Phonon Bose-Einstein condensation in a Hubbard-phonon interacting system with infrared divergence

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2013-08-26

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Abstract

We show that a finite Hubbard-phonon interacting system exhibits phonon BEC at sufficiently low temperature. We also have the gauge symmetry breaking for phonons. The key tools are a unitary transformation introduced by Arai and Hirokawa [5] and the Araki-Woods representation. This system is essentially the same as a free system or the van Hove model.

1 Introduction

An electron-phonon interacting system is an important model for quantum statistical mechanics and quantum field theory, and is fundamental for condensed matter physics. Nevertheless there are few mathematically rigorous researches. We have had several fundamental development in quantum statistical mechanics and quantum field theory, for example, the Pauli-Fierz model for quantum electrodynamics and the Nelson model for a scalar neutron and bosons [1, 3, 12, 13, 14, 16, 19, 24, 25, 26, 21, 22, 31]. The analysis for the Nelson model includes some type of an electron-phonon system, but we have few results on many body electron systems. We have many body electron results for the Pauli-Fierz model [21]: but it is a stability of matter problem and the existence of ground states. This is, of course, a very interesting and important result for condensed matter physics. However I would like to analyze the many body electron problems such as phase transition, especially ferromagnetism. This is proved in my previous paper [35]. In that paper we proved the existence of ferromagnetic phase transition for a finite Hubbard-phonon interacting system at absolute zero temperature.

We continue analyzing an electron-phonon interacting system in this paper, in particular, a finite Hubbard-phonon interacting system at finite temperature. We cannot expect (ferromagnetic) phase transition for a finite Hubbard system at finite temperature, so our main concern is the analysis for bosons. We can expect the occurrence of Bose-Einstein condensation (BEC) for a boson system, and we will actually prove it in this paper.

Araki-Woods’ result [11] is also fundamental for mathematical analysis of BEC in this paper. Our analysis reduces the interacting system to the free one, so we can use the results summarized in [4].

Our result includes the existence of equilibrium states and we have had several fundamental results [15, 17, 18, 27, 28] in this direction. In particular the results by Dereziński, Jakšić and Pillet include our existence result because they
treat the generalized model. However our result is different in some points and stronger. Firstly ours and their physical motivation are totally different: their study focuses general phenomena, return to equilibrium for a small system coupling to a heat bath, but our study focuses the specific condensed matter phenomenon, BEC. In particular sometimes they explicitly assume the non-existence of BEC [28]. Secondly we prove the existence under weaker conditions for infrared regularization, which is due to specification of the model.

Moreover our BEC analysis is a new feature: the analysis for phonons. We have much detailed mathematical analysis for boson only systems [30], but few results for fermion-boson interacting system, as far as the author knows. We can also prove the spontaneously gauge symmetry breaking.

For phonons we prove more: infrared divergence and BEC are not related. Of course infrared divergence for phonons is due to the interaction with electrons and BEC is the phenomenon of a bosonic system. However, as far as the author knows, we have no rigorous results in this direction. Note that we may have some relation between these if we consider infinite Hubbard-phonon system: See the remark 4.8.

Our next studies will be in several directions:

1. the analysis for an infinite Hubbard-phonon system (using an operator-algebraic method),
2. the one for this system using functional integration,
3. construction of “Sobolev representations”.

The first is natural direction to study. The infinite system is expected to exhibit phase transition, ferromagnetism or superconductivity, and infrared divergence by spin wave excitation by electrons, so our system will have two phase transitions and two infrared divergence. The second is interesting in view of mathematics. Our system is relatively easy to understand because of the special form of interaction. Hence it will be easy to compare the results by other methods. The final one is also mathematical and needs explanation.

We review the theory of distribution: quantum mechanics introduces distributions, such as Dirac’s delta function. Take an approximation sequence, \( \psi_t(x) := e^{-x^2/2t}/\sqrt{2\pi t} \). The functions \( \psi_t, t > 0 \) are all in \( L^2(\mathbb{R}) \), but the limit \( \lim_{t\downarrow 0} \psi_t = \delta \) is not in \( L^2(\mathbb{R}) \). Dirac’s delta is in the larger space, \( S'(\mathbb{R}) \). We face this type of phenomena in quantum field theory, that is, infrared divergence. Take ground states \( \Psi_\kappa, \kappa > 0 \) in the original Hilbert space (Fock space), where \( \kappa \) is infrared cutoff. The vectors \( \Psi_\kappa \) has no limit in the Fock space if infrared divergence occurs for the Nelson model. However we expect to have a limit in some larger space. For example, set states \( \psi_\kappa(\cdot) := \langle \Psi_\kappa, \cdot \rangle \) on the \( C^* \)-algebra \( \mathbb{B}(\mathcal{H}) \) and take a limit in the state space. So we can seem the set of states on a \( C^* \)-algebra “the space of distribution for quantum field theory”. But this space is too large so we want smaller and more useful spaces like Sobolev spaces in quantum mechanics. They will be representations rather than spaces in the theory of operator algebras. Araki-Woods’ representation is one of the “Sobolev representations”, I think, but we need more such representations. In particular, for absolute zero temperature situation, Araki-Woods’ representation has no meaning; hence we must search Araki-Woods’ representation for absolute zero temperature.

### 2 Mathematical settings

In this section we summarize the mathematical settings and notations. Our Hilbert spaces are as follows.

\[
\mathcal{F}_L := \mathcal{H}_{c} \bigotimes_{\text{as}} \mathcal{F}_{b,L}, \quad \mathcal{H}_{c} := \bigotimes_{\text{as}} L^2(\Lambda; \mathbb{C}^2), \quad \mathcal{F}_{b,L} := \mathcal{F}_{b}(\mathcal{H}_{b,L}), \quad \mathcal{H}_{b,L} := L^2(C_L; \mathbb{C}^{N_{L}}), \quad C_{L} := \left[ -\frac{L}{2}, \frac{L}{2} \right]^d,
\]

where \( N_{c} \) and \( N_{l} \) are positive integers, \( L \) is a positive real number, and \( \Lambda \) is a finite set. The symbol \( \bigotimes \) is an ordinary tensor product and \( \bigotimes_{\text{as}} \) is a symmetric/anti-symmetric tensor product. A Hilbert space \( \mathcal{F}_{b}(\mathcal{K}) \), called a bosonic Fock space, for a Hilbert space \( \mathcal{K} \) is defined by

\[
\mathcal{F}_{b}(\mathcal{K}) := \bigoplus_{n=0}^{\infty} \bigotimes_{s} \mathcal{K}.
\]

The following Hilbert spaces are for thermodynamic limit spaces:

\[
\mathcal{F} := \mathcal{F}_{\infty} = \mathcal{H}_{c} \bigotimes \mathcal{F}_{b}, \quad \mathcal{F}_{b} := \mathcal{F}_{b}(\mathcal{H}_{b}), \quad \mathcal{H}_{b} := L^2(\mathbb{R}^d; \mathbb{C}^{N_{L}}).
\]
Next we define the Hamiltonians. We formally define operators for thermodynamic limit spaces, such as $\mathcal{F}$. For subspaces $\mathcal{F}_L$, we define them by restricting domains properly, where we use the property $L^2(C_L) \subset L^2(\mathbb{R}^d)$.

