The Dirichlet problem for Monge-Ampère equation for \((n - 1)\)-PSH functions on Hermitian manifolds

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Abstract

We solve the Dirichlet problem for Monge-Ampère equation for \((n - 1)\)-PSH functions possibly with degenerate right-hand side, through deriving a quantitative version of boundary estimate under the assumption of \((n - 1)\)-PSH subsolutions. In addition, we confirm the subsolution assumption on a product of a closed balanced manifold with a compact Riemann surface with boundary.

1 Introduction

A basic work in Kähler geometry is Yau’s [31] proof of Calabi’s conjecture. In [8] Gauduchon showed that every closed Hermitian manifold \((M, J, \omega)\) of complex dimension \(n \geq 2\) admits a unique (up to rescaling) Gauduchon metric in the conformal class \([\omega]\), and he furthermore conjectured in [9] that the Calabi-Yau theorem is also true for Gauduchon metric. On complex surfaces and astheno-Kähler manifolds introduced by Jost-Yau [19], the Gauduchon conjecture was solved by Cherrier [5] and Tosatti-Weinkove [29], respectively. The Gauduchon conjecture in higher dimensions amounts to solving a Monge-Ampère type equation

\[
\left(\bar{\partial} + \frac{1}{n - 1} \left(\Delta u - \sqrt{-1} \partial \bar{\partial} u + Z\right)\right) = \phi^\omega
\]

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where
\[ \tilde{\chi} = \frac{1}{(n-1)!} \ast \omega_0^{n-1}, \quad Z = \frac{1}{(n-1)!} \ast \Re \left( \sqrt{-1} \partial u \wedge \overline{\partial} (\omega^{n-2}) \right), \]
see [24, 29]. Here * is the Hodge star operator with respect to \( \omega \) and \( \omega_0 \) is a Gauduchon metric. Following [16], also [28], we call \( u \) an \( (n-1) \)-plurisubharmonic ((\( n-1 \))-PSH in short) function if
\[ \tilde{\chi} + \frac{1}{n-1} \left( \Delta u \omega - \sqrt{-1} \partial \overline{\partial} u \right) + Z \geq 0 \text{ in } M. \]
Furthermore, we call \( u \) a strictly \( (n-1) \)-PSH function if
\[ \tilde{\chi} + \frac{1}{n-1} \left( \Delta u \omega - \sqrt{-1} \partial \overline{\partial} u \right) + Z > 0 \text{ in } M. \]
Geometrically, the strictly \( (n-1) \)-PSH solution of (1.1) allows one to construct a Gauduchon metric, say \( \Omega_u \), with prescribed volume form
\[ \Omega_u^n = \tilde{\phi} \omega^n \] 
and \( \Omega_u^{n-1} = \omega_0^{n-1} + \sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2} + \Re(\sqrt{-1} \partial u \wedge \overline{\partial} \omega^{n-2}), \) where and hereafter
\[ \tilde{\phi} = \phi^{1/(n-1)}. \]
Recently, Székelyhidi-Tosatti-Weinkove [27] proved Gauduchon’s conjecture for \( n \geq 3 \), thereby extending the Calabi-Yau theorem to Gauduchon metric.

This paper is devoted to extending Székelyhidi-Tosatti-Weinkove’s results to complex manifolds with boundary. To this end, we assume that \( (M, J, \omega) \) is a compact Hermitian manifold of complex dimension \( n \geq 2 \) with boundary \( \partial M, \) \( \bar{M} := M \cup \partial M, \) where \( J \) is the complex structure, \( \omega = \sqrt{-1} g_{ij} dz^i \wedge d\overline{z}^j \) denotes the Kähler form. We consider the Dirichlet problem with prescribed boundary value condition
\[ u = \varphi \text{ on } \partial M. \] 
Before stating our results, we summarize some notion.

**Definition 1.1.** A strictly \( (n-1) \)-PSH function \( u \in C^2(\bar{M}) \) is called a **subsolution** of the Dirichlet problem (1.1) and (1.4), if
\[ \left( \tilde{\chi} + \frac{1}{n-1} \left( \Delta u \omega - \sqrt{-1} \partial \overline{\partial} u \right) + Z \right)^n \geq \tilde{\phi} \omega^n \text{ in } M, \]
\[ u = \varphi \text{ on } \partial M. \]
We say a strictly \((n-1)\)-PSH function \(u \in C^2(\bar{M})\) is a \textit{strict subsolution} of the Dirichlet problem (1.1) and (1.4), if
\[
\left( \hat{\chi} + \frac{1}{n-1} \left( \Delta_m \omega - \sqrt{-1} \partial \bar{\partial} u \right) + Z \right)^n \geq (\phi + \delta) \omega^n \quad \text{in } M
\]
\[
u = \phi \quad \text{on } \partial M
\]
for a positive constant \(\delta\). Here \(Z = \frac{1}{(n-1)!} * \Re(\sqrt{-1} \partial \bar{\partial} u \wedge (\omega^{n-2}))\).

Our main results are stated as follows.

Theorem 1.2. Let \((M, J, \omega)\) be a compact Hermitian manifold with smooth boundary. Assume the given data \(\phi \in C^\infty(\partial M)\) and \(0 < \phi \in C^\infty(\bar{M})\) support an \((n-1)\)-PSH subsolution \(u \in C^{2,1}(\bar{M})\) subject to (1.5). Then the Dirichlet problem (1.1) and (1.4) is uniquely solvable in the class of smooth strictly \((n-1)\)-PSH functions.

We say the boundary is \textit{mean pseudoconcave} if
\[-(\kappa_1 + \cdots + \kappa_{n-1}) \geq 0 \quad \text{on } \partial M\]
where \(\kappa_1, \cdots, \kappa_{n-1}\) are the eigenvalues of Levi form \(L_{\partial M}\) with respect to \(\omega' = \omega|_{T_{\partial M} \cap JT_{\partial M}}\). When the boundary is mean pseudoconcave we can solve the equation with degenerate right-hand side \(\phi \geq 0\).

Theorem 1.3. Let \((M, J, \omega)\) be a compact Hermitian manifold with smooth mean pseudoconcave boundary. Let \(\varphi \in C^{2,1}(\partial M)\) and \(0 \leq \varphi^{1/n} \in C^{1,1}(\bar{M})\). Suppose in addition that there exists a \(C^{2,1}\)-smooth \((n-1)\)-PSH strict subsolution satisfying (1.6). Then the Dirichlet problem (1.1) and (1.4) admits a \((weak)\) \((n-1)\)-PSH solution \(u \in C^{1,\alpha}(\bar{M})\), \(\forall 0 < \alpha < 1\), with \(\Delta u \in L^\infty(\bar{M})\).

As shown in the above theorem, we obtain a \(C^{1,\alpha}\)-smooth \((n-1)\)-PSH weak solution for (1.1) with right-hand side of the form \(\phi = w^a\) for some \(0 \leq w \in C^{1,1}(\bar{M})\). However, it is fairly restrictive even in the case \(n = 2\). In some problems arising from complex geometry and analysis, it would be interesting to build up \(\Omega^\alpha_a = \dot{\phi}\omega^a\) with \(\partial \bar{\partial} \Omega^{-1}_a = 0\) in the weak sense, given a nonnegative \(C^{1,1}\)-smooth function \(\dot{\phi}\).

We make progress on this subject under the assumption that
\[
\frac{\partial \dot{\phi}}{\partial \nu} = 0 \quad \text{at } \{ p \in \partial M : \dot{\phi}(p) = 0 \}
\]
where \(\nu\) is the unit inner normal vector along the boundary. Such a condition is automatically satisfied if the data \((M, \dot{\phi})\) is \textit{extensible} in the sense that...
• \((M, \omega)\) is a complex submanifold of an open Hermitian manifold \((M', \omega)\) of complex dimension \(n, \bar{M} \subset M'\); and
• \(\tilde{\phi}\) can be \(C^{1,1}\)-smoothly extended to a nonnegative function on \(M'\).

**Theorem 1.4.** Let \((M, J, \omega)\) be a compact Hermitian manifold with smooth mean pseudo-concave boundary. Let \(\varphi \in C^{2,1}(\partial M)\) and \(\tilde{\phi} = \phi^{1/(n-1)}\) be as in (1.3) for \(\phi \geq 0\). Assume the data \(\tilde{\phi} \in C^{1,1}(\bar{M})\) satisfies (1.8) and suppose a \(C^{2,1}\) \((n-1)\)-PSH strict subsolution subject to (1.6). Then

\[\Omega^n_u = \tilde{\phi} \omega^n\]

possesses an \((n-1)\)-PSH weak solution \(u \in C^{1,\alpha}(\bar{M})\) with \(\forall 0 < \alpha < 1, u|_{\partial \bar{M}} = \varphi, \Delta u \in L^\infty(\bar{M})\).

The existence of subsolution is imposed as an important ingredient to study the Dirichlet problem, subsequent to the works of [13, 10] concerning Mong-Ampère equations on domains. It would be worthwhile to note that the subsolution assumption is a necessary condition for the solvability of Dirichlet problem.

Let \((X, J_X, \omega_X)\) be a closed balanced manifold (introduced by Michelsohn [22]), let \((S, J_S, \omega_S)\) be a compact Riemann surface with boundary \(\partial S\). We confirm the subsolution assumption on the standard product

\[(M, J, \omega) = (X \times S, J, \pi_1^* \omega_X + \pi_2^* \omega_S),\]  

(1.9)

when the boundary data \(\varphi\) can be extended to a \(C^{2,1}\)-smooth functions on \(\bar{M}\), still denoted by \(\varphi\), satisfying

\[\tilde{\chi} + \frac{1}{n-1} \left(\Delta \varphi \omega - \sqrt{-1} \partial \bar{\partial} \varphi\right) + Z[\varphi] + t\pi_1^* \omega_X > 0\]  

for some \(t \gg 1\).

Here \(\pi_1, \pi_2\) denote the nature projections:

\[\pi_1 : X \times S \to X, \quad \pi_2 : X \times S \to S.\]

The construction starts with the solution of

\[\Delta_S h = 1 \text{ in } S, \quad h = 0 \text{ on } \partial S.\]  

(1.11)

According to Lemma 6.1 below, the obstruction to constructing subsolutions vanishes whenever \(\omega_X\) is balanced. We can verify

\[*(\sqrt{-1} \partial \bar{\partial} h \wedge \omega^{n-2}) = (n-2)! (\Delta h \omega - \sqrt{-1} \partial \bar{\partial} h) = (n-2)! \pi_1^* \omega_X.\]
The \((n - 1)\)-PSH subsolution is given by
\[
\overline{u} = \varphi + th \quad \text{for } t \gg 1.
\]

As a consequence, we can solve Dirichlet problem (1.1) and (1.4) on such products. In addition, combining with Propositions 4.1 and 5.3, we obtain some more delicate results with less regularity assumption.

**Theorem 1.5.** Let \(0 < \beta < 1\), and let \((M, J, \omega) = (X \times S, J, \pi_X^* \omega_X + \pi_S^* \omega_S)\) be as above with \(\partial S \in C^{2\beta}\). For any \(0 \leq \tilde{\phi} \in C^{1,1}(\bar{M})\) satisfying (1.8), the Dirichlet problem
\[
\Omega^n_u = \tilde{\phi} \omega^n \quad \text{in } M, \quad u = 0 \quad \text{on } \partial M
\]
admits a \(C^{1,\alpha}~(n - 1)\)-PSH solution with \(\forall 0 < \alpha < 1, \Delta u \in L^\infty(\bar{M})\) in the weak sense.

**Theorem 1.6.** Let \((M, J, \omega) = (X \times S, J, \pi_X^* \omega_X + \pi_S^* \omega_S)\) be as above with \(\partial S \in C^{2\beta}\) for some \(0 < \beta < 1\). Given a \(0 < \tilde{\phi} \in C^2(\bar{M})\), there is a unique \(C^{2,\alpha}\)-smooth \((n - 1)\)-PSH function for some \(0 < \alpha \leq \beta\) solving (1.12).

Finally, we make some remarks:

1. Our results are valid when replacing \(\tilde{\chi} = \frac{1}{(n-1)!} \ast \omega_0^{n-1}\) by more general real \((1, 1)\)-forms \(\tilde{\chi}\).

2. It is still open to derive gradient estimate directly for Monge-Ampère equation for \((n - 1)\)-PSH functions. Our strategy is to derive a quantitative version of boundary estimate
\[
\sup_{\partial M} \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right).
\]
This is precisely the estimate we prove in Theorem 2.2. In the case of \(Z = 0\) such a boundary estimate was derived by the author in [33]. It would be worthwhile to note that when \(Z \neq 0\) the proof of quantitative boundary estimate is much more complicated and fairly difficult, due to the gradient terms from \(Z\). It requires some new ideas and insights.

