LINEARIZED STABILITY FOR ABSTRACT FUNCTIONAL DIFFERENTIAL EQUATIONS SUBJECT TO STATE-DEPENDENT DELAYS WITH APPLICATIONS

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(Communicated by Shigui Ruan)

Abstract. In this paper, the linearized stability for a class of abstract functional differential equations (FDE) with state-dependent delays (SD) is investigated. In particular, such equations contain more general delay terms which not only cover the discrete delay and distributed delay as special cases, but also extend the SD to abstract integro-differential equation that the states belong to some infinite-dimensional space. The principle of linearized stability for such equations is established, which is nontrivial compared with ordinary differential equations with SD. Moreover, it should be stressed that such topic is untreated in the literatures up to date. Finally, we present an example to show the effectiveness of the proposed results.

1. Introduction. In recent years, the theory and applications of ordinary differential equations (ODE) with SD are emerging as an important area of investigation. Among the topics for such SD are the principle of linearized stability [3, 8, 9, 29], center manifold theorem [9, 16, 27], the Hopf bifurcation theorem [4, 6, 9, 15], and so on. Meanwhile, SD have been incorporated into a variety of ordinary differential equation models, since they are more realistic in the real world problems. For example, in [20], the stability of the equilibria for a stage-structured competitive model with SD is analyzed by using the distribution of the roots of the characteristic equation, the comparison principle and iterative method. Hou and Guo [14] dealt with the Hopf bifurcation problem for a class of predator–prey equations with state-dependent delayed feedback according to the Hopf bifurcation theory. Recently, Getto and Waurick [5] obtained the existence of a local semiflow and a new general sufficient criterion of saturated existence for a class of differential equations with SD from cell population biology. For more results concerning the application of such differential equations we refer the interested reader to [1, 4, 9, 12, 19] and references therein.

However, to the best of our knowledge, there are a limited number of reports ([2, 13, 17, 24]) for the existence of solution for nonlinear partial functional differential
equations (PFDE) with SD that the states belong to infinite-dimensional space, let alone the principle of linearized stability for this matter. Moreover, many dynamical systems theories are inapplicable for such equations due to the complexity of the SD involved. For example, as stated [26], such equations on the space of continuous functions may not generate a dynamical system. And the uniqueness of solutions for such equations with continuous initial function is not guaranteed (see [17, 19]). Moreover, There does not exist applicable variation of constants formula for such equations on the space of continuous functions, and we refer the readers to [10]. These facts are the main motivation of the present work. Although such factors make the analysis of nonlinear PFDE with SD more difficult, we overcome these obstacles to establish the results of linearized stability by using semigroup theory. And the results of linearized instability, which are motivated by Henry [11] for PDEs, are mainly offered in frame of sectorial operator theory. It is worth mentioning that these facts are the main motivation of the present work.

The rest of this paper is organized as follows. In section 2, we mainly give the principle of linearized stability for such equations in frame of sectorial operator theory. Section 3 is devoted to applying the results obtained for FDE to a class of reaction-diffusion equations with SD, then the straightforward stability conditions are provided. In section 4, an example is performed to illustrate the efficiency of the proposed method.

**Notation**: The spaces \( C([a, b], Y) \) and \( C^1([a, b], Y) \) are endowed with the supremum norms \( \| \cdot \|_{C([a, b], Y)} \) and \( \| \cdot \|_{C^1([a, b], Y)} \), respectively. \( B(Y, Y) \) stands for the space of bounded linear operators from \( Y \) to \( Y \) endowed with norm \( \| Y \|_{B(Y, Y)} \). \( B_r(x, Y) \) denotes the closed ball with center \( x \in Y \) and radius \( r \geq 0 \) in \( Y \). Accordingly, \( B_{r_1}(x, Y) \) denotes the open ball. The partial derivatives of a function \( f \) of two variables with respect to its first or second variable is denoted by \( D_1f \) or \( D_2f \), respectively.

### 2. Principle of linearized stability.

#### 2.1. Preliminaries.

Now we present some definitions and the existence and uniqueness of solution for initial value problem (IVP) (1), which will be useful in the sequel.

\[
\begin{align*}
\dot{u}(t) + Au(t) &= F(u_t) \\
u_0 &= \phi \in C([-\tau, 0], X)
\end{align*}
\]
Definition 2.1. A function \( u(t) \in C([-\tau, t_f), X) \) is a mild solution of IVP (1) if \( u_0 = \phi \) and
\[
  u(t) = e^{-At}\phi(0) + \int_0^t e^{-A(t-\theta)}f(u(\theta), \Lambda(u_\theta))d\theta, \forall t \in [0, t_f).
\]

Definition 2.2. A function \( u(t) \in C([-\tau, t_f), X) \cap C^1([0, t_f), X) \) is a classical solution of IVP (1) if \( u(t) \in D(A) \) for \( t \in [0, t_f) \) and (1) holds true.

Set
\[
  X_f = \{ \phi \in C^1([-\tau, 0], X), \phi(0) \in D(A) \}
\]
edowed with
\[
  \|\phi\|_{X_f} = \max_{\theta \in [-\tau, 0]} \|\phi(\theta)\|_X + \max_{\theta \in [-\tau, 0]} \|\phi'(\theta)\|_X + \|A\phi(0)\|_X.
\]

We can follow the streamline of the proof of Lemma 1.2 in [19], (see also [17, Theorem 2]) established by [23, Theorem 3.1 and Corollary 3.3]. Then we get the following Lemma.

Lemma 2.3. Suppose that \( A, f, g, \eta \) are as above. Then there exists \( \bar{t} \) such that Eq. (1) has a unique classical solution \( u(t) \) on \([-\tau, \bar{t})\) with initial value \( u_0 = \phi \in X_f \).

Remark 1. Let \( u(t) = u(t, 0, \phi) \) be a classical solution of IVP (1), then \( u \) is continuously differentiable on \([-\tau, \bar{t})\) and satisfies the compatibility condition \( \phi(0) + A\phi(0) = f(\phi(0), \Lambda(\phi)) \).

In order to obtain the principle of linearized stability, we will make use of some basic results.

Lemma 2.4. ( [11, Theorem 1.3.4] ) If \( A \) is a sectorial operator, then \(-A\) is the infinitesimal generator of an analytic semigroup \( \{e^{-tA}\}_{t \geq 0} \) where
\[
e^{-tA} = \frac{1}{2\pi i} \int_\Gamma (\lambda + A)^{-1} e^{\lambda t}d\lambda
\]
where \( \Gamma \) is a contour in \( \rho(-A) \) with \( \arg \lambda \to \pm \theta \) as \( |\lambda| \to \infty \) for some \( \theta \) in \([\frac{\pi}{2}, \pi)\).

Further \( e^{-tA} \) can be continued analytically into a sector \( \{ t \neq 0 : |\arg t| < \varepsilon \} \) containing the positive real axis, and if \( \text{Re} \sigma(A) > a \), i.e. if \( \text{Re} \lambda > a \) whenever \( \lambda \in \sigma(A) \), then for \( t > 0, x \in D(A) \)
\[
  \|e^{-tA}x\|_X \leq c e^{-at} \|x\|_X, \quad \|Ae^{-tA}x\|_X \leq \frac{c}{t} e^{-at} \|x\|_X
\]
for some constant \( c \).

Finally \( \frac{d}{dt} e^{-tA} = -Ae^{-tA} \) for \( t > 0 \).

Lemma 2.5. ( [11, Theorem 1.3.2] ) If \( A \) is a sectorial operator in \( X \), \( B \) is a linear operator with \( D(A) \subset D(B) \), and for all \( x \in D(A) \),
\[
  \|Bx\|_X \leq \varepsilon \|Ax\|_X + K(\varepsilon) \|x\|_X \quad \text{(for sufficiently small } \varepsilon > 0 \text{)}
\]
where \( K(\varepsilon) > 0 \) is a constant, then \( A + B \) is a sectorial operator.

Definition 2.6. ( [11, Definition 1.5.1] ) If \( A \) is a linear operator with domain and range in a Banach space \( X \), and \( \sigma(A) \) denotes the spectrum, a set \( \sigma \subset \sigma(A) \cup \{\infty\} \) \( \hat{\sigma} \) \( A \) is a spectral set if both \( \sigma \) and \( \hat{\sigma}(A) \setminus \sigma \) are closed in the extended plane \( \mathbb{C} \cup \{\infty\} \).
Lemma 2.7. ([11, Theorem 1.5.2]) Suppose $A$ is a closed linear operator in $X$ and suppose $\sigma_1$ is a bounded spectral set, and $\sigma_2 = \sigma(A) \setminus \sigma_1$ so $\sigma_2 \cup \{\infty\}$ is another spectral sets. Let $E_1$, $E_2$ be the projections associated with these spectral sets, and $X_j = E_j(X), j = 1, 2$. Then $X = X_1 \oplus X_2$, the $X_j$ are invariant under $A$, and if $A_j$ is the restriction of $A$ to $X_j$, then

\[
A_1 : X_1 \to X_1 \text{ is bounded, } \sigma(A_1) = \sigma_1; \quad D(A_2) = D(A) \cap X_2 \text{ and } \sigma(A_2) = \sigma_2.
\]

Lemma 2.8. ([11, Theorem 1.5.3]) Suppose that $A$ is a sectorial operator and $\sigma_1$ is a bounded spectral set; forming the operators $A_1$ and $A_2$ as in Lemma 2.7, $A_1$ is bounded and $A_2$ is a sectorial operator.

