Octonionic bimodule

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Abstract

The structure of octonionic bimodules is formulated in this paper. It turns out that every octonionic bimodule is a tensor product, the category of octonionic bimodules is isomorphic to the category of real vector spaces. We show that there is also a real part structure on octonionic bimodules similar to the quaternion case. Different from the quaternion setting, the octonionic bimodule structure is uniquely determined by its left module structure and hence the real part can be obtained only by left multiplication. The structure of octonionic submodules generated by one element is more involved, which leads to many obstacles to further development of the octonionic functional analysis. We introduce a notion of cyclic decomposition to deal with this difficulty. Using this concept, we give a complete description of the submodule generated by one element in octonionic bimodules. This paper clears the barrier of the structure of O-modules for the later study of octonionic functional analysis.

Keywords: Octonionic bimodule; real part; cyclic decomposition.

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1 Introduction

Theory of quaternion Hilbert spaces has been studied a lot ([7, 13, 14, 17, 18]). The Theory of spherical spectrum of normal operator, continuous slice functional calculus in quaternionic Hilbert spaces had been also established ([3]). The quaternionic vector spaces have also been studied thoroughly ([12]), which makes it well-grounded to do further study on the quaternionic functional analysis. However, in the octonion case, the structure of the one-sided modules and bimodules over octonion are not completely clear, this leads to many obstacles to further development of the octonionic functional analysis, although there are also some results on the study of octonion Hilbert spaces ([4, 5, 10, 11]). Consequently, it is worth discussing the structure of $\mathbb{O}$-modules.

The general case of the structures of bimodules over Jordan algebra and alternative algebra has been studied by Jacobson in [9]. The left-alternative left-modules for the real algebra of octonions have been considered, and the irreducible ones are known to be isomorphic to the regular or conjugate regular modules (See, for example, Chapter 11 of the monograph by Zhevlakov, Slinko, Shirshov, and Shestakov [19]). And we point out that the $\mathbb{O}$-vector space studied in [10] is not an $\mathbb{O}$-bimodule under the definition given in [9]. It is actually a left $\mathbb{O}$-module with an irrelevant right $\mathbb{O}$-module structure. And it seems no proof for the structure of such $\mathbb{O}$-vector spaces being a tensor product, which has been used several times.

In quaternion case, Ng gives a systematic study in [12], which shows the category of quaternion vector spaces, that is, quaternion bimodules, is equivalent to the category of real vector spaces. More precisely, there is a natural structure of real part on each quaternion bimodule, which is the corresponding real vector space. A natural question is whether similar results hold in octonion case.

In previous work [8], we have formulated the structures of left $\mathbb{O}$-modules. It shows that there is an isomorphism between the category $\text{O-Mod}$ and the category $\text{Cl}_7$-$\text{Mod}$, here the object in $\text{O-Mod}$ is left $\mathbb{O}$-module. Each left $\mathbb{O}$-module $M$ will be of the form

$$M = \mathbb{O}\mathcal{A}(M) \oplus \mathbb{O}\mathcal{A}^- (M).$$

Where $\mathcal{A}(M)$ and $\mathcal{A}^- (M)$ represent the subset of associative elements and conjugate associative elements respectively:

$$\mathcal{A}(M) := \{ m \in M \mid [p, q, m] = 0, \forall p, q \in \mathbb{O} \};$$

and

$$\mathcal{A}^- (M) := \{ m \in M \mid (pq)m = q(pm), \forall p, q \in \mathbb{O} \}.$$ 

It is therefore natural to ask whether a given left $\mathbb{O}$-module admits a compatible bimodule structure, and if so, is it unique? if not, what is the condition for a left $\mathbb{O}$-module to admit a compatible bimodule structure. In this paper, we direct ourselves to answering these questions.

In this paper, we show that the necessary and sufficient condition for a left $\mathbb{O}$-module admitting a compatible bimodule structure is just the vanishing of the subset of conjugate associative elements. And if so, the bimodule structure is then uniquely determined by its left multiplication. More precisely, we obtain:

**Theorem 1.1.** A left $\mathbb{O}$-module $M$ admits a compatible $\mathbb{O}$-bimodule structure if and only if it holds $M = \mathbb{O}\mathcal{A}(M)$.

Moreover, if $M$ admits an $\mathbb{O}$-bimodule structure, then it is unique.
There is also a structure of real part on $\mathbb{O}$-bimodules as in quaternion case:

$$Re \ x = \frac{5}{12} x - \frac{1}{12} \sum_{i=1}^{7} e_i xe_i.$$  

And we can rewrite this formula in terms of left multiplication:

$$Re \ x = x + \frac{1}{48} \sum_{i,j,k} \epsilon_{ijk} e_i[e_j, e_k, x],$$

where the symbol $\epsilon_{ijk}$ depends on the multiplication table of the octonions. Using this one easily obtains that an $\mathbb{O}$-bimodule $M$ is isomorphic to the tensor product $Re \ M \otimes \mathbb{O}$, coherent with the quaternion case. Therefore, we get that the category of $\mathbb{O}$-bimodules is isomorphic to the category of $\mathbb{R}$-vector spaces.

Our last topic is about the structure of submodules generated by one element. In contrast to the complex or quaternion setting, some new phenomena occur in the setting of octonions, which has already been known in [4]: If $m$ is an element of an octonionic module, then

- $\mathbb{O}m$ is not a submodule in general.
- The submodule generated by $m$ maybe the whole module.

This means that the structure of octonionic submodules is more involved and such property is crucial for classical functional analysis. We point out that some gaps appear in establishing the octonionic version of Hahn-Banach Theorem by taking $\mathbb{O}m$ as a submodule ([10, Lemma 2.4.2]). The submodule generated by a submodule $Y$ and a point $x$ is not of the form $\{y+px \mid y \in Y, p \in \mathbb{O}\}$, this is wrong even for the case $Y = \{0\}$. It means the proof can not repeat the way in canonical case. The involved structure of octonion submodules accounts for the slow developments of octonion Hilbert spaces. We shall give a new proof in a later paper.

This phenomena motivates us to introduce a new notion of cyclic elements, which play a key role in the study of submodules. An element $m$ in a given module $M$ is called cyclic elements if the submodule generated by it is exactly $\mathbb{O}m$. We next introduce a notion of cyclic decomposition to describe the structure of these submodules generated by one element. It turns out that each element can be decomposed into a sum of some special cyclic elements. More precisely, we obtain:

**Theorem 1.2.** Let $m$ be an arbitrary element of an $\mathbb{O}$-bimodule $M$. Then

$$\langle m \rangle_\mathbb{O} = \bigoplus_{i=1}^{n} \mathbb{O}m_i,$$

where $\{m_i\}_{i=1}^{n} \subseteq \mathcal{C}(M)$ is an arbitrary cyclic decomposition of $m$.

The length of a cyclic decomposition is therefore an invariant of $m$ and by definition at most 8, hence there are only 8 kinds of elements in $\mathbb{O}$-bimodules and each element $m$ will generate a submodule with dimension $\dim_\mathbb{R} \langle m \rangle_\mathbb{O} \leq 64$.

By the way, we point out a mistake in [10]. It appears in the proof of the corollary of Hahn Banach Theorem ([10, Lemma 2.4.2]), which declared every element in an $\mathbb{O}$-module will satisfy $\mathbb{O}x = x\mathbb{O}$. In fact, with the help of the notion of cyclic decomposition, we shall show that only cyclic elements posses such property.
2 Preliminaries

In this section, we review some basic properties of the algebra of the octonions $\mathbb{O}$ and one-sided $\mathbb{O}$-modules, and introduce some fundamental notations.

2.1 The octonions $\mathbb{O}$

The algebra of the octonions $\mathbb{O}$ is a non-associative, non-commutative, normed division algebra over the $\mathbb{R}$. Let $e_1, \ldots, e_7$ be its natural basis throughout this paper, i.e.,

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i, j = 1, \ldots, 7.$$ 

For convenience, we denote $e_0 = 1$.

In terms of the natural basis, an element in octonions can be written as

$$x = x_0 + \sum_{i=1}^{7} x_i e_i, \quad x_i \in \mathbb{R},$$

The conjugate octonion of $x$ is defined by $\overline{x} = x_0 - \sum_{i=1}^{7} x_i e_i$, and the norm of $x$ equals $|x| = \sqrt{x \overline{x}} \in \mathbb{R}$, the real part of $x$ is $\text{Re} x = x_0 = \frac{1}{2}(x + \overline{x})$. The term $\sum_{i=1}^{7} x_i e_i$ will be abbreviated as $\sum x_i e_i$ in this paper. We denote by $\mathbb{S}$ the set of imaginary units in $\mathbb{O}$:

$$\mathbb{S} := \{ J \in \mathbb{O} \mid J^2 = -1 \}.$$ 

Then there is a book structure on octonions:

$$\mathbb{O} = \bigcup_{J \in \mathbb{S}} \mathbb{C}_J,$$

here $\mathbb{C}_J$ represents the complex plane spanned by $\{1, J\}$.

The associator of three octonions is defined as

$$[x, y, z] = (xy)z - x(yz)$$

for any $x, y, z \in \mathbb{O}$, which is alternative in its arguments and has no real part. That is, $\mathbb{O}$ is an alternative algebra and hence it satisfies the so-called R. Monfang identities [16]:

$$(xy)z = x(y(xz)), \quad z(xy) = ((zx)y)x, \quad x(yz)x = (xy)(zx).$$

The commutator is defined as

$$[x, y] = xy - yx.$$ 

One can prove that, for any $J \in \mathbb{S}$ and $x \in \mathbb{C}_J \setminus \mathbb{R}$, we have

$$\{ p \in \mathbb{O} \mid [p, x] = 0 \} = \mathbb{C}_J.$$ 

The full multiplication table is conveniently encoded in the 7-point projective plane, which is often called the Fano mnemonic graph. In the Fano mnemonic graph, the vertices are labeled by $1, \ldots, 7$ instead of $e_1, \ldots, e_7$. Each of the 7 oriented lines gives a quaternionic triple. The product of any two imaginary units is given by the third unit on the unique line connecting them, with the sign determined by the relative orientation.

Fig.1 Fano mnemonic graph
It will be convenient to use an $\epsilon$-notation that will now be introduced (see [2]). This is the unique symbol that is skew-symmetric in either three or four indices. One way to think of this symbol is:

$$
\begin{align*}
\epsilon_i \epsilon_j &= \epsilon_{ijk} \epsilon_k - \delta_{ij} \\
[e_i, e_j, e_k] &= 2\epsilon_{ijkl} \epsilon_l
\end{align*}
$$

The symbol $\epsilon$ satisfies various useful identities. For example (using the summation convention),

$$
\begin{align*}
\epsilon_{ijk} \epsilon_{ijl} &= 6 \delta_{kl} \\
\epsilon_{ijq} \epsilon_{ijkl} &= 4 \epsilon_{qkl} \\
\epsilon_{spq} \epsilon_{ijk} &= \epsilon_{pqjk} + \delta_{pj} \delta_{qk} - \delta_{pk} \delta_{qi} \\
\epsilon_{ipq} \epsilon_{ijkl} &= \delta_{pj} \epsilon_{qkl} - \delta_{jq} \epsilon_{tpl} + \delta_{pk} \epsilon_{jql} - \delta_{ql} \epsilon_{jpk}
\end{align*}
$$

We shall always use the Einstein summation convention when we compute in terms of $\epsilon$-notation.

The 14-dimensional group $G_2$ is the smallest of the five exceptional Lie groups and is closely related to the octonions. In particular, $G_2$ can be defined as the automorphism group of the octonion algebra:

$$
G_2 := \{ g \in GL(O) \mid g(xy) = g(x)g(y) \text{ for all } x, y \in \mathbb{O} \}.
$$

As a fact, $G_2 \subseteq SO(7)$. We refer to [6, 15] for more details.

We can now use octonion multiplication to define a vector cross product $\times$ on $\mathbb{R}^7$ ([16]). Given $u, v \in \mathbb{R}^7$, we regard them as elements in $Im(O)$, then

$$
u \times v := Im(uv).
$$

A 3-dimensional subspace $\Lambda \subset Im(O)$ is called **associative** if the associator bracket vanishes on $\Lambda$, i.e.,

$$
[u, v, w] = 0 \text{ for all } u, v, w \in \Lambda.
$$

If $u, v \in \Lambda$ are linearly independent, then the subspace spanned by the vectors $u, v, u \times v$ is associative (see [16]). This subspace will be denoted by $\Lambda(u, v)$. By definition, it is easy to verify:

$$
\Lambda(u, v) = \{ x \in Im(O) \mid [u, v, x] = 0 \}.
$$

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2.2 The definition of $\mathbb{O}$-modules

It’s well known that the algebra of octonions $\mathbb{O}$ is an alternative algebra, thus that an $\mathbb{O}$-module $M$ is actually an alternative module. From now on let $A$ be a unital alternative algebra over a field $F$. For the sake of completeness, we give the definition of modules over alternative algebra as follows.

**Definition 2.1.** An $F$-vector space $M$ is called a left alternative module over $A$, if there is an $F$-linear map $L : A \to \text{End}_F M$, $a \mapsto L_a$ satisfying $L_1 = \text{id}_M$ and

$$[a, b, x] = -[b, a, x], \quad \forall a, b \in A, \ x \in M.$$

Here the associator is defined by $[a, b, x] := (ab)x - a(bx)$. The definition of right alternative algebraic module is similar. A left alternative $A$-module $M$ is called an alternative bimodule if the associator is alternative:

$$[p, q, m] = [m, p, q] = [q, m, p],$$

for all $p, q \in A$, and for all $m \in M$. Where the middle associator $[q, m, p]$ is defined by

$$[q, m, p] := (qm)p - q(mp),$$

and the right associator $[p, q, m]$ is defined by

$$[p, q, m] := (pq)m - p(qm).$$

**Remark 2.2.** Let $M$ be an $A$-module. For all $m, m' \in M$, $\alpha, \beta \in F$ and all $a, a' \in A$, we have:

(i). $L_a \in \text{End}_F M \Rightarrow a(\alpha m + \beta m') = \alpha(am) + \beta(am')$. In particular, $a(\alpha m) = \alpha(am)$.

(ii). $L \in \text{Hom}_F(A, \text{End}_F M) \Rightarrow (\alpha a + \beta a')m = \alpha am + \beta a'm$.

In particular, $(\alpha a)m = (\alpha a)m = \alpha(am) = a(\alpha m)$ and thus we can write $a\alpha m$ unambiguously.

(iii). Thinking of $M$ as an $F$-vector space, the scalar multiplication over $F$ coincides with $L|_F$ since $L(1) = \text{id}_M$.

Note that the left alterativity requirement of the associator in $M$ is equivalent to the following condition given in [4, 10]

$$a^2 m = a(am), \text{ for all } a \in A, m \in M.$$  

The proof is trivial by polarizing the above relation. And the notion of $A$-bimodules here agrees with the definition given in [9]. However, the following postulates:

$$1m = m1 = m, \quad a^2m = a(am), \quad ma^2 = (ma)a, \quad (am)a = a(ma)$$

cannot deduce that $M$ is an alternative bimodule in general.

