Restricted isometry property of random subdictionaries

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Abstract—We study statistical restricted isometry, a property closely related to sparse signal recovery, of deterministic sensing matrices of size $m \times N$. A matrix is said to have a statistical restricted isometry property (StRIP) of order $k$ if most submatrices with $k$ columns define a near-isometric map of $\mathbb{R}^k$ into $\mathbb{R}^m$. As our main result, we establish sufficient conditions for the StRIP property of a matrix in terms of the mutual coherence and mean square coherence. We show that for many existing deterministic families of sampling matrices, $m = O(k)$ rows suffice for $k$-StRIP, which is an improvement over the known estimates of either $m = \Theta(k \log N)$ or $m = \Theta(k \log k)$. We also give examples of matrix families that are shown to have the StRIP property using our sufficient conditions.

I. INTRODUCTION

A. RIP matrices and binary codes

We study conditioning properties of subdictionaries motivated by the problem of faithful recovery of sparse signals from low-dimensional projections. A universal sufficient condition for reliable reconstruction of sparse signals is given by the restricted isometry property (RIP) of sampling matrices [15]. It has been shown that sparse high-dimensional signals compressed to low dimension using linear RIP maps can be reconstructed using $\ell_1$ minimization procedures such as Basis pursuit and Lasso [19], [17], [15], [12].

Let $x$ be an $N$-dimensional signal and denote by $[N] = \{1, 2, \ldots, N\}$ the set of coordinates. Below we use $\Phi$ to denote the $m \times N$ sampling matrix and write $\Phi_I$ to refer to the $m \times k$ submatrix of $\Phi$ formed of the columns with indices in $I$, where $I = \{i_1, \ldots, i_k\} \subset [N]$ is a $k$-subset of $[N]$. We say $\Phi$ is $(k, \delta)$-RIP if every $k$ columns of $\Phi$ satisfy the following near-isometry property:

$$\|\Phi_{I}^{T} \Phi_{I} - I_d\|_2 \leq \delta$$

(1)

where $I_d$ is the identity matrix, and $\|\cdot\|_2$ is the spectral norm (the largest singular value).

It is known that a $k$-RIP matrix must have at least $m = \Omega(k \log(N/k))$ rows [32], [30]. Moreover, if $x$ is compressed to a sketch $y = \Phi x$ of dimension $m$, then $m = \Omega(k \log(N/k))$ samples are required for any recovery algorithm to provide an approximation of the signal with an error guarantee expressed in terms of the $\ell_1$ or $\ell_2$ norm [33], [25] (this bound applies to signals which are not necessarily $k$-sparse). Matrices with random Gaussian or Bernoulli entries with high probability provide the best known error guarantees for recovery from sketches of dimension $m$ that matches this lower bound [19], [20], [18].

Let $\mu_{i,j} = |\langle \phi_i, \phi_j \rangle|$ be the coherence between columns $i$ and $j$ and denote by $\mu := \max_{i,j \neq i} \mu_{i,j}$ the mutual coherence parameter of the matrix $\Phi$. The relation between the mutual coherence and RIP has served the starting point in a number of studies on RIP matrix construction [41], [26]. One way of constructing incoherent dictionaries begins with taking a binary code, i.e., a set $C$ of binary $m$-dimensional vectors. We say that the code $C$ has small width if all pairwise Hamming distances between distinct vectors of $C$ are close to $m/2$. For instance, if $m/2 - w \leq d(x_i, x_j) \leq m/2 + w$ for every $x_i, x_j \in C, x_i \neq x_j$, we say that the code has width $w$. A real sampling matrix can be generated from a small-width binary code by mapping bits of the codewords to bipolar signals according to $0 \rightarrow 1, 1 \rightarrow -1$. The resulting vectors are normalized to unit length and written in the columns of the matrix $\Phi$. The coherence parameter $\mu(\Phi)$ of the matrix and the width of the code $C$ are connected by the obvious equality $w(C) = \mu(\Phi)/m/2$.

One of the first papers to put forward the idea of constructing RIP matrices from binary vectors was [24]. While it did not make a connection to error-correcting codes, a number of later papers pursued both its algorithmic and constructive aspects [6], [13], [14], [23]. Examples of codes with small width are given in [2], where they are studied under the name of small-bias probability spaces. RIP matrices obtained from the constructions in [2] satisfy $m = O\left(\frac{k \log N}{\log \log(k)}\right)^{2.5}$ for $(\log N)^{-3/2} \leq \mu \leq (\log N)^{-1/2}$. The advantage of obtaining RIP matrices from binary or spherical codes is low construction complexity: in many instances it is possible to define the matrix using only $O(\log N)$ columns while the remaining columns can be computed as their linear combinations. We also note a result of [10] that gave the first (and the only known) construction of RIP matrices with $k$ on the order of $m^{2+\epsilon}$ (i.e., greater than $O(\sqrt{m})$). An overview of the state of the art in the construction of RIP matrices is given in a recent
recent paper [9] states that StRIP is “of great potential interest for applications in sparse recovery. Indeed, papers such as [44] and Plan [16] used the same technique to prove almost exact support sparse recovery of a class of signals. Cand` es or nonlinear regression models.” [9] goes on to investigate for a wide class of problems involving high-dimensional linear sensing, taking a random collection of columns from a general dictionary. The estimates known in the literature for a range of the sparsity signal dimension that satisfy conditions discussed in Sec. [16] In general, Theorem 2.1 extends the currently known region of sufficient conditions for StRIP matrices, and for many standard sampling matrices, ensures that \( m = O(k) \) rows suffice for \( k \)-StRIP, which is an improvement over the known estimates of \( m = \Theta(k \log N) \).

Application of our results to some deterministic matrices popularized in recent literature on sparse recovery, for instance, the Delsarte-Goethals matrices [13, 14], shows that the statistical RIP property is fulfilled for a smaller sketch dimension \( m \) than previously known. We also estimate the dimensions of many other known families of matrices, deriving sufficient conditions for the statistical RIP property. Since the StRIP and statistical incoherence properties suffice for stable recovery with Basis Pursuit, our results, in turn, provide sufficient conditions for sparse recovery for many families of sampling matrices. A more detailed discussion and some further applications of our results appear in an earlier version of this paper in arXiv [7].

II. MAIN RESULT AND DISCUSSION

A. Main result

Theorem 2.1: Let \( \Phi \) be an \( m \times N \) matrix. Let \( \epsilon < \min\{1/k, e^{1-1/\log 2}\} \) and suppose that \( \Phi \) satisfies

\[
\frac{1}{\log^2(1/\epsilon)} \min \left( \frac{(1-a)^2 b^2}{32 \log(2k) \log(e/\epsilon) \epsilon^2} \right) \leq k \mu^4 \leq \frac{ab}{\log(1/\epsilon)},
\]

(2)

and

\[
k \bar{\mu}^2 \leq \frac{ab}{\log(1/\epsilon)},
\]

(3)

where \( a, b, c \in (0, 1) \) are constants such that

\[
\sqrt{a} + \sqrt{2ab} + \sqrt{c} + \frac{2k}{N} \|\Phi\|^2 \leq e^{-1/4} \delta^2 / \log 2.
\]

(4)

Then \( \Phi \) is \((k, \delta, \epsilon)\)-StRIP.

