Construction of gauge invariant effective nucleonic theories: functional approach

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Abstract

Starting from relativistic quantum field theories, describing interacting nucleons and pions coupled to the dynamical electromagnetic field, the pion degrees of freedom are eliminated by means of functional integration. Apart from taking into account some operators perturbatively in $e$, e.g. the vacuum polarization, this procedure is exact, giving effective theories for nucleons and photons. The subsequent nonrelativistic reduction yields the corresponding nonrelativistic quantum field theory. The latter is unique, irrespective of the precise form of the original nucleon-pion interaction. Nucleonic potentials and electromagnetic interactions are mutually consistent. Local gauge invariance is satisfied at any stage of the formal developments.

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1 Introduction

Low energy strong interaction physics can be rather well described in terms of effective theories with phenomenological interactions. Coupling the electromagnetic field, however, is ambiguous since it is essentially impossible to uniquely construct the relevant currents without a deeper understanding of the origin of the interactions. Local gauge invariance clearly shows the necessity of additional, i.e., beyond the standard one-body terms, electromagnetic interaction terms by means of constraints. These gauge conditions for ‘exchange currents’ cannot fix them completely: transverse part of currents are not constrained. It may appear frustrating that these eventually yield the physical amplitudes. Nevertheless, in particular in combination with Lorentz- or rotational invariance, local gauge invariance is a powerful concept in effective models also.

Electromagnetic current conservation is a consequence of global $U(1)$ gauge symmetry. Therefore, the question arises whether the stronger condition of local gauge invariance is necessary in nonrelativistic theories. This difference, global versus local, may have practical consequences, for example with respect to Siegert’s hypothesis \[1\]. The latter can be shown to be a consequence of local $U(1)$ gauge symmetry in nonrelativistic quantum mechanics \[2\]. Thus effective theories where the charge density is modified, e.g. \[3\], are not invariant under the usual local phase transformations although the current is conserved. The breaking seems to be caused by the chosen procedure of eliminating degrees of freedom by means of unitary transformations followed by projection on a subspace of the Hilbert space \[4\]. A more recent example of this method is the derivation of an effective meson-exchange model for pion-nucleon scattering and pion photo- and electroproduction \[5\].

Another method to obtain nuclear Hamiltonians and effective electromagnetic currents from a model with interacting nucleons and mesons is the (extended) $S$-matrix approach \[6\]. Alternatively, Friar developed a perturbation technique to eliminate meson degrees of freedom exploiting the equations of motion \[7\]. Thus, effective baryonic theories are often perturbatively constructed via the elimination of mesons from an interacting model. In this work we address the following problem of this kind: how to get, from a pion-nucleon Lagrangian, to an effective nucleon theory including electromagnetic interactions. In contrast to the examples above, the elimination procedure is nonperturbative. In principle our methods are general and can be applied to other models as well; in practice the feasibility depends on the appearing interactions. It is essential that we couple the electromagnetic field from the very beginning and take this gauge invariant Lagrangian as starting point. Then we derive a relativistic quantum field theory describing only nucleons and photons. The corresponding Lagrangian contains nonlocal interactions. Eventually we arrive at a theory with nonrelativistic nucleons. At any stage local gauge invariance is satisfied and, consequently, Ward-Takahashi identities \[8\] hold. In this way, given the original action, the deduced interactions and electromagnetic currents are mutually consistent and no ambiguities arise.

It should be emphasized that our starting point is a local field theory, describing point-like hadrons. No form factors are put in by hand; the structure of the nucleon is generated by the pions. Since renormalizability is not an issue here, one may of course put in a Pauli-term, which partly accounts for the anomalous magnetic moment of the nucleon. As is well-known, such a term is separately gauge invariant and obviously does not contain the pion field. Consequently, the developments and discussions in this work do not depend on its possible presence.

We work in the path-integral formulation of quantum field theory. The explicit elimination of the pions is done by means of functional integration. We have chosen this approach for several reasons. First, it allows for an ‘exact’, nonperturbative removal of the pion fields. Secondly, the electromagnetic field can be treated dynamically. However, the effective self-coupling of the photons -induced by charged pions- as well as some appearing operators are only calculated perturbatively in $\epsilon$. Thirdly, it is relatively easy to demonstrate that the appearing effective actions reflect the local $U(1)$ gauge symmetry. Finally, by means of sources one can easily identify relevant Green functions, including those with an external pion. In other words, even after integrating out the pions one could still calculate, e.g., pion photoproduction.

In the next section we present the formalism and subsequently derive the effective actions for nucleons interacting with an external electromagnetic field, starting from pseudoscalar, pseudovector and mixed pion-nucleon couplings. Section 3 contains the extension to dynamical photons. Local gauge invariance is explicitly verified in section 4. Section 5 deals with nonrelativistic approximations. Finally, we summarize and discuss possible applications in section 6. Some technical aspects are relegated to Appendices.
2 Functional integration

2.1 Path-integral quantization

Let us first address the notation and some well-known aspects of the formalism. Since we want to concentrate on gauge invariance aspects, we do not explicitly introduce isospin. Recall that the electromagnetic interaction breaks isospin symmetry. Our choices of the strong interaction terms, however, respect this symmetry. The proton and neutron fields are denoted by $\Psi_p, \Psi_n$, respectively. The neutral pion, $\pi^0$, is described with a real scalar field $\phi$. For the charged pions, $\pi^+ \text{ and } \pi^-$, we introduce one complex scalar field $\Phi$. The relation to the real isovector components is given by $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$. We will use pseudoscalar as well pseudovector pion-nucleon interactions; the corresponding coupling constants $g$ and $f$ are related: $\frac{g}{m} = \frac{f}{M}$, with $m$ the (common) pion mass and $M$ the (common) nucleon mass. The electromagnetic field strength tensor $F_{\mu\nu}$ is given in terms of the gauge fields $A_{\mu}$ by $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$.

In the path-integral formulation of quantum field theory the generating functional $Z$ is the central object from which Green’s functions and $S$-matrix elements (in principle) can be obtained. Of course, the generating functional is determined by the Lagrangian of the theory. The general formalism is presented in modern books on field theory, e.g. [9]. Here we briefly sketch the connection for the models under consideration.

The Lagrangian density $\mathcal{L}(x)$ to be specified later- can be extended with sources and gauge-fixing term for the Lorentz-gauge

$$\mathcal{L}_s(x) = \mathcal{L}(x) + \bar{\Psi}_p \gamma^\mu \partial_\mu \Psi_p + \bar{\Psi}_p \gamma^\mu \partial_\mu \Psi_n + \bar{\Psi}_n \gamma^\mu \partial_\mu \Psi_n + \Phi^*J + J^*\Phi + \phi J^0 + J^\mu A_\mu - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2. \quad (1)$$

The (source dependent) action $S_s$ appears in the generating functional $Z$,

$$Z = \mathcal{N} \int DA_{\mu}D\bar{\Psi}_p D\Psi_p D\bar{\Psi}_n D\Psi_n D\Phi^* D\Phi D\phi \exp(iS_s) = \mathcal{N} \int DA_{\mu} \cdots D\phi \exp \left(i \int d^4x \mathcal{L}_s(x) \right). \quad (2)$$

The Faddeev-Popov ghost term has been absorbed in the the normalization $\mathcal{N}$; this is possible because in abelian theories the ghosts do not couple to physical fields [9]. It should be remarked that for the formal developments in this section it is irrelevant whether or not one includes the gauge-fixing term, fermion sources and the photon source. In other words, these terms can also be added in a later stadium, in particular after integrating out the pion fields.

