Infrared Fixed Points in the minimal MOM Scheme

Ryttov, Thomas Aaby

Published in:
Physical Review D

DOI:
10.1103/PhysRevD.89.056001

Publication date:
2014

Citation for published version (APA):
Ryttov, T. (2014). Infrared Fixed Points in the minimal MOM Scheme. Physical Review D, 89, [056001]. https://doi.org/10.1103/PhysRevD.89.056001
Infrared Fixed Points in the minimal MOM Scheme

Thomas A. Ryttov

CP³-Origins & the Danish Institute for Advanced Study DIAS,
University of Southern Denmark, Campusvej 55, DK-5230 Odense M, Denmark.

We analyze the behavior of several renormalization group functions at infrared fixed points for $SU(N)$ gauge theories with fermions in the fundamental and two-indexed representations. This includes the beta function of the gauge coupling, the anomalous dimension of the gauge parameter and the anomalous dimension of the mass. The scheme in which the analysis is performed is the minimal momentum subtraction scheme through third loop order. Due to the fact that scheme dependence is inevitable once the perturbation theory is truncated we compare to previous identical studies done in the minimal subtraction scheme and the modified regularization invariant scheme. We find only mild to moderate scheme dependence.

Preprint: CP³-Origins-2013-043 DNRF 90 & DIAS-2013-43

*Electronic address: ryttov@cp3.dias.sdu.dk
I. INTRODUCTION

The study of the infrared (IR) dynamics of gauge theories has been of considerable interest for the past many decades. Specifically the possibility of whether certain theories may possess the features consistent with conformal symmetry has received much attention [1–23].

Recently two independent investigations studied the evolution of the renormalization group functions from the ultraviolet (UV) to the IR in the $\overline{\text{MS}}$ scheme [24, 25] at three and four loop order. One of the important insights gained by including three and four loops in the calculations was the realization that the anomalous dimension of the mass was lowered significantly compared to the two loop analysis. These results align with virtually all of the lattice simulations studying similar issues. The main theories investigated include: Three colors and a set of flavors in the fundamental representation, two colors and a set of flavors in the fundamental representation, two colors and two flavors in the adjoint representation and three colors and two flavors in the two-indexed symmetric representation. Via [26] one can find an up to date review on all the simulations. On the analytical side more work along these lines can be found in [27, 28].

In supersymmetric theories one has the exact results of Seiberg [29, 30] for the boundary of the conformal window. Therefore since the beta function and anomalous dimension is known to three loop order in the $\overline{\text{DR}}$ scheme a study comparing the exact results with the higher loop results has also appeared [31].

With such results in hand the question of scheme dependence must be asked. Initial steps in this direction were taken in [32–34] where the stability of the higher loop analysis in the $\overline{\text{MS}}$ scheme was investigated. More precisely it was studied by transforming the coupling constant away from its value in the $\overline{\text{MS}}$ scheme using rather generic transformations.

More recently the question of scheme dependence was studied by a comparing the results in the $\overline{\text{MS}}$ scheme with a similar analysis carried out in a scheme known as the modified regularization invariant, $\text{RI}'$, scheme [35]. This enabled a first comparison of various renormalization group functions evaluated at an infrared fixed point in two different and explicit schemes. In the $\text{RI}'$ scheme the analysis was done at the three loop order [36–39].

It is reasonable to say that the work done so far cannot be considered complete. Therefore it is the purpose of this paper to take the investigations one step further by studying the evolution of the gauge coupling and the anomalous dimension towards an IR fixed point in a different scheme known as the minimal MOM scheme [40]. It will provide an important additional check on the size of the scheme dependence of earlier results. Since the method used to estimate the anomalous
dimension is similar to the one used in the RI’ scheme we refer the reader to [35] for more details on the setup of the analysis. The body of this work is devoted to the associated numerical results.

In Section II we introduce our notation while in Section III we discuss specific schemes including the mMOM scheme. We then investigate the IR dynamics and possible fixed points within the mMOM scheme in Section IV. Finally we conclude in Section V. Appendix A provides all the necessary information to do the analysis while Appendix B is a summary of our numerical results.

II. SET UP

We will consider gauge theories with gauge group $G$ and a set of $N_f$ Dirac fermions belonging to a representation $r$ of $G$. We let $d(r)$ denote the dimension of the representation $r$. The adjoint representation is denoted by $G$. The trace normalization factor $T(r)$ and the quadratic Casimir $C_2(r)$ are defined via

\begin{align}
\text{Tr}[T^a_r T^b_r] &= T(r) \delta^{ab} \quad (1) \\
T^a_r T^a_r &= C_2(r) I \quad (2) \\
da, b &= 1, \ldots, d(G) \quad (3)
\end{align}

where $T^a_r$ are the generators of the gauge group in the representation $r$. From these definitions we note the following identity $C_2(r)d(r) = T(r)d(G)$. In Table I we provide the specific values of the group factors for the fundamental and two-indexed representations used throughout this paper.

