G\textsubscript{2}–instantons over Kovalev manifolds II

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Abstract

This is the first nontrivial construction to date of instantons over a compact manifold with Hol = G\textsubscript{2}. The HYM connections on asymptotically stable bundles over Kovalev’s noncompact Calabi-Yau 3-folds, obtained in the first article [SaE\textsubscript{1}], are glued compatibly with a twisted connected sum, to produce a G\textsubscript{2}–instanton over the resulting compact 7–manifold [Kov\textsubscript{1}, Kov\textsubscript{2}]. This is accomplished under a nondegeneracy acyclic assumption on the bundle ‘at infinity’, which occurs e.g. over certain projective varieties X\textsubscript{22} \hookrightarrow \mathbb{C}P\textsuperscript{13} [Isk, Muk\textsubscript{2}] equipped with an asymptotically rigid bundle.

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Introduction

The first paper [SaE] was devoted to the Hermitian Yang-Mills (HYM) problem over A. Kovalev’s noncompact Calabi-Yau 3-folds $W$ with an exponentially asymptotically cylindrical (EAC) end. Its main theorem asserts the existence of such a metric on holomorphic bundles $E \to W$ which are asymptotically stable, i.e., (slope-)stable over the divisor ‘at infinity’ $D \subset W$. Moreover, solutions have $C^\infty$—exponential decay’ along the tubular end to a reference metric, which extends the instanton metric on $E|_D$ in a prescribed way. On the other hand, it is known that a HYM connection $A$ on a holomorphic vector bundle over a Calabi-Yau 3—fold $(W, \omega, \Omega)$ lifts to a $G_2$—instanton on the pull-back bundle $p^*_1 E \to M = W \times S^1$, hence the above result yields a non-trivial solution of the corresponding instanton equation:

$$F_A \wedge \ast \varphi = \frac{1}{2} F_A \wedge (\omega \wedge \omega - 2 \text{Re} \Omega \wedge d\theta) = 0 \quad (1)$$

(where $d\theta$ is the coordinate 1-form on $S^1$ and $\varphi$ is the induced $G_2$—structure).

This paper will be concerned with the natural sequel, the gluing of two such solutions according to Kovalev’s twisted connected sum of the base manifolds. That construction joins two ‘truncated’ EAC products $W_S^{(i)} \times S^1$ in order to obtain a smooth compact $G_2$—manifold

$$M_S = (W_S^{(i)} \times S^1) \cup_{F_S} (W_S^{(i)} \times S^1) \cong W' \#_S W'',$$

where $S$ is the ‘neck-length’ parameter and $F_S$ is the product of a hyper-Kähler rotation on the divisors ‘near infinity’ and a nontrivial identification of the circle components of the boundary. In view of the exponential decay property of solutions over each end, one is led to expect that residual self-dual curvature, i.e., error terms from truncation in the corresponding $G_2$—instanton equation over $M_S$, can be dealt with by a perturbative, ‘stretch-the-neck’-type argument.

This paper’s main result [Theorem 4, Subsection 1.2] posits that this is indeed the case, albeit under a rigidity assumption on the ‘bundle at infinity’. In order to guarantee a right-inverse for the linearisation of the self-dual curvature operator $A \mapsto F_A \wedge \ast \varphi$ around a solution, one requires that the corresponding deformation complex over $E|_D$ be acyclic. In other words, the instanton ‘at infinity’ must be an isolated point in its moduli space. Examples satisfying this hypothesis are obtained from certain base manifolds $X_22 \hookrightarrow \mathbb{C}P^{13}$ admissible by Kovalev’s construction and studied by Iskovskih [Isk] and Mukai [Muk2].

Readers familiar with the 4-dimensional model outlined by Donaldson in [Don1] will find that this paper mimics that source in all its essential aspects.

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1  \(G_2\)–instantons over EAC manifolds

Let us briefly recall the basic facts about the Calabi-Yau 3-folds \(W_i\) that one intends to glue together, via a nontrivial product with \(S^1\). A base manifold for our purposes is a compact, simply-connected Kähler 3-fold \((\bar{W}, \bar{\omega})\) such that the complement \(W = \bar{W} \setminus D\) has \(\pi_1(W)\) finite. Topologically \(W\) looks like a compact manifold \(W_0\) with boundary \(D \times S^1\) and a cylindrical end attached:

\[
W = W_0 \cup W_\infty \\
W_\infty \simeq D \times S^3 \times (\mathbb{R}^+)\,.
\] (2)

The \(K3\) divisor \(D\) is actually hyper-Kähler, with complex structure \(I\) inherited from \(\bar{W}\) and the additional structures \(J\) and \(K = IJ\) satisfying the quaternionic relations; denote by \(\kappa_I, \kappa_J\) and \(\kappa_K\) their associated Kähler forms. Then \([\text{Kov1}]\) Theorem 2.2 \(W\) admits a complete Calabi-Yau structure \(\omega\) and holomorphic volume form \(\Omega\) exponentially asymptotic to the cylindrical model

\[
\begin{align*}
\omega_\infty &= \kappa_I + ds \wedge d\alpha \\
\Omega_\infty &= (ds + i\alpha) \wedge (\kappa_J + i\kappa_K)
\end{align*}
\] (3)

in the sense that

\[
\omega|_{W_\infty} = \omega_\infty + d\psi, \quad \Omega|_{W_\infty} = \Omega_\infty + d\Psi,
\]

where \(\psi\) and \(\Psi\) are smooth and decay exponentially in all derivatives along the tubular end.

As to the gauge-theoretic initial data \([\text{SaE1}]\), let \(z = e^{-s+i\alpha}\) be the holomorphic coordinate along the tube and denote \(D_z\) the corresponding \(K3\) component of the boundary. A bundle \(\mathcal{E} \to W\) is called asymptotically stable (or stable at infinity) if it is the restriction of an indecomposable holomorphic vector bundle \(\mathcal{E}|_{\bar{W}}\) such that \(\mathcal{E}|_{D_z}\) is stable (hence also \(\mathcal{E}|_{D_{\bar{z}}}\) for \(|z| < \delta\)). Moreover, a reference metric \(H_0\) on such \(\mathcal{E} \to W\) is (the restriction of) a smooth Hermitian metric on \(\mathcal{E} \to \bar{W}\) such that \(H_0|_{D_z}\) are the corresponding HYM metrics on \(\mathcal{E}|_{D_z}\), \(0 \leq |z| < \delta\), and \(H_0\) has finite energy: \(\|\hat{F}_{H_0}\|_{L^2(W, \omega)} < \infty\).

Then, given an asymptotically stable bundle with reference metric \((\mathcal{E}, H_0)\), a nontrivial smooth \(G_2\)–instanton on \(p^*\mathcal{E} \to W \times S^1\) is obtained from every solution of the HYM problem over \(W\). Moreover, such solutions have the property of exponential asymptotic decay in all derivatives to \(H_0\) along the tubular end \([\text{ibid.}, \text{Theorem 59}]\):

\[
\begin{align*}
\hat{F}_H &= 0, \\
H &\xrightarrow{C^\infty} H_0.
\end{align*}
\] (4)

Here convergence takes place over cylindrical bands of fixed ‘size’.
Notation 1 Let \( Q \to W \) be a bundle equipped with a fibrewise metric and denote \( W_S \) the truncation of \( W \) at ‘neck length’ \( S \); given \( S > r > 0 \), write \( \Sigma_r(S) \) for the interior of the cylinder \((W_{S+r} \setminus W_{S-r})\) of ‘length’ \( 2r \). We denote the \( C^k \)-exponential tubular limit of an element in \( C^k(\Gamma(Q)) \) by:
\[
\phi \xrightarrow{C^k} \phi_0 \Leftrightarrow \|\phi - \phi_0\|_{C^k(\Sigma_1(S), \omega)} = O(e^{-S}).
\]

Finally, let us fix some vocabulary towards the statement of the main theorem. Denote \( A_0 \) the Chern connection of \( H_0 \); then by definition each \( A_0|_D \) is ASD. In particular, \( A_0|_D \) induces an elliptic deformation complex
\[
\Omega^0(\mathfrak{g}|_D) \xrightarrow{d_{A_0}} \Omega^1(\mathfrak{g}|_D) \xrightarrow{d_{A_0}^2} \Omega^2(\mathfrak{g}|_D)
\]
where \( \mathfrak{g}|_D = \text{Lie}(\mathcal{G}|_D) \) generates the gauge group \( \mathcal{G} = \text{End} E \) over \( D \). Thus, the requirement that \( E \) be indecomposable restricts the associated cohomology:
\[
H^0_{A_0|_D} = 0.
\]
On the other hand, one might restrict attention to acyclic connections, i.e., whose gauge class \([A_0]\) is isolated in \( \mathcal{M}_D = \mathcal{M}_E|_D \). The absence of infinitesimal deformations translates into the vanishing of the other cohomology group:
\[
H^1_{A_0|_D} = 0.
\]

Definition 2 A reference metric on \( E \to W \) is asymptotically rigid if the associated complex \((5)\) over \( D \) has trivial cohomology.

