A PROOF OF THE CONVERGENCE OF THE HEGSELMANN-KRAUSE
DYNAMICS ON THE CIRCLE

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1. Introduction

In this note, we give a complete proof that Hegselmann-Krause systems converge on the circle following the proof strategy developed by Hegarty, Martinsson, and Wedin in [4, 5]. By letting

$$K_s = \sum_{t \in \mathbb{N}} \sum_{i \in [n]} \delta(\tau_i(t), \tau_i(t+1))$$

the s-kinetic energy of the system, their proof strategy consists in showing that (1) the quadratic kinetic energy $K_2$ is finite, (2) the influence graph is eventually constant, and (3) the 1-kinetic energy $K_1$ is finite, which immediately implies the convergence of the sequence of position vectors $(\tau(t))_{t \in \mathbb{N}}$.

To show the finiteness of $K_2$, we present a simple proof in Section 2 which is based on a reduction of the HK dynamics on the circle to the HK dynamics on the line.

Concerning the eventual stability of influence graphs, we are not able to understand the proof outlined in [4], and we give our proof of this key point in Section 3.

For the third point, namely the finiteness of $K_1$, Hegarty, Martinsson, and Wedin introduce the vector of position differences $x^*(t)$, and show that the sequence $(x^*(t))_{t \in \mathbb{N}}$ converges to some limit $x^*_\infty$ such that $\|x^*(t) - x^*_\infty\| = O(e^{-ct})$. The argument given in the first version [4] for the latter point is erroneous, and we fix the argument in Section 4. An alternative argument, based on the finiteness of the quadratic kinetic energy, is provided in [5] which still appeals to the vector of differences $x^*(t)$. In fact, this argument can be directly applied to the position vector as we show in Section 6, which allows to circumvent the arguments of Sections 4 and 5 and prove convergence directly.

Notation.

The circle. Let $p$ be a positive real number and let $\mathbb{T} = \mathbb{R}/p\mathbb{Z}$. We define $\delta$ on $\mathbb{T}$ by:

$$\delta(\tau, \eta) = \min_{x \in \tau, y \in \eta} |x - y|,$$

and we easily check that $\delta$ is a distance on $\mathbb{T}$ such that

$$0 \leq \delta(\tau, \eta) \leq p/2.$$

Let vect$(\tau, \eta)$ be the unique element in $\eta - \tau$ which lies in $]-p/2, p/2]$. Clearly

$$|\text{vect}(\tau, \eta)| = \delta(\tau, \eta).$$

When $\eta, \tau$ are close enough, we have

$$\text{vect}(\tau, \eta) = \text{vect}(\tau, \eta) + \text{vect}(\eta, \tau),$$

but the equality does not hold in general.

This induces a bijection $\phi : \mathbb{T} \to ]-p/2, p/2[$.

The HK dynamics on the circle. The Hegselmann-Krause dynamics on the circle with an influence radius $r \in [0, p/2]$ is defined by:

$$\text{vect}(\tau_i(t), \tau_i(t+1)) = \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} \text{vect}(\tau_i(t), \tau_j(t))$$

where

$$N_i(t) = \{ j \in [n] \mid \text{vect}(\tau_i(t), \tau_j(t)) \in [-r, r] \}.$$

Indeed we have

$$-p/2 < \frac{1}{|N_i(t)|} \sum_{j \in N_i(t)} \text{vect}(\tau_i(t), \tau_j(t)) \leq p/2,$$
and \( \mathbf{1} \) defines a unique element \( \mathbf{\tau}_i(t + 1) \) in \( \mathbf{T} \). The sets \( N_i(t) \) define a directed graph, called the influence graph at time \( t \) and denoted by \( G_t \).

At each time \( t \) and for each agent \( i \), we define the set of left neighbors \( L_i(t) \) and the set of right neighbors \( R_i(t) \):

\[
L_i(t) = \{ j \in [n] : \text{vect}(\mathbf{\tau}_j(t), \mathbf{\tau}_i(t)) \in [0, r] \} \quad \text{and} \quad R_i(t) = \{ j \in [n] : \text{vect}(\mathbf{\tau}_i(t), \mathbf{\tau}_j(t)) \in [0, r] \}.
\]

For each agent \( i \), let \( r_i(t) \) be the (unique) integer at most equal to \( i \) and such that

\[
|R_i(t)| \equiv r_i(t) - i + 1 \mod n.
\]

Similarly we define \( L_i(t) \) for the set \( L_i(t) \).

