Numerical demonstration of fluctuation dynamo at low magnetic Prandtl numbers

A. B. Iskakov,1 A. A. Schekochihin,2,3,4 S. C. Cowley,1,2 J. C. McWilliams,5 and M. R. E. Proctor4

1Department of Physics and Astronomy, UCLA, Los Angeles, California 90095-1547, USA
2Plasma Physics Group, Blackett Laboratory, Imperial College, London SW7 2BW, United Kingdom
3King’s College, Cambridge CB2 1ST, United Kingdom
4DAMTP, University of Cambridge, Cambridge CB3 0WA, United Kingdom
5Department of Atmospheric Sciences, UCLA, Los Angeles, California 90095-1565, USA

Direct numerical simulations of incompressible nonhelical randomly forced MHD turbulence are used to demonstrate for the first time that the fluctuation dynamo exists in the limit of large magnetic Reynolds number \( Rm \gg 1 \) and small magnetic Prandtl number \( Pm \ll 1 \). The dependence of the critical \( Rm_c \) for dynamo on the hydrodynamic Reynolds number \( Re \) is obtained for \( 1 \leq Re \leq 6700 \). In the limit \( Pm \ll 1 \), \( Rm_c \) is about three times larger than for the previously well established dynamo at large and moderate Prandtl numbers: \( Rm_c \lesssim 200 \) for \( Re \geq 6000 \) compared to \( Rm_c \sim 60 \) for \( Pm \gg 1 \). It is as yet possible to determine numerically whether the growth rate of the magnetic energy is \( \propto Rm^{1/2} \) in the limit \( Rm \to \infty \), as it should if the dynamo is driven by the inertial-range motions at the resistive scale.

PACS numbers: 91.25.Cw, 47.65.-d, 95.30.Qd, 96.60.Hv

Introduction. The amplification of magnetic field by turbulent fluid motion, or dynamo, is believed to be the cause of cosmic magnetism \([1,2,3]\). Two types of turbulent dynamo should be distinguished. The first is the mean-field dynamo defined as the growth of magnetic field at scales larger than the outer (energy-containing) scale \( L \) of the turbulent fluid motion. The second, which is the focus of this Letter, is the fluctuation dynamo (or small-scale dynamo) defined as the growth of magnetic-fluctuation energy at or below the outer scale \([25]\).

 Reflexively, one tends to think of turbulence as an effective mixing mechanism rather than a constructive agent. It is then a remarkable idea that random motions can amplify a magnetic field. The currently accepted qualitative explanation of how the fluctuation dynamo is possible is based on the notion of the random Lagrangian stretching of the field by the fluid motion \([4,5,6,7]\). This picture depends on the assumption that the scale of the stretching (which is the viscous scale \( l_v \), because the motions have the largest turnover rate) is larger than the scale of the field that is stretched (the resistive scale \( l_\eta \)). Whether this is true depends on the magnetic Prandtl number, \( Pm = Rm/Re \), which, in most natural systems, is either very large or very small. When \( Pm \gg 1 \), we have indeed \( l_v/l_\eta \sim Pm^{-1/2} \ll 1 \) (see \([5]\) and references therein). In contrast, when \( Pm \ll 1 \) (while both \( Re \gg 1 \) and \( Rm \gg 1 \)), one expects \( l_v/l_\eta \sim Pm^{-3/4} \gg 1 \) \([6]\), so the resistive scale is in the middle of the inertial range, asymptotically far away both from the viscous and outer scales. Can the field still grow? In the absence of a mechanistic model of the field amplification, the problem is quantitative: the stretching and turbulent diffusion are of the same order at each scale in the inertial range, so which of them wins is not obvious. For over 50 years, the resolution of this problem has fascinated and confounded several generations of scholars, who were intermittently convinced that the fluctuation dynamo did or did not exist.

With the advent of modern scientific computing, numerical simulations have been used to build a case for dynamo as a very generic property of random and chaotic flows \([2,3,4,10,11,12]\). This case has recently been strengthened by a successful laboratory dynamo in a geometrically unconstrained turbulence of liquid sodium \([13]\). However, both in the computer and in the laboratory, it is nearly impossible to access the values of \( Re \) and \( Rm \) that are sufficiently large to resemble real astrophysical situations. It has been especially difficult to model the case of \( Re \gg Rm \gg 1 \), corresponding to the limit of low \( Pm \). The computational challenge in this limit is to resolve two scale separations: \( L \gg l_\eta \gg l_v \). The low-\( Pm \) limit is encountered in the liquid-metal cores of planets (\( Pm \sim 10^{-5} \)), the solar convection zone (\( Pm \sim 10^{-7} \sim 10^{-4} \)), protostellar disks, etc. In the absence of a proof of the fluctuation dynamo at low \( Pm \), the case for the dynamo origin of the small-scale magnetic fields in such systems (e.g., the observed fields in the solar photosphere) has been based on simulations done in the opposite regime of \( Pm \geq 1 \) \([10]\).

