Zero Divisors in Associative Algebras over Infinite Fields

MICHAEL SCHWEITZER AND STEVEN FINCH

March 30, 1999

Abstract. Let $F$ be an infinite field. We prove that the right zero divisors of a three-dimensional associative $F$-algebra $A$ must form the union of at most finitely many linear subspaces of $A$. The proof is elementary and written with students as the intended audience.

1. Introductory Example

Consider three-dimensional real space $R^3$ endowed with the following vector multiplication:

$$
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix}
\cdot
\begin{pmatrix}
\delta \\
\epsilon \\
\varphi
\end{pmatrix}
= 
\begin{pmatrix}
\alpha \delta + \frac{1}{2}(\gamma \varphi - \beta \epsilon) - \frac{\sqrt{3}}{2}(\beta \varphi + \gamma \epsilon) \\
(\alpha \epsilon + \beta \delta) - \frac{\sqrt{2}}{8}(5 \gamma \varphi - \beta \epsilon) - \frac{\sqrt{2}}{8}(\beta \varphi + \gamma \epsilon) \\
(\alpha \varphi + \gamma \delta) + \frac{\sqrt{6}}{8}(\gamma \varphi + 3 \beta \epsilon) - \frac{\sqrt{6}}{8}(\beta \varphi + \gamma \epsilon)
\end{pmatrix}
$$

This definition first appeared in [6] and has application to pattern recognition and information processing.

Observe that multiplication is associative, commutative and has identity element $(1,0,0)$. To further understand this algebra, it is natural to study the set $Z$ of all zero divisors, that is, all vectors which are mapped to zero under multiplication by a nonzero element. A determinant argument gives that $Z$ is the set of all $(\alpha, \beta, \gamma)$ satisfying the homogeneous cubic equation

$$6\sqrt{6} \gamma^2 + (6\sqrt{2} \beta - 12 \alpha) \gamma^2 + (6\sqrt{6} \beta^2 + 24\sqrt{3} \alpha \beta) \gamma - 10\sqrt{2} \beta^3 + 12 \alpha \beta^2 + 16 \alpha^3 = 0$$

It may surprise some that this equation factors:

$$\left(\sqrt{6} \gamma + 2 \alpha - \sqrt{2} \beta\right) \left(2 \alpha - \sqrt{2} \beta - \sqrt{6} \gamma\right)^2 + \left(2 \alpha + 2 \sqrt{2} \beta\right)^2 = 0$$

and hence $Z$ is the union of a plane and a line passing through the origin. In fact, this is not surprising to algebraists [4], [7]. For any finite-dimensional, associative, commutative algebra $A$ with identity, the set $Z$ is the union of at most finitely many linear subspaces of $A$. We say $A$ is tame when $Z$ is so simply described. Does tameness hold even if the commutativity hypothesis is discarded? No, the four-dimensional algebra of real $2 \times 2$ matrices is not tame. What can be said about the three-dimensional case? We answer this question in the following sections. As far as is known, this material has not previously appeared in the literature.
2. Main Result

Let $F$ be an infinite field, for example, the real numbers $R$ or the complex numbers $C$. Let $A$ be a finite-dimensional $F$-algebra, that is, $A$ is a finite-dimensional vector space over $F$ together with a bilinear map $A \times A \to A$, known as vector multiplication $xy$, which satisfies $f(xy) = (fx)y = x(fy)$ for all $f \in F$ and $x, y \in A$. Further, let $A$ be associative, that is, $(xy)z = x(yz)$ for all $x, y, z \in A$. We do not assume $A$ is commutative or that $A$ has an identity element. Everything in the following holds true if one systematically interchanges "left" and "right".

A subring $I$ of $A$ is a left ideal if $a \in A$, $x \in I$ implies that $ax \in I$. A left ideal $I \neq A$ is maximal if, whenever $J$ is a left ideal such that $I \subset J \subset A$, then either $J = I$ or $J = A$.

An element $z \in A$ is a right zero divisor if there exists a nonzero $w \in A$ with $wz = 0$. The set of all right zero divisors is denoted by $Z$. Note that we consider $0 \in Z$, which is uncustomary.

The algebra $A$ is right tame if $Z$ is the union of at most finitely many linear subspaces of $A$. We believe this phraseology to be new. Define $A$ to be right proper tame if $A$ is right tame and, additionally, $Z \neq A$. Also define $AA$ to be the set of all products $xy$, where $x \in A$, $y \in A$. Our main result depends on whether $AA$ fills out all of $A$ or not; the condition $AA = A$ may be thought of as a poor man's replacement for the existence of an identity element.

