A method for solving nonlinear differential equations: an application to $\lambda\phi^4$ model

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Abstract

Recently, it has been great interest in the development of methods for solving nonlinear differential equations directly. Here, it is shown an algorithm based on Padé approximants for solving nonlinear partial differential equations without requiring a one-dimensional reduction. This method is applied to the $\lambda\phi^4$ model in 4 dimensions and new solutions are obtained.

Keywords: Integrable Equations in Physics, Integrable Field Theory, Padé approximants, $\lambda\phi^4$ model.

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1 Introduction

For many years, nonlinear differential equations have been an important topic of study and in many branches of knowledge. This interest has led to the development of many techniques through the last few years in order to obtain exact solutions without requiring further properties of the differential equation (for example [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]). In [13], it was proposed the multiple Exp-function method for finding exact solutions of partial differential equation (PDE) without requiring a one-dimensional reduction. Although this method is very powerful, there are a large number of parameters to be determined.

Here, we present a simpler algorithm based on Padé approximants and apply it to the classical equation of the $\lambda \phi^4$ model with $m \neq 0$. There are several papers generalizing Padé approximants for multivariate functions [19, 20, 21, 22, 23, 24, 25]. Here, we focus on the homogeneous Padé approximant, introduced in [21] (see also [24]).

The $\lambda \phi^4$ model is one of the simplest example of a renormalizable scalar field theory and it is defined by the Lagrangian density

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2 - \frac{\lambda}{4} \phi^4,$$

where we use the metric $\eta = (-+++)$ and the notation of repeated indices summed. The Euler-Lagrange equation, i.e.

$$\frac{\partial}{\partial \mu} \frac{\partial L}{\partial \partial_\mu \phi} - \frac{\partial L}{\partial \phi} = 0,$$

of (1) yields the classical equation of motion

$$- \partial_t^2 \phi + \nabla^2 \phi + m^2 \phi + \lambda \phi^3 = 0.$$

(2)

Static solutions of the two-dimensional $\lambda \phi^4$ model was already presented in [27] and [26]. Here, we construct some solutions of equation (2) representing travelling waves and the scattering of two travelling waves.

The paper is organized in two main sections. In section 2, I show a practical introduction to the homogeneous multivariate Padé approximant and present a new approach for solving nonlinear PDEs. The section 3 is devoted to the $\lambda \phi^4$ model.

2 The method

2.1 Homogeneous multivariate Padé approximants

Consider a function $f(z) = f(z_1, ..., z_D)$ regular at origin and with Taylor expansion around $z = 0$ given by

$$f(z) = \sum_{J=0}^{\infty} c_J z^J = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} ... \sum_{j_D=0}^{\infty} c_{j_1, j_2, ..., j_D} \prod_{d=1}^{D} z_d^{j_d}.$$
The homogeneous multivariate Padé approximant consist in rearranging the coefficients such that we can use the Padé approximant in one dimension. Through the map $z \rightarrow \xi z$, the Taylor Expansion can be rearranged as

\[
f(\xi z) = \sum_{n=0}^{L+M} a_n(z)\xi^n + \mathcal{O}(\xi^{L+M+1})
\]

where

\[
a_n(z) = \sum_{j_1=0}^{n} \sum_{j_2=0}^{n-j_1} \cdots \sum_{j_{D-1}=0}^{n-\sum_{r=1}^{D-2} j_r} c_{j_1,j_2,\ldots,j_{D-1},n-\sum_{r=1}^{D-1} j_r} \left( \prod_{d=1}^{D-1} z_d^{j_d} \right) z^{n-\sum_{r=1}^{D-1} j_r}
\]

This rearrangement allows us to compute the univariate Padé approximant of $f(\xi z)$ on $\xi$, i.e.

\[
[L/M]_z(\xi) \equiv \frac{P_{z,L}(\xi)}{Q_{z,M}(\xi)} = \frac{\sum_{j=0}^{L} p_j(z)\xi^j}{1 + \sum_{j=1}^{M} q_j(z)\xi^j},
\]

where the coefficients $p_j(z)$ and $q_j(z)$ are determined such that the Padé approximant agrees with $f(\xi z)$ up to the degree $L + M$, i.e. $f(\xi z) = [L/M]_z(\xi) + \mathcal{O}(\xi^{L+M+1})$. Thus we need to solve the following system of equations

\[
\sum_{r+s=j} a_r(z)q_s(z) - p_j(z) = 0, \quad j = 0, 1, \ldots, L + M.
\]

This system can be easily solved by a symbolic computation software. Concluding, the homogeneous Padé approximant for $f(z)$ is obtained by setting $f(z) = f(\xi z)|_{\xi=1}$.

