Optimization of the geometrical stability in square ring laser gyroscopes

R Santagata1,2, A Beghi3, J Belfi2, N Beverini2,4, D Cuccato3,5, A Di Virgilio2, A Ortolan6, A Porzio7,8 and S Solimeno9

1 Department of Physics, University of Siena, Via Roma 56, Siena, Italy
2 INFN Section of Pisa, Largo Bruno Pontecorvo 3, Pisa, Italy
3 Department of Information Engineering, University of Padova, Via Gradenigo 6/B, Padova, Italy
4 Department of Physics, University of Pisa, Largo Bruno Pontecorvo 3, Pisa, Italy
5 INFN Section of Padova, Via Marzolo 8, 35131, Padova, Italy
6 INFN National Laboratories of Legnaro, Viale dell’Università 2, Legnaro, Padova, Italy
7 CNR-SPIN Section of Napoli, Complesso Universitario Monte Sant’Angelo, Via Cintia, Napoli, Italy
8 INFN Section of Napoli, Complesso Universitario Monte Sant’Angelo, Via Cintia, Napoli, Italy
9 Department of Physics, University of Napoli, Complesso Universitario Monte Sant’Angelo, Via Cintia, Napoli, Italy

E-mail: rosa.santagata@pi.infn.it

Received 9 November 2014, revised 16 December 2014
Accepted for publication 12 January 2015
Published 11 February 2015

Abstract
Ultra-sensitive ring laser gyroscopes are regarded as potential detectors of the general relativistic frame-dragging effect due to the rotation of the Earth. Our project for this goal is called GINGER (gyroscopes in general relativity), and consists of a ground-based triaxial array of ring lasers aimed at measuring the rotation rate of the Earth with an accuracy of $10^{-14}$ rad s$^{-1}$. Such an ambitious goal is now within reach, as large-area ring lasers are very close to the required sensitivity and stability. However, demanding constraints on the geometrical stability of the optical path of the laser inside the ring cavity are required. Thus, we have begun a detailed study of the geometry of an optical cavity in order to find a control strategy for its geometry that could meet the specifications of the GINGER project. As the cavity perimeter has a stationary point for the square configuration, we identify a set of transformations on the mirror positions that allows us to adjust the laser beam steering to the shape of a square. We show that the geometrical stability of a square cavity strongly increases by implementing a suitable system to measure the mirror distances,
and that the geometry stabilization can be achieved by measuring the absolute lengths of the two diagonals and the perimeter of the ring.

Keywords: ring laser gyroscopes, laser beam steering, Fermat’s principle, lense-thirring effect

(Some figures may appear in colour only in the online journal)

1. Introduction

A ring laser (RL) gyroscope is composed of a square or triangular optical cavity, inside which two opposite light beams propagate in a closed loop. In a reference frame rotating at velocity $\Omega$ with respect to an inertial frame, the opposite directions are not equivalent; the two laser emission frequencies are then split by the Sagnac frequency $f_S$:

$$f_S = \frac{4A \cdot \Omega}{\lambda p} = k_S \cdot \Omega,$$

where $k_S = 4A/(\lambda p)$ is the oriented scale factor, $A$ is the vectorial area enclosed by the laser beams, $\lambda$ the optical wavelength and $p$ the length of the round-trip path. The frequency difference can be measured by observing the beat note between the two counter-propagating beams.

The fundamental limit to the angular velocity resolution of a RL is given by the photon shot-noise. The power spectral density of shot noise, converted in equivalent rotational noise and expressed in units of $(\text{rad s})^{-1}$, reads [1]

$$p_{\Omega}^{1/2} = \frac{c}{\lambda \| k_S \| Q} \sqrt{\frac{h \nu}{P_{\text{out}} T}},$$

where $Q$ is the quality factor of the optical cavity, $h$ the Planck constant, $P_{\text{out}}$ the optical power detected by the photodiode, $T$ the measuring time, $c$ the speed of light, and $\nu = c/\lambda$ the light frequency.

