Beyond Quantum Mechanics

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In this paper, an alternative formalism for modeling physics is proposed. The motivation for this approach arises from the tension between the countable (discrete) nature of empirical data and the uncountable sets (continuous functions) that form the foundations of modern physical theories. The foundation of this alternative formalism is the set of all base-2 sequences of length \( n \). While this set is countable for finite \( n \), it becomes uncountable in the limit that \( n \) goes to infinity, providing a viable pathway to correspondence with current theories. The mathematical construction necessary to model physics is developed by considering relationships among different base-2 sequences. Upon choosing a reference base-2 sequence, a relational system of numbers can be defined. Based on the properties of these relational numbers, the rules of angular momentum addition in quantum mechanics can be derived from first principles along with an alternative representation of the Clebsch-Gordan coefficients. These results can then be employed to model basic physics such as spin, as well as simple geometric elements such as directed edges. The simultaneous emergence of these modeling tools within this construction give hope that models for both matter and space-time may be supported by a single formalism.

I. INTRODUCTION

A physical measurement inevitably involves interaction. While one cannot say with certainty if interactions in nature are discrete or continuous at the fundamental level, an observable outcome of any interaction is always discrete. Perhaps the best illustration of this is the photoelectric effect, as interpreted by A. Einstein [1]. For this reason, the results of any conceivable physical experiment can be reduced to counting. Given that a space-time event is simply a representation of one of these interactions, the set of all space-time events that one can observe will also be discrete. Following from the work of G. Cantor, the discreteness of these sets implies that their maximum cardinality will be \( \aleph_0 \), or countably infinite [2]. In other words, one can always use the integers to label each space-time event in one’s causal past, regardless of the cosmological model employed. This fact stands in stark contrast with the uncountable sets universally employed by current physical theories, which are based on continuous functions satisfying differential equations.

This tension between the countable nature of empirical data and the uncountable sets that form the foundations of modern physical theories has been well documented, if not fully appreciated by the broader physics community. This lack of appreciation is justifiable given the productivity of these theories over the past century. However, that productivity has slowed considerably over the past several decades, all while new observations continue to highlight their shortcomings.

The quantum revolution of the 1920’s was a direct consequence of the observed discreteness of interactions. However, every formulation of quantum mechanics (QM) was built with the classical Hamiltonian in mind [3]. This approach resulted in a strong dependence of the resulting theory on uncountable sets. While quantum gravity is generally considered to be the final piece of the quantum revolution, there remain significant questions regarding the nature of the wave function in QM [4–9]. This less appreciated use of uncountable sets in physics is where the approach outlined in this paper begins. The advantage of the formalism to be developed here is the ability to reproduce the uncountable nature of the wave function in QM under a particular limit, while also revealing important geometric and deterministic properties in the pre-limit regime. Of course, this is not to imply that QM is wrong. Rather, it is not yet complete.

The work presented in this paper originated from a purely deductive approach to resolving the tension discussed above. Unlike the original approaches to QM, which retained much of the mathematics of classical physics, a fully constructivist approach is taken here. In other words, the set of mathematical tools comprising this formalism will be built up out of simpler mathematical concepts. Of course, this is an immense undertaking, which is likely why the originators of QM chose to retain as much of the old formalism as possible. Due to the overwhelming nature of this task, the result of this paper will be nothing close to a new theory of physics. Rather, it will highlight the basic principles of an alternative formalism that appears to offer some powerful modeling tools. As with any deductive approach in theoretical physics, a good starting point is critical. Perhaps the most intriguing aspect of this formalism is the simplicity of that starting point, which is the set of all base-2 sequences of length \( n \).

II. SEQUENCES

A base-2 sequence of length \( n \) is sometimes called a binary sequence. Naturally, the words “binary sequence” conjure images of microprocessors, computer code, and perhaps even the notion of simulation. This is not the right picture to have in mind. Rather, one should think of a base-2 sequence as a fundamental mathematical ob-
ject, which will become the fundamental physical object within the conceptual model introduced here. Using this conceptual picture, physical observables can then be modeled by considering relationships between base-2 sequences. Relationships between two and three base-2 sequences will be considered, with comments offered about more complicated cases. A related approach to the one presented here can be found in T.N. Palmer’s work [10]. In this work, a relationship between permutations of base-2 sequences and the complex numbers is proposed. From this, a connection is made between the mathematical model of the Bloch sphere in QM and base-2 sequences. While the work presented here differs from that of Palmer’s in several important ways, the motivating observations, as discussed in section I, are similar.

According to information theory, a base-2 sequence is the simplest structure capable of carrying information [11]. The use of the word information can be problematic within the physics community, especially given the conceptually opaque definitions typically offered in the literature [12]. For the purposes of the work presented here, an operational definition of information is sufficient. Put very simply, information is a measure of the reductive power of a statement or message. Given a set of objects, a statement or message containing a large amount of information points to a relatively small subset of those objects, with the most informative statement or message pointing to just a single object within that set. For example, the word “Human” contains less information than “Alice” (a particular human named Alice), while Alice’s genetic sequence (which is base-4) contains the most information. Rather than using English words such as “Human” to encode information, which are sequences using 26 symbols, one can always use base-2 sequences instead. That is, sequences written in a basis greater than two can always be reduced to base-2 through an encoding procedure.

As previously stated, the fundamental mathematical object in this construction is the base-2 sequence. A base-2 sequence is a list comprised of two distinct symbols, where the symbols may be repeated and order matters. Typically, the symbols chosen to represent these basis elements are 0 and 1. While this will be the symbolic representation used here, one should keep in mind that any other choice is equally valid:

\[
\begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
False \\
True \\
True \\
False
\end{pmatrix}
\]

\[
\begin{pmatrix}
False \\
True \\
True \\
False
\end{pmatrix}
= 
\begin{pmatrix}
+ \\
+ \\
- \\
+
\end{pmatrix}
= \cdots
\tag{1}
\]

A base-2 sequence can be of any length, which is denoted as \(n\). For a given length \(n\), there will be \(2^n\) unique sequences. This set of \(2^n\) unique sequences is denoted as \(S^1(n)\), where the superscript indicates that this set only contains individual base-2 sequences, such as the one shown in equation (1). The base-2 basis elements comprising these sequences are actually members of the group \(Z_2\), where the associated group operation is addition modulo two [13]:

\[
0 \oplus 0 = 1 \oplus 1 = 0
\tag{2}
\]

\[
1 \oplus 0 = 0 \oplus 1 = 1
\tag{3}
\]

The \(\oplus\) operation introduced in equations (2-3) will be discussed further in section VIII and will be relevant when calculating the squares of the Clebsch-Gordan coefficients in section IX. Though, there is a second operation which must be introduced and developed before this application of the \(\oplus\) can be explored. This second operation, which will be called a product within this formalism, allows one to study relationships between base-2 sequences.

