Non-geometric States in a Holographic Conformal Field Theory

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In the AdS$_3$/CFT$_2$ correspondence, we find some conformal field theory (CFT) states that have no bulk description by the Bañados geometry. We elaborate the constraints for a CFT state to be geometric, i.e., having a dual Bañados metric, by comparing the order of central charge of the entanglement/Rényi entropy obtained respectively from the holographic method and the replica trick in CFT. We find the geometric CFT states fulfill Bohr's correspondence principle by reducing the quantum KdV hierarchy to its classical counterpart. We call the CFT states that satisfy the geometric constraints geometric states, and otherwise non-geometric states. We give examples of both the geometric and non-geometric states, with the latter case including the superposition states and descendant states.

INTRODUCTION

The conventional wisdom of the AdS/CFT correspondence [1] suggests a one to one correspondence between the conformal field theory (CFT) states at the boundary and the bulk asymptotically anti-de Sitter (AdS) geometries. Especially a surface/state correspondence has been proposed in [2, 3] which relates a bulk co-dimension two surface to a CFT state. However, the above correspondence usually holds only when the bulk gravity is truly classical. We thereby can imagine that it would fail when there is no obvious classical bulk dual for some CFT property. One example is the linear superposition principle of CFT as the bulk gravity is classical. Thus, we will expect that the superposition of two CFT states with bulk dual geometries cannot be described by bulk geometry since the linear superposition does not work for classical gravity.

The lesson from the above example becomes sharper in three-dimensional (3D) AdS gravity, which is dual to a two-dimensional (2D) CFT [4], since the bulk Bañados geometries [5] are determined by the expectation value of stress tensor of dual 2D CFT in the large central charge $c$ limit, and have the simple closed forms. Thus, each primary state is dual to a Bañados geometry, but the superposed state should have no dual Bañados geometry. It is intriguing to investigate the criterion to tell whether a CFT state has a bulk geometric description.

To this question is to find the distinguishing property which only holds for the CFT states with bulk geometric dual. In this work, we find this property is encoded in the holographic entanglement entropy [6, 7], or holographic Rényi entropy [8]. Given a Bañados geometry dual to a state with order $c$ stress tensor expectation value, it is then straightforward to see that the resultant holographic entanglement/Rényi entropy is of order $c$ because the bulk metric is $c$-independent. Therefore, the entanglement/Rényi entropy of the dual CFT state obtained from the replica trick should be also of order $c$.

Based on the above criterion, in this paper we will elaborate the procedure of extracting the constraints for a generic 2D CFT state to be geometric by evaluating its entanglement/Rényi entropy à la replica trick similar to what have been done in [9–12]. We will then see that the aforementioned superposed states, along with other CFT states are non-geometric.

By Bohr’s correspondence principle of quantum states, these geometric CFT states incarnate its classical nature by having a classical geometric description, and should also reflect this in the pure context of CFT. As we will see this is indeed the case by the classical reduction of quantum Korteweg-de Vries (KdV) hierarchy.

CRITERION FOR GEOMETRIC CFT STATES IN BAÑADOS GEOMETRY

Due to the topological nature of 3D Einstein gravity, i.e., with no bulk propagating degree of freedom, the bulk geometry is completely determined by the asymptotic boundary constraints, this then leads Bañados to conjecture that all the vacuum asymptotically AdS$_3$ solutions of 3D Einstein gravity are completely classified by the boundary conformal symmetries. Applying this conjecture to AdS/CFT correspondence, it leads to the Bañados geometries which are determined by the expectation value of stress tensor with respect to the dual CFT state. More precisely, the form of the Bañados geometry
tongues the form \[5\]

\[
d s^2 = \frac{dy^2}{y^2} + \frac{L_\rho}{2} dz^2 + \frac{\tilde{L}_\rho}{2} d\bar{z}^2 + \left(\frac{1}{y^2} + \frac{\gamma^2}{4} L_\rho \tilde{L}_\rho\right) dz d\bar{z},
\]

where we set the AdS radius to unity \(R = 1\) so that the bulk Newton constant \(G_N\) is related to the central charge \(c\) of the dual CFT by \(c = \frac{3}{2G_N}\) \[1\].

