The quantum phase problem: steps toward a resolution

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Defining the observable $\phi$ canonically conjugate to the number observable $N$ has long been an open problem in quantum theory. The problem stems from the fact that $N$ is bounded from below. In a previous work we have shown how to define the absolute phase observable $\Phi \equiv |\phi|$ by suitably restricting the Hilbert space of $x$ and $p$ like variables. Here we show that also from the classical point of view, there is no rigorous definition for the phase even though it’s absolute value is well defined.

I. INTRODUCTION

Defining the observable $\phi$ which represents the phase of a monochromatic wave has long been an open problem in quantum theory \cite{footnote}. In a previous work \cite{2} the problem has been investigated and solved partly by the construction of the absolute value $\Phi = |\phi|$ of the phase observable which is determined modulus $\pi$ (and not $2\pi$). As Moshinsky and Seligman \cite{4} have shown many years ago, the classical canonical transformations to action and angle variables for the harmonic oscillator (and some other systems) turn out to be non-bijective (not one-to-one onto). Hence, it is not surprising that it is possible to construct only the absolute phase observable and not the phase observable itself. The main purpose of this paper is to extend the idea and to show that even in the classical point of view it is only the absolute value of the phase which is well determined. Our technique looks similar to the one given by Newton \cite{3}, however, in his approach he doubled the Hilbert space of a simple harmonic oscillator.
Problems in the definition of the quantum phase were first addressed by Fritz London in 1926. One year latter Dirac introduced an operator solution which was proved to be incomplete by Susskind and Glogower (for history and measurements see Nieto). Since then a series of workers have made many attempts to resolve the problem (for reviews see and for recent attempts see). However, some of the quantum phase theories do not pass the Barnett and Pegg “acid-test”: not all the number states represent states of random phase. Others suffer from distressing mathematical difficulties. A few of the theories are even incomplete, etc. We shall mention that there is a solution if one generalize the formal description of measurement to include the so-called POM or POVM observables. In the previous work it has been shown that it is possible to define rigrously the absolute value of the phase operator without generalizing the formal description of measurement. As we shall see in this work, the source of all difficulties lies in the fact that the domain of a well defined phase observable must be restricted to half of the domain which is used in most of the quantum phase theories. This is a direct consequence of the fact that the number operator $N$ is bounded from below.

Until the last years, there were minimal experimental works, compared to the huge quantity of theoretical effort. We shall mention here the work of Noh, Fougères and Mandel. By analyzing classical phase measurement configurations they define the “operational phase operators”. There measurements are found to agree very well with the theoretical predictions of these operators and disagree with the predictions of some of the theories mentioned in. There are many other experiments which have been performed in recent years, however, in this paper we shall focus on the theoretical aspect of the quantum phase.

The phase observable should be canonically conjugate to the number operator, and thus represent also the time operator of a simple harmonic oscillator. In classical mechanics it is possible to define the canonical conjugate to the Hamiltonian of an harmonic oscillator by performing a canonical transformation on the standard coordinates $q$ and $p$. In this paper, a careful analysis of such transformation shows that the notion of a phase which is canonically conjugate to the Hamiltonian is also problematic at the classical level. Hence,
the first step towards a definition of a quantum phase observable should be the investigation of the problems as it appear in the classical picture (section II).

The paper is organized as follows. In section II it is shown that the classical phase is determined up to modulus $\pi$ by the standard coordinates $q$ and $p$ of a linear harmonic oscillator. In section III we discuss how problems appear in the construction of a quantum phase and the notion of canonical commutation relation. In section IV we construct the absolute quantum phase and examine its properties. In section V we discuss the quantum phase theories in a finite dimensional space. Finally, in section VI, we present our summary and conclusions.

II. PHASE IN CLASSICAL THEORY

In the Lagrangian formalism, one dimensional classical system can be described by one general coordinate $q(t)$ and its time derivative. The canonical conjugate to $q(t)$ is defined by

$$p(t) = \frac{\partial L}{\partial \dot{q}(t)}$$

where $L = L(q(t), \dot{q}(t), t)$ is the Lagrangian describing the system. Hence, $p(t)$ can be written as a function of $q(t), \dot{q}(t)$ and $t$. In the Hamiltonian formulation, on the other hand, two variables $q(t)$ and $p(t)$ are said to be canonically conjugate if there exists some function (Hamiltonian) $H(q, p, t)$ such that the equations of motion are given in the form

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}.$$  (2)

As it will be shown below, in the case of harmonic oscillator there exist two variables $Q$ (the phase) and $P$ (proportional to the Hamiltonian) which satisfy Eq. (2) even though there is no Lagrangian $L(Q, \dot{Q}, t)$ such that Eq. (1) is satisfied. Hence, $Q$ and $P$ are canonically conjugate according to the Hamiltonian formalism but not according to the Lagrangian formalism.
The Hamiltonian of a simple harmonic oscillator in one dimension can be written as

\[ H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2), \]  

where \( \omega \) is the frequency of oscillations. We shall consider here a canonical transformation \( q, p \rightarrow Q, P \) such that the Hamiltonian in the new coordinates can be written as \( H = \omega P \). Thus, \( P \) represents the classical analog to the number operator. The generating function \( F_1(q, Q) \) of such a transformation is given by

\[ F_1 = \frac{m\omega q^2}{2} \cot Q. \]  

However, there are two transformations derived from \( F_1 \) (a generating function may at times be double-valued), namely

\[
\begin{align*}
  p &= \sqrt{2m\omega P} \cos Q \\
  q &= \sqrt{\frac{2P}{m\omega}} \sin Q
\end{align*}
\quad \text{and} \quad
\begin{align*}
  p &= -\sqrt{2m\omega P} \cos Q \\
  q &= -\sqrt{\frac{2P}{m\omega}} \sin Q.
\end{align*}
\]  

The inverse transformations

\[ P = \frac{1}{2m\omega}(p^2 + m^2\omega^2 q^2) \quad \text{and} \quad Q = \cot^{-1} \left( \frac{p}{m\omega q} \right), \]  

shows that \( q \) and \( p \) determine \( Q \) modulus \( \pi \) and not \( 2\pi \) because a shift in \( Q \) by \( \pi \) corresponds to going from one transformation to the other (see Eq. (5)). Hence, in order to obtain the classical phase which is determined modulus \( 2\pi \), we would have to combine (somewhat artificially) the two transformations in Eq. (5). As we shall see, such a combination has no quantum analog (and thus the definition of a quantum phase is problematic). Therefore, we expect the quantum counterpart of \( Q \) to be determined by the position and momentum operators up to modulus \( \pi \). This explains why it is possible to define only the absolute value of the phase observable \( [2] \) which is restricted to the domain \([0, \pi]\) (assuming the phase itself is defined in the interval \((-\pi, \pi]\)).

