Boundary Conformal Field Theory
and Fusion Ring Representations

TERRY GANNON

Department of Mathematical Sciences, University of Alberta
Edmonton, Canada, T6G 2G1
e-mail: tgannon@math.ualberta.ca

Abstract
To an RCFT corresponds two combinatorial structures: the amplitude of a torus (the
1-loop partition function of a closed string, sometimes called a modular invariant), and
a representation of the fusion ring (called a NIM-rep or equivalently a fusion graph, and
closely related to the 1-loop partition function of an open string). In this paper we develop
some basic theory of NIM-reps, obtain several new NIM-rep classifications, and compare
them with the corresponding modular invariant classifications. Among other things, we
make the following fairly disturbing observation: there are infinitely many (WZW) modular
invariants which do not correspond to any NIM-rep. The resolution could be that those
modular invariants are physically sick. Is classifying modular invariants really the right
 thing to do? For current algebras, the answer seems to be: Usually but not always. For
finite groups à la Dijkgraaf-Vafa-Verlinde-Verlinde, the answer seems to be: Rarely.

1. Introduction
For many reasons, not the least of which is open string theory, we are interested in
boundary conformal field theory. Although it has apparently never been established that
bulk RCFT necessarily requires for its consistency that there be a compatible and complete
system of boundary conditions, the conventional wisdom seems to be that otherwise the
RCFT would have sick operator product expansions. In any case, a boundary CFT would
seem to be a relatively accessible halfway-point between constructing the full CFT from a
chiral CFT (= vertex operator algebra). This paper explores the relation between boundary
and bulk CFT, by comparing the classification of modular invariants with NIM-reps, and
in so doing it probes this ‘conventional wisdom’.

Cardy [1] was the first to explain how conformally invariant boundary conditions in
CFT are related to fusion coefficients. In particular, given a bulk CFT and a choice of
(not necessarily maximal) chiral algebra, consider the set of (conformally invariant)
boundary conditions which don’t break the chiral symmetry. These should be spanned by
the appropriate Ishibashi states $|\mu\rangle$, labelled by the spin-zero primary fields $\phi(\mu, \overline{\mu})$ in
the theory. We know [2] these boundary states need not be linearly independent, but we should
be able to find a (unique) $\mathbb{Z}_2$-basis $|x\rangle \in \mathcal{B}$ for them, equal in number to the Ishibashi
states. Then the 1-loop vacuum amplitude $Z_{x}|y\rangle$, where the two edges of the cylinder are
decorated with boundary conditions $|x\rangle, |y\rangle \in \mathcal{B}$, can be expanded in terms of the chiral characters $\chi_\lambda$:

$$Z_x|y\rangle = \sum_{\lambda} N_{\lambda x}^{\lambda y} \chi_\lambda$$

Cardy explained that these coefficients $N_{\lambda x}^{\lambda y}$ define a representation of the chiral fusion ring with nonnegative integer matrices. Strictly speaking, Cardy only considered the diagonal theory given by the modular invariant partition function $Z = \sum_\mu |\chi_\mu|^2$. The more general theory has been developed by e.g. Pradisi et al [3] (see e.g. [4] for a good review), Fuchs–Schweigert (see e.g. [5]), and Behrend et al (see e.g. [6]). We will review and axiomatise the combinatorial essence of this theory below in Section 3, under the name NIM-reps.

In a remarkable paper, Di Francesco–Zuber [7] sought a generalisation of the A-D-E pattern of $\hat{sl}(2)$ modular invariants, by assigning graphs to RCFT. They were (largely empirically) led to introduce what we now will call fusion graphs. Over the years the definition was refined, and their relations to the lattice models of statistical mechanics, structure constants in CFT, etc were clarified (see the enchanting review in [8]). In particular, their connection with NIM-reps is now fully understood (see e.g. [6]).

The torus partition function (=modular invariant) and the cylinder partition function (=NIM-rep) of an RCFT should be compatible. Roughly, the eigenvalues of the NIM-rep matrices $(N_\lambda)_{xy} = N_{\lambda x}^{\lambda y}$ should be labeled by the Ishibashi states, and the Ishibashi states should be labeled by the spin-0 primaries, i.e. the diagonal ($\lambda = \mu$) terms in the modular invariant $Z = \sum_{\lambda,\mu} M_{\lambda\mu} \chi_\lambda \chi_\mu^*$. (Strictly speaking, all this assumes a choice of ‘pairing’ or ‘gluing automorphism’ $\omega$ — see §2.2 below.)

We call a modular invariant NIMmed if it has a compatible NIM-rep; otherwise we call it NIM-less. In this way, we can use NIM-reps to probe lists of modular invariants. After all, the definition (see §2.2) of modular invariants isolates only certain features of RCFT, and it is not at all obvious that classifying them is really the right thing to do.

NIM-reps and modular invariants, and their compatibility condition, also appear very naturally in the context of braided subfactor theory (see e.g. [9,10] for reviews of this remarkable picture, due originally to Ocneanu). The term ‘NIM-rep’ [10] is short for ‘nonnegative integer matrix representation’.

Even if we restrict attention to the current (=affine Kac-Moody) algebras, i.e. WZW theories, very little is known about NIM-rep classifications. The $\hat{sl}(2)$ theories at all levels $k$, and all $\hat{sl}(n)$ at level 1, are all that have been done [7,6], although Ocneanu [11] has announced a classification of the subset of $\hat{sl}(3)$ and $\hat{sl}(4)$ NIM-reps (any level) of relevance to subfactors. Although there isn’t a perfect match with the corresponding modular invariant classifications, the relation between what superficially seem to be distinct mathematical problems is remarkable.

Our two main results are:

- We classify the NIM-reps for all $\hat{sl}(n)$ and $\hat{so}(n)$ at level 2. Those of $\hat{sl}(n)$ match up well with the corresponding modular invariant classification; those of $\hat{so}(n)$ dramatically do not, and in fact most $\hat{so}(n)$ level 2 modular invariants are NIM-less.

- We develop the basic theory of NIM-reps (see especially Theorem 3 below), and find striking similarities with modular invariants (compare Theorem 1).
In §3.4 we discuss the rationality and nonnegativity of the coefficients $M_{\nu\lambda\mu}$ of the Pasquier algebra and of the dual Pasquier algebra $\hat{N}_{xy}^z$. In §4 we give the level 1 NIM-rep classifications for all current algebras. We relegate the (unpleasant) proofs of the level 2 classifications and Theorem 3 to the Appendix.

The reader less interested in the details may wish to jump to §6, where we find two simple NIM-less modular invariants, then to §7 where we explain using the example of $\hat{\mathfrak{sl}}(3)$ level 8 how the results of §3.3 come together to yield NIM-rep classifications, and finally move to the conclusion, §8, where we give some concluding thoughts and speculations.

What do all these NIM-less modular invariants tell us? One possibility is that this picture of the relation of boundary and bulk CFT is too naive — e.g. it is related to the assumption of the completeness of boundaries first raised in [3]. Another possibility is that there are infinitely many modular invariants which cannot be realised as the torus partition function of a CFT.

What about NIM-reps for higher-rank algebras and levels? We get good control over the eigenvalues of the fusion graphs. Much more difficult is, given these eigenvalues, to draw the possible fusion graphs. We know (proved below) that there will only be finitely many of these, but based on considerations given in (6) in the concluding section, we expect that number to be typically quite large. The classes presently worked out are atypical, because the critical Perron-Frobenius eigenvalues involved are $\leq 2$. As the eigenvalues rise, we expect the number of NIM-reps to grow out of control. In other words, we expect that classifying NIM-reps is probably hopeless (and pointless) for all but the smallest ranks and levels.

On the other hand, [12] suggests that the modular invariant situation for $\hat{\mathfrak{so}}(n)$ level 2 is quite atypical, and that we can expect that all modular invariants for most current algebras $X_{r,k}$ are related to Dynkin diagram symmetries. The corresponding NIM-reps would then be fairly well understood (see e.g. [13,14] and references therein); in particular they are probably all NIMmed. The situation however will probably be much worse for the modular invariants coming from other (i.e. non-WZW) chiral algebras, e.g. the untwisted sector in holomorphic orbifolds by finite groups [15] — see e.g. §6.

There is a tendency in the literature to focus only on $\hat{\mathfrak{sl}}(n)$ (although [16] briefly discussed NIM-reps for $\hat{G}_2$ level $k$). This perhaps is a mistake — $\hat{\mathfrak{sl}}(n)$ is very special, and this limited perspective leads to incorrect intuitions as to characteristic WZW or RCFT behaviour. We see that here: for instance the NIM-rep vrs modular invariant situation for $\hat{\mathfrak{so}}(n)$ level 2 is quite remarkable, and considerably more interesting than that for $\hat{\mathfrak{sl}}(n)$ level 2.

2. Review: Fusion rings and modular invariants

2.1. Modular data of the RCFT.

The material of this subsection is discussed in more detail in the reviews [17,18].

As is well-known, the RCFT characters $\chi_\lambda(q)$ yield a finite-dimensional unitary representation of the modular group $\text{SL}_2(\mathbb{Z})$, given by the natural action of $\text{SL}_2(\mathbb{Z})$ on $\tau = \frac{1}{2\pi i} \ln q$. Denote by $S$ and $T$ the matrices associated to \[ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \] and \[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]. Then $T$
is diagonal, and $S$ is symmetric. The rows and columns of $S$ and $T$ are parametrised by the primary fields $\lambda \in P_+$. One of these, the ‘vacuum’ 0, is distinguished.

In this paper we will be primarily interested in the data coming from current algebras $\hat{\mathfrak{g}}$ (simple), i.e. associated to Wess-Zumino-Witten models. However, unless otherwise stated, everything here holds for arbitrary RCFT.

We will assume for convenience that $S_\lambda 0 > 0$ — this holds in particular for unitary RCFTs, such as the WZW models. The changes required for nonunitary RCFT consist mainly of replacing some appearances of the vacuum with the unique primary $o \in P_+$ with minimal conformal weight. Then $S_\lambda o > 0$. In a unitary theory, we have $o = 0$. The ratio $S_\lambda o / S_0 0$ is called the quantum-dimension of $\lambda$, and plays a central role.

The matrix $S^2$ is a permutation matrix $C$, called charge-conjugation. It obeys $C 0 = 0$, $T C \lambda, \mu = T \lambda, \mu$, and corresponds to complex conjugation:

$$S_{C \lambda, \mu} = S_{\lambda, C \mu} = S^*_{\lambda, \mu} \quad (2.1)$$

The fusion coefficients $N^\nu_{\lambda \mu}$ of the theory are given by Verlinde’s formula [19]:

$$N^\nu_{\lambda \mu} = \sum_{\gamma \in P_+} \frac{S_{\lambda \gamma} S_{\mu \gamma} S^*_{\nu \gamma}}{S_{0 \gamma}} \in \mathbb{Z}_{\geq} := \{0, 1, 2, \ldots\} \quad (2.2)$$

Let $N_\lambda$ denote the fusion matrix, i.e. the matrix with entries $(N_\lambda)_{\mu \nu} = N^\nu_{\lambda \mu}$. Then $N_0 = I$, $N_{C \lambda} = N^t_{\lambda}$, and

$$N_\lambda N_\mu = \sum_{\nu \in P_+} N^\nu_{\lambda \mu} N_{\nu} \quad (2.3)$$

We use the term modular data for any matrices $S$ and $T$ obeying these conditions. The ring with preferred basis $P_+$ and structure constants $N^\nu_{\lambda \mu}$ is called the fusion ring. For example, modular data and a fusion ring exist for every choice of current algebra $\hat{\mathfrak{g}} = X^{(1)}_r$ and positive integer $k$ (called the level) — of course this is precisely what arises in the WZW models. At times we will abbreviate this to $X_{r,k}$. The primaries $\lambda \in P_+$ for this WZW modular data consist of the level $k$ integrable highest weights $\lambda = \lambda_1 \Lambda_1 + \cdots + \lambda_r \Lambda_r$, where the basis vectors $\Lambda_i$ are called fundamental weights. See e.g. Ch.13 of [20] for more details. Explicit formulas for $S_{\lambda \mu}$ are given in [20]; see also [21].

The quantum-dimensions in (unitary) RCFT obey $S_{\lambda 0} / S_{0 0} \geq 1$. When it equals 1, $\lambda$ is called a simple-current [22]. Then $N_\lambda$ will be a permutation matrix, corresponding to a permutation $J$ of $P_+$, and there will be a phase $Q_J : P_+ \to \mathbb{Q}$ such that

$$S_{J \mu, \nu} = e^{2\pi i Q_J(\nu)} S_{\mu \nu} \quad (2.4)$$

The simple-currents form an abelian group, under composition of permutations. Note that

$$N^J_{J', \gamma} \nu = N^\gamma_{\mu \nu} \quad (2.5a)$$

$$N^C_{J, \gamma} \nu = N^\gamma_{\mu \nu} \quad (2.5b)$$

for any simple-currents $J, J'$, where $C$ as usual is charge-conjugation.
For example, for $A_{1,k}$ we may take $P_k = \{0, 1, \ldots, k\}$ (the value of the Dynkin label $\lambda_1$), and then the $S$ matrix is $S_{ab} = \sqrt{\frac{2}{k+2}} \sin(\pi \frac{(a+1)(b+1)}{k+2})$. Charge-conjugation $C$ is trivial here, but $j = k$ is a simple-current corresponding to permutation $Ja = k - a$ and phase $Q_j(a) = a/2$. The fusion coefficients are given by

$$N^c_{ab} = \begin{cases} 1 & \text{if } c \equiv a + b \pmod{2} \text{ and } |a - b| \leq c \leq \min\{a + b, 2k - a - b\} \\ 0 & \text{otherwise} \end{cases}$$

Write $\xi_N$ for the root of unity $\exp[2\pi i/N]$. A fundamental symmetry of modular data is a certain generalisation of charge-conjugation. For any RCFT, the entries $S_{\lambda\mu}$ are sums of roots of unity $\xi_N^m$, all divided by a common integer $L$. For example for $\text{sl}(n)_k$ we can take $N = 4n(n + k + 1)$. We say that the entries $S_{\lambda\mu}$ lie in the cyclotomic number field $\mathbb{Q}[\xi_N]$. The automorphisms $\sigma \in \text{Gal}(\mathbb{Q}[\xi_N]/\mathbb{Q})$ of this field preserve by definition both multiplication and addition and fix the rational numbers; they are parametrised by an integer $\ell$ coprime to $N$ (more precisely, the action of $\sigma_\ell$ is uniquely determined by the relation $\sigma_\ell(\xi_N) = \xi_N^m$, so really $\ell$ is defined modulo $N$). To each such integer $\ell$, i.e. each automorphism $\sigma_\ell$, there is a permutation $\sigma_\ell$ of $P_+$ and a choice of signs $\epsilon_\ell(\lambda) = \pm 1$, such that

$$\sigma_\ell(S_{\lambda\mu}) = \epsilon_\ell(\lambda) S_{\sigma_\ell(\lambda),\mu} = \epsilon_\ell(\mu) S_{\lambda,\sigma_\ell(\mu)} \tag{2.6}$$

This Galois symmetry may sound complicated, but that could be due only to its unfamiliarity. It plays a central role in the theory of modular data, modular invariants, and NIM-reps (see e.g. §7 below), and makes accessible problems which have no right to be so. Algorithms for this Galois symmetry, for the current algebras, are explicitly worked out in [21].

An important ingredient of the theory is that of fusion-generators. We call $\Gamma = \{\gamma^{(1)}, \ldots, \gamma^{(g)}\} \subset P_+$ a fusion-generator if to any $\lambda \in P_+$ there is a $g$-variable polynomial $P_\lambda(x_1, \ldots, x_g)$ such that the fusion matrices obey

$$N_\lambda = P_\lambda(N_{\gamma^{(1)}}, \ldots, N_{\gamma^{(g)}})$$

or equivalently, for any $\lambda, \mu \in P_+$,

$$S_{\lambda\mu} / S_{0\mu} = P_\lambda(\frac{S_{\gamma^{(1)}\mu}}{S_{0\mu}}, \ldots, \frac{S_{\gamma^{(g)}\mu}}{S_{0\mu}}) \tag{2.7}$$

This says that $\gamma^{(1)}, \ldots, \gamma^{(g)}$ generate the fusion ring, and also (we will see) the NIM-reps.

One of the reasons fusion rings for the current algebras are relatively tractable is the existence of small fusion-generators. In particular, because we know that any Lie character for $X_r$ can be written as a polynomial in the fundamental weights $\text{ch}_\Lambda_1, \ldots, \text{ch}_\Lambda_r$, it is easy to show [24] that $\Gamma = \{\Lambda_1, \ldots, \Lambda_r\} \cap P_+$ is a fusion-generator for any $X_r^{(1)}$ level $k$. Smaller fusion-generators usually exist however. The question for $\text{sl}(n)_k$ has been studied quite thoroughly in [25]. For example, $\{\Lambda_1\}$ is a fusion-generator for $\text{sl}(n)_k$ iff both

(i) each prime divisor $p$ of $k + n$ satisfies $2p > \min\{n, k\}$, and

(ii) either $n$ divides $k$, or $\gcd(n, k) = 1$. 


More generally, for $\mathfrak{sl}(n)_k$ the following are always fusion-generators:

$$
\Gamma_\pm = \{ \Lambda_d | 2d \leq n \text{ and } d \text{ divides } k+n \}
$$

$$
\Gamma_\tau^\pm = \begin{cases} 
\{ \Lambda_d | 2d \leq k \text{ and } d \text{ divides } k+n \} & \text{when } k \text{ doesn’t divide } n \\
\{ k\Lambda_1, \Lambda_d | 2d \leq k \text{ and } d \text{ divides } k+n \} & \text{when } k \text{ divides } n
\end{cases}
$$

(Of course, the weight $k\Lambda_1$ in $\Gamma_\tau^\pm$ is a simple-current.) Examples are:

- $\Lambda_1$ is a fusion-generator for $\mathfrak{sl}(2)_k$ and $\mathfrak{sl}(3)_k$, for any level $k$;
- for $\mathfrak{sl}(4)_k$, $\Lambda_1$ is a fusion-generator when $k$ is odd, while both $\{ \Lambda_1, \Lambda_2 \}$ are needed when $k$ is even;
- $\Lambda_1$ is a fusion-generator for $\mathfrak{sl}(n)_1$ for any $n$; it’s also a fusion-generator for $\mathfrak{sl}(n)_2$ when $n$ is odd, while both $\{ \Lambda_1, 2\Lambda_1 \}$ are needed when $n$ is even.

2.2. Modular invariants and their exponents.

The one-loop vacuum-to-vacuum amplitude of a rational conformal field theory is the modular invariant partition function

$$
Z(q) = \sum_{\lambda,\mu \in P_+} M_{\lambda\mu} \chi_\lambda(q) \chi_\mu(q)^* \quad (2.8)
$$

**Definition 1.** By a modular invariant $M$ we mean a matrix with nonnegative integer entries and $M_{00} = 1$, obeying $MS = SM$ and $MT = TM$.

Two examples of modular invariants are $M = I$ and $M = C$ (of course these may be equal). It is known that for any choice of modular data, the number of modular invariants will be finite [26,10]. We identify the function $Z$ in (2.8) with its coefficient matrix $M$.

The coefficient matrix $M$ of an RCFT partition function is a modular invariant, but the converse need not be true. Also, different RCFTs can conceivably have the same modular invariant. *Is the classification of modular invariants the right thing to do? Is there actually a resemblance between the list of modular invariants, and the corresponding list of RCFTs? Or are we losing too much information and structure by classifying not the full RCFTs, but rather the much simpler modular invariants? We return to these questions in the concluding section, §8.*

We have a good understanding now of the modular invariant lists for the current algebras, at least for small rank and/or level. See [12,18] and references therein for the main results and appropriate literature.

The most famous modular invariant list is that of $\hat{\mathfrak{sl}}(2)$, due to Cappelli-Itzykson-Zuber [27]. The trivial modular invariant $M = I$ exists for all levels $k$; a simple-current invariant $M[J]$ (see (2.9) below) exists for all even $k$; and there are exceptionals at $k = 10, 16, 28$. For instance, the level 28 exceptional is

$$
Z_{28}(q) = |\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}|^2 + |\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}|^2
$$
Cappelli-Itzykson-Zuber noticed something remarkable about their list: it falls into the A-D-E pattern. Each of their modular invariants $M$ can be identified with the Dynkin diagram $G(M)$ of a finite-dimensional simply laced Lie algebra (these are the diagrams $A_n, D_n, E_n$ in Figure 1). The level of the modular invariant, plus 2, equals the Coxeter number $h$ of $G(M)$. Each number $1 \leq a \leq k + 1$ will be an exponent of $G(M)$ with multiplicity given by the diagonal entry $M_{a-1,a-1}$. The Coxeter number $h$ and exponents $m_i$ of the diagram $G$ are listed in Table 1. For instance, the modular invariant $Z_{28}$ given above corresponds to the $E_8$ Dynkin diagram: $28 + 2 = 30$, the Coxeter number of $E_8$; and the nonzero diagonal entries $M_{bb}$ of $M$ are at $b = 0, 6, 10, 12, 16, 18, 22, 28$, compared with the $E_8$ exponents $1, 7, 11, 13, 17, 19, 23, 29$ (all multiplicities being 1). Likewise, the $D_8$ Dynkin diagram has Coxeter number 14, and exponents $1, 3, 5, 7, 9, 11, 13$, and corresponds to the $sl(2)_{12}$ modular invariant

$$|\chi_0 + \chi_{12}|^2 + |\chi_2 + \chi_{10}|^2 + |\chi_4 + \chi_8|^2 + 2|\chi_6|^2$$

Table 1. Eigenvalues of graphs in Figure 1

| Graph   | Coxeter number $h$ | Exponents $m_i$ |
|---------|-------------------|-----------------|
| $A_n$, $n \geq 1$ | $n + 1$ | $1, 2, \ldots, n$ |
| $D_n$, $n \geq 4$ | $2n - 2$ | $1, 3, \ldots, 2n - 3, n - 1$ |
| $E_6$ | 12 | $1, 4, 5, 7, 8, 11$ |
| $E_7$ | 18 | $1, 5, 7, 9, 11, 13, 17$ |
| $E_8$ | 30 | $1, 7, 11, 13, 17, 19, 23, 29$ |
| $T_n$, $n \geq 1$ | $2n + 1$ | $1, 3, 5, \ldots, 2n - 1$ |

Because of that observation, [7,16] suggested the following general definition.

**Definition 2.** By the exponents of a modular invariant $M$, we mean the multi-set $E_M$ consisting of $M_{\lambda\lambda}$ copies of $\lambda$ for each $\lambda \in P_+$. (By a ‘multi-set’, we mean a set together with multiplicities, so $E_M \subset P_+ \times \mathbb{Z}_{\geq}$.)

In other words, the exponents are precisely the spin-0 primary fields in the theory (periodic sector). By analogy with the A-D-E classification for $\hat{sl}(2)$, we would like to assign graphs to a modular invariant in such a way that the eigenvalues of the graph (that is to say, the eigenvalues of its adjacency matrix) can be identified with the exponents of the modular invariant. We shall see next section that there is a natural way to do this: NIM-reps!

For example, $M = I$ has exponents $E_I = P_+$, while the exponents $E_C$ of $M = C$ are the self-conjugate primaries $\lambda = C\lambda$. In both $E_I$ and $E_C$, all multiplicities are 1, but simple-current invariants (see (2.9) below) can have arbitrarily large multiplicities.

It is merely a matter of convention whether $M_{\lambda,C\lambda} \neq 0$ or $M_{\lambda\lambda} \neq 0$ is taken as the definition of exponents — it has to do with the arbitrary choice of taking the holomorphic and antiholomorphic (i.e. left-moving and right-moving) chiral algebras to be isomorphic or anti-isomorphic. In the literature both choices can be found. We’ve taken them to be anti-isomorphic, hence our definition of spin-0 primaries.
Implicit in this discussion is the diagonal (i.e. identity) choice of ‘gluing automorphism’ \( \Omega \) [28] or ‘pairing’ \( \omega \) [5]. The pairing can be any permutation of \( P_+ \) which preserves fusions and conformal weights, so it must yield an ‘automorphism invariant’, i.e. a modular invariant \( M \) which is a permutation matrix: \( M_{\lambda\mu} = \delta_{\mu,\omega\lambda} \). For the current algebras, all possible pairings can be obtained from [24]. This pairing tells one how to identify left- and right-moving primaries. Definition 2 can now be generalised to the multi-set \( \mathcal{E}_\omega^\alpha \), where \( \lambda \) occurs with multiplicity \( M_{\lambda,\omega\lambda} \).

