Anomalous diffusion modifies solar neutrino fluxes

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Abstract

Density and temperature conditions in the solar core suggest that the microscopic diffusion of electrons and ions could be nonstandard: Diffusion and friction coefficients are energy dependent, collisions are not two-body processes and retain memory beyond the single scattering event. A direct consequence of nonstandard diffusion is that the equilibrium energy distribution of particles departs from the Maxwellian one (tails goes to zero more slowly or faster than exponentially) modifying the reaction rates. This effect is qualitatively different from temperature and/or composition modification: Small changes in the number of particles in the distribution tails can strongly modify the rates without affecting bulk properties, such as the sound speed or hydrostatic equilibrium, which depend on the mean values from the distribution. This mechanism can considerably increase the range of predictions for the neutrino fluxes allowed by the current experimental values (cross sections and solar properties) and can be used to reduce the discrepancy between these predictions and the solar neutrino experiments.

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I. INTRODUCTION

Nowadays solar modeling seems to have reached a satisfactory stage [1–3]. The inclusion in the latest models of higher-order effects, such as the diffusion of heavy elements, brings the theoretical predictions in good agreement even with the detailed helioseismological data [4–11]. Nevertheless, solar neutrino experiments still contradict the model predictions. The problem appears to be almost independent of the details of solar models [4,12], and it has mostly been interpreted as a hint of new physics [4,14–27,11]. However, not everybody believes that the discrepancy is large enough (3σ effects have often disappeared) and that the solar models, in spite of their successes, are still solid enough to pose a real problem [28].

In this context, there has been a considerable amount of work devoted to answering questions such as: how large are the uncertainties of the solar model input parameters? Has something been left out of standard solar models? How does this affects predictions for the fluxes and the status of solar neutrino problem (SNP) [29–33,15]?

The solar core is a dense, strongly interacting many-body system: no realistic microscopic calculation of such a system exists. However, only three properties are important for the purpose of solar modeling. The first one is the equation of state, i.e., the relation between the average one-body local properties of the system (temperature, pressure, density). It is the equation of state that regulates the local hydrodynamical equilibrium and, in addition, contributes to the interpretation of the helioseismological data. The second one is the local opacity, which controls the energy transmission. In this paper we are interested in the last feature, the two-body relative-energy distribution or two-body correlation between particles. The rates of the most important reactions in the Sun are strongly affected by the high-energy tail of this distribution.

Solar models implicitly assume that the solar core can be described in terms of a gas of particles interacting via two-body short-range forces with no many-body effects apart for mean-field screening. Therefore, the velocity distribution and, in particular, the relative velocity distribution for the particles involved in the reactions are Maxwellian.

Two decades ago, Kocharov and collaborators [34–36], and Clayton and collaborators [37,38] speculated that the high-energy tail of the relative energy distribution could depart from the Maxwell-Boltzmann (MB) exponential form, \( \exp\{-E/kT\} \). At that time the SNP consisted only in the low yield of the Chlorine experiment [39] compared to theoretical predictions [40]. Since only the highest-energy proton can significantly penetrate the Coulomb barrier of the reaction \( ^7\text{Be}(p,\gamma)^8\text{B} \), Clayton suggested that a significantly lower boron-neutrino flux could be obtained by depleting the small number of protons in the tail and proposed to parameterize the small deviation with a Gaussian factor, \( \exp\{-\delta(E/kT)^2\} \): a value of \( \delta = 0.01 \) was sufficient to solve the SNP. This suggestion met some criticism and has been mostly ignored (see, however, a few recent papers [41–44]). In particular, it has been remarked that deviations from the MB energy distribution could not be due to nuclear reactions keeping the distribution out of equilibrium [45–47,2]. In addition, nobody was able to give a microscopical dynamical derivation of a non-Maxwellian distribution for the specific conditions inside the Sun (however, nobody has demonstrated with realistic calculations, which do not already assume the validity of the Boltzmann transport equation, that the distribution in the Sun is Maxwellian either).
Several new developments have convinced us that it is necessary to reconsider the possibility of deviation from a MB distribution. First of all, non-Maxwellian equilibrium energy distributions have been shown theoretically possible [48–52] and relevant to many physical systems [53–63]. Then, it has been recently proved by explicitly solving the Fokker-Planck equation that a velocity-dependent diffusion or friction coefficient results in an equilibrium energy distribution that departs from the MB one [64,65]. One should consider that constant diffusion and friction coefficients are only the first terms of a derivative expansion of the linearized hydrodynamic equations: Few percent contributions from the next terms in the expansion are not unrealistic, and are sufficient to give appreciable deviations from the MB distribution. Moreover, successive scattering events should not be independent (Markovian) in the solar interior, but correlated over periods corresponding to a few scattering processes. Indeed, such correlations have also been found to be related to anomalous diffusion [66,67] and to the Tsallis’ statistics that is a generalization of the Boltzmann-Gibbs statistics [48,49,56]. Time correlations, velocity dependence of the transport coefficients or other deviations from the so-called Navier-Stokes limit of the hydrodynamic equations are all manifestations of multiple particle collisions and of the many-body nature of the system, which is not completely described by two-body short-range interactions. In addition, the new data of the solar neutrino experiments (Chlorine [68], Kamiokande [69,70], Gallex [71–73] and SAGE [74]) have made the actual SNP more puzzling [5,75] and the possibility that Clayton’s suggestion could contribute to its solution should be reassessed.

