Group Theoretical Analysis of the Wave Function of the $[70, 1^-]$ Nonstrange Baryons in the $1/N_c$ Expansion

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Abstract. Using standard group theoretical techniques we construct the exact wave function of the $[70, 1^-]$ multiplet in the orbital, spin and flavor space. This symmetric wave function is compared to that customarily used in the $1/N_c$ expansion, which is asymmetric. The comparison is made by analyzing the matrix elements of various operators entering the mass formula. These matrix elements are calculated by the help of isoscalar factors of the permutation group, specially derived for this purpose as a function of $N_c$. We also compare two distinct methods used in the study of the $[70, 1^-]$ multiplet. In the first method the generators are divided into two parts, one part acting on a subsystem of $N_c - 1$ quarks called core and another on the separated quark. In the second method the system is treated as a whole. We show that the latter is simpler and allows to clearly reveal the physically important operators in the mass formula.

Keywords: Baryon Spectroscopy, $1/N_c$ Expansion, Group Theory

PACS: 12.39.-x, 11.15.Pg, 11.30.Hv

1. INTRODUCTION

For fifteen years the $1/N_c$ expansion of QCD, where $N_c$ is the number of colors [1, 2] has revealed itself to be an interesting and powerful approach for studying baryon spectroscopy. The method is based on an exact contracted SU$_c$(2$N_f$) symmetry appearing in the large $N_c$ limit, $N_f$ being the number of flavors [3, 4]. For large $N_c$ this algebra becomes the SU(2$N_f$) of the constituent quark model. Much work has been devoted to the ground state baryons where the operator reduction rules simplifies the expansion [3, 4]. Usually higher order corrections of order $1/N_c^2$ are neglected.

For excited states the problem is more complicated. To include orbital excitations, by analogy to the quark model, one can classify the large $N_c$ baryons according to an extended symmetry given by the direct product SU(2$N_f$)×O(3). The group O(3) implies the introduction of a spin-orbit and a tensor interaction. It is a phenomenological fact that these contributions are small so that the breaking of this symmetry is also small. An open problem is to investigate the validity of the $1/N_c$ expansion in this extended symmetry.

In the language of the quark model the excited states can be grouped into excitation bands $\mathcal{N}$. In the $1/N_c$ expansion, the baryon masses have been calculated
for the lowest multiplets of all excited bands from $N = 1$ to 4. In these bands the multiplets belong either to the symmetric $[56]$ or to the mixed symmetric $[70]$ representation of SU(6). The symmetry of the wave function of excited baryons belonging to $[56]$ representation allows a similar treatment as that of the ground state. The spin-flavor part being symmetric, the introduction of a symmetric orbital part does not modify the procedure. In the $[70]$ representation (mainly the $[70,1^-]$), the situation turned out to be more complicated. There is a standard scheme [7] where the wave function is written as a product of a written on spa6.log. stancu@ins symmetric ground state core composed of $N_c - 1$ quarks and an excited quark. In this approach, based on a Hartree picture, the $s^{N_c-1}p$ orbital part is not properly symmetrized and the excited quark is always the last quark. The flavor-spin part is also asymmetric and corresponds to a single term of the exact wave function.

Recently, a new scheme which avoids the separation into a core and an excited quark has been suggested [8]. In that case the system is treated as a whole and the orbital-flavor-spin wave function is symmetric under any permutation of $N_c$ quarks. Some convincing quantitative arguments in favor of this procedure can be found in Ref. [9]. The exact $[70,1^-]$ wave function was written as a product of a core and a separated quark. Its orbital and the spin-flavor parts are mixed symmetric such as to recover the exact symmetric orbital-spin-flavor wave function. The procedure is described in Sec. 4.1.

Using group theoretical arguments here we examine the relation between the exact and the customarily used asymmetric wave function. We argue that the description of the system is unsatisfactory when the spin operator $S^2$ and the isospin $T^2$ operators are separated into independent parts in terms of operators acting separately on the core and on the excited quark. Much better results are obtained when we directly consider the operators $S^2$ and $T^2$ acting on the whole system. In addition we examine the role of an operator constructed from the product of $S$, $T$ and $G$ generators of SU(4).

