Predictive model of fermionic dark matter halos with a quantum core and an isothermal atmosphere

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We develop a thermodynamical model of fermionic dark matter halos at finite temperature. Statistical equilibrium states may be justified by a process of violent collisionless relaxation in the sense of Lynden-Bell or from a collisional relaxation of nongravitational origin if the fermions are self-interacting. The most probable state (maximum entropy state) generically has a “core-halo” structure with a quantum core (fermion ball) surrounded by an isothermal atmosphere. The quantum core is equivalent to a polytrope of index $n = 3/2$. The Pauli exclusion principle creates a quantum pressure that prevents gravitational collapse and solves the core-cusp problem of the cold dark matter model. The isothermal atmosphere (which is similar to the NFW profile of cold dark matter) accounts for the flat rotation curves of the galaxies at large distances. We numerically solve the equation of hydrostatic equilibrium with the Fermi-Dirac equation of state and determine the density profiles and rotation curves of fermionic dark matter halos. We impose that the surface density of the dark matter halos has the universal value $\Sigma_0 = \rho_0 r_h = 141 M_\odot/pc^2$ obtained from the observations.

For a fermion mass $m = 165$ eV/c$^2$, the “minimum halo” has a mass $(M_h)_{\text{min}} = 10^7 M_\odot$ and a radius $(r_h)_{\text{min}} = 597$ pc similar to dwarf spheroidals like Fornax. This ultracompact halo corresponds to a completely degenerate fermion ball at $T = 0$. This is the ground state of the self-gravitating Fermi gas. For ultracompact dark matter halos with a mass $(M_h)_{\text{min}} < M_h < (M_h)_{\text{CCP}} = 6.73 \times 10^8 M_\odot$ (canonical critical point), the quantum core is surrounded by a tenuous classical isothermal atmosphere. Dark matter halos with a mass $M_h > (M_h)_{\text{CCP}}$ are dominated by the classical isothermal atmosphere. They may be purely gaseous (similar to the Burkert profile) or harbor a fermion ball. The gaseous solution is stable in all statistical ensembles. The core-halo solution is canonically unstable (having a negative specific heat) but, for small dark matter halos with a mass $(M_h)_{\text{CCP}} < M_h < (M_h)_{\text{MCP}} = 1.08 \times 10^{10} M_\odot$ (microcanonical critical point), it is microcanonically stable. By maximizing the entropy at fixed mass and energy we find that the mass of the quantum core scales with the halo mass as $M_c/(M_h)_{\text{min}} = 1.47 [M_h/(M_h)_{\text{min}}]^{3/8}$. This relation is equivalent to the “velocity dispersion tracing” relation according to which the velocity dispersion in the core $\sigma_v^2 \sim G M_c/R_c$ is of the same order as the velocity dispersion in the halo $\sigma_v^2 \sim G M_h/r_h$. We provide therefore a justification of this relation from thermodynamical arguments. The fermion ball represents a large quantum bulge which is either present now or may have, in the past, triggered the collapse of the surrounding gas, leading to a supermassive black hole and a quasar. When $M_h > (M_h)_{\text{MCP}}$, the quantum core-halo solution is microcanonically unstable. Large dark matter halos may undergo a gravothermal catastrophe leading ultimately to the formation of a small out-of-equilibrium condensed core or, in the case of very large dark matter halos, to a supermassive black hole when the core mass overcomes the Oppenheimer-Volkoff limit. The isothermal halo is left undisturbed and is in agreement with the Burkert profile. Our model has no free parameter (the mass $m = 165$ eV/c$^2$ of the fermionic particle is determined by the minimum halo) so it is completely predictive. It predicts that the Milky Way should harbor a fermionic dark matter bulge of mass $M_c = 9.45 \times 10^6 M_\odot$ and radius $R_c = 240$ pc in possible agreement with the observations.

We also consider another model involving a larger fermion mass $m = 986$ keV/c$^2$, a supermassive black hole of mass $M_{\text{BH}} = 2.10 \times 10^9 M_\odot$ which could account for active galactic nuclei. For an even larger fermion mass $m = 546$ keV/c$^2$, a supermassive black hole of mass $M_{\text{BH}} = 4.20 \times 10^9 M_\odot$ should be formed in the Milky Way instead of a fermion ball. However, models with a fermion mass $m = 546$ keV/c$^2$ predict that ultracompact dark matter halos of mass $\sim 10^8 M_\odot$ should contain a fermionic core of mass $M_c \sim 10^5 M_\odot$ and radius $R_c \sim 5$ mpc similar to intermediate mass black holes, a prediction which may be challenged by observations.

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I. INTRODUCTION

The cold dark matter (CDM) model of cosmology is remarkably successful in explaining the large scale structure of the universe but it experiences several difficulties at small scales: (i) classical CDM simulations lead to a universal cuspy density profile – the Navarro-Frenk-White (NFW) profile [1] which decreases as $r^{-3}$ at large distances and diverges as $r^{-1}$ at the center – while observations rather favor core-like centers – the Burkert [2] profile which also decreases as $r^{-3}$ at large distances but tends to a constant at the center; (ii) the number of sub-
halos obtained in CDM simulations is much larger than the number of satellites observed in the Galaxy [3, 5]. (iii) dissipationless CDM simulations predict that the majority of the most massive subhaloes of the Milky Way are too dense to host any of its bright satellites; (iv) the stellar velocity dispersions measured in CDM simulations are larger than those observed in the satellites of the Galaxy [6]. These problems are called the core-cusp problem [7], the missing satellites problem [3, 5] and the “too big to fail” problem [8]. They are responsible for the small-scale crisis of CDM [9]. To solve these problem one solution might be to take into account baryonic feedback that can transform cusps into cores [1, 10]. A possible alternative is to take into account the quantum (or wave) nature of the particles. Indeed, quantum mechanics creates an effective pressure which can balance gravitational attraction and lead to cores instead of cusps. Moreover, the quantum Jeans length is finite even at $T = 0$ (contrary to the classical Jeans length) and this may solve the missing satellites problem and other small-scale problems experienced by the classical CDM model.

If the DM particle is a boson, like an ultra-light axion (ULA) [14], the quantum pressure is due to the Heisenberg uncertainty principle which is equivalent to an anisotropic pressure or to a quantum potential. If the bosons are self-interacting, there is an additional (isotropic) pressure arising from the self-interaction. At $T = 0$, bosons form Bose-Einstein condensates (BECs). Newtonian self-gravitating BECs are described by the Schrödinger-Poisson (SP) if they are noninteracting or by the Gross-Pitaevskii-Poisson (GPP) equations if they are self-interacting [15]. Numerical simulations of the SP equations [16–24] show that BECDM halos typically have a core-halo structure with a quantum core (soliton) surrounded by a halo made of quantum interferences. The halo typically has a NFW or Burkert profile. It can be approximated in certain cases by an isothermal profile with an effective temperature (see Sec. III.C of [25]). The halo leads to approximately flat rotation curves and the quantum core solves the core-cusp problem. This core-halo structure results from a process of gravitational cooling [26–28] or from a collisional relaxation of non-gravitational origin if the fermions are self-interacting. If the DM particle is a fermion, like a sterile neutrino, the quantum pressure is due to the Pauli exclusion principle which creates an effective isotropic pressure. Fermionic DM halos are described by the Fermi-Dirac distribution at finite (effective) temperature which may be justified by Lynden-Bell’s theory of violent relaxation [30, 31] as argued in [32, 33]. They typically have a core-halo structure with a quantum core (fermion ball) surrounded by an isothermal halo [31]. An isother-

FIG. 1: Normalized density profiles in logarithmic scales (zoom of Fig. 18 in [32]). Solid line: Critical (marginal) King profile; Dotted line: Modified Hubble profile; Dashed line: Burkert profile.

The analogy between bosonic and fermionic DM halos suggests that (i) the process of gravitational cooling is similar to the process of violent relaxation (they may even correspond to the same phenomenon); (ii) the NFW profile (excluding the cusp) and the Burkert profile, which are similar to the isothermal profile, may be physically justified by Lynden-Bell’s theory of violent relaxation; (iii) the fermion ball in the fermionic model is the counterpart of the soliton in the BEC model.

A problem of considerable interest in the physics of quantum (fermionic and bosonic) DM halos is to construct core-halo profiles of DM halos and predict the quantum core mass – halo mass relation $M_c (M_h)$. One

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1 See the Introduction of Ref. [13] for a short review and an exhaustive list of references on bosonic DM.

2 See the Introduction of Ref. [20] for a short review and an exhaustive list of references on fermionic DM.

3 It is shown in Figs. 5 and 6 of [25] that the isothermal profile is almost indistinguishable from the empirical (observational) Burkert profile up to a few halo radii. This is even more true if we account for tidal effects by using the fermionic King model [32, 33]. It is shown in [32] that the critical (marginal) King profile triggering the gravothermal catastrophe at the turning point of energy is relatively close to the Burkert profile (see Fig. [1]).

4 Fermions and bosons behave antisymmetrically regarding their collisional relaxation. The Pauli blocking $f(\rho_0 - f)$ for fermions has the tendency to slow down the relaxation and the Bose enhancement $f(\rho_0 + f)$ for bosons, leading to the formation of “granules” or “quasiparticles”, has the tendency to accelerate the relaxation. Gravitational encounters (“collisions”) are completely negligible in fermionic DM halos. In bosonic DM halos, they manifest themselves on a (secular) timescale of the order of the age of the universe (see [34] and references therein).
can then compare theoretical predictions with direct numerical simulations and observations.

In a previous paper [25], we have developed a predictive model of BECDM halos in the case where the bosons have a strong repulsive self-interaction so that the Thomas-Fermi (TF) approximation can be implemented. We considered a generalized GPP equation\(^5\) which provides a parametrization of the complicated processes of violent relaxation and gravitational cooling. With respect to the ordinary GPP equation, this new wave equation includes an effective thermal term and a source of dissipation. We determined the equilibrium states of this equation and for bosons with a repulsive or attractive self-interaction, we adapt the bosonic model developed in [25] to the case of fermions.\(^6\)

In the recent years, three types of studies have been conducted in the context of fermionic DM:

(i) Chavanis and coworkers studied phase transitions in the self-gravitating Fermi gas in Newtonian gravity \(^{31, 33, 40–43}\) and general relativity \(^{37–39}\). They showed that three possibilities can arise in the caloric curve depending on the size of the system (measured by the so-called “degeneracy parameter” \(\mu\)). For small systems \(\mu < \mu_{\text{CCP}}\), no phase transition occur. For intermediate size systems \(\mu_{\text{CCP}} < \mu < \mu_{\text{MCP}}\), phase transitions can occur in the canonical ensemble but not in the microcanonical ensemble. For large systems \(\mu > \mu_{\text{MCP}}\), both canonical and microcanonical phase transitions can occur. They also showed that above the Oppenheimer-Volkoff limit \(N > N_{\text{OV}}\), a new turning point of energy appears in the caloric curve and triggers a general relativistic instability leading to the formation of a black hole by gravitational collapse.

(ii) de Vega and coworkers \(^{47–52}\) constructed models of DM halos in Newtonian gravity adopting a fermion mass of the order of \(1\,\text{keV}/c^2\). This mass determines a minimum mass of mass \((M_h)_{\text{min}} = 0.39 \times 10^6 M_{\odot}\) and size \((r_h)_{\text{min}} = 33\,\text{pc}\) corresponding to Willman I assumed to be completely degenerate. They argued that larger halos are nondegenerate (without quantum core) so they coincide with the well-known classical isothermal sphere \(^{53}\).

(iii) Arguelles and coworkers \(^{54–60}\) constructed general relativistic models of DM halos adopting a fermion mass of the order of \(48\,\text{keV}/c^2\) and applied these models to the Milky Way. The system has a core-halo structure made of a quantum core (fermion ball) surrounded by an isothermal atmosphere. Reviving the original idea of Bilic and coworkers \(^{61–65}\), they argued that the fermion ball could mimic a supermassive black hole (SMBH) that is purported to exist at the center of the Galaxy.

In the present paper, we develop a general model valid for fermions of arbitrary mass \(m\). Then, we consider specifically the case of a “small” mass \(m = 165\,\text{eV}/c^2\) or \(m \sim 1\,\text{keV}/c^2\) and the case of a “large” mass \(m = 54.6\,\text{keV}/c^2\) or \(m = 386\,\text{keV}/c^2\) and discuss the connection with previous works. A brief and synthetic presentation of our results is given in Ref. \(^{66}\) that the readers may consult in a first reading. The present paper provides a justification and a detailed description of these results.

The paper is organized as follows. In Sec. [1] we explain why fermionic DM halos may be in a maximum entropy state described by the Fermi-Dirac distribution at finite (effective) temperature. In Sec. [III] we recall basic results concerning the thermodynamics of a self-gravitating Fermi gas in a box. In Sec. [IV] we consider the nondegenerate limit of the self-gravitating Fermi gas which explains the external structure of large DM halos. In Sec. [V] we consider the completely degenerate limit of the self-gravitating Fermi gas which explains the structure of ultracompact DM halos and the cores of large DM halos. In Sec. [VI] we consider partially degenerate DM halos. We show that they have a core-halo structure with a quantum core (fermion ball) surrounded by an isothermal halo. The isothermal halo leads to flat rotation curves and the quantum core solves the core-cusp problem. We determine the core – halo mass relation \(M_c(M_h)\) from thermodynamical arguments. In Sec. [VII] we consider astrophysical applications of our model for a fermion mass \(m = 165\,\text{eV}/c^2\). We evidence a bifurcation above a canonical critical point \(\mu_{\text{CCP}}\) between purely gaseous states and core-halo states containing a fermion ball. For the core-halo states, we also evidence a transition at a microcanonical critical point \(\mu_{\text{MCP}}\). We argue that small DM halos with \(M_h < (M_h)_{\text{MCP}}\) should contain a large quantum bulge while large DM halos with \(M_h > (M_h)_{\text{MCP}}\) should rather contain a small out-of-equilibrium quantum core resulting from a

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\(^5\) This equation was introduced heuristically in \(^{45}\) and justified with more precise arguments in \(^{56}\) from a coarse-graining of the Wigner-Poisson equations.

\(^6\) The fact that the results obtained for bosons could be transposed to fermions was mentioned in \(^{25}\). Conversely, some of the results obtained here for fermions can be exported to bosons and can complete the discussion given in \(^{25}\).
gravothermal catastrophe arrested by quantum effects. For very large DM halos, when the core mass \( M_c \) passes above the Oppenheimer-Volkoff limit \( M_{OV} \), quantum mechanics cannot prevent gravitational collapse and the gravothermal catastrophe leads to the formation of a SMBH. The isothermal halo is left undisturbed and is in agreement with the Burkert profile. We argue that the Milky Way should contain a large fermionic bulge of mass \( M = 9.45 \times 10^9 \, M_\odot \) and radius \( R_c = 240 \, \text{pc} \) in possible agreement with the observations. In Sec. VIII, we consider another model with a fermion mass \( m = 54.6 \, \text{keV}/c^2 \) in which the fermion ball could mimic a SMBH at the center of the Milky Way. We argue that, for larger DM halos or for a larger fermion mass, the fermion ball should be replaced by a real SMBH. In Sec. IX, we propose possible solutions to an apparent paradox related to the universal surface density of DM halos. In Sec. X, we discuss the difference between isothermal and quantum cores and their formation process. In Sec. XI, we summarize our main results and conclude.

II. FORMATION AND EVOLUTION OF DM HALOS

In this section, we recall basic elements of kinetic theory related to the formation and the evolution of fermionic DM halos in order to justify our thermodynamical approach (we refer to [34] for a more detailed discussion).

It is well-known that self-gravitating systems experience two successive types of relaxation:

(i) In a first regime, gravitational encounters can be neglected and the evolution of the system is described by the Vlasov (or collisionless Boltzmann) equation. The Vlasov-Poisson equations experience a complicated process of collisionless violent relaxation as described by Lynden-Bell [30] in the context of stellar systems. Through violent relaxation, the system reaches a quasistationary state (virialized state) on the coarse-grained scale which is a stable stationary solution of the Vlasov equation. The Vlasov-Poisson equations admit an infinite number of stationary solutions. In addition, all spherical DFs of the form \( f = f(\epsilon) \) with \( f'(\epsilon) < 0 \), where \( \epsilon = v^2/2 + \Phi(r) \) is the energy of the particles, are dynamically (Vlasov) stable in Newtonian gravity [67]. Lynden-Bell proposed to determine the “most probable state” of the system resulting from a collisionless relaxation by using arguments of statistical mechanics and thermodynamics. This equilibrium state is obtained by maximizing a mixing entropy while taking into account all the constraints of the Vlasov equation. In the single level approximation, the Lynden-Bell entropy is similar to the Fermi-Dirac entropy and the constraints of the Vlasov equation reduce to the conservation of mass and energy. In addition, if the particles are fermions, the Lynden-Bell exclusion principle \( 7 \leq \eta_0 \), where \( \eta_0 \) is the initial DF, coincides with the Pauli exclusion principle \( f \leq 2m^4/\hbar^3 \) up to a numerical factor of order unity (see footnote 34 of [32]). In order to select the most probable structure arising from a violent collisionless relaxation, one has therefore to maximize the Lynden-Bell (or Fermi-Dirac) entropy at fixed mass and energy. The extremization problem leads to the Lynden-Bell (or Fermi-Dirac) DF. Then, we have to make sure that the equilibrium state is an entropy maximum (most probable state) not an entropy minimum or a saddle point.

(ii) In a second regime, collisions must be taken into account. The collisional relaxation of the system is described by a kinetic equation such as the gravitational Boltzmann, Landau or Lenard-Balescu equation. If the particles are fermions, the collisional relaxation leads to a statistical equilibrium state described by the ordinary Fermi-Dirac distribution which maximizes the Fermi-Dirac entropy at fixed mass and energy. This corresponds to the “most probable state” of the system resulting from a collisional relaxation. This is also a stable stationary solution of the kinetic equation.

In the two situations described above we are led to maximizing the Lynden-Bell or Fermi-Dirac entropy at fixed mass and energy. We stress that the justification of this maximization problem is different in the collisionless (Lynden-Bell) and collisional (Fermi-Dirac) regimes. If the system is collisionless, the temperature is effective. We also note that the proper thermodynamical ensemble to consider is the microcanonical ensemble. Indeed,

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8 If several stable equilibrium states (entropy maxima) are found for the same values of mass and energy, they may all be equally relevant. Indeed, metastable states (local entropy maxima) have a very long lifetime and are as much relevant as fully stable states (global entropy maxima). Their selection is related to a notion of basin of attraction and cannot be decided simply by comparing their entropies. Finally, we note that the predictive power of Lynden-Bell’s theory is limited by the problem of incomplete relaxation: the system may reach a dynamically stable quasi-stationary state that is not a maximum entropy state.

9 We consider collisions of all sort. They may correspond to weak gravitational encounters or strong (hard core-like) collisions if the particles have a self-interaction, leading to the notion of self-interacting dark matter (SIDM) halos. DM halos may also experience a stochastic forcing due to the presence of baryons or other external sources that can induce a secular relaxation of the system (see Appendix B of [25]).

10 See the introduction of Ref. [53] for a short review and an exhaustive list of references on the kinetic theory of self-gravitating systems.

11 Actually, because of evaporation and because of its interaction with nearby galaxies, the system is tidally truncated and the Fermi-Dirac distribution must be replaced by the fermionic King distribution [69, 70]. This DF can be derived from a kinetic theory based on the fermionic Landau equation [70]. The fermionic King model is studied in [53] in Newtonian gravity and in [68, 69] in general relativity.
the system is assumed to be isolated so that the energy and the mass are conserved. This remark is important since statistical ensembles may be inequivalent for systems with long-range interactions such as self-gravitating systems.

If the particles interact only via (weak) two-body gravitational encounters, the collisional relaxation time is extremely long, scaling as \( t_R \sim (N/\ln N) t_D \) \(^{67}\), where \( N \) is the number of particles and \( t_D \sim R/v \sim 1/\sqrt{G\rho} \sim 0.1 \text{ Gyrs} \) is the dynamical time (we have taken \( R \sim 20.1 \text{kpc}, \rho \sim 7.02 \times 10^{-3} M_\odot/pc^3 \) and \( v \sim 146 \text{ km/s} \) in a galaxy of mass \( M \sim 10^{12} M_\odot \) like the Milky Way). For fermionic DM halos, \( N \) is huge (\( N \sim 10^{75} \) for keV fermions) so the relaxation time is much larger than the age of the universe \( t_U \sim 13.8 \text{ Gyrs} \) by many orders of magnitude (the Pauli blocking even increases this relaxation time). In that case, the system is essentially collisionless and only the Lynden-Bell type of relaxation is relevant. However, in order to be more general, we consider the possibility that the particles have a (strong) self-interaction that can cause a faster collisional evolution of nongravitational origin. This allows us to consider the possibility of a collisional relaxation (especially in the core of the system where the density is high and the relaxation time short) towards a Fermi-Dirac equilibrium state on a timescale smaller than the age of the universe. For example, if they have a cross section per unit mass \( \sigma_m \equiv \sigma/m = 1.25 \text{ cm}^2/\text{g} \) consistent with the Bullet Cluster constraint \(^{73}\) we get \( t_{\text{coll}} \sim 1/(\rho \sigma_m v) \sim 3.66 \text{ Gyrs} < t_U \). In that case, fermionic DM halos behave similarly to globular clusters with additional quantum effects. Because of collisions and evaporation, they follow a series of equilibria towards configurations of higher and higher central density. If the equilibrium state becomes thermodynamically unstable, a fermionic DM halo may experience a phase transition from a gaseous phase to a condensed phase (with a quantum core and an isothermal envelope) associated with a form of gravothermal catastrophe \(^{74}\) stopped by quantum degeneracy \(^{31}\). This may be followed by a dynamical instability of general relativity origin leading to the formation of a SMBH.

### III. SELF-GRAVITATING FERMI GAS IN A BOX

In this section, we consider the statistical mechanics of a self-gravitating Fermi gas in a box. We summarize the main results obtained in our previous papers \(^{31, 40, 42}\) and detail the theoretical framework that will be needed in the present study.

#### A. Theoretical framework

We consider a gas of nonrelativistic fermions interacting via Newtonian gravity. Let \( f(r, v, t) \) denote its distribution function (DF) in phase space giving the mass density of fermions with position \( r \) and velocity \( v \) at time \( t \). The mass density in configuration space is \( \rho = \int f \, dv \). The total mass of the system is

\[
M = \int f \, dv \tag{1}
\]

and its total energy is

\[
E = \int \frac{v^2}{2} \, d\mathbf{v} + \frac{1}{2} \int \rho \Phi \, d\mathbf{r}, \tag{2}
\]

where the first term is the kinetic energy and the second term is the gravitational energy \( (E = E_{\text{kin}} + W) \). We introduce the Fermi-Dirac entropy

\[
\frac{S}{k_B} = -\frac{\eta_0}{m} \int \left\{ \frac{f}{\eta_0} \ln \frac{f}{\eta_0} + \left( 1 - \frac{f}{\eta_0} \right) \ln \left( 1 - \frac{f}{\eta_0} \right) \right\} \, d\mathbf{v}, \tag{3}
\]

where

\[
\eta_0 = \frac{2 m^4}{h^3} \tag{4}
\]

is the maximum value of the DF fixed by the Pauli exclusion principle (the factor 2 accounts for the multiplicity \( 2s + 1 \) of quantum states for particles of spin \( s = 1/2 \)). The Fermi-Dirac entropy is equal to the logarithm of the number of microstates, specified by the precise position and velocity \( \{r_i, v_i\} \) of all the fermions, corresponding to a given macrostate specified by the DF \( f(r, v) \) giving the density of fermions around the point \( (r, v) \) in phase space.