B. Comparison to earlier work

Most relevant to our results are two papers by Tropp [43, 44]. The first of them proved a nearly optimal sufficient condition for StRIP using mutual coherence and matrix norm, namely that \( \Phi \) is \((k, \delta, \epsilon)\)-StRIP if

\[
\mu = O((\log N)^{-1}) \quad \text{and} \quad \|\Phi\|^2 = O\left(\frac{N}{k \log N}\right).
\]

(5)
where the constants that depend on $\delta$ are absorbed into $O(\cdot)$. For the above result to hold, $\epsilon$ has to be less than $1/k$, just as in Thm. 2.1 above. The restriction on $\mu$ is very mild, while the condition on $\|\Phi\|$ can be further improved. Namely, (44) shows that the conditions
\[
\mu = O((k \log k)^{-1/2}) \text{ and } \|\Phi\|^2 = O \left( \frac{N}{k} \right)
\] (6)
suffice for the $(k, \delta, \epsilon)$-StRIP property. Note that the improvement for $\|\Phi\|$ in (6) over (5) is obtained at the expense of tightening the condition on the coherence. For this reason, conditions (5) are better suited for verifying the StRIP property of deterministic matrices.

Equations (5) and (6) together define the currently known region of sufficient conditions for StRIP matrices. The contribution of Theorem 2.1 is to further extend this region by including matrices that satisfy
\[
\mu = O((k \log k)^{-1/4}), \quad \bar{\mu}^2 = O(1/k) \text{ and } \|\Phi\|^2 = O \left( \frac{N}{k} \right).
\] (7)

We can claim an improvement over the results of [43] when inequality (7) is better than (5) (in the sense that a smaller value of $m$ is required for the conditions to be satisfied). Most known examples of deterministic sampling matrices, including the examples in Sect. IV below, have mean square coherence of order $\bar{\mu}^2(\Phi) = O(1/m)$, coherence $\mu = \frac{1}{\sqrt{m}}$ and spectral norm $\|\Phi\|^2 \leq \frac{N}{m}$. Hence the most restrictive constraint of the three conditions in (7) is the last one, and (7) essentially reduces to the constraint $m = \Theta(k)$ for many standard sampling matrix families. On the other hand, (5) reduces to the constraint $m = \Theta(k \log N)$ for the same reason. Note that the most restrictive condition in (5) is the first one which gives rise to the constraint $m = \Theta(k \log k)$ for the sampling matrices of Sect. IV.

The sufficient condition on the coherence $\mu$ implied by (7) is
\[
\mu = O((k \log k)^{-1/4}),
\] (8)
which by itself is an improvement over the coherence condition of (5) if $k \log k = O(\log^4 N)$. In the next subsection we discuss a concrete family of sampling matrices for which our results yield better parameters than the conditions known previously.

Apart from this, we also note that imposing the StRIP condition together with the statistical incoherence condition, or SINC (defined below), suffices to prove stable sparse recovery by Basis Pursuit. This observation, which is an extension of known results, is included in the Appendix. We list examples of dictionaries that meet the StRIP and SINC conditions in Sect. IV.

C. Example: Delsarte-Goethals codes

A class of sensing matrices that satisfy the condition of Theorem 2.1 comes from a family of binary codes called the Delsarte-Goethals codes which are certain nonlinear subcodes of the second-order Reed-Muller codes; see [35], Ch. 15. Suppose that the length of the chosen code is $m$. Writing the code vectors as columns of the matrix and replacing $0$ with $1/\sqrt{m}$ and $1$ with $-1/\sqrt{m}$, we obtain the following parameters:
\[
m = 2^{2s+2}, \quad N = 2^{-r}m^{r+2}, \quad \mu = 2^r m^{-1/2}
\] (9)
where $s \geq 0$ is any integer, and where for a fixed $s$, the parameter $r$ can be any number in $\{0, 1, \ldots, s-1\}$. If we take $s$ to be such that $s + 1$ is divisible by $3$ and set $r = (s+1)/3$, then we obtain,
\[
m = 2^{6r}, \quad N = 2^{6r^2+11r}, \quad \mu = 2^{-2r} = m^{-1/3}.
\]

An easy calculation that relies on the Pliss identities for binary codes (e.g. [35], p. 132] shows that
\[
\bar{\mu}^2 = \frac{N - m}{m(N - 1)} < \frac{1}{m}.
\] (10)

Using the properties of the Delsarte-Goethals codes, it is easy to see that the norm of the sampling matrix $\Phi$ is $\|\Phi\| = \sqrt{N/m}$. Employing condition (8), we observe that $m = O(k \log k)$ samples suffice for this matrix to satisfy the $(k, \delta, 1/k)$-StRIP condition while (5) requires $m = O(k \log N)$. If $m$ is fixed as above, this implies that using our results we can claim the StRIP property for larger $k$ that was previously known.

III. PROOF OF THE MAIN RESULT

A. Notation

Let $\Phi$ denote the $m \times N$ real sensing matrix with columns of unit norm. By $\mathcal{P}_k(N)$ we denote the set of all $k$-subsets of $[N]$. The usual notation for probability $\Pr$ is used to refer a probability measure when there is no ambiguity. At the same time, we use separate notation for some frequently encountered probability spaces. In particular, we use $P_{R_k}$ to denote the uniform probability distribution on $\mathcal{P}_k(N)$. We also use $P_{R_k}$ to denote the uniform distribution on the set $R_k := \{(I, j) : |I| = k, I \subseteq [N], j \in I^c\}$.

To express our results concisely we introduce the following concept.

Definition 3.1: An $m \times N$ matrix $\Phi$ is said to satisfy a statistical incoherence condition (is $(k, \alpha, \epsilon)$-SINC) if
\[
\Pr_k (\{I \in \mathcal{P}_k(N) : \max_{i\in I} \|\Phi^T \phi_i\|^2 \leq \alpha\}) \geq 1 - \epsilon.
\] (11)

This condition is discussed in [29], [42], and more explicitly in [43]. Following [43], it appears in the proofs of sparse recovery in [16] and below in this paper.