It should be stressed from the beginning that modifications of the shape of the energy distribution are not equivalent for solar models to changes of temperature and/or densities; rather, a new degree of freedom is introduced. For instance, one might think that the effects of a distribution with a depleted high-energy tail could be reproduced by a MB distribution with a lower temperature. However, a lower temperature produces two effects: (1) the rates are reduced by changing the thermal average $\langle v\sigma \rangle$ (most of the contribution to this average comes from the high-energy tail because of the particular energy dependence of the main cross sections $\sigma$); (2) the system finds a new hydrostatic equilibrium because the average momentum of the particles becomes smaller (particles of all energies give comparable contributions to this average). In contrast, a change of shape of the distribution can reduce the rates and, at the same time, maintain the same hydrostatic equilibrium, since the two effects are dominated by particles from different parts of the energy spectrum. Indeed, the effects on solar models of changing the shape of the energy distribution could be reproduced by simultaneous local changes of temperature and cross sections.

Given the widespread misconception about the inevitability of the Maxwell-Boltzmann velocity distribution in the solar core, we briefly review in Sect. [I] the general assumptions under which the MB distribution is derived, show that these assumptions can be only approximately verified in the solar interior, and discuss some of the expected corrections to the standard treatment. In particular we shall consider (Sect. [II]) three concrete examples of anomalous diffusion (velocity-dependent corrections to the diffusion coefficient, to the friction coefficient, and slowly-decaying velocity autocorrelation), and how they generate deviations from the MB energy distribution. Then in Sect. [IV] we illustrate the consequences of small deviations for solar neutrino fluxes and estimate the magnitude of the deviations necessary to change the fluxes by amounts relevant for the SNP. We reserve Sect. [V] to our conclusions.
II. THE MAXWELL-BOLTZMANN DISTRIBUTION AND THE SOLAR CORE

In this Section we briefly review the standard hypotheses that lead to the MB distribution for the single-particle velocity, and the ones under which the two-particle relative-velocity distribution is also Maxwellian. Then we estimate the order of magnitude of the relevant physical quantities and show that these hypotheses are not met, making it likely that the MB distribution be only an approximation to the real distribution. These same “order of magnitude” considerations suggest some of the corrections to the hydrodynamical equations.

A. The ubiquity of the Maxwell-Boltzmann distribution

The velocity distribution can be studied either with a dynamical kinetical approach or with the methods of equilibrium statistical mechanics [76–80].

The kinetical approach yields a hierarchy of differential equations (the Bogoliubov-Born-Green-Kirkwood-Yvon hierarchy). The $n$-th differential equation is an equation of motion for the $n$-body distribution function $f_n(r_1, v_1; \ldots; r_n, v_n; t)$, consisting of a streaming term, which involves $f_n$, and a collision term, which involves the $(n + 1)$-body distribution function. In particular, the one-body distribution function $f_1(r_1, v_1; t)$ verifies a differential equation that involves the two-body distribution function $f_2(r_1, v_1; r_2, v_2; t)$ in the collision integral. The derivation of the BBGKY hierarchy already assumes that (1) the collision time is much smaller than the mean time between collisions and that (2) the interaction is sufficiently local. When these conditions are not met, collisions are not well-defined events in space and time. The rate of change of the $f$’s at time $t$ would depend not only on the $f$’s themselves at time $t$, but also on its previous history (the process is not Markovian), and higher-order spatial derivative cannot be discharged. Assuming that the BBGKY hierarchy is valid, the additional hypothesis of molecular chaos (Boltzmann’s Stosszahlansatz), i.e. that (3) the velocities of two particles at the same point are not correlated $f_2(r, v_1; r, v_2; t) = f_1(r, v_1; t) \times f_1(r, v_2; t)$, allows the truncation of the hierarchy of equations and yields the Boltzmann transport equation for the one-body distribution function $f_1(r, v; t)$ alone. From the Boltzmann transport equation the Boltzmann’s H theorem can be derived whose consequences are that (i) under arbitrary initial conditions $\lim_{t \to \infty} f(v, t) = f_0(v)$, and that (ii) $f_0$ is the equilibrium distribution if and only if $f_0(v_1)f_0(v_2) = f_0(v_1')f_0(v_2')$, where $(v_1, v_2)$ and $(v_1', v_2')$ are the velocities before and after the collision, i.e. $\sum_i \log f_0(v_i)$ is conserved in the collision. Then the additional assumption that (4) energy is locally conserved when using only the degrees of freedom of the colliding particles yields that the energy of the particle contributes linearly to $\log f_0(v)$, and, therefore, the one-particle equilibrium distribution is Maxwellian.

