Mixing and Asymptotically Decoupling Properties in General Probabilistic Theory

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Abstract: We investigate the ergodic and mixing properties for dynamics in the framework of general probabilistic theory as well as the decoupling properties with an asymptotic setting, where the ergodicity is defined as the convergence of the long time average state and the mixing condition is defined as the combination of the ergodicity and aperiodicity. In this paper, we give relations between the mixing and asymptotically decoupling conditions. In addition, we give conditions equivalent to the mixing one. One of them is useful in determining whether a given dynamical map is mixing or not, by calculation because this condition results in a system of linear equations. We apply our result to the studies on the irreducibility and the primitivity.

1. Introduction

In dynamical system, as a fundamental property, we often focus on ergodicity, which means the convergence of the long time average state and has been often discussed in the context of statistical mechanics. The ergodicity does not exclude the periodicity. When the aperiodicity is imposed on a dynamical map in addition to the ergodicity, this condition is called the mixing condition. Although ergodicity and the mixing condition have been discussed in the context of relaxation to the thermal equilibrium, beyond the study of relaxation processes, these properties have found important applications in several fields of quantum information theory. In particular, in quantum control, quantum estimation, quantum communication and in the study of the efficient tensorial representation of critical many-body quantum systems. As another property, the decoupling property attracts an attention in open quantum systems and quantum information.

In this paper, we focus on the asymptotically decoupling property by considering the limiting behavior of a dynamical map in the framework of general proba-
bilistic theory including quantum theory. Further, to handle stochastic operation (including stochastic local operation and classical communication, SLOCC), we discuss a dynamical map without trace preserving condition. We show the equivalence of the asymptotically decoupling and mixing properties. Then, we derive their useful equivalent conditions as well as useful equivalent conditions for the ergodicity.

On the other hand, in the classical case, a dynamical map is described by a transition matrix. For a transition matrix, we often focus on the irreducibility instead of the ergodicity because it can be easily checked in the classical case [27,28]. Notice that the ergodicity is often defined with a different meaning [27,28]. In the quantum setting, there are many equivalent conditions for the irreducibility [23]. Due to one of the equivalent conditions, we find that the irreducibility is stronger than the ergodicity. In addition, when a dynamical map is irreducible and mixing, it is called primitive [23]. In this paper, we state these equivalent conditions for the irreducibility in a general dynamical map with the framework of general probabilistic theory. Although a primitive dynamical map has not been sufficiently studied, we also derive several equivalent conditions for the primitivity.

Finally, applying our result to the classical case, we derive a necessary and sufficient condition for the primitivity of a transition matrix, i.e., the convergence from an arbitrary initial distribution to the stationary distribution with full support. This necessary and sufficient condition is determined by the supports of the output distribution of each input.

The remaining part is organized as follows. Section 2 explains general probabilistic theory. Section 3 discusses the ergodicity in general probabilistic theory. Section 4 studies the asymptotically decoupling and mixing properties, and their equivalent conditions. Section 5 considers the irreducibility and the primitivity.

2. General probabilistic theory with closed convex cone

To discuss several properties of a dynamical map in a general setting, we adopt general probabilistic theory, which requires the following components. Let $V$ be a finite-dimensional real vector space and $V^*$ be the dual space of $V$. The dual space $V^*$ can be naturally identified with $V$ by using an inner product $\langle \cdot, \cdot \rangle$ on $V$.

A convex set $K$ of $V$ is called a convex cone (for short, cone) if any nonnegative number multiple of any vector $x \in K$ is contained in $K$. For any nonempty closed convex cone $K$ of $V$, the dual cone is defined as

$$K^* := \{ y \in V^* \mid \langle y, x \rangle \geq 0 \ (\forall x \in V) \}.$$  

It is well-known that $K^*$ is also a closed convex cone, and the relation $K^{**} = K$ holds [21,22]. When $K^* = K$, the cone $K$ is called self-dual. The norm on $V$ is defined as $\|x\| := \sqrt{\langle x, x \rangle}$ for any vector $x$, and $\partial X$ and $X^\circ$ denote the boundary and the interior of a set $X$, respectively.

Let $K$ be a closed convex cone of $V$ satisfying $K^\circ \neq \emptyset$ and $K \cap (-K) = \{0\}$. Once a unit element $u \in (K^*)^\circ$ is given, the set of all states is given as $S(K, u) := \{ x \in K \mid \langle u, x \rangle = 1 \}$. Any measurement is given as a decomposition $\{e_i\}$ of the unit element $u$, that is, $e_i \in K^*$ and $\sum_i e_i = u$, where $i$ corresponds to an outcome. When a state is given as $x$ and a measurement $\{e_i\}_i$ is performed, the probability to obtain a outcome $i$ is $\langle e_i, x \rangle$. Therefore, once the tuple $V, K, \langle \cdot, \cdot \rangle$,....
and $u$ is given, our general probabilistic theory is established. For two vectors $x, x' \in V$, we define $x \leq x'$ and $x < x'$ to be $x' - x \in K$ and $x' - x \in K^0$, respectively. We give two typical examples for general probabilistic theory.

**Example 1 (Classical probabilistic theory).** To recover the classical probabilistic theory with $d$ outcomes, we choose $V = \mathbb{R}^d$ and $K = [0, \infty)^d$. When the inner product $\langle \cdot, \cdot \rangle$ is chosen to be the standard inner product, $V^* = \mathbb{R}^d$ and $K^* = [0, \infty)^d$, and hence $K$ is self-dual. When we choose $u$ to be $[1, \ldots, 1]^T$, the set of all states equals the set of all probability vectors. Then, we obtain the classical probabilistic theory with $d$ outcomes. In this case, when a $d \times d$ matrix has only nonnegative entries, it maps $K$ to itself and is called a nonnegative matrix.

**Example 2 (Quantum system).** Next, we discuss the quantum system on a Hilbert space $H$. We choose $V$ and $V^*$ commonly as the set $T(H)$ of all Hermitian matrices. The closed convex cone $K$ is chosen as the set $T_+(H)$ of all positive semi-definite matrices, which has nonempty interior and satisfies $T_+(H) \cap (-T_+(H)) = \{0\}$. The inner product is chosen as $\langle y, x \rangle = \text{Tr}yx$ for any $x \in V$ and $y \in V^*$. Then, we find that the dual cone $K^*$ is also the set of all positive semi-definite matrices, i.e., $K$ is self-dual. Choosing $u$ to be the identity matrix, we find that the set of all states equals the set of all density matrices. Also, any measurement is given as a positive operator-valued measure (POVM).

Now, we return to the general probabilistic theory. We focus on two systems written as real vector spaces $V_1$ and $V_2$ with closed convex cones $K_1$ and $K_2$ and the unit elements $u_1$ and $u_2$, respectively. The joint system is given as the tensor product space $V_1 \otimes V_2$ with the natural inner product induced by the inner products of $V_1$ and $V_2$. When states $x_1 \in K_1$ and $x_2 \in K_2$ are prepared in the respective systems, it is natural that the state in the joint system is given as the tensor product element $x_1 \otimes x_2 \in V_1 \otimes V_2$. Since any convex combination of any tensor product elements can also be realized by randomization, we can realize any element of the tensor product cone $K_1 \otimes K_2$ of $V_1 \otimes V_2$ defined as

$$K_1 \otimes K_2 := \left\{ \sum_{i=1}^n x_1^i \otimes x_2^i \in V_1 \otimes V_2 \mid n \in \mathbb{N}, x_1^i \in K_1, x_2^i \in K_2 \ (1 \leq i \leq n) \right\}.$$  

Any two elements $\sum_{i=1}^m x_1^i \otimes x_2^i \in K_1 \otimes K_2$ and $\sum_{j=1}^n y_1^j \otimes y_2^j \in K_1 \otimes K_2$ satisfy the relation $\langle \sum_{i=1}^m x_1^i \otimes x_2^i, \sum_{j=1}^n y_1^j \otimes y_2^j \rangle = \sum_{i=1}^m \sum_{j=1}^n \langle x_1^i, y_1^j \rangle \langle x_2^i, y_2^j \rangle \geq 0$. Then, we have the following lemma.

**Lemma 1.** When $K_1$ and $K_2$ are closed convex cones, the inclusion relation

$$K_1^* \otimes K_2^* \subset (K_1 \otimes K_2)^*$$

holds.

However, even when $K_1$ and $K_2$ are self-dual, the convex cone $K_1 \otimes K_2$ is not necessarily self-dual. To see this fact, we discuss two quantum systems as follows.
Example 3. We consider two quantum systems on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$. Since the composite quantum system is given by the tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$, any state of the composite quantum system is given by an element of $T_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with normalization. Now, the tensor product cone $\text{Sep}(\mathcal{H}_1; \mathcal{H}_2) := T_+(\mathcal{H}_1) \otimes T_+(\mathcal{H}_2)$ has an interior point, satisfies $\text{Sep}(\mathcal{H}_1; \mathcal{H}_2) \cap (-\text{Sep}(\mathcal{H}_1; \mathcal{H}_2)) = \{0\}$, and is strictly contained in $T_+(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Then, we have

$$\text{Sep}(\mathcal{H}_1; \mathcal{H}_2) \subset T_+(\mathcal{H}_1 \otimes \mathcal{H}_2) \subset \text{Sep}(\mathcal{H}_1; \mathcal{H}_2)^*,$$

where the second relation follows by considering the dual cones in the first relation. Indeed, the set \{ $x \in \text{Sep}(\mathcal{H}_1; \mathcal{H}_2)$ | $\langle u_1 \otimes u_2, x \rangle = 1$ \} equals the set of all separable states, and is strictly contained in the set of all density matrices. Further, when $A_j$ is a positive semi-definite matrix on $\mathcal{H}_j$ for $j = 1, 2$, $A_1 \otimes A_2$ is contained in the interior of $\text{Sep}(\mathcal{H}_1; \mathcal{H}_2)$ if and only if both $A_1$ and $A_2$ are strictly positive definite (see Lemma 13 in Appendix).

3. $\mathcal{K}$-ergodicity

3.1. Dynamical map and spectral radius. We discuss a dynamical map on the set $S(\mathcal{K}, \nu)$ of all states on the general probability system given by the tuple $\mathcal{V}, \mathcal{K}, \langle \cdot, \cdot \rangle$, and $\nu$. Since the dynamical map needs to preserve the convex combination structure, it must be given as a linear map $A$ on $\mathcal{V}$, whose restriction to $S(\mathcal{K}, \nu)$ preserves the convex combination structure. In addition, the linear map $A$ must be imposed the following two conditions on.

Definition 1 ($\mathcal{K}$-positivity [11]). If a linear map $A$ on $\mathcal{V}$ satisfies $AK \subset K$, we call the map $A$ a $\mathcal{K}$-positive map.

Definition 2 (Dual unit-preserving). If a linear map $A$ on $\mathcal{V}$ satisfies $A^*u = u$, we call the map $A$ a dual unit-preserving (DUP) map, where $A^*$ denotes the adjoint map of $A$.

Although we need the two conditions to discuss dynamics of states, a $\mathcal{K}$-positive map also plays an important role in large deviation analysis [19] even when it does not satisfy the dual unit-preserving condition. Hence, we focus on a $\mathcal{K}$-positive map $A$. In the quantum case, the dual unit-preserving property is the trace preserving property. If we apply measurement operation and keep the output state only with a specific measurement outcome, the state evolution is given as a $\mathcal{K}$-positive map $A$ that does not necessarily satisfy the trace preserving property.

Let us start the following well-known propety.

Proposition 1. If $A$ is a $\mathcal{K}$-positive map, then $A^*$ is $\mathcal{K}^*$-positive.

