On classification of the solutions for general elliptic equation

Sirendaoreji
Mathematical Science College,Inner Mongolia Normal University,
Huhhot 010022,PR China
Email: siren@imnu.edu.cn

October,11,2018

Abstract: The Bäcklund transformations and the superposition formulas for two sub–equations of the general elliptic equation are constructed from the Riccati equation by using an indirect mapping method. The thirty-six previously known solutions of the general elliptic equation are proved to be equivalent to the another ten solutions. The classification of solutions for the general elliptic equation is obtained based on the equivalence relations. The general elliptic equation expansion method with new classified solutions are used to obtain abundant new exact traveling wave solutions of a modified Camassa-Holm equation.

1 Introduction

Seeking exact solutions of nonlinear evolution equations (NLEEs) is a practical and attractive problem in mathematical physics and nonlinear science. People have made continuous effort on the problem and various direct methods have been proposed to construct exact solutions for NLEEs. Among them the Jacobi elliptic function expansion method [1], the F–expansion method [2], the auxiliary equation method [3] and the sub-equation method [4, 5, 6] are related to the general elliptic equation of the form [7, 8, 9, 10, 11, 12]

\[ F''(\xi) = c_0 + c_1 F(\xi) + c_2 F^2(\xi) + c_3 F^3(\xi) + c_4 F^4(\xi), \] (1)

where \( c_i \) (\( i = 0, 1, \ldots, 4 \)) are constants. Except for the sub–equation method, the Eq. (1) was also chosen as the auxiliary equation in other direct methods. It is the fact that the Eq. (1) with \( c_0 = 1, c_1 = 0, c_2 = -(m^2 + 1), c_3 = 0, c_4 = m^2 \) and \( c_0 = 1-m^2, c_1 = 0, c_2 = 2m^2-1, c_3 = 0, c_4 = -m^2 \) were employed in the Jacobi elliptic function expansion method to seek the snoidal wave solutions and the cnoidal wave solutions of NLEEs, respectively. The F–expansion method for seeking twelve types of Jacobi elliptic function solutions of NLEEs was proposed by using the following sub-equation

\[ F''(\xi) = c_0 + c_2 F^2(\xi) + c_4 F^4(\xi). \] (2)
Another sub-equation \[ F''(\xi) = c_2 F^2(\xi) + c_3 F^3(\xi) + c_4 F^4(\xi), \] (3)

or

\[ G''(\xi) = h_2 G^2(\xi) + h_4 G^4(\xi) + h_6 G^6(\xi), \] (4)

were used in the auxiliary equation method to find different types of exact traveling wave solutions of NLEEs simultaneously. Because of Eq.(3) and Eq.(4) can be transformed into each other by the transformation

\[ G(\xi) = F^\frac{1}{2}(\xi), \quad h_2 = \frac{c_2}{4}, \quad h_4 = \frac{c_3}{4}, \quad h_6 = \frac{c_4}{4}, \] (5)

so we only need to study Eq.(3). In addition, the following sub-equations \[ F''(\xi) = c_0 + c_1 F(\xi) + c_2 F^2(\xi) + c_3 F^3(\xi), \] (6)

\[ F''(\xi) = c_0 + c_1 F(\xi) + c_2 F^2(\xi), \] (7)

were also widely applied to seek exact traveling wave solutions of NLEEs. The Eq.(1) generalized to the case of involving six degree nonlinear term was also studied by many authors \[ \cite{25, 26, 27, 28, 29, 30, 31}, \] but according to the relation (5) their results can be converted into the case of Eq.(3). All these facts show that the Eq.(1) has a strong application background and its solutions are worth to further study. At present, in the area of studying exact solutions of Eq.(1), there exist some problems such as

1. Many different solutions have been constructed and some of them are found to be equivalent, but these equivalence relations have not been proved.
2. The classification of its solutions is not given.
3. Its theoretical system such as the Bäcklund transformation (BT) and the superposition formula (SF) has not been established.

For this reason, the paper aims to give a systematic study of Eq.(1) and try to solve the problems in a comprehensive way. This effort lead us to organize the paper as follows. In Sec.2 we shall construct the BT and the SF for the sub-equations of Eq.(1). This lead us to construct the BT and the SF of Eq.(3) and Eq.(7) from the Riccati equation in terms of an indirect mapping method. In Sec.3 we shall give the proof of the equivalence relations of the known solutions for Eq.(1). As a result, the thirty six previously known solutions are proved to be equivalent to the another ten solutions for Eq.(1). In Sec.4 we shall give the classification of the solutions for Eq.(1). It is shown that the solutions of Eq.(1) are classified into five classes which consist of thirty eight independent solutions. In Sec.5 we shall construct abundant new exact traveling wave solutions for the modified Camassa–Holm equation by means of the general elliptic equation expansion method. Sec.6 is devoted to concluding remarks.
2 Bäcklund transformations and superposition formulas

Although the BT and the SF of Eq.(1) is difficult to establish in a simple way, but it usually can be constructed for its sub-equations through use of some tedious integration procedure. For avoid the complexity, below we shall construct the BT and the SF for Eq.(3) and Eq.(7) by using an indirect mapping method.

2.1 The Bäcklund transformation and superposition formula of Eq.(3)

Introducing the transformation

\[ F(\xi) = \frac{4f^2(\xi) - c_2}{c_3 - 4\sqrt{c_4}f(\xi)}, \]  

which, substituted into the Eq.(3), implies that

\[
\left( \frac{df}{d\xi} \right)^2 - c_2 f^2 - c_3 f^3 - c_4 f^4 = \frac{16\left( \frac{df}{d\xi} - f^2 + \frac{c_4}{c_3} \right) \left( \frac{df}{d\xi} + f^2 - \frac{c_4}{c_3} \right) H(f)}{(4\sqrt{c_4}f - c_3)^4} \]

\[ H(f) = 16c_4 f^4 - 16c_3 \sqrt{c_4} f^3 + (4c_3^2 + 8c_2 c_4) f^2 - 4c_2 c_3 \sqrt{c_4} f + c_2^2 c_4. \]

From this identity we deduce that \( F(\xi) \) is a solution of Eq.(3) if and only if \( f(\xi) \) is a solutions of the Riccati equation

\[ f'(\xi) = f^2(\xi) - \frac{1}{4}c_2. \]

This shows that the BT and the SF of Eq.(3) can be induced from the BT and the SF of the Riccati equation (9). Using the BT and SF for Riccati equation \( 32 \), we obtain the BT and the SF of Eq.(3) as follows

\[ f_n = \frac{f_{n-1} + \frac{\alpha_1 f_2 - \alpha_2 f_1}{\alpha_1 f_1 - \alpha_2 f_2} f_0}{1 + \alpha f_{n-1}} = \frac{4f_{n-1} + \alpha c_2}{4 + 4\alpha f_{n-1}}, \]  

and

\[ f_3 = \frac{\alpha_1 f_2 - \alpha_2 f_1}{\alpha_1 f_1 - \alpha_2 f_2} f_0. \]

where \( f_n, f_{n-1}, f_i \ (i = 0, 1, 2, 3) \) are solutions of Eq.(9) and \( \alpha, \alpha_i \ (i = 1, 2) \) are arbitrary constants. Taking the mapping (3) into consideration, we are confident that two solutions \( F_n \) and \( F_{n-1} \) of Eq.(3) can be written as

\[ F_n = \frac{4f^2_{n-1} - c_2}{c_3 - 4\sqrt{c_4}f_{n-1}}, \]

\[ F_{n-1} = \frac{4f^2_{n-2} - c_2}{c_3 - 4\sqrt{c_4}f_{n-1}}. \]

It solves from (13) that

\[ f_{n-1} = -\frac{1}{2}\sqrt{c_4}F_{n-1} + \frac{c_2}{2}T_{n-1}, \]

\[ T_{n-1} = \sqrt{c_2 + c_3 F_{n-1} + c_4 F^2_{n-1}}, \ \varepsilon = \pm 1. \]
Therefore, substituting (10) with (14) into (12) leads the BT of Eq. (3) of the form

\[ F_n = \frac{H_{n-1}^2 - c_2}{c_3 - 2\sqrt{c_4 H_{n-1}}} , H_{n-1} = \frac{2(\varepsilon T_{n-1} - \sqrt{c_4 F_{n-1}}) + \alpha c_2}{2 + \alpha(\varepsilon T_{n-1} - \sqrt{c_4 F_{n-1}})} , \tag{15} \]

where \( T_{n-1} \) is determined by (14). It is now easy to see from (11) and (12) that the SF of Eq. (3) is given by

\[
\begin{align*}
F_3 &= 4f_2^3 - c_2 \sqrt{c_4 f_3}, \\
F_n &= \frac{\alpha_1 f_2 - \alpha_2 f_1}{\alpha_1 f_1 - \alpha_2 f_2} f_0, \\
f_n &= \frac{1}{2} \left( \varepsilon T_n - \sqrt{c_4 F_n} \right), T_n = \sqrt{c_2 + c_3 F_n + c_4 F_n^2}, n = 0, 1, 2. \tag{16}
\end{align*}
\]

### 2.2 The Bäcklund transformation and superposition formula of Eq. (7)

By virtue of the indirect mapping method as used above, we define a mapping

\[ F(\xi) = \frac{c_1 + 4\sqrt{c_0 g(\xi)}}{4g^2(\xi) - c_2} , \tag{17} \]

and substitute it into Eq. (17) leads

\[
\left( \frac{dF}{d\xi} \right)^2 - c_0 - c_1 F - c_2 F^2 = \frac{16(\frac{\alpha_1 f_2 - \alpha_2 f_1}{\alpha_1 f_1 - \alpha_2 f_2}) + g^2 - \frac{1}{4}}{4g^2(c_2)} K(f),
\]

\[ K(f) = 16c_0 g^4 + 16c_1 \sqrt{c_0 g^3} + (4c_1^2 + 8c_0 c_2) g^2 + 4c_1 c_2 \sqrt{c_0 g} + c_0 c_2^2. \]

This indicates that \( F(\xi) \) is a solution of Eq. (7) if and only if \( g(\xi) \) satisfies the Riccati equation

\[ g'(\xi) = g^2(\xi) - \frac{1}{4} c_2 . \tag{18} \]

Following the same procedure as used to derive the BT and the SF for Eq. (9), we can now obtain the BT and the SF of Eq. (18)

\[ g_n = \frac{4g_{n-1} + \beta c_2}{4 + 4\beta g_{n-1}}, \tag{19} \]

and

\[ g_3 = \frac{\beta_1 g_2 - \beta_2 g_1}{\beta_1 g_1 - \beta_2 g_2} g_0, \tag{20} \]

respectively. It can be induced from the mapping (17) that

\[ F_n = \frac{c_1 + 4\sqrt{c_0 g_n}}{4g_n^2 - c_2}, \tag{21} \]

\[ F_{n-1} = \frac{c_1 + 4\sqrt{c_0 g_{n-1}}}{4g_{n-1}^2 - c_2}. \tag{22} \]