3. We prove in Theorem 1.4 the existence of \(C^{1,\alpha}\)-smooth solution to \(\Omega^n_u = \tilde{\phi} \omega^n\) with the assumption (1.8) but without restriction to the dimension. That is different from the works of Guan [14] and Guan-Li [15] on certain degenerate real Monge-Ampère type equations, in which the assumptions were only completely confirmed in dimensions 2 and 3.
(4) In Theorems 1.5-1.6 we find a significant phenomenon on weakening regularity assumptions on boundary. For the Dirichlet problem with homogeneous boundary data on $M = X \times S$, the regularity assumption on boundary can be weakened to $C^{2,\beta}$; while such $C^{2,\beta}$ regularity assumption is impossible even for Dirichlet problem of nondegenerate real Monge-Ampère equation on certain bounded domains $\Omega \subset \mathbb{R}^2$, as shown by Wang [30], the optimal regularity assumptions on the boundary and boundary data are both $C^3$-smooth. More results with less regularity assumption are established in Section 6.

(5) When $\omega_0$ is a balanced metric and $\omega$ is astheno-Kähler, one obtains a $d$-closed $(n - 1, n - 1)$-form
\[
\omega_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \omega^{n-2} + 2 \Re(\sqrt{-1} \partial \bar{\partial} u \wedge \partial \omega^{n-2}),
\]
the Form-type Calabi-Yau equation introduced in [7] thus falls into a Monge-Ampère type equations analogous to (1.1). By the same argument, we obtain a balanced metric $\Psi_\omega$ with prescribed volume form
\[
\Psi_\omega^n = \tilde{\phi} \omega^n
\]
with the boundary value (1.4); similarly, we can construct the subsolutions on the products as we constructed in (1.9). It was shown by Latorre-Ugarte [20] that there are closed complex manifolds $X$ admitting a non-Kähler balanced metric $\omega_{X,0}$ and a non-Kähler metric $\omega_X$ simultaneously being astheno-Kähler and Gauduchon. On that product $X \times S$ one obtains a balanced metric $\omega_0 = \pi_1^* \omega_{X,0} + \pi_2^* \omega_S$ and an astheno-Kähler metric $\omega = \pi_1^* \omega_X + \pi_2^* \omega_S$. As a result, we can solve the Dirichlet problem for Form-type Calabi-Yau equation on such products.

The paper is organized as follows. In Section 2 we sketch the proof of main theorems. In Section 3 we summarize some lemmas. In Sections 4 and 5 we derive the quantitative boundary estimate, which is the main part of this paper. In Section 6 we also use our estimate to derive more delicate results for Dirichlet problem with less regularity assumptions. In Section 7 we briefly extend our main results to more general equations. In Appendix A we summarize and give the proof of a quantitative lemma proposed in [32], which is a crucial ingredient for quantitative boundary estimate.
2 Sketch of the proof

It is mysterious to derive gradient estimate for equation (1.1) directly, as done by Hanani [17], Błocki [1] and Guan-Li [12] for complex Monge-Ampère equation. In [27, Section 3], Székelyhidi-Tosatti-Weinkove derived gradient estimate for (1.1) on closed Hermitian manifolds via a blow-up argument, establishing the second order estimate of the form

$$\sup_M \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right)$$

left open by Tosatti-Weinkove [29]. Such a blow-up argument using (2.1) appeared in previous works as done by Chen [4], complemented by [2, 23], for Dirichlet problem of complex Monge-Ampère equation, and by Dinew-Kołodziej [6] based on Hou-Ma-Wu’s second estimate [18] for complex k-Hessian equations on closed Kähler manifolds. An extensive extension was obtained by Székelyhidi [26] on closed manifolds, and by the author [33] for the Dirichlet problem.

When $\partial M \neq \emptyset$, Székelyhidi-Tosatti-Weinkove’s estimate yields the following:

**Theorem 2.1** ([27]). Let $u \in C^4(M) \cap C^2(\bar{M})$ be a $(n-1)$-PSH solution to (1.1), then

$$\sup_M \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 + \sup_{\partial M} |\Delta u| \right)$$

where $C$ depends on $|\phi^{1/n}|_{C^2(\bar{M})}$ and other known data but not on $(\inf_M \phi)^{-1}$.

It only requires to prove quantitative boundary estimate.

**Theorem 2.2.** Under the assumptions of Theorem 1.2, any strictly $(n-1)$-PSH solution $u \in C^3(M) \cap C^2(\bar{M})$ to the Dirichlet problem (1.1) and (1.4) satisfies

$$\sup_{\partial M} \Delta u \leq C \left( 1 + \sup_M |\nabla u|^2 \right),$$

where $C$ depends on $|\phi^{1/n}|_{C^1(\bar{M})}$, $|\phi|_{C^3(\bar{M})}$, $|u|_{C^2(\bar{M})}$, $\partial M$ up to third derivatives and other known data.

Furthermore, $C$ is independent of $(\inf_M \phi)^{-1}$ if $\partial M$ is mean pseudoconcave.

With Theorems 2.2 and 2.1 at hand, we derive (2.1) and then establish gradient estimate via the blow-up argument. A somewhat remarkable fact to us is that we can prove the quantitative boundary estimate under a weaker assumption $\bar{\phi} = \phi^{1/(n-1)} \in C^{1,1}(\bar{M})$. We observe in Proposition 5.5 that Székelyhidi-Tosatti-Weinkove’s original proof for second estimate also works under such a weak assumption. As a consequence we obtain Theorem 1.4.
3 Preliminaries and lemmas

3.1 Background of the equation

The equation (1.1) can be reformulated in the following form
\[
\log P_{n-1}(\lambda(\tilde{g}[u])) = \psi \quad \text{and} \quad \lambda(\tilde{g}[u]) \in \mathcal{P}_{n-1}.
\]
(3.1)

Here
\[
P_{n-1}(\lambda) = \mu_1 \cdots \mu_n, \quad \mu_i = \sum_{j \neq i} \lambda_j.
\]
(3.2)
\[
\mathcal{P}_{n-1} = \{ \lambda \in \mathbb{R}^n : \mu_i > 0, \ \forall 1 \leq i \leq n \},
\]
(3.3)
and \( \psi = \log \phi + n \log(n - 1) \),
\[
\tilde{g}_{ij} = u_{ij} + \tilde{\chi}_{ij} + W_{ij},
\]
(3.4)
where \( \tilde{\chi}_{ij} = (\text{tr}_o \tilde{\chi}) g_{ij} - (n - 1) \lambda_i \), \( W_{ij} = (\text{tr}_o Z) g_{ij} - (n - 1)Z_{ij} \),
\[
Z_{ij} = \frac{g^{pq} T_{ij}^{lp} g_{ij} u_p + g^{pq} T_{kp}^{sj} u_q - g^{kl} g_{ij} T_{lj}^{pq} u_k - g^{kl} g_{ij} T_{lj}^{pq} u_l - T_{ij}^{pq} u_{ij} - T_{ij}^{pq} u_{ij} }{2(n - 1)},
\]
(3.5)
where \( T_{ij}^k = g^{k} \left( \frac{\partial g_{ij}}{\partial z_i} - \frac{\partial g_{ij}}{\partial z_j} \right) \), see also [27].

For simplicity we denote
\[
f(\lambda) = \log P_{n-1}(\lambda),
\]
(3.6)
For any \( \lambda \in \mathcal{P}_{n-1} \) we have the following simple properties:
\[
f_i(\lambda) = \sum_{j \neq i} \frac{1}{\mu_j}, \quad \sum_{i=1}^{n} f_i(\lambda) = (n - 1) \sum_{i=1}^{n} \frac{1}{\mu_i},
\]
(3.7)
\[
f_i(\lambda) \geq \frac{\min_{k} \mu_k}{n \max_{k} \mu_k} \sum_{j=1}^{n} f_j(\lambda), \quad \forall i,
\]
(3.8)
\[
\sum_{i=1}^{n} f_i(\lambda) \lambda_i = n, \quad \sum_{i=1}^{n} f_i(\lambda) \lambda_i = \sum_{i=1}^{n} \frac{\mu_i}{\mu_i},
\]
(3.9)
Moreover
\[
\sum_{i=1}^{n} f_i(\lambda) \geq n(n - 1)e^{-\sigma/n}, \quad \text{for} \ f(\lambda) = \sigma,
\]
(3.10)
Lemma 3.1. Let $\lambda \in P_{n-1}$ and $\varepsilon = \inf_M \min_j \mu_j$ (then $\varepsilon > 0$). Suppose that

$$\min_j \mu_j \leq \frac{\varepsilon}{2n}. \quad (3.11)$$

Then

$$\sum_{i=1}^n \frac{1}{\mu_i} \geq \frac{2n}{\varepsilon}, \quad (3.12)$$

$$\sum_{i=1}^n f_i(\lambda)(\lambda_i - \lambda) \geq \frac{\varepsilon}{2} \sum_{i=1}^n \frac{1}{\mu_i} = \frac{\varepsilon}{2(n-1)} \sum_{i=1}^n f_i(\lambda). \quad (3.13)$$

Remark 3.2. It only requires to consider the case when (3.11) holds. Otherwise for $\sum_{i=1}^n \log \mu_i = \sigma$, we have

$$\max_i \mu_i \leq (2n)^{n-1} e^{\sigma}/\varepsilon^{n-1}, \quad (3.14)$$

and

$$\frac{\min_i \mu_i}{\max_i \mu_i} \geq \frac{\varepsilon^n}{(2n)^{n} e^{n\sigma}}. \quad (3.15)$$

Throughout this paper we denote

$\tilde{a} = \tilde{a}[u], \tilde{g} = \tilde{g}[u], Z = Z[u], W = W[u], W = W[u]\,$

for solution $u$ and subsolution $\underline{u}$. In Sections 4 and 5 we denote

$$\lambda = \lambda(\tilde{a}), \underline{\lambda} = \lambda(\tilde{g}). \quad (3.16)$$

3.2 Some computation and notation

Fix $x_0 \in \partial M$ we can choose a local holomorphic coordinate system

$$(z_1, \ldots, z_n), \quad z_i = x_i + \sqrt{-1}y_i, \quad (3.17)$$

centered at $x_0$, such that $g_{ij}(0) = \delta_{ij}, \frac{\partial}{\partial z_\alpha}$ is the inner normal vector at the origin, and $T^{1,0}_{x_0, \partial M}$ is spanned by $\frac{\partial}{\partial z_\alpha}$ for $1 \leq \alpha \leq n-1$. We denote $\sigma(z)$ the distance function from $z$ to $\partial M$ with respect to $\omega$. Also we denote

$$\Omega_\delta := \{ z \in M : |z| < \delta \}, \quad M_\delta := \{ z \in M : \sigma(z) < \delta \}. \quad (3.18)$$
In the computation we use derivatives with respect to Chern connection $\nabla$ of $\omega$, and write $\partial_i = \frac{\partial}{\partial z_i}$, $\bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i}$, $\nabla_i = \nabla \partial_i$, $\nabla_i = \nabla \bar{\partial}_i$.

For a smooth function $v$,

$$v_i := \partial_i v, \quad v_i := \bar{\partial}_i v, \quad v_{ij} := \nabla_j v = \partial_i \bar{\partial}_j v - \Gamma^k_{ij} v_k, \cdots$$

where $\Gamma^k_{ij}$ are the Christoffel symbols defined by $\nabla_{\partial_i} \frac{\partial}{\partial z_j} = \Gamma^k_{ij} \frac{\partial}{\partial z_k}$.

