PROJECTIVE RECONSTRUCTION IN ALGEBRAIC VISION

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Abstract. We discuss the geometry of rational maps from a projective space of an arbitrary dimension to the product of projective spaces of lower dimensions induced by linear projections. In particular, we give a purely algebro-geometric proof of the projective reconstruction theorem by Hartley and Schaffalitzky [HS09].

1. Introduction

Let \( r \) be a positive integer and \( \mathbf{m} = (m_1, \ldots, m_r) \) be a sequence of positive integers. For each \( i = 1, \ldots, r \), take a vector space \( W_i \) of dimension \( m_i + 1 \) over a field \( k \), which we assume to be an algebraically closed field of characteristic zero unless otherwise stated. Let further \( V \) be a vector space satisfying \( n := \dim V - 1 > m_i \) for any \( i = 1, \ldots, r \). A sequence \( s = (s_1, \ldots, s_r) \) of surjective linear maps

\[
s_i : V \rightarrow W_i, \quad i = 1, \ldots, r
\]

induce rational maps

\[
\varphi_i : \mathbb{P}^n \rightarrow \mathbb{P}^{m_i}, \quad i = 1, \ldots, r
\]

from \( \mathbb{P}^n := \mathbb{P}(V) \) to \( \mathbb{P}^{m_i} := \mathbb{P}(W_i) \) called cameras. The loci

\[
Z_i := \mathbb{P}(\ker s_i) \subset \mathbb{P}^n, \quad i = 1, \ldots, r
\]

of indeterminacy are called the focal loci of the cameras. The closure \( X \) of the image of the rational map

\[
\varphi := (\varphi_1, \ldots, \varphi_r) : \mathbb{P}^n \rightarrow \mathbb{P}^m := \prod_{i=1}^r \mathbb{P}^{m_i}
\]

is called the multiview variety in the case \( n = 3 \) and \( \mathbf{m} = (2^r) := (2, \ldots, 2) \) in [AST13], and we use the same terminology for arbitrary \( n \) and \( \mathbf{m} \).

Basic properties of multiview varieties are studied in [Li], where formulas for dimensions, multidegrees and Hilbert polynomials are obtained and the Cohen–Macaulay property is proved. In the case with \( n = 3 \) and \( \mathbf{m} = (2^r) \), a compactification of the space of \( r \)-tuples of cameras is introduced in [AST13] using multigraded Hilbert schemes. The space of cameras is also studied in [LV] from a functorial point of view.

Each camera \( \varphi_i : \mathbb{P}^n \rightarrow \mathbb{P}^{m_i} \) is parametrized by an open subset of the projective space \( \mathbb{P}(V^\vee \otimes W_i) \). Let

\[
\Phi : \prod_{i=1}^r \mathbb{P}(V^\vee \otimes W_i) \rightarrow \text{Hilb}(\mathbb{P}^m)
\]

be the rational map sending a camera configuration \( \varphi \) to the multiview variety \( X \) considered as a point in the Hilbert scheme. The natural right action of \( \text{Aut}(\mathbb{P}^n) = \text{PGL}(n + 1, k) \) on each \( \mathbb{P}(V^\vee \otimes W_i) \) induces the diagonal action on the product \( \prod_{i=1}^r \mathbb{P}(V^\vee \otimes W_i) \). The main result of this paper is the following:

Theorem 1.1. Assume that \( |\mathbf{m}| := m_1 + \cdots + m_r \geq n + 1 \).

(1) If \( \mathbf{m} \neq (1^{n+1}) := (1, \ldots, 1) \), then a general fiber of \( \Phi \) consists of a single \( \text{PGL}(n + 1, k) \)-orbit.

(2) If \( \mathbf{m} = (1^{n+1}) \), then a general fiber of \( \Phi \) consists of two \( \text{PGL}(n + 1, k) \)-orbits.

(3) If \( |\mathbf{m}| \geq 2n - 1 \), then \( \Phi \) is dominant onto an irreducible component of \( \text{Hilb}(\mathbb{P}^m) \).
We assume $|m| \geq n + 1$ so that the multiview variety $X$ is a proper subvariety of $\mathbb{P}^m$. Theorem 1.1.1(1,2) is an algebro-geometric reformulation of the projective reconstruction theorem by Hartley and Schaffalitzky [HS09] with a new purely algebro-geometric proof. Theorem 1.1.3 for $n = 3$ and $m = (2^\ast)$ is proved in [AST13, Theorem 6.3], and the bound $|m| \geq 2n - 1$ is sharp in this case.

The motivation for Theorem 1.1 comes from the following problem in computer vision: Set $k = \mathbb{R}$, $n = 3$ and $m = (2^\ast)$. Consider taking $r$ photos of some object by $r$ distinct cameras. Assume that we do not know the configuration of the cameras, but we can tell the point correspondences, i.e., which point in one photo corresponds to which point in another photo, say, by the feature of the object. This amounts to the assumption that we do not know the linear projection $\varphi$, but we know the multiview variety $X$. Now the projective reconstruction problem asks if $\varphi$ is determined uniquely from $X$, up to the inevitable ambiguity by the action of $\text{PGL}(n + 1, \mathbb{R})$. The projective reconstruction theorem of Hartley and Schaffalitzky states that when $|m|$ is greater than $n$ and $s$ is general, the unique projective reconstruction holds if $m \neq (1^{n+1})$, and the 2-to-1 projective reconstruction holds if $m = (1^{n+1})$.

Note that a real configuration of cameras can naturally be viewed as a complex configuration of cameras, and a pair of real configurations of cameras are related by an action of $\text{PGL}(1, \mathbb{C})$. It follows that the reconstruction over $\mathbb{C}$ in Theorem 1.1.1 implies the reconstruction over $\mathbb{R}$. See also Remark 3.5 for the fact that Theorem 1.1.2 also holds over $\mathbb{R}$.

