Risk-sensitive Dynkin games with heterogeneous Poisson random intervention times

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Abstract

The paper solves constrained Dynkin games with risk-sensitive criteria, where two players are allowed to stop at two independent Poisson random intervention times, via the theory of backward stochastic differential equations. This generalizes the previous work of [Liang and Sun, Dynkin games with Poisson random intervention times, SIAM Journal on Control and Optimization, 2019] from the risk-neutral criteria and common signal times for both players to the risk-sensitive criteria and two heterogenous signal times. Furthermore, the paper establishes a connection of such constrained risk-sensitive Dynkin games with a class of stochastic differential games via Krylov’s randomized stopping technique.

Keywords: Dynkin games, heterogenous Poisson signal times, backward stochastic differential equations, stochastic differential games, randomized stopping

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1 Introduction

Risk-sensitive criteria constitute a genuinely interesting class of performance criteria in optimization problems, in which the linear expectation $E[\cdot]$ is replaced by the nonlinear expectation

$$\tilde{E}[\cdot] := g^{-1}(E[g(\cdot)])$$

for some strictly increasing function $g$ as a risk-sensitive function. The corresponding risk-sensitive control has been developed to reflect an optimizer’s attitudes to risks. In particular, the risk-sensitive function $g$ is chosen to model the optimizer’s attitudes towards risks (e.g. strict concavity of $g$ reflects risk-aversion of maximization players or risk-seeking of minimization players).

In this paper, we are interested in Dynkin games with risk-sensitive criteria, by taking into account of both players’ attitudes to risks. Namely, the two players aim to minimize/maximize some payoff functional $R(\sigma, \tau)$ under the nonlinear expectation $\tilde{E}[\cdot]$:

$$J(\sigma, \tau) = \tilde{E}[R(\sigma, \tau)] = g^{-1}(E[g(R(\sigma, \tau))]),$$

*We thank David Hobson for the suggestion of considering heterogenous signal times for constrained Dynkin games, which motivates the current project.
where $\sigma$ and $\tau$ are the stopping times to be chosen by the respective minimization maximization players. It is called risk-sensitive because

$$J(\sigma, \tau) \approx \mathbb{E}[R(\sigma, \tau)] - \frac{1}{2} l_g(\mathbb{E}[R(\sigma, \tau)]) \text{Var}[R(\sigma, \tau)],$$

where $l_g(x) = -\frac{g''(x)}{g'(x)}$ is the Arrow-Pratt function of absolute risk aversion. The case $g(x) = x$ corresponds to a risk-neutral attitude of both players since $l_g(x) = 0$. For the case of an exponential utility $g(x) = -e^{-\gamma x}$ with $\gamma > 0$, $l_g(x) = \gamma$ is constant and the risk-sensitivity is only expressed through the risk-sensitivity parameter $\gamma$.

The stopping time strategies of the two players are restricted to two independent sequences of Poisson arrival times as the exogenous constraints on the players’ abilities to stop. The constraints may represent liquidity effects, indicating the times at which the underlying stochastic processes are available to stop. Applications of such a liquidity model can be found in [24] for bank runs and [25] for convertible bonds. The constraints can also be seen as information constraints. The players are allowed to make their stopping decisions at all times, but they are only able to observe the underlying stochastic processes at Poisson arrival times. See [11] and [21] for applications to perpetual American options. Due to the introduction of constraints on stopping times and risk-sensitive criteria, we call the Dynkin games considered in this paper the constrained risk-sensitive Dynkin games.

We generalize our previous work [25] on constrained Dynkin games in two aspects: First, it takes into consideration of both players’ attitudes towards risks via the risk-sensitive function $g$; second, there are control constraints for both players and, moreover, the constraints are different in the sense that they are allowed to stop at two heterogeneous sequences of Poisson arrival times. Consequently, since the two players’ stopping time strategies are chosen from two different sequences of signal times, the usual condition of the upper obstacle $U$ dominating the lower one $L$ is not required. In [25], the risk-sensitive function $g(x) = x$ and both players stopat a single sequence of signal times (so $U \geq L$ is assumed therein).

New challenges arise from the above generalizations. Since the two players stop at two different sequences of Poisson arrival times, the first step to solve the constrained risk-sensitive Dynkin game is merging the two Poisson sequences together while still keeping track of their order. This is crucial when we consider a family of constrained risk-sensitive Dynkin games (3.5)-(3.6) starting from different signal times in order to apply the dynamic programming principle. Note that the starting times of the games (3.5)-(3.6) may not be the respective player’s own Poisson signal times; instead they could be from the counterparty’s signal times. To deal with the nonlinear expectation $\mathbb{E}$ arising from the risk-sensitive function $g$, we introduce a new transformation resulting the auxiliary payoff processes (2.7)-(2.9), which enable us to rewrite the payoff functional under the linear expectation $\mathbb{E}$ instead of the nonlinear expectation $\mathbb{E}$. For a special case of exponential risk-sensitive function $g$ (see section 5.2), the representation formula (2.10) of the game value is closely related to Cole-Hopf transformation in the literature of backward stochastic differential equations (BSDEs for short), which is widely used to linearize a class of BSDEs with quadratic growth (see [18]). Our representation formula (2.10) can be regarded as a stochastic control version of Cole-Hopf transformation.

We also make a connection of constrained risk-sensitive Dynkin games with a class of stochastic differential games via Krylov’s randomized stopping technique (see [19]). It is established in [19] that standard optimal stopping problems (without constraints on stopping times) admit stochastic control representation, which can be further solved via the so-called normalized Bellman equations.
The stochastic control representation of the corresponding constrained optimal stopping problems has been established in [23] (see section 4 therein). In a constrained stopping game setting as considered in the current paper, it is natural to expect that a stochastic differential game representation should hold accordingly. Indeed, we show that the two players in the stochastic differential game choose their respective running controls and discount rates with binary values 0 or the Poisson intensity $\lambda$, and the optimal control is the Poisson intensity $\lambda$ whenever the value of the game falls below the lower obstacle process/goes above the upper obstacle process.

Turing to the literature of Dynkin games, there has been a considerable development since the seminal works of Dynkin [12] and Neveu [32]. The continuous time models were developed, among others, by Bismut [5], Alario-Nazaret et al [11], Lepeltier and Maingueneau [22] and Morimoto [30]. In order to relax the dominating condition $U \geq L$ in those papers, Yasuda [36] proposed the strategies of randomized stopping times, and proved that the game value exists under merely an integrability condition. Rosenberg et al [34], Touzi and Vielle [35] and Laraki and Solan [20] further extended his work in this direction. The non-Markovian case was addressed in Cvitanic and Karatzas [7] for a fixed horizon and Hamadene et al [14] for an infinite horizon using the theory of reflected BSDEs. If the two players in the game are with asymmetric payoffs/information, then it gives arise to a nonzero-sum Dynkin game. See, for example, Hamadene and Zhang [15], De Angelis et al [10] and, more recently, De Angelis and Ekstrom [9] with more references therein. A robust version of Dynkin games can be found in Bayraktar and Yao [4] if the players are ambiguous about their probability model.

On the other hand, the risk-sensitive optimal stopping problems have been studied by Nagai [31], Bäuerle and Rieder [3], Bäuerle and Popp [2] and, more recently, Jelito et al [17]. For the risk-sensitive zero-sum and nonzero-sum stochastic differential games, we refer to El-Karoui and Hamadène [13]. To the best of our knowledge, the study of risk-sensitive Dynkin games is still lacking, no matter with or without constraints on stopping time strategies. The current paper offers a first step to understand risk-sensitive Dynkin games, in particular with constraints on the stopping time strategies.

The constrained optimal stopping problems was first studied by Dupuis and Wang [11], where they used it to model perpetual American options exercised at exogenous Poisson arrival times. See also Lempa [21], Menaldi and Robin [27] and Hobson and Zeng [16] for further extensions of this type optimal stopping models. From a different perspective, Liang [23] made a connection between such kind of optimal stopping problems with penalized BSDEs. The corresponding optimal switching (impulse control) models were studied by Liang and Wei [28], and by Menaldi and Robin [25, 29] with more general signal times and state spaces. More recently, Liang and Sun [25] introduced the corresponding constrained Dynkin games (with the risk-sensitive function $g(x) = x$), where both players were allowed to stop at a sequence of random times generated by a single exogenous Poisson process serving as a signal process.

The paper is organized as follows. Section 2 contains the problem formulation and main result, with its proof provided in section 3. In section 4, we establish its connection with a class of stochastic differential games, and in section 5 we further provide two examples. Finally, section 6 concludes the paper.

2 Constrained risk-sensitive Dynkin games

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a $d$-dimensional standard Brownian motion $(W_t)_{t \geq 0}$ with $\mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}$ being the minimal augmented filtration of $W$. The probability
space also supports two independent sequences of Poisson arrival times \( T^{(1)} = \{T^{(1)}_n\}_{n \geq 0} \) and \( T^{(2)} = \{T^{(2)}_n\}_{n \geq 0} \) with their respective intensities \( \lambda^{(1)} \) and \( \lambda^{(2)} \) and the minimal augmented filtration \( \mathbb{H} = \{\mathcal{H}_t\}_{t \geq 0} \), satisfying \( T^{(1)}_0 = T^{(2)}_0 = 0 \) and \( T^{(1)}_\infty = T^{(2)}_\infty = \infty \). Denote the smallest filtration generated by \( \mathbb{F} \) and \( \mathbb{H} \) as \( \mathbb{G} = \{\mathcal{G}_t\}_{t \geq 0} \), i.e. \( \mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t \), and write \( \lambda = (\lambda^{(1)}, \lambda^{(2)}) \).

Let \( T \) be a finite \( \mathbb{F} \)-stopping time representing the (random) terminal time of the game. For each player \( i \in \{1, 2\} \), let us define a random variable \( M_i : \Omega \rightarrow \mathbb{N} \) such that \( T_{M_i} \) is the next arrival time in the Poisson sequence \( T^{(i)} \) following \( T \), i.e. \( M_i(\omega) := \sum_{n \geq 1} n \mathbb{1}_{\{T^{(i)}_{n-1}(\omega) < T(\omega) \leq T^{(i)}_n(\omega)\}} \).

