A computer-friendly construction of the monster

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Preliminary version

Abstract

Let \( M \) be the monster group which is the largest sporadic finite simple group, and has first been constructed in 1982 by Griess. In 1985, Conway has constructed a 196884-dimensional representation \( \rho \) of \( M \) with matrix coefficients in \( \mathbb{Z}[\frac{1}{2}] \). So these matrices may be reduced modulo any (not necessarily prime) odd number \( p \), leading to representations of \( M \) in odd characteristic. The representation \( \rho \) is based on representations of two maximal subgroups \( G_{x0} \) and \( N_0 \) of \( M \). In ATLAS notation, \( G_{x0} \) has structure \( 2^{1+24}.\text{Co}_1 \) and \( N_0 \) has structure \( 2^{2+11+22}.(M_{24} \times S_3) \). Conway has constructed an explicit set of generators of \( N_0 \), but not of \( G_{x0} \).

This paper is essentially a rewrite of Conway’s construction augmented by an explicit construction of an element of \( G_{x0} \setminus N_0 \). This gives us a complete set of generators of \( M \). It turns out that the matrices of all generators of \( M \) consist of monomial blocks, and of blocks which are essentially Hadamard matrices scaled by a negative power of two. Multiplication with such a generator can be programmed very efficiently if the modulus \( p \) is of shape \( 2^k - 1 \).

So this paper may be considered a programmer’s reference for Conway’s construction of the monster group \( M \). We have implemented representations of \( M \) modulo 3, 7, 15, 31, 127, and 255.

Key Words:

Monster group, finite simple groups, group representation

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1 Introduction

Let \( M \) be the monster group, which is the largest sporadic finite simple group, and has first been constructed by Griess [9]. That construction has been greatly simplified by Conway [5], leading to a rational representation \( \rho \) of \( M \) of dimension \( 196883+1 \). This paper is essentially a rewrite of Conway’s construction of \( M \), augmented by an explicit construction of the representations of a complete set of generators of \( M \). Here all generators of \( M \) have coefficients in \( \mathbb{Z}[\frac{1}{2}] \) as in [5], and they can be computed very efficiently. So a programmer may use this paper as a reference for implementing the representation \( \rho \) of \( M \) modulo a small odd number \( p \).

The success of the construction in [5] is due to the fact that the representation \( \rho(N_{x0}) \) of a large subgroup \( N_{x0} \) of \( M \) of structure \( 2^{1+24+11}.M_{24} \) can be made explicit and monomial. We use the notation in the ATLAS [6] for describing the structure of a group, so \( M_{24} \) is the Mathieu group acting as a permutation group on 24 elements. There is a maximal subgroup
The Gottesman-Knill theorem states that stabilizer circuits can be simulated in polynomial order, which is called $4096 \times 2^2$. This explains simpler relations in the group $N_{1+24}$. For computations in the subgroup $N_0$ of $M$, we have to select a specific cocycle for the Parker loop in such a way that the (non-monomial) representation of a certain element $\xi$ of $G_{x0}\setminus N_{x0}$ obtains a reasonably simple structure.

In our construction we take the notation from [5], except for two explicitly stated sign changes required for the construction of a generator $\xi \in G_{x0}\setminus N_{x0}$. One of these changes is explained in section 5 and motivated by Ivanov’s construction of $M$ in [14]. This leads to simpler relations in the group $N_0$.

The construction [5] of $M$ uses the Parker loop, which is a non-associative loop of order $2^{1+12}$. For computations in the Parker loop a cocycle can be used, see e.g. Aschbacher [2], Chapter 4. Such a cocycle is not unique. While the choice of that cocycle is not too important for computing in the subgroup $N_0$ of $M$, we have to select a specific cocycle for the Parker loop in such a way that the (non-monomial) representation of a certain element $\xi$ of $G_{x0}\setminus N_{x0}$ obtains a reasonably simple structure.

It is worth noting that the complex analogue $\mathcal{X}_n$ of the real Clifford group $C_n$, also defined in [15], plays an important role in the theory of quantum computing. A general quantum circuit with $n$ qubits is modelled using a dense subgroup of the complex unitary group $U(2^n, \mathbb{C})$ and cannot be simulated in polynomial time on a classical computer. There is an important class of quantum circuits with $n$ qubits called stabilizer circuits, which can be modelled using the discrete subgroup $\mathcal{X}_n$ of $U(2^n, \mathbb{C})$. Here all quantum gates can be modelled by tensor products of certain well-behaved monomial and of Hadamard matrices.

The hard part of Conway’s construction [5] was to find an algebra invariant under $\rho(M)$, similar to the Griess algebra defined in [3]. Therefore Conway defined an algebra visibly invariant under $\rho(G_0)$ and showed that this algebra is also invariant under $\rho(N_0)$, using a basis of $\rho$ where $N_0$ has a simple representation. In this paper we need an explicit representation $\rho(\xi)$ of a $\xi \in G_0 \setminus N_{x0}$. So we adjust the signs of that basis in order to simplify $\rho(G_0)$.

Define a Hadamard step on $\rho(M)$ to be a multiplication with a matrix that contains just monomial blocks and blocks of $2 \times 2$-Hadamard matrices, i.e. matrices of shape $c(\frac{1}{2}, \frac{-1}{2})$ for $\mathbb{C}$.
$c \in \{\frac{1}{2}, 1\}$. We will see that the representations of all our generators of $M$ can be decomposed into at most 6 Hadamard steps plus some monomial operations.

For any odd natural number $p$ let $\rho_p$ be the representation $\rho$ of the monster, where all coefficients are taken modulo $p$. Computations in $\rho_p$ are very efficient if $p + 1$ is a reasonably small power $2^k$ of two. In this case we may represent an integer modulo $p$ with $k$ bits, putting $(1, \ldots, 1)_2 = (0, \ldots, 0)_2 = 0$. Then negation can be done by complementing all bits, halving can be done by right rotation, and the carry bit of an addition has the same valence as the least significant bit. Using integer additions, and bitwise and shift operations on a 32-bit or 64-bit computer, several components of a vector in $\rho_p$ can be negated, halved, or added modulo a small number $p = 2^k - 1$ simultaneously in a single register.

We have implemented the representations $\rho_p$ of the monster for $p = 3, 7, 15, 31, 127,$ and 255, see [10]. Calculating in the monster modulo different numbers is useful e.g. for distinguishing between classes in $M$ of the same order, see [4].

Let $g \in M$ be the product of an arbitrary element of $G_{40}$ with a power of $\xi$. On the author’s 64-bit Windows computer, the operation of $\rho_p(g)$ on a single vector costs 0.73 ms for $p = 3$ and 1.35 ms for $p = 255$. That computer has an Intel Core i7-8750H CPU running at up to 4.0 GHz. These benchmarks are single-threaded.

## 2 The Golay code $C$ and its cocode $C^*$

### 2.1 Description of the Golay code $C$ and its cocode $C^*$

Let $\tilde{\Omega}$ be a set of size 24 and construct the vector space $\mathbb{F}_2^{24}$ as $\prod_{\alpha \in \tilde{\Omega}} \mathbb{F}_2$. A Golay code $C$ is a 12-dimensional linear subspace of $\mathbb{F}_2^{24}$ whose smallest weight is 8. This characterizes the Golay code up to permutation. A Golay code has weight distribution $0^1 8^7 9^5 12^7 25^3 107^1$. We identify the power set of $\tilde{\Omega}$ with $\mathbb{F}_2^{24}$ by mapping each subset of $\tilde{\Omega}$ to its characteristic function, which is a vector in $\mathbb{F}_2^{24}$. So we may write $\tilde{\Omega}$ for the Golay code word containing 24 ones, and for elements $d, e$ of $\mathbb{F}_2^{24}$ we write $d \cup e, d \cap e$ for their union and intersection and $d + e$ for their symmetric difference. Golay code words of length 8 and 12 are called octads and dodecads, respectively.

The Golay cocode $C^*$ corresponding to a Golay code $C$ is the 12-dimensional quotient space $\mathbb{F}_2^{24}/C$. For each $\delta \in C^*$ define the weight of $\delta$ to be the weight of its lightest representative in $\mathbb{F}_2^{24}$. Then $C^*$ has weight distribution $0^1 1^{24} 2^{759} 3^{2024} 4^{1771}$. Here the lightest representative is unique if its weight is less than 4. If $\delta$ has weight 4, there is a set of six mutually disjoint lightest representatives of $\delta$, and any such subset of $\mathbb{F}_2^{24}$ is called a tetrad. The parity of a cocode element $\delta$ is defined as the parity of its weight $|\delta|$.

There is a natural scalar product $\langle, \rangle$ on $C \times C^*$ given by $\langle d, \delta C \rangle = |d \cap \delta|$ mod 2 for any $d \in C, \delta \in \mathbb{F}_2^{24}$. As usual, a subspace $X^*$ of $C^*$ is orthogonal to a subspace $X$ of $C$ if $\langle d, \delta \rangle = 0$ for all $d \in X, \delta \in X^*$.

The automorphism group of a Golay code $C$ is the Mathieu group $M_{24}$. $M_{24}$ also preserves $C^*$, see [7]. For our construction of the Monster we need a specific instance $C$ of a Golay code. Therefore we assume that the reader is familiar with [7], Chapter 11, section 1–7 and 9.

There is a 3-dimensional linear code of length 6 over $\mathbb{F}_4$ with weight distribution $0^1 4^{50} 18$ called the hexacode. A detailed description of one instance $H_6$ of the hexacode is given in [7], Chapter 11. To be concrete, let $\mathbb{F}_4 = \{0, 1, \alpha, \bar{\alpha}\}$ with $\alpha^2 = \bar{\alpha} = 1 + \alpha$, and let $H_6$ be the subspace of $\mathbb{F}_4^6$ spanned by $(1001\bar{\alpha}, 0101\bar{\alpha}, 001111)$ as in [7].

We number the elements of $\tilde{\Omega}$ from 0 to 23, and arrange them in a table with 4 rows and 6 columns. We also assign an element of $\mathbb{F}_4$ and a colour to each row as follows:

| white | red  | green | blue |
|-------|------|-------|------|
| 0     | 0    | 0     | 0    |
| 1     | 1    | 1     | 1    |
| 2     | 2    | 0     | 3    |
| 3     | 3    | 1     | 6    |
| 4     | 4    | 1     | 7    |
| 5     | 5    | 2     | 10   |
| 6     | 6    | 2     | 10   |
| 7     | 7    | 3     | 11   |
| 8     | 8    | 3     | 11   |
| 9     | 9    | 4     | 12   |
| 10    | 10   | 5     | 13   |
| 11    | 11   | 5     | 13   |
| 12    | 12   | 6     | 14   |
| 13    | 13   | 6     | 14   |
| 14    | 14   | 7     | 15   |
| 15    | 15   | 8     | 16   |
| 16    | 16   | 8     | 16   |
| 17    | 17   | 9     | 17   |
| 18    | 18   | 10    | 18   |
| 19    | 19   | 10    | 18   |
| 20    | 20   | 11    | 20   |
| 21    | 21   | 12    | 21   |
| 22    | 22   | 13    | 22   |
| 23    | 23   | 14    | 23   |

(2.1.1)
This table is called MOG (Miracle Octad Generator) in [7]. We start row and column numbers with 0, so element $m + 4 \cdot n$ of $\Omega$ corresponds to row $m$, column $n$. This is bad for Fortran and good for C programmers.

Each element $x$ of $\mathbb{F}_2^{24}$ has a hexacode value $h(x) \in \mathbb{F}_4^6$ which is calculated as follows. We write the entries of $x$ into the MOG. Then for each column of the MOG we compute a weighted sum of its nonzero entries, where each entry of the MOG has weight corresponding to the element of $\mathbb{F}_4$ associated with its row. The result $h(x)$ is a vector in $\mathbb{F}_4^6$.

Also, for each row or column in the MOG the parity of $x$ in that row or column is the number of the nonzero entries in that row or column taken modulo 2.

Definition 2.2. Let $C$ be the linear subspace of $\mathbb{F}_2^{24}$, characterized by the following properties:

$x$ is in $C$ if and only if $h(x) \in \mathcal{H}_6$ and for all columns of the MOG the parity of $x$ in that column is equal to the parity of $x$ in row 0.

In [7], Chapter 11 it is shown that $C$ is indeed a Golay code. We call an element of $C$ odd or even, depending on its parity in row 0.

2.2 The ‘grey’ and the ‘coloured’ subspaces of $C$ and of $C^*$

This subsection contains material which is not covered by [2], [5], [7] or [14], and which we need in section 7 for the first time. So it may be skipped at first reading.

The construction of the Golay code in [7], Chapter 11.5 motivates the construction of the Golay code as a direct sum $C = G \oplus H$. For reasons to be explained below, the elements of $G$ and $H$ will be called grey and coloured, respectively. We also give a similar decomposition $C^* = G^* \oplus H^*$ of the Golay cocode $C^*$ into a direct sum of a grey and a coloured subspace.

In section 9 we will embed both, the Golay code $C$ and its cocode $C^*$, into the 24-dimensional Leech lattice $\Lambda$ modulo 2. Loosely speaking, we will construct an automorphism $\xi$ of $\Lambda/2\Lambda$ that fixes $H$ and $H^*$, and exchanges $G$ with $G^*$. By construction, $\xi$ is in the automorphism group $\text{Co}_1$ of $\Lambda/2\Lambda$, but not in the automorphism group $M_{24}$ of $C$. Thus $\xi$ can be lifted to an element of the monster group in $2^{1+24}.\text{Co}_1 \setminus 2^{1+24+11}.M_{24}$ as required. We remark that our construction of $\xi$ can also be achieved by embedding two orthogonal copies of the 12-dimensional Coxeter-Todd lattice $K_{12}$ into the Leech lattice, as outlined in [7], Ch. 4.9. Here one copy of $K_{12}$ corresponds to $G \oplus G^*$, and the other copy to $H \oplus H^*$.

We call an element $d$ of $C$ grey if in each column of the MOG all entries of $d$ in that column in rows 1–3 are equal. Different columns may have different entries in rows 1–3. Imagine that each nonzero entry in the MOG switches on a light with a colour as given for its row in the MOG, and that the lights in each column are mixed to a common single colour. Then $d$ is grey if none of these 6 mixed lights switched on by $d$ shows any colour apart from white or black.

The even grey elements of $C$ are precisely those with equal entries in each column, so that the number of nonzero columns is even. So there are 32 of them and they form a 5-dimensional subspace $G^0$ of $C$. There is also an odd grey element of $C$, e.g.:

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Thus the grey elements form a 6-dimensional subspace $G$ of $C$.

We define a monomorphism $h^* : \mathbb{F}_4^6 \rightarrow \mathbb{F}_2^{24}$ with $h^*(h) = h$ for $h \in \mathbb{F}_4^6$ ($\mathbb{F}_4^6$ being considered as a vector space over $\mathbb{F}_2$) as follows. For $h \in \mathbb{F}_4^6$ let $h^*(h)$ be the unique element $x$ of $\mathbb{F}_2^{24}$ that has zeros in row 0 of the MOG and exactly 0 or 2 nonzero elements in each column of the MOG such that the hexacode value $h(x)$ is equal to $h$. We put $\mathcal{H} = \{h^*(x) \mid x \in \mathcal{H}_6\}$. Then we have $C = G \oplus H$. The elements of $H$ are called coloured. So the coloured Golay
code words are those without any white light and with an even number of coloured lights in each column of the MOG.

Let $G^*$ be the orthogonal complement of $H$ and let $H^*$ be the orthogonal complement of $G$ in $C^*$. Elements of $G^*$ and $H^*$ are also called grey and coloured, respectively. The monomorphism $\Phi$ yields a natural isomorphism $F_5^6/H_6 \rightarrow H^*$. Since more than half of the codewords in $H_6$ have weight 4, the following lemma is obvious:

**Lemma 2.3.** $H$ is generated by $\Phi(h)$, where $h$ runs over the code words in $H_6$ of weight 4. $H^*$ is generated by $\Phi(h)$, where $h$ runs over the basis vectors of $F_5^6$ and their scalar multiples.

Let $\omega_\infty$ be the element of $F_2^{24}$ with entries 1 in MOG row 0 and zeros in the other rows. Let $\omega_i, i = 0, \ldots, 5$ be the element of $F_2^{24}$ with entries 1 in MOG column $i$ and zeros in the other columns. Put $g_i = \omega_i + \omega_\infty$. Then $g_0$ is as in [22,1] and $g_i$ is obtained form $g_0$ by exchanging column 0 with column $i$ in the MOG. $g_0, \ldots, g_5$ is a basis of $G$. So $\omega_0, \ldots, \omega_5$ and $\omega_\infty$ represent the same element of the code $C^*$, which we will denote by $\omega$. $\omega$ has minimum weight 4 in $C^*$ and the tetrad corresponding to $\omega$ is $\{\omega_0, \ldots, \omega_5\}$. Then $d \in C$ is even if and only if $\langle d, \omega \rangle = 0$.

Let $\gamma_n \in C^*$ correspond the vector with an entry 1 in MOG row 0, column $n$, and entries 0 elsewhere. Then $(\gamma_0, \ldots, \gamma_5)$ is a basis of $G^*$.

**Definition 2.4.** For $d \in G$ and $\delta \in G^*$ let $w(d)$ and $w(\delta)$ be the weight of $d$ and $\delta$ with respect to the basis $(g_0, \ldots, g_5)$ and $(\gamma_0, \ldots, \gamma_5)$, respectively.

Then $\Omega = \sum_{n=0}^{5} g_n$, $\omega = \sum_{n=0}^{5} \gamma_n$ and we have $w(\Omega) = w(\omega) = 6$. For $d \in G$, $\delta \in G^*$, the weights $w(d)$ and $w(\delta)$ determine the weight $|d|$ and the minimum weight $|\delta|$ as follows:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
w(d), w(\delta) & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
|d| & 0 & 8 & 8 & 12 & 16 & 16 & 24 \\
\min |\delta| & 0 & 1 & 2 & 3 & 4 & 4 & 4 \\
\hline
\end{array}
\]

(2.5.1)

By definition of $g_0, \ldots, g_5$ and $\gamma_0, \ldots, \gamma_5$ we have:

\[
\langle g_m, \gamma_n \rangle = 0 \quad \text{if} \quad m = n \quad \text{and} \quad \langle g_m, \gamma_n \rangle = 1 \quad \text{otherwise}.
\]

(2.5.2)

Thus the reciprocal basis of $(g_0, \ldots, g_5)$ is $(\gamma_0 + \omega, \ldots, \gamma_5 + \omega)$ and the reciprocal basis of $(g_0 + \Omega, \ldots, g_5 + \Omega)$ is $(\gamma_0, \ldots, \gamma_5)$.

The following fact is rather obvious:

\[
\forall d \in G : \quad w(d + \Omega) = 6 - w(d), \quad w(d) = \text{parity}(d) \quad \text{(mod 2)}.
\]

(2.5.3)

## 3 The Parker loop $\mathcal{P}$

### 3.1 The definition of the Parker loop

The Parker loop $\mathcal{P}$ is a non-associative Moufang loop written multiplicatively and operating on the set $\mathcal{P} = C \times F_2$, see [25,14]. For any element $d$ of $\mathcal{P}$ we write $\bar{d}$ for the loop inverse of $d$ in $\mathcal{P}$ and we write $\bar{d}$ for the projection of $d$ into $C$ obtained by dropping the second component of $d$. This projection is a homomorphism from $\mathcal{P}$ to the additive group $(C, +)$. We also write 1, $-1$, $\Omega$ and $-\Omega$ for the elements $(0, 0)$, $(0, 1)$, $(\Omega, 0)$ and $(\Omega, 1)$ of $\mathcal{P}$. Let $d, e, f \in \mathcal{P}$.

The following property characterizes the Parker loop up to isomorphism:

\[
d^2 = (-1)^P(d) \quad \text{with} \quad P(d) = \frac{1}{4}|d|,
\]

\[
(de)(ed)^{-1} = (-1)^C(d,e) \quad \text{with} \quad C(d,e) = \frac{1}{2}|d \cap e|,
\]

\[
(d(e)f)((de)f)^{-1} = (-1)^A(d,e,f) \quad \text{with} \quad A(d,e,f) = |d \cap e \cap f|.
\]

(3.1.1)
In [2] the mappings $P : C \to \mathbb{F}_2$, $A : C^3 \to \mathbb{F}_2$ and $A : C^4 \to \mathbb{F}_2$ given by (3.1.1) are called power map, commutator and associator, respectively. Recall that \( ' + ' \) denotes the symmetric difference of two sets. By [2], Lemma 11.1 and Lemma 11.8 or by direct calculation using the identity

\[
|\tilde{d} + \tilde{e}| = |\tilde{d}| + |\tilde{e}| - 2|\tilde{d} \cap \tilde{e}|,
\]

for finite sets $\tilde{d}, \tilde{e}$ we obtain:

**Lemma 3.2.** The associator $A$ is a symmetric trilinear form on $C$, and we have:

\[
P(\tilde{d} + \tilde{e}) = P(\tilde{d}) + P(\tilde{e}) + C(\tilde{d}, \tilde{e}), \quad (3.2.1)
\]

\[
C(\tilde{d} + \tilde{e}, \tilde{f}) = C(\tilde{d}, \tilde{f}) + C(\tilde{e}, \tilde{f}) + A(\tilde{d}, \tilde{e}, \tilde{f}). \quad (3.2.2)
\]

We obviously have $C(\tilde{d}, \tilde{e}) = C(\tilde{e}, \tilde{d})$ and by Lemma 3.2 we have $A(\tilde{d}, \tilde{d}, \tilde{e}) = 0$. This implies that $P$ is diassociative, i.e. any subloop of $P$ generated by two elements is a group, see [2] [13]. This saves a few brackets in some cases.

