Metric freeness and projectivity for classical and quantum normed modules

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Abstract. In functional analysis, there are several diverse approaches to the notion of projective module. We show that a certain general categorical scheme contains all basic versions as special cases. In this scheme, the notion of free object comes to the foreground, and, in the best categories, projective objects are precisely retracts of free ones. We are especially interested in the so-called metric version of projectivity and characterize the metrically free classical and quantum (= operator) normed modules. Informally speaking, so-called extremal projectivity, which was known earlier, is interpreted as a kind of ‘asymptotical metric projectivity’.

In addition, we answer the following specific question in the geometry of normed spaces: what is the structure of metrically projective modules in the simplest case of normed spaces? We prove that metrically projective normed spaces are precisely the subspaces of $l_1(M)$ (where $M$ is a set) that are denoted by $l_0^1(M)$ and consist of finitely supported functions. Thus, in this case, projectivity coincides with freeness.

Bibliography: 28 titles.

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§ 1. Introduction

There are several substantially different ways of extending the notion of projective module, one of the most important notions in homological algebra, to functional analysis. In our opinion, four approaches are the most important. Two of them take into account only the topology defined by the norm on the module in question, whilst the other two consider the precise value of the norm. (Rather recently, interest in the latter approaches has increased in connection with some problems of operator space theory; cf. [1]–[3].) Here are the respective definitions.

Let $\mathcal{K}$ be an arbitrary category, let $P$, $X$, and $Y$ be objects in $\mathcal{K}$, and let $\tau: Y \to X$ and $\varphi: P \to X$ be morphisms in $\mathcal{K}$. We recall that a morphism
\[ \psi : P \to Y \] is referred to as a lifting of \( \varphi \) with respect to \( \tau \) if the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{\varphi} & X \\
\downarrow{\psi} & & \downarrow{\tau} \\
Y & & 
\end{array}
\] (1.1)

is commutative.

Throughout the paper, \( A \) stands for an arbitrary normed algebra which is assumed to be unital for simplicity. (We suppose that the multiplicative inequality holds and the norm of the identity element is equal to 1.) When saying ‘normed module’ or simply ‘module’, we mean a left normed unital \( A \)-module with the multiplicative inequality for outer multiplication. The identity map of a set \( M \) onto itself is denoted by \( 1_M \).

We begin with the oldest and best-known version of projectivity, introduced (in the context of Banach modules) in [4]; see also, for example, [5]–[7].

**Definition 1.1.** A module \( P \) is said to be relatively projective (or simply projective, as in the majority of publications) if every bounded morphism \( \varphi : P \to X \) admits a bounded lifting with respect to every bounded module morphism \( \tau : Y \to X \) admitting a bounded right inverse.

The following more restrictive condition arose in connection with the study of amenable Banach algebras (cf. [5]). However, it was known for a long time in the context of Banach spaces (= Banach \( \mathbb{C} \)-modules) under the name ‘lifting property’ (see, for instance, [8] and the references therein).

**Definition 1.2.** A module \( P \) is said to be topologically projective if every bounded morphism \( \varphi : P \to X \) admits a bounded lifting with respect to every bounded surjective module morphism \( \tau : Y \to X \).

For a normed space \( E \) we denote by \( \overline{\circ{E}} \) the closed unit ball of \( E \), and by \( \circ{E} \) the open unit ball of \( E \). We recall that an operator \( \tau : E \to F \) between normed spaces is said to be coisometric (respectively strictly coisometric) if it takes \( \circ{E} \) onto \( \circ{F} \) (respectively \( \overline{\circ{E}} \) onto \( \overline{\circ{F}} \)). We recall that the Hahn-Banach theorem admits the following equivalent formulation.

*An operator \( \varphi : E \to F \) between normed spaces is isometric if and only if its adjoint \( \varphi^* : F^* \to E^* \) is coisometric and if and only if this adjoint is strictly coisometric.*

(This fact plays a decisive role in the forthcoming presentation.)

**Definition 1.3.** A module \( P \) is said to be extremally projective if an arbitrary bounded morphism \( \varphi : P \to X \) admits a bounded lifting with respect to \( \tau \) for which \( \| \psi \| < \| \varphi \| + \varepsilon \) for every coisometric module morphism \( \tau : Y \to X \) and every \( \varepsilon > 0 \).

Extremally projective Banach spaces are known in Banach space geometry as ‘spaces with metric lifting property’ (see, for example, [8], §4.9.2.1). These spaces were initially treated by Grothendieck [9]. (We recall and apply his results in §3.) Much later, extremally projective normed and Banach modules were considered in [10]. (For so-called extremally flat and extremally injective modules, see [11].) The term ‘extremally’ is explained below (see Remark 2.8).
The following notion is new, and relates to the main objects of investigation in the present paper. We shall see below that, when considering a series of questions (in particular, from the general categorical viewpoint), this notion behaves better than extremal projectivity.

**Definition 1.4.** A module $P$ is said to be **metrically projective** if every bounded morphism $\varphi: P \to X$ has a bounded lifting $\psi$ for which $\|\psi\| = \|\varphi\|$ with respect to every strictly coisometric module morphism $\tau: Y \to X$.

Each of the above notions of projectivity has both noncompleted (‘normed’) and completed (‘Banach’) versions. (The behaviour of modules in these cases is often very similar; however, sometimes it is substantially different, cf. [10], Theorems I and II.) Moreover, each of these versions has both a ‘classical’ prototype and its ‘quantum’ (= operator) version. In particular, quantum extremal projectivity was introduced and studied by Blecher [12] in 1992 under the title of ‘projectivity’.

We note a useful relationship between the noncompleted and completed versions.

**Proposition 1.5.** Let $P$ be a normed module over a Banach algebra $A$ and let $\overline{P}$ be its completion. Further, let $P$ have a property of projectivity formulated in one of Definitions 1.1–1.4 in its noncompleted version.

Then $\overline{P}$ has the same property in its completed version.

**Proof.** We restrict ourselves to the case of extremal projectivity. Suppose that a coisometric morphism $\tau: Y \to X$ between Banach modules, a bounded morphism $\varphi: P \to X$, and a number $\varepsilon > 0$ are given. We consider the restriction $\varphi_0$ of $\varphi$ to $P$. It admits a lifting $\psi_0: P \to Y$ with respect to $\tau$ for which $\|\psi_0\| < \|\varphi_0\| + \varepsilon$. Since $Y$ is complete, $\psi_0$ has a continuous extension $\psi: \overline{P} \to Y$. The rest of the proof is clear.

The present paper has several interrelated objectives. First, we show that a certain general categorical scheme contains all the above variants of projectivity, as well as their ‘injective’ versions, as special cases. The basic notion is that of a so-called rigged category, which is a generalization of MacLane’s relative Abelian category (see [13], Ch. IX, §5). (We mainly consider categories that are not even additive.) In the framework of this scheme, the notion of free object comes to the foreground. In fact, in this paper, we study projective objects by methods for free objects.

In the majority of examples, this approach is sufficient to describe projective objects as the retracts of free objects. Similarly, the injective objects are the retracts of cofree objects.

The case of extremal projectivity is a notable exception. The ‘asymptotic’ nature of this concept, with an unavoidable $\varepsilon$ in its definition, needs a certain refinement of our approach. This is achieved by equipping a given rigging with an additional (so-called asymptotic) structure. In this situation, the notion of an asymptotically projective object naturally arises, and this object can be characterized as a so-called asymptotic retract of a free object (Proposition 6.12).

This general approach includes extremally projective modules (both classical and quantum) as special cases.

Another objective of the paper is to give an explicit description of the free (and cofree) objects in our main examples of rigged categories of modules. We describe the metrically free and cofree classical modules. Further, we find an appropriate
rigging for the category of quantum modules and describe the corresponding free objects, demonstrating the richness of this class. In particular, in the case of the simplest algebra \( \mathbb{C} \), the free quantum modules (that is, the free quantum spaces with one-point base) are of the form

\[
\mathcal{N}_1 \oplus_1 \mathcal{N}_2 \oplus_1 \cdots \oplus_1 \mathcal{N}_n \oplus_1 \cdots,
\]

where \( \mathcal{N}_n \) stands for the space of trace-class operators on \( \mathbb{C}^n \), which is equipped with the quantum trace norm, and \( \oplus_1 \) is the symbol for the quantum \( l_1 \)-sum of quantum spaces. (For details, see §5.)

I would like to stress that the part of the present paper dealing with quantum spaces and modules was written under the strong influence of [12], already mentioned above. It was Blecher who recognized, among other ideas and results, the decisive role of the spaces \( \mathcal{N}_n \) in the corresponding lifting problems and who characterized arbitrary extremally projective quantum modules in terms of these spaces. (See Corollary 6.13, (ii) and the related remarks.)

The last (but not least) objective of the paper is to give a complete description of metrically projective normed spaces (incomplete in general). (We solve also a similar problem for Banach spaces, but it is simpler.) We show that these spaces are the normed subspaces \( l_0^1(M) \) of \( l_1(M) \) consisting of finitely supported functions; here \( M \) stands for the set of indices. This means, in particular, that, in the context of normed spaces (as well as that of Banach spaces), metric projectivity coincides with metric freeness. We do not know whether or not such a coincidence holds for the class of extremally projective normed spaces, which is wider in general. (We note that, thanks to Grothendieck, we have an affirmative answer in the context of Banach spaces; for details, see §3.)

The order of our presentation differs from the list of our objectives, for the convenience of the reader. In §2 we introduce projective and free objects as derivative notions of the concept of a rigged category, and then describe the free objects in the context of classical normed and Banach modules (Theorem 2.18).

In §3 we leave the ‘abstract nonsense’ for a while and study the entirely concrete problem of characterizing metrically projective normed spaces (Theorem 3.4).

In §4, we study injectivity and cofreeness. We introduce these notions, again in the framework of rigged categories, and characterize the cofree classical modules, using some general categorical observations (Theorem 4.9 and Proposition 4.5).

In §5 we introduce a rigged category which is suitable for working with metrically projective quantum spaces and describe its free objects (Theorem 5.9).

Finally, in §6 we show that the notion of an extremally projective module in both its versions (classical and quantum), together with the results concerning the characterization of these modules, can be included in the general scheme of an asymptotic category (Corollary 6.13).

§2. Projectivity and freeness in rigged categories. ‘Classical’ examples

Definition 2.1. Let \( \mathcal{K} \) be an arbitrary category. A rigging of \( \mathcal{K} \) is a faithful covariant functor \( \Box: \mathcal{K} \to \mathcal{L} \), where \( \mathcal{L} \) is another category. A pair consisting of a category and a rigging of this category is referred to as a rigged category.
Let us fix a rigged category for the time being, say \((\mathcal{K}, \Box: \mathcal{K} \to \mathcal{L})\). A morphism \(\tau\) in \(\mathcal{K}\) is said to be an *admissible epimorphism* if \(\Box(\tau)\) is a retraction in \(\mathcal{L}\). (It is clear that such a \(\tau\) is indeed an epimorphism.)

