Skin-friction critical points in wall-bounded flows.

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Abstract. Critical points in the skin friction field of wall-bounded flows were investigated using data from direct numerical simulations of channels at $Re_\tau = 934$ and $Re_\tau = 1834$. A method for their detection and characterisation is outlined, and a statistical description of their properties is reported. Their lifetime, average distance, velocity and area density were computed. Conditionally averaged fields were calculated in order to examine the average flow in the vicinity of a critical point. It was found that the critical points are connected to surprisingly large features in the channel. A mechanism for the generation of the critical points is postulated, based on the conditional averages and analysis of time sequences in the flow above them.

1. Introduction

Using data from full direct numerical simulations (DNS) of wall-bounded flows (pipes, channels and boundary layers), Ref. [1] have shown that critical points, i.e. points where the skin friction and surface vorticity are zero, exist on the surface of a no-slip wall. They suggested that, since the invariants of the velocity gradient tensor are zero at the wall, the invariants of the ‘no-slip tensor’ can be used instead to investigate turbulent structures at this location. The existence of critical points indicates that there are complex three-dimensional separated flow structures at the wall, resulting in negative streamwise skin friction or shear stresses at the wall. Ref. [2] also reported regions of negative skin-friction events in their numerical simulations. Regions at the wall with critical points are rare and there is a need to establish the significance of these regions in the generation of ‘turbulence’ at the wall. In this paper we will establish accurately the location of the critical points and investigate the statistics and topological properties of these critical points.

2. No-slip critical points

We will adopt a frame of reference where the three orthogonal directions and velocities are labelled as follows:

- streamwise $\rightarrow u, u_1, x, x_1$
- spanwise $\rightarrow v, u_2, y, x_2$
- wall-normal $\rightarrow w, u_3, z, x_3$.

The unit vectors in the three directions are $e_1, e_2, e_3$. 

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Let the velocity gradient tensor be

\[ A_{ij} = \frac{\partial u_i}{\partial x_j}. \]  

(1)

It is easy to show that at a wall, only two components of \( A_{ij} \) do not vanish as a result of the no-slip, impermeability and incompressibility conditions: \( A_{13} \) and \( A_{23} \). It follows that at such a wall, all three similarity invariants of \( A_{ij} \), namely \( P \), \( Q \) and \( R \) as defined by [3]. A local characterisation of the flow field now requires a higher order truncation. Keeping up to second-order terms, the Taylor expansion of the velocity field at a no-slip wall is

\[ u(x, y, z) = \dot{x} = \frac{dx}{dt} = z \frac{\partial u}{\partial z} + z \left( x \frac{\partial^2 u}{\partial x \partial z} + y \frac{\partial^2 u}{\partial y \partial z} + z \frac{\partial^2 u}{2 \partial z^2} \right), \]  

(2)

\[ v(x, y, z) = \dot{y} = \frac{dy}{dt} = z \frac{\partial v}{\partial z} + z \left( x \frac{\partial^2 v}{\partial x \partial z} + y \frac{\partial^2 v}{\partial y \partial z} + z \frac{\partial^2 v}{2 \partial z^2} \right), \]  

(3)

\[ w(x, y, z) = \dot{z} = \frac{dz}{dt} = z \left( x \frac{\partial w}{\partial x} + y \frac{\partial^2 w}{\partial y \partial z} + z \frac{\partial^2 w}{2 \partial z^2} \right), \]  

(4)

with all derivatives estimated at \( z = 0 \). We introduce the no-slip tensor \( \mathcal{A}_{ij} \)

\[ \mathcal{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{\partial y \partial z} & \frac{1}{2} \frac{\partial^2 u}{\partial z^2} \\ \frac{\partial^2 v}{\partial x \partial z} & \frac{\partial^2 v}{\partial y \partial z} & \frac{1}{2} \frac{\partial^2 v}{\partial z^2} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{1}{2} \frac{\partial w}{\partial z} \end{bmatrix}, \]  

(5)

and the rescaling

\[ dt = z dt. \]  

(6)

Equations (2), (3) and (4) can now be written in matrix form as

\[ \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_{13} \\ 0 & 0 & A_{23} \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \]  

(7)

or simply \( \dot{x}_i = A_{ij} + A_{ij} x_j \). We note in passing that \( \dot{x}_i = dx_i/d\tau \neq \dot{x}_i = dx_i/dt \), and that incompressibility at a no-slip wall implies \( A_{31} = A_{32} = 0 \), \( A_{33} = -(A_{11} + A_{22})/2 \).