### III. RELATIONSHIPS

In physics, it is well known that most of the information contained in a composite system does not lie in its subsystems, but actually in the correlations between its subsystems [14]. For this reason, it is useful to introduce the sets \(S^2(n)\) and \(S^3(n)\), which can be thought of as the set of all relationships, or correlations between two and three base-2 sequences respectively. More generally, \(S^d(n)\) is the set of all \(n \times d\) base-2 matrices, which can be constructed from the elements of \(S^1(n)\) through matrix augmentation. For example, an element of the set \(S^2(4)\) is given here:

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 1 \\
1 & 0
\end{pmatrix}
\in S^2(n = 4)
\tag{4}
\]

Individual elements of these sets are denoted as \(s^d \in S^d(n)\), where \(n\) has been suppressed for notational ease. For example, the symbol \(s^1\) represents a single base-2 sequence like the one shown in equation (1). Using the notation just introduced, an element of \(S^2(n)\) can be constructed using two elements of \(S^1(n)\) like so:

\[
s^1 \otimes s^1 = s^2
\tag{5}
\]

In equation (5), the \(\otimes\) symbol has been used to denote the matrix augmentation operation. Within this formalism, this augmentation operation will be referred to as a product. A more explicit representation of the operation shown in equation (5) is given here, where a particular example of \(s^1\) and \(s^1\) has been chosen:
an approach that is rather appealing from the perspective of which one should come first? If one spends enough time ordering sets. That is, given a set of base-2 sequences, using this distribution of points is related to the issue of this formalism is to treat each base-2 sequence as a physical object, where observables emerge by considering relationships between them. It may be useful to imagine this formalism as a relational mathematical language, which emerges from the foundations of this formalism as a relational mathematical model introduced in this paper. These relational numbers play the role of labels, which can be used to distinguish one base-2 sequence from another. That is, one can think of the foundations of this formalism as a relational mathematical language, which emerges from the set of all base-2 sequences of length n. From these four counts, a relational set of measures can be assigned to each sequence. It will be shown that the measures \( j = \frac{\hat{C} + \hat{D}}{2} \) and \( m = \frac{\hat{C} - \hat{D}}{2} \) share important properties with the quantum numbers for total angular momentum and the z-component of angular momentum, re-
spectively [15]. Moreover, the measure \( j \), which is closely related to the Hamming distance in computer science, is a metric. This means that for any choice of three base-2 sequences, one can be placed at each of the vertices of a triangle, where the measure \( j \) defines the length of the edge connecting two vertices. This feature endows this formalism with important geometric properties, which will be discussed further in section VI.

A complete set of measures allows one to determine the number of times each basis element appears within a particular sequence. To make the measures \( j \) and \( m \) complete, the measures \( g = \frac{\hat{A} + \hat{B}}{2} \) and \( l = \frac{\hat{A} - \hat{B}}{2} \), which do not yet have established physical analogues, must be included. Thus, the complete set of measures for a particular base-4 sequence is as follows:

\[
\begin{align*}
  j &= \frac{\hat{C} + \hat{D}}{2} \quad (9) \\
  m &= \frac{\hat{C} - \hat{D}}{2} \quad (10) \\
  g &= \frac{\hat{A} + \hat{B}}{2} \quad (11) \\
  l &= \frac{\hat{A} - \hat{B}}{2} \quad (12) \\
  -j &\leq m \leq j \\
  -g &\leq l \leq g \\
\end{align*}
\]

As with the counts, these measures can be used to distinguish one sequence, or set of sequences, from another. In the case of a base-4 sequence, this can be accomplished notationally by first defining the subset of \( S^2(n) \) containing all base-4 sequences with a particular set of measures like so: \( S^2(j, m, g, l) \subset S^2(n) \). It should be noted that \( n = 2j + 2g = A + B + \hat{C} + \hat{D} \), making explicit mention of \( n \) unnecessary. An element of the subset \( S^2(j, m, g, l) \), which is a base-4 sequence, can then be denoted by including subscripts like so: \( s_{j,m,g,l}^2 \in S^2(j, m, g, l) \). When these measures are used to label a base-4 sequence, as in \( s_{j,m,g,l}^2 \) no particular reference sequence has been singled out. That is, two base-4 sequences with some measures may not share the same reference sequence. On the other hand, when these same measures are used to label a base-2 sequence, one inherently assumes the presence of a particular reference sequence like so: \( s_{j,m,g,l}^4 \in S^2(j, m, g, l) \). When these measures are used to label a base-2 sequence \( s_{j,m,g,l}^1 \), the choice of a particular reference sequence is implied.

V. NON-DETERMINISM

Everything that has been introduced thus far falls within the realm of classical information theory, which is a deterministic framework. To be promising as a tool for modeling physics, this formalism must account for the non-determinism observed in real physical processes. This is accomplished by forcing the user of this formalism to occupy the frame of the reference sequence. That is, the user of this formalism is taken to be an element of the set \( S^4(n) \). By taking this step, the information one may have about another base-2 sequence is limited to relational measures, such as the ones introduced in section IV. In other words, the information associated with the ordering of basis elements is hidden. By specifying a set of four measures, one can only reduce the set of all base-2 sequences to a subset, rather than an individual element of that set. In this way, an epistemic interpretation of the non-deterministic nature of real physical processes emerges within this formalism [16]. However, a decisive statement on this issue cannot be made until a complete description of the wave function is found using this formalism [17]. This issue will be explored further in section VII. Returning to the example offered at the beginning of section II, the step taken in this section is akin to limiting one’s information about Alice to her name, rather than revealing Alice’s genetic sequence. Of course, there is more than one Alice in the world, as there will be more than one base-2 sequence with a particular set measures.

In this paper, the physical interpretation of the product \( s_{j,m,g,l}^0 \otimes s_{j,m,g,l}^1 = s_{j,m,g,l}^2 \) is taken to be the outcome of an experiment, observation, or a measurement. One should read the expression \( s_{j,m,g,l}^0 \otimes s_{j,m,g,l}^1 = s_{j,m,g,l}^2 \) as follows: the sequence to the left of the \( \otimes \) symbol “looks” at the sequence to the right and “sees” the measures \( j, m, g, \) and \( l \). The orientation of this product is important due to the asymmetry of the \( C = 10 \) and \( D = 01 \) basis elements under the commutation operation. Under this operation, the counts \( \hat{C} \) and \( \hat{D} \) are exchanged, implying the measure \( m \) must change sign according to equation (10). That is, the relationship described by the product \( s_{j,m,g,l}^0 \otimes s_{j,m,g,l}^1 = s_{j,m,g,l}^2 \) will differ depending on the definition
of the reference sequence within the relationship. This implies that any graphical representation of this product must take the form of a directed edge. By convention, the reference sequence is placed to the left of the \( \otimes \) symbol. If, for any reason, one must deviate from this convention, then the sign of \( m \) must change and the \( * \) symbol is then used to track this deviation: \( s_0^1 \otimes s_j^1 \rightarrow s_j^{1-m,g,l} \otimes s_0^1 \). With this in hand, a partial ordering on the points shown Figure 1 can be established (Figure 2).

An interesting case of the product \( s_0^1 \otimes s_j^1 = s_j^{2,m,g,l} \) happens when \( j = m = 0 \). This occurs when the sequence to the right of the \( \otimes \) symbol is also the reference sequence. That is, the product \( s_0^1 \otimes s_0^0,g,l = s_0^0,g,l \) corresponds to the case in which the reference sequence “looks” at itself. This is the only case in which \( j = 0 \). The physical interpretation offered for the measure \( j \) in section IV was the quantum number for total angular momentum. In section VI, the case for this interpretation will be strengthened. Under this interpretation, the special product \( s_0^1 \otimes s_0^1,g,l = s_0^0,g,l \) corresponds to a scalar field.

The measures introduced in section IV should be thought of as quantum numbers. In other words, they are just labels that can be used to distinguish one object, or base-2 sequence, from another within this formalism. As noted, this labeling scheme is degenerate, which is just to say that more than one object will have the same quantum numbers. From this picture, an interesting questions arises. Given two objects with the labels \( j_1, m_1, g_1, l_1 \) and \( j_2, m_2, g_2, l_2 \), as determined by a common reference sequence, what set of labels describes their relationship? As will be shown in section VI, the answer to this question contains the rules for angular momentum addition in QM. Given the simplicity of the derivation, this may seem like a trivial result. However, one should remind themselves of the starting point of this deductive approach, which was simply the set of all base-2 sequences. In particular, the formalism developed thus far should be viewed as a careful study of the set \( S^2(n) \) from the perspective of an element of that set, otherwise known as the reference sequence.