For the theories we will consider below, the action without the sources and the gauge-fixing term is locally gauge invariant. The local gauge transformations are explicitly given by

$$A_\mu \rightarrow A_\mu + \partial_\mu \chi, \quad \Psi_p \rightarrow \exp(-ie\chi)\Psi_p, \quad \Phi \rightarrow \exp(-ie\chi)\Phi, \quad (3)$$

where $\chi(x)$ is an arbitrary function. The fields $\Psi_n$ and $\phi$ are invariant since they describe neutral particles. This gauge symmetry leads to relations between vertex functions and propagators, i.e., Ward-Takahashi identities. In the functional formalism, they are readily derived starting from the generating functional $Z$ [9]. Note that the integration measure in $Z$ is also invariant. Therefore one may a priori expect that exactly integrating out the pions yields an effective action which, again without sources and gauge fixing term, is invariant under gauge transformations of the fields left. In turn, this implies Ward-Takahashi identities for the relevant Green’s functions. Nevertheless, it seems to be appropriate to explicitly check the invariance of the appearing effective actions, especially if further approximations are involved.

2.2 Pseudoscalar coupling

The Lagrangian density for nucleons and pions with pseudoscalar coupling interacting with the dynamical electromagnetic field reads

$$\mathcal{L}(x) = i\bar{\Psi}_p \gamma^\mu (\partial_\mu + ieA_\mu)\Psi_p + i\bar{\Psi}_n \gamma^\mu \partial_\mu \Psi_n - M\bar{\Psi}_p \Psi_p - M\bar{\Psi}_n \Psi_n - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial_\mu - ieA_\mu)\Phi^* (\partial^\mu + ieA^\mu)\Phi + \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - m^2\Phi^* \Phi - \frac{1}{2}m^2\phi^2 - ig\sqrt{2}\bar{\Psi}_p \gamma_5 \Psi_p \Phi - ig\sqrt{2}\bar{\Psi}_n \gamma_5 \Psi_n \Phi - ig\bar{\Psi}_p \gamma_5 \Psi_p - \bar{\Psi}_n \gamma_5 \Psi_n \phi. \quad (4)$$
As mentioned in the introduction, one can add Pauli terms for the nucleons; they are proportional to $\Psi \sigma^{\mu \nu} \Psi F_{\mu \nu}$. The action $S = \int d^4x \mathcal{L}(x)$ is locally gauge invariant in either case. In order to prepare the functional integration we rewrite the terms where the covariant derivative acts on the complex field,
\[
\int d^4x \left[ (\partial_\mu - ieA_\mu) \Phi^* (\partial^\mu + ieA^\mu) \Phi - m^2 \Phi^* \Phi \right] = - \int d^4x \Phi^* \mathcal{O}_A \Phi ,
\]
with the differential operator
\[
\mathcal{O}_A = \partial_\mu \partial^\mu + m^2 + 2ieA_\mu \partial^\mu + ie(\partial^\mu A_\mu) - e^2 A^2 .
\]
In the following we will also use the operator $\mathcal{O}$ which can be obtained from $\mathcal{O}_A$ by putting $e = 0$. Now we extend this action by the inclusion of sources and gauge fixing term (cf. eqs. (1, 2)), and rearrange the Lagrangian as follows
\[
\mathcal{L}_s = \mathcal{L}_1 - \Phi^* \mathcal{O}_A \Phi + \Phi^* F + \bar{\Phi} F - \frac{1}{2} \phi O \phi + \phi F_0 ,
\]
with the nucleon-photon Lagrangian
\[
\mathcal{L}_1 (x) = i \bar{\Psi}_{\gamma} (\partial_\mu + ieA_\mu) \Psi_{\mu} + i \bar{\Psi}_{\gamma} \partial_\mu \Psi_{\mu} - M \bar{\Psi} \Psi - M \bar{\Psi} \Psi - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}
\]
\[
+ \bar{\eta} \Psi \eta_{\mu} + \bar{\eta} \Psi \eta_{\nu} + \bar{\eta} \Psi \eta_{\mu} + \eta^\mu A_{\mu} - \frac{1}{2} \phi \phi F_0 ,
\]
and the generalized sources
\[
F = -ig\sqrt{2} \bar{\Psi}_{\gamma} \partial_\mu \Psi_{\mu} + J ,
\]
\[
\bar{F} = -ig\sqrt{2} \bar{\Psi}_{\gamma} \partial_\mu \Psi_{\mu} + J^* ,
\]
\[
F_0 = -ig\sqrt{2} \bar{\Psi}_{\gamma} \partial_\mu \Psi_{\mu} + \Psi_{\gamma} \Psi_{\mu} + J_0 .
\]
The latter effectively account for the pion-nucleon interaction.

We define the effective action $S_{cf}$ by functional integration over the pion fields,
\[
\exp (i S_{cf}) = \mathcal{N} \int \mathcal{D} \Phi^* \mathcal{D} \Phi \exp (i S_s)
\]
\[
= \mathcal{N} \exp \left( i \int d^4x \mathcal{L}_1 (x) \right) \cdot \int \mathcal{D} \phi \exp \left( -i \int d^4x \frac{1}{2} \phi O \phi - \phi F_0 \right)
\]
\[
\cdot \int \mathcal{D} \Phi^* \mathcal{D} \Phi \exp \left( -i \int d^4x \left[ \Phi^* \mathcal{O}_A \Phi - \Phi^* F - \bar{\Phi} F \right] \right) .
\]
Since the generalized sources contain the fermion fields, this integral yields effective fermion-fermion interactions. The exchange currents are generated by the operator $\mathcal{O}_A$, which explicitly depends on the gauge field. Despite these nontrivial dependences the pion fields can now readily be integrated out; the appearing integrals are essentially gaussian.

Let us start with the neutral pions. The functional integral gives
\[
\int \mathcal{D} \phi \exp \left( -i \int d^4x \frac{1}{2} \phi O \phi - \phi F_0 \right) = C \exp \left( \frac{-i}{2} \int d^4x \int d^4y F_0 (x) G_0 (x - y) F_0 (y) \right) ,
\]
where $C \propto (\det O)^{-\frac{1}{2}}$ is an infinite constant, to be absorbed in $\mathcal{N}$. Furthermore we recognize the Feynman propagator $G_0$, which is defined as the inverse of the differential operator $O$,
\[
O (x) G_0 (x - y) = -\delta^4 (x - y) ,
\]
and explicitly reads
\[
G_0 (z) = \int \frac{d^4k}{(2\pi)^4} \frac{\exp (ikz)}{k^2 - m^2 + i\epsilon} .
\]
Note that we (re-)inserted the $i\epsilon$ prescription. In this way we obtain the following nonlocal term in the effective Lagrangian density

$$
\Delta L_0(x) = -\frac{1}{2} \int d^4y \, F_0(x) G_0(x-y) F_0(y). 
$$

(14)

Apart from the four-fermion interaction, it describes the coupling of the $\pi^0$ sources.