The equilibrium statistical mechanics approach uses the concept of most probable value of the distribution, and the large number of degrees of freedom assures that large fluctuations away from the most probable distribution are extremely unlikely in non-critical conditions.

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1For the sake of this discussion, we drop from now on the spatial label $r$, that should be reintroduced when discussing non-local interactions or density gradients.
Probabilities are calculated using the Postulate of Equal a Priori Probability and, when deriving the MB distribution, the assumption (3) that the velocity probabilities of different particles are independent. In addition, it is necessary that the total energy of the system could be expressed as a sum of a term quadratic in the momentum of the particle and independent of the other variables, and a term independent of momentum \[7\] (this second term includes the energy of the rest of the system and the interaction energy). This second condition is related not only to assumption (4) but also to (1) and (2), since if (1) and (2) are not verified the resulting effective two-body interaction is not local and depends on the momentum and energy of the particles.

Finally, even when the one-particle distribution is Maxwellian, assumptions (3) and (4), i.e. that the velocities are uncorrelated and energy is locally conserved, are again necessary to deduce that the relative-velocity distribution is also Maxwellian.

At last, we suggest two of the reasons of the ubiquity of the MB distribution. On the one side, whenever the above assumptions are a good approximation (typical examples are those systems that are dilute in the appropriate variables), the resulting equilibrium distribution can be demonstrated to be Maxwellian independently of the details of the interaction. On the other side, even when conditions are such that one can not rigorously deduce the form of the distribution, considering that most of the experimental measurable observables do not really test the form of the distribution, but only a few of its moments, the assumption of a MB form has several advantages: (a) It becomes exact in the dilute limit; (b) Being determined by a single parameter or scale, the second moment of the velocity distribution, it is consistent with our maximal ignorance (it assumes the least about the distribution); (c) It allows simple analytical treatments.

B. The solar core quasi-plasma

In the light of the above considerations, we examine the situation in the solar core, and discuss what kind of corrections to the simplified picture of a dilute gas could better describe a system where, as we shall argue, multiple particle collisions and many-body correlations between clusters of tens of particles should be present. Of course, our discussion suggests only qualitative answers, since these issues need a more sophisticated theoretical framework, better microscopical models and, perhaps, they could still be quantitatively settled only by a numerical dynamical microscopic calculation, which is not trivial and, to our knowledge, still missing.

Ions in the solar core are completely ionized: \(T_e = 1.36\) keV.

Only the many-body correlations between ions and electrons cut off the range of the electromagnetic interaction. The Debye-Hückel estimate (weak mean-field static approach) gives a screening length \(R_D^2 = kT/(4\pi e^2 \sum Z_i^2 n_i)\) of the order of the average interparticle distance in the center of the Sun: \(R_D \approx n^{-1/3} \approx 3 \times 10^{-9}\) cm, where \(n\) is the average density. In addition, the same Debye-Hückel approach estimates that, at the average interparticle distance, i.e., when the potential energy is at its minimum, the potential energy is already about 4% of the kinetic energy: its contribution is comparable to the kinetic energy for large portions of the particle trajectories.

These “back of the envelope” calculations already suggest that it is not possible to
completely describe the system as streaming particles plus collisions. Since the collision time is of the same order of the mean time between collisions and the size of the quasi-particle (ion plus screening cloud) is of the order of the distance between particles, a description in terms of a gas of compact quasi-particles interacting via local two-body collisions can only be an approximation whose corrections involve higher order derivative terms. In fact, the relevant expansion parameter in the BBGKY hierarchy is \( n r_0^3 \), where \( n \) is the density and \( r_0 \) the range of the interaction; but \( n r_0^3 \approx 1 \) using \( r_0 \sim R_D \).

Moreover, while the Debye-Hückel screening strictly applies only to a charge at rest or travelling at constant velocity, i.e. in the limit that the plasma reaction time is infinitely fast compared to the particle rate of velocity change, the real screening is dynamical and the inverse plasma frequency gives us an estimate of the reaction time:

\[
t_{pl} = \omega_{pl}^{-1} = \sqrt{m/(4\pi ne^2)} \approx 10^{-17} \text{ sec.}
\]

Times of this order of magnitude are necessary to build up screening after a hard collision and should be compared to the typical collision time \( t_{coll} \approx \langle \sigma v n \rangle^{-1} \approx 10^{-17} \text{ sec.} \)

We describe the fact that \( t_{pl} \approx t_{coll} \) by saying that collective effects have a time scale comparable to the average time between collisions and, therefore, several collisions are necessary before the particle looses memory of the initial state. Another consequence of \( t_{pl} \approx t_{coll} \) is that collisions between the quasi-particles (bare particle plus the screening cloud) are inelastic: part of the energy is dissipated by the process of the creation and successive removal of the screening cloud.