One useful identity which holds in any left module $M$ is

$$[p, q, r]m + p[q, r, m] = [pq, r, m] - [p, qr, m] + [p, q, rm]. \quad (2.7)$$

Here $m \in M$ is an arbitrary element and it holds for all $p, q, r \in A$.

The Moufang identities and Artin Theorem hold as before.
Theorem 2.3 (Moufang identities). Let $M$ be an alternative algebraic bimodule over $A$. Then for all $p, q \in A, m \in M$, the Moufang identities hold:

\begin{align*}
(pmp)q &= p(m(pq)) \quad (2.8) \\
q(pmp) &= ((qp)m)p \quad (2.9) \\
(pq)(mp) &= p(qm)p. \quad (2.10)
\end{align*}

Proof. The proof is similar as in classical case. We only prove the first identity.

\[
(pmp)q - p(m(pq)) = [pm, p, q] + (pm)(pq) - p(mpq)
\]
\[
= [pm, p, q] + [p, m, pq]
\]
\[
= -[p, pm, q] - [p, pq, m]
\]
\[
= -p^2mq + p((pm)q) - (p^2q)m + p((pq)m)
\]
\[
= -p^2mq - [p, m, q] + [p, m, q] + [p, pq, m]
\]
\[
= p([p, m, q] + [p, q, m])
\]
\[
= 0
\]

The rest proof runs as classical case.

\[\square\]

Theorem 2.4 (Artin Theorem). Let $M$ be a left alternative algebraic module over $A$. Then $[p^m, p^n, x] = 0$, for all $p \in A$, any $m, n \in \mathbb{N}$ and $x \in M$.

Proof. The proof will be divided into two steps.

Step 1. $[p^m, p, x] = 0$, \forall $p \in A, \forall m \in \mathbb{N}, \forall x \in M$.

Clearly it holds for $m = 1$. Assume the formula holds for degree $k$, we will prove it for $k + 1$. By induction hypothesis, $p(p^{k+1}x) = p(p^{k}(px)) = p^{k+1}(px)$, hence $[p^{k+1}, p, x] = p^{k+2}x - p^{k+1}(px) = p^{k+2}x - p(p^{k+1}x) = [p, p^{k+1}, x]$.

By definition of alternative algebraic bimodule, we thus conclude that $[p^{k+1}, p, x] = 0$.

Step 2. $[p^m, p^n, x] = 0, \forall p \in A, \forall m, n \in \mathbb{N}, \forall x \in M$.

Fix $m$, we prove this by induction on $n$. We have just proved for case $n = 1$. Assume the formula holds for degree $n = k$, we will prove it for $n = k + 1$.

\[
p^m(p^{k+1}x) = p^m((p^k)p) x
\]
\[
= p^m(p^k(px))
\]
\[
= (p^m p^k)(px)
\]
\[
= p^{m+k}(px)
\]
\[
= p^{m+k+1}x
\]

That is, $[p^m, p^{k+1}, x] = 0$. This proves the theorem.

\[\square\]
Our previous work gives a complete classification of left $\mathcal{O}$-modules [8]. The irreducible ones are already known to be isomorphic to the regular or conjugate regular modules [19]. However, using the relation between octonions to Clifford algebra, we can give a more simple proof and classify the structure of left $\mathcal{O}$-modules completely.

It is well-known (for example, [1, 6]) that the octonions have a very close relationship with spinors in 7, 8 dimensions. In particular, multiplication by imaginary octonions is equivalent to Clifford multiplication on spinors in 7 dimensions. It follows that the category of left $\mathcal{O}$-modules is isomorphic to the category of left $\mathcal{C}\ell_7$-modules. Note that $\mathcal{C}\ell_7$ is a semi-simple algebra, we thus obtain that there are only two kinds of irreducible left $\mathcal{O}$-module. They are the regular module $\mathcal{O}$ and the conjugate regular module $\overline{\mathcal{O}}$. Where the left module structure of $\mathcal{O}$ is defined by

$$p \cdot x := px,$$

for all $p \in \mathcal{O}$, and all $x \in \mathcal{O}$. The associator on $\overline{\mathcal{O}}$ is as follows:

$$[p, q, x]_{\overline{\mathcal{O}}} = [p, q, x] + [p, q]x.$$  \hfill (2.11)

The direct sum of their several copies exhaust all octonion modules with finite dimensions. The structure of general left $\mathcal{O}$-modules is then clear:

**Theorem 2.5.** Let $M$ be a left $\mathcal{O}$-module. Then

$$M \cong \mathcal{O}A(M) \oplus \mathcal{O}A^{-}(M).$$

Where $\mathcal{A}(M)$ is the set of all associative elements:

$$\mathcal{A}(M) := \{ m \in M \mid [p, q, m] = 0, \ \forall p, q \in \mathcal{O} \};$$

$\mathcal{A}^{-}(M)$ is the set of all conjugate associative elements:

$$\mathcal{A}^{-}(M) := \{ m \in M \mid (pq)m = q(pm), \forall p, q \in \mathcal{O} \}.$$  

Its proof will depend on the following lemma.

**Lemma 2.6.** Let $M$ be a left $\mathcal{O}$-module, then $\langle m \rangle_\mathcal{O}$ is finite dimensional for any $m \in M$. More precisely, the dimension is at most 128.

**Proof.** $\langle m \rangle_\mathcal{O}$ is such module generated by $e_{i_1}(e_{i_2}(\cdots(e_{i_n}m)))$, where $i_k \in \{1, 2, \ldots, 7\}, n \in \mathbb{N}$. Note that

$$e_i(e_jm) + e_j(e_im) = (e_i e_j + e_j e_i)m = -2\delta_{ij}m,$$

hence the element defined by $e_{i_1}(e_{i_2}(\cdots(e_{i_n}m)))$ for $n > 7$ can be reduced. Thus the vectors $\{ m, e_1m, \ldots, e_7m, e_1(e_2m), \ldots, e_1(e_2(\cdots(e_7m))) \}$ will generate $\langle m \rangle_\mathcal{O}$, we conclude that $\dim_{\mathbb{R}} \langle m \rangle_\mathcal{O} \leq C_0^7 + C_1^7 + \cdots + C_7^7 = 128$. \hfill $\square$

**Remark 2.7.** In fact, this property has already appeared in [4]. However, it is worth stressing the essentiality of this property. It enables us to characterize the structure of general left $\mathcal{O}$-modules in terms of finite dimensional case, which is already clear in view of the structure of the $\mathcal{C}\ell_7$-modules.
3 Bimodule structure on $\mathbb{O}$-modules

As shown in previous work [8], each left $\mathbb{O}$-module $M$ is some copies of $\mathbb{O}$ and $\overline{\mathbb{O}}$. It is natural to ask whether a given left $\mathbb{O}$-module admits a compatible $\mathbb{O}$-bimodule structure, and if so, is it unique? if not, what is the condition for a left $\mathbb{O}$-module to admit a compatible bimodule structure. In this section, we direct ourselves to answering these questions.

3.1 Bimodule structure in low dimensions

In this subsection, we are concerned with the $\mathbb{O}$-bimodule structures in low dimensional cases. The general case will be proved in a similar way in the sequel. We begin this subsection by proving a technical lemma which is useful later.

Lemma 3.1. Let $f \in \text{End}_R(\mathbb{O})$. Then the following are equivalent:

(i) $\text{Re} \left( f(px) - pf(x) \right) = 0$ for all $p, x \in \mathbb{O}$.

(ii) $f(x) = f_0(x) - \sum e_i f_0(e_i x)$, where $f_0(x) = \text{Re} f(x)$.

(iii) There exists an octonion $q \in \mathbb{O}$, such that $f(x) = xq$.

Proof. We prove (i) $\implies$ (ii). Suppose $f(x) = f_0(x) + \sum e_i f_i(x)$, where $f_j(x) \in \mathbb{R}$, $j = 0, 1, \ldots, 7$. Using $\epsilon$-notation, we have

$$e_i f(x) = e_i f_0(x) + e_i \sum e_j f_j(x)$$

$$= -f_1(x) + e_i f_0(x) + \sum e_{jk} e_k f_j(x)$$

and

$$f(e_i x) = f_0(e_i x) + \sum e_j f_j(e_i x).$$

It follows from assertion (i) that $\text{Re} (e_i f(x) - f(e_i x)) = 0$, we infer that $f_i(x) = -f_0(e_i x)$. Thus assertion (ii) holds.

We prove (ii) $\implies$ (iii). Denote by $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ the real inner product on $\mathbb{O} \cong \mathbb{R}^8$, and define

$$\langle x, y \rangle_{\mathbb{O}} := x\overline{y}.$$ 

By straight-forward calculation, we obtain

$$\text{Re} \langle x, y \rangle_{\mathbb{O}} = \langle x, y \rangle_{\mathbb{R}}.$$ 

It follows that,

$$\langle e_i x, y \rangle_{\mathbb{R}} = \text{Re} \langle e_i x, \overline{y} \rangle = \text{Re} \langle e_i (x \overline{y}) \rangle = \text{Re} \langle e_i \langle x, y \rangle_{\mathbb{O}} \rangle.$$ 

This immediately implies

$$\langle x, y \rangle_{\mathbb{O}} = \langle x, y \rangle_{\mathbb{R}} - \sum e_i \langle e_i x, y \rangle_{\mathbb{R}}.$$ 

Thinking of $(\mathbb{O}, \langle \cdot, \cdot \rangle_{\mathbb{R}})$ as a real Hilbert space and $f_0$ a real linear functional, it follows from the Riesz Representation Theorem that, there exists an element $y \in \mathbb{O}$ such that $f_0(x) = \langle x, y \rangle_{\mathbb{R}}$. Therefore by assertion (ii),

$$f(x) = \langle x, y \rangle_{\mathbb{R}} - \sum e_i \langle e_i x, y \rangle_{\mathbb{R}} = x\overline{y}.$$ 

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Setting $q = y$, then $f(x) = xq$ as desired.

We prove (iii) $\implies$ (i). Note that the associator is pure imaginary in $\mathbb{O}$, therefore
\[
f(px) - pf(x) = (px)q - p(xq) = [p, x, q] \in \text{Im}(\mathbb{O}).
\]
This completes the proof. \[\square\]

**Remark 3.2.** We introduce a new notion of linearity in the theory of octonion functional analysis utilizing assertion (i) in a later paper. It turns out that this concept plays a role of “linearity” as in the classical theory.

Utilizing this lemma, we can determine the bimodule structure on $\mathbb{O}^2$.

**Theorem 3.3.** Let the left $\mathbb{O}$-module structure of $\mathbb{O}^2$ is
\[
p(x, y) = (px, py) \quad \text{for all } p \in \mathbb{O}, (x, y) \in \mathbb{O}^2.
\]
Then there is a unique compatible bimodule structure on $\mathbb{O}^2$.

**Proof.** Suppose the right multiplication is given by
\[
(x, 0) \cdot p = (f_p(x), g_p(x)); \quad (0, x) \cdot p = (h_p(x), l_p(x)),
\]
and hence
\[
(x, y) \cdot p = (f_p(x) + h_p(y), g_p(x) + l_p(y)).
\]

**Step 1.** For all $p \in \mathbb{O}$, $f_p, g_p, h_p, l_p \in \text{End}_R(\mathbb{O})$, and they are also real linear on $p$.

In view of Remark 2.2, we have that for all $p \in \mathbb{O}$, and all $r \in \mathbb{R}$,
\[
((x, y) \cdot p)r = (x, y) \cdot (rp) = ((x, y)r) \cdot p
\]
that is,
\[
(r(f_p(x) + h_p(y)), r(g_p(x) + l_p(y))) = (f_{rp}(x) + h_{rp}(y), g_{rp}(x) + l_{rp}(y)) = (f_p(rx) + h_p(ry), g_p(rx) + l_p(ry)).
\]
Let $x$ and $y$ equal zero respectively, we obtain the conclusion.

**Step 2.** Fulfilling the condition $[p, q, (x, y)] = [q, (x, y), p]$.

In order to obtain a compatible bimodule structure, firstly we must have

\[
[p, q, (x, y)] = [q, (x, y), p]
\]
(3.1)
We compute:
\[
[q, (x, y), p] = (qx, qy) \cdot p - q(f_p(x) + h_p(y), g_p(x) + l_p(y)) \\
= (f_p(qx) + h_p(qy), g_p(qx) + l_p(qy)) - (qf_p(x) + qh_p(y), qg_p(x) + ql_p(y)) \\
= (f_p(qx) - qf_p(x) + h_p(qy) - qh_p(y), g_p(qx) - qg_p(x) + l_p(qy) - ql_p(y)).
\]
Let $y = 0$, the equation (3.1) becomes
Equations (3.3) and (3.4) imply that

\[
\begin{align*}
\{ & f_p(qx) - qf_p(x) = [p, q, x] \quad (3.2) \\
g_p(qx) - gg_p(x) = 0 \quad (3.3)
\end{align*}
\]

Let \( x = 0 \), the equation (3.1) becomes

\[
\begin{align*}
\{ & h_p(qy) - qh_p(y) = 0 \quad (3.4) \\
l_p(qy) - gg_p(y) = [p, q, y] \quad (3.5)
\end{align*}
\]

Equations (3.3) and (3.4) imply that \( g_p, h_p \in End_\mathcal{O}(\mathcal{O}) \), it is easily seen that \( Hom_\mathcal{O}(\mathcal{O}, \mathcal{O}) \cong \mathbb{R} \), we thus can assume

\[
g_p(x) = r_p x, \quad h_p(x) = s_p x, \quad r_p, s_p \in \mathbb{R}.
\]

In view of Lemma 3.1, equations (3.2) and (3.5) ensure us to assume

\[
f_p(x) = x\hat{p}, \quad l_p(x) = x\hat{p}, \quad \hat{p}, \hat{p} \in \mathcal{O}.
\]

Since \((x, y) \cdot 1 = (x, y)\), we conclude \( r_1 = s_1 = 0, \hat{1} = \hat{1} = 1 \).