Of specific interest to us are the beta function of the coupling constant and the beta function of the gauge parameter

\begin{align}
\beta_\alpha(\alpha, \xi) &= \frac{\partial \alpha}{\partial \ln \mu}, \quad \beta_\xi(\alpha, \xi) = \frac{\partial \xi}{\partial \ln \mu} \quad (4)
\end{align}

where $\alpha = \frac{g^2}{4\pi}$ is the gauge coupling and $\xi$ is the gauge parameter. It should be noted that in general both renormalization group functions depend on the gauge coupling as well as the gauge parameter. Finally we wish to consider the anomalous dimension of $\bar{\psi}\psi$

\begin{equation}
\gamma(\alpha, \xi) = -\frac{\partial \ln Z_{\bar{\psi}\psi}}{\partial \ln \mu} \quad (5)
\end{equation}

where $Z_{\bar{\psi}\psi}$ is the associated renormalization constant. It is the behavior of these three renormalization group functions that is of our concern. For a more general discussion of the above renormalization group functions see [35].
III. THE MINIMAL MOM SCHEME

Scheme dependence in the renormalization group functions cannot be avoided. As our way of regularizing the divergent integrals in the Greens functions we shall use dimensional regularization. If we by \( d = 4 - 2\epsilon \) denote the number of space-time dimensions the divergencies will then show up as poles in \( \epsilon \). As a subtraction procedure there are several possibilities from which one can choose.

Throughout many years the standard way has been to subtract only the infinite part or the infinite part plus an additional finite term containing the Euler-Mascheroni constant. These two schemes are known as the minimal subtraction, MS, scheme \([41]\) and the modified minimal subtraction, \( \overline{\text{MS}} \), scheme \([42]\) respectively.

The beta function of the gauge coupling and the anomalous dimension of the mass were both computed to four loop order in \([43, 44]\) within these scheme and both results were confirmed in \([45, 46]\). The computations show explicitly that both renormalization group functions are independent of the gauge parameter. A feature not shared by all schemes. Note that one can study the unification of all MS-type schemes as done in \([47, 48]\).

Of specific interest are also the momentum subtraction, MOM, schemes \([49]\). In the MOM schemes the poles in \( \epsilon \) together with all finite pieces are absorbed into the renormalization constant. However this procedure produces several independent schemes the reason being that there are three different vertices one can use to define the coupling constant \([49]\). These are the gluon-gluon-gluon, quark-quark-gluon and ghost-ghost-gluon vertices. For this class of schemes the renormalization group functions have been computed numerically to three loop order in \([50]\) and explicitly to three loop order in \([51]\). The results were derived in the Landau gauge. Finally in \([52]\) they were also derived at three loop order in any gauge using conversion functions connecting the \( \overline{\text{MS}} \) scheme with each of the three MOM schemes.

Recently it was realized that an approach which preserves the definition of the coupling constant could be achieved within the MOM schemes \([40]\). This construction relies on certain properties of the ghost-ghost-gluon vertex. The scheme is known as the minimal MOM, mMOM, scheme. The fact that there is a single coupling associated with the mMOM scheme makes it attractable for the study of IR fixed points as compared to the general MOM schemes. In the original work \([40]\) the beta function of the coupling constant was derived to four loop order while all the renormalization group functions to three loop order and in any gauge were explicitly calculated in \([53]\). In Appendix A we have provided the specific results that will be used throughout this
work using [53].

IV. FIXED POINTS IN THE MINIMAL MOM SCHEME

Conformal dynamics occur when the beta function of the coupling constant vanishes. When there are multiple couplings conformal dynamics occur when all of the beta functions vanish simultaneously. The case of multiple couplings is the one that resembles the situation encountered here where the beta function of the coupling constant and the beta function of the gauge parameter are coupled. We must guarantee that both beta functions vanish simultaneously

\[ \beta_\alpha(\alpha_0, \xi_0) = 0, \quad \beta_\xi(\alpha_0, \xi_0) = 0 \]  (6)

Hence in order to find the fixed point values \( \alpha_0 \) and \( \xi_0 \) of the gauge coupling and gauge parameter we have to solve two coupled equations that are polynomials in \( \alpha \) and \( \xi \).

Having discussed how to find the fixed points of the theory we finally note that value of the anomalous dimension \( \gamma(\alpha_0, \xi_0) \) is a scheme independent quantity. The value is the same within two different schemes provided both beta functions vanish simultaneously [35].

The two beta functions and the anomalous dimension are written as

\[ \beta_\alpha(\alpha, \xi) = -b_{\alpha,1}\left(\frac{\alpha}{4\pi}\right)^2 - b_{\alpha,2}\left(\frac{\alpha}{4\pi}\right)^3 - b_{\alpha,3}\left(\frac{\alpha}{4\pi}\right)^4 + O(\alpha^5) \]  (7)

\[ \beta_\xi(\alpha, \xi) = \xi\left[-b_{\xi,1}\left(\frac{\alpha}{4\pi}\right)^2 - b_{\xi,2}\left(\frac{\alpha}{4\pi}\right)^3 - b_{\xi,3}\left(\frac{\alpha}{4\pi}\right)^4 + O(\alpha^4)\right] \]  (8)

\[ \gamma(\alpha, \xi) = c_1\left(\frac{\alpha}{4\pi}\right) + c_2\left(\frac{\alpha}{4\pi}\right)^2 + c_3\left(\frac{\alpha}{4\pi}\right)^3 + O(\alpha^4) \]  (9)

They have all been computed explicitly to three loop order in the mMOM scheme in [53]. All of the coefficients are reported in Appendix A.

We are now in a position to study the evolution of the beta functions and the anomalous dimension towards an IR fixed point. First we solve the coupled set of beta functions to find the value of the coupling constant and the gauge parameter at the fixed point. We then evaluate the anomalous dimension at this fixed point. Everything is performed in the mMOM scheme and compared to a similar analysis performed in the \( \overline{\text{MS}} \) scheme [24, 25].

