Asymptotic analysis of the Bell polynomials by the ray method

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Abstract
We analyze the Bell polynomials $B_n(x)$ asymptotically as $n \to \infty$. We obtain asymptotic approximations from the differential-difference equation which they satisfy, using a discrete version of the ray method. We give some examples showing the accuracy of our formulas.

Keywords Bell polynomials, asymptotic expansions, Stirling numbers MSC-class: 34E05, 11B73, 34E20

1 Introduction
The Bell polynomials $B_n(x)$ are defined by \cite{1}

$$B_n(x) = \sum_{k=0}^{n} S^n_k x^k, \quad n = 0, 1, \ldots,$$

where $S^n_k$ is a Stirling number of the second kind \cite{2, 24, 1, 4}. They have the generating function

$$\sum_{n=0}^{\infty} B_n(x) \frac{x^n}{n!} = \exp \left[ x (e^t - 1) \right], \quad (1)$$

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from which it follows that

\[ B_0(x) = 1 \]  

(2)

and

\[ B_{n+1}(x) = x [B'_n(x) + B_n(x)] , \quad n = 0, 1, \ldots \]  

(3)

The asymptotic behavior of \( B_n(x) \) was studied by Elbert \[3\], \[4\] and Zhao \[5\], using the saddle point method and \( (1) \). In this paper we will use a different approach and analyze \( (3) \) instead of \( (1) \). The advantage of our method is that no knowledge of a generating function is required and therefore it can be applied to other sequences of polynomials satisfying differential-difference equations \[6\], \[7\].

2 Asymptotic analysis

To analyze \( (3) \) asymptotically as \( n \rightarrow \infty \), we use a discrete version of the ray method \[8\]. Replacing the ansatz

\[ B_n(x) = \varepsilon^{-n} F(\varepsilon x, \varepsilon n) \]  

(4)

in \( (3) \), we get

\[ F(u, v + \varepsilon) = u \left( \frac{\varepsilon \partial F}{\partial x} + F \right) , \]  

(5)

with

\[ u = \varepsilon x , \quad v = \varepsilon n \]  

(6)

and \( \varepsilon \) is a small parameter. We consider asymptotic solutions for \( (5) \) of the form

\[ F(u, v) \sim \exp \left[ \varepsilon^{-1} \psi (u, v) \right] K(u, v) , \]  

(7)

as \( \varepsilon \rightarrow 0 \). Using \( (7) \) in \( (5) \) we obtain, to leading order, the eikonal equation

\[ e^q - u (p + 1) = 0 \]  

(8)

and the transport equation

\[ \frac{\partial K}{\partial v} + \frac{1}{2} \frac{\partial^2 \psi}{\partial v^2} K - u \exp \left( - \frac{\partial \psi}{\partial v} \right) \frac{\partial K}{\partial u} = 0 , \]  

(9)

where

\[ p = \frac{\partial \psi}{\partial x} , \quad q = \frac{\partial \psi}{\partial v} . \]  

(10)

The initial condition \( (2) \), implies

\[ \psi (u, 0) = 0 , \quad K(u, 0) = 1 . \]  

(11)

To solve \( (8) \) we use the method of characteristics, which we briefly review. Given the first order partial differential equation

\[ \mathcal{F} (u, v, \psi, p, q) = 0 , \]
with \( p, q \) defined in (10), we search for a solution \( \psi(u, v) \) by solving the system of “characteristic equations”

\[
\begin{align*}
\dot{u} &= \frac{du}{dt} = \frac{\partial \mathcal{F}}{\partial p}, \\
\dot{v} &= \frac{dv}{dt} = \frac{\partial \mathcal{F}}{\partial q}, \\
\dot{p} &= \frac{dp}{dt} = -\frac{\partial \mathcal{F}}{\partial u} - p \frac{\partial \mathcal{F}}{\partial \psi}, \\
\dot{q} &= \frac{dq}{dt} = -\frac{\partial \mathcal{F}}{\partial v} - q \frac{\partial \mathcal{F}}{\partial \psi}, \\
\dot{\psi} &= \frac{d\psi}{dt} = p \frac{\partial \mathcal{F}}{\partial p} + q \frac{\partial \mathcal{F}}{\partial q},
\end{align*}
\]

where we now consider \( \{u, v, \psi, p, q\} \) to all be functions of the new variables \( t \) and \( s \).

