Possible ground states of D-wave condensates in isotropic space

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A complete and rigorous determination of the possible ground states for D-wave pairing Bose condensates is presented. Using an orbit space approach to the problem, we find 15 allowed phases (besides the unbroken one), with different symmetries, that we thoroughly determine, specifying the group-subgroup relations between bordering phases.

1. INTRODUCTION

Superfluidity and superconductivity are justified on the basis of the macroscopic condensation of Bose quasi-particles. The classical Bardeen, Cooper and Schrieffer theory for superconductivity dates 1957. Soon after, a BCS-type transition was proposed for the Fermi system $^{3}$He by Anderson and Morel\textsuperscript{1}. Cooper pair formation was thought to occur in an $L \neq 0$ state, to take into account the hard core nature of $^{3}$He atoms interaction. The superfluid phases were actually observed\textsuperscript{2}, and the nature of p-wave pairing is now well established for $^{3}$He. The theory of $L \neq 0$-superfluids is also relevant for “unconventional” superconductivity. The high temperature superconducting (HTS) oxides are anomalous in their non-Fermi liquid normal state properties and share with heavy fermion superconductors unconventional d-wave pairing\textsuperscript{3}. The underlying microscopic mechanism inducing superconductivity in these materials is still unclear and is one of today’s major challenges\textsuperscript{4,5}.

Such a situation has motivated the efforts at studying the macroscopic properties of unconventional superconductors through the Landau theory of phase transitions\textsuperscript{6}. Moreover, the identification of the order parameter symmetry may be considered as a preliminary task in the construction of viable models describing the attractive nature of the pairing interaction.

Direct experimental evidence of an order parameter unconventional structure lies on multiple phase transitions. The heavy fermion compounds $U_{1-x}$Th$_x$Be$_{13}$ display four different superconducting phases in the $T$-$x$ phase diagram and UPt$_3$ displays three superconducting phases in a $T$-$H$ phase diagram\textsuperscript{7}. Furthermore, from power-law temperature behaviour of thermodynamic and transport properties (e.g. specific heat, magnetic penetration depth), a non-trivial node structure for the gap function may be inferred, compatible with high $L$-pairing. That is the case also for HTS oxides, for which, at present, clear proofs are lacking of the existence of more than one superconducting state.

Actually, it must be pointed out that, unlike superfluid $^{3}$He, which is an isotropic fermion system, Bloch electrons in a superconductor crystal lattice exhibit, in general, a reduced finite symmetry; in addition, the existence of imperfections, even in the cleanest samples, can partly destroy the gap node structure at low energy. So, the very fact that the experimental observation cannot at present completely unravel the node structure of the gap function, reinforces the necessity of classifying all the possible symmetry breaking schemes.

Moreover, the role of Fermi isotropic space beyond a zero-th order approximation in phenomenological theories, is still supported by some recent studies: In HTS oxides, low symmetrical crystal fields (tetragonal and orthorombic), as well as spontaneous strain, have weak influence on the temperature of the superconducting phase transition\textsuperscript{3} or on the penetration depth\textsuperscript{4}. This also means that low symmetry crystal fields do not directly influence condensate states.

The possible ground states of a high $L$ superfluid has been the object of intense investigations during the 60’s and the 70’s. From the solution of the state equations, Anderson and Morel\textsuperscript{1} and Mermin\textsuperscript{10} identified five different phases, through the minimization of a 4-th degree Landau potential. Schakel and Bias\textsuperscript{11} analyzed the problem using only group theoretical arguments. Capel and Schakel\textsuperscript{12}, taking advantage of the results in\textsuperscript{11}, computed the possible ground states of condensates driven by p-waves. They also investigated the consequences of lowering the residual symmetry resulting from the solutions of this problem by means of strong spin-orbit interactions. Since, in this case, the Cooper pair is in a $J = 2$ state ($S = 1, L = 1$), the order parameter is represented by a traceless symmetric tensor. The same holds true for an ($S = 0, L = 2$) state order parameter, so those results may be directly applied to D-wave pairing. According to the authors of ref.\textsuperscript{12}, eleven different phases are allowed. Their analysis,
however, is questionable, since time reversal symmetry cannot be neglected in the theory of condensate states and the use of a 4-th degree polynomial Landau free energy strongly limits the number of phases that would be allowed by the symmetry of the system.

