HEAVY TAIL PHENOMENON AND CONVERGENCE TO
STABLE LAWS ITERATED LIPSCHITZ MAPS

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ABSTRACT. We consider the Markov chain \( \{X_n^x\}_{n=0}^{\infty} \) on \( \mathbb{R}^d \) defined by the stochastic recursion
\[ X_n^x = \psi_{\theta_n}(X_{n-1}^x), \]
starting at \( x \in \mathbb{R}^d \), where \( \theta_1, \theta_2, \ldots \) are i.i.d. random variables taking their values in a metric space \((\Theta, d)\) and \( \psi_{\theta_n} : \mathbb{R}^d \rightarrow \mathbb{R}^d \) are Lipschitz maps. Assume that the Markov chain has a unique stationary measure \( \nu \). Under appropriate assumptions on \( \psi_{\theta_n} \) we will show that the measure \( \nu \) has a heavy tail with the exponent \( \alpha > 0 \), i.e.,
\[ \nu(\{x \in \mathbb{R}^d : |x| > t\}) \approx t^{-\alpha}. \]
Using this result we show that properly normalized Birkhoff sums \( S_n^x = \sum_{k=0}^{n} X_k^x \), converge in law to an \( \alpha \)-stable law for \( \alpha \in (0, 2] \).

1. INTRODUCTION AND STATEMENT OF RESULTS

We consider the Euclidean space \( \mathbb{R}^d \) endowed with the scalar product \[ \langle x, y \rangle = \sum_{i=1}^{d} x_i y_i \]
and the norm \( |x| = \sqrt{\langle x, x \rangle} \). An iterated random function is a sequence of the form
\[ X_0^x = x, \]
\[ X_n^x = \psi(X_{n-1}^x, \theta_n), \]
where \( n \in \mathbb{N} \) and \( \theta_1, \theta_2, \ldots \in \Theta \) are independent and identically distributed according to the measure \( \mu \) on a metric space \( \Theta = (\Theta, d) \). We assume that \( \psi : \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^d \) is jointly measurable and we write \( \psi_{\theta}(x) = \psi(x, \theta) \). Then the sequence \( \{X_n^x\}_{n \geq 0} \) is a Markov chain with the state space \( \mathbb{R}^d \), the initial distribution \( \delta_x \) and the transition probability \( P \) defined by
\[ P(x, B) = \int_{\Theta} 1_B(\psi_\theta(x)) \mu(d\theta), \]
for all \( x \in \mathbb{R}^d \) and \( B \in \text{Bor}(\mathbb{R}^d) \). Unless otherwise stated we assume throughout this paper that \( \psi_\theta : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a Lipschitz map with the Lipschitz constant \( L_\theta < \infty \).

Matrix recursions
\[ X_n^x = \psi_{\theta_n}(X_{n-1}^x) = M_n X_{n-1}^x + Q_n \in \mathbb{R}^d, \]
where \( \theta_n = (M_n, Q_n) \in GL(\mathbb{R}^d) \times \mathbb{R}^d = \Theta \) and \( X_0^x = x \) are probably the best known examples of the situation we have in mind (4; 5; 12; 21; 22).

We are going to describe the asymptotic behavior of Birkhoff sums \( S_n^x = \sum_{k=0}^{n} X_k^x \) of (non independent) random variables \( X_k^x \). We prove that \( S_n^x \) normalized appropriately converge to a stable law (see Theorem 1.24).

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The problem has been recently studied in (3) for the recursion (1.2) with $M \in \mathbb{R}^*_+ \times O(\mathbb{R}^d)$ and a central limit theorem has been proved. Depending on the growth of $M$, a stable law or a Gaussian law appear as the limit. In the first case the heavy tail behavior of stationary solution at infinity is vital for the proof. (See (4)).

On the other hand being linear is not that crucial for $\psi_\theta$ and so, it is tempting to generalize the result of (3) for a larger class of possible $\psi_\theta$. Lipschitz transformations fit perfectly into the scheme – see examples in section 2 due to Goldie (6) and Borkovec and Klüppelberg (2).

The paper is divided into three parts. In the first one (section 3) we describe the support of the stationary law $\nu$ – see Theorems 1.4 and 1.8 below. Secondly, in section 4 we take care of the tail of $\nu$. (See Theorem 1.13 saying that $\nu(\{x \in \mathbb{R}^d : |x| > t\}) \asymp t^{-\alpha})$. Finally, sections 5–7 are devoted to the proof the limit Theorem 1.24. The limit law is a stable law with exponent $\alpha \in (0, 2]$. Thus we generalize results for one dimensional and multidimensional ”ax+b” model stated in (14) and (3) respectively. The case where $\alpha > 2$ has been widely investigated in the general context of complete separable metric spaces by (1; 16; 17; 25) and (29). Recently, in (19) the authors proved $\alpha$ – stable theorem for $\alpha \in (0, 2)$ for additive functionals on metric spaces using martingale approximation method, but our situation does not fit into their framework.

Now we are ready to formulate assumptions and to state theorems. We start with existence of the stationary solution.

**Assumption 1.3 (For the stationary solution).**

(S1) **Lipschitz constant $L_\theta$ is contracting in average** i.e.
\[
\int_\Theta \log(L_\theta) \mu(d\theta) < 0.
\]

(S2) Moreover,
\[
\int_\Theta |\log(L_\theta)| \mu(d\theta) < \infty.
\]

(S3) For some $x_0 \in \mathbb{R}^d$,
\[
\int_\Theta \log^+(|\psi_\theta(x_0)|) \mu(d\theta) < \infty.
\]

Under above assumption recursion (1.1) has a stationary solution. More precisely we have the following

**Theorem 1.4.** Assume that Lipschitz maps $\psi_\theta : \mathbb{R}^d \mapsto \mathbb{R}^d$ satisfy hypotheses (S1)–(S3). Let $Y_{n}^x = \psi_{\theta_1} \circ \psi_{\theta_2} \circ \ldots \circ \psi_{\theta_n}(x)$, $n \in \mathbb{N}$. Then there exists a unique probability measure $\nu$ defined on $\mathbb{R}^d$ such that
\[
\int_{\mathbb{R}^d} \int_{\Theta} f(\psi_\theta(x)) \mu(d\theta) \nu(dx) = \int_{\mathbb{R}^d} f(x) \nu(dx).
\]
for any bounded and continuous function $f$ on $\mathbb{R}^d$. The measure $\nu$ is the law the random variable $S$
\[
S = S_1 = \lim_{n \to \infty} Y_n^x = \lim_{n \to \infty} \psi_{\theta_1} \circ \psi_{\theta_2} \circ \ldots \circ \psi_{\theta_n}(x),
\]
that does not depend on the choice of the starting point $x \in \mathbb{R}^d$, and $S$ satisfies equation (1.1) in law. Moreover, $S_1 = \psi_{\theta_1}(S_2)$, a.s. where
heavy tail phenomena and convergence to stable laws

(1.7) \[ S_2 = \lim_{n \to \infty} \psi_{\theta_2} \circ \psi_{\theta_3} \circ \ldots \circ \psi_{\theta_n}(x). \]

The proof is standard, see (5) for complete separable metric spaces.

Notice that (S1)–(S3) are straightforward generalizations of analogous conditions for recursion (1.2) with \( \theta = (M, Q) \in GL(\mathbb{R}^d) \times \mathbb{R}^d \) where \( L_\theta \) and \( |\psi_0(0)| \) playing the roles of \(|M|\) and \( |Q| \) respectively (see (4; 5; 12; 21)).

1.1. Support of the stationary measure.

Let \( L^\mu_\Theta = \{ \psi_{\theta_1} \circ \ldots \circ \psi_{\theta_n}(\cdot) : \forall_{n \in \mathbb{N}} \forall_{1 \leq i \leq n} \theta_i \in \text{supp} \mu \} \) i.e. \( L^\mu_\Theta \) is closed subgroup generated by the Lipschitz maps \( \psi_0 \) where \( \theta \in \text{supp} \mu \). Given \( \psi_0 \) with \( L_\theta < 1 \) let \( \psi_\theta^\bullet \) be the unique fixed point of \( \psi_\theta \). Then we have the following

**Theorem 1.8.** Assume that \( \Theta \ni \theta \mapsto \psi_\theta(x) \in \mathbb{R}^d \) is continuous for every \( x \in \mathbb{R}^d \) and

\[ S = \{ \psi_\theta^\bullet \in \mathbb{R}^d : \psi_\theta(\psi_\theta^\bullet) = \psi_\theta^\bullet, \text{ where } \psi_0 \in L^\mu_\Theta \text{ and } L_\theta < 1 \} \subseteq \mathbb{R}^d, \]

then \( \text{supp} \nu = \overline{S} \).

**Theorem 1.8** generalizes similar theorems for affine random walks, see (4) and (12) for more details. The proof is contained in section 3.

1.2. Heavy tail phenomena.

In this section we state conditions that assure a heavy tail of \( \nu \). Contrary to the affine recursion

(1.9) \[ X_n^x = \psi_{\theta_n}(X_{n-1}^x) = M_n X_{n-1}^x + Q_n \in \mathbb{R}, \]

where \( \theta_n = (M_n, Q_n) \in \mathbb{R} \times \mathbb{R} = \Theta \), we need more than just behavior of the Lipschitz constant \( L_\theta \).

For instance, it is easy to see that the iterative model generated by \( \psi_n(x) = \sin(L_n x) \) where \( n \in \mathbb{N} \) and \( L_n \) are i.i.d. random variables does not have a heavy tail. Indeed, suppose that every \( \psi_n \) satisfies (S1)–(S3). Then for \( t > 1 \) we have that \( \nu(\{|x \in \mathbb{R} : |x| > t\}) = \mathbb{P}(\{|S| > t\}) = \mathbb{P}(\{|\sin(LS)| > t\}) = 0 \).

**Assumption 1.10 (Shape of the mappings \( \psi \)).** For every \( t > 0 \), let \( \psi_{\theta,t} : \mathbb{R}^d \mapsto \mathbb{R}^d \) be defined by

\[ \psi_{\theta,t}(x) = t \psi_0(t^{-1} x), \]

where \( x \in \mathbb{R}^d \) and \( \theta \in \Theta \). \( \psi_{\theta,t} \) are called dilatations of \( \psi_\theta \).

(H1) For every \( \theta \in \Theta \), there exist a map \( \overline{\psi}_\theta : \mathbb{R}^d \mapsto \mathbb{R}^d \) such that for every \( x \in \mathbb{R}^d \)

\[ \lim_{t \to 0} \psi_{\theta,t}(x) = \overline{\psi}_\theta(x), \]

where \( \overline{\psi}_\theta(x) = M_\theta x \) for every \( x \in \text{supp} \nu \). Random variable \( M_\theta \) takes its values in group \( G = \mathbb{R}_+^* \times K \), where \( K \) is closed subgroup of orthogonal group \( O(\mathbb{R}^d) \).

(H2) For every \( \theta \in \Theta \), there is a random variable \( N_\theta \) such that \( \psi_\theta \) satisfies cancellation condition i.e.

\[ |\psi_\theta(x) - M_\theta x| \leq |N_\theta|, \]

for every \( x \in \text{supp} \nu \).
To get the idea what is the meaning of (H1)–(H2) the reader may think of the affine recursion (1.9) with \( \theta = (M, Q) \in G \times \mathbb{R}^d = \Theta \) or the recursion \( \psi_0(x) = \max\{Mx, Q\} \) where \( \theta = (M, Q) \in \mathbb{R}^+ \times \mathbb{R} = \Theta \) (see section 2). Then \( \psi_0(x) = Mx \) or \( \psi_0(x) = \max\{Mx, 0\} \) respectively. It is recommended to have in mind \( \psi_0(x) = \max\{Mx, Q\} \) to get the first approximation of what the hypotheses mean. Notice that for the max recursion (H2) is not satisfied on \( \mathbb{R} \) but only on \( [0, \infty) \supset \text{supp}\nu \).

In one dimensional case condition (H2) has a very natural geometrical interpretation, namely it can be written in an equivalent form

\[
M_\theta x - |N_\theta| \leq \psi_0(x) \leq M_\theta x + |N_\theta|,
\]

It means that the graph of \( \psi_0(x) \)'s lies between the graphs of \( M_\theta x - |N_\theta| \) and \( M_\theta x + |N_\theta| \) for every \( x \in \text{supp}\nu \). This allows us to think that the recursion is, in a sense, close to the affine recursion.

For simplicity we write \( X \) instead of \( X_\theta \).

**Assumption 1.11 (Moments condition for the heavy tail).** Let \( \kappa(s) = \mathbb{E}|M|^s \) for \( s \in [0, s_\infty) \), where \( s_\infty = \sup\{s \in \mathbb{R}_+ : \kappa(s) < \infty\} \). Let \( \bar{\mu} \) be the law of \( M \).

(H3) \( G \) is the smallest closed semigroup generated by the support of \( \bar{\mu} \) i.e. \( G = (\text{supp}\bar{\mu})^\ast \).

(H4) The conditional law of \( \log|M| \), given \( M \neq 0 \) is non arithmetic.

(H5) \( M \) satisfies Cramér condition with exponent \( \alpha > 0 \), i.e. there exists \( \alpha \in (0, s_\infty) \) such that

\[
\kappa(\alpha) = \mathbb{E}(|M|^\alpha) = 1.
\]

(H6) Moreover,

\[
\mathbb{E}(|M|^\alpha \log|M|) < \infty.
\]

(H7) For the random variable \( N \) defined in (H2) we have

\[
\mathbb{E}(|N|^\alpha) < \infty.
\]

Conditions (H4)–(H7) are natural in this context, see (2; 3; 4; 6; 8; 9; 10; 14; 15; 21) and (28).

**Remark 1.12.** Conditions (H5), (H6) imply that the function \( \kappa(s) = \mathbb{E}|M|^s \) is well defined on \( [0, \alpha] \) and \( \kappa(0) = \kappa(\alpha) = 1 \). Since \( \kappa \) is convex, we have

\[
\mathbb{E}(\log(|M|)) < 0, \quad \text{and} \quad m_\alpha = \mathbb{E}(|M|^\alpha \log|M|) > 0.
\]

For more details we refer to (6). Clearly, condition \( L \leq |M| \) together with conditions (H2), (H5) and (H7) imply conditions (S1)–(S3).

A closed subgroup of \( \mathbb{R}_+^\ast \times O(\mathbb{R}^d) \) containing \( \mathbb{R}_+^\ast \) is necessarily \( G = \mathbb{R}_+^\ast \times K \), where \( K \) is closed subgroup of orthogonal group \( O(\mathbb{R}^d) \) see e.g. appendix C in (4) and appendix A in (3). Let \( \frac{\text{d} \rho}{\text{d} \lambda} \) be the Haar measure on \( \mathbb{R}_+^\ast \) and \( \rho \) be the Haar measure on \( K \) such that \( \rho(K) = 1 \). Any element \( g \in \mathbb{R}_+^\ast \times K \) can be uniquely written as \( g = rk \), where \( r \in \mathbb{R}_+^\ast \) and \( k \in K \), and so the Haar measure \( \lambda \) on \( \mathbb{R}_+^\ast \times K \) is

\[
\int_G f(g) \lambda(dg) = \int_{\mathbb{R}_+^\ast} \int_K f(rk) \rho(dk) \frac{dr}{r}.
\]

Clearly, \( G \) is unimodular. Now we are ready to formulate the main result.
Theorem 1.13. For $\theta \in \Theta$ assume that $\psi_\theta$ satisfy assumptions 1.3, 1.10 and 1.11. Let $S$ be the stationary solution of (1.1) and let $\nu$ be its law. Then there exists a Radon measure $\Lambda$ on $\mathbb{R}^d \setminus \{0\}$ such that
\begin{equation}
(1.14) \quad \lim_{|g| \to 0} |g|^{-\alpha} \mathbb{E} f(gS) = \lim_{|g| \to 0} |g|^{-\alpha} \int_{\mathbb{R}^d} f(gx) \nu(dx) = \int_{\mathbb{R}^d \setminus \{0\}} f(x) \Lambda(dx),
\end{equation}
for every function $f \in \mathcal{F}$, where
\begin{equation}
\mathcal{F} = \{ f : \mathbb{R}^d \to \mathbb{R} : f \text{ is measurable function such that } \Lambda(\text{Dis}(f)) = 0 \text{ and } (1.15) \quad \sup_{x \in \mathbb{R}^d} |x|^{-\alpha} |\log |x||^{1+\epsilon} |f(x)| < \infty \text{ for some } \epsilon > 0 \}.
\end{equation}
and $\text{Dis}(f)$ is the set of all discontinuities of function $f$. Moreover, the measure $\Lambda$ is homogeneous with degree $\alpha$ i.e. for every $g \in G$,
\begin{equation}
(1.16) \quad \int_{\mathbb{R}^d} f(gx) \Lambda(dx) = |g|^\alpha \Lambda(f).
\end{equation}
There exists a measure $\sigma_\Lambda$ on $\mathbb{S}^{d-1}$ such that $\Lambda$ has the polar decomposition
\begin{equation}
(1.17) \quad \int_{\mathbb{R}^d \setminus \{0\}} f(x) \Lambda(dx) = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(rx) \sigma_\Lambda(dx) \frac{dr}{r^{d+1}},
\end{equation}
where
\begin{equation}
(1.18) \quad \sigma_\Lambda(\mathbb{S}^{d-1}) = \frac{1}{m_\alpha} \mathbb{E} (|\psi(S)|^\alpha - |MS|^\alpha),
\end{equation}
and $m_\alpha = \mathbb{E}(|M|^\alpha \log |M|) \in (0, \infty)$. Furthermore recursion defined in (1.1) has a heavy tail
\begin{equation}
(1.19) \quad \lim_{t \to \infty} t^\alpha \mathbb{P}(\{|S| > t\}) = \frac{1}{\alpha m_\alpha} \mathbb{E} (|\psi(S)|^\alpha - |MS|^\alpha).
\end{equation}
If additionally support of $\nu$ is unbounded and one of the following condition is satisfied
\begin{equation}
(1.20) \quad s_\infty < \infty \quad \text{and} \quad \lim_{s \to s_\infty} \frac{\mathbb{E}(|N|^s)}{\kappa(s)} = 0,
\end{equation}
\begin{equation}
(1.21) \quad s_\infty = \infty \quad \text{and} \quad \lim_{s \to \infty} \left( \frac{\mathbb{E}(|N|^s)}{\kappa(s)} \right)^{\frac{1}{s}} < \infty,
\end{equation}
then the measures $\Lambda$ and $\sigma_\Lambda$ are nonzero.

Remark 1.22. Contrary to Theorems 1.4 and 1.8 the assumption that $\psi_\theta$ are Lipschitz is not necessary for Theorem 1.13. The same conclusion holds if $\psi_\theta : \mathbb{R}^d \to \mathbb{R}^d$ is continuous for every $\theta \in \Theta$ and the map $\Theta \ni \theta \mapsto \psi_\theta(x) \in \mathbb{R}^d$ is continuous for every $x \in \mathbb{R}^d$, 1.10 and 1.11 are satisfied and $S = \lim_{n \to \infty} \psi_{\theta_1} \circ \psi_{\theta_2} \circ \ldots \circ \psi_{\theta_n}(x)$ exists a.s. and does not depend on $x \in \mathbb{R}^d$.

In view of Letac's principle (24) the random variable $S$ with law $\nu$ is a unique stationary solution of the recursion (1.1).

Theorem 1.13 on one hand generalizes Theorem 1.6 of (4) for multidimensional affine recursions and on the other, the results of Goldie (6) for a family of one-dimensional recursions modeled on $ax + b$. (H4)–(H6) were already assumed by Goldie. (H3) was introduced in (4) and the whole proof is based on it. (H1)–(H2) say that asymptotically (1.1) looks like an affine recursion and it allows us to use the methods of (4).
On the other hand, the example below shows that for (1.20) and (1.21) the hypothesis that the support of the measure $\nu$ is unbounded is crucial. Consider $\psi_n(x) = A_n \max\{x, B_n\} + C_n$ and assume that $P\{\{A_n = \frac{1}{2}\}\} = \frac{2}{4}$, $P\{\{A_n = 2\}\} = \frac{1}{2}$ and $P\{\{B_n = \frac{1}{2}\}\} = P\{\{C_n = -1\}\} = 1$. Then $E(\log A_n) < 0$ and $E(A_n^n) = 1$ where $\alpha \approx 1.851$. It is easy to see that the stationary measure $\nu$ is supported by the set $\{-\frac{1}{2}, 0\}$ though the function $\psi_n(x)$ is unbounded.

1.3. Limit theorem for Birhhoff sums. Now we introduce conditions necessary to obtain convergence in law of appropriately normalized sums $S_n^x = \sum_{k=0}^n X_k^x$ to an $\alpha$–stable distribution.

**Assumption 1.23 (For the limit theorem).**

(L1) For every $\theta \in \Theta$ Lipschitz constant $L_\theta \leq |M_\theta|$.

(L2) For every $\theta \in \Theta$, there is a random variable $Q_\theta$, such that $\bar{\psi}_\theta$ satisfies smoothness condition with respect to $t > 0$ i.e.

$$|\psi_{\theta,t}(x) - \bar{\psi}_\theta(x)| \leq |t| |Q_\theta|,$$

for every $x \in \mathbb{R}^d$.

(L3) For the random variable $Q$ we have

$$E(|Q|^\alpha) < \infty.$$

Clearly, if $\bar{\psi}_\theta(x) = M_\theta x$ for every $x \in \text{supp}\nu$, then (L2) implies (H1) and (L2) together with (L3) imply (H2) and (H7). Now we are ready to formulate the limit theorem.