**Definition 2.2.** An object \(P\) in \(\mathcal{K}\) is said to be \(\Box\)-projective (or, if no confusion can occur, simply *projective*) if for any admissible epimorphism \(\tau: Y \to X\) and any morphism \(\varphi: P \to X\) there is a lifting of \(\varphi\) with respect to \(\tau\).

Briefly, \(P\) is projective if the standard covariant functor \(h_{\mathcal{K}}(P,?): \mathcal{K} \to \text{Set}\) takes the admissible epimorphisms to surjective maps.

We single out a simple proposition (which is well-known in essence).

**Proposition 2.3.**

(i) Every retract of a projective object is projective.

(ii) If \(\sigma: X \to P\) is an admissible epimorphism in \(\mathcal{K}\) and \(P\) is projective, then \(\sigma\) is a retraction.

We consider some examples. In what follows, \(A\) is a normed algebra (cf. §1), \(A\)-\text{mod} is the category of normed \(A\)-modules and their bounded morphisms, and \(A\)-\text{mod}_1 is the category of the same modules and their contraction morphisms. As usual, we denote by \(\text{Nor}\) (\(\text{Nor}_1\)) the category of normed spaces (= \(\mathbb{C}\)-modules) and all bounded operators (respectively all contraction operators). We denote by \(A\)-\text{mod}, \(A\)-\text{mod}_1, \text{Ban} and \text{Ban}_1 the similar categories of Banach (= complete) modules over Banach algebras and Banach spaces. As usual, \(\text{Set}\) stands for the category of sets.

**Example 2.4.** We write \(\mathcal{K} := \overline{A\text{-mod}}\) and \(\mathcal{L} := \text{Ban}\) and take for \(\Box\) the corresponding forgetful functor (forgetting outer multiplication). It this case, we certainly obtain the old and well-known definitions of an admissible epimorphism of Banach \(A\)-modules and of a (relatively) projective Banach \(A\)-module (see, for example, [5], [6], or [7]). An obvious noncompleted version of the above rigging with \(\mathcal{K} := \overline{A\text{-mod}}\) and \(\mathcal{L} := \text{Nor}\) leads to the notion of a (relatively) projective normed \(A\)-module, that is, to the form of projectivity mentioned in Definition 1.1.

**Example 2.5.** Let \(\mathcal{K}\) be the category \(\overline{A\text{-mod}}\) again but let \(\mathcal{L}\) be \(\text{Set}\) so that \(\Box\) ‘forgets everything’. Then we obtain all surjective (and hence open) morphisms of Banach \(A\)-modules as admissible epimorphisms and the obvious completed version of topologically projective modules in Definition 1.2 as \(\Box\)-projective objects. These modules were mentioned in [10] (see also [14]); their flat and injective ‘colleagues’ were mentioned under another name many years ago in [5].

In the noncompleted case, we replace \(\overline{A\text{-mod}}\) by \(A\text{-mod}\) in the last rigging and obtain the type of projectivity presented in Definition 1.2. However, in this case, admissible morphisms need not be open, and we obtain too few projective objects as a result. For example, for \(A := \mathbb{C}\), the projective normed spaces are just the finite-dimensional ones.

**Remark 2.6.** This inconvenience can be removed. Gusarov [15] showed that, if we consider as a rigging the forgetful functor from \(A\text{-mod}\) to the category \(\text{Bor}\) of so-called bornological spaces, then the admissible epimorphisms thus obtained are again open, and the family of projective objects is significantly enlarged.

The following rigging occupies a special place in this section.
**Example 2.7.** We consider the functor $\bigcirc: \textbf{A-mod}_1 \to \textbf{Set}$ assigning to a module $X$ its closed unit ball and assigning to any contraction morphism its restriction to the closed unit balls of the corresponding modules. Obviously, in this case, admissible morphisms are precisely strictly coisometric morphisms and, respectively, the projective modules are the metrically projective modules of Definition 1.4. This rigging has an obvious analogue for Banach modules.

**Remark 2.8.** It is easy to see that metrically projective objects can also be defined as follows: a normed $A$-module $P$ is metrically projective if the morphism functor $h_A(P,?): \textbf{A-mod} \to \textbf{Nor}$ preserves the property of a morphism being strictly coisometric. In a similar way, $P$ is extremally projective if this functor preserves the property of a morphism being coisometric or, equivalently, an extremal epimorphism in general categorical terms (cf. [16], Ch. 1, §1.7 or [17], Ch. 1, §5). This is the origin of our term ‘extremally’.

**Remark 2.9.** If we consider an obvious analogue of the rigged category in Example 2.7 by taking the symbol $\otimes$ (see §1) instead of $\bigcirc$, then the admissible morphisms become coisometric in Definition 1.3 as projective modules; moreover, as is well-known in essence, we obtain only one $\otimes$-projective module, namely, the zero module. We know of no rigging which affords the extremally projective modules, and we conjecture that there is no such rigging. Nevertheless, as was mentioned in §1, there is a construction which enables one to consider extremal projectivity from the general categorical viewpoint. We discuss this construction in §6.

We restrict ourselves to the above examples of rigging. However, there are many other riggings, and some of them are rather curious. One of them is the rigging $\textbf{A-mod} \to \textbf{Ban}^\circ$ (where $\circ$ denotes the dual category) which assigns the dual space of the underlying normed space to any module. This rigging is useful because, when compared with the rigging in Example 2.4, it provides a much greater stock of admissible epimorphisms.

We return to the general case and fix a rigged category

$$(\mathscr{K}, \Box: \mathscr{K} \to \mathscr{L}).$$

We present two notions, the first of which must be well known, at least in pure algebra.

**Definition 2.10.** Let $M$ be an object in $\mathscr{L}$. An object $\text{Fr}(M)$ in $\mathscr{K}$ is called a free (or, to be more precise, $\Box$-free) object with base $M$ if for every object $X$ in $\mathscr{K}$ there is a bijection

$$\mathcal{I}_X: h_{\mathcal{K}}(M, \Box(X)) \to h_{\mathcal{K}}(\text{Fr}(M), X), \quad (2.1)$$

which is natural with respect to the second argument.

A rigged category is said to be freedom-loving if every object in $\mathcal{L}$ is a base of some free object in $\mathcal{K}$.

We note that, in a freedom-loving rigged category, the correspondence

$$M \mapsto \text{Fr}(M)$$

is natural with respect to the second argument.
can be extended to a morphism in a suitably ‘consistent’ manner (cf. [18], §4.1).

In this way, we obtain the so-called freedom functor

\[ \text{Fr}: L \to \mathcal{K}, \]

which is left adjoint to \( \Box \); see, for example, [18]. We recall that for arbitrary functors \( \Psi: \mathcal{K} \to L \) and \( \Phi: L \to \mathcal{K} \), the functor \( \Phi \) is said to be left adjoint to \( \Psi \) (or, equivalently, \( \Psi \) is said to be right adjoint to \( \Phi \)) if for any \( X \in \mathcal{K} \) and \( M \in L \) there is a bijection

\[ I_{M,X}: h_L(M, \Psi(X)) \to h_K(\Phi(M), X), \]

which is natural with respect to both the arguments.

The following observations show how useful freeness is. These observations are well-known in essence, and they can be obtained as special cases of simple consequences of general facts presented in [18], Chs. 3 and 4.

\textbf{Proposition 2.11.} (i) If for an object \( X \) in \( \mathcal{K} \) the object \( \Box(X) \) in \( L \) is a base of some free object, then there is an admissible epimorphism of this free object onto \( X \).

As a consequence, in a freedom-loving rigged category \( \mathcal{K} \), every object in \( \mathcal{K} \) is the range of some admissible epimorphism with free domain.

(ii) All free objects in \( \mathcal{K} \) are projective.

(iii) If our rigged category is freedom-loving, then an object in \( \mathcal{K} \) is projective if and only if it is a retract of a free object.

We shall see in specific examples (here, and especially in §5) that it is much simpler to seek projective objects if the category in question has coproducts. We recall briefly what a coproduct is.

A \textit{coproduct} of a family of objects \( X_\nu, \nu \in \Lambda \), in a category \( \mathcal{K} \) is a pair \( (X, \{i_\nu \mid \nu \in \Lambda\}) \), where \( X \) is an object and \( i_\nu: X_\nu \to X \) are morphisms. By definition, this pair has the following property: for every object \( Y \) the map between the set \( h_{\mathcal{K}}(X,Y) \) and the Cartesian product \( X\{h_{\mathcal{K}}(X_\nu,Y) \mid \nu \in \Lambda\} \), taking \( \psi \) to the family \( \{\psi i_\nu\} \), is a bijection. This object \( X \), which is denoted in more detail by

\[ \prod\{X_\nu \mid \nu \in \Lambda\}, \]

is referred to as the \textit{coproduct object} and \( i_\nu \) as the \textit{coproduct injections}, and the above property is called the \textit{universal property of coproducts}. (See, for example, [19], Ch. 2 or [17], Ch. 0, §6.) The morphism \( \psi \) is referred to as the \textit{coproduct of the morphisms} \( \psi i_\nu \).

It readily follows from this definition that, if two coproducts of the same family of objects are given, then there is a categorical isomorphism between the corresponding coproduct objects which agrees in an obvious sense with the coproduct injections.

We say that a category \( \mathcal{K} \) \textit{admits coproducts} if every family of objects of the category has a coproduct.

Let us return to our rigged category.

\textbf{Proposition 2.12.} Let \( (P, \{i_\nu \mid \nu \in \Lambda\}) \) be a coproduct of a family \( P_\nu, \nu \in \Lambda \), of projective objects in \( \mathcal{K} \).

Then the object \( P \) is also projective.
Proof. We take \( \tau \) and \( \varphi \) as in Definition 2.2 and consider the corresponding liftings \( \psi_{\nu} : P_{\nu} \to Y \) of the morphisms \( \varphi_{\nu} := \varphi_{i_{\nu}} \) and their coproduct \( \psi \). We can see that both \( \tau \psi \) and \( \varphi \) are coproducts of the morphisms \( \varphi_{\nu}, \nu \in \Lambda \). Therefore, \( \tau \psi = \varphi \).

**Proposition 2.13.** Let \( \Lambda \) be an index set and let a free object \( F_{\nu} \) with base \( M_{\nu} \) be given for every \( \nu \in \Lambda \). Suppose further that the coproducts \( F := \bigsqcup \{F_{\nu} \mid \nu \in \Lambda\} \) in \( \mathcal{K} \) and \( M := \bigsqcup \{M_{\nu} \mid \nu \in \Lambda\} \) in \( \mathcal{L} \) exist.

Then \( F \) is a free object with base \( M \).

**Proof.** Using the definitions of coproducts and free objects, we consider the following chain of bijections:

\[
h_{\mathcal{K}}(M, \square(X)) = \left\{ h_{\mathcal{K}}(M_{\nu}, \square(X)) \mid \nu \in \Lambda \right\} = \left\{ h_{\mathcal{K}}(F_{\nu}, X) \mid \nu \in \Lambda \right\} = h_{\mathcal{K}}(F, X),
\]

which is obviously natural with respect to \( X \). The rest of the proof is clear.