Notice that it is impossible to express \( P \) as a function of \( Q \) and \( \dot{Q} \) because \( \dot{Q} = \frac{\partial H}{\partial P} = \omega \). Thus, it is impossible to construct a Lagrangian \( L(Q, \dot{Q}, t) \) such that \( P = \frac{\partial L}{\partial \dot{Q}} \). This shows that \( P \) and \( Q \) can be considered as canonically conjugated variables only in the Hamiltonian.
formalism. However, even then, if \( Q \) is restricted to the domain \((-\pi, \pi]\) (by taking \( Q \) mod 2\( \pi \)) then it is no longer the canonical conjugate of \( P \); that is: if \( Q \) satisfy the Hamilton equation \( \dot{Q} = \frac{\partial H}{\partial P} = \omega \), then \( Q = \omega t + \phi_0 \) can not be restricted!

How is the fact that the phase is determined modulus \( \pi \) is consistent with the phase which is determined by the action-angle method? The so called action variable \( J \) is defined as \([14]\)

\[
J = \frac{1}{2\pi} \int p \, dq \tag{7}
\]

where the integration is to be carried over a complete period. Now, the momentum of a linear harmonic oscillator is given by

\[
p = \pm \sqrt{2m\alpha - m^2\omega^2q^2}, \tag{8}
\]

where \( \alpha \equiv H \) represents the constant energy of the oscillator. Hence, substituting the expression for \( p \) in Eq. (7) yields \([14]\)

\[
J = \pm \frac{H}{\omega}, \tag{9}
\]

or solving for the Hamiltonian,

\[
H = \pm \omega J. \tag{10}
\]

Hence, \( J \) itself cannot represent the classical analog for the number operator since it is not bounded from below.

As we can see from Eq. (10) it is the absolute value of \( J \) which is proportional to the Hamiltonian. The angle variable \( w \) is therefore

\[
w = \omega \text{ sign}(J) \, t + \beta \tag{11}
\]

which is related to \( q \) and \( p \) by

\[
q = \sqrt{\frac{2|J|}{m\omega}} \sin w, \quad p = \sqrt{2m\omega|J|} \sin w. \tag{12}
\]

However, in this case the angle variable \( w \) is determined up to modulus 2\( \pi \) because it depends also on the sign of \( J \). This does not conflict with the fact \( Q \) is determined by \( q \) and \( p \) up to modulus \( \pi \) because \( w \) is the canonical conjugate of \( J \) and not of \(|J| = H/\omega\).
III. PROBLEMS IN THE DEFINITION OF A QUANTUM PHASE

In quantum mechanics two observers $q$ and $p$ are said to be canonically conjugate if they satisfy the commutation relation $[q, p] = i\hbar$. Usually, this definition is equivalent to the definition given in the Lagrange formalism

$$p = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}}. \tag{13}$$

As we have seen in the previous section it is impossible to construct a Lagrangian for the harmonic oscillator in terms of the phase observable $\phi$ (the analog of $Q$) and its time derivative. Hence, we may use only the Hamiltonian formalism in order to define the phase observable.

In section II it has also been shown that if $Q$ is restricted to some domain, then $Q$ and $P$ do not satisfy the Hamilton equations. This phenomenon is manifested in the quantum picture by the fact that $[N, \phi] \neq i$ for a restricted phase operator. However, if $N$ and $\phi$ do not satisfy the canonical commutation relation, then how can the phase observable be defined, and in what sense can $N$ and $\phi$ be regarded as canonically conjugate observables? In order to answer these questions we shall first examine why the phase operator defined many years ago by Dirac cannot be Hermitian.

Dirac proposed to decompose the annihilation operator in the form

$$a = \exp(-i\phi)N^{1/2}. \tag{14}$$

However, such a decomposition is problematic since the operator

$$E \equiv (N + 1)^{-1/2}a = \sum_{n=0}^{\infty} |n\rangle\langle n + 1|, \tag{15}$$

(which represents $\exp(-i\phi)$) is not unitary (see Susskind and Glogower [8]). Eq. (14) seems reasonable in view of the classical analog

$$a \equiv \frac{1}{\sqrt{2m\omega}}p - i\sqrt{\frac{m\omega}{2}}q = \sqrt{P} \exp(-iQ), \tag{16}$$
where we have used Eq. (14). However, the above decomposition for $a$ is incomplete since $Q$ is determined by $q$ and $p$ modulus $\pi$ and $Q \in [0, \pi]$ (see section II). That is, the half domain of $Q$ makes it impossible to cover the domain of $a$. Hence, we have to find a different relation between $a$ and $Q$. For example, we can use the relation

$$p = \sqrt{2m\omega P} \cos Q \Rightarrow \cos Q = \frac{1}{2\sqrt{P}}(a + a^*)$$

(17)

since $\cos Q$ can have all values in the domain $[-1, 1]$ ($Q \in [0, \pi]$). The quantum analog of Eq. (17) is given by the operator of Carruthers and Nieto [13]:

$$C \equiv \frac{1}{2}(E + E^\dagger).$$

(18)

However, as we have shown previously [2], this operator does not exactly represent the “cosine” of the quantum phase and a small correction is needed. Now, in the domain $Q \in [0, \pi]$, $\sin Q$ is positive and thus we can use the relation

$$|q| = \sqrt{\frac{2P}{m\omega}} \sin Q \Rightarrow \sin Q = \left|\frac{1}{2i\sqrt{P}}(a^* - a)\right|$$

(19)

since $\sin Q$ can only have values in the positive domain $[0, 1]$. The quantum analog of Eq. (19) is given by the absolute value of the operator of Carruthers and Nieto [15]:

$$|S| \equiv \frac{1}{2i}(E^\dagger - E).$$

(20)

Notice that it is $|S|$ which is the quantum analog of $\sin Q$. However, as we have shown previously [2] (and as we reiterate in the next section), a small correction to $|S|$ is needed.

This explains (also classically) why in the theory of Carruthers and Nieto the phase observable which is determined from $C$ is different from the one determined from $S$, and it is equal (up to a small correction) to the one determined from $|S|$. It is clear that $C$ and $|S|$ cannot represent the exact cosine and sine of the phase observable, respectively, since for example $C^2 + S^2 \neq 1$. A new definition is needed. In the next section we shall obtain the correct $\sin \Phi$ and $\cos \Phi$ ($\Phi \equiv |\pi|$).