At the same time, it solves from (22) that

\[ g_{n-1} = \frac{1}{2F_{n-1}} \sqrt{c_0 + \varepsilon R_{n-1}}, R_{n-1} = \sqrt{c_0 + c_1 F_{n-1} + c_2 F_{n-1}^2}, \varepsilon = \pm 1. \tag{23} \]
The BT of Eq. (7) can be obtained by taking (19) and (23) into Eq. (21), which is now determined to be

\[ F_n = \frac{c_1 + 2\sqrt{c_0}K_{n-1}}{K_{n-1}^2 - c_2}, \quad K_{n-1} = \frac{2\varepsilon R_{n-1} + \beta c_2 F_{n-1} + 2\sqrt{c_0}}{\beta \varepsilon R_{n-1} + 2F_{n-1} + \beta \sqrt{c_0}}. \]  

(24)

Finally, the superposition formula of Eq. (7) is found to be

\[ \begin{cases} F_{3a} = \frac{c_1 + 2\sqrt{c_0}g_{3a}}{4g_{3a}^2 - c_2}, & g_{3a} = \frac{\beta_1 g_2 - \beta_2 g_1}{\beta_1 g_1 - \beta_2 g_2}, \\ g_n = \frac{\sqrt{c_0 + \varepsilon R_n}}{2F_n}, & R_n = \sqrt{c_0 + c_1 F_n + c_2 F_n^2}, \quad n = 0, 1, 2. \end{cases} \] 

(25)

3 Equivalence relations for solutions

Definition 1 Two waves are called equivalence in the sense of wave translation if they have the same waveform but their phase difference is equal to a constant. Here this equivalence relation will be expressed by the notation "\( \sim \)".

Lemma 1 For any real numbers \( \alpha \) and \( \beta \), if \( \alpha > 0, \beta > 0 \), then we hold that

\[ \begin{cases} \alpha e^\xi + \beta e^{-\xi} = 2\sqrt{\alpha \beta} \cosh(\xi + \frac{1}{2} \ln \frac{\alpha}{\beta}), \\ \alpha e^\xi - \beta e^{-\xi} = 2\sqrt{\alpha \beta} \sinh(\xi + \frac{1}{2} \ln \frac{\alpha}{\beta}). \end{cases} \]

Lemma 2 For any real numbers \( A \) and \( B \), if \( A^2 > B^2 \), then we have

\[ \begin{cases} A \sinh \eta \pm B \cosh \eta = \sqrt{A^2 - B^2} \sinh \left( \eta \pm \frac{1}{2} \ln \frac{A+B}{A-B} \right), & A > B, \\ A \sinh \eta \pm B \cosh \eta = -\sqrt{A^2 - B^2} \sinh \left( \eta \pm \frac{1}{2} \ln \frac{A+B}{A-B} \right), & A < B. \end{cases} \]

Lemma 3 For any real numbers \( A \) and \( B \), if \( A^2 < B^2 \), then we have

\[ \begin{cases} A \sinh \eta \pm B \cosh \eta = \pm \sqrt{B^2 - A^2} \cosh \left( \eta \pm \frac{1}{2} \ln \frac{B+A}{B-A} \right), & A > B, \\ A \sinh \eta \pm B \cosh \eta = \pm \sqrt{B^2 - A^2} \cosh \left( \eta \pm \frac{1}{2} \ln \frac{B+A}{B-A} \right), & A < B. \end{cases} \]

Lemma 4 For any real numbers \( A \) and \( B \), we have

\[ \begin{cases} A \sin \eta \pm B \cos \eta = \sqrt{A^2 + B^2} \sin (\eta \pm \theta), \\ A \cos \eta \pm B \sin \eta = \sqrt{A^2 + B^2} \cos (\eta \mp \theta), \end{cases} \]

where \( \theta = \arctan(B/A) \).

The above four lemmas are very important in classifying the traveling wave solutions of NLEEs. The proof of the Lemma 1 and the Lemma 4 are quite evidently. The Lemma 2 and the Lemma 3 can be proved by using the Lemma 1. So the proofs are omitted here.
3.1 The equivalence relations for solutions of Eq. (3)

We first prove the solutions of Eq. (3) given by Table 1 are equivalent to the solutions given by the Lemma 3 and the Lemma 2. Therefore, noting the definition of hyperbolic functions, using the formulas

\[
\begin{align*}
\Delta &> B, \\
\Delta &< B.
\end{align*}
\]

Thus for \( \Delta > 0 \) we have

\[
\begin{align*}
\Delta &> B, \\
\Delta &< B.
\end{align*}
\]

Hence for \( \Delta > 0 \) we obtain that \( A^2 > B^2 \) and \( A > B \). Therefore, noting the definition of hyperbolic functions, using the formulas

\[
\sinh x = 2 \sinh \frac{x}{2} \cosh \frac{x}{2}, \cosh^2 \frac{x}{2} = \frac{\cosh x + 1}{2}, \sinh^2 \frac{x}{2} = \frac{\cosh x - 1}{2},
\]

the Lemma 3 and the Lemma 2 we can derive that

\[
z_1(\xi) = \frac{-2c_2c_4}{c_4^2 + 2c_2c_4 \sinh(\sqrt{\Delta} \xi) + (c_4^2 - 2c_2c_4) \cosh(\sqrt{\Delta} \xi)},
\]

\[
= \begin{cases} 
\frac{2c_2}{\varepsilon \sqrt{\Delta} \cosh \left( \frac{\sqrt{\Delta} \xi + \frac{1}{2} \ln \left( \frac{2c_2}{\sqrt{\Delta}} \right)}{2} \right) - c_3} & \Delta > 0, c_2 > 0, \\
\frac{2c_2}{\varepsilon \sqrt{-\Delta} \sinh \left( \frac{\sqrt{-\Delta} \xi + \frac{1}{2} \ln \left( \frac{2c_2}{\sqrt{-\Delta}} \right)}{2} \right) - c_3} & \Delta < 0, c_2 > 0.
\end{cases}
\]

Following the same procedure we can derive that \( z_2(\xi) \approx F_1(\xi), \Delta > 0, c_2 > 0 \) and \( z_2(\xi) \approx F_2(\xi), \Delta < 0, c_2 > 0 \) and the details of the proof is omitted here.

Set \( A = 2\sqrt{c_2c_4}, B = c_3, \) then \( A^2 - B^2 = -\Delta. \) Thus for \( \Delta > 0 \) we hold \( A^2 < B^2. \) But for \( \Delta < 0, \) we have \( A^2 > B^2. \) Hence for \( c_2 > 0, c_4 > 0, \) the formulas (26), the Lemma 2 and the
Lemma 3 are used to obtain that

\[
z_3(\xi) = \frac{-2c_2}{c_3 + \xi \cosh(\frac{\sqrt{c_3} \xi}{\sqrt{c_2} + c_3})} \approx F_1(\xi), \Delta > 0,
\]

\[
= \begin{cases}
\frac{\varepsilon \Delta \cosh(\frac{\sqrt{c_3} \xi}{\sqrt{c_2} + c_3})}{2c_2} - c_3 \approx F_2(\xi), \Delta < 0.
\end{cases}
\]

Applying the same procedure as used to prove \( z_3(\xi) \), we obtain the results \( z_4(\xi) \approx F_1(\xi), \Delta > 0 \) and \( z_4(\xi) \approx F_2(\xi), \Delta < 0 \).

If \( \Delta = 0 \), then \( 4c_2c_4 = c_3^2 \). In this case, for \( c_2 > 0, c_4 > 0 \) we find

\[
z_3(\xi) = \frac{-c_2}{c_3 + \xi} \left[ 1 - \tanh(\frac{\sqrt{c_3} \xi}{\sqrt{c_2} + c_3}) \right] = -\frac{c_2}{c_3} \left[ 1 + \varepsilon \tanh \left( \frac{\sqrt{c_3} \xi}{\sqrt{c_2} + c_3} \right) \right] = F_4(\xi),
\]

\[
z_4(\xi) = \frac{-c_2}{c_3 + \xi \coth(\frac{\sqrt{c_3} \xi}{\sqrt{c_2} + c_3})} = -\frac{c_2}{c_3} \left[ 1 + \varepsilon \coth \left( \frac{\sqrt{c_3} \xi}{\sqrt{c_2} + c_3} \right) \right] = F_5(\xi).
\]

When \( \Delta > 0, c_2 < 0, c_4 > 0 \), using the expressions \( \sec x = \frac{1}{\cos x}, \csc x = \frac{1}{\sin x} \), the identities

\[
\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}, \sin^2 \frac{x}{2} = \frac{1 - \cos x}{2}, \cos^2 \frac{x}{2} = \frac{1 + \cos x}{2},
\]

(27)

and the Lemma 4, the solutions \( z_5(\xi) \) can be simplified as

\[
z_5(\xi) = \frac{-2c_2}{2\varepsilon \sqrt{-c_2} c_4 \sin(\sqrt{-c_2} \xi) + c_3 \cos(\sqrt{-c_2} \xi) + c_4},
\]

\[
= \begin{cases}
\frac{2c_2}{-\varepsilon \Delta \cosh(\sqrt{-c_2} \xi + \xi_1) - c_3} \approx F_{3a}(\xi), \varepsilon = -1,
\end{cases}
\]

\[
\frac{2c_2}{\varepsilon \Delta \sinh(\sqrt{-c_2} \xi + \xi_2) - c_3} \approx F_{3b}(\xi),
\]

where \( \xi_1 = \arctan \left( \frac{2\sqrt{-c_2} c_4}{c_3} \right), \xi_2 = \arctan \left( \frac{c_3}{2\sqrt{-c_2} c_4} \right) \). Similarly, when \( \Delta > 0, c_2 < 0, c_4 > 0 \), we can prove that \( z_6(\xi) \approx F_{3a}(\xi) \mid \varepsilon = -1 \) and \( z_6(\xi) \approx F_{3b}(\xi) \).