\begin{align}
H^\kappa_{\mu_b} &:= H_{\text{fr},\mu_b} + \alpha H^\kappa_1, \\
H_{\text{fr},\mu_b} &:= H_0 \otimes 1 + 1 \otimes H_{b,\mu_b}, \\
H^\kappa_1 &:= \sum_{x \in \Lambda} n_x \otimes \phi (\lambda^\kappa_x),
\end{align}

where $\kappa \geq 0$ is an infrared cutoff parameter, $\alpha \in \mathbb{R}$ is a coupling constant, and $\mu_b \in \mathbb{R}$ is a chemical potential for bosons. The self-adjoint operators $\phi(\lambda^\kappa_x)$ are Segal's field operators and functions $\lambda^\kappa_x \in \mathcal{H}_b$ are infrared cutoff functions. The Hamiltonians for electrons and phonons are defined by

\begin{align}
H_e &:= d \Gamma_e (T) + U \sum_{x \in \Lambda} n_{x,+} n_{x,-}, \\
H_{b,\mu_b} &:= d \Gamma_b (\omega - \mu_b),
\end{align}

where $T \in \mathcal{B}(\ell^2(\Lambda))$ is self-adjoint, $U > 0$ is a Coulomb repulsion, and $n_{x,\sigma} := c^*_x \sigma c_{x,\sigma}$. The operators $\{c_{x,\sigma}\}$ satisfy the canonical anti-commutation relation (CAR). The multiplication operator defined by the function $\omega: \mathbb{R} \to \mathbb{R}_{\geq 0}$ is nonnegative and self-adjoint.

We impose several conditions for $\lambda^\kappa_x$ and $\omega$. The functions $\lambda^\kappa_x$ have a role for infrared conditions, and the conditions for the operator $\omega$ are for an infrared condition and thermodynamic limit conditions. Here we fix an inverse temperature $\beta > 0$.

**Assumption. 2.1.** For cutoff functions $\lambda^\kappa_x$ we impose the conditions

\begin{align}
\lambda^\kappa_x &\in \text{dom} \omega^{-1/2} (\kappa \geq 0), \\
\lambda^\kappa_x &\in \text{dom} \omega^{-1} (\kappa > 0), \\
\lambda^\kappa_x &\to \lambda^0_x =: \lambda_x.
\end{align}

Set the function $r: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ $r(|k|) := \omega(k)$. For the operator $\omega$ we impose the conditions

\begin{align}
r \in C^1_{+,\infty} &:= C^1_{+,\infty} ([0, \infty)) := \{ f \in C^1 ([0, \infty)) : f'(k) > 0, \quad f(k) \to \infty (k \to \infty) \}, \\
\sup_{k \geq 0} (1 + k)^{d_0} e^{-\beta \omega(k)} &< \infty, \\
\int_{|k| \leq 1} \frac{1}{\omega(k) - \omega(0)} dk &< \infty,
\end{align}

where we assume the existence of the real constant $d_0 > d$. Furthermore set $\omega_0 := \omega(0)$ and $\omega_0 - \mu_b > 0$.

Without further comment we always assume the above in the following.

## 3 Unitary transform of the Hamiltonians

Firstly we define the unitary operators. We can define these on both $\mathcal{F}_L$ and $\mathcal{F}$. The operators

\begin{equation}
S^\kappa := \sum_{x \in \Lambda} n_x \otimes \phi (i \omega^{-1} \lambda^\kappa_x), \quad \kappa > 0
\end{equation}

are self-adjoint thanks to Nelson’s analytic vector theorem. From this we define the demanded unitaries as follows:

\begin{equation}
V^\kappa := e^{i \alpha S^\kappa}.
\end{equation}

**Theorem. 3.1.** We have the following operator equality

\begin{equation}
V^\kappa 1 \otimes H_{b,\mu_b} (V^\kappa)^{-1} = 1 \otimes H_{b,\mu_b} + \alpha H^\kappa_1 + \frac{\alpha^2}{2} R^\kappa_{-1/2} \otimes 1,
\end{equation}

where

\begin{equation}
R^\kappa_m := \sum_{x,y \in \Lambda} (\omega^m \lambda^\kappa_x, \omega^m \lambda^\kappa_y) n_x n_y, \quad m \in \mathbb{R}.
\end{equation}
Proof. We prove this in [35]. For the reader’s sake we show the outline of the proof. Define the subspace

\[ D := \mathcal{H}_c \otimes \mathcal{F}_{b, \text{fin}} (E(H_{b, \mu_b})) , \]

where \( \otimes \) is an algebraic tensor product,

\[ \mathcal{F}_{b, \text{fin}} (E(H_{b, \mu_b})) := \bigoplus_{n=0}^{\infty} \bigotimes_{s} H_{b, \mu_b} , \]

\( \bigoplus \) is an algebraic direct sum, and \( E(H_{b, \mu_b}) \) is the set of entire analytic vectors for \( H_{b, \mu_b} \). Since the LHS is essentially self-adjoint and since \( D \) is a dense set of analytic vectors for the LHS, the following formal calculation is permitted:

\[ V^\kappa 1 \otimes H_{b, \mu_b} (V^\kappa)^{-1} = \sum_{n=0}^{\infty} \frac{(i\alpha)^n}{n!} \delta_\kappa^n (1 \otimes H_{b, \mu_b}) , \quad \delta_A(B) := [A, B] . \]

More calculation shows our theorem. \( \square \)

Corollary. 3.2. \( H_{\mu_b}^\kappa \) and \( \hat{H}_{fr, \mu_b}^\kappa \) are unitarily equivalent for \( \kappa > 0 \), where

\[ \hat{H}_{fr, \mu_b}^\kappa := \hat{H}_c^\kappa \otimes 1 + 1 \otimes H_{b, \mu_b} , \quad \hat{H}_c^\kappa := H_c - \alpha^2 R_{-1/2}^\kappa . \]

From here we work on \( \mathcal{F}_L, \quad L < \infty \) and assume \( \kappa > 0 \). Hence we can use the trace functional for defining the equilibrium state. We denote \( \text{Tr} \) is the trace on \( \mathcal{F}_L \), \( \text{Tr}_c \) is on \( \mathcal{H}_c \), and \( \text{Tr}_b \) is on \( \mathcal{F}_{b, L} \). First we define von-Neumann algebras.