**Theorem 1.24.** For $\theta \in \Theta$ suppose that $\psi_\theta$ satisfies assumptions 1.10, 1.11 and 1.23. Let $h_v(x) = E(e^{i(x, \sum_{k=1}^n \bar{\psi}_k \circ \cdots \circ \bar{\psi}_1)}(x))$ for $x \in \mathbb{R}^d$ and $\nu$ be the stationary measure for recursion (1.1). Then

- if $0 < \alpha < 1$, then $n^{-\alpha} S_n^x$ converges in law to the $\alpha$–stable random variable with characteristic function
  $$\Upsilon_\alpha(tv) = e^{tv C_\alpha(v)},$$
  for any $t > 0$ and $v \in \mathbb{S}^{d-1}$, where
  $$C_\alpha(v) = \int_{\mathbb{R}^d} \left( e^{i(v, x)} - 1 \right) h_v(x) \Lambda(dx).$$

- if $\alpha = 1$ and $\xi(t) = \int_{\mathbb{R}^d} \frac{tv}{1 + |tv|^2} \nu(dx)$, then $n^{-1} S_n^x - n \xi(n^{-1})$ converges in law to the random variable with characteristic function
  $$\Upsilon_1(tv) = e^{i C_1(v) + it(v, \tau(t))},$$
  for any $t > 0$ and $v \in \mathbb{S}^{d-1}$, where
  $$C_1(v) = \int_{\mathbb{R}^d} \left( e^{i(v, x)} - 1 \right) h_v(x) - \frac{i(v, x)}{1 + |x|^2} \Lambda(dx),$$
  and
  $$\tau(t) = \int_{\mathbb{R}^d} \left( \frac{x}{1 + |tx|^2} - \frac{x}{1 + |x|^2} \right) \Lambda(dx).$$
• if $1 < \alpha < 2$ and $m = \int_{\mathbb{R}^d} xv(dx)$, then $n^{-\frac{1}{\alpha}} (S_n^x - nm)$ converges in law to the $\alpha$-stable random variable with characteristic function

$$\Upsilon_\alpha(tv) = e^{i\alpha C_\alpha(v)},$$

for any $t > 0$ and $v \in \mathbb{S}^{d-1}$, where

$$C_\alpha(v) = \int_{\mathbb{R}^d} \left( e^{i(v,x)} - 1 \right) h_v(x) - i\langle v, x \rangle \Lambda(dx).$$

• if $\alpha = 2$ and $m = \int_{\mathbb{R}^d} xv(dx)$, then $(n \log n)^{-\frac{1}{2}} (S_n^x - nm)$ converges in law to the random variable with characteristic function

$$\Upsilon_2(tv) = e^{i2 C_2(v)},$$

for any $t > 0$ and $v \in \mathbb{S}^{d-1}$, where

$$C_2(v) = -\frac{1}{4} \int_{\mathbb{S}^{d-1}} \left( (v, w)^2 + 2\langle v, w \rangle \langle v, \mathbb{E} (\varphi(w)) \rangle \right) \sigma_\Lambda(dw),$$

and $\varphi(x) = \sum_{k=1}^{\infty} \psi_k \circ \cdots \circ \psi_1(x)$ and $\sigma_\Lambda$ is the measure on $\mathbb{S}^{d-1}$ defined in (1.17).

Moreover, $C_\alpha(tv) = t^\alpha C_\alpha(v)$ for every $t > 0$, $v \in \mathbb{S}^{d-1}$ and $\alpha \in (0, 1) \cup (1, 2]$. If $\varphi(x) = \sum_{k=1}^{\infty} M_k \cdots M_1 x$ for every $x \in \text{supp} \Lambda$ and there exist $w_1, \ldots, w_d \in \text{supp} \sigma_\Lambda$ that span $\mathbb{R}^d$ as a linear space, then $\Re C_\alpha(v) < 0$ for every $v \in \mathbb{S}^{d-1}$.

The proof of the above theorem will be based on the spectral method that was initiated by Nagaev in (26) and then used and improved by many authors (see (17) for references). The spectral method is based on quasi-compactness of transition operators $P f(x) = \mathbb{E} (f(\psi(x))) = \int_{\Theta} \int f(\psi(x)) \mu(d\theta)$ on appropriate function spaces (see (3; 14; 16; 17)). They are perturbed by adding Fourier characters.

The standard use of the perturbation theory requires exponential moments of $\mu$, but there is some development towards $\mu$’s with polynomial moments (16), or even fractional moments (14), and (3). They are based on a theorem of Keller and Liverani (20). It says that the spectral properties of the operator $P$ can be approximated by those of its Fourier perturbations

$$(1.25) \quad P_{t,v} f(x) = \mathbb{E} \left( e^{i\langle tv, \psi(x) \rangle} f(\psi(x)) \right) = \int_{\Theta} e^{i\langle tv, \psi(\theta) \rangle} f(\psi(\theta)) \mu(d\theta),$$

(with convention that $P_{0,v} = P$). Indeed,

$$(1.26) \quad P_{t,v} f(x) = k_v(t) \Pi_{P,t} + Q_{P,t},$$

where $\lim_{t \to 0} k_v(t) = 1$, $\Pi_{P,t}$ is a projection on a one dimensional subspace and the spectral radii of $Q_{P,t}$ are smaller than $\rho < 1$ when $t \leq t_0$. To obtain Theorem 1.24 we need to expand the dominant eigenvalue $k_v(t)$ at $0$.

When $\alpha \in (0, 2]$, $k_v(t)$ is neither analytic nor differentiable, hence their asymptotics at zero is much harder to obtain. The method used in (3) does not work here and so we propose another approach which is applicable to general Lipschitz models (see section 6).

2. Examples

The following examples will help the reader to understand the meaning of assumptions formulated in the introduction as well as to feel the breadth of the method.
2.1. **An affine recursion.** Let $G = \mathbb{R}_+^* \times O(\mathbb{R}^d)$ and take the sequence of i.i.d. random pairs $(A_n, B_n)_{n \in \mathbb{N}} \subseteq \Theta = G \times \mathbb{R}^d$ with the same law $\mu$ on $\Theta$ and define the affine map $\psi_n(x) = A_n x + B_n$ where $x \in \mathbb{R}^d$. This example was also widely considered in the context of discrete subgroups of $\mathbb{R}_+^*$, see (4) and (3).

2.2. **An extremal recursion.** Let $G = \mathbb{R}_+^*$ and $\Theta = G \times \mathbb{R}$. We consider the sequence of i.i.d. pairs $(A_n, B_n)_{n \in \mathbb{N}}$ with values in $\Theta$ and with the law $\mu$ satisfying (S1)–(S3). Let $\psi_n(x) = \max\{A_n x, B_n\}$ where $x \in \mathbb{R}$. Then

- $\lim_{t \to 0} \psi_{n,t}(x) = \overline{\psi}_n(x)$ where $\overline{\psi}_n(x) = \max\{M_n x, 0\}$ and $M_n = A_n$.
- The stationary solution $S$ with law $\nu$ is given by the explicit formula,

$$S = \bigvee_{k=1}^{\infty} A_1 A_2 \cdots A_{k-1} B_k,$$

where $A_0 = 1$ a.s. (6).

- If $P(B > 0) > 0$, then the support $\subseteq [0, \infty)$ and is unbounded.

In order to check cancellation condition (H2) notice that $S \geq 0$ a.s and for $x > 0$

$$|\psi_{n,t}(x) - A_n x| = |\max\{A_n x, B_n\} - A_n x| 1\{A_n x < B_n\} \leq (|B_n| + |A_n x|) 1\{A_n x < B_n\} \leq 2|B_n|,$$

so (H2) is fulfilled with $|N_n| = 2|B_n|$ and we assume (H4)–(H7) for $M_n = A_n$ and $N_n = 2B_n$.

- Notice, that $|\psi_{n,t}(x) - \overline{\psi}_n(x)| = |\max\{A_n x, tB_n\} - \max\{A_n x, 0\}| \leq |t||B_n|$, so (L2) is satisfied with $|Q_n| = |B_n|$ and we assume (L3) for $Q_n = B_n$.

2.3. A model due to Letac. Let $G$ be as above and take the sequence of i.i.d. random triples $(A_n, B_n, C_n)_{n \in \mathbb{N}} \subseteq \Theta = G \times \mathbb{R}_+ \times \mathbb{R}_+$ with the same law $\mu$ on $\Theta$. Consider map $\psi_n(x) = A_n \max\{x, B_n\} + C_n$ where $x \in \mathbb{R}$. If $C \geq 0$ a.s. and $P(B > 0) + P(C > 0) > 0$, then the support of the stationary measure $\nu$ is unbounded (6). Similar consideration as above applied to the Letac model show that our assumptions are satisfied.

2.4. Another example. Take the sequence of i.i.d. random triples $(A_n, B_n, C_n)_{n \in \mathbb{N}} \subseteq \Theta = \mathbb{R}_+^* \times \mathbb{R}_+ \times \mathbb{R}_+$ with the same law $\mu$ on $\Theta$, such that $C_n - \frac{B_n^2}{A_n^2} > 0$. Consider map $\psi_n(x) = \sqrt{A_n x^2 + B_n x} + C_n$ where $x \in \mathbb{R}$. If $P(B > 0) + P(C > 0) > 0$, then the support of the stationary measure $\nu$ is unbounded (6). Conditions (H2) and (L2) can be easily verified.

For the above examples statements 1.8, 1.13 and 1.24 apply straightforwardly.

2.5. **An autoregressive process with ARCH(1) errors.** Now we consider an example described by Borkovec and Klüppelberg in (2). For $x \in \mathbb{R}$ let $\psi(x) = |\gamma| x + \sqrt{\beta + \lambda x^2} A$ where $\gamma \geq 0$, $\beta > 0$, $\lambda > 0$ are constants and $A$ is a symmetric random variable with continuous Lebesgue density $p$, finite second moment and with the support equal the whole of $\mathbb{R}$. Moreover, see section 2. in (2) for more details). Now consider the sequence $(\psi_n(x))_{n \in \mathbb{N}}$ of i.i.d. copies of $\psi(x)$ and observe that

- $\lim_{t \to 0} \psi_{n,t}(x) = \overline{\psi}_n(x)$, where $\overline{\psi}_n(x) = M_n |x|$ and $M_n = |\gamma + \sqrt{\lambda} A_n|$.
\begin{itemize}
\item $|\psi_{n,t}(x) - M_n| = |\gamma |x| + \sqrt{\beta^2 + \lambda^2}A_n| = |\gamma + \sqrt{\lambda}A_n| |x|$
\end{itemize}
so (L2) holds with $|Q_n| = \sqrt{\beta} |A_n|$. Notice that (H2) holds for every $x \in [0, \infty)$ with $|N_n| = \sqrt{\beta} |A_n|$. In (2) the authors showed that it is possible to choose parameters $\gamma > 0, \beta > 0, \lambda > 0$ such that $\mathbb{E}(\log M_n) < 0$ and $\mathbb{E}(M_n^\alpha) = 1$ for some $0 < \alpha < 1$. Observe that $\mathbb{P} \{ \{ M_n \in \mathbb{R}_+ \} \} = 1$. We are not able to verify conditions (1.20) and (1.21) to conclude that $\Lambda$ is not zero. The latter follows however from (2) and so Theorem 1.24 applies.

3. Stationary measure

3.1. Support of the stationary measure. Let $C(\mathbb{R}^d)$ be the set of continuous function on $\mathbb{R}^d$ and $C_b(\mathbb{R}^d)$ be the set of bounded and continuous function on $\mathbb{R}^d$.

Before proving Theorem 1.8, we need two lemmas. Given, $\psi_{\theta}$ with $L_{\theta} < 1$, the Banach fixed point theorem ensures existence of a unique fixed point $\psi_{\theta}^n \in \mathbb{R}^d$ of the map $\psi_{\theta}$. Moreover, for every $x \in \mathbb{R}^d$

\begin{equation}
\lim_{n \to \infty} \psi_{\theta}^n(x) = \psi_{\theta}^n.
\end{equation}

**Lemma 3.2.** Assume that for the map $\psi_{\theta}$ we have $L_{\theta} < 1$ and $\psi \in L^\infty_{\Theta}$. Then

\begin{equation}
\lim_{n \to \infty} (\psi \circ \psi_{\theta}^n)^* = \psi(\psi_{\theta}^n)^*,
\end{equation}

where $(\psi \circ \psi_{\theta}^n)^* \in \mathbb{R}^d$ is the fixed point of the map $\psi \circ \psi_{\theta}^n$, for $n \in \mathbb{N}$.

**Proof.** Notice that for $n$ sufficiently large $\psi_{\theta}^n = \psi \circ \psi_{\theta}^n$ is contracting. Fix $\varepsilon > 0$, then there exist $N_{\varepsilon} \in \mathbb{N}$ such that $\frac{L_{\theta}L_{n}^{\alpha}}{1 - L_{\theta}L_{n}^{\alpha}} < \varepsilon$ for all $n \geq N_{\varepsilon}$, where $L_{\psi}$ is Lipschitz constant associated to $\psi$. For every $m \in \mathbb{N}$ we have

\[ (\psi \psi_{\theta}^n)^m(\psi_{\theta}^n)^* - \psi(\psi_{\theta}^n)^* \leq (L_{\psi}L_{n}^{\alpha})^{m-1}(\psi_{\theta}^n)^*(\psi_{\theta}^n)^* - \psi_{\theta}^n + (\psi_{\theta}^n)^{m-1}(\psi_{\theta}^n)^* - \psi_{\theta}^n) \]

\[ \leq ((L_{\psi}L_{n}^{\alpha})^{m-1} + (L_{\psi}L_{n}^{\alpha})^{m-2} + \ldots + (L_{\psi}L_{n}^{\alpha})^{m-\varepsilon} + (L_{\psi}L_{n}^{\alpha})^{m-\varepsilon} + \ldots + (L_{\psi}L_{n}^{\alpha})^{m-\varepsilon}) \cdot |\psi(\psi_{\theta}^n)^* - \psi_{\theta}^n| \]

\[ \leq \left( \sum_{k=1}^{\infty} (L_{\psi}L_{n}^{\alpha})^{k} \right) \cdot |\psi(\psi_{\theta}^n)^* - \psi_{\theta}^n| = \frac{L_{\psi}L_{n}^{\alpha}}{1 - L_{\psi}L_{n}^{\alpha}} \cdot |\psi(\psi_{\theta}^n)^* - \psi_{\theta}^n|. \]

By (3.1) we can find $m \in \mathbb{N}$ such that $|(\psi \psi_{\theta}^n)^m(\psi_{\theta}^n)^* - \psi(\psi_{\theta}^n)^*| < \varepsilon$. Then

\[ |(\psi \psi_{\theta}^n)^* - \psi(\psi_{\theta}^n)^*| \leq |(\psi \psi_{\theta}^n)^* - (\psi \psi_{\theta}^n)^m(\psi_{\theta}^n)^*| + |(\psi \psi_{\theta}^n)^m(\psi_{\theta}^n)^* - \psi(\psi_{\theta}^n)^*| \]

\[ \leq \varepsilon + |(\psi \psi_{\theta}^n)^* - \psi_{\theta}^n| \cdot \frac{L_{\psi}L_{n}^{\alpha}}{1 - L_{\psi}L_{n}^{\alpha}} \leq \varepsilon (1 + |(\psi(\psi_{\theta}^n)^* - \psi_{\theta}^n)|), \]

for all $n \geq N_{\varepsilon}$. Since $\varepsilon$ is arbitrary, (3.3) is established. \hfill \Box

**Lemma 3.4.** If $\psi_{\theta} : \mathbb{R}^d \to \mathbb{R}^d$ is continuous for every $\theta \in \Theta$ (not necessarily Lipschitz) and $\Theta \ni \theta \mapsto \psi_{\theta}(x) \in \mathbb{R}^d$ is continuous for every $x \in \mathbb{R}^d$, then for every $\theta \in \text{supp} \mu$

\[ \psi_{\theta}[\text{supp} \mu] \subseteq \text{supp} \mu. \]
Proof. Suppose for a contradiction that $\psi_0(\text{supp}\nu) \subsetneq \text{supp}\nu$. Then for some $\theta_0 \in \text{supp}\mu$ and $x_0 \in \text{supp}\nu$ there exists an open neighborhood $U$ of $\psi_0(x_0)$ such that $U \cap \text{supp} \nu = \emptyset$. Since $\nu$ is lower semi–continuous i.e. $\liminf_{x \to x_0} 1_U(x) \geq 1_U(x_0)$. Now we show

\begin{equation}
\Theta \ni \psi \mapsto \int_{\mathbb{R}^d} 1_U(\psi(x)) \nu(dx) \text{ is lower semi–continuous.}
\end{equation}

Indeed, take any $(\theta_n)_{n \in \mathbb{N}} \subseteq \Theta$ such that $\lim_{n \to \infty} d(\theta_n, \theta_0) = 0$, then by assumption $\lim_{n \to \infty} \psi_{\theta_n}(x) = \psi_{\theta_0}(x)$ for every $x \in \mathbb{R}^d$ and it implies that $1_U(\psi_{\theta_0}(x)) \leq \liminf_{n \to \infty} 1_U(\psi_{\theta_n}(x))$. Now by Fatou lemma

\[
\int_{\mathbb{R}^d} 1_U(\psi_{\theta_0}(x)) \nu(dx) \leq \liminf_{n \to \infty} \int_{\mathbb{R}^d} 1_U(\psi_{\theta_n}(x)) \nu(dx) \leq \liminf_{n \to \infty} \int_{\mathbb{R}^d} \nu(dx).
\]

Hence by the above inequalities (3.6) holds and it is equivalent with the fact that $\{\theta \in \Theta : \int_{\mathbb{R}^d} 1_U(\psi_{\theta}(x)) \nu(dx) > 0\}$ is open subset of $\Theta$. By the property (3.5) we have $\mu(\{\theta \in \Theta : \int_{\mathbb{R}^d} 1_U(\psi_{\theta}(x)) \nu(dx) > 0\}) = 0$. Since $\psi_{\theta_0}(x)$ is continuous map, then $\psi_{\theta_0}^{-1}[U]$ is open neighborhood of $x_0 \in \text{supp}\nu$, so

\[
0 < \nu(\psi_{\theta_0}^{-1}[U]) = \int_{\mathbb{R}^d} 1_{\psi_{\theta_0}^{-1}[U]}(x) \nu(dx) = \int_{\mathbb{R}^d} 1_U(\psi_{\theta_0}(x)) \nu(dx),
\]

It implies that $\theta_0 \in \{\theta \in \Theta : \int_{\mathbb{R}^d} 1_U(\psi_{\theta}(x)) \nu(dx) > 0\}$ and it is contradiction with the fact that $\theta_0 \in \text{supp}\mu$.

Proof of Theorem (1.8). For $f \in C_b(\mathbb{R}^d)$ by (3.1) we have

\[
\int_{\mathbb{R}^d} f(\psi_0^n(x)) \nu(dx) \to \infty \delta_{\psi_0^n}(f).
\]

If $\psi_0 \not\subseteq \text{supp}\nu$, there exist an open neighborhood $U$ of $\psi_0$ such that $U \cap \text{supp}\nu = \emptyset$. By Lemma 3.4, $U \cap \psi_0^n[\text{supp}\nu] \subseteq U \cap \text{supp}\nu = \emptyset$, for any $n \in \mathbb{N}$. Hence

\[
1 = \delta_{\psi_0^n}(U) \leq \liminf_{n \to \infty} \int_{\mathbb{R}^d} 1_U(\psi_0^n(x)) \nu(dx) = 0.
\]

Therefore, by contradiction $\mathfrak{S} \subseteq \text{supp}\nu$. Now we show the opposite inclusion. By Lemma 3.2 we know that $\psi[\mathfrak{S}] \subseteq \mathfrak{S}$ for every $\psi \in L^p_{\text{loc}}$. Let $\lambda$ be a probability measure on $\mathfrak{S}$. Then for any $f \in C_b(\mathbb{R}^d)$

\[
\lim_{n \to \infty} \int_{\mathbb{R}^d} \ldots \int_{\Theta} f(\psi_{\theta_1} \circ \ldots \circ \psi_{\theta_n}(x)) \mu(d\theta_1) \ldots \mu(d\theta_n) \lambda(dx) = \nu(f),
\]

hence

\[
\nu(\mathfrak{S}) \geq \limsup_{n \to \infty} \int_{\mathbb{R}^d} \ldots \int_{\Theta} 1_{\mathfrak{S}}(\psi_{\theta_1} \circ \ldots \circ \psi_{\theta_n}(x)) \mu(d\theta_1) \ldots \mu(d\theta_n) \lambda(dx)
\]

\[
\geq \limsup_{n \to \infty} \int_{\Theta} \ldots \int_{\Theta} \lambda(\mathfrak{S}) \mu(d\theta_1) \ldots \mu(d\theta_n) = \lambda(\mathfrak{S}) = 1,
\]
and finally, we get $\nu(S) = 1$ i.e. $\text{supp}\nu \subseteq S$.

3.2. Simple properties recursions and their stationary measures.

**Lemma 3.7.** Let $Y_{n,t}^{x} = \psi_{\theta_{1},t} \circ \psi_{\theta_{2},t} \circ \ldots \circ \psi_{\theta_{n},t}(x)$ for any $n \in \mathbb{N}$ and $t > 0$. Then,

(3.8) $|Y_{n,t}^{x} - Y_{n,t}^{y}| \leq \sum_{i=1}^{n} L_{\theta_{i}} |x - y|$, 

(3.9) $|Y_{n,t}^{x} - Y_{n+1,t}^{x}| \leq \sum_{i=1}^{n} L_{\theta_{i}} |x - \psi_{n,t} \circ \ldots \circ \psi_{n+1,t}(x)|$, 

(3.10) $|x - \psi_{n,t} \circ \ldots \circ \psi_{n+1,t}(x)| \leq \sum_{k=1}^{n} \left( \prod_{i=n+1}^{n+k-1} L_{\theta_{i}} \right) |x - \psi_{n+k,t}(x)|$, 

for any $x, y \in \mathbb{R}^{d}$ and $m, n \in \mathbb{N}$.