In the microcanonical ensemble, the statistical equilibrium state of a self-gravitating gas of fermions is obtained by maximizing the Fermi-Dirac entropy \( S \) at fixed energy \( E \) and mass \( M \). One has therefore to solve the optimization problem

\[
\max \{ S \mid E, M \text{ fixed} \}. \tag{5}
\]

This thermodynamic approach is justified in a mean field approximation which is exact in a proper thermodynamic limit \( N \to +\infty \) (see Sec. 7.1 of \(^{42}\) and Appendix B of \(^{44}\)).

An extremum of entropy at fixed energy and mass is determined by the variational principle

\[
\frac{\delta S}{k_B} - \beta \delta E + \frac{\alpha}{m} \delta M = 0, \tag{6}
\]

where \( \beta = 1/(k_B T) \) and \( \alpha = \mu/(k_B T) \) are Lagrange multipliers (\( T \) is the temperature and \( \mu \) is the global chemical potential). This leads to the Fermi-Dirac distribution

\[
f(r, v, t) = \frac{\eta_0}{1 + e^{[(mv^2/2 + m\Phi(r) - \mu)]/k_B T}}. \tag{7}
\]

The density of particles \( \rho = \int f \, dv \) and the pressure \( P = \frac{1}{2} \int f v^2 \, d\mathbf{v} \) are related to the gravitational potential \( \Phi(r) \) by

\[
\rho(r) = \frac{4\pi \sqrt{2} \eta_0}{(2\beta m)^{3/2}} I_{1/2} \left[ \lambda e^{\beta m \Phi(r)} \right], \tag{8}
\]
\[ P(r) = \frac{8\pi \sqrt{2}\eta_0}{3(\beta m)^{3/2}} I_{3/2} \left[ \lambda e^{\beta m \Phi(r)} \right], \quad (9) \]

where \( \lambda = e^{-\beta \mu} \) and \( I_n(t) \) denotes the Fermi integrals

\[ I_n(t) = \int_0^{+\infty} \frac{x^n}{1 + xe^x} \, dx. \quad (10) \]

We recall the identity

\[ I_n'(t) = -\frac{n}{t} I_{n-1}(t) \quad (n > 0), \quad (11) \]

which can be established from Eq. (10) by an integration by parts. Eliminating \( \lambda e^{\beta m \Phi(r)} \) between Eqs. (8) and (9), we obtain the equation of state \( P(\rho) \) of the nonrelativistic Fermi gas at finite temperature in parametric form.

Combining the condition of hydrostatic equilibrium

\[ \nabla P + \rho \nabla \Phi = 0 \quad (12) \]

with the Poisson equation

\[ \Delta \Phi = 4\pi G \rho, \quad (13) \]

we obtain the fundamental differential equation of hydrostatic equilibrium

\[ \nabla \cdot \left( \frac{\nabla P}{\rho} \right) = -4\pi G \rho. \quad (14) \]

Together with the barotropic equation of state \( P(\rho) \) specified by Eqs. (8) and (9), this equation determines the density profile of the self-gravitating Fermi gas at statistical equilibrium.

Alternatively, substituting the density-potential relation from Eq. (8) into the Poisson equation (13), we obtain a differential equation determining the gravitational potential

\[ \Delta \Phi = \frac{16\pi^2 \sqrt{2} G \eta_0}{(\beta m)^{3/2}} I_{1/2} \left[ \lambda e^{\beta m \Phi(r)} \right], \quad (15) \]

which is called the Fermi-Poisson equation or the finite temperature Thomas-Fermi equation. The density is then obtained from Eq. (8). The two equations (14) and (15) are equivalent.

We now assume that the system is spherically symmetric and introduce the dimensionless variables

\[ \psi = \beta m (\Phi - \Phi_0), \quad k = \lambda e^{\beta m \Phi_0}, \quad (16) \]

and

\[ \xi = \left[ \frac{16\pi^2 \sqrt{2} G \eta_0}{(\beta m)^{3/2}} \right]^{1/2} r, \quad (17) \]

where \( \Phi_0 \) is the central value of the gravitational potential. We can then rewrite the density and the pressure under the form

\[ \rho(r) = \frac{4\pi \sqrt{2}\eta_0}{(\beta m)^{3/2}} I_{1/2} \left[ ke^{\psi(\xi)} \right], \quad (18) \]

\[ P(r) = \frac{8\pi \sqrt{2}\eta_0}{3(\beta m)^{3/2}} I_{3/2} \left[ ke^{\psi(\xi)} \right]. \quad (19) \]

On the other hand, Eqs. (14) and (15) lead to the fermionic Emden equation

\[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\psi}{d\xi} \right) = I_{1/2}(ke^{\psi}) \quad (20) \]

with the boundary conditions

\[ \psi(0) = \psi'(0) = 0. \quad (21) \]

This equation determines the structure of the system as a function of the parameter \( k \). For \( k \to +\infty \), the system is nondegenerate: this corresponds to the gaseous phase (see Sec. IV). For \( k \to 0 \), the system is completely degenerate: this corresponds to the condensed phase (see Sec. V). For intermediate values of \( k \), the system is partially degenerate: it typically has a core-halo structure with a quantum core (fermion ball) surrounded by a classical isothermal halo (see Secs. IV and VI). Some density profiles are plotted in Fig. 2 for different values of \( k \).

As is well-known, self-gravitating systems at nonzero temperature have the tendency to evaporate. Therefore, there is no equilibrium state in a strict sense and the statistical mechanics of self-gravitating systems is essentially an out-of-equilibrium problem. However, the evaporation rate is small in general and the system can be found in a quasi-equilibrium state for a relatively long time. In order to describe the thermodynamics of the

\[ \text{FIG. 2: Normalized density profiles of fermionic DM halos for different values of } k (k = 10^{-8}, 10^{-6}, 10^{-4}, 10^{-2} \text{ from bottom to top}). The dashed line corresponds to a completely degenerate fermion ball. The upper curve corresponds to a nondegenerate isothermal DM halo.} \]

\[ P(r) = \frac{8\pi \sqrt{2}\eta_0}{3(\beta m)^{3/2}} I_{3/2} \left[ ke^{\psi(\xi)} \right]. \quad (19) \]

12 See Ref. [31] for a description of this core-halo configuration and some analytical approximations. See also Sec. V of Ref. [25] for similar results obtained in the case of bosonic DM halos which can be directly exported to the case of fermionic DM halos.
self-gravitating Fermi gas rigorously, we shall use an artifice and enclose the system within a spherical box of radius \( R \). The box typically represents the size of the cluster under consideration (see Sec. VII). In that case, the solution of Eq. \( (20) \) is terminated by the box at the normalized radius

\[
\alpha = \left[ \frac{16\pi^2 \sqrt{2} \mu}{3 \beta \eta} \right]^{1/2} R. \tag{22}
\]

Since \( \alpha \) is the value of \( \xi \) at the box radius \( R \) we can write

\[
\xi = \alpha \frac{r}{R}. \tag{23}
\]

Let us first calculate the normalized inverse temperature

\[
\eta = \frac{\beta GM \rho_m}{R}. \tag{24}
\]

For a spherically symmetric distribution of matter, the Poisson equation \( \Box \) is equivalent to Newton’s law

\[
\frac{d\Phi}{dr} = \frac{GM(r)}{r^2}, \tag{25}
\]

where \( M(r) = \int_0^r \rho(r')4\pi r'^2 dr' \) is the mass contained within the sphere of radius \( r \). Applying Newton’s law at \( r = R \) and using Eqs. \( (16) \), \( (3) \) and \( (4) \), we get

\[
\eta = \alpha \psi_k(\alpha). \tag{26}
\]

This equation relates the dimensionless box radius \( \alpha \) and the concentration variable \( k \) to the dimensionless inverse temperature \( \eta \).

On the other hand, according to Eqs. \( (22) \) and \( (24) \), \( \alpha \) and \( k \) are linked to each other by the relation

\[
\alpha^2 \sqrt{\eta} = \mu. \tag{27}
\]

or, more explicitly [using Eq. \( (26) \)]

\[
\alpha^5 \psi_k(\alpha) = \mu^2, \tag{28}
\]

where

\[
\mu = \eta_0 \sqrt{512 \pi^4 G^3 M R^3} \tag{29}
\]

is the so-called degeneracy parameter \( [22] \). We shall give a physical interpretation of this parameter in Sec. VII.

The calculation of the energy is a little more involved. The kinetic energy of a nonrelativistic gas can be written as

\[
E_{\text{kin}} = \frac{3}{2} \int P \, dr. \tag{30}
\]

Using Eq. \( (19) \), we obtain

\[
E_{\text{kin}} = \frac{\alpha^7}{\mu^4} \int_0^\alpha I_{3/2} \left[ ke^{\psi_k(\xi)} \right] \xi^2 d\xi. \tag{31}
\]

In order to determine the potential energy, we can use the Virial theorem (see, e.g., \[29\])

\[
2E_{\text{kin}} + W = 3P_0 V, \tag{32}
\]

where \( P_0 = P(R) \) is the pressure on the boundary of the box and \( V = \frac{4}{3} \pi R^3 \) is the volume of the spherical box. Using the expression of the pressure from Eq. \( (19) \) at the box radius \( R \), we get

\[
WR = \frac{2\alpha^{10}}{3\mu^4} I_{3/2} \left[ ke^{\psi_k(\alpha)} \right] - \frac{2E_{\text{kin}} R}{GM^2}. \tag{33}
\]

Introducing the normalized energy

\[
\Lambda = -\frac{ER}{GM^2} \tag{34}
\]

and combining Eqs. \( (31) \) and \( (33) \), we finally obtain

\[
\Lambda = \frac{\alpha^7}{\mu^4} \int_0^\alpha I_{3/2} \left[ ke^{\psi_k(\xi)} \right] \xi^2 d\xi - \frac{2\alpha^{10}}{3\mu^4} I_{3/2} \left[ ke^{\psi_k(\alpha)} \right]. \tag{35}
\]

The expression of the entropy is derived in Appendix B. Finally, using Eqs. \( (16) \), \( (18) \), \( (24) \), \( (25) \), \( (29) \) and \( (C7) \) the normalized density and velocity profiles can be written as

\[
\frac{\rho(r)}{\rho_0 R} = \frac{\mu}{4\pi \eta^{3/2}} I_{1/2} [ke^{\psi(\xi)}], \tag{36}
\]

\[
\frac{v^2(r)}{GM/R} = 1 + \frac{1}{\eta} \xi \psi'(\xi). \tag{37}
\]

B. Caloric curves and ensembles inequivalence

Using the foregoing formulae, we can obtain the caloric curve \( \eta(\Lambda) \) of the self-gravitating Fermi gas for a specified value of \( \mu \) as follows. For a given value of \( k \), we can solve the ordinary differential equation \( (20) \) with the initial conditions \( (21) \) until the value of \( \alpha \) at which the relation \( (35) \) is satisfied. Then, Eqs. \( (26) \) and \( (35) \) determine the normalized inverse temperature \( \eta \) and the normalized energy \( \Lambda \) of the configuration. By varying the parameter \( k \) from 0 to \( +\infty \), we can determine the full caloric curve \( \eta(\Lambda) \) for the specified value of the degeneracy parameter \( \mu \) (see Fig. 3). We can then study the occurrence of

\[A\] A more rigorous approach would be to use a truncated model (fermionic King model) like in \[22, 23\].

\[B\] It should not be confused with the chemical potential which is denoted by the same symbol. In principle, no confusion should arise.
phase transitions as a function of $\mu$. This study has been made in detail in [42]. Below, we summarize the main results of this study that will be useful in the following.

We have to be careful that, for self-gravitating systems (which have a long-range interaction), the statistical ensembles are inequivalent. In the previous section, we have worked in the microcanonical ensemble. This is the statistical ensemble associated with isolated systems where the energy $E$ is fixed. By contrast, systems in contact with a heat bath fixing the temperature $T$ are described by the canonical ensemble. In the canonical ensemble, the statistical equilibrium state of a self-gravitating gas of fermions is obtained by minimizing the Fermi-Dirac free energy $F = E - TS$ at fixed mass $M$. One has therefore to solve the optimization problem

$$
\min \{ F \mid M \text{ fixed} \}. 
$$

(38)

One can easily show that the equilibrium states in the microcanonical and in the canonical ensembles are the same; an extremum of free energy at fixed mass is also an extremum of entropy at fixed mass and energy [73]. However, their stability may be different in the two ensembles. An equilibrium state that is canonically stable is always microcanonically stable (a minimum of free energy at fixed mass is always a maximum of entropy at fixed mass and energy), but the converse is wrong: a maximum of entropy at fixed mass and energy is not necessarily a minimum of free energy at fixed mass [73]. For example, equilibrium states with a negative specific heat are always unstable in the canonical ensemble while they may be stable in the microcanonical ensemble. This corresponds to the concept of ensembles inequivalence for systems with long-range interaction [42, 71, 72]. As a result, we may miss important solutions if we use the canonical ensemble instead of the microcanonical one.

The self-gravitating Fermi gas presents two critical points, one in each ensemble: the function $\Lambda(\eta)$ becomes multivalued at the canonical critical point $\mu_{\text{CCP}}$ and the function $\eta(\Lambda)$ becomes multivalued at the microcanonical critical point $\mu_{\text{MCP}}$ whose values are [42]

$$
\mu_{\text{CCP}} = 83, \quad \mu_{\text{MCP}} = 2670.
$$

(39)

For $\mu < \mu_{\text{CCP}}$ the series of equilibria $\eta(\Lambda)$ is monotonic (see the curve $\mu = 10$ in Fig. 3, see also Fig. 11 below). The equilibrium states are stable in both canonical and microcanonical ensembles.

For $\mu > \mu_{\text{CCP}}$ the series of equilibria $\eta(\Lambda)$ presents turning points of temperature (see the curves $\mu = 10^2$ and $10^5$ in Fig. 3; see also Figs. 12 and 14 below). Following the series of equilibria towards higher and higher density contrasts, and using the Poincaré-Katz [76, 77] criterion, one can show that the equilibrium states are canonically stable before the first turning point of temperature $\eta_c$ (gaseous phase) and after the last turning point of temperature $\eta^*$ (condensed phase). They are microcanonically unstable in between. In the canonical ensemble, the system undergoes an isothermal collapse at $\eta_c$ from the gaseous phase to the condensed phase and an explosion at $\eta^*$ from the condensed phase to the gaseous phase. The first order phase transition that is expected at $\eta_c$ (at which the free energy of the two phases coincides) does not take place in practice because of the very long lifetime of metastable states for systems with long-range interactions scaling as $e^{N}$ [78].

For $\mu > \mu_{\text{MCP}}$ the series of equilibria $\eta(\Lambda)$ presents turning points of energy (see the curves $\mu = 10^4$ and $10^5$ in Fig. 3; see also Fig. 11 below). Following the series of equilibria towards higher and higher density contrasts, and using the Poincaré-Katz [76, 77] criterion, one can show that the equilibrium states are microcanonically stable before the first turning point of energy $\Lambda_c$ (gaseous phase) and after the last turning point of energy $\Lambda^*$ (condensed phase). They are microcanonically unstable in between. In the microcanonical ensemble, the system undergoes a gravothermal catastrophe at $\Lambda_c$ from the gaseous phase to the condensed phase and an explosion at $\Lambda^*$ from the condensed phase to the gaseous phase. The first order phase transition that is expected at $\Lambda_c$ (at which the entropy of the two phases coincides) does not take place in practice because of the very long lifetime of metastable states for systems with long-range interactions scaling as $e^{N}$ [78].

---

15 Very similar results apply to the case of tidally truncated systems described by the fermionic King model [32, 53].
For $\mu \to +\infty$ we recover the series of equilibria $\eta(\Lambda)$ of a classical isothermal self-gravitating gas (see Fig. 6 below). It has a snail-like structure (spiral). Using the Poincaré-Katz criterion, one can show that the equilibrium states become (and remain) unstable after the first turning point of temperature in the canonical ensemble and after the first turning point of energy in the microcanonical ensemble.

For $\mu_{\text{CCP}} < \mu < \mu_{\text{MCP}}$ (see the curves $\mu = 10^2$ and $10^3$ in Fig. 3 see also Fig. 12 below) all the equilibrium states are stable in the microcanonical ensemble while the equilibrium states between the first turning point of temperature $\eta_c$ and the last turning point of temperature $\eta_s$ (core-halo solution) are unstable in the canonical ensemble. They have a core-halo structure and a negative specific heat. The region of negative specific heat that is allowed in the microcanonical ensemble is replaced by a phase transition in the canonical ensemble. This corresponds to a region of ensembles inequivalence.\(^{17}\)

We now apply these results to DM halos. We have explained that metastable states (local entropy maxima) are as much relevant as fully stable states (global entropy maxima). They are robust and long-lived. Therefore, we shall not make a distinction between fully stable and metastable states in our study.

IV. NONDEGENERATE LIMIT: EXTERNAL STRUCTURE OF LARGE DM HALOS

Let us first consider the nondegenerate limit of the self-gravitating Fermi gas which describes the external structure (envelope) of large DM halos.

A. Isothermal equation of state

In the nondegenerate limit $T \to +\infty$ (or $T \gg T_F$ where $T_F \sim \hbar^2 \rho^{2/3} / m^{5/3} k_B$ is the Fermi temperature) the Fermi-Dirac DF \(^{(7)}\) reduces to the Maxwell-Boltzmann distribution

$$f = \eta_0 e^{\beta \mu} e^{-\beta m \left( \frac{v^2}{2} + \Phi(r) \right)}.$$  \(\text{(40)}\)

In that case, the density and the pressure are given by

$$\rho = \eta_0 e^{\beta \mu} \left( \frac{2\pi}{\beta m} \right)^{3/2} e^{-\beta m \Phi(r)},$$  \(\text{(41)}\)

$$P = \eta_0 e^{\beta \mu} \left( \frac{2\pi}{\beta m} \right)^{3/2} \frac{1}{\beta m} e^{-\beta m \Phi(r)},$$  \(\text{(42)}\)

leading to the classical isothermal equation of state

$$P(r) = \rho(r) \frac{k_B T}{m}.$$  \(\text{(43)}\)

The fundamental differential equation \((\text{14})\) of hydrostatic equilibrium takes the form

$$- \frac{k_B T}{m} \Delta \ln \rho = 4\pi G \rho.$$  \(\text{(44)}\)

It describes the balance between the gravitational attraction and the thermal pressure. It is equivalent to the Boltzmann-Poisson equation

$$\Delta \Phi = 4\pi G \eta_0 e^{\beta \mu} \left( \frac{2\pi}{\beta m} \right)^{3/2} e^{-\beta m \Phi(r)}.$$  \(\text{(45)}\)

obtained by combining Eqs. \((\text{13})\) and \((\text{41})\). Equations \((\text{44})\) and \((\text{45})\) can be reduced to the Emden equation \((\text{D5})\) \(\text{[53]}\).

B. Flat rotation curves

The differential equation of hydrostatic equilibrium \((\text{44})\) has no simple analytical solution and must be solved numerically. However, its asymptotic behavior is known analytically \(\text{[53]}\) \(\text{[67]}\). The density of a self-gravitating isothermal halo decreases as

$$\rho(r) \sim \frac{k_B T}{2\pi G m r^2}$$  \(\text{(46)}\)

for $r \to +\infty$, corresponding to an accumulated mass $M(r) \sim 2k_B T r / (G m)$ increasing linearly with $r$. This leads to flat rotation curves

$$v^2(r) = \frac{GM(r)}{r} \quad \Rightarrow \quad v^2 = \frac{2k_B T}{m}$$  \(\text{(47)}\)

in agreement with the observations \(\text{[67]}\).

C. Thermal core radius

The isothermal density profile has not a compact support so it extends to infinity. Furthermore, its total mass is infinite \(\text{[67]}\). As a result, self-gravitating systems have no statistical equilibrium state in an unbounded domain. In practice, the isothermal equation of state is not valid at arbitrarily large distances and the halo is confined by

\(^{17}\) The physical nature of the core-halo solution is very different in the two ensembles. In the canonical ensemble, the degenerate core represents a “germ” or a “critical droplet” (in the language of phase transitions and nucleation) that the system must form to pass from the gaseous phase to the condensed phase. This is a saddle point of free energy at fixed mass. The probability to form this configuration is very low, scaling with the number of particles as $e^{-N}$. This is a rare event. This explains why metastable gaseous states have a very long lifetime scaling as $e^{-N}$. In the microcanonical ensemble, the core-halo solution is fully stable and corresponds to the most probable state of the system for the corresponding energy. It is therefore expected to be physically selected by the system.
other effects, either incomplete relaxation or tidal confinement. From the scaling of Eq. (44), we can define a characteristic radius

\[ r_0 = \left( \frac{k_BT}{4\pi G \rho_0 m} \right)^{1/2} \]  

that we shall call the thermal core radius. It represents the typical radius of an isothermal halo of central density \( \rho_0 \). The halo mass \( M_h \), the halo radius \( r_h \), the temperature \( T \) and the circular velocity \( v_h \) at the halo radius are defined in Appendix C. For an isothermal profile they are given by (see Appendix D)

\[ r_h = 3.63, \quad \frac{M_h}{\rho_0 r_h^3} = 1.76, \]  

\[ \frac{k_BT}{Gm\rho_0 r_h^2} = 0.954, \quad \frac{v_h^2}{4\pi G \rho_0 r_h^2} = 0.140. \]  

We note that the dimensionless inverse temperature has the value

\[ \eta_T = \frac{\beta GM_h m}{r_h} = 1.84. \]  

This is essentially a consequence of the virial theorem. Equation (51) will be called the “virial condition”.

The density and circular velocity profiles of a purely isothermal halo are plotted in Figs. 2-6 of Ref. 25. The isothermal profile is relatively close to the empirical (observational) Burkert profile up to \( \frac{r}{r_h} = 6 \).

D. The constant surface density

It is an observational evidence that the surface density of DM halos is independent of their mass and size and has a universal value:\[85,83:\]

\[ \Sigma_0 = \rho_0 r_h = 141^{+83}_{-52} M_\odot/pc^2. \]  

This result is valid for all the galaxies even if their sizes and masses vary by several orders of magnitude (up to 14 orders of magnitude in luminosity). The reason for this universality is not known but it is crucial to take this result into account in any modeling of DM halos. Therefore, we shall assume this relation as an empirical fact.\[19\]

\[ E. \text{ Halo mass-radius relation} \]

Substituting the constraint (52) of a uniform surface density into Eqs. (49) and (50), we obtain the relations

\[ M_h = 1.76 \Sigma_0 r_h^2, \quad \frac{k_BT}{m} = 0.954 G \Sigma_0 r_h, \]  

\[ v_h^2 = 1.76 G \Sigma_0 r_h, \quad \rho_0 = \frac{\Sigma_0}{r_h}. \]  

They determine the halo mass, the halo temperature, the halo velocity and the halo central density as a function of the halo radius. The halo mass scales with the size as \( M_h \propto r_h^2 \) and the temperature as \( k_BT/m \propto r_h \) (basically, these scalings stem from the universality of the surface density of DM halos \( M_h/r_h^2 \sim \Sigma_0 \) and from the virial theorem \( k_BT/m \sim v_h^2 \sim GM_h/r_h \sim G \Sigma_0 r_h \)).