2. SU(4) GENERATORS AS TENSOR OPERATORS

The SU(4) generators $S_i$, $T_a$ and $G_{ia}$, globally denoted by $E_{ia}$ [10], are components of an irreducible tensor operator which transforms according to the adjoint representation [211] of dimension 15 of SU(4). We recall that the SU(4) algebra is

$$
\begin{align*}
[S_i, T_a] &= 0, & [S_i, G_{ja}] &= i\varepsilon_{ijk}G_{ka}, \\
[T_a, G_{ib}] &= i\varepsilon_{abc}G_{ic}, \\
[S_i, S_j] &= i\varepsilon_{ijk}S_k, & [T_a, T_b] &= i\varepsilon_{abc}T_c, \\
& & [G_{ia}, G_{jb}] &= \frac{1}{4}\delta_{ij}\varepsilon_{abc}T_c + \frac{1}{4}\delta_{ab}\varepsilon_{ijk}S_k.
\end{align*}
$$

(1)

As one can see, the tensor operators $E_{ia}$ are of three types: $E_i$ ($i = 1,2,3$) which form the subalgebra of SU(2)-spin, $E_a$ ($a = 1,2,3$) which form the subalgebra of SU(2)-isospin and $E_{ia}$ which act both in the spin and the isospin spaces. They are
related to $S_i$, $T_a$ and $G_{ia}$ ($i = 1, 2, 3; a = 1, 2, 3$) by

$$E_i = \frac{S_i}{\sqrt{2}}; \quad E_a = \frac{T_a}{\sqrt{2}}; \quad E_{ia} = \sqrt{2}G_{ia}. \quad (2)$$

The matrix elements of every $E_{ia}$ between states belonging to the representation $[N_c - 1, 1]$ are given by

$$\langle [N_c - 1, 1]I'I_3'S'S'_3|E_{ia}|[N_c - 1, 1]I'I_3S'S'_3 \rangle = \sqrt{C_{[N_c - 1, 1]}(SU(4))} \times \left(\begin{array}{ccc}
S & S' \\ S_3 & S'_3
\end{array}\right) \left(\begin{array}{c}
I \\ I'_3
\end{array}\right) \left(\begin{array}{cc}
I_a \\ I'_a
\end{array}\right) \left(\begin{array}{c}
3 \\ 3
\end{array}\right) \left(\begin{array}{c}
[N_c - 1, 1] \\ [N_c - 1, 1]
\end{array}\right), \quad (3)$$

where $C_{[N_c - 1, 1]}(SU(4)) = N_c(3N_c + 4)/8$ is the eigenvalue of the SU(4) Casimir operator for the representation $[N_c - 1, 1]$. The three factors in the second line are respectively an SU(2)-spin Clebsch-Gordan coefficient (CG), an SU(2)-isospin CG and an SU(4) isoscalar factor. The necessary isoscalar factors for the derivation of the matrix elements of $E_{ia}$ have been calculated by Hecht and Pang [10]. Here, the phases and notations have been adapted to our problem.

3. THE MASS OPERATOR

The mass operator $M$ is defined as a linear combination of independent operators $O_i$

$$M = \sum_i c_i O_i, \quad (4)$$

where the coefficients $c_i$ are reduced matrix elements that encode the QCD dynamics and are determined from a fit to the existing data. Here we are concerned with nonstrange baryons only. The building blocks of the operators $O_i$ are the SU(2Nf) generators $S_i$, $T_a$ and $G_{ia}$ and the SO(3) generators $\ell_i$. Their general form is

$$O_i = \frac{1}{N_{f}^{n-1}} O^{(k)}_i \cdot O^{(k)}_{SF}, \quad (5)$$

where $O^{(k)}_i$ is a $k$-rank tensor in SO(3) and $O^{(k)}_{SF}$ a $k$-rank tensor in SU(2)-spin, but invariant in SU($N_f$). Thus $O_i$ are rotational invariant. For the ground state one has $k = 0$. The excited states also require $k = 1$ and $k = 2$ terms.

The spin-flavor (SF) operators $O^{(k)}_{SF}$ are combinations of SU(2Nf) generators, the lower index $i$ in the left hand side of (5) representing a specific combination. Each $n$-body operator is multiplied by an explicit factor of $1/N_{c}^{n-1}$ resulting from the power counting rules. Some compensating $N_c$ factors may arise in the matrix elements when $O_i$ contains a coherent operator such as $G^{aa}$ or $T^a$. 
3.1. The symmetric core plus excited quark procedure

So far, for the baryons belonging to the $[70, \ell]$ multiplet the general practice was to consider that they consist of one distinguishable excited quark moving in the collective potential generated by $N_c - 1$ ground state quarks [7], the latter subsystem being called core. The wave function of the core is symmetric both in the orbital (O) and flavor-spin (FS) spaces, which makes the treatment of the core analogous to that of ground state baryons. This description is known as the Hartree picture.