We return to our specific examples of rigged categories. To what extent are they freedom-loving?

Throughout the paper, we use the symbols \( \otimes_p \) and \( \hat{\otimes} \) for the noncompleted and completed Grothendieck projective tensor products, respectively. A space of the form \( A \otimes_p E \) or \( A \hat{\otimes} E \) is regarded as a normed or Banach \( A \)-module, respectively, with the outer multiplication well-defined by the equation \( a \cdot (b \otimes x) := ab \otimes x \), where \( a, b \in A \) and \( x \in E \) (cf., for example, [5], Ch. III, §1).

**Example 2.14.** The rigged category in Example 2.4 is freedom-loving, and this fact is well-known. For every Banach space \( E \) the free Banach left \( A \)-module with base \( E \) is \( A \hat{\otimes} E \) (see [5] or [6]). The same holds for the noncompleted version of the rigged category in question; in this case, the free modules are of the form \( A \otimes_p E \).

**Example 2.15.** In contrast to Example 2.14, the rigged category in Example 2.5 is not freedom-loving. One can readily note that a Banach \( A \)-module is free if and only if it is topologically isomorphic to a module of the form \( A \hat{\otimes} \mathbb{C}^n \) for some \( n = 1, 2, \ldots \), and its base set consists of \( n \) points. As a consequence, only finite sets can be bases of free Banach modules, and all these modules are finitely generated.

**Remark 2.16.** If for \( \mathcal{L} \) one uses the category \( \text{Bor} \) rather than \( \text{Set} \) (cf. Remark 2.6), then we obtain a much larger family of free objects and their bases (see Gusarov [15]). In particular, in this case, a module is free if and only if it is topologically isomorphic to a module \( A \hat{\otimes} l_1(M) \) for some set \( M \).

Certainly, the category \( \text{Set} \) admits coproducts (‘disjoint unions’) of given sets. Therefore, Proposition 2.13 immediately implies the following corollary.

**Corollary 2.17.** Let \( \mathcal{K} \) admit coproducts, let \( \mathcal{L} \) be the category \( \text{Set} \), and let there be a free module, say \( F \), with a one-point base.

Then our rigged category is freedom-loving. Moreover, if \( M \) is an arbitrary set, then the coproduct object of the family of copies of \( F \) indexed by the points of \( M \) is a free object with base \( M \).
In particular, this corollary shows that the ‘poorness’ of the family of free objects in Example 2.15 is related to the following well-known demerit of the category $\text{A-mod}$: a family of nonzero objects in the category has a coproduct only if this family is finite (see, for example, [17], Ch. 2, §5).

We recall that, on the contrary, the categories $\text{A-mod}_1$ and $\overline{\text{A-mod}}_1$ admit coproducts. Indeed, it can readily be seen that a coproduct of a family $X_\nu$, $\nu \in \Lambda$, in $\text{A-mod}_1$ (respectively in $\overline{\text{A-mod}}_1$) is the noncompleted (respectively completed) $l_1$-sum of the given modules, together with the natural embeddings of direct summands. We also recall that a noncompleted $l_1$-sum of a family of copies of the module $A$, which is indexed by the points of some set $M$ is simply $A \otimes_p l_0^1(M)$, whereas the completed $l_1$-sum of this family is $A \otimes l_1(M)$. In particular, the space $l_0^1(M)$ (respectively $l_1(M)$) is a coproduct object in $\text{Nor}_1$ (respectively in $\text{Ban}_1$) of the family of copies of $\mathbb{C}$ indexed by the points of $M$.

Provided with these facts, we return to our main rigging $\odot: \text{A-mod}_1 \to \text{Set}$. In what follows, the $\odot$-free objects in $\text{A-mod}_1$ are referred to as metrically free modules.

**Theorem 2.18.** The rigged category $(\text{A-mod}_1, \odot)$ is freedom-loving. Here a metrically free normed A-module with base set $M$ is $A \otimes_p l_1^0(M)$.

**Proof.** By the above considerations, it is necessary only to show that a metrically free module with one-point base set, say $\{t\}$, is equivalent to $A$.

This is indeed the case: for every object $X$ in $\text{A-mod}_1$ the map between $\text{hset}((\{t\}, \odot(X))$ and $b_{\text{A-mod}_1}(A, X)$, taking a map $\varphi$ to the (contraction) morphism $\psi: a \mapsto a \cdot \varphi(t)$, is obviously a bijection, which is natural with respect to $X$.

This theorem has an obvious Banach (= completed) analogue. Namely, the metrically free Banach $A$-modules with base $M$ are the modules $A \otimes l_1(M)$.

**Corollary 2.19.** Every normed (respectively Banach) $A$-module is the image of the module $A \otimes_p l_0^1(M)$ (respectively $A \otimes l_1^0(M)$) under a strictly coisometric morphism for some set $M$.

Moreover, this module is metrically projective if and only if it is a retract for some set $M$ (or, equivalently, a module direct summand with natural projection of norm 1) of the module $A \otimes_p l_1^0(M)$ (respectively $A \otimes l_0^1(M)$).

Certainly, the facts presented above have simple direct proofs; however, we would like to show that these facts, and many other facts, are special cases of the general categorial scheme under consideration.

**Remark 2.20.** It is easy to see that the rigged category

$$(\text{A-mod}_1, \odot: \text{A-mod}_1 \to \text{Set})$$

(cf. Remark 2.9) is not freedom-loving and, moreover, contains no $\odot$-free $A$-module but $0$.

**Remark 2.21.** Many years ago, Semadeni [20] suggested a definition of free object in a wide class of categories, and this definition differs substantially from ours. Semadeni’s approach is useful for many important examples; however, as was noted in [20], pp. 5, 27, it fails for the category of modules over pure algebras.
§ 3. Metrically projective normed spaces are the spaces $l_1^0(M)$

We proceed with a specific problem in the geometry of normed spaces: how to construct metrically projective spaces?

We recall that Grothendieck answered a similar question for extremally projective Banach spaces, and these spaces turned out to be precisely the spaces $l_1(M)$, where $M$ is a set. It is natural to assume that the extremally projective normed spaces are the spaces $l_1^0(M)$, but we do not know whether or not this is true. However, we can describe the metrically projective spaces, both noncompleted and completed. We begin with some preparation.

A Banach space, say $E$, is said to be metrically flat if for any isometry $i: F \to G$ between Banach spaces the operator

$$1_E \hat{\otimes} i: E \hat{\otimes} F \to E \hat{\otimes} G.$$ 

is also an isometry.

**Proposition 3.1.** Every metrically or extremally projective Banach space is metrically flat.

**Proof.** Let $E$ be a given space, and let $i: F \to G$ be as above. We recall (see §1) that the property of an operator $1_E \hat{\otimes} i$ being isometric is equivalent to the property of its adjoint $(1_E \otimes i)^*$ being strictly coisometric and to that of the adjoint being coisometric. By the law of adjoint associativity (we use it now in the simplest form; see, for example, [17], Ch. 2, §7 or [1], §5.1), this adjoint operator is isometrically equivalent to the operator

$$\mathcal{B}(E, i^*): \mathcal{B}(E, G^*) \to \mathcal{B}(E, F^*), \quad \varphi \mapsto i^* \varphi.$$ 

Since $i$ is isometric, it follows that $i^*$ is strictly coisometric, and, what is more, coisometric. Therefore, by the condition on $E$, the operator $\mathcal{B}(E, i^*)$ is coisometric.

In our subsequent considerations, we use Grothendieck’s theorem, as mentioned above. Note that, in the literature, when speaking about this theorem, authors usually cite [9] (see, for example, [21] p. 182). In essence, this is correct. At the same time, although the paper [9] contains all the necessary ingredients for the proof, the theorem in question is not formulated explicitly. By ‘ingredients’ we mean that the following two statements are formulated (certainly, in equivalent terms) and completely proved by Grothendieck.

1. A Banach space is metrically flat if and only if it is isometrically isomorphic to some space $L_1(\Omega, \mu)$, where $(\Omega, \mu)$ is a measure space (see [9], Theorem 1).
2. A space $L_1(\Omega, \mu)$ topologically isomorphic to some closed subspace of $l_1(N)$ (where $N$ is a set) is isometrically isomorphic to $l_1(M)$ for some set $M$ (cf. [9], Proposition 2).

**Proposition 3.2.** Every metrically projective Banach space is isometrically isomorphic to $l_1(M)$ for some set $M$.

**Proof.** By assumption, our space is a retract in $\text{Ban}_1$ of some free Banach space, that is, of $l_1(N)$ for some set $N$ (Corollary 2.19). Hence, it coincides, up to an (isometric) isomorphism, with a closed subspace of $l_1(N)$. At the same time, combining Proposition 3.1 with the ‘ingredient’ (i), we see that our space is isometrically isomorphic to some $L_1(\Omega, \mu)$. ‘Ingredient’ (ii) now applies.
Remark 3.3. The proof of Proposition 3.2 using Proposition 3.1 completes the proof of Grothendieck’s theorem in the form in which it was stated above. (The only difference is that, instead of Corollary 2.19, we use a similar corollary, Corollary 6.13, (i); see below.) Apparently, the traditional way of proving Proposition 3.1 is to use a nontrivial criterion for metric flatness (namely, Proposition 1 in [9]) and, correspondingly, to verify the related condition. To this end, one considers some family of operators (indexed by all numbers $\varepsilon$ in Definition 1.1) and then, applying the Banach-Alaoglu theorem, passes to an adherent point of this family with respect to an appropriate weak* topology. (See, for example, [22], Ch. VII, Sec. 27.4.2.) In our opinion, the argument presented here is simpler.

However, the noncompleted version of Proposition 3.2 needs some additional work. We arrive at the main result of the section.

Theorem 3.4. Every metrically projective normed space is isometrically isomorphic to $l^0_1(M)$ for some set $M$.

Proof. It is clear from Propositions 1.5 and 3.2 that the given space, say $E$, is a dense subspace of some $l_1(M)$ up to isometric isomorphism. Therefore, we are to prove that $E$ contains all vectors in $l^0_1(M)$ and only these vectors.

Combining Proposition 2.11, (iii) and Theorem 2.18, we see that $E$ is a retract in $\text{Nor}_1$ of a space $l^0_1(N)$ for some set $N$. We fix a retraction $\sigma: l^0_1(N) \to E$ and a coretraction $\rho: E \to l^0_1(N)$ such that $\sigma \rho = 1_E$. We also consider the corresponding extensions by continuity, $\sigma: l_1(N) \to l_1(M)$ and $\rho: l_1(M) \to l_1(N)$. For any set $L$ we denote by $\text{bas}(L)$ the set of characteristic functions of the one-point subsets of $L$, this set is certainly a linear basis in $l^0_1(L)$.

We take an arbitrary $e \in \text{bas}(M)$. Obviously, the vector $\rho(e)$ has a unique expansion $\sum_k \lambda_k e'_k$ with an at most countable set of summands such that the vectors $e'_k$ are proportional to pairwise distinct vectors in $\text{bas}(N)$, $\|e'_k\| = 1$, and $\lambda_k > 0$ for any $k$.