(a) If $\text{Re}\sigma_1 = \text{Re}\sigma(A_1) < \alpha$, then there is a constant $c \geq 0$ such that $\|e^{-A_1 t}x\|_X \leq ce^{-\alpha t} \|x\|_X$ for $t \leq 0, x \in D(A_1)$;

(b) If $\text{Re}\sigma_2 = \text{Re}\sigma(A_2) > \beta$, then there is a constant $c \geq 0$ such that $\|e^{-A_2 t}x\|_X \leq ce^{-\beta t} \|x\|_X$ for $t > 0, x \in D(A_2)$.

Lemma 2.9. ([11, Lemma 5.1.4]) Suppose $X$ is a real Banach space, $L$ is a continuous linear operator on $X$ with spectral radius $r > 0$. Given any $\delta > 0$ and $N_0 \geq 0$, there exists an integer $N \geq N_0$ and $u \in X$, $\|u\|_X = 1$, such that

\[
\|L^n u\|_X \leq (\sqrt{2} + \delta)^r \quad \text{for } 0 \leq n \leq N, \\
\|L^n u\|_X \geq (1 - \delta) r^N.
\]

2.2. Linearized stability. Without loss of generality, we suppose that $u^* = 0$ such that

\[
f(0, 0) = 0
\]

which implies that $u^* = 0$ is an equilibrium point of Eq.(1). To study the local stability of the equilibrium point $u^* = 0$, we linearize (1) at $u^* = 0$ by treating $r = r(u^*)$ as a constant and setting $g(0) = 0$ to obtain a usual state-independent linear delay equation (see [9, 30] for more details about linearization of differential equations with SD).

\[
\dot{u}(t) + Au(t) = D_1 f(0, 0) u(t) + D_2 f(0, 0) \int_{-\tau}^0 ds \eta(s, 0) Dg(0) u(t - r(0)).
\]

In the sequel, for the sake of convenience, we assume

\[ (H1) \quad g(0) = 0, f(0, 0) = 0. \]

Define

\[
\mathfrak{G}(\psi) \equiv f(\psi(0), \Lambda(\psi)) - \mathcal{L}\psi
\]

with

\[
\mathcal{L}\psi \equiv D_1 f(u^*, \Lambda(u^*)) \psi(0) + D_2 f(u^*, \Lambda(u^*)) \int_{-\tau}^0 ds \eta(s, u^*) Dg(u^*) \psi(-r(u^*)).
\]

In order to obtain the main results, we need the following proposition.

Proposition 1. Assume (H1) is satisfied. Given any $\mu \in (0, 1]$, there exist constants $R(\mu) > 0$ and $N_i > 0, i = 1, 2$, such that

\[
\|\mathfrak{G}(\psi)\|_X \leq \left(5N_1 + N_2 \|\psi\|_{C([-\tau, 0], X)}\right) \|\psi\|_{C([-\tau, 0], X)}
\]

\[
\Delta = \mathcal{B} \left(\|\psi\|_{C([-\tau, 0], X)}\right) \|\psi\|_{C([-\tau, 0], X)},
\]
for $\psi \in \bar{B}_R \left(0, C^1 \triangleq C^1 \left([-\tau, 0], X \right) \right)$, where $\mathfrak{B} : [0, +\infty) \to [0, +\infty)$ is continuous and nondecreasing.

**Proof.** It follows from (4), (5) and mean value theorem that

\[
\|\mathfrak{G}(\psi)\|_X = \|f(\psi(0), \Lambda(\psi)) - f(0, 0) - D_1 f(0, 0) \psi(0)
- D_2 f(0, 0) \int_{-\tau}^{0} d_s \eta(s, 0) D g(0) \psi(-r(0))\|_X
\leq \sup_{0 \leq \beta \leq 1} \|D_1 f(\beta \psi(0), \Lambda(\beta \psi)) \psi(0) + D_2 f(\beta \psi(0), \Lambda(\beta \psi)) \Lambda(\psi) - D_1 f(0, 0)\|_X \|\psi(0)\|_X
+ \sup_{0 \leq \beta \leq 1} \|D_2 f(\beta \psi(0), \Lambda(\beta \psi)) \Lambda(\psi) - \Lambda(0)\|_X
- D_2 f(0, 0) \int_{-\tau}^{0} d_s \eta(s, 0) D g(0) \psi(-r(0))\|_X
\leq \sup_{0 \leq \beta \leq 1} \|D_1 f(\beta \psi(0), \Lambda(\beta \psi)) - D_1 f(0, 0)\|_X \|\psi(0)\|_X
+ \sup_{0 \leq \beta \leq 1} \|D_2 f(\beta \psi(0), \Lambda(\beta \psi)) \Lambda(\psi) - \Lambda(0)\|_X
- D_2 f(0, 0) \int_{-\tau}^{0} d_s \eta(s, 0) D g(0) \psi(-r(0))\|_X
\leq \sup_{0 \leq \beta \leq 1} \|D_1 f(\beta \psi(0), \Lambda(\beta \psi)) - D_1 f(0, 0)\|_X \|\psi(0)\|_X
+ \sup_{0 \leq \beta \leq 1, 0 \leq \varsigma \leq 1} \|D_2 f(\beta \psi(0), \Lambda(\beta \psi)) \int_{-\tau}^{0} d_s \eta(s, \varsigma \psi(-r(\varsigma \psi(s)))) D g(\varsigma \psi(-r(\varsigma \psi(s)))) \psi(-r(\psi(s)))
+ D_2 f(\beta \psi(0), \Lambda(\beta \psi)) \int_{-\tau}^{0} g(\varsigma \psi(-r(\varsigma \psi(s))))
\|D_2 f(\beta \psi(0), \Lambda(\beta \psi)) \int_{-\tau}^{0} d_s \eta(s, \varsigma \psi(-r(\varsigma \psi(s)))) \psi(-r(\psi(s)))
- D_2 f(0, 0) \int_{-\tau}^{0} d_s \eta(s, 0) D g(0) \psi(-r(0))\|_X
\leq \|\mathfrak{G}_1(\psi)\|_X + \|\mathfrak{G}_2(\psi)\|_X + \|\mathfrak{G}_3(\psi)\|_X + \|\mathfrak{G}_4(\psi)\|_X + \|\mathfrak{G}_5(\psi)\|_X
\] (6)

where

\[
\|\mathfrak{G}_1(\psi)\|_X \triangleq \sup_{0 \leq \beta \leq 1} \|D_1 f(\beta \psi(0), \Lambda(\beta \psi)) - D_1 f(0, 0)\|_X \|\psi(0)\|_X
\]
\[ \| \mathcal{G}_2 (\psi) \|_X \triangleq \sup_{0 \leq \beta \leq 1, 0 \leq \varsigma \leq 1} \| [D_2 f (\beta \psi (0), \Lambda (\beta \psi)) - D_2 f (0, 0)] \\
\int_{-\tau}^{0} d_s \eta (s, c\psi (r (c\psi (s)))) Dg (c\psi (r (c\psi (s)))) \psi (r (0)) \|_X, \]
\[ \| \mathcal{G}_3 (\psi) \|_X \triangleq \sup_{0 \leq \beta \leq 1, 0 \leq \varsigma \leq 1} \| D_2 f (0, 0) \left[ \int_{-\tau}^{0} d_s \eta (s, c\psi (r (c\psi (s)))) Dg (c\psi (r (c\psi (s)))) \\
- \int_{-\tau}^{0} d_s \eta (s, 0) Dg (0) \right] \psi (r (0)) \|_X, \]
\[ \| \mathcal{G}_4 (\psi) \|_X \triangleq \sup_{0 \leq \beta \leq 1, 0 \leq \varsigma \leq 1} \| D_2 f (\beta \psi (0), \Lambda (\beta \psi)) \int_{-\tau}^{0} d_s \eta (s, c\psi (r (c\psi (s)))) Dg (c\psi (r (c\psi (s)))) \psi (r (0)) \|_X \]
\[ \| \mathcal{G}_5 (\psi) \|_X \triangleq \sup_{0 \leq \beta \leq 1, 0 \leq \varsigma \leq 1} \| D_2 f (\beta \psi (0), \Lambda (\beta \psi)) \int_{-\tau}^{0} g (c\psi (r (c\psi (s)))) \\
d_s D\eta (s, c\psi (r (c\psi (s)))) \|_X \psi (r (c\psi (s))) \|_X. \]

