Study on dispersion and wave velocity in 2D elliptic granular crystals by a micromechanics-based micromorphic model

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Abstract
This study considers two-dimensional elliptic granular crystals respectively with different aspect ratios. Using a micromechanics-based micromorphic model, macroscopic micromorphic constitutive modulus tensors of elliptic granular crystals are obtained. Two modes of Taylor expansion of relative displacements are used to establish contact relation between particles. This study obtains micromorphic transverse-rotational and longitudinal waves. Obvious dispersions of micromorphic waves are given in 2D elliptic granular crystals. With increase of aspect ratio, width of frequency band gap increases when wave propagates along $x_1$ direction but decreases when wave propagates along $x_2$ direction. Two modes of Taylor expansion of relative displacements only have influence on dispersions and frequency band gap of transverse-rotational waves. Velocities of longitudinal and transverse-rotational waves first decrease to zero, keep zero for a range of frequency and then increase to a larger wave velocity with increase of frequency for different aspect ratios. The situation with a larger aspect ratio leads to larger transverse-rotational and longitudinal velocities, and ratio between transverse-rotational and longitudinal velocities decreases from 0.7 to 0.59, when wave propagates along $x_1$ direction. However, it leads to smaller transverse-rotational and longitudinal velocities, and ratio between two velocities decreases from 0.88 to 0.56, when wave propagates along $x_2$ direction.

Keywords
Elliptic granular crystals, micromechanics-based micromorphic model, wave propagation, dispersion, frequency band gap, wave velocity

Introduction
Granular crystals are ordered closely packed or disordered packed systems of elastically interacting solid particles1,2 which are arranged in one-, two-, or three-dimensional lattices.3 Nowadays, many studies focus on the metamaterial design through dynamic response characteristics of granular crystals4–8 which can control wave propagation behaviors to obtain some mechanical properties such as energy absorption, heat absorption, noise reduction. It is necessary to deeply understand mechanisms of wave propagation and control in granular crystals. Considering the discrete nature of granular crystals and the complexity of contact behaviors between particles, the dispersion is an important
wave propagation behavior of granular crystals, which is verified by experiments. Due to dispersion characteristics, the wave velocity depends on the frequency (or the wave length) when waves propagating in granular crystals. And the dispersion has a close relevance to the microstructure in granular crystals. How to correctly describe the relationship between the dispersion and the microstructure is an important and difficult issue for granular crystals.

Discrete particle models or continuous medium models are both applied to simulate wave propagations and dispersions in granular crystals. Discrete particle models are closer to discrete natures of granular crystals, therefore, wave propagations can be directly described from the microscopic view using the energy transfer among particles. In this way, it is convenient to consider microstructural information and contact constitutive relationships on dispersions at the micro scale. However, there are seldom studies based on discrete particle models reporting the prediction of frequency band gaps resulting from dispersions. It is noted that frequency band gaps have great significance in designing metamaterials based on granular crystals. Besides, the wave is actually a conception under the continuum framework, which gives the difficulties in how to quantitatively describe wave propagations and how to assess effects of discrete information after homogenization on dispersions by discrete particle models. As for continuous medium models, the conception of wave is naturally fulfilled because it is built on continuum mechanics framework, which provides more accurate quantitative analysis of wave propagation in granular crystals. And numerical methods based on continuous medium models usually have higher computational efficiency than those based on discrete particle models for the computational scale of engineering problems. However, continuous medium models usually lack the microscopic information such as length scales associated with microstructures, which leads to the difficulty in describing relationships between microstructures and dispersion behaviors of granular crystals correctly.

Based on the above analysis, a continuous medium model containing enough microscopic information is an appropriate choice to simulate wave propagations in granular crystals. On the basis of this background, micromechanics-based continuous medium models have been proposed by Chang and Ma first to investigate microstructural effects and interactions on macroscopic mechanical behaviors. Therefore, micromechanics-based continuous medium models are appropriate to present macroscopic measures reflecting discrete nature and develop macroscopic constitutive relationships containing microscopic information for granular crystals. Early, the continuous medium in the micromechanics-based continuous medium models is usually the classical or Cosserat medium. For example, Bacigalupo and Gambarotta proposed an enhanced micropolar homogenization procedure for periodic granular materials to simulate dispersion behaviors. However, classical and Cosserat media lack enough macroscopic measures with respect to microscopic information in the homogenization process. Recently, the micromorphic continuum theory is introduced to develop micromechanics-based continuous medium models. For example, Biswas and Poh proposed a micromorphic computational homogenization framework for heterogeneous materials. Silva et al. presented a multiscale micromorphic model to simulate localization problems of quasi-brittle media. Comparing with classical and Cosserat continuum theories, the micromorphic continuum theory has the capacity of providing complete deformation modes of microstructures and more macroscopic measures relating to microscopic information. Subsequently, some micromechanics-based micromorphic models are developed and applied to describe wave propagations and dispersions in granular materials. Misra and Poorsalhjouy first proposed a micromorphic model based on a micromechanical approach and applied the model to analyze dispersions in elastic granular media and 1D granular crystal. Following this method, Xiu and Chu proposed a first-order micromechanics-based micromorphic model to predict dispersions and frequency band gaps of granular materials. Furthermore, Xiu et al. derived macroscopic micromorphic constitutive modulus tensors for granular crystals with different specific 3D microstructures, instead of granular materials based on a hypothesis of isotropic contact density distribution in above models. In this way, more microscopic information is considered by specific particle arrangements and sizes, void ratios, and coordination numbers. And dispersions predicted by the micromechanics-based micromorphic model are obtained for different granular crystals. But above studies still used the conventional circular or spherical particles which differ from real particle shapes to some extent. Actually, different particle shapes need to be investigated to match the real physical situation. Usually, effects of particle shapes on wave propagations or dispersions in granular crystals are studied by discrete particle models. To our knowledge, there seem no studies associated with particle shapes through the micromechanics-based continuous medium model to investigate wave propagations and dispersion behaviors in granular crystals. And recently, Zhou et al. proposed a micromechanics-based micropolar model considering the elliptic particle assembly to investigate some static problems of granular materials. But there still lack studies based on micromechanics-based micromorphic models considering particle shapes to investigate the static and dynamic problems of granular crystals or materials.
coordinate system, and \( U_i^m, \Omega_3^m, \) and \( R_j^m \) are those of the particle at the \( m \)th contact point. It is noted that \( U_i \) and \( \Omega_3 \) are respectively a sum of products of specified functions of the \( X \) and arbitrary functions of the \( x \) and \( t \): \( U_i = U_i(x, X, t), \Omega_3 = \Omega_3(x, X, t) \). A key to solve equation (1) is to establish the relations of displacement and rotation between particles \( m \) and \( n \), and usually, Taylor expansion is used. With an affine deformation assumption, in this study, two modes of Taylor expansions are used to establish the relations of displacement and rotation between particles \( m \) and \( n \):

(1) Mode A (Taylor expansion at the particle center)

\[
U_i^m = U_i^n + \frac{\partial U_i^n}{\partial X_j} L_j^c = U_i^n + \psi_{ij} L_j^c,
\]

\[
\Omega_3^m = \Omega_3^n + \frac{\partial \Omega_3^n}{\partial X_i} L_i^c = \Omega_3^n + \phi_{3i} L_i^c
\]

where \( L_j^c = X_j^m - X_j^n \) is a branch vector connecting the centroids of \( m \) and \( n \), and \( \psi_{ij}, \phi_{3i} \) are respectively gradients of microscopic displacement and rotation as followings show

\[
\eta_{ij} = \frac{\partial U_i^n}{\partial X_j^n} = \psi_{ij}(x, t), \quad \phi_{3i} = \frac{\partial \Omega_3^n}{\partial X_i^n} = \phi_{3i}(x, t)
\]

Therefore, \( \psi_{ij} \) and \( \phi_{3i} \) are functions of the \( x \) and \( t \) only, which are macroscopic variables.