Converting \( z_7(\xi) \) into the exponential form and then using the Lemma 4, we obtain

\[
z_7(\xi) = \frac{4c_2}{\varepsilon \sqrt{\varepsilon + 2(\varepsilon^2 - 4c_2 c_4) e^{-\varepsilon \sqrt{\varepsilon + 2c_3}}}},
\]

\[
= \begin{cases}
\frac{\varepsilon \Delta \cosh(\varepsilon \sqrt{\varepsilon + 2 \ln(\frac{1}{\varepsilon})}) - c_3}{2c_2} \approx F_1(\xi), \varepsilon > 0, c_2 > 0,
\end{cases}
\]

\[
= \begin{cases}
\frac{\varepsilon \Delta \sinh(\varepsilon \sqrt{\varepsilon + 2 \ln(\frac{1}{\varepsilon})}) - c_3}{2c_2} \approx F_2(\xi), \Delta < 0, c_2 > 0.
\end{cases}
\]

When \( \Delta = 0, c_2 > 0 \), it is obtained from the identities \( \frac{1}{1 + e^x} = \frac{1}{2} \left( 1 - \tanh \frac{x}{2} \right) \) and \( \frac{1}{1 - e^x} = \)
\( \frac{1}{2} \left( 1 - \coth \frac{\phi}{2} \right) \) that

\[
z_7(\xi) = \frac{4e^{\phi^2}}{e^{\sqrt{\frac{\phi^2}{2}} - 2c^2}} = \left\{ \begin{array}{ll}
- \frac{2c^2}{c^3} & 1 - \coth \left( \frac{\sqrt{c^2}}{2} \xi + \ln \left( \frac{1}{c^3} \right) \right), \\
- \frac{2c^2}{c^3} & 1 - \tanh \left( \frac{\sqrt{c^2}}{2} \xi + \ln \left( \frac{1}{c^3} \right) \right), \\
- \frac{2c^2}{c^3} & 1 + \coth \left( \frac{\sqrt{c^2}}{2} \xi + \ln \left( \frac{1}{c^3} \right) \right) \approx F_5(\xi), c_3 > 0, \\
- \frac{2c^2}{c^3} & 1 + \tanh \left( \frac{\sqrt{c^2}}{2} \xi + \ln \left( \frac{1}{c^3} \right) \right) \approx F_4(\xi), c_3 < 0.
\end{array} \right.
\]

For \( c_2 > 0, c_3 = 0 \), it yields from the Lemma [1] that

\[
z_8(\xi) = \frac{4e^{\phi^2}}{e^{\sqrt{\frac{\phi^2}{2}} - 4c^2c_4e^{\frac{\phi^2}{2}}}} = \left\{ \begin{array}{ll}
\frac{4e^{\phi^2}}{4c_2} & 2\sqrt{-4c^2c_4} \cosh \left( \frac{\sqrt{c^2}}{2} \xi + \frac{1}{2} \ln(-4c^2c_4) \right), \\
-2\sqrt{4c^2c_4} \sinh \left( \frac{\sqrt{c^2}}{2} \xi + \frac{1}{2} \ln(4c^2c_4) \right),
\end{array} \right.
\]

The above obtained results on the equivalence relations for solutions of Eq. [3] are consistent with the Liu’s results [33, 34], but our proofs are more simple and clear than that of Liu’s proofs because the proving processes have been simplified by the above four Lemmas.

We next consider the following four solutions [35]

\[
\phi_5(\xi) = \frac{-2c^2 \csc \left( \sqrt{c^2} \xi \right)}{c_3 \csc \left( \sqrt{c^2} \xi \right) + \sqrt{4c^2 - c_3^2 + 4c^2c_4 \tan \left( \sqrt{c^2} \xi \right)} - 2c^2}, \quad c_2 > 0,
\]

\[
\phi_6(\xi) = \frac{-2c^2 \csc \left( \sqrt{c^2} \xi \right)}{c_3 \csc \left( \sqrt{c^2} \xi \right) - \sqrt{4c^2 - c_3^2 + 4c^2c_4 \coth \left( \sqrt{c^2} \xi \right) + 2c^2}}, \quad c_2 > 0,
\]

\[
\phi_7(\xi) = \frac{-2c^2 \cot \left( \sqrt{c^2} \xi \right)}{c_3 \cot \left( \sqrt{c^2} \xi \right) + \sqrt{c^2 - c_3^2 + 4c^2c_4 \tan \left( \sqrt{c^2} \xi \right)} - 2c^2}, \quad c_2 < 0,
\]

\[
\phi_8(\xi) = \frac{-2c^2 \cot \left( \sqrt{c^2} \xi \right)}{c_3 \cot \left( \sqrt{c^2} \xi \right) - \sqrt{c^2 - c_3^2 + 4c^2c_4 \cot \left( \sqrt{c^2} \xi \right) + 2c^2}}, \quad c_2 < 0.
\]

Let \( A = \sqrt{4c^2 - c_3^2 + 4c^2c_4}, B = 2c_2 \), then we get \( A^2 - B^2 = -\Delta \). However, for \( c_2 > 0 \), the condition \( \Delta > 0 \) yields that \( A^2 < B^2 \) and \( A < B \). And the condition \( \Delta < 0 \) gives that \( A^2 > B^2 \) and \( A > B \). Therefore, when \( c_2 > 0 \), it follows from the Lemma [3] and the Lemma [2] that

\[
\phi_5(\xi) = \frac{-2c^2 \csc \left( \sqrt{c^2} \xi \right)}{c_3 \csc \left( \sqrt{c^2} \xi \right) + \sqrt{4c^2 - c_3^2 + 4c^2c_4 \sinh \left( \sqrt{c^2} \xi \right)}},
\]

\[
\begin{array}{ll}
\phi_5(\xi) \approx F_1(\xi)|_{\varepsilon = 1}, \quad \Delta > 0, \\
\phi_5(\xi) \approx F_2(\xi)|_{\varepsilon = -1}, \quad \Delta < 0.
\end{array}
\]

where

\[
\theta_1 = \frac{2c^2 + \sqrt{4c^2 - c_3^2 + 4c^2c_4}}{2c_2 - \sqrt{4c^2 - c_3^2 + 4c_2c_4}}, \quad \theta_2 = \frac{\sqrt{4c^2 - c_3^2 + 4c_2c_4 + 2c_2}}{\sqrt{4c^2 - c_3^2 + 4c_2c_4 - 2c_2}}.
\]
By introducing $A = 2c_2, B = \sqrt{4c_2^2 + c_3^2 - 4c_2c_4}$, we find that $A^2 - B^2 = -\Delta$. When $c_2 > 0$, the condition $\Delta > 0$ implies that $A^2 < B^2$ and $A < B$. On the other hand, from the condition $\Delta < 0$ we can deduce that $A^2 > B^2$ and $A > B$. Hence for $c_2 > 0$, it is obtained from the Lemma 3 and the Lemma 2 that

$$
\phi_0(\xi) = \frac{-2c_2}{c_3 + 2c_2 \sinh(\sqrt{\xi}) - \sqrt{4c_2^2 + c_3^2 - 4c_2c_4} \cosh(\sqrt{\xi})},
$$

$$
= \begin{cases}
\sqrt{\Delta \cos(\sqrt{\xi} + \theta_1) - c_3} \approx F_1(\xi)|_{\varepsilon=1}, \quad \Delta > 0, \\
-\sqrt{\Delta \sin(\sqrt{\xi} - \theta_2) - c_3} \approx F_2(\xi)|_{\varepsilon=-1}, \quad \Delta < 0.
\end{cases}
$$

where

$$
\theta_1 = \frac{\sqrt{4c_2^2 + c_3^2 - 4c_2c_4} + 2c_2}{\sqrt{4c_2^2 + c_3^2 - 4c_2c_4} - 2c_2}, \quad \theta_2 = \frac{2c_2 + \sqrt{4c_2^2 + c_3^2 - 4c_2c_4}}{2c_2 - \sqrt{4c_2^2 + c_3^2 - 4c_2c_4}}.
$$

For $\Delta > 0, c_2 < 0$ we obtain from the Lemma 4 that

$$
\phi_7(\xi) = \frac{-2c_2}{c_3 - 2c_2 \cos(\sqrt{\xi} + \theta_1) + \sqrt{c_3^2 - 4c_2^2 - 4c_2c_4} \sin(\sqrt{\xi} + \theta_2)} ,
$$

$$
= \begin{cases}
\sqrt{\Delta \cos(\sqrt{\xi} + \theta_1) - c_3} \approx F_3a(\xi)|_{\varepsilon=1}, \\
-\sqrt{\Delta \sin(\sqrt{\xi} - \theta_2) - c_3} \approx F_3b(\xi)|_{\varepsilon=-1},
\end{cases}
$$

$$
\phi_8(\xi) = \frac{-2c_2}{c_3 + 2c_2 \sin(\sqrt{\xi} - \theta_2) - \sqrt{c_3^2 - 4c_2^2 - 4c_2c_4} \cos(\sqrt{\xi} - \theta_2)} ,
$$

$$
= \begin{cases}
\sqrt{\Delta \cos(\sqrt{\xi} + \theta_1) - c_3} \approx F_3a(\xi)|_{\varepsilon=1}, \\
-\sqrt{\Delta \sin(\sqrt{\xi} - \theta_2) - c_3} \approx F_3b(\xi)|_{\varepsilon=-1},
\end{cases}
$$

where

$$
\theta_1 = \arctan\left(\frac{\sqrt{c_3^2 - 4c_2^2 - 4c_2c_4}}{2c_2}\right), \quad \theta_2 = \arctan\left(\frac{2c_2}{\sqrt{c_3^2 - 4c_2^2 - 4c_2c_4}}\right).
$$

Third we turn to consider the eight solutions listed in Table 3 which were obtained by Yang et al. [36].
Proceeding as before, we can easily prove that

\[ \text{Condition No} \]

For \( \xi^2 = 0 \), using the identity \( \sinh(2\xi) = 2\sinh(\xi)\cosh(\xi) \), we have

For \( \xi^2 > 0 \),

\[ \varphi(\xi) = \frac{2c_2\text{sech}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{2\sqrt{\Delta - (\Delta + c_3)\text{sech}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} c_2 > 0 \]
\[ \text{Condition No} \]

\[ \varphi(\xi) = \frac{2c_2\text{sech}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{-2\sqrt{\Delta + (\Delta - c_3)\text{sech}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} c_2 > 0 \]
\[ \text{Condition No} \]

\[ \varphi(\xi) = \frac{2c_2\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{2\sqrt{\Delta -(\Delta - c_3)\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} c_2 > 0 \]
\[ \text{Condition No} \]

Using the definition of the hyperbolic functions and trigonometric functions, the formulas (26) and (27), we find that

\[ \varphi_1(\xi) = \frac{2c_2}{2\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - \sqrt{\Delta - c_3}} = \frac{2c_2}{\sqrt{\Delta\cosh(\sqrt{\Delta}c_2) - c_3}} = F_1(\xi)|_{\xi = 1}, \]
\[ \varphi_2(\xi) = \frac{2c_2}{2\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) + \sqrt{\Delta - c_3}} = \frac{2c_2}{\sqrt{\Delta\cosh(\sqrt{\Delta}c_2) - c_3}} = F_1(\xi)|_{\xi = 1}, \]
\[ \varphi_3(\xi) = \frac{2c_2}{2\sqrt{\Delta}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi\right) - \sqrt{\Delta - c_3}} = \frac{2c_2}{\sqrt{\Delta\cosh(\sqrt{\Delta}c_2) - c_3}} = F_1(\xi)|_{\xi = 1}, \]
\[ \varphi_4(\xi) = \frac{2c_2}{-2\sqrt{\Delta}\sinh\left(\frac{\sqrt{\Delta}}{2}\xi\right) + \sqrt{\Delta - c_3}} = \frac{2c_2}{\sqrt{\Delta\cosh(\sqrt{\Delta}c_2) - c_3}} = F_1(\xi)|_{\xi = 1}. \]

Proceeding as before, we can easily prove that \( \varphi_2(\xi) = F_1(\xi)|_{\xi = -1} \), \( \varphi_4(\xi) = F_1(\xi)|_{\xi = -1} \), \( \varphi_6(\xi) = F_30(\xi)|_{\xi = -1} \), \( \varphi_7(\xi) = F_30(\xi)|_{\xi = -1} \).