We consider a holographic CFT on a cylinder with complex coordinate \(w\) and spatial period \(L\) in a state with density matrix \(\rho\), and the cylinder can be mapped to a complex plane with coordinate \(z\) by the conformal transformation \(z = e^{\frac{2\pi i w}{L}}\). The functions \(L_\rho(z), \tilde{L}_\rho(\bar{z})\) in the Bañados geometry are respectively holomorphic and anti-holomorphic, and are related to expectation value of stress tensor on the plane with respect to the dual CFT state

\[
\langle T(z) \rangle_\rho = -\frac{c}{12} L_\rho(z), \quad \langle \bar{T}(\bar{z}) \rangle_\rho = -\frac{c}{12} \tilde{L}_\rho(\bar{z}).
\]

Given a Bañados geometry which is dual to a CFT state \(\rho\), one can then evaluate the holographic entanglement/Rényi entropy á la the prescriptions in \[1-3,12\]. Both the holographic entanglement and Rényi entropies are given by the area law formula. If we consider a CFT state, for which \(\langle T(z) \rangle_\rho\) and \(\langle \bar{T}(\bar{z}) \rangle_\rho\) are of order \(c\), then the metric of the dual Bañados geometry is of order \(c^0\) in the large \(c\) expansion, and should be independent of \(c\) in the large \(c\) limit. Thereby, the area of minimal surface or cosmic brane should be independent of \(c\) so that the holographic entanglement/Rényi entropies should be of order \(c\) due to the relation \(c = \frac{3}{2G_N}\). Based on the above result, we now formulate our criterion for the geometric CFT states:

**For a 2D CFT state of order \(c\) stress tensor expectation value to be holographic dual to a Bañados geometry, the entanglement/Rényi entropy obtained from CFT calculations should be at most order \(c\) in the large \(c\) limit. Otherwise, we call the CFT state non-geometric.**

The above criterion can also be generalized to cases with different scaling of \(c\) for the expectation values of the stress tensor.

\[
\langle T(w) \rangle_\rho = c\alpha(w) + \beta(w) + \frac{\gamma(w)}{c} + O\left(\frac{1}{c^2}\right), \quad \langle A(w) \rangle_\rho = c^2\alpha(w)^2 + c\delta(w) + \epsilon(w) + O\left(\frac{1}{c}\right),
\]

\[
\langle B(w) \rangle_\rho = c^2\left[\alpha'(w)^2 - \frac{4}{5}\alpha(w)\alpha''(w)\right] + c\zeta(w) + O(c^0),
\]

\[
\langle D(w) \rangle_\rho = c^3\alpha(w)^3 + 3c^2\alpha(w)\delta(w) - \alpha(w)\beta(w) + cn(w) + O(c^0),
\]

\[
\langle E(w) \rangle_\rho = c^2\left[\alpha''(w)^2 + \frac{10}{63}[\alpha(w)\alpha(4)(w) - 7\alpha'(w)\alpha(3)(w)]\right] + O(c).
\]

**CONSTRAINTS FOR GEOMETRIC CFT STATES**

Based on our proposed criterion for the geometric CFT states, i.e., that the entanglement/Rényi entropy should be at most of order \(c\) in the large \(c\) limit, we would like to extract the necessary constraints by explicitly evaluating the entanglement/Rényi entropy. The prescription of evaluating entanglement/Rényi entropy is based on the replica trick \[13\], which leads to an \(n\)-fold CFT that we call CFT\(n\). However, there is usually no closed form of entanglement/Rényi entropy for generic excited states. Instead we will evaluate in the short-interval expansion, similar to what has done in \[12\]. By assuming dominance of the vacuum conformal family in the operator product expansion (OPE) of twist operators \[14,16\] in the large \(c\) limit, the entanglement/Rényi entropy takes the formal form in terms of the series of expectation values of CFT\(n\) quasiprimary fields \(\Phi_K\) that are constructed by operators in the vacuum conformal family of the original one-fold CFT. Since the contributions from the holomorphic and anti-holomorphic sectors decouple and are similar, in this paper we only consider the contributions from the holomorphic sector.

We consider the short interval \(A = [w, w + \ell]\) with \(\ell \ll L\), and from OPE of twist operators we get the short interval expansion of the Rényi entropy

\[
S^{(n)}_{A,\rho} = \frac{c(n+1)}{12n} \log \frac{\ell}{c} - \frac{1}{n-1} \log \left(\sum_K d_K \sum_{r=0}^{\infty} \frac{a_{hK+r}^n}{r!} \langle \Phi_K^{(r)}(w) \rangle_\rho\right),
\]

with the summation of \(K\) being over all the CFT\(n\) holomorphic quasiprimary fields \(\Phi_K\) and the coefficient

\[
a_{hK+r}^n = \frac{C_{hK+r-1}^r}{C_{2hK+r-1}^r}, \quad C_{x}^y = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)}.
\]

To order \(\ell^6\), the CFT\(n\) holomorphic quasiprimary operators \(\Phi_K\) and their OPE coefficients \(d_K\) can be found in \[17\], and they are constructed by the holomorphic quasiprimary operators in the original one-fold CFT, i.e., \(T\) at level 2, \(A\) at level 4, \(B, D\) at level 6, and \(E, H, I\) at level 8, for whose explicit forms one can look up in \[17\].