For any integer \( n \geq 0 \), let us define the control set for each player \( i \in \{1, 2\} \) as

\[
\mathcal{R}^{(i)}_n = \{\mathcal{G} \text{-stopping time } \sigma \text{ for } \sigma(\omega) = T^{(i)}_N(\omega) \text{ where } n \leq N \leq M_i(\omega)\},
\]

so the player \( i \) chooses from the Poisson arrival times \( T^{(i)} \) with intensity \( \lambda^{(i)} \), and \( T^{(i)}_n \) is the smallest stopping time allowed.

Consider a constrained risk-sensitive Dynkin game, where the two players choose their respective stopping times \( \sigma \in \mathcal{R}^{(1)}_1 \) and \( \tau \in \mathcal{R}^{(2)}_1 \) in order to minimize/maximize the expected cost functional \( J(\sigma, \tau) = \mathbb{E}[R(\sigma, \tau)] \),

where the nonlinear expectation \( \mathbb{E} : \mathbb{R} \rightarrow \mathbb{R} \) is defined via the risk-sensitive function \( g \), i.e.

\[
\mathbb{E}[\cdot] := g^{-1}(\mathbb{E}[g(\cdot)]).
\]

The discounted payoff functional \( R(\sigma, \tau) \) in \((2.2)\) is defined by

\[
R(\sigma, \tau) = \int_0^{\sigma \wedge \tau \wedge T} e^{-r s} f_s ds + e^{-rT} \xi \mathbb{1}_{\{\sigma \wedge \tau \geq T\}} + e^{-r \tau} L_\tau \mathbb{1}_{\{\tau < T, \tau \leq \sigma\}} + e^{-r \sigma} U_\sigma \mathbb{1}_{\{\sigma < T, \sigma < \tau\}},
\]

where \( r > 0 \) is the discount rate, and \( f \), as a real-valued \( \mathbb{F} \)-progressively measurable process, is the running payoff. The terminal payoff is \( U \) if \( \sigma \) happens firstly, \( L \) if \( \tau \) happens firstly or \( \sigma \) and \( \tau \) happen simultaneously, and \( \xi \) otherwise, where \( L \) and \( U \) are two real-valued \( \mathbb{F} \)-progressively measurable processes, and \( \xi \) is a real-valued \( \mathcal{F}_T \)-measurable random variable.

Let us define the upper and lower values of the constrained risk-sensitive Dynkin game

\[
\underline{v}^\lambda = \inf_{\sigma \in \mathcal{R}^{(1)}_1} \sup_{\tau \in \mathcal{R}^{(2)}_1} J(\sigma, \tau), \text{ and } \overline{v}^\lambda = \sup_{\tau \in \mathcal{R}^{(2)}_1} \inf_{\sigma \in \mathcal{R}^{(1)}_1} J(\sigma, \tau).
\]

The game \((2.4)\) is said to have value \( v^\lambda \) if \( v^\lambda = \underline{v}^\lambda = \overline{v}^\lambda \), and a saddle point \((\sigma^*, \tau^*) \in \mathcal{R}^{(1)}_1 \times \mathcal{R}^{(2)}_1 \) is called an optimal stopping strategy of the game if

\[
J(\sigma^*, \tau^*) \leq J(\sigma^*, \tau) \leq J(\sigma, \tau^*) \leq J(\sigma^*, \tau^*)
\]

for every \((\sigma, \tau) \in \mathcal{R}^{(1)}_1 \times \mathcal{R}^{(2)}_1 \).

Compared with the constrained Dynkin game introduced in \((2.5)\), there are two new features of the game \((2.4)\): First, it takes into consideration of the both players’ attitudes towards risks via the risk-sensitive function \( g \); Second, there are control constraints for both players and, moreover, the constraints are different in the sense that they are allowed to stop at two heterogeneous sequences of Poisson arrival times. As a consequence, since the two players’ stopping time strategies are chosen from two different control sets, the usual dominating condition \( U \geq L \) is not required. In \((2.5)\), the risk-sensitive function \( g(x) = x \) and both players stop at a single sequence of Poisson arrival times (so \( U \geq L \) is a critical assumption therein).
2.1 Main result

To solve the above constrained risk-sensitive Dynkin game, we introduce the characterizing BSDE on a random horizon \([0, T]\):

\[
\overline{Q}^\lambda_{1\wedge T} = \xi + \int_{t\wedge T}^T \left[ -\lambda^{(1)} \left( \overline{Q}^\lambda_s - \overline{U}_s \right)^+ + \lambda^{(2)} \left( \overline{L}_s - \overline{Q}^\lambda_s \right)^+ - r \overline{Q}^\lambda_s \right] ds - \int_{t\wedge T}^T \overline{Z}^\lambda_s dW_s, \tag{2.6}
\]

for \(t \geq 0\), where the auxiliary payoff processes \(\overline{L}, \overline{U}\) and \(\overline{\xi}\) are given by

\[
\overline{L}_t = e^{rt} g(e^{-rt} L_t + \int_0^t e^{-ru} f_u \, du), \tag{2.7}
\]

\[
\overline{U}_t = e^{rt} g(e^{-rt} U_t + \int_0^t e^{-ru} f_u \, du), \tag{2.8}
\]

\[
\overline{\xi} = e^{rt} g(e^{-rt} \xi + \int_0^t e^{-ru} f_u \, du), \tag{2.9}
\]

respectively. And also we set \(\overline{Q}^\lambda_t \equiv \overline{\xi}\) for \(t \geq T\). Moreover, we introduce the following spaces: for any given \(\alpha \in \mathbb{R}\) and \(n \in \mathbb{N}\),

- \(\mathbb{L}^{2, n}_0 : \mathcal{F}_T\text{-measurable random variables } \xi : \Omega \mapsto \mathbb{R}^n\text{ with } \mathbb{E} \left[ e^{2\alpha T} \| \xi \|_2^2 \right] < \infty,

- \(\mathbb{H}^{2, n}_0 : \mathbb{F}\text{-progressively measurable processes } \varphi : [0, T] \times \Omega \mapsto \mathbb{R}^n\text{ with } \mathbb{E} \left[ \int_0^T e^{2\alpha s} \| \varphi_s \|_2^2 \, ds \right] < \infty,

- \(\mathbb{S}^{2, n}_0 : \mathbb{F}\text{-progressively measurable processes } \varphi : [0, T] \times \Omega \mapsto \mathbb{R}^n\text{ with } \mathbb{E} \left[ \sup_{s \in [0, T]} e^{2\alpha s} \| \varphi_s \|_2^2 \right] < \infty,

where we denote \(\mathbb{L}^{2, n}_0, \mathbb{H}^{2, n}_0\) and \(\mathbb{S}^{2, n}_0\) by \(\mathbb{L}_0^{2, n}, \mathbb{H}_0^{2, n}\) and \(\mathbb{S}_0^{2, n}\) for the ease of notation.

We impose the following assumptions on the risk-sensitive function \(g\), the running payoff \(f\) and the terminal payoffs \(L, U\) and \(\xi\) in terms of the auxiliary payoffs \(\overline{L}, \overline{U}\) and \(\overline{\xi}\).

**Assumption 2.1** The deterministic risk-sensitive function \(g : \mathbb{R} \mapsto \mathbb{R}\) is strictly increasing and, moreover, (i) when \(T\) is an unbounded stopping time, \(\overline{L}, \overline{U}\) and \(\overline{\xi}\) are all bounded; (ii) when \(T\) is a bounded stopping time, \(\overline{L} \in \mathbb{S}^{2, 1}, \overline{U} \in \mathbb{S}^{2, 1}\) and \(\overline{\xi} \in \mathbb{L}^{2, 1}\), where \(\overline{L}, \overline{U}\) and \(\overline{\xi}\) are given by (2.7), (2.8) and (2.9), respectively.

On one hand, since the two players’ control sets are different, the usual dominating condition \(U \geq L\) is not required. On the other hand, the conditions (i) and (ii) in Assumption 2.1 guarantee the existence and uniqueness of the solution to BSDE (2.6), which will in turn be used to construct the game value and its associated optimal stopping strategy. Under Assumption 2.1 the solvability of BSDE (2.6) follows from Theorem 3.3 in [6] (when \(T\) is unbounded) and Theorem 4.1 in [33] (when \(T\) is bounded), and thus we omit the proof of the following proposition and refer to [6] and [33] for the details.

**Proposition 2.2** Suppose that Assumption 2.1 holds. Then, there exists a unique solution \((\overline{Q}^\lambda, \overline{Z}^\lambda)\) to BSDE (2.6). Moreover, (i) when \(T\) is an unbounded stopping time, \(\overline{Q}^\lambda\) is continuous and bounded, and \(\overline{Z}^\lambda\) belongs to \(\mathbb{H}^{2, d}_\tau\); (ii) when \(T\) is a bounded stopping time, the solution pair \((\overline{Q}^\lambda, \overline{Z}^\lambda)\) belongs to \(\mathbb{S}^{2, 1} \times \mathbb{H}^{2, d}\).
We are now in a position to state the main result of this paper.

**Theorem 2.3** Suppose that Assumption 2.1 holds. Let \((Q^\lambda, \mathcal{Z}^\lambda)\) be the unique solution to BSDE (2.6), and define the value process

\[
Q^\lambda_t = e^{r(t\wedge T)}g^{-1}(e^{-r(t\wedge T)/Q^\lambda_{t\wedge T}}) - \int_0^{t\wedge T} e^{-r(u\wedge T)}f_u \, du,
\]

for \(t \geq 0\). Then, the value of the constrained risk-sensitive Dynkin game (2.5) exists and is given by

\[
v^\lambda = \mathcal{V}^\lambda = \mathcal{A}^\lambda = Q^\lambda_0.
\]

Moreover, the optimal stopping strategy of the game is given by

\[
\begin{aligned}
\sigma^* &= \inf \{T^{(1)}_{T^1_N} \geq T^{(1)}_1; Q^\lambda_{T^1_N} \geq U_{T^1_N} \} \land T^{(1)}_{M_1}; \\
\tau^* &= \inf \{T^{(2)}_{T^2_N} \geq T^{(2)}_1; Q^\lambda_{T^2_N} \leq L_{T^2_N} \} \land T^{(2)}_{M_2}.
\end{aligned}
\]

**Remark 2.4** For a special case of exponential risk-sensitive function \(g\) (see section 5.2), the representation formula (2.10) is closely related to Cole-Hopf transformation in the BSDE literature, which is widely used to linearize a class of BSDEs with quadratic growth (see [18]). Our representation formula (2.10) can be regarded as a stochastic control version of Cole-Hopf transformation.

