Minor identities for Sklyanin determinants

Naihuan Jing and Jian Zhang

Abstract. We explore the invariant theory of quantum symmetric spaces of orthogonal and symplectic types by employing R-matrix techniques. Our focus involves establishing connections among the quantum determinant, Sklyanin determinants associated with the orthogonal and symplectic cases, and the quantum Pfaffians over the symplectic quantum space. Drawing inspiration from twisted Yangians, we not only demonstrate but also extend the applicability of q-Jacobi identities, q-Cayley’s complementary identities, q-Sylvester identities, and Muir’s theorem to Sklyanin minors in both orthogonal and symplectic types, along with q-Pfaffian analogs in the symplectic scenario. Furthermore, we present expressions for Sklyanin determinants and quantum Pfaffians in terms of quasideterminants.

1. Introduction

One of the central aspects of representation theory revolves around the general linear group and the symmetric group, establishing a close connection with classical invariant theory [32, 10, 8]. Representations of the general linear group can be mostly generalized to the other classical groups (the orthogonal and symplectic groups). All these classical groups and their representations have profound applications in various contexts, especially with several classical combinatorial identities such as Capelli identities, Sylvester identities, etc, for more background see [32]. For instance, the irreducible characters of the general linear group are recognized as the Schur symmetric polynomials indexed by the Young diagrams corresponding to the highest weights, similarly the irreducible characters of the orthogonal and symplectic groups are also realized as the orthogonal and symplectic Schur functions [20, 21], and many of the classical identities are expressed in terms of determinants, Pfaffians and the like.

MSC (2020): Primary: 20G42; Secondary: 15B33, 15A75, 58A17, 15A15, 13A50
Keywords: quantum groups, q-determinants, Sklyanin determinants, q-Pfaffians, q-minor identities, quantum symmetric spaces.

*Corresponding author.
The Yangian algebra $Y(\mathfrak{g})$ was introduced by Drinfeld \cite{4} to solve the quantum Yang-Baxter equation over the finite-dimensional simple Lie algebra $\mathfrak{g}$. It is known that the Yangian algebra $Y(\mathfrak{gl}_n)$ is closely related to the classical representation theory of the general linear group and invariant theory \cite{23}. The quantum determinant of $Y(\mathfrak{gl}_n)$, called the Yangian determinant in this paper, has enjoyed similar properties to the usual determinant, especially the Capelli identity associated with the Yangian determinant gives rise to a complete set of generators for the center $ZY(\mathfrak{gl}_n)$. This and other combinatorial properties of the Yangian determinant have been well studied in \cite{23}, where one can also see quite some classical identities have been generalized to the Yangian $Y(\mathfrak{gl}_n)$ and its determinant.

The twisted Yangians $Y^\pm(n)$ of Olshanski \cite{30} are certain subalgebras of $Y(\mathfrak{gl}_n)$ corresponding to the orthogonal Lie algebra $\mathfrak{o}_n$ and the symplectic Lie algebra $\mathfrak{sp}_n$, which also provide contexts for generalized combinatorial identities \cite{26} associated with their quantum determinants: the Sklyanian determinant. Again the coefficients of the Sklyanin determinant $sdet(Y^\pm(u))$ generate the center $ZY^\pm(n)$ \cite{23, 24}.

The quantum group $GL_q(n)$ was introduced by Faddeev-Reshetikhin-Tacktajan \cite{6} as a quadratic algebra generated by the $t_{ij} (1 \leq i, j \leq n)$ defined by the RTT relation for $T = (t_{ij})$ based on the trigonometric $R$-matrix, and the quantum determinant $det_q(T)$ also generates the center of $GL_q(n)$ (see also \cite{3, 1, 31, 19, 28}). In \cite{13} and \cite{12} we have studied the quantum symmetric spaces corresponding to the orthogonal and symplectic group as certain co-ideals of $GL_q(n)$, and we have shown that the quantum Pfaffian is a special central element in the quantum symmetric space of symplectic type. In \cite{27} Noumi studied spherical functions on the quantum symmetric spaces. He showed that the quantum determinant can be regarded as a quantum Pfaffian in the symplectic case and raised a question on how to represent the square of the quantum determinant $det(T)^2$ inside the quantum symmetric space of orthogonal type.

This paper aims to study a similar quantum invariant theory for the quantum symmetric spaces of orthogonal and symplectic types. In the first part of the paper, we formulate the quantum symmetric spaces using the $R$-matrix subject to certain reflective RTT equations and introduce the Sklyanin determinant in both cases. We show that the quantum symmetric spaces are characterized by the reflective RTT equations, which are very much analogous to twisted Yangian algebras \cite{23}. In particular, we introduce the Sklyanin determinant of the matrix $X$ and show that it generates the center of the quantum symmetric space of orthogonal type. In the symplectic case, the center is generated by quantum Pfaffian. Moreover, we will show that $det_q(T)^2$ is exactly the Sklyanin determinant $sdet_q(X)$ (up to a constant) as a special element in the quantum symmetric space $A_q(X_N)$, thus answering the aforementioned question of Noumi.

In the second part of the paper we give several identities for the quantum Sklyanin determinant $sdet_q(X)$ and the associated quantum Pfaffian.
The key identities can be expressed as minor identities for the quantum Sklyanin determinant, which correspond to the classical identities for the quantum determinant over the quantum general linear group. We will generalize the $q$-Jacobi identities, $q$-Cayley’s complementary identities, the $q$-Sylvester identities and Muir’s theorem to Sklyanin determinants both in the quantum orthogonal and quantum symplectic situations. In a sense, we have generalized several key identities for the general linear, orthogonal, and symplectic groups to their counterparts for the quantum general linear group and quantum symmetric spaces in the orthogonal and symplectic types.

In the case of quantum general linear semigroup $A_q(Mat_N)$, Krob and Leclerc [16] have expressed the quantum determinant $\det_q(T)$ as a product of successive commuting (principal) quasideterminants defined by Gelfand and Retakh [7]. At the end of the paper, we extend Krob-Leclerc’s formula to derive similar identities for the Sklyanin determinants and quantum Pfaffians in terms of quasideterminants associated with the generator matrix $X$ of the quantum symmetric spaces.

### 2. The quantum coordinate algebras

Let $R$ be the matrix in $\text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \simeq \text{End}(\mathbb{C}^N)^{\otimes 2}$:

\begin{equation}
R = q \sum_{i=1}^{N} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i > j} e_{ij} \otimes e_{ji},
\end{equation}

where $e_{ij}$ are the unit matrices in $\text{End}(\mathbb{C}^N)$. It is known that $R$ satisfies the well-known Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

where $R_{ij} \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N \otimes \mathbb{C}^N)$ acts on the $i$th and $j$th copies of $\mathbb{C}^N$ as $R$ does on $\mathbb{C}^N \otimes \mathbb{C}^N$.

Let $P$ be the permutation operator on $\mathbb{C}^N \otimes \mathbb{C}^N$ defined by $P(w \otimes v) = v \otimes w$, $w, v \in \mathbb{C}^N$. We define two $R$-matrices $R^\pm$ associated with $R$ by $R^+ = PRP$, $R^- = R^{-1}$, then

\begin{align*}
R^+ &= q \sum_{i} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i < j} e_{ij} \otimes e_{ji}, \\
R^- &= q^{-1} \sum_{i} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} - (q - q^{-1}) \sum_{i > j} e_{ij} \otimes e_{ji}.
\end{align*}

Viewing $q$ as a variable, let $\mathbb{C}(q)$ be the field of rational functions in $q$. Following [6], we introduce the quantum coordinate algebra $A_q(Mat_N)$ of generic matrices as the unital associative algebra generated by $t_{ij}$, $1 \leq i, j \leq N$ subject to the quadratic relations defined by the matrix equation

\begin{equation}
RT_1T_2 = T_2T_1R
\end{equation}

in $\text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \otimes M_q(N)$, where $T = (t_{ij})$, $T_1 = T \otimes I$, and $T_2 = I \otimes T$. 
The quadratic defining relations are explicitly written as follows:

\begin{align*}
(2.3) & \quad t_{ik} t_{il} = q t_{il} t_{ik}, \\
(2.4) & \quad t_{ik} t_{jk} = q t_{jk} t_{ik}, \\
(2.5) & \quad t_{il} t_{jk} = t_{jk} t_{il}, \\
(2.6) & \quad t_{ik} t_{jl} - t_{jl} t_{ik} = (q - q^{-1}) t_{il} t_{jk},
\end{align*}

where \( i < j \) and \( k < l \).

The algebra \( A_q(Mat_N) \) is a bialgebra under the comultiplication \( A_q(Mat_N) \to A_q(Mat_N) \otimes A_q(Mat_N) \) defined by

\[
\Delta(t_{ij}) = \sum_{k=1}^{N} t_{ik} \otimes t_{kj},
\]

and the counit given by \( \varepsilon(t_{ij}) = \delta_{ij} \). We will briefly write the coproduct as \( \Delta(T) = T \otimes T \).

Let \( I \) and \( J \) be two (ordered) subsets of \( \{1, 2, \cdots, N\} \) with identical cardinality \( r \): \( i_1 < i_2 < \cdots < i_r \in I \) and \( j_1 < j_2 < \cdots < j_r \in J \). The quantum \( r \)-minor is defined as

\[
\xi_{i_1, \cdots, i_r}^{j_1, \cdots, j_r} = \sum_{\sigma \in S_r} \left( -q \right)^{l(\sigma)} t_{i_1, j_{\sigma(1)}} \cdots t_{i_r, j_{\sigma(r)}}
\]

\[
= \sum_{\sigma \in S_r} \left( -q \right)^{l(\sigma)} t_{i_{\sigma(1)}, j_1} \cdots t_{i_{\sigma(r)}, j_r},
\]

where \( l(\sigma) = |\{(i, j) | i < j, \sigma_i > \sigma_j\}| \) is the inversion number of \( \sigma \). The second equality follows from relation (2.5). In particular, the quantum determinant of \( T \) is the \( n \)-minor

\[
\det_q(T) = \xi_{1, \cdots, N}^{1, \cdots, N}.
\]

The center of \( A_q(Mat_N) \) is generated by \( \det_q(T) \) and \( \Delta(\det_q(T)) = \det_q(T) \otimes \det_q(T) \) (cf. [15]). The coordinate ring \( GL_q(N) \) is defined by adjoining the inverse of the quantum determinant \( \det_q(T) \) to \( A_q(Mat_N) \). It has a Hopf algebra structure with the antipode \( S \) of \( GL_q(N) \) given by the anti-automorphism such that

\[
TS(T) = S(T)T = I,
\]

where \( S(T) = S(t_{ij})_{1 \leq i, j \leq N} \).

The quantum coordinate algebra \( A_q(X_N) \) of symmetric (resp. antisymmetric) matrices is defined as a noncommutative algebra generated by \( x_{ij}, 1 \leq i, j \leq N \) subject to the reflection relation (2.11) and symmetry relation (2.12) (resp. (2.13)):

\[
RX_1 R^t X_2 = X_2 R^t X_1 R,
\]
where \( X = (x_{ij})_{N \times N} \) and \( R^t = R^{t_1} \) denotes the partial transpose in the first tensor factor; plus the relations

\[(2.12) \quad \text{Case(O)} : \quad x_{ij} = q x_{ji}; \]
\[(2.13) \quad \text{Case(Sp)} : \quad x_{kk} = 0, \quad x_{ji} = -q x_{ij}. \]

where \( 1 \leq i < j \leq N, \ k \in \{1, \cdots , N\}. \) In the symplectic case, \( N \) is even. Using

\[(2.14) \quad R^{t_1} = q \sum_{1 \leq i \leq N} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i > j} e_{ji} \otimes e_{ji}. \]

the reflection relations are explicitly written as

\[(2.15) \quad q^{\delta_{jk}} x_{ik} x_{jl} - q^{\delta_{kl}} x_{jl} x_{ik} \]
\[= (q - q^{-1}) (\delta_{i<j} q^{\delta_{kl}} x_{ji} x_{lk} - \delta_{j<k} q^{\delta_{ij}} x_{ij} x_{kl}) \]
\[+ (q - q^{-1}) q^{\delta_{ik}} (\delta_{j<k} - \delta_{j<i}) x_{jk} x_{il} \]
\[+ (q - q^{-1})^2 (\delta_{j<k<l} - \delta_{j<i<k}) x_{ji} x_{kl} \]

where \( \delta_{i<j} \) or \( \delta_{i<j<k} \) equals 1 if the subindex inequality is satisfied and 0 otherwise. In the orthogonal case, the relations \[(2.11)\] and \[(2.12)\] can be written as:

\[(2.16) \quad x_{ij} = q x_{ji}, \quad i < j, \]
\[(2.17) \quad x_{ik} x_{jk} = q x_{jk} x_{ik}, \quad i < j < k, \]
\[(2.18) \quad x_{ik} x_{il} = q x_{il} x_{ik}, \quad i < k < l, \]
\[(2.19) \quad x_{ij} x_{jj} = q^2 x_{jj} x_{ij}, \quad i < j, \]
\[(2.20) \quad x_{ii} x_{ij} = q^2 x_{ij} x_{ii}, \quad i < j, \]
\[(2.21) \quad x_{ii} x_{jk} - x_{jk} x_{ii} = q^{-1} (q^2 - q^{-2}) x_{ij} x_{ik}, \quad i < j < k, \]
\[(2.22) \quad x_{ij} x_{kk} - x_{kk} x_{ij} = q^{-1} (q^2 - q^{-2}) x_{ik} x_{jk}, \quad i < j < k, \]
\[(2.23) \quad x_{ii} x_{jj} - x_{jj} x_{ii} = q^{-1} (q^2 - q^{-2}) x_{ij}^2, \quad i < j, \]
\[(2.24) \quad x_{ii} x_{jk} - x_{jk} x_{ii} = x_{jk} x_{il}, \quad i < j \leq k < l, \]
\[(2.25) \quad x_{ij} x_{jk} - q x_{jk} x_{ij} = q (q - q^{-1}) x_{jj} x_{ik}, \quad i < j < k, \]
\[(2.26) \quad x_{ik} x_{jl} - x_{jl} x_{ik} = (q - q^{-1}) x_{il} x_{jk}, \quad i < j < k < l, \]
\[(2.27) \quad x_{ij} x_{kl} - x_{kl} x_{ij} = (q - q^{-1}) (x_{ik} x_{jl} + q^{-1} x_{il} x_{jk}), \quad i < j < k < l. \]