Significant advancements in RL sensitivity have recently driven new applications concerning geophysics, geodesy and the test of general relativity [2–5]. A present challenge of this research is the observation of the relativistic Lense–Thirring effect, a perturbation of a few parts in $10^{10}$ to the Earth rotation rate. Such a measurement requires a calibration of the RL with respect to the local spacetime in order to measure the Earth’s rotation with high accuracy. On the other hand, the time-dependent fluctuation of scale factor, backscattering and laser dynamics-induced non-reciprocal effects have to be controlled in order to fully exploit the RL sensitivity up to the shot noise level in equation (2). In recent papers [7, 8], we addressed the role of laser dynamics on the stability and accuracy of a ring laser used as rotation sensor; in this work we focus on the role of cavity deformations on $\| k_S \|$ and $p$.

The design of the cavity is crucial for the stability of the geometrical scale factor $k_S$. The most stable and sensitive rotation sensor at present, the Gross Ring G (Wettzell, Bavaria) [6], has an optical cavity with a side length of 4 m, and has been constructed on a rigid frame made of Zerodur®. In this case, the geometrical stability is obtained by fixing the mirrors to a very rigid and heavy structure, and by minimizing thermal expansions and pressure variations. This experimental set-up has shown that ring lasers have the potentiality to measure the Lense–Thirring effect on an Earth-based experiment, but the ‘monolithic’ design cannot be extended to a triaxial array of ring lasers with sides of 6–10 m, as required by the GINGER...
A heterolithic design must be considered: the cavity frame can be made of more standard materials with thermal expansion coefficients at the level of several ppm K$^{-1}$, and the mirror positions must be actively controlled in order to compensate the frame deformations by implementing a suitable system to measure the mirror distances based on ultra-stable optical frequency references.

A detailed study of the effects of geometrical distortions on the optical path is the first necessary step toward the active control of mirror positions. In the literature, one can find many papers about the geometry of a ring cavity [9–15]. In this paper, we address for the first time the problem of finding a suitable approach to minimizing the scale factor $k_S$ variations, and pose the basis for the control of a heterolithic ring laser cavity. Our study is based on Fermat’s principle to model the optical cavity geometry starting from the position of the center of curvature of the mirrors. We demonstrate that in a square cavity, the control of the length of the two diagonals plays an important role. In fact, if the two RL diagonals are locked to the same absolute length, the optical cavity length has a stationary point corresponding to the regular square cavity, with two important implications for the control of a heterolithic structure: (i) one can implement a procedure to approach the regular geometry; (ii) in this case, the residual deformations contribute with quadratic terms depending on the ratio between the cavity side-length and the mirror’s radius of curvature.

The paper is organized as follows. In section 2 we used Fermat’s principle to calculate the light path inside a closed optical cavity formed by four equal spherical mirrors. Section 3 is devoted to the decomposition of the cavity deformations in a suitable eigenvectors basis and to the estimation of their contributions to the cavity perimeter and scale factor. Section 4 reports the analysis of the perimeter and scale factor variations due to the residual deformations in a square cavity with fixed diagonal lengths. In section 5 the requirements on a real cavity design are discussed; and the conclusions are drawn in section 6.

2. The geometrical model of the optical cavity

We consider a square ring optical cavity defined by four spherical mirrors. Let $r_k \in \mathbb{R}^3$ and $c_k \in \mathbb{R}^3$ denote the radius and the center of curvature of the $k_{th}$ mirror with $k = \{1, 2, 3, 4\}$. The four light spots can be usefully represented as elements $X$ of $S^{2 \times 4}$, i.e. four points on the spheres representing the mirrors; let $s_k = r_k \mathbf{x}_k + c_k$ be the position vector of the $k_{th}$ spot with respect to a reference frame, and $\mathbf{x}_k \in S^2 = \{ \mathbf{x} \in \mathbb{R}^3, \mathbf{x}^T \mathbf{x} = 1 \}$ the directional cosines of the spot position with respect to its mirror center. We introduce the compact notation for the directional cosine matrix $X = [\mathbf{x}_1, ..., \mathbf{x}_4] \in S^{2 \times 4} \subset \mathbb{R}^{3 \times 4}$, the mirror configuration matrix $C = [c_1, ..., c_4] \in \mathbb{R}^{3 \times 4}$, and the curvature radii matrix $R = \text{diag} [r_1, ..., r_4] \in \mathbb{R}^{4 \times 4}$. In this notation the spot position matrix can be written as $S = XR + C$.