The boundary value condition implies

$$u_\alpha(0) = \bar{u}_\alpha(0), \quad u_{\alpha\beta}(0) = u_{\alpha\beta}(0) + (u - \bar{u})_{x_\alpha}(0)\sigma_{\alpha\beta}(0) \quad (3.19)$$

for $1 \leq \alpha, \beta \leq n - 1$. Let $\bar{u}$ be the solution to

$$\text{tr}_\omega(\bar{g}[\bar{u}]) = 0 \text{ in } M, \quad \bar{u} = \varphi \text{ on } \partial M. \quad (3.20)$$

The existence of $\bar{u}$ follows from standard theory of elliptic equations. By the maximum principle and boundary value condition, one derives the following:

**Lemma 3.3.**

$$u \leq \bar{u} \leq u \text{ in } M, \quad 0 \leq (u - \bar{u})_{x_\alpha}(0) \leq (\bar{u} - u)_{x_\alpha}(0). \quad (3.21)$$

In particular,

$$\sup_M |u| + \sup_{\partial M} |\nabla u| \leq C. \quad (3.22)$$

By $(3.5)$, $(3.19)$ and $W[v]_{ij} = (\text{tr}_\omega Z[v])_g_{ij} - (n - 1)Z[v]_{ij}$, one can verify the following:

**Lemma 3.4.** At the origin ($\{z = 0\}$),

$$\sum_{a=1}^{n-1} \bar{g}[v]_{a\bar{a}} = \sum_{a=1}^{n-1} (v_{a\bar{a}} + \bar{\chi}_{a\bar{a}}) + \sum_{a=1}^{n-1} W[v]_{a\bar{a}}, \quad (3.23)$$

$$2(n - 1)Z[v]_{a\bar{a}} = \sum_{a,b=1}^{n-1} (\bar{T}^\beta_\alpha v_\alpha + T^\beta_\alpha v_{\bar{\alpha}}), \quad (3.24)$$

$$\sum_{a=1}^{n-1} \bar{g}_{a\bar{a}} = \sum_{a=1}^{n-1} \bar{g}_{a\bar{a}} + (u - \bar{u})_{x_\alpha} \sum_{a=1}^{n-1} \sigma_{a\bar{a}}. \quad (3.25)$$

If in addition we take

$$W[v]_{ij} = W^k_{ij} v_k + W^k_{ij} v_{\bar{k}}, \quad (3.26)$$

then at the origin

$$\sum_{a=1}^{n-1} W^a_{a\bar{a}} = \sum_{a=1}^{n-1} W^a_{a\bar{a}} = 0. \quad (3.27)$$
4 Quantitative boundary estimate for pure normal derivative

The goal of this section is to derive a quantitative version of boundary estimate for pure normal derivative. Before stating it, we denote an orthonormal basis of $T_{\partial M}^{1,0} := T_{\partial M}^{1,0} \cap T_{\partial M}^C$ by

$$\xi_1, \cdots, \xi_{n-1}. \quad (4.1)$$

As above, $\nu$ denotes the unit inner normal vector along the boundary. Let

$$\xi_n = \frac{1}{\sqrt{2}} \left( \nu - \sqrt{-1} J \nu \right). \quad (4.2)$$

**Proposition 4.1.** Let $(M, J, \omega)$ be a compact Hermitian manifold with $C^3$ boundary. In addition we assume (1.5) is satisfied. For any strictly $(n-1)$-PSH solution $u \in C^2(M)$ to the Dirichlet problem (1.1) and (1.4), then there is a uniform positive constant $C$ depending on $|u|_{C^0(M)}$, $|\nabla u|_{C^0(\partial M)}$, $\sup_M \phi$, $|u|_{C^2(\partial M)}$, $\partial M$ up to third derivatives and other known data, such that

$$\text{tr}_\omega(\tilde{g})(x_0) \leq C \left( 1 + \sum_{\alpha=1}^{n-1} |\tilde{g}(\xi_\alpha, J\xi_\alpha)(x_0)|^2 \right), \quad \forall x_0 \in \partial M. \quad (4.3)$$

Moreover, if $\partial M$ is mean pseudoconcave then the constant $C$ is independent of $(\inf_M \phi)^{-1}$ and only depends on $\partial M$ up to second derivatives and other known data under control.

4.1 First ingredient and its proof

Let $\eta = (u - u)_x(0)$. We know that $\eta \geq 0$. Let

$$t_0 = -\eta \sum_{\alpha=1}^{n-1} \sigma_{\alpha\alpha}(0) / \sum_{\alpha=1}^{n-1} \tilde{\eta}_{\alpha\alpha}(0). \quad (4.4)$$

Obviously, $t_0 < 1$. From (3.25), we have at the origin

$$\sum_{\alpha=1}^{n-1} \tilde{g}_{\alpha\alpha} = (1 - t_0) \sum_{\alpha=1}^{n-1} \tilde{g}_{\alpha\alpha}. \quad (4.5)$$
Lemma 4.2. There is a uniform positive constant $C$ depending on $(1 - t_0)^{-1}$, $\text{sup}_M \phi$ and other known data such that

$$\text{tr}_\omega(\tilde{g}) \leq C \left(1 + \sum_{\alpha = 1}^{n-1} |\tilde{g}_{\alpha\bar{\alpha}}|^2\right).$$

Proof. The argument is based on Lemma A.1 proposed in [32] (also see [33]). Around $x_0$ we use the local holomorphic coordinates that we have chosen in (3.17); furthermore, we assume that $(\tilde{g}_{\alpha\bar{\beta}})$ is diagonal at the origin $x_0 = \{z = 0\}$. In the proof the discussion is done at the origin, and the Greek letters, such as $\alpha, \beta$, range from 1 to $n-1$. Let’s denote

$$\bar{A}(R) = \begin{pmatrix}
R - \tilde{g}_{11} & \cdots & -\tilde{g}_{1\bar{n}} \\
\vdots & \ddots & \vdots \\
-\tilde{g}_{n1} & \cdots & R - \tilde{g}_{n(n-1)} - \tilde{g}_{n\bar{n}} \sum_{\alpha=1}^{n-1} \tilde{g}_{\alpha\bar{\alpha}}
\end{pmatrix},$$

and

$$\bar{A}(R) = \begin{pmatrix}
R - \tilde{g}_{11} & \cdots & -\tilde{g}_{1\bar{n}} \\
\vdots & \ddots & \vdots \\
-\tilde{g}_{n1} & \cdots & R - \tilde{g}_{n(n-1)} - \tilde{g}_{n\bar{n}} \sum_{\alpha=1}^{n-1} \tilde{g}_{\alpha\bar{\alpha}} - (1 - t_0) \sum_{\alpha=1}^{n-1} \tilde{g}_{\alpha\bar{\alpha}}
\end{pmatrix}.$$

In particular, when $R = \text{tr}_\omega(\tilde{g})$, $\bar{A}(R) = \text{tr}_\omega(\tilde{g}) \omega - \tilde{g}$. By (3.25) and (4.4),

$$\bar{A}(R) = \bar{A}(R).$$

One can see that there is a uniform positive constant $R_0$ depending on $(1 - t_0)^{-1}$ and $(\text{inf}_M \text{dist}(\lambda, \partial \Gamma_n))^{-1}$ (but not on $(\text{inf}_M \phi)^{-1}$) such that

$$f \left(R_0, \cdots, R_0, (1 - t_0) \sum_{\alpha=1}^{n-1} \tilde{g}_{\alpha\bar{\alpha}}\right) > \psi.$$

Therefore, there is a positive constant $\epsilon_0$, depending on $\text{inf}_M \text{dist}(\lambda, \partial \Gamma_n)$, such that

$$\begin{cases}
f \left(R_0 - \epsilon_0, \cdots, R_0 - \epsilon_0, (1 - t_0) \sum_{\alpha=1}^{n-1} \tilde{g}_{\alpha\bar{\alpha}} - \epsilon_0\right) \geq \psi, \\
R_0 > \epsilon_0, \ (1 - t_0) \sum_{\alpha=1}^{n-1} \tilde{g}_{\alpha\bar{\alpha}} > \epsilon_0.
\end{cases} \tag{4.6}$$
Note that 
\[ \tilde{A}(R) = \tilde{A}(R) = RI_n = \begin{pmatrix}
\tilde{g}_{11} & \cdots & \tilde{g}_{1n} \\
\vdots & \ddots & \vdots \\
\tilde{g}_{n1} & \cdots & \tilde{g}_{nn}
\end{pmatrix} \]

here \( I_n = (\delta_{ij}) \). Let’s pick \( \epsilon = \frac{n(1-t_0)}{2(n-1)} \) in Lemma A.1 then set
\[ R_s = \frac{2(n-1)(2n-3)}{\epsilon_0(1-t_0)} \sum_{a=1}^{n-1} |\tilde{g}_{\alpha \bar{\alpha}}|^2 + (n-1) \sum_{a=1}^{n-1} |\tilde{g}_{\alpha \bar{\alpha}}| + (1-t_0) \sum_{a=1}^{n-1} |\tilde{g}_{\alpha \bar{\alpha}}| + R_0, \]

where \( \epsilon_0 \) and \( R_0 \) are constants from (4.6). Let \( \lambda(\tilde{A}(R_s)) = (\lambda_1(R_s), \cdots, \lambda_n(R_s)) \) denote the eigenvalues of \( \tilde{A}(R_s) \). According to Lemma A.1,
\[ \lambda_\alpha(R_s) \geq R_s - \tilde{g}_{\alpha \bar{\alpha}} - \frac{\epsilon_0}{2(n-1)}, \quad \forall 1 \leq \alpha \leq n-1, \]
\[ \lambda_n(R_s) \geq (1-t_0) \sum_{a=1}^{n-1} |\tilde{g}_{\alpha \bar{\alpha}}| - \frac{\epsilon_0}{2}. \]

Therefore
\[ f(\lambda(\tilde{A}(R_s))) \geq \psi. \]
We get
\[ \text{tr}_\omega(\tilde{g}) \leq R_s. \]

4.2 Second ingredient and the proof

According to Lemma 4.2 it requires only to prove that \((1-t_0)^{-1}\) can be uniformly bounded from above. That is,
\[ (1-t_0)^{-1} \leq C. \] (4.7)

Case 1: The boundary \( \partial M \) is mean pseudoconcave.

Note \( \eta = (u - \underline{u})_{\nu_0}(0) \geq 0 \). The mean pseudoconcavity of boundary gives
\[ t_0 \leq 0 \]

which automatically implies (4.7).
Case 2: Without mean pseudoconcave restriction to \( \partial M \).

**Lemma 4.3.** The inequality (4.7) holds for a uniform positive constant \( C \) depending on \( (\inf_M \phi)^{-1}, |u|_{C^0(\tilde{M})}, |\nabla u|_{C^0(\tilde{M})}, |\tilde{u}|_{C^1(\tilde{M})}, \partial M \) up to third derivatives and other known data.

We assume throughout \( \eta > 0 \); otherwise \( t_0 = 0 \) and we have done. From \( \lambda(\tilde{\eta}) \in P_{n-1} \) we know \( \sum_{\alpha=1}^{n-1} \tilde{g}_{n\alpha}(0) > 0 \). In what follows we assume

\[
t_0 > \frac{1}{2}.
\]

Since \( \eta \) has a uniform upper bound, thus at origin

\[
- \sum_{\alpha=1}^{n-1} \sigma_{n\alpha}(0) \geq t_0 \sum_{\alpha=1}^{n-1} \tilde{\sigma}_{n\alpha}(0)/\eta \geq \sum_{\alpha=1}^{n-1} \tilde{\sigma}_{n\alpha}(0)/2\eta \geq a_2 \quad (4.8)
\]

where \( a_2 = \inf_{z \in \partial M} \sum_{\alpha=1}^{n-1} \tilde{g}_{n\alpha}(z)/2 \sup_{\partial M} |\nabla (u - \tilde{w})| \).

Let \( \Omega_\delta \) be as in (3.18). Inspired by an idea of Caffarelli-Nirenberg-Spruck [3] (see also [21]), we set on \( \Omega_\delta \)

\[
d(z) = \sigma(z) + \tau |z|^2
\]

where \( \tau \) is a positive constant to be determined. Let

\[
w(z) = u(z) + (\eta/t_0)\sigma(z) + l(z)\sigma(z) + A\sigma(z)^2. \quad (4.9)
\]

where \( A \) is a positive constant to be determined, and \( l(z) = \sum_{i=1}^{n} (l_i z_i + \bar{l}_i \bar{z}_i) \) where \( l_i \in \mathbb{C}, \bar{l}_i = l_i \) to be chosen as in (4.12).

Let \( T_1(z), \ldots, T_{n-1}(z) \) be an orthonormal basis for holomorphic tangent space of level hypersurface \( \{ w : d(w) = d(z) \} \) at \( z \), so that at the origin \( T_\alpha(0) = \frac{\partial}{\partial x_\alpha} \) for each \( 1 \leq \alpha \leq n-1 \). Furthermore, let \( T_\alpha = \frac{\partial}{\partial x_\alpha} \).

Such a basis exists: We see at the origin \( \partial d(0) = \partial \sigma(0) \). Thus for \( 1 \leq \alpha \leq n-1 \), we can choose \( T_\alpha \) such that at the origin \( T_\alpha(0) = \frac{\partial}{\partial x_\alpha} \).