### 2.2 The functional ansatz

Consider now a system of $N_e$ equations with $N_e$ fields in $D$ dimensions, i.e.

\[
E_k(x^\mu, \phi_i, \partial_\mu \phi_i, \ldots; S_0) = 0, \quad k = 1, \ldots, N_e
\]

where $S_0$ is the space formed by the Cartesian product of the set of parameters, $\mu = 1, \ldots, D$ and $i = 1, \ldots, N_e$. Now, let us suppose at least one solution for this system can be expressed as functionals of a set of functions $\rho = (\rho_1, \ldots, \rho_{N_\rho})$, i.e.

\[
\phi_i(x^\mu) = \hat{\phi}_i(\rho_1, \ldots, \rho_{N_\rho}), \quad i = 1, \ldots, N_e
\]

where $\rho_k = \rho_k(x^\mu)$ for $k = 1, \ldots, N_\rho$. Moreover, suppose the first derivative of all $\rho_k$ are known in terms of $\rho$, i.e.

\[
\partial_\mu \rho_k = F_{\mu,k}(\rho_1, \ldots, \rho_{N_\rho}; S_1), \quad \mu = 1, \ldots, D, \quad k = 1, \ldots, N_\rho
\]
where $S_1$ is the space formed by the Cartesian product of the sets of parameters introduced by $\rho$. The choice of $\rho$ is the first ansatz of the algorithm and it yields the transformation

$$E_k(x^\mu, \phi_i, \partial\mu\phi_i, \ldots; S_0) = \hat{E}_k(\rho_k, \hat{\phi}_i, \partial_k\hat{\phi}_i, \ldots; S_0 \times S_1) = 0, \quad k = 1, \ldots, N_e,$$

(4)

where $\hat{E}_k$ is polynomial or rational in $\rho_k$, $\hat{\phi}_i$ and its derivatives. Now, the system (4) can be worked out as a system in $N_\rho$ dimensions. If at least one particular solution for the set of fields $\hat{\phi}_i$ are regular at origin, we can consider a multivariate Taylor expansion at $\rho = 0$, i. e.

$$\hat{\phi}_i(\rho) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \cdots \sum_{j_{N_\rho}=0}^{\infty} c_{i;j_1,j_2,\ldots,j_{N_\rho}} \prod_{d=1}^{N_\rho} \rho_d^{j_d}, \quad i = 1, \ldots, N_e$$

(5)

where $c_{i;j_1,j_2,\ldots,j_{N_\rho}} = c_{i;j_1,j_2,\ldots,j_{N_\rho}}(S_0 \times S_1)$. Observe that the expansion (5) can be drastically changed due to the particular combination of the parameters in the space formed by $S_0 \times S_1$. Therefore, for obtaining particular solutions, we can consider a set of constraints $\psi_i = \psi_i(S_0 \times S_1) = 0$ acting on $\rho$ and $\hat{E}_k(\rho_k, \hat{\phi}_i, \partial_k\hat{\phi}_i, \ldots; S_0 \times S_1) = 0$. When a constraint $\psi$ is considered, the notation $\bar{\rho} \equiv \rho|_{\psi=0}$ will be employed.

Let us call $S_2$ the space formed by the Cartesian product of all undetermined $c_{i;j_1,j_2,\ldots,j_{N_\rho}}$ and define $S = S_0 \times S_1 \times S_2$. Now, we can use the homogeneous multivariate Padé approximant in $\hat{\phi}_i(\rho)$. Mapping $\rho \to \xi\rho$, we can rearrange the expansion (5) as

$$\hat{\phi}_i(\xi\rho) = \sum_{n=0}^{L_i+M_i} a_{i;n}(\rho)\xi^n + O(\xi^{L_i+M_i+1}), \quad i = 1, \ldots, N_e,$$

$$a_{i;n}(\rho) = \sum_{j_1=0}^{n-j_1} \sum_{j_2=0}^{n-j_2} \cdots \sum_{j_{N_\rho-1}=0}^{n-j_{N_\rho-1}} c_{i;j_1,j_2,\ldots,j_{N_\rho-1},n-j_{N_\rho-1}} \sum_{j_{N_\rho}=0}^{N_\rho-1} \left( \prod_{d=1}^{N_\rho} \rho_d^{j_d} \right)^{n-j_{N_\rho}-j_{N_\rho-1}} j_{N_\rho} = 1, \ldots, n-j_{N_\rho-1}.$$

Thus, we can apply the univariate Padé approximant on $\xi$ in order to obtain an approximation of the solution, i. e.

$$\hat{\phi}_i(\xi\rho) = \frac{P_{p,L_p}(\xi; \tilde{S})}{Q_{p,M_p}(\xi; \tilde{S})} + O(\xi^{L_i+M_i+1}).$$

