A slightly better bound on the crossing number in terms of the pair-crossing number

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Abstract

The crossing number of a graph $G$, $\text{cr}(G)$, is the minimum number of crossings, the pair-crossing number, $\text{pcr}(G)$, is the minimum number of pairs of crossing edges over all drawings of $G$. In this note we show that $\text{cr}(G) = O(\text{pcr}(G)^{3/2} \log \text{pcr}(G))$, which is an improvement of the result of Matoušek, by a log factor.

1 Introduction

By a graph we always mean a simple graph, that is, a graph with no loops and parallel edges. We use the term multigraph if loops and parallel edges are allowed. A drawing of a (multi)graph in the plane is a representation such that vertices are represented by distinct points and its edges by curves connecting the corresponding points. We assume that no edge passes through any vertex other than its endpoints, no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), no three edges cross at the same point, and two edges cross only finitely many times.

The crossing number of a graph $G$, $\text{cr}(G)$, is the minimum number of crossings (crossing points) over all drawings of $G$. The pair-crossing number, $\text{pcr}(G)$, is the minimum number of pairs of crossing edges over all drawings of $G$. In an optimal drawing for $\text{cr}(G)$, any two edges cross at most once. Therefore, it is not easy to see the difference between these two definitions. Indeed, there was some confusion in the literature between these two notions, until the systematic study of their relationship [PT00], [S17]. Clearly, $\text{pcr}(G) \leq \text{cr}(G)$, and in fact, we cannot rule out the possibility, that $\text{cr}(G) = \text{pcr}(G)$ for every graph $G$. Probably it is the most interesting open problem in this area. From the other direction, the best known bound is $\text{cr}(G) = O(\text{pcr}(G)^{3/2} \log^2 \text{pcr}(G))$ [M14]. In this note we slightly improve it.

Theorem 1. For any graph $G$, $\text{cr}(G) = O(\text{pcr}(G)^{3/2} \log \text{pcr}(G))$.

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Proof of Theorem \[\text{(1)}\]

A string graph is the intersection graph of continuous arcs in the plane. Vertices of the graph correspond to continuous curves (strings) in the plane such that two vertices are connected by an edge if and only if the corresponding strings intersect each other.

Suppose that \(G(V, E)\) is a graph of \(n\) vertices. A separator in a graph \(G\) is subset \(S \subset V\) for which there is a partition \(V = S \cup A \cup B\), \(|A|, |B| \leq 2n/3\), and there is no edge between \(A\) and \(B\). According to the Lipton-Tarjan separator theorem, \([LT79]\), every planar graph has a separator of size \(O(\sqrt{m})\). This result has been generalized in several directions, in particular, for string graphs, by Fox and Pach \([FP10]\).

Theorem 2. \([FP10]\) There is a constant \(c\) such that for any string graph \(G\) with \(m\) edges, there is a separator of size at most \(cm^{3/4}\sqrt{\log m}\).

The bound has been improved by Matoušek \([M14]\) to \(cm^{1/2}\log m\) and then by Lee \([L16]\) to \(cm^{1/2}\), which is asymptotically optimal.

Theorem 3. \([L16]\) There is a constant \(c\) such that for any string graph \(G\) with \(m\) edges, there is a separator of size at most \(cm^{1/2}\).

Using the separator theorem of Fox and Pach, Tóth \([T13]\) proved that for any graph \(G\),

\[
\text{cr}(G) = O(\text{pcr}(G)^{7/4} \log^{3/2}(\text{pcr}(G))).
\]

Matoušek \([M14]\) used the same approach, and his stronger separator theorem, and proved that

\[
\text{cr}(G) = O(\text{pcr}(G)^{3/2} \log(\text{pcr}(G))).
\]

To prove Theorem \[\text{(1)}\] we simply apply the separator theorem of Lee \([L16]\) with the same method.

Proof of Theorem \[\text{(1)}\]. Let \(c\) be the constant from the theorem of Lee \([L16]\). In a drawing \(D\) of a graph \(G\) in the plane, call those edges which participate in a crossing crossing edges, and those which do not participate in a crossing empty edges.

