PRODUCT RECURRENT PROPERTIES, DISJOINTNESS AND WEAK DISJOINTNESS

BY

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ABSTRACT

Let $\mathcal{F}$ be a collection of subsets of $\mathbb{Z}_+$ and $(X, T)$ be a dynamical system; $x \in X$ is $\mathcal{F}$-recurrent if for each neighborhood $U$ of $x$, 
\[ \{n \in \mathbb{Z}_+ : T^n x \in U\} \in \mathcal{F}; \]
$x$ is $\mathcal{F}$-product recurrent if $(x, y)$ is recurrent for any $\mathcal{F}$-recurrent point $y$ in any dynamical system $(Y, S)$. It is well known that $x$ is \textit{infinite}-product recurrent if and only if it is minimal and distal. In this paper it is proved that the closure of a \textit{syndetic}-product recurrent point (i.e., weakly product recurrent point) has a dense minimal points; and a \textit{piecewise syndetic}-product recurrent point is minimal. Results on product recurrence when the closure of an $\mathcal{F}$-recurrent point has zero entropy are obtained.

It is shown that if a transitive system is disjoint from all minimal systems, then each transitive point is weakly product recurrent. Moreover, it proved that each weakly mixing system with dense minimal points is disjoint from all minimal PI systems; and each weakly mixing system with a dense set of distal points or an $\mathcal{F}_s$-independent system is disjoint from all minimal systems. Results on weak disjointness are described when considering disjointness.

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1. Introduction

1.1. Dynamical Preliminaries. In the article, integers, nonnegative integers and natural numbers are denoted by \( \mathbb{Z} \), \( \mathbb{Z}^+ \) and \( \mathbb{N} \), respectively. By a **topological dynamical system** (t.d.s.) we mean a pair \((X,T)\), where \( X \) is a compact metric space (with metric \( d \)) and \( T : X \to X \) is continuous and surjective. A non-vacuous closed invariant subset \( Y \subset X \) defines naturally a subsystem \((Y,T)\) of \((X,T)\).

The **orbit** of \( x \), \( \text{orb}(x,T) \) (or simply \( \text{orb}(x) \)), is the set \( \{T^nx : n \in \mathbb{Z}^+\} = \{x, T(x), \ldots\} \). The **\( \omega \)-limit set** of \( x \), \( \omega(x,T) \), is the set of all limit points of \( \text{orb}(x,T) \). It is easy to verify that \( \omega(x,T) = \bigcap_{n \geq 0} \{T^iy : i \geq n\} \).

A t.d.s. \((X,T)\) is **transitive** if for each pair of open (i.e., nonempty and open) subsets \( U \) and \( V \), \( N(U,V) = \{n \in \mathbb{Z}^+ : T^{-n}V \cap U \neq \emptyset\} \) is infinite. It is **point transitive** if there exists \( x \in X \) such that \( \text{orb}(x,T) = X \); such \( x \) is called a **transitive point**, and the set of transitive points is denoted by \( \text{Tran}_T \). It is well known that if a compact metric system \((X,T)\) is transitive, then \( \text{Tran}_T \) is a dense \( G_\delta \) set; \((X,T)\) is **weakly mixing** if \((X \times X, T \times T)\) is transitive.

A t.d.s. \((X,T)\) is **minimal** if \( \text{Tran}_T = X \). Equivalently, \((X,T)\) is minimal if and only if it contains no proper subsystems. By the argument using Zorn’s Lemma any t.d.s. \((X,T)\) contains some minimal subsystem, which is called a **minimal set** of \( X \). A point \( x \in X \) is **minimal** or **almost periodic** if the subsystem \((\text{orb}(x,T),T)\) is minimal.

Let \((X,T)\) be a t.d.s. and \((x,y) \in X^2\). It is a **proximal** pair if there is a sequence \( \{n_i\} \) in \( \mathbb{Z}^+ \) such that \( \lim_{n \to +\infty} T^{n_i}x = \lim_{n \to +\infty} T^{n_i}y \); and it is a **distal** pair if it is not proximal. Denote by \( P(X,T) \) or \( P_X \) the set of all proximal pairs of \((X,T)\). A point \( x \) is said to be **distal** if whenever \( y \) is in the orbit closure of \( x \) and \((x,y)\) is proximal, then \( x = y \). A t.d.s. \((X,T)\) is called **distal** if \((x,x')\) is distal whenever \( x, x' \in X \) are distinct.

A t.d.s. \((X,T)\) is **equicontinuous** if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( d(x_1,x_2) < \delta \) implies \( d(T^n x_1, T^n x_2) < \epsilon \) for every \( n \in \mathbb{Z}^+ \). It is easy to see that each equicontinuous system is distal.

For a t.d.s. \((X,T)\), \( x \in X \) and \( U \subset X \) let \( N(x,U) = \{n \in \mathbb{Z}^+ : T^n x \in U\} \).

A point \( x \in X \) is said to be **recurrent** if for every neighborhood \( U \) of \( x \), \( N(x,U) \) is infinite. Equivalently, \( x \in X \) is recurrent if and only if \( x \in \omega(x,T) \),
i.e., there is a strictly increasing subsequence \( \{ n_i \} \) of \( \mathbb{N} \) such that \( T^{n_i}x \rightarrow x \). Denote by \( R(X,T) \) the set of all recurrent points of \( (X,T) \).

1.2. Product recurrence and weakly product recurrence. The notion of product recurrence was introduced by Furstenberg in [15]. Let \( (X,T) \) be a t.d.s. A point \( x \in X \) is said to be \textbf{product recurrent} if, given any recurrent point \( y \) in any dynamical system \( (Y,S) \), \( (x,y) \) is recurrent in the product system \( (X \times Y,T \times S) \). By associating product recurrence with a combinatorial property on the sets of return times (i.e., \( x \) is product recurrent if and only if it is \( IP^* \) recurrent), Furstenberg proved that product recurrence is equivalent to distality [15, Theorem 9.11]. In [6] Auslander and Furstenberg extended the equivalence of product recurrence and distality to more general semigroup actions. If a semigroup \( E \) acts on the space \( X \) and \( F \) is a closed subsemigroup of \( E \), then \( x \in X \) is said to be \textbf{\( F \)-recurrent} if \( px = x \) for some \( p \in F \), and \textbf{product \( F \)-recurrent} if, whenever \( y \) is an \( F \)-recurrent point (in some space \( Y \) on which \( E \) acts), the point \( (x,y) \) is \( F \)-recurrent in the product system. In [6] it is shown that, under certain conditions, a point is product \( F \)-recurrent if and only if it is a distal point. This subject is also discussed in [12].

In [6], Auslander and Furstenberg posed a question: if \( (x,y) \) is recurrent for all minimal points \( y \), is \( x \) necessarily a distal point? This question is answered in the negative in [20]. Such \( x \) is called a \textbf{weakly product recurrent} point there.

The main purpose of this paper is to study a more general question, i.e., to study a point \( x \) with property that \( (x,y) \) is recurrent for any \( y \) with some special recurrent property. We will also show how this question is related to disjointness and weak disjointness. To be more precise, we need some notions.

1.3. Furstenberg families. Let us recall some notions related to Furstenberg families (for details, see [1, 15]). Let \( \mathcal{P} = \mathcal{P}(\mathbb{Z}_+) \) be the collection of all subsets of \( \mathbb{Z}_+ \). A subset \( \mathcal{F} \) of \( \mathcal{P} \) is a \textbf{(Furstenberg) family} if it is hereditary upwards, i.e., \( F_1 \subset F_2 \) and \( F_1 \in \mathcal{F} \) imply \( F_2 \in \mathcal{F} \). A family \( \mathcal{F} \) is \textbf{proper} if it is a proper subset of \( \mathcal{P} \), i.e., neither empty nor all of \( \mathcal{P} \). It is easy to see that \( \mathcal{F} \) is proper if and only if \( \mathbb{Z}_+ \in \mathcal{F} \) and \( \emptyset \notin \mathcal{F} \). Any subset \( \mathcal{A} \) of \( \mathcal{P} \) can generate a family

\[
[\mathcal{A}] = \{ F \in \mathcal{P} : F \supset A \text{ for some } A \in \mathcal{A} \}.
\]
If a proper family $\mathcal{F}$ is closed under intersection, then $\mathcal{F}$ is called a \textbf{filter}. For a family $\mathcal{F}$, the \textbf{dual family} is

$$\mathcal{F}^* = \{ F \in \mathcal{P} : Z_+ \setminus F \notin \mathcal{F} \} = \{ F \in \mathcal{P} : F \cap F' \neq \emptyset \text{ for all } F' \in \mathcal{F} \}.$$ 

$\mathcal{F}^*$ is a family, proper if $\mathcal{F}$ is. Clearly, $(\mathcal{F}^*)^* = \mathcal{F}$ and $\mathcal{F}_1 \subset \mathcal{F}_2 \implies \mathcal{F}_2^* \subset \mathcal{F}_1^*$.

Denote by $\mathcal{F}_{inf}$ the family consisting of all infinite subsets of $Z_+$.

1.4. $\mathcal{F}$-recurrence and some important families. Let $\mathcal{F}$ be a family and $(X, T)$ be a t.d.s. We say $x \in X$ is $\mathcal{F}$-\textbf{recurrent} if for each neighborhood $U$ of $x$, $N(x, U) \in \mathcal{F}$. So the usual recurrent point is just an $\mathcal{F}_{inf}$-recurrent one.

Recall that a t.d.s. $(X, T)$ is

- an \textbf{E-system} if it is transitive and has an invariant measure $\mu$ with full support, i.e., $supp(\mu) = X$;
- an \textbf{M-system} if it is transitive and the set of minimal points is dense; and
- a \textbf{P-system} if it is transitive and the set of periodic points is dense.

A subset $S$ of $Z_+$ is \textbf{syndetic} if it has a bounded gap, i.e., there is $N \in \mathbb{N}$ such that $\{i, i + 1, \ldots, i + N\} \cap S \neq \emptyset$ for every $i \in Z_+$; $S$ is \textbf{thick} if it contains arbitrarily long runs of positive integers, i.e., there is a strictly increasing subsequence $\{n_i\}$ of $Z_+$ such that $S \supset \bigcup_{i=1}^{\infty} \{n_i, n_i + 1, \ldots, n_i + i\}$. The collection of all syndetic (resp. thick) subsets is denoted by $\mathcal{F}_s$ (resp. $\mathcal{F}_t$). Note that $\mathcal{F}_s^* = \mathcal{F}_t$ and $\mathcal{F}_t^* = \mathcal{F}_s$.

Some dynamical properties can be interrupted by using the notions of syndetic or thick subsets. For example, a classic result of Gottschalk stated that $x$ is a minimal point if and only if $N(x, U) \in \mathcal{F}_s$ for any neighborhood $U$ of $x$, and a t.d.s. $(X, T)$ is weakly mixing if and only if $N(U, V) \in \mathcal{F}_t$ for any non-empty open subsets $U, V$ of $X$ [14, 15].

A subset $S$ of $Z_+$ is \textbf{piecewise syndetic} if it is an intersection of a syndetic set with a thick set. Denote the set of all piecewise syndetic sets by $\mathcal{F}_{ps}$. It is known that a t.d.s. $(X, T)$ is an \textbf{M-system} if and only if there is a transitive point $x$ such that $N(x, U) \in \mathcal{F}_{ps}$ for any neighborhood $U$ of $x$ (see, for example, [28, Lemma 2.1]).
Let \( \{b_i\}_{i \in I} \) be a finite or infinite sequence in \( \mathbb{N} \). One defines
\[
FS(\{b_i\}_{i \in I}) = \left\{ \sum_{i \in \alpha} b_i : \alpha \text{ is a finite non-empty subset of } I \right\}.
\]

\( F \) is an **IP set** if it contains some \( FS(\{p_i\}_{i=1}^{\infty}) \), where \( p_i \in \mathbb{N} \). The collection of all IP sets is denoted by \( F_{ip} \). A subset of \( \mathbb{N} \) is called an **IP*-set**, if it has non-empty intersection with any IP-set. It is known that a point \( x \) is a recurrent point if and only if \( N(x, U) \in F_{ip} \) for any neighborhood \( U \) of \( x \), and \( x \) is distal if and only if \( x \) is IP*-recurrent [15].

Let \( S \) be a subset of \( \mathbb{Z}_+ \). The **upper Banach density** and **lower Banach density** of \( S \) are
\[
BD^*(S) = \limsup_{|I| \to \infty} \frac{|S \cap I|}{|I|} \quad \text{and} \quad BD_*(S) = \liminf_{|I| \to \infty} \frac{|S \cap I|}{|I|},
\]
where \( I \) ranges over intervals of \( \mathbb{Z}_+ \), while the **upper density** of \( S \) is
\[
D^*(S) = \limsup_{n \to \infty} \frac{|S \cap [0, n-1]|}{n}.
\]
Let \( F_{pubd} = \{ S \subseteq \mathbb{Z}_+ : BD^*(S) > 0 \} \) and \( F_{pd} = \{ S \subseteq \mathbb{Z}_+ : D^*(S) > 0 \} \). It is known a t.d.s. \((X, T)\) is an **E-system** if and only if there is a transitive point \( x \) such that \( N(x, U) \in F_{pubd} \) for any neighborhood \( U \) of \( x \) (see, for example, [22, Lemma 3.6]).

1.5. **F-product recurrence and disjointness.** Let \( F \) be a family. For a t.d.s. \((X, T)\), \( x \in X \) is an **F-product recurrent** if, given any \( F \)-recurrent point \( y \) in any t.d.s. \((Y, S)\), \((x, y)\) is recurrent in the product system \((X \times Y, T \times S)\). Note that \( F_{inf} \)-product recurrence is nothing but product recurrence; and \( F_s \)-product recurrence is weak product recurrence. In this paper we will study the properties of \( F \)-product recurrent points, especially when \( F = F_{pubd}, F_{ps}, \) or \( F_s \).

The notion of **disjointness** of two t.d.s. was introduced by Furstenberg in his seminal paper [14]. Let \( (X, T) \) and \( (Y, S) \) be two t.d.s. We say \( J \subseteq X \times Y \) is a **joining** of \( X \) and \( Y \) if \( J \) is a non-empty closed invariant set, and is projected onto \( X \) and \( Y \), respectively. If each joining is equal to \( X \times Y \), then we say that \( (X, T) \) and \( (Y, S) \) are **disjoint**, denoted by \( (X, T) \perp (Y, S) \) or \( X \perp Y \). Note that if \( (X, T) \perp (Y, S) \), then one of them is minimal [14], and if \( (X, T) \) is minimal, then the set of recurrent points of \((Y, S)\) is dense [28].
In [14], Furstenberg showed that each totally transitive system with dense set of periodic points is disjoint from any minimal system; each weakly mixing system is disjoint from any minimal distal system. He left the following question:

**Problem:** Describe the system which is disjoint from all minimal systems.

### 1.6. Main results of the paper.

It turns out that if a transitive t.d.s. \((X, T)\) is disjoint from all minimal t.d.s., then each transitive point of \((X, T)\) is a weak product recurrent one (Theorem 4.3). Thus, by [28] it is not necessarily minimal. Moreover, it is proved that the orbit closure of each weak product recurrent point is an \(M\)-system, i.e., with a dense set of minimal points (Theorem 4.5). Contrary to the above situation it is shown that an \(F_{ps}\)-product recurrent point is minimal (Theorem 3.4).

Results on product recurrence when the closure of an \(F\)-recurrent point has zero entropy are obtained. It is shown that if \((x, y)\) is recurrent for any point \(y\) whose orbit closure is a minimal system having zero entropy, then \(x\) is \(F_{pubd}\)-recurrent (Theorem 5.5); and if \((x, y)\) is recurrent for any point \(y\) whose orbit closure is an \(M\)-system having zero entropy, then \(x\) is minimal (Theorem 5.6). Moreover, it turns out that if \((x, y)\) is recurrent for any recurrent \(y\) whose orbit closure has zero entropy, then \(x\) is distal (Theorem 5.2).