Requiring the Rényi entropy of \(A\) in state \(\rho\) being of at most order \(c\), we get the constraints for the one-point functions
with \(\alpha(w), \beta(w), \gamma(w), \delta(w), \epsilon(w), \zeta(w), \eta(w)\) being arbitrary order \(c^0\) holomorphic functions.

Taking \(n \to 1\) limit of the Rényi entropy, we obtain the entanglement entropy. Requiring the entanglement entropy being of at most order \(c\), we get the constraints

\[
\langle T(w) \rangle_{\rho} = c\alpha(w) + O(c^0),
\]
\[
\langle A(w) \rangle_{\rho} = c^2\alpha(w)^2 + O(c).
\]

Note that to a fixed order \(\ell\), we can get more constraints from the Rényi entropy than from the entanglement entropy.

**EXAMPLES OF GEOMETRIC CFT STATES**

In [18–21], it has been shown that the Rényi entropy for the primary excited state

\[
\rho_\phi = \frac{1}{\alpha_\phi} |\phi\rangle \langle \phi|,
\]

is order \(c\) if the conformal weight \(h_\phi\) is at most of order \(c\), so that they should satisfy all the geometric state constraints [3]. This is also consistent with the calculation [10, 11] from OPE of twist operators to order \(c^0\).

Even without an explicit check as done for the primary states, we can argue that some particular states should satisfy the geometric state constraints. For example, the thermal states which are dual to BTZ black holes, thus should also be geometric. Similarly, the states which are conformally related to the vacuum state on the plane, denoted by \(|0\rangle\) should also be geometric. In the bulk, these states are dual to the Bañados geometries which can be transformed to pure AdS\(_3\) by the coordinate transformation dual to the boundary conformal map. These states include the thermal state and the conical defect state.

In quantum mechanics the coherent states have minimal uncertainty and thus behave quite like the classical states. This motivates us now to check if a “coherent CFT state” can also have the bulk description. Explicitly, the state considered has the density matrix

\[
\rho_{\phi(w_0)} = \frac{1}{\alpha_\phi} \left| \frac{\alpha_{L-1}}{\pi} \sin \left( \frac{\pi \bar{w}_0 - w_0}{L} \right) \right|^{2h_\phi} \langle \phi(w_0) |0\rangle \langle 0 | \phi(\bar{w}_0) \rangle = \frac{1}{\alpha_\phi} \left( \frac{\alpha_{L}}{\bar{z}_0 - z_0} \right)^{2h_\phi} \phi(\bar{z}_0) |0\rangle \langle 0 | \phi(1/\bar{z}_0). \tag{8}
\]

Note that \(w_0\) is a position on the cylinder and \(z_0\) is a position on the plane with the relation \(z_0 = e^{2\pi i w_0}\). Since \(\phi(2\pi z_0) |0\rangle \langle 0 | \phi(z)\) the above state can be understood as a coherent sum of the primary state \(|\phi\rangle\) and its global descendants. We check that the one point functions in the state \(\rho_{\phi(w_0)}\) satisfy the constraints [3]. This is consistent with the fact that on the cylinder the locally excited state is dual to a moving particle in AdS\(_3\) [18 22], i.e., that there exists a bulk geometric description.

**QUANTUM TO CLASSICAL KDV EQUATION AND CHARGES FOR GEOMETRIC CFT STATES**

The geometric state constraints relate the expectation values of operators in the vacuum family quasiprimaries. We will show that these constraints in fact reduce the quantum KdV equation and charges to their classical counterparts.

For demonstration, we write down the quantum KdV currents up to level 8 [23 25]

\[
J_2 = T, \quad J_4 = (TT), \quad J_6 = (T(TT)) - \frac{c + 2}{12} (T'T'),
\]
\[
J_8 = (T(T(TT))) + \frac{c + 8}{6} (T(T'T')) + \frac{c^2 + 4c - 101}{180} (TT^{(4)}),
\]

with the parentheses \((\cdot \cdot \cdot)\) denoting the normal ordering operators. In terms of the quasiprimary operators and their derivatives, we obtain

\[
J_2 = T, \quad J_4 = A + \frac{3}{10} T'', \quad J_6 = D - \frac{25(2c + 7)(7c + 68)}{108(70c + 29)} B - \frac{2c - 23}{108} A'' - \frac{c - 14}{280} T^{(4)},
\]
\[
J_8 = I - \frac{5(3c + 46)(350c^2 + 2315c - 361)}{39(1050c^2 + 3305c - 251)} H + \frac{49(3c + 46)(5c^2 + 46c + 99)}{2860(105c + 11)} E + \frac{5c + 94}{234} D''
\]
These currents form the mutually commuting KdV charges

\[ Q_{2k-1} = \int_0^L \frac{dw}{L} J_{2k}(w), \quad (11) \]

which constitute the integrability hierarchy of the quantum KdV equation

\[ \dot{T} = \frac{1 - c}{6} T''' - 3(TT)' = -\frac{5c + 22}{30} T''' - 3A'. \quad (12) \]

Using the leading order geometric state constraints \[^5\], we set \( \alpha(w) = U(w)/6 \) and get the classical KdV equation

\[ \dot{U} = U''' + 6UU'. \quad (13) \]

Note that \( \partial_i \), which we denote by dot, has been rescaled from the quantum KdV equation to its classical counterpart.

