STABILITY OF PLANAR RAREFACTION WAVE TO 3D FULL COMPRESSIBLE NAVIER-STOKES EQUATIONS

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Abstract. We prove the time-asymptotic stability toward planar rarefaction wave for the three-dimensional full compressible Navier-Stokes equations in an infinite long flat nozzle domain $\mathbb{R} \times \mathbb{T}^2$. Compared with one-dimensional case, the proof here is based on our new observations on the cancellations on the flux terms and viscous terms due to the underlying wave structures, which are crucial to overcome the difficulties due to the wave propagation along the transverse directions $x_2$ and $x_3$ and its interactions with the planar rarefaction wave in $x_1$ direction.

1. Introduction

The motion of compressible viscous and heat-conductive fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ is governed by the following full compressible Navier-Stokes system:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p &= \text{div}\mathcal{T}, \\
(\rho E)_t + \text{div}(\rho Eu + pu) &= \kappa \Delta \theta + \text{div}(u\mathcal{T}),
\end{align*}
\]

where $t \geq 0$ is the time variable and $x = (x_1, x_2, x_3) \in \Omega$ is the spatial variables. In the present paper, we are concerned with the viscous fluid flowing in an infinite long flat nozzle domain $\Omega := \mathbb{R} \times \mathbb{T}^2$ with $\mathbb{R}$ being a real line and $\mathbb{T}^2 := (\mathbb{R}/\mathbb{Z})^2$ being a two-dimensional unit flat torus. The functions $\rho, u = (u_1, u_2, u_3)\,^t, p$ and $\theta$ represent respectively the fluid density, velocity, pressure and absolute temperature and $E = e + \frac{1}{2}|u|^2$ is the specific total energy with $e$ being the internal energy, and $\mathcal{T}$ is the viscous stress tensor given by

\[
\mathcal{T} = 2\mu \mathcal{D}(u) + \lambda \text{div} \mathbb{I}
\]

where $\mathcal{D}(u) = \frac{\nabla u + (\nabla u)^t}{2}$ stands for the deformation tensor, $\mathbb{I}$ is the $3 \times 3$ identity matrix and $\mu$ and $\lambda$ represent the shear and bulk viscosity coefficients of the fluids respectively and they both are constants satisfying the physical constraints:

\[
\mu > 0, \quad 2\mu + 3\lambda \geq 0.
\]

Moreover, the constant $\kappa > 0$ denotes the heat-conductivity coefficient for the fluids. The equations (1.1) then express respectively the conservation of mass, the balance of momentum, and the balance of energy for the flow under the effect of the inner pressure, viscosities and the conduction of thermal energy. Here we investigate the ideal poly-tropic fluids such that the pressure $p$ and the internal energy $e$ are given by the following state equations:

\[
p = R\rho \theta = A\rho^\gamma \exp \left( \frac{\gamma - 1}{R} S \right), \quad e = \frac{R}{\gamma - 1} \theta,
\]

where $S$ is the entropy, $\gamma > 1$ is the adiabatic exponent, and both $A$ and $R$ are positive fluid constants.

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The following initial data is imposed to the system (1.1)

\[(1.5)\quad (\rho, u, \theta)(x,t=0) = (\rho_0, u_0, \theta_0)(x), \quad x \in \Omega.\]

Since we are concerned with the stability of the planar rarefaction wave to the system (1.1), we consider the following far-fields conditions on the \(x_1\)-direction

\[(1.6)\quad (\rho, u, \theta)(x,t) \to (\rho_{\pm}, u_{\pm}, \theta_{\pm}), \quad \text{as} \quad x_1 \to \pm \infty, \quad t > 0,
\]

with \(u_{\pm} = (u_{1\pm}, u_{2}, u_{3})^t\) and \(\rho_{\pm} > 0, \quad u_{1\pm}, \quad \theta_{\pm} > 0\) are prescribed constant states, and the periodic boundary conditions are imposed on \((x_2, x_3) \in T^2\) for the solution \((\rho, u, \theta)(x,t)\). Moreover, the two end states \((\rho_{\pm}, u_{\pm}, \theta_{\pm})\) are connected by the rarefaction wave solution to the Riemann problem of the corresponding 1D compressible Euler system:

\[(1.7)\quad \begin{cases}
\rho_t + (\rho u_1)_{x_1} = 0, \\
(\rho u_1)_t + (\rho u_1^2 + p)_{x_1} = 0, \\
(\rho E)_t + (\rho u_1 u_1 + pu_1)_{x_1} = 0,
\end{cases}\]

with the Riemann initial data

\[(1.8)\quad (\rho, u_1, \theta)(x_1,0) = (\rho_0^{r}, u_1^{r}, \theta_0^{r})(x_1) = \begin{cases}
(\rho_-, u_{1-}, \theta_-), \quad x_1 < 0, \\
(\rho_+, u_{1+}, \theta_+), \quad x_1 > 0.
\end{cases}\]

It could be expected that the large-time behavior of the solution to the compressible Navier-Stokes equations (1.1)-(1.6) is closely related to the Riemann problem to the corresponding three-dimensional compressible Euler equations

\[(1.9)\quad \begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla p = 0, \\
(\rho E)_t + \text{div}(\rho u u + pu) = 0,
\end{cases}\]

with the Riemann initial data

\[(1.10)\quad (\rho, u, \theta)(x,0) = (\rho_0^{r}, u_0^{r}, \theta_0^{r})(x) = \begin{cases}
(\rho_-, u_-, \theta_-), \quad x_1 < 0, \\
(\rho_+, u_+, \theta_+), \quad x_1 > 0.
\end{cases}\]

The inviscid compressible Euler system (1.7) or (1.9) is an ideal fluid model with the dissipative effects being neglected, which is a typical example of the system of hyperbolic conservation laws. The most important feature of the hyperbolic system (1.7) or (1.9) is that its classical solution may blow up, that is, the shock may form, in finite time, no matter how smooth or small the initial data is. Actually, there are three basic wave patterns to the system of hyperbolic conservation laws, i.e., shock and rarefaction waves in the genuinely nonlinear characteristic fields, and contact discontinuity in the linearly degenerate fields. However, the motion of real fluids should take into account the effects of both viscosities and heat-conductivity, which is described by the compressible Navier-Stokes system (1.1), i.e., the corresponding viscous system of inviscid Euler system (1.7) or (1.9). Moreover, it can be expected that the large-time behavior of the solutions to the compressible Navier-Stokes equations (1.1)-(1.6) is governed by the solutions of the corresponding Riemann problem (1.9)-(1.10), which contains planar shock wave, planar rarefaction wave and contact discontinuity in general. It is interesting and important to investigate the time-asymptotic stability of these basic planar wave patterns to the compressible Navier-Stokes equations (1.1) in higher dimension. In the present paper, we first study the nonlinear stability of planar rarefaction wave to the system (1.1) in an infinite long flat nozzle domain \(\Omega := \mathbb{R} \times T^2\).

On one hand, there are essential differences between the one-dimensional Riemann problem (1.7)-(1.8) and the multi-dimensional Riemann problem (1.9)-(1.10) even with the components \(u_2, u_3\) are
continuous on both sides of \( x_1 = 0 \) as in (1.10). Precisely speaking, it is first proved by Chiodaroli, De Lellis and Kreml [3] and Chiodaroli and Kreml [4] that there exist infinitely many bounded admissible weak solutions to (1.9)-(1.10) in two-dimensional isentropic regime satisfying the natural entropy condition for shock Riemann initial data by using the elegant convex integration methods as in De Lellis and Székelyhidi [5]. Meanwhile, the construction of weak solutions in [3, 4] seems essential to the two-dimensional system and can not be applied to one-dimensional problem (1.7)-(1.8). Then Klingen-berg and Markfelder [17] and Brezina, Chiodaroli and Kreml [1] extend the results in [3, 4] to the case when the corresponding Riemann initial data contain shock or contact discontinuity. On the other hand, similar to the one-dimensional case, for the Riemann solution only containing rarefaction waves to (1.9)-(1.10), Chen and Chen [2] and Feireisl and Kreml [6], Feireisl, Kreml and Vasseur [7] independently proved rarefaction wave is unique in the class of bounded weak solution to (1.9)-(1.10) even the rarefaction waves are connected with vacuum states (cf. [2]).

As mentioned before, the inviscid Euler system (1.7) or (1.9) is an ideal fluid model and the real fluids could be described by the viscous system (1.1), which is a typical example of the system of the viscous conservation laws. Deep investigations have been achieved on the nonlinear stability of basic wave patterns for viscous conservation laws in one-dimensional case. For the asymptotic stability of viscous shock profile, it started from Goodman [8] for the uniformly viscous conservation laws and Matsumura and Nishihara [20] for the compressible Navier-Stokes equations with physical viscosities independently by the anti-derivative methods under zero mass condition imposed on the initial perturbation. Then Liu [21] and Szepessy and Xin [33] removed the zero mass condition for the uniformly viscous conservation laws by introducing the suitable shift on the shock profile and diffusion waves in the transverse characteristic fields and Liu and Zeng [25] for the physical viscosity case. For the stability of rarefaction wave, we refer to Matsumura and Nishihara [27, 28] for isentropic compressible Navier-Stokes equations and Liu and Xin [22] for the non-isentropic system. Then Liu and Yu [24] proved the stability of rarefaction wave to one-dimensional general \( n \times n \) conservation laws system with artificial viscosity by point-wise Green function methods. For the stability of viscous contact discontinuity wave, we refer to Liu and Xin [23] and Xin [36] for the uniformly viscous conservation laws and Huang, Matsumura and Xin [12] for the compressible Navier-Stokes equations under the zero mass condition on the perturbation. Then Huang, Xin and Yang [14] removed this zero mass condition in [12] for the 1D compressible Navier-Stokes equations (1.1). For the composite waves, Huang and Matsumura [11] first studied the asymptotic stability of two viscous shock waves under general initial perturbation without zero mass conditions on initial perturbations for the full 1D Navier-Stokes system and Huang, Li and Matsumura [10] justified the stability of a combination wave of a viscous contact wave and rarefaction waves. Recently, Huang and Wang [13] improved the stability result in [10] to a class of large initial perturbations.

Although there have been rather satisfactory results about the stability of basic wave patterns for viscous conservation laws in the one-dimensional case, the stability toward the planar wave patterns for the compressible Navier-Stokes equations (1.1) in multi-dimensional case is still open due to the higher dimensionality. For the scalar viscous conservation laws, Xin [35] proved the asymptotic stability of planar rarefaction waves in multi-dimensional case by elementary \( L^2 \)-energy method in 1990. Then Ito [15] and Nishikawa and Nishihara [31] extended the stability result in [35] by obtaining the decay rate in time. For an artificial \( 2 \times 2 \) system with positively definite viscosity matrix, Hokari and Matsumura [9] proved the stability of the planar rarefaction wave in two-dimensional case, which crucially depends on the strict positivity of the viscosity matrix and can not be applied to the compressible Navier-Stokes system (1.1) with physical viscosities. For the compressible and isentropic Navier-Stokes equations,
which is the special case of the system (1.1) with the entropy being constant and the energy equation can be decoupled and neglected, the first and third author of the present paper Li and Wang [19] proved the stability of planar rarefaction wave in two-dimensional domain $\mathbb{R} \times \mathbb{T}$.

