Power spectrum modelling of galaxy and radio intensity maps including observational effects

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ABSTRACT
Fluctuations in the large-scale structure of the Universe contain significant information about cosmological physics, but are modulated in survey datasets by various observational effects. Building on existing literature, we provide a general treatment of how fluctuation power spectra are modified by a position-dependent selection function, noise, weighting, smoothing, pixelization and discretization. Our work has relevance for the spatial power spectrum analysis of galaxy surveys with spectroscopic or accurate photometric redshifts, and radio intensity-mapping surveys of the sky brightness temperature including generic noise, telescope beams and pixelization. We consider the auto-power spectrum of a field, the cross-power spectrum between two fields and the multipoles of these power spectra with respect to a curved sky, deriving the corresponding power spectrum models, estimators, errors and optimal weights. We note that “FKP weights” for individual tracers do not in general provide the optimal weights when measuring the cross-power spectrum. We validate our models using mock datasets drawn from N-body simulations, and provide the python code we use for these tests at https://github.com/cblakeastro/intensitypower. Our treatment should be useful for modelling and studying cosmological fluctuation fields in observed and simulated datasets.

Key words: large-scale structure of Universe – surveys – methods: statistical

1 INTRODUCTION
The power spectrum of the large-scale structure of the Universe – and its dependence on scale, redshift and direction – contains significant information about the composition of the Universe and the cosmological physics governing the growth of structure with time. Modern cosmological surveys can trace this large-scale structure over large volumes, by mapping the individual redshift-space positions of galaxies or quasars, the cumulative brightness temperature of spectral emission in a region of sky using intensity mapping in radio wavebands, or the spectral absorption of background light by intervening matter.

One of the central problems in cosmological analysis is to relate these measured fluctuations in probes of large-scale structure, which are modulated by various observational effects, to the underlying matter power spectrum which encodes the important cosmological information. Relevant observational effects may include a variation in the mean background level of the fields as a function of position (the survey selection function or mask), noise due to the sampling of discrete objects or in the measured brightness temperature, smoothing of the fields in the mapping process due to the telescope resolution, pixelization or gridding technique used in analysis, and the statistical variations imprinted by the changing line-of-sight direction across the survey volume.

Moreover, we may also utilize the cross-correlation between two different observed fields which trace the same underlying matter fluctuations. Such a multi-tracer analysis offers several benefits: (1) uncorrelated noise components in the two fields will bias the amplitude of their auto-power spectra, but not their cross-power spectrum; (2) an additive systematic component afflicting one of the fields will appear in its auto-correlation but not the cross-correlation; (3) if the fields trace a common sample variance of matter fluctuations, such that their measurement errors are correlated, then the noise in some joint derived
parameters will be reduced (Seljak 2009). A valuable example of the ability of cross-correlation to mitigate systematic errors may arise in the joint analysis of 21-cm intensity mapping performed by radio telescopes and galaxy redshift surveys. Even if intensity-mapping surveys are afflicted by significant residual components of foreground emission, cross-correlation will allow the neutral hydrogen content of galaxies to be studied (Wolz et al. 2016).

The imprint of observational effects in the galaxy power spectrum has been widely modelled in the literature. Peacock & Nicholson (1991) described the additive and multiplicative effects of a survey mask on the observed Fourier coefficients of the density field. These results were extended by Feldman et al. (1994) who, starting from a general model of the galaxy density field including clustering, Poisson noise and survey selection, derived power spectrum estimators, covariance and optimal weights. These weights were extended by Percival et al. (2004) to include the dependence of clustering on luminosity, and by Smith & Marian (2015) to encapsulate the population of halos by galaxies. Related treatments of the cross-power spectrum between two galaxy tracers were presented by Smith (2009) and Blake et al. (2013). Jing (2005) modelled the effect on the estimated power spectrum of how fields are assigned to a Fast Fourier Transform (FFT) grid (see also, Cui et al. 2008). Much recent work has focussed on modelling the multipoles of the power spectrum with respect to a varying line-of-sight direction (Yamamoto et al. 2006; Beutler et al. 2014; Wilson et al. 2017; Beutler et al. 2017; Castorina & White 2018; Blake et al. 2018).

Our paper aims to review and extend these previous results by providing a general formalism relating the 2-point statistics of fluctuations in Fourier space to various observational effects. This framework may be applied to galaxy and intensity-mapping surveys, other 3D cosmological maps, and their cross-correlation. In particular, we extend the literature by deriving the imprint on the auto- and cross-power spectra of smoothing or pixelization schemes which depend on position. Such effects are particularly relevant for radio intensity maps, which may include a telescope beam, frequency channels and angular pixelization across a curved sky. We also extend the results of Feldman et al. (1994) to intensity mapping correlations and cross-correlations, by considering the general optimal weighting of fields in auto- and cross-power spectrum measurements.

Our paper is structured as follows. In Section 2 we present models for the imprint of observational effects on the fluctuation power spectra. After some introductory definitions (Sections 2.1, 2.2), we start by reviewing the relations between the Fourier transform of the fluctuation fields and their underlying power spectra, including the effects of a position-dependent selection function, noise and weights, and considering both auto- and cross-power spectra (Section 2.3). We then derive the impact on the fluctuation power spectra if the fields are smoothed or pixelized in a manner varying with position, for example by a telescope beam, redshift errors, a spherical pixelization scheme or nearest grid-point assignment (Section 2.4). We also review the effect of the discretization of the fields onto an FFT grid (Section 2.5). Finally, we summarize how the power spectra may be analysed in terms of their multipoles with respect to a varying line-of-sight direction (Section 2.6). In Section 3, we review the estimators for the auto- and cross-power spectra and their multipoles (Section 3.1), the variance in these estimators under certain approximations, and derive the general optimal weighting of the fluctuation fields for measurement of these different power spectra, providing examples for galaxy and intensity-mapping surveys and their cross-correlation (Section 3.2). In Section 4 we validate our models by computing the observed and predicted multipole power spectra of mock galaxy and intensity-mapping datasets drawn from an N-body simulation, including a variety of observational effects. We summarize our results in Section 5.

2 POWER SPECTRUM MODELLING

2.1 Fourier conventions

For clarity of the subsequent derivations, we start by noting that we adopt the following conventions for the Fourier transform of a function $f(x)$:

$$\tilde{f}(k) = \frac{1}{V} \int d^3x \, f(x) e^{ik \cdot x}, \quad f(x) = \frac{V}{(2\pi)^3} \int d^3k \, \tilde{f}(k) e^{-ik \cdot x},$$

(1)

such that $f(x)$ and $\tilde{f}(k)$ have the same units, and where $V$ is the volume of the enclosing Fourier cuboid. We define dimensionless Dirac delta functions $\delta_D$ in configuration and Fourier space such that,

$$\delta_D(x) = \frac{1}{V} \int d^3x \, e^{ik \cdot x}, \quad \delta_D(k) = \frac{V}{(2\pi)^3} \int d^3k \, e^{ik \cdot x},$$

(2)

which are applied to functions such that,

$$\frac{1}{V} \int d^3x \, f(x) \delta_D(x - x_0) = f(x_0), \quad \frac{V}{(2\pi)^3} \int d^3k \, \tilde{f}(k) \delta_D(k - k_0) = \tilde{f}(k_0).$$

(3)