Lemma 3.5. $\sigma(e'_k) = e$ for any $k$.

Proof. We fix some $e'_k$. Since $\|e\| = 1$ and $\rho$ is an isometry, together with $\rho$, it follows that $\sum_k \lambda_k = 1$.

Certainly, we may assume that there are at least two summands in the expansion of the vector $\rho(e)$. This implies that $\rho(e)$ is a convex combination of $e'_k$ and $e''$.

$z := \left(\sum_{l:l \neq k} \lambda_l \right)^{-1} \sum_{l:l \neq k} \lambda_l e'_l$.

Hence, $e$ (which is equal to $\sigma \rho(e)$) is a convex combination of $\sigma(e'_k)$ and $\sigma(z)$. Since $\sigma$ (as well as $\sigma$) is a contraction operator, it follows that both the vectors belong to $\bigcup l_1(M)$. However, $e$ is certainly an extreme point of the ball $\bigcup l_1(M)$. Therefore, $\sigma(e'_k) = e$. It remains to recall that $e'_k$ is in the domain of the operator $\sigma$.

We now complete the proof of Theorem 3.4. It follows from Lemma 3.5 that $E$ contains the whole space $l^0_1(M)$. Suppose now that $E$ contains vectors not belonging
to $l^0_1(M)$. Then there is an $x \in E$ of the form $\sum_{k=1}^{\infty} \lambda_k e_k$, where the vectors $e_k$ belong to $\text{bas}(M)$ and are pairwise distinct, and $\lambda_k \neq 0$ for any $k \in \mathbb{N}$. For each $k$ we take an expansion of $\rho(e_k)$ of the form

$$\sum_{l} \lambda_{kl} e'_{kl},$$

where all vectors $e'_{kl}$ are proportional to pairwise distinct vectors in $\text{bas}(N)$, $\|e'_{kl}\| = 1$, and $\lambda_{kl} > 0$ for all indices. By Lemma 3.5, for any $e'_{kl}$ appearing in the last sum we have

$$\sigma(e'_{kl}) = e_k.$$

This obviously implies that the supports of these $\rho(e_k)$ (regarded as functions on $N$) are disjoint for distinct $k$. Hence, passing to the vector

$$\rho(x) = \sum_{k=1}^{\infty} \lambda_k \rho(e_k),$$

we see that its support is the disjoint union of the supports of all $\rho(e_k)$, $k \in \mathbb{N}$. Therefore, this support is an infinite set. However, on the other hand, $\rho(x)$ is in $l^0_1(N)$, and hence its support is finite. A contradiction.

This completes the proof of Theorem 3.4.

Remark 3.6. As was mentioned above, we do not know whether all extremally projective normed spaces are isometrically isomorphic to the spaces $l^0_1(M)$; our approach (using the extreme points of the unit balls) does not work. At the same time, an answer was recently obtained to a similar question on the natural ‘weakened’ version of topological projectivity which is obtained from Definition 1.2 by replacing the word ‘surjective’ by the word ‘open’. Groenbaek [23] showed that every projective (in the above sense) normed space is topologically isomorphic to $l^0_1(M)$. Groenbaek’s theorem is a noncompleted version of an earlier result of Köthe (see [21]) who proved that every topologically projective Banach space is topologically isomorphic to $l_1(M)$.

§ 4. Injectivity and cofreeness

We return to the general categorical scheme. What happens if we define injective objects in a given category $\mathcal{K}$ along with projective objects, or instead of them? To this end, we consider a rigged category

$$(\mathcal{K}, \Box: \mathcal{K} \to \mathcal{M}),$$

where $\mathcal{M}$ is an additional category (which possibly differs from $\mathcal{L}$ in § 2).

A morphism (which is necessarily a monomorphism) $\iota$ in $\mathcal{K}$ is said to be an admissible monomorphism if $\Box(\iota)$ is a coretraction in $\mathcal{M}$. We say that an object $J \in \mathcal{K}$ is injective (to be more precise, $\Box$-injective) if the standard contravariant morphism functor $h_\mathcal{K}(?, J): \mathcal{K} \to \text{Set}$ takes the admissible monomorphisms to surjective maps.
The same can be said in a compressed form by using the notion of the so-called rigged category dual to \((\mathcal{K}, \Box)\), namely, the pair
\[
(\mathcal{K}^\circ, \Box^\circ : \mathcal{K}^\circ \to \mathcal{M}^\circ),
\]
where \(\Box^\circ\) indicates the dual category and the corresponding ‘copy’ of the functor \(\Box\). An object \(J\) is injective with respect to the original rigged category if and only if it is projective with respect to the dual rigged category.

**Example 4.1.** We consider the rigged category
\[
(\text{A-mod}, \Box : \text{A-mod} \to \text{Ban})
\]
of Example 2.4. It is easy to see that we obtain the customary notions of admissible monomorphism between Banach \(A\)-modules and of (relatively) injective Banach \(A\)-module (see [5]–[7] again).

**Example 4.2.** We consider a rigging \(\odot : \text{A-mod} \to \text{Set}^\circ\), taking a module \(X\) to the underlying set of the dual module \(X^*\) and a morphism in \(\text{A-mod}\) to the adjoint morphism regarded simply as a map of sets. Using the Hahn-Banach theorem, one can readily see that the \(\odot\)-admissible monomorphisms are topologically injective morphisms (cf., for example, [17], Ch. 2, §5). As far as the \(\odot\)-injective normed modules are concerned, we refer to them as topologically injective modules. The latter objects (under the name ‘strictly injective modules’) play a certain role in topological homology (see [5], Ch. VII, §10), although their role is lesser than that of relatively injective modules.

**Example 4.3.** We consider the rigged category (as a ‘companion’ of the rigged category in Example 2.7)
\[
(\text{A-mod}_1, \otimes : \text{A-mod}_1 \to \text{Set}^\circ),
\]
where \(\otimes\) denotes the covariant functor taking a module \(X\) to the closed unit ball \(\otimes_X := \bigcap X^*\) of its dual module \(X^*\) and a morphism in \(\text{A-mod}_1\) (which is a contraction, as we remember) to the corresponding restriction of the adjoint morphism to the unit balls.

It follows from the equivalent formulation of the Hahn-Banach theorem (presented in §1) that \(\otimes\)-admissible monomorphisms are precisely the isometric morphisms and the \(\otimes\)-injective objects are precisely the objects that have the ‘Hahn-Banach property’ (the ‘metric extension property’, as specialists in Banach space geometry say). Thus, the term ‘metrically injective’ is justified for a module of this kind.

**Example 4.4.** If we replace the ball \(\bigcap X^*\) by \(\otimes X^*\) in the definition of the last rigging, then the description of the admissible monomorphisms remains the same. The Hahn-Banach theorem provides the same isometries as admissible monomorphisms, and hence the same injective objects.

Passing from projectivity to injectivity, we are led unavoidably to so-called cofreeness. Briefly, an object in \(\mathcal{K}\), say \(\text{Cfr}(M)\), is said to be cofree with respect to a given rigged category \((\mathcal{K}, \Box : \mathcal{K} \to \mathcal{M})\), and \(M \in \mathcal{M}\) is said to be a cobase, if
\textbf{Cfr}(M) is a free object with respect to the dual rigged category \((\mathcal{K}^\circ, \square^\circ)\) and \(M\) is a base of \textbf{Cfr}(M). By analogy with Definition \ref{def:freepro} we define a \textit{cofreedom-loving} category. It is clear that Proposition \ref{prop:projectiveobject} has an obvious ‘twin’; in particular, \textit{an object in a cofreedom-loving category is injective if and only if it is a retract of a cofree object}.

The following general categorical observation, which is formally related to free objects only, actually provides a unified approach for seeking cofree objects in many specific cases.

**Proposition 4.5.** Let \((\mathcal{K}_1, \square: \mathcal{K}_1 \to \mathcal{L}_1)\) and \((\mathcal{K}_2, \square: \mathcal{K}_2 \to \mathcal{L}_2)\) be rigged categories and \(\Psi: \mathcal{K}_1 \to \mathcal{K}_2\) and \(\Upsilon: \mathcal{L}_1 \to \mathcal{L}_2\) be covariant functors making the diagram

\[
\begin{array}{ccc}
\mathcal{K}_1 & \xrightarrow{\square} & \mathcal{L}_1 \\
\downarrow\quad\quad\quad\Psi & & \downarrow\quad\quad\quad\Upsilon \\
\mathcal{K}_2 & \xrightarrow{\square} & \mathcal{L}_2
\end{array}
\]

(4.1)

commutative. Further, suppose that \(\Psi\) and \(\Upsilon\) have left adjoint functors, \(\Phi\) and \(\Delta\), respectively, and that \(F\) is a free object in \(\mathcal{K}_2\) with base \(M\).

Then \(\Phi(F)\) is a free object in \(\mathcal{K}_1\) with base \(\Delta(M)\).

**Proof.** We take an arbitrary object \(Y\) in \(\mathcal{K}_1\) and consider the following chain of bijections:

\[
\text{h}_{\mathcal{K}_1}(\Delta(M), \square(Y)) \to \text{h}_{\mathcal{K}_2}(M, \Upsilon \square(Y)) \to \text{h}_{\mathcal{K}_2}(M, \square \Psi(Y)) \\
\to \text{h}_{\mathcal{K}_1}(\Phi(F), Y),
\]

provided by the conditions on \(\Delta\) and \(\Phi\), by the definition of a free object and by diagram (4.1). The resulting bijection between \(\text{h}_{\mathcal{K}_1}(\Delta(M), \square(Y))\) and \(\text{h}_{\mathcal{K}_1}(\Phi(F), Y)\) is obviously natural with respect to \(Y\). The rest of the proof is clear.

To apply this proposition to our main examples, we consider, along with the category \(\text{A-mod}\), its right-module ‘twin’ \(\text{mod-A}\). Every normed space \(E\) generates the standard contravariant functors

\[
\mathcal{B}(?, E): \text{A-mod} \to \text{mod-A}, \quad \mathcal{B}(?, E): \text{mod-A} \to \text{A-mod}
\]

(cf., for example, \cite{5}, Ch. III, \S 1). Here \(\mathcal{B}(\cdot, \cdot)\) stands, as usual, for the corresponding space of all bounded operators equipped with the operator norm. The first functor takes a module \(X\) to the module \(\mathcal{B}(X, E)\) with right outer multiplication given by the equation

\[
(T \cdot a)(x) := T(a \cdot x), \quad a \in A, \quad x \in X, \quad T \in \mathcal{B}(X, E),
\]

and takes a morphism \(\varphi: X \to Y\) to the morphism

\[
\mathcal{B}(\varphi, E): \mathcal{B}(Y, E) \to \mathcal{B}(X, E), \quad \psi \mapsto \psi \varphi.
\]

The other functor is defined in a similar way.