Two main problems appear in the construction of a phase operator. The first one connected with the fact that the phase operator is an angle operator. Thus it is restricted to
a finite interval which is a problem since, for example, the matrix elements of $[N, \phi]$ in the
number state basis $|n\rangle$,

$$\langle n|[N, \phi]|n'\rangle = (n - n')\langle n|\phi|n'\rangle$$  \hspace{1cm} (21)

vanish for $n = n'$ because $|\langle n|\Phi|n\rangle|$ is bounded. This implies that $[N, \phi] \neq i$. Hence, only
unrestricted operators can satisfy the standard canonical commutation relations. Recall that
classically, the pair $Q \bmod \pi$ and $P$ do not satisfy the Hamilton equations (even though $Q$
itself and $P$ do).

The solution to this problem is simple since only a slight change in the commutation
relations is needed. For example, in the case of a plane rotator, Judge and Lewis [16]
showed many years ago that the angular momentum $J_z$ component and its associated angle
$\Theta$ satisfy the commutation relation

$$[J_z, \Theta] = i\hbar (1 - 2\pi\delta(\Theta - \pi)), \quad -\pi < \Theta \leq \pi,$$  \hspace{1cm} (22)

where $\Theta$ can be expressed as a $2\pi$-periodic function of an unrestricted angle operator [1].

However, Eq. (22) can not be used for the case of number and phase operators. In
addition to the fact that $\phi$ is restricted, a second problem arises because the number operator
$N$ is bounded from below (not so $J_z$). Therefore, a commutation relation like (22) for $N$
and $\phi$ does not hold [4]. For example, by (22) the matrix elements of $[N, \phi]$ taken in the
phase basis $|\phi\rangle$ (assuming $\phi_0 < \phi \leq \phi_0 + \Delta$), would be

$$\langle \phi|[N, \Phi]|\phi'\rangle = (\phi' - \phi)\langle \phi|N|\phi'\rangle = i\delta(\phi - \phi'),$$  \hspace{1cm} (23)

which imply that

$$\langle \phi|N|\phi'\rangle = -\frac{i\delta(\phi - \phi')}{\phi - \phi'} = \frac{i}{d\phi}\delta(\phi - \phi').$$  \hspace{1cm} (24)

Defining a state $|\psi\rangle = \int_{\phi_0}^{\phi_0+\Delta} d\phi \; \psi(\phi)|\phi\rangle$ in the basis of the phase states ($\psi(\phi)$ is a complex
function of $\phi$), we find

$$\langle \psi|N|\psi\rangle = \int_{\phi_0}^{\phi_0+\Delta} d\phi \int_{\phi_0}^{\phi_0+\Delta} d\phi' \; \psi^*(\phi)\psi(\phi')\langle \phi|N|\phi'\rangle = \int_{\phi_0}^{\phi_0+\Delta} d\phi \; \psi^*(\phi) \left(\frac{d}{d\phi}\right) \psi(\phi).$$  \hspace{1cm} (25)
Thus, for the state \( \psi(\phi) = \frac{1}{\Delta} \exp(-i n \phi) \), desired result \( \langle \psi | N | \psi \rangle = n \), but for \( \psi(\phi) = \frac{1}{\Delta} \exp(+i n \phi) \) Eq. (25) implies a negative average \( \langle \psi | N | \psi \rangle = -n \). Thus, Eq. (22) cannot hold for an operator bounded from below.

As shown in section II, the action variable \( J \) of a classical harmonic oscillator is not bounded from below. Thus, its canonical conjugate angle variable \( w \) is well defined. This explains why the problem of defining the phase observable of an harmonic oscillator appears only at the quantum level. Furthermore, in section II we showed, that the domain of the classical phase \( Q \) is determined modulus \( \pi \) and not \( 2\pi \). In the next section we shall see that in the quantum case this follows from the fact that \( N \) is bounded from below.

**IV. THE QUANTUM PHASE**

Starting from the position and momentum variables, \( q \) and \( p \), and then applying the canonical transformation (5) enables one to define the classical phase of the harmonic oscillator (see Eq. (6)). In the quantum picture we start also with the two canonically conjugate operators \( x \) and \( p \). However, here these two observables reside in a one dimensional Hilbert space \( \mathcal{H} \) of a free particle. Then, in order to obtain the spectrum of a number operator for \( p \), one must require for periodicity in \( x \) space and symmetry under reflection in \( p \) space.

**A. The periodic subspace \( \mathcal{H}_L \)**

The periodic subspace \( \mathcal{H}_L \) is defined by the requirement that for any \( |\psi\rangle \in \mathcal{H}_L \) and for any position eigenstate \( |x\rangle \in \mathcal{H} \), \( \langle x - L/2 | \psi \rangle = \langle x + L/2 | \psi \rangle \). Since a position shift of \( L \) does not alter the states in \( \mathcal{H}_L \) we can restrict \( x \) to the range \( (-L/2, L/2] \). Furthermore, the momentum eigenstates \( |p_n\rangle \) in \( \mathcal{H}_L \) have eigenvalues

\[
p_n \equiv \frac{2\pi \hbar n}{L} \quad n = 0, \pm 1, \pm 2, ... \tag{26}
\]

Thus the position and momentum observables in \( \mathcal{H}_L \) can be written as
\[
\mathbf{x}_L = \int_{-L/2}^{L/2} x \langle x \rangle \, dx \quad \mathbf{p}_L = \sum_{n=-\infty}^{\infty} p_n \langle p_n \rangle |p_n|,
\]
(27)
where the subscript \( L \) indicates that these observables describe a particle in a box of size \( L \).

Now, since the position and momentum eigenstates \(|x\rangle, |p\rangle \in \mathcal{H}\) satisfy the relation
\[
\langle x | p \rangle = (2\pi)^{-1/2} \exp \left( -ixp/\hbar \right),
\]
(28)
then \(|x\rangle, |p_n\rangle \in \mathcal{H}_L\) satisfy the relation
\[
\langle x | p_n \rangle = L^{-1/2} \exp \left( -ip_n\hbar \right),
\]
(29)
where the normalization is in accordance with the domain of \( x \in (-L/2, L/2] \). However, equation Eq. (29) implies that the commutation relation between \( x_L \) and \( p_L \) is no longer \( i\hbar \). This is not surprising because the analogous thing happens also in the classical picture.