## 3 Proof of Theorem 2.3

Since the two players stop at two different sequences of Poisson arrival times, the first step to prove Theorem 2.3 is merging the two Poisson sequences together while still keeping track of their order. To this end, for each \(T^{(1)}\) and \(T^{(2)}\), we construct an increasing sequence of \(\mathcal{G}\)-stopping times \(\theta = (\theta_k)_{k \geq 0}\) as follows:

- \(\theta_0 = T^{(1)}_0 = T^{(2)}_0 = 0\),
- \(\theta_1 = \min\left(T^{(1)}, T^{(2)}\right)\),
- \(\theta_2 = \min\left(T^{(1)}_{\{T^{(1)} > \theta_1\}} + T^{(1)}_{\{T^{(1)} \leq \theta_1\}}, T^{(2)}_{\{T^{(2)} > \theta_1\}} + T^{(2)}_{\{T^{(2)} \leq \theta_1\}}\right)\),
- \(\theta_3 = \min\left(T^{(1)}_{\{T^{(1)} > \theta_2\}} + T^{(1)}_{\{T^{(1)} \leq \theta_2\}}, T^{(2)}_{\{T^{(2)} > \theta_2\}} + T^{(2)}_{\{T^{(2)} \leq \theta_2\}}\right)\),
- \(\theta_4 = \min\left(T^{(1)}_{\{T^{(1)} > \theta_3\}} + T^{(1)}_{\{T^{(1)} \leq \theta_3\}}, T^{(2)}_{\{T^{(2)} > \theta_3\}} + T^{(2)}_{\{T^{(2)} \leq \theta_3\}}\right)\),
- \(\theta_5 = \min\left(T^{(1)}_{\{T^{(1)} > \theta_4\}} + T^{(1)}_{\{T^{(1)} \leq \theta_4\}}, T^{(2)}_{\{T^{(2)} > \theta_4\}} + T^{(2)}_{\{T^{(2)} \leq \theta_4\}}\right)\),
- \(\theta_6 = \min\left(T^{(1)}_{\{T^{(1)} > \theta_5\}} + T^{(1)}_{\{T^{(1)} \leq \theta_5\}}, T^{(2)}_{\{T^{(2)} > \theta_5\}} + T^{(2)}_{\{T^{(2)} \leq \theta_5\}}\right)\),
- \(\theta_7 = \min\left(T^{(1)}_{\{T^{(1)} > \theta_6\}} + T^{(1)}_{\{T^{(1)} \leq \theta_6\}}, T^{(2)}_{\{T^{(2)} > \theta_6\}} + T^{(2)}_{\{T^{(2)} \leq \theta_6\}}\right)\),
- \(\theta_8 = \min\left(T^{(1)}_{\{T^{(1)} > \theta_7\}} + T^{(1)}_{\{T^{(1)} \leq \theta_7\}}, T^{(2)}_{\{T^{(2)} > \theta_7\}} + T^{(2)}_{\{T^{(2)} \leq \theta_7\}}\right)\),
- \(\theta_9 = \min\left(T^{(1)}_{\{T^{(1)} > \theta_8\}} + T^{(1)}_{\{T^{(1)} \leq \theta_8\}}, T^{(2)}_{\{T^{(2)} > \theta_8\}} + T^{(2)}_{\{T^{(2)} \leq \theta_8\}}\right)\),
- \(\theta_{10} = \min\left(T^{(1)}_{\{T^{(1)} > \theta_9\}} + T^{(1)}_{\{T^{(1)} \leq \theta_9\}}, T^{(2)}_{\{T^{(2)} > \theta_9\}} + T^{(2)}_{\{T^{(2)} \leq \theta_9\}}\right)\).

In Figure 1, we illustrate the construction of the merged sequence \(\theta\), where the top and the middle line are a realization of \(T^{(1)}\) and \(T^{(2)}\), and the bottom line is the merged sequence \(\theta\). Intuitively,
given any $\mathcal{G}$-stopping time $\theta_{k-1}$, $k \geq 1$, (to be used as the starting times for a family of constrained Dynkin games (3.5)-(3.6) below), we find the first arrival time of each Poisson sequence following $\theta_{k-1}$, say $T^{(1)}_{k_1}$ and $T^{(2)}_{k_2}$ for some $k_1, k_2 \geq 0$, and then define $\theta_k = \min\{T^{(1)}_{k_1}, T^{(2)}_{k_2}\}$. Moreover, given the stopping time $\theta_k$, we define pre-$\theta_k$ $\sigma$-field:

$$\mathcal{G}_{\theta_k} = \left\{ A \in \bigcup_{s \geq 0} \mathcal{G}_s : A \cap \{\theta_k \leq s\} \in \mathcal{G}_s \text{ for } s \geq 0 \right\},$$

and $\tilde{\mathcal{G}} = \{\mathcal{G}_{\theta_k}\}_{k \geq 0}$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{An illustration of a merged Poisson arrival sequence $\theta$.}
\end{figure}

Next, we tackle the nonlinear expectation $\tilde{\mathbb{E}}$ associated with the risk-sensitive function $g$. To this end, introduce the discounted processes

$$\tilde{L}_t = e^{-rt}L_t + \int_0^t e^{-ru}f_u \, du, \quad (3.1)$$

$$\tilde{U}_t = e^{-rt}U_t + \int_0^t e^{-ru}f_u \, du, \quad (3.2)$$

$$\tilde{\xi} = e^{-rT}\xi + \int_0^T e^{-ru}f_u \, du, \quad (3.3)$$

and rewrite the discounted payoff functional $R(\sigma, \tau)$ as

$$\tilde{R}(\sigma, \tau) = \tilde{\xi} \mathbb{1}_{\{\sigma \wedge \tau \geq T\}} + \tilde{L}_\tau \mathbb{1}_{\{\tau < T, \tau \leq \sigma\}} + \tilde{U}_\sigma \mathbb{1}_{\{\sigma < T, \sigma < \tau\}} = R(\sigma, \tau). \quad (3.4)$$

In turn, consider a family of constrained risk-sensitive Dynkin games starting from $\theta_{k-1}$, for $k \geq 1$, whose upper and lower values are defined by

$$\underline{q}_\lambda^\theta_{\theta_{k-1}} = \operatorname{ess \ inf} \operatorname{ess \ sup} \tilde{\mathbb{E}} \left[ \tilde{R}(\sigma, \tau) \mid \mathcal{G}_{\theta_{k-1}} \right], \quad (3.5)$$

$$\overline{q}_\lambda^\theta_{\theta_{k-1}} = \operatorname{ess \ sup} \operatorname{ess \ inf} \tilde{\mathbb{E}} \left[ \tilde{R}(\sigma, \tau) \mid \mathcal{G}_{\theta_{k-1}} \right], \quad (3.6)$$

where

$$\tilde{R}^{(i)}_{\theta_k} = \{\mathcal{G}\text{-stopping time } \sigma \text{ for } \sigma(\omega) = T^{(i)}_N(\omega) \text{ where } T^{(i)}_N(\omega) \geq \theta_k \text{ and } N \leq M_i(\omega)\}. \quad (3.7)$$
Remark 3.1 Note that in the above definition of control set $\tilde{R}_{\theta_k}^{(i)}$, $\theta_k$ is not necessary from the Poisson sequence $T^{(i)}$, so $\tilde{R}_{\theta_k}^{(i)}$ is in general different from $R_k^{(i)}$ in (2.7). However, they do coincide when $k = 1$: $\tilde{R}_{\theta_k}^{(i)} = R_k^{(i)}$.

On the other hand, thanks to the introduction of the discounted processes $\tilde{L}, \tilde{U}$ and $\tilde{\xi}$ in (3.3)-(3.4), the payoff functional in (2.7) can be divided into three disjoint sets and the risk-sensitive function $g$ can be applied to each of them separately. Thus, we can rewrite the payoff in (3.3)-(3.4) under the linear expectation $\tilde{E}$ of the auxiliary payoff processes $\tilde{L}, \tilde{U}$ and $\tilde{\xi}$ as

$$\tilde{E} \left[ \tilde{R}(\sigma, \tau) \mid G_{\theta_k-1} \right] = g^{-1} \left( \tilde{E} \left[ e^{-rT} \mathbb{1}_{\{ T_{\wedge} \geq T \}} + e^{-rT} \mathbb{1}_{\{ \tau < T, \tau \leq \sigma \}} + e^{-rT} \mathbb{1}_{\{ \sigma < T, \sigma < \tau \}} \right] \right).$$

This motivates us to introduce the Cole-Hopf representation formula (2.10).