In the present paper, we shall prove the time-asymptotic stability of the planar rarefaction wave for the three-dimensional full compressible Navier-Stokes equation (1.1) with physical viscosities and heat-conductivity for any adiabatic exponent $\gamma > 1$. Compared with the one-dimensional stability results in [27, 28, 30], the main difference here lies in higher dimensionality and the physical viscosities terms coupled in momentum equation (1.1)$_2$ and the energy equation (1.1)$_3$ and we can not use the technique for one-dimensional fluid by substituting the mass equation (1.1)$_1$ into the momentum equation (1.1)$_2$ directly to obtain the derivative estimates of the density function as in [27, 28, 30]. Compared with the two-dimensional stability result for isentropic flow in [19], the full compressible Navier-Stokes equation (1.1) here is a real physical model involving the thermal conduction and the main difference lies in additionally in three-dimensional domain. Fortunately, we observe some cancellations between the flux terms and viscosity terms for the full compressible Navier-Stokes equations (1.1) such that we can successfully overcome the difficulties due to the planar rarefaction wave propagation in $x_2, x_3$-directions and its interactions with $x_1$-direction and finally we can prove our time-asymptotic stability toward the planar rarefaction wave. More precisely, we prove that if the initial data $(\rho_0, u_0, \theta_0)$ in (1.5) is suitably close to the planar rarefaction wave, then the three-dimensional problem (1.1)–(1.6) admits a global-in-time smooth solution which tends to the planar rarefaction wave as $t \to +\infty$. Note that the rarefaction wave strength $|[(\rho_+ - \rho_-, u_+ - u_-, \theta_+ - \theta_-)]$ here need not to be sufficiently small. The detailed stability result can be found in Theorem 1.1 below.

To state our main result, we first recall the planar rarefaction wave. It is straight to calculate that the Euler system (1.7) for $(\rho, u_1, \theta)$ has three distinct eigenvalues

$$
\lambda_i(\rho, u_1, S) = u_1 + (-1)^{i+1} \sqrt{p(\rho, S)}, \quad i = 1, 3,
$$

$$
\lambda_2(\rho, u_1, S) = u_1,
$$

with corresponding right eigenvectors

$$
r_i(\rho, u_1, S) = ((-1)^{i+1} \rho, \sqrt{p(\rho, S)}, 0)^{\frac{1}{2}}, \quad i = 1, 3,
$$

$$
r_2(\rho, u_1, S) = (p, 0, -p)^{\frac{1}{2}},
$$

such that

$$
r_i(\rho, u_1, S) \cdot \nabla_{(\rho, u_1, \theta)} \lambda_i(\rho, u_1, S) \neq 0, \quad i = 1, 3, \quad \text{and} \quad r_2(\rho, u_1, S) \cdot \nabla_{(\rho, u_1, \theta)} \lambda_2(\rho, u_1, S) \equiv 0.
$$

Thus the two $i$-Riemann invariants $\Sigma_i^{(j)} (i = 1, 3, j = 1, 2)$ can be defined by (cf. [32])

$$
\Sigma_i^{(1)} = u_1 + (-1)^{i+1} \int_0^\rho \sqrt{p(z, S)} \frac{dz}{z}, \quad \Sigma_i^{(2)} = S_{i,j},
$$

such that

$$
\nabla_{(\rho, u_1, S)} \Sigma_i^{(j)} (\rho, u_1, S) \cdot r_i(\rho, u_1, S) \equiv 0, \quad i = 1, 3, \quad j = 1, 2.
$$

Given the right state $(\rho_+, u_{1+}, \theta_+)$ with $\rho_+ > 0, \theta_+ > 0$, the $i$-rarefaction wave curve $(i = 1, 3)$ in the phase space $(\rho, u_1, \theta)$ with $\rho > 0$ and $\theta > 0$ can be defined by (cf. [18]):

$$
R_i(\rho_+, u_{1+}, \theta_+) := \left\{ (\rho, u_1, \theta) \bigg| \lambda_{ix_1}(\rho, u_1, S) > 0, \Sigma_i^{(j)}(\rho, u_1, S) = \Sigma_i^{(j)}(\rho_+, u_{1+}, S_+), \quad j = 1, 2 \right\}.
$$

Without loss of generality, we consider the stability of planar $3$–rarefaction wave to the Euler system (1.7), (1.8) in the present paper and the stability of $1$–rarefaction wave can be done similarly. The
3–rarefaction wave to the Euler system \([1.7], [1.8]\) can be expressed explicitly by the Riemann solution to the inviscid Burgers equation:

\[
\begin{aligned}
& w_t + w w_x = 0, \\
& w(x_1, 0) = w_0(x_1) = \begin{cases} w_-, & x_1 < 0, \\
& w_+, & x_1 > 0. \end{cases}
\end{aligned}
\]  
\(1.13\)

If \(w_- < w_+\), then the Riemann problem \([1.13]\) admits a rarefaction wave solution \(w^r(x_1, t) = \frac{x_1}{t}\) given by

\[
\begin{aligned}
& w_+ \leq \frac{x_1}{t} < w_-, \\
& w^r(x_1, t) = \begin{cases} w_-, & \frac{x_1}{t} < w_-, \\
& w_+, & \frac{x_1}{t} > w_. \end{cases}
\end{aligned}
\]  
\(1.14\)

Then the 3-rarefaction wave solution \((\rho^r, u^r, \theta^r)(\frac{x_1}{t})\) to the compressible Euler equations \([1.7], [1.8]\) can be defined explicitly by

\[
\begin{aligned}
& w_{\pm} = \lambda_3(\rho_{\pm}, u_{1\pm}, \theta_\pm), \\
& w^r(x_1, t) = \lambda_3(\rho^r, u^r_1, \theta^r)(\frac{x_1}{t}), \\
& \Sigma^j_3(\rho^r, u^r_1, \theta^r)(\frac{x_1}{t}) = \Sigma^j_3(\rho_{\pm}, u_{1\pm}, \theta_\pm), \quad j = 1, 2, \quad u^r_{2j} = u^r_3 = 0,
\end{aligned}
\]  
\(1.15\)

where \(\Sigma^j_3\) \((j = 1, 2)\) are the 3-Riemann invariants defined in \([1.11]\).

We construct a smooth 3-rarefaction wave profile to the wave fan defined in \([1.15]\). Motivated by \([28]\), the smooth rarefaction wave can be constructed by the Burgers equation

\[
\begin{aligned}
& \ddot{w}_t + \dot{w} \dot{w} x_1 = 0, \\
& \ddot{w}(x_1, 0) = \ddot{w}_0(x_1) = \frac{w_+ + w_-}{2} + \frac{w_+ - w_-}{2} k_{q} \int_0^{\infty} (1 + y^2)^{-q} dy,
\end{aligned}
\]  
\(1.16\)

where \(\varepsilon > 0\) is a small constant to be determined and \(k_{q}\) is a positive constant such that \(k_{q} \int_0^{\infty} (1 + y^2)^{-q} dy = 1\) for each \(q \geq 2\). Note that the solution \(\ddot{w}(x_1, t)\) of the problem \([1.16]\) can be given explicitly by

\[
\ddot{w}(x_1, t) = \ddot{w}_0(x_0(x_1, t)), \quad x_1 = x_0(x_1, t) + \ddot{w}_0(x_0(x_1, t)) t.
\]  
\(1.17\)

Correspondingly, the smooth rarefaction wave profile \((\dot{\rho}, \dot{\bar{u}}, \dot{\theta})(x_1, t)\) to compressible Euler equations \([1.7], [1.8]\) can be defined by

\[
\begin{aligned}
& w_{\pm} = \lambda_3(\rho_{\pm}, u_{1\pm}, \theta_\pm), \\
& \ddot{w}(x_1, 1 + t) = \lambda_3(\dot{\rho}, \dot{\bar{u}}, \dot{\theta})(x_1, t), \\
& \Sigma^j_3(\dot{\rho}, \dot{\bar{u}}, \dot{\theta})(x_1, t) = \Sigma^j_3(\rho_{\pm}, u_{1\pm}, \theta_\pm), \quad j = 1, 2, \quad \ddot{u}_2 = \ddot{u}_3 = 0,
\end{aligned}
\]  
\(1.18\)

where \(\ddot{w}(x_1, t)\) is the solution of Burgers equation \([1.16]\) defined in \([1.17]\). Then the planar 3-rarefaction wave \((\dot{\rho}, \dot{\bar{u}}, \dot{\theta})(x_1, t)\) satisfies the Euler system

\[
\begin{aligned}
& \dot{\rho}_t + (\dot{\bar{u}}_1) x_1 = 0, \\
& (\dot{\bar{u}}_1)_t + (\dot{\bar{u}}_1^2 + \dot{\rho}) x_1 = 0, \\
& (\dot{\bar{u}}_i)_t + (\dot{\bar{u}}_1 \dot{u}_i)_x_1 = 0, \quad i = 2, 3; \\
& \frac{R}{\gamma - 1} [(\dot{\rho \theta})_t + (\dot{\rho \bar{u}_1 \theta}) x_1] + \dot{\bar{u}}_{1x_1} = 0.
\end{aligned}
\]  
\(1.19\)

Now we can state the main result in this paper as follows.
Theorem 1.1. Suppose \((\bar{\rho}, \bar{u}, \bar{\theta})(x_1, t)\) is the planar 3-rarefaction wave defined in (1.18). For each fixed \((\rho_+, u_+, \theta_+)\), there exists a positive constant \(\varepsilon_0\), such that if \((\rho_-, u_-, \theta_-) \in R_3(\rho_+, u_+, \theta_+)\), and 
\begin{equation}
\varepsilon + \|\|\rho_0 - \bar{\rho}_0, u_0 - \bar{u}_0, \theta_0 - \bar{\theta}_0\|_{H^2}\| \leq \varepsilon_0, 
\end{equation}
then the initial value problem \((1.1)-(1.6)\) admits a unique global smooth solution \((\rho, u, \theta)\) satisfying 
\begin{equation}
\begin{cases}
(\rho - \bar{\rho}, u - \bar{u}, \theta - \bar{\theta}) \in C(0, +\infty; L^2(\Omega)), \\
\nabla (\rho, u, \theta) \in C(0, +\infty; H^1(\Omega)), \\
\nabla^2 \rho \in L^2(0, +\infty; L^2(\Omega)), \\
\nabla^2 (u, \theta) \in L^2(0, +\infty; H^1(\Omega)),
\end{cases}
\end{equation}
and the time-asymptotic stability toward the planar rarefaction wave \((\bar{\rho}, \bar{u}, \bar{\theta})(x_1, t)\) holds true:
\begin{equation}
\lim_{t \to +\infty} \sup_{x \in \Omega} |(\rho, u, \theta)(x, t) - (\bar{\rho}, \bar{u}, \bar{\theta})(x_1, t)| = 0.
\end{equation}

Remark 1.1. This is the first result about nonlinear stability of planar rarefaction wave for the three-dimensional non-isentropic equations, while the corresponding stability result for shock wave or contact discontinuity is still open as far as we know.