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### 2. Projective reconstruction in the case $m \neq (1^{n+1})$

We prove Theorem 1.1.1 in this section. We keep the same notations as in Section 1 and write the projections as $p: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^n$, $q_i: \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{m_i}$, $q := (q_1, \ldots, q_r): \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m$ and $w_i: \mathbb{P}^m \to \mathbb{P}^{m_i}$. We do not assume $|m| \geq n + 1$ unless otherwise stated. We abuse notation and identify $s$ with the corresponding section in

$$H^0 \left( \mathbb{P}^n \times \mathbb{P}^m, \bigoplus_{i=1}^{r} p_i^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes q_i^* T_{\mathbb{P}^{m_i}}(-1) \right) \cong \bigoplus_{i=1}^{r} V^\vee \otimes W_i.$$  \hspace{1cm} (2.1)

Let $\tilde{X} \subset \mathbb{P}^n \times \mathbb{P}^m$ be the closure of the graph of the rational map $\varphi: \mathbb{P}^n \dashrightarrow \mathbb{P}^m$, and set $\tilde{p} := p|_{\tilde{X}}$, $\tilde{q}_i := q_i|_{\tilde{X}}$, and $\tilde{q} := q|_{\tilde{X}}$, so that we have the diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{p}} & \mathbb{P}^n \\
\mathbb{P}^n \xrightarrow{\varphi} \xrightarrow{\tilde{q}} X & \subset & \mathbb{P}^m.
\end{array}$$  \hspace{1cm} (2.2)

Assume that the section $s$ in (2.1) is general.

**Lemma 2.1.** Let $Z_s \subset \mathbb{P}^n \times \mathbb{P}^m$ be the zero locus of the section $s$ in (2.1). For $z \in \mathbb{P}^n$, one has

$$Z_s \cap p^{-1}(z) = \{z\} \times \prod_{i \in \mathbb{Z}_s} \varphi_i(z) \times \prod_{i \in \mathbb{Z}_s} \mathbb{P}^{m_i}. \hspace{1cm} (2.3)$$

**Proof.** Let $v \in V$ be a vector corresponding to $z \in \mathbb{P}^n = \mathbb{P}(V)$. Then $Z_s \cap p^{-1}(z) \subset \{z\} \times \mathbb{P}^m$ coincides with the zero locus of the section

$$\left( s_1(v), \ldots, s_r(v) \right) = \bigoplus_{i=1}^{r} W_i \cong H^0 \left( \mathbb{P}^m, \bigoplus_{i=1}^{r} q_i^* T_{\mathbb{P}^{m_i}}(-1) \right). \hspace{1cm} (2.4)$$

For each $i$, the zero locus of $s_i(v) \in H^0(\mathbb{P}^{m_i}, T_{\mathbb{P}^{m_i}}(-1))$ is $\varphi_i(z)$ (resp. $\mathbb{P}^{m_i}$) if $s_i(v) \neq 0$ (resp. $s_i(v) = 0$). Since $s_i(v) = 0$ if and only if $z \in Z_s$, we have (2.3).  \hspace{1cm} \Box
Lemma 2.2. The zero locus $Z_s$ coincides with $\tilde{X}$.

Proof. Since $\mathbb{P}^m$ is irreducible, the closure $\tilde{X}$ of the graph of $\varphi$ is irreducible of dimension $n$. By Lemma 2.1, the generic point of $\tilde{X}$ and hence $\tilde{X}$ itself are contained in $Z_s$. Since $s$ is a general section of a globally generated bundle, the zero of $s$ is smooth of dimension $n$ by a generalization of the theorem of Bertini [Muk92, Theorem 1.10]. Since $Z_s$ is smooth and all fibers of $Z_s \to \mathbb{P}^m$ are irreducible by Lemma 2.1 $Z_s$ is irreducible as well. It follows that $\tilde{X}$ and $Z_s$ are equal, since $\tilde{X} \subset Z_s$ and they are irreducible of the same dimension. □

We regard the section $q_* s \in H^0(\mathbb{P}^m, V^\vee \otimes \bigoplus_{i=1}^r q_i^* T_{\mathbb{P}^m_i}(-1))$ as a morphism

\begin{equation}
V \otimes \mathcal{O}_{\mathbb{P}^m} \to \bigoplus_{i=1}^r q_i^* T_{\mathbb{P}^m_i}(-1)
\end{equation}

on $\mathbb{P}^m$. It follows from the definition of $Z_s = \tilde{X}$ that

\begin{equation}
\tilde{q}^{-1}(x) = \mathbb{P}(\ker(q_* s)_x) \times \{x\} \subset \mathbb{P}^n \times \mathbb{P}^m
\end{equation}

for any $x \in \mathbb{P}^m$. For $0 \leq j \leq \min\{n+1, |m|\}$, let $X_j \subset \mathbb{P}^m$ be the $j$-th degeneracy locus of $q_* s$ defined as the zero of

\begin{equation}
(q_* s)^{(j+1)}: \bigcup_{j=0}^{j+1} V \otimes \mathcal{O}_{\mathbb{P}^m} \to \bigoplus_{i=1}^r q_i^* T_{\mathbb{P}^m_i}(-1),
\end{equation}

that is, the locus where the rank of $q_* s$ is at most $j$. Hence one has

\begin{equation}
dim \ker((q_* s)_x: V \to \bigoplus_{i=1}^r q_i^* T_{\mathbb{P}^m_i}(-1) \otimes k(x)) = n+1-j
\end{equation}

for $x \in X_j \setminus X_{j-1}$. In particular, one has

\begin{equation}
X = X_n.
\end{equation}

Since $q_* s \in H^0(\mathbb{P}^m, V^\vee \otimes \bigoplus_{i=1}^r q_i^* T_{\mathbb{P}^m_i}(-1))$ is a general section of a globally generated vector bundle, one has $X_j = \emptyset$ or

\begin{equation}
codim (X_j / \mathbb{P}^m) = (n+1-j)(|m|-j)
\end{equation}

by Ott95 Theorem 2.8]. If $|m| \geq n+1$, the dimension of $X_{n-1}$ is at most

\begin{align}
|m| - (n+1) - (n-1)(|m| - (n-1)) &= |m| - 2(|m| - n + 1) \\
&= 2n - |m| - 2 \\
&\leq n - 3,
\end{align}

so that $X = X_n$ is smooth in codimension one. Since $X_n$ is Cohen-Macaulay by ACGHS5 Chapter II], $X$ is normal.