For (8), we have

\[
\mathcal{F}(u, v, \psi, p, q) = e^q + p - 2u
\]

and therefore the characteristic equations are

\[
\begin{align*}
\dot{u} + u &= 0, \\
\dot{v} &= e^q, \\
\dot{p} - p &= 1, \\
\dot{q} &= 0
\end{align*}
\] (12)

Solving (12), subject to the initial conditions

\[
\begin{align*}
u(0, s) &= s, \\
v(0, s) &= 0, \\
p(0, s) &= B(s) - 1,
\end{align*}
\] (13)

we obtain

\[
\begin{align*}
u &= se^{-t}, \\
v &= Bst, \\
p &= Be^t - 1, \\
q &= \ln(Bs)
\end{align*}
\]

where we have used

\[
0 = \mathcal{F}|_{t=0} = e^{q(0,s)} - sB.
\]

From (11) and (13) we have

\[
\psi(0, s) = 0, \\
K(0, s) = 1,
\] (14)

which implies

\[
0 = \frac{d}{ds} \psi(0, s) = p(0, s) \frac{d}{ds} u(0, s) + q(0, s) \frac{d}{ds} v(0, s)
\]

\[
= (B - 1) \times 1 + \ln(Bs) \times 0 = B - 1.
\]

Thus,

\[
\begin{align*}
u &= se^{-t}, \\
v &= st, \\
p &= e^t - 1, \\
q &= \ln(s).
\end{align*}
\] (15)

The characteristic equation for \( \psi \) is

\[
\dot{\psi} = p\dot{u} + q\dot{v} = (e^t - 1) (se^{-t}) + \ln(s) \cdot s,
\]

which together with (14) gives

\[
\psi(t, s) = s \left(1 - t - e^{-t}\right) + \ln(s) \cdot st.
\] (16)

We shall now solve the transport equation (9). From (15), we get

\[
\begin{align*}
\frac{\partial t}{\partial u} &= -\frac{te^t}{s(t+1)}, \\
\frac{\partial t}{\partial v} &= \frac{1}{s(t+1)}, \\
\frac{\partial s}{\partial u} &= \frac{e^t}{t+1}, \\
\frac{\partial s}{\partial v} &= \frac{1}{t+1}
\end{align*}
\] (17)
and therefore,
\[
\frac{\partial^2 \psi}{\partial v^2} = \frac{\partial q}{\partial v} = \frac{\partial q}{\partial t} \frac{\partial t}{\partial v} + \frac{\partial q}{\partial s} \frac{\partial s}{\partial v} = \frac{1}{s(t+1)}.
\] (18)

Using (17)-(18) to rewrite (9) in terms of \(t\) and \(s\), we have
\[
\dot{K} + \frac{1}{2(t+1)} K = 0
\]
with solution
\[
K(t,s) = \frac{1}{\sqrt{t+1}}.
\] (19)

where we have used (14).

Solving for \(t, s\) in (15), we obtain
\[
t = \text{LW} \left( \frac{v}{u} \right), \quad s = \frac{v}{\text{LW} \left( \frac{v}{u} \right)},
\] (20)

where \(\text{LW} (\cdot)\) denotes the Lambert-W function \([9]\), defined by
\[
\text{LW} (z) \exp \left[ \text{LW} (z) \right] = z.
\]

