Research Article

A Modified Dai–Liao Conjugate Gradient Method with a New Parameter for Solving Image Restoration Problems

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One adaptive choice for the parameter of the Dai–Liao conjugate gradient method is suggested in this paper, which is obtained with modified quasi–Newtonequation. So we get a modified Dai–Liao conjugate gradient method. Some interesting features of the proposed method are introduced: (i) The value of parameter \( t \) of the modified Dai–Liao conjugate gradient method takes both the gradient and function value information. (ii) We establish the global convergence property of the modified Dai–Liao conjugate gradient method under some suitable assumptions. (iii) Numerical results show that the modified DL method is effective in practical computation and the image restoration problems.

1. Introduction

In this paper, we consider the following unconstrained optimization problem:

\[
\min f(x), \quad x \in \mathbb{R}^n,
\]

(1)

where \( f: \mathbb{R}^n \to \mathbb{R} \) and \( f \) is a continuously differentiable function. Requiring simplicity and low memory storage [1–5], conjugate gradient (CG) methods [6–11] are useful tools for solving large-scale problems. Thus, we consider solving (1) with the conjugate gradient methods. A sequence of iterates is generated from a given \( x_0 \) with the following simple iteration formula:

\[
x_{k+1} = x_k + \alpha_k d_k, \quad k = 0, 1, 2, \ldots,
\]

(2)

where \( x_k \) is the \( k \)-th iteration point, the step size \( \alpha_k > 0 \) can be usually chosen to satisfy certain line search conditions, and \( d_k \) is the search direction defined by

\[
d_k = \begin{cases} 
-g_k, & \text{if } k = 0, \\
-g_k + \beta_k d_{k-1}, & \text{if } k \geq 1,
\end{cases}
\]

(3)

where \( \beta_k \) is a scalar. Many scholars presented some classical formulas of \( \beta_k \) as follows: the corresponding methods are called the Polak–Ribière–Polyak (PRP) [12, 13], Fletcher–Reeves (FR) [14], Hestense–Stiefel (HS) [15], conjugate descent (CD) [16], Liu–Storey (LS) [17], and Dai–Yuan (DY) [18] CG methods:

\[
\beta_k^{\text{PRP}} = \frac{g_k^T (g_k - g_{k-1})}{\|g_{k-1}\|^2},
\]

\[
\beta_k^{\text{FR}} = \frac{\|g_k\|^2}{\|g_{k-1}\|^2},
\]

\[
\beta_k^{\text{HS}} = \frac{g_k^T (g_k - g_{k-1})}{d_{k-1}^T (g_k - g_{k-1})},
\]

\[
\beta_k^{\text{CD}} = \frac{\|g_k\|^2}{-d_{k-1}^T g_{k-1}}.
\]
According to the above two equations, we have
\[ d_k^T y_{k-1} = d_k^T (H_k s_{k-1}) = -g_k^T s_{k-1}. \] (11)

Based on the above relations, Dai and Liao acquired the following conjugacy condition:
\[ d_k^T y_{k-1} = -t g_k^T s_{k-1}, \] (12)

where \( t \) is a positive parameter. In the case \( t = 0 \), (12) becomes (8). In the case \( t = 1 \), (12) becomes (11). Moreover, \( g_k^T s_{k-1} = 0 \) holds under the exact line search. It is easily observed that both (11) and (12) coincide with (8).

According to the above discussion, (12) can be considered as an extension of (8) and (11). Multiplying the definition of \( d_k \) with \( y_{k-1} \) and using (12), Dai and Liao introduced the new formula in the DL method as follows:
\[ \beta_{k}^{DL} = \frac{g_k^T y_{k-1} - t g_k^T s_{k-1}}{d_k^T y_{k-1}}. \] (13)

In [20], the conjugate gradient method with (13) establishes the global convergence property of uniformly convex functions. Furthermore, in order to ensure general functions also satisfy global convergence, a new formula was presented by Dai and Liao as follows:
\[ \beta_{k}^{DL+} = \max \left\{ \frac{g_k^T y_{k-1}}{d_k^T y_{k-1}}, 0 \right\} - \frac{t g_k^T s_{k-1}}{d_k^T y_{k-1}}. \] (14)