Remark 1.2. If we assume \(|\|\rho_0 - \bar{\rho}_0, u_0 - \bar{u}_0, \theta_0 - \bar{\theta}_0\|_{L^2(\Omega)} + \|\nabla (\rho_0, u_0, \theta_0)\|_{H^1(\Omega)}\) and the wave strength \(|(\rho_+ - \rho_-)u_+ - u_-, \theta_+ - \theta_-|\) is suitably small, then the time-asymptotic stability of the 3-rarefaction wave fan holds true:
\begin{equation}
\lim_{t \to +\infty} \sup_{x \in \Omega} |(\rho, u, \theta)(x, t) - (\rho^r, u^r, \theta^r)(x_1, t)| = 0,
\end{equation}
where \(u^r = (u^r_1, 0, 0)^t\) and \((\rho^r, u^r_1, \theta^r)\) is the 3-rarefaction wave to the Euler system \((1.7), (1.8)\).

The rest part of the paper is arranged as follows. First, we present some properties on the smooth rarefaction wave solution in section 2. Then, The energy estimates will be given in section 3. Finally, in the last section, based on a priori estimates, we proved our main Theorem 1.1.

2. Rarefaction wave

In this section, we present some properties on the planar rarefaction wave constructed in (1.18).

Lemma 2.1 \((28)\). The problem \((1.16)\) has a unique smooth global solution \(\bar{w}(x_1, t)\) such that 
(i) \(w_- < \bar{w}(x_1, t) < w_+, \|\bar{w}_x(x_1, t)\| > 0\), for \(x_1 \in \mathbb{R}, t \geq 0\).
(ii) For any \(t > 0\) and \(p \in [1, \infty]\), there exists a constant \(C_{p,q}\) such that 
\begin{align*}
\|\bar{w}(\cdot, t) - w^r(\cdot)^t\|_{L^p} &\leq C_{p, q} \varepsilon^{\frac{1}{p}} (w_+ - w_-), \\
\|\bar{w}_{x_1}(\cdot, t)\|_{L^p} &\leq C_{p, q} \min\{\varepsilon^{1 - \frac{1}{p}} (w_+ - w_-), (w_+ - w_-)^{-1 + \frac{1}{p}}, \}, \\
\|\bar{w}_{x_1 x_1}(\cdot, t)\|_{L^p} &\leq C_{p, q} \min\{\varepsilon^{2 - \frac{1}{p}} (w_+ - w_-), \varepsilon^{(1 - \frac{1}{p})(1 - \frac{1}{2p})} (w_+ - w_-)^{-\frac{p-1}{2p}} t^{1 - \frac{p}{2p}}, \}, \\
|\bar{w}_{x_1}(x_1, t)| &\leq C_{q} \|\bar{w}_x(x_1, t)\|.
\end{align*}
(iii) The smooth rarefaction wave \(\bar{w}(x_1, t)\) and the original rarefaction wave \(w^r(\cdot)^t\) are time-asymptotically equivalent, i.e., 
\begin{equation}
\lim_{t \to +\infty} \sup_{x_1 \in \mathbb{R}} |\bar{w}(x_1, t) - w^r(\frac{x_1}{t})| = 0.
\end{equation}

Lemma 2.2 \((16), (28)\). Let \(\delta = \|\|\rho_+ - \rho_-)u_+ - u_-, \theta_+ - \theta_-\|\) be the strength of the smooth 3-rarefaction wave \((\bar{\rho}, \bar{u}, \bar{\theta})\) defined in (1.18), then it satisfies the following properties:
(i) \( \bar{u}_{1x_1}(x_1, t) = \frac{2}{\tau + 1} \bar{w}_{x_1} > 0, \) for \( x_1 \in \mathbb{R}, \) \( t \geq 0, \) \( \bar{\rho}_{x_1} = \frac{1}{\sqrt{A\gamma \exp(2/\gamma)S_+}} \rho^{1/\gamma}_{x_1}, \) \( \bar{\theta}_{x_1} = \frac{1}{\sqrt{A\gamma}} \theta_{x_1}. \)

(ii) The following estimates hold for all \( t > 0 \) and \( p \in [1, \infty]: \)

\[
\begin{align*}
&\| (\bar{\rho}, \bar{u}_1, \bar{\theta})(\cdot, t) - (\rho^r, u^r, \theta^r)(\cdot) \|_{L^2} \leq C_{p,q} \delta \varepsilon^{-1}, \\
&\| (\bar{\rho}, \bar{u}_1, \bar{\theta})_{x_1}(\cdot, t) \|_{L^p} \leq C_{p,q} \min\{ \delta \varepsilon^{1-\frac{1}{p}}, \delta \varepsilon^{1+(1+\gamma)} \}, \\
&\| (\bar{\rho}, \bar{u}_1, \bar{\theta})_{x_1x_1}(\cdot, t) \|_{L^p} \leq C_{p,q} \min\{ \delta \varepsilon^{2-\frac{1}{p}}, \delta \varepsilon^{\frac{1}{2}-\frac{1}{pq}} \varepsilon^{1-\frac{1}{p}}(1-\frac{1}{\gamma}) \}, \\
&\| (\bar{\rho}, \bar{u}_1, \bar{\theta})_{x_1x_1x_1}(\cdot, t) \|_{L^p} \leq C_{p,q} \min\{ \delta \varepsilon^{3-\frac{1}{p}}, \delta \varepsilon^{2-\frac{1}{pq}} \varepsilon^{1-\frac{1}{p}}(1-\frac{1}{\gamma})(2-\frac{1}{\gamma}) \}.
\end{align*}
\]

(iii) Time-asymptotically, the smooth rarefaction wave and the inviscid rarefaction wave fan are equivalent, i.e.,

\[
\lim_{t \to +\infty} \sup_{x_1 \in \mathbb{R}} \| (\bar{\rho}, \bar{u}_1, \bar{\theta})(x_1, t) - (\rho^r, u^r_1, \theta^r)(\frac{x_1}{t}) \| = 0.
\]

3. A Priori Estimates

Before we present the energy estimates, we first set

\[(\phi, \psi, \zeta)(x, t) = (\rho - \bar{\rho}, u - \bar{u}, \theta - \bar{\theta})(x, t),\]

Then the solution is sought in the set of functional space \( X(0, +\infty) \) defined by

\[
X(0, T) = \left\{ (\phi, \psi, \zeta) | (\phi, \psi, \zeta) \in C(0, T; H^2), \quad \nabla \phi \in L^2(0, T; H^2), \quad \nabla \psi \in L^2(0, T; H^2) \right\},
\]

with \( 0 \leq T \leq +\infty. \)

Note that if \( \chi \) is suitably small, then the condition \( \sup_{0 \leq t \leq T} \| (\phi, \psi, \zeta) \|_{H^2} \leq \chi \) and Sobolev embedding theorem imply that \( \| (\phi, \psi) \| \leq \frac{1}{2} \rho_-, \| \zeta \| \leq \frac{1}{2} \theta_-, \) and \( \| u \| = \| (u_1, u_2, u_3) \| \leq C \) with \( C \) being a positive constant which only depends on \( \rho_-, u_+ \). Therefore, the density function \( \rho(x, t) := \bar{\rho}(x_1, t) + \phi(x, t) \) and the absolute temperature function \( \theta(x, t) := \theta(x_1, t) + \zeta(x, t) \) satisfy that

\[(3.2) \quad 0 < \frac{1}{2} \rho_- \leq \rho(x, t) \leq \frac{1}{2} \rho_- + \rho_+, \quad 0 < \frac{1}{2} \theta_- \leq \theta(x, t) \leq \frac{1}{2} \theta_- + \theta_+,
\]

since \( 0 < \rho_- \leq \bar{\rho}(x_1, t) \leq \rho_+ \) and \( 0 < \theta_- \leq \bar{\theta}(x_1, t) \leq \theta_+ \). It should be noted that the uniform lower and upper bounds of the density function \( \rho(x, t) \) in \( \{1.4\}_2 \), which are crucial for the local and global-in-time existence of the classical solution to the system \( \{1.1\} \). Hence, for classical solutions, \( \{1.1\} \) can be rewritten as

\[
\begin{align*}
\rho_t + u \cdot \nabla \rho + \rho \div u &= 0, \\
u_t + u \cdot \nabla u + R \theta \nabla \rho + R \nabla \theta &= \frac{1}{\rho} (\mu \Delta u + (\mu + \lambda) \nabla (\div u)), \\
\frac{R}{\gamma - 1} (\theta_t + u \cdot \nabla \theta) + R \theta \div u &= \frac{1}{\rho} \left[ \kappa \Delta \theta + \frac{\mu}{2} |\nabla u + (\nabla u)^T|^2 + \lambda (\div u)^2 \right],
\end{align*}
\]
with the initial data \( (1.5) \) and far fields conditions on the \( x_1 \)-direction \( (1.6) \). From \( (1.19) \) and \( (3.3) \), we can get the perturbation system for \( (\phi, \psi, \zeta) \):

\[
\begin{aligned}
&\phi_t + u \cdot \nabla \phi + \rho \text{div} \psi + \psi \cdot \nabla \rho + \phi \text{div} \bar{u} = 0, \\
&\psi_t + u \cdot \nabla \psi + R \frac{\partial}{\partial \rho} \nabla \phi + R \nabla \zeta + \psi \cdot \nabla \bar{u} + R \left( \frac{\partial}{\rho} \right) \nabla \bar{\rho}, \\
&= \frac{1}{\rho} \left( \mu \Delta \psi + (\mu + \lambda) \nabla \text{div} \psi \right) + \left( \frac{2\mu + \lambda}{\rho} \right) u_{1x_1}, 0, 0 \right)^t, \\
&= \frac{R}{\gamma - 1} \left[ \zeta_t + u \cdot \nabla \zeta \right] + R \text{div} \psi + \psi \cdot \nabla \bar{\theta} + R \zeta \text{div} \bar{u} = \frac{\kappa}{\rho} \Delta \zeta + \frac{\kappa}{\rho} \bar{\theta}_{x_1 x_1} \\
&\quad + \frac{1}{\rho} \left[ \frac{\mu}{2} |\nabla \psi| \right|^2 + \lambda (\nabla \psi)^2 + 2 \bar{u}_{1x_1} (2\mu \partial_1 \psi + \lambda \text{div} \psi) + (2\mu + \lambda) \bar{u}_{1x_1}^2],
\end{aligned}
\]

and the initial data is

\[
(\phi, \psi, \zeta)(x, 0) = (\phi_0, \psi_0, \zeta_0)(x) = (\rho_0 - \bar{\rho}_0, u_0 - \bar{u}_0, \theta_0 - \bar{\theta}_0)(x).
\]

Since the proof for the local-in-time existence and uniqueness of the classical solution to \( (3.4) \)-(3.5) is standard (for instance, one can refer to \([29]\) or \([34]\)), in particular for the suitably small perturbation of the solution around the planar rarefaction wave satisfying the property \( (3.2) \), the details will be omitted. To prove Theorem \( 1.1 \) it suffices to show the following a priori estimates.