In this paper we will limit ourselves to the trivial (=identity) pairing \( \omega \). This is permitted for two reasons. First and most important, \( \mathcal{E}_\omega^\alpha = \mathcal{E}_M^\alpha \), where \( M_{\omega} \) is the modular invariant obtained by matrix multiplication. Second and quite intriguing, in practice it appears that the question of whether or not \( M \) is NIM-less (see \( \S \) 3.2 below) is independent of \( \omega \).

Consider a simple-current \( J \) with order \( n \) (so \( J^n = \text{id.} \)). Then we can find an integer \( R \) obeying \( T_{J_0,J_0} T_{0,0}^* = \exp[2\pi i R \frac{n-1}{2n}] \). Define a matrix \( M[J] \) by [22]

\[
M[J]_{\lambda\mu} = \sum_{\ell=1}^n \delta_{J^\ell,\lambda,\mu} \delta(Q_J(\lambda) + \frac{\ell}{2n}R) \tag{2.9}
\]

where \( \delta(x) = 1 \) if \( x \in \mathbb{Z} \) and 0 otherwise. For example, \( M[\text{id.}] = I \). The matrix \( M[J] \) will be a modular invariant iff \( T_{J_0,J_0} T_{0,0}^* \) is an \( n \)-th root of 1 (this is automatic if \( n \) is odd; for \( n \) even, it’s true iff \( R \) is even); it will in addition be a permutation matrix iff \( T_{J_0,J_0} T_{0,0}^* \) has order exactly \( n \). For example, for sl(2), \( R = k \) so \( M[J] \) is a modular invariant iff \( k \) is even, and when \( k/2 \) is odd it will in fact be a permutation matrix. The modular invariant \( M[J] \) for sl(2)\(_{12} \) is given above.

We call these modular invariants simple-current invariants. This construction can be generalised somewhat when the simple-current group isn’t cyclic, but (2.9) is good enough here. For any current algebra, at any sufficiently large level \( k \), it appears that the only modular invariants are simple-current invariants and their product with \( C \) (except for the algebras so(4\(_n \)), whose Dynkin symmetries permit (2.9) to be slightly generalised, and which have other ‘conjugations’ \( C_i \neq C \)).

We’ll end this subsection by establishing some of the basic symmetries of modular invariants and their exponents. First, note that \( MC = CM \) (since \( C = S^2 \)), so \( \lambda \) and \( C\lambda \) always appear in \( \mathcal{E}_M \) with equal multiplicity. More generally, the Galois symmetry (2.6) of modular data yields an important modular invariant symmetry [23]:

\[
M_{\lambda\mu} = M_{\sigma(\lambda),\sigma(\mu)} \tag{2.10a}
\]

\[
M_{\lambda\mu} \neq 0 \quad \implies \quad \epsilon_{\sigma}(\lambda) = \epsilon_{\sigma}(\mu) \tag{2.10b}
\]

for any Galois automorphism \( \sigma \), i.e. any \( \ell \) coprime to \( N \). One thing (2.10a) implies is that \( \lambda \) and \( \sigma(\lambda) \) will always have the same multiplicity in \( \mathcal{E}_M \). This is quite strong — for instance, the primaries 0, 6, 10, 12, 16, 18, 22, 28 for sl(2)\(_{28} \) all lie in the same Galois orbit, and indeed they all have the same multiplicity as exponents of the \( k = 28 \) exceptional modular invariant (just as they must for the other two \( k = 28 \) modular invariants).
The other fundamental symmetry of modular data is due to simple-currents. Let \( J, J' \) be simple-currents, and suppose that \( M_{J_0,J'_0} \neq 0 \). Then (see e.g. [18]) \( \forall \lambda, \mu \in P_+ \)

\[
M_{J_0,J'_0} = M_{\lambda,\mu} \\
M_{\lambda,\mu} \neq 0 \implies Q_J(\lambda) \equiv Q_{J'}(\mu) \pmod{1} \tag{2.11b}
\]

Thus by (2.11a), \( J \in E_M \) implies that all powers \( J^i \) are in \( E_M \), all with multiplicity 1, and also that \( \lambda \) and \( J\lambda \) have the same multiplicity in \( E_M \) for any \( \lambda \in P_+ \).

It is curious that the selection rules (2.10b) and (2.11b) seem to have no direct consequences for \( E_M \), although they are profoundly important in constraining off-diagonal entries of \( M \).

For later comparison, let’s collect some of the main results on the exponents of modular invariants:

**Theorem 1.** Choose any modular data. Let \( M \) be any modular invariant, and let \( E_M \) be its exponents, with \( m_\mu \) being the multiplicity \( M_{\lambda,\mu} \) in \( E_M \).

(i) There are only finitely many modular invariants for that choice of modular data. They obey the bound

\[
M_{\lambda,\mu} \leq \frac{S_{\lambda,0}}{S_{0,0}} \frac{S_{\mu,0}}{S_{0,0}}.
\]

(ii) \( m_0 = 1 \).

(iii) For any simple-current \( J, m_J = 0 \) or 1; if \( m_J = 1 \) then \( m_{J\lambda} = m_\lambda \) for all \( \lambda \in P_+ \).

(iv) For any Galois automorphism \( \sigma \) and any primary \( \lambda \in P_+ \), \( m_{\sigma(\lambda)} = m_\lambda \).

(v) For any symmetry \( \pi \) of the fusion coefficients, and any \( \lambda \in P_+ \), we get

\[
\sum_{\mu \in E_M} \frac{S_{\lambda,\mu}}{S_{0,\mu}} \in \mathbb{Z}_{\geq}
\]

In (v) we sum over \( E_M \) as a multi-set, i.e. each \( \mu \) appears \( m_\mu \) times. The sum in (v) will typically be very large. By a symmetry of the fusion coefficients, we mean a permutation \( \pi \) of \( P_+ \) for which \( N_{\lambda,\mu}^{\nu,\nu'} = N_{\pi\lambda,\pi\mu}^{\pi\nu,\pi\nu'} \) for all \( \lambda, \mu, \nu, \nu' \in P_+ \). It is equivalent to the existence of a permutation \( \pi' \) for which \( S_{\pi\lambda,\pi'\mu} = S_{\lambda,\mu} \) — all such symmetries for the current algebras were found in [29]. To prove (v), let \( \Pi \) and \( \Pi' \) be the corresponding permutation matrices. Then

\[
\sum_{\mu \in E_M} \frac{S_{\lambda,\mu}}{S_{0,\mu}} = \text{Tr}(M\Pi D_\lambda) = \text{Tr}(S^* S \Pi D_\lambda) = \text{Tr}(M\Pi' S D_\lambda S^*) = \text{Tr}(M\Pi' N_\lambda) \in \mathbb{Z}_{\geq}
\]

where \( D_\lambda \) is the diagonal matrix with entries \( S_{\lambda,\mu}/S_{0,\mu} \). Thm.1(v) seems to be new.

Thm.1 assumes all \( S_{\lambda,0} > 0 \). If instead we have a nonunitary RCFT, let \( o \in P_+ \) be as in §2.1. Then we can show \( m_o \geq 1 \). However the known proofs that there are finitely many modular invariants, all break down, as does the proof of (iii).

In §3.3 as well as paragraph (4) in §8, we are interested in when simple-currents are exponents. Consider any matrix \( M \) which commutes with the \( T \) of \( \text{sl}(2)_k \). That is,

\[
M_{ab} \neq 0 \quad \Rightarrow \quad (a + 1)^2 \equiv (b + 1)^2 \pmod{4(k + 2)}
\]
Thus $a$ is odd iff $b$ is odd, i.e. $Q_J(a) \equiv Q_J(b) \pmod{1}$, provided $M_{ab} \neq 0$. If $M$ is in addition a modular invariant, we get from this that

$$M_{J,J} = \sum_{a,b=0}^k S_{J,a} M_{ab} S_{J,b}^* = M_{00} = 1$$

Thus it is automatic for $sl(2)_k$ that $J \in \mathcal{E}_M$, for all modular invariants $M$.

This argument generalises considerably. The norms of the weights of $sl(n)_k$ satisfy

$$(\lambda|\lambda) \equiv Q_J(\lambda) - Q_J(\lambda)^2/n \pmod{2} \quad (2.12a)$$

where $Q_J(\lambda) = \sum_i i\lambda_i$, for the simple-current $J = k\Lambda_1$. For the basic calculation consider $sl(3)_k$. Then commutation of $M$ with $T$ implies from (2.12a) the selection rule

$$M_{\lambda\mu} \neq 0 \quad \Rightarrow \quad Q_J(\lambda)^2 \equiv Q_J(\mu)^2 \pmod{3} \quad (2.12b)$$

and hence

$$M_{J,J} + M_{J,J^{-1}} = \sum_{\lambda,\mu \in P_+} (\exp[2\pi i \frac{Q_J(\lambda) - Q_J(\mu)}{3}] + \exp[2\pi i \frac{Q_J(\lambda) + Q_J(\mu)}{3}]) S_{0\lambda M_{\lambda\mu} S_{0\mu}}$$

$$= \sum_{\lambda,\mu \in P_+} (\cos[2\pi \frac{Q_J(\lambda) - Q_J(\mu)}{3}] + \cos[2\pi \frac{Q_J(\lambda) + Q_J(\mu)}{3}]) S_{0\lambda M_{\lambda\mu} S_{0\mu}} \quad (2.12c)$$

where we use the reality of the LHS of (2.12c). But every term on the RHS of (2.12c) will be nonnegative: whenever $M_{\lambda\mu} \neq 0$, (2.12b) says that the sum of cosines in (2.12c) will either be $\frac{1}{2}$ or $2$. Thus (2.12c) will be positive, so either $M_{J,J} \neq 0$ or $M_{J,J^{-1}} \neq 0$. In other words, for any $sl(3)_k$ modular invariant $M$, either $J \in \mathcal{E}_M$ or $J \in \mathcal{E}_{MC} = \mathcal{E}_M^\omega$.

What we find, in this way, for an arbitrary current algebra, is:

**Proposition 2.** Let $M$ be a modular invariant for some current algebra $X_{r,k}$ and let $\mathcal{E}_M$ and $\mathcal{E}_M^\omega = \mathcal{E}_{MC}$ be the sets of exponents, where $C$ is charge-conjugation (2.1).

(i) For any $sl(2)_k$, so$(2n+1)_k = B_{n,k}$, $sp(2n)_k = C_{n,k}$, and $E_{\gamma,k}$, we have $J \in \mathcal{E}_M$.

(ii) For $sl(n)_k = A_{n-1,k}$ when $n < 8$, as well as $E_{6,k}$, either $J \in \mathcal{E}_M$ or $J \in \mathcal{E}_M^\omega$.

(iii) More generally, for $sl(n)_k = A_{n-1,k}$, define $n' = n$ or $n/2$ when $n$ is odd or even, resp., and similarly $k' = k$ or $k/2$ when $k$ is odd or even, resp. Define $a_i$ by the prime decomposition $n' = \prod p_i^{a_i}$, and let $s = \prod p_i^{[a_i/2]}$. Assume $\gcd(n',k')$ equals 1 or a power of a single prime. Then there exists an automorphism invariant (=‘pairing’) $\omega$ such that the simple-current $J^\omega = k\Lambda_1$ lies in $\mathcal{E}_M^\omega = \mathcal{E}_{M\omega}$.

(iv) For $so(2n)_k = D_{n,k}$, when $4$ doesn’t divide $n$, the simple-current $J_v = k\Lambda_1$ lies in $\mathcal{E}_M$.

The simple-current $J$ in (i)-(iii) is any generator of the corresponding (cyclic) groups of simple-currents. By ‘$[a_i/2]$’ here we mean to truncate to the nearest integer not greater than $a_i/2$. Note that $s$ is the largest number such that $s^2$ divides $n$ or (if $n$ is even) $n/2$. For instance $s = 1, 2, 3$ for $n = 4, 8, 18$. The automorphism invariants $\pi$ for $sl(n)_k$ are explicitly given in [24]. To prove (iii), use (2.12) to obtain $M_{J^\omega,J^\omega} = 1$ for some integer.
ℓ congruent to ±1 modulo an appropriate power of each prime \( p_i \); the automorphism invariants \( \omega \) (when they exist) can be seen to reverse those signs.

When instead distinct primes divide \( \gcd(n', k') \), a multiple \( s' \) of \( s \) will work in (iii): namely, choose any prime (say \( p_1 \)) dividing both \( n' \) and \( k' \), and define \( s' = p_1^{[a_1]} \prod_i p_i^{[a_i]/2} \prod_j p_j^{a_j} \) where the \( p_i \) don’t divide \( k' \), and the \( p_j \) (\( j \neq 1 \)) do.

More generally, suppose some weight \( \kappa \) has the property that, for any \( \lambda, \mu \), we have

\[
T_{\lambda\lambda} = T_{\mu\mu} \Rightarrow \text{ either } S_{\lambda\kappa} S_{\mu\kappa} \geq 0 \text{ or } S_{\lambda\kappa} S^*_{\mu\kappa} \geq 0 \quad (2.12d)
\]

Then, as in (2.12c), for any modular invariant \( M \) we must have either \( \kappa \in \mathcal{E}_M \) or \( \kappa \in \mathcal{E}_M^C \).

2.3. Quick review of matrix and graph theory.

We will write \( A^t \) for the transpose of \( A \). By a \( \mathbb{Z}_{\geq 0} \)-matrix we mean a matrix whose entries are nonnegative integers. Two \( n \times n \) matrices \( A \) and \( B \) are called equivalent if there is a permutation which, when applied simultaneously to the rows and columns of \( A \), yields \( B \) — i.e. \( B = \Pi A \Pi^t \). The direct sum \( A \oplus B \) of an \( n \times n \) matrix \( A \) and \( m \times m \) matrix \( B \) is the \( (n+m) \times (n+m) \) matrix \( \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \). A matrix \( M \) is called decomposable if it can be written in the form (i.e. is equivalent to) \( A \oplus B \), otherwise it is called indecomposable. A matrix \( N \) is called reducible if it is equivalent to a matrix of the form \( \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \) for submatrices \( A, B, C \) where \( B \neq 0 \). Fortunately, all of our matrices turn out to be irreducible.

For example, two \( n \times n \) permutation matrices \( \Pi \) and \( \Pi' \) are equivalent iff the corresponding permutations \( \pi \) and \( \pi' \) are conjugate in the symmetric group \( S_n \) (i.e. have the same numbers of disjoint 1-cycles, 2-cycles, 3-cycles, etc). They will be indecomposable iff they are transitive, i.e. iff they are equivalent to the \( n \times n \) matrix

\[
\Pi^{(n)} := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}
\]

(2.13)

in which case they will also be irreducible.

The eigentheory (i.e. the Perron-Frobenius theory — see e.g. [30]) of nonnegative matrices is fundamental to the study of NIM-reps. The basic result is that if \( A \) is a square matrix with nonnegative entries, then there is an eigenvector with nonnegative entries whose eigenvalue \( r(A) \) is nonnegative. The eigenvector(resp., -value) is called the Perron-Frobenius eigenvector(-value). This eigenvalue has the additional property that if \( s \) is any other eigenvalue of \( A \), then \( r(A) \geq |s| \). There are many other results in Perron-Frobenius theory that we’ll use, but we’ll recall them as needed.

The matrices with small \( r(A) \) have been classified (see especially [31] for \( r(A) < \sqrt{2 + \sqrt{5} \approx 2.058} \)), but unfortunately with a weaker notion of ‘equivalence’ than we would like. The moral of the story seems to be that these matrix classifications are very difficult,
and will be hopeless as \( r(A) \) gets much larger than 2; the only hope is to simultaneously impose other conditions on the matrix, e.g. some symmetries. Fortunately, we can always find other conditions obeyed by our matrices, besides the value of \( r \).

This is one of the places where ‘irreducibility’ simplifies things. For instance, \( \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \), \( \forall k \), are some of the indecomposable \( \mathbb{Z}_2 \)-matrices with maximum eigenvalue \( r(A) = 1 \), but the only \( 2 \times 2 \) indecomposable irreducible \( \mathbb{Z}_2 \)-matrix with \( r(A) = 1 \) is \( \Pi^{(2)} \).

An irreducible \( \mathbb{Z}_2 \)-matrix will have at most \( r(A) \) nonzero entries in each row, and so for small \( r(A) \) will be quite sparse. A sparse matrix is usually more conveniently depicted as a \((\text{multi-di})\text{graph}\). For example, in Lie theory a Dynkin diagram replaces the Cartan matrix. The same trick is used here, and is responsible for the beautiful pictures scattered throughout the NIM-rep literature (see e.g. [7,9]).

By a \textit{graph} we allow loops (i.e. an edge starting and ending at the same vertex), but its edges aren’t directed and aren’t multiple. A multi-digraph is the generalisation which allows multiple edges and directed edges. We assign a multi-digraph to a \( \mathbb{Z}_2 \)-matrix \( A \) as follows. For any \( i, j \), draw \( A_{ij} \) edges directed from vertex \( i \) to vertex \( j \). Replace each pair \( i \rightarrow j, j \rightarrow i \) of directed edges with an undirected one connecting \( i \) and \( j \) (except we never put arrows on loops).

We are very interested in the spectra of \((\text{multi-di})\text{graphs}\), i.e. the list of eigenvalues (with multiplicities) of the associated adjacency matrix. There has been a lot of work on this in recent years — see e.g. the readable book [32]. We will state the results as we need them. A major lesson from the theory: the eigenvalues usually won’t determine the graph. For example, the graphs \( D^{(1)}_4 \) and \( A^{(1)}_3 \cup A_1 \) have identical spectra.

| Graph | eigenvalues | range |
|-------|-------------|-------|
| \( A^{(1)}_n \), \( n \geq 1 \) | \( 2 \cos(2\pi k/(n + 1)) \) | \( 0 \leq k \leq n \) |
| \( D^{(1)}_n \), \( n \geq 4 \) | \( 0, 0, 2 \cos(\pi k/(n - 2)) \) | \( 0 \leq k \leq n - 2 \) |
| \( E^{(1)}_6 \) | \( \pm 2, \pm 1, \pm 1, 0 \) | |
| \( E^{(1)}_7 \) | \( \pm 2, \pm \sqrt{2}, \pm 1, 0, 0 \) | |
| \( E^{(1)}_8 \) | \( \pm 2, \pm 2 \cos(\pi/5), \pm 1, \pm 2 \cos(2\pi/5), 0 \) | |
| \( 0A^{(1)}_n \), \( n \geq 1 \) | \( 2 \cos(k\pi/n) \) | \( 0 \leq k < n \) |
| \( D^{(1)}_n \), \( n \geq 3 \) | \( 0, 2 \cos(2\pi k/(2n - 3)) \) | \( 0 \leq k \leq n - 2 \) |

Let \( \mathcal{G} \) be any multi-digraph. We’ll write \( r(\mathcal{G}) \) for the Perron-Frobenius eigenvalue of its adjacency matrix. \( \mathcal{G} \) is called \textit{bipartite} if its vertices can be coloured black and white, in
such a way that the endpoints of any (directed) edge are coloured differently. For example, any tree is bipartite. If $G$ is connected and its adjacency matrix is irreducible, it will be bipartite iff the number $-r(G)$ is also an eigenvalue of $G$.

3. NIM-reps

3.1. The physics of NIM-reps.

In this section we introduce the main subject of the paper: NIM-reps. Recall the discussion in the Introduction. Fix an RCFT and a choice of chiral algebra. In other words, we are given modular data $S$ and $T$ and a modular invariant $M$. We are interested here in boundary conditions which are not only conformally invariant, but also don’t break the given chiral algebra.

Let $x \in B$ parametrise the $\mathbb{Z}_\geq$-basis for the boundary states in the RCFT (or Chan-Paton degrees-of-freedom for an open string). Consider the 1-loop vacuum-to-vacuum amplitude of an open string, i.e. the amplitude associated to a cylinder whose boundaries are labelled with states $|x\rangle, |y\rangle$. Then we can write them as

$$Z_{x|y}(q) = \sum_{\lambda \in P_+} \mathcal{N}_{\lambda x}^y \chi_\lambda(q)$$

(3.1a)

where $\mathcal{N}_{\lambda x}^y \in \mathbb{Z}_\geq$ and $\chi_\lambda(q)$ are the usual RCFT (e.g. current algebra) characters. The parameter $0 < q < 1$ parametrises the conformal equivalence classes of cylinders, just as $|q| < 1$ did for tori in (2.8). Depending on how we choose the time direction, we can interpret the cylinder either as a 1-loop open string worldsheet, or a 0-loop closed string worldsheet; using this Cardy [1] derived (at least for $M = I$)

$$Z_{x|y}(q) = \sum_{\lambda \in P_+} \sum_{\mu \in \mathcal{E}} \frac{U_{x\mu} S_{\lambda \mu} U_{y\mu}^*}{S_{0\mu}} \chi_\lambda(q)$$

(3.1b)

Here $\mathcal{E}$ is the exponents $\mathcal{E}_M$ of the modular invariant $M$ for the RCFT. The matrix entries $U_{x\mu}$ (appropriately normalised) give the change-of-coordinates from boundary states $|x\rangle$, $x \in B$, to the Ishibashi states $|\mu\rangle$, $\mu \in \mathcal{E}_M$. The matrices $\mathcal{N}_\lambda$ given by

$$(\mathcal{N}_\lambda)_{xy} = \mathcal{N}_{\lambda x}^y = \sum_{\mu \in \mathcal{E}} \frac{U_{x\mu} S_{\lambda \mu} U_{y\mu}^*}{S_{0\mu}}$$

(3.1c)

constitute what we will soon call a NIM-rep. Note by taking complex conjugation of (3.1c) that $\mathcal{N}_\lambda^* = \mathcal{N}_{\lambda C}$. We will require that $U$ be unitary (although the physical reasons for this are not so clear). This gives us (3.2a) below.
3.2. Basic definitions.

**Definition 3.** By a NIM-rep $\mathcal{N}$ we mean an assignment of a matrix $\mathcal{N}_\lambda$, with nonnegative integer entries, to each $\lambda \in P_+$ such that $\mathcal{N}$ forms a representation of the fusion ring:

$$\mathcal{N}_\lambda \mathcal{N}_\mu = \sum_{\nu \in P_+} \mathcal{N}_\nu^{\lambda \mu} \mathcal{N}_\nu$$

for all primaries $\lambda, \mu, \nu \in P_+$, and also that

$$\mathcal{N}_0 = I$$

$$\mathcal{N}_{C\lambda} = \mathcal{N}_\lambda^t \quad \lambda \in P_+$$

The NIM-rep ‘$\mathcal{N}$’ should not be confused with the fusion ‘$N$’. In (3.2c), ‘$C$’ denotes charge-conjugation (2.1), and ‘$t$’ denotes transpose. Equation (3.2b) isn’t significant, and serves to eliminate from consideration the trivial $\lambda \mapsto (0)$. Further refinements of Definition 3 are probably desirable, and are discussed in paragraphs (4), (5) in §8.

The dimension $n$ of a NIM-rep is the size $n \times n$ of the matrices $\mathcal{N}_\lambda$. Note that our definition is more general (i.e. fewer conditions) than in older papers by (various subsets of) Di Francesco&Petkova&Zuber. The fusion graphs of $\mathcal{N}$ are the multi-digraphs associated to the matrices $\mathcal{N}_\lambda$.

We call two $n$-dimensional NIM-reps $\mathcal{N}, \mathcal{N}'$ equivalent if there is an $n \times n$ permutation matrix $P$ (independent of $\lambda \in P_+$) such that $\mathcal{N}_\lambda' = P \mathcal{N}_\lambda P^{-1}$ for all $\lambda \in P_+$. Obviously we can and should identify NIM-reps equivalent in this sense. More generally, when that same relation holds for some arbitrary invertible (i.e. not necessarily permutation) matrix $P$, we will call $\mathcal{N}$ and $\mathcal{N}'$ linearly equivalent. At least 3 distinct NIM-reps for $\text{sl}(3)_9$ have been found with identical exponents [7], which shows that linear equivalence is strictly weaker than equivalence (similar examples are known [11] for $\text{sl}(4)_8$). In fact, linear equivalence isn’t important, and doesn’t respect the physics.

