mathbb{R}^2} (s *_{sd} \phi)(t) e^{i\pi t^T B^{-1} A t - 2i\pi t^T B^{-1} \xi + i\pi T DB^{-1} \xi} dt
\]
\[
\begin{align*}
&= \frac{1}{\det(iB)} \sum_{k \in \mathbb{Z}^2} \lambda_M(k) s(k) \left( \int_{\mathbb{R}^2} \bar{\lambda}_M(t) \lambda_M(t-k) \phi(t-k) \right. \\
&\quad \times e^{i\pi t^T B^{-1} A t - 2i\pi t^T B^{-1} \xi + i\pi \xi^T D B^{-1} \xi dt} \\
&= \frac{1}{\sqrt{\det(iB)} \sum_{k \in \mathbb{Z}^2} s(k) e^{i\pi k^T B^{-1} A k - 2i\pi k^T B^{-1} \xi + i\pi \xi^T D B^{-1} \xi} \left( \frac{1}{\sqrt{\det(iB)}} \int_{\mathbb{R}^2} \phi(t) \
&\quad \times e^{i\pi t^T B^{-1} A t - 2i\pi t^T B^{-1} \xi + i\pi \xi^T D B^{-1} \xi dt} \right) \\
&= \sqrt{\det(iB)} \sum_{k \in \mathbb{Z}^2} s(k) e^{i\pi k^T B^{-1} A k - 2i\pi k^T B^{-1} \xi + i\pi \xi^T D B^{-1} \xi} \left( \frac{1}{\sqrt{\det(iB)}} \int_{\mathbb{R}^2} \phi(t) 
&\quad \times e^{i\pi t^T B^{-1} A t - 2i\pi t^T B^{-1} \xi + i\pi \xi^T D B^{-1} \xi dt} \right) \\
&= \eta_M(\xi)(L_M s)(\xi)(L_M \phi)(\xi).
\end{align*}
\]
Furthermore, for all \(\xi \in \mathbb{Z}^2\) and any \(l \in \mathbb{Z}^2\), we get
\[
(L_M s)(\xi + B l) \\
= \frac{1}{\sqrt{\det(iB)} \sum_{k \in \mathbb{Z}^2} s(k) e^{i\pi k^T B^{-1} A k - 2i\pi k^T B^{-1} \xi + i\pi \xi^T D B^{-1} \xi}} \left( \frac{1}{\sqrt{\det(iB)}} \int_{\mathbb{R}^2} \phi(t) 
&\quad \times e^{i\pi t^T B^{-1} A t - 2i\pi t^T B^{-1} \xi + i\pi \xi^T D B^{-1} \xi dt} \right) \\
&= (L_M s)(\xi) e^{i\pi (B l)^T D B^{-1} \xi + i\pi \xi^T D l + i\pi (B l)^T D l},
\]
where we use \(e^{-2i\pi k^T l} = 1\) in the last step. Hence,
\[
|(L_M s)(\xi + B l)| = |(L_M s)(\xi)|.
\]
This completes the proof.

Similarly, we introduce the definition and property of the canonical convolution of two sequences in \(l^2(\mathbb{Z}^2)\) associated with the 2D-DT-NS-LCT as follows.

**Definition 2.7.** Let \(s = s(k) \in l^2(\mathbb{Z}^2)\) and \(c = c(k) \in l^2(\mathbb{Z}^2)\). The canonical convolution operator \(\star_d\) of two sequences \(s\) and \(c\) is defined by
\[
h(l) = (s \star_d c)(l) = \frac{\bar{\lambda}_M(l)}{\sqrt{\det(iB)}} \sum_{k \in \mathbb{Z}^2} \lambda_M(k) s(k) \lambda_M(l-k) c(l-k).
\]

