INVARINTS OF STABLE QUASIMAPS WITH FIELDS

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ABSTRACT. For arbitrary smooth hypersurface $X \subset \mathbb{P}^n$ we construct moduli of quasimaps with $\mathbb{P}$ fields. Apply Kiem-Li’s cosection localization we obtain a virtual fundamental class. We show the class coincides, up to sign, with that of moduli of quasimaps to $X$. This generalizes Chang-Li’s [CL] numerical identity to the cycle level, and from Gromov Witten invariants to quasimap invariants.

1. INTRODUCTION

Recently there are more approaches to GW invariants using Landau Ginzburg theory. One example is Jun Li and the first author’s work [CL] identifying quintics’ GW invariants with its GW invariants of maps with fields, which can be regarded as enumerating curves “in” the LG space $(K_{\mathbb{P}^4}, W)$, where $W$ is a regular function on $K_{\mathbb{P}^4}$ induced by quintic polynomials. Such point of view enables some applications: (i) providing an algebraic proof of Li-Zinger relation [CL1] so that $g = 1$ GW invariants can be approached by reduced $g = 1$ GW invariants (recover [LZ] algebraically); (ii) an all genus wall crossing from CY’s GW invariants to FJRW invariants can be realized by the moduli of Mixed-Spin-P fields ([CLL1, CLLL1])

Recently, Ionut Ciocan-Fontanine, Bumsig Kim [FK1] used stable quasimaps to give a mathematical description of $B$-side (also see Clader-Janda-Ruan [CJR]). They use all genus wall crossing to provide relations between GW and quasimap invariants. The relations is interpreted as mirror maps. Then using that $\epsilon = 0^+$-quasimap moduli admits bundle’s euler class description, B. Kim and H. Lho [K-L] also obtain quintics $g = 1$ GW invariants recovering Zinger’s formula. This generalized stability is also used by Fan, Jarvis, Ruan in Gauged Linear Sigma Model [FJR].

In this paper we consider the moduli of $\epsilon$-stable quasimaps to $\mathbb{P}^n$ with fields (also c.f. [FJR]). Similar to [CL] in many aspects, the moduli spaces are not proper for positive genus, and admit virtual cycles by using Kiem-Li’s cosection localization. Our main theorem is that these virtual cycles coincide (up to signs) with the virtual cycles of the $\epsilon$-stable quasimaps moduli of the smooth hypersurface $X = (q(x) = 0) \subset \mathbb{P}^n$ with deg $q = m$. This generalizes the result in [CL] to arbitrary

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1this also gives another way to obtain Zinger’s genus one formula [Zi] of quintic GW;
hypsersurfaces, with markings, and lifts the identity between invariants ([CL, Thm. 1.1]) to identity between virtual cycles.

We briefly outline our construction and the theorem. Let \( q = q(x) \) be a degree \( m \) homogenous polynomial. Assume the hypersurface \( X := (q = 0) \subset \mathbb{P}^n \) is smooth. Let \( \mathcal{M}_{g,\ell}(\mathbb{P}^n, d) \) be the moduli of genus \( g \) degree \( d \) \( \epsilon \)-stable quasimaps to \( \mathbb{P}^n \). Every closed point of \( \mathcal{M}_{g,\ell}(\mathbb{P}^n, d) \) is of the form

\[ \xi = (C, p_1, \ldots, p_\ell \in C, L \in \text{Pic}(C), u \in \Gamma(L^{\oplus n+1})) \]

We let \( \mathcal{M}_{g,\ell}(\mathbb{P}^n, d)^p \) be the moduli of \( (\xi, p) \) where \( p \in \Gamma(L^{\oplus m} \otimes \omega_C) \). It forms a Deligne-Mumford stack. When \( g \) is positive, it is not proper. Parallel to [CL], the homogeneous polynomial \( q(x) \) determines a cosection of its obstruction sheaf:

\[ \sigma : \mathcal{O}_b\mathcal{M}_{g,\ell}(\mathbb{P}^n, d)^p \to \mathcal{O}_d\mathcal{M}_{g,\ell}(\mathbb{P}^n, d)^p. \]

The non-surjective loci (the degeneracy loci) of the cosection is

\[ \mathcal{M}_{g,\ell}(X, d) \subset \mathcal{M}_{g,\ell}(\mathbb{P}^n, d)^p, \]

and is proper. The cosection localization of Kiem-Li induces a virtual cycle

\[ [\mathcal{M}_{g,\ell}(\mathbb{P}^n, d)^p]^{\text{vir}} \in A_\delta \mathcal{M}_{g,\ell}(X, d). \]

**Theorem 1.1.** For \( g \geq 0 \), \( \epsilon > 0 \) and \( \ell \) be a nonnegative integer, then we have

\[ [\mathcal{M}_{g,\ell}(\mathbb{P}^n, d)^p]^{\text{vir}} = (-1)^{md+1-g} [\mathcal{M}_{g,\ell}(X, d)]^{\text{vir}} \in A_\delta \mathcal{M}_{g,\ell}(X, d). \]

In case \( X \) is the quintic CY hypersurface and \( \ell = 0 \), we have virtual dimension \( \delta = 0 \). Taking degree of the identity shows that the quasimap invariant of quintics \( \text{deg}[\mathcal{M}_{g,\ell}(X, d)]^{\text{vir}} \) admits a Landau-Ginzburg alternate construction \( \text{deg}[\mathcal{M}_{g,\ell}(\mathbb{P}^n, d)^p]^{\text{vir}} \). When \( \epsilon = \infty \), this recovers the main theorem in [CL].

**Notation** Every stack in this paper is over \( \mathbb{C} \). Fixing \( g \) and \( \ell \) in whole paper, we use \( \mathcal{D} := \mathcal{D}_{g,\ell} \) to denote the smooth Artin stack of \( \ell \)-pointed nodal curve with a line bundle (c.f. Definition 2.9). The symbol \( \mathbb{T} \) denotes the (relevant) tangent complex, which is the derived dual of cotangent complex \( \mathbb{L} \).

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## 2. Deformation Theory of Quasimaps via Direct Image Cones

### 2.1. Quasimaps to \( \mathbb{P}^n \) and smooth hypersurface.

Fix a smooth hypersurface \( X = (q = 0) \) of \( \mathbb{P}^n \) where \( q \) is a degree \( m \) homogeneous polynomial. We briefly recall the \( \epsilon \)-stable quasimaps (defined in [FK0, Definition 3.1.1]) to \( \mathbb{P}^n \) and to \( X \).

**Definition 2.1.** A prestable quasimap to \( \mathbb{P}^n \) consists of

\[ (C, p_1, \ldots, p_\ell, L, \{u_i\}_{i=1}^{n+1}), \]

where
Definition 2.2. The length at \( z \in C \) of a quasimap \((C, p_1, \ldots, p_\ell, L, \{u_i\}_{i=1}^{n+1})\) is
\[
leng(z) := \text{length}_2(\text{coker}(\mathcal{O}_C^n(\pi) \to L)).
\]

Let \( \epsilon \) be a positive rational number.

**stability**

Definition 2.3. A prestable quasimap \((C, p_1, \ldots, p_\ell, L, \{u_i\}_{i=1}^{n+1})\) is \( \epsilon \)-stable if
1. \( \omega_C(p_1 + \ldots + p_\ell) \otimes L^\epsilon \) is ample,
2. \( \epsilon \cdot \text{leng}(z) \leq 1 \) for every point \( z \in C \).

As in [FK] we denote by \( \epsilon = 0^+ \) when \( \epsilon \to 0 \).

**family**

Definition 2.4. A family of genus \( g \) degree \( d \) \( \epsilon \)-stable quasimaps to \( \mathbb{P}^n \) over a scheme \( S \) consists of the data
\[
(\pi : \mathcal{C}_S \to S, \{p_i : S \to \mathcal{C}_S\}_{i=1, \ldots, \ell}, \mathcal{L}_S, \{u_j\}_{j=1}^{n+1})
\]
where
- \( \pi : \mathcal{C}_S \to S \) is a flat family of connected nodal curves over \( S \),
- \( p_1, \ldots, p_\ell \) are sections of \( \pi \),
- \( \mathcal{L}_S \) is a line bundle over \( \mathcal{C}_S \) with degree \( d \) along fibers of \( \mathcal{C}_S/S \),
- \( u_j \in \Gamma(\mathcal{C}_S, \mathcal{L}_S) \) for \( i = 1, \ldots, n+1 \),

such that the restriction to every geometric fiber \( \mathcal{C}_s \) of \( \pi \) is an \( \epsilon \)-stable \( \ell \)-marked quasimap of genus \( g, \epsilon \) and \( d \). An arrow from \((\mathcal{C}_S/S, \ldots)\) to \((\mathcal{C}'_S/S, \ldots)\) are \( S \)-isomorphisms \( f : \mathcal{C}_S \to \mathcal{C}'_S \) and \( \sigma_f : \mathcal{L}_S \to f^*\mathcal{L}'_S \) preserving markings and sections.

Given a scheme \( S \), denote \( \overline{\mathcal{M}}_{g, \ell}(\mathbb{P}^n, d)(S) \) to be the set of all \((\mathcal{C}_S/S, \{p_i\}, \mathcal{L}_S, \{u_j\})\).

Then, with naturally defined arrows, \( \overline{\mathcal{M}}_{g, \ell}(\mathbb{P}^n, d) \) forms a groupoid fibered over the category of schemes. By [FK0, Theorem 4.0.1], or [FKM, Example 3.1.4] and [FKM, Thm. 7.1.6], we have

**equi**

Lemma 2.5. The groupoid \( \overline{\mathcal{M}}_{g, \ell}(\mathbb{P}^n, d) \) is a DM stack, proper over \( C \).

We recall the notion of the \( \epsilon \)-stable quasimaps to the hypersurface \( X \) (of \( \mathbb{P}^n \)).

**family1**

Definition 2.6. A family of genus \( g \) degree \( d \) \( \epsilon \)-stable quasimaps to \( X \) over a base scheme \( S \) consists of the data
\[
(\pi : \mathcal{C}_S \to S, \{p_i : S \to \mathcal{C}_S\}_{i=1, \ell}, \mathcal{L}_S, \{u_i\}_{i=1}^{n+1}) \in \overline{\mathcal{M}}_{g, \ell}(\mathbb{P}^n, d)(S)
\]
where the section \( q(u_i) \in \Gamma(\mathcal{C}_S, \mathcal{L}_S^m) \) vanishes.

Arrows are defined the same as Definition 2.4.

\[ ^2 \text{when } \epsilon = 0^+ \text{ the definition is exactly the stable quotient to } X \text{ defined in [MOP, Sect. 10.1].} \]
Let $\overline{M}_{g,\ell}(X, d)(S)$ be the set of all families in above definition. With naturally defined arrows $\overline{M}_{g,\ell}(X, d)$ forms a closed substack of $\overline{M}_{g,\ell}(\mathbb{P}^n, d)$.