Our aim is to derive a functional dependence between the spot position matrix $S$ and the mirror configuration matrix $C$. Since each mirror displacement can be described as a variation of the position of its curvature center, the four vectors $c_k$ represent a set of dynamical variables of the system. All the relevant quantities, with respect to the Sagnac signal and its noise, will be given in terms of these variables. The four geometrical constraints $\|s_k - c_k\|^2 = r_k^2$, $k = 1, ..., 4$ ($\mathbf{x}_k$ are points of spherical surfaces), reduce the dimensions of the set of $X$ from dim ($\mathbb{R}^{3 \times 4}$) = $3 \times 4 = 12$ to dim ($S^{2 \times 4}$) = $2 \times 4 = 8$.

The Euclidean distances among the six pairs $(k, j)$ of the four light spots are given by

$$l_{kj}(X; C, R) = \|s_k - s_j\| = \|r_k \mathbf{x}_k - r_j \mathbf{x}_j + c_k - c_j\|,$$

(3)
where $\| \cdot \|$ is the Euclidean norm in $\mathbb{R}^3$. The optical path length $p$ is simply the sum of the lengths of the sides of the, in general non-planar, optical cavity

$$p(X; C, R) = l_{12}(X; C, R) + l_{23}(X; C, R) + l_{34}(X; C, R) + l_{41}(X; C, R),$$

and the modulus of the enclosed area reads

$$\| \mathbf{A}(X, C, R) \| = \frac{1}{2} \| (s_1 - s_3) \wedge (s_2 - s_4) \|$$

$$= \frac{1}{2} l_{13}(X; C, R) l_{24}(X; C, R) \cos \theta,$$

where $\theta$ is the angle between the vectors $s_1 - s_3$ and $s_2 - s_4$. If the four mirror centers lie in the same plane, these vectors represent the diagonals of a polygon. In this case $p(X; C, R)$, $l_{13}(X; C, R)$ and $l_{24}(X; C, R)$ coincide with the perimeter and diagonal lengths of the corresponding planar polygon.

The effective directional cosine matrix $X(C, R)^a$ for a given mirror's configuration $C$ and curvature radii $R$ is found by using Fermat's principle [16], which states that light rays follow extremal optical paths, i.e.

$$V p(X; C, R) = 0,$$  \hspace{1cm} (5)

where $V (\cdot \cdot)$ is the gradient with respect to a coordinate set of $X \in S^{2 \times 4}$. The vector components are referred to a clockwise orthonormal coordinate system $(e_1, e_2, e_3)$ with the origin in the center of the square, the axes $e_1$ and $e_2$ along the two diagonals, and $e_3$ perpendicular to the square (see figure 1).

In this coordinate system, the ideal mirror configuration for the regular square cavity, with side length $L$ and all the mirror curvature radii equal to $r$, is represented by the matrix

$$C_0 = r \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ \end{pmatrix}. \hspace{1cm} (6)$$

Then the matrix

$$X_0 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \end{pmatrix}, \hspace{1cm} (7)$$

is the solution of equation (5) for the spot directional cosines. In general, equation (5) is an irrational algebraic system of equations, and no analytic solution is available. However, for small deformations, we can calculate the Taylor expansion of $p(X; C, R)$ in a neighborhood of $X_0$.

Let us consider the matrix $Z = X_0 + Y$. The columns of $Y$ parametrize the tangent vectors to $X_0$, i.e. each column of $Y$, identified as a vector of $\mathbb{R}^3$, is orthogonal to the corresponding column of $X_0$:

$$Y = \begin{pmatrix} 0 & y_3 & 0 & y_7 \\ y_1 & 0 & y_5 & 0 \\ y_2 & y_4 & y_6 & y_8 \\ \end{pmatrix}. \hspace{1cm} (8)$$

We obtain the new point $X'$ of $S^{2 \times 4}$ using the map $R : S^2 \times \mathbb{R}^3 \to S^2$, $R(x, a) = (x + a)/\|x + a\|$ column-wise on $Z$, so that each column of $X'$ is the corresponding column of $Z$ normalized. The matrix $Z$ can be considered as a function of
the vector $y = (y_1, ..., y_n)^T$, and therefore $X'$ can be also regarded as a function of $y$. A second order geometric model of the function $p(X'(y); C, R)$ is then

$$
p(X'(y); C, R) = p(X_0; C, R) + s(C, R)^T y + \frac{1}{2} y^T F(C, R) y + o(\|y\|^2),
$$

where the vector $s(C, R)$ and the matrix $F(C, R)$ are the gradient and Hessian of the function $p$ with respect to $y$, respectively.

