Dynamics of a quantum quench in an ultra-cold atomic BCS superfluid

Chih-Chun Chien and Bogdan Damski

Theoretical Division, MS B213, Los Alamos National Laboratory, Los Alamos, NM, 87545, U.S.A.

We study dynamics of an ultra-cold atomic BCS superfluid driven towards the BCS superfluid-Fermi liquid quantum critical point by a gradual decrease of the pairing interaction. We analyze how the BCS superfluid falls out of equilibrium and show that the non-equilibrium gap and Cooper pair size reflect critical properties of the transition. We observe three stages of evolution: adiabatic where the Cooper pair size is inversely proportional to the equilibrium gap, weakly non-equilibrium where it is inversely proportional to the non-equilibrium gap, and strongly non-equilibrium where it decouples from both equilibrium and non-equilibrium gap. These phenomena should stimulate future experimental characterization of non-equilibrium ultra-cold atomic BCS superfluids.

PACS numbers: 03.75.Kk, 74.20.Fg, 74.40.Kb

Ultra-cold atoms provide quantum simulators for emulating challenging theoretical models. This offers invaluable opportunities for the study of many-body systems which used to be experimentally inaccessible by conventional condensed matter setups [1]. The ability to make the parameters of the emulated models time-dependent encourages exciting studies of non-equilibrium dynamics of many-body quantum systems [2]. This opens up fresh prospects for interdisciplinary research of dynamics of phase transitions, where universal- ity of the critical behavior links dynamics of distinct systems quenched across a critical point [3]. Examples include superfluids [4], superconductors [5], spin chains [2], cosmological models [6], and others [3].

In conventional superconductors, observations of the dynamics of superconductivity would require a real-time change of either dopant concentration or lattice structure, injection of quasi-particles, or exposition to microwave or light [7,8], which is very difficult. The dynamics of BCS wavefunction is also of interest to nuclear physicists [9]. Such dynamics can be realized in cold atoms by adjusting the inter-atomic interactions, which is routinely done in current experiments. Moreover, the dynamics of a BCS superfluid can be mapped to a central-spin problem [10]. Our goal is to study the non-equilibrium dynamics of an s-wave BCS superfluid resulting from a ramp of attractive pairing interactions.

Previous work on the quench dynamics of a BCS superfluid focused on the non-equilibrium gap following an instantaneous change of the pairing interaction [11-13]. Here we explore finite-rate quench dynamics which, unlike an instantaneous one, allows for an observation of all stages of the evolution ranging from adiabatic to strongly non-equilibrium ones. This can be done in a controllable way by changing the quench rate, which also allows for observations of universal critical properties of the phase transition through the scaling relations of out-of-equilibrium observables. These experimentally accessible features should make a substantial progress in the understanding of non-equilibrium physics of BCS superfluids. A linear ramp of the external magnetic field around the unitary point is predicted to drive the pairing gap out of equilibrium [14].

Several fundamentally important questions will be addressed: What is the non-equilibrium gap and Cooper pair size during the ramp of the pairing interactions and how are the two quantities related? How do those non-equilibrium quantities reflect the critical properties characterizing the BCS superfluid-Fermi liquid (BCS-FL) phase transition?

We consider a three-dimensional homogeneous two-component (σ = ↑, ↓) unpolarized Fermi gas. Although in experiments there is a trap potential, the center of the Fermi gas can be considered as locally homogeneous and the dynamics we investigate should be observable. For weakly attractive or repulsive interactions the ground state is a BCS superfluid [15] or a normal Fermi liquid [16], respectively. The Hamiltonian with a time-dependent BCS-type pairing interaction is

$$\hat{H} = \sum_{k\sigma} \epsilon_k \hat{c}_k^{\dagger}\hat{c}_{k\sigma} - \lambda(t) \sum_{kl} \hat{c}_{k\uparrow}^{\dagger}\hat{c}_{-l\downarrow}\hat{c}_{-l\uparrow}\hat{c}_{k\downarrow},$$

Here $\epsilon_k = \hbar^2 k^2/2m$ and $\lambda(t)$ is the time-dependent coupling constant from interactions between atoms in different hyperfine states, which is regularized as $\lambda^{-1} = m/4\pi\hbar^2 a + \sum_{k<k}(1/2\epsilon_k)$ [17]. $a$ is the two-body s-wave scattering length and $K \gg k_F$ is a momentum cutoff [18]. We choose as our units the Fermi momentum and energy $k_F$ and $E_F$ of a non-interacting Fermi gas with the same density. Experimentally $a$ is tuned by an external magnetic field $B$ via $a = a_{bg}[1 - \delta B/(B - B_0)]$, where $a_{bg}$ is the background scattering length, while $\delta B$ and $B_0$ are the width and position of the Feshbach resonance [15]. The $T = 0$ BCS-FL quantum critical point corresponds to the point where the system becomes a non-interacting (free) Fermi gas. In BCS-Leggett theory the critical point is at $a = 0$ and it can be located numerically for a finite-range potential [19]. This critical point for $^{40}$K atoms is located at 209.9$\text{G}$ [20,21].