Our aim in this paper is to give a definitive answer to the classification of possible symmetry breaking patterns in D-wave pairing Bose condensate, in the framework of the Landau theory of phase transitions. To this end we make use of a geometrical invariant theory approach to the problem, proposed in ref. [3]. The strategy is to exploit a set of basic invariant polynomials of the symmetry group of the system, as fundamental variables in the description of the phase space of the system and in the minimization procedure of the free energy [4].

II. THE ORBIT SPACE APPROACH

Let us briefly recall some basic elements of the orbit space approach [3]. To this end, we shall denote by $x \in \mathbb{R}^n$ a vector order parameter, transforming linearly and orthogonally under the compact real symmetry group $G$, and by $\Phi(\alpha; x)$ the $G$-invariant free energy, expressed also in terms of state variables $\alpha$. The points $x_0(\alpha)$, where the function $\phi_\alpha(x) = \Phi(\alpha; x)$ takes on its absolute minimum, correspond to the stable phase of the system, whose symmetry is determined by the isotropy subgroup, $G_{\alpha_0}$, of $G$ at $x_0$. Owing to $G$-invariance, the stationary points of the free energy are degenerate along $G$-orbits. Since the isotropy subgroups of $G$ at points of the same orbit are conjugate in $G$, only the conjugate class, $[G_{\alpha_0}]$, of $G_{\alpha_0}$ in $G$, i.e. the symmetry (or orbit-type) of the orbit through $x_0$, is physically relevant.

The set of all $G$-orbits, endowed with the quotient topology and differentiable structure, forms the orbit space, $\mathbb{R}^n/G$, of $G$ and the subset of all the $G$-orbits with a given symmetry forms a stratum of $\mathbb{R}^n/G$. Phase transitions take place when, by varying the values of the $\alpha$'s, the point $x_0(\alpha)$ is shifted to an orbit lying on a different stratum. If $\Phi(\alpha; x)$ is a sufficiently general function of the $\alpha$'s, by varying these parameters, it is possible to shift $x_0(\alpha)$ on whichever stratum of $\mathbb{R}^n/G$. So, the strata are in one-to-one correspondence with the symmetry phases allowed by the $G$-invariance of the free energy. On the contrary, extra restrictions on the form of the free energy function, not coming from G-symmetry requirements (e.g., the assumption that the free energy is a polynomial of low degree), can limit the number of allowed phases.

Being constant along each $G$-orbit, the free energy may be conveniently thought of as a function defined in the orbit space of $G$. This fact can be formalized using some basic results of invariant theory. In fact, the $G$-invariant polynomial functions separate the $G$-orbits, meaning that, for any two distinct $G$-orbits, there is at least a polynomial $G$-invariant function assuming different values on them. Moreover, every $G$-invariant polynomials can be built as real polynomial functions of a finite set, $\{p_1(x), \ldots, p_q(x)\}$, of basic polynomial invariants (integrity basis of the ring of $G$-invariant polynomials), which need not, for general compact groups, be algebraically independent. The number of algebraically independent elements in an em minimal set of basic polynomial invariants is $n - \nu$, where $\nu$ is the dimension of the generic (principal) orbits of $G$. Information on the number and degrees of a minimal set of basic invariants, and the degrees of the algebraic relations (syzygies) among them, can be inferred from the Möbius function of $G$.

Let us call $q_0$ the number of independent elements of the set $\{p\}$. The range of the orbit map, $x \mapsto p(x) = (p_1(x), \ldots, p_q(x)) \in \mathbb{R}^q$, yields a realization of the orbit space of the linear group $G$, as a connected semi-algebraic surface, i.e. a subset of $\mathbb{R}^q$, determined by algebraic equations and inequalities. The orbit space of $G$ is, therefore, a closed and connected region of a $q_0$-dimensional algebraic surface, delimited by lower dimensional semi-algebraic surfaces.

Like all semi-algebraic sets, the orbit space of $G$ presents a natural stratification, since it can be considered as the disjoint union of semi-algebraic subsets of various dimensions (geometrical strata), each stratum being in the border of a higher dimensional stratum, but for the highest dimensional one, which is unique (principal stratum). The connected components of the symmetry strata are in one-to-one correspondence with the geometrical strata. The symmetries of two bordering strata are related by a group subgroup relation and the lower dimensional stratum has a larger symmetry.