For a halo of mass \( M_h = 10^{11} M_\odot \), similar to the halo that surrounds our Galaxy, we find \( r_h = 20.1 \text{kpc} \), \( \rho_0 = 7.02 \times 10^{-3} M_\odot/pc^3 \), \( (k_BT/m)^{1/2} = 108 \text{km/s} \), and \( v_h = (GM_h/r_h)^{1/2} = 146 \text{km/s} \) (we also have \( v_\infty = 153 \text{km/s} \)). We stress that these results are independent of the characteristics of the DM particle. The corresponding density and velocity profiles are plotted in Figs. 4 and 5.

Remark: The (effective) temperature of the DM halos depends on the fermion mass \( m \) and on the halo mass \( M_h \) through the law \( k_BT = 0.719 Gm \sqrt{\Sigma_0 M_h} \). Let us consider a halo mass \( M_h = 10^{11} M_\odot \) as before. For \( m = 165 \text{eV}/c^2 \) (see Sec. V C), we get \( T = 0.247 \text{K} \). For \( m = 1 \text{keV}/c^2 \) (see Sec. V C), we get \( T = 1.50 \text{K} \). For \( m = 54.6 \text{keV}/c^2 \) (see Sec. VIII), we get \( T = 81.9 \text{K} \). These values, which are of the order of the Kelvin scale, are much more physical that those obtained in the case of bosonic DM, which are of the order of \( T \sim 10^{-25} \text{K} \) [25].
FIG. 5: Rotation curve of a classical isothermal DM halo of mass \( M_h = 10^{11} M_\odot \) (Milky Way).

F. Classical isothermal gas in a box

Let us finally derive the equations determining the caloric curve of a self-gravitating isothermal gas in a box.

The density profile \( \rho(r) = \rho_0 e^{-\beta m(r) - \Phi_0} \), (55)

where \( \rho_0 \) is the central density and \( \Phi_0 \) is the central potential. The Boltzmann-Poisson equation \[ \partial_t \psi = \frac{\partial^2}{\partial r^2} \psi - \frac{\beta m}{\rho_0} \partial_\phi \psi \] then becomes

\[ \Delta \Phi = 4\pi G \rho_0 e^{-\beta m(r) - \Phi_0}. \] (56)

If we assume that the system is spherically symmetric and introduce the dimensionless variables

\[ \rho = \rho_0 e^{-\psi(\xi)}, \quad \psi = \beta m(\Phi - \Phi_0), \] (57)

and

\[ \xi = (4\pi G \beta m \rho_0)^{1/2} r \] (58)

into Eq. (56), we obtain the Emden equation \[ \frac{\partial^2}{\partial \xi^2} \alpha \phi = \frac{\partial}{\partial \xi} \left( \frac{\partial}{\partial \xi} \alpha \phi \right) \] (59)

Introducing the inverse normalized temperature from Eq. (4) and applying Newton’s law (25) at \( r = R \), we get

\[ \eta = \alpha \psi' \alpha(\xi). \] (60)

To compute the energy, we proceed as follows. The kinetic energy of an isothermal gas is

\[ E_{\text{kin}} = \frac{3}{2} N k_B T. \] (61)

Using the virial theorem from Eq. (32) we can compute the gravitational energy as

\[ W = -2E_{\text{kin}} + 3P_v V = -3N k_B T + \frac{4\pi R^3 \rho(R) k_B T}{m}. \] (62)

The total energy is \( E = E_{\text{kin}} + W \). Introducing the normalized energy from Eq. (44) we obtain

\[ \Lambda = \frac{3}{2\alpha \psi'(\alpha)} - \frac{e^{-\psi(\alpha)}}{\psi'(\alpha)^2}. \] (63)

The expression of the entropy is derived in Appendix B.

The caloric curve \( \eta(\Lambda) \) of the classical self-gravitating gas is represented in Fig. 6. It has the form of a spiral (see Sec. III B). It leads to an isothermal collapse in the microcanonical ensemble above \( \eta_c = 2.52 \) and to a gravothermal catastrophe in the microcanonical ensemble above \( \Lambda_c = 0.335 \).

Remark: The equations of this section can be recovered from the general equations of Sec. III by taking the nondegenerate limit \( k \to +\infty \) and replacing the Fermi integrals by their asymptotic expressions

\[ I_n(t) \sim \frac{1}{t} \Gamma(n+1), \quad (t \to +\infty). \] (64)

V. COMPLETELY DEGENERATE LIMIT: MINIMUM HALO (GROUND STATE) AND QUANTUM CORE OF DM HALOS

We now consider the completely degenerate limit of the self-gravitating Fermi gas which describes (i) ultracompact dwarf spheroidals (dSphs) like Fornax or Willman I and (ii) the quantum core of bigger DM halos.

A. Polytropic equation of state

In the completely degenerate limit \( T = 0 \) (or \( T \ll T_F \)), the Fermi-Dirac DF (7) reduces to the step function

\[ f(r, v) = \eta_0, \quad v \leq v_F(r), \]

\[ f(r, v) = 0, \quad v \geq v_F(r), \] (65)

where

\[ v_F(r) = \sqrt{2 \left[ \frac{\mu}{m} - \Phi(r) \right]} \] (66)
is the Fermi velocity. The density and the pressure are explicitly given by
\[ \rho = \int f \, dv = \int_0^{v_F} \eta_0 4\pi v^2 \, dv = \frac{4\pi}{3} \eta_0 v_F^3(r), \quad (67) \]
\[ P = \frac{1}{3} \int f v^2 \, dv = \frac{1}{3} \int_0^{v_F} \eta_0 v^2 4\pi v^2 \, dv = \frac{4\pi}{15} \eta_0 v_F^5(r). \quad (68) \]
Eliminating the Fermi velocity between these two expressions, we find that the equation of state of a cold Fermi gas is
\[ P = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{m^{8/3}} \rho^{5/3}. \quad (69) \]
This is a polytropic equation of state \( P = K_1 \rho^{5/3} \) of index \( \gamma = 5/3 \) (i.e. \( n = 3/2 \)). The fundamental differential equation of hydrostatic equilibrium determining the density profile of a fermion ball at \( T = 0 \) with the equation of state reads [see Eq. (14)]
\[ \frac{1}{8} \left( \frac{3}{\pi} \right)^{2/3} \frac{\hbar^2}{m^{8/3}} \Delta \rho^{2/3} = -4\pi G \rho. \quad (70) \]
It describes the balance between the gravitational attraction and the quantum pressure. It is equivalent to the Thomas-Fermi equation
\[ \Delta \Phi = \frac{16}{3} \pi^2 G \eta_0 \left[ 2 \left( \frac{\mu}{m} - \Phi \right) \right]^{3/2} \quad (71) \]
obtained by combining Eqs. (13) and (67) with Eq. (66). Equations (70) and (71) can be reduced to the Lane-Emden equation (E5) of index \( n = 3/2 \).

B. Mass-radius relation

The density profile of a fermion ball at \( T = 0 \) (corresponding to a polytrope of index \( n = 3/2 \)) has a compact support: the density vanishes at a finite distance \( r = R \) representing the radius of the object (see Fig. 7 below). The radius, the mass and the central density of the object satisfy the relations (see Appendix E)
\[ R = 0.359 \frac{\hbar}{m^{4/3} G^{1/2} \rho_0^{1/6}}, \quad (72) \]
\[ M = 0.699 \rho_0 R^3, \quad (73) \]
\[ M R^3 = 0.00149 \frac{\hbar^6}{G^3 m^8}. \quad (74) \]

Similarly, the halo mass, the halo radius and the central density satisfy the relations (see Appendix E)
\[ r_h = 0.223 \frac{\hbar}{m^{4/3} G^{1/2} \rho_0^{1/6}}, \quad (75) \]
\[ M_h = 1.99 \rho_0 r_h^3, \quad (76) \]
\[ M_h r_h^3 = 2.45 \times 10^{-4} \frac{\hbar^6}{G^3 m^8}. \quad (77) \]
Therefore
\[ \frac{M}{M_h} = 1.46, \quad \frac{R}{r_h} = 1.61, \quad (78) \]
yielding \( GM/R = 0.907 GM_h/r_h \).

C. Minimum halo

The foregoing equations determine the ground state \( (T = 0) \) of the self-gravitating Fermi gas. This fermion ball corresponds either to the smallest and most compact DM halo of the Universe (which has no isothermal atmosphere) that we call the “minimum halo”, or to the quantum cores of larger DM halos (which are surrounded by an isothermal atmosphere) \[37\]. We consider here the first possibility (minimum halo). Using Eqs. (75)-(77) and the constraint from Eq. (52), we obtain
\[ (r_h)_{\text{min}} = 1.50 \left( \frac{\hbar^6}{G^3 m^8 \Sigma_0} \right)^{1/5}, \quad (79) \]
\[ (M_h)_{\text{min}} = 4.47 \left( \frac{\hbar^{12} \Sigma_0^3}{G^6 m^{16}} \right)^{1/5}, \quad (80) \]
\[ (\rho_0)_{\text{max}} = 0.667 \left( \frac{\Sigma_0 m^{4/3} G^{1/2}}{\hbar} \right)^{6/5}, \quad (81) \]
\[ (v_h^2)_{\text{min}} = 2.98 \left( \frac{\Sigma_0^4 G^2 h^8}{m^8} \right)^{1/5}. \quad (82) \]
These equations determine the radius, the mass, and the central density of the minimum halo as a function of the fermion mass \( m \) and the universal surface density \( \Sigma_0 \). In practice, they are used the other way round in order to determine the fermion mass \( m \). Assuming that the mass \( (M_h)_{\text{min}} \) of the minimum halo is known, we obtain the fermion mass under the form
\[ m = 1.60 \frac{\hbar^{3/4} \Sigma_0^{3/16}}{G^{3/8} (M_h)_{\text{min}}^{5/16}}. \quad (83) \]
If we take \( (M_h)_{\text{min}} = 10^8 M_\odot \) we find that \( m = 165 \text{ eV}/c^2 \). We then obtain \((r_h)_{\text{min}} = 597 \text{ pc}, (\rho_0)_{\text{max}} = 0.236 M_\odot/\text{pc}^3\), and \((v_h)_{\text{min}} = 26.8 \text{ km/s}\). Using Eq. (78), we also have \( M_{\text{min}} = 1.46 \times 10^8 M_\odot \) and \( R_{\text{min}} = 961 \text{ pc} \). The corresponding density and velocity profiles are plotted in Figs. 7 and 8.

The Fermi temperature is defined by

\[
k_B T_F = \frac{\hbar^2 \rho_{1/3}}{m^{1/3}}. \tag{84}
\]

It can be obtained qualitatively by equating Eqs. (43) and (69). For the minimum halo, using Eq. (81), it reads

\[
k_B T_F = \left( \frac{\hbar^2 \Sigma G^2}{m^3} \right)^{1/5}. \tag{85}
\]

For \( m = 165 \text{ eV}/c^2 \), we get \( T_F = 5.15 \times 10^{-3} \text{ K} \). We note that the minimum halo is determined by the condition \( T \sim T_F \), where \( T \) is the temperature given by Eq. (53).

Remark: The choice of the mass \( (M_h)_{\text{min}} = 10^8 M_\odot \) (Fornax) for the minimum halo is a little bit arbitrary and open to criticisms. We shall adopt this value, however, in order to be consistent with our other papers [23, 37, 39, 91]. Nevertheless, our model is perfectible. If a more relevant minimum halo mass is considered, our analytical results remain valid but the numerical applications must be reconsidered. For example, if we take a minimum halo mass \( (M_h)_{\text{min}} = 0.39 \times 10^6 M_\odot \) corresponding to Willman I (as in Refs. [17, 52] and Refs. [33, 92]) we obtain \( m \sim 1 \text{ keV}/c^2 \). We then obtain \((r_h)_{\text{min}} \sim 33 \text{ pc} \) (in very good agreement with the measured value reported in Ref. [47]). \((\rho_0)_{\text{max}} \sim 4.3 M_\odot/\text{pc}^3\), and \((v_h)_{\text{min}} = 6.35 \text{ km/s} \). It is also possible that the concept of a “minimum halo” which would be completely degenerate is wrong (see Sec. VIII). In that case, our general results remain valid but the fermion mass cannot be obtained from the considerations of this section.

D. Quantum core of DM halos

The relations from Eqs. (72)-83 apply to the minimum halo which is a pure fermion ball without isothermal atmosphere. The relations from Eqs. (72)-74 also apply to the quantum core of larger DM halos. If we normalize the core mass \( M_c \) by the minimum halo mass \( (M_h)_{\text{min}} \) and the core radius \( R_c \) by the minimum halo radius \((r_h)_{\text{min}}\), we get

\[
\frac{M_c}{(M_h)_{\text{min}}} \left( \frac{R_c}{(r_h)_{\text{min}}} \right)^3 = 6.09. \tag{86}
\]

We can check that Eq. (86) is verified for the minimum halo for which \( M_c = 1.46 (M_h)_{\text{min}} \) and \( R_c = 1.61 (r_h)_{\text{min}} \) [see Eq. (78)].

E. Maximum mass due to general relativity

The maximum mass and the minimum stable radius of a fermion ball at \( T = 0 \) set by general relativity are

\[
M_{\text{OV}} = 0.384 \left( \frac{\hbar c}{G} \right)^{3/2} \frac{1}{m^2}, \quad R_{\text{OV}} = 8.73 \frac{GM_{\text{max}}}{c^2}. \tag{87}
\]

They were first determined by Oppenheimer and Volkoff [93] in the context of neutron stars. For a fermion of mass \( m = 165 \text{ eV}/c^2 \), we obtain \( M_{\text{OV}} = 2.30 \times 10^{13} M_\odot \) and \( R_{\text{OV}} = 9.61 \text{ pc} \). For \( m = 1 \text{ keV}/c^2 \), we get \( M_{\text{OV}} = 6.26 \times 10^{11} M_\odot \) and \( R_{\text{OV}} = 0.262 \text{ pc} \). The maximum mass is much larger than the typical core mass of any DM halo. Assuming that a fermion ball at \( T = 0 \) describes the quantum core of a DM halo, we conclude that such cores are nonrelativistic since \( M_c \ll M_{\text{max}} \) in general. Since the maximum mass is much larger than the core mass,
gravity can be treated within a Newtonian framework.\textsuperscript{21}

F. Energy of a fermion ball

Let us finally derive the energy of a completely degenerate fermion ball (ground state).

A nonrelativistic fermion ball at \(T = 0\) is equivalent to a polytrope of index \(n = 3/2\) [see Eq. (69)]. Its kinetic energy is given by [see Eq. (80)]

\[
E_{\text{kin}} = \frac{3}{2} \int P \, dr = \frac{3}{2} K_1 \int \rho^{5/3} \, dr. \quad (88)
\]

It gravitational energy is given by the Betti-Ritter formula\textsuperscript{53}

\[
W = -\frac{6 \, GM^2}{7 \, R}. \quad (89)
\]

with the mass-radius relation from Eq. (74). The virial theorem reduces to [see Eq. (52)]

\[
2E_{\text{kin}} + W = 0 \quad (90)
\]

since there is no pressure on the boundary of the fermion ball. Therefore, the total energy \(E = E_{\text{kin}} + W\) of the fermion ball is

\[
E = -E_{\text{kin}} = \frac{W}{2} = -\frac{3 \, GM^2}{7 \, R}. \quad (91)
\]

This is the minimum energy \(E_{\text{min}}\) of the self-gravitating Fermi gas (ground state). Using the mass-radius relation\textsuperscript{74}, we obtain

\[
E_{\text{min}} = -3.75 \frac{G^2 m^{8/3}}{\hbar^2} M^{7/3}. \quad (92)
\]

VI. PARTIALLY DEGENERATE DM HALOS: CORE-HALO STRUCTURE

To study partially degenerate DM halos with a core-halo structure, we shall use the box model of Sec. \textsuperscript{III}. In order to apply this model to real DM halos, we identify the mass \(M\) with the halo mass \(M_h\) and the box radius \(R\) with the halo radius \(r_h\):

\[
M = M_h, \quad R = r_h. \quad (95)
\]

We first have to determine the relation between the degeneracy parameter \(\mu\) and the halo mass \(M_h\).

A. Relation between the degeneracy parameter \(\mu\) and the halo mass \(M_h\)

Sufficiently large DM halos are dominated by their classical isothermal envelope (the quantum core of mass \(M_c \ll M_h\) does not affect their external structure). As a result, the halo mass is related to the halo radius by [see Eq. (53)]

\[
M_h = 1.76 \Sigma_0 r_h^2. \quad (96)
\]

Using this relation, the degeneracy parameter defined by Eq. (29) can be written as

\[
\mu = 1.18 \frac{G^{3/2} m^4 M_h^{5/4}}{\hbar^3 \Sigma_0^{3/4}}. \quad (97)
\]

Normalizing the halo mass \(M_h\) by the minimum halo mass \((M_h)_{\text{min}}\) given by Eq. (80) we obtain

\[
\mu = 7.66 \left(\frac{M_h}{(M_h)_{\text{min}}}\right)^{5/4}. \quad (98)
\]

This equation relates the degeneracy parameter \(\mu\) to the halo mass \(M_h\). As a result, the canonical and microcanonical critical points given by Eq. (39) may be expressed in terms of the halo mass by using \((M_h)_{\text{CCP}} = (\mu_{\text{CCP}}/7.66)^{4/5}(M_h)_{\text{min}}\) and \((M_h)_{\text{MCP}} = (\mu_{\text{MCP}}/7.66)^{4/5}(M_h)_{\text{min}}\) yielding

\[
(M_h)_{\text{CCP}} = 6.73 \, (M_h)_{\text{min}} = 30.1 \left(\frac{h^{12} \Sigma_0^{3/2}}{G^6 m^{16}}\right)^{1/5}, \quad (99)
\]

\[
(M_h)_{\text{MCP}} = 108 \, (M_h)_{\text{min}} = 483 \left(\frac{h^{12} \Sigma_0^{3/2}}{G^6 m^{16}}\right)^{1/5}. \quad (100)
\]

Taking \((M_h)_{\text{min}} = 10^8 \, M_\odot\) (Fornax) corresponding to a fermion mass \(m = 165 \, \text{eV}/c^2\) we get

\[
(M_h)_{\text{CCP}} = 6.73 \times 10^8 \, M_\odot, \quad (M_h)_{\text{MCP}} = 1.08 \times 10^{10} \, M_\odot. \quad (101)
\]

If we take \((M_h)_{\text{min}} = 0.39 \times 10^6 \, M_\odot\) (Willman I) instead, corresponding to a fermion mass \(m \sim 1 \, \text{keV}/c^2\),

\textsuperscript{21} This statement is valid for the fermion mass \(m = 165 \, \text{eV}/c^2\) that we consider here. It is less valid for larger masses, of the order of \(m = 48 \, \text{keV}/c^2\), such as those considered in Sec. \textsuperscript{VIII}.
we get \((M_h)_{\text{CCP}} = 2.62 \times 10^6 M_\odot\) and \((M_h)_{\text{MCP}} = 4.21 \times 10^7 M_\odot\). We note that \((M_h)_{\text{CCP}}\) and \((M_h)_{\text{MCP}}\) are very sensitive to the value of \(m\) since it occurs in their expressions [see Eqs. (99) and (100)] with the power 16/5.

Remark: We can also write the degeneracy parameter \(\mu\) under the form \([42]\)

\[
\mu = 17.3 \left[ \frac{R}{R_F(M)} \right]^{3/2}, \quad (102)
\]

where \(R_F(M)\) is the (Fermi) radius of a completely degenerate fermion ball of mass \(M\) given by Eq. (74). The condition that \(R > R_c\) imposes \(\mu > \mu_{\text{min}} = 17.3\). Applying this inequality to real DM halos, using Eq. (98), we find that \(M_h > 1.92(M_h)_{\text{min}}\). Up to a factor of order unity, we recover the fact that the ground state of the self-gravitating Fermi gas \((T = 0)\) determines the existence of a minimum halo mass \((M_h)_{\text{min}}\).

B. Virial condition

We have seen in Sec. [IV] that the normalized inverse temperature of an isothermal DM halo is \(\eta_v = 1.84^{\pm 0.02}\). Therefore, if we want to make the connection between the box model and real DM halos, we should consider a value of \(\eta_v\) equal to 1.84. It is reassuring to note that \(\eta_v = 1.84\) is smaller than \(\eta_v \approx 2.52\), corresponding to the maximum inverse temperature of the classical isothermal spiral, implying that there always exists a gaseous (non degenerate) equilibrium state with \(\eta_v = 1.84\). Actually, we should not give too much importance on the precise value of \(\eta_v\). It is sufficient to consider that \(\eta_v\) is of the order of unity. Therefore, we shall take

\[
\eta_v \sim 1. \quad (103)
\]

The intersections between the series of equilibria \(\eta(\Lambda)\) and the line level \(\eta = \eta_v \sim 1\) determine the possible equilibrium states of our system of self-gravitating fermions. We can generically have three kinds of solutions: (i) a gaseous solution (G) which is purely isothermal without quantum core; (ii) a core-halo solution (CH) with a quantum core (fermion ball) surrounded by a classical isothermal atmosphere; (iii) a condensed solution (C) which is an essentially quantum object with a tenuous isothermal atmosphere. The gaseous solution has been discussed in Sec. [IV]. The condensed solution is not physical in the case of large DM halos because it would imply that the halo is completely condensed, which is not the case. This solution only applies to the minimum halo (see Sec. [V]). The core-halo solution is the most important one for our purposes. It is similar to the gaseous solution at sufficiently large distances but it contains a small nucleus (fermion ball) at its center. An important question is to determine the core mass \(M_c\) as a function of the halo mass \(M_h\).