2.2. Deformation theory.

2.2.1. Deformation theory via the direct image cone. We recall the direct image cone construction in [CL]. Let $\pi : C \to Y$ be a flat family of connected nodal genus $g$ curves, where $Y$ is a smooth Artin stack. Let $Z$ be an Artin stack representable and quasi-projective over $C$. Define a groupoid $\mathfrak{X}$ as follows. For any scheme $S \to Y$, denote $C_S = C \times_Y S$ and $Z_S = Z \times_C C_S$. Then $Z \to C$ induces the projection $\pi_S : Z_S \to C_S$. Let

$$\mathfrak{X}(S) = \{ s : C_S \to Z_S \mid s \text{ are } C_S \text{-morphisms} \}.$$ 

The arrows are defined by pullbacks.

**Proposition 2.7.** [CL, Prop. 2.3] The groupoid $\mathfrak{X}$ is an Artin stack with a natural representable and quasi-projective morphism to $Y$.

In case $Z = \text{Vb}(\mathcal{E})$ is the underlying vector bundle of some locally free sheaf $\mathcal{E}$ over $D$, the space $C(\pi_\ast \mathcal{E})$ is the moduli of sections, denoted as $C(\pi_\ast \mathcal{E}) \cong \mathfrak{X}$ in [CL, Coro. 2.4]

Let $\pi_X : C_X \to X$ be the universal family of $X$ and let $\epsilon : C_X \to Z$ be the tautological evaluation map. We generalize [CL, Prop. 2.5] to the following.

**Proposition 2.8.** Suppose $Z \to C$ is a flat morphism. Then $X \to \mathcal{Y}$ has a relative obstruction theory

$$\phi_{\mathfrak{X}/\mathcal{Y}} : T_{\mathfrak{X}/\mathcal{Y}} \to R^\ast \pi_{\mathfrak{X}} \ast \epsilon^\ast T_{Z/C}.$$ 

**Proof.** In the case $Z/C$ is smooth, the claim is proved in [CL, Prop. 2.5, Appendix A.3]. All the arguments applies to the case $Z/C$ is flat. For example, the equality in the second lines under [CL, (A.9)] holds by the flat base change property of cotangent complex ([Illusie, Coro. 2.2.3]).

2.2.2. Perfect obstruction theory of the quasimap moduli.

**Definition 2.9.** Let $D := \mathcal{D}_{g,\ell}$ be the groupoid associating to each scheme $S$ the set $\mathcal{D}(S) = \{ (C_S, \{ p_i : S \to C_S \}_{i=1}^\ell, L_S) \}$, where $\pi : C_S \to S$ is a flat family of connected nodal curves and $L_S$ is a line bundle on $C_S$. An arrow from $(C_S, \{ p_i : S \to C_S \}_{i=1}^\ell, L_S)$ to $(C'_S, \{ p'_i : S \to C'_S \}_{i=1}^\ell, L'_S)$ consists of $f : C_S \to C'_S$ and an isomorphism $\tau : L_S \to f^\ast L'_S$, which preserve the markings.

Let $C_D \xrightarrow{\pi_D} D$ and $L_D$ be the universal curve and line bundle of $D$, and set $Z := \text{Vb}(L_D^{\oplus (n+1)})$ over $C_D$. There is an arrow

$$\lambda : \overline{M}_{g,\ell}(\mathbb{P}^n, d) \to \mathfrak{X} \cong C(\pi_D \ast L_D^{\oplus (n+1)}),$$
sending the data in Definition 2.4 to the associated \( S \to D \) with sections given by \( \{u_j\} \)'s. Since the nondegeneracy condition in Definition 2.1 and stability conditions in Definition 2.3 are open conditions, \( \lambda \) is an open embedding.

By Proposition 2.7 and 2.8, \( \mathcal{M}_{g,\ell}(\mathbb{P}^n,d)/D \) has an obstruction theory
\[
\phi_{\mathcal{M}_{g,\ell}(\mathbb{P}^n,d)/D} : T_{\mathcal{M}_{g,\ell}(\mathbb{P}^n,d)/D} \to E_{\mathcal{M}_{g,\ell}(\mathbb{P}^n,d)/D} := R^*\pi_X\ast\mathbb{T}\mathbb{Z}/\mathcal{C}_D.
\]
Since \( \mathcal{Z} \) is smooth over \( \mathcal{C}_D \), we know \( R^*\pi_X\ast\mathbb{T}\mathbb{Z} \) has amplitude contained in \([0,1]\) and \( \phi_{\mathcal{M}_{g,\ell}(\mathbb{P}^n,d)/D} \) is a relative perfect obstruction theory.

Let \( \mathcal{Q} \) be the universal \( \mathbb{G}_m \) principal bundle over \( \mathcal{C}_D \), then \( \mathcal{Z} \cong \mathcal{Q} \times_{\mathbb{G}_m} \mathbb{C}^{n+1} \).

We have the following canonical map
\[
\Psi : \mathcal{Z} = \text{Vb}(\mathcal{L}_D^{\oplus(n+1)}) \to [\mathbb{C}^{n+1}/\mathbb{G}_m],
\]
Let \( C_X \) be the cone \((q(x_i) = 0) \subset \mathbb{C}^{n+1} \). Denote
\[
(2.1) \quad \mathcal{Z}_X := \mathcal{Q} \times_{\mathbb{G}_m} C_X \cong \text{Vb}(\mathcal{L}_D^{\oplus(n+1)}) \times [\mathbb{C}^{n+1}/\mathbb{G}_m] [C_X/\mathbb{G}_m] \subset \text{Vb}(\mathcal{L}_D^{\oplus(n+1)}).
\]

Let \( \mathcal{X}_X \) be the stack of sections constructed in Proposition 2.7 with \( \mathcal{Z} \) replaced by \( \mathcal{Z}_X \). There by definition we have a Cartesian diagram
\[
\begin{array}{ccc}
\mathcal{M}_{g,\ell}(X,d) & \longrightarrow & \mathcal{M}_{g,\ell}(\mathbb{P}^n,d) \\
\downarrow & & \downarrow \\
\mathcal{X}_X & \longrightarrow & \mathcal{X},
\end{array}
\]
Thus \( \lambda_X \) is also an open embedding. Then Proposition 2.8 induces
\[
\phi_{\mathcal{M}_{g,\ell}(X,d)/D} : T_{\mathcal{M}_{g,\ell}(X,d)/D} \to E_{\mathcal{M}_{g,\ell}(X,d)/D} := R^*\pi_X\ast\mathbb{T}\mathbb{Z}_X/\mathcal{C}_D.
\]
Since \( E_{\mathcal{M}_{g,\ell}(X,d)/D} \) above is a two-term complex concentrated in \([0,1]\) by Theorem 4.52 in [FKM], \( \phi_{\mathcal{M}_{g,\ell}(X,d)/D} \) is a relative perfect obstruction theory. It is identical to the perfect obstruction theory of \( \mathcal{M}_{g,\ell}(X,d)/D \) defined by I. Ciocan-Fontanine, B. Kim and D. Maulik in [FKM, Sect. 4.5].

3. \( \epsilon \)-stable quasimaps with \( P \)-fields

We enlarge the moduli of \( \epsilon \)-stable quasimaps to \( \mathbb{P}^n \) by adding \( P \)-fields, and define its localized virtual cycle. This section is parallel to [CL, Sect. 3].

3.1. Moduli of \( \epsilon \)-stable quasimaps with \( P \)-fields. Let \( M := \mathcal{M}_{g,\ell}(\mathbb{P}^n,d) \) be the moduli of genus \( g \) degree \( d \) \( \epsilon \)-stable quasimaps to \( \mathbb{P}^n \). Let \( (\mathcal{C}_M,\pi_M) \) be the universal family of \( \mathcal{M}_{g,\ell}(\mathbb{P}^n,d) \), and \( \mathcal{L}_M \) be the university line bundle. Denote
\[
\mathcal{R}_M := \mathcal{L}_M^{\oplus m} \otimes \omega_{\mathcal{C}_M/M}.
\]
Using direct image cone construction, we form
\[
\mathcal{P} := \mathcal{M}_{g,\ell}(\mathbb{P}^n,d)^P := C(\pi_M,\mathcal{R}_M).
\]
It is represented by a DM stack. We call it the moduli of genus \( g \) degree \( d \) \( \epsilon \)-stable quasimaps with \( P \)-fields.

Let \( \pi_P : \mathcal{C}_P \to \mathcal{D} \) be the universal family. Let \( \mathcal{L}_P \) be the universal line bundle on \( \mathcal{C}_P \). Then we have an invertible sheaf and a vector bundle
\[
\mathcal{R}_P := \mathcal{L}_P^{\oplus m} \otimes \omega_{\mathcal{C}_P/\mathcal{D}},
\]
\[
\mathfrak{z} := \text{Vb}(\mathcal{L}_P^{\oplus (n+1)} \oplus \mathcal{R}_P)
\]
over \( \mathcal{C}_P \). If \( \mathfrak{X} \) be the moduli stack of sections of \( \mathfrak{z}/\mathcal{C}_P \) in Proposition 2.7, \( \mathcal{P} \) is an open substack of \( \mathfrak{X} \). The tautological evaluation morphism \( \mathcal{C}_X \to \mathfrak{z} \) induces an evaluation morphism \( \tilde{\mathcal{C}} : \mathcal{C}_P \to \mathfrak{z} \). Let \( \pi_P : \mathcal{C}_P \to \mathcal{P} \) and \( \mathcal{L}_P \) be the universal curve and line bundle over \( \mathcal{P} \) defined by the pull-back of \( \mathcal{C}_P \) and \( \mathcal{L}_P \). Denote \( \mathcal{R}_P := \mathcal{L}_P^{\oplus m} \otimes \omega_{\mathcal{C}_P/\mathcal{P}} \).