In the large \( c \) limit, a natural definition of the classical counterpart of quantum KdV currents with respect to state \( \rho \) is

\[ J_{2k}^\rho(w) = \lim_{c \to \infty} \frac{6^k}{cT} \langle J_{2k}(w) \rangle_\rho. \quad (14) \]

Using the leading order of \[^5\] we can then turn \( J_{2k}^\rho \) into the standard classical form

\[ J_2^\rho = U, \quad J_4^\rho = U^2, \quad J_6^\rho = U^3 - \frac{1}{2} U'^2, \]
\[ J_8^\rho = U^4 + U^2 U''' + \frac{1}{5} UU(4). \quad (15) \]

Their associated KdV charges constitute the integrability hierarchy of the classical KdV equation \[^13\]. This reflects Bohr’s correspondence principle for these geometric states by reducing these KdV conserved currents into their classical counterparts.

**EXAMPLES OF NON-GEOMETRIC CFT STATES**

From our discussions we see that there are an infinite tower of constraints for a state to be geometric. Then it seems that it should be quite easy to have non-geometric states by violating one of the infinite number of constraints. The reason why we did not know any example of non-geometric states is partly due to lack of principle of check as proposed in this work, and partly due to the technical involvement of evaluating the geometric state constraints. In the following we will consider some examples of non-geometric states, for which we know how to evaluate the associated one-point functions of the vacuum family quasiprimary operators to check \[^5\].

As discussed in the introduction, one expects the superposition of primary states will not be geometric because the bulk gravity is classical so that the superposition principle does not work. Now we would like to check this explicitly.

Let us choose \( |\phi_1 \rangle \) and \( |\phi_2 \rangle \) as two primary states with conformal weights \( h_{\phi_1} = c\epsilon_{\phi_1} + O(c^0), h_{\phi_2} = c\epsilon_{\phi_2} + O(c^0), \) and \( \epsilon_{\phi_1} \neq \epsilon_{\phi_2}. \) We consider the superposition state

\[ \cos(\theta)|\phi_1 \rangle + e^{i\psi} \sin(\theta)|\phi_2 \rangle. \quad (16) \]

The constraints \[^5\] are satisfied separately for the states \( |\phi_1 \rangle \) and \( |\phi_2 \rangle \), however they are violated for the superposition state \[^{10}\]. This means that the superposition of two primary states is non-geometric as we expect. It is straightforward to generalize the above result to superposition states \( \sum_i c_i |\phi_i \rangle \) with \( |\phi_i \rangle \)’s being different primary states.

Other examples that do not satisfy the constraints \[^5\] are some descendant states

\[ |\phi^{(m)} \rangle \text{ with } h_{\phi} + m \sim O(c), \]
\[ |\tilde{\phi}^{(m)} \rangle \text{ with } h_{\phi} \sim O(c), \]
\[ |\tilde{\phi}^{(m)} \rangle \text{ with } h_{\phi} + m \sim O(c), \]
\[ |T^{(m)} \rangle \text{ with } m \sim O(c), \]
\[ |A^{(m)} \rangle \text{ with } m \sim O(c), \quad (17) \]

where \( \phi \) is a primary operator and \( \tilde{\phi} \) is a quasiprimary operator with the definition \( \tilde{\phi} \equiv (T \phi) - \frac{a_\phi}{a_{\tilde{\phi}}} \phi'' \). Note that we have not yet normalize these descendant states properly.

Among the examples of non-geometric states, the superposition states can be understood intuitively. On the other hand, we have no immediate understanding why the descendant states lack the bulk geometric descriptions. We may then ask if these states will turn to be geometric if quantum gravity effects are taken into account. In the context of perturbative quantum gravity by including higher derivative curvature terms, the answer is no because these terms are of higher orders in \( G_N \sim 1/c \) so that they can only yield subleading order \( 1/c \) corrections to the Bañados geometry, and the holographic entanglement/Rényi entropies remain order \( c \). Therefore, we are forced to accept the existence of these non-geometric states, or the quantum gravity correction should be non-perturbative.

Moreover, in the context of quantum thermalization and canonical typicality \[^{24, 27}\] the non-geometric states are obviously the atypical states because their entanglement/Rényi entropies are quite different from the ones
of thermal states. If density of the non-geometric states is comparable or even larger than the one of the geometric CFT states, one would then expect the canonical typicality to fail for 2D large $c$ CFTs.

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