The mapping \( \phi \) naturally induces an ordering on the initial positions, and we assume that

\[
\phi(\mathbf{\tau}_1(0)) \leq \ldots \leq \phi(\mathbf{\tau}_n(0)).
\]

Like for the line, the HK dynamics on the circle cannot cause agents to cross (even if there may be some cyclic shifts in the ordering with respect to \( \phi \)). We say that there is a cut at time \( t \) if there are two consecutive agents \( i \) and \( i + 1 \) such that \( i + 1 \notin R_i(t) \). We easily observe that if there is a cut at time \( t \), the system remains cut forever and the dynamics is similar to the HK dynamics on the line.

In the rest of this note, we fix the influence radius \( r = 1 \).

### 2. Lyapunov Function

We consider an HK system on the circle with \( n \) agents and an \( n \)-vector of initial positions. From this system, we construct an HK system on the line with \( nN \) agents by “unrolling” \( N \) samples of the HK system on the circle, and we denote the position of agent \( i \) at time \( t \) for this system by \( y_i(t) \). Then we introduce

\[
V_N(t) = \sum_{(i,j) \in [nN]^2} \min \{1, |y_i(t) - y_j(t)|^2 \}
\]

and

\[
W(t) = \sum_{(i,j) \in [n]^2} \min \{1, \delta(\mathbf{\tau}_i(t), \mathbf{\tau}_j(t))^2 \}
\]

for the HK systems on the line and the circle, respectively.

If \( N \geq 4 \), then we easily check that

\[
V_N(0) = (N - 2)W(0) + R_0 \quad \text{with} \quad 0 \leq R_0 \leq 2n^2 \\
V_N(1) = (N - 4)W(1) + R_1 \quad \text{with} \quad 0 \leq R_1 \leq 4n^2.
\]

Moreover by definition of \( W \), we have

\[
0 \leq W(0) \leq n^2.
\]

From \( [2] \) we know that

\[
V_N(0) - V_N(1) \geq 4 \sum_{i \in [nN]} |y_i(1) - y_i(0)|^2 \geq 4 \sum_{i=n+1}^{(N-1)n} |y_i(1) - y_i(0)|^2.
\]

We let

\[
S = \sum_{i \in [n]} \delta(\mathbf{\tau}_i(0), \mathbf{\tau}_i(1))^2.
\]

Since \( R_0 \leq 2n^2 \), \( 0 \leq R_1 \), and \( W(0) \leq n^2 \), for every integer \( N \geq 4 \) we have

\[
W(0) - W(1) \geq 4 \left( \frac{N - 2}{N - 4} S - \frac{n^2}{N - 4} \right).
\]

When \( N \) tends to \(+\infty\), that gives

\[
W(0) - W(1) \geq 4S.
\]

Because the above inequality holds whatever the initial positions of the agents on the circle, we derive the following proposition.

**Proposition 1.** At each time \( t \in \mathbb{N} \),

\[
W(t) - W(t + 1) \geq 4 \sum_{i \in [n]} \delta(\mathbf{\tau}_i(t), \mathbf{\tau}_i(t + 1))^2.
\]
Therefore the sequence \((W(t))_{t \in \mathbb{N}}\) is decreasing and nonnegative, and so converges to some \(W(\infty)\). Following [3], we define the *s-kinetic energy* of the HK system on the circle by
\[
K_s = \sum_{t \in \mathbb{N}} \sum_{i \in [n]} \delta(\mathbf{s}_i(t), \mathbf{s}_i(t+1))^s
\]
where \(s\) is a real number. Obviously the finiteness of \(K_1\) enforces the convergence of the sequence \((\mathbf{s}(t))_{t \in \mathbb{N}}\).

Since at each time \(t\), we have \(0 \leq W(t) \leq n^2\), we derive the following theorem from Proposition [1].

**Theorem 2.** The 2-kinetic energy of an HK system on the circle with \(n\) agents satisfies
\[
K_2 \leq n^2/4 .
\]

Unfortunately the finiteness of \(K_2\) is not sufficient to enforce the convergence of the sequence \((\mathbf{s}(t))_{t \in \mathbb{N}}\). The proof below consists in showing that for the HK dynamics on the circle, the 1-kinetic energy \(K_1\) is also finite, which does imply the convergence of \((\mathbf{s}(t))_{t \in \mathbb{N}}\).