The reason that (11) is less restrictive than the constraint on the coherence parameter $\mu(\Phi)$ is as follows. The columns of $\Phi$ can be considered as points in the real projective space $\mathbb{R}^{P_m-1}$. Recall that $\mu(\Phi) = \min_{i \neq j} \|\phi_i, \phi_j\|$. The columns of a matrix $\Phi$ with small $\mu(\Phi)$ form a packing of the space with large pairwise separation between the points. Such a packing cannot contain too many elements as not to contradict universal bounds on packings of $\mathbb{R}^{P_m-1}$. At the same time, for the norm $\|\Phi^T \phi_i\|_2$ to be large it is necessary that a given column is close to the majority of the $k$ vectors from the set $I$, which is easier to rule out.
B. Sufficient conditions for statistical incoherence properties

We begin with establishing a sufficient condition for the SINC property in terms of the coherence parameters of \( \Phi \). This result is not necessarily stronger than the result of [43], but is essential in proving our main theorem.

**Theorem 3.1:** Let \( \Phi \) be an \( m \times N \) matrix with unit-norm columns, coherence \( \mu \) and mean square coherence \( \mu^2 \).

\[
\mu^4 \leq \frac{(1-a)2\beta^2}{32k(\log 2N/e)^3} \quad \text{and} \quad \mu^2 \leq \frac{a\beta}{k\log(2N/e)},
\]

where \( \beta > 0 \) and \( 0 < a < 1 \) are any constants. Then \( \Phi \) has the \((k, \alpha, e)\)-SINC property with \( \alpha = \beta/\log(2N/e) \).

Before proving this theorem we will introduce some notation. Fix \( j \in [N] \) and let \( I_j = \{i_1, i_2, \ldots, i_k\} \) be a random \( k \)-subset such that \( j \notin I_j \). The subsets \( I_j \) are chosen from the set \([N]\setminus j\) with uniform distribution. Define random variables \( Y_{j,t} = \mu_{j}^2 I_{j,i} l = 1, \ldots, k \). Next define a sequence of random variables \( Z_{j,t}, t = 0, 1, \ldots, k \), where

\[
Z_{j,0} = \mathbb{E}_j \left( \sum_{l=1}^{k} Y_{j,l} \right),
\]

\[
Z_{j,t} = \mathbb{E}_j \left( \sum_{l=1}^{k} Y_{j,l} \mid Y_{j,1}, Y_{j,2}, \ldots, Y_{j,t-1} \right), \quad t = 1, 2, \ldots, k.
\]

For \( t = 1, \ldots, k \), let

\[
Z_{t} = E_{j} Z_{j,t} = E_{R^*_k} \left( \sum_{l=1}^{k} Y_{j,l} \mid Y_{j,1}, Y_{j,2}, \ldots, Y_{j,t} \right),
\]

where \( R^*_k \) is defined in Section [III-A].

Let us show that the random variables \( Z_t \) form a Doob martingale. Begin with defining a sequence of \( \sigma \)-algebras \( \mathcal{F}_t, t = 0, 1, \ldots, k \), where \( \mathcal{F}_0 = \{0, [N]\} \) and \( \mathcal{F}_t, t \geq 1 \) is the smallest \( \sigma \)-algebra with respect to which the variables \( Y_{j,1}, \ldots, Y_{j,t} \) are measurable (thus, \( \mathcal{F}_t \) is formed of all subsets of \([N]\) of size \( \leq t+1 \)). Clearly, \( \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_k \), and for each \( t \), \( Z_t \) is a bounded random variable that is measurable with respect to \( \mathcal{F}_t \). Observe that

\[
Z_0 = E_{j} Z_{j,0} = E_{R^*_k} \left( \mu_{j}^2 \right) = \sum_{l=1}^{k} E_{R^*_k} \mu_{j,l}^2 \leq k \mu^2. \tag{13}
\]

The next two lemmas are useful in proving Theorem [3.1].

**Lemma 3.2:** The sequence \( (Z_t, \mathcal{F}_t)_{t=0,1,\ldots,k} \) forms a bounded-differences martingale, namely \( E_{R^*_k} (Z_t \mid Z_0, Z_1, \ldots, Z_{t-1}) = Z_{t-1} \) and

\[
|Z_t - Z_{t-1}| \leq 2 \mu^2 \left( 1 + \frac{k}{N-k-2} \right), \quad t = 1, \ldots, k.
\]

**Proof:** In the proof we write \( \mathbb{E} \) instead of \( E_{R^*_k} \). We have

\[
Z_t = \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_t \right) = \sum_{l=1}^{k} Y_{j,l} + \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_t \right)
\]

\[
= Z_{t-1} + Y_{j,t} + \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_t \right) - \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1} \right).
\]

Next,

\[
\mathbb{E}(Z_t \mid Z_0, Z_1, \ldots, Z_{t-1}) = Z_{t-1} + \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid Z_0, Z_1, \ldots, Z_{t-1} \right)
\]

\[
+ \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_t \right) - \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1} \right) - \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1} \right) = Z_{t-1}.
\]

\[
= Z_{t-1} + \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid Z_0, Z_1, \ldots, Z_{t-1} \right)
\]

\[
+ \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_t \right) - \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1} \right) = Z_{t-1},
\]

which is what we claimed.

Next we prove a bound on the random variable \( |Z_t - Z_{t-1}| \).

We have

\[
|Z_t - Z_{t-1}| = \left| \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_t \right) - \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1} \right) \right|
\]

\[
\leq \max_{a,b} \left| \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = a \right) - \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = b \right) \right|
\]

\[
= \max_{a,b} \left| \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = a \right) - \mathbb{E} \left( \sum_{l=1}^{k} Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = b \right) \right|
\]

\[
= \max_{a,b} \left| a - b + \sum_{l=1}^{k} \left( \mathbb{E} \left( Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = a \right) - \mathbb{E} \left( Y_{j,l} \mid \mathcal{F}_{t-1}, Y_{t,l} = b \right) \right) \right|
\]

\[
\leq 2 \mu^2 + \sum_{l=1}^{k} \frac{2 \mu^2}{N-l-2}
\]

\[
= 2 \mu^2 \frac{N-2}{N-k-2}.
\]

**Proposition 3.3:** (Azuma-Hoeffding, e.g., [38]) Let \( X_0, \ldots, X_{k-1} \) be a martingale with \( |X_i - X_{i-1}| \leq a_i \) for each \( i \), for suitable constants \( a_i \). Then for any \( \nu > 0 \),

\[
\Pr \left( \left| \sum_{i=1}^{k-1} (X_i - X_{i-1}) \right| \geq \nu \right) \leq 2 \exp \left( -\frac{\nu^2}{2 \sum a_i^2} \right).
\]

**Proof of Theorem [3.7]** Bounding large deviations for the sum \( \sum_{i=1}^{k} (Z_k - Z_{k-1}) = |Z_k - Z_0| \), we obtain

\[
\Pr(|Z_k - Z_0| > \nu) \leq 2 \exp \left( -\frac{\nu^2}{8 \mu^4 k (N-k-2)^2} \right), \tag{14}
\]

where the probability is computed with respect to the choice of ordered \((k+1)\)-tuples in \([N]\) and \( \nu > 0 \) is any constant.
Using (13) and the inequality \((N - 2)/(N - k - 2) < 2\) valid for all \(k < \frac{N}{2} - 1\), we obtain

\[
\Pr(Z_k \geq \nu + k\bar{\mu}^2) \leq \Pr(|Z_k - k\bar{\mu}| \geq \nu) \leq 2 \exp \left(-\frac{\nu^2}{32\mu^4_k}\right).
\]