VI. ANGULAR MOMENTUM ADDITION

In this section, the measures associated with the relationship between two elements of \( S^2(n) \), which are denoted as \( s_{j_1,m_1,g_1,l_1}^2 \) and \( s_{j_2,m_2,g_2,l_2}^2 \), are obtained. Because the given measures provide no information regarding the ordering of the underlying basis elements, this relationship will be non-deterministic. However, if it is required that both base-4 sequences share a common reference sequence \( s_0^1 \), one can significantly reduce the uncertainty regarding this relationship. Thus, the relationships of interest involve three base-2 sequences, or one base-8 sequence, which are written as \( s_{j_1,m_1,g_1,l_1}^1 \otimes s_{j_2,m_2,g_2,l_2}^1 \). Note that the sign of \( m_1 \) for \( s_{j_1,m_1,g_1,l_1}^1 \) and \( s_{j_1,m_1,g_1,l_1}^2 \) differs due to the different orientations of the reference sequence. The objective is to define the measures between the base-2 sequences \( s_{j_1,m_1,g_1,l_1}^1 \) and \( s_{j_2,m_2,g_2,l_2}^1 \), which are denoted using the capitalized symbols \( J, M, G, \) and \( L \). A graphic representation of this three point correlation among base-2 sequences is offered in Figure 3. By simple algebraic arguments (see appendix A), one can prove the following relationships between the measures \( j_1, m_1, g_1, l_1, j_2, m_2, g_2, \) and \( l_2 \) and the measures \( J, M, G, \) and \( L \), where it is assumed that \( n \geq 2j_1 + 2j_2 \):

\[
n = 2(j_1 + g_1) = 2(j_2 + g_2)
\]

\[
M = m_1 + m_2 = l_2 - l_1
\]

\[
L = l_1 + m_2 = l_2 - m_1
\]

\[
|j_1 - j_2| \leq J \leq j_1 + j_2
\]

\[
\frac{n}{2} - j_1 - j_2 \leq G \leq \frac{n}{2} - |j_1 - j_2|
\]

Equations (13), (16), and (18) are the rules for angular momentum addition in QM [18]. While a complete description of entanglement within this model would require a definition of space-time, equation (16) represents a critical first step. The implication of this equation is that given the measures \( M \) and \( m_1 \), one can infer \( m_2 \).
without performing an additional experiment. This is an important feature of entanglement experiments and constitutes a conservation law within this formalism. More generally, equations (16) and (17) represent deterministic relationships between two base-2 sequences, provided their measures have been defined with respect to a common reference sequence. The emergence of these deterministic relationships from an otherwise random selection of base-2 sequences is striking. For example, this implies that classical computations can be performed using a set of random base-2 sequences, so long as they are ordered relationally.

On the other hand, the non-deterministic nature of the measure \( J \) highlights the manner in which this formalism can be used to model non-deterministic physical systems. The fact that \( J \) is non-deterministic, yet also qualifies as a metric, gives rise to the possibility of modeling quantum geometry. The base-4 and base-8 sequences introduced thus far can be used to model simple geometric elements, which take the form of directed graphs. The rules obeyed by these graphs include the rules for angular momentum addition. More complicated graphs will be considered in future work. However, geometric elements involving four or more base-2 sequences will be far less trivial than the cases introduced thus far. For example, given a set of four randomly chosen base-2 sequences, one will not typically be able to construct a tetrahedron using \( j \) as the distance measure. That is, unless one does not require the angles of each face of the tetrahedron to sum to \( \pi \). In this way, the formalism introduced here shares many similarities with other approaches to quantum geometry [19–23].

VII. MODELING A SPIN \( \frac{1}{2} \) SYSTEM

Simply introducing non-determinism is insufficient to claim a relationship with QM. There are many important properties of quantum systems, such as interference, that must be present within this formalism to justify further interest [24]. To make progress along these lines, one must assign a conceptual meaning to the relationships between base-2 sequences. As stated in section V, these relationships are interpreted as measurements. Under this interpretation, simple quantum states can be thought of as ensembles of base-4 sequences, each labeled using the four relational measures previously defined [25]. In the model presented in this section, a state with the quantum number \( j = \frac{1}{2} \) can be described by the set \( S^2(j = \frac{1}{2}, m, g, l) \). Within this ensemble, there are two possible values for \( m \) according to equations (9) and (10), which can be taken to be the two possible states of a qubit, for example. The degrees of freedom associated with \( g \) and \( l \) can then be used to vary the composition of the ensemble, allowing one to control the probability of obtaining a particular spin state. In the case of a spin \( \frac{1}{2} \) system, an arbitrary ensemble can be constructed with the appropriate choices of \( g, l_+, \) and \( l_- \), where \( l_+ \) and \( l_- \) each represent a unique set of \( l \) values constrained by equation (14):

\[
\sum_{l \in l_+} S^2\left(\frac{1}{2}, +\frac{1}{2}, g_0, l\right) + \sum_{l \in l_-} S^2\left(\frac{1}{2}, -\frac{1}{2}, g_0, l\right)
\]

Using the ensemble in equation (20) to model quantum states requires one to consider two distinct epistemic issues. First, the principle of superposition is introduced, which arises by temporarily restricting the knowledge one has about the measures associated with a particular physical system. For example, the spin state of a free electron could either be up or down before an experiment, where the quantum state gives the probability of each measurement outcome. This restriction in knowledge is of course temporary. That is, one can always perform an experiment to gain this knowledge, which is modeled in QM by updating the wave function, or as updating equation (20) in the model presented here. The second issue is a direct consequence of the step outlined at the beginning of section V, which is the fundamental epistemic issue within this formalism. Using equation (20) to calculate the probability of obtaining a particular spin state requires one to count the number of elements of \( S^2(n) \) associated with each spin state within the chosen ensemble. Because the measures \( j, m, g, \) and \( l \) are complete, one can always use them to determine the counts \( A, B, C, \) and \( D \). Within the field of combinatorics, these counts enable one to determine the cardinality, or size of the set \( S^2(j, m, g, l) \). This is done using the following tool [26]:

\[
\Phi(j, m, g, l) = \frac{n!}{A!B!C!D!}
\]

Where \( n = 2j + 2g \) is the length of the underlying sequences and equations (9-10) and (11-12) have been used to determine the counts \( A, B, C, \) and \( D \). Equation (21) represents an important step in this formalism, both mathematically and physically. Mathematically, it represents an efficient means of counting the number of elements within a given ensemble. Physically, it will play the central role in defining probabilities. Generally, \( \Phi \) will be referred to as the counting function and can be used to determine the cardinality of a set of sequences, regardless of the number of basis elements.