### 3. Topological Changes

In this section, we study the impact on the kinetic energy of changes in the influence graph.

Suppose that the influence graph \(G_{t+1} = \mathcal{T}\) at time \(t+1\) contains a link that is not a link at time \(t\), in \(\mathcal{T}\).

In the case of HK on the line, the agent with a new left neighbor that has the greatest identity has no new right neighbor since influence graphs are bidirectional. Actually we show that this key point for the study of the HK dynamics on the line also holds on the circle.

**Proposition 3.** If \(G_{t+1}\) contains a link that is not a link in \(\mathcal{T}\), then there is an agent that has a new left neighbor but no new right neighbor at time \(t+1\).

**Proof.** We proceed by contradiction: suppose that such an agent does not exist. Because of the above remark, the system is not cut at time \(t+1\). Let \(i_1\) be the agent with the smallest identity and a new left neighbor; let \(i_2\) be one of the new right neighbor of \(i_1\). We repeat the construction and obtain an infinite chain of agents \(i_1, i_2, \ldots\).

By the pigeonhole principle, this chain is closed, i.e., there exist two indices \(k\) and \(\ell\) such that \(i_k = i_\ell\). That gives two closed chains of elements in \(\mathbb{T}\), namely \(x_{i_k}(t), \ldots, x_{i_k}(t)\) and \(x_{i_\ell}(t+1), \ldots, x_{i_\ell}(t+1)\), with the same length which is a multiple of \(p\), say \(\nu p\) since the system is not cut. By construction, we have:
\[
(k - \ell)r < \nu p \leq (k - \ell)r ,
\]
a contradiction. \(\square\)

**Proposition 4.** If agent \(i\) has a new left neighbor but no new right neighbor at time \(t+1\) in the influence graph, i.e., \(L_i(t) \subseteq L_i(t+1)\) and \(R_i(t+1) \subseteq R_i(t)\), then there is at least one agent \(j\) which moves by more than \(1/6n\) at time \(t+1\) or \(t+2\)
\[
\delta(\mathbf{s}_i(t), \mathbf{s}_i(t+1)) > \frac{1}{6n} \quad \text{or} \quad \delta(\mathbf{s}_i(t+1), \mathbf{s}_i(t+2)) > \frac{1}{6n} .
\]

**Proof.** We proceed by contradiction and assume that no agent moves from time \(t\) to time \(t+2\) by more than \(\mu = \frac{1}{3n}\), i.e., for each agent \(\ell \in [n],\)
\[
\delta(\mathbf{s}_i(t+1), \mathbf{s}_i(t+2)) + \delta(\mathbf{s}_\ell(t), \mathbf{s}_\ell(t+1)) < \mu .
\]

By hypothesis, we have
\[
L_i(t) \subseteq L_i(t+1) \quad \text{and} \quad R_i(t+1) \subseteq R_i(t) .
\]
Without loss of generality we assume that \(\mathbf{s}_i(t) = 0\), which allows us to identify \(\mathbf{s}_j(t)\) with \(\phi(\mathbf{s}_j(t))\) in the rest of this proof. It is,
\[
(2) \quad \mathbf{s}_i(t+2) = \left( \sum_{k \in L_i(t+1)} \mathbf{s}_k(t+1) \right) / \left( |N_i(t+1)| \right) .
\]

We will show that \(\mathbf{s}_i(t+2) < -\mu\), which contradicts the assumption that no agent moves by more than \(\mu\) from time \(t\) to time \(t+2\) since \(\mathbf{s}_i(t) = 0\).
For every agent \( j \in L_i(t + 1) \setminus L_i(t) \), it holds that \( \pi_j(t) < -1 \) and thus \( \pi_j(t + 1) < -1 + \mu \). Let \( m = |L_i(t + 1) \setminus L_i(t)| \). We have

\[
\pi_i(t + 2) \leq -\frac{m}{|N_i(t + 1)|} + \frac{\sum_{k \in L_i(t) \cup R_i(t + 1)} \pi_k(t)}{|N_i(t + 1)|}.
\]