For every \( \mu \in (0, 1] \), by the continuous differentiability of \( f, g \) and \( r \), we can choose a constant \( R (\mu) > 0 \) such that for \( \psi \in B_R (0, C^1) \), \( i = 1, 2 \), we have
\[ \| D_i f (\psi (0), \Lambda (\psi)) - D_i f (0, 0) \|_X \leq \mu \]
\[ \| Dg (\psi (r (\psi (0)))) - Dg (0) \|_X \leq \mu. \]

This yields that
\[ \| \mathcal{G}_1 (\psi) \|_X \leq \mu \| \psi \|_{C([-\tau, 0], X)}, \]
\[ \| \mathcal{G}_2 (\psi) \|_X \leq \mu \| \eta \|_{NBV} \left( \| Dg (0) \|_X + 1 \right) \| \psi \|_{C([-\tau, 0], X)} \]
\[ \| \mathcal{G}_3 (\psi) \|_X \leq \sup_{0 \leq \beta \leq 1, 0 \leq \varsigma \leq 1} \| D_2 f (0, 0) \int_{-\tau}^{0} d_s \eta (s, c\psi (r (c\psi (s)))) \\
\left[ Dg (c\psi (r (c\psi (s)))) - Dg (0) \right] \psi (r (0)) \|_X \]
\[ + \sup_{0 \leq \beta \leq 1} \| D_2 f (0, 0) \left[ \int_{-\tau}^{0} d_s \eta (s, c\psi (r (c\psi (s)))) \right] \right] \left[ - \int_{-\tau}^{0} d_s \eta (s, 0) \right] Dg (0) \psi (r (0)) \|_X \]
\[ \leq \mu \| D_2 f (0, 0) \|_X \| \eta \|_{NBV} \| \psi \|_{C([-\tau, 0], X)} + \| D_2 f (0, 0) \|_X \| \eta \|_{NBV} \| Dg (0) \|_X \| \psi \|_{C([-\tau, 0], X)}, \]
\[ \| \mathcal{G}_4 (\psi) \|_X \]
\[ \leq (\| D_2 f (0, 0) \|_X + 1) \| \eta \|_{NBV} (\| Dg (0) \|_X + 1) l_1 \| \psi \|_{C([-\tau, 0], X)} \| \psi \|_{C([-\tau, 0], X)}. \]

Here, \( l_1 \) is Lipschitz constant for the function \( r \).
In terms of (4) and (H1), we can assume that for $\rho > 0$, $FDE$ with SD is stated in the following. For the sake of convenience, we assume

$$
(1)
$$

i.e., the origin is a locally asymptotically stable steady-state of $\mathcal{L}$ and $h > 0$.

Suppose that (H1) and (H2) are satisfied. Given $\psi \in X$, the solution $\phi$ of the linear system $\phi(0) = \psi$ is stated in (5).

$$
\begin{align*}
\|D_2\eta(0,0)\|_{X} + 1 & \leq \sup_{0<\xi<1} \left| \int_{\tau}^{0} \left[ g \left( \frac{\xi}{\nu} \left( - g \left( \frac{\xi}{\nu} (s) \right) \right) \right) - g(0) \right] \right| \\
& \leq \left( \|D_2f(0,0)\|_{X} + 1 \right) l_2 \|\psi\|_{C([-\tau, 0], X)} \|D_2\eta\|_{NVB} \|\psi\|_{C([-\tau, 0], X)}
\end{align*}
$$

where $l_2$ is Lipschitz constant for the function $g$,

$$
\|D_2\eta\|_{NVB} \triangleq \sup_{\psi \in \mathcal{B}_{R}(0, C^1), 0<\xi<1} \{V_{-\xi}^\tau(D_2\eta(\cdot, \psi(-r(\mathcal{G}(\cdot))))\} 
$$

with $V_{-\xi}^\tau(D_2\eta(\cdot, \psi(-r(\mathcal{G}(\cdot))))\}$ is the total variation of $D_2\eta(\cdot, \psi(-r(\mathcal{G}(\cdot))))\}$ on $[-\tau, 0]$.

By combining (6) – (11), we obtain for $\psi \in \mathcal{B}_{R}(0, C^1)$

$$
\|\mathcal{G}(\psi)\|_{X} \leq \mu \|\psi\|_{C([-\tau, 0], X)} + \mu \|\eta\|_{NVB} \left( \|D_2\eta(0)\|_{X} + 1 \right) \|\psi\|_{C([-\tau, 0], X)} \\
+ \|D_2f(0,0)\|_{X} \|Dg(0)\|_{X} \|\eta\|_{NVB} \|\psi\|_{C([-\tau, 0], X)} \\
+ \|D_2f(0,0)\|_{X} \|\psi\|_{C([-\tau, 0], X)} \|\eta\|_{NVB} \left( \|Dg(0)\|_{X} + 1 \right) l_1 R, \\
+ \|D_2f(0,0)\|_{X} \|\psi\|_{C([-\tau, 0], X)} \|\eta\|_{NVB} \left( \|Dg(0)\|_{X} + 1 \right) l_2 \|\psi\|_{C([-\tau, 0], X)} \\
\leq \left( 5N_1 + N_2 \|\psi\|_{C([-\tau, 0], X)} \right) \|\psi\|_{C([-\tau, 0], X)}
$$

where

$$
\begin{align*}
\max & \{\mu, \mu \|\eta\|_{NVB} \left( \|Dg(0)\|_{X} + 1 \right), \|D_2f(0,0)\|_{X} \|\eta\|_{NVB} \} \\
& \left( \|D_2f(0,0)\|_{X} + 1 \right) \|\eta\|_{NVB} \left( \|Dg(0)\|_{X} + 1 \right) l_1 R, \\
& \|D_2f(0,0)\|_{X} \|\psi\|_{C([-\tau, 0], X)} \|\eta\|_{NVB} \left( \|Dg(0)\|_{X} + 1 \right) l_2 \|\psi\|_{C([-\tau, 0], X)}
\end{align*}
$$

This completes the proof. \hfill \Box

Based on the above arguments, the linearized stability result for a class of abstract FDE with SD is stated in the following. For the sake of convenience, we assume (H2) $\varpi = \sup \{Re \lambda \mid \lambda \in \mathbb{C}, \lambda \in \text{Dom}(A) \setminus \{0\} \}$ s.t. $\lambda e^{\lambda x} x - Ax = \lambda x < 0$, where $\mathcal{L}$ is stated in (5).

**Theorem 2.10.** Suppose that (H1) and (H2) are satisfied. Given $K \geq 1, \gamma > 0$ and $h > 0$, then there is a solution $u$ of (1) existing on $t \geq -\tau$ and satisfying

$$
\|u_t\|_{C([-\tau, 0], X)} \leq Ke^{-\gamma t} \|\phi\|_{X_f}, \phi \in \mathcal{B}_{h}(0, X_f), t \geq 0,
$$

i.e., the origin is a locally asymptotically stable steady-state of (1).

**Proof.** Let $\{S(t)\}_{t \geq 0}$ be the solution semigroup defined by the linear system

$$
u(t) = e^{-At} u(0) + \int_{0}^{t} e^{-A(t-\theta)} \mathcal{L} u_0 d\theta.
$$

By assumption (H2), we can find $M \geq 1$ and $\beta > 0$ such that

$$
\|S(t)\phi\|_{C([-\tau, 0], X)} \leq Me^{-\beta t} \|\phi\|_{X_f}, t \geq 0, \phi \in X_f.
$$

In terms of (4) and (H1), we can assume that for $\rho > 0, \psi \in B_{\rho}(0, C^1)$

$$
f(\psi(0), \Lambda(\psi)) = f(0,0) + \mathcal{L} \psi + \mathcal{G}(\psi).
$$
Moreover, it follows from Proposition 1 that there exists a continuous nondecreasing function \( \mathcal{B} : [0, \rho] \to [0, \infty) \) such that \( \mathcal{B}(0) \geq 0 \) and
\[
\| \mathcal{G}(\psi) \|_X \leq \mathcal{B} \left( \|\psi\|_{C([\tau, 0], X)} \right) \|\psi\|_{C([\tau, 0], X)}.
\]

Let \( \kappa \in (0, \rho) \) be chosen such that
\[
\mathcal{B}(\kappa) < \frac{\beta}{2M}
\]  
and define
\[
h \triangleq \min \left\{ \rho, \frac{\kappa}{2M} \right\}.
\]