Several results on disjointness are obtained, and results on weak disjointness are described when considering disjointness. For example, it is proved that a weakly mixing system with dense minimal points is disjoint from all minimal PI systems (Theorem 7.10); and a weakly mixing system with a dense set of distal points or an \(F_s\)-independent t.d.s. is disjoint from any minimal t.d.s. (Theorem 7.14 and 7.21). Moreover, it is shown that if a transitive t.d.s. is disjoint from all minimal weakly mixing t.d.s., then it is an \(M\)-system (Proposition 7.32).

### 1.7. Organization of the paper.

The paper is organized as follows: In Section 2 we discuss recurrence and product recurrence. We begin with the Hindman Theorem and review Furstenberg’s result about product recurrence. In Section 3 we study \(F_{ps}\)-product recurrence and show that any \(F_{ps}\)-product recurrent point is minimal. In Section 4 we aim to show that the closure of an \(F_s\)-product recurrent point is an \(M\)-system. On the way to doing this, we show that if \((X, T)\) is a transitive t.d.s. which is disjoint from any minimal system, then each point in \(Tran_T\) is \(F_s\)-product recurrent. In Section 5 we study \(F\)-product recurrence with zero entropy. We discuss properties concerning
extensions and factors in Section 6. We study disjointness and weak disjointness in Section 7. Section 8 contains two tables and in Section 9 we discuss some more generalizations of the notions concerning product recurrence. Finally, in the Appendix in Section 10 we discuss relative proximal cells.

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2. Recurrence and product recurrence

It is known that $x$ is distal if and only if $(x, y)$ is recurrent for any recurrent point $y$ [15]. The usual proof uses the Auslander–Ellis theorem which states that if $(X, T)$ is a t.d.s. and $x \in X$, then there is a minimal point $y \in \text{orb}(x, T)$ such that $(x, y)$ is proximal. Usually one proves the Auslander–Ellis theorem by using the Ellis semigroup theory. In this section we give a proof of the theorem without using the Ellis semigroup theory.

2.1. Recurrence and the IP-set. In this subsection the Hindman Theorem is used to prove the Auslander–Ellis Theorem. Also, some interesting relations between recurrence and the IP-set will be built.

Theorem 2.1 (Hindman, [21]): For any finite partition of an IP-set, one of the cells of the partition is an IP-set.

The following lemma is basically due to Furstenberg; see [15].

Lemma 2.2: Let $(X, T)$ be a compact metric t.d.s. If $x \in R(X, T)$ and $\{V_i\}_{i=1}^\infty$ is a collection of neighborhoods of $x$, then there is some IP-set $FS(\{p_i\}_{i=1}^\infty)$ such that $FS(\{p_i\}_{i=n}) \subset N(x, V_n)$ for all $n \in \mathbb{N}$. Especially, each recurrent point is $\mathcal{F}_{i_p}$-recurrent.

Proof. We prove the lemma using induction. Since $V_1$ is a neighborhood of $x$ and $x$ is recurrent, there is some $p_1 \in \mathbb{N}$ such that

$$T^{p_1}x \in V_1.$$
As $V_1, T^{-p_1}V_1, V_2$ are neighborhoods of $x$, so is their intersection $V_1 \cap T^{-p_1}V_1 \cap V_2$. And by the recurrence of $x$ there is some $p_2 \in \mathbb{N}$ such that

$$T^{p_2}x \in V_1 \cap T^{-p_1}V_1 \cap V_2.$$ 

Hence

$$T^{p_1}x, T^{p_2}x, T^{p_1+p_2}x \in V_1,$$

and

$$T^{p_2}x \in V_2.$$ 

Now for $n \in \mathbb{N}$ assume that we have a finite sequence $p_1, p_2, \ldots, p_n$ such that

$$FS(\{p_i\}_{i=1}^{n}) \subseteq N(x, V_j), \quad j = 1, 2, \ldots, n. \quad (2.1)$$

That is, for each $j = 1, 2, \ldots, n$

$$T^m x \in V_j, \quad \forall m \in FS(\{p_i\}_{i=1}^{n}).$$

Hence \( \left( \bigcap_{j=1}^{n} \bigcap_{m \in FS(\{p_i\}_{i=1}^{n})} T^{-m}V_j \right) \cap \bigcap_{i=1}^{n+1} V_i \) is a neighborhood of $x$. Take $p_{n+1} \in \mathbb{N}$ such that

$$T^{p_{n+1}}x \in \left( \bigcap_{j=1}^{n} \bigcap_{m \in FS(\{p_i\}_{i=1}^{n})} T^{-m}V_j \right) \cap \bigcap_{i=1}^{n+1} V_i.$$

Then for each $j = 1, 2, \ldots, n + 1$

$$T^m x \in V_j, \quad \forall m \in FS(\{p_i\}_{i=1}^{n+1}).$$

That is

$$FS(\{p_i\}_{i=1}^{n+1}) \subseteq N(x, V_j), \quad j = 1, 2, \ldots, n + 1.$$ 

So inductively we have an IP-set $FS(\{p_i\}_{i=1}^{\infty})$ such that $FS(\{p_i\}_{i=n}) \subset N(x, V_n)$ for all $n \in \mathbb{N}$. And the proof is completed. 

Let $(X, T)$ be a t.d.s. and $A \subseteq \mathbb{Z}_+$ be a sequence. Write

$$T^A x = \{T^nx : n \in A\}$$

and let $A - n = \{m - n : m \in A, m - n \geq 1\}$ for $n \in \mathbb{Z}_+$.

Using the method from [11], we have

**Lemma 2.3:** Let $(X, T)$ be a compact metric t.d.s. and $Q = FS(\{p_i\}_{i=1}^{\infty})$. For any $x \in X$ there is some $y \in \overline{TQ} \cap R(X, T)$ and $\{p_{n_i}\}_{i=1}^{\infty} \subseteq \{p_i\}_{i=1}^{\infty}$ such that, for any neighborhood $U$ of $y$, there is some $j$ with $FS(\{p_{n_i}\}_{i=j}^{\infty}) \subseteq N(y, U)$ and $(x, y) \in P(X, T)$.  

Proof. Set $K_1 = \overline{T^P x}$, $P_1 = Q$ and $p_{n_i} \in \{p_i\}_{i=1}^{\infty}$. Then
$$
P_1 \cap (P_1 - p_{n_i}) \supseteq FS(\{p_i\}_{i \neq n_1}).$$
Hence
$$K_1 \cap T^{-p_{n_1}} K_1 \supseteq \overline{T^{P_1 \cap (P_1 - p_{n_1})} x}.$$ Let $K_1 \cap T^{-p_{n_1}} K_1 = \bigcup_{i=1}^{r_1} K_{1,i}$, where $K_{1,i}$ is compact and $diam K_{1,i} < \frac{1}{2}$. So we have
$$P_1 \cap (P_1 - p_{n_1}) = \bigcup_{i=1}^{r_1} \{n \in P_1 \cap (P_1 - p_{n_1}) : T^n x \in K_{1,i}\}.$$ By the Hindman Theorem there is some $j$ such that
$$P_2 = \{n \in P_1 \cap (P_1 - p_{n_1}) : T^n x \in K_{1,j}\}$$ is an IP-subset of $P_1 \cap (P_1 - p_{n_1})$. And we set $K_2 = K_{1,j}$. Clearly, $K_2 \subseteq K_1$, $diam K_2 < \frac{1}{2}$, $T^{p_{n_1}} K_2 \subseteq K_1$ and $T^{P_2} x \subseteq K_2$.

Continuing inductively, we have $\{p_{n_i}\} \subseteq \{p_i\}$, IP-sets $P_1 \supseteq P_2 \supseteq \cdots$ and compact sets $K_1 \supseteq K_2 \supseteq \cdots$ such that $diam K_j < \frac{1}{j}$, $p_{n_j} \in P_j$, $T^{p_{n_j}} K_{j+1} \subseteq K_j$ and $T^{P_j} x \subseteq K_j$. Let $y \in \bigcap_{i=1}^{\infty} K_i$. It is easy to check that $y$ is the point we look for.

**Proposition 2.4:** Let $(X, T)$ be a compact metric t.d.s. If $(Y, S)$ is another t.d.s. and $z \in R(Y, S)$, then for any $x \in X$ there is some $y \in orb(x, T)$ such that $(x, y) \in P(X, T)$ and $(y, z)$ is a recurrent point of $X \times Y$.

**Proof.** Let $\{V_n\}_{n=1}^{\infty}$ be a neighborhood basis of $z$. By Lemma 2.2 there is some IP set $Q = FS(\{p_i\}_{i=1}^{\infty})$ such that $FS(\{p_i\}_{i=n}^{\infty}) \subset N(z, V_n)$ for all $n \in \mathbb{N}$. Let $y$ be the recurrent point described in Lemma 2.3. Then for any neighborhoods $U, V$ of $y, z$ we have
$$N((y, z), U \times V) = N(y, U) \cap N(z, V) \neq \emptyset.$$ Hence $(y, z)$ is a recurrent point of $X \times Y$.

**Theorem 2.5 (Auslander–Ellis):** Let $(X, T)$ be a compact metric t.d.s. Then for any $x \in X$ there is some minimal point $y \in orb(x, T)$ such that $(x, y)$ is proximal.

**Proof.** Without loss of generality, we assume $x$ is not minimal. Then there is some minimal set $Y$ in $\overline{orb(x)}$. Now we will find a thick $A$ such that $\overline{T^A x} \setminus T^A x \subseteq Y$. Then taking any IP-subset $Q$ from $A$, by Lemma 2.3 there is
some $y \in \overline{TQx} \cap R(X,T)$ and $(x,y) \in P(X,T)$. Since $y \in \overline{TQx} \setminus TQx \subseteq Y$, $y$ is a minimal point. Thus we finish our proof.

It remains to find a thick $A$ such that $\overline{T^A x} \setminus T^A x \subseteq Y$. Let

$$V_n = \{ z \in X : d(z,Y) < 1/n \}$$

and then $\{V_n\}_{n=1}^{\infty}$ is a neighborhood basis of $Y$. Let $\delta_n > 0$ such that $d(T^i x', T^i x'') < 1/n$, $i = 0, 1, \ldots, n-1$ if $d(x', x'') < \delta_n$. As $Y \subseteq \overline{\text{orb}(x,T)}$ there is some $i_n$ such that $d(T^{i_n} x, Y) < \delta_n$. Then by the invariance of $Y$, $d(T^{i_n+j} x, Y) < 1/n$, $j = 0, 1, \ldots, n-1$. Set $A = \bigcup_{n=1}^{\infty} \{i_n + j\}_{j=0}^{n-1}$. By our construction we have $\overline{T^A x} \setminus T^A x \subseteq Y$.  

Remark 2.6: (1) The previous proofs of Theorem 2.5 involve the use of Zorn’s Lemma. Here for a compact metric space we get a proof only using the Hindman Theorem. Note that usually, to show that any t.d.s. $(X,T)$ contains some minimal subsystem, we use the well-known Zorn’s Lemma argument. But for the case when $X$ is metric and the action semigroup is $\mathbb{Z}^+$ Weiss [36] gave a constructive proof.

(2) From the Auslander–Ellis Theorem, Furstenberg introduced the notion of a central set. A subset $S \subseteq \mathbb{Z}^+$ is a central set if there exists a system $(X,T)$, a point $x \in X$ and a minimal point $y$ proximal to $x$, and a neighborhood $U_y$ of $y$ such that $N(x, U_y) \subseteq S$. It is known that any central set is an IP-set [15, Proposition 8.10].

(3) By Lemma 2.2, $x$ is a recurrent point if and only if it is $\mathcal{F}_{ip}$-recurrent. In [15, Theorem 2.17] it is also shown that for any IP-set $R$ there exists a t.d.s. $(X,T)$, a recurrent $x \in X$ and a neighborhood $U$ of $x$ such that $N(x, U) \subseteq R \cup \{0\}$.

2.2. Product recurrence. The following proposition was proved in [15, Theorem 9.11.] and we give a proof for completeness.

Proposition 2.7: Let $(X,T)$ be a t.d.s. The following statements are equivalent:

1. $x$ is distal.
2. $x$ is product recurrent.
3. $(x,y)$ is minimal for each minimal point $y$ of a system $(Y,S)$.
4. $x$ is $IP^*$-recurrent.
Proof. Denote $X = \overline{\text{orb}(x,T)}$. First, by Remark 2.6 it is easy to see that (2) $\iff$ (4).

(1) $\implies$ (4). If $x$ is not $IP^*$-recurrent, then there is a neighborhood $U$ of $x$ such that $N(x,U)$ is not an $IP^*$-set, i.e., there exists an IP-set $Q$ such that $TQ \cap U = \emptyset$. By Lemma 2.3, we know that there is a point $y \in TQ$, i.e., $y \not\in U$, such that $(x,y) \in P(X,T)$, which contradicts the assumption that $x$ is distal.

(4) $\implies$ (1). As any thick set contains an IP-set, we get that $x$ is a minimal point. If $x$ is not distal, there exists a different point $x' \in X$ such that $(x,x') \in P(X,T)$. Let $U$ and $U'$ be any neighborhoods of $x$ and $x'$ which are disjoint; $N(x,U')$ is a central set and contains an IP-set, so $N(x,U) \cap N(x,U') \neq \emptyset$, which implies $x = x'$.

(1) $\implies$ (3). Let $y$ be a minimal point of $(Y,S)$. If $(x,y)$ is not minimal, by Theorem 2.5 there exists a minimal point $(x',y') \in \text{orb}((x,y),T \times S)$ which is proximal to $(x,y)$. It follows that $x'$ is proximal to $x$, which implies $x = x'$. For any neighborhood $U \times V$ of $(x,y)$, $N(x,U)$ is an $IP^*$-set and $N(y',V)$ is a central set as $y'$ is proximal to minimal point $y$, so we know that $N(x,U) \cap N(y',V) \neq \emptyset$, i.e., $(x,y) \in \text{orb}((x,y'),T \times S)$, which implies that $(x,y)$ is a minimal point.

(3) $\implies$ (1). It is easy to see that $x$ is a minimal point. If there exists a point $x' \in \text{orb}(x,T)$ which is proximal to $x$, then there exists a point $(y,y) \in \text{orb}((x,x'),T \times T)$. As $(x,x')$ is a minimal point, then $(x,x') \in \text{orb}((y,y),T \times T)$, which implies $x = x'$, so $x$ is distal.

3. $F_{ps}$-product recurrent points

In this section we aim to show that if $x$ is an $F_{ps}$-product recurrent point, then it is minimal.

Definition 3.1: Let $(X,T)$ be a t.d.s. and $F$ be a family. Then $x \in X$ is $F$-product recurrent ($F$-PR for short) if, given any $F$-recurrent point $y$ in any t.d.s. $(Y,S)$, $(x,y)$ is recurrent in the product system $(X \times Y,T \times S)$.

By definition we have the following observation immediately.

Lemma 3.2: Let $F_1,F_2$ be two families with $F_1 \subseteq F_2$. Then each $F_2$-PR point is $F_1$-PR.
It is clear that
\[ F_{inf} - PR \Rightarrow F_{pubd} - PR \Rightarrow F_{ps} - PR \Rightarrow F_{s} - PR. \]

It was shown in [20] that an \( F_{s} \)-PR point is not necessarily minimal (more examples will be given in the next section). A natural question is: if \( x \) is \( F_{ps} \)-PR, is \( x \) minimal? Before continuing discussion, we need some preparation about symbolic dynamics. Let \( \Sigma_2 = \{0, 1\}^{\mathbb{Z}^+} \) and \( \sigma: \Sigma_2 \rightarrow \Sigma_2 \) be the shift map, i.e., the map
\[(x_0, x_1, x_2, x_3, \ldots) \mapsto (x_1, x_2, x_3, \ldots) \in \Sigma_2.\]

A shift space \((X, \sigma)\) is a subsystem of \((\Sigma_2, \sigma)\). For any \( S \subset \mathbb{Z}^+ \), we denote by \( 1_S \in \{0, 1\}^{\mathbb{Z}^+} \) the indicator function of \( S \), i.e., \( 1_S(a) = 1 \) if \( a \in S \) and \( 1_S(a) = 0 \) if \( a \not\in S \). For finite blocks \( A = (a_1, \ldots, a_n) \in \{0, 1\}^n \) and \( B = (b_1, \ldots, b_n) \in \{0, 1\}^n \) we say \( A \leq B \) if \( a_i \leq b_i \) for each \( i \in \{1, 2, \ldots, n\} \). For finite blocks \( A \) and \( B \) we denote the length of \( A \) by \(|A|\), \( A \cdots A \) by \( A^n \) for \( n \in \mathbb{N} \) (in particular, \( 0^n = 00\cdots0 \)), and the concatenation of \( A \) and \( B \) by \( AB \). If \((X, \sigma)\) is a shift space, let
\[ [i] = [i]_X = \{x \in X : x(0) = i\} \]
for \( i = 0, 1 \), and \([A] = [A]_X = \{x \in X : x_0 x_1 \cdots x_{(|A| - 1)} = A\}\) for any finite block \( A \).