C. The \(M_c - M_h\) relation for the CH solution

The core mass-halo mass relation \(M_c(M_h)\) can be obtained from the box model as follows. The halo mass \(M_h\) determines the value of the degeneracy parameter \(\mu\). We can then plot the series of equilibria \(\eta(\Lambda)\) parametrized by the concentration parameter \(k\) (see Figs. [11] [12] and [14] below). For \(\mu > \mu_{\text{CCP}}\), the intersections between the caloric curve \(\eta(\Lambda)\) and the virial condition \(\eta_v \sim 1\) determine three solutions (G), (CH) and (C) (see Fig. [II] for an illustration). We select the core-halo solution (CH) and compute the corresponding concentration parameter \(k = k_{\text{CH}}(\mu, \eta_v)\). The density profile of the core-halo solution is then given by Eq. (18). Its central density is

\[
\rho_0 = \frac{4\pi \sqrt{2} \eta_0}{(\beta \mu)^{3/2}} \eta_0^{1/2}(k). \quad (104)
\]

Introducing the dimensionless variables defined in Sec. [III A] we get

\[
\frac{4\pi \rho_0 R^3}{M} = \frac{\mu}{\eta_v^{3/2}} I_{1/2}(k). \quad (105)
\]

Now, the core-halo configuration can be decomposed into a fermion ball at \(T = 0\) and a classical isothermal atmosphere. For a completely degenerate fermion ball, representing the quantum core of the DM halo, the relation between the core mass and the central density is given by [see Eqs. (72) (73)]

\[
M_c = 8.01 \frac{k^3 \rho_0^{1/2}}{G^{3/2} m^4}. \quad (106)
\]

Combining Eqs. (105) and (106) we obtain

\[
\frac{M_c}{M} = 4.07 \frac{\sqrt{I_{1/2}(k)}}{\mu^{1/2} \eta_v^{3/4}}. \quad (107)
\]

Recalling that \(M = M_h\), \(\eta = \eta_v \sim 1\) and \(k = k_{\text{CH}}(\mu, \eta_v)\) we get

\[
\frac{M_c}{M_h} = 4.07 \frac{\sqrt{I_{1/2}[k_{\text{CH}}(\mu, \eta_v)]}}{\mu^{1/2} \eta_v^{3/4}}. \quad (108)
\]

Together with the relation [98] between \(\mu\) and \(M_h\), Eq. (108) determines the core mass-halo mass relation \(M_c(M_h)\). In principle, the function \(I_{1/2}[k_{\text{CH}}(\mu, \eta_v)]\) has to be determined numerically as a function of \(\mu\). However, it turns out that, for the CH solution, \(I_{1/2}[k_{\text{CH}}(\mu, \eta_v)]\) changes slowly (logarithmically) with \(\mu\). As a result, up to logarithmic corrections (see Sec. [VII D],

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22 Coincidentally, this value turns out to be very close to the value \(\eta_2 = 1.84\) corresponding to the minimum inverse temperature of the classical isothermal spiral (see Sec. [IV]).
it can be taken as a constant. Therefore, we obtain the scaling
\[
\frac{M_c}{M_h} \propto \frac{1}{\mu^{1/2}}.
\]
(109)

More precisely, recalling Eq. (108), Eq. (108) can be written as
\[
\frac{M_c}{(M_h)_{\text{min}}} = 1.47 \sqrt{\frac{1}{2} [k_{\text{CH}}(\mu, \eta_c)]} \left( \frac{M_h}{(M_h)_{\text{min}}} \right)^{3/8}.
\]
(110)

If we take \( \eta_c = 1 \) and \( I_{1/2}(k_{\text{CH}}) = 1 \) we get
\[
\frac{M_c}{(M_h)_{\text{min}}} = 1.47 \left( \frac{M_h}{(M_h)_{\text{min}}} \right)^{3/8}.
\]
(111)

This relation shows that the core mass \( M_c \) scales with the halo mass as \( M_c^{3/8} \). The prefactor, obtained from our model, is of order unity implying that \( M_c \sim M_h \) for the minimum halo, as expected. Actually, for \( M_h = (M_h)_{\text{min}} \) we get \( M_c = 1.47 (M_h)_{\text{min}} \) in very good agreement with Eq. (78). Returning to the original variables, using Eq. (80), we can rewrite Eq. (111) as
\[
M_c = 3.75 \frac{h^{3/2}}{m^2} \left( \frac{M_h \Sigma_0}{G^2} \right)^{3/8}.
\]
(112)

Once we have \( M_c \) by Eq. (111) or (112) we can get \( R_c \) by Eq. (74) or (86) and \( \rho_0 \) by Eq. (106). Explicitly,
\[
R_c = 4.51 \frac{h^2}{G m^{8/3} M_c^{1/3}} = 2.90 \frac{h^{3/2}}{G^{3/4} m^{2} M_h^{1/8} \Sigma_0^{1/8}}.
\]
(113)

\[
\rho_0 = 0.0156 \frac{G^{3/4} M_c^{3/2}}{h^0} = 0.219 \frac{G^{3/4} m^4 M_h^{3/4} \Sigma_0^{3/4}}{h^3}.
\]
(114)

**Remark:** In this paper, we have defined the halo mass \( M_h \) such that the density at the halo radius \( r_h \) is equal to the central density divided by 4 (see Appendix C). However, other authors work in terms of a halo mass \( M_v \) defined in another manner as explained in Sec. V. C of Ref. [37]. The relation between \( M_h \) and \( M_v \) is (see Eq. (146) of Ref. [37])
\[
\frac{M_h}{M_v} = 6.01 \times 10^{-6} \left( \frac{M_c}{M_v} \right)^{4/3}.
\]
(115)

Combining Eqs. (112) and (115), we obtain the core mass – halo mass relation \( M_c(M_h) \). It exhibits the fundamental scaling \( M_c \propto M_h^{1/2} \). This theoretical scaling, first obtained in the form of Eq. (112) in Appendix H of Ref. [33], is consistent with the scaling found numerically by Ruffini et al. [37] (they find an exponent equal to 0.52 instead of 1/2).

D. Logarithmic corrections

In the previous section, we have assumed that \( I_{1/2}(k_{\text{CH}}) \) is approximately constant and we have taken \( I_{1/2}(k_{\text{CH}}) \approx 1 \). A more precise expression of \( I_{1/2}(k_{\text{CH}}) \) can be obtained as follows. We see on Fig. 9 that \( k_{\text{CH}} \sim 1/\mu \) for large values of \( \mu \). Using the asymptotic expression \( I_{1/2}(k) \sim \frac{2}{3} (-\ln k)^{3/2} \) of the Fermi integral \( I_{1/2}(k) \) for \( k \to 0 \) [see Eq. (94)], we obtain
\[
I_{1/2}(k_{\text{CH}}) \sim \frac{2}{3} (\ln \mu)^{3/2}.
\]
This behavior is confirmed by the plot of Fig. 10. If we take this logarithmic correction into account we have to multiply \( M_c \) [given by Eq. (111) or (112)] by the factor
\[
A = \sqrt{\frac{2}{3}} (\ln \mu)^{3/4}.
\]
(116)

On the other hand, \( R_c \) [given by Eq. (113)] has to be divided by \( A^{1/3} \) and \( \rho_0 \) [given by Eq. (114)] has to be multiplied by \( A^2 \).

E. Velocity dispersion tracing relation

The core mass – halo mass relation can also be obtained from a simple analytical model of self-gravitating fermions enclosed within a box as detailed in Sec. IV of Ref. [37]. In that model, the fermion ball is represented by a polytrope of index \( n = 3/2 \) and the classical isothermal atmosphere is assumed to be uniform. Under these approximations, one can compute the energy and the entropy analytically. The mass of the fermion ball \( M_c \) is

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23 This is not rigorously the case but this approximation is sufficient for our purposes since \( k \) arises in a logarithm.
then obtained by maximizing the entropy $S(M_c)$ for a
given value of $E_h$, $M_h$ and $r_h$. This leads to a relation
$M_c(M_h)$ similar to that of Eq. (109) with Eq. (116) (see
Eq. (123) of Ref. 37). This relation is obtained from
a thermodynamical approach (maximum entropy princi-
ple) determining the “most probable” core mass $M_c$. It
is shown furthermore in Sec. V of 37 that this relation
is equivalent to the “velocity dispersion tracing” relation

$$ v_c^2 \sim v_h^2 \quad \text{or} \quad M_c \sim \frac{R}{r_h} M_h $$

(117)

stating that the velocity dispersion in the core $v_c^2 \sim
GM_c/R_c$ is of the same order as the velocity dispersion
in the halo $v_h^2 \sim GM_h/r_h$. This is the reason why Eq.
111 is similar to Eq. (169) of 37. It is interesting to
note that the prefactors appearing in these relations are
almost the same although these relations are obtained
from substantially different calculations. Therefore, the
present approach provides an additional justification of
the “velocity dispersion tracing” relation from thermo-
dynamical arguments.

\section{VII. ASTROPHYSICAL APPLICATIONS}

We now consider astrophysical applications of our
model and discuss several scenarios that are suggested
by the previous results.

\subsection{A. Minimum halo with $M_h = (M_h)_\text{min}$}

The minimum halo has a mass $(M_h)_\text{min} = 10^8 M_\odot$
and a radius $(r_h)_\text{min} = 597$ pc. It corresponds to
the ground state (minimum energy state) of the self-
gravitating Fermi gas. A completely degenerate fermion
ball at $T = 0$ is equivalent to a polytrope of index $n = 3/2
\frac{1}{2}(\ln \mu)^{3/2}$ in good approximation.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig10}
\caption{This curve confirms that $I_{1/2}(k_{\text{CH}}) \sim \frac{1}{2}(\ln \mu)^{3/2}$}
\end{figure}

\section{B. Ultracompact DM halos with $(M_h)_\text{min} < M_h < (M_h)_{\text{CCP}}$}

Ultracompact DM halos have a mass in the range
$(M_h)_\text{min} = 10^8 M_\odot < M_h < (M_h)_{\text{CCP}} = 6.73 \times 10^8 M_\odot$. Since $\mu < \mu_{\text{CCP}}$ the caloric curve $\eta(\Lambda)$ is monotonic (see
Fig. 11). There is only one equilibrium state with $\eta_\text{e} \sim 1$
It corresponds to a completely degenerate fermion ball
surrounded by a tenuous classical isothermal atmosphere.
This quantum solution (Q) is thermodynamically stable
in all statistical ensembles.

\section{C. Small DM halos with $(M_h)_{\text{CCP}} < M_h < (M_h)_{\text{MCP}}$}

Small DM halos have a mass in the range
$(M_h)_{\text{CCP}} = 6.73 \times 10^8 M_\odot < M_h < (M_h)_{\text{MCP}} = 1.08 \times 10^{10} M_\odot$. Specifically, we consider a DM halo characterized by a
degeneracy parameter $\mu = 10^3$. It has a mass $M_h =
4.93 \times 10^9 M_\odot$ [see Eq. (98)] and a radius $r_h = 4.46$ kpc
[see Eq. (96)]. The corresponding caloric curve (see Sec.
III A) is represented in Fig. 12 Since $\mu_{\text{MCP}} < \mu < \mu_{\text{MCP}}$, The
calic curve has an N-shape structure (see Sec. III B). The intersection of this curve with the line $\eta_{\text{e}} \sim 1$
(see Sec. VI B) determines three solutions: a gaseous
solution (G), a core-halo solution (CH) and a condensed
solution (C) that we do not consider here (see Sec. VI B).
The gaseous solution (G) has a concentration parameter
$k_G = 206$ (see Fig. 9). The corresponding density pro-
file is plotted as a dashed line in Fig. 13. It represents
a purely classical isothermal DM halo without quantum
core as investigated in Sec. VI This solution lies in the
region of the caloric curve where the specific heat is posi-
tive ($C = dE/dT > 0$). It is thermodynamically stable.
The solutions (G), (CH) and (C) that we consider have different concentrations parameter $k$ fixed mass). The core-halo solution (CH) has a concentration parameter $k_{CH} = 1.12 \times 10^{-3}$ (see Fig. 9). The corresponding density profile (see Sec. III A) is plotted as a solid line in Fig. 13. It represents a DM halo with a quantum core (fermion ball) of mass $M_c = 2.21 \times 10^9 M_\odot$, radius $R_c = 389 \text{ pc}$ and central density $\rho_0 = 53.6 M_\odot/\text{pc}^3$ [we have used Eq. (111) to obtain $M_c$, Eqs. (113) and (114) to obtain $R_c$ and $\rho_0$, and we have taken into account the logarithmic correction $A = 3.48$ from Eq. (116)] surrounded by a classical isothermal atmosphere. This core-halo solution lies in the region of the caloric curve where the specific heat is negative ($C = dE/dT < 0$). It is thermodynamically unstable in the canonical ensemble (saddle point of free energy at fixed mass) but it is thermodynamically stable in the microcanonical ensemble (entropy maximum at fixed mass and energy) which is the relevant ensemble to consider (see Sec. II).\footnote{The solutions (G), (CH) and (C) that we consider have different energies but the same temperature. Indeed, the temperature is more relevant than the energy to characterize a DM halo since, according to the virial theorem, $\eta \sim 1$. However, we stress that we must analyze the thermodynamical stability of the system in the microcanonical ensemble, not in the canonical ensemble.}

For small DM halos with $(M_h)_{\text{CCP}} < M_h < (M_h)_{\text{MCP}}$, the gaseous solution (G) and the core-halo solution (CH) are both thermodynamically stable in the microcanonical ensemble (maximum entropy state at fixed mass and energy). Therefore, they are both likely to result from a natural evolution in a thermodynamical sense. Let us consider different scenarios of formation and evolution in line with the general discussion given in Sec. II:

(i) The core-halo solution (CH) may arise naturally from a process of violent collisionless relaxation (following Jeans instability and free fall) since it is a maximum entropy state in the sense of Lynden-Bell. This is a fast process taking place on a dynamical timescale. If the evolution is collisionless, the system remains in that state.

(ii) The gaseous solution (G) may also arise naturally from a process of violent collisionless relaxation since it is a maximum entropy state in the sense of Lynden-Bell. Then, there are two possibilities:

(ii-a) If the evolution is collisionless, the system remains in that state.

(ii-b) If the evolution is collisional, the system may evolve along the series of equilibria (see Fig. 12). Indeed, because of collisions\footnote{These collisions between DM particles are not two-body gravitational encounters because the relaxation time would be too long \cite{32,33}, but they can have another origin such as short-range interactions (SIDM model) like in, e.g., Ref. [35].} and evaporation the central density increases and the energy decreases. The temperature first decreases in the region of positive specific heat ($C = dE/dT > 0$) then increases in the region of negative specific heat ($C = dE/dT < 0$). The whole series of equilibria represented in Fig. 12 is stable in the microcanonical ensemble. Therefore, if the DM halo evolves adiabatically under the effect of collisions, it can progressively pass from the gaseous solution (G) to the core-halo solution (CH). This is a slow relaxation taking place on a secular timescale. This may be a mechanism – alternative to violent relaxation – which explains how

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig12}
\caption{Caloric curve of self-gravitating fermions for $\mu = 10^3$. When $(M_h)_{\text{CCP}} < M_h < (M_h)_{\text{MCP}}$, the caloric curve has an N-shape structure. In the canonical ensemble, the gaseous phase (G) and the condensed phase (C) are stable while the core-halo phase (CH) is unstable (it represents a “critical droplet” that the system must create to pass from one phase to the other). In the microcanonical ensemble, all the equilibrium states are stable. The system may directly reach the core-halo phase (CH) through a process of collisionless violent relaxation. It may also evolve collisionally, following the arrows, from the gaseous solution (G) to the core-halo solution (CH).}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{fig13}
\caption{Density profile of the core-halo solution (CH) ($k_{CH} = 1.12 \times 10^{-3}$) for $\mu = 10^3$. For comparison, we have represented in dashed line the gaseous solution (G) which corresponds to a classical isothermal halo ($k_G = 206$).}
\end{figure}
the system reaches the core-halo solution (CH).

In conclusion, small DM halos with \((M_h)_{CCP} < M_h < (M_h)_{MCP}\) can be in two types of configuration:

(I) The gaseous solution (G) coinciding with the classical isothermal sphere. This simple solution is consistent with the observations because we have shown in Sec. III.C of [25] that an isothermal DM halo is almost indistinguishable from the observational Burkert profile.

(II) The core-halo solution (CH) made of a quantum core (fermion ball) of mass \(M_c = 2.21 \times 10^9 \, M_\odot\) and radius \(R_c = 389\, \text{pc}\) surrounded by a classical isothermal atmosphere. The quantum core cannot mimic a SMBH because it is too big (its radius \(R_c = 389\, \text{pc}\) is much larger than its Schwarzschild radius \(R_s = 2GM/c^2 = 2.11 \times 10^{-4}\, \text{pc}\)). However, it can represent a large quantum bulge made of DM. This quantum bulge may possibly exist at present at the center of some galaxies or may have existed in the past as a temporary state, and has disappeared since then. Indeed, a large bulge may provide a favorable environment for triggering the formation of a SMBH that can then grow by accretion. The final outcome of this scenario would then be a classical isothermal halo containing either a quantum bulge (large fermion ball) or a SMBH that would be the remnant of the original bulge.

D. Large DM halos with \(M_h > (M_h)_{MCP}\)

Large DM halos have a mass \(M_h > (M_h)_{MCP} = 1.08 \times 10^{10}\, M_\odot\). Specifically, we consider a DM halo characterized by a degeneracy parameter \(\mu = 10^5\). It has a mass \(M_h = 1.96 \times 10^{11}\, M_\odot\) [see Eq. (98)] and a radius \(r_h = 28.1\, \text{kpc}\) [see Eq. (96)]. The corresponding caloric curve (see Sec. IIIA) is represented in Fig. 14 Since \(\mu > \mu_{MCP}\), the caloric curve has a Z-shape structure similar to a dinosaur’s neck (see Sec. III-B). As before, the intersection of this curve with the line \(\eta_c \sim 1\) (see Sec. VI-B) determines two physical solutions: a gaseous solution (G) and a core-halo solution (CH).

The gaseous solution (G) has a concentration parameter \(k_G = 2.05 \times 10^4\) (see Fig. 9). The corresponding density profile is plotted as a dashed line in Fig. 15. It represents a purely classical isothermal DM halo of mass \(M_h\) and radius \(r_h\) without quantum core as investigated in Sec. IV. It lies in a region of positive specific heat. It is thermodynamically unstable in all statistical ensembles (maximum entropy state at fixed mass and energy and minimum free energy state at fixed mass). The core-halo solution (CH) has a concentration parameter \(k_{CH} = 1.38 \times 10^{-5}\) (see Fig. 9). The corresponding density profile (see Sec. IIIA) is plotted as a solid line in Fig. 15. It represents a DM halo with a quantum core (fermion ball) of mass \(M_c = 1.29 \times 10^9\, M_\odot\), radius \(R_c = 216\, \text{pc}\) and central density \(\rho_0 = 1820\, M_\odot/\text{pc}^3\) [we have used Eq. (111) to obtain \(M_c\), Eqs. (113) and (114) to obtain \(R_c\) and \(\rho_0\), and we have taken into account the logarithmic correction \(\Lambda = 5.10\) from Eq. (116)] surrounded by a classical isothermal atmosphere of mass \(\sim M_h\) and radius \(r_h\). Since this solution is located between the first and the last turning points of energy, it is thermodynamically unstable in all statistical ensembles (saddle point of entropy at fixed mass and energy and saddle point of free energy at fixed mass). Note that it lies in a region of the caloric curve with a positive specific heat, showing that a positive specific heat does not necessarily imply that the system is stable.

For large DM halos with \(M_h > (M_h)_{MCP} = 1.08 \times 10^{10}\, M_\odot\) the gaseous solution (G) is thermodynamically stable but the core-halo solution (CH) is thermodynamically unstable. Therefore, the gaseous solution is likely to result from a natural evolution in a thermodynamical sense while the core-halo solution (CH) should not be observed.26 Let us consider different scenarios of formation

\[\text{FIG. 14: Caloric curve of self-gravitating fermions for } \mu = 10^5. \text{ For } M_h > (M_h)_{MCP}, \text{ the caloric curve has a Z-shape structure (dinosaur’s neck). In the microcanonical ensemble, the gaseous phase (G) and (G’) before the first turning point of energy and the condensed phase (C’) and (C) after the last turning point of energy are stable. By contrast, the core-halo phase (CH) in the intermediate branch between the first and the last turning points of energy is unstable. The system can evolve collisionally in the gaseous phase (G) and (G’) up to the turning point of energy \(E_c\) and collapse in the condensed phase (C’) (see arrows). This corresponds to the gravothermal catastrophe [74] arrested by quantum effects. Another possibility is that the gravothermal catastrophe triggers a dynamical instability of general relativistic origin leading to the formation of a SMBH [95] (see Fig. 16 below).}\]
and evolution in line with the general discussion given in Sec. II.

The gaseous solution (G) may arise naturally from a process of violent collisionless relaxation (following Jeans instability and free fall) since it is a maximum entropy state in the sense of Lynden-Bell. This is a fast process taking place on a few dynamical times. Then, there are two possibilities:

(a) If the evolution is collisionless, the system remains in that state.

(b) If the evolution is collisional, the system may slowly evolve along the series of equilibria (see Fig. 14). The beginning of the collisional evolution is similar to that described previously. The temperature first decreases in the region of positive specific heat \( C = dE/dT > 0 \) then increases in the region of negative specific heat \( C = dE/dT < 0 \). However, when the system reaches the turning point of energy (corresponding to the minimum energy \( E_c \)) it becomes thermodynamically unstable and undergoes the gravothermal catastrophe \( 74 \). At that point, there are several possibilities:

(b.1) We first assume that the gravothermal catastrophe is eventually halted by quantum mechanics (Pauli’s exclusion principle) and that the system reaches an equilibrium state. This takes the system from the gaseous phase \( (G') \) to the condensed phase \( (C') \) in which only a fraction (typically \( \sim 1/4 \)) of the mass of the DM halo forms a compact fermion ball while the rest of the mass constitutes a hot halo. The hot halo has a uniform density so that it is strongly held by the box (see Fig. 16 in 42). As discussed in 42 44, if we remove the box, the halo should be expelled at large distances in a process reminiscent of a supernova explosion 96 98, This is because the collapse of the core heats the halo which thus extends at large distances. Although this mechanism could be at work for fermion stars such as white dwarfs and neutron stars, it may not be relevant for DM halos. Therefore, we shall prefer the following scenarios.

(b.2) We assume that the gravothermal catastrophe is eventually halted by quantum mechanics as before, but the system does not reach the equilibrium solution \( (C') \). It may reach an out-of-equilibrium core-halo structure \( (CH)_{\text{out}} \) that is not described by the Fermi-Dirac distribution. This out-of-equilibrium state \( (CH)_{\text{out}} \) may be made of a slowly evolving quantum core surrounded by a classical atmosphere that is not as much extended as the classical atmosphere of the equilibrium solution \( (C') \). Actually, in this scenario, the initial isothermal halo \( (\text{at criticality}) \) is essentially left undisturbed. Since the solution \( (CH)_{\text{out}} \) is an out-of-equilibrium structure, we expect that the core-halo mass relation \( M_c(M_h) \) is different from the one predicted by Eq. (111). In particular, the quantum core resulting from the gravothermal catastrophe should be more compact and more massive than the quantum core composing the \( (CH) \) solution. The occurrence of this out-of-equilibrium state \( (CH)_{\text{out}} \) is due to the fact that the exchange of energy between the core and the halo, and the process of thermalization, may take a very long time. Therefore, the equilibrium state \( (C') \) of scenario (b.1) may not be reached on relevant timescales.

(b.3) Finally, we assume that the halo undergoes a gravothermal catastrophe at \( E_c \) but we consider another evolution in which quantum mechanics cannot prevent gravitational collapse (the validity of this hypothesis is considered in Sec. VII E). This scenario, already advocated in Refs. 32, 33, is based on the SIDM model of Balberg et al. 95 who developed the idea of an “avalanche-type contraction” towards a SMBH initially suggested by Zeldovich and Podurets 99, improved by Fackerell et al. 100, and confirmed numerically by Shapiro and Teukolsky 101 103. The initial stage of the gravothermal catastrophe is well-known. The core collapses and reaches high densities and high temperatures while the halo is not sensibly affected by the collapse of the core and maintains its initial structure. Now, Balberg et al. 95 argue that during the gravothermal catastrophe, when the central density and the temperature increase above a critical value, the system undergoes a dynamical instability of general relativistic origin leading to the formation of a SMBH on a dynamical time scale. Only the central region of the DM halo (not its outer part) is affected by this collapse so the final outcome of this scenario is a classical isothermal halo at criticality containing a central SMBH.