As we have shown in section II, if \( Q \) and \( P \) (see there definitions in Eq. (3)) satisfy the Hamilton equations then \( Q \) must be unrestricted. This means that a restricted \( Q \) (i.e. \( Q \mod 2\pi \)) is not canonically conjugate to \( P \). Thus, a restricted position operator \( x_L \) and the momentum operator \( p_L \) satisfy a non-standard commutation relation:
\[
\left[ p_L, x_L \right] = \int_{-L/2}^{L/2} x e^{i\frac{\pi}{\hbar} (p_n - p_n')} x \, dx
\]
\[
= \frac{1}{L} \left[ \delta_{n,n'} - e^{-i\frac{\pi}{\hbar} (p_n - p_n')} \right] |p_n \rangle \langle p_n' |
\]
\[
= \frac{i\hbar}{L} \left[ 1 - L \delta \left( x = L/2 \right) \langle x = L/2 \rangle \right] = \frac{i\hbar}{L} \left[ 1 - \delta \left( x - L/2 \right) \right].
\]
(30)
This commutation relation also implies a novel uncertainty principle. That is, for a general state
\[
|\psi \rangle = \sum_{n=-\infty}^{\infty} \psi_n |p_n \rangle
\]
(31)
in \( \mathcal{H}_L \) we find (using the fact that \( \langle x = L/2 | p_n \rangle = L^{-1/2} (-1)^n \))
\[
(\Delta x_L)_{\psi}(\Delta p_L)_{\psi} \geq \frac{1}{2} |\langle [p_L, x_L] | \psi \rangle| = \frac{\hbar}{2} |1 - L \langle \psi | \delta \left(x - \frac{L}{2}\right) | \psi \rangle|
\]

\[
= \frac{\hbar}{2} \left|1 - \left| \sum_{n=-\infty}^{\infty} (-1)^n \psi_n \right|^2\right|,
\]  

(32)

where \((\Delta x_L)_{\psi} \equiv \sqrt{\langle \psi | x^2 | \psi \rangle - \langle \psi | x | \psi \rangle^2}\). Hence, Eq. (32) shows clearly why \(x_L\) is not canonically conjugate to \(p_L\) in the strong standard way. However, in this paper we shall call \(x_L\) and \(p_L\) canonically conjugate observables because of their physical interpretation: they describe the position and momentum of a particle in a box of size \(L\). But note that it is possible to define a dense subspace \(C \subset \mathcal{H}_L\)

\[
C : |\psi\rangle = \sum_{n=-\infty}^{\infty} \psi_n |p_n\rangle,
\]

such that \(\sum_{n=-\infty}^{\infty} \psi_n (-1)^n = 0,\)

(33)
in which \([p_L, x_L] = i\). That is, in the subspace \(C\), \(x_L\) and \(p_L\) satisfy the strong form of the canonical commutation relation.

The dimensionless operator \(\Theta \equiv \frac{2\pi}{L} x_L\) and the operator \(J_z \equiv \frac{L}{2\pi} p_L\) can be interpreted as the angle and angular momentum observables of the plan rotator. Thus the canonically conjugate to the angular momentum of a plan rotator is given by the matrix elements of \(\Theta\) in the angular momentum state basis \(|j_n\rangle \equiv |p_n\rangle\)

\[
\langle j_n| \Theta |j_{n'} \rangle = \frac{2\pi}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} x \langle p_n | x | p_{n'} \rangle dx = \frac{i(-1)^{n-n'}(1 - \delta_{nn'})}{n-n'},
\]

(34)

where we have used Eq. (29). The same matrix elements have been obtained by Galindo [17] for the phase observable, but as we see here, these matrix elements describe the canonical conjugate to the angular momentum observable, and not to the number operator.

In order to obtain a positive number-momentum operator we have to add a further restriction on \(\mathcal{H}_L\). Recall that the position \(x\)-space has been divided into infinite identical boxes each of size \(L\), and then, since a position shift of \(L\) does not alter the states in \(\mathcal{H}_L\) it was possible to restrict \(x\) to the domain \((-L/2, L/2]\). In exactly the same way we can divide the momentum \(p\)-space into two identical half spaces and then restrict \(p\) to the domain \(p \geq 0\).
B. The subspace $\mathcal{H}_L^+$

The subspace $\mathcal{H}_L^+ \subset \mathcal{H}_L$ is defined as follows: if $|\psi\rangle \in \mathcal{H}_L^+$ then $\langle p_n|\psi\rangle = \langle p_{-n}|\psi\rangle$ for any $|p_n\rangle \in \mathcal{H}_L$. Hence the momentum $p$-space in $\mathcal{H}_L^+$ is divided into two identical half spaces, namely, $p \geq 0$ and $p \leq 0$. However, notice that $|p_n\rangle$ itself do not belong to $\mathcal{H}_L^+$ and a different definition for a momentum eigenstate is needed. The new momentum eigenstates are defined by

$$|p_n\rangle^+ = \frac{1}{2}(|p_n\rangle + |p_{-n}\rangle)$$

since $^+\langle p_n|\psi\rangle = \langle p_n|\psi\rangle$ for any $|\psi\rangle \in \mathcal{H}_L^+$. Note that any $|\psi\rangle$ in $\mathcal{H}_L^+$ can be written as a superposition of $|p_n\rangle^+$'s states, that is $\mathcal{H}_L^+ = \text{span}\{|p_n\rangle^+\}$. We shall now examine the $x$-space of $\mathcal{H}_L^+$.

For any $|\psi\rangle \in \mathcal{H}_L^+$ and for any $|x\rangle \in \mathcal{H}_L$, $\langle x|\psi\rangle = \langle -x|\psi\rangle$. This follows because $|\psi\rangle$ can be written as $|\psi\rangle = \int_{-L/2}^{L/2} dx \, \psi(x)|x\rangle$ since $|\psi\rangle$ also belongs to $\mathcal{H}_L$. The requirement $\langle p_n|\psi\rangle = \langle p_{-n}|\psi\rangle$ implies that

$$\int_{-L/2}^{L/2} dx \, \psi(x) \exp(ip_nx/\hbar) = \int_{-L/2}^{L/2} dx \, \psi(x) \exp(-ip_nx/\hbar)$$

for all $n$ and, therefore, $\psi(x) = \psi(-x)$. Thus also the eigenstates of $x_L$ itself do not belong to $\mathcal{H}_L^+$. The new position eigenstates are defined analogously to Eq. (35)

$$|x\rangle^+ \equiv \frac{1}{2}(|x\rangle + |-x\rangle)$$

since $^+\langle x|\psi\rangle = \langle x|\psi\rangle$ for any $|\psi\rangle \in \mathcal{H}_L^+$.