(2) Mode B (Taylor expansion at the contact point)

\[
U_i^m = U_i^n + \frac{\partial U_i^n}{\partial X_j} \left( X_j^m - X_j^n \right) = U_i^n + \psi_{ij} \left( X_j^m - X_j^n \right),
\]

\[
U_i^n = U_i^n + \frac{\partial U_i^n}{\partial X_j} \left( X_j^m - X_j^n \right) = U_i^n + \psi_{ij} \left( X_j^m - X_j^n \right)
\]

\[
\Omega_3^m = \Omega_3^n + \frac{\partial \Omega_3^n}{\partial X_i} \left( X_i^m - X_i^n \right) = \Omega_3^n + \phi_{3i} \left( X_i^m - X_i^n \right),
\]

\[
\Omega_3^n = \Omega_3^n + \frac{\partial \Omega_3^n}{\partial X_i} \left( X_i^m - X_i^n \right) = \Omega_3^n + \phi_{3i} \left( X_i^m - X_i^n \right)
\]

where the superscript \( c \) denotes the contact point between particles \( m \) and \( n \). \( \psi_{ij} \) and \( \phi_{3i} \) are expressed by

\[
\psi_{ij} = \frac{\partial U_i^n}{\partial X_j^n} = \psi_{ij}(x, t), \quad \phi_{3i} = \frac{\partial \Omega_3^n}{\partial X_i^n} = \phi_{3i}(x, t)
\]

Then, in the micromorphic theory, the most important hypothesis, the decomposition of motion, needs to be introduced. In our previous studies, the micromorphic actual motion (displacement and rotation) of a

The micromechanics-based micromorphic model

Firstly, a globe coordinate system \( x \) and a local coordinate system \( X \) are given in Figure 1. The origin of \( X \) is at the barycenter of \( V \) and its axes are parallel to \( x \). The motion of every particle in \( V \) includes its independent motion and the motion of the material point \( P \), and it is also divided into translation and rotation. We use \( c \) to denote the contact point between a reference particle \( n \) and its contact particle \( m \), and the relative displacement and rotation at the contact point are obtained in the 2D form for simplifying derivations as

\[
\delta_i^c = U_i^m - U_i^n + \epsilon_{3hk} \left( \Omega_3^{nk} R_j^k - \Omega_3^c R_j^c \right), \quad \theta_3^c = \Omega_3^n - \Omega_3^m
\]

(1)

where \( i, j = 1, 2 \), the subscript \( 3 \) shows that the rotation can only occur around the \( x_3 \) axis. The upper-case \( U_i^n, \Omega_3^c, R_j^n \), respectively denote the microscopic displacement, rotation and radius of the particle \( n \) at \( X \)
The microscopic displacement is equal to the average displacement, while the macroscopic rotation is a sum of the rigid body rotation and the average rotation, that is, \( \gamma_i = \omega_i^{\text{macro}} = e_{i3} \dot{u}_{i,j} + \omega_3 \).

Then, we can respectively decompose \( \gamma_i \) and \( \omega_i \) into macro-gradients \( \dot{u}_{i,j} \) and \( \omega_3 \), and fluctuations' gradients \( \gamma_i \) and \( \alpha_3 \) of displacement and rotation which are given by

\[
\gamma_i = \psi_i - \dot{u}_{i,j}, \quad \alpha_3 = \varphi_3 - \omega_3.
\]

(7)

The macroscopic displacement is equal to the average displacement, while the macroscopic rotation is a sum of the rigid body rotation and the average rotation, that is, \( \gamma_i = \omega_i^{\text{macro}} = e_{i3} \dot{u}_{i,j} + \omega_3 \). And the strain, micro-curvature, relative strain, and relative micro-curvature measures are respectively expressed by

\[
e_{ij} = \frac{1}{2} (\dot{u}_{i,j} + \dot{u}_{j,i}), \quad \kappa_3 = \omega_3.
\]

(8)

Substituting equations (2)-(5) into equation (1), \( \delta_i \) and \( \theta_3 \) are respectively obtained by

(1) Mode A (Taylor expansion at the particle center)

\[
\delta_i^c = \psi_i \dot{L}_j^c + e_{i3} \left[ (\Omega_k^c + \varphi_{3k} \dot{L}_k^c) R_j^m - \Omega_j^c R_j^m \right]
\]

\[
= \dot{u}_{i,j}^c + \gamma_i \dot{L}_j^c + e_{i3} \varphi_{3k} \dot{L}_k^c + e_{i3} \Omega_k^c \left( R_j^m - R_j^m \right)
\]

\[
= \dot{u}_{i,j}^c + \gamma_i \dot{L}_j^c + e_{i3} \varphi_{3k} \dot{L}_k^c + e_{i3} \Omega_k^c \left( R_j^m - R_j^m \right)
\]

where \( N_k^c = R_j^m L_k^c \) is seen as a fabric tensor.

(2) Mode B (Taylor expansion at the contact point)

\[
\delta_i^c = \psi_i \left( X_j^m - X_j^m \right) + e_{i3} \left[ (\Omega_k^c + \varphi_{3k} \left( X_j^m - X_j^m \right)) R_j^m - (\Omega_j^c + \varphi_{3k} \left( X_j^m - X_j^m \right)) R_j^m \right]
\]

\[
= \dot{u}_{i,j}^c + \gamma_i \dot{L}_j^c + e_{i3} \varphi_{3k} \dot{L}_k^c + e_{i3} \Omega_k^c \left( R_j^m - R_j^m \right) + e_{i3} \Omega_k^c \left( R_j^m - R_j^m \right)
\]

(9)

where \( N_k^c = R_j^m L_k^c \) is seen as a fabric tensor.

\[
\delta_i^c = \psi_i \left( X_j^m - X_j^m \right) + e_{i3} \left[ (\Omega_k^c + \varphi_{3k} \left( X_j^m - X_j^m \right)) R_j^m - (\Omega_j^c + \varphi_{3k} \left( X_j^m - X_j^m \right)) R_j^m \right]
\]

(10)

where \( N_k^c = R_j^m L_k^c \) is seen as a fabric tensor.