Corresponding to the case \( c_4 = 0 \), we shall now prove the following two solutions (37) of Eq.(33)

\[ f_1(\xi) = \frac{4c_2(\cosh(\sqrt{\Delta}c_2)+\sinh(\sqrt{\Delta}c_2))}{(b+cosh(\sqrt{\Delta}c_2)+\sinh(\sqrt{\Delta}c_2))^2}, c_2 > 0, \]
\[ f_2(\xi) = \frac{4c_2(\cosh(\sqrt{\Delta}c_2)+\sinh(\sqrt{\Delta}c_2))}{(-b+cosh(\sqrt{\Delta}c_2)+\sinh(\sqrt{\Delta}c_2))^2}, c_2 > 0, \]

satisfy the equivalent relation \( f_1(\xi) \cong F_1(\xi)|_{\xi = -1} \) and \( f_2(\xi) \cong F_1(\xi)|_{\xi = -1} \).

For \( c_4 = 0 \), using the identity \( \sinh x + \cosh x = e^x \) and the Lemma 1 yields

\[ f_1(\xi) = -\frac{4c_2}{e^{\sqrt{\Delta}c_2} + c_3 e^{-\sqrt{\Delta}c_2}} = \begin{cases} -\frac{4c_2}{2\sqrt{c_3}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi - \ln(\sqrt{\Delta}c_3)\right)} \quad & c_3 > 0, \\
\frac{4c_2}{2\sqrt{-c_3}\sinh\left(\frac{\sqrt{\Delta}}{2}\xi - \ln(-\sqrt{-c_3})\right)} \quad & c_3 < 0, \end{cases} \]

\[ f_2(\xi) = \begin{cases} -\frac{4c_2}{2\sqrt{c_3}\cosh\left(\frac{\sqrt{\Delta}}{2}\xi - \ln(\sqrt{\Delta}c_3)\right)} \quad & c_3 > 0, \\
\frac{4c_2}{2\sqrt{-c_3}\sinh\left(\frac{\sqrt{\Delta}}{2}\xi - \ln(-\sqrt{-c_3})\right)} \quad & c_3 < 0, \end{cases} \]

Table 3: Solutions of Eq.(3) with \( \Delta = c_3^2 - 4c_2c_4 > 0. \)

| No | \( \varphi(\xi) \) | Condition | No | \( \varphi(\xi) \) | Condition |
|----|-----------------|-----------|----|-----------------|-----------|
| 1  | \( \frac{2c_2\text{sech}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{2\sqrt{\Delta-(\Delta+c_3)\text{sech}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} \) | \( c_2 > 0 \) | 5 | \( \frac{2c_2\text{sec}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{2\sqrt{\Delta-(\Delta+c_3)\text{sec}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} \) | \( c_2 < 0 \) |
| 2  | \( \frac{2c_2\text{sech}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{-2\sqrt{\Delta+(\Delta-c_3)\text{sech}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} \) | \( c_2 > 0 \) | 6 | \( \frac{2c_2\text{sec}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{-2\sqrt{\Delta+(\Delta-c_3)\text{sec}^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} \) | \( c_2 < 0 \) |
| 3  | \( \frac{2c_2\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{2\sqrt{\Delta-(\Delta+c_3)\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} \) | \( c_2 > 0 \) | 7 | \( \frac{2c_2\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{-2\sqrt{\Delta+(\Delta-c_3)\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} \) | \( c_2 < 0 \) |
| 4  | \( \frac{2c_2\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{-2\sqrt{\Delta-(\Delta+c_3)\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} \) | \( c_2 > 0 \) | 8 | \( \frac{2c_2\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}{-2\sqrt{\Delta+(\Delta-c_3)\csc^2\left(\frac{\sqrt{\Delta}}{2}\xi\right)}} \) | \( c_2 < 0 \) |
Finally, we shall prove the twelve solutions [38] of Eq.(7) listed in Table 4 are equivalent to the following solutions

\[ F_5(\xi) = -\frac{c_1}{2c_2} + \frac{\sqrt{c_2}}{c_2} \cos(\sqrt{c_2} \xi), \delta > 0, c_2 > 0, \]
\[ F_9(\xi) = -\frac{c_1}{c_2} + \frac{\sqrt{c_2}}{2c_2} \sinh(\sqrt{c_2} \xi), \delta < 0, c_2 > 0, \]
\[ F_{10a}(\xi) = -\frac{c_1}{c_2} + \frac{\sqrt{c_2}}{2c_2} \cos(\sqrt{c_2} \xi), \delta > 0, c_2 < 0, \]
\[ F_{10b}(\xi) = -\frac{c_1}{c_2} + \frac{\sqrt{c_2}}{2c_2} \sin(\sqrt{c_2} \xi), \delta > 0, c_2 < 0. \]
where $\delta = c_1^2 - 4c_0c_2$.

Let $A = \sqrt{\frac{c_1}{c_2}}$, $B = \frac{c_1}{2c_2}$ then $A^2 - B^2 = -\frac{\delta}{4c_1^2}$. Thus, the condition $\delta > 0$ gives that $A^2 < B^2$, and the condition $\delta < 0$ implies that $A^2 > B^2$. Hence, for $c_0 > 0$, $c_2 > 0$ we obtain from the formulas (26), the Lemma 3 and the Lemma 2 that

\[
Z_1(\xi) = -\frac{c_1}{2c_2} - \left(\sqrt{\frac{c_1}{c_2}} \sinh (\sqrt{c_2}\xi) - \frac{c_1}{2c_2} \cosh (\sqrt{c_2}\xi)\right),
\]

\[
Z_2(\xi) = -\frac{c_1}{2c_2} + \sqrt{\frac{c_1}{c_2}} \cosh (\sqrt{c_2}\xi),
\]

\[
Z_3(\xi) = -\frac{c_1}{2c_2} - \left(\sqrt{\frac{c_1}{c_2}} \sin (\sqrt{c_2}\xi) + \frac{c_1}{2c_2} \cos (\sqrt{c_2}\xi)\right),
\]

\[
Z_4(\xi) = -\frac{c_1}{2c_2} + \sqrt{\frac{c_1}{c_2}} \sin (\sqrt{c_2}\xi),
\]

\[
Z_5(\xi) = -\frac{c_1}{2c_2} + \frac{1}{\sqrt{2}} \left[(1 - c_1^2 + 4c_0c_4) \sinh (\sqrt{c_2}\xi) - (1 + c_1^2 - 4c_0c_4) \cosh (\sqrt{c_2}\xi)\right],
\]

\[
Z_6(\xi) = -\frac{c_1}{2c_2} + \frac{1}{\sqrt{8c_2}} \left[4c_2^2e^{\sqrt{c_2}\xi} + (c_1^2 - 4c_0c_4)e^{-\sqrt{c_2}\xi}\right],
\]

Using the identities (27) and the Lemma 4 yields

\[
Z_3(\xi) = -\frac{c_1}{2c_2} - \left(\sqrt{\frac{c_1}{c_2}} \sin (\sqrt{c_2}\xi) + \frac{c_1}{2c_2} \cos (\sqrt{c_2}\xi)\right),
\]

\[
Z_4(\xi) = -\frac{c_1}{2c_2} + \sqrt{\frac{c_1}{c_2}} \sin (\sqrt{c_2}\xi),
\]

\[
Z_5(\xi) = -\frac{c_1}{2c_2} + \frac{1}{\sqrt{2}} \left[(1 - c_1^2 + 4c_0c_4) \sinh (\sqrt{c_2}\xi) - (1 + c_1^2 - 4c_0c_4) \cosh (\sqrt{c_2}\xi)\right],
\]

\[
Z_6(\xi) = -\frac{c_1}{2c_2} + \frac{1}{\sqrt{8c_2}} \left[4c_2^2e^{\sqrt{c_2}\xi} + (c_1^2 - 4c_0c_4)e^{-\sqrt{c_2}\xi}\right],
\]

The following identities are useful to prove the equivalence relations for solutions of the auxiliary equations and we can prove these identities by means of the trigonometric and hyperbolic
function identities with the aid of some direct calculations.

\[ \tan \eta + \sec \eta = \tan \left( \frac{\eta}{2} + \frac{\pi}{4} \right), \tan \eta - \sec \eta = -\cot \left( \frac{\eta}{2} + \frac{\pi}{4} \right), \quad (28) \]

\[ \cot \eta + \csc \eta = \cot \left( \frac{\eta}{2} \right), \cot \eta - \csc \eta = -\tan \left( \frac{\eta}{2} \right) \quad (29) \]

\[ \tanh \eta + \text{sech} \eta = \tanh \left( \frac{\eta}{2} + \frac{\pi}{4} \right), \tanh \eta - \text{sech} \eta = \coth \left( \frac{\eta}{2} + \frac{\pi}{4} \right). \quad (30) \]

Now the identities (28), (29), (30) and the trigonometric identities are employed to obtain

\[ Z_7(\xi) = -\frac{c_1 + 2\sqrt{-c_0c_2} \tan \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}{c_2 \left( 1 + \tan^2 \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right) \right)} = -\frac{c_1 + 2\sqrt{-c_0c_2} \tan \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}{c_2 \sec^2 \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}, \]

\[ Z_8(\xi) = -\frac{c_1 + 2\sqrt{-c_0c_2} \cot \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}{c_2 \left( 1 + \cot^2 \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right) \right)} = -\frac{c_1 + 2\sqrt{-c_0c_2} \cot \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}{c_2 \csc^2 \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}, \]

\[ Z_9(\xi) = -\frac{c_1 + 2\sqrt{-c_0c_2} \csc \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}{c_2 \left( 1 + \csc^2 \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right) \right)} = -\frac{c_1 + 2\sqrt{-c_0c_2} \csc \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}{c_2 \csc^2 \left( \frac{\sqrt{c_0} \xi + \pi}{2} \right)}, \]

\[ Z_{10}(\xi) = -\frac{c_1 - 2\sqrt{-c_0c_2} \sinh \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)}{c_2 \left( 1 - \sinh^2 \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right) \right)} = -\frac{c_1 - 2\sqrt{-c_0c_2} \sinh \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)}{c_2 \cosh^2 \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)}, \]

\[ Z_{11}(\xi) = -\frac{c_1 - 2\sqrt{-c_0c_2} \cosh \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)}{c_2 \left( 1 - \cosh^2 \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right) \right)} = -\frac{c_1 - 2\sqrt{-c_0c_2} \cosh \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)}{c_2 \sinh^2 \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)}, \]

\[ Z_{12}(\xi) = -\frac{c_1 - 2\sqrt{-c_0c_2} \coth \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)}{c_2 \left( 1 - \coth^2 \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right) \right)} = -\frac{c_1 - 2\sqrt{-c_0c_2} \coth \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)}{c_2 \coth^2 \left( \frac{\sqrt{c_0} \xi + \pi}{4} \right)} , \]

\[ \approx Z_1(\xi). \]

In accordance with the transitivity, we have proved the equivalence relations \( Z_i(\xi) \cong F_{10a}(\xi) \mid_{\varepsilon=-1} \) (\( c_0 > 0, c_2 < 0, \delta > 0, i = 7, 8, 9, 10 \)), \( Z_i(\xi) \cong F_9(\xi) \) (\( c_0 > 0, c_2 > 0, \delta > 0, i = 11, 12 \)) and \( Z_i(\xi) \cong F_9(\xi) \) (\( c_0 > 0, c_2 > 0, \delta < 0, i = 11, 12 \)).