1.1 Suitable pairs and gluing

A 7–dimensional product \( W'_{S} \times S^1 \), where \( W' \) is of the above form, has boundary \( D' \times S^1 \times S^1 \). Comparing the asymptotic model \((6)\) with the standard form of the \( G_2 \)-structure on a product \( CY \times S^1 \) we find that \( W'_{S} \times S^1 \) carries a \( G_2 \)-structure on a collar neighbourhood of the boundary that is asymptotic to:
\[
\varphi'_S = \kappa'_J \wedge d\alpha + \kappa'_J \wedge d\theta + \kappa'_K \wedge ds + d\alpha \wedge d\theta \wedge ds.
\]
Since the inclusion of the set of all \( G_2 \)-structures \( \mathcal{P}^3 \left(W' \times S^1\right) \subset \Omega^3 \left(W' \times S^1\right) \) is open \([\text{Joy}] \ p. \ 243\), \( \varphi'_S \) is itself a \( G_2 \)-structure on \( W' \times S^1 \) for large \( S \).

Condition 3 Two manifolds \( W' \) and \( W'' \) as above will be suitable for the gluing procedure if there is a hyper-Kähler isometry
\[
f : D'_J \to D''
\]
between \( D'' \) and the hyper-Kähler rotation of \( D' \) with complex structure \( J \). In this case the (pull-back) action on Kähler forms is
\[
f^* : \kappa''_{J} \mapsto \kappa'_J, \quad \kappa''_{J} \mapsto \kappa'_J, \quad \kappa''_{K} \mapsto -\kappa'_K.
\]
Assuming this holds, define a map between collar neighbourhoods of the boundaries by

$$F_S : D' \times S^1 \times S^1 \times [S - 1, S] \rightarrow D'' \times S^1 \times S^1 \times [S - 1, S]$$

$$(y, \alpha, \theta, s) \mapsto (f(y), \theta, \alpha, 2S - 1 - s).$$

This identification gives a compact oriented 7-manifold

$$M_S = (W'_S \times S^1) \cup_{F_S} (W''_S \times S^1) \cong W''_S \#_S W'.$$

Figure 1: The compact 7-manifold $M_S$

The matching of Kähler forms guarantees that the respective $G_2$-structures on $W'_S \times S^1$ and $W''_S \times S^1$ agree along the gluing region $[S - 1, S]$:

$$F_S^* \varphi_S'' = F_S^* (\kappa''_I^J \wedge \alpha'' + \kappa''_I \wedge \delta \theta'' + \kappa''_K \wedge \delta s'' + \delta \alpha'' \wedge \delta \theta'' \wedge \delta s'')$$

$$= \kappa'_I \wedge \delta \alpha' + \kappa'_J \wedge \delta \theta' + \kappa'_K \wedge \delta s' + \delta \alpha' \wedge \delta \theta' \wedge \delta s'$$

$$= \varphi'_S.$$

so we obtain a globally well-defined $G_2$-structure $\varphi_S$ on $M_S$. Thus, for large enough $S$, there is a 1-parameter family $(M_S, \varphi_S)$ of compact oriented manifolds $M_S$ equipped with $G_2$-structures $\varphi_S$.

While it is possible to arrange $d\varphi_S = 0$ for any $S$ [Kov2, eq. (4.23)], a pair $(M_S, \varphi_S)$ is not in principle a $G_2$-manifold, as one has yet to satisfy the co-closedness condition:

$$d *_{\varphi_S} \varphi_S = 0.$$

In fact, although the cut-off functions involved in the asymptotic approximations leading to [Kov2, Lemma 4.25] add error terms to $d *_{\varphi_S} \varphi_S$, these are controlled by the estimate

$$\|d *_{\varphi_S} \varphi_S\|_{L^p} \leq C_{p,k} e^{-\lambda S},$$

with $0 < \lambda < 1$. This exponential decay implies that, by ‘stretching the neck’ up to large enough $S_0$, one can make the error so small as to be compensated by

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a suitably small perturbation of $\varphi_S$ in $P^3(M_S)$, $S > S_0$ [Kov1]. Hence one achieves a 1–parameter family of compact oriented $G_2$–manifolds:

$$(M_S, \tilde{\varphi}_S), \quad S > S_0.$$ 

1.2 Statement of the instanton gluing theorem

Let $(M_S, \tilde{\varphi}_S)$ be a compact $G_2$–manifold with $\text{Hol}(\tilde{\varphi}_S) = G_2$ as above, obtained from a Fano pair by Kovalev’s construction:

$$M_S = W'\tilde{\#}_S W'' \cong (W'_S \times S^1) \cup_{F_S} (W''_S \times S^1).$$

Furthermore, let $E^{(i)} \to W^{(i)}$ be asymptotically stable holomorphic bundles with same structure group $G = \text{Aut}(E)$, such that there is a $G$–isomorphism

$$g : E'|_{D'} \cong E''|_{D''}.$$

One can define a holomorphic bundle over $M_S$ by an induced bundle gluing

$$E_S^g \cong E'|_{\tilde{\#}_S} E'' \to M_S$$

of the following form. First, fix holomorphic trivialisations over neighbourhoods of infinity $U^{(i)} \subset W^{(i)}_\infty$ along the ends. Then, spreading $g$ along $U^{(i)}$ via pull-back by the fibration maps $\tau^{(i)} : W^{(i)} \to D^{(i)}$, we identify the fibres of $p^{(i)}_1 E'$ and $p^{(i)}_1 E''$ across the gluing zone of operation $\tilde{\#}_S$. Having said that, I will omit henceforth the superscript $g$ as well as any reference to the particular choice of trivialisations. This paper proves the following Gluing theorem:

**Theorem 4** Let $E^{(i)} \to W^{(i)}$, $i = 1, 2$, be asymptotically stable bundles of same semi-simple structure group $G$, with asymptotically rigid reference metrics $H^{(i)}_0$, admitting a $G$–isomorphism $g : E'|_{D'} \cong E''|_{D''}$.

There exists $S_0 > 0$ such that the bundle $E_S \to M_S \cong W'\tilde{\#}_S W''$ admits a $G_2$–instanton, for every $S \geq S_0$. 

6
2 Approximate instantons over $M_S$

In order to produce a $G_2$—instanton over the compact 7-manifold $M_S$, I use cut-offs to obtain a connection on $\mathcal{E}$ which is ‘approximately’ an instanton, then show that a certain surjectivity requirement is satisfied, in order to perturb it into a true solution.