**Theorem.** (i) If $AA \neq A$, then $Z = A$ and hence $A$ is right tame.

(ii) If $AA = A$, then $Z$ is the union of all maximal left ideals of $A$. Further, $A$ is right proper tame if and only if $A$ possesses at most finitely many maximal left ideals.

**Corollary.** If $\dim(A) = 3$, then $A$ is right tame.

The proofs of these assertions and preliminary lemmas are given in the next section. For the benefit of students and for completeness's sake, we provide many underlying details. The required background is essentially covered in [4].

3. Detailed Proofs

Under the conditions on $A$ described in section 2, if $S$ is a subset of $A$, define

\[
L(S) = \text{the linear subspace generated by } S
\]

\[
= \text{the intersection of all linear subspaces which contain } S
\]

and

\[
I(S) = \text{the left ideal generated by } S
\]

\[
= \text{the intersection of all left ideals which contain } S.
\]
In the following, if $k$ is a positive integer and $s \in S$, then $ks$ is repeated addition: $ks = \sum_{j=1}^{k} s$, $0s = s$ and $(-k)s = -(ks)$.

Lemma 1. $L(S) = \left\{ \sum_{i=1}^{n} f_{i}s_{i} : f_{i} \in F, s_{i} \in S, \text{any integer } n \geq 1 \right\}$ and $I(S) = \left\{ \sum_{i=1}^{n} a_{i}s_{i} + \sum_{j=1}^{m} k_{j}t_{j} : a_{i} \in A, s_{i}, t_{j} \in S, \text{any integers } k_{j}, n \geq 1, m \geq 1 \right\}$.

Proof. Focus only on the second formula. First, show that the right-hand side is a left ideal containing $S$. It clearly contains $S$ since 1 is an integer and $1s = s$ for any $s \in S$. It is a left ideal since, given $b \in A$, 

$$b \cdot \left( \sum_{i} a_{i}s_{i} + \sum_{j} k_{j}t_{j} \right) = \sum_{i} b(a_{i}s_{i}) + \sum_{j} b(k_{j}t_{j}) = \sum_{i} (ba_{i})s_{i} + \sum_{j} (k_{j}b)t_{j}$$

by associativity and since both $ba_{i} \in A$ and $k_{j}b \in A$. Conversely, show that the right-hand side is contained in every left ideal containing $S$.

Lemma 2. If $I$ is a left ideal, then $L(I)$ is also a left ideal and $A \cdot L(I)$ is contained in $I$.

Proof. If $f_{i} \in F, x_{i} \in I$ and $a \in A$, then $a \cdot \left( \sum_{i} f_{i}x_{i} \right) = \sum_{i} f_{i}(ax_{i}) \in L(I)$ since $ax_{i} \in I$, so $L(I)$ is a left ideal. Also, $\sum_{i} f_{i}(ax_{i}) = \sum_{i} (f_{i}a)x_{i} \in I$, so $A \cdot L(I) \subset I$.

Let $x \in A$ and define $\langle x \rangle = I(\{x\})$, the left ideal generated by $\{x\}$. Define too a map $R(x) : A \longrightarrow A$ by $R(x)(y) = yx$, the right-multiplication map by $x$.

Lemma 3. The map $R(x)$ is a linear transformation, $Im(R(x)) \subset \langle x \rangle$ and $A \cdot \langle x \rangle \subset Im(R(x))$. In particular, if $\langle x \rangle = A$, then $Im(R(x)) = AA$.

Proof. The first two assertions are clear. By Lemma 1, $\langle x \rangle$ is the set of all $ax + kx$ where $a \in A$ and $k$ is an integer, so for any $b \in A$, 

$$b \cdot (ax + kx) = (ba + kb) \cdot x = R(x)(ba + kb)$$

by associativity, hence $A \cdot \langle x \rangle \subset Im(R(x))$. If $\langle x \rangle = A$, then $AA = A \cdot \langle x \rangle$ is contained in $Im(R(x))$, but $Im(R(x))$ is trivially contained in $AA$. 


Lemma 4. Let $x \in A$. Then $x$ is a right zero divisor if and only if $\text{Im}(R(x)) \neq A$.