To proceed, one defines separate SU($2N_f$) generators that act on the excited quark $s^i$, $t^a$ and $g^{ia}$ and $S^i_c$, $T^a_c$ and $G^{ia}_c$ that act on the core. Thus one has

$$S^i = s^i + S^i_c, \quad T^a = t^a + T^a_c, \quad G^{ia} = g^{ia} + G^{ia}_c. \quad (6)$$

As a consequence, the number of linearly independent operators $O_i$ increases tremendously and the number of coefficients $c_i$, to be determined, becomes much larger than the experimental data available. For example, for the $[70, 1]$ multiplet with $N_f = 2$ one has 12 linearly independent operators up to order $1/N_c$ included [3]. For example, there is one operator of order $N^1_c$: $N_c \mathbb{1}$, three operators of order $N^0_c$: $\ell \cdot s$, $1/N_c \ell \cdot t \cdot G$, $1/N_c \ell \cdot (2) \cdot g \cdot G$ and 8 operators of order $N^{-1}_c$: $1/N_c t \cdot T$, $1/N_c \ell \cdot S_c$, $1/N_c \ell \cdot g \cdot T_c$, $1/N_c S^2_c$, $1/N_c s \cdot S_c$, $1/N_c \ell \cdot (2) \cdot s \cdot S_c$, $1/N_c^2 \ell \cdot (2) \cdot t \cdot \{S_c, G_c\}$ and $1/N_c^2 \ell \cdot g \cdot \{S_c, G_c\}$. Then, in making the fit to the data, one faces the difficult situation of selecting among them the physically most dominant operators. We recall that there are only 7 nonstrange resonances belonging to this band. So one must select 7 out of 12 operators. Consequently, in selecting the operators one risks to make an arbitrary choice [3]. A much simpler method can be found, as shown below.

3.2. A simpler procedure

A simpler procedure is to avoid the splitting of the generators and the decoupling of the wave function and to consider instead only the global generators $S^i$, $T^a$ and $G^{ia}$ acting on the whole system of $N_c$ quarks. However, the approach is not free of difficulties as the derivation of the matrix elements of the operators is more involved for a mixed symmetric wave function. Presently, the study of strange baryons is not possible. In the case of three flavors, one needs the analogue of Eq. (3) containing the corresponding SU(6) isoscalar. These factors have not been calculated yet.

In addition to the fact that it uses an exact wave function, this approach implies only seven independent operators up to order $O(1/N_c)$ appearing in the mass operator: the order $N_c$ operator $N_c \mathbb{1}$, three operators of order $1$, $\ell \cdot s$, $1/N_c \ell \cdot (2) \cdot G \cdot G$ and $1/N_c \ell \cdot T \cdot G$ and three operators of order $O(1/N_c)$, namely $1/N_c S^2$, $1/N_c T^2$ and $1/N_c^2 S \cdot T \cdot G$. 

4. THE EXACT WAVE FUNCTION

The total wave function is the product of the orbital (O), the spin (S), the flavor (F) and the color (C) parts. The color part being always antisymmetric, in order to fulfill the Fermi statistics, the orbital-spin-flavor must be symmetric. As the mass operator does not involve color operators, the color being integrated out, we are concerned with the orbital-spin-flavor part only.

Here, as we are interested in the multiplet \([70,1^-]\), the orbital and the spin-flavor parts must have both the mixed symmetry \([N_c - 1, 1]\). In terms of inner products of the permutation group \(S_{N_c}\), the wave function takes the form

\[
|([N_c]1) = \frac{1}{\sqrt{N_c - 1}} \sum_Y |[N_c - 1, 1]Y\rangle_O |[N_c - 1, 1]Y\rangle_{FS},
\]

where \(Y\) is the corresponding Young tableau. Here we sum over the \(N_c - 1\) possible standard Young tableaux. The factor \(1/\sqrt{N_c - 1}\) represents the CG coefficient of \(S_{N_c}\) needed to construct a symmetric wave function \([N_c]\) from its mixed symmetric parts.

4.1. The decoupled wave function

In order to calculate matrix elements of the operators listed in the Sec. 3.1 one must decouple the wave function as well. Using the Racah’s factorization lemma, it is possible to decouple \(N_c\)th quark from the rest. The \(S_{N_c}\) CG coefficients can be factorized into an isoscalar factor times a CG coefficient of \(S_{N_c-1}\). In the following, we need to know the position of the \(N_c\)th quark inside a given Young tableau. In that purpose, one introduces the integer \(p\) which denotes the row where is the \(N_c\)th quark is located inside the Young tableau.

The exact \([70,1^-]\), but decoupled, wave function reads

\[
|\ell S J J_3; I I_3\rangle = \sum_{p, p', p'', \ell, \ell_q, \ell c, \ell q, m_c, m_q, m_{\ell c}, m_{\ell q}} a(p, \ell_c, \ell_q) \left( \begin{array}{c|c} \ell_c & \ell_q \\ m_c & m_q \end{array} \right) \left( \begin{array}{c|c} \ell & S \\ m_{\ell} & m_s \end{array} \right) \left( \begin{array}{c} J \\ J_3 \end{array} \right)
\times K([f']^p[f'']^p'|[N_c - 1, 1]p) \left( \begin{array}{c|c} S_c & 1 \frac{1}{2} \\ m_1 & m_2 \end{array} \right) \left( \begin{array}{c} I_c \\ i_1 \end{array} \right) \left( \begin{array}{c} 1 \frac{1}{2} \\ i_2 \end{array} \right) \left| I_3 \right|
\times |\ell_c m_c \rangle |S_c m_1 \rangle |I_c i_1 \rangle |\ell_q m_q \rangle |1/2 m_2 \rangle |1/2 i_2 \rangle,
\]