**Proposition 3.1.** (A priori estimates) Suppose that the reformulated problem \( (3.4) \)-(3.5) admits a solution \( (\phi, \psi, \zeta) \in X(0, T) \) for some \( T > 0 \). Then there exist positive constants \( \chi \leq 1 \) and \( C \) independent of \( T \), such that if

\[
\sup_{0 \leq t \leq T} \| (\phi, \psi, \zeta)(\cdot, t) \|_{H^2} \leq \chi,
\]

then it follows the estimates:

\[
\sup_{0 \leq t \leq T} \| (\phi, \psi, \zeta)(\cdot, t) \|_{H^2}^2 + \int_0^T \left[ \| \sqrt{u_{1x_1}} (\phi, \psi, \zeta) \|_{H^1}^2 + \| \nabla (\psi, \zeta) \|_{H^2}^2 \right] d\tau \\
\leq C \left( \| (\phi_0, \psi_0, \zeta_0) \|_{H^2}^2 + \varepsilon \right).
\]

From now on, we always assume that \( \chi + \varepsilon \leq 1 \). Proposition \( 3.1 \) is an easy consequence of the following lemmas. We first give the following \( L^2 \) estimate.

**Lemma 3.1.** For \( T > 0 \) and \( (\phi, \psi, \zeta) \in X(0, T) \) satisfying a priori assumption \( 3.6 \) with suitably small \( \chi + \varepsilon \), we have for \( t \in [0, T] \),

\[
\| (\phi, \psi, \zeta)(t) \|_{H^2}^2 + \int_0^t \left[ \| \sqrt{u_{1x_1}} (\phi, \psi, \zeta) \|_{H^1}^2 + \| \nabla (\psi, \zeta) \|_{H^2}^2 \right] d\tau \leq C \| (\phi_0, \psi_0, \zeta_0) \|_{H^2}^2 + C \varepsilon \frac{1}{\gamma}.
\]

**Proof:** For ideal polytropic fluids, it holds

\[
S = -R \ln \rho + \frac{R}{\gamma - 1} \ln \theta + \frac{R}{\gamma - 1} \ln R, \quad p = R \rho \theta = A \rho^\gamma \exp \left( \frac{\gamma - 1}{R} S \right).
\]

Denote

\[
X = \left( \rho, \rho u_1, \rho u_2, \rho u_3, \rho \left( \frac{R}{\gamma - 1} \theta + \frac{|u|^2}{2} \right) \right)^t,
\]

\[
Y = \left( \rho u, \rho u u_1 + p \delta_1, \rho u u_2 + p \delta_2, \rho u u_3 + p \delta_3, \rho u \left( \frac{R}{\gamma - 1} \theta + \frac{|u|^2}{2} + p u \right) \right)^t.
\]
where $\mathbb{I}_1 = (1, 0, 0)^t$, $\mathbb{I}_2 = (0, 1, 0)^t$, $\mathbb{I}_3 = (0, 0, 1)^t$. Then the system (1.1) can be rewritten as

$$X_t + \text{div} Y = \begin{pmatrix} 0 \\ \mu \Delta u_1 + (\mu + \lambda) \partial_1 \text{div} u \\ \mu \Delta u_2 + (\mu + \lambda) \partial_2 \text{div} u \\ \mu \Delta u_3 + (\mu + \lambda) \partial_3 \text{div} u \\ \kappa \Delta \theta + \text{div}(u T) \end{pmatrix},$$

where $\partial_j = \partial_{x_j}$ ($j = 1, 2, 3$). We define a relative entropy-entropy flux pair $(\eta, q)$ as

$$\eta = \hat{\eta} \left\{ -\rho S + \bar{\rho} \bar{S} + \nabla X(\rho S) \right\}_{X = \bar{X}} \cdot (X - \bar{X}),$$

$$q_j = \hat{\eta} \left\{ -\rho u_j S + \bar{\rho} \bar{u_j} \bar{S} + \nabla X(\rho S) \right\}_{X = \bar{X}} \cdot (Y_j - \bar{Y}_j) \quad j = 1, 2, 3.$$

Here, we can compute that

$$(\rho S)_\rho = S + \frac{|u|^2}{2\theta} - \frac{R\gamma}{\gamma - 1}, \quad (\rho S)_m_i = \frac{u_i}{\theta}, \quad i = 1, 2, 3, \quad (\rho S)_\theta = \frac{1}{\theta},$$

where $m_i = \rho u_i$ ($i = 1, 2, 3$) and $E = \rho(\frac{R}{\gamma - 1}\theta + \frac{|u|^2}{2})$, then

$$\eta = R\rho \bar{\theta} \Psi \left( \frac{\bar{\rho}}{\rho} \right) + \frac{R}{\gamma - 1} \rho \bar{\theta} \Psi \left( \frac{\theta}{\bar{\theta}} \right) + \frac{1}{2\rho}|u - \bar{u}|^2,$$

where $\Psi(\cdot)$ is the convex function

$$\Psi(s) = s - \ln s - 1.$$

Then, for $X$ in any closed bounded region in $\sum = \{ X : \rho > 0, \theta > 0 \}$, there exists a positive constant $C_0$ such that

$$C_0^{-1}|(\phi, \psi, \zeta)|^2 \leq \eta \leq C_0|(\phi, \psi, \zeta)|^2.$$

Direct computations yield that

(3.9)

$$\eta_t + \text{div} q + \frac{\bar{\theta}}{\theta} \left( \frac{\mu}{2} |\nabla \psi + (\nabla \psi)^t|^2 + \lambda (\text{div} \psi)^2 \right) + \frac{\kappa \bar{\theta}}{\theta^2} |\nabla \zeta|^2 - \left[ \nabla \bar{\theta} \cdot \bar{\theta} \nabla \psi \cdot (\nabla \psi)^t \right] + (\mu + \lambda) \psi \bar{u}_1 = 0,$$

where $\bar{\theta} = \sqrt{\theta}$, $\bar{\theta} = \sqrt{\theta}$.

There exists a positive constant $C > 0$ such that (cf. [20])

$$- \left[ \nabla \Bar{(\rho, u, S)} \cdot (\rho \bar{u}, S) \right] + \nabla \Bar{(\rho, u, S)} q \cdot (\rho \bar{u}, S) \geq C^{-1} \bar{u}_1 \left( \rho \psi_1^2 + R(\gamma - 1) \rho \bar{\theta} \Psi \left( \frac{\bar{\rho}}{\rho} \right) + R \bar{\rho} \bar{\theta} \Psi \left( \frac{\theta}{\bar{\theta}} \right) \right) + \bar{\theta}_x \rho \psi_1 \left( R \ln \bar{\rho} + \frac{R}{\gamma - 1} \ln \bar{\theta} \right) \geq C^{-1} \bar{u}_1 \left( \phi^2 + \psi_1^2 + \zeta^2 \right).$$
Integrating \((3.9)\) with respect to \(x, t\) over \(\Omega \times (0, t)\) yields that

\[
\|\phi, \psi, \zeta(t)\|^2 + \int_0^t \left[ \|\nabla \phi, \nabla \zeta\|^2 + \|\sqrt{u_{1x_1}} \phi, \psi_1, \zeta\|^2 \right] dt \\
\leq C \|\phi_0, \psi_0, \zeta_0\|^2 + C \int_0^t \left[ \frac{2C}{\theta} (2\mu \partial_1 \psi_1 + \lambda \text{div} \psi) \bar{u}_{1x_1} + \frac{\kappa}{\theta^2} \zeta \partial_1 \zeta \bar{g}_{x_1} \right] dx dt
\]

\[
+ C \int_0^t \int \left[ \psi_1 \bar{u}_{1x_1 x_1} + \frac{\zeta}{\theta} \bar{u}_{1x_1} + \frac{\kappa}{\theta} \zeta \partial_1 \zeta \right] dx dt,
\]

where we have used the following fact

\[
\int \mu |\nabla \psi|^2 + (\mu + \lambda) (\text{div} \psi)^2 dx = \int \frac{\mu}{2} \nabla \psi + (\nabla \psi)^2 + \lambda (\text{div} \psi)^2 dx.
\]

First, by the Cauchy’s inequality and Lemma 2.2 it holds that

\[
C \left| \int_0^t \int \left[ \frac{2C}{\theta} (2\mu \partial_1 \psi_1 + \lambda \text{div} \psi) \bar{u}_{1x_1} + \frac{\kappa}{\theta^2} \zeta \partial_1 \zeta \bar{g}_{x_1} \right] dx dt \right|
\leq \frac{1}{2} \int_0^t \|\nabla (\psi, \zeta)\|^2 dt + C \epsilon \int_0^t \|\sqrt{u_{1x_1}} \zeta\|^2 dt.
\]

By Sobolev’s inequality, Hölder’s inequality, Young’s inequality, Lemma 2.2 and assumption \((3.6)\), we have

\[
C \left| \int_0^t \int \psi_1 \bar{u}_{1x_1 x_1} dx dt \right| \leq C \int_0^t \int_{T^2} \|\psi_1\|_{L^\infty(\mathbb{R})} \|\bar{u}_{1x_1 x_1}\|_{L^1(\mathbb{R})} dx dt
\]

\[
\leq C \epsilon \frac{1}{2} \int_0^t (1 + \tau)^{-\frac{7}{8}} \left( \int_{T^2} \|\psi_1\|_{L^2(\mathbb{R})} \|\partial_1 \psi_1\|_{L^2(\mathbb{R})} dx dt \right) d\tau
\]

\[
\leq C \epsilon \frac{1}{2} \int_0^t (1 + \tau)^{-\frac{7}{8}} \|\partial_1 \psi_1\|_{L^2(\mathbb{R})} \left( \int_{T^2} \|\psi_1\|_{L^2(\mathbb{R})} dx dt \right)^{\frac{3}{8}} d\tau
\]

\[
\leq C \epsilon \frac{1}{2} \int_0^t (1 + \tau)^{-\frac{7}{8}} \left( \frac{\zeta}{\theta} \right)^{\frac{3}{8}} \|\partial_1 \psi_1\|_{L^2(\mathbb{R})} \|\psi_1\|_{L^2(\mathbb{R})} d\tau \leq C \epsilon \frac{1}{2} \int_0^t \|\partial_1 \psi_1\|^2 dt + C \epsilon \frac{1}{8}.
\]

Similarly, one has

\[
C \left| \int_0^t \int \left[ \frac{\zeta}{\theta} \bar{u}_{1x_1} + \frac{\kappa}{\theta} \zeta \partial_1 \zeta \bar{g}_{x_1} \right] dx dt \right| \leq C \epsilon \frac{1}{8} \int_0^t \|\nabla \zeta\|^2 dt + C \epsilon \frac{1}{8}.
\]

Substituting the estimates \((3.11)-(3.13)\) into \((3.10)\) gives \((3.8)\), and the proof of Lemma 3.1 is completed.