2.1 Preliminary moduli theory

Following [Don1, pp.83-87], given an asymptotically stable bundle $\mathcal{E} \to W$, we are interested in gauge classes of connections on $\tilde{\mathcal{E}} = p_1^\ast \mathcal{E} \to M = W \times S^1$ which are ‘asymptotically HYM’ [cf. (5)]. For every (smooth) reference metric $H_0$, we denote its Chern connection $A_0$ and pose [cf. Notation 4]

$$\mathcal{A} \doteq \left\{ p_1^\ast (A_0 + a) \mid A_0|_D \in \mathcal{M}_D, \ |a|, |\nabla_{A_0} a| \in L^p_k, \ a \xrightarrow{S \to \infty} 0 \right\},$$

(8)

taking, for suitable integers $p,k$ [cf. (14) below], the $L^p_k$—norm induced by $\varphi$:

$$\|f\| \doteq \|f\|_{L^p_k} = \left( \int_M \sum_{l=0}^k |\nabla^{i_1} \ldots \nabla^{i_l} f| \, d\text{Vol} \right)^{\frac{1}{p}}.$$  (9)

This has in view the use of Sobolev’s embedding (Lemma 24) in Subsection 4.1.

For notational clarity I will henceforth leave implicit the pull-back $p_1^\ast$.

Posing the gauge-equivalence condition

$$A_1 \sim A_2 \iff A_2 = g(A_1) \overset{loc}{=} A_1 - d_1 g \cdot g^{-1}, \quad g \in L^p_{k+1,loc}(\text{Aut } \mathcal{E}),$$

and adopt accordingly the gauge group

$$\mathcal{G} \doteq \left\{ g \in \text{Aut } \tilde{\mathcal{E}} \mid |\nabla_0 g \cdot g^{-1}| \in L^p_k, \ g \xrightarrow{S \to \infty} 1 \right\},$$

whose (bundle of) Lie algebra(s) we denote $\mathfrak{g}$. Since every $A_0|_D$ is assumed irreducible [cf. p\textsuperscript{3}], $\mathcal{G}$ is in fact a Banach Lie group and the action $\mathcal{G} \times \mathcal{A} \to \mathcal{A}$ is smooth. Finally, the Coulomb gauge condition provides transversal slices for the action [cf. (21), below]:

$$U_\varepsilon (A) = A + \left\{ a \in \Omega^1 (g) \mid d^*_A a = 0, \ |a| < \varepsilon, \ a \xrightarrow{S \to \infty} 0 \right\},$$

(10)

so that the quotient $B = \mathcal{A}/\mathcal{G}$ is a Banach manifold [D-K Prop. 4.2.9, p.132]:

**Proposition 5** If $A$ is irreducible then, for small $\varepsilon$, the projection from $\mathcal{A}$ to $B$ induces a homeomorphism from $T_\varepsilon (A)$ to a neighbourhood of $[A]$ in $B$.

**Definition 6** The moduli space of (irreducible) $G_2$—instantons on $\mathcal{E}$ is:

$$\mathcal{M}^+ \doteq \{ [A] \in B \mid p_+(F_A) \doteq F_A \wedge * \varphi = 0 \}.$$
NB.: In particular, the instantons obtained in the first paper [SaE], as solutions
of the HYM problem, decay exponentially in all derivatives to $H_0$ along the
cylindrical end, hence $\mathcal{M}^+ \neq \emptyset$ [cf. (4) and (8)].

For the local description of $\mathcal{M}^+$, define around a solution $A$ the map
\[
\psi : U_\varepsilon(A) \subset \mathcal{A} \to \Omega^6(g_E) \\
\quad a \mapsto \psi(a) = p_+ (F_{A+a}) = (d_A a + a \wedge a) \wedge * \varphi
\]  
and write $Z(\psi) \subset T_\varepsilon(A)$ for its zero set. Slicing out by Coulomb gauge indeed
makes $\psi$ a Fredholm map, and we have:

**Proposition 7** If $A \in \mathcal{A}$ is an irreducible $G_2-$instanton, then an
$\varepsilon-$neighbourhood of $[A] \in \mathcal{M}^+$ is modelled on $Z(\nu)$, where
$\nu$ is the invertible map between finite-dimensional spaces defined by
\[
\nu : \ker (d_A^* + d_A^+) \cap \Omega^1(g) \to \coke (d_A^+ \cap \ker d_A) \cap \Omega^6(g) \\
\quad a \mapsto \nu(a) = \sigma(0,a)
\]
and $\sigma$ is the non-linear part of the local Fredholm decomposition of $\psi$.

The proof of **Proposition 7** is postponed to Section 4 as part of a more
detailed discussion of the moduli theory.

### 2.2 Truncating instantons with decaying error term

We may now start in earnest the proof of **Theorem 4**. Let $A'$ be a $G_2-$instanton
on $p_1^*E'$ [cf. (4)]. Along a neighbourhood of infinity down the tubular end of
$W'$, we write $A' = A_0' + a'$, where $A_0'$ is the lifted Chern connection associated
to the reference metric $H_0'$ and $a' \overset{C_\infty}{\longrightarrow} S \to 0$. Fix a smooth cut-off
\[
\chi : \mathbb{R}^+ \to [0,1] \\
\chi(s) = \begin{cases}
1, & s < 0 \\
0, & s \geq 2
\end{cases}
\]
and truncate to
\[
A'_S \doteq A_0' + \chi(s - S + 3) a',
\]
which agrees with $A_0'$ over the gluing region $[S - 1, S]$ and has self-dual part
\[
F_{A'_S}^+ \simeq F_{A'_S} \wedge * \varphi
\]
supported in the (topological) cylinder segment $\Sigma_1(S - 1)$. Clearly
\[
\|F_{A'_S} \wedge * \varphi\|_{L^p(M')} \leq C_{p,k} e^{-S}
\]
for $M' \doteq W' \times S^1$. Repeating the construction for $A''_S$ on $p_1^*E''$, we may assume
that the truncated connections $A''_S^{(i)}$ match (via the bundle isomorphism $g$) over
\([S - 1, \infty]\), hence in particular over \([S - 1, S]\), so they glue together to define a smooth connection

\[A_S = A' \#_S A'' \quad \text{on} \quad \mathcal{E}_S.\]

Since the hyper-Kähler rotation at infinity is assumed to be an isometry, we still have the asymptotic decay of the self-dual part

\[\| F_{A_S} \wedge \ast \varphi \|_{L^p(M_S)} \leq C_{p,k} e^{-S}\]

from which we see that \(A_S\) is almost an instanton.

### 2.3 Non-degeneracy under acyclic limits

Let \(A = A^{(i)}\) as above, say, be a \(G_2\)–instanton on the pull-back bundle

\[\tilde{\mathcal{E}} = p^* \mathcal{E} \to M \cong W \times S^1.\]

Then the hypothesis that the connection at infinity is acyclic implies that \(A\) itself is acyclic, with respect to its own deformation complex:

\[\begin{array}{c}
\Omega^0 (g) \xrightarrow{d_A} \Omega^1 (g) \xrightarrow{d_A} \Omega^2 (g) \xrightarrow{\ast \varphi} \Omega^6 (g) \xrightarrow{d_A} \Omega^7 (g).
\end{array}\]  \(12\)

The goal of this Subsection is to prove that claim, in the following terms:

**Proposition 8** When the reference metric \(H_0\) is asymptotically rigid, then the induced \(G_2\)–instanton \(A\) lifted from a HYM solution [cf. (1) and (4)] is acyclic, i.e., \(H^0_A = 0\) and \(H^1_A = 0\) in the deformation complex (12).