One way to build new NIM-reps from old ones $\mathcal{N}', \mathcal{N}''$ is direct sum $\mathcal{N} = \mathcal{N}' \oplus \mathcal{N}''$:

$$\mathcal{N}_\lambda := \mathcal{N}'_\lambda \oplus \mathcal{N}''_\lambda = \begin{pmatrix} \mathcal{N}'_\lambda & 0 \\ 0 & \mathcal{N}''_\lambda \end{pmatrix}$$

We call such a representation $\mathcal{N}$ decomposable (or reducible); $\mathcal{N}$ is indecomposable when the $\mathcal{N}_\lambda$’s cannot be simultaneously put into block form. Obviously, an arbitrary NIM-rep can always be written as (i.e. is equivalent to) a direct sum of indecomposable NIM-reps, so it suffices to consider the indecomposable ones. Physically, decomposable NIM-reps would correspond to completely decoupled blocks of boundary conditions. We will show in §3.3 that for any given choice of modular data, there are only finitely many indecomposable NIM-reps $\mathcal{N}$.

Two obvious examples of NIM-reps are given by fusion matrices, namely $\mathcal{N}_\lambda = N_\lambda$, and $\mathcal{N}_\lambda = N_\lambda^t$. Both are indecomposable, but they are equivalent: in fact, $N_\lambda^t = CN_\lambda C^{-1}$. We call this obvious NIM-rep the regular one.

The matrices $\mathcal{N}_\lambda$ of §3.1 define a NIM-rep. Thus to any RCFT should correspond a NIM-rep, and it should play as fundamental a role as the modular invariant.
Let $N$ be any NIM-rep. Equation (3.2a) tells us that the matrices $N_{\lambda}$ pairwise commute; (3.2c) then tells us that they are normal. Thus they can be simultaneously diagonalised, by a unitary matrix $U$. Each eigenvalue $e_\lambda(a)$ defines a 1-dimensional representation $\lambda \mapsto e_\lambda(a)$ of the fusion ring, so $e_\lambda(a) = \frac{S_{\lambda\mu}}{S_{\mu\mu}}$ for some $\mu \in P_+$. Thus any NIM-rep will necessarily obey the Verlinde-like decomposition (3.1c), for some multi-set $E = E(N)$. We will call $E$ the exponents of the NIM-rep, by analogy with the A-D-E classification of $\hat{sl}(2)$. Note that $N$ and $N'$ are linearly equivalent iff their exponents $E(N), E(N')$ are equal (including multiplicities).

At first glance, there doesn’t seem to be much connection between NIM-reps and modular invariants. But the discussion in §1 tells us that the NIM-rep $N$ and modular invariant $M$ of an RCFT should obey the compatibility relation

$$\mathcal{E}(N) = \mathcal{E}_M$$

Thus we want to pair up the NIM-reps with the modular invariants so that (3.3) is satisfied; any NIM-rep (resp. modular invariant) without a corresponding modular invariant (resp. NIM-rep) can be regarded as having questionable physical merit (more precisely, before a modular invariant is so labelled, all possible pairings $\omega$ should be checked — see §2.2).

**Definition 4.** We call a modular invariant $M$ NIMmed if there exists a NIM-rep $N$ compatible with $M$ in the sense of (3.3). Otherwise we call $M$ NIM-less.

For instance the regular NIM-rep $N_\Lambda = N_\lambda$ has exponents $E = P_+$, as does the modular invariant $M = I$. Thus they are paired up. It is easy to verify that the only modular invariant $M$ with $E_M \supseteq P_+$ is $M = I$, so it is the unique modular invariant which can be paired with the regular NIM-rep. It would be interesting to find other indecomposable NIM-reps with $E(N) \supseteq P_+$. The Cardy ansatz [1] is essentially the statement that $E(N) = P_+$ implies $N$ is the regular NIM-rep.

Suppose the RCFT has a discrete symmetry $G$. We can consider fields in the theory with twisted, nonperiodic boundary conditions induced by the action of $G$. The resulting partition functions $Z_{g,g'}(\tau)$ (one for each twisted sector of the theory) are submodular invariants. The philosophy of [33] is that what one can do (e.g. study NIM-reps) with the modular invariant $E_{\epsilon,\epsilon}$, can be done as well for the submodular invariants $Z_{g,g'}$ — indeed this is a way of probing the global structure of the theory. The material of this paper, e.g. the Thm.1$\leftrightarrow$Thm.3 correspondence, should be generalised to this more general situation.

Let $\Gamma = \{\gamma^{(1)}, \ldots, \gamma^{(g)}\}$ be any fusion-generator of $P_+$. From (3.1c) and (2.7) it is easy to see that

$$N_{\lambda} = P_\lambda(N_{\gamma^{(1)}}, \ldots, N_{\gamma^{(g)}}) \quad \forall \lambda \in P_+$$

for any NIM-rep $N$. Thus for $\hat{sl}(2)$ and $\hat{sl}(3)$ NIM-reps, the entire $N$ is uniquely determined by knowing the first-fundamental $N_{\Lambda_1}$, or equivalently its fusion graph. But for most choices of $sl(n)_k$ (see §2.1 for the complete answer), knowing $N_{\Lambda_1}$ is not enough to determine all of $N$.

Several fusion graphs for $\hat{sl}(3)$ are given in [7]. They make no claims for the completeness of their lists, and in fact it is not hard to find missing ones. To give the simplest example, the 1-dimensional $sl(3)_3$ NIM-rep (given by the quantum-dimension 1, 2 or 3) is
missing. Incidentally, 1-dimensional \( \text{sl}(n)_k \) NIM-reps exist only for \( \text{sl}(n)_1 \), \( \text{sl}(2)_2 \), \( \text{sl}(2)_4 \), \( \text{sl}(3)_3 \), and \( \text{sl}(4)_2 \) (for a proof, see p.691 of [34]).

3.3. The basic theory of NIM-reps.

This section is central to the paper. Most of the results here are new. For simple examples using them, see §6,7. Although we go much further, some consequences were already obtained in especially [35], using more restrictive axioms.

Let \( \mathcal{N} \) be any indecomposable NIM-rep. Let \( \mathcal{E}(\mathcal{N}) \) be its exponents, and for any exponent \( \mu \in \mathcal{E}(\mathcal{N}) \), let \( m_\mu \) denote the multiplicity. Then the matrix \( \sum_{\lambda \in P_+} N_\lambda \) is strictly positive, and \( m_0 = 1 \). More generally, the value of \( m_0 \) tells you how many indecomposable summands \( \mathcal{N}^i \) there are in a decomposable \( \mathcal{N} = \bigoplus_i \mathcal{N}^i \).

To see this, write \( x \sim y \) if \( N^y_{\lambda x} \neq 0 \) for some \( \lambda \). Then this defines an equivalence relation on \( \mathcal{B} \): \( x \sim x \) because \( N^0_0 = I \); if \( x \sim y \) then \( y \sim x \), because \( N^y_{\lambda x} = N^x_{C \lambda y} \); if \( x \sim y \) (say \( N^y_{\lambda x} \neq 0 \)) and \( y \sim z \) (say \( N^z_{xy} \neq 0 \)) then \( x \sim z \), because \( (N_{\lambda x} N_{\mu y})_{xz} \neq 0 \). So we get a partition \( \mathcal{B}_i \) of \( \mathcal{B} \) such that \( \sum_{\lambda \in P_+} N_\lambda \) restricted to each \( \mathcal{B}_i \) is strictly positive, but \( (\sum_{\lambda \in P_+} N_\lambda)_{ij} = 0 \) when \( x \in \mathcal{B}_i, y \in \mathcal{B}_j \) belong to different classes. This tells us that \( \mathcal{N} \) is the direct sum of the \( \mathcal{N}^{(i)} \) (the restriction of \( \mathcal{N} \) to the subset \( \mathcal{B}_i \)), so \( \mathcal{N} \) being indecomposable requires that there be only one class \( \mathcal{B}_i \) (i.e. that \( \mathcal{B} = \mathcal{B}_i \)). The reason this forces \( m_0 = 1 \) is because of Perron-Frobenius theory [30]: the multiplicity of the Perron-Frobenius eigenvalue for a strictly positive matrix (e.g. \( \sum_{\lambda \in P_+} N_\lambda \)) is 1.

Consider \( \mathcal{N} \) indecomposable. The Perron-Frobenius eigenspace of \( \sum_{\lambda} N_\lambda \) will then be one-dimensional, spanned by a strictly positive vector \( \vec{v} \). Now the simultaneous eigenspaces of the matrices \( N_\lambda \) will necessarily be a partition of the eigenspaces of e.g. \( \sum_{\lambda} N_\lambda \). Thus \( \vec{v} \) must be an eigenvector (hence a Perron-Frobenius eigenvector) of all \( N_\lambda \). Suppose \( \vec{v} \) corresponds to exponent \( \mu \in \mathcal{E}(\mathcal{N}) \). Then its eigenvalues \( S_{\lambda \mu}/S_{00} \) must all be positive. The only primary \( \mu \in P_+ \) with this property for all \( \lambda \in P_+ \), is \( \mu = 0 \). This means that we know the Perron-Frobenius eigenvalue of any matrix \( N_\lambda \): it’s simply the quantum-dimension

\[
r(\mathcal{N}_\lambda) = \frac{S_{\lambda 0}}{S_{00}} \tag{3.5}
\]

Let \( U \) be a unitary diagonalising matrix of all \( N_\lambda \), as in (3.1c) (its existence was proved last subsection). \( U \) will not be unique, but it can be chosen to have properties reminiscent of \( S \). In particular the column \( U_{\lambda 0} \) can be chosen to be the Perron-Frobenius eigenvector \( \vec{v} \) (normalised), so each entry obeys \( U_{\lambda 0} > 0 \). We will discuss \( U \) in more detail in §3.4.

This argument also tells us the important fact that if the matrix \( N_\lambda \) is a direct sum of indecomposable matrices \( A_i \), then each \( A_i \) (equivalently each component of the fusion graph of \( \lambda \)) must have the same maximal eigenvalue \( r(A_i) = S_{\lambda 0}/S_{00} \). The reason is that \( N_\lambda \) has a strictly positive eigenvector, namely \( \vec{v} \). Moreover, these matrices \( A_i \) will be irreducible (see §2.3 for the definition). This follows for example from Corollary 3.15 of [30] — in particular, the Perron-Frobenius vector for both \( N_\lambda \) and \( N_{\lambda}^i = N_{C \lambda}^i \) is the vector \( \vec{v} > 0 \).

Clearly, all \( N_\lambda \) are symmetric iff all exponents \( \mu \in \mathcal{E} \) satisfy \( \mu = C \mu \). More generally,

\[
N_\lambda = N_{\bar{\lambda}} \quad \text{iff} \quad S_{\lambda \mu} = S_{\bar{\lambda} \mu} \quad \forall \mu \in \mathcal{E}(\mathcal{N}) \tag{3.6}
\]
So for any simple-current $J$, $N_J = I$ iff $Q_J(\mu) \in \mathbb{Z} \forall \mu \in \mathcal{E}$, in which case $N_{J\lambda} = N_\lambda \forall \lambda \in P_+$. More generally, by (3.5) $N_J$ will be a permutation matrix. If we let $j$ denote the permutation of the vertices $\mathcal{B}$, corresponding to $N_J$, then the order of $j$ will be the least common multiple of all the denominators of $Q_J(\mu)$ as $\mu$ runs over $\mathcal{E}$. Thus the order of $j$ will always divide the order of $J$. Moreover,

$$ (N_{J\lambda})_x^y = N_{\lambda,j,x}^{-1} = N_{\lambda,x}^{-1}y $$

We will prove in Theorem 3 below the very useful and nontrivial facts that the multiplicity $m_J$ of any simple-current must be 0 or 1, and if it is 1 then $J$ will be a symmetry of $\mathcal{E}$ — i.e. $m_{J\mu} = m_{\mu}$ for all $\mu \in \mathcal{E}$. It follows from this that the simple-currents in $\mathcal{E}$ form a group, which we’ll denote $\mathcal{E}_{sc}$.

Fix any vertex $1 \in \mathcal{B}$. By an $N_1$-grading $g$ we mean a colouring $g(x) \in \mathbb{Q}$ of the vertices $\mathcal{B}$ and colouring $g_\lambda \in \mathbb{Q}$ of the primaries $P_+$, such that $g(1) \in \mathbb{Z}$ and

$$ N_{\lambda,x}^y \neq 0 \quad \Rightarrow \quad g_\lambda + g(x) \equiv g(y) \pmod{1} \quad (3.7) $$

Clearly the $N_1$-gradings form a group under addition; different choices of ‘1’ yield isomorphic groups. Thm.3(viii) says that this group is naturally isomorphic to the group of simple-currents in $\mathcal{E}$. In particular, to any simple-current $J \in \mathcal{E}$ we get an $N_1$-grading as follows. Define $g_\lambda = Q_J(\lambda)$, and put $g(y) = Q_J(\lambda)$ if $N_{\lambda,x}^y \neq 0$ for some $\lambda \in P_+$. This defines an $N_1$-grading, and we learn in Thm.3(viii) that all $N_1$-gradings arise in this way.

Let $A$ be any matrix, and let $m_s$ be the multiplicity of eigenvalue $s$. If all entries of $A$ are rational, then each eigenvalue $s$ will be an algebraic number (since it’s the root of a polynomial over $\mathbb{Q}$). If $\sigma$ is any Galois automorphism (of the splitting field of the characteristic polynomial of $A$), and $s$ is any eigenvalue, then the image $\sigma(s)$ will also be an eigenvalue of $A$, and the multiplicities $m_s$ and $m_{\sigma(s)}$ will be equal.

Now, $\sigma S_{0\mu}^\mu = S_{\lambda,\sigma\mu}/S_{0,\sigma\mu}$, by (2.6). So what this means is that the multiplicities $m_{\mu}, m_{\sigma\mu}$ of $\mu$ and $\sigma\mu$ in the exponents $\mathcal{E}(N)$ must be equal — that is, the exponents $\mathcal{E}(N)$ obey the same Galois symmetry as the exponents $\mathcal{E}(M)$ (see Thm.1(iv)).

A special case of this is that $\lambda$ and $C\lambda$ have the same multiplicity. That follows from (is equivalent to) the fact that the entries $N_{\lambda,x}^y$ are all real. The much more general Galois symmetry follows from (and together with (3.10a) is equivalent to) the much stronger statement that each $N_{\lambda,x}^y$ is rational.

Note that for any $\lambda \in P_+$,

$$ \text{Tr}(N_\lambda) = \sum_{\mu \in \mathcal{E}(N)} \frac{S_{\lambda\mu}}{S_{0\mu}} S_{0\mu} \in \mathbb{Z}_{\geq} \quad \forall \lambda \in P_+ \quad (3.8) $$

This is a strong condition for a multi-set $\mathcal{E}$ to obey — see e.g. §7. If $\mathcal{E}$ obeys the Galois condition $m_\lambda = m_{\sigma(\lambda)}$, as it must, then the sum in (3.8) will automatically be integral, so the important thing in (3.8) is nonnegativity.

Suppose $\mathcal{N}$ is indecomposable. Then

$$ \sum_{\lambda \in P_+} \text{Tr}(N_\lambda) = \sum_{\lambda \in P_+} \sum_{\mu \in \mathcal{E}} S_{0\lambda} \frac{S_{\lambda\mu}}{S_{0\mu}} = \frac{1}{S_{00}} \quad (3.9a) $$

17
All LHS terms are nonnegative. By considering the contribution to the LHS by \( \lambda = 0 \), we find that the dimension of an indecomposable NIM-rep is bounded above by \( S_{00}^{-2} \).

Moreover, each entry of \( \mathcal{N}_\lambda \) must be bounded above by the quantum-dimension \( S_{\lambda 0}/S_{00} \). To see this, note that the matrix \( \mathcal{N}_\lambda \mathcal{N}_\lambda^t \) has largest eigenvalue \( r = (S_{\lambda 0}/S_{00})^2 \); by Perron-Frobenius theory any diagonal entry \( A_{ii} \) of a nonnegative matrix \( A \) will be bounded above by \( r(A) \). Thus for each \( i, j \) we get

\[
(\mathcal{N}_\lambda)_{ij}^2 \leq (\mathcal{N}_\lambda \mathcal{N}_\lambda^t)_{ii} \leq (S_{\lambda 0}/S_{00})^2
\]  

(3.9b)

Together, these two bounds tell us that the number of indecomposable NIM-reps, for a fixed choice of modular data, must be finite.

We collect next the main things we’ve established.

**Theorem 3.** Choose any modular data, and let \( \mathcal{N} \) be any indecomposable NIM-rep, with exponents \( \mathcal{E} \) and multiplicities \( m_\lambda \).

(i) For the given modular data, there are only finitely many different indecomposable NIM-reps. We have the bounds \( (\mathcal{N}_\lambda)_{ij} \leq S_{\lambda 0}/S_{00} \) and \( \text{dim}(\mathcal{N}) \leq S_{00}^{-2} \).

(ii) \( m_0 = 1 \).

(iii) For any simple-current \( J \), either \( m_J = 0 \) or 1; if \( m_J = 1 \) then \( m_J \lambda = m_\lambda \) for any primaries \( \lambda \in P_+ \).

(iv) For any Galois automorphism \( \sigma \) and primary \( \lambda \in P_+ \), \( m_{\sigma(\lambda)} = m_\lambda \).

(v) For any primary \( \lambda \in P_+ \), inequality (3.8) holds.

(vi) For any primary \( \lambda \in P_+ \), each indecomposable submatrix of \( \mathcal{N}_\lambda \) will be irreducible and have largest eigenvalue equal to the quantum-dimension \( S_{\lambda 0}/S_{00} \) of \( \lambda \). The number of indecomposable components will precisely equal the number of \( \mu \in \mathcal{E} \) such that \( S_{\lambda \mu}/S_{0\mu} = S_{\lambda 0}/S_{00} \). The number of these components which have a \( \mathbb{Z}_m \)-grading is precisely the number of \( \mu \in \mathcal{E} \) with \( S_{\lambda \mu}/S_{0\mu} = \epsilon^{2\pi i/m} S_{\lambda 0}/S_{00} \).

(vii) No row or column of any matrix \( \mathcal{N}_\lambda \) can be identically 0.

(viii) Fix any vertex \( 1 \in B \). The \( \mathcal{N}_{1x} \)-gradings of the NIM-rep are in a natural one-to-one correspondence with the simple-currents \( J \in \mathcal{E} \).

(ix) Let \( \mathcal{E}_{sc} \) denote the set of all simple-currents in \( \mathcal{E} \), \( S_{sc} \) denote all simple-currents in \( P_+ \), and \( \mathcal{S}_0 \) be the set of all simple-currents \( J \in P_+ \) such that \( Q_J(J') \in \mathbb{Z} \) for all \( J' \in \mathcal{E}_{sc} \). Then \( \|S_{sc}\| \) must divide \( \|S_0\| \text{dim}(\mathcal{N}) \).

(x) If a primary \( \lambda \in P_+ \) has \( Q_J(\lambda) \notin \mathbb{Z} \) for some simple-current \( J \in \mathcal{E} \), then \( \mathcal{N}_{\lambda x}^c = 0 \) for all \( x \in B \).

Note that the grading in (vi) applies to an individual matrix \( \mathcal{N}_\lambda \), whereas that of (viii) refers to a grading valid simultaneously for all matrices \( \mathcal{N}_\lambda \). Part (vii) comes from applying nonnegativity to \( (\mathcal{N}_\lambda) (\mathcal{N}_\lambda)^t = I + \cdots \). Part (x) comes from (3.8) and Thm.3(iii). The remainder of the proof of Thm.3 is relegated to the end of the appendix.

Compare Theorems 1 and 3: surprisingly, the general properties obeyed by the exponents of a modular invariant, and those of a NIM-rep, match remarkably well. It would be nice to obtain a simple, general, and effective test for the NIM-lessness of a modular invariant. One candidate is Thm.3(ix): this author has managed to show for modular invariants, only the weaker statement that \( \|\mathcal{E}(M)_{sc}\| \text{ must divide } \|\mathcal{E}(M)_{sc} \cap S_0\| \text{ Tr}(M) \), where \( \mathcal{E}(M)_{sc} \) equals the number of simple-currents in \( \mathcal{E}_M \).
Thm. 3 assumes all $S_{\lambda 0} > 0$. For nonunitary RCFT, let $o \in P_+$ be as in §2.1. Then 3(ii) becomes $m_o = 1$, but $m_o$ seems unconstrained. The bound on $\dim(\mathcal{N})$ is now $S_{00}^{-2}$. In 3(iii) replace $m_J$ with $m_{J0}$.

There are several generic constructions of NIM-reps, and a systematic study of these should probably be made. We will only mention one, which seems to have been overlooked in the literature. It involves the notion of fusion-homomorphism, i.e. a map $\pi : P_+ \to P'_+$ between the primaries of two (possibly identical) fusion rings, which defines a ring homomorphism of the corresponding fusion rings: that is,

$$\pi(\lambda) \boxtimes' \pi(\mu) = \sum_{\nu \in P_+} N_{\lambda\mu}^\nu \pi(\nu)$$

where $\boxtimes'$ is the fusion product for $P'_+$. See Prop. 3 of [18] for its basic properties. In particular, there exists a map $\pi' : P'_+ \to P_+$ such that [18]

$$\frac{S_{\lambda\mu'}}{S_{00'}^{\lambda\mu'}} = \frac{S_{\lambda,\pi'}^{\lambda\mu'}}{S_{0,\pi'}^{\lambda\mu'}}$$

Also, $\pi\lambda = \pi\mu$ iff $\mu = J\lambda$ for some simple-current $J$ with $\pi(J) = 0$.

Suppose $\pi : P_+ \to P'_+$ is a fusion-homomorphism, and $\mathcal{N}$ is a NIM-rep of $P'_+$. Then $\mathcal{N}^\pi$ defined by $(\mathcal{N}^\pi)_\lambda = N_{\pi\lambda}$ is a (usually decomposable) NIM-rep of $P_+$. For a trivial example, when $\pi$ is a fusion-isomorphism, and $\lambda \mapsto N_\lambda$ is the regular (=fusion matrix) NIM-rep, then $\lambda \mapsto N_{\pi\lambda}$ will be equivalent to the regular NIM-rep (permute the rows and columns by $\pi$).

The exponents $\mathcal{E}(\mathcal{N}^\pi)$ of $\mathcal{N}^\pi$ is the multi-set $\pi'(\mathcal{E}(\mathcal{N}))$. If $\pi$ is onto, then it can be shown using [18] that $\mathcal{N}^\pi$ will be indecomposable iff $\mathcal{N}$ is.

3.4. The diagonalising matrix $U$ and the Pasquier algebra.

Consider now the diagonalising matrix $U$ of (3.1c). In the event where some multiplicities $m_\mu$ are greater than 1, it will be convenient at times to introduce the following explicit notation for the entries of $U$: write $U_{x,(\mu,i)}$, where $1 \leq i \leq m_\mu$.

We would expect the diagonalising matrix $U$ to obey essentially the same properties as $S$, except symmetry $S = S^t$ of course (the columns and rows are labelled by completely different sets $P_+$ and $B$).

However, the unitary matrix $U$ is not uniquely determined by (3.1c): for an exponent $\mu \in \mathcal{E}$ with multiplicity $m_\mu$, we can choose for the $m_\mu$ columns corresponding to $\mu$ any orthogonal basis of the corresponding eigenspace — i.e. the freedom is parametrised for each $\mu \in \mathcal{E}$ by an $m_\mu \times m_\mu$ unitary matrix $A^{(\mu)} \in U(m_\mu)$. Explicitly, an alternate matrix $U'$ would be given by the formula

$$U'_{x,(\mu,i)} = \sum_{j=1}^{m_\mu} A^{(\mu)}_{ij} U_{x,(\mu,j)}$$

The question we address in this subsection is, is there a preferred choice for $U$ which realises most of the symmetries of the $S$ matrix which we saw in §2.1?
We claim only that the ‘preferred’ matrix $U$ constructed below, diagonalises the $N_{\lambda}$ as in (3.1c). Its relation to the change-of-coordinate matrix $U$, which goes from the boundary condition basis $|x\rangle$ to the Ishibashi states $|\mu\rangle$, is uncertain, although the following properties are all natural.