Replacing (20) in (16) and (19), we get
\[
\psi (u,v) = v \text{LW} \left( \frac{v}{u} \right) + v \ln \left[ \frac{v}{\text{LW} \left( \frac{v}{u} \right)} \right] - (u + v),
\]
\[
K(u,v) = \frac{1}{\sqrt{\text{LW} \left( \frac{v}{u} \right) + 1}}
\]

and from (7) we find that
\[
F(u,v) \sim \exp \left\{ \frac{v/\varepsilon}{\text{LW} \left( \frac{v}{u} \right)} + \frac{v}{\varepsilon} \ln \left[ \frac{v}{\text{LW} \left( \frac{v}{u} \right)} \right] - \left( \frac{u + v}{\varepsilon} \right) \right\} \frac{1}{\sqrt{\text{LW} \left( \frac{v}{u} \right) + 1}},
\] (21)
as \(\varepsilon \to 0\). Using (6) and (21) in (4), we conclude that
\[
B_n(x) \sim \exp \left\{ \frac{n}{\text{LW} \left( \frac{n}{x} \right)} + n \ln \left[ \frac{n}{\text{LW} \left( \frac{n}{x} \right)} \right] - (x + n) \right\} \frac{1}{\text{LW} \left( \frac{n}{x} \right) + 1},
\] (22)
as \(n \to \infty\).

**Remark 1** The function \(\text{LW} (z)\) has two real-valued branches for \(-e^{-1} \leq z < 0\), denoted by \(\text{LW}_0 (z)\) (the principal branch of \(\text{LW}\)) and \(\text{LW}_{-1} (z)\), satisfying
\[
\text{LW}_0 : [-e^{-1},0) \to [-1,0), \quad \text{LW}_{-1} : [-e^{-1},0) \to (-\infty,-1],
\]
with
\[
LW_0(-e^{-1}) = -1 = LW_{-1}(-e^{-1})
\]

For \( z \geq 0 \), \( LW(z) \) has only one real-valued branch
\[
LW_0 : [0, \infty) \rightarrow [0, \infty)
\]
and for \( z < -e^{-1} \), \( LW_0(z) \) and \( LW_{-1}(z) \) are complex conjugates. Therefore, for (22) to be well defined, we need to consider three separate regions:

1. An exponential region for \( x > 0 \) or \( x < -en \). Here we have
   \[
   B_n(x) \sim \Phi_n(x; 0), \quad n \rightarrow \infty,
   \]
   where
   \[
   \Phi_n(x; k) = \exp \left\{ \frac{n}{LW_k(\frac{x}{n})} + n \ln \left[ \frac{n}{LW_k(\frac{x}{n})} \right] - (x + n) \right\} \frac{1}{\sqrt{LW_k(\frac{x}{n}) + 1}}.
   \]

2. An oscillatory region for \( -en < x < 0 \). In this interval,
   \[
   B_n(x) \sim \Phi_n(x; 0) + \Phi_n(x; -1), \quad n \rightarrow \infty.
   \]

3. A transition region for \( x \approx -en \). We will analyze this region in the next section.

In Figure 1 (a) we plot \( B_5(x) \) and the asymptotic approximations (23) (+++) and (24) (ooo), all multiplied by \( e^{-|x|} \) for scaling purposes, in the interval \((-10, 10)\). We see that our formulas are quite accurate even for small values of \( n \) and that the transition between (23) and (24) is smooth.

In Figure 1 (b) we plot \( B_5(x) \) and (23) (+++) and (24) (ooo), all multiplied by \( e^x \), in the interval \((-20, 0)\). We observe that the approximations (23) and (24) break down in the neighborhood of \(-e5 \approx -13, 59\).