**Proposition 3.1.** The pair \( \mathcal{P} \to \mathcal{D} \) admits a perfect relative obstruction theory
\[
\phi_{\mathcal{P}/\mathcal{D}} : T_{\mathcal{P}/\mathcal{D}} \longrightarrow \mathcal{E}_{\mathcal{P}/\mathcal{D}} := R^\bullet \pi_P^*(\mathcal{L}_P^{\oplus (n+1)} \oplus \mathcal{R}_P).
\]

**Proof.** It follows from Proposition 2.8 by applying it to the morphism \( \tilde{\mathcal{C}} \), using that \( T_{\mathfrak{z}/\mathcal{C}_P} = \omega^* \mathcal{L}_P^{\oplus (n+1)} \oplus \omega^* \mathcal{R}_P \), where \( \omega : \mathfrak{z} \to \mathcal{C}_P \) denotes the projection. \( \square \)

### 3.2. Construction of the cosection

Define a bundle morphism
\[
\Lambda : \text{Vb}(\mathcal{L}_P^{\oplus (n+1)} \oplus \mathcal{R}_P) \longrightarrow \text{Vb}(\omega_{\mathcal{C}_P/\mathcal{D}}), \quad \Lambda(x, p) = p \cdot q(x),
\]
for \( (x, p) = ((x_i), p) \in \text{Vb}(\mathcal{L}_P^{\oplus (n+1)} \oplus \mathcal{R}_P) \). The product \( p \cdot q(x) \) is given by \( \mathcal{L}_P^{\oplus m} \otimes \mathcal{R}_P \to \omega_{\mathcal{C}_P/\mathcal{D}} \). The \( \Lambda \) induces a morphism
\[
d\Lambda : T_{\text{Vb}(\mathcal{L}_P^{\oplus (n+1)} \oplus \mathcal{R}_P)/\mathcal{C}_P} \to \Lambda^* T_{\text{Vb}(\omega_{\mathcal{C}_P/\mathcal{D}})/\mathcal{C}_P}.
\]

Pull back to \( \mathcal{C}_P \) via evaluation map \( \tilde{\mathcal{C}} \) and apply \( R^\bullet \pi_P \). We have
\[
\sigma_1^\bullet : \mathcal{E}_{\mathcal{P}/\mathcal{D}} \longrightarrow R^\bullet \pi_P^*(\tilde{\mathcal{C}}^* \Lambda^* \Omega^\vee_{\text{Vb}(\omega_{\mathcal{C}_P/\mathcal{D}})/\mathcal{C}_P}) \cong R^\bullet \pi_P^*(\omega_{\mathcal{C}_P/\mathcal{P}}).
\]

It induces the following morphism:
\[
\sigma_1 : \mathcal{O}b_{\mathcal{P}/\mathcal{D}} = R^1 \pi_P^* \mathcal{L}_P^{\oplus (n+1)} \oplus R^1 \pi_P^* \mathcal{R}_P \longrightarrow \mathcal{O}_P.
\]

A coordinate expression of \( \sigma_1 \) is as follows. Let \( u_1, \ldots, u_{n+1} \in \Gamma(\mathcal{C}_P, \mathcal{L}_P) \) and \( p \in \Gamma(\mathcal{C}_P, \mathcal{R}_P) \) be the tautological section of \( \mathcal{P} \). For any chart \( T \to \mathcal{P} \), let \( \mathcal{C}_T = \mathcal{C}_P \times_{\mathcal{P}} T \). And let \( p \) and \( u_i \) be the pull-back of \( p \) and \( u_i \) to \( \mathcal{T} \), and \( u := (u_1, \ldots, u_{n+1}) \). Then for arbitrary
\[
\dot{p} \in H^1(\mathcal{C}_T, \mathcal{R}_P) \quad \text{and} \quad \dot{u} = (\dot{u}_i) \in H^1(\mathcal{C}_T, \mathcal{L}_P^{\oplus (n+1)}),
\]
\[
\sigma_1(\dot{p}, \dot{u}) = p \sum_i \frac{\partial q(u)}{\partial u_i} \dot{u}_i + \dot{p} q(u).
\]
3.3. Degeneracy locus of the cosection. We define the degeneracy loci of $\sigma_1$
\[\text{Deg}(\sigma_1) := \{\xi \in \mathcal{P} \mid |\sigma_1|_{\xi} : \mathcal{O}_{\mathcal{P}/D} \otimes \mathcal{O}_p \to \mathbb{C} \to \mathbb{C}(\xi) \text{ vanishes}\}.
\]
As the definition in section 2, we can embed
\[\overline{M}_{g,\ell}(X, d) \subset \overline{M}_{g,\ell}(\mathbb{P}^n, d) \subset \mathcal{P},\]
where the second inclusion is by assigning zero $P$-fields.

**Proposition 3.2.** The degeneracy loci of $\sigma_1$ is $\overline{M}_{g,\ell}(X, d) \subset \mathcal{P}$; it is proper.

**Proof.** At each $\eta = (C, p_1, \ldots, p_{\ell}, L, u, p) \in \mathcal{P}$, where $u = (u_i) \in H^0(C, L^{\oplus n+1})$, we have $\sigma_1|_{\eta}(\hat{p}, \hat{u}) = p \sum_i \frac{\partial q(u)}{\partial u_i} \hat{u}_i + \hat{q}(u)$. If $\eta \in \text{Deg}(\sigma_1)$, $\hat{q}(u) = 0$ for arbitrary $\hat{p}$ implies $q(u) = 0$ by Serre duality. Similarly $p \sum_i \frac{\partial q(u)}{\partial u_i} \hat{u}_i = 0$ for arbitrary $\{\hat{u}_i\}$ implies $p \frac{\partial q(u)}{\partial u_i} = 0$ for each $i$. By definition the set of common zeros of $(u_i)$ is a finite set $B \subset C$. As $C_X - \{0\}$ is smooth, $(\frac{\partial q(u)}{\partial u_1}, \ldots, \frac{\partial q(u)}{\partial u_{n+1}})$ is nowhere zero on $C - B$. Thus $p|_{C-B} = 0$ and $p = 0$. Therefore $\text{Deg}(\sigma_1)$ is the set of $(C, p_1, \ldots, p_{\ell}, L, u, p)$ with $q(u) = 0$ and $p = 0$. This set is $\overline{M}_{g,\ell}(X, d) \subset \mathcal{P}$. \hfill $\square$

Let $q : \mathcal{P} \to D$ be the tautological morphism. We form the triangle

\[\text{(3.3)} \quad q^* \mathbb{L}_D \longrightarrow \mathbb{L}_P \longrightarrow \mathbb{L}_{\mathcal{P}/D} \stackrel{\zeta}{\longrightarrow} q^* \mathbb{L}_D[1].\]

Composed with $\phi_{\mathcal{P}/D} : T_{\mathcal{P}/D} \to E_{\mathcal{P}/D}$, let

\[\text{(3.4)} \quad \eta := H^0(\phi_{\mathcal{P}/D} \circ \zeta^\vee) : q^* H^0(T_D) \longrightarrow H^1(T_{\mathcal{P}/D}) \longrightarrow H^1(E_{\mathcal{P}/D}) = \mathcal{O}_{\mathcal{P}/D}.
\]

Set the absolute obstruction sheaf $\mathcal{O}_{\mathcal{P}} := \text{cokernel} \eta$. The lemma below follows from exactly same argument as the proof of [CL, Lemm. 3.6].

**Lemma 3.3.** The following composition is trivial:
\[0 = H^1(\sigma_1 \circ \phi_{\mathcal{P}/D}) : H^1(T_{\mathcal{P}/D}) \longrightarrow H^1(E_{\mathcal{P}/D}) \longrightarrow R^1\pi_{\mathcal{P}*}\omega_{\mathcal{P}/D}.\]

**Corollary 3.4.** The cosection $\sigma_1 : \mathcal{O}_{\mathcal{P}/D} \to \mathcal{O}_P$ lifts to a $\bar{\sigma}_1 : \mathcal{O}_{\mathcal{P}} \to \mathcal{O}_P$.

**Proof.** The composition of $\sigma_1$ with (3.4) is the $H^1$ of the composition
\[q^* T_D[-1] \overset{\sigma_1}{\longrightarrow} T_{\mathcal{P}/D} \overset{\phi_{\mathcal{P}/D}}{\longrightarrow} E_{\mathcal{P}/D} \overset{\sigma_1^*}{\longrightarrow} R^1 \pi_{\mathcal{P}*}\omega_{\mathcal{P}/D},\]
which vanishes by Lemma 3.3. \hfill $\square$

3.4. The virtual dimension and the virtual cycle. Pick arbitrary closed point $\xi = (C, \{p_j\}_{j=1}^\ell, L, \phi, p)$ of $\mathcal{P}$. The virtual dimension of $\mathcal{P}/D$ is
\[\text{rank}(E_{\mathcal{P}/D} \otimes \mathcal{O}_P \mathcal{C}(\xi)) = \chi(C, L^{\oplus (n+1)}) + \chi(C, L^{\oplus m} \otimes \omega_C) = (n+1-m)d+n(1-g).
\]

One then calculates
\[\text{dim } D = \text{dim } D/\overline{M}_{g,\ell} + \text{dim } \overline{M}_{g,\ell} = g - 1 + 3g - 3 + \ell = 4(g-1) + \ell,\]
\[\delta := \text{vdim } \mathcal{P} = \text{dim } D + \text{vdim } \mathcal{P}/D = (n+1-m)d+(n-4)(1-g) + \ell.\]
Let $\mathcal{U} \subset \mathcal{P}$ be the locus where $\sigma_1$ is surjective; and denote $\text{Deg}(\sigma_1) = \mathcal{P} - \mathcal{U}$. Since $\sigma_1|_{\mathcal{U}}$ is surjective, it induces a surjective bundle homomorphism

$$
\sigma_1|_{\mathcal{U}} : h^1/h^0(\mathcal{E}_{\mathcal{P}/\mathcal{D}}) \times \mathcal{U} \longrightarrow \mathcal{C}_{\mathcal{U}},
$$

where $\mathcal{C}_{\mathcal{U}} = \mathcal{C} \times \mathcal{U}$. Let $\ker(\sigma_1|_{\mathcal{U}})$ be the kernel bundle-stack of $\sigma_1|_{\mathcal{U}}$. Endow

$$
\tag{3.5}
\text{cone-stack}
$$

with the reduced structure. It is a closed substack of $h^1/h^0(\mathcal{E}_{\mathcal{P}/\mathcal{D}})$.

By [KL, Thm. 5.1] the normal cone cycle $[\mathcal{C}_{\mathcal{P}/\mathcal{D}}]$ lies in $Z h^1/h^0(\mathcal{E}_{\mathcal{P}/\mathcal{D}})$ lies in $\pi_*h^1/h^0(\mathcal{E}_{\mathcal{P}/\mathcal{D}})(\sigma_1)$. Kiem-Li’s localized Gysin map gives

$$
0_{g,1,\text{loc}} : A_*h^1/h^0(\mathcal{E}_{\mathcal{P}/\mathcal{D}})(\sigma_1) \longrightarrow A_*\text{Deg}(\sigma_1).
$$

**Definition-Proposition 3.5.** We define the localized virtual cycle of $\mathcal{P}$ to be

$$
[\mathcal{P}]^{\text{vir}} = [\mathcal{M}_{g,\ell}(\mathbb{P}^n,d)]^{\text{vir}} := 0_{g,1,\text{loc}}([\mathcal{C}_{\mathcal{P}/\mathcal{D}}]) \in \overline{\mathcal{M}}_{g,\ell}(X,d).
$$

4. **Degeneration of moduli of $\epsilon$-stable quasimaps with $P$-fields**

In the following sections, we use degenerations in [CL] to prove $[\mathcal{P}]^{\text{vir}}$ coincides up to a sign with $[\mathcal{M}_{g,\ell}(X,d)]^{\text{vir}}$. The setup is close to [CL] but the perfectness of family obstruction theories requires the quasimap conditions (see Proposition 4.6). We also give a new proof of the constancy of virtual cycles in the degeneration (Theorem 4.6), which greatly simplifies that in [CL1].