As mentioned in §3.3, the $\mu = 0$ column can (and will) be chosen to be strictly positive. Fix any $\mu \in \mathcal{E}$. Let $K_{\mu}$ be the number field generated by $Q$ and all ratios $S_{\lambda \mu}/S_{0\mu}$, for $\lambda \in P_+$. Then for each $1 \leq i \leq m_{\mu}$, we can require that all entries $U_{x, (\mu, i)}$ lie in a quadratic extension $K_{\mu}^i$ of $K$. For any Galois automorphism $\sigma \in \text{Gal}(K_{\mu}^i/Q)$, we can require

$$\sigma U_{x, (\mu, i)} = \epsilon_{\sigma}(\mu, i) U_{x, (\mu, i)} \quad (3.10a)$$

where $\mu \mapsto \sigma \mu$ is the permutation of (2.6), and where $\epsilon_{\sigma}(\mu, i) \in \{\pm 1\}$. We will prove this shortly. Also, fix any vertex $1 \in \mathcal{B}$; we can require $U$ to satisfy

$$U_{x, (J\mu, i)} = e^{2\pi i g(x)} U_{x, (\mu, i)} \quad \forall J \in \mathcal{E}, \mu \in \mathcal{E}, x \in \mathcal{B}, 1 \leq i \leq m_{\mu} \quad (3.10b)$$

where $g$ is the $N_1$-grading associated to $J$ by Thm.3(viii). Conversely, let $J \in P_+$ be any simple-current, and write $N_{Jx, \mu} = \delta_{y, jx}$ for the appropriate permutation $j$ of $\mathcal{B}$. Then each column $U_{x, (\mu, i)}^{\ast}$ is an eigenvector of $N_J$ with eigenvalue $e^{2\pi i Q_{J}(\mu)}$, that is to say

$$U_{jx, (\mu, i)} = e^{2\pi i Q_{J}(\mu)} U_{x, (\mu, i)} \quad (3.10c)$$

Incidentally, the relation (3.10a) allows us to prove the rationality of the coefficients of the so-called dual Pasquier algebra (or $\hat{N}$-algebra). Assume there is some vertex $1 \in \mathcal{B}$ such that the row $U_{1, (\mu, i)} \neq 0$ for all $\mu, i$. Define [16]

$$\hat{N}_{xy} := \sum_{\substack{\mu \in \mathcal{E} \\mu \leq i \leq m_\mu}} U_{x, (\mu, i)} U_{y, (\mu, i)} U_{z, (\mu, i)} U_{1, (\mu, i)}$$

Then for any such choice of $1 \in \mathcal{B}$, (3.10a) tells us

$$\sigma \hat{N}_{xy} = \sum_{\substack{\mu \in \mathcal{E} \\mu \leq i \leq m_\mu}} \epsilon_{\sigma}(\mu, i) U_{x, (\mu, i)} \epsilon_{\sigma}(\mu, i) U_{y, (\mu, i)} \epsilon_{\sigma}(\mu, i) U_{z, (\mu, i)} \epsilon_{\sigma}(\mu, i) U_{1, (\mu, i)} = \hat{N}_{xy}$$

for all Galois automorphisms $\sigma$. This is precisely the statement that each coefficient $\hat{N}_{xy}$ is rational. This result is new, although it had been empirically observed in e.g. [16] that the coefficients $\hat{N}_{xy}$ for each of the then-known NIM-reps always seemed to be rational.

Ideally, we would like the coefficients $\hat{N}$ to be nonnegative integers. In this case the Perron-Frobenius eigenvalue of $\hat{N}_x$ would be given by $U_{x, (0, 1)}/U_{1, (0, 1)}$, and hence we would have the inequalities

$$U_{1, (0, 1)} \leq U_{x, (0, 1)} \quad (3.11a)$$

$$\left| U_{x, (\mu, i)}/U_{1, (\mu, i)} \right| \leq U_{x, (0, 1)}/U_{1, (0, 1)} \quad (3.11b)$$
In particular, (3.11a) is the statement that a normal $\mathbb{Z}_\geq$-matrix $A \neq 0$ must have $r(A) \geq 1$, and (3.11b) says that whenever $A \geq 0$ then $|s| \leq r(A)$ for any eigenvalue $s$ of $A$. The inequality (3.11a) justifies the empirical rule of [16] for choosing the vertex $1 \in \mathcal{B}$.

For example, consider the sl(2)$_{16}$ NIM-rep called $E_7$: its diagonalising matrix $U$ is given in [6]. We can see by inspection that its dual Pasquier coefficients cannot all be in $\mathbb{Z}_\geq$. In particular, (3.11a) identifies the vertex 1, and then we find (3.11b) is not satisfied.

On the other hand, the coefficients of the Pasquier algebra (or $\mathcal{M}$ algebra) [16]

$$\mathcal{M}^{(\nu,k)}_{(\lambda,i),(\mu,j)} := \sum_{x \in \mathcal{B}} \frac{U_{x,(\lambda,i)} U_{x,(\mu,j)} U^*_{x,(\nu,k)}}{U_{x,(0,1)}}$$

will in general not be rational — they will be rational iff the analogue of (3.10a) holds for rows. The coefficients $\mathcal{M}$ can be rational, only when all entries of $U$ lie in a cyclotomic field (the proof in [23] for $S$ works here). We will return to this shortly.

See e.g. [6] for a discussion of the (dual) Pasquier algebra. Note that our matrix $U$ is denoted there by $\psi$, and our $\mathcal{B}$ is denoted there by $\mathcal{V}$. It has appeared in other related contexts — see e.g. the classifying algebra in e.g. [14] and references therein. Unlike fusion coefficients, neither the coefficients $\mathcal{M}$ nor $\hat{N}$ need be integral or nonnegative, and both depend on the choice of $U$.

To see (3.10a), first note that finding an orthogonal basis of eigenvectors for the $\mu$-eigenspace amounts to solving a system of linear equations with coefficients in the cyclotomic field $\mathbb{K}_\mu$. Find any such basis $\tilde{u}_{(\mu,i)}$, so that each 1-coordinate $(\tilde{u}_{(\mu,i)})_1$ is rational. Then hit these vectors $\tilde{u}_{(\mu,i)}$ componentwise by $\sigma$ to yield an orthogonal basis of eigenvectors for the $\sigma(\mu)$-eigenspace. Note that $\sigma(\mu) = \mu$ iff the automorphism $\sigma$ is trivial in $\mathbb{K}_\mu$, so these bases will be well-defined. When $\sigma(\mu) = J\mu$ for some simple-current $J \in \mathcal{E}$, then (3.10b) will be automatic; otherwise note from the proof of Thm.3(viii) given in the appendix that the vectors $(\tilde{u}_{(\mu,i)}^J)_x := e^{2\pi i g(x)}(\tilde{u}_{(\mu,i)})_x$ are orthogonal eigenvectors for $J\mu$. Run this construction through a set of representatives $\mu$ of the orbits in $\mathcal{E}$ of the group $\langle \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathcal{E}_{sc} \rangle$; normalising the resulting eigenbases (this is where the quadratic extensions $\mathbb{K}_\mu^j$ and the signs $\epsilon_\sigma$ arise), gives a unitary diagonalising matrix $U$ satisfying (3.10).

Unlike the entries of $S$, those of $U$ will not in general lie in a cyclotomic field, and there won’t in general be a Galois action on the rows of $U$. A simple example of this is the sl(2)$_{10}$ exceptional called $E_6$, whose diagonalising matrix is [6]

$$U = \frac{1}{2} \begin{pmatrix} a & 1 & b & b & 1 & a \\ b & 1 & a & -a & -1 & -b \\ c & 0 & -d & -d & 0 & c \\ b & -1 & a & -a & 1 & -b \\ a & -1 & b & b & -1 & a \\ d & 0 & -c & c & 0 & -d \end{pmatrix}$$

where $a, b$ equal $\sqrt{(3 + \sqrt{3})/6}$, respectively, and $c, d$ equal $\sqrt{2a}, \sqrt{2b}$, respectively. Note first that $\sqrt{3 + \sqrt{3}}$ does not lie in any cyclotomic field, and so neither do $a, b, c, d$. In
fact the smallest normal extension of $\mathbb{Q}$ containing $\sqrt{3} + \sqrt{3}$ is $\mathbb{Q}[\sqrt{2}, \sqrt{3} + \sqrt{3}]$ (note that $\sqrt{3} + \sqrt{3} \sqrt{3} - \sqrt{3} = \sqrt{3} \sqrt{2}$), and the corresponding Galois group is the nonabelian quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. The Galois automorphism sending $\sqrt{2}$ to itself and $\sqrt{3} \pm \sqrt{3}$ to $\sqrt{3} \mp \sqrt{3}$ interchanges for instance columns 1 and 3 with $\epsilon = -1$, but doesn’t send the first row anywhere. ($U$ here is unique, up to phases for each column; no choice of phases however will give us a cyclotomic field.)

Of course, the simplest and most important example of a Galois automorphism is complex conjugation $z \mapsto z^*$. Eq.(3.10a) becomes

$$U^*_{x,(\mu,i)} = U_{x,(C\mu,i)} \quad (3.12a)$$

where $\mu \mapsto C\mu$ is charge-conjugation (2.1) — the parity $\epsilon_+(\mu,i)$ in (3.10a) will be $+1$ here because the normalisation of the columns of $U$ only involves rescaling by a real number. Using the facts that $U$ is unitary and $C$ is an involution, we get that $U^tU$ is a permutation matrix:

$$(U^tU)_{(\mu,i),(\nu,j)} = \delta_{\nu,C\mu} \delta_{j,i} \quad (3.12b)$$

In many examples, the analogue of (3.12a) for rows also holds: that is, there is an invertible involution $\iota$ of $\mathcal{B}$ such that

$$U^*_{x,(\mu,i)} = U_{x,(\iota \mu,i)} \quad (3.12c)$$

When this holds, we get $N^y_{\mathcal{C}\lambda,i \iota x} = N^y_{\mathcal{C}\lambda,i x}$ and $(UU^t)_{xy} = \delta_{y,\iota x}$. Since $\text{Tr}(U^tU) = \text{Tr}(UU^t)$, the number of fixed-points of $\iota$ would equal the number of $\mu \in \mathcal{E}$ with $C\mu = \mu$, counting multiplicities. It is easy to show that $\iota$ exists iff the NIM-rep $\lambda \mapsto N_{\mathcal{C}\lambda}$ is equivalent to $\lambda \mapsto N_\lambda$ — even when $\iota$ doesn’t exist, they will be linearly equivalent. Also, $\iota$ exists iff the corresponding Pasquier algebra $\mathcal{M}$ has real structure constants. The existence of $\iota$ is assumed in the axioms of [7,16] and it holds in all examples of NIM-reps known to this author, but probably NIM-reps without an $\iota$ can be found for $\mathfrak{sl}(3)_k$ or $\mathfrak{sl}(4)_k$.

### 4. The current algebras at level 1

In the next two sections we obtain several new NIM-rep classifications for the current algebras, and compare them to the corresponding modular invariant classifications.

We begin in §4.1 by finding all NIM-reps for any modular data obeying the restrictive property that all primaries are simple-currents. This allows us immediately to do all simply-laced current algebras at level 1. The NIM-reps for the $\mathfrak{B}^{(1)}$- and $\mathfrak{C}^{(1)}$-series at level 1 follow from the $\hat{\mathfrak{sl}}(2)$ classification, so we repeat the $\hat{\mathfrak{sl}}(2)$ classification in §4.3.

In all these cases, the NIM-rep and modular invariant classifications match up fairly well: each modular invariant has a unique NIM-rep, and most NIM-reps are paired with a unique modular invariant. The only interesting situation here is $so(8n)_1$, where different modular invariants correspond to identical NIM-reps.

Note that NIM-reps (unlike modular invariants) depend only on the fusion ring. When two fusion rings are isomorphic, their NIM-reps will be identical. In [29] we found all isomorphisms $X_{r,k} \cong X'_{r',k'}$ among the fusion rings of current algebras. The complete list
is: $\text{sp}(2n)_k \cong \text{sp}(2k)_n$ for all $n, k$; all $\text{so}(2n+1)$ at level 1 are isomorphic to $\text{sl}(2)_2 \cong \text{sp}(4)_1 \cong E_{8,2}$; $\text{sl}(2)_k \cong \text{sp}(2k)_1$; so$(2n)_1 \cong \text{so}(2m)_1$ whenever $n \equiv m \pmod{2}$, and in addition odd $m$ are isomorphic to $\text{sl}(2)_2$; $\text{sl}(3)_1 \cong E_{6,1}$; $\text{sl}(2)_1 \cong E_{7,1}$; $F_{4,1} \cong G_{2,1}$; $F_{4,3} \cong G_{2,4}$; and finally $E_{8,3} \cong F_{4,2}$.

Coincidentally, when the fusion rings of $X_{r,k}$ and $X_{r',k'}$ are isomorphic, it turns out that their modular invariant classifications will usually be identical. The only exception is so$(4n)_1$, which has either 2 or 6 modular invariants, depending on whether or not $n$ is odd.

4.1. All primaries are simple-currents.

The simple-currents (i.e. the primaries with quantum-dimension 1 — see §2.1) always form an abelian group, called the centre of the modular data. Any NIM-rep, when restricted to the centre, yields a group-representation of the centre by permutation matrices. In this subsection we consider the special case where all primaries $\lambda \in P_+$ are simple-currents (the modular data though is otherwise general — it may or may not come from a current algebra).

Proposition 4. Consider any modular data. Suppose all primaries in $P_+$ are simple-currents.

(a) The indecomposable NIM-reps are in one-to-one correspondence with the subgroups $\mathcal{J}$ of the centre: $\mathcal{J} \leftrightarrow \mathcal{N}(\mathcal{J})$. The exponent $\mathcal{E}$ of the NIM-rep $\mathcal{N}(\mathcal{J})$ is $\mathcal{J}$. (We will explicitly construct $\mathcal{N}(\mathcal{J})$ below.) The NIM-rep is uniquely specified by its exponents.

(b) The exponent of any modular invariant is a subgroup of the centre. Thus any modular invariant is NIMmed. However, some subgroups (hence NIM-reps) may be realised by none or by several modular invariants. There may be more/less/the same number of modular invariants as NIM-reps.

In particular, choose any subgroup $\mathcal{J}$ of the centre $P_+$, and put $k = ||\mathcal{J}||$. Define a $k$-dimensional NIM-rep as follows. Let $\mathcal{J}'$ be the subset (in fact subgroup) of $P_+$, consisting of all primaries $J'$ for which $Q_J(J') \in \mathbb{Z}$ for all $J \in \mathcal{J}$. There will be $||P_+||/k$ such $J'$. This is a subgroup because of the relation $Q_J(J'J'') = Q_J(J') + Q_J(J'')$ which holds for any simple-currents $J, J', J''$, and which follows immediately from (2.4). Now consider the quotient group $P_+/\mathcal{J}' = \{[J_0], [J_1], \ldots, [J_{k-1}]\}$. It will in fact be isomorphic to $\mathcal{J}$. Define the NIM-rep $\mathcal{N}(\mathcal{J})$ by

$$(\mathcal{N}(\mathcal{J}))_{ij} = \delta_{[J_i],[J_j]} \quad \forall J \in P_+$$

So the rows and columns of $\mathcal{N}(\mathcal{J})$ are essentially labelled by the elements of $P_+/\mathcal{J}'$. To get that the exponents of $\mathcal{N}(\mathcal{J})$ are $\mathcal{J}$, use the fact that $J' \in P_+$ is sent to $I$ iff $J' \in \mathcal{J}'$, and so $Q_J(J') \in \mathbb{Z}$ for any exponent $J$ and any $J' \in \mathcal{J}'$.

The two extremes are when the subgroup is all of $P_+$, in which case the NIM-rep is given by fusion matrices, and when the subgroup is $\{0\}$, in which case the NIM-rep is the constant $\mathcal{N}_J = 1$.

It is clear from Thm.1(iii) and Thm.3(iii) that the exponents of a modular invariant and a NIM-rep must both form a subgroup of the centre $P_+$. It is not obvious that there is only one NIM-rep realising that subgroup. To see the general argument, it is perhaps
easiest to consider an example: \( P_+ \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \cong \mathcal{J} \). Let \( J_1, J_2, J_3 \) be the corresponding generators. Let \( \mathcal{N} \) be any NIM-rep with exponents \( P_+ \). We know from Thm.3(x) that \( \text{Tr}(\mathcal{N}_J) = 0 \) provided \( J \neq 0 \), so the permutation associated to \( \mathcal{N}_J \), for any \( J \neq 0 \), can have no fixed-points. Thus the permutation associated to \( \mathcal{N}_{J_1} \) must be a disjoint product of nine 4-cycles. By relabelling the rows/columns appropriately, we may take it to send \( i + 4j + 12k \) (\( i \in \mathbb{Z}_4, j \in \mathbb{Z}_3, k \in \mathbb{Z}_3 \)) to \((i + 1 \text{mod } 4) + 4j + 12k \). Likewise, \( \mathcal{N}_{J_2} \) must be a disjoint product of 12 3-cycles, and it must commute with \( \mathcal{N}_{J_1} \), so we may take the corresponding permutation to send \( i + 4j + 12k \) to \((i + 4(j + 1 \text{mod } 3)) + 12k \). The matrix \( \mathcal{N}_{J_3} \) is handled similarly; none of its 3-cycles can coincide with those of \( \mathcal{N}_{J_2} \) because otherwise \( \mathcal{N}_{J_3} \mathcal{N}_{J_2}^{-1} = \mathcal{N}_{J_3 J_2}^{-1} \) would have fixed points and nonzero trace. So we can likewise fix \( \mathcal{N}_{J_3} \). Manifestly, the resulting NIM-rep is the regular NIM-rep corresponding to the fusion matrices.

4.2. The simply-laced algebras at level 1.

The algebra \( \tilde{\mathfrak{sl}}(n) = A^{(1)}_{n-1}, n \geq 2 \), at level 1 has \( n \) primaries, \( P_+ = \{0, \Lambda_1, \ldots, \Lambda_{n-1}\} \). Put \( \Lambda_0 = 0 \), then \( \Lambda_i = J^i \) for the simple-current \( J = \Lambda_1 \). The centre of \( \mathfrak{sl}(n)_1 \) is the cyclic group \( \mathbb{Z}_n \), so there is an indecomposable NIM-rep corresponding to each divisor \( d \) of \( n \). In particular, the exponents will be generated by \( J^d \), the subgroup \( J' \) defined above will be generated by \( J^{n/d} \), and the resulting NIM-rep will be \( n/d \)-dimensional. This classification is given in [6].

There is a modular invariant, namely \( M[J^d] \) in (2.9), for any divisor \( d \) of \( n \) for which \( (n-1)d \) is even [36]. It has exponents \( \langle J^{n/d} \rangle \) and corresponds to the NIM-rep \( \mathcal{N}(\langle J^{n/d} \rangle) \).

The algebra \( \tilde{\mathfrak{so}}(2r) = D^{(1)}_r, r \geq 4 \), at level 1 has 4 primaries \( P_+ = \{0, J_v = \Lambda_1, J_s = \Lambda_r, J_c = \Lambda_{r-1}\} \), all of which are simple-currents. For \( r \) odd they define the cyclic group \( \langle J_s \rangle \cong \mathbb{Z}_4 \), while for \( r \) even they define the group \( \langle J_v, J_s \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Thus there are precisely three indecomposable NIM-reps for \( r \) odd — one for each choice of exponents \( \mathcal{E} = \{0\}, \{0, J_v\}, \{0, J_v, J_s, J_c\} \). For \( r \) even, there are precisely five indecomposable NIM-reps — one for each choice of exponents

\[
\mathcal{E} = \{0\}, \{0, J_v\}, \{0, J_s\}, \{0, J_c\}, \{0, J_v, J_s, J_c\}
\]

For \( D_{r,1} \), when 4 does not divide \( r \), there are only two modular invariants [26]: \( M = I \) (which has exponents \( \{0, J_v, J_s, J_c\} \)) and \( M = C_1 \), the permutation fixing 0 and \( \Lambda_1 \) and interchanging \( \Lambda_r \leftrightarrow \Lambda_{r-1} \) (which has exponents \( \{0, J_v\} \)). When 4 divides \( r \), there are six modular invariants [26]: along with \( I \) and \( C_1 \), these are \( M[J_s], C_1 M[J_s], M[J_s] C_1 \), and \( C_1 M[J_s] C_1 \) (with exponents \( \{0, J_s\}, \{0\}, \{0\}, \{0, J_c\} \), resp.). In particular, both

\[
C_1 M[J_s] = (\chi_0 + \chi_{\Lambda_{r-1}}) (\chi_0^* + \chi_{\Lambda_r}^*), \quad M[J_s] C_1 = (\chi_0 + \chi_{\Lambda_r}) (\chi_0^* + \chi_{\Lambda_{r-1}}^*)
\]
correspond to the identical NIM-rep (namely \( \mathcal{N}_J = 1 \forall J \)).

The algebra \( E_{6,1} \) has centre \( \{0, \Lambda_1, \Lambda_5\} \cong \mathbb{Z}_3 \), two indecomposable NIM-reps, and two modular invariants (\( M = I \) and \( M = C \)). The algebra \( E_{7,1} \) has centre \( \{0, \Lambda_6\} \cong \mathbb{Z}_2 \), two indecomposable NIM-reps, and one modular invariant (\( M = I \)). The algebra \( E_{8,1} \) has trivial centre \( \{0\} \), one indecomposable NIM-rep, and one modular invariant.
4.3. The algebra $\widehat{\mathfrak{sl}(2)} = A_1^{(1)}$, at level $k$.

Because we’ll be needing it in the next two subsections, we repeat here the NIM-rep classification for $\widehat{\mathfrak{sl}(2)}$, which was first given in [7].

Let $\mathcal{N}$ be any indecomposable NIM-rep of $A_{1,k}$. Its modular data is given in §2.1. A fusion generator for $A_{1,k}$ is $\Lambda_1$, so it suffices to give $\mathcal{N}_1 = \mathcal{N}_{\Lambda_1}$. For $k$ odd, the fusion graph for $\mathcal{N}_1$ is either $A_{k+1}$ or the tadpole $T_{(k+1)/2}$ (see Figure 1). For $k$ even, the possible fusion graphs are $A_{k+1}$ and $D_{k/2+2}$, except for $k = 10, 16$ or 28 where in addition there are $E_6, E_7, E_8$ respectively.

The modular invariants for $A_{1,k}$ were found in [27]. Each corresponds to a unique NIM-rep, namely one of A-D-E type, as is well-known.

4.4. The algebra $\widehat{\mathfrak{so}(2r+1)} = B_r^{(1)}$, for $r \geq 3$ at level 1.

The weights here are $P_+ = \{0, \Lambda_1, \Lambda_r\}$. For $B_{r,1}$ the only modular invariant [26] is the identity matrix $I$. We learned above that its fusion ring is isomorphic to that of $\mathfrak{sl}(2)_2$ (the isomorphism sends $\Lambda_r$ to the fusion generator $\Lambda_1$ of $\mathfrak{sl}(2)_2$) and so we can read off its NIM-reps from the classification of §4.3: we find that there is only the ‘regular’ one, given by the fusion matrices, which assigns to the generator $\Lambda_r$ the fusion graph $A_3$.

4.5. The algebra $\widehat{\mathfrak{sp}(2r)} = C_r^{(1)}$, for $r \geq 2$ at level 1.

Here, $P_+ = \{0, \Lambda_1, \ldots, \Lambda_r\}$. Write $\Lambda_0$ for 0. The fusion-isomorphism between $C_{r,1}$ and $A_{1,r}$ identifies the primary $\Lambda_i$ of $C_{r,1}$ with the primary $i\Lambda_1$ of $A_{1,r}$. The NIM-reps for $C_{r,1}$ are thus of A-D-E or tadpole type, exactly as in §4.3.

The modular invariants for $C_{r,1}$ [26] fall into the A-D-E pattern, and are in a natural one-to-one correspondence with those of $A_{1,r}$ (again using the identification $\Lambda_i \leftrightarrow i\Lambda_1$).

Thus the NIM-rep ↔ modular invariant situation for $C_{r,1}$ is identical to that of $A_{1,r}$.