For a more concrete picture, one can imagine an urn filled with red and blue balls with some value of \( l \) written on each ball (Figure 4). The red balls represent elements of \( S^2(n) \) with the quantum number \( m = +\frac{1}{2} \), the blue balls represent elements of \( S^2(n) \) with the quantum number \( m = -\frac{1}{2} \), and the content of the urn represents the ensemble given in equation (20). Equation (21) can be used to calculate the relative frequency with which a ball with some chosen color and value of \( l \) occurs within the ensemble:

\[
\alpha_l = \frac{\Phi(\frac{1}{2}, +\frac{1}{2}, g_0, l)}{\sum_{l \in l_+} \Phi(\frac{1}{2}, +\frac{1}{2}, g_0, l) + \sum_{l \in l_-} \Phi(\frac{1}{2}, -\frac{1}{2}, g_0, l)}
\]
The probability of obtaining \( (23) \) will always be rational numbers by construction. The numerical values generated by equations (22) and (23) will always be rational numbers by construction. Importantly, the numerical values generated by equations (22) and (23) will always be rational numbers by construction. The probability of obtaining \( m = +\frac{1}{2} \) or \( m = -\frac{1}{2} \) can be found by summing over these rational numbers:

\[
\alpha = \sum_{i \in \ell_+} \alpha_i
\]

\( \beta = \sum_{i \in \ell_-} \beta_i \)  

For systems with finite total angular momentum \( j \), in the limit that \( n \) goes to infinity, \( g \) also goes to infinity due to the relation \( n = 2j + 2g \). In this limit, the measure \( l \) becomes unbounded according to equation (14), allowing the sets \( \ell_+ \) and \( \ell_- \) to also become infinite. This implies that the sums given in equations (24) and (25) can become infinite sums of rational numbers, where each term in the sum is associated with an element of the set \( \ell_+ \) or \( \ell_- \). It is well known that infinite sums of rationals map to the real number line. Therefore, by manipulating the composition and size of the sets \( \ell_+ \) and \( \ell_- \), the probability of obtaining a base-4 sequence with the measures \( j = \frac{1}{2}, m = +\frac{1}{2}, g = g_0, \) and \( l = -\frac{2}{2} \). In general, the size of the ensemble within this urn will become uncountable in the limit that \( n \) goes to infinity.

VIII. PATHS

To be a promising formalism for modeling physics, a means of incorporating dynamics must be introduced. While a full dynamical model is well beyond the scope of this paper, the purpose of this section is to introduce a general set of tools that may be used to support this future dynamical model. To begin, the notion of an initial sequence and a final sequence is introduced, which are distinguished from one another using the subscripts \( i \) and \( f \) like so: \( s_{j_i,m_i,g_i,l_i} \) and \( s_{j_f,m_f,g_f,l_f} \). Within this formalism, any unique combination of an initial sequence and a final sequence is taken to be a path, where the associated map connecting them represents a transformation. The operation used to perform this transformation is addition modulo two, which is the operation associated with the 0 and 1 representation of the group \( \mathbb{Z}_2 \) and is denoted by the \( \oplus \) symbol. That is, given the proper map, any initial sequence can be mapped to any final sequence like so:

\[
s_{\text{initial}}^2 \oplus s_{\text{map}}^2 = s_{\text{final}}^2 \tag{27}
\]
is made, where \( s_{\text{initial}}^2 = s_{\text{initial}}^2 \frac{j_1}{2} + \frac{j_2}{2} \) and \( s_{\text{final}}^2 = s_{\text{final}}^2 \frac{j_1}{2} - \frac{j_2}{2} \):

\[
\begin{pmatrix}
A \\
C \\
B \\
A \\
\end{pmatrix} \oplus 
\begin{pmatrix}
A \\
C \\
A \\
D \\
\end{pmatrix} = 
\begin{pmatrix}
B \\
A \\
B \\
D \\
\end{pmatrix}
\tag{28}
\]

By restricting the information one has about the initial and final sequence to measures alone, the operation depicted in equations (27) and (28) becomes far more intricate. However, important information can still be extracted about these paths without considering these intricacies. For example, the number of unique paths between two states can be found by forming the product of the counting functions associated with the initial and final states, where the counting function is given in equation (21):

\[
\Phi_{\text{path}} = \Phi(j_i, m_i, g_i, l_i)\Phi(j_f, m_f, g_f, l_f)
\tag{29}
\]

In some cases, two distinct paths being counted in equation (29) will share a common map. This implies that the set of unique maps necessary to execute the transformation between the initial and final state may be smaller than the number of unique paths. A particularly simple example of this phenomenon arises when the initial and final measures are the same. For example, the identity map, which is a sequence containing only \( A \) basis elements, will appear \( \Phi(j, m, g, l) \) times as the map between distinct pairs of initial and final sequences. However, the advantage of the path counting method introduced in equation (29) is that much of the complexity associated with these transformations can be ignored. Within this formalism, this tool can be used to study a special class of paths involving measure preserving transformations. For example, one can easily count the number of paths between an initial and final state such that \( j_i = j_f = \frac{1}{2} \) like so, where \( g \) is fixed by the requirement that all sequences share a common length:

\[
\Phi_{\text{path}} = \sum_{m_i, l_i} \Phi\left(\frac{1}{2}, m_i, g_0, l_i\right) \sum_{m_f, l_f} \Phi\left(\frac{1}{2}, m_f, g_0, l_f\right)
\tag{30}
\]

As will be seen in section IX, the method of path counting introduced in equation (30) will be relevant when calculating the squares of the Clebsch-Gordan coefficients, which are probabilities in QM. However, it will also become apparent within that section that the simplified view of the path counting concept introduced here is insufficient. To complete the formalism associated with paths, one must develop a method of distinguishing one set of paths from another by introducing a new class of measures, while also incorporating interference. This new class of measures must then be studied to determine their relevance as potential modeling tools. In the case of paths connecting base-4 sequences, one will be required to fully develop the formalism associated with base-16 sequences, for example. While the 16 measures necessary to label these paths will be strongly coupled to the base-4 measures associated with the initial and final states, there will be 9 additional degrees of freedom that must be considered. Furthermore, the geometric interpretation of these paths will be non-trivial, as discussed at the end of section VI.

IX. THE CLEBSCH-GORDAN COEFFICIENTS

The squares of the Clebsch-Gordan coefficients (CGC) are probabilities obeyed by real non-deterministic physical systems. They can be derived in several ways within the formalism of QM, but their most technical derivation comes from the concept of tensor decomposition [13]. This conceptually opaque method is often replaced by a recursive approach, usually involving ladder operators [28]. For some, it may be surprising to find that a closed form, or non-recursive expression exists for calculating these probabilities in QM [29, 30]. This closed form expression is often omitted from pedagogical reviews of the subject of angular momentum addition due to the complexity of the expression. However, within the context of the formalism introduced here, the existence of such an expression represents an important opportunity to obtain a new understanding of the nature of probabilities in QM.

The most important thing to understand about this closed form CGC expression is that its only arguments are quantum numbers. That is, the probabilities that are generated using this expression are not elements of an uncountable set, so long as the total angular momentum is finite. Given the motivations of the work presented here, this fact makes the closed form CGC expression an excellent resource. In particular, the phenomena of interference and locality are relevant in physical scenarios where the CGC’s are applicable. Therefore, studying them carefully within the context of this formalism may offer insight into how these phenomena may be modeled and interpreted. To calculate the squares of CGC’s within this formalism, it is necessary to count the number of paths connecting base-8 sequences, such that certain measures remain invariant. An efficient method for counting these paths was introduced in section VIII. However, there are several complicating issues, such as path interference and the requirement of a static background, which may be related to the issue of locality.

The issue of path interference arises when two base-8 sequences with identical angular momentum measures have differing compositions of the base-8 basis elements 010 and 101, which are denoted as \( k_0 \) and \( k_1 \) respectively. The counts \( k_0 \) and \( k_1 \) are related to \( J \) through the equation \( J = j_1 + j_2 - k_0 - k_1 \), where the allowed range for
\( \Phi_{BG} = 2^{\tilde{k}_0 + \tilde{k}_1} \frac{(2j_1 + 2j_2)!}{(2j_1 + 2j_2 - \tilde{k}_0 - \tilde{k}_1)!(\tilde{k}_0 + \tilde{k}_1)!} \) \hspace{2cm} (36)

\[ \Phi_{\text{ind}}(j_1, j_2, J) = \sum_{i=0}^{2J} \frac{(2J + i)!}{i!(2J)!} \] \hspace{2cm} (37)

While the terms in equations (36) and (37) lack a clear first principles justification, each can be interpreted as counting degrees of freedom in a path between two base-8 sequences. Although, each term permits a variety of possible interpretations. Issues of interpretation aside, equations (32-37) can be composed to calculate the squares of the Clebsch-Gordan coefficients (see appendix C):

\[ |CG|^2(j_1, j_2, m_1, m_2, J) = \frac{\Phi_{BG}}{\Phi_{\text{ind}}^{m_1, m_2, J, M}} \] \hspace{2cm} (38)

The expression offered in equation (38) represents a critical resource for improving the understanding of this model. In particular, the path interference exhibited in equation (32) provides an important opportunity to understand interference generally within this formalism.