First it is only within a limited region of theory space that theories have the potential to develop an IR fixed point. It is clear that the theory should be asymptotically free and hence we shall only consider a number of flavors for which \( N_f < \frac{11}{4} \frac{C_2(G)}{C_2} \).

As the number of flavors is decreased the critical value of the coupling constant at the fixed point increases. The number of flavors is then bounded from the below by only allowing values
of the coupling constant that are less than order unity since at this point the theory is instead
expected to form the chiral condensate and break chiral symmetry [3–7].

Lastly we note that at the three loop level we are bound to have many solutions to the set of
coupled fixed point equations. Many of these however will be discarded. We will only keep the
solutions in the coupling constant that are positive while we shall allow both positive and negative
solutions of the gauge parameter.

A. Results

At two loops there is a solution for a vanishing value of the coupling constant and for any
value of the gauge parameter. This is the UV fixed point. In addition there is one negative and
two complex solutions of the value of the coupling constant which are all discarded on physical
grounds. We are then left with two IR fixed points \((α_2^{\ell,1}, ξ_2^{\ell,1})\) and \((α_2^{\ell,2}, ξ_2^{\ell,2})\) which follow the
pattern\(^1\)

- The first fixed point \((α_2^{\ell,1}, ξ_2^{\ell,1})\) is a saddle point. It is located at \(ξ_2^{\ell,1} = 0\) being stable in the
  \(α\) direction. This fixed point is therefore only reached along the trajectory \(ξ(μ) = 0\) for all
  scales \(μ\).

- The second fixed point \((α_2^{\ell,2}, ξ_2^{\ell,2})\) is stable in all directions. It exists as an IR fixed point in
  a limited range of the number of flavors just below where asymptotic freedom is lost. The
  value of the gauge parameter is \(ξ_2^{\ell,2} ≤ -3\) in the entire range.

At three loops the solutions follow the same pattern as in the two loop case with the addition
of two negative and four complex solutions which are all discarded. There is also a solution for
a positive value of the coupling constant. However this solution does not tend to zero as the
number of flavors approach the critical value where asymptotic freedom is lost. Hence it is also
discarded.\(^2\) The results can be found in Tables II-IX in Appendix B.

\(^1\) For a few isolated theories the negative solution for the coupling constant is positive. This is the case for \(N = 3, r = 1, N_f = 16\) and \(N = 4, r = 1, N_f = 21\) and \(N = 3, r = 3, N_f = 3\) and \(N = 4, r = 3, N_f = 3\) and
\(N = 4, r = 1, N_f = 10\). However the associated value is large and cannot be trusted within perturbation theory. The
solution is therefore discarded.

\(^2\) For a few isolated theories two additional positive zeros of coupling constant exist. This is the case for \(N = 3, r = 1, N_f = 11\) and \(N = 4, r = 1, N_f = 7\). However since these solution do not persist in the limit where the number of
flavors approach the critical value where asymptotic freedom is lost they are discarded. Also for the specific theory
\(N = 4, r = 1, N_f = 7\) there are in total eight complex, two negative, one vanishing and two positive solutions for the
value of the coupling constant.
It is clear from these tables that the difference between the anomalous dimension at the two IR fixed points is very small even though it is evaluated at rather different values of the gauge parameter. This is very similar to the results obtained in the RI’ scheme [35] in which the renormalization group functions also depend on the gauge parameter. In addition the value of the anomalous dimension is lowered when including the three loop contributions. This is seen both in the $\overline{\text{MS}}$ scheme [24, 25] and in the RI’ scheme [35]. Finally we observe a mild scheme dependence among the three different schemes mild showing an overall quite remarkable stability of the analysis.

V. CONCLUSION

An analysis of the infrared evolution of various renormalization group functions was carried out in the mMOM scheme. Since the beta functions and anomalous dimension of the $\bar{\psi}\psi$ operator depend on the gauge parameter we had to use the method developed in [35] in order for us to investigate the IR fixed points. Our results indicated a mild scheme dependence when compared to the $\overline{\text{MS}}$ scheme [24, 25] and slightly larger deviations when compared to the RI’ scheme [35].

Acknowledgments

The author would like to thank C. Pica, F. Sannino and R. Shrock for discussions and/or careful reading of the manuscript. The CP³-Origins centre is partially funded by the Danish National Research Foundation, grant number DNRF90.
Appendix A: Renormalization Group Functions in the RI’ Scheme

The coefficients of the coupling constant beta function are

\[ b_{\alpha,1} = -\frac{11}{3} C_2(G) - \frac{4}{3} T(r) N_f \] (A1)

\[ b_{\alpha,2} = -\frac{1}{12} \left( -3\xi^3 C_2(G)^2 + 10\xi^2 C_2(G)^2 - 8\xi C_2(G) T(r) N_f + 13\xi C_2(G)^2 - 8\xi C_2(G) T(r) N_f - 136 C_2(G)^2 + 80 C_2(G) T(r) N_f + 48 C_2(T(r) N_f) \right) \] (A2)

\[ b_{\alpha,3} = -\frac{1}{288} \left( -165\xi^4 C_2(G)^3 + 24\xi^4 C_2(G)^2 T(r) N_f + 108\xi(3)\xi^3 C_2(G)^3 - 189\xi^3 C_2(G)^3 - 144\xi^3 C_2(G)^2 T(r) N_f - 684\xi^2 C_2(G) C_2(G) T(r) N_f - 1188\xi C_2(G)^3 + 3291\xi C_2(G)^3 - 38620 C_2(G)^3 \right) \]

\[ + 6576\xi(3) C_2(G)^2 T(r) N_f + 3124 C_2(G)^2 T(r) N_f - 16896\xi(3) C_2(G) C_2(G) T(r) N_f \]

\[ + 20512 C_2(G) C_2(T(r) N_f - 3072\xi(3) C_2(G) T(r) N_f) - 4416 C_2(G) T(r) N_f \]

\[ - 576 C_2(G)^2 T(r) N_f + 6144\xi(3) C_2(C_2(T(r) N_f) - 5888 C_2(T(r) N_f) \right) \] (A3)

