Research Article
On Perimeters and Volumes of Fattened Sets

Andrea C. G. Mennucci

Scuola Normale Superiore, Pisa, Italy

Correspondence should be addressed to Andrea C. G. Mennucci; andrea.mennucci@sns.it

Received 21 November 2018; Accepted 17 January 2019; Published 24 April 2019

Academic Editor: Jean Michel Rakotoson

Copyright © 2019 Andrea C. G. Mennucci. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper we analyze the shape of fattened sets; given a compact set $C \subset \mathbb{R}^N$, let $C_r$ be its $r$-fattened set; we prove a general bound $rP(C_r) \leq N \mathcal{L}(C_r \setminus C)$ between the perimeter of $C_r$ and the Lebesgue measure of $C_r \setminus C$. We provide two proofs: one elementary and one based on Geometric Measure Theory. Note that, by the Fleming–Rishel coarea formula, $P(C_r)$ is integrable for $r \in (0, a)$. We further show that for any integrable continuous decreasing function $\psi : (0, 1/2) \to (0, \infty)$ there exists a compact set $C \subset \mathbb{R}^N$ such that $P(C_r) \geq \psi(r)$. These results solve a conjecture left open in (Mennucci and Duci, 2015) and provide new insight in applications where the fattened set plays an important role.

1. Introduction

For any $A \subseteq \mathbb{R}^N$ closed, let $u_A$ be the distance function from $A$

$$u_A(x) \overset{\text{def}}{=} \inf_{y \in A} |y - x|.$$  \hspace{1cm} (1)

Let $\{u_A \leq r\} = \{x \in \mathbb{R}^N : u_A(x) \leq r\}$ be the fattened set of $A$ of radius $r > 0$. It is equal to the Minkowski sum of $A$ and a closed ball $D_r(0)$ of radius $r$ with center in the origin. It is also called the parallel set or the tubular neighborhood.

Let $\mathcal{L}$ be the $N$-dimensional Lebesgue measure. Let $P(E)$ be the perimeter of a Borel set $E \subseteq \mathbb{R}^N$.

1.1. Main Results. In this short essay we will prove some geometrical properties of the function $u_A$ and of the fattened set.

The main result is as follows.

**Theorem 1.** Let $C \subset \mathbb{R}^N$ be a compact set. For all $r > 0$ we have

$$rP(\{u_C \leq r\}) \leq N \mathcal{L}(\{0 < u_C \leq r\}).$$  \hspace{1cm} (2)

Note that (2) is sharp (consider the case $C = \{0\}$; see relations (16)).

We will provide two proofs of Theorem 1. An elementary proof is in Section 3; it is based on simple geometrical properties of sets in $\mathbb{R}^N$. Another proof is in Section 5, it is based on semiconcavity of $u_C^2$ and the Gauss–Green formula. It may be appreciated that the first method of proof is simpler.

A corollary of the above theorem is that, for any $r > 0$, the perimeter of the fattened set $\{u_C \leq r\}$ is finite—even when the perimeter of $C$ is not finite. We elaborate on this fact further.

**Remark 2.** Let $\psi(r) = P(\{u_C \leq r\})$ for convenience; by the Fleming–Rishel coarea formula (see Proposition 14 and Lemma 24 here)

$$\mathcal{L}(\{0 < u_C \leq r\}) = \int_0^r \psi(s) \, ds < \infty$$  \hspace{1cm} (3)

(so the function $\psi$ is locally in $L^1$); then thesis (2) can be rewritten as

$$\psi(r) \leq \frac{N}{r} \int_0^r \psi(s) \, ds;$$  \hspace{1cm} (4)

the above implies

$$\psi(r) = o\left(\frac{1}{r}\right)$$  \hspace{1cm} (5)

for $r \to 0$. 


The above properties (5) and (3) are again "sharp" in this sense.

**Theorem 3.** If \( \psi : (0, 1/2) \rightarrow (0, \infty) \) is a continuous decreasing function satisfying \( \lim_{\epsilon \downarrow 0} \psi(\epsilon) = 0 \) and \( \int_0^{1/2} \psi(s) \, ds < \infty \) (that is requirement (3) in the above remark); then \( \psi(r) = o(1/r) \) (as in (5)), and we can construct a compact set \( C \) such that \( P(\{ u_C \leq r \}) \geq \psi(r) \) for \( r \in (0, 1/2) \).

This holds in any \( \mathbb{R}^N \) (for \( N \geq 1 \)). The proof is in Section 4.1.

The interest on these results was spurred by the use of \( u_A \) in [1]. We will discuss the connection to [1] in Section 6. The main theorem will be as follows.

**Theorem 4.** Let \( p \in [1, \infty) \) and \( \varphi : [0, \infty) \rightarrow [0, \infty) \) a Borel function such that

\[
\int_0^{\infty} t^{N-1} \varphi(t)^p \, dt < \infty,
\]

\[
\exists \epsilon > 0, \quad \text{ess}\sup_{0 < t < \epsilon} \varphi(t) < \infty;
\]

then for any compact \( C \subseteq \mathbb{R}^N \) we have \( \varphi * u_C \in L^p(\mathbb{R}^N) \).

Request (7) is justified by the existence of compact sets with properties as described in Theorem 3.

**1.2. Motivation**

**1.2.1. Banach-Like Distances.** Let \( \mathcal{M} \) be the family of all nonempty compact subsets of \( \mathbb{R}^N \). In 2015 Duci and Menucci [1] studied a family of distances on \( \mathcal{M} \), defined by means of the distance function \( u_A \). Some natural questions came from that study; one of them was eventually answered by Theorem 4. See Section 6 for more details.

The study of fattened set though is quite wide and interesting; we provide two examples.

**1.2.2. Steiner Formulas.** In 2004 Hug et al. [2] generalized the Steiner formulas as follows. Let \( \mathcal{M} \subseteq \mathbb{R}^N \) closed, and let \( \partial C \) be the topological boundary.

**Definition 5.** Define the normal bundle \( N(C) \subseteq C \times S^1 \) as the set of pairs \((x, u) \in C \times S^1 \) for which there exists a \( t > 0 \) such that \( x \) is the unique point of \( C \) at minimum distance from \( x + tu \) (this definition is equivalent to the definition in Section 2.1 in [1], but it is simplified to avoid introducing further definitions and notations that are not needed in this paper); in that case, let \( \delta(x, u) \) be the supremum of such \( t \); let \( \delta(x, u) = 0 \) for all \((x, u) \in (\mathbb{R}^N \times S^1) \setminus N(C) \).