Lemma 1. Suppose that \(D\) is a drawing of a graph \(G\) in the plane with \(l > 0\) crossing edges and \(k \geq 0\) crossing pairs of edges. Then \(G\) can be redrawn such that (i) empty edges are drawn the same way as before, (ii) crossing edges are drawn in the neighborhood of the original crossing edges, and (iii) there are at most \(4ck^{3/2}/2\) edge crossings.

Proof of Lemma. The proof is by induction on \(l\). For \(l = 0\) the statement is trivial. Suppose that the statement has been proved for all pairs \((l', k')\), where \(l' < l\) and consider a drawing of \(G\) with \(k\) crossing pairs of edges, such that \(l\) edges participate in a crossing. Obviously, \(\binom{l}{2} \geq k\), and \(2k \geq l\), therefore, \(2k \geq l > \sqrt{l}\).

We define a string graph \(H\) as follows. The vertex set \(\overline{F}\) of \(H\) corresponds to the crossing edges of \(G\). Two vertices are connected by an edge if the corresponding edges cross each other. Note that the endpoints do not count; if two edges do not cross, the corresponding vertices are not connected even if the edges have a common endpoint. The graph \(H\) is a string graph, it can be represented by the crossing edges of \(G\), as strings, with their endpoints removed. It has \(l\) vertices, and \(k\) edges. By Theorem \[\text{(3)}\] \(H\) has a separator of size \(ck^{1/2}\) that is, the vertices can be decomposed into three sets, \(\overline{F}_0, \overline{F}_1, \overline{F}_2\), such that (i) \(|\overline{F}_0| \leq ck^{1/2}\), (ii) \(|\overline{F}_1|, |\overline{F}_2| \leq 2l/3\), (iii) there is no edge of \(H\) between \(\overline{F}_1\) and \(\overline{F}_2\).

This corresponds to a decomposition of the set of crossing edges \(F\) into three sets, \(F_0, F_1, F_2\) such that (i) \(|F_0| \leq ck^{1/2}\), (ii) \(|F_1|, |F_2| \leq 2l/3\), (iii) in drawing \(D\), edges in \(F_1\) and in \(F_2\) do not cross each other.
For \( i = 0, 1, 2 \), let \( |F_i| = l_i \). Let \( G_1 = G(V, E \cup F_1) \) and \( G_2 = G(V, E \cup F_2) \), then in the drawing \( D \) of the graph \( G_i \) has \( l_i \) crossing edges. Denote by \( k_i \) the number of crossing pairs of edges of \( G_i \) in drawing \( D \). Then we have \( k_1 + k_2 \leq k, l_1, l_2 \leq 2l/3, l_1 + l_2 + l_0 = l \).

For \( i = 1, 2 \), apply the induction hypothesis for \( G_i \) and drawing \( D \). We obtain a drawing \( D_i \) satisfying the conditions of the Lemma: (i) empty edges drawn the same way as before, (ii) crossing edges are drawn in the neighborhood of the original crossing edges, and (iii) there are at most \( 4ck_i^{3/2} \log l_i \) edge crossings.

Consider the following drawing \( D_3 \) of \( G \). (i) Empty edges are drawn the same way as in \( D, D_1 \), and \( D_2 \). (ii) For \( i = 1, 2 \), edges in \( F_i \) are drawn as in \( D_i \), (iii) Edges in \( F_0 \) are drawn as in \( D \). Now count the number of edge crossings (crossing points) in the drawing \( D_3 \). Edges in \( E \) are empty, edges in \( F_1 \) and in \( F_2 \) do not cross each other, there are at most \( 4ck_i^{3/2} \log l_i \) crossings among edges in \( F_i \). The only problem is that edges in \( F_0 \) might cross edges in \( F_1 \cup F_2 \) and each other several times, so we can not give a reasonable upper bound for the number of crossings of this type. Color edges in \( F_1 \) and \( F_2 \) blue, edges in \( F_0 \) red. For any piece \( p \) of an edge of \( G \), let \( \text{BLUE}(p) \) (resp. \( \text{RED}(p) \)) denote the number of crossings on \( p \) with blue (resp. red) edges of \( G \). We will apply the following transformations.