Since our asymptotically stable bundle is, by definition, indecomposable, we have already \(H^0_A = 0\), so one only needs to check the non-degeneracy condition \(H^1_A = 0\). On the other hand, the complex is self-dual (under the Hodge star), so this is equivalent to showing \(H^2_A = 0\), which means precisely that \(A\) is an isolated point in its moduli space \(\mathcal{M}^+\), in the light of the local model given by Proposition 7. To check this fact, we resort to the Chern-Simons functional

\[\rho(b)_A = \int_{W \times S^1} \text{tr} F_A \wedge b_A \wedge \ast \varphi.\]

For any given direction \(a \in T_{[A]} \mathcal{B} \subset \Omega^1 (g)\) with \(\|a\| = 1\) and possibly short length \(\varepsilon > 0\), we have

\([A + ha] \in \mathcal{M}^+ \iff 0 \equiv \rho(b)_{A+\varepsilon a} = \rho(b)_A + \varepsilon D [\rho(b)]_A (a) + O(\varepsilon^2)\)
for vector fields \( b \in \Gamma (T\mathcal{A}) \). Explicitly, the first order variation is

\[
D [\rho(b)]_A (a) \quad = \quad \int_{W \times S^1} \operatorname{tr} \left\{ d_A a \wedge b_A + (Db)_A (a) \wedge F_A \right\} \wedge *\varphi
\]

\[
= \int_{W \times S^1} \operatorname{tr} a \wedge d_A b_A \wedge *\varphi + \lim_{S \to \infty} \int_{\partial W \times S^1} \operatorname{tr} \left\{ a \wedge b_A \right\} \wedge *\varphi
\]

\[
= \int_{W \times S^1} \operatorname{tr} a \wedge d_A^+ b_A
\]

since \( H_0 \) is asymptotically rigid and so \( |a| \xrightarrow{S \to \infty} 0 \), for small enough \( \varepsilon \). Notice in passing that this is zero for any direction \( a \in \text{img} d_A \subset \Omega^1 (g) \) along gauge orbits, corresponding to the intuitive fact that the only ‘meaningful’ perturbations are those which descend nontrivially to \( \mathcal{B} \). Choose now \( 0 \neq \xi \in \Omega^9 (g) \) such that

\[
b_A = (d_A^+)^* \xi \neq 0 \quad \text{in} \quad \Omega^1 (g).
\]

By the orthogonal decomposition (21), this can be done in such a way that \( d_A^+ b_A \neq 0 \), so for any direction \( a \in \Omega^1 (g) \) (transverse to gauge orbits) the number

\[
D [\rho(b)]_A (a) = N_A (a, \xi) \in \mathbb{R}
\]

is not zero for a generic choice of \( \xi \). Rescaling \( \tilde{b}_A = \frac{1}{N_A (a, \xi)} b_A \), we find

\[
\rho(\tilde{b})_{A+\varepsilon a} = \varepsilon + O(\varepsilon^2) O(\|\tilde{b}_A\|) \neq 0, \quad \text{for} \quad 0 \neq \varepsilon \ll 1.
\]

Now, since \( \mathcal{M}^+ \) is finite-dimensional, there are tangent vectors (1–forms on the base) \( u_1, \ldots, u_n \in T_{[A]} \mathcal{B} \) such that any such perturbation is written as

\[
\varepsilon a = \varepsilon (a^1 u_1 + \cdots + a^n u_n), \quad a^i \in \mathbb{R}
\]

But in the above way we can find, respectively for \( u_1, \ldots, u_n \), vector fields \( \tilde{b}_1, \ldots, \tilde{b}_n \) such that \( \rho(\tilde{b}_i)_{A+\varepsilon a} = \varepsilon + O(\varepsilon^2) \). Consequently, for a generic linear combination \( \tilde{b} = \beta_1 \tilde{b}_1 + \cdots + \beta_n \tilde{b}_n \), one has

\[
\rho(\tilde{b})_{A+\varepsilon a} = \varepsilon (a^1 \beta_1 + \cdots + a^n \beta_n) + O(\varepsilon^2).
\]

Hence there exists a (possibly small) value \( \varepsilon_0 > 0 \) such that \( \rho_{A+\varepsilon a} \neq 0 \), as a 1–form on \( \mathcal{A} \), for any \( \varepsilon a \in U_{\varepsilon_0} ([A]) \). In other words, there are no instantons in the open ball of radius \( \varepsilon_0 \) around \([A]\) in the moduli space \( (q.e.d.)\).
3 Perturbation theory over long tubular ends

Following the standard approach, we may now look for a nearby exact solution $A = A_S + a$ to the $G_2$–instanton equation

$$d_{A_S}^+ a + (a \wedge a) \wedge \ast \varphi = - F_{A_S} \wedge \ast \varphi = \epsilon(S).$$

We adopt all along the acyclic assumption that the operators $d_{A(i)}^+$ have trivial cokernel, i.e., the (irreducible) connections $A^{(i)}$ are isolated points in the respective moduli spaces $\mathcal{M}_{\mathbb{C}(i)}^+$. [Proposition 8]

3.1 Noncompact Sobolev estimates

For briefness, let us refer to ordered integers $k > l \geq 0$ and $q \geq p$ as suitable if they satisfy the Sobolev condition:

$$\frac{1}{p} - \frac{1}{q} < \frac{k - l}{7}.$$  \hfill (14)

In particular, the pair $p, k$ will be suitable when $k = l + 1$ and $2p = q$ are suitable.

**Lemma 9** Let $W$ be an asymptotically cylindrical $3$–fold; given suitable $k \geq l$ and $q \geq p$, there exists a constant $C = C_{W,p,q,k,l} > 0$ such that, for sections of any bundle over $W \times S^1$ (with metric and compatible connection),

$$\|f\|_{L^q} \leq C \|f\|_{L^p}.$$  \hfill (15)

**Proof** Following [Don1, pp.70-72], set $B_0 = W_0 \times S^1$ and consider for $n \geq 1$ the tubular segments of ‘length one’ $B_n = (W_n \setminus W_{n-1}) \times S^1$ along the tubular end. Then, using the usual Sobolev estimate for compact domains,

$$\|f\|_{L^q}^q = \sum_{n \in \mathbb{N}} \int_{B_n} |f|^q \leq \sum_{n \in \mathbb{N}} C_n \left( \int_{B_n} |\nabla f|^p + |f|^p \right)^{q/p} \leq \check{C} \left( \sum_{n \in \mathbb{N}} \int_{B_n} |\nabla f|^p + |f|^p \right)^{q/p} = \check{C} \|f\|_{L^p}^q$$

with $\check{C} = \limsup C_n < \infty$, since the segments are asymptotically cylindrical. This proves the statement, by induction on $l$. \[\blacksquare\]
3.2 Gluing right inverses

We now investigate the behaviour of right-inverses under truncation and gluing:

Lemma 10 For $S \gg 0$, the operators $d_{A_S}^{(i)}$ admit bounded right inverses $Q_S^{(i)}$ satisfying

$$\|Q_S^{(i)}\|_{L_k^p} \leq C^{(i)}_{p,k} \|\xi\|_{L_{k-1}^p}.$$  

for suitable $p,k \in \mathbb{N}$, where the bound $C^{(i)}_{p,k}$ depends only on $A^{(i)}$, not on $S$.

Proof The operators $d_{A_S}^{(i)}$ correspond to the original instantons over each tubular component $W^{(i)} \times S^1$, hence by the acyclic assumption [cf. Proposition 5] they admit bounded right inverses $Q^{(i)}$, independent of $S$.

The crucial fact is that $a_S^i = A_S^{(i)} - A^{(i)} = O(e^{-S})$ so, for $S \gg 0$, a right inverse for $d_{A_S}^{(i)}$ gives a right inverse $Q^{(i)}_S$ for $d_{A_S}^{(i)}$ with (approximately) the same uniform Lipschitz bound. 