2.2 Fluctuation fields

We now provide some definitions related to fluctuation fields, their correlation functions and power spectra. Consider a function $\delta(x)$ which represents the fluctuations of a field $f(x)$ with position $x$, relative to its mean “background” value across many
realizations of an ensemble (indicated by angled brackets) such that,
\[ \delta(x) = f(x) - \langle f(x) \rangle, \]  
and \[ \langle \delta(x) \rangle = 0. \] The field could represent the galaxy number density distribution \( f(x) = V n_g(x) \) (which is dimensionless) or HI brightness temperature \( f(x) = T_b(x) \) (with dimensions of temperature).\(^1\) In a simple linear bias model neglecting redshift-space distortions, the fluctuations in galaxy number density and temperature may be described by,
\[ n_g(x) = \langle n_g(x) \rangle \left[ 1 + b_g \delta_m(x) \right], \quad T_b(x) = \langle T_b(x) \rangle \left[ 1 + b_{HI} \delta_m(x) \right], \]
where \( b_g \) and \( b_{HI} \) are the linear bias of the galaxies and HI-emitting objects, respectively, and \( \delta_m \) is the underlying matter overdensity. Hence, the corresponding fluctuations are:
\[ \delta_g(x) = \langle n_g(x) \rangle b_g \delta_m(x), \quad \delta_T(x) = \langle T_b(x) \rangle b_{HI} \delta_m(x). \]
The dimensionless auto-correlation function of the field between two positions \( x \) and \( x' \) with separation \( s = x - x' \) is defined by,
\[ \xi(s) = \frac{\langle f(x) f(x') \rangle - \langle f(x) \rangle \langle f(x') \rangle}{\langle f(x) \rangle \langle f(x') \rangle} = \frac{\langle \delta(x) \delta(x') \rangle}{\langle f(x) \rangle \langle f(x') \rangle}. \]
Re-arranging Equation 7 and adding uncorrelated noise to the field with variance \( \sigma^2(x) \) as a function of position, the 2-point statistics of the fluctuations can be written in the form,
\[ \langle \delta(x) \delta(x') \rangle = \langle f(x) \rangle \langle f(x') \rangle \xi(x - x') + \sigma^2(x) \delta_{DL}(x - x'). \]
Similarly, the cross-correlation of two fluctuation fields \( \delta_1(x) = f_1(x) - \langle f_1(x) \rangle \) and \( \delta_2(x) = f_2(x) - \langle f_2(x) \rangle \), assuming that the noise in the fields is uncorrelated, is given by,
\[ \langle \delta_1(x) \delta_2(x') \rangle = \langle f_1(x) \rangle \langle f_2(x') \rangle \xi_c(x - x'), \]
in terms of the cross-correlation function \( \xi_c(s) \).

The correlation functions of the fields can be related to their auto-power spectra \( P(k) \) and cross-power spectra \( P_c(k) \) by,
\[ P(k) = \int d^3 s \xi(s) e^{i k \cdot s}, \quad P_c(k) = \int d^3 s \xi_c(s) e^{i k \cdot s}, \]
defined here in volume units \( (k^3 \text{ Mpc}^{-3}) \), including the appropriate temperature unit for the intensity map. As exemplified by Equation 6, our measured fields trace fluctuations in the matter overdensity \( \delta_m(x) \), which we model in terms of the matter power spectrum \( P_m(k) \). For the purposes of this study, we assume that the redshift-space power spectra of the fields, in the absence of any observational effects, may be described by a simple 3-parameter redshift-space distortion model (Hatton & Cole 1998) combining the large-scale Kaiser effect (Kaiser 1987) imprinted by the growth rate \( f \), exponential damping from random pairwise velocities with dispersion \( \sigma_v \), and a linear bias \( b \): 
\[ P(k) = P(k, \mu) = \frac{(b + f \mu^2)^2 P_m(k)}{1 + (k \mu \sigma_v/H_0)^2}, \quad P_c(k) = P_c(k, \mu) = \frac{(b_1 + f \mu^2)^2 P_m(k)}{1 + (k \mu \sigma_v/H_0)^2}, \]
where \( \mu \) is the cosine of the angle between \( k \) and the line of sight.

In the following subsections we build a model connecting the Fourier transform of the fluctuation fields to their underlying auto-power spectra \( P(k) \) and cross-power spectra \( P_c(k) \), which contains cosmological information as described by Equation 11. We include a number of practical observational and measurement effects:

- A selection function which varies with position, \( \langle f(x) \rangle = \langle n_g(x) \rangle V \) or \( \langle f(x) \rangle = \langle T_b(x) \rangle \),
- Uncorrelated noise in the field as a function of position, described by \( \sigma^2(x) \) in Equation 8, including the specific example of Poisson noise,
- A weight \( w(x) \) applied to the field to optimize the signal-to-noise ratio of the measurement,
- A smoothing function which can vary with position, with specific examples provided for a Gaussian telescope beam, frequency channels in radio observations, redshift errors, HEALPix pixelization\(^2\) and nearest grid point assignment,
- Discretization of the field onto an FFT grid.

### 2.3 Relating the fluctuation fields to the power spectra

We now start developing the relationship between the observed fluctuations and their underlying power spectra, building on existing literature. We can relate the fluctuation fields to their power spectra by considering the Fourier transform of the

\(^1\) It is appropriate to consider number density and temperature on the same footing, since both quantities do not change with the resolution of the pixelization.

\(^2\) http://healpix.sourceforge.net
weighted fields,
\[ \tilde{\delta}(k) = \frac{1}{V} \int d^3 x \, w(x) \delta(x) e^{i k \cdot x}, \]
where \( w(x) \) is a general position-dependent weight (which has inverse units to those of the field, considering the definition presented after Equation 4 – i.e., dimensionless for a galaxy survey and inverse temperature for an intensity map), which may be applied to optimize the signal-to-noise ratio of the measurement. The average of \( |\tilde{\delta}(k)|^2 \) across realizations is,
\[ \langle |\tilde{\delta}(k)|^2 \rangle = \frac{1}{V^2} \int d^3 x \int d^3 x' \, w(x) w(x') \langle \delta(x) \delta(x') \rangle e^{i k \cdot (x-x')} . \] (13)
Substituting in Equations 7, 8 and 10 to Equation 13 we find, in a slight generalization of the results of Feldman et al. (1994) to include a general noise term,
\[ \langle |\tilde{\delta}(k)|^2 \rangle = \int \frac{d^3 k'}{(2\pi)^3} P(k') |\tilde{W}(k-k')|^2 + \frac{1}{V} \int d^3 x \, w^2(x) \sigma^2(x) = P \ast |\tilde{W}|^2 + S, \] (14)
where we have defined a window function \( W(x) = w(x) \langle f(x) \rangle \). Hence, \( \langle |\tilde{\delta}(k)|^2 \rangle \) is the sum of the convolution (which we denote by \( \ast \)) of the underlying power spectrum \( P(k) \) and \( |\tilde{W}(k)|^2 \), and a noise term,
\[ S = \frac{1}{V} \int d^3 x \, w^2(x) \sigma^2(x). \] (15)
Repeating this process for the Fourier transform of two different fluctuation fields (see also Smith 2009; Blake et al. 2013), weighted by functions \( w_1(x) \) and \( w_2(x) \), we find that,
\[ \langle \tilde{\delta}_1(k) \tilde{\delta}_2^*(k) \rangle = \int \frac{d^3 k'}{(2\pi)^3} P_c(k') \tilde{W}_1(k-k') \tilde{W}_2^*(k-k') = P_c \ast \tilde{W}_1 \tilde{W}_2^*, \] (16)
where \( W_i(x) = w_i(x) \langle f_i(x) \rangle \). If we adopt an approximation where the power spectrum does not vary significantly over the width of \( |\tilde{W}(k)|^2 \), such that we can take it outside the integral over \( k' \) in Equations 14 and 16, and we use Parseval’s theorem \( \int \frac{d^3 k}{(2\pi)^3} |\tilde{W}(k)|^2 = \frac{1}{V} \int d^3 x \, W^2(x) \), we find,
\[ \langle |\tilde{\delta}_1(k)|^2 \rangle \approx Q \frac{P(k)}{V} + S, \quad \langle \tilde{\delta}_1(k) \tilde{\delta}_2^*(k) \rangle \approx Q_c \frac{P_c(k)}{V} , \] (17)
where we have defined dimensionless quantities,
\[ Q = \frac{1}{V} \int d^3 x \, W^2(x), \quad Q_c = \frac{1}{V} \int d^3 x \, W_1(x) W_2(x). \] (18)
In the following subsections we consider two important special cases of fluctuation fields, which will be relevant in the subsequent analysis.