Proof. Let $A$ be a $\mathcal{K}$-positive map, and $y \in \mathcal{K}^*$. Any vector $x \in \mathcal{K}$ satisfies $\langle A^*y, x \rangle = \langle y, Ax \rangle \geq 0$, whence $A^*y \in \mathcal{K}^*$.

The $\mathcal{K}$-positivity ensures a few good properties. To discuss further properties, we recall an existing result from the paper [11]. To state it, we focus on the
spectral radius $r(A)$ of a linear map $A$, which is defined as the maximum of the absolute values of all eigenvalues of $A$ \[^{[23]}\]. The spectral radius satisfies

$$r(A) = \limsup_{n \to \infty} \|A^n\|^{1/n},$$

where $\|A\|$ is the operator norm of $A$ based on the norm on $V$. Since two arbitrary norms on a finite-dimensional vector space are uniformly equivalent to each other, we can select any norm as the above one. Moreover, the relations $r(A_1 \otimes A_2) = r(A_1)r(A_2)$ and $r(A^*) = r(A)$ hold. As for the spectral radius, see the lecture notes \[^{[23]}\].

**Definition 3 (\[^{[11]}\] Definition 3.1).** If $\lambda$ is an eigenvalue of a linear map $A$, the degree of $\lambda$ is the size of the largest Jordan block associated with $\lambda$, in the Jordan canonical form of $A$.

For example, the following matrix

$$
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{bmatrix},
$$

which is already a Jordan canonical form, has the eigenvalues zero and one. The degree of zero is two, and that of one is one. Now, we have been ready to state the following proposition. As for the Jordan canonical form, see linear algebra textbooks \[^{[24]}^{[25]}^{[26]}\] or the lecture notes \[^{[23]}\].

**Proposition 2 (\[^{[11]}\] Theorem 3.1).** If $A$ is a $K$-positive map, then the following conditions hold.

- $r(A)$ is an eigenvalue of $A$.
- $K$ contains an eigenvector of $A$ associated with $r(A)$.
- The degree of $r(A)$ is not smaller than the degree of any other eigenvalue having the same absolute value.

When the eigenvalue $r(A)$ of $A$ has the algebraic multiplicity 1, the Jordan block associated with $r(A)$ is the matrix $[r(A)]$ alone. Hence, the above proposition implies that the Jordan block associated with any eigenvalue $\lambda$ whose absolute value is $r(A)$, equals the matrix $[\lambda]$. This argument is used in the proof of Theorem \[^{[1]}\].

### 3.2. $K$-ergodicity

We define the $K$-ergodicity as follows.

**Definition 4 ($K$-ergodicity).** A $K$-positive map $A$ is called $K$-ergodic when the following conditions hold.

- $r(A) > 0$.
- There exist two vectors $x_0 \in K$ and $y_0 \in K^*$ such that $\langle y_0, x_0 \rangle = 1$, and any vector $x$ satisfies

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (r(A)^{-1}A)^k x = \langle y_0, x \rangle x_0. \quad (1)$$
The vectors $x_0$ and $y_0$ are called a stationary vector and a dual stationary vector of $A$, respectively.

The pair of the stationary vector and the dual one is unique up to a positive number multiple, i.e., once we take a pair of a stationary vector $x_0$ and a dual one $y_0$, any pair of those is the pair of $\alpha x_0$ and $\alpha^{-1} y_0$, where $\alpha$ is an arbitrary positive number. This fact follows from Proposition 3 and Condition 2 in Theorem 1. It is not important in this paper, but we have stated it in order to help some reader to understand the $K$-ergodicity.

Next, to show that the stationary vector and the dual one are eigenvectors, we prove the following proposition, whose proof is based on the proof of [15, Theorem 2].

**Proposition 3.** If a $K$-positive map $A$ is $K$-ergodic, then $A^*$ is $K^*$-ergodic, and moreover a stationary vector and a dual one of $A^*$ are a dual one and a stationary vector of $A$, respectively.

**Proof.** Let $A$ be a $K$-ergodic map. Taking the inner product of any vector $y$ and the equation (1), we find that any vector $x$ satisfies

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle (r(A)^{-1} A^*)^k y, x \rangle = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \langle y, (r(A)^{-1} A^*)^k x \rangle = \langle y_0, x \rangle \langle y, x_0 \rangle.
$$

Therefore, $\lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} (r(A)^{-1} A^*)^k y = \langle y, x_0 \rangle y_0$ for any vector $y \in V^*$.

**Lemma 2.** If $A$ is a $K$-ergodic map, then a stationary vector $x_0$ and a dual one $y_0$ of $A$ are eigenvectors of $A$ and $A^*$ associated with $r(A)$, respectively.

**Proof.** Let $A$ be a $K$-ergodic map. Thanks to Proposition 3, $A$ and $A^*$ have eigenvectors $x_1 \in K$ and $y_1 \in K^*$ associated with $r(A)$, respectively. Since $A$ is $K$-ergodic, any vector $x$ satisfies the equation (1). Putting $x = x_1$ in the equation (1), we have

$$
x_1 = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (r(A)^{-1} A^*)^k x_1 = \langle y_0, x_1 \rangle x_0.
$$

The relation $x_1 \neq 0$ leads to $\langle y_0, x_1 \rangle \neq 0$, and thus the stationary vector $x_0$ is an eigenvector associated with $r(A)$ of $A$. Due to Proposition 3, the remaining part is also shown by the same way.

Definition 4 cannot be checked by numerical calculation. To realize its numerical check, we have necessary and sufficient conditions for the $K$-ergodicity. As stated at the end of this section, their conditions for a DUP map become simpler.

**Theorem 1.** For any $K$-positive map $A$, the following conditions are equivalent.

1. $A$ is $K$-ergodic.
2. The eigenvalue $r(A) > 0$ of $A$ has the geometric multiplicity 1. In addition, there exist eigenvectors $x_0 \in K$ and $y_0 \in K^*$ associated with $r(A)$ of $A$ and $A^*$, respectively, such that $\langle y_0, x_0 \rangle = 1$.
3. The eigenvalue $r(A) > 0$ of $A$ has the algebraic multiplicity 1.
Proof. \( 1 \Rightarrow 2 \). Assume Condition 1. The latter condition of Condition 2 follows from Lemma 2. The former condition of Condition 2 is shown by the same way as the proof of Lemma 2.

\( 2 
\Rightarrow 3 \). Assume Condition 2. We prove Condition 3 by contradiction. Suppose that a nonzero vector \( x \) satisfies \( Ax = r(A)x + x_0 \), we have

\[
r(A) \langle y_0, x \rangle = \langle y_0, Ax \rangle = \langle y_0, r(A)x + x_0 \rangle = r(A) \langle y_0, x \rangle + 1,
\]
whose contradiction proves Condition 3.

\( 3 \Rightarrow 1 \). Assume Condition 3, and let \( \lambda \neq r(A) \) be an eigenvalue of \( A \) with \( |\lambda| = r(A) \). Since the degree of \( r(A) \) is one, Proposition 2 implies that the degree of \( \lambda \) is one. Consider the Jordan canonical form of \( A \). Since any Jordan block associated with \( \lambda \) equals the matrix \( [\lambda] \), and

\[
\frac{1}{n} \sum_{k=0}^{n-1} (r(A)^{-1}\lambda)^k = \frac{1}{n} \frac{1 - (r(A)^{-1}\lambda)^n}{1 - r(A)^{-1}\lambda} \xrightarrow{n \to \infty} 0,
\]

there exist two vectors \( x_0 \) and \( y_0 \) such that any vector \( x \) satisfies the equation (1). All we need to do is show that \( x_0 \in K \), \( y_0 \in K^* \), and \( \langle y_0, x_0 \rangle = 1 \). By the same way as the proof of Lemma 2, it is verified that the vectors \( x_0 \) and \( y_0 \) can be taken to be \( x_0 \in K \setminus \{0\} \) and \( y_0 \in K^* \setminus \{0\} \), respectively. Putting \( x = x_0 \) in the equation (1), we obtain

\[
x_0 = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (r(A)^{-1}A)^k x_0 = \langle y_0, x_0 \rangle x_0,
\]
which implies \( \langle y_0, x_0 \rangle = 1 \). Therefore, Condition 1 holds.

The latter condition of Condition 2 in Theorem 1 cannot be removed. To explain it, we give a \( K \)-positive map \( A \) that is not \( K \)-ergodic but satisfies the former condition of Condition 2 as follows.

Example 4. Letting \( V = \mathbb{R}^2 \), \( K = [0, \infty)^2 \), and

\[
A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},
\]
we discuss the classical case. The \( K \)-positive map \( A \) has the eigenvalue \( r(A) = 1 \) with the geometric multiplicity 1. However, since

\[
A^n = \begin{bmatrix} 1^n & n \\ 0 & 1 \end{bmatrix},
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} (r(A)^{-1}A)^k \text{ does not converge, i.e., the equation (1) does not hold. Note that } A \text{ and } A^* \text{ have unique eigenvectors } x_0 = [1, 0]^\top \in K \text{ and } y_0 = [0, 1]^\top \in K = K^* \text{ up to a scalar multiple, respectively, which satisfy } \langle y_0, x_0 \rangle = 0. \]
3.3. Application to DUP $\mathcal{K}$-positive map. We focus on the $\mathcal{K}$-ergodicity for a DUP $\mathcal{K}$-positive map $A$. We pay attention to the following two facts on $A$. First, we show $r(A) = 1$. From Proposition 2, $A$ has an eigenvector $x_0 \in \mathcal{K}$ associated with $r(A)$. Since $A^*u = u$ and $u \in (\mathcal{K}^*)^G$, we have $\langle u, x_0 \rangle = \langle A^*u, x_0 \rangle = \langle u, Ax_0 \rangle = r(A) \langle u, x_0 \rangle$ and $\langle u, x_0 \rangle > 0$. Thus, $r(A) = 1$. Second, if $A$ is $\mathcal{K}$-ergodic, then $u$ must be a dual stationary vector of $A$. It is proved by using Theorem 1 immediately. Thanks to the first fact, we need not calculate the special radius. In addition, the second fact ensures the existence of the eigenvector $x_0 \in \mathcal{K}$ and $y_0 = u \in (\mathcal{K}^*)^G$ satisfying $\langle y_0, x_0 \rangle = 1$. Hence, rewriting the conditions in Theorem 1 in forms of simple matrix algebra, we reproduce the following existing result in the case of a DUP $\mathcal{K}$-positive map. That is, Theorem 1 can be regarded as a generalization of the result by [14, Appendix], [15, Corollary 2].

**Corollary 1.** Let $A$ be a DUP $\mathcal{K}$-positive map. Then, $A$ is $\mathcal{K}$-ergodic if and only if $\dim \ker (A - I) = 1$. Here, $I$ denotes the identity map on $\mathcal{V}$.

We can calculate $\dim \ker (A - I)$ by solving a system of linear equations so that we can strictly determine whether a given DUP $\mathcal{K}$-positive map $A$ is $\mathcal{K}$-ergodic or not.

4. $\mathcal{K}$-mixing and asymptotically decoupling conditions

4.1. $\mathcal{K}$-mixing condition. As already shown in Theorem 1, a $\mathcal{K}$-ergodic map $A$ has good properties, but $(r(A)^{-1}A)^n$ might periodically behave as $n \to \infty$. To exclude it from the $\mathcal{K}$-ergodicity, we introduce the $\mathcal{K}$-mixing condition as follows.

**Definition 5 ($\mathcal{K}$-mixing).** A $\mathcal{K}$-positive map $A$ is called $\mathcal{K}$-mixing when the following conditions hold.

- $r(A) > 0$.
- There exist two vectors $x_0 \in \mathcal{K}$ and $y_0 \in (\mathcal{K}^*)^G$ such that $\langle y_0, x_0 \rangle = 1$, and any vector $x$ satisfies

$$\lim_{n \to \infty} (r(A)^{-1}A)^n x = \langle y_0, x \rangle x_0.$$ (2)

The vectors $x_0$ and $y_0$ are called a stationary vector and a dual stationary vector of $A$, respectively.