A morphism is said to be small if it is an isomorphism in codimension one.

Lemma 2.3. If $|m| \geq n+1$, the morphism \( \tilde{q}: \tilde{X} \to X \) is small.

Proof. It follows from (2.6) that $\tilde{q}$ is an isomorphism over $X_n \setminus X_{n-1}$ and

\begin{align}
\dim \tilde{q}^{-1}(X_{n-1}) &= \max \{\dim (X_j \setminus X_{j-1}) + n - j \mid 0 \leq j \leq n - 1\} \\
&\leq \max \{|m| - (n+1) - (n-1)(|m| - j) + n - j \mid 0 \leq j \leq n - 1\} \\
&= \max \{j - (n-j)(|m| - j - 1) \mid 0 \leq j \leq n - 1\} \\
&= 2n - |m| - 1 \\
&\leq n - 2,
\end{align}

hence $\tilde{q}$ is small. □
Since $s_i : p^*\mathcal{O}_{\mathbb{P}^n}(-1) \to q_i^*T_{\mathbb{P}^m_i}(-1)$ is zero on $\tilde{X}$, its restriction $s_i|_{\tilde{X}}$ is a global section of $\tilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \tilde{q}_i^*\mathcal{O}_{\mathbb{P}^m_i}(-1)$. Let $E_i \subset \tilde{X}$ be the Cartier divisor defined by this section. The image of the projection $\tilde{p} \times \tilde{q}_i : \tilde{X} \to \mathbb{P}^n \times \mathbb{P}^m_i$ is the blow-up of $\mathbb{P}^n$ along $Z_i$, and $E_i$ is the total transform of the exceptional divisor of this blow-up. In particular, $E_i = \tilde{p}^{-1}(Z_i)$ holds.

**Lemma 2.4.** (1) The restriction of $\tilde{p}$ to $\tilde{X} \setminus \bigcup_{i=1}^r E_i$ is an isomorphism onto $\mathbb{P}^n \setminus \bigcup_{i=1}^r Z_i$.

(2) For each $i = 1, \ldots, r$, the divisor $E_i$ is irreducible.

(3) One has $\tilde{q}(E_i) = \mathbb{P}^m_i \times (\phi_2, \ldots, \phi_r)(Z_1 \setminus \bigcup_{j \neq i} Z_j)$, and similarly for $\tilde{q}(E_i)$ for $i = 2, \ldots, r$.

**Proof.** (1) holds since the rational map $\varphi$ is defined on $\mathbb{P}^n \setminus \bigcup_{i=1}^r Z_i$ and $\tilde{X}$ is the graph of $\varphi$.

(2) It follows from Lemma 2.1 that $\tilde{p}^{-1}(Z_i \setminus \bigcup_{j \neq i} Z_j)$ is irreducible of dimension $n - 1$ and $\dim \tilde{p}^{-1}(Z_i \cap \bigcup_{j \neq i} Z_j) < n - 1$. Since $E_i$ is a Cartier divisor, all irreducible components are $n - 1$-dimensional, and hence $E_i$ is irreducible.

(3) By Lemma 2.1 the image of $\tilde{p}^{-1}(Z_i \setminus \bigcup_{j \neq i} Z_j)$ by $\tilde{q}$ is $\mathbb{P}^m_i \times (\phi_2, \ldots, \phi_r)(Z_1 \setminus \bigcup_{j \neq i} Z_j)$. Since $\tilde{p}^{-1}(Z_1 \setminus \bigcup_{j \neq i} Z_j)$ is dense in $E_i$, we have (3).

From now on, we assume $|m| \geq n + 1$. Lemma 2.3 implies that $\tilde{q}(E_i)$ is a prime Weil divisor on $X$. We set

$$L_i := \omega_i^*\mathcal{O}_{\mathbb{P}^m_i}(1)|_X$$

for $i = 1, \ldots, r$.

**Lemma 2.5.** The rational map defined by $|\mathcal{O}_X(\tilde{q}(E_1)) \otimes L_1|$ is inverse to the rational map $\varphi : \mathbb{P}^n \dashrightarrow X$.

**Proof.** Since $X$ is normal and $\tilde{q}$ is small, we have

$$H^0(X, \mathcal{O}_X(\tilde{q}(E_1)) \otimes L_1) \cong H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(E_1) \otimes \tilde{q}_1^*\mathcal{O}_{\mathbb{P}^m_1}(1)).$$

Since $\mathcal{O}_{\tilde{X}}(E_1) \cong \tilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes \tilde{q}_1^*\mathcal{O}_{\mathbb{P}^m_1}(-1)$, we have

$$H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(E_1) \otimes \tilde{q}_1^*\mathcal{O}_{\mathbb{P}^m_1}(1)) \cong H^0(\tilde{X}, \tilde{p}^*\mathcal{O}_{\mathbb{P}^n}(1)).$$

Hence $|\mathcal{O}_X(\tilde{q}(E_1)) \otimes L_1|$ is inverse to the rational map $\varphi : \mathbb{P}^n \dashrightarrow X$.

**Lemma 2.6.** Assume $|m| \geq n + 2$ or $|m| = n + 1$ and $m_1 \geq 2$. Then $\tilde{q}(E_1)$ is the unique Weil divisor on $X$ of the form $\mathbb{P}^m_1 \times Y$ for some $Y \subset \prod_{i \neq 1} \mathbb{P}^m_i$.