Now take \(\beta > 0\) and \(\nu = \frac{\beta}{\log(2N/\epsilon)} - k\bar{\mu}^2\). Suppose that for some \(a \in (0, 1)\)

\[
k_k^4 \leq \frac{(1 - \alpha/\beta)^2}{32} \left(\frac{2N}{\epsilon}\right)^{-3} \quad \text{and} \quad k_k^2 \leq \frac{a\beta}{\log(2N/\epsilon)},
\]

then we obtain

\[
\Pr \left(\|\Phi^T_l \phi_j\|^2 \geq \frac{\beta}{\log(2N/\epsilon)}\right) \leq 2 \exp \left(-\frac{\nu^4}{32\mu^4_k}\right) \leq \frac{\epsilon}{N}.
\]

Now the first claim of Theorem 3.1 follows by the union bound with respect to the choice of the index \(j\).

The above proof contains the following statement.

**Corollary 3.4:** Let \(\Delta\) be an \(m \times N\) matrix with mutual coherence \(\mu\) and mean square coherence \(\mu^2\). Let \(a \in (0, 1)\) and \(\beta > 0\) be any constants. Suppose that for \(\alpha < \beta\log_2 e\),

\[
\mu^4 \leq \frac{(1 - \alpha)^2\alpha^3}{32\beta^2}, \quad k_k^2 \leq a\alpha.
\]

Then \(P_{R_k}(\sum_{l=1}^k \vec{R}_l^2 \geq \alpha) \leq 2e^{-\alpha/\alpha}\).

**Proof:** Denote \(\alpha = \beta/\log(2N/\epsilon)\), then \(\epsilon/N = 2e^{-\alpha/\alpha}\). The claim is obtained by substituting \(\alpha\) in (15)-(16).

C. **Proof of Theorem 2.1**

We are now ready to prove the main Theorem 2.1. The proof relies on several results from [44]. The following theorem is a modification of Theorem 25 in that paper. Below \(\Delta\) denotes a linear operator that performs a restriction to \(k\) coordinates chosen according to some rule (e.g., randomly). Its domain is determined by the context. Its adjoint \(\Delta^\ast\) acts on \(\mathbb{R}^k\) by padding the \(k\)-vector with the appropriate number of zeros.

**Theorem 3.5:** (Decoupling of the spectral norm) Let \(\Delta\) be a \(2N \times 2N\) symmetric matrix with zero diagonal. Let \(\eta \in \{0, 1\}^{2N}\) be a random vector with \(\mathbb{N}\) components equal to one. Define the index sets \(T_1(\eta) = \{i : \eta_i = 0\}\) and \(T_2(\eta) = \{i : \eta_i = 1\}\). Let \(\Delta\) be a random restriction to \(k\) coordinates. For any \(q \geq 1\) we have

\[
(\mathbb{E}[\|\Delta R_k \|^q])^{1/q} \leq 2 \max_{k_1 + k_2 = k} \mathbb{E}_\eta(\mathbb{E}[\|R_1 A_T(\eta) T(\eta) R_2\|^q])^{1/q},
\]

(17)

where \(A_T(\eta) T(\eta)\) denotes the submatrix of \(A\) indexed by \(T_1(\eta) \times T_2(\eta)\) and the matrices \(R_k\) are independent restrictions to \(k_1\) coordinates from \(T_1, i = 1, 2\).

When \(\Delta\) has order \((2N + 1) \times (2N + 1)\), then an analogous result holds for partitions into blocks of size \(N\) and \(N + 1\). Inequality (17) appeared in the proof of the decoupling theorem, Theorem 9 in [44]. The ideas behind it are due to [44].

The next lemma is due to Tropp [43] and Rudelson and Vershinin [40].

**Lemma 3.6:** Suppose that \(\Delta\) is a matrix with \(N\) columns and let \(R\) be a random restriction to \(k\) coordinates. Let \(q \geq 2, p = \max(2, 2 \log(\text{rk} \Delta A^\ast), q/2)\). Then

\[
(\mathbb{E}[\|A^\ast\|^q])^{1/q} \leq 3\sqrt{p}(\mathbb{E}[\|A^\ast\|^q_{1\rightarrow 2}])^{1/q} + \sqrt{\frac{k}{N}}\|A\|
\]

where \(\|\cdot\|_{1\rightarrow 2}\) is the maximum column norm.

The following lemma is a simple generalization of Proposition 10 in [44]. The only difference is that we allow the \(\xi_\eta\) below to be a function of \(q\) instead of a constant.

**Lemma 3.7:** Let \(q, \lambda > 0\) and let \(\xi_\eta\) be a positive function of \(q\). Suppose that \(Z\) is a positive random variable whose \(q\)th moment satisfies the bound

\[
(\mathbb{E} Z^q)^{1/q} \leq \xi_\eta \sqrt{q} + \lambda.
\]

Then

\[
P(Z \geq e^{1/q}(\xi_\eta \sqrt{q} + \lambda)) \leq e^{-q/4}.
\]

**Proof:** By the Markov inequality,

\[
P \left(Z \geq e^{1/q}(\xi_\eta \sqrt{q} + \lambda)\right) \leq \frac{\mathbb{E}(Z^q)}{e^{1/q}(\xi_\eta \sqrt{q} + \lambda)^q}
\]

\[
\leq \left(\frac{\xi_\eta \sqrt{q} + \lambda}{e^{1/q}(\xi_\eta \sqrt{q} + \lambda)}\right)^q = e^{-q/4}.
\]

The main part of the proof is contained in the following lemma.