\[ \mathcal{M} := \mathcal{M}_c \otimes \mathcal{M}_b , \quad (3.9) \]

\[ \mathcal{M}_c := \mathbb{B} (\mathcal{H}_c) , \quad (3.10) \]

\[ \mathcal{M}_b := \text{vN} \left\{ W(f) : f \in \text{dom} \omega^{-1/2} \right\} , \quad W(f) := e^{i\phi(f)} , \quad (3.11) \]

where \( \text{vN} \mathcal{A} \) is the von-Neumann algebra generated by \( \mathcal{A} \subset \mathbb{B}(\mathcal{K}) \) for a Hilbert space \( \mathcal{K} \). Next we define various equilibrium states as follows:

\[ \psi_{\beta, \mu_b}^\kappa (A_c) := \frac{1}{Z_{\beta, L, \mu_b}^\kappa} \text{Tr} \left[ A_c e^{-\beta H_{\mu_b}^\kappa} \right] , \quad A_c \in \mathcal{M}_c , \quad Z_{\beta, L, \mu_b}^\kappa := \text{Tr} \left[ e^{-\beta H_{\mu_b}^\kappa} \right] , \quad (3.12) \]

\[ \tilde{\psi}_{\beta, \mu_b}^\kappa := \tilde{\psi}_{\beta, \mu_b}^\kappa \otimes \psi_{fr, \beta, \mu_b}, \quad (3.13) \]

where

\[ \tilde{\psi}_{\beta, \mu_b}^\kappa (A_c) := \frac{1}{Z_{\beta, \mu_b}^\kappa} \text{Tr}_c \left[ A_c e^{-\beta \hat{R}_c^\kappa} \right] , \quad A_c \in \mathcal{M}_c , \quad Z_{\beta, \mu_b}^\kappa := \text{Tr}_c \left[ e^{-\beta \hat{R}_c^\kappa} \right] , \quad (3.14) \]

\[ \psi_{fr, \beta, \mu_b} (A_b) := \frac{1}{Z_{fr, \mu_b, \beta}^\kappa} \text{Tr}_b \left[ A_b e^{-\beta H_{\mu_b}^\kappa} \right] , \quad A_b \in \mathcal{M}_b , \quad Z_{fr, \mu_b, \beta}^\kappa := \text{Tr}_b \left[ e^{-\beta H_{\mu_b}^\kappa} \right] . \]

We prove the following important result as in [35]. We can derive BEC from this because we can use the arguments for free boson’s BEC.

Theorem. 3.3. For \( A_c \in \mathcal{M}_c \) and \( f \in \text{dom} \omega^{-1/2} \), we have

\[ \psi_{\beta, \mu_b}^\kappa (A_c \otimes W(f)) = \tilde{\psi}_{\beta, \mu_b}^\kappa (e^{i\tilde{n}_c^\kappa (f)} A_c) \psi_{fr, \beta, \mu_b} (W(f)) , \quad (3.16) \]

where

\[ \tilde{n}_c^\kappa (f) := \sum_{x \in A} \text{Re} (\omega^{-1/2} f, \omega^{-1/2} \lambda_c^\kappa) n_x . \]

Proof. Recall the operator \( S^c = \sum_{x \in A} n_x \otimes \phi (i\omega^{-1/2} \lambda_c^\kappa) \) and the unitary transform using it. Then we obtain

\[ \text{Tr} \left[ A_c \otimes W(f) e^{-\beta H_{\mu_b}^\kappa} \right] = \text{Tr} \left[ e^{-i\alpha S^c} A_c \otimes W(f) e^{i\alpha S^c} e^{-\beta H_{\mu_b}^\kappa} e^{i\alpha S^c} \right] = \text{Tr}_c \left[ e^{i\alpha \tilde{n}_c^\kappa (f)} A_c \right] \text{Tr}_b \left[ W(f) \right] . \]

\( \square \)
4 Existence of BEC

In this section we take the thermodynamic limit and assume infrared cutoff, i.e., \( \kappa > 0 \). Here \( \kappa \) is fixed. See the chapter 9 in [4] for the detailed analysis. We define a lattice \( \Gamma_L \) in a wave number space for finite regions in a real space:

\[
\Gamma_L := \frac{2\pi}{L} \mathbb{Z}.
\]  
(4.1)

Firstly we show the exact expression for the boson number expectation.

**Theorem.** 4.1. Assume \( \omega(0) > \mu_b \), the operator \( e^{-\beta \omega} \) is in the trace class, and the spectrum of the operator \( \omega \) is \( \sigma(\omega) = \{ \omega(k) : k \in \Gamma_L^d \} \). Then we obtain the following exact expression,

\[
N_{b,L}^\kappa := \psi_{b,\nu,0,\rho_b} \left( \hat{N}_b \right) = N_i \left[ \sum_{k \in \Gamma_L^d} \frac{1}{e^{\beta \omega(k) - \mu_b}} + \alpha^2 \hat{\psi}_{e,\beta}^\kappa (R_{-1}) \right],
\]  
(4.2)

where \( \hat{N}_b \) is the boson number operator.

**Proof.** Use the same arguments as in Theorem 3.3.

**Remark.** 4.2. We must impose infrared cutoff for massless bosons in this lattice approximation otherwise the second term diverges even if we consider the number density.

We define some symbols before the detailed analysis.

\[
\omega_0 := \omega(0) > \mu_b,
\]
(4.3)

\[
y := e^{\beta \omega_0 - \mu_b},
\]
(4.4)

\[
F(k) := \omega(k) - \omega_0 \geq 0,
\]
(4.5)

\[
N_{b,L}^\kappa (\beta, y) := N_{b,L}^\kappa = N_i \left[ \sum_{k \in \Gamma_L^d} \frac{1}{e^{\beta \omega(k) - \mu_b}} + \alpha^2 \hat{\psi}_{e,\beta}^\kappa (R_{-1}) \right],
\]
(4.6)

\[
f_L^\kappa (y) := \rho_{b,L}^\kappa (\beta, y) := \frac{1}{L^d} N_{b,L}^\kappa, \quad y > 1.
\]
(4.7)

Furthermore we set the following decompositions:

\[
N_{b,L}^\kappa := N_{b,L}^\kappa (0) + N_{b,L,1}^\kappa (\beta, \mu_b),
\]
(4.8)

\[
N_{b,L}^\kappa (0) := \frac{N_i}{y - 1} + N_{b,L,1}^\kappa (\beta, \mu_b),
\]
(4.9)

\[
N_{b,L,1}^\kappa (\beta, y) := \frac{1}{L^d} \sum_{k \in \Gamma_L^d \setminus \{0\}} \frac{1}{y e^{\beta F(k)} - 1}.
\]
(4.10)

Using this decomposition, we set

\[
\rho_{b,L}^\kappa (\beta, y) = \rho_{b,L,0}^\kappa (y) + \rho_{b,L,1}^\kappa (\beta, y),
\]
(4.12)

\[
\rho_{b,L,0}^\kappa := \frac{1}{L^d} N_{b,L}^\kappa (0),
\]
(4.13)

\[
\rho_{b,L,1}^\kappa (\beta, y) := \frac{1}{L^d} N_{b,L,1}^\kappa (\beta, y),
\]
(4.14)

\[
\rho_{b,L,1}^\kappa (\beta, y) := \frac{1}{L^d} N_{b,L,1}^\kappa (\beta, y).
\]
(4.15)