The integration over the complex fields is more involved. The reason is the $A$-dependence of the differential operator $O_A$. It implies that, for dynamical photon fields, its determinant cannot be absorbed in the normalization. Therefore, this determinant has to be explicitly evaluated. We postpone this issue and first restrict ourselves to external electromagnetic fields. Thus we do not consider here the full nucleon-photon Lagrangian $L_1$, but rather

$$
L^{\text{ex}}_1(x) = L_1(x) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu + \frac{1}{2\alpha} (\partial_\mu A^\mu)^2. 
$$

(15)

Then the formal $\Phi, \Phi^*$ integration is again straightforward

$$
\int D\Phi^* D\Phi \exp(-i \int d^4x \, [\Phi^* O_A \Phi - \Phi^* F - F \Phi]) = D \exp(-i \int d^4x \, \int d^4y \, F(x) G_A(x,y) F(y)), 
$$

where $G_A$ satisfies

$$
O_A(x) G_A(x,y) = -\delta^4(x-y). 
$$

(17)

This result corresponds to the nonlocal term

$$
\Delta L_A(x) = -\int d^4y \, F(x) G_A(x,y) F(y), 
$$

in the effective Lagrangian density. The latter is given by

$$
L^{\text{eff},\text{ex}}_1(x) = L^{\text{ex}}_1(x) + \Delta L_0(x) + \Delta L_A(x). 
$$

(19)

The construction of the effective action for pseudoscalar pion-nucleon coupling and external electromagnetic fields is complete at this point. However, the quantity $G_A$, describing the propagation of charged pions in a (given) electromagnetic field, has only been defined via a differential equation and—in contrast to $G_0$—has not been explicitly given yet. For general $A$-fields we were not able to construct an exact solution in a closed form. Nevertheless, one can calculate $G_A$ perturbatively in the electromagnetic coupling $e$. Note that the operator $O_A$ actually contains first and second order terms:

$$
O_A = O + eD - e^2 A^2, 
$$

(20)

with

$$
D = 2i A_\mu \partial^\mu + i(\partial_\mu A_\mu). 
$$

(21)

In Appendix A we will derive a method to construct $G_A$ to any order in the charge $e$. Herewith, the electromagnetic interactions of the nucleons can also be determined to arbitrary order in $e$.

### 2.3 Pseudovector coupling

Let us now consider pseudovector pion-nucleon coupling. In this case the interaction contains a derivative of the pion field. For example, instead of the pseudoscalar term $\sqrt{2}i g \bar{\Psi}_p \gamma_5 \gamma^\mu (\partial_\mu \Phi) \Psi_n$. Since the derivatives also act on the complex fields, describing charged pions, minimal coupling of the electromagnetic fields generates additional electromagnetic interactions. These are the well-known ‘contact terms’; indeed they are required by local gauge invariance. The full pseudovector Lagrangian density reads

$$
L^{PV}(x) = i \bar{\Psi}_p \gamma^\mu (\partial_\mu + ieA_\mu) \Psi_p + i \bar{\Psi}_n \gamma^\mu \partial_\mu \Psi_n - M \bar{\Psi}_p \Psi_p - M \bar{\Psi}_n \Psi_n - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} 
$$

$$
+ (\partial_\mu - ieA_\mu) \Phi^* (\partial^\mu + ieA^\mu) \Phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m^2 \Phi^* \Phi - \frac{1}{2} m^2 \phi^2 
$$

$$
- \frac{\sqrt{2} f}{m} \bar{\Psi}_p \gamma_5 \gamma_\mu (\partial^\mu \phi) \Psi_p - \bar{\Psi}_n \gamma_5 \gamma_\mu (\partial^\mu \phi) \Psi_n 
$$

$$
- \frac{\sqrt{2} f}{m} \bar{\Psi}_p \gamma_5 \gamma_\mu (\partial^\mu \Phi^*) \Psi_p - \bar{\Psi}_n \gamma_5 \gamma_\mu (\partial^\mu \Phi^*) \Psi_n. 
$$

(22)
2.4 Mixed coupling

Analogous to the pseudoscalar case, we consider the action including sources and gauge fixing term and rewrite $S_{PV}$ in terms of the operators $O_A$ and $O$. Furthermore, after more integrations by parts, one can define the pseudovector generalized sources as

$$F_{PV}^A = \frac{\sqrt{2}f}{m} \left[ \bar{\Psi}_n \gamma_5 \gamma_\mu (\partial^\mu \Psi_p) + (\partial^\mu \bar{\Psi}_n) \gamma_5 \gamma_\mu \Psi_p + ie \bar{\Psi}_n \gamma_5 \gamma_\mu A^\mu \Psi_p \right] + J,$$

$$\tilde{F}_{PV}^A = \frac{\sqrt{2}f}{m} \left[ \bar{\Psi}_p \gamma_5 \gamma_\mu (\partial^\mu \Psi_n) + (\partial^\mu \bar{\Psi}_p) \gamma_5 \gamma_\mu \Psi_n - ie \bar{\Psi}_p \gamma_5 \gamma_\mu A^\mu \Psi_n \right] + J^*,

$$F_{0}^{PV} = \frac{f}{m} \left[ \bar{\Psi}_p \gamma_5 \gamma_\mu (\partial^\mu \Psi_p) + (\partial^\mu \bar{\Psi}_p) \gamma_5 \gamma_\mu \Psi_p - \bar{\Psi}_n \gamma_5 \gamma_\mu (\partial^\mu \Psi_n) - (\partial^\mu \bar{\Psi}_n) \gamma_5 \gamma_\mu \Psi_n \right] + J_0 .$$

In contrast to pseudoscalar coupling, these generalized sources explicitly depend on the gauge field. Though the elimination of the pions was completely analogous, the resulting effective interactions are of course different for pseudoscalar and pseudovector coupling. Concomitantly, the additional electromagnetic interactions are consistently generated. Exchange currents are not only contained in $G_A$, due to the coupling of the photon to a propagating charged pion, but also in $F_{PV}^A$ in the form of contact terms.

