Abstract

Geometric reduction of the Newtonian planar three-body problem is investigated in the framework of equivariant Riemannian geometry, which reduces the study of trajectories of three-body motions to the study of their moduli curves, that is, curves which record the change of size and shape, in the moduli space of oriented mass-triangles. The latter space is a Riemannian cone over the shape sphere \( S^2 \), and the shape curve is the image curve on this sphere. It is shown that the time parametrized moduli curve is in general determined by the relative geometry of the shape curve and the shape potential function. This also entails the reconstruction of time, namely the geometric shape curve determines the time parametrization of the moduli curve, hence also of the three-body motion itself, modulo a fixed rotation of the plane.

The first version of this work is an unpublished paper from 2012, and the present version is an editorial revision of this.
Traditionally, the 3-body problem in celestial mechanics is most often studied in the framework of Hamiltonian mechanics, cf. e.g. [11]. However, since 1994 (cf. [3]) the authors have studied the kinematic geometry of 3-body systems and geometric reduction of 3-body motions, in the framework of equivariant Riemannian geometry and inspired by Jacobi’s geometrization of Lagrange’s least action principle. For a general introduction to this reduction approach we refer to the monograph [4].

In the general study of the 3-body problem, the Newtonian dynamical equations are formulated at the level of the configuration space \( M \), which consists of all ordered positions of the three bodies in a Euclidean 3-space, called \( m \)-triangles, and the solutions \( t \to \gamma(t) \) of the 3-body motions are the trajectories of these equations. The group of rigid motions acts naturally on \( m \)-triangles, whose orbits are the congruence classes of \( m \)-triangles, and we shall denote by \( \overline{M} \) the \textit{moduli space} consisting of all congruence classes. The image \( \overline{\gamma}(t) \) of the trajectory \( \gamma(t) \) in \( \overline{M} \) is referred to as the \textit{moduli curve}. This is the first step of the geometric reduction, namely the reduction of the 3-body problem to dynamics and analysis in the quotient space \( \overline{M} \) of \( M \).

Indeed, the reconstruction of the 3-body motion \( \gamma(t) \) from the curve \( \overline{\gamma}(t) \) is a purely geometric problem, essentially the lifting problem of a fiber bundle \( M \to \overline{M} \). Generally, the trajectory is uniquely determined, modulo a global
congruence, by the moduli curve, but in this paper we shall not be concerned with the associated lifting procedure.

In the second step of the reduction procedure, \( \tilde{M} \) is replaced by a subspace \( M^* \) called the shape space, consisting of congruence classes of \( m \)-triangles of unit size, and hence the points of \( M^* \) represent the (nonzero) similarity classes of \( m \)-triangles. It is a crucial fact that \( \tilde{M} \) naturally identifies with the 3-space \( \mathbb{R}^3 \) and with \( M^* \) as its (unit) sphere \( S^2 \), with respect to the naturally induced metric (see below). The projected image \( \gamma^*(t) \) of the moduli curve \( \tilde{\gamma}(t) \) in \( M^* \) is referred to as the shape curve. By a geometric curve on the sphere we shall mean a curve parametrized by arc-length (or with no specific parametrization).

Along these lines, in a previous paper [5] the authors have investigated sub-tle questions pertaining to 3-body motions with vanishing angular momentum, which should be regarded as Part I of the present study. In Part I we showed that the moduli curves \( \tilde{\gamma}(t) \) representing 3-body motions of vanishing angular momentum are the geodesics of the induced Jacobi-metric on the moduli space \( \tilde{M} \), or equivalently, they are also the solutions of the reduced Lagrange’s least action principle. Among the major results from Part I we mention the following characteristics for the case of zero angular momentum:

- The moduli curve \( \tilde{\gamma}(t) \) of a 3-body motion is determined by the associated shape curve \( \gamma^*(t) \) on the 2-sphere.
- The unique parametrization theorem asserts that the time parametrized curve \( \tilde{\gamma}(t) \) is (essentially) determined by the oriented geometric shape curve \( \gamma^* \). In turn, the latter is uniquely determined by a few curvature invariants representing the relative geometry between \( \gamma^* \) and the gradient flow of the potential function \( U^* \) on the 2-sphere at a point.
- A remarkable property of the shape curve on the 2-sphere is expressed by the Monotonicity Theorem, describing the piecewise monotonic behavior of its (mass-modified) latitude between two succeeding local maxima or minima, which must lie on different hemispheres.

The purpose of this paper is to establish the first two of the above stated properties to planar motions in general. On the other hand, the monotonicity theorem is no longer valid.

We start with a description in the next two subsections of the geometric reduction procedure, following the basic setting from [5], and a summary with the major results, Theorem A and Theorem B, is presented in Section 1.3. In Section 2 we present the reduced Newtonian ODE system, at the level of the moduli space \( \tilde{M} \simeq \mathbb{R}^3 \). We shall also point out the subtle distinction between this system of differential equations and the geodesic equations of the dynamical Riemannian metric on \( \tilde{M} \) associated with a possible reduction applied to Jacobi’s geometrization approach. The two systems are identical if and only if the angular momentum vanishes. But they are distinguished by some terms depending linearly on the angular momentum, representing the effect of a fictitious ”Coriolis force” as if the sphere \( M^* \) is rotating.
In Section 3 we show how a few geometric invariants of the shape curve $\gamma^*$, in general at a single regular point, provide enough information to determine the initial data for the moduli curve $\bar{\gamma}(t)$ as the unique solution of the reduced ODE system. This also completes the proofs of theorems stated in Section 1.3.

1.1 The basic kinematic quantities and the potential function

The classical 3-body problem in celestial mechanics studies the local and global geometry of the trajectories of a 3-body system, namely the motion of three point masses (bodies) of mass $m_i > 0, i = 1, 2, 3$, under the influence of the mutual gravitational forces. This system constitutes a conservative mechanical system with the Newtonian potential function

$$U = \sum_{i<j} \frac{m_im_j}{r_{ij}}, \quad r_{ij} = |a_i - a_j|, \quad a_i \in \mathbb{R}^3,$$

and potential energy $-U$. We introduce the notion of an $m$-triangle, which we shall identify with the vector $\delta = (a_1, a_2, a_3) \in \mathbb{R}^{3 \times 3}$ which records the position of the system in a barycentric inertial frame, namely we also assume $\sum m_i a_i = 0$.

A trajectory is a time parametrized curve $\gamma(t) = (a_1(t), a_2(t), a_3(t))$ representing a motion of the 3-body system, locally characterized by Newton’s equation

$$\frac{d^2}{dt^2} \gamma = \nabla U(\gamma) = \left( \frac{1}{m_1} \frac{\partial U}{\partial a_1}, \frac{1}{m_2} \frac{\partial U}{\partial a_2}, \frac{1}{m_3} \frac{\partial U}{\partial a_3} \right)$$

However, the trajectories can also be characterized globally as solutions of a suitable boundary value problem, namely as extremals of an appropriate least action principle, such as the two principles due to Lagrange and Hamilton.

Let us also recall the basic kinematic quantities which are the (polar) moment of inertia, kinetic energy and angular momentum, respectively defined by

$$I = \sum m_i |a_i|^2, \quad T = \frac{1}{2} \sum m_i |\dot{a}_i|^2, \quad \Omega = \sum m_i (a_i \times \dot{a}_i)$$

The dynamics of the 3-body problem is largely expressed by their interactions with the potential function $U$ via the equation (2), and the invariance of the total energy

$$h = T - U$$

is a simple consequence of (2) and the definition of $T$, whereas the invariance of the vector $\Omega$ also follows from (2), but is essentially due to the rotational symmetry of $U$. 

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1.2 Reduction to the moduli space and the shape space

In this article we shall be concerned with planar three-body motions, namely the individual position vectors \( \mathbf{a}_i \) are confined to a fixed plane \( \mathbb{R}^2 \) and hence the trajectory \( \gamma(t) \) is a curve in the configuration space

\[
M \simeq \mathbb{R}^4 : \sum_{i=1}^{3} m_i \mathbf{a}_i = 0, \quad \mathbf{a}_i \in \mathbb{R}^2
\]  \( \text{(5)} \)

We assume the plane \( \mathbb{R}^2 \) is positively oriented by the unit normal vector \( \mathbf{k} \), and the angular momentum of a trajectory \( \gamma(t) \) is written as

\[
\Omega = \omega \mathbf{k},
\]  \( \text{(6)} \)

so that the scalar angular momentum \( \omega \in \mathbb{R} \) is a constant of the motion. Thus, each trajectory \( \gamma(t) \) belongs to a specific energy-momentum level \((h, \omega)\). Consequently, due to the conservation of energy and angular momentum, the Newtonian system \( \text{(2)} \) for planar motions reduces to a system of differential equations of total order \( 8 - 2 = 6 \).

We assume \( M \) has the Euclidean kinematic metric \( M \), with the inner product of \( m \)-triangles \( \delta = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3), \delta' = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) \) defined by

\[
\delta \cdot \delta' = \sum_{i=1}^{3} m_i \mathbf{a}_i \cdot \mathbf{b}_i,
\]  \( \text{(7)} \)

and then the right side of Newton’s equation \( \text{(2)} \) is the gradient field \( \nabla U \). Moreover, the squared norm is the moment of inertia, \( I = I(\delta) = |\delta|^2 \), and the hyperradius \( \rho = \sqrt{I} \) is the natural size function which also measures the distance from the origin.

The linear group \( SO(2) \) acts orthogonally on \( M \) by rotating \( m \)-triangles, and the orbit space of \( M \) and its unit sphere \( M^1 \)

\[
\bar{M} = M/\text{SO}(2), \quad M^* = M^1/\text{SO}(2)
\]  \( \text{(8)} \)

are the (congruence) moduli space and the shape space, respectively. The points in \( \bar{M} \) represent congruence classes \( \delta \) of \( m \)-triangles \( \delta \), and points in \( M^* \) represent the shapes (or similarity classes) \( \delta^* \) of \( m \)-triangles \( \delta \neq 0 \).

Next, we recall that the above orbit spaces and related orbit maps is actually the classical Hopf map construction \( h: \mathbb{R}^4 \to \mathbb{R}^3 \) in disguise, which is illustrated by the following diagram

\[
\begin{align*}
M & \simeq \mathbb{R}^4 \; h \to \; \mathbb{R}^3 \simeq \bar{M} \\
M^1 & \simeq \mathbb{S}^3 \; h \cup \mathbb{S}^2 \simeq M^*
\end{align*}
\]  \( \text{(9)} \)

where \( M \simeq \mathbb{R}^4 = \mathbb{R}^{2 \times 2} \) is a chosen \( \text{SO}(2) \)-equivariant isometry between \( M \) and the matrix space \( \mathbb{R}^{2 \times 2} \), whose column vectors \( \mathbf{x}_1, \mathbf{x}_2 \) provides a choice of coordinates for \( m \)-triangles and are also referred to as Jacobi vectors in the literature. In the sequel we shall identify the pair \((\bar{M}, M^*)\) with \((\mathbb{R}^3, \mathbb{S}^2)\).
Similar to the pair \((M, M^1)\), \( \bar{M} \) is a cone over \( M^* \) with \( \rho \) as the radial coordinate, and the sphere \( M^* \) is the subset \((\rho = 1)\). These spaces have the naturally induced orbital distance metric, making the Hopf map a Riemannian submersion. With this geometry \( M^* = S^2(1/2) \) is the round sphere of radius \(1/2\), namely with the metric
\[
d\sigma^2 = \frac{1}{4}(d\varphi^2 + \sin^2(\varphi)d\theta^2),
\]
in terms of spherical polar coordinates \((\varphi, \theta)\) on \( S^2 \), whereas \((\bar{M}, d\bar{s}^2)\) is best understood as a Riemannian cone over \( M^* \):
\[
\bar{M} = C(M^*): d\bar{s}^2 = d\rho^2 + \frac{\rho^2}{4}(d\varphi^2 + \sin^2(\varphi)d\theta^2)
\]
Note, however, the representation of the various shapes of m-triangles on a fixed model sphere \( S^2 \), with a distinguished equator circle representing collinear shapes, depends on some choice of conventions together with the mass distribution \( \{m_i\} \), via the mass dependence of the Jacobi vectors.

