SO(5) as a Critical Dynamical Symmetry in the SU(4) Model of High-Temperature Superconductivity

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An SU(4) model of high-temperature superconductivity and antiferromagnetism has recently been proposed. The SO(5) group employed by Zhang is embedded in this SU(4) as a subgroup. In order to understand the relationship between the two models, we have used generalized coherent states to analyze the nature of the SO(5) subgroup. By constructing coherent-state energy surfaces, we demonstrate that the SU(4) ⊂ SO(5) symmetry can be interpreted as a critical dynamical symmetry interpolating between superconducting and antiferromagnetic phases, and that this critical dynamical symmetry has many similarities to critical dynamical symmetries identified previously in other fields of physics. More generally, we demonstrate with this example that the mathematical techniques associated with generalized coherent states may have powerful applications in condensed matter physics because they provide a clear connection between microscopic many-body theories and their broken-symmetry approximate solutions. In addition, these methods may be interpreted as defining the most general Bogoliubov transformation subject to a Lie group symmetry constraint, thus providing a mathematical connection between algebraic formulations and the language of quasiparticle theory. Finally, we suggest that the identification of the SO(5) symmetry as a critical dynamical symmetry implies deep algebraic connections between high-temperature superconductors and seemingly unrelated phenomena in other field of physics.

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I. INTRODUCTION

Data for cuprate high-temperature superconductivity (SC) suggests \textit{d}-wave singlet pairing in the superconducting state and that superconductivity in these systems is closely related to the antiferromagnetic (AF) insulator properties of the undoped compounds. This proximity of superconducting and antiferromagnetic order is unusual and suggests that a correct description of cuprate superconductors must permit the SC and AF order to enter the theory on a similar footing.

A. Unified Models

S.-C. Zhang et al\textsuperscript{(1,3,4)} proposed to unify AF and SC states by assembling their order parameters into a 5-dimensional vector and constructing an SO(5) group that rotates AF order into \textit{d}-wave SC order. Recently, we introduced an SU(4) model\textsuperscript{(5)} of high-temperature SC and AF order having SO(5) as a subgroup. The motivation for and methodology of the SU(4) model are superficially rather different from those of Zhang and collaborators. However, the appearance of the Zhang SO(5) as a subgroup of the SU(4) symmetry implies that the two models must have a strong physical relationship, as we have already suggested\textsuperscript{(6)}.

One difficulty in understanding this relationship is that the two models are formulated using different approaches that employ different languages. In the SU(4) model the five operators responsible for AF order (three staggered magnetization operator components) and SC order (two \textit{d}-wave pair operators) enter the theory as quantum mechanical operators, on exactly the same footing as all of the other ten generators of SU(4)\textsuperscript{(5)}. Thus, the SU(4) model is a many-body, fully quantum mechanical theory in which the charge and spin are exactly conserved, so there is no spontaneously broken symmetry. This is, of course, as it should be: the charge and spin are rigorously conserved in a full many-body formulation of the problem as they are in nature. In the Zhang SO(5) model, the five operators corresponding to the staggered magnetization and the \textit{d}-wave singlet pairing are instead treated as order parameters (expectation values of operators in a broken symmetry state). Therefore, in the SO(5) model the antiferromagnetic phase and the superconducting phase are associated with \textit{approximate solutions} of the many-body problem that break charge and spin symmetry spontaneously.

The methodology employed in the development of the SU(4) model (systematic application of principles of dynamical symmetry) has found broad application in other fields such as nuclear structure\textsuperscript{(4,5)}, molecular physics\textsuperscript{(6)} or particle physics\textsuperscript{(7)}. This implies a deep algebraic connection between high temperature supercon-
ductivity and a variety of phenomena in other fields of physics that bear no superficial resemblance to high-temperature superconductors. For example, we have pointed out previously that there is an almost perfect mathematical analogy at the algebraic level between the AF-SC competition in cuprates and the competition between quadrupole deformation of the nuclear mean field and nucleon pairing that is a central organizing principle of nuclear structure physics. The appearance of the $SO(5)$ subgroup in the dynamical symmetry of the $SU(4)$ model for high-temperature superconductors is very similar to that of the $SO(7)$ subgroup in the $SO(8)$ fermion dynamical symmetry model that describes nuclear structure.

B. Generalized Coherent States

There is a well-developed theoretical approach to relating a many-body algebraic theory with no broken symmetry to an approximation of that theory that exhibits spontaneously broken symmetry: the method of generalized coherent states. This method may be viewed as the extension of Glauber coherent state theory (which is built on an $SU(2)$ Lie algebra) to a more complex system having an arbitrary Lie algebra structure. It has also been shown to be equivalent to the most general Hatree-Fock-Bogoliubov variational method under symmetry constraints, and has been applied extensively in various areas of physics and mathematical physics, though not to our knowledge in condensed matter.

The result of such a coherent state analysis is a set of energy surfaces that represent an approximation to the original theory in which order parameters appear as independent variables. In the general case, these energy surfaces can exhibit (possibly multiple) minima and these minima may appear at non-zero values of the order parameters, implying spontaneous symmetry breaking. Thus, the generalized coherent state method is a systematic approach to relating a many-body algebraic theory to its approximate symmetric and broken symmetry solutions.

C. Critical Dynamical Symmetries

The concept of a critical dynamical symmetry appears naturally in applications of generalized coherent state techniques to other fields of physics. A critical dynamical symmetry is a dynamical symmetry having eigenstates that vary smoothly with a parameter (usually particle-number related) such that the eigenstates approximate one phase of the theory on one end of the parameter range and a different phase of the theory at the other end of the parameter range, with eigenstates in between exhibiting large softness against fluctuations in the order parameters describing the two phases.

We shall demonstrate here what is, to our knowledge, the first example in condensed matter physics of such a symmetry. In this case, the critical dynamical symmetry will be shown to be based on the $SO(5)$ subgroup of $SU(4)$, and it will be shown to interpolate between AF and SC order as the hole doping parameter is varied. Thus, we shall propose that, within the context of the more general $SU(4)$ model, the $SO(5)$ symmetry employed by Zhang serves as a doorway between AF and SC order in a manner that can be specified in precise terms using the language of Lie algebras and generalized coherent state theory, and that is related to critical dynamical symmetries that have been found in other fields of physics.

D. Symmetry Breaking

We shall use these results to derive a result that has received considerable attention for the $SO(5)$ model: that an exact $SO(5)$ symmetry cannot account for the detailed phenomenology of the cuprate superconductors and that it is necessary to break $SO(5)$ (explicitly, not spontaneously) in a particular way in order to recover Mott insulator normal states at half filling, as is required by the data. As we shall show, the embedding of $SO(5)$ as a subgroup of $SU(4)$ implies naturally that $SO(5)$ must be broken in this manner in order to produce the correct normal states at half filling.