Analogously, using the notation \(\text{mod-A}_1\) for the corresponding category of right modules, we obtain contravariant functors from \(\text{A-mod}_1\) to \(\text{mod-A}_1\) and from \(\text{mod-A}_1\) to \(\text{A-mod}_1\). We preserve the notation \(\mathcal{B}(?, E)\) and \(\mathcal{B}(?, E)\) for them.
Proposition 4.6. The functor $\mathcal{B}(?, r, E)$, regarded as a covariant functor from $\text{mod-}A$ to $\text{A-mod}^\circ$, is a left adjoint to the functor $\mathcal{B}(?, l, E)$, regarded as a covariant functor from $\text{A-mod}^\circ$ to $\text{mod-}A$.

The same assertion holds if we replace $\text{mod-}A$ by $\text{mod-}A_1$ and $\text{A-mod}^\circ$ by $\text{A-mod}_{1}^\circ$.

Proof. Let $X$ and $Y$ be left and right normed modules, respectively. By the law of adjoint associativity (cf., for example, [6], Ch. 6, §3), there is a bijection (in fact, an isometric isomorphism) between

$$h_{\text{mod-}A}(Y, \mathcal{B}(X, E)) = h_{\text{mod-}A}(Y, \mathcal{B}(?, r, E)(X))$$

and

$$h_{\text{A-mod}}(X, \mathcal{B}(Y, E)) = h_{\text{A-mod}}(\mathcal{B}(?, l, E)(Y), X)$$

which is natural with respect to $X$ and $Y$. The rest of the proof is clear.

Example 4.7. We derive from Proposition 4.5 a description of cofree Banach modules which is contained in the equivalent formulation in [5], Ch. III, §1 or [6], Ch. 7, §1, and a similar description of cofree normed modules.

We consider the rigged category $(\text{A-mod}, \Box)$ which provides the ‘customary’ projective and injective modules; see Examples 2.4 and 4.1. We take an arbitrary normed space $E$ and consider the diagram

$$
\begin{array}{ccc}
\text{A-mod}^\circ & \xrightarrow{\Box^\circ} & \text{Nor}^\circ \\
\mathcal{B}(?, l, E) & \downarrow & \mathcal{B}(?, l, E) \\
\text{mod-}A & \xrightarrow{\Box} & \text{Nor}
\end{array}
$$

where we use the notation $\Box$ for the obvious forgetful functor and the notation $\mathcal{B}(?, r, E)$ for the functor defined above for $A$-modules and (as a special case) for the normed spaces. We consider this diagram as a special case of the diagram (4.1) and the functor $\mathcal{B}(?, r, E)$ and its specialization for the normed spaces as special cases of $\Phi$ and $\Delta$, respectively. By Proposition 4.6, the conditions of Proposition 4.5 are satisfied.

It is clear that $A$ is a free right normed module with base normed space $C$. Therefore, it follows from Proposition 4.5 that the module $\mathcal{B}(?, r, E)(A)$ is a free object in $\text{A-mod}^\circ$ with respect to the rigged category $(\text{A-mod}^\circ, \Box^\circ)$, and its base object in $\text{Nor}^\circ$ is $\mathcal{B}(?, r, E)(C)$. Since $\mathcal{B}(C, E) = E$, this means that every normed space $E$ is a cobase space of a cofree left normed $A$-module, namely, the module $\mathcal{B}(A, E)$ with the outer multiplication $(a \cdot T)(b) := T(ba), a, b \in A, T \in \mathcal{B}(A, E)$.

We see that every cofree module in $\text{A-mod}$ is of the form $\mathcal{B}(A, E)$ for a suitable $E$, up to topological isomorphism.

A similar argument works in the completed case. The cofree left Banach modules are the same modules $\mathcal{B}(A, E)$, but this time $E$ ranges over the category of Banach spaces.

However, we are now interested in cofreeness in the rigged category $(\text{A-mod}_{1}, \oplus)$ of Example 4.3. In what follows, we refer to the $\oplus$-cofree objects in $\text{A-mod}_{1}$ as metrically cofree normed $A$-modules. We make a simple observation.
Proposition 4.8. Let the assumptions of Proposition 4.5 be satisfied and let the functor $\Delta$ have a right inverse functor $\nabla : \mathcal{L}_1 \to \mathcal{L}_2$.

In this case, if the rigged category $(\mathcal{K}_2, \Box)$ is freedom-loving, then so is the category $(\mathcal{K}_1, \Box)$, and $\Phi(Fr(\nabla(M)))$ is a free object in $\mathcal{K}_1$ with base $M$.

We consider the so-called star functors

$$\left( \begin{array}{c} * \\ l \end{array} \right) : A\text{-mod}_1 \to \text{mod}-A_1 \quad \text{and} \quad \left( \begin{array}{c} * \\ r \end{array} \right) : \text{mod}-A_1 \to A\text{-mod}_1$$

that take a normed module to its dual module and a morphism to its adjoint morphism. (These are obvious special cases of the functors $\mathcal{B}(?, E)$ and $\mathcal{B}(?, E)$ for $E := \mathbb{C}$.)

Theorem 4.9. The rigged category $(A\text{-mod}_1, \otimes : A\text{-mod}_1 \to \text{Set}^o)$ is cofreedom-loving, and a cofree module with cobase set $M$ is the module $\mathcal{B}(A, l_{\infty}(M))$ with outer multiplication defined by the equation

$$[a \cdot T](b) := T(ba), \quad a, b \in A, \quad T \in \mathcal{B}(A, E).$$

Proof. We consider the diagram

$$\begin{array}{ccc}
A\text{-mod}_1 & \xrightarrow{\otimes^o} & \text{Set} \\
\downarrow \scriptstyle \ell & & \downarrow \scriptstyle 1_{\text{Set}} \\
\text{mod}-A_1 & \xrightarrow{\Box} & \text{Set}
\end{array}$$

where the upper row is the rigging dual to that used in the statement and the lower row is the obvious right-module version of the rigging in Example 2.7. As an obvious right-module analogue of Theorem 2.18, the rigged category $(\text{mod}-A_1, \Box)$ is freedom-loving. Further, a right free module with base set $M$ is $l_1^0(M) \otimes p A$ with the outer multiplication well-defined by the equation

$$(x \otimes b) \cdot a := x \otimes ba, \quad a \in A, \quad b \in A, \quad x \in l_1^0(M).$$

With regard to what was said above, we can readily see that the conditions of Proposition 4.8 are satisfied if we set

$$\Phi := \left( \begin{array}{c} * \\ r \end{array} \right), \quad \Psi := \left( \begin{array}{c} * \\ l \end{array} \right), \quad \Delta := \Upsilon := \nabla := 1_{\text{Set}}.$$
§ 5. Metrically projective and free quantum spaces

We pass from ‘classical’ to ‘quantum’ functional analysis. We assume that the main notions and results of this branch of mathematics are known in the form presented in the books [24]–[27] (in the framework of the customary ‘matrix’ or ‘coordinate’ presentation). The only terminological modification is as follows: when speaking about the objects that these books call an ‘abstract operator space’ and an ‘operator space structure’, we say instead ‘quantum space’ and ‘quantum norm’, respectively (thereby avoiding the adjective ‘operator’, whose meaning varies). Thus, a quantum norm on a linear space $E$ is a sequence of norms that are given, for any $n \in \mathbb{N}$, on the space $M_n(E)$ of $(n \times n)$ matrices with entries in $E$ and satisfying Ruan’s axioms. A quantum space is a linear space equipped with a quantum norm.

If $\varphi : F \to E$ is an operator, then its $n$-amplification is the operator

$$\varphi_n : M_n(E) \to M_n(F)$$

taking a matrix $(a_{kl})$ to $(b_{kl}) := (\varphi(a_{kl}))$. An operator $\varphi$ between quantum spaces is said to be completely bounded if

$$\sup\{\|\varphi_n\| \mid n \in \mathbb{N}\} < \infty;$$

we refer to this supremum as the completely bounded norm of the operator $\varphi$ and denote it by $\|\varphi\|_{cb}$. The same operator $\varphi$ is said to be completely contractive, completely isometric, a completely isometric isomorphism, completely coisometric, or completely strictly coisometric if the operator $\varphi_n$ is a contraction, an isometry, an isometric isomorphism, and so on, for every $n$.

The space of completely bounded operators between quantum spaces $E$ and $F$, which admits a natural quantum norm by itself (see [24], §3.2), is denoted by $\mathcal{CB}(E,F)$.

If $E$ is a quantum space, we identify $M_m(M_n(E))$ with $M_{mn}(E)$, and thus equip $M_n(E)$ with the structure of a quantum space. We note that $\|\varphi_n\|_{cb} = \|\varphi\|_{cb}$.

In what follows, the symbols $\mathcal{QNor}_1$ and $\mathcal{QNor}$ denote the categories in which the objects are quantum spaces (not necessarily complete) and the morphisms are the completely contractive operators in the first category and arbitrary completely bounded operators in the other. The completed versions of these categories are denoted by $\mathcal{QBan}_1$ and $\mathcal{QBan}$, respectively.

Let $A$ be an algebra equipped (as a linear space) with a quantum norm. We refer to $A$ as a quantum algebra if the bilinear multiplication operator is completely contractive in the sense of [24], §7.1. A module over a quantum algebra is referred to as a quantum module if the bilinear operator of outer multiplication is completely contractive. The category of quantum modules and completely contractive morphisms (respectively, of completely bounded morphisms) is denoted by $\mathcal{QA}\text{-mod}_1$ (respectively, by $\mathcal{QA}\text{-mod}$). Thus, $\mathcal{QNor}_1 = \mathcal{QC}\text{-mod}_1$ and $\mathcal{QNor} = \mathcal{QC}\text{-mod}$.

We can define relatively projective and topologically projective quantum modules by ‘quantum’ versions of Definitions 1.1 and 1.2. These are projective objects of the category $\mathcal{QA}\text{-mod}$ with respect to the rigging $\Box : \mathcal{QA}\text{-mod} \to \mathcal{QNor}$ (respectively, $\Box : \mathcal{QA}\text{-mod} \to \text{Set}$), where $\Box$ is an appropriate forgetful functor. It can readily be seen that the first of these rigged categories (in contrast to the second) is freedom-loving, and its free object with base $E$ is $A \otimes_{\text{op}} E$, where $\otimes_{\text{op}}$ denotes...
the noncompleted operator-projective tensor product of quantum spaces (see [24], §7.1). The corresponding outer multiplication is defined by analogy with the ‘classical’ case (cf. §2 of the present paper).

However, the following kind of projectivity is the most interesting for us.

**Definition 5.1.** A quantum space $P$ is said to be **metrically projective** if for every completely strictly coisometric operator $\tau: F \to E$ between quantum spaces and an arbitrary completely bounded operator $\varphi: P \to E$ there is a completely bounded operator $\psi$ which is a lifting of the operator $\varphi$ with respect to $\tau$, and $\|\psi\|_{cb} = \|\varphi\|_{cb}$.

In a compressed form: $P$ is metrically projective if the functor of morphisms $\mathcal{CB}(P,?): \mathcal{QNor} \to \mathcal{QNor}$ preserves the property of an operator being completely strictly coisometric.

Similarly, with obvious modifications, one can give a definition of a metrically projective quantum module over a quantum algebra.