From now on we follow the convention in [5], using the same notation for elements of $P$ and of $C$. If a function $F$ has domain $C^*$ then $F(d, e, f, \ldots)$ will mean $F(\tilde{d}, \tilde{e}, \tilde{f}, \ldots)$ for $d, e, f, \ldots \in P$. E.g. $A(d, e, f)$ means $A(\tilde{d}, \tilde{e}, \tilde{f})$, $(d, \delta)$ means $(\tilde{d}, \tilde{\delta})$ for $\delta \in C^*$. But we still distinguish between $d \in P$ and $\tilde{d} \in C$ whenever we consider $d$ as a pair $d = (\tilde{d}, \lambda) \in C \times \mathbb{F}_2$.

We use this convention also for functions defined on subsets of $\Omega$, with the embedding $C \to \Omega$. So $i \in d$ means $i \in \tilde{d}$, $|d \cap e| = |\tilde{d} \cap \tilde{e}|$, etc. We also use the conventions in [2] for denoting elements of $P$, $C$ and $C^*$:

\[
a, b, c, d, e, f, h \quad \text{denote elements of } P \text{ or, loosely, of } C,
\]

\[
\delta, \epsilon, \varphi, \eta \quad \text{denote elements of } C^*;
\]

\[
i, j, k \quad \text{denote elements of } \Omega, \text{ also considered as elements of } C^* \text{ of weight 1},
\]

\[
ij \quad \text{and } d \cap e \quad \text{denote the sets } \{i, j\} \text{ and } \tilde{d} \cap \tilde{e}, \text{ considered as elements of } C^*.
\]

We will also write $\tilde{d}$ for the inverse $d^{-1}$ of $d$ in $P$. We have $\tilde{d} = (-1)^{|d|/4}d$. For $d = (\tilde{d}, \mu) \in P = C \times \mathbb{F}_2$ we put $\text{sign}(d) = (1)^{\mu}$. Note that this sign mapping is not a homomorphism from $P$ to $\{\pm 1\}$.

### 3.2 Cocycles for the Parker loop

Let $V$ be an $n$-dimensional vector space over $\mathbb{F}_2$ and $q : V \to \mathbb{F}_2$ be a function with $q(0) = 0$. Let $\beta_q : V \times V \to \mathbb{F}_2$ be given by $\beta_q(x, y) = q(x + y) + q(x) + q(y)$. If $\beta_q$ is bilinear then $q$ is called a quadratic form on $V$ and $\beta_q$ is the bilinear form associated with $q$. Two quadratic forms on $V$ have the same associated bilinear form if they differ by a linear function on $V$. The bilinear form $\beta_q$ associated with $q$ is alternating, i.e. $\beta_q(x, x) = 0$ for all $x \in V$. Any alternating bilinear form on $V$ is also symmetric.

For computations in $P$ it is convenient to use a cocycle $\theta : C \times C \to \mathbb{F}_2$ such that for $P = C \times \mathbb{F}_2$ we have:

\[
(\tilde{d}_1, \lambda_1) \cdot (\tilde{d}_2, \lambda_2) = (\tilde{d}_1 + \tilde{d}_2, \lambda_1 + \lambda_2 + \theta(d_1, d_2)) , \quad \tilde{d}_1, \tilde{d}_2 \in C, \lambda_1, \lambda_2 \in \mathbb{F}_2 . \quad (3.2.3)
\]

Cocycles for certain types of loops have been studied in [2] [8] [11]. A cocycle for $P$ satisfying (3.1.1) which is quadratic in the first and linear in the second argument, has been constructed in [7], Ch. 29, Appendix 2. This can be summarized as follows:

**Lemma 3.3.** There is a cocycle $\theta : C \times C \to \mathbb{F}_2$ for $P$ with the following properties:

\[
P(d) = \theta(d, d), \quad (3.3.1)
\]

\[
C(d, e) = \theta(d, e) + \theta(e, d), \quad (3.3.2)
\]

\[
\theta(d + e, f) = \theta(d, f) + \theta(e, f) + A(d, e, f), \quad (3.3.3)
\]

\[
\theta(d, e + f) = \theta(d, e) + \theta(d, f). \quad (3.3.4)
\]
Sketch Proof

Let \( b_0, \ldots, b_{11} \) be a basis of \( C \). Define

\[
\theta(b_i, b_j) = \begin{cases} 
0 & \text{if } i < j \\
P(b_i) & \text{if } i = j \\
C(b_i, b_j) & \text{if } i > j
\end{cases}
\]

Then \( \theta(b_i, f), f \in C \) is uniquely determined by \( \theta(b_i, e + f) = \theta(b_i, e) + \theta(b_i, f) \). We put

\[
\theta(\sum_i \mu_i b_i, f) = \left( \sum_i \mu_i \theta(b_i, f) \right) + \sum_{i<j} \mu_i \mu_j A(b_i, b_j, f).
\]

By construction of \( \theta \) and trilinearity of \( A \), cocycle \( \theta \) satisfies (3.3.4). Put \( d = \sum_i \mu_i b_i, e = \sum_i \nu_i b_i \). By construction of \( \theta \) and trilinearity of \( A \) we have

\[
\theta(d + e, f) + \theta(d, f) + \theta(e, f) = \sum_{i \neq j} \mu_i \nu_j A(b_i, b_j, f),
\]

so together with \( A(b_i, b_j, f) = 0 \), we obtain (3.3.3).

Put \( C'(d, e) = \theta(d, e) + \theta(e, d) \). (3.3.3) and (3.3.4) imply \( C'(d+e, f) = C'(d, f) + C'(e, f) + A(d, e, f) \). We have \( C'(b_i, b_j) = C(b_i, b_j) \). So by (3.2.2) and induction over the basis vectors we obtain \( C' = C \), i.e. (3.3.2). A similar argument using \( \theta(b_i, b_i) = P(b_i) \), (3.2.1) and induction over the basis vectors establishes (3.3.1).

\[ \square \]

An immediate consequence of Lemma 3.3 is:

**Lemma 3.4.** Let \( \theta_1 \) be a fixed cocycle satisfying Lemma 3.3. Then the cocycles satisfying Lemma 3.3 are precisely the functions \( \theta_1 + \beta \), with \( \beta \) an alternating bilinear form on \( C \).

Since a cocycle given by Lemma 3.3 is linear in its second argument, it may also be interpreted as a function \( C \to C^* \), and we let \( \theta(d_1) \) be the element of \( C^* \) such that \( \langle c, \theta(d) \rangle = \theta(d, e) \) holds for all \( e \in C \). Similarly, we let \( A(d, e) \) be the element of \( C^* \) such that \( \langle f, A(d, e) \rangle = A(d, e, f) \) holds for all \( f \in C \). Then for \( d, e \in C \) (or \( d, e \in \mathcal{P} \)) we have:

\[
\theta(d + e) = \theta(d) + \theta(d) + A(d, e), \quad \text{with } A(d, e) = d \cap e \in C^*.
\]

There are cocycles satisfying (3.2.2) but not Lemma 3.3. But we only consider cocycles which satisfy Lemma 3.3.

### 3.3 Selecting a suitable cocycle for the Parker loop

In this subsection we extend the decomposition of \( C \) into a grey and a coloured subspace to the Parker loop \( \mathcal{P} \). Then we construct a cocycle for \( \mathcal{P} \) that has some specific properties regarding that decomposition as stated in Lemma 3.9. This is needed in section 9 for the first time and may be skipped at first reading.

We also talk about **grey** and **coloured** elements of \( \mathcal{P} \). Let \( \mathcal{P}_G \) and \( \mathcal{P}_H \) be the set of elements of \( \mathcal{P} \) that are mapped to \( G \) and \( H \), respectively, by the homomorphism \( \gamma : \mathcal{P} \to C \).

We need a cocycle such that in our representation of \( \mathcal{M} \) the non-monomial generator \( \xi \) of the monster \( M \) to be constructed in section 9 becomes as simple as possible. More specifically, we will select a cocycle that is related to the decomposition of \( C \) and \( C^* \) into a grey and a coloured subspace as indicated below.

Let \( \gamma_{(m, i)} \) be the element of \( F_2^3 \) with an entry 1 in MOG row \( m \), column \( i \) and zero elsewhere. Thus \( \gamma_{(m, 0)} = \gamma_m \). We define a function \( \gamma : F_2^3 \to F_2^3 \) by:

\[
\gamma \left( \sum_{i=0}^5 \sum_{m=0}^3 \mu_{m,i} \gamma_{(m,i)} \right) = \sum_{i=0}^5 \left( \mu_{1,i} + \mu_{2,i} + \mu_{3,i} \right) 2^{\gamma_i}, \quad \text{for } \mu_{m,i} \in \{0, 1\}.
\]
The natural text is as follows:

So γ(x) has entry one in row 0, column n of the MOG, if x has at least two nonzero entries in column n, ignoring the entry in row 0. We usually consider γ as a function \( C \rightarrow G^* \) with γ(d) equal to the cost \( C_\gamma(d) \). But we occasionally write \( \gamma(d) \cap \gamma(e) \), which is meaningful only for \( \gamma(d) \) and \( \gamma(e) \) considered as elements of \( \mathbb{F}_{2^4}^2 \). The restriction of \( \gamma \) to \( G \) is an linear bijection \( G \rightarrow G^* \) with \( \gamma(g_m) = \gamma_m \). We have

\[
\gamma(g_i) = \gamma(\omega_i) = \gamma_i.
\]

**Lemma 3.6.** \( \gamma(d + e) = \gamma(d) + \gamma(e) \) for \( d \in C, e \in G \).

**Proof**

Let \( \phi_j : \mathbb{F}_{2^4}^2 \rightarrow \mathbb{F}_2 \) be the function that maps \( x \in \mathbb{F}_{2^4}^2 \) to the coefficient of the function \( \gamma(x) \) corresponding to \( \gamma_j \). Then \( \gamma = \sum_{j=0}^{5} \gamma_j \phi_j \). Since \( (g_0, \ldots, g_5) \) is a basis of \( G \), it suffices to show

\[
\phi_j(d + g_i) = \phi_j(d) + \phi_j(g_i), \quad \text{for } i, j = 0, \ldots, 5. \tag{3.6.1}
\]

\( \phi_j \) depends only on the co-ordinates of \( \mathbb{F}_{2^4}^2 \) corresponding to \( \gamma(m,j) \) for \( m = 1, 2, 3 \). Since all these coordinates of \( g_i \) are zero in case \( i \neq j \), (3.6.1) is established for \( i \neq j \).

Assume \( i = j \). Then by definition of \( \phi_j \) we have:

\[
\phi_j (\mu_1 \gamma(1,j) + \mu_2 \gamma(2,j) + \mu_3 \gamma(3,j)) = \left( \frac{\mu_1 + \mu_2 + \mu_3}{2} \right), \quad \text{for } \mu_1, \mu_2, \mu_3 \in \{0, 1\}.
\]

These three co-ordinates of \( g_j \) corresponding to \( \gamma(1,j), \gamma(2,j), \gamma(3,j) \) are all equal to one. Thus (3.6.1) follows from \( \binom{3}{2} = \binom{3}{i} + \binom{3}{j} \) (mod 2), for \( k \in \{0, 1, 2, 3\} \).

Define \( w_2 : G \cup G^* \rightarrow \mathbb{F}_2 \) by

\[
w_2(d) = \left( \frac{w(d)}{2} \right) \pmod{2}, \quad \text{with } w \text{ as in Definition 2.4} \tag{3.7.1}
\]

Since \( w(\Omega + d) = 6 - w_2(d) \) we have

\[
w_2(\Omega + d) = w_2(d) + w(d) + 1, \quad \text{for } d \in G. \tag{3.7.2}
\]

A quadratic form \( q : \mathbb{F}_{2^4}^2 \rightarrow \mathbb{F}_2 \) is called non-singular if its associated form \( \beta_q \) is non-singular, i.e. \( \det(\beta_q(v_i,v_j)) = 1 \) for a basis \( (v_1, \ldots, v_n) \) of \( \mathbb{F}_{2^4}^n \). A bilinear form which is non-singular and alternating is called symplectic.

**Lemma 3.8.** Define \( \langle \langle \cdot, \cdot \rangle \rangle : G \times G \rightarrow \mathbb{F}_2 \) by \( \langle \langle d, e \rangle \rangle = \langle e, \gamma(d) \rangle \). Then \( \langle \langle \cdot, \cdot \rangle \rangle \) is symplectic and it is the bilinear form associated with the quadratic form \( w_2 \) on \( G \). We also have:

\[
\langle e, \gamma(d) \rangle = w(\gamma(d)) \cdot w(\gamma(e)) + w(\gamma(d) \cap \gamma(e)) \quad \text{for } d \in C, e \in G \tag{3.8.1}
\]

**Proof**

Both sides of (3.8.1) are linear in \( \gamma(d) \) and also in \( e \). So it suffices to show (3.8.1) for \( e = g_i \) and \( \gamma(d) \) being substituted by \( \gamma_i \). Thus we have to show \( \langle g_i, \gamma_j \rangle = 1 - \delta_{i,j} \), with \( \delta_{i,j} \) the Kronecker delta. But this is an immediate consequence of (2.5.2).

Hence \( \langle g_i, g_j \rangle = 1 - \delta_{i,j} \). For \( i = 1, 2 \) assume \( d_i = \sum_{n=0}^{5} \mu_{i,n} g_n \), \( \mu_{i,n} \in \{0, 1\} \). Since \( \binom{3}{2} = \sum_{m=0}^{5} \mu_{i,m} g_m \), we have \( w_2(d_i) = \sum_{0 \leq m < n \leq 5} \mu_{i,m} \mu_{i,n} \). So \( w_2 \) is a quadratic form on \( G \). Let \( \beta^2 : G \times G \rightarrow \mathbb{F}_2 \) be given by \( (d_1, d_2) \rightarrow w_2(d_1) + w_2(d_2) + w_2(d_1 + d_2) \). Then \( \beta^2(d_1, d_2) = \sum_{m \neq n} \mu_{1,m} \mu_{2,n} \). Thus \( \beta^2 \) is linear in both arguments and we have \( \beta^2(g_m, g_n) = 1 - \delta_{m,n} \). Hence \( \langle \langle \cdot, \cdot \rangle \rangle = \beta^2 \), i.e. \( \langle \langle \cdot, \cdot \rangle \rangle \) is associated with \( w_2 \).

We have just shown that \( \det \beta^2 \) (with respect to the basis \( g_1, \ldots, g_5 \)) has entry \( 1 - \delta_{m,n} \) in row \( m \), column \( n \), so direct calculation yields \( \det \beta^2 = 1 \pmod{2} \).

Ref. 3.8.1
Thus we have:

\[ \theta(d + \Omega) = \theta(d) \quad \text{for} \quad d \in \mathbb{C}, \]
\[ \theta(e) = (w(e) - 1)\gamma(e) + w_2(e)\omega \quad \text{for} \quad e \in \mathbb{G}, \quad \omega \text{ as in section 2} \]
\[ \theta(e, h) = 0 \quad \text{for} \quad e \in \mathbb{G}, \quad h \in \mathbb{H}, \]
\[ \theta(h, e) = \langle e, \gamma(h) \rangle \quad \text{for} \quad e \in \mathbb{G}, \quad h \in \mathbb{H}. \]

**Proof**

Choose a basis \((b_0, \ldots, b_{11})\) of \(\mathcal{C} = \mathbb{G} \oplus \mathbb{H}\) with \(b_m = g_m \in \mathbb{G}\) for \(m = 0, \ldots, 5\), \(b_6, \ldots, b_{11} \in \mathbb{H}\), and \(g_m\) as in section 2. Since two cocycles for \(\mathcal{C}\) differ by an alternating bilinear form, a cocycle \(\theta\) is uniquely determined by decreasing the values \(\theta(b_m, b_n)\) for all \(0 \leq m < n < 12\). So we put \(\theta(b_m, b_n) = 0\) for \(m < n\). We have \(|g_m \cap g_n| = 4\) for \(m \neq n\) So for \(0 \leq m < n < 6\) we have:

\[ \theta(b_m, b_n) + \theta(b_n, b_m) = C(g_m, g_n) = \frac{1}{2}|g_m \cap g_n| = 0 \quad (\text{mod } 2), \]

by \((3.1.1)\) and Lemma 3.4. For \(m < 6\) we have \(\theta(b_m, b_m) = P(g_i) = \frac{1}{4}|g_m| = 0 \quad (\text{mod } 2)\). Thus \(\theta(b_m, b_n) = 0\) for all \(m < 6, n < 12\). Hence \(\theta(b_m) = 0\) for \(m < 6\). Thus by \((3.5.1)\) the restriction of \(\theta\) (as a mapping \(\mathcal{C} \rightarrow \mathbb{C}^*\)) to \(\mathbb{G}\) is symmetric under permutations of the basis vectors \(b_0, \ldots, b_5\). \(\theta(e)\) can be computed for all \(e \in \mathbb{G}\) by \((3.5.1)\). Due to the permutation symmetry of \(\theta\) it suffices to verify the formula for \(\theta(e)\) for the cases \(\theta(e_{m,n})\) with \(e_{m,n} = \sum_{i=0}^{m} g_n\), \(m = 1, \ldots, 5\). This can easily be done by hand calculation. (Row 0 of \(\theta(e_{m,n})\) in the MOG is 000000, 001111, 111111, 111100, 000000 for \(m = 1, \ldots, 5\); the other rows are zero.)

That way we obtain \(\theta(\Omega) = 0\), so we have \(\theta(d + \Omega) = \theta(d)\) by \((3.5.1)\). We have \(\theta(e, h) = 0\) for \(e \in \mathbb{G}, \quad h \in \mathbb{H}\), by construction of \(\theta\). Thus \(\theta(h, e) = C(e, h) = \frac{1}{2}|e \cap h|\). Note that \(e \cap h\) has 0 or 2 nonzero entries in each column of the MOG and zeros in row 0. Hence

\[ \theta(h, e) = \frac{1}{2}|e \cap h| = |\gamma(e) \cap \gamma(h)| = w(\gamma(e) \cap \gamma(h)) = \langle e, \gamma(h) \rangle + w(\gamma(e)) \cdot w(\gamma(h)). \]

by definition of \(\gamma\) and \((3.8.1)\). Since \(\gamma(h)\) is even, this proves the formula for \(\theta(h, e)\).

\[ \Box \]

**Corollary 3.10.** For every \(h \in \mathbb{H}\) there is a cocycle \(\theta\) on \(\mathcal{C}\) satisfying Lemma 3.9 with \(\theta(h) = \gamma(h) \in \mathbb{G}^*\).

**Proof**

In the proof of Lemma 3.9 we simply choose a basis \(b_0, \ldots, b_{11}\) of \(\mathcal{C}\) with \(b_6 = h\).

\[ \Box \]

\((2.5.1)\) and Lemma 3.4 imply that for every \(e \in \mathbb{G}, h \in \mathbb{H}\) we have:

\[ P(e) = w(e) \cdot w_2(e), \quad P(h) = w_2(\gamma(h)), \quad C(e, h) = \langle e, \gamma(h) \rangle. \]

(3.10.1)

From now on we assume that the cocycle for \(\mathcal{P}\) satisfies the conditions in Lemma 3.4.

4 Automorphisms of the Parker loop \(\mathcal{P}\)

The *center* \(Z(G)* of a group or a loop \(G* is the set of elements \(d* of \(G* such that \(d* commutes and associates with all elements of \(G* Any element of the Parker loop \(\mathcal{P}\) squares to \(\pm 1\) and we have \(Z(\mathcal{P}) = \{\pm 1, \pm \Omega\}\). Thus any automorphism \(\pi\) of \(\mathcal{P}\) fixes \(\{\pm 1\}\) and maps \(\Omega\) to \(\pm \Omega\). \(\pi\) is called *even* if \(\pi(\Omega) = \Omega* and *odd* if \(\pi(\Omega) = -\Omega* Since \(\pi\) fixes \(\{\pm 1\}, it maps to a unique automorphism \(\tilde{\pi}\) of \(\mathcal{C} = \mathcal{P}/\{\pm 1\}\). Then \(\tilde{\pi}\) preserves power map \(P\), commutator \(C\) and associator \(A\), but it need not preserve the Golay code on the vector space \(\mathcal{C}\). An automorphism \(\pi\) of \(\mathcal{P}\) is called a *standard automorphism* in \([5]\), if \(\tilde{\pi}\) preserves the Golay code. E.g. for a nonzero even \(\delta \in \mathbb{C}^*\) the mapping \(d \mapsto d \cdot \Omega (d, \delta)\) is a non-standard automorphism of \(\mathcal{P}\), see \([14]\), section 1.6 for background. In the sequel we only deal with the group \(\text{Aut}_{\mathcal{P}}\).
of standard automorphisms $\pi$ of $\mathcal{P}$. For any $\pi \in \text{Aut}_{\mathcal{S}} \mathcal{P}$ we have $\tilde{\pi} \in M_{24}$. For any $\tilde{\pi} \in M_{24}$ there are precisely $2^{12}$ standard automorphisms $\pi$ of $\mathcal{P}$ mapping to $\tilde{\pi}$, see [5,7].