2.4 Mixed coupling

With the apparatus developed so far, the extension to mixed, i.e., pseudoscalar and pseudovector pion-nucleon terms is straightforward. We will take as Lagrangian density

$$\mathcal{L}(x) = i \bar{\Psi}_p \gamma^\mu (\partial_\mu + ieA_\mu) \Psi_p + i \bar{\Psi}_n \gamma^\mu \partial_\mu \Psi_n - M \bar{\Psi}_p \Psi_p - M \bar{\Psi}_n \Psi_n - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}

+ (\partial_\mu - ieA_\mu) \Phi^*(\partial^\mu + ieA^\mu) \Phi + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - m^2 \Phi^* \Phi - \frac{1}{2} m^2 \phi^2

- ig \beta_1 \left[ \sqrt{2} \bar{\Psi}_p \gamma_5 \Psi_n \Phi + \sqrt{2} \bar{\Psi}_n \gamma_5 \Psi_p \Phi^* + (\bar{\Psi}_p \gamma_5 \psi_p - \bar{\Psi}_n \gamma_5 \psi_n) \phi \right]

- \frac{\beta_2 f \sqrt{2}}{m} \left[ \bar{\Psi}_p \gamma_5 \gamma_\mu [(\partial^\mu - ieA^\mu) \Phi] \Psi_n + \sqrt{2} \bar{\Psi}_n \gamma_5 \gamma_\mu [(\partial^\mu - ieA^\mu) \Phi^*] \Psi_p \right]

- \frac{\beta_3 f}{m} \left[ \bar{\Psi}_p \gamma_5 \gamma_\mu (\partial^\mu \phi) \Psi_p - \bar{\Psi}_n \gamma_5 \gamma_\mu (\partial^\mu \phi) \Psi_n \right],$$

with $\beta_1 + \beta_2 = 1$. The generalized sources simply are

$$F_{PV}^M = \beta_1 F + \beta_2 F_{PV}^A ,

\tilde{F}_{PV}^M = \beta_1 \tilde{F} + \beta_2 \tilde{F}_{PV}^A ,

F_{0}^M = \beta_1 F_0 + \beta_2 F_{0}^{PV} .$$
Herewith the results easily follow: just replace the earlier generalized sources by the mixed ones. In particular, one obtains for the effective Lagrangian
\[ \mathcal{L}_M^{eff}(x) = \mathcal{L}_M^+(x) + \Delta \mathcal{L}_0^M(x) + \Delta \mathcal{L}_A^M(x), \]
with
\[ \Delta \mathcal{L}_0^M(x) = -\frac{1}{2} \int d^4y \, F_0^M(x) G_0(x-y) F_0^M(y), \]
and
\[ \Delta \mathcal{L}_A^M(x) = -\int d^4y \, F_A^M(x) G_A(x,y) F_A^M(y). \]
This concludes the formalism for external electromagnetic fields.

### 3 Dynamical photons

The effective actions derived in the previous section describe nucleons in an external electromagnetic field. For dynamical photons the formalism needs to be extended. However, an exact extension is not possible yet and therefore perturbation theory in the electromagnetic coupling is applied. We present explicit results up to (and including) $O(\varepsilon^2)$.

The first step towards a full dynamical theory is trivial. In the effective Lagrangians (cf. eqs. [13][23][30]) one merely replaces the ‘external field’ nucleon-photon Lagrangian, eq. [13], by the original expression, eq. [8]. (Although the latter Lagrangian contains a gauge fixing term, in the following we will do a completely gauge invariant calculation.) The problem arises due to the integration of the complex pion field. Apart from the terms already given, the determinant of the operator $O_A$ contributes to the effective action. It represents a self-interaction term of the electromagnetic field. In other words, the photon propagator is modified. This is exactly what one physically expects: the charged pions polarize the vacuum and also after integrating them out the effect should be there.

Exploiting his proper-time formalism, Schwinger [10] explicitly calculated the analogous effective action due to the spin 1/2 field. Moreover, he gave the result for the spin zero charged field. Nevertheless, we believe it to be useful to present our alternative derivation, which is -lacking the genius of Schwinger- more straightforward but tedious and uses some modern techniques. Thus, in this respect, it may render the present paper not only self-contained but the result also more accessible.

Our calculation starts with the famous trace-log formula for an operator $Q$, $\det Q = \exp(\text{tr} \ln Q)$. Application to the problem in question yields the effective photon action
\[ \Delta \mathcal{S}_A^{ef} \propto \text{tr} \ln(-G_A) - \text{tr} \ln(-G_0) = \text{tr} \ln(G_0^{-1}G_A), \]
where we have subtracted the corresponding expression for $G_0$, in order to get rid of the first, trivial divergence. We insert the perturbative result for $G_A$ derived in Appendix A, and further expand in $\varepsilon$
\[ \Delta \mathcal{S}_A^{ef} \propto e \text{tr} (DG_0) + e^2 \text{tr} \left( \frac{1}{2} DG_0 DG_0 - A^2 G_0 \right) + O(\varepsilon^3). \]

Since the appearing explicit expressions contain infinities one has to regularize the theory. We choose the dimensional regularization scheme which preserves gauge invariance. The appearing momentum as well as space-time integrals are defined in $n$ dimensions and expanded around $n = 4$. For dimensional reasons, one also replaces $\varepsilon$ by $\varepsilon M_0^{2-n/2}$, where $M_0$ is an arbitrary reference mass. We refer to the textbooks, e.g. [11], for more details and the original references.

Let us start with the first order term. The appearing operator follows from eqs. [13] and [21], and since $\text{tr} Q = \int d^n x \, Q(x,x)$ we get
\[ \text{tr} (DG_0) = \int d^n x \left[ (i\partial_\mu^2 A_\mu) \int \frac{d^nk}{(2\pi)^n} \frac{1}{k^2 - m^2 + i\varepsilon} - 2A_\mu(x) \int \frac{d^nk}{(2\pi)^n} \frac{k_\mu}{k^2 - m^2 + i\varepsilon} \right] = 0. \]
The first term vanishes because $\int d^n x \, (i\partial_\mu^2 A_\mu) = 0$, the second one because $\int \frac{d^nk}{(2\pi)^n} \frac{k_\mu}{k^2 - m^2 + i\varepsilon} = 0$. Consequently, no $O(\varepsilon)$ term in the effective photon action is generated. In other words, the vacuum is not polarized in first order.
The calculation of the second order term is more strenuous. Putting in the operators and integration by parts yields

\[
\text{tr}\left(\frac{1}{2}DG_0DG_0 - A^2G_0\right) = \int d^n x \int d^n z \int d^n l (2\pi)^n A_{\mu}(x)A_{\nu}(z) \frac{\exp il(x-z) \exp iq(z-x)}{l^2 - m^2 + i\epsilon q^2 - m^2 + i\epsilon} \\
\times \left[ 2\gamma^\nu q^\mu - l^\mu(q - l)^\nu - q^\nu(l - q)^\mu \right] \\
- \int d^n x A^2(x) \int d^n k (2\pi)^n \frac{1}{k^2 - m^2 - i\epsilon}.
\]

(36)

These integrations can be done and the results expanded around \(4-n\). We refer to Appendix B for the technical details and immediately proceed to the result, the effective regularized action \(S_A^{\text{eff}} = S_A^0 + \Delta S_A^{\text{ef}}\), where (in momentum space, omitting \(O(4-n)\) and \(O(\epsilon^3)\))

\[
\Delta S_A^{\text{ef}} = \frac{e^2\pi^2}{3(2\pi)^4} \int \frac{d^np}{(2\pi)^n} A_{\mu}(-p)A_{\nu}(p) [p^\mu p^\nu - p^2 g^{\mu\nu}] \\
\times \left[ \frac{1}{4-n} - \frac{1}{2}\gamma_E - \frac{3}{2} \int_0^1 du (1-2u)^2 \ln \left( \frac{m^2 - u(1-u)p^2 - i\epsilon}{4\pi M_0^2} \right) \right],
\]

(37)

with Euler’s constant \(\gamma_E = 0.577\ldots\). Recall the free action

\[
S_A^0 = -\frac{1}{4} \int d^n x F_{\mu\nu}(x)F^{\mu\nu}(x) = -\frac{1}{4} \int \frac{d^nk}{(2\pi)^n} F_{\mu\nu}(-k)F^{\mu\nu}(k),
\]