Briefly, in this article we shall focus on the two-step reduction
\[
M \to \bar{M}, \quad \bar{M} - \{0\} \to M^*, \quad \gamma(t) \to \bar{\gamma}(t) \to \gamma^*(t)
\]
by which a trajectory \( \gamma(t) \) of a planary 3-body motion is projected to its moduli curve \( \bar{\gamma}(t) \) and further to its shape curve \( \gamma^*(t) \) on the 2-sphere. In Section 2.1 we shall put the above reduction in the framework of Riemannian geometry, and the relative geometry of the shape curve and the gradient flow of the function \( U^* \), the restriction of \( U \) to the sphere, will be our primary concern. For basic information on this geometric reduction approach we refer to [4], [5].

1.3 A summary of the main results

The Hopf map construction [14] makes it convenient to use a Euclidean model, \( \bar{M} = \mathbb{R}^3 \), for the moduli space with the unit sphere \( S^2(1) \) as the shape space \( M^* \). In this way one can express all kinematic quantities and dynamical equations in terms of the usual spherical geometry, and hence take the full advantage of the cone structure of \( \bar{M} \) over \( M^* \) using the coordinates \((\rho, \varphi, \theta)\) on \( \bar{M} \), \( 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi \).

Besides the \( SO(2) \)-invariance of the Newtonian potential function \( U \), another crucial property of \( U \) that we shall exploit is its homogeneity, namely it is of type
\[
U = \frac{U^*(\varphi, \theta)}{\rho}
\]
For a given curve \( \gamma(t) \) in \( M \), the two curves
\[
\gamma^*(t) = (\varphi(t), \theta(t)), \quad \bar{\gamma}(t) = (\rho(t), \gamma^*(t))
\]
are the associated shape and moduli curve, respectively. Let us consider trajectories \( \gamma(t) \) of Newton’s equation [2] at a given energy-momentum level \((h, \omega)\).
In Section 2.2 we shall focus on the reduced Newton’s equations in $\bar{M}$ in the coordinates $(\rho, \varphi, \theta)$, see (34), which by spherical geometry can be presented as the pair

\begin{align*}
0 &= \ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho} \left( \frac{1}{\rho} U^* + 2h \right) \\
0 &= \ddot{\gamma}^* + P\dot{\gamma}^* + Q\nu^* + R\nabla U^*
\end{align*}

(15)

Here the first equation in (15) is the so-called Lagrange-Jacobi equation, which in terms of the moment of inertia $I = \rho^2$ expresses as

$$
\ddot{I} = 2U + 4h.
$$

The second equation is a vector equation on the unit 2-sphere, expressing the covariant acceleration $\ddot{\gamma}^*$ of the shape curve as a sum of three “forces”, $\nu^*$ is the oriented unit normal of the curve, and $\nabla U^*$ is the gradient field of $U^*$. The three coefficients can be expressed as

\begin{align*}
P &= 2\frac{\dot{\rho}}{\rho}, \\
Q &= \frac{2\omega v}{\rho^2}, \\
R &= -\frac{4}{\rho^3},
\end{align*}

(16)

where $v = |\dot{\gamma}^*|$ is the speed of the shape curve.

The scalar version of (15) is stated as the system (34). We may imagine the component $Q\nu^*$ in (15) to be the Coriolis “force” caused by some fictitious rotation of the shape sphere, and we note that its magnitude is proportional to the speed as well as the angular momentum. As an ODE system, (15) should be augment (15) with the energy integral (4), viewed as a first order equation in $\bar{M}$

\begin{align*}
\frac{1}{2}\dot{\rho}^2 + \frac{\rho^2 v^2}{8} + \frac{\omega^2}{2\rho^2} \frac{U^*}{\rho} = h
\end{align*}

(17)

where $h$ is a constant, called the energy level. In fact, combined with (17) any of the three scalar equations in (34) can be derived from the other two, so the total order of the system is actually 5.

As a curve on the 2-sphere, the basic geometric invariant of the shape curve $\gamma^*$ is its geodesic curvature function $K^* = K^*(s)$, where $s$ is the arc-length. In general, crucial information of the 3-body motion is encoded into this function, and our problem is rather to detect the code and extract the hidden information in an appropriate way.

Let $U^*_\nu$ denote the normal derivative of the function $U^*$ along the curve $\gamma^*$. We shall derive the following formula for the curvature

\begin{align*}
K^* &= 4 \frac{U^*_\nu}{\rho^3 v^2} - \frac{2\omega}{\rho^2 v}
\end{align*}

(18)

This identity is the key to the understanding of how the relative geometry between $\gamma^*$ and the gradient flow of $U^*$, in fact, provides the data for the initial value problem of the ODE (15) and thus determine the moduli curve $\bar{\gamma}(t)$. This
is the major issue we shall be addressing, and the main results can be formulated neatly as the following two theorems:

**Theorem A** For a given total energy and angular momentum, a planar three-body motion is completely determined up to congruence by its time parametrized shape curve (which records only the changing of the similarity class).

**Theorem B** (Elimination of time) The time parametrized shape curve is determined by the oriented geometric (i.e. non-parametrized) shape curve.

**Remark 1.1**
(i) The main purpose of the present paper is to give the complete proofs of the two theorems.
(ii) The theorems and their proof do not apply to the case of exceptional shape curves (as defined in Section 2.4.4). Uniqueness of time parametrization means, of course, unique modulo time translation.
(iii) The simpler case of vanishing angular momentum \((\omega = 0)\) was treated in the paper \([5]\), where the same two theorems are proved, with special attention to the case of \((h, \omega) = (0, 0)\).
(iv) Planar 3-body motions were also investigated in \([6]\), attempting to generalize results from \([5]\). However, in \([6]\) the proof of the result corresponding to the above two theorems is incorrect when \(\omega \neq 0\), since the Coriolis term \(Q\nu^*\) in (15) was missing; we refer to the discussion in Section 2.1 and 2.2.

# 2 Geometric reduction

## 2.1 Riemannian structures on the moduli space

In his famous lectures \([8]\), Jacobi introduced the concept of a *kinematic metric* \(ds^2\) on the configuration space \(M\) of a mechanical system with kinetic energy \(T\). For example, in the case of an \(n\)-body system with total mass \(\sum m_i = 1\),

\[
ds^2 = 2Tdt^2 = \sum_i m_i(dx_i^2 + dy_i^2 + dz_i^2) \tag{19}\]

which is clearly equivalent to the definition (7). Now, for a system with potential energy \(-U\) and a fixed total energy \(h\), set

\[
M_h = \{p \in M; h + U(p) \geq 0\} \tag{20}
\]

\[
ds_h^2 = (h + U)ds^2
\]

where \(ds_h^2\) will be referred to as the *dynamical metric* on \(M_h\). By writing

\[
ds_h = \sqrt{h + U}ds = \sqrt{T}ds = \sqrt{2T}dt
\]

Jacobi transformed Lagrange’s action integral (on the left side of (21)) into an arc-length integral, namely

\[
J(\gamma) = \int_\gamma T\, dt = \frac{1}{\sqrt{2}} \int_\gamma ds_h \tag{21}
\]
and hence the least action principle becomes the following simple geometric statement:

\[
\text{Trajectories with total energy } h \text{ are along the geodesic curves in the space } M_h \text{ with the dynamical metric } ds^2_h. \tag{22}
\]

Nowadays, the metric spaces \((M, ds^2), (M_h, ds^2_h)\) are called Riemannian manifolds, and the dynamical metric is a conformal modification of the kinematic metric by the scaling factor \((h + U)\). In general, and as exemplified by a Riemannian metric on a manifold \(N\) amounts to the choice of a kinetic energy function on the tangent bundle, \(T : TN \to \mathbb{R}\), which is a positive definite quadratic form on each tangent plane \(T_p N\). This allows us to define the speed \(\left| \frac{d\Gamma}{dt} \right|\) along a time parametrized curve \(\Gamma(t)\) in \(N\) and hence an arc-length function \(u(t)\) along the curve by

\[
T \left( \frac{d\Gamma}{dt} \right) = \frac{1}{2} \left| \frac{d\Gamma}{dt} \right|^2 = \frac{1}{2} \left( \frac{du}{dt} \right)^2 \tag{23}
\]

Next, we would like to inquire further into the above Lagrange-Jacobi approach to dynamics, to check whether the dynamics in \(M\) and characterization of the trajectories can be pushed down to \(\bar{M}\), with a similar geometric description of the moduli curves of 3-body motions on a fixed energy-momentum level \((h, \omega)\).

First of all, \(\bar{M}\) already has the orbital distance metric \(d\bar{s}^2\) as in \((11)\) and hence a corresponding notion of kinetic energy \(\bar{T}\) as indicated in \((23)\), namely

\[
d\bar{s}^2 = 2\bar{T} dt^2 = d\rho^2 + \rho^2 d\sigma^2 = d\rho^2 + \frac{\rho^2}{4} (d\varphi^2 + \sin^2(\varphi) d\theta^2) \tag{24}
\]

On the other hand, for a curve \(\gamma(t)\) in \(M\) there is the orthogonal decomposition \(\dot{\gamma} = \dot{\gamma}^h + \dot{\gamma}^\omega\) of its velocity, and the corresponding splitting of kinetic energy

\[
T = \frac{1}{2} |\dot{\gamma}^h|^2 + \frac{1}{2} |\dot{\gamma}^\omega|^2 = T^h + T^\omega, \tag{25}
\]

where \(\dot{\gamma}^\omega\) is tangential to the \(SO(2)\)-orbit. Hence, \(T^\omega\) is the kinetic energy due to purely rotational motion of the m-triangle, and for planar 3-body motions this energy term can be expressed as

\[
T^\omega = \frac{\omega^2}{2\rho^2}. \tag{26}
\]

By definition of the metric \(d\bar{s}^2\), the orbit map \(M \to \bar{M}\) is a Riemannian submersion and hence maps the "horizontal" velocity \(\dot{\gamma}^h\) of \(\dot{\gamma}\) isometrically to the velocity vector of \(\dot{\gamma}\). This shows \(T^h = \bar{T}\) is naturally the kinetic energy at the level of \(\bar{M}\), that is, the kinematic metric \(d\bar{s}^2\) on \(\bar{M}\) naturally identifies with the orbital distance metric. Therefore, by \((25)\) and \((26)\), the kinematic metric
on $\bar{M}$ can be finally expressed as in $[21]$

$$ds^2 = 2T^h dt^2 = 2(T - T^ω)dt^2 = 2(T - \frac{ω^2}{2ρ^2})dt^2$$ \hspace{1cm} (27)

$$= d\bar{s}^2 = 2\bar{T} dt^2 = d\rho^2 + \frac{ρ^2}{4}(d\varphi^2 + \sin^2\varphi d\theta^2)$$

Now, in the spirit of Lagrange-Jacobi, let us turn to the description of $\bar{M}$ as the configuration space of a simple classical mechanical system, with the kinetic energy $\bar{T}$ of the kinematic metric as above, and effective potential energy $-\bar{U}$ defined so that the system is conservative, namely by setting

$$\bar{U} = U - \frac{ω^2}{2ρ^2}, \quad T - \bar{U} = T - U = h$$ \hspace{1cm} (28)

Then the Lagrange function is $\bar{L} = \bar{T} + \bar{U}$, and according to Lagrange’s least action principle the trajectories of the mechanical system should be the solutions of Lagrange’s equations

$$\frac{d}{dt}(\frac{∂\bar{L}}{∂\dot{ρ}}) = \frac{∂\bar{L}}{∂ρ}, \quad \frac{d}{dt}(\frac{∂\bar{L}}{∂\dot{ϕ}}) = \frac{∂\bar{L}}{∂ϕ}, \quad \frac{d}{dt}(\frac{∂\bar{L}}{∂\dot{θ}}) = \frac{∂\bar{L}}{∂θ}$$ \hspace{1cm} (29)

Alternatively, Lagrange’s approach may well be modified according to Jacobi’s geometrization idea, leading to the dynamical Riemannian metric $d\bar{s}^2_{h,ω}$:

$$\bar{J}(\bar{γ}) = \sqrt{2} \int_{\bar{γ}} \bar{T} dt = \sqrt{2} \int_{\bar{γ}} (\bar{U} + h) dt = \int_{\bar{γ}} \sqrt{\bar{U} + h} d\bar{s} = \int d\bar{s}_{h,ω},$$

$$d\bar{s}^2_{h,ω} = \bar{T} ds^2 = (\bar{U} + h)ds^2 = (U + h - \frac{ω^2}{2ρ^2})d\bar{s}^2$$ \hspace{1cm} (30)