To settle down the question we need the following notion. By an md-set \( A \) we mean there is an \( M \)-system \((Y, S)\), a transitive point \( y \in Y \) and a neighborhood \( U \) of \( y \) such that \( A = N(y, U) \).

**Proposition 3.3:** Every thick set containing 0 contains an md-set.

**Proof.** Let \( C \subset \mathbb{Z}^+ \) be a thick set with \( 0 \in C \). Let \( x = 1_C = (x_0, x_1, \ldots) \in \{0, 1\}^{\mathbb{Z}^+} \).

By the assumption \( x_0 = 1 \), and there are \( p_n < q_n \in \mathbb{N} \) with \( 11 \cdots 1 \leq (x_{p_n}, \ldots, x_{q_n}) \) for any \( n \in \mathbb{N} \). It is clear that there is \( a_1 \geq 1 \) such that
\[ A_1 = 10^{a_1} 1 \leq (x_0, \ldots, x_{l_1}) \]
with \( l_1 = |A_1| - 1 \). By the same reasoning there is \( a_2 > a_1 \) and \( a_2 \) can be divided by \(|A_1|\) with
\[ A_2 = A_1 0^{a_2} A_1 \leq (x_0, \ldots, x_{l_2}), \]
where $l_2 = |A_2| - 1$. Then $|A_2|$ can be divided by $|A_1|$.

Inductively assume that $A_1, \ldots, A_k$ are defined. Then there is $a_{k+1} > a_k$ and $a_{k+1}$ can be divided by $|A_k|$ with

$$A_{k+1} = A_k 0^{a_k} A_k A_{k-1}^{n_{k-1}} \cdots A_2^{n_2} A_1^{n_1} \leq (x_0, \ldots, x_{l_{k+1}})$$

where $|A_1|^{n_1} = |A_2|^{n_2} = \cdots = |A_{k-1}|^{n_{k-1}} = |A_k|$ and $l_{k+1} = |A_{k+1}| - 1$. Then $|A_{k+1}|$ can be divided by $|A_j|$ for $1 \leq j \leq k$. It is easy to see that $\forall i \in \mathbb{N}, n_i \to \infty$ when $j \to \infty$.

Let $y = \lim_{k \to \infty} A_k \in \{0, 1\}^{\mathbb{Z}_+}$; then $y$ is a recurrent point under the shift $\sigma$. It is clear that $N(y, [A_n])$ is piecewise syndetic. Thus the orbit closure of $y$ is an $M$-system (in fact it is a $P$-system). At the same time,

$$N(y, [1]) = \{ n \in \mathbb{Z}_+ : \sigma^n y \in [1] \} \subset C.$$ 

This completes the proof. ■

Now we give a positive answer to the question.

**Theorem 3.4:** Let $(X, T)$ be a t.d.s. If $x$ is $F_{ps}$-PR, then it is minimal.

**Proof.** If $x$ is not minimal, then there is a neighborhood $U$ of $x$ such that $N(x, U)$ is not syndetic. Thus, $\mathbb{Z}_+ \setminus N(x, U)$ is thick. Let $C = \{0\} \cup \mathbb{Z}_+ \setminus N(x, U)$. By Proposition 3.3, $C$ contains a subset $A = N(y, V)$, where $y$ is a transitive point of some $M$-system, which is $F_{ps}$-recurrent, and $V$ is a neighborhood of $y$. Then

$$N((x, y), U \times V) = N(x, U) \cap N(y, V) \subset \{0\},$$

which implies that $(x, y)$ is not recurrent, a contradiction. Thus $x$ is minimal. ■

Since each $F_{pubd}$-PR point is an $F_{ps}$-PR one, as a corollary of Theorem 3.4, each $F_{pubd}$-PR point is minimal. Generally, we have

**Corollary 3.5:** Let $F$ be a family with $F_{ps} \subseteq F$. Then each $F$-PR point is minimal.

### 4. $F_s$-product recurrent points

In this section we aim to show that the orbit closure of an $F_s$-product recurrent point is an $M$-system. On the way to doing this, we show that if $(X, T)$ is a transitive t.d.s. which is disjoint from any minimal system, then each transitive
point of \((X, T)\) is \(\mathcal{F}_s\)-PR. Thus combining results from [28] we reprove that an \(\mathcal{F}_s\)-PR point is not necessarily minimal, which was obtained in [20]. Note that weak product recurrence has also been discussed in [31] recently.

4.1. \(\mathcal{F}_s\)-PRODUCT RECURRENCE.

**Definition 4.1:** A subset \(A\) of \(\mathbb{Z}_+\) is called an **m-set**, if there exist a minimal system \((Y, S), y \in Y\) and a non-empty open subset \(V\) of \(Y\) such that \(A \supset N(y, V)\). Note that \(y\) need not be a point of \(V\). The family consisting of all \(m\)-sets is denoted by \(\mathcal{F}_{mset}\).

A subset \(A\) of \(\mathbb{Z}_+\) is called an **sm-set** (standing for standard \(m\)-set), if there exist a minimal system \((Y, S), y \in Y\) and a non-empty neighborhood \(V\) of \(y\) such that \(A \supset N(y, V)\). The family consisting of all \(sm\)-sets is denoted by \(\mathcal{F}_{smset}\).

It is clear that \(\mathcal{F}_{smset} \subset \mathcal{F}_{mset}\) and hence \(\mathcal{F}_{smset}^* \subset \mathcal{F}_{mset}^*\). We will show (Proposition 4.4) that \(\mathcal{F}_{smset}^* \subset \mathcal{F}_{ps}\). Moreover, we have the following observation.

**Proposition 4.2:** The following statements hold.

1. Let \((X, T)\) be transitive and \(x \in \text{Tran}_T\). Then \((X, T)\) is disjoint from any minimal t.d.s. if and only if \(N(x, U) \cap A \neq \emptyset\) for each neighborhood \(U\) of \(x\) and each \(m\)-set \(A\), i.e., \(N(x, U) \in \mathcal{F}_{mset}^*\).

2. A point \(x\) is \(\mathcal{F}_s\)-PR if and only if, for each open neighborhood \(U\) of \(x\) and each \(sm\)-set \(A\), \(N(x, U) \cap A \neq \emptyset\), i.e., \(N(x, U) \in \mathcal{F}_{smset}^*\).

**Proof.** (1) is proved in [28]. (2) follows from the definitions. 

So we have

**Theorem 4.3:** Let \((X, T)\) be a transitive t.d.s. which is disjoint from any minimal system. Then each point in \(\text{Tran}_T\) is \(\mathcal{F}_s\)-PR and non-minimal.

**Proof.** It follows by Proposition 4.2 directly. We give a direct argument here. Let \(x \in \text{Tran}_T\) and \((Y, S)\) be a given minimal t.d.s. For \(y \in Y\) let \(A = \text{orb}((x, y), T \times S)\). It is clear that \(A\) is a joining and hence \(A = X \times Y\). This implies that \((x, y)\) is a recurrent point of \((X \times Y, T \times S)\) and hence \(x\) is \(\mathcal{F}_s\)-PR.

For a t.d.s. \((X, T)\), \(x \in X\) is a **regular minimal point** if, for each neighborhood \(U\) of \(x\), there is \(k = k(U)\) such that \(N(x, U) \supset k\mathbb{Z}_+\). In [28] Huang and
Ye showed that any weakly mixing t.d.s. with a dense regular minimal points is disjoint from any minimal t.d.s. There are a lot of non-minimal systems with this properties, for example the full shift and the example constructed in [28]. Thus an $\mathcal{F}_s$-PR point is not necessarily minimal. We note that this result was also obtained in [20]. So naturally one would ask: if $x$ is $\mathcal{F}_s$-PR and not minimal, what can we say about the properties of such point? In fact we will show that the closure of $x$ is an $M$-system, i.e., it has a dense minimal point.

The way we answer the question is that we will show every thickly syndetic set containing $\{0\}$ contains an m-set. Note that a subset $A$ of $\mathbb{Z}_+$ is thickly syndetic if it has non-empty intersection with any piecewise syndetic set. More precisely, a subset of $\mathbb{Z}_+$ is thickly syndetic if for each $n \in \mathbb{N}$ there is a syndetic subset $S_n = \{s_1^n, s_2^n, \ldots\}$ such that $S \supset \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \{s_{i+1}^n + 1, s_i^n + 2, \ldots, s_i^n + n\}$.

For a transitive system, whether it is disjoint from all minimal systems can be checked through m-sets; for the details see [28]. Particularly, the authors showed that every thickly syndetic set contains an m-set. To solve our question we need to show

**Proposition 4.4:** Every thickly syndetic set containing $\{0\}$ contains an m-set.

Since the proof of Proposition 4.4 is a little long, we leave it to the next subsection. Now we have

**Theorem 4.5:** The orbit closure of an $\mathcal{F}_s$-PR point is an $M$-system.

**Proof.** Let $x$ be an $\mathcal{F}_s$-PR point and $U$ be an open neighborhood of $x$. If $N(x, U)$ is not piecewise syndetic, then $A = \mathbb{Z}_+ \setminus N(x, U)$ is thickly syndetic. Then by Proposition 4.4, $A \cup \{0\}$ contains $N(y, V)$, where $(Y, S)$ is a minimal set, $y \in Y$ and $V$ is an open neighborhood of $y$. Thus we have

\[ N((x, y), U \times V) = N(x, U) \cap N(y, V) \subset \{0\}, \]

which implies that $(x, y)$ is not recurrent, a contradiction. 

**Remark 4.6:** Recall that two t.d.s. $(X, T)$ and $(Y, S)$ are weakly disjoint if $(X \times Y, T \times S)$ is transitive. A t.d.s. is scattering if it is weakly disjoint from all minimal t.d.s. [9]. We remark that a transitive point in a non-minimal scattering t.d.s. is not necessarily weakly product recurrent, since there is an almost equicontinuous scattering t.d.s., and such a system is never an $M$-system;
see [25, Theorem 4.6]. It is noteworthy that when considering weak disjointness the return time sets \( N(U, V) \) play the crucial role, but this is not the case when considering disjointness or weak product recurrence, where sets \( N(x, U) \) play the role.

We also have the following remark.

**Remark 4.7:** It is easy to see that if \( x \) is weakly product recurrent and \( y \) is distal, then \((x, y)\) is also weakly product recurrent. This implies that \( \text{orb}(x, y) \) is not necessarily weakly mixing. Thus, the collection of sm-sets is strictly contained in the collection of m-sets, since if \((X, T)\) is transitive and is disjoint from all minimal t.d.s., then \((X, T)\) is weakly mixing; see [28].

4.2. Proof of Proposition 4.4. Let \( F \subset \mathbb{Z}_+ \) be a thickly syndetic subset containing \( \{0\} \). We will construct \( y^n = 1_{F_n} \in \{0, 1\}^{\mathbb{Z}_+} \) such that \( F_n \subset F \) and \( y = \lim_{n \to \infty} y^n = 1_A \) is a minimal point. Then let \( Y = \text{orb}(y, \sigma) \) and \( [1] = \{x \in Y : x(0) = 1\} \). Since \( A \subset F \) and \( A = N(y, [1]) \), we have the theorem.

To obtain \( y^n \) we construct a finite word \( A_n \) such that \( y_n \) begins with \( A_n \). The reason we can do this is that \( 1^n = (1, \ldots, 1) \) \((n \text{ times})\) appears in \( 1_F \) syndetically for each \( n \in \mathbb{N} \).

More precisely we do as follows.

**Step 1:** Construct \( A_1 \) and \( F_1 \subset F \) such that \( A_1 \) appears in \( y_1 \) with gaps bounded by \( l_1 \) and \( y_1 \) begins with \( A_1 \).

- Let \( \min F = a_1 - 1 \) and \( A_1 = 1_F[0; a_1 - 1] \). Set \( B_1 = A_1 A_1 0 A_1 \) and \( r_1 = b_1 = |B_1| = 3a_1 + 1 \). As \( F \) is thickly syndetic, \( 1^{r_1} \) appears in \( F \) at a syndetic set \( W_1 = \{w_1^1, w_2^1, \ldots\} \). Without loss of generality assume that \( 2r_1 \leq w_{j+1}^1 - w_j^1 \leq l_1 \) and \( 2k_1 \leq w_1^1 \leq l_1 \), where \( l_1 \) is some number in \( \mathbb{N} \). Put \( u_i^1 = w_i^1, i \in \mathbb{N} \).
- Choose \( y^1 \in \{0, 1\}^{\mathbb{Z}_+} \) such that
  - \( y^1[0; a_1 - 1] = A_1, y^1[u_i^1; u_i^1 + b_1 - 1] = B_1 \) and
  - \( y^1(j) = 0 \) if \( j \in \mathbb{Z}_+ \setminus ([0; a_1 - 1] \cup \bigcup_{i=1}^\infty [u_i^1; u_i^1 + b_1 - 1]) \).

It is easy to see that \( B_1 \) as well as \( A_1 \) appear in \( y^1 \) with gaps bounded by \( l_1 \) and \( F_1 \subset F \), where \( 1_{F_1} = y^1 \).

**Step 2:** Construct \( A_2 \) and \( F_2 \subset F \) such that

1. \( A_2 \) has the form of \( A_1 V_1 B_1 \) and, if \( a_2 = |A_2| \), then \( A_2 = y^1[0; a_2 - 1] \).
2. \( y^2[0; a_2 - 1] = A_2 \) and \( A_1, A_2 \) appear in \( y^2 \) syndetically with gaps bounded by \( l_1 \) and \( l_2 \), respectively.
3. \( F_2 = \{i \in \mathbb{Z}_+ : y^2(i) = 1\} \subset F \).
Set \(a_2 = u_1^1 + b_1\) and let \(A_2 = y^1[0; a_2 - 1]\), \(B_2 = A_2A_20A_2\), \(b_2 = |B_2| = 3a_2 + 1\). Then \(A_2\) has the form of \(A_1V_1B_1\). Let \(r_2 = 2l_1 + 2b_1 + b_2\). As \(F\) is thickly syndetic, \(1^{r_2}\) appears in \(F\) at a syndetic set \(W_2 = \{w_1^2, w_2^2, \ldots\}\). Without loss of generality assume that \(2r_2 \leq w_2^2 - w_i^2 \leq l_2 - (l_1 + b_1) \) and \(2a_2 \leq w_i^2 \leq l_2 - (l_1 + b_1)\), where \(l_2\) is some number in \(\mathbb{N}\).

To get \(y^2\) we change \(y^1\) at places \([w_1^2; w_i^2 + r_2 - 1]\) for each \(i \in \mathbb{N}\). To illustrate the idea, it is enough to show what we do at \([w_1^2; w_i^2 + r_2 - 1]\).

Let \(k, j\) satisfy that \(u_{k-1}^1 < w_1^2 \leq u_k^1\) and \(u_1^1 + b_1 - 1 \leq w_1^2 + r_2 - 1 < u_{j+1}^1 + b_1 - 1\). Let \(l\) be the integer part of \((u_1^1 - 1 - u_k^1 - b_1 - b_2)/b_1\).