Let $(\mathcal{C}_M, \pi_M)$ be the universal family of $\mathcal{M}_{g,\ell}(\mathbb{P}^n,d)$, and let $\mathcal{L}_M$ be the universality line bundle as before. We form a separated DM stack

$$
F = C(\pi_M^*(\mathcal{L}_M^{\otimes m} \oplus \mathcal{O}_{\mathcal{C}_M})).
$$

We define $\mathcal{K}_{g,\ell}(V,d)$ to be the subgroupoid of $F$, such that $\mathcal{K}_{g,\ell}(V,d)(S)$ is the set of all families:

$$
\tag{4.1}
\text{obj}
$$

subject to the equation

$$
q(u_1, \ldots, u_{n+1}) - ty = 0 \in \Gamma(\mathcal{C}_S, \mathcal{L}_S^{\otimes m}),
$$

where $t \in \Gamma(\mathcal{C}_S, \mathcal{O}_{\mathcal{C}_S})$ and $y \in \Gamma(\mathcal{C}_S, \mathcal{L}_S^{\otimes m})$. Clearly $\mathcal{K}_{g,\ell}(V,d)$ is a closed substack of $F$, and thus is a separated DM stack.

Now we introduce the stable quasimaps with $p$-field. Let $(\tilde{\mathcal{C}}, \tilde{\pi})$ be the universal family of $\mathcal{K}_{g,\ell}(V,d)$, and let $\tilde{\mathcal{L}}$ be the universal line bundle over $\tilde{\mathcal{C}}$. Let $\tilde{\mathcal{R}} = \tilde{\mathcal{L}}^{\otimes m} \otimes \omega_{\tilde{\mathcal{C}}/\mathcal{K}_{g,\ell}(V,d)}$ be the tautological invertible sheaves. Define the moduli of stable morphisms coupled with $P$-fields to be

$$
\mathcal{V} := \mathcal{K}_{g,\ell}(V,d)^P := C(\tilde{\pi}_*\tilde{\mathcal{R}}),
$$
the direct image cone. Sending (4.1) to \( S \to \mathbb{A}^1 \) induces a map \( \mathcal{V} \to \mathbb{A}^1 \), with fibers canonically described as

\[
\mathcal{V}_c := \mathcal{V} \times_{\mathbb{A}^1} c \cong \mathcal{P}, \quad c \neq 0;
\]

\[
\mathcal{V}_0 := \mathcal{V} \times_{\mathbb{A}^1} 0 =: \mathcal{N}; \quad c = 0.
\]

By construction \( \mathcal{V}_0 \) represents the groupoid where \( \mathcal{V}_0(S) \) is the set of all \( (C_S, L_S, \cdots, p, y) \) where \( (C_S, L_S, \cdots) \in \overline{\mathcal{M}}_{g, \ell}(X, d)(S), p \in \Gamma(C_S, L_S^{\otimes m} \otimes \omega_{C_S/S}) \) and \( y \in \Gamma(C_S, L_S^{\otimes m}) \).

4.1. The evaluation maps. We construct a natural obstruction theory of \( \mathcal{V} \) relative to \( D := D \times \mathbb{A}^1 \). Denote the universal curve by

\[
\pi_D : \mathcal{C}_D := \mathcal{C}_D \times \mathbb{A}^1 \to D \times \mathbb{A}^1 = D;
\]

and \( \mathcal{L}_D \) the pull-back of \( \mathcal{L}_D \) via \( \mathcal{C}_D \to \mathcal{C}_D \). We have a bundle over \( \mathcal{C}_D \)

\[
\text{Vb}(\mathcal{L}_D^{\oplus (n+1)}) \times_{\mathcal{C}_D} \text{Vb}(\mathcal{L}_D^{\otimes m}) \longrightarrow \mathcal{C}_D.
\]

Let \( E_2 = \mathcal{C}_{\mathbb{A}^1} \) (resp. \( E_1 = \mathcal{C}_{\mathbb{A}^1}^{\oplus (n+1)} \)) be the trivial line bundle (resp. rank \( n + 1 \) trivial vector bundle) over \( \mathbb{A}^1 \). We consider the rank \( n + 2 \) bundle

\[
\text{pr}_{\mathbb{A}^1} : E_1 \times_{\mathbb{A}^1} E_2 \longrightarrow \mathbb{A}^1
\]

with the \( \mathbb{G}_m \)-action: \( \mathbb{G}_m \) acts on the base \( \mathbb{A}^1 \) trivially and acts on fibers of \( E_1 \) via \( g \cdot (x_1, \cdots, x_{n+1}) = (gx_1, \cdots, gx_{n+1}) \), and acts on fibers of \( E_2 \) as \( g \cdot (y_0) = (g^n y_0) \).

Denote \( \Delta := q(x_i) - t \cdot y_0 \) and let \( C(V) = (\Delta = 0) \subset E_1 \times_{\mathbb{A}^1} E_2 \), where \( t \) denotes the coordinate of \( \mathbb{A}^1 \). There is a canonical morphism

\[
\text{Vb}(\mathcal{L}_D^{\oplus (n+1)}) \times_{\mathcal{C}_D} \text{Vb}(\mathcal{L}_D^{\otimes m}) \longrightarrow [(E_1 \times_{\mathbb{A}^1} E_2)/\mathbb{G}_m].
\]

We define

\[
\mathcal{X}' = \left( \text{Vb}(\mathcal{L}_D^{\oplus (n+1)}) \times_{\mathcal{C}_D} \text{Vb}(\mathcal{L}_D^{\otimes m}) \right) \times [(E_1 \times_{\mathbb{A}^1} E_2)/\mathbb{G}_m] [C(V)/\mathbb{G}_m],
\]

and

\[
\mathcal{X} = \mathcal{X}' \times_{\mathcal{C}_D} \text{Vb}(\mathcal{R}_D).
\]

Let \( \pi_V : \mathcal{C}_V \to \mathcal{V} \) be the universal family. The natural evaluation morphism

\[
e : \mathcal{C}_F \longrightarrow \text{Vb}(\mathcal{L}_D^{\oplus (n+1)}) \times_{\mathcal{C}_D} \text{Vb}(\mathcal{L}_D^{\otimes m}),
\]

(where \( \mathcal{C}_F \) is the universal family of \( F \)) restricts to give the evaluation morphism

\[
e_V : \mathcal{C}_V \longrightarrow \mathcal{X}.
\]
4.2. The obstruction theory of $\mathcal{V}/\mathbb{D}$. We begin with a description of the tangent complex $T_{\mathcal{X}/\mathbb{C}_d}$. Let $\rho : \mathcal{X}' \to \mathbb{C}_d$ be the tautological projection. By the defining equation of $C(V)$,

$$T_{C(V)/\mathbb{A}} \cong \left[ \mathcal{O}_{C(V)}^{\oplus(n+1)} \oplus \mathcal{O}_{C(V)} \xrightarrow{d\Delta} \mathcal{O}_{C(V)} \right],$$

where $d\Delta$ at $((x_i), y_0, t) \in C(V)$ sends $((\dot{x}_i), \dot{y}_0)$ to $\sum \frac{\partial y(x_i)}{\partial x_i} \dot{x}_i - t\dot{y}_0$. Thus

$$T_{\mathcal{X}'/\mathbb{C}_d} \cong T_{\mathcal{P} \times \mathbb{C}_d} C(V)/\mathbb{C}_d \cong \left[ \rho^* \mathcal{L}_{\mathbb{D}}^{\oplus(n+1)} \oplus \rho^* \mathcal{L}_{\mathbb{D}}^{\oplus m} \xrightarrow{d\Delta} \rho^* \mathcal{L}_{\mathbb{D}}^{\oplus m} \right],$$

where $d\Delta$ at $((z_i), y, t) \in \mathcal{X}'$ sends $((\dot{z}_i), \dot{y})$ to $\sum \frac{\partial y(z_i)}{\partial z_i} \dot{z}_i - t\dot{y}$. Let $\mathcal{L}_V$ be the universal line bundle over $\mathcal{C}_V$, then

$$\mathcal{L}_{\mathcal{V}} \cong \left[ \mathcal{L}_{\mathcal{V}}^{\oplus(n+1)} \oplus \mathcal{L}_{\mathcal{V}}^{\oplus m} \xrightarrow{d\kappa} \mathcal{L}_{\mathcal{V}}^{\oplus m} \right] \oplus \left[ \mathcal{L}_{\mathcal{V}}^{\oplus m} \otimes \omega_{\mathcal{C}_V/\mathcal{V}} \to 0 \right],$$

where $d\kappa$ restricted to $((\phi_i), \tilde{i}, b, p) \in \mathcal{C}_V$ sends $((\dot{\phi}_i), \dot{b})$ to $\sum \frac{\partial \phi(x_i)}{\partial x_i} \dot{\phi}_i - t\dot{b}$. Denote

$$\mathcal{K}^1 = \left[ \mathcal{L}_{\mathcal{V}}^{\oplus(n+1)} \oplus \mathcal{L}_{\mathcal{V}}^{\oplus m} \xrightarrow{d\kappa} \mathcal{L}_{\mathcal{V}}^{\oplus m} \right].$$

Lemma 4.1. Let $C$ be a geometric fiber of $\mathcal{C}_V \to \mathcal{V}$ over arbitrary closed point $\xi \in \mathcal{V}$, then $\mathcal{H}^1(\mathcal{L}_{\mathcal{V} \mathcal{X}/\mathbb{C}_d}|_C)$ is a torsion sheaf on $C$.

Proof. By the definition of $\mathcal{V}$, for each point $\xi = ((u_1), t, y, p) \in \mathcal{V}$ there exists a finite set $B \subset C$ such that $(u_1(z)) \neq 0$ for every $z \notin B$. Therefore $d\kappa$ is surjective when restricted on $C \setminus B$. \(\square\)

Proposition 4.2. The $\mathcal{V} \to \mathbb{D}$ has a relative perfect obstruction theory

$$\phi_{\mathcal{V}/\mathbb{D}} : T_{\mathcal{V}/\mathbb{D}} \to \mathcal{E}_{\mathcal{V}/\mathbb{D}} := R^\bullet \pi_{\mathcal{V}*} \mathcal{E}_{\mathcal{X}/\mathbb{C}_d}.$$

Its specialization at $c \neq 0 \in \mathbb{A}^1$ (resp. $0 \in \mathbb{A}^1$) is the $\phi_{\mathcal{P}/\mathbb{D}}$ (resp. $\phi_{\mathcal{X}/\mathbb{D}}$).

Proof. Since $\mathcal{V}$ is an open substack of the direct image cone $C(\mathcal{P}, \mathcal{X})$, Proposition 2.8 provides a canonical obstruction theory $\phi_{\mathcal{V}/\mathbb{D}}$. Let $C$ be a geometric fiber of $\mathcal{C}_V \to \mathcal{V}$ over arbitrary closed point $\xi \in \mathcal{V}$. As $\mathcal{L}_{\mathcal{V} \mathcal{X}/\mathbb{C}_d}|_C$ is two-term, the sequence

$$\mathcal{H}^0(\mathcal{E}_{\mathcal{V} \mathcal{X}/\mathbb{C}_d}|_C) \to \mathcal{E}_{\mathcal{V} \mathcal{X}/\mathbb{C}_d}|_C \to \mathcal{H}^1(\mathcal{E}_{\mathcal{V} \mathcal{X}/\mathbb{C}_d}|_C)[-1] \to$$

is a distinguished triangle.

Since $H^1(C, \mathcal{H}^1(\mathcal{E}_{\mathcal{V} \mathcal{X}/\mathbb{C}_d}|_C))$ vanishes by Lemma 4.1, $H^2(\mathcal{E}_{\mathcal{V} \mathcal{X}/\mathbb{C}_d}|_C) = 0$. Thus $R^\bullet \pi_{\mathcal{V}*} \mathcal{E}_{\mathcal{X}/\mathbb{C}_d}$ is perfect of amplitude $[0, 1]$.