Then we have the following

**Lemma.** 4.3. ([4] Lemma. 9.22) Set a real number \( \bar{\rho}_b^\kappa \) as \( \bar{\rho}_b^\kappa > \rho_{b,L,1}^\kappa (\beta) \). Then, for each fixed \( L \), there exists a unique number \( y_L^\kappa \geq 1 \) such that \( \bar{\rho}_b^\kappa = \rho_{b,L}^\kappa (\beta, y_L^\kappa) \). Furthermore there exists a monotonically increasing sequence \( \{L_n\} \) such that

\[
y_L^\kappa \to y_\infty^\kappa \geq 1.
\]
(4.16)
Proof. Clearly we have
\[
\lim_{y \to 1} f^L_k(y) = \infty, \quad \lim_{y \to \infty} f^L_k(y) = \rho^L_{b,L,i,r,a}(\beta),
\] (4.17)
and \( f^L_k \) is monotone decreasing. Since the function \( f^L_k \) is continuous and bijection from \((1, \infty)\) to \( \left( \rho^L_{b,L,i,r,a}(\beta), \infty \right) \), there exists a unique number \( y^L_k > 1 \) such that \( f^L_k(y^L_k) = \rho^L_b \).

Next we prove the existence of the sequence \( \{L_n\} \). Note that, for any fixed \( L \),
\[
\tilde{\rho}_b^L = \frac{N_i}{L^d} \left( \frac{1}{y^L_k - 1} + \sum_{k \in \Gamma^L} e^{-\beta F(k)} \right) + \rho^L_{b,L,i,r,a}(\beta),
\] (4.18)
\[
\leq \frac{N_i}{L^d (y^L_k - 1)} \left( 1 + \sum_{k \in \Gamma^L} e^{-\beta F(k)} \right) + \rho^L_{b,L,i,r,a}(\beta).
\] (4.19)
Then we have
\[
y^L_k - 1 \leq \frac{N_i}{\tilde{\rho}_b^L - \rho^L_{b,L,i,r,a}(\beta)} \frac{1}{L^d} \left( 1 + \sum_{k \in \Gamma^L} e^{-\beta F(k)} \right).
\] (4.20)
Hence the sequence \( \{y^L_k\} \) is bounded and have a convergent subsequence for the limit \( L \to \infty \). □

Lemma 4.4. ([4] Lemma 9.24) Take numbers \( L > L_0 > 0 \) and \( a > 0 \). Then, for any number \( y_1, y_2 \geq 1 + a \), we have
\[
|f^L_k(y_1) - f^L_k(y_2)| \leq C |y_1 - y_2|,
\] (4.21)
where the constant \( C > 0 \) is dependent on only \( L_0 \) and \( a \).

Proof. By definition we have
\[
|f^L_k(y_1) - f^L_k(y_2)| = |y_2 - y_1| \frac{N_i}{\rho^L_{b,L,i,r,a}(\beta)} \sum_{k \in \Gamma^L} \frac{e^{\beta F(k)}}{(y_1 e^{\beta F(k)} - 1)(y_2 e^{\beta F(k)} - 1)}.
\] (4.22)
Furthermore the following estimate holds:
\[
\frac{e^{\beta F(k)}}{(y_1 e^{\beta F(k)} - 1)(y_2 e^{\beta F(k)} - 1)} = e^{-\beta F(k)} \frac{e^{\beta F(k)}}{y_1 e^{\beta F(k)} - 1} \frac{e^{\beta F(k)}}{y_2 e^{\beta F(k)} - 1}
\] (4.23)
\[
\leq e^{\beta F(k)} \frac{1}{y_1 - 1} \frac{1}{y_2 - 1}
\] (4.24)
\[
\leq \frac{1}{a^2} e^{-\beta F(k)}.
\] (4.25)
From this estimate we obtain
\[
\lim_{L \to \infty} \frac{1}{L^d} \sum_{k \in \Gamma^L} \frac{e^{\beta F(k)}}{(y_1 e^{\beta F(k)} - 1)(y_2 e^{\beta F(k)} - 1)} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{e^{\beta F(k)}}{(y_1 e^{\beta F(k)} - 1)(y_2 e^{\beta F(k)} - 1)} dk < \infty.
\] (4.26)
Thus the desired result is proved. □

For later use we define the following 3 symbols:
\[
\rho_{b,L,i}(\beta, y) := r_{b,L}(\beta, y) + R_{b,L}(\beta, y),
\] (4.27)
where we set
\[
r_{b,L}(\beta, y) := \frac{N_i}{L^d} \sum_{k_j \neq 0, j=1,...,d} \frac{1}{y e^{\beta F(k)} - 1},
\] (4.28)
\[
R_{b,L}(\beta, y) := \frac{N_i}{L^d} \sum_{j=1}^d \sum_{k_j \neq 0, k_j \in \Gamma^L} \frac{1}{y e^{\beta F(k)} - 1}.
\] (4.29)
Here we set \( R_{b,L}(\beta, y) = 0 \) if \( d = 1 \). See [4] for details for the following 3 facts.

\(^1\)In [4], this part of the proof is wrong.
Fact. 4.5. ([4] Lemma. 9.25) For any \( y > 1 \) we have
\[
0 < \rho_{b,L,1}(\beta, y) \leq \frac{N_i}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{ye^{\beta F(k)} - 1} + R_{b,L}(\beta, y). \tag{4.30}
\]
Furthermore let \( \varepsilon \) be a positive number. There exists a positive number \( L_0(\beta, y, \varepsilon) \) such that if \( L \geq L_0(\beta, y, \varepsilon) \) then
\[
0 < R_{b,L}(\beta, y) \leq \frac{dN_i}{(2\pi)^d L} \left( \sum_{j=1}^{d} \int_{\mathbb{R}^{d-1}} \frac{1}{ye^{\beta \omega(\|k\|) - \omega(\varepsilon)}} - 1 \right) \quad (d \geq 2). \tag{4.31}
\]

Fact. 4.6. ([4] Lemma. 9.26) The integral
\[
\rho_{b,c,fr}(\beta) := \frac{N_i}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{e^{\beta F(k)} - 1} \, dk \tag{4.32}
\]
is a finite positive number.