For a real \((1, 1)\)-form \( \Theta = -1\Theta_{ij}dz_i \wedge d\bar{z}_j \), we denote by \( \lambda(\Theta) \) the eigenvalues of \( \Theta \) (with respect to \( \omega \)) with order \( \lambda_1(\Theta) \leq \cdots \leq \lambda_n(\Theta) \). Since \( \lambda_n(\Theta) \geq \Theta(T_n, J\tilde{T}_n) \), one has

\[
\sum_{\alpha=1}^{n-1} \lambda_\alpha(\Theta) \leq \sum_{\alpha=1}^{n-1} \Theta(T_\alpha, J\tilde{T}_\alpha). \quad (4.10)
\]

We then define \( \Lambda(\Theta) := \sum_{\alpha=1}^{n-1} T_\alpha^t \tilde{T}_\alpha \Theta_{ij} \) for \( \Theta = -1\Theta_{ij}dz_i \wedge d\bar{z}_j \).

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Lemma 4.4. There are \( \tau, \delta, A \) and \( l(z) \) depending on \( |u|_{C^3(M)} \), \( |\nabla u|_{C^3(\partial M)} \), \( |u|_{C^2(M)} \), \( \partial M \) up to third derivatives and other known data so that

\[
\Lambda(\tilde{g}[w]) \leq 0 \text{ in } \Omega_0, \quad u \leq w \text{ on } \partial \Omega_0.
\]

Proof. In the proof, Lemma 3.4 plays key roles in treating the gradient terms from \( Z \) in equation. Direct computations give

\[
w_i = u_i + \frac{\eta}{t_0} \sigma_i + l_i \sigma + l(z) \sigma_i + 2A d_i,
\]

\[
w_{ij} = u_{ij} + \frac{\eta}{t_0} \sigma_{ij} + l(z) \sigma_{ij} + (l_i \sigma_j + \sigma_i l_j) + 2A d_{ij} + 2A d_i d_j.
\]

Then

\[
\Lambda(\tilde{g}[w]) = \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}((\tilde{\chi}_{ij} + \tilde{w}_{ij} + W_{ij}^{p}w_p + W_{ij}^{q}w_q + \frac{\eta}{t_0} \sigma_{ij})
\]

\[
= \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}(\tilde{\chi}_{ij} + u_{ij} + W_{ij}^{p}u_p + W_{ij}^{q}u_q + \frac{\eta}{t_0} \sigma_{ij})
\]

\[
+ l(z) \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}\sigma_{ij} + \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}(\sigma_i l_j + l_i \sigma_j)
\]

\[
+ 2A d(z) \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}d_{ij} + \frac{\eta}{t_0} \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}(W_{ij}^{p} \sigma_p + W_{ij}^{q} \sigma_q)
\]

\[
+ l(z) \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}(W_{ij}^{p} \sigma_p + W_{ij}^{q} \sigma_q) + \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}(W_{ij}^{p} l_p + W_{ij}^{q} l_q)\sigma
\]

\[
+ 2A d(z) \sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}(W_{ij}^{p} d_p + W_{ij}^{q} d_q).
\]

- At origin \( z = 0 \), \( T_{a}^{i} = \delta_{ai}\)

\[
\sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}(\tilde{\chi}_{ij} + u_{ij} + W_{ij}^{p} u_p + W_{ij}^{q} u_q + \frac{\eta}{t_0} \sigma_{ij})(0)
\]

\[
= \sum_{a=1}^{n-1} \tilde{g}_{ai}(0) + \frac{\eta}{t_0} \sum_{a=1}^{n-1} \sigma_{ai}(0) = 0.
\]

Thus there are complex constants \( k_i \) such that on \( \Omega_{\sigma} \),

\[
\sum_{a=1}^{n-1} T_{a}^{i} \tilde{t}_{a}^{j}(\tilde{\chi}_{ij} + u_{ij} + W_{ij}^{p} u_p + W_{ij}^{q} u_q + \frac{\eta}{t_0} \sigma_{ij}) = \sum_{i=1}^{n} (k_i z_i + \tilde{k}_i \bar{z}_i) + O(|z|^2).
\]
Next, we see

\[ 2Ad(z) \sum_{a=1}^{n-1} T^i_a \bar{T}^j_a d_{ij} \leq -\frac{Aa^2 d(z)}{2}, \]

provided \( 0 < \delta, \tau \ll 1 \). Here we use

\[
\sum_{a=1}^{n-1} T^i_a \bar{T}^j_a d_{ij} = (\sum_{a=1}^{n-1} T^i_a \bar{T}^j_a - \sum_{a=1}^{n-1} T^i_a \bar{T}^j_a(0))d_{ij} + \sum_{a=1}^{n-1} \sigma_{a\dot{a}}(z) + (n - 1)\tau
\]

\[
\leq (n - 1)\tau - a_2 + O(|z|) \leq -\frac{a_2}{4}
\]

by \( \sum_{a=1}^{n-1} \sigma_{a\dot{a}}(z) = \sum_{a=1}^{n-1} \sigma_{a\dot{a}}(0) + O(|z|), \ (4.8) \) and

\[
\sum_{a=1}^{n-1} T^i_a \bar{T}^j_a(z) = \sum_{a=1}^{n-1} T^i_a \bar{T}^j_a(0) + O(|z|). \quad (4.11)
\]

At the origin,

\[
\sum_{a=1}^{n} T^i_a \bar{T}^j_a(W^p_{ij}\sigma_p + W^\dot{q}_{ij}\sigma_{\dot{q}})(0)
\]

\[
= \sum_{a=1}^{n-1} (W^a_{\alpha\dot{a}}\sigma_{\beta} + W^\dot{a}_{\alpha\dot{a}}\sigma_{\beta})(0) + \sum_{a=1}^{n-1} (W^a_{\alpha\dot{a}}\sigma_n + W^\dot{a}_{\alpha\dot{a}}\sigma_n)(0) = 0,
\]
since \( \sigma_\beta(0) = 0 \), and by (3.27)
\[
\sum_{a=1}^{n-1} W_{a\alpha}^\beta(0) = 0, \sum_{a=1}^{n-1} W_{a\alpha}^\theta(0) = 0.
\]
Thus on \( \Omega_\alpha \),
\[
l(z) \sum_{a=1}^{n-1} T^i_a \bar{T}^i_a(W_{ij}^p \sigma_p + W_{ij}^q \sigma_q)(z) = O(|z|^2),
\]
and there are complex constants \( m_i \) such that
\[
\frac{n}{t_0} \sum_{a=1}^{n-1} T^i_a \bar{T}^i_a(W_{ij}^p \sigma_p + W_{ij}^q \sigma_q)(z) = \sum_{i=1}^{n-1} (m_i z_i + \bar{m}_i \bar{z}_i) + O(|z|^2).
\]

• Similarly, \( \sum_{a=1}^{n-1} T^i_a \bar{T}^i_a(W_{ij}^p d_p + W_{ij}^q d_q)(0) = 0 \), thus on \( \Omega_\alpha \),
\[
\sum_{a=1}^{n-1} T^i_a \bar{T}^i_a(W_{ij}^p d_p + W_{ij}^q d_q)(z) = O(|z|)
\]
so
\[
2Ad(z) \sum_{a=1}^{n-1} T^i_a \bar{T}^i_a(W_{ij}^p d_p + W_{ij}^q d_q)(z) = Ad(z)O(|z|).
\]

• Finally
\[
\sum_{a=1}^{n-1} T^i_a \bar{T}^i_a(W_{ij}^p l_p + W_{ij}^q l_q)\sigma(z) \leq C_1 \sigma(z).
\]
Therefore, we get
\[
\Lambda(\bar{\beta}[w]) \leq 2\Re \sum_{a=1}^{n-1} \left[ z_a \left( k_a + m_a + l_a \left( \sum_{\beta=1}^{n-1} \sigma_{\beta\beta}(0) - \tau \right) \right) \right] + 2\Re \left[ z_n \left( k_n + m_n + \sum_{\beta=1}^{n-1} \sigma_{\beta\beta}(0) \right) \right] - \frac{A_2 Ad(z)}{2} + Ad(z)O(|z|) + C_1 \sigma(z) + O(|z|^3).
\]
We complete the proof if \( 0 < \tau, \delta \ll 1, A \gg 1 \), and we set
\[
l_a = -\frac{k_a + m_a}{\sum_{\beta=1}^{n-1} \sigma_{\beta\beta}(0) - \tau} \text{ for } 1 \leq \alpha \leq n - 1, \quad l_n = \frac{k_n + m_n}{\sum_{\beta=1}^{n-1} \sigma_{\beta\beta}(0)}.
\]
We can see each \(|l_i|\) is uniformly bounded, since \(\sum_{\beta=1}^{n-1} \sigma_{\beta\beta}(0) \leq -a_2 < 0\).

Furthermore, on \(\partial M \cap \Omega_\delta\), \(u(z) - w(z) = -A\tau^2|z|^4\). On \(M \cap \partial \Omega_\delta\),

\[
u(z) - w(z) \leq |u - u| c_{c_0} - (2A\tau\delta^2 + \frac{\eta}{t_0} - 2n \sup_i |l_i|)\sigma(z) = -A\tau^2\delta^4
\]

provided \(A \gg 1\).

\(\Box\)

4.3 Completion of proof of Lemma 4.3

Let \(w\) be the function as in Lemma 4.4. From the construction above, we know that there is a uniform positive constant \(C_0\) such that

\[
\sup_M |\tilde{g}[w]| \leq C_0.
\]

Let \(\lambda(\tilde{g}[w]) = (\lambda_1[w], \ldots, \lambda_n[w])\), let \(\mu_i[w] = \sum_{j \neq i} \lambda_j[w]\) and we assume \(\lambda_1[w] \leq \cdots \leq \lambda_n[w]\). Denote by

\[
\mathcal{P}_{n-1}^{\inf_M, \psi} = \left\{ \lambda \in \mathcal{P}_{n-1} : \sum_{i=1}^n \log \mu_i \geq \inf_M \psi \right\}.
\]

Near the origin \(x_0 = \{z = 0\}\), there are complex valued constants \(b_{ij}, a_{ij}\) with \(\bar{a}_{ij} = a_{ji}\) such that

\[
\sigma(z) = x_n + \sum_{i,j=1}^n a_{ij}z_i\bar{z}_j + \Re \sum_{i,j=1}^n b_{ij}z_i\bar{z}_j + O(|z|^3).
\]

One can choose a positive constant \(C'\) such that \(x_n \leq C'|z|^2\) on \(\partial M \cap \Omega_\delta\), there is a positive constant \(C_2\) depending only on \(M\) and \(\delta\) so that

\[
x_n \leq C_2|z|^2 \text{ on } \partial \Omega_{\sigma}.
\]

Let \(C_2\) be as above we set \(h(z) = w(z) + \epsilon(|z|^2 - \frac{2n}{C_2})\). Thus

\[
u \leq h \text{ on } \partial \Omega_\delta.
\]

Lemma 4.4 and (4.10) give

\[
\sum_{a=1}^{a=n-1} \lambda_a[w] \leq 0 \text{ in } \Omega_\delta.
\]
That is, in $\Omega_\delta$, 
$$
\lambda[w] \notin P_{n-1}, \text{ i.e. } \mu[w] \notin \Gamma_n.
$$
In other words, $\lambda[w] \in X$, where
$$
X = \{ \lambda \in \mathbb{R}^n : \lambda \notin P_{n-1} \} \cap \{ \lambda \in \mathbb{R}^n : \|\lambda\| \leq C_0 \}.
$$
Notice $P_{n-1}$ is open so $X$ is a compact subset; moreover $X \cap P_{n-1}^{\inf M} = \emptyset$. So we can deduce that the distance between $P_{n-1}^{\inf M}$ and $X$ is greater than some positive constant depending on $\inf_M \phi$ and other known data. Therefore, there exists an $0 < \epsilon \ll 1$ depending on $\inf_M \phi, \lambda[w], \text{ torsion tensor and other known data}$ such that
$$
\lambda[h] \notin P_{n-1}^{\inf M}.
$$
By [3, Lemma B], we have
$$
u \leq h \text{ in } \Omega_\delta.
$$
Notice $u(0) = \varphi(0)$ and $h(0) = \varphi(0)$, we have $(u - h)_x(0) \leq 0$ then
$$
(1 - t_0)^{-1} \leq 1 + \frac{\eta C_2}{\epsilon}.
$$

5 Quantitative boundary estimate for tangential-normal derivatives

The remaining goal is to derive quantitative boundary estimate for tangential-normal derivatives.