The orbit space can be identified with the semi-algebraic variety, $S$, formed by the points $p \in \mathbb{R}^q$, satisfying the following conditions i) and ii) [3] [8]:

i) $p$ lies on the surface, $Z$, defined by the syzygies;

ii) the $q \times q$ matrix $\hat{P}(p)$, defined by the relation

$$\hat{P}_{ab}(p(x)) = \sum_{j=1}^{n} \partial_j p_a(x) \partial_j p_b(x), \quad \forall x \in \mathbb{R}^n$$ (1)
is positive semidefinite at $p$.

The relations defining the strata can be obtained as positivity and rank conditions on the matrix $\hat{P}(p)$ and the minimum of $\Phi(\alpha; x)$ can be computed as a constrained minimum in orbit space of the function $\tilde{\Phi}(\alpha; p)$,

$$\tilde{\Phi}(\alpha; p(x)) = \Phi(\alpha; x), \forall x \in \mathbb{R}^n$$  \hspace{1cm} (2)

or from the solutions of the equation \cite{13}

$$\sum_{\alpha=1}^{q} \hat{P}_{\alpha}(p) \partial_{\alpha} \tilde{\Phi}(\alpha; p) = 0, \quad a = 1, \ldots, q,$$  \hspace{1cm} (3)

which is equivalent to the state equation, $\partial \Phi(\alpha; x)/\partial x_j = 0, j = 1, \ldots, n$.

### III. SYMMETRY OF THE ALLOWED D-WAVE CONDENSATE STATES IN ISOTROPIC SPACE

The formation of $D$-wave condensate states breaks the symmetry of the isotropic 3-dimensional space, which corresponds to the group $O_3 \otimes U_1 \times \langle T \rangle$, where $O_3$ is the complete rotation group, $U_1$ is the group of gauge transformations and $\langle T \rangle$ is the group generated by the time reversal operator $T$.

The symmetry of the allowed $D$-wave condensate ground states is defined by the relative values of the complex coefficients in the decomposition of the gap-function, $\Delta$, in terms of spherical harmonics with $L = 2$:

$$\Delta(\theta, \phi) = \sum_{m=-2}^{2} D_m Y_2^m(\theta, \phi)$$  \hspace{1cm} (4)

The set of functions $\{Y_2^m, Y_2^{m*}\}$ yields a basis of a ten-dimensional (10 D) space hosting a real representation of the symmetry group $O_3 \otimes U_1 \times \langle T \rangle$. A general element, $\gamma$, of the group will be denoted by a triple $\gamma = (\rho, \epsilon, \epsilon')$, where, $\rho \in O_3, 0 \leq \phi < 2\pi$ and $\epsilon = -1, \text{or} +1$ according as a time reflection is involved in the transformation, or not.

The action of $G$ can be transferred to a real irreducible action on the 10D vector formed by the coefficients $\{D_2, \ldots, D_{-2}, D_{-2}^*, \ldots, D_2^*\}$. The representation of $G$ thus obtained can be realized in the 10 D real vector space of a couple of two independent, real, second rank, symmetric, traceless tensors, $X_1^{(i)}$ and $X_2^{(i)}$, $i, j = 1, 2, 3$, which can be considered as the real and imaginary parts of a complex $3 \times 3$ matrix $\psi$, whose elements will be written in terms of five complex coordinates, $z_j$:

$$z_j = x_j + i x_{5+j}, \quad j = 1, \ldots, 5; \quad x_1 \in \mathbb{R},$$  \hspace{1cm} (5)

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} z_2 + \frac{z_3}{\sqrt{3}} & z_1 & z_3 \\ z_1 & -z_2 + \frac{z_3}{\sqrt{3}} & -z_3 \\ z_3 & z_2 & -z_3 \end{pmatrix}.$$  \hspace{1cm} (6)

The coordinates $D_\alpha$ are connected to the $z_j$ by the following relations:

$$D_2 = \frac{i z_1 + z_2}{\sqrt{2}}, \quad D_1 = \frac{i z_3 + z_4}{\sqrt{2}}, \quad D_0 = z_5, \quad D_{-1} = \frac{i z_3 - z_4}{\sqrt{2}}, \quad D_{-2} = \frac{i z_1 - z_2}{\sqrt{2}}.$$  \hspace{1cm} (7)

The matrix $\psi$ transforms in the following way under a general transformation $\gamma = (\rho, \phi, \epsilon) \in G$:

$$\gamma \cdot \psi = e^{i\phi} \rho \psi' \rho^T, \quad \gamma \in G,$$  \hspace{1cm} (8)

where $\psi' = \psi$ or $\psi^*$, according as $\epsilon = +1$ or $-1$ and the apex $T$ denotes transposition. As a consequence, the group $G$ acts as a group of linear, real, orthogonal transformations on the vector order parameter $x \in \mathbb{R}^{10}$.