It is easily observed that the values of the first term in (13) are nonnegative. They proved that the modified DL method with (14) satisfies global convergence for general functions under some suitable conditions. In the DL method, different parameter \( t \) shows different numerical performance. Based on the singular value study, Hager and Zhang [21] presented the new choices for \( t \):
\[ t = \frac{2 \| y_k \|^2}{s_k y_k}, \] (15)

and they proved that \( d_k \) in [21] satisfies the sufficient descent condition. In addition, Babaie-Kafaki and Ghanbari [22] presented a new value for \( t \):
\[ t_1 = \frac{s_k y_k}{\left( s_k y_k \right)^2} \] (16)

In [22], they proved that the DL method with \( t = t_1 \) has better numerical results than the methods proposed by Dai and Kou [23]. In last several years, some scholars tried to find the new choice for the nonnegative parameter \( t \) in (13) [24, 25]. The remainder of this paper is organized as follows: in Section 2, based on the new conjugacy condition, a modified DL gradient method is proposed with a new value for the parameter. We establish the global convergence property of the presented method in Section 3. Finally, we do some numerical experiments in Section 4.
2. New DL Conjugate Gradient Method

It is well known that the objective function can be regarded as a quadratic equation, so, if a point $x_k$ is close enough to a local minimizer, a good direction is the Newton direction:

$$d_k = -\nabla^2 f(x_k)^{-1}g_k.$$  

(17)

Hence, from $d_k$ of CG methods and the formula of $\beta_k$ in (13), we can compute the values of parameter $t$ as follows:

$$-\nabla^2 f(x_k)^{-1}g_k = -g_k + \frac{\partial^2}{\partial x^2}f(x_k)_{j,k}d_{k-1} - \frac{\partial^2}{\partial x^2}f(x_k)_{j,k}d_{k-1}.$$

(18)

By some algebra, we obtain

$$t = \frac{s_k^Tg_k - \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x^j \partial x^j} f(x_k)_{i,j} s_{k-1}^T}{s_k^Tg_k + \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x^j \partial x^j} f(x_k)_{i,j} s_{k-1}^T}.$$

(19)

Thoughtfully, because computing the Hessian matrix $\nabla^2 f(x_k)$ is inefficient, a quasi–Newton method is used in this paper. In the quasi–Newton methods [26], the search direction $d_k$ is computed by solving unconstrained optimization problems, $B_k d_k = -g_k$, where $B_k$ is the approximation matrix of the Hessian $\nabla^2 f(x_k)$ such that the matrix $B_k$ satisfies the following important equation:

$$B_k s_{k-1} = y_{k-1}.$$  

(20)

Theorem 1. Assume that the function $f(x)$ is sufficiently smooth and $\|s_{k-1}\|$ is sufficiently small, and then, we have

$$s_{k+1}^T\nabla^2 f(x_k) s_{k-1} - s_{k-1}^Tg_k = \frac{1}{2} s_{k-1}^T (T_k s_{k-1}) s_{k-1} + O\left(\|s_{k-1}\|^2\right).$$

(21)

where $T_k$ is the tensor of $f$ at $x_k$ and

$$\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x^j \partial x^j} f(x_k)_{i,j} s_{k-1}^T = \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x^j \partial x^j} f(x_k)_{i,j} s_{k-1}^T s_{k-1}^T d_k.$$

(22)

Proof. Doing Taylor expansion for the objective function $f(x)$, we have

$$f_{k-1} = f_k - g_k^T s_{k-1} + \frac{1}{2} s_{k-1}^T \nabla^2 f(x_k) s_{k-1}$$

$$- \frac{1}{6} s_{k-1}^T (T_k s_{k-1}) s_{k-1} + O\left(\|s_{k-1}\|^4\right).$$

$$g_k^T s_{k-1} = g_k s_{k-1} - s_{k-1} \nabla^2 f(x_k) s_{k-1} + \frac{1}{2} s_{k-1}^T (T_k s_{k-1}) s_{k-1}$$

$$+ O\left(\|s_{k-1}\|^4\right).$$

(23)

By the definitions of $y_{k-1}$, we completed the proof.

