Ground-states for systems of $M$ coupled semilinear Schrödinger equations with attraction-repulsion effects: characterization and perturbation results

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Abstract

We focus on the study of ground-states for the system of $M$ coupled semilinear Schrödinger equations with power-type nonlinearities and couplings. We extend the characterization result in [2] to the case where both attraction and repulsion are present and cannot be studied separately. Furthermore, we derive some perturbation and classification results to study the general system where components may be out of phase. In particular, we present several conditions to the existence of nontrivial ground-states.

Keywords: Coupled semilinear Schrödinger equations; ground-states; nontrivial solutions; perturbations.

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1 Introduction

In this work, we consider the system of $M$ coupled semilinear Schrödinger equations

$$i(v_i)_t + \Delta v_i + \sum_{j=1}^{M} k_{ij} |v_j|^{p+1} |v_i|^{p-1} v_i = 0, \quad i = 1, \ldots, M$$

where $V = (v_1, \ldots, v_M) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^M$, $\Omega \subset \mathbb{R}^N$ open with smooth boundary, $k_{ij} \in \mathbb{R}$, $k_{ij} = k_{ji}$, and $0 < p < 2/(N-2)^+$ (we use the convention $2/(N-2)^+ = +\infty$ if $N = 1, 2$, and $2/(N-2)^+ = 2/(N-2)$ if $N \geq 3$). Given $1 \leq i \neq j \leq M$, if $k_{ij} \geq 0$, one says that the coupling between the components $v_i$ and $v_j$ is attractive; if $k_{ij} < 0$, it is repulsive.

When we look for nontrivial periodic solutions of the form $V = e^{i\omega t} U$, with $U = (u_1, \ldots, u_M) \in (H^1_0(\Omega))^M$ (called bound-states), we are led to the study of the system

$$\Delta u_i - \omega u_i + \sum_{j=1}^{M} k_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i = 0, \quad i = 1, \ldots, M.$$  

On the other hand, one may also consider periodic solutions where the time-frequency is not necessarily the same for each component (one then says that the components are out of phase).
These solutions are of the form $V = (e^{i\omega t}u_1, ..., e^{i\omega t}u_M)$ and the stationary system is

$$\Delta u_i - \omega_i u_i + \sum_{j=1}^{M} k_{ij}|u_j|^{p+1}|u_i|^{p-1}u_i = 0 \quad i = 1, ..., M. \quad \text{(M-NLS')}
$$

Notice that, if $M = 1$ and $\Omega = \mathbb{R}^N$, the presence of $\omega > 0$ may be eliminated by a suitable scaling. However, in any other case, such a procedure is no longer possible.

In any case, for both physical and mathematical reasons, one is interested in bound-states which have minimal action among all bound-states, the so-called ground-states. The set of such solutions is noted $G$. For $\Omega = \mathbb{R}^N$, in the scalar case, one may prove that there is a unique ground-state $Q$ (modulo translations and rotations, see [1]). For a general $\Omega$, the problem has not been completely solved. However, it is known, for example, that there exists a ground-state if

$$\omega > -\lambda_1(\Omega), \quad \lambda_1(\Omega) = \begin{cases} \text{first eigenvalue of } -\Delta \text{ on } H_0^1(\Omega), & \Omega \text{ bounded} \\ 0 & \Omega \text{ infinite parallelepiped} \end{cases}. \quad \text{(1.2)}
$$

The vector case is much more complex. The existence of ground-states for system (M-NLS) on $\Omega = \mathbb{R}^N$ has been proven under the sufficient and necessary condition

$$\exists U = (u_1, ..., u_M) \in (H^1(\mathbb{R}^N))^M : \sum_{i,j=1}^{M} k_{ij} \int |u_i|^{p+1}|u_j|^{p+1} > 0 \quad \text{(1.3)}
$$

using a suitable variational formulation. We note that the result is still true for any $\Omega$. In fact, one proves that the set of ground-states is the set of minimizers of

$$\inf \left\{ \int \sum_{i=1}^{M} \omega_i |u_i|^2 + |\nabla u_i|^2 : \sum_{i,j=1}^{M} k_{ij} \int |u_i|^{p+1}|u_j|^{p+1} = \lambda \right\}, \quad \text{(1.4)}
$$

for a precise and explicit $\lambda$. To prove existence of minimizers, the main difficulty is the strong compactness of the minimizing sequence in $L^{2p+2}$. In $\Omega$ bounded, this is trivial, since one has the compact injection $H_0^1(\Omega) \hookrightarrow L^{2p+2}(\Omega)$. For $\Omega = \mathbb{R}^N$, one uses the concentration-compactness principle and proves the compactness alternative. For $\Omega$ an infinite parallelepiped, one simply extends the minimizing sequence to $\mathbb{R}^N$ by 0 and apply the technique for the whole space. Since the existence of ground-states for general $\Omega$ is an open problem, we shall make the following assumption

"The set of all ground-states for (M-NLS) over $\Omega$, $G$, is nonempty." \quad \text{(Exist)}

One then may pose a number of questions: is there a unique positive ground-state? Does a ground-state have all components different from 0 (called nontrivial ground-states)? Can we obtain a simple characterization of the family of ground-states? Are the solutions positive and radially decreasing?

Regarding system (M-NLS), for $\Omega = \mathbb{R}^N$, a recent work [2] has answered to these questions for a very large family of matrices $K = (k_{ij})_{1 \leq i,j \leq M}$. Essentially, if one may group the components in such a way that two components attract each other if and only if they belong to the same group, one may answer all questions above in a satisfactory fashion. One may also prove that, in the case where all components attract each other, the result is extendible to any $\Omega$. If this grouping hypothesis fails, the situation becomes much more difficult. The reason is that two components may repel each other directly but, by transitivity, they also attract each
Figure 1: The simplest balanced system: the signs indicate whether the components attract or repel each other. Though components 1 and 3 repel each other, they are both attracted to component 2. This case was studied for the first time in [4].

other (see figure 1). Then the balance between these forces is not clear and the analysis is not straightforward.

For the general system \((M\text{-NLS}')\), it is impossible in general to obtain a characterization similar to the one for \((M\text{-NLS})\). Results obtained so far consider mostly the case \(M = 2, \Omega = \mathbb{R}^N\). For this case, the behaviour of the system with respect to the parameters is very well understood. Uniqueness of positive radial solution has been considered in [9] and [16], mostly through a careful analysis of the system of ODE’s that one obtains when considering radial solutions. The question that has attracted more attention in the past few years is the existence of nontrivial ground-states. We advise the reader to check [10], [7], [12] (and references therein).

In this work, we shall consider both systems \((M\text{-NLS})\) and \((M\text{-NLS}')\). For system \((M\text{-NLS})\), we complete the work started in [2] and obtain the characterization of ground-states regardless of coupling coefficients. We note by \(G^+\) the set of nontrivial ground-states.