Using relation \[(2.26),\] Eq. \[(2.27)\] can be rewritten as

\[(2.28) \quad x_{ij} x_{kl} - x_{kl} x_{ij} = q x_{ik} x_{jl} - q^{-1} x_{jl} x_{ik}, \quad i < j < k < l. \]
In the symplectic case, the relations (2.11) and (2.13) can be written as
(2.29) \( x_{ii} = 0 \),
(2.30) \( x_{ji} = -qx_{ij} \), \( i < j \),
(2.31) \( x_{ikx_{il}} = qx_{ijx_{ik}} \), \( k < l \),
(2.32) \( x_{ikx_{jk}} = qx_{jx_{ik}} \), \( i < j \),
(2.33) \( x_{il}x_{jk} = x_{jkx_{il}} \), \( i < j < k < l \),
(2.34) \( x_{ikx_{il}} - x_{jlx_{ik}} = (q - q^{-1})x_{ilx_{jk}} \), \( i < j < k < l \),
(2.35) \( x_{ijx_{kl}} - x_{klx_{ij}} = (q - q^{-1})(x_{ikx_{il}} - qx_{ilx_{jk}}) \), \( i < j < k < l \).

Using relation (2.34), Eq. (2.35) can be rewritten as
(2.36) \( x_{ijx_{kl}} - x_{klx_{ij}} = qx_{ilx_{jk}} - q^{-1}x_{ikx_{jl}} \), \( i < j < k < l \).

The following lemma follows from the explicit relations.

**Lemma 2.1.** The monomials
(2.37) \( C = x^{c_{11}x^{c_{12}} \cdots x^{c_{1N}}x^{c_{22}}x^{c_{23}} \cdots x^{c_{2N}} \cdots x^{c_{NN}}} \),
Case(O): \( C = x^{c_{11}x^{c_{12}} \cdots x^{c_{1N}}x^{c_{22}}x^{c_{23}} \cdots x^{c_{2N}} \cdots x^{c_{NN}}} \),
Case(Sp): \( C = x^{c_{12}x^{c_{13}} \cdots x^{c_{1N}}x^{c_{23}}x^{c_{24}} \cdots x^{c_{2N}} \cdots x^{c_{NN-1,N}}} \),

span the algebra \( A_q(X_N) \), where \( C = (c_{ij})_{1 \leq i,j \leq N} \) are (or strictly) upper triangular matrices with nonnegative integers.

We define the matrix \( J(a) \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \) by
(2.38) \( J(a) = \sum_{i=1}^{N} a_i e_{ii} \),
Case(O): \( J(a) = \sum_{i=1}^{N} a_i e_{ii} \),
Case(Sp): \( J(a) = \sum_{i=1}^{N} a_i (e_{2i-1,2i-1} - qe_{2i,2i-1}) \),

where \( a_i \in \mathbb{C} \) (1 \( \leq i \leq N \)) are all nonzero numbers, and \( n = N/2 \) in the symplectic case.

**Theorem 2.2.** The map \( X \mapsto TJ(a)T^t \) is a homomorphic embedding \( \phi : A_q(X_N) \rightarrow A_q(\text{Mat}_N) \).

**Proof.** Write \( TJ(a)T^t = X = (\tilde{x}_{ij}) \). In the symplectic case, \( \tilde{x}_{ii} = 0 \). \( \tilde{x}_{ij} = \sum_{k=1}^{N} \det_q(T^j_{2k-1,2k})a_k \) and \( \tilde{x}_{ji} = -q\tilde{x}_{ij} \) for \( i < j \). In the orthogonal case, \( \tilde{x}_{ij} = \sum_{k=1}^{N} a_k t_{ik} t_{jk} \) and \( \tilde{x}_{ij} = q\tilde{x}_{ij} \) for \( i < j \).

Further, we need to verify that
(2.39) \( RT_1J_1(a)T_1^tR^tT_2J_2(a)T_2^t = T_2J_2(a)T_2^tR^tT_1J_1(a)T_1^tR \).

The relations \( R^t_1t_2 = R^t_2t_1 \) and (2.2) imply that
(2.40) \( T_1^tR_1^tT_2 = T_2R_1^tT_1^t \),
(2.41) \( RT_1^tT_2^t = T_2^tT_1^tR \),
(2.42) \( T_1^tR_1^tT_2^t = T_2^tR_1^tT_1^t \).
By direct computation, one has that
\[ RJ_1(a)R^t J_2(a) = J_2(a)R^t J_1(a)R. \]
Consequently
\[ RT_1 J_1(a)T^t_1 R^t J_2(a)T^t_2 = T^t_2 T_1 R J_1(a)R^t J_2(a)T^t_1 T^t_2. \]
\[ = T^t_2 T_1 J_2(a)R^t J_1(a)R T^t_1 T^t_2. \]
\[ = T^t_2 J_2(a)T^t_2 R^t J_1(a)R^t T^t_1. \]
\[ (2.43) \]

This proves that \( \phi \) is an algebra homomorphism. We now check that the images of the monomials in Lemma 2.1 are linearly independent under \( \phi \).

Let \( A_q(\text{Mat}_N)^t \) be the \( \mathbb{C}[q, q^{-1}] \)-subalgebra of \( A_q(\text{Mat}_N) \) generated by the elements \( t_{i,j}, 1 \leq i, j \leq N \). Then there exists an isomorphism
\[ A_q(\text{Mat}_N)^t \otimes \mathbb{C}[q, q^{-1}] \mathbb{C} \cong A(\text{Mat}_N) \]
with the action of \( \mathbb{C}[q, q^{-1}] \) on \( A_q(\text{Mat}_N)^t \) via the evaluation \( q = 1 \). Suppose there is a nontrivial linear relation among ordered monomials in the \( x^C \):
\[ \sum_C a_C x^C = 0, \]
where \( a_C \in \mathbb{C}[q, q^{-1}] \). We can assume that at least one coefficient \( a_C \) does not vanish at \( q = 1 \). Take the image of \( \sum C x^C = 0 \) under \( \phi \), a nontrivial linear combination of the image of \( \phi(x^C) \) in \( A_q(\text{Mat}_N) \) equal to zero. This is a contradiction. \[ \square \]

The following proposition follows from the proof of the Theorem 2.2.

**Proposition 2.3.** The monomials
\[ x^C = x_{c11}^{11} x_{c12}^{12} \cdots x_{c1N}^{1N} x_{c22}^{22} x_{c23}^{23} \cdots x_{c2N}^{2N} \cdots x_{cNN}^{NN} \]
form a basis of the algebra \( A_q(X_N) \), where \( C = (c_{ij})_{1 \leq i,j \leq N} \) runs through (or strictly) upper triangular matrices with nonnegative integers.

Moreover, one has that \( \Delta(\tilde{x}_{ij}) \in A_q(\text{Mat}_N) \otimes A_q(X) \). Explicitly,
\[ Case(O): \Delta(\tilde{x}_{ij}) = \sum_{r,s=1}^{N} t_{ir} t_{js} \otimes \tilde{x}_{rs}, \]
\[ Case(Sp): \Delta(\tilde{x}_{ij}) = \sum_{r<s} \det_q(T^t_{rs}) \otimes \tilde{x}_{rs}. \]
\[ (2.48) \]

Therefore, the algebras \( A_q(X) \) can be regarded as the left coideal subalgebras of \( A_q(\text{Mat}_N) \) via the embedding. Similarly, the map \( X \mapsto T^t J(a)T \) also defines a homomorphic embedding, and the image of \( A_q(X_N) \) is a right coideal subalgebra of \( A_q(\text{Mat}_N) \).
We remark that the quantum coordinate algebras $A_q(X_N)$ were studied in [13, 12] as certain invariant subalgebras of $A_q(Mat_N)$ annihilated by $q$-differential operators and they are quadratic quantum algebras in the sense of Manin [22]. Their dual pictures are important examples of quantum symmetric pairs studied in general [18] associated to the quantum enveloping algebra [4, 11].

3. The Sklyanin determinant

We introduce the spectral-dependent $R$-matrix $R(\lambda) = \lambda R^+ - \lambda^{-1} R^-$, which satisfies the Yang-Baxter equation:

$$R_{12}(\lambda/\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda/\mu).$$

Let $\hat{R}(\lambda) = R(\lambda)P$, then the Yang-Baxter equation is equivalent to the braid relation

$$\hat{R}_{12}(\lambda/\mu)\hat{R}_{23}(\lambda)\hat{R}_{12}(\mu) = \hat{R}_{23}(\mu)\hat{R}_{12}(\lambda)\hat{R}_{12}(\lambda/\mu),$$

and relation (2.11) is then equivalent to

$$\hat{R}(\lambda)X_1R^tX_2 = X_1R^tX_2\hat{R}(\lambda).$$

In this paper, we use the $v$-based quantum number $[n]_v = 1+v+\cdots+v^{n-1}$ and the quantum factorial $[n]_v! = [1]_v[2]_v\cdots[n]_v$ for any natural number $n \in \mathbb{N}$. In particular, $[0]! = 1$. Let $A_r$ be the $q$-antisymmetrizer:

$$A_m = \frac{1}{[m]_q^2!} \sum_{c_1 < c_2 < \cdots < c_m, \sigma, \tau \in S_m} (-q)^{l(\sigma)+l(\tau)} e_{c_\sigma(1)c_\tau(1)} \otimes \cdots \otimes e_{c_\sigma(m)c_\tau(m)},$$

and it is an idempotent: $A_r^2 = A_r$. It follows from the Yang-Baxter equation that

$$A_2 = \frac{1}{q^2 - q^{-2}} \hat{R}(q^{-1}), \quad A_{m+1} = \frac{1}{q^{m+1} - q^{-m-1}} A_m \hat{R}_{m,m+1}(q^{-m})A_m.$$

For any permutation $i_1, i_2, \ldots, i_m$ of $1, 2, \ldots, m$ we denote

$$\langle X_{i_1}X_{i_2}\cdots X_{i_m} \rangle = X_{i_1}(R_{i_1i_2}^t \cdots R_{i_{m-1}i_m}^t)X_{i_2}(R_{i_2i_3}^t \cdots R_{i_{m-2}i_{m-1}}^t)\cdots X_{i_m}.$$

The following proposition follows from the reflection relation (3.3) and the variant form of the Yang-Baxter equation (3.1):

$$\hat{R}(\lambda)_{ij}R_{ik}^tR_{jk}^t = R_{ik}^tR_{jk}^t\hat{R}(\lambda)_{ij},$$

$$\hat{R}(\lambda)_{jk}R_{ij}^tR_{ik}^t = R_{ij}^tR_{ik}^t\hat{R}(\lambda)_{jk}.$$

Proposition 3.1. One has that

$$A_m(X_1, \ldots, X_m) = \langle X_1, \ldots, X_m \rangle A_m.$$

The element $[m]_q^2!A_m(X_1, \ldots, X_m)$ can be written as

$$\sum X_{j_1 \cdots j_m}^{i_1 \cdots i_m} e_{i_1j_1} \otimes \cdots \otimes e_{i_mj_m}.$$
where the sum is taken over all \( i_k, j_k \in \{1, 2, \cdots, N\} \). We call \( X^{i_1 \cdots i_m}_{j_1 \cdots j_m} \) the Sklyanin minor associated to the rows \( i_1 \cdots i_m \) and the columns \( j_1 \cdots j_m \).