where \(\ell_c\) and \(\ell_q\) represent the angular momenta of the core and of the decoupled quark respectively and where \(a(p, \ell_c, \ell_q)\) are the one-body fractional parentage coefficients to decouple the \(N_c\)th quark from the rest in the orbital part. These are given by \[9\]

\[a(2, \ell_c = 0, \ell_q = 1) = \sqrt{\frac{N_c - 1}{N_c}},\]
The isoscalar factors $K([f']p'[f''p'']|\ell_q = 0)$ used in Eq. (8) are given in Appendix A (Tables 10, 11 and 12). The columns corresponding to $p = 1$ have been derived in Ref. [9]. If we compare Eq. (8) with Eq. (3.4) of Ref. [7] one can notice that in the latter only the terms with $p = 2$ have been taken into account. Furthermore, as the core was assumed to be in the ground state, the authors had considered $a(2, \ell_c = 1, \ell_q = 0) = 0$ and $a(2, \ell_c = 0, \ell_q = 1) = 1$. Thus the wave function of Ref. [7] breaks $S_{N_c}$ symmetry. As it represents only one part from the exact wave function we shall call it approximate or asymmetric.

Tables 1 and 2 show the matrix elements for some spin and the isospin operators respectively calculated with the exact and with the approximate wave function. One can notice that the analytic expressions are different. Consequently, one expects the $c_i$ coefficients determined from the fit to the data to be different if we use the exact or the approximate wave function.

**TABLE 1.** Matrix elements of the spin operators calculated with the approximate (Ref. [7]) and the exact, Eq. (8), wave functions, with $s$ and $S_c$ defined by Eq. (6).

|        | Approx. w.f. | Exact w.f. | Approx. w.f. | Exact w.f. |
|--------|--------------|------------|--------------|------------|
| $\langle s \cdot S_c \rangle$ | $-\frac{N_c+3}{4N_c}$ | $\frac{N_c+3}{4N_c}$ | $\frac{3(N_c-1)}{2N_c}$ | $\frac{3(N_c-1)}{2N_c}$ |
| $\langle S_c^2 \rangle$ | $\frac{3(N_c-1)}{2N_c}$ | $\frac{3(N_c-1)}{2N_c}$ | $\frac{3(N_c-5)}{2N_c}$ | $\frac{3(N_c-5)}{2N_c}$ |

**TABLE 2.** Matrix elements of the isospin operators calculated with the approximate (Ref. [7]) and the exact, Eq. (8), wave functions, with $t$ and $T_c$ defined by Eq. (6).

|        | Approx. w.f. | Exact w.f. | Approx. w.f. | Exact w.f. |
|--------|--------------|------------|--------------|------------|
| $\langle t \cdot T_c \rangle$ | $-\frac{N_c+3}{4N_c}$ | $\frac{N_c+3}{4N_c}$ | $\frac{3(N_c-1)}{2N_c}$ | $\frac{3(N_c-1)}{2N_c}$ |
| $\langle T_c^2 \rangle$ | $\frac{3(N_c-5)}{2N_c}$ | $\frac{3(N_c-5)}{2N_c}$ | $\frac{3(N_c-1)}{2N_c}$ | $\frac{3(N_c-1)}{2N_c}$ |

4.2. The global wave function

As already mentioned above, one can write the exact $[70, 1^-]$ states without decoupling them into a core and an excited quark. If there is no decoupling, there is no need to specify $Y$, the matrix elements being identical for all $Y$'s, due to
Weyl’s duality between a linear group and a symmetric group in a given tensor space. Then the explicit form of a wave function of total angular momentum \( \vec{J} = \vec{\ell} + \vec{S} \) and isospin \( I \) is

\[
|\ell S J J_3; I I_3\rangle = \sum_{m_\ell, m_s} \left( \begin{array}{cc} \ell & S \\ m_\ell & m_s \end{array} \right) [N_c - 1, 1] \ell m_\ell) [N_c - 1, 1] S m_s I I_3),
\]

(12)
each term containing an SU(2) CG coefficient, an orbital part \([N_c - 1, 1] \ell m_\ell)\) and a spin-flavor part \([N_c - 1, 1] S m_s I I_3)\).

5. RESULTS

Here we present the results obtained from different fits to the experimental data. In the fits, the seven nonstrange resonances have been taken into account:

- \( ^2N_{1/2}(1538 \pm 18) \),
- \( ^4N_{1/2}(1660 \pm 20) \),
- \( ^2N_{3/2}(1523 \pm 8) \),
- \( ^4N_{3/2}(1700 \pm 50) \),
- \( ^4N_{5/2}(1678 \pm 8) \),
- \( ^2\Delta_{1/2}(1645 \pm 30) \) and
- \( ^2\Delta_{3/2}(1720 \pm 50) \).