We will refer to the following two-dimensional vector field at the wall as the skin-friction field:

\[ s = A_{13} e_1 + A_{23} e_2. \]  

(8)

The flow’s vorticity at the wall reduces to a two-component vector field \( \omega = A_{23} e_1 - A_{13} e_2 \). It is easy to see that at the wall, \( s \cdot \omega = 0 \). Hence, the two fields are always mutually orthogonal.

2.1. Definition and characterisation of critical points

A linear approximation to \( s \) around a point \( x_0 \) yields \( s_i (x) = A_{ij} (x_0) + x_j A_{ij} (x_0) \), which is valid only in the immediate vicinity of \( x_0 \). A critical point (CP) in \( s \) occurs where both its components vanish: \( A_{13} = A_{23} = 0 \). The topology of the critical point is given by the similarity invariants of the matrix \( A_{ij} \) computed at the critical point. This topology is representative of the CP’s immediate vicinity, i.e. of the region where the linear approximation holds. Subscripts \( i \) and \( j \) now span the range \( 1 \to 2 \) only, so that there are two invariants to characterise the local topology of a CP in \( s \): its trace and determinant. We adopt the convention to label them as \( p = -A_{ij} \) and \( q = \text{det}(A) \). A detailed description of the possible local topologies based on \( p \) and \( q \) can be found in [4]. Here, we will only make the distinction between two types of CPs: saddles (\( q < 0 \)) and nodes (\( q > 0 \)). Using this characterisation, it can be shown that a CP in \( s \) is also a CP of the same type in \( \omega \) because their corresponding \( q \) is equal.
3. Data sets
The data used for the present study comes from DNS of turbulent channel flow. The numerical scheme integrates the fluid flow equations following the decomposition described in [5]. On wall-parallel planes, space is discretised using Fourier expansions while Chebychev polynomials are used in the wall-normal direction. Two data sets are used, M950 and S1900. The latter is the same as case W1900 from [6], whereas the former is a smaller version of case L950 in [7] – but time-resolved. The details of the simulations are summarised in table 1.

The set S1900 contains 700 independent fields. They were used for standard statistics, i.e. statistics that do not require time resolution. The data set M950 was run at constant time step $\delta t$, instead of constant CFL as is usually done. The chosen $\delta t$ of 0.02$^+$ corresponded to a CFL of approximately 0.189. One in every fifteen fields was stored, thus creating a time series where consecutive fields are separated by 0.3$^+$. The full time series stored spans 889.5$^+$, which is almost one large-eddy turnover time. It was used for standard statistics as well as for statistical analysis of time-resolved events. For the two data sets, we stored data from both walls. This meant 1400 independent wall realisations from S1900 and two independent time series from M950, each spanning $t^+ = 889.5$.

Table 1. Properties of the two DNS data sets used. $\Delta x$ and $\Delta y$ are the streamwise and spanwise resolutions, respectively, after dealiasing. $L_x$ and $L_y$ are the domain extents in those two directions. $N_x$ and $N_y$ are the number of coefficients in the Fourier expansions in those two directions, while $N_z$ is the number of Chebychev polynomials of the wall-normal expansion. If $h$ is the channel half-height, $h/Re\tau$ yields the wall length unit with which all lengths with superscript “+$” are normalised. $u_\tau$ is the usual friction velocity.

|               | $Re\tau$ | $L_x^+$ | $L_y^+$ | $\Delta x^+$ | $\Delta y^+$ | $N_x$ | $N_y$ | $N_z$ | $u_\tau \times 10^2$ |
|---------------|----------|---------|---------|---------------|---------------|-------|-------|-------|-------------------------|
| M950          | 934      | 5870    | 2935    | 11.5          | 5.7           | 512   | 511   | 385   | 4.539                   |
| S1900         | 1834     | 2880    | 1440    | 11.3          | 5.6           | 256   | 255   | 769   | 4.079                   |

3.1. Finding critical points
In a numerical data base, the components of $s$ are only available at discrete locations on a regular grid and stored with finite precision. Naively looking for a zero in both $A_{13}$ and $A_{23}$ yields no CPs. Instead, methods based on thresholding and local minima can be thought of, but we used a different approach which we will now describe in some detail.