In summary, the subregion $\bar{M}_{h,ω} \subset \bar{M}$ defined by $\bar{U} + h \geq 0$ can be regarded as a classical mechanical system, with kinematic metric $d\bar{s}^2 = 2\bar{T} dt^2$, potential function $\bar{U}$, and energy conservation $h = \bar{T} - \bar{U}$. But it is also a Riemannian manifold with the dynamical metric $d\bar{s}^2_{h,ω} = (\bar{U} + h)d\bar{s}^2$. Thus we arrive at the following geometric statement similar to $[22]$

Trajectories of the simple mechanical system on $\bar{M}_{h,ω}$ are the solutions of the Lagrange equations $[29]$, and the curves coincide with the geodesic curves of the dynamical metric $d\bar{s}^2_{h,ω}$. In terms of the coordinates $(ρ, ϕ, θ)$ on $\bar{M}$, both ways of calculating the associated differential equations lead to the following ODE system
\( (i) \quad 0 = \ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho}(U^* + 2h) \)

\( (ii) \quad 0 = \ddot{\varphi} + 2\frac{\dot{\rho}}{\rho}\dot{\varphi} - \frac{1}{2}\sin(2\varphi)\dot{\theta}^2 - \frac{4}{\rho^3}U^*_\varphi \)

\( (iii) \quad 0 = \ddot{\theta} + 2\frac{\dot{\rho}}{\rho}\dot{\theta} + 2\cot(\varphi)\dot{\varphi}\dot{\theta} - \frac{4}{\rho^3}\frac{1}{\sin^2 \varphi}U^*_\theta \)  

where \( U^*_\varphi = \frac{\partial}{\partial \varphi}U^* \) etc., and the following first integral is the energy conservation law

\( (iv) \quad h = \bar{T} - \bar{U} = \frac{1}{2}\dot{\rho}^2 + \frac{1}{8}(\dot{\varphi}^2 + (\sin^2 \varphi)\dot{\theta}^2) + \frac{\omega^2}{2\rho^2} - \frac{U^*}{\rho} \)

Therefore, the system (i)-(iv) of differential equations has total order 5.

**Remark 2.1** The above system was derived in the same way in [5] for the special case of vanishing angular momentum (\( \omega = 0 \)), and then the equations actually yield the moduli curves of the 3-body motions. However, this fails when \( \omega \neq 0 \), namely the ordinary Lagrange-Jacobi approach does not yield the correct reduced Newton’s equations (34) at the level of \( \bar{M} \). The subtle difference between the latter and the above system (32) is conspicuous by direct comparison, namely the Coriolis term is missing in the system (32).

### 2.2 The reduced Newton’s equations on the moduli space

The induced Newton’s equations on the moduli space \( \bar{M} \simeq \mathbb{R}^3 \), for given values of \( (h, \omega) \), can be expressed in spherical coordinates \( (\rho, \varphi, \theta) \) as the following ODE system:

\( (i) \quad 0 = \ddot{\rho} + \frac{\dot{\rho}^2}{\rho} - \frac{1}{\rho}(U^* + 2h) \)

\( (ii) \quad 0 = \ddot{\varphi} + 2\frac{\dot{\rho}}{\rho}\dot{\varphi} - \frac{1}{2}\sin(2\varphi)\dot{\theta}^2 - 2\frac{\sin \varphi}{\rho^2}\dot{\theta} - \frac{4}{\rho^3}U^*_\varphi \)

\( (iii) \quad 0 = \ddot{\theta} + 2\frac{\dot{\rho}}{\rho}\dot{\theta} + 2\cot(\varphi)\dot{\varphi}\dot{\theta} + 2\sin \theta \frac{1}{\rho^2\sin^2 \varphi}\dot{\varphi} - \frac{4}{\rho^3}\frac{1}{\sin^2 \varphi}U^*_\theta \)

which also has the first integral (iv) stated in (33), namely the conservation of energy. Again, the total order of the system (i)-(iv) is 5.

**Remark 2.2** These equations are valid for any shape potential function \( U^* \), and the results in this paper do not depend on specific properties of \( U^* \). For the Newtonian case there is the explicit expression (35) below.

Clearly, for \( \omega = 0 \) the two systems (32) and (34) coincide, and the choice of spherical polar coordinates \( (\varphi, \theta) \) on the shape sphere \( M^* = S^2 \) is immaterial.
However, for $\omega \neq 0$ the explicit form of the Coriolis term in equation (ii) and (iii) depends on the polar coordinates $(\varphi, \theta)$ to be centered at the north pole $(\varphi = 0)$, and then the equator circle $(\varphi = \pi/2)$ is the locus representing the eclipse shapes. On this circle there are three distinguished points $b_i$, of longitude angle $\theta_i$, $i = 1, 2, 3$, representing the collision of point masses $m_2$ and $m_3$ etc. Choosing the zero meridian to be $\theta_1 = 0$ say, we shall assume positive direction of $\theta$ so that

$$(\theta_1, \theta_2, \theta_3) = (0, \beta_3, -\beta_2), \quad \cos \beta_i = \frac{\mu_j \mu_k - \mu_i}{(1 - \mu_j)(1 - \mu_k)}, \quad \text{cf. (133) in [4]}$$

where the angle $\beta_i$ is the longitude distance between $b_j$ and $b_k$, for different $i, j, k$, and we have introduced the normalized masses

$$\mu_i = m_i/\bar{m}; \quad \bar{m} = \sum m_i$$

For convenience, the Newtonian shape potential function $U^*$ can be expressed as (cf. (169) in [4])

$$U^*(\varphi, \theta) = \bar{m}^{5/2} \sum_{i=1}^{3} \frac{(\mu_j \mu_k)^{3/2}(\mu_i^*)^{-1/2}}{\sqrt{(1 - \sin \varphi \cos(\theta - \theta_i))}}, \quad \mu_i^* = \frac{1}{2}(1 - \mu_i). \quad (35)$$

For example, in the special case of $m_1 = m_2 = m_3 = 1$ we obtain

$$U^*(\varphi, \theta) = \frac{3}{\sqrt{(1 - \sin \varphi \cos(\theta - \theta_i))}} \quad (36)$$

For a derivation of the above ODE system, we shall recall the proof given in Sydnes[2013], adapted to the planar case. It is based upon singular value decomposition of $M$ as the space of matrices $X = [\mathbf{x}_1][\mathbf{x}_2]$ with column (Jacobi) vectors $\mathbf{x}_i$,

$$\Phi : \mathcal{G} = SO(2) \times D \times SO(2)' \rightarrow \mathbb{R}^{2 \times 2} \simeq M,$$

$D \simeq \mathbb{R}^2$ consists of diagonal matrices $\text{diag}(r_1, r_2)$, and $\Phi$ is the surjective map given by matrix multiplication and is locally an analytic diffeomorphism at generic points $(P, R, Q) \in \mathcal{G}$. With the following parametrization

$$P = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad R = \rho \begin{pmatrix} \sin(\frac{\varphi}{2} + \frac{\theta}{2}) & 0 \\ 0 & \cos(\frac{\varphi}{2} + \frac{\theta}{2}) \end{pmatrix},$$

$$Q = \begin{pmatrix} \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \\ -\sin \frac{\theta}{2} & \cos \frac{\theta}{2} \end{pmatrix},$$

we can use $(\alpha, \rho, \varphi, \theta)$ as (local) coordinates in $M$, where $\rho, \varphi, \theta$ have the previous geometric interpretation as coordinates in the moduli space $M = M/\text{SO}(2)$, and the angle $\alpha$ parametrizes the "congruence" rotation group $\text{SO}(2)$ acting by multiplication on the left side of $\mathcal{G}$.

Moreover, the columns $\mathbf{u}_1, \mathbf{u}_2$ of $P$ are
the eigenvectors of the inertia tensor of the 3-body system, hence constitute an intrinsic moving frame of the 3-body motion, and by definition of $\Phi$

$$[x_1|x_2] = \rho|u_1|u_2| \left( \begin{array}{cc} \sin(\frac{\phi}{2} + \frac{\pi}{4}) \cos \frac{\theta}{2} & \sin(\frac{\phi}{2} + \frac{\pi}{4}) \sin \frac{\theta}{2} \\ -\cos(\frac{\phi}{2} + \frac{\pi}{4}) \sin \frac{\theta}{2} & \cos(\frac{\phi}{2} + \frac{\pi}{4}) \cos \frac{\theta}{2} \end{array} \right)$$

Now, the kinetic energy can be expressed as

$$T = \frac{1}{2}\text{tr}(\dot{X}\dot{X}^t) = \frac{1}{2}(||\dot{x}_1||^2 + ||\dot{x}_2||^2)$$

$$= \frac{1}{2}\rho^2 + \frac{\rho^2}{8}(\dot{\phi}^2 + \dot{\theta}^2) + \frac{1}{2}\rho^2\dot{\alpha}^2 - \frac{1}{2}\rho^2(\cos \varphi)\dot{\alpha}\dot{\theta},$$

and the angular momentum as

$$\omega = \frac{\partial T}{\partial \dot{\alpha}} = \rho^2(\dot{\alpha} - \frac{1}{2} \cos \varphi \dot{\theta}), \quad (37)$$

which also equals the cross-product $X \times \dot{X}$ (appropriately defined).

The equations of the Newtonian motion $t \rightarrow X(t)$ are equivalent to the Euler-Lagrange equations associated with the Lagrange function

$$L = T + U = T + \frac{U^*(\varphi, \theta)}{\rho},$$

namely the following equations

$$\dot{\omega} = \frac{d}{dt}(\frac{\partial T}{\partial \dot{\alpha}}) = \frac{d}{dt}(\frac{\partial L}{\partial \dot{\alpha}}) = \frac{\partial L}{\partial \dot{\alpha}} = 0$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\rho}}) = \frac{dL}{d\dot{\varphi}} \frac{d}{dt}(\frac{\partial L}{\partial \dot{\varphi}}) = \frac{dL}{d\dot{\theta}} \frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}}) = \frac{\partial L}{\partial \dot{\theta}}$$

The first equation simply says $\omega$ is constant. In the last three equations the occurrence of $\dot{\alpha}$ and $\ddot{\alpha}$ can be eliminated by (37), namely using

$$\dot{\alpha} = \frac{\omega}{\rho^2} + \frac{1}{2} \cos \varphi \dot{\theta},$$

and this yields, in fact, the three equations of (34).

### 2.3 The initial value problem in the moduli space

The ODE (34) in the moduli space $\tilde{M} \simeq \mathbb{R}^3$ is an analytic system which depends analytically on the scalar angular momentum $\omega$, and moreover, there is a first integral (33) whose value at a given integral curve is the total energy $h$. Thus we shall refer to the pair of constants $(h, \omega)$ as the energy-momentum level of the curve. These curves, which we shall refer to as moduli curves, are therefore analytic curves

$$t \rightarrow \tilde{\gamma}(t) = (\rho(t), \varphi(t), \theta(t)), \quad \text{13}$$
with power series expansions

\[
\begin{align*}
\rho(t) &= \rho_0 + \rho_1 t + \rho_2 t^2 + \ldots + \\
\varphi(t) &= \varphi_0 + \varphi_1 t + \varphi_2 t^2 + \ldots + \\
\theta(t) &= \theta_0 + \theta_1 t + \theta_2 t^2 + \ldots +
\end{align*}
\]

at a chosen initial point \((\rho_0, \varphi_0, \theta_0)\). The initial value problem in \(\tilde{M}\) is to determine the solution \(\tilde{\gamma}(t)\) from a given initial data set

\[
\rho_0, \varphi_0, \theta_0; \rho_1, \varphi_1, \theta_1
\]

We shall regard \(\omega\) as a given constant (or parameter) whereas the value of \(h\) is determined by \(\omega\) and the initial data (39) by evaluation of the expression (33) at initial time \(t = 0\).

In contrast to this, the spherical initial data

\[
\varphi_0, \theta_0; \varphi_1, \theta_1
\]

would not suffice to determine the associated shape curve

\[
t \to \gamma^*(t) = (\varphi(t), \theta(t))
\]
on the sphere, since the curve is not the solution of a second order differential equation on the sphere. Anyhow, elimination of \(\rho\) and solving an initial value problem purely on the sphere does not seem to be a tractable approach for our purpose.

Let us briefly consider the initial value problem for the system (34) and the dependence on the parameter \(\omega\).