From the definitions, any $\mathcal{K}$-mixing map is $\mathcal{K}$-ergodic. Like Definition 4, Definition 5 cannot be checked by numerical calculation. To realize its numerical check, we derive a necessary and sufficient condition for the $\mathcal{K}$-mixing condition, which is simpler than the conditions in Theorem 1. Before deriving it, we prove the following theorem, which is interesting in itself and is used to attain our purpose.

**Theorem 2.** Let $\tilde{\mathcal{K}}$ be a closed convex cone with $\mathcal{K}_1 \otimes \mathcal{K}_2 \subset \tilde{\mathcal{K}} \subset (\mathcal{K}_1^* \otimes \mathcal{K}_2^*)^*$, $A_j$ be a $\mathcal{K}_j$-mixing map for $j = 1, 2$, and $A_1 \otimes A_2$ be a $\tilde{\mathcal{K}}$-positive map. Then, $A_1 \otimes A_2$ is $\tilde{\mathcal{K}}$-mixing.

**Proof.** Any vector $x^j \in \mathcal{V}_j$ satisfies that

$$\lim_{n \to \infty} (r(A_j)^{-1}A_j)^n x^j = \langle y_0^j, x^j \rangle x_0^j,$$
where \(x_j^0\) and \(y_j^0\) are a stationary vector and a dual one of \(A_j\), respectively, for \(j = 1, 2\). Therefore, any vector \(x \in \mathcal{V}_1 \otimes \mathcal{V}_2\) satisfies
\[
\lim_{n \to \infty} (r(A_1)^{-1} r(A_2)^{-1} A_1 \otimes A_2)^n x = \langle y_0^1 \otimes y_0^2, x \rangle x_0^1 \otimes x_0^2.
\]
Due to Lemma 13 in Appendix, the relations \(x_0^1 \otimes x_0^2 \in \tilde{K}^\circ\) and \(y_0^1 \otimes y_0^2 \in (\tilde{K}^*)^\circ\) hold. Noting \(r(A_1) r(A_2) = r(A_1 \otimes A_2)\), we find that the \(\tilde{K}\)-positive map \(A_1 \otimes A_2\) is \(\tilde{K}\)-mixing.

Now, we have been ready to prove the following theorem, which is very simple, fortunately.

**Theorem 3.** For any \(\mathcal{K}\)-positive map \(A\), the following conditions are equivalent.

1. \(A\) is \(\mathcal{K}\)-mixing.
2. The eigenvalue \(r(A)^2 > 0\) of \(A^{\otimes 2}\) has the geometric multiplicity 1.

**Proof.** \(\Box\) Assume Condition 1. Then, Theorem 2 implies that \(A^{\otimes 2}\) is \(\mathcal{K}^{\otimes 2}\)-mixing, whence \(A^{\otimes 2}\) is \(\mathcal{K}^{\otimes 2}\)-ergodic. Thus, Theorem 1 implies Condition 2.

Assume Condition 2. This condition implies that the eigenvalue \(r(A)\) of \(A\) has the geometric multiplicity 1; otherwise the eigenvalue \(r(A)^2\) of \(A^{\otimes 2}\) would have the geometric multiplicity not smaller than 2. By contradiction, we prove that the eigenvalue \(r(A)\) of \(A\) has the algebraic multiplicity 1. Due to Proposition 2, we can take an eigenvector \(x_0 \in \mathcal{K}\) of \(A\). Suppose that a nonzero vector \(x\) satisfies \(Ax = r(A)x + x_0\), the equation
\[
A^{\otimes 2}(x \otimes x_0 - x_0 \otimes x) = (r(A)x + x_0) \otimes r(A)x_0 - r(A)x_0 \otimes (r(A)x + x_0) = r(A)^2 (x \otimes x_0 - x_0 \otimes x)
\]
holds. Since the vector \(x_0^{\otimes 2}\) is an eigenvector associated with \(r(A)^2\) of \(A^{\otimes 2}\), Condition 2 implies that \(\alpha x_0^{\otimes 2} = x \otimes x_0 - x_0 \otimes x\) for a real number \(\alpha\). Hence, \(\alpha x_0 = x - \alpha' x_0\) for a real number \(\alpha'\), which means \(x \in \text{span}(x_0)\). Here, \(\text{span}(A^{\otimes 2})\) denotes the linear span of a subset \(\mathcal{K}\) of \(\mathcal{V}\). However, the relation \(r(A)x + x_0\) is a contradiction.

Next, by contradiction, we prove that any eigenvalue \(\lambda \neq r(A)\) of \(A\) satisfies \(|\lambda| < r(A)\). Suppose that \(\lambda \neq r(A)\) is an eigenvalue of \(A\) with \(|\lambda| = r(A)\), and let \(\mathcal{V}^C\) be the complexification of \(\mathcal{V}\). Then, \(A\) can be naturally identified with the linear map on \(\mathcal{V}^C\) to be \(A(x_1 + \sqrt{-1}x_2) = Ax_1 + \sqrt{-1}Ax_2\) for any vectors \(x_1, x_2 \in \mathcal{V}\). Take an eigenvector \(z \in \mathcal{V}^C\) associated with \(\lambda\) of \(A\). Then, \(A\) has the eigenvector \(z\) with \(A^2(z) = \lambda^2 z\), whereas \(A^{\otimes 2}\) has the eigenvector \(x_0^{\otimes 2}\) associated with \(r(A)^2\). Hence, Condition 2 implies that \(x_0^{\otimes 2} = \xi z \otimes \xi^*\) for a complex number \(\xi\). Thus, \(x_0 = \xi' z\) for a complex number \(\xi'\). Since \(x_0\) and \(z\) are eigenvectors associated with \(r(A)\) and \(\lambda\) of \(A\), respectively, it contradicts the assumption \(\lambda \neq r(A)\). Therefore, \(|\lambda| < r(A)\).

Finally, considering the Jordan canonical form of \(A\), we find that the above statements on eigenvalues implies that there exist two vectors \(x_0\) and \(y_0\) such that any vector \(x\) satisfies the equation (2). The remaining part, namely, \(x_0 \in \mathcal{K}\), \(y_0 \in \mathcal{K}^*\), and \(\langle y_0, x_0 \rangle = 1\), is shown by the same way as the corresponding part of the proof of Theorem 1.
Using the above theorem, we obtain the following corollary immediately.

**Corollary 2.** Let $A$ be a DUP $\mathcal{K}$-positive map. Then, $A$ is $\mathcal{K}$-mixing if and only if $\dim \text{Ker } (A^{\otimes 2} - I^{\otimes 2}) = 1$.

We can calculate $\dim \text{Ker } (A^{\otimes 2} - I^{\otimes 2})$ by solving a system of linear equations so that we can strictly determine whether a given DUP $\mathcal{K}$-positive map $A$ is $\mathcal{K}$-mixing or not.

**Remark 1.** As already stated at the end of Section 3.3, Corollary 1 is well-known for the quantum system. On the other hand, Corollary 2 is not known at all. As already shown Theorem 3, it holds independently of general probabilistic theory. For the quantum system, there is another condition that is equivalent to the mixing condition, in the paper [14, Theorem 7]. Let $A$ be a trace-preserving positive map. Then, that condition is as follows: any eigenvalue except 1 of $A$ has the absolute value that is smaller than 1, and the eigenvalue 1 has the algebraic multiplicity 1. We call it the spectral condition. The spectral condition is well-known and equivalent to the mixing condition in general (consider the Jordan canonical form). However, when we determine whether a given $\mathcal{K}$-positive map is $\mathcal{K}$-mixing or not by using the spectral condition, we must find all eigenvalues of $A$, i.e., must solve its characteristic equation. On the other hand, using Corollary 2 we have only to solve a system of linear equations, which is an advantage.

The following theorem describes our main idea well, namely the equivalence of the $\mathcal{K}$-mixing condition of $A$ and the $\mathcal{K}^{\otimes 2}$-ergodicity of $A^{\otimes 2}$.

**Theorem 4.** Let $\tilde{\mathcal{K}}$ be a closed convex cone with $\mathcal{K}^{\otimes 2} \subset \tilde{\mathcal{K}} \subset ((\mathcal{K}^*)^{\otimes 2})^*$, $A$ be a $\mathcal{K}$-positive map, and $A^{\otimes 2}$ be a $\tilde{\mathcal{K}}$-positive map. Then, $A$ is $\mathcal{K}$-mixing if and only if $A^{\otimes 2}$ is $\tilde{\mathcal{K}}$-ergodic.

**Proof.** Let $A$ be $\mathcal{K}$-mixing. Theorem 2 implies that $A^{\otimes 2}$ is $\tilde{\mathcal{K}}$-mixing, whence $A^{\otimes 2}$ is $\tilde{\mathcal{K}}$-ergodic. Conversely, let $A^{\otimes 2}$ be $\tilde{\mathcal{K}}$-ergodic. Owing to Theorem 1, the eigenvalue $\tau(A)^2$ of $A^{\otimes 2}$ has the geometric multiplicity 1. Theorem 3 means that $A$ is $\mathcal{K}$-mixing.

**Remark 2.** Although we have focused on the 2-fold tensor product of a $\mathcal{K}$-positive map, similar statements hold for the $N$-fold tensor product of a $\mathcal{K}$-positive map with $N \geq 3$. Indeed, the $N$-fold tensor product version of Theorem 2 is proved by the same way. Then, the $N$-fold tensor product version of Theorem 4 is proved if we replace “$x \otimes x_0 - x_0 \otimes x$” and “$z \otimes \overline{z}$” with “$(x \otimes x_0 - x_0 \otimes x) \otimes x_0^{\otimes N-2n}$” and “$z \otimes \overline{z} \otimes x_0^{\otimes N-2n}$”, respectively, in the proof of Theorem 4. Finally, the $N$-fold tensor product version of Theorem 4 is proved by the same way.

**4.2. Asymptotically decoupling condition.** Let the tuple $\mathcal{V}_j$, $\mathcal{K}_j$, $\langle \cdot, \cdot \rangle_j$, and $w^j$ be a general probabilistic system for $j = 1, 2$. We discuss a $\tilde{\mathcal{K}}$-positive map on the joint system $\mathcal{V}_1 \otimes \mathcal{V}_2$, where $\tilde{\mathcal{K}}$ is a closed convex cone with $\mathcal{K}_1 \otimes \mathcal{K}_2 \subset \tilde{\mathcal{K}} \subset (\mathcal{K}_1^* \otimes \mathcal{K}_2^*)^*$, and we adopt the general probabilistic system given by the tuple $\mathcal{V}_1 \otimes \mathcal{V}_2$, $\tilde{\mathcal{K}}$, $\langle \cdot, \cdot \rangle$, and $w^1 \otimes w^2$. Before proceeding, we define the reduced state $\pi_1(x)$ of $x \in \mathcal{S}(\tilde{\mathcal{K}}, w^1 \otimes w^2)$ on the system $\mathcal{V}_1$ as

$$
\pi_1(x) := \sum_{i=1}^l \langle u^2_i, x^2_i \rangle x^1_i,
$$
where \( x_i^1 \) and \( x_i^2 \) with \( i = 1, \ldots, l \) are vectors of \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \), respectively, satisfying \( x = \sum_{i=1}^{l} x_i^1 \otimes x_i^2 \). This definition is well-defined, which is proved by the same way as the definition of the partial trace. The reduced state \( \pi_2(x) \) of \( x \) on the system \( \mathcal{V}_2 \) is also defined in a similar way to \( \pi_1(x) \).