**Theorem 1.** Consider system \((M\text{-NLS})\) and suppose \((\text{Exist})\). Define \(f : (\mathbb{R}_0^+)^M \to \mathbb{R}\),

\[
f(X) = \sum_{i,j=1}^{M} k_{ij} x_i^{p+1} x_j^{p+1} \tag{1.5}
\]

and let \(X \subset (\mathbb{R}_0^+)^M\) be the set of solutions of

\[
f(X_0) = f_{\text{max}} := \max_{|X| = 1} f(X), \quad |X_0| = 1. \tag{1.6}
\]

Then \(U \in G\) if and only if there exist \(a_i \in \mathbb{C}, 1 \leq i \leq M\), such that \((f_{\text{max}})^{1/2p}([a_1],...,[a_M]) \in X\) and \(u_0\) ground-state of

\[
\Delta u - \omega u + |u|^{2p} u = 0 \quad \text{on } \Omega \tag{1.7}
\]

such that

\[
U = (a_i u_0)_{1 \leq i \leq M}. \tag{1.8}
\]

In particular, \(G^+ \neq \emptyset\) if and only if there exists \(X \in X\) such that \(X_i \neq 0, i = 1,...,M\). Moreover \(G = G^+\) if and only if all elements of \(X\) have no zero components.

**Remark 1.** The fact that the constants appearing in (1.8) do not depend on \(\Omega\) is a remarkable property. As a consequence, the question of whether \(G^+\) is empty or not is also independent on \(\Omega\). For example, we know that, for \(M = 3, k_{12} > 0, k_{13}, k_{23} < 0\) and \(\Omega = \mathbb{R}^N\), either \(a_1 = a_2 = 0\) or \(a_3 = 0\) (see [2]). This has been proven by arguing that translating the third component of a ground-state to the infinite decreases the action, which is not an available argument for \(\Omega\) bounded. Now, however, we see that the result is also true for any \(\Omega\) for which \((\text{Exist})\) holds, in particular over bounded domains.
Remark 2. In [4], it is considered the case \( \Omega = \mathbb{R}^N, M = 3, p = 1, k_{12}, k_{23} > 0 \) and \( k_{13} < 0 \). They prove that if \( k_{ij} = 1, i = 1, 2, 3, k_{12}, k_{23} \approx \delta^2 \) and \( k_{13} \approx -\delta, \delta > 0 \) small, any nontrivial ground-state is not radial. This implies that such a ground-state cannot be of the form \( U = (a_i Q)_{1 \leq i \leq 3} \), where \( Q \) is the unique ground-state for the scalar equation, since \( Q \) is radial. Together with the above theorem, one sees that there are no nontrivial ground-states for this system. We claim that it is possible to obtain such a conclusion in a more general setting. In [2], it is proved that for \( p > 1 \) and \( k_{ij} > 0, \forall i, j \), there exists \( \epsilon > 0 \) such that, if \( \max_{i,j} k_{ij} < \epsilon \), there are no nontrivial ground-states. To prove this, one uses the implicit function theorem to determine the constants of the characterization formula as a perturbation of the system where \( p \) is not radial. This implies that such a ground-state cannot be of the form \( \omega \). Afterwards, the computation of the action proves that the ground-state is semitrivial. We now notice that this proof still works without the restriction \( k_{ij} > 0 \). In fact, this restriction was made only because the characterization result available needed such an hypothesis.

For \((\text{M-NLS}')\), since a reduction to the scalar case is impossible, the main questions are about existence of nontrivial ground-states (one might also discuss uniqueness, but that is a difficult matter even for the (M-NLS) system, where we have a complete characterization). Our results focus on two approaches: the first considers perturbation of the parameters of the system, while the second considers a real-valued function on the parameters whose properties determine the emptiness of \( G^+ \). We now explain the main ideas.

**Approach 1: Perturbation theory**

First of all, a scaling reduces any \((\text{M-NLS}')\) system to the case \( \omega \geq 1 \). Given a nonempty symmetric subset \( P \) of \( \{1, \ldots, M\}^2 \), \( \beta \in \mathbb{R} \) and \( \eta > 0 \), consider, for \( i = 1, \ldots, M \),

\[
\Delta u_i - (1 + \eta(\omega_i - 1))u_i + \sum_{(i,j) \in P} k_{ij}|u_j|^{p+1}|u_i|^{p-1}u_i + \sum_{(i,j) \in P} \beta k_{ij}|u_j|^{p+1}|u_i|^{p-1}u_i = 0 \quad (1.9)
\]

For the sake of simplicity, suppose that \( k_{ij} > 0, \forall (i,j) \in P \). If one considers the ground-state action level, \( T_\beta^0 \), and the semitrivial ground-state action level, \( (T_\beta^0)_{sem} \), then \( T_\beta^0 < (T_\beta^0)_{sem} \) is equivalent to \( G = G^+ \). The continuity of these action levels with respect to \( \beta \) and \( \eta \) leads to perturbation results: if, for some \( \beta_0, \eta_0 \), one proves that the ground-state action level is strictly lower than the semitrivial action level, then the same inequality is valid for \( \beta, \eta \) close to \( \beta_0, \eta_0 \). We exemplify such an argument with two corollaries:

**Corollary 2.** Consider system \((\text{M-NLS}')\).

1. If \( M = 2 \) and \( 0 < k_{11}, k_{22} \ll k_{12}, G = G^+ \); 
2. For \( M \geq 3 \), if \( k_{ii} = -1, \forall i \) and \( k_{ij} = \beta, \forall i \neq j \), there exists \( \epsilon > 0 \) such that, if \[
\frac{2}{M-1} < \beta < \frac{2}{M-2} + \epsilon, \quad (1.10)
\]
then \( G = G^+ \).

**Corollary 3.** Consider system \((\text{M-NLS}')\), \( M \geq 3 \). Suppose that \( \Omega = \mathbb{R}^N, p \leq 1, \omega_1 \leq \omega_2 \leq \ldots \leq \omega_M \) and \( k_{ij} = b > 0, \forall i \neq j \). Assume that

\[
\frac{M - 1}{(M - 2)^{1/p}} > \frac{M}{(M - 1)^{1/p}} \left( \frac{\omega_M}{\omega_1} \right)^{\frac{z-p(N-2)}{2p}} \quad (1.11)
\]

Then there exists \( \delta > 0 \) such that, if \( \max_i |k_{ii}| < \delta b, G = G^+ \).
Note that the first part of corollary 2 is already known (see [12] and [10]). Also, in corollary 3, if \( \omega_1 = \omega_M \), the result is a particular case of [8] and [14]. Even so, we prove these results for two reasons: first, the proof is very simple when one looks from this perturbative perspective; second, the approach is rather different in nature and it deals only with continuity properties, which may have a greater capacity of generalization to other systems.

Regarding corollary 3, a comment is in need: it might be expected that the restriction \( p \leq 1 \) would be technical. In fact, for \( M = 2 \), the result is valid for any \( p > 0 \). By contrast, we prove Proposition 4. Consider system (3-NLS), with \( p = 3 \), \( k_{ii} = 0, \forall i \) and \( k_{ij} = 1, \forall i \neq j \). Then \( G^+ = \emptyset \).

We conjecture that the above result applies for more general \( M \) and \( p \), with \( k_{ii} = \mu, \forall i \), and \( k_{ij} = b, \forall i \neq j, \mu < b \). In fact, a necessary and sufficient condition for the existence of nontrivial ground-states should be (see equation (4.33))

\[
\frac{M - 1}{(M - 1)^{1/p}} \leq \frac{M - 1}{(M - 2)^{1/p}}. \tag{1.12}
\]

In fact, if the only possible nontrivial ground-state is the one with all components equal, this condition determines whether it truly is a ground-state. Numerical simulations suggest that this uniqueness should hold for any \( p, M \). We advise the reader to compare this hypothesis with the condition for existence of nontrivial ground-states that appears in [8].