Put \(u_1^2 = u_k^1 + b_1\). Let \(y^2[w_i^2; u_i^2 + b_2 - 1] = B_2\) and

\[
y^2[w_1^2 + b_2 + pb_1; u_1^2 + b_2 + (p + 1)b_1 - 1] = B_1 \quad \text{for} \quad p = 0, 1, \ldots, l - 1.
\]

That is, first we put \(B_2\) at place \(u_1^2\) and then we put as many \(B_1\) as we can. We do the same at all places \([w_1^2; w_i^2 + r_2 - 1]\); we get \(u_i^2 \in [w_1^2, w_i^2 + r_2 - 1]\) with \(y^2[w_i^2; u_i^2 + b_2 - 1] = A_2\), \(i = 1, 2, \ldots\).

In such a way we get \(y^2\). It is easy to see that \(y^1\) and \(y^2\) differ possibly at \([w_i^2; w_i^2 + r_2 - 1]\). Thus

\[
F_2 = \{i \in \mathbb{Z}_+: y^2(i) = 1\} \subset F_1 \cup \bigcup_{i=1}^{\infty} [w_i^2; w_i^2 + r_2 - 1].
\]

At the same time \(B_1, B_2\) appear in \(y^2\) syndetically with gaps bounded by \(l_1\) and \(l_2\), respectively, by the construction and so are \(A_1, A_2\).

**Step 3:** Construct \(A_{m+1}\) and \(F_{m+1}\) inductively such that

1. \(A_{m+1}\) has the form of \(A_mB_mB_m\) and, if \(a_{m+1} = |A_{m+1}|\), then \(A_{m+1} = y^{m}[0; a_{m+1} - 1]\).
2. \(y^{m+1}[0; a_{m+1} - 1] = A_{m+1}\) and \(A_i\) appear in \(y^{m+1}\) syndetically with gaps bounded by \(l_i\) for each \(1 \leq i \leq m + 1\).
3. \(F_{m+1} = \{i \in \mathbb{Z}_+: y^{m+1}(i) = 1\} \subset F\).

Set \(a_{m+1} = u_1^{m} + b_m\) and let \(A_{m+1} = y^m[0; a_{m+1} - 1]\), \(B_{m+1} = A_{m+1}A_{m+1}0A_{m+1}\), and \(b_{m+1} = |B_{m+1}| = 3a_{m+1} + 1\). Then \(A_{m+1}\) has the form of \(A_mB_mB_m\). Let \(r_{m+1} = 2l_m + 2b_m + b_{m+1}\). As \(F\) is thickly syndetic, \(1^{r_{m+1}}\) appears in \(F\) at a syndetic set \(W_{m+1} = \{w_1^{m+1}, w_2^{m+1}, \ldots\}\). Without loss of generality assume that \(2r_{m+1} \leq w_j^{m+1} - w_i^{m+1} \leq l_{m+1} - (l_m + b_m)\) and \(2k_{m+1} \leq w_i^{m+1} \leq l_{m+1} - (l_m + b_m)\), where \(l_{m+1}\) is some number in \(\mathbb{N}\).

To get \(y^{m+1}\) we change \(y^m\) at places \([w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1]\) for each \(i \in \mathbb{N}\). To illustrate the idea it suffices to show what we do at \([w_1^{m+1}, w_1^{m+1} + r_{m+1} - 1]\).
Let \( k, j \) satisfy that \( u_{k-1}^m < w_1^{m+1} \leq u_k^m \) and \( u_j^m + b_m - 1 \leq w_1^{m+1} + r_{m+1} - 1 < u_{j+1}^m + b_m - 1 \).

Put \( u_1^{m+1} = u_k^m + b_m \). Let \( y^{m+1} = u_1^{m+1} \leq u_j^m + b_m + 1 \) and \( y^{m+1} + [u_1^{m+1}; u_j^m + b_m + 1 - 1] = B_{m+1} \) and

\[
y^{m+1}[u_1^{m+1}, u_j^m - 1] = B_{m+1}(B_m)^{p_m}(B_{m-1})^{p_{m-1}} \cdots (B_1)^{p_1}C_{m+1},
\]

where \( C_{m+1} \) is a word, and \( p_1, \ldots, p_m \) are natural numbers with

1. \( |C_{m+1}| < b_1 \),
2. \( |C_{m+1}| + b_1 p_1 < b_2 \), and
3. \( |C_{m+1}| + b_1 p_1 + \ldots + b_i p_i < b_{i+1} \) for each \( 1 \leq i \leq m - 1 \).

That is, first we put \( B_{m+1} \) at place \( u_1^{m+1} \) and, starting from \( u_1^{m+1} + k_{m+1} \) to \( u_j^m \), we put as many \( B_m \) as we can and then we put as many \( B_{m-1} \) as we can, and so on. We do the same at all places \([u_i^{m+1}; u_i^{m+1} + r_{m+1} - 1] \); we get \( u_i^{m+1} \in [u_i^{m+1}; u_i^{m+1} + r_{m+1} - 1] \) with \( y^{m+1}[u_1^{m+1}, u_i^{m+1} + b_m + 1 - 1] = B_{m+1}, i = 1, 2, \ldots \).

In such a way we get \( y^{m+1} \). It is easy to see that \( y^{m+1} \) and \( y^{m} \) differ possibly only at \([w_i^{m+1}; w_i^{m+1} + r_{m+1} - 1], i = 1, 2, \ldots \). Thus

\[
F_{m+1} = \{ i \in \mathbb{Z}_+ : y^{m+1}(i) = 1 \} \subset F_m \cup \bigcup_{i=1}^{\infty} [w_i^{m+1}, w_i^{m+1} + r_{m+1} - 1].
\]

At the same time \( B_i \) appears in \( y^{m+1} \) syntetically with gaps bounded by \( l_i \) for each \( 1 \leq i \leq m + 1 \) by the construction, and so is \( A_i \) for each \( 1 \leq i \leq m + 1 \).

In such a way for each \( m \in \mathbb{N} \) we defined a finite word \( A_m \). Let \( y = \lim A_m = \lim y^m \). By the construction, \( A_m \) appears in \( y \) with gaps bounded by \( l_m \) for each \( m \in \mathbb{N} \). That is, \( y \) is a minimal point for the shift. It is obvious that \( y \neq (0, 0, \ldots) \). Let \( Y = \overline{\text{orb}(y, \sigma)} \) and \( U = [1] = \{ x \in Y : x(0) = 1 \} \). Then

\[
\emptyset \neq N(y, U) = \bigcup_{i=1}^{\infty} \{ i \in \mathbb{Z}_+ : A_n(i) = 1, 0 \leq i \leq k_n - 1 \} \subset \bigcup_{i=1}^{\infty} F_n \subset F.
\]

Thus \( F \) contains the m-set \( N(y, U) \). The proof is completed.

Remark 4.8: In fact, in the proof of Proposition 4.4, \((Y, \sigma)\) is a weakly mixing system. Indeed, for each \( m \in \mathbb{N} \), \( A_{m+1} \) has the form \( A_m V_m B_m \), i.e., the form \( A_m V_m A_m A_m 0 A_m \), so we know that \( N([A_m], [A_m]) = N(y, [A_m]) - N(y, [A_m]) \supset \{ a_m, a_m + 1 \} \), which implies that \( Y \) is weakly mixing (see Lemma 4.9 below).
4.3. THE CONDITION IN [20]. In this subsection we will show that there is no minimal t.d.s. satisfying the sufficient condition in [20, Theorem 3.1]. First we state this condition as follows: Let \((X, T)\) be a t.d.s. Say \(x \in X\) satisfies the property \((\star)\):

\[(\star) \text{ if for each neighborhood } V \text{ of } x, \text{ there exists } n = n(V) \text{ such that if } S \subset \mathbb{Z}_+ \text{ is a finite subset with } |s - t| \geq n \text{ for all distinct } s, t \in S, \text{ then there exists } \ell \in \mathbb{Z}_+ \text{ such that } T^{s+\ell} x \in V \text{ for all } s \in S.\]

We will show that if \((X, T)\) is a transitive system with a transitive point \(x\) satisfying \((\star)\), then it is weakly mixing. Note that the orbit closure of an \(F_s\)-PR point need not be weakly mixing (see Remark 4.7).

First we require the following lemma.

**Lemma 4.9** ([28, Lemma 5.1]): Let \((X, T)\) be a transitive system. If for any open non-empty subset \(U\) of \(X\) there is \(s = s_U \in \mathbb{Z}_+\) such that \(s, s + 1 \in N(U, U)\), then \((X, T)\) is weakly mixing.

Let \(F_{rs}\) be the smallest family containing \(\{n \mathbb{Z}_+ : n \in \mathbb{N}\}\). The following notion was introduced in [28]. Let \((X, T)\) be a t.d.s. We say \((X, T)\) has **dense small periodic sets** if, for any open and non-empty subset \(U\) of \(X\), there exists a non-empty closed \(A \subset U\) and \(k \in \mathbb{N}\) such that \(A\) is invariant for \(T^k\). Now we are ready to show

**Lemma 4.10:** Let \((X, T)\) be a transitive system with a transitive point \(x\) satisfying \((\star)\). Then \((X, T)\) is weakly mixing, and it has dense small periodic sets.

**Proof.** First we show \((X, T)\) is weakly mixing. Let \(U\) be a non-empty open subset of \(X\) and \(V\) be a neighborhood of \(x\) such that \(T^m V \subset U\) for some \(m \in \mathbb{N}\). Assume \(n = n(V)\) is the number appearing in the definition of \((\star)\). Then there is \(\ell \in \mathbb{Z}_+\) such that \(\{\ell + n, \ell + 2n, \ell + 3n + 1\} \subset N(x, V)\). That is, \(T^{\ell+n} x, T^{\ell+2n} x, T^{\ell+3n+1} x \in V\), which implies that \(T^{m+\ell+n} x, T^{m+\ell+2n} x, T^{m+\ell+3n+1} x \in U\). Thus \(\ell + n, \ell + 2n, \ell + 3n + 1 \in N(T^m x, U)\). We have

\[N(U, U) = N(T^m x, U) - N(T^m x, U) \supset \{n, n + 1\}.\]

By Lemma 4.9, \((X, T)\) is weakly mixing.

Now we show \((X, T)\) has dense small periodic sets. Let \(V\) be a neighborhood of \(x\) and \(n = n(V)\) be the number appearing in the definition of \((\star)\). By \((\star)\), for all \(k \in \mathbb{Z}_+\) there is some \(l = l(k) \in \mathbb{Z}_+\) such that \(\bigcap_{j=0}^k T^{-jn-1} V \neq \emptyset\). That is, \(\bigcap_{j=0}^k T^{-jn} V = \emptyset\). By a compactness argument we have \(\bigcap_{j=0}^\infty T^{-jn} V \neq \emptyset\).
This implies that there is \( y \in \bigcap_{j=0}^{\infty} T^{-jn} V \) such that \( T^{jn} y \in \overline{V} \) for all \( j \in \mathbb{Z}_+ \). Thus, \((X, T)\) has dense small periodic sets since \( x \) is transitive. ■

With the help of Lemma 4.10 we have

**Theorem 4.11:** There is no minimal t.d.s. with points satisfying (\(*\)).

**Proof.** Assume the contrary, that there is a minimal t.d.s. \((X, T)\) with points satisfying (\(*\)). By Lemma 4.10, \((X, T)\) has dense small periodic sets. If \( A \) is a proper closed \( T^k \) invariant subset, then \( \bigcup_{i=0}^{k-1} T^i(A) = X \). It follows that \((X, T)\) has a finite factor and so it is not totally transitive. But by Lemma 4.10, \((X, T)\) is weak mixing and so is totally transitive, a contradiction. ■

5. \( \mathcal{F} \)-product recurrence for zero entropy

Entropy is a measurement of complexity or chaos of a t.d.s. For a t.d.s. \((X, T)\) the entropy of \((X, T)\) will be denoted by \( h(T) \). For the definitions and basic properties of entropy and how to compute the entropy of a symbolic system, we refer to [34]. In this section we investigate the properties of points, whose product with points whose orbit closure have zero entropy is recurrent. We show that if \((x, y)\) is recurrent for any point \( y \) whose orbit closure is a minimal system having zero entropy, then \( x \) is \( \mathcal{F}_{pubd} \)-recurrent, and if \((x, y)\) is recurrent for any point \( y \) whose orbit closure is an \( M \)-system having zero entropy, then \( x \) is minimal. Moreover, it turns out that if \((x, y)\) is recurrent for any recurrent \( y \) whose orbit closure has zero entropy, then \( x \) is distal.

5.1. \( \mathcal{F} \)-PR\(_0\).

**Definition 5.1:** Let \((X, T)\) be a t.d.s. and \( \mathcal{F} \) be a family. Then \( x \in X \) is \( \mathcal{F} \)-PR\(_0\) if, for any t.d.s. \((Y, S)\) and any \( \mathcal{F} \)-recurrent point \( y \in Y \) whose orbit closure \( \text{orb}(y, S) \) has zero entropy, \((x, y)\) is a recurrent point of \((X \times Y, T \times S)\).

It is clear that

\[
\begin{align*}
\mathcal{F}_{inf} - PR &\longrightarrow \mathcal{F}_{pubd} - PR \longrightarrow \mathcal{F}_{ps} - PR \longrightarrow \mathcal{F}_s - PR \\
\mathcal{F}_{inf} - PR_0 &\longrightarrow \mathcal{F}_{pubd} - PR_0 \longrightarrow \mathcal{F}_{ps} - PR_0 \longrightarrow \mathcal{F}_s - PR_0
\end{align*}
\]

where “\(\longrightarrow\)” means implication.
Recall that $x$ is $F_{inf}$-PR if and only if $x$ is distal. We have

**Theorem 5.2**: Let $(X, T)$ be a t.d.s. and $x \in X$. Then $x$ is $F_{inf}$-PR$_0$ if and only if it is distal.

**Proof.** If $x$ is distal, then it is clear that it is $F_{inf}$-PR$_0$. Now assume that $x$ is $F_{inf}$-PR$_0$. Let $A$ be an IP-set. Then $A$ contains a sub IP-set $B$ with zero entropy (see, for example, [22]). Then $N(x, U)$ is IP$^*$ for each neighborhood $U$, and $x$ is distal by Proposition 2.7.

Similar to Theorem 4.3 we have

**Theorem 5.3**: Let $(X, T)$ be a transitive t.d.s. which is disjoint from any minimal system with zero entropy. Then each point in $\text{Tran}_T$ is $F_s$-PR$_0$.

Let $E(X, T)$ be the set of all entropy pairs (see [8]). A t.d.s. $(X, T)$ is **diagonal** if $\{(x, Tx) : x \in X\} \subset E(X, T)$ and **u.p.e.** if $E(X, T) = X^2 \setminus \Delta$. It was proved in [8] that a transitive diagonal system is disjoint from all minimal t.d.s. with zero entropy. Thus if $(X, T)$ is a transitive diagonal t.d.s., then each transitive point $x$ is in $F_s$-PR$_0$. It was proved in [22] that every subset of $\mathbb{Z}_+$ with lower Banach density 1 contains an m-set $A$ such that the orbit closure of $1_A$ has zero entropy. With a small modification we have the following proposition.

**Proposition 5.4**: Every subset of $\mathbb{Z}_+$ with lower Banach density 1 containing $\{0\}$ contains an sm-set $A$ such that the orbit closure of $1_A$ has zero entropy.

Using the same argument as in Theorem 4.5 we have

**Theorem 5.5**: The orbit closure of an $F_s$-PR$_0$ point is an $E$-system.

**Proof.** Let $x$ be an $F_s$-PR$_0$ point and $U$ be an open neighborhood of $x$. If $N(x, U)$ has zero Banach density, then the lower Banach density of $A = \mathbb{Z}_+ \setminus N(x, U)$ is 1. Then by Proposition 5.4, $A \cup \{0\}$ contains $N(y, V)$, where $(Y, S)$ is a minimal set, $y \in Y$, $V$ is an open neighborhood of $y$ and $h(S) = 0$. Thus we have $N((x, y), U \times V) = N(x, U) \cap N(y, V) \subset \{0\}$, a contradiction.