The constrained risk-sensitive Dynkin game (3.5)-(3.6) is said to have value $Q_{\theta_k-1}^\lambda$ if $Q_{\theta_k-1}^\lambda = \tilde{g}_{\theta_k-1}^\lambda = \tilde{g}_{\theta_k-1}^\lambda$, and $(\sigma_k^*, \tau_k^*) \in \tilde{R}_{\theta_k}^{(1)} \times \tilde{R}_{\theta_k}^{(2)}$ is called an optimal stopping strategy of the game if

$$\tilde{E} \left[ \tilde{R}(\sigma^*, \tau) \mid G_{\theta_k-1} \right] \leq \tilde{E} \left[ \tilde{R}(\sigma^*, \tau^*) \mid G_{\theta_k-1} \right] \leq \tilde{E} \left[ \tilde{R}(\sigma, \tau) \mid G_{\theta_k-1} \right],$$

for every $(\sigma, \tau) \in \tilde{R}_{\theta_k}^{(1)} \times \tilde{R}_{\theta_k}^{(2)}$. In particular, when $k = 1$, (3.5)-(3.6) corresponds to the original constrained Dynkin game (2.3). Thus, to prove Theorem 2.3, it is sufficient to show that

$$Q_{\theta_k-1}^\lambda = \tilde{g}_{\theta_k-1}^\lambda = \tilde{g}_{\theta_k-1}^\lambda = \hat{Q}_{\theta_k-1}^\lambda,$$

and the optimal stopping strategy is given by

$$\begin{aligned}
\{ \sigma_k^* = \inf \{ T_N^{(1)} \geq \theta_k : \tilde{Q}_t^\lambda \geq \tilde{U}_t^\lambda, \} \wedge T_M^{(1)} \}, \\
\{ \tau_k^* = \inf \{ T_N^{(2)} \geq \theta_k : \tilde{Q}_t^\lambda \leq \tilde{L}_t^\lambda, \} \wedge T_M^{(2)} \},
\end{aligned}$$

(3.8)

where $\hat{Q}^\lambda$ is given by

$$\hat{Q}_t^\lambda = g^{-1} (e^{-r(t \wedge T)} \tilde{Q}_t^\lambda),$$

(3.9)

with $\tilde{Q}^\lambda$ being the first component of the solution to BSDE (2.6). In turn, the value process $Q^\lambda$ in (2.3) is given via the discounted process $\hat{Q}^\lambda$ via the relationship

$$Q_t^\lambda = e^{r(t \wedge T)} \hat{Q}_t^\lambda - \int_0^{t \wedge T} e^{-r(u \wedge T)} f_u du.$$

(3.10)

Note that, for $t \geq T$,

$$Q_t^\lambda = e^{r(T)} g^{-1}(e^{-rT} \tilde{\xi}) - \int_0^{T} e^{-r(u-T)} f_u du = \tilde{\xi}.$$

Remark 3.2 For the reader’s convenience, we recall the notations that have been introduced thus far. For the payoff processes $h = L, U, \xi$, we have defined the discounted processes $\hat{h}_t = e^{-rt} \hat{h}_t + \int_0^t e^{-ru} f_u du$, and auxiliary payoff processes $\tilde{h}_t = e^{rt} g(\tilde{h}_t)$. In terms of the value process $Q^\lambda$, likewise we have $\hat{Q}_t^\lambda = e^{-rt} Q_t^\lambda + \int_0^t e^{-ru} f_u du$, and $\tilde{Q}_t^\lambda = e^{rt} g(\hat{Q}_t^\lambda)$, for $t \in [0, T]$. 8
To prove the above assertions (and therefore Theorem 2.3), we start with the following lemma.

**Lemma 3.3** Suppose that Assumption (A) holds. Then, $\tilde{Q}_{\theta_{k-1}}^\lambda$ given in (3.9) satisfies the recursive equation

$$
\tilde{Q}_{\theta_{k-1}}^\lambda = \tilde{E} \left[ \xi |_{(\theta_{k-1}, \theta_k]} + \left( \text{min}\{\tilde{U}_{\theta_k}, \tilde{Q}_{\theta_k}^\lambda\} \mathbb{1}_{\{\theta_k \in T^{(1)}\}} + \text{max}\{\tilde{L}_{\theta_k}, \tilde{Q}_{\theta_k}^\lambda\} \mathbb{1}_{\{\theta_k < T\}} \right) \mathbb{1}_{\{\theta_k < T\}} \mathbb{1}_{\{\theta_k \in T^{(2)}\}} \bigg| \mathcal{G}_{\theta_{k-1}} \right],
$$

(3.11)

for $k \geq 1$.

**Proof.** It is equivalent to prove that

$$
g(\tilde{Q}_{\theta_{k-1}}^\lambda) = \mathbb{E} \left[ g(\tilde{\xi}) \mathbb{1}_{\{\theta_{k-1} \geq T\}} + \left( \min\{g(\tilde{U}_{\theta_k}), g(\tilde{Q}_{\theta_k}^\lambda)\} \mathbb{1}_{\{\theta_k \in T^{(1)}\}} + \max\{g(\tilde{L}_{\theta_k}), g(\tilde{Q}_{\theta_k}^\lambda)\} \mathbb{1}_{\{\theta_k \in T^{(2)}\}} \right) \mathbb{1}_{\{\theta_k < T\}} \bigg| \mathcal{G}_{\theta_{k-1}} \right],
$$

(3.12)

where $g(\tilde{\xi}) = e^{-rT} g(\tilde{T})$, $g(\tilde{L}) = e^{-rT} g(\tilde{L})$, and $g(\tilde{U}) = e^{-rT} g(\tilde{U})$. For $k$ such that $\theta_{k-1} > T$, it follows from (3.9) that $g(\tilde{Q}_{\theta_{k-1}}^\lambda) = g(\tilde{\xi})$, and thus (3.12) holds. In the rest of the proof, we only focus on the cases where $\theta_{k-1} \leq T$.

By applying Itô’s formula to $\alpha_t g(\tilde{Q}_{\theta_{k-1}}^\lambda)$, where $\alpha_t = e^{-((\lambda^{(1)} + \lambda^{(2)})t)}$, we can obtain that

$$
\alpha_t \mathbb{E} g(\tilde{Q}_{\theta_{k-1}}^\lambda) = \alpha_T g(\tilde{\xi}) + \int_{\lambda T}^{T} \mathbb{E} g(\tilde{Q}_{\theta_{k-1}}^\lambda) \left( \lambda^{(1)} + \lambda^{(2)} \right) g(\tilde{Q}_{\theta_{k-1}}^\lambda) - \lambda^{(1)} g(\tilde{Q}_{\theta_{k-1}}^\lambda) - g(\tilde{U}_s) + \int_{\lambda T}^{T} \mathbb{E} \left[ \lambda^{(1)} \min\{g(\tilde{U}_s), g(\tilde{Q}_{\theta_{k-1}}^\lambda)\} + \lambda^{(2)} \max\{g(\tilde{L}_s), g(\tilde{Q}_{\theta_{k-1}}^\lambda)\} \right] ds
$$

$$
- \int_{\lambda T}^{T} \mathbb{E} \left[ \lambda^{(1)} \min\{g(\tilde{U}_s), g(\tilde{Q}_{\theta_{k-1}}^\lambda)\} + \lambda^{(2)} \max\{g(\tilde{L}_s), g(\tilde{Q}_{\theta_{k-1}}^\lambda)\} \right] ds,
$$

for $t \geq 0$. By choosing $t = \theta_{k-1}$ and taking the conditional expectation with respect to $\mathcal{G}_{\theta_{k-1}}$, we further have

$$
g(\tilde{Q}_{\theta_{k-1}}^\lambda) = \mathbb{E} \left[ e^{-((\lambda^{(1)} + \lambda^{(2)})t - \theta_{k-1})} \right] g(\tilde{\xi})
$$

$$
+ \int_{\theta_{k-1}}^{T} e^{-((\lambda^{(1)} + \lambda^{(2)})t - \theta_{k-1})} \left( \lambda^{(1)} \min\{g(\tilde{U}_s), g(\tilde{Q}_{\theta_{k-1}}^\lambda)\} + \lambda^{(2)} \max\{g(\tilde{L}_s), g(\tilde{Q}_{\theta_{k-1}}^\lambda)\} \right) ds \bigg| \mathcal{G}_{\theta_{k-1}} \bigg],
$$

(3.13)

for any $k \geq 1$.

On the other hand, by defining $\hat{T}_t^{(i)}$ as the first arrival time in $T^{(i)}$ following any fixed time $t$, i.e. $\hat{T}_t^{(i)} = \inf\{T_N^{(i)} \geq \hat{T}_t^{(i)} : T_N^{(i)} > t\}$, we can rewrite the right-hand-side of (3.12) as

$$
\mathbb{E} \left[ g(\tilde{\xi}) \mathbb{1}_{\{\hat{T}_{\theta_{k-1}}^{(1)} \land \hat{T}_{\theta_{k-1}}^{(2)} \geq T\}} + \min\{g(\tilde{U}_{\theta_{k-1}}), g(\tilde{Q}_{\theta_{k-1}}^\lambda)\} \mathbb{1}_{\{\hat{T}_{\theta_{k-1}}^{(1)} \land \hat{T}_{\theta_{k-1}}^{(2)} \geq T\}} \right.
$$

$$
+ \max\{g(\tilde{L}_{\theta_{k-1}}), g(\tilde{Q}_{\theta_{k-1}}^\lambda)\} \mathbb{1}_{\{\hat{T}_{\theta_{k-1}}^{(2)} < T, \hat{T}_{\theta_{k-1}}^{(2)} \leq \hat{T}_{\theta_{k-1}}^{(1)}\}} \bigg| \mathcal{G}_{\theta_{k-1}} \bigg].
$$

(3.14)
Indeed, applying the joint probability density function of \((\tilde{T}_{\theta_{k-1}}^{(1)}, \tilde{T}_{\theta_{k-1}}^{(2)})\) conditional on \(G_{\theta_{k-1}}\),

\[
p_{\theta_{k-1}}(S, U) = \lambda^{(1)} e^{-\lambda^{(1)}(S-\theta_{k-1})} \lambda^{(2)} e^{-\lambda^{(2)}(U-\theta_{k-1})},
\]

yields that

\[
\mathbb{E} \left[ g(\tilde{\xi}) 1_{\{\tilde{T}_{\theta_{k-1}}^{(1)} T_{\theta_{k-1}}^{(2)} \geq T\}} \right] = \mathbb{E} \left[ g(\tilde{\xi}) \int_{S \land U \geq T} p_{\theta_{k-1}}(S, U) \, dS \, dU \right] \left| G_{\theta_{k-1}} \right.
\]

\[
= \mathbb{E} \left[ g(\tilde{\xi}) \int_{U \geq S \geq T} \lambda^{(1)} e^{-\lambda^{(1)}(S-\theta_{k-1})} \lambda^{(2)} e^{-\lambda^{(2)}(U-\theta_{k-1})} \, dS \, dU \right] \left| G_{\theta_{k-1}} \right.
\]

\[
= \mathbb{E} \left[ g(\tilde{\xi}) \int_{S \geq U \geq T} \lambda^{(1)} e^{-\lambda^{(1)}(S-\theta_{k-1})} \lambda^{(2)} e^{-\lambda^{(2)}(U-\theta_{k-1})} \, dS \, dU \right] \left| G_{\theta_{k-1}} \right. \tag{I}
\]

\[
+ \mathbb{E} \left[ g(\tilde{\xi}) \int_{S \geq U \geq T} \lambda^{(1)} e^{-\lambda^{(1)}(S-\theta_{k-1})} \lambda^{(2)} e^{-\lambda^{(2)}(U-\theta_{k-1})} \, dS \, dU \right] \left| G_{\theta_{k-1}} \right. \tag{II}
\]

where the first integral

\[
(I) = \lambda^{(1)} \int_{T}^{\infty} e^{-\lambda^{(1)}(S-\theta_{k-1})} \left( \int_{S}^{\infty} \lambda^{(2)} e^{-\lambda^{(2)}(U-\theta_{k-1})} \, dU \right) \, dS = \frac{\lambda^{(1)}}{\lambda^{(1)} + \lambda^{(2)}} e^{-\lambda^{(1)}(S-\theta_{k-1})},
\]

and, similarly, the second integral

\[
(II) = \frac{\lambda^{(2)}}{\lambda^{(1)} + \lambda^{(2)}} e^{-\lambda^{(2)}(U-\theta_{k-1})}.
\]