Fix a cocycle $\theta$ for $\mathcal{P}$ that satisfies Lemma 3.3. For any $\pi \in \mathcal{P}$ define $\theta_{\pi} : \mathcal{C} \times \mathcal{C} \to \mathbb{F}_2$ by

$$\theta_{\pi}(d, e) = \theta(d^{\pi}, e^{\pi}) + \theta(d, e).$$

(4.0.1)

Then the following lemma is useful for effective computations in the group $\text{Aut}_{\mathcal{S}} \mathcal{P}$:

**Lemma 4.1.** For any standard automorphism $\pi$ of $\mathcal{P}$ the mapping $\theta_{\pi}$ is an alternating bilinear form on $\mathcal{C}$ depending on $\tilde{\pi}$ only. There is a unique quadratic form $q_{\pi}$ on $\mathcal{C}$ with associated bilinear form $\theta_{\pi}$, such that for any $d = (\tilde{d}, \lambda) \in \mathcal{P}$, $\tilde{d} \in \mathcal{C}$, $\lambda \in \mathbb{F}_2$ we have

$$(\tilde{d}, \lambda)^{\pi} = (d^{\pi}, \lambda + q_{\pi}(\tilde{d})).$$

**Proof**

Define $\theta^{(\pi)} : \mathcal{C} \times \mathcal{C} \to \mathbb{F}_2$ by $\theta^{(\pi)}(d, e) = \theta(d^{\pi}, e^{\pi})$. Since $M_{24}$ acts as a group of linear transformations on $\mathcal{C}$ and preserves $\mathcal{P}$, $\mathcal{C}$ and $\Lambda$, the function $\theta^{(\pi)}$ satisfies the conditions for a cocycle in Lemma 3.3. So $\theta_{\pi} = \theta^{(\pi)} + \theta$ is an alternating bilinear form by Lemma 3.3. By construction, $\theta_{\pi}$ depends on $\tilde{\pi}$ only. Let $q_{\pi}$ be any quadratic form with associated form $\theta_{\pi}$. Write $(\tilde{d}, \lambda)^{\pi, \phi}$ for $(d^{\pi}, \lambda + q_{\pi}(\tilde{d}))$. For $\tilde{d}, \tilde{e} \in \mathcal{C}$ and $\lambda, \mu \in \mathbb{F}_2$ we have:

$$(\tilde{d}, \lambda)^{\pi, \phi} \cdot (\tilde{e}, \mu)^{\pi, \phi} = (d^{\pi} + e^{\pi}, \lambda + \mu + q_{\pi}(\tilde{d} + \tilde{e}) + \theta(d^{\pi}, e^{\pi})).$$

$$= (d^{\pi} + e^{\pi}, \lambda + \mu + q_{\pi}(\tilde{d} + \tilde{e}) + \theta_{\pi}(d, e) + \theta(d^{\pi}, e^{\pi})).$$

$$= (\tilde{d} + \tilde{e}, \lambda + \mu + \theta(d, e))^{\pi, \phi}$$

$$= (\tilde{f}, \nu)^{\pi, \phi}, \quad \text{with} \quad (\tilde{f}, \nu) = (\tilde{d}, \lambda) \cdot (\tilde{e}, \mu).$$

So the mapping $(\tilde{d}, \lambda) \mapsto (\tilde{d}, \lambda)^{\pi, \phi}$ is a standard automorphism of $\mathcal{P}$ which maps to the element $\pi$ of $M_{24}$. There are $2^{12}$ different standard automorphisms $\pi$ of $\mathcal{P}$ mapping to the same element $\pi$ of $M_{24}$, and there are $2^{12}$ different quadratic forms on $\mathcal{C}$ with the same associated bilinear form $\theta_{\pi}$. Hence for any such $\pi$ there is a unique $q_{\pi}$ satisfying the lemma. □

A standard automorphism $\delta$ of $\mathcal{P}$ that maps to the neutral element $\tilde{\delta} = 1$ of $M_{24}$ is called a **diagonal automorphism**. In that case the bilinear form $\theta_{\delta}$ in Lemma 4.1 is 0 and a quadratic form associated with the bilinear form 0 is a linear form in $\text{hom}(\mathcal{C}, \mathbb{F}_2) \cong \mathcal{C}^{*}$, which we also denote by $\delta$. So a diagonal automorphism $\delta \in \mathcal{C}^{*}$ maps $d$ to $d \cdot (-1)^{(d, \delta)}$. The parity of $\delta$ as a diagonal automorphism agrees with the parity of $\delta$ as an element of $\mathcal{C}^{*}$.

$\text{Aut}_{\mathcal{S}} \mathcal{P}$ is a non-split extension with normal subgroup $\mathcal{C}^{*}$ and factor group $M_{24}$. Even if we assume that calculations in $\mathcal{C}$, $\mathcal{C}^{*}$, $\mathcal{P}$ and $M_{24}$ are easy, the calculations in $\text{Aut}_{\mathcal{S}} \mathcal{P}$ are still quite technical. In the remainder of this section we study such calculations.

Using Lemma 4.1 calculation in $\text{Aut}_{\mathcal{S}} \mathcal{P}$ can be done as follows. Choose a basis $(\tilde{b}_0, \ldots, \tilde{b}_{11})$ of $\mathcal{C}$ and for $\pi \in M_{24}$ let $[\pi] \in \text{Aut}_{\mathcal{S}} \mathcal{P}$ be defined by

$$(\tilde{b}_i, 0) \xrightarrow{[\pi]} (\tilde{b}_i^{\pi}, 0), \quad i = 0, \ldots, 11.$$ 

Then any element of $\text{Aut}_{\mathcal{S}} \mathcal{P}$ can uniquely be written in the form $\delta \cdot [\pi]$, $\delta \in \mathcal{C}^{*}$, $\pi \in M_{24}$, and we have $[\pi] \cdot \delta^{\pi} = \delta \cdot [\pi]$.

For $\pi, \pi' \in M_{24}$ we have $[\pi \pi'] = \vartheta(\pi, \pi') \cdot [\pi' \pi']$, where $\vartheta(\pi, \pi') \in \mathcal{C}^{*}$ is given by

$$(\tilde{b}_i, \vartheta(\pi, \pi')) = q_{[\pi \pi']}^{\pi}(\tilde{b}_i^{\pi}), \quad i = 0, \ldots, 11,$$ 

(4.2.1)

and $q_{[\pi \pi']}$ is the unique quadratic form on $\mathcal{C}$ with associated bilinear form $\theta_{\pi'}$ satisfying $q_{[\pi \pi']}(\tilde{b}_i) = 0$ for $i = 0, \ldots, 11$. Here $\theta_{\pi'}$ is as in (4.0.1). Note that $\theta_{\pi'}$ and hence also $q_{[\pi \pi']}$, $i = 0, \ldots, 11$. 

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The proof of (4.2.1) is a simple calculation based on the equation

\[(\tilde{d}, \lambda)|\pi|\pi'| = (\tilde{d}^\tau, \lambda + q[\pi](\tilde{d}) + q[\pi'](\tilde{d}^\tau)) \quad , \quad \tilde{d} \in C, \quad \lambda \in F_2, \quad \pi, \pi' \in \text{Aut}_\tilde{\mathcal{L}}\mathcal{P}, \]

with \(q[\pi]\) and \(q[\pi']\) as in Lemma 4.3. (4.2.2) is a consequence of Lemma 4.1.

5 The code loop group \(N_0\) of the Parker loop \(\mathcal{P}\)

We define a group \(N\) which acts as a permutation group on the elements of \(\mathcal{P}^3\). It will turn out that \(N\) is a fourfold cover of a maximal subgroup \(N_0\) of the monster \(\mathbb{M}\). The elements of \(\mathcal{P}^3\) are called triples. \(N_0\) has structure \(2^2.2^{11}.2^{22}.(S_3 \times M_{24})\) and is the normalizer of a certain four-group \(\{1, x, y, z\}\) in \(\mathbb{M}\), see [5]. Following Conway’s construction of the Monster \(\mathbb{M}\) in [5], we define maximal subgroups \(G_{x0}, G_{y0}\) and \(G_{z0}\) of \(\mathbb{M}\), each of structure \(2^{1+24}.\text{Co}_1\), which centralize the elements \(x, y, z\) of the four-group, respectively.

We use the ATLAS conventions [6] for group structures. \(G_1.G_2\) denotes a group extension with normal subgroup \(G_1\) and factor group \(G_2\). \(G_1 : G_2\) denotes a split extension and \(G_1 \times G_2\) a direct product. All these operators associate to the left with the same precedence, and they imply that the given decomposition is invariant. E.g. \(G = G_1 \times G_2 : G_3.G_4\) means \(G = ((G_1 \times G_2) : G_3).G_4\) and implies the existence of normal subgroups \(G_1\) and \(G_1 \times G_2\) of \(G\). \(2^n\) is an elementary Abelian group of order \(2^n\), \(S_n\) is the symmetric permutation group of \(n\) elements, \(\text{Co}_1\) is the automorphism group of the Leech lattice \(\Lambda\) modulo 2, and \(\text{Co}_0 = 2.\text{Co}_1\) is the automorphism group of \(\Lambda\).

The Leech lattice \(\Lambda\) is discussed in section [6]. Following [5] we construct a 196884-dimensional real monomial representation 196884x of the group \(N_{x0} = N_0 \cap G_{x0}\), with \(|N_0|/|N_{x0}| = 3\), in section [7]. Then in section [8] we extend 196884x to a representation of \(N_0\) by adding a trivity element \(\tau\) which (by conjugation) cyclically exchanges the elements \(x, y, z\) of the four-group and also their centralizers \(G_{x0}, G_{y0}, G_{z0}\).

The group \(2^{1+24}\) is an extraspecial 2-group, which we will discuss in section [9]. There we explicitly construct the representation of an element \(\xi \in G_{x0} \setminus N_{x0}\) in the \(N_0\)-module 196884x. Since \(N_{x0}\) is a maximal subgroup of \(G_{x0}\), this extends 196884x to a representation of a group generated by \(G_{x0}\) and \(N_0\), which is visibly equivalent to the representation of \(\mathbb{M}\) in [5]. Essentially, the main added value of this paper compared to [5] is an explicit description of the representation of an element \(\xi\) of \(G_{x0} \setminus N_{x0}\) of the Monster, so that a programmer may implement this construction with not too much effort.

As in [5], we define various subgroups of \(\mathbb{M}\) shown in Figure 1. All these subgroups of \(\mathbb{M}\) agree with the corresponding subgroups of \(\mathbb{M}\) in [5]. In our construction we change a few signs compared to [5] in order to simplify the representation of the element \(\xi\).

We start with the definition of a group \(\hat{N}\) which has a simpler structure than \(N\) and we will define \(N\) as a subgroup of \(\hat{N}\) of index 2. We define the following generators \(x_d, y_d, z_d, \nu_\pi, \tau, x_\pi, y_\pi, z_\pi\) of \(\hat{N}\) by their action on a triple \((a, b, c)\) in \(\mathcal{P}^3:\)

\[x_d : (\tilde{d}a, \tilde{d}b, \tilde{d}c) \quad y_d : (\tilde{a}d, \tilde{b}d, \tilde{c}d) \quad z_d : (\tilde{d}a, \tilde{bd}, \tilde{d}c),\]

\[\nu_\pi : (a^\pi, b^\pi, c^\pi) \quad \tau : (c, a, b),\]

\[x_\pi : (\tilde{a}, \tilde{c}, \tilde{b}) \quad y_\pi : (\tilde{c}, \tilde{b}, \tilde{a}) \quad z_\pi : (\tilde{b}, \tilde{a}, \tilde{c}),\]

with \(d\) ranging over \(\mathcal{P}\) and \(\pi\) ranging over \(\text{Aut}_\tilde{\mathcal{L}}\mathcal{P}\). We put

\[x_\pi = x_\pi^{\pi} \nu_\pi, \quad y_\pi = y_\pi^{\pi} \nu_\pi, \quad z_\pi = z_\pi^{\pi} \nu_\pi, \quad \text{with } |\pi| = 1 \text{ for odd } |\pi| = 0 \text{ for even } |\pi|.

So e.g. \(x_\pi\) maps \((a, b, c)\) to \((a^\pi, c^\pi, b^\pi)\) for odd \(\pi\). We define \(N\) to be the subgroup of \(\hat{N}\) generated by \(x_d, y_d, z_d, x_\pi, y_\pi, z_\pi\).

Next we show some relations in \(\hat{N}\). For elements \(u, v\) of a group we write \([u, v]\) for the commutator \(u^{-1}v^{-1}uv\) and \(u^v\) for \(v^{-1}uv\). We abbreviate \(x_{-1}, y_{-1}\) and \(z_{-1}\) to \(x, y\) and \(z\).
respectively, as in [5]. The groups $\tilde{N}$ and $N$ have a visible symmetry with respect to cyclic permutations of the letters $x, y$ and $z$, which we will call triality. Conjugation with $\tau$ just performs such a cyclic permutation. We often just write only one of the three possible cyclic permutations in a formula or a definition, and we state that the others are obtained by triality. We use the symbol $(&x^y^z)$ to indicate that a relation remains valid if the letters $x, y, z$ are cyclically exchanged, e.g.:

\[ x_d y_d = z_d x^{P(d)} \quad (\&x^y^z) \quad \text{means} \quad x_d y_d = z_d x^{P(d)}, \quad y_d z_d = x_d x^{P(d)}, \quad z_d x_d = y_d y^{P(d)}. \]

**Theorem 5.1.** The following relations define the group $\tilde{N}$:

\[
\begin{align*}
[x_d, y_d] &= \nu_{A(d, e)} z_{C(d, e)}, & x_d y_d z_d &= 1, & [x_d, \nu_{\delta}] &= x^{(d, \delta)}, \\
x_d \nu &= \nu d, & \nu = \nu_{A(d, e)} z_{C(d, e)}, & x_d \nu &= \nu d, & [x_d, \nu_{\pi}] &= x^{(d, \delta)}, \\
\tau = x_{\tau}^{2} &= 1, & \nu_{\tau} &= \nu_{\tau} \nu_{\pi}, & \tau = x_{\tau}^{2} = 1, \quad \nu_{\pi} = x_{\pi}^{2}, \\
[x_{\nu} \pi, \tau] &= 1, & \tau = x_{\tau}^{2} &= 1, & x_{\tau} &= x_{\tau}^{2} = 1, & x_{\tau} &= x_{\tau}^{2} = 1, \\
[x_{\nu} \pi, \tau] &= [x_d, x_{\tau}] = 1, & x_{\tau} &= x_{\tau}^{2} &= 1, & x_{\tau} &= x_{\tau}^{2} &= 1, \quad x_{\tau} = x_{\tau}^{2} = 1.
\end{align*}
\]

\((&x^y^z)\)

**Proof**

Most of these relations can be shown by calculations within associative subloops of $\mathcal{P}$ generated by two elements, which we leave to the reader. We first show $x_d x_e = x_{de} x_{A(d, e)}$. We have

\[
\begin{align*}
(a, b, c) &\xrightarrow{x_d x_e} (e, d, a, e) \quad \text{by (3.2.2)} \\
&= \left( (1)^{A(a, d) + A(a, e)} a, (1)^{A(b, d, e)} b, (1)^{A(c, d, e)} c(d) \right) \\
&= \left( (1)^{A(a, d) + A(a, e)} a, (1)^{A(b, d, e)} b, (1)^{A(c, d, e)} c(d) \right) \\
&= \left( (1)^{A(a, d) + A(a, e)} a, (1)^{A(b, d, e)} b, (1)^{A(c, d, e)} c(d) \right).
\end{align*}
\]

For the last step, note that $C(a, d) + C(a, e) = C(a, d + e) + A(a, d, e)$ by (3.2.2) Clearly, $x_d x_e$ maps $(a, b, c)$ to the same triple.
We also show \([xd, ye] = z_{A(d,e)}z^{C(d,e)}\). We have
\[
(a, b, c) \left[ \frac{[x, yc]}{d} \right] \left((-1)^{C(a,d)+C(a,e,d)}a, (-1)^{C(b+d,e)+C(b,e)}b, c \right) = \left((-1)^{C(d,e)+A(a,d,e)}a, (-1)^{C(d,e)+A(b,d,e)}b, (-1)^{A(c,d,e)}c \right).
\]
Note that \(A(c + d, d, e) = A(c, d, e)\). Clearly, \(z_{A(d,e)}z^{C(d,e)}\) maps \((a, b, c)\) to the same triple.

Using the relations shown we may convert any word in the generators of \(\hat{N}\) to the form:
\[
xdyex^\pi m x_n^\pi, \quad 0 \leq m < 3, 0 \leq n < 2.
\]

Such a word with \((m, n) \neq (0, 0)\) does not fix a triple \((1, \Omega, d)\) with \(d \notin \{\pm1, \pm\Omega\}\). A word of shape \(x^\pi y^\pi x^\pi\) permutes the elements of each component \(P\) in \(P^3\). The stabilizers of components 1, 2 and 3 of \(P\) are \(\{x_{11}, x_{1\Omega} \}, \{y_{11}, y_{1\Omega} \}\) and \(\{z_{11}, z_{1\Omega} \}\). The intersection of these stabilizers is \(\{1\}\).

An immediate consequence of Theorem 5.1 is:
\[
[x_d, x_e] = x^{C(d,e)}, \quad x^2_d = x^{P(d)}, \quad x^2_e = [\nu^\pi, \nu_e] = 1. \quad (\& \hat{\chi^y}x^z) \quad (5.2.1)
\]

Note that \([xd, xe] = (xe^\pi x)_d^\pi x^e\) and \(x_{A(d,e)}^\pi\) and \(x_{A(d,e)}^\pi\) are \(\delta\) even parity to \(x, y, z\) and to \(\nu_e\), \(\pi\) odd, and "even" parity to the other generators of \(\hat{N}\), we see that the generators of \(N\) have even parity and that the relations in Theorem 5.1 preserve that parity. Thus \(N\) is the subgroup of the "even" elements of \(\hat{N}\) and we have \(|\hat{N}/N| = 2\). Note also that \(\tau \in N\).

Our generators of \(N\) agree with Conway’s generators in \([5]\) with the following exceptions: Our generators \(x_d, y_d, z_d\) correspond to \(x_d^\pi \cdot y_{-1}^\pi, y_d^\pi \cdot x_{-1}^\pi, z_d^\pi \cdot x_{-1}^\pi\) in \([5]\). But our group \(N\) agrees with that in \([5]\). Our definition of \(x_d, y_d, z_d\) leads to simpler relations in \(N\), and it agrees with Ivanov’s construction of \(G_2\) in \([14]\), section 2.7, with \(G_2\) in \([14]\) corresponding to Conway’s and our group \(N_0\).

As in \([5]\), we define \(K_1 = y_{11}z_{-\Omega}, K_2 = z_{\Omega}x_{-\Omega}, K_3 = x_{\Omega}y_{-\Omega}\), and we put:
\[
K_1 \equiv \{1, K_1\}, \quad K_2 \equiv \{1, K_2\}, \quad K_3 \equiv \{1, K_3\}, \quad K_0 \equiv \{1, K_1, K_2, K_3\}.
\]

Then \(K_0\) is normal in \(N\) and \(N_0 = N/K_0\) is the normalizer of a certain fourgroup in \(M\). For a subgroup \(\Gamma\) of \(\Gamma\) we define \(\Gamma_n = \Gamma/\Gamma \cap K_n\), \(n = 0, 1, 2, 3\). Note that \(K_0\) is not normal in \(\hat{N}\).

The symbol \((k \hat{\chi}^y x^z)\) will also indicate that a statement remains valid if the indices 1, 2, 3 in the definitions above are cyclically permuted together with \(x, y, z\). E.g.:
\[
K_4 = x_{\Omega}z \in K_1 \quad (k \hat{\chi}^y x^z) \quad \text{implies} \quad K_2 = y_{11}x \in K_2 \quad \text{and} \quad K_3 = z_{\Omega}y \in K_3.
\]

As a corollary of the proof of Theorem 5.1 we obtain the following structure of \(N\):
\[
N = \frac{2^2}{K_1, K_2, K_3} \times \frac{2^2}{x, y, z} \times \frac{2^{11}}{x = yz = z = \delta \text{ even}} \times \frac{2^{22}}{x_d, y_d, z_d} \times \frac{2^{22}}{x, y, z} \times \frac{2^{11}}{x = yz = z = \delta \text{ odd}} \times \frac{M_24}{x = y = z = \pi \text{ even}}. \quad (5.3.1)
\]

By omitting the \(M_{24}\) generators we obtain a normal subgroup \(Q\) and by omitting the \(S_3\) generators we obtain a normal subgroup \(N_{xyz}\) of \(N\).

The centralizer of \(x\) in \(N\) is called \(N_x\) and has structure
\[
N_x = \frac{2}{K_1} \times \frac{2}{K_2, K_3} \times \frac{2^{1+24}}{x_d, x_e} \times \frac{2^{11}}{x_d, x_e} \times \frac{M_{24}}{x} \quad (\text{This is } Q_x). \quad (5.3.2)
\]
Let $Q_x$ be the subgroup of $N_x$ generated by $x_d, x_\delta$ with $d \in \mathcal{P}, \delta \in \mathbb{C}^*$. Then $Q_x \cap K_0 = 1$, so we have $Q_x \cong Q_{x_1} \cong Q_{x_0}$. $Q_x$ is not normal in $N_x$, but $K_1Q_x$ is. To see this, it suffices to verify $[x_\delta, K_1] = 1$ and $K_2x_\delta = x_\delta K_3$ for odd $\delta$ (apart from trivial checks).