The coefficients of the gauge parameter beta function are

\[ b_{\xi,1} = \frac{1}{6} \left( 3\xi C_2(G) - 13 C_2(G) + 8 T(r) N_f \right) \] (A4)

\[ b_{\xi,2} = \frac{1}{24} \left( -6\xi^2 C_2(G)^2 + 17\xi^2 C_2(G)^2 - 16\xi C_2(G) T(r) N_f + 17\xi C_2(G)^2 - 16\xi C_2(G) T(r) N_f - 170 C_2(G)^2 + 136 C_2(G) T(r) N_f + 96 C_2(T(r) N_f) \right) \] (A5)

\[ b_{\xi,3} = \frac{1}{288} \left( -165\xi^4 C_2(G)^3 + 24\xi^4 C_2(G)^2 T(r) N_f + 54\xi(3)\xi^3 C_2(G)^3 - 126\xi^3 C_2(G)^3 - 144\xi^3 C_2(G)^2 T(r) N_f - 844\xi^2 C_2(G) C_2(G) T(r) N_f - 774\xi(3)\xi C_2(G)^3 + 102\xi C_2(G)^3 \right) \]

\[ - 228\xi(3) C_2(G)^2 T(r) N_f - 600\xi C_2(G) T(r) N_f - 1152\xi C_2(G) C_2(G) T(r) N_f + 3456\xi(3) C_2(G)^3 \]

\[ - 23032 C_2(G)^3 + 6288\xi(3) C_2(G)^2 T(r) N_f + 21320 C_2(G)^2 T(r) N_f - 16896\xi(3) C_2(G) C_2(T(r) N_f) \]

\[ 19648 C_2(G) C_2(T(r) N_f - 3072\xi(3) C_2(G) T(r) N_f) - 576 C_2(G)^2 T(r) N_f + 6144\xi(3) C_2(C_2(T(r) N_f) - 5888 C_2(T(r) N_f) \right) \] (A6)
The coefficients of $\gamma(\alpha, \xi)$ are

\begin{align*}
c_1 &= 6C_2(r) \quad \text{(A7)} \\
c_2 &= -\frac{1}{2} \left[ \xi^2 C_2(G) - 67C_2(G) - 6C_2(r) + 8T(r)N_f \right] C_2(r) \quad \text{(A8)} \\
c_3 &= -\frac{1}{24} \left[ -3\xi^3 C_2(G)^2 + 24\xi^3 C_2(G)C_2(r) - 54\xi(3)\xi^2 C_2(G)^2 + 411\xi^2 C_2(G)^2 + 108\xi^2 C_2(G)C_2(r) \\
&\quad -48\xi^2 C_2(G)T(r)N_f + 396\xi(3)\xi C_2(G)^2 + 15\xi C_2(G)^2 + 72\xi C_2(G)C_2(r) + 48\xi C_2(G)T(r)N_f \\
&\quad +5634\xi(3)C_2(G)^2 - 10095C_2(G)^2 - 4224\xi(3)C_2(G)C_2(r) + 244C_2(G)C_2(r) \\
&\quad -1152\xi(3)C_2(G)T(r)N_f + 3888C_2(G)T(r)N_f - 3096C_2(r)^2 + 1536\xi(3)C_2(r)T(r)N_f \\
&\quad +736C_2(r)T(r)N_f - 384T(r)^2 N_f \right] C_2(r) \quad \text{(A9)}
\end{align*}

Appendix B: Tables
TABLE II: Values of the IR zeros $\alpha_n$ and $\xi_n$ with $N_f$ fermions in the fundamental representation and $N = 2, 3, 4$.
The loop order is denoted by $n$.