In turn, we obtain

\[
\mathbb{E} \left[ g(\tilde{\xi}) 1_{\{\tilde{T}_{\theta_{k-1}}^{(1)} \land \tilde{T}_{\theta_{k-1}}^{(2)} \geq T\}} \right] = \mathbb{E} \left[ e^{-\lambda^{(1)}(S-\theta_{k-1})} g(\tilde{\xi}) \right] \bigg| G_{\theta_{k-1}} \bigg. \right) \tag{3.15}
\]

Similarly, we have

\[
\mathbb{E} \left[ \min \left\{ g(\tilde{U}_{\tilde{T}_{\theta_{k-1}}^{(1)}}), g(\tilde{Q}_{\tilde{T}_{\theta_{k-1}}^{(1)}}) \right\} 1_{\{\tilde{T}_{\theta_{k-1}}^{(1)} < T, \tilde{T}_{\theta_{k-1}}^{(2)} < \tilde{T}_{\theta_{k-1}}^{(2)}\}} \right] \left| G_{\theta_{k-1}} \right.
\]

\[
= \mathbb{E} \left[ \min \left\{ g(\tilde{U}_{\tilde{T}_{\theta_{k-1}}^{(1)}}), g(\tilde{Q}_{\tilde{T}_{\theta_{k-1}}^{(1)}}) \right\} p_{\theta_{k-1}}(S, U) \, dS \, dU \right] \left| G_{\theta_{k-1}} \right.
\]

\[
= \mathbb{E} \left[ \int_{S}^{T} \lambda^{(1)} e^{-\lambda^{(1)}(S-\theta_{k-1})} \min \left\{ g(\tilde{U}_{\tilde{T}_{\theta_{k-1}}^{(1)}}), g(\tilde{Q}_{\tilde{T}_{\theta_{k-1}}^{(1)}}) \right\} \left( \int_{S}^{\infty} \lambda^{(2)} e^{-\lambda^{(2)}(U-\theta_{k-1})} \, dU \right) \, dS \right] \left| G_{\theta_{k-1}} \right. \tag{3.16}
\]
Lemma 3.4 \textit{Suppose that Assumption 2.1 holds. Then, for any } k \geq 1, \textit{the value of the auxiliary constrained risk-sensitive Dynkin game } (3.20) - (3.21) \textit{starting from } \theta_k - 1 \textit{exists. Its value } \hat{Q}_\theta^\lambda_{k-1} \textit{is reached by choosing } \hat{\sigma}^*_{k-1} \textit{and } \hat{\tau}^*_{k-1}.  

As a direct consequence of Lemma 3.3, we deduce that \( \hat{Q}_\theta^\lambda_{k-1} \), which is given by (3.9), satisfies the recursive equation (3.11), for \( k \geq 1 \).  

As a direct consequence of Lemma 3.3, we deduce that \( \hat{Q}_\theta^\lambda_{k-1} \) defined by 

\[ \hat{Q}_\theta^\lambda_{k-1} := \hat{\xi} \mathbb{1}_{\{\theta_k - 1 \geq T\}} + \left( \min\{\tilde{U}_{\theta_k - 1}, \hat{Q}_\theta^\lambda_{k-1}\} \mathbb{1}_{\{\theta_k - 1 \in T(1)\}} + \max\{\tilde{L}_{\theta_k - 1}, \hat{Q}_\theta^\lambda_{k-1}\} \mathbb{1}_{\{\theta_k - 1 \in T(2)\}} \right) \mathbb{1}_{\{\theta_k - 1 < T\}}, \]  

(3.18)  

for \( k \geq 1 \).  

We will show that \( \hat{Q}_\theta^\lambda_{k-1} \) in (3.18) is actually the unique solution of the recursive equation (3.11). The uniqueness is proved by showing that \( \hat{Q}_\theta^\lambda_{k-1} \) is the value of an auxiliary constrained risk-sensitive Dynkin game starting from \( \theta_k - 1 \), whose upper and lower values are defined by 

\[ \underline{q}_{\theta_k - 1}^\lambda = \text{ess inf}_{\sigma \in \tilde{\mathcal{R}}_{\theta_k - 1}^{(1)}} \text{ess sup}_{\tau \in \tilde{\mathcal{R}}_{\theta_k - 1}^{(2)}} \mathbb{E}\left[ \tilde{R}(\sigma, \tau) | \mathcal{G}_{\theta_k - 1} \right], \]  

(3.20) 

\[ \overline{q}_{\theta_k - 1}^\lambda = \text{sup}_{\tau \in \tilde{\mathcal{R}}_{\theta_k - 1}^{(2)}} \text{ess inf}_{\sigma \in \tilde{\mathcal{R}}_{\theta_k - 1}^{(1)}} \mathbb{E}\left[ \tilde{R}(\sigma, \tau) | \mathcal{G}_{\theta_k - 1} \right], \]  

(3.21) 

where the payoff functional \( \tilde{R}(\sigma, \tau) \) is given by (3.4) and the control set \( \tilde{\mathcal{R}}_{\theta_k - 1}^{(i)} \) is given by (3.7).  

The auxiliary game (3.20) - (3.21) is said to have value \( \hat{q}_{\theta_k - 1}^\lambda \) if \( \hat{q}_{\theta_k - 1}^\lambda = \underline{q}_{\theta_k - 1}^\lambda = \overline{q}_{\theta_k - 1}^\lambda \), and \((\hat{\sigma}^*_{k-1}, \hat{\tau}^*_{k-1}) \in \tilde{\mathcal{R}}_{\theta_k - 1}^{(1)} \times \tilde{\mathcal{R}}_{\theta_k - 1}^{(2)} \) is called an optimal stopping strategy of the game (3.20) - (3.21) if 

\[ \mathbb{E}\left[ \tilde{R}(\sigma^*_{k-1}, \tau^*_{k-1}) | \mathcal{G}_{\theta_k - 1} \right] \leq \mathbb{E}\left[ \tilde{R}(\sigma^*_{k-1}, \tau^*_{k-1}) | \mathcal{G}_{\theta_k - 1} \right] \leq \mathbb{E}\left[ \tilde{R}(\sigma^*_{k-1}, \tau^*_{k-1}) | \mathcal{G}_{\theta_k - 1} \right], \]  

for every \((\sigma, \tau) \in \tilde{\mathcal{R}}_{\theta_k - 1}^{(1)} \times \tilde{\mathcal{R}}_{\theta_k - 1}^{(2)} \).  

The difference between (3.20) - (3.21) and (3.9) - (3.10) is that the players first make their stopping decisions and then move forward in the former game, while in the latter game they first move forward and then make their decisions.
The unique solution of the recursive equation $q_{k-1}$. Hence, $q_{k-1} = q_{k-1}$, where the latter is given by $3.18$. The optimal stopping strategy of the auxiliary constrained risk-sensitive Dynkin game $(3.20)-(3.21)$ is given by $(3.18)$. The optimal stopping strategy of the auxiliary constrained risk-sensitive Dynkin game $(3.20)-(3.21)$ is given by $(3.18)$.

**Proof.** Step 1. Let $q_{k-1}$ be a solution of the recursive equation $(3.19)$ for $k \geq 1$. We claim the following martingale properties hold:

(i) $(q_{k-1}^m \wedge q_{k-1})_{m \geq k-1}$ is a $\mathcal{G}$-martingale under the nonlinear expectation $\tilde{E}$;

(ii) $(q_{k-1}^m \wedge q_{k-1})_{m \geq k-1}$ is a $\mathcal{G}$-supermartingale under $\tilde{E}$, for any $\tau \in \tilde{R}_{\theta_{k-1}}^{(2)}$;

(iii) $(q_{k-1}^m \wedge q_{k-1})_{m \geq k-1}$ is a $\mathcal{G}$-submartingale under $\tilde{E}$, for any $\sigma \in \tilde{R}_{\theta_{k-1}}^{(1)}$.

If the martingale property (i) holds, then, for $k \geq 1$,

$$ q_{k-1} = q_{k-1}^m \wedge q_{k-1}^m = \tilde{E} \left[ q_{k-1}^m \wedge q_{k-1}^m | \mathcal{G}_{k-1} \right], $$

and the definition of $(q_{k-1}^m, q_{k-1}^m)$ in $(3.22)$ further yields that

$$ q_{k-1} = \tilde{E} \left[ q_{k-1} \wedge q_{k-1} | \mathcal{G}_{k-1} \right], $$

Using the similar arguments, if the supermartingale property (ii) and the submartingale property (iii) hold, then we have, for any $\tau \in \tilde{R}_{\theta_{k-1}}^{(2)}$,

$$ q_{k-1} = \tilde{E} \left[ q_{k-1} | \mathcal{G}_{k-1} \right], $$

and, for any $\sigma \in \tilde{R}_{\theta_{k-1}}^{(1)}$,

$$ q_{k-1} = \tilde{E} \left[ q_{k-1} | \mathcal{G}_{k-1} \right]. $$
It follows from (3.21) and (3.24) that

$$\hat{Q}_{\theta_{k-1}}^\lambda \geq \text{ess sup}_{\tau \in \bar{R}_{\theta_{k-1}}^{(2)}} \hat{E}\left[\hat{R}(\hat{\sigma}_{k-1}^*, \tau) | \mathcal{F}_{\theta_{k-1}}\right] \geq \text{ess inf}_{\sigma \in \bar{R}_{\theta_{k-1}}^{(1)}} \text{ess sup}_{\tau \in \bar{R}_{\theta_{k-1}}^{(2)}} \hat{E}\left[\hat{R}(\sigma, \tau) | \mathcal{F}_{\theta_{k-1}}\right] = \lambda_{\theta_{k-1}},$$

and

$$\hat{Q}_{\theta_{k-1}}^\lambda \leq \text{ess inf}_{\sigma \in \bar{R}_{\theta_{k-1}}^{(1)}} \hat{E}\left[\hat{R}(\sigma, \hat{\tau}_{k-1}^*) | \mathcal{F}_{\theta_{k-1}}\right] \leq \text{ess sup}_{\tau \in \bar{R}_{\theta_{k-1}}^{(2)}} \text{ess inf}_{\sigma \in \bar{R}_{\theta_{k-1}}^{(1)}} \hat{E}\left[\hat{R}(\sigma, \tau) | \mathcal{F}_{\theta_{k-1}}\right] = \lambda_{\theta_{k-1}}.$$