### 2.3.1 Poisson point process

If \( f(x) \) is generated by a Poisson point process from a galaxy number density distribution \( n_g(x) \), then \( \xi(x, x') = 0 \) and
\[ \langle \delta(x) \delta(x') \rangle = V \langle n_g(x) \rangle \delta_D(x-x'), \] (19)
and from the definition \( \delta(x) = V \left[ n_g(x) - \langle n_g(x) \rangle \right] \),
\[ \langle \delta(x) \delta(x') \rangle = V^2 \left[ \langle n_g(x) n_g(x') \rangle - \langle n_g(x) \rangle \langle n_g(x') \rangle \right] . \] (20)
To justify this formula, we can use \( N = \int d^3 x \, n_g(x) \) and consider
\[ \langle N^2 \rangle = \int d^3 x \int d^3 x' \langle n_g(x) n_g(x') \rangle = \int d^3 x \int d^3 x' \left[ \langle n_g(x) \rangle \langle n_g(x') \rangle + \frac{\langle n_g(x) \rangle}{V} \delta_D(x-x') \right] = \langle N \rangle^2 + \int d^3 x \langle n_g(x) \rangle = \langle N \rangle^2 + \langle N \rangle , \] (21)
as expected from Poisson statistics. Comparing Equations 8 and 19, we identify \( \sigma^2(x) = V \langle n_g(x) \rangle \) and hence assuming weight \( w = 1 \),
\[ \langle |\tilde{\delta}(k)|^2 \rangle = S = \int d^3 x \, \langle n_g(x) \rangle = N. \] (22)

### 2.3.2 Uniform window and noise

Suppose that a field is sampled from a constant mean \( \langle f(x) \rangle = f_0 \) with a constant noise \( \sigma^2(x) = \sigma_0^2 \), within a fraction of the cuboid defined by \( w = 1 \), such that the observed volume is \( V_w = \int d^3 x \, w(x) \). In this case, we find from Equations 15 and 18 that \( Q = f_0^2 V_w / V \) and \( S = \sigma_0^2 V_w / V \) such that Equation 17 takes the form,
\[ \langle |\tilde{\delta}(k)|^2 \rangle = \frac{f_0^2 V_w}{V^2} \left[ P(k) + \frac{V \sigma_0^2}{f_0^2} \right] . \] (23)
In this scenario, the equivalent power spectrum due to noise is hence the second term in the bracket, 
\[ P_{\text{noise}}(k) = \frac{V\sigma_0^2}{f_0^2}. \] 
(24)

We note that for Poisson statistics, \( f_0 = Vn_0 \) in terms of the number density \( n_0 \), and \( \sigma_0^2 = Vn_0 \), such that \( P_{\text{noise}} = 1/n_0 \), as expected.

### 2.4 Smoothing

We now extend the power spectrum model to describe the effect of a general smoothing of the fields, such as might result from a telescope beam in radio observations, redshift errors in optical observations, or a general pixelization. We suppose that the smoothed field may be written in the form,
\[ \tilde{\delta}^{sm}(x) = \frac{1}{V} \int d^3x' \tilde{\delta}(x') B(x - x', x), \] 
(25)

where the dimensionless smoothing function \( B \) is a compact function that depends on the separation \( x - x' \), but may also vary with position \( x \). The smoothing function is normalized such that \( \frac{1}{V} \int d^3s B(s, x) = 1 \) for all \( x \), and we define the Fourier transform of the smoothing function at each location as \( \hat{B}(k, x) = \frac{1}{V} \int d^3s B(s, x) e^{ik\cdot s} \), where \( \hat{B}(k = 0, x) = 1 \). Substituting in these expressions we find that the Fourier transform of Equation 25 is,
\[ \tilde{\delta}^{sm}(k) = \int d^3x \int \frac{d^3k'}{(2\pi)^3} \hat{\delta}(k') \hat{B}(k', x) e^{i(k-k')\cdot x}. \] 
(26)

which reproduces the standard result of the convolution theorem, that \( \tilde{\delta}^{sm}(k) = \hat{B}(k) \tilde{\delta}(k) \), if \( B(s, x) \) is independent of position \( x \). Assuming that the smoothing function is compact in configuration space compared to the variation of the survey window function and weights, this leads to the result that,
\[ \langle |\tilde{\delta}^{sm}(k)|^2 \rangle \approx \langle |\tilde{\delta}(k)|^2 \rangle \frac{1}{V} \int d^3x |\hat{B}(k, x)|^2 = \langle |\tilde{\delta}(k)|^2 \rangle D^2(k), \] 
(27)

such that the power spectra (including both the signal and noise) are modulated by a damping function \( D^2(k) = \frac{1}{V} \int d^3x |\hat{B}(k, x)|^2 \), which is the volume average of \( |\hat{B}(k, x)|^2 \). If a smoothed field \( \delta^{sm}_i(x) \) is correlated with an unsmoothed field \( \delta_2(x) \), the modulation of the resulting cross-power spectrum is,
\[ \langle \delta^{sm}_i(k) \delta_2(k) \rangle \approx \langle \delta_1(k) \delta_2(k) \rangle \frac{1}{V} \int d^3x \hat{B}(k, x), \] 
(28)

or for the cross-correlation of two fields smoothed with different functions \( B_1(s, x) \) and \( B_2(s, x) \),
\[ \langle \delta^{sm}_1(k) \delta^{sm}_2(k) \rangle \approx \langle \delta_1(k) \delta_2(k) \rangle \frac{1}{V} \int d^3x B_1(k, x) \hat{B}_2(k, x). \] 
(29)

In the following subsections we consider some special cases of these results, which will be utilized in the subsequent N-body simulation tests.

#### 2.4.1 Pixelization

A special case of the smoothing operation described by Equation 25 occurs when a field is pixelized into distinct “cells”, such that the average value of the field at each cell is assigned to all positions within the cell. This behaviour can be modelled by Equation 25 if the pixel assignment function near position \( x \) is offset by a function \( \epsilon(x) \) to enforce averaging within each cell; this offset results in a phase change in the Fourier transform such that,
\[ \hat{B}(k, x) \rightarrow \hat{B}(k, x) e^{i k . \epsilon(x)}. \] 
(30)

This behaviour does not change the value of \( |\hat{B}(k, x)|^2 \), hence the overall effect on the auto-power spectrum can still be evaluated using Equation 27. However, the damping of the cross-power spectrum as computed by Equation 28 is changed by this type of smoothing, by the volume average of \( e^{i k . \epsilon(x)} \). Given that the offsets \( \epsilon \) will be distributed across space with the same profile as the cells, this volume average is well-approximated by \( \hat{B}(k, x) \). Therefore, even though only one of the two fields is smoothed, the cross-power spectrum is damped due to pixelization by approximately the same factor \( D^2(k) \) as the auto-power spectrum.