| $N$ | $N_f$ | $\alpha_{2f,1}$ | $\xi_{2f,1}$ | $\alpha_{2f,2}$ | $\xi_{2f,2}$ | $\alpha_{3f,1}$ | $\xi_{3f,1}$ | $\alpha_{3f,2}$ | $\xi_{3f,2}$ |
|-----|-------|-----------------|---------------|-----------------|---------------|-----------------|---------------|-----------------|---------------|
| 2   | 7     | 2.83            | 0             | -               | -             | 0.854           | 0             | -               | -             |
| 2   | 8     | 1.26            | 0             | -               | -             | 0.588           | 0.612         | -3.34           |
| 2   | 9     | 0.595           | 0             | 0.421           | -3.25         | 0.377           | 0.386         | -3.18           |
| 2   | 10    | 0.231           | 0             | 0.202           | -3.11         | 0.187           | 0.190         | -3.01           |
| 3   | 10    | 2.21            | 0             | -               | -             | 0.621           | 0             | -               | -             |
| 3   | 11    | 1.23            | 0             | -               | -             | 0.485           | 0.539         | -3.51           |
| 3   | 12    | 0.754           | 0             | -               | -             | 0.377           | 0.393         | -3.32           |
| 3   | 13    | 0.468           | 0             | 0.312           | -3.29         | 0.283           | 0.291         | -3.21           |
| 3   | 14    | 0.278           | 0             | 0.219           | -3.19         | 0.198           | 0.203         | -3.14           |
| 3   | 15    | 0.143           | 0             | 0.127           | -3.10         | 0.118           | 0.120         | -3.08           |
| 3   | 16    | 0.0416          | 0             | 0.0402          | -3.03         | 0.0392          | 0.0394        | -3.03           |
| 4   | 13    | 1.85            | 0             | -               | -             | 0.490           | 0             | -               | -             |
| 4   | 14    | 1.16            | 0             | -               | -             | 0.406           | 0             | -               | -             |
| 4   | 15    | 0.783           | 0             | -               | -             | 0.338           | 0.365         | -3.45           |
| 4   | 16    | 0.546           | 0             | -               | -             | 0.278           | 0.291         | -3.32           |
| 4   | 17    | 0.384           | 0             | -               | -             | 0.226           | 0.233         | -3.23           |
| 4   | 18    | 0.266           | 0             | 0.195           | -3.24         | 0.177           | 0.182         | -3.17           |
| 4   | 19    | 0.175           | 0             | 0.143           | -3.17         | 0.131           | 0.134         | -3.12           |
| 4   | 20    | 0.105           | 0             | 0.0929          | -3.10         | 0.0868          | 0.0882        | -3.08           |
| 4   | 21    | 0.0472          | 0             | 0.0448          | -3.05         | 0.0432          | 0.0436        | -3.04           |
TABLE III: Values of the anomalous dimension $\gamma_{nf}$ with $N_f$ fermions in the fundamental representation and $N = 2, 3, 4$. The loop order is denoted by $n$. We also include the values in the $\overline{\text{MS}}$ scheme.

| $N$ | $N_f$ | $\gamma_{2f,1}$ | $\gamma_{2f,2}$ | $\gamma_{3f,1}$ | $\gamma_{3f,2}$ | $\gamma_{4f,1}$ | $\gamma_{4f,2}$ |
|-----|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2   | 7     | 3.12            | -               | 0.524           | -               | 2.67            | 0.457           | 0.0325          |
| 2   | 8     | 0.849           | -               | 0.300           | 0.283           | 0.752           | 0.272           | 0.204           |
| 2   | 9     | 0.299           | 0.185           | 0.169           | 0.164           | 0.275           | 0.161           | 0.157           |
| 2   | 10    | 0.0950          | 0.0801          | 0.0748          | 0.0744          | 0.0910          | 0.0738          | 0.0748          |
| 3   | 10    | 4.89            | -               | 0.735           | -               | 4.19            | 0.674           | 0.156           |
| 3   | 11    | 1.85            | -               | 0.493           | 0.476           | 1.61            | 0.439           | 0.250           |
| 3   | 12    | 0.867           | -               | 0.340           | 0.324           | 0.773           | 0.312           | 0.253           |
| 3   | 13    | 0.443           | 0.250           | 0.233           | 0.226           | 0.404           | 0.220           | 0.210           |
| 3   | 14    | 0.227           | 0.164           | 0.151           | 0.149           | 0.212           | 0.146           | 0.147           |
| 3   | 15    | 0.104           | 0.0887          | 0.0836          | 0.0832          | 0.0997          | 0.0826          | 0.0836          |
| 3   | 16    | 0.0276          | 0.0264          | 0.0259          | 0.0259          | 0.0272          | 0.0258          | 0.0259          |
| 4   | 13    | 6.28            | -               | 0.857           | -               | 5.38            | 0.755           | 0.192           |
| 4   | 14    | 2.82            | -               | 0.623           | -               | 2.45            | 0.552           | 0.259           |
| 4   | 15    | 1.50            | -               | 0.467           | 0.447           | 1.32            | 0.420           | 0.281           |
| 4   | 16    | 0.871           | -               | 0.354           | 0.338           | 0.778           | 0.325           | 0.269           |
| 4   | 17    | 0.529           | -               | 0.267           | 0.258           | 0.481           | 0.251           | 0.234           |
| 4   | 18    | 0.325           | 0.212           | 0.197           | 0.193           | 0.301           | 0.189           | 0.187           |
| 4   | 19    | 0.194           | 0.148           | 0.138           | 0.136           | 0.183           | 0.134           | 0.136           |
| 4   | 20    | 0.106           | 0.0914          | 0.0864          | 0.0860          | 0.102           | 0.0854          | 0.0865          |
| 4   | 21    | 0.0449          | 0.0420          | 0.0408          | 0.0408          | 0.0440          | 0.0407          | 0.0409          |

TABLE IV: Values of the IR zeros $\alpha_{sf}$ and $\xi_{sf}$ with $N_f = 2$ fermions in the adjoint representation and $N = 2, 3, 4$. The loop order is denoted by $n$.

| $N$ | $N_f$ | $\alpha_{2f,1}$ | $\alpha_{2f,2}$ | $\alpha_{3f,1}$ | $\alpha_{3f,2}$ | $\xi_{2f,1}$ | $\xi_{2f,2}$ | $\xi_{3f,1}$ | $\xi_{3f,2}$ |
|-----|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 2   | 2     | 0.628           | 0               | 0.424           | -               | 0               | 0.447           | -3.23           |
| 3   | 2     | 0.419           | 0               | 0.283           | 0.136           | 0.183           | 0.298           | -3.23           |
| 4   | 2     | 0.314           | 0               | 0.212           | 0.151           | 0.212           | 0.223           | -3.23           |
**TABLE V**: Values of the anomalous dimension \( \gamma_{nl} \) with \( N_f = 2 \) fermions in the adjoint representation and \( N = 2, 3, 4 \). The loop order is denoted by \( n \). We also include the values in the \( \overline{\text{MS}} \) scheme.