**Proof.** It is easy to see that $|\psi_{n,t}(x) - \psi_{n,t}(y)| = |t \psi_{0}(t^{-1} x) - t \psi_{0}(t^{-1} y)| \leq L_{0} |x - y|$ for any $x, y \in \mathbb{R}^{d}$, so (3.8), (3.9) and (3.10) follow by induction.

**Lemma 3.11.** Under the assumptions of Remark 1.22 for every $\beta \in (0, \alpha)$

$$\left( \mathbb{E}|S|^\beta \right)^{\frac{1}{\beta}} < \infty.$$ 

**Proof.** Observe that by (H2) $|\psi_{0}(x)| \leq |M_{0} x| + |N_{0}|$ for any $x \in \text{supp}\nu$. Notice also that for any $n \in \mathbb{N}$

$$|\psi_{\theta_{1}} \circ \psi_{\theta_{2}} \circ \ldots \circ \psi_{\theta_{n}}(x)| \leq |M_{\theta_{1}} \psi_{\theta_{2}} \circ \ldots \circ \psi_{\theta_{n}}(x)| + |N_{\theta_{1}}|,$$

since $\psi_{\theta_{2}} \circ \ldots \circ \psi_{\theta_{n}}(x) \in \text{supp}\nu$ for any $x \in \text{supp}\nu$. By Theorem 1.4 we know that $\lim_{n \to \infty} \psi_{\theta_{1}} \circ \psi_{\theta_{2}} \circ \ldots \circ \psi_{\theta_{n}}(x) = S$ a.e. hence by induction we obtain

$$\left( \mathbb{E}|\psi_{\theta_{1}} \circ \psi_{\theta_{2}} \circ \ldots \circ \psi_{\theta_{n}}(x)|^{\beta} \right)^{\frac{1}{\beta}} \leq \left( 1 + (\kappa(\beta))^{\frac{1}{\beta}} + \ldots + (\kappa(\beta))^{\frac{m}{\beta}} \right) \left( \mathbb{E}|N|^{\beta} \right)^{\frac{1}{\beta}} = \frac{1 - (\kappa(\beta))^{\frac{1}{\beta}}}{1 - (\kappa(\beta))^{\frac{1}{\beta}}} \left( \mathbb{E}|N|^{\beta} \right)^{\frac{1}{\beta}} < \infty.$$ 

Now by the Fatou lemma

$$\left( \mathbb{E}|S|^\beta \right)^{\frac{1}{\beta}} \leq \liminf_{n \to \infty} \frac{1 - (\kappa(\beta))^{\frac{1}{\beta}}}{1 - (\kappa(\beta))^{\frac{1}{\beta}}} \left( \mathbb{E}|N|^{\beta} \right)^{\frac{1}{\beta}} = \frac{1}{1 - (\kappa(\beta))^{\frac{1}{\beta}}} \left( \mathbb{E}|N|^{\beta} \right)^{\frac{1}{\beta}} < \infty.$$ 

Repeating the above argument we obtain the following

**Lemma 3.12.** If (H2), (H5), (H7) and (L1) are satisfied, then for every $\beta \in (0, \alpha)$, $x \in \mathbb{R}^{d}$,

$$\sup_{n \in \mathbb{N}} \left( \mathbb{E}|Y_{n,t}^{x}|^{\beta} \right)^{\frac{1}{\beta}} = \sup_{n \in \mathbb{N}} \left( \mathbb{E}|Y_{n,t}^{x}|^{\beta} \right)^{\frac{1}{\beta}} < \infty,$$

where $Y_{n,t}^{x} = \psi_{\theta_{1},t} \circ \psi_{\theta_{2},t} \circ \ldots \circ \psi_{\theta_{n},t}(x)$ for any $n \in \mathbb{N}$ and $t > 0$.
4. The tail measure

This section deals with heavy tail phenomenon for Lipschitz recursions satisfying assumptions 1.9 and 1.10 modeled on analogous hypotheses needed for matrix recursions (1.9). (H1) and (H2) say that recursion (1.1) is in a sense close to an affine recursion with the linear part $M \in \mathbb{R}_+^d \times \mathbb{K}$. This allows us to treat the multidimensional situation using techniques of (4), in particular a generalized renewal theorem. Conditions in assumption 1.11 are typical in considerations of this type and decide of asymptotic behaviour of stationary measure, especially condition (H5) is crucial. Goldie and Grübel (7) show that $\mathbb{P}(|S > t|)$ can decay exponentially fast to zero if (H5) is not satisfied.

Proof of Theorem (1.13). It is a direct consequence of Theorem 4.1 (existence of the tail measure $\Lambda$), Theorem 4.23 (property (1.14) for $f \in \mathcal{F}$), Theorem 4.25 (polar decomposition for the tail measure $\Lambda$) and Theorem 4.30 (nontriviality of the tail measure $\Lambda$).

Define convolution of a function $f$ with measure $\mu$ on group $G$ as

$$f * \mu(g) = \int_G f(gh)\mu(dh).$$

4.1. Existence of the tail measure $\Lambda$. Given $f \in C_b(\mathbb{R}^d)$ let

$$\tilde{f}(g) = \mathbb{E}(f(gS)) \text{ and } \chi_f(g) = \tilde{f}(g) - \tilde{f} * \tilde{\mu}(g).$$

The functions $\tilde{\mu}$ and $\chi_f$ are bounded and continuous. We are going to express function $\tilde{f}$ in the terms the potential $U = \sum_{k=0}^{\infty} \tilde{\mu}^k$. Notice that for any $n \in \mathbb{N} \cup \{0\}$

$$\mathbb{E}(f(gM_{\theta_1} M_{\theta_2} \cdots M_{\theta_n} S)) = \int_G \mathbb{E}(f(ghS))\tilde{\mu}^n(dh) = \tilde{f} * \tilde{\mu}^n(g).$$

Now, for an $\varepsilon \in (0,1]$, we define the set of Hölder functions by

$$\mathcal{H}_\varepsilon = \{f \in C_b(\mathbb{R}^d) : \forall x,y \in \mathbb{R}^d \ |f(x) - f(y)| \leq C_f |x - y|^{\varepsilon} \text{ and } f \text{ vanish in a neighbourhood of 0}\}.$$

Let $\bar{\mu}_\alpha(dg) = |g|^{\alpha}\tilde{\mu}(dg)$. In view of Remark 1.12 $\bar{\mu}_\alpha$ is a probability measure with positive mean and $\bar{\mu}_\alpha^n(dg) = |g|^n\tilde{\mu}^n(dg)$ for all $n \in \mathbb{N}$. Let $U_\alpha = \sum_{k=0}^{\infty} \bar{\mu}_\alpha^k$ be the potential kernel with respect to measure $\bar{\mu}_\alpha$.

The aim of this section is to prove the following

**Theorem 4.1.** Given $f \in \mathcal{H}_\varepsilon$ for some $\varepsilon \in (0,1]$, we write $\chi_{f,\alpha}(g) = |g|^{-\alpha}\chi_f(g)$. Under the assumptions of Remark 1.22 we have

$$\lim_{|g| \to 0} |g|^{-\alpha}\tilde{f}(g) = \lim_{|g| \to 0} U_\alpha(\chi_{f,\alpha})(g) = \frac{1}{m_\alpha} \int_G \chi_{f,\alpha}(g)\lambda(dg).$$

The formula

$$\Lambda(f) = \frac{1}{m_\alpha} \int_G \chi_{f,\alpha}(g)\lambda(dg) = \frac{1}{m_\alpha} \int_G |g|^{-\alpha}(\mathbb{E}(f(gS)) - f(gMS))\lambda(dg),$$

defines a nonnegative Radon measure on $\mathbb{R}^d \setminus \{0\}$, which is $\alpha$ homogeneous i.e.

$$\int_{\mathbb{R}^d} f(gx)\Lambda(dx) = |g|^\alpha \Lambda(f), \quad g \in G.$$
Every \( f \in \mathcal{H}_\varepsilon \) is \( \Lambda \) integrable. Furthermore,
\[
\sup_{t > 0} t^\alpha \nu(\{ x \in \mathbb{R}^d : |x| > t \}) < \infty,
\]
and
\[
\sup_{t > 0} t^\alpha \Lambda(\{ x \in \mathbb{R}^d : |x| > t \}) < \infty.
\]

To prove Theorem 4.1 we will apply a generalized renewal theorem for closed subgroups of \( \mathbb{R}^*_+ \times K \), where \( K \) is a metrizable group not necessarily abelian. Let \( D \) be the closed subgroup of \( \mathbb{R}^*_+ \times K \) and let \( \Delta_n = \{ g \in D : n < \log |g| \leq n + 1 \} \) for \( n \in \mathbb{Z} \).

**Definition 4.7.** A bounded Borel function \( h \) is \( d\text{Ri} \) (direct Riemann integrable) on \( D \) if
- the set of discontinuities of \( h \) is negligible with respect to the Haar measure on \( D \),
- \( \sum_{n \in \mathbb{Z}} \sup_{g \in \Delta_n} |h(g)| < \infty \).

In this context we have the following theorem

**Theorem 4.8.** Assume that \( \mu \) is a probability measure on \( \mathbb{R}^*_+ \times K \) such that
\[
m = \int_{\mathbb{R}^*_+ \times K} \log(\operatorname{pr}_{\mathbb{R}^*_+}(g)) \mu(dg) > 0,
\]
where \( \operatorname{pr}_{\mathbb{R}^*_+} : \mathbb{R}^*_+ \times K \to \mathbb{R}^*_+ \) is a natural projection onto \( \mathbb{R}^*_+ \). Then the potential \( U = \sum_{k=0}^{\infty} \mu^{*k} \) is a Radon measure supported by \( D_\mu \), where \( D_\mu \) is closed subgroup generated by \( \text{supp} \mu \). Furthermore, for any \( d\text{Ri} \) function \( f \) on \( D_\mu \) we have
\[
\lim_{\operatorname{pr}_{\mathbb{R}^*_+}(g) \to 0, g \in D_\mu} U f(g) = \frac{1}{m} \int_{\mathbb{R}^*_+ \times K} f(g) \lambda(dg),
\]
where \( \lambda \) is Haar measure on \( \mathbb{R}^*_+ \times K \).

Proof of above theorem can be found in appendix A of (4), see also (11) and (27).

**Lemma 4.9.** Let assume that \( 0 < \varepsilon < s \leq s_\infty \), \( f \in \mathcal{H}_\varepsilon \) and \( \eta > 0 \) such that \( \text{supp} f \cap B_\eta(0) = \emptyset \), where \( B_\eta(0) \) is ball with center 0 and radius \( \eta \). If \( \kappa(s) < \infty \), \( \mathbb{E}(|N|^s) < \infty \) and assumption 1.10 holds, then the function
\[
\chi_{f,s}(g) = |g|^{-s} \chi_f(g),
\]
is \( d\text{Ri} \) on \( G \) and
\[
\sum_{n \in \mathbb{Z}} \sup_{g \in \Delta_n} |\chi_{f,s}(g)| \leq C C_f \eta^{-s},
\]
where constant \( C \) does not depend on function \( f \) and \( \eta \).

**Proof.** By the cancellation condition (H2) and Hölder continuity we have
\[
|g|^{-s} |f(gS_1) - f(gM_1S_2)| \leq C_f |g|^{-s} |N_1|^s,
\]
Since function \( f \) vanish on some neighborhood of 0, we can define a family of random sets \( P_n \) for \( n \in \mathbb{Z} \) such that \( f(gS_1) = 0 \) and \( f(gM_1S_2) = 0 \) on \( P_n^c \). Let
\[
P_n = \{ e^{n+1}(|N_1| + |M_1S_2|) > \eta \} = \{ \omega \in \Omega : n \geq n_0(\omega) \},
\]
where \( n_0 = \log \eta - \log(|N_1| + |M_1S_2|) - 1. \)

Then for \( |g| \in (e^n, e^{n+1}] \), by (4.12) we obtain
\[
|\chi_{f,s}(g)| \leq \mathbb{E} \left( (|f(gS_1)| + |f(gM_1S_2)|)1_{P_n} \right) + C_f \mathbb{E}(e^{-n(s-\varepsilon)}|N_1|^\varepsilon 1_{P_n})
\]
\[
= C_f \mathbb{E}(e^{-n(s-\varepsilon)}|N_1|^\varepsilon 1_{P_n}).
\]
Therefore,
\[
(4.14) \quad \sum_{n \in \mathbb{Z}} \sup_{g \in \Delta_n} |\chi_{f,s}(g)| \leq C_f \mathbb{E} \left( |N_1|^\varepsilon \sum_{n \geq n_0} e^{-n(s-\varepsilon)}1_{P_n} \right).
\]

By (4.13) we estimate
\[
(4.15) \quad \sum_{n \geq n_0} e^{-n(s-\varepsilon)} = e^{-n_0(s-\varepsilon)} \sum_{n \geq 0} e^{-n(s-\varepsilon)} = \frac{e^{-n_0(s-\varepsilon)}}{1 - e^{-s(\varepsilon)}} = \frac{e^{s-\varepsilon} \eta^{s-\varepsilon}}{1 - e^{-s\varepsilon}} \cdot (|N_1| + |M_1S_2|)^{s-\varepsilon}.
\]

In view of Hölder inequality and independence \( M_1 \) of \( S_2 \) we have
\[
(4.16) \quad \mathbb{E} \left( |N_1|^\varepsilon \cdot (|N_1| + |M_1S_2|)^{s-\varepsilon} \right) \leq D < \infty.
\]

Finally combining (4.14), (4.15) and (4.16) we have
\[
\sum_{n \in \mathbb{Z}} \sup_{g \in \Delta_n} |\chi_{f,s}(g)| \leq C_f \mathbb{E} \left( |N_1|^\varepsilon \sum_{n \geq n_0} e^{-n(s-\varepsilon)}1_{P_n} \right)
\]
\[
\leq C_f \frac{e^{s-\varepsilon}}{1 - e^{-s-\varepsilon}} \eta^{s-\varepsilon} \mathbb{E} \left( |N_1|^\varepsilon \cdot (|N_1| + |M_1S_2|)^{s-\varepsilon} \right)
\]
\[
\leq C_f \frac{e^{s-\varepsilon}}{1 - e^{-s-\varepsilon}} D \eta^{s-\varepsilon} = C_f C \eta^{s-\varepsilon},
\]
where \( C = \frac{e^{s-\varepsilon}}{1 - e^{-s-\varepsilon}} D. \)

\( \square \)

**Lemma 4.17.** Given \( f \in \mathcal{H}_\varepsilon \) for some \( \varepsilon \in (0, 1) \), under the assumptions of Remark 1.22 the function \( \chi_{f,\alpha} \) is \( U_\alpha \) integrable and for every \( g \in G \)
\[
(4.18) \quad \tilde{f}(g) = \sum_{n=1}^{\infty} \chi_f \ast \bar{\mu}^{*n}(g) = U(\chi_f)(g) \quad \text{and}
\]
\[
(4.19) \quad |g|^{-\alpha} \tilde{f}(g) = \sum_{k=0}^{\infty} \chi_{f,\alpha} \ast \bar{\mu}_\alpha^{*k}(g) = U_\alpha(\chi_{f,\alpha})(g).
\]

\( \square \)

**Proof.** For the proof we refer to the (4).

**Proof of Theorem (4.1).** Formula (4.19) and the Renewal Theorem 4.8 applied to the potential associated with the measure \( \bar{\mu}_\alpha \) give
\[
\lim_{|g| \to 0} |g|^{-\alpha} \tilde{f}(g) = \lim_{|g| \to 0} U_\alpha(\chi_{f,\alpha})(g) = \frac{1}{m_\alpha} \int_G \chi_{f,\alpha}(g) \lambda(dg),
\]
and so (4.2) holds. Theorem 4.8 ensures also that the linear functional
\[
f \to \frac{1}{m_\alpha} \int_G \chi_{f,\alpha}(g) \lambda(dg),
\]
defines a nonnegative Radon measure $\Lambda$ on $\mathbb{R}^d \setminus \{0\}$, given by the explicit formula

$$\Lambda = \frac{1}{m_\alpha} \left( (|\cdot|^{-\alpha})^\ast (\nu \ast \bar{\mu}) \right).$$

If $f \in \mathcal{H}_\varepsilon$, then $|f| \in \mathcal{H}_\varepsilon$ and by Lemma 4.9 function $\chi_{|f|,\alpha}$ is $d\mathcal{R}i$ hence it is $\lambda$ integrable. In order to show that $\Lambda$ is $\alpha$ homogeneous we define the measure $\Lambda^s$ on $\mathbb{R}^d \setminus \{0\}$ by

$$\Lambda^s(f) = \frac{1}{m_\alpha} \int_G |g|^{-s} (\mathbb{E}(f(gS) - f(gMS))) \lambda dg.$$

We will show that the measures $\Lambda^s$ converge weakly to measure $\Lambda$ when $s \not< \alpha$. Since $f$ vanishes in a neighborhood of 0 and $\rho(K) = 1$, then for $0 < \varepsilon < s < \alpha$

$$\int_G |g|^{-s} \mathbb{E}\left( \left| (gS)^\ast 1_{B_\alpha(0)}(gS) \right| \lambda (dg) \right) \leq \mathbb{E} \left( |S|^\varepsilon \int_{\mathbb{R}^d} x^{\varepsilon-s-1} dx \right) = \frac{\eta^{\varepsilon-s}}{s-\varepsilon} \mathbb{E}(\|S\|) < \infty,$$

hence (4.21) implies that $\int_G |g|^{-s} \mathbb{E}(f(gS) - f(gMS)) \lambda (dg)$ is finite for every $s < \alpha$, and converge to $\Lambda(f)$. Notice that the measures $\Lambda^s$ are also $s$ homogeneous. Indeed for every $f \in \mathcal{H}_\varepsilon$

$$\int_{\mathbb{R}^d} f(hx) \Lambda^s(dx) = \frac{1}{m_\alpha} \int_G |g|^{-s} \mathbb{E}(f(hgS) - f(hgMS)) \lambda (dg) = |h|^s \int_{\mathbb{R}^d} f(x) \Lambda^s(dx),$$

so (4.22) implies (4.4). In order to show (4.5) and (4.6) take the function $h \in \mathcal{H}_\varepsilon$ such that $h(x) \geq 1_{B_\alpha(0)}(x)$ for any $x \in \mathbb{R}^d$. Then

$$\lim_{|g| \to 0} |g|^{-\alpha} \mathbb{P}\left( \|S\| > |g|^{-1} \right) \leq \lim_{|g| \to 0} |g|^{-\alpha} \mathbb{E} h(gS) = \Lambda(h) < \infty.$$ 

In a similar way we obtain

$$|g|^{-\alpha} \Lambda\left( \{x \in \mathbb{R}^d : |x| > |g|^{-1} \} \right) \leq |g|^{-\alpha} \int_{\mathbb{R}^d} h(gx) \Lambda(dx) = \Lambda(h) < \infty,$$

since $\Lambda$ is $\alpha$ homogeneous.

\begin{proof}
For the proof we refer to the (4).
\end{proof}

\section{Polar decomposition for the measure $\Lambda$}

Being homogeneous $\Lambda$ can be nicely expressed in polar coordinates. More precisely, we have the following

\begin{theorem}
Under the assumptions of Remark 1.22 measure $\Lambda$ can be expressed in the following form

$$\int_{\mathbb{R}^d \setminus \{0\}} f(x) \Lambda(dx) = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(rx) \sigma_\Lambda(dx) \frac{dr}{r^{\alpha+1}},$$
\end{theorem}
where \( \sigma_\Lambda \) is a Radon measure on \( S^{d-1} \), and

\[
\sigma_\Lambda(S^{d-1}) = \frac{1}{m_\alpha} \mathbb{E} (|\psi(S)|^\alpha - |MS|^\alpha) .
\]

Furthermore,

\[
\lim_{t \to \infty} t^\alpha \mathbb{P}(|S| > t) = \frac{1}{\alpha} \sigma_\Lambda(S^{d-1}) = \frac{1}{\alpha m_\alpha} \mathbb{E} (|\psi(S)|^\alpha - |MS|^\alpha) .
\]

Proof. Let \( \Phi : \mathbb{R}^d \setminus \{0\} \to (0, \infty) \times S^{d-1} \) be defined as follows \( \Phi(x) = \left( |x|, \frac{x}{|x|} \right) \) and its inverse \( \Phi^{-1} : (0, \infty) \times S^{d-1} \to \mathbb{R}^d \setminus \{0\} \) by \( \Phi^{-1}(r, z) = rz \). Notice that

\[
\int_{\mathbb{R}^d \setminus \{0\}} f(x) \Lambda^s(dx) = \int_{(0, \infty) \times S^{d-1}} f(\Phi^{-1}(r, z)) (\Lambda^s \circ \Phi^{-1}) (dr, dz),
\]

For \( s < \alpha \) we define the measures \( \sigma^s \) on \( S^{d-1} \)

\[
\sigma^s(F) = s \Lambda^s (\Phi^{-1} [[1, \infty) \times F]) ,
\]

where \( F \in \mathcal{B}(S^{d-1}) \). Now we express the measure \( \Lambda^s \circ \Phi^{-1} \) in the terms of polar coordinates i.e.

\[
\int_{\mathbb{R}^d \setminus \{0\}} f(x) \Lambda^s(dx) = \int_0^\infty \int_{S^{d-1}} f(rx) \sigma^s(dr, \sigma^s(dx) , \frac{dr}{r^{s+1}}),
\]

Fix \( 0 < \beta < \gamma \) and notice that for any \( \alpha \times \beta \times F \in \mathcal{B}((0, \infty)) \otimes \mathcal{B}(S^{d-1}) \),

\[
(\Lambda^s \circ \Phi^{-1} )([[\beta, \gamma) \times F]) = \Lambda^s (\Phi^{-1}[[\beta, \infty) \times F]) - \Lambda^s (\Phi^{-1}[[1, \infty) \times F])
\]

\[
= \Lambda^s (\beta \Phi^{-1}[[1, \infty) \times F]) - \Lambda^s (\gamma \Phi^{-1}[[1, \infty) \times F])
\]

\[
= \frac{1}{s^{3s}} \sigma^s (\Phi^{-1}[[1, \infty) \times F]) - \frac{1}{s^{3s}} \sigma^s (\Phi^{-1}[[1, \infty) \times F])
\]

\[
= \sigma^s(F) \left( \frac{1}{s^{3s}} - \frac{1}{s^{3s}} \right) = \sigma^s(F) \int_\beta^\gamma \frac{dr}{r^{s+1}} .
\]

The above calculation proves (4.29). Now we compute \( \sigma^s(S^{d-1}) \)

\[
\sigma^s(S^{d-1}) = s \Lambda^s (\Phi^{-1} [[1, \infty) \times S^{d-1}]) = s \int_{\mathbb{R}^d \setminus \{0\}} \mathbf{1}_{\Phi^{-1}[[1, \infty) \times S^{d-1}]}(x) \Lambda^s(dx)
\]

\[
= \frac{s}{m_\alpha} \int_{\mathbb{R}^d \setminus \{0\}} \int_G \mathbf{1}_{\Phi^{-1}[[1, \infty) \times S^{d-1}]} \left( \frac{gx}{|g|^2} \right) |x|^s |g|^{-s} \lambda(dg)(\nu - \bar{\mu} \ast \nu)(dx)
\]

\[
= \frac{s}{m_\alpha} \int_{\mathbb{R}^d \setminus \{0\}} \int_G \mathbf{1}_{\Phi^{-1}[[1, \infty) \times \mathbb{R}^{d-1}} \left( |g|, \frac{gx}{|g|^2} \right) |x|^s |g|^{-s} \lambda(dg)(\nu - \bar{\mu} \ast \nu)(dx)
\]

\[
= \frac{1}{m_\alpha} \int_{\mathbb{R}^d \setminus \{0\}} |x|^s (\nu - \bar{\mu} \ast \nu)(dx) = \frac{1}{m_\alpha} \mathbb{E} (|\psi(S)|^s - |MS|^s) .
\]

Hence (4.26) and (4.27) hold. Furthermore,

\[
\lim_{t \to \infty} t^\alpha \mathbb{P}(|S| > t) = \lim_{t \to \infty} \int_t^\infty \frac{dr}{r^{s+1}} \sigma_\Lambda(S^{d-1}) = \frac{1}{\alpha m_\alpha} \mathbb{E} (|\psi(S)|^\alpha - |MS|^\alpha) ,
\]

(4.28) holds and the proof is finished. \( \square \)
4.3. Nontriviality of the tail measure. If $\supp \nu$ is bounded, then $\mathbb{P}(|S| > t) = 0$ for $t$ large enough and so $\Lambda$ is trivial. If $\nu$ has an unbounded support, it is natural to ask whether $\Lambda$ is not zero. It is so under some extra conditions.