Since the plasma parameter, i.e., the number of particles in the Debye sphere, is not large (\( N_D \equiv (4\pi/3)R_D^3 n \approx 1 \)), a description in terms of a high-temperature plasma \( (1/N_D \sim n^{(1/2)}T^{-(3/2)} \rightarrow 0) \) is not a good approximation either. We could name it a quasi-plasma: strong many-body effects on the scale of a few average interparticle distances appear to be necessary, and clusters of a few (or tens) of correlated particle participate in the collisions, the number depending on the time scale considered.

In the next Section, we show that when the lowest-order kinetic equations, which are valid in the low-density limit, are supplemented with terms that have the form suggested by the above considerations, i.e. higher derivative terms (spatial nonlocality), time correlations (temporal nonlocality), or nonlinearity, the resulting equilibrium distribution is not Maxwellian.

### III. ANOMALOUS DIFFUSION, TIME CORRELATIONS AND NONSTANDARD STATISTICS

For concreteness we consider three possible corrections: i) A correction to the lowest order friction coefficient \( J(v) \); ii) a correction to the lowest order diffusion coefficient \( D(v) \); and iii) a modification of the two-body time correlations. These three types of corrections affect the Maxwell-Boltzmann statistical distribution. Indeed, similar corrections have already been shown to exist in hydrodynamic systems \([81,80,82]\): direct microscopic calculations could prove this possibility also in the solar context.

It is clear that such corrections imply other consequences. For instance, it is well-known that the present approach to the slow diffusion of heavy elements could also be modified \([83,84]\). In the present paper, we are interested in only one of these consequences: the
actual *equilibrium* statistical distribution of the relative-energy departs from the Maxwell-Boltzmann equilibrium distribution.

Corrections i) and ii) have already been considered in a more general context [64,65], and we only recall the main points. We assume that the system is not too far from the standard regime that leads to the MB distribution, so that an expansion starting from the usual formalism makes sense.

The Fokker-Planck equation, given in the Landau form, is

\[ \frac{\partial}{\partial t} f(t, v) = \frac{\partial}{\partial v} \left( J(v) f(t, v) + \frac{\partial}{\partial v} D(v) f(t, v) \right), \]

where \( f(t, v) \) is the distribution probability of particles with velocity \( v \) at time \( t \) and \( J(v) \) and \( D(v) \) are the dynamical friction and diffusion coefficients. The stationary distributions are the asymptotic solutions of the above equation. To lowest order \( J(v) = v/\tau \) and \( D(v) = \epsilon/\tau \), where the constant \( \tau > 0 \) has dimension of time (\( m/\tau \) is the friction constant) and \( \sqrt{\epsilon} \) has dimension of a velocity (\( \epsilon = kT/m \) for Brownian motion). At equilibrium one obtains the well-known Maxwellian distribution

\[ f(v) \equiv \lim_{t \to \infty} f(t, v) \sim \exp \left\{ -\frac{v^2}{2\epsilon} \right\} = \exp \left\{ -\frac{mv^2}{2kT} \right\}. \]

We can generalize the standard Brownian kinetics considering the expressions of the quantities \( J(v) \) and \( D(v) \) to the next order in the velocity variable: \( J(v) = v/\tau (1 + \beta_1 v^2/\epsilon) \) and \( D(v) = \epsilon/\tau (1 + \gamma_1 v^2/\epsilon) \); these higher derivative terms can be interpreted as signals of nonlocality in the Fokker-Planck equation.

If \( \beta_1 = 0 \) and \( \gamma_1 \neq 0 \) we find the Tsallis’ distribution

\[ f(v) = \left[ 1 + (q - 1) \frac{mv^2}{2kT} \right]^{1/(1-q)} \Theta \left[ 1 + (q - 1) \frac{mv^2}{2kT} \right], \]

where \( q - 1 = 2\gamma_1/(2\gamma_1 + 1) \), \( \Theta \) is the Heaviside step-function, and \( kT/m \equiv \epsilon(2 - q) \). When the characteristic parameter \( q \) is smaller that 1 (\( -1/2 < \gamma_1 < 0 \)), this distribution has a upper cut-off: \( mv^2/2 \leq kT/(1 - q) \) (the tail is depleted). The distribution correctly reduces to the exponential Maxwell-Boltzmann distribution in the limit \( q \to 1 \) (\( \gamma_1 \to 0 \)). When the parameter \( q \) is greater than 1 (\( \gamma_1 > 0 \)), there is no cut-off and the (power-law) decay is slower than exponential (the tail is enhanced).