As we have seen above the Hilbert space $\mathcal{H}_L^+$ is symmetric both in the position and momentum space; it does not distinguish between $x$ and $-x$ and between $p$ and $-p$. Hence, the orthonormality conditions of the bases $\{|x\rangle^+\}$ and $\{|p_n\rangle^+\}$ are given by

$$^+\langle x'|x\rangle^+ = \frac{1}{2}(\delta(x-x') + \delta(x+x'))$$

and $^+\langle p_n|p_{n'}\rangle^+ = \frac{1}{2}(\delta_{n,n'} + \delta_{n,-n'})$. \hspace{1cm} (38)

Furthermore, in a symmetric space the projection of $|p_n\rangle^+$ on $|x\rangle^+$ is no longer like Eq. (29) since it should be an even function of both $x$ and $p$. That is,
\begin{equation}
\langle x|p_n \rangle^+ = \frac{1}{4} \left( \langle x|p_n \rangle + \langle -x|p_n \rangle + \langle x| -p_n \rangle + \langle -x| -p_n \rangle \right) = \sqrt{1/L} \cos \left( \frac{xp_n}{\hbar} \right),
\end{equation}

where we have used Eq. (29).

The next step is to restrict the Hilbert space \( \mathcal{H}_L^+ \) to the domain of \( p \geq 0 \) and \( x \geq 0 \). However, since \( \mathcal{H}_L^+ \) is already symmetric we just have to change the normalization conditions to be in accordance with the new domain of \( x \) and \( p \). That is,

\begin{equation}
|x\rangle^+ \rightarrow \sqrt{2} |x\rangle^+ \quad \text{and} \quad |p_n\rangle^+ \rightarrow \sqrt{2} |p_n\rangle^+
\end{equation}

for \( x \) greater than zero and \( n \geq 1 \). For \( |x = 0\rangle^+ \) and \( |p_{n=0}\rangle^+ \) no change is needed. With these changes, the position and momentum observables in the half bounded space \( 0 \leq x \leq L/2 \) are given by

\begin{equation}
x_L^\pm = \int_0^{L/2} x |x\rangle^+ \langle x|dx \quad \text{and} \quad p_L^\pm = \sum_{n=0}^{\infty} p_n |p_n\rangle^+ \langle p_n|,
\end{equation}

where the orthonormality conditions in half space are given by

\begin{align}
+\langle x|x' \rangle^+ &= \delta(x - x') \quad \text{for} \quad x, x' > 0 \quad \text{and} \quad +\langle x|x' = 0 \rangle^+ = 2\delta(x) \\
+\langle p_n|p_{n'} \rangle^+ &= \delta_{n,n'} \quad \text{for} \quad n, n' \geq 0.
\end{align}

The projection of \( |p_n\rangle^+ \) on \( |x\rangle^+ \) in the half space is given by:

\begin{equation}
+\langle x|p_{n \geq 1} \rangle^+ = \sqrt{4/L} \cos \left( \frac{xp_n}{\hbar} \right) \quad \text{and} \quad +\langle x|p_{n=0} \rangle^+ = \sqrt{2/L}.
\end{equation}

C. The number and absolute value of phase observables

The dimensionless number and absolute value of phase observables are defined by

\begin{equation}
N \equiv \frac{L}{2\pi\hbar} p_L^\pm = \sum_{n=0}^{\infty} n |n\rangle \langle n| \quad \text{and} \quad \Phi \equiv \frac{2\pi}{L} x_L^\pm = \int_0^\pi \phi |\phi\rangle \langle \phi|d\phi,
\end{equation}

where \( |n\rangle \equiv |p_n\rangle^+ \) and \( |\phi\rangle \equiv \sqrt{\frac{L}{2\pi}} |x = \frac{L}{2\pi} \phi\rangle^+ \). In a dimensionless form, Eq. (43) can be written as
\[ |\phi \rangle = \frac{1}{\sqrt{\pi}} |n = 0 \rangle + \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} \cos(n\phi) |n \rangle, \tag{45} \]

which is consistent with the normalization conditions \( \langle \phi | \phi' \rangle = \delta(\phi - \phi') \) for \( 0 < \phi, \phi' \leq \pi \) and \( \langle \phi | \phi = 0 \rangle = 2\delta(\phi). \) Furthermore, Eq. (45) together with the definition of the phase observable (44) implies that the matrix elements of \( \Phi \) in the \( |n \rangle \) basis are:

\[
\langle n' | \Phi | n \rangle = \langle n | \Phi | n' \rangle = \begin{cases} 
\pi/2 & \text{for } n' = n \\
-2\sqrt{2}\pi^{-1}n^{-2} & \text{for } n' = 0, \text{ odd } n \\
-2\pi^{-1} ((n + n')^{-2} + (n - n')^{-2}) & \text{for } n, n' > 0, \text{ odd } n + n' \\
0 & \text{otherwise.} 
\end{cases} \tag{46} \]

We shall now examine the absolute value of the phase observable \( \Phi \equiv |\phi| \) defined by Eq. (44) and Eq. (46). First, the range of the \( \Phi \) is from 0 to \( \pi \). Although the unrestricted phase operator can take values from \( -\infty \) to \( +\infty \), the values which correspond to actual states will be in this \( \pi \) range. This domain of the phase observable is in a complete agreement with the classical picture, especially with Eq. (6). The fact that the number operator cannot have negative eigenvalues implies that the phase is also bounded to half range (i.e. it is possible to define only the absolute value of the phase observable).

The second characteristic of \( \Phi = |\phi| \) is that \( \phi \) itself is complementary to the number operator. That is, all the number states, including the vacuum, are states of random phase \( \phi \). This is reasonable since a number state should not have a preferred phase. This idea is known as the Barnett and Pegg “acid-test” for quantum phase theories. In our theory, the distribution of the phase in a number state \( |n \rangle \) is given by

\[
P_{n=0}(\phi) \equiv |\langle n = 0 | \phi \rangle|^2 = \frac{1}{\pi} \\
P_{n \geq 1}(\phi) \equiv |\langle n | \phi \rangle|^2 = \frac{2}{\pi} \cos^2(n\phi) = \frac{1}{\pi}(1 + \cos(2n\phi)), \tag{47} \]

where we have used Eq. (45). Not surprisingly this distribution is not uniform since it is the distribution of the absolute value of the phase, and not of the phase itself. In the case
of the plane rotator for example, the state \(|\theta >_+ \equiv 1/\sqrt{2}(|\theta > + |-\theta >)|\) is an eigenstate of the absolute value of the angle operator (I owe this point to M. J. W. Hall), and it gives a similar distribution as in Eq. (47). Thus, Eq. (47) yields a non-uniform phase distribution. However, in the classical limit \(n \to \infty\), the average of \(\Phi^m\) \((m = 0, 1, 2...\)) in a number state is given by:

\[
\langle n|\Phi^m|n \rangle \equiv \langle \Phi^m \rangle_n = \frac{1}{\pi} \int_0^\pi d\Phi (1 + \cos(2n\Phi))\Phi^m \to \frac{1}{\pi} \int_0^\pi d\Phi \Phi^m
\]

which is identical to the average of \(\Phi^m\) in a classical uniform phase distribution \(P(\Phi) = 1/\pi\).