Next, we want to get the estimation of \(\nabla \phi\). Compared with the one-dimensional stability results in \([27, 28, 30]\), the physical viscosity in momentum equation \((3.4)_2\) has the form: \(\mu \Delta \psi + (\mu + \lambda) \text{div} \psi\) in high dimensions. Therefore, we can not substitute mass equation \((3.4)_1\) into momentum equation \((3.4)_2\) to obtain the derivative estimate of density perturbation \(\nabla \phi\) as in \([27, 28, 30]\). Our new observation is that we find some cancellation between the flux terms and viscosity terms for system \((3.4)\), by which we successfully overcome the difficulty when the planar rarefaction wave propagate in \(x_2, x_3\)-directions may interact with \(x_1\)-direction, and derive the derivative estimates of density perturbation \(\nabla \phi\). The following lemma is crucial to get a priori estimates \((3.7)\).
Lemma 3.2. For $T > 0$ and $(\phi, \psi, \zeta) \in X(0, T)$ satisfying a priori assumption (3.6) with suitably small \( \chi + \varepsilon \), it holds that for \( t \in [0, T] \),

\[
\|\nabla \phi(t)\|^2 + \int_0^t \|\nabla \phi\|^2 d\tau \leq C\|\phi_0, \psi_0, \zeta_0, \nabla \phi_0\|^2 + C\varepsilon^{2} + C(\chi + \varepsilon) \int_0^t \|\nabla^2 u\|^2 d\tau.
\]

**Proof:** For this, we multiply (3.4) by \( \rho \nabla \phi \) and integrate by parts with respect to \( x \) to obtain

\[
\int \rho \psi_t \cdot \nabla \phi \, dx + \int \rho u \cdot \nabla \psi \cdot \nabla \phi \, dx + \int R \theta |\nabla \phi|^2 \, dx\, dt
\]

\[
= -\int \rho \left[ R \nabla \phi + \nabla u \cdot \nabla \phi + \frac{\theta}{\rho} \frac{\phi_t}{\nabla \rho} \right] \cdot \nabla \phi \, dx
\]

\[
+ \int \left[ \mu \Delta \phi + (\mu + \lambda) \nabla \psi \cdot \nabla \phi \right] \cdot \nabla \phi \, dx + (2\mu + \lambda) \int \tilde{u}_{1x_1} \partial_1 \phi \, dx,
\]

By using the following two facts:

\[
\begin{align*}
\int \rho \psi_t \cdot \nabla \phi \, dx + \int \rho u \cdot \nabla \psi \cdot \nabla \phi \, dx \\
= \frac{d}{dt} \int \rho \psi \cdot \nabla \phi \, dx + \int \nabla (\rho u) \cdot \nabla \phi \, dx + \int \rho \psi_t \cdot \nabla \phi \, dx + \int \rho u \cdot \nabla \psi \cdot \nabla \phi \, dx
\end{align*}
\]

and

\[
\int \left[ \mu \Delta \phi + (\mu + \lambda) \nabla \psi \cdot \nabla \phi \right] \cdot \nabla \phi \, dx = (2\mu + \lambda) \int \nabla \phi \cdot \nabla \psi \, dx,
\]

the equality (3.15) becomes

\[
\begin{align*}
\frac{d}{dt} \int \rho \psi \cdot \nabla \phi \, dx + \int R \theta |\nabla \phi|^2 \, dx\, dt = \int \nabla (\rho u) \cdot \nabla \phi + \rho \nabla \psi \cdot \nabla \phi + \phi \nabla u \cdot \nabla \phi \, dx
\end{align*}
\]

\[
- \int \partial_j (\rho u \psi) \cdot \nabla \phi \, dx - \int \rho \left[ R \nabla \phi + \nabla u \cdot \nabla \phi + \frac{\theta}{\rho} \frac{\phi_t}{\nabla \rho} \right] \cdot \nabla \phi \, dx
\]

\[
+ (2\mu + \lambda) \int \tilde{u}_{1x_1} \partial_1 \phi \, dx,
\]

In order to close the a priori assumption (3.6), we need to get rid of the higher order term \((2\mu + \lambda) \int \nabla \phi \cdot \nabla \psi \, dx\) in (3.18). Otherwise, the first-order derivative estimate in (3.18) will depend on the second order derivative \(\nabla^2 \psi \) and deductively one can not close the a priori assumption (3.6). For this, we first apply \( \partial_i \) \((i = 1, 2, 3)\) to the equation (3.11) to derive

\[
\partial_i \phi_t + u \cdot \nabla \partial_i \phi + \rho \partial_i \psi \partial_2 \phi + \partial_i u \cdot \nabla \phi + \rho \partial_2 \psi + \partial_i \psi \cdot \nabla \phi + \partial_i \phi \nabla \psi + \phi \partial_i \phi \nabla \psi = 0.
\]

Then multiplying the above equation by \( \frac{2\mu + \lambda}{\rho} \partial_i \phi \), integrating over the domain \( \Omega \) with respect to \( x \) and summing \( i \) from 1 to 3 yield

\[
\frac{d}{dt} \int \frac{2\mu + \lambda}{2\rho} |\nabla \phi|^2 \, dx = (2\mu + \lambda) \int \frac{\rho u}{\rho} |\nabla \phi|^2 \, dx - \int \frac{2\mu + \lambda}{\rho} \partial_i \phi \left( \partial_i u \cdot \nabla \phi + \partial_i \rho \partial_2 \psi + \partial_i \phi \nabla \psi + \phi \partial_i \phi \nabla \psi \right) \, dx - (2\mu + \lambda) \int \nabla \phi \cdot \nabla \psi \, dx,
\]
where we have the following equality:

\[
\int_0^t \frac{2\mu + \lambda}{\rho} \partial_t \phi \left( \partial_t \phi_t + u \cdot \nabla \partial_t \phi \right) = \frac{d}{dt} \int_0^t \frac{2\mu + \lambda}{2\rho} \left| \partial_t \phi \right|^2 dx - \frac{2\mu + \lambda}{2} \int_0^t \left[ \frac{1}{(\rho^2)} \right] \partial_t \phi^2 dx \\
= \frac{d}{dt} \int_0^t \frac{2\mu + \lambda}{2\rho} \left| \partial_t \phi \right|^2 dx - (2\mu + \lambda) \int_0^t \frac{\text{div} u}{\rho} \left| \partial_t \phi \right|^2 dx.
\]

Thus adding the equalities (3.18) and (3.20) together and the higher order term \((2\mu + \lambda) \int \nabla \phi \cdot \nabla \text{div} \psi dx\) will be cancelled as desired, and then integrating the resulted equation with respect to the time \(t\) over \((0, t)\) to give

\[
\int_0^t \left( \frac{2\mu + \lambda}{2\rho} \left| \nabla \phi \right|^2 + \rho \psi \cdot \nabla \phi \right) dx \bigg|_{\tau=0}^{\tau=t} + \int_0^t \int R\theta |\nabla \phi|^2 dxd\tau \\
= (2\mu + \lambda) \int_0^t \int \frac{\text{div} u}{\rho} |\nabla \phi|^2 dxd\tau - \int_0^t \int \frac{2\mu + \lambda}{\rho} \partial_t \phi (\partial_t u + \nabla \phi + \partial_t \rho \text{div} \psi) dxd\tau \\
+ \int_0^t \int \text{div} (\rho \psi) (u \cdot \nabla \phi + \rho \text{div} \psi + \nabla \rho + \phi \text{div} \nabla \psi) dxd\tau \\
- \int_0^t \int \rho \left[ R\nabla \psi + \nabla \nabla \psi + R \left( \frac{\theta}{\rho} \right) \nabla \rho \right] \cdot \nabla \phi dxd\tau + (2\mu + \lambda) \int_0^t \int \bar{u}_{1x_1} \partial_1 \phi dxd\tau.
\]

We just estimate the first term on the right hand side of (3.21) as follows and the other terms can be done similarly and the details for estimating these terms will be omitted for brevity. By Sobolev’s inequality Lemma 2.2 and assumption (3.6), one has

\[
(2\mu + \lambda) \int_0^t \int \frac{\text{div} u}{\rho} |\nabla \phi|^2 dxd\tau \leq C \int_0^t \int |\text{div} u| |\nabla \phi|^2 dxd\tau \\
\leq \int_0^t \| \nabla u \|_{L^\infty} \| \nabla \phi \|^2 d\tau \leq C \int_0^t \| \nabla u \|_{H^2} \| \nabla \phi \|^2 d\tau \\
\leq C \int_0^t \left( \| \nabla u \| + \| \nabla^2 u \| \right) \| \nabla \phi \|^2 d\tau + C \int_0^t \| \nabla^3 u \| \| \nabla \phi \|^2 d\tau \\
\leq C \sup_{0 \leq \tau \leq t} \| \nabla u, \nabla^2 u(\tau) \| \int_0^t \| \nabla \phi \|^2 d\tau + C \sup_{0 \leq \tau \leq t} \| \nabla \phi(\tau) \| \int_0^t \| \nabla^3 u \| \| \nabla \phi \| d\tau \\
\leq C(\chi + \varepsilon) \int_0^t \left( \| \nabla \phi \|^2 + \| \nabla^3 u \|^2 \right) d\tau.
\]

By Cauchy’s inequality and the estimates as in (3.22), it follows from (3.21) that

\[
\| \nabla \phi(t) \|^2 + \int_0^t \| \nabla \phi \|^2 d\tau \leq C \| (\psi_0, \nabla \phi_0) \|^2 + C \varepsilon + C \| \psi(t) \|^2 + C \int_0^t \| \nabla (\psi, \zeta) \|^2 d\tau \\
+ C \varepsilon \int_0^t \| \sqrt{\bar{u}_{1x_1}(\phi, \psi_1, \zeta)} \|^2 d\tau + C(\chi + \varepsilon) \int_0^t \| \nabla^3 u \|^2 d\tau,
\]

which together with (3.8) leads to (3.14), and the proof of Lemma 3.2 is completed. □
Lemma 3.3. For $T > 0$ and $(\phi, \psi, \zeta) \in X(0, T)$ satisfying a priori assumption (3.6) with suitably small $\chi + \varepsilon$, we have for $t \in [0, T]$,

$$
(3.24) \quad \|\nabla (\psi, \zeta)(t)\|^2 + \int_0^t \|\nabla^2 (\psi, \zeta)\|^2 d\tau \leq C\|(\phi_0, \psi_0, \zeta_0)\|_{H^1}^2 + C\varepsilon + C(\chi + \varepsilon) \int_0^t \|\nabla^3 u\|^2 d\tau.
$$