Then, the relative displacement and rotation can be decomposed into the following equations:

\[
\delta_i^c = \dot{u}_{i,j}^c L_j^c, \quad \delta_i^c = \gamma_i \dot{L}_j^c, \quad \delta_i^c = e_{i3} \varphi_{3k} N_k^c \theta_3^c, \quad \theta_3^c = e_{i3} \alpha_3 L_i^c
\]

(13)

There is an important method in the micromechanics-based micromorphic model to establish the relationships between strain and stress measures through a hypothesis that the macroscopic deformation energy density \( W \) of a particle cell is a sum of the microscopic deformation energy density \( \psi^c \) in the particle cell, that is,

\[
W = \frac{1}{V} \sum_{i=1}^{n} \psi^c \left( \delta_i^c, \delta_i^c, \delta_i^c, \delta_i^c, \theta_3^c, \theta_3^c \right)
\]

(14)

The detailed discussion can be seen in our previous study. And 2D contact constitutive equations are used by

\[
f_i^c = K_i^c \delta_i^c, \quad K_i^c = K_n n_i n_j + K_t t_i t_j
\]

(15)

\[
m_i^c = G \theta_3^c
\]

(16)

where \( f_i^c \) and \( m_i^c \) are the contact forces and moments, and \( K_n \) and \( K_t \) are the normal and tangential contact stiffness parameters for contact forces, and \( G \) is the contact stiffness parameter for contact moments. \( n_i \) and \( t_i \) are the unit contact normal and contact tangential vectors at the contact point, which are expressed by

\[
n = \frac{\partial X}{\sqrt{(\partial X)^2 + (\partial Y)^2}} \quad t = \frac{\partial Y}{\sqrt{(\partial X)^2 + (\partial Y)^2}}
\]

(17)

\[
\psi^c = f_i^c n_i \quad \psi^c = m_i^c \theta_3^c
\]

(18)

In which \( X \) and \( Y \) are the local coordinates of the contact point, and \( a \) and \( b \) are half lengths of axes in the elliptic particle along local \( X \) and \( Y \) coordinate axes respectively. And the contact force measures can be derived by derivatives of the microscopic deformation energy with respect to the components \( \delta_i^c, \delta_i^c, \delta_i^c, \delta_i^c, \theta_3^c, \theta_3^c \):

\[
\frac{\partial \psi^c}{\partial \theta_3^c} = f_i^c n_i, \quad \frac{\partial \psi^c}{\partial \theta_3^c} = m_i^c \theta_3^c
\]

(19)

Due to the decomposition, the contact force \( f_i^c \) and moment \( m_i^c \) in equations (15) and (16) turn into \( f_i^c \) and \( m_i^c \).
Then, we can derive the macroscopic stress measures in 2D granular crystals. And brief derivation processes are given as

\[
\sigma_{ij} = \frac{\partial W}{\partial e_{ij}} = \frac{1}{V} \sum_{c=1}^{n} \frac{\partial w}{\partial \delta_{ik}} \frac{\partial \epsilon_{kl}}{\partial e_{ij}} = \frac{1}{V} \sum_{c=1}^{n} \frac{\partial w}{\partial \epsilon_{ik}} \frac{\partial \delta_{kl}}{\partial e_{ij}} = \frac{1}{V} \sum_{c=1}^{n} f_{ij}^{(c)} \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) L_{k}^{c} \\
\tau_{ij} = \frac{\partial W}{\partial \gamma_{ij}} = \frac{1}{V} \sum_{c=1}^{n} \frac{\partial w}{\partial \gamma_{kl}} \frac{\partial \gamma_{ij}}{\partial \gamma_{kl}} = \frac{1}{V} \sum_{c=1}^{n} f_{ij}^{(c)} \gamma_{kl} \\
\kappa_{ij} = \frac{\partial W}{\partial \kappa_{ij}} = \frac{1}{V} \left( \sum_{c=1}^{n} \frac{\partial w}{\partial \kappa_{ij}} \frac{\partial \kappa_{ij}}{\partial \kappa_{ij}} + \sum_{c=1}^{n} \frac{\partial w}{\partial \kappa_{ij}} \frac{\partial \kappa_{ij}}{\partial \kappa_{ij}} \right)
\]

where 2D micromorphic constitutive modulus tensors are expressed by discrete summation forms:

\[
C_{ijkl} = \frac{1}{V} \sum_{c=1}^{n} (K_{ijkl}^{c} n_{kl} + K_{ijlk}^{c} t_{kl}) L_{k}^{c} L_{l}^{c} \\
D_{ijkl} = \frac{1}{V} \sum_{c=1}^{n} (K_{ijkl}^{c} n_{kl} + K_{ijlk}^{c} t_{kl}) e_{ijkl} e_{klmn} N_{mn}^{c} N_{ij}^{c} + \frac{1}{V} \sum_{c=1}^{n} G_{ijkl}^{c} L_{k}^{c} L_{l}^{c} \\
A_{ijkl} = \frac{1}{V} \sum_{c=1}^{n} (K_{ijkl}^{c} n_{kl} + K_{ijlk}^{c} t_{kl}) e_{ijkl} e_{klmn} N_{mn}^{c} N_{ij}^{c} + \frac{1}{V} \sum_{c=1}^{n} G_{ijkl}^{c} L_{k}^{c} L_{l}^{c} \\
B_{ijkl} = \frac{1}{V} \sum_{c=1}^{n} (K_{ijkl}^{c} n_{kl} + K_{ijlk}^{c} t_{kl}) e_{ijkl} e_{klmn} N_{mn}^{c} N_{ij}^{c} + \frac{1}{V} \sum_{c=1}^{n} G_{ijkl}^{c} L_{k}^{c} L_{l}^{c}
\]
It is noted that these micromorphic constitutive moduli tensors of ellipsoidal granular crystals can be actually obtained for the 3D problem as

\[
\begin{align*}
C_{ijkl} & = \frac{1}{\nu} \sum_{c=1}^{n} K_{cij} L_{ik} L_{cj} \\
D_{ijkl} & = \frac{1}{\nu} \sum_{c=1}^{n} K_{mij} e_{nij} e_{mkl} N_{ci} N_{cj} + \frac{1}{\nu} \sum_{c=1}^{n} G_{cij} L_{ik} L_{cj} \\
A_{ijkl} & = \frac{1}{\nu} \sum_{c=1}^{n} \frac{K_{cij} L_{nk} L_{cn}}{\theta_{k}} \\
B_{ijkl} & = \frac{1}{\nu} \sum_{c=1}^{n} \frac{K_{cij} e_{nk} e_{mkl} N_{ci} N_{cj}}{\theta_{k}} + \frac{1}{\nu} \sum_{c=1}^{n} \frac{G_{cij} L_{nk} L_{cn}}{\theta_{k}}
\end{align*}
\]

(25)

And the derivation process is similar with that in our previous study. However, these equations in equation (25) are too hard to be solved. Therefore, we investigate the 2D problem for elliptic granular crystals in this study instead. Besides, our previous study has given the micromorphic constitutive modulus tensors for 3D spherical granular crystals, and equation (25) can degenerate into that in our previous study if 3D ellipsoidal particle degenerates into a sphere particle.