Clearly if \( i_k = i_l \) or \( j_k = j_l \) for \( k \neq l \), then \( X^{i_1 \cdots i_m}_{j_1 \cdots j_m} = 0 \). Suppose that \( i_1 < i_2 < \cdots < i_m \) and \( j_1 < j_2 < \cdots < j_m \). Then the Sklyanin minors satisfy the relations:

\[
X^{i_{\sigma(1)} \cdots i_{\sigma(m)}}_{j_{\tau(1)} \cdots j_{\tau(m)}} = (-q)^{l(\sigma)+l(\tau)} X^{i_1 \cdots i_m}_{j_1 \cdots j_m}.
\]

Note that \( X_j^i = x_{ij} \). In particular, \( X^{1 \cdots N}_{1 \cdots N} \) is called the Sklyanin determinant and denoted by \( \text{sdet}_q(X) \). Sometimes we will simply write \( \text{sdet}(X) \) if the parameter \( q \) is clear from the context.

**Proposition 3.2.** Regarding \( A_q(X_N) \) as subalgebra of \( A_q(\text{Mat}_N) \) through the embedding \( \phi \) in Thm. 2.2, we have that

\[
\text{sdet}_q(X) = \gamma_{N,J(a)} \det_q(T)^2,
\]

where \( \gamma_{N,J(a)} = a_1 a_2 \cdots a_N \) for case (O) and \( q^{\text{dim}(a_1 a_2 \cdots a_n)} \) for case (Sp).

**Proof.** The image of \( A_N(X_1, \ldots, X_N) = \text{sdet}_q(X)A_N \) under \( \phi \) takes the following form

\[
A_N T_1 J_1(a) T_1^t (R_{12}^t \cdots R_{1N}^t) T_2 J_2(a) T_2^t (R_{23}^t \cdots R_{2N}^t) \cdots T_N J_N(a) T_N^t.
\]

Applying the relation \( T_i^t R_{ij} T_j = T_j R_{ij} T_i \), we get that

\[
\text{sdet}_q(X)A_N = A_N T_1 \cdots T_N \langle J_1(a), \ldots, J_N(a) \rangle T_1^t \cdots T_N^t.
\]

By relation (2.2), we have

\[
RT_1^t T_2^t = T_2^t T_1^t R.
\]

Therefore,

\[
A_N T_1^t \cdots T_N^t = \det_q(T)A_N,
\]

and

\[
\text{sdet}_q(X)A_N = (\det_q(T))^2 A_N \langle J_1(a), \ldots, J_N(a) \rangle.
\]

In the orthogonal case, it is easy to see that

\[
A_N \langle J_1(a), \ldots, J_N(a) \rangle = a_1 \cdots a_N A_N.
\]

Now we consider symplectic case. Let \( J = \sum_{i=1}^n e_{2i-1,2i} - q e_{2i,2i-1} \), and \( U = \sum_{i=1}^n u_i (e_{2i-1,2i-1} + e_{2i,2i}) \), where \( u_i^2 = a_i \). Then \( \det(U) = a_1 \cdots a_n \), \( J(a) = UJU^t \) and

\[
A_N \langle J_1(a), \ldots, J_N(a) \rangle = \det(U)^2 A_N \langle J_1, \ldots, J_N \rangle.
\]

Apply \( A_N \langle J_1, \ldots, J_N \rangle \) to the basis vector

\[
v = e_{2n-1} \otimes e_{2n-3} \otimes \cdots \otimes e_1 \otimes e_2 \otimes e_4 \otimes \cdots \otimes e_{2n},
\]

which can be written as

\[
A_N J_1(R_{12}^t \cdots R_{1N}^t) J_2(R_{23}^t \cdots R_{2N}^t) \cdots J_n(R_{n,n+1}^t \cdots R_{n,N}^t) w.
\]
where
\begin{equation}
(3.21) \quad w = e_{2n-1} \otimes e_{2n-3} \otimes \cdots \otimes e_1 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{2n-1}.
\end{equation}

Let $A'_{N-1}$ be the $q$-antisymmetrizer on the indices $\{i+1, \ldots, N\}$. Then (3.20) can also be rewritten as
\begin{equation}
A_N J_1 (R_{23}^t \cdots R_{1N}^t) A'_{N-1} J_2 (R_{23}^t \cdots R_{2N}^t) A'_{N-2} \cdots A'_{n+1} J_n (R_{n,n+1}^t \cdots R_{n,N}^t) w.
\end{equation}
Since
\begin{equation}
(3.22) \quad R_{ij} e_k \otimes e_k = q e_k \otimes e_k + (q - q^{-1}) \sum_{l<k} e_l \otimes e_l,
\end{equation}
we conclude that
\begin{equation}
(3.23) \quad A_N \langle J_1, \ldots, J_N \rangle v = (-q^2)^n A_N w'
\end{equation}
where
\begin{equation}
(3.24) \quad w' = e_{2n} \otimes e_{2n-2} \otimes \cdots \otimes e_2 \otimes e_1 \otimes e_3 \otimes \cdots \otimes e_{2n-1}.
\end{equation}
Since $A_N w' = (-q)^n A_N v$, we have that $A_N \langle J_1, \ldots, J_N \rangle v = q^{3n} A_N v$. Therefore,
\begin{equation}
(3.25) \quad \gamma_{N,J(a)} = q^{3n} (a_1 a_2 \cdots a_n)^2.
\end{equation}

We remarked that the proposition solved the question raised in [27, Rem. 4.12]. Since the quantum determinant $\det_q(T)$ can be regarded as a quantum Pfaffian [27], the proposition confirms that $\text{Pf}_q(X)^2 = \text{sdet}_q(X)$ up to a constant, which also solves the puzzle between $\det_q(T)$ and $\text{Pf}_q(X)$ [15].

**Corollary 3.3.** The Sklyanin determinant $\text{sdet}_q(X)$ belongs to the center of $A_q(X_N)$.

**Proof.** It follows from Theorem 2.2 and Theorem 3.2. \hfill \Box

**Proposition 3.4.** Let $I = \{i_1, \ldots, i_m\}$ and $J = \{j_1, \ldots, j_m\}$ be two subsets of $\{1, 2, \ldots, N\}$ such that $1 \leq i_1 < i_2 < \cdots < i_m \leq N$ and $1 \leq j_1 < j_2 < \cdots < j_m \leq N$. Suppose that $a, b \in I \cap J$, then
\begin{equation}
(3.26) \quad x_{ab} X^I_j = X^I_j x_{ab}.
\end{equation}

**Proof.** Let $A_m$ be the $q$-antisymmetrizer on the indices $\{1, \ldots, m\}$. It follows from the Yang-Baxter equation and the reflection relation that
\begin{equation}
(3.27) \quad R_{0m} \cdots R_{01} X_0 R_{01}^t \cdots R_{0m}^t A_m (X_1, \ldots, X_m) = A_m (X_1, \ldots, X_m) R_{01}^t \cdots R_{0m}^t X_0 R_{0m} \cdots R_{01}.
\end{equation}
By applying both sides to the vector $e_b \otimes e_{j_1} \otimes \cdots \otimes e_{j_m}$ and comparing the coefficients of $e_a \otimes e_{i_1} \otimes \cdots \otimes e_{i_m}$ we have that
\begin{equation}
(3.28) \quad x_{ab} X^I_j = X^I_j x_{ab}.
\end{equation}
\hfill \Box
We define the auxiliary minor $\tilde{X}_{j_1,\ldots,j_m}^{i_1,\ldots,i_m}$ by

$$
[m]_{q!}A_m(X_1,\ldots,X_{m-1})R^t_{1m} \cdots R^t_{m-1,m} = \sum \tilde{X}_{j_1,\ldots,j_m}^{i_1,\ldots,i_m} c_{i_1j_1} \cdots c_{i_mc},
$$

where the sum is taken over all $i_k, j_k, c \in \{1, 2, \ldots, N\}$.

If $i_k = i_l$ or $j_k = j_l$ for $k \neq l$, then $\tilde{X}_{j_1,\ldots,j_m}^{i_1,\ldots,i_m} = 0$. Suppose that $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_m$. Then

$$
\tilde{X}_{j_1,\ldots,j_m}^{i_{\sigma(1)},\ldots,i_{\sigma(m)}} = (-q)^{l(\sigma)+l(\tau)} \tilde{X}_{j_1,\ldots,j_{m-1}}^{i_1,\ldots,i_m},
$$

where $\sigma \in S_m, \tau \in S_{m-1}$.

**Proposition 3.5.** Suppose that $i_1 < i_2 < \cdots < i_m$ and $j_1 < j_2 < \cdots < j_{m-1}$. Then

$$
X_{j_1,\ldots,j_m}^{i_1,\ldots,i_m} = \sum_{c=1}^N X_{j_1,\ldots,j_{m-1},c}^{i_1,\ldots,i_m} c_{ejm}
$$

**Proof.** The identity follows from the formula

$$
A_m(X_1,\ldots,X_m) = A_m(X_1,\ldots,X_{m-1})R^t_{1m} \cdots R^t_{m-1,m}X_m.
$$

□

**Proposition 3.6.** Suppose that $i_1 < i_2 < \cdots < i_{m-1}, j_2 < \cdots, j_{m-1}, j_1 \in \{i_1, \ldots, i_m\}$ and $c \notin \{j_2, \ldots, j_{m-1}\}$. Then

$$
\tilde{X}_{j_1,\ldots,j_{m-1},c}^{i_1,\ldots,i_m} = 0, \quad \text{if } c \notin \{i_1, \ldots, i_m\},
$$

$$
\tilde{X}_{j_1,\ldots,j_{m-1},c}^{i_1,\ldots,i_m} = \pm(-q)^{2(l)} \sum_{r=1}^{m-1} (-q)^{r+1} x^t_{i_rj_1} X_{i_1,\ldots,i_r,\ldots,i_{m-1}}^{i_1,\ldots,i_m}
$$

if $c = i_m$, where $x^t_{ij}$ denote the $(i,j)$-th entry of the matrix $X^t$ and $l(I)$ is the number of inversions of $i_1, i_2, \ldots, i_m$.

**Proof.** As $c \notin \{j_2, \ldots, j_{m-1}\}$, we have that

$$
R^t_{2m} \cdots R^t_{m-1,m} e_{j_2} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c = e_{j_2} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c
$$

Now let’s compute

$$
A_m(X_1,\ldots,X_{m-1})R^t_{1m} \cdots R^t_{m-1,m} e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c = A_m X^t_{12} \cdots X^t_{1m}(X_2,\ldots,X_{m-1}) e_{j_1} \otimes \cdots \otimes e_{j_{m-1}} \otimes e_c
$$

Let $A'_{m-1}$ be the $q$-antisymmetrizer numbered by indices $\{2, \ldots, m\}$.

Then $A_m = A_mA'_{m-1}$, and we have the following relation:

$$
A'_{m-1} R^t_{12} \cdots R^t_{1m} = R^t_{12} \cdots R^t_{1m} A'_{m-1},
$$
which follows from the variant form of the Yang-Baxter equation (3.7). Then (3.35) is continued to

\begin{equation}
A_m X_1 R^t_{12} \cdots R^t_{i,m} A'_m (X_2, \ldots, X_{m-1}) e_1 \otimes \cdots \otimes e_{j-1} \otimes e_c
\end{equation}

where the sum runs through \( k_2 < \cdots < k_{m-1} \). Now let us divide it into the following cases:

(i) If \( c \notin \{i_1, \ldots, i_m\} \), then \( c \neq j_1 \) and

\begin{equation}
R^t_{1m} e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c = e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c,
\end{equation}

So the basis vectors \( e_{r_1} \otimes \cdots \otimes e_{r_m} \) in the expansion of (3.37) only contain those with \( c \in \{r_1, \ldots, r_m\} \), thus \( X^{i_1 \cdots i_m}_{j_1 \cdots j_{m-1}, c} = 0 \).