In [22] a transitive diagonal t.d.s. with a unique minimal point was constructed (see [23] for more examples). Thus we have

$$F_s - PR_0 \not\Rightarrow F_s - PR.$$
We remark that there is a minimal point $x$ which is $F_s$-PR$_0$ and is not $F_s$-PR. In fact by [10], if $h(T) > 0$ then there are asymptotic pairs $(x, y)$ with $x \neq y$, and by [19] or [29] there are minimal u.p.e. systems.

5.2. $F_{ps}$-PR$_0$. In Theorem 3.4 we have shown that if a point $x$ is $F_{ps}$-PR, then $x$ is minimal. Here is a natural question: if $x$ is $F_{ps}$-PR$_0$, is $x$ minimal? The answer is affirmative. That is, we have

**Theorem 5.6:** Let $(X, T)$ be a t.d.s. If $x \in X$ is $F_{ps}$-PR$_0$, then it is minimal.

**Proof.** According to the proof of Theorem 3.4 it remains to show that the point $y$ constructed in Proposition 3.3 has zero entropy.

Recall that

$$A_{k+1} = A_k a_k A_{k-1} \cdots A_2 n_{k+1}^2 A_1 n_{k+1}^1 \leq (x_0, \ldots, x_{l_{k+1}})$$

with $|A_1| n_{k+1}^1 = |A_2| n_{k+1}^2 = \cdots = |A_{k-1}| n_{k+1}^{k-1} = |A_k|$, $a_{k+1}$ can be divided by $|A_k|$ and $y = \lim_{k \to \infty} A_k$. Let $X = \text{orb}(y, \sigma)$ and $m_k = |A_k|$. We are going to show that $h(X, \sigma) = 0$. Let

$$B_k(y) = \# \{ u \in \{0, 1\}^k : \exists i \in \mathbb{Z}_+ \text{ such that } u = y[i; i + k - 1] \},$$

where $\#(\cdot)$ means the cardinality of a set. Then $h(X, \sigma) = \lim_{k \to \infty} \frac{1}{m_k} \log B_{m_k}(y)$. Let $u \in \{0, 1\}^{m_k}$ appear in $y$. Then there exists $i > k$ such that $u$ appears in $A_i$. By the way of the construction of $A_j$, $j \in \mathbb{N}$, it is known that $A_i = W_0 W_1 \cdots W_s$, where $W_j$ has the form of $0^{m_k} A_k A_{k-1}^{n_{k+1}} \cdots A_2^{n_{k+1}} A_1^{n_{k+1}}$ with $|0^{m_k}| = |A_k| = |A_{k-1}| = \cdots = |A_2^{n_{k+1}}| = |A_1^{n_{k+1}}|$. So we have

$$B_{m_k}(y) \leq (m_k + 1)(k + 1)k \leq (m_k + 1)^3.$$

It follows that

$$h(X, \sigma) = \lim_{k \to \infty} \frac{1}{m_k} \log B_{m_k}(y) = 0.$$

This ends the proof. □

5.3. Summary and Some Questions. Let $\mathcal{E}_0$ be the collection of all $E$-systems with zero entropy, and $\mathcal{M}_0$ be the collection of all $M$-systems with zero entropy. The following proposition is from [22]. Recall that a t.d.s. $(X, T)$ is c.p.e. if the factor induced by the smallest closed invariant equivalence relation containing $E(X, T)$ is trivial.
Proposition 5.7: The following statements hold.

1. If $X \perp \mathfrak{E}_0$ (i.e., $X$ is disjoint from each element of $\mathfrak{E}_0$), then $X$ is minimal and has c.p.e.
2. If $X$ is minimal and for each $\mu \in M(X, T)$, $(X, \mathcal{B}_X, \mu, T)$ is a measurable $K$-system, then $X \perp \mathfrak{E}_0$.
3. If $X$ is a minimal diagonal system, then $X \perp \mathfrak{M}_0$.

Thus we have

Theorem 5.8: The following statements hold.

1. $\mathcal{F}_{pubd} - PR_0 \neq \mathcal{F}_{inf} - PR_0$.
2. $\mathcal{F}_{ps} - PR_0 \neq \mathcal{F}_{ps} - PR$.
3. $\mathcal{F}_{pubd} - PR_0 \neq \mathcal{F}_{pubd} - PR$.

Proof. (1) Let $(X, T)$ be a minimal t.d.s. such that there is $\mu \in M(X, T)$ with $(X, \mathcal{B}_X, \mu, T)$ being a measurable $K$-system. Then each point of $X$ is in $\mathcal{F}_{pubd} - PR_0$. Since, in such a system, there exist asymptotic pairs, we have $\mathcal{F}_{pubd} - PR_0 \neq \mathcal{F}_{inf} - PR_0$.

(2) and (3) follow from Proposition 5.7. □

The following question is open:

$\mathcal{F}_{ps} - PR_0 \neq \mathcal{F}_{pubd} - PR_0$?

Note that it is an open question if there is a t.d.s. in $M_0^\perp \setminus E_0^\perp$; see [22].

To sum up we have

For minimal systems we have
6. Factors and extensions

In this section we investigate product recurrent properties for a family under factors or extensions. In this section and the next section we will use some tools from the theory of Ellis semigroups; see [5, 17, 32, 33] for details.

6.1. Definitions on factors. A homomorphism \( \pi : X \to Y \) between the t.d.s. \((X, T)\) and \((Y, S)\) is a continuous onto map which intertwines the actions; one says that \((Y, S)\) is a factor of \((X, T)\) and that \((X, T)\) is an extension of \((Y, S)\), and one also refers to \( \pi \) as a factor map or an extension. The systems are said to be conjugate if \( \pi \) is bijective. An extension \( \pi \) is determined by the corresponding closed invariant equivalence relation \( R_\pi = \{ (x_1, x_2) : \pi x_1 = \pi x_2 \} = (\pi \times \pi)^{-1} \Delta_Y \subset X \times X \).

An extension \( \pi : (X, T) \to (Y, S) \) is called proximal if \( R_\pi \subset P(X, T) \), \( \pi \) is distal when \( R_\pi \cap P(X, T) = \Delta_X \); \( \pi \) is distal if and only if every \( x \in X \) is \( \pi \)-distal. Note that when \( Y \) is trivial, the map \( \pi \) is distal if and only if \((X, T)\) is distal.

An extension \( \pi \) is equicontinuous if for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that \( (x, y) \in R_\pi \) and \( d(x, y) < \delta \) implies \( d(T^n x, T^n y) < \epsilon \), for every \( n \in \mathbb{N} \). And \( \pi \) is almost one-to-one if the \( G_\delta \) set \( X_0 = \{ x \in X : \pi^{-1}(\pi(x)) = \{ x \} \} \) is dense.

6.2. Product recurrent properties under factors or extensions. In this subsection we will use the following basic result frequently: \( x \) is recurrent if and only if there is an idempotent \( u \) such that \( ux = x \) (please refer to [1, 2, 12, 15] etc. for details).

**Proposition 6.1:** Let \( \pi : X \to Y \) be a factor map. If \( x \in R(X, T) \) then \( \pi(x) \in R(Y, S) \). Conversely, if \( y \in R(Y, S) \) then there is \( x \in \pi^{-1}(y) \cap R(X, T) \).

**Proof.** Let \( y \in R(Y, S) \). Then there is an idempotent \( u \) with \( uy = y \). Take \( x' \in \pi^{-1}(y) \) and set \( x = ux' \). Then \( x \in R(X, T) \) and \( \pi(ux') = u\pi(x') = y \).

**Corollary 6.2:** Let \((Y, S)\) be a t.d.s. and \( y \in Y \) be recurrent. Then for any t.d.s. \((X, T)\), there is \( x \in X \) such that \((x, y)\) recurrent.

**Proof.** One can get the corollary from Proposition 6.1 or Proposition 2.4.
Theorem 6.3: Let \( \mathcal{F} \) be a family, \((X, T), (Y, S)\) be two t.d.s. and \( \pi : X \to Y \) be a factor map.

1. If \( x \) is \( \mathcal{F} \)-PR, then \( \pi(x) \) is \( \mathcal{F} \)-PR.
2. If \( x \) is \( \pi \)-distal and \( y = \pi(x) \) is \( \mathcal{F} \)-PR, then \( x \) is \( \mathcal{F} \)-PR.
3. If \( y \in Y \) satisfies \( \pi^{-1}(y) = \{x\} \) for some \( x \in X \) and \( y \) is \( \mathcal{F} \)-PR, then \( x \) is \( \mathcal{F} \)-PR.

Proof. (1) Let \( x \) be \( \mathcal{F} \)-PR and \( X_1 \) be the orbit closure of \( x \). Assume that \( z \) is a \( \mathcal{F} \)-recurrent point and \( Z = \text{orb}(z) \). Then \( \pi \times Id : X_1 \times Z \to Y \times Z \) is a factor map. Since \( x \) is \( \mathcal{F} \)-PR, \((x, z)\) is a recurrent point. It follows that \((\pi(x), z)\) is a recurrent point, and thus \( \pi(x) \) is \( \mathcal{F} \)-PR.

(2) Assume \( y \) is \( \mathcal{F} \)-PR. Let \( z \) be a \( \mathcal{F} \)-recurrent point. Then \((y, z)\) is recurrent, and hence there exists an idempotent \( u \) such that \( u(y, z) = (y, z) \). Now we have \( \pi(ux) = u\pi(x) = uy = y = \pi(x) \) and note that \((x, ux) \in P(X, T) \). Since \( x \) is \( \pi \)-distal, we have \( ux = x \). Thus \( u(x, z) = (x, z) \), i.e., \((x, z)\) is recurrent. Hence \( x \) is \( \mathcal{F} \)-PR.

(3) is a special case of (2).

Theorem 6.4: Let \((X, T), (Y, S)\) be t.d.s.

1. If \((X, T)\) and \((Y, S)\) have dense sets of minimal points (resp. \( E \)-systems, \( P \)-systems), then so does \( X \times Y \).
2. If \((X, T)\) has a measure with full support and \((Y, S)\) has a dense set of recurrent points, then \( X \times Y \) has a dense set of recurrent points.
3. There are transitive t.d.s. \((X, T)\) and \((Y, S)\) such that \( X \times Y \) does not have a dense set of recurrent points.

Proof. If \((X, T)\) and \((Y, S)\) have dense sets of periodic points, or have measures with full support, then it is clear that so does \((X \times Y, T \times S)\).

If \( X \) and \( Y \) are minimal, then there is a minimal point \((x, y) \in X \times Y \). Since \( T^n \times S^m : X \times Y \to X \times Y \) is a factor map, it follows that \((T^n x, S^m y)\) is minimal for each pair \((n, m) \in \mathbb{Z}_+ \times \mathbb{Z}_+ \). Thus the set of minimal points in \( X \times Y \) is dense. This implies that if \( X \) and \( Y \) have dense sets of minimal points, then so does \( X \times Y \).

Now assume that \( X \) has a measure with full support and \( Y \) has a dense set of recurrent points. Without loss of generality we assume that \( X \) is an \( E \)-system and \( Y \) is transitive. For non-empty open sets \( U \subset X \) and \( V \subset Y \), pick transitive
points \( x \in U \) and \( y \in V \). Then
\[
N(U \times V, U \times V) = N(U, U) \cap N(V, V) = (N(x, U) - N(x, U)) \cap (N(y, V) - N(y, V)).
\]

Since \( N(x, U) \in F_{pubd} \), \( N(x, U) - N(x, U) \) is an IP*-set [15, Theorem 3.18]. This implies that \( N(U \times V, U \times V) \) is infinite. That is, \( X \times Y \) is non-wandering, which implies that the set of recurrent points in \( X \times Y \) is dense [15, Theorem 1.27].

Let \( F_1 \) and \( F_2 \) be two disjoint thick sets. Let \( A_1 \) and \( A_2 \) be two IP-sets contained in \( F_1 \) and \( F_2 \), respectively. Moreover, we may assume that \( A_i \) is generated by \( \{ p^i_j \} \) with
\[
p^i_{j+1} > p^i_1 + \cdots + p^i_j
\]
for all \( j \in \mathbb{N} \) and \( A_i - A_i \subset F_i \) for \( i = 1, 2 \). Let \( X_i = \text{orb}(1_{A_i}, \sigma) \subseteq \{0, 1\}^{\mathbb{Z}^+}, i = 1, 2 \). Then \( X_1 \times X_2 \) does not have a dense set of recurrent points, since
\[
N([1]X \times [1]Y, [1]X \times [1]Y) = N([1]X, [1]X) \cap N([1]Y, [1]Y)
\]
\[
= (A_1 - A_1) \cap (A_2 - A_2) \subset F_1 \cap F_2 = \emptyset.
\]

7. Disjointness and weak disjointness

Let \( \mathcal{T} \) be a class of t.d.s. and \( (X, T) \) be a t.d.s. If \( (X, T) \perp (Y, S), \forall (Y, S) \in \mathcal{T} \), then we denote it by \( (X, T) \perp \mathcal{T} \) or \( (X, T) \in \mathcal{T}^\perp \), where
\[
\mathcal{T}^\perp = \{(X, T) : (X, T) \perp \mathcal{T}\}.
\]

Let \( \mathcal{M} \) be the class of all minimal systems and \( \mathcal{M}_0 \) be the class of all minimal systems with zero entropy. Let \( \mathcal{M}_{eq} \) (resp. \( \mathcal{M}_d \) and \( \mathcal{M}_{wm} \)) be the class of all minimal equicontinuous (resp. distal and weakly mixing) systems. In [14], Furstenberg asked the question: Describe the classes \( \mathcal{M}^\perp \) and \( \mathcal{M}_d^\perp \). We extend the question:

**Question 7.1:** Which t.d.s. is disjoint from \( \mathcal{M}, \mathcal{M}_0, \mathcal{M}_{eq}, \mathcal{M}_d \) and \( \mathcal{M}_{wm} \)? Or determine \( \mathcal{M}^\perp, \mathcal{M}_0^\perp, \mathcal{M}_{eq}^\perp, \mathcal{M}_d^\perp \) and \( \mathcal{M}_{wm}^\perp \).

A related question is about the weak disjointness. In this section we will summarize what one knows concerning the above question and give additional new results.
7.1. Some basic properties on disjointness. Let \( \pi : (X, T) \to (Y, S) \) be an extension between two t.d.s. \((X, T)\) and \((Y, S)\). \( \pi \) is called **minimal** if the only closed invariant subset \( K \) of \( X \) such that \( \pi(K) = Y \) is \( X \) itself. Clearly, \( X \) is minimal if and only if \( \pi \) is minimal and \( Y \) is minimal. More generally, let \( \pi : X \to Y, \ \psi : Y \to Z \) be extensions; then \( \psi \circ \pi \) is a minimal extension if and only if both \( \psi \) and \( \pi \) are minimal extensions.

By definitions it is easy to get the following important observation:

**Lemma 7.2:** Let \((X, T)\) be a t.d.s. and let \((Y, S)\) be minimal. Then

\[
(X, T) \perp (Y, S)
\]

if and only if the projection map \( \pi_1 : X \times Y \to X \) is a minimal extension.

An extension \( \pi : X \to Y \) is said to be **semi-distal** if \((x, y) \in R_\pi \) is both recurrent and proximal; then \( x = y \).

**Lemma 7.3** ([2, Theorem 2.14]): Let \( \pi : (X, T) \to (Y, S) \) be a factor map. If \( X \) is transitive and \( \pi \) is semi-distal, then \( \pi \) is minimal.

Since each equicontinuous or distal extension is semi-distal, we have

**Corollary 7.4:** Let \( \pi : (X, T) \to (Y, S) \) be a factor map. If \( X \) is transitive and \( \pi \) is equicontinuous or distal, then \( \pi \) is minimal.

The following proposition concerns the ‘lifting’ of disjointness by semi-distal extensions.

**Proposition 7.5:** Let \((X, T)\) be a t.d.s. and \( \pi : (Y', S') \to (Y, S) \) be an extension of minimal systems. If \( \pi \) is semi-distal (resp. distal, equicontinuous) and \((X \times Y', T \times S')\) is transitive, then

\[
X \perp Y' \quad \text{if and only if} \quad X \perp Y.
\]

**Proof.** It follows from Lemma 7.2 and Lemma 7.3.

The following proposition concerns the ‘lifting’ of disjointness by proximal extensions.