We will show that if \( \phi \in \tilde{B}_h(0, X_f) \) then the solution \( u \) of (1) exists and \( \|u_t\|_{C([\tau, 0], X)} < \kappa \) for all \( t \geq 0 \). For a contradiction, if there is \( \vartheta > 0 \) such that
\[
\|u_t\|_{C([\tau, 0], X)} < \kappa, \quad \text{for } t \in [0, \vartheta], \quad \text{and } \|u_\vartheta\|_{C([\tau, 0], X)} = \kappa,
\]
then for \( t \in [0, \vartheta] \), we have
\[
u (t) = e^{-At}\phi(0) + \int_0^t e^{-A(t-\theta)} f(u_\theta(0), \Lambda(u_\theta)) d\theta
\]
\[
= e^{-At}\phi(0) + \int_0^t e^{-A(t-\theta)} \left[ f(0, 0) + \mathcal{L}u_\theta + \mathcal{G}(u_\theta) \right] d\theta.
\]

This leads to the following formula since \( f(0, 0) = 0 \)
\[
u (t) = e^{-At}\phi(0) + \int_0^t e^{-A(t-\theta)} \left[ \mathcal{L}u_\theta + \mathcal{G}(u_\theta) \right] d\theta.
\]

By the variation of constants formula, it follows that
\[
u_t = S(t)\phi + \int_0^t S(t-s)X_0\mathcal{G}(u_s) ds, \quad t \in [0, \vartheta]
\]
where \( X_0 : [-\tau, 0] \to B(X, X) \) is defined as
\[
X_0(\theta) = \begin{cases} 
0 & \theta \in [-\tau, 0) \\
1d & \theta = 0.
\end{cases}
\]

Then, we have
\[
\|u_t\|_{C([\tau, 0], X)} = \|S(t)\phi\|_{C([\tau, 0], X)} + \int_0^t \|S(t-s)X_0\mathcal{G}(u_s)\|_{C([\tau, 0], X)} ds
\]
\[
\leq Me^{-\beta t}\|\phi\|_{X_f} + \int_0^t Me^{-\beta(t-s)}\mathcal{B} \left( \|u_s\|_{C([\tau, 0], X)} \right) \|u_s\|_{C([\tau, 0], X)} ds
\]
\[
\leq Me^{-\beta t}\|\phi\|_{X_f} + M\mathcal{B}(\kappa) \int_0^t e^{-\beta(t-s)} ds
\]
\[
< Me^{-\beta t}\|\phi\|_{X_f} + M\mathcal{B}(\kappa) \frac{\kappa}{\beta}.
\]  

In virtue of (12), (13) and (15), it follows for \( \phi \in \tilde{B}_h(0, X_f) \)
\[
\|u_t\|_{C([\tau, 0], X)} < Mh + M\mathcal{B}(\kappa) \frac{\kappa}{\beta}
\]
\[
< M\frac{\kappa}{2M} + M \frac{\beta}{2M} \frac{\kappa}{\beta}
\]
which contradicts to the definition of $\vartheta$. Therefore $\|u_t\|_{C([\tau, t_0], X)} < \kappa$ for all $t \in [0, \vartheta]$, but this implies that $\vartheta = \infty$. It has been substantiated that $\|u_t\|_{C([\tau, t_0], X)} < \kappa$ for all $t \geq 0$. Thus,

$$\|u_t\|_{C([\tau, t_0], X)} \leq M e^{-\beta t} \|\phi\|_{X_f} + M \mathcal{B}(\kappa) \int_0^t e^{-\beta(t-s)} \|u_s\|_{C([\tau, t_0], X)} ds$$

holds for all $t \geq 0$. By applying Gronwall-Bellman’s inequality for the function $\|u_t\|_{C([\tau, t_0], X)}$, we obtain

$$\|u_t\|_{C([\tau, t_0], X)} \leq M e^{-\beta t} e^{M \mathcal{B}(\kappa) t} \|\phi\|_{X_f} \leq M e^{\frac{\beta}{2} t} \|\phi\|_{X_f}, \quad \phi \in \bar{B}_h(0, X_f).$$

It is evident to see that the theorem holds. \qed

**Remark 2.** Many of the standard tools of dynamical systems theory is inapplicable for partial functional equations with SD. As a result, it leads to that the study of the instability of the origin is nontrivial compared with ODE subject to SD.

2.3. Linearized instability. In the sequel, for the sake of convenience, we always assume that $A, f, g, \eta, \mathcal{G}$ are as above and (H1) holds, unless otherwise stated.

In the position, we state the linearized instability result for abstract FDE with SD.

**Theorem 2.11.** Assume

$$\|\mathcal{G}(z_1) - \mathcal{G}(z_2)\|_X \leq k(\rho) \|z_1 - z_2\|_{C([\tau, t], X)}; \quad \forall \|z_i\|_{C([\tau, t], X)} \leq \rho, \quad i = 1, 2, \tag{16}$$

where $k(\rho) \to 0$ as $\rho \to 0^+$. Define $A_S \triangleq \mathcal{G} - A$. If $\sigma(A_S) \cap \{\lambda \in \mathbb{C} | \text{Re}\lambda > 0\}$ is nonempty spectral set and $v_0 = 0$ is an equilibrium. Then the equilibrium $v_0$ is unstable. Specifically, there exist $\varepsilon_0 > 0$ and $\{v_n\}_{n \geq 1}$ with $\|v_n - v_0\|_X \to 0$ as $n \to \infty$, but for all $n$

$$\sup_{t \geq t_0} \|v(t; t_0, v_n) - v_0\|_X \geq \varepsilon_0 > 0.$$

Here the suprema is taken over the maximal interval of existence of $v(\cdot; t_0, v_n)$.

**Proof.** This theorem can be proved by the method analogous to that used in Henry [11] for abstract differential equations. However, since our argument involves certain technicalities caused by the complexity of the state-dependent delays, we provide the details of the proof here.

It is easy to check that $\mathcal{G}$ is a bounded linear map from $C$ to $C$ since

$$\|\eta\|_{BV} < V^0_{\tau}(\eta(\cdot, 0)) \tag{17}$$

holds. By Lemma 2.5, it follows that $A_S$ is sectorial.

Set $\sigma_1 \triangleq \sigma(A_S) \cap \{\lambda \in \mathbb{C} | \text{Re}\lambda > 0\}$ and $\sigma_2 \triangleq \sigma(A_S) \setminus \sigma_1$. Based on Lemma 2.7, we have $C = C_1 \oplus C_2$ being the corresponding decomposition into $A_S$-invariant subspaces, and $\sigma(A_S^j) = \sigma_j$, where $A_S^j$ is the restriction of $A_S$ onto $C_j = E_j(C)$ with $E_j (j = 1, 2)$ being the projections associated with these spectral sets.
For some $\beta > 0$, $\hat{M} \geq 1$, it yields the following results in terms of Lemma 2.8: for $t > 0$ and $x \in C$, $A_S^1$ is a sectorial operator and
\[
\left\| e^{A_S^1 t} E_2 x \right\|_X \leq \hat{M} e^{-\beta t} \left\| x(t) \right\|_X ,
\] (18)
for $t \leq 0$, $A_S^1$ is a bounded operator and
\[
\left\| e^{A_S^1 t} E_1 x \right\|_X \leq \hat{M} e^{\beta t} \left\| x(t) \right\|_X .
\] (19)

Given $\bar{a} : [-\tau, \theta] \rightarrow C_1$ defined by $\bar{a}(s) = a$ with $a$ a small constant, we consider the integral equation
\[
y = e^{A_S^1 (t-\theta)} \bar{a}(\theta) + \int_0^t e^{A_S^1 (t-s)} E_1 \Pi_0 \Theta(y) \, ds
\]
\[
+ \int_{-\tau}^t e^{A_S^1 (t-s)} E_2 \Pi_0 \Theta(y) \, ds , \quad \text{for } -\tau \leq t \leq \theta ,
\] (20)
where $\Pi_0 : [-\tau, \theta] \rightarrow B(X, X)$ is given by
\[
\Pi_0(\omega) = \begin{cases} 1, & \omega = \theta , \\ 0 , & \omega \in [-\tau, \theta) . \end{cases}
\]

Next, we choose $c > 0$ such that $\frac{\rho}{c} \leq \| a \|_X \leq \frac{\rho}{2\hat{M}}$ and prove that the integral equation (20) has a unique solution $y$ on $t \in [-\tau, \theta]$ with $\| y \|_{C([-\tau, \theta], X)} \leq \rho e^{2\beta(t-\theta)}$.

Define $Y(t) \triangleq y(t) e^{-2\beta(t-\theta)}$ and
\[
G(Y) \triangleq e^{-2\beta(t-\theta)} e^{A_S^1 (t-\theta)} a + \int_0^t e^{-2\beta(t-s)} e^{A_S^1 (t-s)} E_1 \Pi_0 \Theta(y) \, ds
\]
\[
+ \int_{-\tau}^t e^{-2\beta(t-s)} e^{A_S^1 (t-s)} E_2 \Pi_0 \Theta(y) \, ds .
\]

Let
\[
S \triangleq \left\{ Y : [-\tau, \theta] \rightarrow X \text{ is continuous, } \| Y \|_{C([-\tau, \theta], X)} \leq 2\hat{M} \| a \|_X \right\}
\] (21)
be the space endowed with the uniform convergence topology.