(38)

with \(F_{\mu\nu}(k) = ik_{\mu}A_{\nu}(k) - ik_{\nu}A_{\mu}(k)\). Verifying \(F_{\mu\nu}(-k)F^{\mu\nu}(k) = 2A_{\mu}(-k)A_{\nu}(k) [k^2g^{\mu\nu} - k^\mu k^\nu]\), explicitly demonstrates the local gauge invariance of \(\Delta S_A^{\text{ef}}\), including the \((n = 4)\) divergent terms. It also enables us to rewrite the effective action as

\[
S_A^{\text{ef}} = -\frac{1}{4} \int \frac{d^nk}{(2\pi)^n} F_{\mu\nu}(-k)F^{\mu\nu}(k) \\
\times \left[ 1 + \frac{e^2\pi^2}{3(2\pi)^4} \left[ \frac{1}{4-n} - \frac{1}{2}\gamma_E - \frac{3}{2} \int_0^1 du (1-2u)^2 \ln \left( \frac{m^2 - u(1-u)p^2 - i\epsilon}{4\pi M_0^2} \right) \right] \right].
\]

(39)

The latter form is also suited to renormalize the theory. It shows that the term we need to subtract, \(i.e.,\) the counterterm, indeed has the same structure as the original action. As is usual in electrodynamics we perform ‘on-shell’ renormalization. Thus we choose the subtraction point to coincide with the physical photon mass, \(k^2 = 0\), and (apart from cancelling the divergence) demand that the pole of the renormalized propagator remains at \(k^2 = 0\) with residue 1. This uniquely fixes the subtraction to be

\[
\delta S_A = -\frac{1}{4} \int \frac{d^nk}{(2\pi)^n} F_{\mu\nu}(-k)F^{\mu\nu}(k) \frac{2e^2\pi^2}{3(2\pi)^4} \left[ \frac{1}{4-n} - \frac{1}{2}\gamma_E + \frac{1}{2} \ln \left( \frac{m^2}{4\pi M_0^2} \right) \right].
\]

(40)

In this way we get for the effective renormalized action

\[
S_A^{\text{ef}} = -\frac{1}{4} \int \frac{d^nk}{(2\pi)^n} F_{\mu\nu}(-k)F^{\mu\nu}(k) \\
\times \left[ 1 + \frac{e^2\pi^2}{3(2\pi)^4} \left[ \int_0^1 du (1-2u)^2 \ln \left( \frac{m^2 - u(1-u)p^2 - i\epsilon}{4\pi M_0^2} \right) \right] - \frac{1}{3} \ln \left( \frac{m^2}{4\pi M_0^2} \right) \right].
\]

(41)

After integration by parts, hereby cancelling the reference mass \(M_0\), and by changing the integration variable to \(v = 2u - 1\), we finally obtain the finite and gauge invariant result

\[
S_A^{\text{ef}} = -\frac{1}{4} \int \frac{d^nk}{(2\pi)^n} F_{\mu\nu}(-k)F^{\mu\nu}(k) \left[ 1 + \frac{k^2}{6m^2} \frac{e^2}{(4\pi)^2} \int_0^1 dv \frac{v^4}{1 - \frac{v^2}{4m^2}(1-v^2) - i\epsilon} \right].
\]

(42)

We kept the \(i\epsilon\) prescription because the denominator in the equation above can vanish for \(k^2 \geq 4m^2\). This corresponds to pion pair production by the electromagnetic field.
4 Local gauge invariance

The results of the previous section, in particular the effective photon action, is manifestly gauge invariant. The local gauge invariance of the nonlocal Lagrangians derived in section (3) is not that obvious. Therefore, we want to verify this; it reduces to explicitly check the invariance of the additional terms $Δ_L^0, Δ_L^A, Δ_L^{PV}, Δ_L^{PV}^A, Δ_L^M, Δ_L^M_A$ without external sources.

It is easily seen that the generalized sources (with the external ones put to zero) have the following transformation properties under local gauge transformations:

\[
\begin{align*}
(F, F^{PV}_A, F^M_A) (x, j, 0) & \rightarrow \exp(-i e \chi(x)) (F, F^{PV}_A, F^M_A) (x, j, 0), \\
(\tilde{F}, \tilde{F}^{PV}_A, \tilde{F}^M_A) (x, j, 0) & \rightarrow \exp(i e \chi(x)) (\tilde{F}, \tilde{F}^{PV}_A, \tilde{F}^M_A) (x, j, 0), \\
(F_0, F^{PV}_0, F^M_0) (x, j_0, 0) & \rightarrow (F_0, F^{PV}_0, F^M_0) (x, j_0, 0).
\end{align*}
\]

The third relation immediately guarantees the invariance of $Δ_L^0, Δ_L^A, Δ_L^{PV}, Δ_L^{PV}^A, Δ_L^M, Δ_L^M_A$, corresponding to neutral pions. The first two transformations, however, require

\[G_A(x, y) \rightarrow G_{A+\partial \chi} = \exp(-i e \chi(x)) G_A(x, y) \exp(i e \chi(y)),\]

in order that $Δ_L^A, Δ_L^{PV}, Δ_L^{PV}^A, Δ_L^M, Δ_L^M_A$ are locally gauge invariant.

We first verify this relation nonperturbatively by only using the definition of $G_A$, eq. (47). After a gauge transformation one has

\[O_{A+\partial \chi} G_{A+\partial \chi}(x, y) = -\delta^4(x-y).\]  

Combining the latter two equations yields

\[O_A [\exp(i e \chi(x)) G_{A+\partial \chi} \exp(-i e \chi(y))] = \exp(i e \chi(x)) \exp(-i e \chi(y)) O_{A+\partial \chi} G_{A+\partial \chi}(x, y).\]

where also uniqueness of the Greens function has been used. This completes the proof.

However, recall that we only can provide a perturbative construction of the operator $G_A$. Do we indeed satisfy local gauge invariance order by order in $e$? In order to answer this question one needs to expand the gauge condition, eq. (44), using the expansion for $G_A$ (see Appendix A). For the gauge transformed $G_n$ we introduce the notation

\[G_n \rightarrow G_n + \delta G_n.\]

Local gauge invariance to zeroth order is trivial; in first order we need to show that

\[i [\chi(x) - \chi(y)] G_0(x-y) + \delta G_1(x, y) = 0.\]

Explicitly we have for $\delta G_1$,

\[\delta G_1(x, y) = i \int d^4z G_0(x-z) \left[ 2 \left( \partial_\mu^\nu \chi(z) \right) \partial_\nu^\mu + \left( \partial_\mu^\nu \partial_\nu^\mu \chi(z) \right) \right] G_0(z-y).\]