**Step 3.** *Fulfilling the condition* \([p, q, (x, y)] = [(x, y), p, q]*.

In order to obtain a compatible bimodule structure, we need the following equation as well:

\[
[p, q, (x, y)] = [(x, y), p, q] \quad (3.6)
\]

for all \( p, q \in \mathcal{O} \). We compute:

\[
[(x, y), p, q] = (f_p(x) + h_p(y), g_p(x) + l_p(y)) \cdot q - (x, y) \cdot (pq)
\]

\[
\quad = (x\hat{p} + s_p y, r_p x + y\hat{p}) \cdot q - (x\hat{p}) + s_p q y, r_p q x + y\hat{p})
\]

\[
\quad = ((x\hat{p} + s_p y)\hat{q} + s_q (r_p x + y\hat{p}), r_q (x\hat{p} + s_p y) + (r_p x + y\hat{p})\hat{q}) - (x\hat{p}) + s_p q y, r_p q x + y\hat{p})
\]

\[
\quad = ((x\hat{p} + s_p y)\hat{q} + s_q (r_p x + y\hat{p}) - x\hat{p}) - s_p q y, r_q (x\hat{p} + s_p y) + (r_p x + y\hat{p})\hat{q} - r_p q x - y\hat{p})).
\]

Let \( x = 0 \) and \( y = 0 \) respectively, then we have:

\[
\begin{align*}
\{ & (x\hat{p})\hat{q} + s_q r_p x - x\hat{p}) = [p, q, x] \quad (3.7) \\
& r_q \hat{p} + r_p \hat{q} - r_p q = 0 \quad (3.8) \\
& s_p \hat{q} + s_q \hat{p} - s_p q = 0 \quad (3.9) \\
& r_q s_p y + (y\hat{p})\hat{q} - y\hat{p}) = [p, q, y] \quad (3.10)
\end{align*}
\]

**Step 4.** Claim: \( r_p = s_p = 0, \) for all \( p \in \mathcal{O} \).

If there exists \( p_0 \in \mathcal{O} \), such that \( r_{p_0} \neq 0 \), then by equation (3.8), we obtain:

\[
\hat{q} = r_{p_0}^{-1} (r_{p_0} q - r_q \hat{p}_0), \quad \forall q \in \mathcal{O}.
\]

Let \( \hat{p}_0 \in \mathcal{C}_J \) for some imaginary unit \( J \in S \), we conclude \( \hat{q} \in \mathcal{C}_J \) for all \( q \). Let \( y = J \) in equation (3.10), we thus get

\[
[p, q, J] \in \mathcal{C}_J, \quad \forall p, q \in \mathcal{O}.
\]

However this is impossible. Indeed, we can choose \( p \in \mathcal{O} \) orthogonal to \( J \), and then choose \( q \) orthogonal to \( p \) and \( J \), then \([p, q, J] \notin \mathcal{C}_J \). This forces that \( r_p = 0 \) for all \( p \in \mathcal{O} \). We can prove \( s_p = 0 \) for all \( p \in \mathcal{O} \) in the same way.
Step 5. Define $\sigma : p \mapsto \tilde{p}$ and $\tau : p \mapsto \hat{p}$. Claim: $\sigma = \tau = \text{id}$.

Let $x = 1$ in equation (3.7), we obtain $\sigma(pq) = \sigma(p)\sigma(q)$ and hence $\sigma \in G_2$. Suppose that $\sigma \neq \text{id}$, that is, there exists $p \in O$, such that $\sigma(p) = \tilde{p} \neq p$. Note that $\sigma(\text{Im}(O)) \subseteq \text{Im}(O)$, which yields $\text{Re} \sigma(p) = \sigma(\text{Re} p) = \text{Re} p$, we can assume $\text{Re} p = 0$. Let $x = \tilde{p}$ in equation (3.7), we obtain

$$0 = [p, q, \tilde{p}], \quad \forall q \in O.$$ 

However, $\sigma$ is an automorphism of $O$, we can certainly choose $q \in O$ such that $[p, q, \tilde{p}] \neq 0$, a contradiction. Similar argument apply to $\tau$. In summary, the right multiplication is just given by $(x, y) \cdot p = (xp, yp)$. Therefore, $O^2$ admits a unique compatible bimodule structure.

Next we consider the case of $O \oplus \overline{O}$.

Theorem 3.4. Let the left $O$-module structure on $O \oplus \overline{O}$ is as follows:

$$p(x, y) = (px, \overline{py}).$$

Then $O \oplus \overline{O}$ admits no compatible bimodule structures.

The proof of Theorem 3.4 will rely on the following two lemmas, which are also important in the sequel.

Lemma 3.5. Let $f \in \text{End}_R(O)$ satisfy $f(px) = \overline{p}f(x)$ for all $p, x \in O$, then $f = 0$.

Proof. Let $f(1) = x_0 + \sum x_je_i$, where $x_j \in \mathbb{R}, j = 0, 1, \ldots, 7$. Fix $i \neq j, i, j \in \{1, \ldots, 7\}$. We compute:

$$f(e_ie_j) = f(\epsilon_{ijk}e_k - \delta_{ij})$$

$$= -\epsilon_{ijk}e_kf(1)$$

$$= -\epsilon_{ijk}e_kx_0 - \epsilon_{ijk}x_m(\epsilon_{kmn}e_n - \delta_{kn})$$

$$= -\epsilon_{ijk}e_kx_0 - \epsilon_{ijk}\epsilon_{kmn}x_me_n + \epsilon_{ijk}x_k,$$

and

$$f(e_ie_j) = \overline{f(e_j)}$$

$$= \overline{f(\epsilon_{ijk}e_k - \delta_{ij})}$$

$$= e_i(\epsilon_{j}x_0 + \epsilon_{j}e_mx_m)$$

$$= \epsilon_{ijk}e_kx_0 + e_i(x_m(\epsilon_{jmn}e_n - \delta_{jn}))$$

$$= \epsilon_{ijk}e_kx_0 + x_m\epsilon_{jmn}(\epsilon_{imi}\epsilon_l - \delta_{in}) - e_ix_j$$

$$= \epsilon_{ijk}e_kx_0 + x_m\epsilon_{jmn}\epsilon_{imi}\epsilon_l - x_m\epsilon_{jmi} - e_ix_j.$$

Taking the real part of both equalities infers that:

$$\epsilon_{ijk}x_k = -x_m\epsilon_{jmi} = -\epsilon_{ijk}x_k.$$
This yields \( x_k = 0 \), where \( k \) is determined by \( i, j \) uniquely. Since \( i, j \) are fixed arbitrarily, we conclude \( f(1) = x_0 \in \mathbb{R} \). Hence

\[
\overline{p}(\overline{x}x_0) = \overline{p}f(x) = f(px) = \overline{p}x_0 = (\overline{x} \overline{p})x_0.
\]

That is

\[
x_0[\overline{x}, \overline{p}] = 0, \quad \forall x, p \in \mathcal{O}.
\]

This leads to \( x_0 = 0 \) and hence \( f = 0 \).

**Lemma 3.6.** The left module \( \mathcal{O} \) admits no compatible bimodule structures.

**Proof.** Suppose there exits an \( \mathcal{O} \)-bimodule structure on \( \mathcal{O} \) with a right scalar multiplication defined by an \( \mathbb{R} \)-linear map \( R \in \text{End}_\mathbb{R}(\mathcal{O}) \). Write \( R_p(x) = x \cdot p \). By the definition of \( \mathcal{O} \)-bimodule, we have

\[
[p, q, x]_\mathcal{O} = [q, x, p]_\mathcal{O}, \quad \forall p, q, x \in \mathcal{O}.
\]

Note the equation (2.11), we obtain:

\[
[p, q, x] + [q, p] = R_p(qx) - R_p(xq).
\]

Replacing \( q \) with \( x \), it becomes:

\[
R_p(qx) = qR_p(x) - [p, q, x] + [q, p]x
\]

Let \( x = 1 \) in (3.11), we get:

\[
R_p(q) = qR_p(1) + [q, p] \tag{3.12}
\]

It follows that

\[
R_p(qx) = qR_p(x) - [p, q, x] + [q, p]x
\]

and

\[
R_p(qx) = qxR_p(1) + [qx, p].
\]

Hence we conclude

\[
0 = (qx)R_p(1) + [qx, p] - (q(xR_p(1) + [x, p]) - [p, q, x] + [q, p]x)
\]

\[
= [q, x, R_p(1)] + (qx)p - \overline{p}(qx) - q(x\overline{p} - \overline{p}x) - (q\overline{p} - \overline{p}q)x - [\overline{p}, q, x]
\]

\[
= [R_p(1), q, x] + 2[q, p, x]
\]

\[
= [R_p(1) - 2p, q, x].
\]

Since the above equation holds for any \( p, q, x \in \mathcal{O} \), this yields

\[
R_p(1) - 2p \in \mathbb{R}, \quad \forall p \in \mathcal{O}.
\]
Hence we can assume $R_{e_1} = 2e_1 + r$ for some $r \in \mathbb{R}$. Note that formula (3.12) ensures $R_p(p) = pR_p(1)$, it follows that
\[
R_{e_1}(R_{e_1}1) = R_{e_1}(2e_1 + r) = 2e_1(e_1 + r) + r(2e_1 + r) = -4 + 4re_1 + r^2.
\]
However, $R_{e_1}(R_{e_1}1) = R_{e_1}1 = -1$, and hence we obtain
\[
-1 = -4 + 4re_1 + r^2,
\]
for some $r \in \mathbb{R}$, this is impossible. This proves the lemma.

**Proof of Theorem 3.4.** Suppose $\mathbb{O} \oplus \overline{\mathbb{O}}$ admits a compatible bimodule structure, and the right multiplication is as follows:
\[
(x, y) \cdot p = (f_p(x) + h_p(y), g_p(x) + l_p(y)).
\]
Similar as before, we can derive that $f_p, g_p, h_p, l_p \in \text{End}_{\mathbb{R}}(\mathbb{O})$ for all $p \in \mathbb{O}$, and are all real linear on $p$. Let $[p, q, (x, y)] = [q, (x, y), p]$, we obtain
\[
\begin{align*}
&f_p(qx) - qf_p(x) = [p, q, x] \quad (3.13) \\
g_p(qx) - \overline{g_p(x)} = 0 \quad (3.14) \\
h_p(qy) -qh_p(y) = 0 \quad (3.15) \\
l_p(qy) - \overline{l_p(y)} = [p, q, y]_\mathbb{O} \quad (3.16)
\end{align*}
\]
Hence by Lemma 3.5, $g_p = h_p = 0$ for all $p \in \mathbb{O}$. Let $[p, q, (x, y)] = [(x, y), p, q]$, we obtain
\[
l_q(l_p(y)) - l_{pq}(y) = [p, q, y]_\mathbb{O} \quad (3.17)
\]
The fact that $l_p(x)$ is real linear on $p$ and $x$, along with the equation (3.17) imply that
\[
y \cdot p := l_p(y)
\]
defines a right $\mathbb{O}$-module structure on $\overline{\mathbb{O}}$ and satisfies $[p, q, y]_\mathbb{O} = [y, p, q]_\mathbb{O}$. Note that equation (3.16) yields $[p, q, y]_\mathbb{O} = [q, y, p]_\mathbb{O}$, this means that it defines an $\mathbb{O}$-bimodule structure on $\overline{\mathbb{O}}$, which contradicts the Lemma 3.6. 

### 3.2 Bimodule structures on finite dimensional $\mathbb{O}$-modules

In this subsection, we will formulate the structure of finite dimensional $\mathbb{O}$-bimodules. As is shown in [8], each finite dimensional left $\mathbb{O}$-module $M$ is of the form:
\[
M \cong \mathbb{O}^n \oplus \overline{\mathbb{O}}^m.
\]
$\mathbb{O}^n$ is a left $\mathbb{O}$-module endowed with the left multiplication:
\[
p(x_1, \ldots, x_n) = (px_1, \ldots, px_n).
\]
$\overline{\mathbb{O}}^m \oplus \overline{\mathbb{O}}^m$ is a left $\mathbb{O}$-module endowed with the left multiplication:
\[
p(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) = (\overline{p}x_1, \ldots, \overline{p}x_n, px_{n+1}, \ldots, px_{n+m}).
\]
We are first concerned with the case $\mathbb{O}^n$. The following theorem asserts that it admits a unique compatible bimodule structure.
\textbf{Theorem 3.7.} There exists a unique compatible bimodule structure on $\mathcal{O}^n$.

\textit{Proof.} Suppose there exists a compatible bimodule structure on $\mathcal{O}^n$ and the right multiplication is as follows:
\[
(0, \ldots, 0, x_i, 0, \ldots, 0) \cdot p = (f_{i1}(p; x_i), \ldots, f_{in}(p; x_i)).
\]
Then
\[
(x_1, \ldots, x_n) \cdot p = (\sum f_{i1}(p; x_i), \ldots, \sum f_{in}(p; x_i)).
\]
Similar as before, $f_p, g_p, h_p, l_p \in \text{End}_R(\mathcal{O})$ are real linear maps for all $p \in \mathcal{O}$ and also real linear on $p$.

\textbf{Step 1.} $[p, q, (x_1, \ldots, x_n)] = [q, (x_1, \ldots, x_n), p]$.

Let $[p, q, (x_1, \ldots, x_n)] = [q, (x_1, \ldots, x_n), p]$, we obtain
\[
[p, q, x_j] = \sum f_{ij}(p; qx_i) - qf_{ij}(p; x_i), \quad j = 1, \ldots, n.
\]
Let $i_0 \in \{1, \ldots, n\}$, set
\[
x_i = \begin{cases} 
  x, & i = i_0 \\
  0, & i \neq i_0
\end{cases}
\]
then we obtain:
\[
\begin{cases} 
  f_{i_0,j}(p; qx) - qf_{i_0,j}(p; x) = 0, & j \neq i_0 \\
  f_{i_0,i_0}(p; qx) - qf_{i_0,i_0}(p; x) = [p, q, x], & i_0 \in \{1, \ldots, n\}
\end{cases}
\]
(3.18) (3.19)

Since $i_0$ is fixed arbitrarily, we conclude from equations (3.18) that $f_{ij}(p; x)$ is $\mathcal{O}$-homomorphism when $i \neq j$. hence we can assume as before
\[
f_{ij}(p; x) = r_{ij}(p)x, \quad r_{ij}(p) \in \mathbb{R}, \ i \neq j.
\]
Equations (3.19) enable us to assume
\[
f_{ii}(p; x) = xx_{ii}(p), \quad r_{ii}(p) \in \mathcal{O}.
\]

\textbf{Step 2.} $[p, q, (x_1, \ldots, x_n)] = [(x_1, \ldots, x_n), p, q]$.

Let $[p, q, (x_1, \ldots, x_n)] = [(x_1, \ldots, x_n), p, q]$, we obtain
\[
[p, q, x_i] = \sum (x_i r_{ik}(p)) r_{kl}(q) - \sum x_i r_{il}(pq).
\]
Let
\[
x_i = \begin{cases} 
  x, & i = l_0 \\
  0, & i \neq l_0
\end{cases}
\]
then we obtain:
\[
\begin{cases} 
  [p, q, x] = \sum (x r_{ik}(p)) r_{kl}(q) - x r_{il}(pq), & l_0 \in \{1, \ldots, n\} \\
  0 = \sum (x r_{ik}(p)) r_{kl}(q) - x r_{il}(pq), & l \neq l_0
\end{cases}
\]
(3.20) (3.21)

Note that $r_{ij}(p) \in \mathbb{R}$ for any distinct indices $i$ and $j$, hence equations (3.21) are equivalent to
\[
\sum r_{ik}(p) r_{kl}(q) - r_{il}(pq) = 0, \quad l \neq l_0
\]
(3.22)
**Step 3.** Claim: \( r_{jl}(p) = 0 \) for all \( p \in \mathcal{O}, \ j \neq l \).