Corollary 11 For $S \gg 0$, there exist an ‘approximate’ right inverse $Q_S$ and a true right inverse $P_S$ for $d_{A_S}^{+}$:

$$P_S = Q_S \left(d_{A_S}^{+} Q_S\right)^{-1}.$$  

Moreover, for suitable $p,k \in \mathbb{N}$, there is a uniform bound $C_{p,k}$ on the operator norm of $P_S$:

$$\|P_S(\xi)\|_{L_k^p} \leq C_{p,k} \|\xi\|_{L_{k-1}^p}.$$  

Proof Following a standard argument [Don1 §3.3 & §4.4], we may take truncation functions $\chi_S^{(i)} : W^{(i)} \rightarrow [0,1]$ satisfying

$$\left(\chi_S^{(i)}\right)^2 + (\chi_S^{(r)})^2 = 1, \quad \text{supp } \chi_S^{(i)} \subset W_S^{(i)}, \quad \|\nabla \chi_S^{(i)}\|_{L^\infty} = O(e^{-S}).$$  

Then, denoting $r_S^{(i)} : M_S \rightarrow W^{(i)} \times S^1$ the maps given by restriction for $s \leq 2S$ and extended by zero along the rest of the tubular end, we form

$$Q_S = (\chi_S^{(i)}')^2 (Q_S \circ r_S^{(i)}) + (\chi_S^{(r)})^2 (Q_S \circ r_S^{(r)})$$  

and we check that this is an approximate right inverse in the sense that $\|d_{A_S}^{+} Q_S - I\| = O(e^{-S})$; indeed we have

$$d_{A_S}^{+} Q_S = \sum_{i=1,2} \left\{ (\chi_S^{(i)})^2 d_{A_S}^{+} (Q_S^{(i)} \circ r_S^{(i)}) + 2 (\chi_S^{(i)} \nabla \chi_S^{(i)}) \odot (Q_S^{(i)} \circ r_S^{(i)}) \right\}_{O(e^{-S})}$$  

where $\odot$ denotes an algebraic operation. It follows from the Lemma that the second summand is dominated by the decay of $\|\nabla \chi_S^{(i)}\|_{L^\infty}$. Then

$$P_S \equiv Q_S \left(d_{A_S}^{+} Q_S\right)^{-1}$$  

is a true right inverse for $d_{A_S}^{+}$, with uniformly bounded norm determined by $Q_S$ and $d_{A_S}^{+}$ itself. 

12
3.3 Exact solution via the contraction principle

For a solution of the form $a = P\xi$, equation (13) reads

$$(I + G) (\xi) = \epsilon(S)$$

(16)

where $G(\xi) \doteq P(\xi) \land P(\xi) \land \ast \varphi$ and $\|\epsilon(S)\|$ is small, as $S \gg 0$. Thus if, given suitable $p, k$, the map \text{cf. Lemma 9 for last inclusion} $G : L^2_{k-1} (\Omega^6) \rightarrow L^p_{k-1} (\Omega^6)$ is Lipschitz with constant strictly smaller than 1, then by the Banach contraction principle $I + G$ is continuously invertible between neighbourhoods of $(I + G) (0) = 0$ and we obtain, for large $S$ say, a solution as

$$a = P (I + G)^{-1} \epsilon(S).$$

In order to prove this, we need a uniform bound on exterior multiplication, which is an immediate consequence of Hölder’s inequality:

\textbf{Proposition 12} For suitable $p, k \in \mathbb{N}$, there exists a uniform constant $M_{p,k}$ such that

$$\|a \land b\|_{L^2_{k-1}} \leq M_{p,k} \|a\|_{L^p_{k-1}} \|b\|_{L^p_{k-1}}.$$

From this we infer that operator $G$ is bounded uniformly in $S$ for any suitable Sobolev norm, but so far we have no definite control over the actual bound. We can scale away this apparent difficulty using the fact that $G$ is homogeneous of degree 2; letting $\lambda = MC^2$, $\tilde{\xi} = \lambda \xi$ and $\tilde{G} = \frac{1}{\lambda} G$, equation (16) becomes

$$
(I + \tilde{G}) (\tilde{\xi}) = \lambda \epsilon(S).
$$

The point here is that now $\tilde{G}$ is a contraction over the ball $\tilde{B}_1 \doteq \{\|\tilde{\xi}\| \leq 1\}$:

$$\|\tilde{G}(\tilde{\xi})\| \leq \frac{MC^2}{\lambda} \|\tilde{\xi}\|^2 \leq \|\tilde{\xi}\|.$$

Then indeed $I + \tilde{G}$ is a homeomorphism onto an interior domain $0 \in U \subset \tilde{B}_1$, and one can choose $S \gg 0$ so that $\lambda \epsilon(S) \in U$. One may check, by bootstrapping [Dom], that $A$ is in fact smooth. We have thus proved \textbf{Theorem 4}.

In conclusion, it should be noticed that the acyclic hypothesis is a rather strong, non-generic requirement. The fact that it is not void should therefore be illustrated:

\textbf{Example 13} The prime Fano 3-folds of type $X_{22}$ were discovered by Iskovskikh [Isk] and further studied by Mukai [Muk], §3. On one hand, these appear in Kovalev’s list of suitable blocks for the gluing construction; in fact, they are extremal in the sense that a pair of base manifolds of type $X_{22}$ realises the lower bound on the third Betti number $b_3(M) = 71$ [Kov, pp.158-159].

On the other hand, crucially, these come equipped with an asymptotically stable bundle $E \rightarrow X_{22}$ which is rigid [Muk, §3] over a divisor $D \in |−K_{X_{22}}|$. In other words, the holomorphic bundle $E|_D$ corresponds to an isolated point in its moduli space, hence its associated HYM metric $H_0$ is indeed acyclic.
4 Local model for the moduli space

The moduli space $M^{+}$ of $G_{2}−$instantons on $E \rightarrow M$ is locally described as the zero set of a map $\psi$ [cf. (11)] between the Banach spaces $U_{e}(A) \subset A$ and $\Omega^{6}(g)$. Therefore, if our map $\psi$ is Fredholm on $Z(\psi)$, it is a matter of standard theory to model a neighbourhood of $[A]$ in $M^{+}$ on the finite-dimensional set $\nu^{-1}(0)$ [cf. Corollary 22 in the Appendix]. Foreseeing Proposition 17 we restrict attention to irreducible connections on an $SU(n)$–bundle $E$.

4.1 Noncompact Fredholm theory

As defined before, the map $\psi$ is just the self-dual part of the curvature, so $\psi(a) − \psi(0) = (p_{+} \circ d_{A}) a + O(|a|^{2})$ and

$$(D\psi)_{0} = p_{+} \circ d_{A}.$$ 

Moreover, by the ‘slicing’ condition (10) across orbits, we consider in fact the restriction

$$p_{+} \circ d_{A} : \ker d^{*}_{A} \rightarrow \Omega^{2}_{+}(g).$$

Since the map $L_{*\varphi} = \star \varphi \wedge \cdot$ now plays the role of ‘SD projection’, we denote henceforth

$$d_{A}^{+} = L_{*\varphi} \circ d_{A} : \Omega^{1}(g) \rightarrow \Omega^{6}(g)$$

and consider the extended deformation complex

$$\begin{array}{ccccccc}
\Omega^{0}(g) & \xrightarrow{d_{A}} & \Omega^{1}(g) & \xrightarrow{d_{A}^{+}} & \Omega^{2}(g) & \xrightarrow{\ast \varphi \wedge \cdot} & \Omega^{6}(g) & \xrightarrow{d_{A}} & \Omega^{7}(g).
\end{array}$$

Using $d \ast \varphi = 0$, we find

$$[L_{*\varphi}, d_{A}] = 0,$$

so, when $A$ is an instanton, (19) is indeed a complex and the identification of the self-dual 2–forms with the 6–forms is consistent with the relevant differential operators (for more on elliptic complexes under the condition $d \ast \varphi = 0$, see [Fe-U]). Moreover, this complex is elliptic:

Lemma 14 The operator $d_{A}^{+}$ defined by (18) has formal adjoint

$$(d_{A}^{+})^{*} = \ast d_{A}^{*} : \Omega^{6}(g) \rightarrow \Omega^{1}(g).$$

Proof For $a \in \Omega^{1}(g)$ and $\eta \in \Omega^{6}(g)$, we have pointwise:

$$\langle d_{A}^{+} a, \eta \rangle (\cdot \cdot) = \langle \ast \varphi \wedge d_{A} a \wedge \ast \eta \rangle = (d_{A} a) \wedge \ast (\ast \varphi \wedge \ast \eta)$$

$$= \langle d_{A} a, \ast (L_{*\varphi} \ast \eta) \rangle = \langle a, d_{A}^{*} (\ast L_{*\varphi} \ast \eta) \rangle (\cdot \cdot)$$

$$= \langle a, \ast (d_{A} L_{*\varphi} \ast \eta) \rangle (\cdot \cdot)$$

$$= \langle a, (\ast d_{A}^{*} \ast \eta) \rangle (\cdot \cdot)$$

$$= \langle a, \ast (L_{*\varphi} d_{A} \ast \eta) \rangle (\cdot \cdot)$$

$$= \langle a, \ast d_{A}^{+} \ast \eta \rangle (\cdot \cdot)$$
Proposition 15 When $A$ is a $G_2$-instanton, the complex $(19)$ is elliptic.