4.6. The algebras $G_2^{(1)}$ and $F_4^{(1)}$ at level 1.

$G_{2,1}$ has $P_+ = \{0, \Lambda_2\}$. We compute the quantum-dimension: $\frac{S_{\Lambda_2,0}}{S_{0,0}} = \frac{1+\sqrt{5}}{2}$, the Golden Mean. Thus there’s a Galois automorphism $\sigma$ for which $\sigma\frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} = \frac{-2}{1+\sqrt{5}}$. Applying that $\sigma$ to the quantum-dimension and using (2.6), we see that $\sigma 0 = \Lambda_2$. Thus for any (indecomposable) NIM-rep of $G_{2,1}$, $m_{\Lambda_2} = m_0 = 1$, and the NIM-rep must be 2-dimensional. It is now trivial to find it:

$$\mathcal{N}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathcal{N}_{\Lambda_2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and the fusion graph of $\Lambda_2$ is the tadpole $T_2$.

The only modular invariant [26] is $M = I$, which is paired with $T_2$.

The situation is completely identical for $F_{4,1}$: $P_+ = \{0, \Lambda_4\}$ here, and the fusion-isomorphism identifies $\Lambda_4$ with $\Lambda_2$. There is again only one NIM-rep and one modular invariant, and again the graph is the tadpole $T_2$. 

25
5. The unitary and orthogonal algebras at level 2

5.1. \(\hat{\mathfrak{sl}}(n)\) at level 2.

Consider next \(\hat{\mathfrak{sl}}(n) = A_{n-1}^{(1)}\) at level 2. The weights \(\lambda\) are all of the form \(\lambda(ab) := \Lambda_a + \Lambda_b\), for \(0 \leq a, b < n\). Since \(\lambda(ab) = \lambda(ba)\), we will usually require \(a \leq b\).

The simple-current \(J\) and charge-conjugation \(C\) act on \(P_+\) by:

\[
J\lambda(ab) = \lambda(a + 1, b + 1) , \quad C\lambda(ab) = \lambda(n - b, n - a)
\]

\(J\) has order \(n\). For any divisor \(d\) of \(n\), we get the modular invariant \(M[J^d]\) for \(\mathfrak{sl}(n)_2\) given in (2.9), where \(Q_{J^d}(\lambda(ab)) = d(a + b)/n\) and \(R_{J^d} = 2d\). For example, \(M[J^n] = I\) and \(M[J] = C\).

The remaining, exceptional, \(\mathfrak{sl}(n)_2\) modular invariants \(\mathcal{E}^{(n,2)}\) are [37]

\[
\mathcal{E}^{(10,2)} = \sum_{i=0}^{9} |\chi_{\lambda}(i,i) + \chi_{\lambda}(i+3,i+7)|^2 + \sum_{i=0}^{4} |\chi_{\lambda}(i,i+3) + \chi_{\lambda}(i+5,i+8)|^2
\]

\[
\mathcal{E}^{(16,2)} = \sum_{i=0}^{7} (|\chi_{\lambda}(i,i) + \chi_{\lambda}(i+8,i+8)|^2 + |\chi_{\lambda}(i,i+4) + \chi_{\lambda}(i+8,i-4)|^2 + |\chi_{\lambda}(i,i+8)|^2
\]

\[
+ |\chi_{\lambda}(i,i+6) + \chi_{\lambda}(i+8,i-2)|^2 + (\chi_{\lambda}(i+3,i+5) + \chi_{\lambda}(i-5,i-3)) \chi_{\lambda}^*(i,i+8)
\]

\[
+ \chi_{\lambda}(i,i+8) (\chi_{\lambda}(i+3,i+5) + \chi_{\lambda}(i-5,i-3))^*)
\]

\[
\mathcal{E}^{(28,2)} = \sum_{i=0}^{13} (|\chi_{\lambda}(i,i) + \chi_{\lambda}(i+14,i+14) + \chi_{\lambda}(i+5,i-5) + \chi_{\lambda}(i-9,i+9)|^2
\]

\[
+ |\chi_{\lambda}(i+3,i-3) + \chi_{\lambda}(i-11,i+11) + \chi_{\lambda}(i+6,i-6) + \chi_{\lambda}(i-8,i+8)|^2)
\]

together with the matrix products \(C \cdot \mathcal{E}^{(10,2)}\), \(C \cdot \mathcal{E}^{(16,2)}\), \(\frac{1}{2}M[J^4] \cdot \mathcal{E}^{(16,2)}\), and \(C \cdot \mathcal{E}^{(28,2)}\).

Note the strong resemblance of the exceptional modular invariants here to the so-called \(E_6, E_7, E_8\) exceptional of \(\hat{\mathfrak{sl}}(2)\) [27]. This is not a coincidence, and is a consequence of a duality between \(\hat{\mathfrak{sl}}(n)\) level \(k\), and \(\hat{\mathfrak{sl}}(k)\) level \(n\). See also the resemblance between (A.1) and the \(S\) matrix for \(\hat{\mathfrak{sl}}(2)\) level \(n\).

We next turn to the NIM-reps. The proof that our list is complete, is given in §A.1. Write \(n = 2^h m\) where \(m\) is odd. We know \(J = \lambda(11)\) and \(\Lambda_1 = \lambda(01)\) are fusion-generators, so are \(J^m, J^{2^h}\) and \(\lambda := J^{(m-1)/2}\Lambda_1 = \lambda(m-1, m+1)/2\). Thus, the NIM-rep is uniquely defined once the matrices \(A := \mathcal{N}_\lambda, P' := \mathcal{N}_{J^m}\) and \(P'' := \mathcal{N}_{J^{2^h}}\) are known. The reason it is more convenient to use these fusion-generators is Lemma A in the appendix — roughly, the matrix \(A\) is nearly symmetric, and its failure to be symmetric is governed by the permutation matrix \(P'\).

The matrix \(A\) comes from the disjoint union of equivalent diagrams taken from Figure 3. Each of those diagrams \(X_n(k)\) corresponds to a digraph, as follows. Number the diagram nodes from 1 to \(n\), say from left to right, top to bottom. The weight \(k\) or \(2k\) of each node tells how many vertices are represented by that node. So each vertex in the digraph will
be labelled by a pair \((v, i)\), where \(v\) is the number of the node in the diagram, and \(i\) runs from 1 to the weight of that node (we take it modulo the weight). Suppose nodes \(v < v'\) are adjacent to each other in the diagram. If they have identical weight (\(k\) say), put a directed edge from \((v, i)\) to \((v', i)\), and from \((v', i)\) to \((v, i + 1)\). If nodes \(v < v'\) have weights \(2k\) and \(k\), respectively, then draw directed edges from \((v, i)\) to \((v', i)\), and from \((v', i)\) to both \((v, i + 1)\) and \((v, i + k + 1)\). If nodes \(v < v'\) have weights \(k\) and \(2k\), respectively, then draw directed edges from \((v, i)\) to both \((v', i)\) and \((v', i + k)\), and from \((v', i)\) to \((v, i + 1)\). The digraph corresponding to \(A_3(4)\) is given in Figure 4 — note there the \(k = 4\) vertical copies of \(A_3\).

![Figure 4. The digraph \(A_3(4)\)](image)

The matrix \(A\) will consist of \(d\) disconnected copies of a digraph \(X_s(2^\ell)\) taken from Figure 3, for some divisor \(d\) of \(m\) and some \(\ell \leq h\). The order-\(d\) permutation \(P''\) takes vertex \((v, i)\) of the \(j\)th component digraph, to vertex \((v, i)\) of the \((j + 1)\)-th component digraph (where \(j + 1\) is taken mod \(d\)). The order \(2^\ell\) or \(2^{\ell + 1}\) permutation \(P'\) maps each component to itself, and takes the vertex \((v, i)\) to the vertex \((v, i + 1)\), where \(i + 1\) is taken modulo the weight \(2^\ell\) or \(2^{\ell + 1}\) of the node \(v\).

So once we know the matrix \(A = N_\lambda\), or equivalently the digraph \(X_s(2^\ell)\) from Figure 3 and its multiplicity \(d\), then in principle we know the entire NIM-rep, using the above prescription and the \(\mathfrak{sl}(n)\) fusions in \((A.2)\).

The complete list of indecomposable NIM-reps are:

(i) **Valid whenever 4 divides \(n\):** Choose any divisor \(2^\ell d\) of \(n/2\) (where \(d\) is odd). Then the matrix \(A\) corresponds to \(d\) copies of the digraph \(D_{(n+4)/2}(2^\ell)\). The result is a \(2^{\ell - 1}d(n + 4)\)-dimensional NIM-rep which we will denote by \(N_{\ell}(\frac{n+4}{2}; 2^\ell d)\). It has exponents consisting of all \(\lambda(i, j)\) for which \(n/(2^{\ell}d)\) divides \(i + j\). Each of these \(\lambda(i, j)\) has multiplicity 1 except for the fixed-points \(j - i = n/2\), which have multiplicity 2.

For another class, choose any divisor \(2^\ell d\) of \(n/4\) (again \(d\) is odd). Then the matrix \(A\) corresponds to \(d\) copies of the digraph \(C_{(n+2)/4}(2^\ell)\). The resulting \(d2^{\ell - 1}(n + 4)\)-dimensional NIM-rep is denoted \(N_C(\frac{n+2}{2}; 2^\ell d)\). Its exponents consist of all \(\lambda(i, j)\) for which \(n/(2^{\ell}d)\) divides \(i + j\), together with the fixed-points \(\lambda(\frac{m}{2^{\ell}d}, i, \frac{m}{2^{\ell}d}, i + \frac{n}{2})\); all have multiplicity 1.

(ii) **Valid whenever \(n\) is odd:** Choose any divisor \(d\) of \(n\). Then the matrix \(A\) corresponds to \(d\) copies of the tadpole \(T_{(n+1)/2}\). The resulting \(d\)-dimensional NIM-rep will be denoted \(N_T(\frac{n+1}{2}; d)\). It has exponents consisting of all \(\lambda(i, j)\) for which \(n/d\) divides \(i + j\) — all with multiplicity 1.
(iii) Valid whenever n is even: Choose any odd divisor d of n. Then the matrix A corresponds to d copies of the digraph $B_{(n+2)/2}(2^{h-1})$. The resulting $d(n+1)2^{h-1}$-dimensional NIM-rep will be denoted $N_B(\frac{n+2}{2}; d)$. It has exponents consisting of all $\lambda(i,j)$ for which $\frac{n}{d}$ divides $i + j$ — all with multiplicity 1.

(iv) Valid whenever n/2 is odd: Choose any odd divisor d of n. Then the matrix A corresponds to d copies of the digraph $C_{(n+2)/2}(1)$. The resulting $d\frac{n+4}{2}$-dimensional NIM-rep will be denoted $N_C(\frac{n+2}{2}; d)$. It has exponents consisting of all $\lambda(i,j)$ for which $\frac{n}{d}$ divides $i + j$, together with the fixed-points $\lambda(\frac{n}{d}i, \frac{n}{d}i + \frac{n}{2})$; all have multiplicity 1.

(v) Only for $\widehat{sl}(10)$: Choose either $d = 1$ or $d = 5$. Then the matrix A corresponds to d copies of the digraph $F_4(1)$. The resulting $6d$-dimensional NIM-rep will be denoted $N_{F_4}(d)$. It has exponents

$$\mathcal{E} = \{\lambda(\frac{5}{d}i, \frac{5}{d}i), \lambda(\frac{5}{d}i + 1, \frac{5}{d}i + 4), \lambda(\frac{5}{d}i + 2, \frac{5}{d}i + 8) | 0 \leq i < 2d\}$$

All those primaries have multiplicity 1.

(vi) Only for $\widehat{sl}(16)$: Choose any $0 \leq \ell \leq 3$. Then the matrix A corresponds to the digraph $E_7(2^\ell)$. The resulting $2^\ell 7$-dimensional NIM-rep will be denoted $N_{E_7}(2^\ell)$. It has exponents

$$\mathcal{E} = \{\lambda(2^{3-\ell}i, 2^{3-\ell}i), \lambda(2^{3-\ell}i+2, 2^{3-\ell}i+14), \lambda(2^{3-\ell}i+3, 2^{3-\ell}i+13), \lambda(2^{3-\ell}i+4, 2^{3-\ell}i+12)\}$$

where $i,j$ range over $0 \leq i < 2^{\ell+1}$ and $0 \leq j < 2^\ell$. All those primaries have multiplicity 1.

(vii) Only for $\widehat{sl}(28)$: Choose any divisor $2^\ell d$ of 14 (take d odd). Then the matrix A corresponds to d copies of the digraph $E_8(2^\ell)$. The resulting $2^{3+\ell}d$-dimensional NIM-rep will be denoted $N_{E_8}(2^\ell d)$. It has exponents

$$\mathcal{E} = \{\lambda(\frac{14}{2^\ell d}i, \frac{14}{2^\ell d}i), \lambda(\frac{14}{2^\ell d}i + 3, \frac{14}{2^\ell d}i + 25), \lambda(\frac{14}{2^\ell d}i + 5, \frac{14}{2^\ell d}i + 23), \lambda(\frac{14}{2^\ell d}i + 6, \frac{14}{2^\ell d}i + 22)\}$$

where $i$ ranges over $0 \leq i < 2^{\ell+1}d$. All those primaries have multiplicity 1.

For example, sl(10)2 has six NIM-reps: $N_B(6; 5), N_B(6; 1), N_C(6; 5), N_C(6; 1), N_{F_4}(5)$ and $N_{F_4}(1)$. We will find next that these are in precise one-to-one correspondence with the six sl(10)2 modular invariants.

All sl(n)2 modular invariants are uniquely NIMmed. In particular, for n odd, $M[J^d]$ corresponds to the tadpole NIM-rep $N_T(\frac{n+1}{2}; d)$. When both n − 1 and n/d are odd, $M[J^d]$ corresponds to $N_B(\frac{n+2}{2}; d_o)$. Otherwise d divides n/2: when d is even or odd, respectively, $M[J^d]$ corresponds to $N_D(\frac{n+4}{2}; d)$ and $N_C(\frac{n+2}{2}; d)$. We use here ‘$d_o$’ to denote the odd part of d, i.e. d/d_o is a power of 2.

Note that $N_D(\frac{n+4}{2}; odd)$ and $N_C(\frac{n+2}{2}; even)$ aren’t paired to any modular invariant, when 4 divides n. Our classification overlaps the A-D-E-T one of sl(2)k, at sl(2)2; note that in our notation the single NIM-rep there corresponds to diagram $C_2(1)$ and not $A_3(1)$, because the simple-current $N_J$ should have order 2, not 1.

The sl(10)2 exceptional modular invariants $E^{(10,2)}$ and $C \cdot E^{(10,2)}$ correspond to the NIM-reps $N_{F_4}(5)$ and $N_{F_4}(1)$, respectively. The sl(16)2 exceptions $E^{(16,2)}, C \cdot E^{(16,2)}$ and
\(\frac{1}{2} M[J^4] \cdot E^{(16,2)}\) correspond to \(N_{E7}(8), N_{E7}(4)\) and \(N_{E7}(2)\) respectively. Finally, the \(sl(28)\) exceptions correspond to \(N_{E8}(14)\) and \(N_{E8}(2)\), respectively.

The only remarkable thing about this NIM-rep ↔ modular invariant classification is how well they match: all but the exceptional NIM-reps \(N_{E8}(1)\) and \(N_{E8}(7)\) are paired to a unique modular invariant, except when \(n\) is a multiple of 4.

5.2. \(\hat{so}(\text{odd})\) at level 2.

Consider \(\hat{so}(n) = B_r^{(1)}\), where \(n = 2r + 1\). The set \(P_+\) consists of precisely \(r+4\) weights, which we’ll name as follows: 0, 2\(\Lambda_1\), \(\Lambda_r\), \(\Lambda_1 + \Lambda_r\), \(\gamma_i := \Lambda_i\) for \(i < r\), and \(\gamma^r := 2\Lambda_r\). Write \(\gamma^0\) for the weight 0. The simple-current is 2\(\Lambda_1 = J\); it fixes all \(\gamma^1, \ldots, \gamma^r\). The spinors are \(\Lambda_r\) and \(J\Lambda_r = \Lambda_1 + \Lambda_r\). For the additional modular invariants existing when \(n\) is a perfect square, the following notation is convenient: if \(8 \mid r\), write \(\lambda^r := \Lambda_r\) and \(\mu^r := J\Lambda_r\); otherwise write \(\lambda^r := J\Lambda_r\) and \(\mu^r := \Lambda_r\). Also write \(C = \{\gamma^a \neq 0 \mid \sqrt{n}\) divides \(a\}\).

Very atypically for the current algebras, the list of modular invariants for so(\(n\)) is messy. Define matrices \(\mathcal{B}(d, \ell), \mathcal{B}(d_1, \ell_1|d_2, \ell_2), \mathcal{B}^i, \mathcal{B}^{ii}, \mathcal{B}^{iii}, \mathcal{B}^{iv}\) by:

\[
\mathcal{B}(d, \ell)_{J^i\gamma^a, J^j\gamma^b} = \begin{cases} 
2 & \text{if } d|a, d|b, \text{ and both } a \neq 0, \ b \neq 0 \\
0 & \text{if either } n \not{|} da \text{ or } b \not{|} \pm a\ell \text{ (mod } d) \\
1 & \text{otherwise}
\end{cases}
\]

\[
\mathcal{B}(d, \ell)_{J^i\Lambda_r, J^j\Lambda_r} = 1
\]

where \(a, b \in \{0, 1, \ldots, r\}\) and \(i \in \{0, 1\}\), and make all other matrix entries 0;

\[
\mathcal{B}(d_1, \ell_1|d_2, \ell_2) = \frac{1}{2}(\mathcal{B}(d_1, \ell_1) + \mathcal{B}(d_2, \ell_2)) M[J]
\]

\[
\mathcal{B}^i_{00} = \mathcal{B}^{ii}_{00} = \mathcal{B}^{iii}_{00} = \mathcal{B}^{iv}_{00} = 1
\]

and all other entries are 0, where \(\gamma, \gamma' \in C\). Finally, \(\mathcal{B}^{iii} := \mathcal{B}^i M[J]\) and \(\mathcal{B}^{iv} := M[J] \mathcal{B}^i\).

By ‘\(n \not{|} da\)’ in the definition of \(\mathcal{B}(d, \ell)\), we mean that \(n\) does not divide \(da\). By ‘\(b \neq \pm a\ell \) (mod \(d)\)’ there we mean that \(b\) is congruent mod \(d\) to neither \(a\ell\) nor \(-a\ell\).

In [38] we proved that the modular invariants of \(\hat{so}(n)\) are \(B_{r,2}\) precisely:

(a) \(\mathcal{B}(d, \ell)\) for any divisor \(d\) of \(n = 2r + 1\) obeying \(n|d^2\), and for any integer \(0 \leq \ell < \frac{d^2}{2n}\) obeying \(\ell^2 \equiv 1 \text{ (mod } \frac{d^2}{n})\);  

(b) \(\mathcal{B}(d_1, \ell_1|d_2, \ell_2)\) for any divisors \(d_i\) of \(n\) obeying \(n|d_i^2\), and for any integers \(0 \leq \ell_i < \frac{d_i^2}{2n}\) obeying \(\ell_i^2 \equiv 1 \text{ (mod } \frac{d_i^2}{n})\);  

(c) when \(n\) is a perfect square, there are four remaining modular invariants: \(\mathcal{B}^i, \mathcal{B}^{ii}, \mathcal{B}^{iii}, \text{ and } \mathcal{B}^{iv}\).

The only redundancy here is that \(\mathcal{B}(d_1, \ell_1|d_2, \ell_2) = \mathcal{B}(d_2, \ell_2|d_1, \ell_1)\). The simple-current invariants are \(\mathcal{B}(n, 1) = I\) and \(\mathcal{B}(n, 1|n, 1) = M[J]\). For example, when \(3 \leq r \leq 10\),
respectively, there are precisely 2, 9, 2, 5, 2, 2, and 5 modular invariants for \( B_{r,2} \). The nine \( B_{4,2} \) modular invariants are: \( B(9, 1) = I \),

\[
B(9, 1) | 9, 1) = |\chi_{0000} + \chi_{2000}|^2 + 2|\chi_{1000}|^2 + 2|\chi_{0100}|^2 + 2|\chi_{0010}|^2 + 2|\chi_{0002}|^2 \\
B(3, 1) = |\chi_{0000} + \chi_{0010}|^2 + |\chi_{2000} + \chi_{0010}|^2 + |\chi_{0001}|^2 + |\chi_{1001}|^2 \\
B(3, 1) | 3, 1) = |\chi_{0000} + \chi_{2000} + 2\chi_{0010}|^2 \\
B(3, 1) | 9, 1) = |\chi_{0000} + \chi_{2000} + \chi_{0010}|^2 + 2|\chi_{0010}|^2 + |\chi_{1000}|^2 + |\chi_{0002}|^2 \\
B^i = |\chi_{0000} + \chi_{0010}|^2 + (\chi_{2000} + \chi_{0010}) \chi_{1001} \chi_{1001}^* + \chi_{1001} (\chi_{2000} + \chi_{0010})^* + |\chi_{0001}|^2 \\
B^ii = |\chi_{0000} + \chi_{0010} + \chi_{1001}|^2 \\
B^iii = (\chi_{0000} + \chi_{0010} + \chi_{1001}) (\chi_{0000} + \chi_{2000} + 2\chi_{0010})^* \\
B^iv = (\chi_{0000} + \chi_{2000} + 2\chi_{0010}) (\chi_{0000} + \chi_{0010} + \chi_{1001})^*
\]

In expressing our NIM-reps as explicitly as possible, we will use the following matrices. By \( \ell \times \ell \), we mean the \( \ell \times \ell \) matrix, all of whose entries equal 1. Write \( 0_{\ell m} \) for the \( \ell \times m \) zero-matrix, and \( I_m \) for the \( m \times m \) identity. Write \( B_{\ell m}(a) \) for the \( \ell \times m \) matrix defined by

\[
B_{\ell m}(a) = \begin{pmatrix}
2a \cdot 1_{\ell - 2, m - 2} & a \cdot 1_{\ell - 2, 1} & a \cdot 1_{\ell - 2, 1} \\
1_{1, m - 2} & x & x - 1 \\
1_{1, m - 2} & x - 1 & x
\end{pmatrix} = \begin{pmatrix}
2a & \cdots & 2a & a & a \\
\vdots & \ddots & \vdots & \vdots \\
2a & \cdots & 2a & a & a \\
\vdots & \ddots & 2a & a & x \\
a & \cdots & a & x & x - 1 \\
a & \cdots & a & x - 1 & x
\end{pmatrix}
\] (5.1)

where \( x = \frac{2a + 1}{\ell} \). For any integers \( m, i \), define \( M^{(m|i)} \) to be the \( m \times m \) ‘off-diagonal’ matrix with entries \( M^{(m|i)}_{ab} = \delta_{b, a + i \text{ mod } m} \). Put \( M^{(m|i, j)} = M^{(m|i)} + M^{(m|j)} \). So \( M^{(m, 1, -1)} \) is the adjacency matrix for the circle, i.e. the graph \( A_{m-1}^{(1)} \). Put \( \tilde{M}(\ell) \) for the \( \ell \times \ell \) adjacency matrix for the \( D_\ell^0 \) graph (see Figure 2), where the loop is at 1, the branch is at \( \ell - 2 \), the degree-1 vertices are at \( \ell - 1 \) and \( \ell \), and the other vertices are numbered sequentially in the obvious way. For example,

\[
M^{(4|1, -1)} = \begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0
\end{pmatrix} \quad \text{and} \quad \tilde{M}^{(4)} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}
\]

Recall the graphs in Figure 2. For later convenience in this subsection, we will identify \( A_{0,1}^{(1)} \) with \( \mathcal{A}_1^0 \), i.e. the matrix (2), and \( D_2^0 \) with \( \mathcal{A}_2^0 \), i.e. the matrix \( \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \).