The BFGS-type methods have been proven to satisfy ideal global convergence of uniformly convex functions, but the methods may fail to establish the convergence property of nonconvex functions. So, a new version of the BFGS method is proposed by Wei et al. [27] to overcome its convergence failure for more general objective function. In their method, the matrix $B_k$ is obtained from the following modified secant equation:

$$B_k s_{k-1} = y_{k-1}^*,$$

$$y_{k-1}^* = y_{k-1} + y_{k-1} s_{k-1},$$

(24)

where $y_{k-1} = (\max(\theta_{k-1}, 0)/\|s_{k-1}\|^2)$ and $\theta = 2(f_k - f_{k-1})/(g_{k-1} + g_k)^T s_{k-1}$. In order to get the powerful theoretical and numerical properties of the modified BFGS method, use (24) to simplify (19) and the parameter $t$ is proposed as follows:

$$t = \frac{(1 - \theta_{k-1}) s_{k-1}^T g_k + \theta_{k-1} s_{k-1}^T g_k}{g_k^T s_{k-1} + \theta_{k-1} g_k^T s_{k-1}}.$$

(25)

Then, to ensure that the new DL method satisfies the descent condition, we present a modified definition of the value of parameter $t$ similar to [28] as follows:

$$t^* = \max\left(t_2, \theta \frac{\|y_{k-1}\|^2}{s_{k-1}^T y_{k-1}}\right),$$

(26)

where $\theta > (1/4)$; it means that we have $d_k$ as follows:

$$d_k = -g_k + P_k^D t^* d_{k-1},$$

(27)

where $\theta_k^D = (g_k^T y_{k-1} - t^* s_{k-1}^T y_{k-1})/(s_{k-1}^T y_{k-1})$. Based on the above discussion, we can collect some of the advantages of these formulas:

(i) The new DL method possesses the information about function values.

(ii) The global convergence for nonconvex functions is established under some assumptions.

(iii) The new formulas are applied to the engineering Muskingum model and image restoration problems. Finally, we end this section by presenting a modified DL algorithm, which is listed in Algorithm 1.

3. Convergence Analysis

In the following paper, there are some indispensable assumptions for the global convergence of the algorithm on objective functions.

Assumption 1. (i) The level set $Y = \{x \mid f(x) \leq f(x_0)\}$ is bounded.

(ii) In some neighbourhood $\Psi$ of $Y$, $f$ is differentiable, and its gradient function $g$ is the Lipschitz continuous, namely,
Step 0: choose an initial point \( x_0 \in \mathbb{R}^n \), \( \delta \in (0, 1) \), \( \delta_i \in (0, \delta) \), \( \sigma \in (\delta, 1) \), \( \varepsilon > 0 \). Set \( d_0 = -g_0 = -\nabla f(x_0) \), \( k = 0 \).

Step 1: stop if \( \|g_k\| \leq \varepsilon \).

Step 2: determine a stepsize satisfying the following line search by [4]:

\[
\begin{align*}
&f(x_k + \alpha_k d_k) \leq f_k + \delta g_k^T d_k + \alpha_k \min \left\{ -\delta_1 g_k^T d_k, \delta (\alpha_k/2) \|d_k\|^2 \right\}, \\
&\text{and } g(x_k + \alpha_k d_k)^T d_k \geq \sigma g_k^T d_k + \min \left\{ -\delta_1 g_k^T d_k, \delta \alpha_k \|d_k\|^2 \right\}.
\end{align*}
\]

where \( \delta \in (0, (1/2)) \), \( \delta_i \in (0, \delta) \), and \( \sigma \in (\delta, 1) \).

Step 3: let \( x_{k+1} = x_k + \alpha_k d_k \).

Step 4: if \( \|d_{k+1}\| \leq \varepsilon \), then the modified DL algorithm stops.

Step 5: calculate the search direction by (27).

Step 6: set \( k = k + 1 \) and go to step 3.

**Algorithm 1: YWLDL algorithm.**

\[
\|g(x) - g(y)\| \leq L\|x - y\|. \tag{28}
\]

where \( L > 0 \) is a constant and any \( x, y \in \Psi \).

**Remark 1.** Assumption 1 implies that there exists a constant \( \eta > 0 \) satisfying

\[
\|g(x)\| \leq \eta, \quad \forall x \in \Psi. \tag{29}
\]

**Remark 2.** In order to establish the convergence property of new method, we assume the new proposed parameter is bounded. For this purpose, we set

\[
t^* \leq H, \tag{30}
\]

where \( H \) is a large positive constant.