\( N(C) \) is a Borel subset of \( (\partial C) \times S^1 \). By Lemma 2.3 in [2], it is countably \((N - 1)\)-rectifiable. There are examples of compact sets such that \( \mathcal{H}^{N-1}(N(C)) = \infty \) (the examples in this paper will do).

**Definition 6.** A reach measure is a real function \( \mu \) with domain the Borel subsets of \( \mathbb{R}^N \times S^1 \), such that

- (i) \( \mu(A) = 0 \) for all Borel subsets of \( (\mathbb{R}^N \times S^1) \setminus N(C) \);
- (ii) at the same time, for any fixed \( a > 0 \), \( \mu(A) \) is a signed measure (of bounded variation) when evaluated on the family of Borel sets A contained in

\[
\left\{(x, u) \in N(C) : |x| \leq a, \delta(x, u) \geq \frac{1}{a}\right\}.
\]

**Theorem 7 (Theorem 2.1 in [2]).** For any closed set \( C \) there exist reach measures \( \mu_0, \mu_1 \ldots \mu_{N-1} \) such that

\[
\int_{x \in B, u \in S} (\delta(x, u) \wedge r)^{N-i} \, d\mu_j(x, u) < \infty
\]

for any \( r > 0 \) and \( B \subseteq \mathbb{R}^N \) compact, and, for any \( f : \mathbb{R}^N \rightarrow \mathbb{R} \) bounded Borel,

\[
\int_{\mathbb{R}^N \setminus C} f(x) \, dx = \sum_{i=0}^{N-1} (N - i) \cdot \omega_{N-i} \int_{N(C)} (\delta(x, u))^i t^{N-1-i} f(x + tu) \, dt \, d\mu_i(x, u).
\]

**Remark 8.** A closed set \( C \) is a set of positive reach [3] when there exists a \( \rho > 0 \) such that for any \( x \in \mathbb{R}^N \) with \( u_C(x) < \rho \) there exists a unique \( y \in C \) at minimum distance from \( x \); the reach is the largest such \( \rho \) (possibly infinite). For example, smooth manifolds embedded in \( \mathbb{R}^N \) have positive reach, as well as convex subsets. It is easily verified that \( \rho = \lim\sup_{r \downarrow 0} \delta(x, u) \in N(C) \), where \( N(C), \delta \) were defined in Definition 5. When the set \( C \) has positive reach then for small \( r > 0 \) formula (2) can be verified (through relation (3)) using results in [3]: see Theorem 5.5 in [3] that provides an explicit formula for \( \mathcal{L}(\{ u_C \leq r \}) = \rho(r) \), where \( p(r) \) is a polynomial in \( r \) of degree at most \( N \).

In this respect it is worth noting this result by Fu et al [4]. For any \( r > 0 \) that is a regular value for the distance function \( u_C \), the set \( \mathbb{R}^N \setminus C \) is a set of positive reach (see Corollary 3.4 in [4]). When \( N = 2 \) or \( N = 3 \) a Sard-type result shows that the set of critical values is small, in an appropriate sense (see Thm. 4.1 in [4]).

**1.2.3. Minkowski Content.** The study of the fattened set is linked to the Minkowski content

\[
\lim_{r \downarrow 0} \mathcal{L}\left(\{ u_A \leq r \}\right) \frac{1}{r}
\]

which has wide applications in the theory of Stochastic Differential Equations. See Ambrosio et al. [2, 5] and references therein.

**2. Notation**

We will write \( B_r(x) \) for the open ball

\[
B_r(x) \overset{\text{def}}{=} \{ y \in \mathbb{R}^N : |x - y| < r \}
\]
of center \(x\) and radius \(r > 0\) in \(\mathbb{R}^N\); we will write \(B_r\) for \(B_r(0)\). \(D_r(x)\) will be the disk

\[
D_r(x) \overset{\text{def}}{=} \{ y \in \mathbb{R}^N : |x - y| \leq r \}
\]

of center \(x\) and radius \(r > 0\) in \(\mathbb{R}^N\) and \(D_r = D_r(0)\); \(S_r(x)\) will be the sphere

\[
S_r(x) \overset{\text{def}}{=} \{ y \in \mathbb{R}^N : |x - y| = r \}
\]

of center \(x\) and radius \(r > 0\) in \(\mathbb{R}^N\).

For \(A \subseteq \mathbb{R}^N\) closed, let \(u_A\) be the distance function from \(A\) (defined in (1)). Let \(|u_A| \leq r\) be the fattened set of \(A\) of radius \(r > 0\).

Let \(d_{H}\) be the Hausdorff distance of compact sets in \(\mathbb{R}^N\); it can be defined (as shown in Sec. C in Chap. 4 in [6] and Sec. 2.2 in Chap. 4 in [7]) as

\[
d_{H}(A, B) \overset{\text{def}}{=} \sup_{x \in \mathbb{R}^N} |u_A(x) - u_B(x)|.
\]

For \(d \geq 0\) let \(\mathscr{H}^d\) be the \(d\)-dimensional Hausdorff measure; let \(\mathcal{L}\) be the \(N\)-dimensional Lebesgue measure, and \(\omega_N \overset{\text{def}}{=} \mathcal{L} (B_1)\). Let \(P(E)\) be the perimeter of a Borel set \(E \subseteq \mathbb{R}^N\), as defined in Definition 1.6 in [8]. We define \(\omega_N = \mathcal{L}(B_1)\) and consequently for \(r > 0\)

\[
\mathcal{L} (B_r) = \omega_N r^N,
\]

\[
P(B_r) = \mathcal{H}^{N-1} (S_r) = N \omega_N r^{N-1},
\]

\[
\mathcal{L} (B_r) = \omega_N r^N = \frac{r}{N} \mathcal{H}^{N-1} (S_r).
\]

3. Area and Perimeter of Fattened Sets

These facts are known; see Sec. 4 and 5 in [1].

**Proposition 9.** Let \(A \subseteq \mathbb{R}^N\) be a compact set.

1. Let \(r > 0\). Let \(F = \{u_A \leq r\}\) and \(E = \{u_A = r\}\) for convenience.

   (i) The boundary \(\partial F\) of \(F\) is contained in the set \(E\). (Equality may fail, consider \(A = \{ |x| \in [1,2], r = 1\}\).

   (ii) \(E\) is Lebesgue negligible (hence \(\partial F\) is as well).