Approach 2: Mandel’s characteristic function

Once again, given a nonempty symmetric subset \( P \) of \( \{1, ..., M\}^2 \) and \( \beta \in \mathbb{R} \), consider the following system:

\[
\Delta u_i - \omega_i u_i + \sum_{(i,j) \in P} k_{ij} |u_j|^{p+1}|u_i|^{p-1} u_i + \sum_{(i,j) \in P} \beta k_{ij} |u_j|^{p+1}|u_i|^{p-1} u_i = 0, \quad i = 1, ..., M. \tag{1.13}
\]

In section 5, we shall build a mapping \( \beta \mapsto \hat{\beta} \) such that, if

- \( \beta < \hat{\beta} \), then \( G^+_\beta = \emptyset \);
- \( \beta > \hat{\beta} \), then \( G^+_\beta = G_\beta \);
- \( \hat{\beta} = \beta \), then \( G_\beta \setminus G^+_\beta \neq \emptyset \).

This approach was introduced by R. Mandel ([12]) for the system with two equations to study the existence of nontrivial ground-states as a function of the coupling coefficient \( k_{12} \). The (very important) feature of the case \( M = 2 \) is that any semitrivial bound-state is never influenced by the coupling coefficient. This implies that \( \hat{\beta} \) is constant and therefore it defines in a very precise way when does \( G^+ \neq \emptyset \). For more equations, \( \hat{\beta} \) is not that well-behaved (for more details, see section 5 and the last example in section 6). Using this mapping, we may however prove two results:

**Proposition 5.** Let \( U \) be a semitrivial bound-state for (M-NLS’) and suppose that \( P \subset \{1, ..., M\} \) is such that

\[
(i, j) \in P \Rightarrow U_i U_j = 0. \tag{1.14}
\]

Then, for \( \beta \) large, \( U \notin G \).
**Proposition 6.** Consider system (M-NLS') and fix $p \geq 1$. Suppose that $k_{ij} = \beta > 0$, $\forall i \neq j$, and $k_{ii} = \mu > 0$, $\forall i$. If $\beta \ll \mu$, any ground-state has exactly one nonzero component.

This work is organized as follows: in section 2, we give a few definitions and fix some notations. In section 3, we focus on the results regarding system (M-NLS). In sections 4 and 5, we study system (M-NLS’), the first using perturbation theory, the second using Mandel’s characteristic function. Finally, in section 6, we give three examples: one to see the application of theorem 1; the rest to show the complexity of these systems for $M = 3$. We recall that we shall always assume (Exist).

2 Preliminaries

**Definition 7.** (Bound-states and ground-states of (M-NLS))

1. We define bound-state of (M-NLS) as any element $(u_1, \ldots, u_M) \in (H_0^1(\Omega))^M \setminus \{0\}$ solution of (M-NLS) and define $A_{(M-NLS)}$ to be the set of all bound-states of (M-NLS).

2. A nontrivial bound-state is a bound-state such that $u_i \neq 0$, $\forall i$. The set of such bound-states is called $A^+_{(M-NLS)}$. On the other hand, a bound-state which is not nontrivial is called semitrivial.

3. Given $U = (u_1, \ldots, u_M) \in (H_0^1(\Omega))^M$, set

$$I_M(U) = \sum_{i=1}^M \int |\nabla u_i|^2 + \int \omega_i |u_i|^2, \quad J_M(U) = \sum_{i,j=1}^M k_{ij} \int |u_i|^p+1 |u_j|^p+1$$ (2.1)

and define the action of $U$,

$$S_M(U) = \frac{1}{2} I_M(U) - \frac{1}{2p+2} J_M(U).$$ (2.2)

4. The set of ground-states of (M-NLS) is defined as

$$G_{(M-NLS)} = \{ U \in A_{(M-NLS)} : S_M(U) \leq S_M(W), \forall W \in A_{(M-NLS)} \} \subset A_{(M-NLS)},$$ (2.3)

and the set of nontrivial ground-states is

$$G^+_{(M-NLS)} = G_{(M-NLS)} \cap A^+_{(M-NLS)}.$$ (2.4)

**Remark 3.** If $U \in A_{(M-NLS)}$, $I_M(U) = J_M(U)$ (one multiplies the $i$-th equation by $u_i$ and integrates over $\mathbb{R}^N$). Therefore

$$S_M(U) = \left( \frac{1}{2} - \frac{1}{2p+2} \right) I_M(U) = \left( \frac{1}{2} - \frac{1}{2p+2} \right) J_M(U).$$ (2.5)

Hence a ground-state is a bound-state with $I_M$ (or $J_M$) minimal.

**Remark 4.** Throughout this work, we shall assume that $k_{ij}$ are such that

$$\{ U \in (H_0^1(\Omega))^M : J_M(U) > 0 \} \neq \emptyset.$$ (P1)

This hypothesis is necessary for the existence of bound-states, since $J_M(U) = I_M(U) > 0$, for any $U \in A_{(M-NLS)}$. Furthermore, we shall assume that $\omega > 0$. 

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Remark 5. Since $M \geq 2$ will always be fixed, to simplify notations, we write
\[
A := A_{(M, \text{NLS})}, \quad G := G_{(M, \text{NLS})}, \quad G^+ := G^+_{(M, \text{NLS})}
\] (2.6)

and
\[
I := I_M, \quad J := J_M, \quad S := S_M.
\] (2.7)

Let
\[
\lambda_G := \left( \inf_{J(U)=1} I(U) \right)^{\frac{1}{p+1}}.
\] (2.8)

The following lemma, which may be found in [2], gives a variational characterization of the set of ground-states:

**Lemma 8.** Under hypothesis (P1) and (Exist), $G$ is the set of solutions of the minimization problem
\[
I(U) = \min_{J(W) = \lambda_G} I(W), \quad J(U) = \lambda_G.
\] (2.9)

Moreover, if $\Omega = \mathbb{R}^N$, (Exist) holds.

3 Proof of theorem 1

Take $U \in G$. Define $\hat{U}(x) = (|u_1(x)|, ..., |u_M(x)|)$ and $u(x) = |\hat{U}(x)|$. Since $J(U) = J(\hat{U})$ and $I(U) \geq I(\hat{U})$, $\hat{U}$ is a minimizer. Fix $X_0 \in \mathcal{X}$. Now notice that
\[
J(\hat{U}) = \int f(\hat{U}(x)) dx = \int f \left( \frac{\hat{U}(x)}{u(x)} \right) u(x)^{2p+2} dx \leq \int f(X_0)u(x)^{2p+2} dx
\]
and that, from Cauchy-Schwarz inequality,
\[
I(X_0u) = \int \omega u(x)^2 |X_0|^2 + |\nabla u(x)|^2 |X_0|^2 = \sum_{i=1}^M \omega |u_i|^2 + \left( \sum_{i=1}^M |u_i|^2 \right)^2 \left( \sum_{i=1}^M |u_i|^2 \right)^2
\]
\[
\leq \int \sum_{i=1}^M |\nabla u_i|^2 + |\nabla |u_i||^2 = I(\hat{U}).
\]

Let $a \leq 1$ be such that $J(aX_0u) = J(\hat{U})$. Then
\[
I(aX_0u) \leq I(X_0u) \leq I(\hat{U})
\] (3.1)

By the minimality of $U$, the above inequalities must be equalities:
\[
a = 1, \quad I(X_0u) = I(U).
\] (3.2)

Therefore $X_0u$ is also a ground-state. Note that $J(U) = J(X_0u)$ implies that $\hat{U}(x) = u(x)X(x)$ a.e. $x \in \mathbb{R}^N$, where $X(x) \in \mathcal{X}$.