**Proof.** By Lemma 2.4 (3), $\tilde{q}(E_1)$ is a Weil divisor of such form.

Assume there exists a subvariety $Y \subset \prod_{i \neq 1} \mathbb{P}^m_i$ of dimension $n - 1 - m_1$ such that $D = \mathbb{P}^m_1 \times Y$ is contained in $X$ and $D \neq \tilde{q}(E_1)$. Let $\tilde{X}^\dagger \subset \mathbb{P}^n \times \prod_{i \neq 1} \mathbb{P}^m_i$ and $X^\dagger \subset \prod_{i \neq 1} \mathbb{P}^m_i$ be the subvarieties obtained from $(s_2, \ldots, s_r)$ in the same way as $\tilde{X}$ and $X$. Consider the diagram

$$\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{q}} & X \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
\tilde{X}^\dagger & \xrightarrow{\tilde{q}^\dagger} & X^\dagger,
\end{array}$$

where $\pi, \tilde{\pi}, \tilde{q}^\dagger$ are induced by the natural projections. Lemma 2.4 shows that $E_1$ is the unique divisor contracted by $\tilde{\pi}$. Since $\tilde{q}$ is small and $D \neq \tilde{q}(E_1)$, we have a divisor $\tilde{D}^\dagger \subset \tilde{X}^\dagger$, which is the strict transform of $D$. Since $D = \mathbb{P}^m_1 \times Y$, one has $\pi(D) = Y$ and hence $\tilde{q}^\dagger(\tilde{D}^\dagger) = Y$. 


Since \( \dim \tilde{D} - \dim \bar{Y} = m_1 \), it follows from (2.6) and (2.8) for \( q_1, \tilde{X}, X^\dagger \) that \( Y \) must be contained in \( X^\dagger_{n-m_1} \), where we define the degeneracy locus \( X^\dagger_{n-m_1} \) in the same way as \( X_j \). On the other hand, the dimension of \( X^\dagger_{n-m_1} \) is at most

\[
\sum_{i \neq 1} m_i - (n + 1 - (n - m_1)) \left( \sum_{i \neq 1} m_i - (n - m_1) \right) = \sum_{i \neq 1} m_i - (m_1 + 1)(|m| - n).
\]

Hence we have

\[
n - 1 - m_1 = \dim Y \leq \dim X^\dagger_{n-m_1} \leq \sum_{i \neq 1} m_i - (m_1 + 1)(|m| - n),
\]

which implies \( m_1(|m| - n) \leq 1 \). This contradicts the assumption \(|m| \geq n + 2 \) or \(|m| = n + 1 \) and \( m_1 \geq 2 \).

Proof of Theorem 1.1. (1). Let \( X \) be the multiview variety for general \( \varphi \). In order to show that \( X \subset \mathbb{P}^m \) determines the rational map \( \varphi \) uniquely up to the action of \( \text{Aut}(\mathbb{P}^n) \cong \text{PGL}(n+1, \mathbb{k}) \), it suffices to see that the inverse \( \varphi^{-1} \) is uniquely determined by \( X \subset \mathbb{P}^m \) up to \( \text{PGL}(n+1, \mathbb{k}) \).

Assume \(|m| \geq n + 1 \) and \( m \neq (1^{n+1}) := (1, \ldots, 1) \). Relabeling the indexes of \( m \), if necessary, we may assume that \(|m| \geq n + 2 \) or \(|m| = n + 1 \) and \( m_1 \geq 2 \). Then Lemma 2.6 states that \( X \subset \mathbb{P}^m \) uniquely determines \( q(E_1) \subset X \) without using \( \varphi, \tilde{q}, \) etc. Hence \( X \subset \mathbb{P}^m \) uniquely determines \( \varphi^{-1} \) by Lemma 2.5. The inevitable ambiguity by the action of \( \text{PGL}(n+1, \mathbb{k}) \) comes from the choice of the isomorphisms (2.20) and (2.21).

\[ \square \]

3. Projective Reconstruction in the Case \( m = (1^{n+1}) \)

We prove Theorem 1.1. (2) in this section. Assume \( r = n + 1 \) and \( m_i = 1 \) for any \( 1 \leq i \leq n + 1 \). Note that \( \mathbb{P}(W_i) \) can be canonically identified with \( \mathbb{P}(W_i^\vee) \) since \( \dim W_i = 2 \). Set

\[
V' := (\text{coker } \mathbf{s})^\vee,
\]

which is \((n + 1)\)-dimensional since \( \mathbf{s} = (s_1, \ldots, s_{n+1}): V \rightarrow \bigoplus_{i=1}^{n+1} W_i \) is general. The canonical inclusion

\[
\mathbf{s}': V' \rightarrow \left( \bigoplus_{i=1}^{n+1} W_i \right)^\vee = \bigoplus_{i=1}^{n+1} W_i^\vee
\]

defines a hypersurface

\[
X' \subset \prod_{i=1}^{n+1} \mathbb{P}(W_i^\vee) = (\mathbb{P}^1)^{n+1}
\]

in the same way as \( X \). We also define \( \tilde{X}', E_i', \tilde{q}' \), etc. in the same way as \( X \).

Lemma 3.1. The hypersurfaces \( X \) and \( X' \) coincide under the canonical identifications \( \mathbb{P}(W_i) = \mathbb{P}(W_i^\vee) \) for \( i = 1, \ldots, n + 1 \).
Proof. On $\prod_{i=1}^{n+1} \mathbb{P}(W_i)$, we have a diagram

$\begin{array}{ccc}
0 & \xrightarrow{\oplus_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1)} & 0 \\
\oplus_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) & \xrightarrow{(3.4)} & V \otimes \mathcal{O} \\
0 & \xrightarrow{s} & (\oplus_{i=1}^{n+1} W_i) \otimes \mathcal{O} \quad \xrightarrow{(s')^\vee} \quad (V')^\vee \otimes \mathcal{O} \quad 0.
\end{array}$

(3.4)

A point $x \in \prod_{i=1}^{n+1} \mathbb{P}(W_i)$ is contained in $X = X_n$ if and only if the rank of the linear map

(3.5)

$$ (q_s s)_x : V \rightarrow \bigoplus_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) \otimes k(x) $$

is at most $n$, that is, $(q_s s)_x$ is not injective. By (3.4), this is equivalent to the condition that

(3.6)

$$ s(V) \cap \bigoplus_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) \otimes k(x) \neq \{0\}, $$

where we take the intersection as subspaces of $\bigoplus_{i=1}^{n+1} W_i$. By (3.4) again, this is equivalent to the condition that the rank of the linear map