   (iii) As a consequence, for any \(r > 0\),

   \[
P \left( \{ u_C \leq r \} \right) = P \left( \{ u_C < r \} \right).
\]

2. For any fixed \(A \in \mathcal{M}\), the fattening map \(\lambda \mapsto \{u_A \leq \lambda\}\) is Lipschitz (of constant one) as a map from \([0,\infty)\) to \((\mathcal{M}, d_{H})\).

3. For any fixed \(\lambda > 0\), the “fattened area map” \(L_\lambda : \mathcal{M} \rightarrow \mathbb{R}\) defined by \(L_\lambda (A) \overset{\text{def}}{=} \mathcal{L}(\{u_A \leq \lambda\})\) is continuous on \((\mathcal{M}, d_{H})\).

We add a further result in the same spirit.

**Lemma 10.** If \(A_n, A\) are compact and \(A_n \rightarrow A\) according the Hausdorff distance and \(\bar{A} = \{u \leq \lambda\}, \bar{A}_n = \{u_{A_n} \leq \lambda\}\) are the fattened sets, then \(\mathcal{L}(\bar{A}_n \Delta \bar{A}) \rightarrow 0\) where \(\Delta\) is the symmetric difference of sets.

**Proof.** Let again \(E = \{x : u_A(x) = \lambda\}\); then for any \(x \in \bar{A}_n \Delta \bar{A}\) we have that

(i) if \(u_A(x) > \lambda\) then \(u_{A_n}(x) > \lambda\) for \(n\) large (by (15)), so \(x \notin \bar{A}_n \Delta \bar{A}\) for \(n\) large;

(ii) if \(u_A(x) < \lambda\) then \(u_{A_n}(x) < \lambda\) for \(n\) large, so \(x \notin \bar{A}_n \Delta \bar{A}\) for \(n\) large;

(iii) if \(u_A(x) = \lambda\) then \(x \in E\), that is negligible (by Proposition 9).

The proof then follows from the Lebesgue dominated convergence theorem.

We will also need this simple inequality.

**Lemma 11.** For each \(r, R \geq 0\) and \(N \geq 1\) integer we have

\[
(r + R)^N - R^N \geq r (r + R)^{N-1}.
\]

**Proof.** We have

\[
(r + R)^N - R^N = \sum_{j=0}^{N-1} \binom{N}{j} r^{N-j} R^j,
\]

\[
r (r + R)^{N-1} = \sum_{j=0}^{N-1} \binom{N-1}{j} r^{N-j-1} R^j,
\]

\[
\left(\frac{N}{N-j}\right) = \frac{N}{N-j} \geq 1 \quad \forall j = 0, \ldots, N-1.
\]

We now come to the proof of the main contribution of this paper, Theorem 1.

**Proof.** The proof is in three steps.

1. Let \(z, \omega \in \mathbb{R}^N, r, \rho > 0\) and \(D_r(z), D_\rho(\omega)\) two disjoint disks (as defined in (13)). The locus of points \(x \in \mathbb{R}^n\) at the same distance from the two disks is a (connected sheet of) a quadric. Choose appropriate coordinates where \(z = (-a, 0, \ldots, 0), \omega = (b, 0, \ldots, 0)\), with \(0 < r < a, 0 < \rho < b < a\) and \(a - r = b - \rho\) so that the locus contains the origin; then the locus can be written as

\[
x_1 \geq 0 \wedge 4abx_1 (x_1 + a - b) = (a - b)^2 \sum_{j=2}^{N} x_j^2.
\]

See Figure 1

Suppose that \(C = \bigcup_{j=1}^{N} D_{r_j}(z_j)\) is the union of finitely many disjoint disks. Let for simplicity

\[
\delta_1 (x) = u_{D_{r_j}}(x) (x) = (|x - z_j| - r_j)^+.
\]

(20)
be the distance to one such disk. Note that
\[ u_C(x) = \min_{i=1,...,n} \delta_i(x). \]  
(22)

Fix \( k \in 1,\ldots,n \) and then consider the region \( R_k \) of points \( x \) that are nearer to \( D_k(z_k) \) than to any other disk; \( R_k \) is usually called a Voronoi cell. \( R_k \) is defined by the inequalities
\[ R_k = \left\{ x : \forall j \neq k, \delta_k(x) < \delta_j(x) \right\}. \]  
(23)

These can be reduced to inequalities involving first and second-degree polynomials in \( x = (x_1,\ldots,x_n) \). For example, the three inequalities
\[ \begin{align*}
|x-z| > r_j, \\
|x-z_k| > r_k, \\
\delta_k(x) < \delta_j(x)
\end{align*} \]  
(24)

when \( r_k < r_j \) can be reduced (in appropriate coordinates as above, setting \( z = z_k, w = z_j, \rho = r_k, r = r_j \)) to
\[ \begin{align*}
(x_1-b)^2 &+ \sum_{j=2}^{n} x_j^2 > \rho^2 & \land & x_1 > 0 \\
&\land 4ab x_1 (x_1 + a - b) > (a-b)^2 \sum_{j=2}^{n} x_j^2.
\end{align*} \]  
(25)

So region \( R_k \) is an open semialgebraic set. Its boundary is a semialgebraic set. It is contained in the finite union of quadrics as described in (20).

The set \( \{u_C = r\} \) is contained in the union of spheres \( \bigcup_{j=1}^{n} S_{r_j}(z_j) \). A sphere can intersect a quadric such as (20) in a set of at most dimension \( N-2 \). So when evaluating \( \mathcal{H}^{N-1}(\{u_C = r\}) \) we only consider the parts of \( \{u_C = r\} \) that are inside the regions \( R_k \).

Inside each region \( R_k \) the set \( \{u_C = r\} \) is given by the equality \( |x-z| = r + r_k \), so it is a part of a sphere. For each point \( x \in \{u_C = r\} \cap R_k \), the projection of \( x \) to \( C \) is an unique point \( p \) contained in \( S_{r_k}(z_k) \). The segment from \( x \) to \( z_k \) passes through \( p \), and it is contained in \( R_k \). This follows from well-known theory for distance functions, but in this case can also be easily checked with simple geometrical arguments. We denote by \( J \) the union of all segments \( xz_k \) for \( x \in \{u_C = r\} \cap R_k \). See Figure 2.