The kernel of the representation of $G$ just defined is the group generated by the space reflection. So, it will not be restrictive to assume that the symmetry group is $G = SO_3 \otimes U_1 \times \langle T \rangle$ and, when speaking of $G$, in the following, we shall always refer to this linear group acting in the vector space $\mathbb{R}^{10}$.

3
The linear group \( G \) has a trivial principal isotropy subgroup (the isotropy subgroup of generic points of \( \mathbb{R}^{10} \)), thus the principal \( G \)-orbits have the same dimensions as \( G \) and its orbit space, i.e., the quotient space \( \mathbb{R}^{10}/G \), has dimensions \( q_0 = 10 - 4 = 6 \).

The Möbius function of \( G \), \( M(\eta) \), can be calculated in the form of an invariant Haar integral over \( G \) (see, for instance, [18]):

\[
M(\eta) = \int_G \frac{d\mu(g)}{\det(1 - \eta g)}, \quad |\eta| < 1,
\]

where \( \mu(g) \) is a normalized invariant measure on the group \( G \), the integration is over the whole group \( G \) and \( g \in G \).

An explicit calculation of the integral leads to

\[
M(\eta) = \frac{\eta^{20} + \eta^{12} + \eta^{10} + \eta^8 + 1}{(1 - \eta^2)(1 - \eta^4)^2(1 - \eta^6)(1 - \eta^8)}.
\]

Equation (10) yields the following indications, whose validity has been checked through direct calculations:

1. A minimal integrity basis for the linear group \( G \) contains nine elements, \( \{p_1, \ldots, p_9\} \) with degrees \((d_1, \ldots, d_9) = (2, 4, 4, 6, 6, 8, 8, 10, 12)\).

2. The invariants \( p_i \) are connected by five independent syzygies of degrees 16, 18, 20, 22 and 24.

3. The most general \( G \)-invariant polynomial, like a general non-equilibrium polynomial Landau potential, \( \hat{\Phi} (\alpha; p) \), can be written as a polynomial function of the elements of the integrity basis \( \{p_i\}_{i=1 \ldots 9} \), in terms of five arbitrary polynomials \( Q_i \), \( Q_i = Q_i (\alpha; p_1, \ldots, p_9) \), \( i = 0, \ldots, 4 \):

\[
\hat{\Phi} = Q_0 + Q_1 p_7 + Q_2 p_8 + Q_3 p_9 + Q_4 p_7 p_9.
\]

The elements of the minimal integrity basis can be chosen in the following form:

\[
\begin{align*}
p_1 &= \text{Tr}(\psi^*) = \sum_{i=1}^{10} x_i^2, & p_6 &= \Re \left[ \text{Tr}(\psi^2) \text{Tr}(\psi^2 \psi^*) \text{Tr}(\psi^*^3) \right], \\
p_2 &= \text{Tr}(\psi^*^2)^2, & p_7 &= \Re \left[ \text{Tr}(\psi^*^2)^2 \right], \\
p_3 &= \text{Tr}(\psi^2)^2, & p_8 &= \Re \left[ \text{Tr}(\psi^2 \psi^*)^2 \right], \\
p_4 &= \text{Tr}(\psi^3)^2, & p_9 &= \Re \left[ \text{Tr}(\psi^3)^3 \right].
\end{align*}
\]

Using these definitions, we have determined the explicit form of the syzygies, of the \( \hat{P} \)-matrix elements, of the equations and inequalities determining the strata in the orbit space of \( G \). For each stratum, denoted by \( S^{(d,r)} \), where \( d \) denotes the dimension and \( r \) is an enumeration index, we have picked up a “typical point” and determined the corresponding isotropy subgroup of \( G \). All these results are essential pre-requisites for a rigorous calculation of the minima of a \( G \)-invariant polynomial along the lines indicated in the Introduction.