There are four classes of NIM-reps:

(i) **Provided \( n \) is a perfect square:** choose any integer \( m \geq 1 \) such that \( m \) divides \( \sqrt{n} \). Put \( \mathcal{N}(m)_{\gamma_j} = M^{(m|i, -1)} \), \( \mathcal{N}(m)_{j} = I_m \), and

\[
\mathcal{N}(m)_{\Lambda_1} = \mathcal{N}(m)_{\Lambda_1 + \Lambda_r} = \frac{\sqrt{n}}{m} \cdot 1_{mm}
\]
This defines an $m$-dimensional NIM-rep with exponents $E = \{0, 2\langle \gamma^{n/m} \rangle\}$, where we use the short-hand $\langle \gamma^d \rangle := \{\gamma^d, \gamma^{2d}, \ldots, \gamma^{(n-d)/2}\}$ for $d$ dividing $n$ (i.e. all $\gamma^i$, $1 \leq i \leq r$, where $d$ divides $i$). The coefficient ‘2’ in this $E$ means each of these $\gamma^{in/m}$’s appear with multiplicity 2.

The fusion graph of $N(m)_{\Lambda_i}$ is the circle $A^{(1)}_{m-1}$. The simplest example is the 1-dimensional NIM-rep given by quantum-dimension: $\gamma^i \mapsto 2$, $\Lambda_r \mapsto \sqrt{n}$.

**(ii)** Provided $n$ is a perfect square: Choose either spinor $\sigma = \Lambda_r$ or $\Lambda_1 = \Lambda_r$, and choose any integer $m \geq 2$ such that $2m - 3$ divides $\sqrt{n}$. Put $N'(m,\sigma)_{\Lambda_i} = \tilde{M}(m)$. The other matrices $N'(m,\sigma)_{\gamma^i}$ can now be constructed recursively from (A.4a),(A.4b) — more on this shortly. Put $N'(m,\sigma)_{J} = I_{m-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $N'(m,\sigma)_{\sigma} = B_{mm}(a)$ (see (5.1)), where $a = \sqrt{n}/(2m - 3)$. The matrix $N'(m,\sigma)_{J_{\sigma}}$ for the other spinor is the same except with $x$ and $x - 1$ interchanged in the bottom right $2 \times 2$ block of (5.1).

$N'(m,\sigma)$ is an $m$-dimensional NIM-rep with exponents $E = \{0, \sigma, \langle \gamma^{n/(2m - 3)} \rangle\}$. $\Lambda_1$ has fusion graph $D^0_{m}$. The simplest example is $\gamma^i \mapsto \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $J \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\Lambda_r \mapsto \begin{pmatrix} x & x - 1 \\ x - 1 & x \end{pmatrix}$ where $x = (\sqrt{n} + 1)/2$.

**(iii)** Valid for any $n$: Choose any $m \geq m' \geq 1$ such that $\sqrt{n/m'm'} \in \mathbb{Z}$. Put $N(m, m')_{\gamma^i} = M^{(m_ii,i)} \oplus M^{(m'i,i)}, N(m, m')_{J} = I_{m + m'}$, and

$$N(m, m')_{\Lambda_r} = N(m, m')_{\Lambda_1 + \Lambda_r} = \begin{pmatrix} 0_{mm} & b \cdot 1_{m'm} \\ b \cdot 1_{m'm} & 0_{m'm} \end{pmatrix}$$

where $b = \sqrt{n/m'm'}$.

This is an $(m + m')$-dimensional NIM-rep with exponents $E = \{0, J, 0, 2\langle \gamma^{n/m'} \rangle, 2\langle \gamma^{n/m'} \rangle\}$. The fusion graph of $\Lambda_1$ consists of two circles: $A^{(1)}_{m-1} \cup A^{(1)}_{m'-1}$. The simplest possible example of this NIM-rep is $\gamma^i \mapsto \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\Lambda_r \mapsto \begin{pmatrix} 0 & \sqrt{n} \\ \sqrt{n} & 0 \end{pmatrix}$.

**(iv)** Valid for any $n$: Choose any integers $m \geq m' \geq 2$ such that $\sqrt{(2m - 3)(2m' - 3)} \in \mathbb{Z}$. The matrices $N'(m, m')_{\gamma^i}$ are the direct sums $N'(m, \Lambda_r)_{\gamma^i} \oplus N'(m', \Lambda_r)_{\gamma^i}$ of matrices of NIM-rep (ii). Put $N'(m, m')_{J} = I_{m-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus I_{m'-2} \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $N'(m, m')_{\Lambda_r} = \begin{pmatrix} 0 & B \\ B^t & 0 \end{pmatrix}$, where $B = B_{mm'}(a)$ for $a = \sqrt{n/(2m - 3)(2m' - 3)}$. The matrix for the other spinor, $\Lambda_1 + \Lambda_r$, is the same except with $x$ and $x - 1$ interchanged in both $B$ and $B^t$ (see (5.1)).

$N'(m, m')$ is an $(m + m')$-dimensional NIM-rep, with exponents $E = \{0, J, \Lambda_r, \Lambda_1 + \Lambda_r, \langle \gamma^{n/(2m - 3)} \rangle, \langle \gamma^{n/(2m' - 3)} \rangle\}$. The fusion graph of $\Lambda_1$ is $D^0_{m} \cup D^0_{m'}$. It may seem that we’ve ‘broken the symmetry’ between $\Lambda_r$ and $\Lambda_1 + \Lambda_r$, and so there should be another NIM-rep with the images of $\Lambda_r$ and $\Lambda_1 + \Lambda_r$ interchanged (as we did in (ii)). However, these two NIM-reps are equivalent here (and they aren’t in (ii)).
The simplest example is $\gamma^i \mapsto \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right) \oplus \left( \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right)$, $J \mapsto \left( \begin{array}{ccc} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{array} \right)$, and $\Lambda_r \mapsto \left( \begin{array}{ccc} 0 & 0 & x & x-1 \\ 0 & 0 & x-1 & x \\ x & x-1 & 0 & 0 \\ x-1 & x & 0 & 0 \end{array} \right)$

for $x = \sqrt{\frac{n+1}{2}}$.

The proof of this NIM-rep classification is deferred to Appendix A.2. Note that each of these $B_{r,2}$ NIM-reps has a different set $E$ of exponents.

In (i) and (iii) we could explicitly write all matrices $N_{\gamma^i}$. This is much harder in (ii) (though they are constructable recursively by (A.4a),(A.4b)). We’ll make only the following remark: the graph for $N'(m, \sigma)_{\gamma^i}$ will consist of one $D_0^0$-type component containing the nodes $m-1, m$, and precisely $\frac{gcd(i,2m-3)-1}{2} A_0^0$-type components of equal size.

Which of the modular invariants are NIMmed? We find that the exponents for $B(d, \ell)$ are $\{0, J_0, \Lambda_r, J\Lambda_r, \langle \gamma^m \rangle, \langle \gamma^{n/m} \rangle\}$ where $m = gcd(d, (\ell-1)\frac{n}{\ell})$. There is one and only one NIM-rep corresponding to $B(d, \ell)$, namely $N'(\frac{m+3}{2}, \frac{n+3m}{2m})$. Note that the square-root condition is automatically obeyed. For example, the identity modular invariant $I = B(n, 1)$ corresponds to $N'(r+2, 2)$.

The exponents of $B(d, \ell|d', \ell')$ are $\{0, J_0, \langle \gamma^m \rangle, \langle \gamma^{n/m} \rangle, \langle \gamma^m' \rangle, \langle \gamma^{n/m'} \rangle\}$ where $m = gcd(d, (\ell-1)\frac{m}{\ell})$ and $m' = gcd(d', (\ell'-1)\frac{m}{\ell'})$. These have a corresponding NIM-rep iff both $d = d'$ and $\ell = \ell'$, in which case the NIM-rep is given by $N'(m, n/m)$. The simple-current extension $M[J] = B(n, 1|n, 1)$ corresponds to $N'(n, 1)$.

When $\sqrt{n}$ is a perfect square, we get the four additional modular invariants $B^i, \ldots, B^{iv}$. Note that $B^i$ and $B^{ii}$ have exponents $\{0, \mu^*, \langle \gamma^{n/m} \rangle\}$ and $\{0, \lambda^*, \langle \gamma^{n/m} \rangle\}$ respectively, and so correspond to $N'(\frac{\sqrt{n}+3}{2}, \mu^*)$ and $N'(\frac{\sqrt{n}+3}{2}, \lambda^*)$, respectively (both $\lambda^*$ and $\mu^*$ are defined at the beginning of this subsection). Both $B^{iii}$ and its transpose $B^{iv}$ have the same exponents, namely $\{0, 2, \langle \gamma^{\sqrt{n}} \rangle\}$, and so correspond to NIM-rep $N'(\sqrt{n})$.

In other words, only the following NIM-reps will have an associated modular invariant: $N'(\sqrt{n}), N'(\sqrt{n}, \Lambda_r), N'(\sqrt{n}, \Lambda_1 + \Lambda_r), N'(\sqrt{n}, m, \frac{a}{m})$ and $N'(\sqrt{n}, m, \frac{a}{m})$ (for any divisor $m$ of $n$).

For example, we gave earlier explicitly the nine different modular invariants for $so(9)$. For each of these, in the order given above, the fundamental weight $\Lambda_1$ corresponds to the fusion graph $D_0^0 \cup 0 A_2^0, A_8^{(1)} \cup 0 A_1^0, D_3^0 \cup D_3^0, A_2^{(1)} \cup A_2^{(1)}, -, D_0^0, D_3^0, A_2^{(1)}$ and $A_2^{(1)}$. Only the last two have identical NIM-reps. Only $B(3, 1|9, 1)$ is NIM-less — an elementary proof of this is given in §6.

For larger rank, the number of NIM-less $so(n)_2$ modular invariants will typically far exceed the NIMmed ones: the former grows like $D^2$ while the latter grows like $D$, where $D$ is the number of divisors of $n$. For example, when $n$ is a power $p^a$ of a prime, the number of NIM-reps grows like $a^2/4$, while the NIMmed modular invariants grow like $a$ and the NIM-less ones grow like $a^2/8$.

All modular invariants for $so(n)_2$ will be NIMmed, iff $n$ is a prime. Other small ranks with NIM-less modular invariants are $so(15)_2$ (for $B(15, 1|15, 4)$), $so(21)_2$ (for $B(21, 1|21, 8)$), $so(25)_2$ (for $B(5, 1|25, 1)$), and $so(27)_2$ (for $B(9, 1|27, 1)$).
5.3. $\text{so}(even)$ at level 2.

Consider $\text{so}(n)_2 = D_{r,2}$, where $n = 2r$. There are $r + 7$ weights: 0, $2\Lambda_1$, $2\Lambda_{r-1}$, $2\Lambda_r$, $\Lambda_r$, $\Lambda_1 + \Lambda_{r-1}$, $\Lambda_{r-1}$, $\Lambda_1 + \Lambda_r$, $\lambda^i := \Lambda_i$ for $1 \leq i \leq r - 2$, and $\lambda^{r-1} := \Lambda_{r-1} + \Lambda_r$. Write $\lambda^0$ for the weight 0 and $\lambda^r$ for $2\Lambda_r$. There are three (nontrivial) simple-currents, namely $J_v = 2\Lambda_1$, $J_s = 2\Lambda_r$ and $J_c = 2\Lambda_{r-1}$. The simple-current $J_v$ fixes the $\lambda^i$ ($1 \leq i < r$). The four spinors are $\Lambda_r$, $\Lambda_{r-1}$, $J_v\Lambda_r = \Lambda_1 + \Lambda_{r-1}$, $J_v\Lambda_{r-1} = \Lambda_1 + \Lambda_r$. For the additional modular invariants occurring when $r$ is a perfect square, it is convenient to write $C_j = \{\lambda^b \neq 0 | 2\frac{b \cdot \mu}{r} \equiv \pm j \text{ (mod 8)}\}$ for $j = 0, 1, 2, 3, 4$.

Write $C_0 = I$, and let $C_1$ be the permutation of $P_+$ interchanging $\Lambda_{r-1} \leftrightarrow \Lambda_r$, $2\Lambda_{r-1} \leftrightarrow 2\Lambda_r$, and $\Lambda_1 + \Lambda_{r-1} \leftrightarrow \Lambda_1 + \Lambda_r$, and fixing all other weights. We call these $C_i$ conjugations because they correspond to symmetries of the unextended Dynkin diagram. Charge-conjugation $C$ for $D_{odd,2}$ is $C_1$, and for $D_{even,2}$ is $I$. When $r = 4$, there are four other conjugations, corresponding to so(8) triality.

Define the matrices $D(d, \ell)$, $D(d_1, \ell_1|d_2, \ell_2)$, $D^i$, $D^{ii}$, and $D^{iii}$, as follows:

\[
D(d, \ell)_{J_v^i \lambda^a, J_v^j \lambda^b} = \begin{cases} 
2 & \text{if } d|a, d|b, 2d|(a + b), \text{ and } \{a, b\} \subseteq \{1, \ldots, r - 1\} \\
0 & \text{if either } r \nmid da \text{ or } b \neq \pm a \ell \text{ (mod 2d)} \\
1 & \text{otherwise}
\end{cases}
\]

and all other entries are 0, where $a, b \in \{0, 1, \ldots, r\}$, $i \in \{0, 1\}$, and $\lambda_s$ is any spinor;

\[
D(d_1, \ell_1|d_2, \ell_2) = \frac{1}{2}(D(d_1, \ell_1) + D(d_2, \ell_2)) M[J_v]
\]

\[
D^i_{J_v^i \lambda^a, J_v^j \lambda^b} = D^i_{\lambda^a, \mu} = D^i_{\mu, \lambda^b} = D^i_{J_v^i \lambda^a, \mu'} = D^i_{\mu', J_v^i \lambda^b} = D^i_{\lambda^a, \mu'} = D^i_{\mu, \lambda^b} = D^i_{\lambda^a, \mu'} = D^i_{\mu, \lambda^b} = D^i_{\lambda^a, \mu'} = D^i_{\mu, \lambda^b} = 1
\]

where $\lambda, \lambda' \in C_0 \cup C_4$, $\mu \in C_1$, $\mu' \in C_3$, $\gamma, \gamma' \in C_2$, $\nu \in C_0$, $\nu' \in C_4$, $J', J'' \in J_s$, and $j \in \{0, 1\}$. All other entries equal 0. Finally, $D^{ii} = D^i M[J_v]$ and $D^{iii} = M[J_v] D^i$.

In [38] we proved that the modular invariants for $\text{so}(n)_2 = D_{r,2}$ are: (for arbitrary conjugations $C_i, C_j$)

(a) $C_i D(d, \ell) C_j$ for any divisor $d$ of $r$ obeying $r|d^2$, and for any integer $1 \leq \ell \leq \frac{d^2}{r}$ obeying $\ell^2 \equiv 1 \text{ (mod } \frac{4r^2}{r})$;

(b) $D(d_1, \ell_1|d_2, \ell_2)$ for any divisors $d_i$ of $r$ obeying $r|d_i^2$, as well as the additional property that $2d_1|r$ iff $2d_2|r$, and for any integers $1 \leq \ell_i \leq \frac{d_i^2}{r}$ obeying $\ell_i^2 \equiv 1 \text{ (mod } \frac{4r^2}{r})$;

(c) when $r$ is a perfect square and $16|r$, there are 8 other modular invariants: $C_i D^i C_j$, $C_i D^{ii} C_j$, and $D^{iii} C_j$.

Take $C_i = I$ in (a) unless $2d|r$. The number of these grows asymptotically with the square of the number of divisors of $r$. The following are simple-current invariants: $D(r, 1) = I$ and $D(r, 1|r, 1) = M[J_v]$ (for all $r$), and $D(r, r - 1) = M[J_v]$ and $D(r, r - 1|r, r - 1) =$
$M[J_v] M[J_s]$ (when $\frac{r}{2}$ is odd), and $\mathcal{D}(\frac{r}{2}, 1) = M[J_s]$ and $\mathcal{D}(\frac{r}{2}, 1|\frac{r}{2}, 1) = M[J_v] M[J_s]$ (when $4|r$).

For example, for $4 \leq r \leq 16$, respectively, there are precisely $16, 3, 7, 3, 8, 7$, and $7$ modular invariants. Of these, $0, 0, 1, 0, 0, 4$, and $1$ are exceptional. The seven modular invariants for $D_{6,2}$ are: the identity $\mathcal{D}(6, 1) = I$;

$$
\mathcal{D}(6, 5) = |\chi_{000000}|^2 + |\chi_{000020}|^2 + |\chi_{000001}|^2 + |\chi_{001000}|^2 + |\chi_{000100}|^2 + |\chi_{100000}|^2 + |\chi_{000001}|^2 + |\chi_{100001}|^2
$$

as well as the conjugates $C_1$ and $C_1 \mathcal{D}(6, 5)$.

Recall the matrices $I_m, 1_{mm'}$, $0_{mm'}$, and $M^{(m[i,j])}$ from §5.2. Let $C_{mn}$ and $C'_{mn}$ be the $m \times n$ checkerboard matrices, i.e. their $(i, j)$th entries are $1$ if $i + j$ is even/odd respectively (all other entries are $0$). Write $I^*_m$ for the $m \times m$ skew-identity:

$$
I^*_m = \begin{pmatrix}
0 & \cdots & 0 & 1 \\
\vdots & & 1 & 0 \\
0 & \ddots & \vdots & \\
1 & 0 & \cdots & 0
\end{pmatrix}
$$

The circle $A_3^{(1)}$ should also be interpreted here as the graph $'D_3^{(1)}'$, with adjacency matrix

$$
\begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix}
$$

The NIM-props of $\mathfrak{so}(2r) = D_r^{(1)}$ level 2 are as follows.

(i) Provided $r$ is a perfect square: Choose any odd divisor $m \geq 3$ of $\sqrt{r}$ (the case of even divisors will be treated shortly). Define $\mathcal{N}(m)_{\lambda'} = M^{(m[i,j])}$ and $\mathcal{N}(m)_{\lambda} = I_m$ for any of the simple-currents $J$. For all of the four spinors $\sigma$, put $\mathcal{N}(m)_{\sigma} = \sqrt{\frac{r}{m}} \cdot I_{mm'}$.

If $r$ is even, choose any even divisor $m$ of $2\sqrt{r}$, as well as either simple-current $J' = J_s$ or $J' = J_c$. Define $\mathcal{N}(m, J')_{\lambda'}$ and $\mathcal{N}(m, J')_{\lambda}$ as for $\mathcal{N}(k)$ (odd). Write $\sigma = \Lambda_r$ or $\sigma = \Lambda_{r-1}$ depending on whether or not $J' = J_s$. Then $\mathcal{N}(m, J')_{\lambda} = \mathcal{N}(m, J')_{\lambda} J_{c, \sigma} = \frac{2\sqrt{r}}{m} C_{mm'}$ and $\mathcal{N}(m, J')_{c, \sigma} = \mathcal{N}(m, J')_{c, \sigma} J_{c, \sigma} = \frac{2\sqrt{r}}{m} C_{mm'}$. 

34
These are $m$-dimensional NIM-reps. The fusion graph of $\Lambda_1$ is the circle $A_{m-1}^{(1)}$. When $m$ is odd, the exponents are $E = \{0, 2\langle \lambda^{2r/m} \rangle \}$, and when $m$ is even the exponents are $\{0, J', 2\langle \lambda^{2r/m} \rangle \}$. (We write $\langle \lambda^d \rangle$ for $\{\lambda^d, \lambda^{2d}, \ldots, \lambda^{-d/2} \}$ when $d|r$, and for $\{\lambda^d, \lambda^{2d}, \ldots, \lambda^{-d/2} \}$ when otherwise $d|2r$; the coefficient ‘2’ means all those primaries appear with multiplicity 2.)

(ii) Provided $r$ is a perfect square: Choose any divisor $m$ of $\sqrt{r}$. Let $N^0(m)_{\Lambda_1}$ be the adjacency matrix of $A_m^0$. How to get the other $N^0(m)_{\Lambda_i}$ will be explained shortly. Define $N^0(m)_{J_\sigma} = I_m$, and for any spinor $\sigma$ put $N^0(m)_{\sigma} = \sqrt{r/m} \cdot 1_{mm}$. If $r$ is even, then $N^0(m)_{J_\sigma} = N^0(m)_{J_\sigma} = I_m$, otherwise for odd $m$ $N^0(m)_{J_\sigma} = N^0(m)_{J_\sigma}$ will be the unique order-2 symmetry of $A_m^0$, namely the skew-identity $I_m^s$. $N^0(m)$ is an $m$-dimensional NIM-rep. The fusion graph of $\Lambda_1$ will be $A_m^0$. The exponents are $\{0, \langle \lambda^{r/m} \rangle \}$. The simplest example is quantum-dimension $\lambda \mapsto S_{\lambda_0}/S_{00}$.

(iii) Provided $r$ is even and a perfect square: Choose any $m \geq 5$ so that $\sqrt{r}/(m-3)$ is an odd integer, and choose either simple-current $J' = J_\sigma$ or $J' = J_\sigma$. Define $\sigma = \Lambda_r$ or $\sigma = \Lambda_{r-1}$, as in (i). Put $N'(m, J')_{\Lambda_1}$ to be the adjacency matrix of $D_{m-1}^{(1)}$. We’ll discuss how to obtain the matrices $N'(m, J')_{\Lambda_1}$ shortly. Put $N'(m, J')_{J_\sigma} = N'(m, J')_{J_\sigma} = I_2 \oplus I_{m-4} \oplus I_2^s$. Define

$$N'(m, J')_{J_\sigma} = \left( \begin{array}{ccc} X & aY_m & X \\ aY_m^t & 2aC_{m-4, m-4} & aY_m^t \\ X & aY_m & X' \end{array} \right)$$

$$N'(m, J')_{C_1\sigma} = N'(m, J')_{J_\sigma} = \sigma \left( \begin{array}{ccc} 0_{22} & Y'_m & 0_{22} \\ Y'_m^t & 2C'_{m-4, m-4} & Y'_m^t \\ 0_{22} & 0_{22} & 0_{22} \end{array} \right)$$

where $X = \left( \begin{array}{ccc} x & x' \\ x' & x \end{array} \right)$, $X' = \left( \begin{array}{ccc} x' & x \\ x & x' \end{array} \right)$, $Y_m = \left( \begin{array}{ccc} 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \cdots & & & \cdots \end{array} \right)$, $Y'_m = \left( \begin{array}{ccc} 1 & 0 & 1 & \cdots \\ 1 & 0 & 1 & \cdots \\ \cdots & & & \cdots \end{array} \right)$.

This NIM-rep is $m$-dimensional. The fusion graph for $\Lambda_1$ is $D_{m-1}^{(1)}$. The exponents are $\{0, J', \sigma, J_\sigma, \langle \lambda^{r/(m-3)} \rangle \}$.

(iv) Valid for any $r$: Choose any $m, m' \geq 1$ such that $\sqrt{r/mm'} \in \mathbb{Z}$. Define $N^{00}(m, m')_{\mu} = N^0(m)_{\mu} \oplus N^0(m')_{\mu}$ (see (ii) above) for any $\mu = \lambda^i$, and any simple-current $\mu$. For any spinor $\sigma$, put $N^{00}(m, m')_{\sigma} = \left( \begin{array}{ccc} 0 & A \\ A^t & 0 \end{array} \right)$ where $A = \sqrt{r/mm'} \cdot 1_{mm'}$.

Next, when in addition $m$ is odd and $\geq 3$, define another NIM-rep by $N^0(m, m')_{\mu} = N^0(m)_{\mu} \oplus N'(m')_{\mu}$ (see also (i) above) for any $\mu = \lambda^i$, and any simple-current $\mu$. For any spinor $\sigma$, put $N^0(m, m')_{\sigma} = N^{00}(m, m')_{\sigma}$.

Next, when both $m$ and $m'$ are odd and $\geq 3$, define $N(m, m')_{\mu} = N(m)_{\mu} \oplus N'(m')_{\mu}$ for any $\mu = \lambda^i$, and any simple-current $\mu$. For any spinor $\sigma$, put $N(m, m')_{\sigma} = N^{00}(m, m')_{\sigma}$. 

35
Finally, if \( m \) and \( m' \) are both even, we can weaken the condition \( \sqrt{r/mm'} \in \mathbb{Z} \) to \( \sqrt{4r/mm'} \in \mathbb{Z} \). Define \( \mathcal{N}(m,m')_\mu = \mathcal{N}(m)_\mu \oplus \mathcal{N}(m')_\mu \) for any \( \mu = \lambda^i \). The simple currents are \( \mathcal{N}(m,m')_{J_e} = I_{m+m'} \) and \( \mathcal{N}(m,m')_{J_s} = \mathcal{N}(m,m')_{J_c} = M^{(m|r)} \oplus M'(m'|r) \). For \( r \) even put
\[
\mathcal{N}(m,m')_{A_r} = \mathcal{N}(m,m')_{A_{1+r}} = a \begin{pmatrix}
0_{mm} & C_{mm'} \\
C_{m'm} & 0_{m'm'}
\end{pmatrix}
\]
while for \( r \) odd put
\[
\mathcal{N}(m,m')_{A_r} = \mathcal{N}(m,m')_{A_{1+r}} = a \begin{pmatrix}
0_{mm} & C_{mm'} \\
C'_{m'm} & 0_{m'm'}
\end{pmatrix}
\]
In both cases, \( a = 2\sqrt{r/mm'} \).