In a first stage, we describe the fits obtained when we use the exact decoupled wave function. Afterwards, the results obtained with the global wave function are presented.

In each case, we follow the spirit of the Hartree picture which leads to a one-body spin-orbit operator \( \ell \cdot s \). Its matrix elements are naturally of order \( N_c^0 \).

5.1. With the decoupled wave function

Tables 3–6 show the four different fits considered. Each time, the results obtained with the exact decoupled wave function are compared to the ones obtained with the approximate wave function.

In Table 3 we decouple the spin and the isospin operators. The operator \( 1/N_c T_c \cdot T_c \) is not present because its matrix elements are identical to those \( 1/N_c S_c \cdot S_c \) for the approximate wave function (see Tables 1 and 2). This is apparently a practical advantage in the decoupling scheme but it has considerable physical disadvantages. One can notice that even if the \( \chi^2_{\text{dof}} \) is satisfactory, the fit is very bad. Indeed, the value of \( c_1 \) is under-evaluated with respect to the commonly found value of around 500 MeV and the values of \( c_3 \) and \( c_4 \) are exceedingly large and of opposite signs, which suggest some compensation.

The fits presented in Tables 4 and 5 seem better. Here the linear combinations \( 2s \cdot S_c + S_c \cdot S_c + 3/4 = S^2 \) or \( 2t \cdot T_c + T_c \cdot T_c + 3/4 = T^2 \) have been introduced. The coefficient \( c_1 \) has recovered its common value and the coefficients \( c_3' \) or \( c_5' \) have reasonable sizes, being about 70 MeV smaller for the exact wave function than for the approximate one. The quark-core operators \( 1/N_c s \cdot S_c \) or \( 1/N_c t \cdot T_c \) are still

1 see Ref. [11], Sec 4.5.
problematic because the value of their respective coefficients are too high, of order 500 MeV.

The last fit shown in Table 6 correct these problems. All the coefficients have their natural sizes. This shows the necessity to consider the isospin-isospin operator on the same footing as the spin-spin operator. The values obtained with the exact wave function and the approximate one are identical in this case because the matrix elements of the operators considered are the same for the two wave functions. By construction, in both cases they are eigenfunctions of the total spin and isospin operators.

**TABLE 3.** List of operators $O_i$ and coefficients $c_i$ obtained in the numerical fit to the 7 known experimental masses of the lowest negative parity resonances (see text). For the operators $O_3$, $O_4$ and $O_5$ we use the matrix elements from Tables 1 and 2.

| $O_i$                | $c_i$(MeV) with approx. w.f. | $c_i$(MeV) with w.f. Eq. (8) |
|----------------------|------------------------------|-------------------------------|
| $O_1 = N_c \mathbb{1}$ | 211 ± 23                     | 299 ± 20                      |
| $O_2 = \ell^i s^i$   | 3 ± 15                        | 3 ± 15                        |
| $O_3 = \frac{1}{N_c} s^i S_c^i$ | −1486 ± 141                 | −1096 ± 125                   |
| $O_4 = \frac{1}{N_c} S_c^i S_c^i$ | 1182 ± 74                   | 1545 ± 122                    |
| $O_5 = \frac{1}{N_c} t^a T^a_c$ | −1508 ± 149                  | 417 ± 79                      |
| $\chi^2_{\text{dof}}$ | 1.56                         | 1.56                          |

**TABLE 4.** Same as Table 3 but for $O'_3$, which combines $O_3$ and $O_4$ instead of using them separately.

| $O_i$                | $c_i$(MeV) with approx. w.f. | $c_i$(MeV) with w.f. Eq. (8) |
|----------------------|------------------------------|-------------------------------|
| $O_1 = N_c \mathbb{1}$ | 513 ± 4                      | 519 ± 5                       |
| $O_2 = \ell^i s^i$   | 3 ± 15                        | 3 ± 15                        |
| $O'_3 = \frac{1}{N_c} (2s^i S_c^i + S_c^i S_c^i + \frac{3}{2})$ | 219 ± 19                     | 150 ± 11                      |
| $O_5 = \frac{1}{N_c} t^a T^a_c$ | 417 ± 80                     | 417 ± 80                      |
| $\chi^2_{\text{dof}}$ | 1.04                         | 1.04                          |