Suppose on the contrary, there exists \( p \in \mathcal{O} \), and \( j_0 \neq i_0 \), such that \( r_{i_0j_0}(p) \neq 0 \). Let \( l = j_0 \) and take imaginary part on both sides of equations (3.22), we obtain

\[
\text{Im} \left( r_{i_0i_0}(p)r_{i_0j_0}(q) + r_{i_0j_0}(p)r_{j_0j_0}(q) \right) = 0,
\]

thus

\[
\text{Im} \ r_{j_0j_0}(q) = -r_{i_0j_0}(p)^{-1}r_{i_0j_0}(q)\text{Im} \ r_{i_0i_0}(p).
\]

Suppose \( r_{i_0i_0}(p) \in \mathcal{C}J \) for some imaginary unit \( J \), then we conclude that

\[
r_{j_0j_0}(q) \in \mathcal{C}J, \quad \forall q \in \mathcal{O}.
\]

Replacing \( l_0 \) with \( j_0 \) and \( x \) with \( J \) in equations (3.20), we get

\[
[p, q, J] \in \mathcal{C}J, \quad \forall p, q \in \mathcal{O}.
\]

Thus we have arrived at a contradiction.

Now equations (3.20) become

\[
[p, q, x] = (xr_{ll}(p))r(q) - xr_{ll}(pq), \quad l = 1, \ldots, n.
\]

As in the proof of Theorem 3.3 of the case \( n = 2 \), we can deduce \( r_{ll} = \text{id} \) for \( l = 1, \ldots, n \). This completes the proof. \( \square \)

By similar argument, we can prove:

**Theorem 3.8.** There exist no bimodule structures on \( \mathcal{O}^n \oplus \mathcal{O}^m \) when \( n > 0 \).

**Proof.** Suppose on the contrary there exists an \( \mathcal{O} \)-bimodule structure on \( \mathcal{O}^n \oplus \mathcal{O}^m \) and the right scalar multiplication is given by:

\[
(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}) \cdot p = \left( \sum_{i=1}^{N} f_{i1}(p; x_i), \ldots, \sum_{i=1}^{N} f_{in}(p; x_i) \right),
\]

where \( N = n + m \).

**Step 1.** \([p, q, (x_1, x_2, x_3, \ldots, x_{n+m})] = [q, (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}), p] \).

We first compute \([q, (x_1, \ldots, x_N), p] \).

\[
[q, (x_1, \ldots, x_N), p] = (q x_1, \ldots, q x_n, q x_{n+1}, \ldots, q x_{n+m}) \cdot p - q \left( \sum_{i=1}^{N} f_{i1}(p; x_i), \ldots, \sum_{i=1}^{N} f_{in}(p; x_i) \right)
\]

\[
= \left( \sum_{i=1}^{N} f_{ij}(p; q x_1) - q f_{ij}(p; x_i) + \sum_{i=n+1}^{N} f_{ij}(p; q x_i) - q f_{ij}(p; x_i) \right)_{j=1}^{N} +
\]

\[
\left( \sum_{i=1}^{N} f_{ij}(p; q x_1) - q f_{ij}(p; x_i) + \sum_{i=n+1}^{N} f_{ij}(p; q x_i) - q f_{ij}(p; x_i) \right)_{j=n+1}^{N},
\]

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where \((x_j)_{j=1}^n := (x_1, \ldots, x_n, 0, \ldots, 0) \in \mathbb{D}^n \oplus \mathbb{O}^m\), similar for \((x_j)_{j=n+1}^N\).

By the definition of \(\mathbb{D}\)-bimodule, we have
\[
[p, q, (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})] = [q, (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})], p].
\]

Fix \(j_0 \in \{1, \ldots, N\}\) and let
\[
x_i = \begin{cases} x, & i = j_0 \\ 0, & i \neq j_0 \end{cases}.
\]

If \(j_0 \in \{1, \ldots, n\}\), we obtain:
\[
\begin{aligned}
f_{j_0 j_0}(p, q x) - q f_{j_0 j_0}(p; x) &= [p, q, x]_{pq}, & j_0 &\in \{1, \ldots, n\} \\
f_{j_0 j_0}(p; qx) - q f_{j_0 j_0}(p; x) &= 0, & j_0 &\neq j \in \{1, \ldots, n\} \\
f_{j_0 j_0}(p; qx) - q f_{j_0 j_0}(p; x) &= 0, & j &\in \{n + 1, \ldots, N\}
\end{aligned}
\]

If \(j_0 \in \{n+1, \ldots, N\}\), we obtain:
\[
\begin{aligned}
f_{j_0 j_0}(p; qx) - q f_{j_0 j_0}(p; x) &= [p, q, x]_{pq}, & j_0 &\in \{n+1, \ldots, N\} \\
f_{j_0 j_0}(p; qx) - q f_{j_0 j_0}(p; x) &= 0, & j &\in \{1, \ldots, n\} \\
f_{j_0 j_0}(p; qx) - q f_{j_0 j_0}(p; x) &= 0, & j_0 &\neq j \in \{n+1, \ldots, N\}
\end{aligned}
\]

By Lemma 3.5 and equations (3.25) and (3.27), we conclude that \(f_{ij} = 0\) for \(i \in \{1, \ldots, n\}\), \(j \in \{n+1, \ldots, N\}\) or \(j \in \{1, \ldots, n\}\), \(i \in \{n+1, \ldots, N\}\); the same as before, we can assume
\[
f_{j_0 j_0}(p; x) = r_{j_0 j_0}(p), \quad r_{j_0 j_0}(p) \in \mathbb{R}
\]

for \(j_0, j \in \{1, \ldots, n\}, j_0 \neq j\) and \(j_0, j \in \{n+1, \ldots, N\}, j_0 \neq j\).

**Step 2.** \([p, q, (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})] = [(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m}), p, q].\)

Now we compute \([(x_1, \ldots, x_N), p, q].\)

\[
[(x_1, \ldots, x_N), p, q] = \left( \sum_{i=1}^{N} f_{ij}(p; x_i) \right)^{N} \cdot q - \left( \sum_{i=1}^{N} f_{ij}(pq; x_i) \right)^{N} \\
= \left( \sum_{k=1}^{N} f_{kj} \left( q \sum_{i=1}^{N} f_{ik}(p; x_i) \right) - \sum_{i=1}^{N} f_{ij}(pq; x_i) \right)^{N}.
\]

Similar as before, we have:
\[
\begin{aligned}
\sum_{k=1}^{N} f_{j_0 k}(q; f_{j_0 k}(p; x)) - f_{j_0 j_0}(pq; x) &= [p, q, x]_{pq}, & j_0 &\in \{1, \ldots, n\} \\
\sum_{k=1}^{N} f_{j_0 k}(q; f_{j_0 k}(p; x)) - f_{j_0 j_0}(pq; x) &= [p, q, x], & j_0 &\in \{n+1, \ldots, N\} \\
\sum_{k=1}^{N} f_{kj} (q; f_{j_0 k}(p; x)) - f_{j_0 j_0}(pq; x) &= 0, & j &\neq j_0, j, j_0 \in \{1, \ldots, N\}
\end{aligned}
\]
Note what we have just proved, we can rewrite equations (3.29) as follows:

\[ \sum_{k=1, k \neq j_0}^{n} r_{k,j_0}(q) r_{j_0,k}(p)x + f_{j_0,j_0}(p;x) - f_{j_0,j_0}(pq;x) = [p,q,x]_{\mathbb{C}} \quad j_0 \in \{1, \ldots, n\} \]  

(3.32)

Rewrite equations (3.31) as follows:

\[ \sum_{k=1, k \neq j_0, k \neq j}^{n} r_{k,j_0}(q) r_{j,j_0}(p)x + f_{j,j}(q;r_{j,j}(p)x) + r_{j,j}(q)f_{j,j}(p;x) - r_{j,j}(pq)x = 0 \]  

(3.33)

The equations (3.33) hold for \( j \neq j_0, j, j_0 \in \{1, \ldots, N\} \).

**Step 3.** \( r_{ij}(p) = 0 \) for all \( i \neq j \) in \( \{1, \ldots, n\} \).

Taking imaginary part on both sides of equations (3.33), we get

\[ \text{Im} \left( r_{j_0,j}(p)f_{j,j}(q;x) + r_{j,j}(q)f_{j,j}(p;x) \right) = 0. \]

If there exists an octonion \( p \in \mathbb{O} \), and \( l_0, l \in \{1, \ldots, n\}, l_0 \neq l \) such that \( r_{i,l}(p) \neq 0 \), let \( x = 1 \) and \( f_{l_0,l_0}(p;1) \in \mathbb{C}_{j} \), we then conclude as before

\[ f_{i}(q;1) \in \mathbb{C}_{j}, \quad \forall q \in \mathbb{O}. \]

However, replacing \( j_0 \) by \( l \) in equations (3.32), we conclude

\[ [p,q,1]_{\mathbb{C}} = [p,q] \in \mathbb{C}_{j}, \quad \forall p, q \in \mathbb{O}. \]

Obviously this is impossible. Thus we have arrived at a contradiction. This shows \( r_{ij}(p) = 0 \) for all \( i \neq j \) in \( \{1, \ldots, n\} \).

**Step 4.** For each \( j = 1, \ldots, n \), \( f_{j,j}(p;x) \) defines a bimodule structure on \( \mathbb{O} \).

Thanks to Step 3, equations (3.32) become

\[ f_{j_0,j_0}(q;f_{j_0,j_0}(p;x)) - f_{j_0,j_0}(pq;x) = [p,q,x]_{\mathbb{O}}, \quad j_0 \in \{1, \ldots, n\}. \]

This imply that we get a right \( \mathbb{O} \)-module structure on \( \mathbb{O} \) with the right multiplication defined by \( x \cdot j p := f_{j,j}(p;x) \) for each \( j = 1, \ldots, n \). Moreover, combining equations (3.23) and (3.32) yields an \( \mathbb{O} \)-bimodule structure on \( \mathbb{O} \), which contradicts the Lemma 3.6. This proves the theorem.

\[ \square \]

### 3.3 The structure of general \( \mathbb{O} \)-bimodule

In this subsection, we are in a position to deal with the bimodule structure of general left \( \mathbb{O} \)-modules. We have shown that each left \( \mathbb{O} \)-module has a “basis” in a previous paper [8], this loosely means that each left \( \mathbb{O} \)-module is a free module. In much the same way as finite dimensional case, we can prove that a left \( \mathbb{O} \)-module \( M \) admits a compatible bimodule structure if and only if \( M = \mathbb{O}_{\text{af}}(M) \). Moreover, the bimodule structure is unique if it exists.

In view of identity (2.7), it holds \( [p,q,x]_{\mathbb{O}} \in \mathbb{O} \) for any associative element \( x \in \mathbb{O} \). We now give a similar formula for conjugate associative element. For convention, we define a
new associator, denoted by \([p, q, r] := [p, q, r] + r[p, q]\). Then by direct calculation, we have for any conjugate associative element \(x \in \mathcal{A}^{-}(M)\):

\[
[p, q, rx] = [p, q, r]x.
\]  

(3.34)

In fact, let \(x \in \mathcal{A}^{-}(M)\),

\[
[p, q, rx] = (pq)(rx) - p(q(rx))
\]

\[
= (r(pq) - (rq)p)x
\]

\[
= (r[p, q] - [r, q, p])x
\]

\[
= [p, q, r]x.
\]

By the way, we can give an alternative derivation of the associator of \(\mathcal{O}\) as follows:

\[
[p, q, x]\mathcal{O} = [p, q, x]\mathcal{O}1 = [p, q, x]1 = [p, q, x] = [p, q, x] + [p, q]x.
\]

(3.35)

**Theorem 3.9.** A left \(\mathcal{O}\)-module \(M\) admits a compatible bimodule structure if and only if \(M = \mathcal{O}\mathcal{A}(M)\).

Moreover, in this case, the right scalar multiplication on \(\mathcal{A}(M)\) coincides with the left scalar multiplication:

\[
xp = px, \quad \forall p \in \mathcal{O}, \forall x \in \mathcal{A}(M).
\]

And this determines the right scalar multiplication on \(M\).

We first prove a simple lemma which will be used later.

**Lemma 3.10.** Let \(f \in \text{End}_R(\mathcal{O})\). If it holds \(f(qx) = qf(x)\) for all \(q, x \in \mathcal{O}\), then \(f = 0\).

**Proof.** This is a simple deformation of Lemma 3.5. We define \(g(x) := f(x)\), then we obtain:

\[
g(px) = f(px) = pf(x) = pg(x).
\]

It thus follows from Lemma 3.5 that \(g = 0\) and hence \(f = 0\). \(\square\)

**Proof of Theorem 3.9.** Suppose \(M \cong (\oplus_{i \in \Lambda_1} \mathcal{O}) \oplus (\oplus_{\alpha \in \Lambda_2} \mathcal{O})\). Hence there is a canonical basis \(\{\epsilon_i, \epsilon_\alpha\}_{i \in \Lambda_1, \alpha \in \Lambda_2}\), such that \(\epsilon_i \in \mathcal{A}(M)\) and \(\epsilon_\alpha \in \mathcal{A}^{-}(M)\) for each \(i \in \Lambda_1\) and \(\alpha \in \Lambda_2\). We assume there exists an \(\mathcal{O}\)-bimodule structure and for any \(x \in \mathcal{O}\), the right multiplication is supposed to be:

\[
(\epsilon_i) \cdot p = \sum_{j \in \Lambda_1} f_{ij}(p;x)\epsilon_j + \sum_{\beta \in \Lambda_2} f_{i\beta}(p;x)\epsilon_\beta;
\]

\[
(\epsilon_\alpha) \cdot p = \sum_{j \in \Lambda_1} f_{\alpha j}(p;x)\epsilon_j + \sum_{\beta \in \Lambda_2} f_{\alpha \beta}(p;x)\epsilon_\beta.
\]
Note that these sums here are all finite sums. Therefore,

$$\left( \sum_{i \in \Lambda_1} x_i \epsilon_i + \sum_{\alpha \in \Lambda_2} x_\alpha \epsilon_\alpha \right) \cdot p = \sum_{j \in \Lambda_1} \left( \sum_{i \in \Lambda_1} f_{ij}(p; x_i) + \sum_{\alpha \in \Lambda_2} f_{\alpha j}(p; x_\alpha) \right) \epsilon_j + \sum_{\beta \in \Lambda_2} \left( \sum_{i \in \Lambda_1} f_{i\beta}(p; x_i) + \sum_{\alpha \in \Lambda_2} f_{\alpha \beta}(p; x_\alpha) \right) \epsilon_\beta.$$  