**Lemma 7.6:** Let \( \pi : (X, T) \to (Y, S) \) be an extension. If \( X \) has a dense set of minimal points and \( \pi \) is proximal, then \( \pi \) is minimal.

**Proof.** Let \( J \) be a closed invariant subset of \( X \) with \( \pi(J) = Y \). Let \( x \) be a minimal point of \( X \). Since \( \pi(J) = Y \), there is \( x' \in J \) such that \( \pi(x) = \pi(x') \).
Now as $\pi$ is proximal, $x, x'$ are proximal. Hence by minimality of $x$,
\[ x \in \text{orb}(x, T) \subset J. \]
Since the set of minimal points of $X$ is dense, $J = X$. That is, $\pi$ is minimal. \hfill \blacksquare

**Proposition 7.7:** Let $(X, T)$ be a t.d.s. and $\pi : (Y', S') \to (Y, S)$ be an extension of minimal systems. If $\pi$ is proximal and $(X \times Y', T \times S')$ has a dense set of minimal points, then
\[ X \perp Y' \quad \text{if and only if} \quad X \perp Y. \]

**Proof.** It follows from Lemma 7.2 and Lemma 7.6. \hfill \blacksquare

Finally, we have the following property:

**Proposition 7.8:** [3] Disjointness is a residual property, i.e., it is inherited by factors, irreducible lifts and inverse limits.

7.2. A note on $\mathbb{Z}_+\text{-actions and } \mathbb{Z}\text{-actions.}$ In the sequel we will deal with the structure theorem of minimal systems. This theory was mainly developed for group actions, and accordingly we assume that $T$ is a homeomorphism when we use the related results.

To get the results for surjective maps we need to consider the natural extensions. For a t.d.s. $(X, T)$ with a metric $d$, we say $(\tilde{X}, \tilde{T})$ is the **natural extension** of $(X, T)$, if $\tilde{X} = \{(x_1, x_2, \ldots) : T(x_{i+1}) = x_i, x_i \in X, i \in \mathbb{N}\}$, which is a subspace of the product space $\prod_{i=1}^{\infty} X$ with the compatible metric $d_T$ defined by
\[ d_T((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i}. \]
Moreover, $\tilde{T} : \tilde{X} \to \tilde{X}$ is the shift homeomorphism, i.e.,
\[ \tilde{T}(x_1, x_2, \ldots) = (T(x_1), x_1, x_2, \ldots). \]

The important fact is that: $(X, T) \perp (Y, S)$ if and only if $(\tilde{X}, \tilde{T}) \perp (\tilde{Y}, \tilde{S})$, where $(\tilde{X}, \tilde{T})$ and $(\tilde{Y}, \tilde{S})$ are the natural extensions of $(X, T)$ and $(Y, S)$, respectively [28, Proposition 1.1]. Hence when considering disjointness of two systems, we can assume both of them are homeomorphisms.

Another problem is that the traditional structure theory of minimal systems is developed for group actions, and that means here it works for $\mathbb{Z}\text{-actions.}$ But till now we only confront $\mathbb{Z}_+\text{-actions.}$ This is not a big problem here, since
by definition it is easy to verify that for two homeomorphism systems they are
disjoint under the $\mathbb{Z}_+\times\mathbb{Z}$-actions if and only if they are under the $\mathbb{Z}$-actions. Note
that when we consider $\mathbb{Z}$-actions, the notions defined before are a little different.
For example, for $\mathbb{Z}$-actions $(x, y)$ of $X$ is proximal if there is a subsequence $\{n_i\}$
in $\mathbb{Z}$ such that $\lim_{n\to\infty} T^{n_i}x = \lim_{n\to\infty} T^{n_i}y$. We deal with other notions in a
similar way. It is easy to check that all results of Subsection 7.1 still hold when
considering $\mathbb{Z}$-actions.

To sum up, in the sequel when we deal with the structure theorem of minimal
systems, we assume that $T$ is a homeomorphism and use related results freely.

7.3. STRUCTURE THEOREM FOR MINIMAL SYSTEMS. In this subsection we
briefly review the structure theorem of minimal systems.

We say that a minimal system $(X, T)$ is a strictly PI system if there is an
ordinal $\eta$ (which is countable when $X$ is metrizable) and a family of systems
\[\{(W_t, w_t)\}_{t \leq \eta}\] such that (i) $W_0$ is the trivial system, (ii) for every $\iota < \eta$ there
exists a homomorphism $\phi_t : W_{t+1} \to W_t$ which is either proximal or equicon-
tinuous, (iii) for a limit ordinal $\nu \leq \eta$ the system $W_\nu$ is the inverse limit of the
systems $\{W_t\}_{t < \nu}$, and (iv) $W_\eta = X$. We say that $(X, T)$ is a PI-system if
there exists a strictly PI-system $\tilde{X}$ and a proximal homomorphism $\theta : \tilde{X} \to X$.

If in the definition of PI-systems we replace proximal extensions by almost
one-to-one extensions, we get the notion of HPI systems. If we replace the
proximal extensions by trivial extensions (i.e., we do not allow proximal exten-
sions at all), we have I systems. These notions can be easily relativized and
we then speak about I, HPI and PI extensions.

We have the following structure theorem for minimal systems; for details see
[5, 13, 17, 32, 33] etc.

**Theorem 7.9** (Structure theorem for minimal systems): *Given a homomor-
phism $\pi : X \to Y$ of a minimal dynamical system, there exists an ordinal $\eta$
(countable when $X$ is metrizable) and a canonically defined commutative dia-
gram (the canonical PI-Tower)*

\[
\begin{array}{cccccccccccc}
X & \xleftarrow{\pi_0} & X_0 & \xleftarrow{\sigma_0^1} & X_1 & \cdots & X_\nu & \xleftarrow{\sigma_\nu^1} & X_{\nu+1} & \cdots & X_\eta = X_
fty \\
\downarrow{\pi} & & \downarrow{\sigma_1} & & \downarrow{\pi_1} & & \downarrow{\sigma_\nu} & & \downarrow{\pi_\nu} & & \downarrow{\pi_\infty} \\
Y & \xleftarrow{\theta_0} & Y_0 & \xleftarrow{\rho_1} & Z_1 & \cdots & Y_\nu & \xleftarrow{\rho_\nu} & Z_{\nu+1} & \cdots & Y_\eta = Y\nfty \\
\end{array}
\]
where, for each $\nu \leq \eta$, $\pi_\nu$ is RIC, $\rho_\nu$ is isometric, $\theta_\nu, \theta^*_\nu$ are proximal and $\pi_\infty$ is RIC and weakly mixing of all orders. For a limit ordinal $\nu$, $X_\nu, Y_\nu, \pi_\nu$ etc. are the inverse limits (or joins) of $X_\iota, Y_\iota, \pi_\iota$ etc. for $\iota < \nu$.

Thus if $Y$ is trivial, then $X_\infty$ is a proximal extension of $X$ and a RIC weakly mixing extension of the strictly PI-system $Y_\infty$. The homomorphism $\pi_\infty$ is an isomorphism (so that $X_\infty = Y_\infty$) if and only if $X$ is a PI-system.

Recall an extension $\pi : X \to Y$ of minimal systems is a relatively incontractible (RIC) extension if it is open, and for every $n \geq 1$ the minimal points are dense in the relation

$$R^n_\pi = \{(x_1, \ldots, x_n) \in X^n : \pi(x_i) = \pi(x_j), \; \forall \; 1 \leq i \leq j \leq n\}.$$  

### 7.4. Disjointness for $\mathcal{M}_{pi}$

In this subsection we discuss disjointness for $\mathcal{M}_{pi}$, which is the collection of all minimal PI-systems. It is known [14] that $\mathcal{M}_{eq}^+ \cap \mathcal{M} = \mathcal{M}_{wm}$, which implies that $\mathcal{M}_{pi}^+ \cap \mathcal{M} = \mathcal{M}_{wm}$ (see Theorem 7.10). In this subsection we will show that each weakly mixing t.d.s. with dense minimal points is disjoint from all minimal PI-systems. We remark that a weakly mixing t.d.s. (even scattering) is disjoint from all HPI minimal t.d.s. (using Propositions 7.5 and 7.8).

**Theorem 7.10:** Each weakly mixing t.d.s. with dense minimal points is disjoint from all minimal PI-systems.

**Proof.** Since a PI-system is constructed by equicontinuous and proximal extensions, the result follows from Propositions 7.5, 7.7 and 7.8 and the well known facts:

- a weakly mixing t.d.s. is weakly disjoint from all minimal t.d.s. [9] (since a weakly mixing t.d.s. is scattering),
- the product of two systems with dense sets of minimal points still has a dense set of minimal points (Theorem 6.4),
- a weakly mixing t.d.s. is disjoint from all minimal equicontinuous t.d.s. [14].

**Remark 7.11:** Note that a weakly mixing system with dense minimal points is not necessarily disjoint from all minimal systems. Let $(X, T)$ be a minimal weakly mixing t.d.s. and $(Y, S) = (X \times X, T \times T)$. Then $(Y, S)$ is weakly mixing and has a dense set of minimal points. We claim that $(Y, S) \not\subseteq (X, T)$. In fact, $J = \{(x, y, x) : x, y \in X\}$ is a joining and it is clear that $J \neq X \times X \times X$. 

**Remark 7.12:** By the structure theorem of a minimal t.d.s. and the result in [28], to obtain the necessary and sufficient condition for disjointness from all minimal t.d.s. (for a transitive t.d.s.) is equivalent to finding such a condition (implying weakly mixing, dense minimal points and something more) such that if $X$ satisfies the condition, and $X$ is disjoint from a minimal t.d.s. $Y'$, then $X$ is disjoint from all minimal t.d.s. $Y$ satisfying that $\pi : Y \to Y'$ is a weakly mixing extension.

We think that the following question has an affirmative answer.

**Question 7.13:** Assume $(X,T)$ is transitive and $(X,T) \in M_{\pi}^\perp$. Is it true that $(X,T)$ is a weakly mixing $E$-system?

The difficulty in answering the question is that we do not know if each subset of $\mathbb{Z}_+$ having lower Banach density 1 and containing 0 contains a subset $A$ such that the orbit closure of $1_A$ is a minimal PI-system (there is such a set which does not contain any subset $A$ such that the orbit closure of $1_A$ is a minimal HPI system, since otherwise we have that scattering implies weak mixing).

### 7.5. Disjointness and weak disjointness for $M$.

In [28] it was shown that a weakly mixing system with a dense set of regular minimal points is disjoint from any minimal t.d.s. Now we improve the result by showing that each weakly mixing system with a dense set of distal points is disjoint from all minimal systems. We give two proofs, where the first one is provided by W. Huang and the second one relies on the structure theorem for minimal systems. After that we will give another result on disjointness: each $\mathcal{F}_s$-independent t.d.s. is disjoint from any minimal t.d.s.

First we will prove

**Theorem 7.14:** Each weakly mixing system with a dense set of distal points is disjoint from all minimal systems.

To prove it we need the following Lemma 7.15 concerning proximal cell (see [4, 24]). Note that for a t.d.s. $(X,T)$ and $x \in X$, $P[x]$ denotes the **proximal cell**, i.e., $P[x] = \{y \in X : y \text{ is proximal to } x\} = \{y \in X : (x,y) \in P(X,T)\}$.

**Lemma 7.15:** Let $(X,T)$ be a weakly mixing t.d.s. Then for each $x \in X$, $P[x]$ is a dense $G_\delta$ subset of $X$.

**Proof of Theorem 7.14.** Let $(X,T)$ be a weakly mixing system with a dense set of distal points and $\{x_s\}_{s=1}^\infty$ be a dense set of distal points. By Lemma 7.15
there is \( x \in \bigcap_{s=1}^{\infty} P[x_s] \). Let \((Y,S)\) be a minimal t.d.s. and \( J \subset X \times Y \) be a joining. Then there is \( y \in Y \) such that \((x,y) \in J\). For each \( x_s \), \((x,x_s)\) is proximal, thus for each \( \epsilon > 0 \),
\[
\{ n \in \mathbb{Z}_+ : d(T^n x, T^n x_s) < \epsilon / 2 \}
\]
is thick. Since \( x_s \) is a distal point, \((x_s,y)\) is minimal and hence
\[
\{ n \in \mathbb{Z}_+ : d(T^n x_s, x_s) < \epsilon / 2, d(T^n y, y) < \epsilon \}
\]
is syndetic. Thus, for a given \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) such that
\[
d(T^n x, T^n x_s) < \epsilon / 2, \quad d(T^n x_s, x_s) < \epsilon / 2 \quad \text{and} \quad d(T^n y, y) < \epsilon.
\]
That is, \( d(T^n x, x_s) < \epsilon \) and \( d(T^n y, y) < \epsilon \). This implies that \((x_s,y) \in W =: \text{orb}((x,y), T \times S)\), and thus \( X \times \{y\} \subset W \subset J \). It follows that \( J = X \times Y \) since \((Y,S)\) is minimal. Hence \((X,T)\) is disjoint from \((Y,S)\).

Now we give the second proof. Since by Theorem 7.10 each weakly mixing system with a dense set of distal points is disjoint from any PI minimal system, by the structure theorem for minimal systems (Theorem 7.9) we need to deal with weakly mixing RIC extensions.

**Lemma 7.16:** Let \( \pi: (Y', S') \to (Y, S) \) be a weakly mixing RIC extension of minimal systems. Then there is a dense \( G_\delta \) subset \( Y_0 \subset Y \) such that, for each \( y \in Y_0 \) and each \( x \in \pi^{-1}(y) \), \( P_{Y'} [x] \) is dense in the fibre \( \pi^{-1}(y) \).

**Proof.** See Appendix.

The following proposition concerns the “lifting” of disjointness by weakly mixing RIC extensions. Note that each t.d.s. \((X,T)\) has a natural extension \((X', T')\) such that \( T' \) is a homeomorphism. We may assume that all t.d.s. are invertible when considering disjointness; see [28, Proposition 1.1].

**Proposition 7.17:** Let \((X,T)\) be a t.d.s. with a dense set of distal points and let \( \pi: (Y', S') \to (Y, S) \) be a weakly mixing RIC extension of minimal systems. Then
\[
X \perp Y' \quad \text{if and only if} \quad X \perp Y.
\]

**Proof.** It suffices to show that if \( X \perp Y \), then \( X \perp Y' \). Let \( J \subset X \times Y' \) be a joining of \( X \) and \( Y' \). Let \( x \) be a distal point of \( X \) and \( y \in Y_0 \), where \( Y_0 \) is defined in Lemma 7.16. We remark that \( Y_0 \) is residual in \( Y \). Since \( X \perp Y \), \( \text{id} \times \pi(J) = X \times Y \). Thus there is some \( y_0 \in Y' \) such that \((x,y_0) \in J \) and
\( \pi(y_0) = y. \) Let \( y' \in P_y[y_0] \cap \pi^{-1}(y). \) Then \( (x, y_0), (x, y') \) are proximal. Since \( x \) is distal, \( (x, y') \) is minimal. And this implies that
\[
(x, y') \in \text{orb}((x, y_0), T \times S') \subset J.
\]

By Lemma 7.16, such \( y' \) is dense in \( \pi^{-1}(y). \) Thus \( \{x\} \times \pi^{-1}(y) \subset J. \) Since \( y \in Y_0 \) is arbitrary and \( Y_0 \) is residual, we have \( \{x\} \times Y' \subset J. \) Finally, by the density of distal points in \( X, \) we have \( J = X \times Y'. \) \( \blacksquare \)

Now Theorem 7.14 follows from the structure theorem (Theorem 7.9), Theorem 7.10 and Proposition 7.17.

To prove another disjointness result we need some notions and results from [23].