Considering the configuration of 2D elliptic granular crystals as seen in Figure 1, the specific constitutive modulus tensors can be obtained for equation (24). Note that the fabric tensor \( N_{ci} \) are expressed in two forms, therefore, \( D_{123} \) and \( B_{312} \) both have two expressions accordingly, while \( C_{ijkl} \) and \( A_{ijkl} \) only have one expression. Then, micromorphic constitutive modulus tensors of 2D elliptic granular crystals are derived for Modes A and B by

\[
\begin{align*}
C_{1111} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{1} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
C_{2221} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{2} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
C_{1212} & = C_{2112} = C_{1212} = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{2} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
C_{1122} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{1} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
C_{2222} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{2} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right)
\end{align*}
\]

(26)

\[
\begin{align*}
A_{1111} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{1} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
A_{2221} & = A_{1212} = A_{2112} = A_{1122} = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{1} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
A_{1212} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{1} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
A_{2222} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{2} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right)
\end{align*}
\]

(27)

\[
\begin{align*}
D_{3131} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{1} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
D_{3232} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{2} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) \\
D_{3131} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{2} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) + \frac{\nu}{\lambda} G_{n} \\
D_{3232} & = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{1} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) + \frac{\nu}{\lambda} G_{n}
\end{align*}
\]

Mode A: \( B_{3131} = \frac{\nu}{\lambda} G_{n} \) and \( B_{3232} = \frac{\nu}{\lambda} G_{n} \)

Mode B: \( B_{3131} = \frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{2} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) + \frac{\nu}{\lambda} G_{n} \)

\( B_{3232} = -\frac{\nu}{\lambda} \left( K_{n} - \frac{\alpha_{1} (K_{n} - K_{1})}{3\nu + \theta_{b}} \right) + \frac{\nu}{\lambda} G_{n} \)

(29)

Wave equations and dispersion equations in granular crystals

Firstly, a plane wave propagates along \( x_{1} \) and \( x_{2} \) axes respectively, therefore, the kinematic measures are expressed by

\[
\begin{align*}
\bar{u}_{i} & = \bar{u}_{i}(x_{1}, t) \\
\Gamma_{ij} & = \Gamma_{ij}(x_{1}, t) \\
\varphi_{ij} & = \varphi_{ij}(x_{1}, t) \\
\varphi_{ij} & = \varphi_{ij}(x_{2}, t) \\
\varphi_{ij} & = \varphi_{ij}(x_{2}, t)
\end{align*}
\]

(30)

Then, substituting equation (30) and constitutive equations (20)–(23) into balance equations, we can derive wave equations for 2D elliptic granular crystals as follows show

(1) Along \( x_{1} \) direction

\[
\begin{align*}
\rho \ddot{u}_{2} & = (C_{2121} + A_{2121}) \ddot{u}_{2,11} - (D_{3131} + B_{3131}) \ddot{u}_{2,111} \\
& - A_{2121} \psi_{2,11} + (D_{3131} + B_{3131}) \Gamma_{3,111} - B_{3131} \varphi_{3,111} \\
\rho \ddot{u}_{3} & = (D_{3131} + B_{3131}) \ddot{u}_{3,11} - B_{3131} \varphi_{3,11} \\
& - (D_{3131} + B_{3131}) \ddot{u}_{2,111} \\
\rho \ddot{u}_{1} & = (C_{1111} + A_{1111}) \ddot{u}_{1,11} - A_{1111} \psi_{1,11} \\
\rho \ddot{u}_{1} & = A_{1111} \ddot{u}_{1,11} - A_{1111} \psi_{1,1} \\
\rho \ddot{u}_{1} & = A_{1111} \ddot{u}_{1,11} - A_{1111} \psi_{1,1} \\
\rho \ddot{u}_{1} & = A_{1111} \ddot{u}_{1,11} - A_{1111} \psi_{1,1}
\end{align*}
\]

(31)

(2) Along \( x_{2} \) direction

Wave equations for situations along \( x_{2} \) direction are those in which the subscripts 1 and 2 are exchanged in equations (31)–(32), which can be omitted here.

Solutions of wave equations are obtained in harmonic wave forms as follows show

(32)

\[
\begin{align*}
\bar{u}_{i} & = S_{i, \exp} [i(k_{1}x_{1} - \omega t)] \\
\psi_{ij} & = T_{ij, \exp} [i(k_{1}x_{1} - \omega t)] \\
\varphi_{ij} & = P_{ij, \exp} [i(k_{2}x_{2} - \omega t)] \\
\varphi_{ij} & = Q_{ij, \exp} [i(k_{2}x_{2} - \omega t)]
\end{align*}
\]

(33)
where \( k \) is the wave number, \( \omega \) is the circular frequency, and \( S_i, T_{ij}, P_3, Q_{3i} \) denote wave amplitudes.

Substituting equation (33) into equations (31)–(32), equations about relations between \( k \) and \( \omega \), that is, dispersion equations are obtained for 2D granular crystals as following equations (34)–(36) show, including two micromorphic wave modes, that is, a coupled transverse – rotational wave (T-R wave) and a longitudinal wave (L wave).

(1) Along \( x_1 \) direction

\[
T - R \text{ wave: } \begin{vmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ \vdots & A^{22} & A^{23} & A^{24} \\ \vdots & \vdots & \ddots & \vdots \\ \text{sym} & \cdots & \cdots & A^{44} \end{vmatrix} = 0 \quad (34-1)
\]

where

\[
\begin{align*}
A^{11} &= (D_{3131} + B_{3131})k^4 + (C_{3121} + A_{2121})k^2 - \rho\omega^2 \\
A^{12} &= (D_{3131} + B_{3131})k^3 \\
A^{13} &= A_{2121}k \\
A^{14} &= -B_{3131}k^2 \\
A^{21} &= A^{12} \\
A^{22} &= (D_{3131} + B_{3131})k^2 - \rho\omega^2 \\
A^{23} &= 0 \\
A^{24} &= -B_{3131}k \\
A^{31} &= A^{13} \\
A^{32} &= A^{23} \\
A^{33} &= A_{2121} - \rho' T' \omega^2 \\
A^{34} &= 0 \\
A^{41} &= A^{14} \\
A^{42} &= A^{24} \\
A^{43} &= A^{34} \\
A^{44} &= B_{3131} - \rho' J' \omega^2 \\
L \text{ wave: } \begin{vmatrix} (C_{1111} + A_{1111})k^2 - \rho\omega^2 & A_{1111}k \\
A_{1111}k & A_{1111} - \rho' T' \omega^2 \end{vmatrix} &= 0 \quad (35)
\end{align*}
\]

(2) Along \( x_2 \) direction

Similarly, the dispersion equations for situations along \( x_2 \) direction are those in which the subscripts 1 and 2 are exchanged in equations (34)–(35).