A \( \tilde{K} \)-positive map \( \tilde{A} \) on \( \mathcal{V}_1 \otimes \mathcal{V}_2 \) with \( \text{Ker} \tilde{A} \cap \tilde{K} = \{0\} \) is called decoupling on \( \mathcal{S}(\tilde{K}, u^1 \otimes u^2) \) when any state \( x \in \mathcal{S}(\tilde{K}, u^1 \otimes u^2) \) satisfies

\[
\tilde{A}x = \pi_1\left( \frac{\tilde{A}x}{\langle u^1 \otimes u^2, Ax \rangle} \right) \otimes \pi_2\left( \frac{\tilde{A}x}{\langle u^1 \otimes u^2, Ax \rangle} \right).
\]

As its asymptotic version, we introduce the following condition.

**Definition 6 (Asymptotically decoupling condition).** A \( \tilde{K} \)-positive map \( \tilde{A} \) on \( \mathcal{V}_1 \otimes \mathcal{V}_2 \) with \( \text{Ker} \tilde{A} \cap \tilde{K} = \{0\} \) is called asymptotically decoupling on \( \mathcal{S}(\tilde{K}, u^1 \otimes u^2) \) if any state \( x \in \mathcal{S}(\tilde{K}, u^1 \otimes u^2) \) satisfies

\[
\lim_{n \to \infty} \left\| \tilde{A}^n x - \pi_1\left( \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} \right) \otimes \pi_2\left( \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} \right) \right\| = 0. \tag{3}
\]

When \( \mathcal{V}_1 \otimes \mathcal{V}_2 = \mathcal{V}^{\otimes 2} \), \( \tilde{A} = \mathcal{A}^{\otimes 2} \), and \( \mathcal{A} \) is DUP, the equation \( \text{(3)} \) can be rewritten as

\[
\lim_{n \to \infty} \left\| (\mathcal{A}^{\otimes 2})^n x - \pi_1((\mathcal{A}^{\otimes 2})^n x) \otimes \pi_2((\mathcal{A}^{\otimes 2})^n x) \right\| = 0.
\]

Similarly, when \( \mathcal{V}_1 \otimes \mathcal{V}_2 = \mathcal{V}^{\otimes 2} \), \( \tilde{A} = \mathcal{A} \otimes I \), and \( \mathcal{A} \) is DUP, the equation \( \text{(3)} \) can be rewritten as

\[
\lim_{n \to \infty} \left\| (\mathcal{A} \otimes I)^n x - \pi_1((\mathcal{A} \otimes I)^n x) \otimes \pi_2((\mathcal{A} \otimes I)^n x) \right\| = 0.
\]

Although Definition \( \text{(6)} \) is dependent on the cone \( \tilde{K} \) and the unit elements \( u^1 \) and \( u^2 \), the asymptotically decoupling condition is independent of the unit elements \( u^1 \) and \( u^2 \) as follows.

**Lemma 3.** Let \( \tilde{A} \) be a \( \tilde{K} \)-positive map on \( \mathcal{V}_1 \otimes \mathcal{V}_2 \) with \( \text{Ker} \tilde{A} \cap \tilde{K} = \{0\} \) and \( u^j \in (\mathcal{K}_j^+)^c \) for \( j = 1, 2 \). If \( \mathcal{A} \) is asymptotically decoupling on \( \mathcal{S}(\tilde{K}, u^1 \otimes u^2) \), then \( \tilde{A} \) is asymptotically decoupling on \( \mathcal{S}(\tilde{K}, u^1 \otimes u^2) \).

**Proof.** Let \( \tilde{A} \) be an asymptotically decoupling map on \( \mathcal{S}(\tilde{K}, u^1 \otimes u^2) \). We define the reduced state \( \pi_1'(x) \) of a state \( x \in \mathcal{S}(\tilde{K}, u^1 \otimes u^2) \) on the system \( \mathcal{V}_1 \) as

\[
\pi_1'(x) := \sum_{i=1}^{l'} \langle u_i^2, x_i^2 \rangle x_i^1,
\]

where \( x_i^1 \) and \( x_i^2 \) with \( i = 1, \ldots, l' \) are vectors of \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \), respectively, satisfying \( x = \sum_{i=1}^{l'} x_i^1 \otimes x_i^2 \). The reduced state \( \pi_2'(x) \) of \( x \) on the system \( \mathcal{V}_2 \) is also defined
in a similar way to \( \pi_1'(x) \). Take an arbitrary state \( x \in S(\tilde{K}, u^1 \otimes u^2) \), and put \( c_n := (\langle u^1 \otimes u^2, \tilde{A}^n x \rangle / \langle u^1 \otimes u^2, \tilde{A}^n x \rangle)^{1/2} \), \( x_n^1 := c_n^{-1} \pi_1(\tilde{A}^n x / \langle u^1 \otimes u^2, \tilde{A}^n x \rangle) \), and \( x_n^2 := c_n^{-1} \pi_2(\tilde{A}^n x / \langle u^1 \otimes u^2, \tilde{A}^n x \rangle) \). Since \( \tilde{A} \) is asymptotically decoupling on \( S(\tilde{K}, u^1 \otimes u^2) \), we have

\[
   c_n^2 \left( \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} - x_n^1 \otimes x_n^2 \right) = \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} - \pi_1 \left( \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} \right) \otimes \pi_2 \left( \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} \right) \to 0, \quad (4)
\]

As \( \tilde{A}^n x / \langle u^1 \otimes u^2, \tilde{A}^n x \rangle \) is a state with respect to the unit element \( u^1 \otimes u^2 \) and the set \( S(\tilde{K}, u^1 \otimes u^2) \) is compact, the sequence \( (c_n) \) of positive numbers is bounded. Thus, the equation \( (4) \) implies

\[
   \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} - x_n^1 \otimes x_n^2 \to 0, \quad (5)
\]

\[
   1 - \langle u^1, x_n^1 \rangle \langle u^2, x_n^2 \rangle \to 0. \quad (6)
\]

The equation \( (5) \) guarantees that the sequence \( (x_n^1 \otimes x_n^2) \) is bounded. Using the equations \( (5) \) and \( (6) \), we obtain

\[
   \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} - \pi_1' \left( \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} \right) \otimes \pi_2' \left( \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} \right) = \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} - \langle u^1, x_n^1 \rangle \langle u^2, x_n^2 \rangle x_n^1 \otimes x_n^2 + o(1)
\]

\[
   = \frac{\tilde{A}^n x}{\langle u^1 \otimes u^2, \tilde{A}^n x \rangle} - \langle u^1, x_n^1 \rangle \langle u^2, x_n^2 \rangle x_n^1 \otimes x_n^2 + o(1) \to 0.
\]

Therefore, \( \tilde{A} \) is asymptotically decoupling on \( S(\tilde{K}, u^1 \otimes u^2) \).

In order to state relations between the \( K \)-mixing and asymptotically decoupling conditions, we need the following lemma, which is proved by combining \cite{[13]} Lemma 5 with Lemma \cite{[7]} in Appendix.

**Lemma 4.** Let \( A \) be a \( K \)-positive map whose adjoint map has an eigenvalue \( \gamma_0 \in (K^*)^2 \) associated with \( \tau(A) = 1 \). If \( A \) has only one fixed state, then \( A \) is \( K \)-ergodic.

**Proof.** Assume that \( A \) has only one fixed state. Since the map \( x \mapsto (x, y_0)^{-1} x \) from \( S(K, u) \) to \( S(K, y_0) \) is bijective, \( A \) also has only one fixed point in \( S(K, y_0) \). Combining \cite{[13]} Lemma 5 with Lemma \cite{[7]} in Appendix, we obtain our assertion.

In addition, we also need the following lemma, which is a generalization of the trace norm.

Lemma 5 (u-norm). For any vector \( x \in \mathcal{V} \), the value \( \|x\|_u \) is defined as
\[
\|x\|_u := \max_{-u \leq y \leq u} |\langle y, x \rangle|,
\]
where the order is induced by the dual cone \( \mathcal{K}^* \). Then, \( \| \cdot \|_u \) is a norm on \( \mathcal{V} \), which we call u-norm. If \( A \) is a DUP \( \mathcal{K} \)-positive map, then the any vector \( x \) satisfies \( \|Ax\|_u \leq \|x\|_u \).

Proof. First, we show that \( \| \cdot \|_u \) is a norm. From the definition, the absolute homogeneity and the triangle inequality follow immediately. We show that the equation \( \|x\|_u = 0 \) implies \( x = 0 \). The equation \( \|x\|_u = 0 \) means that \( \langle y, x \rangle = 0 \) for any vector \( -u \leq y \leq u \). Take an arbitrary vector \( y' \in \mathcal{V}^* \). Since the unit element \( u \) is an interior point of \( \mathcal{K}^* \), the inequality \( -u \leq \epsilon y' \leq u \) holds for a sufficiently small number \( \epsilon > 0 \). Thus, \( \epsilon \langle y', x \rangle = 0 \) and so \( \langle y', x \rangle = 0 \). Since \( y' \) is an arbitrary vector, we have \( x = 0 \). Therefore, \( \| \cdot \|_u \) is a norm.

In order to show the remaining part, let \( A \) be a DUP \( \mathcal{K} \)-positive map. Since the inequality \( -u \leq y \leq u \) yields \( -u = -A^*u \leq A^*y \leq A^*u = u \), we have
\[
\|Ax\|_u = \max_{-u \leq y \leq u} |\langle y, Ax \rangle| = \max_{-u \leq y \leq u} |\langle A^*y, x \rangle| \leq \max_{-u \leq y \leq u} |\langle y, x \rangle| = \|x\|_u.
\]

Now, we have been ready to prove the following theorem, which clarifies relations between the \( \mathcal{K} \)-mixing and asymptotically decoupling conditions.

Theorem 5. Let \( A \) be a \( \mathcal{K} \)-positive map whose adjoint map has an eigenvector \( \gamma_0 \in (\mathcal{K}^*)^0 \) associated with \( r(A) > 0 \), and \( A^{\otimes 2} \) be a \( \tilde{\mathcal{K}} \)-positive map. Then, the following conditions are equivalent.

1. \( A \) is \( \mathcal{K} \)-mixing.
2. The eigenvalue \( r(A)^2 \) of \( A^{\otimes 2} \) has the geometric multiplicity 1.
3. \( A^{\otimes 2} \) is \( \tilde{\mathcal{K}} \)-ergodic.
4. \( A^{\otimes 2} \) is \( \tilde{\mathcal{K}} \)-mixing.
5. \( A^{\otimes 2} \) is asymptotically decoupling on \( \mathcal{S}(\tilde{\mathcal{K}}, u^{\otimes 2}) \).

Moreover, if \( A \otimes I \) and \( I \otimes A \) are \( \tilde{\mathcal{K}} \)-positive maps, then the following condition is equivalent to the above conditions.

6. \( A \otimes I \) is asymptotically decoupling on \( \mathcal{S}(\tilde{\mathcal{K}}, u^{\otimes 2}) \).

Remark 3. The assumption of the above theorem implies \( (\text{Ker} \ A^{\otimes 2}) \cap \tilde{\mathcal{K}} = \{0\} \) and \( (\text{Ker} \ A \otimes I) \cap \tilde{\mathcal{K}} = \{0\} \). Indeed, letting \( x \in (\text{Ker} \ A^{\otimes 2}) \cap \tilde{\mathcal{K}} \), we have \( 0 = (\gamma_0^{\otimes 2}, A^{\otimes 2}x) = \langle (A^*)^{\otimes 2}y_0^{\otimes 2} \rangle, x \rangle = r(A)^2 \langle y_0^{\otimes 2} \rangle, x \rangle \). The relations \( r(A) > 0 \) and \( y_0^{\otimes 2} \in (\mathcal{K}^*)^0 \) yield \( x = 0 \). Similarly, the other relation \( (\text{Ker} \ A \otimes I) \cap \tilde{\mathcal{K}} = \{0\} \) is proved.