\begin{itemize}
  \item \text{ReduceCrossings}(e, f) Suppose that two crossing edges, \( e \) and \( f \) cross twice, say, in \( X \) and \( Y \). Let \( e' \) (resp. \( f' \)) be the piece of \( e \) (resp. \( f \)) between \( X \) and \( Y \). If \( \text{BLUE}(e') < \text{BLUE}(f') \), or \( \text{BLUE}(e') = \text{BLUE}(f') \) and \( \text{RED}(e') \leq \text{RED}(f') \), then redraw \( f' \) along \( e' \) from \( X \) to \( Y \). Otherwise, redraw \( e' \) along \( f' \) from \( X \) to \( Y \). See Figure 3.
\end{itemize}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{reduce-crossings}
\caption{\text{ReduceCrossings}(e, f)}
\end{figure}

Observe that \text{ReduceCrossings} might create self-crossing edges, so we need another transformation.

\begin{itemize}
  \item \text{RemoveSelfCrossings}(e) Suppose that an edge \( e \) crosses itself in \( X \). Then \( X \) appears twice on \( e \). Remove the part of \( e \) between the first and last appearance of \( X \).
\end{itemize}

Start with drawing \( D_3 \) of \( G \), and apply \text{ReduceCrossings} and \text{RemoveSelfCrossings} recursively, as long as there are two crossing edges that cross at least twice, or there is a self-crossing edge.

Let \( BB \) (resp. \( BR, RR \)) denote the number of blue-blue (resp. blue-red, red-red) crossings in the current drawing of \( G \). Observe, that the triple \((BB, BR, RR)\) lexicographically decreases with each of the transformations. Indeed,

- if \( e \) and \( f \) are both blue edges then \text{ReduceCrossings}(e, f) decreases \( BB \),
• if \( e \) is blue and \( f \) is red then either \( BB \) decreases, or if it stays the same then \( BR \) decreases,

• if \( e \) and \( f \) are both red edges then \( BB \) stays the same, and either \( BR \) decreases, or if it also stays the same
  then \( RR \) decreases,

• if \( e \) is blue then \( \text{RemoveSelfCrossings}(e) \) decreases \( BB \),

• and finally, if \( e \) is red then \( BB \) does not change, \( BR \) does not increase, and \( RR \) decreases.

Therefore, after finitely many steps we arrive to a drawing \( D_4 \) of \( G \), where any two edges cross at most once, and \((BB, BR, RR)\) is lexicographically not larger than originally. That is, in the drawing \( D_4 \), \( BB \leq 4ck^{3/2} \log l_1 + 2ck^{3/2} \log l_2 \), and any two edges cross at most once, therefore, \( BR + RR \leq l_0l \). So, for the total number of crossings we have

\[
4ck^{3/2} \log l_1 + 4ck^{3/2} \log l_2 + l_0l \\
\leq 4ck^{3/2} \log (2l/3) + 4ck^{3/2} \log (2l/3) + l_0l \\
= 4c(k_1^{3/2} + k_2^{3/2})(\log l + \log (2/3)) + l_0l \\
\leq 4ck^{3/2}(\log l + \log (2/3)) + l_0l \\
\leq 4ck^{3/2} \log l - 2ck^{3/2} + l_0l \\
\leq 4ck^{3/2} \log l - 2ck^{3/2} + 2ck^{3/2} \\
= 4ck^{3/2} \log l.
\]

\( \square \)

Now consider a graph \( G \) and let \( \text{pcr}(G) = k \). Take a drawing of \( G \) with exactly \( k \) crossing pairs of edges. Let \( l \) be the total number of crossing edges. By Lemma 1, \( G \) can be redrawn with at most \( 4ck^{3/2} \log l \) crossings. Since \( 2k \geq l \), \( \text{cr}(G) \leq 4ck^{3/2} \log l < 8ck^{3/2} \log k \). This concludes the proof of Theorem 1. \( \square \)

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