Thus substituting equation (9) in equation (5) we find that the stationary point for this geometric model is given by

$$
y^* = -F(C, R)^{-1} s(C, R).
$$

The corresponding extremal optical length is

$$
p^*(C, R) = p(X_0; C, R) = \frac{s(C, R)^T F^{-1}(C, R) s(C, R)}{2},
$$

and the new spot positions are $S(C, R) = X'(y^*) R + C$. The matrix $F^{-1}(C, R)$ has a straightforward interpretation as the ray matrix of the RL cavity, while the vector $y$ can be interpreted as the linear response of the optical cavity to a displacement $\delta C$ of the centers of curvature from the ideal mirror configuration.

### 3. The eigenvector basis of the cavity deformations

The geometry of the cavity is uniquely determined by the mirror configuration matrix $C$, representing the position of the centers of curvature of the four mirrors. Consider a generic curve $C(t)$ in $\mathbb{R}^{3 \times 4}$, starting at $C_0$. To classify the cavity deformation we differentiate $C(t)$ with respect to $t$, and then express the components of the columns of $C(0)$ with respect to a suitable basis of the tangent space in $C_0$.
\[ \dot{C}(0) = \sum_{\alpha=1}^{12} \tau_\alpha \alpha \]  \tag{12} 

where \( \alpha \) is the \( \alpha \)th element of the basis of the \( 3 \times 4 \) matrices, and \( \tau_\alpha \in \mathbb{R} \). With reference to the polygon defined by the position of the centers of curvature of the four mirrors, the basis elements can be classified depending on their effects on the geometry of the optical cavity (see figure 2) as follows:

- common and differential stretching of the diagonals

\[
E_1 = \frac{1}{2} \begin{pmatrix}
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
E_2 = \frac{1}{2} \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix};
\]

- shear planar deformations \( E_3 \) (left), \( E_4 \) (right).

- diagonal tilt \( E_5 \) (left); out of plane tilt \( E_6 \) (right).
• shear planar deformations

\[ E_3 = \frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_4 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \]

• diagonal tilts and out of plane motions

\[ E_5 = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_6 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 \end{pmatrix}; \]

• rigid body translations

\[ E_7 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_8 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

\[ E_9 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}; \]

• infinitesimal rigid body rotations

\[ E_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad E_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{pmatrix}. \]

\[ E_{12} = \frac{1}{2} \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \]

By definition, the six distances \( l_i(X; C, R) \) and their functions, such as \( p \) and \( A \), do not depend, to the first order in \( \tau_\alpha \), on the rigid body translation and rotations represented by the \( E_\alpha \) with \( \alpha = 7, \ldots, 12 \). Therefore, the mirror configuration matrix for a general deformation of the cavity is given by

\[ C(\tau_1, \ldots, \tau_6) = C_0 + \sum_{\alpha=1}^{6} \tau_\alpha E_\alpha. \tag{13} \]

The expansion of \( p^\alpha(C, R) \) in the power series of the \( \tau_\alpha \), retaining only second order terms, reads

\[ p^\alpha(C, R) = 4L \left[ 1 - \frac{\tau_1}{\sqrt{2} L} + \frac{\tau_2^2}{4L^2} - \frac{\tau_1^2 + \tau_4^2}{2L\left(\sqrt{2} r - 2L\right)} - \frac{\tau_6^2}{2L\left(2\sqrt{2} r - L\right)} \right] + o\left(\tau_\alpha^2\right). \tag{14} \]