The defining relations in the group $Q_x$ given by Theorem 5.1 and (5.3.1) simplify to:

$$x_d^2 = x^{P(d)}, \quad x_\delta^2 = 1, \quad x_d x_\epsilon = x_d x_{A(d, \epsilon)}, \quad x_\delta x_\epsilon = x_{\delta+\epsilon}, \quad [x_d, x_\delta] = x^{(d, \delta)},$$

(5.3.3)

so every element of $Q_x$ can uniquely be written in the form $x_d x_\delta$. We often write $x_r, x_s, ...$ for elements of $Q_x$. We put

$$\tilde{x}_d = x^{\text{sign}(d)} x_d x_{\theta(d)} \quad \text{for} \quad d \in \mathcal{P}, \quad \text{and cocycle} \quad \theta, \quad \text{i.e.} \quad \theta(d) \in \mathbb{C}^*.$$

(5.3.4)

$\tilde{x}_d$ does not depend on the sign of $d$, so $\tilde{x}_d$ is also well defined for $d \in \mathcal{P}$. Then the group $Q_x$ generated by $x_d, x_\delta$, $d \in \mathcal{P}, \delta \in \mathbb{C}^*$ is also generated by $\tilde{x}_d, x_\delta$. It is visibly extraspecial of type $2^{1+24}_+$, satisfying the even simpler relations:

$$\tilde{x}_d^2 = x_\delta^2 = 1, \quad \tilde{x}_d \tilde{x}_\epsilon = \tilde{x}_d \tilde{x}_\epsilon, \quad x_d x_\epsilon = x_{\delta+\epsilon}, \quad [\tilde{x}_d, x_\delta] = x^{(d, \delta)},$$

(5.3.5)

which are easy consequences of (3.2.3), (3.3), and (5.3.3). We will discuss extraspecial 2-groups in section 9.1. So the structure description (5.3.2) of $N_x$ should now be clear.

Similarly, we let $N_y$ and $N_z$ be the centralizers of $y$ and $z$ in $N$. We let $Q_y$ and $Q_z$ be the subgroups of $N_y$ and $N_z$ generated by $y_d, y_\delta$ and $z_d, z_\delta$, respectively.

Our definition of $Q_x$ differs from that in [5], since the definition of $N$ in [5] does not easily lead to a split extension corresponding to our extension $K_0 : Q_x$. We put

$$Q_{xyz} = K_0Q_x \cap K_0Q_y \cap K_0Q_z.$$  

$Q_{xyz}$ is elementary Abelian of structure $2^2 \cdot 2^2 \cdot 2^{11}$ and generated by $K_0, x, y, z$ and $x_\delta, \delta$ even.

6 The relation between $Q_x$ and the Leech lattice $\Lambda$

6.1 The homomorphism from $Q_x$ to the $\Lambda/2\Lambda$

The 24-dimensional Leech lattice $\Lambda$ is defined as follows. Consider an Euclidean vector space $\mathbb{R}^{24}$ with scalar product $\langle ., . \rangle$ and basis vectors labelled by the elements of $\tilde{\Omega}$. Assume that a Golay code $C$ is given on $\mathbb{F}_2^{24} = \bigoplus_{i \in \tilde{\Omega}} \mathbb{F}_2$. Then the Leech lattice is the set of vectors $u$ in $\mathbb{R}^{24}$ with co-ordinates $(u_i, i \in \tilde{\Omega})$, such that there is an $m \in \{0, 1\}$ and a $d \in C$ with

$$\forall i \in \tilde{\Omega} : \quad u_i = m + 2 \cdot \langle d, i \rangle \quad \text{(mod 4)},$$

$$\sum_{i \in \tilde{\Omega}} u = 4m \quad \text{(mod 8)}.$$

See e.g. [7], Chapter 4, section 11 for background. The Leech lattice $\Lambda$ is an even unimodular lattice, i.e. $\langle u, u \rangle$ is even for all $u \in \Lambda$ and det $\Lambda = 1$. Therefore we have to scale the basis vectors of the underlying space $\mathbb{R}^{24}$ so that they have norm $1/\sqrt{8}$, not 1. Then for vectors $u, v$ with co-ordinates $u_i, v_i$ (where $i$ ranges over $\tilde{\Omega}$) we have $\langle u, v \rangle = \frac{1}{16} \sum_i u_i v_i$. For $u \in \Lambda$ we define $\text{type}(u) = \frac{1}{2} \langle u, u \rangle = \frac{1}{16} \sum_i u_i^2$. The Leech lattice can be characterized as the (unique) 24-dimensional even unimodular lattice without any vectors of type 1, see e.g. [7].

Theorem 6.1. There is an isomorphism from $Q_x/\{1, x\}$ to $\Lambda/2\Lambda$ given by:

$$x_i \mapsto \lambda_i = (-3_{\text{on } i}, 1_{\text{elsewhere}}),$$

$$x_d \mapsto \lambda_d = (2_{\text{on } d}, 0_{\text{elsewhere}}) \quad \text{if } |d| = 0 \text{ mod 8},$$

$$x_d \mapsto \lambda_d = (0_{\text{on } d}, 2_{\text{elsewhere}}) \quad \text{if } |d| = 4 \text{ mod 8},$$

which satisfies $[x_r, x_s] = x^{(\lambda_r, \lambda_s)}$, $x_r^2 = x^{\text{type}(\lambda_r)}$, with $\text{type}(\lambda_r) = \frac{1}{2} \langle \lambda_r, \lambda_r \rangle$.  


The more relaxed condition 

\[ \text{All these checks can be done by easy calculations using 5.3.3.} \]

\[ \delta \leq \Lambda \]

\[ 6.2 \text{ Short vectors in } \Lambda/2\Lambda \]

\[ \text{Proof of Theorem 6.1} \]

The proof is along the lines of the proof of Theorem 2 in [5]. The definitions of \( \lambda_i, \lambda_d \) imply:

\[ x_\Omega \mapsto -2\lambda_i + \lambda_\Omega = (8_{on}, 0_{\text{elsewhere}}) \quad \text{for } i \in \tilde{\Omega}, \]

\[ x_\Omega |d|/4 \mapsto (2_{on}, 0_{\text{elsewhere}}), \]

\[ x_\delta |\delta|/2 \mapsto (4_{on}, 0_{\text{elsewhere}}) \quad \text{for any } \delta \subset \tilde{\Omega} \text{ with } |\delta| \text{ even}. \]

For any \( d \in C \) the value \(|d|/2\) is even, so that the product \( \Pi_{i \in \tilde{d}} x_i \) maps to \( 2 \cdot \lambda_d \) or to \( 2 \cdot \lambda_d + \Omega \). Thus for any \( \delta \in C^* \) the image of \( x_\delta \) is well defined in \( \Lambda/2\Lambda \), even if \( \delta \) is defined modulo \( C \) only. To show that the mapping \( x_\delta \mapsto \lambda_\delta \) is a homomorphism, it suffices to check that it preserves the relations 5.3.3. Since \( x_{-1} \) is mapped to 0, all these checks are easy except for the relation 

\[ x_d x_e = x_{de} x_{A(d,e)} . \]

We have

\[ x_d x_e |d|/4 \mapsto (4_{on}, 0_{\text{elsewhere}}). \]

Clearly, \( x_{d+e} |d+e|/4 \mapsto x_{d+e} |d|/4 \mapsto x_{d+e} |d|/2 \) maps to the same element of \( \Lambda/2\Lambda \). Thus the relation 

\[ x_d x_e = x_{de} x_{A(d,e)} x_{A(d,e)} |d|/4 \]

is preserved for \( m = |d+e|/4 + |d|/4 + |e|/4 + |d \cap e|/2 = 0 \pmod{2} \). The mapping \( x_\delta \mapsto \lambda_\delta \) is surjective by construction and both, its origin and its image, have size \( 2^{24} \); hence it is an isomorphism.

In an extraspecial group the commutator is bilinear and we have \( \text{type}(u+v) = \text{type}(u) + \text{type}(v) + \langle u, v \rangle \) and \( (x_\tau x_\delta)^2 = x_\tau^2 x_\delta^2 [x_\tau, x_\delta] \). So it suffices to check 

\[ x_\tau^2 = x_\tau \text{type}(\lambda_\tau) \] and 

\[ [x_\tau, x_\delta] = x_\tau^{\delta_\tau, \lambda_\delta} \]

with \( x_\tau, x_\delta \) running through all the generators \( x_d, x_\delta \) given in the Theorem. All these checks can be done by easy calculations using 5.3.3.

\[ \square \]

An immediate consequence of Theorem 6.1 is \( P(d) = \text{type}(\lambda_d) \pmod{2} \) for all \( d \in P \).

We will abbreviate \( x_d \cdot x_\delta \) to \( x_{d+\delta} \) and \( \lambda_d + \lambda_\delta \) to \( \lambda_{d+\delta} \) for \( d \in P, \delta \in C^* \) as in [5]. For \( r = (d, \delta) \in P \times C^* \) we will define \( x_r = x_{d+\delta} \), and for \( f \in \{ \pm 1, \pm \Omega \} \) we put \( x_f r = x_f \cdot x_r \). The more relaxed condition \( f \in P \) could have the funny effect \( x_{f \cdot d} = x_f x_d \neq x_{f \cdot d} \).

6.2 Short vectors in \( \Lambda/2\Lambda \)

A vector in \( \Lambda/2\Lambda \) is called short, if it is congruent modulo \( 2\Lambda \) to a vector of type \( 2 \) in \( \Lambda \), i.e. to a shortest nonzero vector in \( \Lambda \). \( x_r \) is called short if its image \( \lambda_r \) in \( \Lambda/2\Lambda \) is short. We define \( x_\Omega^+ \) to be \( x_\Omega r \) if this is short and \( x_r \) otherwise. The short vectors in \( \Lambda/2\Lambda \) are given as follows, see [2] or [7], Ch. 4.11.

\[ \lambda_{ij} : (4_{on}, -4_{on}, 0_{\text{elsewhere}}), \]

\[ \lambda_{ij}^+ : (4_{on}, j, 0_{\text{elsewhere}}), \]

\[ \lambda_{dx}^+ : \lambda_{0d} : (2_{on}, 2_{on}, 0_{\text{elsewhere}}), \]

\[ n = |\delta|/2, |d| = 8, \text{ and } \delta \text{ is an even subset of } d, \]

\[ \lambda_{dx}^+ : \lambda_{0d} : (\mp 3_{on}, \pm 1_{\text{elsewhere}}), \]

\[ m = (d, i) + P(d), \text{ and the lower sign is taken on } d. \]

For \( d \in C \) (or in \( P \)) with \(|d| = 8 \) we define \( A(d, C) = \{ A(d, c) \mid c \in C \} \). It follows from standard facts about the Golay code that \( A(d, C) \) is the set of all even elements of \( C^* \) which can be represented as a subset of \( d \). An element of \( A(d, C) \) has exactly two representatives \( \delta \) and \( d + \delta \) which are subsets of \( d \). If \( \delta, \epsilon \) are given as elements of \( A(d, C) \), then the expressions \(|\delta|/2\) and \(|\delta \cap \epsilon|\) are well defined (modulo 2) under the assumption that subsets of \( d \) are
chosen as representatives of \( \delta \) and \( \epsilon \). Thus \( \lambda_{d,\delta}^+ \), \(|d| = 8\), \( \delta \) even, is short if and only if \( \delta \in A(d, \mathbb{C}) \). For each octad \( d \) (i.e. \( d \in \mathbb{C} \), \(|d| = 8\)) there are 64 short vectors of that shape.

We obviously have \( \lambda_{-r} = \lambda_r \) for all short vectors \( \lambda_r \). We also have \( \lambda_{d,\delta}^+ = \lambda_{\Omega d,\Omega \delta}^+ \). \( \mathcal{C} \) contains 759 octads. So the numbers of the short vectors as given above are:

\[
|\lambda_0| = |\lambda_1^+| = \binom{24}{2} = 276, \quad |\lambda_4^+| = 759 \cdot 64 = 48576, \quad |\lambda_4^-| = 2048 \cdot 24 = 49152.
\]

Altogether we have 98280 short vectors in \( \Lambda/2\Lambda \).

## 7 Representations of the groups \( N_{x0}, N_{y0}, N_{z0} \)

In this section we construct a 196884-dimensional real monomial representation 196884 \( x \) of the group \( N_{x0} = N_0 \cap G_{x0} \). We give a basis of the vector space 196884 and we state the operations of the generators of \( N_{x0} \). In section 8 we extend 196884 \( x \) to \( N_0 \) and in section 9 we extend 196884 \( x \) to \( G_{x0} \) so that we eventually obtain a representation of the monster \( \mathcal{M} \). Similar representations 196884 \( y \) and 196884 \( z \) of \( N_{y0} \) and \( N_{z0} \) may be obtained in the same way.

### 7.1 Representations 98280 \( x \), 98280 \( y \), 98280 \( z \) of \( N_{x0}, N_{y0}, N_{z0} \)

Since \( Q_{x0} \cong Q_x \) and \( Q_{x0} \) is normal in \( N_{x0} \), the group \( N_{x0} \) acts on \( Q_x \) via conjugation, preserving the paring between \( x_r \) and \( x_{-r} \) in \( Q_x \). This leads to a representation of \( N_{x0} \) on a real vector space 98280 \( x \) of dimension 196884 spanned by short vectors \( X_r, r \in P \times \mathbb{C}^* \) with the relation \( -(X_r) = X_{-r} \), so that \( N_{x0} \) acts on \( X_r \) in the same way as on \( x_r \) by conjugation. We will use the same symbol 98280 \( x \) for that vector space and the action of the group \( N_{x0} \) on that vector space. Since \( N_{x0} \) is a quotient of \( N_{x1} \) and \( N_x \), this gives us also a representation on \( N_{x1} \) and \( N_x \).

We give 98280 \( x \) the structure of an Euclidean space by assigning norm 1 to all vectors \( X_r \), and by declaring all pairs of such vectors perpendicular unless they are equal or opposite.

Considering 98280 \( x \) as a representation of \( N_x \), its kernel is the group generated by \( K_0 \) and \( x \). The elements \( y \) and \( z \) of \( N_x \) act on the basis vector \( X_{d,\delta} \) of 98280 \( x \) as \((-1)^{|\delta|}\). Using triality, we may define similar representations 98280 \( y \) and 98280 \( z \) of \( N_y \) and \( N_z \).

### 7.2 Representations 4096 \( x \), 4096 \( y \), 4096 \( z \) of \( N_{x1}, N_{y2}, N_{z3} \)

In this section we construct representations 4096 \( x \), 4096 \( y \), and 4096 \( z \) of the groups \( N_{x1}, N_{y2} \) and \( N_{z3} \). On the way we also obtain a representation \( (6 \cdot 2048)_N \) of \( N \) which will be useful for combining the first three representations to a representation of \( N_0 \).

We construct these representations from the action of \( N \) on \( P^3 \) given in section 5. We augment \( P \) by an element 0 by decreeing \( 0 \cdot d = d \cdot 0 = 0 \cdot 0 = 0 \), \( d \in P \) and we write \( P_0 \) for \( P \cup 0 \) with that operation. Put \( P_0^{(3)} = \{(d, 0, 0), (0, d, 0), (0, 0, d) \mid d \in P \} \). By definition \( N \) acts as a permutation group on \( P_0^{(3)} \). The action of \( N \) on \( P_0^{(3)} \) preserves the pairing between \( (a, b, c) \) and \((-a, -b, -c) \). So we have a monomial representation of \( N \) on the real vector space \( (6 \cdot 2048)_N \), spanned by the basis vectors in \( P_0^{(3)} \) with the identification \( -(a, b, c) = (-a, -b, -c) \). Let \( K_0, K_1 \), and \( K_2 \) be as in section 5. We prefer a different basis of \( (6 \cdot 2048)_N \), so that we can easily identify the subspaces of \( (6 \cdot 2048)_N \) where the representation of \( K_0, K_1 \), or \( K_2 \) is trivial. For all \( d \in P \) we define

\[
\begin{align*}
  d_1^- &= (0, d, 0) + (0, -\Omega d, 0), & d_1^+ &= (0, 0, d) + (0, 0, \Omega d), \\
  d_2^- &= (0, 0, d) + (0, -\Omega d, 0), & d_2^+ &= (d, 0, 0) + (\Omega d, 0, 0), \\
  d_3^- &= (d, 0, 0) + (-\Omega d, 0, 0), & d_3^+ &= (0, d, 0) + (0, \Omega d, 0).
\end{align*}
\]
Here we make another deviation from Conway’s original definition in [9]: \( d_m^\pm \) in [9] corresponds to our \( \bar{d}_m^\pm \). The motivation for this change is given in section \[9.10\]. This change greatly simplifies the representation of the element \( \xi \) of \( G_{x0} \setminus N_{x0} \) in \( 4096_x \), as stated in Corollary \[9.14\].

The set \( \{d_m^\pm\}, d \in \mathcal{P}, m = 1, 2, 3 \) is a basis of \( (6 \cdot 2048)_N \) with the obvious identifications

\[
(d)_m = (-d)_m = (- \Omega d)_m = (- \Omega d)_m = -(d)_m = (\Omega d)_m = -(d)_m . \tag{7.1.1}
\]

For \( m = 1, 2, 3 \) we write \( \langle d_1^+ \rangle \) and \( \langle d_2^- \rangle \) for the subspaces of \( (6 \cdot 2048)_N \) generated by the corresponding basis vectors. All these 2048-dimensional subspaces are invariant under the action of \( N_{xyz} \). They are permuted by the cosets of \( N_{xyz} \) as indicated in Table \[1\] for the coset representatives \( x_1 \) and \( \tau \).

We give \( (6 \cdot 2048)_N \) the structure of an Euclidean space by assigning the norm \( \frac{1}{\sqrt{2}} \) to all vectors \((d, 0, 0), (0, d, 0), \) and \((0, 0, d)\), and by declaring all pairs of such vectors perpendicular unless they are equal or opposite. Then all vectors \( (d_m^\pm) \), \( m = 1, 2, 3 \) have norm 1 and pairs such vectors are also perpendicular unless they are equal or opposite. In the sequel \( (6 \cdot 2048)_N \) means the representation of \( N \) on this Euclidean space with orthonormal basis vectors taken from the vectors \( (d_m^\pm) \), \( m = 1, 2, 3 \). Then \( (6 \cdot 2048)_N \) is an orthogonal and monomial representation of \( N \).

We define the representations \( 4096_x, 4096_y \) and \( 4096_z \) of \( N_x, N_y \) and \( N_z \), respectively, by their natural action on the following subspaces of \( (6 \cdot 2048)_N \):

\[
4096_x : \langle d_1^- \rangle \oplus \langle d_1^+ \rangle, \quad 4096_y : \langle d_2^- \rangle \oplus \langle d_2^+ \rangle, \quad 4096_z : \langle d_3^- \rangle \oplus \langle d_3^+ \rangle,
\]

where \( \oplus \) denotes the direct sum. The kernels of \( 4096_x, 4096_y, 4096_z \), and elements acting as \(-1\) are displayed in Table \[1\]. From these kernels we see that \( 4096_x, 4096_y, 4096_z \) are also representations of \( N_{x1}, N_{y2}, N_{z3} \), respectively.

| Subspace of \( (6 \cdot 2048)_N \) | \( 4096_x \) | \( 4096_y \) | \( 4096_z \) |
|----------------------------------|-----------------|-----------------|-----------------|
| \( \langle d_1^- \rangle \) | \( \langle d_1^+ \rangle \) | \( \langle d_2^- \rangle \) | \( \langle d_2^+ \rangle \) |
| \( \langle d_3^- \rangle \) | \( \langle d_3^+ \rangle \) | \( \langle d_3^- \rangle \) | \( \langle d_3^+ \rangle \) |
| Kernel of \( 4096_{x,y,z} \) | \( 1, K_1, y_1, z_1 \) | \( 1, K_2, z_2, x_1 - \Omega \) | \( 1, K_3, x_1, y_1 - \Omega \) |
| Elements acting as \(-1\) | \( x_1 - 1, K_2, K_3 \) | \( y_1 - 1, K_1, K_3 \) | \( z_1 - 1, K_1, K_2 \) |
| \( x_1 \) maps subspace to | \( \langle d_1^+ \rangle \) | \( \langle d_1^- \rangle \) | \( \langle d_2^- \rangle \) | \( \langle d_2^+ \rangle \) |
| \( \tau \) maps subspace to | \( \langle d_2^- \rangle \) | \( \langle d_3^- \rangle \) | \( \langle d_3^+ \rangle \) | \( \langle d_1^+ \rangle \) |

Table 1: The kernels of and the action of \( S_3 \) on the subspaces of \( (6 \cdot 2048)_N \)

### 7.3 Representations \( 24_x, 24_y, 24_z \) of \( N_{x1}, N_{y2}, N_{z3} \)

In this section we construct representations \( 24_x, 24_y \) and \( 24_z \) of the groups \( N_{x1}, N_{y2} \) and \( N_{z3} \). On the way we also obtain a representation \( (3 \cdot 24)_N \) of \( N \) which will be useful for combining the first three representations to a representation of \( N_0 \).

Now we construct a representation \( (3 \cdot 24)_N \) of \( N \). For each \( i \in \hat{\Omega} \) we define \( i_1 \) by:

\[
i_1 = \sum_{d \in \mathcal{P}} (-1)^{\langle d, i \rangle} (d, 0, 0),
\]

without any identification of \( (d, 0, 0) \) and \(- (d, 0, 0)\).

We define \( i_2, i_3 \) similarly by using the triples \((0, d, 0)\) and \((0, 0, d)\). Then it is easy to check that the subgroup \( Q_{xyz}^{-1} \) of \( Q_{xyz} \) generated by \( x_\delta, x, y, z, \delta \) even, preserves \( i_1, i_2, i_3 \) and that \( N \) operates on \( i_1 \) as follows:

\[
i_1 \xrightarrow{x_\delta} i_1, \quad i_1 \xrightarrow{y_\delta z} i_1, \quad (-1)^{\langle d, i \rangle} i_1, \quad i_1 \xrightarrow{x_\delta} i_1, \quad i_1 \xrightarrow{\tau} i_2, \quad (\& x_\delta y_\delta z).
\]
Here the symbol ($\&xy\hat{z}$) means that $x, y, z$ and indices 1, 2, 3 must be cyclically permuted. We define $24_x$ to be the representation of $N_x$ with basis $i_1, i \in \hat{\Omega}$. Representations $24_y$ and $24_z$ are defined similarly, based on $i_2$ and $i_3$. Then

$$(3\cdot24)_N = 24_x \oplus 24_y \oplus 24_z$$

is a representation of $N$ with kernel $Q^{(x)}_{xy}$. $x_\delta$ preserves $i_1$ and exchanges $i_2$ with $i_3$ for odd $\delta$, and $\tau$ cyclically permutes $i_1, i_2$ and $i_3$.