#### 2.4.2 Noise applied to cells

We now consider a scenario where the noise in the fluctuation field is generated by drawing a random variable in a series of cells of volume \( \Delta V_i \), with zero mean and variance \( \sigma_i^2 \). In this case, the 2-point statistics of the noise in Equation 8 is modified...
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from \( \langle \delta(x) \delta(x') \rangle = \sigma^2(x) \delta_D(x-x') \) to,

\[
\langle \delta(x) \delta(x') \rangle = \sum_i \sigma_i^2 B_i(x) B_i(x'),
\]

where \( B_i(x) = 1 \) if \( x \) is in cell \( i \), and zero otherwise, such that the noise is uncorrelated between different cells. Substituting this relation in Equation 13 we find,

\[
\langle |\delta(k)|^2 \rangle = \frac{1}{V^2} \sum_i \sigma_i^2 \int d^3 \mathbf{x} \int d^3 \mathbf{x}' w(x) w(x') B_i(x) B_i(x') e^{i k \cdot (x-x')} = \frac{1}{V^2} \left( \sum_i \sigma_i^2 (\Delta V_i)^2 \sigma_i^2 \right) D^2(k),
\]

after applying the same arguments as in the previous subsection. By comparing this result with Equation 15, the appropriate noise power can be recovered by taking,

\[
\sigma^2(x) = \frac{1}{V} \sum_i \sigma_i^2 \Delta V_i B_i(x).
\]

Poisson noise, \( \sigma^2(x) = V(n_g(x)) \), is obtained if \( \sigma_i^2 = V^2(n_{g,i})/\Delta V_i \), where \( (n_{g,i}) \) is the galaxy number density in cell \( i \). This result will be useful in Section 4, when adding a Poisson noise component to model an intensity map constructed by binning a simulation catalogue of discrete objects in cells.

2.4.3 Telescope beam

For Gaussian smoothing perpendicular to the line of sight, such as would result from a radio telescope beam, we have a smoothing kernel \( B_i(x) \propto e^{-x_i^2/2\sigma_i^2} \) as a function of perpendicular spatial separation \( s_x \), where \( \sigma_x \) is the spatial standard deviation of the beam. The Fourier transform of this function is,

\[
\tilde{B}_{\text{beam}}(k) = e^{-k_x^2/2\sigma_x^2/2},
\]

where \( k_x = k_x - \mu^2 \) and \( \mu \) is the cosine of the angle between \( k \) and the line of sight. Hence, the beam damps power at small perpendicular separations. For a beam of constant angular standard deviation \( \sigma_{\theta} \) on the sky in units of radians, the corresponding spatial smoothing scale will vary with position as \( \sigma_x(x) = \frac{|x|}{\sigma_{\theta}} \). In this case we would derive the damping factor using Equation 27 as,

\[
D^2(k) = \frac{1}{V} \int d^3 \mathbf{x} \ e^{-k_x^2|x|^2/\sigma_{\theta}^2}.
\]

2.4.4 Frequency channels

Smoothing in the radial direction results from the width of the frequency channels in which radio intensity mapping data is collected. For a frequency channel of spatial width \( s_{\|} \), the Fourier transform of the top-hat assignment function is,

\[
\tilde{B}_{\text{chan}}(k) = \frac{\sin(k_{\|} s_{\|}/2)}{k_{\|} s_{\|}/2},
\]

where \( k_{\|} = k_x \). If the frequency width of the channel is \( \Delta f \) then, for line measurements with rest frequency \( v_0 \) such that \( v = v_0/(1+z), s_{\|}(x) = [c/H(z)](1+z)^2 (\Delta f/v_0) \) in terms of the speed of light \( c \) and Hubble parameter \( H(z) \). In this case we would derive the damping factor using Equation 27 as,

\[
D^2(k) = \frac{1}{V} \int d^3 \mathbf{x} \left( \frac{\sin(k_{\|} s_{\|}(x)/2)}{k_{\|} s_{\|}(x)/2} \right)^2.
\]

2.4.5 Redshift errors

Damping of power in the radial direction can also result from errors in measured galaxy redshifts, for example due to photometric redshift estimates with error \( \Delta z \) (which may vary with redshift). Assuming that these errors are Gaussian, the Fourier transform of the smoothing kernel follows Equation 30,

\[
\tilde{B}_{\Delta z}(k) = e^{-k_z^2/2\sigma_z^2},
\]

where \( \sigma_z(x) = [c/H(z)] \Delta z \) is the spatial radial error in each location. In this case the overall damping factor can be computed using,

\[
D^2(k) = \frac{1}{V} \int d^3 \mathbf{x} \ e^{-k_z^2/2\sigma_z^2(z)}.
\]
2.4.6 Angular pixelization

A process of angular pixelization, for example using a scheme such as HEALPix (Górski et al. 2005), results in a damping of power as a function of $k \perp$. The associated damping of the angular power spectrum $C_\ell$ of the field as a function of multipole $\ell$ can be described in terms of the pixel window function $W_{\text{ang}}(\ell)$, such that the damping is described by $C_\ell \rightarrow C_\ell W_{\text{ang}}^2(\ell)$ and

$$W_{\text{ang}}^2(\ell) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} |w_{lm}|^2,$$

where $w_{lm} = \sum_{\Omega} \Omega Y_{lm}(\Omega)$ is the spherical harmonic transform of a pixel in terms of the spherical harmonic functions $Y_{lm}$. In the case of a 3D survey smoothed using angular pixelization, the contribution to the damping factor at position $x$ relative to the observer is determined by identifying $\ell = k \perp |x|$ such that $B_{\text{ang}}(k, x) = W_{\text{ang}}(k \perp |x|)$. In this case we would derive the damping factor for the 3D power spectrum as,

$$D^2(k) = \frac{1}{V} \int d^3 x W_{\text{ang}}^2(k \perp |x|).$$

2.4.7 Nearest grid point assignment

For nearest grid point assignment, the Fourier transform of the assignment function is,

$$\tilde{B}_{\text{NGP}}(k) = \frac{\sin(k_x H/2) \sin(k_y H/2) \sin(k_z H/2)}{(k_x H/2) (k_y H/2) (k_z H/2)},$$

where $H$ is the grid spacing (e.g., Jing 2005). This special case will be useful in the following section.

2.5 Discretization

We now consider the effect on the power spectrum if a continuous field $\delta(x)$, possibly having been smoothed using one of the schemes described in the previous section, is sampled on a regular FFT grid at positions $x_n = H n$, where $n$ is a vector of integers. This case is important for efficient power spectrum estimators. Following Jing (2005), we can conveniently describe this process using the sampling function $\Pi(x) = \sum_n \delta_D(x - n)$, an array of $\delta$-functions placed at integers $n$, such that the gridded field can be written in the form,

$$\delta^{\text{grid}}(x) = \Pi(x/H) \delta(x).$$

We can see that Equation 43 produces the correct result for the Fourier-transformed field by considering,

$$\tilde{\delta}^{\text{grid}}(k) = \frac{1}{V} \int d^3 x \delta^{\text{grid}}(x) e^{ik \cdot x} = \sum_n \frac{1}{V} \int d^3 x \delta_D(x - nH) \delta(x) e^{ik \cdot x} = \sum_n \delta(x_n) e^{ik \cdot x_n},$$

as expected when evaluating an FFT. The Fourier transform of Equation 43 may also be obtained using the convolution theorem,

$$\tilde{\delta}^{\text{grid}}(k) = \sum_n \tilde{\delta}(k_n),$$

where the Fourier transform of the sampling function is given by,

$$\tilde{\Pi}(k) = \frac{1}{V} \int d^3 x \Pi(x/H) e^{ik \cdot x} = \sum_n e^{i2\pi \frac{k_n}{H}} = \sum_n \delta_D(k - 2k_N n),$$

where $k_N = \pi / H$ is the Nyquist frequency of the grid. Hence, Equation 45 may be simplified as,

$$\tilde{\delta}^{\text{grid}}(k) = \sum_n \tilde{\delta}(k_n),$$

where $k_n = k + 2k_N n$ such that,

$$\langle |\tilde{\delta}^{\text{grid}}(k)|^2 \rangle = \sum_n \langle |\tilde{\delta}(k_n)|^2 \rangle.$$