We start by assuming the linear approximation $s_i(x) = A_{i3}(x_0) + x_j A_{ij}(x_0)$ to be valid. In particular, we ask the following question at each discrete point $x_0$ of the data sets $A_{13}$ and $A_{23}$: at which distance $x$ from $x_0$ does the linear approximation predict a zero in $s$? In matrix form, this question is equivalent to solving the system

$$
\begin{bmatrix}
  s_1 \\
  s_2 
\end{bmatrix}
= \begin{bmatrix}
  A_{13} \\
  A_{23}
\end{bmatrix}
+ \begin{bmatrix}
  A_{11} & A_{12} \\
  A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}.
$$

A solution for $x_1$ and $x_2$ is found at every grid point in terms of $A_{13}$ and $A_{ij}$ at those points. However, we only keep those solutions found which comply to the requirement that $\sqrt{x_1^2 + x_2^2} < l$, where $l$ is the data’s grid spacing on a wall plane. Any CP found further away than this from the grid point is considered outside of the region where the linear approximation applies. The latter is assumed to be a circle of radius $l$ centred at the interrogated grid point – see figure 1. Each CP thus found should be detected from at least two contiguous grid points – see figure 2. Since each grid point predicts the location of the CP according to the linear
approximation valid at the interrogated grid point, two slightly different CP locations will be predicted from two different contiguous grid points for the same CP. We use this redundancy in our favour, and discard any CP found which does not possess at least one neighbour within a circle of radius \(|r| = l\) centred at the CP – see figure 2. Most of the time, CPs are found in sets of four: one from each grid point within a distance \(l\). The CP finding algorithm then screens each “cloud” of such CPs which fit within a circle of radius \(l\) and discards all but one – chosen randomly. In the case where a CP was found to have no neighbours within this circle, it is discarded as spurious.

Once the position of the CP is known, the values of \(A_{ij}\) need to be estimated at the CP location. Since its position does not correspond to a grid point, we carry out a subgrid interpolation. This involves the use of a square of \(11 \times 11\) neighbouring grid points which has the CP location at its centre. The square patch of variables \(A_{ij}\) is given to Matlab®’s \texttt{interp2} function which applies cubic spline interpolation to infer their value at the CP location.

Figure 1. Region of assumed validity for a CP location predicted by the linear expansion at grid point \(x_0\).

Figure 2. Neighbouring grid points predicting two different CP locations. CPs within \(|r| < 2l\) from each other are considered to be the same CP.

The method that has been described relies heavily on the assumption that the grid spacing, \(l\), is small. This aspect will now be discussed.

3.2. Data interpolation
Section 3.1 concluded by emphasising the strong dependence of the CP-finding routine on \(l\), the streamwise and spanwise spacing between data points. We artificially decreased this spacing by taking advantage of the periodicity in the streamwise and spanwise directions. The data sets of \(A_{13}\) and \(A_{ij}\) were Fourier transformed, padded with zeros in Fourier space, and then transformed back into real space. The fields obtained in this manner differed from the original ones in several ways which are perhaps best explained with a specific example as follows.