(6)

Finally, we can apply the second ansatz. Let us assume that there is a particular subset $\tilde{S} \subset S$, such that expression (6) yields an exact solution when $\xi = 1$, i. e.

$$\hat{\phi}_i(\rho) = \frac{P_{p,L_p}(\xi; \tilde{S})}{Q_{p,M_p}(\xi; \tilde{S})} \bigg|_{\xi=1}.$$  

(7)

This idea was used in [18] for the simpler case when $D = N_e = N_\rho = 1$ and $\rho_1 = z$ (where $z$ was the one-dimensional variable). In order to determine $\tilde{S}$, let us substitute (7) in (4). This yields

$$\hat{E}_k(\rho_k, \hat{\phi}_i, \partial_k\hat{\phi}_i, \ldots; S_0 \times S_1) = \sum_{n=0}^{\Lambda} \frac{\hat{E}_{k;n}(\tilde{S})}{D_k(S)} = 0,$$
where $\Lambda$ is determined by the choices of $L_i$, $M_i$, the set $\rho$ and the differential equation under consideration. Hence, in order to determine all elements of $\hat{S}$, we need to solve the following algebraic system:

\[
\tilde{E}_{k,j_1,j_2,...,j_{n_{\rho}}}(\hat{S}) = 0, \quad k = 1,...,N_\epsilon, \quad n = 0,...,\Lambda, \quad j_i = 0,...,n - \sum_{r=1}^{l-1} j_r \quad (9a)
\]

\[
D_k(\hat{S}) \neq 0, \quad k = 1,...,N_\epsilon \quad (9b)
\]

Step (9a) may require a huge computational power for some models, but it yields a system of algebraic equations smaller than the one we should solve by using the multiple Exp-function method [13].

3 An application to the $\lambda\phi^4$ theory in 4 dimensions

Here, we apply the algorithm presented in section 2.2 for the $\lambda\phi^4$ theory, i.e.

\[- \partial_i^2 \phi + \nabla^2 \phi + m^2 \phi + \lambda \phi^3 = 0, \quad (10)\]

by using two different functional ansatz:

(i) $\phi(x^\mu) = \hat{\phi}(\rho_1), \quad \rho_1 = e^{i(k_{1,0}t+k_{1,1}x+k_{1,2}y+k_{1,3}z)} \quad (11)$

(ii) $\phi(x^\mu) = \hat{\phi}(\rho_1, \rho_2), \quad \rho_1 = e^{i(k_{1,0}t+k_{1,1}x+k_{1,2}y+k_{1,3}z)}, \quad \rho_2 = e^{i(k_{2,0}t+k_{2,1}x+k_{2,2}y+k_{2,3}z)} \quad (12)$

For simplicity, let us use the notation $\mathbf{k}_j = (k_{j,1}, k_{j,2}, k_{j,3})$, where $\mathbf{k}_i \cdot \mathbf{k}_j = k_{i,1}k_{j,1} + k_{i,2}k_{j,2} + k_{i,3}k_{j,3}$.

3.1 Ansatz (i)

First, consider the ansatz (i). Obviously, this set for $\rho$ satisfies condition [13], namely

\[
\partial_t \rho_1 = ik_{1,0} \rho_1, \quad \partial_x \rho_1 = ik_{1,1} \rho_1, \quad \partial_y \rho_1 = ik_{1,2} \rho_1, \quad \partial_z \rho_1 = ik_{1,3} \rho_1,
\]

and yields the equation

\[
\tilde{E}(\rho_1, \hat{\phi}, \partial_t \hat{\phi}, \partial_x \hat{\phi}, \partial_y \hat{\phi}, \partial_z \hat{\phi}; \mathcal{S}_0 \times \mathcal{S}_1) \equiv (k_{1,0}^2 - |k_1|^2)(\rho_1^2 \partial_t^2 \hat{\phi} + \rho_1 \partial_t \hat{\phi}) + m^2 \hat{\phi} + \lambda \hat{\phi}^3 = 0. \quad (13)
\]