Previous numerical investigations of the onset of the fluctuation dynamo at low \( Pm \) \([14,15,16]\) revealed that the critical magnetic Reynolds number \( Rm_c \) for dynamo increased with \( Re \). In this Letter, we report that \( Rm_c \) eventually reaches a finite limit as \( Re \to \infty \), i.e., we demonstrate for the first time that the fluctuation dynamo at asymptotically low \( Pm \) exists. The most important outstanding issue is whether the dynamo we have found is driven by the inertial-range motions at the resistive scale — if it is, its growth rate should be proportional to \( Rm^{1/2} \), which would make it a dominant (and universal!) field-amplification effect compared to any mean-field dynamo due to the outer-scale motions.
**Existence of the Dynamo.** In Fig. 1, we show the growth/decay rates of the magnetic energy $\langle B^2 \rangle$ vs. $Pm$ for five sequences of runs, each with a fixed value of $\eta$. Thus, decreasing $Pm$ is achieved by increasing $Re$ while keeping $Rm$ fixed. The growth rates are calculated via a least-squares fit to the evolution of $\ln (\langle B^2 \rangle)$ vs. time. We find that as $Pm$ is decreased, the growth rate decreases, passes through a minimum and then saturates a constant value, i.e., at fixed $Rm$, $\gamma(Rm, Re) \rightarrow \gamma_\infty(Rm) = \text{const as } Re \rightarrow \infty$. It is natural that such a limit exists because $l_\eta \gg l_\nu \sim LR^{-3/4} \rightarrow 0$ and one cannot expect the magnetic field to “know” exactly how small the viscous scale is. The nontrivial result is that, as $Rm$ increases, the entire curve $\gamma(Re, Rm)$ is lifted upwards, so the asymptotic values $\gamma_\infty(Rm) > 0$. Although we cannot at current resolutions determine these positive asymptotic values, our judgement is that Fig. 1 provides sufficient evidence for claiming that such values exist and are positive.

The Stability Curve. In Fig. 2, we show the reconstructed stability curve $Rm_\infty(Re)$: each point on the curve is obtained by linear interpolation between a decaying and a growing case. We see that $Rm_\infty$ increases with $Re$, reaches a maximum $Rm_\infty^{(\text{max})} \sim 350$ at $Re \sim 3000$, and then decreases. Since the exact value of the viscous scale cannot matter when $Re \gg Rm, Rm_\infty(Re) \rightarrow Rm_\infty(\infty) = \text{const as } Re \rightarrow \infty$. We cannot as yet obtain this

**Numerical Set Up.** The equations of incompressible magnetohydrodynamics

\begin{align}
\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla p - \nu_0 |\nabla|^{\eta} \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{B} + \mathbf{f},
\end{align}

\begin{align}
\partial_t \mathbf{B} + \mathbf{u} \cdot \nabla \mathbf{B} &= -\mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B},
\end{align}

are solved in a periodic code of size 1, using a standard pseudospectral code. Here $\mathbf{u}$ is the velocity and $\mathbf{B}$ the magnetic field (in velocity units). The density is constant and equal to 1. The velocity is forced by a random nonhelical homogeneous white-noise body force, which injects energy into the velocity components with wave numbers $|k| \leq \sqrt{2} k_0$, where $k_0 = 2\pi$ is the box wave number. Because the forcing is a white noise, the average injected power is fixed: $\varepsilon = \langle \mathbf{u} \cdot \mathbf{f} \rangle = 1$, where the angle brackets stand for volume and time averaging. The numerical integration is continued only for as long as is needed to obtain a converged value of the growth/decay rate. In all of our runs, the magnetic field is energetically much weaker than the velocity at all times, so the Lorentz force is never important.