X. SUMMARY AND DISCUSSION

The motivating observation of the work presented here is the tension between the countability of empirical data and the uncountable sets employed by the theories tasked with modeling that data. What perhaps makes the approach taken here unique is that one need not choose between these two views of nature. By choosing base-2 sequences as the fundamental object of an alternative formalism, the discrete nature of empirical data can be rectified with the apparent relevance of uncountable sets. As the length of these sequences is taken to infinity, the number of unique sequences becomes uncountable, while the number of unique measures used to distinguish them remains discrete.

In section II, the foundations of this formalism were introduced. These foundations rest on the assumed existence of the set \( S^1(n) \), or the set of all base-2 sequences of length \( n \), where the basis elements are the members of the group \( Z_2 \). Two and three point correlations on this set were then introduced, which were denoted as \( S^2(n) \) and \( S^3(n) \) and interpreted as the set of all base-4 and base-8 sequences, respectively. In section IV, the notion of counting basis elements within a particular sequence was introduced. These counts were then used as a relational system of numbers, from which the measures \( j, m, g \), and \( l \) were derived. These measures, which can be used to distinguish elements of \( S^2(n) \) up to permutations, were then

\( \tilde{k}_0 \) and \( \tilde{k}_1 \) can be found in appendix B. Paths connecting base-8 sequences with identical angular momentum measures, but with opposite parity of \( \Delta \tilde{k}_0 = \tilde{k}_0, f - \tilde{k}_0, i \), interfere destructively. The issue of a static background concerns the base-8 basis elements 000 and 111, which do not contribute to the angular momentum measures in any way. A requirement of the expression offered here is that all paths leave these basis elements unchanged. This implies that the counting expression one must define need only consider six of the eight basis elements. That is, one can count the number of base-8 sequences with the measures \( j_1, j_2, m_1, m_2, J, \tilde{k}_0 \), and a particular background, using the following combinatorial expression, which is denoted as \( \Phi(j_1, j_2, m_1, m_2, J, \tilde{k}_0) \):

\[ \frac{(2j_1 + 2j_2 - \tilde{k}_0 - \tilde{k}_1)!}{\tilde{k}_0!\tilde{k}_1!(C_1 - \tilde{k}_1)!(C_2 - \tilde{k}_0)!(D_1 - \tilde{k}_0)!(D_2 - \tilde{k}_1)!} \] \hspace{2cm} (31)

In equation (31), the counts \( C_1, D_1, C_2, \) and \( D_2 \) can be derived from \( m_1, j_1, m_2, j_2 \) using equations (9-10), and \( \tilde{k}_1 \) can be derived from \( \tilde{k}_0, j_1, j_2, \) and \( J \). The term in the numerator of equation (31) is just the effective length of the base-8 sequence after eliminating the 000 and 111 basis elements as degrees of freedom. The six terms in the denominator are the number of times the basis elements 010, 101, 100, 110, 011, and 001 occur in the base-8 sequence, respectively. Using equation (31), an expression counting the number of paths between base-8 sequences with identical backgrounds and angular momentum measures, while also accounting for interference, takes the following form:

\[ \Phi_{\text{sym}} = \sum_{\tilde{k}_0, i, \tilde{k}_0, f} (-1)^{\Delta \tilde{k}_0} \Phi_{\tilde{k}_0, i} \Phi_{\tilde{k}_0, f} \] \hspace{2cm} (32)

The normalization scheme for the expression in equation (32) requires one to count the paths between base-8 sequences with the measures \( j_1, j_2, m_1, m_2, \) and \( J = j_1 + j_2 \), and base-8 sequences with the measures \( j_1, j_2, J, \) and \( M \). This can be accomplished by forming a product out of the following two combinatorial expressions, where \( m'_2 \) can be found using equation (16):

\[ \Phi_{m_1, m_2, J, M} = \Phi(j_1, j_2, m_1, m_2, J, j_1 + j_2, 0) \] \hspace{2cm} (33)

\[ \Phi_{j, M} = \sum_{m'_1} \sum_{\tilde{k}_0} \Phi(j_1, j_2, m'_1, m'_2, J, \tilde{k}_0) \] \hspace{2cm} (34)

\[ \Phi_{m_1, m_2, J, M} = \Phi_{m_1, m_2, J, M} \Phi_{j, M} \] \hspace{2cm} (35)

The remaining terms necessary to recover a closed form expression for the squares of the Clebsch-Gordan coefficients are as follows:

The normalization scheme for the expression in equation (32) requires one to count the paths between base-8 sequences with identical angular momentum measures, but with opposite parity of \( \Delta \tilde{k}_0 = \tilde{k}_0, f - \tilde{k}_0, i \), interfere destructively. The issue of a static background concerns the base-8 basis elements 000 and 111, which do not contribute to the angular momentum measures in any way. A requirement of the expression offered here is that all paths leave these basis elements unchanged. This implies that the counting expression one must define need only consider six of the eight basis elements. That is, one can count the number of base-8 sequences with the measures \( j_1, j_2, m_1, m_2, J, \tilde{k}_0 \), and a particular background, using the following combinatorial expression, which is denoted as \( \Phi(j_1, j_2, m_1, m_2, J, \tilde{k}_0) \):

\[ \frac{(2j_1 + 2j_2 - \tilde{k}_0 - \tilde{k}_1)!}{\tilde{k}_0!\tilde{k}_1!(C_1 - \tilde{k}_1)!(C_2 - \tilde{k}_0)!(D_1 - \tilde{k}_0)!(D_2 - \tilde{k}_1)!} \] \hspace{2cm} (31)

In equation (31), the counts \( C_1, D_1, C_2, \) and \( D_2 \) can be derived from \( m_1, j_1, m_2, j_2 \) using equations (9-10), and \( \tilde{k}_1 \) can be derived from \( \tilde{k}_0, j_1, j_2, \) and \( J \). The term in the numerator of equation (31) is just the effective length of the base-8 sequence after eliminating the 000 and 111 basis elements as degrees of freedom. The six terms in the denominator are the number of times the basis elements 010, 101, 100, 110, 011, and 001 occur in the base-8 sequence, respectively. Using equation (31), an expression counting the number of paths between base-8 sequences with identical backgrounds and angular momentum measures, while also accounting for interference, takes the following form:

\[ \Phi_{\text{sym}} = \sum_{\tilde{k}_0, i, \tilde{k}_0, f} (-1)^{\Delta \tilde{k}_0} \Phi_{\tilde{k}_0, i} \Phi_{\tilde{k}_0, f} \] \hspace{2cm} (32)

The normalization scheme for the expression in equation (32) requires one to count the paths between base-8 sequences with the measures \( j_1, j_2, m_1, m_2, \) and \( J = j_1 + j_2 \), and base-8 sequences with the measures \( j_1, j_2, J, \) and \( M \). This can be accomplished by forming a product out of the following two combinatorial expressions, where \( m'_2 \) can be found using equation (16):

\[ \Phi_{m_1, m_2, J, M} = \Phi(j_1, j_2, m_1, m_2, J, j_1 + j_2, 0) \] \hspace{2cm} (33)

\[ \Phi_{j, M} = \sum_{m'_1} \sum_{\tilde{k}_0} \Phi(j_1, j_2, m'_1, m'_2, J, \tilde{k}_0) \] \hspace{2cm} (34)

\[ \Phi_{m_1, m_2, J, M} = \Phi_{m_1, m_2, J, M} \Phi_{j, M} \] \hspace{2cm} (35)

The normalizat
used to label elements of $S^1(n)$, which is a procedure that requires one to choose a particular element of $S^1(n)$ to serve as a reference sequence. As discussed in section V, restricting the user of this formalism to the frame of the reference sequence has significant epistemological consequences. While this step should raise questions about the compatibility of this formalism with empirical observations made in Bell type tests, a complete description of space-time within this formalism must be developed before a definitive statement on this issue can be made. One should also note that the nature of the hidden information within this formalism, which is stored in the ordering of basis elements, is distinct from other proposed hidden variable models.