Hence, discretization involves the aliasing of power to scale $k$ from a series of scales $k_n$ spaced by $2k_N$ (Jing 2005). Discretization is often combined with smoothing, in which case combining Equations 27 and 48 yields,

$$\langle |\tilde{\delta}^{\text{grid}}(k)|^2 \rangle = \sum_n \langle |\tilde{\delta}(k_n)|^2 \rangle D^2(k_n).$$

We note an interesting special case in which nearest grid point assignment is applied to a Poisson noise spectrum $\langle |\tilde{\delta}(k)|^2 \rangle = S$, in which case the power spectrum is unchanged:

$$\langle |\tilde{\delta}^{\text{grid}}(k)|^2 \rangle = S \sum_n D_{\text{NGP}}^2(k_n) = S.$$
where we have used the identity $\sum_{k_n} |\tilde{B}_{\text{NGP}}(k_n)|^2 = 1$, where $\tilde{B}_{\text{NGP}}(k)$ is defined by Equation 42.\(^3\)

Combining the results of the above sections, we can describe the joint effects of the window function, noise, smoothing and discretization on the auto-power spectrum of the fluctuation field by,

$$
\langle |\delta(k)|^2 \rangle = \sum_{k_n} \left[ (P \ast |\tilde{W}|^2)(k_n) + \delta \right] D^2(k_n),
$$

and on the cross-power spectrum by,

$$
\langle \tilde{\delta}_i(k) \tilde{\delta}_j^*(k) \rangle = \sum_{k_n} \left[ (P_i \ast \tilde{W}_1 \tilde{W}_2^*)(k_n) \right] D^2(k_n).
$$

2.6 Power spectrum multipoles

As the above sections demonstrate, the contribution of a fluctuation field to its power spectrum in a volume may vary as a function of $x$ owing to variations in the statistical properties of the field such as its clustering, noise, smoothing or weighting. We can encapsulate these effects by writing the power spectrum as an integral over $x$,

$$
P(k) = \frac{1}{V} \int d^3 x \; P(k, x),
$$

where $P(k, x)$ represents the contribution to the power spectrum originating from position $x$.

This concept is useful when considering a further cause of position-dependence: the changing line-of-sight direction across the volume. Effects such as redshift-space distortions, the telescope beam and angular/radial smoothing will cause the amplitude of the power spectra $P(k)$ to depend on the direction of $k$ with respect to a global line of sight. Assuming azimuthal symmetry, the power spectra will only depend on $\mu$, the cosine of the angle between $k$ and the line-of-sight direction. In this case the 2D function $P(k, \mu)$ may be conveniently quantified by power spectrum multipoles, $P_\ell(k)$:

$$
P(k, \mu) = \sum_\ell P_\ell(k) L_\ell(\mu) = \sum_\ell P_{\ell}(k) L_\ell(\hat{k}, \hat{\mu}),
$$

where $L_\ell$ are the Legendre polynomials, and in the last expression we are describing the varying line-of-sight, given that in a region of space around position $x$ from the observer we can write $\mu = \hat{k} \cdot \hat{x}$. Inverting Equation 54 and averaging the statistic over all positions using Equation 53, we can model the power spectrum multipoles as,

$$
P_\ell(k) = \frac{2\ell + 1}{4\pi} \int d^3 \Omega \; P(k, x) L_\ell(\hat{k}, \hat{x}) = \frac{2\ell + 1}{4\pi} \int \frac{d^3 \Omega}{4\pi} \frac{1}{V} \int d^3 x \int d^3 x' \xi(x, x') e^{i k \cdot (x-x')} L_\ell(\hat{k}, \hat{x}'),
$$

where $d\Omega$ integrates over all angles $\hat{k}$. The equivalent relation to Equation 13 for power spectrum multipole $\ell$ can then be written as,

$$
\langle |\delta(k)|^2 \rangle = \frac{1}{V^2} \int d^3 x \int d^3 x' w(x) w(x') \langle \delta(x) \delta(x') \rangle e^{i k \cdot (x-x')} L_\ell(\hat{k}, \hat{x}'),
$$

where $\langle |\delta(k)|^2 \rangle$ describes the contribution of wavenumber $k$ to the power spectrum multipole $P_\ell(k)$. Blake et al. (2018) develop the equivalent expressions to Equations 14 and 16 for modelling the auto- and cross-power spectrum multipoles. For the auto-power spectrum:

$$
\langle |\delta(k)|^2 \rangle = \sum_\ell \int \frac{d^3 k'}{(2\pi)^3} P_\ell(k') \tilde{W}(k - k') W_{\ell \ell_0}^*(k, k') + \frac{1}{V} \int d^3 x w^2(x) n^2(x) L_\ell(\hat{k}, \hat{x}),
$$

where

$$
W_{\ell \ell_0}(k, k') = \frac{1}{V} \int d^3 x W(x) L_\ell(\hat{k}, \hat{x}) L_{\ell_0}(\hat{k}', \hat{x}'),
$$

The equivalent expression for the cross-power spectrum multipoles is,

$$
\langle \tilde{\delta}_i(k) \tilde{\delta}_j^*(k) \rangle = \sum_\ell \int \frac{d^3 k'}{(2\pi)^3} P_{\ell i}(k') \tilde{W}_1(k - k') W_{\ell j}^*(k, k').
$$

These equations reduce to the results of Equations 14 and 16 for $\ell = \ell' = 0$.\(^3\)

\(^3\) This identity is known as Glaisher’s series.
Power spectrum modelling of galaxy and intensity maps

3 POWER SPECTRUM MEASUREMENT

In this section we consider the estimators for the auto- and cross-power spectra, the variance in these estimators, and the optimal weighting of the fluctuation fields which minimizes this variance under certain approximations. These results are useful for practical power spectrum analysis.

3.1 Estimators

Equation 17 motivates an estimator for the power spectrum in terms of the Fourier transform of the weighted fluctuation field, \( \tilde{\delta}(k) \):

\[
\hat{P}(k) = \frac{\left| \langle \tilde{\delta}(k) \rangle \right|^2 - S}{Q} V,
\]

such that \( \langle \hat{P}(k) \rangle \approx P(k) \). Similarly for the cross-power spectrum,

\[
\hat{P}_c(k) = \frac{\text{Re}\langle \tilde{\delta}(k) \tilde{\delta}_c(k) \rangle V}{Q_c},
\]

where \( \langle \hat{P}_c(k) \rangle \approx P_c(k) \) and \( \text{Re}\{\} \) indicates we are taking the real part of the expression, noting that Equation 61 is symmetric in the two fields since \( \text{Re}\langle \tilde{\delta}_1 \tilde{\delta}_2 \rangle = \langle \tilde{\delta}_1 \tilde{\delta}_2 + \tilde{\delta}_1^* \tilde{\delta}_2 \rangle / 2 \). The estimator for the power spectrum multipoles is,

\[
\hat{P}_\ell(k) = \frac{(2\ell + 1) \left| \langle \tilde{\delta}(k) \rangle \right|^2 - S_\ell(k)}{Q} V,
\]

where \( S_\ell(k) = \frac{1}{2} \int d^3 x \, w^2(x) \sigma^2(x) L_\ell(k, \hat{x}) \). Bianchi et al. (2015) provide an FFT-based method for evaluating Equation 62 (see also, Scoccimarro 2015). The estimator for the cross-power spectrum multipoles is,

\[
\hat{P}_{c,\ell}(k) = \frac{(2\ell + 1) \text{Re}\langle \tilde{\delta}(k) \tilde{\delta}_c(k) \rangle V}{Q_c},
\]

which can be evaluated using an adapted version of the Bianchi et al. (2015) method. Equations 60 to 63 all provide estimates of the corresponding power spectrum for a mode with wavenumber \( k \). We can then bin these estimates in spherical shells of \( k = |k| \), to extract measurements of isotropic power spectra.