So we can establish the relations
\[ \mathcal{L}(J \setminus C) = \frac{\alpha}{N} ((r + r_k)^N - r_k^N), \]  
(26)

\[ \mathcal{H}^{N-1}(\{u_C = r\} \cap R_k) = (r + r_k)^{N-1} \alpha. \]

(\( \alpha > 0 \) is the solid angle under which \( z_k \) sees \( \{u_C = r\} \cap R_k \). By Lemma 11 then
\[ r \mathcal{H}^{N-1}(\{u_C = r\} \cap R_k) \leq N \mathcal{L}(J \setminus C). \]  
(27)

Since
\[ (J \setminus C) \subseteq \{0 < u_C \leq r\} \cap R_k \]  
(28)

then a fortiori
\[ r \mathcal{H}^{N-1}(\{u_C = r\} \cap R_k) \leq N \mathcal{L}(\{0 < u_C \leq r\} \cap R_k). \]  
(29)

Summing in \( k \) we obtain the relation
\[ r \mathcal{H}^{N-1}(\{u_C = r\}) \leq N \mathcal{L}(\{0 < u_C \leq r\}). \]  
(30)

That is the same as (2) for the case when \( C \) is an union of spheres.

(2) Let \( C \) be a compact set. Define \( A_n = \{u_C < 1/n\} \); this is a decreasing sequence of open sets \( A_n \) such that \( A_n \supseteq C, \bigcap_{n=1}^{\infty} A_n = C, \mathcal{L}(A_n) \rightarrow_n \mathcal{L}(A), \) and \( d_H(A_n, C) \leq 1/n. \)

Fix \( n \); by Vitali’s covering theorem we can choose finitely many disjoint disks inside \( A_n \) so that their union is \( C_n \) and it satisfies \( \mathcal{L}(A_n \setminus C_n) \leq 1/n; \) we also have that \( d_H(A_n, C_n) \leq 1/n. \)

Summarizing we obtain a sequence \( C_n \) such that each \( C_n \) is the union of finitely many disjoint disks, \( d_H(C_n, C) \rightarrow_n 0 \) and \( \mathcal{L}(C \setminus C_n) \rightarrow_n 0. \)
(3) Since \(dH(C', C) \to 0\) then by Lemma 10
\[
\mathcal{L}(\{u_{C'} \leq r\} \Delta \{u_C \leq r\}) \to 0. \tag{31}
\]
Using moreover the fact that \(\mathcal{L}(C \Delta C_n) \to 0\) then
\[
\mathcal{L}(\{0 < u_{C'} \leq r\} \Delta \{0 < u_C \leq r\}) \to 0. \tag{32}
\]
Since \(\mathcal{L}(C \Delta C, n) \to 0\) then, by Thm. 1.9 in [8],
\[
P(C) \leq \liminf_n P(C_n). \tag{33}
\]
So (2) follows. \(\square\)

4. Examples

We first present the simplest case, of a compact subset \(C \subseteq \mathbb{R}\).

Example 1 (Let \(N = 1\)). For \(h \in \mathbb{N}, h \geq 1\) let \(r_h > 0\) be a monotone nonincreasing sequence, such that \(\sum_{h=1}^{\infty} r_h < \infty\). Let \(a_1 = 0\) and \(a_h = 4 \sum_{j=1}^{h-1} r_j\) for \(h \geq 2\). Consider a compact set
\[
C = \{a_{\infty}\} \cup \bigcup_{h=1}^{\infty} [a_h - r_h, a_h + r_h]. \tag{34}
\]

\(C\) is composed by countably many disjoint closed intervals spaced as in Figure 3, and the limit point \(\{a_{\infty}\}\) is added. Then
\[
\mathcal{L}(C) = 2\beta < \infty, \tag{35}
\]
\[
P(C) = \infty. \tag{36}
\]

Let \(r > 0, r < r_1\), define
\[
\chi(r) \overset{\text{def}}{=} \max\{h \geq 1 : r < r_h\}; \tag{37}
\]
note that \(\chi(r) < \infty\) since \(\lim_{h \to \infty} r_h = 0\). We can estimate for \(r < r_1\) that
\[
P(\{u_C \leq r\}) \geq 2\chi(r), \tag{38}
\]
and indeed (setting \(k = \chi(r)\) for convenience)
\[
\{u_C \leq r\} \cap (-\infty, a_{\infty}) = [a_k - r_h - r_h, a_k + r_h + r], \tag{39}
\]
where the latter part is a disjoint union.

The example can be built in higher dimensions as well. Let \(N = 2\) for simplicity (the case \(N \geq 3\) is similar, by repeating spheres along the extra dimensions and changing some constants).

Example 2. Let \((m_h)_{h=1}\) be a monotone nondecreasing sequence of integers, such that \(m_h \geq 1\) and \(\sum_{h=1}^{\infty} 1/m_h < \infty\). Let \(r_h = 4/(4m_h - 2)\). (Note that \(2 \geq r_h, m_h \geq 1\), and \(\sum_{h=1}^{\infty} r_h < \infty\). Let \(a_1 = r_1\) and \(a_h = a_1 + 4 \sum_{j=1}^{h-1} r_j\) for \(h \geq 2\). Consider a compact set \(C_h\) union of a family of \(m_h\) disks with centers in \(x_{i,j} = (a_h, r_h(1 + j))\) for \(j = 0, 1, \ldots, m_h - 1\), each of radiuses \(r_h\), as follows:
\[
C_h \overset{\text{def}}{=} \bigcup_{j=0}^{m_h-1} D(x_{h,j}, r_h). \tag{40}
\]

It is easily proven that
\[
\mathcal{L}(C_h) = \pi r_h^2 m_h \leq 2\pi r_h, \tag{41}
\]
\[
P(C_h) = 2\pi r_h m_h \geq 2\pi. \tag{42}
\]

Let then the compact set be
\[
C = \left\{(a_{\infty}, t) : t \in [0, 4]\right\} \cup \bigcup_{h=1}^{\infty} C_h \tag{43}
\]

\(C\) is compact since we include in \(C\) the line that contains limit points of \(\bigcup_{h=1}^{\infty} C_h\). This set \(C\) is tightly contained in the rectangle of corners \((0, 0)\) and \((a_{\infty}, 4)\). See Figures 5 and 6 for two examples.