In conclusion, large DM halos with \( M_h > (M_h)_{\text{MCP}} \) can be in three types of configuration:

(I) A purely classical isothermal halo \( (G) \) without quantum core.

(II) An out-of-equilibrium core-halo solution \( (CH)_{\text{out}} \), resulting from the gravothermal catastrophe, which is different from the solution \( (CH) \) that is unstable or from the solution \( (C') \) that is unphysical. It is made of a compact 

![FIG. 15: Density profile of the core-halo (CH) solution \( (k_{\text{CH}} = 1.38 \times 10^{-5}) \) for \( \mu = 10^5 \). For comparison, we have represented in dashed line the gaseous (G) solution which corresponds to a classical isothermal halo \( (k_G = 2.05 \times 10^4) \).](image-url)
Large DM halos may contain a SMBH but they should not contain a fermion ball because the core-halo solution (CH) is thermodynamically unstable. By contrast, small DM halos may contain a large quantum bulge (fermion ball) but they should not contain a SMBH because the gravothermal catastrophe is inhibited by quantum mechanics.

Remark: The scenarios (II) and (III) may be particularly interesting especially if we account for tidal effects. Indeed, it has been shown in [32, 33] that the King profile at criticality (i.e. at the verge of the gravothermal catastrophe) is very close to the observational Burkert profile (see, e.g., Figs. 18 and 27 of [32] and Fig. 1). Therefore, the structure of large DM halos could consist of a fermion ball or a SMBH surrounded by an envelope with a marginal (critical) King profile unaffected by the collapse of the core [32, 33]. The conditions for forming a SMBH at the center of a DM halo are discussed in Sec. VII E based on the results of Alberti and Chavanis [44, 45].

E. Criterion for the existence of a SMBH at the center of a galaxy

According to the above scenario, the formation of a SMBH at the center of a galaxy is possible only if the system can experience the gravothermal catastrophe and if, during core collapse, the core can reach sufficiently high densities and high temperatures to trigger a general relativistic dynamical instability leading to the formation of a SMBH. This may happen in sufficiently large systems. By contrast, in small systems, quantum mechanics (Pauli’s exclusion principle for fermions) prevents the gravothermal catastrophe and leads to a large fermion ball (bulge) instead of a SMBH. In conclusion, a SMBH can form only if the degeneracy parameter $\mu$ is sufficiently large so that the gravothermal catastrophe is efficient. Therefore, we expect that DM halos harbor a SMBH if $\mu \gg \mu_{\text{MCP}} = 2670$ i.e.

$$M_h \gg (M_h)_{\text{MCP}} = 1.08 \times 10^{10} M_\odot$$

and we expect that DM halos harbor a large quantum bulge (fermion ball) in the opposite case.$^{27}$

This result is qualitatively consistent with the conclusion reached by Ferrarese [104] on the basis of observations. She found that black holes can form only in sufficiently large galaxies, above a typical mass $\sim 5 \times 10^{11} M_\odot$. This limit may correspond to the micro-canonical critical point $(M_h)_{\text{MCP}}$ of our model. To facilitate further comparisons, using Eq. (117), we rewrite this criterion as

$$M_h \gg (M_h)_{\text{MCP}} = 483 \left(\frac{h^3 \Sigma_0^{3/4}}{G^{3/2} m^4}\right)^{4/5}.$$  \hspace{1cm} (119)

Actually, things are more complicated than the scenario just exposed. Indeed, as shown by Alberti and Chavanis [44, 45], when general relativity is taken into account, the caloric curves of the self-gravitating Fermi gas depend not only on $\mu$, but also on the value of the particle number $N$ with respect to $N_{\text{OV}}$. When $N < N_{\text{OV}}$, the caloric curve is similar to the one reported in Fig. 14. In particular, there is an equilibrium state for any value of the energy since quantum mechanics (Pauli’s exclusion principle) can prevent gravitational collapse even at $T = 0$. By contrast, when $N > N_{\text{OV}}$, a new turning point of energy appears [14, 45] as shown in Fig. 16. In that case, there is no equilibrium state below a minimum energy $E_{\min}^{\text{OV}}$ and the system collapses towards a black hole. These results suggest that the correct criterion for the existence of a SMBH at the center of a galaxy is that $\mu \gg \mu_{\text{MCP}}$ (in order to trigger the gravothermal catastrophe) and $N > N_{\text{OV}}$ (in order to have a gravitational collapse towards a SMBH). The first condition yields Eq. (119). It signals the instability of the core-halo solution (CH) with respect to the gravothermal catastrophe. If we approximate the second condition by $M_h > M_{\text{OV}}$, where $M_{\text{OV}}$ is given by Eq. (87), we obtain the condition

$$M_h > M_{\text{OV}} = 0.384 \left(\frac{\hbar c}{G}\right)^{3/2} \frac{1}{m^2}.$$ \hspace{1cm} (120)

If $(M_h)_{\text{MCP}} < M_h < M_{\text{OV}}$ we expect that the halo experiences the gravothermal catastrophe but does not form a SMBH. It may rather form an out-of-equilibrium fermion ball (CH)$_{\text{out}}$ (scenario b.2). By contrast, if $M_h > M_{\text{OV}}$, the halo may form either an out-of-equilibrium fermion ball (CH)$_{\text{out}}$ (scenario b.2) or a SMBH (scenario b.3). A necessary condition to form a SMBH is that $M_{\text{OV}} > (M_h)_{\text{MCP}}$. This yields

$$m > 383 \left(\frac{h^3 \Sigma_0^{3/4} G^{1/4}}{c^3 \Sigma_0^{3/4}}\right) = 0.278 \text{eV}/c^2.$$ \hspace{1cm} (121)

This condition is always fulfilled in practice.

For $m = 165 \text{eV}/c^2$ we find that $(M_h)_{\text{MCP}} = 1.08 \times 10^{10} M_\odot$ and $M_{\text{OV}} = 2.30 \times 10^{13} M_\odot$. In that case, the OV mass is very large, much larger than the mass $M_h = 10^{11} M_\odot$ of a DM halo comparable to the Milky Way.
Way. As a result, the gravothermal catastrophe should be stopped by quantum mechanics and a SMBH cannot be formed. This suggests that the Milky Way contains an out-of-equilibrium fermion ball rather than a SMBH. However, if we consider a larger fermion mass \( m \sim 1 \text{ keV/c}^2 \) we find \((M_{h})_{\text{MCP}} = 3.38 \times 10^{7} M_{\odot}\) and \( M_{\text{OV}} = 6.26 \times 10^{11} M_{\odot}\), which are closer to the conditions required to form a SMBH (see, however, the Remark below).

Remark: It is natural to expect that the gravitational collapse at \( E_c'' \) leads to a SMBH mass \( M_{\text{OV}} \) because the instability of the DM halo occurs precisely at the moment where the core mass becomes critical \( (M_c = M_{\text{OV}}) \) \[44, 60\]. In that case, we find for \( m = 165 \text{ eV/c}^2 \) and \( m \sim 1 \text{ keV/c}^2 \) that the SMBH mass would be \( M_{\text{OV}} = 2.30 \times 10^{13} M_{\odot}\) and \( M_{\text{OV}} = 6.26 \times 10^{11} M_{\odot}\) respectively. These very large masses are not consistent with the observations of SMBHs. The OV mass is more relevant if the fermion has a larger mass \( m \) as considered in Sec. \[VIII\]. For example, for \( m = 54.6 \text{ keV/c}^2 \), we get \( M_{\text{OV}} = 2.10 \times 10^{8} M_{\odot}\) which is of the order of the mass of SMBHs observed in AGNs. On the other hand, for \( m = 386 \text{ keV/c}^2 \) we get \( M_{\text{OV}} = 4.2 \times 10^{6} M_{\odot}\) which is of the order of the mass of Sagittarius A*.

In that case, according to the scenario (III) discussed above, the Milky Way could consist of a SMBH of mass \( M_{\text{OV}} = 4.2 \times 10^{6} M_{\odot}\) (Sagittarius A*) resulting from the gravothermal catastrophe surrounded by an envelope with a marginal King profile similar to the Burkert profile (see Sec. \[VIII\]).

\[ F. \text{ Application to the Milky Way} \]

We now specifically apply our fermionic model to the Milky Way. We consider a DM particle mass \( m = 165 \text{ eV/c}^2 \) so that the minimum halo has a mass \( (M_h)_{\text{min}} = 10^{8} M_{\odot}\) and a radius \( (r_h)_{\text{min}} = 597 \text{ pc} \) (see Sec. \[V\]). To be specific, we consider a DM halo of mass \( M_h = 10^{11} M_{\odot}\) (corresponding to \( M_c \sim 10^{12} M_{\odot}\)) and radius \( r_h = 20.1 \text{ kpc} \) similar to the one that surrounds our Galaxy (see Sec. \[IV\]). Using Eq. (108) we find that the corresponding degeneracy parameter is \( \mu = 4.31 \times 10^{8} \). The corresponding caloric curve has a Z-shape structure like in Fig. \[14\]. The gaseous solution (G) corresponding to a purely classical isothermal halo is plotted as a dashed line in Figs. \[17\] and \[18\]. Then, considering the core-halo solution (CH) and using Eqs. (86), (106), (111) and (116), we find that the DM halo should contain a quantum core of mass \( M_c = 9.45 \times 10^{9} M_{\odot}\), radius \( R_c = 240 \text{ pc} \) and central density \( \rho_0 = 983 M_{\odot}/\text{pc}^3 \). The density and velocity profiles given by Eqs. (86) and (87) are represented as a solid lines in Figs. \[17\] and \[18\]. Clearly, the fermion ball is too extended to mimic a black hole. It is more likely to represent a large quantum bulge as discussed in Sec. \[VIII\].

The halo mass \( M_h = 10^{11} M_{\odot}\) is above the microcanonical critical point \( (M_h)_{\text{MCP}} = 1.08 \times 10^{10} M_{\odot}\). The gaseous solution (G) is thermodynamically stable and should not be observed. It should be replaced by an out-of-equilibrium core-halo structure \( (\text{CH}_{\text{out}})\) with a compact quantum core as discussed in Sec. \[VIII\]. However, since we are relatively close to the microcanonical critical point, the core-halo solution (CH) may be marginally relevant, especially if \( \eta_0 \) is smaller than expected, e.g., if we select the solution (CH)_g of Fig. \[14\].

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\[29\] Comparatively, for \( m = 1 \text{ keV/c}^2 \) we get \( \mu = 4.42 \times 10^{7} \), \( M_c = 4.30 \times 10^{9} M_{\odot}\), \( R_c = 5.84 \text{ pc} \) and \( \rho_0 = 3.71 \times 10^{6} M_{\odot}/\text{pc}^3 \).
In that case, we must also add a primordial black hole in the model in order to account for the observation of a large central mass at the center of the Milky Way corresponding to Sgr A*.

31 As noted by C. Argüelles (private communication), De Martino et al. [105] have to add “by hand” a Plummer component of bulge stars to reduce the central dispersion because otherwise the BECDM model overestimates the data. On the other hand, Bar et al. [106] argue in their Sec. III that the central mass component could well be due to ordinary baryonic matter rather than a DM soliton.
could be fermion balls [57, 58, 61, 63, 65] or boson stars [114, 115] that could mimic a SMBH.

Let us consider this possibility in the framework of the fermionic model. More precisely, let us investigate if a fermion ball can mimic a SMBH at the center of the Galaxy.

B. Standard Fermi-Dirac distribution

Bilic et al. [57] developed a general relativistic model of fermionic DM halos at finite temperature with a fermion mass $m = 15 \text{ keV}/c^2$ that describes both the center and the halo of the Galaxy in a unified manner. The density profile has a core-halo structure with a quantum core (fermion ball) and a classical isothermal atmosphere. By using the usual Fermi-Dirac distribution and choosing parameters so as to fit observational data at large distances, they found a fermion ball of mass $M_c = 2.27 \times 10^6 M_\odot$ and radius $R_c = 18 \text{ mpc}$. Unfortunately, its radius is larger by a factor 100 than the bound $R_p = 6 \times 10^{-4} \text{ pc}$ set by observations [110]. This is why Bilic and coworkers abandoned this fermion ball scenario (R. Viollier, private communication). The same problem was encountered later by Ruffini et al. [57] who developed a similar model with a fermion mass $m \sim 10 \text{ keV}/c^2$.

Let us check that their results are consistent with our analytical box model. Following Bilic et al. [57], we take a DM particle mass $m = 15 \text{ keV}/c^2$. The corresponding minimum halo (see Sec. IV) has a mass $(M_h)_{\text{min}} = 54.0 M_\odot$ and a radius $(r_h)_{\text{min}} = 0.439 \text{ pc}$. If we consider a DM halo of mass $M_h = 10^{11} M_\odot$ and radius $r_h = 20.1 \text{ kpc}$ similar to the one that surrounds our Galaxy (see Sec. IV) we find that the corresponding degeneracy parameter is $\mu = 2.94 \times 10^{12}$ [see Eq. (68)]. Considering the core-halo solution (CH) and using Eqs. (86), (106), (111) and (116), we find that this DM halo should contain a quantum core of mass $M_c = 2.39 \times 10^6 M_\odot$, radius $R_c = 22.7 \text{ mpc}$ and central density $\rho_0 = 2.95 \times 10^{11} M_\odot/\text{pc}^3$ in good agreement with the numerical results of Bilic et al. [65] and Ruffini et al. [67]. The corresponding density and velocity profiles given by Eqs. (36) and (37) are represented in Figs. 19 and 20. They are in good agreement with Fig. 3 of Bilic et al. [65] and Figs. 1 and 3 of Ruffini et al. [57]. Therefore, our semi-analytical model [see in particular Eq. (111)] can reproduce their numerical results.

Let us now discuss the thermodynamical stability of the core-halo solution considered by Bilic et al. [57] and Ruffini et al. [57]. The caloric curve (see Sec. III A) corresponding to $\mu = 2.94 \times 10^{12}$ is similar to the one represented in Fig. 21. For large values of $\mu$, a spiral appears in the caloric curve at the location of the “head” of the dinosaur. As $\mu$ increases, the spiral winds more and more before unwinding. For $\mu \gtrsim 1$ the direct and reversed spirals are very close to each other. The intersections of the caloric curve with the line $r_h \sim 1$ (see Sec. VII B) determines two physical solutions: a gaseous solution (G) and a core-halo solution (CH). The gaseous solution represents a purely classical isothermal DM halo of mass $M_h$ and radius $r_h$ without quantum core as investigated in Sec. IV (see Figs. 4 and 5). This solution lies on the caloric curve in the region of positive specific heat. This solution (G) is thermodynamically stable in all statistical ensembles (maximum entropy state at fixed mass and energy and minimum free energy state at fixed mass).

The core-halo solution (CH) (see Sec. III A) represents a DM halo with a quantum core (fermion ball) of mass $M_c = 2.39 \times 10^6 M_\odot$, radius $R_c = 22.7 \text{ mpc}$ and central density $\rho_0 = 2.95 \times 10^{11} M_\odot/\text{pc}^3$ surrounded by a classical isothermal atmosphere of mass $\sim M_h = 10^{11} M_\odot$ and radius $r_h = 20.1 \text{ kpc}$. The corresponding density and velocity profiles are plotted in Figs. 19 and 20. Since this

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32 The fermion ball is weakly general relativistic because $M_c = 2.27 \times 10^6 M_\odot \ll M_{\text{QV}} = 2.78 \times 10^9 M_\odot$ [see Eq. (67)].
solution lies between the first and the last turning points of energy, it is thermodynamically unstable in all statistical ensembles (saddle point of entropy at fixed mass and energy and saddle point of free energy at fixed mass). This solution lies in a region of the caloric curve with a positive specific heat.

The discussion about the thermodynamical stability of the core-halo solution (CH) is essentially the same as in Sec. VII D. The main differences are the followings:

(i) According to the Poincaré-Katz criterion, the system loses more and more modes of stability, one at each turning point of energy, as we progress clockwise into the spiral. However, when the spiral unwinds the modes of stability are progressively regained. Indeed, one mode of stability is regained at each turning point of energy as we follow the spiral anticlockwise. As a consequence, the core-halo solutions that lie on the spiral are very unstable since they have several modes of instability. We note, however, that the core-halo solution (CH) has only one mode of instability as before.

(ii) The energy of the core-halo solution (CH) almost coincides with the energy of the gaseous solution (G). This is because their external structure is exactly the same. The core-halo solution (CH) only differs from the gaseous solution (G) by the presence of a small core with a small mass, a small radius and a very high density (see Fig. 19). The core and the halo are separated by a large plateau where the density is approximately constant. Therefore, when $\mu \gg \mu_{\text{MCP}}$, the core-halo solution (CH) almost coincides with the gaseous solution (G) except that it contains a small nucleus (fermion ball). For smaller values of $\mu$, the plateau is reduces and finally disappears.

For example, in Fig. 15 the core-halo solution (CH) does not show a very pronounced separation between the quantum core and the halo. Furthermore, the halo is perturbed by the presence of the core (unlike in Fig. 19). As a result, the energy of the core-halo and gaseous solutions on the caloric curve of Fig. 14 are relatively different (unlike in Fig. 21).

In conclusion, the models of Bilic et al. [65] and Ruffini et al. [76] that are based on the standard Fermi-Dirac DF lead to DM halos with a core-halo structure made of a small quantum core (fermion ball) of mass $M_c = 2.39 \times 10^6 M_\odot$ and radius $R_c = 22.7$ mpc surrounded by a classical isothermal atmosphere. The core and the halo are separated by an extended plateau. The quantum core describes a very compact central object not very different from Sagittarius A*. However, the quantum core is not small enough to account for the observational constraints. Furthermore, this core-halo configuration is thermodynamically unstable so it is not expected to result from a natural evolution (in the sense of Lynden-Bell). Therefore, the original models of Bilic et al. [65] and Ruffini et al. [76] have to be rejected.

C. Fermionic King Model

More recently, Argüelles et al. [58] considered the general relativistic fermionic King model accounting for a tidal confinement. They applied this model to the Milky Way and determined the parameters by fitting the core-halo profile to the observations. For a fermion mass $m = 48$ keV/c² they obtained a fermion ball of mass $M_c = 4.2 \times 10^6 M_\odot$ and radius $R_c = R_P = 6 \times 10^{-4}$ pc which, this time, is consistent with the observational constraints.

Let us see if their results are consistent with our analytical box model. Following Argüelles et al. [58], we take a DM particle mass $m = 48$ keV/c². The corresponding minimum halo (see Sec. V) has a mass ($M_{h\text{min}} = 1.30 M_\odot$) and a radius ($r_h\text{min} = 0.0683$ pc). If we consider a DM halo of mass $M_h = 10^{11} M_\odot$ and radius $r_h = 20.1$ kpc similar to the one that sur-

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33 See [42] and Appendix C of [14] for a detailed discussion of the Poincaré-Katz criterion.

34 See Sec. V of [25] for a detailed discussion of the structure of quantum DM halos involving a quantum core, a plateau, and a classical isothermal atmosphere. The core-halo profiles of DH halos with a “small” $\mu$ do not show a plateau while an extended plateau is present in DM halos with $\mu \gg \mu_{\text{MCP}}$.

35 The claim that the core-halo solution of Refs. [57][65] is thermodynamically unstable was first made in [23][43][44].

36 The fermionic King model was heuristically introduced by Ruffini and Stella [69] as a generalization of the classical King model [80]. It was also introduced independently by Chavanis [70] who derived it from a kinetic theory based on the fermionic Landau equation. The nonrelativistic fermionic King model was studied by Chavanis et al. [33] who showed that the density profiles typically have a core-halo structure with a quantum core (fermion ball) and a tidally truncated isothermal halo leading to flat rotation curves. They also studied the caloric curves and the thermodynamical stability of the equilibrium states. The name “fermionic King model” was introduced in [33][43].

37 The fermion ball is weakly general relativistic because $M_c = 4.2 \times 10^6 M_\odot \ll M_{\text{OV}} = 2.71 \times 10^8 M_\odot$ [see Eq. (87)].
rounds our Galaxy (see Sec. IV) we find that the corresponding degeneracy parameter is $\mu = 3.09 \times 10^{14}$ [see Eq. (98)]. Considering the core-halo solution (CH) and using Eqs. (86), (106), (111) and (116), we find that this halo should contain a quantum core of mass $M_c = 2.61 \times 10^5 M_\odot$, radius $R_c = 2.13 \text{ mpc}$ and central density $\rho_0 = 3.87 \times 10^{13} M_\odot/\text{pc}^3$. Our analytical results are not consistent with the results of Argüelles et al. [58] because we find that the mass $M_c$ of the fermion ball is about 10 times smaller than their value. Since our analytical model is consistent with the results of Bilic et al. [57] and Ruffini et al. [67] that are based on the usual Fermi-Dirac DF but not with the results of Argüelles et al. [58] that are based on the fermionic King model we deduce that the difference comes from the fact that tidal effects – not taken into account in our analytical model – are important (a priori, the difference does not come from general relativity effects which are small as we have indicated in footnote 36).

Therefore, in order to obtain accurate results, it is important to use the fermionic King model [58, 65] instead of the usual fermionic model [42, 57, 65]. Argüelles et al. [58] managed to fit the density profile and the rotation curve of the Milky Way with the fermionic King distribution and argued that a fermion ball can mimic the effect of a SMBH. This scenario is very attractive because it can explain the whole structure of the galaxy, the supermassive central object and the isothermal halo, by a single DF (the fermionic King model [65, 70]).

Let us now discuss the thermodynamical stability of the core-halo solution considered by Argüelles et al. [58]. The caloric curves of the fermionic King model in Newtonian gravity for arbitrary values of $\mu$ were first studied by Chavanis et al. [83]. The caloric curve corresponding to a large value of $\mu$ (i.e. having the characteristics of the Milky Way) is plotted in Fig. 30 of [33]. In that paper, we have focused on the density profiles of the solutions located in the region of the spiral (see Fig. 44 of [33]). For a given energy in that region, we found a gaseous solution (G’), a core-halo solution (CH’) and a condensed solution (C’). These results are reproduced in Figs. 22 and 23 for convenience. The gaseous solution (G’) corresponds to the classical isothermal sphere. Since it lies before the first turning point of energy, it is thermodynamically stable in the microcanonical ensemble (maximum entropy state at fixed mass and energy). The condensed solution (C’) is also stable in the microcanonical ensemble because it lies after the last turning point of energy. However, this solution is not astrophysically relevant because it has a too extended halo that is not consistent with the structure of DM halos (see Fig. 23). The core-halo solution (CH’) is similar to the solution found by Ruffini et al. [57] which was claimed to reproduce the structure of the Milky Way. It consists in a large nondegenerate isothermal atmosphere harboring a small “fermion ball” with a high density, a large mass and a small radius that could mimic a SMBH. Since this solution lies between the first and the last turning points of energy, it is thermodynamically unstable in the microcanonical ensemble (saddle point of entropy at fixed mass and energy). Therefore, we concluded in [33] that this type of solution is not likely to result from a natural evolution and, consequently, we questioned the possibility that a fermion ball could mimic a central SMBH.