**Lemma 3.8:** Let \(\Delta\) be an \(m \times N\) matrix with mutual coherence parameter \(\mu\). Suppose that for some \(0 < \epsilon_1, \epsilon_2 < 1\)

\[
P_{R_k}(\{i, j : \|\Phi^T_l \phi_i\|^2 \geq \epsilon_1\} \mid i) \leq \epsilon_2.
\]

(18)

Let \(R\) be a random restriction to \(k\) coordinates and \(H = \Phi^T \Phi - Id\). For any \(q \geq 2, p = \max(2, 2 \log(\text{rk} \Delta R_k^\ast), q/2)\) we have

\[
(\mathbb{E}[\|R_k \|^q])^{1/q} \leq \frac{6\sqrt{p}(\sqrt{q} + (k\epsilon_2)^{1/q})\mu\sqrt{k}}{2 + 2k\mu^2} \leq \frac{2k^2}{N}\|\Phi\|^2.
\]

(19)

**Proof:** We begin with setting the stage to apply Theorem 3.5. Let \(\eta \in \{0, 1\}^N\) be a random vector with \(N/2\) ones and let \(R_1, R_2\) be random restrictions to \(k_1\) coordinates in the sets \(T_i(\eta), i = 1, 2\), respectively. Denote by \(\text{supp}(R_k)\) and \(R\) be a set of indices selected by \(R_k\) and \(H(\eta) := H_T(\eta) \times T_2(\eta)\). Let \(q \geq 1\) and let us bound the term \(\mathbb{E}_\eta(\mathbb{E}[\|R_1 H(\eta) R_2\|^q])^{1/q}\) that appears on the right side of (17). The expectation in the \(q\)-norm is computed for two random restrictions \(R_1, R_2\) that are conditionally independent given \(\eta\). Let \(\mathbb{E}_k\) be the expectation with respect to \(R_k, i = 1, 2\). Given \(\eta\) we can evaluate these expectations in succession and apply Lemma 3.6 to \(\mathbb{E}_k\) :
Recall that computed with respect to it. We can write that the last expression. Let \( R \) be the random vector conditional on the choice of \( k_2 \) coordinates. The sample space for \( \eta(R_2) \) is formed of all the vectors \( \eta \in \{0,1\}^N \) such that \( \text{supp}(R_2) \subset T_2(\eta) \). In other words, this is a subset of the sample space \( \{0,1\}^N \) that is compatible with a given \( R_2 \). The random restriction \( R_1 \) is still chosen out of \( T_1(\eta) \) independently of \( R_2 \). Denote by \( R \) a random restriction to \( k_1 \) indices in the set \( \{\text{supp}(R_2)\}^\perp \) and let \( E \) be the expectation computed with respect to it. We can write

\[
E_\eta \left( E_1 E_2 \left\| R_1 H(\eta) R_2^* \right\|_{1 \to 2}^{q/2} \right)^{1/q} \leq \left( E_\eta E_1 E_2 \left\| R_1 H(\eta) R_2^* \right\|_{1 \to 2}^{q/2} \right)^{1/q}
\]

Recall that \( H_{ij} = \mu_{ij} I_{\{i \neq j\}} \) and that \( R \) and \( R_2 \) are 0-1 matrices. Using this in the last equation, we obtain

\[
E_\eta \left( \text{supp}(R) \right) \left( \sum_{i \in \text{supp}(R)} \mu_{ij} \right)^{q/2}
\]

Now let us invoke assumption \( 18 \). Recalling that \( k_1 < k_2 \), we have

\[
P_{R_2,\tilde{R}} \left( \max_{j \in \text{supp}(R_2)} \sum_{i \in \text{supp}(\tilde{R})} \mu_{ij}^2 \geq \epsilon_1 \right) \leq k_2 \epsilon_2.
\]

Thus with probability \( 1 - k_2 \epsilon_2 \) the sum in \( 21 \) is bounded above by \( \epsilon_1 \). For the other instances we use the trivial bound \( k_1 \mu^2 \). We obtain

\[
3 \sqrt{p} E_\eta \left( E_1 \left\| R_1 H(\eta) R_2^* \right\|_{1 \to 2}^{q/2} \right)^{1/q}
\]

where in the last step we used the inequality \( a^q + b^q \leq (a + b)^q \) valid for all \( q \geq 1 \) and positive \( a, b \). Let us turn to the second term on the right-hand side of \( 20 \). We observe that

\[
\left\| H(\eta)^* R_1^* \right\|_{1 \to 2} = \max_{j \in \text{supp}(\eta)} \left\| H_j, T_2(\eta) \right\|_2
\]

where \( H_j \) denotes the \( j \)-th row of \( H \) and \( H_j, T_2(\eta) \) is a restriction of the \( j \)-th row to the indices in \( T_2(\eta) \).

Finally, the third term in \( 20 \) can be bounded as follows:

\[
\sqrt{\frac{4k_1 k_2}{N^2}} E_\eta \left\| H(\eta) \right\| \leq \sqrt{\frac{(k_1 + k_2)^2}{N^2}} \left\| H \right\| = \frac{k}{N} \Phi^T \Phi - I_N
\]

where the last step uses the fact that the columns of \( \Phi \) have unit norm, and so \( \Phi^2 \geq N/m > 1 \).

Combining all the information accumulated up to this point in \( 20 \), we obtain

\[
E_\eta \left( E_1 E_2 \left\| R_1 H(\eta) R_2^* \right\|_{1 \to 2}^{q/2} \right)^{1/q} \leq 3 \sqrt{p} \left( \sqrt{\epsilon_1} + (k_2 \epsilon_1)^{1/4} \sqrt{\frac{1}{k} \Phi^T \Phi} \right)
\]

Finally, use this estimate in \( 17 \) to obtain the claim of the lemma.

**Proof of Theorem 2.1** The strategy is to fix a triple \( a, b, c \in (0,1) \) that satisfies \( 4 \) and to prove that \( 3 \) implies \((k,\beta,c)\text{-SiRIP}\). Let \( \epsilon_1 = \frac{\log b}{k-1+\log c} \) and \( \epsilon_2 = k^{-1+\log c} \). In Corollary 3.4 set \( \alpha = \epsilon_1 \) and \( \beta = \alpha \log(2/\epsilon_2) \). Under the assumptions in \( 4 \) this corollary implies that

\[
P_{R^*} \left( \sum_{m=1}^k \mu_{m,m}^2 > \epsilon_1 \right) < \epsilon_2.
\]

Introducing the following quantities:

\[
\xi_q = 3 \sqrt{2} \left( \sqrt{\epsilon_1} + (k_2 \epsilon_1)^{1/4} \sqrt{k} \right)
\]

Now \( 22 \) matches the assumption of Lemma 3.7 and we obtain

\[
P_{R^*} \left( \left\| RHR^* \right\| \geq e^{1/4} (\xi_q \left\| \Phi \right\| + \lambda) \right) \leq e^{-q/4}.
\]

Choose \( q = 4 \log(1/\epsilon) \), which is consistent with our earlier assumptions on \( k, q, \) and \( \epsilon \). With this, we obtain

\[
P_{R^*} \left( \left\| RHR^* \right\| \geq e^{1/4} (\xi_q \left\| \Phi \right\| + \lambda) \right) \leq \epsilon.
\]

Now observe that \( \left\| RHR^* \right\| \leq \delta \) is precisely the RIP property for the support identified by the matrix \( R \). Let us verify that the inequality

\[
6 \sqrt{2} \left( \sqrt{\epsilon_1} + (k_2 \epsilon_1)^{1/4} \sqrt{k} \mu^2 \right)
\]

is equivalent to \( 4 \). This is shown by substituting \( \epsilon_1 \) and \( \epsilon_2 \) with their definitions, and \( \mu \) and \( \mu^2 \) with their bounds in statement of Theorem. Thus, \( P_{R^*} \left( \left\| RHR^* \right\| \geq \delta \right) \leq \epsilon \), which establishes the SiRIP property of \( \Phi \).