The maximum resolution we could afford was $512^3$. In order to increase the range of accessible Reynolds numbers, we performed simulations both with the Laplacian viscosity ($n = 2$ in Eq. \ref{eq:laplacian}) and with the 8th-order hyperviscosity ($n = 8$). We define $Rm = \langle u^2 \rangle^{1/2} / \nu k_0$, $Re = \langle u^2 \rangle^{1/2} / \nu k_0$, $Pm = \nu / \eta$, where $\nu = \nu_2$ for the Laplacian runs and $\nu = \nu_{\text{eff}} = \varepsilon / \langle |\nabla u|^2 \rangle$ for the hyperviscous ones. We believe that using hyperviscosity is justified for $Pm \ll 1$ because the resistive scale in this limit is much larger than the viscous scale, $l_\eta \gg l_\nu$, and the magnetic properties of the system should be independent of the form of viscous regularization.

FIG. 1: Growth/decay rate of $\langle B^2 \rangle$ vs. $Pm$ for $\eta = 4 \times 10^{-3}$ ($Rm \sim 60$), $\eta = 2 \times 10^{-3}$ ($Rm \sim 110$), $\eta = 10^{-3}$ ($Rm \sim 230$), $\eta = 5 \times 10^{-4}$ ($Rm \sim 450$), and $\eta = 2.5 \times 10^{-4}$ ($Rm \sim 830$).

FIG. 2: The stability curve $Rm_\infty(Re)$: “Error bars” connect (Re, Rm) for decaying and growing runs used to obtain points on the stability curve. Stability curves based on the Laplacian and hyperviscous runs are shown separately. For comparison, we also plot the $Rm_\infty(Re)$ curve obtained in simulations employing TG1 [17] and TG2 forcing [12] (the three highest-Re points in the latter case were obtained by large-eddy simulations). The values of Re and Rm are recalculated according to our definitions, using the forcing wavenumber $k_0$, rather than the dynamical integral scale as in [12, 17].
FIG. 3: Normalized spectra of kinetic energy (compensated by $k^{5/3}$) and (growing) magnetic energy for fixed $\eta = 5 \times 10^{-4}$ (the $Rm \sim 450$ sequence from Fig. 1) and increasing $Re$: (a) Laplacian runs, (b) hyperviscous runs.

asymptotic value. Unless the stability curve has multiple extrema (which we consider unlikely), $Rm_{c(\infty)} \lesssim 200$. Note that this value is only about 3 times larger than the well-known critical value $Rm_{c} \simeq 60$ for the fluctuation dynamo at $Pm \geq 1$ [7, 9, 16].

Considering that there is a small but measurable difference between the stability curves for the Laplacian and hyperviscous simulations, how universal are our results? We believe it is plausible to argue, as we did above, that the existence of the dynamo in the limit $Rm \ll Re \to \infty$ does not depend on the nature of the viscous regularization. We also think that the asymptotic value $Rm_{c(\infty)}$ is likely to be robust. However, the full functional dependence $Rm_{c}(Re)$ is certainly not universal. Indeed, let us consider what determines the shape of the stability curve in the transition region between the high- and low-$Pm$ limits. When $Re < Rm$, $l_{v} > l_{\eta}$. As $Re$ is increased, the spectral bottleneck associated with the viscous cutoff moves past the resistive scale $l_{\eta}$ until finally $l_{v} \ll l_{\eta}$ and the resistive scale is in the inertial range. This transition is illustrated by Fig. 3. The properties of the velocity field around the viscous scale are obviously not independent of the viscous regularization. Therefore, the $\gamma(Re,Rm)$ and $Rm_{c}(Re)$ cannot be universal in the transition region. In particular, since the bottleneck is narrower for the hyperviscous runs, the transition in the parameter space should be sharper.

Magnetic-Energy Spectrum. The shape of the magnetic-energy spectrum is qualitatively different for $Pm \geq 1$ and $Pm \ll 1$ (see Fig. 3). At $Pm$ above and just below unity, the spectrum has a positive slope and its peak is at the resistive scale. This is typical for the fluctuation dynamo at $Pm \geq 1$ — in the limit $Pm \gg 1$, the Kazantsev [18] $k^{+3/2}$ spectrum emerges [1]. As $Pm$ is decreased, the spectrum flattens and then appears to develop a negative slope in the inertial range. At current resolutions, it is not possible to determine definitively what the asymptotic spectral slope is and whether the spectral peak is independent of $Rm$ or moves with the resistive scale as $k_{\text{peak}} \propto Rm^{3/4}$.