(3.7)

$$ \bigoplus_{i=1}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1) \otimes k(x) \rightarrow (V')^\vee $$

is at most $n$. Under the identification $\mathbb{P}(W'_i) = \mathbb{P}(W_i)$, the sheaf $\mathcal{O}_{\mathbb{P}(W_i)}(1)$ is identified with $\mathcal{O}_{\mathbb{P}(W_i)}(1)$. Hence the rank of the linear map (3.7) is at most $n$ if and only if $x$ is contained in $X'$. Thus $X = X'$ holds. \qed

Recall from Lemma 2.4 that $\overline{q}(E_1) = \mathbb{P}(W_1) \times (\phi_2, \ldots, \phi_r)(Z_1)$. Similarly, one has $\overline{q}'(E'_1) = \mathbb{P}(W'_1) \times (\phi_2', \ldots, \phi_r')(Z_1')$. \vspace{0.5cm}

**Lemma 3.2.** The closure $(\overline{\phi}_2, \ldots, \overline{\phi}_r) (Z_1) \subset \prod_{i=2}^{n+1} \mathbb{P}(W_i)$ is the $(n-2)$-th degeneracy locus of the composite map

(3.8)

$$ (\ker s_1) \otimes \mathcal{O}_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} \hookrightarrow V \otimes \mathcal{O}_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} \xrightarrow{(s_2, \ldots, s_{n+1})} \bigoplus_{i=2}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1), $$

where we use the same letter $\varpi_i$ for the projection $\prod_{i=2}^{n+1} \mathbb{P}(W_i) \rightarrow \mathbb{P}(W_i)$. On the other hand, the closure $(\overline{\phi}_2', \ldots, \overline{\phi}_r') (Z_1) \subset \prod_{i=2}^{n+1} \mathbb{P}(W_i') = \prod_{i=2}^{n+1} \mathbb{P}(W_i)$ is the $(n-1)$-th degeneracy locus of

(3.9)

$$ (s_2, \ldots, s_{n+1}) : V \otimes \mathcal{O}_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} \rightarrow \bigoplus_{i=2}^{n+1} \varpi_i^* \mathcal{O}_{\mathbb{P}(W_i)}(-1). $$

**Proof.** The first statement follows by applying (2.9) to $(\phi_2, \ldots, \phi_r)|_{Z_1} : Z_1 = \mathbb{P}(\ker s_1) \rightarrow \prod_{i=2}^{n+1} \mathbb{P}(W_i)$. Since $s_1 : V \rightarrow W_1$ is surjective, we have an exact sequence

(3.10)

$$ 0 \rightarrow \ker s_1 \rightarrow \bigoplus_{i=2}^{n+1} W_i \rightarrow (V')^\vee \rightarrow 0, $$

Since $s_2, \ldots, s_{n+1}$ are surjective, we have an exact sequence

$$ 0 \rightarrow \ker s_2 \rightarrow \bigoplus_{i=2}^{n+1} W_i \rightarrow (V')^\vee \rightarrow 0, $$
which gives a diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & (\ker s_1) \otimes \mathcal{O} & \rightarrow & (\bigoplus_{i=2}^{n+1} W_i) \otimes \mathcal{O} & \rightarrow & (V')^\vee \otimes \mathcal{O} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\bigoplus_{i=2}^{n+1} \omega^i T_{\mathbb{P}(W_i)}(-1) & & & & & & & & \\
\end{array}
\]

(3.11)

By an argument similar to that in the proof of Lemma 3.1, we see that the \((n-2)\)-th degeneracy locus of (3.8) coincides with the \((n-1)\)-th degeneracy locus of \(\bigoplus_{i=2}^{n+1} \omega^i \mathcal{O}_{\mathbb{P}(W_i)}(-1) \rightarrow (V')^\vee \otimes \mathcal{O}\), that is, the \((n-1)\)-th degeneracy locus of \(V' \otimes \mathcal{O} \rightarrow \bigoplus_{i=2}^{n+1} \omega^i \mathcal{O}_{\mathbb{P}(W_i)}(1)\) on \(\prod_{i=2}^{n+1} \mathbb{P}(W_i)\).

By replacing \(X\) with \(X'\), we see that \((\phi_2', \ldots, \phi_r')(Z'_1) \subset \prod_{i=2}^{n+1} \mathbb{P}(W_i)\) is the \((n-1)\)-th degeneracy locus of (3.9) since \(V'\) and \(\omega^i \mathcal{O}_{\mathbb{P}(W_i)}(1)\) are replaced by \(V\) and \(\omega^i T_{\mathbb{P}(W_i)}(-1)\) respectively. \(\square\)

**Lemma 3.3.** One has \(\tilde{\mathbf{q}}(E_1) \neq \tilde{\mathbf{q}}'(E'_1)\).

**Proof.** It suffices to see \((\phi_2, \ldots, \phi_r)(Z_1) \neq (\phi_2', \ldots, \phi_r')(Z'_1)\). Take a general point \(y \in (\phi_2, \ldots, \phi_r)(Z_1)\).

By Lemma 3.2, the rank of

\[
(s_2, \ldots, s_{n+1})_y: V \rightarrow \bigoplus_{i=2}^{n+1} \omega^i T_{\mathbb{P}(W_i)}(-1) \otimes k(y)
\]

is \(n-1\) since \(s_2, \ldots, s_{n+1}\) and \(y\) are general. Hence \(\ker(s_2, \ldots, s_{n+1})_y \subset V\) is two-dimensional. Then \(\ker(s_2, \ldots, s_{n+1})_y \cap \ker s_1 = \{0\} \subset V\) since \(\ker s_1 \subset V\) is of codimension two and general. This means that (3.8) has rank \(n-1\) at \(y\). By Lemma 3.2, we have \(y \notin (\phi_2, \ldots, \phi_r)(Z_1)\). \(\square\)

Lemma 3.3 and Lemma 3.4 below show that we have exactly two reconstructions:

**Lemma 3.4.** The exceptional locus of the birational morphism \(X \rightarrow \prod_{i=2}^{n+1} \mathbb{P}(W_i)\) is the union of \(\tilde{\mathbf{q}}(E_1)\) and \(\tilde{\mathbf{q}}'(E'_1)\).