**IV. EXAMPLES AND EXTENSIONS**

**A. Examples of sampling matrices.**

It is known [27] that experimental performance of many known RIP sampling matrices in sparse recovery is far better than predicted by the theoretical estimates. Theorems 3.1 and 2.1 provide some insight into the reasons for such behavior. As an example, take binary matrices constructed from the Delsarte-Goethals codes mentioned previously. The sampling
matrices \( \Phi \) obtained from them are coherence-invariant. If we take \( s \) to be an odd integer and set \( r = (s + 1)/2 \), then we obtain for this family of matrices the parameters
\[
m = 2^r, \quad N = 2^{4r^2 + 7r}, \quad \mu = m^{-1}/4.
\]
As noted above, we have \( \tilde{\mu}^2 < 1/m \) and \( \| \Phi \| = \sqrt{N/m} \). Thus for \( \mu \) and \( \tilde{\mu}^2 \) to satisfy the assumptions in Theorems 3.1 and 2.1 we need \( m, N, \) and \( k \) to satisfy the relation
\[
m = \Theta(k \log^3 N/\tau)
\]
which is nearly optimal for sparse-recovery. Note that to satisfy just the assumptions of Thm. 2.1 we can construct a Delsarte-Goethals matrix with shorter column length of \( m = O(k \log k) \), see Section II.C.

Similar logic leads to derivations of such relations for other matrices. We summarize these arguments in the next proposition, which shows that matrices with nearly optimal sketch length support high-probability recovery of sparse signals chosen from the generative signal model (more on sparse recovery in the Appendix; see in particular Theorem A.1).

Definition 4.1: We say that a signal \( x \in \mathbb{R}^N \) is drawn from a generic random signal model \( S_k \) if
1) The locations of the \( k \) coordinates of \( x \) with largest magnitudes are chosen among all \( k \)-subsets \( I \subset [N] \) with a uniform distribution;
2) Conditional on \( I \), the signs of the coordinates \( x_i, i \in I \) are i.i.d. uniform Bernoulli random variables taking values in the set \( \{1, -1\} \).

Proposition 4.1: Let \( \Phi \) be an \( m \times N \) sampling matrix. Suppose that it has coherence parameters \( \mu = O(m^{-1/4}) \), \( \tilde{\mu}^2 = O(m^{-1}) \), and
\[
\| \Phi \| = O(\sqrt{N/k}).
\]
If \( m = \Theta(k (\log(N/\epsilon)^3) \) and \( k < 1/\epsilon \), then \( \Phi \) supports sparse recovery under Basis Pursuit for all but an \( \epsilon \) proportion of \( k \)-sparse signals chosen from the generic random signal model \( S_k \).

We remark that the conditions on mean square coherence are generally easy to achieve. As seen from Table I below, they are satisfied by most examples considered in the existing literature, including both random and deterministic constructions. The most problematic quantity is the mutual coherence parameter \( \mu \). It might either be large itself, or have a large theoretical bound. Compared to earlier work, our results rely on a more relaxed condition on \( \mu \), enabling us to establish near-optimality for new classes of matrices. For readers’ convenience, we summarize in Table 1 a list of such optimal matrices along with several of their useful properties. A systematic description of all but the last two classes of matrices can be found in [4]. Therefore we limit ourselves to giving definitions and performing some not immediately obvious calculations of the newly defined parameter, the mean square coherence.

**Normalized Gaussian Frames.** A normalized Gaussian frame is obtained by normalizing each column of a Gaussian matrix with independent, Gaussian-distributed entries that have zero mean and unit variance. The mutual coherence and spectral norm of such matrices were characterized in [4] (see Table I). These results together with the relation \( \tilde{\mu}^2 < \mu^2 \) lead to a trivial upper bound on \( \tilde{\mu}^2 \), namely \( \tilde{\mu}^2 \leq 15 \log N/m \). Since this bound is already tight enough for \( \tilde{\mu}^2 \) to satisfy the assumption of Proposition 4.1 and to avoid distraction from the main goals of the paper, we made no attempt to refine it here.

**Random Harmonic Frames.** Let \( F \) be an \( N \times N \) discrete Fourier transform matrix, i.e., \( F_{jk} = \frac{1}{\sqrt{N}} e^{2 \pi i jk/N} \). Let \( \eta_i, i = 1, \ldots, N \), be a sequence of independent Bernoulli random variables with mean \( \frac{1}{2} \). Set \( M = \{ i : \eta_i = 1 \} \) and use \( F_M \) to denote the submatrix of \( F \) whose row indices lies in \( M \). Then the random matrix \( \sqrt{m} F_M \) is called a random harmonic frame [29, 17]. In the next proposition we compute the mean square coherence for all realizations of this matrix.

**Proposition 4.2:** All instances of the random harmonic frames are coherence invariant with the following mean square coherence
\[
\tilde{\mu}^2 = \frac{N - |M|}{(N - 1)|M|}.
\]
Proof: For each \( t \in [|M|] \), let \( a_t \) with be the \( t \)-th member of \( M \). To prove coherence invariance, we only need to show that \( \{ \mu_{jk} : k \in [N] \setminus j \} = \{ \mu_{N,k} : k \in [N - 1] \} \) holds for all \( j \in [N] \). This is true since
\[
\mu_{j,k} = \frac{1}{|M|} \left| \sum_{t=1}^{\left| M \right|} e^{2 \pi i (j - k) a_t / N} \right| = \mu_{N, (k-j+N) \mod N} \quad \text{for all } k \neq j.
\]
In words, the \( k \)-th coherence in the set \( \{ \mu_{j,k} : k \in [N] \setminus j \} \) is exactly the \( (k-j+N \mod N) \)-th coherence in \( \{ \mu_{N,k} : k \in [N-1] \} \), therefore the two sets are equal. We proceed to calculate the mean square coherence,
\[
\tilde{\mu}^2 = \frac{1}{N(N-1)|M|^2} \sum_{j \neq k, k, k=1}^{N} \left| \sum_{t=1}^{\left| M \right|} e^{2 \pi i (j-k) a_t / N} \right|^2 = \frac{1}{N(N-1)|M|^2} \sum_{j \neq k, k, k=1}^{N} \sum_{t=1}^{\left| M \right|} e^{2 \pi i (j-k) (a_t_1-a_t_2) / N} + \sum_{t_1 \neq t_2, t_1, t_2=1}^{\left| M \right|} \sum_{k=1}^{N} e^{2 \pi i (j-k) (a_{t_1}-a_{t_2}) / N} = \frac{1}{N(N-1)|M|^2} (N(N-1)|M| - |M|(|M|-1)N) = \frac{N - |M|}{(N - 1)|M|}.
\]