**Proposition 5.1.** Let $(M, J, \omega)$ be a compact Hermitian manifold with $C^3$-smooth boundary. In addition we assume (1.5) holds. Then for any strictly $(n - 1)$-PSH solution $u \in C^3(M) \cap C^2(\bar{M})$ to the Dirichlet problem (1.1) and (1.4), there is a uniform positive constant $C$ depending on $|\varphi|_{C^1(\bar{M})}, |\varphi|^{1/n}_{C^1(\bar{M})}, |\varphi|_{C^3(\bar{M})}, |\nabla u|_{C^0(\partial M)}$, $\partial M$ up to third derivatives and other known data (but neither on $(\inf_M \phi)^{-1}$ nor on $\sup_M |\nabla u|$) such that
$$
|\tilde{g}(\xi_\alpha, J \tilde{\xi}_\alpha)(x_0)| \leq C(1 + \sup_M |\nabla u|), \quad \forall 1 \leq \alpha \leq n - 1, \quad \forall x_0 \in \partial M.
$$

(5.1)
5.1 Tangential operators on the boundary

For a given point \( x_0 \in \partial M \), we choose local holomorphic coordinates (3.17) centered at \( x_0 \) in a neighborhood which we assume to be contained in \( M_\delta \), such that \( x_0 = \{ z = 0 \} \), \( g_{ij}(0) = \delta_{ij} \) and \( \frac{\partial}{\partial x_0} \) is the interior normal direction to \( \partial M \) at \( x_0 \). For convenience we set

\[
t_{2k-1} = x_k, \quad t_{2k} = y_k, \quad 1 \leq k \leq n-1; \quad t_{2n-1} = y_n, \quad t_{2n} = x_n.
\]

We define the tangential operator on \( \partial M \)

\[
T = \nabla_{\frac{\partial}{\partial t}} - \eta \nabla_{\frac{\partial}{\partial x_n}} \quad \text{for each fixed } 1 \leq \alpha < 2n,
\]

where \( \eta = \frac{\partial z}{\partial t} \), \( \sigma_{x_0}(0) = 1 \), \( \sigma_{y_0}(0) = 0 \). One has \( T(u - \varphi) = 0 \) on \( \partial M \cap \bar{\Omega}_\delta \). The boundary value condition also gives for each \( 1 \leq \alpha, \beta < n \)

\[
(u - \varphi)_{t_{ij}}(0) = (u - \varphi)_{x_0}(0)\sigma_{t_{ij}}(0) \quad \forall 1 \leq i, j < 2n.
\]

Let’s turn our attention to the setting of complex manifolds with holomorphically flat boundary. Given \( x_0 \in \partial M \), one can pick local holomorphic coordinates

\[
(z_1, \cdots, z_n), \quad z_i = x_i + \sqrt{-1} y_i,
\]

centered at \( x_0 \) such that \( \partial M \) is locally of the form \( \Re(z_n) = 0 \) and \( g_{ij}(x_0) = \delta_{ij} \). Under the holomorphic coordinate (5.4), we can take

\[
T = D := \frac{\partial}{\partial x_\alpha}, \quad \frac{\partial}{\partial y_\alpha}, \quad 1 \leq \alpha \leq n-1.
\]

It is noteworthy that such local holomorphic coordinate system (5.4) is only needed in the proof of Proposition 5.3. In addition, when \( M = X \times S, \ D = \frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha} \),

where \( z' = (z_1, \cdots, z_{n-1}) \) is a local holomorphic coordinate of \( X \).

For simplicity we denote the tangential operator on \( \partial M \) by

\[
T = \nabla_{\frac{\partial}{\partial \alpha}} - \gamma \eta \nabla_{\frac{\partial}{\partial x_n}}.
\]

Here \( \gamma = 0 \) (i.e. \( T = \nabla_{\frac{\partial}{\partial \alpha}} = D \)) if \( \partial M \) is holomorphically flat, while for general boundary we take \( \gamma = 1 \).

From (4.13) we have \( |\eta| \leq C' |z| \) on \( \Omega_\delta \). The boundary value condition \( (u - \varphi)_{t_{ij}}|_{\partial M} = 0 \) gives \( T(u - \varphi)_{t_{ij}}|_{\partial M} = 0 \). Combining with (3.22), we have

\[
T(u - \varphi) = 0 \quad \text{and} \quad |(u - \varphi)_{t_{ij}}| \leq C |z| \quad \text{on} \quad \partial M \cap \bar{\Omega}_\delta, \quad \forall 1 \leq \alpha < 2n.
\]
5.2 Completion of proof of Proposition 5.1

Let \( F(A) = f(\lambda(A)) \), and we denote
\[
F^{ij}(A) = \frac{\partial F}{\partial a_{ij}}, \quad A = (a_{ij}).
\]

The linearized operator of equation (1.1) at \( u \) is given by
\[
\mathcal{L}v = F^{ij}(\bar{\tilde{g}}[u])(v_{ij} + W^k_{ij}v_k + W^k_{ij}\bar{v}_k),
\]
where \( W^k_{ij} \) and \( W^k_{ij} \) are defined in (3.26). For simplicity, we denote
\[
F^{ij} = F^{ij}(\bar{\tilde{g}}[u]).
\]

We derive quantitative boundary estimates for tangential-normal derivatives by using barrier functions. This type of construction of barrier functions goes back at least to [13, 10]. We shall point out that the constants in proof of quantitative boundary estimates, such as \( C, C_\Phi, C_1, C_2, A_1, A_2, A_3, \) etc, depend on neither \( |\nabla u| \) nor \( (\inf_M \phi)^{-1} \), nor \( |\nabla \psi| \).

By direct calculations, one derives
\[
\begin{align*}
\u_{xi} &= u_{lx} + T^p_{ki}u_p, \quad \u_{yi} = u_{ly} + \sqrt{-1}T^p_{ki}u_p, \\
(u_x)_j &= u_{xj} + \Gamma^l_{kj}u_l, \quad (u_y)_j = u_{yj} - \sqrt{-1}\Gamma^l_{kj}u_l, \\
(u_x)_{ij} &= u_{xij} + \Gamma^l_{ik}u_{ij} + \Gamma^l_{jk}u_{il} - g^{lp}R_{ijkl}u_l, \\
(u_y)_{ij} &= u_{yij} + \sqrt{-1}(\Gamma^l_{ik}u_{ij} - \Gamma^l_{jk}u_{il}) - \sqrt{-1}g^{lp}R_{ijkl}u_l,
\end{align*}
\]
where
\[
R_{ijkl} = -\frac{\partial^2 g_{kl}}{\partial z_i \partial \bar{z}_j} + g_{pq} \frac{\partial g_{k\bar{p}}}{\partial z_i} \frac{\partial g_{l\bar{q}}}{\partial \bar{z}_j}.
\]

As a consequence
\[
\mathcal{L}(\pm u_{\lambda_\alpha}) \geq \pm \psi_{\lambda_\alpha} - C(1 + |\nabla u|) \sum_{i=1}^n f_i - C \sum_{i=1}^n f_i|\lambda_i|. \quad (5.8)
\]

We denote
\[
b_1 = 1 + \sup_M |\nabla u|^2.
\]
Lemma 5.2. Given $x_0 \in \partial M$. Let $u$ be a $C^3$-smooth $(n-1)$-PSH solution to equation (1.1), and $\Phi$ is defined as

$$\Phi = \pm T(u - \varphi) + \frac{\gamma}{\sqrt{b_1}}(u_{y_n} - \varphi_{y_n})^2 \text{ in } \Omega_\delta. \quad (5.9)$$

Then there is a positive constant $C_\Phi$ depending on $|\varphi|_{C^3(M)}$, $|\chi|_{C^1(M)}$ and other known data such that

$$\mathcal{L}\Phi \geq -C_\Phi \sqrt{b_1} \sum_{i=1}^n f_i - C_\Phi \sum_{i=1}^n f_i|\lambda_i| - C_\Phi |\nabla \psi| \text{ on } \Omega_\delta$$

for some small positive constant $\delta$. In particular, if $\partial M$ is holomorphically flat and $\varphi \equiv \text{constant}$ then $C_\Phi$ depends on $|\chi|_{C^1(M)}$ and other known data.

Proof. Together with (5.8) and Cauchy-Schwarz inequality, one can compute and obtain

$$\mathcal{L}(\pm T u) \geq - C \sqrt{b_1} \sum_{i=1}^n f_i - C \sum_{i=1}^n f_i|\lambda_i| - \frac{\gamma}{\sqrt{b_1}} F^{ij} u_{y_i} u_{y_j} \pm T \psi,$$

$$F^{ij}(\tilde{\eta}_i(u_{y_i})) \leq C \sum_{i=1}^n f_i|\lambda_i| + \frac{1}{\sqrt{b_1}} F^{ij} u_{y_i} u_{y_j} + C \sqrt{b_1} \sum_{i=1}^n f_i,$$

$$\mathcal{L}(u_{y_n} - \varphi_{y_n})^2 \geq F^{ij} u_{y_i} u_{y_j} - C \left(1 + |\nabla u|^2\right) \sum_{i=1}^n f_i - C|\nabla u| \sum_{i=1}^n f_i|\lambda_i| - C|\nabla \psi|(1 + |\nabla u|).$$

Putting these inequalities together we complete the proof. \(\square\)

To estimate the quantitative boundary estimates for mixed derivatives, we should employ barrier function of the form

$$v = (u - u) - t \sigma + N \sigma^2 \text{ in } \Omega_\delta, \quad (5.10)$$

where $t, N$ are positive constants to be determined.

Let $\delta > 0$ and $t > 0$ be sufficiently small with $N\delta - t \leq 0$, such that, in $\Omega_\delta$, $v \leq 0$, $\sigma$ is $C^2$ and

$$\frac{1}{4} \leq |\nabla \sigma| \leq 2, \quad |\mathcal{L}\sigma| \leq C_2 \sum_{i=1}^n f_i. \quad (5.11)$$

We construct the barrier function as follows:

$$\bar{\Psi} = A_1 \sqrt{b_1} v - A_2 \sqrt{b_1}|z|^2 + \frac{1}{\sqrt{b_1}} \sum_{\tau < n} |(u - \varphi)_\tau|^2 + A_3 \Phi \text{ in } \Omega_\delta. \quad (5.12)$$
Proof of Proposition 5.1. If $A_2 \gg A_3 \gg 1$ then one has $\tilde{\Psi} \leq 0$ on $\partial \Omega$, here we use (5.7). Note $\tilde{\Psi}(x_0) = 0$. It suffices to prove

$$\mathcal{L}\tilde{\Psi} \geq 0 \text{ on } \Omega,$$

which yields $\tilde{\Psi} \leq 0$ in $\Omega$, and then $(\nabla_v \tilde{\Psi})(x_0) \leq 0$.

By a direct computation one has

$$\mathcal{L}v \geq F^{ij}(\tilde{\gamma}_{ij} - \tilde{g}_{ij}) - C_2|2N\sigma - t| \sum_{i=1}^n f_i + 2N F^{ij} \sigma_i \sigma_j.$$

Applying $[3, \text{Lemma 6.2}]$, with a certain order of $\lambda$

$$F^{ij} \tilde{\gamma}_{ij} \geq \sum_{i=1}^n f_i(\lambda) \lambda_i = \sum_{i=1}^n f_i \lambda_i.$$

By $[11, \text{Proposition 2.19}]$ there is an index $r$ so that

$$\sum_{\tau < \lambda} F^{ij} \tilde{g}_{r \tau} \tilde{g}_{r j} \geq \frac{1}{4} \sum_{i \neq \tau} f_i \lambda_i^2.$$

In what follows we denote $\tilde{u} = u - \varphi$. By straightforward computations

$$\mathcal{L}\left(\sum_{\tau < \lambda} [\tilde{u}]^2 \right) \geq \frac{1}{2} \sum_{\tau < \lambda} F^{ij} \tilde{\gamma}_{r \tau} \tilde{g}_{r j} - C_1 \sqrt{b_1} \sum_{i=1}^n f_i |\lambda_i| - C_1 b_1 \sum_{i=1}^n f_i - C_1 \sqrt{b_1} |\nabla \psi|$$

$$\geq \frac{1}{8} \sum_{i \neq \tau} f_i \lambda_i^2 - C_1 \sqrt{b_1} \sum_{i=1}^n f_i |\lambda_i| - C_1 b_1 \sum_{i=1}^n f_i - C_1 \sqrt{b_1} |\nabla \psi|.$$

We are going to deal with $\sum_{i=1}^n f_i |\lambda_i|$:

1. $\sum_{i=1}^n f_i |\lambda_i| = \sum_{i=1}^n f_i \lambda_i - 2 \sum_{i, i \neq \lambda_i} f_i \lambda_i = n - 2 \sum_{i, i \neq \lambda_i} f_i \lambda_i$;
2. $\sum_{i=1}^n f_i |\lambda_i| = 2 \sum_{i \geq 0} f_i \lambda_i - \sum_{i=1}^n f_i \lambda_i = 2 \sum_{i \geq 0} f_i \lambda_i - n$.