Proof. $x$ is a right zero divisor iff there exists $y \neq 0$ with $R(x)(y) = 0$. This is possible iff $\text{Ker}(R(x)) \neq \{0\}$. This, in turn, is possible iff $\text{Im}(R(x)) \neq A$.

Proof of Theorem, part (i). We prove, assuming $A A \neq A$, that every element of $A$ is both a right and a left zero divisor. Let $x \in A$, $x \neq 0$. Then $\text{Im}(R(x)) \subset A A \neq A$, which implies immediately that $x$ is a right zero divisor by Lemma 4. Likewise, $x$ is a left zero divisor.

Having dealt with the easy case for which $A A \neq A$, we devote attention to the more interesting case $A A = A$. Given two sets $S$ and $T$, define $S + T$ to be the set of all sums $s + t$, where $s \in S$, $t \in T$.

Lemma 5. Assume $A A = A$. If $I$ is a maximal left ideal, then $I = L(I)$. That is, a maximal left ideal is necessarily a linear subspace. If $I$ and $J$ are maximal left ideals and $I \neq J$, then $I + J = A$.

Proof. By Lemma 2, $L(I)$ is a left ideal and $I \subset L(I) \subset A$. By maximality, it follows that $I = L(I)$ or $L(I) = A$. The latter is impossible since otherwise $A = A A = A \cdot L(I) \subset I$ by Lemma 2, but $I \neq A$ by definition. Thus $I = L(I)$. Also, $I + J$ is a left ideal, $I \subset I + J$, and $I + J \neq I$ (since otherwise $J = \{0\} + J \subset I$, contradicting maximality of $J$). Therefore $I + J$ contains $I$ properly and, by maximality of $I$, this implies that $I + J = A$.

Lemma 6. Let $V$ be a vector space over $F$. Then $V$ is not the union of finitely many proper linear subspaces.

Proof. Suppose not. Among all possible such decompositions of $V$, choose one

$$V = \bigcup_{i=1}^{n} L_i$$

with $n$ minimal. Then $n \geq 2$ and no $L_i$ is contained in the union of the other $L_j$’s. Select vectors $v_1$ and $v_2$ such that $v_1 \in L_1$ but $v_1 \notin L_i$ for $i \neq 1$, and $v_2 \in L_2$ but $v_2 \notin L_i$ for $i \neq 2$. Consider the affine line $K = \{v_1 + f v_2 : f \in F\}$ and suppose that it intersects $L_i$ in two distinct points $p = v_1 + f v_2$ and $q = v_1 + g v_2$, $f \neq g$. Then $p - q = (f - g) \cdot v_2$ and $q p - f q = (g - f) \cdot v_1$ are also in $L_i$, but this is impossible since $v_1$ is in $L_1$ only and $v_2$ is in $L_2$ only. So $K$ intersects each $L_i$ in at most one
point. Since $V$ is the union of $L_1, L_2, ..., L_n$, it follows that $K$ is a finite set. This is absurd since $F$ is infinite.

**Proof of Theorem, part (ii).** We first prove, assuming $AA = A$, that the set $Z$ of all right zero divisors is the union of all maximal left ideals of $A$. Let $x \in Z$, then $Im(R(x)) \neq A$ by Lemma 4. If $\langle x \rangle = A$, then $Im(R(x)) = AA = A$ by Lemma 3, which is a contradiction. Hence $\langle x \rangle \neq A$ and $\langle x \rangle$ must be contained in some maximal left ideal. Conversely, let $x$ be an element of a maximal left ideal. Thus $\langle x \rangle \neq A$. It follows from Lemma 3 that $Im(R(x)) \subset \langle x \rangle$, hence $Im(R(x)) \neq A$. Therefore by Lemma 4, $x \in Z$.

If there are only finitely many maximal left ideals, then by Lemmas 5 and 6, each of them is a linear subspace and thus $A$ is right proper tame. Conversely, if $A$ is right proper tame, then there exist linear subspaces $L_1, L_2, ..., L_n$ such that

$$Z = \bigcup_{i=1}^{n} L_i = \text{the (possibly infinite) union of maximal left ideals.}$$

Let $M$ be a maximal left ideal and let $M_i = M \cap L_i$. Then

$$M = \bigcup_{i=1}^{n} M_i$$

By Lemma 5, $M$ is itself a vector space. A finite decomposition of $M$ as linear subspaces $M_i$ is impossible, by Lemma 6, unless $M = M_i$ for some $i$. Thus $M \subset L_i$. We wish to show that $L_i$ contains no other maximal left ideal. Suppose $N \subset L_i$ for some maximal left ideal $N \neq M$. Then $A = M + N \subset L_i$ by Lemma 5 and hence $Z = A$, which cannot be true. Therefore every maximal left ideal is contained in some $L_i$ and no two are contained in the same $L_i$, so they must be only finite in number.