Proof: Multiplying the equation (3.4) by $(-\Delta \psi)$, and integrating over $\Omega \times (0, t)$ lead to

$$
(3.25) \quad \frac{1}{2} \int_0^t \|\nabla \psi\|^2 d\tau + \int_0^t \frac{(\mu + \lambda)}{\rho} |\nabla \psi|^2 d\tau + \int_0^t \frac{(\mu + \lambda)}{\rho^2} \left(\partial_j \rho \partial_j \text{div} \psi - \partial_i \rho \text{div} \psi \Delta \psi_i\right) d\tau
$$

Substituting (3.27)-(3.29) into (3.25) yields

$$
\sigma \|\nabla (\psi, \zeta)(t)\|^2 + \int_0^t \|\nabla^2 (\psi, \zeta)\|^2 d\tau \leq C\|(\phi_0, \psi_0, \zeta_0)\|_{H^1}^2 + C\varepsilon + C(\chi + \varepsilon) \int_0^t \|\nabla^3 u\|^2 d\tau.
$$

Now we will estimate each $I_i$ $(i = 1, 2, 3)$ on the right hand side of (3.25). By Cauchy’s inequality and Lemma 2.2 one has

$$
(3.27) \quad |I_1| \leq \sigma \int_0^t \|\nabla^2 \psi\|^2 d\tau + C_\sigma \int_0^t \|\nabla (\phi, \psi, \zeta)\|^2 d\tau + C_\sigma \varepsilon \int_0^t \|\sqrt{u_{1x_1}}(\phi, \psi_1, \zeta)\|^2 d\tau,
$$

where $\sigma$ is a suitably small positive constant to be determined and $C_\sigma$ is a positive constant depending on $\sigma$. It follows from Cauchy’s inequality, Sobolev’s inequality, Lemma 2.2 and assumption (3.6) that

$$
(3.28) \quad |I_2| \leq C \int_0^t \int \left(\|\nabla \phi\| + |\vec{\rho}_{x_1}|\right) |\nabla \psi| |\nabla^2 \psi| dxd\tau
$$

$$
\leq C \int_0^t \|\nabla \phi\|_{L^4} \|\nabla \psi\|_{L^4} \|\nabla^2 \psi\| d\tau + C\varepsilon \int_0^t \|\nabla \psi\| \|\nabla^2 \psi\| d\tau
$$

$$
\leq C \int_0^t \|\nabla \phi\| \|\nabla^2 \psi\| d\tau + C\varepsilon \int_0^t \|\nabla \psi\| \|\nabla^2 \psi\| d\tau
$$

$$
\leq C(\chi + \varepsilon) \int_0^t \left(\|\nabla \psi\|^\frac{7}{4} \|\nabla^2 \psi\|^\frac{2}{7} + \|\nabla \psi\| \|\nabla^2 \psi\|\right) d\tau \leq C(\chi + \varepsilon) \int_0^t \|\nabla \psi, \nabla^2 \psi\|^2 d\tau.
$$

By Cauchy’s inequality and Lemma 2.2 we obtain

$$
(3.29) \quad |I_3| \leq \sigma \int_0^t \|\nabla^2 \psi\|^2 d\tau + C_\sigma \int_0^t \|\sqrt{u_{1x_1}}\|^2 d\tau \leq \sigma \int_0^t \|\nabla^2 \psi\|^2 d\tau + C_\sigma \varepsilon.
$$

Substituting (3.27) (3.29) into (3.25) yields

$$
(3.30) \quad \|\nabla \psi(t)\|^2 + \int_0^t \|\nabla^2 \psi\|^2 d\tau \leq C\|\nabla \psi_0\|^2 + C\varepsilon + C \int_0^t \|\nabla (\phi, \psi, \zeta)\|^2 d\tau + C\varepsilon \int_0^t \|\sqrt{u_{1x_1}}(\phi, \psi_1, \zeta)\|^2 d\tau.
$$
Next, we estimate \( \|\nabla \zeta\| \). We multiply the equation (3.4) by \((-\Delta \zeta)\), and integrate by parts over \(\Omega \times (0, t)\), similar as (3.26), it holds

\[
\frac{R}{\gamma - 1} \int_{\Omega} |\nabla \zeta|^2 \, dx \bigg|_{\tau = 0}^{\tau = t} + \int_{0}^{t} \int_{\Omega} \frac{\kappa}{\rho} |\nabla^2 \zeta|^2 \, dx \, d\tau = \int_{0}^{t} \int_{\Omega} \left[ \frac{R}{\gamma - 1} u \cdot \nabla \zeta + R \theta \text{div} \psi 
+ \psi \cdot \nabla \theta + R \zeta \text{div} \bar{u} \right] \Delta \zeta \, dx \, d\tau - \int_{0}^{t} \int_{\Omega} \left( \frac{\mu}{2} |\nabla \psi + (\nabla \psi)^T|^2 + \lambda (\text{div} \psi)^2 \right) \, dx \, d\tau 
- \int_{0}^{t} \int_{\Omega} 2\Delta \zeta \cdot \bar{u}_{x1} \, dx \, d\tau - \int_{0}^{t} \int_{\Omega} \left( \frac{\kappa}{\rho} \tilde{\theta}_{x1x1} + (2\mu + \lambda) \bar{u}_{x1x1}^2 \right) \, dx \, d\tau
+ \int_{0}^{t} \int_{\Omega} \kappa (\partial_i \rho \partial_i \partial_j \zeta - \partial_j \rho \partial_j \zeta \Delta \zeta) \, dx \, d\tau := \sum_{i=4}^{8} I_i.
\]

(3.31)

We just estimate \( I_5 \), other terms are similar to \((3.27) - (3.29)\). By Hölder’s inequality, Sobolev’s inequality, Cauchy’s inequality, Lemma 2.2 and assumption (3.6),

\[
|I_5| \leq C \int_{0}^{t} \|\nabla^2 \zeta\| \|\nabla \psi\|^2 \, d\tau \leq C \int_{0}^{t} \|\nabla^2 \zeta\| \|\nabla \psi\| \|\nabla^2 \psi\|^\frac{1}{2} \, d\tau
\leq C(\chi + \varepsilon) \int_{0}^{t} \|\nabla^2 \zeta\| \|\nabla^2 \psi\| \, d\tau \leq C(\chi + \varepsilon) \int_{0}^{t} (\|\nabla^2 \psi\|, \|\nabla^2 \zeta\|) \, d\tau.
\]

(3.32)

Similar to (3.30), it follows from (3.31) and (3.32) that

\[
\|\nabla \zeta(t)\|^2 + \int_{0}^{t} \|\nabla^2 \zeta\|^2 \, d\tau \leq C \|\nabla \zeta_0\|^2 + C\varepsilon + C \|\nabla(\psi, \zeta)\|^2 \, d\tau
+ C\varepsilon \int_{0}^{t} \|\sqrt{\bar{u}_{x1}}(\psi_1, \zeta)\|^2 \, d\tau + C(\chi + \varepsilon) \int_{0}^{t} \|\nabla \psi\|^2 \, d\tau,
\]

which together with (3.30) leads to

\[
\|\nabla(\psi, \zeta)(t)\|^2 + \int_{0}^{t} \|\nabla^2(\psi, \zeta)\|^2 \, d\tau
\leq C \|\nabla(\psi_0, \zeta_0)\|^2 + C\varepsilon + C \int_{0}^{t} \|\nabla(\phi, \psi, \zeta)\|^2 \, d\tau + C\varepsilon \int_{0}^{t} \|\sqrt{\bar{u}_{x1}}(\phi, \psi_1, \zeta)\|^2 \, d\tau.
\]

(3.34)

Finally, combining (3.34) with (3.8) and (3.14) implies (3.24), and the proof of Lemma 3.3 is completed. 

The following lemmas are concerned with the higher order estimates of the perturbation \((\phi, \psi, \zeta)\). In order to obtain these estimates, we prefer to consider the system (3.3) of \((\rho, u, \theta)\) rather than the perturbation system (3.4) of \((\phi, \psi, \zeta)\) due to the fact \(\|\tilde{\rho}, \tilde{u}_1, \tilde{\theta}\|_{x1x1}, (\bar{\rho}, \bar{u}_1, \bar{\theta})_{x1x1} \|_1^2 \approx \varepsilon(1 + t)^{-2}\), which is integrable with respect to the time \(t\) on \(\mathbb{R}^+\). Therefore, we can use system (3.3) to derive Lemmas 3.4 and 3.6. We start from Lemma 3.3 concerning the second order derivatives estimates for \(\phi\).

Lemma 3.4. For \(T > 0\) and \((\phi, \psi, \zeta) \in X(0, T)\) satisfying a priori assumption (3.6) with suitably small \(\chi + \varepsilon\), it holds that for \(t \in [0, T]\),

\[
\|\nabla^2 \phi(t)\|^2 + \int_{0}^{t} \|\nabla^2 \phi\|^2 \, d\tau \leq C(\|(\phi_0, \psi_0, \zeta_0)\|_{H^1}^2 + \|\nabla^2 \phi_0\|^2 + \varepsilon^\frac{3}{2}) + C(\chi + \varepsilon) \int_{0}^{t} \|\nabla^3 u\|^2 \, d\tau.
\]

(3.35)
Proof: Applying \( \partial_j \partial_i \) \((i, j = 1, 2, 3)\) to the mass equation (3.3) \(_1\) and \( \partial_j \) \((j = 1, 2, 3)\) to the \(i\)–th \((i = 1, 2, 3)\) component of the momentum equation (3.3) \(_2\), we have

\[
\begin{align*}
\partial_j \partial_i \rho_i + u \cdot \nabla \partial_j \partial_i \rho + \rho \partial_j \partial_i \rho + \rho \partial_j \partial_i \rho + \partial_j u \cdot \nabla \partial_i \rho + \partial_i u \cdot \nabla \partial_j \rho + \text{div} u \partial_j \partial_i \rho \\
+ (\partial_j \rho \partial_i \text{div} u + \partial_j \partial_i u \cdot \nabla \rho + \partial_i \rho \partial_j \text{div} u) = 0,
\end{align*}
\]

(3.36)