It is clear that $\lambda_{\theta_{k-1}} \geq \hat{q}_{\theta_{k-1}}$, and therefore the value of the auxiliary constrained risk-sensitive Dynkin game (3.20)-(3.21) exists, i.e.

$$\hat{Q}_{\theta_{k-1}}^\lambda = \lambda_{\theta_{k-1}} = \hat{q}_{\theta_{k-1}} = \hat{q}_{\theta_{k-1}}.$$

This also implies the recursive equation (3.19) admits a unique solution. Furthermore, since $\hat{Q}_{\theta_{k-1}}^\lambda$, given by (3.18) satisfies the recursive equation (3.19), it is actually the unique solution of (3.19).

As a direct consequence of (3.21)-(3.24), we can obtain that $(\hat{\sigma}_{k-1}^*, \hat{\tau}_{k-1}^*)$, which is given by (3.22), is indeed an optimal stopping strategy of the auxiliary constrained risk-sensitive Dynkin game (3.20)-(3.21).

Step 2. It remains to prove the martingale property (i), the supermartingale property (ii) and the submartingale property (iii) in Step 1.

Indeed, for $m \geq k - 1$, we have

$$\hat{E}\left[\hat{Q}_{\theta_{m+1}}^\lambda \mathbb{1}_{\sigma_{k-1}^* \wedge \hat{\sigma}_{k-1}^* \land T} | \mathcal{F}_{\theta_m}\right] = \hat{E}\left[\mathbb{1}_{\sigma_{k-1}^* \wedge \hat{\sigma}_{k-1}^* \land T} \hat{Q}_{\sigma_{k-1}^* \land \hat{\sigma}_{k-1}^*}^\lambda \mathbb{1}_{\sigma_{k-1}^* \land \hat{\sigma}_{k-1}^* \land T} | \mathcal{F}_{\theta_m}\right]$$

where the second last equality follows from the definition (3.22) of $(\hat{\sigma}_{k-1}^*, \hat{\tau}_{k-1}^*)$, and thus the martingale property (i) has been proved.

To prove the supermartingale property (ii), for any $\tau \in \bar{R}_{\theta_{k-1}}^{(2)}$, we have

$$\hat{E}\left[\hat{Q}_{\theta_{m+1}}^\lambda | \mathcal{F}_{\theta_m}\right] = \mathbb{1}_{\sigma_{k-1}^* \land \hat{\sigma}_{k-1}^* \land T} \hat{Q}_{\sigma_{k-1}^* \land \hat{\sigma}_{k-1}^*}^\lambda \mathbb{1}_{\sigma_{k-1}^* \land \hat{\sigma}_{k-1}^* \land T} \mathbb{E}\left[\hat{Q}_{\theta_{m+1}}^\lambda | \mathcal{F}_{\theta_m}\right].$$

Conditional on the set $\{\sigma_{k-1}^* \land \tau \geq \theta_{m+1}\} \cap \{\theta_m < T\}$, we have

$$\hat{Q}_{\theta_m}^\lambda = \hat{E}\left[\hat{Q}_{\theta_{m+1}}^\lambda | \mathcal{F}_{\theta_m}\right] \mathbb{1}_{\{\theta_m \in T^{(1)}\}} + \max\left\{\hat{L}_{\theta_m}, \hat{E}\left[\hat{Q}_{\theta_{m+1}}^\lambda | \mathcal{F}_{\theta_m}\right] \mathbb{1}_{\{\theta_m \in T^{(2)}\}}\right\} \mathbb{1}_{\{\theta_m \in T^{(2)}\}} \geq \hat{E}\left[\hat{Q}_{\theta_{m+1}}^\lambda | \mathcal{F}_{\theta_m}\right].$$
and thus
\[
\hat{q}_{\theta_{m+1} \wedge \sigma_{m+1} \wedge \tau | \mathcal{F}_m} = \mathbb{E} \left[ \hat{q}_{\theta_{m+1} \wedge \sigma_{m+1} \wedge \tau} \mathbb{1}_{\{\sigma_{m+1} \wedge \tau \leq \theta_m\}} \hat{q}_{\sigma_{m+1} \wedge \tau} + \mathbb{1}_{\{\sigma_{m+1} \wedge \tau \geq \theta_m\}} \left( \hat{q}_{\theta_m \wedge \tau | \mathcal{F}_m} + \hat{q}_{\theta_m \wedge \tau} \mathbb{1}_{\{\theta_m \geq \tau\}} \right) \right],
\]
which proves the supermartingale property (ii). Likewise, the submartingale property (iii) can be proved in a similar way, and the proof of this lemma is thus completed. ■

We are now in a position to prove Theorem 3. Let \( \hat{q}_{\theta_{k-1}} \) be a solution of the recursive equation
\[\hat{q}_{\theta_{k-1}} = \mathbb{E} \left[ \hat{q} \mathbb{1}_{\{\theta_k \geq T\}} + \hat{q}_{\theta_{k-1}} \mathbb{1}_{\{\theta_k < T\}} \right],\]
and
\[\hat{q}_{\theta_{k-1}} = \mathbb{E} \left[ \hat{q} \mathbb{1}_{\{\theta_k \geq T\}} + \mathbb{E} \left( \hat{R}(\hat{\theta}_k, \hat{\tau}_k) | \mathcal{F}_\theta \right) \mathbb{1}_{\{\theta_k < T\}} \right],\]
which completes the proof. \(\blacksquare\)

Using the relationship \(\{\theta_k \geq T\} \subseteq \{\hat{\sigma}_k \wedge \hat{\tau}_k \geq T\}, \{\hat{\tau}_k < T, \hat{\sigma}_k \leq \hat{\tau}_k \} \subseteq \{\theta_k \geq T\}, \{\hat{\sigma}_k < T, \hat{\tau}_k \leq \hat{\tau}_k \} \subseteq \{\theta_k < T\}\), we can further obtain that
\[\hat{q}_{\theta_k - 1} = \mathbb{E} \left[ \hat{R}(\hat{\sigma}_k, \hat{\tau}_k) | \mathcal{F}_{\theta_k - 1} \right], \tag{3.26}\]
where \((\hat{\sigma}_k, \hat{\tau}_k)\) is the optimal stopping strategy of the auxiliary constrained risk-sensitive Dynkin game starting from \(\theta_k\) given in (3.22). Similarly, we can obtain that, for any \(T \in \mathcal{R}^{(2)}_{\theta_k}\)
\[\hat{q}_{\theta_k - 1} \geq \mathbb{E} \left[ \hat{R}(\hat{\sigma}_k, \hat{\tau}_k) | \mathcal{F}_{\theta_k - 1} \right], \tag{3.27}\]
and, for any \(\sigma \in \mathcal{R}^{(1)}_{\theta_k}\)
\[\hat{q}_{\theta_k - 1} \leq \mathbb{E} \left[ \hat{R}(\sigma, \hat{\tau}_k) | \mathcal{F}_{\theta_k - 1} \right]. \tag{3.28}\]

It follows from (3.27) and (3.28) that
\[\hat{q}_{\theta_k - 1} \geq \mathbb{E} \left[ \hat{R}(\hat{\sigma}_k, \hat{\tau}_k) | \mathcal{F}_{\theta_k - 1} \right] \geq \mathbb{E} \left[ \hat{R}(\sigma, \tau) | \mathcal{F}_{\theta_k - 1} \right] = \mathbb{E} \left[ \hat{R}(\sigma, \tau) | \mathcal{F}_{\theta_k - 1} \right], \tag{3.29}\]
and
\[\hat{q}_{\theta_k - 1} \leq \mathbb{E} \left[ \hat{R}(\sigma, \hat{\tau}_k) | \mathcal{F}_{\theta_k - 1} \right] \leq \mathbb{E} \left[ \hat{R}(\sigma, \tau) | \mathcal{F}_{\theta_k - 1} \right] = \mathbb{E} \left[ \hat{R}(\sigma, \tau) | \mathcal{F}_{\theta_k - 1} \right].\]
It is clear that $\tilde{Q}^{\lambda}_{\theta_{k-1}} \geq q^{\lambda}_{\theta_{k-1}}$, and therefore the value of the constrained risk-sensitive Dynkin game starting from $\theta_{k-1}$ (3.5)-(3.6) exists, i.e.

$$\tilde{Q}^{\lambda}_{\theta_{k-1}} = q^{\lambda}_{\theta_{k-1}} = \tilde{Q}^{\lambda}_{\theta_{k-1}}.$$  

This also implies the recursive equation (3.11) admits a unique solution. Furthermore, since $\tilde{Q}^{\lambda}_{\theta_{k-1}}$ given by (3.9) satisfies the recursive equation (3.11), it is actually the unique solution of (3.11). We conclude the proof by proving $(\hat{\sigma}^*_k, \hat{\tau}^*_k)$ are actually $(\sigma^*_k, \tau^*_k)$ in (3.8). Indeed,

$$\hat{\sigma}^*_k = \inf\{T^{(1)}_N \geq \theta_k : \tilde{Q}^{\lambda}_{T^{(1)}_N} = \tilde{U}^{T^{(1)}_N}_M\} \land T^{(1)}_M = \sigma^*_k,$$

and, similarly, $\hat{\tau}^*_k = \tau^*_k$.

4 Connection with stochastic differential games via randomized stopping

In this section, we connect constrained risk-sensitive Dynkin games with a class of stochastic differential games via randomized stopping first introduced by Krylov (see [19]). In particular, we generalize the optimal control representation of constrained optimal stopping problems in [23] (see section 4 therein).