In what follows, for the sake of the simplicity of our presentation, we restrict ourselves to the case of quantum spaces (= quantum $\mathbb{C}$-modules). All subsequent constructions and results can be extended to the case of a general algebra $A$; cf. also the end of this section.

We now introduce a rigged category which makes it possible to study projectivity using freeness. We consider the covariant functor

$$\bigcirc: \mathcal{QNor}_1 \to \text{Set},$$

which acts as follows. It assigns, to a quantum space $E$, the set $\bigotimes_{n=1}^{\infty} \bigcirc M_n(E)$ which is the Cartesian product of the closed unit balls of the normed spaces $M_n(E)$ (see above). Thus, the elements of the set $\bigcirc(E)$ are sequences $(v_1, \ldots, v_n, \ldots)$, where $v_n \in \bigcirc M_n(E)$. As far as morphisms are concerned, our functor takes a completely contractive operator $\varphi: G \to E$ to the map

$$\bigcirc(\varphi): \bigcirc(G) \to \bigcirc(E), \quad (u_1, \ldots, u_n, \ldots) \mapsto (\varphi_1(u_1), \ldots, \varphi_n(u_n), \ldots).$$

It is clear that this functor is a rigging. We note the following obvious statement.

**Proposition 5.2.** A completely contractive operator $\tau: G \to E$ between quantum spaces is an admissible morphism with respect to the rigged category $(\mathcal{QNor}_1, \bigcirc)$ if and only if it is completely strictly coisometric.

**Corollary 5.3.** The $\bigcirc$-projective quantum spaces are precisely the metrically projective quantum spaces.

We denote by $\mathcal{N}_n$, $n \in \mathbb{N}$, the spaces of trace-class operators on $\mathbb{C}^n$ that are equipped with the trace norm. We recall that, by the Schatten-von Neumann theorem, there are isometric isomorphisms $\mathcal{N}_n \to (\mathcal{B}(\mathbb{C}^n))^*$ which assign to any operator $b$ the functional

$$f: a \mapsto \text{tr}(ba), \quad a \in \mathcal{B}(\mathbb{C}^n).$$

(Here and below, $\text{tr}(\cdot)$ stands for the trace.) This enables one to equip all spaces $\mathcal{N}_n$, $n \in \mathbb{N}$, with quantum norms taken from the corresponding dual spaces; see, for example, [24], §3.2.
We fix a quantum space $E$ for the time being. The existence of a completely isometric isomorphism between the quantum spaces $M_n(E)$ and $\mathcal{CB}(\mathcal{N}_n, E)$ is the main statement of Proposition 5.4 below; this fact was mentioned by Blecher [12] (according to [12], this statement follows from the results of [28]).

We recall that $M_n(E)$ can be identified, as a linear space, with the space $M_n \otimes E$. Further, as a quantum space, $M_n := M_n(\mathbb{C})$ is identified with $B(\mathbb{C}^n)$. Thus, $\mathcal{N}_n = M_n^*$. Here $(\mathcal{N}_n)^* = M_n$ by [24], Proposition 3.2.1 because $\dim \mathcal{N}_n < \infty$.

**Proposition 5.4.** There is a completely isometric isomorphism

$$\iota_n^E : M_n(E) \to \mathcal{CB}(\mathcal{N}_n, E),$$

which is well-defined by the condition that it takes every elementary tensor $a \otimes x \in M_n \otimes E$ to the operator

$$b \mapsto \text{tr}(ab)x, \quad a \in M_n = B(\mathbb{C}^n), \quad b \in \mathcal{N}_n, \quad x \in E.$$

Here the morphism $\iota_n^E$ is natural with respect to $E$, that is, in more detail, for every completely bounded operator $\varphi : G \to E$ between quantum spaces the diagram

$$
\begin{array}{ccc}
M_n(G) & \xrightarrow{\iota_n^G} & \mathcal{CB}(\mathcal{N}_n, G) \\
\varphi_n \downarrow & & \downarrow \varphi_n^* \\
M_n(E) & \xrightarrow{\iota_n^E} & \mathcal{CB}(\mathcal{N}_n, E)
\end{array}
$$

where $\varphi_n^*$ is induced by the operator $\varphi$ (that is, it acts by the rule $\psi \mapsto \varphi \psi$), is commutative.

**Proof.** We denote by $\mathcal{I}$ the canonical embedding of $E$ in $E^{**}$ and consider the following chain of quantum spaces and operators:

$$
M_n(E) \xrightarrow{\iota_1} M_n(E^{**}) \xrightarrow{\iota_2} \mathcal{CB}(E^*, M_n) \xrightarrow{\iota_3} \mathcal{CB}(\mathcal{N}_n, E^{**}).
$$

Our operators are as follows:

$\iota_1$ is the $n$-amplification of $\mathcal{I}$, and therefore it is completely isometric together with $\mathcal{I}$ (see [24], Proposition 3.2.1);

$\iota_2$ is the completely isometric isomorphism appearing in the definition of the quantum dual space (see [24], §3.2);

$\iota_3$ takes $\varphi : E^* \to M_n = (\mathcal{N}_n)^*$ (see above) to $\psi : f \mapsto \beta$, where

$$\beta : g \mapsto [\varphi(g)](f), \quad f \in \mathcal{N}_n, \quad \beta \in (E^*)^*, \quad g \in E^*.$$

By the rule of so-called quantum adjoint associativity (see [24], §7.1), this is a completely isometric isomorphism.

We see that the composition of these operators, which we denote by

$$\iota_0 : M_n(E) \to \mathcal{CB}(\mathcal{N}_n, E^{**}),$$

is a complete isometry. However, considering the elementary tensors in $M_n \otimes E$, we can readily see that $\iota_0 = \iota_4 \iota_n^E$, where the operator $\iota_4 : \mathcal{CB}(\mathcal{N}_n, E) \to \mathcal{CB}(\mathcal{N}_n, E^{**})$
is induced by the operator $\mathcal{J}$. Since $\iota_0$ and $\iota_4$ are complete isometries (the latter by [24], Proposition 3.2.1 again), the same holds for $\iota_n^E$.

Moreover, the identification of $(\mathcal{N}_n)^*$ and $M_n$ obviously implies that every one-dimensional operator in $\mathcal{C}\mathcal{B}(\mathcal{N}_n, E)$ acts by the rule $b \mapsto \text{tr}(ab)x$ for some $a \in M_n = \mathcal{B}(\mathbb{C}^n)$ and $x \in E$, and hence coincides with $\iota_n^E(a \otimes x)$. Since $\dim \mathcal{N}_n < \infty$, this implies that $\iota_n^E$ is a surjection. Hence, this is a completely isometric isomorphism.

As far as the diagram (5.1) is concerned, one can immediately verify the commutativity of the diagram at the elementary tensors in $M_n(G)$.

In what follows, we use the following well-known statement heavily (see Blecher [12]): the category $\text{QNor}_1$ (as well as the category $\text{A-mod}_1$ in §2) admits coproducts.

Following [12], we denote the coproduct object of a family $E_\nu, \nu \in \Lambda$, of quantum spaces by $\bigoplus_1 \{E_\nu \mid \nu \in \Lambda\}$. The universal property of this quantum space is explicitly indicated by Pisier (see [26], §2.6), where the notation $l_1(\{E_\nu \mid \nu \in \Lambda\})$ is used; see also the book of Blecher and Le Merdy [27], §1.4. We note only that all these authors discuss Banach (= complete) operator spaces, while we prefer to consider the noncompleted case. However, both cases are quite similar; the Banach space coproduct is simply the completion of the coproduct space considered here.

**Remark 5.5.** We use only the existence of coproducts in $\text{QNor}_1$ rather than their explicit construction. Nevertheless, for the convenience of the reader, we recall one of the possible realizations. As a linear space, $\bigoplus_1 \{E_\nu \mid \nu \in \Lambda\}$ is an algebraic sum $\bigoplus \{E_\nu \mid \nu \in \Lambda\}$, and the injections of the coproduct are the natural embeddings of the direct summands.

To introduce a quantum norm, that is, ‘classical’ norms that satisfy the Ruan axioms on the spaces $M_n(\bigoplus \{E_\nu \mid \nu \in \Lambda\})$, we consider the index set $\Upsilon$ consisting of all possible pairs of the form

$$(H_{\Upsilon}, \{\varphi_\nu : E_\nu \to \mathcal{B}(H_{\Upsilon}) \mid \nu \in \Lambda\}),$$

where $H_{\Upsilon}$ is a Hilbert space (one and the same for all $\nu$) and $\varphi_\nu, \nu \in \Lambda$, are completely contractive operators.

We take $u \in M_n(\bigoplus \{E_\nu \mid \nu \in \Lambda\})$. Since the last space coincides, up to a linear isomorphism, with $\bigoplus \{M_n(E_\nu) \mid \nu \in \Lambda\}$, it follows that our element $u$ can be represented as a sum $\sum_\nu u_\nu$, where $u_\nu \in M_n(E_\nu)$, with finitely many nonzero summands. We now consider the space $\mathcal{B}(H_{\Upsilon})$ equipped with the standard quantum norm and write

$$\|u\| := \sup \left\{ \left\| \sum_\nu (\varphi_\nu)_n(u_\nu) \right\| \right\},$$

where the supremum is taken over all families $\{\varphi_\nu \mid \nu \in \Lambda\}$ belonging to pairs in $\Upsilon$. All the properties we need can readily be verified.

Combining the information on coproducts presented here with Proposition 2.12, we obtain the following corollary.

**Corollary 5.6.** The $\bigoplus_1$-sum of an arbitrary family of metrically projective quantum spaces is metrically projective.
To move forward, we need two preliminary statements. The first of them is related to linear algebra.

**Proposition 5.7.** Let $E$, $F$ and $G$ be linear spaces, $n \in \mathbb{N}$, $\dim F = \dim G = n$, and $e'_1, \ldots, e'_n$, $e'_1, \ldots, e'_n$ be linear bases in $F$ and $G$, respectively. We write $u := \sum_{k=1}^n e'_k \otimes e'_k \in G \otimes F$.

Then for every $v \in G \otimes E$ there is a unique operator $\varphi : F \to E$ such that $[1_G \otimes \varphi](u) = v$.

**Proof.** As we know, the vector $v$ can be represented in the form $\sum_{k=1}^n e'_k \otimes x_k$, where the $x_k \in E$ are uniquely defined. On the other hand, for every operator $\psi : F \to E$ we have $[1_G \otimes \psi](u) = \sum_{k=1}^n e'_k \otimes \psi(e'_k)$.

Hence by the linear independence of the vectors $e'_1, \ldots, e'_n$, the desired $\varphi$ is the unique operator taking $e'_k$ to $x_k$, $k = 1, \ldots, n$.

We concentrate now on the special case of Proposition 5.4 in which $E$ is $\mathcal{N}_n$. A completely isometric isomorphism arises,

$$i_n^{\mathcal{N}_n} : M_n(\mathcal{N}_n) \to \mathcal{CB}(\mathcal{N}_n, \mathcal{N}_n).$$

Further, we single out a special element $I_n$ in $M_n(\mathcal{N}_n)$ which is uniquely defined by the condition that $i_n^{\mathcal{N}_n}(I_n) = 1_{\mathcal{N}_n}$.

