Towards optimal integration of external information in a multi-trigger population study analysis

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Abstract. We present an algorithm for population study which optimally uses information obtained from the electromagnetic (or particle) counterpart of gravitational wave (GW) events. Existing methods do not associate suitable weight factors to the triggers besides trivial ones like a hard cutoff on flux. However, the assignment of weights needs to take into account the background astrophysical distribution to be estimated. This is done using a likelihood-based approach where electromagnetic (or particle) and gravitational wave data are incorporated into a common likelihood before the parameters of a given population distribution are estimated. We present preliminary results from this method using simulated data corresponding to simplified models of GW and electromagnetic detectors.

1. Introduction
Consider a set of triggers, such as gamma-ray bursts (GRBs), with which we would like to test the association of gravitational waves (GWs). This can be done by suitably combining data from multiple triggers, such that the signal-to-noise ratio (SNR) of the association is increased. This idea, called population study, was developed and analyzed quantitatively in the context of GWs and GRBs by Finn, Mohanty and Romano in [1] (FMR).

An improved version of the FMR method was derived in [2] from first principles using a likelihood-based approach. In [3] it was shown that a population study analysis can significantly increase our belief in the association of GWs with a set of triggers when we do not have a single-trigger detection from that set, or our belief in such a detection is weak at best. Population study based observational results have been reported in [4, 5, 6, 7].

There is significant room for improvement in the sensitivity of population study methods. One expects the biggest improvement from applying suitable weights to the triggers such that the ones likely to contribute mostly to noise are suppressed in the analysis. Given an astronomical model, this may be done by using additional information available for the triggers.

In this paper we introduce the first steps towards such a method, which is to be extended for use in realistic burst searches. To demonstrate that this method suppresses noisy triggers and incorporates non-GW information, we apply it to a simple toy model of a triggered search. The results from this simple exercise support our expectations about how this method may work in realistic cases.

The rest of this paper is organized as follows. In section 2 we introduce our method and then derive the combined likelihood function. Section 3 gives an implementation of the method for
the toy model, the results of which are given in section 4. In section 5 we give our conclusions and thoughts on future work.

2. The method

Consider a set of measurements, \( S \), consisting of the information we know about a collection of trigger events. The trigger events are assumed to be samples from a population parameterized by \( \Gamma_1 \) and \( \Gamma_2 \), where \( \Gamma_1 \) is defined to be the set of parameters related to the GWs, and \( \Gamma_2 \) is defined to be the set of parameters related to the redshift distribution. In practice, there will be more information available about some trigger events than others.

In this paper we assume that each and every trigger event has an associated GW detection statistic, \( \hat{C}C \), and estimated sky location, \( \hat{\theta}, \hat{\phi} \), where a \( \hat{A} \) denotes a measured value of the quantity \( A \). In addition, trigger events may or may not have an associated measured redshift, \( \hat{Z} \). The set of information that we will use to estimate the population parameters is of the form \( S = \{\{\hat{C}C_1, \hat{Z}_1, \hat{\theta}_1, \hat{\phi}_1\}, \{\hat{C}C_2, \hat{\theta}_2, \hat{\phi}_2\}, ...\} \), where, in this example, the first trigger event has a measured redshift and the second does not. Thus, our method incorporates trigger events with varying information content.

We begin the derivation by assuming that the trigger events are statistically independent of one another. This is the case with GRBs, which are widely separated in space, but may not be applicable to triggers from the same object, such as soft-gamma-repeaters (SGRs) [8]. The probability of the combined event, \( P(S; \Gamma_1, \Gamma_2) \), is then simplified as the multiplication of the probability of the individual trigger events. There are then two different probability types that need to be derived: When the redshift is known, and when it is not. We begin with the former in 2.1, and the latter in 2.2. The combined likelihood function is then derived in 2.3.

2.1. Case I: Trigger event with known redshift, \( \hat{Z} \)

The likelihood function for the data associated with a single trigger can be expressed as follows. Here, \( Z, \theta, \phi \) denote the true “redshift”, and location on the sky, and \( \hat{C}C \) denotes the value of the statistic used for GW detection. (Here \( Z \) is simply a scaling factor for the observed GW signal amplitude).

\[
P(\hat{C}C, \hat{Z}, \hat{\theta}, \hat{\phi}; \Gamma_1, \Gamma_2) = \int d\theta d\phi dZ P(\hat{C}C, \hat{Z}, \hat{\theta}, \hat{\phi}, Z, \theta, \phi; \Gamma_1, \Gamma_2) = \int d\theta d\phi dZ P(\hat{C}C | \hat{\theta}, \hat{\phi}, Z, \theta, \phi; \Gamma_1) \frac{P(\hat{Z} | Z)P(Z)}{P(\hat{\theta}, \hat{\phi}, \theta, \phi)}.
\]  

(2)

Note that \( P(\hat{Z} | Z) \) is not part of the population model. It is determined by the measurement error of the method for measuring redshift from the electromagnetic information.