(ii) If \( c = j_1 = i_m \), then

\begin{equation}
A'_m e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c = 0
\end{equation}

if \( k_r = j_1 \) for some \( 2 \leq r \leq m - 1 \). Assume that \( i_p < j_1 < i_{p+1} \), then the coefficient of \( e_{i_1} \otimes \cdots \otimes e_{i_m} \) is

\begin{equation}
-(-q)^{2(l-1)} \sum_{r=1}^{m-1} (-q)^r x^{i_1 \cdots i_m}_{j_1 \cdots j_{m-1}}
\end{equation}

\begin{equation}
+(-q)^{2(l-1)(q-q^{-1})} \sum_{r=1}^{p} (-q)^r x^{i_1 \cdots i_m}_{j_1 \cdots j_{m-1}}
\end{equation}

(iii) Suppose \( c = i_m \) and \( j_1 = i_p \) for some \( 1 \leq p \leq m - 1 \). If \( j_1 \notin \{k_2, \ldots, k_{m-1}\} \), then

\begin{equation}
A_m X_1 R^t_{12} \cdots R^t_{i,m} X^k_{j_2 \cdots j_{m-1}} e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c
\end{equation}

\begin{equation}
= A_m \sum_{k_1} x^{k_1 j_1}_{k_2 \cdots k_{m-1}} e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c
\end{equation}

If \( j_1 = k_r \) for \( 2 \leq r \leq m - 1 \), then

\begin{equation}
A_m X_1 R^t_{12} \cdots R^t_{i,m} e_{j_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c
\end{equation}

\begin{equation}
= (-q)^{2-r} A_m X_1 R^t_{12} \cdots R^t_{i,m} e_{j_1} \otimes e_{k_r} \otimes e_{k_2} \cdots e_{k_{m-1}} \otimes e_c
\end{equation}

\begin{equation}
= A_m \left( \sum_{k_1} q x^{k_1 j_1}_{k_2 \cdots k_{m-1}} e_{k_1} \otimes e_{k_2} \otimes \cdots \otimes e_{k_{m-1}} \otimes e_c \right)
\end{equation}

\begin{equation}
+(-q)^{2-r}(q-q^{-1}) \sum_{k_1, j < j_1} x^{k_1 j}_{k_1 j_1} e_{k_1} \otimes e_{j_1} \otimes e_{k_r} \cdots e_{k_{m-1}} \otimes e_c \right).
The coefficient of $e_{i_1} \otimes \cdots \otimes e_{i_m}$ in the expansion of (3.37) is
\begin{equation}
(-q)^{p-1}(-q)^{2l} x_{i_{p',j_1}} x_{j_{2}}, \ldots, j_{m-1} \\
- (-q)^{2l} \sum_{r \neq p} (-q)^{r} x_{i_{r,j}}, x_{j_{2}}, \ldots, j_{m-1} \\
+ (q - q^{-1}) (-q)^{2l} \sum_{r = 1}^{p-1} (-q)^{r} x_{j_{i}, i}, x_{j_{2}}, \ldots, j_{m-1}.
\end{equation}
(3.43)

In the cases (ii) and (iii), both coefficients of $e_{i_1} \otimes \cdots \otimes e_{i_m}$ can be written as
\begin{equation}
\pm (-q)^{2l} \sum_{r = 1}^{m-1} (-q)^{r+i} x_{i_{r,j}, i}, x_{j_{2}}, \ldots, j_{m-1}.
\end{equation}
(3.44)

\[\square\]

In order to produce an explicit formula for the Sklyanin determinant in the orthogonal case we introduce a map
\[\pi_N : S_N \rightarrow S_N, \ p \mapsto p'\]
which was used in the formula for the Sklyanin determinant for the twisted Yangians \[23, 25\]. The map $\pi_N$ is defined inductively as follows. Given a set of positive integers $\omega_1 < \cdots < \omega_N$, regard $S_N$ as the symmetric group of these indices. If $N = 2$ we define $\pi_2$ as the identity map of $S_2 \rightarrow S_2$. For $N > 2$ define a map from the set of ordered pairs $(\omega_k, \omega_l)$ with $k \neq l$ into itself by the rule
\begin{equation}
(\omega_k, \omega_l) \mapsto (\omega_l, \omega_k), \ k, l < N, \ \\
(\omega_k, \omega_N) \mapsto (\omega_{N-1}, \omega_k), \ k < N - 1, \ \\
(\omega_N, \omega_k) \mapsto (\omega_k, \omega_{N-1}), \ k < N - 1, \ \\
(\omega_{N-1}, \omega_N) \mapsto (\omega_{N-1}, \omega_{N-2}), \ \\
(\omega_N, \omega_{N-1}) \mapsto (\omega_{N-1}, \omega_{N-2}).
\end{equation}
(3.45)

Let $p = (p_1, \ldots, p_N)$ be a permutation of the indices $\omega_1, \ldots, \omega_N$. Its image under the map $\pi_N$ is the permutation $p' = (p'_1, \ldots, p'_{N-1}, \omega_N)$, where the pair $(p'_1, p'_{N-1})$ is the image of the ordered pair $(p_1, p_N)$ under the map $\pi_N$. The pair $(p'_2, p'_{N-2})$ is found as the image of $(p_2, p_{N-1})$ under the map $\pi_N$ which is defined on the set of ordered pairs of elements obtained from $(\omega_1, \ldots, \omega_N)$ by deleting $p_1$ and $p_N$. The procedure is completed in the same manner by determining consecutively the pairs $(p'_i, p'_{N-i})$.

**Theorem 3.7.** The Sklyanin determinant $\text{sdet}_q(X)$ can be written explicitly as
\begin{equation}
\text{sdet}_q(X) = \gamma_N \sum_{p \in S_N} (-q)^{l(p)} x_{p_1}^{t} x_{p_2}^{t} \cdots x_{p_{N-1}}^{t} x_{p_N}^{t} p_1 p_2 \cdots x_{p_{N-1}} p_N.
\end{equation}
(3.46)
where $x_{ij}^t = x_{ji}$ and

$$
\gamma_N = \begin{cases} 
1 & \text{Case (O)}, \\
(-1)^n q^{2n} & \text{Case (Sp)}.
\end{cases}
$$

**Proof.** For $i_1 < i_2 \cdots < i_m$, we can write

$$
X_{i_1, \ldots, i_m}^{i_1, \ldots, i_m} = \sum_{k=1}^m X_{i_1, \ldots, i_{m-1}, i_k}^{i_1, \ldots, i_m} x_{i_k j_m}.
$$

Applying Proposition 3.6, we get that

$$
X_{i_1, \ldots, i_m}^{i_1, \ldots, i_m} = \gamma_2 x_{i_1, \ldots, i_m}^{i_1, \ldots, i_{m-1}} X_{i_1, \ldots, i_m}^{i_1, \ldots, i_{m-2}} x_{i_m j_m}
+ \gamma_2 \sum_{l=1}^{m-2} (-q)^{2l-2m+3} x_{i_1, i_{m-1}}^{i_1, i_l} X_{i_1, \ldots, i_{m-2}, i_l}^{i_1, \ldots, i_m} x_{i_l j_m}
+ \gamma_2 \sum_{l=1}^{m-2} \sum_{k=1}^{m-k} (-q)^{2l-2k+2} x_{i_1, i_k}^{i_1, i_l} X_{i_1, \ldots, i_{m-1}, i_l}^{i_1, \ldots, i_k} x_{i_k j_m}
+ \gamma_2 \sum_{l=1}^{m-1} \sum_{k=1}^{m-l} (-q)^{2l-2k} x_{i_1, i_k}^{i_1, i_l} X_{i_1, \ldots, i_{m-1}, i_l}^{i_1, \ldots, i_k} x_{i_k j_m}
+ \gamma_2 \sum_{l=1}^{m-1} (-q)^{2m-2k-1} x_{i_1, i_k}^{i_1, i_l} X_{i_1, \ldots, i_{m-1}, i_l}^{i_1, \ldots, i_k} x_{i_k j_m}
+ \gamma_2 \sum_{l=1}^{m-1} (-q)^{2l-2k} x_{i_1, i_k}^{i_1, i_l} X_{i_1, \ldots, i_{m-1}, i_l}^{i_1, \ldots, i_k} x_{i_k j_m}
$$

Starting with $X_{1, \ldots, N}^1$, we apply the recurrence relation repeatedly to write the Sklyanin determinant $sdet_q(X)$ in terms of the generator $x_{ij}$:

$$
sdet_q(X) = \gamma_N \sum_{p \in S_N} (-q)^{l(p)-l(p')} x_{p_1 p_1'}^t \cdots x_{p_n p_n'}^t \cdot x_{p_{n+1} p_{n+1}'} \cdots x_{p_{N-1} p_{N-1}'}^t
$$

The coefficient $\gamma_N$ is fixed by examining the leading term according to the two cases. \hfill \Box

The following theorem describes the center of $A_q(X_N)$ in the orthogonal case (c.f. Theorem 5.3).

**Theorem 3.8.** *In the orthogonal case, the center of algebra $A_q(X_N)$ is generated by $sdet_q(X)$ and isomorphic to the polynomial ring in one variable.*

**Proof.** It follows from Corollary 3.3 that $sdet_q(X)$ belongs to the center. We introduce a total order among basic vectors

$$
x^A = x_{11}^a_{11} x_{12}^a_{12} \cdots x_{1N}^a_{1N} x_{22}^a_{22} x_{23}^a_{23} \cdots x_{2N}^a_{2N} \cdots x_{N,N}^a_{N,N}, \quad A \in Mat_N(\mathbb{Z}_+),
$$

of $A_q(X_N)$ by comparing the associated sequences

$$
\sum_{1 \leq i < j \leq N} a_{ij}, a_{11}, a_{12}, \ldots, a_{1N}, a_{22}, \ldots, a_{N,N} \in \mathbb{N}^{N(N+1)/2+1}
$$
in the lexicographic order. Let \( p \) be the permutation

\[
(N - 1, N - 3, \cdots, 4, 2, 1, 3, \cdots, N) \quad \text{if } N \text{ is odd},
\]

\[
(N - 1, N - 3, \cdots, 3, 1, 2, 4, \cdots, N) \quad \text{if } N \text{ is even}.
\]

Then the image \( p' \) under the image of \( \pi_N \) is \( p \). So the leading term of \( \text{sdet}_q(X) \) is \( x^J \) where \( J \) is the identity matrix and the leading term of \( (\text{sdet}_q(X))^m \) is \( x^{mJ} \). Let \( y \) be any element in the center of \( A_q(X_N) \) with leading term \( cx^A \), \( c \neq 0 \). Then \( yx_{ij} = x_{ij}y \) for any \( 1 \leq i \leq j \leq N \). In particular, we consider

\[
yx_{ii} \equiv q^{-2} \sum_{k<i} a_{ki} x^A + e_{ii} \mod \text{lower terms.}
\]

(3.53)

modulo lower terms. On the other hand,

\[
x_{ii}y \equiv q^{-2} \sum_{k>i} a_{ik} x^A + e_{ii} \mod \text{lower terms.}
\]

(3.54)

Then we have that

\[
\sum_{k<i} a_{ki} = \sum_{k>i} a_{ik}
\]

(3.55)

Taking \( i = 1 \), we obtain that \( \sum_{k>1} a_{ik} = 0 \). It implies that \( a_{1k} = 0 \) for \( k > 1 \).

For \( i = 2, \cdots, N \), by repeating the same argument we obtain that \( a_{ij} = 0 \) for any \( i < j \).

For \( i < j \), we have that

\[
yx_{ij} \equiv q^{-2a_{jj} - \sum_{k>j} a_{ik} - \sum_{i<k<j} a_{kj} x^A + e_{ij} + \text{lower terms},}
\]

(3.56)

\[
x_{ij}y \equiv q^{-2a_{ii} - \sum_{k<i} a_{kj} - \sum_{i<k<j} a_{ik} x^A + e_{ij} + \text{lower terms}}.
\]

Since \( a_{ij} = 0 \) for \( i < j \), we obtain that \( a_{ii} = a_{jj} \) for \( i < j \).

Thus \( y \equiv x^{mJ} \) for some \( m \). Let \( y' = y - c(\text{sdet}_q(X))^m \). Then \( y' \) also belongs to the center and its leading term is strictly lower than that of \( y \). By induction, we conclude that \( y' \) is a polynomial in \( \text{sdet}_q(X) \), so is \( y \). The powers of \( \text{sdet}_q(X)^m \) are linearly independent since they have linear independent leading terms. Therefore, the center of \( A_q(X_N) \) is isomorphic to the polynomial ring in one variable.

\[\square\]

4. Minor identities for Sklyanin determinants

In the following, we will work with the localizations of \( A_q(X_N) \) by \( \text{sdet}_q(X) \) and \( A_q(\text{Mat}_N) \) by \( \text{det}_q(T) \). In particular, we derive minor identities for the Sklyanin determinants. Let’s define the Sklyanin comatrix \( \hat{X} \) by

\[
\hat{X} X = \text{sdet}_q(X) I,
\]

(4.1)

so the entries of \( X^{-1} \) belong to \( A_q(X_N)[\text{sdet}_q(X)^{-1}] \).
Proposition 4.1. The matrix elements \( \hat{x}_{ij} \) are given by
\[ \hat{x}_{ij} = (-q)^{i-N} X_{1,i_1, \ldots, N,j} \]
Moreover,
\[ \hat{x}_{ii} = X_{1,i_1, \ldots, N} \]

Proof. Multiplying \( X_N^{-1} \) from the right of the formulas
\[ A_N\langle X_1, \ldots, X_N \rangle = A_N\langle X_1, \ldots, X_{N-1} \rangle R_{1N}^t \cdots R_{N-1,N}^t X_N = A_N sdet_q(X) \]
we get that
\[ A_N\langle X_1, \ldots, X_{N-1} \rangle R_{1N}^t \cdots R_{N-1,N}^t = A_N \hat{X}_N. \]
Applying both sides to the vector
\[ v_{ij} = e_1 \otimes \cdots \otimes e_i \otimes e_N \otimes e_j \]
and comparing the coefficients of \( e_1 \otimes \cdots \otimes e_N \) we get the first formula. Using
\[ R_{kN}^t v_{ii} = v_{ii} \text{ for } 1 \leq k \leq N - 1, \]
applying the operators to the vector \( v_{ii} \) we obtain the second formula. \( \Box \)

The matrix \( X^{-1} = sdet_q(X)^{-1} \hat{X} \) is neither a \( q \)-symmetric nor \( q^{-1} \)-symmetric (resp. antisymmetric) matrix in orthogonal (resp. symplectic) case.

Let \( Q \) be the \( N \times N \) diagonal matrix with \( q_{ii} = (-q)^i, 1 \leq i \leq N \) and \( Y = Q^{-1} X^{-1} Q \). The following result shows that \( Y \) is a \( q^{-1} \)-symmetric (antisymmetric) matrix in the orthogonal (resp. symplectic) case and satisfies the \( q^{-1} \)-reflection relation.