3.2 Errors and optimal weighting

We now consider the variance in these power spectrum estimators. Assuming Gaussian statistics, the covariance of the power spectrum estimator between two modes \( k \) and \( k' \) is given by (see Feldman et al. 1994; Blake et al. 2013),

\[
\langle \delta \hat{P}(k) \delta \hat{P}(k') \rangle \approx \frac{P(k) \dot{Q}(k) + V S(k)}{Q^2} \langle \tilde{\delta}(k) \rangle^2,
\]

where \( \dot{Q}(k) \) and \( \dot{S}(k) \) are the Fourier transforms of \( Q(x) = w^2(x) \langle f(x) \rangle^2 \) and \( S(x) = w^2(x) \sigma^2(x) \), respectively, such that \( \dot{Q} = \dot{Q}(0) \). If we average the power spectrum estimates in a bin of Fourier space of volume \( V_k \) near wavenumber \( k \), this produces a variance in a bin which may be approximately evaluated as (Feldman et al. 1994),

\[
\sigma_P^2 \approx \frac{1}{V_k} \int d^3 k' \frac{P(k) \dot{Q}(k') V \dot{S}(k')}{Q^2} \langle \tilde{\delta}(k') \rangle^2,
\]

assuming the width of the bin is large compared to the correlation length in \( k \)-space. Using Parseval’s theorem, together with the expression for the number of unique modes in the bin \( N_m = V_k V/(2\pi)^3 \), this yields,

\[
\sigma_P^2 \approx \frac{1}{N_m V} \int d^3 x \frac{P(k) Q(x) V S(x)}{Q^2} = \frac{V^3}{N_m} \int d^3 x \frac{P(k) \langle f(x) \rangle^2 + \sigma^2(x)}{[\int d^3 x \, w^2(x) \langle f(x) \rangle^2]^2}. \tag{66}
\]

In the special case corresponding to Section 2.3.2, where the field is sampled from a constant mean \( \langle f(x) \rangle = f_0 \) with a constant noise \( \sigma^2(x) = \sigma_0^2 \), within a subset of the cuboid defined by \( w = 1 \), we find,

\[
\sigma_P^2 = \frac{1}{N_m w} \frac{V}{V_w} \left( \frac{P(k)}{V} + \frac{V \sigma_0^2}{V_w \sigma_0^2} \right)^2, \tag{67}
\]

noting that for Poisson statistics, the second term in the bracket in Equation 67 reduces to \( 1/n_0 \).

Following Feldman et al. (1994), we can identify the weight function \( w(x) \) in Equation 66 which minimizes \( \sigma_P^2 \) by solving the equation \( \partial \sigma_P^2/\partial w = 0 \). We find:

\[
\frac{\partial \sigma_P^2}{\partial w} \propto \left[ \int d^3 x \, 4 w^3 \left( \frac{P}{V} \langle f \rangle^2 + \sigma^2 \right)^2 \right] \left[ \int d^3 x \, w^2 \langle f \rangle^2 \right]^{-2} - 2 \left[ \int d^3 x \, w^2 \langle f \rangle^2 \right] \left[ \int d^3 x \, w^2 \langle f \rangle^2 \right]^{-3} \left[ \int d^3 x \, 2 w \langle f \rangle^2 \right] = 0.
\]
which may be re-arranged to yield,

$$\frac{\int d^3x \, w^3 \left( \frac{P_V(f)}{V} \right)^2}{\int d^3x \, w^4 \left( \frac{P_V(f)}{V} \right)^2} = \frac{\int d^3x \, w \left( \frac{\sigma}{V} \right)^2}{\int d^3x \, w^2 \left( \frac{\sigma}{V} \right)^2}.$$  \hfill (69)

By inspection, we find that this equation is satisfied if,

$$w^2 \left( \frac{P_V(f)}{V} \right)^2 = \left( \frac{\sigma}{V} \right)^2 = \langle f(x) \rangle^2,$$  \hfill (70)

or,

$$w(x) = \frac{\langle f(x) \rangle}{\frac{P_V(k)}{V} \langle f(x) \rangle^2 + \sigma^2(x)}.$$  \hfill (71)

In the case of a galaxy survey with Poisson statistics, we have $\langle f(x) \rangle = \sigma^2(x) = V \langle n_g(x) \rangle$, in which case we recover the usual dimensionless FKP weighting,

$$w(x) = \frac{1}{1 + \langle n_g(x) \rangle \rho(k)}.$$  \hfill (72)

For an intensity map with temperature variance $\sigma^2_T(x)$ and mean brightness temperature $\langle f(x) \rangle = \langle T_b(x) \rangle$ we have,

$$w(x) = \frac{\langle T_b(x) \rangle}{\frac{P_V(k)}{V} \langle T_b(x) \rangle^2 + \sigma^2_T(x)},$$  \hfill (73)

which has dimensions of inverse temperature as required (see the discussion after Equation 12).

The expression equivalent to Equation 64 for the cross-power spectrum is (Smith 2009; Blake et al. 2013),

$$\langle \delta P_c(k) \delta P_c(k') \rangle \approx \left[ \frac{P_c(k) \tilde{Q}_c(\delta(k))}{V} \right]^2 + \Re \left[ \left[ P_c(k) \tilde{Q}_c(\delta(k)) + V S_1(\delta(k)) \right] \left[ P_c(k) \tilde{Q}_c(\delta(k)) + V S_2(\delta(k)) \right] \right],$$  \hfill (74)

where $\tilde{Q}_c(k)$ is the Fourier transform of $Q_c(x) = w_1(x) w_2(x) \langle f_1(x) \rangle \langle f_2(x) \rangle$ and $Q_c \equiv Q_c(0)$. This leads to,

$$\sigma^2_{P_c} \approx \frac{1}{N_{mV}} \int d^3x \left( \frac{P_{c_1}^2(k) Q_{c_1}^2(x)}{V^2} + \left[ P_{c_1}(k) Q_{c_1}(x) + V S_1(x) \right] \left[ P_{c_2}(k) Q_{c_2}(x) + V S_2(x) \right] \right) \frac{2 Q^2_c}{V^3} \langle f_1(x) \rangle \langle f_2(x) \rangle^2. \hfill (75)

In the special case where the weights, means and noise are position-independent we find,

$$\sigma^2_{P_c} \approx \frac{1}{2 N_{mV}} \frac{V}{V_{w,c}} \left[ \frac{N_{mV}}{V_{w,c}} \left( P_{c_1}^2 + \frac{P_{c_1} V \sigma^2_{c_1}}{f_{c_1}^2} \right) - \frac{V}{V_{w,c}} \left( P_{c_2} + \frac{V \sigma^2_{c_2}}{f_{c_2}^2} \right) \right],$$  \hfill (76)

where $V_{w,c} = \int d^3x w(x) x^2$ is the overlap volume of the two datasets. Following the same method as above, we find that $\partial \sigma^2_{P_c} / \partial w_1 = \partial \sigma^2_{P_c} / \partial w_2 = 0$, such that the error in the cross-power spectrum is minimized, if the product of the individual weights satisfies,

$$w_1(x) w_2(x) = \frac{\langle f_1(x) \rangle \langle f_2(x) \rangle}{P_{c_1}^2(k) \langle f_1(x) \rangle^2 + \langle f_2(x) \rangle^2 \sigma^2_{c_1}(x)} P_{c_2}^2(k) \langle f_2(x) \rangle^2 + \sigma^2_{c_2}(x).$$  \hfill (77)