To prove the Corollary, we need some properties of polynomials in $n$ variables with coefficients in $F$. An ideal $P$ in the ring $F[X_1, X_2, X_3, ..., X_n]$ (in fact, any ring) is **prime** if for all $p, q \in F[X_1, X_2, X_3, ..., X_n]$, $p \cdot q \in P$ implies $p \in P$ or $q \in P$.

**Lemma 7.** If a polynomial $p \in F[X_1, X_2, X_3, ..., X_n]$ vanishes over all of $F^n$, then $p = 0$.

**Proof.** If $n = 1$ and $p \neq 0$, then $p$ can have at most finitely many zeros, but $F$ is infinite. Use induction on $n$ to complete the argument.
Lemma 8. If \( p, q \in F[X_1, X_2, X_3, ..., X_n] \), \( p \) is linear and every zero of \( p \) is also a zero of \( q \), then \( p \) divides \( q \).

**Proof.** Make an invertible affine change of coordinates so that \( p \) becomes \( X_1 \). Then the zero set of \( p \) is the set of all \( (0, f_2, ..., f_n) \), \( f_i \in F \). Write 
\[
q = q_0(X_2, ..., X_n) + q_1(X_2, ..., X_n) \cdot X_1 + ...
\]
then 
\[
0 = q(0, f_2, ..., f_n) = q_0(f_2, ..., f_n)
\]
for all \( f_2, ..., f_n \in F \). By Lemma 7, \( q_0 = 0 \) and hence \( X_1 \) divides \( q \).

Lemma 9. Assume \( p, q, r \in F[X_1, X_2, X_3, ..., X_n] \) and \( p \) and \( q \) are both linear. If \( p \) and \( q \) both divide \( r \) and \( q \) is not a scalar multiple of \( p \), then \( p \cdot q \) divides \( r \).

**Proof.** Again, change coordinates so that \( p \) becomes \( X_1 \). By assumption, \( r = p \cdot u = q \cdot v \) for some polynomials \( u \) and \( v \), hence \( X_1 \cdot u = q \cdot v \). The ideal generated by \( X_1 \) is prime since the quotient is \( F[X_2, X_3, ..., X_n] \), an integral domain. We deduce \( q \) is not in this ideal since \( q \) is linear but is not a scalar multiple of \( X_1 \). Thus \( v = X_1 \cdot w \) for some polynomial \( w \). Therefore \( r = q \cdot X_1 \cdot w \) and so \( X_1 \cdot q \) divides \( r \).

The above propositions can be strengthened, of course, but these are all we need.

Consider now the **determinant form** \( D(\xi_1, ..., \xi_n) \) of the algebra \( A \), that is, the determinant of right multiplication \( R(x) \) relative to a fixed basis \( b_1, ..., b_n \). We have 
\[
x = \sum \xi_i b_i \in Z \text{ iff } D(\xi_1, ..., \xi_n) = 0 \text{ by Lemma 4.}
\]
From Lemma 7, clearly \( Z = A \) iff \( D = 0 \). In the following, assume \( AA = A \) as always and \( dim(A) = n \). A linear subspace of \( A \) is said to have **codimension** \( k \) if it has dimension \( n - k \). The guiding principle is that it is easy to bound the number of maximal left ideals of either low dimension or low codimension.

Lemma 10. The algebra \( A \) possesses at most \( n \) maximal left ideals of codimension 1.

**Proof.** Let \( M \) be a maximal left ideal of codimension 1. \( M \) is the zero set of a linear form \( f \). Every element of \( M \) is a right zero divisor by the Theorem, hence a zero of \( D \). By Lemma 8, \( f \) divides \( D \). If \( N \) is another maximal left ideal of codimension 1, it is the zero set of some linear form \( g \). Since \( M \neq N \), \( g \) is not a scalar multiple of \( f \) and \( g \) also divides \( D \). By Lemma 9, \( f \cdot g \) divides \( D \). Since the degree of \( D \) is \( \leq n \), there can be at most \( n \) maximal left ideals of codimension 1.
Lemma 11. The algebra $A$ possesses at most one maximal left ideal of dimension $< \frac{n}{2}$.