We define a number:
\[
\rho_{b,fr}(\beta, y) := \frac{N_i}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{1}{ye^{\beta F(k)} - 1} \, dk, \quad \beta > 0, \quad y > 1. \tag{4.33}
\]
Then we get the

Fact. 4.7. ([4] Lemma. 9.27) For any fixed \( \beta > 0 \) the function \( \rho_{b,fr}(\beta, y) \) is strictly monotone decreasing, continuous, and
\[
\lim_{y \downarrow 1} \rho_{b,fr}(\beta, y) = \rho_{b,c,fr}(\beta), \quad \lim_{y \to \infty} \rho_{b,fr}(\beta, y) = 0. \tag{4.34}
\]
In particular we have
\[
\rho_{b,fr}(\beta, y) < \rho_{b,c,fr}(\beta), \quad y > 1. \tag{4.35}
\]

Remark. 4.8. Since \( \lim_{L \to \infty} \rho_{b,L,ir,\alpha}^* (\beta) = 0 \) for a fixed \( \kappa \) we can define the above \( \rho_{b,c}^* \) without the term from \( \rho_{b,L,ir,\alpha}^* (\beta) \) here. However, if we take the limits \( \kappa \) and \( L \) simultaneously, the number \( \lim_{L \to \infty} \rho_{b,L,ir,\alpha}^* (\beta) \) may not vanish. Moreover, for an infinite Hubbard system, the limit \( \lim_{L \to \infty} \rho_{b,L,ir,\alpha}^* (\beta) \) may not be zero, either. Hence we add the suffix \( \kappa \) for \( \rho_{b,c}^* \).

We may have to think the order or the way of taking the thermodynamic limit and the infrared cutoff. As far as the author knows we have no studies on this situation.

Fact. 4.9. ([4] Lemma. 9.28) Assume the relation
\[
\bar{\rho}_b^* > \rho_{b,c}^*. \tag{4.37}
\]
Then we obtain \( y_{\infty} = 1 \), where \( y_{\infty} \) is defined in Lemma 4.3.

Fact. 4.10. ([4] Theorem 9.29) Assume \( \bar{\rho}_b^* > \rho_{b,c}^* (\beta) \). Then we obtain the relation
\[
\lim_{n \to \infty} \frac{N_{b,0}(y_{L_n}^+)}{L_n^d} = \bar{\rho}_b^* - \rho_{b,c,fr}(\beta). \tag{4.38}
\]

We define the critical inverse temperature \( \beta_c \) as
\[
\rho_{b,c}^*(\beta_c) = \bar{\rho}_b^*. \tag{4.39}
\]
and define the critical temperature \( T_c \) as
\[
T_c = \frac{1}{\beta_c}. \tag{4.40}
\]
Define the temperature as \( T := 1/\beta \). Then the condition (4.37) is equivalent to the condition
\[
0 < T < T_c. \tag{4.41}
\]
Furthermore Fact. 4.10 shows that the mean boson number behaves asymptotically like
\[
N_0(y_{L_n}^+) \sim (\bar{\rho}_b^* - \rho_{b,c}^*(\beta)) L^d \quad (L \to \infty). \tag{4.42}
\]
Hence this phenomenon is Bose-Einstein condensation.

Next we summarize the behavior for \( T \geq T_c \).
Fact. 4.11. ([4] Lemma 9.32) Suppose $0 < \rho_{b,c}^\kappa \leq \rho_{b,c}^\kappa(\beta)$.

1. There exists a real number $b$ such that $\rho_{b}^\kappa = \rho_{b,fr}(\beta, b)$. If $\rho_{b}^\kappa = \rho_{b,fr}(\beta)$, then $b = 1$.
2. We have $y_{\infty} = b$, where $y_{\infty}$ is defined in Lemma 4.3.

Fact. 4.12. ([4] Lemma 9.33) Suppose $0 < \rho_{b}^\kappa \leq \rho_{b,c}^\kappa(\beta)$. Then we obtain

$$\lim_{n \to \infty} \frac{1}{L_n} N_0(y_L^f) = 0.$$ (4.43)

This means that Bose-Einstein condensation does not occur in sufficiently high temperature, of course.

5 Analysis of the BEC state

In this section we consider the infinite system in detail. We omit the super/subscript $\kappa$ in variables because this plays no role in this section. Furthermore, we remove the infrared cutoff after thermodynamic limit, i.e., take $\kappa$ limit after $L$ limit here.

5.1 Definitions and notations

We use the famous Araki-Woods representation[11]. First we set several notations. Our new Hilbert space is $\mathcal{F}_{b}^{AW}$, defined by

$$\mathcal{F}_{b}^{AW} := \mathcal{F}_b \big( \mathcal{H}_b \bigotimes \mathcal{H}_b \big).$$ (5.1)

The conjugation $C$ on $\mathcal{H}_b$ is defined by

$$C f := (\bar{f}_1, \ldots, \bar{f}_N), \quad f \in \mathcal{H}_b,$$ (5.2)

where $\bar{f}_j$ means a complex conjugation of $f_j$. The following vacuum vector $\Omega_{b}^{AW}$ is important because this forms our equilibrium state:

$$\Omega_{b}^{AW} := \Omega_b \otimes \Omega_b.$$ (5.3)

Next we define operators:

$$L_{fr} := \tilde{H}_c^e \otimes 1 + 1 \otimes L_{b,fr},$$ (5.4)
$$L_{b,fr} := \tilde{H}_{b,\mu_b} \otimes 1 - 1 \otimes \tilde{H}_{b,\mu_b},$$ (5.5)
$$W_{p,1}(f) := W_{p(\beta, \mu_b)}(f) := e^{i \phi_{p,1}(f)},$$ (5.6)
$$\phi_{p,1}(f) := \phi((1 + \rho)^{1/2} f) \otimes 1 + 1 \otimes \phi(C \rho^{1/2} f), \quad f \in \text{dom} \left( 1 - e^{-\beta(\omega - \mu_b)} \right)^{-1} \cap \text{dom} \omega^{-1/2},$$ (5.7)
$$\rho := \rho(\beta, \mu_b) := e^{-\beta(\omega - \mu_b)}(1 - e^{-(\omega - m\mu)})^{-1} = \left( e^{\beta(\omega - \mu_b)} - 1 \right)^{-1},$$ (5.8)

where the operator $L_{fr}$ is the Liouvillian (Hamiltonian) for our full dynamics, $L_{b,fr}$ is the Liouvillian for (free) phonons, $W_{p,1}(f)$ is the Weyl operator for the left Araki-Woods algebra, $\phi_{p,1}$ is the Segal’s field operator for left Araki-Woods algebra, and $\rho$ is the phonon density operator. The operator $\tilde{A}$ for a closable operator $A$ is the closure of $A$, here.