The excitation gap on the BCS side ($a < 0$) closes exponentially near the critical point. It is well approximated for $k_F a > -1$ by [17]

$$\Delta_{\text{eq}} \approx 8E_F e^{-\pi/2k_F|a|/e^2}.$$

Importantly, the excitation gap in BCS theory serves as the order parameter as well. This is a feature that distinguishes a BCS superfluid from other systems previously studied in the field of dynamics of quantum phase transitions [2].
The BCS coherence length (after adjusting an overall constant) behaves near the quantum critical point as [7]

\[ k_F \xi^{eq} = E_F / \sqrt{2 \Delta^{eq}}. \] (3)

Equations (2) and (3) set the critical exponents to \( z = 1 \) and \( \nu = \infty \) through the relations \( \xi^{eq} \sim (\Delta^{eq})^{-1/z} \) and \( \xi^{eq} \sim |a|^{-\nu} \). The infinite \( \nu \) exponent makes a BCS superfluid distinct from typical systems considered in the dynamics of phase transitions where \( \nu = O(1) \) [2]. To account for this singular behavior we will propose a quench protocol allowing for efficient tests of quench-induced scaling relations.

To numerically study the effect of driving, we employ the generalized BCS-Leggett wavefunction [17]

\[ |\Psi(t)\rangle = \prod_k (u_k(t) + v_k(t)e^{i\theta} \hat{c}_k - \hat{c}_k^\dagger |0\rangle), \] (4)

which provides a consistent description of BCS-Bose Einstein condensate (BEC) crossover but does not include medium effects [11,13,15,17,22], which renormalize the equilibrium gap. The medium effects can introduce corrections to the dynamics, which may be evaluated non-perturbatively beyond BCS-Leggett theory [15]. Minimizing the quantum action \[ \int dt \langle \dot{\Psi} | \hat{H} | \Psi \rangle / 2 + c.c. - \langle \Psi | \hat{H} | \Psi \rangle \] [23], we obtain the following equations:

\[ \hat{h}_k = 2ig_k \Delta + 2id_k c_k, \quad \hat{g}_k = id_k \Delta^* - id_k^* \Delta. \] (5)

Here \( d_k = 2u_k v_k^* \), \( g_k = |v_k|^2 - |u_k|^2 \), and the non-equilibrium gap function \( \Delta(t) = \lambda(t) \sum_k d_k(t)/2 \). Eqs. (5) are consistent with those in Ref. [13]. Here \( \tau_0 = \hbar / E_F \) is the unit of time. Eqs. (4) and (5) can be applied to the whole BCS-BEC crossover, though here we focus on the quench dynamics close to the \( a = 0 \) critical point.

The quench dynamics proceeds as follows. Initially the system is in its ground state at \( k_F a \ll -1 \), and the pairing interaction is slowly decreased towards \( a = 0 \) (the quantum critical point) by tuning away from the Feshbach resonance. In the beginning the system evolves adiabatically. At some point, say \( a = -\bar{a} \), the gap becomes so small that the system will fall out of equilibrium. This happens when the reaction time \( \hbar / \Delta^{eq} \) of the BCS superfluid becomes comparable to the inverse of the quench rate \( \Delta^{eq} / |\Delta^{eq}| \) [24]:

\[ \frac{\hbar}{\Delta^{eq}(-\bar{a})} = \mathcal{C} \frac{\Delta^{eq}}{|\Delta^{eq}|} \bigg|_{a = \bar{a}}, \] (6)

where \( \mathcal{C} \) is a constant and the dot stands for \( d/dt \).

Basic understanding of what happens next comes from the Kibble-Zurek theory [3,6], which has not been verified for a BCS superfluid yet. This theory predicts that a system switches suddenly from adiabatic to diabatic dynamics: its state is frozen near the critical point. Thus in the non-equilibrium stage of the evolution, the order parameter \( \Delta \) and coherence length \( \xi \) should be given by their values \( \Delta \) and \( \xi \) at the border between adiabatic and diabatic regimes: \( \Delta = \Delta^{eq}(-\bar{a}) \), \( \xi = \xi^{eq}(-\bar{a}) \). In reality there is no sudden freeze out, but one can investigate if the following scaling relations hold near the critical point:

\[ \Delta(t) = \hat{\Delta} f(t/\hat{\tau}), \quad \xi(t) = \hat{\xi} g(t/\hat{\tau}), \] (7)

where \( f \) and \( g \) are unknown functions, \( \hat{\tau} \) is the time left to reaching the critical point at the onset of non-equilibrium dynamics, \( a(-t) = -\bar{a} \), and the critical point is reached at \( t = 0 \). Once the quench protocol, \( a(t) \), is specified, quantitative predictions about system dynamics can be made.