Proof. If $M$ and $N$ are maximal left ideals of dimension $< \frac{n}{2}$ and $M \neq N$, then on one hand $M + N = A$ by Lemma 5. On the other hand, $\dim(M + N) \leq \dim(M) + \dim(N) < n$, which is a contradiction. Hence $M = N$.

Lemma 12. If $A$ possesses a maximal left ideal of dimension 1, then $A$ has at most $n + 1$ maximal left ideals. In particular, $A$ is right tame.

Proof. Let $M$ be that ideal. If $N$ is some other maximal left ideal, then $M + N = A$ by Lemma 5. Since

$$n = \dim(M + N) \leq \dim(M) + \dim(N) = 1 + \dim(N)$$

it follows that $\dim(N) \geq n - 1$. Since $N$ is a proper subspace (being a maximal ideal), $\dim(N) = n - 1$. Hence $N$ has codimension 1. Now use Lemma 10.

Proof of Corollary. A maximal left ideal has dimension 0, 1 or 2. If there is one with dimension 0, then it is the only one by Lemma 11. If there is one with dimension 1, then there are at most four by Lemma 12. Otherwise all are of dimension 2 and there are at most three by Lemma 10.

4. Closing Remarks
We have proved that a three-dimensional associative $F$-algebra is right tame, where $F$ is infinite. For the one or two-dimensional cases, the associativity requirement may be dropped and right tameness still follows. The case when $F$ is finite is also trivial.

Using some tools from algebraic geometry [3], we can prove the following general result:

Theorem. Let $A$ be a finite-dimensional algebra over an algebraically closed field $F$. Let $D$ be the determinant form for right multiplication with respect to some basis of $A$. Then $A$ is right tame iff $D$ splits into a product of linear forms over $F$.

Proof. Assuming $A$ is right tame and $D \neq 0$, let $f$ be an irreducible factor of $D$. The ideal $(f)$ generated by $f$ is prime, thus the zero set $V$ of $f$ is a variety. Since $V$ is contained in a finite union of linear subspaces, it must be contained in one of them. Call this subspace $L$. Under the correspondence between algebraic sets and ideals of polynomial rings, the ideal $I(L)$ is contained in the ideal $I(V)$, but $I(V) = (f)$ because $(f)$ is prime. $I(V)$ is generated by linear forms, hence these must be multiples.
of $f$. Therefore $f$ is linear.

We ask the following:

**Open Question.** Let $A$ be a finite-dimensional algebra over $R$ and let $D$ be as before. Is $A$ right tame iff $D$ splits into a product of linear forms over $C$?

For an infinite-dimensional algebra, there is in general no close relationship between zero divisors and maximal ideals. For example, let $A$ be the group algebra for the group of (rational) integers over a field $F$. Then $A$ has no zero divisors ($\neq 0$) but it has maximal ideals $\neq 0$. So in the infinite-dimensional case, one can assume the existence of an identity element and commutativity, etc., but all this doesn’t help.

Finally, returning to the three-dimensional real case, there is computational evidence that the associativity requirement may be weakened somewhat \[\text{\cite{2}}\] and yet right tameness still holds. An successful elementary treatment (as above) under such extended circumstances is not likely.

**References**

[1] Atiyah, M. F. and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[2] Finch, S. R., Zero divisor structure in real algebras, MathSoft Inc., website URL [http://www.mathsoft.com/asolve/zerodiv/zerodiv.htm], 1998.

[3] Hartshorne, R., *Algebraic Geometry*, Springer-Verlag, 1977.

[4] Herstein, I. N., *Topics in Algebra*, Wiley, 1975.

[5] Hungerford, T. W., *Algebra*, Springer-Verlag, 1974.

[6] Lucas, D., A multiplication in n-space, *Proc. Amer. Math. Soc.* 74 (1979) 1-8.

[7] Zariski, O. and P. Samuel, *Commutative Algebra*, vol. 1, Van Nostrand, 1965.

Michael Schweitzer
Alt-Heiligensee 51A
13503 Berlin, Germany
106664.726@compuserve.com

Steven Finch
MathSoft Inc., 101 Main Street
Cambridge, MA, USA 02142
sfinch@mathsoft.com