We declare $\{i_m | i \in \hat{\Omega}, m = 1, 2, 3\}$ to be an orthonormal basis of the Euclidean space $(3\cdot24)_N$. Then $N$ acts orthogonally and monomially on that space.

Representations $24_x, 24_y, 24_z$ are invariant under the action of $N_{xyz}$. But they are permuted by the cosets of $N_{xy}$ in $N$, as indicated by the action of the coset representatives $x_i$ and $\tau$ in the following Table 2. This table also displays the kernels of these representations and the elements acting as $-1$.

| Subspace of $(3\cdot24)_N$ | $24_x$ | $24_y$ | $24_z$ |
|---------------------------|--------|--------|--------|
| basis vectors             | $i_1, i \in \hat{\Omega}$ | $i_2$ | $i_3$ |
| Kernel generated by       | $k_1, x\delta, x\delta, y, z$ | $k_2, y\delta, y\delta, x, z$ | $k_3, y\delta, z\delta, x, y$ |
| Elements acting as $-1$   | $k_1, k_3, y\Omega, z\Omega$ | $k_2, k_\delta, x\Omega, z\Omega$ | $k_1, k_2, x\Omega, y\Omega$ |
| $x_i$ maps subspace to    | $24_x$ | $24_y$ | $24_z$ |
| $\tau$ maps subspace to   | $24_y$ | $24_z$ | $24_x$ |

Table 2: Kernels of and action of $S_3$ on the subspaces of $(3\cdot24)_N$

### 7.4 Representation 196884

We will now construct a representation 196384 of subgroup $N_{x0}$, $N_{y0}$, $N_{z0}$ of $N_x$, $N_y$, $N_z$. We define the representation

$$196884 = 300_x \oplus 98280_x \oplus 98304_x$$

of $N_{x0}$ to be the direct sum of representations $300_x$, $98280_x$, $98304_x$, where $300_x$ is the symmetric tensor square $24_x \otimes_{sym} 24_x$ of $24_x$, $98280_x$ is defined as above, and $98304_x$ is the tensor product $4096_x \otimes 24_x$.

$98280_x$ is a representation of $N_{x0}$ by construction. For both representations, $300_x$ and $98304_x$, the group $K_1$ is in their kernel, and $K_2, K_3$ act as $-1$. Thus $K_0$ is contained in the kernel of the tensor products $24_x \otimes 24_x$ and $4096_x \otimes 24_x$, so that $300_x$ and $98304_x$ are indeed representations of $N_{x0}$.

We introduce some abbreviations for the basis vectors that we will use for $196884_x$:

- for $300_x$: $(ii)_1 = i_1 \otimes i_1$, of norm 1,
- $(ij)_1 = i_1 \otimes j_1 + j_1 \otimes i_1$ ($i \neq j$), of squared norm 2,
- for $98280_x$: $X_r$ (r short), of norm 1,
- for $98304_x$: $d^\pm \otimes i = d^\pm \otimes i_1$ ($d \in \mathcal{P}, i \in \hat{\Omega}$), of norm 1.

These basis vectors are mutually orthogonal, except when equal or opposite. For representations $196884_y$ and $196884_z$, we use the corresponding notation ($\&xy\hat{z}$) obtained by cyclically permuting $(x, y, z)$ and $(1, 2, 3)$. The action of the generators of $N$ on the basis vectors of $196884_x$, $196884_y$ and $196884_z$ can be obtained from Table 3.
The action of the generating elements $g$ of $N$ on the basis vectors $i_2, d_2^+, Y_{d\times\delta}$ and $i_3, d_3^+, Z_{d\times\delta}$ can be obtained by applying the triality operation $(\&x_\delta y_\delta z_\delta)$ to the corresponding entries in Table 3. E.g. from the action of $z_\delta$ on $d_1^+$ and $i_1$ we may deduce

$$d_2^+ \otimes i_2 \xrightarrow{z_\delta} (-1)^{(c,i)}(\tilde{e}d_2^+) \otimes i_2 \quad \text{and} \quad d_3^+ \otimes i_3 \xrightarrow{y_\delta} (-1)^{(c,i)}(\tilde{e}d_3^+) \otimes i_3$$

by applying the triality operation $(\&x_\delta y_\delta z_\delta)$. In the sequel the phrase "from Table 3 we obtain . . ." means that the reader has to apply the triality operation $(\&x_\delta y_\delta z_\delta)$ by himself, if necessary.

The action of $g$ on the basis vectors $X_{d,\delta}, X_{d,i}$ is obtained by conjugation of the corresponding basis vector with $g$ and taking the result modulo the kernel $K_1$.

The basis vector $X_{d,i}^+$ in Table 3 is short for all $d \in P, i \in \bar{\Omega}$ and all short basis vectors $X_{d,i}$ for odd $\delta$ are of that shape. We have $X_{d,i}^+ = X_{d,\delta}^{i+\delta}$. The notation $X_{d,i}^+$ in Table 3 is consistent with the notation $x_{d,i}^{+\delta}$ in section 6.2.

The following table is helpful for computing the last two columns in Table 3.

| $g$ | Action of $g$ on the basis vectors | $X_{d,\delta}, \delta$ even | $X_{d,i}^+, i \in \bar{\Omega}$ |
|-----|----------------------------------|----------------------------|----------------------------|
| $x_e$ | $i_1$ | $(\tilde{e}d)^-_{1}$ | $(\tilde{e}d)^+_{1}$ | $(-1)^{m}X_{d,\delta}$ |
| $y_e$ | $(1)^{(c,i)}i_1$ | $(\tilde{e}d)c^-_{1}$ | $(\tilde{e}d)c^+_{1}$ | $X_{d,\delta}$ |
| $z_e$ | $(1)^{(c,i)}i_1$ | $(\tilde{e}d)c^-_{1}$ | $(\tilde{e}d)c^+_{1}$ | $(-1)^{(c,i)}X_{d,\delta}$ |

Table 3: Action of the generators of $N$ on $24_x, 4096_x$ and $98280_x$.

Notation: $n = C(d,e) + (c,\delta), \delta' = A(d,e), X_{d,i}^+ = X_{d,\delta}^{i+\delta}$, with $m = P(d) + (d, i)$.

The action of the generating elements $g$ of $N$ on the basis vectors $i_2, d_2^+, Y_{d\times\delta}$ and $i_3, d_3^+, Z_{d\times\delta}$ can be obtained by applying the triality operation $(\&x_\delta y_\delta z_\delta)$ to the corresponding entries in Table 3. E.g. from the action of $z_\delta$ on $d_1^+$ and $i_1$ we may deduce

$$d_2^+ \otimes i_2 \xrightarrow{z_\delta} (-1)^{(c,i)}(\tilde{e}d_2^+) \otimes i_2 \quad \text{and} \quad d_3^+ \otimes i_3 \xrightarrow{y_\delta} (-1)^{(c,i)}(\tilde{e}d_3^+) \otimes i_3$$

by applying the triality operation $(\&x_\delta y_\delta z_\delta)$. In the sequel the phrase "from Table 3 we obtain . . ." means that the reader has to apply the triality operation $(\&x_\delta y_\delta z_\delta)$ by himself, if necessary.

The action of $g$ on the basis vectors $X_{d,\delta}, X_{d,i}$ is obtained by conjugation of the corresponding basis vector with $g$ and taking the result modulo the kernel $K_1$.

The basis vector $X_{d,i}^+$ in Table 3 is short for all $d \in P, i \in \bar{\Omega}$ and all short basis vectors $X_{d,i}$ for odd $\delta$ are of that shape. We have $X_{d,i}^+ = X_{d,\delta}^{i+\delta}$. The notation $X_{d,i}^+$ in Table 3 is consistent with the notation $x_{d,i}^{+\delta}$ in section 6.2.

The following table is helpful for computing the last two columns in Table 3.

| $g$ | Value of $g^{-1}Bg$ for |
|-----|-------------------------|
| $x_e$ | $x \xrightarrow{C(d,e)} x_d$ |
| $y_e$ | $y \xrightarrow{C(d,e)} x_d,x\delta$ |
| $z_e$ | $z \xrightarrow{C(d,e)} x_d,x\delta$ |
| $x_{\pi}$ | $x \xrightarrow{d} x\delta$ |
| $y_{\pi}$ | $y \xrightarrow{d} x\delta$ |
| $z_{\pi}$ | $z \xrightarrow{d} x\delta$ |

Table 4: Action of generators of $N$ by conjugation.

8 Extending the representation $196884_x$ from $N_{x0}$ to $N_0$

8.1 The action of the triality element $\tau$ on $196884_x$

In order to extend the representation $196884_x$ from $N_{x0}$ to $N_0$ it suffices to state the action of the triality element $\tau$ on the basis vectors of $196884_x$. We first state the result, which
is the most important information for a programmer. Proofs are deferred to the next two subsections.

The operation of \( \tau \) on the basis vectors \((ij)_1, X_{ij} \) and \( X_{ij}^+ \) is given by:

\[
(ii)_1 \xrightarrow{\tau} (ii)_1, \quad (ij)_1 \xrightarrow{\tau} X_{ij} - X_{ij}^+ \xrightarrow{\tau} X_{ij} + X_{ij}^+ \xrightarrow{\tau} (ij)_1, \quad i \neq j .
\] (8.1.1)

The operation of \( \tau \) on the basis vectors \( X_{d,i}^+ \) and \( d^\pm \otimes_1 i \) is given by:

\[
X_{d,i}^+ = X_{\Omega(d,i) + \tau(d,i),i} \xrightarrow{\tau} (-1)^{(d,i)} \cdot d^- \otimes_1 i \quad \xrightarrow{\tau} d^+ \otimes_1 i \quad \xrightarrow{\tau} X_{d,i}^+. \] (8.1.3)

Now we specify the action of \( \tau \) on the remaining basis vectors \( X_{d,\delta} \) of 196884\(_x\). We put \( X_{d,\delta} = X_{\Gamma(d,\delta)} \) for \( n = |\delta|/2, |d| = 8, \delta \in A(d,C) \), with \( A(d,C) = \{ A(d,e) \mid e \in C \} \), in accordance with the notation in section 6.2. Clearly all remaining basis vectors are of that shape. Let \( V_T \) be the subspace of 196884\(_x\) generated by these basis vectors. We extend \( V_T \) from a representation of \( N_\tau \), with kernel \( K_T \) generated by \( xf, yf, z_f \) for \( f \in \{-1, \pm 1\} \), to a representation of \( \tilde{N} \). Then \( V_T \) is also a representation of \( N \). It is also a representation of \( K_0 \), because \( K_0 \) is in the kernel of \( V_T \).

The operations of the generators \( x_e, y_e, z_e, \nu_\pi, x_\tau, y_\tau \) of \( \tilde{N} \) on the basis vector \( X_{d,\delta}^+ \) of \( V_T \) are given by:

\[
x_e : (-1)^{C(d,e)} X_{d,\delta}^+, \quad y_e : (-1)^{C(d,e)} X_{d,\delta}^+, \quad \nu_\pi : (-1)^{\delta|/2} X_{d,\delta}^+, \quad x_\tau : (-1)^{\delta/2} X_{d,\delta}^+, \quad y_\tau : \frac{1}{8} \sum_{e \in A(d,e)} (-1)^{\delta(e)\epsilon} X_{d,e}^+ .
\] (8.1.4)

Here \( \delta/2 \) and \( \delta \cap e \) are defined (modulo 2) for \( \delta \) and \( e \) as even subsets of \( d \) as in section 5.2 and \( |\pi| \) is equal to 1 for odd and to 0 for even \( \pi \). The sum in the expression for \( y_\tau \) runs over all even subsets \( e \) of \( d \), identifying \( d \setminus e \) with \( e \), so that it has 64 terms. \( x_\tau \) is monomial and the action of \( y_\tau \) resembles that of a Hadamard matrix, thus allowing a very efficient implementation.

The operations of \( x_e, y_e, z_e \) and \( x_\pi \) are obviously equal to the corresponding operations in Table 3. In section 5.3 we will show that the remaining operations in \( V_T \) are consistent with the relations in \( \tilde{N} \). By Theorem 5.1 we have \( \tau = y_\tau x_\tau \) and \( \tau^2 = \tau^{-1} = x_\tau y_\tau \). We obtain the operation of \( x_\pi \) in Table 3 by using \( x_\pi = x_\tau |\pi| \nu_\pi \).

The operation of the triality element \( \tau \) leads to an identification of the three spaces 196884\(_x\), 196884\(_y\) and 196884\(_z\), which is called the dictionary in [5], Table 2. In our notation we have the following identification of these three spaces:

| Subspace     | 196864\(_x\) | 196864\(_y\) | 196864\(_z\) |
|--------------|--------------|--------------|--------------|
| \( V_A \)    | \((ij)_1\)   | \( Y_{ij} + Y_{ij}^+ \) | \( Z_{ij} - Z_{ij}^+ \) |
| \( V_B \)    | \( X_{ij} - X_{ij}^+ \) | \((ij)_2\) | \( Z_{ij} + Z_{ij}^+ \) |
| \( V_C \)    | \( X_{ij} + X_{ij}^+ \) | \( Y_{ij} - Y_{ij}^+ \) | \((ij)_3\) |
| \( V_D \)    | \((ii)_1\)   | \((ii)_2\) | \((ii)_3\) |
| \( V_X \)    | \( X_{d,i}^+, i \) | \( d^+ \otimes_2 i \) | \( (-1)^{(d,i)} d^- \otimes_3 i \) |
| \( V_Y \)    | \( d^+ \otimes_1 i \) | \( (-1)^{(d,i)} d^- \otimes_2 i \) | \( Z_{d,i}^+ \) |
| \( V_Z \)    | \( X_{d,i}^+ \) | \( \frac{1}{2} \sum (-)_{e,\delta} Y_{d,\delta}^+ \) | \( \frac{1}{2} \sum (-)_{e,\delta} Z_{d,\delta}^+ \) |
| \( V_T \)    | \( \frac{1}{8} \sum (-)_{\delta,\epsilon} X_{d,\delta}^+ \) | \( Y_{d,\delta}^+ \) | \( Z_{d,\delta}^+ \) |

Table 5: The dictionary

Here \( (-)_{\delta,\epsilon} = (-1)^{|\delta|\epsilon + |\delta|/2} \) and the sum in the last three lines runs over \( \epsilon \in A(d,C) \).

Following [5], we also give names to the subspaces of 196884\(_x\) as indicated in table 5.
8.2 Proofs for the monomial and almost-monomial actions of $\tau$

The purpose of this subsection is to establish (8.1.1), (8.1.2) and (8.1.3).

Using the information in Table 3, it is easy to check that the identification $(ii)_1 = (ii)_2 = (ii)_3$ in line 4 of Table 5 is compatible with the action of $N_{x0}$. Together with the obvious action $(ii)_1 \mapsto (ii)_2 \mapsto (ii)_3 \mapsto (ii)_1$ we obtain (8.1.1).

Next we show (8.1.2). The group ring $\mathbb{R}Q_{xyz0}$ is the real algebra with basis vectors labelled by $Q_{xyz}$, and multiplication of the basis vectors given by the group operation in $Q_{xyz0}$. Since $Q_{xyz0}$ is normal in $N_0$, the group $N_0$ operates on $\mathbb{R}Q_{xyz0}$ by conjugation.

Let $V$ be the subspace of $98280$, the kernel elements in tables 1 and 2 show that $K_{xyz0}$ is a representation of $N$, and the subspaces in the definition of (6.·) in line 4 of Table 5 is compatible with the action of $N_{x0}$. Together with the obvious action $(ii)_1 \mapsto (ii)_2 \mapsto (ii)_3 \mapsto (ii)_1$ we obtain (8.1.2).

Next we show (8.1.3). This is a consequence of lines 5–7 of the dictionary in Table 5, 98280 given by $K_0$ and $x_{ij} = y_{ij} = z_{(ij)}$ we obtain:

$$Y_{ij} = Y_{ij}^\dagger = Y_{ij} + Y_{ij}^\dagger = y_{1(ij)} + y_{1(ij)} - y_{-1(ij)} = z_{1(ij)} + z_{-1(ij)} = Z_{ij} - Z_{ij}^\dagger = x_{1(ij)} + x_{-1(ij)} - x_{-1(ij)} - x_{-1(ij)} .$$

Hence we may identify $Y_{ij} = Y_{ij}^\dagger$ with $Z_{ij} = Z_{ij}^\dagger$. A similar calculation yields $Z_{ij} + Z_{ij}^\dagger = X_{ij} - X_{ij}^\dagger$ and $X_{ij} + X_{ij}^\dagger = Y_{ij} - Y_{ij}^\dagger$. Put $(x_{ij}) = x_{1(ij)} + x_{-1(ij)} - x_{-1(ij)} - x_{-1(ij)}$, and define $(Y_{ij})$, $(Z_{ij})$ similarly, using triality. From Table 3 we see that the action of the generators $x_{a}$, $x_{e}$ and $\tau$ of $N_{x0}$ on the vectors $(X_{ij})$, $(Y_{ij})$ and $(Z_{ij})$ is the same as their action on $(ij)_1$, $(ij)_2$ and $(ij)_3$, respectively. So we have just established the identifications in the first three lines of Table 5. Using these identifications and the obvious action $(ij)_1 \mapsto (ij)_2 \mapsto (ij)_3 \mapsto (ij)_1$ we obtain (8.1.3).

Next we show (8.1.3). This is a consequence of lines 5–7 of the dictionary in Table 5 and of the obvious operation of the triality element $\tau$. Lines 5–7 in Table 5 are direct consequences of the following two lemmas.

**Lemma 8.2.** The group $N_0$ has a monomial representation $(3\cdot49152)_N$ with basis vectors $d^\pm \otimes_m i$, $d \in P$, $i \in \Omega$, $m = 1, 2, 3$, and the identifications

$$(-d)^\pm = -(d^\pm), \quad (\pm \Omega d)^\pm = d^\pm, \quad d^+ \otimes_m i = (-1)^{(d,i)} d^- \otimes_{m+1} i ,$$

where $m$ is to be taken modulo 3, and the action of the generators of $N_0$ is given by table 5 and triality.

**Proof**

For $m = 1, 2, 3$ let $\langle d^+ \otimes_m i \rangle$ and $\langle d^- \otimes_m i \rangle$ be the vector spaces spanned by the basis vectors $d^+ \otimes_m i$ and $d^- \otimes_m i$, respectively, for $d \in P$, $i \in \Omega$, with the identifications given by (7.1.1). Define $(6\cdot49152)_N$ by:

$$(6\cdot49152)_N = \langle d^+ \otimes_1 i \rangle \oplus \langle d^- \otimes_1 i \rangle \oplus \langle d^+ \otimes_2 i \rangle \oplus \langle d^- \otimes_2 i \rangle \oplus \langle d^+ \otimes_3 i \rangle \oplus \langle d^- \otimes_3 i \rangle$$

The six subspaces in the definition of $(6\cdot49152)_N$ are invariant under $N_{xyz}$, and the action of the kernel elements in tables 1 and 2 show that $K_0$ acts as identity everywhere, so $(6\cdot49152)_N$ is a representation of $N_{xyz0}$. From the actions of $\tau$ and $x_i$ in these tables we see that both of them preserve $(6\cdot49152)_N$ so that this is indeed a representation of $N_0$.

Let $i$ be the involution on $(6\cdot49152)_N$ given by:

$$d^\pm \otimes_m i \mapsto (-1)^{(d,i)} d^\mp \otimes_{m+1} i , \quad \text{with } m \text{ to be taken modulo } 3 ,$$

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Involving $i$ trivially preserves the identification between $(-d)^\pm \otimes m$ and $-(d^\pm \otimes m)$. Since $\langle \Omega, i \rangle = 1$, we have

$$i((\Omega d)^+_m \otimes m) = -(1)((d, i) \Omega d^- \otimes m)$$

$$= -(1)((d, i) d^+ \otimes m).$$

A similar calculation shows $i([-\Omega d^+_m \otimes i m]) = i((-d^- \otimes m))$. Thus these identifications $\langle \Omega d^+_m \otimes m \rangle = d^+_m$, and $\langle [-\Omega d^-_m \otimes i m] \rangle = d^-_m$ are also preserved, so that $i$ is well defined on $(3\cdot 49152)_N$.

For the proof of the Lemma, it suffices to show that $i$ commutes with the action of the generators of $N_0$. Using Table 3, it is trivial to check that $i$ commutes with $\pi$ for even $\pi$. So it remains to check that $i$ commutes with $x_\tau$ and $x_\pi$ for odd $\pi$, which is a tedious but straightforward calculation using Table 3. The relevant action of $x_\epsilon$ is given in Table 6.