**Lemma 1.** Consider the proposed modified DL method in the form of (27). If the line search procedure guarantees \( d_k^T y_k \geq 0 \), for all \( k \geq 0 \), then, we have

\[
g_k^T d_k \leq -\lambda \|g_k\|^2, \tag{31}
\]

where \( \lambda = (1 - (1/4\theta)) \).

**Proof.** Similar to the proof of Theorem 1 [29]. \( \square \)

**Lemma 2.** Suppose that Assumption 1 holds. Consider the proposed modified DL method in the form of (27), and the step size \( \alpha_k \) obtained by the equations in step 2 in Algorithm 1. If \( f \) be a nonconvex function on the level set \( \Psi \), then, we have

\[
\sum_{k=0}^{\infty} \left( \frac{g_k^T d_k}{\|d_k\|^2} \right)^2 < \infty. \tag{32}
\]

**Proof.** From Lemma 1, \( d_k^T g_k < 0 \) holds. By the second equation in Step 2 in Algorithm 1 and (28),

\[
\begin{align*}
\alpha_k L\|d_k\|^2 &\geq (g_{k+1} - g_k)^T d_k \geq (\sigma - 1) g_k^T d_k \\
&\quad + \min \left\{ -\delta_1 g_k^T d_k, \delta \alpha_k \|d_k\|^2 \right\} \geq (\sigma - 1) g_k^T d_k.
\end{align*}
\]

So the following equation holds:

\[
\alpha_k \geq \frac{(\sigma - 1) g_k^T d_k}{L\|d_k\|^2}. \tag{34}
\]

From the first equation in Step 2 in Algorithm 1, we obtain

\[
\begin{align*}
f(x_k) - f(x_k + \alpha_k d_k) &\geq -\delta g_k^T d_k \\
&\quad - \alpha_k \min \left\{ -\delta_1 g_k^T d_k, \delta \frac{\|d_k\|^2}{2} \right\} \\
&\geq -(\delta - \delta_1) \alpha_k g_k^T d_k.
\end{align*}
\]

Summing up both sides of the above inequalities from \( k = 0 \) to \( k = \infty \), we obtain

\[
\sum_{k=0}^{\infty} (-\alpha_k g_k^T d_k) < \infty. \tag{35}
\]

Combining with (34), (32) holds, which completes the proof. \( \square \)

**Lemma 3.** Suppose that Assumption 1 is true and then the sequence \( \{x_k, d_k, \alpha_k, g_k\} \) is generated by Algorithm YWLDL. If

\[
\|g_k\| \geq \tau, \tag{37}
\]

we obtain

\[
\sum_{k=0}^{\infty} \frac{1}{\|d_k\|^2} < \infty. \tag{38}
\]

**Proof.** From Lemma 1 and (37), we have

\[
\frac{1}{\tau} \sum_{k=0}^{\infty} \frac{\|g_k\|^4}{\|d_k\|^2} \leq \sum_{k=0}^{\infty} \frac{1}{\|d_k\|^2} \leq \sum_{k=0}^{\infty} \frac{1}{\|d_k\|^2} \left( g_k^T d_k \right)^2. \tag{39}
\]

Combining with Lemma 2, such that

\[
\sum_{k=0}^{\infty} \frac{1}{\|d_k\|^2} < \infty. \tag{40}
\]

The proof is complete. \( \square \)
Theorem 2. Suppose that Assumption 1 holds. Consider that the proposed modified DL method in the form of (27) and the step size \( \alpha_k \) obtained the equations in step 2 in Algorithm 1. If \( f \) be a nonconvex function on the level set \( \Psi \), then we obtain

\[
\lim_{k \to \infty} \inf \|g_k\| = 0.
\]

Proof. First, we suppose that equation (41) is not true. Namely, equation (37) holds. From the equations in step 2 in Algorithm 1, in case 1, \(-\delta_1 g_k^T d_k \geq \delta \alpha_k \|d_k\|^2 \) holds, combining with (36), we have

\[
\sum_{k=0}^{\infty} \|s_k\|^2 = \sum_{k=0}^{\infty} \|\alpha_k d_k\|^2 \leq \sum_{k=0}^{\infty} \left( \frac{\delta_1}{\delta} \alpha_k g_k^T d_k \right) < \infty.
\]