Part of our results are resumed in Tables I, II and III and in Fig. 1, where the group–subgroup relations among the symmetries of bordering strata are also specified.

The explicit expressions of the syzygies and of the elements of the \( \hat{P}(p) \) matrix and the relations defining higher dimensional strata would require too much space to be written down here.

As a simple example of the effectiveness of our approach, we have also repeated Mermin’s calculation of the minimum of a general fourth degree polynomial \( G \)-invariant free energy

\[
\hat{\Phi}^{(4)}(p) = \alpha_0 \frac{p_1^2}{2} + \sum_{j=1}^{3} \alpha_j p_j,
\]

in the additional assumptions that it is bounded below and has a local maximum at the origin (\( \alpha_1 < 0 \)). The \( \alpha_i \)'s are connected to the parameters \( \alpha, \beta_1, \beta_2, \beta_3 \) used by Mermin by the following relations:

\[
\alpha_0 = 2 \beta_2 + \beta_3, \quad \alpha_1 = \alpha, \quad \alpha_2 = -\frac{\beta_2}{2}, \quad \alpha_3 = \beta_1 + \frac{\beta_3}{4}.
\]

With the definitions:
\[ \hat{p}_i = \frac{p_i}{p_1^{d/2}}, \quad \hat{p} = (\hat{p}_2, \ldots, \hat{p}_9), \quad \] (15)

and

\[ \Delta(\hat{p}) = \alpha_0 + 2 \alpha_2 \hat{p}_2 + 2 \alpha_3 \hat{p}_3, \quad \] (16)

the polynomial \( \tilde{\Phi}(p) \) can be put in the following convenient form:

\[ \tilde{\Phi}(p) = \frac{p_1^2}{2} \Delta(\hat{p}) + \alpha_1 p_1. \quad \] (17)

Since, owing to its definition, \( p_1 \) ranges over the whole non negative real numbers, \( \tilde{\Phi}(p) \) is bounded below \((\text{stability condition})\) if and only if the minimum, \( \delta \), of \( \Delta(\hat{p}) \) is positive. Being the minimum of the r.h.s. of (17), thought of as a function only of \( p_1 \geq 0 \), equal to \(-\alpha_1/(2\Delta)\), the absolute minimum of \( \tilde{\Phi}(p) \) is \(-\alpha_1^2/(2\delta)\). In this way, the problems of rendering explicit the stability condition and evaluating the minimum of \( \tilde{\Phi}(p) \) are reduced to the calculation of \( \delta \).

The absolute minimum of \( \delta \) in each singular stratum can be easily obtained from the equations of the strata, listed in Table IV and Table II for strata with dimensions \( < 3 \). For the principal stratum, it is easier to solve the projection of equation (6) in the unit sphere \((\sum_{i=1}^{10} x_i^2 = p_1 = 1)\), which, using (15), can be written in the form:

\[ \sum_{j=2}^{9} P_{ij}(\hat{p}) \frac{\partial \Delta(\hat{p})}{\partial p_j} = 0 \quad \] (18)

and to select, subsequently, the solutions lying in the principal stratum.

A comparison of the values of the minima in the different strata, obtained in this way, leads to the results resumed in Table IV and illustrated in Figure 2. Owing to the low degree of the polynomial defining \( \tilde{\Phi}(p) \) in (13), and the consequent low number of free parameters \( \alpha \), the absolute minimum presents strong degeneracy, particularly for special values of the \( \alpha \)'s. If these special values are excluded, spontaneous breaking of the symmetry can generate only five distinct phases out of fifteen permitted by the \( G \)-symmetry; some of them are unstable. For no non trivial values of \((\alpha_2, \alpha_3)\) does the absolute minimum lie on the stratum \( S^{(2,2)} \).