**Lemma 2.3** Suppose there is a planar 3-body motion

\[
t \to \gamma(t) = (a_1(t), a_2(t), a_3(t))
\]
in the configuration space \(\tilde{M}\), starting at \(\gamma(0) = \delta_0\) with the initial velocity \(\dot{\gamma}(0) = \delta_1\) and angular momentum vector \(\omega k\), cf. (5) and (6). Let the corresponding initial data for the moduli curve \(\tilde{\gamma}(t)\) in \(\tilde{M}\) be the numbers in (39). Then for any number \(\omega_0\) there is a planar 3-body motion in \(M\) with angular momentum \(\omega_0 k\), whose moduli curve \(\tilde{\gamma}_0(t)\) has the same initial data (39) as \(\tilde{\gamma}(t)\).

Namely, the moduli curves \(\tilde{\gamma}(t)\) and \(\tilde{\gamma}_0(t)\) are determined by the ODE (34) with the same initial data, but with parameters \((h, \omega)\) and \((h_0, \omega_0)\) respectively, where \(h_0\) is the total energy (33) determined by \(\omega_0\) and the above initial data.

**Proof.** The initial velocity of the original 3-body motion has an orthogonal decomposition

\[
\delta_1 = \delta_1^\parallel + \delta_1^\perp, \quad \delta_1^\perp = \tilde{\omega} \times \delta_0 = (\tilde{\omega} \times a_1(0), \tilde{\omega} \times a_2(0), \tilde{\omega} \times a_3(0))
\]

where \(\tilde{\omega}\) is the (instantaneous) rotational velocity vector, which in the case of planar motions is \(\tilde{\omega} = \omega \rho_0^2 \mathbf{k}\). Evidently, we may freely modify the velocity component \(\delta_1^\perp\) by changing the scalar \(\omega\) correspondingly, keeping \(\delta_0\) and \(\delta_1^\parallel\) unchanged. But on the other hand, the initial data (39) in the moduli space \(\tilde{M}\) depend only on \((\delta_0, \delta_1^\parallel)\), so this proves the lemma.
Remark 2.4  (i) The various shape curves $\gamma^*_+ \text{ with the same initial data (40) at a point } p \in M^*$ are uniquely distinguished by the parameter $\omega$, or equivalently by their curvature $K^*$ at $p$, see (18).

(ii) The various moduli curves $\bar{\gamma}(t)$ with the same initial data (39) are also distinguished by the parameter $\omega$. For $\omega = 0$ the curves are geodesics with respect to the dynamical metric $ds^2_{h,0}$, cf. (30), so $\omega$ acts as a deformation parameter on this family of curves.

Remark 2.5 The 3-body motions in $M$ have a 1-parameter symmetry group $\{\Phi_k, k \neq 0\}$, namely time-size scaling symmetries $\Phi_k$, which transform a trajectory $t \rightarrow \gamma(t)$ to the trajectory $t \rightarrow \gamma^{(k)}(t) = k^{-2/3}\gamma(kt)$.

The effect of the transformation $\Phi_k$ on initial data in $\bar{M}$ and parameters $(h, \omega)$ is as follows

\[(\rho_0, \varphi_0, \theta_0; \rho_1, \varphi_1, \theta_1) \rightarrow (k^{-2/3}\rho_0, \varphi_0, \theta_0; k^{1/3}\rho_1, k\varphi_1, k\theta_1); \quad (41)\]

\[\omega \rightarrow k^{-1/3}\omega, h \rightarrow k^{2/3}h\]

In particular, there is the time reversal transformation $\Phi_{-1}$ which converts a 3-body motion $t \rightarrow \gamma(t)$ to the motion $t \rightarrow \gamma^{(-1)}(t) = \gamma(-t)$ in the opposite direction. This changes the sign of $\omega$, but the sign of $h$ is invariant. Also note that the quantity $H = h\omega^2$ is an invariant of the symmetry group, and so is the (unoriented, geometric) shape curve $\gamma^*$.

2.4 Geometry on the shape sphere

2.4.1 Temporal and intrinsic invariants and their order

We shall make effective usage of the fact that the shape space is the round sphere $S^2$, and moreover, essential information about the moduli curve $\bar{\gamma}(t)$ in $\bar{M}$ is encoded into the relative geometry between the shape curve $\gamma^*$ and the gradient field $\nabla U^*$ on $S^2$. Two types of quantities (also called invariants) are involved in this interplay and we shall refer to them as being either temporal or intrinsic.

The temporal invariants are associated with $\bar{\gamma}(t)$ and differentiation with respect to time $t$, whereas the intrinsic ones depend on the geometric curve $\gamma^*$ or the relative geometry between this curve and the gradient flow of $U^*$, which may involve differentiations with respect to the arc-length of $\gamma^*$. So generally, we shall define the order of the invariant to be the highest order of differentiations with respect to $t$ or $s$. For example, the coefficients $\rho_k, \varphi_k, \theta_k$ of the expansions (38) are temporal invariants of order $k$, but we shall regard $\varphi_0, \theta_0$ as intrinsic since they simply specify the chosen initial point $p \in \gamma^*$.

In a more general setting, let us start with a given time parametrized analytic curve $\gamma^*(t) = (\varphi(t), \theta(t))$ on the sphere $S^2$, and let $s \geq 0$ be its arc-length parameter measured from an initial point $p = \gamma^*(0) = (\varphi_0, \theta_0)$. The linkage
between the parameters $t$ and $s$, which is a one-to-one correspondence in general, is given by the speed function

$$v = v(t) = ds/dt \geq 0$$

(42)

and corresponding differential operators are related by

$$\frac{d}{ds} = \frac{1}{v} \frac{d}{dt}, \quad \frac{d^2}{ds^2} = \frac{1}{v^2} \frac{d^2}{dt^2} - \frac{\dot{v}}{v^3} \frac{d}{dt}, \text{ etc.}$$

(43)

Since the sphere has the Riemannian metric $ds^2 = d\varphi^2 + (\sin^2 \varphi) d\theta^2$, the speed and its time derivative have the expressions

$$v = \sqrt{\dot{\varphi}^2 + (\sin^2 \varphi) \dot{\theta}^2}, \quad \dot{v} = \frac{d}{dt} \frac{1}{v} [\dot{\varphi}^2 + (\sin \varphi \cos \varphi) \dot{\varphi} \dot{\theta}^2 + \sin^2 (\varphi) \dot{\theta}^3]$$

(44)

and they are viewed as temporal invariants of order 1 and 2, respectively.

**Definition 2.6** A direction element on the sphere $S^2$ consists of a point $p \in S^2$ together with a tangential direction at $p$, and it is denoted by $(J_\varphi, J_\theta)_p$ or simply $(J_\varphi, J_\theta)$ when the point $p$ is tacitly understood. It is said to be regular if $\nabla U^*$ is transversal to it.

In fact, the direction is represented by the pair

$$(J_\varphi, J_\theta) = \left( \frac{\partial \varphi}{\partial s}, \frac{\partial \theta}{\partial s} \right)$$

which is an intrinsic invariant of order 1, and the following identities hold

$$\dot{\varphi} = J_\varphi v, \quad \dot{\theta} = J_\theta v, \quad J_\varphi^2 + \sin^2 (\varphi) J_\theta^2 = 1$$

(45)

Next, consider the two positively oriented orthonormal moving frames $(\tau^*, \nu^*)$ and $(\frac{\partial}{\partial \varphi}, \frac{1}{\sin \varphi} \frac{\partial}{\partial \theta})$ along the oriented spherical curve $\gamma^*$, where $\tau^*$ (resp. $\nu^*$) is the unit tangent (resp. normal) vector. Writing the frames formally as column vectors we can express their relationship at each point $p$ by a rotation matrix defined by the direction element, namely

$$(\tau^*, \nu^*)^T = \begin{pmatrix} J_\varphi & J_\theta \sin \varphi \\ -J_\theta \sin \varphi & J_\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \varphi} & \frac{1}{\sin \varphi} \frac{\partial}{\partial \theta} \end{pmatrix}^T,$$

(46)

On the other hand, the gradient field of $U^*$ on the sphere is

$$\nabla U^* = U^*_\varphi \frac{\partial}{\partial \varphi} + \frac{U^*_\theta}{\sin^2 \varphi} \frac{\partial}{\partial \theta},$$

and its inner product with $\tau^*$ and $\nu^*$ yields the tangential and normal derivative of $U^*$ along the curve, namely the pair $(U^*_\varphi, U^*_\theta)$. The latter is, in fact, related to the pair of partial derivatives $(U^*_\varphi, \frac{1}{\sin \varphi} U^*_\theta)$ via the same matrix as in (46), as follows

$$(U^*_\varphi, U^*_\theta)^T = \begin{pmatrix} J_\varphi & J_\theta \sin \varphi \\ -J_\theta \sin \varphi & J_\varphi \end{pmatrix} \begin{pmatrix} U^*_\varphi & \frac{1}{\sin \varphi} U^*_\theta \end{pmatrix}^T$$

(47)

In particular, the left side is an intrinsic invariant of order 1 which represents the gradient field $\nabla U^*$ along the curve $\gamma^*$. 

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2.4.2 Geodesic curvature of the shape curve

The geometry of the spherical curve \( \gamma^* \) itself is encoded into its geodesic curvature function \( K^*(s) \), and in general \( \gamma^*(s) \) is in fact completely determined by the intrinsic function \( K^* \) and the initial direction of \( \gamma^* \). Below we shall calculate and deduce an expression for \( K^*(s) \) in terms of simpler invariants of order \( \leq 1 \).

One way to calculate \( K^* \) is to express \( \gamma^* \) in Euclidean coordinates as \( x(s) = (x(s), y(s), z(s)) \) and use the formula

\[
K^*(s) = x(s) \times x'(s) \cdot x''(s)
\]

where differentiation is with respect to arc-length \( s \). Then, by returning to spherical coordinates and writing \( \phi' = \frac{d\phi}{ds} \) etc.,

\[
K^* = (\cos \phi) \hat{\theta}(v^2 + \dot{\phi}^2) + (\sin \phi) \hat{\varphi} \left( -2 \frac{\dot{\phi}}{\rho} \hat{\varphi} - \frac{2\omega}{\rho^2} \sin \varphi \hat{\varphi} + \frac{4}{\rho^3} \frac{1}{\sin^2 \varphi} U^*_\theta \right)
\]

where in the first line the expression for \( K^* \) is intrinsic and the second is an expression in temporal invariants.

Next, let us eliminate the second order terms \( \ddot{\phi} \) and \( \ddot{\theta} \) in the expression (48), using equations (ii), (iii) of the ODE system (34). The ensuing calculations

\[
K^*v^3 = (\cos \varphi) \ddot{\theta}(v^2 + \dot{\varphi}^2) + (\sin \varphi) \ddot{\varphi} \left( -2 \frac{\dot{\phi}}{\rho} \hat{\varphi} - \frac{2\omega}{\rho^2} \sin \varphi \hat{\varphi} + \frac{4}{\rho^3} \frac{1}{\sin^2 \varphi} U^*_\theta \right)
\]

lead us to the fundamental curvature formula (18), which we may also write as

\[
\rho^3 v^2 + \frac{2\omega}{K^*} \rho v = 4 \mathfrak{S}
\]

where

\[
\mathfrak{S} = \frac{U^*_\nu}{K^*}
\]

is the intrinsic Siegel function introduced in [5]. This function neatly encodes the relative geometry of the pair \( (\gamma^*, \nabla U^*) \), and it is also independent of the orientation of \( \gamma^* \).

In the special case that \( K^* \) vanishes, the above calculation of \( K^* \) yields the identity

\[
\rho v = \frac{2U^*_\nu}{\omega}
\]
2.4.3 Power series expansions of functions on the shape sphere

Of primary interest are the solution curves \( \gamma(t) \) of the ODE (34), whose coordinate functions \( \rho(t), \phi(t), \theta(t) \) are temporal invariants, and by definition, so are the coefficients of their series expansions (38). Furthermore, evaluation of functions on the shape sphere along the curve \( \gamma^*(t) = (\phi(t), \theta(t)) \) yields series expansions in time \( t \), such as

\[
U^* = u_0 + u_1 t + u_2 t^2 + ...
\]

\[
U_\phi^* = u_0 + u_1 t + \mu_2 t^2 + ...
\]  

(52)

\[
U_\theta^* = \eta_0 + \eta_1 t + \eta_2 t^2 + ....
\]

The coefficients are temporal invariants, except the leading terms which are, by definition, intrinsic and of order zero.