The linear ramp of the external magnetic field proposed in Ref. [14] results in a non-linear drive of \( k_F a(t) \) and makes it difficult to predict the dynamics. A more typical quench protocol assumes a linear quench [2]. This corresponds here to having \( k_F a(t) \propto t / \tau_Q \), where \( \tau_Q \) provides the quench time-scale: the slower the system is driven, the larger \( \tau_Q \) is. Such a quench can be realized by a non-linear (in time) sweep of \( B(t) \) near a Feshbach resonance. For a system with an exponentially closing gap, however, it introduces logarithmic corrections (in \( \tau_Q \)) to the non-equilibrium length scale \( \xi \) [25], which demands very long quench times for verification of scaling relations. To overcome this problem, we propose the following quench protocol

\[ \Delta^{eq}(a(t)) e^2 / 8E_F = -t / \tau_Q, \] (8)

where time goes from \( t_i < 0 \) to \( 0 \) such that \( k_F a(t_i) < -1 \), and the \( e^2 / 8 \) prefactor is for convenience. This can be inverted
for \( k_F a > -1 \) using Eq. (2)

\[
k_F a(t) = \pi / \ln(t^2 / \tau_Q^2).
\]

(9)

Such a ramp may be induced by \( B(t) = B_0 + \delta B/[1 - \pi/|k_F a(t)| \ln(t^2 / \tau_Q^2)] \). To avoid essential complications with the exact inversion of Eq. (8), we use the protocol shown in Eq. (9) through the whole evolution and choose the quench times so slow that the system is brought to \( k_F a = -1 \) nearly adiabatically. Therefore, significant excitations in our calculations only happen at \( k_F a > -1 \), where Eqs. (8) and (9) become consistent. From Eqs. (8) and (9) one gets

\[
k_F a = \pi / \ln(8C \tau_Q / \tau_0 \pi^2), \quad \hat{\Delta} \propto \sqrt{\tau_0 / \tau_Q E_F},
\]

\[
\hat{t} \propto \sqrt{T_Q / \tau_0}, \quad k_F \xi \propto \sqrt{\tau_0 / \tau_Q}.
\]

(10)

Eqs. (10) complement the scaling relations shown in Eqs. (7). A simple power-law dependence of \( \Delta \), \( \hat{t} \) and \( \xi \) on the quench time scale will serve as a verification of our predictions.

Typical dynamics of the gap function is illustrated in Fig. 1. Away from the critical point the gap is large enough to enforce adiabatic evolution in which \( \Delta \) obtained from Eq. (5) matches \( \Delta^\text{eq} \). As the gap becomes small enough, the relative deviation, \( (|\Delta| - \Delta^\text{eq}) / \Delta^\text{eq} \), grows fast (see the upper inset of Fig. 1).

From the simulations, this happens when the time left to reach the critical point, \( \hat{t} \), behaves as \( \tau_Q^{0.5} \), which agrees with the prediction of Eq. (10). Moreover, the difference \( |\Delta| - \Delta^\text{eq} \) is controlled by the quench rate and increases as \( \tau_Q \) decreases. Finally, we have checked that the scaling relation for \( \Delta \) near the critical point, Eq. (7), holds. Very close to the \( \alpha = 0 \) critical point both \( \Delta \) and \( \Delta^\text{eq} \) approach zero. This reflects the vanishing coupling constant \( \lambda(t) \) and does not imply that the system has reached an equilibrium state.

Equilibrium gap can be inferred from radio-frequency (RF) spectroscopy. Modification of this technique to non-equilibrium systems has been proposed [26]. Therefore we expect that the non-equilibrium dynamics of the gap function described above will be measured experimentally.

Next we study the Cooper pair size, which is defined as

\[
\xi_c = \sqrt{\langle \phi | r^2 | \phi \rangle / \langle \phi | \phi \rangle} = \sqrt{\langle \phi | - \nabla_E^2 | \phi \rangle / \langle \phi | \phi \rangle},
\]

where \( | \phi \rangle \) stands for the Cooper pair wave-function: \( \langle k | \phi \rangle = u_k v_k^* \) [13, 27]. In equilibrium the Cooper pair size \( \xi^\text{eq}_c \) is the same as the coherence length: \( \xi^\text{eq}_c = \xi^\text{eq} \). In a non-equilibrium situation it is interesting to address if the Cooper pair size encodes a non-equilibrium length scale and remains finite despite the fact that \( \xi^\text{eq}_c \) diverges near the critical point.