For general values of the \( \alpha \)'s, our results are in agreement with Mermin's ones. Let us add a few words about the perturbative stability of the three degenerate phases in the region \( R_2 \) (see Table IV). For \((\alpha_0, \alpha_2, \alpha_3) \in R_2 \), the addition to the free energy, \( \Phi(\alpha_4, \alpha_5, p_1, \alpha_2, \alpha_3) \), of a "small" perturbation, consisting in an invariant polynomial of degree six, \( \theta_6(p_1) = \alpha_4 p_1 + \alpha_5 p_5 \), splits the three degenerate minima determined by the 4-th degree term.\( \theta_6(p_1) \). This is easy to check, at least in the additional assumption that the perturbation leaves the absolute minimum in one of the strata corresponding to the degenerate phases. In fact, at the first perturbative order, one obtains from Tables II and IV the following shifts, \( \theta_6(p_1) \), in the values of the 6-th order free energy at the points where \( \Phi(\alpha_4, \alpha_5, p_1, \alpha_2, \alpha_3) \) takes on its degenerate absolute minimum under consideration:

\[ \theta_6^{(1,2)} = \left( \frac{\alpha_1}{\delta} \right)^3 \frac{\alpha_4 + \alpha_5}{6}, \quad \theta_6^{(1,3)} = 0, \quad \theta_6^{(2,4)} = \left( \frac{\alpha_1}{\delta} \right)^3 (\alpha_4 + \alpha_5) \xi, \quad \] (19)

where \( 0 < \xi < 1/6 \).

Since \(-\alpha_1/\delta > 0\), the absolute minimum will be perturbatively stable on \( S^{(1,2)} \) or, respectively, \( S^{(1,3)} \), according as \( \alpha_4 + \alpha_5 \) is negative or positive.

As stressed in the Introduction, the difficulties mentioned above can be overcome if one puts less restrictive upper limits to the degree of the polynomial describing the free energy. It is trivial, for instance, to realize that the following class of bounded below polynomial functions have a vanishing maximum at the origin of \( R^{10} \) and display an absolute minimum at the arbitrarily chosen point \( \bar{p} \in S \):

\[ \sum_{i=1}^{9} \alpha_i \left( p_i - \bar{p}_i \right)^{2n_i - 2} \quad \] (20)

\(^1\)The inclusion of terms of 6-th degree, depending only on \( p_1, p_2 \) and \( p_3 \) would be useless for splitting the degenerate minima, so these terms will be neglected, with no loss of generality.
where the $\alpha$’s are positive constants and the $n$’s are positive integers.
The physical implications of our results and the derivation of a more realistic form of the free energy will be discussed in forthcoming papers.

ACKNOWLEDGMENTS

This paper is partially supported by RFBR, INFN and MURST.

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| $r$ | $\bar{p}_2$ | $\bar{p}_3$ | $\bar{p}_4$ | $\bar{p}_5$ | $\bar{p}_6$ | $\bar{p}_7$ | $\bar{p}_8$ | $\bar{p}_9$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1   | 1           | 0           | 0           | 0           | 0           | 0           | 0           | 0           |
| 2   | $\frac{1}{2}$  | 1           | $\frac{1}{6}$  | $\frac{1}{6}$  | $\frac{1}{6}$  | $\frac{1}{6}$  | $\frac{1}{6}$  | $\frac{1}{6}$  |
| 3   | $\frac{1}{2}$  | 1           | 0           | 0           | 0           | 0           | 0           | 0           |
| 4   | $\frac{1}{3}$  | 0           | $\frac{1}{3}$  | 0           | 0           | 0           | 0           | 0           |
| 5   | $\frac{1}{2}$  | 0           | 0           | 0           | 0           | 0           | 0           | 0           |
TABLE II. Relations defining strata, \(S^{(2,r)}\), of dimensions 2 in orbit space. The \(\tilde{p}_i\)'s are defined as in Table III and \(\epsilon = \pm 1\).

| \(\tilde{p}_r\) | 1 | 2 | 3 | 4 | 5 |
|----------------|---|---|---|---|---|
| \(\tilde{p}_2\)  | \((2 + \xi^2)/6\) | \(1/2\) | \((2 + \xi^2)/6\) | \(1/2\) | \(\xi\) |
| \(\tilde{p}_3\)  | 0 | \(\xi\) | \(\xi^2\) | 1 | \(2 - 2\xi\) |
| \(\tilde{p}_4\)  | \((2 - \xi)^2(1 + \xi)/12\) | 0 | \((2 - \xi)^2(1 + \xi)/12\) | \(\xi\) | 0 |
| \(\tilde{p}_5\)  | 0 | \(\xi^2(1 + \xi)/12\) | \(\xi\) | 0 |
| \(\tilde{p}_6\)  | 0 | \((2 - \xi)(1 + \xi)\xi^2/12\) | \(\xi\) | 0 |
| \(\tilde{p}_7\)  | 0 | \((1 + \xi)\xi^3/12\) | \(\xi\) | 0 |
| \(\tilde{p}_8\)  | 0 | \((2 - \xi)^2(1 + \xi)\xi^3/12\) | \(\xi\) | 0 |
| \(\tilde{p}_9\)  | 0 | \((2 - \xi)^2(1 + \xi)\xi^3/12\) | \(\xi\) | 0 |
| \(\xi\) range    | \([0, \frac{1 + 3\xi}{2}]\) | \([0,1]\) | \([0,\epsilon]\) | \([0,\frac{\epsilon}{2}]\) | \([\frac{3}{2},1]\) |