Proof First of all, since $(d_A^*)^\ast = *d_A^* / \text{Lemma } 14$ and $d_A^* = *d_A^*$, notice that our complex is self-dual with respect to the Hodge star:

\[
\begin{align*}
\Omega^0 & \overset{d_A}{\longrightarrow} \Omega^1 & \Omega^6 & \overset{d_A}{\rightarrow} \Omega^7 \\
*\Omega^7 & \overset{d_A^*}{\longleftarrow} *\Omega^6 & (d_A^*)^* & \overset{d_A^*}{\longrightarrow} & *\Omega^1 & \overset{d_A^*}{\longrightarrow} *\Omega^0.
\end{align*}
\]

By Corollary 20, it suffices to show ellipticity at $\Omega^1 (g)$, as that is equivalent to the ellipticity of the dual $*\Omega^7 \overset{d_A^*}{\longleftarrow} *\Omega^6 (d_A^*)^* \overset{d_A^*}{\longrightarrow} *\Omega^1$, which is just $\Omega^1 \overset{d_A^*}{\longrightarrow} \Omega^6 \overset{d_A^*}{\longrightarrow} \Omega^7$. Fixing a section $\xi$ of $T' M$ (the cotangent bundle minus its zero section), we have symbol maps

\[
0 \rightarrow \pi^* (\Omega^0 (g)) \xi \overset{\xi \ast (-)}{\longrightarrow} \pi^* (\Omega^1 (g)) \xi \overset{\ast \varphi \wedge \xi \wedge (-)}{\longrightarrow} \pi^* (\Omega^6 (g)) \xi \longrightarrow \ldots
\]

For $\alpha \in \Omega^1 (g)$ such that $\ast \varphi \wedge \xi \wedge \alpha = 0$, exactness means $\alpha$ has to lie in $\xi \Omega^0 (g)$. Since $G_2$ acts transitively on $S^6$, take $g \in G_2$ such that $g^* \xi = ||\xi||e^1$ and denote $\tilde{\alpha} = g^* \alpha$, so that

\[
\ast \varphi \wedge e^1 \wedge \tilde{\alpha} = 0.
\]

That is just the statement that $e^1 \wedge \tilde{\alpha}$ is anti-self-dual, but this cannot occur unless $e^1 \wedge \tilde{\alpha} = 0$, as

\[
(e^1 \wedge \tilde{\alpha}) \wedge \varphi = \tilde{\alpha} \wedge (e^{1567} - e^{1345} - e^{1426} + e^{1237})
\]

has non-vanishing components involving $e^1$ and $\ast (e^1 \wedge \tilde{\alpha})$ obviously has not. Therefore $\tilde{\alpha} = f e^1$ for some $f \in \Omega^0 (g)$, and

\[
\alpha = (g^*)^{-1} (f e^1) = \frac{f}{||\xi||} \xi \in \xi \Omega^0 (g).
\]

In view of the isomorphism $L_{\ast \varphi} |_{\Omega^2_+} : \Omega^2_+ \rightarrow \Omega^6$, taking the self-dual part of curvature via the $G$-equivariant map $L_{\ast \varphi} \circ F^+ : A \rightarrow \Omega^6 (g)$ defines a section $\Psi ([A]) = F_A \wedge \ast \varphi$ of the Hilbert bundle

\[
A \times G \Omega^6 (g) \rightarrow B.
\]

The intrinsic derivative, i.e., the component of the total derivative tangent to the gauge-fixing slices in $\Omega^1 (g_E)$, of $\Psi$ at $[A]$ is

\[
(D\Psi)_{[A]} : \ker d_A^* \rightarrow \ker d_A \subset \Omega^6 (g)
\]

\[
a \mapsto d_A^* a.
\]
To see that \((D\Psi)_{[A]}\) is Fredholm over \(Z(\Psi)\), consider the extended operator

\[
D_A : \Omega^1(g) \oplus \Omega^7(g) \to \Omega^0(g) \oplus \Omega^6(g)
\]

\[
(a, f) \mapsto (d_A^* a, d_A^* a + d_A^* f).
\]

If the base manifold was compact, then standard elliptic theory would imply

\[
D_A = d_A^* \oplus (d_A^+ \oplus d_A^-)
\]

is Fredholm, as the ‘Euler characteristic’ of an elliptic complex, by Proposition 15. However, on our manifolds with cylindrical ends \(W \cong W_0 \cup W_\infty\), the parametrix patching method over the compact piece \(W_0\) must be combined with tubular theory over \(W_\infty\) under the acyclic assumption. This follows in all respects the proof of \([\text{Don}1]\), Prop. 3.6 and its preceding discussion, except that in this case we are in the much simpler situation where the exponential decay of \((a, f) \in \ker D_A\) is guaranteed from the outset by our definitions [cf. (8)] and the fact that the bundle is indecomposable. Then we have:

**Claim 16** If \([A] \in Z(\Psi)\) is asymptotically rigid [cf. Definition 2], \(D_A\) is a Fredholm operator.

In particular, \(D_A\) has closed range, so there is an orthogonal decomposition:

\[
\Omega^6(g) = \ker d_A \oplus \im d_A^*
\]

and this implies that \((D\Psi)_{[A]}\) is also Fredholm:

\[
\ker (D\Psi)_{[A]} \hookrightarrow \ker D_A
\]

\[
\coker (D\Psi)_{[A]} = \coker d_A^+ \cap \coker (d_A^*|\Omega^7) \hookrightarrow \coker D_A.
\]

\[
\ker d_A = \ker d_A^*
\]

Finally, the moduli space of \(G_2\)-instantons [Definition 6] is cut out as its zero set \(Z(\Psi)\). As an immediate consequence of (20) and the Bianchi identity, we have \(\Psi([A]) \in \ker d_A \subset \Omega^6(g_E)\), so, intuitively, the image of \(\Psi\) lies in the ‘subbundle of kernels of \(d_A\)’:

\[
\mathcal{A} \times_G \ker d_A \to \mathcal{B}
\]

\[
\Omega^6(g)
\]

When \(\mathcal{E}\) is an \(SU(n)\)-bundle, say, one can use the orthogonal projections \(p_a : \ker d_A \to \ker d_{A_0}\) to trivialise the fibres onto \(\ker d_{A_0}\) over a neighbourhood \(U_\varepsilon ([A_0]) \subset \mathcal{B}\), where \(\varepsilon\) is a small global constant:
Proposition 17 Let $\mathcal{E}$ be an $SU(n)$–bundle over a compact $G_2$–manifold $(M, \varphi)$ and $A_0$ an irreducible connection; then there exists $\varepsilon > 0$ such that the orthogonal projection
\[
p_a : \ker d_A \to \ker d_{A_0}
\]
in $\Omega^6(g)$ is an isomorphism for all $A = A_0 + a \in U_{\varepsilon}(A_0)$.