**TABLE 5.** Same as Table 4 but combining isospin operators instead of spin operators.

| $O_i$                | $c_i$(MeV) with approx. w.f. | $c_i$(MeV) with w.f. Eq. (8) |
|----------------------|------------------------------|-------------------------------|
| $O_1 = N_c \mathbb{1}$ | 516 ± 3                      | 522 ± 3                       |
| $O_2 = \ell^i s^i$   | 3 ± 15                        | 3 ± 15                        |
| $O_3 = \frac{1}{N_c} s^i S_c^i$ | 450 ± 33                     | 450 ± 33                      |
| $O'_5 = \frac{1}{N_c} (2t^a T^a_c + T^a_c T^a_c + \frac{3}{2})$ | 214 ± 28                     | 139 ± 27                      |
| $\chi^2_{\text{dof}}$ | 1.04                         | 1.04                          |
TABLE 6. Fit with global operators proportional to the SU(2)-spin and SU(2)-isospin Casimir operators acting on the whole system (see text).

| $O_i$ | $c_i$(MeV) with approx. w.f. | $c_i$(MeV) with w.f. Eq. (8) |
|-------|-----------------------------|-------------------------------|
| $O_1 = N_c \mathbb{1}$ | $484 \pm 4$ | $484 \pm 4$ |
| $O_2 = \ell s^i$ | $3 \pm 15$ | $3 \pm 15$ |
| $O_3' = \frac{1}{N_c} (2s^i S_c^i + S_c^i S_c^i + \frac{1}{3})$ | $150 \pm 11$ | $150 \pm 11$ |
| $O_5' = \frac{1}{N_c} (2t^a T_a^c + T_a^c T_a^c + \frac{1}{3})$ | $139 \pm 27$ | $139 \pm 27$ |
| $\chi^2_{\text{dof}}$ | $1.04$ | $1.04$ |

5.2. With the global wave function

With the simplified procedure described in Sec. 3.2 we can analyse the role of every of the seven independent operators introduced there. Table 7 shows six different fits to the experiment. The operators $O_5$, $O_6$ and $O_7$ are normalized to allow their coefficients $c_i$ to have a natural size. As already emphasized, the Fits 1–4 indicate that the coefficients of $O_3$ and $O_4$ have similar values.

The partial contributions and the theoretical masses obtained from the Fits 1 and 6 are presented in Tables 8 and 9 respectively. From Table 8 one can notice that the isospin-isospin operator in $\Delta$ masses plays a comparable role to the spin-spin operator in $N^*$ resonances. This was impossible to observe in the symmetric core + excited quark procedure where the isospin-isospin operators were always ignored, for reasons explained above. From Table 9 one can see that the operator $O_7$, never included before, is dominant in all resonances except $^2N_J$. This is a new finding, to be algebraically understood.

The operators $O_5$ and $O_6$ do not seem to play an important role because, in addition to the fact that their coefficients are small and have an error bar comparable to their central values, their removal from the fit does not deteriorate it too badly. This justifies the previous choice presented in Section 5.1 where $O_5$ and $O_6$ were neglected. Of course, Fit 4 is identical to the one shown in Table 6.

Table 7 does not include a fit with $O_3$, $O_4$ and $O_7$ together. In our calculations we found that the simultaneous presence of $O_3$, $O_4$ and $O_7$ leads to a $\chi^2_{\text{dof}} \approx 2$. In this case the coefficients $c_3$ and $c_4$ become a bit higher, of the order 270 MeV and $c_7$ becomes negative suggesting a possible compensation with the contributions of $O_3$ and $O_4$. This suggests that, by construction, $O_7$ contains part of the contribution of the spin-spin and isospin-isospin interactions. As mentioned above, the role of $O_7$ needs more investigation.

6. CONCLUSIONS

In principle, both the core + excited quark (Sec. 3.1) or the global (Sec. 3.2) procedures are legitimate as long as they are combined with adequately constructed wave functions. The core + excited quark procedure was the first attempt to study
TABLE 7. List of operators and the coefficients resulting from numerical fits using the global wave function. The values of $c_i$ are indicated under the headings Fit n, in each case.

| Operator                      | Fit 1 (MeV) | Fit 2 (MeV) | Fit 3 (Mev) | Fit 4 (MeV) | Fit 5 (MeV) | Fit 6 (MeV) |
|-------------------------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $O_1 = N_c \mathbb{1}$       | 481 ± 5     | 482 ± 5     | 484 ± 4     | 484 ± 4     | 498 ± 3     | 495 ± 3     |
| $O_2 = \ell^i s^i$           | –31 ± 26    | –20 ± 23    | –12 ± 20    | 3 ± 15      | 38 ± 34     | –30 ± 25    |
| $O_3 = \frac{1}{N_c} S^i S^i$ | 161 ± 16    | 149 ± 11    | 163 ± 16    | 150 ± 11    | 156 ± 16    |             |
| $O_4 = \frac{1}{N_c} T^a T^a$ | 169 ± 36    | 170 ± 36    | 141 ± 27    | 139 ± 27    |             |             |
| $O_5 = \frac{1}{N_c} \ell^{(2)ij} G^{ia} G^{ja}$ | –29 ± 31    | –34 ± 30    | –34 ± 31    | –32 ± 29    |             |             |
| $O_6 = \frac{3}{N_c} \ell^i T^a G^{ia}$ | 32 ± 26     | 35 ± 26     | –67 ± 30    | 28 ± 20     |             |             |
| $O_7 = \frac{3}{N_c^2} S^i T^a G^{ia}$ |             |             |             |             | 649 ± 61    |             |