For any \(h < k\) any two points \(x \in C_h, y \in C_k\) in different families are at a distance \(|x - y| \geq 2r_h\); moreover two different disks composing \(C_h\) are at a distance at least \(2r_h\) (see Figure 4); hence
\[
\mathcal{L}(C) < \infty, \tag{44}
\]
\[
P(C) = \infty. \tag{45}
\]

Let \(r > 0, r < r_1\), and define (as before)
\[
\chi(r) \overset{\text{def}}{=} \max\{h \geq 1 : r < r_h\}; \tag{46}
\]
note that \(\chi(r) < \infty\) since \(\lim_{h \to \infty} r_h = 0\). Let \(M_h = \sum_{j=1}^{h} m_j\). We can estimate for \(r < r_1\) that
\[
P(\{u_C \leq r\}) \geq 2\pi \sum_{h=1}^{\infty} (r + r_h) m_h \geq 2\pi \chi(r), \tag{47}
\]
and indeed for the first inequality we note that when $r < r_h$ and $k > h$, any two
\begin{align}
  x &\in \{ u_{C_k} \leq r \}, \\
y &\in \{ u_{C_h} \leq r \}
\end{align}
are at a distance $|x - y| \geq 2(r_h - r) > 0$. See Figure 4 again.

In general for $N \geq 1$ the above examples will satisfy
\begin{equation}
P \left( \{ u_C \leq r \} \right) \geq N\omega_N \chi(r).
\end{equation}

We will need a lemma.

**Lemma 12.** Suppose that $f : (0,1) \rightarrow (0,\infty)$ is continuous and decreasing and $\int_0^1 f(t)dt < \infty$ then $\lim_{r \to 0} rf(r) = 0$.  

Equivalent (setting $f(t) = g(1/t)$), suppose that $g : (1, \infty) \to (0, \infty)$ is continuous and increasing and 
\[
\int_1^\infty g(t)t^{-2}dt < \infty \text{ and then } \lim_{t \to \infty} g(t)/t = 0.
\]

Proof. Suppose instead that \( \limsup_{t \to \infty} g(t)/t > 0 \) and let \( \varepsilon > 0 \) and \( t_j > 0 \) be an increasing sequence such that \( t_j \to \infty \) and for all \( j \) we have \( g(t_j) > \varepsilon t_j \) then
\[
\frac{1}{\varepsilon} \int_1^\infty g(t)t^{-2}dt \geq \sum_{j=1}^{\infty} \int_{t_j}^{t_{j+1}} t_j^{-2}dt = \sum_{j=1}^{\infty} \frac{1}{t_j} (\frac{1}{t_j} - \frac{1}{t_{j+1}}) = \infty
\]
where the last step is explained in exercise 11 in Section 3 in [9].

The above example can be finetuned as follows.

**Proposition 13.** Let \( \sigma : [1, \infty) \to [1, \infty) \) be a continuous increasing function such that \( \sigma(1) = 1 \), \( \lim_{t \to \infty} \sigma(t) = \infty \), and
\[
\int_1^1 s^{-1} \sigma \left( \frac{1}{t} \right) dt = \int_1^\infty \sigma(t)t^{-2}dt < \infty,
\]
and we will build a compact set, following the previous example, such that for \( r < 1/2 \)
\[
\Pr \left( \{u_C \leq r\} \right) \geq N \omega_N \sigma \left( \frac{1}{r} - 1\right).
\]

Proof. Let \( \tau : [1, \infty) \to [1, \infty) \) be the inverse \( \tau = \sigma^{-1} \), and then \( \tau(1) = 1 \) and \( \lim_{t \to \infty} \tau(t)/t = \infty \). The subgraph
\[
\{(s, t) : 0 < t < 1, 1 < s < \sigma \left( \frac{1}{\tau(s)} \right)\}
\]
coincides with the subgraph
\[
\{(s, t) : 0 < t < 1, 1 < s < \sigma \left( \frac{1}{\tau(s)} \right)\}
\]
so
\[
\int_1^\infty \frac{1}{\tau(s)} ds = \int_0^1 \frac{1}{\tau(s)} \frac{1}{\sigma \left( \frac{1}{\tau(s)} \right)} ds < \infty.
\]

Let \( m_h = \lceil \tau(h) \rceil \); note that \( m_1 = 1 \). Since \( 1/\tau(t) \geq 1/m_h \) for \( h \geq 2 \) and \( t \in [h-1, h] \) so \( \sum_{h=1}^\infty 1/m_h < \infty \); so we can build \( C \) as in the previous examples.

When \( N \geq 2 \) we let \( r_h = 4/(4m_h - 2) \) as before; note that \( r_h m_h \geq 1 \). We define \( \chi \) as in (43) and note that \( k = \chi(r) \) is characterized by
\[
r_{k+1} \leq r < r_k.
\]
Since \( \tau(h) + 1 \geq m_h \geq \tau(h) \) then
\[
\tau(k + 1) + 1 \geq m_{k+1} \geq \frac{1}{r_{k+1}} \geq \frac{1}{r}
\] (54)
and hence for \( r < 1/2 \)
\[
\psi(r) = \sigma\left(\frac{1}{r} - 1\right) - 1.
\] (55)
Combining this with relation (46) we obtain (49).

When \( N = 1 \) then we just set \( r_h = 1/m_h \) and proceed similarly.

4.1. Proof. We eventually prove Theorem 3.

Proof. Possibly adding a constant to \( \psi \) and large disks to \( C \), we assume that \( \psi(1/2) = N\omega_N - 1 \). We relate
\[
t = \frac{1}{r} - 1,
\] (56)
for \( t \geq 1 \) and \( 0 < r \leq 1/2 \); we let
\[
\sigma(t) = \frac{1 + \psi(1/(t + 1))}{N\omega_N}
\] (57)
and then \( \sigma : [1, \infty) \to [1, \infty) \) is continuous and increasing, \( \sigma(1) = 1 \), \( \lim_{t \to \infty} \sigma(t) = \infty \), and
\[
\int_0^1 \sigma\left(\frac{1}{t}\right) dt = \int_0^1 \frac{1 + \psi(t/(t + 1))}{N\omega_N} dt
\] (58)
where
\[
\left\{ \begin{array}{l}
\|Df\|_\Omega \overset{\text{def}}{=} \sup \left\{ \int_\Omega f \text{ div } (g) \, dx : g \in C^0_c(\Omega) \to \mathbb{R}^n \right\} \|g(x)| \leq 1 \forall x \in \mathbb{R}^n \left. \right) \\
\text{is the total mass of the distribution } Df \text{ and }
\end{array} \right.
\] (61)
\[
P(E, \Omega) \overset{\text{def}}{=} \|Df\|_\Omega
\] (62)
is the perimeter of a Borel set \( E \) inside \( \Omega \).

(For a proof, see also Theorem 3.40 in [11]). We will write \( P(E) \) for \( P(E, \mathbb{R}^N) \).

