Few-body Physics in a Many-body World

Received: date / Accepted: date

Abstract The study of quantum mechanical few-body systems is a century old pursuit relevant to countless subfields of physics. While the two-body problem is generally considered to be well-understood theoretically and numerically, venturing to three or more bodies brings about complications but also a host of interesting phenomena. In recent years, the cooling and trapping of atoms and molecules has shown great promise to provide a highly controllable environment to study few-body physics. However, as is true for many systems where few-body effects play an important role the few-body states are not isolated from their many-body environment. An interesting question then becomes if or (more precisely) when we should consider few-body states as effectively isolated and when we have to take the coupling to the environment into account. Using some simple, yet non-trivial, examples I will try to suggest possible approaches to this line of research.

Keywords Few-body bound states · Many-body physics · Cold atomic gases · Strong interactions

1 Introduction

Bound states of atomic particles are synonymous with the birth of modern quantum mechanics through the pursuit of a solution to the Hydrogen atom and later more complicated atoms and molecules. From a mathematical point of view, bound states are studied by establishing criteria a given potential must satisfy in order for the system to allow a bound state. Here we refer to a bound state in the usual sense, i.e. a state of negative energy with a normalizable wave function. Already in introductory quantum mechanics classes we learn that the answer to this question depends very strongly on dimensionality. In one dimension (1D), any amount of attraction will bind a particle, or more precisely, if the integral of the potential in all space (a line in this case) is non-positive. In two dimensions (2D) the situation is similar for the case of weak potentials since a bound state occurs when the integral of the potential in the plane is non-positive [1]. However, the formula for the binding energy is very different from the case of 1D [2]. In 3D this is no longer true. The simple example of an attractive square well potential demonstrates this fact as a finite depth is required to obtain a bound state. Intuitively this suggests that binding of two particles is harder to achieve in 3D. With this in mind, it must have been a bit of shock when Vitaly Efimov in 1970 announced that he had found an infinite number of three-body bound states in a 3D system with three bosonic particles in the limit where each two-body subsystem
Fig. 1 (Left panel) An Efimov three-body state (three (red) balls surrounded by solid (black) circle) in an environment where the density is low ($L > R$) so that all other particles ((blue) balls outside the circle) are far away on average. (Right panel) At higher densities ($L \approx R$), the interparticle distance in the total system becomes comparable to the interparticle in the three-body bound state and we expect a modification of its binding energy and/or character.

has a bound state with zero binding energy [3]. A flurry of theoretical interest followed and it was shown that it happens only in 3D and not in pure 1D or 2D [4; 5; 6] (mixed dimensional examples have been found, see [7]). Efimov's finding is a key insight into the field of few-body physics as it demonstrates the intricacies beyond two-particle systems.

Alas, in spite of many efforts to verify the predictions of Efimov in nuclear physics experiments no unambiguous signals have been found [8]. The observation was instead made in cold alkali atomic gases [9] about a decade after the production of the first atomic condensates [10; 11; 12]. It was made possible by the advanced controllability of interactions that these systems provide through the use of Feshbach resonances [13; 14]. These resonances can be used to give the system an effective short-range interaction with a tunable strength, or more precisely a tunable scattering length, $a$. In this notation, the Efimov effect occurs in the limit where $|a| \to \infty$, i.e. exactly on the Feshbach resonance. Since no quantity can depend on $a$ when it diverges, this is called the universal regime. An illustrative example of what this means can be seen in a (homogeneous) Fermi gas; when $|a| \to \infty$ the only scale left is the density (or Fermi wave vector), ergo the total energy of the system must be proportional to some constant times the Fermi energy (see [15] for a review of the Fermi gas in the strongly interaction regime).

A question now arises at the borderline of few- and many-body physics. The Efimov effect is observed in many cold atomic gas experiments as a resonant peak in loss rates. This comes about since the presence of bound states modifies the recombination rates as function of temperature and for given value of the probability to populate tightly-bound molecular two-body bound states which are abundantly present in alkali systems and cause losses from the trap [16; 17]. But what about the density of particles? Naturally, the loss rate will scale with the density since shorter interparticle distances increase the probability for reactions. The density could, however, also have an effect of the position of the resonance peaks since the Efimov states may feel the presence of other particles in their surroundings and endure a shift in binding energy. At low density we do not expect this to happen (left panel in Fig. 1), but as the density increases there should be a point at which the interparticle distance in the many-body system becomes comparable to the interparticle distance in the few-body state (right panel in Fig. 1) and the characteristics of the few-body state should change; it is no longer an isolated state. Below we discuss examples of how to approach this issue when the background is a degenerate Fermi or Bose gas.
Fig. 2 A system with three distinct particles, $|1\rangle$, $|2\rangle$, and $|3\rangle$, with identical two-body interacting potentials in all subsystems (illustrated by the dashed lines). Particle $|3\rangle$ has a Fermi sea (large (blue) filled sphere) that causes Pauli blocking of states in momentum space.