Proposition 4.2. The matrix \( Y \) satisfies the relation
\[ R^{-1} Y_1 (R^{-1})^t Y_2 = Y_2 (R^{-1})^t Y_1 R^{-1}. \]
and
\[ \text{Case(O): } y_{ij} = q^{-1} y_{ji} \ (1 \leq i < j \leq N), \]
\[ \text{Case(Sp): } y_{ii} = 0 \ (1 \leq i \leq N), y_{ij} = -q^{-1} y_{ji} \ (1 \leq i < j \leq N). \]

Proof. If follows from Eq. (2.11) that
\[ R^{-1} X_1^{-1} (R^t)^{-1} X_2^{-1} = X_2^{-1} (R^t)^{-1} X_1^{-1} R^{-1}. \]
Substituting \( QYQ^{-1} \) for \( X^{-1} \) we get that
\[ R^{-1} Q_1 Y_1 Q_1^{-1} (R^t)^{-1} Q_2 Y_2 Q_2^{-1} = Q_2 Y_2 Q_2^{-1} (R^t)^{-1} Q_1 Y_1 Q_1^{-1} R^{-1}. \]
Multiplying \( Q_1^{-1} Q_2 \) from the left and \( Q_2 Q_1 \) from the right of both sides and noting the relation
\[ RQ_1 Q_2 = Q_2 Q_1 R \]
we get that
\[(4.13) \quad R^{-1}Y_1Q_2^{-1}Q_1^{-1}(R^{t_1})^{-1}Q_2Q_1Y_2 = Y_2Q_1^{-1}Q_2^{-1}(R^{t_1})^{-1}Q_1Q_2Y_1R^{-1}.\]

Then relation (4.14) follows as \( Q \) satisfies the equation: \( Q_1^{-1}Q_2^{-1}(R^{t_1})^{-1}Q_1Q_2 = (R^{-1})^{t_1}Q_1. \)

Now let’s consider the second part. Let \( J(a) \in \text{End}(\mathbb{C}^N \otimes \mathbb{C}^N) \) be the matrix:
\[(4.14) \quad \text{Case(O)}: J(a) = J = \sum_{i=1}^N e_{ii},\]
\[(4.15) \quad \text{Case(Sp)}: J(a) = J = \sum_{i=1}^n (e_{2i-1,2i} - qe_{2i,2i-1}).\]

Regard \( A_q(X_N) \) as subalgebra of \( A_q(\text{Mat}_N) \) by
\[(4.16) \quad X = TJT^t.\]

Then \( Y = Q^{-1}(T^t)^{-1}J^{-1}T^{-1}Q \). The matrix \( T^{-1} \) satisfies the relation
\[(4.17) \quad R^{-1}T_1^{-1}T_2^{-1} = T_2^{-1}T_1^{-1}R^{-1}.\]

Denote the \( ij \)-th entry of \( T^{-1} \) by \( \hat{t}_{ij} \), then \( \hat{t}_{ij} = \det_q(T)^{-1}(-q)^{(i-j)\xi_1^{1,\cdots,i,\cdots,N}} \)
and \( ij \)-th entry of \( (T^t)^{-1} \) is given by \( det(T)^{-1}(-q)^{(i-j)\xi_1^{1,\cdots,i,\cdots,N}}. \) Therefore, we have that
\[(4.18) \quad (T^t)^{-1} = Q^2(T^{-1})^tQ^{-2}.\]

The matrix \( Y \) can be written as \( Q(T^{-1})^tQ^{-2}J^{-1}T^{-1}Q \).

In the orthogonal case,
\[(4.19) \quad y_{ij} = \sum_{k=1}^N (-q)^{i+j-2k}\hat{t}_{ki}\hat{t}_{kj}.\]

It is easy to see that \( y_{ij} = q^{-1}y_{ji} \) for \( i < j \).

In the symplectic case,
\[(4.20) \quad y_{ij} = \sum_{k=1}^n (-q)^{i+j-4k}(q\hat{t}_{2k-1,i}\hat{t}_{2k,j} - \hat{t}_{2k,i}\hat{t}_{2k-1,j}).\]

Therefore \( y_{ii} = 0 \) and \( y_{ji} = -q^{-1}y_{ij} \) for \( i < j \). \( \square \)

**Proposition 4.3.** Let \( A \) be the \( N \times N \) antidiagonal matrix with \( a_{i,N+1-i} = 1 \), \( 1 \leq i \leq N \). The map \( X \mapsto AYA \) defines an algebra automorphism \( \omega \) of the localization of \( A_q(X_N) \) by \( sdet_q(X) \).

**Proof.** Since the matrix \( Y \) satisfies the \( q^{-1} \)-relations \((2.29, 2.35)\), the matrix \( AYA \) satisfies the \( q \)-relations. Therefore, the map \( X \mapsto AYA \) defines
an algebra homomorphism. Next, we show that $\omega$ is an involution. Applying $\omega$ to the equation
\begin{equation}
XQA(AYA)A^{-1}Q^{-1} = I
\end{equation}
we get that
\begin{equation}
(AYA)QA\omega^2(X)A^{-1}Q^{-1} = I,
\end{equation}
which implies that
\begin{equation}
\omega^2(X) = (QA)^{-2}X(QA)^2.
\end{equation}
Since $(QA)^2 = (-q)^{N+1}I$, we conclude that $\omega^2(X) = X$.

The classical determinant holds significance in the realm of linear algebra and finds extensive applications in both mathematics and physics. Throughout its extensive history, numerous classical determinant identities have been discovered, often linked to prominent figures such as Cauchy, Jacobi, Muir, Sylvester, and others. For a review of these classical identities, the reader is referred to [2] [17], where the second reference also gave quasideterminant analogs. In the subsequent discussion, we extend these classical identities to encompass Sklyanin determinants and quantum Pfaffians.

The following theorem is the Sklyanin determinant analog of Jacobi’s theorem.

**Theorem 4.4.** Let $I = \{i_1 < i_2 < \cdots < i_k\}$ be a subset of $[1, N]$ and $I^c = \{i_{k+1} < \cdots < i_N\}$ the complement of $I$. Then
\begin{equation}
\text{sdet}_q(Y_{I^c}) = \text{sdet}_q(X_I)\text{sdet}_q(X)^{-1}.
\end{equation}

**Proof.** By the relation
\begin{equation}
A_N(X_1, \ldots, X_N) = \text{sdet}_q(X)A_N
\end{equation}
and the definition of $(X_1, \ldots, X_N)$ we have that
\begin{equation}
A_N(X_1, \ldots, X_k)\prod_{1 \leq i < j \leq N} R_{ij}^t
= \text{sdet}_q(X)A_NX_N^{-1}(R_{N-1,N}^t)^{-1}X_{N-1}^{-1}\cdots(R_{k+1,k+2}^t)^{-1}X_{k+1}^{-1}
\end{equation}
Since $RQ_1Q_2 = Q_2Q_1R$ and $Q_1^{-1}Q_2^{-1}(R^t)^{-1}Q_1Q_2 = (R^{-1})^t$ we get that
\begin{equation}
A_N(X_1, \ldots, X_k)\prod_{1 \leq i < j \leq N} R_{ij}^t
= \text{sdet}_q(X)A_NQ_{k+1} \cdots Q_NY_N(R_{N-1,N}^{-1})^tY_{N-1}^{-1}\cdots(R_{k+1,k+2}^{-1})^tY_{k+1}^{-1}
\end{equation}
Applying both sides to the vector $e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes e_{i_{N}} \otimes \cdots \otimes e_{i_{k+1}}$ and comparing the coefficient of $e_{i_1} \otimes \ldots \otimes e_{i_k} \otimes e_{i_{N}} \otimes \cdots \otimes e_{i_{k+1}}$ we obtain that
\begin{equation}
\text{sdet}_q(X_I) = \text{sdet}_q(X)\text{sdet}_q(Y_{I^c}).
\end{equation}
**Proposition 4.5.** For any $1 \leq a, b \leq k$, one has that

\[(4.29)\]

\[X_{b,k+1,\ldots,N}^{a,k+1,\ldots,N} = (-q)^{k-b} \text{sdet}_q(X) Y_{1,\ldots,a,\ldots,k,b}^1, \ldots, k\]

where $Y = Q^{-1}X^{-1}Q$.

**Proof.** The proof is similar to Jacobi’s theorem. Using the relation

\[(4.30)\]

\[A_N(X_1,\ldots,X_{k+1}) \prod_{1 \leq i \leq k,k+2 \leq j \leq N} R_{ij}^{-1} = \text{sdet}_q(X)A_NQ_{k+1} \cdots Q_NY_N(R_{N-1,N}^{-1})^{t} \cdots Y_{k+2} \cdot (R_{k+1,N})^{t} \cdots (R_{k+1,k+2})^{t} Q_{k+1}^{-1} \cdots Q_{N}^{-1}\]

Applying both sides to the vector $e_{k+1} \otimes \ldots e_N \otimes e_b \otimes e_k \otimes \cdots \otimes e_a \otimes \cdots e_1$ and comparing the coefficient of $e_1 \otimes e_2 \otimes \cdots \otimes e_N$ we obtain that

\[(4.31)\]

\[X_{b,k+1,\ldots,N}^{a,k+1,\ldots,N} = (-q)^{k-b} \text{sdet}_q(X) Y_{1,\ldots,a,\ldots,k,b}^1, \ldots, k\]

\[\Box\]

Using Jacobi’s theorem we obtain the following analog of Cayley’s complementary identity for the Sklyanin determinant.

**Theorem 4.6.** Suppose a minor identity for the Sklyanin determinant is given:

\[(4.32)\]

\[\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \text{sdet}_q(X_{I_{ij}}) = 0,\]

where $I_{ij}$'s are subsets of $[1, N]$ and $b_i \in \mathbb{C}(q)$. Then the following identities hold

\[(4.33)\]

\[\sum_{i=1}^{k} b_i' \prod_{j=1}^{m_i} \text{sdet}_q(X)^{-1} \text{sdet}_q(X_{I_{ij}}) = 0,\]

where $b_i'$ is obtained from $b_i$ by replacing $q$ by $q^{-1}$.

**Proof.** The matrix $Y$ satisfies the $q^{-1}$ relations. Applying the minor identity to $Y$ we get that

\[(4.34)\]

\[\sum_{i=1}^{k} b_i' \prod_{j=1}^{m_i} \text{sdet}_{q^{-1}}(Y_{I_{ij}}) = 0.\]

It follows from Theorem 4.5 that $\text{sdet}_{q^{-1}}(Y_{I_{ij}})$ can be replaced by $\text{sdet}_q(X)^{-1} \cdot \text{sdet}_q(X_{I_{ij}})$. This completes the proof.

\[\Box\]

The following theorem is an analog of Muir’s law for the Sklyanin determinant.
Theorem 4.7. Suppose a minor Sklyanin determinant identity is given:

\[
\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \text{sdet}_q(X_{I_{ij}}) = 0,
\]

where \(I_{ij}\)s are subsets of \(I = \{1, 2, \ldots, N\}\) and \(b_i \in \mathbb{C}(q)\). Let \(J\) be the set \(\{N, \ldots, N+M\}\). Then the following identities hold

\[
\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \text{sdet}_q(X_J)^{-1}\text{sdet}_q(X_{I_{ij} \cup J}) = 0.
\]

Proof. Applying Cayley’s complementary identity concerning the set \(I\), we get that

\[
\sum_{i=1}^{k} b_i' \prod_{j=1}^{m_i} \text{sdet}_q(X_I)^{-1}\text{sdet}_q(X_{I_{ij} \setminus I}) = 0,
\]

Applying Cayley’s complementary identity for the set \(I \cup J\), we obtain that

\[
\sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \text{sdet}_q(X_J)^{-1}\text{sdet}_q(X_{I_{ij} \cup J}) = 0.
\]

Let \(A\) be a \(N \times N\) matrix. For any subset \(I\) of \([1, N]\), we denote by \(A_I\) the principal submatrix of \(A\) with rows and columns indexed by the elements of \(I\). The following relation involving the determinants and the permanents were first established by Muir:

\[
\sum_{k=0}^{N} (-1)^k \sum_{I \subset [1, N], |I| = k} \det(A_I)\text{per}(A_{[1,N]\setminus I}) = 0.
\]

In the following, we give an analog of Muir’s identity for the Sklyanin determinant. Denote \((n)_q = \frac{q^n - q^{-n}}{q - q^{-1}}\). Let \(S_r\) be the \(q\)-symmetrizer:

\[
S_2 = \frac{1}{q^2 - q^{-2}} \hat{R}(q), \quad S_{m+1} = \frac{1}{q^{m+1} - q^{-m-1}} S_m \hat{R}_{m,m+1}(q^m) S_m.
\]

Theorem 4.8. One has that

\[
\sum_{r=0}^{k} (-1)^r \text{tr}_{1,\ldots,k} A'_k S'_{k-r}(X_1, \ldots, X_k) = 0,
\]

\[
\sum_{r=0}^{k} (-1)^r \text{tr}_{1,\ldots,k} A_r S'_{k-r}(X_1, \ldots, X_k) = 0,
\]

where \(A'_k\) and \(S'_{k-r}\) denote the antisymmetrizer and symmetrizer over the copies of \(\text{End}(\mathbb{C}^k)\) labeled by \(\{r+1, \ldots, k\}\).
Thus we have that
\[ tr_{1,\ldots,k}S_r A'_{k-r}(X_1,\ldots,X_k) \]
\[ = tr_{1,\ldots,k} \frac{(r)q(k-r+1)q}{(k)q} S_r A'_{k-r+1}(X_1,\ldots,X_k) \]
\[ + tr_{1,\ldots,k}S_r A'_{k-r+1} \frac{(k-r)q(r+1)q}{(k)q} S_{r+1} A'_{k-r}(X_1,\ldots,X_k). \]

The element \( \langle X_1,\ldots,X_k \rangle \) can be written as
\[ \langle X_1,\ldots,X_k \rangle \prod_{1 \leq i \leq r \atop r+1 \leq j \leq k} R_{ij} \langle X_{r+1},\ldots,X_k \rangle, \]
where the product is taken in the lexicographical order on the pairs \((i,j)\).