E. Goals of Paper

Let us conclude this introduction by enumerating concisely the primary goals of this paper:

1. This paper serves to introduce the generalized coherent state method to issues in condensed matter physics.
2. We shall show how to relate the generalized coherent state to the most general variational quasiparticle states that can be constructed subject to the constraints of $SU(4)$ symmetry.
3. This paper introduces into the high-temperature superconductor discussion in particular and condensed matter in general, the concepts of critical dynamical symmetries that have been applied with considerable success in other fields of physics.
4. We shall demonstrate that the coherent state solution of the $SU(4)$ model identifies $SO(5)$ as a critical dynamical symmetry. This critical dynamical symmetry will be shown to interpolate between AF and SC order as the hole doping of the system is varied. This doping dependence is a natural consequence of the $SU(4)$ symmetry, without the introduction of a chemical potential ansatz.
5. We shall show that the AF and SC phases themselves are more economically described, not by \(SO(5)\), but by dynamical symmetries built on \(SO(4)\) and \(SU(2)\) subgroups of \(SU(4)\), respectively. Thus, we shall demonstrate that \(SU(4)\) accounts for both the origin of the AF and SC order parameters and the \(SO(5)\) “rotation” of the superspin vector between these phases.

6. We shall demonstrate that as a fundamental consequence of the \(SU(4)\) structure the \(SO(5)\) subgroup must be broken in order to produce Mott insulator normal states. Furthermore, we shall demonstrate that the required symmetry breaking terms and the doping dependence of the solutions occur naturally within the \(SU(4)\) parent algebra and need not be introduced by hand as is required in the Zhang \(SO(5)\) model.

7. We shall show that, because of the nature of the critical dynamical symmetry, an AF perturbed \(SU(4) \supset SO(5)\) Hamiltonian is able to approximate various symmetry limit solutions depending on the doping: the solutions are close to the \(SO(4)\) limit presenting AF order around half filling, and approach the \(SU(2)\) limit presenting SC order as the hole doping increases. Thus the \(SU(4)\) model with a perturbed \(SO(5)\) Hamiltonian is able to account for the essential features of high \(T_c\) superconductors.

Our approach will be to introduce the basic features of the coherent state technique in sections II and III. In section IV we derive the coherent state energy surfaces for the \(SU(4)\) model, and in Section V we discuss the \(SU(4)\) energy surfaces in various dynamical symmetry limits. In Section VI, we use \(SU(4)\) energy surfaces to examine the properties of broken \(SO(5)\) symmetry, and then use these results in Section VII to argue that with a small \(SO(5)\) “rotation” of the superspin vector between these phases.

A. Algebraic Structure

A convenient way to analyze these generalized coherent states is in terms of their geometry, which is in one-to-one correspondence with the coset space, that is the space of cosets of \(SU(4)\) by \(SU(2)\). Let us begin with the algebraic structure of the \(SU(4)\) model [5]. We introduce 16 bilinear fermion operators:

\[
\begin{align*}
\rho_{12} &= \sum_k g(k)c_{k\uparrow}c_{-k\downarrow}, \\
\rho_{12}^* &= \sum_k g^*(k)c_{-k\downarrow}c_{k\uparrow}, \\
q_{ij} &= \sum_k g(k)c_{k+Q,i}c_{-k,j}, \\
q_{ij}^* &= (q_{ij})^*, \\
Q_{ij} &= \sum_k c_{k+Q,i}c_{k,j} - \frac{i}{2}\Omega\delta_{ij}
\end{align*}
\]

where \(c_{k\uparrow}\) creates a fermion of momentum \(k\) and spin projection \(i, j = 1\) or 2, \(\Omega = (\pi, \pi, \pi)\) is an AF ordering vector, \(\Omega/2\) is the electron-pair degeneracy, and following Refs. [4, 5] we define

\[
g(k) = \text{sgn}(\cos k_x - \cos k_y)
\]

with the constraints

\[
|g(k)| = 1.
\]

Under commutation the operator set \(\rho\) closes a \(U(4)\) algebra corresponding to the group structure

\[
\begin{align*}
\rho &\supset SO(4) \times U(1) \supset SU(2)_s \times U(1) \\
SU(4) &\supset SU(5) \supset SU(2)_s \times U(1) \\
SU(2)_p &\times SU(2)_s \supset SU(2)_s \times U(1)
\end{align*}
\]

where we require each subgroup chain to end in the subgroup

\[
SU(2)_s \times U(1)
\]

representing spin (the \(SU(2)_s\) factor) and charge (the \(U(1)\) factor) conservation, because the physical states of the system obey these conservation laws.

In Ref. [4] we discussed the representation structure of \(S(4)\) and showed that the \(SO(4)\) subgroup is associated with antiferromagnetism, the \(SU(2)_p\) subgroup is associated with \(d\)-wave superconductivity, and the \(SU(5)\) subgroup is associated with a transitional symmetry interpolating between the other two. In this paper, we further provide the full mathematical justification for interpreting the \(SO(5)\) subgroup as a symmetry interpolating dynamically between SC and AF phases.

II. COHERENT STATES AND THE SU(4) MATRIX REPRESENTATION

Gilmore [12, 13] and Perelomov [14] (see also earlier work by Klauder [15]) demonstrated that Glauber coherent states [16] for the electromagnetic field could be generalized to define coherent states associated with the structure of an arbitrary Lie group. In particular, they observed that the original Glauber theory for coherent photon states may be expressed in terms of an \(SU(2)\) Lie algebra by examining the commutation properties of the second-quantized operators of the theory. Once the theory has been formulated in terms of an \(SU(2)\) algebra generated by combinations of creation and annihilation operators, the formalism may be generalized to encompass a set of such operators closed under any Lie algebra.
B. A Convenient Basis for Generators

It is convenient to take as the generators of \( U(4) \to U(1)_{cd} \times SU(4) \) the new combinations

\[
Q_+ = Q_{11} + Q_{22} = \sum_k (c^\dagger_{k+Q_1} c_{k+Q_1} + c^\dagger_{k+Q_1} c_k)
\]

\[
\mathcal{S} = \left( \frac{S_{12} + S_{21}}{2}, -i \frac{S_{12} - S_{21}}{2}, S_{11} - S_{22} \right)
\]

\[
\mathcal{Q} = \left( \frac{Q_{12} + Q_{21}}{2}, -i \frac{Q_{12} - Q_{21}}{2}, \frac{Q_{11} - Q_{22}}{2} \right)
\]

(4)

\[
\pi^\dagger = \left( \frac{q_{11}^\dagger - q_{22}^\dagger}{2}, \frac{q_{11}^\dagger + q_{22}^\dagger}{2}, -i \frac{q_{12}^\dagger + q_{21}^\dagger}{2} \right)
\]

\[
\tilde{\pi} = (\pi^\dagger)^\dagger \quad D^\dagger = p_{12}^\dagger \quad D = p_{12} \quad M = \frac{1}{2}(S_{11} + S_{22})
\]

(5)

where \( Q_+ \) generates the \( U(1)_{cd} \) factor and is associated with charge density waves (do not confuse this \( U(1) \) factor with the one appearing in Eq. (3) that is associated with charge conservation), \( \mathcal{S} \) is the spin operator, \( \mathcal{Q} \) is the staggered magnetization, the operators \( \pi^\dagger \) and \( \tilde{\pi} \) are those of Ref. [3], the operators \( D^\dagger \) and \( D \) are associated with \( d \)-wave pairs, and

\[
2M = n - \Omega
\]

is the charge operator. Because of the direct product structure \( U(4) \to U(1)_{cd} \times SU(4) \), we can without loss of generality analyze the \( U(4) \) structure in terms of its subgroup \( SU(4) \), with the \( U(1)_{cd} \) factor considered separately. Hence all subsequent discussion will deal with the \( SU(4) \) subgroup of \( U(4) \).