By studying three point correlations on the set $S^1(n)$, the rules for angular momentum addition were derived from first principles in section VI. While this result helps justify the applicability of this formalism to real physical systems, the implications of this derivation go beyond angular momentum. On the one hand, the emergence of classical deterministic addition enables one to model conserved quantities within this formalism. That is, quantities which evolve deterministically from one space-time event to another. On the other hand, the non-deterministic nature of the metric $j$ gives rise to hopes for a quantum model of geometry. Of course, non-determinism alone does not imply a connection with quantum systems. In section VII, two general features of quantum systems were explored. These features were the uncountable nature of the wave function and interference. In both cases, exploiting the additional degrees of freedom associated with the measures $g$ and $l$ lead to potential explanations for these features within this formalism. While a robust description of interference within the spin model offered in this paper is lacking, a sound argument was presented that such a description exists.

In section VIII, a brief introduction to the notion of paths was presented. This conceptual step allows one to begin discussing the notion of dynamics within the context of this formalism. In particular, a base-4 sequence, which is taken to be the outcome of an experiment, or a measurement in section V, can be thought of as a path between two base-2 sequences. In other words, a measurement is taken to be a discrete process within this formalism, rather than an object. The mediators of these paths are maps, which are themselves sequences combined with the $\oplus$ operation. While the work presented here falls well short of a full dynamical model, a naive method of path counting was defined, which takes the form of a product of two counting functions as introduced in section VII. This path counting technique provides an efficient means of counting measure preserving transformations between sequences. In section IX, a more advanced version of this counting method was employed. The result of this application was a reproduction of the closed form expression for the squares of the CGC’s, which are probabilities in QM with a close relationship to classical rotations in space. While a great deal of work remains to be done, the closed form expression for the squares of the CGC’s represents an excellent opportunity to connect abstract concepts within this formalism to real physical systems.

We reiterate the most important insights obtained here. First, elementary particles, as states labeled with quantum numbers $j$ and $m$, might just be emergent phenomena for an observer who can observe only the relationships between sequences and not their exact composition. Second, within this formalism, the probabilistic nature of QM is a simple consequence of obscuring information about the underlying sequences. Third, while a frequentist interpretation of probabilities in QM is not obvious at all, it is clear in this formalism that one must square the usual CGC’s to reduce the calculation of probabilities to path counting. Finally, the role of the reference sequence is crucial, which may shed some new light on the role of the observer and/or the vacuum in QM. In particular, an observer is an integral part of the system which is observed. While significant work remains to be done, it is our belief that the conceptual simplicity of this approach, together with its apparent ability to model fundamental physical systems, makes this line of research promising.

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Appendix A: Derivation of the Rules of Angular Momentum Addition

The definition of the base-4 basis elements $A$, $B$, $C$, and $D$, along with our convention of placing the reference sequence to the left of the $\otimes$ symbol imply that the reference sequence contributes a 0 basis element to both the $A$ and $D$ basis elements, while it contributes a 1 basis element to both the $B$ and $C$ basis elements. These observations imply the following relations, where $\hat{0}_0$ and $\hat{1}_0$ represent the number of 0 and 1 basis elements within the reference sequence, as indicated by the subscript:

\[ \hat{A} + \hat{D} = \hat{0}_0 \]  \hspace{1cm} (A1)

\[ \hat{B} + \hat{C} = \hat{1}_0 \]  \hspace{1cm} (A2)

These observations can be extended to the case of two base-4 sequences which share a common base-2 reference sequence. In this case, subscripts are added to the base-4 counts to indicate which base-4 sequence they are associated with:

\[ \hat{A}_1 + \hat{C}_1 = \hat{A}_2 + \hat{D}_2 = \hat{0}_0 \]  \hspace{1cm} (A3)

\[ \hat{B}_1 + \hat{D}_1 = \hat{B}_2 + \hat{C}_2 = \hat{1}_0 \]  \hspace{1cm} (A4)

The left hand side of equations (A3-A4) differ from equations (A1-A2) due to the commutation of the first base-4 sequence necessary to form the base-8 sequence $s_j^{1,1}_1 \otimes s_0^1 \otimes s^1_{2,2,m_2,g_2,l_2}$. Using equations (10) and (12), we substitute $2\hat{m}_1 + \hat{D}_1$ for $\hat{C}_1$ and $2\hat{l}_1 + \hat{B}_1$ for $\hat{A}_1$ in equation (A3) to generate the following:

\[ 2\hat{l}_1 + \hat{B}_1 + 2\hat{m}_1 + \hat{D}_1 = \hat{A}_2 + \hat{D}_2 \]  \hspace{1cm} (A5)

We then use equation (A4) and substitute $\hat{B}_2 + \hat{C}_2$ for $\hat{B}_1 + \hat{D}_1$ to generate:

\[ 2\hat{l}_1 + 2\hat{m}_1 = \hat{A}_2 - \hat{B}_2 + \hat{D}_2 - \hat{C}_2 \]  \hspace{1cm} (A6)

Using equations (10) and (12) once again, we substitute $2\hat{l}_2$ for $\hat{A}_2 - \hat{B}_2$ and $-2\hat{m}_2$ for $\hat{D}_2 - \hat{C}_2$ to generate the following result, which will be used to derive the rules of angular momentum addition:

\[ m_2 + m_1 = l_2 - l_1 \]  \hspace{1cm} (A7)

1. Derivation of $n = 2(j_1 + g_1) = 2(j_2 + g_2)$

Adding equations (9) and (11) yields the following:

\[ j + g = \frac{\hat{C} + \hat{D}}{2} + \frac{\hat{A} + \hat{B}}{2} \]  \hspace{1cm} (A8)

The length of a particular sequence is given by the total number of basis elements contained within that sequence. In the case of a base-4 sequence, that is given by $\hat{A} + \hat{B} + \hat{C} + \hat{D} = n$. Substituting this result into equation (A8) yields:

\[ j + g = \frac{n}{2} \]  \hspace{1cm} (A9)

An obvious consequence of two base-4 sequences sharing a common base-2 reference sequence is that both base-4 sequences must be the same length. This fact, together with equation (A9) yields the result in equation (15):

\[ n = 2(j_1 + g_1) = 2(j_2 + g_2) \]  \hspace{1cm} (A10)

2. Derivation of $M = m_1 + m_2 = l_2 - l_1$

We shall use the base-2 representation of the base-8 basis elements. For example, the number of times the basis element $101$ appears in a particular base-8 sequence will be denoted by $\hat{1}01$. This notation can be extended to all eight base-8 basis elements. The measure $M$ associated with the relationship between base-4 sequences $s_j^{1,1,m_1,g_1,l_1}$ and $s_j^{1,2,m_2,g_2,l_2}$, which is an instance of the base-8 sequence $s_j^{1,1,m_1,g_1,l_1} \otimes s_0^1 \otimes s^1_{2,2,m_2,g_2,l_2}$, can be defined in terms base-8 counts like so:

\[ M = \frac{1}{2} \left( \hat{1}00 + \hat{1}10 + \hat{0}01 - \hat{0}11 \right) \]  \hspace{1cm} (A11)