**Lemma 8.3.** The subspace of $196884_x \oplus 196884_y \oplus 196884_z$ spanned by the basis vectors $X_{d,i}$, $Y_{d,i}$ and $Z_{d,i}$, $d \in \mathcal{P}$, $i \in \Omega$, is odd, $(d, i) = P(d)$ is a monomial representation of $N_0$. There is an isomorphism from that space into $(3\cdot 49152)_N$ given by

$$X_{d,i} \mapsto \bar{d}^+ \otimes 2 i, \quad Y_{d,i} \mapsto \bar{d}^+ \otimes 3 i, \quad Z_{d,i} \mapsto \bar{d}^+ \otimes 1 i.$$

**Proof**

Let $V$ be the space spanned by $X_{d,i}$, $Y_{d,i}$ and $Z_{d,i}$. The last column in Table 3 shows that $V$ is a representation of $N_0$. So we have to show that the mapping $V \rightarrow (3\cdot 49152)_N$ given in the Lemma preserves the action of the generators, say $x_\epsilon, x_\pi$ and $x_\tau$. This is trivial for $x_\tau$ and $x_\pi$. The operation of $x_\pi$ on $V$ and on $(3\cdot 49152)_N$ is stated in Table 6. □

| $B$ | $B \mapsto B_{x_\epsilon}$ for the element $x_\epsilon$ of $N_0$ on basis vectors $B$ |
|-----|------------------------------------------------|
| $X_{d,i}^+ \mapsto (-1)^{(d, i)} X_{d,i}^+$ | $d^+ \otimes 2 i \mapsto (-1)^{(d, i)} (\bar{d}e^d) \otimes 2 i$ |
| $Y_{d,i}^+ \mapsto (-1)^{(d, i)} Y_{d,i}^+$ | $d^+ \otimes 3 i \mapsto (-1)^{(d, i)} (\bar{d}e^d) \otimes 3 i$ |
| $Z_{d,i}^+ \mapsto Z_{d,i}^+$ | $d^+ \otimes 1 i \mapsto (\bar{d}e) \otimes 1 i$ |

Table 6: Operation of $x_\epsilon$ on some basis vectors of $196884_x \oplus 196884_y \oplus 196884_z$.

Entries in the same row are equivalent (up to sign) with respect to the dictionary in Table 3.

### 8.3 The proof for the non-monomial action of $\tau$

In this subsection we will show that $V_\tau$ is a representation of $N$ with the operation of the generators of $N$ given by (8.1) and (8.4). We already know from section 8.1 that $V_\tau$ represents $N_x$ with kernel $K_\tau$. In the sequel the generators $x_d, y_d, z_d, x_\delta, x_\tau, y_\tau$ denote the linear operation of the corresponding generator on $V_\tau$, and we put $\nu_\tau(x) = x x_\tau|_x$. Our goal is to establish the defining relations for $N/K_\tau$ in $V_\tau$, with the relations between the generators of $N_x$ already established in $V_\tau$.

By construction $x_\tau$ and $y_\tau$ act as real symmetric orthogonal matrices on $V_\tau$, establishing $x_\tau^2 = y_\tau^2 = 1$. Note that $y_\tau$ consists of $64 \times 64$ blocks, one for each $d$, and each of these blocks can easily be checked to be symmetric and orthogonal.

From $\nu_\tau(x) = x_\tau x_\tau|_x$ and the operation of $x_\pi$ and $x_\tau$ in (8.1) and (8.4) we conclude that $\nu_\tau$ maps $X_{d,\delta}^+ \otimes X_{d',\delta'}^+$ to $X_{d,\delta}^+ \otimes X_{d',\delta'}^+$ for odd $\pi \in \text{Aut}_{2\pi} \mathcal{P}$. Thus for any odd diagonal automorphism $\varphi \in \mathcal{C}^*$ the element $\nu_{\varphi}$ acts by multiplication with $+\mathbb{1}$ on each 64-dimensional block of $V_\tau$ corresponding to a fixed $d$. Thus $\nu_{\varphi}$ commutes with $x_d, y_d, z_d, x_\delta$ for all $d \in \mathcal{D}$, $\delta \in \mathcal{C}^*$ and also with $x_\tau$ and $y_\tau$. From the operation of $\nu_{\tau}$ on $X_{d,\delta}^+$ in $V_\tau$ we conclude $\nu_{\tau} \nu_{\tau}' = \nu_{\tau} \nu_{\tau}'$ for all $\pi, \pi' \in \text{Aut}_{2\pi} \mathcal{P}$. With this information we easily check that the involution $\nu_{\varphi}$ acts on the representation $V_\tau$
of \( N_x \) by conjugation in the same way the element \( \nu_\varphi \) of \( \hat{N} \) acts on the group \( N_x \). Thus \( V_T \) represents the subgroup \( \hat{N}_x \) of \( \hat{N} \) generated by \( N_x \) and \( \nu_\varphi, \varphi \) odd.

So \( V_T \) also represents the normal subgroup \( \hat{N}_{x y z} \) of \( \hat{N} \) generated by \( N_{x y z} \) and \( \nu_\varphi, \varphi \) odd. We claim that the involution \( y_T \) acts on the representation \( V_T \) of \( \hat{N}_{x y z} \) by conjugation in the same way as the element \( y_T \) of \( \hat{N} \) acts on the normal subgroup \( \hat{N}_{x y z} \) of \( \hat{N} \). By (8.1.4), \( y_T \) commutes with all \( x_T, \pi \) even. We have already shown that \( y_T \) commutes with an odd \( x_\varphi \), hence it commutes with all \( \nu_\varphi \), establishing the claim for all generators \( \nu_\varphi \).

To prove our claim, we will show that \( y_T x_T = z_T y_T \) holds in \( V_T \). To see this, note that

\[
X^+_d \xrightarrow{x_T y_T z_T} \frac{1}{8} \sum_{\epsilon \in \mathbb{A}(d,\mathbb{C})} (-1)^{|\epsilon|+|\delta|} X^+_d e \epsilon \tan \mathbb{A}(d,\mathbb{C}) = \frac{1}{8} \sum_{\epsilon \in \mathbb{A}(d,\mathbb{C})} (-1)^{|\epsilon|} X^+_d e \epsilon
\]

with \( m(\epsilon) = |\delta \cap (\epsilon + \mathbb{A}(d,\mathbb{C}))| + |\epsilon, \delta| = |\delta \cap \epsilon| + |\delta \cap d \cap \epsilon| + |\delta \cap \epsilon| = |\delta \cap \epsilon| \pmod{2} \).

Hence \( x_T y_T z_T = y_T \), implying \( y_T x_T = z_T y_T \). Together with the relations already established in \( V_T \) we obtain \( y_T x_T y_T = z_T, y_T z_T y_T = x_T \) and \( y_T y_T y_T = y_T \), implying our claim. So \( V_T \) also represents the group \( \hat{N}_y \) generated by \( \hat{N}_{x y z} \) and \( y_T \).

Thus \( V_T \) represents an extension with normal subgroup \( \hat{N}_{x y z} \) and factor group generated by the involutions \( x_T \) and \( y_T \), where \( x_T \) and \( y_T \) operate on \( \hat{N}_{x y z} \) by conjugation in \( V_T \) in the same way as in \( \hat{N} \). \( \hat{N} \) has structure \( \hat{N}_{x y z} : S_3 \) with \( S_3 \) the symmetric permutation group of three elements generated by \( x_T \) and \( y_T \). So in order to show that \( V_T \) represents \( \hat{N} \) it suffices to show \( \tau^3 = 1 \) (with \( \tau = x_T y_T \)) in \( V_T \).

Since \( \tau^3 = 1 \) in \( \hat{N} \) and conjugation with \( \tau \) in \( V_T \) is the same as in \( \hat{N} \), the matrix \( \tau^3 \) in \( V_T \) centralizes all generators in \( V_T \). By construction the generators \( x_T, y_T \) and \( \tau \) consist of blocks of \( 64 \times 64 \) matrices, with one block for each octad \( d \). Thus for each \( d \) the block of \( \tau^3 \) corresponding to \( d \) centralizes the blocks of all matrices \( x_T \) and \( y_T \) corresponding to the same \( d \). It is easy to see that for any such \( d \) the corresponding blocks of the matrices \( x_T \) and \( y_T \) generate the complete ring of all \( 64 \times 64 \) matrices. So the block of \( \tau^3 \) corresponding to \( d \) is a multiple of the identity matrix, and since \( \tau \) is orthogonal, that block of matrix \( \tau^3 \) is equal to \( \pm 1 \). The trace of that block of matrix \( \tau \) is equal to \( \frac{1}{8} \sum_{\epsilon \in \mathbb{A}(d,\mathbb{C})} (-1)^{|\epsilon|/2} = \frac{1}{8}(36 - 28) = 1 \). It is easy to check that a real \( 64 \times 64 \)-matrix \( \tau \) with \( \tau^3 = -1 \) has eigenvalues \( -1, (1 \pm \sqrt{-3})/2 \) and hence trace equal to \(-1 \pmod{3} \). This establishes \( \tau^3 = 1 \) in \( V_T \).

\[ \square \]

9 Extending representation 196884\(_x\) from \( N_{x0} \) to \( 2^{1+24} \cdot \text{Co}_1 \)

In this section we extend the representation of 196884\(_x\) from \( N_{x0} \) to a maximal subgroup \( G_{x0} \) (with structure \( 2^{1+24} \cdot \text{Co}_1 \)) of the monster \( \mathbb{M} \). It turns out that such an extension is possible for each of the building blocks \( 24_x, 4096_x, \) and \( 98280_x \) of 196884\(_x\). For an optimized computer construction of \( \mathbb{M} \) we need an explicit representation of a \( \xi \in G_{x0} \setminus N_{x0} \) in all these building blocks such that \( N_{x0} \) and \( \xi \) generate \( G_{x0} \). Note that [5] contains an explicit construction of \( N_0 \), but no construction any specific \( \xi \in \mathbb{M} \setminus N_0 \). Here the hardest part is the extension of \( N_0 \) to \( G_{x0} \) and the construction of a suitable element \( \xi \) in the representation 4096\(_x\).

We first describe the representation theory of the extraspecial 2 group \( 2^{1+2n} \) and of certain extensions of such groups called \textit{holomorphs} in [12] and [13]. In section 9.2 we present \( N_{x0} \) as a central quotient of a fiber product corresponding to the faithful representation 4096\(_x \otimes 24_x \) of \( N_{x0} \). In section 9.3 we extend that fiber product to the group \( G_{x0} \). In sections 9.4 ff. we give an explicit construction of a suitable \( \xi \in G_{x0} \setminus N_{x0} \) of order 3 and of its representation. The action of \( \xi \) and \( \xi^2 \) on the components of 196884\(_x\) is stated in Lemma 9.2 for 98304\(_x\), in Corollary 9.11 for 24\(_x\), and in in Corollary 9.16 for 4096\(_x\).
9.1 Extraspecial 2-groups and their representations

Recall the definition of quadratic, bilinear any symplectic forms on vector spaces over $\mathbb{F}_2$ from section 3.3. Symplectic bilinear or non-singular quadratic forms exist only on spaces $\mathbb{F}_2^n$ of even dimension. A quadratic form $q$ on $\mathbb{F}_2^n$ is of plus type if it is non-singular and there is an $n$-dimensional subspace $V$ of $\mathbb{F}_2^n$ with $q(x) = 0$ for all $x \in V$. It is known that all quadratic forms of plus type on $\mathbb{F}_2^n$ are equivalent under the linear group $\text{SL}_{2n}(2)$.

A finite 2-group $E$ is said to be extraspecial if its center $Z(E)$ has order 2, the factor group $E/Z(E)$ is elementary abelian, and the commutator group $[E, E]$ of $E$ is contained in $Z(E)$. We usually write $-1$ for the non-identity element in $Z(E)$, so $Z(E) = \{\pm 1\}$. The elementary Abelian 2-group $E/\{\pm 1\}$ may be regarded as vector space over $\mathbb{F}_2$.

Define $P : E/\{\pm 1\} \to \mathbb{F}_2$ by $x^2 = (-1)_E P(x)$, $x \in E$. Then $P$ is well-defined on $E/\{\pm 1\}$, and by (23.10) in [3], $P$ is a non-singular quadratic form on the vector space $E/\{\pm 1\}$, and we have $[x, y] = (-1)^{\beta_P(x, y)}$ for all $x, y \in E$, with $\beta_P$ symmetric. Thus $E$ has order $2^{2n+1}$.

An extraspecial 2-group $E$ is of plus type if the corresponding quadratic form $P$ on $E/\{\pm 1\}$ is of plus type. For any $n \in \mathbb{N}$ there is a unique extraspecial 2-group of order $2^{1+2n}$ of plus type denoted by $2^{1+2n}_+$. The remainder of this subsection deals with the representation theory of $2^{1+2n}_+$ and of certain extensions of that group.

Let $E$ be an extraspecial 2-group of type $2^{1+2n}_+$. We write $\mathbb{R}E$ for the group ring of $E$ over the real field $\mathbb{R}$. So $\mathbb{R}E$ is a real algebra with the elements of $E$ as basis vectors, and the basis vectors are multiplied as in $E$. Let $\mathbb{R}E^{\pm}$ be the quotient algebra of $\mathbb{R}E$, where we identify the real number $-1$ with element $-1$ of $Z(E)$. It is well known that the irreducible complex representations of the group $2^{1+2n}_+$ consist of a unique faithful real $2^n$-dimensional representation and of $2^{2n}$ real one-dimensional representations of the elementary Abelian 2-group $G/\{\pm 1\}$, see e.g. [10]. So it is obvious that $\mathbb{R}E^{\pm}$ is isomorphic to the unique faithful irreducible real representation of $E$.

A holomorph of a group $E$ is an extension with normal subgroup $E$ and factor group $\text{Out} E$, where $\text{Out} E$ is the group of outer automorphisms of $E$, see [10][12][14]. We remark that the meaning of the term 'holomorph' in [12] and [14] differs from the meaning of that term in the older group-theoretic literature. If $E$ is of type $2^{1+2n}_+$, the group $\text{Out} E$ is the orthogonal group $O_{2n}^+(2)$. Here the orthogonal group $O_{2n}^+(2)$ preserves the quadratic form $P$ of plus type on the vector space $E/\{\pm 1\}$ defined above. There is a unique holomorph $\mathcal{H}(E)$ of the group $E = 2^{1+2n}_+$ which has a faithful $2^n$-dimensional real representation, called the standard holomorph of $E$, see [10], Appendix 1 or [14], Lemma 1.4.2. Since $\mathbb{R}E^{\pm}$ is isomorphic to the unique faithful irreducible representation of $E$ of dimension $2^n$, it is the restriction of the $2^n$-dimensional real representation of $\mathcal{H}(E)$ to $E$. Schur’s lemma implies that an element of $\mathbb{R}E^{\pm}$ which acts as an automorphism on $E$ is determined by that action up to a scalar multiple, which must be $\pm 1$.

The following lemma helps us to construct elements of the standard holomorph $\mathcal{H}(E)$:

**Lemma 9.1.** Let $E = 2^{1+2n}_+$ and $H \subset E$ an elementary Abelian 2-group not containing $-1$. We consider $H$ and $E/\{\pm 1\}$ as vector spaces over $\mathbb{F}_2$. Let

$$\xi = |H|^{-1/2} \sum_{z \in H} (-1)^q(z) z$$

be an element of $\mathbb{R}E^{\pm}$, with $q$ a non-singular quadratic form on $H$ with associated bilinear form $\beta_q$.

Then $\xi^2 = 1$ and there is a unique linear mapping $\phi : E/\{\pm 1\} \to H$ with $[x, y] = (-1)^{\beta_q(\phi(x), y)}$ for all $x \in E$, $y \in H$. For all $x \in E$ we have

$$[x, \xi] = (-1)^q(\phi(x)) \phi(x) \in Z(E) \cdot H.$$
Proof

\[ |H|^{-1} x \xi = \sum_{y, z \in H} x^{-1} (-1)^{q(y)} y x (-1)^{q(z)} z = \sum_{y, z \in H} (-1)^{q(y) + q(z)} [x, y] y z \]

\[ = \sum_{y, z \in H} (-1)^{q(y) + q(yz)} [x, y] z = \sum_{z \in H} (-1)^{q(z)} z \sum_{y \in H} (-1)^{\beta_q(z, y) + C(x, y)} , \]

where \( C \) is the bilinear form on \( E/\{\pm 1\} \) with \([x, y] = (-1)^{C(x, y)}\). Since both, \( \beta_q(z, y) \) and \( C(x, y) \) are linear in \( y \), the last sum over \( y \) is equal to \(|H|\) if \( \beta_q(z, y) = C(x, y) \) for all \( y \in H \) and zero otherwise. Since \( \beta_q \) is non-singular, it follows from linear algebra that for each \( x \in E/\{\pm 1\} \) there is a unique \( z \in H \) with \( \beta_q(z, y) = C(x, y) \) for all \( y \in H \), and that the mapping \( \phi \) which maps each \( x \in E/\{\pm 1\} \) to that value \( z \) is linear. This proves \( x^{-1} x \xi = (-1)^{q(z)} z \), with \( z = \phi(x) \). In case \( x = 1 \) we have \( \phi(x) = 0 \), obtaining \( \xi^2 = 1 \), and hence \([x, \xi] = (-1)^{q(z)} z\).

\[ \square \]

Conjugation with an element \( \xi \) of \( \mathbb{R}E^\pm \) constructed in Lemma 9.1 is an automorphism of \( E \), since \([x, \xi] \in E\) for all \( x \in E \). Hence \( \xi \) represents an element of \( \mathcal{F}(E) \). \( \mathbb{R}E^\pm \) is a faithful irreducible representation of \( E \) and also of \( \mathcal{F}(E) \). So we may identify the element \( \xi \) of \( \mathbb{R}E^\pm \) with the element of \( \mathcal{F}(E) \) represented by \( \xi \).

Remark

In Lemma 9.1 \( \xi \) operates on the vector space \( V = E/\{\pm 1\} \) by conjugation as an orthogonal transformation with \( \text{im}(\xi - 1) = H \). The coset \( \xi E \) of \( E \) can be considered as an element of \( \mathcal{F}(E)/E \cong O_n^+ \), and the bilinear form \( \beta_q \) is uniquely determined by the coset \( \xi E \). For each element \( X \) of an orthogonal group operating on a vector space \( V \), Wall [17] has defined a nondegenerate bilinear form \( K_X \) on the image of \( X^{-1} \) in \( V \) called the parametrization of \( X \). It can be shown that \( \beta_q \) is just Wall’s parametrization of \( \xi E \).

9.2 \( N_{x0} \) is a central quotient of a fiber product

In section 7 we have constructed representations \( 4096_x \) and \( 24_x \) of the group \( N_x \) which both are also representations of \( N_{x1} = N_x/K_1 \). By Tables 1 and 2 the kernels \( K(4096_x) \) and \( K(24_x) \) of these two representations of \( N_x \) intersect in \( K_1 \). Let \( N(4096_x) = N_x/K(4096_x) \) and \( N(24_x) = N_x/K(24_x) \) be the quotients of \( N_x \) for which the representations \( 4096_x \) and \( 24_x \) are faithful. Put \( N_{x2} = N_x/(K(4096_x)K(24_x)) \). Then

\[ N_{x2} \cong N(4096_x)/(K(24_x)/K_1) \cong N(24_x)/(K(4096_x)/K_1) , \]

where e.g. the natural injection from \( K(24_x)/K_1 \) into \( N(4096_x) \) is obtained by extending the coset \( xK_1 \) to \( xK_{4096} \) for \( x \in K(24_x) \).

Then \( N_{x1} \) is isomorphic to the fiber product \( N(4096_x) \triangleleft N_x \triangleleft N(24_x) \). If \( G_1, G_2 \) are groups with a common factor group \( H \) and homomorphisms \( \phi_i : G_i \rightarrow H \), \( i = 1, 2 \), then the fiber product \( G_1 \triangleleft H \) \( G_2 \) is the subgroup of the direct product \( G_1 \times G_2 \) defined by:

\[ G_1 \triangleleft H \ G_2 = \{ (x, y) \in G_1 \times G_2 \mid \phi_1(x) = \phi_2(y) \} . \]

If \( G_i \) has a center \( \{ \pm 1 \} \) of order 2 for \( i = 1, 2 \), then we write \( \frac{1}{2}(G_1 \triangleleft H \ G_2) \) for the quotient of \( G_1 \triangleleft H \ G_2 \) by \( \{ (1, 1), (-1, -1), (1, -1), (-1, 1) \} \) as in [9]. Since \( K_2 \) and \( K_3 \) act as \( -1 \) in both, \( 4096_x \) and \( 24_x \), the group \( N_{x2} \) is isomorphic to \( \frac{1}{2}(N(4096_x) \triangleleft N_x \triangleleft N(24_x)) \).

The following diagram, which is essentially a copy of Fig. 2 in [9], depicts the homomorphisms from \( N_x \) to the various factor groups of \( N_x \) defined above. For each homomorphism we show a generating system and the structure of the corresponding factor group.
From (5.3.2) we see that \( N(4096_x) \) has structure \( 2^{1+24} \cdot 2^{11} \cdot M_{24} \), \( N(24_x) \) has structure \( 2^{12} : M_{24} \) and \( N_x^* \) has structure \( 2^{11} : M_{24} \). The relations (5.3.3) show that the subgroup \( Q_x \) of \( N_x \) generated by \( x_d, x_\delta \) has structure \( 2^{1+24} \). Note that \( |Q_x \cap K_1| = 1 \), so \( Q_x \) is also isomorphic to a subgroup of \( N_x^* \).

By construction, \( N(24_x) \) acts as a matrix group on \( \mathbb{R}^{24} \) with coordinates labelled by \( \Omega \). \( N(24_x) \) has an elementary Abelian normal subgroup \( E \) of order \( 2^{12} \). An element of \( E \) corresponds to the negation of the coordinates given by a codeword of \( C \), hence \( E \cong C \). The permutation representation of \( M_{24} \) acts as a complement of \( E \) in \( N(24_x) \).