Any flow variable can be obtained from the Fourier and Chebychev coefficients of the wall-normal vorticity (\(\text{vor}\)) and Laplacian of wall-normal velocity (\(\phi\)), which are the two variables the DNS code solves for [5]. A single field of \(A_{13}\) from the series M950 would usually yield a real-space representation of \(768 \times 768 \times 385\) points for a single quantity \((x \times y \times z)\) when using the settings outlined in table 1. However, we obtain a single field representation of \(A_{13}\) on a \(1536 \times 1536 \times 769\) grid after zero-padding the coefficients of \(\text{vor}\) and \(\phi\) in the post-processing routine which generates \(A_{13}\) and \(A_{ij}\). This has doubled the number of points in each direction. Only the bottom \((z = 0)\) and top \((z = 2)\) planes of \(A_{ij}\) and \(A_{ij}\) are stored to disk. A patch of field \(s\) extending \(12^+ \times 12^+\) in the \(x\) and \(y\) directions is shown on figure 3(a), as it is obtained
directly from the 1536 × 1536 slice that is stored to disk. It can be seen that the streamwise grid spacing is twice the one in the spanwise direction, which is a consequence of the DNS settings – see table 1. In the next step, the planes of fields $A_{ij}$ and $A_{ij}$ are then Fourier-transformed. Zero padding is applied to the transforms of $A_{ij}$ and $A_{ij}$ in order to add $n - 1$ data points in $y$ between two contiguous grid points of the original grid, and $2n - 1$ additional points in $x$. The resulting grid now features a constant spacing in both directions. Figure 3(b) shows the same patch of $s$ as in figure 3(a) after applying this interpolation procedure with $n = 5$. Two CPs have become more evident in the interpolated data set. The Fourier-based interpolation was done prior to running the CP finding routine, and hence before applying the final stage cubic spline interpolation on the patch surrounding a detected CP.

As explained in [1], the number of saddles has to be equal to the number of nodes by a topological constraint which follows from the Poincaré-Hopf index theorem. Using the CP finding routine outlined in this section, it was found that the data set M950 obeyed this one-to-one ratio between saddles and nodes in 85.9% of the fields analysed. In 13.41% there was one more CP of one type than the other, in 0.72% there were 2 more CPs of one type than the other and in 0.02% there was an imbalance of three. Within the data set S1900, 95.14% of the fields had the same number of saddles as nodes, 4.72% had an imbalance of one CP, 0.07% had an imbalance of two CPs and 0.07% of three CPs. No trend was observed of one type of CP being found more than the other.

![Figure 3](image)

**Figure 3.** Vector field $s = A_{ij}e_1 + A_{ij}e_2$ over a wall. Selected region contains two CPs. (a) Original field from rectangular DNS grid; (b) Field after interpolation using zero-padding in Fourier space. 4 and 9 additional points have been added by interpolation between 2 original points in $y$ and $x$, respectively.

4. Statistics at the wall

We proceed to present various CP statistics measured at the walls of both flows: S1900 and M950. Before doing so, however, it is perhaps best to illustrate at this point what kind of data is being used in a statistical manner.

The data set S1900 is made of instantaneous snapshots of independent fields. The CPs found in M950 can be used for statistics like average number of CPs per wall plane, average distance between CPs, etc.... M950 contains a time-resolved data series, which extends for long enough in time so as to extract the same type of statistics as from S1900. In addition, a look at figure 4
shows what other possibilities this data set offers. Figure 4(a) displays the typical life of 2 CPs. A saddle and a node are born simultaneously and near each other. During their life, they travel predominantly in the streamwise direction before coming close again and disappearing together. The topological constraint that there be an equal number of saddles and nodes is thus satisfied. However, this condition is also compatible with more elaborate life patterns. Figure 4(b) depicts such a scenario extracted from M950. The lives of three CP pairs overlap, without any of them disappearing next to the CP they were born with. From this type of data, we can extract dynamical information like average life times and average convection velocity of the CPs, which we will report in the next sections.

![Figure 4(a): A single saddle/node pair.](image1)

![Figure 4(b): Three saddle/node pairs with overlapping lives.](image2)

Figure 4. Examples of patterns in CP pair lives. Symbols represent nodes (◦) and saddles (+). The time gap between two consecutive points in the series is 0.3 wall time units. (a) A single saddle/node pair. (b) Three saddle/node pairs with overlapping lives.