It follows from (3.7) that
\[ A'_{k-r} \prod_{1 \leq i \leq r \atop r+1 \leq j \leq k} R_{ij} = \prod_{1 \leq i \leq r \atop r+1 \leq j \leq k} R_{ij} A'_{k-r}, \]
\[ S_r \prod_{1 \leq i \leq r \atop r+1 \leq j \leq k} R_{ij} = \prod_{1 \leq i \leq r \atop r+1 \leq j \leq k} R_{ij} S_r. \]

Then
\[ A'_{k-r} \langle X_1,\ldots,X_k \rangle = \langle X_1,\ldots,X_k \rangle A'_{k-r}, \]
\[ S_r \langle X_1,\ldots,X_k \rangle = \langle X_1,\ldots,X_k \rangle S_r. \]

Thus we have that
\[ tr_{1,\ldots,k}S_r A'_{k-r+1}(X_1,\ldots,X_k) \]
\[ = \frac{1}{q^{k-r+1} - q^{r-k-1}} tr_{1,\ldots,k}S_r A'_{k-r} \tilde{R}_{r,r+1}(q^{r-k}) A'_{k-r}(X_1,\ldots,X_k) \]
\[ = \frac{1}{q^{k-r+1} - q^{r-k-1}} tr_{1,\ldots,k}S_r \tilde{R}_{r,r+1}(q^{r-k}) A'_{k-r}(X_1,\ldots,X_k). \]

Similarly,
\[ tr_{1,\ldots,k}S_r A'_{k-r-1}(X_1,\ldots,X_k) \]
\[ = \frac{1}{q^{r+1} - q^{-r+1}} tr_{1,\ldots,k}S_r \tilde{R}_{r,r+1}(q^r) S_r A'_{k-r}(X_1,\ldots,X_k) \]
\[ = \frac{1}{q^{r+1} - q^{-r+1}} tr_{1,\ldots,k}S_r \tilde{R}_{r,r+1}(q^r) A'_{k-r}(X_1,\ldots,X_k). \]

Using the equation \( R^+ - R^- = (q - q^{-1})P \), we have
\[ \frac{(r)q}{(k)q} \tilde{R}_{r,r+1}(q^{r-k}) + \frac{(k-r)q}{(k)q} \tilde{R}_{r,r+1}(q^r) = q - q^{-1} \]

These imply the equation (4.43). Therefore we have shown the first equation. The second equation can be proved by the same arguments. \( \square \)
The following is an analog of Sylvester’s theorem for the Sklyanin determinant.

**Theorem 4.9.** Let $I = \{1, \ldots, N\}$, $J = \{N + 1, \ldots, N + M\}$, where $N$ and $M$ are positive integers such that $N$ and $M$ are even in the symplectic case. Then the mapping $x_{ij} \mapsto X_{i,M+1}, \ldots, M+N$ defines an algebra morphism $A_q(X_N) \to A_q(X_{N+M})$. Denote $\tilde{x}_{ij}$ by the image of $x_{ij}$ and $\tilde{X} = (\tilde{x}_{ij})$.

Then

\[(4.52) \quad sdet_q(\tilde{X}) = sdet_q(X_J)^{N-1}sdet_q(X).\]

**Proof.** Let $X$ be the $(M+N) \times (M+N)$ matrix for $A_q(X_{N+M})$, we write $Y = Q^{-1}X^{-1}Q$ as

\[(4.53) \quad \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix},\]

where $Y_{11}$ is a $N \times N$ matrix and $Y_{22}$ is a $M \times M$ matrix and the inverse of $Y_{11}$ is $(sdet_{q^{-1}}Y_{11})^{-1}$. Let $Z = QY_{11}^{-1}Q^{-1}$, then $Z$ satisfies the $q$-reflection relation and the $q$-symmetry relation. It follows from Proposition 4.1 that the $(i, j)$-th entry of $\tilde{Y}_{11}$ is $(-q)^{N-i}Y_{1,i \ldots, N,j}$. By Proposition 4.5 we have that

\[(4.54) \quad \tilde{x}_{ij} = X_{i,M+1, \ldots, M+N}^{j,M+1, \ldots, M+N} = sdet_q(X)sdet_{q^{-1}}(Y_{11})z_{ij} \]

Since $sdet_q(X)$ and $sdet_{q^{-1}}(Y_{11})$ commute with $z_{ij}$ for any $1 \leq i, j \leq N$, $\tilde{X}$ satisfies the $q$-reflection relation and the $q$-symmetry relation. This proves the first statement.

By Jacobi’s theorem,

\[(4.55) \quad sdet_q(X)sdet_{q^{-1}}(Y_{11}) = sdet_q(X_J),\]

then $\tilde{x}_{ij} = sdet_q(X_J)z_{ij}$. Using the explicit formula for Sklyanin determinants, we have that

\[(4.56) \quad sdet_q(\tilde{X}) = sdet_q(X_J)^{N}sdet_q(Z)\]

It follows from Jacobi’s identity that

\[(4.57) \quad sdet_{q^{-1}}(Y_{11})sdet_q(Z) = 1.\]

This implies that

\[(4.58) \quad sdet_q(Z) = sdet_q(X_J)^{-1}sdet_q(X).\]

Therefore,

\[(4.59) \quad sdet_q(\tilde{X}) = sdet_q(X_J)^{N-1}sdet_q(X).\]
5. Quantum Pfaffians

A matrix $A$ is an $N \times N$ $q$-antisymmetric if $a_{ii} = 0$ and $a_{ij} = -qa_{ij}$, $i < j$. The quantum Pfaffian (or $q$-Pfaffian) of a $q$-antisymmetric matrix $A$ is defined by

$$\text{Pf}_q(A) = \frac{1}{(1 + q^2)^n} \sum_{\sigma \in S_{2n}} (-q)^{l(\sigma)} a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)} \cdots a_{\sigma(2n-1)}a_{\sigma(2n)}.$$

Let $I = \{i_1, i_2, \ldots, i_{2r}\}$ be a subset of $[1, 2n]$ with $i_1 < i_2 < \cdots < i_{2r}$. Denote the complement of $I$ by $I^c$. Denote by $A_I$ the matrix obtained from $A$ by picking up the rows and columns indexed by $I$. We denote the quantum Pfaffian of $A_I$ by $\text{Pf}_q(A_I) = [i_1, i_2, \ldots, i_{2r}]$.

Denote by $\Pi$ the set of 2-shuffles, consisting of all $\sigma$ in $S_{2n}$ such that $\sigma_{2k-1} < \sigma_{2k}$, $1 \leq k \leq n$ and $\sigma_1 < \sigma_3 < \cdots < \sigma_{2n-1}$.

**Proposition 5.1.** [13] If the $q$-antisymmetric matrix $A$ satisfies the condition:

$$a_{ij}a_{kl} + (-q)a_{ik}a_{jl} + (-q)^2a_{il}a_{jk} = a_{kl}a_{ij} + (-q)^{-1}a_{jl}a_{ik} + (-q)^{-2}a_{jk}a_{il},$$

where $i < j < k < l$, then the quantum Pfaffian can be computed by

$$\text{Pf}_q(A) = \sum_{\pi \in \Pi} (-q)^{l(\pi)} [i_1, j_1][i_2, j_2] \cdots [i_n, j_n]$$

$$= \sum_{j=2}^{2n} (-q)^{j-2} [1, j][2, 3, \cdots, j, \cdots, 2n].$$

It is easy to verify that in the symplectic case the matrix $X = (x_{ij})_{1 \leq i, j \leq N}$ is $q$-antisymmetric and satisfies the condition

$$x_{ij}x_{kl} + (-q)x_{ik}x_{jl} + (-q)^2x_{il}x_{jk} = x_{kl}x_{ij} + (-q)^{-1}x_{jl}x_{ik} + (-q)^{-2}x_{jk}x_{il}$$

for $i < j < k < l$. Thus for any sub-square matrix, these conditions are also met. Like in determinant, we introduce the cofactor $X_{ij}$ by $X_{ii} = 0$ and

$$X_{ij} = (-q)^{i-j}[1, \cdots, \hat{i}, \cdots, \hat{j}, \cdots, 2n], i < j$$

$$X_{ij} = (-q)^{i-j-1}[1, \cdots, \hat{j}, \cdots, \hat{i}, \cdots, 2n], i > j.$$ 

**Theorem 5.2.** The cofactors of the Pfaffian satisfy the orthogonality relations:

$$\sum_{j=1}^{2n} [i, j]X_{jk} = \delta_{ik}\text{Pf}_q(X),$$

$$\sum_{j=1}^{2n} X_{kj}[j, i] = \delta_{ik}\text{Pf}_q(X).$$
PROOF. Both identities are shown by induction on $n$ similarly, so we just check the first one. The case of $n = 1$ is trivial. Expanding $[2, 3, \cdots, j, \cdots, 2n]$ in the $q$-Laplace expansion (5.2) of the Pfaffian, we have that for any fixed $k$

$$\text{Pf}_q(X) = (-q)^{k-2}[1, k][2, 3, \cdots, \hat{k}, \cdots, 2n]$$

$$+ \sum_{1 < i < j < k} (-q)^{i+j-k-2} ([1, i][k, j] - q[1, j][k, i]) \text{Pf}_q(X_{1,i,j,k} \cdots)$$

$$+ \sum_{1 < i < k < j} (-q)^{i+j-k-3} ([1, i][k, j] - q[1, j][k, i]) \text{Pf}_q(X_{1,i,k,j} \cdots)$$

$$+ \sum_{1 < k < i < j} (-q)^{i+j-k-4} ([1, i][k, j] - q[1, j][k, i]) \text{Pf}_q(X_{1,k,i,j} \cdots)$$

where the sums are taken over all $i, j$ satisfying the corresponding conditions. By relations (2.29-2.35), the factor in front of $\text{Pf}_q(X_{1,i,j,k} \cdots)$ etc. can be expressed as follows.

$$[1, i][k, j] - q[1, j][k, i]$$

$$= \begin{cases} [k, j][1, i] - q^{-1}[k, i][1, j] - (q^2 - q^{-2})[k, 1][i, j], & 1 < i < j < k \\ [k, j][1, i] - (1 - q^{-2})[k, 1][i, j] - q^{-1}[k, i][1, j], & 1 < i < k < j \\ [k, j][1, i] - q^{-1}[k, i][1, j], & 1 < k < i < j \end{cases}$$

Denote the sum of all items with the first factor $[k, i]$ by $\alpha_i$. Then

$$\alpha_1 = (-q)^{k-3}[k, 1][2, 3, \cdots, \hat{k}, \cdots, 2n]$$

$$- \sum_{1 < i < j < k} (-q)^{i+j-k-2}(q^2 - q^{-2})[k, 1][i, j] \text{Pf}(X_{1,i,j,k} \cdots)$$

$$- \sum_{1 < i < k < j} (-q)^{i+j-k-3}(1 - q^{-2})[k, 1][i, j] \text{Pf}(X_{1,i,k,j} \cdots)$$

$$= (-q)^{k-3}[k, 1][2, 3, \cdots, \hat{k}, \cdots, 2n]$$

$$+ (q - q^{-1})[k, 1] \sum_{i=2}^{k-1} (-q)^{2i-k-2} \sum_{j \notin \{1, i, k\}} (-q)^{j-i+\beta_j} \text{Pf}(X_{1,i,k,j} \cdots)$$

where $\beta_j = 0$ for $2 \leq j \leq i - 1$, $\beta_j = -1$ for $i + 1 \leq j \leq k - 1$, and $\beta_j = -2$ for $k + 1 \leq j \leq 2n$.