Proposition 15 (Federer coarea formula [3]). Let \( \Omega \subseteq \mathbb{R}^N \) be a Borel set and \( f, g : \Omega \to \mathbb{R} \) with \( f \) Lipschitzian and \( g \) integrable, and then
\[
\int_\Omega g(x) |\nabla f(x)| \, dx = \int_\mathbb{R} \left( \int_{\{f = t\} \cap \Omega} g(y) \, d\mathcal{H}^{N-1}(y) \right) \, dt.
\] (63)
In particular
\[
\int_\Omega |\nabla f(x)| \, dx = \int_\mathbb{R} \mathcal{H}^{N-1}(\{f = t\} \cap \Omega) \, dt.
\] (64)

Proposition 16. Let \( \Omega \subseteq \mathbb{R}^N \) be an open set; let \( E \) be Borel and \( P(E, \Omega) < \infty \). Let \( \partial^* E \) be the reduced boundary of \( E \) and \( \nu \) the inward normal, and then
\[
D1_E = \nu \|D1_E\|.
\] (65)

For definitions and proofs see, e.g., those in Section 3.5 in [11] or in Section 3 in [8].

Proposition 17. Let \( \Omega \subseteq \mathbb{R}^N \) be an open set; let \( E \) be Borel and \( P(E, \Omega) < \infty \). Let \( \partial^* E \) be the reduced boundary of \( E \), and then
\[
\partial^* E = \partial E;
\] (i)
\[
P(E, A) = \int_A |D1_E| = \mathcal{H}^{n-1}(A \cap \partial^* E)
\] (66)
for any \( A \subseteq \Omega \) open.

These results are due to De Giorgi [12]; for a proof see, e.g., in Section 3.5 in [11] or in Section 4 in [8]. Combining the above with the Fleming–Rishel coarea formula we obtain the following.

Proposition 18 (structure theorem). If \( E \) is a Borel set such that \( P(E) < \infty \) then
\[
\int_E \partial_x g(x) \, dx = -\int_{\partial^* E} g(x) \nu_i(x) \, d\mathcal{H}^{n-1}(x)
\] (67)
for all \( g \in C^1_0(\mathbb{R}^N) \) and \( i = 1, \ldots, n \), where \( \partial^* E \) is the reduced boundary and \( \nu(x) \) is the inward normal. (For a definition, see also Theorem 3.54 in [11]).
5.2. Lemmas

Lemma 19. Let $\Omega$ be open; suppose that $f : \Omega \to \mathbb{R}$ is Lipschitz; let $A \subseteq \Omega$ open. For almost all $r > 0$ we have that

$$ P \{ \{ f \leq r \} \cap A \} = \mathcal{H}^{N-1} ( \{ f = t \} \cap A ) = \mathcal{H}^{N-1} ( \{ f = t \} \cap A ). \quad (68) $$

The proof follows comparing the Fleming–Rishel and the Federer coarea formula.

Lemma 20. Let $\Phi : \mathbb{R}^N \to [0, \infty)$ be an integrable compactly supported function. Let $\phi : \mathbb{R}^N \to \mathbb{R}$ be convex. Then the convolution $\Phi * \phi$ is convex.

This is easily proved, e.g., by showing that the derivatives of $t \in \mathbb{R} \to (\Phi * \phi)(x + tv)$ are monotone nondecreasing.

Let $C$ be compact. It is well known that $u_C^2$ is semiconcave; we provide a simple quantitative proof.

Lemma 21. $u_C^2(x) - |x|^2$ is concave.

Proof. Indeed

$$ \varphi (x) = u_C^2(x) - |x|^2 = \min_{y \in C} \{ |x - y|^2 - |y|^2 \} = \min_{y \in C} \{ |x - y|^2 - |y|^2 \} = \min_{y \in C} \{ -2x \cdot y + |y|^2 \} \quad (69) $$

but

$$ x \mapsto (-2x \cdot y + |y|^2) \quad (70) $$

is affine hence $\varphi$ is concave. \(\square\)

Lemma 22. Let again $\psi(x) = u_C^2(x)$; it is well known that

(i) $\nabla u_C(x)$ and $\psi(x)$ are locally Lipschitz (by direct proof)
(ii) hence for almost any $x$ the differentials exist,
(iii) when these functions are differentiable and $x \notin C$ then

(a) $|\nabla u_C(x)| = 1$ and $|\nabla \psi(x)| = 2\psi(x)$,
(b) the point $y \in C$ closest to $x$ is unique and is $y = x - \nabla\psi(x)/2$;

for these last two, see in [13] (in particular Theorem 3.1 and Remark 3.6).

5.3. Proof. Here is then a "geometric measure theory proof" of Theorem 1.

Proof. In the first part of this proof we assume that $P(C) < \infty$.

Let $F_t = \{ u_C \leq t \}$ and $E_t = \{ u_C = t \}$ in the following. The idea of the proof is as follows. Set $\psi(x) = u_C^2(x)$. Let $K = \{ 0 < u_C \leq r \} = F_t \setminus C$, supposing that the boundary of $K$ is smooth (in this case $\partial F_t = E_t$), and we note that $\partial K = \partial C \cup \partial F_t$; we have that $\nabla \psi(x) = 0$ for $x \in \partial C$ whereas $\nabla \psi(x) = -2u_C(x)\nu = -2r\nu(x)$ for $x \in \partial F_t$, where $\nu(x)$ is the inward normal to $\partial F_t$; so we use the Gauss–Green formula and write

$$ 2N \mathcal{L}(K) \geq \int_K \Delta \psi(x) \, dx \quad (71) $$

where the first inequality follows from Lemma 21 that implies that $\Delta(u_C^2) \leq 2N$ in the sense of distributions.

We now provide the general proof.

We sketch the following facts:

(1) For almost all $t > 0$ we have that $P(F_t) < \infty$, by the coarea formula. (Note that, by the proof in Section 3, this is actually true for every $t > 0$.)

(2) $\partial F_t \subseteq E_t$ by Proposition 9, so $\partial^* F_t \subseteq E_t$ for all $t > 0$ such that $P(F_t) < \infty$.

(3) By Lemma 19 for almost any $r > 0$ we have $\mathcal{H}^{N-1}(E_t \setminus \partial^* F_t) = 0$.

(4) Hence for almost any $r > 0$ and $\mathcal{H}^{N-1}$-almost any $x \in E_t$ we have that

$$ \nabla u_C(x) = \frac{\nabla \psi(x)}{2\psi(x)} = -\nu(x) \quad (72) $$

that is the outward normal. This follows from the existence of a "weak" tangent space of $\partial^* F_t$ near $x$; see Theorem 3.8 in [8].