\( p^\alpha(C, R) \) depends linearly on the diagonal common stretching, while it depends quadratically on differential stretching, shears and tilts. Assuming that the cavity satisfies the stability condition for the confinement of the optical rays \( 0 \leq L/r \leq \sqrt{2} \) [17], the coefficients of \( \tau_3^2, \tau_4^2 \) and \( \tau_6^2 \) are negative.
Combining equations (14) and (4), and assuming the cavity oriented along the direction of maximum rotational signal \((\mathbf{A}^{(1)}u_{\Omega})\), the ring laser scale factor reads

\[
\frac{\delta k_s}{k_s} \equiv \frac{4}{\lambda_p \mathbf{A}^0(C, R) \cdot u_{\Omega}} \left[ \frac{1 - \frac{r_1^2}{2L^2} - \frac{r_2^2}{4L^2} - \frac{2L + \sqrt{2}r}{4L(r - \sqrt{2}L)}(r_2^2 + r_2^2)}{L + \frac{2\sqrt{2}r}{L(r - \sqrt{2}L)}r_2^2} \right] + o(r_2^2). \tag{15}
\]

From equations (14) and (15) we note that, to obtain an accuracy of 1 part in \(10^{10}\) on \(k_s\), the relative amplitude of \(E_1\) deformation has to be \(10^{-10}\), while the requirements on the other cavity deformation amplitudes are drastically reduced by the quadratic dependence.

4. Discussion

The measurement of the length of the two diagonals provides a very precise observable to constrain the \(E_1\) deformation [18]. The model has shown that the stabilization of the length of the two Fabry–Perot resonators, and consequently of the distance between the centers of curvature of the diagonally opposite mirrors, provides a high rejection of the mirror perturbations, even if their values are biased by some systematic errors.

In figure 3 we report the result of a simulation of a single square cavity with a 6 m side length, where the position of each corner mirror is varied randomly in the three directions of space with a standard deviation \(\delta\) under the constraint of equal diagonals. This shows that to obtain a stability of 1 part in \(10^{10}\) in the RL scale factor \(k_s\), needs to keep the mirror positions within a few \(\mu\)m. The relative variation of the scale factor is also plotted for different values of the unbalance \(\delta D\) between the two diagonal absolute lengths. Note that even in the case of
a systematic error in the difference between the two diagonal cavity lengths of $10^{-6}$ m [18], the scale factor stability is compliant with the requirements of GINGER ($\Delta k_S/k_S < 10^{-10}$).

The results of equations (14) and (15) provide a procedure to approach the regular square geometry, starting from a generic mirror configuration. Let us consider the distances $\|e_3 - e_1\|$ and $\|e_4 - e_2\|$ between the centers of curvature of the two pairs of opposite mirrors; imposing that the two diagonals have equal distance $d_0$, i.e.

$$\|e_3 - e_1\| = \|e_4 - e_2\| = \sqrt{(2r - \sqrt{2}L + \tau_1 + \tau_2)^2 + \tau_2^2} = d_0$$  \hspace{1cm} (16)

we have that $\tau_2 = 0$ and $\tau_1 = \tau_2^2/d_0$.

The radius of curvature also plays an important role: if we define $\tilde{\tau}_u = \tau_u/d_0$, the expressions for the constrained perimeter and scale factor, normalized with respect to their nominal values of the square $4L$ and $L/\lambda$, can be rewritten as:

$$\hat{p}^*(C, R) = 1 - \frac{h}{\sqrt{2} - 2h}\left(\tilde{\tau}_3^2 + \tilde{\tau}_4^2\right) - \tilde{\tau}_5^2 - \frac{\sqrt{2}h}{4 - \sqrt{2}h}\tilde{\tau}_6^2$$  \hspace{1cm} (17)

and

$$\hat{k}^*(C, R) = 1 - \frac{h\left(2h + \sqrt{2}\right)}{2\left(1 - \sqrt{2}h\right)^2}\left(\tilde{\tau}_3^2 + \tilde{\tau}_4^2\right) - \tilde{\tau}_5^2 - \frac{2h\left(h + 2\sqrt{2}\right)}{4 - \sqrt{2}h^2}\tilde{\tau}_6^2.$$  \hspace{1cm} (18)