By the induction hypothesis, the above can be simplified as follows.

$$\alpha_1 = (-q)^{k-3}[k, 1][2, 3, \cdots, \hat{k}, \cdots, 2n]$$

$$+ (q - q^{-1})[k, 1] \sum_{i=2}^{k-1} (-q)^{2i-k-2} \text{Pf}(X_{1,k} \cdots)$$

$$= (-q)^{1-k}[k, 1][2, 3, \cdots, \hat{k}, \cdots, 2n]$$

(5.7)
Similarly, we also have
\begin{equation}
\alpha_i = \begin{cases} 
(-q)^{i-k}[k,i][2,3,\ldots,k,\ldots,2n], & 2 \leq i \leq k - 1 \\
(-q)^{i-k-1}[k,i][2,3,\ldots,k,\ldots,2n], & k + 1 \leq i \leq 2n
\end{cases}
\end{equation}

Therefore,
\begin{equation}
\sum_{j=1}^{2n} [k,j]X_{jk} = Pf_{q}(X).
\end{equation}

If \( i \neq k \), \( X_{jk} \) can be expanded as
\begin{equation}
X_{jk} = \sum_{l \notin \{i,j,k\}} (-q)^{a_{jl}}[i,l]Pf_{q}(X_{\{i,j,k,l\}c}),
\end{equation}

where \( a_{ij} \in \mathbb{Z} \). Then we have that
\begin{equation}
\sum_{j=1}^{2n} [i,j]X_{jk} = \sum_{j=1}^{2n} \sum_{l \notin \{i,j,k\}} (-q)^{a_{jl}}[i,j][i,l]Pf_{q}(X_{\{i,j,k,l\}c}).
\end{equation}

Note that \( a_{ij} = a_{ji} + 1 \) for \( j < l \), therefore, \( \sum_{j=1}^{2n} [i,j]X_{jk} = 0 \).

THEOREM 5.3. The center of the algebra \( A_{q}(X_{N}) \) is generated by \( Pf_{q}(X) \) and is isomorphic to the polynomial ring in one variable.

PROOF. Let \( X^{*} = (X_{ij}) \), the Pfaffian analog of the adjoint matrix of \( X \). It follows from the orthogonality relations that
\begin{equation}
Pf_{q}(X)X = XX^{*}X = XPf_{q}(X),
\end{equation}

which implies that \( Pf_{q}(X) \) belongs to the center of \( A_{q}(X_{N}) \).

We now order the monomials \( x^{A} \) as follows. To each \( A \in Mat_{N}(\mathbb{Z}_{+}) \) we associate a sequence of integers
\begin{equation}
( \sum_{1 \leq i < j \leq N} a_{ij}, a_{12}, a_{13}, \ldots, a_{1N}, a_{23}, \ldots, a_{N-1,N} ) \in \mathbb{N}^{N(N-1)/2+1}
\end{equation}

and order \( x^{A}, A \in Mat_{N}(\mathbb{Z}_{+}) \) by the lexicographic order of these sequences. This order gives rise to a total order among the basic vectors of \( A_{q}(X_{N}) \):
\begin{equation}
x^{A} = x_{12}^{a_{12}}x_{13}^{a_{13}} \cdots x_{1N}^{a_{1N}}x_{23}^{a_{23}}x_{24}^{a_{24}} \cdots x_{2N}^{a_{2N}} \cdots x_{N-1,N}^{a_{N-1,N}}.
\end{equation}

The quantum Pfaffian \( Pf_{q}(X) \) has the leading term \( x^{J} \), where \( J = \sum_{k=1}^{n} e_{2i-1,2i} \), subsequently the leading term of \( (Pf_{q}(X))^{m} \) is \( x^{mJ} \). Let \( y \) be any element in the center of \( A_{q}(X_{N}) \) with the leading term \( cx^{A}, c \neq 0 \). Then \( yx_{ij} = x_{ij}y \) for any \( 1 \leq i < j \leq N \). Now
\begin{equation}
yx_{ij} \equiv q^{-(\sum_{k>j}(a_{jk}+a_{ik})+\sum_{i<k<j}a_{kj})}x^{A+e_{ij}}
\end{equation}

modulo lower terms. Similarly
\begin{equation}
x_{ij}y \equiv q^{-(\sum_{k<i}(a_{ki}+a_{kj})+\sum_{i<k<j}a_{ik})}x^{A+e_{ij}}
\end{equation}
modulo lower terms. Then we have that
\begin{equation}
(5.17) \quad \sum_{k>j} (a_{jk} + a_{ik}) + \sum_{i<k<j} a_{kj} = \sum_{k<i} (a_{ki} + a_{kj}) + \sum_{i<k<j} a_{ik}
\end{equation}
for any $1 \leq i < j \leq N$.

Taking $(i, j) = (1, 2)$, relation (5.17) implies that $a_{1k} = a_{1j} = 0$ for $j \geq 3$. Eventually, one gets that $a_{2k-1,j} = a_{2k,j} = 0$ for $j \geq 2k + 1$ by repeating this argument. Taking $(i, j) = (2k - 1, 2k + 1)$, we get that $a_{2k+1,2k+2} = a_{2k-1,2k}$. Thus $y \equiv x^m$ for some $m$. Let $y' = y - c(Pf_q(X))^m$. Then $y'$ also belongs to the center with the leading term strictly lower than that of $y$. By induction concerning the order of the basis of $A_q(X_N)$, we conclude that $y$ is a polynomial in the variable $Pf_q(X)$. Powers of Pfaffian $(Pf_q)^m$ are linearly independent since they have linear independent leading terms. Therefore, the center of $A_q(X_N)$ is isomorphic to the polynomial ring in one variable. \hfill $\Box$

Let $\Lambda_N$ be the quantum exterior algebra $\mathbb{C}(y_1, \ldots, y_N)/I$, where $I$ is the ideal $(y_i^2, qy_iy_j + y_jy_i(i < j))$. For simplicity we still use the same symbol $y_i$ for the quotient $y_i + I$. We will simply write the element $x \otimes y$ as $xy$ or $yx$ for $x \in A_q(X_N)$ and $y \in \Lambda_N$.

Let $\Omega = \sum_{1 \leq i,j \leq N} x_{ij}y_{ij} \in A_q(X_N) \otimes \Lambda_N$, then
\begin{equation}
(5.18) \quad \Omega^n = (1 + q^2)^n [n]_q! Pf_q(X)y_1y_2 \cdots y_{2n}.
\end{equation}

**Proposition 5.4.** Under the homomorphic injection $\phi : A_q(X_N) \rightarrow A_q(\text{Mat}_N)$ in Theorem 4.2 we have that
\begin{equation*}
\phi(Pf_q(X)) = a_1a_2 \cdots a_n \text{det}_q(T).
\end{equation*}

**Proof.** Define the algebra homomorphism $\phi' : A_q(X_N) \otimes \Lambda_N \rightarrow A_q(\text{Mat}_N) \otimes \Lambda_N$ by $x \otimes y \mapsto \phi(x) \otimes y$. Denote $Y = (y_1, \ldots, y_n)^t$. Then $\Omega$ can be written as $Y^tXY$ and $\phi'(\Omega) = (Y^t)^tJ(a)(Y^t)Y$.

Let $\omega_i = \sum_{j=1}^N t_{ji} \otimes y_j$. Then
\begin{align*}
\omega_j &\omega_i = -q\omega_i \omega_j, \quad i < j, \\
\omega_i &\omega_i = 0.
\end{align*}

As $T^tX = (\omega_1, \ldots, \omega_N)^t$, one has that
\begin{equation*}
\phi'(\Omega)^n = (1 + q^2)^n [n]_q! Pf_q(J(a))\omega_1\omega_2 \cdots \omega_N
\end{equation*}
\begin{equation*}
= (1 + q^2)^n [n]_q! Pf_q(J(a))\text{det}_q(T)y_1y_2 \cdots y_N.
\end{equation*}

Therefore, $\phi(Pf_q(X)) = Pf_q(J(a))\text{det}_q(T) = a_1a_2 \cdots a_n \text{det}_q(T)$. \hfill $\Box$

**Theorem 5.5.** In the symplectic case, the Sklyanin determinant $s\det_q(X)$ is explicitly given by
\begin{equation}
(5.19) \quad s\det_q(X) = q^{3n} Pf_q(X)^2.
\end{equation}
This gives a formula for the square of Pfaffian:

\[
\Pf_q(X)^2 = (-q)^{-n} \sum_{p \in S_N} (-q)^{l(p')-l(p)} x^t_{p_1p_1'} \cdots x^t_{p_np_n'} x_{p_{n+1}p_{n+1}'} \cdots x_{p_Np_N'},
\]

where \( x^t_{ij} = x_{ji} \).

**Proof.** The result follows from \( \Pf_q(X) = a_1 a_2 \cdots a_n \det_q T \), Proposition 3.2 and Theorem 3.7. \( \square \)

### 6. Minor identities for quantum Pfaffians

For the quantum Pfaffian we have the following analog of Jacobi’s theorem.

**Theorem 6.1.** Let \( I = \{ i_1 < i_2 < \cdots < 2k \} \) be a subset of \([1, 2n]\), \( I^c = \{ i_{2k+1} < \cdots < i_{2n} \} \) be the complement of \( I \). Then

\[
\Pf_{q^{-1}}(Y_{I^c}) = \Pf_q(X_I) \Pf_q(X)^{-1}
\]

**Proof.** It follows from the Theorem 4.4 that

\[
\Pf_{q^{-1}}(Y_{I^c}) = \pm \Pf_q(X_I) \Pf_q(X)^{-1}
\]

Let \( J \) be the matrix with entries \( J_{ij}, 1 \leq i, j \leq 2n \) such that \( J_{i_{2k+1}i_{2k+1}} = 1, J_{i_{2k+1}i_{2k+1}} = -q \) all other entries 0. The mapping \( X \mapsto J \) defines an algebra homomorphism. For matrix \( J \) we have that

\[
\Pf_q(J) = \Pf_q(J_I) = \Pf_{q^{-1}}(J_{I^c}) = 1.
\]

Therefore, \( \Pf_{q^{-1}}(Y_{I^c}) = \Pf_q(X_I) \Pf_q(X)^{-1} \).

\( \square \)

Using the same arguments in the proofs of minor identities for Sklyanin determinants we obtain Theorems 6.2, 6.4 for the quantum Pfaffians.

**Theorem 6.2 (Cayley’s complementary identity for quantum Pfaffians).** Suppose a quantum minor Pfaffian identity is given:

\[
\sum_{i=1}^k b_i \prod_{j=1}^{m_i} \Pf_q(X_{I_{ij}}) = 0,
\]

where \( I_{ij} \)'s are subsets of \([1, 2n]\) with even cardinality and \( b_i \in \mathbb{C}(q) \). Then the following identity holds

\[
\sum_{i=1}^k b'_i \prod_{j=1}^{m_i} \Pf_q(X)^{-1} \Pf_q(X_{I_{ij}^c}) = 0,
\]

where \( b'_i \) is obtained from \( b_i \) by replacing \( q \) by \( q^{-1} \).
Theorem 6.3 (Muir’s law). Suppose given a quantum minor Pfaffian identity
\[ k \sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \text{Pf}_q(X_{I_{ij}}) = 0, \]
where \( I_{ij} \)s are subsets of \( I = \{1, 2, \ldots, 2n\} \) with even cardinality and \( b_i \in \mathbb{C}(q) \). Let \( J \) be the set \( \{2n+1, \ldots, 2n+2m\} \). Then the following identities holds
\[ k \sum_{i=1}^{k} b_i \prod_{j=1}^{m_i} \text{Pf}_q(X_J)^{-1}\text{Pf}_q(X_{I_{ij} \cup J}) = 0. \]

Theorem 6.4 (Sylvester-type Theorem). Let \( I = \{1, 2, \ldots, 2n\} \) and \( J = \{2n+1, \ldots, 2n+2m\} \) the mapping \( x_{ij} \mapsto \text{Pf}_q(X_{I \cup \{j\}}) \) defines an algebra morphism \( A_q(X_{2n}) \rightarrow A_q(X_{2n+2m}) \). Denote \( \tilde{x}_{ij} \) by the image of \( x_{ij} \) and \( \tilde{X} = (\tilde{x}_{ij}) \). Then
\[ \text{Pf}_q(\tilde{X}) = \text{Pf}_q(X_J)^{m-1}\text{Pf}_q(X_{I \cup J}). \]

The following is an analogue of the Grassmann-Plücker relation for the quantum Pfaffian.

Theorem 6.5. Let \( n \) and \( m \) be odd numbers, and \( I = \{1, 2, \ldots, n\} \), \( J = \{n+1, 2, \ldots, n+m\} \). Then the following relation holds.
\[ \sum_{j=1}^{n} (-q)^{n-j}\text{Pf}_q(X_{I \setminus \{j\}})\text{Pf}_q(X_{J \cup \{j\}}) = \sum_{j=n+1}^{n+m} (-q)^{j-n}\text{Pf}_q(X_{I \cup \{j\}})\text{Pf}_q(X_{J \setminus \{j\}}). \]

Proof. The element \( \text{Pf}_q(X_{J \cup \{j\}}) \) can be expanded as
\[ \sum_{k=n+1}^{n+m} (-q)^{k-n-1}x_{jk}\text{Pf}_q(X_{J \setminus \{k\}}) \]

The left-hand side of (6.9) can be written as
\[ \sum_{j=1}^{n} \sum_{k=n+1}^{n+m} (-q)^{k-j-1}\text{Pf}_q(X_{I \setminus \{j\}})x_{jk}\text{Pf}_q(X_{J \setminus \{k\}}) \]

Similarly, we expand \( \text{Pf}_q(X_{I \cup \{j\}}) \) on the right-hand side and get that
\[ \sum_{j=n+1}^{n+m} \sum_{l=1}^{n} (-q)^{j-l-1}\text{Pf}_q(X_{I \setminus \{l\}})x_{lj}\text{Pf}_q(X_{J \setminus \{j\}}). \]

This completes the proof. \( \Box \)
7. Quasideterminant, sdet$_q$ and Pf$_q$

The usual determinant can be written as a product of successive principal quasideterminants \[ sdet(q). \] As the quasideterminants can be defined for matrices over noncommutative rings, Krob and Leclerc \[ 16 \] found that the quantum determinant \( det_q(T) \) also obeys this type of identity (cf. \[ 5 \]). In this subsection, we prove that similar identities hold for the Sklyanin determinants and quantum Pfaffians.