(In fact, $I_n = \sum_{i,j=1}^n e_{ij} \otimes e_{ji}$, where $e_{ij}$ stands for the elementary matrix with 1 at the $(i,j)$th place. However, we do not use this fact.) Certainly, $\|I_n\| = 1$.

**Proposition 5.8.** For any quantum space $E$, $n \in \mathbb{N}$, and $x \in \bigotimes M_n(E)$ there is a unique operator $\varphi : \mathcal{N}_n \to E$ such that $\varphi_n : M_n(\mathcal{N}_n) \to M_n(E)$ takes $I_n$ to $x$. Moreover, $\varphi$ is completely contractive.

**Proof.** The first statement is an obvious special case of Proposition 5.7 with $\mathcal{N}_n$ instead of $G$, and so on. As far as the second statement is concerned, we take $\mathcal{N}_n$ for $G$ again, apply diagram (5.1), and use the operator $\varphi \in \mathcal{CB}(\mathcal{N}_n, E)$. We see that $\varphi = \varphi_n^*(1_{\mathcal{N}_n}) = i_n^E \varphi_n(I_n) = i_n^E(x)$.

Therefore, $\|\varphi\|_{cb} = \|x\| \leq 1$ holds, because $i_n^E$ is isometric.

Finally, we can present $\bigcirc$-free objects, which are referred to in what follows as *metrically free quantum spaces*. We denote by $(\mathcal{N}_\infty, \{i^n \mid n = 1, 2, \ldots\})$ the coproduct of the family $\{\mathcal{N}_n \mid n = 1, 2, \ldots\}$ in $\textbf{QNor}_1$. In other words,

$$\mathcal{N}_\infty := \mathcal{N}_1 \oplus_1 \mathcal{N}_2 \oplus_1 \cdots \oplus_1 \mathcal{N}_n \oplus_1 \cdots.$$
Here, obviously, \(i^n : \mathcal{N}_n \to \mathcal{N}_\infty\) are coretractions in \(\mathbb{Q}\text{Nor}_1\), and the same is true for their amplifications \((i^n)_n : M_n(\mathcal{N}_n) \to M_n(\mathcal{N}_\infty)\). In particular, these operators are completely isometric. For each \(n\) we write
\[
I^n := (i^n)_n(I_n);
\]
it is clear that \(\|I^n\| = 1\).

**Theorem 5.9.** The rigged category \(\mathbb{Q}\text{Nor}_1, \bigcirc\) is freedom-loving. Moreover,

(i) the metrically free quantum space with a one-point base, say \(\{t\}\), is \(\mathcal{N}_\infty\);

(ii) if \(M\) is a set, then a metrically free quantum space with base \(M\) is the coproduct of a family of copies of the space \(\mathcal{N}_\infty\) indexed by the points of \(M\).

**Proof.** (i) For every quantum space \(E\) we are to find a bijection
\[
\mathcal{J}_E : h\text{Set}(\{t\}, \bigcirc(E)) \to h\text{Nor}_1(\mathcal{N}_\infty, E), \tag{5.2}
\]
which is natural with respect to \(E\). We consider a map \(\varphi^0 : \{t\} \to \bigcirc(E)\) taking \(t\) to some sequence \((x_1, \ldots, x_n, \ldots)\), \(x_n \in M_n(E)\). By Proposition 5.8, for every \(n \in \mathbb{N}\), there is a unique completely contractive operator \(\varphi^n : \mathcal{N}_n \to E\) such that
\[
(\varphi^n)_n(I_n) = x_n.
\]
By the universal property of coproducts (see §2), there is a unique completely contractive operator \(\varphi : \mathcal{N}_\infty \to E\) such that
\[
\varphi_i = \varphi^n.
\]
We obtain a well-defined map \(\mathcal{J}_E : \varphi^0 \mapsto \varphi\) between our sets of morphisms.

Further, let a contraction operator \(\varphi : \mathcal{N}_\infty \to E\) be given. We assign to this operator the sequence
\[
(\ldots, x_n, \ldots) \in \bigcirc(E),
\]
where \(x_n := \varphi_n(I^n)\), and then the map from \(\{t\}\) to \(\bigcirc(E)\) taking our point \(t\) to this sequence. We obtain a map \(\mathcal{J}_E\) from the second set in (5.2) to the first. Since
\[
\varphi_n(I^n) = \varphi_n(i^n)_n(I_n) = (\varphi^n)_n(I_n),
\]
we can readily see that \(\mathcal{J}_E\) and \(\mathcal{J}_E\) are mutually inverse maps.

Thus, \(\mathcal{J}_E\) is a bijection. It is clear that \(\mathcal{J}_E\) is natural with respect to \(E\).

The statement (ii) follows immediately from (i) and Corollary 2.17.

A quantum space is said to be *trace-class composed* if it is of the form \(\oplus_1 \{E_\nu \mid \nu \in \Lambda\}\), where every summand is of the form \(\mathcal{N}_n\) for some \(n \in \mathbb{N}\). Theorem 5.9 shows that such a space is metrically free if and only if the cardinality of the set of summands \(\mathcal{N}_n\) with a fixed \(n\) is the same for all \(n\).

Combining Theorem 5.9 and Proposition 2.11, we obtain the following corollary.

**Corollary 5.10.** (i) Every quantum space is the image of a strictly coisometric operator on a trace-class-composed quantum space.

(ii) A quantum space is metrically projective if and only if it is a retract in \(\mathbb{Q}\text{Nor}_1\) of some trace-class-composed quantum space (or, equivalently, a direct summand of a space of this kind with the corresponding natural projection of unit completely bounded norm). In particular, all spaces \(\mathcal{N}_n, n \in \mathbb{N}\), are metrically projective.
This corollary is a metric version of results in [12] (Proposition 3.1 and Theorem 3.10) concerning extremally projective quantum spaces (see §6). The consideration in [12] is based on work with quantum duals of coproducts mentioned above.

Certainly, the metric projectivity of the spaces $\mathcal{N}_n$ (the ‘bricks’) follows immediately from the identification of $M_n(E)$ with $C\mathcal{B}(\mathcal{N}_n, E)$ given by Proposition 5.4 (see [12], Proposition 3.7).

Theorem 5.9 can be readily extended from quantum spaces to quantum modules (see the beginning of the section). The corresponding rigging acts from the category $\textup{QA-mod}$ to $\textup{Set}$ and, as for the rigging $\boxtimes$ treated above, takes an $A$-module $X$ to $X_{n=1}^\infty \bigodot M_n(X)$. The rigged category thus obtained is also freedom-loving, and its free modules are $\oplus_1$-sums of families of copies of the quantum $A$-module $A \otimes_{\text{op}} N_\infty$.

Another direction of investigation is related to the study of injective and cofree quantum spaces (and modules) with respect to a quantum version of the rigging $\circ$ used in §4. In particular, the cofree quantum spaces turn out to be quantum $l_\infty$-sums of quantum spaces $\mathcal{B}(\mathbb{C}^n)$, $n = 1, 2, \ldots$ (with one copy for every $n$).

§6. Asymptotic structure in categories and extremal projectivity

The definitions and results related to extremal projectivity necessarily have, so to speak, an asymptotic nature. We suggest here a general scheme including this kind of projectivity. In what follows, $\mathcal{K}$ is an arbitrary category.

**Definition 6.1.** Let $\{\mathbb{J}_\nu \mid \nu \in \Lambda\}$ be a family of natural transformations of the identity functor taking $\mathcal{K}$ onto itself. (We recall that, in an expanded form, this means that for every object $X$ in $\mathcal{K}$ there is a morphism $\mathbb{J}^X_\nu : X \to X$, and for every $\nu \in \Lambda$ and every morphism $\varphi : X \to Y$ the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow^{\mathbb{J}^X_\nu} & & \downarrow^{\mathbb{J}^Y_\nu} \\
X & \xrightarrow{\varphi} & Y
\end{array}
\]

is commutative.)

A family of this kind is said to be an **asymptotic structure in $\mathcal{K}$** if it satisfies the following two conditions:

(i) for every $\nu \in \Lambda$ and any $X$ the morphism $\mathbb{J}^X_\nu$ is a bimorphism;

(ii) for every $\nu \in \Lambda$ there are $\lambda, \mu \in \Lambda$ such that $\mathbb{J}_\nu = \mathbb{J}_\lambda \mathbb{J}_\mu$.

**Definition 6.2.** A triple consisting of a category, a rigging on the category, and an asymptotic structure on the category is said to be an **asymptotic category**.

From now on, we assume that an asymptotic category is given,

$$\left(\mathcal{K}, \square : \mathcal{K} \to \mathcal{L}, \{\mathbb{J}_\nu \mid \nu \in \Lambda\}\right).$$

**Definition 6.3.** A morphism $\varphi : X \to Y$ in $\mathcal{K}$ is said to be **permitted** (with respect to a given asymptotic structure) if there are $\nu \in \Lambda$ and a morphism $\overline{\varphi} : X \to Y$ such that $\varphi = \overline{\varphi} \mathbb{J}^X_\nu$ (or, equivalently, $\varphi = \mathbb{J}^Y_\nu \overline{\varphi}$).
Definition 6.4. A morphism $\tau: Y \to X$ in $\mathcal{H}$ is said to be an asymptotically admissible epimorphism if for every $\nu \in \Lambda$ there is a morphism $\rho_\nu: \Box(X) \to \Box(Y)$ in $\mathcal{L}$ such that

$$\Box(\tau) \rho_\nu \Box(J^X_\nu) = \Box(J^X_\nu).$$

The family $\{\rho_\nu \mid \nu \in \Lambda\}$ is said to be an asymptotic right inverse to $\Box(\tau)$.

The word ‘epimorphism’ can be justified. Indeed, if $\varphi_\tau = \psi_\tau$, then

$$\Box(\varphi) \Box(\tau) \rho_\nu \Box(J^X_\nu) = \Box(\psi) \Box(\tau) \rho_\nu \Box(J^X_\nu).$$

Hence,

$$\Box(\varphi) \Box(J^X_\nu) = \Box(\psi) \Box(J^X_\nu), \quad \Box(\varphi J^X_\nu) = \Box(\psi J^X_\nu).$$

Since $\Box$ is a faithful functor and $\Box(J^X_\nu)$ is an epimorphism (see Definition 6.1), it follows that $\varphi = \psi$.

Proposition 6.5. Every admissible epimorphism is asymptotically admissible.

As a consequence, if our asymptotic category is freedom-loving when viewed as a rigged category, then every object in $\mathcal{H}$ is the target of an asymptotically admissible morphism from a free object.

Proof. We take for any $\nu \in \Lambda$ a ‘genuine’ right inverse to $\Box(\tau)$ for the morphism $\rho_\nu$. The rest of the proof is clear.

Finally, we can introduce the following definition.

Definition 6.6. An object $P$ in $\mathcal{H}$ is said to be asymptotically projective if every permitted morphism $\varphi: P \to X$ admits a lifting in $\mathcal{H}$ with respect to any asymptotically admissible epimorphism.