Select $\rho > 0$ so small that
\[
\frac{\hat{M} k(\rho)}{\beta} < \frac{1}{8\hat{M}} < \frac{1}{4} .
\] (22)

We can now show that $G$ maps $S$ into $S$.

For small $a$ and $\| a \|_X \leq \frac{\rho}{2\hat{M}}$, it yields that from (16) and (18)–(22)
\[
\| y \|_{C([-\tau, \theta], X)} \leq e^{2\beta(t-\theta)} \| Y \|_{C([-\tau, \theta], X)}
\]
\[
\leq 2\hat{M} \| a \|_X e^{2\beta(t-\theta)} \leq \rho e^{2\beta(t-\theta)} \text{ for } t \in [-\tau, \theta] ,
\] (23)
\[
\left\| G(Y) \right\|_{C([-\tau, \theta], X)} \leq \left\| e^{-2\beta(t-\theta)} e^{A_S^1 (t-\theta)} a + \int_0^t e^{-2\beta(t-s)} e^{A_S^1 (t-s)} E_1 \Pi_0 \Theta(y) \, ds
\]
\[
+ \int_{-\tau}^t e^{-2\beta(t-s)} e^{A_S^1 (t-s)} E_2 \Pi_0 \Theta(y) \, ds \right\|_{C([-\tau, \theta], X)}
\]
\[
\leq \left\| e^{-2\beta(t-\theta)} e^{A_S^1 (t-\theta)} a \right\|_{C([-\tau, \theta], X)}
\]
\[
\begin{align*}
+ \left\| \int_{\theta}^{t} e^{-2\beta(t-\theta)} e^{A_1(t-s)} E_1 \Pi_0 \mathfrak{G}(y) \right\|_{C([-\tau, \theta], X)} \leq \hat{M} e^{\beta(t-\theta)} \left\| a \right\|_X + \left\| \int_{\theta}^{t} e^{-2\beta(t-\theta)} \hat{M} e^{3\beta(t-s)} \left\| \mathfrak{G}(y) \right\|_X \right\|_X \\
+ \left\| \int_{-\tau}^{t} e^{-2\beta(t-\theta)} \hat{M} e^{-\beta(t-s)} \right\| \left\| \mathfrak{G}(y) \right\|_X \right\|_X \\
\leq \hat{M} \left\| a \right\|_X + \left\| \int_{\theta}^{t} \hat{M} k(\rho) e^{3\beta(t-s)} e^{-2\beta(t-\theta)} \left\| y(s) \right\|_X \right\|_X \\
+ \left\| \int_{-\tau}^{t} e^{-\beta(t-s)} \right\| \hat{M} k(\rho) e^{-2\beta(t-\theta)} 2\hat{M} \left\| a \right\|_X e^{2\beta(t-\theta)} \right\|_X \\
&= \hat{M} \left\| a \right\|_X + \hat{M} k(\rho) \left\| e^{\beta(t-\theta)} ds \right\| - \frac{1}{\beta} \cdot 2\hat{M} \left\| a \right\|_X \\
&+ \hat{M} k(\rho) \left\| e^{-\beta(t+\tau)} ds \right\| - \frac{1}{\beta} \cdot 2\hat{M} \left\| a \right\|_X \\
&\leq \hat{M} \left\| a \right\|_X + \frac{2\hat{M} k(\rho)}{\beta} \cdot 2\hat{M} \left\| a \right\|_X \\
&\leq 2\hat{M} \left\| a \right\|_X,
\end{align*}
\]

Remind that \( Y(t) = y(t) e^{-2\beta(t-\theta)} \). Moreover, for \( Y, Z \in S \), we have

\[
\| G(Y) - G(Z) \|_{C([-\tau, \theta], X)} \leq \| \int_{\theta}^{t} e^{-2\beta(t-\theta)} e^{A_1(t-s)} E_1 \Pi_0 \left[ \mathfrak{G}(y) - \mathfrak{G}(z) \right] ds \\
+ \int_{-\tau}^{t} e^{-2\beta(t-\theta)} e^{A_1(t-s)} E_2 \Pi_0 \left[ \mathfrak{G}(y) - \mathfrak{G}(z) \right] ds \|_{C([-\tau, \theta], X)} \leq \| \int_{\theta}^{t} e^{-2\beta(t-\theta)} e^{A_1(t-s)} E_1 k(\rho) \| y - z \|_{C([-\tau, \theta], X)} ds \|_X \\
+ \| \int_{-\tau}^{t} e^{-2\beta(t-\theta)} e^{2\beta(t-s)} E_2 k(\rho) \| y - z \|_{C([-\tau, \theta], X)} ds \|_X \\
\leq \| \int_{\theta}^{t} e^{-2\beta(t-\theta)} \hat{M} e^{3\beta(t-s)} k(\rho) \| y - z \|_{C([-\tau, \theta], X)} ds \|_X
\]
Then, due to the Banach fixed point theorem, $G$ has a unique fixed point in $S$ for $-\tau \leq t \leq \theta$. We obtain that the integral equation has a unique solution $y \triangleq y^*(t; \theta, a)$ (note that the notation does not imply $y^*(\theta; \theta, a) = a$) and

$$\|y^*(\theta; \theta, a) - a\|_X = \left\| \int_{-\tau}^{\theta} e^{-2\beta(t-\theta)} e^{A_2^* (\theta-s)} E_2 \Theta (y^*) \, ds \right\|_X$$

$$\leq \hat{M} \left\| \int_{-\tau}^{\theta} e^{-2\beta(t-\theta)} e^{-\beta(\theta-s)} k(\rho) \|y^*\| \, ds \right\|_X$$

$$\leq \hat{M} k(\rho) \left\| \int_{-\tau}^{\theta} e^{-2\beta(t-\theta)} e^{-\beta(\theta-s)} 2\hat{M} \|a\| \, e^{2\beta(t-\theta)} ds \right\|_X$$

$$\leq \hat{M} k(\rho) \int_{-\tau}^{\theta+\tau} e^{-\beta u} du \cdot 2\hat{M} \|a\|_X$$

$$= \hat{M} k(\rho) \left( \frac{1}{\beta} - \frac{e^{-\beta(\theta+\tau)}}{\beta} \right) 2\hat{M} \|a\|_X$$

$$\leq \frac{\hat{M} k(\rho)}{\beta} 2\hat{M} \|a\|_X$$

$$\leq \frac{1}{8\hat{M}} 2\hat{M} \|a\|_X \leq \frac{1}{4} \|a\|_X.$$  \hspace{1cm} (24)

It follows that

$$\|a\|_X - \|y^*(\theta; \theta, a)\|_X \leq \|y^*(\theta; \theta, a) - a\|_X \leq \frac{1}{4} \|a\|_X$$

i.e.,

$$\frac{3}{4} \|a\|_X \leq \|y^*(\theta; \theta, a)\|_X.$$
Assume that \( y^* (\cdot ; \theta, a) \) is a solution of
\[
\frac{dz}{dt} - A_S z = \mathcal{G} (z) \quad \text{for } t_0 \leq t \leq \theta.
\]
(25)

It yields that the solution of (25) can be rewritten as
\[
z (t; t_0, z_n) = y^* (t; t_0 + n, a), \quad t \in [t_0, t_0 + n]
\]
with the initial value \( z_n = y^* (t_0; t_0 + n, a) \). Moreover, one can obtain
\[
\| z_n \|_X = \| y^* (t_0; t_0 + n, a) \|_X
\]
\[
\leq \rho e^{2\beta(t_0 - (t_0 + n))}
\]
\[
\leq \rho e^{-2\beta n} \to 0 \text{ as } n \to \infty.
\]

Thus, by \( \frac{\rho}{c} \leq \| a \|_X \), it is easy to verify that
\[
\sup_{t \in [t_0, t_0 + n]} \| z (t; t_0, z_n) \|_X \geq \| z (t_0 + n; t_0, z_n) \|_X
\]
\[
= \| y^* (t_0 + n; t_0 + n, a) \|_X
\]
\[
\geq \frac{3}{4} \| a \|_X \geq \frac{3 \rho}{4c}.
\]

Now the integral equation above defines a strict contraction on the space of all continuous functions \( y : [-\tau, \theta] \to X \) with \( E_1 y (\theta) = a \) and \( \| y \|_{C([-\tau, \theta], X)} \leq \rho e^{2\beta(t-\theta)} \) for \( t \in [-\tau, \theta] \). Next, we shall show that solving (25) is equivalent to solving the integral equation (20).