To facilitate comparison with the \( SO(5) \) symmetry, the \( SO(6) \sim SU(4) \) generators may be expressed as

\[
L_{ab} = \begin{pmatrix}
0 & \frac{1}{2}(D + D^\dagger) \\
\pi_x & -Q_x & 0 \\
\pi_y & -Q_y & -S_z \\
iD_z & M & i\pi_x & i\pi_y & -i\pi_z & 0
\end{pmatrix}
\]

(5)

where we define

\[
D_\pm = \frac{1}{2}(D \pm D^\dagger) \quad \pi_{i\pm} = \pi_i \pm \pi_i^\dagger
\]

(6)

with \( L_{ab} = -L_{ba} \) and with commutation relations

\[
[L_{ab}, L_{cd}] = i(\delta_{ac}L_{bd} - \delta_{ad}L_{bc} - \delta_{be}L_{ad} + \delta_{bd}L_{ac}).
\]

(7)

C. Faithful Matrix Representation

The coherent state method requires a faithful matrix representation of \( SU(4) \). Explicit multiplication verifies that the following mapping preserves the algebra of (7):

\[
p_{12} \to \begin{pmatrix}
0 & i\sigma_y \\
0 & 0
\end{pmatrix} \quad p_{12} \to \begin{pmatrix}
0 & 0 \\
0 & -i\sigma_y & 0
\end{pmatrix}
\]

\[
q_{12} \to \begin{pmatrix}
0 & \sigma_x \\
0 & 0
\end{pmatrix} \quad q_{12} \to \begin{pmatrix}
0 & 0 \\
\sigma_x & 0
\end{pmatrix}
\]

\[
q_{11} \to \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \quad q_{11} \to \begin{pmatrix}
0 & I + \sigma_z \\
0 & 0
\end{pmatrix}
\]

\[
q_{22} \to \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix} \quad q_{22} \to \begin{pmatrix}
0 & I - \sigma_z \\
0 & 0
\end{pmatrix}
\]

(8)

\[
S_{12} \to \begin{pmatrix}
\sigma_+ & 0 \\
0 & -\sigma_+
\end{pmatrix} \quad S_{21} \to \begin{pmatrix}
\sigma_- & 0 \\
0 & -\sigma_-
\end{pmatrix}
\]

\[
S_{11} \to \begin{pmatrix}
\frac{I + \sigma_+}{2} & 0 \\
0 & \frac{I - \sigma_+}{2}
\end{pmatrix} \quad S_{22} \to \begin{pmatrix}
\frac{I - \sigma_+}{2} & 0 \\
0 & \frac{I + \sigma_+}{2}
\end{pmatrix}
\]

\[
Q_{12} \to \begin{pmatrix}
\sigma_+ & 0 \\
0 & \sigma_-
\end{pmatrix} \quad Q_{21} \to \begin{pmatrix}
\sigma_- & 0 \\
0 & \sigma_+
\end{pmatrix}
\]

\[
Q_{11} \to \begin{pmatrix}
\frac{I + \sigma_+}{2} & 0 \\
0 & \frac{I - \sigma_+}{2}
\end{pmatrix} \quad Q_{22} \to \begin{pmatrix}
\frac{I - \sigma_+}{2} & 0 \\
0 & \frac{I + \sigma_+}{2}
\end{pmatrix}
\]

where

\[
\tilde{Q}_{ii} \equiv Q_{ii} + \frac{\Omega}{2},
\]

the \( \sigma_x, \sigma_y, \) and \( \sigma_z \) are Pauli matrices in the standard representation,

\[
\sigma_\pm = \frac{1}{2}(\sigma_x \pm i\sigma_y),
\]

and \( I \) is a unit matrix. Likewise, it is easily verified that in terms of this representation,

\[
D^\dagger \to \begin{pmatrix}
0 & i\sigma_y \\
0 & 0
\end{pmatrix} \quad D \to \begin{pmatrix}
0 & 0 \\
0 & -i\sigma_y & 0
\end{pmatrix}
\]

\[
\pi_x^\dagger \to \begin{pmatrix}
0 & i\sigma_z \\
0 & 0
\end{pmatrix} \quad \pi_x \to \begin{pmatrix}
0 & 0 \\
0 & -i\sigma_z & 0
\end{pmatrix}
\]

\[
\pi_y^\dagger \to \begin{pmatrix}
0 & I \\
0 & 0
\end{pmatrix} \quad \pi_y \to \begin{pmatrix}
0 & 0 \\
I & 0
\end{pmatrix}
\]

\[
\pi_z^\dagger \to \begin{pmatrix}
0 & 0 \\
0 & \sigma_x
\end{pmatrix} \quad \pi_z \to \begin{pmatrix}
0 & 0 \\
0 & \sigma_x
\end{pmatrix}
\]

\[
\tilde{Q} \to \begin{pmatrix}
\tilde{S} & 0 \\
0 & \tilde{S}
\end{pmatrix} \quad \tilde{S} \to \begin{pmatrix}
\tilde{S} & 0 \\
0 & -\tilde{S}
\end{pmatrix}
\]

\[
M \to \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & -\frac{1}{2}
\end{pmatrix} \quad \tilde{Q}_+ \to \begin{pmatrix}
I & 0 \\
0 & I
\end{pmatrix}
\]

where \( \tilde{Q}_+ = Q_+ + \Omega \).

D. Collective Subspace

We take as a Hilbert space

\[
|\Psi\rangle = |n_x n_y n_z n_d\rangle = (\pi_x^\dagger)^{n_x}(\pi_y^\dagger)^{n_y}(\pi_z^\dagger)^{n_z}(D^\dagger)^{n_d}|0\rangle
\]
which is a collective subspace associated with \( SO(6) \) irreps of the form 
\[
(\sigma_1, \sigma_2, \sigma_3) = (\frac{1}{2}, 0, 0).
\]

In this notation we use the well-known isomorphism of \( SU(4) \) and \( SO(6) \) to label the irreducible representations with the standard \( SO(6) \) quantum numbers \((\sigma_1, \sigma_2, \sigma_3)\) \cite{17}. Physically, this irrep represents a “maximally stretched” state in the representation space that is in turn associated with maximal collectivity; as such, it is the obvious candidate for a collective subspace describing the lowest states of the system.

One sees immediately that the expectation value of \( Q_+ \) is zero for any state in this collective subspace: the matrix representation of \( \tilde{Q} \) is zero for any state in this collective subspace \cite{18}. The matrix representation of \( \tilde{Q} \) is a unit matrix, and thus \( Q_+ \) commutes with all the \( SU(4) \) generators, leading to
\[
\langle S | Q_+ | S \rangle = 0.
\]

In the symmetry limit, this implies that charge density wave excitation are excluded from the \( SU(4) \) model restricted to this subspace \cite{18}.