2 Efimov states in a Fermi gas

The first example we consider is the presence of a degenerate Fermi sea as a background. To simplify the problem we assume that we have three distinct particle and where one of the three feel the presence of a background of other particles of the same species. This implies that there will be Pauli blocking of the momentum states of this particle. This is illustrated in Fig. 2. In the Born-Oppenheimer limit, where the mass of the particle with the Fermi sea is much smaller than the masses of the two other particles, this problem was studied by Nishida [18], by MacNeill and Zhou [19], and more recently for several heavy particles by Endo and Ueda [20]. Here we will consider the case of equal masses, although we note that the formalism discussed here can be applied for any choice of masses and agrees with the Born-Oppenheimer result in the appropriate limit.

Our approach will be top-down, i.e. we consider a formalism that can describe the Efimov states accurately and then introduce the Fermi sea background [21]. Given that a Fermi sea is most easily described in momentum-space, we will work with the momentum-space formalism using the so-called Skornyakov-Ter-Martirosian equations [22], suitably modified to avoid the Thomas collapse that is otherwise present in the limit where the two-body interaction has zero range [23]. Notice that we will consider the Fermi sea inert, i.e. it will not contain any fluctuations in the form of particle-hole pairs. In the Born-Oppenheimer limit, it was argued [19] that fluctuations should not change the qualitative effect discussed here. Similar arguments can be made for general masses (this will be presented elsewhere). The key issue is to introduce the proper Pauli blocking in the two-body propagator and then implement this in the three-body equations which are subsequently solved for negative (bound) energy states using a pole and residue expansion [16].

In Fig. 3 we show the results of a three-body calculation with a finite Fermi sea background in one of the components. The spectrum demonstrates the general modification of the binding energies in the presence of a background. For instance, the appearance points (filled (red) circles along the A+A+A continuum line) for Efimov three-body states are moved as the Fermi sea increases and eventually they are eaten by the atom-dimer (A+D) continuum. This can be understood intuitively by comparing the binding energy on resonance ($|a| \to \infty$), $E_B$, with the Fermi energy, $E_F$. When $E_F << E_B$ no spectral flow is seen, but once $E_F \sim E_B$ the states start moving. In fact, we have found that this happens in a manner that also displays Efimov scaling, but now in the Fermi energy or Fermi wave vector [21]; another state gets swallowed by the A+D continuum every time the Fermi wave vector scales by the Efimov factor, $e^{\gamma}/s_0 \sim 22.7$. This scaling factor can be changed by taking different masses.

A prominent experimental feature of the Efimov effect is the loss rate as the three-body state merges with the three-atom continuum. As seen in Fig. 3 our theory predicts that these points are moving as a function of $E_F$. As discussed in Ref. [21], this implies that the loss peak could be displaced due to the presence of the Fermi sea. An ideal system to study this effect is the three-component $^6$Li gas which has produced Efimov features in several experiments around the world [24, 25, 26]. Our calculations indicate that the peak should be affected at densities of about $10^{12}$ cm$^{-3}$ and above. This is about one order of magnitude larger than the reported experiments and could be within reach for
Increasing $k^*$ is a parameter that describes the short-range details of the interparticle interactions that are not captured by a zero-range approximation. The three atom continuum (A+A+A) is at the top, the atom-dimer (A+D) continuum on the right side, while the trimer states (T) live below both continua. As indicated, the presence of a finite Fermi sea induces a spectral change or flow toward the atom-dimer continuum with increasing Fermi wave vector $k_F$.

3 Efimov states in a Bose gas background

Consider now the case where one of the particles in a three-body Efimov state has a Bose background. In the limit where $|a| \to \infty$, we again need to have some scale to compare to the three-body energy in order to estimate whether we can expect an effect of the background. Some recent discussions on the few- and many-body effects of the resonant Bose gas for equal mass bosons can be found in Refs. [27, 28, 29]. In an ideal Bose gas we could construct an energy scale from the density, $n$, and the mass, $m$. A short moment of contemplation reveals that this scale is really $E_F$, and thus our estimates above would apply. However, experimental alkali Bose gases typically have a short-range interaction between the identical bosons given by some scattering length, $a_B$. This implies that they will also have a superfluid length or coherence length, $\xi = 1/\sqrt{8\pi na_B}$. This implies that for excitation with wavelengths longer than $\xi$ the collective properties of the Bose condensate are dominant, or in terms of energy for modes with energies lower than about $\hbar^2/m\xi^2$. Since Efimov three-body states are low-energy states one may expect that a modification of the dispersion relation due to a Bose condensate background would have an effect.