| \( N_f \) | \( N \) | \( \gamma_{2l,1} \) | \( \gamma_{2l,2} \) | \( \gamma_{3l,1} \) | \( \gamma_{3l,2} \) | \( \gamma_{4l} \) | \( \gamma_{3l} \) | \( \gamma_{4l} \) |
|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 0.885 | - | 0.569 | 0.570 | 0.820 | 0.543 | 0.500 |
| 3 | 2 | 0.885 | - | 0.569 | 0.570 | 0.820 | 0.543 | 0.523 |
| 4 | 2 | 0.885 | - | 0.569 | 0.570 | 0.820 | 0.543 | 0.532 |

**TABLE VI**: Values of the IR zeros \( \alpha_{nl} \) and \( \xi_{nl} \) with \( N_f \) fermions in the two-indexed symmetric representation and \( N = 3, 4 \). The loop order is denoted by \( n \).

| \( N_f \) | \( N \) | \( \alpha_{2l,1} \) | \( \alpha_{2l,2} \) | \( \alpha_{3l,1} \) | \( \alpha_{3l,2} \) | \( \xi_{3l,1} \) | \( \xi_{3l,2} \) |
|---|---|---|---|---|---|---|---|
| 3 | 2 | 0.842 | 0 | - | 0.460 | 0 | (22.5) | -3.10 |
| 3 | 3 | 0.0849 | 0 | 0.0793 | 0.0771 | 0 | 0.0781 | 3.05 |
| 4 | 2 | 0.967 | 0 | - | 0.451 | 0 | - | - |
| 4 | 3 | 0.152 | 0 | 0.128 | 0.123 | 0 | 0.126 | -3.12 |

**TABLE VII**: Values of the anomalous dimension \( \gamma_{nl} \) with \( N_f \) fermions in the two-indexed symmetric representation and \( N = 3, 4 \). The loop order is denoted by \( n \). We also include the values in the \( \overline{\text{MS}} \) scheme.

| \( N_f \) | \( N \) | \( \gamma_{2l,1} \) | \( \gamma_{2l,2} \) | \( \gamma_{3l,1} \) | \( \gamma_{3l,2} \) | \( \gamma_{4l} \) | \( \gamma_{3l} \) | \( \gamma_{4l} \) |
|---|---|---|---|---|---|---|---|---|
| 3 | 2 | 2.69 | - | 1.42 | (27055) | 2.44 | 1.28 | 1.12 |
| 3 | 3 | 0.147 | 0.135 | 0.133 | 0.133 | 0.144 | 0.133 | 0.133 |
| 4 | 2 | 5.37 | - | 2.44 | - | 4.82 | 2.08 | 1.79 |
| 4 | 3 | 0.400 | 0.318 | 0.319 | 0.319 | 0.381 | 0.313 | 0.315 |
**TABLE VIII:** Values of the IR zeros $\alpha_n$ and $\xi_n$ with $N_f$ fermions in the two-indexed antisymmetric representation and $N = 4$. The loop order is denoted by $n$.

| $N$ | $N_f$ | $\alpha_{2f,1}$ | $\xi_{2f,1}$ | $\alpha_{2f,2}$ | $\xi_{2f,2}$ | $\alpha_{3f,1}$ | $\xi_{3f,1}$ | $\alpha_{3f,2}$ | $\xi_{3f,2}$ |
|-----|-------|-----------------|--------------|-----------------|--------------|----------------|--------------|----------------|--------------|
| 4   | 6     | 2.16            | 0            | -               | -            | 0.557          | 0            | -              | -            |
| 4   | 7     | 0.890           | 0            | -               | -            | 0.376          | 0            | 0.478          | -3.73        |
| 4   | 8     | 0.449           | 0            | -               | -            | 0.255          | 0            | 0.268          | -3.29        |
| 4   | 9     | 0.225           | 0            | 0.174           | -3.21        | 0.161          | 0            | 0.165          | -3.15        |
| 4   | 10    | 0.0904          | 0            | 0.0818          | -3.09        | 0.0775         | 0            | 0.0787         | -3.07        |

**TABLE IX:** Values of the anomalous dimension $\gamma_n$ with $N_f$ fermions in the two-indexed antisymmetric representation and $N = 4$. The loop order is denoted by $n$. We also include the values in the $\overline{\text{MS}}$ scheme.