In conclusion, combining with Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^n f_i |\lambda_i| \leq \frac{\epsilon}{8 \sqrt{b_1}} \sum_{i \neq \tau} f_i \lambda_i^2 + \frac{8 \sqrt{b_1}}{\epsilon} \sum_{i=1}^n f_i + n.$$
Taking $\epsilon = \frac{1}{c_1 + A_3 C_\Phi}$. Putting the above inequalities together we have

$$\mathcal{L} \tilde{\Psi} \geq A_1 \sqrt{b_1} \sum_{i=1}^{n} f_i(\lambda_i - \lambda_i) + 2A_1 N \sqrt{b_1} F^{ij} \sigma_i \sigma_j$$

- \{C'_1 + A_2 + A_3 C_\Phi + A_1 C_2|2N\sigma - t| + 8(C'_1 + A_3 C_\Phi)^2

$$+ n(C'_1 + A_3 C_\Phi)/ \sqrt{b_1} \sqrt{b_1} \sum_{i=1}^{n} f_i - (C'_1 + A_3 C_\Phi)|\nabla \psi|.$$  \hspace{1cm} (5.13)

Let’s take $\epsilon = \inf_M \min_i \mu_i$ as in Lemma 3.1, let $\theta_0 = \frac{2n^2}{(2n)^{p+e}} e^{-\max_M \psi}$. 

**Case 1**: Suppose $\min_j \mu_j \leq \frac{\epsilon}{2n}$. 

By Lemma 3.1 we have

$$\sum_{i=1}^{n} f_i(\lambda_i - \lambda_i) \geq \frac{\epsilon}{2(n-1)} \sum_{i=1}^{n} f_i.$$  \hspace{1cm} (5.14)

The bad term $-(C'_1 + A_3 C_\Phi)|\nabla \psi|$ can be controlled according to (3.12). In addition, we can choose $\delta$ and $t$ small enough such that

$$|2N\delta - t| \leq \min \left\{ \frac{\epsilon}{8C_2}, \frac{\theta_0}{16C_2} \right\}.$$  \hspace{1cm} (5.15)

Taking $A_1 \gg 1$ we can derive

$$\mathcal{L} \tilde{\Psi} \geq 0 \text{ on } \Omega_\delta.$$ 

**Case 2**: If $\min_j \mu_j > \frac{\epsilon}{2n}$, then $\max_i \mu_i \leq (2n)^{p+e}/e^{n-1}$. By (3.8) we have

$$f_i \geq \theta_0 \sum_{i=1}^{n} f_j, \hspace{1cm} \forall 1 \leq i \leq n.$$  \hspace{1cm} (5.16)

All the bad terms containing $\sum_{j=1}^{n} f_i$ in (5.13) can be controlled by

$$A_1 N \sqrt{b_1} F^{ij} \sigma_i \sigma_j \geq A_1 N \theta_0 \sqrt{b_1} \sum_{i=1}^{n} f_i \text{ on } \Omega_\delta.$$  \hspace{1cm} (5.17)

On the other hand, the bad term $-(C'_1 + A_3 C_\Phi)|\nabla \psi|$ in last term of (5.13) can be dominated by combining (3.10) with (5.17). Then $\mathcal{L} \tilde{\Psi} \geq 0$ on $\Omega_\delta$, if one chooses $A_1 N \gg 1$. \hspace{1cm} $\Box$
When the boundary is holomorphically flat and the boundary data is a constant, we have a slightly delicate result.

**Proposition 5.3.** Let \((M, J, \omega)\) be a compact Hermitian manifold with holomorphically flat boundary. Suppose, in addition to \((1.5)\) and \(\phi^{1/n} \in C^1(\bar{M})\), that the boundary data \(\varphi\) is a constant. Then every strictly \((n - 1)\)-PSH solution \(u \in C^3(M) \cap C^2(\bar{M})\) of Dirichlet problem \((1.1)\) and \((1.4)\) satisfies

\[
|\tilde{g}(\xi, J\tilde{\xi})| \leq C \left( 1 + \sup_M |\nabla u| \right),
\]

where \(C\) depends on \(|\phi^{1/n}|_{C^1(\bar{M})}, |u|_{C^2(\bar{M})}, |\nabla u|_{C^0(\partial M)}, \partial M\) up to second derivatives and other known data, but neither on \(\sup_M |\nabla u|\) nor on \((\inf_M \phi)^{-1}\).

### 5.3 Further results under appropriate conditions on \(\phi^{1/(n-1)}\)

As in \((1.3)\) we denote \(\tilde{\phi} = \phi^{1/(n-1)}\). If \(0 \leq \tilde{\phi} \in C^{1,1}(\bar{M})\) satisfies \((1.8)\), then

\[
\nabla \tilde{\phi} = 0 \text{ at points } p \in \bar{M} \text{ where } \tilde{\phi} = 0. \tag{5.18}
\]

This implies that there exists a uniform positive constant \(C\) such that

\[
|\nabla \tilde{\phi}| \leq C \sqrt{\tilde{\phi}} \text{ in } \bar{M}. \tag{5.19}
\]

#### 5.3.1 Stability of condition \((5.19)\)

Fix \(0 < \epsilon < 1\). Suppose \(\tilde{\phi}_\epsilon \in C^2(\bar{M})\) and it satisfies

\[
|\tilde{\phi}_\epsilon - (\tilde{\phi} + \epsilon)|_{C^{1,1}(\bar{M})} \leq \frac{\epsilon}{2}.
\]

Then

\[
\tilde{\phi} + \frac{\epsilon}{2} \leq \tilde{\phi}_\epsilon \leq \tilde{\phi} + \frac{3\epsilon}{2},
\]

\[
|\nabla \tilde{\phi}_\epsilon| \leq |\nabla \tilde{\phi}| + \frac{\epsilon}{2} \leq C \sqrt{\tilde{\phi} + \frac{\epsilon}{2}} \leq (1 + C) \sqrt{\tilde{\phi} + \frac{\epsilon}{2}} \leq (1 + C) \sqrt{\tilde{\phi}_\epsilon}.
\]
5.3.2 Quantitative boundary estimate revisited

**Proposition 5.4.** Let \( 0 < \tilde{\phi} = \frac{\phi}{\phi^{1/(n-1)}} \in C^1(\bar{M}) \) satisfy (1.8). Suppose the other assumptions in Proposition 5.1 hold. Then we have the quantitative boundary estimates (5.1). In addition (1.8) can be removed when \( M = X \times S \).

*Sketch of proof.* Together with (3.10), (5.19) implies

\[
|\nabla \psi| = \frac{(n-1)|\nabla \phi|}{\phi^{1/(n-1)}} \leq \frac{C}{\phi^{1/2(n-1)}} \leq \frac{C}{n} \sum_{i=1}^{n} f_i.
\]

When \( M = X \times S \) we always have the following inequality even if (1.8) does not hold,

\[
|\nabla_\xi \phi^{1/(n-1)}| \leq C \phi^{1/2(n-1)}
\]

(5.20)

where \( \xi \in T_X^{1,0} \) and \( |\xi| = 1 \). On the other hand, the tangential operator is

\[
\mathcal{T} = \frac{\partial}{\partial z_\alpha} \frac{\partial}{\partial \bar{z}_\sigma}, \quad 1 \leq \alpha \leq n-1
\]

where \( z' = (z_1, \cdots, z_{n-1}) \) is local holomorphic coordinate of \( X \). And the barrier function is

\[
\overline{\Psi} = A_1 \sqrt{b_1} v - A_2 \sqrt{b_1} |z|^2 + \frac{1}{\sqrt{b_1}} \sum_{\tau < n} |(u - \varphi)_\tau|^2 + A_3 \pm \mathcal{T}(u - \varphi).
\]

Similar to (5.13) we obtain

\[
\mathcal{L} \overline{\Psi} \geq A_1 \sqrt{b_1} \sum_{i=1}^{n} f_i (\lambda_i - \lambda_i) + \frac{1}{8 \sqrt{b_1}} \sum_{i \neq r} f_i \lambda_i^2
\]

\[
+ 2A_1 N \sqrt{b_1} F^{ij} \sigma_i \sigma_j - C \sqrt{b_1} \sum_{i=1}^{n} f_i - C \sum_{\tau=1}^{n-1} |\psi_\tau|.
\]

This completes the proof by using (5.20).

\[ \square \]

5.3.3 Global second estimate revisited

One can check Székelyhidi-Tosatti-Weinkove’s proof of second estimate works under such weaker assumptions on \( \phi^{1/(n-1)} \).
Proposition 5.5. Let \( 0 < \phi = \phi^{1/(n-1)} \in C^2(\bar{M}) \) satisfy (5.19), we assume that there is a \((n-1)\)-PSH function \( u \in C^2(\bar{M}) \). Then there is a positive constant \( C \) depending on \( \|\phi^{1/(n-1)}\|_{C^2(M)} \), \( \inf_M \min_i \mu_i \), \( |\phi|_{C^2(M)} \), and other known data (but neither on \( (\inf_M \phi)^{-1} \) nor on \( \sup_M |\nabla u| \)) such that (2.1) and (2.2) hold.

Sketch of proof. Let \( \mu_i = \sum_{j \neq i} \lambda_j, \psi = \log \phi + n \log(n-1) \). Generalized Newton-Maclaurin inequality gives

\[
\sum_{i=1}^{n} \frac{1}{\mu_i} \geq n^{\frac{n-2}{n-1}} \left( \sum_{i=1}^{n} \frac{\mu_i}{\lambda_1 \cdots \lambda_n} \right)^{1/(n-1)}. \tag{5.21}
\]

By straightforward computations we get

\[
(\log \phi)_{ij} = \frac{\phi_{ij}}{\phi} - \frac{\phi_i \phi_j}{\phi^2},
\]

\[
(\phi^{1/(n-1)})_{ij} = \frac{1}{n-1} \phi^{1/(n-1)} \left( \frac{\phi_{ij}}{\phi} - \frac{n-2 \phi_i \phi_j}{\phi^2} \right).
\]

Combining with (5.21) and concavity of \( f \), for

\[
\sum_{i=1}^{n} \log \mu_i = \log \phi + n \log(n-1),
\]

we obtain

\[
\sum_{i=1}^{n} f_i \geq n^{\frac{n-2}{n-1}} \left( \sum_{i=1}^{n} \lambda_i \right)^{\frac{1}{n-1}} \phi^{-\frac{n-1}{n-1}}, \tag{5.22}
\]

\[
|\psi_k| = |(\log \phi)_k| = (n-1)\phi^{-1/(n-1)}((\phi^{1/(n-1)})_k), \tag{5.23}
\]

\[
\psi_{kk} = (\log \phi)_{kk} = (n-1)\phi^{-\frac{n-1}{n-1}}((\phi^{1/(n-1)})_{kk} - \frac{1}{n-1} \frac{|\phi_k|^2}{\phi^2}). \tag{5.24}
\]

By (5.19) again,

\[
\frac{|\phi_k|^2}{\phi^2} \leq \frac{|\nabla \phi|^2}{\phi^2} = (n-1)^2 \frac{|\nabla \phi^{1/(n-1)}|^2}{\phi^{2n}} \leq C \phi^{-\frac{1}{n-1}}. \tag{5.25}
\]

Thus

\[
|\psi_k|, -\psi_{kk} \leq C \left( \sum_{i=1}^{n} \lambda_i \right)^{-\frac{1}{n-1}} \sum_{i=1}^{n} f_i.
\]
With those conditions at hand, one can prove the second order estimates (2.2), following the original proof of Székelyhidi-Tosatti-Weinkove almost words by words.

\[\square\]

**Remark 5.6.** When \(\partial M = \emptyset\), each \(0 \leq \tilde{\phi} = \phi^{1/(n-1)} \in C^2(M)\) satisfies (5.19). Thus one can improve Székelyhidi-Tosatti-Weinkove’s second estimate on closed Hermitian manifolds.

## 6 The Dirichlet problem with less regular boundary and boundary data

The purpose of this section is to investigate the equations on complex manifolds with less regular boundary.*

We first state an observation.