2.2. Case II: Trigger event with unknown redshift

The probability for a trigger that does not have associated redshift information, \( \hat{Z} \), is derived similarly to the previous case.

\[
P(\hat{C}C, \hat{\theta}, \hat{\phi}; \Gamma_1, \Gamma_2) = \int d\theta d\phi dZ P(\hat{C}C, \hat{\theta}, \hat{\phi}, Z, \theta, \phi; \Gamma_1, \Gamma_2) = \int d\theta d\phi dZ P(\hat{C}C | \hat{\theta}, \hat{\phi}, Z, \theta, \phi; \Gamma_1) P(Z; \Gamma_2)P(\hat{\theta}, \hat{\phi}, \theta, \phi).
\]  

(4)

We see that the expression is similar to the case for known redshift, except that the \( P(\hat{Z} | Z) \) term is understandably absent.
2.3. The likelihood function

We can now write down the loglikelihood function for the set of measurements, $S$:

$$
\log(P(S; \Gamma_1, \Gamma_2)) = \log(P(\hat{C}C_1, \hat{\phi}_1, \hat{\theta}_1, \hat{Z}_1, \hat{C}C_2, \hat{\phi}_2, \hat{\theta}_2, \hat{C}C_3, \hat{\phi}_3, \hat{\theta}_3, \hat{Z}_3, \ldots ; \Gamma_1, \Gamma_2))
$$

$$
= \sum_{m=1}^{M} \log(P(\hat{C}C_m, \hat{\phi}_m, \hat{\theta}_m, \hat{Z}_m ; \Gamma_1, \Gamma_2))
$$

$$
+ \sum_{n=M+1}^{N} \log(P(\hat{C}C_n, \hat{\phi}_n, \hat{\theta}_n ; \Gamma_1, \Gamma_2)).
$$

The first summation is over those trigger events with a measured redshift, which are labelled 1 to $M$. The second is over those trigger events lacking a measured redshift, which are labelled $M+1$ to $N$. Thus there are a total of $N$ trigger events whose information is used for the combined loglikelihood.

Once a specific model is chosen for the population, as well as a model for the measured redshift given the true redshift, then the population parameters can be estimated by maximizing the likelihood function just obtained. This is illustrated in the next section.

Note especially that the expressions derived in the present case take into account the presence and absence of redshift information, but the procedure can similarly be performed for other types of information such as electromagnetic flux and neutrino flux.

3. Implementation

To demonstrate and explore the method we choose a simplified toy model: The GW waveforms are sine-gaussians with parameterization $\Gamma_1 = \{A\}$, where $A$ is the intrinsic GW amplitude. The distribution of the triggers on the sky is a delta function centered at a fixed sky location. This reduces the computational demands as well as allows us to understand the impact of missing information better.

The redshift distribution is taken to be a scaled (across redshift) beta distribution [9] parameterized by $\Gamma_2 = \{\alpha, \beta\}$, where the $\alpha$ and $\beta$ values are positive real values that define a specific member of the beta distribution family. Examples of this distribution are shown in Figures 1 and 3, where in both cases it has been scaled to be from $Z = 1$ to $Z = 4$. This distribution was chosen because it is finitely bounded, and so we can be sure that no random trigger will ever have a redshift lower than some chosen minimum. This allows a clean test to be made of the population without contamination from extra loud triggers. Finally, for some choices of $\Gamma_2$, the beta distribution resembles predicted astrophysical redshift distributions in the sense that it has a single mode round which it is asymmetric.

The measured sky location, $(\hat{\theta}, \hat{\phi})$, is assumed to be normally distributed about the true sky location. The measured redshift, $\hat{Z}$, if known, is equal to the true redshift (which means that the probability $P(\hat{Z} \mid Z)$ is a delta function). Equation (2) is simplified by integrating over the delta function, which gives

$$
P(\hat{C}C, \hat{Z}, \hat{\theta}, \hat{\phi}; \Gamma_1, \Gamma_2) = \int P(\hat{C}C \mid \hat{Z}, \hat{\theta}, \hat{\phi}, \hat{\theta}, \hat{\phi}; \Gamma_1)P(\hat{Z}; \Gamma_2) P(\hat{\theta}, \hat{\phi}, \hat{\theta}, \hat{\phi}) \, d\theta \, d\phi.
$$

The set of trigger information, $S$, is simulated by picking random samples from the toy model described above for some fixed choice of $\Gamma_1, \Gamma_2$.