**Theorem 4.30.** Assume that $\alpha < s_\infty$ and the hypothesis stated in Remark 1.22 are satisfied. Additionally assume that $\mathbb{E}(|N|^s) < \infty$ for every $s < s_\infty$. If support of $\nu$ is unbounded and one of the following condition is satisfied

\[
(4.31) \quad s_\infty < \infty \quad \text{and} \quad \lim_{s \to s_\infty} \frac{\mathbb{E}(|N|^s)}{\kappa(s)} = 0,
\]

\[
(4.32) \quad s_\infty = \infty \quad \text{and} \quad \lim_{s \to \infty} \left(\frac{\mathbb{E}(|N|^s)}{\kappa(s)}\right) = C < \infty,
\]

then the measure $\Lambda$ is nontrivial.

The proof goes along the same lines as in (4) Proposition 3.12, but it is not so easy to extract it from section 3 there containing a more general argument. Therefore, and to show how our assumptions 1.10 and 1.11 do work, we include here the proof of Theorem 4.30. Conditions (4.31) and (4.32) are very restrictive. For many concrete stochastic recursion these conditions can be relaxed for details we refer (2; 3; 6; 12).

In order to prove that measure $\Lambda$ is nontrivial in view of Theorem 4.25 we will show that

\[ \sigma_\Lambda(S^{d-1}) \neq 0 \]

Before proving the theorem we need some lemmas. In the proofs the following inequalities will be used

\[
(4.33) \quad |x - y|^r \leq C_r (|x|^r + |y|^r) \quad \text{where} \quad r > 0 \quad \text{and} \quad x, y \in \mathbb{R}^d,
\]

and

\[
(4.34) \quad ||x|^r - |y|^r| \leq \begin{cases} |x - y|^r, & \text{if} \ 0 < r \leq 1, \\ r|x - y| (\max(|x|, |y|))^r - 1, & \text{if} \ r > 1,
\end{cases}
\]

where $x, y \in \mathbb{R}^d$. Moreover, by (1.6) and (1.7), for every $s < \alpha$,

\[
(4.35) \quad \mathbb{E}(|S|^s) = \frac{\mathbb{E}(\psi_1(S_2)^s - |M_1S_2|^s)}{1 - \kappa(s)},
\]

**Remark 4.36.** Notice that, in view of (H2), for $s \leq s_\infty \leq 1$,

\[
(4.37) \quad ||\psi_1(S_2)|^s - |M_1S_2|^s| \leq |\psi_1(S_2) - M_1S_2|^s \leq |N_1|^s,
\]

and for $s > 1$,

\[
(4.38) \quad ||\psi_1(S_2)|^s - |M_1S_2|^s| \leq s|\psi_1(S_2) - M_1S_2| \max(|\psi_1(S_2)|^{s-1}, |M_1S_2|^{s-1})
\]

\[
\leq s|N_1| \max \left(|\psi_1(S_2)|^{s-1}, (|\psi_1(S_2) - M_1S_2|^s), (|\psi_1(S_2)|^{s-1}) \right)
\]

\[
\leq s|N_1| (|N_1| + |S_1|)^{s-1},
\]

For reader’s convenience we formulate the following theorem due to Landau that will be used in the proof of the next lemma.

**Theorem 4.39.** Let $\gamma$ be a positive measure on $\mathbb{R}^*_+$ and let $\hat{\gamma}(s) = \int_{\mathbb{R}^*_+} x^s \gamma(dx)$ be its Mellin transform which is well defined for $0 < s < \theta_\infty$. $\theta_\infty$ is called the abscissa of convergence of $\hat{\gamma}$. Then $\hat{\gamma}$ extends holomorphically to $\mathcal{R}(\theta_\infty) = \{z \in \mathbb{C} : \Re z < \theta_\infty\}$ and cannot be extended holomorphically to a neighborhood of $\theta_\infty$. 
Let $\mathcal{R}(s) = \{z \in \mathbb{C} : \Re z < s\}$ for $s < s_\infty$.

**Lemma 4.40.** If $\sigma_N(S^{d-1}) = 0$, then $\mathbb{E}(|S|^s) < \infty$ for $s < s_\infty$, where $S$ is the stationary solution of recursion (1.1).

**Proof.** We will show that function $s \mapsto \mathbb{E}(|\psi_1(S_2)|^s - |M_1S_2|^s)$ has a holomorphic extension to the set $\mathcal{R}(\alpha + \varepsilon)$ where $\alpha + \varepsilon < s_\infty$ and $\varepsilon > 0$. It suffices to show that the function $s \mapsto \mathbb{E}(|\psi_1(S_2)|^s - |M_1S_2|^s)$ is well defined for $s < \alpha + \varepsilon$, where $\varepsilon > 0$ we will choose later. If $s \leq s_\infty \leq 1$, then by (4.37)

$$\mathbb{E}||\psi_1(S_2)|^s - |M_1S_2|^s| \leq \mathbb{E}|N_1|^s < \infty.$$  

If $s > 1$, then by (4.38)

$$\mathbb{E}||\psi_1(S_2)|^s - |M_1S_2|^s| \leq s2^{s-1}\mathbb{E}(|N_1|^s + |N_1||S_1|^{s-1}) \leq s^{2s-1}\mathbb{E}(|N_1|^s) \mathbb{E}(|S_1|^{s-1}) < \infty.$$

By assumption, $\mathbb{E}|N_1|^s < \infty$ for $s < s_\infty$ and $\mathbb{E}|S_1|^s < \infty$ for $s < \alpha$, hence $\mathbb{E}|S_1|^{s-1} < \infty$ for $s < \alpha + \varepsilon$ where $0 < \varepsilon = \frac{s_\infty - s}{s_\infty} < 1$. Therefore, we can extend the function $s \mapsto \mathbb{E}(|\psi_1(S_2)|^s - |M_1S_2|^s)$ holomorphically to the set $\mathcal{R}(\alpha + \varepsilon)$.

Now we will show that also the function $s \mapsto \mathbb{E}(|S|^s)$ has a holomorphic extension to the set $\mathcal{R}(\alpha + \eta)$ for some $\eta > 0$. Indeed, let $\Lambda(z) = \mathbb{E}(|\psi_1(S_2)|^z - |M_1S_2|^z)$. By above $\Lambda(z)$ is holomorphic for $z \in \mathcal{R}(\alpha + \varepsilon)$. Since $\kappa(z) - 1$ has simple zero at $z = \alpha$, $\Lambda(\alpha) = 0$ and $\Lambda(s) = (1 - \kappa(s))\mathbb{E}(|S|^s)$ for any $s < \alpha$, hence function $h(z) = \frac{\Lambda(z)}{1 - \kappa(z)}$ defines a holomorphic extension of $\mathbb{E}(|S|^s)$ on some ball $B_\eta(\alpha)$ with center $\alpha$ and radius $\eta > 0$. Since $\gamma(s) = \int |x|^s\nu(dx)$ is Mellin transform of some positive measure, then Landau theorem 4.39 ensures us that $\gamma(s)$ does not extend beyond to abscissia of convergence. But $\mathbb{E}(|S|^s)$ extends holomorphically to $B_\eta(\alpha)$, so an abscissia of convergence has to be greater than $\alpha + \eta$.

Now we are ready show that $\mathbb{E}(|S|^s) < \infty$ for $s < s_\infty$. Let $s_0 = \sup\{s < s_\infty : \mathbb{E}(|S|^s) < \infty\}$. Suppose for a contradiction that $s_0 < s_\infty$. If $s_1 = s_0 \leq 1$ notice, that

$$\mathbb{E}||\psi_1(S_2)|^{s_1} - |M_1S_2|^{s_1}| \leq \mathbb{E}|N_1|^{s_1} < \infty.$$  

If $s_0 > 1$, we take $s_1 < s_\infty$ such that $0 < s_1 - 1 < s_0 < s_1$, so $\mathbb{E}(|S|^{s_1-1}) < \infty$. By (4.38) we have

$$\mathbb{E}||\psi_1(S_2)|^{s_1} - |M_1S_2|^{s_1}| < \infty.$$  

It means that in both cases $\Lambda(s_1)$ is well defined, hence it has a holomorphic extension. Now using Landau theorem we argue in a similar way as above. Finally we obtain that $\mathbb{E}(|S|^{s_1}) < \infty$ which contradicts with the definition of $s_0$ and the lemma follows.

**Lemma 4.41.** There exist $\xi > 0$ such that $\kappa(s) \geq C_\xi(1 + \xi)^s$ for any $s < s_\infty$. If $s_\infty = \infty$, then for sufficiently large $s > 0$

$$\frac{2s(1 + \xi)^s \xi^{-1} \xi}{\kappa(s) - 1} \leq \frac{1}{2}.$$  

**Proof.** In view of (H5) $\mathbb{P}(|M_1| \in (0,1]) < 1$. Hence there is $\xi_0 > 0$ such that $C_\xi = \mathbb{P}(|M_1| \in [1 + \xi_0, \infty)) < 0$ for any $\xi \leq \xi_0$ and so

$$\kappa(s) = \mathbb{E}(|M|^s) \geq \int_{\{M| \in [1 + \xi, \infty]\}} |M|^s \mathbb{P}(d\omega) \geq C_\xi (1 + \xi)^s.$$
Taking $\xi \leq \xi_0$ we get the Lemma. 

\[ \square \]

**Proof of Theorem (4.30).** In order to prove the theorem suppose for a contradiction that $\sigma_{\lambda}(S^{d-1}) = 0$. We are going to show that $|S|_\infty < \infty$. With out loss of generality we can assume that $|N_1|$ is not identically equal 0. Hence $\limsup_{s \to s_\infty} E(|N_1|^s) > 0$. At first assume $0 < s < s_\infty \leq 1$, then by Lemma 4.40

\[ \mathbb{E}(|S_1|^s) = \frac{\mathbb{E}(|\psi_1(S_2)|^s - |M_1S_2|^s)}{1 - \kappa(s)}, \]

for $s < s_\infty$. By above and (4.31),

\[ \mathbb{E}(|S_1|^s) \leq \frac{E(|N_1|^s)}{\kappa(s)-1} \text{ for } s > \alpha, \]

Since $\limsup_{s \to s_\infty} E(|N_1|^s) > 0$, condition (4.31) ensures us that $\lim_{s \to s_\infty} \kappa(s) = 0$ and it implies that $\lim_{s \to s_\infty} E(|S|^s) = 0$. For $s > 1$ by (4.38) we obtain

\[ E\left(|\psi_1(S_2)|^s - |M_1S_2|^s\right) \leq s\mathbb{E}\left(|N_1|(|N_1| + |S_1|)^{s-1}\mathbb{1}_{\{|N_1| \leq \xi |S_1|\}}\right) \]

\[ \leq s \xi \left(\mathbb{E}|S_1|(|S_1| + 1)^{s-1}\right) \]

\[ \leq s \xi (1 + \frac{1}{\xi})^{s-1} \mathbb{E}(|N_1|^s), \]

hence, combining two above inequalities we obtain

\[ \mathbb{E}(|S_1|^s) \leq \mathbb{E}\left(|\psi_1(S_2)|^s - |M_1S_2|^s\right) \mathbb{1}_{\{|N_1| \leq \xi |S_1|\}} \]

\[ + \mathbb{E}\left(|\psi_1(S_2)|^s - |M_1S_2|^s\right) \mathbb{1}_{\{|S_1| \leq \xi |N_1|\}} \]

\[ \leq \frac{s \xi (1 + \frac{1}{\xi})^{s-1}}{\kappa(s)-1} \mathbb{E}(|S_1|^s) + \frac{s (1 + \frac{1}{\xi})^{s-1}}{\kappa(s)-1} \mathbb{E}(|N_1|^s). \]

Now we consider two cases

- $1 < s_\infty < \infty$, by (4.31) $\lim_{s \to s_\infty} \kappa(s) = \infty$, then for sufficiently large $s > \alpha$

\[ \frac{2s(1 + \frac{1}{\xi})^{s-1}}{\kappa(s)-1} \leq \frac{1}{2}, \]

Hence,

\[ \mathbb{E}(|S_1|^s) \leq \frac{2s(1 + \frac{1}{\xi})^{s-1}}{\kappa(s)-1} \mathbb{E}(|N_1|^s), \]

so $\lim_{s \to s_\infty} \mathbb{E}|S_1|^s = 0$.

- $s_\infty = \infty$, then by condition (4.32) and Lemma 4.41 for sufficiently large $s > \alpha$
\[ \frac{2s(1 + \xi)^{s-1} \xi}{\kappa(s) - 1} \leq \frac{1}{2}, \]

hence,

\[ \mathbb{E}(|S_1|^s) \leq \frac{2s \left(1 + \frac{1}{\xi}\right)^{s-1}}{\kappa(s) - 1} \mathbb{E}(|N_1|^s) \leq \frac{2s \left(1 + \frac{1}{\xi}\right)^{s-1}}{\kappa(s) - 1} C^s \kappa(s), \]

then

\[ \mathbb{E}(|S_1|^s)^{\frac{1}{s}} \leq C(2s)^{\frac{1}{s}} \left(1 + \frac{1}{\xi}\right)^{\frac{1}{s} - 1} \left(\frac{\kappa(s)}{\kappa(s) - 1}\right)^{\frac{1}{s}}, \]

and finally

\[ |S_1|_\infty = \lim_{s \to \infty} \mathbb{E}(|S_1|^s)^{\frac{1}{s}} \leq C \left(1 + \frac{1}{\xi}\right) < \infty. \]

$|S|_\infty < \infty$ means that $S$ is bounded which is equivalent with the fact that support of measure $\nu$ is bounded. This contradicts our hypothesis, hence it proves that $\mathbb{E}||\psi_1(S_2)||^a - |M_1S_2|^a| \neq 0$. \( \square \)

5. Fourier operators and their properties

This section is devoted to study operators $P$ and their perturbations $P_{t,v}$. Properties (L1)–(L3) allows us to proceed along the same lines as in (3) with one major difference—operators $T_{t,v}$. Auxiliary operators $T_{t,v}$ are used in (3) to obtain an explicit expression for the peripherical eigenfunctions corresponding to the eigenvalues $k_\epsilon(t)$ and they are written there by the formula that does not work beyond the affine recursion. Let $\delta_t$ be the dilatation acting on functions as follows $(f \circ \delta_t)(x) = f(tx)$. Here we prove that

\[ T_{t,v}f = P_{t,v}(f \circ \delta_t) \circ \delta_{t^{-1}}, \tag{5.1} \]

do the same job making the method applicable to a much more general context (see Lemma 5.30).

We start by introducing two Banach spaces $C_\rho(\mathbb{R}^d)$ and $B_{\rho,\epsilon,\lambda}(\mathbb{R}^d)$ of continuous functions (23) (see also (16) and (17)).

\[ C_\rho = C_\rho(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) : |f|_\rho = \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{(1 + |x|^\rho)} < \infty \right\}, \]

\[ B_{\rho,\epsilon,\lambda} = B_{\rho,\epsilon,\lambda}(\mathbb{R}^d) = \left\{ f \in C(\mathbb{R}^d) : \|f\|_{\rho,\epsilon,\lambda} = |f|_\rho + |f|_{\epsilon,\lambda} < \infty \right\}, \]

where

\[ |f|_{\epsilon,\lambda} = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\epsilon}(1 + |x|^\lambda)(1 + |y|^\lambda)}. \]

Remark 5.2. If $\epsilon + \lambda < \rho$, then $|f|_{\epsilon,\lambda} < \infty$ implies $|f|_\rho < \infty$. As a simple application of Arzelà–Ascoli theorem we obtain that the injection operator $B_{\rho,\epsilon,\lambda} \hookrightarrow C_\rho$ is compact.

From now on we assume that $\psi_\theta$ satisfies 1.10, 1.11 and 1.23 for every $\theta \in \Theta$. On $C_\rho$ and $B_{\rho,\epsilon,\lambda}$ we consider the transition operator

\[ Pf(x) = \mathbb{E}(f(\psi(x))) = \int_{\Theta} f(\psi_\theta(x)) \mu(d\theta), \]
and its perturbations

\[ P_{t,v} f(x) = \mathbb{E} \left( e^{i(t,v,\psi(x))} f(\psi(x)) \right) = \int_{\Theta} e^{i(t,v,\psi_\theta(x))} f(\psi_\theta(x)) \mu(d\theta) , \]

where \( x \in \mathbb{R}^d \), \( v \in \mathbb{S}^{d-1} \) and \( t \in [0,1] \). We will also use the following Fourier operators

\[ T_{t,v} f(x) = \mathbb{E} \left( e^{i(v,\psi_\theta(x))} f(\psi_\theta(x)) \right) = \int_{\Theta} e^{i(v,\psi_\theta(\cdot))} f(\psi_\theta(x)) \mu(d\theta) , \]

and

\[ \mathcal{F}_{0,t,v} f(x) = \mathbb{E} \left( e^{i(v,\overline{\psi}(x))} f(\overline{\psi}(x)) \right) = \int_{\Theta} e^{i(v,\overline{\psi}_\theta(x))} f(\overline{\psi}_\theta(x)) \mu(d\theta) , \]

for \( x \in \mathbb{R}^d \), \( t \in [0,1] \) and \( v \in \mathbb{S}^{d-1} \). The above operators will allow us to study the expansion of \( k_v(t) \) at 0. Later on we will show connection between operators \( P_{t,v} \) and \( T_{t,v} \). In particular we are going to show that an appropriate dilation of the projections of the eigenfunction \( h_v \) of \( T_v \) with the eigenvalue 1 approximates well peripheral eigenvectors of \( P_{t,v} \). Later on we will show connection between operators \( P_{t,v} \) and \( T_{t,v} \). To treat both \( P_{t,v} \) and \( T_{t,v} \) in a unified way we write

\[ \mathcal{F}_{s,t,v} f(x) = \mathbb{E} \left( e^{i(s,v,\psi_\theta(x))} f(\psi_\theta(x)) \right) = \int_{\Theta} e^{i(s,v,\psi_\theta(\cdot))} f(\psi_\theta(x)) \mu(d\theta) . \]

Notice that

\[ \mathcal{F}_{s,0,v} f(x) = \mathbb{E} \left( e^{i(s,v,\overline{\psi}(x))} f(\overline{\psi}(x)) \right) = \int_{\Theta} e^{i(s,v,\overline{\psi}_\theta(x))} f(\overline{\psi}_\theta(x)) \mu(d\theta) , \]

and

\[ \mathcal{F}_{0,0,v} f(x) = \mathbb{E} \left( f(\psi_\theta(x)) \right) = \int_{\Theta} f(\psi_\theta(x)) \mu(d\theta) , \]

for \( x \in \mathbb{R}^d \), \( s, t \in [0,1] \) and \( v \in \mathbb{S}^{d-1} \). Observe that, \( \mathcal{F}_{s,1,v} = P_{s,v} \) and \( \mathcal{F}_{1,1,v} = T_{1,v} \).

Now we show the connection between operators \( P_{t,v} \) and \( T_{t,v} \) in the lemma below.