All of these \( \text{NIM-reps} \) are \( (m+m') \)-dimensional. Their exponents are, respectively,
\[
\mathcal{E}^{00} = \{0, J_v, \langle \lambda^{r/m}\rangle, \langle \lambda^{r/m'}\rangle\}
\]
\[
\mathcal{E}^0 = \{0, J_v, \langle \lambda^{r/m}\rangle, 2\langle \lambda^{r/m'}\rangle\}
\]
\[
\mathcal{E} = \{0, J_v, 2\langle \lambda^{r/m}\rangle, 2\langle \lambda^{r/m'}\rangle\}
\]
\[
\mathcal{E} = \{0, J_v, J_s, J_c, 2\langle \lambda^{r/m}\rangle, 2\langle \lambda^{r/m'}\rangle\}
\]
The fusion graph of \( \Lambda_1 \) is \( 0A_{m}^0 \cup 0A_{m'}^0, 0A_{m}^0 \cup A_{m-1}^{(1)}, A_{m-1}^{(1)} \cup A_{m-1}^{(1)}, A_{m-1}^{(1)} \cup A_{m-1}^{(1)} \), resp.

For instance, the conjugation \( C_1 \) corresponds to \( \mathcal{N}^{00}(r,1) \) and \( M[J_v] \) to \( \mathcal{N}(2r,2) \).

(v) Valid for any \( r \): Choose any \( m, m' \geq 4 \) such that \( a := \sqrt{r/(m-3)(m'-3)} \) is an odd integer — when \( r \) is even we require in addition that \( m \) be odd and \( m' \) even. Let \( \mathcal{N}''(m,m')_{\Lambda_1} \) be the adjacency matrix of the graph \( D_{m-1}^{(1)} \cup D_{m'-1}^{(1)} \). We’ll discuss shortly how to obtain the other matrices \( \mathcal{N}''(m,m')_{\Lambda_i} \). Put
\[
\mathcal{N}''(m,m')_{J_e} = I_2^s \oplus I_{m-4} \oplus I_2^s \oplus I_{m-4}^s \oplus I_2^s \oplus I_{m-4}^s \oplus I_2^s
\]
For \( r \) even, \( \mathcal{N}''(m,m')_{J_e} = I_{m} \oplus I_{m'}^s \) (see (5.2)), while for \( r \) odd \( \mathcal{N}''(m,m')_{J_s} = I_{m} \oplus I_{m'} \), where \( I_{m'} \) is the order-4 symmetry of the Dynkin diagram of \( D_{m-1}^{(1)} \), i.e. the \( m \times m \) matrix
\[
I_{m'} := \begin{pmatrix}
0_{2,m-2} & I_2 \\
I_2 & 0_{m-2,2}
\end{pmatrix}
\]
For \( r \) even, put \( \mathcal{N}''(m,m')_{A_r} = \begin{pmatrix}
0_{mm} & D \\
D^t & 0_{m'm'}
\end{pmatrix} \) and \( \mathcal{N}''(m,m')_{A_{r-1}} = \begin{pmatrix}
0_{mm} & E \\
E^t & 0_{m'm'}
\end{pmatrix} \)
where
\[
D = \begin{pmatrix}
X & aY_{m'} & 0_{22} \\
aY_m & 2aC_{m-4,m'-4} & aY_m^t \\
X & aY_{m'} & 0_{22}
\end{pmatrix}
\]
\[
E = \begin{pmatrix}
0_{22} & aY_m^t & X \\
aY_m & 2aC_{m-4,m'-4} & aY_{m'}^t \\
0_{22} & aY_{m'} & X
\end{pmatrix}
\]
For \( r \) odd, put \( \mathcal{N}''(m, m')_{\Lambda_r} = \begin{pmatrix} 0_{m m} & P \\ Q^t & 0_{m' m'} \end{pmatrix} = (\mathcal{N}''(m, m')_{\Lambda_{r-1}})^t \) where

\[
P = \begin{pmatrix} X \\ aY_m \\ 2aC_{m-4, m'-4} \\ 0_{22} \\ aY_m' \\ aY_m' \\ aY_m' \\ X' \end{pmatrix} \quad Q = \begin{pmatrix} 0_{22} & aY_m' \\ aY_m' & 2aC_{m-4, m'-4} \\ aY_m' & aY_m' \\ 0_{22} \\ X \end{pmatrix}
\]

Here, \( x = (a+1)/2, x' = (a-1)/2 \), and the matrices \( X, Y, \ldots \) are as in (iii). The matrices \( \mathcal{N}''(m, m')_{J_c}, \mathcal{N}''(m, m')_{J_s\Lambda_{r-1}} \) and \( \mathcal{N}''(m, m')_{J_s\Lambda_r} \) are obtained by the obvious matrix products.

These are \((m + m')\)-dimensional NIM-reps, with exponents

\[
\mathcal{E} = \{0, J_v, J_s, J_c; \langle \lambda^{r/(m-3)} \rangle, \langle \lambda^{r/(m'-3)} \rangle, \Lambda_r, \Lambda_{r-1}, \Lambda_1 + \Lambda_r, \Lambda_1 + \Lambda_{r-1}\}
\]

The fusion graph of \( \Lambda_1 \) is \( D^{(1)}_{m-1} \cup D^{(1)}_{m'-1} \) (recall that \( D^{(1)}_3 = A^{(1)}_3 \)).

(vi) Valid whenever \( 4 | r \): Choose any odd \( m, m' \geq 5 \) such that \( a := \sqrt{r/(m-3)(m'-3)} \) is an odd integer, and pick either \( \sigma \in \{\Lambda_r, \Lambda_{r-1}\} \). Put \( \mathcal{N}''(m, m', \sigma)_{J'} = I_{m+m'} \), where \( J' \) denotes \( J_s \) or \( J_c \) when \( \sigma = \Lambda_r \) or \( \Lambda_{r-1} \), respectively. Put \( \mathcal{N}''(m, m', \sigma)_{J_s\sigma} = \begin{pmatrix} 0_{m m} & D \\ E^t & 0_{m' m'} \end{pmatrix} \) and \( \mathcal{N}''(m, m', \sigma)_{C_c\sigma} = \mathcal{N}''(m, m', \sigma)_{J_sC_c\sigma} = \begin{pmatrix} 0_{m m} & E \\ E^t & 0_{m' m'} \end{pmatrix} \) where now

\[
D = \begin{pmatrix} X \\ aY_m \\ 2aC_{m-4, m'-4} \\ 0_{22} \\ aY_m' \\ aY_m' \\ aY_m' \\ X' \end{pmatrix} \quad E = \begin{pmatrix} 0_{22} & aY_m' \\ aY_m' & 2aC_{m-4, m'-4} \\ aY_m' & aY_m' \\ 0_{22} \\ X \end{pmatrix}
\]

Then \( \mathcal{N}''(m, m', \sigma)_{J_v} \) and the other matrices are as in (v).

This is an \((m + m')\)-dimensional NIM-rep, with exponents

\[
\mathcal{E} = \{0, J_v, J_s, J_c; \langle \lambda^{r/(m-3)} \rangle, \langle \lambda^{r/(m'-3)} \rangle, \sigma, \sigma, J_v, J_v\}
\]

The fusion graph of \( \Lambda_1 \) is \( D^{(1)}_{m-1} \cup D^{(1)}_{m'-1} \). The simple-current invariant \( M[J_s] \) corresponds to \( \mathcal{N}''(5, \frac{r}{2} + 3, \Lambda_r) \).

For any NIM-rep, the matrices for \( \lambda^i \) are obtained recursively from (A.6a),(A.6b). For the NIM-reps \( \mathcal{N}^0 \) and \( \mathcal{N}^{00} \) based on the \( 0A^0_m \) diagram, these are most easily found by using the explicit formula \( M[2m[i, -i]] \) for the \( A^{(1)}_{2m-1} \) graph, and then folding the result in the obvious way. For the NIM-reps \( \mathcal{N}' \) and \( \mathcal{N}'' \) based on the diagram \( D^{(1)}_{m-1} \), the matrices for \( \lambda^i \) will be a union of graphs from Figure 2. For \( i \) odd there will be a total of \((1 + \gcd(i, m - 3))/2 \) components, all bipartite. When \( i \) is even, and the exact power of 2 dividing \( i \) also divides \( m - 3 \), then there will be \( \gcd(i, m - 3)/2 \) bipartite components, together with one graph of type \( 0A^0 \). Otherwise, when the power of 2 dividing \( i \) exceeds that of \( m - 3 \), there will be \( 1 + \gcd(i, m - 3) \) components, none of them bipartite, and a total of two loops.
For instance, there are precisely eight NIM-reps for $D_{6,2}$: namely, $N^{00}(6,1)$, $N^0(6,1)$, $N^{00}(2,3)$, $N^0(2,3)$, $N(2,12)$, $N^0(4,6)$, $\mathcal{N}''(9,4)$, $\mathcal{N}''(5,6)$, of dimensions 7,7,5,5,14,10 respectively. Only $N^0(6,1)$ and $N^0(2,3)$ fail to have a corresponding modular invariant. Only the modular invariant $D(6,1)[6,5]$ is NIM-less.

More generally, the exceptions $D$ and $C_1D_iC_1$ correspond to $N''(\sqrt{r} + 3, J_s)$ and $N''(\sqrt{r} + 3, J_c)$, respectively, while both $C_1D^r$ and $D^rC_1$ correspond to $N^0(\sqrt{r})$. Both $D^{ii}$ and $D^{iii}$ correspond to $N(2\sqrt{r}, J_s)$, while both $C_1D^{ii}$ and $D^{iii}C_1$ correspond to $N(2\sqrt{r}, J_c)$.

Given parameters $d, \ell$, define $m$ as follows: if $4|\ell$, then $m = \gcd(d, \frac{(\ell+1)r}{2d})$. When $2d|r$, the modular invariant $D(d, \ell)$ corresponds to $N''(m + 3, \frac{r}{m} + 3, \Lambda_r)$ and $C_1D(d, \ell)C_1$ to $N''(m + 3, \frac{r}{m} + 3, \Lambda_r)$. Otherwise, $D(d, \ell)$ corresponds to $N''(m + 3, \frac{r}{m} + 3)$. In both cases, both $C_1D(d, \ell)$ and $D(d, \ell)C_1$ correspond to $N^0(m, r/m)$. The modular invariant $D(d_1, \ell_1|d_2, \ell_2)$ is NIMmed iff both $d_1 = d_2$ and $\ell_1 = \ell_2$, in which case the corresponding NIM-rep is $N(2m, \frac{2r}{m})$.

Triality for $so(8)$ introduces some additional $so(8)_2$ modular invariants, but all are NIMmed. In particular, the two order-3 conjugations both correspond to $N^0(2)$, while the other two additional conjugations correspond to $N''(5, J_s)$ and $N''(5, J_c)$. The additional modular invariants arising from conjugations of $D(2,1) = M[J_s]$ correspond to $N(4, J_s)$ and $N(4, J_c)$.

The current algebra $so(2r)_2 = D_{r,2}$ will have NIM-less modular invariants, unless either $r$ is prime, or $r = 4$ or 8.

6. The simplest NIM-less modular invariants

There is modular data canonically associated to finite groups [15]. The $S$ and $T$ matrices, and a list of modular invariants, is given in [34] for the symmetric group $S_3$. For convenience label the primary fields here 0, 1, \ldots, 7 as in [34]. Then:

$$S = \frac{1}{6} \begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 \\ 1 & 1 & 2 & 2 & 2 & -3 & -3 \\ 2 & 2 & 4 & -2 & -2 & -2 & 0 \\ 2 & 2 & -2 & 4 & -2 & -2 & 0 \\ 2 & 2 & -2 & -2 & 4 & 0 & 0 \\ 3 & -3 & 0 & 0 & 0 & 3 & -3 \\ 3 & -3 & 0 & 0 & 0 & -3 & 3 \end{pmatrix}$$

Some of its modular invariants don’t have an associated NIM-rep. For example consider

$$\mathcal{Z} = (c_0 + c_1 + c_2 + c_3)(c_0 + c_2 + c_6)^*$$

Its exponents(=spin-0 primaries) are the primaries ‘0’ and ‘2’, so we’re looking for a 2-dimensional NIM-rep. Let’s consider the existence of the NIM-rep matrix $\mathcal{N}_3 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, corresponding to primary ‘3’. Like all the $S_3$ primaries, ‘3’ is self-conjugate, so $b = c$. So we’re looking for a $2 \times 2$ symmetric $\mathbb{Z}_2$-matrix, with eigenvalues $S_{30}/S_{00} = 2$ and $S_{32}/S_{02} = \ldots$. \ldots
−1. One way to see such a matrix can’t exist is to consider its trace and determinant: $\text{Tr}(N_0) = 1 = a + d$ (so either $a$ or $d$ vanishes), and $\det(N_0) = -2 = ad - b^2 = -b^2$, i.e. $b = \sqrt{2} \not\in \mathbb{Z}$.

So no NIM-rep can correspond to that modular invariant $\mathbb{Z}$. More generally, this probably accounts for the abundance of modular invariants arising for finite groups [34].

The simplest example of a NIM-less WZW modular invariant is $\mathcal{B}(3,1|9,1)$ for $\mathfrak{so}(9)$ level 2, given explicitly in $\S$5.2. It has exponents $\{0, 2\Lambda_1, \Lambda_1, \Lambda_2, \Lambda_3, \Lambda_3, 2\Lambda_4\}$, all with multiplicity 1. Here’s a simple argument that it is NIM-less:

Let $A = N_{\Lambda_1}$ be the matrix for the first-fundamental weight. It is an $8 \times 8 \mathbb{Z}_2$-matrix. Since charge-conjugation is trivial here, we know $A = A^t$. The quantum-dimension of $\Lambda_1$ is 2, so this must be the maximal eigenvalue $r(A)$ of $A$. The traces of a NIM-rep matrix $N_{\mu}$ can be obtained in terms of the exponents $\mathcal{E}$ by (3.8). Using this we see that $\text{Tr}(A) = 1$. We want to show that no such matrix can exist, and also respect the fusion

$$\Lambda_1 \boxtimes \Lambda_1 = 0 \boxtimes (2\Lambda_1) \boxtimes \Lambda_2$$

Looking at the exponents, we see that all eigenvalues of both the vacuum 0 and simple-current $2\Lambda_1$ are +1, and thus $N_0 = N_{2\Lambda_1} = I$. Nonnegativity of $N_{\Lambda_1}$ thus requires $2 \leq (A^2)_{ii} = \sum_j (A_{ij})^2$ for all $i$ and so each row sum $\sum_j A_{ij} \geq 2$. But a standard fact of Perron-Frobenius theory is that for any nonnegative matrix $B$, the minimum row-sum can equal the maximum eigenvalue $r(A)$ iff all row-sums equal $r(A)$. Thus all row-sums equal 2, and $\sum_{i,j} A_{ij}$ is even. However, since $A$ is symmetric, $\sum_{i,j} A_{ij} \equiv \text{Tr}(A) = 1 \pmod{2}$.

This contradiction means that no such matrix $N_{\Lambda_1}$ can exist, and so we can’t have a NIM-rep for this modular invariant. As was proved last section, most of the modular invariants for $\mathfrak{so}(n)$ level 2 are likewise NIM-less.

7. How to make your own NIM-rep classifications

In this short section we explain how to put some of the ideas of $\S$3.3 together, in order to obtain NIM-rep classifications (of a sort) for any choice of modular data. For definiteness, consider $\widehat{\mathfrak{sl}}(3)$ level $k = 8$ (our methods though are completely general). It has 45 weights=primaries. We’ll find all possible sets of exponents.

At first glance this seems challenging, since the generator $N_{\Lambda_1}$ will have largest eigenvalue about 2.6825, significantly beyond any known matrix or graph classification. However, the Galois symmetry enormously simplifies this task, making it essentially do-able by hand. In particular, Thm.3(iv) says that the exponent $\mathcal{E}(N)$ is a union of Galois orbits. $P_+$ here has only four orbits with respect to its Galois group $(\mathbb{Z}/33\mathbb{Z})^\times$. They are

$$\mathcal{O}_0 = \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4)\}$$
$$\mathcal{O}_1 = \{(0, 3), (0, 6), (1, 4), (1, 7), (2, 5), (3, 0), (4, 1), (5, 2), (6, 0), (7, 1)\}$$

and $\mathcal{O}_2 = J\mathcal{O}_0 \cup J^2\mathcal{O}_0$ and $\mathcal{O}_3 = J\mathcal{O}_1 \cup J^2\mathcal{O}_1$, using obvious notation, where the action of the simple-current $J$ here is given by $J(a, b) = (8 - a - b, a)$. It suffices then to determine the four multiplicities $m_0, m_1, m_2, m_3$. However, the multiplicity of the vacuum must be
if the NIM-rep is to be indecomposable, so we must have \( m_0 = 1 \). A simple-current can only have multiplicities 0 or 1 (Thm.3(iii)), so \( m_2 = 0, 1 \).

Consider first \( m_2 = 1 \), i.e. \( J \in \mathcal{E}(N) \). Then by Thm.3(iii), all \( J \)-orbits must have constant multiplicity, i.e. \( m_1 = m_3 \). Now the trace (3.8) for \( \lambda = (0, 6) \) gives us

\[
8m_0 - 6m_1 + 16m_2 - 12m_3 \geq 0
\]

that is, \( 24 \geq 18m_1 \), i.e. \( m_1 = 0, 1 \). So two possible exponents are \((m_0, m_1, m_2, m_3) = (1, 0, 1, 0)\) and \((1, 1, 1, 1)\).

Now consider \( m_2 = 0 \). Then the same trace inequality now becomes \( 8 \geq 6m_1 + 12m_3 \), i.e. \( m_3 = 0 \) and \( m_1 = 0, 1 \). So the two remaining possible exponents are \((m_0, m_1, m_2, m_3) = (1, 0, 0, 0)\) and \((1, 1, 0, 0)\).

Each of these four possible exponent multi-sets are in fact realised by each of the four \( sl(3)_8 \) modular invariants. For example, charge-conjugation corresponds to \((1, 0, 0, 0) = \mathcal{O}_0\). Corresponding NIM-reps are given in [7].

Apparently no other NIM-reps are known for \( \hat{sl}(3) \) level 8, but that may be simply because no one has looked really hard (e.g. we give at the end of §3.2 a NIM-rep for \( sl(3)_3 \) which seems to be new). The Perron-Frobenius eigenvalue of \( N \lambda_1 \) here is large enough to conceivably allow more than one realisation for a given exponent.

Incidentally, the same method severely constrains some off-diagonal entries of any \( sl(3)_8 \) modular invariant. For instance, we likewise get four possibilities for each of the multi-sets \( \{m_\mu^\pi = M_{\mu, \pi \mu}\}_{\mu \in P_+} \) where \( \pi \) is any of the four fusion-automorphisms of \( sl(3)_8 \). The fusion-automorphisms for any current algebra were classified in [29]; for \( sl(n)_k \) they are given by \( \lambda \mapsto C^j J^a \sum_{i=1}^{n-1} i^{\lambda_i} \lambda \), where \( j = 0, 1 \) and \( \text{gcd}(ak + 1, n) = 1 \).

8. Final remarks: speculations and questions

To get the main thrust of the paper with a minimum of effort, read §§6,7 and this conclusion. Our main results are the \( sl(n)_2 \) and \( so(n)_2 \) NIM-rep classifications, as well as Thm.3 and its comparison to Thm.1.

(1) We’ve found infinitely many NIM-reps lacking a corresponding modular invariant (this is very typical behaviour). We’ve found infinitely many modular invariants lacking a NIM-rep (e.g. \( \hat{so}(n) \) level 2). We’ve found infinitely many pairs of distinct modular invariants which correspond to identical NIM-reps (e.g. \( so(8n) \) level 1, or triality and its inverse for \( so(8) \) at any level) — this refutes a hope expressed in §2.4 of [33]. There are also different NIM-reps corresponding to identical modular invariants (e.g. the \( \hat{sl}(3) \) level 9 NIM-reps called \( \mathcal{E}_i^{(12)} \), \( i = 1, 2, 3 \), in [7]).

Incidentally, different modular invariants can correspond to identical RCFTs!* A simple example is WZW \( \hat{so}(16) \) level 1, where the four distinct modular invariants \( C_i^1 M[J_i] C_i^1 \) \((i, j = 0, 1)\) all correspond to the WZW \( \hat{E}_8 \) level 1 theory (see §4.2 if this notation seems obscure) — there is, after all, only one \( c = 8 \) holomorphic theory! More precisely, the

* On this simple but (to me) surprising point, as well as paragraph (2) below, I’ve benefitted from conversations with M. Gaberdiel. The referee informs me that ‘it is already known for years’, but not to me!
partition functions $\mathcal{Z}$ of these seemingly different modular invariants are indeed different functions of the modular parameter $q = e^{2\pi i \tau}$ and the left- and right-moving Cartan angles $\vec{z}_L, \vec{z}_R \in \mathbb{C}^8$ — though their $q$-dependence is the same, their $\vec{z}$-dependence differs by a change-of-basis. Different so(16)’s sit inside $E_8$, and they yield different decompositions $\mathcal{H} = \bigoplus_{\lambda, \mu} M_{\lambda \mu} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\mu$ of the state space in terms of so(16)$_1$ modules, and hence express the same $E_{8,1}$ theory by distinct so(16)$_1$ modular invariants!

It’s long been known that different RCFTs can have identical partition functions $\mathcal{Z}$, but this always seems to be because $\mathcal{Z}$ isn’t taken with full variable dependence. For instance the $q$-functions of the two holomorphic $c = 16$ theories are identical, but can be distinguished when their Cartan angles are considered. Or a more interesting example: the simple-current modular invariant $M[J]$ for $\mathfrak{sl}(3)_3$ is indistinguishable from its charge-conjugate, even when all $\mathfrak{sl}(3)$ Cartan angles are included; by contrast we get 6 different modular invariants (all restricting to $M[J]$) when the Cartan angles of so(8)$_1$ (the maximally extended chiral algebra here) are considered. Incidentally, other hints that this $\mathfrak{sl}(3)_3$ modular invariant is ‘degenerate’ come from NIM-reps [7] and twisted partition functions $\mathcal{Z}_{g,g'}$ [33].

But what this new observation tells us is that there can be different ways to introduce the Cartan angles into a given RCFT, and they result in different ‘full-variable’ partition functions. Potentially, a similar problem can arise any time the modular invariant is written in terms of characters of a nonmaximal chiral algebra.

(2) What does it mean when a modular invariant is NIM-less? The simplest guess is that it is nonphysical (i.e. can’t be realised as the 1-loop partition function in a consistent RCFT). In fact, Verstegen argued in [39] that the so(9)$_2$ modular invariant $\mathcal{B}(3,1|9,1)$ is nonphysical, by saying that no chiral extension with modular data could be found in which $\mathcal{B}(3,1|9,1)$ would be diagonal. A similar claim is made in [40], regarding the so(15)$_2$ modular invariant we call $\mathcal{B}(15,1|15,4)$. It is tempting to conjecture that any NIM-less modular invariant for so(n)$_2$ will have a similar problem: its ‘maximal chiral extension’, if it exists in some form, won’t have healthy $S$ and $T$ matrices. It should be emphasised though that the requirement that a consistent RCFT have a compatible NIM-rep is not as solid as for instance the requirement that its torus partition function be modular invariant.

(3) The dual question is: What about the NIM-reps that fail to correspond to a modular invariant? Most notable among these are the tadpoles $T_n$ of $\hat{\mathfrak{sl}}(2)$ level $2n - 1$. In fact these correspond to the submodular invariant $M_{\lambda \mu} = \delta_{\mu \cdot J \lambda \cdot \lambda}$ where $J$ is the simple-current, taking $\lambda = (\lambda_0, \lambda_1)$ to $(\lambda_1, \lambda_0)$. This $M$ is not a true modular invariant (e.g. it commutes with $T^4$ but not $T$); because it’s invariant under a (small-index) subgroup of the modular group, we call it a submodular invariant. Similar remarks apply to the NIM-reps for $\mathfrak{sl}(n)_1$ which don’t have a corresponding modular invariant (see §4.2).