According to dispersion equations, we can obtain cutoff frequencies \( \omega_k \) at vanishing wave numbers (\( k = 0 \)) as followings show

\[
\omega_k^{TR} = \begin{cases} 
\sqrt{B_{3131}/\rho' J'} \rightarrow TR/O \\
\sqrt{A_{2121}/\rho' T'} \rightarrow TR/O \\
0 \rightarrow TR/A \\
0 \rightarrow TR/A
\end{cases} \quad (36-1)
\]

where \( O \) and \( A \) indicates the optical and acoustic branches. Then, referring to dispersion equations (34) and (35) and degrees of freedom (DOFs) in wave equations (31) and (32), we can obtain DOF \( \tilde{u}_2 \) responsible for T-R wave’s acoustic branches, \( \Gamma_3, \psi_{21}, \phi_{31} \) responsible for T-R wave’s optical branches, \( \tilde{u}_1 \) responsible for L wave’s acoustic branches, \( \text{DOF } \psi_{11} \) responsible for L wave’s optical branches. Therefore, the macroscopic DOFs of displacement and rotation \( \tilde{u}_1, \tilde{u}_2 \) are associated with acoustic branches of micromorphic waves, while the gradients of microscopic displacement and rotation \( \psi_{11}, \psi_{21}, \) and \( \phi_{31} \) and macroscopic rotation \( \Gamma_3 \) associated with optical branches. Then, it can conclude that the optical branches are resulted from the rotation and the microscopic deformation of the volume element for material point. Furthermore, if there is no microscopic deformation, that is, the micromorphic medium degenerates into a micropolar one, the DOFs \( \tilde{u}_1 \) and \( \tilde{u}_2 \) lead to the acoustic branches of micropolar waves, and rotational DOF \( \Gamma_3 \) also leads to a dispersive optical branch. This matches the conclusions in previous studies.\(^{19,31} \) And if the rotation of material point is not considered, that is, the micropolar medium degenerates into a classical one, the non-dispersive acoustic branches are resulted from the DOFs \( \tilde{u}_1 \) and \( \tilde{u}_2 \). To summarize, the optical mode of wave comes from two aspects: the rotation of material point and the microscopic deformation of material point.

**Dispersion behaviors in 2D elliptic granular crystals**

According to dispersion equations, dispersion curves can be obtained. It is noted that microscopic parameters have effects on dispersion equations, and they are referred from Xiu et al.\(^{31} \) and given in Table 1.

**Dispens in different aspect ratio a/b**

In above discussions, we considered two modes to describe the relative displacement by Taylor expansions. In this part, Mode A, that is, using Taylor expansion at the center of particle, is investigated, and the comparisons of dispersion behaviors between using Mode A and Mode B are considered in the next part.

Firstly, Figure 2 gives dispersion curves and frequency band gaps of elliptic granular crystals for different aspect ratio \( a/b = 0.1, 0.2, 0.5, 1, 2, 5, 10 \) by Mode A. It shows that there are two branches for L wave containing an acoustic one and an optic one and four branches for T-R wave containing two acoustic ones and two optic ones. All branches of L and T-R waves can exhibit obvious dispersion behaviors. The width of
Table 1. Values of microscopic parameters.

| Parameters | Values       | Parameters   | Values       | Parameters | Values       |
|-----------|--------------|--------------|--------------|-----------|--------------|
| $K_U^n$   | 200 MN/m     | $K_U^t$     | 100 MN/m     | $b$       | $0.5 \times 10^{-3}$ m |
| $K_R^n$   | 200 MN/m     | $K_R^t$     | 100 MN/m     | $\rho'$   | 3000 kg/m$^3$ |
| $K_I^n$   | 0.02 MN/m    | $K_I^t$     | 0.01 MN/m    | $\rho$    | 1570 kg/m$^3$ |
| $G^R$     | $10^{-1}$ N$\cdot$m | $G'$       | $10^{-2}$ N$\cdot$m |           |              |

Figure 2. Dispersion curves and frequency band gaps of elliptic granular crystals for different $a/b$ by Mode A: (a) $a/b = 0.1$; (b) $a/b = 0.2$; (c) $a/b = 0.5$; (d) $a/b = 1$; (e) $a/b = 2$; (f) $a/b = 5$; (g) $a/b = 10$. 

Advances in Mechanical Engineering
frequency band gap of elliptic granular crystals (called total frequency band gap below) has a positive correlation with the aspect ratio $a/b$. The average frequency in the band gap also increases with the increase of $a/b$, and the detailed results can be seen in Table 2 below. Furthermore, for a clearer presentation, we respectively investigate the dispersion behaviors of $L$ and T-R waves, as shown in Figures 3 to 5. Similarly, widths of frequency band gaps of $L$, high-frequency and low-frequency T-R waves are positively correlated with $a/b$. And the width of high-frequency T-R wave’s frequency band gap has a greatest increase, because the frequency of T-R/O branch gradually increases while that of T-R/A branch almost increases first and then decreases with the increase of $a/b$. Both the widths of frequency band gaps of $L$ and high-frequency T-R waves are greater than that of total frequency band gap in Figure 2. Besides, the low-frequency T-R wave’s frequency band gap has the $10^3$ rad/s magnitude, which is about 1/1000~1/100 of that of high-frequency T-R wave. Specifically, Table 2 gives widths of frequency band gaps and average frequencies in their band gaps for different $a/b$.

Here, we record the width of frequency band gap and the average frequency in band gap for the situation $a/b = 1$ respectively as $\omega_{1b}^L$ ($0.072 \times 10^6$ rad/s) and $\omega_{1b}^T$ ($0.353 \times 10^6$ rad/s). Then, normalized widths of frequency band gaps and average frequencies in band gaps can be obtained for different $a/b$ by widths $\omega_{a/b}^L$ ($\omega_{a/b}^{L/L,T-R}$) and average frequencies $\bar{\omega}_{a/b}^L$ ($\bar{\omega}_{a/b}^{L/L,T-R}$) in Table 2 dividing $\omega_{1b}^L$ and $\omega_{1b}^T$ respectively. Figure 6 gives curves of aspect ratio $a/b$ versus normalized widths of total frequency band gap, and $L$’s and high-frequency T-R’s frequency band gaps, and Figure 7 gives curves of aspect ratio $a/b$ versus their normalized average frequencies. Note that a detailed discussion about the dispersion of low-frequency T-R wave is given in the next part, considering its smaller magnitude. In the above discussion, it says that the width of frequency band gap and its average frequency are positively correlated with $a/b$. From Figures 6 and 7, positive correlations seem 1/2 power relationships. And we can give power expressions for the width of total frequency band gap $\omega_{a/b}^{L,T-R}$ (i.e. the width of frequency band gap for granular crystals) and its average frequency $\bar{\omega}_{a/b}^L$ with respect to the aspect ratio $a/b$:

$$\omega_{a/b}^{L,T-R} = \frac{1.6 \times (a/b)^{0.5} - 0.6}{\bar{\omega}_{1b}^L}$$