After integration by parts and putting in the definition of the free propagator, eq. (12), we readily obtain eq. (50). The second order term is more involved. The gauge condition reads

\[\frac{1}{2} [\chi(x) - \chi(y)]^2 G_0(x, y) + i [\chi(x) - \chi(y)] G_1(x, y) + \delta G_2(x, y) = 0.\]
Since $G_2 = G_0 D G_1 - G_0 A^2 G_0$ (see Appendix A), we get

$$
\delta G_2(x, y) = \int d^3 z G_0(x - z) \{ -i \mathcal{D}(z) (\chi(z) - \chi(y)) G_0(z - y) \\
+ i \left( 2(\partial_\mu^0 \chi(z)) \partial_\nu^0 + (\partial_\mu^0 \partial_\nu^0 \chi(z)) \right) G_1(z, y) \\
+ \left[ (2(\partial_\mu^0 \chi(z)) \partial_\nu^0 + (\partial_\mu^0 \partial_\nu^0 \chi(z)) \right) \chi(z) - \chi(y)) \\
- 2(\partial_\mu^0 \chi(z)) A^\mu(z) + (\partial_\mu^0 \chi(z)) \} G_0(z - y) \}.
$$

Now it is straightforward but somewhat tedious to verify eq. (52). Most conveniently one separates linear and quadratic terms in $\chi$. Furthermore, only integration by parts and the defining differential equations for $G_0$ and $G_1$ are needed. If some reader feels the inspiration, he/she is invited to show the local gauge invariance of the higher order terms. Note, however, that we gave a nonperturbative proof; in other words, these explicit verifications only serve as a check of the perturbative calculations.

We conclude this section by re-emphasizing that we have achieved to eliminate the pion degrees of freedom while maintaining local gauge invariance. Moreover, the induced effective strong and electromagnetic interactions are consistent and (given the original action) unambiguous. Furthermore, the results obtained so far are formally exact, in particular nonperturbative in the strong coupling constant. Because we have obtained a nonlocal field theory with four-fermion interactions, its applicability as a relativistic quantum field theory may be limited.

## 5 Nonrelativistic reduction

### 5.1 Propagators

In the nonrelativistic limit ($c \to \infty$) the differential operators $\mathcal{O}$ and $\mathcal{O}_A$ reduce to

$$
\mathcal{O} \to \mathcal{O}^N = m^2 - \Delta, \\
\mathcal{O}_A \to \mathcal{O}_A^N = \mathcal{O}^N + e\mathcal{D}^N - e^2 \vec{A}^2,
$$

where $O(1/c^2)$ terms have been neglected and

$$
\mathcal{D}^N = 2i A_k \partial^k + i(\partial^k A_k).
$$

Note that $\mathcal{O}_A^N$ is time dependent but does not contain time derivatives. Concomitantly, it does not depend on the scalar potential $A_0$; the indices $A$ below therefore refer to the vector potential $\vec{A}$ only. The inverse operators satisfy

$$
\mathcal{O}_A^N G_0^N(x, y) = -\delta^4(x - y), \\
\mathcal{O}_A^N G_0^N(x, y) = -\delta^4(x - y).
$$

We define nonrelativistic propagators by separating the delta function in time:

$$
G_0^N(x, y) = -\delta(x^0 - y^0) g_0(\vec{x}, \vec{y}), \\
G_A^N(x, y) = -\delta(x^0 - y^0) g_A(\vec{x}, \vec{y}, t),
$$

which immediately yields

$$
\mathcal{O}_A^N g_0(\vec{x}, \vec{y}) = \delta^3(\vec{x} - \vec{y}), \\
\mathcal{O}_A^N g_A(\vec{x}, \vec{y}, t) = \delta^3(\vec{x} - \vec{y}).
$$

Here we ‘divided out’ $\delta(x^0 - y^0)$, which apparently renders this procedure to be nonunique since the nonrelativistic propagators can be multiplied by a function $f(x^0, y^0)$ with $f(t, t) = 1$. However, these ambiguities will disappear via the integration over $y^0$.

The solution for the free static propagator is well-known,

$$
g_0(\vec{x}, \vec{y}) = \int \frac{d^3k}{(2\pi)^3} \frac{\exp(-i\vec{k}(\vec{x} - \vec{y}))}{k^2 + m^2} = \frac{\exp(-m|\vec{x} - \vec{y}|)}{4\pi|\vec{x} - \vec{y}|}.
$$
For arbitrary electromagnetic fields we again can construct $g_A$ only perturbatively in $e$. The actual construction is completely analogous to the relativistic case (see Appendix A) and starts with the expansion

$$g_A(\vec{x}, \vec{y}, t) = \sum_{n=0}^{\infty} e^n g_n(\vec{x}, \vec{y}, t).$$

(60)

We readily obtain

$$g_1 = -g_0 D^N g_0.$$  

(61)

Here and in the following the integrations are three-dimensional; thus explicitly $g_1$ reads

$$g_1(\vec{x}, \vec{y}, t) = -\int d^3z g_0(\vec{x}, \vec{z}) D^N(\vec{z}, t) g_0(\vec{z}, \vec{y}).$$

(62)

For $n \geq 2$ we get the recursion relation

$$g_n = -g_0 (D^N g_{n-1} - \vec{A}^2 g_{n-2}),$$

(63)

allowing for a calculation of $g_A$ to arbitrary order in $e$.

Let us address local gauge transformations. The transformation property of the relativistic propagator $G_A$, given in eq. (44), actually suggests

$$g_A(\vec{x}, \vec{y}, t) \rightarrow g_{A+\partial\chi} = \exp (-ie\chi(\vec{x}, t)) g_A(\vec{x}, \vec{y}, t) \exp (ie\chi(\vec{y}, t)).$$

(64)

Analogous to the relativistic case one can verify this relation either exactly, by using the defining differential equation (58), or perturbatively by means of the explicit construction given above.

5.2 Fermion fields

The nonrelativistic approximation of the nucleon fields is obtained in the standard way, see e.g. [11]. One considers the Dirac equation in an electromagnetic field and writes the fields as

$$\Psi_{p,n} = \begin{pmatrix} \phi_{p,n}(x) \\ \chi_{p,n}(x) \end{pmatrix} = e^{-iMt} \begin{pmatrix} \phi_{p,n}(x) \\ \chi_{p,n}(x) \end{pmatrix},$$

(65)

where $\phi$ and $\chi$ are two-component spinors. Then one approximately has

$$\chi_p(x) \simeq \frac{\vec{\sigma} \cdot \vec{p}}{2M} \phi_p(x), \quad \chi_n(x) \simeq -\frac{\vec{\sigma} \cdot \vec{p}}{2M} \phi_n(x),$$

(66)

$$\chi^\dagger_p(x) \simeq -\frac{\vec{\sigma} \cdot \vec{p}}{2M} \phi^\dagger_p(x), \quad \chi^\dagger_n(x) \simeq -\frac{\vec{\sigma} \cdot \vec{p}}{2M} \phi^\dagger_n(x),$$

with $\vec{\pi} = \vec{p} - e\vec{A}(x), \vec{\sigma} = \vec{p} + e\vec{A}(x)$ and $\vec{p} = -i\vec{\nabla}$. In case there are time derivatives, factors $M$ appear; for instance:

$$e^{-iMt} \phi_p = Me^{-iMt} \phi_p + e^{-iMt} (i\partial_t - eA_0) \phi_p,$$

$$e^{-iMt} \chi_p = -Me^{-iMt} \chi_p + e^{-iMt} (\vec{\sigma} \cdot \vec{\pi}) \phi_p.$$  

(67)

Analogous expressions for neutron and/or hermitian conjugate fields can readily be derived.