**Definition 7.18:** Let \((X, T)\) be a t.d.s. For a tuple \( A = (A_1, \ldots, A_k) \) of subsets of \( X, \) we say that a subset \( F \subseteq \mathbb{Z}_+ \) is an independence set for \( A, \) if for any nonempty finite subset \( J \subseteq F, \) we have
\[
\bigcap_{j \in J} T^{-j} A_{s(j)} \neq \emptyset
\]
for any \( s \in \{1, \ldots, k\}^J. \) Denote the collection of all independence sets for \( A \) by \( \text{Ind}(A_1, \ldots, A_k) \) or \( \text{Ind} A. \)

**Definition 7.19:** Let \( \mathcal{F} \) be a family, \( k \in \mathbb{N} \) and \((X, T)\) be a t.d.s. A tuple \((x_1, \ldots, x_k) \in X^k\) is called an \( \mathcal{F} \)-independent tuple if, for any neighborhoods \( U_1, \ldots, U_k \) of \( x_1, \ldots, x_k, \) respectively, one has \( \text{Ind}(U_1, \ldots, U_k) \cap \mathcal{F} \neq \emptyset. \)

A t.d.s. \((X, T)\) is said to be \( \mathcal{F} \)-independent of order \( k \) if, for each tuple of non-empty open subsets \( U_1, \ldots, U_k \) of \( X, \) \( \text{Ind}(U_1, \ldots, U_k) \cap \mathcal{F} \neq \emptyset, \) and \((X, T)\) is said to be \( \mathcal{F} \)-independent, if it is \( \mathcal{F} \)-independent of order \( k \) for each \( k \in \mathbb{N}. \)

It is proved in [23] that an \( \mathcal{F}_s \)-independent t.d.s. is weakly mixing, has positive entropy and has a dense set of minimal points. Moreover, the following lemma is proved.

**Lemma 7.20:** For every minimal subshift \( X \subseteq \Sigma_2, \) \( \text{Ind}([0]_X, [1]_X) \) does not contain any syndetic set.

An easy consequence of Lemma 7.20 is that there is no non-trivial minimal t.d.s. which is \( \mathcal{F}_s \)-independent. Now we are ready to show
THEOREM 7.21: Each $\mathcal{F}_s$-independent of order 2 t.d.s. is disjoint from all minimal systems.

Proof. Since it is an open question if an $\mathcal{F}_s$-independent pair can be lifted by extensions, the proof of [8] cannot be applied here directly. We will use ideas of the proof in [8] and Lemma 7.20.

Let $(X, T)$ be an $\mathcal{F}_s$-independent t.d.s. and $(Y, S)$ be minimal. Assume the contrary that $X \not\perp Y$. Then there is a joining $J \neq X \times Y$. We may assume that $J$ is minimal, i.e., if $J'$ is a joining and $J' \subset J$ then $J' = J$. For $x \in X$ let $J[x] = \{y \in Y : (x, y) \in J\}$. We claim that there exists $x \in X$ such that $J[x] \cap J[Tx] = \emptyset$.

Now suppose that $J[x] \cap J[Tx] \neq \emptyset$ for all $x \in X$. Let

$$J' = \bigcup_{x \in X} \{x\} \times (J[x] \cap J[Tx]).$$

It is easy to check that $J' \subset J$ is a joining, and hence by minimality $J' = J$. This implies that $J = X \times Y$, a contradiction. So there exists $x \in X$ such that $J[x] \cap J[Tx] = \emptyset$.

There exist disjoint closed neighborhoods $W_0$ and $W_1$ of $x$ and $Tx$ such that $J[W_0] \cap J[W_1] = \emptyset$, since $J$ is closed and $J[x] \cap J[Tx] = \emptyset$. So there is a syndetic subset $S \in \text{Ind}(W_0, W_1)$. Let $\pi_X : J \to X$ and $\pi_Y : J \to Y$ be the projections. It is clear that $S \in \text{Ind}(\pi_X^{-1}(W_0), \pi_X^{-1}(W_1))$ and $S \in \text{Ind}(\pi_Y \pi_X^{-1}(W_0), \pi_Y \pi_X^{-1}(W_1))$. Since $J[W_0] \cap J[W_1] = \emptyset$ we know that $\pi_Y \pi_X^{-1}(W_0) \cap \pi_Y \pi_X^{-1}(W_1) = \emptyset$. Let $V_0$ and $V_1$ be disjoint closed neighborhoods of $\pi_Y \pi_X^{-1}(W_0)$ and $\pi_Y \pi_X^{-1}(W_1)$, respectively. It is clear that $S \in \text{Ind}(V_0, V_1)$.

It is well known that we can find a minimal t.d.s. $(X_1, T_1)$ and a factor map $\pi : (X_1, T_1) \to (Y, S)$ such that $X_1$ is a closed subset of a Cantor set. It is easy to see that $\text{Ind}(V_0, V_1) = \text{Ind}(\pi^{-1}(V_0), \pi^{-1}(V_1))$. Write $X_1$ as the disjoint union of clopen subsets $U_0$ and $U_1$ such that $U_j \supseteq \pi^{-1}(V_j)$ for $j = 0, 1$. Then $\text{Ind}(V_0, V_1) \subseteq \text{Ind}(U_0, U_1)$.

Define a coding $\phi : X_1 \to \Sigma_2$ such that for each $x \in X_1$, $\phi(x) = (x_0, x_1, \ldots)$, where $x_i = j$ if $T_i^j(x) \in U_j$ for all $i \in \mathbb{Z}_+$. Then $Z = \phi(X_1)$ is a minimal subshift contained in $\Sigma_2$ and $\phi : X_1 \to Z$ is a factor map. It is easy to verify that $\text{Ind}(U_0, U_1) \subseteq \text{Ind}([0]_Z, [1]_Z)$.

By Lemma 7.20 we know that $\text{Ind}([0]_Z, [1]_Z)$ does not contain any syndetic set. This contradicts the fact that $S \in \text{Ind}([0]_Z, [1]_Z)$. So $X$ and $Y$ are disjoint. ■
We remark that the assumption of \( F_s \)-independence cannot be weakened significantly, since there exists an \( F_{pd} \)-independent t.d.s. with only one minimal point [23]. So combining the result in [28] we have

**Proposition 7.22:** The following statements hold:

1. Each weakly mixing system with a dense set of distal points is disjoint from all minimal systems; and each \( F_s \)-independent t.d.s. is disjoint from all minimal systems.
2. If \((X, T)\) is transitive and is disjoint from any minimal t.d.s., then \((X, T)\) is weakly mixing and has a dense set of minimal points.

Recall that a t.d.s. is scattering if it is weakly disjoint from \( \mathcal{M} \). In [9] the following proposition was proved. Recall that a cover is non-trivial if each element of the cover is not dense in \( X \), and for a cover \( U \), \( N(U) = \min\{|V| : V \text{ is a subcover of } U\} \).

**Proposition 7.23:** A t.d.s. is scattering if and only if, for any non-trivial open cover \( U \), \( N(\bigvee_{i=0}^{n-1} T^{-i}U) \to \infty \).

### 7.6. Disjointness and weak disjointness for \( \mathcal{M}_{eq} \)

Recall that a t.d.s. is **weakly scattering** if it is weakly disjoint from \( \mathcal{M}_{eq} \). The following proposition is known; see, for example, [3].

**Proposition 7.24:** A transitive t.d.s. is disjoint from \( \mathcal{M}_{eq} \) if and only if it is weakly scattering.

Let \((X, T)\) and \((Y, S)\) be two transitive t.d.s. If there exists a continuous map \( \phi : \text{Tran}_T(X) \to \text{Tran}_S(Y) \) with \( \phi(Tx) = S\phi(x) \) for \( x \in \text{Tran}_T(X) \), then we say \( \phi \) is a **generic homomorphism** from \((X, T)\) to \((Y, S)\). \((Y, S)\) is a **generic factor** of \((X, T)\) and \((X, T)\) is a **generic extension** of \((Y, S)\). It is not hard to see that if \((X, T)\) is minimal and \( \phi : (X, T) \to (Y, S) \) is a generic homomorphism, then \( \phi \) is a factor map.

In [27] the authors considered weakly scattering t.d.s. The following proposition was a result in [27] combined with a simple observation.

**Proposition 7.25:** The following hold

1. A transitive t.d.s. is weakly scattering if and only if it has no non-trivial generic equicontinuous factors.
2. A minimal t.d.s. is disjoint from \( \mathcal{M}_{eq} \) if and only if it is weakly mixing.
Proof. (1) was proved in [27]. To show (2), note that if a minimal t.d.s. is disjoint from $\mathcal{M}_{eq}$, then the maximal equicontinuous factor of $(X,T)$ is trivial, which implies that $(X,T)$ is weakly mixing. There are several ways to show a weakly mixing t.d.s. is disjoint from $\mathcal{M}_{eq}$, say, for example, [9, 14].

It is clear that scattering implies weak scattering. To end the subsection we recall an open question:

**Question 7.26:** Does weak scattering imply scattering?

### 7.7. Disjointness for $\mathcal{M}_0$.

The following proposition was proved in [22].

**Proposition 7.27:** The following statements hold:

1. If a transitive $(X,T) \perp \mathcal{M}_0$, then it is weakly mixing and is an $E$-system.
2. If $(X,T)$ is a transitive diagonal t.d.s., then $(X,T) \perp \mathcal{M}_0$.
3. If $(X,T)$ is minimal and $(X,T) \perp \mathcal{M}_0$, then $(X,T)$ has c.p.e.; and if $(X,T)$ is minimal and diagonal, then $(X,T) \perp \mathcal{M}_0$.

### 7.8. Disjointness for $\mathcal{M}_{wm}$.

Since $\mathcal{M}_d \cap \mathcal{M} = \mathcal{M}_{eq} \cap \mathcal{M} = \mathcal{M}_{wm}$ [14], it implies that $\mathcal{M}_{wm} \supset \mathcal{M}_d$. The following proposition is known.

**Proposition 7.28:** [7] A minimal t.d.s. is in $\mathcal{M}_{wm}$ if and only if every non-trivial quasi-factor of $X$ has a non-trivial distal factor.

Recall that a **quasi-factor** of $X$ is a minimal subset of $(2^X, T)$, where $2^X$ is the collection of all non-empty closed subsets of $X$ equipped with the Hausdorff metric.

**Definition 7.29:** A minimal point $x$ is a **quasi-distal point** if $(x, y)$ is minimal for every minimal $y$ whose orbit closure is weakly mixing.

It is clear that a distal point is quasi-distal. Moreover, if $(X,T)$ is minimal and $(X,T) \in \mathcal{M}_{wm}^+$, then each point of $X$ is quasi-distal, since two minimal t.d.s. are disjoint in which case the product is minimal. By [18, Theorem 2.2], there exists a quasi-distal point which is not distal. Since any almost one-to-one extension of a minimal equicontinuous systems is in $\mathcal{M}_{wm}^+$ (say the Denjoy minimal t.d.s.), it follows that there is a quasi-distal point which is not weakly product recurrent. It is not clear if a minimal weakly product recurrent point is quasi-distal. We have the following theorem.
**Theorem 7.30:** Let \((X, T)\) be a weakly mixing t.d.s. with dense quasi-distal points; then \((X, T) \in \mathcal{M}_{wm}^\perp\).

**Proof.** Apply the proof of Theorem 7.14. □

It is well known that a t.d.s. \((X, T)\) is weakly mixing if and only if \(N(U, V)\) is thick \([14]\). Weiss \([35]\) showed that if \(F \subset \mathbb{Z}_+\) is a thick set, then there is a weakly mixing t.d.s. \((X, T) \subset (\{0, 1\}^{\mathbb{Z}_+}, \sigma)\) such that \(N([1], [1]) \subset F\). Huang and Ye \([26]\) showed that if \((X, T)\) is minimal, then \((X, T)\) is weakly mixing if and only if \(N(U, V)\) has lower Banach density 1. By Remark 4.8 we have

**Lemma 7.31:** Let \(F \subset \mathbb{Z}_+\) be thickly syndetic. Then there are a minimal weakly mixing \((X, T) \subset (\{0, 1\}^{\mathbb{Z}_+}, \sigma)\) and \(x \in X\) such that \(N(x, [1]) \subset F\).

So in the transitive case we have the following corollary:

**Proposition 7.32:** If a transitive \((X, T)\) is disjoint from all minimal weakly mixing t.d.s., then it is an \(M\)-system.

Since a minimal equicontinuous systems is in \(\mathcal{M}_{wm}^\perp\), \((X, T) \in \mathcal{M}_{wm}^\perp\) does not imply that it is weak mixing.

The following question remains open:

**Question 7.33:** Is it true that a transitive t.d.s. \((X, T)\) is disjoint from any minimal t.d.s. if and only if \((X, T)\) is weakly mixing and has a dense set of quasi distal points?

8. Tables

| \(\mathcal{F}\) | \(\mathcal{F}_{inf}\) | \(\mathcal{F}_{pubd}\) | \(\mathcal{F}_{ps}\) | \(\mathcal{F}_s\) |
|-----------------|-------------------|-------------------|-------------------|-------------------|
| Orbit closure of a \(\mathcal{F}\)-PR point | minimal distal | minimal | minimal | \(M\)-system |
| Orbit closure of a \(\mathcal{F}\)-PR\(_0\) point | minimal distal | minimal | minimal | \(E\)-system |
### Table 2. Disjointness and weak disjointness

|                  | $\mathcal{F}$ | $\mathcal{M}_{eq}$ | $\mathcal{M}_{hpi}$ |
|------------------|---------------|---------------------|---------------------|
| Properties of transitive systems in $\mathcal{F}$ | weak scattering | weak scattering | weak scattering |
| Systems in $\mathcal{F}^\perp$ | weak scattering | weak scattering | scattering |
| Minimal systems in $\mathcal{F}^\perp$ | weak mixing | weak mixing | weak mixing |
| Systems weakly disjoint from $\mathcal{F}$ | weak scattering | weak scattering | ?? |

| $\mathcal{M}_{pi}$ | $\mathcal{M}_{wm}$ | $\mathcal{M}$ |
|---------------------|---------------------|----------------|
| weak mixing + $E$-system | $M$-system | weak mixing + $M$-system |
| weak mixing + $M$-system | weak mixing + dense quasi-distal points | w.m. + dense distal points; $\mathcal{F}_s$-independent |
| weak mixing | every non-trivial quasi-factor | trivial |
| ?? | ?? | scattering |

### 9. More discussions

#### 9.1. ($\mathcal{F}_1, \mathcal{F}_2$)-PRODUCT RECURRENCE

In this subsection we discuss some generalizations of the notions concerning product recurrence.

**Definition 9.1:** Let $\mathcal{F}_1, \mathcal{F}_2$ be families and $(X, T)$ be a t.d.s. A point $x \in X$ is called **($\mathcal{F}_1, \mathcal{F}_2$)-product recurrent** if $(x, y)$ is $\mathcal{F}_2$-recurrent for any $\mathcal{F}_1$-recurrent point $y$ in some t.d.s. $(Y, S)$.

By the definition it is obvious that $\mathcal{F}$-product recurrence is nothing but $(\mathcal{F}, \mathcal{F}_{inf})$-product recurrence. As we have seen in this paper, for a family the property $\mathcal{F}$-PR may be very complex. Hence it is more difficult to discuss the general case ($\mathcal{F}_1, \mathcal{F}_2$)-PR. But if we assume $\mathcal{F}_1 = \mathcal{F}_2$, then we can use the results from [6, 12]. To see this, let us recall some notions first.

Now we consider the Stone–Čech compactification of the semigroup $\mathbb{Z}_+$ with the discrete topology. The set of all ultrafilters on $\mathbb{Z}_+$ is denoted by $\beta \mathbb{Z}_+$. Let $A \subset \mathbb{Z}_+$ and define $\overline{A} = \{ p \in \beta \mathbb{Z}_+ : A \in p \}$. The set $\{ \overline{A} : A \subset \mathbb{Z}_+ \}$ forms a basis for the open sets (and also a basis for closed sets) of $\beta \mathbb{Z}_+$. Under this topology, $\beta \mathbb{Z}_+$ is the **Stone–Čech compactification** of $\mathbb{Z}_+$. See [1, 2, 12] etc. for details.
For $F \subset \mathbb{Z}_+$, the hull of $F$ is $h(F) = \overline{F} = \{p \in \beta \mathbb{Z}_+ : F \subseteq p\}$. For a family $\mathcal{F}$, the hull of $\mathcal{F}$ is defined by $h(\mathcal{F}) = \bigcap_{F \in \mathcal{F}} h(F) = \bigcap_{F \in \mathcal{F}} \overline{F} = \{p \in \beta \mathbb{Z}_+ : \mathcal{F} \subseteq p\} \subseteq \beta \mathbb{Z}_+$.