(37)

$$\bar{\omega}_{a/b}^L = \frac{[(a/b)^{0.5} - 0.02]}{\bar{\omega}_{1b}^T}$$

(38)

And maximal errors of equations (37) and (38) are about 4%. Accordingly, we can also obtain the 1/2 power equations for $L$’ and T-R/high’s widths of frequency band gaps $\omega_{a/b}^{L,T-R}$ and $\bar{\omega}_{a/b}^L$ and their $\bar{\omega}_{a/b}^{L,T-R}$ and $\bar{\omega}_{a/b}^T$, and they are omitted here for space. And it can be seen from Figures 6 and 7, the T-R/high’s width of frequency band gap increases greatly almost to three times that of the total one with the increase of $a/b$, while its average frequency increases slowly. The $\omega_{a/b}^{L,T-R}$ and $\bar{\omega}_{a/b}^{L,T-R}$ of $L$ wave are about 1.5 and 1.1 times the total $\omega_{a/b}^{L,T-R}$ and $\omega_{a/b}^L$ when $a/b$ increase to 10.

Furthermore, $a/b$ increasing from 0.1 to 10 represents the fabric of elliptic granular crystals, as seen in Figure 8. Here, the contact fabric tensor $F_{ij}$ is defined by

| Aspect ratio | L wave (10^6 rad/s) | High-frequency T-R wave (10^6 rad/s) | Low-frequency T-R wave (10^3 rad/s) | Total frequency band gap (10^6 rad/s) |
|-------------|---------------------|-------------------------------------|------------------------------------|-------------------------------------|
| $a/b = 0.1$ | 0.048               | 0.036                               | 0.744                              | 0.004                               |
| $a/b = 0.2$ | 0.068               | 0.053                               | 1.052                              | 0.008                               |
| $a/b = 0.5$ | 0.105               | 0.118                               | 1.664                              | 0.032                               |
| $a/b = 1$   | 0.144               | 0.240                               | 2.352                              | 0.072                               |
| $a/b = 2$   | 0.200               | 0.439                               | 3.322                              | 0.122                               |
| $a/b = 5$   | 0.315               | 0.808                               | 5.211                              | 0.206                               |
| $a/b = 10$  | 0.444               | 1.195                               | 7.252                              | 0.293                               |

**Table 2.** Widths and average frequencies of frequency band gaps of elliptic granular crystals for different $a/b$ by Mode A.
Considering the wave propagation along $x_1$ direction, we can obtain this component

$$F_{ij} = \frac{1}{V} \sum_{c=1}^{a} n_i n_j$$  \hspace{1cm} (39)

$$F_{11} = \frac{(a/b)^2 + 1}{3(a/b)^2 + 1}$$  \hspace{1cm} (40)

$F_{11}$ decreases from 1 to 1/3 when $a/b$ increase from zero to the positive infinity. Therefore, the width of total frequency band gap and its average frequency have correlations with the contact fabric $F_{11}$ by

$$\tilde{\omega}_F^{Ba} = 1.6 \times \left( \frac{F_{11} + 1}{3F_{11} - 1} \right)^{0.25} - 0.6 \times \tilde{\omega}_1^{Ba}$$  \hspace{1cm} (41)

$$\tilde{\omega}_F^{Av} = \left[ \frac{F_{11} + 1}{3F_{11} - 1} \right]^{0.25} - 0.02 \times \tilde{\omega}_1^{Av}$$  \hspace{1cm} (42)

And more in-depth and detailed relationship between dispersion behaviors of elliptic granular crystals and their fabric information will be discussed in next studies.
Dispersions by Mode A and Mode B

As mentioned above, there are two modes to describe the relative displacement by Taylor expansions at the particle center (Mode A) or the contact point (Mode B). In this part, we investigate dispersion behaviors of elliptic granular crystals by these two modes.

We can also obtain dispersion curves and frequency band gaps of elliptic granular crystals for different \( a = b = 0.1, 0.2, 0.5, 1, 2, 5, 10 \) by Mode B. But the curves are similar with those in the last part, therefore, the figures are not presented here to save space. And we obtain a result that the total frequency band gaps are same with those in Figure 2 for different \( a/b \), and dispersion curves of L and high-frequency T-R waves are quite similar between the two situations by Mode A and Mode B. Those of low-frequency T-R waves are different comparing with Figure 5. Based on equations (26)–(29), dispersions of T-R waves are related to microscopic parameters \( k_{R}^{*}, k_{T}^{*} \). Therefore, we increase these parameters by 1000 times and name this situation as Mode B1. Then, we further discuss dispersion behaviors among different modes and give comparisons of frequency band gaps in Table 3. It

Figure 4. Dispersion curves and frequency band gaps of high-frequency branches of T-R waves for different \( a/b \) by Mode A:
(a) \( a/b = 0.1 \); (b) \( a/b = 0.2 \); (c) \( a/b = 0.5 \); (d) \( a/b = 1 \); (e) \( a/b = 2 \); (f) \( a/b = 5 \); (g) \( a/b = 10 \).
obviously shows that the total frequency band gaps are same among different modes for different a/b. With the increase of a/b, the T-R/high’s frequency band gap for Mode B1 is gradually getting smaller than those for the other two modes. And the T-R/low’s frequency band gap for Mode B is about 0.5% larger than that for Mode A, while the T-R/low’s frequency band gap for Mode B1 has an increase from 1.5 times to 3 times that for Mode B with the increase of a/b. It illustrates that Taylor expansion at the particle center or the contact point mainly has an influence on dispersions of low-frequency T-R waves and has no influence on total dispersion behaviors of elliptic granular crystals.

**Dispersions in x₂ versus x₁ directions**

In above parts, we investigate the wave propagation along x₁ direction. Considering the anisotropy of elliptic granular crystals, the wave propagation along x₂ direction is also meaningful. Note that only Mode A is considered in this part to save space.