E. \( SU(4) \) Casimir Operator

The Casimir operator of \( SU(4) \sim SO(6) \)
\[
C_{su(4)} = \pi^\dagger \cdot \pi + D^\dagger D + \hat{S} \cdot \hat{S} + \hat{\mathcal{Q}} \cdot \hat{\mathcal{Q}} + M(M - 4)
\]
(10)
is an invariant and its expectation value in this collective subspace
\[
\langle S | C_{su(4)} | S \rangle = \frac{\Omega}{2}(\frac{5}{2} + 4)
\]
(11)
is a constant.

III. SYMMETRY-CONSTRAINED BOGOLIUBOV TRANSFORMATION

Utilizing the methods of Ref. \cite{11}, the coset space is
\[
SU(4)/SO(4) \times U(1),
\]
where the \( SO(4) \) subgroup is generated by \( \hat{\mathcal{Q}} \) and \( \hat{\mathcal{S}} \), and \( U(1) \) is generated by the charge operator \( M \). The coherent state may be written as
\[
| \psi \rangle = \mathcal{T} | 0^\ast \rangle.
\]
(12)
The operator \( \mathcal{T} \) is defined by
\[
\mathcal{T} = \exp(\eta_{00}p_{12}^\dagger + \eta_{01}q_{12}^\dagger - \text{h. c.}),
\]
(13)
where \( | 0^\ast \rangle \) is the physical vacuum (the ground state of the system), the real parameters \( \eta_{00} \) and \( \eta_{10} \) are symmetry-constrained variational parameters, and \( \text{h. c.} \) means the hermitian conjugate. Since the variational parameters weight the elementary excitation operators \( p_{12}^\dagger \) and \( q_{12}^\dagger \) in Eq. (13), they represent collective state parameters for a subspace truncated under the \( SU(4) \) symmetry. The most general coherent state corresponds to a 4-dimensional, complex, compact manifold parameterized by 8 real variables. The reduction of the coherent state parameters to only two in Eq. (13) follows from requiring time reversal symmetry and assuming the conservation of spin projection \( S_z \) for the wavefunction.

It is often simpler to view the coherent states as Hatree-Fock-Bogoliubov (HFB) variational states constrained by the dynamical symmetry. The symmetry-constrained HFB coherent state method is discussed in Refs. \cite{11, 19, 20, 21, 22}. It may be viewed as a type of mean-field approximation to the underlying many-body problem that is particularly useful in the present context because it leads to easily visualized energy surfaces. This identification provides a natural connection to spontaneously broken symmetries and effective Lagrangian field theories on the one hand, and to quasiparticle language on the other.

From the coset representative expressed in the 4-dimensional matrix representation \cite{8}, the transformation operator \( \mathcal{T} \) defined in Eq. (12) may be written as
\[
\mathcal{T} = \begin{bmatrix} Y_1 & X \\ -X^\dagger & Y_2 \end{bmatrix} \quad X \equiv \begin{bmatrix} 0 & \alpha + \beta \\ -(\alpha - \beta) & 0 \end{bmatrix}
\]
where \( Y_1 \) and \( Y_2 \) are determined by the requirement that \( \mathcal{T} \) be unitary, and \( \alpha \) and \( \beta \) are variational parameters related to \( \eta_{00} \) and \( \eta_{10} \) in Eq. (13) (see Ref. \cite{11}). Introducing
\[
v_+ \equiv \alpha + \beta, \quad v_- \equiv \alpha - \beta, \quad (14)
\]
the requirement of unitarity gives
\[
X = \begin{bmatrix} 0 & v_+ \\ -v_- & 0 \end{bmatrix} \quad Y_1 = \begin{bmatrix} u_+ & 0 \\ 0 & u_- \end{bmatrix} \quad Y_2 = \begin{bmatrix} u_- & 0 \\ 0 & u_+ \end{bmatrix}
\]
with the constraint that
\[
u_+^2 + v_-^2 = 1 \quad (15)
\]

A. Quasi-fermion Transformation

The existence of the 4-dimensional matrix representation of the \( SU(4) \) algebra implies the existence of a representation in which the single-particle basis can be written in the form
\[
\{ c_{\sigma \uparrow}, c_{\sigma \downarrow}^\dagger, c_{\bar{\sigma} \uparrow}, c_{\bar{\sigma} \downarrow} \},
\]
where \( \bar{\sigma} \) is a state conjugate to \( \sigma \). In this representation, any generator \( \hat{O} \) may be written as
\[
\begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \rightarrow \hat{O} = \sum_{\sigma, i, j} \left[ O_{ij}^{(11)} c_{\sigma i}^\dagger c_{\sigma j} + O_{ij}^{(22)} c_{\bar{\sigma} i} c_{\bar{\sigma} j} \right. \\
\left. + O_{ij}^{(12)} c_{\sigma i}^\dagger c_{\bar{\sigma} j} + O_{ij}^{(21)} c_{\bar{\sigma} i} c_{\sigma j} \right],
\]

and Eq. (12) is seen to be a Bogoliubov type transformation, but one that is constrained to preserve the $SU(4)$ symmetry.

Through the operator $T$, the physical vacuum state $|0^+\rangle$ (the ground state of the system) is transformed to a quasiparticle vacuum state $|\psi\rangle$, with the parameters $\alpha$ and $\beta$ (or $v_\pm$) determined by minimizing the energy of the system. Likewise, the basic fermion operators $c_{\sigma\lambda}$ are transformed to quasifermion operators $a_{\sigma\lambda}$ through

$$T \begin{pmatrix} c_{\sigma\lambda} \\ c_{\bar{\sigma}\lambda} \\ c_{\sigma\bar{\lambda}} \\ c_{\bar{\sigma}\bar{\lambda}} \end{pmatrix} = \begin{pmatrix} a_{\sigma\lambda} \\ a_{\bar{\sigma}\lambda} \\ a_{\sigma\bar{\lambda}} \\ a_{\bar{\sigma}\bar{\lambda}} \end{pmatrix}. \quad (16)$$

### B. One-Body and Two-Body Operators

Using the transformation (16), one can express any one-body operator in the quasiparticle space as

$$T^\dagger T^{-1} = \begin{bmatrix} O^{(11)} & O^{(12)} \\ O^{(21)} & O^{(22)} \end{bmatrix}$$

$$\rightarrow \hat{O} = \sum_{\sigma,i} O^{(22)}_{\sigma,i} + \sum_{\sigma,j} \left\{ O^{(11)}_{ij} a_{\sigma i}^\dagger a_{\sigma j} - O^{(22)}_{ji} a_{\bar{\sigma} i}^\dagger a_{\bar{\sigma} j} \right\}$$

$$+ O^{(12)}_{i,j} a_{\sigma i}^\dagger a_{\bar{\sigma} j} + O^{(21)}_{i,j} a_{\bar{\sigma} i}^\dagger a_{\sigma j} \right\}$$

where the $O^{(\mu\nu)}$’s are fixed by the transformation properties of the operator $\hat{O}$:

$$O^{(\mu\nu)} = \sum_{m,n} [T^{(mn)}O^{(mn)}(T^{-1})^{(nv)}]_{ij} \quad (17)$$

and $T^{(mn)}$ and $O^{(mn)}$ are two-dimensional submatrices of $T$ and $\hat{O}$, respectively.