where $h = L/r$. Now $\hat{p}^*(C, R)$ and $\hat{k}^*(C, R)$ are quadratic forms of $\tilde{\tau}_u$ with no cross terms and the regular square geometry corresponds to a saddle-point of $\hat{p}^*(C, R)$. The importance of operating the ring with a geometry as close as possible to a square, which for a large apparatus cannot be obtained by construction, has already been pointed out. The presence of a saddle-point means that an experimental procedure to approach the square configuration, by monitoring the perimeter length, exists; this can be made with very high accuracy through the measurement of the laser emission frequency. The present analysis also shows that the spot of the beams on the mirrors should be in an ideal position with tolerances of microns.

It is straightforward to see that the coefficients of the $\tau_u$ have a very similar behaviour in both equations; in particular they are constant and equal to 1, for $\alpha = 5$; while the others, $\alpha = 4, 6$, depend on $h$. The ideal condition should be to work with a $h$ value so that the coefficients of $\tau_u$ are large in equation (17) (which means that we can approach the regular geometry more accurately), while the corresponding coefficients in equation (18) are small (which means that the rotational sensitivity is affected less by the geometry nonregularities). From these considerations, a large radius of curvature seems to offer the best compromise. Values of $h$ around $1/\sqrt{2}$ must be avoided; actually, for $h = 1/\sqrt{2}$ the cavity becomes instable, being confocal in the sagittal plane [14].

The design of the RL optical cavity should also consider the features of the resulting Gaussian beam, taking into account the active medium inside the cavity. In particular, the beam radius in the waist is of great interest, because the gain medium is usually located in it, and the optimal waist radius should be selected to be a tradeoff between low losses (small beam radius) and high output power (large beam radius). The precision in the realization of the mirror coating and the dielectric shift linked to it should also be taken into account. In this respect, the model is not complete. The authors intend to make a more detailed analysis in subsequent work.
5. Requirements on cavity design

In order to be able to measure the Lense–Thirring effect, a stabilization of the scale factor (equation (1)) better than $10^{-10}$ is required. The design study of GINGER, that has been published in [5], satisfies this requirement. The basic idea of the project is to use a rigid and stable multi-axial system of large RL, in order to reconstruct the modulus of the Earth’s rotational vector and compare the observed data to those provided by IERS (the International Earth Rotation and Reference Systems Service).

To test the limits of a single ring laser of such an array, an intermediate prototype, called GP2, has been designed and installed at INFN laboratories in Pisa [19]. This is the ring where the control strategies will be tested, as regards cavity geometry and laser active medium parameters. As we have shown in this paper, the constraint of fixed diagonal lengths implies that, in order to obtain a stability of 1 part in $10^{10}$ on $k_0$, it is sufficient to set the residual mirror degrees of freedom and the unbalance between the two diagonal lengths with an accuracy of few $\mu$m (see figure 3).

The building material, mechanical tolerances and precision on mirror active driving have been selected to work up to this purpose. GP2 is a single-axis He–Ne square RL with a side-length of 1.60 m; it is oriented along the local latitude in order to maximize the Sagnac signal and minimize the orientation errors on the scale factor. Its mirrors are fixed in four mirror holders mounted at the corners of a black Afrika granite slab machined with a precision better than 10 $\mu$m. Black granite has been chosen because of the following properties: dimensional stability (being the material free of internal tension), thermal stability (the linear expansion is much lower than for steel or cast iron, being the thermal expansion coefficient of $6.5 \times 10^{-6}$), hardness (being comparable to tempered steel), and accuracy (the flatness of the surfaces is better than the one obtained with traditional materials).

To get a regular square configuration of the mirrors, this prototype is equipped with a piezo nano-positioning system with a large dynamic range (80 $\mu$m). A feedback mechanism will be implemented through six translators attached to mirror holders: one of these is provided with a 3-axial PZT, while the other three with a 1-axial PZT along the diagonal. The stabilization of the diagonal cavities will be implemented controlling their lengths against an accurate and stable optical frequency standard together with the ring cavity perimeter [18]. To this aim the GP2 vacuum chamber encloses not only the four mirrors and the optical path of the circulating beams, but also the optical path along the two linear diagonal cavities. Spherical supermirrors are used; a special optical coating guarantees a reflectivity >99.999% at 45 degree angle of incidence and of about 99.9% at normal incidence.