When $c \neq 0$ using $\iota_c : \mathcal{V} = \mathcal{V} \times_{\mathcal{A}_\mathbb{D}} c \to \mathcal{V}$, the functoriality of the construction implies that $\phi_{\mathcal{P}/\mathbb{D}}$ is the composition of $T_{\mathcal{P}/\mathbb{D}} \to \iota_c^* T_{\mathcal{V}/\mathbb{D}}$ with

$$\iota_c^* (\phi_{\mathcal{V}/\mathcal{P}}) : \iota_c^* T_{\mathcal{V}/\mathbb{D}} \to \iota_c^* \mathcal{E}_{\mathcal{V}/\mathbb{D}} \cong \mathcal{E}_{\mathcal{P}/\mathbb{D}}.$$

In case $c = 0$, we define $\mathcal{E}_{\mathcal{N}/\mathbb{D}} := \iota_0^* \mathcal{E}_{\mathcal{V}/\mathbb{D}}$. This proves the Proposition. \(\square\)
Let $\mathcal{C}_N = \mathcal{C}_V \times V, N$ and $\pi_N : \mathcal{C}_N \to N$ be the restriction of $\pi_V$. If $c = 0$,
$$
\mathcal{K}_1^\bullet \cong \left[ \mathcal{L}_V^{\oplus(n+1)} \xrightarrow{d\kappa_1} \mathcal{L}_V^{\oplus m} \right]|_{\mathcal{C}_N} + \left[ \mathcal{L}_V^{\oplus m} \to 0 \right]|_{\mathcal{C}_N},
$$
where $d\kappa_1$ restricted to $\mathcal{C}_N$ sends $(\phi_i)$ to $\sum \frac{\partial (\phi_i)}{\partial \phi_i} \cdot \phi_i$. Denote by
$$
\mathcal{K}_2^\bullet = \left[ \mathcal{L}_V^{\oplus(n+1)} \xrightarrow{d\kappa_1} \mathcal{L}_V^{\oplus m} \right]|_{\mathcal{C}_N}.
$$

Then
$$
H^1(\mathbb{E}_{V/\mathcal{D}}) \cong R^1\pi_N^*(\mathcal{K}_2^\bullet) \oplus R^1\pi_N^*(\mathcal{L}_V^{\oplus m}|_{\mathcal{C}_N}) \oplus R^1\pi_N^*(\mathcal{L}_V^{\oplus m} \otimes \omega_{\mathcal{C}_V/V}|_{\mathcal{C}_N}).
$$

4.3. Family cosection of $\mathcal{O}b_{\mathcal{V}/\mathcal{D}}$. Let $h$ be a bi-linear morphism of bundles

$$
h : \text{Vb}(\mathcal{L}_D^{\oplus(n+1)} \oplus \mathcal{L}_D^{\oplus m} \oplus \mathcal{R}_D) \xrightarrow{pr_1,pr_2} \text{Vb}(\mathcal{L}_D^{\oplus m}) \times_{\mathcal{D}} \text{Vb}(\mathcal{R}_D) \to \text{Vb}(\omega_{\mathcal{C}_D/\mathcal{D}}).
$$

where $pr_i$ is the $i$-th projection, and the second arrow is induced by $\mathcal{L}_D^{\oplus m} \otimes \mathcal{R}_D \to \omega_{\mathcal{C}_D/\mathcal{D}}$. Using that the family $\mathcal{X} \to \mathcal{C}_D$ in (4.2) is a subfamily

$$
\mathcal{X} \subset \text{Vb}(\mathcal{L}_D^{\oplus(n+1)} \oplus \mathcal{L}_D^{\oplus m} \oplus \mathcal{R}_D),
$$

composing with $h$, we obtain a $\mathcal{C}_D$-morphism $\mathcal{X} \to \text{Vb}(\omega_{\mathcal{C}_D/\mathcal{D}})$.

**Lemma 4.3.** The homomorphism $\mathcal{X} \to \text{Vb}(\omega_{\mathcal{C}_D/\mathcal{D}})$ induces a homomorphism

$$
\sigma^* : \mathbb{E}_{V/\mathcal{D}} \to R^\bullet \pi_V^* \omega_{\mathcal{C}_V/V}
$$

whose restriction to $V \times A_l \ c \cong \mathcal{P}$, $c \neq 0$, is proportional (by an element in $\mathbb{C}^*$) to $\sigma_1^*$ in (3.2).

**Proof.** The proof is exactly as in Section 3.2. We omit it here. \[\Box\]

We denote

$$
\sigma = H^1(\sigma^*) : \mathcal{O}b_{\mathcal{V}/\mathcal{D}} := H^1(\mathbb{E}_{V/\mathcal{D}}) \to R^1\pi_V^* \omega_{\mathcal{C}_V/V} \cong \mathcal{O}_V.
$$

For $c \neq 0$, the restriction of $\sigma$ over $V \times A_l \ c \cong \mathcal{P}$ with $c \neq 0$ equals to $\sigma_1$. For $c = 0$ the restriction $\sigma_0 := \sigma|_{c=0}$ admits the following simple expression.

Every closed point $\xi \in \mathcal{N}$ can be represented by

$$
((\phi_i), b, p, 0) \in H^0(L^{\oplus(n+1)}) \times H^0(L^{\oplus m}) \times H^0(L^{\oplus m} \otimes \omega_C) \times H^0(C, \mathcal{O}_C)
$$

where $(C, p_1, \ldots, p_\ell, L) \in \mathcal{D}$ denotes the point under $\xi$. The restriction

$$
\sigma_0|_{\xi} : \mathcal{O}b_{\mathcal{N}/\mathcal{D}}|_{\xi} \to \mathbb{C}
$$

is the composite of

$$
\mathcal{O}b_{\mathcal{N}/\mathcal{D}}|_{\xi} \xrightarrow{\mathcal{S}_C} H^1(L^{\oplus(n+1)}) \oplus H^1(L^{\oplus m}) \oplus H^1(L^{\oplus m} \otimes \omega_C)
$$

with the pairing

$$
H^1(L^{\oplus(n+1)}) \oplus H^1(L^{\oplus m}) \oplus H^1(L^{\oplus m} \otimes \omega_C) \to H^1(\omega_C)
$$

defined via $((\phi_i), b, \tilde{p}) \to \tilde{b} \cdot p + b \cdot \tilde{p}$. 

Thus \( \sigma_0|_\xi = 0 \) if and only if \( p = 0 \) and \( b = 0 \). The loci \( \text{Deg}(\sigma_0) \) is \( N \cap (p = b = 0) = \overline{M}_{g,\ell}(X, d) \). Since \( \sigma|_{c \not= 0} \cong \sigma_1 \) we conclude

\[
\text{Deg}(\sigma) = \overline{M}_{g,\ell}(X, d) \times \mathbb{A}^1 \subset \mathcal{V}.
\]

Let \( \tilde{q} : \mathcal{V} \to \mathbb{D} \) be the projection. The natural \( \tilde{q}^*\mathbb{T}_{\mathbb{D}} \to \mathbb{T}_{\mathbb{V}/\mathbb{D}}[1] \) composed with \( \phi_{\mathbb{V}/\mathbb{D}} : \mathbb{T}_{\mathbb{V}/\mathbb{D}} \to \mathbb{E}_{\mathbb{V}/\mathbb{D}} \) gives \( \eta : \tilde{q}^*\mathbb{T}_{\mathbb{D}} \to \mathbb{E}_{\mathbb{V}/\mathbb{D}}[1] \). Set

\[
\mathcal{O}_{\mathcal{V}} := \text{coker}\{H^0(\eta) : \tilde{q}^*\mathbb{O}_{\mathbb{D}} \to H^1(\mathbb{E}_{\mathbb{V}/\mathbb{D}})\}.
\]

By same reasons as Lemma 3.3 and Corollary 3.4, we have the following.

**Lemma 4.4.** The composite vanishes

\[
\tilde{q}^*H^0(\mathbb{T}_{\mathbb{D}}) \xrightarrow{H^0(\eta)} H^1(\mathbb{E}_{\mathbb{V}/\mathbb{D}}) \xrightarrow{\sigma} R^1\pi_{\mathcal{V}}\omega_{\mathcal{V}/\mathcal{V}}.
\]

Thus the cosection \( \sigma : \mathcal{O}_{\mathcal{V}/\mathbb{D}} \to \mathcal{O}_{\mathcal{V}} \) lifts to a cosection \( \tilde{\sigma} : \mathcal{O}_{\mathcal{V}} \to \mathcal{O}_{\mathcal{V}} \).

**4.4. The constancy of the virtual cycles.** One verifies directly the virtual dimension of \( \mathcal{V} \) is \( \delta + 1 \). Using Lemma 4.4, following the convention introduced in Subsection 3.3, we denote by

\[
h^1/h^0(\mathbb{E}_{\mathbb{V}/\mathbb{D}})(\sigma) \subset h^1/h^0(\mathbb{E}_{\mathbb{V}/\mathbb{D}})
\]

the kernel of a cone-stack morphism \( h^1/h^0(\mathbb{E}_{\mathbb{V}/\mathbb{D}}) \to \mathbb{C}_\mathcal{V} \) induced by \( \tilde{\sigma} \) defined as in (3.5).\(^3\) By [KL] we have the localized Gysin map

\[
0^1_{\tilde{\sigma}, \text{loc}} : A_*(h^1/h^0(\mathbb{E}_{\mathbb{V}/\mathbb{D}})(\sigma)) \to A_*(\overline{M}_{g,\ell}(X, d) \times \mathbb{A}^1).
\]

**Definition 4.5.** We define the localized virtual cycle of \( (\mathcal{V}, \tilde{\sigma}) \) be

\[
[\mathcal{V}]^\text{vir} := 0^1_{\tilde{\sigma}, \text{loc}}([\mathbb{C}_\mathcal{V}]) \in A_{\delta + 1}(\overline{M}_{g,\ell}(X, d) \times \mathbb{A}^1).
\]

Let \( j_c : c \to \mathbb{A}^1 \) be the inclusion, then we have the following theorem.