The first element of the Taylor expansion of $\phi$ can be $c_0 = 0$ or $c_0 = \frac{i\mu m}{\sqrt{\lambda}}$ where $\mu = \pm 1$. Without imposing any constraint $\psi$, this expansion yields two trivial solutions, namely,

\[
\hat{\phi} = 0 \quad \text{and} \quad \hat{\phi} = \frac{i\mu m}{\sqrt{\lambda}}, \quad \mu = \pm 1. \quad (14)
\]
However, a convenient choice for $\psi$ leads us to a more interesting expansion. If we impose
\[
\psi = -k_{1,0}^2 + k_1^2 - m^2 = 0
\]
on equation (13) and the set $\rho$ by eliminating $k_{1,0}$, we get
\[
-m^2 (\bar{\rho}_1^2 \partial_{\rho_1}^2 \hat{\phi} + \bar{\rho}_1 \partial_{\rho_1} \hat{\phi}) + m^2 \hat{\phi} + \lambda \hat{\phi}^3 = 0,
\]
where $\bar{\rho}_1 = e^{i(\nu \sqrt{(k_1^2 - m^2) \xi + k_{1,1}x + k_{1,2}y + k_{1,3}z})}$ and $\nu = \pm 1$. The expansion of $\hat{\phi}$ starting with $c_0 = 0$ then has the form
\[
\hat{\phi} = (c_1 \bar{\rho}_1) \xi + \left( \frac{c_1^2 \lambda}{8m^2 \bar{\rho}_1^2} \right) \xi^3 + \left( \frac{c_1^4 \lambda^2}{64m^4 \bar{\rho}_1^5} \right) \xi^5 + \left( \frac{c_1^7 \lambda^3}{512m^6 \bar{\rho}_1^7} \right) \xi^7 + \ldots, \quad m \neq 0
\]
while the expansion starting with $c_0 = \frac{i \mu m}{\sqrt{\lambda}}$ truncate at the constant term. By employing the Padé approximant $[1/1]_{(\rho_1)}(\xi)|_{\xi=1}$ of expansion (17), ansatz (7) yields
\[
\hat{\phi} = c_1 \bar{\rho}_1,
\]
whose substitution into equation (20) leads us to the conditions:
\[
E_{1,3}(\hat{\xi}) = c_1^3 \lambda = 0,
\]
\[
D_{1,3}(\hat{\xi}) = 1 \neq 0.
\]
Here, we see that the condition for (18) to be a solution of (20) is $c_1 = 0$ or $\lambda = 0$, which represents the vacuum solution and the Klein-Gordon limit respectively. Now, let us consider the Padé approximant $[2/2]_{(\rho_1)}(\xi)|_{\xi=1}$. In this case, the ansatz (7) yields
\[
\hat{\phi} = \frac{8c_1 m^2 \bar{\rho}_1}{8m^2 - c_1^4 \lambda \bar{\rho}_1^2}
\]
By substituting (19) into (20), we can check that expression (19) already is an exact solution without requiring any further conditions. We also obtain (19) if we use the ansatz $[3/3]_{(\rho_1)}(\xi)|_{\xi=1}$, $[4/4]_{(\rho_1)}(\xi)|_{\xi=1}$ or $[5/5]_{(\rho_1)}(\xi)|_{\xi=1}$.
Now, consider the constraint
\[
\psi = -k_{1,0}^2 + k_1^2 + 2m^2 = 0.
\]
By eliminating $k_{1,0}$, equation (13) yields
\[
2m^2 (\bar{\rho}_1^2 \partial_{\rho_1}^2 \hat{\phi} + \bar{\rho}_1 \partial_{\rho_1} \hat{\phi}) + m^2 \hat{\phi} + \lambda \hat{\phi}^3 = 0,
\]
where $\bar{\rho}_1 = e^{i(\nu \sqrt{(k_1^2 + 2m^2) \xi + k_{1,1}x + k_{1,2}y + k_{1,3}z})}$ for $\nu = \pm 1$. Using this constraint, the expansion of $\hat{\phi}$ starting with $c_0 = 0$ truncates at the constant term, while the expansion starting with $c_0 = \frac{i \mu m}{\sqrt{\lambda}}$ has the form
\[
\hat{\phi} = \frac{i \mu m}{\sqrt{\lambda}} + (c_1 \bar{\rho}_1) \xi - \left( \frac{i \mu c_1 \lambda}{2m \bar{\rho}_1} \xi^2 - \frac{c_1^2 \lambda}{4m^2 \bar{\rho}_1^2} \xi^4 + \left( \frac{c_1^7 \lambda^3}{8m^6 \bar{\rho}_1^7} \right) \xi^7 + \ldots, \quad m \neq 0.
\]
By employing the Padé approximant \([1/1]_{(\rho_1)}(\xi)|_{\xi=1}\) and \([2/2]_{(\rho_1)}(\xi)|_{\xi=1}\) of expansion (21), ansatz (7) yields
\[
\hat{\phi} = [1/1]_{(\rho_1)}(\xi)|_{\xi=1} = \frac{m(c_1\sqrt{\lambda_1} + 2i\mu m)}{\sqrt{\lambda(2m + i\mu c_1\sqrt{\lambda_1})}}, \tag{22}
\]
\[
\hat{\phi} = [2/2]_{(\rho_1)}(\xi)|_{\xi=1} = \frac{4c_1\sqrt{\lambda m^2\rho_1} + i\mu(4m^3 - c_1^2\lambda m\rho_1^2)}{\sqrt{\lambda(4m^2 + c_1^2\lambda\rho_1^2)}}, \tag{23}
\]
Both (22) and (23) are exact solutions without requiring further conditions on the space of constants \(S\). The solution (23) is also obtained if we consider the ansatz \([3/3]_{(\rho_1)}(\xi)|_{\xi=1}\), \([4/4]_{(\rho_1)}(\xi)|_{\xi=1}\) or \([5/5]_{(\rho_1)}(\xi)|_{\xi=1}\).