4.1. Lifetimes

Once we have computed the CP positions at each snapshot of the series M950, we still need to obtain for each CP the list of locations it occupied during its life. This involves inferring if a CP at time $t$ is the same as a CP already detected at time $t - \Delta t$ earlier on. To carry out this task, a post-processing routine made several assumptions in order to establish if two CPs at different instants are actually the same CP. These assumptions are listed below, where $\Delta t = 0.3^+$:

- At time $t$, a CP is of the same type (saddle or node) as at $t + \Delta t$.
- At $t + \Delta t$, a CP’s new position has to be within a radius of up to $15^+$ from its position at time $t$.
- At $t + \Delta t$, a CP’s new streamwise position can be at most $3^+$ upstream from its position at time $t$. 

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If more than one CP at $t + \Delta t$ meets the above conditions to be the successor of a CP detected at time $t$, then the closest CP at $t + \Delta t$ to the CP at time $t$ is chosen to be the valid successor.

The data sampling rate is not sufficient to infer any lifetimes shorter than 0.3$^+$. With all these constraints in mind, the probability density function (PDF) of the CP lifetimes looks as that shown on figure 5. The mean lifetime of 6.6$^+$ appears to be representative of events similar to those on figure 4(a), while lifetimes as long as 25$^+$ are, although uncommon, occurring every now and then – see for example figure 4(b). In computing the PDF on figure 5, we took care not to include the CPs born before the data series’ first 60$^+$ wall time units or during the last 60$^+$. These edges in time could have introduced CPs which were short lived as a consequence of being already born when the series started, or which disappeared not because of natural death but because of the data series’ end.

4.2. Convection velocity

The velocity at which the CPs move along the wall is a question we analysed based on the post-processed data of M950. The displacements measured from snapshot to snapshot lead to velocity estimates which are an average over the time lapse of $\Delta t = 0.3^+$. This way of measuring CP velocities is not ideal, but it yields nevertheless a range of velocities as that seen on figure 6. As an a posteriori justification of the maximum displacements assumed in section 4.1, we argue that the PDF does not exhibit abrupt cuts at its edges. On the contrary, the PDF’s tails tend smoothly towards zero, hinting at the fact that no particle displacement reached the maxima allowed by the post-processing routine – see section 4.1. Moreover, the likelihood of a slightly negative streamwise velocity appears to be small but non-vanishing, as confirmed by some CP trajectories of the kind displayed on figure 4 but not shown here. The mean streamwise velocity of 9.5$^+$ corresponds to the flow’s mean velocity at $z^+ \approx 9$. The CPs are therefore advecting – in the mean – at the same velocity as the structures in the buffer layer. The mean spanwise velocity measured was 0.017$^+$ – which will be considered as zero – and the mean total velocity was 9.8$^+$. 

![Figure 5. PDF of the CP lifetimes. The computed mean is 6.6$^+$.](image1)

![Figure 6. PDF of the streamwise velocity of CPs. The computed mean is 9.5$^+$.](image2)
4.3. Average distance between CPs and CP density

We now turn our attention to the PDF of the distance between a CP and all other CPs present at the wall. In computing this, we tried to avoid obtaining distance histograms which are biased. Particular problems arise when, for example, a CP is located at the corner of the channel’s rectangular wall plane. Short distances are only available within a quarter of a circle around such a point, as opposed to within a full circle if the point was located in the middle of the wall. On the other hand, a point located in the middle of the wall can be at a maximum distance from another CP of half a diagonal length, whereas a point in a corner can be spaced by a full diagonal from another one. In order to remove the disparity in possibilities depending on where the point is located, we proceeded as follows. Taking the wall dimensions to be \( L_x \times L_y \) in the streamwise and spanwise directions, respectively, we constructed a rectangle of length \( L_x - L_y / 2 \) and width \( L_y / 2 \) with its lower-left corner located at \((L_y/4; L_y/4)\) – see figure 7. Only the CPs located inside the inner rectangle were used to compute the distances from, while all points within a radius of \( L_y / 4 \) from such a CP were used to compute the distances to – this includes distances to points outside of the inner rectangle. In this manner, any distance away from the reference CP is equally likely within that radius. The price paid is that no distance above \( L_y / 4 \) is measured. We discarded the repeated distances which inevitably arise when following this procedure.