At this point the nonrelativistic reduction of the effective Lagrangians is straightforward. Some remarks, however, may be appropriate. Since the Dirac equation including electromagnetic field has been used above, the reduction is locally gauge invariant. On the other hand, the procedure obviously does not take into account the strong interaction terms in the equation of motion. Thus it is implicitly assumed that the strong interaction energy is small compared to the mass and therefore can be neglected in this order. Although this is common practice, it renders this nonrelativistic approximation somewhat uncontrolled in this case.
5.3 Nonrelativistic action

With the ingredients derived in the previous sections we obtain as nonrelativistic limit of the nucleon-photon Lagrangian $L_1$, eq. (8),

$$L_{NR}^N = \phi_p^\dagger (i\partial_0 - eA_0) \phi_p - \frac{1}{2M} \phi_p^\dagger (\vec{\sigma} \cdot \vec{\pi}) (\vec{\sigma} \cdot \vec{\pi}) \phi_p + \phi_p^\dagger i\partial_0 \phi_p - \frac{1}{2M} \phi_n^\dagger (\vec{\sigma} \cdot \vec{p}) (\vec{\sigma} \cdot \vec{p}) \phi_n - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$= \phi_p^\dagger (i\partial_0 - eA_0) \phi_p - \frac{1}{2M} \phi_p^\dagger (\vec{\pi}^2 - e\vec{\sigma} \cdot \vec{B}) \phi_p + \phi_p^\dagger i\partial_0 \phi_n - \frac{1}{2M} \phi_n^\dagger \vec{p} \cdot \vec{\phi} + \phi_n^\dagger \vec{p} \cdot \vec{\phi} + \frac{1}{2} (\vec{E}^2 - \vec{B}^2). \quad (68)$$

Here and in the following we neglect $O(1/c^2)$ contributions.

We proceed by approximating the generalized sources; for pseudoscalar coupling and zero external sources we easily get

$$F \simeq -ig \sqrt{2} \left( \phi_n^\dagger \vec{\sigma} \cdot \vec{\pi} \phi_p + \frac{\vec{p}}{2M} \phi_p^\dagger \vec{\sigma} \phi_p \right),$$

$$\bar{F} \simeq -ig \sqrt{2} \left( \phi_p^\dagger \vec{\sigma} \cdot \vec{p} \phi_n + \frac{\vec{\pi}}{2M} \phi_n^\dagger \vec{\sigma} \phi_n \right),$$

$$F_0 \simeq -ig \left( \phi_p^\dagger \vec{\sigma} \cdot \vec{p} \phi_n + \frac{\vec{\pi}}{2M} \phi_p^\dagger \vec{\sigma} \phi_p - \phi_n^\dagger \vec{\sigma} \cdot \vec{p} \phi_n + \frac{\vec{\pi}}{2M} \phi_p^\dagger \vec{\sigma} \phi_p \right). \quad (69)$$

Note that these nonrelativistic sources depend on the gauge field. This is due to the (gauge-invariant) elimination of $\chi_p$.

The analogous calculation for pseudovector coupling is more involved because these sources contain derivatives. Nevertheless, the final result is simple. In fact, using the relation between the coupling constants $g$ and $f$, i.e., $\frac{g}{f} = \frac{1}{\sqrt{2}m}$ yields

$$F_{PV}^A = F + O(1/c^2),$$

$$\bar{F}^A_{PV} = \bar{F} + O(1/c^2),$$

$$F_0^{PV} = F_0 + O(1/c^2). \quad (70)$$

Therefore, also mixed coupling produces the same result.

As a consequence, irrespective of chosen strong coupling (pseudoscalar, pseudovector or mixed), we find an unique lowest order result for the nonrelativistic reduction of the effective Lagrangians:

$$L_{NR}^N(x) = L_{NR}^N(x) + \frac{1}{2} \int d^3y F_{0}(x)g_0(\vec{x} - \vec{y})F_0(y) + \int d^3y \bar{F}(x)g_A(\vec{x}, \vec{y}, t)F(y), \quad (71)$$

with $x^0 = y^0 = t$. We can include the effective photon action derived in section (3),

$$S_{NR}^A = S_{eff}^A + \int d^4x L_{NR}^N(x) + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (72)$$

where we need to subtract the free photon term from $L$ because it is already contained in $S_{eff}^A$. This action is locally gauge invariant. We may add the nonrelativistic form of the gauge invariant Pauli terms in order to account for the anomalous magnetic moments of the nucleons. Furthermore, one may reintroduce the external sources and/or the gauge fixing term. In any case, this final result represents a nonrelativistic field theory for interacting nucleons, which also interact with the electromagnetic field.

6 Summary and outlook

Using functional integrations, we have shown a way to eliminate pion degrees of freedom from a relativistic quantum field theory describing nucleons, pions and dynamical photons. This elimination can be done exactly, i.e., nonperturbatively in the strong coupling constant, as long as the pion-nucleon interactions are linear in the pion fields. Indeed we have considered pseudoscalar, pseudovector and ‘mixed’ pion-nucleon interactions. The induced photon-photon interaction is only calculated up to order $c^2$ and agrees
with the result of Schwinger [10]. Another appearing operator, accounting for charged pion propagation in an electromagnetic field, is constructed perturbatively in $\epsilon$.

After having integrated out the pions, we have obtained a nonlocal relativistic quantum field theory which is nevertheless locally gauge invariant. Effective, mutually consistent, strong and electromagnetic interactions have appeared. Furthermore, the nucleons have acquired structure due to the pion cloud. At this point, it is not clear whether this intermediate result is suited for detailed practical calculations. For instance, the nonlocalities may prevent applications beyond the tree level. Further studies, addressing regularization and renormalization of the nonlocal action, are therefore desirable.

Here we have subsequently made a nonrelativistic reduction in order to arrive at a nonrelativistic quantum field theory for nucleons interacting with the (dynamical) electromagnetic field. In this nonrelativistic limit, the resulting action is unique irrespective of choosing pseudoscalar, pseudovector or mixed pion-nucleon coupling in the original action. Potentials, electromagnetic structure as well as exchange currents are consistently generated. The action is invariant under $U(1)$ local gauge transformations. This symmetry has several consequences. First, no ambiguities concerning exchange contributions in order to restore gauge invariance arise. Secondly, the nonrelativistic Ward-Takahashi [12] holds. Finally, as also can be seen explicitly, Siegert’s hypothesis is satisfied.

The resulting nonrelativistic field theory can be used to derive the Hamiltonian of the $N$-nucleon system, including the electromagnetic interactions. For $N = 1$ electromagnetic structure, i.e., form factors, will be isolated whereas for $N = 2$ the effective interaction and two-body currents will explicitly appear. In particular, it will be interesting to compare to other, existing, results for these exchange contributions. The formalism presented here may also provide a systematic way to derive relativistic corrections. Moreover, it can be applied to the analogous scalar and vector meson exchange models. We hope to address these issues in the nearby future.

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Appendix A: Pion propagation in an electromagnetic field

We perturbatively construct the inverse operator of $O_A$, i.e., $G_A(x, y)$ (cf. eq. (17)). Since the result is inductive, this method yields $G_A$ to any arbitrary order in the electromagnetic coupling constant $\epsilon$.