3.2 Ansatz (ii)

Now, consider the ansatz (ii). Clearly, this set for \(\rho\) also satisfies condition (3), namely
\[
\partial_t \rho_1 = i k_{1,0} \rho_1, \quad \partial_x \rho_1 = i k_{1,1} \rho_1, \quad \partial_y \rho_1 = i k_{1,2} \rho_1, \quad \partial_z \rho_1 = i k_{1,3} \rho_1,
\]
\[
\partial_t \rho_2 = i k_{2,0} \rho_2, \quad \partial_x \rho_2 = i k_{2,1} \rho_2, \quad \partial_y \rho_2 = i k_{2,2} \rho_2, \quad \partial_z \rho_2 = i k_{2,3} \rho_2,
\]
yields the equation
\[
\hat{E}(\rho_1, \rho_2, \hat{\phi}, \ldots; S_0 \times S_1) = (k_{1,0}^2 - k_{1}^2)(\rho_1^2 \partial_{\rho_1} \hat{\phi} + \rho_1 \partial_{\rho_1} \hat{\phi}) + (k_{2,0}^2 - k_{2}^2)(\rho_2^2 \partial_{\rho_2} \hat{\phi} + \rho_2 \partial_{\rho_2} \hat{\phi}) + 2(k_{1,0} k_{2,0} - k_{1} k_{2}) \rho_1 \rho_2 \partial_{\rho_1} \partial_{\rho_2} \hat{\phi} + m^2 \hat{\phi} + \lambda \hat{\phi}^3 = 0. \tag{24}
\]