For the following two solutions [7] of Eq. (7)

\[ y_1(\xi) = -\frac{c_1 + \sqrt{c_0} c_2 + \sqrt{c_0} \sinh^2 \left( \frac{\sqrt{c_0} \xi}{2} \right)}{2c_2} , \delta = c_1^2 - 4c_0c_2 > 0, c_2 > 0, \]

\[ y_2(\xi) = -\frac{c_1 - \sqrt{c_0} c_2 - \sqrt{c_0} \sinh^2 \left( \frac{\sqrt{c_0} \xi}{2} \right)}{2c_2} , \delta = c_1^2 - 4c_0c_2 > 0, c_2 > 0, \]

we have \( y_1(\xi) = F_8(\xi) \mid_{\varepsilon=1} \) and \( y_2(\xi) = F_8(\xi) \mid_{\varepsilon=-1} \) which can be obtained by substituting the identity \( \sinh^2 \left( \frac{\sqrt{c_0} \xi}{2} \right) = \frac{1}{2} \left( \cosh \left( \frac{\sqrt{c_0} \xi}{2} \right) - 1 \right) \) into \( y_1(\xi) \) and \( y_2(\xi) \).
4 The classification of the solutions

The classification of the solutions of Eq. (1) is very important to clarify whether the solutions given in literatures are new or equivalent in the sense of wave translation. If this problem is clear then we can choose the new solutions and avoid the repeated solutions effectively. Based on the equivalence relations of the solutions for Eq. (1) as proved in Sec. 3, by removing all those repeated solutions and collecting the simple independent solutions presented in the literatures, we now conclude that the solutions of Eq. (1) can be classified into six classes such as the hyperbolic function solutions, the trigonometric function solutions, the elliptic function solutions, the exponential function solutions, the polynomial solutions and the rational solutions. In other words, the six classes of solutions containing thirty eight independent solutions can be divided into the five cases as below.

Case 1 $c_0 = c_1 = 0$.

\[
F_1(\xi) = \frac{2c_2}{\sqrt{\Delta \cosh(\sqrt{c_2\xi})} - c_3}, \Delta > 0, c_2 > 0,
\]

\[
F_2(\xi) = \frac{2c_2}{\sqrt{\Delta \sinh(\sqrt{c_2\xi})} - c_3}, \Delta < 0, c_2 > 0,
\]

\[
F_{3a}(\xi) = \frac{2c_2}{\sqrt{\Delta \cos(\sqrt{-c_2\xi})} - c_3},
\]

\[
F_{3b}(\xi) = \frac{2c_2}{\sqrt{\Delta \sin(\sqrt{-c_2\xi})} - c_3}, \Delta > 0, c_2 < 0,
\]

\[
F_4(\xi) = -\frac{c_2}{c_3} \left[ 1 + \varepsilon \tanh \left( \frac{\sqrt{c_2}}{2} \xi \right) \right], \Delta = 0, c_2 > 0,
\]

\[
F_5(\xi) = -\frac{c_2}{c_3} \left[ 1 + \varepsilon \coth \left( \frac{\sqrt{c_2}}{2} \xi \right) \right], \Delta = 0, c_2 > 0,
\]

\[
F_6(\xi) = \frac{c_2}{\sqrt{c_2}\xi}, c_2 = c_3 = 0, c_4 > 0,
\]

\[
F_7(\xi) = \frac{c_4}{c_3^2 - 4c_2}, c_2 = 0,
\]

where $\Delta = c_3^2 - 4c_2c_4, \varepsilon = \pm 1$.

Case 2 $c_3 = c_4 = 0$.

\[
F_8(\xi) = -\frac{c_1}{2c_2} + \frac{\varepsilon c_2}{2c_2} \cosh \left( \sqrt{c_2\xi} \right), \delta > 0, c_2 > 0,
\]

\[
F_9(\xi) = -\frac{c_1}{2c_2} + \frac{\varepsilon c_2}{2c_2} \sinh \left( \sqrt{c_2\xi} \right), \delta < 0, c_2 > 0,
\]

\[
F_{10a}(\xi) = -\frac{c_1}{2c_2} + \frac{\varepsilon c_2}{2c_2} \cos \left( \sqrt{-c_2\xi} \right),
\]

\[
F_{10b}(\xi) = -\frac{c_1}{2c_2} + \frac{\varepsilon c_2}{2c_2} \sin \left( \sqrt{-c_2\xi} \right), \delta > 0, c_2 < 0,
\]

\[
F_{11}(\xi) = -\frac{c_1}{2c_2} + e^{\varepsilon \sqrt{c_2} \xi}, \delta = 0, c_2 > 0,
\]

\[
F_{12}(\xi) = \varepsilon \sqrt{c_0\xi}, c_1 = c_2 = 0,
\]

\[
F_{13}(\xi) = -\frac{c_1}{c_2} + \frac{\varepsilon c_2}{4} \xi^2, c_2 = 0,
\]

where $\delta = c_1^2 - 4c_0c_2, \varepsilon = \pm 1$. 
Case 3 $c_1 = c_3 = 0$.

\[ F_{14}(\xi) = \varepsilon \sqrt{-\frac{c_2}{2c_4}} \tanh \left(\sqrt{-\frac{c_2}{2}} \xi\right), \Delta_1 = 0, c_2 < 0, c_4 > 0, \]
\[ F_{15}(\xi) = \varepsilon \sqrt{-\frac{c_2}{2c_4}} \coth \left(\sqrt{-\frac{c_2}{2}} \xi\right), \Delta_1 = 0, c_2 < 0, c_4 > 0, \]
\[ F_{16a}(\xi) = \varepsilon \sqrt{-\frac{c_2}{2c_4}} \tan \left(\sqrt{-\frac{c_2}{2}} \xi\right), \]
\[ F_{16b}(\xi) = \varepsilon \sqrt{-\frac{c_2}{2c_4}} \cot \left(\sqrt{-\frac{c_2}{2}} \xi\right), \Delta_1 = 0, c_2 > 0, c_4 > 0, \]
\[ F_{17}(\xi) = \sqrt{-\frac{c_2c_3}{c_4(m^2+1)}} \sin \left(\sqrt{-\frac{c_2}{2}} \xi\right), c_0 = \frac{c_2^2m^2}{c_4(m^2+1)^2}, c_2 < 0, c_4 > 0, \]
\[ F_{18}(\xi) = \sqrt{-\frac{c_2c_3}{c_4(2m^2-1)}} \csc \left(\sqrt{\frac{c_2}{2}} \xi\right), c_0 = \frac{c_2^2m^2(2m^2-1)}{c_4(2m^2-1)^2}, c_2 > 0, c_4 < 0, \]
\[ F_{19}(\xi) = \sqrt{-\frac{c_2c_3}{c_4(2-m^2)}} \sec \left(\sqrt{\frac{c_2}{2}} \xi\right), c_0 = \frac{c_2^2(1-m^2)}{c_4(2-m^2)^2}, c_2 > 0, c_4 < 0, \]
\[ F_{20}(\xi) = \varepsilon \left(-\frac{4c_0}{c_4}\right)^{\frac{1}{4}} \mathrm{ds} \left((-4c_0c_4)^{\frac{1}{4}} \xi, \sqrt{\frac{c_2}{2}}\right), c_2 = 0, c_0c_4 < 0, \]
\[ F_{21}(\xi) = \varepsilon \left(-\frac{4c_0}{c_4}\right)^{\frac{1}{4}} \left[\mathrm{ns} \left(2(c_0c_4)^{\frac{1}{4}} \xi, \sqrt{\frac{c_2}{2}}\right) + \mathrm{cs} \left(2(c_0c_4)^{\frac{1}{4}} \xi, \sqrt{\frac{c_2}{2}}\right)\right], c_2 = 0, c_0c_4 > 0, \]

where $\Delta_1 = c_2^2 - 4c_0c_4$, $\varepsilon = \pm 1$.

Case 4 $c_2 = c_4 = 0$.

\[ F_{22}(\xi) = \wp \left(\sqrt{\frac{c_3}{2}} \xi, g_2, g_3\right), g_2 = -\frac{4c_1}{c_3}, g_3 = -\frac{4c_0}{c_3}, c_3 > 0. \]
Case 5 \( c_0 = 0 \).

\[
F_{23}(\xi) = -\frac{8c_2 \tanh^2\left(\sqrt{\frac{12}{13}} \xi\right)}{3c_3 \left(3 + \tanh^2\left(\sqrt{\frac{12}{13}} \xi\right)\right)}, \quad c_2 < 0, c_1 = \frac{8c_2^2}{27c_3}, c_4 = \frac{c_2^4}{4c_2},
\]

\[
F_{24}(\xi) = -\frac{8c_2 \coth^2\left(\sqrt{\frac{12}{13}} \xi\right)}{3c_3 \left(3 + \coth^2\left(\sqrt{\frac{12}{13}} \xi\right)\right)}, \quad c_2 < 0, c_1 = \frac{8c_2^2}{27c_3}, c_4 = \frac{c_2^4}{4c_2},
\]

\[
F_{25}(\xi) = \frac{8c_2 \tan^2\left(\sqrt{\frac{12}{13}} \xi\right)}{3c_3 \left(3 - \tan^2\left(\sqrt{\frac{12}{13}} \xi\right)\right)}, \quad c_2 > 0, c_1 = \frac{8c_2^2}{27c_3}, c_4 = \frac{c_2^4}{4c_2},
\]

\[
F_{26}(\xi) = \frac{8c_2 \cot^2\left(\sqrt{\frac{12}{13}} \xi\right)}{3c_3 \left(3 - \cot^2\left(\sqrt{\frac{12}{13}} \xi\right)\right)}, \quad c_2 > 0, c_1 = \frac{8c_2^2}{27c_3}, c_4 = \frac{c_2^4}{4c_2},
\]

\[
F_{27}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 > 0, c_1 = \frac{c_3^2 (m^2 - 1)}{32m^4c_4}, c_2 = \frac{c_3^2 (5m^2 - 1)}{16m^4c_4},
\]

\[
F_{28}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \frac{\varepsilon}{\sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)}\right], \quad c_4 > 0, c_1 = \frac{c_3^2 (m^2 - 1)}{32m^4c_4}, c_2 = \frac{c_3^2 (5m^2 - 1)}{16m^4c_4},
\]