Proof. The equivalences [1]–[3] and [4]–[5] result from Theorem 6 and Theorem 4, respectively. The implication [4]–[3] follows from the definitions. Hence, the equivalence of Conditions [1]–[4] holds.

[1]–[3] This implication is proved in a similar way to the proof of Theorem 2. Assume Condition [5]. Without loss of generality, we may assume \( r(\tilde{A}) = 1 \). First, we show that \( A \) has only one fixed state. Let \( Ax_i = x_i \) and \( x_i \in S(\tilde{\mathcal{K}}, u) \).
for $i = 1, 2$, and put $x := (x_1^{\otimes 2} + x_2^{\otimes 2})/2 \in \mathcal{S}(\mathcal{K}, u^{\otimes 2})$. The equations $A^{\otimes 2}x = x$ and $\pi_j(x) = (x_1 + x_2)/2$ hold for $j = 1, 2$. Then, Condition $\text{[5]}$ implies

$$0 = \lim_{n \to \infty} \left[ (A^{\otimes 2})^{n,x} \right] = \pi_1 \left( (A^{\otimes 2})^{n,x} \right) \otimes \pi_2 \left( (A^{\otimes 2})^{n,x} \right)$$

$$= x - \pi_1(x) \otimes \pi_2(x) = (1/4)(x_1 - x_2)^{\otimes 2}.$$  

Therefore, $x_1 = x_2$.

Next, we show that $A^{\otimes 2}$ has only one fixed state. Let $A^{\otimes 2}x = x$ and $x \in \mathcal{S}(\mathcal{K}, u^{\otimes 2})$. Then, Condition $\text{[5]}$ implies

$$0 = \lim_{n \to \infty} \left[ (A^{\otimes 2})^{n,x} \right] = \pi_1 \left( (A^{\otimes 2})^{n,x} \right) \otimes \pi_2 \left( (A^{\otimes 2})^{n,x} \right)$$

$$= x - \pi_1(x) \otimes \pi_2(x).$$

Combining the above equation with $A^{\otimes 2}x = x$, we have $A\pi_j(x) = \pi_j(x)$ for $j = 1, 2$. As already shown, $A$ has only one fixed state $x_0$. Hence, $\pi_1(x) = \pi_2(x) = x_0$ and $x = x_0^{\otimes 2}$, which means that $A^{\otimes 2}$ has only one fixed state. Therefore, Lemma $\text{[4]}$ implies that $A^{\otimes 2}$ is $\mathcal{K}$-ergodic.

This implication is proved in a similar way to the proof of Theorem $\text{[2]}$. Assume Condition $\text{[6]}$. First, we show that $A \otimes I$ is asymptotically decoupling on $\mathcal{S}(\mathcal{K}, y_0^{\otimes 2})$. Note that $A$ is DUP with respect to the unit element $y_0$. Let $x \in \mathcal{S}(\mathcal{K}, y_0^{\otimes 2})$. Since any two norms on a finite-dimensional vector space are uniformly equivalent to each other, Condition $\text{[6]}$ means $\|(A^n \otimes I)(x - \pi_1(x) \otimes \pi_2(x))\|_{y_0^{\otimes 2}} \to 0$. Using Lemma $\text{[5]}$ we have

$$\|(A^{\otimes 2})^{n,x} - \pi_1(x) \otimes \pi_2(x)\|_{y_0^{\otimes 2}} = \|(I \otimes A^n)(A^n \otimes I)(x - \pi_1(x) \otimes \pi_2(x))\|_{y_0^{\otimes 2}} \leq \|(A^n \otimes I)(x - \pi_1(x) \otimes \pi_2(x))\|_{y_0^{\otimes 2}} \to 0.$$ 

Thus, $A^{\otimes 2}$ is asymptotically decoupling on $\mathcal{S}(\mathcal{K}, y_0^{\otimes 2})$. Condition $\text{[5]}$ follows from Lemma $\text{[3]}$.

We have shown the above theorem under the assumption $y_0 \in (\mathcal{K}^*)^0$. In particular, any DUP $\mathcal{K}$-positive map satisfies this assumption. However, the above theorem without this assumption does not necessarily hold. To explain it, we give an asymptotically decoupling map on $\mathcal{S}(\mathcal{K}^{\otimes 2}, u^{\otimes 2})$ that is not $\mathcal{K}$-ergodic as follows.

**Example 5.** Letting $\mathcal{V} = \mathbb{R}^2$, $\mathcal{K} = [0, \infty)^2$, $u = [1, 1]^T$, and

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},$$

we discuss the classical case. As already shown Example $\text{[4]}$, $A$ is not $\mathcal{K}$-ergodic, and a fortiori, is not $\mathcal{K}$-mixing. We show that $A^{\otimes 2}$ is asymptotically decoupling on $\mathcal{S}(\mathcal{K}^{\otimes 2}, u^{\otimes 2})$, where $\mathcal{V}^{\otimes 2} = \mathbb{R}^4$ and $\mathcal{K}^{\otimes 2} = [0, \infty)^4$. Let $x = [x_1, x_2, x_3, x_4]^T$ be a probability vector. Since

$$A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}, \quad (A^{\otimes 2})^n = (A^n)^{\otimes 2} = \begin{bmatrix} 1 & n & n^2 \\ 1 & 0 & n \\ 1 & n & 1 \end{bmatrix},$$

we have $A^{\otimes 2}$ is asymptotically decoupling on $\mathcal{S}(\mathcal{K}^{\otimes 2}, \mathcal{V}^{\otimes 2})$. However, as shown Example $\text{[4]}$, $A$ is not $\mathcal{K}$-mixing, and a fortiori, is not $\mathcal{K}$-ergodic.
we have
\[
\lim_{n \to \infty} \frac{(A^{\otimes 2})^n}{(u^{\otimes 2}, (A^{\otimes 2})^n u)} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]
Therefore, \(A^{\otimes 2}\) is asymptotically decoupling on \(S(K^{\otimes 2}, u^{\otimes 2})\). Similarly, the equation
\[
(A \otimes I)^n = A^n \otimes I = \begin{bmatrix} 1 & n \\ 1 & n \end{bmatrix}
\]
holds. If \(x_2 + x_4 > 0\),
\[
\lim_{n \to \infty} \frac{(A \otimes I)^n}{(u^{\otimes 2}, (A \otimes I)^n u)} = \frac{1}{x_2 + x_4} \begin{bmatrix} x_2 \\ 0 \\ x_4 \\ 0 \end{bmatrix} = \frac{1}{x_2 + x_4} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}.
\]
If \(x_2 = x_4 = 0\),
\[
\lim_{n \to \infty} \frac{(A \otimes I)^n}{(u^{\otimes 2}, (A \otimes I)^n u)} = \frac{1}{x_1 + x_3} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \frac{1}{x_1 + x_3} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}.
\]
Therefore, \(A \otimes I\) is asymptotically decoupling on \(S(K^{\otimes 2}, u^{\otimes 2})\). \(\square\)

5. \(K\)-irreducibility and \(K\)-primitivity

In classical probabilistic theory, the following condition on a transition matrix \(A\) is important: for any pair of integers \(1 \leq i, j \leq d\), there exists a positive integer \(n\) such that \(\langle j | A^n | i \rangle > 0\), where \(\{|i\rangle\}_{i=1}^{d}\) is the standard basis of \(\mathbb{R}^d\). This condition is called irreducibility [19,28]. The irreducibility is easily checked by discussing the support of outputs for the finite number of inputs. When we extend the above definition to the case with a general closed convex cone \(K\), we need to consider the condition \(\langle y, A^n x \rangle > 0\) for any nonzero extremal vectors \(x \in K\) and \(y \in K^*\). When a cone \(K\) has the finite number of the extremal rays, this condition can be easily checked. Here, an extremal ray of a closed convex cone \(K\) means a subset \(\{\alpha x \mid \alpha \geq 0\}\) of \(K\) with a nonzero extremal vector \(x \in K\); a vector \(x \in K\) is called extremal when any two vectors \(x_1, x_2 \in K\) with \(x = x_1 + x_2\) satisfy \(x_1, x_2 \in \text{span}(x)\) [11 Section 2]. Certainly, in classical probabilistic theory, the cone \(K = [0, \infty)^d\) has exact \(d\) extremal rays. However, a general cone does not necessarily have the finite number of the extremal rays. This is the reason why the \(K\)-irreducibility is not easily checked for a general cone \(K\).

In this section, as stronger conditions than the \(K\)-ergodicity and the \(K\)-mixing condition, we introduce the \(K\)-irreducibility and the \(K\)-primitivity, respectively; nevertheless the \(K\)-irreducibility is equivalent to a natural extension of the classical irreducibility (see Condition [1-31] in Theorem [1]).
Table 1. Relations among ergodic, mixing, irreducible, and primitive conditions

| Relation with convergence | $x_0 > 0$ | $x_0 > 0$ |
|---------------------------|-----------|-----------|
| $(1/n) \sum_{k=0}^{n-1} A^k x \to x_0$ | Ergodic | Irreducible |
| $A^n x \to x_0$ | Mixing | Primitive |

| Relation with eigenspace | $x_0 > 0$ | $x_0 > 0$ |
|--------------------------|-----------|-----------|
| $\dim \ker (A-I) = 1$ | Ergodic | Irreducible |
| $\dim \ker (A^{\otimes 2} - I^{\otimes 2}) = 1$ | Mixing | Primitive |

$A$ is a DUP $\mathcal{K}$-positive map. $x_0 \in \mathcal{S}(\mathcal{K}, u)$ is the stationary vector, and $x$ is any initial state. $I$ denotes the identity matrix. The limit $n \to \infty$ is taken in the upper table.

**Definition 7** ($\mathcal{K}$-irreducibility). A $\mathcal{K}$-positive map $A$ is called $\mathcal{K}$-irreducible when the following conditions hold.

- $A$ is $\mathcal{K}$-ergodic.
- A stationary vector $x_0$ and a dual one $y_0$ of $A$ are contained in $\mathcal{K}^\circ$ and $(\mathcal{K}^*)^\circ$, respectively.

**Definition 8** ($\mathcal{K}$-primitivity). A $\mathcal{K}$-positive map $A$ is called $\mathcal{K}$-primitive when the following conditions hold.

- $A$ is $\mathcal{K}$-mixing.
- A stationary vector $x_0$ and a dual one $y_0$ of $A$ are contained in $\mathcal{K}^\circ$ and $(\mathcal{K}^*)^\circ$, respectively.

Theorem 4 leads to the following corollary immediately.

**Corollary 3.** Let $\tilde{\mathcal{K}}$ be a closed convex cone with $\mathcal{K}^{\otimes 2} \subset \tilde{\mathcal{K}} \subset ((\mathcal{K}^*)^{\otimes 2})^*$, $A$ be a $\mathcal{K}$-positive map, and $A^{\otimes 2}$ be a $\tilde{\mathcal{K}}$-positive map. Then, $A$ is $\mathcal{K}$-primitive if and only if $A^{\otimes 2}$ is $\tilde{\mathcal{K}}$-irreducible.

Definitions 7 and 8 are clearly related to the $\mathcal{K}$-ergodicity and the $\mathcal{K}$-mixing condition. The upper table in Table 1 summarizes a relation among the ergodicity, irreducibility, mixing condition, and primitivity of a DUP $\mathcal{K}$-positive map. Corollaries 1 and 2 lead to another relation described by the lower table in Table 1. On the other hand, many other conditions equivalent to the $\mathcal{K}$-irreducibility are well-known. Including these existing conditions, we obtain Theorem 6 which summarizes equivalent conditions to the $\mathcal{K}$-irreducibility. Further, as seen in Corollary 3 by using equivalent conditions to the $\mathcal{K}^{\otimes 2}$-irreducibility, Corollary 3 produces equivalent conditions to the $\mathcal{K}$-primitivity.