Figure 9 shows dispersion curves and frequency band gaps for different a/b by Mode A along x₂ direction. Comparing with Figure 2, we can obtain an opposite rule that the width of frequency band gap has a negative correlation with the aspect ratio a/b when the wave propagates along x₂ direction. Because the width
of frequency band gap is related to the fabric as shown in equation (41), it is respectively related to $F_{11}$ or $F_{22}$ when the wave propagates along $x_1$ or $x_2$ direction, and note that $F_{22} = 1 - F_{11}$, therefore, the opposite rule for width of frequency band gap versus $a/b$ is obtained when wave propagating along $x_1$ and $x_2$ directions. Specifically, Table 4 gives widths of frequency band gaps for different $a/b$ by Mode A when the wave propagates along $x_2$ direction. It is noticed that widths of frequency band gaps are same for $a/b = 1$ when wave propagating along $x_1$ and $x_2$ directions. Then, normalized widths of frequency band gaps are obtained for wave propagating along $x_2$ direction, as shown in Figure 10. When $a/b < 1$, the L’ width of frequency band gap coincides with that of total frequency band gap, while $a/b > 1$, it tends to equal to the T-R/high’s width of frequency band gap. And a power equation for the width of total frequency band gap $\omega_{a/b}^{Ba}$ is also obtained with respect to $a/b$ in this situation as

$$\omega_{a/b}^{Ba} = \left[2.5 \times (a/b)^{-0.4} - 1.2\right] \times \omega_{1}^{Ba} \quad (43)$$

Different from equation (37), $\omega_{a/b}^{Ba}$ here is related to $a/b$ in a negative power form. Similarly, we can also obtain expressions of L’ and T-R’ widths of total frequency band gaps, etc., which are not discussed here to save space.

**Wave velocity in 2D elliptic granular crystals**

Because of dispersions in elliptic granular crystals, the wave velocity changes accordingly with the frequency when wave propagating in granular crystals. It has great significance of investigating the wave velocity in elliptic granular crystals. And the wave velocity can be obtained by taking a derivative of the circular frequency $\omega$ with respect to the wave number $k$. In this section, only the situation of Mode A is considered to save space.

**Wave velocity in different aspect ratio $a/b$**

Firstly, Figures 11 to 13 respectively give the wave velocities of L, high-frequency and low-frequency T-R waves versus the frequency for different $a/b = 0.2, 0.5, 1, 2, 5$. It is noted that we use the frequency $f = \omega/2\pi$ in this section different from the circular frequency $\omega$ in the last section, because the discussion about the $f - \nu$ relationship seems more intuitive and common. Here, $v_L$ and $v_{T-R}$ represent the wave velocities of L and T-R waves. It can be seen that both $v_L$ and $v_{T-R}$ first decrease to zero, keep zero for a range of frequency and then increase to a larger wave

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**Figure 6.** Curves of aspect ratio $a/b$ versus normalized width of frequency band gap.

**Figure 7.** Curves of aspect ratio $a/b$ versus normalized average frequency in band gap.

**Figure 8.** Evolution of fabric of elliptic granular crystals.
velocity with the increase of frequency for different a/b. And the lasting zero for a range of frequency means the frequency band gap corresponding to that shown in Figure 2. Then, we use \( v^0 \) and \( v^f \) to represent the wave velocities when \( f = 0 \) and \( f \) reaches the maximum, respectively. Note that \( v^0 \) only belongs to the acoustic branches, and \( v^f \) only belongs to the optic ones. For L, high-frequency and low-frequency T-R waves, \( v^0 \) is always smaller \( v^f \). And \( v_{T-R/low} \) in Figure 13 is quite smaller than \( v_L \) and \( v_{T-R/high} \), therefore, \( v_{T-R/low} \) is not a central to this discussion. Furthermore, Figure 14 gives \( v^0 \) and \( v^f \) of L and high-frequency T-R waves with respect to a/b. It shows that \( v^0_L \), \( v^f_L \) and \( v^f_{T-R/high} \) have power relationships with a/b, while \( v^0_{T-R/high} \) increases first and then decreases with the increase of a/b. And it is noticed that the ratio between \( v^0_L \) and \( v^f_L \) keeps about 0.73 for different a/b. Comparing \( v^f_{T-R/high} \) with \( v^f_L \) for different a/b, the ratios between \( v^f_{T-R/high} \) and \( v^f_L \) are about 0.7, 0.67, 0.63, 0.6, and 0.59, respectively, which indicates that the ratio between T-R’ and L’ velocities is in the range of 0.59–0.7 for elliptic granular crystals. And experiments\(^{12,43}\) have reported that the ranges of the wave velocity ratio between transverse and longitudinal waves are about 0.62–0.68 in unconsolidated porous sands\(^{42}\) and 0.54–0.68 in dry sands.\(^{43}\) Our micromorphic theoretic predictions are basically consistent with these experiment data.

### Table 3. Widths of frequency band gaps of elliptic granular crystals for different a/b by different modes.

| Aspect ratio | Modes | High-frequency T-R wave \( (10^6 \text{ rad/s}) \) | Low-frequency T-R wave \( (10^3 \text{ rad/s}) \) | Total frequency band gap \( (10^6 \text{ rad/s}) \) |
|--------------|-------|-----------------------------------------------|-----------------------------------------------|-----------------------------------------------|
| a/b = 0.2    | A     | 0.053                                        | 1.052                                        | 0.008                                        |
|              | B     | 0.053                                        | 1.053                                        | 0.008                                        |
|              | BI    | 0.053                                        | 1.530                                        | 0.008                                        |
| a/b = 0.5    | A     | 0.118                                        | 1.664                                        | 0.032                                        |
|              | B     | 0.118                                        | 1.668                                        | 0.032                                        |
|              | BI    | 0.117                                        | 3.884                                        | 0.032                                        |
| a/b = 1      | A     | 0.240                                        | 2.352                                        | 0.072                                        |
|              | B     | 0.240                                        | 2.361                                        | 0.072                                        |
|              | BI    | 0.234                                        | 6.921                                        | 0.072                                        |
| a/b = 2      | A     | 0.439                                        | 3.322                                        | 0.122                                        |
|              | B     | 0.439                                        | 3.337                                        | 0.122                                        |
|              | BI    | 0.417                                        | 10.676                                       | 0.122                                         |
| a/b = 5      | A     | 0.808                                        | 5.211                                        | 0.206                                        |
|              | B     | 0.808                                        | 5.237                                        | 0.206                                        |
|              | BI    | 0.749                                        | 16.980                                       | 0.206                                         |

**Wave velocity in x₂ versus x₁ directions**

Similar with the rules for dispersions in x₂ versus x₁ directions, the wave velocity in x₂ direction is opposite to that in x₁ direction with the increase of a/b. We can also obtain velocities of L and high-frequency T-R waves versus the frequency for different a/b = 0.2, 0.5, 1, 2, 5. The curves are similar with those in Figures 13 and 14, therefore, the figures are not presented here to save space. And Table 5 and Figure 15 give the wave velocities \( v^0 \) and \( v^f \) of L and high-frequency T-R waves with respect to a/b. As we can see, the situation with smaller a/b leads to the larger \( v_L \) and \( v_{T-R/high} \). \( v_{T-R/high} \) is always smaller than \( v_L \) for the acoustic or the optic branch, and \( v^0_{T-R/high} \) increases first and then decreases with the increase of a/b, as shown in Figure 15. It also shows that the ratio between \( v^0_L \) and \( v^f_L \) keeps about 0.73 for different a/b, which is same with that in the last part. Differently, the ratios between \( v^f_{T-R/high} \) and \( v^f_L \) are about 0.88, 0.73, 0.63, 0.58, and 0.56, respectively, and this range is about three times that in the last part. It indicates that ratios of velocities between transverse and longitudinal waves are different when the wave propagates along different directions in elliptic granular crystals, because of the anisotropy.