**Lemma 2.8.** For \(s, c \in l^2(\mathbb{Z}^2)\), let \(h(l) = (s \star_d c)(l)\). Then, we have
\[
(L_M h)(\xi) = \eta_M(\xi)(L_M s)(\xi)(L_M c)(\xi).
\]
Proof. By (7) and (12), we have
\[(L_M h)(\xi) = \frac{1}{\sqrt{\det (iB)}} \sum_{l \in \mathbb{Z}^2} (s \ast c)(l) e^{i\pi T_B^{-1}A_l - 2i\pi T_B^{-1}\xi + i\pi \xi^T DB^{-1}\xi} \]
\[= \frac{1}{\det (iB)} \sum_{l \in \mathbb{Z}^2} \lambda_M(l) \left( \sum_{k \in \mathbb{Z}^2} \lambda_M(k) s(k) \lambda_M(l - k) c(l - k) \right) \times e^{i\pi T_B^{-1}A_l - 2i\pi T_B^{-1}\xi + i\pi \xi^T DB^{-1}\xi} \]
\[= \frac{1}{\det (iB)} \sum_{l \in \mathbb{Z}^2} \lambda_M(l) \left( \sum_{l \in \mathbb{Z}^2} \lambda_M(l) \lambda_M(l - k) c(l - k) \right) \times e^{i\pi T_B^{-1}A_l - 2i\pi T_B^{-1}\xi + i\pi \xi^T DB^{-1}\xi} \]
\[= \eta_M(\xi)(L_M s)(\xi)(L_M c)(\xi), \]
which completes the proof. \(\square\)

On the basis of the above lemmas, we propose to explore the multivariate dynamical sampling in the \(l^2(\mathbb{Z}^2)\) space associated with the 2D-DT-NS-LCT and shift-invariant spaces associated with the 2D-NS-LCT, respectively, in what follows.

3. Multivariate dynamical sampling in \(l^2(\mathbb{Z}^2)\) associated with 2D-DT-NS-LCT

In this section, we focus on the multivariate dynamical sampling in \(l^2(\mathbb{Z}^2)\) (i.e., a sequence space) associated with the 2D-DT-NS-LCT. Specifically, we derive a necessary and sufficient condition under which signals (or sequences) in \(l^2(\mathbb{Z}^2)\) can be stably reconstructed from their dynamical sampling values associated with the 2D-DT-NS-LCT.

Let \(M\) be a \(2 \times 2\) non-singular real matrix (not necessarily integer matrix) with \(m = |\det(M)|\), and \(MZ^2 \triangleq \{ Mn \mid n \in \mathbb{Z}^2 \}\) be a lattice generated by \(M\) \([38, 39]\). Let us define \(T^2 \triangleq \{ [x_1, x_2]^T \mid x_1 \in [0, 1) \text{ and } x_2 \in [0, 1) \}\) \(\subset \mathbb{R}^2\). The fundamental parallelepiped of \(MZ^2\) is defined as the region
\[G(M) \triangleq \{ Mx \mid x \in T^2 \}.\]
One can see that \( G(M) \) and its shifted copies (called the other lattice cells) constitute the whole real vector space \( \mathbb{R}^2 \), i.e.,
\[
\bigcup_{n \in \mathbb{Z}^2} \{ M(x + n) \mid x \in \mathbb{T}^2 \} = \mathbb{R}^2.
\]
(14)

When \( M \) is further an integer matrix, we define
\[
N(M) \triangleq \{ k \mid k = Mx, x \in \mathbb{T}^2, \text{and } k \in \mathbb{Z}^2 \}.
\]

The number of elements in \( N(M) \) equals \( m \). Without loss of generality, let \( \gamma_0 = [0, 0]^T, \gamma_1, \cdots, \gamma_{m-1} \) be the \( m \) distinct elements in \( N(M) \). It is clear that \( \gamma_k + M\mathbb{Z}^2(k = 0, 1, \cdots, m-1) \) constitute the whole integer vector space \( \mathbb{Z}^2 \), i.e., \( \bigcup_{k=0}^{m-1} \{ \gamma_k + M\mathbb{Z}^2 \} = \mathbb{Z}^2 \). Analogously, let \( M^T\mathbb{Z}^2 \triangleq \{ M^T n \mid n \in \mathbb{Z}^2 \} \) be a lattice generated by \( M^T \), and \( \eta_0 = [0, 0]^T, \eta_1, \cdots, \eta_{m-1} \) be the \( m \) distinct elements in \( N(M^T) \). Obviously, \( \bigcup_{j=0}^{m-1} \{ \eta_j + M^T\mathbb{Z}^2 \} = \mathbb{Z}^2 \).

Let
\[
S_M : l^2(\mathbb{Z}^2) \to l^2(\mathbb{Z}^2)
\]
be the subsampling operator by some fixed dilation matrix \( M \) (i.e., a \( 2 \times 2 \) non-singular integer matrix), that is to say, for any \( c \in l^2(\mathbb{Z}^2) \), and \( k \in \mathbb{Z}^2 \),
\[
(S_M c)(k) = c(M^T k).
\]

In the beginning, we introduce some propositions and lemmas, which are useful for tackling the multivariate dynamical sampling problem in \( l^2(\mathbb{Z}^2) \) associated with the 2D-DT-NS-LCT.