### 9.3 The maximal subgroup \( G_{x0} = 2^{1+24}.C_{01} \) of \( \mathbb{M} \)

As in (5.1), we will enlarge \( N(4096_x) \) and \( N(24_x) \) to larger groups of \( G(4096_x) \) and \( G(24_x) \) of structure \( 2^{1+24}.C_{01} \) and \( 2.C_{01} \), respectively. From this we obtain the group \( G_{x0} = \frac{1}{2}(G(4096_x) \triangle C_{01}, G(24_x)) \), which is a maximal subgroup of the monster.

The group \( N(4096_x) = N_x^*/K(4096_x) \) defined in section (5.2) has the faithful irreducible real representation \( 4096_x \). \( 4096_x \) is also faithful and irreducible for the extraspecial subgroup \( Q_x \) of type \( 2^{1+24} \) of \( N(4096_x) \) and hence also for the standard holomorph \( \delta(Q_x) \) of \( Q_x \).

By Theorem 6.1 the quotient of \( Q_x \) by its center is isomorphic to \( \Lambda/2\Lambda \), and the squaring map in \( Q_x \) is equal to the mapping \( \Lambda/2\Lambda \to \mathbb{F}_2 \) given by \( \lambda \mapsto \text{type}(\lambda) = (\lambda, \lambda)/2 \). This 'type' mapping is a quadratic form on \( \Lambda/2\Lambda \) by construction, which is invariant under the automorphism group \( C_{01} \) of \( \Lambda/2\Lambda \). It is of plus type by (5.3.3), so \( C_{01} \) is a subgroup of \( O_+^+(24) \). The factor group \( N_x^* \) of \( N(4096_x) \) with structure \( 2^{11}.M_{24} \) is isomorphic to the monomial subgroup of the automorphism group \( C_{01} \) of \( \Lambda/2\Lambda \). This leads to a chain of inclusions

\[
\begin{align*}
Q_x &\subseteq N(4096_x) &\subseteq G(4096_x) &\subseteq \delta(Q_x) \\
2^{1+24} &\subseteq 2^{1+24}.2^{11}.M_{24} &\subseteq 2^{1+24}.C_{01} &\subseteq 2^{1+24}.O_{24}^+(2).
\end{align*}
\]

There is a similar chain of inclusions of groups

\[
\begin{align*}
E &\subseteq N(24_x) &\subseteq G(24_x) &\subseteq S_{024}(\mathbb{R}) \\
2^{12} &\subseteq 2^{12}.M_{24} &\subseteq 2.C_{01}.
\end{align*}
\]

Here the group \( G(24_x) \) is the automorphism group of the Leech lattice \( \Lambda \), and \( N(24_x) \) is the monomial subgroup of that group. So the groups \( G(4096_x) \) and \( G(24_x) \) possess natural homomorphisms onto \( C_{01} \) which are extensions of the homomorphisms from \( N(4096_x) \) and \( N(24_x) \) onto \( N_x^* = 2^{12} : M_{24} \). We can therefore extend the fiber product \( N_{x1} \) of \( N(4096_x) \) and \( N(24_x) \) to the fiber product

\[
G_{x1} = G(4096_x) \triangle C_{01}, G(24_x).
\]

This group has representations of degrees 4096 and 24 extending the representations 4096, and 24, which we will also call 4096, and 24. The tensor product \( 4096_x \otimes 24_x \) identifies the centers \( \{\pm 1\} \) of both of its factors and is hence is a representation of

\[
G_{x0} = \frac{1}{2}(G(4096_x) \triangle C_{01}, G(24_x)).
\]
The group $G_{x_0}$ is the maximal subgroup of the monster $\mathcal{M}$ constructed in [5]. The groups $G_{x_0}$ and $G(4096_x)$ are both of structure $2^{1+24}\cdot C_1$, but not isomorphic, see e.g. [10][14].

We have found representations $4096_x$ and $24_x$ of $G_{x_1}$. This leads to representations $98304_x = 4096_x \otimes 24_x$ and $300_x = 24_x \otimes \text{sym} 24_x$ of $G_{x_0}$. $G_{x_0}$ (as a extension of its normal subgroup $Q_x$) permutes the short elements $x_0$ of $Q_x$ via conjugation. Thus the representation $98280_x$ of $N_{x_0}$ can also be extended to a monomial representation of $G_{x_0}$, which we will also call $98280_x$. So we may build a representation $196884_x$ of $G_{x_0}$ from its components in the same way as in the case of $N_{x}$.

### 9.4 Construction of a $\xi \in G_{x_0} \setminus N_{x_0}$ and operation of $\xi$ on $98280_x$

It remains to construct a specific element $\xi \in G_{x_0} \setminus N_{x_0}$. By [5], $2^{11} : M_{24}$ is maximal in $C_{11}$, so $N_{x_0}$ is maximal in $G_{x_0}$, and hence any $\xi \in G_{x_0} \setminus N_{x_0}$ together with $N_{x_0}$ generates $G_{x_0}$.

$4096_x$ is also a faithful irreducible representation of $Q_x = 2^{1+24}$, so $\text{hom}(4096_x, 4096_x)$ is isomorphic to $\mathbb{R} Q^+_x$ as an algebra. Thus we may define $\xi$ as an element $\xi = \xi_{4096} \otimes \xi_{24}$ of $\mathbb{R} Q^+_x \otimes \text{hom}(24_x, 24_x)$. Conjugation of $Q_x$ with $\xi_{4096}$ may be computed by Lemma 9.1. The action of $\xi_{4096}$ determines the action of $\xi$ in $98280_x$, uniquely and in $24_x$ up to sign. We will construct a specific element $\xi_{4096}$ or order 3. Then we let $\xi_{24}$ be the unique element of order 3 corresponding to $\xi_{4096}$, i.e. $-\xi_{24}$ has order 6. In the sequel we will abbreviate $\xi_{4096}$ to $\xi$.

The decompositions $C = G \oplus H$ and $C^* = G^* \oplus H^*$ of $C$ and $C^*$ into grey and coloured subspaces discussed in section 2.2 is useful for describing the action of $\xi$.

Recall from section 2.2 that $G$ has a natural basis $g_0, \ldots, g_5$ and that $G^*$ has a natural basis $\gamma_0, \ldots, \gamma_5$. Let $w : G \cup G^* \to \mathbb{Z}$ be the weight of a vector in the corresponding natural basis, as in Definition 2.2. Let $w_2 : G \cup G^* \to \mathbb{F}_2$ as in (3.7.1). So $w_2(d) = (w(d) \mod 2)$. Let $\gamma : C \to G^*$ be as in (3.5.2). By Lemma 3.8, $w_2$ is a non-singular quadratic form on $G$ with associated bilinear form

$$\beta_{w_2} = \langle \ldots \rangle,$$

where $\langle d, e \rangle = \langle d, \gamma(e) \rangle = \langle e, \gamma(d) \rangle$.

By (3.5.3) the restriction of $\gamma$ to $G$ is an isomorphism $G \to G^*$ with $\gamma(g_i) = \gamma_i$. Thus the mapping $w_2 : G \cup G^* \to \mathbb{F}_2$ is also a non-singular quadratic form on $G^*$. We also define the bilinear form $\langle \ldots \rangle$ on $G^*$ by decreeing $\langle d, e \rangle = \beta_{w_2}(\delta, e)$.

For $d \in \mathcal{P}$ (or $d \in C$) let $\tilde{x}_d = x_{-1}^d, x_{0}^d$ be as in (5.3.1). Then $\{x_e | e \in G^*\}$ and $\{\tilde{x}_e | e \in G\}$ are elementary Abelian subgroups of $Q_x$ not containing the central element $x_{-1}$ of $Q_x$. In order to construct the three-bases element $\xi$ we put $\xi = \xi_2 \xi_0 \in \mathbb{R} G^\pm$ with

$$\xi_2 = 1/8 \sum_{e \in G^*} (-1)^{w_2(e)} x_e,$$

$$\xi_0 = 1/8 \sum_{e \in G} (-1)^{w_2(e)} \tilde{x}_e.$$  

By Lemma 3.8 and (9.2.1), $w_2$ is a non-singular quadratic form on $G$ and also on $G^*$. Thus by Lemma 9.1, $\xi_2$ and $\xi_0$ are involutions in the holomorph $\mathfrak{H}(Q_x)$, so they both normalize $Q_x$.

$\xi_2$ and $\xi_0$ are in $\mathfrak{H}(Q_x)$ but not in $G(4096_x)$. In section 9.5 we will show that the product $\xi = \xi_2 \xi_0$ is in $G(4096_x)$ as required. The relevant relations for $\xi_2$ and $\xi_0$ are given by:

**Lemma 9.4.**

$$\xi_2^2 = \xi_0^2 = (\xi_2 \xi_0)^3 = [\tilde{x}_e, \xi_2], [x_e, \xi_2] = 1, \quad e \in C, e \in C^*;$$

$$[\tilde{x}_h, \xi_2] = [x_\eta, \xi_0] = 1, \quad h \in H, \eta \in H^*;$$

$$\tilde{x}_d^\xi_2 = x_d^\xi_0 = (-1)^{w_2(d)} \tilde{x}_d x_d, \quad [\tilde{x}_d, x_d] = 1, \quad d \in G, \delta = \gamma(d) \in G^*.$$
Proof
We have already shown \( \xi_\gamma^2 = \xi_g^2 = 1 \). Put \( \tilde{x}_r = \tilde{x}_e \) for \( r = (e, c) \in C \times C^* \). We define a scalar product \( \langle \cdot, \cdot \rangle \) on \( (C \times C^*) \times (C \times C^*) \) by decreeing \( \langle (e, c), (f, \varphi) \rangle = \langle e, \varphi \rangle + \langle f, e \rangle \). Then (5.3.5) implies
\[
[\tilde{x}_r, \tilde{x}_s] = (-1)^{(r,s)}, \quad \text{for } r, s \in C \times C^*.
\]
Thus \( \tilde{x}_r, r \in H \times C^* \), commutes with every term \( \tilde{x}_s, s \in G^* \) in the sum that defines \( \xi_\gamma \). Hence \( \tilde{x}_r \) commutes with \( \xi_\gamma \). A similar argument shows that \( \tilde{x}_r, r \in C \times H^* \), commutes with \( \xi_g \).

Next we show the formula for \( \tilde{x}_d^{\xi_\gamma} \). Let \( \phi_\gamma : Q_x / \{ \pm 1 \} \to G^* \) be the linear mapping given by \( \phi_\gamma(\tilde{x}_g) = \gamma_n, \phi_\gamma(\tilde{x}_r) = 0, r \in H \times C^* \). Then
\[
[\tilde{x}_r, x_\delta] = (-1)^{(r,\delta)} = (-1)^{\langle \phi_\gamma(\tilde{x}_r), \delta \rangle)} \quad \text{for } r \in C \times C^*.
\]
Since \( \langle \cdot, \cdot \rangle \) is the bilinear form associated with \( w_2 \), the last equation and Lemma 9.1 imply:
\[
[\tilde{x}_d, \xi_\gamma] = (-1)^{w_2(\phi_\gamma(\tilde{x}_d))} x_{\phi_\gamma(\tilde{x}_d)} = (-1)^{w_2(d)} x_\delta.
\]
The proof of the formula for \( \tilde{x}_d^{\xi_g} \) is similar. Let \( \phi_g : Q_x / \{ \pm 1 \} \to G^* \) be the linear mapping given by \( \phi_g(\tilde{x}_g) = g_n, \phi_g(\tilde{x}_r) = 0, r \in C \times H^* \). Then
\[
[\tilde{x}_r, x_\delta] = (-1)^{(r,\delta)} = (-1)^{\langle \phi_g(\tilde{x}_r), \delta \rangle)} \quad \text{for } r \in C \times C^*.
\]
This equation together with Lemma 9.1 implies:
\[
[x_\delta, \xi_\gamma] = (-1)^{w_2(\phi_g(\tilde{x}_d))} x_{\phi_g(\tilde{x}_d)} = (-1)^{w_2(\delta)} x_d = (-1)^{w_2(d)} x_\delta.
\]
Since \( \langle \cdot, \cdot \rangle \) is associated with the quadratic form \( w_2 \) and hence alternating we have:
\[
[\tilde{x}_d, x_\delta] = (-1)^{(d,\delta)} = (-1)^{\langle d, \delta \rangle} = 1.
\]
It remains to show \( \langle \xi_\gamma, \xi_g \rangle = 1 \). This follows from \( \xi_\gamma^2 = \xi_g^2 = 1 \) and
\[
8\xi_\gamma^2 = \sum_{d \in G} (-1)^{w_2(\delta)} x_\delta = \sum_{d \in G} x_\delta x_\delta = \sum_{\delta \in G^*} (-1)^{w_2(\delta)} x_\delta^2 = 8\xi_g^2.
\]

By Lemma 9.4, \( \xi_\gamma \) operates on \( \tilde{x}_d, \xi_g \) operates on \( \tilde{x}_d, \xi_g \) operates on \( \tilde{x}_d, \xi_g \) operates on \( \tilde{x}_d \) as follows:
\[
\begin{align*}
\tilde{x}_d & \xrightarrow{\xi_\gamma} x_\gamma(\tilde{x}_d) & \xrightarrow{(-1)^{w_2(\delta)}} & \tilde{x}_d x_\gamma(\tilde{x}_d) & \xrightarrow{\xi_\gamma} & \tilde{x}_d
\end{align*}
\]
\[
\xi_g
\]
\[
\begin{align*}
\tilde{x}_d & \xrightarrow{\xi_\gamma} x_\gamma(\tilde{x}_d) & \xrightarrow{(-1)^{w_2(\delta)}} & \tilde{x}_d x_\gamma(\tilde{x}_d) & \xrightarrow{\xi_g} & x_\gamma(\tilde{x}_d)
\end{align*}
\]

Figure 2: Action of \( \xi_\gamma \) and \( \xi_g \) on \( \tilde{x}_d \in G \) and on \( x_\delta = \gamma(x_d) \in G^* \).

Lemma 9.5. Let \( d, e \in G, h \in \mathcal{P}_H, \eta \in H^* \). Put \( \delta = \gamma(d), \epsilon = \gamma(e) \). Then \( \xi \) operates by conjugation on \( Q_x \) as follows:
\[
\begin{align*}
\tilde{x}_d \tilde{x}_h x_\epsilon x_\eta & \xrightarrow{\xi_\gamma} (-1)^{w_2(\epsilon)} \tilde{x}_d \tilde{x}_h x_\delta x_\eta \\
\tilde{x}_d \tilde{x}_h x_e x_\eta & \xrightarrow{\xi_g} (-1)^{w_2(\delta)} \tilde{x}_d \tilde{x}_h x_\delta x_\eta \\
x_\delta x_\gamma(h) & \xrightarrow{\xi_\gamma} x_\delta x_\gamma(h)
\end{align*}
\]
We define \( \Lambda \) and \( E \). Then \( \Lambda \) vectors entries 2 is divisible by 4. Thus there is a vector \( \xi \) column with equal entries divisible by 4 or two columns with equal entries in each column of the MOG. Hence \( \theta \) may assume \( \theta(h) = \gamma(h) \) and hence \( \tilde{x}_h = x_h x_{\gamma(h)} \). Then \( \xi \) also gives us the operation of \( \xi \) and \( \xi^2 \) on 98280.

### 9.5 The operation of \( \xi \) on 24

We also define a linear transformation \( \xi_{24} = \xi_{24a} \xi_{24b} \) on 24, where \( \xi_{24a} \) and \( \xi_{24b} \) are 24 \times 24 matrices operating on 24, by right multiplication. Matrices \( \xi_{24a} \) and \( \xi_{24b} \) consist of six identical \( 4 \times 4 \)-blocks \( \xi_{4a} \) and \( \xi_{4b} \), respectively, where each of the \( 4 \times 4 \)-blocks transforms the four basis vectors of \( 24 \) labelled by a column of the MOG, as given in (2.11). We put:

\[
\xi_{4a} = \frac{1}{2} \begin{pmatrix}
1 & -1 & -1 & -1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1
\end{pmatrix}, \quad \xi_{4b} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

(9.5.4)

We define \( \Lambda^E = \{ \lambda \in \Lambda \mid \langle \lambda_{ij}, \lambda \rangle = \langle \lambda, \omega \rangle = 0 \mod 2 \} \), with \( \Omega \in \mathcal{G}, \omega \in \mathcal{G}^* \) as in section 2. Then \( \Lambda^E \) is a sublattice of \( \Lambda \) of index 4.

\( \xi_{4a} \) and \( \xi_{4b} \) are orthogonal by construction. Thus \( \xi_{24a} \xi_{24b} \) and hence also \( \xi_{24} \) has order 3. Alternatively, we may check that matrix \( \xi_{4a} \xi_{4b} \) has trace 1 and eigenvectors \((0,1,0,0)\) and \((1,0,0,0)\) with eigenvalue 1. So orthogonality forces the other eigenvalues to \(-\frac{1\pm\sqrt{-3}}{2}\) and hence order 3.

**Lemma 9.6.** \( \Lambda^E \) is invariant under \( \xi_{24a} \) and \( \xi_{24b} \).

**Proof**

Using the isomorphism in Theorem 6.1 it is easy to see that \( \Lambda^E \) is generated by the vectors

\[
\lambda^E_d : \begin{cases} 
4_{on} i, \pm 4_{on} j, 0 \, \text{else}, & i, j \in \tilde{\Omega}, \\
2_{on} d, 0 \, \text{else}, & d \in C, \langle d, \omega \rangle \, \text{even}.
\end{cases}
\]

Here the condition \( \langle d, \omega \rangle = 0 \mod 2 \) implies that \( \lambda^E_d \) has an even number of entries 2 in each column and also in row 0 of the MOG.

The operation of \( \xi_{24b} \) is negation of row 0 in the MOG. Thus \( \lambda^E_{ij} - \xi_{24b}(\lambda^E_{ij}) \) has entries divisible by 8 and \( \lambda^E_d - \xi_{24b}(\lambda^E_d) \) has an even number of entries divisible by 4 in row 0 of the MOG and zeros elsewhere. All these differences are in \( \Lambda^E \), so \( \Lambda^E \) is invariant under \( \xi_{24b} \).

Matrix \( 1 - \xi_{4a} \) contains entry \( \frac{1}{2} \) everywhere, so the operation of \( 1 - \xi_{4a} \) may be described as follows: For each column of the MOG calculate half the sum of its entries and write the result into each entry of that column. Performing this operation on \( \lambda^E_{ij} \) we obtain either one column with equal entries divisible by 4 or two columns with equal entries \pm 2. These results are all in \( \Lambda^E \).

Any \( \lambda^E_d \) has an even number of entries 2 in each MOG column and the total number of entries 2 is divisible by 4. Thus there is a vector \( e \in \Lambda^E \) with an even number of nonzero entries, which are all equal to 4, such that \( \lambda^E_d - e \) has the same number of entries 2 and \( -2 \) in each column of the MOG. Hence \( \lambda^E_d - e \) is invariant under \( \xi_{4a} \). We have already shown \( \xi_{4a}(e) \in \Lambda^E \), thus \( \xi_{24a}(\lambda^E_d) \in \Lambda^E \). Hence \( \Lambda^E \) is also invariant under \( \xi_{24a} \).
By (2.2.1) and Theorem 6.1 we have the following Leech lattice vectors in MOG coordinates:

\[
\begin{align*}
\lambda_{g_0} &= \begin{bmatrix} 0 & 2 & 2 & 2 & 2 \\
2 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0
\end{bmatrix}, \quad \lambda_{\gamma_0} &= \begin{bmatrix} -3 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix},
\end{align*}
\]

where \(g_0, \ldots, g_5\) is the natural basis of \(G\) and \(\gamma_0, \ldots, \gamma_5\) is the natural basis of \(G\). \(g_n\) and \(\gamma_n\) are obtained from \(g_0\) and \(\gamma_0\) by exchanging MOG column 0 with column \(n\). We have:

\[
\begin{pmatrix}
-3 & 3 & 0 & 1 & 1 & -2 \\
1 & 1 & -2 & 1 & -1 & 0 \\
1 & 1 & -2 & 1 & -1 & 0 \\
1 & 1 & -2 & 1 & -1 & 0
\end{pmatrix}^\top \cdot \xi_{4a} \cdot \xi_{4b} = \begin{pmatrix}
3 & 0 & -3 & 1 & -2 & 1 \\
1 & -2 & 1 & -1 & 0 & 1 \\
1 & -2 & 1 & -1 & 0 & 1 \\
1 & -2 & 1 & -1 & 0 & 1
\end{pmatrix}^\top,
\]

and hence

\[
\lambda_{\gamma_i} \cdot \xi_{24} \cdot \lambda_{g_i} = -\lambda_{\gamma_i} \cdot \xi_{24} \cdot \lambda_{g_i} = \xi_{24} \cdot \lambda_{\gamma_i} - \xi_{24} \cdot \lambda_{\gamma_i} - \xi_{24} \cdot \lambda_{\gamma_i} - \xi_{24} \cdot \lambda_{\gamma_i}.
\]

Since \(\Lambda\) is generated by \(\Lambda^E\), \(g_0\) and \(\gamma_0\), Lemma 9.6 and (9.7.2) imply:

**Corollary 9.8.** The Leech lattice \(\Lambda\) is invariant under \(\xi_{24}\).

**Lemma 9.9.** The isomorphism \(Q_x/Z(Q_x) \rightarrow \Lambda/2\Lambda\) given by Theorem 6.1 maps conjugation with \(\xi\) on \(Q_x\) to the automorphism \(\xi_{24}\) of the Leech lattice \(\Lambda\) (modulo \(2\Lambda\)).