However, in our analysis, we did not consider the stable solution (CH)∗ located just after the turning point of energy $E_*$, believing that this solution would be unreachable by a natural evolution or that it would look like the solution (C’) which has a too extended halo. Recently, Argüelles et al. [60] computed the caloric curves of the fermionic King model in general relativity. For not too negative energies they obtained a caloric curve

\[ \begin{align*}
\text{FIG. 22: Caloric curve of the fermionic King model for large DM halos (from [33]).} \\
\text{The core-halo solution (CH') considered in [33] is unstable. The core-halo solution (CH)∗ considered in [60] has a similar structure and is stable (being located after the last turning point of energy $E_*$). It may account for the structure of the Milky Way in which the fermion ball mimics a SMBH.}
\end{align*} \]

\[ \begin{align*}
\text{FIG. 23: Density profiles of the gaseous (G’), core-halo (CH’) and condensed (C’) solutions identified in Fig. 22 (from [33]).}
\end{align*} \]

\[ \text{38 For smaller energies, it becomes crucial to take general relativity} \]
similar to the one represented in Fig. 22. They confirmed the instability of the (CH') solutions in the region of the spiral previously considered by Chavanis et al. 23 but they also investigated the solution (CH)∗ close to E∗, and showed that this solution actually corresponds to the density profile obtained in their previous work 55 which provides a good agreement with the structure of the Milky Way. Since this solution is located after the last turning point of energy it is thermodynamically stable in the microcanonical ensemble. This is a very interesting result because it shows that the core-halo structure found by Arguelles et al. 55 is thermodynamically stable and can, therefore, arise from a natural evolution.

In conclusion, when we use the ordinary Fermi-Dirac DF, the core-halo solution purported to reproduce the structure of the Milky Way is thermodynamically unstable but when we use the fermionic King model, this core-halo solution is thermodynamically stable. Therefore, this core-halo configuration may result from a natural evolution in the sense of Lynden-Bell. This gives further support to the scenario according to which a fermion ball could mimic a SMBH at the centers of the galaxies.

Remark: The discovery 60 that the core-halo solution (CH)∗ with a compact fermion ball mimicking a SMBH is thermodynamically stable is a very important result. However, it does not prove that this structure will effectively arise from a natural evolution. The reason is that violent relaxation is in general incomplete 80, 79. In particular, the fluctuations of the gravitational potential that are the engine of the collisionless relaxation can die out before the system has reached statistical equilibrium. Therefore, it is not clear if violent relaxation can produce this type of structures with a very high central density. 39

In order to vindicate this scenario, the next step would be to perform direct numerical simulations of collisionless fermionic matter to see if they spontaneously generate fermion balls with the characteristics of SMBHs. Indeed, it is not clear why the system should spontaneously reach an equilibrium state that is just in the bend after the turning point of energy Λ∗. The purely gaseous solution (G') without a quantum core, which is also a maximum entropy state, may be easier to reach through a violent relaxation process and is consistent with the observations. However, it does not account for a massive central object at the centers of the galaxies. In that case, we either have to introduce a primordial SMBH “by hand” or advocate a scenario of gravitational collapse such as the one discussed in the following section.

D. General relativistic collapse towards a SMBH

For a fermion mass \( m = 48 \, \text{keV}/c^2 \), the mass \( M_h = 10^{11} M_\odot \) of the Milky Way is larger than the OV mass \( M_{OV} = 2.71 \times 10^8 M_\odot \), so we have to take into account general relativity effects in the caloric curve. As first shown by Alberti and Chavanis 44, 45 for box-confined systems, and recovered by Arguelles et al. 60 for tidally truncated models, relativistic effects create a new turning point of energy in the caloric curve at which the condensed branch terminates (see Fig. 16). Below \( E_{c''} \) the system collapses towards a black hole. As we have seen previously, two stable equilibrium states are relevant in the structure of DM halos: the gaseous solution (G') equivalent to the classical isothermal sphere and the core-halo solution (CH)∗ which contains a fermion ball mimicking a SMBH. Only direct numerical simulations can tell us which metaequilibrium state will be reached in practice from a violent collisionless relaxation. Since these numerical results are not available yet, we shall consider the two possibilities:

(A) Suppose that violent relaxation selects the gaseous solution (G'). On a secular timescale, the system follows the upper series of equilibria from point (G') to the point of minimum energy \( E_c \). At that point, it becomes thermodynamically unstable and undergoes a gravitational catastrophe up to point (C') where the collapse is stopped by quantum mechanics, leading to the formation of a fermion ball. Then, if the energy keeps decreasing, the system follows the lower series of equilibria up to the point of minimum energy \( E_{c''} \) where it becomes thermodynamically and dynamically unstable (in a general relativistic sense) and collapses towards a SMBH. 40

As discussed in 44, there are two possible evolutions:
(i) If the particle number \( N \) is below a critical value \( N'_1 \), then \( \Lambda_c < \Lambda''_c \) and the system is first arrested by quantum mechanics before collapsing towards a BH. (ii) If the particle number \( N \) is above a critical value \( N'_2 \), then \( \Lambda_c > \Lambda''_c \) and the system directly collapses towards a BH. These two possibilities are illustrated in Fig. 24.

(B) Suppose that violent relaxation selects the core-halo solution (CH)∗ where the fermion ball mimics a SMBH. On a secular timescale, the system follows the series of equilibria from point (CH)∗ to the point of min-

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39 It may be easier to form core-halo configurations with a very high central density if the fermions are self-interacting and if the Fermi-Dirac equilibrium state results from a collisional evolution of nongravitational origin as discussed in Sec. 11.

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40 This requires that the core mass increases until it reaches the critical OV value. The increase of the core mass may take place through an accretion process.
imum energy $E_{0''}$. At that point, it becomes thermodynamically and dynamically unstable (in a general relativistic sense) and collapses towards a SMBH.

In the two cases, the ultimate fate of the system is to form a SMBH surrounded by an envelope. This picture may be just qualitative because it is not clear if the lower branch of equilibrium states is astrophysically relevant. Indeed, we have indicated that the envelope of the solutions $(C')$ is too much extended to match the characteristics of DM halos. Therefore, the collisional evolution of the system from point $(G')$ or from point $(CH)_c$ up to the formation of a SMBH at $E_{0''}$ may involve out-of-equilibrium states $(CH)_{out}$ instead of following the series of equilibrium states $(C')$.

For a fermion mass $m = 48\,\text{keV}/c^2$, the OV mass $M_{OV} = 2.71 \times 10^8 M_\odot$ is too large to account for the mass of a SMBH like Sgr $\Lambda^*$ at the center of the Milky Way. Either the mass of the SMBH resulting from gravitational collapse is smaller than $M_{OV}$ or there is no gravitational collapse and Sgr $\Lambda^*$ is a fermion ball $(CH)_c$ as suggested by Arguelles et al. [60]. Therefore, a fermion ball is favored in medium size galaxies like the Milky Way. However, for very large halos it is shown by Alberti and Chavanis [44] that the condensed branch disappears (see the last panel of Fig. 24). In that case, there is no solution with a fermion ball such as $(CH)_c$ and the system necessarily collapses towards a SMBH. Therefore, medium size galaxies ($N < N'_c$) like the Milky Way may harbor a fermion ball of mass $M = 4.2 \times 10^6 M_\odot$ while very large galaxies ($N > N'_c$) like ellipticals may harbor a SMBH of mass $M_{OV} = 2.71 \times 10^8 M_\odot$ that could even grow by accretion. This could account for the mass of SMBHs in AGNs like the one recently photographed in M87 ($M_\bullet \sim 10^{13} M_\odot$ and $M_{BH} \sim 10^{10} M_\odot$).

For a fermion mass $m = 386\,\text{keV}/c^2$, the OV mass $M_{OV} = 4.2 \times 10^6 M_\odot$ is comparable to the mass of Sgr $\Lambda^*$. Furthermore, the caloric curve is similar to the one reported in the last panel of Fig. 24 and there is no possibility to have a solution $(CH)_c$ involving a fermion ball. In that case, the Milky Way could have undergone a gravitational collapse leading to a SMBH of mass $M_{OV} = 4.2 \times 10^6 M_\odot$. The halo surrounding the SMBH is left undisturbed and could correspond to a marginal classical King profile which gives a good agreement with the Burkert profile (see Ref. [52] and Fig. 1).

E. Potential problems with a DM model involving a fermion mass $m = 48\,\text{keV}/c^2$ or $m = 386\,\text{keV}/c^2$

In Sec. [44] we have determined the mass $m$ of the DM particle by arguing that the smallest halo observed

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41 It would be interesting to determine precisely the condition of disappearance of the condensed branch in the framework of the relativistic fermionic King model, i.e., the value of $N'_c$. 

FIG. 24: Caloric curve of the general relativistic Fermi gas in a box as a function of the particle number $N$ (adapted from [44]). For $N < N_{OV}$, the gravothermal catastrophe at $E_c$ leads to a fermion ball surrounded by a hot halo. For $N_{OV} < N < N'_c$ the system first takes a quantum core-halo structure resulting from the gravothermal catastrophe at $E_c$ (as before) then collapses towards a SMBH at $E''_c$. For $N > N'_c$ the condensed branch disappears so that only the collapse at $E''_c$ towards a SMBH is possible. [These caloric curves are valid for relatively small DM halos. For larger halos a spiral develops in the head of the dinosaur but the phenomenology remains the same].
in the universe (“minimum halo”) with a typical mass $M \sim 10^8 M_\odot$ and a typical radius $R \sim 1 \text{ kpc}$ (Fornax) represents the ground state of the self-gravitating Fermi gas at $T = 0$. This yields $m = 165 \text{ eV}/c^2$. This value (previously given in Appendix D of [91]) is of the order of magnitude of the fermion mass obtained by other authors [116–118] using more detailed comparisons with observations. Alternatively, Argüelles et al. [58, 60] determined the mass of the fermionic DM particle in such a way that the fermion ball that composes the core-halo structure of a large DM halo like the Milky Way, obtained in the framework of the fermionic King model, mimics the effect of a SMBH at the center of the Galaxy. This leads to a much larger mass $m = 48 \text{ keV}/c^2$. In very recent works, Becerra-Vergara et al. [119, 120] showed that the gravitational potential of a fermion ball (with a particle mass $m = 56 \text{ keV}/c^2$) leads to a better fit of the orbits of all the 17 best resolved S-stars orbiting Sgr A* (including the S2 and G3 objects) than the one obtained by the central SMBH model.

A possible problem with this model is the following. If the DM particle had a mass $m = 48 \text{ keV}/c^2$, the minimum halo (ground state) would be too small: it would have a mass $(M_h)_{\text{min}} = 1.30 M_\odot$ and a radius $(r_h)_{\text{min}} = 0.0683 \text{ pc}$. This would imply the formation of structures at very small scales, up to $\sim 1 M_\odot$. Therefore, DM halos should exist up to very small scales, like in the CDM model. Indeed, (bosonic or fermionic) quantum models with a large particle mass behave essentially as CDM. This is not what we observe. There is apparently no DM halos with a mass below $\sim 10^8 M_\odot$, leading to the missing satellite problem [3–5]. This is why quantum models of DM with a small particle mass have been introduced. Namely, they have been introduced precisely in order to have a ground state (minimum halo) with a typical mass $M \sim 10^8 M_\odot$ and a typical radius $R \sim 1 \text{ kpc}$, corresponding to dSphs like Fornax, not smaller. Accordingly, a fermionic model with $m = 48 \text{ keV}/c^2$ may not be able to solve the missing satellite problem.

If we disregard this difficulty, another consequence of the model of Argüelles et al. [58, 60] is that dSphs should have a very pronounced core-halo structure (since they do not correspond to the ground state of the self-gravitating Fermi gas). For example, a compact DM halo of mass $M_h = 10^8 M_\odot$ (Fornax) should have a core-halo structure with a small central fermion ball (possibly mimicking an intermediate mass BH) and an atmosphere. Using Eqs. [86], [106], [111] and [116], we find that this DM halo should contain a quantum core of mass $M_c = 1.57 \times 10^3 M_\odot$, radius $R_c = 5.42 \text{ mpc}$ and central density $\rho_0 = 1.40 \times 10^{11} M_\odot/pc^3$. The corresponding density and velocity profiles are plotted in Figs. 25 and 26. To our knowledge, this core-halo structure has not been observed in ultracompact DM halos. dSphs are rather expected to correspond to pure fermion balls at $T = 0$ (possibly surrounded by a tenuous atmosphere). Therefore, they are expected to have a profile similar to Figs. 7 and 8 instead of Figs. 25 and 26. It would be extremely important to clarify this issue by applying the model of Argüelles et al. [58, 60] to ultracompact halos in order to determine which of the two scenarios (the scenario of Argüelles et al. [58, 60] with $m = 48 \text{ keV}/c^2$ or the one developed in the present paper with $m = 165 \text{ eV}/c^2$ or $m \sim 1 \text{ keV}/c^2$) is the most relevant.

There is also a problem related to the validity of the Fermi-Dirac (or Lynden-Bell) DF as discussed further in Sec. X. Indeed, for a large fermion mass $m \gg 1 \text{ keV}/c^2$, the DM halo is essentially classical except in a very small quantum core (fermion ball). Away from the core, we should recover the NFW profile leading to cusps. It is precisely in order to avoid these cusps that quantum models of DM with a small particle mass $m \lesssim 1 \text{ keV}/c^2$ have been introduced.

Domcke and Urbano [116] model dSphs as a completely degenerate fermionic system and find that $m = 200 \text{ eV}/c^2$ provides the best fit to observations of velocity dispersion. Randall et al. [117] show that self-gravitating fermions under full-degeneracy do not fit well the velocity dispersion data of some local dwarfs and introduce by hand a Boltzmannian tail (i.e. finite temperature effects) in order to better reproduce the data. They find good agreement for $70 \text{ eV}/c^2 < m < 400 \text{ eV}/c^2$. Bar et al. [118] study the globular cluster timing problem in Fornax assuming that the core is a completely degenerate fermion ball. They find $m = 135 \text{ eV}/c^2$ but point out that this mass violates the Lyman-alpha limit.

If we use the nonrelativistic mass-radius relation (74) of a fermion ball at $T = 0$ and take $M_c = 4.2 \times 10^5 M_\odot$ and $R_c = 6 \times 10^{-4} \text{ pc}$, corresponding to the characteristics of the massive object at the center of our Galaxy (see Sec. VII A), we get $m = 54.6 \text{ keV}/c^2$.

Interestingly, these values obtained from our semi-analytical model [see in particular Eq. (111)] are comparable to the values obtained numerically in [59].
IX. POSSIBLE SOLUTIONS TO AN APPARENT PARADOX RELATED TO THE UNIVERSAL SURFACE DENSITY OF DM HALOS

A. The apparent paradox

The mass-radius relation of a completely degenerate fermion ball (ground state of the self-gravitating Fermi gas at $T = 0$) is given by [see Eq. (74)]

$$R = 0.114 \frac{h^2}{G m^{8/3} M_{1/3}^3}. \quad (122)$$

The radius decreases like $M^{-1/3}$ as the mass increases. Therefore, if we identify $M$ with the halo mass $M_h$ and $R$ with the halo radius $r_h$, this result is in contradiction with the universality of the surface density of DM halos [see Eq. (52)] implying that the radius increases with the mass as $M^{1/2}$ [see Eq. (53)]. A similar problem arises in the BECDM model.

This apparent paradox was pointed out by the author at several occasions in the case of fermions and bosons (see, e.g., Appendix F of Ref. [33], the Introduction of Ref. [35] and Appendix L of [23]). It has also been recently emphasized by Deng et al. [12] and Burkert [122] in the case of bosons. A possible implication of this paradox is that the quantum (fermionic and bosonic) models of DM are ruled out because they are not consistent with the constraint from Eq. (52). This is essentially the conclusion reached by Deng et al. [121] and Burkert [122] for the BECDM model. Below, we discuss several possible solutions to this apparent paradox that were suggested in [24] for bosonic DM and that can be straightforwardly adapted to fermionic DM.

Remark: The constant surface density of DM halos $\Sigma_0 = 0.01955 c^2 \Lambda / G = 133 M_0 / pc^2$ may be explained by the logotropic model developed in [84–86] which involves a logotropic envelope instead of an isothermal one. This model not only explains why the surface density of DM halos is constant but it also determines its universal value (in agreement with the observations) in terms of fundamental constants $\Sigma_0^h = 0.01955 c \sqrt{\Lambda / G} = 133 M_0 / pc^2$ without adjustable parameter. At the same time, in a cosmological context, it correctly accounts for the accelerating expansion of the Universe with a single dark fluid.

B. Model I: purely gaseous solution

A first possible solution to this problem is that DM halos do not have a quantum core such as a fermion ball or such as a soliton (in the BECDM model). Indeed, the DM halos could be in the purely gaseous phase (G) corresponding to the classical isothermal sphere (see Sec. V). This solution is always thermodynamically stable (maximum entropy state) so it represents the most probable state of the system. Furthermore, it is always possible to satisfy the constraint from Eq. (52) by adapting the temperature [see Eq. (53)]. This leads to the mass-radius relation from Eq. (53). As shown in [25], the classical isothermal distribution (without quantum core) is fully consistent with the observational Burkert profile and can therefore represent a satisfying description of DM halos. It is nevertheless crucial to take quantum mechanics into account in the case of ultracompact DM halos with a small mass, corresponding to dSphs like Fornax. This leads to the Model I of Ref. [24] in which the DM halos are purely isothermal (without quantum core) except near the ground state. More precisely:

(i) At $(M_h)_{\min}$ the DM halo is completely degenerate (see Sec. V). The values of $M_h$ and $r_h$ for this minimum halo are consistent with the constraint from Eq. (52).

(ii) Ultracompact DM halos with a mass $(M_h)_{\min} \leq M_h \leq (M_h)_{CCP}$ have a quantum core surrounded by a tenuous isothermal atmosphere. The presence of a small isothermal halo allows us to satisfy the constraint from Eq. (52) as discussed in Sec. VI of [25] for BECDM halos. All the profiles constructed in Sec. VI of [24] satisfy the constraint from Eq. (52). The same results apply to fermionic DM halos.

(iii) DM halos with a mass $M_h \geq (M_h)_{CCP}$ are purely isothermal without quantum core. Indeed, as shown in [25] for BECDM halos, if we enforce the constraint from Eq. (52) in Model I we find that the core mass decreases as the halo mass increases so that large DM halos are essentially classical without quantum core.

This model leads to the mass-radius relation reported in Fig. 16 of [25] and reproduced in Fig. 27 (adapted to fermions). It coincides with the classical law from Eq. (53) except at small halo masses. Quantum mechanics just determines the ground state of DM halos at $(M_h)_{\min} = 10^8 M_0$ and $(r_h)_{\min} = 597 pc$. This scenario does not account for the presence of a compact object, such as a SMBH, at the centers of the galaxies. Of course, we can always add “by hands” a primordial SMBH at the center of a classical isothermal halo but this is almost assuming the result. In order to explain self-consistently the presence of a SMBH at
the center of the galaxies, we can consider the following scenarios.

C. Model II: core-halo solution with a large quantum bulge

Another possibility to solve the paradox of Sec. [XXA] and “save” the quantum core-halo solution (CH) from Sec. [VI] is to assume that the constraint from Eq. (52) should be replaced by

$$\Sigma_0 = \rho_c r_h = 141^{+83}_{-52} M_\odot / pc^2,$$

where $\rho_c$ is the central density of the quantum core but rather an “apparent” central density. It corresponds to the density at the separation between the quantum core and the classical halo in configurations such as those from Fig. [17] Similarly, $r_h$ is the radius at which $\rho_c$ (instead of $\rho_0$) is divided by 4. The idea underlying this replacement is that observations may not be able to resolve the presence of a quantum bulge at the centers of galaxies. Therefore, we have to distinguish between the true central density $\rho_0$ (which is the central density of the quantum core) and the apparent central density $\rho_c$ (which is the “central” density of the classical isothermal halo surrounding the quantum core). Similarly, we have to distinguish the halo radius $r_h$ which typically corresponds to the distance where the apparent central density $\rho_c$ is divided by 4 from the core radius $r_c$ which is of the order of the distance where the core central density $\rho_0$ is divided by 4. The distinction between these quantities is explicitly shown in Fig. [17]. It is clear that the radius of large DM halos is given by $r_h$ not by the quantum core radius $R_c$.

This leads to Model II of [25] in which the DM halos of large mass have a core-halo structure. More precisely:

(i) At $(M_h)_{\text{min}}$, the DM halo is completely degenerate (see Sec. [V]). The values of $M_h$ and $r_h$ for this minimum halo are consistent with the constraint from Eq. (52).

(ii) Ultracompact DM halos with a mass $(M_h)_{\text{CCP}} \leq M_h \leq (M_h)_{\text{MCP}}$ have a core-halo structure made of a large quantum bulge surrounded by a tenuous isothermal atmosphere. The presence of a small isothermal halo allows us to satisfy the constraint from Eq. (52) as discussed in Sec. VI of [25] for BECDM halos. All the profiles constructed in Sec. VI of [25] satisfy the constraint from Eq. (52). The same results apply to fermionic DM halos.

(iii) Small DM halos with a mass $(M_h)_{\text{CCP}} \leq M_h \leq (M_h)_{\text{MCP}}$ have a core-halo structure made of a large quantum bulge surrounded by a classical isothermal halo. The core mass $M_c$ increases with the halo mass $M_h$ according to Eq. (111). The constraint from Eq. (52) is satisfied provided that we replace the central density $\rho_0$ by the apparent central density $\rho_c$, i.e., provided that we use Eq. (123). From the outside (i.e., considering the external structure of the DM halo and ignoring the quantum bulge), the system looks like a classical isothermal sphere. This leads to the mass-radius relation from Eq. (53) which is consistent with the observations.

(iv) For large DM halos with a mass $M_h \geq (M_h)_{\text{MCP}}$, the quantum bulge is replaced by a small out-of-equilibrium quantum core or by a SMBH. In that case, the replacement of Eq. (52) by Eq. (123) is even more justified. What is relevant is not the central density of the compact object but rather the density of the classical halo at the contact with this object. [45]

This model leads to the mass-radius relation from Fig. [27] It coincides with the classical law from Eq. (53) except at small halo masses. Quantum mechanics determines the ground state of DM halos at $(M_h)_{\text{min}} = 10^6 M_\odot$ and $(r_h)_{\text{min}} = 597 \ p_c$. It also implies the existence of a large quantum bulge of mass $M_c$ that increases with the halo mass $M_h$ according to Eq. (111). For very large halos, the fermionic bulge is replaced by an out-of-equilibrium compact quantum core or by a SMBH.

These arguments may solve, or alleviate, the problem reported in Sec. [XXA]. The crucial point is to know if observations are able to detect a large quantum bulge of typical mass $M_c = 9.45 \times 10^9 M_\odot$ and size $R_c = 240 \ p_c$ at the center of the Milky Way that is predicted by our model. This possibility is discussed in Sec. [VI].

