Large Deviations Analysis for Distributed Algorithms in an Ergodic Markovian Environment

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Abstract We provide a large deviations analysis of deadlock phenomena occurring in distributed systems sharing common resources. In our model transition probabilities of resource allocation and deallocation are time and space dependent. The process is driven by an ergodic Markov chain and is reflected on the boundary of the $d$-dimensional cube. In the large resource limit, we prove Freidlin-Wentzell estimates, we study the asymptotic of the deadlock time and we show that the quasi-potential is a viscosity solution of a Hamilton-Jacobi equation with a Neumann boundary condition. We give a complete analysis of the colliding 2-stacks problem and show an example where the system has a stable attractor which is a limit cycle.

Keywords Large deviations · Distributed algorithm · Averaging principle · Hamilton-Jacobi equation · Viscosity solution

1 Introduction

Distributed algorithms are related to resource sharing problems. Colliding stacks problems and the banker algorithm are among the examples which have attracted
large interest over the last decades in the context of deadlock prevention on multiprocessor systems. Knuth [22], Yao [31], Flajolet [15], Louchard, Schott et al. [25–27] have provided combinatorial or probabilistic analysis of these algorithms in the 2-dimensional case under the assumption that transition probabilities (of allocation or deallocation) are constant. Maier [28] proposed a large deviations analysis of colliding stacks for the more difficult case where the transition probabilities are non-trivially state-dependent. More recently Guillotin-Plantard and Schott [17, 18] analyzed a model of exhaustion of shared resources where allocation and deallocation requests are modeled by time-dependent dynamic random walks. In [8], the present authors provided a probabilistic analysis of the $d$-dimensional banker algorithm when transition probabilities evolve, as time goes by, along the trajectory of an ergodic Markovian environment, whereas the spatial parameter just acts on long runs. The analysis in [8] relies on techniques from stochastic homogenization theory. In this paper, we consider a similar dynamics, but in a stable regime instead of a neutral regime as in our previous paper, and we provide an original large deviations analysis in the framework of Freidlin-Wentzell theory. Given the environment, the process of interest is a Markov process depending on the number $m$ of available resource, with smaller and more frequent jumps as $m \to \infty$, see (3.1). A number of monographs and papers have been written on this theory: [4, 14, 16] and [19] for random environment, [10, 12, 21] and [1, 20] for reflected processes, [2, 3, 9] and [29] for homogeneous Markov processes. However, our framework, including both reflections on the boundary and averaging on the Markovian environment, is not covered by the current literature, and we establish here the large deviations principle. We prove that the time of resource exhaustion then grows exponentially with the size of the system—in place of polynomially in the neutral regime of [8]—and has exponential law as limit distribution. Then, we study the quasi-potential, which solves, according to general wisdom, some Hamilton-Jacobi equation: in view of the reflection on the hypercube, which boundary is non-regular, we prove this fact in the framework of viscosity solution, and study the optimal paths (so-called instantons).

We investigate in details a particular situation introduced in a beautiful paper of Maier [28], where the motion in each direction depends on the corresponding coordinate only, with the additional dependence in the Markovian environment. In fact, we discover the quasi-potential by observing that the discrete process has an invariant measure, for which we study the large deviations properties. We can then use the characterization in terms of Hamilton-Jacobi equation to bypass the Hamiltonian mechanics approach of [28]. For the deadlock phenomenon, we finally obtain a (even more) complete picture (after an even shorter work). To the best of our knowledge, this is first such analysis developed for space-time inhomogeneous distributed algorithms.

The organization of this paper is as follows: we discuss our probabilistic model in Sect. 2. In Sect. 3 we prove a Large Deviations Principle. Deadlock phenomenon analysis is done rigorously with much details in Sect. 4. In Sect. 5 we illustrate with the two-stacks model. In Sect. 6 we work out an example where the system has (in the large scale resource limit $m \to \infty$) a stable attractor which is a limit cycle. Some technical proofs of results stated in Sect. 4 are deferred to Appendix 6.