Given \( m = \sum_{i \in \Lambda_1} x_i \epsilon_i + \sum_{\alpha \in \Lambda_2} x_\alpha \epsilon_\alpha \), we compute \([q, m, p]\). Note that \( \epsilon_i \in \mathcal{A}(M) \) and \( \epsilon_\alpha \in \mathcal{A}^-(M) \), which means for all \( p, q \in \mathbb{O} \), it holds

\[
p(q\epsilon_i) = (pq)\epsilon_i, \quad p(q\epsilon_\alpha) = (qp)\epsilon_\alpha
\]

for every \( i \in \Lambda_1 \) and \( \alpha \in \Lambda_2 \). Consequently,

\[
[q, m, p] = \left( q \sum_{i \in \Lambda_1} x_i \epsilon_i + q \sum_{\alpha \in \Lambda_2} x_\alpha \epsilon_\alpha \right) p - q \sum_{j \in \Lambda_1} \left( \sum_{i \in \Lambda_1} f_{ij}(p; x_i) + \sum_{\alpha \in \Lambda_2} f_{\alpha j}(p; x_\alpha) \right) \epsilon_j -
\]

\[
q \sum_{\beta \in \Lambda_2} \left( \sum_{i \in \Lambda_1} f_{i\beta}(p; x_i) + \sum_{\alpha \in \Lambda_2} f_{\alpha \beta}(p; x_\alpha) \right) \epsilon_\beta
\]

\[
= \sum_{j \in \Lambda_1} \left[ \sum_{i \in \Lambda_1} \left( f_{ij}(p; qx_i) - qf_{ij}(p; x_i) \right) + \sum_{\alpha \in \Lambda_2} \left( f_{\alpha j}(p; x_\alpha q) - qf_{\alpha j}(p; x_\alpha) \right) \right] \epsilon_j +
\]

\[
\sum_{\beta \in \Lambda_2} \left[ \sum_{i \in \Lambda_1} \left( f_{i\beta}(p; qx_i) - f_{i\beta}(p; x_i)q \right) + \sum_{\alpha \in \Lambda_2} \left( f_{\alpha \beta}(p; x_\alpha q) - f_{\alpha \beta}(p; x_\alpha)q \right) \right] \epsilon_\beta.
\]

As before, we obtain:

\[
\begin{align*}
  f_{jj}(p; qx) - qf_{jj}(p; x) &= [p, q, x_j], & j & \in \Lambda_1 \\
  f_{ij'}(p; qx) - qf_{ij'}(p; x) &= 0, & j & \neq j', j, j' \in \Lambda_1 \\
  f_{\beta j}(p; qx) - f_{\beta j}(p; x)q &= 0, & j & \in \Lambda_1, \beta \in \Lambda_2 \\
  f_{\beta \beta'}(p; qx) - f_{\beta \beta'}(p; x)q &= 0, & \beta & \neq \beta', \beta, \beta' \in \Lambda_2 \\
  f_{j\beta}(p; qx) - qf_{j\beta}(p; x) &= 0, & j & \in \Lambda_1, \beta \in \Lambda_2
\end{align*}
\]

where \( x \) is an arbitrary octonion.

Thanks to Lemma 3.10, we deduce from the equations (3.38) and (3.41) that \( f_{j\beta} = f_{j\beta} = 0 \) for all \( j \in \Lambda_1, \beta \in \Lambda_2 \) as before. What seems slightly different from before is the equations (3.39). However, if we define

\[ g_{\beta\beta}(p; x) = \overline{f_{\beta\beta}(p; x)}, \]

we then have:

\[
\begin{align*}
  g_{\beta\beta}(p; \overline{qx}) - \overline{g_{\beta\beta}(p; x)} &= \overline{f_{\beta\beta}(p; \overline{x}q)} - \overline{qf_{\beta\beta}(p; \overline{x})} \\
  &= \overline{f_{\beta\beta}(p; \overline{x})} - f_{\beta\beta}(p; \overline{x})q \\
  &= [p, q, \overline{x}]_{\mathbb{O}}.
\end{align*}
\]
Where we have used the equality (3.35) in the last line. The rest proof runs completely in the same manner as in Theorem 3.7 and Theorem 3.8.

3.4 Some consequences

Let $M$ be an $O$-bimodule and define the communicating center of $M$

$$\mathcal{Z}(M) := \{x \in M \mid px = xp, \text{ for all } p \in O\}.$$

Then it turns out that the communicating center is exactly the set $\mathcal{A}(M)$.

**Proposition 3.11.** Let $M$ be an $O$-bimodule. Then $\mathcal{A}(M) = \mathcal{Z}(M)$.

**Proof.** Let $x \in \mathcal{Z}(M)$, then for any $p, q \in O$, we have

$$[p, q, x] = (pq)x - p(qx) = x(pq) - p(xq) = (xp)q - [x, p, q] - p(xq) = (px)q - p(xq) - [p, q, x] = -2[p, q, x].$$

Thus $[p, q, x] = 0$ for any $p, q \in O$ and hence $x \in \mathcal{A}(M)$. On the other hand, if $x \in \mathcal{A}(M)$, we clearly have $x \in \mathcal{Z}(M)$ in view of Theorem 3.9. This proves the proposition.

In order to avoid any confusion, we use the prefixes $l$- and $r$- to indicate that the module under consideration is a left or right module. For example, let $M$ and $M'$ be two $O$-bimodules, we use $l$-$\text{Hom}_O(M, M')$ to denote the set of all left homomorphisms over $O$-bimodules $M$ and $M'$, similar notation $r$-$\text{Hom}_O(M, M')$ for right homomorphisms, and use $\text{Hom}_O(M, M')$ to denote the set of bihomomorphisms:

$$\text{Hom}_O(M, M') := \{f \in \text{Hom}_R(M, M') \mid f(px) = pf(x), f(xp) = f(x)p, \text{ for all } x \in M, p \in O\}.$$

That is,

$$\text{Hom}_O(M, M') = l$\text{-Hom}_O(M, M') \cap r$\text{-Hom}_O(M, M').$$

It turns out that the three sets above are all the same in bimodule case.

**Proposition 3.12.** Suppose $M$ and $M'$ are two $O$-bimodules. Then

$$l$\text{-Hom}_O(M, M') = r$\text{-Hom}_O(M, M') = \text{Hom}_O(M, M') \quad (3.42)$$

**Proof.** Let $f \in l$-$\text{Hom}_O(M, M')$. For any $x \in \mathcal{A}(M)$, in view of Theorem 3.9, we deduce

$$f(xp) = f(px) = pf(x),$$

Since $x \in \mathcal{A}(M)$, we conclude that $f(x) \in \mathcal{A}(M')$ and hence $pf(x) = f(x)p$, namely,

$$f(xp) = f(x)p, \text{ for all } x \in \mathcal{A}(M). \quad (\ast)$$

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Now for arbitrary \( x \in M \), suppose \( \{ x_j \}_{j \in \Lambda} \subseteq \mathcal{A}(M) \) is a basis of \( M \), then we can write \( x = \sum_{i=1}^{n} r_i x_i \), where \( r_i \in \mathbb{O} \), \( i = 1, \ldots, n \). It follows that

\[
\begin{align*}
f(xp) &= \sum_{i=1}^{n} f(x_i(r_i p)) \\
&= \sum_{i=1}^{n} f(x_i)(r_i p) \\
&= \sum_{i=1}^{n} (f(x_i)r_i)p
\end{align*}
\]

which means that \( f \in r\text{-Hom}_{\mathbb{O}}(M, M') \) and thus \( f \in \text{Hom}_{\mathbb{O}}(M, M') \) as desired. Therefore, we obtain that \( l\text{-Hom}_{\mathbb{O}}(M, M') = \text{Hom}_{\mathbb{O}}(M, M') \). This completes the proof.

**Remark 3.13.** Proposition 3.12 shows there is no difference between left \( \mathbb{O} \)-homomorphisms and \( \mathbb{O} \)-bihomomorphisms when \( M \) is an bimodule. Therefore it is no need to consider them separately as in [10]. What’s more, this proposition actually shows that if \( M \) and \( M' \) are two \( \mathbb{O} \)-bimodules such that they are isomorphic as left \( \mathbb{O} \)-modules, then they are isomorphic as \( \mathbb{O} \)-bimodules.

## 4 The real part of \( \mathbb{O} \)-bimodules

In this section, we shall introduce the structure of real part on \( \mathbb{O} \)-bimodules in a similar way as in quaternion case [12]. It turns out the category of \( \mathbb{O} \)-bimodules is also isomorphic to the category of \( \mathbb{R} \)-vector spaces.

Let \( M \) be an \( \mathbb{O} \)-bimodule. For any given \( x \in M \), since \( M = \mathbb{O}\mathcal{A}(M) \), then for any \( m \in M \), we can write

\[
m = \sum_{i=1}^{n} r_i x_i,
\]

for some \( r_i \in \mathbb{O} \) and some \( x_i \in \mathcal{A}(M) \). Let \( r_i = r_{i0} + \sum r_{ij} e_j, r_{ij} \in \mathbb{R} \) for \( j = 0, \ldots, 7 \) and \( i = 1, \ldots, n \). Hence the above equality can be rewritten as

\[
m = m_0 + \sum_{i=1}^{n} c_i m_i,
\]

for some associative elements \( m_j \in \mathcal{A}(M) \), \( j = 0, 1, \ldots, 7 \).

**Lemma 4.1.** Let \( M \) be an \( \mathbb{O} \)-bimodule. Then we have the following equations:

\[
4[e_i, x] = \sum_{j,k} \epsilon_{ijk} [e_j, e_k, x], \quad i = 1, \ldots, 7.
\]

**Proof.** For any given \( x \in M \), let

\[
x = x_0 + \sum e_i x_i, \quad \text{where} \ x_j \in \mathcal{A}(M), \ j = 0, 1, \ldots, 7.
\]
We first compute $\sum_{j,k} \epsilon_{ijk}[e_j, e_k, x]$. Using Einstein summation convention,
\[
\sum_{j,k} \epsilon_{ijk}[e_j, e_k, x] = \epsilon_{ijk}[e_j, e_k, e_m x_m]
\]
\[
= \epsilon_{ijk}[e_j, e_k, e_m]x_m
\]
\[
\overset{(2.2)}{=} 2\epsilon_{ijk}\epsilon_{ikmn}e_n x_m
\]
\[
\overset{(2.4)}{=} 8\epsilon_{imn}e_n x_m.
\]
Whereas
\[
4[e_i, x] = 4[e_i, e_m x_m]
\]
\[
= 4((e_i e_m)x_m - e_m(x_m e_i))
\]
\[
= 4(e_i e_m - e_m e_i)x_m
\]
\[
= 4(\epsilon_{imn} e_n - \epsilon_{min} e_n)x_m
\]
\[
= 8\epsilon_{imn} e_n x_m.
\]

This proves the equations (4.1) as desired. 
\[\square\]

**Corollary 4.2.** Let $M$ be an $\mathbb{O}$-bimodule. Then the right multiplication is uniquely determined by its left module structure. More precisely, for any $x \in M$, the right multiplication is given by
\[
x e_i = e_i x - \frac{1}{4} \sum_{j,k} \epsilon_{ijk}[e_j, e_k, x], \quad i = 1, \ldots, 7.
\]

**Remark 4.3.** This lemma gives a new proof of the fact that $\mathcal{A}(M) \subseteq \mathcal{Z}(M)$.

We wish to define a real part structure on $\mathbb{O}$-bimodule $M$, which plays a role of the real number in $\mathbb{O}$. Recall in the quaternion setting, Ng gives a structure of real part on a quaternion bimodule $X$ as follows ([12]):
\[
Re x = \frac{1}{4} \sum_{e \in B} e x e \quad (x \in X).
\]
Where $B := \{1, i, j, k\}$ is a basis of the quaternions $\mathbb{H}$.

In the octonion case, it turns out that there is a real part structure on $\mathbb{O}$-bimodules as well. We define the real part for an arbitrary bimodule $M$ as follows:
\[
Re x := \frac{5}{12} x - \frac{1}{12} \sum_{i=1}^{7} e_i x e_i \quad (x \in M).
\]

The operator $Re : M \to M$ is called the real part structure of $M$, and we call an element $m \in M$ real if $m \in Re M$. Combining the equations (4.1), we conclude that $Re x = x + \frac{1}{48} \epsilon_{ijk} e_i[e_j, e_k, x]$.

It turns out that the subset of all real elements coincides with the subset $\mathcal{A}(M)$ of associative elements.

**Theorem 4.4.** If $M$ is an $\mathbb{O}$-bimodule, then for all $x \in M$, the following hold:
(i) \(Re^2 x = Re x\).

(ii) \(x = Re x - \sum e_i Re(e_i x)\).

(iii) \(Re M = \mathcal{Z}(M) = \mathcal{O}(M)\).

(iv) For all \(x \in Re M, p \in \mathcal{O}\), we have \(Re(px) = (Re p)x\).

(v) \(M = Re M \oplus \bigoplus_{i=1}^7 e_i Re M\).

Proof. We prove assertion (i). The proof is straightforward.

\[
Re^2 x = Re \left( \frac{5}{12} x - \frac{1}{12} \sum e_i x e_i \right)
= \frac{5}{12} \left( \frac{5}{12} x - \frac{1}{12} e_j x e_j \right) - \frac{1}{12} e_j \left( \frac{5}{12} x - \frac{1}{12} e_i x e_i \right) e_j
= \frac{5^2}{12^2} x - 10 \frac{1}{12^2} e_j x e_i + \frac{1}{12^2} e_j (e_i x e_i) e_j.
\]

Using Moufang identities, we obtain

\[
e_j (e_i x e_i) e_j = \left( (e_j e_i x) e_i \right) e_j
= \left( (e_{ijk} e_k - \delta_{ji}) x \right) e_i e_j
= (e_{ijk} e_k x - \delta_{ji}) (e_{ijm} e_m - \delta_{ji}) + e_{ijk} [e_k x, e_i, e_j]
= -6 \delta_{km} (e_k x) e_m + 7 x + e_{ijk} [e_i, e_j, e_k x]
\]

where we have used identity (2.3) in the last line. Note that

\[
e_{ijk} [e_k x, e_i, e_j] = e_{ijk} [e_i, e_j, e_k x]
= e_{ijk} ([e_i, e_j, e_k] x + e_i [e_j, e_k, x] - [e_i, e_j, e_k, x] + [e_i, e_j, e_k, x])
= e_{ijk} (2 e_{ijkm} e_m x + e_i [e_j, e_k, x] - e_{ijm} [e_m, e_k, x] + e_{jkm} [e_i, e_m, x])
= e_{ijk} e_i [e_j, e_k, x]
= -4 e_i [e_j, x]
= \sum_{i=1}^7 -4 e_i (e_i x - x e_i)
= 28 x + 4 e_i x e_i,
\]

where we have used the equations (4.1). Hence

\[
Re^2 x = \frac{5^2}{12^2} x - 10 \frac{1}{12^2} e_j x e_i + \frac{1}{12^2} (-6 \delta_{km} (e_k x) e_m + 7 x + 28 x + 4 e_i x e_i)
= \frac{5}{12} x - \frac{1}{12} \sum e_i x e_i
= Re x.
\]
This proves assertion (i).