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[45] The same comment holds if the quantum bulge in scenario (iii) has led to the formation of a SMBH by accretion of the gas (see Sec. [VII]).
D. Model III: core-halo solution mimicking a SMBH

Finally, we note that the apparent paradox reported in Sec. IX A does not arise in the model of Argüelles et al. [58] [60] where the fermion ball mimics a SMBH of negligible extent. Indeed, in that case, there is a clear separation between the quantum core and the classical isothermal halo as depicted in Fig. [19]. It is clear that the central density to consider in Eq. (52) is not the central density \( \rho_0 \) of the fermion ball but rather the density \( \rho_c \) of the plateau that connects the fermion ball to the classical isothermal halo. Similarly, the halo radius \( r_h \) is not the radius where the central density is divided by 4 (which would coincide with the radius of the quantum core \( R_c \)) but the radius where the density of the plateau is divided by 4.

In model III, the DM halo behaves from the outside as a classical isothermal halo but harbors a tiny massive fermion ball mimicking a SMBH. The halo mass-radius relation is similar to that reported in Fig. [27] except that it starts at a much smaller minimum halo mass \( (M_h)_{\text{min}} = 1.30 M_\odot \) corresponding to \( (r_h)_{\text{min}} = 0.0683 \text{pc} \).

X. THERMAL OR QUANTUM CORE?

In this paper, we have assumed that a fermionic DM halo reaches a statistical equilibrium state described by the Fermi-Dirac DF (see Sec. H). The same assumption was made by other authors [47] [50] [65] [110] [117]. However, this is a strong assumption and the establishment of a statistical equilibrium state for self-gravitating systems is far from trivial.

Let us first consider a collisionless system of classical self-gravitating particles. According to the statistical theory of Lynden-Bell [30], it should reach a Fermi-Dirac-like DF, reducing to the classical isothermal DF in the dilute limit. The corresponding density profile has a core due to effective thermal effects (in the sense of Lynden-Bell). However, this prediction is not consistent with numerical simulations of classical collisionless self-gravitating systems. Indeed, such simulations lead to NFW profiles [1] presenting a \( r^{-1} \) central cusp, not a core. This demonstrate that, for classical collisionless self-gravitating systems, the Lynden-Bell prediction does not work in the central part of the system. Therefore, classical collisionless self-gravitating systems are not in a maximum entropy state. Since observations show that DM halos possess a core rather than a cusp, we conclude that DM halos are either quantum or collisional, two features that are not accounted for in NFW numerical simulations [1].

Let us now consider a collisionless system of quantum self-gravitating particles (fermions). If the fermion mass is small \( (m \lesssim 1 \text{keV}/c^2) \), as assumed in Sec. VII, DM halos should harbor a large quantum bulge of radius \( R_c \approx 240 \text{pc} \) (Model II) according to the Lynden-Bell prediction. In that case, the \( r^{-1} \) cusps are prevented by the Pauli exclusion principle which forbids high densities. As a result, the classical cusp is replaced by a large quantum bulge. It is possible that, for quantum systems with a small fermion mass \( m \lesssim 1 \text{keV}/c^2 \), the Lynden-Bell prediction works well in the central part of the system. This is not in contradiction with NFW numerical simulations [1] since they do not take into account quantum mechanics. Quantum effects may facilitate the collisionless relaxation of the system towards a maximum entropy state. Therefore, when \( m \lesssim 1 \text{keV}/c^2 \), quantum effects can solve the core-cusp problem.

By contrast, if the fermion mass is large \( (m \gg 1 \text{keV}/c^2) \), as assumed in Sec. VIII the quantum core predicted by the Lynden-Bell theory is very small \( (R_c \lesssim 6 \times 10^{-4} \text{pc}) \). In the main part of the DM halo excluding the tiny fermion ball at the very center (i.e. for \( r \gg R_c \)) the system is essentially in the classical regime. In that case, we should recover the NFW profile which displays a \( r^{-1} \) cusp while Argüelles et al. [58] find a classical isothermal profile with a thermal core. This is because their model assumes that the Lynden-Bell DF is valid everywhere (even in the classical region) while we have just seen that the Lynden-Bell DF is not valid in a classical system. Therefore, when \( m \gg 1 \text{keV}/c^2 \) (e.g. \( m \sim 50 \text{keV}/c^2 \)), quantum effects cannot solve the core-cusp problem if the system is collisionless.

One possibility to solve this problem and “save” the scenario of Argüelles et al. [58] is to assume that the fermions are self-interacting and that the evolution of DM halos is collisional. In that case, the Fermi-Dirac DF is established through a collisional relaxation of non-gravitational origin, not through a collisionless relaxation (see Sec. H). In the classical (nonquantum) regime, collisions lead to an isothermal core of size \( r_h \) instead of a \( r^{-1} \) cusp. The classical core is due to thermal effects like in the SIDM model. This is not in contradiction with NFW numerical simulations [1] since they do not take into account self-interaction and collisions among the particles. In the quantum + collisional regime, we should both have a quantum core of size \( R_c \) and an isothermal core of size \( r_h \). Therefore, quantum and/or thermal (collisional) ef-

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46 It does not work well neither in the outer part of the system since it predicts a density profile decaying as \( r^{-2} \) instead of \( r^{-3} \). However, we have previously argued that the difference is not very strong and that it can even be reduced if we take into account tidal effects [32] [33].

47 In this respect, Yunis et al. [123] have taken into account self-interaction in fermionic DM halos with \( m = 48 \text{keV}/c^2 \) and shown that the extended hydrostatic equilibrium equations for tidally truncated systems (which account for such interactions) do not spoil the rotation curve fittings for typical cross-sections.
fects can solve the core-cusp problem. In particular, collisions can establish a Maxwell-Boltzmann DF for classical particles and a Fermi-Dirac DF for fermions.

XI. SUMMARY AND CONCLUSIONS

In this paper, we have developed a predictive model of fermionic DM halos. We have considered different scenarios depending on the fermion mass $m$ and on the DM halo mass $M_h$. We have discussed the case of a collisionless evolution and the case of a collisional evolution of nongravitational origin possibly due to SIDM. Below, we recall our basic assumptions, summarize our main results, present synthetic phase diagrams, and conclude.

A. Assumptions

We have used the following observational results:

(i) The surface density of DM halos has a universal value $\Sigma_0 = 141 M_\odot/pc^2$ \cite{51, 53}.

(ii) There exists a minimum halo of mass $(M_h)_{\text{min}} \sim 10^8 M_\odot$, corresponding to dSphs like Fornax. Observations reveal that there is no DM halo below this typical mass.\footnote{If this observational result were not (exactly) valid, our model could be generalized but it would depend on more parameters.}

(iii) We assumed that the minimum halo of mass $(M_h)_{\text{min}} = 10^8 M_\odot$ (Fornax) is completely degenerate, i.e., it corresponds to the ground state of the self-gravitating Fermi gas (see Sec. III). This automatically determines the fermion mass $m = 165$ eV/c$^2$.\footnote{We have taken this value in order to be consistent with our previous papers. It is possible that this mass is overestimated. Some authors \cite{52} (see also Refs. \cite{53, 52}) argue that the minimum halo mass is $(M_h)_{\text{min}} = 0.39 \times 10^6 M_\odot$, corresponding to Willman I. Numerical applications have also been given in that case for a comparison.}

We have made the following assumptions:

(i) We assumed that DM is made of fermions and that DM halos are in a statistical equilibrium state of the self-gravitating Fermi gas (see Sec. II). This statistical equilibrium state may result from a process of collisionless violent relaxation in the sense of Lynden-Bell or, possibly, from a collisional relaxation of nongravitational origin if the fermions are self-interacting (see Sec. II). Note that this is a strong assumption. It is possible that DM halos are in an out-of-equilibrium state in which case predictions become more complicated (or even impossible).

(ii) We assumed that the minimum halo of mass $(M_h)_{\text{min}} = 0.39 \times 10^6 M_\odot$ (Willman I) we get $m \sim 1$ keV/c$^2$ and $(r_h)_{\text{min}} = 33$ pc (see Sec. V C). A fermion mass in the keV scale (or slightly larger) is more consistent with the constraints coming from cosmological observations such as the Lyman $\alpha$ forest.

33 pc (see Sec. V C). A fermion mass in the keV scale (or slightly larger) is more consistent with the constraints coming from cosmological observations such as the Lyman $\alpha$ forest.

B. Methodology

The structure of fermionic DM halos, and the mass $M_\odot$ of the quantum core, are obtained by maximizing the Fermi-Dirac entropy at fixed mass and energy (microcanonical ensemble). To solve this problem, we proceeded as follows. We considered the thermodynamics of a gas of self-gravitating fermions in a box of radius $R = r_h$ containing a mass $M = M_h$ (see Sec. III). The caloric curve depends on a unique parameter $\mu$ which is a measure of the mass $M_h$ of the DM halo [see Eq. \cite{95}]. We then used the fact that the DM halos are virialized so that the dimensionless inverse temperature $\eta = \beta G M m/R$ is of order unity. The intersection between the line $\eta = 1$ and the caloric curve $\eta(\Lambda)$ determines the possible equilibrium states of the system consistent with the virial condition. For $M_h < (M_h)_{\text{CCP}}$ there is only one solution. However, for $M_h > (M_h)_{\text{CCP}}$ we found two relevant solutions:

(i) A purely gaseous solution (G) without quantum core corresponding to the classical isothermal sphere \cite{53}. Its density profile is consistent with the observational Burkert profile (see Sec. III C of \cite{25}). This corresponds to Model I of Sec. IX B.

(ii) A core-halo solution (CH) with a quantum core surrounded by a classical isothermal halo. The core is relatively large ($M_c > 9.45 \times 10^9 M_\odot$ and $R_c = 240$ pc for the Milky Way) so it can represent a quantum bulge.
The core mass increases with the halo mass as $M_{h}^{3/8}$ [see Eq. (111)]. This corresponds to Model II of Sec. [CX] 51.

The gaseous solution (G) is always thermodynamically stable. The core-halo solution (CH) is thermodynamically stable for $M_{h} < (M_{h})_{MCP}$ and unstable for $M_{h} > (M_{h})_{MCP}$ (as explained before, we consider the thermodynamical stability in the microcanonical ensemble).

The formation of DM halos arises from a process of collisionless violent relaxation. The metaequilibrium state resulting from violent relaxation must be a maximum entropy state in the sense of Lynden-Bell. It turns out that the Lynden-Bell DF coincides with the Fermi-Dirac DF. For $M_{h} < (M_{h})_{MCP}$ the gaseous solution (G) and the core-halo solution (CH) are both entropy maxima in the sense of Lynden-Bell. They could naturally arise from a process of violent relaxation. For $M_{h} > (M_{h})_{MCP}$ only the gaseous solution (G) is a maximum entropy state in the sense of Lynden-Bell. Therefore, violent relaxation should lead to the gaseous solution (G), not to the core-halo solution (CH). Actually, there exist stable core-halo solutions (CH), close to the last turning point of energy that may be physically relevant. Violent relaxation may also lead to a stable quasistationary state, resulting from incomplete relaxation, that is not a maximum entropy state.

If the DM halos are collisionless, they remain in the state resulting from violent relaxation. If the DM halos are collisional, they follow the series of equilibria determined by the caloric curve towards states of higher and higher density. When $M_{h} < (M_{h})_{MCP}$ the system can evolve from the gaseous solution (G) to the core-halo solution (CH). For $M_{h} > (M_{h})_{MCP}$ the system can follow the series of equilibria from the gaseous solution (G) up to the point of minimum energy $E_{c}$. At that point, it becomes thermodynamically unstable and undergoes a gravothermal catastrophe. If the DM halo is small enough ($M_{h} < M_{OV}$), the gravothermal catastrophe stops when the core becomes degenerate. In that case, gravitational collapse is prevented by quantum mechanics (Pauli’s exclusion principle). The system achieves a core-halo state with a small quantum core. This state does not correspond to a state of statistical equilibrium such as solution (C') which would have a too extended halo [scenario (b.1)] but it could be an out-of-equilibrium structure (CH) [scenario (b.2)]. Alternatively, if the DM halo is large enough ($M_{h} > M_{OV}$), the gravothermal catastrophe can lead to the formation of a SMBH by the mechanism discussed in Secs. VII D and VII F [scenario (b.3)].

Remark: For small halos $M_{h} < (M_{h})_{MCP}$, the core-halo solution (CH) is stable (see Fig. [12]) and the scaling $M_{c} \propto M_{h}^{3/8}$ from Eq. (111) is reliable. However, for large DM halos $M_{h} > (M_{h})_{MCP}$, this is no more the case because the core-halo solution (CH) is unstable and is replaced by an out-of-equilibrium core-halo solution (CH) [see Fig. 14]. It is possible, in that case, that DM halos of the same mass $M_{h}$ may contain cores of different masses $M_{c}$ depending on their evolution. The core mass $M_{c}$ may evolve from a small value $M_{c}^\ast$ corresponding to the beginning of the condensed branch at $\Lambda_{c}$ up to the value $M_{OV}$ corresponding to the end of the condensed branch at $\Lambda'_{c}$ at which it becomes unstable and collapses towards a SMBH (see Fig. [16]).

C. Results for $m = 165 eV/c^2$

For a fermion mass $m = 165 eV/c^2$, we obtained the following results:

(i) There exists a minimum halo of mass $(M_{h})_{min} = 10^{6} M_{\odot}$ and radius $(r_{h})_{min} = 597 pc$ corresponding to the ground state ($T = 0$) of the self-gravitating Fermi gas. This DM halo is a purely quantum object (fermion ball) without atmosphere. It is completely degenerate. It is equivalent to a polytrope of index $n = 3/2$. It is fully stable. Quantum mechanics (Pauli’s exclusion principle) determines the minimum mass and the minimum radius of fermionic DM halos. 52 This minimum halo can be assimilated with ultracompact dSphs like Fornax.

(ii) For $(M_{h})_{min} = 10^{6} M_{\odot} < M_{h} < (M_{h})_{CCP} = 6.73 \times 10^{8} M_{\odot}$, the caloric curve is monotonic ($\mu < \mu_{CCP}$; see Fig. [3]). There is only one solution with $\eta \sim 1$: A quantum object (Q) corresponding to a fermionic ball surrounded by a tenuous isothermal atmosphere. This equilibrium state is fully stable. This situation may describe dSphs. Even if collisions allow the system to evolve along the series of equilibria, no instability occurs.

(iii) For $(M_{h})_{CCP} = 6.73 \times 10^{8} M_{\odot} < M_{h} < (M_{h})_{MCP} = 1.08 \times 10^{10} M_{\odot}$, the caloric curve has an N-shape structure ($\mu_{CCP} < \mu < \mu_{MCP}$; see Fig. [12]). There are two physical solutions with $\eta \sim 1$: A gaseous solution (G) corresponding to a purely classical isothermal halo without quantum core and a core-halo (CH) solution with a quantum core surrounded by a massive atmosphere. The fermion ball may mimic a large bulge but not a SMBH because it is too much extended (see Sec. VII F). The gaseous solution and the core-halo solution are both stable. If the system evolves adiabatically

51 de Vega and coworkers [47–52] only considered the gaseous (non-degenerate) solution (G). A merit of our study is to have evidenced a bifurcation above the canonical critical point $\mu_{CCP}$ yielding a new branch of solutions (CH) possessing a quantum core.

52 Actually, the mass and the size of the DM halos should be determined by a theory of structure formation. The first stage of this theory is the Jeans instability, leading to the formation of clumps in the linear regime. When the density of the clumps has grown significantly, we enter in the nonlinear regime of structure formation where the overdensity regions experience free fall, violent relaxation, nonlinear Landau damping, merging and accretion, leading to the DM halos that we observe today.
along the series of equilibria under the effect of collisions, it can pass from the gaseous solution to the core-halo solution without collapsing. The gravothermal catastrophe is prevented by quantum mechanics. This situation may describe small and medium spiral galaxies. They may have a core-halo structure made of a quantum core (representing a bulge) and an isothermal atmosphere. The bulge may provide a favorable environment to induce the formation of a SMBH on a long timescale by an accretion process (see Sec. V.II.C).

(iv) For $M_h > (M_h)_{\text{MCP}} = 1.08 \times 10^{10} M_\odot$, the caloric curve has a Z-shape structure ($\mu > \mu_{\text{MCP}}$; see Fig. 14). There are two physical solutions for a given value of $\eta \sim 1$ as before. The gaseous solution (G) is stable while the core-halo solution (CH) is unstable [we must also keep in mind the potentially relevant solution (CH)$_*$]. In principle, only the gaseous solution may result from a process of violent collisionless relaxation because the core-halo solution is not a maximum entropy state in the sense of Lynden-Bell. If, starting from the gaseous phase, the system evolves adiabatically along the series of equilibria under the effect of collisions, it can undergo a gravothermal catastrophe at the point of minimum energy $E_c$. Then, there are two possibilities:

(iv-a) For $(M_h)_{\text{MCP}} < M_h < M_{\text{OV}} = 2.30 \times 10^{13} M_\odot$ the gravothermal catastrophe is stopped by quantum degeneracy. This leads to a possibly out-of-equilibrium small quantum core (CH)$_{\text{out}}$ (different from a large quantum bulge) surrounded by an envelope. In that case, the core mass – halo mass relation from Eq. (111) may not be valid anymore.

(iv-b) For $M_h > M_{\text{OV}} = 2.30 \times 10^{13} M_\odot$ a new turning point of energy occurs in the caloric curve [14, 15] below which the condensed branch disappears and the core of the DM halo collapses towards a SMBH of mass $M_{\text{OV}}$ (presumably). If $M_{\text{OV}} < M_h < M'_h$ the DM halo may either harbor a fermion ball or a SMBH. If $M_h > M'_h$ there is no condensed branch at all and the DM halo cannot harbor a fermion ball. It can just harbor a SMBH of mass $M_{\text{OV}}$.

This situation may apply to large spiral and elliptical galaxies. Therefore, large spiral and elliptical galaxies are expected to contain a small quantum core or a SMBH resulting from the gravothermal catastrophe instead of a large quantum bulge. During the gravothermal catastrophe, their envelope is left undisturbed and should correspond to a marginal King profile (if we take into account tidal effects) which is in good agreement with the Burkert profile (see Refs. 22, 23 and Fig. 1).

The main results of our study for $m = 165 \text{ eV}/c^2$ are summarized in the phase diagram of Fig. 28. The bullet corresponds to the minimum halo of mass $(M_h)_{\text{min}} = 10^8 M_\odot$. For $(M_h)_{\text{min}} = 10^8 M_\odot < M_h < (M_h)_{\text{CCP}} = 6.73 \times 10^8 M_\odot$, there is only one solution, the quantum solution (Q). The canonical critical point $(M_h)_{\text{ICCP}} = 6.73 \times 10^8 M_\odot$ determines a bifurcation between the branch of purely gaseous solutions (G) and the branch of core-halo solutions (CH) where the core represents a large quantum bulge. This bifurcation is associated with the occurrence of a region of negative specific heat in the caloric curve. The microcanonical critical point $(M_h)_{\text{MCP}} = 1.08 \times 10^{10} M_\odot$ determines the moment at which the DM halo can experience a gravothermal catastrophe. For $(M_h)_{\text{MCP}} < M_h < M_{\text{OV}}$ the gravothermal catastrophe is stopped by quantum mechanics and the DM halo harbors a possibly out-of-equilibrium small quantum core. Therefore, the microcanonical critical point $(M_h)_{\text{MCP}} = 1.08 \times 10^{10} M_\odot$ determines the transition between DM halos possessing a large quantum bulge and DM halos harboring a small quantum core (CH)$_{\text{out}}$ [there are also potentially relevant solutions (CH)$_*$]. This transition is associated with the instability of the large

\[ m = 165 \text{ eV}/c^2 \]

\[ \text{Large halo} + \text{out-of-equilibrium fermion ball} \]

\[ \text{or (CH)$_{out}$} \]

\[ \text{or SMBH} \]

\[ \text{Large halo + out-of-equilibrium fermion ball (CH)$_{out}$} \]

\[ \text{or (CH)$_{out}$} \]

\[ \text{or SMBH} \]

\[ \text{MW} \]

\[ \text{Large halo + out-of-equilibrium fermion ball} \]

\[ \text{or (CH)$_{out}$} \]

\[ \text{or SMBH} \]

\[ \text{Large halo + out-of-equilibrium fermion ball} \]

\[ \text{or (CH)$_{out}$} \]

\[ \text{or SMBH} \]

\[ \text{MW} \]

\[ \text{Large halo + out-of-equilibrium fermion ball} \]

\[ \text{or (CH)$_{out}$} \]

\[ \text{or SMBH} \]

\[ \text{MW} \]

\[ \text{Large halo + out-of-equilibrium fermion ball} \]

\[ \text{or (CH)$_{out}$} \]

\[ \text{or SMBH} \]
quantum bulge with respect to the gravothermal catastrophe. The mass \( (M_h)_{\text{MCP}} \approx 1.08 \times 10^{10} M_\odot \) also determines the moment at which the behavior of the core mass - halo mass relation changes. On the other hand, the mass \( M_{OV} = 2.30 \times 10^{13} M_\odot \) determines the moment at which the core of the DM halo may collapse towards a SMBH. For \( M_{OV} < M_h < M'_s \) the DM halo may either harbor a fermion ball or a SMBH. For \( M_h > M'_s \) the DM halo can only harbor a SMBH.

For \( m = 165 \text{ eV/c}^2 \), the value of \( M_{OV} = 2.30 \times 10^{13} M_\odot \) is very large and may not be astrophysically relevant. The value of \( M_{OV} \) is reduced if the fermion mass is larger. For \( m \sim 1 \text{ keV/c}^2 \) we find \( M_{OV} = 6.26 \times 10^{11} M_\odot \) but this value is still too large. It is comparable to the mass the whole Milky Way instead of being comparable to the mass of Sgr A\(^*\). Therefore, the fermionic DM model with a mass \( m = 165 \text{ eV/c}^2 \) or \( m \sim 1 \text{ keV/c}^2 \) cannot account for the presence of a supermassive compact object (either a SMBH or a fermion ball) of mass \( M_\star \approx 4.2 \times 10^6 M_\odot \) and radius \( R_c < 6 \times 10^{-4} \text{ pc} \) at the center of the Milky Way. It rather predicts the existence of a large fermion ball (bulge) of mass \( M_\star = 9.45 \times 10^{10} M_\odot \) and radius \( R_c = 240 \text{ pc} \) (Model II), or no fermion ball at all (Model I). In that case, in order to account for the observation, we have to generalize the fermionic DM model by introducing a primordial SMBH (Sgr A\(^*\)) at the center of the Milky Way.

We note that the fermionic model developed in the present paper is based on the same ideas as those developed in Ref. [25] for bosonic DM halos. The general scenario is the same (the fermion ball replacing the soliton in the BEC model) but the \( M_\star(M_h) \) relation and the values of the characteristic masses and radii are different. Therefore, a detailed comparison between the two models may help determining whether DM is made of fermions or bosons.

### D. Results for \( m = 48 \text{ keV/c}^2 \)

We have also considered the possibility suggested by other authors [57, 65] that the fermion ball may mimic a SMBH at the center of the galaxies. This scenario requires a larger particle mass \( m = 48 \text{ keV/c}^2 \) (see footnote 42).