![Figure 7](image)

Figure 7. Sketch illustrating which distances were measured within a wall plane. At each CP inside the inner rectangle, a circle of radius \( L_y / 4 \) centred at the CP gave the region within which to measure distances to other CPs. The CPs inside both rectangles were used to measure distances to, but only the CPs inside the inner rectangle were used to measure distances from.

The PDF of the distance between a CP and all other CPs within the allowed radius is shown on figure 8. The fact that the S1900 curve is higher than that for M950 is entirely due to the difference in the domain size. For S1900, \( L_y^+ / 4 = 360 \), while for M950 \( L_y^+ / 4 = 734 \). This means the range of distances in M950 extends for longer. Hence, the normalisation in order to yield a total area of 1 under each curve varies, leaving the S1900 curve slightly higher than that of M950.

Two separate regimes are distinguished in the curves shown on figure 8. At lower spacing values \( \Delta^+ \), the CP separation is governed by the typical distance between CPs which are somehow related, or which interact. In other words, the CPs that appear and/or disappear together, like those shown on figure 4. There is a peak in the distance likelihood at \( 20^+ \), which is consistent with what is observed on figure 4. If we call a cloud of interacting CPs a cluster of CPs, then the CP spacing within a cluster is given by the left-hand hump of the PDF in figure 8. The right-hand side of the curves, or the apparently never-ending increase in likelihood of distance, is governed by the inter-cluster spacing. This part can increase as much as the domain allows it to, and is perhaps less informative as a result.
Finally, the average CP density estimated using all CPs over the entire domain available yields 2 CPs per million wall square units. The same density was found at both Reynolds numbers. This leads to the estimate that within a wall plane of M950 and S1900, there are on average 34 and 8 CPs, respectively.

5. The flow field around critical points

Although critical points are spatially compact and only rarely found in the flow field, it will be shown in this section that they are associated with extreme events in the flow and a relatively large-scale surrounding flow feature.

5.1. Conditional shear stresses

Firstly, PDFs of the wall shear stress are shown in figure 9 normalised by their peak value. The unconditional PDFs show the well-known characteristics of viscous-scaled wall shear stress: the streamwise component has a mean value of 1 and is positively skewed while the spanwise component has a mean of zero and a lower standard deviation. The PDFs conditioned on the existence of a critical point were determined by analysing the wall shear stress in a circle of radius 20 wall units around a critical point. The PDF of the streamwise component shows that most of the shear stress fluctuations that occur around a critical point are at the extreme ends of the unconditional PDFs. In other words, the wall shear stress is at its extreme lowest around critical points. The variance is also much higher in the conditional sense, illustrating the extreme fluctuations about a critical point. The spanwise conditional average confirms this with a much larger variance (longer tails in the conditional PDF) than the overall flow.

The average skin friction field conditioned on the existence of a critical point at (0,0) is shown in figure 10. The contours are scaled with viscous-scaling so that the mean streamwise wall shear stress is unity. Near the critical point, the wall shear stress is considerably lower than the mean on average. Surrounding this low shear stress field, there are lobes of streamwise wall shear stress greater than the mean value. The pattern thus left is one of sign-alternating large spanwise wall shear stresses. This points to a converging-diverging behaviour in the flow near a critical point. Upstream and downstream of the critical point, the wall shear stress is also lower than the mean. Somewhat surprisingly, there is a signature in the flow downstream of the critical point persisting hundreds of wall units downstream of the event. The lower-than-average wall shear stress is accompanied by opposite-sign spanwise wall shear stress, suggesting
5.2. Conditional velocity fields

In figure 11 a fluctuating streamwise velocity field is shown. The velocity fluctuations, \( u' \), are calculated by averaging velocity fluctuations (about the unconditional local mean streamwise velocity) above a critical point. Conditional averages for both Reynolds numbers are shown and there is little difference between them. Since the Reynolds number effect appears minimal, no further analysis of the high Reynolds number data will be included in this paper (the low Reynolds number simulation was selected for further analysis owing to the larger box size and availability of time-resolved data).