The number of 0 basis elements within $s_j^{1,1,m_1,g_1,l_1}$ and $s_j^{1,2,m_2,g_2,l_2}$, which we shall denote as $\hat{0}_1$ and $\hat{0}_2$ respectively, permit the following relation between base-4 and base-8 elements:

\[ \hat{0}_0 + \hat{0}10 + \hat{0}11 + \hat{0}01 = \hat{A}_1 + \hat{D}_1 = \hat{0}_1 \]  \hspace{1cm} (A12)

\[ \hat{0}00 + \hat{0}10 + \hat{1}10 + \hat{1}00 = \hat{A}_2 + \hat{C}_2 = \hat{0}_2 \]  \hspace{1cm} (A13)

Using equation (A12), we substitute $\hat{0}00 + \hat{0}10 - \hat{A}_1 - \hat{D}_1$ for $-\hat{0}11 - \hat{0}01$ in equation (A11):

\[ 2M = \hat{1}00 + \hat{1}10 + \hat{0}00 + \hat{0}10 - \hat{A}_1 - \hat{D}_1 \]  \hspace{1cm} (A14)

We then use equation (A13) and substitute $\hat{A}_2 + \hat{C}_2 - \hat{1}10 - \hat{1}00$ for $\hat{0}00 + \hat{0}10$:
Using equation (A1), we then substitute $\tilde{C}_1 - \tilde{D}_2$ for $\tilde{A}_2 - \tilde{A}_1$:

$$2M = \tilde{C}_1 - \tilde{D}_1 + \tilde{C}_2 - \tilde{D}_2$$  \hspace{1cm} (A16)

Finally, using equations (10) and (A7), we obtain equation (16):

$$M = m_1 + m_2 = l_2 - l_1$$  \hspace{1cm} (A17)

\section{Derivation of $L$}

The number of basis elements within $s_{j_2,m_2,g_2,l_2}^1$, which we shall denote as $\hat{1}_2$, permit the following relation between base-4 and base-8 elements:

$$\tilde{111} + \tilde{010} + \tilde{001} = \tilde{B}_2 + \tilde{D}_2 = \tilde{1}_2$$  \hspace{1cm} (A19)

Using equation (A12), we substitute $\hat{A}_1 + \hat{D}_1 - \tilde{011} - \tilde{001}$ for $000 + 010$ in equation (A18):

$$2L = \hat{A}_1 + \hat{D}_1 - \tilde{011} - \tilde{001} - \tilde{111} - \tilde{101}$$  \hspace{1cm} (A20)

We then use equation (A19) and substitute $\tilde{111} + \tilde{010} - \tilde{B}_2 - \tilde{D}_2$ for $-\tilde{011} - \tilde{001}$:

$$2L = \hat{A}_1 + \hat{D}_1 - \tilde{B}_2 - \tilde{D}_2$$  \hspace{1cm} (A21)

Using equation (A3), we then substitute $\tilde{D}_2$ for $\hat{A}_1 + \hat{C}_1 - \hat{A}_2$:

$$2L = \hat{D}_1 - \hat{C}_1 + \hat{A}_2 - \hat{B}_2$$  \hspace{1cm} (A22)

Finally, using equations (10), (12), and (A7), we obtain equation (17):

$$L = l_2 - m_1 = l_1 + m_2$$  \hspace{1cm} (A23)

\section{Derivation of $J$}

When defining the measure $J$ between the sequences $s_{j_1,m_1,g_1,l_1}^1$ and $s_{j_2,m_2,g_2,l_2}^1$, we are asking how many times the base-8 basis elements $\{000, 110, 001, 011\}$ appear in the base-8 sequence $s_{j_1,m_1,g_1,l_1}^1 \otimes s_{j_2,m_2,g_2,l_2}^1$. In terms of base-8 counts, the measure $J$ and its $n$-compliment $G$ are defined as follows:

$$J = \frac{1}{2}(\tilde{100} + \tilde{110} + \tilde{001} + \tilde{011})$$  \hspace{1cm} (A24)

$$G = \frac{1}{2}(\tilde{000} + \tilde{010} + \tilde{111} + \tilde{101})$$  \hspace{1cm} (A25)

As an explicit example, we offer the following element of $S^3(n = 4)$ to keep in mind during this discussion, where we have added brackets around the base-2 elements to make clear which elements $J$ and $G$ count:

$$\left( \begin{array}{c} 0 \ 0 \ 1 \ 1 \\ 1 \ 1 \ 1 \\ 0 \ 0 \ 0 \\ 1 \ 1 \ 1 \end{array} \right)$$  \hspace{1cm} (A26)

This element of $S^3(n = 4)$ is just another possible instance of the product $s_{j_1,m_1,g_1,l_1}^1 \otimes s_{j_2,m_2,g_2,l_2}^1 \otimes s_{j_3,m_3,g_3,l_3}^1$. The difference here is that one of the bracketed base-2 elements from $s_{j_1,m_1,g_1,l_1}^1$ now overlaps one from $s_{j_3,m_3,g_3,l_3}^1$. This implies that the measure $J$ between $s_{j_1,m_1,g_1,l_1}^1$ and $s_{j_2,m_2,g_2,l_2}^1$ is now $J = j_1 + j_2 - 1 = \frac{(2+1)}{2} - 1 = \frac{1}{2}$. In other words, the product $s_{j_1,m_1,g_1,l_1}^1 \otimes s_{j_2,m_2,g_2,l_2}^1$ can result in $J = \frac{3}{2}$ or $J = \frac{1}{2}$. In general, the allowed range of the measure $J$ is as follows, which is equation (18):

$$|j_1 - j_2| \leq J \leq j_1 + j_2$$  \hspace{1cm} (A28)

In the case that $n < 2(j_1 + j_2)$, it is guaranteed that
an overlap will occur. Because $s_{21,m_1,g_1,l_1}^2$ and $s_{22,m_2,g_2,l_2}^2$ share a common reference sequence, the base-4 basis elements that may legally overlap in the resulting base-8 sequence $s_{21,m_1,g_1,l_1}^{1\times} \otimes s_{22,m_2,g_2,l_2}^{1\times}$ are $(A_1, A_2), (B_1, B_2), (C_1, A_2), (D_1, B_2), (A_1, D_2), (B_1, C_2), (C_1, D_2),$ and $(D_1, C_2)$. The $(C_1, D_2)$ and $(D_1, C_2)$ cases correspond to the base-8 basis elements $101$ and $100$ respectively, which are precisely the overlap scenarios we are interested in when considering $J$. Therefore, the maximum number of overlaps that may occur are limited by the sum \[ \min \left\{ D_1, C_2 + \min \left[ \tilde{C}_1, \tilde{D}_2 \right] \right\}. \] Each overlap leads to a reduction in $J$ by one, leading to the following expression for $J_{\text{min}}$:

\[ J_1 + J_2 - \min \left\{ \tilde{D}_1, 2 \right\} - \min \left\{ \tilde{C}_1, \tilde{A}_2 \right\} \quad (A29) \]