Because the quasiparticle annihilation operator acting on the quasiparticle vacuum $|\psi\rangle$ is zero, the expectation values for one-body operators $\hat{O}$ are given by

$$\langle \hat{O} \rangle = \langle \psi | \hat{O} | \psi \rangle = \sum_{\sigma,i} O^{(22)}_{\sigma,i} = \sum_{\sigma} \text{Tr}(O^{(22)}) \quad (18)$$

and for two-body operators $\hat{O}_A \hat{O}_B$,

$$\langle \hat{O}_A \hat{O}_B \rangle = \langle \psi | \hat{O}_A \hat{O}_B | \psi \rangle$$

$$= \sum_{\sigma} \text{Tr}(O^{(22)}_A) \sum_{\sigma'} \text{Tr}(O^{(22)}_B)$$

$$+ \sum_{\sigma} \text{Tr}(O^{(21)}_A O^{(12)}_B) \quad (19)$$

### C. Expectation Value of Generators

Utilizing equations (1) and (17)–(19), and noting that the summation $\sum_{\sigma}$ provides a factor of $\Omega/2$ since the matrix elements of Eq. (17) do not depend on $\sigma$, one obtains the expectation values for all the generators and their scalar products in the coherent state representation:

$$\langle D^\dagger \rangle = \langle D \rangle = \frac{1}{2} \Omega (u_{+} v_{-} + u_{-} v_{+}), \quad (20)$$

$$\langle \pi^\dagger \rangle = \langle \pi \rangle = \frac{1}{2} \Omega (u_{-} v_{-} - u_{+} v_{+}), \quad (21)$$

$$Q \equiv \langle Q_z \rangle = \frac{1}{2} \Omega (v_{+}^2 - v_{-}^2), \quad (22)$$

$$\langle \hat{n} \rangle \equiv n = \Omega (v_{+}^2 + v_{-}^2), \quad (23)$$

$$\langle \pi_x \rangle = \langle \pi_y \rangle = \langle S \rangle = \langle Q_y \rangle = 0, \quad (24)$$

$$\langle D^\dagger D \rangle = \frac{1}{2} \Omega^2 (u_{+} v_{+} + u_{-} v_{-})^2 + \frac{\Omega}{2} (v_{+}^2 + v_{-}^2), \quad (25)$$

$$\langle \nabla^\dagger \cdot \nabla \rangle = \frac{1}{4} \Omega^2 (u_{+} v_{-} - u_{-} v_{+})^2 + \frac{\Omega}{2} (v_{+}^2 + v_{-}^2), \quad (26)$$

$$\langle \nabla^\dagger \cdot \nabla \rangle = \frac{1}{4} \Omega^2 (v_{+}^2 - v_{-}^2)^2 + \frac{\Omega}{2} (u_{+} v_{+}^2 + (u_{-} v_{-})^2 + (u_{-} v_{+}^2)^2 + (u_{+} v_{-})^2), \quad (27)$$

$$\langle \nabla \cdot \nabla \rangle = \frac{1}{2} \Omega [(u_{+} v_{+})^2 + (u_{-} v_{-})^2], \quad (28)$$

$$\langle M^2 \rangle = \frac{1}{2} (n - \Omega)^2 + \frac{\Omega}{2} [(u_{+} v_{+})^2 + (u_{-} v_{-})^2]. \quad (29)$$

Using the above results, one can also verify Eq. (14) explicitly for the expectation value of the Casimir operator $C_{su(4)}$.

### D. Order Parameters

By virtue of the unitarity condition (15), there are only two independent variational parameters in the above equations. They may be chosen as either $v_+$ and $v_-$, or as $\alpha$ and $\beta$, using (14). However, from Eq. (22) the squares of $v_\pm$ (or of $\alpha$ and $\beta$) are constrained by the equation of a circle since

$$n = \langle \hat{n} \rangle = \Omega (v_{+}^2 + v_{-}^2) = 2\Omega (\alpha^2 + \beta^2). \quad (30)$$

Thus, for a fixed particle number $n$ we may evaluate matrix elements with only a single variational parameter, say $\beta$, which may in turn be related to standard order parameters by comparing matrix elements. For example, the $z$ component of the staggered magnetization is related to $\beta$ and $v_\pm$ by

$$Q \equiv \langle Q_z \rangle = \frac{1}{2} \Omega (v_{+}^2 - v_{-}^2)$$

$$= 2\Omega \beta (n/(2\Omega) - \beta^2)^{1/2}. \quad (31)$$

These measures of antiferromagnetic order are in turn related to the superconducting order parameter $\alpha$ through Eq. (30). From Eqs. (30) and (31), the ranges of $\beta$ and $\alpha$ are

$$0 \leq \beta \leq \sqrt{n/4\Omega}, \quad \sqrt{n/4\Omega} \leq \alpha \leq \sqrt{n/2\Omega}.$$

Using Eqs. (20)–(31) one can then evaluate the energy surface as a function of $Q$ or $\beta$ or $\alpha$, and study the ground state properties of the $SU(4)$ model.
IV. COHERENT STATE ENERGY SURFACES

The most general Hamiltonian for the $SU(4)$ model\cite{4} is

$$H = \varepsilon \hat{n} - v \hat{n}^2 - G_0 D^\dagger D - G_1 \hat{\pi}^\dagger \cdot \hat{\pi} - \chi \hat{\vec{Q}} \cdot \hat{\vec{Q}} + g \hat{\vec{S}} \cdot \hat{\vec{S}},$$

where $\varepsilon, v, G_0, G_1, \chi$, and $g$ are parameters defining the strengths of single-particle and interaction terms. Since $C_{su(4)}$ is an $SU(4)$ invariant, if we assume for the ground state spin that $\langle \hat{S} \rangle = 0$ and that the number of particles $n$ is a good quantum number, Eqs. (11)–(13) imply that the Hamiltonian (32) may be parameterized without loss of generality as

$$H = H_0 - \tilde{G}_0 [ (1 - p) D^\dagger D + p \hat{\vec{Q}} \cdot \hat{\vec{Q}} ],$$

where $p$ lies in the interval 0 to 1 and

$$G^{(0)}_{\text{eff}} = (1 - p) \tilde{G}_0 = G_0 - G_1,$n_{\text{eff}} = p \tilde{G}_0 = \chi - G_1,$n_{0} = \varepsilon n - vn^2 - G_1 \frac{1 - x^2}{4}.$

$G^{(0)}_{\text{eff}}$ and $\chi_{\text{eff}}$ are the effective strengths of pairing and the $\hat{\vec{Q}} \cdot \hat{\vec{Q}}$ interactions, respectively.

From (10)–(11), one can show that

$$\nu^2 = \frac{n}{2\Omega} \pm \frac{Q}{\Omega},$$

Eqs. (20) and (21) can be written as

$$\Delta = \langle D^\dagger \rangle = \langle D \rangle = \sqrt{D^\dagger D} = \Delta_+ + \Delta_-,$n_{\Pi} = \langle \pi_{±}^\dagger \rangle = \langle \pi_{±} \rangle = \sqrt{\pi^\dagger \cdot \pi} = \Delta_+ - \Delta_-,$

where

$$\Delta_{±} = \frac{1}{2} \sqrt{\frac{1}{4} - \left(\frac{Q}{\Omega} + \frac{x}{2}\right)^2}$$

and $x$ is the effective hole concentration

$$x = 1 - \frac{n}{\Omega}.$$

(By “effective” we mean that $x$ is a ratio of the hole-pair number $(\Omega - n)/2$ to the pair degeneracy $\Omega/2$, rather than the ratio of hole number to the total number of lattice sites.) It can be estimated that to avoid hole-pair collapse, $\Omega$ is required to be roughly one-third of the total lattice sites, and in turn the true hole concentration is one-third of $x$\cite{24}.