Since $X_0u$ is a bound-state for (M-NLS), one easily checks that
\[
- \Delta u + \omega u = f_{\max} |u|^{2p} u
\] (3.3)
and so, setting \( c = (f_{\max})^{1/2p} \), \( cu \) is a bound-state for (1.7). The fact that \( X_0 u \) is a ground-state clearly implies that \( cu \) is a ground-state for (1.7). Hence \( u = e^{-1} u_0 \), with \( u_0 \) ground-state of (1.7). From the maximum principle, \( u > 0 \) in \( \mathbb{R}^N \).

Since \( \hat{U}(x) = u(x) X(x) \) is a bound-state, inserting this expression into system (M-NLS), one obtains

\[
2 \nabla u \cdot \nabla X_i + u \Delta X_i = 0, \quad i = 1, \ldots, M
\]

By integration by parts,

\[
\int u X_i \nabla u \cdot \nabla X_i = - \int u X_i \nabla u \cdot \nabla X_i - \int |u|^2 |\nabla X_i|^2 - \int |u|^2 X_i \Delta X_i \\
= - \int u X_i \nabla u \cdot \nabla X_i - \int |u|^2 |\nabla X_i|^2 + 2 \int u X_i \nabla u \cdot \nabla X_i \\
= \int u X_i \nabla u \cdot \nabla X_i - \int |u|^2 |\nabla X_i|^2
\]

Hence

\[
\int |u|^2 |\nabla X_i|^2 = 0, \quad i = 1, \ldots, M.
\]

which implies that \( X_i \) is constant. Therefore

\[
\hat{U} = u X, \quad X \in \mathcal{X}.
\]

Finally, since \( u > 0 \), one may write \( u_i(x) = |u_i(x)| e^{i \theta_i(x)} = u(x) \omega_i e^{i \theta_i(x)} \). Then, since \( I(U) = I(\hat{U}) \),

\[
\int \sum_{i=1}^{M} \omega |u_i|^2 + |\nabla u_i|^2 = \int \sum_{i=1}^{M} \omega |\hat{U}_i|^2 + |\nabla \hat{U}_i|^2 = I(\hat{U}) = I(U) \\
= \int \sum_{i=1}^{M} \omega |u_i|^2 + |\nabla u_i|^2 = \int \sum_{i=1}^{M} \omega |u_i|^2 + |\nabla u_i|^2 + |u_i|^2 |\nabla \theta_i(x)|^2.
\]

One then concludes that \( \theta_i \) is constant, which ends the proof. \( \square \)

Remark 6. As expected, this approach is only possible since the norms inside functional \( I \) are the same. This is not the case for the general system (M-NLS').

4 System (M-NLS'): Perturbation theory

Lemma 9 (Monotonicity of the action with respect to \( \omega \)). Let \( \omega = (\omega_1, \ldots, \omega_M) \) and \( \omega' = (\omega'_1, \ldots, \omega'_M) \) be such that \( \omega \triangleright \omega' \). Fix a matrix \( K = (k_{ij})_{1 \leq i, j \leq M} \in \mathbb{R}^{M \times M} \). Let \( U^{\omega} \) be a ground-state of

\[
\Delta u_i - \omega_i u_i + \sum_{j=1}^{M} k_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i = 0 \quad i = 1, \ldots, M
\]

and \( U^{\omega'} \) be a ground-state of

\[
\Delta u_i - \omega'_i u_i + \sum_{j=1}^{M} k_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i = 0 \quad i = 1, \ldots, M.
\]

Then \( J(U^{\omega'}) \triangleright J(U^{\omega'}) \).
Proof. Simply recall that
\[
J(U^\omega) = \left( \inf_{J(U) = 1} \sum_{i=1}^M \omega_i \|u_i\|^2_2 + \|\nabla u_i\|^2_2 \right)^{p+1} \geq \left( \inf_{J(U) = 1} \sum_{i=1}^M \omega_i' \|u_i\|^2_2 + \|\nabla u_i\|^2_2 \right)^{p+1} = J(U^\omega).
\]
(4.3)

Analogously, we may obtain the following:

Lemma 10 (Monotonicity of the action with respect to \(K\)). Fix \(\omega \in (\mathbb{R}^+)^M\). Consider matrices \(K = (k_{ij})_{1 \leq i, j \leq M} \in \mathbb{R}^{M \times M}\) and \(K' = (k'_{ij})_{1 \leq i, j \leq M} \in \mathbb{R}^{M \times M}\) such that \(K \geq K'\). Let \(K^X\) be a ground-state of
\[
\Delta u_i - \omega_i u_i + \sum_{j=1}^M k_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i = 0 \quad i = 1, \ldots, M
\]
and \(K'^X\) be a ground-state of
\[
\Delta u_i - \omega_i u_i + \sum_{j=1}^M k'_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i = 0 \quad i = 1, \ldots, M.
\]

Then \(I(U'^X) \geq I(U^K)\).

Suppose that one wishes to study \(G^{+}\) in function of a given set of couplings. Let \(P\) a nonempty symmetric subset of \(\{1, \ldots, M\}^2\) and fix a matrix \(K \in \mathbb{R}^{M^2}\). Given \(\beta \in \mathbb{R}\), consider the system
\[
\Delta u_i - \omega_i u_i + \sum_{(i,j) \notin P} k_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i + \sum_{(i,j) \in P} \beta k_{ij} |u_j|^{p+1} |u_i|^{p-1} u_i = 0, \quad i = 1, \ldots, M
\]
Suppose, for the sake of simplicity, that \(k_{ij} > 0, (i, j) \in P\). Everytime a functional, a set or a solution depends on \(\beta\), we shall place a subscript \(\beta\).

Set
\[
\mathcal{I}_{\beta} = \left( J_{\beta(U) = 1} I(U) \right)^{\frac{p+1}{p}}.
\]
(4.7)

For any \(X \subset \{1, \ldots, M\}\), define
\[
\mathcal{I}_{\beta}^X := \left( J_{\beta(U) = 1, U_i = 0, i \notin X} I(U) \right)^{\frac{p+1}{p}}, \quad \mathcal{I}_{\beta}^{sem} := \min_{X \subseteq \{1, \ldots, M\}} \mathcal{I}_{\beta}^X.
\]
(4.8)

Notice that \(\mathcal{I}_{\beta} = \mathcal{I}_{\beta}^{\{1, \ldots, M\}}\). Then \(G^{+} = G\) if \(\mathcal{I}_{\beta} < \mathcal{I}_{\beta}^{sem}\).

From the results regarding existence of ground-states, we know that, for each \(X \subset \{1, \ldots, M\}\), there exists \(\beta_X\) such that \(\beta \leq \beta_X\) if \(\mathcal{I}_{\beta}^X = +\infty\). Define
\[
\beta_{\text{sem}} := \min_{X \subseteq \{1, \ldots, M\}} \beta_X, \quad \beta := \beta_{\{1, \ldots, M\}}.
\]
(4.9)

Then

1. If \(\beta \leq \beta\), there are no ground-states;
2. If \(\beta < \beta \leq \beta_{\text{sem}}\), all ground-states are nontrivial.
3. If $\bar{\beta}^{sem} < \beta$, both $I_\beta$ and $I_\beta^{sem}$ are finite.

**Proposition 11.** For any $X \subset \{1, ..., M\}$, the mapping $\beta \mapsto I_\beta^X$, $\beta \in \mathbb{R}$, is continuous (in $\mathbb{R}$). In particular, $I_\beta$ and $I_\beta^{sem}$ are continuous with respect to $\beta$.