If \( \beta_1 \neq 0 \) and \( \gamma_1 = 0 \), we find a Druyvenstein-like distribution:

\[ f(v) \sim \exp \left\{ -\frac{v^2}{2\epsilon} - \beta_1 \left( \frac{v^2}{2\epsilon} \right)^2 \right\}, \]

which has also the functional form suggested by Clayton to parameterize a small deviation (depletion) from the Maxwellian statistics.

The statistical distribution of Eq. (3) has an additional appealing feature: it naturally appears in the context of the generalized Boltzmann-Gibbs statistics obtained by introducing a new non-extensive entropy (Tsallis’ entropy):

\[ S_q = \frac{k}{q - 1} \sum_i p_i (1 - p_i^{(q-1)}) \].

7
The formal structure of the conventional thermostatistics is maintained and its results are also naturally generalized [48,56,52]. Apart from the formal aspect, this distribution is also attractive because of the many systems where it plays a role [53–63]. Non-extensivity \((q \neq 1)\) arises in systems with long-range interactions (gravitational systems [63], plasmas [85,86], condensed matter [87]) or with long memory at the microscopic level.

We mention also the possible connection between time correlations and extended statistics (other mechanisms are also possible). For a Markovian scattering process, the time correlation between particle velocities is by definition proportional to a delta function in time: \(\langle v(0)v(t) \rangle \sim \delta(t)\). As discussed above, we expect that particles lose memory of the initial state only after a few scattering processes; we can model the long-time asymptotical behavior of the velocity-correlation as \(\langle v(0)v(t) \rangle \sim t^{-(1+\gamma)}\). If \(\gamma \geq 1\), i.e., the correlation decays sufficiently fast, the diffusion process is no qualitative different from the delta-function case: \(\langle x^2(t) \rangle \sim t\). The same standard result holds if \(0 < \gamma < 1\) and \(0 < \int \langle v(0)v(t) \rangle < \infty\).

However, it has been shown [66,67] that, if \(0 < \gamma < 1\) and \(\int \langle v(0)v(t) \rangle = 0\) (or very small), or if \(-1 < \gamma \leq 0\), the diffusion is anomalous \(\langle x^2(t) \rangle \sim t^{1+\gamma} \sim t \log t\), if \(\gamma = 0\). Indeed, Tsalallis [56] shows that the generalized entropy \(S_q\) quite naturally generate anomalous diffusion \((\langle x^2(t) \rangle \sim t^{1+\gamma})\): this same generalized entropy leads also to the non-Maxwellian probability distribution for the velocities given by Eq. (3).

**IV. NONSTANDARD STATISTICS AND SOLAR NEUTRINOS**

From the considerations above we infer that generalized distributions, among which Tsalallis’ distribution has a special theoretical appeal, could better approximate the situation in the solar interior.

One could proceed and study the effect of using generalized distributions on solar models, distributions with both depleted and enhanced tails, for small and large deviations from the Maxwellian statistics. In particular, it would be interesting to study how the relative balance among the different reaction chains would change because of non-Maxwellian statistics; for instance, a distribution with an enhanced high-energy tail could make the CNO cycle important at relatively lower temperatures. This study would be very interesting, since it could experimentally constrain deviations from the MB statistics, given the high sensibility of these reaction rates to the tail of the distribution. However, such a systematic study, which should be performed by consistently including nonstandard statistics in solar model calculations, is not the purpose of this paper.

We shall only consider small deviations from the Maxwellian distribution and, for the purpose of illustration, use Clayton’s parameterization with the factor \(e^{-\delta(E/kT)^2}\). For instance, the Tsalallis’ distribution can be also approximated to first order in \((1-q)\) by Clayton’s form with \(\delta = (1-q)/2\) and a renormalized temperature \(T' = T + T(1-q)\). The usual asymptotic expansion of the integrand over the velocity distribution around the most effective energy, \(E_0\), yields an analytical expression for the rate changes. This analytical expression is valid also for \(\delta < 0\) as an asymptotic expansion around the \(\delta = 0\) case, in spite of the fact that the distribution is unbounded at high energy. Instead, a numerical integration should be performed with a suitable cut-off. In fact, the integrand decays exponentially after the Gamow peak when \(\delta \geq 0\), and a sufficiently large cut-off does not changes the numerical
value of the integral. When $\delta > 0$, the integral still decays after the Gamow peak, but it goes back up at energies $E \approx E_0/\delta$: this contribution to the enhanced tail is “unphysical” and comes from the choice of the parameterization. If $\delta \ll 1$, the large window between $E_0$ and $E_0/\delta$ allows the unambiguous elimination of this contribution. The alternative is to use a distribution that has an enhanced tail but still decays at high energy, such as the Tsallis’ distribution for $q > 1$. For small deviations from the MB distribution, the two descriptions give the same numerical results with the appropriate reparameterization, since one single number characterizes the deviation to first order.