**Theorem 6.** For any $\mathcal{K}$-positive map $A$, the following conditions are equivalent.

I-i. $A$ is $\mathcal{K}$-irreducible.

I-ii. $A$ and $A^*$ have eigenvectors $x_0 \in \mathcal{K}^\circ$ and $y_0 \in (\mathcal{K}^*)^\circ$ associated with $r(A) > 0$, respectively. In addition, the eigenvalue $r(A)$ of $A$ has the geometric multiplicity 1.

I-iii. For any vector $x \in \mathcal{K} \setminus \{0\}$, there exists a positive number $t$ such that $e^{tA} x \in \mathcal{K}^\circ$.

I-iv. Any vector $x \in \mathcal{K} \setminus \{0\}$ satisfies $(I + A)^{d-1} x \in \mathcal{K}^\circ$. 

Definitions 7 and 8 are clearly related to the $\mathcal{K}$-ergodicity and the $\mathcal{K}$-mixing condition. The upper table in Table 1 summarizes a relation among the ergodicity, irreducibility, mixing condition, and primitivity of a DUP $\mathcal{K}$-positive map. Corollaries 1 and 2 lead to another relation described by the lower table in Table 1. On the other hand, many other conditions equivalent to the $\mathcal{K}$-irreducibility are well-known. Including these existing conditions, we obtain Theorem 6 which summarizes equivalent conditions to the $\mathcal{K}$-irreducibility. Further, as seen in Corollary 3 by using equivalent conditions to the $\mathcal{K}^{\otimes 2}$-irreducibility, Corollary 3 produces equivalent conditions to the $\mathcal{K}$-primitivity.
If a vector $x \in \mathcal{K} \setminus \{0\}$ and a nonnegative number $\alpha$ satisfy $Ax \leq \alpha x$, then $x \in \mathcal{K}$.

A has no eigenvector on the boundary of $\mathcal{K}$.

For any nonzero extremal vectors $x$ and $y$ of $\mathcal{K}$ and $\mathcal{K}^*$, respectively, there exists a natural number $n$ such that $\langle y, A^n x \rangle > 0$.

Here, $d$ denotes the dimension of the vector space $\mathcal{V}$.

Proof. \[\text{[III-1]} \Rightarrow \text{[III-3]}\] This implication follows from Theorem I.

\[\text{[III-1]} \Rightarrow \text{[III-3]}\] Assume Condition [III-1] and let $x_0$ and $y_0$ be a stationary vector and a dual one, respectively, and $x \in \mathcal{K} \setminus \{0\}$. Since $\lim_{n \to \infty} (1/n) \sum_{k=0}^{n-1} (r(A)^{-1}A)^k x = \langle y_0, x \rangle x_0 \in \mathcal{K}$, a sufficiently large number $n$ satisfies $\sum_{k=0}^{n} (r(A)^{-1}A)^k x \in \mathcal{K}$.

Hence,

$$e^{r(A)^{-1}A}x \geq \sum_{k=0}^{n} \frac{r(A)^{-k}}{k!} A^k x \geq \frac{1}{n^k} \sum_{k=0}^{n} (r(A)^{-1}A)^k x > 0.$$
the inequality $\langle y, e^{tA}x \rangle > 0$ holds. Using the expansion $e^{tA} = \sum_{n=0}^{\infty} (t^n/n!)A^n$, we have $\langle y, A^n x \rangle > 0$ for a nonnegative integer $n$. Conversely, assume Condition (i), and let $x$ be a nonzero extremal vector of $\mathcal{K}$. Since for any nonzero extremal vector $y$ of $\mathcal{K}^*$, there exists a nonnegative integer $n$ such that $\langle y, A^n x \rangle > 0$, any nonzero extremal vector $y$ of $\mathcal{K}^*$ satisfies $\langle y, e^{tA}x \rangle > 0$. Since $\mathcal{K}^*$ is generated by extremal vectors of $\mathcal{K}^*$ (proved by using the Krein-Milman theorem; see the paper [11] for details), any vector $y \in \mathcal{K}^* \setminus \{0\}$ satisfies $\langle y, e^{tA}x \rangle > 0$, which means $e^{tA}x \in \mathcal{K}^\circ$. Since $\mathcal{K}$ is generated by extremal vectors of $\mathcal{K}$, any vector $x \in \mathcal{K} \setminus \{0\}$ satisfies $e^{tA}x \in \mathcal{K}^\circ$.

Any DUP map satisfies “$y_0 \in (\mathcal{K}^\circ)^\circ$” in Condition (ii), automatically. However, if we replace “$y_0 \in (\mathcal{K}^\circ)^\circ$” with “$y_0 \in \mathcal{K}^\circ$” in Condition (ii), the equivalence does not hold. In addition, we cannot also replace “$\partial \mathcal{K}$” with “$\mathcal{K}^\circ \setminus \text{span}(x_0)$” in Condition (ii). To explain them, we give a $\mathcal{K}$-mixing map that is not $\mathcal{K}$-irreducible. It has a stationary vector $x_0 \in \mathcal{K}^\circ$ and a dual one $y_0 \in \partial(\mathcal{K}^\circ)$ as follows.

Example 6. Letting $\mathcal{V} = \mathbb{R}^2$, $\mathcal{K} = [0, \infty)^2$ and

$$A := \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix},$$

we discuss the classical case. $A$ is $\mathcal{K}$-mixing because the two relations $r(A) = 2$ and

$$(r(A)^{-1}A)^n = \begin{bmatrix} 1 & 0 \\ 1 - \frac{1}{2^n} & 1/2^n \end{bmatrix}$$

hold for any natural number $n$. Here, its stationary vector and dual one are $x_0 = [1, 1] \top$ and $y_0 = [1, 0] \top$, respectively. On the other hand, $A$ is not $\mathcal{K}$-irreducible due to $y_0 \in \partial \mathcal{K} = \partial(\mathcal{K}^\circ)$. They are easily checked that $A$ has the unique eigenvector $x_0$ in $\mathcal{K}^\circ$ up to a scalar multiple, and that $A$ has the eigenvalue $[0, 1] \top \in \partial \mathcal{K}$. Hence, this example shows the importance of Conditions (ii) and (iii) $\square$

Remark 4. There are many studies on $\mathcal{K}$-irreducibility. The papers [11,12,13] defined a $\mathcal{K}$-irreducible map as a $\mathcal{K}$-positive map having no invariant face of $\mathcal{K}$. We call this condition the face condition. As for the definition of the face, see the papers [11,12,13]. Theorem 4.1 of [11] is the equivalence of the face condition and Condition (vi). The paper [12] has the equivalence of the face condition and Condition (v) (Proposition 6), that of the face condition and Condition (iii) (Theorem 7), and that of Conditions (ii) and (iii) (Sawashima’s Theorem). Condition (iii) is called semi-nonsupporting in the paper [12]. Barker and Schneider [13] proved the equivalence of the face condition and Condition (iv) under a more general setting.

Remark 5. For the quantum system, there are many studies on the irreducibility. Schrader [16] defined the irreducibility, which Schrader called ergodicity, as Condition (iii). The paper [16] has the equivalence of Conditions (iii) and (iv). Carbone and Pautrat [17] adopted the following definition of an irreducible map $A$: (*i) $A$ is a positive map, and no orthogonal projections $P \neq 0, I$ on $\mathcal{H}$ satisfy $A(P(T(H) + \sqrt{1 - T(H)})P) \subset P(T(H) + \sqrt{1 - T(H)})P$. Proposition 3.4 of [17] means the equivalence of Conditions (*i) and (iii). The lecture notes [23] proved
the following equivalences: the equivalence of Conditions (*), (**), (***) and (****) for any positive map (Theorems 6.2 and 6.4); that of Conditions (***), (**), and (****) for any trace-preserving positive map (Corollary 6.3).

Combining Corollary 3 and Theorem 6, we characterize the primitivity as follows.

**Corollary 4.** For any $\mathcal{K}$-positive map $A$, the following conditions are equivalent.

- **P-i.** $A$ is $\mathcal{K}$-primitive.
- **P-ii.** $A^{\otimes 2}$ and $(A^{\otimes 2})^*$ have eigenvectors in the interiors of $\mathcal{K}^{\otimes 2}$ and $(\mathcal{K}^{\otimes 2})^*$ associated with $r(A)^2 > 0$, respectively. In addition, the eigenvalue $r(A)^2$ of $A^{\otimes 2}$ has the geometric multiplicity 1.
- **P-iii.** For any vector $x \in \mathcal{K}^{\otimes 2} \setminus \{0\}$, there exists a positive number $t$ such that $e^{tA^{\otimes 2}}x \in (\mathcal{K}^{\otimes 2})^\circ$.
- **P-iv.** Any vector $x \in \mathcal{K}^{\otimes 2} \setminus \{0\}$ satisfies $(I^{\otimes 2} + A^{\otimes 2})^{-1}x \in (\mathcal{K}^{\otimes 2})^\circ$.
- **P-v.** If a vector $x \in \mathcal{K}^{\otimes 2} \setminus \{0\}$ and a nonnegative number $\alpha$ satisfy $A^{\otimes 2}x \leq \alpha x$, then $x \in (\mathcal{K}^{\otimes 2})^\circ$.
- **P-vi.** $A^{\otimes 2}$ has no eigenvectors on the boundary of $\mathcal{K}^{\otimes 2}$.

**Remark 6.** There are fewer studies on $\mathcal{K}$-primitivity. Barker [12] defined a $\mathcal{K}$-primitive map as a $\mathcal{K}$-positive map $A$ which satisfies that for any vector $x \in \partial \mathcal{K} \setminus \{0\}$, there exists a natural number $n$ such that $A^n x \in \mathcal{K}^\circ$. Proposition 2 of [12] means that the number $n$ can be taken independently of the vector $x$.

**Remark 7.** For the quantum system, fewer conditions that are equivalent to the $\mathcal{K}$-primitivity are well-known. The condition stated in Remark 6 can be rewritten as follows: (***) a positive map $A$ satisfies that for any state $\rho$, there exists a natural number $n$ such that $A^n(\rho)$ is strictly positive definite. As already stated in Remark 6 the number $n$ can be taken independently of the state $\rho$. Furthermore, for any trace-preserving and completely positive map, the natural number $n$ can be taken so that $n \leq (\dim \mathcal{H})^2((\dim \mathcal{H})^2 - N + 1)$ [18], where the natural number $N$ is the number of linearly independent Kraus operators of $A$. This fact is called a quantum Wielandt’s inequality in the paper [18]. For any trace-preserving positive map, the equivalence of Definition 5 and Condition (**) was proved in the lecture notes [24] Theorem 6.7. There is another condition which is based on the spectral condition (see Remark 4 [24] Condition 4 in Theorem 6.7). For any trace-preserving positive map, it is equivalent to Definition 5 [24] Theorem 6.7]. In this paper, we have explained that Corollary 3 makes many other conditions that are equivalent to the $\mathcal{K}$-primitivity, under a general setting.

In the paper [19] and the textbook [27] of classical probabilistic theory, the ergodicity means our $\mathcal{K}$-primitivity with $\mathcal{K} = [0, \infty)^d$. However, papers in quantum theory use the ergodicity to express Definition 4. Hence, we have adopted Definition 4 in Section 3. Applying Corollary 3 to the classical case, we have the following corollary.
Corollary 5. A transition matrix $W$ is primitive if and only if $W^\otimes 2$ is irreducible.

Due to this corollary, we find that the primitivity of a transition matrix $W$ can be determined by the support of the output distribution for $W^\otimes 2$ of each input.