**Proof.** Notice that we only need to prove the proposition for $X = \{1, ..., M\}$, since any other case may be reduced to this one. Fix $\beta_0 \in \mathbb{R}$. If $\beta_0 < \beta$, then $I_\beta = +\infty$ in a neighbourhood of $\beta_0$ and so it is continuous. If $\beta_0 > \beta$, let $\beta_n \to \beta_0$. By definition, there exists $\{U_n\} \subset (H^1(\mathbb{R}^N))^M$ such that

$$I(U_n) = I_{\beta_n}, \quad J_{\beta_n}(U_n) = 1. \quad (4.10)$$

Let $\lambda_n = J_{\beta_n}(U_n)^{-1/2p}$. Then $J_{\beta_n}(\lambda_n U_n) = 1$. Moreover,

$$|\lambda_n^{-1/2p} - 1| = |J_{\beta_n}(U_n) - J_{\beta_n}(U_n)| = |\beta_n - \beta_0| |J_P(U_n)| \quad (4.11)$$

Since

$$|J_P(U_n)| \leq C \|U_n\|^2_{H^1} \leq CI(U_n)^{2p+2} \leq C(I_{\beta_n})^{2p+2} < C, \quad (4.12)$$

we obtain $\lambda_n \to 1$. Therefore,

$$\lim \inf I_{\beta_n} = \lim \inf I(U_{\beta_n}) = \lim \inf I(\lambda_n U_{\beta_n}) \geq I_{\beta_0}. \quad (4.13)$$

On the other hand, for $n > 0$, let $U$ be such that

$$I_{\beta_0} = I(U), \quad J_{\beta_0}(U) = 1. \quad (4.14)$$

Define $\lambda^n = J_{\beta_n}(U)^{-1/2p}$. As before, $J_{\beta_n}(\lambda^n U) = 1$ and $\lambda^n \to 1$. Hence

$$I_{\beta_0} = I(U) = \lim I(\lambda^n U) \geq \lim \sup I_{\beta_n}. \quad (4.15)$$

Therefore $I_\beta$ is continuous for $\beta > \beta$. If $\beta_0 = \beta$ and $\beta_n \to \beta_0^+$, consider $U_n$ as above. Then

$$1 = J_{\beta_n}(U) = J_{\beta_0}(U_n) + (\beta_n - \beta_0) J_P(U_n) \leq C(\beta_n - \beta_0) I(U_n)^{2p+2}, \quad (4.16)$$

and so $I_{\beta_n} = I(U_n) \to \infty = I_{\beta_0}$. \qed

**Lemma 12.** Suppose that $\beta < \bar{\beta}^{sem}$. For $\beta$ sufficiently close to $\bar{\beta}^{sem}$, $G = G^+$. 

**Proof.** Since $\beta < \bar{\beta}^{sem}$, there exists $U$ nontrivial such that, for some $m > 0$ and for $\beta$ close to $\bar{\beta}^{sem}$,

$$J_\beta(U) > m. \quad (4.17)$$

This implies that

$$I_\beta \leq \left( \frac{1}{m^{\frac{1}{p+1}}} I(U) \right)^{\frac{p+1}{p}}. \quad (4.18)$$

On the other hand, since $I_\beta^{sem}$ is continuous, for $\beta$ sufficiently close to $\bar{\beta}^{sem}$,

$$I_\beta^{sem} \geq \left( \frac{1}{m^{\frac{1}{p+1}}} I(U) \right)^{\frac{p+1}{p}} \geq I_{\beta^+}. \quad (4.19)$$

Therefore $G = G^+$. \qed

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Proof of corollary 2

1. First part: take \( P = \{(1, 1), (2, 2)\} \). One easily observes that \( \beta_{\text{sem}} = 0 \) and that \( \beta < 0 \). Therefore, using the previous lemma, for \( \beta > 0 \) small enough, \( G = G^+ \).

2. Second part: take \( P = \{(i, j), 1 \leq i, j \leq M, i \neq j\} \). A simple calculation shows that \( \beta(M) = 2/(M - 1) \) and \( \beta_{\text{sem}}(M) = \beta(M - 1) = 2/(M - 2) \). Therefore, by the previous lemma, there exists \( \epsilon > 0 \) such that, for \( 2/(M - 1) < \beta < 2/(M - 2) + \epsilon, G = G^+ \). \( \square \)

The same procedure may be applied to study the dependence of \( G^+ \) on \( \omega = (\omega_1, \ldots, \omega_M) \). Suppose that \( \omega_i > 1, \forall i \) (this condition is not restraining at all, since any case may be reduced to this one by a simple scaling). Define

\[
\eta = - \min_{1 \leq i \leq M} 1/(\omega_i - 1). \tag{4.20}
\]

For \( \eta > \eta \), consider the system

\[
\Delta u_i - (1 + \eta(\omega_i - 1))u_i + \sum_{j=1}^{M} k_{ij}|u_j|^{p+1}|u_i|^{p-1}u_i = 0, \quad i = 1, \ldots, M. \tag{4.21}
\]

Now we write the dependence on \( \eta \) as a superscript. If one defines

\[
T^n = \left( \inf_{I(U) = 1} \frac{I^n(U)}{I^n(U)} \right)^{\frac{p+1}{p-1}}, \tag{4.22}
\]

and, for any \( X \subset \{1, \ldots, M\} \),

\[
(T^n)^X := \left( \inf_{I(U) = 1, U_i = 0, \forall X} \frac{I^n(U)}{I^n(U)} \right)^{\frac{p+1}{p-1}}, \quad (T^n)^{\text{sem}} := \min_{X \subseteq \{1, \ldots, M\}} (T^n)^X, \tag{4.23}
\]

we have once again \( G = G^+ \) iff \( T^n < (T^n)^{\text{sem}} \). As before, we may show that

**Proposition 13.** For any \( X \subset \{1, \ldots, M\} \), the mapping \( \eta \mapsto (T^n)^X, \eta > \eta \), is continuous. In particular, \( T^n \) and \( (T^n)^{\text{sem}} \) are continuous with respect to \( \eta \).

**Proof of corollary 3.** First of all, notice that, if \( U \) is a ground-state, \( V = b^{1/2p}U \) is a ground-state of

\[
\Delta u_i - \omega_i u_i + \sum_{j=1}^{M} \frac{k_{ij}}{b}|u_j|^{p+1}|u_i|^{p-1}u_i = 0, \quad i = 1, \ldots, M. \tag{4.24}
\]

Therefore, we may consider that \( b = 1 \) and that the diagonal terms are small. Then such a system may be seen as a \( \beta \)-perturbation of

\[
\Delta u_i - \omega_i u_i + \sum_{j=1, i \neq j}^{M} |u_j|^{p+1}|u_i|^{p-1}u_i = 0, \quad i = 1, \ldots, M. \tag{4.25}
\]

We note by \( I(\omega_1, \ldots, \omega_M) \) the corresponding ground-state action level. By the monotonicity properties, \( I(\omega_1, \ldots, \omega_M) \leq I(\omega_M, \ldots, \omega_M) \). On the other hand, if \( I^{\text{sem}}(\omega_1, \ldots, \omega_M) \) is the semitrivial ground-state action level (that is, the lowest action among semitrivial bound-states), then \( I^{\text{sem}}(\omega_1, \ldots, \omega_M) \geq I^{\text{sem}}(\omega_1, \ldots, \omega_1) \). The proof will be concluded if one proves that

\[
I^{\text{sem}}(\omega_1, \ldots, \omega_1) > I(\omega_M, \ldots, \omega_M). \tag{4.26}
\]