TABLE III. Possible symmetry strata, \(S^{(d,r)}\), for D-wave driven pairing in isotropic space (\(d\) denotes the dimension of the stratum and \(r\) is a secondary enumeration index). From left to right, the columns refer to phase reference numbers according to our\((d,r)\) and, respectively, ref.[12] \((N)\) classification, order \((|H|)) and residual symmetry group \((H)\), complex coordinates \((z_j = x_j + i x_j, j = 1, \ldots, 5)\) of the \(H\)-invariant superconducting vector order parameter and corresponding values of the partial wave amplitudes \((d_2,\ldots,d_{-2})\). In columns five and six, the \(t's\) are real coordinates, while the \(\epsilon's\) are complex ones. The notations are the same as in [19], \(R_z(\phi)\) denotes a clockwise proper rotation of the vectors, about the \(z\)-axis and \(O_z^0 = \{R_z(\phi)\}_{0 \leq \phi < 2\pi} \cup \{C_{2\alpha} R_z(\phi)\}_{0 \leq \phi < 2\pi}\).

\[
\begin{array}{cccc}
(d, r) & N & |H| & (z_1, \ldots, z_5) & \sqrt{2}(D_2, D_1, D_0, D_{-1}, D_{-2}) \\
\hline
(1,1) & II & \infty & (C_2 T) \times (R_z \phi) U_1(2\phi) \alpha & (it_1, -t_{1,2}, 0) \times (2\pi, 0, 0,0,0) \\
(1,2) & V HI & \infty & O_z^5(\pi) & (0, 0, 0, t) \times (0, 0, 0, 0, 0) \\
(1,3) & IX.X & 16 & (C_2 T, C_4 U_1(\pi)) & (0, 0, 0, 0, 0) \\
(1,4) & XI & 24 & (C_2 T, C_4 T, C_4 U_1(4\pi/3)) & (0, -\pi, 0, 0, t) \times (0, 0, 0, 0, 0) \\
(1,5) & I & \infty & (C_2 T) \times (R_z \phi) U_{1(-\phi)} \alpha & (0, 0, 0, -t, 0) \times (0, 0, 0, 0, 0) \\
(2,1) & V & 6 & (C_2 T, C_3 U_1(4\pi/3)) & (it_1, -t_{1,2}, 0) \times (1, 0, 0, 0, 0) \\
(2,2) & IV & 4 & (C_2 T, C_3 U_1(\pi)) & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
(2,3) & -- & 8 & (C_2 T, C_4 T, C_4 U_1(2\phi)) & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
(2,4) & V HI & 8 & (C_2 T, C_4 T) & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
(2,5) & VI & 8 & (C_2 T, C_4 U_1(\pi)) & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
(3,1) & III & 4 & (C_2 T, C_2 T, C_2 T) & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
(3,2) & -- & 4 & (C_2 T, C_2 T) & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
(4,1) & -- & 2 & (C_2 T) & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
(4,2) & -- & 2 & (C_2 T, C_2 T) & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
(6,1) & -- & 1 & \{1\} & (0, 0, 0, 0, 0) \times (0, 0, 0, 0, 0) \\
\end{array}
\]

TABLE IV. Absolute minimum, \(\tilde{\Phi}^{(d,r)}_{\min} = -\alpha_2^2/(24)\), of a general, bounded below, \(G\)-invariant 4-th degree polynomial, \(\tilde{\Phi}^{(d,r)}(\alpha, p) = \alpha_0 p_r^2/2 + \sum_{j=1}^3 \alpha_j p_j, \alpha_1 < 0\), and hosting strata, \(S^{(d,r)}\), as functions of the coefficients \(\alpha\). The denomination of the strata is the same as in Table III.