**Lemma 5.3.** If \( f \in C_\rho \), then for every \( n \in \mathbb{N}, x \in \mathbb{R}^d \) and \( t \in [0,1] \)

\[ P^n_{t,v}(f \circ \delta_t)(x) = T^n_{t,v} f(tx) . \]

Moreover if \( f \in C_\rho \) is eigenfunction of operator \( T_{t,v} \) with eigenvalue \( k_v(t) \), then \( f \circ \delta_t \) is eigenfunction of operator \( P_{t,v} \) with the same eigenvalue.

**Proof.** For \( n = 1 \) formula (5.4) is obvious. Then we proceed by induction.

\[
\begin{align*}
P^{n+1}_{t,v}(f \circ \delta_t)(x) &= \int_{\Theta} e^{i(t,v,\psi_\theta(x))} P^n_{t,v}(f \circ \delta_t)(\psi_\theta(x)) \mu(d\theta) \\
&= \int_{\Theta} e^{i(v,t\psi_\theta(t^{-1}tx))} T^n_{t,v} f(t\psi_\theta(t^{-1}tx)) \mu(d\theta) \\
&= \int_{\Theta} e^{i(v,\psi_{\theta,t}(tx))} T^n_{t,v} f(\psi_{\theta,t}(tx)) \mu(d\theta) = T^{n+1}_{t,v} f(tx). 
\end{align*}
\]

If \( T_{t,v} f(x) = k_v(t) f(x) \), then

\[ P_{t,v}(f \circ \delta_t)(x) = T_{t,v} f(tx) = k_v(t) f(tx) = k_v(t)(f \circ \delta_t)(x). \]
Proposition 5.5. Assume that \(0 < \epsilon < 1, \lambda > 0, \lambda + 2\epsilon < \rho = 2\lambda \) and \(2\lambda + \epsilon < \alpha\), then there exist \(0 < \varrho < 1, \delta > 0 \) and \(t_0 > 0\) such that \(\varrho < 1 - \delta\) and for every \(t \in [0, t_0]\) and every \(v \in S^{d-1}\)

- \(\sigma(P_{t,v})\) and \(\sigma(T_{t,v})\) are contained in \(D = \{z \in \mathbb{C} : |z| \leq \varrho\} \cup \{z \in \mathbb{C} : |z| - 1| \leq \delta\}\).
- The sets \(\sigma(P_{t,v}) \cap \{z \in \mathbb{C} : |z| - 1| \leq \delta\}\) and \(\sigma(T_{t,v}) \cap \{z \in \mathbb{C} : |z| - 1| \leq \delta\}\) consist of exactly one eigenvalue \(k_v(t)\), the corresponding eigenspace is one dimensional and \(\lim_{t \to 0} k_v(t) = 1\).
- For any \(z \in D^c\) and every \(f \in B_{\rho,\epsilon,\lambda}\)
  \[
  \| (z - P_{t,v})^{-1}f \|_{\rho,\epsilon,\lambda} \leq D \| f \|_{\rho,\epsilon,\lambda},
  \]
  where \(D > 0\) is universal constant which does not depend on \(t \in [0, t_0]\).
- Moreover, we can express operators \(P_{t,v}\) and \(T_{t,v}\) in the following form
  \[
  P_{t,v}^n = k_v(t)^n \Pi_{P,t} + Q_{P,t}^n,
  \]
  \[
  T_{t,v}^n = k_v(t)^n \Pi_{T,t} + Q_{T,t}^n,
  \]
  for every \(n \in \mathbb{N}\). Where \(\Pi_{P,t}\) and \(\Pi_{T,t}\) are projections onto mentioned above one dimensional eigenspaces. \(Q_{P,t}\) and \(Q_{T,t}\) are complemented operators to projections \(\Pi_{P,t}\) and \(\Pi_{T,t}\) respectively, such that \(\Pi_{P,t}Q_{P,t} = Q_{P,t}\Pi_{P,t} = 0\) and \(\Pi_{T,t}Q_{T,t} = Q_{T,t}\Pi_{T,t} = 0\), furthermore \(\|Q_{P,t}\|_{B_{\rho,\epsilon,\lambda}} \leq \varrho\) and \(\|Q_{T,t}\|_{B_{\rho,\epsilon,\lambda}} \leq \varrho\).
- The above operators can be expressed in the terms of the resolvents of \(P_{t,v}\) and \(T_{t,v}\). Indeed, for appropriately chosen parameters \(\xi_1 > 0 \) and \(\xi_2 > 0\)
  \[
  k(t)\Pi_{F,t} = \frac{1}{2\pi i} \int_{|z-1|=\xi_1} z(z - F_{t,v})^{-1}dz,
  \]
  \[
  \Pi_{F,t} = \frac{1}{2\pi i} \int_{|z-1|=\xi_1} (z - F_{t,v})^{-1}dz,
  \]
  \[
  Q_{F,t} = \frac{1}{2\pi i} \int_{|z|=\xi_2} z(z - F_{t,v})^{-1}dz,
  \]
  where \(F = P\) or \(F = T\).

Proposition 5.5 is a consequence of the perturbation theorem of Keller and Liv-erani (20). Before we apply their theorem we will check in a number of Lemmas that its assumptions are satisfied.

Lemma 5.6. For every \(n \in \mathbb{N}\)

\[
(5.7) \quad F_{n,t,v}^n f(x) = \mathbb{E}\left( e^{i\langle sv, S_{n,t}^x \rangle} f\left( X_{n,t}^x \right) \right).
\]

Proof. For \(n = 1\) (5.7) coincides with the definition of \(F_{n,t,v}^n\). Assume that the above holds for some \(n \in \mathbb{N}\). Let \(\psi_1(x)\) be independent of \(S_{n,t}^x\), then

\[
F_{n+1,t,v} f(x) = \mathbb{E}\left( e^{i\langle sv, \psi_1(x) \rangle} F_{n,t,v} f(\psi_1(x)) \right)
= \mathbb{E}\left( e^{i\langle sv, \psi_1(x) \rangle} e^{i\langle sv, S_{n,t}^x \rangle} f\left( X_{n,t}^x \right) \right) = \mathbb{E}\left( e^{i\langle sv, S_{n+1,t}^x \rangle} f\left( X_{n+1,t}^x \right) \right).
\]

□
We need also following inequalities

**Lemma 5.8.** For every $x, y \in \mathbb{R}^d$ and $0 < \epsilon \leq 1$,

\[
|e^{i(x,y)} - 1| \leq 2|x|^\epsilon |y|^\epsilon,
\]

and more generally for $n \in \mathbb{N}$,

\[
|e^{i(x,y)} - \sum_{k=0}^{n-1} \frac{(i(x,y))^k}{k!}| \leq 2|x|^{\epsilon+n-1}|y|^{\epsilon+n-1}.
\]

Let denote $\Pi_n = L_{\theta_1} \cdots L_{\theta_n}$ for $n \in \mathbb{N}$ and $\Pi_0 = 1$.

**Lemma 5.11.** Assume that $0 < \rho < \alpha$. Then there exists a constant $C_1 > 0$ independent of $s, t \in [0, 1]$ and $v \in \mathbb{S}^{d-1}$ such that for every $n \in \mathbb{N}$

\[
|\mathcal{F}^n_{s,t,v} f|_\rho \leq C_1 |f|_\rho.
\]

**Proof.** By the (3.9) we have $|X^n_{n,t} - X^n_{n,t}| \leq L_{\theta_1} \cdots L_{\theta_n} |x - y|$. Since $X^n_{n,t} = tX^n_0$, we have

\[
|X^n_{n,t}| \leq |t| \cdot |X^n_0| + L_{\theta_1} \cdots L_{\theta_n} |x|,
\]

then by Lemma 5.6, definition of $| \cdot |_\rho$ and (5.13) we have

\[
\frac{|\mathcal{F}^n_{s,t,v} f(x)|}{(1 + |x|)^\rho} \leq \mathbb{E} \left( \frac{|f(X^n_{n,t})|}{(1 + |X^n_{n,t}|)^\rho} \cdot \frac{(1 + |X^n_{n,t}|)^\rho}{(1 + |x|)^\rho} \right)
\leq |f|_\rho \mathbb{E} \left( \frac{(1 + |t| \cdot |X^n_0| + \Pi_n |x|)^\rho}{(1 + |x|)^\rho} \right)
\leq 3^\rho \left( 1 + |t| |X^n_0| + \kappa(\rho) n \right) |f|_\rho \leq C_1 |f|_\rho.
\]

and by Lemma 3.12 $C_1 = 3^\rho \sup_{n \in \mathbb{N}} (1 + \mathbb{E} (|X^n_0|) + \kappa(\rho)^n)$ is finite which gives (5.12). \hfill \Box

**Lemma 5.14.** Assume that $0 < \epsilon < 1$, $\lambda > 0$, $2\lambda + \epsilon < \alpha$, and $\rho = 2\lambda$. Then there exist constants $C_2 > 0$, $C_3 > 0$ and $0 < \rho < 1$ independent of $s, t \in [0, 1]$ and $v \in \mathbb{S}^{d-1}$ such that for every $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$ and $n \in \mathbb{N}$

\[
[\mathcal{F}^n_{s,t,v}]_{\epsilon,\lambda} \leq C_2 \rho^n |f|_{\epsilon,\lambda} + C_3 |f|_\rho.
\]

**Proof.** By the definition of the seminorm $[ \cdot ]_{\epsilon,\lambda}$ we have

\[
\mathcal{F}^n_{s,t,v} f(x) - \mathcal{F}^n_{s,t,v} f(y) = \mathbb{E} \left( e^{i(sv, S^n_{x,t})} (f(X^n_{x,t}) - f(X^n_{y,t})) \right)
+ \mathbb{E} \left( e^{i(sv, S^n_{x,t})} - e^{i(sv, S^n_{y,t})} \right) f(X^n_{y,t}).
\]
To obtain (5.15) we have to estimate (5.16) and (5.17) separately. Indeed,

\[
\frac{|\mathbb{E}(e^{i(sv,\xi_\nu,t)}(f(X^n_{x,t}) - f(X^n_{y,t}))|}{|x-y|^{\rho}(1 + |x|)^{\lambda}(1 + |y|)^{\lambda}} \leq |f|_{\rho,\lambda} \cdot \mathbb{E}
\]

\[
\frac{|X^n_{x,t} - X^n_{y,t}|^{\rho}(1 + |X^n_{x,t}|)^{\lambda}(1 + |X^n_{y,t}|)^{\lambda}}{|x-y|^{\rho}(1 + |x|)^{\lambda}(1 + |y|)^{\lambda}} \leq |f|_{\rho,\lambda} \cdot \mathbb{E}
\]

\[
\frac{(\Pi_n (1 + |X^n_{0,t}| + \Pi_n|x|) \lambda (1 + |X^n_{0,t}| + \Pi_n|y|)^{\lambda}}{(1 + |x|)^{\lambda}(1 + |y|)^{\lambda}} \leq |f|_{\rho,\lambda} \cdot \mathbb{E}
\]

\[
(\Pi_n (|X^n_{0,t}| + 1)^{2\lambda}) \leq 3^{2\lambda}|f|_{\rho,\lambda} \cdot \left( \mathbb{E}(\Pi_n^{2\lambda+\epsilon}) + \mathbb{E}(\Pi_n^0 |X^n_{0,t}|^{2\lambda}) \right) \leq C_2 g^\alpha |f|_{\rho,\lambda},
\]

where by Lemma 3.12 constant $C_2 = 3^{2\lambda} \sup_{n \in \mathbb{N}} \left( 2 + \mathbb{E} \left( |X^n_0|^{2\lambda+\epsilon} \frac{2n}{2n+\epsilon} \right) \right)$ is finite.

In order to estimate (5.17) notice that by (3.9) we have

\[
|\xi_{n,t} - \xi_{n,t}^0| \leq \sum_{k=1}^n \Pi_k |x - y| \leq B_n |x - y|,
\]

where $B_n = \sum_{k=0}^n \Pi_k$. Assume that $|y| \leq |x|$, then

\[
\frac{\mathbb{E}\left( \left| e^{i(sv,\xi_{n,t})} - e^{i(sv,\xi_{n,t}^0)} \right| f(X^n_{y,t}) \right)}{|x-y|^{\rho}(1 + |x|)^{\lambda}(1 + |y|)^{\lambda}} \leq |f|_{\rho,\lambda} \cdot \mathbb{E}
\]

\[
\frac{|e^{i(sv,\xi_{n,t} - \xi_{n,t}^0)} - 1| (1 + |X^n_{y,t}|)^{\rho}}{|x-y|^{\rho}(1 + |x|)^{\lambda}(1 + |y|)^{\lambda}} \leq 2|s|^\rho |f|_{\rho,\lambda} \cdot \mathbb{E}
\]

\[
\frac{B_n (1 + |X^n_{0,t}| + \Pi_n)^{\rho} (1 + |y|)^{\rho}}{(1 + |y|)^{2\lambda}} \leq 2 \cdot 3^{\rho}|s|^\rho |f|_{\rho,\lambda} \cdot \mathbb{E}
\]

\[
(B_n^\epsilon + |X^n_0|^{\rho} + \Pi_n^0) \leq C_3 |f|_{\rho,\lambda},
\]

where constant $C_3 = \sup_{n \in \mathbb{N}} 2 \cdot 3^{\rho} \cdot \mathbb{E}(B_n^\epsilon + B_n^0 |X^n_0|^{\rho} + B_n^0 \Pi_n^0)$ is finite. Indeed, for every $0 < \eta < \min\{\alpha, 1\}$ we have that $B_n^0 \leq 1 + \Pi_n^0 + \ldots + \Pi_n^0$ and for every $0 < \beta < \alpha$, $\mathbb{E} \left( \sum_{n=0}^{\infty} \Pi_n^0 \right) = \frac{1}{1 - \epsilon(\eta)}$ is finite. Hence the Hölder inequality and Lemma 3.12 applied to $\mathbb{E}(B_n^\epsilon + B_n^0 |X^n_0|^{\rho} + B_n^0 \Pi_n^0)$ gives $C_3 < \infty$. Combining estimates (5.18) and (5.19) with (5.20) we obtain the inequality in (5.15). □
Lemma 5.21. Assume that $0 < \epsilon < 1$, $\lambda > 0$, $2\lambda + \epsilon < \alpha$, $\rho = 2\lambda$ and $\lambda + 2\epsilon < \rho$. Then there exist finite constants $C_4 > 0$ and $C_5 > 0$ independent of $s, t \in [0, 1]$ and of $v \in S^{d-1}$ such that for every $f \in B_{\rho, \epsilon, \lambda}$

\begin{align}
(5.22) & \quad |(\mathcal{F}_{s,t,v} - \mathcal{F}_{s,0,v}) f|_\rho \leq C_4 |t|^{\rho} \|f\|_{\rho, \epsilon, \lambda}, \\
(5.23) & \quad |(\mathcal{F}_{s,t,v} - \mathcal{F}_{0,t,v}) f|_\rho \leq C_5 |s|^{\rho} \|f\|_{\rho, \epsilon, \lambda}.
\end{align}

**Proof.** In order to prove (5.22) notice that

\begin{align}
(5.24) & \quad (\mathcal{F}_{s,t,v} - \mathcal{F}_{s,0,v}) f(x) = \mathbb{E} \left( e^{i(sv, \psi_t(x))} (f(\psi_t(x)) - f(\overline{\psi}(x))) \right) \\
& \quad + \mathbb{E} \left( e^{i(sv, \psi_t(x))} - e^{i(sv, \overline{\psi}(x))} \right) f(\overline{\psi}(x)).
\end{align}

Now we estimate (5.24) and (5.25) separately. By the definition of map $\overline{\psi}$ we know that $\overline{\psi}(0) = 0$, so $|\overline{\psi}(x)| \leq |Mx|$. Then condition (L2) implies that $|\psi_t(x)| \leq |t||Q| + |M||x|$ and so

\begin{align}
(5.26) & \quad \mathbb{E} \left( e^{i(sv, \psi_t(x))} (f(\psi_t(x)) - f(\overline{\psi}(x))) \right) \\
& \quad \leq |t|^{\rho} \cdot E \left( |Q|^\epsilon (1 + |t||Q| + |M||x|)^{2\lambda} \right) \lesssim 3^{2\lambda} |t|^{\rho} \cdot E (|Q|^\epsilon + |t|^{2\lambda} |Q|^\epsilon |M|^{2\lambda}).
\end{align}

It is easy to see that the H"older inequality, (H5) and (L3) applied to $D_1 = 3^{2\lambda} \cdot \mathbb{E} (|Q|^\epsilon + |Q|^{2\lambda+\epsilon} + |Q|^\epsilon |M|^{2\lambda})$ ensures that $D_1 < \infty$. For (5.25), we have

\begin{align}
(5.27) & \quad \mathbb{E} \left( e^{i(sv, \psi_t(x))} - e^{i(sv, \overline{\psi}(x))} \right) f(\overline{\psi}(x)) \\
& \quad \leq |t|^{\rho} \cdot E \left( |e^{i(sv, \psi_t(x))} - 1| (1 + |M||x|)^{\rho} \right) \\
& \quad \lesssim 2 |s|^{\rho} |t|^{\rho} \cdot E (|Q|^\epsilon + |M|^{\rho}) \\
& \quad \lesssim 2^{\rho+1} |s|^{\rho} |t|^{\rho} \cdot E (|Q|^\epsilon + |M|^{\rho}) \leq D_2 |t|^{\rho} |f|_\rho,
\end{align}

where the constant $D_2 = 2^{\rho+1} \cdot \mathbb{E} (|Q|^\epsilon + |Q|^\epsilon |M|^{\rho})$ is also finite by the H"older inequality, (H5) and (L3). Combining (5.26) with (5.27) we obtain (5.22) with $C_4 = \max\{D_1, D_2\}$. 
In order to prove (5.23) notice that
\[
\frac{|(F_{s,t,v} - F_{0,t,v}) f(x)|}{(1 + |x|)^\rho} \leq \mathbb{E} \left( \frac{|e^{i(s,v,\psi_i(x))} f(\psi_i(x)) - f(\psi_i(x))|}{(1 + |x|)^\rho} \right)
\]
\[
\leq \mathbb{E} \left( \frac{|e^{i(s,v,\psi_i(x))} - 1||f(\psi_i(x)) - f(0)|}{(1 + |x|)^\rho} \right) + \mathbb{E} \left( \frac{|e^{i(s,v,\psi_i(x))} - 1||f(0)|}{(1 + |x|)^\rho} \right)
\]
\[
\leq 2|s| \mathbb{E} \left( |f|_{c,\lambda} \cdot \mathbb{E} \left( \frac{|\psi_i(x)|^{2\rho} (1 + |\psi_i(x)|)^\lambda}{(1 + |x|)^\rho} \right) + |f|_{c,\lambda} \cdot \mathbb{E} \left( \frac{|\psi_i(x)|^{\lambda}}{(1 + |x|)^\rho} \right) \right)
\]
\[
\leq 2^{\lambda + 1} |s|^\rho \mathbb{E} \|f\|_{c,\lambda} \mathbb{E} \left( \frac{|\psi_i(x)|^{2\rho} + |\psi_i(x)|^{\lambda + 2\rho} + |\psi_i(x)|^\lambda}{(1 + |x|)^\rho} \right)
\]
\[
\leq C_5 |s|^\rho \mathbb{E} \|f\|_{c,\lambda},
\]
where $C_5 = 2^{\lambda + 1} \mathbb{E} \left( (1 + |M| + |Q|)^{2\rho} + (1 + |M| + |Q|)^{\lambda + 2\rho} + (1 + |M| + |Q|)^\lambda \right)$ is finite by (H5) and (L3). Hence (5.28) proves (5.23) and finally it completes the proof of the Lemma.
\[\square\]

**Lemma 5.29.** The unique eigenvalue of modulus 1 for operator $P$ acting on $C_\rho$ is 1 and the eigenspace is one dimensional. The corresponding projection on $C \cdot 1$ is given by the map $f \mapsto \nu(f)$.

**Proof.** The proof is the same as in Lemma 5.30. \[\square\]

Recall, that for every $n \in \mathbb{N}$,
\[
T_v^n f(x) = \mathbb{E} \left( e^{i(v, \sum_{k=1}^n \bar{\psi}_k \circ \cdots \circ \bar{\psi}_1(x))} f(\bar{\psi}_n \circ \cdots \circ \bar{\psi}_1(x)) \right),
\]

**Lemma 5.30.** The unique eigenvalue of modulus 1 for operator $T_v$ acting on $C_\rho$ is 1 with the eigenspace $C \cdot h_v(x)$, where
\[
h_v(x) = \mathbb{E} \left( e^{i(v, \sum_{k=1}^\infty \bar{\psi}_k \circ \cdots \circ \bar{\psi}_1(x))} \right).
\]

**Proof.** Observe that the random variables $\sum_{k=1}^\infty \bar{\psi}_k \circ \cdots \circ \bar{\psi}_1(x)$ and $\sum_{k=2}^\infty \bar{\psi}_k \circ \cdots \circ \bar{\psi}_2(x)$ have the same law, hence
\[
T_v h_v(x) = \mathbb{E} \left( e^{i(v, \bar{\psi}_1(x))} h_v(\bar{\psi}_1(x)) \right) = \mathbb{E} \left( e^{i(v, \bar{\psi}_1(x))} e^{i(v, \sum_{k=2}^\infty \bar{\psi}_k \circ \cdots \circ \bar{\psi}_2(x))} \right) = \mathbb{E} \left( e^{i(v, \bar{\psi}_1(x))} e^{i(v, \sum_{k=2}^\infty \bar{\psi}_k \circ \cdots \circ \bar{\psi}_2(x))} \right) = h_v(x).
\]

This proves that 1 is eigenvalue for $T_v$ and by Lemma 5.51 we know that $T_v$ acts on $C_\rho$. Let $f \in C_\rho$ be such that $T_v^n f(x) = f(x)$. Since $h_v(0) = 1$ and $\lim_{n \to \infty} \bar{\psi}_n \circ \cdots \circ \bar{\psi}_1(x) = 0$ a.e., we have
\[
|f(x) - f(0)h_v(x)| \leq \mathbb{E} \left( |f(\bar{\psi}_n \circ \cdots \circ \bar{\psi}_1(x)) - f(0)h_v(\bar{\psi}_n \circ \cdots \circ \bar{\psi}_1(x))| \right)
\]
\[
\leq \mathbb{E} \left( |f(0) - f(0)h_v(0)| \right) = 0.
\]

Hence $f(x) = f(0)h_v(x)$. Now assume that for a $z$ of modulus 1 and a nontrivial $f \in C_\rho$ we have $T_v f(x) = z f(x)$. Then for every $x$ such that $f(x) \neq 0$
\[
z^n f(x) = T_v^n f(x) = \mathbb{E} \left( e^{i(v, \sum_{k=1}^n \bar{\psi}_k \circ \cdots \circ \bar{\psi}_1(x))} \left( f(\bar{\psi}_n \circ \cdots \circ \bar{\psi}_1(x)) - f(0) \right) \right)
\]
\[
+ \mathbb{E} \left( e^{i(v, \sum_{k=1}^n \bar{\psi}_k \circ \cdots \circ \bar{\psi}_1(x))} f(0) \right) \leq \mathbb{E} \left( f(0) \right) = f(0)h_v(x),
\]
i.e.\[
\lim_{n \to \infty} z^n = \frac{f(0)}{f(x)} h_v(x)
\]
but this is impossible. 