And in case 2, similar to the proof of Theorem 3.1 [30], the relation holds as follows:

\[
\alpha_k \|d_k\|^2 \leq -\delta g_k^T d_k,
\]

where \( \delta > 0 \). From (36) and (43), the following equation holds in case 2:

\[
\sum_{k=0}^{\infty} \|s_k\|^2 \leq \sum_{k=0}^{\infty} \|\alpha_k d_k\|^2 \leq \sum_{k=0}^{\infty} \left( \frac{1}{\delta} \alpha_k g_k^T d_k \right) < \infty.
\]

Thus,

\[
\|s_k\| \to 0, \quad \forall k > 0.
\]

We have

\[
\|s_k\| < z,
\]

where \( 0 < z < 1 \) holds. And from the second equation in Step 2 in Algorithm 1, we obtain

\[
y_k^T d_k = g_k^T d_k - \sigma k \geq (\sigma - 1) g_k^T d_k + \min \left\{ -\delta_1 g_k^T d_k, \delta \alpha_k \|d_k\|^2 \right\} \geq (\sigma - 1) g_k^T d_k.
\]

By (31) and (37), we have

\[
y_k^T d_k \geq (1 - \sigma) \lambda r^2.
\]

Combining with (30)–(30), (46), and (48), the following equation holds:

\[
\|d_k\| = \frac{\|g_k - \beta_k^* d_{k-1}\|}{\delta^T d_{k-1} Y_{k-1}} \leq \frac{(L + H) \|g_k\| \|s_{k-1}\|}{\|d_{k-1} Y_{k-1}\|} \leq \frac{(L + H) \|g_k\| \|s_{k-1}\|}{\|d_{k-1} Y_{k-1}\|} \leq \frac{(L + H) \eta z}{(1 - \sigma) \lambda r^2} = Z,
\]

where \( Z = (L + H) \eta z / (1 - \sigma) \lambda r^2 \). From (27), we have

\[
\|d_k\| = \|\eta Z - Z\| / Z = Z
\]

and

\[
\|d_k\| = \|\eta Z - Z\| / Z = Z
\]

Let \( v = \max \{\|d_1\|, \|d_2\|, \ldots, \|d_k\|, (\eta / (1 - Z)) + \|d_k\|\} \), the following holds:

\[
\|d(x_k)\| \leq v, \quad \forall k \geq 0.
\]

This contradicts (38), so the proof is completed. \( \Box \)

4. Numerical Experiments

This section reports some numerical results. We do some numerical experiments in order to investigate the performance of our proposed algorithm and compare it with others. In the next subsections, all the numerical tests were run in a 2.30 GHz CPU and 8.00 GB of memory in a Windows 10 operating system.

4.1. Normal Unconstrained Optimization Problems. In this section, we test the effect of some algorithms on numerical problems, and the unconstrained optimization problems are listed in Table 1. We will report on various numerical implementations of our presented method (YWLDL algorithm) with the modified Wolfe–Powell line search (YWLDL) on these problems and compare its performance with the one of the following algorithms: HZ algorithm, the BG algorithm, and the DL+ algorithm, for the problems. The proposed algorithm is listed as follows:

- HZ algorithm: the DL method with parameter \( t \) in [21] under the weak Wolfe–Powell line search
- BG algorithm: the DL method with parameter \( t \) in [22] under the weak Wolfe–Powell line search
- DL+ algorithm: the DL method with parameter \( t \) in [20] under the strong Wolfe–Powell line search

In this part of the numerical experiment, we introduce the stopping rules, dimensions, and some parameters as follows:

Stop rules (the Himmelblau stop rule [31]): if \( |f(x_k)| > e_1 \), let stop \( 1 =(|f(x_k) - f(x_{k+1})|) / |f(x_k)| \) or stop \( 1 = |f(x_k) - f(x_{k+1})| \). If the condition \( \|g(x_k)\| < \varepsilon \) or stop \( 1 < \varepsilon \) is satisfied, where \( e_1 = e_2 = 10^{-6}, e = 10^{-7} \). The algorithm will stop if the number of iterations is greater than 1000.