### 2.1 The transition region

When \( x = -en \), the quantity \( LW(\frac{2}{3}) + 1 \) vanishes and (23) is no longer valid. To find an asymptotic approximation in a neighborhood of \(-en\), we introduce the stretched variable \( \beta \) defined by
\[
x = -en - \beta n \frac{2}{3}, \quad \beta = O(1).
\]

For values of \( z \) close to \( z_0 = -e^{-1} \), the Lambert-W function can be approximated by [9, (4.22)]
\[
LW(z) \sim -1 + \sqrt{2e(z - z_0)} - \frac{2}{3}e(z - z_0) + \frac{11}{36} \sqrt{2e^3(z - z_0)^3}, \quad z \rightarrow -e^{-1}.
\]

Using (25) in (26), we have,
\[
LW \left( \frac{n}{-en - \beta n \frac{2}{3}} \right) \sim -1 + \sqrt{2e^{-1} \beta n^{-\frac{2}{3}}} - \frac{2}{3}e^{-1} \beta n^{-\frac{2}{3}} - \frac{7}{36} \sqrt{2e^{-3} \beta^3 n^{-1}}, \quad \beta \rightarrow 0.
\]
Figure 1: A comparison of the exact (solid curve) and asymptotic (ooo), (+++) values of $B_5(x)$.

Hence,

$$\exp\left\{ \frac{n}{\text{LW}_k \left( \frac{x}{n} \right)} + n \ln \left[ \frac{n}{\text{LW}_k \left( \frac{x}{n} \right)} \right] - (x + n) \right\} \sim \varphi(\beta, n), \quad \beta \to 0,$$

for $k = 0, 1$ with $x = -en - \beta n^{\frac{1}{3}}$ and

$$\varphi(\beta, n) = (-1)^n \exp \left\{ [\ln(n) + e - 2]|n - (e^{-1} - 1)\beta n^{\frac{1}{3}}] \right\}.$$  (28)

We now consider solutions for (3) of the form

$$B_n(x) = \varphi(\beta, n) \Lambda(\beta) = \varphi \left[ - \left( e + \frac{x}{n} \right) n^{\frac{2}{3}}, n \right] \Lambda \left[ - \left( e + \frac{x}{n} \right) n^{\frac{2}{3}} \right],$$  (29)

for some function $\Lambda(\beta)$. Replacing (29) in (3) and using (25) we obtain, to leading order

$$\Lambda'' - 2e^{-3} \beta \Lambda = 0,$$

with solution

$$\Lambda(\beta) = C_1 \text{Ai} \left( 2^\frac{2}{3} e^{-1} \beta \right) + C_2 \text{Bi} \left( 2^\frac{2}{3} e^{-1} \beta \right),$$  (30)

where $\text{Ai}(\cdot)$, $\text{Bi}(\cdot)$ are the Airy functions.

To determine the constants $C_1, C_2$ in (30), we shall match (23) with (29). Using (25) and (27) in (23), we have

$$B_n(x) \sim \varphi(\beta, n) \exp \left( -\frac{2}{3} \sqrt{2e^{-\frac{2}{3}} \beta \frac{3}{2}} \right) \left( 2e^{-1} \beta \right)^\frac{1}{2} n^{-\frac{1}{6}}, \quad \beta \to 0^+. $$  (31)
On the other hand, the Airy functions have the well known asymptotic approximations \[2\]
\[
\text{Ai}(z) \sim \frac{1}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}z^{\frac{3}{2}}\right) z^{-\frac{1}{4}}, \quad z \to \infty,
\]
\[
\text{Bi}(z) \sim \frac{1}{\sqrt{\pi}} \exp\left(\frac{2}{3}z^{\frac{3}{2}}\right) z^{-\frac{1}{4}}, \quad z \to \infty.
\]

and therefore we conclude that
\[
C_1 = \sqrt{\pi} 2^{\frac{3}{4}} n^{\frac{1}{6}}, \quad C_2 = 0. \tag{32}
\]

Replacing (30) and (32) in (29), we find that for \(x \simeq -\text{en}\), we have
\[
B_n(x) \sim \sqrt{\pi} 2^{\frac{3}{4}} n^{\frac{1}{6}} \varphi(\beta,n) \text{Ai}\left(2^{\frac{3}{4}} e^{-1} \beta\right), \quad n \to \infty.
\]

This concludes the asymptotic analysis of \(B_n(x)\) for large \(n\).

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