(5) Let $\Phi_h : \mathbb{R}^N \to [0, \infty)$ a family of $C^\infty$ mollifiers, with support in $D_{1/h}$ and $\int_{\mathbb{R}^N} \Phi_h(x) \, dx = 1$. Let $\psi_h \overset{\text{def}}{=} \psi * \Phi_h$; then (possibly passing to a subsequence) for almost all $t > 0$ we have

$$ \lim_{h \to \infty} \int_{\partial^* F_t} |\partial \psi_h \cdot \nu| \, d\mathcal{H}^{N-1}(x) = 0. \quad (73) $$

Indeed we know that $\lim_{h \to \infty} \nabla \psi_h = \nabla \psi$ in $L^1(F_t)$ (for any $R$ large), and we use the coarea formula

$$ \lim_{h \to \infty} \int_0^R g_h(t) \, dt = 0, \quad (74) $$

and then (up to a subsequence) $g_h(t) \to 0$ for almost all $t > 0$.

Let $r > 0$ be such that all above properties are satisfied. Let $K = \{ 0 < u_C \leq r \} = F_t \setminus C$. Note that $P(K) < \infty$ (since the boundaries of $C$ and $F_t$ are separated); and the essential boundary of $K$ is the union of the essential boundaries of $F_t$.\(\square\)
and of $C$. For any point $y$ in the topological boundary of $C$, \( \lim_{x \to y} \nabla \psi(x) = 0 \), and then
\[
\int_{\partial^* C} |\nabla \psi_h(x)| \, d\mathcal{H}^{n-1}(x) \to_h 0 . \tag{75}
\]

By Proposition 18
\[
\int_K \Delta \psi_h(x) \, dx = \sum_{i=1}^n \int_{\partial^* F_i} \nabla_x \psi_h(x) \cdot \nu_i(x) \, dx \tag{76}
\]
the right hand side passes to the limit (by (75) and by (73))
\[
\lim_{h \to \infty} \int_{\partial^* K} \sum_{i=1}^n \nabla_x \psi_h(x) \cdot \nu_i(x) \, d\mathcal{H}^{n-1}(x) = - \int_{\partial^* K} \sum_{i=1}^n \nabla_x \psi_h(x) \cdot \nu_i(x) \, d\mathcal{H}^{n-1}(x) ;
\]

5.4. Almgren Taylor Wang. It is worth mentioning that the above proof follows closely part of the proof by Ambrosio (lectures notes Minimizing Movements held in Padova, June 20-24, 1994) of this result by Almgren, Taylor, and Wang [14].

**Theorem 23.** Let $C \subset \mathbb{R}^N$ be a compact set; suppose that there are $\theta > 0$, $\tau > 0$ such that
\[
\mathcal{H}^{n-1}(C \cap B_r(x)) \geq \theta r^{N-1} \quad \forall x \in C, \, \rho \in (0, \tau). \tag{83}
\]
Then for all $R > \tau$
\[
\text{ess-} \sup_{0 < r < R} \mathcal{H}^{n-1}(\{x \in C \in \langle y \rangle \}) \leq \alpha R^{N-1} \mathcal{H}^{n-1}(C) \tag{84}
\]
where $\alpha = \gamma (1 - N)$ and $\gamma > 0$ is an appropriate positive constant dependent only on the dimension $N$.

6. Banach-Like Metric Spaces of Compact Sets

Let $\mathcal{M}$ be the family of all nonempty compact subsets of $\mathbb{R}^N$. In 2015 Duci and Mennucci [1] studied a family of distances $d_{p,q}$ on $\mathcal{M}$, where $p \in [1, \infty]$ (11) also includes the case $p = \infty$, which will not be considered here) and $\varphi : [0, \infty) \to (0, \infty)$ is a fixed given function. The distance was then defined, for $A, B \in \mathcal{M}$, as
\[
d_{p,q}(A, B) \overset{\text{def}}{=} \left\| \varphi \circ u_A - \varphi \circ u_B \right\|_{L^p(\mathbb{R}^N)} . \tag{85}
\]

**Lemma 24** (Lemma 6.2 in [1]). Suppose that $\varphi$ is monotonically strictly decreasing and
\[
\varphi(|x|) \in L^p\left(\mathbb{R}^N\right) . \tag{86}
\]

Let $C \subset \mathbb{R}^N$ be closed and nonempty; then the following are equivalent:

(a) $\varphi \circ u_C \in L^p(\mathbb{R}^N)$.

(b) $C$ is bounded (and then $C$ is compact).

So, for any $\varphi$ satisfying the above hypotheses, the formula in (85) would properly define a distance on $\mathcal{M}$.

6.1. Further Results on $(\mathcal{M}, d_{p,q})$. Assuming appropriate hypotheses on $\varphi$, [1] proved many interesting properties of the metric space $(\mathcal{M}, d_{p,q})$.

**Hypotheses 25** (Hypotheses 6.1 in [1]). Suppose that $\varphi$ is monotonically strictly decreasing and of class $C^1$, such that $\varphi(|x|) \in L^p\left(\mathbb{R}^N\right)$.

The resulting metric spaces enjoy many interesting properties.

(i) $d_{p,q}$ is Euclidean invariant.

(ii) The topology induced by $d_{p,q}$ over the space $\mathcal{M}$ coincides with the topology induced by the Hausdorff distance (Theorem 6.11 in [1]).

(iii) The metric space $(\mathcal{M}, d_{p,q})$ is complete (Theorem 6.12 in [1]).
6.2. Induced Distance and Geodesics. Given a metric space \((M,d)\), we define the length \(\text{Len}^d \gamma\) of a continuous path \(\gamma : [\alpha,\beta] \to M\) by using the total variation

\[
\text{Len}^d \gamma \overset{\text{def}}{=} \sup_T \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)),
\]

where the supremum is computed over all finite subsets \(T = \{t_0, \ldots, t_n\}\) of \([\alpha,\beta]\) and \(t_0 \leq \cdots \leq t_n\). We then define the induced distance \(d^\rho\) by

\[
d^\rho(x,y) \overset{\text{def}}{=} \inf \text{Len}^\gamma,
\]

where the infimum is taken in the class of all continuous paths \(\gamma\) connecting \(x\) to \(y\). If the infimum is a minimum, the path providing the minimum is called a geodesic.

With further hypotheses on \(\varphi(t)\), more can be said on the metric space \((\mathcal{M}, d_{p,q})\).