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Using a suitable scaling, we have

\[ I^{sem}(\omega_1, ..., \omega_1) = \omega_1^{\frac{2 - \kappa N}{2\kappa}} I^{sem}(1, ..., 1), \quad I(\omega_M, ..., \omega_M) = \omega_M^{\frac{2 - \kappa N}{2\kappa}} I(1, ..., 1). \quad (4.27) \]

Therefore we only have to compare the ground-state and semitrivial ground-state actions levels for

\[ \Delta u_i - u_i + \sum_{j=1, j \neq i}^{M} |u_j|^p|u_i|^{p-1}u_i = 0, \quad i = 1, ..., M. \quad (4.28) \]

We claim that any nontrivial ground-state of must be of the form

\[ U = a_i u_0, \quad a_i > 0 \quad (4.29) \]

Inserting this information in the system, we have

\[ a_i^{p-1} \sum_{j \neq i} a_j^{p+1} = 1, \quad i = 1, ..., M \quad (4.30) \]

Suppose, w.l.o.g., that \( a_1 < a_2 \) and \( a_1 \leq a_i, i \geq 2 \). Then

\[ \begin{cases} \sum_{j \neq 1} a_j^{p+1} = a_1^{1-p} \\ \sum_{j \neq 2} a_j^{p+1} = a_2^{1-p} \end{cases} \quad (4.31) \]

This is a contradiction, since the first sum is larger than the second. Therefore \( a_1 = ... = a_M = a \) and so

\[ a^{2p}(M - 1) = 1, \quad (4.32) \]

yielding the claim. Now compute the action for such a ground-state:

\[ I(U) = \frac{M}{(M - 1)\frac{p}{2}} I(u_0). \quad (4.33) \]

Since the mapping \( M \mapsto M/(M - 1)^{\frac{p}{2}} \) is strictly decreasing and any semitrivial ground-state is a nontrivial ground-state for the same system with \( M - L \) equations, for some \( L \in \mathbb{N} \), we have

\[ I(1, ..., 1) = \frac{M}{(M - 1)^{\frac{p}{2}}} I(u_0) \quad (4.34) \]

\[ I^{sem}(1, ..., 1) = \min_{1 \leq L \leq M - 2} \left\{ \frac{M - L}{(M - L - 1)^{\frac{p}{2}}} I(u_0) \right\} = \frac{M - 1}{(M - 2)^{\frac{p}{2}}} I(u_0). \quad (4.35) \]

The result follows from (4.27) and hypothesis (1.11). \( \square \)

**Proof of proposition**. First of all, using the characterization result, any nontrivial ground-state is of the form \( U = (a_i u_0)_{1 \leq i \leq 3} \), with \( u_0 \) a scalar ground-state. Inserting this formula in the system and writing \( b_i = a_i^{-1} \),

\[ \begin{cases} b_1 (b_1^2 + b_2^2) = 1 \\ b_2 (b_1^2 + b_2^2) = 1 \\ b_3 (b_1^2 + b_2^2) = 1 \end{cases} \quad (4.36) \]
Suppose, w.l.o.g., that \( b_1 \neq b_3 \). Multiply the first equation by \( b_1 \), the third by \( b_3 \) and take the difference. Then

\[
b_2^2(b_1^2 - b_3^2) = b_1 - b_3, \quad \text{i.e.,} \quad b_2^2(b_1 + b_3) = 1. \quad (4.37)
\]

Define \( x = b_1/b_2 \), \( y = b_3/b_2 \). The above and the second equation imply

\[
x^2 + y^2 = 1/b_2^2 = x + y. \quad (4.38)
\]

Now divide the system by \( b_2^2 \) and take the difference between the two last equations:

\[
x^2 + y^2 = y(x^2 + 1). \quad (4.39)
\]

Hence \( x = yx^2 \) and so \( y = 1/x \). Therefore \( x^4 + 1 = x^3 + x \). One easily checks that \( x = 1 \) is the only positive solution to this equation. Therefore \( x = y \) and \( b_1 = b_3 \), which is absurd. Therefore \( b_1 = b_2 = b_3 \) and so \( a_i = a =: 2^{-1/6} \).

We observe that \( V = (1, 1, 0)u_0 \) is also a bound-state. Now compute the action of \( U \) and \( V \):

\[
I(U) = \frac{3}{2^{1/6}} I(u_0) > 2I(u_0) = I(V). \quad (4.40)
\]

This means that \( U \) cannot be a ground-state, which ends the proof. \( \square \)

5 System (M-NLS'): Mandel’s characteristic function

Once again, consider system (4.6): for a given nonempty symmetric subset \( P \) of \( \{1, \ldots, M\}^2 \) and \( \beta \in \mathbb{R} \),

\[
\Delta u_i - \omega_i u_i + \sum_{(i,j) \notin P} k_{ij} |u_j|^{p+1}|u_i|^{p-1} u_i + \sum_{(i,j) \in P} \beta k_{ij} |u_j|^{p+1}|u_i|^{p-1} u_i = 0, \quad i = 1, \ldots, M \quad (5.1)
\]

For the sake of simplicity, we suppose that \( k_{ij} > 0 \), \( \forall (i, j) \notin P \). We define

\[
J_P(U) = \sum_{(i,j) \notin P} k_{ij} \int |u_i|^{p+1}|u_j|^{p+1}, \quad J_{NP}(U) = \sum_{(i,j) \in P} k_{ij} \int |u_i|^{p+1}|u_j|^{p+1}. \quad (5.2)
\]

As before, we shall place a subscript \( \beta \) whenever a solution, function or set depends on \( \beta \). Suppose that \( G_\beta \neq \varnothing \). Therefore there exists \( U_\beta \) nontrivial bound-state such that

\[
I(U_\beta) \leq I_\beta. \quad (5.3)
\]

Since \( I(U_\beta) = J_\beta(U_\beta) \),

\[
(T_{\beta \text{sem}})^p \geq \frac{I(U_\beta)^{p+1}}{J_\beta(U_\beta)}, \quad \text{i.e.} \quad J_{NP}(U_\beta) + \beta J_P(U_\beta) \geq I(U)^{p+1}(T_{\beta \text{sem}})^{-p}. \quad (5.4)
\]

Hence

\[
\beta \geq \frac{I(U_\beta)^{p+1}(T_{\beta \text{sem}})^{-p} - J_{NP}(U_\beta)}{J_P(U_\beta)} =: B_\beta(U_\beta). \quad (5.5)
\]

Define

\[
\hat{\beta} = \inf_{U \in (H^1(\mathbb{R}^N) \setminus \{0\})^M} B_\beta(U). \quad (5.6)
\]
Then $\beta < \hat{\beta}$ clearly implies $G_\beta^+ = \emptyset$. Moreover, it is not hard to check that, if $\beta > \hat{\beta}$, $G_\beta^+ = G_\beta$.

Also, if $\hat{\beta} = \beta$, $G_\beta \cap G_\hat{\beta} = \emptyset$.