**Proof.** Since \(X \subset \mathbb{P}(W_1) \times \prod_{i=2}^{n+1} \mathbb{P}(W_i)\), the exceptional locus of \(X \rightarrow \prod_{i=2}^{n+1} \mathbb{P}(W_i)\) is

\[
\mathbb{P}(W_1) \times \left\{ y \in \prod_{i=2}^{n+1} \mathbb{P}(W_i) \mid \mathbb{P}(W_1) \times \{y\} \subset X \right\} \subset X.
\]

Hence we need to show

\[
\left\{ y \in \prod_{i=2}^{n+1} \mathbb{P}(W_i) \mid \mathbb{P}(W_1) \times \{y\} \subset X \right\} = (\phi_2, \ldots, \phi_r)(Z_1) \cup (\phi_2', \ldots, \phi_r')(Z'_1).
\]

Since \(\tilde{\mathbf{q}}(E_1) = \mathbb{P}(W_1) \times (\phi_2, \ldots, \phi_r)(Z_1)\) and \(\tilde{\mathbf{q}}'(E'_1) = \mathbb{P}(W_1) \times (\phi_2', \ldots, \phi_r')(Z'_1)\), the inclusion \(\supseteq\) in (3.14) is clear. To show the converse inclusion, we take \(y \notin (\phi_2, \ldots, \phi_r)(Z_1) \cup (\phi_2', \ldots, \phi_r')(Z'_1)\) and show \(\mathbb{P}(W_1) \times \{y\} \not\subset X\). By Lemma 3.2, the linear map

\[
(s_2, \ldots, s_{n+1})_y: V \rightarrow U := \bigoplus_{i=2}^{n+1} \omega^i T_{\mathbb{P}(W_i)}(-1) \otimes k(y)
\]

has rank \(n\) and the restriction \((s_2, \ldots, s_{n+1})_y|_{\ker s_1}\) has rank \(n-1\). Recall that the dimensions of \(V, U,\) and \(\ker s_1\) are \(n+1, n,\) and \(n-1\) respectively. Hence \(\ker(s_2, \ldots, s_{n+1})_y \subset V\) is one-dimensional and
ker(s_2, \ldots, s_{n+1}) \cap \ker s_1 = \{0\} \subset V. Let K \subset W_1 be the image of \ker(s_2, \ldots, s_{n+1}) by s_1: V \to W_1. Then we have a diagram

$$
\begin{array}{c}
0 \to \ker(s_2, \ldots, s_{n+1}) \otimes \mathcal{O} \xrightarrow{i} V \otimes \mathcal{O} \xrightarrow{(s_2, \ldots, s_{n+1})} U \otimes \mathcal{O} \to 0
\end{array}
$$

(3.16)
on \mathbb{P}(W_1) \times \{y\}, where c: K \otimes \mathcal{O} \to W_1 \otimes \mathcal{O} \to T_{\mathbb{P}(W_1)}(-1) is the canonical map. Then (c, 0) is injective outside of the point x_0 \in \mathbb{P}(W_1) corresponding to the one-dimensional subspace K \subset W_1. Hence the rank of s|_{\mathbb{P}(W_1) \times \{y\}} is n + 1 at any x \neq x_0 \in \mathbb{P}(W_1), which means that (\mathbb{P}(W_1) \setminus \{x_0\}) \times \{y\} is not contained in X. Thus y is not contained in the left hand side of (3.19).

**Proof of Theorem 1.1 (2).** Lemmas 3.3 and 3.4 show that \(q(E_1) \subset X\) is one of the exceptional prime divisors of the birational morphism \(X \to \prod_{i=2}^{n+1} \mathbb{P}(W_i)\). If we choose one of such divisors, we can reconstruct \(\varphi^{-1}\) or \(\varphi'^{-1}\) by Lemma 2.5 as in the proof of Theorem 1.1 (1).

**Remark 3.5.** If \(\varphi\) is defined over \(\mathbb{R}\), so is \(\varphi'.\) This follows from the construction of \(s'\) in (3.2).

We have the diagram

$$
\begin{array}{c}
\tilde{X} \xrightarrow{\tilde{p}} X \\
\mathbb{P}(V) \xrightarrow{\varphi} X = X' \xrightarrow{\varphi'} \mathbb{P}(V').
\end{array}
$$

(3.17)

In the rest of this section, we describe the birational map \(\varphi'^{-1} \circ \varphi: \mathbb{P}(V) \to \mathbb{P}(V)\). Recall the definition of \(L_i\) from (2.19).

**Lemma 3.6.** The divisor \(\tilde{q}(E_1) + \tilde{q}'(E_1')\) on \(X\) is linearly equivalent to \(-L_1 + \sum_{i=2}^{n+1} L_i\).

**Proof.** Since the exceptional locus of the birational morphism

$$
(\varpi_2, \ldots, \varpi_{n+1})|_X: X \to \prod_{i=2}^{n+1} \mathbb{P}(W_i)
$$

(3.18)is \(\tilde{q}(E_1) \cup \tilde{q}'(E_1')\) by Lemma 3.4, we can write

$$
K_X = (\varpi_2, \ldots, \varpi_{n+1})^*K_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} + a\tilde{q}(E_1) + a'\tilde{q}'(E_1')
$$

(3.19)for some integers \(a\) and \(a'.\) By the birational map

$$
(\varphi_2, \ldots, \varphi_{n+1}): \mathbb{P}(V) \to \prod_{i=2}^{n+1} \mathbb{P}(W_i),
$$

(3.20)the birational morphism (3.18) can be identified with the blow-up \(\text{Bl}_{Z_1} \mathbb{P}(V) \to \mathbb{P}(V)\) over the generic point of \(Z_1\). Hence the integer \(a\) in (3.19), which is the coefficient of \(\tilde{q}(E_1) = \mathbb{P}(W_1) \times (\varphi_2, \ldots, \varphi_r)(Z_1)\), is one. Similarly one has \(a' = 1\), and

$$
\tilde{q}(E_1) + \tilde{q}'(E_1') = K_X - (\varpi_2, \ldots, \varpi_{n+1})^*K_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)}
$$