The quantities $\Delta$ and $\Pi$ present the spin-singlet and spin-triplet pairing correlations. The former is proportional to the superconducting pairing gap; the latter can be regarded as a measure of the $SO(5)$ correlation since the $SO(5)$ Casimir operator is

$$C_{so5} = \hat{\pi}^\dagger \cdot \hat{\pi} + \hat{\vec{S}} \cdot \hat{\vec{S}} + M(M - 3).$$

By utilizing Eqs. (25)–(27), a general expression for the energy surface of the $SU(4)$ Hamiltonian as a function of the antiferromagnetic order parameter $Q$ may be obtained. In the $\Omega \to \infty$ limit, the energy surface is defined by

$$E(Q) = \langle H \rangle - H_0 = -\tilde{G}_0 (1 - p) \Delta^2 + PQ^2.$$

Converting $Q$ into the alternative order parameter $\beta$ allows us to express the energy surface as a function of $\beta$ and $n$,

$$E(\beta) = \langle H \rangle - H_0 = -\frac{\tilde{G}_0 \Omega^2}{4}$$

$$\times \left\{ (24p - 8) \beta^2 \left(\frac{n}{2\Omega} - \beta^2\right) + 2(1 - p) \left[ \frac{n}{2\Omega} \left(1 - \frac{n}{2\Omega}\right) \right] \right\}$$

$$+ \left(\frac{n}{2\Omega} - 2\beta^2\right) \sqrt{\left(1 - \frac{n}{2\Omega}\right)^2 - 4\beta^2 \left(\frac{n}{2\Omega} - \beta^2\right)}$$

which may also be expressed in terms of the superconducting order parameter $\alpha$ using (33).

V. COHERENT STATES AND SU(4) SUBGROUPS

Assuming $\tilde{G}_0 > 0$ (suggested by phenomenology), $p = 1/2$ in Eq. (13) corresponds to $SO(5)$ symmetry, while the extreme values 0 and 1 correspond to $SU(2)$ and $SO(4)$ symmetries, respectively (see Ref. [4]). Other values of $p$ respect $SU(4)$ symmetry but break the $SO(5), SO(4),$ and $SU(2)$ subgroups. In Fig. 1 we illustrate the ground-state energy $E(\beta)$ of Eq. (42) as a function of the order parameter $\beta$ for different electron occupation fractions $n/\Omega$ with $p = 0, 1/2, 1$.

A. SU(2) Limit

For $p = 0$ [$SU(2)$ limit; see Fig. 1a], the minimum energy occurs at $\beta = 0$ (equivalently, $Q = 0$) for all values of $n$. Thus, $\Delta$ reaches its maximum value of

$$\Delta_{\text{max}} = \frac{1}{2} \Omega \sqrt{1 - x^2},$$

indicating superconducting order.

B. SO(4) Limit

For $p = 1$ [$SO(4)$ limit; see Fig 1c], the opposite situation occurs: $\beta = 0$ is an unstable point and an infinitesimal fluctuation will drive the system to the energy
FIG. 1: Coherent state energy surfaces. The energy units are $G(0)^2/4$ for figures (a) and (b), and $\chi_{\text{eff}}\Omega^2/4$ for (c). $H_0$ is taken as the energy zero point. Numbers on curves are the effective lattice occupation fractions, with $n/\Omega = 1$ corresponding to half filling and $0 < n/\Omega < 1$ to finite hole doping. $SO(5)$ symmetry corresponds to $p = 1/2$ and the allowed range of $\beta$ is $[-\frac{1}{2}\sqrt{n/\Omega}, \frac{1}{2}\sqrt{n/\Omega}]$, which depends on $n$. The order parameter $\beta$ is related to the order parameter $Q \equiv \langle Q_z \rangle$ (staggered magnetization) and the electron number $n$ through Eq. (31).

Thus, $|Q|$ reaches its maximum value of $n/2$, indicating the presence of AF order.

D. $SO(5)$ As a Critical Dynamical Symmetry

Dynamical symmetries that, within the dynamical symmetry itself, exhibit a transition between qualitatively different energy surfaces as a parameter (usually related to particle number) is varied have been termed critical dynamical symmetries [19]. The $SO(5)$ dynamical symmetry, within the context of its $SU(4)$ parent symmetry, exhibits such transitional properties. At half filling the energy surface is completely flat under variations of the antiferromagnetic order parameter $\beta$ (see the $n = 1.0$ curve of Fig. 1b), implying large fluctuations in the order parameters. But as hole doping is increased the $SO(5)$ energy surface changes smoothly into one localized around $\beta = 0$ (see the $n = 0.1$ curve of Fig. 1b). Thus, $SO(5)$ is an example of a critical dynamical symmetry.

Such symmetries are well known in nuclear structure physics [11, 19, 20]. The $SO(5)$ critical dynamical symmetry discussed here in a condensed matter context has many formal similarities with the $SO(8) \supset SO(7)$ critical dynamical symmetry of the (nuclear) Fermion Dynamical Symmetry Model [6]. The condition for realization of the $SO(5)$ critical dynamical symmetry is that the strength of $Q \cdot Q$ equals that of $D^\dagger D$ in the Hamiltonian; This is similar to the $SO(7)$ nuclear critical dynamical symmetry, which is realized when there is an overall $SO(8)$ symmetry and the monopole pairing and the quadrupole interaction terms are of equal strength in the

\[ \beta = \pm \frac{1}{2}\sqrt{n/\Omega}. \]
nuclear Hamiltonian. In the $SO(7)$ case of nuclear physics, the order parameter analogous to $\beta$ presents nuclear deformation: nuclei around midshell (half filling of a shell by nucleons) are soft against shape fluctuations and transform into a spherical shape (the $SO(5) \times SU(2)$ dynamical symmetry limit of the $SO(8)$ symmetry) as the number of nucleons increases.

VI. $SO(5)$ SYMMETRY BREAKING

Under exact $SO(5)$ symmetry ($p = \frac{1}{2}$), the AF and SC states are degenerate at half filling. There is no barrier between AF and SC states, and one can fluctuate into the other at zero cost in energy (see the $n/\Omega = 1$ curve of Fig. 1b). This situation is inconsistent with Mott insulating behavior at half-filling. The Zhang $SO(5)$ model has been challenged because under exact symmetry it does not fully respect the phenomenological requirements of “Mottness”. As Zhang has recognized, for antiferromagnetic insulator properties to exist at half filling, it is necessary to break $SO(5)$ symmetry. Such breaking of the $SO(5)$ subgroup symmetry is implicit in the $SU(4)$ model, occurring naturally in the $SU(4)$ model if $p > 1/2$ in the Hamiltonian. Furthermore, the $SU(4)$ symmetry leads to the following constraint

$$\langle D^\dagger D + \vec{Q} \cdot \vec{Q} + \vec{\pi} \cdot \vec{\pi} \rangle = \frac{1 - x^2}{4} \Omega^2. \tag{45}$$

This ensures a doping dependence in the solutions, which is necessary for describing the transition from AF to SC in the cuprates.