$\chi^2_{\text{dof}}$ | 0.43 | 0.68 | 0.94 | 1.04 | 11.5 | 0.24 |
TABLE 8. The partial contribution and the total mass (MeV) predicted by the $1/N_c$ expansion using Fit 1 and the global wave function. The last two columns give the empirically known masses, name and status.

|       | Part. contrib. (MeV) | Total (MeV) | Exp. (MeV) | Name, status |
|-------|----------------------|-------------|------------|--------------|
| $^2N_\uparrow$ | $c_1O_1$ 1444 $c_2O_2$ 10 $c_3O_3$ 40 $c_4O_4$ 42 $c_5O_5$ 0 $c_6O_6$ -8 | 1529 ± 11 | 1538 ± 18 | $S_{11}(1535)^{****}$ |
| $^4N_\uparrow$ | $c_1O_1$ 1444 $c_2O_2$ 26 $c_3O_3$ 201 $c_4O_4$ 42 $c_5O_5$ -31 $c_6O_6$ -20 | 1663 ± 20 | 1660 ± 20 | $S_{11}(1650)^{****}$ |
| $^2N_\downarrow$ | $c_1O_1$ 1444 $c_2O_2$ -5 $c_3O_3$ 40 $c_4O_4$ 42 $c_5O_5$ 0 $c_6O_6$ 4 | 1525 ± 8 | 1523 ± 8 | $D_{13}(1520)^{****}$ |
| $^4N_\downarrow$ | $c_1O_1$ 1444 $c_2O_2$ 10 $c_3O_3$ 201 $c_4O_4$ 42 $c_5O_5$ 25 $c_6O_6$ -8 | 1714 ± 45 | 1700 ± 50 | $D_{13}(1700)^{****}$ |
| $^4N_\downarrow$ | $c_1O_1$ 1444 $c_2O_2$ -16 $c_3O_3$ 201 $c_4O_4$ 42 $c_5O_5$ -6 $c_6O_6$ 12 | 1677 ± 8 | 1678 ± 8 | $D_{15}(1675)^{****}$ |
| $^2\Delta_\downarrow$ | $c_1O_1$ 1444 $c_2O_2$ -10 $c_3O_3$ 40 $c_4O_4$ 211 $c_5O_5$ 0 $c_6O_6$ -40 | 1645 ± 30 | 1645 ± 30 | $S_{31}(1620)^{****}$ |
| $^2\Delta_\downarrow$ | $c_1O_1$ 1444 $c_2O_2$ 5 $c_3O_3$ 40 $c_4O_4$ 211 $c_5O_5$ 0 $c_6O_6$ 20 | 1720 ± 50 | 1720 ± 50 | $D_{33}(1700)^{****}$ |
We have shown that the separation core + quark procedure fails to emphasize the role of the isospin operator. This is due to the inherent structure of the asymmetric ground state core + excited quark wave function [7] which leads to equal matrix elements for \(S^2c\) and \(T^2c\). Then the remaining part of the isospin interaction \(1/N_c t \cdot T c\) becomes exceedingly large if included in the fit (Table 3) and it is not surprising that in all previous studies (see Ref. [9] for a review) it has been totally ignored.

In conclusion, the simple procedure we advocate here brings much more physical insight into the study of the nonstrange \([70, 1−]\) baryons in the \(1/N_c\) expansion. It is an urgent need to determine the isoscalar factors of SU(6) for mixed symmetric representations \([N_c − 1, 1]\) in order to extend Eq. (3) to SU(6) and apply it to orbit and the smallness of the tensor interaction.

### TABLE 9. The partial contribution and the total mass (MeV) predicted by the \(1/N_c\) expansion using Fit 6 and the global wave function. The last two columns give the empirically known masses, name and status.