Recall that the essential spectral radius \( r_e(T) \) of the operator \( T \) is the smallest nonnegative number \( l \) for which elements of the spectrum outside of the disk of radius \( l \) centered at the origin are isolated eigenvalues of finite multiplicity.

**Lemma 5.31.** If \( z \in \sigma(P_{t,v}) \) or \( z \in \sigma(T_{t,v}) \) and \( |z| > q \) where \( 0 < q < 1 \) is defined as in Lemma 5.14, then \( z \) does not belong to the residual spectrum of operator \( P_{t,v} \) or \( T_{t,v} \).

**Proof.** We have to show that \( r_e(P_{t,v}) \leq q \) and \( r_e(T_{t,v}) \leq q \) for any \( t \in [0, 1] \). It means that if \( z \in \sigma(P_{t,v}) \) or \( z \in \sigma(T_{t,v}) \) and \( |z| > q \), then \( z \) belongs to the point spectrum of operator \( P_{t,v} \) or \( T_{t,v} \). In order to prove above consider two cases:

- \( r(T_{t,v}) \leq q \), then \( r_e(T_{t,v}) \leq r(T_{t,v}) \leq q \) and the conclusion follows.
- \( r(T_{t,v}) > q \), then by Lemmas 5.11 and 5.14, Remark 5.2 and Theorem of Ionescu Tulcea and Marinescu (18), the operator \( T_{t,v} \) is quasi-compact and \( r_e(T_{t,v}) \leq q \).

In a similar way the conclusion follows for operators \( P_{t,v} \). It is easy to see that operators \( P \) and \( T \) are quasi-compact. 

**Proof of Proposition (5.5).** In view of Lemmas 5.11, 5.14, 5.21, 5.29, 5.30 and 5.31 we may use the perturbation theorem of Keller and Liverani (20) for the operators \( P_{t,v} \) and \( T_{t,v} \) to get Proposition. 

6. **Rate of convergence**

In all the lemmas and theorems below we assume that hypotheses of Proposition 5.5 hold. To write down an expansion of \( k_v(t) \) sufficiently good for the limit Theorem 1.24 we approximate the peripheral eigenfunction \( \Pi_{T,v} h_v \) by \( \Pi_{T,0} h_v = h_v \). Section 6 is the main novelty in the proof of Theorem 1.24.

6.1. **Rate of convergence of projections.** Now we want to know what is the rate of convergence of \( \| (\Pi_{T,v} - \Pi_{T,0}) h_v \|_{\rho, \epsilon, \lambda} \), where \( h_v \) is the peripheral eigenfunction of \( T_v \). More precisely we will prove following

**Theorem 6.1.** Let \( h_v \) be the eigenfunction for operator \( T_v \) defined in Lemma 5.30. Then for any \( 0 < \delta \leq 1 \) such that \( \epsilon < \delta < \alpha \) there exists \( C > 0 \) such that

\[
||(\Pi_{T,v} - \Pi_{T,0}) h_v) \circ \delta_t||_{\rho, \epsilon, \lambda} \leq C|t|^\delta,
\]

for every \( |t| \leq t_0 \). Moreover, for every \( x \in \mathbb{R}^d \) and every \( |t| \leq t_0 \)

\[
|\Pi_{T,v}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)| \leq C|t|^\delta (1 + |x|)^\alpha.
\]

For affine recursions, (6.2) and (6.3) where obtained by very particular computations based on the fact that the Fourier transform sees dilatations, modulations and translations. In (14) and (3) the authors expressed explicitly eigenvectors associated with dominant eigenvalues in terms of the Fourier transform and in this way they got sufficiently good estimates of the rate of convergence in the fractional expansions. Their elegant and very tricky proof is not applicable to general non affine recursions. We will proceed differently. Our method is based on spectral
properties of operators $T_{t,v}$ and $T_v$ that were defined in the previous section and which are strongly connected with operators $P_{t,v}$ and $P$. First we prove a number of lemmas.

**Lemma 6.4.** Assume that $\zeta : \mathbb{R}^d \to \mathbb{R}^d$ is a Lipschitz map with Lipschitz constant $L_{\zeta}$. Then $h_v \circ \zeta \in B_{\rho,\varepsilon,\lambda}$, where $h_v$ is the eigenfunction of $T_v$ defined in Lemma 5.30. Moreover,

$$
|h_v(\zeta(x))| \leq 1, \tag{6.5}
$$

$$
|h_v(\zeta(x)) - h_v(\zeta(y))| \leq \frac{2}{1 - \kappa(\delta)} |L_{\zeta}|^\delta |x - y|^{\delta}, \tag{6.6}
$$

for every $x, y \in \mathbb{R}^d$ and every $0 < \delta \leq 1$ such that $0 < \delta < \alpha$.

**Proof.** Observe, that

$$
|h_v(\zeta(x))| \leq \mathbb{E} \left( \left| e^{i \sum_{k=1}^\infty \overline{\psi}_k \circ \cdots \circ \overline{\psi}_1 (\zeta(x))} \right| \right) = 1,
$$

and

$$
|h_v(\zeta(x)) - h_v(\zeta(y))| \leq \mathbb{E} \left( \left| e^{i \sum_{k=1}^\infty \overline{\psi}_k \circ \cdots \circ \overline{\psi}_1 (\zeta(x))} - e^{i \sum_{k=1}^\infty \overline{\psi}_k \circ \cdots \circ \overline{\psi}_1 (\zeta(y))} \right| \right)
\leq 2 \sum_{k=1}^\infty \left| \overline{\psi}_k \circ \cdots \circ \overline{\psi}_1 (\zeta(x)) - \overline{\psi}_k \circ \cdots \circ \overline{\psi}_1 (\zeta(y)) \right|^{\delta}
\leq \frac{2}{1 - \kappa(\delta)} |L_{\zeta}|^\delta |x - y|^{\delta}.
$$

This proves (6.5) and (6.6). Moreover,

$$
\|h_v \circ \zeta\|_{\rho,\varepsilon,\lambda} \leq 1 + \frac{2}{1 - \kappa(\epsilon)} |L_{\zeta}|^\epsilon,
$$

and so $h_v \circ \zeta \in B_{\rho,\varepsilon,\lambda}$. \qed

**Lemma 6.7.** Assume that the function $f$ satisfies $|f(x)| \leq C$ for any $x \in \mathbb{R}^d$ and $|f(x) - f(y)| \leq C|x - y|^{\delta}$ for any $0 < \delta \leq 1$ and $x, y \in \mathbb{R}^d$, where constant $C > 0$ depends on $\delta$. Then for every $\delta \in (\epsilon, \alpha)$

$$
|(T_{t,v} - T_v)f|_{\rho,\varepsilon,\lambda} \leq C_1 |t|^{\delta - \epsilon}, \tag{6.8}
$$

$$
|(T_{t,v} - T_v)f|_\rho \leq C_2 |t|^\delta, \tag{6.9}
$$

where $C_1 > 0$ and $C_2 > 0$ does not depend on $t$.

**Proof.** In order to show (6.8) we have to estimate the seminorm $|(T_{t,v} - T_v)f|_{\rho,\varepsilon,\lambda}$. Notice, that

$$
|(T_{t,v} - T_v)f|_{\rho,\varepsilon,\lambda} \leq \sup_{x \neq y, |x - y| \leq t} \frac{|(T_{t,v} - T_v)f(x) - (T_{t,v} - T_v)f(y)|}{|x - y|^\epsilon (1 + |x|)^\lambda (1 + |y|)^\lambda} + \sup_{x \neq y, |x - y| > t} \frac{|(T_{t,v} - T_v)f(x) - (T_{t,v} - T_v)f(y)|}{|x - y|^\epsilon (1 + |x|)^\lambda (1 + |y|)^\lambda},
$$

where $C_1 > 0$ and $C_2 > 0$ does not depend on $t$. \qed
For the first term in (6.10) \(|x - y| \leq t\) we observe that
\[
(T_{t,v} - T_v)f(x) - (T_{t,v} - T_v)f(y) = \]
\[
\mathbb{E} \left( \left( e^{i(v, \psi_t(x))} - e^{i(v, \psi_t(y))} \right) f(\psi_t(x)) \right) + \mathbb{E} \left( e^{i(v, \psi_t(y))} (f(\psi_t(x)) - f(\psi_t(y))) \right) \]
\[
- \mathbb{E} \left( \left( e^{i(v, \psi_t(x))} - e^{i(v, \psi_t(y))} \right) f(\psi_t(x)) \right) - \mathbb{E} \left( e^{i(v, \psi_t(y))} (f(\psi_t(x)) - f(\psi_t(y))) \right). \]
\[
(6.11) \hspace{1cm} \hspace{1cm} (6.12) \hspace{1cm} \hspace{1cm} (6.13) \hspace{1cm} \hspace{1cm} (6.14)
\]
We will estimate (6.11), (6.12), (6.13) and (6.14) separately. By the assumptions on the function \(f\) observe, that for every \(0 < \delta \leq 1\) such that \(\epsilon < \delta < \alpha\) we have
\[
\mathbb{E} \left( \left| e^{i(v, \psi_t(x))} - e^{i(v, \psi_t(y))} \right| \right) \leq 2CE \left( \frac{|\psi_t(x) - \psi_t(y)|^\delta}{|x - y|^\epsilon} \right) \leq 2CE \left( |M|^\delta \right) |x - y|^{\delta - \epsilon} \leq 2CE \left( |M|^\delta \right) |t|^{\delta - \epsilon}.
\]
\[
(6.15)
\]
Similarly we obtain the estimate of the second term. Indeed,
\[
\mathbb{E} \left( \frac{|f(\psi_t(x)) - f(\psi_t(y))|}{|x - y|^\epsilon (1 + |x|)^\lambda (1 + |y|)^\lambda} \right) \leq \mathbb{E} \left( \frac{|f(\psi_t(x)) - f(\psi_t(y))|}{|x - y|^\epsilon} \right) \leq 2CE \left( |M|^\delta \right) |x - y|^{\delta - \epsilon} \leq 2CE \left( |M|^\delta \right) |t|^{\delta - \epsilon}.
\]
\[
(6.16)
\]
Remaining (6.13) and (6.14) are estimated in the similar way. Now consider the second term of (6.10) \(|x - y| > t\) and notice, that
\[
(T_{t,v} - T_v)f(x) - (T_{t,v} - T_v)f(y) = \]
\[
\mathbb{E} \left( \left( e^{i(v, \psi_t(x))} - e^{i(v, \psi_t(y))} \right) f(\psi_t(x)) \right) + \mathbb{E} \left( e^{i(v, \psi_t(y))} (f(\psi_t(x)) - f(\psi_t(y))) \right) \]
\[
- \mathbb{E} \left( \left( e^{i(v, \psi_t(x))} - e^{i(v, \psi_t(y))} \right) f(\psi_t(x)) \right) - \mathbb{E} \left( e^{i(v, \psi_t(y))} (f(\psi_t(x)) - f(\psi_t(y))) \right). \]
\[
(6.17) \hspace{1cm} \hspace{1cm} (6.18) \hspace{1cm} \hspace{1cm} (6.19) \hspace{1cm} \hspace{1cm} (6.20)
\]
As before we will estimate (6.17), (6.18), (6.19) and (6.20) separately using (L2) and (L3). Indeed, for every \(0 < \delta \leq 1\) such that \(\epsilon < \delta < \alpha\) we have
\[
\mathbb{E} \left( \left| e^{i(v, \psi_t(x))} - e^{i(v, \psi_t(x))} \right| \right) \leq 2CE \left( \frac{|\psi_t(x) - \psi_t(x)|^\delta}{|x - y|^\epsilon} \right) \leq 2CE \left( \frac{|t|^\delta}{x - y|^\epsilon} \right) \leq 2CE \left( |Q|^\delta \right) |t|^{\delta - \epsilon}.
\]
\[
(6.21)
\]
Similarly we obtain the estimate for the second term. Indeed,

\[
\mathbb{E} \left( \frac{|e^{i\langle \nu, \psi(x) \rangle} (f(\psi_t(x)) - f(\overline{\psi}(x)))|}{|x-y|^\delta} \right) \leq \mathbb{E} \left( \frac{|f(\psi_t(x)) - f(\overline{\psi}(x))|}{|x-y|^\delta} \right) \leq 2C \mathbb{E} \left( \frac{|t|^\delta |Q|^\delta}{|x-y|^\rho} \right) \leq 2C \mathbb{E} \left( |Q|^\delta |t|^\delta \right).
\]

Also remaining (6.19) and (6.20) can be estimated similarly. Hence, in view of (6.15), (6.16), (6.21) and (6.22), we obtain (6.8). For (6.9) notice that

\[
(T_{t,v} - T_v)f(x) = \mathbb{E} \left( \left( e^{i\langle \nu, \psi(x) \rangle} - e^{i\langle \nu, \overline{\psi}(x) \rangle} \right) f(\psi_t(x)) \right)
+ \mathbb{E} \left( e^{i\langle \nu, \overline{\psi}(x) \rangle} (f(\psi_t(x)) - f(\overline{\psi}(x))) \right).
\]

Therefore,

\[
\mathbb{E} \left( \left| e^{i\langle \nu, \psi(x) \rangle} - e^{i\langle \nu, \overline{\psi}(x) \rangle} \right| \frac{|f(\psi_t(x))|}{(1 + |x|)^\rho} \right) \leq 2C \mathbb{E} \left( \frac{|\psi_t(x) - \overline{\psi}(x)|^\delta}{(1 + |x|)^\rho} \right) \leq 2C \mathbb{E} \left( |Q|^\delta |t|^\delta \right).
\]

and

\[
\mathbb{E} \left( \left| e^{i\langle \nu, \psi(x) \rangle} (f(\psi_t(x)) - f(\overline{\psi}(x))) \right| \frac{(1 + |x|)^\rho}{(1 + |x|)^\rho} \right) \leq 2C \mathbb{E} \left( \frac{|t|^\delta |Q|^\delta}{(1 + |x|)^\rho} \right) \leq 2C \mathbb{E} \left( |Q|^\delta |t|^\delta \right).
\]

Combining (6.24) with (6.25) we obtain (6.9) which completes the proof of the Lemma.

**Lemma 6.26.** Assume that $f \in B_{p,t} \cap D^c$, $x \in \mathbb{R}^d$ and $t \in [0, t_0]$. Then

\[
((z - P_t) f \circ \delta_t)(x) = ((z - T_t) f \circ \delta_t)(x),
\]

for every $z \in D^c$.

**Proof.** If $f \in B_{p,t} \cap D^c$, then $f \circ \delta_t \in B_{p,t} \cap D^c$ and $(z - T_t) f \circ \delta_t$ are holomorphic for every $z \in D^c$. Moreover $((z - P_t) f \circ \delta_t) \in B_{p,t}$ and $((z - T_t) f \circ \delta_t) \in B_{p,t}$. Furthermore, when $x \in \mathbb{R}^d$ and $t \in [0, t_0]$ are fixed, the maps

\[
B_{p,t} \ni f \rightarrow f(x) \in \mathbb{C}, \quad \text{and} \quad B_{p,t} \ni f \rightarrow f(tx) \in \mathbb{C},
\]

are continuous linear functional on $B_{p,t}$. Therefore,

\[
D^c \ni z \rightarrow ((z - P_t) f \circ \delta_t)(x),
\]

\[
D^c \ni z \rightarrow ((z - T_t) f)(tx),
\]
are holomorphic in $D^c$. In order to prove (6.27) notice that $D^c$ is connected open subset of $\mathbb{C}$. Since $r(P_{t,v}) \leq 1$ and $r(T_{t,v}) \leq 1$, then
\[
(z - P_{t,v})^{-1} = \sum_{n=0}^{\infty} \frac{P^*_n}{z^{n+1}} \quad \text{and} \quad (z - T_{t,v})^{-1} = \sum_{n=0}^{\infty} \frac{T^*_n}{z^{n+1}},
\]
for every $|z| > 1$. Let $z \in \mathcal{Z} = \{z \in \mathbb{C} : |z| = 2\}$, then by Lemma 5.3
\[
((z - T_{t,v})^{-1}f)(tx) = \sum_{n=0}^{\infty} \frac{(T^*_n f)(tx)}{z^{n+1}} = \sum_{n=0}^{\infty} \frac{(P^*_n(f \circ \delta_t))(x)}{z^{n+1}} = ((z - P_{t,v})^{-1}(f \circ \delta_t))(x).
\]
Since $\mathcal{Z}$ has all its accumulation points in $D^c$ and the holomorphic functions (6.28) and (6.29) coincide on the set $\mathcal{Z}$, then they have to coincide on $D^c$.

**Lemma 6.30.** Let $h_v$ be the eigenfunction for operator $T_v$ as in Lemma 5.30, then
\[
(\Pi_{T,v} - \Pi_{T,0})h_v = \frac{1}{2\pi} \int_0^{2\pi} (\xi e^{is} + 1 - T_{t,v})^{-1}((T_{t,v} - T_v)h_v) \, ds.
\]

**Proof.** Notice that
\[
(z - T_v)^{-1}h_v = \frac{1}{z - 1}h_v,
\]
then
\[
(z - T_{t,v})^{-1} - (z - T_v)^{-1} = (z - T_{t,v})^{-1}(T_{t,v} - T_v)(z - T_v)^{-1},
\]
then
\[
(\Pi_{T,v} - \Pi_{T,0})h_v = \frac{1}{2\pi i} \int_{|z-1| = \xi} ((z - T_{t,v})^{-1} - (z - T_v)^{-1}) h_v \, dz
\]
\[
= \frac{1}{2\pi i} \int_{|z-1| = \xi} ((z - T_{t,v})^{-1}(T_{t,v} - T_v)(z - T_v)^{-1}) h_v \, dz
\]
\[
= \frac{1}{2\pi i} \int_{|z-1| = \xi} \frac{1}{z - 1}((z - T_{t,v})^{-1}(T_{t,v} - T_v)h_v) \, dz
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} (\xi e^{is} + 1 - T_{t,v})^{-1}((T_{t,v} - T_v)h_v) \, ds,
\]
which completes the proof of (6.31). \qed

**Proof of Theorem (6.1).** For every $f \in B_{p,c,\lambda}$ we have
\[
\|f \circ \delta_t\|_{p,c,\lambda} \leq \begin{cases} |f|_p + \frac{t|\xi|^c}{2(1 + t^{2\lambda + c})|\xi|}, & \text{if } |t| \leq 1 \\ |t|^\beta|f|_p + |t|^{2\lambda + c}|f|_{c,\lambda}, & \text{if } |t| > 1 \end{cases},
\]
In view of (6.31) and (6.27) we have
\[
((\Pi_{T,v} - \Pi_{T,0})h_v)(tx) = 
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} ((\xi e^{is} + 1 - T_{t,v})^{-1}(T_{t,v} - T_v)h_v)(tx) \, ds
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} ((\xi e^{is} + 1 - P_{t,v})^{-1}((T_{t,v} - T_v)h_v \circ \delta_t))(x)ds.
\]
A straightforward application of (6.33), Proposition 5.5, inequalities (6.32), (6.8) and (6.9)
\[\|((\Pi_{T,t} - \Pi_{T,0})h_v) \circ \delta_t\|_{\rho,\varepsilon,\lambda} \leq\]
\[\leq \frac{1}{2\pi} \int_0^{2\pi} \left\| \left( e^{it\alpha} + 1 - P_{t,v} \right)^{-1} \left( ((T_{t,v} - T_v)h_v) \circ \delta_t \right) \right\|_{\rho,\varepsilon,\lambda} ds\]
\[\leq D \|((T_{t,v} - T_v)h_v) \circ \delta_t\|_{\rho,\varepsilon,\lambda}\]
\[\leq D((T_{t,v} - T_v)h_v) \circ \delta_t|_{\rho} + |((T_{t,v} - T_v)h_v) \circ \delta_t|_{\varepsilon,\lambda}\]
\[\leq D(|(T_{t,v} - T_v)h_v|_{\rho} + |t|^\rho |(T_{t,v} - T_v)h_v|_{\varepsilon,\lambda})\]
\[\leq D(C_2|t|^\delta + |t|^\varepsilon C_1|t|^\beta - \varepsilon)\]
\[\leq C|t|^\beta,\]
for every $|t| \leq t_0$ and the proof is finished. □

6.2. Rate of convergence of eigenvalues.

Lemma 6.34. For every $f \in B_{\rho,\varepsilon,\lambda}$, $\Pi_{T,t}(f) \circ \delta_t$ is an eigenfunction for operator $P_{t,v}$ corresponding to the eigenvalue $k_v(t)$. Furthermore,

\[(k_v(t) - 1) \cdot \nu((\Pi_{T,t}(f) \circ \delta_t) = \nu \left( e^{it\langle v, \cdot \rangle} - 1 \right) \cdot (\Pi_{T,t}(f) \circ \delta_t)\]

where $\nu$ is the stationary measure for the operator $P_v$.