Remark 8. Corollary 5 is also derived from an existing result in graph theory. To explain it, we rewrite the irreducibility and the primitivity of a transition matrix to terms of graph theory. A directed graph is called strongly connected when any two vertices are connected by a path in each direction [20]. A transition matrix $W$ is irreducible if and only if the directed graph corresponding to $W$ is strongly connected [26]. The directed graph corresponding to $W$ has the $d$ vertices $0, \ldots, d-1$, and it has directed edges $(j, i)$ if $⟨j|W|i⟩ > 0$. Fig. 1 is an example of a transition matrix, the directed graph corresponding to the transition matrix, and the adjacency matrix of the graph. Since the path $0 \rightarrow 1 \rightarrow 0 \rightarrow 2 \rightarrow 3 \rightarrow 0$ passes all the vertices and returns to the first vertex, the graph in Fig. 1 is strongly connected. Furthermore, we call the greatest common divisor of the length of all cycles contained in a strongly connected graph, the period. The period of an irreducible transition matrix is defined as the period of the directed graph corresponding to the transition matrix [29]. The textbook [26] call it the index of imprimitivity. The period of an irreducible transition matrix $W$ equals one if and only if $W$ is primitive (as for the only if part, see the textbooks [26,29]; the if part is proved from the definition). In graph theory, the tensor product of two directed graph $G_1$ and $G_2$ is defined as the directed graph corresponding to the adjacency matrix $A_1 \otimes A_2$, where $A_1$ and $A_2$ are the adjacency matrices of $G_1$ and $G_2$, respectively. McAndrew [20] calls it “the product” simply, and proved the following statement [20, Theorem 1]: if two directed graphs $G_1$ and $G_2$ are strongly connected, then their tensor product has the exact $d_{12}$ strongly connected components, where $d_{12}$ is the greatest common divisor of the periods of $G_1$ and $G_2$. When both $G_1$ and $G_2$ are the directed graph corresponding to an irreducible transition matrix $W$, this statement implies Corollary 5 because in this case, $d_{12}$ equals the period of the directed graph corresponding to $W$.

Finally, we show that the $K$-irreducibility and the $K$-primitivity are preserved under a proper deformation of a map. The deformation is used in considering a cumulant generating function.

Corollary 6. Let $Ω$ be a nonempty finite set, $A_ω$ be a $K$-positive map and $a_ω$ be a positive number for each $ω \in Ω$. If $A := \sum_ω A_ω$ is $K$-irreducible, then so is $A_a := \sum_ω a_ω A_ω$. If $A := \sum_ω A_ω$ is $K$-primitive, then so is $A_a := \sum_ω a_ω A_ω$. 

}\textbf{Fig. 1.} The left matrix is a transition matrix, the center graph is the directed graph corresponding to the transition matrix, and the right matrix is the adjacency matrix of the center graph.
Proof. We adopt Condition I-v. Suppose $A$ be a $K$-irreducible map, and let $x \in K \setminus \{0\}$, $\alpha \geq 0$, $A_\alpha x \leq \alpha x$, and $a_{\min} := \min_{\omega} a_\omega > 0$. Then, the inequality $a_{\min} A x \leq A_\alpha x \leq \alpha x$ holds, and so the relation $x \in K^\circ$ results from the $K$-irreducibility of $A$. Therefore, $A_\alpha$ is also $K$-irreducible.

We adopt Condition I-v again. Suppose $A$ be a $K$-primitive map. Since $A \otimes 2 = \sum_{\omega_1, \omega_2} A_{\omega_1} \otimes A_{\omega_2}$ and $A_\alpha \otimes 2 = \sum_{\omega_1, \omega_2} a_{\omega_1, \omega_2} A_{\omega_1} \otimes A_{\omega_2}$, we obtain the second assertion from the first one in this theorem.

Unfortunately, the $K$-ergodic or $K$-mixing version of Proposition 3 does not hold. To explain it, we give two $K$-positive maps $A_1$ and $A_2$ that satisfy that $A_1 + A_2$ is $K$-mixing, and that $(1/2)A_1 + A_2$ is not $K$-ergodic.

Example 7. Letting $V = \mathbb{R}^2$ and $K = [0, \infty)^2$, we discuss the classical case. The two matrices

$$A_1 := \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

satisfy that $A_1 + A_2$ is $K$-mixing because it has the two eigenvalues 2 and 1. However, as already shown Example 4, $(1/2)A_1 + A_2$ is not $K$-ergodic. ⊓⊔

6. Conclusion

We have clarified the relations between the mixing and asymptotically decoupling properties. Since the mixing condition of a dynamical map $A$ is equivalent to the ergodicity of $A \otimes 2$, conditions equivalent to the ergodicity can be applied to those equivalent to the mixing condition. For example, we have proved that the mixing property of a given dynamical map $A$ is determined by the dimension of the kernel of $A \otimes 2 - I \otimes 2$. Furthermore, we have stated that conditions equivalent to the primitivity are made of those equivalent to the irreducibility.

There are many existing results on the ergodicity and the mixing condition [14, 15], but no papers discussed the 2-fold tensor product of a dynamical map. Some existing results will be more easily shown by using our ideas in this paper.

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A. Lemmas for convex cones

As technical preparation, we prove the following lemmas for convex cones.

Lemma 6. For any closed convex cone $K \neq \emptyset$, the following relations hold:

$$\partial(K^*) = \{ y \in V^* \mid (y, x) = 0 \ (\exists x \in K \setminus \{0\}) \},$$

$$\partial K = \{ x \in V \mid (y, x) = 0 \ (\exists y \in K^* \setminus \{0\}) \},$$

$$\langle K^* \rangle^\circ = \{ y \in V^* \mid (y, x) > 0 \ (\forall x \in K \setminus \{0\}) \},$$

$$K^\circ = \{ x \in V \mid (y, x) > 0 \ (\forall y \in K^* \setminus \{0\}) \}.$$
Proof. Let \( S \) be the unit sphere of \( V \). We show
\[
(K^*)^o = \{ y \in V^* \mid \langle y, x \rangle > 0 \ (\forall x \in K \cap S) \}.
\] (7)

Let a vector \( y \in V^* \) satisfy \( \langle y, x \rangle > 0 \) for any vector \( x \in K \cap S \). Then, \( y \neq 0 \).

Any vector \( y' \in V^* \) with \( \|y'\| < \min_{x \in K \cap S} \langle y, x \rangle \) satisfies
\[
\min_{x \in K \cap S} \langle y + y', x \rangle \geq \min_{x \in K \cap S} \langle y, x \rangle - \|y'\| > 0.
\]

This means that \( y + y' \in K^* \) whenever \( \|y'\| < \min_{x \in K \cap S} \langle y, x \rangle \), and so \( y \in (K^*)^o \).

Thus, \((K^*)^o\) contains the right-hand side of the relation (7).

Let \( y \in (K^*)^o \). We prove the remaining part of the relation (7) by contradiction. Suppose \( \langle y, x \rangle = 0 \) for a vector \( x \in K \cap S \). Since \( y \in (K^*)^o \), we can take a sufficiently small number \( \epsilon > 0 \) such that \( y - \epsilon x \in K^* \). However, the inequality \( 0 \leq \langle y - \epsilon x, x \rangle = -\epsilon < 0 \) is a contradiction. Hence, the relation (7) holds. Since \( K \) is a convex cone, we obtain the 3rd relation.

Since \( K \neq \emptyset \) is a closed convex cone, the relation \( K^{**} = K \) holds. Using the 3rd relation, we obtain the 4th relation. The 1st and 2nd relations follows from the 3rd and 4th ones, respectively.

**Lemma 7.** Let \( K \) be a convex cone having nonempty interior. Then, \( V = K + (-K) := \{ x_1 - x_2 \mid x_1, x_2 \in K \} \).

Proof. From the definition, \( K + (-K) \) is a linear subspace of \( V \). Since \( K \) has the nonempty interior, so does \( K + (-K) \). Therefore, \( V = K + (-K) \).

**Lemma 8.** Let \( K \) be a closed convex cone satisfying \( K^o \neq \emptyset \) and \( K \cap (-K) = \{ 0 \} \).

Then, there exists a unit vector \( e \in K^o \) such that
\[
K \setminus \{ 0 \} \subset \{ x \in V \mid \langle x, e \rangle > 0 \}.
\]

In particular, \( K^o \cap (K^*)^o \neq \emptyset \).

Proof. Let \( S \) be the unit sphere of \( V \), \( \overline{B} \) be the closed unit ball of \( V \), and \( K_0 := \text{conv}(K \cap S) \subset K \cap \overline{B} \), where \( \text{conv}(X) \) denotes the convex hull of a subset \( X \) of \( V \). \( K_0 \) is a compact convex set. The zero vector is an extreme point of \( K \) owing to \( K \cap (-K) = \{ 0 \} \), whence \( 0 \notin K_0 \). Therefore,
\[
\sup_{x \in K \cap S} \min_{y \in K \cap S} \langle x, y \rangle \overset{(a)}{=} \max_{y \in K \cap \overline{B}} \min_{x \in K \cap S} \langle y, x \rangle \overset{(b)}{=} \max_{y \in K_0} \min_{x \in K_0} \langle y, x \rangle \geq \max_{x \in K_0} \min_{y \in K_0} \langle x, y \rangle \overset{(c)}{=} \min_{y \in K_0} \max_{x \in K_0} \langle x, y \rangle \geq \max_{y \in K_0} \|y\|^2 > 0.
\]

This inequality implies our assertion.

We check this inequality. The equality (b) holds, because the function \( \langle x, y \rangle \) of \( y \in K_0 \) can be minimized when the vector \( y \) is an extreme point of \( K_0 \). The equality (c) results from the minimax theorem. The equality (a) can be shown as follows: defining \( f(x) := \min_{y \in K \cap S} \langle x, y \rangle \) for any vector \( x \), the function \( f \) is concave. From \( K^o \neq \emptyset \), we can take a vector \( x_0 \in K^o \cap \overline{B} \). Taking any vector \( x_1 \in \partial K \cap \overline{B} \), we find that the vector \( x_1 \) satisfies that \( x_1 := (1-t)x_0 + tx_1 \in K^o \cap \overline{B} \) for any number \( 0 \leq t < 1 \). From the concavity of \( f \), we have
\[
f(x_1) \geq (1-t)f(x_0) + tf(x_1), \quad \lim_{t \downarrow 1} f(x_1) \geq f(x_1).
\]

Hence, the equality (a) holds. (We have shown the \( \geq \) part. The \( \leq \) part is trivial.)
Corollary 7. Let $\mathcal{K}$ be a closed convex cone with $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. Then, there exists a unit vector $e \in \mathcal{K}$ such that
\[ \mathcal{K} \setminus \{0\} \subset \{ x \in \mathcal{V} \mid \langle x, e \rangle > 0 \}. \]
In particular, $\mathcal{K} \cap (\mathcal{K}^*)^\circ \neq \emptyset$.

Proof. Our assertion is more readily shown than Lemma 8. Indeed, the same inequality
\[ \max_{x \in \mathcal{K} \cap \mathcal{B}} \min_{y \in \mathcal{K} \cap \mathcal{S}} \langle x, y \rangle \leq \min_{y \in \mathcal{K} \cap \mathcal{S}} \max_{x \in \mathcal{K} \cap \mathcal{B}} \langle x, y \rangle \]
holds. This inequality implies our assertion.

Lemma 9. Let $\mathcal{K}$ be a closed convex cone satisfying $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$, and $x_1, x_2 \in \mathcal{V}$. If any real number $\alpha$ satisfies $x_1 + \alpha x_2 \in \mathcal{K}$, then $x_2 = 0$.