| \(\alpha\) range | \(d, r\) |
|-----------------|---------|
| \(R_1\) : Max(0, -3\alpha_0/2, -6\alpha_3) < \alpha_2 | \(\alpha_0 + 2\alpha_2/3\) |
| \(R_2\) : -6\alpha_3 > \alpha_2 > Max(-\alpha_0 - 2\alpha_2, 2\alpha_3) | \(\alpha_0 + \alpha_2 + 2\alpha_3\) |
| \(R_3\) : -\alpha_0/2 < \alpha_2 < Min(0, 2\alpha_3) | \(\alpha_0 + 2\alpha_2\) |
| \(R_{13}\) : \(0 = \alpha_2 < Min(\alpha_0, \alpha_3)\) | \(\alpha_0\) |
| \(R_{12}\) : Max(-3\alpha_0/2, 0) < \alpha_2 = -6\alpha_3 | \(\alpha_0 - 4\alpha_3\) |
| \(R_{23}\) : -\alpha_0/2 < \alpha_2 = 2\alpha_3 < 0 | \(\alpha_0 + 4\alpha_3\) |
| \(R_{423}\) : \(\alpha_2 = \alpha_3 = 0 < \alpha_0\) | \(\alpha_0\) | all, except (0, 1) |
FIG. 1. Possible phase transitions between bordering strata, connected, in the figure, by continuous sequences of one or more arrows. The notations are the same as in Table III.

\[ S^{(0)} = \{0\} \]
\[ SO_3 \otimes U_1 \times T \]

\[ S^{(1,5)} \]
\[ \{\langle C_{2x} T, C_{3z} U_1(\frac{\pi}{2}) \rangle\} \]

\[ S^{(1,4)} \]
\[ \{\langle C_{2x}, C_{3z} U_1(\frac{\pi}{2}), C_{2x} T \rangle\} \]

\[ S^{(1,3)} \]
\[ \{\langle C_{2x}, C_{4z} U_1(\pi), T \rangle\} \]

\[ S^{(1,2)} \]
\[ \{\langle C_{2x} T \rangle\} \]

\[ S^{(1,1)} \]
\[ \{\langle C_{2x} T \rangle\} \]

\[ S^{(2,1)} \]
\[ \{\langle C_{2x} T, C_{3z} U_1(\frac{\pi}{2}) \rangle\} \]

\[ S^{(2,2)} \]
\[ \{\langle C_{2x} T, C_{2z} U_1(\pi) \rangle\} \]

\[ S^{(2,3)} \]
\[ \{\langle C_{2x}, C_{4z} T \rangle\} \]

\[ S^{(2,4)} \]
\[ \{\langle C_{2x}, C_{4z}, T \rangle\} \]

\[ S^{(2,5)} \]
\[ \{\langle C_{4z} U_1(\pi), C_{2x} T \rangle\} \]

\[ S^{(3,2)} \]
\[ \{\langle C_{2x}, C_{2z} \rangle\} \]

\[ S^{(3,1)} \]
\[ \{\langle C_{2x}, C_{2z} T \rangle\} \]

\[ S^{(4,1)} \]
\[ \{\langle C_{2x} T \rangle\} \]

\[ S^{(4,2)} \]
\[ \{\langle C_{2z} \rangle\} \]

\[ S^{(6,1)} = S_{\text{principal}} \]
\[ \{\{\}\} \]

FIG. 2. Localization of the absolute minimum of a fourth degree G-invariant polynomial, \( \tilde{\Phi}^{(4)}(\alpha, p) = \alpha_0 p_1^2/2 + \sum_{j=1}^{3} \alpha_j p_j, \)
\( \alpha_1 < 0, \) as a function of its coefficients. For values of \( (\alpha_0, \alpha_2, \alpha_3) \) in \( \mathcal{R}_1 \) or in \( \mathcal{R}_2 \) or in \( \mathcal{R}_3, \) the absolute minimum lies, respectively, in the strata \( S^{(1,4)} \) or \( \{S^{(1,2)}, S^{(1,3)}, S^{(2,4)}\} \) (degenerate minimum) or \( S^{(1,1)} \). For particular values of the \( \alpha \)'s, see Table IV.
\( R_1: \alpha_0 > -2\alpha_2 \)

\( R_2: \alpha_0 > -2\alpha_2/3 \)

\( R_3: \alpha_0 > -\alpha_2 - 2\alpha_3 \)