Without imposing any constraint \(\phi\), this functional ansatz yields solution (14). In addition, if we impose the constraint
\[
\psi_1 = -k_{1,0}^2 + k_{1}^2 - m^2 = 0 \tag{25}
\]
or
\[
\psi_2 = -k_{2,0}^2 + k_{2}^2 - m^2 = 0 \tag{26}
\]
we obtain solution (19) up to a redefinition of an arbitrary constant. However, if we employ constraints (25) and (26) and eliminate \(k_{1,0}\) and \(k_{2,0}\), equation (24) yields
\[
-m^2(p_1^2 \partial_{\rho_1} \hat{\phi} + \rho_1 \partial_{\rho_1} \hat{\phi} + p_2^2 \partial_{\rho_2} \hat{\phi} + \rho_2 \partial_{\rho_2} \hat{\phi}) + 2(k_{1,0} k_{2,0} - k_{1} k_{2}) \rho_1 \rho_2 \partial_{\rho_1} \partial_{\rho_2} \hat{\phi} + m^2 \hat{\phi} + \lambda \hat{\phi}^3 = 0 \tag{27}
\]
Observe that we did not eliminate the linear terms of \(k_{1,0}\) and \(k_{2,0}\) at this stage in order to avoid mistakes with the sign of the root square. The multivariate Taylor expansion of (27) yields
\[
\hat{\phi} = (c_{1,0} \rho_1 + c_{0,1} \rho_2) \xi + \left(\frac{\lambda(c_{1,0}^2 \rho_1^2 + c_{0,1}^2 \rho_2^2)}{8m^2} - \frac{3\lambda(c_{1,0} c_{0,1} \rho_1 \rho_2 + c_{1,0} c_{0,1} \rho_1 \rho_2)}{4(k_{1,0} k_{2,0} - k_{1} k_{2} - m^2)}\right) \xi^3 + ..., \tag{28}
\]
for \(m \neq 0\) and \(k_{1,0} k_{2,0} - k_{1} k_{2} - m^2 \neq 0\).
The Padé approximant \([1/1]_{(\rho_1, \rho_2)}(\xi)\) of expansion (28) yield the ansatz
\[
\hat{\phi} = c_{1,0}\bar{\rho}_1 + c_{0,1}\bar{\rho}_2, \tag{29}
\]
such that the conditions for (29) to become an exact solution are
\[
\begin{align*}
\tilde{E}_{1;3,0}(\hat{S}) &= c_{1,0}\lambda = 0, \\
\tilde{E}_{1;2,1}(\hat{S}) &= 3c_{1,0}^2c_{0,1}\lambda = 0, \\
\tilde{E}_{1;1,2}(\hat{S}) &= 3c_{1,0}c_{0,1}^2\lambda = 0, \\
\tilde{E}_{1;0,3}(\hat{S}) &= c_{0,1}\lambda = 0, \\
D_1(\hat{S}) &= 1 \neq 0.
\end{align*}
\]
Hence, the only possibility for (29) to be a solution of (24) is in the Klein-Gordon limit, i.e. \(\lambda = 0\). Employing the ansatz \([2/2]_{(\rho_1, \rho_2)}(\xi)\) for (30) to be a solution of (24), we obtain
\[
\hat{\phi} = \frac{8m^2(-k_{1,0}k_{2,0} + k_{1,1}k_{2}^2 + m^2)(c_{1,0}\tilde{\rho}_1 + c_{0,1}\tilde{\rho}_2)}{(k_{1,0}k_{2,0} - k_{1,1}k_{2} + m^2)(\lambda(c_{1,0}^2\tilde{\rho}_1^2 - c_{1,0}c_{0,1}\tilde{\rho}_1\tilde{\rho}_2 + c_{0,1}^2\tilde{\rho}_2^2) - 8m^2) - 6m^2c_{1,0}c_{0,1}\lambda\tilde{\rho}_1\tilde{\rho}_2}
\tag{30}
\]
By substituting ansatz (30) into (24), we obtain the conditions
\[
\begin{align*}
\tilde{E}_{1;3,1}(\hat{S}) &= 64c_{1,0}^4c_{0,1}\lambda^2m^2(k_{1,0}k_{2,0} - k_{1,1}k_{2} - m^2)^2(k_{1,0}k_{2,0} - k_{1,1}k_{2} + m^2) \\
&= (k_{1,0}k_{2,0} - k_{1,1}k_{2} + 2m^2) = 0, \\
\tilde{E}_{1;3,2}(\hat{S}) &= -96c_{1,0}^3c_{0,1}\lambda^2m^2(k_{1,0}k_{2,0} - k_{1,1}k_{2} - m^2)^2(k_{1,0}k_{2,0} - k_{1,1}k_{2} + m^2) = 0, \\
\tilde{E}_{1;2,3}(\hat{S}) &= -96c_{1,0}^3c_{0,1}\lambda^2m^2(k_{1,0}k_{2,0} - k_{1,1}k_{2} - m^2)^2(k_{1,0}k_{2,0} - k_{1,1}k_{2} + m^2) = 0, \\
\tilde{E}_{1;1,4}(\hat{S}) &= 64c_{1,0}^4c_{0,1}\lambda^2m^2(k_{1,0}k_{2,0} - k_{1,1}k_{2} - m^2)^2(k_{1,0}k_{2,0} - k_{1,1}k_{2} + m^2) \\
&= (k_{1,0}k_{2,0} - k_{1,1}k_{2} + 2m^2) = 0, \\
D_1(\hat{S}) &= [(k_{1,0}k_{2,0} - k_{1,1}k_{2} - m^2)^2(c_{1,0}^2\tilde{\rho}_1^2 - c_{1,0}c_{0,1}\lambda\tilde{\rho}_1\tilde{\rho}_2 + c_{0,1}^2\lambda\tilde{\rho}_2^2 - 8m^2) \\
&+ 6m^2c_{1,0}c_{0,1}\lambda\tilde{\rho}_1\tilde{\rho}_2]^3 \neq 0.
\end{align*}
\]
The system of conditions above has four possibilities for solutions, namely \(c_{1,0} = 0\), \(c_{0,1} = 0\), \(\lambda = 0\) or
\[
k_{1,0}k_{2,0} - k_{1,1}k_{2} + m^2 = 0. \tag{31}
\]
It is easy to see that conditions \(c_{1,0} = 0\) and \(c_{0,1} = 0\) yield the same solution (19) up to a redefinition of the arbitrary constants, while the condition \(\lambda = 0\) yields the solution (29). We can also use condition (31) and the constraints (25) and (26) for eliminating three constants, for example
\[
\begin{align*}
k_{1,0} &= \frac{\nu_1k_{2,1}\sqrt{m^2\sum_{j=2}^3(k_{1,j} - k_{2,j})^2 - (k_{1,3}k_{2,2} - k_{1,2}k_{2,3})^2 + \nu_2\sqrt{k_{2}^2 - m^2}(\sum_{j=2}^3k_{1,j}k_{2,j} - m^2)}}{\sum_{j=2}^3k_{2,j}^2 - m^2} \\
k_{2,0} &= \nu_2\sqrt{k_{2}^2 - m^2} \\
k_{1,1} &= \frac{k_{2,1}(\sum_{j=2}^3k_{1,j}k_{2,j} - m^2) + \nu_1\nu_2\sqrt{(k_{2}^2 - m^2)(\sum_{j=2}^3(k_{1,j} - k_{2,j})^2 - (k_{1,3}k_{2,2} - k_{1,2}k_{2,3})^2)}}{\sum_{j=2}^3k_{2,j}^2 - m^2}
\end{align*}
\tag{32}
\tag{33}
\tag{34}
where \( \nu_1 = \pm 1 \) and \( \nu_2 = \pm 1 \). Therefore, expression (30) is a solution of (10) with \( \tilde{\rho}_1 = e^{i(k_1,0,1+k_1,1,1+x+k_1,2,0+k_1,3,0)} \), \( \tilde{\rho}_2 = e^{i(k_2,0,1+k_2,1,1+x+k_2,2,0+k_2,3,0)} \) and the constants (32), (33) and (34).