Next, multiplying the equation (3.3) \(_2\) by \( \rho \partial_j \partial_i \rho \) and integrating with respect to \(x\) lead to

\[
\frac{d}{dt} \int \rho \partial_j u_i \partial_j \partial_i \rho \, dx + \int \nabla^2 \partial_j \partial_i \rho \, dx = -\int \text{div} (\rho u) \partial_j u_i \partial_j \partial_i \rho \, dx
\]

\[
+ \int \partial_j (\rho \partial_j u_i) (u \cdot \nabla \partial_i \rho + \rho \partial_i \text{div} u + \partial_i u \cdot \nabla \rho + \partial_i \rho \text{div} u) \, dx
\]

(3.37)

where we have used the following two facts:

\[
\int \rho \partial_j u_i \partial_j \partial_i \rho \, dx = \frac{d}{dt} \int \rho \partial_j u_i \partial_j \partial_i \rho \, dx - \int \rho \partial_j u_i \partial_j \partial_i \rho \, dx - \int \rho \partial_j u_i \partial_j \partial_i \rho \, dx
\]

(3.38)

\[
= \frac{d}{dt} \int \rho \partial_j u_i \partial_j \partial_i \rho \, dx + \int \text{div} (\rho u) \partial_j u_i \partial_j \partial_i \rho \, dx + \int \partial_j (\rho \partial_j u_i) \partial_i \rho \, dx
\]

and

\[
\int \frac{1}{\rho} (\mu \Delta \partial_j u_i + (\mu + \lambda) \partial_j \partial_i \text{div} u) \rho \partial_j \partial_i \rho \, dx = (2\mu + \lambda) \int \partial_j \partial_i \rho \partial_j \partial_i \text{div} u \, dx.
\]

(3.39)

Next, multiplying the equation (3.3) \(_1\) by \( \frac{2\mu + \lambda}{\rho} \partial_j \partial_i \rho \) and integrating with respect to \(x\) yield

\[
\frac{d}{dt} \int \left( \frac{2\mu + \lambda}{2\rho} \right) \partial_j \partial_i \rho \, dx = -(2\mu + \lambda) \int \partial_j \partial_i \rho \partial_j \partial_i \text{div} u \, dx + (2\mu + \lambda) \int \frac{\text{div} u}{\rho} \partial_j \partial_i \rho \, dx
\]

(3.40)

\[
- \int \frac{2\mu + \lambda}{\rho} \partial_j \partial_i \rho (\partial_j u \cdot \nabla \partial_i \rho + \partial_i u \cdot \nabla \partial_j \rho + \text{div} u \partial_j \partial_i \rho) \, dx
\]

\[
- \int \frac{2\mu + \lambda}{\rho} \partial_j \partial_i \rho (\partial_j \partial_i \text{div} u + \partial_j \partial_i u \cdot \nabla \rho + \partial_i \partial_j \text{div} u) \, dx.
\]
Finally, we add (3.37) and (3.40) together, summate \(i, j\) from 1 to 3 and integrate the resulted equation over \((0, t)\) to give

\[
\int_0^t \left( \frac{2\mu + \lambda}{2\rho} |\nabla^2 \rho|^2 + \rho \partial_j u_i \partial_j \partial_i \rho \right) dt \leq \int_0^t \int \frac{\theta}{|\nabla^2 \rho|^2} dxd\tau + \int_0^t \int R \theta |\nabla^2 \rho|^2 dxd\tau
\]

\[
= - \int_0^t \int \text{div}(\rho u_i \partial_j \partial_i \rho) dx d\tau + \int_0^t \int \partial_j (\rho \partial_j u_i) (u \cdot \nabla \partial_i \rho + \partial_i u \cdot \nabla \rho + \partial_t \rho \text{div} u) dxd\tau
\]

\[
+ \partial_t \rho \text{div} u dxd\tau - \int_0^t \int \left[ u \cdot \nabla \partial_j u_i + R \partial_j \partial_i \theta + \partial_j u \cdot \nabla u_i + \partial_j \left( \frac{R \theta}{\rho} \partial_i \rho \right) \rho \partial_j \partial_i \rho \right] dxd\tau
\]

\[
+ \int_0^t \int \frac{2\mu + \lambda}{\rho} \partial_j \partial_i \rho (\partial_j \rho \partial_i u + \partial_i \rho \partial_j u) dxd\tau
\]

\[
\int_0^t \int \frac{2\mu + \lambda}{\rho} \partial_j \partial_i \rho (\partial_j \rho \partial_i \text{div} u + \partial_i \rho \partial_j \text{div} u) dxd\tau := \sum_{i=1}^{15} I_i.
\]

By Hölder’s inequality, Cauchy’s inequality, Sobolev’s inequality, Lemma 2.2 and assumption (3.6), it holds

\[
|I_9| \leq C \int_0^t \left( |\nabla \rho| |\nabla u||\nabla^2 \rho| dxd\tau \right) \leq C \int_0^t \left( |\nabla \rho|^2 + |\nabla u|^2 \right) |\nabla^2 \rho| dxd\tau
\]

\[
\leq C \int_0^t \left( |\nabla \rho| L^4 + |\nabla u| L^4 \right) |\nabla^2 \rho| dtd\tau \leq C \int_0^t \left( |\nabla \rho|^2 \frac{\nabla^2 \rho}{\nabla \rho} + |\nabla u|^2 \frac{\nabla^2 \rho}{\nabla \rho} \right) |\nabla^2 \rho| dtd\tau
\]

\[
\leq C(\chi + \varepsilon) \int_0^t \left( |\nabla^2 \rho|^2 + |\nabla^2 u|^2 \right) dtd\tau.
\]

Similar to \(I_9\), we have

\[
|I_{10}| \leq \int_0^t \int \left( |\nabla \rho| |\nabla u| + |\nabla^2 \rho| dxd\tau \right) \leq (\sigma + C(\chi + \varepsilon)) \int_0^t |\nabla^2 \rho|^2 dtd\tau + C \int_0^t |\nabla^2 u|^2 dtd\tau.
\]

It follows from Young’s inequality, Sobolev’s inequality, Lemma 2.2 and assumption (3.6) that,

\[
|I_{10}| \leq \int_0^t \int \left( |\nabla \rho| |\nabla u| + |\nabla^2 u| \right) |\nabla^2 u| dxd\tau \leq C \int_0^t \left( |\nabla \rho| L^4 |\nabla u| L^4 |\nabla^2 u| + |\nabla^2 u|^2 \right) dtd\tau
\]

\[
\leq C \int_0^t \left( |\nabla \rho|^2 \frac{\nabla^2 \rho}{\nabla \rho} \frac{\nabla \rho}{\nabla \rho} + |\nabla u|^2 \frac{\nabla^2 \rho}{\nabla \rho} \frac{\nabla \rho}{\nabla \rho} \right) dtd\tau
\]

\[
\leq C(\chi + \varepsilon) \int_0^t |\nabla^2 \rho|^2 dtd\tau + C \int_0^t |\nabla^2 u|^2 dtd\tau,
\]

\[
|I_{10}| + |I_{10}| \leq C \int_0^t \int \left( |\nabla \rho| |\nabla u| + |\nabla^2 u| \right) |\nabla \rho| |\nabla u| dxd\tau
\]

\[
\leq C \int_0^t |\nabla \rho|^2 L^4 |\nabla u|^2 L^4 dtd\tau + C \int_0^t |\nabla^2 u|^2 |\nabla \rho| |\nabla u| L^4 dtd\tau
\]

\[
\leq C(\chi + \varepsilon) \int_0^t \left( |\nabla^2 \rho|^2 + |\nabla^2 u|^2 \right) dtd\tau,
\]
By Hölder’s inequality, Sobolev’s inequality, Young’s inequality, Lemma 2.2 and assumption (3.6), we have
\[
|I_{11}^1| + |I_{11}^2| \leq \sigma \int_0^t \|\nabla^2 \rho\|^2 d\tau + C_\sigma \int_0^t \|\nabla^2 (u, \theta)\|^2 d\tau.
\]
Similar to $I_9$, one has
\[
|I_{11}^4| + |I_{11}^4| \leq \int_0^t \int (|\nabla u|^2 + |\nabla (\rho, \theta)| |\nabla \rho|) |\nabla^2 \rho| dx d\tau \leq C \int_0^t \|\nabla (\rho, u, \theta)\|^2 \|\nabla^2 \rho\| d\tau
\]
\[
\leq C(\chi + \varepsilon) \int_0^t (\|\nabla^2 \rho\|^2 + \|\nabla^2 u\|^2 + \|\nabla^2 \theta\|^2) d\tau.
\]
By Hölder’s inequality, Sobolev’s inequality, Young’s inequality, Lemma 2.2 and assumption (3.6), we have
\[
|I_{12}| + |I_{15}| \leq C \int_0^t \|\nabla \rho\|_{L^4} \|\nabla^2 \rho\|_{L^4} \|\nabla^2 \rho\| d\tau \leq C \int_0^t \|\nabla \rho\| \|\nabla^2 \rho\| \|\nabla^2 u\|^2 \|\nabla^3 u\|^2 d\tau
\]
\[
\leq C(\chi + \varepsilon) \int_0^t \|\nabla^2 \rho\|^2 + \|\nabla^3 u\|^2 d\tau.
\]
The same as (3.22), it holds
\[
|I_{13}| + |I_{14}| \leq C(\chi + \varepsilon) \int_0^t (\|\nabla^2 \rho\|^2 + \|\nabla^3 u\|^2) d\tau.
\]
Substituting (3.42) into (3.41) leads to
\[
\|\nabla^2 \rho(t)\|^2 + \int_0^t \|\nabla^2 \rho\|^2 d\tau \leq C\|\nabla u_0, \nabla_0 \rho\|^2 + C\|\nabla u(t)\|^2
\]
\[
+ C \int_0^t \|\nabla^2 (u, \theta)\|^2 d\tau + C(\chi + \varepsilon) \int_0^t \|\nabla^3 u\|^2 d\tau,
\]
which along with Lemma 2.2 and (3.24) implies (3.35), and the proof of Lemma 3.4 is completed. \qed

Thus, Lemma 3.3 and 3.4 imply

**Lemma 3.5.** For $T > 0$ and $(\phi, \psi, \zeta) \in X(0, T)$ satisfying a priori assumption 3.6 with suitably small $\chi + \varepsilon$, we have for $t \in [0, T]$,
\[
\| (\nabla \phi, \nabla \zeta, \nabla^2 \phi)(t) \|^2 + \int_0^t \|\nabla^2 (\phi, \psi, \zeta)\|^2 d\tau
\]
\[
\leq C(\|\phi_0, \psi_0, \zeta_0\|_{H^1}^2 + \|\nabla^2 \phi_0\|^2 + \varepsilon^{\frac{1}{4}}) + C(\chi + \varepsilon) \int_0^t \|\nabla^3 u\|^2 d\tau.
\]

Finally, we want to derive the highest order derivatives of $\psi$ and $\zeta$. It holds