(3.21)holds. By (2.17), \(X\) is linearly equivalent to \(\sum_{i=1}^{n+1} \varpi_i^*\mathcal{O}(1)\) on \(\prod_{i=1}^{n+1} \mathbb{P}(W_i)\). Hence we have

$$
K_X = -\sum_{i=1}^{n+1} L_i
$$

by the adjunction formula. Since \((\varpi_2, \ldots, \varpi_{n+1})^*K_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} = -2\sum_{i=2}^{n+1} L_i\), one has

$$
\tilde{q}(E_1) + \tilde{q}'(E_1') = K_X - (\varpi_2, \ldots, \varpi_{n+1})^*K_{\prod_{i=2}^{n+1} \mathbb{P}(W_i)} = -L_1 + \sum_{i=2}^{n+1} L_i,
$$

(3.22)and Lemma 3.6 is proved. \(\square\)

For each \(i = 1, \ldots, n + 1\), let \(F_i \subset \tilde{X}\) be the strict transform of the divisor \(\tilde{q}'(E'_i) \subset X' = X\).
Corollary 3.8. (1) The birational map $\varphi^{-1} \circ \varphi : \mathbb{P}(V) \dashrightarrow \mathbb{P}(V')$ is obtained by the linear system

$$\mathcal{O}_{\mathbb{P}(V)}(n) \otimes \bigoplus_{i=1}^{n} Z_i.$$  

(2) For each $i$, the image $\widetilde{\varphi}(F_i) \subset \mathbb{P}(V)$ is the unique hypersurface of degree $n-1$ containing $Z_j$ for all $j \in \{1, \ldots, n+1\} \setminus \{i\}$.

(3) The birational map $\varphi^{-1} \circ \varphi$ contracts the hypersurface $\widetilde{\varphi}(F_i)$ to $Z'_i$ for each $i$.

Proof. (1) By Lemma 2.5, the birational map $\varphi^{-1}$ is obtained by the linear system $\mathcal{O}_{\mathbb{P}(V)}(n) \otimes \bigoplus_{i=1}^{n} Z_i$. This follows from Lemma 3.7.

$$F_1 + \widetilde{q}_i \mathcal{O}_{\mathbb{P}(V)}(1) \sim \widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(n-1) - \sum_{i=2}^{n+1} E_i + \widetilde{p}^* \mathcal{O}_{\mathbb{P}(V)}(1) - E_i$$

follows from Lemma 3.7 that $\mathcal{O}_{\mathbb{P}(V)}(n) \otimes \bigoplus_{i=1}^{n} Z_i$ is identified with $\mathcal{O}_{\mathbb{P}(V)}(n) \otimes I_{\bigcup_{i=1}^{n+1} Z_i}$ by $\overline{p}$. Hence $\varphi^{-1} \circ \varphi = \varphi^{-1} \circ \varphi \circ \overline{p}^{-1}$ is obtained by $\mathcal{O}_{\mathbb{P}(V)}(n) \otimes I_{\bigcup_{i=1}^{n+1} Z_i}$.

(2) It suffices to show this statement for $i = 1$. The linear system on $\mathbb{P}(V)$ consisting of divisors of degree $n-1$ containing $Z_1$, $\ldots$, $Z_{n+1}$ is identified with $\mathcal{O}_{\mathbb{P}(V)}(n-1) - \sum_{i=2}^{n+1} E_i$ on $\mathbb{F}$ by $\overline{p}$. By Lemma 3.7, we have $\overline{p}^* \mathcal{O}_{\mathbb{P}(V)}(n-1) - \sum_{i=2}^{n+1} E_i = \overline{F}_1$, which in turn is identified with the linear system $\mathcal{E}_1'$ on $\mathbb{F}$. The linear system $\mathcal{E}_1'$ is 0-dimensional since $\mathcal{E}_1'$ is an exceptional divisor. Hence $\overline{p}(F_1)$ is the unique such divisor.

(3) This statement holds since each $E'_i \subset \mathbb{F}'$ is contracted to $Z'_i$. $\square$

4. DOMINANCE FOR $|m| \geq 2n-1$

We prove Theorem 1.1 (3) in this section. We use the same notation as in Section 2.

Lemma 4.1.

$$\tilde{\Phi} : \prod_{i=1}^{r} \mathbb{P}(V^i \otimes W_i) \dashrightarrow \text{Hilb}(\mathbb{P}^n \times \mathbb{P}^m), \quad \varphi = (\varphi_1, \ldots, \varphi_r) \mapsto [\tilde{X}]$$

is a birational map onto an irreducible component.

Proof. Since we can recover $\varphi$ from the graph $\tilde{X}$ of $\varphi$, the rational map $\tilde{\Phi}$ is generically injective. Hence it suffices to show that the differential

$$\left(d\tilde{\Phi}\right)_{\varphi} : T_{\varphi} \prod_{i=1}^{r} \mathbb{P}(V^i \otimes W_i) \rightarrow T_{[\tilde{X}]} \text{Hilb}(\mathbb{P}^n \times \mathbb{P}^m)$$

is surjective for general $\varphi$.

Recall that $\tilde{X} \subset \mathbb{P}^n \times \mathbb{P}^m$ is the zero locus of a general section

$$s \in H^0\left(\mathbb{P}^n \times \mathbb{P}^m, \bigoplus_{i=1}^{r} p^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes q_i^* T_{\mathbb{P}^m}, (-1)\right).$$