Now, let us consider equation (24) and the set \( \rho \) with the constraints
\[
\psi_1 = -k_{1,0}^2 + k_1^2 + 2m^2 = 0, \quad (35) \\
\psi_2 = -k_{2,0}^2 + k_2^2 + 2m^2 = 0. \quad (36)
\]
If we consider only one of these constants, we will obtain the solution (22) again, up to a redefinition of the arbitrary constants. However, if both constraints are considered we can eliminate \( k_{1,0}, k_{1,0} \), such that (24) yields
\[
2m^2(\rho_1^2 \partial_1 \hat{\phi} + \rho_1 \partial_1 \hat{\phi} + \rho_2^2 \partial_2 \hat{\phi} + \rho_2 \partial_2 \hat{\phi}) + 2(k_{1,0}k_{2,0} - k_1.k_2)\rho_1 \rho_2 \partial_1 \rho_1 \partial_2 \hat{\phi} + m^2 \hat{\phi} + \lambda \hat{\phi}^3 = 0. \quad (37)
\]
Observe that we did not substitute linear terms of \( k_{1,0} \) and \( k_{2,0} \) again. The multi-variate Taylor expansion of (37) yields
\[
\hat{\phi} = \frac{i u m}{\sqrt{\lambda}} + (c_{1,0} \tilde{\rho}_1 + c_{0,1} \tilde{\rho}_2) \xi - \frac{i u m c_{1,0}^2 \tilde{\rho}_1^2 + c_{0,1}^2 \tilde{\rho}_2^2}{2m} + \frac{3 i u m \lambda c_{1,0} c_{0,1} \tilde{\rho}_1 \tilde{\rho}_2}{(k_{1,0}k_{2,0} - k_1.k_2 + m^2)} \xi^2 + \ldots. \quad (38)
\]
for \( m \neq 0, \lambda \neq 0 \) and \( k_{1,0}k_{2,0} - k_1.k_2 + m^2 \neq 0 \). The Padé approximant \( [1/1]_{(\rho_1, \rho_2)}(\xi)|\xi=1 \) of (38) yields the ansatz
\[
\hat{\phi} = m \left( (k_{1,0}k_{2,0} - k_1.k_2 + m^2)(\sqrt{\lambda}(c_{1,0}^2 \tilde{\rho}_1^2 + c_{0,1}^2 \tilde{\rho}_2^2 + 4 c_{1,0} c_{0,1} \tilde{\rho}_1 \tilde{\rho}_2) + 2 i u m (c_{1,0} \tilde{\rho}_1 + c_{0,1} \tilde{\rho}_2)) \right.
\]
\[
- 6m^2 \sqrt{\lambda} c_{1,0} c_{0,1} \tilde{\rho}_1 \tilde{\rho}_2 \bigg) / \left( (k_{1,0}k_{2,0} - k_1.k_2 + m^2)(i u \lambda(c_{1,0}^2 \tilde{\rho}_1^2 + c_{0,1}^2 \tilde{\rho}_2^2) + 2m \sqrt{\lambda} c_{1,0} \tilde{\rho}_1 + c_{0,1} \tilde{\rho}_2) \right)
\]
\[
+ c_{0,1} \tilde{\rho}_2) + 6 i u \lambda m^2 c_{1,0} c_{0,1} \tilde{\rho}_1 \tilde{\rho}_2 \bigg), \quad (39)
\]
and the conditions for (39) to become an exact solution are
\[
\hat{E}_{1,5,1}(\hat{S}) = 8c_{1,0}^2 c_{0,1} \lambda^{3/2} m(k_{1,0}k_{2,0} - k_1.k_2 + m^2)(k_{1,0}k_{2,0} - k_1.k_2 + m^2)^2
\]
\[
(k_{1,0}k_{2,0} - k_1.k_2 - 2m^2) = 0
\]
\[
\hat{E}_{1,4,2}(\hat{S}) = -96c_{1,0}^4 c_{0,1}^2 \lambda^{3/2} m^5(k_{1,0}k_{2,0} - k_1.k_2 + m^2)(k_{1,0}k_{2,0} - k_1.k_2 - 2m^2) = 0
\]
\[
\hat{E}_{1,3,3}(\hat{S}) = 16c_{1,0}^3 c_{0,1}^3 \lambda^{3/2} m(k_{1,0}k_{2,0} - k_1.k_2 + m^2)(k_{1,0}k_{2,0} - k_1.k_2 - 2m^2)(8m^2(k_{1,0}k_{2,0} - k_1.k_2 - 2m^2)
\]
\[
= 3((k_{1,0}k_{2,0} - k_1.k_2))^2 + 5m^4 \big) = 0
\]
\[
\hat{E}_{1,2,4}(\hat{S}) = -96c_{1,0}^4 c_{0,1} \lambda^{3/2} m^5(k_{1,0}k_{2,0} - k_1.k_2 + m^2)(k_{1,0}k_{2,0} - k_1.k_2 - 2m^2)
\]
\[
\hat{E}_{1,1,5}(\hat{S}) = 8c_{1,0}^5 c_{0,1} \lambda^{3/2} m(k_{1,0}k_{2,0} - k_1.k_2 - 2m^2)(k_{1,0}k_{2,0} - k_1.k_2 + m^2)^2
\]
\[
(k_{1,0}k_{2,0} - k_1.k_2 - 2m^2) = 0
\]
\[
D_1(\hat{S}) = [(k_{1,0}k_{2,0} - k_1.k_2 + m^2)(i u \lambda(c_{1,0}^2 \lambda \tilde{\rho}_1^2 + c_{0,1}^2 \lambda \tilde{\rho}_2^2) + 2m \sqrt{\lambda} (c_{1,0} \tilde{\rho}_1 + c_{0,1} \tilde{\rho}_2)
\]
\[
+ 6 i u \lambda m^2 c_{1,0} c_{0,1} \tilde{\rho}_1 \tilde{\rho}_2) \big) \neq 0.
\]