Proof. By Proposition 5.5 we know that $\Pi_{T,t}(f)$ is an eigenfunction of $T_{t,v}$ with the eigenvalue $k_v(t)$ and $\Pi_{T,t}(f) \circ \delta_t$ is an eigenfunction for operator $P_{t,v}$ with the same eigenvalue by Lemma 5.3. Now we show that (6.35) holds. Indeed, on one hand,

\[\nu(P_{t,v}((\Pi_{T,t}(f) \circ \delta_t))) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{it\langle v, \psi(\theta) \rangle} ((\Pi_{T,t}(f) \circ \delta_t)(\psi(\theta)) \mu(\theta)\nu(\delta x)\]

\[= \int_{\mathbb{R}^d} e^{it\langle v, x \rangle} ((\Pi_{T,t}(f) \circ \delta_t)(x) \nu(dx) = \nu(e^{it\langle v, \cdot \rangle} (\Pi_{T,t}(f) \circ \delta_t))\]

and on the other,

\[\nu(P_{t,v}((\Pi_{T,t}(f) \circ \delta_t))) = k_v(t)\nu((\Pi_{T,t}(f) \circ \delta_t))\]

Hence,

\[k_v(t)\nu((\Pi_{T,t}(f) \circ \delta_t)) = \nu(e^{it\langle v, \cdot \rangle} (\Pi_{T,t}(f) \circ \delta_t))\]

Finally, subtracting $\nu((\Pi_{T,t}(f) \circ \delta_t)$ from both sides of (6.36) we obtain (6.35). □

Condition 6.37. Assume that $0 < \epsilon < 1$, $\lambda > 0$, $\lambda + 2\epsilon < \rho = 2\lambda$ and $2\lambda + \epsilon < \alpha$ as in Proposition 5.5 and additionally

– If $0 < \alpha \leq 1$, take any $0 < \beta < \frac{1}{\lambda}$ such that $\rho + 2\beta < \alpha$.
– If $1 < \alpha \leq 2$, take any $\lambda > 0$ such that $\rho = 2\lambda < 1$ and $\rho + 1 < \alpha$.

Proposition 6.38. Let $h_v$ be eigenfunction of operator $T_v$ defined in Lemma 5.30. If $0 < \alpha < 2$, then

\[\lim_{t \to 0} \frac{1}{|t|^\alpha} \int_{\mathbb{R}^d} \left( e^{it\langle v, x \rangle} - 1 \right) (\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)) \nu(dx) = 0.\]
If $\alpha = 2$, then

\begin{equation}
\lim_{t \to 0} \frac{1}{t^{2|\log t|}} \int_{\mathbb{R}^d} \left( e^{it(v,x)} - 1 \right) (\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)) \nu(dx) = 0.
\end{equation}

Proof. In estimations below in view of Condition 6.37 we have to use appropriate parameters $\epsilon, \lambda, \rho, \delta$ and $\eta$ which are determined by $\alpha$.

- If $0 < \alpha \leq 1$, we take $\delta = \alpha - \beta > \rho + \beta > \epsilon$ and $\eta = 2\beta$.
- If $1 < \alpha \leq 2$, we take $\delta = 1 > \epsilon$ and $\eta = 1$.

In view of (5.9) and (6.3), then for every $0 < t \leq t_0 \leq 1$

\begin{equation}
\left| \frac{1}{t^n} \int_{\mathbb{R}^d} \left( e^{it(v,x)} - 1 \right) (\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)) \nu(dx) \right| \leq C_1 |t|^\eta \delta - \alpha \int_{\mathbb{R}^d} |x|^\eta (1 + |x|)^{\rho} \nu(dx) \leq C_2 |t|^\eta - \alpha.
\end{equation}

Notice that, if

- $0 < \alpha \leq 1$, then $\eta + \delta - \alpha = 2\beta + \alpha - \alpha = \beta > 0$ and $\rho + \eta = \rho + 2\beta < \alpha$.
- $1 < \alpha \leq 2$, then $\eta + \delta - \alpha = 1 + 1 - \alpha = 2 - \alpha \geq 0$ and $\rho + \eta = \rho + 1 < \alpha$.

This justifies inequalities in (6.41) and completes the proof of (6.39) and (6.40). $\square$

**Proposition 6.42.** For every $x \in \mathbb{R}^d$, $t > 0$ and $v \in \mathbb{S}^{d-1}$ we have $h_v(tx) = h_{tv}(x)$ where $h_v$ is eigenfunction defined in Lemma 5.30. Moreover for $0 < \delta \leq 1$ and $\alpha > \delta$ we have

\begin{equation}
\left| E \left( e^{i(v, \sum_{k=1}^{\infty} \varphi_k \circ \cdots \circ \varphi_1(x))} - 1 \right) \right| \leq \frac{2}{1 - \kappa(\delta)} |x|^\delta,
\end{equation}

and for $0 < \delta \leq 1$ and $\alpha > 1 + \delta$

\begin{equation}
\left| E \left( e^{i(v, \sum_{k=1}^{\infty} \varphi_k \circ \cdots \circ \varphi_1(x))} - 1 - i \left( v, \sum_{k=1}^{\infty} \varphi_k \circ \cdots \circ \varphi_1(x) \right) \right) \right| \leq \frac{2}{(1 - \kappa(\delta + 1))^{1+\delta}} |x|^{1+\delta}.
\end{equation}

Proof. In order to prove the first formula it is enough to show that for a fixed $s > 0$ $\psi(sx) = s\psi(x)$ for any $x \in \mathbb{R}^d$. For every $\varepsilon > 0$ there exist $\delta > 0$ such that $|t\psi(t^{-1}sx) - \psi(sx)| < \varepsilon$ for every $0 < t < s\delta$. Hence if $t = rs$ and $0 < r < \delta$ then $|s\psi(r^{-1}x) - s\psi(sx)| < \varepsilon$. Letting $r$ tend to 0 we obtain $s\psi(x) = \psi(sx)$. Hence,

$h_v(tx) = E \left( e^{i(v, \sum_{k=1}^{\infty} \varphi_k \circ \cdots \circ \varphi_1(tx))} \right) = E \left( e^{i(v, \sum_{k=1}^{\infty} \varphi_k \circ \cdots \circ \varphi_1(x))} \right) = h_{tv}(x)$.

Now by (5.9) we have

\begin{equation}
\left| E \left( e^{i(v, \sum_{k=1}^{\infty} \varphi_k \circ \cdots \circ \varphi_1(x))} - 1 \right) \right| \leq 2E \left( \sum_{k=1}^{\infty} \varphi_k \circ \cdots \circ \varphi_1(x) \right) ^\delta \leq 2 \sum_{k=1}^{\infty} \kappa^k(\delta) |x|^\delta \leq \frac{2}{1 - \kappa(\delta)} |x|^\delta.
\end{equation}
Finally, by (5.10) and $\alpha > 1 + \delta$ we have

$$\left| \mathbb{E} \left( e^{i \langle \psi, \sum_{k=1}^{\infty} \psi_k \circ \ldots \circ \psi_1(x) \rangle} - 1 - i \left( \psi, \sum_{k=1}^{\infty} \psi_k \circ \ldots \circ \psi_1(x) \right) \right) \right|$$

$$\leq 2 \mathbb{E} \left( \left| \sum_{k=1}^{\infty} \psi_k \circ \ldots \circ \psi_1(x) \right|^{1+\delta} \right)^{1+\delta} \leq 2 \left( \sum_{k=1}^{\infty} \mathbb{E} \left( \left| \psi_k \circ \ldots \circ \psi_1(x) \right|^{1+\delta} \right)^{1+\delta} \right)^{1+\delta}$$

$$\leq 2 \left( \sum_{k=1}^{\infty} K^{1+\delta} (1+\delta) \right)^{1+\delta} |x|^{1+\delta} \leq \frac{2}{(1 - \kappa^\delta (1+\delta))^{1+\delta}} |x|^{1+\delta}.$$ 

Lemma 6.45. Let $h_v$ be eigenfunction of operator $T_v$ defined in Lemma 5.30. If $0 < \delta \leq 1$ such that $0 < \delta < \alpha$, then

$$(6.46) \quad \nu(\Pi_{T,t}(h_v) \circ \delta_t - 1) \leq D|t|^\delta.$$ 

for some constant $D > 0$.

Proof. Observe that for $\delta$ as above by Proposition 6.42, (6.3) and (6.43) for every $0 < t \leq t_0 \leq 1$ we have

$$|\Pi_{T,t}(h_v)(tx) - 1| \leq |\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)| + |\Pi_{T,0}(h_v)(tx) - 1|$$

$$\leq |t|^\delta \left( C(1 + |x|)^\rho + \frac{2}{1 - \kappa^\delta |x|^\delta} \right).$$

Above and the Lebesgue dominated convergence theorem imply (6.46). \qed

7. Proof of the limit theorem for Birkhoff sums.

The purpose of this section is to give a proof of the limit Theorem 1.24 for Birkhoff sums. We will see that the behavior of $S_n = \sum_{k=0}^{n} X_k$ is strongly related to the asymptotics of the stationary measure $\nu$ at infinity. Before we prove convergence of underlying characteristic functions we write fractional expansions of the eigenvalues $k_v(t)$ when $t$ goes to 0. Later on in view of Levy–Cramer theorem it is enough to justify that the characteristic functions converge pointwise to a continuous function at zero. We shall use radial coordinates in $\mathbb{R}^d$ i.e. every point can be expressed as $tv$ where $t > 0$ and $v \in \mathbb{S}^{d-1}$.

The lemmas are analogues of those in (3) and the proofs follow the scheme there with the function $h_v$ playing the role of $\tilde{h}_v$ there. Therefore, we have omitted as much as we could. However, some arguments here contain slight modifications or are just simpler since the group $G$ in (3) is more general than $\mathbb{R}^+ \times K$. Therefore, we include some proofs here for reader’s convenience.

Proof of Theorem 1.24. It is a direct consequence of Lemma 7.3 (Case $0 < \alpha < 1$), Lemma 7.7 (Case $\alpha = 1$), Lemma 7.16 (Case $1 < \alpha < 2$), Lemma 7.21 (Case $\alpha = 2$) and Theorem 7.32 (nondegeneracy of the limit variable $C_\alpha(v)$ for $\alpha \in (0,2]$ and $v \in \mathbb{S}^{d-1}$). \qed
7.1. Case $0 < \alpha < 1$.

**Lemma 7.1.** Assume that $0 < \alpha < 1$. Then for every $v \in S^{d-1}$

\[
\lim_{t \to 0} \frac{k_v(t) - 1}{|t|^{\alpha}} = C_\alpha(v)
\]

where

\[
C_\alpha(v) = \int_{\mathbb{R}^d} \left( e^{i(v,x)} - 1 \right) h_v(x) \Lambda(dx).
\]

Moreover, $C_\alpha(tv) = t^\alpha C_\alpha(v)$ for $t > 0$.

**Lemma 7.3.** Assume that $\alpha < 1$. Let $\Delta^n_\alpha$ be the characteristic function of the random variable $n^{-\frac{1}{\alpha}} S^n_\alpha$ and let $t_n = t n^{-\frac{1}{\alpha}}$ for $n \in \mathbb{N}$, then

\[
\lim_{n \to \infty} \Delta^n_\alpha(t_n) = \Upsilon_\alpha(tv),
\]

where $\Upsilon_\alpha(tv) = e^{tc_\alpha(v)}$.

**Proof of Lemmas (7.1) and (7.3).** For the proof we refer to (3). See also the proof of Lemmas 7.7, 7.18 and 7.21. \( \square \)

7.2. Case $\alpha = 1$.

**Lemma 7.5.** Assume that $\alpha = 1$ and $\xi(t) = \int_{\mathbb{R}^d} \frac{t}{1 + |tx|^2} \nu(dx)$. Then for every $v \in S^{d-1}$

\[
\lim_{t \to 0} \frac{k_v(t) - 1 - i(v, \xi(t))}{|t|} = C_1(v),
\]

where

\[
C_1(v) = \int_{\mathbb{R}^d} \left( e^{i(v,x)} - 1 \right) h_v(x) \cdot \frac{i(v,x)}{1 + |x|^2} \Lambda(dx).
\]

**Lemma 7.7.** Assume that $\alpha = 1$. Let $\Delta^n_\alpha$ be the characteristic function of the random variable $n^{-1} S^n_\alpha - n \xi(n^{-1})$ and define $t_n = t n^{-1}$ for $n \in \mathbb{N}$, then

\[
\lim_{n \to \infty} \Delta^n_1(t_n) = \Upsilon_1(tv),
\]

where $\Upsilon_1(tv) = e^{ic_1(v)+it(v,\tau(t))}$ and $\tau(t) = \int_{\mathbb{R}^d} \left( \frac{t}{1 + |tx|^2} - \frac{t}{1 + |x|^2} \right) \Lambda(dx)$.

**Proof.** In order to prove (7.8) notice that by Lemma 5.6 and Proposition 5.5 we have

\[
\Delta^n_\alpha(tv) = E \left( e^{itn(v,S^n_\alpha - n \xi(n^{-1})))} \right) = e^{-itn(v,\xi(n^{-1}))} E \left( e^{itn(v,S^n_\alpha)} \right)
\]

\[
= e^{-itn(v,\xi(n^{-1}))} \cdot (k^n_\alpha(t_n) (\Pi_{P,t_n}(1)) (x) + (Q^n_{P,t_n}(1))(x)).
\]

Proposition 5.5 ensures that $\lim_{n \to \infty} \|Q^n_{P,t_n} \|_{s_{\rho,\epsilon,\lambda}} = 0$, because $\|Q_{P,t} \|_{s_{\rho,\epsilon,\lambda}} < 1$.

Observe that

\[
\lim_{n \to \infty} e^{-itn(v,\xi(n^{-1}))} k^n_\alpha(t_n) =
\]

\[
= \lim_{n \to \infty} \left( 1 + e^{-i(tn,\xi(n^{-1}))} k_\alpha(t_n) - 1 \right) e^{-itn(v,\xi(n^{-1}))} k_\alpha(t_n) - 1
\]

\[
= \lim_{n \to \infty} e^{i(tn,v,\xi(n^{-1}))} k_\alpha(t_n) - 1 = e^{iC_1(v)+it(v,\tau(t))}.
\]
Indeed, 

\[ \lim_{n \to \infty} \left( t e^{-it\langle v, \xi(n^{-1}) \rangle} k_v(t_n) - 1 \right) = \lim_{n \to \infty} \left( te^{-it\langle v, \xi(n^{-1}) \rangle} \frac{k_v(t_n) - 1 - i\langle v, \xi(t_n) \rangle}{t_n} \right) \]

\[ + \lim_{n \to \infty} \left( ne^{-it\langle v, \xi(n^{-1}) \rangle} (1 + i\langle v, \xi(t_n) \rangle) - n \right) = tC_1(v) + \lim_{n \to \infty} \left( n \cdot (1 - it\langle v, \xi(n^{-1}) \rangle + O (t^2\langle v, \xi(n^{-1}) \rangle)^2) \cdot (1 + i\langle v, \xi(t_n) \rangle) - n \right) \]

\[ = tC_1(v) + \lim_{n \to \infty} \left( \langle v, \xi(t_n) \rangle - \langle v, \xi(n^{-1}) \rangle \right) \]

Notice that the limit in (7.10) is equal to 0 by Lemma 7.7. By (7.12) we have

\[ \lim_{n \to \infty} \left( \langle v, \xi(t_n) \rangle - \langle v, \xi(n^{-1}) \rangle \right) = \lim_{n \to \infty} \left( \langle v, n^{-1} x \rangle - \langle v, n^{-1} x \rangle \right) \nu(dx) = it\langle v, \tau(t) \rangle, \]

hence the limit in (7.9) is equal to \( tC_1(v) + it\langle v, \tau(t) \rangle \) and the (7.8) follows. Finally to prove continuity of \( \mathcal{Y}_1 \) at zero, it is enough to observe that for \( |x| < 1 \),

\[ \left| \left( e^{i\langle v, x \rangle} - 1 \right) h_v(x) - \frac{i\langle v, x \rangle}{1 + |x|^2} \right| \leq C|x|^{1+\delta}, \]

for any \( 0 < \delta < 1 \), and some \( C > 0 \) independent of \( v \in S^{d-1} \). This completes the proof of the Lemma. 

\[ \square \]

**Lemma 7.11.** For every \( t \in \mathbb{R} \) and \( v \in S^{d-1} \)

\[ \lim_{s \to 0} \frac{1}{s} \int_{\mathbb{R}^d} \left( \frac{\langle v, stx \rangle}{1 + |stx|^2} - \frac{\langle v, stx \rangle}{1 + |sx|^2} \right) \nu(dx) = t\langle v, \tau(t) \rangle, \]

where

\[ \tau(t) = \int_{\mathbb{R}^d} \left( \frac{x}{1 + |tx|^2} - \frac{x}{1 + |x|^2} \right) \Lambda(dx). \]

Moreover, there exists a constant \( C > 0 \) such that

\[ |t\langle v, \tau(t) \rangle| \leq \begin{cases} C|t|\log |t|, & \text{for } |t| < \frac{1}{2} \\ C|t|, & \text{for } |t| \geq \frac{1}{2} \end{cases}. \]

**Proof.** For the proof we refer to (3). \( \square \)

**7.3. Case** \( 1 < \alpha < 2 \).

**Lemma 7.14.** Assume that \( 1 < \alpha < 2 \) and \( m = \int_{\mathbb{R}^d} xv(dx) \). Then for every \( v \in S^{d-1} \)

\[ \lim_{t \to 0} \frac{k_v(t) - 1 - i\langle v, tm \rangle}{|t|^{\alpha}} = C_\alpha(v), \]

where

\[ C_\alpha(v) = \int_{\mathbb{R}^d} \left( \left( e^{i\langle v, x \rangle} - 1 \right) h_v(x) - i\langle v, x \rangle \right) \Lambda(dx). \]

Moreover, \( C_\alpha(tv) = t^\alpha C_\alpha(v) \) for \( t > 0 \).
Lemma 7.16. Assume that $1 < \alpha < 2$. Let $\Delta_n^\alpha$ be the characteristic function of the random variable $n^{-\frac{1}{\alpha}}(S_n^\alpha - nm)$ and define $t_n = tn^{-\frac{1}{\alpha}}$ for $n \in \mathbb{N}$, then

$$\lim_{n \to \infty} \Delta_n^\alpha(t_n) = \Upsilon_\alpha(t),$$

where $\Upsilon_\alpha(t) = e^{tC_\alpha(v)}$.

Proof of Lemma (7.14) and (7.16). For the proof we refer to (3). See also the proof of Lemmas 7.7, 7.18 and 7.21.

7.4. Case $\alpha = 2$.

Lemma 7.18. Assume that $\alpha = 2$ and $m = \int_{\mathbb{R}^d} x\nu(dx)$. Then for every $v \in \mathbb{R}^{d-1}$

$$\lim_{t \to 0} \frac{k_n(t) - 1 - i\langle v, tm \rangle}{t^2|\log t|} = 2C_2(v),$$

where

$$C_2(v) = -\frac{1}{4} \int_{\mathbb{R}^{d-1}} ((v, w)^2 + 2\langle v, w \rangle \langle v, \mathbb{E}(w) \rangle) \sigma_\Lambda(dw),$$

and $\varphi(x) = \sum_{k=1}^{\infty} \psi_k \circ \ldots \circ \psi_1(x)$. Moreover, $C_2(tv) = t^2C_2(v)$ for $t > 0$.