**Proposition 3.1** (33). As stated above, let \( \{ \gamma_k \}_{k=0}^{m-1} \in \mathbb{Z}^2 \) and \( \{ \eta_j \}_{j=0}^{m-1} \in \mathbb{Z}^2 \) be the \( m \) distinct elements of \( N(M) \) and \( N(M^T) \), respectively, with \( \gamma_0 = \eta_0 = [0, 0]^T \). Then,
\[
\sum_{k=0}^{m-1} e^{-i2\pi \eta_j^T M^{-1} \gamma_k} = m \delta_j, \quad j = 0, 1, \cdots, m - 1,
\]
(15)
where \( m = |\det(M)| = |\det(M^T)| \), and
\[
\delta_j = \begin{cases} 1, & j = 0, \\ 0, & j \neq 0. \end{cases}
\]

Based on Proposition 3.1, we next obtain the Poisson summation formula in the 2D-DT-NS-LCT domain as follows.
Lemma 3.2. Let a sequence \( c \in l^2(\mathbb{Z}^2) \). Then, we have

\[
\frac{m}{\sqrt{\det(iB)}} \{ \mathcal{F}[S_M(c \lambda_M)] \}(\xi) = \sum_{k=0}^{m-1} \eta_M[BM^{-1}(\xi + \gamma_k)](L_Mc)[BM^{-1}(\xi + \gamma_k)],
\]

where \( \mathcal{F} \) stands for the discrete time Fourier transform on \( T^2 \), i.e., for any \( c \in l^2(\mathbb{Z}^2) \),

\[
(\mathcal{F}c)(\xi) = \sum_{k \in \mathbb{Z}^2} c(k)e^{-2i\pi k^T \xi}, \ \xi \in T^2.
\]

Proof. Combining (7) and (15), we have

\[
\sum_{k=0}^{m-1} \eta_M[BM^{-1}(\xi + \gamma_k)](L_Mc)[BM^{-1}(\xi + \gamma_k)]
\]

\[
= \frac{1}{\sqrt{\det(iB)}} \sum_{k=0}^{m-1} e^{-i\pi[BM^{-1}(\xi+\gamma_k)]^T DM^{-1}(\xi+\gamma_k)} \sum_{n \in \mathbb{Z}^2} c(n)e^{i\pi n^T B^{-1} A n}
\]

\[
\times e^{-2i\pi n^T B^{-1} [BM^{-1}(\xi+\gamma_k)] e^{i\pi[BM^{-1}(\xi+\gamma_k)]^T DB^{-1} [BM^{-1}(\xi+\gamma_k)]}}
\]

\[
= \frac{1}{\sqrt{\det(iB)}} \sum_{n \in \mathbb{Z}^2} c(n)e^{i\pi n^T B^{-1} A n} e^{-2i\pi n^T M^{-1}(\xi+\gamma_k)}
\]

\[
= \frac{1}{\sqrt{\det(iB)}} \sum_{n' \in \mathbb{Z}^2} c(n')e^{i\pi n'^T B^{-1} A n'} e^{-2i\pi n'^T M^{-1}\xi} \sum_{k=0}^{m-1} e^{-2i\pi n^T M^{-1}\gamma_k}
\]

\[
+ \frac{1}{\sqrt{\det(iB)}} \sum_{n' \in \mathbb{Z}^2} c(n')e^{i\pi n'^T B^{-1} A n'} e^{-2i\pi n'^T M^{-1}\xi} \sum_{j=1}^{m-1} \sum_{n = M^T n' + \eta_j} e^{-2i\pi n^T M^{-1}\gamma_k}
\]

\[
= \frac{m}{\sqrt{\det(iB)}} \sum_{n' \in \mathbb{Z}^2} c(M^T n')e^{i\pi(M^T n')^T B^{-1} A (M^T n')} e^{-2i\pi [M^T n']^T M^{-1}\xi}
\]

\[
= \frac{m}{\sqrt{\det(iB)}} \{ \mathcal{F}[S_M(c \lambda_M)] \}(\xi).
\]

This completes the proof. \qed
Let \( a \in l^2(\mathbb{Z}^2) \) be the kernel of an evolution operator, \( a^j = a \ast_d \cdots \ast_1 a, j = 1, 2, \cdots, m - 1, \) and \( y_0(k) = c(M^T k), y_j(k) = (a^j \ast_d c)(M^T k), j = 1, 2, \cdots, m - 1. \) Then, based on Lemma 3.2 and Lemma 2.8, we have the following theorem.