**Chirp Matrices.** Let \( m \) be a prime. An \( m \times m \) “chirp matrix” \( \Phi \) is defined by \( \Phi_{t,am+b} = \frac{1}{m} e^{2 \pi i (bt + at)/m} \) for \( t, a, b = 1, \ldots, m \). The coherence between each pairs of column vectors is known to be
\[
\mu_{jk} = \frac{1}{\sqrt{m}} \quad (j \neq k),
\]
from which we immediately obtain the inequalities \( \mu \leq 1/\sqrt{m} \) and \( \tilde{\mu}^2 \leq 1/m \). More details on these frames are given, e.g.,
Equiangular tight frames (ETFs): A matrix $\Phi$ is called an ETF if its columns $\{\varphi_i \in \mathbb{R}^m, i = 1, \ldots, N\}$ satisfy the following two conditions:

1. $\|\varphi_i\|_2 = 1$, for $i = 1, \ldots, N$.
2. $\mu_{ij} = \frac{\sqrt{N-m}}{m(N-1)}$, for $i \neq j$.

From this definition we obtain $\mu = \frac{\sqrt{N-m}}{m(N-1)}$ and $\theta = \frac{\mu^2}{\sqrt{m(N-1)}}$. The entry in the table also covers the recent construction of ETFs from Steiner systems [28].

Reed-Muller matrices: In Table II we list two tight frames obtained from binary codes. The Reed-Muller matrices are obtained from certain special subcodes of the second-order Reed-Muller codes [35]; their coherence parameter $\mu$ is found in [4] and the mean square coherence is found from [10]. The Delsarte-Goethals matrices are also based on some subcodes of the second order Reed-Muller codes and were discussed earlier in this section. Both dictionaries form unit-norm tight frames (the rows of the matrix $\Phi$ are pairwise orthogonal), with a consequence that $\|\Phi\|_2 = \sqrt{N/m}$. We include these two examples out of many other possibilities based on codes because they appear in earlier works, and because their parameters are in the range that fits well our conditions.

We note that the quaternary version of these frames is also of interest in the context of sparse recovery; see in particular [13].

Deterministic sub-Fourier Construction [37]: Let $p > 2$ be a prime, and let $f(x) \in \mathbb{F}_p[x]$ be a polynomial of degree $d > 2$ over the finite field $\mathbb{F}_p$. Suppose that $m$ is some integer satisfying $p^{1/(d-1)} \leq m \leq p$. Then we can construct an $m \times p$ deterministic RIP matrix from a $p \times p$ DFT matrix by keeping only the rows with indices in $\{f(m)(\text{mod } p), m = 1, \ldots, m\}$, and normalizing the columns of the resulting matrix. These submatrices form tight frames, and so their spectral norms can be easily verified to be $\sqrt{p/m}$. It is known [31] that this matrix has mutual coherence no greater than $\epsilon^{-3d}m^{-1/(9d^2 \log d)}$. Even though this bound is an artifact of the proof technique used in [31], there seem to be no obvious ways of improving it.

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**APPENDIX**

Among the most studied estimators for sparse recovery is the Basis Pursuit algorithm [22]. This is an $\ell_1$-minimization algorithm that provides an estimate of the signal through solving a convex programming problem

$$\hat{x} = \arg \min \| \tilde{x} \|_1 \text{ subject to } \Phi \tilde{x} = y.$$  

In this section we prove approximation error bounds for recovery by Basis Pursuit from linear sketches obtained using deterministic matrices with the StRIP and SINC properties.

It was proved in [44] that random sparse signals sampled using matrices with the StRIP property can be recovered with high probability from low-dimensional sketches using linear programming. Theorem A.1 below generalizes this result to signals that are not necessarily sparse. Its proof essentially follows from [20] with an extra calculation of the failure rate stemming from replacing the hard RIP condition with its statistical version. It is presented here for reader’s convenience.

**Theorem A.1:** Suppose that $x$ is a generic random signal from the model $S_k$. Let $y = \Phi x$ and let $\hat{x}$ be the approximation of $x$ by the Basis Pursuit algorithm. Let $I$ be the set of $k$ largest coordinates of $x$. If

1. $\Phi$ is $(k, \delta, \epsilon)$-StRIP;
2. $\Phi$ is $(k, (1-\delta)^2, \epsilon)$-SINC,

then with probability at least $1 - 3\epsilon$

$$\|x_I - \hat{x}_I\|_2 \leq \frac{1}{2\sqrt{2\log(2N/\epsilon)}} \min_{x' \text{ is } k \text{-sparse}} \|x - x'\|_1 \quad (25)$$

and

$$\|x_{I^c} - \hat{x}_{I^c}\|_2 \leq 4 \min_{x' \text{ is } k \text{-sparse}} \|x - x'\|_1 \leq \frac{1}{\epsilon} \quad (26)$$

This theorem implies that if the signal $x$ itself is $k$-sparse then the basis pursuit algorithm will recover it exactly. Otherwise, its output $\hat{x}$ will be a tight sparse approximation of $x$. Note that it is easy to join the estimates (25) and (26) into a single inequality that gives an $l_2/l_1$ error guarantee.

Theorem A.1 will follow from the next three lemmas. Some of the ideas involved in their proofs are close to the techniques used in [20]. Let $h = x - \hat{x}$ be the error in recovery of basis pursuit. In the following $I \subset [N]$ refers to the support of the $k$ largest coordinates of $x$.

**TABLE I**

| Name                      | $|\Phi|$ | Restrictions | Probability | Requirement for StRIP: $m = O(\cdot)$ |
|---------------------------|---------|--------------|-------------|-------------------------------------|
| Normalized Gaussian (G)   | $\mathbb{R}$ | $m \times N$ | $\leq \frac{\sqrt{14 \log N}}{m - \sqrt{12 \log N}}$ | $\leq \mu^2$ |
| Random harmonic (RH)      | $\mathbb{C}$ | $|\mathcal{M}| \times N$, $\frac{1}{2}m \leq |\mathcal{M}| \leq \frac{m}{2}$ | $\leq \sqrt{\frac{118(N-m) \log N}{m \cdot N}}$ | $\leq \frac{N - |\mathcal{M}|(N-1)}{|\mathcal{M}|}$ |
| Chirp (C)                 | $\mathbb{C}$ | $m \times m^2$ | $\leq \sqrt{\frac{N - m}{m(N-1)}}$ | $\mu^2$ |
| ETF (including Steiner)   | $\mathbb{C}$ | $\sqrt{N} \leq m \leq N$ | $\leq \sqrt{\frac{N-m}{m(N-1)}}$ | $\leq 2^{-s}$ |
| Reed-Muller (RM)          | $\mathbb{R}$ | $2^s \times 2^{s(1+s)}$ | $\leq 2^{r - s - 1}$ | $\leq 2^{-2s-2}$ |
| Delsarte-Goethals set (DG) | $\mathbb{R}$ | $2^{2s+2} \times 2^{s(s+1)(r+2)-r}$ | $\leq \mu d^2$ |
| Deterministic subFourier (SF) | $\mathbb{C}$ | $m \times p$ | $\leq \frac{1}{m}$ | $\leq \frac{1}{m}$ |

For more details, see [36] A. Mazumdar and A. Barg, General constructions of deterministic $(s)$ RIP matrices for compressive sampling, Proc. IEEE International Symposium on Information Theory Proceedings (ISIT), pp. 676–682, 2011.