Let \( \gamma (t) \triangleq \mathcal{G} (y) \) and pick any \( t_0 \leq \theta \), then for \( t_0 \leq t \leq \theta \)
\[
E_1 y \triangleq e^{A_S^1 (t-\theta) a} + \int_{\theta}^{t} e^{A_S^1 (t-s) E_1 \Pi_0 \gamma (s) d s}
\]
\[
E_2 y \triangleq \int_{-\tau}^{t} e^{A_S^2 (t-s) E_2 \Pi_0 \gamma (s) d s}
\]
\[
= e^{A_S^2 (t-t_0)} \int_{-\tau}^{0} e^{A_S^2 (t-s) E_2 \Pi_0 \gamma (s) d s} + \int_{-\tau}^{t} e^{A_S^2 (t-s) E_2 \Pi_0 \gamma (s) d s}.
\]

The solution \( y = E_1 y + E_2 y \) of integral equation also solves the differential equation (25) due to Lemma 2.4:
\[
\frac{d}{dt} e^{-A^i_{*} t} = -A^i_{*} e^{-A^i_{*} t}, \quad i = 1, 2.
\]

Also we have
\[
y = A_S^1 e^{A_S^1 (t-\theta) a} + A_S^1 \int_{\theta}^{t} e^{A_S^1 (t-s) E_1 \Pi_0 \gamma (s) d s}
\]
\[
+ A_S^2 \int_{-\tau}^{t} e^{A_S^2 (t-s) E_2 \Pi_0 \gamma (s) d s} + E_1 \gamma (t) + E_2 \Pi_0 \gamma (t)
\]
\[
= A_S (e^{A_S^1 (t-\theta) a} + \int_{\theta}^{t} e^{A_S^1 (t-s) E_1 \Pi_0 \gamma (s) d s}
\]
\[
+ \int_{-\tau}^{t} e^{A_S^2 (t-s) E_2 \Pi_0 \gamma (s) d s} + \Pi_0 \gamma (t)
\]
\[
= A_S y + \Pi_0 \gamma (t)
\]
i.e.,
\[ \frac{dy}{dt} - A_S y(t) = \gamma(t) \]
for \( t_0 \leq t \leq \theta \).

On the other hand, based on Lemma 2.3 and the uniqueness of solution the integral equation (20), it can easily be verified that \( y = y^*(t; \theta, a) \) is satisfied. The proof is complete. 

In the following, Theorem 2.11 can be extended to more general case. Specially, the closedness conditions on the set \( \sigma \left( A_S \right) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda > 0 \} \) can be relaxed.

**Theorem 2.12.** Suppose

\[ \| \mathcal{G}(z) \|_X = O \left( \| z \|_{C([-\tau, \tau], X)}^p \right), \quad p > 1 \]

as \( \| z \|_{C([-\tau, \tau], X)} \to 0 \), where

\[ \mathcal{G}(z) = f(z + v_0, \Lambda(z + v_0)) - f(v_0, \Lambda(v_0)) - \mathcal{L}z. \]  

(27)

If \( \sigma \left( A_S \right) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda > 0 \} \) is nonempty, then the equilibrium \( v_0 = 0 \) is unstable.

**Proof.** SD makes many of the standard tools of dynamical systems theory inapplicable for partial functional equations at first sight. Motivated by the hint of Corollary 5.1.6 [11] for abstract differential equation, we try to work out such matter.

Based on (27) and the fact

\[ \lim_{n \to \infty} v(t_0 + n; t_0 + n - 1, v_0 + z) - v_0 \]

\[ = A_S v(t_0 + n - 1; t_0 + n - 1, v_0 + z) + \mathcal{G}(v(t_0 + n - 1; t_0 + n - 1, v_0 + z)) \]

\[ = A_S (v_0 + z) + \mathcal{G}(v_0 + z) \]

\[ = A_S z + \mathcal{G}(z), \]

we have

\[ T_n(z) = A_S z(t) + \mathcal{G}(z), \quad n \to \infty \]

(28)

where

\[ T_n(z) \triangleq v(t_0 + n; t_0 + n - 1, v_0 + z) - v_0. \]

In terms of \( \| \mathcal{G}(z) \|_X = O \left( \| z \|_{C([-\tau, \tau], X)}^p \right), \quad p > 1 \), there exist \( \omega > 0, b > 0 \) such that

\[ \| T_n(z) - A_S z(t) \|_X \leq b \| z \|_{C([-\tau, \tau], X)}^p, \quad \text{for} \quad \| z \|_{C([-\tau, \tau], X)} \leq \omega. \]

Based on the nonempty sets \( \sigma \left( A_S \right) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda > 0 \} \) and difference equation (28), we have that \( r_{A_S} \) the spectral radius of \( A_S \), is greater than one. Moreover, we may select \( d > 0 \) such that \( r_{A_S}^p > r_{A_S} + d \). By employing \( r_{A_S} = \lim_{n \to \infty} \|(A_S)^n\|_{B(C, C)}^p \), it yields that there exists \( K < \infty \) so that \( \|(A_S)^n\|_{B(C, C)} \leq K (r_{A_S} + d)^n \) for all \( n \geq 0 \).

Put \( 0 < \delta < \frac{1}{2} \) and \( \sigma > 0 \) with

\[ \sigma \leq \frac{\omega}{R}, \quad \frac{bKR^p}{r_{A_S}^p - r_{A_S} - d} \sigma^{p-1} \leq \frac{1}{2} \]
where \( R = 2 (\sqrt{2} + \delta) \). We show that there exists \( x_0 \) arbitrarily small so \( x_n = T_n (x_{n-1}) \) \( (n \geq 1) \) in \( \| x_n \|_X \leq \omega \) for \( 1 \leq n \leq N \) while \( \| x_N \|_X \geq \left( \frac{1}{2} - \delta \right) \sigma \), which proves the instability.

Choose \( N_0 \geq 0 \), \( \| u \|_X = 1 \) and \( N \geq N_0 \). Let \( x_0 = \varepsilon u \), \( \varepsilon \equiv \frac{\sigma}{r_A S} \). Since \( r_A S > 1 \), it is obvious that \( \| x_0 \|_X = \varepsilon \) can be arbitrarily small.

Define

\[
x_j = T_j (x_{j-1}) \quad \text{for } j \geq 1,
\]

and we claim

\[
x_j = (A_S)^j x_0 + \sum_{k=0}^{j-1} (A_S)^{j-k-1} (T_{k+1} (x_k) - A_S x_k) \quad \text{for } j \geq 1. \tag{29}
\]

Now (26) and (28) imply that

\[
x_1 = T_1 (x_0) = A_S x_0 + \sigma \left( x_0 \right).
\]

It is easy to check that (29) holds for \( j = 1 \).

Next, let us assume that (29) is satisfied for \( j = n \) and deduce by induction that (29) hold for all \( j \geq 1 \).

For \( j = n + 1 \), we have

\[
x_{n+1} = T_{n+1} (x_n)
= A_S x_n + \sigma (x_n)
= (A_S)^{n+1} x_0 + \sum_{k=0}^{n-1} (A_S)^{n-k} (T_{k+1} (x_k) - A_S x_k) + (T_{n+1} (x_n) - A_S x_n)
= (A_S)^{n+1} x_0 + \sum_{k=0}^{n} (A_S)^{n-k} (T_{k+1} (x_k) - A_S x_k)
\]

which gives (29) for \( j \geq 1 \).

It can easily be verified that \( \| x_k \|_X \leq \varepsilon R r_A^k \) holds for \( k = 0 \), and if it holds for \( 0 \leq k < n \leq N \) then we obtain the following results in view of Lemma 2.9

\[
\| x_n \|_X \leq \| (A_S)^n \| x_0 \|_X + \left\| \sum_{k=0}^{n-1} (A_S)^{n-k-1} (T_{k+1} (x_k) - A_S x_k) \right\|_X
\leq \left( \sqrt{2} + \delta \right) r_A^n \varepsilon + \sum_{k=0}^{n-1} K (r_A S + d)^{n-k-1} b (\varepsilon R r_A^k)^p
= \left( \sqrt{2} + \delta \right) r_A^n \varepsilon + K b (\varepsilon R)^p \sum_{k=0}^{n-1} (r_A S + d)^{n-k-1} (r_A^k)^p
= \left( \sqrt{2} + \delta \right) r_A^n \varepsilon + K b (\varepsilon R)^p r_A^{p(n-1)} \sum_{k=0}^{n-1} (r_A S + d)^{n-k-1} \frac{r_A^{kp}}{r_A^{p(n-1)}}
\]

\[
= \left( \sqrt{2} + \delta \right) r_A^n \varepsilon + K b (\varepsilon R)^p r_A^{p(n-1)} \sum_{k=0}^{n-1} \left( \frac{r_A S + d}{r_A^p} \right)^{n-k-1}
\]
As already mentioned in the above (29), it suffices to prove
\[ \left\| (A_S^n)^n x_0 \right\|_X - \left\| x_n \right\|_X \leq \left\| \sum_{k=0}^{n-1} (A_S)^{n-k-1} (T_{k+1} (x_k) - A_S x_k) \right\|_X \]
i.e. \[ \left\| x_n \right\|_X \geq \left\| (A_S^n)^n x_0 \right\|_X - \left\| \sum_{k=0}^{n-1} (A_S)^{n-k-1} (T_{k+1} (x_k) - A_S x_k) \right\|_X \].