**Proof** We consider, throughout, the operators [cf. (19) & (21)]:
\[
\Omega^6(g) \xrightarrow{d_{A_0}, d_A} \Omega^7(g).
\]
Writing $\rho = d_{A_0}^* f$ for some $f \in \Omega^7(g)$, we denote elements of $\Omega^6(g)$ by
\[
\eta = (\eta_0 \oplus \rho) \in (\ker d_{A_0} \oplus \img d_{A_0}^*) = \Omega^6(g).
\]

**Surjectivity**
Given $\eta_0 \in \ker d_{A_0}$, write $g_0 = -a \wedge \eta_0$; surjectivity of $p_a$ means finding $\rho \in \img d_{A_0}^* \subset \Omega^6(g)$ such that $\eta = \eta_0 \oplus \rho \in \ker d_A$, i.e., solving for $\rho$ the equation
\[
d_A \rho = g_0.
\]
Since $A_0$ is irreducible, one has $(\img d_{A_0})^\perp = \ker d_{A_0}^* = \{0\}$, therefore $\img d_{A_0} = \Omega^7(g)$. Thus one may think of the restriction of $d_A$ to $\img d_{A_0}^*$ as
\[
d_A : \img d_{A_0}^* \to \img d_{A_0}.
\]

Bijectivity of linear maps between Banach spaces is an open condition [Lemma 23], so one can show that (24) is invertible by checking that, for suitably small $a$, this map is arbitrarily close to the isomorphism $d_{A_0}^* : \img d_{A_0}^* \sim \img d_{A_0}$.

Indeed, writing $L_a : \eta \mapsto a \wedge \eta$, there exists a global constant such that
\[
\|d_A - d_{A_0}\| = \|L_a\| \leq C\|a\| < C\varepsilon.
\]
Here we used Lemma 24 since our choice of $\|\|_p$ suits Sobolev’s embedding theorem. So (24) is also an isomorphism for $\varepsilon$ small enough, and we can find a unique $\rho \in \ker d_{A_0}^*$ solving (23).

**Injectivity**
Let $\eta \in \ker d_A \subset \Omega^6(g)$; then
\[
p_a (\eta) = 0 \iff \rho = \eta \in \ker d_A \iff d_A \rho = 0 \iff \rho = 0
\]
since $\rho \in \img d_{A_0}^*$ and we have just seen that $d_A : \img d_{A_0}^* \sim \img d_{A_0}$ is an isomorphism (for suitably small $a$); so $\eta = \rho = 0$.

Hence Proposition 7 is proved, in the terms of Corollary 23 [cf. Appendix].
4.2 Final comments: gluing families and transversality

We achieved in Theorem 4 this paper’s main goal of constructing a solution $A$ of the instanton equation over the compact $G_2$–manifold $M_S = W'' \# S W''$. To conclude, I will briefly outline two natural extensions of this theory.

First, one may consider instanton families, i.e., given (pre)compact sets $N^{(i)} \subset \mathcal{M}^+_S$ of regular points on the moduli spaces over each end, define - for large neck length $S$ - an operation

$$\tau_S : N' \times N'' \to \mathcal{M}^+_S.$$ 

This should be a diffeomorphism over its image, consisting itself of regular points. Moreover, given an adequate notion of ‘distance’ between a connection $A^\#_s$ on $E$ and $A = A' \#_S A''$, any ‘nearby’ instanton is also obtained from such a sum. Namely, for a given integer $q$, one defines

$$\ell^q_S (A^\#_s ; A', A'') \doteq \inf_{g \in G'} \| g.A^\#_s - A' \|_{L^q(W'_s)} + \inf_{g \in G''} \| g.A^\#_s - A'' \|_{L^q(W''_s)} \ (25)$$

where $W'_S^{(i)}$ are the truncations of $W^{(i)}$ at ‘length’ $S$. Then one should expect the following to hold:

**Conjecture 18** Let $N^{(i)} \subset \mathcal{M}^+_S$ be compact sets of regular points in the moduli spaces of $G_2$–instantons over $W^{(i)} \times S^1$; for sufficiently small $\delta > 0$ and large $S > 0$, there are neighbourhoods $V^{(i)} \supset N^{(i)}$ and a smooth map

$$\tau_S : V' \times V'' \to \mathcal{M}^+_S$$

such that, for appropriate choices of $q > 0$ (independent of $\delta$ and $S$),

1. $\tau_S$ is a diffeomorphism to its image, which consists of regular points;
2. $\ell^q_S (\tau_S (A', A''); A', A'') \leq \delta$, $\forall A^{(i)} \in V^{(i)}$;
3. any instanton $A \in \mathcal{M}^+_S$ such that $\ell^q_S (\tau_S (A', A''); A', A'') \leq \delta$ for some $A^{(i)} \in V^{(i)}$ lies in the image of $\tau_S (V' \times V'')$.

Finally, under a generic Morse-Bott assumption, one may envisage relaxing the non-degeneracy requirement and deal with reducible connections via the ‘gluing parameter’ over the divisor at infinity. If the images of the obvious restriction maps as $r^{(i)} : \mathcal{M}^+_S \to \mathcal{M}_D$ meet transversally, one would expect the moduli space $\mathcal{M}^+_S$ over the compact base $M_S$ to be smoothly modelled on the product

$$\mathcal{M}^+_S \times_{\mathcal{M}^+_D} \mathcal{M}^+_D = \{(A', A'') \mid r'(A') = r''(A'')\}$$

for large values of the neck length $S$.

All of this follows strictly, of course, the general ‘programme’ of [Don1]. There are essentially two reasons to expect the analogy to carry through to our
$G_2$ setting: the bounded geometry of Kovalev's manifolds, which implies the uniform (Sobolev) bounds for the relevant operator norms [cf. Lemma 4 and Proposition 12]; and the fact that our original solutions decay exponentially in all derivatives to a HYM connection over the divisor at infinity [cf. (4)], so they are in $L^p_k(W)$ for any choice of $p,k$.

A Chern-Simons formalism under holonomy $G_2$

In (3+1)-dimensional gauge theory [Don1], §2.5, the Chern-Simons functional is defined on $B = A/G$, with integer periods, its critical points being precisely the flat connections. A similar theory can be formulated in higher dimensions given a suitable closed $(n-3)$–form [D-T] [Tho]. Here, for suitable connections over a $G_2$–manifold $(M,\varphi)$, we use the Hodge-dual $*\varphi$.

Recall that the set of connections $\mathcal{A}$ is an affine space modelled on $\Omega^1(g_P)$ so, fixing a reference $A_0 \in \mathcal{A}$, we have $\mathcal{A} = A_0 + \Omega^1(g_P)$ and we can define

$$\vartheta(A) = \frac{1}{2} \int_M \text{tr} \left( dA_0 a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge *\varphi,$$

fixing $\vartheta(A_0) = 0$. Note in passing that, since only the condition $d*\varphi = 0$ is required, the discussion extends to cases in which the $G_2$–structure $\varphi$ is not necessarily torsion-free. Moreover, the theory remains essentially unaltered if the compact base manifold is replaced by a manifold-with-boundary, say, under a setting of connections with suitable decay towards the boundary.

The above function is obtained by integration of the analogous 1–form

$$\rho(a)_A = \int_M \text{tr} (F_A \wedge a) \wedge *\varphi.$$

(26)

We find $\vartheta$ explicitly by integrating $\rho$ over paths $A(t) = A_0 + ta$:

$$\vartheta(A) - \vartheta(A_0) = \int_0^1 \rho_{A(t)} \left( A(t) \right) dt$$

$$= \int_0^1 \int_M \text{tr} \left( (F_{A_0} + td_{A_0}a + t^2 a \wedge a) \wedge a \right) \wedge *\varphi$$

$$= \frac{1}{2} \int_M \text{tr} \left( dA_0 a \wedge a + \frac{2}{3} a \wedge a \wedge a \right) \wedge *\varphi.$$

It remains to check that (26) is closed, so that this doesn’t depend on the path $A(t)$. Using Stokes’ theorem and $d*\varphi = 0$, the leading term of $\rho$,

$$\rho(a)_{A+b} - \rho(a)_A = \int_M \text{tr} (dA b \wedge a) \wedge *\varphi + O \left( |b|^2 \right),$$

is indeed symmetric:

$$\int_M \text{tr} (dA b \wedge a - b \wedge dA a) \wedge *\varphi = \int_M (d \text{tr} (b \wedge a) \wedge *\varphi) = 0.$$
Hence
\[ \rho(a)_{A+b} - \rho(a)_A = \rho(b)_{A+a} - \rho(b)_A + O(|b|^2), \]
and we check that \( \rho \) is closed comparing the reciprocal Lie derivatives on parallel vector fields \( a, b \) around a point \( A \):

\[
d\rho(a,b)_A = (\mathcal{L}_b \rho(a))_A - (\mathcal{L}_a \rho(b))_A
= \lim_{h \to 0} \frac{1}{h} \left\{ (\rho(a)_{A+hb} - \rho(a)_A) - (\rho(b)_{A+hb} - \rho(b)_A) \right\}
= \lim_{h \to 0} \frac{1}{h^2} \left\{ (\rho(ha)_{A+hb} - \rho(ha)_A) - (\rho(hb)_{A+hb} - \rho(hb)_A) \right\} + O(|h|^3)
= 0.
\]

At least locally, then, the functional \( \vartheta \) descends to the orbit space \( \mathcal{B} \).