**Theorem 3.3.** Let \( a \) be a sequence with \( (L_M a)(\xi) \in L^\infty(\mathbb{R}^2). \) Suppose that \( e_k^j = \pi_M^{j+1}[BM^{-1}(\xi + \gamma_k)], k = 0, 1, \cdots, m - 1, \) and \( A_M(\xi) \) is denoted as

\[
A_M(\xi) = \begin{pmatrix}
    e_0^0(\xi) & e_1^0(\xi) & \cdots & e_m^0(\xi) \\
    e_0^1(\xi + \gamma_1) & e_1^1(\xi + \gamma_1) & \cdots & e_m^1(\xi + \gamma_1) \\
    \vdots & \vdots & \ddots & \vdots \\
    e_0^{m-1}(\xi + \gamma_{m-1}) & e_1^{m-1}(\xi + \gamma_{m-1}) & \cdots & e_m^{m-1}(\xi + \gamma_{m-1})
\end{pmatrix}
\]

Any \( c \in l^2(\mathbb{Z}^2) \) can be recovered in a stable way, i.e., the inverse of \( A_M(\xi) \) is bounded, from the dynamical sampling measurements \( y_0(k) = c(M^T k), y_j(k) = (a^j \ast_d c)(M^T k), j = 1, 2, \cdots, m - 1, k \in \mathbb{Z}^2, \) if and only if there exists \( \alpha > 0 \) so that the set \( \{ \xi \mid | \det(A_M(\xi)) | < \alpha \} \) has zero measure.

**Proof.** Combining Lemma 2.8 and Lemma 3.2 for any \( j = 1, 2, \cdots, m - 1, \) we get

\[
\frac{m}{\sqrt{\det(iB)}} \{ \mathcal{F}[y_j S_M(\lambda_M)] \}(\xi) = \frac{m}{\sqrt{\det(iB)}} \{ \mathcal{F}[S_M((a^j \ast_d c)\lambda_M)] \}(\xi)
\]

\[
= \sum_{k=0}^{m-1} \pi_M^{j+1}[BM^{-1}(\xi + \gamma_k)](L_M(a^j \ast_d c))[BM^{-1}(\xi + \gamma_k)]
\]

\[
= \sum_{k=0}^{m-1} \pi_M^{j+1}[BM^{-1}(\xi + \gamma_k)](L_M(a^j))[BM^{-1}(\xi + \gamma_k)](L_M c)[BM^{-1}(\xi + \gamma_k)]
\]

\[
= \sum_{k=0}^{m-1} \pi_M^{j+1}[BM^{-1}(\xi + \gamma_k)](L_M a^j)[BM^{-1}(\xi + \gamma_k)](L_M c)[BM^{-1}(\xi + \gamma_k)].
\]

Let

\[
Y(\xi) = \begin{bmatrix}
    \frac{m}{\sqrt{\det(iB)}} \{ \mathcal{F}[y_0 S_M(\lambda_M)] \}(\xi) \\
    \frac{m}{\sqrt{\det(iB)}} \{ \mathcal{F}[y_1 S_M(\lambda_M)] \}(\xi) \\
    \vdots \\
    \frac{m}{\sqrt{\det(iB)}} \{ \mathcal{F}[y_{m-1} S_M(\lambda_M)] \}(\xi)
\end{bmatrix},
\]

(20)
and

\[
\mathcal{C}(\xi) = \begin{bmatrix}
(L_M c)(BM^{-1}\xi) \\
(L_M c)[BM^{-1}(\xi + \gamma_1)] \\
\vdots \\
(L_M c)[BM^{-1}(\xi + \gamma_{m-1})]
\end{bmatrix}.
\]

In short notation, from (19), we can equivalently rewrite (20) as

\[
\mathcal{Y}(\xi) = \mathcal{A}_\mathcal{M}(\xi)\mathcal{C}(\xi).
\]  
(21)

Thus, we can solve this equation (21) with respect to \(C(\xi)\) (which we use to produce \(c\)), if \(\mathcal{A}_\mathcal{M}(\xi)\) is invertible.

\[\square\]

4. Multivariate dynamical sampling in shift-invariant spaces associated with 2D-NS-LCT

In this section, we naturally study the multivariate dynamical sampling in shift-invariant spaces associated with the 2D-NS-LCT. Shift-invariant spaces are typical spaces of functions considered in sampling theory.