In order to study this issue quantitatively, we have employed the Born-Oppenheimer approximation with a light particle Bose condensate containing two heavy impurity atoms that interact with all the light bosons through a short-range interaction [30]. This is similar to the Fermi gas study in Ref. [19], but in the case of a Bose condensate background instead of Pauli blocking we get a modified dispersion at low energy that has to be taken into account. Again this is a top-down approach where we use a formalism that reproduces the Efimov states when the condensate density goes to zero. In the Born-Oppenheimer limit, the effective potential for $|a| \to \infty$ (with $a$ the heavy-light scattering length) between the two heavy particles in 3D is

$$V(R) = -\frac{\hbar^2 \Omega^2}{mR^2}. \quad (1)$$

This system. Of course, finite temperature effects or the unitarity bound (close to the resonance peaks are washed out) must be considered and further analysis of these effects is necessary.
where $R$ is the distance between the two impurities, and $\Omega$ solves the equation $e^{-\Omega} = \Omega$ so that $\Omega \sim 0.567$. This potential produces the Efimov effect with the number of three-body states given by

$$N_T \approx \frac{1}{\pi} \log \left[ \frac{a}{R_0} \right].$$

Here $R_0$ is a short-range cut-off which in the Born-Oppenheimer limit is obtained from the short-range properties of the interaction of the two heavy particles (which is most likely given by the van der Waals length $31, 32, 33, 34, 35$). The presence of a Bose condensate modifies this formula by a replacement of $a$ by $\xi$, i.e.

$$N_T \approx \frac{1}{\pi} \log \left[ \frac{\xi}{R_0} \right].$$

This implies that the Efimov states are modified at long distances (or low energy) by the deformation of the dispersion controlled by $\xi$. As discussed in Ref. [30], mixtures of metastable $^4\text{He}^*\text{ and } ^{87}\text{Rb}$ [36, 37] could be a candidate system for observing this effect by a modification of the number of loss peaks as the condensate density is varied.

### 4 Outlook

The examples discussed above represent a first step in the exploration of the fate of Efimov states in a many-body environment. We assumed inert backgrounds with no particle-hole pairs in the Fermi sea and no excitations out of the condensate in the Bose condensate background case. Further analysis has to be done to investigate the qualitative and quantitative influence of fluctuations. This is similar in spirit to the Fermi polaron where a single spin down interacts with a Fermi sea of many spin up particles. There only single particle-hole pairs are important due to higher order cancellations [38].

Scaling analysis of the Skornyakov-Ter-Martirosian equations with a Fermi sea suggest that at least the Efimov scaling with the Fermi energy $E_F$ should be robust when including fluctuations but further studies on the quantitative effects are needed.

The examples presented here are for 3D, and we would expect changes as we move to 2D setups. Here the Efimov effect does not occur in a strict sense, but there are universal bound states in the zero-range interaction limit and they will be modified by background effects. Another interesting direction would be Efimov three-body physics in a superfluid Fermi gas where the spectrum is now gapped due to pairing. Finally, one may also ask what happens in the case of interactions with longer ranges. For instance, for cold neutral heteronuclear molecules with dipole-dipole interactions, few-body bound states are quite prolific and can have strong influence on the many-body ground state of the system [39, 40, 41, 42]. In 2D the two-body bound state is always present for purely attractive potentials or dipole-dipole interactions in layered geometries [43]. A goal would be to engineer a system with no bound two-body but bound three-body systems, a quantum trimer system.

### Acknowledgements

I would like to thank Nicolai Gayle Nygaard for his pioneering work on the Efimov effect in the presence of a Fermi sea upon which much of this discussion is based. I would also like to thank Aksel S. Jensen, Dmitri V. Fedorov, Tobias Frederico, and Marcelo Yamashita for discussions and for teaching me most of what I know about the Efimov effect. I also thank our students Peder K. Sørensen, Artem G. Volosniev, and Filipe F. Bellotti for their tireless work on various aspects of few-body physics and the Efimov effect.

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