The $CC$ detector statistic was chosen to be the cross-correlation between two GW detectors. Both detectors have white, Gaussian noise with variance given by $\sigma_n^2 = 1$. We appealed to the law of large numbers, which resulted in the conditional cross-correlation probability distribution, $P(CC \mid \theta, \phi, Z, \theta, \phi; \Gamma_1)$, being approximately a normal distribution with variance, $\sigma_{CC}^2$, and
mean, $\mu_{CC}$, given by

$$
\mu_{CC} = \langle h_1, h_2 \rangle
$$

$$
\sigma^2_{CC} = \langle (h_1, h_1) + (h_2, h_2) \rangle \sigma_n^2 + K \sigma_n^4,
$$

where $K$ is the number of time samples per time series, $h_1$ is the GW time series from detector 1, $h_2$ is the GW time series from detector 2, $\langle \ldots \ldots \rangle$ is the dot product.

The sine-gaussians have a frequency of 250 Hz, and the standard deviation of the Gaussian envelope is 0.003 seconds. Each GW detector sampled at 4096 Hz. Each strain time series consisted of 650 simulated time samples. Based on the sky location, the two time series were shifted relative to one another and then the cross-correlation was taken in a window consisting of 256 window samples ($K = 256$).

The cross-correlation SNR (CSNR) is defined to be the mean of the cross-correlation in the presence of a signal, divided by the standard deviation of the cross-correlation in the absence of a signal (only noise). Applying the definition to the implementation given above results in

$$
\text{CSNR} = \frac{A^2}{\sigma^2} \frac{\langle q_1, q_2 \rangle}{\sqrt{K \sigma_n^2}},
$$

where $q_1 = h_1/(A/d)$, and $q_2 = h_2/(A/d)$. Thus the maximum CSNR will be at the closest distance, and where $\langle q_1, q_2 \rangle$ is greatest. Since we used a scaled beta function for the population distance distribution, $P(z; \Gamma_2)$, no trigger in the study has a CSNR greater than some CSNR$_{\text{max}}$, which we treat as study variable in the results that follow.

Finally, the numerical integration itself was performed over a fine grid in redshift, $Z$. For the probabilities themselves, we binned their pdf values so that they could be used on the integration grid. This explains, for example, the non-zero value at $Z = 1$ in Figure 1, and for the non-zero value at $Z = 4$ in Figure 3.

The implementation was programmed in C99, and used the Message Passing Interface (MPI) in order to reduce the computation times when run on a computer cluster. The processing times often take hours due to the loglikelihood involving integrals, and the need to recompute the integrals during the maximization stage when estimating parameters.

4. Results

We sought to evaluate how well our method performs in reconstructing the GW signal amplitude parameter, $\Gamma_1 = A$, which is related to the energy of the GWs. This should depend on a number of factors, including the number of triggers composing the set of information, $S$, the loudest possible trigger as specified by CSNR$_{\text{max}}$, and the number of known versus unknown redshift triggers. To explore our method in terms of these factors, we ran a number of separate analyses where each analysis differed by one of the factors mentioned.

In the discussion so far we have not specified $\Gamma_2$, which determines the population’s redshift distribution, $P(z; \Gamma_2)$. Since this population parameter plays such a large role, we used two different values for it. The first, where $\Gamma_2 = \{2, 4\}$, gives a population redshift distribution that has most triggers with low redshift, as shown in Figure 1. The second, where $\Gamma_2 = \{4, 2\}$, gives a population redshift distribution where most of the triggers are at high redshifts, as shown in Figure 3. The key qualitative difference between the two is that the latter will have more triggers buried in the noise than the former. Using these two extremes allows us to better qualitatively explore the method.

To obtain statistics for the error in estimation we performed each unique analysis 4096 times, and so obtained 4096 estimates of $\Gamma_1$ for each specific analysis. For example, in one analysis we had $\Gamma_2 = \{2, 4\}$, CSNR$_{\text{max}} = 1.5$, and the number of triggers equal to 64. We then performed that analysis 4096 times, and so obtained 4096 estimations of $\Gamma_1$, which we denote by $\Gamma_{1n}$.
where \( n \) ranges from 1 to 4096. From those 4096 estimations we then computed one relative mean squared error (RMSE), which we then plotted as a single data point in Figure 4.