The third characteristic of \(\Phi\) is that it represents the absolute value of the non-Hermitian phase operator defined long time ago by Dirac [6]. The exponent of this operator can be represented by the Susskind and Glogower [8] non-unitary operators

\[
E \equiv (N + 1)^{-1/2}a = \sum_{n=0}^\infty |n\rangle\langle n + 1|, \quad E^\dagger \equiv a^\dagger(N + 1)^{-1/2} = \sum_{n=0}^\infty |n + 1\rangle\langle n|.
\]

In order to compare these operators to our approach we shall use the definition of Carruthers and Nieto [15] for the Hermitian sine \(S \equiv \frac{1}{2}(E - E^\dagger)\) and cosine \(C \equiv \frac{1}{2}(E + E^\dagger)\) operators. Note that \(S\) and \(C\) cannot represent the exact sine and cosine of the phase observables since, for example, \([S, C] \neq 0\). However, in states of large average number occupations, \(\langle N \rangle \gg 1\), \(S\) and \(C\) can be treated approximately as the sine and cosine of the phase. Thus, our theory should produce (small) corrections to \(S\) and \(C\). Since we have the absolute value of the phase in the range \([0, \pi]\), the sine of \(\Phi\) is always positive, and therefore should be compared with the absolute value of \(S\). This has been explained using the classical picture in section III.

The cosine of the phase operator is defined by:

\[
\cos \Phi = \int_0^\pi \cos \phi \, |\phi\rangle\langle \phi| \, d\phi.
\]

Taking its matrix element in the \(|n\rangle\) basis gives

\[
\langle n|\cos \Phi|n' \rangle = \int_0^\pi \cos \phi \, \langle n|\phi\rangle \langle \phi|n' \rangle \, d\phi = \frac{1}{2} (\delta_{n, n'+1} + \delta_{n, n'-1}) + \frac{1}{2}(\sqrt{2} - 1)\delta_{1, n+n'},
\]

15
where we have used Eq. (45) for the value of $\langle n|\phi \rangle$. Now, since the first term with the parenthesis in Eq. (51) is exactly the matrix element $\langle n|C|n' \rangle$, the cosine of $\Phi$ can be written in the form

$$
\cos \Phi = C + \frac{1}{2}(\sqrt{2} - 1)(|0\rangle\langle 1| + |1\rangle\langle 0|).
$$

(52)

where the projectors involving the number eigenstates $|0\rangle$ and $|1\rangle$ can be neglected for states with $\langle N \rangle \gg 1$.

In order to compare $|S|$ with the sine of $\Phi$, it is enough to find the relation between the square of these operators. In this way we avoid the need to calculate the absolute value of $S$. Using the same technique as for $\cos \Phi$ we find that

$$
\sin^2 \Phi = \int_0^\pi \sin^2 \phi \langle \phi |d\phi = S^2 + \frac{1}{4}(1 - \sqrt{2})(|0\rangle\langle 2| + |2\rangle\langle 0|) + \frac{1}{4}(|0\rangle\langle 0| - |1\rangle\langle 1|),
$$

(53)

where we have used Eq. (45). In our theory it is obvious from its definition that $\sin \Phi$ and $\cos \Phi$ commute and satisfy the trigonometric relation $\sin^2 \Phi + \cos^2 \Phi = 1$.

### D. The classical limit

We shall show that our quantum phase is in a complete agreement with the classical limit. The coherent states which correspond to the classical picture are the eigenstates of the annihilation operator $a$. A coherent state can be written as

$$
|\gamma \rangle = \exp(-\frac{1}{2}|\gamma|^2) \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{n!}} |n\rangle,
$$

(54)

where $\gamma = \sqrt{N}e^{i\theta}$ is the eigenvalue of the annihilation operator $a$. Thus, the average of $\Phi$ in a coherent state can be written as

$$
\langle \gamma |\Phi|\gamma \rangle = \exp(-|\gamma|^2) \sum_{n,n'=0}^{\infty} \frac{\gamma^{n'}(\gamma^*)^n}{\sqrt{n!n'!}} \langle n'|\Phi|n \rangle
$$

$$
= \exp(-|\gamma|^2) \sum_{n=0}^{\infty} \frac{|\gamma|^{2n}}{n!} \langle n|\Phi|n \rangle + \exp(-|\gamma|^2) \sum_{n \neq n'} \frac{\gamma^{n'}(\gamma^*)^n}{\sqrt{n!n'!}} \langle n'|\Phi|n \rangle
$$

$$
= \frac{\pi}{2} + \exp(-|\gamma|^2) \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \left[ \frac{|\gamma|^{2n} \gamma^s}{\sqrt{n!(n+s)!}} + \frac{|\gamma|^{2n} \gamma^{-s}}{\sqrt{n!(n-s)!}} \right] \langle n + s|\Phi|n \rangle
$$