As seen in the conditionally averaged wall shear stress plots, the streamwise coherence
persists hundreds of wall units downstream of the critical point. Above the wall, there is a large, low-speed region as might be expected above the large, low-shear-stress producing event. Interestingly, the wall-normal coherence also extends well into the logarithmic region. A high-speed region centred at the base of the logarithmic region was found above and downstream of a critical point. The demarcation between the high-speed and low-speed regions is approximated by a line of angle $14^\circ$ to the wall. This angle is similar to the well-documented ‘structure angle’ of high Reynolds number wall-turbulence [8], providing the first evidence that critical points are somehow associated with large-scale structures.

Conditionally averaged wall-normal velocity fluctuations are shown in figure 12. Although the coherence is somewhat weaker, it is clear that there is an upward flow upstream of a critical point, followed by a downward flow downstream. The downward flow begins in the logarithmic region and intensifies as the critical point is approached. In the far field, there is a general, albeit weak, upward flow in the low streamwise velocity region downstream of the critical point.

The combination of the streamwise and wall-normal fluctuation behaviour near a critical point points to a spanwise vortical structure above a critical point. In figure 13 the fractional difference of spanwise vorticity, $\Delta \Omega_{yc}^+$ is shown. The fractional difference is defined as

$$\Delta \Omega_{yc}^+ = \frac{\langle \Omega_{yc}^+ \rangle - \langle \Omega_{yc} \rangle^+}{\langle \Omega_{yc} \rangle^+},$$

where $\langle \rangle$ represents the unconditional average and $\langle \rangle_c$, the conditional average. The increase in spanwise vorticity is both large (>100% of the unconditional mean spanwise vorticity) and
extended over a long distance in both streamwise and wall-normal directions. Clearly there is enhanced vorticity directly above the critical point. Ref. [2] also found a vortex directly above regions of negative streamwise wall shear stress. This is perhaps not too surprising since there must be a region of negative wall shear stress in the near vicinity of a critical point, however, the converse is not true (there need not be a critical point near all regions of negative streamwise wall shear stress). As might also have been expected from the conditional averages of streamwise velocity, there exists a shear layer of intense vorticity downstream of the critical point, extending into the logarithmic region and in excess of 700 wall units downstream.

![Figure 13. Conditionally averaged spanwise vorticity field at Re_τ = 934.](image)

Finally, the decomposition of the Reynolds shear stress into conditional Q2 ($\overline{uw} < 0, w > 0$) and Q4 ($\overline{uw} < 0, w < 0$) is shown in figure 14. Note that the contours shown display the Reynolds shear stress component as a fraction of the unconditional Reynolds shear stress (in the manner of equation 10). There are clearly strong ejection events immediately upstream and far downstream of the critical point, even hundreds of wall units downstream. Sweep events are less pronounced, except immediately downstream and above the critical point. In fact, the sweep events can occur high in the logarithmic region above the critical point (corresponding with the high-streamwise-velocity region and the negative-wall-normal-velocity region).

![Figure 14. Conditionally averaged (a) Q2 and (b) Q4 events at Re_τ = 934.](image)
5.3. Vorticity transport

For turbulent channel flow, the Reynolds-averaged mean momentum balance results in

$$\frac{\partial^2 U^+}{\partial x_3^+2} + \frac{1}{Re_T} \frac{\partial \Pi^+}{\partial x_3^+} = 0. \quad (11)$$

It can be shown that the third term is simply the streamwise component of the Lamb vector:

$$-\frac{\partial \Pi^+}{\partial x_3^+} = u\Omega_y^+ - v\Omega_z^+. \quad (12)$$

The right-hand side of this equation can be interpreted as the difference between velocity-vorticity correlations. The first term represents the wall-normal transport of spanwise vorticity. A negative value leads to an increased momentum source, as can be seen from equation 11. Figure 15(a) shows the transport of spanwise vorticity in the vicinity of a critical point. Also shown in figure 15(b) is the spanwise vorticity fluctuation about the unconditional mean, with the blue region highlighting the vortex above a critical point and inclined shear layer downstream. The sign of vorticity associated with this vortex is negative such that the vortex is ‘prograde’ ($\Omega_y < 0$). The vortex above the critical point is associated with regions of positive and negative transport of vorticity. However, the average transport, $\langle \omega \Omega_y \rangle_c$, over the whole vortex is positive, indicating a tendency for the vortex to be transported toward the wall ($w < 0, \Omega_y < 0$) thus acting as a momentum source.