Let us start by assuming the following expansion

$$ G_A(x, y) = \sum_{n=0}^{\infty} \epsilon^n G_n(x, y) . $$

The $G_n$ are to be determined from the differential equation (cf. eqs. [17, 20, 21])

$$ [O + \epsilon D - \epsilon^2 A^2] \sum_{n=0}^{\infty} \epsilon^n G_n(x, y) = -\delta^4(x - y) . $$

The zeroth order solution is indeed nothing else than the free propagator,

$$ G_0(x, y) = G_0(x - y) , $$

which has been explicitly given in eq. (13). To first order we find

$$ G_1 = G_0 D G_0 , $$

which explicitly means

$$ G_1(x, y) = \int d^4 z G_0(x - z) D(z) G_0(z - y) . $$
The next term follows as
\[ G_2 = G_0 D G_1 - G_0 A^2 G_0 = G_0 D G_0 D G_0 - G_0 A^2 G_0 . \]  
(A.6)

Finally, one can easily verify the following recursion relation \((n \geq 2)\)
\[ G_n = G_0 D G_{n-1} - G_0 A^2 G_{n-2} . \]  
(A.7)

This completes the construction of the propagator \(G_A.\)

**Appendix B: Integrations**

First we use the Fourier-representation of the vector field, \(A_\mu(x) = \int \frac{d^n k'}{(2\pi)^n} A_\mu(k') \exp ik' x,\) in order to do the integrals over space-time in eq. (36)

\[
\text{tr} (...) = \int \frac{d^n l}{(2\pi)^n} \int \frac{d^n q}{(2\pi)^n} A_\mu(q-l) A_\nu(l-q) \frac{1}{l^2 - m^2 + i\epsilon} \frac{1}{q^2 - m^2 + i\epsilon} \cdot \left[ 2^{l\mu} q^\nu + l^\mu (l-q)^\nu + q^\nu (q-l)^\mu + \frac{1}{2} (l-q)^\mu (q-l)^\nu \right] - \int \frac{d^n p}{(2\pi)^n} A_\mu(-p) A_\nu(p) \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 - i\epsilon} . \]  
(B.1)

Introducing the new variables \(p = l - q, \ 2k = l + q,\) and combining the denominators by means of Feynman’s trick, yield

\[
\text{tr} (...) = 2 \int \frac{d^n p}{(2\pi)^n} A_\mu(-p) A_\nu(p) \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2 + 2(x - \frac{1}{2})kp + \frac{1}{4} p^2 - m^2 + i\epsilon)^2} \frac{k^\mu k^\nu}{2} \left[ (x - \frac{1}{2})^2 p^\mu p^\nu + \frac{f}{n - 2} g^{\mu\nu} \right] , \]  
(B.2)

These momentum integrals indeed can be evaluated in \(n\) dimensions \([9]\). We obtain

\[
\int \frac{d^n k}{(2\pi)^n} \frac{k^\mu k^\nu}{(k^2 + 2(x - \frac{1}{2})kp + \frac{1}{4} p^2 - m^2 + i\epsilon)^2} = \frac{i\pi^{n/2} \Gamma(2 - \frac{n}{2})}{(2\pi)^n} \frac{f}{\Gamma(2)} \left[ (x - \frac{1}{2})^2 p^\mu p^\nu + \frac{f}{n - 2} g^{\mu\nu} \right] , \]  
(B.3)

with

\[ f = f(p, x) = p^2 (x - \frac{1}{2})^2 - \frac{1}{4} p^2 + m^2 - i\epsilon = m^2 - x(1-x)p^2 - i\epsilon . \]  
(B.4)

At this point it is convenient to include the factor \(M_0^{4-n}\), which stems from the coupling constant, and the factor 2. Moreover, we also consider the \(x\)-integration; we get

\[
2 M_0^{4-n} \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \frac{k^\mu k^\nu}{(k^2 + 2(x - \frac{1}{2})kp + \frac{1}{4} p^2 - m^2 + i\epsilon)^2} = \frac{i\pi^2}{(2\pi)^4} \int_0^1 dx \left[ \frac{1}{2} \Gamma(2 - \frac{n}{2})(1 - 2x)^2 p^\mu p^\nu - \Gamma(1 - \frac{n}{2}) f g^{\mu\nu} \right] . \]  
(B.5)

The second momentum integral gives

\[
g^{\mu\nu} M_0^{4-n} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 - m^2 + i\epsilon} = g^{\mu\nu} \frac{i\pi^2}{(2\pi)^4} \Gamma(1 - \frac{n}{2}) m^2 \left[ \frac{m^2}{4\pi M_0^2} \right] ^{\frac{n-4}{2}} . \]  
(B.6)

As is clear from power counting, both integrals diverge for \(n \to 4\). Here these divergences manifest themselves as poles in the appearing gamma functions.
In order to isolate these poles, we expand the expressions above around \( n = 4 \). We immediately add the two integrals and obtain for the pole term \( P \):

\[
P = \frac{i\pi^2}{3(2\pi)^4} \frac{1}{4-n} \left[ p^\mu p'^\nu - p^2 g^{\mu\nu} \right].
\] (B.7)

As a consequence of the dimensional regularization scheme, it is gauge invariant. Neglecting \( O(4-n) \) yields the finite contribution \( F \),

\[
F = \frac{i\pi^2}{(2\pi)^4} \left( \frac{1}{6} \gamma_E + \frac{1}{2} \int_0^1 dx \, (1-2x)^2 \ln \left( \frac{f}{4\pi M_0^2} \right) \right) \left[ p^\mu p'^\nu - p^2 g^{\mu\nu} \right]
+ \frac{i\pi^2}{(2\pi)^4} \left[ \frac{p^2}{6} - m^2 \ln \left( \frac{m^2}{4\pi M_0^2} \right) + \int_0^1 dx \, \left( \frac{1}{2} p^2 (1-2x)^2 + f \right) \ln \left( \frac{f}{4\pi M_0^2} \right) \right] g^{\mu\nu}.
\] (B.8)

We separated the gauge invariant term. Adding the pole term given above to it, yields the result presented in the main text.

Thus it remains to be shown that the second, gauge variant, contribution \( \propto g^{\mu\nu} \) vanishes. In order to get rid of the logarithmic function one integrates by parts and finds

\[
[...] = p^2 \left[ \frac{1}{6} - \int_0^1 dx \, \frac{x(2x-1)}{f} \left( m^2 + p^2 (x^2 - \frac{3}{2} x + \frac{1}{2}) \right) \right].
\] (B.9)

Note that this expression is independent on the reference mass. We substitute \( z = 2x - 1 \) and indeed obtain for the integral

\[
\int_0^1 dx \, \frac{x(2x-1)}{f} = \frac{1}{2} \int_{-1}^1 dz \, \frac{z(z+1)}{m^2 - \frac{1}{4} p^2 (1-z^2)} - i \epsilon \left( m^2 + \frac{1}{4} p^2 (z^2 - z) \right)
= \frac{1}{4} \int_{-1}^1 dz \, z^2 = \frac{1}{6}.
\] (B.10)

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