First, we derive a sufficient and necessary condition under which a function \(\phi(t) \in L^2(\mathbb{R}^2)\) can be a generator for a shift-invariant space \(V\) associated with the 2D-NS-LCT.

**Theorem 4.1.** Let a sequence \(s \in l^2(\mathbb{Z}^2)\) and a function \(\phi \in L^2(\mathbb{R}^2)\). Assume that the chirp-modulated subspace of \(L^2(\mathbb{R}^2)\) is given by

\[
V(\phi) = \{f \in L^2(\mathbb{R}^2) : f(t) = (s \ast_{sd} \phi)(t)\}.
\]

Then, \(\{e^{-2i\pi(t-k)^T B^{-1}(t-k)}\phi(t-k)\}\) is a Riesz basis for \(V(\phi)\), if and only if there exist two constants \(\eta_1, \eta_2 > 0\) such that

\[
\eta_1 \leq \sum_{k \in \mathbb{Z}^2} |(L_M(\phi))(t + Bk)|^2 \leq \eta_2
\]  
(22)

for all \(t \in \{Bx \mid x \in \mathbb{T}^2\}\).

**Proof.** For \(f(t) = (s \ast_{sd} \phi)(t)\), by Lemma 2.6, we have

\[
(L_M f)(\xi) = e^{-i\pi \xi^T DB^{-1} \xi}(L_M s)(\xi)(L_M \phi)(\xi).
\]

Thus,

\[
|(L_M f)(\xi)|^2 = |(L_M s)(\xi)|^2 |(L_M \phi)(\xi)|^2.
\]
Since \(|(L_M s)(\xi)|\) is periodic with periodicity matrix \(B\), we have, from (14),

\[
\| (L_M f)(\xi) \|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} |(L_M s)(\xi)|^2 |(L_M \phi)(\xi)|^2 \, d\xi
\]

\[
= \sum_{k \in \mathbb{Z}^2} \int_{\{B(t+k) \mid t \in \mathbb{T}^2\}} |(L_M s)(\xi)|^2 |(L_M \phi)(\xi)|^2 \, d\xi
\]

\[
= \sum_{k \in \mathbb{Z}^2} \int_{\{Bt \mid t \in \mathbb{T}^2\}} |(L_M s)(\xi + Bk)|^2 |(L_M \phi)(\xi + Bk)|^2 \, d\xi
\]

\[
= \int_{\{Bt \mid t \in \mathbb{T}^2\}} |(L_M s)(\xi)|^2 \sum_{k \in \mathbb{Z}^2} |(L_M \phi)(\xi + Bk)|^2 \, d\xi
\]

\[
:= \int_{\{Bt \mid t \in \mathbb{T}^2\}} |(L_M s)(\xi)|^2 G_{\phi,M}(\xi) \, d\xi,
\]

where \(G_{\phi,M}(\xi) = \sum_{k \in \mathbb{Z}^2} |(L_M \phi)(\xi + Bk)|^2\) is the Grammian of \(\phi\) associated with the 2D-NS-LCT. Notice that

\[
\int_{\{Bt \mid t \in \mathbb{T}^2\}} |(L_M s)(\xi)|^2 \, d\xi
\]

\[
= \frac{1}{\det(B)} \sum_{n \in \mathbb{Z}^2} \sum_{k \in \mathbb{Z}^2} s(n) \overline{s(k)} e^{i\pi n^T B^{-1} A n - i\pi k^T B^{-1} A k} \int_{\{Bt \mid t \in \mathbb{T}^2\}} e^{-2i\pi (n-k)^T B^{-1} \xi} \, d\xi
\]

\[
= \sum_{n \in \mathbb{Z}^2} \sum_{k \in \mathbb{Z}^2} s(n) \overline{s(k)} e^{i\pi n^T B^{-1} A n - i\pi k^T B^{-1} A k} \delta_{n,k}
\]

\[
= \sum_{n \in \mathbb{Z}^2} |s(n)|^2 = \|s\|_2^2.
\]

From (23), we know that

\[
0 < \eta_1 \leq G_{\phi,M}(\xi) \leq \eta_2 < +\infty \text{ for all } \xi \in \{Bx \mid x \in \mathbb{T}^2\}
\]

is equivalent to

\[
\eta_1 \|L_M s\|^2 = \eta_1 \|s[k]\|_2^2 \leq \|(L_M f)(\xi)\|_{L^2(\mathbb{R}^2)}^2 \leq \eta_2 \|s[k]\|_2^2 \leq \eta_2 \|L_M s\|^2.
\]

This completes the proof.