Let us look deeper into the properties of $\hat{\beta}$. Suppose, for instance, that $\hat{\beta}_0 > \beta_0$. Then

$$\frac{I(U)^{p+1}}{J_{\beta_0}(U)} > (I_{\beta_0}^{\text{sem}})^p, \quad \forall U : J_P(U) \neq 0. \quad (5.7)$$

Take $\beta_0 \leq \beta \leq \hat{\beta}_0$. If $U_{\beta_0}^{\text{sem}}$ (the best semitrivial bound-state) satisfies $J_P(U_{\beta_0}^{\text{sem}}) \neq 0$, then

$$(I_{\beta_0}^{\text{sem}})^p = \frac{I(U_{\beta_0}^{\text{sem}})^{p+1}}{J_{\beta_0}(U_{\beta_0}^{\text{sem}})} \geq \frac{I(U_{\beta_0}^{\text{sem}})^{p+1}}{J_{\beta_0}(U_{\beta_0}^{\text{sem}})} > (I_{\beta_0}^{\text{sem}})^p,$$  

which is absurd, by the monotonicity properties. Therefore $J_P(U_{\beta_0}^{\text{sem}}) = 0$, for all $\beta \in [\beta_0, \hat{\beta}_0]$. In turn, by the definition of $T_{\beta_0}^{\text{sem}}$, we see that it is constant in this interval and so the function $\beta \mapsto \hat{\beta}$ is constant on $[\beta_0, \hat{\beta}_0]$. Moreover, since $\beta < \hat{\beta}_0 = \hat{\beta}$, $G_\beta^+ = \emptyset$ for all $\beta \in [\beta_0, \hat{\beta})$.

So condition $\beta > \hat{\beta}$ has more implications than the simple dichotomy seen in [12]. Precisely because of this fact, is not as powerful when studying the nonemptiness of $G^+$ as one would desire: for example, if $P$ contains all diagonal terms, one has $\beta \geq \hat{\beta}$ for all $\beta$.

**Proof of proposition 5.** Suppose that $U$ is a ground-state for a sequence $\beta_n \to \infty$. The hypothesis on $P$ implies that

$$\frac{I(U)^{p+1}}{J_{\beta}(U)} = \frac{I(U)^{p+1}}{J(U)}. \quad (5.9)$$

Since, for each $\beta_n$, $U$ is the semitrivial bound-state with the lowest action, this implies that $\hat{\beta}_n = \hat{1}_n, \forall n$. Taking $n_0$ large, $\beta_{n_0} > \hat{1} = \hat{\beta}_{n_0}$, which implies that $G = G^+$, contradicting $U \in G$. $\square$

**Proof of proposition 6.** Through a normalization, one may assume $\mu = 1$. From [12], the property is true for $M = 2$. We now proceed by induction: suppose that the result is true for $M - 1$ equations. Then there exists $\beta_M - 1$ and $U_0$ with only one nonzero component such that

$$\frac{I(U)^{p+1}}{J_{\beta}(U)} \geq \frac{I(U_0)^{p+1}}{J(U_0)} = I(U_0)^p, \forall \beta \in (0, \beta_{M-1}). \quad (5.10)$$

Consider the function $\beta \mapsto \hat{\beta}$. Since $U_0$ has only one nonzero component, $\hat{\beta}$ is constant on $(0, \beta_{M-1})$. Take any $U$ such that $J_P(U) \neq 0$. W.l.o.g., assume that the last component has the largest $L^{p+2}$ norm. For each $1 \leq i \leq M$, define

$$r_i = \frac{\|u_i\|_{L^{p+2}}}{\|u_M\|_{L^{p+2}}} \leq 1, \quad V_i = ((v_1)_i, ..., (v_M)_i), \quad (v_i)_j = u_i \delta_{ij}. \quad (5.11)$$

Then, using (5.10) and $J_{\beta}(V_i) = J(V_i)$,

$$\frac{I(U)^{p+1}I(U_0)^{-p} - J_{NP}(U)}{J_{\beta}(U)} = \frac{\left(\sum_{i=1}^{M} I(V_i)\right)^{p+1}}{J(U)} \leq \frac{\left(\sum_{i=1}^{M-1} I(V_i)\right)^{p+1}}{J(U)} - \frac{\sum_{i=1}^{M-1} J(V_i)}{J(U)} \leq g(r_1, ..., r_{M-1})$$

$$\geq \frac{1 + \sum_{i=1}^{M-1} r_i^{p+1} - 1 - \sum_{i=1}^{M-1} r_i^{2p+2}}{2 \sum_{i=1}^{M-1} r_i^{p+1} + \sum_{i=1}^{M-1} r_i^{2p+1}} = g(r_1, ..., r_{M-1}).$$
Since \( p \geq 1 \), \( g \) is bounded below over the set \([0,1]^{M-1}\) by a constant \( m > 0 \). Hence \( B_{\beta}(U) \geq m > 0, \forall U \). Therefore, taking \( \beta_M = \min\{\beta_{\beta_{\beta-1}}, m\} \), we see that \( \beta \geq m > \beta, \) for \( 0 < \beta < \beta_M \). The properties of \( \hat{\beta} \) imply that the result is true for \( M \) equations. \( \square \)

6 Examples

Example 1. Consider \( M = 3, p = 1 \) and suppose that the coefficient matrix \( K \) is of the form

\[
K = \begin{bmatrix}
0 & a & b \\
a & 0 & c \\
b & c & 0
\end{bmatrix}, \quad a \leq b \leq c \in \mathbb{R}\setminus\{0\}.
\] (6.1)

Now one must divide in several cases:

- \( c < 0 \): in this case, the condition for the existence of ground-states is not verified and so there are no ground-states;

- \( c > 0, b < 0 \): applying theorem 6 of [2], any ground-state satisfies either \( u_1 = 0 \) or \( u_2, u_3 = 0 \). The second possibility implies that \( u_1 \) satisfies \(-\Delta u_1 + u_1 = 0\), which is impossible. Therefore \( u_1 = 0 \). Since \( U = (a_i u_1)_{1 \leq i \leq 3} \), a direct substitution on the system gives \( a_2 = a_3 = c^{-1/2} \).

- \( b > 0 \): suppose that \( U \) is a nontrivial ground-state. Then, inserting the characterization formula in the system (M-NLS), we obtain

\[
KX = (1,1,1)^T, \quad X = (a_1^2, a_2^2, a_3^2)^T.
\] (6.2)

The determinant of \( K \) is \( 2abc \). Now, using Cramer’s rule,

\[
a_1^2 = \frac{a + b - c}{2ab}, \quad a_2^2 = \frac{a + c - b}{2ac}, \quad a_3^2 = \frac{b + c - a}{2bc}.
\] (6.3)

This implies that \((a + b - c)a > 0\) and \((a + c - b)a > 0\). Now, if \( V \) is a semitrivial ground-state, using the characterization and the fact that \( c \geq a, b \),

\[
V = \left(0, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{c}}\right) u_0.
\] (6.4)

Now, comparing the actions of these two solutions, the condition for the existence of non-trivial ground-states is

\[
-\frac{a^2 - b^2 - c^2 + 2ab + 2bc + 2ac}{2ab} \leq 2
\] (6.5)

which, for \( a > 0 \), simplifies to

\[
2c(a + b) \leq (a + b)^2 + c^2, i.e., (a + b - c)^2 \geq 0.
\] (6.6)

Therefore, for \( a > 0 \), one has the following:

- if \( a + b \leq c \), \( G^+ = \emptyset \), since system (6.2) has no positive solutions. This implies

\[
G = \left\{ \left(0, \frac{1}{\sqrt{c}}, \frac{1}{\sqrt{c}}\right) u_0 : u_0 \in G_{(1-NLS)} \right\}.
\] (6.7)
if \(a + b > c\), then

\[
G = \left\{ \left( \sqrt{\frac{a + b - c}{2ab}}, \sqrt{\frac{a + c - b}{2ac}}, \sqrt{\frac{b + c - a}{2bc}} \right) u_0 : u_0 \in G_{(1-NLS)} \right\} .
\]  

(6.8)

For \(a < 0\), inequality (6.6) is reversed and strict, hence \(G^+ = \emptyset\).