**Lemma 6.1.** Let \(Z = \frac{1}{(n-1)!} \ast \Re(\sqrt{-1} du \wedge \bar{\delta}(\omega^{n-2}))\) be as in (1.1). For any \(C^1\)-smooth function \(v\) on \(\bar{S}\),

\[
Z[v] = 0.
\]

**Proof.** Note that \(\omega_X\) is balanced, and \(v\) is a function on \(\bar{S}\), we see \(\bar{\partial}\omega^{n-2}_X = 0\) and \(\partial v \wedge \omega_S = 0\); thus \(\partial v \wedge \bar{\partial}\omega^{n-2} = 0\). \(\square\)

A somewhat remarkable fact to us is that the regularity assumptions on boundary and boundary data can be further weakened under certain assumptions. The motivation is mainly based on the estimates which state that, if \(\partial M\) is holomorphically flat and boundary value is a constant, then the constant in quantitative boundary estimate (2.3) depends only on \(\partial M\) up to second derivatives and other known data (see Propositions 5.3 or 5.4 and 4.1). Besides, we can use a result due to Silvestre-Sirakov [25] to derive the \(C^{2,\alpha}\) boundary regularity with only assuming \(C^{2,\beta}\) boundary.

As consequences, we can prove Theorems 1.5-1.6 and the following theorem.

---

*We emphasize that the geometric quantities of \((M, \omega)\) (the curvature \(R_{ijkl}\) and torsion \(T^i_{ij}\)) keep bounded as approximating to \(\partial M\), and all derivatives of \(\chi_{ij}\) has continues extensions to \(\bar{M}\), whenever \(M\) has less regularity boundary.

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Theorem 6.2. Let \((M, J, \omega)\) be a product as in (1.9), and we assume \(\partial S \in C^{2, \beta}\) for some \(0 < \beta < 1\). If \(0 < \phi^{1/n} \in C^{1,1}(M)\) then the Dirichlet problem (1.1) with homogeneous boundary data admits a \(C^{1, \alpha}\)-smooth \((n-1)\)-PSH solution with \(\forall 0 < \alpha < 1\) and \(\Delta u \in L^\infty(\bar{M})\) in the weak sense.

Sketch of proof of Theorems 1.5, 1.6 and 6.2. It only requires to consider the nondegenerate case:

\[ \phi > \delta_0 \text{ in } \bar{M} \text{ for some } \delta_0 > 0. \]

Let \(h\) be the solution to (1.11), and we denote \(u = th\). For \(t \gg 1\) we have

\[ \left( \tilde{\chi} + \frac{1}{n-1}(\Delta u \omega - \sqrt{-1} \partial \bar{\partial} u) + Z[u] \right)^n \geq (\phi + \delta_1) \omega^n \text{ in } M \quad (6.1) \]

for some \(\delta_1 > 0\). In fact \(Z[u] = 0\) by Lemma 6.1. Since we know that \(h \in C^\infty(S) \cap C^{2, \beta}(\bar{S}), h|_S < 0, \frac{\partial h}{\partial \nu}|_\partial S < 0\), we can carefully choose \(\{\alpha_k\}\) with \(\alpha_k \to 0^+\) as \(k \to +\infty\) then use a sequence of level sets \(\{h = -\alpha_k\}\) to enclose a smooth Riemann surface, denoted by \(S_k\) such that \(\cup S_k = S\) and \(\partial S_k\) converge to \(\partial S\) in the norm of \(C^{2, \beta}\). Let’s denote \(M_k = X \times S_k\). For any \(k \geq 1\), there exists a \(\phi^{(k)} \in C^\infty(\bar{M}_k)\) such that

\[ |\phi - \phi^{(k)}|_{C^2(\bar{M}_k)} \leq \beta_k \to 0^+ \text{ as } k \to +\infty. \]

For \(k \gg 1\) we have \(\beta_k < \min\{\delta_0, \delta_1\}\), then

\[ \left( \tilde{\chi} + \frac{1}{n-1}(\Delta u^{(k)} \omega - \sqrt{-1} \partial \bar{\partial} u^{(k)}) + Z[u^{(k)}] \right)^n \geq \phi^{(k)} \omega^n \quad \text{in } M_k, \]

\[ u^{(k)} = -t\alpha_k \quad \text{on } \partial M_k. \]

According to Theorem 1.2 we have a unique smooth \((n-1)\)-PSH function \(u^{(k)} \in C^\infty(\bar{M}_k)\) to solve

\[ \left( \tilde{\chi} + \frac{1}{n-1}(\Delta u^{(k)} \omega - \sqrt{-1} \partial \bar{\partial} u^{(k)}) + Z[u^{(k)}] \right)^n = \phi^{(k)} \omega^n \quad \text{in } M_k, \]

\[ u^{(k)} = -t\alpha_k \quad \text{on } \partial M_k. \]

Moreover, Propositions 5.5, 4.1 and 5.3 or 5.4, and Theorem 2.1 yield

\[ \sup_{M_k} \Delta u^{(k)} \leq C_k (1 + \sup_{M_k} |\nabla u^{(k)}|^2), \]

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where $C_k$ depends on $|\nabla u^{(k)}|_{C^0(\partial M_k)}$, $|u^{(k)}|_{C^0(M_k)}$, $|\partial M_k|$ to second order derivatives and other known data (but not on $\inf_M \phi^{(k)}$).

If there is a uniform positive constant $C$ depending not on $k$, such that

$$|u^{(k)}|_{C^0(M_k)} + \sup_{\partial M_k} |\nabla u^{(k)}| \leq C,$$  \hspace{1cm} (6.2)

then

$$\sup_{M_k} \Delta u^{(k)} \leq C' (1 + \sup_{M_k} |\nabla u^{(k)}|^2) \text{ independent of } k.$$  

Thus we have $|u|_{C^2(M_k)} \leq C$ depending not on $k$ (here we use blow up argument to derive gradient estimate). Finally, we are able to apply Silvestre-Sirakov’s [25] result to derive $C^2,\alpha'$ estimates on the boundary, while the convergence of $\partial M_k$ in the norm $C^2,\beta$ allows us to take a limit ($\alpha'$ can be uniformly chosen).

It only requires to prove (6.2). Let $w^{(k)}$ be the solution of

$$\Delta w^{(k)} + \text{tr}_\omega \tilde{\chi} + \text{tr}_\omega (Z[w^{(k)}]) = 0 \text{ in } M_k, \quad w^{(k)} = -t \alpha_k \text{ on } \partial M_k.$$  

By maximum principle and the boundary value condition, we have

$$u \leq u^{(k)} \leq w^{(k)} \text{ in } M_k, \quad \frac{\partial u}{\partial \nu} \leq \frac{\partial u^{(k)}}{\partial \nu} \leq \frac{\partial w^{(k)}}{\partial \nu} \text{ on } \partial M_k.$$  \hspace{1cm} (6.3)

It remains to prove

$$\sup_{M_k} w^{(k)} + \sup_{\partial M_k} \frac{\partial w^{(k)}}{\partial \nu} \leq C \text{ independent of } k.$$  \hspace{1cm} (6.4)

For $t \gg 1$,

$$\Delta(-u - 2t \alpha_k) + \text{tr}_\omega \tilde{\chi} + \text{tr}_\omega (Z[-u]) = -t + \text{tr}_\omega \tilde{\chi} \leq 0.$$  

Here Lemma 6.1 implies $Z[-u] = 0$. Applying comparison principle,

$$w^{(k)} \leq -u - 2t \alpha_k \text{ in } M_k, \quad \frac{\partial w^{(k)}}{\partial \nu} \leq -\frac{\partial u}{\partial \nu} \text{ on } \partial M_k$$  

as required. We then obtain a $C^2,\alpha'$-smooth $(n-1)$-PSH function to solve

$$\left(\tilde{\chi} + \frac{1}{n-1}(\Delta u \omega - \sqrt{-1} \partial \bar{\partial} u) + Z\right)^n = \phi \omega^n \text{ in } M, \; u = 0 \text{ on } \partial M.$$  

\[ \square \]
Further discussion on more general equations

Our method works for more general equations generated by smooth symmetric functions $f$ defined on $\Gamma \subset \mathbb{R}^n$, dating to the work of Caffarelli-Nirenberg-Spruck [3]. We consider the Dirichlet problem

$$f(\lambda(*\Phi[u])) = \psi \text{ in } M, \quad u = \varphi \text{ on } \partial M$$  \hspace{1cm} (7.1)

where $*\Phi[u] = \chi + \Delta u \omega - \sqrt{-1} \partial \overline{\partial} u + \varphi Z[u]$, and $\varphi$ is a smooth function, i.e.

$$\Phi[u] = *\chi + \frac{1}{(n-2)!} \sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2} + \frac{\partial}{(n-1)!} \text{Re}(\sqrt{-1} \partial \overline{\partial} u \wedge \overline{\partial} \omega^{n-2}).$$

Here $\Gamma$ is an open symmetric convex cone containing positive cone

$$\Gamma_n := \{ \lambda \in \mathbb{R}^n : \text{each component } \lambda_i > 0 \} \subseteq \Gamma$$

with vertex at the origin and with boundary $\partial \Gamma \neq \emptyset$. In addition we assume

$$f_i(\lambda) := \frac{\partial f}{\partial \lambda_i}(\lambda) > 0 \text{ in } \Gamma, \quad \forall 1 \leq i \leq n,$$  \hspace{1cm} (7.2)

$$f \text{ is concave in } \Gamma,$$  \hspace{1cm} (7.3)

$$\text{For any } \lambda \in \Gamma, \quad \lim_{t \to +\infty} f(t\lambda) > -\infty,$$  \hspace{1cm} (7.4)

$$\lim_{t \to +\infty} f(\lambda_1 + t, \cdots, \lambda_{n-1} + t, \lambda_n) = \sup f, \quad \forall \lambda \in \Gamma.$$  \hspace{1cm} (7.5)

The equation (7.1) is nondegenerate when the right-hand side satisfies

$$\inf_M \psi > \sup_{\partial \Gamma} f.$$  \hspace{1cm} (7.6)

Also it is called a degenerate equation if $\inf_M \psi = \sup_{\partial \Gamma} f$ and $f \in C^\infty(\infty) \cap C(\Gamma)$, where

$$\sup f := \sup_{\lambda_0 \in \partial \Gamma} \lim_{\lambda \to \lambda_0} \sup f(\lambda).$$

**Theorem 7.1.** Let $(M, J, \omega)$ be a compact Hermitian manifold with smooth boundary. Suppose, in addition to (7.2)-(7.6), that $\varphi, \psi$ are all smooth and there is a $C^{2,1}$ function such that

$$f(\lambda(*\Phi[u])) \geq \psi, \quad \lambda(*\Phi[u]) \in \Gamma \text{ in } \bar{M}, \quad u = \varphi \text{ on } \partial M.$$  \hspace{1cm} (7.7)

Then Dirichlet problem (7.1) admits a unique smooth solution with $\lambda(*\Phi[u]) \in \Gamma \text{ in } \bar{M}$. 

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Remark 7.2. Let’s denote
\[ \Gamma_{\mathbb{R}}^{\infty} := \{ t \in \mathbb{R} : (R, \cdots, R, t) \in \Gamma \text{ for some } R > 0 \}. \]

The case \( \Gamma \neq \Gamma_n \) is relatively simple, since \( \Gamma_{\mathbb{R}}^{\infty} = \mathbb{R} \) in this case. While for the case \( \Gamma = \Gamma_n \), the proof is almost parallel to that of Monge-Ampère equation for \((n-1)\)-PSH functions. Moreover, we can solve the degenerate Dirichlet problem, when the Levi form satisfies
\[ -(\kappa_1 + \cdots + \kappa_{n-1}) \in \Gamma_{\mathbb{R}}^{\infty}. \]

Remark 7.3. In the presence of (7.2)-(7.4), (7.5) is automatically satisfied if \( \Gamma \neq \Gamma_n \).

Remark 7.4. On the product \( M = X \times S \) with closed balanced factor, we can use the solution of (1.11) to construct the strictly subsolutions.

A A quantitative lemma

The following lemma proposed in earlier works [32], is a key ingredient in proof of Proposition 4.1.