**Theorem 4.6.** \( j_c^[\mathcal{V}]^\text{vir} = [\mathcal{V}_c]^\text{vir} \).

**Proof.** Let \( \mathcal{V}/\mathbb{D} \) be the composition of \( \mathcal{V} \to \mathbb{D} \) with the projection \( \mathbb{D} = \mathbb{D} \times \mathbb{A}^1 \to \mathbb{D} \). Let \( j \) be the composition of \( \tilde{q}^*\mathbb{T}_{\mathbb{D}/\mathbb{D}}[-1] \to \mathbb{T}_{\mathbb{V}/\mathbb{D}} \) with \( \phi_{\mathbb{V}/\mathbb{D}} \). We have a diagram of distinguished triangles

\[
\begin{array}{ccc}
\tilde{q}^*\mathbb{T}_{\mathbb{D}/\mathbb{D}}[-1] & \xrightarrow{j} & \mathbb{E}_{\mathbb{V}/\mathbb{D}} \xrightarrow{\phi_{\mathbb{V}/\mathbb{D}}} \mathbb{E}_{\mathbb{V}/\mathbb{D}} \\
\| & & \uparrow \phi_{\mathbb{V}/\mathbb{D}} \downarrow \\
\tilde{q}^*\mathbb{T}_{\mathbb{D}/\mathbb{D}}[-1] & \longrightarrow & \mathbb{T}_{\mathbb{V}/\mathbb{D}} \longrightarrow \mathbb{T}_{\mathbb{V}/\mathbb{D}}
\end{array}
\]

\(^3\)It is \( h^1/h^0(\mathbb{E}_{\mathbb{V}/\mathbb{D}}) \) along the degeneracy loci and is the kernel of \( h^1/h^0(\mathbb{E}_{\mathbb{V}/\mathbb{D}}) \to \mathcal{O}_{\mathcal{V}} \) induced by \( \sigma \) away from the degeneracy loci.
where $E_{V/D}$ is the mapping cone of $f$, and $\phi_{V/D}$ is given by mapping cone axiom. From diagram 4.4 we have the following commutative diagram

$$
\begin{array}{c}
0 \rightarrow h^0(E_{V/D}) \rightarrow h^0(E_{V/D}) \\
\uparrow \phi_{V/D}^0 \uparrow \phi_{V/D}^0 \\
0 \rightarrow h^0(T_{V/D}) \rightarrow h^0(T_{V/D}) \\
\uparrow \| \uparrow \phi_{V/D} \\
h^1(\tilde{q}^*T_{D/\mathcal{D}}[-1]) \rightarrow h^1(E_{V/D}) \rightarrow h^1(E_{V/D}) \rightarrow 0
\end{array}
$$

By chasing diagrams we know $\phi_{V/D}$ is a relative perfect obstruction theory. As in the proof of [KKP, Prop. 3], we have a short exact sequence of cone stack

$$
h^1/h^0(\tilde{q}^*T_{D/\mathcal{D}}[-1]) \rightarrow \mathcal{C}_{V/D} \rightarrow \mathcal{C}_{V/D},
$$

as well as the similar exact sequence relating $h^1/h^0(E_{V/D})$ with $h^1/h^0(E_{V/D})$. Note $\mathcal{C}_{V/D}$ is the pullback of $\mathcal{C}_{V/\mathcal{D}}$ along the projection $h^1/h^0(E_{V/D}) \rightarrow h^1/h^0(E_{V/D})$. Let $q_1 : \mathcal{D} \rightarrow \mathcal{D}$ and $q_2 : \mathcal{D} \rightarrow \mathcal{A}^1$ be the projections, then $L_{\mathcal{D}} = q_1^*L_{\mathcal{D}} \oplus q_2^*\Omega_{\mathcal{A}^1}$. Therefore $\tilde{q}^*T_{D/\mathcal{D}}[-1] = \tilde{q}^*q_2^*\Omega_{\mathcal{A}^1}$, then by Lemma 4.4, $\sigma$ descended to a cosection $\sigma_{V/D}$ for $Ob_{V/\mathcal{D}} := H^1(E_{V/D})$. Therefore the cosection localized virtual cycle of such $V/\mathcal{D}$ coincide with that defined by $0^1_{\sigma, loc}([C_{V/D}])$.

Restrict (4.4) on $V_c$ we obtain a diagram

$$
\begin{array}{c}
\tilde{q}^*T_{D/\mathcal{D}}[-1]|_{V_c} \rightarrow E_{V/\mathcal{D}}|_{V_c} = E_{V_c/\mathcal{D}} \xrightarrow{g} E_{V/D}|_{V_c} \\
\uparrow \alpha \uparrow \phi_{V/D}|_{V_c} \\
T_{V_c/D} \rightarrow T_{V_c/\mathcal{D}} \rightarrow T_{V_c/D}
\end{array}
$$

where all arrows in the second and the third rows are the natural morphisms induced by the fiber product of $D$-stacks $V_c = \mathcal{D} \times_\mathcal{D} V$ (here $\mathcal{D} \rightarrow \mathcal{D}$ is the product of $c \rightarrow \mathcal{A}^1$ with $\mathcal{D} \rightarrow \mathcal{A}^1$). The morphism $\alpha$ is constructed by mapping cone axiom applied to the second and the third rows of (4.5). The first and the third row of (4.5) then give the compatibility between the deformation theories of $V/\mathcal{D}$ and $V_c/\mathcal{D}$ required in [KL, (5.1)]. Apply [KL, Thm. 5.2], we have $j^1_{\sigma}[\mathcal{V}]^\text{vir} = [\mathcal{V}]^\text{vir}$. □

**Corollary 4.7.** Under the shriek operation of cycles, for $c \neq 0$,

$$
j^1_{\sigma}[\mathcal{V}]^\text{vir} = [\mathcal{P}]^\text{vir} \in A_0\overline{M}_{g,c}(X, d), \quad j^0([\mathcal{V}]^\text{vir}) = [\mathcal{N}]^\text{vir} \in A_0\overline{M}_{g,c}(X, d).
$$

Here $[\mathcal{N}]^\text{vir}$ is the localized virtual cycle using the obstruction theory of $\mathcal{N}$ induced by the restriction of $\phi_{V/D}$ (Prop 4.2) and the cosection $\sigma_0 = \sigma|_{\mathcal{N}}$. 

\[\text{dia5}\]

\[\text{shriek}\]
5. The virtual cycles of $\mathcal{N}$ and $\overline{M}_{g,\ell}(X,d)$

In the special case of [CL, Thm. 5.7] there are no markings and $X$ is the quintic CY hypersurface of $\mathbb{P}^4$. These imply that the virtual dimension $\delta = 0$, and [CL, Thm. 5.7] proved the numerical identity $\deg[\mathcal{N}]^\text{vir} = \deg[\overline{M}_{g}(X,d)]^\text{vir}$, where the proof uses the vanishing of virtual dimension essentially.

In this section we drop all the conditions and prove a more general property of virtual cycles (Proposition 5.5) to compare $[\mathcal{N}]^\text{vir}$ with $[\overline{M}_{g,\ell}(X,d)]^\text{vir}$.

5.1. The virtual cycle of $\mathcal{N}$. By the construction of section 4, we have an evaluation morphism

$$\epsilon_\mathcal{N} : C_\mathcal{N} \longrightarrow X_0 = X \times_{A^1} 0.$$ 

and a perfect relative obstruction theory

$$\phi_{\mathcal{N}/D} : T_{\mathcal{N}/D} \longrightarrow E_{\mathcal{N}/D} = \iota_0^*E_\mathcal{V} = R^}\pi_\mathcal{N}T_{X_0/C_D}$$

which is the restriction of $\phi_\mathcal{V/D\times A^1}$ to the fiber over $0 \in A^1$.

\textbf{Proposition 5.1.} The cosection $\sigma_0$ (c.f. (4.3)) lifts to a cosection $\bar{\sigma}_0 : \mathcal{O}_\mathcal{N} \rightarrow \mathcal{O}_\mathcal{N}$. The degeneracy loci of $\bar{\sigma}_0$ is $\overline{M}_{g,\ell}(X,d) \subset \mathcal{N}$, and is proper.

\textit{Proof.} This follows from the description of $\text{Deg}(\sigma)$ and Lemma 4.4. \hfill \Box

As $[\mathcal{V}]^\text{vir}$ has dimension $\delta + 1$, by Corollary 4.7, the dimension of $[\mathcal{N}]^\text{vir}$ is $\delta$.

\textbf{Definition-Proposition 5.2.} We define the localized virtual cycle of $\mathcal{N}$ to be

$$[\mathcal{N}]^\text{vir} := 0_{0,\text{loc}}^!([C_\mathcal{N}/D]) \in A_\delta \overline{M}_{g,\ell}(X,d);$$

5.2. Virtual cycles of $\mathcal{N}$ and $\overline{M}_{g,\ell}(X,d)$. In the fiber diagram

$$\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\gamma} & B := C(\pi_D^*(\mathcal{L}_D^{\otimes m} + \mathcal{R}_D)) \\
\downarrow \gamma & & \downarrow \\
Q := \overline{M}_{g,\ell}(X,d) & \longrightarrow & \mathcal{D},
\end{array}$$

the morphism $\gamma$ pulls back the relative perfect obstruction theory

$$\phi_{\mathcal{N}/Q} : T_{\mathcal{N}/Q} \longrightarrow E_{\mathcal{N}/Q} := \gamma^*E_{B/D}. $$

The evaluation maps of $\mathcal{N}$ and $Q$ fit in (for $Z_X$ c.f. (2.1))

$$\begin{array}{ccc}
C_\mathcal{N} & \xrightarrow{e_{\mathcal{N}}} & \mathcal{X}_0 \longrightarrow \text{Vb}(\mathcal{L}_D^{\otimes m}) \times_{C_D} \text{Vb}(\mathcal{R}_D) \\
\downarrow \nu_C & & \downarrow \\
C_Q & \xrightarrow{e_Q} & Z_X \longrightarrow C_D
\end{array}$$

where the right square is a fiber product of flat morphisms, and $\nu_C$ is induced by the vertical arrow $v$ in diagram (5.1).
From the above diagram we have a morphism between exact triangles
\[
\begin{array}{cccc}
e^*_N T_{X_0} / \mathbb{Z}_X & \longrightarrow & e^*_N T_{X_0} / \mathcal{C}_D & \longrightarrow & v^*_C e^*_N T_{X_0} / \mathcal{C}_D \\
\uparrow & & \uparrow & & \uparrow \\
T_{\mathcal{C}_N / \mathcal{Q}} & \longrightarrow & T_{\mathcal{C}_N / \mathcal{C}_D} & \longrightarrow & v^*_C T_{\mathcal{C}_N / \mathcal{C}_D} \\
\end{array}
\]
by the projection formula we have \( E_{\mathcal{N} / \mathcal{Q}} \cong R^* \pi_N^* e^*_N T_{X_0} / \mathbb{Z}_X \), and
\[
\begin{array}{cccc}
E_{\mathcal{N} / \mathcal{Q}} & \longrightarrow & E_{\mathcal{N} / \mathcal{D}} & \longrightarrow & h^* E_{\mathcal{D} / \mathcal{D}} \\
\uparrow \phi_{\mathcal{N} / \mathcal{Q}} & & \uparrow \phi_{\mathcal{N} / \mathcal{D}} & & \uparrow v^* \phi_{\mathcal{D} / \mathcal{D}} \\
T_{\mathcal{N} / \mathcal{Q}} & \longrightarrow & T_{\mathcal{N} / \mathcal{D}} & \longrightarrow & v^* T_{\mathcal{D} / \mathcal{D}} \\
\end{array}
\].
Thus \( \phi_{\mathcal{N} / \mathcal{Q}} : T_{\mathcal{N} / \mathcal{Q}} \longrightarrow E_{\mathcal{N} / \mathcal{Q}} \) is a perfect obstruction theory. Composing the cosection \( \sigma_0 : \mathcal{O} b_{\mathcal{N} / \mathcal{D}} \rightarrow \mathcal{O}_N \) with \( H^1(E_{\mathcal{N} / \mathcal{Q}}) \rightarrow H^1(E_{\mathcal{N} / \mathcal{D}}) \), we obtain
\[
\tilde{\sigma}_0 : \mathcal{O} b_{\mathcal{N} / \mathcal{Q}} := H^1(E_{\mathcal{N} / \mathcal{Q}}) \longrightarrow \mathcal{O}_N.
\]
Using the argument parallel to Proposition 5.1, one sees that the degeneracy loci of \( \tilde{\sigma}_0 \) equals \( \mathcal{Q} \subset \mathcal{N} \).