6. Conclusions

We presented a suitable formalism based on Fermat’s principle of optics to define the beam path geometry in a square cavity RL. This allowed us to identify the rigid body motion of the cavity and then to classify the residual optical cavity deformations $E_\alpha$. We demonstrated that when the configuration of the mirrors is close to that of the regular square cavity, the scale factor and the perimeter length of an RL depend linearly only on the cavity isotropic expansion $E_1$. It follows that, once the diagonal lengths are set to the same value, the effects of the remaining deformations on the RL scale factor are quadratic in their amplitudes $\tau_\alpha$. This observation is at the basis of a possible saddle-point optimization of the different deformations $E_\alpha$; keeping the mirror positions within a few $\mu$m, the stability of 1 part in $10^{10}$ in the RL
scale factor $k$, can be obtained. The important conclusion is that a heterolithic structure can be utilized for high sensitivity and stability RL.

Acknowledgments

We would like to thank A Saccon for useful discussions on matrix manifold theory. DC also thanks the Eindhoven University of Technology, Netherlands, for kind hospitality. Finally, the authors thank G Cella for his constructive suggestions.

References

[1] Stedman G E 1997 Ring laser tests of fundamental physics and geophysics Rep. Progr. Phys. 60 615–88
[2] Schreiber K U and Wells J-P R 2013 Invited review article: Large ring lasers for rotation sensing Rev. Sci. Instrum. 84 041101
[3] Belfi J et al 2012 Horizontal rotation signals detected by G-Pisa ring laser for the Mw = 9.0, March 2011, Japan Earthquake J. Seismol. 16 767–76
[4] Schreiber K U et al 2011 How to detect the Chandler and the annual wobble of the Earth with a large ring laser gyroscope Phys. Rev. Lett. 107 173904
[5] Bosi F et al 2011 Measuring gravito-magnetic effects by multi ring-laser gyroscope Phys. Rev. D 84 1220022
[6] Schreiber K U et al 2009 The large Ring Laser G for continuous Earth rotation monitoring J. Pure Appl. Geophys. 166 1485
[7] Cuccato D et al 2014 Controlling the non-linear intracavity dynamics of large He-Ne laser gyroscopes Metrologia 51 97
[8] Beghi A et al 2012 Compensation of the laser parameter fluctuations in large ring-laser gyros: a Kalman filter approach Appl. Opt. 51 7518
[9] Yuan J et al 2007 Nonplanar ring resonator modes: generalized Gaussian modes Appl. Opt. 46 2980
[10] Yuan J et al 2011 Generalized ray matrix for a spherical mirror reflection and its application in square ring resonators and monolithic triaxial ring resonators Opt. Exp. 19 6762
[11] Lu G et al 2013 Mirrors movement-induced equivalent rotation effect in ring laser gyros Opt. Exp. 21 14458
[12] Hurst R B et al 2007 An elementary proof of the geometrical dependence of the Sagnac effect J. Opt. A: Pure and Applied Optics 9 10
[13] Currie B E, Stedman G E and Dunn W 2002 Laser stability and beam steering in a nonregular polygonal cavity Appl. Opt. 41 1689
[14] Bilger H R and Stedman G E 1987 Stability of planar ring lasers with mirror misalignment Appl. Opt. 26 3710
[15] Rodloff R 1987 A laser gyro with optimized resonator geometry IEEE Quantum Electron. 23 438
[16] Born M and Wolf E 1999 Principles of Optics (Cambridge: Cambridge University Press)
[17] Saleh B E A and Teich M C 1991 Fundamentals of Photonics (New York: J. Wiley)
[18] Belfi J et al 2014 Interferometric length metrology for the dimensional control of ultra-stable ring laser gyroscopes Class. Quantum Grav. 31 225003
[19] Belfi J et al 2013 Absolute control of the scale factor in the GP2 laser gyroscope: toward a ground based detector of the Lense-Thirring effect EFTF/IFC-21-25 July 2013, Prague 795–8 INSPEC Accession Number 14025882