Lemma A.1 ([32, 33]). Let \( A \) be an \( n \times n \) Hermitian matrix
\[
\begin{pmatrix}
  d_1 & a_1 \\
  d_2 & a_2 \\
   & \ddots \\
  d_{n-1} & a_{n-1} \\
  \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & a
\end{pmatrix}
\]
(A.1)

with \( d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1} \) fixed, and with \( a \) variable. Denote \( \lambda = (\lambda_1, \cdots, \lambda_n) \) by the eigenvalues of \( A \). Let \( \epsilon > 0 \) be a fixed constant. Suppose that the parameter \( a \) in \( A \) satisfies the quadratic growth condition
\[ a \geq \frac{2n - 3}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n - 1) \sum_{i=1}^{n-1} |d_i| + \frac{(n - 2)\epsilon}{2n - 3}. \] (A.2)

Then the eigenvalues (possibly with a proper order) behave like
\[ d_\alpha - \epsilon < \lambda_\alpha < d_\alpha + \epsilon, \ \forall 1 \leq \alpha \leq n - 1, \]
\[ a \leq \lambda_\alpha < a + (n - 1)\epsilon. \]

†The results in [32] were removed to [33]. More precisely, the paper [33] is essentially extracted from [32], and the first parts of [arXiv:2001.09238] and [arXiv:2106.14837].
For convenience we give the proof of Lemma A.1 in this appendix. We start with the case of \( n = 2 \). For \( n = 2 \), the eigenvalues of \( A \) are

\[
\lambda_1 = \frac{a + d_1 - \sqrt{(a - d_1)^2 + 4|a_1|^2}}{2} \quad \text{and} \quad \lambda_2 = \frac{a + d_1 + \sqrt{(a - d_1)^2 + 4|a_1|^2}}{2}.
\]

We can assume \( a_1 \neq 0 \); otherwise we are done. If \( a \geq \frac{|a_1|^2}{\epsilon} + d_1 \) then one has

\[
0 \leq d_1 - \lambda_1 = \lambda_2 - a = \frac{2|a_1|^2}{\sqrt{(a - d_1)^2 + 4|a_1|^2 + (a - d_1)}} < \frac{|a_1|^2}{a - d_1} \leq \epsilon.
\]

The following lemma enables us to count the eigenvalues near the diagonal elements via a deformation argument. It is an essential ingredient in the proof of Lemma A.1 for general \( n \).

**Lemma A.2** ([32, 33]). *Let \( A \) be an \( n \times n \) Hermitian matrix

\[
\begin{pmatrix}
d_1 & a_1 \\
d_2 & a_2 \\
\ddots & \ddots \\
\bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_{n-1} & a_n
\end{pmatrix}
\]

with \( d_1, \cdots, d_{n-1}, a_1, \cdots, a_{n-1} \) fixed, and with \( a \) variable. Denote \( \lambda_1, \cdots, \lambda_n \) by the eigenvalues of \( A \) with the order \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Fix a positive constant \( \epsilon \). Suppose that the parameter \( a \) in the matrix \( A \) satisfies the following quadratic growth condition

\[
a \geq \frac{1}{\epsilon} \left( \sum_{i=1}^{n-1} |a_i|^2 + \sum_{i=1}^{n-1} [d_i + (n-2)|d_i|] + (n-2) \epsilon \right). \quad (A.3)
\]

Then for any \( \lambda_\alpha \) (\( 1 \leq \alpha \leq n-1 \)) there exists \( d_{i_\alpha} \) with lower index \( 1 \leq i_\alpha \leq n-1 \) such that

\[
|\lambda_\alpha - d_{i_\alpha}| < \epsilon, \quad (A.4)
\]

\[
0 \leq \lambda_n - a < (n-1) \epsilon + |\sum_{\alpha=1}^{n-1} (d_\alpha - d_{i_\alpha})| \quad (A.5)
\]
Proof. Without loss of generality, we assume \( \sum_{i=1}^{n-1} |a_i|^2 > 0 \) and \( n \geq 3 \) (otherwise we are done, since A is diagonal or \( n = 2 \)). Note that in the assumption of the lemma the eigenvalues have the order \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). It is well known that, for a Hermitian matrix, any diagonal element is less than or equals to the largest eigenvalue. In particular,

\[ \lambda_n \geq a, \quad (\text{A.6}) \]

It only requires to prove (A.4), since (A.5) is a consequence of (A.4), (A.6) and

\[ \sum_{i=1}^{n} \lambda_i = \text{tr}(A) = \sum_{a=1}^{n-1} d_a + a. \quad (\text{A.7}) \]

Let’s denote \( I = \{1, 2, \cdots, n-1\} \). We divide the index set \( I \) into two subsets by

\[ B = \{ \alpha \in I : |\lambda_\alpha - d_\alpha| \geq \epsilon, \forall i \in I \} \]

and \( G = I \setminus B = \{ \alpha \in I : \text{There exists an } i \in I \text{ such that } |\lambda_\alpha - d_\alpha| < \epsilon \} \).

To complete the proof we need to prove \( G = I \) or equivalently \( B = \emptyset \). It is easy to see that for any \( \alpha \in G \), one has

\[ |\lambda_\alpha| < \sum_{i=1}^{n-1} |d_i| + \epsilon. \quad (\text{A.8}) \]

Fix \( \alpha \in B \), we are going to estimate \( \lambda_\alpha \). The eigenvalue \( \lambda_\alpha \) satisfies

\[ (\lambda_\alpha - a) \prod_{i=1}^{n-1} (\lambda_\alpha - d_i) = \sum_{i=1}^{n-1} (|a_i|^2 \prod_{j \neq i} (\lambda_\alpha - d_j)). \quad (\text{A.9}) \]

By the definition of \( B \), for \( \alpha \in B \), one then has \( |\lambda_\alpha - d_i| \geq \epsilon \) for any \( i \in I \). We therefore derive

\[ |\lambda_\alpha - a| = \left| \sum_{i=1}^{n-1} \frac{|a_i|^2}{\lambda_\alpha - d_i} \right| \leq \sum_{i=1}^{n-1} \frac{|a_i|^2}{|\lambda_\alpha - d_i|} \leq \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2, \quad \text{if } \alpha \in B. \quad (\text{A.10}) \]

Hence, for \( \alpha \in B \), we obtain

\[ \lambda_\alpha \geq a - \frac{1}{\epsilon} \sum_{i=1}^{n-1} |a_i|^2. \quad (\text{A.11}) \]
We shall use proof by contradiction to prove \( B = \emptyset \). For a set \( S \), we denote \(|S|\) the cardinality of \( S \). Assume \( B \neq \emptyset \). Then \(|B| \geq 1\), and so \(|G| = n - 1 - |B| \leq n - 2\).

In the case of \( G \neq \emptyset \), we compute the trace of the matrix \( A \) as follows:

\[
\text{tr}(A) = \lambda_n + \sum_{\alpha \in B} \lambda_\alpha + \sum_{\alpha \in G} \lambda_\alpha \\
> \lambda_n + |B| (a - \frac{1}{\varepsilon} \sum_{i=1}^{n-1} |a_i|^2) - |G| (\sum_{i=1}^{n-1} |d_i| + \varepsilon) \\
\geq 2a - \frac{1}{\varepsilon} \sum_{i=1}^{n-1} |a_i|^2 - (n - 2) (\sum_{i=1}^{n-1} |d_i| + \varepsilon) \\
\geq \sum_{i=1}^{n-1} d_i + a = \text{tr}(A),
\]

where we use (A.3), (A.6), (A.8) and (A.11). This is a contradiction.

In the case of \( G = \emptyset \), one knows that

\[
\text{tr}(A) \geq a + (n - 1) (a - \frac{1}{\varepsilon} \sum_{i=1}^{n-1} |a_i|^2) > \sum_{i=1}^{n-1} d_i + a = \text{tr}(A).
\]

Again, it is a contradiction. Thus \( B = \emptyset \) as required.

\[\square\]

We apply Lemma A.2 to prove Lemma A.1 via a deformation argument.

**Proof of Lemma A.1.** Without loss of generality, we assume \( n \geq 3 \) and \( \sum_{i=1}^{n-1} |a_i|^2 > 0 \) (otherwise \( n = 2 \) or the matrix \( A \) is diagonal, and then we are done). Fix \( a_1, \ldots, a_{n-1}, d_1, \ldots, d_{n-1} \). Denote \( \lambda_1(a), \ldots, \lambda_n(a) \) by the eigenvalues of \( A \) with the order \( \lambda_1(a) \leq \cdots \leq \lambda_n(a) \). Clearly, the eigenvalues \( \lambda_i(a) \) are all continuous functions in \( a \). For simplicity, we write \( \lambda_i = \lambda_i(a) \).

Fix \( \varepsilon > 0 \). Let \( I_\alpha' = (d_\alpha - \frac{\varepsilon}{2n-3}, d_\alpha + \frac{\varepsilon}{2n-3}) \) and

\[
P_0' = \frac{2n - 3}{\varepsilon} \sum_{i=1}^{n-1} |a_i|^2 + (n - 1) \sum_{i=1}^{n-1} |d_i| + \frac{(n - 2)\varepsilon}{2n - 3}.
\]

In what follows we assume \( a \geq P_0' \) (i.e. (A.2) holds). The connected components of \( \bigcup_{\alpha=1}^{n-1} I_\alpha' \) are as in the following:

\[
J_1 = \bigcup_{\alpha=1}^{j_1} I_\alpha', J_2 = \bigcup_{\alpha=j_1+1}^{j_2} I_\alpha', \ldots, J_l = \bigcup_{\alpha=j_{l-1}+1}^{j_l} I_\alpha', \ldots, J_m = \bigcup_{\alpha=j_{m-1}+1}^{n-1} I_\alpha'.
\]

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Moreover

\[ J_i \bigcap J_k = \emptyset, \text{ for } 1 \leq i < k \leq m. \]

Let

\[ \widehat{\text{Card}}_k : [P'_0, +\infty) \to \mathbb{N} \]

be the function that counts the eigenvalues which lie in \( J_k \). (Note that when the eigenvalues are not distinct, the function \( \widehat{\text{Card}}_k \) denotes the summation of all the algebraic multiplicities of distinct eigenvalues which lie in \( J_k \)). This function measures the number of the eigenvalues which lie in \( J_k \).

The crucial ingredient is that Lemma A.2 yields the continuity of \( \widehat{\text{Card}}_i(a) \) for \( a \geq P'_0 \). More explicitly, by using Lemma A.2 and

\[ \lambda_n \geq a \geq P'_0 > \sum_{i=1}^{n-1} |d_i| + \frac{\epsilon}{2n-3} \]

we conclude that if \( a \) satisfies the quadratic growth condition (A.2) then

\[ \lambda_n \in \mathbb{R} \setminus \left( \bigcup_{k=1}^{n-1} \overline{J_k} \right) = \mathbb{R} \setminus \left( \bigcup_{i=1}^m \overline{J_i} \right), \]

\[ \lambda_{\gamma} \in \bigcup_{i=1}^{n-1} I'_i = \bigcup_{i=1}^m J_i \text{ for } 1 \leq \alpha \leq n - 1. \quad (A.14) \]

Hence, \( \widehat{\text{Card}}_i(a) \) is a continuous function in the variable \( a \). So it is a constant. Together with the line of the proof of [3, Lemma 1.2] we see that \( \widehat{\text{Card}}_i(a) = j_i - j_{i-1} \) for sufficiently large \( a \). Here we denote \( j_0 = 0 \) and \( j_m = n - 1 \). The constant of \( \widehat{\text{Card}}_i \) therefore follows that

\[ \widehat{\text{Card}}_i(a) = j_i - j_{i-1}. \]

We thus know that the \( (j_i - j_{i-1}) \) eigenvalues

\[ \lambda_{j_{i-1}+1}, \lambda_{j_{i-1}+2}, \ldots, \lambda_{j_i} \]

lie in the connected component \( J_i \). Thus, for any \( j_{i-1} + 1 \leq \gamma \leq j_i \), we have \( I'_\gamma \subset J_i \) and \( \lambda_\gamma \) lies in the connected component \( J_i \). Therefore,

\[ |\lambda_\gamma - d_\gamma| < \frac{(2(j_i - j_{i-1}) - 1)\epsilon}{2n - 3} \leq \epsilon. \]

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Here we also use the fact that \( d_{\gamma} \) is midpoint of \( I'_{\gamma} \) and every \( J_i \subset \mathbb{R} \) is an open subset.

To be brief, if for fixed index \( 1 \leq i \leq n - 1 \) the eigenvalue \( \lambda_i(P'_0) \) lies in \( J_{\alpha} \) for some \( \alpha \), then Lemma A.2 implies that, for any \( a > P'_0 \), the corresponding eigenvalue \( \lambda_i(a) \) lies in the same interval \( J_{\alpha} \). The computation of \( \text{Card}_k \) can be done by letting \( a \to +\infty \).

\[ \square \]

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