\[
F_{29}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 > 0, c_1 = \frac{c_3^2 (1 - m^2)}{32c_4}, c_2 = \frac{c_3^2 (5 - m^2)}{16c_4},
\]

\[
F_{30}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 > 0, c_1 = \frac{c_3^2 (1 - m^2)}{32c_4}, c_2 = \frac{c_3^2 (5 - m^2)}{16c_4},
\]

\[
F_{31}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 < 0, c_1 = \frac{c_3^2 (m^2 - 1)}{32m^4c_4}, c_2 = \frac{c_3^2 (4m^2 + 1)}{16m^4c_4},
\]

\[
F_{32}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 < 0, c_1 = \frac{c_3^2 (m^2 - 1)}{32m^4c_4}, c_2 = \frac{c_3^2 (4m^2 + 1)}{16m^4c_4},
\]

\[
F_{33}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 < 0, c_1 = \frac{c_3^2 (m^2 - 1) (m^2 - 2)}{16c_4}, c_2 = \frac{c_3^2 (5m^2 - 4)}{16c_4},
\]

\[
F_{34}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 < 0, c_1 = \frac{c_3^2 (m^2 - 1)}{32c_4}, c_2 = \frac{c_3^2 (5m^2 - 4)}{16c_4},
\]

\[
F_{35}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 > 0, c_1 = \frac{c_3^2 (m^2 - 1)}{32c_4}, c_2 = \frac{c_3^2 (4m^2 - 5)}{16c_4},
\]

\[
F_{36}(\xi) = -\frac{c_1 c_4}{4c_2} \left[1 + \varepsilon \sin\left(\frac{c_3}{4\sqrt{c_4}} \xi\right)\right], \quad c_4 > 0, c_1 = \frac{c_3^2 (m^2 - 1)}{32c_4}, c_2 = \frac{c_3^2 (4m^2 - 5)}{16c_4},
\]

where \( \varepsilon = \pm 1 \).
5 New traveling wave solutions for mCH equation

As an illustrative example below we consider the modified Camassa–Holm (mCH) equation in the form

\[ u_t - u_{xx} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx}, \]  

\( (31) \)

was due to Wazwaz [39] obtained by using the term \( u^2u_x \) instead of the nonlinear convection term \( uu_x \) in Camassa-Holm equation. Wazwaz [39] also found some exact traveling wave solutions of wave speed \( c = 1/2, 1, 2 \) for Eq. (31). Some traveling wave solutions of wave speed \( c = 1/3 \) and some peakon solutions of wave speed \( c = 3 \) for Eq. (31) were obtained by Wang and Tang in [41]. A rational solution and more exact traveling wave solutions of Eq. (31) were also constructed by Liang and Jeffrey in [42].

In the present paper we are interested in finding more new exact traveling wave solutions for Eq. (31) through use of the general elliptic equation expansion method equipped with our new classified solutions in Sec.4. In order that we make the wave transformation \( u(x, t) = u(\xi), \xi = x - \omega t \) and change the Eq. (31) into the form

\[ -\omega u' + \omega u''' + 3u^2u' - 2u'u'' - uu''' = 0. \]  

\( (32) \)

According to the balancing principle to balance \( uu''' \) with \( u^2u' \) yields \( m = 2 \), therefore the Eq. (32) admits the solution in the form

\[ u(\xi) = a_0 + a_1F(\xi) + a_2F^2(\xi), \]  

\( (33) \)

where \( F(\xi) \) expresses the solution of Eq. (1) and \( a_k \) (\( k = 0, 1, 2 \)) are constants to be determined.

Substituting (33) into (32) along with Eq. (1) and setting the coefficients of \( F^jF' \) \((j = 0, 1, 2, 3, 4, 5)\) to zero, we obtain the following set of algebraic equations

\[ \begin{cases} 
6a_0^3 - 48a_0^2c_4 = 0, \\
15a_1a_2^2 - 50a_1a_2c_4 - 35a_2^2c_3 = 0, \\
12a_0a_2^2 - 24a_0a_2c_4 + 12a_1^2a_2 - 10a_1^2c_4 - 34a_1a_2c_3 - 24a_2^2c_2 + 24a_2c_3 + 12a_2c_2 - 2a_2c_2 + 6a_2c_4 = 0, \\
3a_0^3a_4 - a_0a_1a_2 + 3a_0a_2c_3 - a_1^2c_4 - 4a_1a_2c_3 + a_0a_2c_4 + 3a_0a_2c_4 + a_1a_2c_4 = 0, \\
18a_0a_1a_2 - 6a_0a_1c_4 - 15a_0a_2c_3 + 3a_1^2 - 6a_1^2c_3 - 21a_1a_2c_2 + 6a_1c_4 - 15a_2c_2 + 15a_2c_4 = 0, \\
a_0a_2 + 6a_0a_1^2 - 3a_0a_1c_3 - 8a_0a_2c_2 - 3a_1^2c_2 - 2a_2c_2 - 11a_1a_2c_1 + 3a_1c_3c_3 - 8a_2c_3 + 8a_2c_4 = 0.
\end{cases} \]  

\( (34) \)

A. When \( c_0 = c_1 = 0 \), the Eq. (34) solves that

\[ a_0 = -\frac{1}{3}\left(-\omega + \sqrt{2\omega(3 - \omega)}\right), a_1 = 0, a_2 = 8c_4, c_2 = \frac{1}{9}\sqrt{2\omega(3 - \omega)}, c_3 = 0, \]  

\( (35) \)

\[ a_0 = -\frac{1}{3}\left(-\omega - \sqrt{2\omega(3 - \omega)}\right), a_1 = 0, a_2 = 8c_4, c_2 = -\frac{1}{9}\sqrt{2\omega(3 - \omega)}, c_3 = 0, \]  

\( (36) \)

\[ a_0 = \frac{1}{3}\left(-\omega + \sqrt{2\omega(3 - \omega)}\right), a_1 = 2c_3, a_2 = 0, c_2 = \frac{1}{9}\sqrt{2\omega(3 - \omega)}, c_4 = 0, \]  

\( (37) \)

\[ a_0 = \frac{1}{3}\left(-\omega - \sqrt{2\omega(3 - \omega)}\right), a_1 = 2c_3, a_2 = 0, c_2 = -\frac{1}{9}\sqrt{2\omega(3 - \omega)}, c_4 = 0, \]  

\( (38) \)

\[ a_0 = \frac{1}{3}\left(-\omega + \sqrt{2\omega(3 - \omega)}\right), a_1 = \pm (8c_4)^{1/3} (2\omega(3 - \omega))^{1/3}, a_2 = 8c_4, \]  

\( (39) \)
\[ c_2 = \frac{1}{2} \sqrt{2\omega(3-\omega)}, \quad c_4 = \pm \frac{1}{2} (8c_4)^{\frac{1}{8}} (2\omega(3-\omega))^{\frac{1}{4}} \]

\[ a_0 = \frac{1}{3} \left( -\omega + \sqrt{2\omega(3-\omega)} \right), \quad a_1 = 2c_3, \quad c_2 = \frac{1}{2} \sqrt{2\omega(3-\omega)}, \quad c_4 = 0 \]

\[ a_0 = \frac{1}{3} \left( -\omega - \sqrt{2\omega(3-\omega)} \right), \quad a_1 = 2c_3, \quad c_2 = -\frac{1}{2} \sqrt{2\omega(3-\omega)}, \quad c_4 = 0. \]  

Inserting (35)–(41) together with the solutions given by Case 1 into (33), we obtain the exact traveling wave solutions of Eq. (31) as follows

\[ u_{1a}(x, t) = -\frac{\omega}{3} + \frac{1}{3} \sqrt{2\omega(3-\omega)} - \sqrt{2\omega(3-\omega)} \text{sech}^2 \eta, \]

\[ u_{1b}(x, t) = -\frac{\omega}{3} + \frac{1}{3} \sqrt{2\omega(3-\omega)} + \sqrt{2\omega(3-\omega)} \text{csch}^2 \eta, \]

\[ u_{2a}(x, t) = -\frac{\omega}{3} - \frac{1}{3} \sqrt{2\omega(3-\omega)} + \sqrt{2\omega(3-\omega)} \sec^2 \eta, \]

\[ u_{2b}(x, t) = -\frac{\omega}{3} - \frac{1}{3} \sqrt{2\omega(3-\omega)} - \sqrt{2\omega(3-\omega)} \csc^2 \eta, \]

\[ u_{3a}(x, t) = -\frac{\omega}{3} + \frac{1}{3} \sqrt{2\omega(3-\omega)} + \frac{\sqrt{2\omega(3-\omega)}}{\cosh \eta - 1}, \]

\[ u_{3b}(x, t) = -\frac{\omega}{3} - \frac{1}{3} \sqrt{2\omega(3-\omega)} - \frac{\sqrt{2\omega(3-\omega)}}{\cosh \eta - 1}, \]

\[ u_{5a}(x, t) = -\frac{\omega}{3} - \frac{1}{3} \sqrt{2\omega(3-\omega)} + \frac{\sqrt{2\omega(3-\omega)}}{\sinh \eta - 1}, \]

\[ u_{5b}(x, t) = -\frac{\omega}{3} + \frac{1}{3} \sqrt{2\omega(3-\omega)} - \frac{\sqrt{2\omega(3-\omega)}}{\sinh \eta - 1}, \]

\[ u_{6a}(x, t) = u_{1a}(x, t), \quad u_{6b}(x, t) = u_{1b}(x, t), \quad c_4 = 0, \]

\[ u_{7a}(x, t) = u_{2a}(x, t), \quad u_{7b}(x, t) = u_{2b}(x, t), \quad c_4 = 0, \]

\[ \eta = \frac{1}{2} \left( \frac{\omega(3-\omega)}{2} \right)^{\frac{1}{4}} (x - \omega t), \quad 0 < \omega < 3. \]

B. When \( c_3 = c_4 = 0 \) we cannot find the solutions of Eq. (31) so we cannot obtain the exact traveling wave solutions for Eq. (31).