Because $(C_1, A_2), (D_1, B_2), (A_1, D_2),$ and $(B_1, C_2)$ correspond to the basis elements $100, 011, 001,$ and $110$ respectively, these overlap scenarios all contribute to $J$. However, if $n < 2(J_1 + J_2)$, then it is guaranteed that either $D_1 > B_2$ or $C_1 > A_2$, or equivalently $B_1 < \tilde{C}_2$ or $A_1 < \tilde{D}_2$. This implies that $(C_1, D_2)$ and or $(D_1, C_2)$ overlap scenarios must occur. This allows us to define the following expression for $J_{\text{max}}$:

\[ J_1 + J_2 - \max \left\{ 0, \tilde{D}_1 - \tilde{B}_2 \right\} - \max \left\{ 0, \tilde{C}_1 - \tilde{A}_2 \right\} \quad (A30) \]

**Appendix B: Derivation of $k_{0,\text{min}}$ and $k_{0,\text{max}}$**

We introduce the measure $\tilde{X} = k_0 + \tilde{k}_1$, where $J = J_1 + J_2 - \tilde{X}$. These relations imply that for fixed $J_1, J_2$, and $J$, summing over the allowed range of $k_0$ is sufficient because $k_1$ is determined by $\tilde{X}$. Following from the derivation in appendix A 4, the number of times the overlaps $(C_1, D_2)$ and $(D_1, C_2)$ occur within an element of $S^4(n)$ correspond to $k_1$ and $\tilde{k}_0$ respectively. Ignoring $\tilde{X}$, we have $k_{1,\text{max}} = \min \left[ \tilde{C}_1, \tilde{D}_2 \right]$. For a given $\tilde{X}$, $k_{0,\text{min}}$ must be equivalent to $\tilde{X} - k_{1,\text{max}}$. This allows us to define $k_{0,\text{min}}$:

\[ k_{0,\text{min}} = \max \left\{ 0, \tilde{X} - \min \left[ \tilde{C}_1, \tilde{D}_2 \right] \right\} \quad (B1) \]

Again ignoring $\tilde{X}$, we have $k_{0,\text{max}} = \min \left[ \tilde{C}_2, \tilde{D}_1 \right]$, which implies the following:

\[ k_{0,\text{max}} = \min \left[ \tilde{X}, \min \left[ \tilde{C}_2, \tilde{D}_1 \right] \right] \quad (B2) \]

**Appendix C: Derivation of Standard Clebsch-Gordan Coefficient Expression**

In this derivation, we will take a step by step approach, beginning with equation (38) and ending with the result found in [29]. In several cases, we make use of combinatorial identities, some of which are well known and some of which were found by the authors. While proofs of these identities are beyond the scope of this derivation, we will offer some justification when appropriate.

$\Phi_{m_1, m_1, J, M}$, which is given in equation (35), can be broken up into two components as follows:

\[ \Phi_{m_1, m_1} = \Phi(j_1, j_2, m_1, m_2, j_1 + j_2, 0) \quad (C1) \]

\[ \Phi_{J, M} = \sum_{m_1} \sum_{k_0} \Phi(j_1, j_2, m_1', m_2', J, k_0) \quad (C2) \]

We can factor $\Phi_{J, M}$ into three combinatorial terms:

\[ \Phi'_{J, M} = \frac{(2j_1 + 2j_2 - k_0 - \tilde{k}_1)!}{(\tilde{C}_1 + \tilde{C}_2 - \tilde{X})! (\tilde{D}_1 + \tilde{D}_2 - \tilde{X})! \tilde{X}!} \quad (C3) \]

\[ \Phi_J = \frac{(2J)!}{(2j_1 - k_0 - \tilde{k})! (2j_2 - k_0 - \tilde{k})!} \quad (C4) \]

\[ \Phi_{\tilde{X}} = \sum_{k_0=0}^{\tilde{X}} \frac{(\tilde{k}_0 + \tilde{k}_1)!}{(\tilde{k}_1)!} = 2^{\tilde{k}_0 + \tilde{k}_1} \quad (C5) \]

\[ \Phi_{J, M} = \Phi'_{J, M} \Phi_J \Phi_{\tilde{X}} \quad (C6) \]

These combinatorial terms count permutations of base-2 or base-3 sequences, with lengths $2j_1 + 2j_2 - k_0 - \tilde{k}_1$, $2J$, and $k_0 + \tilde{k}_1$ respectively. For $\Phi'_{J, M}$, we use the symbols $C_3$, $D_3$, and $X$ for each of the three basis elements, where $X = k_0 + \tilde{k}_1$, $C_3 = \tilde{C}_1 + \tilde{C}_2 - \tilde{X}$, and $D_3 = \tilde{D}_1 + \tilde{D}_2 - \tilde{X}$. For $\Phi_J$, we use the symbols $L$ and $R$, where $L = 2j_1 - \tilde{X}$ and $R = 2j_2 - \tilde{X}$. Finally, for $\Phi_{\tilde{X}}$, we use the symbols $k_0$ and $\tilde{k}_1$. A particular example of each of these three sequences is given below, where we have aligned the three sequences to convey how they can be taken together to produce a single base six sequence:

\[ \begin{pmatrix} C_3 \\ D_3 \\ X \end{pmatrix} \begin{pmatrix} L \\ R \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 100 \\ 001 \\ 110 \\ 010 \\ 101 \\ 010 \end{pmatrix} \quad (C7) \]

In the example offered in equation (C7), the basis el-
elements comprising the sequences to the left of the arrow may be thought of as containing partial information about the base-6 basis elements in the sequence to the right of the arrow. By taking all three sequences together, one can infer the position of each base-6 basis element. By performing all possible permutations of the sequences to the left of the arrow, as well as summing over all combinations of \(k_0\) and \(k_1\), the set of all base six sequences of length \(2j_1 + 2j_2 - \tilde{k}_0 - \tilde{k}_1\) with the measures \(J\) and \(M\) can be generated. Thus, the factored expression \(\Phi_{J,M}\Phi_J\Phi_{\tilde{X}}\) is equivalent to \(\Phi_{J,M, \tilde{X}}\). \(\Phi_{\tilde{X}}\), which is given in equation (37) and expressed as a sum, can be expressed in an alternative form by way of the parallel summation identity:

\[
\Phi_{\tilde{X}} = \sum_{i=0}^{j} \frac{\tilde{X} (J+i)!}{i!J!} = \frac{(j_1 + j_2 - \tilde{X} + 1)!}{(J+1)!\tilde{X}!} \tag{C8}
\]

With these alternative forms of \(\Phi_{J,M}\) and \(\Phi_{\tilde{X}}\), after cancellation and taking a square root, equation (38) becomes:

\[
\sqrt{|CG|^2(j_1,j_2,m_1,m_2,J)} = \sqrt{(2J+1)(2j_1 - \tilde{X})!(2j_2 - \tilde{X})!\tilde{X}!} \times \tag{C9}
\]

\[
\sum_{k_0} (-1)^{\tilde{k}_0} \frac{(\tilde{C}_1 + \tilde{C}_2 - \tilde{X})!(\tilde{D}_1 + \tilde{D}_2 - \tilde{X})!(\tilde{C}_1!\tilde{D}_1!\tilde{C}_2!\tilde{D}_2)!}{k_0!k_1!(\tilde{C}_1 - k_1)!(\tilde{D}_1 - k_0)!(\tilde{C}_2 - k_0)!(\tilde{D}_2 - k_1)!}
\]

All that is left is to convert each count appearing in this expression to an associated measure. We offer some examples below, where we include an effective definition for the parameter \(r\) used in [29]:

\[
\left(2j_1 - \tilde{X}\right) = (j_1 - j_2 + J) \tag{C10}
\]

\[
\left(\tilde{C}_1 + \tilde{C}_2 - \tilde{X}\right) = (J + M) \tag{C11}
\]

\[
\tilde{k}_1 = r \tag{C12}
\]