Since \( m \geq 1 \) and \( |N_i(t + 1)| \leq n \),

\[
\pi_i(t + 2) \leq -\frac{1}{n} + \frac{\sum_{k \in L_i(t) \cup R_i(t + 1)} \pi_k(t)}{|N_i(t + 1)|}.
\]

Claim 5. Denote by \( \text{avg}(X) \) the (equal weight) average of a multiset \( X \) with elements in \( \mathbb{R} \). Let \( A \) and \( B \) be two multisets with elements in \( \mathbb{R} \). If \( A \subseteq B \) and all elements in multiset \( B \setminus A \) are greater or equal than \( \max(A) \), then \( \text{avg}(B) \geq \text{avg}(A) \).

We now apply the above claim with \( A \) defined by the elements \( \pi_k(t) \) with \( k \in L_i(t) \cup R_i(t + 1) \) and \( 0 \) repeated \( m = |L_i(t + 1) \setminus L_i(t)| \) times, and \( B \) defined with the elements in \( A \) and the \( \pi_k(t) \)'s with \( k \in R_i(t) \setminus R_i(t + 1) \). Thus we obtain

\[
\frac{\sum_{k \in L_i(t) \cup R_i(t + 1)} \pi_k(t)}{|N_i(t + 1)|} \leq \frac{|N_i(t)|}{m + |N_i(t)|} \pi_i(t + 1).
\]

Therefore

\[
\frac{\sum_{k \in L_i(t) \cup R_i(t + 1)} \pi_k(t)}{|N_i(t + 1)|} \leq \frac{|N_i(t)|}{m + |N_i(t)|} \pi_i(t + 1).
\]

It follows that

\[
\pi_i(t + 2) - \pi_i(t + 1) \leq -\frac{1}{n} + \frac{|N_i(t)|}{m + |N_i(t)|} \pi_i(t + 1) \leq -\frac{1}{n} + \frac{|N_i(t)|}{m + |N_i(t)|} \mu,
\]

and so

\[
\pi_i(t + 2) - \pi_i(t + 1) < 2\mu - \frac{1}{n}.
\]

For \( \mu = \frac{1}{m} \), we actually obtain \( \pi_i(t + 2) - \pi_i(t + 1) < -\mu \), a contradiction. \( \square \)

Combining Proposition 4 and Theorem 2, we derive that the number of times a link is added, and so the number of changes in the influence graph, is finite.

Theorem 6. In the HK dynamics on the circle, the influence graph is eventually constant.

That leads us to decompose the HK dynamics into two periods: the first one during which the influence graph changes and the second one with a fixed influence graph. Theorem 2 shows that the first period is finite while the second one may be infinite as exemplified in 4. Observe that if there is a cut, then the dynamics on the circle is the same as on the line, in which case the second phase is of length one and the system freezes just after the first period.

4. THE VECTOR OF POSITION DIFFERENCES

We define

\[
x^*_i(t) = \begin{cases} 
\text{vect}(\pi_i(t), \pi_{i+1}(t)) & \text{if } i \in [n-1] \\
\text{vect}(\pi_n(t), \pi_1(t)) & \text{if } i = n.
\end{cases}
\]

If there is no cut at time \( t \), then we have the fundamental identity

\[
x^*_i(t) + \cdots + x^*_n(t) = p.
\]

Proposition 7. At each time \( t \), there exists a non-negative matrix \( A_1 \) such that

\[
x^*(t + 1) = A_1 x^*(t).
\]

Proof. We fix time \( t \) and we often omit \( t \) in the notation as no confusion can arise. First we observe that under the condition \( r < p/6 \), we have

\[
x^*_i(t + 1) = x^*_i(t) + \frac{1}{|N_{i+1}(t)|} \sum_{k \in N_{i+1}(t)} \text{vect}(\pi_{i+1}(t), \pi_k(t)) - \frac{1}{|N_i(t)|} \sum_{k \in N_i(t)} \text{vect}(\pi_i(t), \pi_k(t)).
\]

From this expression of \( x^*_i(t + 1) \), we derive that \( x^*_i(t + 1) = \sum_{k \in [n]} A_{ik} x^*_k(t) \) with
arise. We consider two cases:

Claim 8. If \( \ell, \ell', r, r', j \) are five integers such that \( r' \geq r, \ell' \geq \ell \) and \( r \geq j \geq \ell - 1 \), then
\[
(r' - j)(r - \ell + 1) \geq (r - j)(r' - \ell' + 1).
\]