We have first considered the case of the usual Fermi-Dirac DF. We have shown that our simple semi-analytical box model, leading to the relation from Eq. (111), reproduces the numerical results of B"{u}l"{u}c et al. [33] and Ruffini et al. [57]. However, the size of the fermion ball is too large to satisfy the observational constraints corresponding to Sgr A\(^*\). On the other hand, in line with our previous claims [25, 33, 44], we showed that the core-halo solution in these models is thermodynamically unstable. Therefore, it cannot result from a process of violent relaxation.

We then mentioned the recent results of Arg"{u}elles et al. [63, 64] based on the fermionic King model. In a first work, Arg"{u}elles et al. [58] obtained a density profile with a core-halo structure that satisfies the observational constraints of Sgr A\(^*\). In a second work, Arg"{u}elles et al. [60] showed that this solution is thermodynamically stable in the microcanonical ensemble so that it is likely to result from a process of violent relaxation. However, it is not clear if the process of violent relaxation can lead to a core-halo solution with such a high value of the central density because of the problem of incomplete relaxation [35, 69]. The purely gaseous solution (without quantum core) whether stable or metastable may be reached more easily. This issue can be settled only with direct numerical simulations.

According to the work of Arg"{u}elles et al. [61], medium size galaxies like the Milky Way may harbor a fermion ball mimicking a SMBH of mass \( M_\star \approx 4.2 \times 10^6 M_\odot \) and radius radius \( R_c = 6 \times 10^{-4} \text{ pc} \). This corresponds to a stable configuration (CH), located on the condensed branch of the caloric curve of the self-gravitating Fermi gas close to the last turning point of energy \( E^*_2 \) (see Fig. 22). Using the results of Alberti and Chavanis [44], we have argued that larger galaxies cannot harbor a fermion ball because, above a critical mass \( M_h > M'_s \), the condensed branch disappears and the system forms a SMBH of mass \( M_{OV} \sim 48 \text{ keV/c}^2 \), very large galaxies are likely to contain a SMBH of mass \( M_{OV} = 2.71 \times 10^8 M_\odot \) possibly accounting for AGNs. Medium size galaxies like the Milky Way may also follow the branch of condensed states up to the turning point of energy \( E^*_2 \) and undergo core collapse towards a SMBH. However the mass of the SMBH should be much smaller than \( M_{OV} = 2.71 \times 10^8 M_\odot \) in order to account for the characteristics of Sgr A\(^*\). This may be achieved with a larger fermion mass. For a fermion mass \( m = 386 \text{ keV/c}^2 \) the disappearance of the condensed branch and the collapse of the core of the system towards a SMBH already occur in galaxies like the Milky Way and lead to a SMBH of mass \( M_{OV} = 4.2 \times 10^6 \text{ keV/c}^2 \) similar to Sgr A\(^*\).

We also mentioned potential difficulties with the model of Arg"{u}elles et al. [60]. If the fermion mass is \( m = 48 \text{ keV/c}^2 \), the mass of the minimum halo (ground state) is \( (M_h)_{\text{min}} = 1.30 M_\odot \). Therefore, there should exist DM halos with a mass much below \( 10^8 M_\odot \), up to \( 1 M_\odot \). On the other hand, under the same conditions, DM halos of mass \( M_h = 10^8 M_\odot \) such as dSphs like Fornax should have a core-halo structure (see Figs. 25 and 26). More precisely, the fermionic DM model with a fermion mass \( m = 48 \text{ keV/c}^2 \) predicts that dSphs of mass \( M_h = 10^8 M_\odot \) should contain

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54 We have mentioned that this problem could be alleviated if the fermions are self-interacting.
a fermion ball of mass $M_c = 1.57 \times 10^4 M_\odot$ and radius $R_c = 5.42 \text{ mpc}$ possibly mimicking an intermediate mass BH (see Sec. VIII E). This is either a very important prediction (if confirmed by observations) or the evidence that this model is incorrect (if invalidated by observations). The main results of our study for $m = 48 \text{ keV}/c^2$ are summarized in the phase diagram of Fig. 29.

### E. The importance of the DM particle mass

There is a strong structural difference between the core-halo density profile corresponding to a large fermion mass $m = 48 \text{ keV}/c^2$ or a small fermion mass $m = 165 \text{ eV}/c^2$ when the fermionic model is applied to a DM halo of mass $M_h = 10^{11} M_\odot$ (Milky Way). In the first case, the degeneracy parameter $\mu$ is very large ($\mu = 3.09 \times 10^{14}$) and the core-halo profile presents a strong separation between the quantum core and the classical halo (see Fig. 19). They are separated by an extended plateau. Furthermore, the fermion ball has a mass $M_c = 4.2 \times 10^6 M_\odot$ and a radius $R_c = 6 \times 10^{-4} \text{ pc}$, mimicking a small SMBH (Sgr A*). In the second case, $\mu$ is relatively small ($\mu = 4.31 \times 10^4$) and the separation between the core and the halo is mild with no clear plateau between them (see Fig. 17). Furthermore, the fermion ball has a mass $M_c = 9.45 \times 10^9 M_\odot$ and a radius $R_c = 240 \text{ pc}$ mimicking a large quantum bulge, not a small SMBH. Therefore, depending on the DM particle mass, the fermionic model predicts very different types of structures. Comparison with observations of the Milky Way should determine which type of structure (a large quantum bulge or a small quantum core mimicking a SMBH) is the most relevant in the fermionic model. In this respect, we note that the BECDM model also leads to core-halo configurations in which the fermion ball is replaced by a soliton. For the commonly adopted boson mass $m \sim 10^{-22} \text{ eV}/c^2$ the core-halo profiles obtained in direct numerical simulations [10,21] do not show a very pronounced separation between a core and a halo (there is no extended plateau) and look similar to Fig. 17 rather than Fig. 19. In addition, the soliton mimics a large quantum bulge rather than a SMBH. Such a large quantum bulge seems to be necessary to account for the dispersion velocity peak observed in the Milky Way [105]. This may be a strong observational evidence for the presence of a large quantum bulge (bosonic or fermionic) at the center of the galaxies. Therefore, the comparison of the fermionic and bosonic models tends to favor a fermion mass of the order of $m = 165 \text{ eV}/c^2$ (or $1 \text{ keV}/c^2$) instead of $m = 48 \text{ keV}/c^2$. It would be interesting to consider BECDM models with a boson mass much larger than $m \sim 10^{-22} \text{ eV}/c^2$ to see if they can lead to a soliton mimicking a SMBH like in the model of Argüelles et al. [58,60]. Considering a noninteracting boson for simplicity, we find that its mass should be $m = 1.84 \times 10^{-18} \text{ eV}/c^2$ (see Sec. V.A of [39]). However, for such a large mass, BECDM is expected to behave like CDM and present a central cusp instead of a core as demonstrated by Mocz et al. [20] (see also the discussion in Sec. X). Similarly, for a large fermion mass $m \sim 50 \text{ keV}/c^2$ such as the one considered in the models of Argüelles et al. [58,60] DM should behave like CDM and may not be described by the Lynden-Bell DF as assumed by these authors. The Lynden-Bell DF may be valid only for a smaller fermion mass $m \sim 1 \text{ keV}/c^2$ where the cusps are prevented by the Pauli exclusion principle. However, these difficulties may disappear if DM is both quantum and self-interacting (see the discussion in Sec. X). In that case, the Fermi-Dirac DF may be justified by the self-interaction (collisions) of the fermions, not by a process of collisionless violent relaxation. The same remarks apply to the bosonic model: One should consider a repulsive self-interaction like in Ref. [25].

The comparison between the bosonic and fermionic models that we have initiated in this paper and in [15,25,35,37,38] may help determining the DM particle mass and whether it is a fermion or a boson. We would like to close this paper by suggesting that DM may be made of different types of particles (fermions and bosons) with different characteristics (mass, scattering length...). Some family of particles may be responsible for creating a large quantum bulge (fermion ball or soliton) [16,17,25,32] at the center of the galaxies which could explain the dispersion velocity peak observed in the Milky Way [105] while other family of particles...
may be responsible for creating a very compact object (fermion ball or soliton) at the very center of the galaxies mimicking a SMBH [58, 60, 65], or even leading to the formation of a real SMBH. If this suggestion is correct, it would give interest to all kinds of research made on quantum (fermionic and bosonic) DM and SIDM. If not, some physically interesting theoretical models may be ruled out by the observations.

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Appendix A: Approximate equations of state

Instead of using the exact equation of state for an ideal Fermi gas at finite temperature, Eqs. [8] and [9], we could consider the approximate equation of state

\[ P = \frac{1}{20} \left( \frac{3}{\pi} \right)^{2/3} \frac{h^2}{m^{2/3}} \frac{\rho^{5/3}}{\beta} + \frac{K_B T}{m}, \]  

(A1)

which is simply the sum of the polytropic equation of state [69] valid at high densities and the isothermal equation of state [43] valid at low densities.

Similarly, in the case of self-interacting BECDM halos in the TF limit, we have used in Ref. [25] an approximate equation of state of the form

\[ P = \frac{2\pi a_s h^2}{m^3} \rho^2 + \frac{K_B T}{m}, \]  

(A2)

where \( a_s \) is the scattering length of the bosons.

Finally, in the case of noninteracting BECDM halos, we have used in Refs. [35, 124] an approximate equation of state of the form

\[ P = \left( \frac{2\pi G M^2}{9 m^2} \right)^{1/2} \rho^{3/2} + \frac{K_B T}{m}, \]  

(A3)

where the first term mimics the quantum potential (see Appendix E of [37] for the justification of this equation of state).

These equations of state are of the generic form

\[ P = K \rho^\gamma + \frac{K_B T}{m}, \quad (\gamma = 1 + 1/n), \]  

(A4)

involving a polytropic equation of state and an isothermal (linear) equation of state. In the models discussed above, the polytropic index is \( n = 3/2, \) \( n = 1 \) and \( n = 2 \) respectively [37]. DM halos described by the mixed equation of state [A4] have been studied in [25]. They are governed by a generalized Lane-Emden equation introduced in Appendix E of [25]. The mixed equation of state [A4] has also been introduced and studied in a cosmological context (in the framework of general relativity) in Refs. [125, 127].

Appendix B: Entropy

Using the Gibbs-Duhem formula (see Eqs. (40), (47) and (58) of [29]), the entropy of the nonrelativistic self-gravitating Fermi gas is given by

\[ S = -\frac{\mu}{T} N + \frac{5E_{\text{kin}}}{3T} + \frac{2W}{T}. \]  

(B1)

Using the virial theorem from Eq. (32) and introducing the total energy \( E = E_{\text{kin}} + W \), we obtain

\[ \frac{S}{N k_B} = -\frac{\mu}{k_B T} + \frac{7E}{3N k_B T} - \frac{P_b V}{N k_B T}. \]  

(B2)

Using

\[ k = e^{-\beta \mu}, \]  

and

\[ \psi(\alpha) = \beta m (\Phi(R) - \Phi_0) = \beta m \left( -\frac{GM}{R} - \Phi_0 \right) = -\Delta - \beta m \Phi_0 \]  

(B4)

from Eq. (14), we get

\[ \beta \mu = -\ln k - \nu - \psi(\alpha). \]  

(B5)

Substituting Eq. (B5) into Eq. (B2) and introducing the dimensionless variables defined in Sec. III we finally obtain

\[ \frac{S}{N k_B} = \ln k + \eta + \psi_k(\alpha) - \frac{7}{3} \Delta \eta - \frac{2\alpha^6}{9 \mu^2} I_{3/2}(ke^{\psi_k(\alpha)}). \]  

(B6)

where \( \mu \) denotes the degeneracy parameter from Eq. (29) (we use the notation \( \bar{\mu} \) here to distinguish it from the chemical potential \( \mu \)). This returns in a more direct manner the result obtained in [40].

The entropy of the nonrelativistic self-gravitating Boltzmann gas is also given by Eq. (B1) (see footnote 55). Using Eq. (B1) and introducing the total energy \( E = E_{\text{kin}} + W \), we obtain

\[ S = -\frac{\mu}{T} N + \frac{2E}{T} - \frac{1}{2} N k_B. \]  

(B7)

55 The compact object at the center of the Galaxy (Sgr A* of mass \( M = 4.2 \times 10^6 M_\odot \)) could be a mixed structure made of a SMBH surrounded by a compact fermion or boson ball. In that case, the mass of the SMBH could be smaller than commonly thought (\( M_{\text{BH}} \leq 4.2 \times 10^6 M_\odot \)) since part of the mass of the compact object (Sgr A*) would be in the fermion or boson ball.

56 It is shown in [29] that this expression is valid for an arbitrary form of entropy.
On the other hand, applying Eq. (41) at \( r = R \) and using Eqs. (57)-(59) and \( \Phi(R) = -GM/R \) we find that

\[
\beta \mu = 2 \ln \alpha + \frac{1}{2} \ln \eta - \psi(\alpha) - \eta - \ln \mu + \ln 2 - \frac{1}{2} \ln \pi. \tag{B8}
\]

Substituting Eq. (B8) into Eq. (B7), we finally obtain

\[
\frac{S}{N \kappa_B} = -\frac{1}{2} \ln \eta - 2 \ln \alpha + \psi(\alpha) + \eta - 2 \Lambda \eta + \ln \mu + \frac{1}{2} \ln \pi - 2 - \frac{1}{2}. \tag{B9}
\]

**Appendix C: Basic equations and definitions**

For classical self-gravitating systems, or for quantum self-gravitating systems in the TF approximation (where the quantum potential can be neglected), the condition of hydrostatic equilibrium reads

\[
\nabla P + \rho \nabla \Phi = 0. \tag{C1}
\]

Combined with the Poisson equation

\[
\Delta \Phi = 4\pi G \rho, \tag{C2}
\]

we obtain the fundamental differential equation

\[
\nabla \cdot \left( \frac{\nabla P}{\rho} \right) = -4\pi G \rho. \tag{C3}
\]

This equation determines the density profile \( \rho(r) \) of a DM halo described by a barotropic equation of state \( P(\rho) \). The halo radius \( r_h \) is defined as the distance at which the central density \( \rho_0 \) is divided by 4:

\[
\frac{\rho(r_h)}{\rho_0} = \frac{1}{4}. \tag{C4}
\]

The mass \( M(r) \) contained within a sphere of radius \( r \) is given by

\[
M(r) = \int_0^r \rho(r')4\pi r'^2 \, dr'. \tag{C5}
\]

The halo mass is

\[
M_h = M(r_h). \tag{C6}
\]

The circular velocity is defined by

\[
v^2(r) = \frac{GM(r)}{r}. \tag{C7}
\]

The circular velocity at the halo radius is

\[
v_h^2 = v^2(r_h) = \frac{GM_h}{r_h}. \tag{C8}
\]

We note the identity

\[
\frac{v_h^2}{\frac{1}{2}G\rho_0 r_h^2} = \frac{M_h}{\rho_0 r_h^2}. \tag{C9}
\]

**Appendix D: Isothermal profile**

1. **Emden equation**

We consider a DM halo with an isothermal equation of state

\[
P = \rho \frac{k_B T}{m}, \tag{D1}
\]

where \( T \) is the temperature \[53\]. The fundamental differential equation of hydrostatic equilibrium \[C3\] takes the form

\[
\frac{k_B T}{m} \Delta \ln \rho = -4\pi G \rho. \tag{D2}
\]

Writing

\[
\rho = \rho_0 e^{-\psi}, \tag{D3}
\]

where \( \rho_0 \) is the central density, introducing the normalized radial distance

\[
\xi = r/r_0, \quad r_0 = \left( \frac{k_B T}{4\pi G \rho_0 m} \right)^{1/2}, \tag{D4}
\]

where \( r_0 \) is the thermal core radius, and assuming that the DM halo is spherically symmetric, we obtain the Emden equation \[53\]

\[
\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\psi}{d\xi} \right) = e^{-\psi} \tag{D5}
\]

with the boundary conditions

\[
\psi(0) = \psi'(0) = 0. \tag{D6}
\]

The density profile has the self-similar (homology) form \( \rho(r)/\rho_0 = e^{-\psi(r/r_0)} \). Using Eqs. (C5), (D3), (D4) and (D5), the mass contained within the sphere of radius \( r \) is given by

\[
M(r) = 4\pi \rho_0 \rho_0^{3/2} \xi^2 \psi'(\xi). \tag{D7}
\]

According to Eqs. (C7), (D4) and (D7), the circular velocity is

\[
v^2(r) = 4\pi G \rho_0 r_0^2 \xi \psi'(\xi). \tag{D8}
\]

Using Eq. (D4), we find that the temperature satisfies the relation

\[
\frac{k_B T}{m} = 4\pi G \rho_0 r_0^2. \tag{D9}
\]

Therefore, we can rewrite Eq. (D8) as

\[
\frac{mv^2(r)}{k_B T} = \xi \psi'(\xi). \tag{D10}
\]
Appendix E: Polytropic profiles

1. Lane-Emden equation

We consider a DM halo with a polytropic equation of state of the form

\[ P = K \rho^n, \]

where \( K \) is the polytropic constant and \( n = 1 + 1/n \) is the polytropic index [53]. The fundamental differential equation of hydrostatic equilibrium (E3) takes the form

\[ K(n + 1) \Delta \rho^{1/n} = -4\pi G \rho. \]

In the following, we restrict ourselves to spherically symmetric distributions. We also assume \( K > 0 \) and \( 6/5 < \gamma < +\infty \) (i.e. \( 0 \leq n < 5 \)) in order to have density profiles with a compact support (see below).

Writing

\[ \rho = \rho_0 \theta^n, \]

where \( \rho_0 \) is the central density, introducing the normalized radial distance

\[ \xi = r/r_0, \quad r_0 = \left[ \frac{K(n + 1)}{4\pi G \rho_0} \right]^{1/2}, \]

where \( r_0 \) is the polytropic core radius, and assuming that the DM halo is spherically symmetric, we obtain the Lane-Emden equation [53]

\[ \frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n \]

with the boundary conditions

\[ \theta(0) = 1, \quad \theta'(0) = 0. \]

The density profile has the self-similar (homology) form \( \rho(r)/\rho_0 = \theta^n(r/r_0) \). Using Eqs. (C5), (E3), (E4) and (E5), the mass contained within the sphere of radius \( r \) is given by

\[ M(r) = -4\pi \rho_0 r_0^3 \xi^2 \theta'(\xi). \]

According to Eqs. (C7), (E4) and (E7), the circular velocity is

\[ v^2(r) = -4\pi G \rho_0 r_0^2 \xi \theta'(\xi). \]

2. Halo mass and halo radius

When \( n < 5 \), the polytropes are self-confined (their density has a compact support) [53, 128]. We denote by \( \xi_1 \) the normalized radius at which the density vanishes: \( \theta_1 = \theta(\xi_1) = 0 \). Their radius \( R \) and their total mass \( M \) are given by

\[ R = \xi_1 r_0, \quad M = -4\pi \rho_0 r_0^3 \xi_1^2 \theta_1' \]

or, more explicitly, by

\[ R = \xi_1 \left[ \frac{K(n + 1)}{4\pi G} \right]^{1/2} \frac{1}{\rho_0^{(n-1)/2n}}, \]

\[ M = -4\pi \frac{\theta_1'}{\xi_1} \rho_0 R^3. \]

Eliminating the central density between these two equations, we obtain the mass-radius relation [53]

\[ M^{(n-1)/n} R^{(3-n)/n} = \frac{K(n + 1)}{G(4\pi)^{1/n}} \omega_n^{(n-1)/n}, \]

where \( \omega_n = -\xi_1^{(n+1)/(n-1)} \theta_1' \) is a constant determined by the Lane-Emden equation (E5). It can be shown that a polytrope of index \( n \) is dynamically stable with respect to the Euler-Poisson equations if \( n < 3 \) and linearly unstable if \( n > 3 \) [53]. On the other hand, the gravitational energy of a polytrope of index \( n \) is given by the Betti-Ritter formula [53]

\[ W = -\frac{3}{5 - n} \frac{GM^2}{R}. \]
where 

\[ \xi_1 = 3.65375, \quad \theta'_1 = -0.203032. \]  

(E14)

This polytrope represents a nonrelativistic fermion star at \( T = 0 \). This leads to Eqs. (72)-(74) quoted in the main text. The mass-radius relation may be written as

\[ MR^3 = \frac{9\omega_{3/2} h^6}{8192\pi^3 G^3 m^8}, \]  

(E15)

where \( \omega_{3/2} = 132.3843 \).

3. Halo mass and halo radius

The halo radius defined by Eq. (C4) is given by \( r_h = \xi_h r_0 \), where \( \xi_h \) is determined by the equation

\[ \theta(\xi_h)^n = \frac{1}{4}. \]  

(E16)

The value of \( \xi_h \) can be obtained by solving the Lane-Emden equation (E5) for a given value of \( n \). The normalized halo mass is

\[ \frac{M_h}{\rho_0 r_h^3} = -4\pi \theta'(\xi_h). \]  

(E17)

The normalized circular velocity at the halo radius is

\[ \frac{v_h^2}{4\pi G \rho_0 r_h^3} = -\frac{\theta'(\xi_h)}{\xi_h}. \]  

(E18)

The halo radius \( r_h \) and the halo mass may be written more explicitly as

\[ r_h = \xi_h \left[ \frac{K(n+1)}{4\pi G} \right]^{1/2} \frac{1}{\rho_0^{(n-1)/2n}}, \]  

(E19)

\[ M_h = -4\pi \frac{\theta'(\xi_h)}{\xi_h} \rho_0 r_h^3. \]  

(E20)

Eliminating the central density between Eqs. (E19) and (E20), we obtain the halo mass-radius relation

\[ M_h T_h^{(3-n)/(n-1)} = -4\pi \theta'(\xi_h) \xi_h^{(n+1)/(n-1)} \times \left[ K(n+1) \right]^{n/(n-1)} \frac{1}{4\pi G} \right]. \]  

(E21)

Let us assume that the minimum halo corresponds to a polytrope of index \( n \) (this includes the case of fermions corresponding to \( n = 3/2 \), the case of noninteracting bosons corresponding to \( n = 2 \) and the case of self-interacting bosons corresponding to \( n = 1 \)). Using Eqs. (E19) and (E20) and introducing the universal surface density of DM halos from Eq. (E5) we find that the minimum halo radius, the minimum halo mass, and the maximum halo central density are given by

\[ (r_h)_\text{min} = \xi_h^{2n/(n+1)} \left[ \frac{K(n+1)}{4\pi G} \right]^{n/(n+1)} \frac{1}{\Sigma_0^{(n-1)/(n+1)}}, \]  

(E22)

\[ (M_h)_\text{min} = -4\pi \theta'(\xi_h) \xi_h^{(3n-1)/(n+1)} \times \left[ K(n+1) \right]^{2n/(n+1)} \frac{1}{\Sigma_0^{(3-n)/(n+1)}}, \]  

(E23)

\[ (\rho_0)\text{max} = \frac{1}{\xi_h^{2n/(n+1)}} \left[ \frac{4\pi G}{K(n+1)} \right]^{n/(n+1)} \Sigma_0^{2n/(n+1)}. \]  

(E24)

For the polytrope \( n = 3/2 \), solving the Lane-Emden equation (E5) numerically, we find

\[ \xi_h = 2.27, \quad \theta'_h = -0.360. \]  

(E25)

If the minimum halo corresponds to a fermion ball at \( T = 0 \) (equivalent to a polytrope \( n = 3/2 \)) we obtain Eqs. (75)-(81) quoted in the main text.

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