We note that if the weights for the two datasets are assigned according to the optimal single-tracer weights (Equation 71), this does not produce the optimal weight for the cross-power spectrum unless $P_c = 0$. The optimal error in the cross-power spectrum, which also satisfies Equation 77, can be produced if the single-tracer optimal weights are modified such that $w_1(x) = w_c(x) w_1(x)$ where,

$$w_c(x) = \left[ 1 + w_1(x) w_2(x) \frac{P_{c_1}^2(k)}{V^2} \left( f_1(x) \right) \langle f_2(x) \rangle^2 \right]^{-1/2}. \hfill (78)

In the case of galaxy surveys with Poisson statistics, these optimal weights are

$$w_1(x) = \frac{1}{1 + \langle n_1(x) \rangle P_1(k)}, \quad w_2(x) = \frac{1}{1 + \langle n_2(x) \rangle P_2(k)}, \quad w_c(x) = \sqrt{1 + w_1(x) w_2(x) \langle n_1(x) \rangle \langle n_2(x) \rangle P_{c_1}^2(k)}.$$  \hfill (79)

and in the case of the auto- and cross-correlations of a galaxy survey and an intensity mapping survey,

$$w_g(x) = \frac{1}{1 + \langle n_g(x) \rangle P_g(k)}, \quad w_T(x) = \frac{\langle T_b(x) \rangle}{P_V(k) \langle T_b(x) \rangle^2 + \sigma^2_T(x)}, \quad w_c(x) = \frac{1}{\sqrt{1 + w_g(x) w_T(x) \langle n_g(x) \rangle \langle T_b(x) \rangle P_{c_1}^2(k)}}.$$
the cone.

For the purposes of this paper, we assume an approximation of the error in the power spectrum multipoles obtained by propagating the errors in the auto- and cross-power spectra, given by Equations 67 and 76, using $P_L(k) = (2\ell + 1) \int_0^1 d\mu P(k, \mu) L_{\ell}\left(\mu\right)$:

$$\sigma^2_{P_\ell} = (2\ell + 1)^2 \frac{1}{N_m} \frac{V}{V_w} \int_0^1 d\mu \left[ P(k, \mu) + \frac{V_0^2}{f_0^2} \right]^2 L_{\ell\theta}^2(\mu).$$

$$\sigma^2_{P_{L,\ell}} = (2\ell + 1)^2 \frac{1}{2N_m} \frac{V}{V_{w,c}} \int_0^1 d\mu \left[ P_{L}(k, \mu)^2 + \left( P_{L}(k, \mu) + \frac{V_0^2}{f_1^2} \right) \left( P_{L}(k, \mu) + \frac{V_0^2}{f_2^2} \right) \right] L_{\ell\theta}^2(\mu).$$

We note that Blake et al. (2018) provide more exact expressions for the covariance of the auto- and cross-power spectrum multipoles, capturing the full spherical geometry, which we do not reproduce here.

4 SIMULATION TEST

We created a simple simulated dataset to test our power spectrum models. Our mock dataset was built from the $z = 0$ dark matter distribution of the GiggleZ Simulation (Poole et al. 2015), an N-body simulation consisting of $2160^3$ particles evolving under gravity in a periodic box of side $1\, h^{-1}$ Gpc. The initial conditions of the simulation were generated using a fiducial flat $\Lambda$CDM cosmological model based on the Wilkinson Microwave Anisotropy Probe (WMAP) 5-year results (Komatsu et al. 2009), with matter density $\Omega_m = 0.273$, baryon density $\Omega_b = 0.0456$, Hubble parameter $h = 0.705$, normalization $\sigma_8 = 0.812$ and spectral index $n_s = 0.96$. We applied the following series of steps to convert the particle distribution into mock galaxy and intensity mapping datasets, whose auto- and cross-correlations could be analysed with the above theory:

(i) We subsampled the particle distribution with a number density $n_p = 10^{-3} \, h^3$ Mpc$^{-3}$ within a survey cone defined by right ascension range $165^\circ < R.A. < 195^\circ$, declination range $-15^\circ < \text{Dec.} < 15^\circ$ and redshift range $0.3 < z < 0.7$, using the simulation fiducial cosmology. This cone has a closest-fitting Fourier cuboid of volume $V = 0.84 \times 10^9 \, h^{-3}$ Mpc$^3$, of which a fraction $V_w/V = 0.54$ is observed.

(ii) We converted the co-moving co-ordinates of the particles into redshift-space positions with respect to the observer at $z = 0$, using the components of the particle peculiar velocities along the line-of-site, whose direction varies across the cone in a curved-sky geometry.

(iii) We split the particles inside the survey cone into two random subsamples, which respectively form the overlapping galaxy and intensity-mapping datasets, with a selection function $W_{\text{cone}}(x)$ which is constant inside the cone, and zero outside the cone.
(iv) We binned the galaxy particles in a 128×128 FFT grid, where each FFT grid cell has side length ~ 7 h⁻¹ Mpc, volume \( \Delta V_{\text{FFT}} = 399.7 h^{-3} \) Mpc³, and associated Nyquist frequency in each dimension \( k_N \sim 0.4 h \) Mpc⁻¹.

(v) We binned the intensity-mapping particles in a spherical grid of HEALPix cells with \( N_{\text{side}} = 128 \) and redshift bins of width \( \Delta z = 0.0025 \). At the centre of the survey cone, an angular pixel subtended \( \sim 10.7 h^{-1} \) Mpc and a redshift channel extended \( \sim 5.8 h^{-1} \) Mpc. The angular footprint of the survey cone was covered by 4026 HEALPix pixels. We normalized the gridded intensity map such that pixels within the survey cone had a mean value of unity, \( f(x) = 1 \).

(vi) We added uncorrelated noise to each spherical pixel \( i \) of the simulated intensity map, drawn from a Gaussian distribution of zero mean and standard deviation \( \sigma_i \). We chose to create unique noise \( \sigma^2(x) \) across the survey cone by varying \( \sigma^2_i \) with the volume of each pixel \( \Delta V_i \) in accordance with Equation 33, such that \( \sigma_i^2 = \sigma_{\text{fid}}^2 \times \sqrt{\Delta V_{\text{fid}}/\Delta V_i} \), where we chose \( \sigma_{\text{fid}} = 1 \) and \( \Delta V_{\text{fid}} = \Delta V_{\text{FFT}} \). From Equation 33, the uniform noise value is \( \sigma_i^2 = \sigma_{\text{fid}}^2 \Delta V_{\text{fid}}/V \).

(vii) We smoothed each redshift slice of the intensity-mapping dataset with a Gaussian beam of standard deviation \( \sigma_B = 0.25^\circ \), using the HEALPix function `sphtfunc.smoothing`.

(viii) We binned the smoothed, noisy intensity-mapping dataset discretized in spherical pixels onto the same cubic FFT grid as the galaxy dataset. We performed this step using a Monte Carlo algorithm, in which we generated a large number (~ 10⁸) of random points across the survey cone, and binned the random points in both the spherical pixels and the FFT pixels. We then used the spherical binning to transfer the values of the intensity-mapping dataset in the spherical pixels to the random points in the FFT grid, and averaged these values on the FFT grid.