Moreover, we also need to expand the shape speed

\[
v = v_0 + v_1 t + v_2 t^2 + ...
\]

whose first two coefficients are readily obtained from (44)

\[
v_0 = \sqrt{\phi_1^2 + g_0 \theta_1^2},
\]

\[
v_1 = \frac{1}{v_0} [2\phi_1 \phi_2 + f_0 \phi_1 \theta_1^2 + 2g_0 \theta_1 \theta_2],
\]

(53)

(54)

where we have simplified notation by setting

\[
f_0 = \sin 2\phi_0, \quad g_0 = \sin^2 \phi_0
\]

Clearly, \( v_0 \) and \( v_1 \) are invariants of order 1 and 2, respectively.

The change of direction of a shape curve is expressed by functions such as

\[
J_\phi' = \frac{d}{ds}J_\phi, \quad J_\theta' = \frac{d}{ds}J_\theta, \quad J_\phi'' = \frac{d^2}{ds^2}J_\phi, \quad \text{etc.}
\]

and we shall use the same notation for their evaluation at \( s = 0 \), for example by (45)

\[
J_\phi' = \frac{1}{v_0^2} (2\phi_2 - v_1 J_\phi), \quad J_\theta' = \frac{1}{v_0^2} (2\theta_2 - v_1 J_\theta), \quad \text{etc.}
\]

(55)

These are intrinsic invariants of order 2, although the above expressions involve temporal ones \( \phi_2, \theta_2, v_1 \) of order 2.

Next, the coefficients of series expansions with respect to arc-length \( s \) of \( \gamma^* \), such as

\[
U^* = u_0 + u_1 s + u_2 s^2 + ...
\]

\[
U_\phi^* = \tau_0 + \tau_1 s + \tau_2 s^2 + ... \quad \text{(note: } \tau_{k-1} = k u_k \text{)}
\]

\[
U_\theta^* = \psi_0 + \psi_1 s + \psi_2 s^2 + ...
\]

(56)

\[
K^* = K_0 + K_1 s + K_2 s^2 + ...
\]

\[
\Theta = \Theta_0 + \Theta_1 s + \Theta_2 s^2 + ...
\]
are intrinsic invariants. The beginning coefficients are calculated using \((46)-(47)\) and \((48)\):

\[
\begin{align*}
    u_0 &= U^*(\varphi_0, \theta_0), \quad \tau_0 = u_1 = U^*_\varphi J_\varphi + U^*_\theta J_\theta \\
    \tau_1 &= U^*_\varphi J'_\varphi + U^*_\theta J'_\theta + (U^*_\varphi J^2_\varphi + 2U^*_\varphi J_\varphi J_\theta + U^*_\theta J^2_\theta) \\
    w_0 &= \frac{U^*_\theta}{g_0} J_\varphi - U^*_\varphi J_\theta \sin \varphi_0 \\
    w_1 &= \frac{U^*_\theta}{\sin \varphi_0} J'_\varphi - U^*_\varphi \sin \varphi_0 J'_\theta - \frac{U^*_\theta \cos \varphi_0 J^2_\varphi}{g_0} + \left( \frac{U^*_\theta}{\sin \varphi_0} - U^*_\varphi \sin \varphi_0 \right) J^2_\theta \\
    &\quad + \frac{U^*_\theta}{\sin \varphi_0} - U^*_\varphi \sin \varphi_0 - U^*_\theta \cos \varphi_0 J_\varphi J_\theta \\
    K_0 &= J_\theta (1 + J^2_\varphi) \cos \varphi_0 + (J_\varphi J'\varphi - J_\theta J'_\theta) \sin \varphi_0 \\
    K_1 &= [-J_\varphi J_\theta (1 + J^2_\varphi) + J_\varphi J'_\varphi - J_\theta J'_\theta] \sin \varphi_0 + [J'_\varphi (1 + 2J^2_\varphi) + J_\varphi J_\theta J'_\varphi] \cos \varphi_0 \\
    \mathcal{S}_0 &= \frac{w_0}{K_0}, \quad \mathcal{S}_1 = \frac{K_0 w_1 - K_1 w_0}{K_0^2}
\end{align*}
\]

Clearly the order of a coefficient \(a_k\) is \(k\) plus the order of \(a_0\), and we note that \(u_0\) has order 0, \(\tau_0, v_0\) and \(w_0\) have order 1, but \(K_0\) has order 2.

### 2.4.4 Singularities of the shape curve

A point \(p = (\varphi_0, \theta_0)\) on the spherical curve \(\gamma^*\) is said to be **regular** if \(K^* \neq 0\) and \(U^*_\varphi \neq 0\) at \(p\), otherwise it is called **singular**, and a curve with no regular point is called **exceptional**. Note the geometric meaning of \(U^*_\varphi = 0\), namely the curve \(\gamma^*\) is tangential to the gradient flow of \(U^*\) at \(p\).

Now consider the time parametrization of \(\gamma^*\). A point \(p = \gamma^*(t_0)\) is a **cusp** of the curve \(\gamma^*(t)\) if the speed \(v\) vanishes at \(t = t_0\). Assuming (for simplicity) that \(K^*\) is defined (or bounded) at \(p\), it follow from \((49)\)

\[
\lim_{t \to t_0} \frac{U^*_\nu}{v} = \frac{1}{2} \omega \rho_0,
\]

so \(U^*_\nu = 0\) but possibly \(K^* \neq 0\) at \(p\). In particular, \(p\) is a singular point. Conversely, by \((49)\) and assuming \(\omega \neq 0\), a point where both \(U^*_\nu\) and \(K^*\) vanish must be a cusp. This does not hold for \(\omega = 0\); for example, the shape curve of the figure eight periodic motion (cf. \((2)\)) passes through the Euler points on the equator circle with \(v \neq 0\) and \(K^* = U^*_\nu = 0\).

Examples of 3-body motions with exceptional shape curve arise from the isosceles solutions of the 3-body problem, which are fairly well understood (cf. e.g. \((1)\)). The isosceles m-triangle has two equal masses at the base, and the shapes \(\gamma^*\) of these m-triangles constitute a longitude circle on the shape sphere \(M^*\), and hence \(K^*\) vanishes along the curve. In fact, \(U^*_\nu\) also vanishes on this circle since it is a gradient line for \(U^*\). We point out, however, an isosceles triangle motion in the plane must have \(\omega = 0\), so in our study these motions are excluded at the outset.
The shape curves of collinear motions, being confined to the equatorial circle of \( M^* \), satisfy \( K^* = U^*_{\nu} = 0 \), so they must be regarded as exceptional. It is not difficult to see (purely kinematically) that a collinear motion is planar, and is even confined to a fixed line if \( \omega = 0 \).

On the other hand, taking a closer look at the Newtonian case with \( \omega \neq 0 \), we observe that a collinear motion must have constant shape and hence the shape curve is a single point, namely one of the three Euler points. In fact, from the identity (49) it follows that \( v = 0 \) at each point, so \( \gamma^* \) must be a single point. The vanishing of \( v \) also follows directly from the ODE (34), where in equation (ii) we have, by assumption, \( \varphi = \pi/2, \dot{\varphi} = 0, U^*_\varphi = 0 \), hence also \( v = \dot{\theta} = 0 \). Then, from the ODE it follows that \( U^*_\varphi = U^*_\theta = 0 \), namely the point is critical for \( U^* \) and hence, by definition, is an Euler point.

3 A three-body motion is essentially determined by its geometric shape curve

The purpose of this section is to interpret properly and provide evidence for the following:

**Conjecture 3.1** The geometric shape curve \( \gamma^* \) of a planar 3-body motion determines the time parametrized moduli curve \( \bar{\gamma}(t) \), and hence determines the 3-body motion modulo a fixed rotation of the plane.

This should hold in general, with some "obvious" exceptions. So, we are aiming at an analytical reconstruction of the moduli curve \( \bar{\gamma}(t) \), based on purely geometric data concerning the shape curve \( \gamma^* \) and its interaction with the gradient field \( \nabla U^* \). This is summarized as follows:

**Theorem 3.2** Assume an oriented geometric curve \( \gamma^* \) on the 2-sphere is realizable as the shape curve of a planar 3-body motion \( \gamma(t) \) at a given energy-momentum level \( (h, \omega) \). Then the relative geometry of \( (\gamma^*, \nabla U^*) \) on the shape sphere in the neighborhood of a generic point \( p \) of \( \gamma^* \) yields the information needed to determine the moduli curve \( \bar{\gamma}(t) \) as a solution of the ODE (34).

The proof will be elaborated in the following subsections, through local analysis which eventually reduces the problem to solving an algebraic system (80) involving three equations and three variables. This system depends only on intrinsic invariants, and it is the solution of this system, with some ambiguity, which enables us to determine the initial data (39) which generate the curve \( \bar{\gamma}(t) \) as a solution of the system (34).

3.1 Explicit calculation of the moduli curve as an initial value problem

Since the ODE system (34) is analytic, one can develop recursively the power series expansion of the solution \( \bar{\gamma}(t) = (\rho(t), \varphi(t), \theta(t)) \) by the method of undetermined coefficients. Namely, starting from the initial data set (39), the
temporal invariants $\rho_k, \varphi_k, \theta_k$ of order $k \geq 2$ are calculated recursively and expressed in terms of temporal invariants of lower order.

More specifically, repeated differentiation of the system (34) yields identities of type

$$E_{1,n} : 0 = (n + 2)(n + 1)\rho_0^2 \rho_{n+2} + .... \quad (58)$$

$$E_{2,n} : 0 = (n + 2)(n + 1)\rho_0^3 \varphi_{n+2} + ....$$

$$E_{3,n} : 0 = (n + 2)(n + 1)g_0 \rho_0^3 \theta_{n+2} + ....$$

Thus, the identity $E_{i,n}$ expresses a unique temporal invariant of highest order $n + 2$ in terms of lower order invariants. This procedure starts with the case $n = 0$ which yields the following identities stated for convenience

$$E_{10} : 0 = 2\rho_0^2 \rho_{2} + \rho_0 \rho_1^2 - 2h\rho_0 - u_0$$

$$E_{20} : 0 = 2\rho_0^3 \varphi_2 + 2\rho_0^2 \rho_1 \varphi_1 - 2\omega(\sin \varphi_0)\rho_0 \theta_1 - \frac{1}{2}f_0 \rho_0^3 \theta_1^2 - 4\nu_0$$

$$E_{30} : 0 = 2g_0 \rho_0^3 \theta_2 + 2g_0 \rho_0^2 \rho_1 \theta_1 + 2\omega(\sin \varphi_0)\rho_0 \varphi_1 + f_0 \rho_0^3 \varphi_1 \theta_1 - 4\eta_0 \quad (59)$$

However, we want to determine the initial data (39) solely in terms of intrinsic invariants and in the simplest way. A natural first step in this direction is the replacement of the six-tuple (39) by another equivalent "six-tuple"

$$[\rho_0, \varphi_0, \theta_0; \rho_1, \varphi_1, \theta_1] \longleftrightarrow [\rho_0, \rho_1, v_0; (J_\varphi, J_\theta)]$$

namely the initial data (39) is replaced by the temporal invariants $\rho_0, \rho_1, v_0$ and the direction element of $\gamma^*$ at $p = (\varphi_0, \theta_0)$, the latter being an intrinsic invariant. This is so because the pair $(\varphi_1, \theta_1)$, representing the velocity of the shape curve $\gamma^*(t)$ at $p$, is determined by the initial speed $v_0$ and direction $(J_\varphi, J_\theta)$ at $p$. The triple $(\rho_0, \rho_1, v_0)$ shall be referred to as the basic temporal invariants.

For convenience, we state the following preliminary observation:

**Proposition 3.3** For a given direction element $(J_\varphi, J_\theta)_p$ and basic temporal invariants $(\rho_0, \rho_1, v_0)$, we can determine the initial data set (39) and hence calculate successively, using the equations (58), the power series expansion of the time parametrized moduli curve $\bar{\gamma}(t)$. In other words, the direction element and the triple $(\rho_0, \rho_1, v_0)$ determine a unique solution $\bar{\gamma}(t)$ of the ODE system (34).

### 3.1.1 Reduction of order

The notion of order of an invariant, as defined in Section 2.4.1, makes little sense when the coordinate functions $\rho(t), \varphi(t), \theta(t)$ solve the ODE system (34). Indeed, by means of this system a differential invariant $Q$ of order 2 reduces its order to 1 by substituting the expressions for $\dot{\rho}, \dot{\varphi}, \dot{\theta}$ into $Q$. Furthermore, expressions for higher order derivatives of $\rho, \varphi, \theta$ are found by successive differentiation of the differential equations, and so they can be substituted into higher order invariants to reduce their total order. Thus, any given differential
invariant \( Q \) of order \( n \) can be reduced stepwise to a functional expression of order \( \leq 1 \). Then by evaluation at \( t = 0 \) the resulting terms involve only temporal invariants of order \( \leq 1 \), possibly also intrinsic invariants of order 0, that is, the function \( U^* \) or its partial derivatives (of any "order") evaluated at \( p \).