(55)
where \( s \equiv |n - n'| \) and \(<n | \Phi | n> = \pi/2 \) for all \( n \). Now, in the classical limit \( N = |\gamma|^2 \to \infty \), \(<n | \gamma> \approx 0 \) for \( n \ll N \). Thus, the main contribution to the sum in Eq. (53) comes from the elements with \( n \gg 1 \). Furthermore, since \(<n + s | \Phi | n> \to 0 \) for \( s \to \infty \) the main contribution comes from elements with \( n \gg s \). Thus, we can approximate

\[
<n + s | \Phi | n> = -\frac{1}{\pi}(1 - (-1)^s) \left( \frac{1}{s^2} + \frac{1}{(2n + s)^2} \right) \approx -\frac{1}{\pi}(1 - (-1)^s)\frac{1}{s^2}, 
\]

and

\[
\frac{|\gamma|^{2n} \gamma^s}{\sqrt{n!(n + s)!}} \approx \frac{|\gamma|^{2n}}{n!} \sqrt{\frac{\gamma^s}{n^s}}, \quad \frac{|\gamma|^{2n} \gamma^{-s}}{\sqrt{n!(n - s)!}} \approx \frac{|\gamma|^{2n}}{n!} \frac{\sqrt{n^s}}{\gamma^s} 
\]

for \( n \gg s \). Substituting these approximations in Eq. (53) gives

\[
<\gamma | \Phi | \gamma> \approx \frac{\pi}{2} - \frac{2}{\pi} \exp(-|\gamma|^2) \sum_{n=0}^{\infty} \sum_{s=1,3,5,...} \frac{1}{s^2} \left[ \frac{|\gamma|^{2n}}{n!} \frac{\gamma^s}{n^s} + \frac{|\gamma|^{2n} n^{s/2}}{n!} \frac{\gamma^s}{\gamma^s} \right] 
\]

\[
\approx \frac{\pi}{2} - \frac{2}{\pi} \sum_{s=1,3,5,...} \frac{1}{s^2} \left[ \frac{\gamma^s}{N^{s/2}} + \frac{N^{s/2}}{\gamma^s} \right], 
\]

where \( N \equiv <\gamma | N | \gamma> = |\gamma|^2 \). We have used the fact that in the classical limit \(<N^{s/2} > \approx N^{s/2} \).

Now, after substituting \( \gamma^s = N^{s/2} e^{i\theta} \) in Eq. (58), we find that in the classical limit

\[
<\gamma | \Phi | \gamma> \to \frac{\pi}{2} - \frac{4}{\pi} \sum_{s=1,3,5,...} \frac{\cos(\theta s)}{s^2} = |\theta|, 
\]

since the r.h.s. is exactly the Fourier series of of the function \( f(\theta) = |\theta| (-\pi < \theta < \pi) \). This result proves useful for establishing that \( \Phi \) has the correct large-field correspondence limit.

The agreement with the classical limit, given by Eq. (59), can be shown not just for the average of \( \Phi \) itself, but for the average of any analytical function of \( \Phi \). To show this we first calculate the average of \( \sin \Phi \). In the same way as Eq. (59) was obtained, it can be shown that

\[
<\gamma | \sin \Phi | \gamma> = \frac{2}{\pi} + \exp(-|\gamma|^2) \sum_{n=0}^{\infty} \sum_{s=1}^{\infty} \left[ \frac{|\gamma|^{2n} \gamma^s}{\sqrt{n!(n + s)!}} + \frac{|\gamma|^{2n} \gamma^{-s}}{\sqrt{n!(n - s)!}} \right] <n + s | \sin \Phi | n>, 
\]

since \(<n | \sin \Phi | n> = 2/\pi \). Now, it can be shown that for \( n \gg s \)

\[
<n + s | \sin \Phi | n> \approx -\frac{1 + (-1)^s}{\pi} \frac{1}{s^2 - 1}. 
\]
Substituting this in Eq. (60) leads to

\[ \langle \gamma | \sin(\Phi) | \gamma \rangle \rightarrow \frac{2}{\pi} - \frac{4}{\pi} \sum_{s=2,4,6,...} \frac{\cos(\theta s)}{s^2 - 1} = | \sin \theta |, \]

in the limit \( N \rightarrow \infty \). The r.h.s. is exactly the Fourier series of of the function \( f(\theta) = | \sin \theta | \), so we have the expected result. Furthermore, Eq. (52) and Eq. (53) imply that for \( N \rightarrow \infty \)

\[ \langle \gamma | \cos(\Phi) | \gamma \rangle \rightarrow \cos \theta \]

\[ \langle \gamma | \cos^2(\Phi) | \gamma \rangle \rightarrow \cos^2 \theta \]

\[ \langle \gamma | \sin^2(\Phi) | \gamma \rangle \rightarrow \sin^2 \theta \]

(63)
since \( \langle C \rangle \rightarrow \cos \theta \) and \( \langle S \rangle \rightarrow \sin \theta \). Thus, all these results imply that any power of \( \sin \Phi \) and \( \cos \Phi \) has the correct classical limit, so that \( \langle \gamma | f(\Phi) | \gamma \rangle \rightarrow f(|\theta|) \) for any analytic function \( f(x) \).

V. THE QUANTUM PHASE IN A FINITE DIMENSIONAL HILBERT SPACE

It is also interesting to discuss the quantum phase problem in a finite dimensional space \([18]\). One can raise the question whether if instead of defining \( \mathcal{H}^+ \), we could define a finite dimensional subspace of \( \mathcal{H} \) in which the phase operator is well defined, and then take the limit of dimensionality to infinity. Is this procedure equivalent to our earlier construction of \( \mathcal{H}^+ \)?

According to our formalism, a finite dimensional Hilbert space can be obtained from \( \mathcal{H} \) by adding the requirement of periodicity in momentum space. That is, instead of working with \( \mathcal{H}^+_L \subset \mathcal{H} \) we shall define a finite dimensional Hilbert space \( \mathcal{H}^m_L \subset \mathcal{H} \) as follows: for any \( |\psi\rangle \in \mathcal{H}^m_L \) and for any \( |p_n\rangle \in \mathcal{H} \), \( \langle p_n | \psi \rangle = \langle p_{n+m+1} | \psi \rangle \) \((m \text{ is an integer})\). Thus, in the same way as the domain of \( x \) has been restricted to \((-L/2,L/2]\) by the requirement of periodicity in the \( x \)-space, the domain of \( p_n \), in \( \mathcal{H}^m_L \), can be restricted to the range \( n = 0, 1, 2, ..., m \).

That is,

\[ \mathcal{H}^m_L = \text{span}\{ |p_n\rangle \}_{n=0}^m, \]

(64)
and therefore $\mathcal{H}_L^m$ is a finite Hilbert space of dimension $m + 1$.

Notice that the domain of $p_n$ could be chosen differently. For example, assuming $m$ is an even number, the domain of $p_n$ can be restricted also to the range $n = -m/2, -m/2 + 1, ..., m/2$. Using this range we can define

$$\mathcal{H}_L^\pm = \text{span}\{|p_n\rangle\}_{n=-m/2}^{m/2},$$

where the subscript $\pm$ just indicates that $n$ can be also negative. It is clear that $\mathcal{H}_L^\pm \equiv \mathcal{H}_L^m$, i.e. both Hilbert spaces are equivalent. However, as we shall see in the following, the difference between the two Hilbert spaces appears in the limit process $m \to \infty$.