Claim 9. If \( \ell, \ell', r, r', j \) are five integers such that \( r' \geq r, \ell' \geq \ell \) and \( r' + 1 \geq j \geq \ell' \), then
\[
(j - \ell)(r' - \ell' + 1) \geq (j - \ell')(r - \ell + 1).
\]

Observe that some entries \( A_{ik} \) with \( \ell \geq k \geq r' - 1 \) may be null if \( \ell = \ell' \) or \( r = r' \). Moreover the \( i \)-th line of \( A \) is null if both \( \ell = \ell' \) and \( r = r' \), in which case agents \( i \) and \( i + 1 \) have merged by time \( t + 1 \) since they have the same neighbors at time \( t \).

Proposition 10. If there is no cut at time \( t \), then \( A_t \) is column-stochastic.

Proof. If there is no cut at time \( t + 1 \) and so at time \( t \), we use the identity (1) to obtain
\[
\sum_{k \in [n]} x_k^i(t) = \sum_{k \in [n]} S_k(t) x_k^i(t),
\]
where \( S_k(t) \) is the sum of \( A_t \)'s entries in the \( k \)-th column.

We now prove that for every \( k \in [n] \), \( S_k(t) = 1 \). Again we omit \( t \) in the notation as no confusion can arise. We consider two cases:

1. For each index \( k \mod n \), \( \delta(\overline{x}_{k-1}, \overline{x}_k) < 1 \) or \( \delta(\overline{x}_k, \overline{x}_{k+1}) < 1 \).

We now fix an index \( i \) and show that \( S_{i-1} = S_i \). For sufficiently small but non-null variations of \( i \)'s position on the circle, there is no change in the influence graph. Formally, in the case \( \delta(\overline{x}_{i-1}, \overline{x}_i) < 1 \), there exists \( \epsilon > 0 \) such that for the position vector \( \overline{y} \) whose each entry \( y_k \) is equal to \( \overline{x}_k \), except \( \overline{y}_i = \overline{x}_i - \epsilon \), the influence graph is the same as for \( \overline{x} \). Hence
\[
y^* = \begin{cases} 
x_k^i - \epsilon & \text{if } k = i - 1 \\
x_k^i + \epsilon & \text{if } k = i \\
x_k^i & \text{otherwise.}
\end{cases}
\]
From (1) and \( \epsilon > 0 \) it follows that \( S_{i-1} = S_i \).

Since \( p \) is positive, then we conclude that for every \( k \in [n] \), \( S_k = 1 \).

2. For some index \( k \mod n \), \( \delta(\overline{x}_{k-1}, \overline{x}_k) = \delta(\overline{x}_k, \overline{x}_{k+1}) = 1 \). Let us denote
\[
d = \min\{\delta(\overline{x}_i, \overline{x}_j) \mid i \in [n], j \notin N_i \}
\]
and let us consider the influence graph for the position vector \( \overline{x} \) and the influence radius \( \rho = \frac{d + 1}{2} \).

Since there is no cut, all agents have not merged and \( d > 1 \). Hence we have
\[
d > \rho > 1.
\]

Therefore
\[
\delta(\overline{x}_i, \overline{x}_j) \leq 1 \Rightarrow \delta(\overline{x}_i, \overline{x}_j) \leq \rho,
\]
and by definition of \( d \),
\[
\delta(\overline{x}_i, \overline{x}_j) > 1 \Rightarrow \delta(\overline{x}_i, \overline{x}_j) \geq d > \rho.
\]

In other words, the influence graph is the same with the influence radius 1 and \( \rho \) when the position vector is \( \overline{x} \). Since for each index \( k \), \( \delta(\overline{x}_{k-1}, \overline{x}_k) < \rho \) and \( \delta(\overline{x}_k, \overline{x}_{k+1}) < \rho \), we conclude as in case 1.

Importantly the above proposition does not hold anymore in the case of a cut in the circle.
Theorem 14. If there is a cut, the dynamics on the circle is the same as on the line, in which case the convergence is well-known.

Proposition 11. If there is no cut at time \( t \), then the directed graph associated to \( A_t \) is rooted.