A complete reduction of an invariant is achieved when all invariants in the reduced expression have order \( \leq 1 \). Let us work out complete reductions of the basic curvature invariants \( K_0, K_1, w_1 \) listed in (57). First of all, recall that the expression for \( K_0 \) in (57) follows directly from its definition, as an intrinsic invariant of order 2. However, its complete reduction follows immediately from (18) or (49), so the following two expressions for \( K_0 \) must be equal

\[
K_0 = \frac{-2}{\rho_0^2 \theta_0^2} (\omega - \frac{2u_0}{\rho_0 \theta_0}) = J_0(1 + J_\varphi^2) \cos \varphi_0 + (J_\varphi J_\rho - J_0 J_\rho^*) \sin \varphi_0, \tag{61}
\]

which can also be verified directly by reducing the 2nd order invariants \( J_\varphi \) and \( J_\rho \), see below.

**Lemma 3.4** The 2nd order temporal invariant \( v_1 \) has the following complete reduction

\[
v_1 = \frac{2}{\rho_0^2} (-\rho_0^2 \theta_0 \varphi_1 + 2u_1) \tag{62}
\]

**Proof.** Starting from the expression (63), let us further reduce the following quantity

\[
R_1 = \frac{\rho_0^3 \varphi_1 \varphi_2}{2} = \rho_0^3 \left[ \varphi_1 \varphi_2 + \frac{1}{4} \rho_0 \varphi_1 \theta_1^2 + g_0 \theta_1 \theta_2 \right]
\]

in terms of temporal invariants of lower order, as follows. In the last expression for \( R_1 \), the second order invariants \( \varphi_2 \) and \( \theta_2 \) can be reduced using the identities (59), and the invariants \( \varphi_1 \) and \( \theta_1 \) can be "reduced" to \( v_0 \) by (61):

\[
R_1 = \varphi_1 [-\rho_0^2 \rho_1 \varphi_1 + \omega a_0 \rho_0 \theta_1 + \frac{1}{4} f_0 \rho_0^3 \theta_1^2 + 2 \mu_0] + \theta_1 [-g_0 \rho_0 \rho_1 \theta_1 - \omega a_0 \rho_0 \varphi_1 - \frac{1}{2} f_0 \rho_0^3 \varphi_1 \theta_1 + 2 \eta_0] + \frac{1}{4} f_0 \rho_0^3 \varphi_1 \theta_1^2
\]

\[
= -\frac{1}{4} f_0 \rho_0^3 \varphi_1 \theta_1^2 - \rho_0^2 \rho_1 (\varphi_1^2 + g_0 \theta_1^2) + 2(\mu_0 \varphi_1 + \eta_0 \theta_1) + \frac{1}{4} f_0 \rho_0^3 \varphi_1 \theta_1^2
\]

\[
= -\rho_0^2 \rho_1 \varphi_1^2 + 2 \varphi_1 (\mu_0 J_\varphi + \eta_0 J_\theta) = -\rho_0^2 \rho_1 \varphi_1^2 + 2 \varphi_1 u_1,
\]

where at the end we have used the relation \( u_1 = \mu_0 J_\varphi + \eta_0 J_\theta \). Recall that \( u_1 = \tau_0 \) is the tangential derivative \( U^*_\tau \) of \( U^* \) at \( t = 0 \), which is an intrinsic invariant of order 1. Comparison of the two expressions for \( R_1 \) yields the identity (62).

Now, to calculate the reductions of \( J_\varphi^* \) and \( J_\rho^* \) from their expressions in (55), we substitute the expression (62) for \( v_1 \) and the expressions (59) for \( \varphi_2 \) and \( \theta_2 \),
which yields

\[ J'_{\varphi} = \frac{1}{v_0^2}(2\varphi_2 - v_0v_1J_{\varphi}) = \frac{1}{2}f_0J_\theta^2 + \frac{2\omega \sin \varphi_0}{\rho_0^2v_0}J_\theta + \frac{4}{\rho_0^2v_0}(\mu_0 - u_1J_{\varphi}) \]  

\[ J'_{\vartheta} = \frac{1}{v_0^2}(2\vartheta_2 - v_0J_{\vartheta}) = -\frac{f_0}{g_0}J_{\varphi}J_{\vartheta} - \frac{2\omega \sin \varphi_0}{g_0\rho_0^2v_0}J_{\varphi} + \frac{4}{\rho_0^2v_0}(\eta_0 - u_1J_{\vartheta}) \]

This enables us to reduce the following intrinsic invariant

\[ U_{\varphi}^* J_{\varphi}' + U_{\vartheta}^* J_{\vartheta}' \]

\[ = U_{\varphi}^*\left[\frac{1}{2}f_0J_\theta^2 + \frac{2\omega \sin \varphi_0}{\rho_0^2v_0}J_\theta + \frac{4}{\rho_0^2v_0}(\mu_0 - u_1J_{\varphi})\right] \]

\[ + U_{\vartheta}^*\left[-\frac{f_0}{g_0}J_{\varphi}J_{\vartheta} - \frac{2\omega \sin \varphi_0}{g_0\rho_0^2v_0}J_{\varphi} + \frac{4}{\rho_0^2v_0}(\eta_0 - u_1J_{\vartheta})\right] \]

\[ = f_0J_\theta\left(\frac{1}{2}U_{\varphi}^*J_{\varphi} - \frac{U_{\varphi}^*}{g_0}J_{\varphi} - \frac{2\omega \sin \varphi_0}{\rho_0^2v_0}(U_{\varphi}^* - \frac{U_{\vartheta}^*}{g_0}J_{\varphi})\right) \]

\[ + \frac{4}{\rho_0^2v_0}\left(\mu_0^2 + \frac{\eta_0^2}{g_0} - u_1(U_{\varphi}^*J_{\varphi} + U_{\vartheta}^*J_{\vartheta})\right) \]

\[ = -\frac{1}{2}f_0\omega u_0^2 - \frac{f_0}{\sin \varphi_0}w_0J_\theta - \frac{2\omega \sin \varphi_0}{\rho_0^2v_0}w_0 - \frac{4}{\rho_0^2v_0}(\tau_0^2 + \omega^2) \]

\[ = \frac{4u_0^2}{\rho_0^2v_0} - \frac{2\omega \sin \varphi_0}{\rho_0^2v_0}f_0J_\theta\left(\frac{w_0}{\sin \varphi_0} + \frac{1}{2}U_{\varphi}^*J_{\varphi}\right) \]

\[ = u_0K_0 - f_0J_\theta\left(\frac{w_0}{\sin \varphi_0} + \frac{1}{2}U_{\varphi}^*J_{\varphi}\right) \]

and hence the expression for \( \tau_1 \) in (57) reduces to

\[ \tau_1 = u_0K_0 + J \]  

where

\[ J = -f_0J_\theta\left(\frac{u_0}{\sin \varphi_0} + \frac{1}{2}U_{\varphi}^*J_{\varphi}\right) + (U_{\varphi}^*J_{\varphi}^2 + 2U_{\vartheta}^*J_{\varphi}J_{\vartheta} + U_{\vartheta}^*J_{\vartheta}^2) \]

is a first order intrinsic invariant. Similarly, the expression for \( w_1 \) in (57) reduces to

\[ w_1 = \frac{2u_1}{\rho_0^2v_0}(\omega \cos \varphi_0 - \frac{2u_0}{\rho_0^2v_0} + \eta_0 \cos \varphi_0(J_{\varphi}^2 + J_{\vartheta}^2) + \left(\frac{U_{\varphi}^*}{\sin \varphi_0} - U_{\varphi}^* \sin \varphi_0\right)J_{\varphi}^2 \]

\[ + (\mu_0 \cos \varphi_0 + \frac{U_{\vartheta}^*}{\sin \varphi_0} - U_{\vartheta}^* \sin \varphi_0)J_{\vartheta} \]

It would be rather laborious to calculate a complete reduction of \( K_1 \) starting from its order 3 expression in (57). However, a reduction is given implicitly by the second curvature equation \( Eq2 \) calculated below (see (75), where the
occurrence of $K_1$ lies in the coefficients $J_2$ and $J_4$. Solving the equation with respect to $K_1$ yields the first of the following three expressions

$$K_1 = 2K_0 \frac{2K_0u_1p_0^2v_0-w_1p_0^2v_0-w_0p_0p_1+2\omega u_1}{\rho_0^2v_0(\omega p_0v_0-2w_0)}$$

(66)

$$= -\frac{2K_0u_1p_0^2v_0-w_1p_0^2v_0-w_0p_0p_1+2\omega u_1}{\rho_0^2v_0}$$

$$= \frac{16u_1}{\rho_0^2v_0}(\omega - \frac{2w_0}{\rho_0v_0} + \frac{4w_1}{\rho_0^2v_0} + \frac{4(\rho_0p_1w_0-2\omega u_1)}{\rho_0^2v_0})$$

(67)

and then elimination of $K_0$ twice using (61) yields the other two expressions. Consequently, a complete reduction follows by substituting the reduced expression (65) for $w_1$, but we shall not need this and hence it is omitted.

3.2 Derivation of the basic algebraic system of equations

According to Proposition 3.3, the crucial problem indicated as follows

\[
\text{[intrinsic data]}_p \rightarrow (\rho_0, \rho_1, v_0),
\]

(67)

is to calculate the basic temporal invariants $(\rho_0, \rho_1, v_0)$ from intrinsic data representing the local relative geometry of the pair $(\gamma^*, \nabla U^*)$. The intrinsic data that we shall utilize consist of the direction element $(J_\varphi, J_\theta)_p$ and the following six basic intrinsic invariants

$$u_0, u_1, w_0; w_1, K_0, K_1 \ (w_0 \neq 0, K_0 \neq 0),$$

(68)

also referred to as the basic 6-tuple at $p$. As indicated, the non-vanishing of $w_0$ and $K_0$ is always assumed in the sequel to avoid cases of singular behavior. Note that $w_0 \neq 0$ means that $\gamma^*$ is transversal to the gradient flow of $U^*$ at $p$. We remark that by (64) we do not need the second order invariant $\tau_1$ in (65) since it can be derived from the others, cf. (64).

The basic 6-tuple (68) is the union of two triples of intrinsic invariants, namely the basic $U^*$-invariants $(u_0, \tau_0, w_0)$ depending only on $U^*$ and $(J_\varphi, J_\theta)_p$

$$u_0 = U^*(p), \ \tau_0 = u_1 = U^*_\tau(p), \ w_0 = U^*_\nu(p),$$

(69)

and the basic curvature invariants $(w_1, K_0, K_1)$ reflecting some of the local geometry of $(\gamma^*, \nabla U^*)$ at $p$.