According to Lemma 2.9, it follows that
\[ \left\| x_N \right\|_X \geq (1 - \delta) r_{A_S}^N \epsilon - \frac{1}{2} r_{A_S}^N \epsilon \]
\[ = \left( \frac{1}{2} - \delta \right) r_{A_S}^N \epsilon = \left( \frac{1}{2} - \delta \right) \sigma. \]

This completes the proof. \( \square \)

3. Reaction-diffusion equations with state-dependent delays. In this section, the above results are applied to the following reaction-diffusion equations with SD.
\[
\begin{cases}
\frac{\partial u (t,y)}{\partial t} = D \Delta u + f (y, u (t,y), \Xi (u_t (y))), & y \in \Omega, \quad t > 0 \\
Q (y, u (t,y) + \frac{\partial u}{\partial \eta} (t,y)) = c, & y \in \partial \Omega, \quad t \geq 0 \\
u (\zeta, y) = \phi (\zeta, y), & y \in \Omega, \quad \zeta \in [-\tau, 0]
\end{cases}
\]
where \( \Xi (u_t (y)) \triangleq \int_0^t d \eta, u_t (y) (-r (u_t (y) (s))) g (y, u_t (y) (-r (u_t (y) (s)))) \) and \( u_t (y) : [-\tau, 0] \rightarrow \mathbb{R}^m \) is given by \( u_t (y) (s) = u (t + s, y) \), \( y \in \Omega \), \( r : \mathbb{R}^m \rightarrow (0, \tau] \) and \( \phi (\zeta, y) : [-\tau, 0] \times \Omega \rightarrow \mathbb{R}^m \) are continuously differentiable with \( \tau > 0 \). \( \Omega \) is a fixed, bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \). \( Q (y) |_{\partial \Omega} \geq 0 \) and \( \vec{n} \) is the outward unit normal vector on the boundary. \( D = \text{diag} (d_1 \cdots d_m) \), where \( d_i > 0 \) is the diffusion coefficient. The functions \( f : \Omega \times \mathbb{R}^m \times \mathbb{R}^l \rightarrow \mathbb{R}^m \),
\( g : \Omega \times \mathbb{R}^m \to \mathbb{R}^q \) are continuously differentiable with \( f (y,0,0) = 0 \) and \( g(y,0) = 0 \). \( \eta(\cdot, \psi) : [-\tau,0] \to \mathbb{R}^{q \times q} \) is of bounded variation for all \( \psi \in C^1([-\tau,0] \times \Omega, \mathbb{R}^m) \), and \( \eta \) is continuously differentiable with respect to \( \psi \).

We now transform system (30) into the IVP (1), which is convenient to adopt the previous theoretical results. Let \( X = L^2(\Omega) \) and define \( \hat{f} : X \times X \to X \) by
\[
\hat{f} (w, \hat{\Xi}(w)) (y) \triangleq f (y, w(t, y), \Xi(w(t))) , \ y \in \Omega \ 	ext{where} \ w_t(s) = w(t+s) \ s \in [-\tau,0] \text{ and} \]
\[
\hat{\Xi}(w) \triangleq \int_{-\tau}^0 ds \hat{\eta}(s,w_{t}(-\hat{r}(w_{t}(s)))) \hat{g}(w_{t}(-\hat{r}(w_{t}(s))))
\]
with
\[
\hat{\eta}(s,w_{t}(-\hat{r}(w_{t}(s))))(y) \triangleq \eta(s,w_{t}(y)(-r(w_{t}(y)(s)))) \\
\hat{g}(w_{t}(-\hat{r}(w_{t}(s))))(y) \triangleq g(y,w_{t}(y)(-r(w_{t}(y)(s)))) \\
\hat{r}(w_{t}(s))(y) \triangleq r(w_{t}(y)(s)).
\]

According to the hypothesis on \( r, f \) and \( g \), we have \( \hat{r} : X \to (0,\tau], \hat{f} : X \times X \to X \text{ and } \hat{g} : X \to X \) are continuously differentiable with \( \hat{f}(0,0) = 0 \) and \( \hat{g}(0) = 0 \). Moreover the derivatives of \( \hat{r}, \hat{f} \) and \( \hat{g} \) are Lipschitz continuous on \( B_{\xi}(\varphi,X) \triangleq \{ \psi \in X : \| \psi - \varphi \|_{X} < \xi \} \). Set \( Av = -D\Delta v \) with \( Dom(A) \triangleq \{ v \mid v \in H^1(\overline{\Omega}) \cap H^2(\Omega) : Q(y)v(y) + \frac{\partial v}{\partial n}(y) = c \ 	ext{on} \ \partial \Omega \} \). It is easy to see that \( -A \) is the infinitesimal generator of an analytic compact semigroup \( \{ T(t) \}_{t \geq 0} \) on \( X \) (see [33] Theorem 2.2 of chapter 1 for more details). Furthermore, \( A \) has discrete spectrum, and the eigenvalues can be represented by \( \mu_i, \ i = 0,1,2, \cdots \).

Consequently, from the above \( A, \hat{f}, \hat{g}, \hat{\eta}, \hat{r}, \phi \) and Lemma 2.3, we can obtain that Eq.(30) has a unique classical solution in \( [-\tau,\hat{t}] \) (by assuming initial value \( \phi \in C^1([-\tau,0],X) \) and \( \phi(0) \in Dom(A) \)). Moreover, it can easily be checked that (H1) is satisfied. Then it is evident to see that Proposition 1 holds.

Let \( u^* = 0 \) be an equilibrium point. Next the linearization at an equilibrium point \( u^* \) is
\[
\hat{u}(t) + Au(t) = \mathcal{L}u_t.
\]
The characteristic equation is
\[
\mathcal{L}e^\lambda x - Ax = \lambda x, \ \exists \ x \in Dom(A) \setminus \{0\}.
\]

Summarizing the above results, we can obtain the following theorem.

**Theorem 3.1.** Assume \( A, \hat{f}, \hat{g}, \hat{\eta}, \hat{r}, \phi \) are as above. Then the following results hold.

1. The equilibrium point \( u^* \) of Eq.(30) is asymptotically stable if all the roots of the equation with respect to \( z \)
\[
(z + \Sigma) e^z + W = 0
\]
have negative real parts, where
\[
\begin{align*}
z & \triangleq \lambda \hat{r}(u^*) \\
\Upsilon & \triangleq D_{1} \hat{f}\left(u^*, \hat{\Xi}(u^*)\right) \\
\Sigma & \triangleq (\mu_i - \Upsilon) \hat{r}(u^*) \\
W & \triangleq -\hat{r}(u^*) D_2 \hat{f}\left(u^*, \hat{\Xi}(u^*)\right) \int_{-\tau}^0 ds \hat{\eta}(s,u^*) D\hat{g}(u^*)
\end{align*}
\]
where $\Theta = \{ \Theta \}$ we refer to \cite{18, 25, 32, 34} and references therein.

Example. To apply the previous results, a model of Hematopoiesis is discussed in this section. The model of Hematopoiesis when $u$ does not depend on the spatial variable $y$ was first proposed by Mackey and Glass \cite{21}. For more results on such model we refer to \cite{18, 25, 32, 34} and references therein.

(ii) The equilibrium point $u^*$ of Eq. (30) is unstable if equation (33) has at least one root with positive real part.

Let us first consider the statement (i) of the theorem. By a simple calculation, we obtain (33) from (32). Next it is easy to verify that (H2) holds if the assumption of the statement (i) is satisfied. Thus the statement (i) follows Theorem 2.10 immediately. The statement (ii) is now evident from Theorem 2.11. Because of the smoothness properties of $\hat{f}$, $\hat{g}$, $\hat{h}$ and $\hat{r}$, it leads to (16). Moreover, the assumption of the statement (ii) implies that $\sigma (A_S) \cap \{ \lambda \in \mathbb{C} | \Re \lambda > 0 \}$ is nonempty spectral set.

In virtue of the statement (i) of Theorem 3.1, the straightforward stability conditions are provided as follows, which results Theorem A.5 of the appendix \cite{7} to deal with the stability problem of the nonlinear FDE with SD.