Proof. Assume that any real number $\alpha$ satisfies $x_1 + \alpha x_2 \in \mathcal{K}$. Since any vector $y \in \mathcal{K}^*$ and any real number $\alpha$ satisfy
\[ 0 \leq \langle y, x_1 + \alpha x_2 \rangle = \langle y, x_1 \rangle + \alpha \langle y, x_2 \rangle, \]
and so $\langle y, x_2 \rangle = 0$ for any vector $y \in \mathcal{K}^*$. Corollary 7 implies $(\mathcal{K}^*)^\circ \neq \emptyset$, which implies the existence of a basis of $\mathcal{V}$ composed of elements of $\mathcal{K}^*$. Therefore, $x_2 = 0$.

Lemma 10. Let $\mathcal{K}$ be a closed convex cone. If a nonzero vector $e$ satisfies
\[ \mathcal{K} \setminus \{0\} \subset \mathcal{H}_+ := \{ x \in \mathcal{V} \mid \langle x, e \rangle > 0 \}, \]
then $\mathcal{K}_1 := \mathcal{K} \cap \mathcal{H}_1$ is a compact convex set, where $\mathcal{H}_1 := \{ x \in \mathcal{V} \mid \langle x, e \rangle = 1 \}$.

Proof. Since $\mathcal{K}_1$ is a closed convex set, all we need to do is show that $\mathcal{K}_1$ is bounded. We prove it by contradiction. Suppose that a sequence $(x_n) \subset \mathcal{K}_1$ with $\|x_n\| \to \infty$ exists, we have
\[ 0 < \min_{x \in \mathcal{K} \cap \mathcal{S}} \langle x, e \rangle \leq \langle x_n, e \rangle / \|x_n\| = 1 / \|x_n\| \to 0, \]
where $\mathcal{S}$ is the unit sphere of $\mathcal{V}$. This contradiction implies that $\mathcal{K}_1$ is bounded.

Lemma 11. Let $\mathcal{K}$ be a closed convex cone of $\mathcal{V}$ satisfying $\mathcal{K}^\circ \neq \emptyset$ and $\mathcal{K} \cap (-\mathcal{K}) = \{0\}$. Then, $\mathcal{K}^*$ also satisfies the same assumptions as $\mathcal{K}$.

Proof. From the definition, $\mathcal{K}^*$ is a closed convex cone. Lemma 8 leads to $(\mathcal{K}^*)^\circ \neq \emptyset$. We show $\mathcal{K}^* \cap (-\mathcal{K}^*) = \{0\}$. Let $y \in \mathcal{K}^* \cap (-\mathcal{K}^*)$. Then, any vector $x \in \mathcal{K}$ satisfies $\langle y, x \rangle = 0$. Taking a vector $x \in \mathcal{K}^*$, we find that the equation $\langle y, x \rangle = 0$ yields $y = 0$.

Lemma 12. Let $\mathcal{K}_j$ be a closed convex cone of $\mathcal{V}_j$ satisfying $\mathcal{K}_j^\circ \neq \emptyset$ and $\mathcal{K}_j \cap (-\mathcal{K}_j) = \{0\}$ for $j = 1, 2$. Then, $\mathcal{K}_1 \otimes \mathcal{K}_2$ also satisfies the same assumptions as $\mathcal{K}_1$. 
Proof. From the definition, $K_1 \otimes K_2$ is a convex cone. First, we show $(K_1 \otimes K_2)^\circ \neq \emptyset$. Take a basis $(x_{ij}^k)_{i=1}^l \subset K_j$ of $V_j$ for $j = 1, 2$. Then, the tuple $(x_{i_1}^1 \otimes x_{i_2}^2)_{i_1,i_2} \subset K_1 \otimes K_2$ is also a basis of $V_1 \otimes V_2$. Hence, $(K_1 \otimes K_2)^\circ \neq \emptyset$.

Next, we show that $K_1 \otimes K_2$ is closed. From Lemma 8 we can take a unit vector $e^l \in K_j$ satisfying $K_j \setminus \{0\} \subset H_{j,l} := \{ x^l \in V_j \mid \langle x^l, e^l \rangle > 0 \}$ for $j = 1, 2$. Put $H_{j,l} := \{ x^l \in V_j \mid \langle x^l, e^l \rangle = 1 \}$ and $K_{j,l} := K_j \cap H_{j,l}$ for $j = 1, 2$. Then, Lemma 10 yields that $K_{1,l}$ and $K_{2,l}$ are compact, and so is $K_{1,l} \times K_{2,l}$, where $K_{1,l} \times K_{2,l}$ denotes the direct product of $K_{1,l}$ and $K_{2,l}$.

Defining the map $f : K_{1,l} \times K_{2,l} \to V_1 \otimes V_2$ to be $(x^1, x^2) \mapsto x^1 \otimes x^2$, we find that the map $f$ is continuous, whence $f(K_{1,l} \times K_{2,l})$ is compact. Hence, $\text{conv}(f(K_{1,l} \times K_{2,l}))$ is also compact, where $\text{conv}(X)$ denotes the convex hull of a set $X$.

Putting $\tilde{H}_l := \{ x \in V_1 \otimes V_2 \mid \langle x, e^1 \otimes e^2 \rangle = 1 \}$, we show $\text{conv}(f(K_{1,l} \times K_{2,l})) = (K_1 \otimes K_2) \cap \tilde{H}_l$. From the definitions, the inclusion relation $\text{conv}(f(K_{1,l} \times K_{2,l})) \subset (K_1 \otimes K_2) \cap \tilde{H}_l$ holds. Let $x \in (K_1 \otimes K_2) \cap \tilde{H}_l$. Then, the vector $x$ can be written as

$$x = \sum_{i=1}^l \alpha_i x_i^1 \otimes x_i^2,$$

where $x_i^j \in K_{j,l}$ and $\alpha_i \geq 0$ for any integers $1 \leq i \leq l$ and $j = 1, 2$. The equation $1 = \langle x, e^1 \otimes e^2 \rangle = \sum_{i=1}^l \alpha_i$ yields $x \in \text{conv}(f(K_{1,l} \times K_{2,l}))$. Therefore, $\text{conv}(f(K_{1,l} \times K_{2,l})) = (K_1 \otimes K_2) \cap \tilde{H}_l$, whence the set $(K_1 \otimes K_2) \cap \tilde{H}_l$ is compact.

Letting $(x_n) \subset K_1 \otimes K_2$ be a convergent sequence whose limit is a vector $x$, we show $x \in K_1 \otimes K_2$. Due to $0 \in K_1 \otimes K_2$, we may assume $x \neq 0$. Put $\tilde{H}_+ := \{ x \in V_1 \otimes V_2 \mid \langle x, e^1 \otimes e^2 \rangle > 0 \}$. Then, the inclusion relation $(K_1 \otimes K_2) \setminus \{0\} \subset \tilde{H}_+$ holds. From $x \neq 0$, any sufficiently large number $n$ satisfies $(x_n, e^1 \otimes e^2) > 0$. Since $(K_1 \otimes K_2) \cap \tilde{H}_1$ is compact, there exists a subsequence $(x_{n(k)})$ of $(x_n)$ such that the sequence $((x_{n(k)}), e^1 \otimes e^2)^{-1}$ converges to a vector $x' \in (K_1 \otimes K_2) \cap \tilde{H}_1$. Since $(x_{n(k)}), e^1 \otimes e^2) \to \langle x, e^1 \otimes e^2 \rangle \geq 0$, we have $x_{n(k)} \to x \to \langle x, e^1 \otimes e^2 \rangle x' \in K_1 \otimes K_2$. Therefore, $K_1 \otimes K_2$ is closed.

We show $(K_1 \otimes K_2) \cap (-K_1 \otimes K_2) = \{0\}$. From Lemma 8 we can take a unit vector $e^l \in K_j$ satisfying $K_j \setminus \{0\} \subset \{ x^l \in V_j \mid \langle x^l, e^l \rangle > 0 \}$ for $j = 1, 2$. If $x \in (K_1 \otimes K_2) \cap (-K_1 \otimes K_2)$, the vector $x$ can be written as

$$x = \sum_{i=1}^l x_i^1 \otimes x_i^2 = - \sum_{i=1}^{l'} x_i'^1 \otimes x_i'^2,$$

where $x_i^j$ and $x_i'^j$ with $i = 1, \ldots, l$ and $i' = 1, \ldots, l'$ are elements of $K_j$ for $j = 1, 2$. Therefore, it is sufficient to show that the relations $\sum_{i=1}^{l''} x_i^1 \otimes x_i^2 = 0$ and $x_i^1 \in K_j$ with $i = 1, \ldots, l''$ and $j = 1, 2$ imply that $x_i^1 \otimes x_i^2 = 0$ for any
integer $1 \leq i \leq l''$. We prove it by contradiction. Suppose an integer $1 \leq i_0 \leq l''$ satisfies $x^1_{i_0} \otimes x^2_{i_0} \neq 0$, we have

$$0 = \sum_{i=1}^{l''} \langle e^1_i, x^1_i \rangle \langle e^2_i, x^2_i \rangle \geq \langle e^1_{i_0}, x^1_{i_0} \rangle \langle e^2_{i_0}, x^2_{i_0} \rangle > 0.$$ 

This contradiction implies that $x^1_i \otimes x^2_i = 0$ for any integer $1 \leq i \leq l''$.

**Lemma 13.** Let $\mathcal{K}_1$ and $\mathcal{K}_2$ satisfy the same assumptions as Lemma 12. Let $\tilde{\mathcal{K}}$ be a closed convex cone with $\mathcal{K}_1 \otimes \mathcal{K}_2 \subset \tilde{\mathcal{K}} \subset (\mathcal{K}_1^* \otimes \mathcal{K}_2^*)^*$, and $x^j \in \mathcal{K}_j$ for $j = 1, 2$. Then, $x^1 \otimes x^2 \in \tilde{\mathcal{K}}^\circ$ if and only if $x^j \in \mathcal{K}_j^\circ$ for $j = 1, 2$.

**Proof.** We prove the only if part by contradiction. Suppose $x^1 \otimes x^2 \in \tilde{\mathcal{K}}^\circ$ and $x^1 \in \partial \mathcal{K}_1$. Taking a vector $y^1 \in (\mathcal{K}_1^\circ \setminus \{0\})$ with $\langle x^1, y^1 \rangle = 0$ and a vector $y^2 \in \mathcal{K}_2 \setminus \{0\}$, we have $x^1 \otimes y^1 = 0$. Since $y^1 \otimes y^2 \in (\mathcal{K}_1^* \otimes \mathcal{K}_2^*) \setminus \{0\} \subset \mathcal{K}^* \setminus \{0\}$, this contradicts our assumption $x^1 \otimes x^2 \in \tilde{\mathcal{K}}^\circ$. Hence, $x^1 \in \mathcal{K}_1^\circ$. The other relation $x^2 \in \mathcal{K}_2^\circ$ is also shown by the same way.

Conversely, let $x^j \in \mathcal{K}_j^\circ$ for $j = 1, 2$, we can take a basis $(x^j_i)_{i=1}^{d_j} \subset \mathcal{K}_j$ of $\mathcal{V}_j$ with $\sum_{i=1}^{d_j} x^j_i = x^j$ for each $j = 1, 2$. Since the tuple $(x^1_i \otimes x^2_{i_2})_{i_1, i_2} \subset \mathcal{K}_1 \otimes \mathcal{K}_2$ is a basis of $\mathcal{V}_1 \otimes \mathcal{V}_2$, the set

$$\mathcal{O} := \left\{ \sum_{i_1, i_2} \alpha_{i_1, i_2} x^1_{i_1} \otimes x^2_{i_2} \left| \alpha_{i_1, i_2} > 0 \right\} \subset \mathcal{K}_1 \otimes \mathcal{K}_2 \subset \tilde{\mathcal{K}} \right.$$ 

is open in $\mathcal{V}_1 \otimes \mathcal{V}_2$, and so $x^1 \otimes x^2 \in \mathcal{O} \subset \tilde{\mathcal{K}}^\circ$.

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