The system for the above conditions above has three possibilities of solutions, namely \( c_{1,0} = 0, c_{0,1} = 0, \) or
\[
k_{1,0}k_{2,0} - k_1.k_2 - 2m^2 = 0. \quad (40)
\]
The conditions $c_{1,0} = 0$ and $c_{0,1} = 0$ yield the solution (22) up to a redefinition of the arbitrary constants, while condition (40) and the constraints (35) and (36) eliminate three constants, for example,

$$
k_{1,0} = \nu_1 k_{2,1} \sqrt{-2m^2 \sum_{j=2}^{3} (k_{1,j} - k_{2,j})^2 - (k_{1,3} k_{2,2} - k_{1,2} k_{2,3})^2} + \nu_2 \sqrt{k_2^2 + 2m^2 (\sum_{j=2}^{3} k_{1,j} k_{2,j} + 2m^2)}
$$

$$
k_{2,0} = \nu_2 \sqrt{k_2^2 + 2m^2}
$$

$$
k_{1,1} = \frac{k_{2,1} (\sum_{j=2}^{3} k_{1,j} k_{2,j} + 2m^2) + \nu_1 \nu_2 \sqrt{(k_2^2 + 2m^2)[-2m^2 \sum_{j=2}^{3} (k_{1,j} - k_{2,j})^2 - (k_{1,3} k_{2,2} - k_{1,2} k_{2,3})^2]}}{\sum_{j=2}^{3} k_{2,j}^2 + 2m^2}
$$

(41)

(42)

(43)

where $\nu_1 = \pm 1$ and $\nu_2 = \pm 1$. Therefore, the expression (39) is a solution of the $\lambda \phi^4$ model with $\bar{\rho}_1 = e^{i(k_{1,0} t + k_{1,1} x + k_{1,2} y + k_{1,3} z)}$, $\bar{\rho}_2 = e^{i(k_{2,0} t + k_{2,1} x + k_{2,2} y + k_{2,3} z)}$ and the constants (11), (12) and (13).

The algorithm presented here could also be used with Padé approximants of higher degree or different functional ansatz. However, we will not consider other ansatz due computational limitations.

4 Conclusions

In this paper, it was shown an algorithm for solving nonlinear partial differential equations based on Padé approximants. The algorithm was applied to the $\lambda \phi^4$ model in 4 dimensions by using two functional ansatzes and it lead us to new solutions for the model. There are many recent papers proposing methods for solving differential equations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and the approach presented here could be an easier algorithm for applying to more complicated model.

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