We prove assertion (ii). First using Moufang identities again, we get
\[ e_j(e_i x) = (e_j e_i)(xe_j) = \epsilon_{ijk}e_k(xe_j) - xe_i. \]

Hence
\[
Re(e_i x) = \frac{5}{12}e_i x - \frac{1}{12}e_j(e_i x) e_j
= \frac{5}{12}e_i x - \frac{1}{12}(\epsilon_{ijk}e_k(xe_j) - xe_i)
\]
and
\[
Re x - \sum e_i Re(e_i x) = \frac{5}{12}x - \frac{1}{12}\sum e_i xe_i - e_i\left(\frac{5}{12}e_i x - \frac{1}{12}(\epsilon_{ijk}e_k(xe_j) - xe_i)\right)
= \frac{40}{12}x - \frac{2}{12}e_i xe_i + \frac{1}{12}\epsilon_{ijk}e_i(e_k(xe_j))
\]
Similar as above,
\[
\epsilon_{ijk}e_i(e_k(xe_j)) = \epsilon_{ijk}\left(\epsilon_j \epsilon_k e_m(xe_j) - [e_i, e_k, xe_j]\right)
= \epsilon_{ijk}\left(\epsilon_{ikm}e_m - \delta_{ik}\right)(xe_j) - [xe_j, e_i, e_k]
= 6\delta_{jm}e_m(xe_j) - \epsilon_{ijk}x[e_j, e_i, e_k]
= 6e_i xe_i - 4[e_k, x]e_k
= 2e_i xe_i - 28x.
\]

Hence
\[
Re x - \sum e_i Re(e_i x) = x.
\]

We prove assertion (iii).
\[
Re x \in \mathcal{Z}(M) \iff e_j Re x = (Re x)e_j, \quad j = 1, \ldots, 7
\]
\[
\iff \frac{5}{12}[e_j, x] = \frac{1}{12}\left((e_i xe_i)e_j - e_j(e_i xe_i)\right), \quad j = 1, \ldots, 7
\]
\[
\iff \frac{5}{12}[e_j, x] = \frac{1}{12}\left(e_i(e_i xe_j) - (e_j e_i x)e_i\right), \quad j = 1, \ldots, 7
\]
\[
\iff 4[e_j, x] = \sum_{i,k} \epsilon_{ijk}[e_i, e_k, x], \quad j = 1, \ldots, 7.
\]

Then it follows from Lemma 4.1 that \(Re M \subseteq \mathcal{Z}(M)\). On the other hand, for any \(x \in \mathcal{Z}(M)\),
\[ Re x = \frac{5}{12}x - \frac{1}{12}\sum e_i xe_i = x \] and thus \(\mathcal{Z}(M) \subseteq Re M\). Hence \(Re M = \mathcal{Z}(M) = \mathcal{A}(M)\). This proves assertion (iii).

We prove assertion (iv). Note that \(Re M = \mathcal{A}(M)\) and \(Re x = x + \frac{1}{48}\epsilon_{ijk}e_i[e_j, e_k, x]\), we
conclude for every $x \in \text{Re} M$, $m = 1, \ldots, 7$,

$$\text{Re} (e_m x) = e_m x + \frac{1}{48} \epsilon_{ijk} e_i [e_j, e_k, e_m] x$$

$$= e_m x + \frac{1}{48} \epsilon_{ijk} \cdot 2 \epsilon_{jkmn} (\epsilon_{inl} e_l - \delta_{in}) x$$

$$= e_m x + \frac{1}{6} \epsilon_{mn} \epsilon_{inl} e_l x$$

$$= e_m x - \frac{1}{6} \delta_{ml} e_l x$$

$$= 0$$

where we have used identities (2.3) and (2.4). This proves assertion (iv).

Assertion (v) follows from assertion (ii) and assertion (iv) directly. \qed

An immediate consequence is a concrete form of the subset of associative elements, which only depends on the left multiplication.

**Corollary 4.5.** Let $M$ be an $\mathbb{O}$-bimodule, then we have

$$\mathcal{A}(M) = \left\{ x + \frac{1}{48} \epsilon_{ijk} e_i [e_j, e_k, x] \mid x \in M \right\}.$$

For any given $m \in M$, it follows from the above theorem that there exists a unique decomposition:

$$m = m_0 + \sum e_i m_i,$$

where $m_j = \text{Re}(\overline{e_i} x) \in \mathcal{A}(M)$, for all $j = 0, 1, \ldots, 7$. We call $m_0 = \text{Re} m$ the **real part** of $m$. $\text{Re} M$ is the unique real vector space (up to isomorphism) whose octonionization, $\text{Re} M \otimes \mathbb{O}$, is isomorphic to $M$ as $\mathbb{O}$-bimodule. We can show that the properties of this real part structure are almost the same with that in octonion. We assemble some elementary properties now.

**Proposition 4.6.** Let $M$ be an $\mathbb{O}$-bimodule. Then for all $p, q \in \mathbb{O}$, for all $x \in M$, we have

(i). $\text{Re} [p, q, x] = 0$;

(ii). $\text{Re} [p, x] = 0$;

(iii). $\text{Re} (pq)x = \text{Re} (qx)p = \text{Re} x(pq)$.

**Proof.** Suppose $x = x_0 + \sum e_i x_i$, $x_j \in \text{Re} M$ for $j = 0, 1, \ldots, 7$. Then in view of identity (2.7) and assertion (iv) in Theorem 4.4, we obtain

$$\text{Re} [p, q, x] = \text{Re} [p, q, \sum_0^7 e_i x_i] = \sum_0^7 \text{Re} ([p, q, e_i] x_i) = \sum_0^7 (\text{Re} [p, q, e_i]) x_i = 0.$$ 

This proves assertion (i). Assertion (ii) follows from the assertion (i) that we just proved and Lemma 4.1. We prove assertion (iii). Thanks to assertion (ii), we get

$$\text{Re} (pq)x = \text{Re} (pq)x - \text{Re} [pq, x] = \text{Re} x(pq),$$

and

$$\text{Re} (pq)x = \text{Re} (pq)x - \text{Re} [p, q, x] = \text{Re} p(qx) = \text{Re} (qx)p.$$ 

This proves the proposition. \qed

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Similar as in quaternion case ([12]), we can prove the following lemma. The proof is much the same and is omitted here.

**Lemma 4.7.** Let $X$ and $Y$ be two $\mathbb{O}$-bimodules. If $f \in \text{Hom}_\mathbb{O}(X, Y)$, then $f|_{\text{Re}(X)} \in \text{Hom}_\mathbb{R}(\text{Re}X, \text{Re}Y)$. This induces an $\mathbb{R}$-linear isomorphism $\Psi$ from $\text{Hom}_\mathbb{O}(X, Y)$ onto $\text{Hom}_\mathbb{R}(\text{Re}X, \text{Re}Y)$.

In exactly the same way as quaternion case, it holds:

**Theorem 4.8.** The category of $\mathbb{O}$-bimodules is isomorphic to the category of $\mathbb{R}$-vector spaces.

**Remark 4.9.** The notion of real part on an $\mathbb{O}$-bimodule will play a crucial role in the development of octonionic functional analysis. We hope to indicate some of these applications in subsequent papers.

## 5 Submodules generated by one element

The submodule $\langle m \rangle_\mathbb{O}$ generated by one point will be very different from the classical case. As is known that $\mathbb{O}m$ is not always a submodule and the submodule generated by one element may be the whole module [4]. We introduce the notion of cyclic element and cyclic decomposition to describe this phenomenon.

### 5.1 Cyclic elements

Let $M$ be a left $\mathbb{O}$-module only in this subsection. We collect some basic properties and lemmas first.

**Definition 5.1.** An element $m \in M$ is said to be cyclic if $\langle m \rangle_\mathbb{O} = \mathbb{O}m$. Denote by $\mathcal{C}(M)$ the set of all cyclic elements in $M$.

The following lemma is crucial to set up the structure of $\mathcal{C}(M)$.

**Lemma 5.2.** Let $x$ be any given nonzero element in $M$. Then

$$x \in \mathcal{C}(M) \iff \dim_\mathbb{R} \langle x \rangle_\mathbb{O} = 8 \iff \langle x \rangle_\mathbb{O} \cong \mathbb{O} \text{ or } \overline{\mathbb{O}}.$$

**Proof.** Let $x \in \mathcal{C}(M)$, then $\langle x \rangle_\mathbb{O} = \mathbb{O}x$ and hence $\dim_\mathbb{R} \langle x \rangle_\mathbb{O} = 8$. On the other hand, since $\langle x \rangle_\mathbb{O}$ is a nonzero $\mathbb{O}$-module of finite dimension, thus $\dim_\mathbb{R} \langle x \rangle_\mathbb{O} \geq 8$, therefore $\dim_\mathbb{R} \langle x \rangle_\mathbb{O} = 8$, this means $\langle x \rangle_\mathbb{O}$ is a simple $\mathbb{O}$-module and hence $\langle x \rangle_\mathbb{O} \cong \mathbb{O}$ or $\overline{\mathbb{O}}$. Suppose $\langle x \rangle_\mathbb{O} \cong \mathbb{O}$ or $\overline{\mathbb{O}}$. Assume $\langle x \rangle_\mathbb{O} \cong \mathbb{O}$ first. Let $\varphi$ denote an isomorphism: $\varphi : \langle x \rangle_\mathbb{O} \to \mathbb{O}$. For any $m \in \langle x \rangle_\mathbb{O}$, suppose $\varphi(x) = p$, $\varphi(m) = q$. Then

$$\varphi(m) = q = qp^{-1}p = qp^{-1}\varphi(x) = \varphi((qp^{-1})x),$$

according to that $\varphi$ is isomorphism, we get $m = (qp^{-1})x$, hence $\langle x \rangle_\mathbb{O} = \mathbb{O}x$ which means $x \in \mathcal{C}(M)$. If $\langle x \rangle_\mathbb{O} \cong \overline{\mathbb{O}}$, still let $\varphi$ denote the isomorphism: $\langle x \rangle_\mathbb{O} \to \overline{\mathbb{O}}$. For any $m \in \langle x \rangle_\mathbb{O}$, suppose $\varphi(x) = p$, $\varphi(m) = q$. Then

$$\varphi(m) = q = (qp^{-1})p = (qp^{-1})\varphi(x) = \varphi((qp^{-1})x),$$

then we get $m = (qp^{-1})x$, hence $x \in \mathcal{C}(M)$. □
According to the above lemma, we define

\[ C^+(M) := \{ x \in C(M) \mid \langle x \rangle_\mathbb{O} \cong \mathbb{O} \} \cup \{ 0 \}, \]

\[ C^-(M) := \{ x \in C(M) \mid \langle x \rangle_\mathbb{O} \cong 0 \} \cup \{ 0 \}. \]

Therefore \( C(M) = C^+(M) \cup C^-(M) \). We shall show that all the cyclic elements are determined by the associative subset \( A(M) \) and the conjugate associative subset \( A^-(M) \).

**Theorem 5.3.** Let \( M \) be a left \( \mathbb{O} \)-module, then:

(i) \( C^+(M) = \bigcup_{p \in \mathbb{O}} p \cdot A(M) \);

(ii) \( C^-(M) = \bigcup_{p \in \mathbb{O}} p \cdot A^-(M) \).

**Proof.** We prove assertion (i). We first show \( \bigcup_{p \in \mathbb{O}} p \cdot A(M) \subseteq C^+(M) \). Given any \( x \in A(M) \). Without loss of generality we can assume \( x \neq 0 \). Define a map \( \phi : \langle x \rangle_\mathbb{O} \rightarrow \mathbb{O} \) such that \( \phi(px) = p \) for \( p \in \mathbb{O} \). This is a homomorphism in \( \text{Hom}_\mathbb{O}(\langle x \rangle_\mathbb{O}, \mathbb{O}) \), since

\[ \phi(q(px)) = \phi((qp)x) = qp = q\phi(px). \]

Thus \( \langle x \rangle_\mathbb{O} \cong \mathbb{O} \). This proves \( x \in C^+(M) \). Because \( px \in \langle x \rangle_\mathbb{O} \) and \( x = p^{-1}(px) \in \langle px \rangle_\mathbb{O} \) for \( p \neq 0 \), that is, \( \langle x \rangle_\mathbb{O} = \langle px \rangle_\mathbb{O} \) whenever \( p \neq 0 \). This implies \( \bigcup_{p \in \mathbb{O}} p \cdot A(M) \subseteq C^+(M) \). On the contrary, let \( 0 \neq x \in C^+(M) \), hence there is an isomorphism \( \phi \in \text{Hom}_\mathbb{O}(\mathbb{O}, \langle x \rangle_\mathbb{O}) \). Suppose \( \phi(1) = y \in \langle x \rangle_\mathbb{O} \), since \( \phi \) is an isomorphism, choose \( 0 \neq r \in \mathbb{O} \) such that \( y = rx \). Note that \( [p,q,y] = \phi[p,q,1] = 0 \) for all \( p, q \in \mathbb{O} \), we thus get \( y = r^{-1}y \in \bigcup_{p \in \mathbb{O}} p \cdot A(M) \). This proves assertion (i). Similarly we can prove assertion (ii).

The following lemma will be useful later. The proof can be found in [8].

**Lemma 5.4.** Let \( \{ x_i \}_{i=1}^n \) be an \( \mathbb{R} \)-linearly independent set of associative elements of \( M \), then \( \{ x_i \}_{i=1}^n \) is also \( \mathbb{O} \)-linearly independent. Further if \( y = \sum_{i=1}^n r_i x_i \in A(M) \), then \( r_i \in \mathbb{R} \) for each \( i \in \{1, \ldots, n \} \).

### 5.2 The Cyclic Decomposition

In this subsection, we will introduce a notion of **cyclic decomposition**. With the help of this concept, we will formulate the structure of the submodule generated by one element. Let \( M \) be an \( \mathbb{O} \)-bimodule throughout this subsection. It follows from Theorem 3.9 and Theorem 5.3 that \( M = \text{Span}_\mathbb{R} C(M) \).

For any nonzero element \( m \in C(M) \), Theorem assures that 5.3 that there exists an octonion \( p \in \mathbb{O} \) such that \( m = px_m \) for some associative element \( x_m \in A(M) \). Such an octonion \( p \) is called a characteristic value of \( m \) and the real vector \( x_m \) is called a characteristic vector of \( m \).

If we have another \( p' \in \mathbb{O} \) and \( x' \in A(M) \), such that \( m = p'x' \). Note that \( m \neq 0 \), we thus obtain

\[ x_m = p^{-1}(p'x') = (p^{-1}p')x'. \]

Thanks to Lemma 5.4, we conclude \( p^{-1}p' \in \mathbb{R} \). That is, viewed \( \mathbb{O} \) as the real vector space \( \mathbb{R}^8 \), the vectors \( p \) and \( p' \) are parallel. This induces a map

\[ \sigma : C(M) \rightarrow \mathbb{R}^8 \quad m \mapsto [p]. \]
Definition 5.5. Let \( m \in M \) be any given element. Let \( m = \sum_{i=1}^{n} m_i = \sum_{i=1}^{n} r_i x_i \) be a decomposition of cyclic elements, where \( \{x_i\}_{i=1}^{n} \) and \( \{r_i\}_{i=1}^{n} \) are the collection of corresponding characteristic vectors and corresponding characteristic values respectively. Then it is called a cyclic decomposition of \( m \) if it satisfies:

(i). \( \{x_i\}_{i=1}^{n} \) is \( \mathbb{R} \)-linearly independent;

(ii). \( \{r_i\}_{i=1}^{n} \) is \( \mathbb{R} \)-linearly independent in \( \mathbb{C} \).