If one computes the thermal average $\langle v\sigma \rangle$ with the modified distribution for a two-body reaction with Coulomb barrier, one finds that to the leading order in $\delta$

$$\frac{\langle v\sigma_i \rangle_\delta}{\langle v\sigma_i \rangle_0} = e^{-\delta(E_0^{(i)}/kT)^2} \equiv e^{-\delta\gamma_i},$$

where $E_0$ is the most effective energy (maximum at the Gamow peak) [1]

$$\frac{E_0}{kT} \approx 5.64 \left( Z_1^2 Z_2^2 \frac{A_1 A_2}{A_1 + A_2} \frac{T_c}{T} \right)^{1/3},$$

which depends on the reaction $i$ through the charges $Z$ and weights $A$ of the ions, and on the relevant average temperature $T$. Here, $T_c = 1.36$ keV is the temperature at the center of the Sun. In Table I we report the values of $\gamma_i \equiv (E_0^{(i)}/kT)^2$ for the five most relevant reactions in the Sun: $p + p$ ($i = 1.1$), $p + ^7$Be ($i = 1.7$), $p + ^{14}$N ($i = 1.14$), $^3$He + $^3$He ($i = 3.3$) and $^3$He + $^4$He ($i = 3.4$). Changing $\langle v\sigma \rangle$ for the $i$th reaction will affect the whole solar model and, in general, all fluxes will change. We estimate the effect on the fluxes by using power-law dependences

$$R_j \equiv \frac{\Phi_j}{\Phi_j^{(0)}} = \prod_i \left( \frac{\langle v\sigma_i \rangle_\delta}{\langle v\sigma_i \rangle_0} \right)^{\alpha_{ji}} = e^{-\sum_i \delta_i \gamma_i \alpha_{ji}},$$

for the fluxes $j = ^7$Be, $^8$B, $^{13}$N and $^{15}$O, while we have used the solar luminosity constraint [12] to determine the pp flux, $R_{pp} = 1 + 0.087 \times (1 - R_{^7}$Be) + 0.010 \times (1 - R_{^8}$N) + 0.009 \times (1 - R_{^{15}$O}), and kept fixed the ratio $\xi \equiv \Phi_{pep}/\Phi_{pp} = 2.36 \times 10^{-3}$. The exponents $\alpha_{ij} = \partial \ln \Phi_j / \partial \ln \langle v\sigma_i \rangle$ (see Table I) have been taken from Ref. [14], where it is also discussed why solar models depend on $\langle v\sigma \rangle_{^3}$ and $\langle v\sigma \rangle_{^4}$ only through the combination $\langle v\sigma \rangle_{^4}/\sqrt{\langle v\sigma \rangle_{^3}}$ and why it is a good approximation to keep the ratio $\xi$ constant.

In principle $\delta$ should be determined by a direct calculation of the complex many-body system and could be different for every reaction ($\delta \to \delta_i$). The energy distribution can be influenced by the specific properties of the ion (charge and mass) and by the different conditions of the environment in those parts of the Sun where each of the reactions mostly takes place. However, a direct calculation is not simple and it does not exist for the solar interior. Therefore, for the only purpose of estimating the potential effect of nonstandard distributions, we consider two simple models and use the corresponding $\delta$’s as free parameter(s). The first model assumes the same deviation $\delta$ for all distributions, while the second model assumes that only the $p + ^7$Be relative-energy distribution (parameterized by $\delta_{^7}$Be) and the two helium reactions (parameterized by $\delta_{^3}$He) are non standard.
In the first case, one finds by substituting Eq. (6) in Eq. (8) that
\[
\frac{\Phi_j}{\Phi_j^{(0)}} = e^{-\delta \beta_j},
\] (9)
where \(\beta_j = \sum_i \alpha_{ji} \gamma_i\) are reported in Table II. This dependence of the fluxes on \(\delta\) is in good agreement with Clayton’s numerical calculation [38]. Using the model of Ref. [3] as reference model and the latest experimental results (see Table III), we obtain the best fit for \(\delta = 0.005\) with a \(\chi^2 = 35\).