Typical evolution of \( \xi_c \) is depicted in Fig. 2. It consists of three consecutive stages. In the first one, the system is away from the critical point and it evolves adiabatically: \( \xi_c \approx \xi^\text{eq}_c \). In the second one, the system enters non-equilibrium dynamics and so \( \xi_c \neq \xi^\text{eq}_c \). Interestingly, we find \( k_F \xi_c \sim E_F / |\Delta| \) there. Thus, the relation between the non-equilibrium Cooper pair size and \( |\Delta| \) is the same as that between \( \xi^\text{eq}_c \) and \( \Delta^\text{eq} \). Importantly, it implies that the measurement of the non-equilibrium gap proposed in [26] can be implemented to estimate the non-equilibrium Cooper pair size.

In the third stage both \( k_F \xi^\text{eq}_c \) and \( E_F / |\Delta| \) diverge, but the non-equilibrium pair size stays finite. Now \( \xi_c \) decouples from the gap function and depends solely on the non-equilibrium scales \( \xi \) and \( t \) through the relation shown in Eq. (7), which is depicted in Fig. 3. The inset of Fig. 3 quantitatively confirms that the non-equilibrium length scale \( \xi \) persists in the system till the end of evolution and scales as \( \tau_Q^{0.5} \), which agrees with the prediction of Eqs. (10). Surprisingly, the existence of the two stages of non-equilibrium dynamics is not predicted by the Kibble-Zurek theory.

How to measure the non-equilibrium Cooper pair size is an important open question. There are experiments “measuring” Cooper pair size by inverting the gap obtained from RF spectroscopy [22]. This will not work close to the critical point (third stage of evolution on Fig. 3) where \( \xi_c \) and \( \Delta \) are decoupled. One possibility is to generalize to a non-equilibrium setting a scheme from [26], where \( \xi_c \) is inferred from the damping rate of the collective mode of a coexisting BEC.

We have shown that close to the BCS-FL quantum phase transition at \( T = 0 \), BCS superfluids exhibit multi-stage dynamics with scaling behavior consistent with Kibble-Zurek theory. At finite temperature two additional effects show up. First, there is a classical (thermal) phase transition from a BCS superfluid to a normal Fermi gas at \( k_B T_c \approx 0.57 \Delta^\text{eq}(T = 0) \) [3], where \( k_B \) is the Boltzmann constant. Second, quasi-particles (QPs) from broken Cooper pairs exist at \( T > 0 \). Their relaxation time is estimated to be \( \tau_{\text{QP}} \approx h E_F / (\Delta^\text{eq}_{\text{GL}})^2 \) [11], where the Ginzburg-Landau gap function \( \Delta^\text{eq}_{\text{GL}} \) corresponds to the effective gap in the presence of both QPs and Cooper pairs. Below we assume that the temperature changes insignificantly during the slow driving and estimate the influence of these finite temperature effects.

If we start the quench dynamics from an equilibrium state at \( T \ll T_c(a) \) and \( k_F a \ll 0 \), \( \Delta^\text{eq}_{\text{GL}} \approx \Delta^\text{eq} \) initially and the
expected scaling is critical point at time \( \tau_0 \) quasi-particle relaxation time \( \tau_{qp} \) reach the quantum critical point is following the quench dynamics of Eq. (9) when the time left to perfluid and QPs follow the ramp adiabatically. Comparing data are shown as pluses. They are fitted by densate, this to the onset of the non-equilibrium dynamics of the con- that there is still a condensate at the time when quantum criti- will eventually be swept out of the BCS superfluid phase. Since the equilibrium \( T_c \) vanishes as \( a \to 0 \), the system will eventually be swept out of the BCS superfluid phase. This raises another important question: Is the BCS superfluid-Fermi gas classical phase transition still well defined when both QPs and the superfluid are out of equilibrium? We point out that crossing a classical critical point may also imprint additional non-equilibrium length scale encoding the critical exponents of the classical transition \([3]\).

In summary, we have shown that a finite-rate quench allows for controlled (through the quench time scale \( \tau_Q \)) studies of non-equilibrium dynamics of a BCS superfluid. The quench imprints non-equilibrium energy and length scales visible through the gap and Cooper pair size. Our results call for further development of experimental techniques for studies of non-equilibrium cold Fermi gases and point out that finite-temperature effects may induce additional scaling behavior of non-equilibrium BCS superfluids in a finite-rate quench.

This work is supported by U.S. Department of Energy through the LANL/LDRD Program.

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