To obtain the periods of \( \vartheta \) under gauge action, take \( g \in G \) and consider a path \( \{ A(t) \}_{t \in [0,1]} \subset \mathcal{A} \) connecting \( A \) to \( g.A \). The natural projection then induces a bundle \( E_g \twoheadrightarrow E \) over \( M \times [0,1] \):

\[
\begin{array}{ccc}
E_g & \xrightarrow{\tilde{p}_1} & E \\
\downarrow & & \downarrow \\
M \times [0,1] & \xrightarrow{p_1} & M
\end{array}
\]

and, using \( g \) to identify the fibres \((E_g)_0 \cong (E_g)_1\), we think of \( E_g \) as a bundle over \( M \times S^1 \). Moreover, in some trivialisation, the path \( A(t) = A_i(t) \, dx^i \) gives a connection \( A = A_0 dt + A_i dx^i \) on \( E_g \):

\[
(A_0)_{(t,p)} = 0, \quad (A_i)_{(t,p)} = A_i(t)_p.
\]

The corresponding curvature 2–form is \( F_A = (F_A)_{0i} \, dt \wedge dx^i + (F_A)_{jk} \, dx^j \wedge dx^k \):

\[
(F_A)_{0i} = \dot{A}_i(t), \quad (F_A)_{jk} = (F_A)_{jk}.
\]

The periods of \( \vartheta \) are then of the form

\[
\vartheta (g.A) - \vartheta (A) = \int_0^1 \rho_{A(t)} \left( \dot{A} (t) \right) dt
= \int_{M \times [0,1]} \text{tr} (F_{A(t)} \wedge \dot{A}_i(t) \, dx^i) \wedge dt \wedge \ast \varphi
= \int_{M \times S^1} \text{tr} F_A \wedge F_A \wedge \ast \varphi
= \langle c_2 (E_g) \cup [\ast \varphi], M \times S^1 \rangle.
\]
The Künneth formula for the cohomology of $M \times S^1$ gives
\[
H^4(M \times S^1) = H^4(M) \oplus H^3(M) \otimes H^1(S^1)
\]
and obviously $H^4(M) \sim [\ast \varphi] = 0$, so, denoting $c'_2(E_g)$ the component lying in $H^3(M)$ and $S_g = [c'_2(E_g)]^{PD}$ its Poincaré dual, we are left with
\[
\vartheta(g.A) - \vartheta(A) = \langle [\ast \varphi], S_g \rangle.
\]
Consequently, the periods of $\vartheta$ lie in the set
\[
\left\{ \int_{S_g} \ast \varphi \mid S_g \in H_4(M, \mathbb{R}) \right\}.
\]
That may seem odd because in general this set is dense (there is no reason to expect $\ast \varphi$ to be an integral class). Nonetheless, as long as our interest remains in the study of the moduli space $M^+ = Z(\rho)$ of $G_2$-instantons, as the critical set of $\vartheta$, there is nothing to worry, for the gradient $\rho = d\vartheta$ is unambiguously defined on $B$.

B Theory of operators

Here is a preliminary to the proofs of Proposition 15 and Proposition 17.

**Lemma 19** For a sequence $A \xrightarrow{S} B \xrightarrow{T} C$ of linear operators (of dense domain) between Hilbert spaces, one has:
\[
\ker T = \img S \iff \ker S^* = \img T^*.
\]

**Proof** I claim $B = \img S \oplus \ker S^*$:
\[
b \in (\img S)^\perp \subset B \iff \langle b, Sa \rangle = 0, \ \forall a \in A \\
\iff \langle S^*b, a \rangle = 0, \ \forall a \in A \\
\iff b \in \ker S^*.
\]

Since $\Dom(S) = A$, $S^*$ is closed [Bre II.16], so $\ker S^* \subset B$ is closed and $B = (\ker S^*)^\perp \oplus \ker S^*$. Mutatis mutandis, $B = \ker T \oplus \img T^*$, which yields the claim by uniqueness of the orthogonal complement. □

**Corollary 20** Let $F \xrightarrow{L_1} G \xrightarrow{L_2} H$ be a complex of differential operators between vector bundles with fibrewise inner products; if the associated symbols satisfy $\sigma(L_i^*) = (\sigma(L_i))^*$, then
\[
F \xrightarrow{L_1} G \xrightarrow{L_2} H \text{ is elliptic} \iff H \xrightarrow{L_2^*} G \xrightarrow{L_1^*} F \text{ is elliptic}.
\]
Elliptic complexes are closely related to Fredholm theory, which provides a local model for sets of moduli given as zeroes of sections, as discussed in Subsection 2.1 and the whole of Section 4. I adopt the notation for the Fredholm decomposition of a map [D-K 4.2.5]:

**Proposition 21** A Fredholm map $\Xi$ from a neighbourhood of $0$ is locally right-equivalent to a map of the form

$$\Xi: \quad U \times F \rightarrow V \times G$$

$$\Xi(\xi, \eta) = (L(\xi), \sigma(\xi, \eta))$$

where $L = (D\Xi)_0: U \rightarrow V$ is a linear isomorphism, $F = \ker L$ and $G = \coker L$ are finite-dimensional and $(D\sigma)_0 = 0$.

**Corollary 22** A neighbourhood of $0$ in $Z(\Xi)$ is diffeomorphic to $Z(\nu)$, where

$$\nu: F \rightarrow G$$

$$\nu(\eta) = \sigma(0, \eta).$$

Finally, a result borrowed from [Fin] is essentially the generalisation to Banach spaces of the fact that the determinant of a linear map is continuous:

**Lemma 23** Let $D: B_1 \rightarrow B_2$ be a bounded invertible linear map of Banach spaces with bounded inverse $Q$. If $L: B_1 \rightarrow B_2$ is another linear map with

$$\|L - D\| \leq (2\|Q\|)^{-1},$$

then $L$ is also invertible with bounded right inverse $P$ satisfying

$$\|P\| \leq 2\|Q\|.$$

For its application in the proof of Proposition 17 we are also going to need the following Lemma, saying that the norm of the operator ‘multiplication by a function’ on $L^p$ is controlled by a suitable Sobolev norm of that function.

**Lemma 24** On a base manifold $M = W \times S^1$, fix $f \in L^p_k(M)$ with $k \geq \frac{7}{p}$; then there exists a constant $c = c(M)$ such that

$$\|L_f\| \leq c\|f\|_{L^p_k},$$

where

$$L_f: L^p(M) \rightarrow L^p(M)$$

$$g \mapsto f.g.$$

**Proof** As a direct consequence of Sobolev’s embedding [Lemma 9], one has

$$\|f.g\|_{L^p} = \left(\int_M |f|^p \cdot |g|^p \ d\text{Vol}\right)^{\frac{1}{p}} \leq \|f\|_{C^0} \cdot \|g\|_{L^p}$$

and so

$$\|L_f\| = \sup_{\|g\|_{L^p} = 1} \|f.g\|_{L^p} \leq \|f\|_{C^0}.$$

$\blacksquare$
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