**Lemma 3.6.** For $T > 0$ and $(\phi, \psi, \zeta) \in X(0, T)$ satisfying a priori assumption 3.6 with suitably small $\chi + \varepsilon$, it holds for $t \in [0, T]$,
\[
\|\nabla^2 (\psi, \zeta)(t)\|^2 + \int_0^t \|\nabla^3 (\psi, \zeta)\|^2 d\tau \leq C(\|\phi_0, \psi_0, \zeta_0\|_{H^2}^2 + C\varepsilon^{\frac{1}{4}}).
Proof: First, applying $\partial_i$ ($i = 1, 2, 3$) to the equation (3.33) gives

$$
\partial_i u + u \cdot \nabla \partial_i u + R^\theta \rho \nabla \partial_i \rho + R \nabla \partial_i \theta = \partial_i \left( R^\theta \nabla \rho \right).
$$

(3.53)

$$
= \frac{1}{\rho} (\mu \Delta \partial_i u + (\mu + \lambda) \nabla \partial_i \text{div} u) - \frac{\partial_i \rho}{\rho^2} (\mu \Delta u + (\mu + \lambda) \nabla \text{div} u).
$$

Multiplying the above equation by $(-\Delta \partial_i u)$, similar to (3.25), we have

$$
\int_0^t \int \frac{|\nabla u|^2}{2} \, dx \tau = t + \int_0^t \int \left( \frac{\mu}{\rho^2} |\nabla u|^2 + \frac{\mu + \lambda}{\rho^2} |\nabla \text{div} u|^2 \right) \, dx \tau
$$

$$
= \int_0^t \int \left[ u \cdot \nabla \partial_i u + R^\theta \rho \nabla \partial_i \rho + R \nabla \partial_i \theta + \partial_i u \cdot \nabla u + \partial_i \left( R^\theta \rho \nabla \rho \right) \cdot \Delta \partial_i u \, dx \tau
$$

$$
+ \int_0^t \int \left[ \frac{1}{\rho^2} (\mu \rho \cdot \nabla \partial_j \partial_i u \cdot \partial_j \partial_i u - \mu \partial_j \rho \partial_j \partial_i u \cdot \Delta \partial_i u + (\mu + \lambda) \rho \cdot \nabla \partial_i \text{div} u \partial_i \text{div} u
$$

$$
- (\mu + \lambda) \Delta \partial_i u \cdot \nabla \rho \partial_i \text{div} u \) \, dx \tau + \int_0^t \int \left[ \frac{\partial_i \rho}{\rho^2} (\mu \Delta u + (\mu + \lambda) \nabla \text{div} u) \cdot \Delta \partial_i u \, dx \tau := \sum_{i=16}^{18} I_i.
$$

It follows from Young’s inequality, Sobolev’s inequality, Lemma 2.2 and assumption (3.6) that

$$
|I_{16}| \leq (\sigma + (\chi + \varepsilon)) \int_0^t \|\nabla^3 u\|^2 \, d\tau + C \sigma \int_0^t \|\nabla^2 (\rho, u, \theta)\|^2 \, d\tau
$$

and

$$
|I_{17}| + |I_{18}| \leq C \int_0^t \|\nabla \rho\|_{L^2} \|\nabla^2 u\|_{L^2} \|\nabla^3 u\| \, d\tau \leq C \int_0^t \|\nabla \rho\|^{\frac{1}{2}} \|\nabla^2 \rho\|^{\frac{1}{2}} \|\nabla^2 u\|^2 \|\nabla^3 u\|^2 \, d\tau
$$

$$
\leq C(\chi + \varepsilon) \int_0^t \|\nabla^2 u\|^{\frac{3}{2}} \|\nabla^3 u\|^2 \, d\tau \leq C(\chi + \varepsilon) \int_0^t (\|\nabla^2 u\|^2 + \|\nabla^3 u\|^2) \, d\tau.
$$

Substituting (3.55), (3.56) into (3.54) yields

$$
\int_0^t \|\nabla^2 u(t)\|^2 \, d\tau + \int_0^t \|\nabla^3 u\|^2 \, d\tau \leq C \|\nabla^2 u_0\|^2 + C \int_0^t \|\nabla^2 (\rho, u, \theta)\|^2 \, d\tau.
$$

Next, applying $\partial_i$ ($i = 1, 2, 3$) to the equation (3.33) gives

$$
\frac{R}{\gamma - 1} (\partial_i \rho + u \cdot \nabla \partial_i \theta) + R \theta \partial_i \text{div} u + \frac{R}{\gamma - 1} \partial_i u \cdot \nabla \theta + R \theta \partial_i \text{div} u = \frac{\kappa}{\rho^2} \Delta \partial_i \theta
$$

$$
+ \frac{1}{\rho} (\mu (\nabla u + (\nabla u)^t) \cdot \partial_i (\nabla u + (\nabla u)^t) + 2 \lambda \text{div} u \partial_i \text{div} u) - \frac{\partial_i \rho}{\rho^2} (\mu \Delta \theta + \frac{\mu}{2} |\nabla u + (\nabla u)^t|^2 + \lambda (\text{div} u)^2).
$$

Multiplying the above equation by $(-\Delta \partial_i \theta)$, similar to (3.54), we have

$$
\int_0^t \frac{|\nabla^2 \theta|^2}{2} \, dx \tau = t + \int_0^t \int \frac{\kappa}{\rho^2} |\nabla^3 \theta|^2 \, dx \tau = \int_0^t \int \left[ \frac{R}{\gamma - 1} u \cdot \nabla \partial_i \theta + R \theta \partial_i \text{div} u \right] \, dx \tau
$$

$$
+ \int_0^t \int \left[ \frac{\kappa}{\rho^2} (\nabla \rho \cdot \nabla \partial_j \partial_i \theta \partial_j \partial_i \theta - \partial_j \rho \partial_j \partial_i \theta \Delta \partial_i \theta
$$

$$
+ \partial_i \rho \Delta \theta \Delta \partial_i \theta) \, dx \tau - \int_0^t \int \left[ \frac{1}{\rho} (\mu (\nabla u + (\nabla u)^t) \cdot \partial_i (\nabla u + (\nabla u)^t) + 2 \lambda \text{div} u \partial_i \text{div} u) \right] \Delta \partial_i \theta \, dx \tau
$$

$$
+ \int_0^t \int \left[ \frac{\partial_i \rho}{\rho^2} \frac{\mu}{2} (\nabla u + (\nabla u)^t)^2 + \lambda (\text{div} u)^2 \right] \Delta \partial_i \theta \, dx \tau := \sum_{i=19}^{22} I_i.
By Young’s inequality, Sobolev’s inequality, Lemma 2.2 and assumption (3.6), one has
\begin{equation}
|I_{19}| \leq (\sigma + (\chi + \varepsilon)) \int_0^t \|
abla^3 \theta\|^2 \, dt + C_\sigma \int_0^t \|
abla^2 (u, \theta)\|^2 \, dt.
\end{equation}

Similar to (3.56), it holds,
\begin{equation}
|I_{20}| \leq C \int_0^t \|
abla \rho\|_{L^4} \|
abla^2 \theta\|_{L^4} \|
abla^3 \theta\| \, dt \leq C(\chi + \varepsilon) \int_0^t (\|
abla^3 \theta\|^2 + \|
abla^2 \theta\|^2) \, dt.
\end{equation}

It follows from Hölder’s inequality, Sobolev’s inequality, Young’s inequality, Lemma 2.2 and assumption (3.6) that
\begin{equation}
|I_{21}| \leq C \int_0^t \|
abla u\|_{L^4} \|
abla^2 u\|_{L^4} \|
abla^3 \theta\| \, dt \leq C \int_0^t \|
abla u\| \|
abla^2 u\| \|
abla^3 u\| \|
abla^3 \theta\| \, dt
\end{equation}
\begin{equation}
\leq C(\chi + \varepsilon) \int_0^t \|
abla^2 u\|^{\frac{4}{3}} \|
abla^3 u\|^{\frac{1}{3}} \|
abla^3 \theta\| \, dt.
\end{equation}

and
\begin{equation}
|I_{22}| \leq C \int_0^t \|
abla \rho\|_{L^4} \|
abla u\|_{L^8} \|
abla^3 \theta\| \, dt \leq C \int_0^t \|
abla \rho\| \|
abla^2 \rho\| \|
abla^2 u\| \|
abla^3 u\| \|
abla^3 \theta\| \, dt
\end{equation}
\begin{equation}
\leq C(\chi + \varepsilon) \int_0^t \|
abla^2 u\| \|
abla^3 u\| \|
abla^3 \theta\| \, dt.
\end{equation}

Substituting (3.60), (3.63) into (3.59) gives
\begin{equation}
\|\nabla^2 \theta(t)\|^2 + \int_0^t \|
abla^3 \theta\|^2 \, dt \leq C\|\nabla^2 \theta_0\|^2 + C \int_0^t \|
abla^2 (u, \theta)\|^2 \, dt + C(\chi + \varepsilon) \int_0^t \|
abla^3 u\|^2 \, dt.
\end{equation}

Combining (3.57) and (3.64), we derive
\begin{equation}
\|\nabla^2 (u, \theta)(t)\|^2 + \int_0^t \|
abla^3 (u, \theta)\|^2 \, dt \leq C\|\nabla^2 (u_0, \theta_0)\|^2 + C \int_0^t \|
abla^2 (\rho, u, \theta)\|^2 \, dt,
\end{equation}
which along with Lemma 2.2 and (3.51) leads to (3.52), and the proof of Lemma 3.6 is completed.

**Proof of Proposition 3.1** Combining (3.8), (3.11), (3.51) and (3.52), together, we can obtain (3.7), the proof of Proposition 3.1 is completed.

4. **Proof of Theorem 1.1**

**Proof of Theorem 1.1** We now finish the proof of the main result in Theorem 1.1. The global existence result follows immediately from Proposition 3.1 (A priori estimates) and local existence which can be obtained similarly as in [29] and [31]. To complete the proof of Theorem 1.1, we only need to justify the time-asymptotic behavior (1.22). In fact, from the estimates (4.1), it holds that
\begin{equation}
\int_0^\infty \left( \|\nabla (\phi, \psi, \zeta)\|^2 + \left| \frac{d}{dt} \nabla (\phi, \psi, \zeta) \right|^2 \right) \, dt < \infty,
\end{equation}
which implies
\begin{equation}
\lim_{t \to \infty} \|\nabla (\phi, \psi, \zeta)(t)\|^2 = 0.
\end{equation}

By three-dimensional Sobolev’s inequality, one has
\begin{equation}
\|\phi, \psi, \zeta\|_{L^\infty} \leq C\|\nabla (\phi, \psi, \zeta)(t)\|\|\nabla^2 (\phi, \psi, \zeta)(t)\|,
\end{equation}
which together with (3.7) and (4.2) yields
\[
\lim_{t \to \infty} \| (\phi, \psi, \zeta)(t) \|_{L^\infty} = 0.
\]
Hence we obtain (1.22) and finish the proof of Theorem 1.1.

\[\square\]

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