Dimension: 3000, 6000, and 9000 variables.
Parameters: \( \delta = 0.2, \delta_1 = 0.05, \) and \( \sigma = 0.9 \).

The columns of Table 1 have the following meanings:

- Nr: the number of tested problems
- Test problems: the name of problems
- Parameters: all the numerical tests were run in a 2.30 GHz CPU and 8.00 GB of memory in a Windows 10 operating system.
The comparison data can be shown as follows:

- **NI**: the number of iterations
- **NFG**: the total of the function and gradient evaluations
- **CPU**: the CPU time of the algorithm spent in seconds

We used the tool presented by Dolan and Moré [32] in these numerical reports, the new tool to show their performance in order to analyze the efficiency of the YWLDL algorithm, HZ algorithm, BG algorithm, and DL+ algorithm, and Figures 1–3 show the profiles relative to the CPU time, NI, and NFG, respectively. Figure 1 presents that YWLDL successfully solves about 35 percent of the test problems with the CPU time in seconds, HZ solves about 18 percent of the test problems, BG solves about 24 percent of the test problems, and DL+ solves about 6 percent of the test problems. We can conclude that YWLDL is more competitive than the HZ algorithm, BG algorithm, and DL+ algorithm since its performance curves corresponding to the CPU time are best. Figures 2 and 3 show that the robustness of YWLDL algorithm is worse than that of the BG algorithm. The YWLDL algorithm can successfully solve most of the test problems. Altogether, it is clear that the YWLDL algorithm is efficient based on the experimental results and the YWLDL algorithm with modified Wolfe–Powell line search is competitive with the other four algorithms for these test problems.

### 4.2 Muskingum Model in Engineering Problems

It is well known that some algorithms for solving optimization problems have been reassessed as a major challenge in engineering problems. Many scholars hope to present some effective algorithms to solve these engineering problems. Parameter estimation is one of the important tools for the research of a well-known hydrologic engineering application problem called the nonlinear Muskingum model. The Muskingum model, depending on the water inflow and outflow, is a popular flood routing model defined as follows:

The Muskingum model [33] is defined by
\begin{align*}
P_p: r(p, s) &\leq \tau \\
\text{YWLDL} &\quad \text{HZ} \\
\text{BG} &\quad \text{DL+}
\end{align*}

Figure 1: Performance profiles of these methods (CPU).

Figure 2: Performance profiles of these methods (NI).

Figure 3: Performance profiles of these methods (NFG).
Table 2: Results of the algorithms.

| Algorithm     | $x_1$  | $x_2$  | $x_3$  |
|---------------|--------|--------|--------|
| BFGS [35]     | 10.8156| 0.9826 | 1.0219 |
| HIWO [33]     | 13.2813| 0.8001 | 0.9933 |
| Algorithm NDL | 11.1937| 0.9999 | 0.9993 |

Figure 4: Performance of data in 1960.

Figure 5: Performance of data in 1961.

Figure 6: Performance of data in 1964.
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\[
\begin{align*}
\min f(x_1, x_2, x_3) & = \sum_{i=1}^{n-1} \left(\left(1 - \frac{\Delta t}{6}\right)x_1(x_2I_{i+1} + (1 - x_2)Q_{i+1})^{\gamma_1} \right) \\
& - \left(1 - \frac{\Delta t}{6}\right)x_1(x_2I_i + (1 - x_2)Q_i)^{\gamma_1} \\
& - \frac{\Delta t}{2}(I_i - Q_i) + \frac{\Delta t}{2}\left(1 - \frac{\Delta t}{3}\right)(I_{i+1} - Q_{i+1})^2.
\end{align*}
\]

at time \( t_k \), where \( k = 1, 2, \ldots, n \), \( n \) denotes the total number of times; \( x_1, x_2, \) and \( x_3 \) denote the storage time constant; the weighting factor, and an additional parameter, respectively; \( \Delta t \) denotes the time step; \( I_i \) denotes the observed inflow discharge; and \( Q_i \) denotes the observed outflow discharge. In Chenggouwan and Linqing of Nanyunhe in the Haihe Basin, Tianjin, China, using actual observation data of flood runoff process, \( \Delta t = 12 \text{ (h)} \) and the initial point \( x = [0, 1, 1]^T \) are presented. The detailed data for \( I_i \) and \( Q_i \) in 1960, 1961, and 1964 can be found in [34]. The tested results of the final points are listed in Table 2.