| Part. contrib. (MeV) | Total (MeV) | Exp. (MeV) | Name, status |
|----------------------|-------------|------------|--------------|
|                      | \(c_1 O_1\) | \(c_2 O_2\) | \(c_3 O_5\) | \(c_6 O_6\) | \(c_7 O_7\) |
| \(2\gamma_4\)       | 1486        | 10         | 0           | -7          | 41          | 1529 ± 11   | 1538 ± 18   | \(S_{11}(1535)^{****}\) |
| \(1\gamma_2\)       | 1486        | 25         | -33         | -18         | 203         | 1663 ± 20   | 1660 ± 20   | \(S_{11}(1650)^{****}\) |
| \(2\gamma_4\)       | 1486        | -5         | 0           | 4           | 41          | 1525 ± 7    | 1523 ± 8    | \(D_{13}(1520)^{****}\) |
| \(1\gamma_4\)       | 1486        | 10         | 26          | -7          | 203         | 1718 ± 41   | 1700 ± 50   | \(D_{13}(1700)^{****}\) |
| \(1\gamma_4\)       | 1486        | -15        | 7           | 11          | 203         | 1677 ± 8    | 1678 ± 8    | \(D_{15}(1675)^{****}\) |
| \(2\Delta_4\)       | 1486        | -10        | 0           | -35         | 203         | 1643 ± 29   | 1645 ± 30   | \(S_{31}(1620)^{****}\) |
| \(2\Delta_3\)       | 1486        | 5          | 0           | 18          | 203         | 1711 ± 24   | 1720 ± 50   | \(D_{33}(1700)^{****}\) |

In all cases, the mass is given with its experimental uncertainty.
strange baryons.

A. ISOSCALAR FACTORS

Here we reproduce the isoscalar factors needed to construct the exact decoupled wave function (see Eq. (8)). Detailed information can be found in Refs. [9, 11, 12]

| TABLE 10. | Isoscalar factors $K([f']p'[f''][f]|p)$ for $S = I = 1/2$, corresponding to $^5\!8$ when $N_c = 3$. The second column gives results for $p = 1$ and the third for $p = 2$. |
|------------------|------------------|------------------|
| $[f']p'[f''][f'']$ | $[N_c - 1, 1]1$ | $[N_c - 1, 1]2$ |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]1\ [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]1$ | 0 | $-\sqrt{\frac{3(N_c - 1)}{4N_c}}$ |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]2\ [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]2$ | $\sqrt{\frac{N_c - 3}{2(N_c - 2)}}$ | $\sqrt{\frac{N_c + 3}{4N_c}}$ |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]2\ [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]1$ | $-\frac{1}{2}\sqrt{\frac{N_c - 1}{N_c - 2}}$ | 0 |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]1\ [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]2$ | $-\frac{1}{2}\sqrt{\frac{N_c - 3}{N_c - 2}}$ | 0 |

| TABLE 11. | Isoscalar factors $K([f']p'[f''][f'']|p)$ for $S = I = 1/2$, corresponding to $^4\!8$ when $N_c = 3$. The second column gives results for $p = 1$ and the third for $p = 2$. |
|------------------|------------------|------------------|
| $[f']p'[f''][f'']$ | $[N_c - 1, 1]1$ | $[N_c - 1, 1]2$ |
| $[\frac{N_c + 3}{2}, \frac{N_c - 3}{2}]1\ [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]1$ | $\frac{1}{2}\sqrt{\frac{(N_c - 1)(N_c + 3)}{N_c(N_c - 2)}}$ | 0 |
| $[\frac{N_c + 3}{2}, \frac{N_c - 3}{2}]2\ [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]2$ | $\frac{1}{2}\sqrt{\frac{3(N_c - 1)(N_c - 3)}{2N_c(N_c - 2)}}$ | 0 |
| $[\frac{N_c + 3}{2}, \frac{N_c - 3}{2}]1\ [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]2$ | $\frac{1}{2}\sqrt{\frac{(N_c - 3)(N_c + 3)}{2N_c(N_c - 2)}}$ | 1 |
| $[\frac{N_c + 3}{2}, \frac{N_c - 3}{2}]2\ [\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]1$ | 0 | 0 |

| TABLE 12. | Isoscalar factors $K([f']p'[f''][f'']|p)$ for $S = 1/2$, $I = 3/2$, corresponding to $^2\!10$ when $N_c = 3$. The second column gives results for $p = 1$ and the third for $p = 2$. |
|------------------|------------------|------------------|
| $[f']p'[f''][f'']$ | $[N_c - 1, 1]1$ | $[N_c - 1, 1]2$ |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]1\ [\frac{N_c + 3}{2}, \frac{N_c - 3}{2}]1$ | $\frac{1}{2}\sqrt{\frac{(N_c - 1)(N_c + 3)}{N_c(N_c - 2)}}$ | 0 |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]2\ [\frac{N_c + 3}{2}, \frac{N_c - 3}{2}]2$ | $\frac{1}{2}\sqrt{\frac{3(N_c - 1)(N_c - 3)}{2N_c(N_c - 2)}}$ | 0 |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]2\ [\frac{N_c + 3}{2}, \frac{N_c - 3}{2}]1$ | $\frac{1}{2}\sqrt{\frac{(N_c - 3)(N_c + 3)}{2N_c(N_c - 2)}}$ | 1 |
| $[\frac{N_c + 1}{2}, \frac{N_c - 1}{2}]1\ [\frac{N_c + 3}{2}, \frac{N_c - 3}{2}]2$ | 0 | 0 |
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