C. When \( c_1 = c_3 = 0 \), we have

1. For \( c_0 = \frac{c_1^2 m^2}{c_4 (m^2 + 1)^2}, \quad c_2 < 0, \quad c_4 > 0 \), the Eq. (34) solves that

\[ a_0 = -\frac{\omega}{3} + \frac{m^2 + 1}{3} \left( \frac{2\omega(3-\omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{2}}, \quad a_1 = 0, \]

\[ a_2 = 8c_4, \quad c_2 = -\frac{m^2 + 1}{8} \left( \frac{2\omega(3-\omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{2}}, \]

which leads the Jacobian snoidal wave solution of Eq. (31) as

\[ u(x, t) = -\frac{\omega}{3} - \frac{m^2 + 1}{3} \left( \frac{2\omega(3-\omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{4}} + m^2 \left( \frac{2\omega(3-\omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{2}} \text{sn}^2 (\eta, m), \]

\[ \eta = \frac{1}{2} \left( \frac{\omega(3-\omega)}{2} \right)^{\frac{1}{4}} (x - \omega t), \quad 0 < \omega < 3. \]  

2. For \( c_0 = \frac{c_1^2 m^2 (m^2 - 1)}{c_4 (2m^2 - 1)^2}, \quad c_2 > 0, \quad c_4 < 0 \), the solution of Eq. (34) is found to be

\[ a_0 = -\frac{\omega}{3} + \frac{2m^2 - 1}{3} \left( \frac{2\omega(3-\omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{2}}, \quad a_1 = 0, \]

\[ a_2 = 8c_4, \quad c_2 = \frac{2m^2 - 1}{3} \left( \frac{2\omega(3-\omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{2}}, \]
which gives the Jacobian cnoidal wave solution of Eq. (41) as following

\[ u(x, t) = -\frac{\omega}{3} + m^2 \left( \frac{2\omega (3 - \omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{3}} - m^2 \left( \frac{2\omega (3 - \omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{2}} \text{cn}^2(\eta, m), \]

\[ \eta = \frac{1}{2} \left( \frac{\omega (3 - \omega)}{2(m^4 - m^2 + 1)} \right)^{\frac{1}{3}} (x - \omega t), 0 < \omega < 3. \]  

(3) For \( c_0 = \frac{c_0^2(1-m^2)}{c_0^2(2-m^2)}, c_2 > 0, c_4 < 0, \) the solution of Eq. (44) is obtained that

\[ a_0 = -\frac{\omega}{3} + \frac{m^2 - 2}{3} \left( \frac{2\omega (3 - \omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{3}}, a_1 = 0, \]

\[ a_2 = 8c_4, c_2 = \frac{m^2 - 2}{8} \left( \frac{2\omega (3 - \omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{3}}, \]

which implies the Jacobian elliptic function solution of Eq. (41) as

\[ u(x, t) = -\frac{\omega}{3} - \frac{m^2 - 2}{3} \left( \frac{2\omega (3 - \omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{3}} - \left( \frac{2\omega (3 - \omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{2}} \text{dn}^2(\eta, m), \]

\[ \eta = \frac{1}{2} \left( \frac{\omega (3 - \omega)}{2(m^4 - m^2 + 1)} \right)^{\frac{1}{3}} (x - \omega t), 0 < \omega < 3. \]  

(4) For \( c_2 = 0, \) the Eq. (43) solves

\[ a_0 = -\frac{\omega}{3}, a_1 = 0, a_2 = 8c_4, c_0 = \frac{\omega (3 - \omega)}{6c_4}, \]

which gives the Jacobian elliptic function solution of Eq. (41) as following

\[ u(x, t) = -\frac{\omega}{3} + \frac{2}{3} \sqrt{6\omega (3 - \omega)} \text{ds}^2 \left( \eta, \frac{\sqrt{2}}{3} \right), \]

\[ \eta = \frac{1}{6} (54\omega (3 - \omega))^{\frac{1}{3}} (x - \omega t), 0 < \omega < 3, \]

\[ u(x, t) = -\frac{\omega}{3} + \frac{\sqrt{2}}{3} \sqrt{6\omega (3 - \omega)} \left[ \text{ns}(\eta, \frac{\sqrt{2}}{3}) + \text{cs}(\eta, \frac{\sqrt{2}}{3}) \right]^{2}, \]

\[ \eta = 384 (6\omega (3 - \omega))^{\frac{1}{3}} (x - \omega t), \omega < 0 \text{ or } \omega > 3. \]  

D. When \( c_2 = c_4 = 0, \) the solution of Eq. (42) is found to be

\[ a_0 = -\frac{\omega}{3}, a_1 = 2c_3, a_2 = 0, c_1 = \frac{\omega (3 - \omega)}{6c_3}, \]

which gives the Weierstrass elliptic function solution of Eq. (41) as

\[ u(x, t) = 2c_3 \wp \left( \frac{\sqrt{c_3}}{2}(x - \omega t), \frac{2\omega (3 - \omega)}{3c_3}, -\frac{4c_0}{c_3} \right). \]  

E. When \( c_0 = 0, \) we have

(1) From the condition of \( F_{27} \) we obtain the solution of Eq. (43) as following

\[ a_0 = -\frac{\omega}{3} + \frac{2m^2 - 1}{3} \left( \frac{2\omega (3 - \omega)}{m^4 - m^2 + 1} \right)^{\frac{1}{3}}, a_1 = 4c_3, \]

\[ a_2 = \frac{c_0^2}{m^2} \left( \frac{2m^4 - m^2 + 1}{\omega (3 - \omega)} \right)^{\frac{1}{3}}, c_4 = \frac{c_0^2}{4m^2} \left( \frac{2m^4 - m^2 + 1}{\omega (3 - \omega)} \right)^{\frac{1}{3}}, \]
from which we obtain the Jacobian snoidal wave solution of Eq. (31) as

\[ u(x, t) = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} + m^2 \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{sn}^2 (\rho(x - \omega t, m),
\]
\[ \rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, 0 < \omega < 3. \] (48)

(2) Under condition of $F_{28}$, the Eq. (34) solves that

\[ a_0 = -\frac{\omega}{3} + \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, a_1 = 4c_3, \]
\[ a_2 = \frac{2c_3^2}{m^2} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, c_4 = \frac{c_3^2}{4m^2} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, \]

from which we obtain the Jacobian elliptic function solution of Eq. (31) in the form

\[ u(x, t) = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} + \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{ns}^2 (\rho(x - \omega t, m),
\]
\[ \rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, 0 < \omega < 3. \] (49)

(3) The solution of Eq. (31) induced by the condition of $F_{29}$ is that

\[ a_0 = -\frac{\omega}{3} - \frac{m^2-2}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, a_1 = 4c_3, \]
\[ a_2 = \frac{2c_3^2}{m^2} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, c_4 = \frac{c_3^2}{4m^2} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, \]

which leads the Jacobian snoidal wave solution of Eq. (31) as

\[ u(x, t) = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} + m^2 \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{sn}^2 (\rho(x - \omega t, m),
\]
\[ \rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, 0 < \omega < 3. \] (50)

(4) Using the condition of $F_{30}$ we find the solution of Eq. (34) as following

\[ a_0 = -\frac{\omega}{3} - \frac{m^2-2}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, a_1 = 4c_3, \]
\[ a_2 = \frac{2c_3^2}{m^2} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, c_4 = \frac{c_3^2}{4m^2} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, \]

from which we obtain the Jacobian elliptic function solution of Eq. (31) in the form

\[ u(x, t) = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} + \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{ns}^2 (\rho(x - \omega t, m),
\]
\[ \rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, 0 < \omega < 3. \] (51)

(5) From the condition of $F_{31}$ we yields the solution of Eq. (34) as following

\[ a_0 = -\frac{\omega}{3} + \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, a_1 = 4c_3, \]
\[ a_2 = \frac{2c_3^2}{m^2} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, c_4 = \frac{c_3^2}{4m^2} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, \]
which gives the Jacobian snoidal wave solution of Eq.(31) in the form

$$u(x,t) = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} + m^2 \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{sn}^2 \left( \rho(x-\omega t, m) \right),$$

$$\rho = \frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, 0 < \omega < 3. \quad (52)$$

(6) By the condition of $F_{32}$ we get the solution of Eq.(34) as following

$$a_0 = -\frac{\omega}{3} - \frac{m^2-1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, a_1 = 4c_3,$$

$$a_2 = \frac{2c_4^2}{m^2-1} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, c_4 = \frac{c_3^2}{4(m^2-1)} \left( \frac{m^4-m^2+1}{\omega(3-\omega)} \right)^{\frac{1}{2}},$$

which leads the Jacobian elliptic function solution of Eq.(31) in the form

$$u(x,t) = -\frac{\omega}{3} - \frac{m^2-2}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} - \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{dn}^2 \left( \rho(x-\omega t, m) \right),$$

$$\rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, 0 < \omega < 3. \quad (53)$$

(7) The solution of Eq.(34) induced by the condition of $F_{33}$ reads

$$a_0 = -\frac{\omega}{3} + \frac{m^2-1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, a_1 = 4c_3,$$

$$a_2 = \frac{2c_4^2}{m^2-1} \left( \frac{m^4-m^2+1}{\omega(3-\omega)} \right)^{\frac{1}{2}}, c_4 = \frac{c_3^2}{4(m^2-1)} \left( \frac{m^4-m^2+1}{\omega(3-\omega)} \right)^{\frac{1}{2}},$$

from which we obtain the Jacobian elliptic function solution of Eq.(31) as

$$u(x,t) = -\frac{\omega}{3} - \frac{m^2-2}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} - \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{dn}^2 \left( \rho(x-\omega t, m) \right),$$

$$\rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, 0 < \omega < 3. \quad (54)$$

(8) The condition of $F_{34}$ leads the solution of Eq.(34) as following

$$a_0 = -\frac{\omega}{3} + \frac{m^2-1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, a_1 = 4c_3,$$

$$a_2 = \frac{2c_4^2}{m^2-1} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, c_4 = \frac{c_3^2}{4(m^2-1)} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}},$$

which gives the Jacobian elliptic function solution of Eq.(31) in the form

$$u(x,t) = -\frac{\omega}{3} - \frac{m^2-2}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} - \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{sn}^2 \left( \rho(x-\omega t, m) \right) + \left( \frac{m^2-2}{3} \right) \text{sn}^2 \left( \rho(x-\omega t, m) \right),$$

$$\rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, 0 < \omega < 3. \quad (55)$$
The solution of Eq. (34) obtained by the condition of $F$ leads the Jacobian elliptic function solution of Eq. (31) in the form

$$a_0 = -\frac{\omega}{3} - \frac{m^2-2}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, \quad a_1 = 4c_3,$$

$$a_2 = \frac{2c_3^2}{m^4-1} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, \quad c_4 = \frac{c_3^2}{4(m^4-1)} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}},$$

from which we obtain the Jacobian elliptic function solution of Eq. (31) as

$$u(x, t) = \left[ -\frac{\omega}{3} - \frac{m^2-2}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \right] \text{nc}^2(\rho(x - \omega t, m)) + \left[ \frac{\omega}{3} - \frac{m^2-1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \right]^2 \rho(x - \omega t, m),$$

$$\rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, \quad 0 < \omega < 3.$$  \hspace{1cm} (56)

(10) By the condition of $F_{36}$ we get the solution of Eq. (34) as following

$$a_0 = -\frac{\omega}{3} - \frac{m^2-2}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, \quad a_1 = 4c_3,$$

$$a_2 = \frac{2c_3^2}{m^4-1} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, \quad c_4 = \frac{c_3^2}{4(m^4-1)} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}},$$

which leads the Jacobian elliptic function solution of Eq. (31) in the form

$$u(x, t) = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} + \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{ns}^2(\rho(x - \omega t, m)),$$

$$\rho = -\frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, \quad 0 < \omega < 3.$$  \hspace{1cm} (57)