Now, we shall set up a system of three algebraic equations used to calculate $(\rho_0, \rho_1, v_0)$. The coefficients of the system are rational functions of the invariants in (68), together with the parameters $(\omega, h)$. For notational convenience, let us introduce the following four intrinsic invariants

$$J_1 = 2u_0 - \mathcal{S}_0, \ J_2 = \frac{2u_1 - \mathcal{S}_1}{\mathcal{S}_0}, \ J_3 = \frac{2u_1}{w_0}, \ J_4 = \frac{-K_1}{2K_0w_0}$$

(70)
Our first equation is

\[ Eq1 : \rho_0^3 v_0^2 + 2 \omega \frac{\nu_0 \rho_0}{K_0} = 4 \mathcal{S}_0, \]  

(71)

which is just the leftmost identity in (61). Another equation is derived from the energy integral \[53\] evaluated at time \( t = 0 \), which combined with \( Eq1 \) yields

\[ Eq3 : \rho_0 (\rho_1^2 - 2h) + \frac{\omega^2}{\rho_0} - \frac{\omega}{2K_0} \rho_0 v_0 = 2u_0 - \mathcal{S}_0 \]  

(72)

Finally, to obtain last equation \( Eq2 \), let us differentiate the curvature equation \[18\] and evaluate at \( t = 0 \), which yields the identity

\[ 2\rho_0^3 v_0 v_1 + 3\rho_0^2 v_0^2 \rho_1 - 4v_0 \mathcal{S}_1 + \frac{2\omega}{K_0} \left[ v_0 \rho_1 + \rho_0 v_1 - \frac{K_1}{K_0} \rho_0 v_1^2 \right] = 0 \]  

(73)

Here we replace the temporal invariant \( v_1 \) by its reduced expression \[62\] and make use of \( 71 \) to calculate

\[
0 = \left[ 4v_0 + \frac{4\omega}{K_0} \frac{1}{\rho_0^2} \left( \frac{1}{2} \rho_0^3 v_1 \right) + 3\rho_0^2 v_0^2 \rho_1 - 4v_0 \mathcal{S}_1 \right. \\
+ \frac{2\omega}{K_0} \rho_0 v_1 - \frac{2\omega K_1}{K_0^2} \rho_0 v_0^2 \\
= \left( 4v_0 + \frac{4\omega}{K_0} \frac{1}{\rho_0^2} \left( -\rho_0^2 v_0^2 \rho_1 + 2v_1 \right) + 3\rho_0^2 v_0^2 \rho_1 - 4v_0 \mathcal{S}_1 \right. \\
+ \frac{2\omega}{K_0} v_0 \rho_1 - \frac{2\omega K_1}{K_0^2} \rho_0 v_0^2 \\
= -\rho_0^2 v_0^2 \rho_1 + 4(2u_1 - \mathcal{S}_1) v_0 + \frac{2\omega}{K_0} (4u_1 \frac{1}{\rho_0} - \frac{K_1}{K_0} \rho_0 v_0^2) \\
= \frac{-\rho_1}{\rho_0} (\rho_0^2 v_0^2 + \frac{2\omega}{K_0} \rho_0 v_0) + 4(2u_1 - \mathcal{S}_1) v_0 + \frac{2\omega}{K_0} (4u_1 \frac{1}{\rho_0} - \frac{K_1}{K_0} \rho_0 v_0^2) \\
\]

Using \( Eq1 \) to substitute the first term in the last line, we further simplify the last identity as follows:

\[
\mathcal{S}_0 \frac{\rho_1}{\rho_0} = (2u_1 - \mathcal{S}_1) v_0 + \frac{\omega}{K_0} (2u_1 \frac{1}{\rho_0} - \frac{K_1}{2K_0} \rho_0 v_0^2) \\
\frac{\rho_1}{\rho_0 v_0} = \frac{2u_1 - \mathcal{S}_1}{\mathcal{S}_0} + \omega \left[ \frac{2u_1}{K_0 \mathcal{S}_0 \rho_0 v_0} - \frac{K_1}{2K_0^2 \mathcal{S}_0} \rho_0 v_0 \right] \\
Eq2 : \frac{\rho_1}{\rho_0 v_0} = J_2 + \omega [J_3 \frac{1}{\rho_0^2 v_0} + J_4 \rho_0 v_0] \\
\]  

(74)

**Definition 3.5** For a given energy-momentum pair \((h, \omega)\), the **basic algebraic system**, affiliated with the direction element \((J_p, J_\theta)_p\) and the basic curvature
invariants \((w_1, K_0, K_1)\) at \(p\), consists of the following three algebraic equations with the basic temporal invariants \((\rho_0, \rho_1, v_0)\) as the variables:

\[
\begin{align*}
\text{Eq1} : & \quad \rho_0^3 v_0^2 + 2 \omega \frac{\rho_0 v_0}{K_0} = 4 \mathcal{S}_0 \\
\text{Eq2} : & \quad \frac{\rho_1}{\rho_0 v_0} - \omega \left[ J_3 \frac{1}{\rho_0 v_0} + J_4 \rho_0 v_0 \right] = J_2 \\
\text{Eq3} : & \quad \rho_0 (\rho_1^2 - 2h) + \frac{\omega^2}{\rho_0} - \frac{\omega}{2K_0} \rho_0 v_0 = J_1
\end{align*}
\]

Remark 3.6 The angular momentum and energy constants are, in fact, expressible at the moduli space level, namely in terms of intrinsic invariants and the basic temporal invariants, as follows:

\[
\begin{align*}
\omega &= \frac{2v_0}{\rho_0 v_0} - \frac{1}{2} K_0 \rho_0 v_0^2 \\
h &= \frac{1}{2} \rho_1^2 + \frac{1}{8} \rho_0^2 v_0^2 + 2 \frac{w_0^2}{\rho_0 v_0} - \frac{w_0 K_0 + u_0}{\rho_0} + \frac{1}{8} K_0^2 \rho_0^2 v_0^2
\end{align*}
\]

The expression for \(\omega\) is simply a restatement of Eq1 in (75), whereas the expression for \(h\) follows from Eq3 or more simply from (17), by inserting the expression for \(\omega\).

3.3 Solution of the basic algebraic system of equations

We may as well consider the system of rational equations (75) from a purely algebraic point of view, with the three real variables \((\rho_0, \rho_1, v_0)\). The coefficients of the equations are themselves rational expressions involving eight real constants (parameters), namely a pair \((h, \omega)\) and a 6-tuple \((68)\). In Definition 3.5 we also say that the system (75) is affiliated with a given 6-tuple \((68)\), irrespective of the interpretation of the latter. Anyhow, we shall be interested only in admissible solutions \((\rho_0, \rho_1, v_0)\), meaning that \(\rho_0 > 0\) and \(v_0 > 0\).

3.3.1 The special case of vanishing angular momentum

For comparison reasons, let us recall the special case of three-body motions with \(\omega = 0\) discussed in [5], in which case the equations (75) simplify to

\[
\begin{align*}
\rho_0^3 v_0^2 = 4 \mathcal{S}_0, \quad & \frac{\rho_1}{\rho_0 v_0} = J_2, \quad \rho_0 (\rho_1^2 - 2h) = J_1
\end{align*}
\]

Then there can be at most one admissible solution, uniquely given by

\[
v_0 = 2 \frac{\sqrt{\mathcal{S}_0}}{\rho_0} \quad \rho_1 = 2J_2 \frac{\sqrt{\mathcal{S}_0}}{\rho_0} \quad \rho_0 = \frac{1}{2h} (4J_2^2 \mathcal{S}_0 - J_1) \text{ when } h \neq 0.
\]
For completeness, let us also recall the special case of \((h, \omega) = (0, 0)\). Then the scaling symmetry of 3-body motions, which scales \(\rho\) and the time \(t\), keeps \((h, \omega) = (0, 0)\) invariant and does not affect the geometric shape curve. Thus we can choose \(\rho_0\) freely, and the above expressions for \(v_0\) and \(\rho_1\) still hold. In any case, it follows from the formulae (79) that positivity of \(\rho_0\) and \(v_0\) poses some obvious conditions on the basic intrinsic invariants (68).

### 3.3.2 The general case \(\omega \neq 0\)

To study the basic algebraic system (75) in general, let us for convenience introduce the variables

\[ x = \rho_0, \quad y = \rho_0 v_0, \quad z = \rho_1 \]

Then the system (75) becomes

\[
\begin{align*}
(i) & \quad xy^2 + \frac{2\omega}{K_0} y = 4\mathcal{E}_0 \\
(ii) & \quad \frac{z}{y} - \omega \left[ J_3 \frac{1}{xy} + J_4 y \right] = J_2 \\
(iii) & \quad x(z^2 - 2h) + \frac{\omega^2}{x} - \frac{\omega}{2K_0} y = J_1
\end{align*}
\]

As an approach to solving the system, we propose to eliminate \(x\) and \(z\) and seek an equation solely involving \(y\), as follows. From equation (i)

\[ x = \frac{4\mathcal{E}_0}{y^2} - \frac{2\omega}{K_0} \frac{1}{y}, \quad (81) \]

which by substitution into equation (ii) yields

\[
z = J_2 y + \omega y^2 \left\{ \frac{J_3}{4\mathcal{E}_0 - \frac{2\omega}{K_0} y} + \frac{J_4}{4\mathcal{E}_0 - \frac{2\omega}{K_0} y} \right\}
\]

\[
= J_2 y + \omega y^2 \left\{ \frac{J_5 + \omega J_6 y}{4\mathcal{E}_0 - \frac{2\omega}{K_0} y} \right\}
\]

\[
= J_2 y \left( 4\mathcal{E}_0 - \frac{2\omega}{K_0} y \right) + \omega y^2 \left( J_5 + \omega J_6 y \right)
\]

\[
= \frac{y(4\mathcal{E}_0 J_2 + \omega (J_5 - \frac{2\omega}{K_0} y + \omega^2 J_6 y^2)}{4\mathcal{E}_0 - \frac{2\omega}{K_0} y}
\]

Thus, we arrive at the expression

\[ z = \frac{y(a + b\omega y + c\omega^2 y^2)}{d + e\omega y}, \quad (82) \]
where we have introduced for notational convenience the intrinsic invariants
\[ J_5 = J_3 + 4 \mathfrak{g}_0 J_4 = \frac{2u_1}{w_0} - \frac{2K_1}{K_0^2}, \quad J_6 = -\frac{2J_2}{K_0} = \frac{K_1}{K_0^2 w_0} \]  
\[ a = 4 \mathfrak{g}_0 J_2, \quad b = J_5 - \frac{2J_2}{K_0}, \quad c = J_6, \quad d = 4 \mathfrak{g}_0, \quad e = -\frac{2}{K_0} \]  

Substitution of the expressions (81) and (82) for \( x \) and \( z \) into equation (iii) yields
\[
\frac{(a + \omega by + \omega^2 cy^2)^2}{d + \omega ey} - 2h \frac{(d + \omega ey)}{y^2} + \frac{\omega^2 y^2}{(d + \omega ey)} + \frac{\omega e}{4} y = J_1,
\]
which we state as the following polynomial equation of degree \( \leq 6 \)
\[
\Pi(y) = y^2 \left[ \beta_0 + \beta_1 \omega y + \beta_2 \omega^2 y^2 + \beta_3 \omega^3 y^3 + \beta_4 \omega^4 y^4 \right] + H \left[ \alpha_0 + \alpha_1 \omega y + \alpha_2 \omega^2 y^2 \right] = 0
\]  
where the role of \( (h, \omega) \) is made more transparent, and \( \alpha_i, \beta_i \) are the following intrinsic invariants
\[
\alpha_0 = -2d^2, \quad \beta_0 = a^2 - J_1 d, \quad \alpha_1 = -4de, \quad \beta_1 = \frac{de}{4} + 2ab - J_1 e, \\
\alpha_2 = -2e^2, \quad \beta_2 = 1 + 2ac + b^2 + \frac{e^2}{4}, \quad \beta_3 = 2bc, \quad \beta_4 = c^2
\]  
We may also prefer to study the equation (85) as a polynomial equation in \( Y = \omega y \):
\[
P(Y) = Y^2 \left[ \beta_0 + \beta_1 Y + \beta_2 Y^2 + \beta_3 Y^3 + \beta_4 Y^4 \right] + H \left[ \alpha_0 + \alpha_1 Y + \alpha_2 Y^2 \right] = 0
\]  
with the explicit dependence on the parameters \( (h, \omega) \) concentrated in the single parameter
\[ H = h \omega^2, \]
as expected in view of Remark 2.5. For a given value of \( H \), a root \( Y \) of the polynomial \( P(Y) \) is said to be admissible if the following inequality holds
\[ \text{sign}(K_0)Y < \text{sign}(K_0)2w_0 \]  
Now, let us fix the pair \( (h, \omega) \) and hence also \( H \). Since the polynomial in (85) has at most six roots, it is clear that the algebraic system (80) has at most six solutions \( (x, y, z) \), obtained by expressing \( x \) and \( z \) in terms of \( y = \omega^{-1}Y \) as explained above.
Moreover, a solution is real, namely \( x, y, z \) are real, if and only if \( Y \) is a real root. Finally, a solution \( (x, y, z) = (\rho_0, \rho_0 v_0, \rho_1) \) is admissible if and only if \( x > 0 \) and \( y > 0 \). In fact, by formula (84) positivity of \( x \) is simply the condition
that $Y$ is admissible, whereas positivity of $y$ demands $Y$ to have the same sign as $\omega$.

For the applications involving shape curves, we prefer to regard the given 6-tuple $(u_0, u_1, w_0; w_1, K_0, K_1)$ as the basic 6-tuple at a regular point $p$ of an oriented curve $\gamma^*$ on the shape sphere, and hence the 6-tuple is the union of the $U^*$-triple $(u_0, u_1, w_0)$ and the basic curvature invariants $(w_1, K_0, K_1)$ of $\gamma^*$ at $p$. In fact, the direction element $(J_\varphi, J_\theta)_p$ and the $U^*$-triple determine each other.