Comparison with Simulations with a Mean Flow. Several authors [12, 17, 19, 20, 21] have been motivated by the liquid-metal dynamo experiments to investigate the dynamo action at low $Pm$ in numerical simulations where the forcing was spatially inhomogeneous and constant in time rather than random. The velocity field in these simulations consisted of a time-independent mean flow and an energetically a few times weaker fluctuating component (turbulence). The stability curves $Rm_{c}(Re)$ obtained in these studies have an entirely different origin than ours. In order to illustrate the difference, Fig. 2 shows the stability curves for simulations with Taylor-Green forcing, using published data [12, 17]. We see that the dynamo threshold for the simulations with a mean flow is much lower than for our homogeneous simulations. The difference is not merely quantitative. The mean flows in question are mean-field dynamos (even in the case of the nonhelical Taylor-Green forcing). This is confirmed by the ordered box-scale structure of the growing magnetic field reported for these simulations at $Pm \geq 1$ (the lower part of their stability curve). For $Pm \geq 1$, the threshold for the field amplification is $Rm_{c} \sim 10$, which is a typical situation for mean-field dynamos [11]. The presence of magnetic energy at small scales is probably due to the random tangling of the mean field by turbulence, rather than to the fluctuation dynamo, because
Rm is below the fluctuation-dynamo threshold. The increase of Rm with Re in these simulations is due to the interference by the turbulence with the dynamo properties of the mean flow [19, 22]. It has not been checked whether the turbulence in these simulations might itself be a dynamo. Comparison of the two stability curves in Fig. 2 suggests that this can only happen at much larger Rm than studied so far.

Outstanding Questions. The most important factual issue that remains unresolved is whether the low-Pm dynamo we have found is due to the inertial-range motions. The most important physical question is what is the physical mechanism that makes the dynamo possible.

If the local (in scale space) interaction of the inertial-range motions with the magnetic field is capable of amplifying the field, the dominant contribution to such a dynamo should be from the motions at the resistive scale \( \eta \sim (\eta^3/\varepsilon)^{1/4} \sim LRm^{-3/4} \), where the stretching rate is maximal. The growth rate of the magnetic energy should scale as the stretching rate: \( \gamma \sim (\varepsilon/\eta)^{1/2} \sim (u_{\text{rms}}/L)Rm^{1/2} \). For Rm \( \gg 1 \), such a dynamo would always be faster than a mean-field or any other kind of dynamo associated with the outer-scale motions, because the latter cannot amplify the field faster than at the rate \( u_{\text{rms}}/L \). Thus, the most pressing task for future numerical studies is to determine whether \( \gamma \) scales as Rm\(^{1/2}\) or reaches an Rm-independent limit. If the latter is the case, one will have to conclude that it is the outer-scale motions that act as a dynamo despite (or in concert with) the turbulence in the inertial range. Unlike the inertial-range dynamo, the characteristics of such a dynamo would not be universal.

Theoretical predictions for a Gaussian white-noise velocity field (the Kazantsev [18] model) strengthen the case for an inertial-range dynamo. For a certain range of scaling exponents of the velocity correlation function, it is possible to prove that the Kazantsev field is a dynamo [23, 24]. The dynamo threshold for Pm \( \ll 1 \) is predicted to be Rm\(^{\infty} \sim 400 \) (using our definition of Rm) — an overestimate by only (or by as much as) a factor of 2. The difficulty in deciding whether this theory applies lies in the unknown effect that assuming zero correlation time has on the dynamo properties of the inertial-range velocity field (in real turbulence, this correlation time is not only not small but also scale dependent). The main problem is that the result of Kazantsev theory is purely mathematical and that we do not have a physical model of the inertial-range dynamo.

Conclusions. We have established that the fluctuation dynamo exists in nonhelical randomly forced homogeneous turbulence of a conducting fluid with low Pm. The critical Rm in this regime is approximately 3 times larger than for Pm \( \geq 1 \). The nature of the dynamo and its stability curve Rm\(_c\) (Re) are different from the low-Pm dynamo found in simulations and liquid-metal experiments with a mean flow. The physical mechanism that enables the sustained growth of magnetic fluctuations in the low-Pm regime is unknown. Is it not as yet possible to determine numerically whether the fluctuation dynamo is driven by the inertial-range motions at the resistive scale and consequently has a growth rate \( \propto Rm^{1/2} \), or rather is an outer-scale effect and has a constant growth rate comparable to the turnover rate of the outer-scale motions.

We have benefited from discussions with S. Fauve, N. Kleeorin, J.-F. Pinton, and I. Rogachevskii. We thank V. Decyk who kindly let us use his FFT libraries. A.B.I. was supported by the USDOE Center for Multi-scale Plasma Dynamics, A.A.S. by PPARC. Simulations were done on NCSA (Illinois), UKAFF (Leicester), and the Dawson cluster (UCLA).