(11) The condition of $F_{37}$ gives the solution of Eq. (34) as following

$$a_0 = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, \quad a_1 = 4c_3,$$

$$a_2 = \frac{2c_3^2}{m^4-1} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, \quad c_4 = \frac{c_3^2}{4(m^4-1)} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}},$$

which gives the Jacobian snoidal wave solution of Eq. (31) in the form

$$u(x, t) = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} + m^2 \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \text{sn}^2(\rho(x - \omega t, m)),$$

$$\rho = \frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, \quad 0 < \omega < 3.$$  \hspace{1cm} (58)

(12) The solution of Eq. (34) obtained by using the condition of $F_{38}$ reads

$$a_0 = -\frac{\omega}{3} - \frac{m^2+1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}}, \quad a_1 = 4c_3,$$

$$a_2 = \frac{2c_3^2}{m^4-1} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}}, \quad c_4 = \frac{c_3^2}{4(m^4-1)} \left( \frac{2(m^4-m^2+1)}{\omega(3-\omega)} \right)^{\frac{1}{2}},$$

from which we obtain the Jacobian elliptic function solution of Eq. (31) as

$$u(x, t) = \left[ -\frac{\omega}{3} + \frac{2m^2-1}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \right] \text{nd}^2(\rho(x - \omega t, m)) + \left[ \frac{m^2\omega}{3} + \frac{m^2(m^2-2)}{3} \left( \frac{2\omega(3-\omega)}{m^4-m^2+1} \right)^{\frac{1}{2}} \right] \text{sd}^2(\rho(x - \omega t, m),$$

$$\rho = \frac{1}{2} \left( \frac{\omega(3-\omega)}{2(m^4-m^2+1)} \right)^{\frac{1}{2}}, \quad 0 < \omega < 3.$$  \hspace{1cm} (59)
6 Concluding remarks

We constructed the BT and the SF of two kinds of sub-equations for Eq.(1) in terms of the indirect mapping method for simplicity. As a matter of fact, we can establish the BT and the SF for most of those sub-equations for Eq.(1) by the elementary integration method which were not considered here and will be reported elsewhere. It is also clear that the new solutions of Eq.(1) obtained from the BT or SF can also be used to find new solutions of NLEEs which is under consideration. But an open problem as how to construct multiple traveling wave solutions of NLEEs from the solutions of NLEEs obtained by using the auxiliary equations is to be solved.

It should be noticed that the four lemmas presented in Sec.3 can be widely used to prove the equivalence relations of numerous solutions for other auxiliary equations. At the same time, the four lemmas can also be applied to prove the equivalence relations of some direct methods. Therefore, the proof of the equivalence relations of solutions for the auxiliary equations is more important than that of discussing the equivalence relations of those obtained solutions for a given nonlinear equation. This is the reason why we studied the equivalence relations of solutions for Eq.(1) here in detail.

In Sec.5 we obtained some exact traveling wave solutions of wave speed defined on the region $\omega \in (0, 3)$ and $\omega \in (-\infty, 0) \cup (3, +\infty)$ for mCH equation that was not considered and obtained before so they are the new solutions for mCH equation. Although some of them are same in form but they are obtained by using different conditions and therefore they can be regarded as new solutions as well. In the limit of $m \to 1$, the solitary wave solutions for mCH equation can be obtained from the Jacobian elliptic function solutions constructed in Sec.5 and which were omitted.

References

[1] S.K.Liu, Z.T. Fu, S.D. Liu, and Q. Zhao, “Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations,” Phys. Lett. A 289, 69–74 (2001).

[2] M.L. Wang and X.Z. Li, “Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation,” Chaos, Solitons and Fractals 24, 1257–1268 (2005).

[3] Sirendaoreji and J. Sun, “Auxiliary equation method for solving nonlinear partial differential equations,” Phys. Lett. A 309, 387–396 (2003).

[4] E.G. Fan, “Uniformly constructing a series of explicit exact solutions to nonlinear equations in mathematical physics,” Chaos, Solitons and Fractals 16, 819–839 (2003).

[5] E.Yomba, “The extended Fan’s sub-equation method and its application to KdV-MKdV, BKK and variant Boussinesq equations,” Phys. Lett. A 336, 463–476 (2005).

[6] A.A. Soliman and M.A. Abdou, “Exact travelling wave solutions of nonlinear partial differential equations,” Chaos, Solitons and Fractals 32, 808–815 (2007).

[7] S.K. Liu and S.D. Liu, “Nonlinear Equations in Physics,” 32–53. Peking University Press, Peking (2000).
[8] E.G. Fan, “Integrable Systems and Computer Algebra,” 29–55. Science Press, Peking (2004).

[9] Z.B. Li, “Traveling Wave Solutions of Nonlinear Equations in Mathematical Physics,” 147–153. Science Press, Peking (2007).

[10] Y.C. Guo, “Nonlinear Partial Differential Equations,” 223–226. Tsinghua University Press, Peking (2008).

[11] N.Taghizadeh, Q.Zhou, M.Ekici, and M.Mirzazadeh, “Soliton solutions for Davydov solitons in α–helix proteins,” Superlattices and Microstructures 102, 323–341 (2017).

[12] M.M. El–Borai, H.M. El–Owaidy, H.M. Ahmed, and A.H. Arnous, “Exact and soliton solutions to nonlinear transmission line model,” Nonlinear Dyn. 87, 767–773 (2017).

[13] Sirendaoreji, “New exact traveling wave solutions for the Kawahara and the modified Kawahara equations,” Chaos, Solitons and Fractals 19, 147–150 (2004).

[14] Sirendaoreji, “Auxiliary equation method and new solutions of Klein-Gordon equations,” Chaos, Solitons and Fractals 31: 943–950 (2007).

[15] E. Yomba, “The sub-ODE method for finding exact travelling wave solutions of generalized nonlinear Camassa-Holm, and generalized nonlinear Schrödinger equations,” Phys. Lett. A 372, 215–222 (2008).

[16] M.A. Abdou, “A generalized auxiliary equation method and its applications,” Nonlinear Dyn. 52: 95–102 (2008).

[17] N. Cheema and M. Younis, “New and more exact traveling wave solutions to integrable (2+1)-dimensional Maccari system,” Nonlinear Dyn. 83: 1395–1401 (2016).

[18] Sirendaoreji, “Exact travelling wave solutions for four forms of nonlinear Klein–Gordon equations,” Phys. Lett. A 363: 440–447 (2007).

[19] H. Triki, H. Leblond, and D. Mihalache, “Soliton solutions of nonlinear diffusion–reaction–type equations with time–dependent coefficients accounting for long–range diffusion,” Nonlinear Dyn. 86: 2115–2126 (2016).

[20] J.P. Yu, D.S. Wang, Y.L. Sun, and S.P. Wu, “Modified method of simplest equation for obtaining exact solutions of the Zakharov–Kuznetsov equation, the modified Zakharov–Kuznetsov equation, and their generalized forms,” Nonlinear Dyn. 85: 2449–2465 (2016).

[21] X.Z. Li and M.L. Wang, “A sub–ODE method for finding exact solutions of a generalized KdV–mKdV equation with high-order nonlinear terms,” Phys. Lett. A 361, 115–118 (2007).

[22] C.S. Liu, “Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations,” Comput. Phys. Commun. 181, 317–324 (2010).
[23] J.Y.Hu, "Classification of single traveling wave solutions to the nonlinear dispersion Drinfel’d–Sokolov system," Appl. Math. Comput. 219, 2017–2025 (2012).

[24] Taogetusang, “Several auxiliary equations and infinite sequence exact solutions to nonlinear evolution equations,” Acta Phys. Sin. 60, 050201:1–13 (2011).

[25] X.Zeng and X.L.Yong, “A new mapping method and its applications to nonlinear partial differential equations,” Phys. Lett. A 372, 6602–6607 (2008).

[26] S.Zhang, “A generalized auxiliary equation method and its application to (2 + 1)-dimensional Korteweg–de Vries equations,” Comput. Math. Appl. 54, 1028–1038 (2007).

[27] S.Zhang and T.C.Xia, “A generalized auxiliary equation method and its application to (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equations,” J. Phys. A: Math. Theor. 40: 227–248 (2007).

[28] D.J.Humag, D.S.Li and H.Q.Zhang, “Explicit and exact travelling wave solutions for the generalized derivative Shrödinger equation,” Chaos, Solitons and Fractals 31, 586–593 (2007).

[29] G.Q.Xu, “Extended auxiliary equation method and its applications to three generalized NLS equations,” Abstract and Appl. Anal. 2014, 1–7 (2014).

[30] E.M.E.Zayed and K.A.E.Alurrfi, “Extended auxiliary equation method and its applications for finding the exact solutions for a class of nonlinear Schröinger–type equations,” Appl. Math. Comput. 289, 111-131 (2016).

[31] Z.Pinar and T. Özis, “An observation on the periodic solutions to nonlinear physical models by means of the auxiliary equation with a sixth-degree nonlinear term. Commun. Nonl. Sci. Numer. Simulat. 18, 2177–2187 (2013).

[32] Sirendaoreji, “Unified Riccati equation expansion method and its application to two new classes of Benjamin–Bona–Mahony equations. Nonlinear Dyn. 89, 333–344 (2017).

[33] C.P.Liu and X.P.Liu, “A note on the auxiliary equation method for solving nonlinear partial differential equations,” Phys. Lett. A 348, 222–227 (2006).

[34] X.P.Liu and C.P.Liu, “The relationship among the solutions of two auxiliary ordinary differential equations,” Chaos Solitons and Fractals 39, 1915–1919 (2009).

[35] C.D.Tian, “The rational solutions for a kind of mCH equation,” J. Qilu Norm. Univ. 26, 119–123 (2011).

[36] X.L.Yang and J.S.Tan, “New travelling wave solutions for combined KdV–mKdV equation and (2+1)-dimensional Broer–Kaup–Kupershmidt system,” Chin.Phys. 16, 310–317 (2007).

[37] A.H.Lu and X.K.Shao, “Using the elliptical auxiliary equation method to obtain exact solutions of the variable coefficients Boussinesq equation,” J. Chongqing Univ. Art. Sci.33, 11–13 (2014).
[38] Taogetusang and Sirendaoreji, “New type of exact solitary wave solutions for dispersive long–wave equation and Benjamin equation,” Acta Physica Sinica 55,3246–3254 (2006).

[39] A.M.Wazwaz, “Solitary wave solutions for modified forms of Degasperis–Procesi and Camassa–Holm equations,” Phys. Lett. A, 352,500-504 (2006).

[40] A.M.Wazwaz, “New solitary wave solutions to the modified forms of Degasperis-Procesi and Camassa-Holm equations,” Appl. Math. Comput. 186,130–141 (2007).

[41] Q.D.Wang and M.Y.Tang, “New exact solutions for two nonlinear equations,” Phys. Lett. A,372,2995–3000 (2008).

[42] S.X.Liang and D.J.Jeffrey, “New traveling wave solutions to modified CH and DP equations,” Comp.Phys.Commun.180,1429–1433 (2009).