Proof. Let us write,

$$\int_{\mathbb{R}^d} (e^{it<v,x>-1})\Pi_{T,t}(h_v)(tx)\nu(dx) =$$

$$\int_{\mathbb{R}^d} (e^{it<v,x>-1}) \cdot (\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)) \nu(dx)$$

$$+ \int_{\mathbb{R}^d} (e^{it<v,x>-1}) \cdot \mathbb{E}(e^{it<v,\varphi(x)>} - 1 - it\langle v, \varphi(x) \rangle) \nu(dx)$$

$$+ \int_{\mathbb{R}^d} (e^{it<v,x>-1}) \cdot \mathbb{E}(it\langle v, \varphi(x) \rangle) \nu(dx)$$

$$+ \int_{\mathbb{R}^d} (e^{it<v,x>-1} - 1 - it\langle v, x \rangle) \nu(dx) + it\langle v, m \rangle$$

where $\varphi(x) = \sum_{k=1}^{\infty} \psi_k \circ \ldots \circ \psi_1(x)$. In view of Lemma 6.40 the first term divided by $t^2|\log t|$ goes to 0. By inequalities (5.9) and (6.44) it is easy to see that the function $f_v(x) = (e^{it<v,x>-1}) \mathbb{E}(e^{it<v,\varphi(x)>} - 1 - it\langle v, \varphi(x) \rangle) \in \mathcal{F}$. Hence by (1.14) the second one divided by $t^2$ has a finite limit. So divided by $t^2|\log t|$ goes to 0. To handle with the third and fourth expression we will use Lemma 7.25. Notice, that

$$\lim_{x \to 0} \frac{(e^{it<v,x>-1}) \cdot \mathbb{E}(i\langle v, \varphi(x) \rangle)}{\langle v, x \rangle \langle v, \mathbb{E}(\varphi(x)) \rangle} = -1,$$

$$\lim_{x \to 0} \frac{e^{it<v,x>-1} - 1 - i\langle v, x \rangle}{\langle v, x \rangle^2} = -\frac{1}{2}.$$

All the assumptions of Lemma 7.25 are satisfied, thus

$$\lim_{t \to \infty} \frac{1}{t^2|\log t|} \int_{\mathbb{R}^d} (e^{it<v,x>-1}) \cdot \mathbb{E}(it\langle v, \varphi(x) \rangle) \nu(dx) =$$

$$= - \int_{\mathbb{R}^{d-1}} \langle v, w \rangle \langle v, \mathbb{E}(\varphi(w)) \rangle \sigma_\Lambda(dw),$$
and
\[
\lim_{t \to \infty} \frac{1}{t^2 |\log t|} \int_{\mathbb{R}^d} \left( e^{it\langle v, x \rangle} - 1 - it\langle v, x \rangle \right) \nu(dx) = -\frac{1}{2} \int_{\mathbb{R}^{d-1}} \langle v, w \rangle^2 \sigma_\Lambda(dw),
\]
hence,
\[
(7.20) \quad \lim_{t \to \infty} \frac{1}{t^2 |\log t|} \left( \int_{\mathbb{R}^d} \left( e^{it\langle v, x \rangle} - 1 \right) \Pi_{T,t}(h)(tx) \nu(dx) - i\langle v, tm \rangle \right) = 2C_2(v).
\]
Applying formula (6.35), (7.20) and Lemma 6.45, we write
\[
\lim_{t \to 0} \frac{k_v(t) - 1 - i\langle v, tm \rangle}{t^2 |\log t|}
\]
\[
= \lim_{t \to 0} \frac{\nu \left( \left( e^{it\langle v, \cdot \rangle} - 1 \right) \cdot (\Pi_{T,t}(h_v) \circ \delta_t) \right) - i\langle v, tm \rangle \nu(\Pi_{T,t}(h_v) \circ \delta_t)}{\nu(\Pi_{T,t}(h_v) \circ \delta_t)t^2 |\log t|}
\]
\[
= \lim_{t \to 0} \left( \frac{\nu \left( \left( e^{it\langle v, \cdot \rangle} - 1 \right) \cdot (\Pi_{T,t}(h_v) \circ \delta_t) \right) - i\langle v, tm \rangle}{\nu(\Pi_{T,t}(h_v) \circ \delta_t)t^2 |\log t|} \right)
\]
\[
+ \lim_{t \to 0} \left( \frac{i \left( 1 - \nu(\Pi_{T,t}(h_v) \circ \delta_t) \right) \langle v, tm \rangle}{\nu(\Pi_{T,t}(h_v) \circ \delta_t)t^2 |\log t|} \right) = 2C_2(v).
\]
This completes the proof of (7.19). □

**Lemma 7.21.** Assume that \( \alpha = 2 \). Let \( \Delta^n_2 \) be the characteristic function of the random variable \((n \log n)^{-\frac{1}{2}} (S_n^x - nm)\) and define \( t_n = t(n \log n)^{-\frac{1}{2}} \) for \( n \in \mathbb{N} \), then
\[
(7.22) \quad \lim_{n \to \infty} \Delta^n_2(tv) = \Upsilon_2(tv),
\]
where \( \Upsilon_2(tv) = e^{itC_2(v)} \).

**Proof.** In order to prove (7.22) notice that
\[
\Delta^n_2(tv) = E \left( e^{i t_n \langle v, S^n_x - nm \rangle} \right) = e^{-i t_n \langle v, m \rangle} E \left( e^{i t_n \langle v, S^n_x \rangle} \right)
\]
\[
= e^{-i t_n \langle v, m \rangle} \cdot \left( k_v(t_n) \left( \Pi_{T,t_n}(1) \right)(x) + (Q_{p,t_n}(1))(x) \right).
\]
By similar argument as in the previous cases
\[
\lim_{n \to \infty} e^{-i t_n \langle v, m \rangle} k_v^n(t_n) = e^{itC_2(v)}.
\]
Indeed,
\[
\lim_{n \to \infty} \left( n \left( e^{-i t_n \langle v, m \rangle} k_v(t_n) - 1 \right) \right) =
\]
\[
= \lim_{n \to \infty} \left( n t_n^2 |\log t_n| e^{-i t_n \langle v, m \rangle} \cdot k_v(t_n) - 1 - i t_n \langle v, m \rangle \right)
\]
\[
+ \lim_{n \to \infty} \left( n e^{-i t_n \langle v, m \rangle} (1 + i t_n \langle v, m \rangle) - n \right)
\]
\[
= \lim_{n \to \infty} n t_n^2 |\log t_n| \cdot 2C_2(v)
\]
\[
+ \lim_{n \to \infty} \left( n t_n^2 \langle v, m \rangle^2 + nO \left( t_n^2 \right) \cdot (1 + i t_n \langle v, m \rangle) \right) = t^2C_2(v).
\]
Notice that
\[
nt_n^2 |\log t_n| = \frac{t^2}{2} \log n \left| \log t - \frac{1}{2} \left( \log n + \log(\log n) \right) \right| \xrightarrow{n \to \infty} \frac{t^2}{2}.
\]
Hence the limit in (7.23) is equal to \( t^2C_2(v) \) and the second one in (7.24) is equal to 0. Finally to prove continuity of \( Y_2 \) at zero, we proceed as in previous cases. It completes the proof of the Lemma.

\[ \text{Lemma 7.25. Suppose we are given Lipschitz maps } \varphi_i : \mathbb{R}^d \rightarrow \mathbb{R}^d \text{ for } i = 1, 2 \text{ and functions } f \text{ and } h \text{ on } \mathbb{R}^d, \text{ such that} \]

- \( t\varphi_i(x) = \varphi_i(tx) \) for any \( t > 0 \) and \( \varphi_i(0) = 0 \) for \( i = 1, 2 \).
- \( h(x) = \langle \varphi_1(x), v_1 \rangle \langle \varphi_2(x), v_2 \rangle \) for some \( v_1, v_2 \in \mathbb{R}^d \).
- \( \lim_{x \to 0} \frac{f(tx)}{h(x)} = C \) for some \( C \in \mathbb{R} \).
- \( |f(x)| \leq D|x|^{1+\eta} \) for some \( D > 0 \) and \( 0 < \eta < 1 \).

Then,

\[
\lim_{t \to 0} \frac{1}{t^2|\log t|} \int_{\mathbb{R}^d} f(tx)\nu(dx) = C \int_{S^{d-1}} h(w)\sigma_\Lambda(dw),
\]

where \( \sigma_\Lambda \) is the measure on the \( S^{d-1} \) defined in (1.17).

**Proof.** Fix \( \beta > 1 \) and define the annulus \( U = \{ x \in \mathbb{R}^d : 1 < |x| \leq 2 \} \). Next observe that for fixed \( \varepsilon > 0 \) there exist \( \delta > 0 \) such that

\[
\left| \frac{f(x)}{h(x)} - C \right| < \varepsilon,
\]

for any \( |x| < \delta \). In view of (1.14) there exists \( A > 0 \) such that for \( 0 < t \leq \frac{1}{A} \)

\[ (7.26) \quad \left| \frac{1}{t^2} \int_{\mathbb{R}^d} 1_U(tx)h(tx)\nu(dx) - \int_U h(x)\Lambda(dx) \right| \leq \varepsilon,
\]

\[ (7.27) \quad \left| \frac{1}{t^2} \int_{\mathbb{R}^d} 1_U(tx)|h(tx)|\nu(dx) - \int_U |h(x)|\Lambda(dx) \right| \leq \varepsilon.
\]

From now, we assume that \( 0 < t < \frac{\delta}{A} \) and consider

\[ (7.28) \quad \int_{\mathbb{R}^d} f(tx)\nu(dx) = \left( \int_{B_1} + \int_{B_2} + \int_{B_3} \right) f(tx)\nu(dx),
\]

where \( B_1 = \{ x \in \mathbb{R}^d : |x| \leq A \}, \ B_2 = \{ x \in \mathbb{R}^d : A < |x| \leq \frac{\delta}{t} \} \) and \( B_3 = \{ x \in \mathbb{R}^d : |x| > \frac{\delta}{t} \} \). Notice first, that

\[
\left| \int_{B_1} f(tx)\nu(dx) \right| \leq (|C| + \varepsilon) \int_{B_1} |h(tx)|\nu(dx) \leq (|C| + \varepsilon)t^2L_\varphi L_{\varphi_2}A^2|v_1||v_2|,
\]

and by (1.14)

\[
\lim_{t \to 0} \frac{1}{t^2} \int_{B_3} f(tx)\nu(dx) = \int_{\{ x \in \mathbb{R}^d : |x| > \delta \}} f(x)\Lambda(dx).
\]

Hence

\[
\lim_{t \to 0} \frac{1}{t^2|\log t|} \left( \int_{B_1} f(tx)\nu(dx) + \int_{B_3} f(tx)\nu(dx) \right) = 0.
\]

We will prove that

\[ (7.29) \quad \lim_{t \to 0} \frac{1}{t^2|\log t|} \int_{B_2} f(tx)\nu(dx) = \frac{C}{\log \beta} \int_U h(x)\Lambda(dx).
\]
We estimate the first expression. For this purpose define $K$ for measure $\Lambda$ to complete the proof. Indeed, in view of our assumptions

\[(7.30) \quad \left| \frac{1}{t^2 |\log t|} \int_{B_2} f(tx)\nu(dx) - \frac{C}{\log \beta} \int_U h(x)\Lambda(dx) \right| \]

\[\leq |C| \left| \frac{1}{t^2 |\log t|} \int_{B_2} h(tx)\nu(dx) - \frac{1}{\log \beta} \int_U h(x)\Lambda(dx) \right| \]

\[+ \frac{\varepsilon}{t^2 |\log t|} \int_{B_2} |h(tx)|\nu(dx).\]

We estimate the first expression. For this purpose define $K = \lfloor \log_\beta \frac{\delta}{\theta \pi} \rfloor - 1$. Let $t_n = A\beta^n$ and annulus $U_n = \{ x \in \mathbb{R}^d : A\beta^n < |x| \leq A\beta^{n+1} \} = t_n U$. Notice that $t_n > A$, therefore applying (7.26) and (7.27) we obtain

\[\left| \frac{1}{t^2 |\log t|} \int_{B_2} h(tx)\nu(dx) - \frac{1}{\log \beta} \int_U h(x)\Lambda(dx) \right| \]

\[\leq \left| \frac{1}{|\log t|} \sum_{n=0}^K t_n^2 \int_{B^d} 1_U(x)h(t_{n-1}x)\nu(dx) - \frac{1}{\log \beta} \int_U h(x)\Lambda(dx) \right| \]

\[+ \frac{(t_{K+1})^2}{|\log t|} \int_{B^d} 1_U(x)h(t_{K+1}x)\nu(dx) \]

\[\leq \left( \frac{K+1}{|\log t|} - \frac{1}{\log \beta} \right) \cdot \int_U |h(x)\Lambda(dx)| \]

\[+ \frac{\varepsilon(K+1)}{|\log t|} \cdot \left( \int_U |h(x)|\Lambda(dx) + \varepsilon \right).\]

Exactly the same arguments as above allows us to estimate the second term in (7.30). Thus, we obtain

\[\frac{\varepsilon}{t^2 |\log t|} \int_{B_2} |h(tx)|\nu(dx) \leq \frac{\varepsilon(K+2)}{|\log t|} \cdot \left( \int_U |h(x)|\Lambda(dx) + \varepsilon \right).\]

Therefore, passing to the limit in (7.30) we have

\[\limsup_{t \to 0} \left| \frac{1}{t^2 |\log t|} \int_{B_2} f(tx)\nu(dx) - \frac{C}{\log \beta} \int_U h(x)\Lambda(dx) \right| \]

\[\leq \frac{\varepsilon}{|\log \beta|} \left( 1 + \varepsilon + \int_U |h(x)|\Lambda(dx) \right).\]

Since $\varepsilon$ can be arbitrary small we obtain (7.29). Now we use polar decomposition for measure $\Lambda$ to complete the proof. Indeed,

\[\frac{1}{\log \beta} \int_U h(x)\Lambda(x) = \frac{1}{\log \beta} \int_1^\beta \int_{S^{d-1}} h(rw)\sigma_\Lambda(dw) \frac{dr}{r^d} \]

\[= \int_1^\beta \int_{S^{d-1}} h(w)\sigma_\Lambda(dw) \frac{dr}{r} = \int_{S^{d-1}} h(w)\sigma_\Lambda(dw).\]
7.5. Nondegeneracy of the limit law. To state a nondegeneracy of the limit variable we need also the following family of affine recursions

\[ W_0 = 0, \]
\[ W_{n+1} = M_{n+1}^* (W_n + v). \]

and their stationary solutions with law \( \eta_v \)

\[ W = \sum_{n=1}^{\infty} M_1^* M_2^* \ldots M_n^* v. \]

In this part we will show following

**Theorem 7.32.** Assume that \( \Lambda \neq 0, \varphi(x) = \sum_{k=1}^{\infty} M_k \ldots M_1 x \) for every \( x \in \mathrm{supp}\Lambda \) and additionally that there are \( w_1, \ldots, w_d \in \mathrm{supp}\sigma_\Lambda \) such that \( \text{span} \mathbb{R}^d \). If \( \mathcal{R}_\alpha(v) = 0 \) for \( 0 < \alpha < 2 \), then \( E (|Wv + v|^\alpha - |Wv|^\alpha) = 0. \)

Notice that with this notation in view of Theorem 7.32

\[ h_v(x) = E (e^{i(v, \varphi(x))}) = \int_{\mathbb{R}^2} e^{i(y, x)} \eta_v(dy). \]

Notice that, for \( 0 < \alpha < 2 \) we have

\[
\mathcal{R}_\alpha(v) = \mathbb{R} \left( \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (e^{i(y, x)} - 1) e^{i(y, x)} \eta_v(dy) \Lambda(dx) \right)
\]

\[
= \mathbb{R} \left( \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{S}^{d-1}} (e^{i(y+tv, tw) - e^{i(y, tw)})} \sigma_\Lambda(dw) \frac{dt}{t^{\alpha+1}} \eta_v(dy) \right)
\]

\[
= \int_{\mathbb{R}^d} \int_0^\infty \int_{\mathbb{S}^{d-1}} (\cos(t\langle y + v, w \rangle) - \cos(t\langle y, w \rangle)) \sigma_\Lambda(dw) \frac{dt}{t^{\alpha+1}} \eta_v(dy),
\]

and

\[ \mathcal{R}_\alpha(v) = C(\alpha) \cdot \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (|\langle y + v, w \rangle|^\alpha - |\langle y, w \rangle|^\alpha) \sigma_\Lambda(dw) \eta_v(dy) \]

\[ = C(\alpha) \cdot \int_{\mathbb{S}^{d-1}} E (|\langle Wv + v, w \rangle|^\alpha - |\langle Wv, w \rangle|^\alpha) \sigma_\Lambda(dw), \]

where

\[ C(\alpha) = \int_0^\infty \frac{\cos t - 1}{t^{\alpha+1}} dt < 0. \]

Indeed, for \( x \in \mathbb{R} \)

\[ \int_0^\infty \frac{\cos(tx) - 1}{t^{\alpha+1}} dt = |x|^\alpha \int_0^\infty \frac{\cos t - 1}{t^{\alpha+1}} dt. \]

Notice also that for \( \alpha = 2 \)

\[ C_2(v) = -\frac{1}{4} \int_{\mathbb{S}^{d-1}} E \left( |\langle Wv + v, w \rangle|^2 - |\langle Wv, w \rangle|^2 \right) \sigma_\Lambda(dw). \]

It is easy to see that \( M \in G \) satisfies assumption of Theorem 1.13 and recursion (7.31) has no fixed points. Then \( E (|Wv + v|^\alpha - |Wv|^\alpha) > 0 \) by Theorem 1.13.

Hence under the assumptions of Theorem 7.32 in view of above \( \mathcal{R}_\alpha(v) < 0 \) for every \( \alpha \in (0, 2] \) and \( v \in \mathbb{S}^{d-1}. \)

To prove Theorem 7.32 we need two Lemmas.
Lemma 7.33. If \( x_1, \ldots, x_d \in \mathbb{R}^d \) are linearly independent, then there exists \( \varepsilon > 0 \) such that for any \( y_1 \in B_\varepsilon(x_1), \ldots, y_d \in B_\varepsilon(x_d) \), the set \{ \( y_1, \ldots, y_d \) \} is linearly independent.

Proof. Suppose for a contradiction that for every \( \varepsilon > 0 \) there exist \( y_1 \in B_\varepsilon(x_1), \ldots, y_d \in B_\varepsilon(x_d) \) such that \( y_1, \ldots, y_d \in \mathbb{R}^d \) are not linearly independent and \( \beta_1, \ldots, \beta_d \in \mathbb{R} \) not all zero such that \( \beta_1 y_1 + \ldots + \beta_d y_d = 0 \). Then \( \beta_1 (y_1 - x_1) + \ldots + \beta_d (y_d - x_d) = -(\beta_1 y_1 + \ldots + \beta_d y_d) \), hence \( \| \beta_1 y_1 + \ldots + \beta_d y_d \| = \| \beta_1 (y_1 - x_1) + \ldots + \beta_d (y_d - x_d) \| \leq \varepsilon \). Let \( \{ a = (a_1, \ldots, a_d) \in \mathbb{R}^d : |a_1| + \ldots + |a_d| = 1 \} \) and \( f : E \to \mathbb{R} \) such that \( f(a_1, \ldots, a_d) = a_1 x_1 + \ldots + a_d x_d \). It is easy to see that \( E \) is compact and \( f \) is continuous. Observe that for every \( n \in \mathbb{N} \) there exist \( a_n \in E \) such that \( f(a_n) < \frac{1}{n} \), hence \( \inf_{a \in E} f(a) = 0 \). Since \( E \) is compact there exist \( a \in E \) such that \( f(a) = 0 \), but it means that \( a_1 x_1 + \ldots + a_d x_d = 0 \) and it is contradiction with fact that \( x_1, \ldots, x_d \in \mathbb{R}^d \) are linearly independent.

\( \square \)

Lemma 7.34. Under the assumptions of Theorem 7.32, there exist the set \{ \( w_1, \ldots, w_d \) \} of vectors spanning \( \mathbb{R}^d \) and an increasing sequence \( (s_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \) converging to \( \alpha \) such that

\[
\lim_{n \to \infty} (1 - \kappa(s_n)) \mathbb{E}(\langle (Wv + v, w_i) \rangle^{s_n}) = 0,
\]

for every \( 1 \leq i \leq d \).

Proof. On one side, \( C_\alpha(v) = 0 \) hence

\[
\int_{S^{d-1}} \mathbb{E}(\langle (Wv + v, w) \rangle^\alpha - \langle (Wv, w) \rangle^\alpha) \sigma_\Lambda(dw) = 0.
\]

On the other side, since \( Wv = M^*(Wv + v) \), we have

\[
0 = \lim_{s \to \alpha} \int_{S^{d-1}} \mathbb{E}(\langle (Wv + v, w) \rangle^\alpha - \langle (Wv, w) \rangle^\alpha) \sigma_\Lambda(dw)
\]

\[
= \lim_{s \to \alpha} \int_{S^{d-1}} \mathbb{E}(\langle (Wv + v, w) \rangle^\alpha - \langle M^*(Wv + v), w \rangle^\alpha) \sigma_\Lambda(dw)
\]

\[
= \lim_{s \to \alpha} \int_{S^{d-1}} (1 - \kappa(s)) \mathbb{E}(\langle (Wv + v, w) \rangle^\alpha) \sigma_\Lambda(dw).
\]

By assumption we know that there are \( x_1, \ldots, x_d \in \text{supp}\sigma_\Lambda \) such that span \( \mathbb{R}^d \). By Lemma 7.33 there exist \( \varepsilon > 0 \) such that for any \( y_1 \in B_\varepsilon(x_1), \ldots, y_d \in B_\varepsilon(x_d) \), the set \{ \( y_1, \ldots, y_d \) \} is linearly independent. Since \( \sigma_\Lambda(S^{d-1} \cap B_\varepsilon(x_i)) > 0 \) for every \( 1 \leq i \leq d \) and

\[
\lim_{s \to \alpha} \int_{S^{d-1} \cap B_\varepsilon(x_i)} (1 - \kappa(s)) \mathbb{E}(\langle (Wv + v, w) \rangle^\alpha) \sigma_\Lambda(dw) = 0,
\]

then there exist an increasing sequence \( (s^i_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \) converging to \( \alpha \) such that

\[
\lim_{n \to \infty} (1 - \kappa(s^i_n)) \mathbb{E}(\langle (Wv + v, w) \rangle^{s^i_n}) = 0,
\]

for \( \sigma_\Lambda \)–almost every \( w \in S^{d-1} \cap B_\varepsilon(x_i) \). Now it is easy to see that we can choose an universal increasing sequence \( (s_n)_{n \in \mathbb{N}} \subseteq \mathbb{R} \) converging to \( \alpha \) and \( w_i \in S^{d-1} \cap B_\varepsilon(x_i) \).
such that
\[
\lim_{n \to \infty} (1 - \kappa(s_n)) \mathbb{E} \left( |\langle Wv + v, w_i \rangle|^{s_n} \right) = 0,
\]
for every \(1 \leq i \leq d\). □

Proof of Theorem (7.32). In view of the previous Lemma we know that there exist an increasing sequence \((s_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}\) converging to \(\alpha\) and vectors \(w_1, \ldots, w_d \in \mathbb{S}^{d-1}\) spanning \(\mathbb{R}^d\) such that
\[
\lim_{n \to \infty} (1 - \kappa(s_n)) \mathbb{E} \left( |\langle Wv + v, w_i \rangle|^{s_n} \right) = 0,
\]
for every \(1 \leq i \leq d\). This implies that
\[
0 = \lim_{n \to \infty} (1 - \kappa(s_n)) \mathbb{E} \left( |Wv + v|^{s_n} \right) = \lim_{n \to \infty} \mathbb{E} \left( |Wv + v|^{s_n} - |Wv|^{s_n} \right)
= \mathbb{E} \left( |Wv + v|^{\alpha} - |Wv|^{\alpha} \right).
\]
and the proof is completed. □

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