The following lemma guarantees the existence of cyclic decomposition.

Lemma 5.6. Each element in an \( \mathbb{O} \)-bimodule has a cyclic decomposition.

Proof. The proof will be divided into three steps. Let \( m \) be any given element in \( M \).

Step 1. There is a collection \( \{m_i\}_{i=1}^{n} \subseteq \mathcal{C}(M) \), such that \( m = \sum_{i=1}^{n} m_i \).

This follows from the fact that \( M = \text{Span}_{\mathbb{R}} \mathcal{C}(M) \).

Step 2. We can assume the collection of corresponding characteristic vectors \( \{x_i\}_{i=1}^{n} \) is \( \mathbb{R} \)-linearly independent.

By Step 1, we can assume \( m = \sum_{i=1}^{n} m_i \), and \( m_i = r_i x_i \) for \( i = 1, \ldots, n \). Suppose that \( \{x_i\}_{i=1}^{n} \) is \( \mathbb{R} \)-linearly dependent. Without loss of generality we can assume

\[
\sum_{i=1}^{n-1} t_i x_i = x_n, \quad t_i \in \mathbb{R} \text{ for } i = 1, \ldots, n - 1.
\]

Set

\[
r'_i = r_i + t_i r_n, \quad x'_i = x_i, \quad m'_i = r'_i x'_i, \quad \text{for } i = 1, \ldots, n - 1.
\]

Obviously \( m'_i \in \mathcal{C}(M) \) with corresponding characteristic vectors \( x'_i \) and characteristic values \( r'_i \). It is easy to verify that

\[
\sum_{i=1}^{n-1} m'_i = \sum_{i=1}^{n-1} (r_i + t_i r_n) x_i = \sum_{i=1}^{n-1} r_i x_i + \sum_{i=1}^{n-1} t_i r_n x_i = \sum_{i=1}^{n} m_i = m.
\]

Note that \( \{x'_i\}_{i=1}^{n-1} \) is a subset of \( \{x_i\}_{i=1}^{n} \) and hence we can proceed in this way until the collection of corresponding characteristic vectors turns into an \( \mathbb{R} \)-linearly independent set.

Step 3. We can assume the collection of corresponding characteristic values \( \{r_i\}_{i=1}^{n} \) is \( \mathbb{R} \)-linearly independent.

By Step 2, we can assume \( m = \sum_{i=1}^{n} m_i \), \( m_i = r_i x_i \) for \( i = 1, \ldots, n \), and \( \{x_i\}_{i=1}^{n} \) is \( \mathbb{R} \)-linearly independent. Suppose that \( \{r_i\}_{i=1}^{n} \) is \( \mathbb{R} \)-linearly dependent. Without loss of generality we can assume

\[
\sum_{i=1}^{n-1} t_i r_i = r_n, \quad t_i \in \mathbb{R} \text{ for } i = 1, \ldots, n - 1.
\]

Set

\[
x'_i = x_i + t_i x_n, \quad r'_i = r_i, \quad m'_i = r'_i x'_i, \quad \text{for } i = 1, \ldots, n - 1.
\]
Obviously \( m'_i \in \mathcal{C}(M) \) with corresponding characteristic vectors \( x'_i \) and characteristic values \( r'_i \). It is easy to verify that

\[
\sum_{i=1}^{n-1} m'_i = \sum_{i=1}^{n-1} r_i (x_i + t_i x_n) = \sum_{i=1}^{n-1} r_i x_i + \sum_{i=1}^{n-1} t_i r_i x_n = \sum_{i=1}^{n} m_i = m.
\]

We claim \( \{x'_i\}_{i=1}^{n-1} \) is also an \( \mathbb{R} \)-linearly independent set. Indeed, let \( \sum_{i=1}^{n-1} s_i x'_i = 0 \) for some \( s_i \in \mathbb{R}, \ i = 1, \ldots, n - 1 \). Then we obtain

\[
0 = \sum_{i=1}^{n-1} s_i (x_i + t_i x_n) = \sum_{i=1}^{n-1} s_i x_i + \left( \sum_{i=1}^{n-1} s_i t_i \right) x_n,
\]

it then follows from the \( \mathbb{R} \)-linear independence of \( \{x_i\}_{i=1}^{n} \) that \( s_i = 0 \) for each \( i = 1, \ldots, n - 1 \). This shows that the collection \( \{x'_i\}_{i=1}^{n-1} \) is \( \mathbb{R} \)-linear independent. Note that \( \{r'_i\}_{i=1}^{n} \) is a subset of \( \{r_i\}_{i=1}^{n} \) and hence we can proceed in this way until the collection of corresponding characteristic values turns into an \( \mathbb{R} \)-linearly independent set.

\( \square \)

**Lemma 5.7.** Let \( m_i \in \mathcal{C}(M) \) and \( x_i \) be the characteristic vectors of \( m_i \) for \( i = 1, 2 \). Then the following hold:

\( (i) \) if \( \{x_1, x_2\} \) is \( \mathbb{R} \)-linearly dependent, then \( m_1 + m_2 \in \mathcal{C}(M) \);

\( (ii) \) if \( \{x_1, x_2\} \) is \( \mathbb{R} \)-linearly independent, then \( m_1 + m_2 \in \mathcal{C}(M) \) \( \iff \) \( \sigma(m_1) = \sigma(m_2) \);

\( (iii) \) if \( \{x_1, x_2\} \) is \( \mathbb{R} \)-linearly independent and \( \sigma(m_1) \neq \sigma(m_2) \), then \( \langle m_1 + m_2 \rangle_\mathbb{O} = \mathbb{O}m_1 \oplus \mathbb{O}m_2 \).

**Proof.** Suppose that \( m_i = r_i x_i \) for \( i = 1, 2 \).

We prove assertion \( (i) \). We can assume \( x_1 = rx_2 \) for some \( r \in \mathbb{R} \) by hypothesis, then \( m_1 + m_2 = (r_1 r + r_2)x_2 \in \bigcup_{p \in \mathbb{O}} p \cdot \mathcal{A}(M) \). Then the conclusion follows by Theorem 5.3.

We prove assertion \( (ii) \). Clearly both \( m_1, m_2 \) are nonzero element by hypothesis. If \( \sigma(m_1) = \sigma(m_2) \), we can assume \( r_1 = r_2 \) for some \( r \in \mathbb{R} \). Hence

\[
m_1 + m_2 = rr_2 x_1 + r_2 x_2 = r_2 (rx_1 + x_2) \in \bigcup_{p \in \mathbb{O}} p \cdot \mathcal{A}(M).
\]

This shows that \( m_1 + m_2 \in \mathcal{C}(M) \). Now suppose \( m_1 + m_2 \in \mathcal{C}(M) \). Then we can choose \( 0 \neq p \in \mathbb{O} \) such that \( m_1 + m_2 = px \) for some \( x \in \mathcal{A}(M) \). It follows that

\[
x = (p^{-1}r_1)x_1 + (p^{-1}r_2)x_2,
\]

in view of Lemma 5.4, we conclude \( p^{-1}r_1 \in \mathbb{R}, p^{-1}r_2 \in \mathbb{R} \), which implies \( \sigma(m_1) = \sigma(m_2) \).

We prove assertion \( (iii) \). Thanks to assertion \( (ii) \) that we have just proved, we deduce

\[
\dim_{\mathbb{R}}(\langle m_1 + m_2 \rangle_\mathbb{O}) > 8.
\]

However \( \langle m_1 + m_2 \rangle_\mathbb{O} \) is a submodule included by \( \mathbb{O}m_1 \oplus \mathbb{O}m_2 \), which yields

\[
\dim_{\mathbb{R}}(\langle m_1 + m_2 \rangle_\mathbb{O}) \leq 16.
\]

This forces \( \langle m_1 + m_2 \rangle_\mathbb{O} = \mathbb{O}m_1 \oplus \mathbb{O}m_2 \). \( \square \)
Now we can describe the structure of these submodules generated by one element.

**Theorem 5.8.** Let $m$ be an arbitrary element of an $\mathcal{O}$-bimodule $M$. Then

$$\langle m \rangle_\mathcal{O} = \bigoplus_{i=1}^n \mathcal{O}m_i,$$

where $\{m_i\}_{i=1}^n \subseteq \mathcal{C}(M)$ is an arbitrary cyclic decomposition of $m$.

**Remark 5.9.** In particular, the length $n$ of a cyclic decomposition $\{m_i\}_{i=1}^n$ is an invariant of $m$, just called the length of $m$, and denoted by $l_n$. It then follows that $\langle m \rangle_\mathcal{O} \cong \mathcal{O}^{l_n}$. Clearly the number of $\mathbb{R}$-linearly independent vectors in $\mathcal{O}$ is at most 8, hence for any element $m$ of an $\mathcal{O}$-bimodule $M$, we infer that $\dim_{\mathbb{R}} (\langle m \rangle_\mathcal{O}) \leq 64$. In terms of the concept of length, we have only 8 kinds of elements in an $\mathcal{O}$-bimodule.

**Proof of Theorem 5.8.** We prove this in six steps. Throughout the proof, let $\{m_i\}_{i=1}^n$ be an arbitrary cyclic decomposition of $m$ with corresponding characteristic vectors $x_i$ and characteristic values $r_i$ for $i = 1, \ldots, n$. One direction is obvious, it remains to show that $\mathcal{O}m_i \subseteq \langle m \rangle_\mathcal{O}$ for each $i$. Note that the theorem holds for $n = 1$ trivially and has been proved for $n = 2$ in Lemma 5.7.

**Step 1.** We can assume $r_1 = 1$.

If not, replacing $m$ with $r_1^{-1}m$ and letting $r_i' = r_1^{-1}r_i$, $x_i' = x_i$, $m_i' = r_i'x_i$, then neither the hypothesis nor the conclusion is affected since $\langle m \rangle_\mathcal{O} = \langle r_1^{-1}m \rangle_\mathcal{O}$.

**Step 2.** We can assume $r_i \in \text{Im}(\mathcal{O})$ for $i = 2, \ldots, n$.

In fact, let $r_i = r_{i0} + \sum r_{ij}e_j$, where $r_{ij} \in \mathbb{R}$ for $j = 0, 1, \ldots, 7$. Set

$$
m_i' = \sum_{i=1}^n r_{i0}x_i,
$$

$$
m_i' = (r_i - r_{i0})x_i, \quad r_i' = r_i - r_{i0}, \quad i = 2, \ldots, n.
$$

Clearly $r_i' \in \text{Im}(\mathcal{O})$ for $i = 2, \ldots, n$. We next show that $\{m_i'\}_{i=1}^n$ is another cyclic decomposition of $m$. Note that we have assumed that $r_1 = 1$ from Step 1, we hence conclude that

$$
\sum_{i=1}^n m_i' = \sum_{i=1}^n r_{i0}x_i + \sum_{i=2}^n (r_i - r_{i0})x_i = m.
$$

Let $\sum_{i=1}^n t_ir_i' = 0$ for some $t_i \in \mathbb{R}$. That is,

$$
0 = t_1 + \sum_{i=2}^n t_i(r_i - r_{i0}) = (t_1 - \sum_{i=2}^n t_ir_{i0})r_1 + \sum_{i=2}^n t_ir_i.
$$

It follows from the linear independence of $\{r_i\}_{i=1}^n$ that $t_i = 0$ for $i = 1, \ldots, n$. Hence $\{r_i'\}_{i=1}^n$ is $\mathbb{R}$-linearly independent. So $\{m_i'\}_{i=1}^n$ is another cyclic decomposition of $m$ with $r_i \in \text{Im}(\mathcal{O})$ for $i = 2, \ldots, n$. Moreover, it is easy to verify that

$$
\bigoplus_{i=1}^n \mathcal{O}m_i = \bigoplus_{i=1}^n \mathcal{O}m_i'.
$$

This means it does not affect the conclusion as well.
**Step 3.** We prove the case \( n = 3 \).

Let \( \alpha \) be any element orthogonal to the associative subspace

\[
\Lambda(r_2, r_3) = \{ x \in \text{Im}(D) \mid [r_2, r_3, x] = 0 \}.
\]

Then

\[
[r_2, \alpha, m] = [r_2, \alpha, r_3]x_3 \in \langle m \rangle_O.
\]

Since \([r_2, \alpha, r_3] \neq 0\), we conclude that \( Ox_3 \subseteq \langle m \rangle_O \) and hence \( r_3x_3 \in \langle m \rangle_O \), this yields \( r_1x_1 + r_2x_2 \in \langle m \rangle_O \) and it then follows from Lemma 5.7, i.e., the case \( n = 2 \).

**Step 4.** We prove the case \( n = 4 \).

- If \( r_4 \in \Lambda(r_2, r_3) \).
  According to \( r_2, r_3, r_4 \in \Lambda(r_2, r_3) \) and noting the dimension of associative subspaces are all 3, we conclude \( \Lambda(r_2, r_4) = \Lambda(r_2, r_3) = \Lambda(r_3, r_4) \). Choose \( \alpha \in \Lambda(r_2, r_3)^\perp \) again, then

\[
[r_2, \alpha, m] = [r_2, \alpha, r_3]x_3 + [r_2, \alpha, r_4]x_4 \in \langle m \rangle_O.
\]

Let \( t_1[r_2, \alpha, r_3] + t_2[r_2, \alpha, r_4] = [r_2, \alpha, t_1r_3 + t_2r_4] = 0 \). If \( t_1r_3 + t_2r_4 \neq 0 \), it follows from the linear independence of \( \{r_i\}_{i=2}^4 \), we know \( \{r_2, t_1r_3 + t_2r_4\} \) is \( \mathbb{R} \)-linearly independent and hence we conclude

\[
\alpha \in \Lambda(r_2, t_1r_3 + t_2r_4) = \Lambda(r_2, r_3).
\]

This is a contradiction and hence \( t_1r_3 + t_2r_4 = 0 \). This immediately implies \( t_1 = t_2 = 0 \), which means \( \{r_2, \alpha, r_3, [r_2, \alpha, r_4]\} \) is \( \mathbb{R} \)-linearly independent. It then turns to the case \( n = 2 \).

- If \( r_4 \notin \Lambda(r_2, r_3) \).
  It follows that \([r_2, r_3, m] = [r_2, r_3, r_4]x_4 \in \langle m \rangle_O \) and then turns to the case \( n = 3 \).

**Step 5.** We prove the case \( n = 5 \).

Obviously, it is impossible that both \( r_4, r_5 \) are in \( \Lambda(r_2, r_3) \) since otherwise the dimension of \( \Lambda(r_2, r_3) \) will exceed 3.

- If \( r_4 \in \Lambda(r_2, r_3) \).
  We must have \( r_5 \notin \Lambda(r_2, r_3) \), since \([r_2, r_3, m] = [r_2, r_3, r_5]x_5 \in \langle m \rangle_O \), we deduce \( r_5x_5 \in \langle m \rangle_O \).
  It then follows from the case \( n = 4 \).

- The case \( r_5 \in \Lambda(r_2, r_3) \) is similar.