Proof. Let \( H_t \) denote the directed graph associated to \( A_t \). As no confusion can arise, we omit \( t \) in the notation \( H_t, G_t, r_i(t), \ldots \).

From the expression (5) of \( A_t \)'s entries, we derive that there is a link \( (i, i + 1) \) in \( H \) if and only if \( r_{i+1} - 1 \geq i + 1 \). Because there is no cut, the latter inequality holds if and only if \( i \) and \( i + 1 \) have not merged by time \( t + 1 \). That proves the following lemma.

Lemma 12. There is a link \( (i, i + 1) \) in \( H \) if and only if \( i \) and \( i + 1 \) have not merged by time \( t + 1 \).

Since there is no cut, all the agents have not merged at time \( t + 1 \); let \( i_1 \) be the first \( i \in [n] \) such that agents \( i \) and \( i + 1 \) have not merged. Thus \((i_1, i_1 + 1)\) is an edge in \( H \).

Now we inductively construct a spanning tree rooted at \( i_1 \) contained in \( H \). For easier notation, let \( i_1 = 1 \).

1. At the first step, we have the subtree \( T_1 \) over the set of nodes \( \{1, 2\} \) and the link \((1, 2)\).
2. Suppose that at the end of the \((i - 1)\)-th step, the resulting directed graph \( T_{i-1} \) over the set of nodes \( \{1, \ldots, i\} \) is a tree rooted at 1 with the last level composed of the sole links \((i - k, i - k + 1), \ldots, (i - k, i)\) if \( k \) denotes the unique integer in \( \{1, \ldots, i - 1\} \) such that

\[
\varpi_{i-k+1}(t+1) = \cdots = \varpi_i(t+1) \quad \text{and} \quad \varpi_{i-k}(t+1) \neq \varpi_i(t+1)
\]

(cf. Figure 1).

(a) If \( \varpi_{i+1}(t+1) = \varpi_i(t+1) \), then \( \varpi_{i-k}(t+1) \neq \varpi_{i+1}(t+1) \) and \((i - k, i + 1)\) is a link in \( H \).

(b) Otherwise \( i \) and \( i + 1 \) have not merged by time \( t + 1 \), and \((i, i + 1)\) is a link in \( H \).

Then we extend the subtree \( T_{i-1} \) to the set of nodes \( \{1, \ldots, i + 1\} \) by adding the link \((i - k, i + 1)\) or \((i, i + 1)\), accordingly.

By construction, the resulting directed graph \( T_{i-1} \) is a tree, rooted at 1, and all its links are in \( H \). \( \square \)

As a consequence of the theorem on the backward product of line-stochastic matrices with oriented associated graphs (or equivalently, on the forward product of column-stochastic matrices with rooted associated graphs) proved by Cao, Morse, and Anderson [1], we derive the following result on the sequence \( (x^*(t))_{t \in \mathbb{N}} \), taking into account the fact that matrix \( A_t \) is eventually constant.

Corollary 13. If there is never a cut, then the sequence \( (x^*(t))_{t \in \mathbb{N}} \) is convergent and \( \|x^*(t) - v\| = O(g') \) for some \( g \in [0, 1] \) if \( v = \lim x^*(t) \).

In [2], we proved that Corollary 13 holds with \( g = 1 - n^{-n} \).

5. Convergence of the HK Dynamics on the Circle

We now put all the pieces together to show the convergence of the HK dynamics on the circle.

Theorem 14. An HK system on the circle converges asymptotically.

Proof. If there is a cut, the dynamics on the circle is the same as on the line, in which case the convergence is well-known.

In the case no cut ever occurs, we prove a refinement of Theorem 2. Let \( t_0 \) be the time at which the influence graph does not change anymore, and let \( E \) denote the set of links in the final influence graph.