(ix) We computed the fluctuation fields of the gridded galaxy and intensity-mapping datasets as \( \delta(x) = f(x) - \langle f(x) \rangle \), and then estimated their auto- and cross-power spectrum multipoles \( \{P_{gg}, P_{gT}, P_{TT}\} \) using Equations 62 and 63. Since the datasets are uniformly distributed within the survey cone, we assumed weights \( w(x) = 1 \) within the cone, and set \( w(x) = 0 \) outside the cone. We binned our power spectrum estimates in 15 Fourier bins of width \( \Delta k = 0.02 h \) Mpc⁻¹ in the range \( 0 < k < 0.3 h \) Mpc⁻¹.

(x) We subtracted Poisson noise from the galaxy power spectrum, but do not subtract the noise power from the intensity-mapping power spectrum.

(xi) We assigned errors to the measured multipole power spectra using Equation 81.

We computed the model power spectra to compare with these measurements as follows:

(i) We generated the underlying power spectrum \( P(k) \) of the fields using the RSD power spectrum of Equation 11. We used a non-linear matter power spectrum \( P_m(k) \) generated using CAMB (Lewis et al. 2000) and halofit (Smith et al. 2003; Takahashi et al. 2012), and adopted parameters \( b = 1 \) (given these are dark matter particles), \( f = O_{\text{m}}^{\text{55}} = 0.49 \) and \( \sigma_v = 400 \) km s⁻¹, which produced a good match to the redshift-space power spectrum of the original particle distribution in the full simulation cube.

(ii) We computed the noise component of the auto-power spectra using Equation 24, \( P_{\text{noise}} = \sqrt{\sigma_i^2/f_0^2} \). For both the galaxy dataset and intensity map, we included the contribution of the Poisson noise resulting from the discreteness of the original particle distribution. This case corresponds to \( \sigma^2_i = f_0 = \sqrt{n_p/2} \). For the intensity map, we also included the additional noise, with \( \sigma^2_{\text{fid}} = \sigma^2_{\text{fid}}/V \) and \( f_0 = 1 \) as above.

(iii) For the intensity-mapping auto-power spectrum and the cross-power spectrum, we included a damping function, \( P(k) \rightarrow P(k) D^2(k) \) to model the smoothing and pixelization. The damping function for the auto-power spectrum of the intensity map is given by Equation 27, \( D^2(k) = \frac{\theta}{\sqrt{2\pi}} \int d^2 x |\tilde{B}(\mathbf{k}, x)|^2 \), where \( \tilde{B} \) combines the effects of the telescope beam and spherical pixelization such that:

\[
\tilde{B}(\mathbf{k}, x) = \tilde{B}_{\text{beam}} \tilde{B}_{\text{chan}} \tilde{B}_{\text{ang}} = e^{-k_\perp^2 |x|^2/2} \times \frac{\sin(k_\|=s||x||/2)}{k_\|=s||x||/2} \times W_{\text{ang}}(k_\| |x|),
\]

where \( (k_\perp, k_\|) = (k\sqrt{1 - \mu^2}, k\mu) \) are the components of \( \mathbf{k} \) perpendicular and parallel to the line of sight, \( |x| \) is the distance of each position from the observer, \( \sigma_B = 0.25^\circ \) is the Gaussian telescope beam, \( s_0(x) = c \Delta z/H(z) \) is the spatial width of the redshift bin at redshift \( z \), and \( W_{\text{ang}}(\ell) \) is the multipole pixel window function of the HEALPix pixelization for \( N_{\text{side}} = 128 \). For the cross-power spectrum, we evaluate \( D^2(k) = \frac{\theta}{\sqrt{2\pi}} \int d^2 x \tilde{B}_{\text{beam}} \tilde{B}_{\text{chan}}^2 \tilde{B}_{\text{ang}}^2 \), using only one power of the telescope beam since only the intensity map is smoothed, but retaining two powers of the pixelization for the reasons discussed at the end of Section 2.4.1. The relative contributions of these different smoothing terms to the overall power spectrum damping is illustrated by Figure 1.

(iv) We convolved the damped, model power spectra with the window function of the survey cones, \( P(k) \rightarrow P \ast |W_{\text{cone}}|^2 \).

(v) To allow for the discretization onto the FFT grid, we summed the resulting power spectra over modes \( k_n = k + 2k \Lambda n \) using Equation 48, taking a 3³ grid of \( n = \{-1,0,1\} \).

(vi) We averaged the model power spectra in the same Fourier bins as the measurements.

The resulting power spectrum multipole measurements and models for this test case are displayed in Figure 2, and are in good general agreement. The measurements of the galaxy power spectrum \( P_{gg} \) for \( \ell = \{0,2,4\} \), and the measurements of the hexadecapole \( \ell = 4 \) for \( \{P_{gg}, P_{gT}, P_{TT}\} \), all produce a satisfactory \( \chi^2 \) statistic with \( \chi^2/\text{dof} \sim 1 \). The measurements of the
monopole and quadrupole intensity power spectrum $P_{T T}$ and cross-power spectrum $P_{g T}$ show some deviations from the model with respect to the small error bars shown in Figure 2. The average deviation in the range $k < 0.3 \, h \, \text{Mpc}^{-1}$ is 8%, which we attribute to the approximations implemented when deriving the damping of the model due to spherical pixelization, as described in Section 2.4. However, we conclude that our model is satisfactory to within 10% accuracy, which is likely to remain sufficient for the analysis of near-future intensity mapping datasets.

### 5 SUMMARY

In this paper we have provided a general framework connecting the measured 2-point auto- and cross-correlations of fluctuation fields to their underlying cosmological power spectra, in the presence of a variety of observational effects. Our framework can be applied to the analysis of galaxy spectroscopic redshift surveys, datasets with accurate photometric redshifts, radio intensity-mapping surveys or other 3D cosmological maps.

The observational effects we considered are the variation with position of the background level of the field, measurement noise, the smoothing and discretization of the field, and the changing line-of-sight direction. We extended previous literature by deriving that if a field is smoothed by a position-dependent kernel, $B(s, x)$, where $s$ is the kernel separation with respect to position $x$, then the power spectrum is damped by the volume-average of the Fourier transform of the kernel at each position, 

$$\frac{1}{V} \int d^3 x |\tilde{B}(k, x)|^2.$$ 

We applied this result to the cases of averaging a field in irregular cells, applying noise in these cells, a telescope beam, redshift errors, and binning data in frequency channels and angular pixels.

We reviewed the direct estimators of the auto- and cross-power spectra, their multipoles, and the variance in these statistics, extending the results of Feldman et al. (1994) to present optimal weights for measuring the auto- and cross-power spectra of general cosmological fluctuation fields, with application to galaxy surveys and intensity maps. FKP weights for individual tracers do not in general provide the optimal weights when measuring the cross-power spectrum. We validated our model by demonstrating that it reproduces the power spectrum multipoles describing the auto- and cross-correlations of a mock galaxy and intensity-mapping dataset drawn from an N-body simulation, including several of these observational effects, within 10% accuracy.

We hope that this study has provided a set of recipes and derivations which will be useful for modelling and studying the Fourier-space statistics of cosmological fluctuation fields in future observed and simulated datasets. Accompanying power spectrum code for producing our mock dataset and evaluating the measurements and models is available at [https://github.com/cblakeastro/intensitypower](https://github.com/cblakeastro/intensitypower).

*Figure 2.* The auto-power spectrum multipoles of the simulated galaxy survey $P_{g g}$ (black error bars) and intensity-mapping survey $P_{T T}$ (red errors), and the cross-power spectrum multipoles $P_{g T}$ (green errors). The left-hand, middle and right-hand panels show the monopole ($P_0$), quadrupole ($P_2$) and hexadecapole ($P_4$), respectively, and the solid lines display the computed models in each case. The power spectra are scaled by a factor of $k$ for clarity of presentation.
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