Let us introduce the basic idea of randomized stopping in a two-player setting as follows. Consider a nonnegative control process $(a_t)_{t \geq 0}$ (resp. $(b_t)_{t \geq 0}$), and let Player I (resp. II) stop with probability $a_t \Delta$ (resp. $b_t \Delta$) in an infinitesimal interval $(t, t + \Delta)$. Then the probability that Player I (resp. II) does not stop before time $t$ is

$$e^{-\int_0^t a_u \, du} \left( \text{resp. } e^{-\int_0^t b_u \, du} \right),$$

and the probability that both players do not stop before time $t$ and Player I (resp. II) does stop in the infinitesimal interval $(t, t + \Delta)$ is

$$e^{-\int_0^t (a_u + b_u) \, du} a_t \Delta \left( \text{resp. } e^{-\int_0^t (a_u + b_u) \, du} b_t \Delta \right).$$

Recall that $T$ is a finite $\mathbb{F}$-stopping time representing the (random) terminal time of the game, and $r > 0$ represents the discount rate. The discounted payoff is assumed to be $e^{-rT} T^T_t$ if Player I stops firstly at time $t < T$, $e^{-rT} T^T_t$ if Player II stops firstly at time $t < T$, and $e^{-rT} T^T_t$ if neither players stop in the time interval $[0, T]$, where the auxiliary payoff processes $\overline{U}, \overline{L}$ and $\overline{\xi}$ are given in (2.8), (2.7), and (2.9), respectively. Thus, the discounted payoff functional associated with the control processes $a$ and $b$ is given by

$$J(a, b) = \int_0^T e^{-\int_0^t (a_u + b_u + r) \, du} (a_t \overline{U}_t + b_t \overline{L}_t) \, dt + e^{-\int_0^T (a_u + b_u + r) \, du} \overline{\xi},$$

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or in terms of the original processes \( L \), \( U \) and \( \xi \),
\[
J(a, b) = \int_0^T e^{-\int_0^t (a_u + b_u)du} \left[ a_t g(e^{-rT}U_t + \int_0^t e^{-ru}f_u du) + b_t g(e^{-rT}L_t + \int_0^t e^{-ru}f_u du) \right] \\
+ e^{-\int_0^t (a_u + b_u)du} g(e^{-rT}\xi + \int_0^T e^{-ru}f_u du).
\]

Let us define the control set \( \mathcal{A}(\lambda^{(1)}) \) (resp. \( \mathcal{B}(\lambda^{(2)}) \)) for Player I (resp. II) as
\[
\mathcal{A}(\lambda^{(1)}) = \{ \mathbb{F}\text{-adapted process } (a_t)_{t \geq 0} : a_t = 0 \text{ or } \lambda^{(1)} \}
\]
(resp.
\[
\mathcal{B}(\lambda^{(2)}) = \{ \mathbb{F}\text{-adapted process } (b_t)_{t \geq 0} : b_t = 0 \text{ or } \lambda^{(2)} \},
\]
and the upper and lower values of the stochastic differential game as
\[
\underline{v}^{\lambda,SDG} = \inf_{a \in \mathcal{A}(\lambda^{(1)})} \sup_{b \in \mathcal{B}(\lambda^{(2)})} g^{-1}(\mathbb{E}[J(a, b)]), \quad \overline{v}^{\lambda,SDG} = \sup_{b \in \mathcal{B}(\lambda^{(2)})} \inf_{a \in \mathcal{A}(\lambda^{(1)})} g^{-1}(\mathbb{E}[J(a, b)]),
\]
where \( g^{-1} \) is the inverse function of the risk-sensitive function \( g \). The game (1.1) is said to have value \( \underline{v}^{\lambda,SDG} \) if \( \underline{v}^{\lambda,SDG} = \overline{v}^{\lambda,SDG} = v^\lambda \), and \( (a^*, b^*) \in \mathcal{A}(\lambda^{(1)}) \times \mathcal{B}(\lambda^{(2)}) \) is said to be an optimal pair of controls if \( \underline{v}^{\lambda,SDG} = g^{-1}(\mathbb{E}[J(a^*, b^*)]) \).

We are now in a position to present the main result of this section.

**Proposition 4.1** Suppose that Assumption 2.1 holds. Let \( (Q^\lambda, Z^\lambda) \) be the unique solution to BSDE (2.6). Then, the value of the stochastic differential game (4.1) exists and equals the value \( v^\lambda \) of the constrained risk-sensitive Dynkin game (2.5), i.e.
\[
v^{\lambda,SDG} = v^{\lambda,SDG} = \underline{v}^{\lambda,SDG} = \overline{v}^{\lambda,SDG} = v^\lambda = g^{-1}(Q^\lambda_0).
\]
Moreover, the optimal pair of controls is given by
\[
a^*_t = \lambda^{(1)} \mathbb{1}_{\{Q^\lambda_t \geq U_t\}}, \quad b^*_t = \lambda^{(2)} \mathbb{1}_{\{Q^\lambda_t \leq L_t\}}
\]
for \( t \geq 0 \).

**Proof.** Following the similar arguments to the proof of Lemma 3.3 it can be shown that, for any pair of controls \( (a, b) \in \mathcal{A}(\lambda^{(1)}) \times \mathcal{B}(\lambda^{(2)}) \), \( \mathbb{E}[J(a, b)] = V^\lambda_0(a, b) \), where the latter is the first component of the unique solution to the following BSDE with a random terminal time \( T \):
\[
V^\lambda_{t\wedge T}(a, b) = \xi + \int_{t \wedge T}^T \left[ a_u (Q^\lambda_u - V^\lambda_u(a, b)) + b_u (T_u - V^\lambda_u(a, b)) - r V^\lambda_u(a, b) \right] du - \int_{t \wedge T}^T Z^\lambda_u(a, b) dW_u,
\]
for \( t \geq 0 \). On the other hand, recall that \( \overline{Q}^\lambda \) is the first component of the solution to BSDE (2.6):
\[
\overline{Q}^\lambda_{t\wedge T} = \xi + \int_{t \wedge T}^T \left[ -\lambda^{(1)} (Q^\lambda_u - \overline{Q}^\lambda_u)^+ + \lambda^{(2)} (\overline{L}_u - Q^\lambda_u)^+ - r Q^\lambda_u \right] du - \int_{t \wedge T}^T \overline{Z}^\lambda_u dW_u,
\]
for \( t \geq 0 \).
for $t \geq 0$. By letting $b^*_t = \lambda(2) \mathbb{1}_{\{Q^\lambda_t \leq t\}}$, we obtain the inequality

$$
-\lambda^{(1)}(Q^\lambda_u - \mathcal{U}^\lambda_u)^+ + \lambda^{(2)}(\mathcal{L}^\lambda_u - Q^\lambda_u)^+ - rQ^\lambda_u \leq a_u(\mathcal{U}^\lambda_u - Q^\lambda_u) + b^*_u(\mathcal{L}^\lambda_u - Q^\lambda_u) - rQ^\lambda_u
$$

holds for any control $a \in \mathcal{A}(\lambda^{(1)})$, and thus, the BSDE comparison result (see Corollary 4.4.2 in [S]) yields that

$$
Q^\lambda_{t \wedge T} \leq V^\lambda_{t \wedge T}(a, b^*)
$$

for $t \geq 0$ and any control $a \in \mathcal{A}(\lambda^{(1)})$. Similarly, by letting $a^*_t = \lambda^{(1)} \mathbb{1}_{\{Q^\lambda_t \geq U^\lambda_t\}}$, we obtain

$$
Q^\lambda_{t \wedge T} \geq V^\lambda_{t \wedge T}(a^*, b)
$$

for $t \geq 0$ and any control $b \in \mathcal{B}(\lambda^{(2)})$, and by letting $a^*_t = \lambda^{(1)} \mathbb{1}_{\{Q^\lambda_t \geq U^\lambda_t\}}$ and $b^*_t = \lambda^{(2)} \mathbb{1}_{\{Q^\lambda_t \leq L^\lambda_t\}}$, we obtain the equality

$$
Q^\lambda_{t \wedge T} = V^\lambda_{t \wedge T}(a^*, b^*).
$$

It follows from (4.4) that

$$
g^{-1}(Q^\lambda_0) \leq \inf_{a \in \mathcal{A}(\lambda^{(1)})} g^{-1}(V^\lambda(a, b^*)) = \inf_{a \in \mathcal{A}(\lambda^{(1)})} g^{-1}(\mathbb{E}[J(a, b^*)])
$$

$$
\leq \sup_{b \in \mathcal{B}(\lambda^{(2)})} \inf_{a \in \mathcal{A}(\lambda^{(1)})} g^{-1}(\mathbb{E}[J(a, b)]) = \lambda^{SDG}.
$$

Likewise, (4.5) yields that $g^{-1}(Q^\lambda_0) \geq \lambda^{SDG}$. Hence, it follows from $\lambda^{SDG} \geq \lambda^{SDG}$ that (4.2) holds. As a direct consequence of (4.4)-(4.6), we can obtain $(a^*, b^*)$ in (4.3) is an optimal pair of controls. $\blacksquare$

5 Examples

5.1 Example I: Constrained risk-neutral Dynkin games

As the first example, we take the risk-sensitive function to be $g(x) = x$. This means both players are risk neutral and, therefore, the corresponding games are called constrained risk-neutral Dynkin games. In this case, the cost functional in (2.2) is evaluated under the linear expectation $\mathbb{E}$:

$$
\mathbb{E}[R(\sigma, \tau)] = \mathbb{E}[R(\sigma, \tau)]
$$

with the payoff functional $R(\sigma, \tau)$ given by (2.4). Hence, the upper and lower values of the constrained risk-neutral Dynkin game are defined as

$$
\lambda^{RN} = \inf_{\sigma \in \mathcal{R}_1^{(1)}} \sup_{\tau \in \mathcal{R}_1^{(2)}} \mathbb{E}[R(\sigma, \tau)], \quad \lambda^{RN} = \sup_{\tau \in \mathcal{R}_1^{(2)}} \inf_{\sigma \in \mathcal{R}_1^{(1)}} \mathbb{E}[R(\sigma, \tau)].
$$

The game (5.1) is said to have value $v^{RN}$ if $v^{RN} = \lambda^{RN} = \lambda^{RN}$, and $(\sigma^{RN}, \tau^{RN}) \in \mathcal{R}_1^{(1)} \times \mathcal{R}_1^{(2)}$ is called an optimal stopping strategy of the game if

$$
\mathbb{E}[R(\sigma^{RN}, \tau)] \leq \mathbb{E}[R(\sigma^{RN}, \tau^{RN})] \leq \mathbb{E}[R(\sigma, \tau^{RN})]
$$

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for every \((\sigma, \tau) \in \mathcal{R}^{(1)}_1 \times \mathcal{R}^{(2)}_1\).