Thus, the coherent state analysis indicates that the phenomenologically required $SO(5)$ symmetry breaking and the doping dependence in the solutions occur naturally in the $SU(4)$ model. They need not be introduced empirically as proposed in the original Zhang $SO(5)$ model. Recently, a projected $SO(5)$ model has been introduced. Its essence is a patch to the original $SO(5)$ model that implements the Gutzwiller projection in order to satisfy the large-$U$ Hubbard (non-double-occupancy or the Mott-insulator) constraint. In our $SU(4)$ model, there is no need to introduce such a projection artificially because the $SU(4)$ symmetry constraint already implies a constraint of non-double-occupancy with charge density localized on sites of the underlying lattice. We shall demonstrate this explicitly and give a detailed discussion of the consequences in a subsequent paper.

To see in more detail how in the $SU(4)$ model a broken $SO(5)$ symmetry can interpolate between AF and SC states as particle number varies, let us perturb slightly away from the $SO(5)$ limit of $p = 1/2$ in Eq. 45. In Fig. 2a, $SU(4)$ coherent state results for $p = 0.52$ are shown. We denote the value of $\beta$ minimizing $\langle H \rangle$ as $\beta_0$. The corresponding variation of the AF order parameter $Q = \langle Q_z \rangle$ with $n$, and its comparison with the variation in various symmetry limits are summarized in Fig. 2b. The variations of the AF, SC and $SO(5)$ correlations ($Q$, $\Delta$ and $\Pi$) with the hole doping $x$ are shown in Fig. 2c, while the variations of the contributions of each term in the Hamiltonian (the pairing $D^\dagger D$ and the AF interaction $\vec{Q} \cdot \vec{Q}$) to the total energy are shown in Fig. 2d. In Fig. 2, there is an important quantity, the critical doping $x_c$, which can be expressed analytically as

$$x_c = \sqrt{1 - \frac{G_{\text{eff}}^{(0)}}{\chi_{\text{eff}}}} = \sqrt{1 - \frac{1 - p}{p}}. \tag{46}$$

For $p = 0.52$, we have $x_c = 0.277$.

A. Antiferromagnetic Order

One sees from Fig. 2 that if $n$ is near $\Omega$ (half filling), $\beta_0 \simeq \pm 0.5$; this corresponds to AF order, since the staggered magnetization reaches its maximum, $Q = \Omega/2$, and there are no pairing or $SO(5)$ correlations ($\Delta = \Pi = 0$). With the onset of hole doping, $n/\Omega$ decreases ($x$ increases). The AF correlation $Q$ quickly diminishes and the pairing and $SO(5)$ correlations, $\Delta$ and $\Pi$, increase.

B. Underdoped $SO(5)$ Fluctuations

Before $Q$ vanishes at the critical doping $x_c = 0.277$ ($n/\Omega = 0.723$), the system has an energy surface almost flat for broad ranges of $\beta$, implying the presence of large-amplitude fluctuations in AF order (and equivalently in SC order). Meanwhile, the $SO(5)$ correlation $\Pi$ increases and reaches its maximum at the doping where the pairing and AF correlations become equal to each other (see Figs. 2c and 2d). This is the underdoped SC region. The coexistence of these three correlations competing with each other is consistent with the complexity and variety of experimental phenomena in this region.

C. Superconducting Order

For small values of $n$ ($x > x_c$), the stable point is $\beta_0 = 0$. This corresponds to SC order, since both the AF and $SO(5)$ correlations vanish ($Q = \Pi = 0$), and only the pairing correlation remains ($\Delta > 0$). The critical doping $x_c$ is the optimal doping point since $\Delta$ is maximum at $x_c$ and decreases as hole-doping increases. Thus the doping range $x > x_c$ may be considered to be the overdoped region. The critical doping $x_c$ depends on the ratio of the pairing and the $\vec{Q} \cdot \vec{Q}$ strengths (see Eq. 46). The larger the pairing strength $G_{\text{eff}}^{(0)}$ relative to the $\vec{Q} \cdot \vec{Q}$ strength $\chi_{\text{eff}}$, the smaller the critical doping value $x_c$. 
VII. HAMILTONIAN FOR HIGH $T_c$ SUPERCONDUCTIVITY

From the above discussion one can see that, with a perturbed $SO(5)$ symmetry, the system can undergo phase transitions from the AF order at half filling to the SC order at smaller filling as particle number varies. This picture is at least qualitatively consistent with the observations. The $SO(5)$ symmetry breaking in the Hamiltonian ($p$ larger than $1/2$) is crucial. Only when $SO(5)$ is broken does the energy surface interolate between AF and SC order as doping is varied (compare the surfaces for $p = 1/2$ and $p = 0.52$ in Figs. 1 and 2). We thus conclude that high temperature superconductivity may be described by a Hamiltonian that conserves $SU(4)$ but breaks (explicitly) $SO(5)$ symmetry in a direction favoring AF order over SC order.

The deviation from the $SO(5)$ symmetry need not be large. Experimentally, it is known that the optimal doping $P_c$ is around 0.16, suggesting that $x_c \simeq 0.48$ (note that $x_c \simeq 3P_c$). This leads to $p = 0.56$ according to Eq. [46], which is formally quite close to the $SO(5)$ symmetry limit.

However, it should be stressed that because of the critical nature of the $SO(5)$ symmetry, a slightly perturbed $SO(5)$ Hamiltonian may have solutions that are close to the other symmetry limits of the $SU(4)$ model for particular electron occupation ratios, even though the Hamiltonian itself is not formally in any of the dynamical symmetry limits. For example, one can see from Figs. 2c and 2d that the Hamiltonian near half filling actually behaves like an $SO(4)$ Hamiltonian since it effectively contains only the $\vec{Q} \cdot \vec{Q}$ correlation term ($\vec{Q} \cdot \vec{Q}$ is the primary component of the $SO(4)$ Casimir). Likewise the perturbed $SO(5)$ Hamiltonian approximates the $SU(2)$ Hamiltonian containing only the $D^ \dagger D$ term when $x > x_c$ (see Figs 2c and 2d) . Only in the intermediate doping range ($0 < x < x_c$), where both $D^ \dagger D$ and $\vec{Q} \cdot \vec{Q}$ correlations have significant contributions, does the $p = 0.52$ Hamiltonian behave like an approximate $SO(5)$ Hamiltonian, as it should formally. In particular, near the region where the $D^ \dagger D$ and $\vec{Q} \cdot \vec{Q}$ terms have equivalent contribu-
tions and the $SO(5)$ correlations (the operator $\Pi$) reach their maximum, the $p = 0.52$ solution lies very close to the $SO(5)$ symmetry limit, as one would expect.

The present analysis implies that the underdoped regime is naturally associated with the $SU(4) \supset SO(5)$ dynamical symmetry interpolating between antiferromagnetic and superconducting order. Likewise, optimally doped and overdoped superconductors are naturally associated with the $SU(4) \supset SU(2)$ dynamical symmetry and AF insulators near half filling are associated with the $SU(4) \supset SO(4)$ dynamical symmetry (or small perturbations around these symmetries).