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Lemma A.2: Let $s = 8 \log(2N/\epsilon)$. Suppose that 
\[\|\Phi_T^T \Phi_I^{-1}\| \leq \frac{1}{1-\delta}\] 
and 
\[\|\Phi_I^T \phi_i\|_2 \leq s^{-1}(1-\delta)^2\] 
for all $i \in I^c := [N] \setminus I$.

Then 
\[\|h_I\|_2 \leq s^{-1/2} \|h_{I^c}\|_1.\]

Proof: Clearly, $\Phi h = \Phi \hat{x} - \Phi x = 0$, so $\Phi_I h_I = -\Phi_{I^c} h_{I^c}$ and 
\[h_I = -\Phi_T^T \Phi_I^{-1} \Phi_T^T \Phi_I h_{I^c}.\]

We obtain
\[\|h_I\|_2 \leq \|\Phi_T^T \Phi_I^{-1}\| \|\Phi_T^T \Phi_I h_{I^c}\|_2 \leq \frac{1}{1-\delta} \sum_{i \in I^c} \|\Phi_T^T \phi_i\|_2 |h_i| \leq s^{-1/2} \|h_{I^c}\|_1,\]
as required. \hfill \blacksquare

Next we show that the error outside $I$ cannot be large. Below sgn($u$) is a ±1-vector of signs of the argument vector $u$.

Lemma A.3: Suppose that there exists a vector $v \in \mathbb{R}^N$ such that
(i) $v$ is contained in the row space of $\Phi$, say $v = \Phi_T w$;
(ii) $v_I = \text{sgn}(x_I)$;
(iii) $\|v_I\|_{\ell_\infty} \leq 1/2$.

Then 
\[\|h_{I^c}\|_1 \leq 4 \|x_{I^c}\|_1.\] (27)

Proof: By (24) we have
\[\|x\|_1 = \|x + h_I\| = \|x_I + h_I\|_1 + \|x_{I^c} + h_{I^c}\|_1 \geq \|x_I\|_1 + \langle \text{sgn}(x_I), h_I \rangle + \|h_{I^c}\|_1 - \|x_{I^c}\|_1.\]

Here we have used the inequality $\|a + b\|_1 \geq \|a\|_1 + \langle \text{sgn}(a), b \rangle$ valid for any two vectors $a, b \in \mathbb{R}^N$ and the triangle inequality. From this we obtain
\[\|h_{I^c}\|_1 \leq \langle \text{sgn}(x_I), h_I \rangle + 2 \|x_{I^c}\|_1.\]

Further, using the properties of $v$, we have
\[\langle \text{sgn}(x_I), h_I \rangle = \langle v_I, h_I \rangle = \|v_I, h_I\| \leq \|\Phi_T w, h_I\| + \|v_I, h_{I^c}\| \leq \|w, \Phi h_I\| + \|v_I\|_{\ell_\infty} \|h_{I^c}\|_1 \leq \frac{1}{2} \|h_{I^c}\|_1.\]

The statement of the lemma is now evident. \hfill \blacksquare

Now we prove that such a vector $v$ as defined in the last lemma indeed exists.

Lemma A.4: Let $x$ be a generic random signal from the model $S_k$. Suppose that the support $I$ of the $k$ largest coordinates of $x$ is fixed. Under the assumptions of Lemma A.2 the vector 
\[v = \Phi_T \Phi_I (\Phi_T^T \Phi_I)^{-1} \text{sgn}(x_I)\]
satisfies (i)-(iii) of Lemma A.3 with probability at least $1 - \epsilon$.

Proof: From the definition of $v$ it is clear that it belongs to the row-space of $\Phi$ and $v_I = \text{sgn}(x_I)$. We have $v_i = \phi_i^T \Phi_I (\Phi_T^T \Phi_I)^{-1} \text{sgn}(x_I) = (s_i, \text{sgn}(x_I))$, where 
\[s_i = (\Phi_T^T \Phi_I)^{-1} \Phi_T^T \phi_i \in \mathbb{R}^k.\]

We will show that $|v_i| \leq \frac{1}{2}$ for all $i \in I^c$ with probability $1 - \epsilon$.

Since the coordinates of $\text{sgn}(x_I)$ are i.i.d. uniform random variables taking values in the set $\{\pm 1\}$, we can use Hoeffding’s inequality to claim that 
\[P_{H^k}(\|v_i\|_1 > 1/2) \leq 2 \exp \left( -\frac{1}{8} \frac{1}{s^2} \right).\] (28)

On the other hand, for all $i \in I^c$, 
\[\|s_i\|_2 = \|\Phi_T^T \Phi_I^{-1} \Phi_T^T \phi_i\| \leq \|\Phi_T^T \Phi_I^{-1}\| \|\Phi_T^T \phi_i\|_2 \leq 1 - \delta \leq \frac{1 - \delta}{\sqrt{8 \log(2N/\epsilon)}},\]

Using the union bound, we now obtain the following relation: 
\[P_{H^k}(\|v_I\|_1 > 1/2) \leq \epsilon.\] (30)

Hence $|v_i| \leq \frac{1}{2}$ for all $i \in I^c$ with probability at least $1 - \epsilon$. \hfill \blacksquare

Now we are ready to prove Theorem A.1.

Proof of Theorem A.1: The matrix $\Phi$ is $(k, \delta, \epsilon)$-SRIP. Hence, with probability at least $1 - \epsilon$, $\|\Phi_T^T \Phi_I^{-1}\| \leq \frac{1}{1-\delta}$. At the same time, from the SINC assumption we have, with probability at least $1 - \epsilon$ over the choice of $I$, 
\[\|\Phi_T \phi_i\|_2 \leq \frac{(1-\delta)^2}{8 \log(2N/\epsilon)},\]
for all $i \in I^c$. Thus, $\Phi_I$ will have these two properties with probability at least $1 - 2\epsilon$. Then from Lemma A.2 we obtain that
\[\|h_I\|_2 \leq \frac{1}{\sqrt{8 \log(2N/\epsilon)}} \|h_{I^c}\|_1,\]
with probability $\geq 1 - 2\epsilon$. Furthermore, from Lemmas A.3, A.4 
\[\|h_{I^c}\|_1 \leq 4 \|x_{I^c}\|_1,\]
with probability $1 - \epsilon$. This completes the proof. \hfill \blacksquare

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