**Discussion and conclusion**

Dispersion is an important wave propagation behavior in granular media such as granular materials, crystals, and structures, and it is closely related to the microstructure of granular media. Discrete particle models\(^{15–18}\) can conveniently simulate the relationship between dispersion and microstructure of granular media, but they are limited by the computational scale. Traditional continuous medium models\(^{28}\) can quantitatively describe dispersion behaviors, but they are not enough to accurately simulate the effect of microstructure. The micromechanics-based continuum models\(^{27,31,33}\) include...
advantages of discrete and continuum models in describing dispersion and the effect of microstructure. And among these models, the micromechanics-based micromorphic model can provide more macroscopic variables and deformation modes with respect to microstructural information in the process of homogenization for granular assembly, which is seen as a complete model. The microstructural information such as particle arrangement and size, void ratio, and contact stiffness are analyzed for their effects on dispersion behaviors. But previous studies on micromechanics-based micromorphic models lack discussions about the particle shape and its effect on dispersion behaviors.

One of major objects in this study is to introduce the factor of particle shape into the micromechanics-based micromorphic model and investigate the effect of particle shape on dispersion behaviors of granular crystals. Then, a micromechanics-based micromorphic model is derived for 2D elliptic granular crystals in this study, where macroscopic micromorphic constitutive modulus tensors are identified by summation expressions of contact stiffness and microstructural parameters in 2D closely packed elliptic particle assembly. This study uses two modes of Taylor expansion of relative displacement and rotation to establish the contact relation between two particles. And two sets of micromorphic constitutive modulus tensors

Figure 9. Dispersion curves and frequency band gaps for different a/b by Mode A along x_2 direction.
are derived respectively based on the two modes of Taylor expansion. Two micromorphic wave modes, that is, coupled transverse – rotational (T-R) and longitudinal (L) waves are predicted. Then, dispersion equations of micromorphic waves are derived for 2D elliptic granular crystals, and accordingly, dispersions and wave velocities are obtained in 2D elliptic granular crystals. The main conclusions are given as followings:

1. Obvious dispersion behaviors are predicted for elliptic granular crystals with different aspect ratio $a/b$. With the increase of $a/b$, the width of frequency band gap increases when the wave propagates along $x_1$ direction, and the width of frequency band gap has a $1/2$ power relationship with $a/b$ and a $1/4$ power relationship with the contact fabric $F_{11}$.

2. Two modes of Taylor expansion of relative displacement and rotation have little effect on the total frequency band gap of elliptic granular crystals, which can influence dispersions and frequency band gaps of T-R waves when multiply increasing microscopic parameters with respect to the rotation.

3. Dispersions and the frequency band gap have the opposite rules when it considers the wave propagation along $x_2$ direction comparing with wave propagation along $x_1$ direction, which can be attributed to the anisotropy of elliptic granular crystals.

4. The velocities $v_L$ and $v_{T-R}$ of L and T-R waves first decrease to zero, keep zero for a range of frequency and then increase to a larger wave velocity with the increase of frequency, for situations with different $a/b$.

5. The situation with a larger $a/b$ leads to larger $T-R$ and $L$’ velocities, and the ratio between $T-R$ and $L$’ velocities decreases from 0.7 to 0.59, when the wave propagates along $x_1$ direction. However, the situation with a larger $a/b$ leads to smaller $v_L$ and $v_{T-R}$, and the ratio between the two velocities decreases from 0.88 to 0.56, when the wave propagates along $x_2$ direction.

### Table 4. Widths and average frequencies of frequency band gaps of elliptic granular crystals for different $a/b$ by Mode A along $x_2$ direction.

| Aspect ratio | L wave (10^6 rad/s) | High-frequency T-R wave (10^6 rad/s) | Low-frequency T-R wave (10^3 rad/s) | Total frequency band gap (10^6 rad/s) |
|--------------|----------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| a/b = 0.1    | 0.349                | 1.432                               | 7.349                               | 0.349                               |
| a/b = 0.2    | 0.256                | 0.933                               | 5.241                               | 0.256                               |
| a/b = 0.5    | 0.184                | 0.452                               | 3.325                               | 0.184                               |
| a/b = 1      | 0.144                | 0.240                               | 2.352                               | 0.072                               |
| a/b = 2      | 0.106                | 0.143                               | 1.663                               | 0.021                               |
| a/b = 5      | 0.069                | 0.084                               | 1.052                               | 0.006                               |
| a/b = 10     | 0.049                | 0.058                               | 0.744                               | 0.003                               |

**Figure 10.** Curves of aspect ratio $a/b$ versus normalized width of frequency band gap.
Figure 11. Curves of frequency versus wave velocity of L wave for different $a/b$: (a) $a/b = 0.2$; (b) $a/b = 0.5$; (c) $a/b = 1$; (d) $a/b = 2$; (e) $a/b = 5$.

Figure 12. Curves of frequency versus wave velocity of high-frequency branches of T-R waves for different $a/b$: (a) $a/b = 0.2$; (b) $a/b = 0.5$; (c) $a/b = 1$; (d) $a/b = 2$; (e) $a/b = 5$. 

Xiu and Chu
Figure 13. Curves of frequency versus wave velocity of low-frequency branches of T-R waves for different \( a/b \): (a) \( a/b = 0.2 \); (b) \( a/b = 0.5 \); (c) \( a/b = 1 \); (d) \( a/b = 2 \); (e) \( a/b = 5 \).

Figure 14. Curves of \( v^0 \) and \( v^f \) of L and high-frequency T-R waves with respect to \( a/b \).

Figure 15. Curves of \( v^0 \) and \( v^f \) of L and high-frequency T-R waves with respect to \( a/b \) along \( x_2 \) direction.


### Table 5. Wave velocities $v^0$ and $v^f$ of L and high-frequency T-R waves with respect to $a/b$.

| Aspect ratio $a/b$ | L wave $v^0$ (m/s) | L wave $v^f$ (m/s) | High-frequency T-R wave $v^0$ (m/s) | High-frequency T-R wave $v^f$ (m/s) |
|-------------------|-------------------|-------------------|-----------------------------------|-----------------------------------|
| $a/b = 0.2$       | 438.6             | 601.5             | 85.3                              | 527.2                             |
| $a/b = 0.5$       | 315.1             | 432.1             | 99.6                              | 317.2                             |
| $a/b = 1$         | 246.6             | 338.2             | 93.2                              | 212.4                             |
| $a/b = 2$         | 182.8             | 250.7             | 73.1                              | 145.4                             |
| $a/b = 5$         | 117.5             | 161.2             | 47.8                              | 90.8                              |

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