Recall
\[
Q^\lambda_t = Q^\lambda_t - \int_0^{t \wedge T} e^{-r(u-T)} f_u \, du
\]
in (2.11), where \((Q^\lambda, Z^\lambda)\) is the unique solution to the characterizing BSDE (2.6). Thus, we deduce the so-called penalized BSDE with double obstacles on a random horizon \([0, T]\) (see [7] for the case of a fixed terminal time \(T\)),

\[
Q^\lambda_{t \wedge T} = \xi + \int_{t \wedge T}^T \left[ f_s - \lambda^{(1)} (Q^\lambda_s - U^s) + \lambda^{(2)} (L^s - Q^\lambda_s) + rQ^\lambda_s \right] \, ds - \int_{t \wedge T}^T Z^\lambda_s \, dW_s,
\]
and
\[
Q^\lambda_t = \xi - \int_0^T e^{-r(u-T)} f_u \, du = \xi \quad \text{for} \ t \geq T.
\]

**Assumption 5.1** The risk-sensitive function \(g(x) = x\). Moreover, (i) when \(T\) is an unbounded stopping time, \(f, L, U\) and \(\xi\) are all bounded; (ii) when \(T\) is a bounded stopping time, \(f \in C^0\), \(L \in C_B^2\), \(U \in C_B^2\) and \(\xi \in C_4^2\).

Note that the above assumption implies Assumption 2.1 and, therefore, it follows from Theorem 2.3 that BSDE (5.2) admits a unique solution \((Q^\lambda, Z^\lambda)\). Moreover, the value of the constrained risk-neutral Dynkin game (5.1) exists and is given by

\[
u^{\lambda, RN} = v^{\lambda, RN} = \xi^{\lambda, RN} = Q^\lambda_0.
\]

The optimal stopping strategy is given by

\[
\left\{
\begin{array}{l}
\sigma^{*, RN} = \inf\{T^{(1)}_N \geq T_1^{(1)} : Q^\lambda_{T^{(1)}_N} \geq U^{(1)}_{T^{(1)}_N} \} \wedge T^{(1)}_{M_1}; \\
\tau^{*, RN} = \inf\{T^{(2)}_N \geq T_1^{(2)} : Q^\lambda_{T^{(2)}_N} \leq L^{(2)}_{T^{(2)}_N} \} \wedge T^{(2)}_{M_2};
\end{array}
\right.
\]

**Remark 5.2** The special case \(g(x) = x\) generalizes the results obtained in [23] and [24]. To be more specific, when \(\lambda^{(1)} = 0\) (resp. \(\lambda^{(2)} = 0\)), Player I (resp. II) is with a zero intensity control set and is never allowed to stop, so the value of the constrained risk-neutral Dynkin game (5.1) equals to the value of the one-player optimal stopping problem with Poisson intervention times introduced in [23]. On the other hand, when the two intensities coincide, i.e. \(\lambda^{(1)} = \lambda^{(2)}\), the value of the constrained risk-neutral Dynkin game (5.1) equals to the value of the Dynkin game with Poisson intervention times introduced in [25].

### 5.2 Example II: Constrained Dynkin games with exponential utility

The second example for the risk-sensitive function \(g\) is an exponential utility: \(g(x) = -e^{-\gamma x}\) for \(\gamma > 0\). In this case, the cost functional in (2.2) becomes

\[
\mathbb{E}[R(\sigma, \tau)] = -\frac{1}{\gamma} \ln \mathbb{E}[\exp(-\gamma R(\sigma, \tau))]
\]
with the payoff functional \( R(\sigma, \tau) \) given by \([24]\). Hence, the upper and lower values of the constrained risk-sensitive Dynkin game are defined as

\[
\underline{v}^{\lambda,EU} = \inf_{\sigma \in R_1^{(1)}} \sup_{\tau \in R_1^{(2)}} -\frac{1}{\gamma} \ln E[\exp(-\gamma R(\sigma, \tau))], \quad (5.3)
\]

\[
\bar{v}^{\lambda,EU} = \sup_{\sigma \in R_1^{(1)}} \inf_{\tau \in R_1^{(2)}} -\frac{1}{\gamma} \ln E[\exp(-\gamma R(\sigma, \tau))]. \quad (5.4)
\]

The game \((5.3) - (5.4)\) is said to have value \( v^{\lambda,EU} \) if \( v^{\lambda,EU} = \underline{v}^{\lambda,EU} = \bar{v}^{\lambda,EU} \), and \((\sigma^{*,EU}, \tau^{*,EU}) \in R_1^{(1)} \times R_1^{(2)}\) is called an optimal stopping strategy of the game if

\[
\mathbb{E} [R(\sigma^{*,EU}, \tau)] \leq \mathbb{E} [R(\sigma^{*,EU}, \tau^{*,EU})] \leq \mathbb{E} [R(\sigma, \tau^{*,EU})]
\]

for every \((\sigma, \tau) \in R_1^{(1)} \times R_1^{(2)}\).

Recall

\[
Q_0^{\lambda} = -\frac{1}{\gamma} e^{\lambda(T-t)} \ln(-e^{-\gamma (T-t)} \tilde{Q}_0^{\lambda}) - \int_0^{T\wedge T} e^{-\gamma (u-T)} f_u du
\]

in \([21]\), where \((\tilde{Q}^{\lambda}, Z^{\lambda})\) is the unique solution to the characterizing BSDE \([26]\). Thus, we deduce the following BSDE with quadratic growth on a random horizon \([0, T]\) (see \([18]\) for the case of a fixed maturity \(T\)):

\[
Q_{t\wedge T}^{\lambda} = \xi + \int_{t\wedge T}^{T} \left[ f_u - \frac{\lambda(1)}{\gamma} e^{\lambda(T-u)} (e^{\gamma (T-u)} Q_u^{\lambda} - e^{-\gamma T} U_u) - 1)^+ + \frac{\lambda(2)}{\gamma} e^{\lambda(T-u)} (1 - e^{\gamma (T-u)} Q_u^{\lambda} - e^{-\gamma T} U_u)^+ \\
- \frac{\gamma}{2} e^{-\gamma t} ||Z_u^{\lambda}||^2 \right] du - \int_{t\wedge T}^{T} Z_u^{\lambda} dW_u,
\]

for \( t \geq 0 \), where \( Z_u^{\lambda} = -e^{-\gamma u} Z_u^{\lambda}/(\gamma \tilde{Q}_u^{\lambda}), u \in [0, T] \). Note that, for \( t \geq T \),

\[
Q_T^{\lambda} = -\frac{1}{\gamma} e^{\lambda T} \ln(-e^{-\gamma T \xi}) - \int_0^{T} e^{-\gamma (u-T)} f_u du = \xi.
\]

**Assumption 5.3** The risk-sensitive function \( g(x) = -e^{-\gamma x} \) for \( \gamma > 0 \), and \( f, L, U \) and \( \xi \) are all bounded.

Note that the above assumption implies Assumption \([24]\) and, therefore, it follows from Theorem \([23]\) that BSDE \([5.5]\) admits a unique solution \((Q^{\lambda}, Z^{\lambda})\). Moreover, the value of the constrained risk-sensitive Dynkin game \([5.3]-[5.4]\) exists and is given by

\[
\underline{v}^{\lambda,EU} = \underline{v}^{\lambda,EU} = \bar{v}^{\lambda,EU} = Q_0^{\lambda}.
\]

The optimal stopping strategy is given by

\[
\begin{align*}
\sigma^{*,EU} &= \inf \{ T^{(1)}_N \geq T^{(1)}_M : Q^{\lambda_{T^{(1)}_N,M^{(1)}_N}} \geq U_{T^{(1)}_M} \} \land T^{(1)}_M, \\
\tau^{*,EU} &= \inf \{ T^{(2)}_N \geq T^{(2)}_M : Q^{\lambda_{T^{(2)}_N,M^{(2)}_N}} \leq L_{T^{(2)}_M} \} \land T^{(2)}_M.
\end{align*}
\]

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6 Conclusions

In this paper, we have solved a new class of Dynkin games with a general risk-sensitive criterion function $g$ and two heterogeneous Poisson arrival times as the permitted stopping time strategies for the two players. Moreover, we have made a connection with a class of stochastic differential games via the so-called randomized stopping technique.

The approach and the results herein may be extended in various directions. First, one may consider stochastic intensity models, an undoubtedly important case since the two players’ signal times may affect each other’s intensities. For example, for $i \in \{1,2\}$, if the player $i$’s first signal time $T_1^{(i)}$ occurs, it will have an impact (either positive or negative) on the other player $(3-i)$’s intensity:

$$\lambda_t^{(1)} = \lambda^{(1)} + \lambda^{(1)} \mathbb{1}_{\{T_1^{(2)} \leq t\}}, \quad \lambda_t^{(2)} = \lambda^{(2)} + \lambda^{(2)} \mathbb{1}_{\{T_1^{(1)} \leq t\}},$$

for some constants $\lambda^{(i)}, \lambda^{(i)}$ such that the process $(\lambda_t^{(i)})_{t \geq 0}$ is always nonnegative. However, various nontrivial technical difficulties arise. In particular, the resulting characterizing BSDEs will become a family of recursive equations, whose solvability is far from clear yet.

Second, one may consider that the two players have different attitudes towards risks and are associated with different information sets. For example, one player is risk-neutral with $g^{(1)}(x) = x$ and the other has an exponential utility with $g^{(2)}(x) = -e^{-\gamma x}$. This leads to heterogeneous payoff functionals and, therefore a nonzero-sum constrained Dynkin game arises. The corresponding characterizing equations will become a BSDE system. Both extensions will be left for the future research.

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