| $N$ | $N_f$ | $\gamma_{2f,1}$ | $\gamma_{2f,2}$ | $\gamma_{3f,1}$ | $\gamma_{3f,2}$ | $\gamma_{2f}$ | $\gamma_{3f}$ | $\gamma_{4f}$ |
|-----|-------|-----------------|-----------------|-----------------|----------------|--------------|--------------|--------------|
| 4   | 6     | 11.3            | -               | 1.57            | -              | 9.78         | 1.38         | 0.293        |
| 4   | 7     | 2.48            | -               | 0.770           | 0.889          | 2.19         | 0.695        | 0.435        |
| 4   | 8     | 0.885           | -               | 0.430           | 0.419          | 0.802        | 0.402        | 0.368        |
| 4   | 9     | 0.354           | 0.248           | 0.236           | 0.232          | 0.331        | 0.228        | 0.232        |
| 4   | 10    | 0.121           | 0.106           | 0.102           | 0.102          | 0.117        | 0.101        | 0.103        |
[1] W. E. Caswell, “Asymptotic Behavior of Nonabelian Gauge Theories to Two Loop Order,” Phys. Rev. Lett. 33, 244 (1974).
[2] T. Banks and A. Zaks, “On the Phase Structure of Vector-Like Gauge Theories with Massless Fermions,” Nucl. Phys. B 196, 189 (1982).
[3] B. Holdom, “Techniodor,” Phys. Lett. B 150, 301 (1985).
[4] K. Yamawaki, M. Bando and K. -i. Matumoto, “Scale Invariant Technicolor Model and a Technidilaton,” Phys. Rev. Lett. 56, 1335 (1986).
[5] T. W. Appelquist, D. Karabali and L. C. R. Wijewardhana, “Chiral Hierarchies and the Flavor Changing Neutral Current Problem in Technicolor,” Phys. Rev. Lett. 57, 957 (1986).
[6] T. Appelquist and L. C. R. Wijewardhana, “Chiral Hierarchies and Chiral Perturbations in Technicolor,” Phys. Rev. D 35, 774 (1987).
[7] T. Appelquist and L. C. R. Wijewardhana, “Chiral Hierarchies from Slowly Running Couplings in Technicolor Theories,” Phys. Rev. D 36, 568 (1987).
[8] T. Appelquist, K. D. Lane and U. Mahanta, “On The Ladder Approximation For Spontaneous Chiral Symmetry Breaking,” Phys. Rev. Lett. 61, 1553 (1988).
[9] T. Appelquist and S. B. Selipsky, “Instantons and the chiral phase transition,” Phys. Lett. B 400, 364 (1997) [hep-ph/9702404].
[10] S. J. Brodsky and R. Shrock, “Maximum Wavelength of Confined Quarks and Gluons and Properties of Quantum Chromodynamics,” Phys. Lett. B 666, 95 (2008) [arXiv:0806.1535 [hep-th]].
[11] F. Sannino and K. Tuominen, “Orientifold theory dynamics and symmetry breaking,” Phys. Rev. D 71, 051901 (2005) [hep-ph/0405209].
[12] D. D. Dietrich and F. Sannino, “Conformal window of SU(N) gauge theories with fermions in higher dimensional representations,” Phys. Rev. D 75, 085018 (2007) [hep-ph/0611341].
[13] T. A. Ryttov and F. Sannino, “Conformal House,” Int. J. Mod. Phys. A 25 (2010) 4603 [arXiv:0906.0307 [hep-ph]].
[14] T. A. Ryttov and R. Shrock, “Infrared Evolution and Phase Structure of a Gauge Theory Containing Different Fermion Representations,” Phys. Rev. D 81, 116003 (2010) [Erratum-ibid. D 82, 059903 (2010)] [arXiv:1006.0421 [hep-ph]].
[15] M. Mojaza, C. Pica, T. A. Ryttov and F. Sannino, “Exceptional and Spinorial Conformal Windows,” Phys. Rev. D 86, 076012 (2012) [arXiv:1206.2652 [hep-ph]].
[16] O. Antipin, M. Gillioz and F. Sannino, “a New Conformal Window Bound from the a theorem,” arXiv:1303.1547 [hep-ph].
[17] T. A. Ryttov and F. Sannino, “Supersymmetry inspired QCD beta function,” Phys. Rev. D 78, 065001 (2008) [arXiv:0711.3745 [hep-th]].
[18] C. Pica and F. Sannino, “Beta Function and Anomalous Dimensions,” Phys. Rev. D 83, 116001 (2011)
15

[arXiv:1011.3832 [hep-ph]].

[19] A. L. Kataev and K. V. Stepanyantz, “NSVZ scheme with the higher derivative regularization for $\mathcal{N} = 1$ SQED,” Nucl. Phys. B 875 (2013) 459 [arXiv:1305.7094 [hep-th]].

[20] F. Sannino, “Conformal Dynamics for TeV Physics and Cosmology,” Acta Phys. Polon. B 40, 3533 (2009) [arXiv:0911.0931 [hep-ph]].

[21] A. L. Kataev, “Conformal symmetry limit of QED and QCD and identities between perturbative contributions to deep-inelastic scattering sum rules,” arXiv:1305.4605 [hep-th].

[22] S. G. Gorishnii, A. L. Kataev, S. A. Larin and L. R. Surguladze, “The Analytical four loop corrections to the QED Beta function in the MS scheme and to the QED psi function: Total reevaluation,” Phys. Lett. B 256, 81 (1991).

[23] A. L. Kataev and K. V. Stepanyantz, “Scheme independent consequence of the NSVZ relation for N=1 SQED with $N_f$ flavors,” arXiv:1311.0589 [hep-th].

[24] T. A. Ryttov and R. Shrock, “Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions,” Phys. Rev. D 83, 056011 (2011) [arXiv:1011.4542 [hep-ph]].

[25] C. Pica and F. Sannino, “UV and IR Zeros of Gauge Theories at The Four Loop Order and Beyond,” Phys. Rev. D 83, 035013 (2011) [arXiv:1011.5917 [hep-ph]].

[26] For a recent review of the latest results see talks at the “Lattice BSM Workshop” at CP3-Origins at the University of Southern Denmark, August 2013.

[27] R. Shrock, “Higher-Loop Structural Properties of the $\beta$ Function in Asymptotically Free Vectorial Gauge Theories,” Phys. Rev. D 87, 105005 (2013) [arXiv:1301.3209 [hep-th]].