Figure 15. (a) Conditionally averaged Velocity-Vorticity correlation, $\langle \omega \Omega_y \rangle_c$ at $Re_T = 934$. (b) Conditional spanwise vorticity, $\Omega_y^+$, relative to the unconditional local mean spanwise vorticity. Color key applies to both subfigures.

6. Time sequence

The vorticity transport deduced in the previous section was also observed by viewing movies of the flow above a critical point. Although this information is difficult to convey in static images, an attempt is made in figure 16, where contours of spanwise vorticity are plotted in the $x_1 - x_3$ plane around a critical point. The time sequence is separated by $t^+ = 5$ and the abscissa is relative to an observer moving with the flow at a velocity of $U_c/u_e = 10$. A shear layer exists initially, even before the critical point pair is created, with a notable vortex centred at approximately $x_3^+ = 20$. Shortly after (but before $\Delta t^+ = 5$) the critical point pair is born as the noted vortex travels toward the wall. At $\Delta t^+ = 10$ the vortex has moved even closer to the wall and intensified significantly. Although not shown for brevity, the wall shear stress around the critical points similarly intensifies as the vortex moves closer to the wall.
Figure 16. Three snapshots illustrating the evolution of the shear layer above a critical point. A critical point pair was created between $\Delta t^+=0$ and 5. The location of the critical point is roughly indicated by the red dashed line. Contours show spanwise vorticity and vectors are instantaneous velocity vectors in the streamwise-wall-normal plane.

The overall picture, then, is that a critical point marks the tail end of some large-scale structure (not necessarily all large-scale structures). The critical point is formed when a negatively signed (prograde) vortex at the tail end of the large-scale structure is transported toward the wall. As the vortex moves closer to the wall, it also intensifies, causing larger fluctuations in all flow variables on the wall and above it. Eventually the vortex decays until the critical points below it come together and die.

7. Summary and conclusions

There are locations on the wall of wall-bounded turbulent flows where the skin friction vector has zero magnitude. These locations are called critical points. This study employed the time-resolved data from two relatively high Reynolds number channel flow simulations to answer some important questions regarding the statistical nature and the life of critical points.

It was found that critical points are always born in saddle-node pairs at the same spatial location. During their life they may encounter other saddle-node pairs, possibly swapping partners, before the saddles and nodes come together and die collocated in space (as they were born). The average lifetime of the critical points is short at $t^+=6.6$, however, some critical points can last more than 30 viscous time units. The critical points are typically spaced by 20 viscous wall units and convect at around $U_c/u_\tau=9.5$. The number of critical points per unit area appears to be fixed with Reynolds number such that there is, on average, one pair of critical points per million square wall units.

Conditional averages confirm the extreme nature of critical points. Moreover, the critical points appear to be at the tail of a large-scale structure that extends from the wall to the outer edge of the logarithmic region. In the streamwise direction, a low-streamwise-velocity region extends over 800 wall units downstream of a critical point. This low-speed region is bounded by a shear layer and is the site of ejection events. There is also a strong spanwise vortex directly above the critical point which is transported toward the wall on average. It is postulated that critical points are created when such a vortex, which is at the tail of a larger collection of vortices, gets pushed sufficiently close to the wall.

Somewhat surprisingly, no Reynolds number dependence was found in the large-scale coherence of the flow around a critical point. This despite the fixed concentration of critical points with Reynolds number. This suggests that only some of the large-scale structures in the flow are tailed by critical points. Understanding the connection between large-scales and the critical points generated will be the focus of ongoing study.
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