Figures 4–6 can draw at least three conclusions: (i) It is not different from the BFGS method and HIWO method, the presented algorithm provides a good approximation for these data, and the YWLDL algorithm can effectively be used to study this nonlinear Muskingum model. (ii) In the parameter estimation of the Muskingum model, YWLDL Algorithm shows good approximation. (iii) The final points \((x_1, x_2, \) and \( x_3)\) are competitive with the final points of similar algorithms.
4.3. Image Restoration Problems. In this section, we will use the proposed algorithm to recover the original image from the image destroyed by impulse noise. It has important practical significance in the field of optimization. The selection of parameters is similar to that in the section of Numerical Experiments. The stop condition is \( \frac{\|f_{k+1}\| - \|f_k\|}{\|f_k\|} < 10^{-3} \) or \( \frac{\|x_{k+1} - x_k\|}{\|x_k\|} < 10^{-3} \) holds. The experiments choose Barbara (512 × 512), Baboon (512 × 512), and Lena (512 × 512) as the test images. The detailed performance results are given in Figures 7–9. It is not difficult to observe that both HZ algorithm, BG algorithm, and DL+ algorithm are tested in the image restoration of tested images. The expenditure of the CPU time is listed in Table 3 to compare the YWLDL algorithm with the other algorithms.

From Table 3 and Figures 7–9, we may draw the following conclusions: (i) It pretty obviously takes less time to restore an image with 25% noise than one with 65% noise. (ii) When the salt-and-pepper noise increases, the cost to restore the image increases. (iii) The YWLDL algorithm shows advantage to the HZ algorithm, BG algorithm, and DL+ algorithm in image restoration problems.

5. Conclusion

In this paper, the YWLDL conjugate gradient algorithm that combines with the modified WWP line search technique is proposed. The theory of other methods for constructing parameter \( t \) is one of the interesting works, and we think that some modified DL methods with other parameter \( t \) under
Figure 9: Restoration of the Babara, Baboon, and Lena images by the HZ algorithm, BG algorithm, DL+ algorithm, and YWLDL algorithm. From left to right: a noisy image with 65% salt-and-pepper noise, restorations obtained by minimizing $z$ with the HZ algorithm, BG algorithm, and DL+ algorithm and YWLDL algorithm.

Table 3: The CPU time of HZ algorithm, BG algorithm, DL+ algorithm, and YWLDL algorithm in seconds.

| Noise Level | Barbara   | Baboon    | Lena      | Total     |
|-------------|-----------|-----------|-----------|-----------|
| 25% noise   |           |           |           |           |
| HZ algorithm| 0.84375   | 1.218750  | 1.046875  | 3.109375  |
| BG algorithm| 1.000000  | 0.921875  | 1.031250  | 2.953125  |
| DL+ algorithm| 1.015625 | 0.953125  | 0.953125  | 2.921875  |
| YWLDL algorithm| 0.937500 | 0.906250  | 0.937500  | 2.78125   |
| 45% noise   |           |           |           |           |
| HZ algorithm| 1.328125  | 1.312500  | 1.343750  | 3.984375  |
| BG algorithm| 1.359375  | 1.285250  | 1.187500  | 3.832125  |
| DL+ algorithm| 1.250000 | 1.265625  | 1.28125   | 3.796875  |
| YWLDL algorithm| 1.203125| 1.125000  | 1.15625   | 3.484375  |
| 65% noise   |           |           |           |           |
| HZ algorithm| 1.812500  | 1.906250  | 1.671875  | 5.390625  |
| BG algorithm| 1.921875  | 1.546875  | 1.78125   | 5.25025   |
| DL+ algorithm| 1.65625  | 1.578125  | 1.765625  | 5.00000   |
| YWLDL algorithm| 1.546875| 1.500000  | 1.562500  | 4.609375  |
the modified WWP line search technique can also solve the Muskingum model, and we may do this work in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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