The local behavior and global decay of \(\phi\) can be described in terms of the Wiener amalgam spaces as follows. A measurable function \(f\) belongs to the Wiener amalgam space \(W(L^p(\mathbb{R}^2))\), \(1 \leq p < \infty\), if it satisfies

\[
\|f\|_{W(L^p(\mathbb{R}^2))}^p := \sum_{k \in \mathbb{Z}^2} \text{ess sup} \{|f(x + k)|^p ; x \in \mathbb{T}^2\} < +\infty.
\]
If $p = \infty$, a measurable function $f$ belongs to $W(L^\infty(\mathbb{R}^2)) = L^\infty(\mathbb{R}^2)$, if it satisfies
\[
\|f\|_{W(L^\infty(\mathbb{R}^2))}^p := \sup_{k \in \mathbb{Z}^2} \left\{ \text{ess sup}\{ |f(x + k)|; x \in \mathbb{T}^2 \} \right\} < +\infty. \tag{26}
\]

Considering that ideal sampling makes sense only for continuous functions, we therefore focus on the amalgam space $W_0(L^p(\mathbb{R}^2)) := W(L^p(\mathbb{R}^2)) \cap C(\mathbb{R}^2)$, where $C(\mathbb{R}^2)$ denotes the space of continuous functions on $\mathbb{R}^2$.

The multivariate dynamical sampling problem in shift-invariant spaces is to recover a function $f \in V(\phi)$ from its dynamical sampling measurements associated with the 2D-NS-LCT, i.e.,
\[
\{(a^j \ast_c f)(M^T k) : j = 1, 2, \cdots, m - 1, \; k \in \mathbb{Z}^2 \},
\]
where $a^j = a \ast_c \cdots \ast_c a$ and $a \in W(L^1(\mathbb{R}^2))$.

First, we present an important lemma, which will be used later.

**Lemma 4.2.** Let a sequence $s \in l^2(\mathbb{Z}^2)$ and two functions $f, g \in L^2(\mathbb{R}^2)$. Then, we have
\[
f \ast_c (s \ast_{sd} g) = s \ast_{sd} (f \ast_c g) \tag{27}
\]

**Proof.** Letting $h_1(t) = (s \ast_{sd} g)(t)$ and $h_2(t) = (f \ast_c g)(t)$, we have
\[
(f \ast_c h_1)(t) = \frac{1}{\sqrt{\det (iB)}} e^{-i\pi t^T B^{-1} A t} \int_{\mathbb{R}^2} f(t - x) e^{i\pi (t-x)^T B^{-1} A (t-x)} p_1(x) e^{i\pi x^T B^{-1} A x} dx
\]
\[
= \frac{1}{\sqrt{\det (iB)}} e^{-i\pi t^T B^{-1} A t} \int_{\mathbb{R}^2} f(t - x) e^{i\pi (t-x)^T B^{-1} A (t-x)} g(x) e^{i\pi x^T B^{-1} A (x-k)} dx
\]
\[
= \frac{1}{\det (iB)} e^{-i\pi t^T B^{-1} A t} \sum_{k \in \mathbb{Z}^2} s(k) e^{i\pi k^T B^{-1} A k} \left( \int_{\mathbb{R}^2} f(t - k - y) e^{i\pi (t-k-y)^T B^{-1} A (t-k-y)} dy \right)
\]
\[
= \frac{1}{\sqrt{\det (iB)}} e^{-i\pi t^T B^{-1} A t} \sum_{k \in \mathbb{Z}^2} s(k) e^{i\pi k^T B^{-1} A k} h_2(t - k) e^{i\pi (t-k)^T B^{-1} A (t-k)}
\]
\[
= (s \ast_{sd} h_2)(t).
\]
This completes the proof. \qed
In the following, we propose a sufficient and necessary condition for stably recovering \( f \) from its dynamical sampling measurements \( f(M^Tk), (a^j \ast_c f)(M^Tk), j = 1, 2, \cdots, m - 1, k \in \mathbb{Z}^2 \) associated with the 2D-NS-LCT.