Hence the normal bundle $N_{\tilde{X}/\mathbb{P}^n \times \mathbb{P}^m}$ is isomorphic to $\bigoplus_{i=1}^{r} p^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes q_i^* T_{\mathbb{P}^m}, (-1)$. From the Euler sequences on $\mathbb{P}^m$'s, we have an exact sequence on $\tilde{X}$

$$0 \rightarrow \bigoplus_{i=1}^{r} \overline{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \overline{q}_i^* T_{\mathbb{P}^m}, (-1) \rightarrow \bigoplus_{i=1}^{r} \overline{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes W_i \rightarrow \bigoplus_{i=1}^{r} \overline{p}^* \mathcal{O}_{\mathbb{P}^n}(1) \otimes \overline{q}_i^* T_{\mathbb{P}^m}, (-1) \rightarrow 0.$$
Since $\tilde{p}^*O_{\mathbb{P}^n}(1) \otimes \tilde{q}_i^*O_{\mathbb{P}^m}(-1) \simeq O_N(E_i)$ and $h^1(\tilde{X}, O_{\tilde{X}}(E_i)) = 0$, we see that the linear map
\begin{equation}
(4.5) \quad \bigoplus_{i=1}^r V^\vee \otimes W_i = H^0\left(\tilde{X}, \bigoplus_{i=1}^r \tilde{p}^*O_{\mathbb{P}^n}(1) \otimes W_i\right) \to H^0\left(\tilde{X}, \bigoplus_{i=1}^r \tilde{p}^*O_{\mathbb{P}^n}(1) \otimes \tilde{q}_i^*T_{\mathbb{P}^m}(-1)\right)
\end{equation}
is surjective with the kernel $\bigoplus_{i=1}^r H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(E_i)) = \bigoplus_{i=1}^r k s_i$. The induced isomorphism
\begin{equation}
(4.6) \quad \bigoplus_{i=1}^r V^\vee \otimes W_i/k s_i \simeq H^0\left(\tilde{X}, \bigoplus_{i=1}^r \tilde{p}^*O_{\mathbb{P}^n}(1) \otimes \tilde{q}_i^*T_{\mathbb{P}^m}(-1)\right)
\end{equation}
of vector spaces is identified with the differential $(4.2)$ under the isomorphisms $(4.11)$, where the middle term $(4.7)$ can be identified with $(4.13)$ induced by $s_i$'s, and Lemma 4.1 is proved.

**Proof of Theorem 1.1 (3).** Take general $\varphi$. We study the tangent space of $\text{Hilb}(\mathbb{P}^m)$ at $[X]$, which is isomorphic to $H^0(X, N_{X/\mathbb{P}^m})$. By $|m| \geq 2n-1$ and (2.12), $q: \tilde{X} \to X$ is an isomorphism in this case. The diagram
\begin{equation}
(4.9) \quad \begin{array}{ccc}
0 & \longrightarrow & T_{\tilde{X}} \longrightarrow T_{\mathbb{P}^n \times \mathbb{P}^m}|_{\tilde{X}} \longrightarrow N_{X/\mathbb{P}^n \times \mathbb{P}^m} \longrightarrow 0 \\
0 & \longrightarrow & \tilde{q}^* T_{\mathbb{P}^m} \longrightarrow \tilde{q}^* (T_{\mathbb{P}^m}|_{X}) \longrightarrow \tilde{q}^* N_{X/\mathbb{P}^m} \longrightarrow 0
\end{array}
\end{equation}
induces an exact sequence
\begin{equation}
(4.10) \quad 0 \to \tilde{p}^* T_{\mathbb{P}^n} \to N_{X/\mathbb{P}^n \times \mathbb{P}^m} \to \tilde{q}^* N_{X/\mathbb{P}^m} \to 0
\end{equation}
on $\tilde{X}$. Since
\begin{equation}
(4.11) \quad H^0\left(\tilde{X}, \tilde{p}^* T_{\mathbb{P}^n}\right) \cong H^0\left(\mathbb{P}^n, T_{\mathbb{P}^n}\right) \cong V^\vee \otimes V/k \text{id}_V
\end{equation}
for $\text{id}_V \in \text{Hom}(V, V) \cong V^\vee \otimes V$ and $h^1(\tilde{X}, \tilde{p}^* T_{\mathbb{P}^n}) = 0$, we have an exact sequence
\begin{equation}
(4.12) \quad 0 \to V^\vee \otimes V/k \text{id}_V \to \bigoplus_{i=1}^r V^\vee \otimes W_i/k s_i \to H^0(X, N_{X/\mathbb{P}^m}) \to 0,
\end{equation}
where the middle term
\begin{equation}
(4.13) \quad H^0\left(\tilde{X}, N_{X/\mathbb{P}^n \times \mathbb{P}^m}\right) \cong \bigoplus_{i=1}^r V^\vee \otimes W_i/k s_i
\end{equation}
can be identified with $T_{\varphi}(\mathbb{P}^n \times \mathbb{P}^m)$ as in the proof of Lemma 4.1. Then the map $d$ in (4.12) can be identified with $(d \Phi)_{\varphi}$, and $V^\vee \otimes V/k \text{id}_V$ can be identified with the tangent space of the PGL($n+1, k$)-orbit of $\varphi$. By Theorem 1.1 (1), a general fiber of $\Phi$ consists of at most two PGL($n+1, k$)-orbits. Note that the dimension of the PGL($n+1, k$)-orbits is equal to that of PGL($n+1, k$) since $\varphi$ is birational. Hence we have $\text{dim} \text{Im}(\Phi) = \text{dim} \bigoplus_{i=1}^r \mathbb{P}(V^\vee \otimes W_i) - \text{dim} \text{PGL}(n+1, k)$. Thus $\text{dim} \text{Im}(\Phi)$ is equal to $h^0(X, N_{X/\mathbb{P}^m})$ by (4.12), which is the dimension of the tangent space of $\text{Hilb}(\mathbb{P}^m)$ at $[X]$. This means that $\text{Hilb}(\mathbb{P}^m)$ is smooth at $[X]$ of dimension $h^0(X, N_{X/\mathbb{P}^m}) = \text{dim} \text{Im}(\Phi)$. Hence $\Phi$ is dominant onto an irreducible component.

**Remark 4.2.** In the case $n = 3$ and $m = (2^r)$, the condition $|m| \geq 2n-1$ is $2r \geq 5$, that is $r \geq 3$. As in [AST13] Section 6, the closure of the image of $\Phi$ is a cubic hypersurface in an irreducible component $\mathbb{P}^8 = \mathbb{P}(W^\vee_1 \otimes W^\vee_2) \subset \text{Hilb}(\mathbb{P}^m)$ for $r = 2$, so this is sharp in this case. We do not know whether the condition $|m| \geq 2n-1$ is sharp or not in general.
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