We are in a situation that fits the setup of [CLK, Sect. 2.2], and apply [CLK, Def. 2.8] below. Let \( \mathcal{C}_{\mathcal{N} / \mathcal{Q}} \) be the intrinsic normal cone of \( \mathcal{N} \) relative to \( \mathcal{Q} \). Then \( \mathcal{C}_{\mathcal{N} / \mathcal{Q}} \) is a closed substack of \( h^1/h^0(E_{\mathcal{N} / \mathcal{Q}}) \) by (5.2). We denote
\[
\Omega := h^1/h^0(E_{\mathcal{N} / \mathcal{Q}}) \times_{\mathcal{N}} \mathcal{Q} \cup \ker{\tilde{\sigma}_0 : h^1/h^0(E_{\mathcal{N} / \mathcal{Q}}) \longrightarrow \mathcal{C}_{\mathcal{N}}}.
\]

The virtual pullback morphism of cosection localized classes
\[
v^!_{\text{loc}} : A_* \mathcal{Q} \longrightarrow A_* \mathcal{Q}
\]
defined as the composite of
\[
A_* \mathcal{Q} \longrightarrow A_* \mathcal{C}_{\mathcal{N} / \mathcal{Q}} \longrightarrow A_* \Omega \quad \longrightarrow \quad A_* \mathcal{Q},
\]
where the first arrow sends \( \sum n_i[V_i] \) to \( \sum n_i[\mathcal{C}_{V_i \times_{\mathcal{Q} \times \mathcal{N}} \mathcal{Q} / \mathcal{V}_i}] \), the second arrow is induced by the inclusion \( \mathcal{C}_{\mathcal{N} / \mathcal{Q}} \subset \Omega \), and the third is the localized Gysin map defined in [KL] via \( \tilde{\sigma}_0 \). By [CLK, Thm. 2.9] we have

**Lemma 5.3.**
\[
v^!_{\text{loc}} ([\mathcal{Q}]^{\text{vir}}) = [\mathcal{V}]^{\text{vir}} \in A_* \mathcal{Q}.
\]

**5.3. A general comparison.** To simplify \( v^!_{\text{loc}} ([\mathcal{Q}]^{\text{vir}}) \), we establish a general comparison, from which the Theorem 1.1 easily follows.

Let \( D \) be a smooth Artin stack over \( \mathcal{C} \), \( \pi : \mathcal{C} \rightarrow D \) be a flat family of connected, nodal, genus \( g \) curves\(^4\). Let \( \mathcal{E} \) be a locally free sheaf on \( \mathcal{C} \) together with an isomorphism
\[
\tau : \mathcal{E} \overset{\cong}{\longrightarrow} \mathcal{E}^\vee \otimes \omega_{\mathcal{C} / D}
\]

\(^4\)it is allowed to be a family of twisted curves (c.f. [AV, Def. 4.1.2])
where \( \omega_{C/D} \) is the dualizing sheaf. Let
\[
B := C(\pi_*\mathcal{E})
\]
be the direct image cone stack and \( \rho : B \to D \) the natural morphism. Let \((C_B, \pi_B)\) be the universal family. Then by Proposition 2.8 there is a relative perfect obstruction theory
\[
\phi_{B/D} : \mathbb{T}_{B/D} \to \mathcal{E}_{B/D} := \rho^*R^\bullet\pi_*(\mathcal{E}).
\]
Composing \( \tau \) with the pairing \( \text{Vb}(\mathcal{E}) \times \mathcal{C} \text{Vb}(\mathcal{E}^\vee \otimes \omega_{C/D}) \to \text{Vb}(\omega_{C/D}) \) induces canonically a representable morphism of stacks
\[
\tilde{h} : \text{Vb}(\mathcal{E}) \to \text{Vb}(\omega_{C/D}), \quad \tilde{h}(x) = \frac{1}{2}(x, \tau(x)),
\]
for \( x \in \text{Vb}(\mathcal{E}) \). The \( \tilde{h} \) also induces a morphism between tangent complexes
\[
d\tilde{h} : T_{\text{Vb}(\mathcal{E})/\mathcal{C}} \to \tilde{h}^*T_{\text{Vb}(\omega_{C/D})/\mathcal{C}}.
\]
Pull back to \( \mathcal{C}_B \) via evaluation map \( \epsilon_B : \mathcal{C}_B \to \text{Vb}(\mathcal{E}) \) and apply \( R^\bullet\pi_B \). We have
\[
\sigma_B^* : \mathcal{E}_{B/D} \to R^\bullet\pi_B (\epsilon_B^*\tilde{h}^*\Omega^\vee_{\text{Vb}(\omega_{C/D})/\mathcal{C}}) \cong R^\bullet\pi_B (\omega_{C_B/B}).
\]
It induces the following morphism:
\[
\sigma_B : \mathcal{O}_{B/D} = \rho^*R^1\pi_*(\mathcal{E}) \to \mathcal{O}_B.
\]
A coordinate expression of \( \sigma_B \) is as follows. For each scheme \( T \to D \), let \( \mathcal{C}_T = \mathcal{C} \times_D T \to T \) and \( \mathcal{E}_T \) over \( \mathcal{C}_T \) be the family along \( T \) via pullback. For every
\[
u \in H^0(\mathcal{C}_T, \mathcal{E}_T), \quad \tau(u) \in H^0(\mathcal{C}_T, \mathcal{E}_T^\vee \otimes \omega_{C_T/T}),
\]
and the Serre pairing
\[
H^0(\mathcal{C}_T, \mathcal{E}_T^\vee \otimes \omega_{C_T/T}) \times H^1(\mathcal{C}_T, \mathcal{E}_T) \to H^1(\mathcal{C}_T, \omega_{C_T/T}).
\]
For arbitrary \( \bar{u} \in H^1(\mathcal{C}_T, \mathcal{E}_T) \), one has \( \tau(\bar{u}) \in H^1(\mathcal{C}_T, \mathcal{E}_T^\vee \otimes \omega_{C_T/T}) \), and
\[
\sigma_B(u, \bar{u}) = \frac{(\tau(u), \bar{u}) + (\tau(\bar{u}), u)}{2} = (\tau(u), \bar{u}) \in H^1(\mathcal{C}_T, \omega_{C_T/T}) \cong \Gamma(T, \mathcal{O}_T),
\]
where we used \( (\tau(u), \bar{u}) = (\tau(\bar{u}), u) \) by direct checks from definition. By Serre duality and (5.4), the (reduced part of) degeneracy loci of \( \sigma_B \) equals that of \( D \). The distinguished triangle \( \rho^*\mathbb{L}_D \to \mathbb{L}_B \to \mathbb{L}_{B/D} \to \rho^*\mathbb{L}_D[1] \) gives a morphism \( \rho^*\mathbb{T}_D \to \mathbb{T}_{B/D}[1] \), which composed with \( \phi_{B/D} : \mathbb{T}_{B/D} \to \mathcal{E}_{B/D} \) gives
\[
\eta : \rho^*\mathbb{T}_D \to \mathcal{E}_{B/D}[1].
\]
Taking the cokernel of the \( H^0(\eta) \) we obtain the absolute obstruction sheaf
\[
\mathcal{O}_{B/D} := \text{coker}\{H^0(\eta) : \rho^*\mathbb{O}^\vee_D \to H^1(\mathcal{E}_{B/D})\}.
\]

**Corollary 5.4.** The cosection \( \sigma_B : \mathcal{O}_{B/D} \to \mathcal{O}_B \) lifts to \( \bar{\sigma}_B : \mathcal{O}_{B/D} \to \mathcal{O}_B \).

**Proof.** The proof is parallel to Corollary 3.4, we omit it here. \( \square \)
Denote $E := h^1/h^0(\mathcal{E}_{B/D})$. Suppose we have a representable morphism $R \to D$ from a DM stack $R$. Over the fiber product

\[
\begin{array}{ccc}
N & \xrightarrow{b} & B \\
\downarrow \rho_R & & \downarrow \rho \\
R & \xrightarrow{\iota} & D
\end{array}
\]

let $E_R := b^*E$ and let $\tilde{\sigma} : E_R \to \mathcal{O}_N$ be the pullback of $\sigma_B$. We define a cosection localized virtual pullback $\rho_{R,loc}^! : A_*(R) \to A_*(R)$ as the composition of

\[
\rho_{R,loc}^! : A_*(R) \xrightarrow{\Upsilon} A_*(C_{N/R}) \xrightarrow{\iota^*} A_*G \xrightarrow{0_{E_R,loc}^!} A_*R,
\]

where $\Upsilon$ sends a cycle class $\sum n_i[V_i]$ to $\sum n_i[C_{V_i \times_R N/V_i}]$, $G := N \times_B h^1/h^0(\mathcal{E}_{B/D})(\sigma_B)$, and $0_{E_R,loc}^!$ is the localized Gysin map defined in [KL] with respect to the pullback of $\sigma_B$.

**Proposition 5.5.** For each integral closed substack $Z$ of $R$, we have

\[
\rho_{R,loc}^!(\lbrack Z \rbrack) = (-1)^\mu \lbrack Z \rbrack,
\]

where $\mu = \min_{z \in Z}\{h^0(C_z, \mathcal{E}|_{C_z})\}$ where for $z$ runs as closed points of $Z$, $C_z = z \times_D \mathcal{C}$, and $\mathcal{E}|_{C_z}$ denotes the pullback via $C_z \to \mathcal{C}$.