By definition of the Lyapunov function \( W \), for any \( t \) and \( t' \), \( t_0 \leq t \leq t' \),

\[
W(t) - W(t') = \sum_{(i,j) \in E} \delta(\varpi_i(t), \varpi_j(t))^2 - \delta(\varpi_i(t'), \varpi_j(t'))^2.
\]

We have

\[
\delta(\varpi_i(t), \varpi_j(t)) + \delta(\varpi_i(t'), \varpi_j(t')) \leq 2r \leq p.
\]
Moreover since \( r < p/2 \), we obtain
\[
|\delta(\mathbf{r}_i(t), \mathbf{r}_j(t)) - \delta(\mathbf{r}_i(t'), \mathbf{r}_j(t'))| \leq |\delta(\mathbf{r}_i(t), \mathbf{r}_{i+1}(t)) - \delta(\mathbf{r}_i(t'), \mathbf{r}_{i+1}(t'))| + \cdots + \\
+ |\delta(\mathbf{r}_j(t), \mathbf{r}_j(t')) - \delta(\mathbf{r}_{j-1}(t'), \mathbf{r}_j(t'))| \\
\leq |x^*_i(t) - x^*_i(t')| + \cdots + |x^*_{j-1}(t) - x^*_{j-1}(t')|.
\]

Then we use the above inequalities to bound each term in (10).
From Corollary [13] it follows that if there is never a cut, then
\[
W(t) - W(\infty) \leq C n^3 \rho^t
\]
for some positive constant \( C \). By Proposition [1] for each \( i \in [n] \)
\[
\delta(\mathbf{r}_i(t), \mathbf{r}_i(t + 1))^2 \leq W(t) - W(\infty).
\]

Therefore
\[
\delta(\mathbf{r}_i(t), \mathbf{r}_i(t + 1)) \leq \sqrt{C} n^{3/2} \rho^{t/2}
\]
and so \( \delta(\mathbf{r}_i(t), \mathbf{r}_i(t + 1)) = O(\rho^{t/2}) \).
This establishes the convergence of each sequence \( \langle \mathbf{r}_i(t) \rangle \) for each \( i \in [n] \).

6. AN ALTERNATIVE PROOF OF CONVERGENCE

In the second version of their paper [5], Hegarty et al. give an alternative proof of Corollary [13]
that uses neither the column-stochasticity of the matrix \( A_k \) nor the rootedness of its associated graph
(Propositions [10] and [11] respectively). In fact, their new argument to prove the exponential convergence
of \( x^* (t) \) also works for the position vectors as we will show below. Thus the resulting proof of the
convergence of the HK dynamics on the circle directly follows from the finiteness of the quadratic kinetic
energy \( K_2 \) and the eventual stability of influence graphs.

For simplicity, let us denote by \( \dot{x}(t) \) the vector whose \( i \)-th component is \( \mathbf{r}_i(t), \mathbf{r}_i(t + 1) \), i.e.,
\[
\dot{x}_i(t) = \frac{1}{|N_i(t)|} \sum_{k \in N_i(t)} \mathbf{r}_i(t), \mathbf{r}_k(t).
\]

Under the condition \( r < p/6 \), for each \( i \)'s neighbor \( k \) at time \( t \) we have
\[
\mathbf{r}(t), \mathbf{r}_k(t) = \mathbf{r}(t), \mathbf{r}_i(t - 1) + \mathbf{r}(t - 1), \mathbf{r}_k(t - 1) + \mathbf{r}(t - 1), \mathbf{r}_k(t).
\]

Therefore,
\[
\dot{x}_i(t) = -\dot{x}_i(t - 1) + \frac{1}{|N_i(t)|} \sum_{k \in N_i(t)} \mathbf{r}_i(t), \mathbf{r}_k(t - 1) + \frac{1}{|N_i(t)|} \sum_{k \in N_i(t)} \mathbf{r}(t - 1), \mathbf{r}_k(t).
\]

From time \( t_0 \), the influence graph is constant and the neighborhood of each agent does not vary anymore.
For \( t > t_0 \), it follows that \( \frac{1}{|N_i(t)|} \sum_{k \in N_i(t)} \mathbf{r}(t - 1), \mathbf{r}_k(t - 1) = \dot{x}_i(t - 1) \), and thus
\[
(11) \quad \dot{x}(t) = B \dot{x}(t - 1)
\]
where \( B \) denotes the line-stochastic matrix whose associated graph is the influence graph at time \( t_0 \)
and with positive entries in the \( i \)-th line equal to \( 1/|N_i(t_0)| \).

By Theorem [2] the sequence of vectors \( \dot{x}(t) \) tends to the zero vector. Using (11) and the Jordan
normal form of \( B \), it follows that each component of \( \dot{x}(t) \) is in \( O(e^{-\alpha t}) \), which proves the finiteness of
\( K_1 \), and thus the convergence of the HK dynamics on the circle.

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