[28] R. Shrock, “Higher-Loop Calculations of the Ultraviolet to Infrared Evolution of a Vectorial Gauge Theory in the Limit $N_c \to \infty, N_f \to \infty$ with $N_f/N_c$ Fixed,” Phys. Rev. D 87, 116007 (2013) [arXiv:1302.5434 [hep-th]].

[29] N. Seiberg, “Electric - magnetic duality in supersymmetric nonAbelian gauge theories,” Nucl. Phys. B 435, 129 (1995) [hep-th/9411149].

[30] T. A. Ryttov and F. Sannino, “Conformal Windows of SU(N) Gauge Theories, Higher Dimensional Representations and The Size of The Unparticle World,” Phys. Rev. D 76, 105004 (2007) [arXiv:0707.3166 [hep-th]].

[31] T. A. Ryttov and R. Shrock, “Comparison of Some Exact and Perturbative Results for a Supersymmetric SU($N_c$) Gauge Theory,” Phys. Rev. D 85, 076009 (2012) [arXiv:1202.1297 [hep-ph]].

[32] T. A. Ryttov and R. Shrock, “Scheme Transformations in the Vicinity of an Infrared Fixed Point,” arXiv:1206.2366 [hep-ph].

[33] T. A. Ryttov and R. Shrock, “An Analysis of Scheme Transformations in the Vicinity of an Infrared Fixed Point,” arXiv:1206.6895 [hep-th].

[34] R. Shrock, “Study of Scheme Transformations to Remove Higher-Loop Terms in the $\beta$ Function of a Gauge Theory,” Phys. Rev. D 88, 036003 (2013) [arXiv:1305.6524 [hep-ph]].

[35] T. A. Ryttov, “Higher Loop Corrections to the Infrared Evolution of Fermionic Gauge Theories in the
RI' Scheme,” Phys. Rev. D 89, 016013 (2014) [arXiv:1309.3867 [hep-ph]].

[36] G. Martinelli, C. Pittori, C. T. Sachrajda, M. Testa and A. Vladikas, “A General method for nonperturbative renormalization of lattice operators,” Nucl. Phys. B 445, 81 (1995) [hep-lat/9411010].

[37] J. A. Gracey, “Three loop anomalous dimension of nonsinglet quark currents in the RI-prime scheme,” Nucl. Phys. B 662, 247 (2003) [hep-ph/0304113].

[38] E. Franco and V. Lubicz, “Quark mass renormalization in the MS-bar and RI schemes up to the NNLO order,” Nucl. Phys. B 531, 641 (1998) [hep-ph/9803491].

[39] K. G. Chetyrkin and A. Retey, “Renormalization and running of quark mass and field in the regularization invariant and MS-bar schemes at three loops and four loops,” Nucl. Phys. B 583, 3 (2000) [hep-ph/9910332].

[40] L. von Smekal, K. Maltman and A. Sternbeck, “The Strong coupling and its running to four loops in a minimal MOM scheme,” Phys. Lett. B 681, 336 (2009) [arXiv:0903.1696 [hep-ph]].

[41] G. ’t Hooft, “Dimensional regularization and the renormalization group,” Nucl. Phys. B 61, 455 (1973).

[42] W. A. Bardeen, A. J. Buras, D. W. Duke and T. Muta, “Deep Inelastic Scattering Beyond the Leading Order in Asymptotically Free Gauge Theories,” Phys. Rev. D 18, 3998 (1978).

[43] T. van Ritbergen, J. A. M. Vermaseren and S. A. Larin, “The Four loop beta function in quantum chromodynamics,” Phys. Lett. B 400, 379 (1997) [hep-ph/9701390].

[44] J. A. M. Vermaseren, S. A. Larin and T. van Ritbergen, “The four loop quark mass anomalous dimension and the invariant quark mass,” Phys. Lett. B 405, 327 (1997) [hep-ph/9703284].

[45] M. Czakon, “The Four-loop QCD beta-function and anomalous dimensions,” Nucl. Phys. B 710, 485 (2005) [hep-ph/0411261].

[46] K. G. Chetyrkin, “Quark mass anomalous dimension to O (alpha-s**4),” Phys. Lett. B 404, 161 (1997) [hep-ph/9703278].

[47] M. Mojaza, S. J. Brodsky and X. -G. Wu, “A Systematic All-Orders Method to Eliminate Renormalization-Scale and Scheme Ambiguities in PQCD,” Phys. Rev. Lett. 110, 192001 (2013) [arXiv:1212.0049 [hep-ph]].

[48] S. J. Brodsky, M. Mojaza and X. -G. Wu, “Systematic Scale-Setting to All Orders: The Principle of Maximum Conformality and Commensurate Scale Relations,” arXiv:1304.4631 [hep-ph].

[49] W. Celmaster and R. J. Gonsalves, “The Renormalization Prescription Dependence of the QCD Coupling Constant,” Phys. Rev. D 20, 1420 (1979).

[50] K. G. Chetyrkin and T. Seidensticker, “Two loop QCD vertices and three loop MOM beta functions,” Phys. Lett. B 495, 74 (2000) [hep-ph/0008094].

[51] J. A. Gracey, “Three loop QCD MOM beta-functions,” Phys. Lett. B 700, 79 (2011) [arXiv:1104.5382 [hep-ph]].

[52] J. A. Gracey, “Two loop QCD vertices at the symmetric point,” Phys. Rev. D 84, 085011 (2011) [arXiv:1108.4806 [hep-ph]].

[53] J. A. Gracey, “Renormalization group functions of QCD in the minimal MOM scheme,” J. Phys. A 46,
225403 (2013) [arXiv:1304.5347 [hep-ph]].