Hence the necessary and sufficient condition for the existence of nontrivial ground-states with \(a \leq b \leq c \in \mathbb{R}\) is \(a + b > c\).

**Example 2.** Consider \(M = 3\), \(p = 1\) and suppose that the coefficient matrix \(K\) is of the form

\[
K = \begin{bmatrix}
1 & a & b \\
a & 1 & c \\
b & c & 1
\end{bmatrix}, \quad 1 \ll a \leq b \leq c
\]

(6.9)

The previous example may be seen as a limit when \(a, b, c\) are very large. With some computations, one derives the following:

- The possible semitrivial ground-state is given by

\[
V = \left( 0, \frac{1}{\sqrt{1 + c}}, \frac{1}{\sqrt{1 + c}} \right).
\]

(6.10)

- The possible nontrivial ground-state, \(U = (a_i u_0)_{1 \leq i \leq 3}\), is given by

\[
a_1^2 = \frac{1 + (a + b)c - a - b - c^2}{1 + 2abc - a^2 - b^2 - c^2}, \quad a_2^2 = \frac{1 + (a + c)b - a - c - b^2}{1 + 2abc - a^2 - b^2 - c^2}, \quad a_3^2 = \frac{1 + (c + b)a - c - b - a^2}{1 + 2abc - a^2 - b^2 - c^2}.
\]

(6.11)

We assume that \(a, b, c\) are such that all numerators and denominators above are positive. Notice that this is true for \(a, b, c\) large enough and \(a + b > c\).

As in the previous example, if one compares the corresponding action levels, one has \(G^+ \neq \emptyset\) iff

\[
0 \leq (a + b - c)(a + b - c - 2).
\]

(6.12)

Since we assumed that \(a + b > c\), the condition is simply \(a + b \geq c + 2\). We see that, even for systems where the couplings \(k_{ij}, i \neq j\), are large comparing to the diagonal terms \(k_{ii}\), one may have \(G^+ = \emptyset\). This does not go against the conclusion of corollary 3 and the perturbation arguments: the problem here is that \(a, b\) and \(c\) are not close to each other. This example shows that, in order for one to have \(G^+ \neq \emptyset\), one must take into account the relation between coupling coefficients.

**Example 3.** Consider system (3-NLS), \(p = 1\) and the coefficient matrix

\[
K = \begin{bmatrix}
0 & b & 1 \\
b & 0 & 2 \\
1 & 2 & \mu
\end{bmatrix}, \quad b > 0, \mu \in \mathbb{R}.
\]

(6.13)

Using the characterization, everything is reduced to the study of the proportionality constants \(a_1, a_2\) and \(a_3\). For the sake of simplicity, \(x = a_1^2, y = a_2^2, z = a_3^2\). It is now a simple calculation to obtain the following:
Semitrivial A: The possible ground-state with $x = 0$ satisfies $y = (2 - \mu)/4$, $z = \frac{1}{b}$. This solution only exists if $2 > \mu$.

Semitrivial B: Analogously, the possible ground-state with $x = 0$ satisfies $x = 1 - \mu$, $z = 1$. This solution only exists if $1 > \mu$.

Semitrivial C: The possible ground-state with $z = 0$ satisfies $x = y = 1/b$.

Semitrivial D: If $x = y = 0$, then $z = 1/\mu$. This solution only exists if $\mu > 0$.

Nontrivial E: For the possible nontrivial ground-state,

$$x = \frac{\mu b - 2b + 2}{b(\mu b - 4)}, y = \frac{\mu b - b - 1}{b(\mu b - 4)}, z = \frac{b - 3}{\mu b - 4}.$$

(6.14)

This solution exists only if $b > 3$ and $\mu > 2/(b - 1)/b$ or if $b < 3$ and $\mu < 2(b - 1)/b$.

The action for each of these solutions is (up to a constant)

$$A : \frac{4 - \mu}{4}, \quad B : 2 - \mu, \quad C : 2/b, \quad D : 1/\mu \quad E : \frac{b^2 + (2\mu - 6)b + 1}{b(\mu b - 4)}.$$

(6.15)

Now we compare the various actions, whenever the solutions exist:

1. First of all, $A$ is always lower than $B$ so one may discard this solution;

2. The solution $D$ is best if $\mu > 2, b/2$;

3. A tedious computation shows that $E$ is the ground-state if $b < 3$ and $\mu < 2(b - 1)/b$;

4. In the remaining area, $A$ is better than $C$ if $2 > \mu > 4 - 8/b$.

Intersecting these comparisons with the domains where each solution exists, we obtain diagram 2, which is already revealing of the complexity of this problem.

Several remarks are necessary:

1. First of all, we see that, for $b < 3$, when $-\mu$ is very large, the ground-state is nontrivial. Moreover, if $b > 3$, no value of $\mu$ produces nontrivial ground-states;

2. One might think that some solutions (for example, the nontrivial one), if they exist, would always have minimal action. However, the reader may check that this is not true for this system;

3. It is natural that one never has $u_1 \neq 0$ and $u_2 = 0$, since the coupling coefficients associated with the second component are larger than the coefficients associated with the first component. In fact, this is a consequence of the monotonicity with respect to the coefficient matrix;

4. Take, for example, $b = 4$. One sees that the mapping $\mu \mapsto \hat{\mu}$ is constant up to $\mu = 2$ and $\hat{\mu} = \mu$ for $\mu > 2$. On the other hand, if one sets $\mu = 1$, one observes that mapping $b \mapsto \hat{b}$ is constant up to $b = 2$, $b > \hat{b}$ for $2 < b < 3$ and $\hat{b} = b$ for $b > 3$. In any case, one may observe that these mappings are continuous.
Further comments

One of the main ideas that should be clear at the end of this work is that the system (M-NLS') for \( M = 2 \) has a much simpler structure than the case \( M = 3 \). The examples we presented put in evidence the complexity of this problem. Using the characterization theorem, one should build more examples to see which properties one may expect or not. It would be especially interesting to build a nontrivial example for \( p > 1 \) and see how does the set \( G^+ \) evolve as a function of the parameters.

Another problem related with system (M-NLS') is the existence of bound-states with the lowest action among nontrivial bound-states. This is not trivial at all, especially because it lacks a suitable variational formulation. Some attempts, using generalized Nehari manifolds, have proven the existence of such bound-states. It would be interesting to see if one may extend the characterization theorem to this case.

One of the reasons for which ground-states are an interesting object to study is because they give rise to periodic solutions for (1.1). In this context, one may study the stability of these solutions. It is known (see [1], [3], [11]) that the variational properties of the ground-states influence deeply their stability. We would like to point out the following: using the characterization theorem, we see that there exists a bijection between the set of ground-states and the set of solutions of a constrained maximization problem over \( \mathbb{R}^M \). Now consider local solutions of the same constrained problem in \( \mathbb{R}^M \). These solutions give rise to bound-states, which may or may not be ground-states. However, the local maximization property should be enough to prove results on local stability.

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