**Theorem 4.3.** Let \( \phi = \phi_0 \in W_0(L^1(\mathbb{R}^2)), a \in W(L^1(\mathbb{R}^2)) \), and \( \phi_j = a^j \ast_{c} \phi \), then \( L_M\phi^j \in C(\mathbb{R}^2) \) for \( l, j = 0, 1, \cdots, m - 1 \), where \( \phi^j_l(r) = \phi_j(M^Tr - \eta_l)e^{i\pi(M^Tr - \eta_l)^TB^{-1}A(M^Tr - \eta_l)} \). Any \( f \in V(\phi) \) can be recovered in a stable way, i.e., the inverse of \( \mathcal{B}(\xi) \) is bounded, from the dynamical sampling measurements \( f(M^Tk), a^j \ast_c f(M^Tk), j = 1, 2, \cdots, m - 1, k \in \mathbb{Z}^2 \), if and only if \( \det(\mathcal{B}(\xi)) \neq 0 \) for any \( \xi \in \mathbb{R}^2 \), where \( \mathcal{B}(\xi) \) is defined by

\[
\mathcal{B}(\xi) = \begin{bmatrix}
(L_M\phi^0_l)(\xi) & (L_M\phi^0_l)(\xi) & \cdots & (L_M\phi^0_{m-1}(\xi) \\
(L_M\phi^1_l)(\xi) & (L_M\phi^1_l)(\xi) & \cdots & (L_M\phi^1_{m-1}(\xi)) \\
\vdots & \vdots & \ddots & \vdots \\
(L_M\phi^{m-1}_0)(\xi) & (L_M\phi^{m-1}_1)(\xi) & \cdots & (L_M\phi^{m-1}_{m-1}(\xi))
\end{bmatrix}.
\]

**Proof.** Given a function \( f \in V(\phi) \), we have

\[
f = (s \ast_sd \phi)(t).
\]

By Lemma 4.2 we obtain

\[
v_j(k) \triangleq (a^j \ast_c f)(M^Tk)e^{i\pi(M^Tk)^TB^{-1}AM^Tk-i\pi k^TB^{-1}A}k
\]

\[
= [a^j \ast_c (s \ast_sd \phi)](M^Tk)e^{i\pi(M^Tk)^TB^{-1}AM^Tk-i\pi k^TB^{-1}A}k
\]

\[
= [s \ast_sd (a^j \ast_c \phi)](M^Tk)e^{i\pi(M^Tk)^TB^{-1}A}k \ast (a^j \ast_c \phi)(M^Tk)
\]

\[
= \frac{1}{\sqrt{\det(AB)}}e^{-i\pi k^TB^{-1}A}k \sum_{n \in \mathbb{Z}^2} s(n)e^{i\pi n^TB^{-1}A}n \ast (a^j \ast_c \phi)(M^Tk - n)
\]

\[
= \frac{1}{\sqrt{\det(AB)}}e^{-i\pi k^TB^{-1}A}k \sum_{n \in \mathbb{Z}^2} s(n)e^{i\pi n^TB^{-1}A}n \ast \phi_j(M^Tk - n)
\]

\[
= \frac{1}{\sqrt{\det(AB)}}e^{-i\pi k^TB^{-1}A}k \sum_{l=0}^{m-1} \sum_{r \in \mathbb{Z}^2} s(M^Tr + \eta_l)e^{i\pi(M^Tr + \eta_l)^TB^{-1}A}k \ast (a^j \ast_c \phi)(M^Tk)
\]

\[
= \sum_{l=0}^{m-1} (s_l \ast_sd \phi^j_l)(k),
\]

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where \( s_l(r) = s(M^T r + \eta_l)e^{i\pi(M^T r + \eta_l)^TB^{-1}A(M^T r + \eta_l)}. \)

Then, by Lemma 2.6 we readily have, for \( j = 1, 2, \cdots, m - 1, \)

\[
(L_Mv_j)(\xi) = \sum_{l=0}^{m-1} e^{-i\pi \xi^TDB^{-1}\xi} (L_Ms_l)(\xi)(L_M\phi_l^j)(\xi).
\] (28)

Let

\[
(L_Mv)(\xi) = \begin{bmatrix}
(L_Mv_0)(\xi) \\
(L_Mv_1)(\xi) \\
\vdots \\
(L_Mv_{m-1})(\xi)
\end{bmatrix}
\]

and

\[
(L_Ms)(\xi) = e^{-i\pi \xi^TDB^{-1}\xi} \begin{bmatrix}
(L_Ms_0)(\xi) \\
(L_Ms_1)(\xi) \\
\vdots \\
(L_Ms_{m-1})(\xi)
\end{bmatrix}.
\]

Hence, from (28), we get

\[
(L_Mv)(\xi) = B(\xi)(L_Ms)(\xi).
\] (29)

That is to say, we can solve the equation (29) with respect to \((L_Ms)(\xi)\), if \(B(\xi)\) is invertible.