Hypotheses 26 (Hypotheses 6.17 in [1]). Let \(\varphi\) be as defined in Hypotheses 25. We moreover suppose that there is a constant \(T > 0\) such that \(\varphi(t)\) is convex for \(t \geq T\), and

\[
\varphi'((x)|_{\mathcal{L}}) \in L^p(\mathbb{R}^N).
\]

Then these results follow.

(i) The space \((\mathcal{M}, d_{p,q})\) is Lipschitz–arc connected (Prop. 6.18 in [1]).

(ii) For any \(\rho > 0\), \(A \in \mathcal{M}\),

\[
\mathcal{D}^\rho(A, \rho) \overset{\text{def}}{=} \{B \in \mathcal{M} | d_{p,q}(A,B) \leq \rho\}
\]

is compact in the \((\mathcal{M}, d)\) topology (where \(d_{p,q}\) is the distance induced by \(d_{p,q}\) (Theorem 6.20 in [1]).

(iii) Hence any two compact sets are connected by a geodesic in \((\mathcal{M}, d_{p,q})\).

(iv) When \(p \in (0, \infty]\) there is a variational description of the geodesics (Sec. 6.4 in [1]), a weak formulation of a “tangent bundle” (Sec. 6.5 in [1]).

(v) For \(p = 2\) there is an interpretation of the metric \(d_{p,q}\) as a “Riemannian metric” when smoothly deforming smooth boundaries of compact sets (Sec. 6.6 in [1]).

6.3. Open Problem. All the above properties are quite interesting for their applications in Shape Analysis and Optimization. In this context, the set of Hypotheses 25 and Hypotheses 26 can be satisfied by simple functions such as \(\varphi(t) = \exp(-t)\) or \(\varphi(t) = (1 + t)^{-a}\) for \(a > N/p\).

From a theoretical point of view, it may be interesting instead to understand more in detail which hypotheses are strictly needed to prove each of the above statements. This requires a deeper understanding of the properties of the distance function and the fattened sets. The theorems in this paper are a step in this direction.

In proving Lemma 24 it was assumed that \(\varphi\) be monotonically strictly decreasing. This was a useful ingredient in writing a simple direct proof of the implication \(b \Rightarrow a\). Some geometrical and analytical considerations though suggested that this hypothesis was not strictly needed.

Indeed as a consequence of Theorem 1 we can now prove Theorem 4.

**Proof.** By Lemma 19 and Theorem 1 for almost any \(r > 0\),

\[
r\mathcal{H}^{N-1}(\{u_C = r\}) \leq \mathcal{N}2^{N-1} \{u_C \leq r\}.
\]

Let then \(\tau > 0\) be such that \(C \subseteq D\varphi(0)\), and then \(\{u_C = r\} \subseteq D_{(r, \tau)}\); so for almost all \(r > 0\)

\[
r\mathcal{H}^{N-1}(\{u_C = r\}) \leq \frac{N}{r}(r + \tau)^N.
\]

Let \(\varphi : [0, \infty) \to [0, \infty)\) be as in hypotheses; we know (Lemma 22) that \(\mathcal{V}u_C(x)\) is 1 for almost all \(x \in \mathbb{R}^N \setminus C\), and then by the Federer coarea formula Proposition 15

\[
\int_{\{u_C(x) > \epsilon\}} \varphi(u_C(x))^p \, dx
\]

\[
= \int_{\{u_C(x) > \epsilon\}} \varphi(u_C(x))^p |\nabla u_C(x)| \, dx
\]

\[
= \int_0^\infty \left( \int_{\{u_C = t\}} \varphi(t)^p \, d\mathcal{H}^{N-1}(y) \right) \, dt
\]

\[
= \int_0^\infty \varphi(t)^p \mathcal{H}^{N-1}(\{u_C = t\}) \, dt
\]

\[
\leq \int_0^\infty \varphi(t)^p \frac{N}{t} (t + \tau)^N \, dt \leq \infty
\]

by relation (6), whereas

\[
\int_{\{u_C(x) \leq \epsilon\}} \varphi(u_C(x))^p \, dx < \infty
\]

by (7); so \(\varphi * u_C \in L^p(\mathbb{R}^N)\).

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares no conflicts of interest.

**Acknowledgments**

The author thanks Prof. Luigi Ambrosio (Scuola Normale Superiore) for the useful discussions. This work was supported by the PRIN 2015PA5MP7 (Calcolo delle Variazioni).

**References**

[1] A.C. Mennucci and A. Duci, “Banach-like distances and metric spaces of compact sets,” SIAM Journal on Imaging Sciences, vol. 8, no. 1, pp. 19–66, 2015.
[2] D. Hug, G. Last, and W. Weil, "A local steiner-type formula for general closed sets and applications," *Mathematische Zeitschrift*, vol. 246, no. 1, pp. 237–272, 2004.

[3] H. Federer, "Curvature measures," *Transactions of the American Mathematical Society*, vol. 93, pp. 418–491, 1959.

[4] J. How and G. Fu, "Tubular neighborhoods in euclidean spaces," *Duke Mathematical Journal*, vol. 52, no. 4, pp. 1025–1046, 1985.

[5] L. Ambrosio, A. Colesanti, and E. Villa, "Outer Minkowski content for some classes of closed sets," *Mathematische Annalen*, vol. 342, no. 4, pp. 727–748, 2008.

[6] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, vol. 317 of *A Series of Comprehensive Studies in Mathematics*, Springer–Verlag, 1998.

[7] M. C. Delfour and J.-P. Zolésio, "Shape and geometries," *Advances in Design and Control*. SIAM, 2001.

[8] E. Giusti, *Minimal Surfaces and Functions of Bounded Variation*, vol. 80 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1984.

[9] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill Book Co, New York, NY, USA, 1964.

[10] W. H. Fleming and R. Rishel, "An integral formula for total gradient variation," *Archiv der Mathematik*, vol. 11, no. 1, pp. 218–222, 1960.

[11] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, NY, USA, 2000.

[12] E. De Giorgi, "Nuovi teoremi relativi alle misure (r – 1)-dimensionali in uno spazio ad r dimensioni," *Ricerche di Matematica*, vol. 4, pp. 95–113, 1955.

[13] C. Mantegazza and A. C. Mennucci, "Hamilton-Jacobi equations and distance functions on Riemannian manifolds," *Applied Mathematics & Optimization*, vol. 47, no. 1, pp. 1–25, 2003.

[14] F. Almgren, J. E. Taylor, and L. Wang, "Curvature-driven flows: a variational approach," *SIAM Journal on Control and Optimization*, vol. 31, no. 2, pp. 387–438, 1993.