We define three states:

$$\psi_{c,\beta}^e(A_c) := \frac{\text{Tr}_e \left[ A_c e^{-\beta \tilde{H}_c^e} \right]}{\text{Tr}_e \left[ e^{-\beta \tilde{H}_c^e} \right]} := \langle \Psi_{c,\beta}^e, A_c \otimes 1 \Psi_{c,\beta}^e \rangle, \quad \Psi_{c,\beta}^e \in \mathcal{H}_e \otimes \overline{\mathcal{H}_e},$$ (5.9)
$$\psi_{b,fr,\beta,\mu_b}(A_b) := \langle \Omega_{b}^{AW}, A_b \Omega_{b}^{AW} \rangle,$$ (5.10)
$$\psi_{\beta,\mu_b}(A_c \otimes A_b) := \psi_{c,\beta}^e(A_c) \psi_{b,fr,\beta,\mu_b}(A_b), \quad A_c \in \mathcal{M}_c, \quad A_b \in \mathcal{M}_b.$$ (5.11)

The state $\psi_{c,\beta}^e$ is the equilibrium one for Hubbard electrons, the vector $\Psi_{c,\beta}^e$ is the vector state for $\psi_{c,\beta}^e$, the state $\psi_{b,fr,\beta,\mu_b}$ is the equilibrium state for free phonons in the Araki-Woods representation, and $\psi_{\beta,\mu_b}$ is the equilibrium state for the coupling system.
We define several functionals and the set for the next theorem:

\[
C_{e,\omega,\beta}^{-1/2} := \left\{ f \in C_c(\mathbb{R}^d; \mathbb{C}^N) : f \in \text{dom} \left( 1 - e^{-\beta(\omega - \mu_b)} \right)^{-1} \cap \text{dom} \omega^{-1/2} \right\}
\]  
(5.12)

\[
I_L(f) := \psi_{b,fr,\beta,\mu_b}(W(f)) = \langle f, \left( 1 + e^{-\beta(\omega - \mu_b)} \right)^{-1} f \rangle_{L^2(C_L; \mathbb{C}^N)}
\]
(5.13)

\[
I_L^{(1)}(f) = I_L^{(2)}(f),
\]
(5.14)

\[
I_L^{(1)}(f) := \left( \frac{2\pi}{L} \right)^d |\tilde{f}(0)|^2 \frac{y_L + 1}{y_L - 1},
\]
(5.15)

\[
I_L^{(2)}(f) := \left( \frac{2\pi}{L} \right)^d \sum_{k \in \mathbb{Z}^d \setminus \{0\}} |\tilde{f}(k)|^2 \frac{y_L \epsilon^{F(k)} + 1}{y_L \epsilon^{F(k)} - 1},
\]
(5.16)

where a function \( f \) in the above \( I_L \)'s is in \( C_{e,\omega,\beta}^{-1/2} \).

### 5.2 Facts

We use the following facts. See [4] for proofs.

**Fact 5.1.** *(Lemma. 10.8 [4])* Let \( f, g \in C_{e,\omega,\beta}^{-1/2} \).

1. Let \( \rho_b > \rho_{b,c}(\beta) \) and \( q_0(f) := \frac{2(2\pi)^d}{N_1} |\tilde{f}(0)|^2 \rho_{b,0}(\beta) \). Then we have

\[
\lim_{L \to \infty} I_L^{(1)}(f) = q_0(f), \quad \lim_{L \to \infty} I_L^{(2)}(f) = \int_{\mathbb{R}^d} |\tilde{f}(0)|^2 \frac{1 + e^{-BF(k)}}{1 - e^{-BF(k)}} dk.
\]
(5.17)

2. Let \( \rho_b \leq \rho_{b,c}(\beta) \). Then we have

\[
\lim_{L \to \infty} I_L^{(1)}(f) = 0, \quad \lim_{L \to \infty} I_L^{(2)}(f) = \int_{\mathbb{R}^d} |\tilde{f}(0)|^2 \frac{\psi + e^{-BF(k)}}{\psi - e^{-BF(k)}} dk.
\]
(5.18)

We set the following functionals.

\[
q_0(f) := \frac{2(2\pi)^d}{N_1} |\tilde{f}(0)|^2 \rho_{b,0}(\beta),
\]
(5.19)

\[
q_1(f) := \| (1 + e^{-\beta(\omega)}) (1 - e^{-\beta(\omega)}) f \|_2,
\]
(5.20)

\[
\psi_{b,BEC,\beta,fr}(W(f)) := e^{-\frac{1}{4}(q_0(f) + q_1(f))}, \quad f \in D_{b,1},
\]
(5.21)

\[
q_2(f) := \langle f, (y_\infty + e^{-B\omega}) (y_\infty - e^{-B\omega})^{-1} f \rangle_{L^2(\mathbb{R}^d; \mathbb{C}^N)}, \quad f \in L^2(\mathbb{R}^d; \mathbb{C}^N),
\]
(5.22)

\[
\psi_{b,\beta,fr,2}(W(f)) := e^{-\frac{1}{4}q_2(f)}, \quad f \in L^2(\mathbb{R}^d; \mathbb{C}^N)
\]
(5.23)

where

\[
D_{b,1} := L^1(\mathbb{R}^d; \mathbb{C}^N) \cap \left( 1 - e^{-\beta(\omega)} \right)^{-1/2} \cap \text{dom} \omega^{-1/2}
\]
(5.24)

**Fact 5.2.** *(4) Theorem 10.9)*

1. Let \( \rho_b > \rho_{b,c}(\beta) \) and \( f \in C_{e,\omega,\beta}^{-1/2} \). Then we have

\[
\lim_{n \to \infty} \tilde{\psi}_{\beta,L_n,\mu_b}(A \otimes W^B(f)) = \tilde{\psi}_{e,\beta,\mu_b}(A \otimes 1 e^{i\alpha_\xi}(f)) \psi_{b,BEC,\beta,fr}(W(f)).
\]
(5.25)

2. Let \( 0 < \rho_b \leq \rho_{b,c}(\beta) \), \( f \in C_{e,\omega,\beta}^{-1/2} \). Then we have

\[
\lim_{n \to \infty} \tilde{\psi}_{\beta,L_n,\mu_b}(A \otimes W^B(f)) = \tilde{\psi}_{e,\beta,\mu_b}(A \otimes 1 e^{i\alpha_\xi}(f)) \psi_{b,BEC,\beta,2}(W(f)).
\]
(5.26)

3. Let \( \rho_b = \rho_{b,c}(\beta) \). Then we have

\[
\lim_{n \to \infty} \tilde{\psi}_{\beta,L_n,\mu_b}(A \otimes W^B(f)) = \tilde{\psi}_{e,\beta,\mu_b}(A \otimes 1 e^{i\alpha_\xi}(f)) \psi_{b,BEC,\beta,fr}(W(f)).
\]
(5.27)
Proposition. 5.3. ([4] Prop.10.10) For any \( t \in \mathbb{R} \), we get
\[
\psi_{\beta} \left( e^{itR_{\alpha}} A_{e} \otimes 1 \right) W^{B}(f) e^{-itR_{\alpha}} = \psi_{\beta}(A_{e} \otimes 1) W^{B}(f).
\] (5.28)

Proof.
\[
\psi_{\beta} \left( e^{itR_{\alpha}} A_{e} \otimes 1 \right) W^{B}(f) e^{-itR_{\alpha}} = \psi_{e,\beta} \left( e^{itR_{e}} A_{e} e^{i\alpha_{e}(f)} e^{-itR_{e}} \right) \psi_{\beta,\text{BEC},\beta,\text{fr}} \left( e^{itR_{\beta,\text{fr}}} W(f) e^{-itR_{\beta,\text{fr}}} \right).
\] (5.29)