- If \( r_4, r_5 \notin \Lambda(r_2, r_3) \).
  We have \([r_2, r_3, m] = [r_2, r_3, r_4]x_4 + [r_2, r_3, r_5]x_5 \in \langle m \rangle_O \). Let \( r_i' = [r_2, r_3, r_i] \) for \( i = 4, 5 \). If \( \{r_4', r_5'\} \) is \( \mathbb{R} \)-linearly independent, then by the case of \( n = 2 \), we conclude \( Ox_4 \oplus Ox_5 \subseteq \langle m \rangle_O \) and then by the case \( n = 3 \), we deduce \( \bigoplus_1^3 \mathbb{R}x_i \subseteq \langle m \rangle_O \). Suppose \( \{r_4', r_5'\} \) is \( \mathbb{R} \)-linearly dependent. Without loss of generality we can assume \( r_5' = tr_4' \) for some \( 0 \neq t \in \mathbb{R} \). Thus \( r_4'(x_4 + tx_5) \in \langle m \rangle_O \), this implies \( r_4'(x_4 + tx_5) \in \langle m \rangle_O \). It follows that

\[
\sum_{i=1}^3 r_ix_i + (r_5 - tr_4)x_5 \in \langle m \rangle_O.
\]

Clearly \( \{r_1, r_2, r_3, r_5 - tr_4\} \) is also \( \mathbb{R} \)-linearly independent, hence this turns to the case \( n = 4 \).
Step 6. We prove for \( n > 5 \).

Suppose there is an associative subspace spanned by \( \{r_{i_k}\}_{k=1}^3 \) for some \( i_k \in \{2, \ldots, n\} \), we simply assume \( \Lambda = \text{Span}_{\mathbb{R}}(r_2, r_3, r_4) \) is an associative space. We claim that \( \{r_2, r_3, r_i\}_{i=5}^n \) is \( \mathbb{R} \)-linearly independent. Indeed, since \( \text{Im}(\mathcal{O}) = \Lambda \oplus \Lambda^\perp \), we have,

\[
 r_i = \alpha_i + \beta_i, \quad \alpha_i \in \Lambda, \beta_i \in \Lambda^\perp,
\]

for each \( i \in \{5, \ldots, n\} \). Suppose \( \sum_{i=5}^n t_i [r_2, r_3, r_i] = 0 \) for some \( t_i \in \mathbb{R}, i = 5, \ldots, n \). Hence we obtain \( [r_2, r_3, \sum_{i=5}^n t_i \beta_i] = 0 \), which implies \( \sum_{i=5}^n t_i \beta_i \in \Lambda \cap \Lambda^\perp \), and thus \( \sum_{i=5}^n t_i \beta_i = \sum_{i=5}^n t_i (r_i - \alpha_i) = 0 \).

Let \( \alpha_i = t_i t_2 r_2 + t_3 r_3 + t_4 r_4 \), we conclude

\[
 \sum_{i=5}^n t_i r_i - \sum_{i=5}^n t_i (t_i t_2 r_2 + t_3 r_3 + t_4 r_4) = 0.
\]

In view of the linear independence of \( \{r_i\}_{i=2}^n \), we deduce that \( t_i = 0 \) for \( i = 5, \ldots, n \) and hence \( \{r_2, r_3, r_i\}_{i=5}^n \) is \( \mathbb{R} \)-linearly independent as desired. By above claim, we can deduce

\[
 \bigoplus_{i=5}^n \mathcal{O} x_i \subset \langle m \rangle_{\mathcal{O}}.
\]

Then the rest of the proof runs as before.

Now suppose every subspace spanned by \( \{r_{i_k}\}_{k=1}^3 \) is not associative. Denote \( r'_i = [r_2, r_3, r_i] \) and hence \( r'_i \neq 0 \) for \( i = 4, \ldots, n \). It follows that \( \sum_{i=1}^n r'_i x_i \in \langle m \rangle_{\mathcal{O}} \).

- If \( \{r'_i\}_{i=4}^n \) is \( \mathbb{R} \)-linearly independent.

  It follows from the case \( n - 3 \) that \( \bigoplus_{i=1}^n \mathcal{O} x_i \subset \langle m \rangle_{\mathcal{O}} \), and hence \( \sum_{i=4}^n r_i x_i \in \langle m \rangle_{\mathcal{O}} \) which implies \( \sum_{i=1}^3 r_i x_i \in \langle m \rangle_{\mathcal{O}} \). Therefore \( \bigoplus_{i=1}^3 \mathcal{O} x_i \subset \langle m \rangle_{\mathcal{O}} \).

- If \( \{r'_i\}_{i=4}^n \) is \( \mathbb{R} \)-linearly dependent.

  Without loss of generality we can assume

\[
 r'_n = \sum_{i=4}^{n-1} t_i r'_i, \quad t_i \in \mathbb{R} \text{ for each } i.
\]

Let \( x'_i = x_i + t_i x_n \). Clearly \( \{x'_i\}_{i=4}^{n-1} \) is \( \mathbb{R} \)-linearly independent and

\[
 \sum_{i=4}^n r'_i x_i = \sum_{i=4}^{n-1} r'_i x_i + \sum_{i=4}^{n-1} t_i r'_i x_n = \sum_{i=4}^{n-1} r'_i x'_i \in \langle m \rangle_{\mathcal{O}}.
\]

* If \( \{r'_i\}_{i=4}^{n-1} \) is \( \mathbb{R} \)-linearly independent.

  It follows from the case \( n - 4 \) that \( \bigoplus_{i=4}^{n-1} \mathcal{O} x'_i \subset \langle m \rangle_{\mathcal{O}} \), and hence

\[
 \sum_{i=4}^{n-1} r_i x'_i = \sum_{i=4}^{n-1} r_i x_i + \left( \sum_{i=4}^{n-1} r_i t_i \right) x_n \in \langle m \rangle_{\mathcal{O}}.
\]
Then
\[ m - \sum_{i=1}^{n-1} r_i x_i' = \sum_{i=1}^3 r_i x_i + \left( r_n - \sum_{i=1}^{n-1} r_i t_i \right) x_n \in \langle m \rangle_0. \]

It is easy to see that \( \{ r_1, r_2, r_3, r_n - \sum_{i=1}^{n-1} r_i t_i \} \) is \( \mathbb{R} \)-linearly independent, according to the case \( n = 4 \), we infer that \( r_n x_n \in \langle m \rangle_0 \) and then reduces to the case \( n - 1 \).

* If \( \{ r_i' \}_{i=4}^{n-1} \) is \( \mathbb{R} \)-linearly dependent.

Without loss of generality we can assume
\[ r_{n-1}' = \sum_{i=4}^{n-2} s_i r'_i, \quad s_i \in \mathbb{R} \text{ for each } i. \]

Let \( x_i'' = x_i' + s_i x_{n-1}' \). Clearly \( \{ x_i'' \}_{i=4}^{n-2} \) is \( \mathbb{R} \)-linearly independent and
\[ \sum_{i=1}^{n-1} r_i x_i'' = \sum_{i=1}^{n-2} r_i x_i' + \sum_{i=4}^{n-2} s_i r'_i x_{n-1}' = \sum_{i=4}^{n-2} r_i x_{n-1}' \in \langle m \rangle_0. \]

* If \( \{ r_i' \}_{i=4}^{n-2} \) is \( \mathbb{R} \)-linearly independent.

It follows from the case \( n = 5 \) that \( \bigoplus_{i=4}^{n-2} \mathbb{R} x_i'' \subset \langle m \rangle_0 \), and hence
\[ \sum_{i=4}^{n-2} r_i x_i'' = \sum_{i=4}^{n-2} r_i x_i' + \sum_{i=4}^{n-2} s_i r'_i x_{n-1}' = \sum_{i=4}^{n-2} r_i x_{n-1}' \in \langle m \rangle_0. \]

Then
\[ m - \sum_{i=4}^{n-2} r_i x_i'' = \sum_{i=1}^{3} r_i x_i + \left( r_{n-1}' - \sum_{i=4}^{n-2} r_i s_i \right) x_{n-1} + \left( r_n - \sum_{i=4}^{n-2} (r_i t_i + r_i s_i t_{n-1}) \right) x_n \in \langle m \rangle_0. \]

It is easy to see that \( \{ r_1, r_2, r_3, r_{n-1} - \sum_{i=1}^{n-2} r_i s_i, r_n - \sum_{i=1}^{n-2} (r_i t_i + r_i s_i t_{n-1}) \} \) is \( \mathbb{R} \)-linearly independent, it follows from the case \( n = 5 \) that \( r_n x_n \in \langle m \rangle_0 \) and then reduces to the case \( n - 1 \). Note that we have already proved the case \( n = 6 \).

* If \( \{ r_i' \}_{i=4}^{n-2} \) is \( \mathbb{R} \)-linearly dependent.

Apply the argument similar to above twice, we can obtain an \( \mathbb{R} \)-linearly independent subset of \( \{ r_i' \}_{i=4}^{n} \) since \( r_i' \neq 0 \) for each \( i \). Thus it can always reduce to the preceding case.

This proves the theorem. \( \square \)

**Example 5.10.** Let \( M = \mathbb{O}^3 \). Let \( m = (e_1, e_2, e_1 + e_2) \), then \( m = e_1(1, 0, 1) + e_2(0, 1, 1) \) and clearly this is a cyclic decomposition of \( m \), hence \( \langle m \rangle_0 = \mathbb{O}(1, 0, 1) \oplus \mathbb{O}(0, 1, 1) \). Consider the example in [4], let \( m = (e_1, e_2, e_3) \), then \( m = e_1(1, 0, 0) + e_2(0, 1, 0) + e_3(0, 0, 1) \) and this is a cyclic decomposition of \( m \), hence \( \langle m \rangle_0 = \mathbb{O}^3 = M \).

**Example 5.11.** Let \( M = \mathbb{O}^3 \) and consider the element \( m = (1, 1 + e_1, e_1) \). Then choose \( m_1 = (1, 0, 0), m_2 = (0, 1 + e_1, 0), m_3 = (0, 0, e_1) \) in \( \mathcal{C}(M) \), it clearly holds \( m = \sum_{i=1}^{3} m_i \) and they are
real linear independent. However, this is not a cyclic decomposition in the sense of Definition 5.5.

On the other hand, we can choose \( m'_1 = (1, 1, 0), \) \( m'_2 = e_1(0, 1, 1) \) in \( \mathcal{C}(M) \), one can verify that this is indeed a cyclic decomposition of \( m \). Thus it holds

\[
\langle (1, 1 + e_1, e_1) \rangle_{\mathcal{O}} = \mathcal{O} \cdot (1, 1, 0) + \mathcal{O} \cdot e_1(0, 1, 1) \cong \mathcal{O}^2.
\]

Using Theorem 5.8, we point out a mistake in [10, Lemma 2.4.2], which claims each element in an \( \mathcal{O} \)-bimodule will satisfy \( \mathcal{O}x = x\mathcal{O} \). We shall show that only cyclic elements possess such property.

**Corollary 5.12.** Let \( m \) be an arbitrary element in an \( \mathcal{O} \)-bimodule \( M \). Then we have

\[
\mathcal{O}m = m\mathcal{O} \iff m \in \mathcal{C}(M).
\]

**Proof.** Suppose \( m \in \mathcal{C}(M) \), then we have \( m = px \) for some \( p \in \mathcal{O} \) and some \( x \in \mathcal{A}(M) \). It is easy to check that

\[
\mathcal{O}m = \mathcal{O}x = x\mathcal{O} = m\mathcal{O}.
\]

On the other hand, if it holds \( \mathcal{O}m = m\mathcal{O} \), suppose on the contrary that \( m \notin \mathcal{C}(M) \). This means the length \( l_m > 1 \), for brief, write \( l_m = n \).

Let \( \{m_i\}_{i=1}^n \) be a cyclic decomposition of \( m \), the corresponding characteristic vectors \( \{x_i\}_{i=1}^n \) and characteristic values \( \{r_i\}_{i=1}^n \) are both \( \mathbb{R} \)-linearly independent. It follows from the hypothesis \( \mathcal{O}m = m\mathcal{O} \) that, for any \( p \in \mathcal{O} \), there is an octonion \( q \in \mathcal{O} \) such that \( pm = mq \), note that \( x_i \in \mathcal{A}(M) \), we have,

\[
0 = pm - mq = \sum_{i=1}^n p(r_i x_i) - \sum_{i=1}^n (r_i x_i)q = \sum_{i=1}^n x_i (pr_i - r_i q).
\]

Since \( \{x_i\}_{i=1}^n \) is \( \mathbb{R} \)-linearly independent, it follows from Lemma 5.4 that

\[
pr_i - r_i q = 0 \quad (5.1)
\]

for each \( i = 1, \ldots, n \). Fix \( p \) arbitrarily, then \( r_i^{-1}pr_i = q \) is a constant for all \( i \in \{1, \ldots, n\} \). If there exists \( r_i \in \mathbb{R} \), it follows that this constant is the fixed octonion \( p \) and hence

\[
r_j^{-1}pr_j = p, \quad \text{for each } j \neq i.
\]

Since \( p \) is arbitrarily fixed, we thus obtain \( r_j \in \mathbb{R} \) for each \( j = 1, \ldots, n \). This contradicts the fact that \( \{r_i\}_{i=1}^n \) is \( \mathbb{R} \)-linearly independent.

We can assume \( r_i \notin \mathbb{R} \) for each \( i \). Suppose \( r_i \in \mathbb{C}J_i \setminus \mathbb{R} \) for some \( J_i \in \mathcal{S} \). Substituting \( p = r_j \) in (5.1) for \( j = 1, \ldots, n \), we obtain \( r_ir_j = r_jr_i \) for all \( i, j \in \{1, \ldots, n\} \). Hence \( r_i \in \cap_j \mathbb{C}J_j \setminus \mathbb{R} \) for each \( i \in \{1, \ldots, n\} \). Consequently, there exists an imaginary unit \( J \in \mathcal{S} \), such that \( r_i \in \mathbb{C}J \) for every \( i \in \{1, \ldots, n\} \). We conclude immediately from the \( \mathbb{R} \)-linearity independence of \( \{r_i\}_{i=1}^n \) that the length \( n \) is no more than 2, it the follows from \( n > 1 \) that \( n = 2 \). Suppose \( r_1 = a + bJ, \ r_2 = c + dJ, \) where \( a, b, c, d \in \mathbb{R} \). Let \( r'_1 = 1, \ r'_2 = J \) and \( x'_1 = ax_1 + cx_2, \ x'_2 = bx_1 + dx_2 \), then

\[
m = r_1x_1 + r_2x_2 = r'_1x'_1 + r'_2x'_2.
\]

The \( \mathbb{R} \)-linearity independence of \( \{x_i\}_{i=1}^2 \) and \( \{r_i\}_{i=1}^2 \) yields the \( \mathbb{R} \)-linearity independence of \( \{x'_i\}_{i=1}^2 \). It follows that \( \{r'_ix'_i\}_{i=1}^2 \) is another cyclic decomposition of \( m \) and satisfies \( r'_1 = 1 \in \mathbb{R} \), this contradicts the assumption above. We thus derive that \( m \in \mathcal{C}(M) \).

\[\square\]
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