In the second case, we proceed similarly, but we use \(\delta_{\text{Be}}\) for the reaction \(p+^7\text{Be}\) and \(\delta_{\text{He}}\) for the two reactions \(\text{He} + \text{He}\): the corresponding \(\beta_{j,\text{Be}}^\text{Be} = \alpha_{ji} \gamma_i|_{i=1.7}\) and \(\beta_{j,\text{He}}^\text{He} = \alpha_{j,3.4} (\gamma_{3.4} - \gamma_{3.3}/2)\) are also reported in Table II. As shown in Table III the best fit is obtained for \(\delta_{\text{Be}} = -0.018\) (negative \(\delta\) corresponds to an enhanced tail, \(q > 1\) in Tsallis’ distribution) and \(\delta_{\text{He}} = 0.030\) with a \(\chi^2 = 20\).

From previous analyses we already knew that it is not possible to obtain a very good fit to all experiments even when the fluxes are used as free parameters [11,88–90,15]: a fit that has a low probability could only be obtained by reducing the boron flux by a factor of two and the beryllium and CNO fluxes as much as possible. Therefore, we are not surprised that we have not been able to obtain good fits, however we have been able to give a specific and physically motivated mechanism that greatly reduces the discrepancy between theory and experiment (the SSM has a \(\chi^2 = 74\)). Another physical mechanism that produces similar results is by introducing arbitrary screening factors [15]. It is also consistent that the second case, which produces the best fit, results in a depletion of the energy tail of the \(^3\text{He}\) and \(^4\text{He}\) ions, so that \(^7\text{Be}\) and \(^8\text{B}\) fluxes are strongly suppressed, and that the energy tail of the \(p+^7\text{Be}\) reaction is enhanced so to bring up the \(^8\text{B}\) flux towards the measured value.

We do not claim that this result is a solution to the SNP, in the sense of providing a model to fit the experimental results within one (a few) sigma. Our point is only that deviations from standard statistics corresponding to values of \(\delta\) of about 1% can change the neutrino fluxes of factors comparable to those that constitute the SNP. Such values of \(\delta\), or even larger values, cannot be excluded by the present knowledge of the strong interacting quasi-plasma in the solar interior. It could turn out that the actual values of the neutrino fluxes coming out of the Sun could result from the interplay of several mechanisms that are disregarded in the standard picture [28].

In the light of the above considerations, the uncertainties of the neutrino fluxes are considerably underestimated by not considering the possibility of non-extensive distributions. Finally, we wish to comment on the fact that limits on the reaction rates that come from determinations of the sound speed through helioseismological measurements do not automatically apply to changes of the rates through the present mechanism. In fact, given the cross sections, the reaction rates change because the densities and/or the thermal averages, \(\langle \sigma v \rangle\), change. However, if the statistics is not changed, the thermal averages change when the temperature changes, and changes of temperature and/or density clearly affect the structure of the solar model and the sound speed. In contrast, nonstandard energy distributions make it possible to change the thermal averages (at least for those reactions whose main contribution comes from the high-energy tail) without affecting the properties that depend on the bulk of the distribution, such as the sound speed and/or the equation of state.
Therefore, non standard velocity distributions affect the helioseismological measurements only insomuch as the consequent changes of the rates modify the solar structure.

V. CONCLUSIONS

New developments in generalized statistics combined with several qualitative hints that the standard approach to the solar interior is only a first approximation to the real situation, make it worthwhile to reconsider the early suggestion by Clayton that the energy distribution in the Sun could depart from the Maxwell distribution.

In particular, we recall that:

(1) The conditions in the solar core (density and temperature) do not satisfy those requirements that would guarantee standard diffusion and Maxwell-Boltzmann velocity distribution.

(2) Nonstandard diffusion is most likely present: Lowest order dynamical friction and diffusion coefficients are not sufficient; the diffusion mechanism is not described by a Markovian chain of independent two-body scattering events and correlations persist for time intervals longer than the mean time of one scattering process (memory effect).

(3) The direct consequence of these corrections to standard diffusion is that the equilibrium energy distributions of electrons and ions are not Maxwellian. In particular, the tails of the distributions are not exponential and, therefore, the small number of particles that have energies large enough to participate in those reactions that are hindered by Coulomb barrier can be much less (more) than the one expected in the standard distribution.

(4) Tsallis’ statistics should also be considered.

(5) Non-Maxwellian energy distributions modify the reaction rates. This phenomenon has the potential of increasing the range of possible values of the reaction rates well beyond the ones allowed by the uncertainties in the corresponding cross sections.

(6) Unlike a change in the temperature, which has a direct effect on the hydrostatic equilibrium and on the the sound speed, modifications of the distribution that affect only the high-energy tail do not change the solar model and the sound speed: the range of neutrino fluxes from models that verify helioseismological constraints could also be increased.

(7) If one modifies the standard distribution by a Clayton’s factor $e^{-\delta(E/kT)^2}$ with $\delta$ of the order of 1% (such modification cannot be excluded by the present knowledge of the strong interacting quasi-plasma in the solar interior) the neutrino fluxes change of amounts comparable to those that constitute the solar neutrino problem, even if it is not possible to solve the SNP by only modifying the energy distributions.