**Proof.** For arbitrary étale $j : U \to Z$ we take fiber products

\[
\begin{array}{ccc}
V & \xrightarrow{j_U} & W \xrightarrow{j_Z} & N \\
j & & \downarrow \rho_R & \\
U & \xrightarrow{j} & Z \xrightarrow{\iota} & R
\end{array}
\]

Let $E_Z := j_Z^*E_R \xrightarrow{\tilde{\sigma}_Z} \mathcal{O}_W$ and $\mathfrak{g}_U := j_U^*j_Z^*E_R \xrightarrow{\tilde{\sigma}_Z} \mathcal{O}_V$ be the pullback of $E_R \xrightarrow{\tilde{\sigma}} \mathcal{O}_N$. Denote $G_Z := G \times_N W$ and $G_U := G \times_N V$. Then analogous to (5.3), we have a commutative diagram

\[
\begin{array}{ccc}
\rho_{Z,loc}^! : A_*(Z) & \xrightarrow{\iota^*} & A_*(C_{W/Z}) \xrightarrow{\iota^*} A_*(G_Z) \xrightarrow{0_{E_Z,loc}^!} A_*(Z) \\
\downarrow j^* & & \downarrow & & \downarrow j^* \\
\rho_{U,loc}^! : A_*(U) & \xrightarrow{\iota^*} & A_*(C_{V/U}) \xrightarrow{\iota^*} A_*(G_U) \xrightarrow{0_{E_U,loc}^!} A_*(U)
\end{array}
\]

The first square above commutes trivially. The second square commutes by [Ful, Prop. 1.7]. The third commutes because in [KL, (2.1)] the steps (i) intersection with divisors and (ii) pushforward via cosection-regularizing map are both commutative with the pullback by étale morphisms. Therefore

\[j^*\rho_{Z,loc}^!(\lbrack Z \rbrack) = \rho_{U,loc}^!(\lbrack U \rbrack).\]
In Proposition 5.5 let us pick $U$ as follows. The morphism $Z \to D$ induces $\pi_Z: C := C \times_D Z \to Z$ and let $E_Z$ be the pull-back of $\mathcal{E}$. Then $E_Z$ is a locally free sheaf on $C$ together with, by (5.4),
$$\tau_Z : E_Z \xrightarrow{\cong} E_Z^! \otimes \omega_{C/Z}.$$ The function $\psi: Z \to \mathbb{Z}$ sending $x \in Z$ to
$$\psi(x) = \dim H^1(C_x, E_Z|C_x)$$
is upper semi-continuous. Thus we can pick an étale chart $U \to Z$ such that $\psi|_U$ is constant and is the smallest number in $\psi(Z)$. This implies the $E_U := R^*\pi_Z^*E_Z$ has cohomologies $H^0(E_U), H^1(E_U)$ locally free. By shrinking $U$ smaller we may assume $U$ is a smooth variety and
$$E_U \cong [H^0(E_U) \xrightarrow{0} H^1(E_U)].$$
Denote $j : U \to Z$ to be the inclusion.

We see that in (5.7) $V$ is the total space of the locally free sheaf $(\pi_Z, E_Z)|_U = H^0(E_U)$. By $\tau_Z$ and Serre duality
$$H^1(E_U) \cong (R^1\pi_Z, E_Z)|_U \cong (\pi_Z, E_Z)|_U \cong V^!$$implies $E_U \cong [V \xrightarrow{0} V^!]$, and also
$$\mathfrak{N}_U = j^*_u j^*_x E_R \cong f^*h^1/h^0(E_U) = [f^*V^!/(f^*V)]$$
where the quotient is by zero action. Applying the composition (5.3) step by step, that $Y(U) = [V/f^*V]$ implies
$$\rho^!_{R, \text{loc}}(U) = 0^!_{\mathfrak{N}_U, \text{loc}}([V/f^*V]) = 0^!_{f^*V^!, \sigma_U}([V]),$$
(c.f. the proof in [CL1, Prop. 5.6]), where $\sigma_U : H^1(E_U) = f^*V^! \to \mathcal{O}_U$ is the pullback of $\sigma_B$. By (5.5) $\sigma_U$ is exactly the pairing with the tautological section $x \in \Gamma(V, f^*V)$. To calculate the cosection localization $0^!_{f^*V^!, \sigma_U}([V])$, we follow the definition [KL, Def. 2.3]. Briefly, $F = f^*V$ and $r = \text{rank } F$, then $\sigma_U$ is a cosection $\sigma_U : F^! \to \mathcal{O}_V$ induced by $\mathcal{O}_V \xrightarrow{\pi} F$. Let $R = Bl_U V$ and $\alpha : R \to V$ be the blow up of $V$ along the zero section $U$ of $V \to U$. Denote $E = \alpha^\perp(U)$ as the exceptional divisor. Then the zero loci of $\alpha^\perp x \in \Gamma(R, \alpha^\perp f^*V)$ is $E$. Denote $\alpha(\sigma) : E \cong \mathbb{P}(V) \to V$ as the restriction of $\alpha$. The exact sequence $F^! \xrightarrow{\sigma_U} \mathcal{O}_V \to \mathcal{O}_U \to 0$ pullbacks to the exact
$$\alpha^* F^! \xrightarrow{\alpha^* \sigma_U} \mathcal{O}_R \to \mathcal{O}_E \to 0.$$
Thus $\alpha^* \sigma_U$ factors through a surjective $\tilde{\sigma} : \tilde{F}^! := \alpha^* F^! \to \mathcal{O}_R(-E)$. Let $\tilde{G} = \ker \tilde{\sigma}$. Then
$$\tilde{G}^! |_E \cong \text{coker} \{ \mathcal{O}_R(\sigma)|_E \to \alpha(\sigma)^* F \} \cong T_{E/U} \oplus E \mathcal{O}(-1)$$
and by the construction in [KL, (2.1)]
$$0^!_{f^*V^!, \sigma_U}([V]) = \alpha(\sigma)_*([-E] \cap 0^!_{\tilde{G}^! |_E}) = (-1)\alpha(\sigma)_* e(\tilde{G}^! |_E \cap [E])$$
$$= (-1)(-1)^{r-1}\alpha(\sigma)_* e(T_{E/U} \otimes E \mathcal{O}(-1)) = (-1)^r[U].$$
Therefore

\[(5.8) \quad \rho_{1,\text{loc}}(U) = (-1)^r[U].\]

As \(Z\) is integral by dimension reason \(\rho_1(Z) = c[Z]\) for some \(c \in \mathbb{Q}\), where (5.7) and (5.8) forces \(c = (-1)^r\). This proves Proposition 5.5. \(\square\)

5.4. We are ready to prove Theorem 1.1 from the theorem 5.5.

**Proof of Theorem 1.1.** We claim

\[\mathcal{P}^\text{vir} = \mathcal{N}^\text{vir} = (-1)^{md+1-g}[\mathcal{Q}]^\text{vir}.\]

The first equality is from Corollary 4.7. For the second identity, compare (5.1) with (5.6) by letting \(D = D, B = B, R = Q, E = \mathcal{L}_{\mathcal{C}}^\otimes \otimes \mathcal{R}_D\), and \(\tau : \mathcal{E} \to \omega_{\mathcal{C}/D} \otimes \mathcal{E}^\vee\) is determined by \(\mathcal{L}_{\mathcal{C}/D}^\otimes \omega_{\mathcal{C}/D} \otimes \mathcal{P}^{-1}_D, \mathcal{P}\mathcal{D}^\otimes \omega_{\mathcal{C}/D} \otimes \mathcal{L}_{-m}^{-1}\). Under such setup \(N = \mathcal{N}\) and \(\mathcal{E}_R \cong h^0(\mathcal{E}_N/Q)\).

The \(\sigma_B\) defined in (5.5) admits the following description. For each closed point \(\xi \in \mathcal{B}\) represented by

\[(u_1, u_2) \in H^0(C, L^\otimes m) \times H^0(C, L^\otimes m) \otimes \omega_C\]

where \((C, p_1, \ldots, p_r, L) \in \mathcal{D}\) is the point under \(\xi\), for arbitrary \(u_1 \in H^1(C, L^\otimes m)\) and \(u_2 \in H^1(C, L^\otimes m \otimes \omega_C)\), the cosection is \(\sigma_B|_\xi(u_1, u_2) = u_1 \cdot u_2 + u_2 \cdot u_1\). A family version of above implies \(\gamma^* \sigma_B\) coincides with the \(\sigma_0\) defined below (4.3) (c.f. Section 4.3). Denote \([\mathcal{Q}]^\text{vir} = \sum_i m_i[\mathcal{Q}_i]\), where \(\mathcal{Q}_i\) is an integral substack. By Lemma 5.3 and Theorem 5.5

\[\mathcal{N}^\text{vir} = v^i_\text{loc}([\mathcal{Q}]^\text{vir}) = \sum_i (-1)^{\mu_i} m_i[\mathcal{Q}_i] = (-1)^{md+1-g}[\mathcal{Q}]^\text{vir},\]

where \(\mu_i = h^0(C, \mathcal{E}|_C) = \chi(C, L^\otimes m) = md + 1 - g\). \(\square\)

6. **Appendix**

All the arguments in the paper can be extended to the complete intersection in \(\mathbb{P}^n\) cases.

Let \(X \subset \mathbb{P}^n\) be a smooth complete intersection defined by equations \(q_1, \ldots, q_r\) with \(\deg q_i = m_i\). Let \(M := \overline{\mathcal{M}}_{g, \ell}(\mathbb{P}^n, d)\) be the moduli of genus \(g\) degree \(d\) \(\epsilon\)-stable quasimaps to \(\mathbb{P}^n\). We denote by \((\mathcal{C}_M, \pi_M)\) be the universal family of \(\overline{\mathcal{M}}_{g, \ell}(\mathbb{P}^n, d)\), and \(\mathcal{L}_M\) be the universal line bundle. Denote

\[\mathcal{P}_M := \mathcal{L}_{\mathcal{C}_M}^\otimes \otimes \omega_{\mathcal{C}_M/M}, \quad i = 1, \ldots, r.\]

Using direct image cone construction, we form the DM stack

\[\overline{\mathcal{P}} := C(\oplus \pi_M* \mathcal{P}_M^i).\]

Letting \(\pi_{\overline{\mathcal{P}}} : \mathcal{C}_{\overline{\mathcal{P}}} \to \overline{\mathcal{P}}\) and \(\mathcal{L}_{\overline{\mathcal{P}}}\) be the universal curve and line bundle. The obstruct theory of \(\overline{\mathcal{P}}\) is
\[
\phi_{\mathcal{P}/\mathcal{D}} : T_{\mathcal{P}/\mathcal{D}} \rightarrow E_{\mathcal{P}/\mathcal{D}} := R^*\pi_{\mathcal{P}*}(L_{\mathcal{P}}^{(n+1)} \bigoplus \oplus_i \mathcal{P}_i^{\mathcal{P}}).
\]

We define a multi-linear bundle morphism from \( q_i \)

\[ h_1 : \text{Vb}(L_{\mathcal{P}}^{(n+1)} \bigoplus \oplus_i \mathcal{P}_i^{\mathcal{P}}) \rightarrow \text{Vb}(\omega_{\mathcal{D}/\mathcal{P}}), \quad h_1(x,p) = \sum_i p_i \cdot q_i(x), \]

It induces the following cosection:

\[ \sigma_m : \text{Ob}_{\mathcal{P}/\mathcal{D}} = R^1\pi_{\mathcal{P}*}(L_{\mathcal{P}}^{(n+1)} \bigoplus \oplus_i \mathcal{P}_i^{\mathcal{P}}) \rightarrow \text{Ob}_{\mathcal{P}}. \]

the degeneracy loci of \( \sigma_m \) is \( \overline{M}_{g,\ell}(X,d) \). The same argument as Theorem 1.1 implies

**Theorem 6.1.** For \( g \geq 0 \), \( \epsilon > 0 \) and \( \ell \) be a nonnegative integer,

\[ [\mathcal{P}]_{\text{vir}}^{\text{loc}} = (-1)^{(\sum_i m_i)d+1-g} [\overline{M}_{g,\ell}(X,d)]_{\text{vir}}^{\text{loc}} \in A_* \overline{M}_{g,\ell}(X,d). \]

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