The RMSE was calculated in terms of \( \Gamma_1 \),

\[
\text{RMSE} = \frac{1}{\Gamma_1} \sqrt{\frac{1}{N-1} \sum_{n=1}^{N} (\Gamma_{n1} - \Gamma_1)^2}
\]  

(11)

where we had \( N = 4096 \), and where each \( \Gamma_1 \) is associated with a corresponding CSNR\(_{\text{max}}\).

The full analysis results are presented in two separate figures. Figure 2, is for analyses with \( \Gamma_2 = \{2, 4\} \), while Figure 4 is for analyses with \( \Gamma_2 = \{4, 2\} \). We expect that the RMSE will be lower in general for the former, than the latter, due to the former having more triggers at lower redshifts and so having a correspondingly stronger population signature. We also expect that knowing the redshift will result in a lower RMSE over not knowing it.

As qualitatively expected, the results show that the RMSE decreases with increasing number of triggers, and with increasing CSNR\(_{\text{max}}\). Importantly, we see that incorporating redshift information into the parameter estimation results in a lower RMSE in all of the cases studied. Thus, the method behaves as expected in that it produces a better parameter estimation when the extra redshift information is included. Since we used a toy model for the simulations, and because we are aiming to show that our method can successfully incorporate non-GW information, in this case redshift, into the parameter estimation, we do not focus on the specific values obtained.

**Figure 1.** Shown in the solid line is an example redshift distribution, \( P(z; \Gamma_2) \), where we have used a scaled beta distribution with \( \Gamma_2 = \{2, 4\} \). The GW amplitude parameter, \( \Gamma_1 \), has been chosen such that at the minimum redshift possible, which is \( Z = 1 \), there will be a CSNR\(_{\text{max}} = 4 \). In this way, we guarantee that in the simulations there will not be any trigger with a larger CSNR than 4. The CSNR for the different redshifts is shown in the dashed curve, and uses the axes on the right. We see that the maximum CSNR occurs at the smallest \( Z \).

5. Conclusions
The main question we ask at this stage is whether our proposed method can correctly take non-GW information into account. To determine this we constructed a toy model on which we tested our method. The results obtained show that the error in estimating the strength of the gravitational waves was lower when redshift information was incorporated, which is what is qualitatively expected, and demonstrates that this approach is on the right track.
Figure 2. Shown are the errors in estimating the GW amplitude parameter $\Gamma_1$ versus CSNR$_{\text{max}}$, where we have fixed $\Gamma_2 = \{2, 4\}$ and whose corresponding redshift distribution is shown in Figure 1. Each series shown differs in the number of triggers in the set (denoted by the number in the series label), and whether or not the redshift information was known (indicated by the ‘k’), versus unknown (indicated by the ‘u’).

Figure 3. Shown is an example plot of redshift distribution, which has been parameterized the same as that for Figure 1, with the only exception that this time we have $\Gamma_2 = \{4, 2\}$. The shape of the distribution is now peaked more strongly towards higher redshifts. Thus triggers whose redshift is generated from this distribution will tend to have a lower CSNR, than those generated from the distribution shown in Figure 1.
Figure 4. Shown are the errors in estimating the GW amplitude parameter $\Gamma_1$ versus CSNR$_{\text{max}}$, where we have fixed $\Gamma_2 = \{4, 2\}$ and whose corresponding redshift distribution is shown in Figure 3. Each series shown differs in the number of triggers in the set (denoted by the number in the series label), and whether or not the redshift information was known (indicated by the ‘k’), versus unknown (indicated by the ‘u’).

In this present study, we used redshift as the non-GW information, but the method itself is not limited to redshift. It can also incorporate other non-GW information such as neutrino fluxes, for instance. Also, we estimated $\Gamma_1$, which is related to the energy of the GWs, but the method can also be used to estimate $\Gamma_2$, which parameterizes the population’s redshift distribution.

The important point is the existence of a gain in performance when non-GW information is included, rather than the actual magnitude obtained, as the approximations used to construct the toy model will not be the same as those used for a full population study.

The computations involved in our method can be time consuming. Due to the fact that the likelihoods derived involve integrals, Equations (2) and (4), the numerical computations often require hours of computation. The parallel nature of the method should lend itself well to an implementation using graphics programming units, which can greatly reduce the computation times for parameter estimation.

In the future we intend to use a GW network statistic, instead of GW cross-correlation, which we expect to perform better. Also, we plan to explore simulated maximum likelihood methods [10], as they can better handle the larger parameter space of full astrophysical models.

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