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**TABLE I.** Most effective energies for thermonuclear reactions and exponents $\gamma$ that characterize the change of the thermal average $\langle v\sigma \rangle$ to the leading order in $\delta$, when the energy distribution changes by a factor $\exp\{-\delta(E/kT)^2\}$: $\langle v\sigma \rangle_\delta = \langle v\sigma \rangle_0 \exp\{-\delta\gamma\}$.

| reaction | $E_0/kT$ | $\gamma = (E_0/kT)^2$ |
|----------|----------|-------------------------|
| $\langle v\sigma \rangle_{11}$: | $p + p \rightarrow ^2\text{H} + e^+ + \nu$ | 4.8 | 23 |
| $\langle v\sigma \rangle_{17}$: | $p + ^7\text{Be} \rightarrow ^8\text{B} + \gamma$ | 13.8 | 190 |
| $\langle v\sigma \rangle_{33}$: | $^3\text{He} + ^3\text{He} \rightarrow \alpha + 2p$ | 16.8 | 281 |
| $\langle v\sigma \rangle_{34}$: | $^3\text{He} + ^4\text{He} \rightarrow ^7\text{Be} + \gamma$ | 17.4 | 303 |
| $\langle v\sigma \rangle_{1,14}$: | $p + ^{14}\text{N} \rightarrow ^{15}\text{O} + \gamma$ | 20.2 | 407 |

**TABLE II.** The first four rows show $\alpha_{ij} = \partial \ln \Phi_j / \partial \ln \langle v\sigma \rangle_i$, the logarithmic partial derivative of neutrino fluxes with respect to the parameter shown at the left of the row. These numbers are discussed in Ref. [15]. The last three rows show $\beta_j$, $\beta_{\text{Be}}^j$ and $\beta_{\text{He}}^j$, the logarithmic partial derivative of the fluxes with respect to the parameters $\delta$‘s; as discussed in the text, they are linear combinations of the $\alpha$‘s weighted by the factors $\gamma$ of Table I.

| | $^7\text{Be}$ | $^8\text{B}$ | CNO |
|---|---|---|---|
| $\langle v\sigma \rangle_{11}$ | -1.0 | -2.7 | -2.7 |
| $\langle v\sigma \rangle_{34}/\sqrt{\langle v\sigma \rangle_{33}}$ | +0.86 | +0.92 | -0.04 |
| $\langle v\sigma \rangle_{17}$ | 0 | 1 | 0 |
| $\langle v\sigma \rangle_{1,14}$ | 0 | 0 | 1 |
| $\beta_j$ | 117 | 277 | 338.5 |
| $\beta_{\text{Be}}^j$ | 0 | 190 | 0 |
| $\beta_{\text{He}}^j$ | 140 | 150 | -6.5 |
TABLE III. The first three columns show the predicted fluxes, and the predicted gallium and chlorine signals in the SSM and in the two models with nonstandard distribution described in the text. The last column shows the present experimental results. For the three models is also given the $\chi^2$ resulting by the comparison with the experimental data.

| Models | SSM $\left(\delta = 0\right)$ | case I $\left(\delta = 0.005\right)$ | case II $\left(\delta_{\text{Be}} = -0.018, \delta_{\text{He}} = 0.030\right)$ | Experiment |
|--------|--------------------------------|-----------------------------------|---------------------------------|------------|
| $\left[10^9 \text{ cm}^{-2} \text{s}^{-1}\right]$ | | | | |
| $\Phi_{pp}$ | 59.1 | 62.2 | 63.7 | |
| $\Phi_{7\text{Be}}$ | 5.15 | 2.87 | 0.08 | |
| $\Phi_{13\text{N}}$ | 0.62 | 0.11 | 0.75 | |
| $\Phi_{15\text{O}}$ | 0.55 | 0.10 | 0.67 | |
| $\left[10^6 \text{ cm}^{-2} \text{s}^{-1}\right]$ | | | | |
| $\Phi_{8\text{B}}$ | 6.62 | 1.65 | 2.25 | 2.55 ± 0.21 $^a$ |
| $[\text{SNU}]$ | | | | |
| gallium | 137.0 | 100 | 97 | 75 ± 5 $^b$ |
| chlorine | 9.3 | 2.84 | 3.34 | 2.54 ± 0.20 $^c$ |
| $\chi^2$ | 74 | 35 | 20 | |

$^a$Weighted average of 2.80 ± 0.38 $^6$ and 2.44 ± 0.26 $^7$

$^b$Weighted average of 76 ± 8 $^73$ and 72 ± 13 $^74$

$^c$Ref. $^68$