As an immediate consequence of the above observations we state the following results, assuming $w_0 \neq 0$, $K_0 \neq 0$ as before.

**Proposition 3.7** For a given pair $(h, \omega)$, consider the basic algebraic system (80) affiliated with a direction element $(J_\varphi, J_\theta)_p$ and triple $(w_1, K_0, K_1)$ of real numbers. Then there is a one-to-one correspondence between the admissible solutions $(x, y, z)$ of the system and solutions $\tilde{\gamma}(t)$ of the ODE (34) at the energy-momentum level $(h, \omega)$, whose shape curves $\gamma^*$ passing through $p$ have i) the given direction $(J_\varphi, J_\theta)_p$, and ii) the triple $(w_1, K_0, K_1)$ as the basic curvature invariants at $p$.

**Proposition 3.8** For a fixed value of $H = h\omega^2$, the negative admissible roots $Y$ of the polynomial equation (87) are in one-to-one correspondence with the admissible solutions $(x, y, z)$ in Proposition 3.7, for any fixed choice $(h, \omega)$ with $\omega < 0$. Similarly, the positive admissible roots $Y$ correspond to the admissible solutions $(x, y, z)$ when $\omega > 0$.

Thus we end up with the following two cases:

- The basic algebraic system has no admissible solution $(\rho_0, \rho_1, v_0)$. Then there is no 3-body motion $\gamma(t)$ whose shape curve passes through $p$ with the given direction $(J_\varphi, J_\theta)_p$ and with $(w_1, K_0, K_1)$ as the basic curvature invariants.

- The basic algebraic system at $p$ has an admissible solution $(\rho_0, \rho_1, v_0)$. As shown in Section 3.1, an admissible solution together with the direction element $(J_\varphi, J_\theta)_p$ yields the initial data (39) and hence a solution $\tilde{\gamma}(t)$ of the ODE (34). As a consequence, $\tilde{\gamma}(t)$ is the moduli curve of a 3-body motion, whose shape curve $\gamma^*$ has the direction $(J_\varphi, J_\theta)_p$ and basic curvature invariants $(w_1, K_0, K_1)$ at $p$. We remark that the number of admissible solutions cannot exceed 6; in the examples we have studied so far the number is at most 2.

### 3.4 Case studies with examples

Without specifying the direction element $(J_\varphi, J_\theta)_p$ and the (mass dependent) potential function $U^*$ on the shape sphere, we shall start from a list of six numbers viewed as the basic 6-tuple at $p$ of the shape curve $\gamma^*$ of a fictitious three-body motion, for a given energy-momentum pair $(h, \omega)$. Then we
inquire whether the affiliated basic algebraic system (75) has admissible solutions \((\rho_0, \rho_1, v_0)\). For each such solution we know there is a solution \(\gamma(t)\) of the ODE \((34)\) in the moduli space \(\bar{M}\) realizing the same basic 6-tuple \((68)\).

Note that once an admissible root \(Y\) of the polynomial \(P(Y)\) is calculated, one obtains the solution \((x, y, z)\) of the system \((80)\) by using the formulas in \((70), (81) - (84)\). We perform the calculations using Maple, and high precision is needed.

3.4.1 Example

Let us choose (randomly) the following six numbers

\[
\begin{align*}
    u_0 &= 0.3, \quad u_1 = 0.4, \quad w_0 = 1.4, \quad w_1 = -0.3, \quad K_0 = 0.4, \quad K_1 = 0.2;
\end{align*}
\]

and consider the basic algebraic system \((80)\) affiliated with this 6-tuple, and with \((h, \omega)\) unspecified. As before, we put \(H = h\omega^2\) and turn to the associated polynomial equation \((87)\).

Let us first choose \(H = 1\). The polynomial has four complex and two real roots

\[
Y_1 = -1.165697412, \quad Y_2 = 1.521207930
\]

Since both \(Y_i\)satisfy the inequality \(Y < 2w_0 = 2.8\), they are admissible by \((88)\). Moreover, \(\omega\) must have the same sign as \(Y_i\), so this yields the following triples of basic temporal invariants

**Case 1:** \(\omega < 0\);

\[
(\rho_0, \rho_1, v_0) = (14.59210391 \cdot \omega^2, -1.302577877 \cdot \omega^{-1}, -0.0798854281 \cdot \omega^{-3})
\]

**Case 2:** \(\omega > 0\);

\[
(\rho_0, \rho_1, v_0) = (2.763075661 \cdot \omega^2, 1.227863217 \cdot \omega^{-1}, 0.5505487785 \cdot \omega^{-3})
\]

In each case we are free to choose the energy-momentum pair \((h, \omega)\) subject to the constraint \(h\omega^2 = 1\), and for each choice there is a unique solution \(\gamma(t)\) of the ODE \((34)\) whose shape curve \(\gamma^*\) has basic 6-tuple \((u_0, ..., K_1)\) as given above, at any given direction element \((J_\phi, J_\theta)\) compatible with the \(U^*\)-invariants \((u_0, u_1, w_0)\).

As long as \(H > 0\) the polynomial \(P(Y)\) will have two admissible roots, and of different sign, so the case \(H = 1\) is typical for energy \(h > 0\). But for \(H \leq 0\) all roots are complex, hence there is no admissible solution in the case of \(h \leq 0\).

3.4.2 Example

The following 6-tuple and associated polynomial \((87)\) are calculated:

\[
\begin{align*}
    u_0 &= 1, \quad u_1 = -0.01, \quad w_0 = 0.2, \quad w_1 = 0, \quad K_0 = 0.1, \quad K_1 = 0.0567 \\
    P(Y) &= 0.00199 - 0.0241 \cdot Y + 0.0864 \cdot Y^2 - 0.128 \cdot Y^3 + 0.0804 \cdot Y^4 = 0
\end{align*}
\]
Let us proceed as in the previous example. The polynomial has two admissible roots $Y_i$, hence for fixed $\omega > 0$ we have an example with two admissible solutions (basic temporal invariants)

\[
Y_1 = 0.137; \quad (\rho_0, \rho_1, v_0) = (279.8 \cdot \omega^2, 0.049 \cdot \omega^{-1}, 0.00049 \cdot \omega^{-3}) \\
Y_2 = 0.390; \quad (\rho_0, \rho_1, v_0) = (0.510 \cdot \omega^2, -0.198 \cdot \omega^{-1}, 0.776 \cdot \omega^{-3})
\]

### 3.4.3 Example

Consider the special case $h = 0$, so the polynomial (85) has the factor $y^2$, which after cancelling yields an equation of degree 4

\[
P(y) = \beta_0 + \beta_1 \omega y + \beta_2 \omega^2 y^2 + \beta_3 \omega^3 y^3 + \beta_4 \omega^4 y^4 = 0,
\]

and equation (87) with $H = 0$ reads

\[
P(Y) = \beta_0 + \beta_1 Y + \beta_2 Y^2 + \beta_3 Y^3 + \beta_4 Y^4 = 0.
\]

Moreover, $P(Y)$ becomes a quadratic polynomial if and only if $K_1 = 0$, by (83), (84), (86).

Now, choose the following 6-tuple (68)

\[
u_0 = 1, u_1 = -0.1, w_0 = 0.2, w_1 = 0, K_0 = 0.1, K_1 = 0,
\]

which yields a quadratic polynomial with two admissible roots

\[
P(Y) = 6.4 \cdot 10^{-3} - 0.416 \cdot Y + 1.02 \cdot Y^2
\]

\[Y_1 \approx 0.016, \quad Y_2 \approx 0.392\]

Hence, for $\omega > 0$ the algebraic system (75) has the following two admissible solutions $(\rho_0, \rho_1, v_0)$:

- Case 1: $\rho_0 \approx 295 \cdot 10^4 \cdot \omega^2$, $\rho_1 \approx -0.0016 \cdot \omega^{-1}$, $v_0 \approx 5.35 \cdot 10^{-7} \omega^{-3}$
- Case 2: $\rho_0 \approx 1.06 \cdot \omega^2$, $\rho_1 \approx -0.979 \cdot \omega^{-1}$, $v_0 \approx 0.368 \cdot \omega^{-3}$

As a control, we verify that $h = 0$ by inserting these values into the energy integral (using high precision arithmetic)

\[
h = \frac{1}{2} \rho_1^2 + \frac{\rho_0^2}{8} v_0^2 + \frac{\omega^2}{2 \rho_0} - \frac{u_0}{\rho_0}
\]

### 3.4.4 Example (linear motions)

Observe that in order to obtain an algebraic system with three independent equations such as (80), the nonvanishing of some of the $U^*$-invariants (69) is crucial. But for comparison reason, consider the extreme case $U^* = 0$ where
this fails, namely linear 3-body motions (in absence of forces). Now, Eq1 of the system (75) reads
\[ K^* = \frac{2\omega}{\rho^2 v}, \]
and this holds everywhere along the motion. Therefore, in the case \( \omega = 0 \) the shape curve \( \gamma^* \) has vanishing curvature and is therefore confined to a geodesic circle on the 2-sphere. In the case \( \omega \neq 0 \), \( K^* \) cannot vanish, but Eq2 on the form (73) together with (62) amount to \( K_1 = 0 \), namely \( K^* \) is a constant. Consequently, \( \gamma^* \) is confined to a circle which is not a geodesic circle. Reconstruction of the moduli curve \( \tilde{\gamma}(t) \) from the shape curve alone is clearly not possible; there is little information "stored" in the constant function \( K^* \).

On the other hand, the simple example with \( U^* \) a nonzero constant can be solved explicitly. Now equation (i) of the ODE system (75) decouples from the other two and can be solved explicitly, as a 1-dimensional Kepler problem with initial values \( \rho_0, \rho_1 \). Then substitution into the energy integral, for given values of \( (h, \omega) \), yields the value of \( v_0^2 \), and consequently one obtains the triple \( (\rho_0, \rho_1, v_0) \). As before, together with a given point and direction element on the shape sphere this yields the data to solve the initial value problem in \( \mathcal{M} \).

### 3.4.5 Example (Henon 2)

We consider a concrete example with three bodies of unit mass in the plane at initial positions \( (x_i, 0) \) on the x-axis and initial velocities \( (0, y_i) \) in the y-axis direction, namely
\[
\begin{align*}
x_1 &= -1.0207041786, \quad x_2 = 2.0532718983, \quad x_3 = -1.0325677197 \\
y_1 &= 9.1265693140, \quad y_2 = 0.0660238922, \quad y_3 = -9.1925932061 \\
h &= -1.040039, \quad \omega = 0.312013
\end{align*}
\]

We infer from these data that the shape curve \( \gamma^* \) starts out at a point \( p = (\varphi_0, \theta_0) \) on the equator circle \( \varphi = \pi/2 \) and with initial velocity perpendicular to this circle. Moreover, we infer
\[ \rho_1 = \theta_1 = u_1 = w_1 = \beta_3 = \beta_4 = 0 \]

Calculation of the basic 6-tuple of \( \gamma^* \) at \( p \):
\[
(u_0, u_1, w_0, w_1, K_0, K_1) = (213.6058, 0, -31771.876, 0, -75.30872, 0)
\]

Now, we can determine the associated polynomial \( P(Y) \), which has degree 4 and has two negative and two positive roots \( Y_i \). The positive roots are admissible and there are two admissible solutions \( (\rho_0, \rho_1, v_0) \) of the basic algebraic system (75), as follows:
\[
\begin{align*}
Y_1 &= 89.01744668991 : \rho_0 = 0.02076, \rho_1 = 0, \quad v_0 = 13741.798 \\
Y_2 &= 8.083161335258 : \rho_0 = 2.51475, \rho_1 = 0, \quad v_0 = 10.30184
\end{align*}
\]

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Remark 3.9 In the above example, we observe that the original 3-body motion is one of the two 3-body motions generated by the roots $Y_i$, namely by the root $Y_2$. Calculation of the corresponding initial data set (39) for the ODE (34) can be checked to be the same as the initial data set deduced from the information (71). On the other hand, the root $Y_1$ yields another 3-body motion, but we claim its shape curve is different from the shape curve of the original motion, although they have the same basic 6-tuple and hence are close to each other in the vicinity of $p$. We shall refer to them as a pair of companion solutions.

The original shape curve $\gamma^*$ has, in fact, been "found" by numerical experiments to be periodic, see [http://three-body.ipb.ac.rs/henon.php](http://three-body.ipb.ac.rs/henon.php) and the example Henon 2. But we leave open the question of whether its companion is also periodic or if the two shape curves are globally close to each other.

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