Note that it holds \( e^{itR_{\beta,\text{fr}}} W(f) e^{-itR_{\beta,\text{fr}}} = W(e^{it\omega} f) \).

\[\square\]

5.3 Direct integral decomposition of BEC state and gauge symmetry breaking

Throughout this section we the following

Assumption. 5.1.
\[
\rho_{b,0}(\beta) > 0.
\] (5.30)

We denote \( \Omega_{b,\text{fr}} \in \mathcal{F}_{b}(\mathbb{C}) = L^{2}(\mathbb{R}) \) which is the Fock vacuum for \( \mathcal{F}_{b}(\mathbb{C}) \). The operator \( \phi(z), z \in \mathbb{C} \) is Segal’s field operator for \( \mathcal{F}_{b}(\mathbb{C}) \). For simplicity we set the constant \( c(\hat{\rho}_{b}, \beta), \)
\[
c(\hat{\rho}_{b}, \beta) := \frac{2(2\pi)^{d} \rho_{b,0}(\beta)}{N_{1}}.
\] (5.31)

Then we get the following relation:
\[
q_{0}(f) = c(\hat{\rho}_{b}, \beta)|\hat{f}(0)|^{2}, \quad f \in \mathcal{D}_{b,1},
\] (5.32)
\[
\langle \Omega_{b,\text{fr}}, e^{i\phi(z)} \Omega_{b,\text{fr}} \rangle = e^{-\frac{1}{2}|z|^{2}},
\] (5.33)
\[
e^{-\frac{1}{2}q_{0}(f)} = \langle \Omega_{b,\text{fr}}, e^{i\phi(z)^{T}} \Omega_{b,\text{fr}} \rangle, \quad z_{f}^{\theta} := c(\hat{\rho}_{b}, \beta)e^{i\theta} \hat{f}(0), \theta \in \mathbb{R}.
\] (5.34)

In the following we use the following relation:

Lemma. 5.4. ([4] Lemma. 10.13) For \( a, b > 0 \) we have
\[
\int_{0}^{\infty} e^{-ar} J_{0}(\sqrt{br}) dr = \frac{1}{a} e^{-b/4a},
\] (5.35)
where the function \( J_{0} \) is the 0-th Bessel function.

Lemma. 5.5. ([4] Lemma. 10.15) For \( p, q > 0 \), we have
\[
\frac{1}{2\pi} \int_{0}^{2\pi} e^{i(p\cos \theta + q \sin \theta)} d\theta = J_{0}(\sqrt{p^{2} + q^{2}}).
\] (5.36)

Lemma. 5.6. For any function \( f \in L^{1}(\mathbb{R}^{d}, \mathbb{C}^{N_{1}}) \), we have
\[
e^{-\frac{1}{2}q_{0}(f)} = \int_{0}^{\infty} dr \int_{0}^{2\pi} d\theta e^{-r} \exp \left[ i \sqrt{c(\rho_{b,0}, \beta)r} \left( e^{i\theta} \hat{f}(0) + e^{-i\theta} \hat{f}(0) \right) \right]
\] (5.37)

We define the following algebra.
\[
\mathcal{M}_{b,0} := \ast \text{-alg} \{ W(f) : f \in \mathcal{D}_{b,1} \}.
\] (5.38)

For \( (r, \theta) \in [0, \infty) \times [0, 2\pi] \) we define the state
\[
\psi_{\beta,\text{fr},\theta}(W(f)) := \exp \left[ i \sqrt{c(\rho_{b,0}, \beta)r} \Re \left( e^{i\theta} \hat{f}(0) \right) \right] e^{-\frac{1}{2}q_{0}(f)}.
\] (5.39)

We define the probability measure \( \chi \) on \([0, \infty) \times [0, 2\pi]\) as follows:
\[
\chi(B \times C) := \frac{1}{2\pi} \int_{B} e^{-r} dr \int_{C} d\theta, \quad B \in \mathcal{B}([0, \infty)), \quad C \in \mathcal{B}([0, 2\pi]),
\] (5.40)
where \( \mathcal{B}(A) \) is the set of Borel sets on \( A \).
Lemma. 6.1. Suppose then the function \( \Omega_{b}^{f} \) is defined, we have
\[
\Omega_{b}^{f}(W(t)W(s)) = e^{i\alpha(t_w + s_w)}\psi_{b,fr,\beta}(W(t)W(s)),
\]
where we set \( a = \sqrt{\rho c(\rho, \beta)}r, w_f := \Re \left[ e^{i\theta}f(0) \right], f \in D_{b,1}, \) and
\[
\psi_{b,fr,\beta}(W(f)) := e^{-\frac{1}{2}q(f)}, \quad f \in D_{b,1}.
\]

Thus two-point functions take the form, for \( f, g \in D_{b,1}, \)
\[
\langle f, g \rangle = \langle f, e^{-\beta}f(1 - e^{-\beta}f) \rangle = \int_{\mathbb{R}^{2d}} \frac{c(\rho, \beta)r}{2(2\pi)^d}f(x)g(y)dxdy.
\]

Hence we can define
\[
\rho_{b}^{\alpha,\beta}(x) := \frac{N_{c}(\rho, \beta)r}{2(2\pi)^d} + \frac{N_{\rho}}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{1}{e^{\beta\omega(k)} - 1}dk = \frac{N_{c}(\rho, \beta)r}{2(2\pi)^d} + \rho_{b,c,fr}(\beta).
\]

6 GNS representation of the BEC state

We define the new Hilbert space.
\[
\mathcal{F}_{b,\text{BEC}} := L^{2}([0, \infty) \times [0, 2\pi], d\chi; \mathcal{F}_{b}^{AW})
\]

For \( f \in D_{b,1}, \) set
\[
e_{f}(r, \theta) := \exp \left[ i\sqrt{c(\rho, \beta)}r \Re(e^{i\theta}f(0)) \right], \quad r \geq 0, \quad \theta \in [0, 2\pi].
\]

We define operators \( W_{\rho,1,\text{BEC}}(f) \) on \( \mathcal{F}_{b,\text{BEC}} \) as follows:
\[
(W_{\rho,1,\text{BEC}}\Psi)(r, \theta) := e_{f}(r, \theta)(W_{\rho,1}(f)\Psi)(r, \theta).
\]

Setting
\[
\Omega_{b,\text{BEC}} : [0, \infty) \times [0, 2\pi] \rightarrow \mathcal{F}_{b}^{\text{AW}}; \quad \Omega_{b,\text{BEC}}(r, \theta) := \Omega_{b}^{\text{AW}},
\]
then the function \( \Omega_{b,\text{BEC}} \) is in \( \mathcal{F}_{b,\text{BEC}}, ||\Omega_{b,\text{BEC}}|| = 1, \) and
\[
\psi_{b,\text{BEC}}(W(f)) = \langle \Omega_{b,\text{BEC}}, W_{\rho,1,\text{BEC}}(f)\Omega_{b,\text{BEC}} \rangle.
\]