Example 6.7 (‘classical’). We consider the rigged category from Example 2.7. We set $\Lambda := (0, 1)$ and write

$$J^X_t: X \to X, \quad x \mapsto tx,$$

for any $t \in (0, 1)$ and normed $A$-module $X$. This is an asymptotic structure in $\mathbf{A-mod}_1$. Indeed, the properties of natural transformations imposed in Definition 6.1, as well as condition (i), obviously hold, and condition (ii) also holds, at least because $J^X_t = (J^X_{\sqrt{t}})^2$. We obtain an asymptotic category with $\mathbf{A-mod}_1$ playing the role of $\mathcal{H}$ and with the functor $\Box$ for $\Box$. It is clear that permitted morphisms are morphisms whose norm is less than 1. Asymptotically admissible epimorphisms are precisely coisometries. Indeed, if a coisometry $\tau: Y \to X$ is given, then for $\rho_\nu: \Box_X \to \Box_Y$ we use a map taking the vector $x$ to $y$, where $y$ is an arbitrary vector in $\Box_Y$ such that $\tau(y) = x$ if $\|x\| \leq t$; an arbitrary vector in $\Box_Y$ otherwise.

It can readily be seen that the asymptotically projective modules are extremally projective ones in the sense of Definition 1.1.

Example 6.8 (‘quantum’). We take the rigged category $(\mathbf{QNor}_1, \Box)$ from §5. We set $\Lambda := (0, 1)$ again and write

$$J^E_t: E \to E, \quad x \mapsto tx,$$
for any \( t \in (0, 1) \) and quantum space \( E \). As in the previous example, we obtain an asymptotic category, this time with \( Q\text{Nor} \) playing the role of \( \mathcal{H} \) and with the functor \( \odot \) for \( \square \). An operator \( \varphi \) is now a permitted morphism precisely when \( \| \varphi \|_{cb} < 1 \), and the asymptotically admissible epimorphisms are the completely coisometric morphisms. Indeed, if a complete coisometry \( \tau: G \to E \) is given, then for

\[
\rho_t: X\{M_n(E) \mid n = 1, 2, \ldots \} \to X\{M_n(G) \mid n = 1, 2, \ldots \}
\]

we use a map taking every sequence \((\ldots, x_n, \ldots)\) to a sequence \((\ldots, y_n, \ldots)\), where \( y_n \) is

- any vector in \( \bigodot M_n(G) \) such that \( \tau(y_n) = x_n \) if \( \| x_n \| \leq t \);
- any vector in \( \bigodot M_n(G) \) otherwise.

It can readily be seen that a quantum space \( P \) is asymptotically projective if and only if it has the following property: for any completely coisometric operator \( \tau: G \to E \), every completely bounded operator \( \varphi: P \to E \), and each number \( \varepsilon > 0 \) there is a completely bounded operator \( \psi: P \to G \) such that

1. it is a lifting of the operator \( \varphi \) with respect to \( \tau \),
2. \( \| \psi \|_{cb} < \| \varphi \|_{cb} + \varepsilon \).

These quantum spaces were introduced in [12] and were referred to there as projective spaces; we say that they are \textit{extremally projective}.

This asymptotic structure (as well as Corollary 6.13, (ii) below) can be extended from quantum spaces to quantum modules over an arbitrary quantum algebra.

We return to the general scheme. When speaking about free objects in an asymptotic category and freedom-loving asymptotic categories, we always mean the underlying rigged category.

**Proposition 6.9.** Every free object in an asymptotic category is asymptotically projective.

**Proof.** Let \( \text{Fr}(M) \) be a \( \square \)-free object with base object \( M \). Suppose that an asymptotically admissible epimorphism \( \tau: Y \to X \) with corresponding asymptotic right inverse \( \rho_\nu, \nu \in \Lambda \), and a permitted morphism \( \varphi: \text{Fr}(M) \to X \) are given; thus, \( \varphi \) is of the form \( J X \mu \widetilde{\varphi} \) for some \( \mu \in \Lambda \) and \( \widetilde{\varphi}: \text{Fr}(M) \to X \).

Recalling the bijections appearing in Definition 2.10, we set

\[
\alpha := \mathcal{J}_{\text{Fr}(M)}^{-1}(1_{\text{Fr}(M)}): M \to \square(\text{Fr}(M)), \quad \psi^0 := \rho_\mu \square(\varphi)\alpha: M \to \square(Y)
\]

and, finally,

\[
\psi := \mathcal{J}_Y(\psi^0): \text{Fr}(M) \to Y.
\]

Since the bijections mentioned above are natural with respect to the second argument, we have

\[
\tau \psi = \tau \mathcal{J}_Y(\psi^0) = \mathcal{J}_X(\square(\tau)\psi^0) = \mathcal{J}_X(\square(\tau)\rho_\mu \square(\varphi)\alpha) = \mathcal{J}_X(\square(\tau)\rho_\mu \square(\mathcal{J}_\mu \widetilde{\varphi})\alpha)
\]

\[
= \mathcal{J}_X(\square(\mathcal{J}_\mu \widetilde{\varphi})\square(\varphi)\alpha) = \mathcal{J}_X(\square(\varphi)\alpha) = \varphi \mathcal{J}_{\text{Fr}(M)}(\alpha)
\]

\[
= \varphi \mathcal{J}_{\text{Fr}(M)} \mathcal{J}_{\text{Fr}(M)}^{-1}(1_{\text{Fr}(M)}) = \varphi.
\]
Proposition 6.10. Let $P$ be an asymptotically projective object, $\tau: Y \to X$ be an asymptotically admissible epimorphism, and $\varphi: P \to X$ be a permitted morphism. Then the set of all liftings of the morphism $\varphi$ with respect to $\tau$ contains a permitted morphism.

Proof. The condition on $\varphi$, combined with Definition 6.1, gives $\lambda, \mu \in \Lambda$ and $\bar{\varphi}: P \to X$ such that

$$\varphi = \mathbb{J}_X^\lambda \mathbb{J}_X^\mu \bar{\varphi}.$$  

Since $\mathbb{J}_X^\mu \bar{\varphi}$ is permitted, the condition on $P$ gives $\psi: P \to Y$ such that

$$\tau \psi = \mathbb{J}_X^\mu \bar{\varphi}.$$  

We write $\psi := \tilde{\psi} \mathbb{J}_X^P$. Then

$$\tau \psi = \tau \tilde{\psi} \mathbb{J}_X^P = \mathbb{J}_X^\mu \bar{\varphi} \mathbb{J}_X^P = \mathbb{J}_X^\lambda \mathbb{J}_\mu \bar{\varphi} = \varphi.$$  

Definition 6.11. A morphism $\sigma: U \to V$ in $\mathcal{K}$ is said to be an asymptotic retraction if for every $\nu \in \Lambda$ there is a morphism $\zeta_\nu: V \to U$ such that $\sigma \zeta_\nu = \mathbb{J}_V^\nu$. An object $V$ in $\mathcal{K}$ is said to be an asymptotic retract of an object $U$ if there is an asymptotic retraction from $U$ to $V$.

We consider the asymptotic category from Example 6.7. It can readily be seen that a contraction morphism of normed $A$-modules is an asymptotic retraction if and only if for any $\varepsilon > 0$ this morphism admits a right inverse module morphism whose norm is less than $1 + \varepsilon$. Thus, an asymptotic retraction is precisely an almost retraction (or a near-retraction in the terminology of [10]).

One can similarly describe the asymptotic retractions in the context of the asymptotic category from Example 6.8. Namely, a completely contractive operator $\sigma: U \to V$, where $U$ and $V$ are quantum spaces, is an asymptotic retraction if and only if for any $\varepsilon > 0$ this operator admits a right inverse completely bounded operator with completely bounded norm less than $1 + \varepsilon$. We again refer to such an operator as an almost retraction or a near-retraction (this time in $\text{QNor}$). It can readily be seen that $V$ is an asymptotic retract of some $U$ if and only if it is an almost direct summand of this space $U$ in the terminology of [12].

Proposition 6.12. Suppose that our asymptotic category is freedom-loving. In this case, an object $P$ in $\mathcal{K}$ is asymptotically projective if and only if it is an asymptotic retract of a free object.

Proof. Necessity. By Propositions 2.11, (i) and 6.5, there are a free object $F$ and an asymptotically admissible epimorphism $\tau: F \to P$. We take $\nu \in \Lambda$. Since $\mathbb{J}_X^P = \mathbb{J}_X^P \mathbb{1}_P: P \to P$ is a permitted morphism, it admits a lifting, say $\zeta_\nu$, with respect to $\tau$. The rest of the proof is clear.

Sufficiency. Let $\sigma: F \to P$ be an asymptotic retraction with free domain, let $\tau: Y \to X$ be an asymptotically admissible epimorphism, and $\varphi: P \to X$ a permitted morphism. In this case, one can readily see (using the diagram in Definition 6.1) that $\varphi \sigma: F \to X$ is also permitted. Hence, by Propositions 6.9 and 6.10, there is a $\chi: F \to Y$ such that $\tau \chi = \varphi \sigma$ and $\chi = \mathbb{J}_Y^\nu \tilde{\chi}$ for some $\tilde{\chi}: F \to Y$ and $\nu \in \Lambda$. 

We write \( \psi := \hat{\chi} \zeta \nu \), where \( \zeta \nu : P \to F \) is such that \( \sigma \zeta \nu = \mathbb{I}^P \nu \) (see Definition 6.11). Then
\[
\mathbb{J}^X \nu \tau \psi = \mathbb{J}^X \nu \tau \hat{\chi} \zeta \nu = \tau \mathbb{J}^Y \nu \hat{\chi} \zeta \nu = \tau \chi \zeta \nu = \varphi \sigma \zeta \nu = \varphi \mathbb{J}^P \nu = \mathbb{J}^X \nu \varphi.
\]
However, \( \mathbb{J}^X \nu \) is a monomorphism (see Definition 6.1), and hence \( \tau \psi = \varphi \).

Combining this proposition with what was said above about the asymptotic categories in Examples 6.7 and 6.8, and also with Theorems 2.18 and 5.9, we immediately obtain the following corollary.

**Corollary 6.13.** (i) A normed module \( P \) over a normed algebra \( A \) is extremally projective if and only if it is a near-retract of a module of the form \( A \otimes_p l^0_1(M) \), where \( M \) is a set.

The same is true for Banach modules over Banach algebras after replacing \( \otimes_p \) by \( \hat{\otimes} \) and \( l^0_1 \) by \( l^1 \).

(ii) (Blecher [12], Theorem 3.10.) A quantum space \( P \) is extremally projective if and only if it is a near-retract of a coproduct in \( \mathbb{Q} \text{Nor}_1 \) of some family of spaces \( N_n, n \in \mathbb{N} \).

The same is true after replacing the words ‘quantum space’ by ‘quantum Banach space’ and \( \mathbb{Q} \text{Nor}_1 \) by \( \mathbb{Q} \text{Ban}_1 \).

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