**Proof.** We also write \(g_i\) for the preimage of \(g_i\) in \(P\) with positive sign. We have \(\theta(g_i) = 0\) by Lemma 5.3.9 and hence \(\tilde{x}_{g_i} = x_{g_i}\) by (5.3.3). So by (9.4.1), \(\xi\) operates on \(x_{g_i}\) and \(x_{\gamma_i}\) by conjugation as follows:

\[
\begin{align*}
\lambda_{\gamma_i} &\mapsto \lambda_{g_i} - \lambda_{\gamma_i}, \\
\xi_{24} &\mapsto -\xi_{24}, \\
\xi_{24} &\mapsto -\xi_{24}, \\
\lambda_{\gamma_i} &\mapsto \lambda_{\gamma_i}.
\end{align*}
\]

Comparing this operation of \(\xi\) with the operation of \(\xi_{24}\) on \(\lambda_{g_i}\) and \(\lambda_{\gamma_i}\) given by (9.7.2), we see that these operations are compatible. So by linearity the operation of \(\xi_{24}\) on \(\lambda_{r}\) (modulo \(2\Lambda\)) is the compatible with the operation of \(\xi\) of \(x_r\) on \(Q_x\) (modulo the center of \(Q_x\)) for all \(r\) in \(G \oplus G^*\).

By Lemma 2.3 the space \(H^*\) is generated by vectors \(ij \in F_{24}^2\), where \(i\) and \(j\) are in the same column of the MOG, and not in row 0. By Theorem 6.1, the vector \(l_{ij}\) has entries 4 and -4 in the corresponding positions. Thus \(\lambda_{ij}\) is not changed by \(\xi_{24a}\) or by \(\xi_{24b}\). By Lemma 2.4 conjugation with \(\xi_k\) or \(\xi_g\) does not change \(x_{ij}\). So by linearity the operations of \(\xi_{24}\) on \(\lambda_{ij}\) and of \(\xi\) on \(x_{ij}\) are trivial for \(\delta \in H^*\) and hence compatible. Hence the operations of \(\xi\) on \(x_r\) and of \(\xi_{24}\) on \(\lambda_r\) are compatible for all \(r \in G \oplus G^*\).

For any \(h \in P_H\) of weight 8 let \(\tilde{x}_h = x_{h}x_{\gamma(h)}\). Then \(\tilde{x}_h\) is invariant under \(\xi\) by Lemma 9.5. Let \(\hat{\lambda}_h \in \Lambda\) be any representative of the image of \(\tilde{x}_h\) under the mapping \(Q_x \rightarrow \Lambda/2\Lambda\) given by Theorem 6.1. Since the elements \(\tilde{x}_h, h \in P_H, |h| = 8\) and \(x_r, r \in P_G \oplus G^*\) generate \(Q_x\), it suffices to show \(\xi_{24}(\hat{\lambda}_h) \in \hat{\lambda}_h + 2\Lambda\). (Note that 45 of the 64 elements of \(H\) have weight 8; so these elements generate \(H\).)

By Theorem 6.1 for every \(h \in P_H\) there is a representative \(\lambda_h\) of the image of \(x_h\) with precisely no or two nonzero entries of value 2 in each column of the MOG and zeros in row 0. There is also a representative \(\lambda_{\gamma(h)}\) of the image of \(x_{\gamma(h)}\) with entries 4 in row 0 in the columns where \(\lambda_h\) does no vanish, and zero entries elsewhere. Thus \(\hat{\lambda}_h = \lambda_h - \lambda_{\gamma(h)}\) is a representative of the image of \(\tilde{x}_h\).

For \(\hat{\lambda}_h\) in each column of the MOG the sum of the entries is zero. Thus \(\hat{\lambda}_h\) is invariant under \(\xi_{24a}\). Since \(\hat{\lambda}_h\) has zeros on row 0 of the MOG and \(\lambda_{\gamma(h)}\) has nonzero entries in row 0 of the MOG only, \(\xi_{24b}\) maps \(\hat{\lambda}_h = \lambda_h - \lambda_{\gamma(h)}\) to \(\lambda_h + \lambda_{\gamma(h)} = \hat{\lambda}_h + 2\lambda_{\gamma(h)}\).

An immediate consequence of Corollary 9.8 and Lemma 9.9 is:
Theorem 9.10. \( \xi \in G(4096_2), \xi_{24} \in G(24_2), \xi \otimes \xi_{24} \in G_{20}. \)

Using \([3.24, 31]\) and Theorem \([3.10]\) and identifying the basis vector \(i_\xi\) of \(24_2\) with the position of entry \(i\) in the MOG, we obtain the following action of \(\xi\) and \(\xi^2\) on the space \(24_2\):

Corollary 9.11. For the \(n\)-th column vector \(c_n\) in the MOG we have:

\[
\begin{align*}
C_n \xrightarrow{\frac{1}{2}c_n^T} \left( \begin{array}{cccc}
-1 & -1 & -1 & -1 \\
1 & 1 & -1 & -1 \\
-1 & -1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{array} \right) \quad \text{and} \quad \xi \xrightarrow{\frac{1}{2}c_n^T} \left( \begin{array}{cccc}
1 & 1 & 1 & 1 \\
-1 & 1 & -1 & -1 \\
-1 & -1 & -1 & -1 \\
-1 & -1 & -1 & 1
\end{array} \right).
\end{align*}
\]

9.6 The operation of \(\xi\) on \(4096_2\)

Let \(\xi_\gamma\) and \(\xi_\varphi\) be as in \([3.31]\). We will now derive the operation of \(\xi = \xi_\gamma \xi_\varphi\) on \(4096_2\).

Definition 9.12.

- 9.12.1) Let \(g_0, \ldots, g_5\) be the standard basis of the grey subspace \(G\) of the Golay code as in section 2.2. We also write \(g_i\) for the preimage of \(g_i\) in \(P\) with positive sign.

- 9.12.2) Let \(P_0^G = \left\{ \left( \prod_{i=0}^{3} g_i^{\alpha_i}, 0 \right) \mid \sum_{i=0}^{3} \alpha_i \equiv 0 \pmod{2} \right\} \).

- 9.12.3) For \(d \in G\) and \(\sigma \in \mathbb{Z}\) let \(d^{\sigma} = d^{\dagger}\) if \(\sigma\) is even and \(d^{\sigma} = d^{\dagger}\) if \(\sigma\) is odd.

Then \(P_0^G\) is a subset of \(P_G\) with 16 even elements. Any element \(e\) of \(P_0^G\) has a unique decomposition \(e = \pm \Omega^\sigma g_0^\xi d, \; d \in P_0^G, \; \sigma, \kappa \in \{0, 1\}\). For the unit vectors \(f_i^{\sigma}\) in \(4096_2\), \(f \in P\) we have \(\Omega f_i^{\sigma} = (-1)^\sigma f_i^{\sigma}\). It is easy to see that we have:

Lemma 9.13. The vectors \((g_0^\xi dh)_i^{\sigma}\), \(\sigma, \kappa = 0, 1, d \in P_0^G, \; h \in P_H\) form a basis of \(4096_2\), with \(w(g_0^\xi d) = \kappa \pmod{2}\), \(w_2(g_0^\xi d) = w_2(d)\), and \(A(g_0^\xi d, h) = \theta(g_0^\xi d) = \theta(g_0, h) = \theta(d, h) = 0\).

So the decomposition of a basis vector of \(4096_2\) in Lemma 9.13 is easy to compute and independent of the association of the factors given in the Lemma.

Lemma 9.14. Let \(d \in P_0^G\) and \(h \in P_H\). Then \(\xi_\gamma\) maps \((g_0^\xi dh)_i^{\sigma}\) to \((-1)^{w_2(d)+1}(g_0^\xi dh)_i^{\sigma+\kappa+1}\).

Proof

For any \(f \in P\) put \(f' = \frac{1}{2}f_1^{\dagger} + \frac{1}{2}f_1^{-}\). From \((7.1.1)\) we obtain:

\[
(\sigma f) = f' + \langle \Omega f, \sigma f \rangle, \quad (\sigma f) = f' - \langle \Omega f, \sigma f \rangle, \quad \text{and hence} \quad f_i^{\sigma} = f' + (-1)^\sigma(\langle \Omega f, \sigma f \rangle) \quad (9.14.1)
\]

Let \(e \in G, \; \varphi \in G^*\). From Table \[X\] we see that \(x_\varphi\) maps \((eh)'\) to \((-1)^{\langle eh, \varphi \rangle}(eh)'\). We have \(\langle h, \varphi \rangle = 0\). So by \([3.31]\) and Lemma 9.13 we obtain:

\[
(\sigma eh)' \xrightarrow{\xi_\gamma} \frac{1}{8}(eh)'(S(e), \sigma e) = \sum_{\varphi \in G^*} (-1)^{w_2(\varphi) + \langle e, \varphi \rangle} = \sum_{f \in G} (-1)^{w_2(f) + \langle e, f \rangle}.
\]

By Lemma 3.8 we have \(w_2(f) + \langle e, f \rangle = w_2(ef) + w_2(e)\) and hence:

\[-1)^{w_2(e)}S(e) = \sum_{f \in G} (-1)^{w_2(ef)} = \sum_{f \in G} (-1)^{w_2(f)} = (6)_m \left( -1 \right)^{m(m-1)/2} = -8.
\]

This proves:

\[(eh)' \xrightarrow{\xi_\gamma} (-1)^{w_2(e)+1}(eh)' \quad (9.14.2)
\]
We have $w_2(\Omega e) = w_2(e) + w(e) + 1$ by (3.7.2) and hence:

\[(\Omega e) h', f \xrightarrow{\xi} (-1)^{w_2(e) + w(e)} (\Omega e) h'. \tag{9.14.3}
\]

From 9.14.1, 9.14.2 and 9.14.3 we obtain:

\[(e h)_1^{[\sigma]} \xrightarrow{\xi} (-1)^{w(e) + 1} (e h)_1^{[\sigma + 1]}, \quad \text{where } \kappa \text{ is the parity of } e. \]

Putting $e = g_0^* d$, the lemma now follows from Lemma 9.13.

The following Lemma states the operation of $\xi_g$ on these basis vectors:

**Lemma 9.15.** Let $d \in \mathcal{P}_G^0$ and $h \in \mathcal{P}_H$. Then $\xi_g$ maps $(g_0^* dh)^{[\sigma]}$ to

\[
\frac{1}{d} \sum_{e \in \mathcal{P}_G^0} (-1)^{w_2(de)} (g_0^* + \sigma + 1) h_1^{[\sigma]}.
\]

**Proof**

Let $\mathcal{P}_G^+ = \{ (f, 0) \mid f \in G \}$. Then $\mathcal{P}_G^0 \subset \mathcal{P}_G^+ \subset \mathcal{P}_G$. By (9.3.1) and (5.3.3) we have:

\[\xi_g = \frac{1}{d} \sum_{e \in \mathcal{P}_G^+} (-1)^{w_2(fe)} \bar{x}_f \bar{x}_e, \quad \text{for any } f \in \mathcal{P}_G^+.
\]

We have $\langle h, \theta(f) \rangle = \langle h, \theta(f) \rangle = 0$ by Lemma 5.3.9. Using Table 3 and $e^2 = (-1)^{w(e)}$, we obtain:

\[\langle fh \rangle_1^{[\sigma]} \xrightarrow{x_f} h_1^{[\sigma]} \xrightarrow{x_e} \langle \varepsilon h \rangle_1^{[\sigma]} \xrightarrow{x_{\varepsilon e}} (-1)^{w(\varepsilon e)} (\varepsilon h)_1^{[\sigma]} = (eh)_1^{[\sigma]}.
\]

Thus $\bar{x}_f \bar{x}_e$ maps $(fh)_1^{[\sigma]}$ to $(eh)_1^{[\sigma]}$ and we obtain:

\[\langle fh \rangle_1^{[\sigma]} \xrightarrow{\xi_g} \frac{1}{d} \sum_{e \in \mathcal{P}_G^+} (-1)^{w_2(fe)} (eh)_1^{[\sigma]}.
\]

(9.15.1)

To finish the proof, we put $f = g_0^* d$.

Case $\sigma = 1$

If $fe$ is odd then $w_2(fed) = w_2(fe)$ by (3.7.2), and we have $(eh)_1^- = - (\Omega e) h_1^- \bysame$ (7.1.1), so that the terms for $e \in \Omega e$ in the sum (9.15.1) cancel. If $fe$ is even, these two terms are equal. The set $\mathcal{P}_G^0$ contains exactly one of the two elements $d \in \mathcal{P}_G^0$ for each even $d \in \mathcal{P}_G^0$, so that summing (9.15.1) over the even elements of $\mathcal{P}_G^+$ proves the Lemma. We remark that $w_2(g_0 de) = w_2(de)$ by Lemma 9.13 for $d, e \in \mathcal{P}_G^0$.

Case $\sigma = 0$

If $fe$ is even then $w_2(fed) = w_2(fe)$ by (3.7.2), and we have $(eh)_1^+ = (\Omega e) h_1^+ \bysame$ (7.1.1), so that the terms for $e \in \Omega e$ in the sum (9.15.1) cancel. If $fe$ is odd, these two terms are equal. Now the same argument as in case $\sigma = 0$ shows that summing over the odd elements of $\mathcal{P}_G^0$ in (9.15.1) proves the Lemma.

We have $\xi = \xi_\gamma \xi_g$. Combining Lemmas 9.14 and 9.15 and using Lemma 3.3, we obtain:

**Corollary 9.16.** Let $d \in \mathcal{P}_G^0, h \in \mathcal{P}_H$. Then:

\[\langle g_0^* dh \rangle_1^{[\sigma]} \xrightarrow{\xi_g} \frac{1}{d} \sum_{e \in \mathcal{P}_G^0} (-1)^{w_2(de) + w(e) + 1} (g_0^* e h)_1^{[\sigma + \gamma + 1]},
\]

\[\langle g_0^* dh \rangle_1^{[\sigma]} \xrightarrow{\xi_g} \frac{1}{d} \sum_{e \in \mathcal{P}_G^0} (-1)^{w_2(de) + w_2(d) + 1} (g_0^* + \sigma + 1 e h)_1^{[\sigma]}.
\]
The following facts are easy to show and of some practical value for the implementation.

The set $G^0 = \{ \bar{e} \mid e \in P^0 \}$ is a 4-dimensional subspace of $G$. Any four different elements $b_i, i = 1, \ldots, 4, \in G^0$ with $w(b_i) = 4$ form a basis of $G^0$ with $\langle\langle b_i, b_j \rangle\rangle = \delta_{i,j}$. If $d$ is a sum of $k$ different basis vectors $b_i$ then $w_2(d) = \binom{k}{2} \pmod{2}$. Thus a suitable basis of the grey part $\mathcal{G}$ of the Golay code is $(g_0, b_1, b_2, b_3, b_4, \Omega)$. 
## Notation

| Symbol | Description | Section |
|--------|-------------|---------|
| $a, b, c, d, e, f, h$ | Elements of the Parker loop $\mathcal{P}$ or of the Golay code $\mathcal{C}$ | 3.1 |
| $A(d, e, f)$ | Associator map of the elements $d, e, f$ of the Parker loop $\mathcal{P}$ | 3.1 |
| $\text{Aut}_{S_4} \langle \mathcal{P} \rangle$ | The group of standard automorphisms of the Parker loop $\mathcal{P}$ | 0 |
| $\mathcal{C}(d, e)$ | Commutator map of the elements $d$ and $e$ of the Parker loop $\mathcal{P}$ | 3.1 |
| $C, \mathcal{C}^*$ | $C$ is the 12-dimensional Golay code in $\mathbb{F}_2^{24}$, $\mathcal{C}^*$ its cocode $\mathbb{F}_2^{24}/\mathcal{C}$ | 2.1 |
| $\delta, \varepsilon, \varphi, \eta$ | Elements of the Golay cocode $\mathcal{C}^*$ | 3.1 |
| $g_0, \ldots, g_5$ | Standard basis of the "grey" subspace $\mathcal{G}$ of the Golay code $\mathcal{C}$; $g_i, i = 0, \ldots, 5$, is also considered as the element $(g_i, 0)$ of $\mathcal{P}_g$, $\mathcal{G}, \mathcal{G}^*$ | 2.2 |
| $\mathcal{G}, \mathcal{G}^*$ | The subspaces of the "grey" elements of $\mathcal{C}$ and $\mathcal{C}^*$, respectively | 2.2 |
| $G_{x0}$ | A maximal subgroup of $\mathcal{M}$ of structure $2^{1+24}: \text{Co}_1$ | 5 |
| $\gamma$ | A specific function | 3.3 |
| $\mathcal{G}^*$ | $\mathcal{G}^*$ | 3.1 |
| $\gamma_0, \ldots, \gamma_5$ | Standard basis of the "grey" subspace $\mathcal{G}^*$ of the Golay cocode $\mathcal{C}^*$ | 2.2 |
| $\mathcal{H}, \mathcal{H}^*$ | The subspace of the "coloured" elements of $\mathcal{C}$ and $\mathcal{C}^*$, respectively | 2.2 |
| $i, j, k$ | Elements of $\Omega$, also considered as elements of $\mathcal{C}^*$ of weight 1 | 3.1 |
| $ij$ | Shorthand for $i \cup j$, considered as an element of $\mathcal{C}^*$ of weight 2 | 3.1 |
| $\Lambda$ | The 24-dimensional Leech lattice. | 6.1 |
| $\mathcal{M}$ | The monster group, i.e. the largest sporadic simple group | 1 |
| $M_{24}$ | Mathieu group, acts on $\Omega$ as the automorphism group of $\mathcal{C}$. | 2.1 |
| MOG | Miracle Octad Generator; a tool for calculations in $\mathcal{C} \subset \mathbb{F}_2^{24}$. | 2.1 |
| $N$ | A fourfold cover of the maximal subgroup $N_0$ of $\mathcal{M}$ | 5 |
| $N_0$ | A maximal subgroup of $\mathcal{M}$ of structure $2^{2+1+2d}.(M_{24} \times S_3)$ | 5 |
| $N_{x0}$ | Subgroup of structure of $\mathcal{M}$ with $G_{x0} \cap N_0 = N_{x0}$ | 5 |
| $\Omega$ | The element $(\Omega, 0)$ of the Parker loop $\mathcal{P}$ | 2.1 |
| $\tilde{\Omega}$ | A set of size 24 used for labeling the basis vectors of $\mathbb{F}_2^{24}$, its power set $2^{\tilde{\Omega}}$ is identified with $\mathbb{F}_2^{24}$, and we have $\mathcal{C} \subset \mathbb{F}_2^{24}$ | 2.1 |
| $\omega$ | A specific "grey" element in the subset $\mathcal{G}^*$ of the cocode $\mathcal{C}^*$ | 2.2 |
| $\mathcal{P}(d)$ | The squaring map in $\mathcal{P}$, with $d^2 = (0, \mathcal{P}(d))$ for $d \in \mathcal{P}$ | 3.1 |
| $\mathcal{P}$ | The Parker loop, any $d \in \mathcal{P}$ has the form $(\bar{d}, \mu)$, $\bar{d} \in \mathcal{C}, \mu \in \mathbb{F}_2$ | 3.1 |
| $\mathcal{P}_G, \mathcal{P}_H$ | Subsets of $\mathcal{P}$: origins of $\mathcal{G}$ and $\mathcal{H}$ of the mapping $\gamma : \mathcal{P} \to \mathcal{C}$. | 3.3 |
| $\pi, \pi', \pi''$ | Standard automorphisms of the Parker Loop $\mathcal{P}$ in $\text{Aut}_{S_4} \mathcal{P}$ | 1 |
| $Q_{x0}$ | Subgroup of structure $2^{1+2d}.\text{Co}_1$ of the group $G_{x0}$ | 0 |
| $\text{sign}(d)$ | "Sign" of an element $d$ of the Parker loop $\mathcal{P}$ | 3.1 |
| $\theta$ | Cocycle of Parker loop $\mathcal{P}$, with $(d, 0) \cdot (\bar{e}, 0) = (\bar{d} + \bar{e}, \theta(d, e))$ | 3.2 |
| $w(d)$ | Weight of vector $d \in \mathcal{G}$ with respect to the basis $g_0, \ldots, g_5$ | 2.2 |
| $\bar{w}(\delta)$ | Weight of vector $\delta \in \mathcal{G}$ with respect to the basis $\gamma_0, \ldots, \gamma_5$ | 2.2 |
| $w_2(d), w_2(\delta)$ | Equal to $(w(d)), (w(\delta))$ modulo 2, for $d \in \mathcal{C}, \delta \in \mathcal{C}^*$ | 3.2 |
| $Z(\mathcal{G})$ | The center of a group or a loop $\mathcal{G}$ | 1 |
| $d$ | Image of $d \in \mathcal{P}$ in $\mathcal{C}$ under the natural homomorphism $\mathcal{P} \to \mathcal{C}$ | 3.1 |
| $\bar{x}_d$ | The element $x_{d-1}^{\text{sign}(d)} x_d x_{\theta(d)}$ of the group $Q_{x0}$, for $d \in \mathcal{P}$ or $d \in \mathcal{C}$ | 5 |
| $\bar{d}$ | Inverse of $d$ in the Parker loop $\mathcal{P}$; $\bar{d} = d^{-1}$ | 3.1 |
| $|d|, |\delta|$ | Weight of code word $d \in \mathcal{C}$, min. weight of cocode word $\delta \in \mathcal{C}^*$ | 2.1 |
| $\langle d, \varepsilon \rangle$ | The scalar product, e.g. on $\mathcal{C} \times \mathcal{C}^*$ | 2.1 |
| $\langle(d, e)\rangle$ | equal to $\langle d, \gamma(e) \rangle$ and to $w_2(de) + w_2(d) + w_2(e)$ for $d, e \in \mathcal{G}$ | 3.2 |
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