Let $X$ be a compact metric space and $S$ a semigroup. Let $\Phi : S \times X \to X$ be an action, i.e., for any $p, q \in S$, $\Phi^p \circ \Phi^q = \Phi^{pq}$. For $(p, x) \in S \times X$, denote $px = \Phi(p, x) = \Phi^p(x) = \Phi_x(p)$.

$\Phi^\# : S \to \mathcal{L}(X)$ is defined by $p \mapsto \Phi^p$. Hence $px = \Phi^\#(p)(x)$. An Ellis semigroup $S$ is a compact Hausdorff semigroup such that the right translation map $R_p : S \to S$, $q \mapsto qp$ is continuous for every $p \in S$. An Ellis action of an Ellis semigroup $S$ on a space $X$ is a map $\Phi : S \times X \to X$ which is an action such that the adjoint map $\Phi^\#$ is continuous, or equivalently, $\Phi_x$ is continuous for each $x \in X$.

Now let $(X, T)$ be a t.d.s. Then $\Phi : \mathbb{Z}_+ \times X \to X, (n, x) \mapsto T^n x$ is an action and it can be extended to an Ellis action $\Phi : \beta \mathbb{Z}_+ \times X \to X$. Hence we have a continuous map $\Phi^\# : \beta \mathbb{Z}_+ \to \mathcal{L}(X)$.

Define $H(\mathcal{F}) = H(X, \mathcal{F}) = \Phi^\#(h(\mathcal{F})) \subset \mathcal{L}(X)$.

It is easy to see that for a family $\mathcal{F}$, $H(\mathcal{F}) \neq \emptyset$ if and only if $\mathcal{F}$ has finite intersection property. Moreover, let $(X, T)$ be a t.d.s. and $\mathcal{F}$ be a filter. Then $H(\mathcal{F}) = \bigcap_{F \in \mathcal{F}} \overline{T^F} \subseteq \mathcal{L}(X)$, where $T^F = \bigcup\{T^n|n \in F\}$.

Now we generalize the notion of $\omega$-limit set. Let $(X, T)$ be a t.d.s. and $\mathcal{F}$ be a family. Define $\omega_{\mathcal{F}}(x, T) = \bigcap_{F \in \mathcal{F}^*} \overline{T^{\mathcal{F}}(x)}$.

It is easy to show that if $\mathcal{F}$ is a filter, then $\omega_{\mathcal{F}}^\#(x, T) = H(\mathcal{F})x$. By the definition one has that a point $x \in X$ is $\mathcal{F}$-recurrent if and only if $x \in \omega_{\mathcal{F}}(x, T)$.

Now let $\mathcal{F}$ be a filter dual (i.e., its dual is a filter). Then a point $x$ is $(\mathcal{F}, \mathcal{F})$-product recurrent if and only if $(x, y) \in \omega_{\mathcal{F}}((x, y), T \times S)$ for any $y$ in some t.d.s. $(Y, S)$ satisfying $y \in \omega_{\mathcal{F}}(y, S)$. That is, $x$ is $H(\mathcal{F}^*)$-product recurrent defined in [6]. Thus we can use the results in [6, 12] to study $(\mathcal{F}, \mathcal{F})$-PR points.

9.2. Questions. Here are some more questions. First we restate the following question in [20].

**Question 9.2:** Is each weakly product minimal point distal?
We conjecture that the above question has a negative answer. The next question concerns disjointness.

**Question 9.3:** Let \((X_1, T_1), (X_2, T_2)\) be t.d.s., and \((Y, S)\) be a minimal t.d.s. If \((X_1, T_1) \perp (Y, S)\) and \((X_2, T_2) \perp (Y, S)\), then is it true that 

\[(X_1 \times X_2, T_1 \times T_2) \perp (Y, S)\]?

Or, for a class \(\mathcal{T}\) of minimal systems, is the finite product closed in \(\mathcal{T}^\perp\)?

10. Appendix: Relative proximal cells

In this appendix we study the relative proximal cell for independent interest, and on the way we give a proof of Lemma 7.16. Here we will use some results from the theory of minimal flows. This theory was mainly developed for group actions and accordingly we assume that \(T\) is a homeomorphism in this appendix. Much of this work can be done for a general locally compact group action. We refer the reader to [5, 17, 32, 33] for details.

10.1. RIM extension. Let \(X\) be a compact metric space and let \(M(X)\) be the collection of regular Borel probability measures on \(X\) provided with the weak star topology. Then \(M(X)\) is a compact metric space in which \(X\) is embedded by the mapping \(x \mapsto \delta_x\), where \(\delta_x\) is the dirac measure at \(x\). If \(\phi : X \to Y\) is a continuous map between compact metric spaces, then \(\phi\) induces a continuous map \(\phi^* : M(X) \to M(Y)\) which extends \(\phi\), where \((\phi^* \mu)(A) = \mu(\phi^{-1}A)\) for all Borel sets \(A \subseteq Y\).

Let \((X, T)\) be a t.d.s. For each \(\mu \in M(X)\), define \((T\mu)(A) = \mu(T^{-1}A)\) for all Borel sets \(A \subseteq X\). Then \((M(X), T)\) is a t.d.s. too. And if \(\pi : X \to Y\) is an extension of t.d.s., then \(\pi^* : M(X) \to M(Y)\) is also an extension.

An extension \(\pi : X \to Y\) of t.d.s. is said to have a relatively invariant measure (RIM for short) if there exists a continuous homomorphism \(\lambda : Y \to M(X)\) of t.d.s. such that \(\pi^* \circ \lambda : Y \to M(Y)\) is just the (dirac) embedding. In other words: \(\pi\) is a RIM extension if and only if, for every \(y \in Y\), there is a \(\lambda_y \in M(X)\) with \(\text{supp}\lambda_y \subseteq \pi^{-1}(y)\) and the map \(y \mapsto \lambda_y : Y \to M(X)\) is a homomorphism of t.d.s.; this map \(\lambda\) is called a section for \(\pi\). Note that \(\pi : X \to \{\star\}\) has a RIM if and only if \(X\) has an invariant measure if and only if \(M(X)\) has a fixed point, where \(\{\star\}\) stands for the trivial system. An extension \(\pi : X \to Y\) is called strongly proximal if, for every pair \(\mu \in M(X)\) and \(y \in Y\)
with suppμ ⊆ π⁻¹(y), a sequence {nᵢ} can be found such that lim Tⁿᵢμ is a point mass. It is easy to see that each strongly proximal extension is proximal.

Every extension of minimal systems can be lifted to a RIM extension by strongly proximal modifications. To be precise, for every extension π : X → Y of minimal systems there exists a canonically defined commutative diagram of extensions (called the G-diagram [16])

\[
\begin{array}{ccc}
X & \xrightarrow{\sigma} & X^# \\
\downarrow{\pi} & & \downarrow{\pi^#} \\
Y & \xleftarrow{\tau} & Y^#
\end{array}
\]

with the following properties:

(a) σ and τ are strongly proximal;
(b) π# is a RIM extension;
(c) X# is the unique minimal set in \( R_{\pi\tau} = \{ (x, y) \in X \times Y^# : \pi(x) = \tau(y) \} \) and σ and π# are the restrictions to X# of the projections of X × Y# onto X and Y#, respectively.

By a small modification we can assume that π# is an open RIM extension. We refer to [16, 33] for the details of the construction.

10.2. Relative regionally proximal relation. Let \( \pi : (X, T) \to (Y, T) \) be t.d.s. For \( \epsilon > 0 \), let \( \Delta_\epsilon = \{ (x, y) \in X \times X : d(x, y) < \epsilon \} \). Then the relative proximal relation is

\[
P_\pi = \bigcap_{n=1}^{\infty} \left( \bigcup_{i \in \mathbb{Z}} T^i \Delta_{1/n} \right) \cap R_\pi,
\]

and the relative regionally proximal relation is

\[
Q_\pi = \bigcap_{n=1}^{\infty} \left( \bigcup_{i \in \mathbb{Z}} T^i \Delta_{1/n} \right) \cap R_\pi.
\]

For \( R \subseteq X \times X \) and \( x \in X \), define \( R[x] = \{ x' \in X : (x, x') \in R \} \). Define

\[
U_\pi[x] = \bigcap_{n=1}^{\infty} \left( \bigcup_{i \in \mathbb{Z}} T^i \Delta_{1/n} \right)[x] \cap \pi^{-1}(\pi(x)).
\]

In other words: \( x' \in U_\pi[x] \) if and only if there are sequences \( \{x'_i\} \) in \( \pi^{-1}(\pi(x)) \) and \( \{n_i\} \) in \( \mathbb{Z} \) such that

\[
x'_i \to x' \quad \text{and} \quad (T \times T)^{n_i}(x, x'_i) \to (x, x).
\]
It is clear that $P_\pi[x] \subseteq U_\pi[x] \subseteq Q_\pi[x]$. Define

$$U_\pi = \{(x, x') \in R_\pi : x' \in U_\pi[x]\}.$$ 

The following is an open question [32]:

**Question 10.1:** If $\pi : X \to Y$ is an open Bronstein extension (i.e., $R_\pi$ has a dense set of minimal points), does $U_\pi[x] = Q_\pi[x]$ for all $x \in X$?

One does not have an answer for this question, but one has the following result.

**Proposition 10.2** ([30, Theorem 1.5]): Let $\pi : X \to Y$ be a RIM extension of minimal systems with section $\lambda$, and let $y \in Y$ be such that $\text{supp}\lambda_y = \pi^{-1}(y)$. Then for all $x \in \pi^{-1}(y)$ we have $U_\pi[x] = Q_\pi[x]$.

The following lemma guarantees that there are lots of such $y$ in Proposition 10.2.

**Lemma 10.3** ([16, Lemma 3.3]): Let $\pi : X \to Y$ be a RIM extension of minimal systems with section $\lambda$. Then there is a residual set $Y_0 \subseteq Y$ such that $y \in Y_0$ implies $\text{supp}\lambda_y = \pi^{-1}(y)$.

10.3. **Relative proximal cell.** Let $(X, T)$ be a weakly mixing t.d.s. Then for each $x \in X$, the proximal cell $P[x]$ is a dense $G_\delta$ subset of $X$ [4, 24] (under the minimality assumption this result was obtained in [15]). Now we consider the relative case. Let $\pi : X \to Y$ be an extension of t.d.s. and $x \in X$. Call $P_\pi[x]$ the **relative proximal cell** of $x$.

**Question 10.4:** If $\pi : X \to Y$ is an open weakly mixing extension of minimal systems, is the relative proximal cell $P_\pi[x]$ a residual subset of $\pi^{-1}(\pi(x))$ for all $x \in X$?

We do not have the full answer to this question. But we have the following results.

**Theorem 10.5:** Let $\pi : X \to Y$ be a weakly mixing and RIM extension of minimal systems. Then there is a residual set $Y_0 \subseteq Y$ such that, for all $y \in Y_0$ and all $x \in \pi^{-1}(y)$, we have that $P_\pi[x]$ is residual in $\pi^{-1}(y)$.

**Proof.** By Proposition 10.2 and Lemma 10.3, there is a residual set $Y_0 \subseteq Y$ such that, for all $y \in Y_0$ and all $x \in \pi^{-1}(y)$, we have $U_\pi[x] = Q_\pi[x]$. Fix such
y and x. Now π is weakly mixing, hence $Q_\pi = R_\pi$. Thus $U_\pi[x] = Q_\pi[x] = R_\pi[x] = \pi^{-1}(y)$. Since $U_\pi[x] = \bigcap_{n=1}^{\infty} \left( \bigcup_{i \in \mathbb{Z}} T_i \Delta_1/n \right)[x] \cap \pi^{-1}(y)$, we have
\[
\left( \bigcup_{i \in \mathbb{Z}} T_i \Delta_1/n \right)[x] \cap \pi^{-1}(y) = \pi^{-1}(y), \quad \forall n \in \mathbb{N}.
\]
Hence
\[
P_\pi[x] = \bigcap_{n=1}^{\infty} \left( \bigcup_{i \in \mathbb{Z}} T_i \Delta_1/n \right)[x] \cap \pi^{-1}(y)
\]
is a residual subset of $\pi^{-1}(y)$.

Applying the above theorem we have

**Theorem 10.6**: Let $\pi : X \to Y$ be an extension of minimal systems. If $\pi$ is weakly mixing and Bronstein (i.e., $R_\pi$ has a dense set of minimal points), then there is a residual set $Y_0 \subseteq Y$ such that, for all $y \in Y_0$ and all $x \in \pi^{-1}(y)$, we have $P_\pi[x]$ is residual in $\pi^{-1}(y)$.

**Proof.** To apply Theorem 10.5, we consider the following G-diagram:

\[
\begin{array}{ccc}
X & \xleftarrow{\sigma} & X^# \\
\pi \downarrow & & \downarrow \pi^# \\
Y & \xleftarrow{\tau} & Y^#
\end{array}
\]

First we claim that $(\sigma \times \sigma)R_{\pi^#} = R_\pi$. By the commutativity of the diagram, we have $(\sigma \times \sigma)R_{\pi^#} \subseteq R_\pi$. Now we show the converse. Since the set of minimal points of $P_\pi$ are dense in $R_\pi$, it is sufficient to show that every minimal point of $R_\pi$ is an element of $(\sigma \times \sigma)R_{\pi^#}$. Let $(x_1, x_2) \in R_\pi$ be minimal; then there is a minimal point $(x'_1, x'_2) \in X^# \times X^#$ such that $(\sigma \times \sigma)(x'_1, x'_2) = (x_1, x_2)$. Hence $(\pi^#(x'_1), \pi^#(x'_2))$ is a minimal point of $Y^# \times Y^#$. But $\tau(\pi^#(x'_1)) = \tau(\pi^#(x'_2))$ and $\tau$ is proximal, and hence we have $\pi^#(x'_1), \pi^#(x'_2)$ are proximal. To conclude we have $\pi^#(x'_1) = \pi^#(x'_2)$, i.e., $(x'_1, x'_2) \in R_{\pi^#}$.

Since $\pi$ is weakly mixing, it can be shown that $\pi^#$ is also weakly mixing (for example, see [33, VI(3.19)]). Now $\pi^#$ is weakly mixing and RIM; by Theorem 10.5, there is a residual set $Y_0^# \subseteq Y^#$ such that for all $y^# \in Y_0^#$ and all $x^# \in \pi^{-1}(y^#)$ we have $P_{\pi^#}[x^#]$ is residual in $(\pi^#)^{-1}(y^#)$. Let $Y_0 = \tau(Y_0^#)$. Since $Y^#$ is minimal and hence $\tau$ is semi-open, $Y_0$ is also a residual subset of $Y$. Let $y \in Y_0$ and $y^# \in Y_0^#$ with $\tau(y^#) = y$. Let $x \in \pi^{-1}(y)$. Since $(\sigma \times \sigma)R_{\pi^#} = R_\pi$, we have $\sigma((\pi^#)^{-1}(y^#)) = \pi^{-1}(y)$. There is some $x^# \in (\pi^#)^{-1}(y^#)$ such that
\[ \sigma(x^\#) = x. \] Since \( P_{\pi^\#}[x^\#] \) is dense in \( (\pi^\#)^{-1}(y^\#) \), \( P_\pi[x] \) is dense in \( \pi^{-1}(y) \). But \( P_\pi[x] \) always is a \( G_\delta \) subset of \( \pi^{-1}(y) \), and hence it is residual in \( \pi^{-1}(y) \). The proof is completed.

Lemma 7.16 now follows from Theorem 10.6, since each RIC extension is Bronstein.

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