Let’s recall the definition of quasideterminants \[ 7 \]. Suppose that matrix \( X \) (with possibly noncommutative entries) is invertible, \( Y = X^{-1} \), and \( y_{ji} \) (the \( ji \)-entry of \( Y \)) is invertible. The \( ij \)-th quasideterminant \( |X|_{ij} \) is the following element:

\[
|X|_{ij} = (y_{ji})^{-1}.
\]

For any \( I \subset [1, N] \), let \( X_I \) denote the submatrix with rows and columns indexed by \( I \). Then we have the following result.

**Theorem 7.1.** In the orthogonal case, the Sklyanin determinant can be expressed as the product of quasideterminants

\[
sdet_q(X) = x_{11} |X_{\{1,2\}}|_2 \cdots |X_{\{1,\ldots,N\}}|_N
\]

and the quasideterminants on the right-hand side commute with each other. More generally, for any permutation \( \sigma \in S_N \),

\[
sdet_q(X) = x_{\sigma_1,\sigma_1} |X_{\{\sigma_1,\sigma_2\}}|_{\sigma_2,\sigma_2} \cdots |X_{\{\sigma_1,\ldots,\sigma_N\}}|_{\sigma_N,\sigma_N}
\]

and the quasideterminants on the right-hand side commute with each other.

**Proof.** It follows from the generalized quantum Cramer relation \( \hat{X}X = sdet_q(X)I \) that

\[
\hat{X} = sdet_q(X)X^{-1}.
\]

Taking the \( \sigma_N\sigma_N \)-th entry, we get that

\[
\hat{X}_{\sigma_N\sigma_N} = sdet_q(X)(X^{-1})_{\sigma_N\sigma_N}.
\]

By Proposition \[ 4.1 \],

\[
\hat{X}_{\sigma_N\sigma_N} = X_{1,\ldots,\sigma_N,\ldots,N}^{1,\ldots,\sigma_N,\ldots,N}.
\]

Therefore,

\[
sdet_q(X) = X_{1,\ldots,\sigma_N,\ldots,N}^{1,\ldots,\sigma_N,\ldots,N} |X_{\{\sigma_1,\ldots,\sigma_N\}}|_{\sigma_N\sigma_N}.
\]

By induction on \( N \) we obtain that

\[
sdet_q(X) = x_{\sigma_1,\sigma_1} |X_{\{\sigma_1,\sigma_2\}}|_{\sigma_2,\sigma_2} \cdots |X_{\{\sigma_1,\ldots,\sigma_N\}}|_{\sigma_N\sigma_N}.
\]

The quasideterminant \( |X_{\{\sigma_1,\ldots,\sigma_N\}}|_{\sigma_N\sigma_N} \) commutes with all \( x_{ij} \) with \( i, j \neq \sigma_N \). Therefore it commutes with \( |X_{\{\sigma_1,\ldots,\sigma_k\}}|_{\sigma_k\sigma_k} \) for \( 1 \leq k \leq N - 1 \). By induction on \( N \), the quasideterminants in the right-hand side commute with each other. \( \square \)
For any permutation $\sigma \in S_N$ with $\sigma_{2k-1} < \sigma_{2k}$, we define $\theta_\sigma(k) = \#\{i|1 \leq i \leq 2k-2, \sigma_{2k-1} < \sigma_i < \sigma_{2k}\}$, and $\theta_\sigma = \sum_{k=1}^{n} \theta_\sigma(k)$. 

**Theorem 7.2.** In the symplectic case, the quantum Pfaffian can be expressed as a product of quasideterminants

\begin{equation}
Pf_q(X) = x_{12} |X_{\{1,2,3,4\}}|_3 \cdots |X_{\{1,\ldots,N\}}|_{N-1,N}
\end{equation}

and the quasideterminants in the right-hand side commute with each other. More generally, for any permutation $\sigma \in S_N$ with $\sigma_{2k-1} < \sigma_{2k}$

\begin{equation}
Pf_q(X) = (-q)^{\theta_\sigma} x_{\sigma_1 \sigma_2} |X_{\{\sigma_1,\ldots,\sigma_4\}}|_{\sigma_3 \sigma_4} \cdots |X_{\{\sigma_1,\ldots,\sigma_N\}}|_{\sigma_{N-1}\sigma_N}
\end{equation}

and the quasideterminants in the right-hand side commute with each other.

**Proof.** Recall the quantum Pfaffian orthogonality

\begin{equation}
X^*X = Pf_q(X)I,
\end{equation}

then we have

\begin{equation}
X^* = Pf_q(X)^{-1}.
\end{equation}

Taking the $\sigma_N \sigma_{N-1}$-th entry, we get that

\begin{equation}
X^*_{\sigma_N \sigma_{N-1}} = Pf_q(X)^{-1})_{\sigma_N \sigma_{N-1}}.
\end{equation}

By Theorem 5.2

\begin{equation}
X^*_{\sigma_N \sigma_{N-1}} = (-q)^{\sigma_N - \sigma_{N-1} - 1} Pf_q(X_{\{\ldots,\sigma_{N-1},\ldots,\sigma_N\}}).
\end{equation}

Therefore,

\begin{equation}
Pf_q(X) = (-q)^{\sigma_N - \sigma_{N-1} - 1} Pf_q(X_{\{\ldots,\sigma_{N-1},\ldots,\sigma_N\}}) |X_{\{\ldots,\sigma_N\}}|_{\sigma_{N-1}\sigma_N}.
\end{equation}

By induction on $N$ we obtain that

\begin{equation}
Pf_q(X) = (-q)^{\theta_\sigma} x_{\sigma_1 \sigma_2} |X_{\{\sigma_1,\ldots,\sigma_4\}}|_{\sigma_3 \sigma_4} \cdots |X_{\{\sigma_1,\ldots,\sigma_N\}}|_{\sigma_{N-1}\sigma_N}.
\end{equation}

The quasideterminant $|X_{\{\sigma_1,\ldots,\sigma_N\}}|_{\sigma_{N-1}\sigma_N}$ commute with all $x_{ij}$ with $i, j \notin \{\sigma_{N-1}\sigma_N\}$, then it commutes with $|X_{\{\sigma_1,\ldots,\sigma_k\}}|_{\sigma_{2k-1}\sigma_{2k}}$ for $\leq k \leq n - 1$. By induction on $N$, the quasideterminants in the right-hand side commute with each other. \qed

**Acknowledgments**

The work is supported in part by the National Natural Science Foundation of China grant nos. 12171303 and 12001218, the Humboldt Foundation, the Simons Foundation grant no. MP-TSM-00002518, and the Fundamental Research Funds for the Central Universities grant no. CCNU22QN002.
REFERENCES

[1] Ken A. Brown, Ken R. Goodearl, *Lectures on algebraic quantum groups*, Birkhäuser, 2012.
[2] Richard A. Brualdi, Hans Schneider, *Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir, and Cayley*, Lin. Alg. Appl. 52-53 (1983), 769-791.
[3] Richard Dipper, Stephen Donkin, *Quantum GL_n*, Proc. London Math. Soc. (3) 63 (1991), 165-211.
[4] V. G. Drinfeld, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR (in Russian). 283 (5) (1985), 1060-1064.
[5] Pavel Etingof, Vladimir Retakh, *Quantum determinants and quasideterminants*, Asian J. Math. 3 (1999), 345-351.
[6] L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtajan, *Quantization of Lie groups and Lie algebras*, Algebraic analysis, Vol. I, pp. 129-139, Academic Press, Boston, MA, 1988.
[7] Israel Gelfand, Vladimir Retakh, *Quasideterminants*, I, Selecta Math. (N.S.) 3 (1997), 517-546.
[8] Roe Goodman, Nolan Wallach, *Representations and invariants of the classical groups*. Encyclopedia of Mathematics and its Applications 68, Cambridge University Press, Cambridge, 1998.
[9] Mitsuyasu Hashimoto, Takahiro Hayashi, *Quantum multilinear algebra*, Tohoku Math. J. (2) 44 (1992), 471-521.
[10] Roger Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. 313 (1989), 539-570; Erratum, ibid. 318 (1990), 823.
[11] Michio Jimbo, *A q-analogue of U(gl(N+1)), Hecke algebra, the Yang-Baxter equation*, Lett. Math. Phys. 11 (1986), 247-252.
[12] Naihuan Jing, Rob Ray, *Zonal polynomials and quantum antisymmetric spaces*, Bull. Inst. Math. Acad. Sinica (N. S.) 7 (2012), 1-31.
[13] Naihuan Jing, Hirofumi Yamada, *Zonal polynomials on quantum general linear groups*, in Nankai Workshop on Quantum Groups ed. by M.L. Ge, World Scientific, 1995, pp. 62-76.
[14] Naihuan Jing, Jian Zhang, *Quantum Pfaffians and hyper-Pfaffians*, Adv. Math. 265 (2014), 336-361.
[15] Naihuan Jing, Jian Zhang, *Quantum permanents and Hafnians via Pfaffians*, Lett. Math. Phys. 106 (2016), 1451-1464.
[16] Daniel Krob, Bernard Leclerc, *Minor identities for quasi-determinants and quantum determinants*, Comm. Math. Phys. 169 (1995), 1-23.
[17] Bernard Leclerc, *On identities satisfied by minors of a matrix*, Adv. in Math. 100 (1993), 101-132.
[18] Gail Letzter, *Symmetric pairs for quantized enveloping algebras*, J. Algebra 220 (1999), 729-767.
[19] T. Levasseur, J. T. Stafford, *The quantum coordinate ring of the special linear group*, J. Pure Appl. Algebra 86 (1993), 181-186.
[20] Dudley E. Littlewood, *The theory of group characters and matrix representations of groups*, 2nd ed., Oxford University Press, London, 1950.
[21] I. G. Macdonald, *Symmetric functions and Hall polynomials*, 2nd. ed., Clarendon Press, Oxford, 1998.
[22] Yu. I. Manin, *Notes on quantum groups and quantum de Rham complexes*. Teoret. Mat. Fiz. 92 (1992), 425–450; English transl. in: Theoret. Math. Phys. 92 (1992), 997-1023.
[23] Alexander Molev, *Yangians and classical Lie algebras*, Amer. Math. Soc., Providence, RI, 2007.
[24] Alexander Molev, *Sugawara operators for classical Lie algebras*, Math. Surv. and Monograph, vol. 229, Amer. Math. Soc., Providence, RI, 2018.

[25] A. I. Molev, E. Ragoucy, P. Sorba, *Coideal subalgebras in quantum affine algebras*, Rev. Math. Phys. 15 (2003), 789-822.

[26] M. L. Nazarov, *Quantum Berezinian and the classical Capelli identity*, Lett. Math. Phys. 21 (1991), 123-131.

[27] Masatoshi Noumi, *Macdonald’s symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces*, Adv. Math. 123 (1996), 16-77.

[28] Masatoshi Noumi, Toru Umeda, Masato Wakayama, *A quantum analog of the Capelli identity and an elementary differential calculus on GL_q(n)*, Duke J. Math. 76 (1994), 567-594.

[29] M. Noumi, H. Yamada, and K. Mimachi, *Finite-dimensional representations of the quantum group GL_q(n; C) and the zonal spherical functions on U_q(n − 1)\backslash U_q(n)*, Japan. J. Math. (N.S.) 19 (1993), 31–80.

[30] G. I. Olshanski, *Twisted Yangians and infinite-dimensional classical Lie algebras*, in: Quantum Groups (Leningrad, 1990), eds. by P. P. Kulish, Lecture Notes in Math. 1510, Springer, Berlin-Heidelberg, 1992, pp. 104-119.

[31] N. Yu. Reshetikhin, L. A. Takhtadzhyan, L. D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra i Analiz 1 (1989), 178-206; transl. in Leningrad Math. J. 1 (1990), 193-225.

[32] Hermann Weyl, *The classical groups, their invariants and representations*, Princeton University Press, Princeton, 1946.

Department of Mathematics, North Carolina State University, Raleigh, NC 27695, USA

Email address: jing@ncsu.edu

School of Mathematics and Statistics, Central China Normal University, Wuhan, Hubei 430079, China

Email address: jzhang@ccnu.edu.cn