Lemma. 6.1. Suppose \( f \in L^{1}([0, \infty) \times [0, 2\pi], d\chi), \) the numbers \( k_{1} \) and \( k_{2} \) are in \( \mathbb{R}. \) Then the relation
\[
\int_{[0,\infty) \times [0, 2\pi]} f(r, \theta)e^{i\sqrt{c(\rho, \beta)}r \Re(e^{i\theta}f(0))}d\chi(r, \theta) = 0
\]
leads \( f = 0. \)

Proof. For \( x = \sqrt{r} \cos \theta \) and \( y = \sqrt{r} \sin \theta, \) we obtain
\[
\int_{\mathbb{R}^{2}} e^{-(x^{2} + y^{2})}f(\sqrt{x^{2} + y^{2}}, \arg(x + iy))e^{i(k_{1}x + k_{2}y)}dxdy = 0.
\]
Hence \( f = 0. \) \hfill \Box

Lemma. 6.2.
\[
\text{span} \{ W_{\rho,1,\text{BEC}}(f)\Omega_{b,\text{BEC}} : f \in D_{b,1} \} = \mathcal{F}_{b,\text{BEC}}.
\]
Proof. Assume $\Psi \in \mathcal{F}_{b,\text{BEC}}$, $f \in \mathcal{D}_{b,1}$, and $\langle \Psi, W_{\rho,1,\text{BEC}}\Omega_{b,\text{BEC}} \rangle = 0$. Then we have

$$0 = \int_{[0,\infty) \times [0,2\pi]} (\Psi(r,\theta), W_{\rho,1}(f)\Omega^\text{AW}_b) e^{i\sqrt{c(\rho_0,\beta)}\rho z(e^{i\rho} f(0))/2} d\chi(r,\theta).$$

(6.9)

For $z \in \mathbb{C}$, set

$$\mathcal{D}_{b,1,z} := \left\{ f \in \mathcal{D}_{b,1} : \hat{f}(0) = z \right\}. \quad (6.10)$$

This is dense in $L^2(\mathbb{R}^d; \mathbb{C}^{N_1})$. From here we get

$$\overline{\text{span}} \left\{ W_{\rho,1}(f)\Omega^\text{AW}_b : f \in \mathcal{D}_{b,1,z} \right\} = \mathcal{F}_{b}^\text{AW}. \quad (6.11)$$

Thus it follows that, for any $\Phi \in \mathcal{F}_{b}^\text{AW}$, $z \in \mathbb{C}$,

$$\int_{[0,\infty) \times [0,2\pi]} d\chi(r,\theta) (\Psi(r,\theta), \Phi) e^{i\sqrt{c(\rho_0,\beta)}\rho z(e^{i\rho} z)/2} = 0. \quad (6.12)$$

Putting $\sqrt{c(\rho_0,\beta)} z = k_1 - ik_2$, for $k_1, k_2 \in \mathbb{R}$, we get

$$\int_{[0,\infty) \times [0,2\pi]} d\chi(r,\theta) (\Psi(r,\theta), \Phi) e^{i\sqrt{2}(k_1 \cos \theta + k_2 \sin \theta)} = 0. \quad (6.13)$$

By the previous lemma, it follows that $\langle \Psi(r,\theta), \Phi \rangle = 0$ a.e. for all $\Phi \in \mathcal{F}_{b}^\text{AW}$. \hfill \Box

**Theorem. 6.3.** The algebra $\ast$-alg $\{W_{\rho,1,\text{BEC}}(f) : f \in \mathcal{D}_{b,1}\}$ is a GNS representation of $\mathcal{M}_{b,0}(\mathcal{D}_{b,1})$ for $\psi_{b,\text{BEC}}$ and its cyclic vector is $\Omega_{b,\text{BEC}}$.

For $(r, \theta) \in [0, \infty) \times [0, 2\pi]$ and $f \in \mathcal{D}_{b,1}$ we define operators

$$W^{(r,\theta)}_{\beta,1}(f) := e_f(r, \theta) W_{\beta,1}(f). \quad (6.14)$$

The following representation $\pi^r,\theta_b$ is a GNS representation of $\mathcal{M}_{b,0}(\mathcal{D}_{b,1})$ for the state $\psi^{r,\theta}_{b,\text{fr,}\beta}$, and its cyclic vector is $\Omega^\text{AW}_b$:

$$\pi^r,\theta_b(W(f)) := W^{(r,\theta)}_{\beta,1}(f). \quad (6.15)$$

**Proposition. 6.4.** Assume $(r, \theta), (r', \theta') \in [0, \infty) \times [0, 2\pi]$, and $(r, \theta) \neq (r', \theta')$. Then $\pi^r,\theta_b$ and $\pi^{r',\theta'}_b$ are mutually unitary-in equivalent.

Suppose there exists a unitary $U$ such that

$$U\Omega^\text{AW}_b = \Omega^\text{AW}_b, \quad U\pi^r,\theta_b(W(f)) U^{-1} = \pi^{r',\theta'}_b(W(f)), \quad f \in \mathcal{D}_{b,1}. \quad (6.16)$$

Take the vacuum expectation, and then we have $e_f(r, \theta) = e_f(r', \theta')$.

Next set $\sqrt{c(\rho_0, \beta)} f(0) = k - ip, k, p \in \mathbb{R}$. Then it follows that

$$\sqrt{r}(k \cos \theta + p \sin \theta) = \sqrt{r'}(k \cos \theta' + p \sin \theta') + 2\pi n, \quad n \in \mathbb{Z}. \quad (6.17)$$

Since $k, p$ is real, the followings must hold:

$$n = 0, \quad \sqrt{r} \cos \theta = \sqrt{r'} \cos \theta', \quad \sqrt{r} \sin \theta = \sqrt{r'} \sin \theta'. \quad (6.18)$$

As $\theta$ and $\theta'$ are in $[0, 2\pi)$, we get $r = r'$ and $\theta = \theta'$.

**Remark. 6.5.** Using the constant fiber direct integral we get the following decompositions:

$$\mathcal{F}_{b,\text{BEC}} \equiv \int_{[0,\infty) \times [0,2\pi]} \mathcal{F}^\text{AW}_{b} d\chi(r,\theta), \quad (6.19)$$

$$\Omega_{b,\text{BEC}}(f) = \int_{[0,\infty) \times [0,2\pi]} W^{r,\theta}_{\rho,1}(f) d\chi(r,\theta). \quad (6.20)$$

From this we summarize this section as follows: The GNS representation of $\ast$-algebra $\mathcal{M}_{b,0}(\mathcal{D}_{b,1})$ on the BEC state is given by the cyclic representations of mutually disjoint representations.
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