5. Example of multivariate dynamical sampling

In this section, to make the main results obtained above more transparent and more complete, we give a simple example to show that the necessary and sufficient condition in Theorem 3.3 is feasible.

Let matrix parameters \( A, B, C, D \) of the 2D-DT-NS-LCT be

\[
A = \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 1 \\
1 & 3
\end{bmatrix}, \quad C = \begin{bmatrix}
-0.5 & 0.5 \\
0.5 & 0.5
\end{bmatrix}, \quad D = B,
\]

which obviously satisfy (2). Let the dilation matrix \( M \in \mathbb{Z}^{2 \times 2} \) be given by \( M = B \). Then, we have \( m = |\det(M)| = 2 \), and \( \gamma_0 = [0, 0]^T, \gamma_1 = [1, 2]^T. \) A sequence \( a \) is given by

\[
a(k) = \begin{cases}
c_1, & k = [-1, -1]^T, \\
c_2, & k = [-1, -2]^T, \\
0, & \text{otherwise},
\end{cases}
\]
where \( c_1, c_2 \in \mathbb{R} \) and \( c_2 \neq 0 \). Let \( \xi = [\xi_1, \xi_2]^T \), then we have

\[
\begin{align*}
  e_0^0 &= \eta_{ \mathcal{M} } (\xi) = e^{-i \pi (\xi_1^2 + \xi_2^2)}, \\
  e_0^1 &= \eta_{ \mathcal{M} } (\xi + \gamma_1) = e^{-i \pi [(\xi_1 + 1)^2 + (\xi_2 + 2)^2]}, \\
  e_1^0 &= \eta_{ \mathcal{M} } (\xi) = e^{-2i \pi (\xi_1^2 + \xi_2^2)}, \\
  e_1^1 &= \eta_{ \mathcal{M} } (\xi + \gamma_1) = e^{-2i \pi [(\xi_1 + 1)^2 + (\xi_2 + 2)^2]}, \\
  (L_{ \mathcal{M} } a)(\xi) &= -\frac{c_1}{\sqrt{2i}} e^{i \pi (2 \xi_1 + \xi_1^2 + \xi_2^2)} - \frac{c_2}{\sqrt{2}} e^{i \pi (\xi_1 + \xi_2 + \xi_1^2 + \xi_2^2)}, \\
  (L_{ \mathcal{M} } a)(\xi + \gamma_1) &= -\frac{c_1}{\sqrt{2i}} e^{i \pi [2 \xi_1 + (\xi_1 + 1)^2 + (\xi_2 + 2)^2]} + \frac{c_2}{\sqrt{2}} e^{i \pi [\xi_1 + \xi_2 + (\xi_1 + 1)^2 + (\xi_2 + 2)^2]}.
\end{align*}
\]

Therefore, through easy computation, we get

\[
|\det (A_{ \mathcal{M} } (\xi))| = \left| \det \left( \begin{bmatrix} e_0^0 & e_1^0 \\ e_0^1 (L_{ \mathcal{M} } a)(\xi) & e_1^1 (L_{ \mathcal{M} } a)(\xi + \gamma_1) \end{bmatrix} \right) \right| = \sqrt{2} |c_2| > 0.
\]

That is to say, the sequence \( a \) we choose can make \( A_{ \mathcal{M} } (\xi) \) invertible for any \( \xi \), thereby Theorem 3.3 holds.

6. Conclusion

In this paper, we investigate the multivariate dynamical sampling in the \( l^2(\mathbb{Z}^2) \) space associated with the 2D-DT-NS-LCT and shift-invariant spaces associated with the 2D-NS-LCT, respectively. More specifically, we obtain a sufficient and necessary condition under which a sequence in \( l^2(\mathbb{Z}^2) \) (or a function in a shift-invariant space \( V(\phi) \) that is generated by \( \phi \in L^2(\mathbb{R}^2) \)) can be stably recovered from its dynamical sampling measurements associated with the 2D-DT-NS-LCT (or the 2D-NS-LCT). Our results extend the original ones in the FT domain.

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