The spectrum of $x_L$ is no longer continuous since $\langle p_n | x \rangle = \langle p_{n+m+1} | x \rangle$. This requirement implies that the spectrum of the position operator in $\mathcal{H}_L^m \equiv \mathcal{H}_L^\pm$ is given by

$$x_l = \frac{l}{m + 1} L, \quad l = -m/2, -m/2 + 1, ..., m/2,$$

where we have used Eq. (29). Hence, the position $x_L^m$ and momentum $p_L^m$ operators in $\mathcal{H}_L^m$ are given by

$$x_L^m = \sum_{l=-m/2}^{m/2} x_l |x_l\rangle\langle x_l| \quad \text{and} \quad p_L^m = \sum_{n=0}^{m} p_n |p_n\rangle\langle p_n| \quad \text{or} \quad p_L^m = \sum_{n=-m/2}^{m/2} p_n |p_n\rangle\langle p_n|.$$

The dimensionless angle operator is defined by $\Theta^m \equiv \frac{2\pi}{L} x_L^m$ and the dimensionless number (angular momentum) operator by $N^m = \frac{L}{2\pi\hbar} p_L^m$.

We shall now examine the operators $\Theta^m$ and $N^m$ in the limit $m \to \infty$. This limit exists only in the case where $n = -m/2, -m/2 + 1, ..., m/2$. This can be proved by showing that the matrix elements of $\Theta^m$ in the momentum $|n\rangle$ basis are given by

$$\langle n'|\Theta^m |n\rangle = \frac{(-1)^{n-n'}}{\frac{m+1}{2\pi} \left[ 1 - \exp\left( \frac{2\pi i (n' - n)}{m+1} \right) \right]}.$$

for both ranges of $n$. However, in the limit $m \to \infty$ these matrix elements coincide with the matrix elements of an angle observable of plane rotator (see Eq. (34)). Thus, in the limit $m \to \infty$, $\Theta^m$ represents the canonical conjugate to the angular momentum (and not to the number operator). That is, $n$ must be unbounded from below.
Furthermore, if one is calculating some physical quantities in $\mathcal{H}_L^m \equiv \mathcal{H}_L^{\pm m}$ (such as the variance of $\Theta^m$ in a momentum eigenstate) and then takes the limit $m \to \infty$, one will get the result obtained for a plane rotator, not for a linear harmonic oscillator. This explains why, for example, finite dimensional quantum phase theories pass the Barnett and Pegg “acid-test” [13]: They just prove that an angular momentum (not number) eigenstate represents a state of indeterminate angle (not phase).

VI. SUMMARY AND CONCLUSIONS

Only the absolute value of the phase observable of a single field mode is well defined. The definition of the phase turns out to be problematic also from the classical point of view. It appears as a result of the fact that the photon number is bounded from below. In the example of a plane rotator, the angle is well defined since the angular momentum can also have negative eigenvalues.

The phase is complementary to the excitation (or photon number) and thus the number eigenstates represent states of random phase. However, since only the absolute value of the quantum phase is well defined, number states do not correspond to a uniform absolute phase distribution.

The time operator which is the canonical conjugate to the Hamiltonian of an harmonic oscillator is determined by: $|t| = \omega^{-1} \Phi$. Hence, our theory suggests that in order to define the complementary operator for the Hamiltonian of a general system, one should seek the absolute value of the time observable.

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REFERENCES

[1] P. Carruthers and M. M. Nieto, Rev. Mod. Phys. 40, 411 (1968); R. Lynch, Phys. Rep. 256, 367 (1995); D. A. Dubin, M. A. Dubin, M. A. Hennings and T. B. Smith, Int. J. Mod. Opt. 44, 225 (1995); D. T. Pegg and S. M. Barnett, J. Mod. Opt. 44, 225 (1997); D. G. Welsch, W. Vogel and T. Opatrny, Prog. in Opt. 39, 63 (1999).

[2] G. Gour, quant-ph/0102092.

[3] R. G. Newton, Ann. Phys. 124, 327 (1980).

[4] M. Moshinsky and T. H. Seligman, Ann. Phys. 114, 243 (1978).

[5] F. London, Zeitschrift fuer Physik 37, 915 (1926); Zeitschrift fuer Physik 40, 193 (1926).

[6] P. A. M. Dirac, Proc. Roy. Soc. (London) A114, 243 (1927).

[7] M. M. Nieto, Physica Scripta, Special Issue devoted to: Quantum Phase and Phase Dependent Measurements, T 48, 5 (1993).

[8] L. Susskind and J. Glogower, Physics 1, 49 (1964).

[9] H. Kastrup, quant-ph/0005033; M. Bojowald and T. Strobl, J. Math. Phys. 41, 2537 (2000); T. Hakioglu and E. Tepedelenlioglu, J. Phys. A 33, 6357 (2000).

[10] J. H. Shapiro and S. R. Shepard, Phys. Rev. A 43, 3795 (1991); M. J. W. Hall, J. Mod. Opt. 40, 809 (1993).

[11] J. W. Noh, A. Fougeres and L. Mandel, Phys. Rev. Lett. 67, 1426 (1991); Phys. Rev. Lett. 71, 2579 (1993).

[12] M. G. Raymer, M. Beck and D. F. McAlister, Phys. Rev. Lett. 72, 1137 (1994); M. G. Raymer, J. Cooper and M. Beck, Phys. Rev. A 48, 4617 (1993); G. Breitenbach, S. Schiller and J. Mlynek, Nature, 387, 471 (1997); G. Breitenbach and S. Schiller, J. Mod. Opt. 44, 2207 (1997).
[13] S. M. Barnett and D. T. Pegg, *J. Mod. Opt.* **39**, 2121 (1992).

[14] H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, 1980).

[15] P. Carruthers and M. M. Nieto, *Phys. Rev. Lett.* **14**, 387 (1965).

[16] D. Judge, *Phys. Lett.* **5**, 189 (1963); *Nuovo Cimento* **31**, 332 (1964); D. Judge and J. T. Lewis, *Phys. Lett.* **5**, 190 (1963).

[17] A. Galindo, *Lett. Math. Phys.* **8**, 495 (1984).

[18] D. T. Pegg and S. M. Barnett, *Europhys. Lett.* **6**, 483 (1988); *Phys. Rev. A* **39**, 1665 (1989); S. M. Barnett and D. T. Pegg, *J. Mod. Opt.* **36**, 7 (1989).