As we shall discuss in a separate publication \[24\], the appearance of pseudogap behavior \[27\] can be described, and much of the quantitative phase diagram in cuprates can be reproduced rather well \[24\], if a fixed Hamiltonian with slightly broken $SO(5)$ symmetry but preserving $SU(4)$ overall symmetry is adopted. This again supports our interpretation of $SO(5)$ symmetry as an critical dynamical symmetry. The small $SO(5)$ symmetry breaking distorts the completely flat energy surface at half filling to stabilize AF character in the system; the critical nature of the $SO(5)$ dynamical symmetry (that the system interpolates between two phases) remains.

VIII. SUMMARY AND CONCLUSIONS

The present paper serves to introduce into the condensed matter literature the technology of generalized coherent states. As we have shown in the example discussed here, these methods provide a systematic way to relate a many-body theory to its approximate broken symmetry solutions. This approach may be viewed as a standardized technology for constructing energy surfaces for many-body theories defined in terms of the algebra of their second-quantized operators, or equivalently as the most general Bogoliubov transformation permitted, subject to a symmetry constraint on the Hamiltonian of a system.

To illustrate the power of these techniques, we have used generalized coherent states to understand the relationship between the $SU(4)$ model of superconductivity \[4\] and the Zhang $SO(5)$ model \[1\]. The use of $SU(4)$ coherent states to analyze the energy surface of its $SO(5)$ subgroup permits us to interpret the $SO(5)$ as a critical dynamical symmetry that interpolates between AF and $d$-wave SC order as doping is varied, and suggests similarities with analogous critical dynamical symmetries well known from nuclear structure physics. This permits the $SO(5)$ symmetry to be understood dynamically as a critical phase that, for a range of doping, has an energy surface extremely soft against AF fluctuations and therefore having much of the character of a spin glass (or possible stripe phases in the presence of a spatially modulated perturbation).

Thus, the coherent state analysis suggests that $SO(5)$ is the appropriate symmetry of the underdoped regime, but that the AF phase at half filling and the optimal and overdoped SC phases are described by two other $SU(4)$ subgroups: $SO(4)$ and $SU(2)$, respectively. The coherent state analysis also shows clearly that the requirement of small deviations from $SO(5)$ symmetry and the necessary doping dependence of the solutions that are inserted in the Zhang model occur naturally when $SO(5)$ is a subgroup of $SU(4)$.

In addition, we note that the results obtained here may have some broader implications. Although the present application is specifically to the high-temperature superconductor problem, we may anticipate that these mathematical techniques could find use for any application in condensed matter physics where it is important to understand the relationship between an exact many-body theory and the order parameter(s) characterizing its approximate broken symmetry solutions. Clearly there are many such possibilities.

Finally, the concept of a critical dynamical symmetry that we have introduced here in a condensed matter context is one that has already found important application in other areas of physics. This implies that there may be deep algebraic analogies between various condensed matter systems and superficially different systems appearing in other fields of many-body physics. We have suggested one such analogy here between the physics of high-temperature superconductors and the physics of collective states in heavy atomic nuclei.

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[1] S.-C. Zhang, Science 275, 1089 (1997).
[2] C. L. Henley, Phys. Rev. Lett. 80, 3590 (1998).
[3] S. Rabello, H. Kohno, E. Demler, and S.-C. Zhang, Phys. Rev. Lett. 80, 3586 (1998).
[4] M. W. Guidry, L.-A. Wu, Y. Sun, and C.-L. Wu, Phys. Rev. B63, 134516 (2001).
[5] The $SU(4)$ model employs 16 generators closed under commutation that define the $U(4)$ Lie algebra and a corresponding $U(4)$ group. In addition to the three components of staggered magnetization and two for $d$-wave pairing, there are three spin operators, one charge operator, six spin-1 operators analogous to the Zhang $\pi$ operators, and one operator that may be interpreted as generating charge density waves. The $U(4)$ group has the subgroup $U(4) \supset U(1) \times SU(4)$, where the $U(1)$ factor may be identified with charge density waves. Thus, because of the direct product structure, one can without loss of generality view the theory as an $SU(4)$ theory describing superconductivity and antiferromagnetism, and global charge and spin conservation, with the $U(1)$ charge-density wave sector treated independently (but the overall group structure implies that the lowest states of the symmetry-limit
theory have no charge-density wave excitations).

[6] C.-L. Wu, D.H. Feng and M. W. Guidry, Adv. in Nucl. Phys. 21, 227 (1994).
[7] F. Iachello and A. Arima, The Interacting Boson Model (Cambridge University Press, Cambridge, 1987).
[8] F. Iachello and R.D. Levine, Algebraic Theory of Molecules (Oxford University Press, Oxford, 1995).
[9] F. Iachello and P. Truini, Ann. Phys. (N.Y.) 276, 120 (1999).
[10] R. Bijker, F. Iachello, and A. Leviatan, Ann. Phys. (N.Y.) 236, 69 (1994).
[11] W.-M. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. 62, 867 (1990).
[12] R. Gilmore, Ann. Phys. 74, 391 (1972).
[13] R. Gilmore, Rev. Mex. de Fisica, 23, 142 (1974).
[14] A. M. Perelomov, Commun. Math. Phys. 26, 222 (1972).
[15] J. R. Klauder, J. Math. Phys. 4, 1055, 1058 (1963).
[16] R. J. Glauber, Phys. Rev. Lett. 10, 277 (1963).
[17] The representation structure and relationship of SU(4) and SO(6) is discussed, for example, in J. N. Ginocchio, Ann. Phys. 126, 234 (1980).
[18] Charge density waves are forbidden in the unbroken-pair subspace with exact SU(4) symmetry. A relaxation of the exact symmetry or breaking pairs can lead to charge density wave excitations in a more realistic theory. However, the clear tendency of the SU(4) symmetry is to suppress charge density waves in the lowest-lying states.

[19] W.-M. Zhang, D. H. Feng, and J. N. Ginocchio, Phys. Rev. Lett. 59, 2032 (1987).
[20] W.-M. Zhang, D. H. Feng, and J. N. Ginocchio, Phys. Rev. C37, 1281 (1988).
[21] W.-M. Zhang, C.-L. Wu, D. H. Feng, J. N. Ginocchio, and M. W. Guidry Phys. Rev. C38, 1475 (1988).
[22] W.-M. Zhang, D. H. Feng, C.-L. Wu, H. Wu, and J. N. Ginocchio, Nucl. Phys. A505, 7 (1989).
[23] P. Ring and P. Schuck, The Nuclear Many-Body Problem, (Springer-Verlag, New York, 1980).
[24] C.-L. Wu et al. (unpublished).
[25] S.-C. Zhang, J. P. Hu, E. Arrigoni, W. Hanke, and A. Auerbach, Phys. Rev. B60, 13070 (1999).
[26] Y. Sun, M. Guidry, L.-A. Wu, and C.-L. Wu, cond-mat/9912072 (unpublished).
[27] T. Timusk and B. Statt, Rep. Prog. Phys. 62, 61 (1999).