**Theorem 3.2.** Assume $\hat{f}$, $\hat{g}$, $\hat{h}$, $\mu_i$ as in Theorem 3.1. All roots of the equations

$$
\left\{ z + \left[ \left( \mu_i - D_1 \hat{f} \left( u^*, \hat{z} (u^*) \right) \right) \hat{r} (u^*) \right] \right\} e^z
$$

$$
- \hat{r} (u^*) D_2 \hat{f} \left( u^*, \hat{z} (u^*) \right) \int_{-\tau}^0 d_s \hat{h} (s, u^*) D_\hat{g} (u^*) = 0
$$

have negative real parts if and only if for $i = 0, 1, 2, \ldots$,

$$
\left( D_1 \hat{f} \left( u^*, \hat{z} (u^*) \right) - \mu_i \right) \hat{r} (u^*) < 1
$$

$$
\left( D_1 \hat{f} \left( u^*, \hat{z} (u^*) \right) - \mu_i \right) \hat{r} (u^*)
$$

$$
+ \hat{r} (u^*) D_2 \hat{f} \left( u^*, \hat{z} (u^*) \right) \int_{-\tau}^0 d_s \hat{h} (s, u^*) D_\hat{g} (u^*) < 0
$$

$$
\left( \mu_i - D_1 \hat{f} \left( u^*, \hat{z} (u^*) \right) \right) \hat{r} (u^*) \cos \Theta - \Theta \sin \Theta
$$

$$
- \hat{r} (u^*) D_2 \hat{f} \left( u^*, \hat{z} (u^*) \right) \int_{-\tau}^0 d_s \hat{h} (s, u^*) D_\hat{g} (u^*) < 0
$$

where $\Theta = \frac{\pi}{2}$ if $\left( \mu_i - D_1 \hat{f} \left( u^*, \hat{z} (u^*) \right) \right) \hat{r} (u^*) = 0$ or $\Theta$ is the root of the following equation

$$
\Theta = \left[ D_1 \hat{f} \left( u^*, \hat{z} (u^*) \right) - \mu_i \right] \hat{r} (u^*) \tan \Theta, \quad \Theta \in (0, \pi)
$$

if $\left( \mu_i - D_1 \hat{f} \left( u^*, \hat{z} (u^*) \right) \right) \hat{r} (u^*) \neq 0$.

**Remark 3.** More specifically, for each $\mu_i$, $i = 0, 1, 2, \ldots$, the exact region of stability of (30) in the space of the parameters $\Upsilon$, $W$ and $\hat{r} (u^*)$ can be shown.
Consider the diffusive model of Hematopoiesis with SD
\begin{equation}
\begin{aligned}
&\frac{\partial u(t, y)}{\partial t} = D\Delta u(t, y) + F(u(t, y)), \quad y \in [0, \pi], \ t > 0 \\
u(t, 0) = u(t, \pi) = 0, \ t \geq 0 \\
u(\zeta, y) = \phi(\zeta, y), \ y \in [0, \pi], \ \zeta \in [-5, 0]
\end{aligned}
\end{equation}

where \(F(u(t, y)) \triangleq -du(t, y) + \frac{Bu(t - r(u(t, y)))}{1 + u^m(t - r(u(t, y)))}, \ d, B \in (0, +\infty), m \geq 1. \quad D > 0\) is the diffusion coefficient. \(u(t, y)\) denotes the density of blood circulation mature cells at time \(t\) and position \(y\), and the cells are lost from the circulation at the rate \(d\). The flux \(\frac{Bu(t - r(u(t, y)))}{1 + u^m(t - r(u(t, y)))}\) of the cells into the circulation from the stem cell compartment depends on \(u(t - r(u(t, y)))\) at time \(t - r(u(t, y))\). \(r(u(t, y))\) is the time between the production of immature cells in the bone marrow and their maturation for release in circulating bloodstreams, which is a nonlinear function of density of cells. Due to the limited food resources, the period of maturity is longer if the number of mature cells is larger in a closed environment.

Now, we want to transform system (37) into the abstract IVP (1). Set \(X = L^2([0, \pi]), \hat{r}(v)(y) = r(v(t, y))\) and \(Av = -D\hat{v}\) with Dom \(A\) \(\triangleq \{v \in H^2((0, \pi)) \cap H^1_0([0, \pi])\}\). It is easy to see that \(-A\) is the infinitesimal generator of an analytic compact semigroup \(\{T(t)\}_{t \geq 0}\) on \(X\). Furthermore, the eigenvectors of \(A\) on \(X\) are \(\Psi_n(y) = \sin(ny)\) with the corresponding eigenvalues \(Dn^2, n = 1, 2, \ldots\)

Define
\begin{equation}
\hat{r}(\psi) \triangleq \arctan(\psi) + 1
\end{equation}
\begin{equation}
\eta(s, \psi) \triangleq -B\chi_{[-5, 0]}(s) = \begin{cases} 0 \quad s \geq 0 \\
-B \quad s \in [-5, 0]
\end{cases}
\end{equation}
\begin{equation}
\Lambda(\psi) \triangleq \int_{-5}^{0} ds\eta(s, \psi(-\hat{r}(\psi(s)))) \frac{\psi(-\hat{r}(\psi(s)))}{1 + \psi^m(-\hat{r}(\psi(s)))} = B \frac{\psi(-\hat{r}(\psi(0)))}{1 + \psi^m(-\hat{r}(\psi(0)))},
\end{equation}

where \(\chi_{[-5, 0]}(s)\) is the characteristic function of the interval \([-5, 0]\).

Thus, the system (37) can be rewritten as the following
\begin{equation}
\begin{aligned}
\dot{v}(t) = -Av_t(0) - dv_t(0) + \frac{Bu_t(-\hat{r}(v_t(0)))}{1 + v_t^m(-\hat{r}(v_t(0)))}.
\end{aligned}
\end{equation}

According to (38)-(40) and
\begin{align*}
g(\psi(-\hat{r}(\psi(s)))) &= \frac{\psi(-\hat{r}(\psi(s)))}{1 + \psi^m(-\hat{r}(\psi(s)))}, \\
f(\psi(0), \Lambda(\psi)) &= -d\psi(0) + \frac{B\psi(-\hat{r}(\psi(0)))}{1 + \psi^m(-\hat{r}(\psi(0)))},
\end{align*}

it is obvious that the hypotheses about \(A, f, g, \eta\) are satisfied. Thus, based on Lemma 2.3, we can get that the system (37) has a unique classical solution (under assumptions initial value \(\phi \in C^1([-5, 0], X)\) and \(\phi(0) \in Dom(A)\)).

Note that \(u^* = 0\) is an equilibrium point of (37), then one obtains that (H1) holds and the linearization at \(u^*\) is
\begin{equation}
\dot{u}(t) = -Au_t(0) + Lu_t
\end{equation}
Figure 1. Numerical simulations of system (37) with the initial value $\phi(\zeta, y) = 0.1 \sin^2(y)$. (a) The equilibrium point $u^* = 0$ is asymptotically stable with the system parameters $m = 3$, $D = 0.01$, $B = 0.1$, $d = 0.2$. (b) The equilibrium point $u^* = 0$ is unstable with the system parameters $m = 3$, $D = 0.01$, $B = 0.6$, $d = 0.2$.

\begin{equation}
\frac{dx}{dt} = -Au(t) - du(t) + Bu(t - \hat{r}(u^*)).
\end{equation}

The characteristic values of (41) are determined by the equation

\begin{equation}
-Ax - dx + Be^{-\lambda \hat{r}(u^*)} x = \lambda x, \ \exists \ x \in \text{Dom}(A) \setminus \{0\}.
\end{equation}

Moreover, since the eigenvalues of $A$ are $Dn^2$, $n = 1, 2, \ldots$, we have

\begin{equation}
-Dn^2 - d + Be^{-\lambda} = \lambda, \ n = 1, 2, \ldots.
\end{equation}

Thus, we can obtain the following results.

(i) If $B < D + d$, then all the roots of the equations (43) have negative real parts. The conclusion is from Theorem 3.2 directly.

(ii) If $B > D + d$, then (43) has at least one positive root. In fact, defining $P(\lambda) \triangleq \lambda + D + d - Be^{-\lambda}$, we have $P(0) < 0$ and $\lim_{\lambda \to +\infty} P(\lambda) = +\infty$. This implies that the first equation in (43) has at least one positive root.

Summarizing the above analysis, we perform numerical simulations of system (37) with the parameters $m = 3$, $D = 0.01$, $B = 0.1$, $d = 0.2$. By substituting the parameters, it follows that the condition (H2) is satisfied. Consequently, in terms of Theorem 2.10 or Theorem 3.1, we infer that the equilibrium point $u^* = 0$ is asymptotically stable which is illustrated in Figure 1 (a). On the other hand, for the parameters $m = 3$, $D = 0.01$, $B = 0.6$, $d = 0.2$, it is evident to see $B > D + d$. It yields that the equilibrium point $u^* = 0$ is unstable (see Figure 1 (b)).

Acknowledgments. The authors are grateful to the anonymous referee for his or her helpful comments and valuable suggestions, which led to the improvement of the manuscript.

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Received April 2018; 1st revision September 2018; 2nd revision December 2018.

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