So a natural question is: can an RCFT (or string theoretic) interpretation be given to the assignment of NIM-reps to certain submodular functions?

(4) It is also natural to ask: Find a simple explanation (there are many complicated ones) for why there is no $\hat{\mathfrak{sl}}(2)$ modular invariant at level $2n - 1$, corresponding to the tadpole $T_n$. Then this could give rise to an additional NIM-rep axiom, permitting us to automatically dismiss nonphysical ones.
An original axiom of $\text{sl}(n)_k$ fusion graphs [7,35] was that there be a $\mathbb{Z}_n$-grading on the vertices of the graph, compatible with the $n$-ality $t(\lambda) := \sum j \lambda_j$. This was introduced because for $\text{sl}(2)_k$ it threw away the unwanted tadpoles and retained the A-D-E NIM-reps. This axiom has now been dropped, because we now understand it to be too restrictive — Thm.3(viii) tells us that it is equivalent to demanding that the simple-current $J$ be an exponent, which isn’t always true of healthy modular invariants.

However, the most appropriate NIM-rep axiomatisation may be in between these two extremes. As mentioned in Prop.2, for most current algebras (including every $\text{sl}(n)_k$), we know that all modular invariants (known and unknown) are required to have certain simple-currents as exponents, and hence the corresponding NIM-reps will necessarily have nontrivial gradings.

If we are interested only in NIM-reps which correspond to modular invariants (for suitable pairing $\omega$ — this is discussed in §2.2), then we should dismiss from any consideration those NIM-reps which will necessarily fail for an elementary reason. In this view, it was correct to require for $\hat{\text{sl}}(2)$ that the fusion graphs be bipartite. If we permit ourselves the freedom of choosing an appropriate pairing $\omega$, as apparently we should [5,28], then for example we can also demand that the NIM-reps for $\hat{\text{sl}}(n)$ be $\mathbb{Z}_n$-graded, for $n < 8$ (for $\hat{\text{sl}}(8)$ we can only demand the NIM-reps to be $\mathbb{Z}_4$-graded).

More generally, we could demand that any weight $\kappa$ satisfying (2.12d) be an exponent of our NIM-rep.

(5) A property (hence a possible additional axiom for NIM-reps) which any physically realised NIM-rep must obey, has been suggested recently [41]. Namely, there must exist a vertex $1 \in B$ such that, for all $\lambda \in P_+$,

$$\min_{x \in B} N_{\lambda x}^x = N_{\lambda 1}^1$$

It would clearly be interesting to test the spurious NIM-reps obtained here (and elsewhere) with this relation, and also to derive some consequences in the spirit of §3. Two quick examples are:

(i) $U_{1,0} = \min_x U_{x,0}$, where $U_{x,0}$ is the common Perron-Frobenius eigenvector of all $N_{\lambda x}$;

(ii) for any $\lambda \in P_+$, the norm-squared $\sum_y (N_{\lambda x}^{xy})^2$ of any row of $N_{\lambda}$ will be minimal for $x = 1$.

To get (i), consider the sum $\sum_\lambda S_{0\lambda} N_{\lambda x}^x$. To get (ii), consider the product $N_{\lambda} N_{\lambda}^T$.

(6) Some authors (e.g. [6]) have suggested that the study of NIM-reps may shed light on modular invariant classifications. However our view is that, although NIM-rep classifications are extremely pleasant in the simplest cases, their complexity rises much quicker than that of modular invariants. For instance we get an immediate understanding of the A-D-E in $\text{sl}(2)$ NIM-reps, while the corresponding explanation is still lacking in the $\hat{\text{sl}}(2)$ modular invariant classification. On the other hand, it is possible to obtain fairly easily the full modular invariant classification for e.g. $\hat{E}_8$ level 380 [12] (the answer is simply $M = I$), although it would be completely hopeless to determine its NIM-reps — certainly we would expect enormous numbers of them. For $\text{sl}(2)_k$ there is a single generating primary, and its quantum-dimension is $< 2$; for $E_{8,380}$ we need 8 generating primaries, and the smallest has quantum-dimension very nearly 248.
Decades ago, it was conjectured that a graph was uniquely determined by its eigenvalues. By now many pairs of *cospectral* graphs (graphs with identical eigenvalues) are known. The simplest pair is $A_3^{(1)} \cup A_1$ and $D_4^{(1)}$. It turns out that 5.9% of all graphs with 5 vertices, are not determined by their eigenvalues; the percentage is 6.4% for 6 vertices, 10.5% for 7, 13.9% for 8, and 18.6% for graphs with 9 vertices. It is now conjectured that this percentage rises to 100% as the number of vertices increases — in other words, to almost every graph there would be at least one other with exactly the same eigenvalues and multiplicities. This is already known to be true for trees [32]. And the situation is far worse if you allow (as typically we must) directed edges and loops — even for 2 vertices, almost never do the eigenvalues identify the multi-digraph. What this seems to suggest is that, for more typical modular data, there will be several NIM-reps possessing the same exponents.

Although it is a natural instinct of the mathematically inclined to classify, in hindsight the resulting lists rarely seem to be of much value. What we seek are classifications which have structure and in that way suggest new questions. Or we want to classify something which is so interesting or useful that even if its classification were a complicated tangle, it would still be of value. We suspect further NIM-rep classifications will typically be not worth the trouble. Similar comments apply to the modular invariants corresponding to the finite group modular data of [15]. By contrast, a typical current algebra has a list of modular invariants which is simple and structured — see e.g. the Tables in [18,12].

That said, the handful of NIM-rep classifications we now have do cast light on the modular invariant ↔ NIM-rep correspondence. It would be interesting to study the NIM-reps for a ‘typical’ current algebra whose smallest nontrivial quantum-dimension is much larger than 2. This would test our speculation that its number of NIM-reps would be large. Also, it would be interesting to study the NIM-reps for the finite group ‘pre-orbifold’ modular data [15], say for the symmetric group $S_3$ and the dihedral group $D_4$. This would test our speculation that most of the remarkable numbers of modular invariants there are spurious.

So our view is that the value of NIM-reps to modular invariant classifications is indirect: eliminating spurious modular invariants. However this inefficacy of the NIM-rep hypothesis could change if someone would find a simple property of a NIM-rep spectrum which isn’t automatically obeyed by modular invariants.

(7) Why is $\text{so}(n)_2$ so special here? Because it has so many modular invariants. One reason for this is that rank-level duality associates $\text{so}(n)_2$ with $\text{u}(1)_{n+2}$, and $\hat{\text{u}}(1)$ has a relatively rich variety of modular invariants coming from its simple-currents. However, a better reason is that the $\text{so}(n)_2$ matrix $S$ formally looks like the character table of the dihedral group and for some $r$ actually equals the Verlinde matrix $S$ associated to the dihedral group $D_n$ twisted by an appropriate 3-cocycle [34]. Finite group modular data yields swarms of modular invariants. The critical factor is the impotence of the Galois parity condition (2.10b) here as most (for $\text{so}(n)_2$) or all (for finite groups) of the parities $\epsilon_\ell$ are identically +1. This is very different from the other current algebras.

(8) As mentioned in (6), we suspect that classifying NIM-reps is probably hopeless for all but the smallest ranks or levels. There will be too many of them. (This is in marked contrast to modular invariants, at least for the current algebras.) This speculation leads to
an intriguing question: Could this be hinting that there will typically (e.g. current algebras of large rank and level) be several different RCFTs for a given modular invariant?

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Appendix A. Proofs

A.1. The $\widehat{\mathfrak{sl}}(n)$ level 2 proof.

Recall the parametrisation $\lambda(ab)$ of $P_+$ given in §5.1. The $S$ entries are given by the formula

$$S_{\lambda(ab),\lambda(cd)} = \frac{2}{\sqrt{n\kappa}} \exp[i \frac{(a+b)(c+d)}{n}] \sin(\pi \frac{(b-a+1)(d-c+1)}{\kappa})$$

(A.1)

where we require $0 \leq a \leq b < n$ and $0 \leq c \leq d < n$, and put $\kappa = n+2$.

For $\widehat{\mathfrak{sl}}(n)$ level 2 and any $1 \leq \ell < n-1$, we have the fusion product

$$\Lambda_1 \boxtimes \Lambda_\ell = \Lambda_{\ell+1} \boxplus (\Lambda_1 + \Lambda_\ell)$$

(A.2)

Let $\mathcal{N}$ be any NIM-rep. Write $n = 2^h m$ where $m$ is odd. Write $P = \mathcal{N}_J$ and $A = \mathcal{N}_\lambda$, where $\lambda := J^{(m-1)/2} \Lambda_1 = \lambda(m-1, m+1)$; $P$ is a permutation matrix corresponding to a permutation $\pi$ of the vertices, and $A$ will correspond to a multi-digraph $\mathcal{G}$. As together $\lambda$ and $J$ are fusion-generators, it suffices to find both $\pi$ and $\mathcal{G}$. The point is that $A^t = P^{-m} A$ and that $r(\mathcal{G}) = 2 \cos(\pi/\kappa) < 2$, so Lemma A of Appendix A.4 applies: we find that the components of $\mathcal{G}$ are digraphs corresponding to the diagrams of Figure 3. Those diagrams are explained in §5.1.

The eigenvalues of these digraphs are given in Table 3. The $m_i$ in the first six rows come from Table 1. The multiplicity of 0 for $C_n(k)$ is $k$ (for $n$ even) and $2k$ (for $n$ odd — the extra $k$ coming from $2m-1 = n$). We obtained these eigenvalues by twisting by roots of 1 the eigenvectors for Figure 1.
Write $2^a d$ ($d$ odd) for the order of $P$. So $a \leq h$ and $d$ divides $m$. The permutation $\pi$ must interchange all components of $\mathcal{G}$, since our NIM-rep $\mathcal{N}$ is indecomposable. Since $P$ and $A$ commute, we get $A_{\pi_i, \pi_j} = A_{ij} = \text{i.e. each component of } \mathcal{G} \text{ is equivalent.}$

Now, $P^m$ permutes vertices within each component, so so must $P^d$, and the number of components must then divide $d$. If it’s a proper divisor ($d'$ say), then $P^{2^a d'}$ will also permute the vertices of each component, and so would constitute an odd-order symmetry of each component. But of the diagrams $X_s(2^\ell)$ in Figure 3, only $D_4(2^\ell)$ has a nontrivial odd-order symmetry. However $r(D_4(2^\ell)) = 2 \cos(\pi/6) = 2 \cos(\pi/(n + 2))$, so $D_4(2^\ell)$ corresponds to $\mathfrak{s}(4)$, in which case $d = 1 = d'$. Thus for any $\mathcal{N}$, the number of components $d'$ of $\mathcal{G}$ must exactly equal $d$.

Write $X_s(2^\ell)$ for the common name of the components of $\mathcal{G}$, as given in Figure 3. Since $P^m$ has order $2^a$, and $A^t = P^{-m} A$, either the order $a = \ell + 1$ (if both weights ‘$k$’ and ‘$2k$’ appear in Figure 3), or $a = \ell$ (otherwise).

Given the matrix $A$, i.e. $d$ copies of the digraph $X_s(2^\ell)$, and the matrix $P^m$, i.e. $d$ copies of $\Pi^t$, we can uniquely determine the permutation matrix $P$ as follows. The order $d$ permutation $P^{2^h}$ must permute the $d$ different components, because otherwise the NIM-rep would be decomposable. Ordering the components appropriately, we can require that $P^{2^a}$ takes the $j$th component to the $(j + 1)$th one. We can label the vertices of each component compatibly, in the sense that $P^{2^h}$ takes vertex $(v, i)$ of one component to $(v, i)$ of the next one. By fixing $P^m$ and $P^{2^h}$ in this way, we’ve determined $P$. Thus, the whole NIM-rep $\mathcal{N}$ is uniquely determined by the component diagram $X_s(2^\ell)$ and the number $d$.

Let’s now run through the possibilities:

**Case 1:** Suppose the components are $X_s(2^\ell) = A_s(2^\ell)$. The largest eigenvalue tells us $s = n + 1$. Counting the simple-current exponents of $A_{n+1}(2^\ell)$, we get precisely $2^{\ell+1} d$; they all have multiplicity 1 and form a subgroup of $\mathbb{Z}_n$, so $\ell < h$ and hence $\kappa = n + 2$ must be even. Then

$$2 \exp[\pi i (a + b)/2^h] \cos[\pi (b - a + 1)/\kappa] = 2 \cos[\pi m_i/\kappa]$$

### Table 3. Eigenvalues of Graphs in Figure 3

| Graph   | # vertices | eigenvalues                                      | range               |
|---------|------------|--------------------------------------------------|---------------------|
| $A_n(k)$, $n \geq 1$ | $kn$       | $2 \exp[\pi i \ell/k] \cos(\pi m_i/(n + 1))$   | $0 \leq \ell < k$, $1 \leq i \leq n$ |
| $D_n(k)$, $n \geq 4$ | $kn$       | $2 \exp[\pi i \ell/k] \cos(\pi m_i/(2n - 2))$   | $0 \leq \ell < k$, $1 \leq i \leq n$ |
| $E_6(k)$  | $6k$       | $2 \exp[\pi i \ell/k] \cos(\pi m_i/12)$        | $0 \leq \ell < k$, $1 \leq i \leq 6$    |
| $E_7(k)$  | $7k$       | $2 \exp[\pi i \ell/k] \cos(\pi m_i/18)$        | $0 \leq \ell < k$, $1 \leq i \leq 7$    |
| $E_8(k)$  | $8k$       | $2 \exp[\pi i \ell/k] \cos(\pi m_i/30)$        | $0 \leq \ell < k$, $1 \leq i \leq 8$    |
| $T_n$, $n \geq 1$ | $n$        | $2 \cos(\pi m_i/(2n + 1))$                      | $1 \leq i \leq n$   |
| $B_n(k)$, $n \geq 3$ | $(2n - 1)k$ | $2 \exp[\pi i \ell/k] \cos(\pi (2m - 1)/2n)$, $2 \exp[\pi (2\ell + 1)/2k] \cos(\pi m'/n)$ | $0 \leq \ell < k$, $1 \leq m' < n$ |
| $C_n(k)$, $n \geq 2$ | $(n + 1)k$ | $2 \exp[\pi i \ell/k] \cos(\pi (2m - 1)/2n)$, $0 \text{ (mult } k)$ | $0 \leq \ell < k$, $1 \leq m \leq n$ |
| $F_4(k)$  | $6k$       | $2 \exp[\pi i \ell/k] \cos(\pi m/12)$, $\pm \exp[\pi i (2\ell + 1)/2k]$ | $m \in \{1, 5, 7, 11\}$, $0 \leq \ell < k$. |
has no solution $a, b$ when $m_i$ is even (except for $n = 2$, which fails because $N_j$ would have order $2^\ell = 1$, even though $\Lambda_1$ would be an exponent). This impossibility means that $A_{n+1}(2^\ell)$ can never generate a representation of our fusion ring, so $c$ can’t appear in an $\mathfrak{sl}(n)_2$ NIM-rep.

**Case 2:** Suppose the components are $D_s(2^\ell)$. Then $s = (n + 4)/2$ (so $n$ is even). For the same reason as in Case 1, we must have $\ell < h$. Again, $m_i = s - 1$ cannot be even, so $4|n$. The rest is trivial.

**Case 3:** Suppose the components are $T_s$. Then $s = (n + 4)/2$ (so $n$ is even). For the same reason as in Case 1, we must have $\ell < h$. Again, $m_i = s - 1$ cannot be even, so $4|n$. The rest is trivial.

**Case 4:** Suppose the components are $B_s(2^\ell)$. Then $s = (n + 2)/2$ and $\ell < h$, as usual. In fact, we can fix $\ell$: no $a, b$ can be found obeying

$$2 \exp[\pi i (a + b)/2^h] \cos[\pi (b - a + 1)/\kappa] = 2 \exp[\pi i (2j - 1)/2^{\ell+1}] \cos[2\pi m'/\kappa]$$

for $m' = j = 1$, unless $\ell = h - 1$.

**Case 5:** Suppose the components are $C_s(2^\ell)$. Then $s = (n + 2)/2$, and everything else proceeds as in Case 2. When $4|n$ and $\ell > n - 2$, what we find though is that $n/2^\ell d$ will divide $i + j$ for all exponents $\lambda(ij) \in \mathcal{E}$, which would mean by (3.5) that $P = \mathcal{N}_j$ would have order $2^\ell d$, not $2^{\ell+1}d$ as it should here. When $n/2$ is odd, we are saved by the fixed-points.

**Case 6:** The exceptional digraphs are all handled in similar ways. For instance, suppose the components are $E_6(2^\ell)$. Then $\ell = 0$, and the graph eigenvalue for $m_i = 4$ won’t equal any Verlinde eigenvalue $S_{\lambda,\lambda(ab)}/S_{0,\lambda(ab)}$. So $E_6(2^\ell)$ cannot appear here.

### A.2. The $\mathfrak{so}(odd)$ level 2 proof.

Recall the weights $\gamma^a$ parametrised in §5.2. The $S$ matrix entries for $\mathfrak{so}(n)_2 = B_{r,2}$, where $n = 2r + 1$, are [38]

$$S_{J^i, J^j} = \frac{1}{2} S_{J^i, \gamma^a} = \frac{(-1)^i}{\sqrt{n}} S_{J^i, J^\Lambda_r} = \frac{1}{2\sqrt{n}}$$

$$S_{\Lambda_r, \Lambda_r} = S_{J\Lambda, J\Lambda_r} = -S_{\Lambda_r, J\Lambda_r} = 0.5$$

$$S_{\gamma^a, \gamma^b} = \frac{2}{\sqrt{n}} \cos \frac{2\pi ab}{n}$$

$$S_{\Lambda_r, \gamma^a} = S_{J\Lambda, \gamma^a} = 0$$

for each $a, b \in \{1, \ldots, r\}, \, i, \, j \in \{0, 1\}$. The fusion products we need are

$$\Lambda_1 \boxtimes \Lambda_1 = 0 \boxplus (2\Lambda_1) \boxplus \Lambda_2$$

$$\Lambda_1 \boxtimes \gamma^i = \gamma^{i-1} \boxplus \gamma^{i+1}$$

$$\Lambda_1 \boxtimes \Lambda_r = \Lambda_r \boxplus (\Lambda_1 + \Lambda_r)$$

for $1 < i < r$. Hence the obvious fusion-generator consists of $\Lambda_1$, the spinor $\Lambda_r$, and the simple-current $J = 2\Lambda_1$. 

46
Let \( \mathcal{N} \) be any indecomposable NIM-rep of \( \text{so}(n)_2 \). Put \( \mathcal{N}_i := \mathcal{N}_{\Lambda_i} \). Write \( m_\mu \) for the multiplicities of its exponents \( \mu \in \mathcal{E} \). The charge-conjugation \( C \) is trivial here, so all matrices \( \mathcal{N}_\lambda \) are symmetric. Let’s try to find \( \mathcal{N}_1 \): its quantum-dimension is \( S_{\Lambda_0}/S_{00} = 2 \). Now, the connected multigraphs \( \mathcal{G} \) with maximum eigenvalue 2 are given in Figure 2. The proof that this list is complete is given in §A.4, and their eigenvalues are given in Table 2. Hence \( \mathcal{N}_1 \) will be the adjacency matrix of a disjoint union of graphs from Figure 2.

Now, there are only two \( \lambda \in P_+ \) with \( S_{\Lambda_1 \lambda}/S_{0\lambda} = 2 \): namely \( \lambda = 0, J \). Moreover, \( m_0 = 1 \) and \( m_J = 0, 1 \), so \( \mathcal{N}_1 \) is made up of at most two connected graphs (Thm.3(vi)).

**Case 1:** \( \mathcal{N}_1 \) is a single connected graph \( \mathcal{G}_0 \). Then \( m_J = 0 \).

**Case 2:** \( \mathcal{N}_1 \) has precisely two components, \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \). Then \( m_J = 1 \), i.e. \( J \) is an exponent.

Now, from (A.3) any eigenvalues of the graph \( \mathcal{G}_i \) will be either 0, or of the form \( 2 \cos(2\pi a/n) \), for \( 0 \leq a \leq r \). In particular, \(-2\) is not an allowed value, which excludes anything bipartite (e.g. trees). We find that the only possibilities for the components \( \mathcal{G}_i \), are \( A_m^{(1)} \) when \( m + 1 \) divides \( n \), \( D_m^0 \) when \( 2m - 3 \) divides \( n \), and \( 0A_1^0 = (2) \) and \( 0A_2^0 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \).

For later convenience, denote \( 0A_1^0 \) by \( A_1^{(1)} \), and \( 0A_2^0 \) by \( D_2^0 \).

By (A.3), the multiplicity of eigenvalue 0 will equal the number \( m_{\Lambda_r} + m_{\Lambda_{1+\Lambda_r}} \) of spinors in \( \mathcal{E} \). If \( J \notin \mathcal{E} \), then \( m_{\Lambda_r} = m_{\Lambda_{1+\Lambda_r}} \). Also, note that the \( A_1^{(1)} \) do not have 0 as an eigenvalue, while it is an eigenvalue of \( D_m^0 \) with multiplicity 1. There are spinors in \( \mathcal{E} \), iff \( \mathcal{N}_J \neq I \), in which case \( \mathcal{N}_J \) will be an order-2 permutation matrix. Note that \( \mathcal{N}_J \mathcal{N}_1 = \mathcal{N}_1 \), while \( \mathcal{N}_{\Lambda_{1+\Lambda_r}} = \mathcal{N}_J \mathcal{N}_r \).

Thus we get the following refinement of our cases:

**Case 1(a):** \( J \notin \mathcal{E} \), and no spinors are in \( \mathcal{E} \). \( \mathcal{N}_1 \) is the adjacency matrix of \( \mathcal{G}_0 = A_m^{(1)} \), for some \( 1 \leq m \) dividing \( n \). Also, \( \mathcal{N}_J = I \) and \( \mathcal{N}_r = \mathcal{N}_{\Lambda_1+\Lambda_r} \).

**Case 1(b):** \( J \notin \mathcal{E} \), but one spinor (call it \( \sigma \)) is in \( \mathcal{E} \); it has multiplicity \( m_\sigma = 1 \). \( \mathcal{N}_1 \) is the adjacency matrix of \( \mathcal{G}_0 = D_m^0 \), for some \( 2 \leq m \) obeying \( 2m - 3 \) divides \( n \). Also, \( \mathcal{N}_J \neq I \) and corresponds to an order-2 symmetry of \( D_m^0 \).

**Case 2(a):** \( J \in \mathcal{E} \), but no spinors are in \( \mathcal{E} \). \( \mathcal{N}_1 \) is given by the direct sum of the adjacency matrices of \( \mathcal{G}_1 = A_{m-1}^{(1)} \) and \( \mathcal{G}_2 = A_{m'}^{(1)} \), where \( 1 \leq m \leq m' \) and both \( m, m' \) divide \( n \). Also, \( \mathcal{N}_J = I \) and \( \mathcal{N}_r = \mathcal{N}_{\Lambda_1+\Lambda_r} \).

**Case 2(b):** \( J \in \mathcal{E} \), and both spinors are in \( \mathcal{E} \) with multiplicity 1. \( \mathcal{N}_1 \) is given by the direct sum of the adjacency matrices of \( \mathcal{G}_1 = D_m^0 \) and \( \mathcal{G}_2 = D_{m'}^0 \), where \( 2 \leq m \leq m' \) and both \( 2m - 3, 2m' - 3 \) divide \( n \). \( \mathcal{N}_J \neq I \) and corresponds to an order-2 symmetry of the graph \( \mathcal{G}_1 \cup \mathcal{G}_2 \).

Consider first Case 1(a). Recall the definition of the matrices \( M^{(m|i)}, M^{(m|i,j)}, \hat{M}^{(m)} \) from §5.2. We may put \( \mathcal{N}_1 = M^{(m|1,-1)} \). Note that \( M^{(m|i)} M^{(m|j)} = M^{(m|i+j)} \) so \( M^{(m|i,-i)} M^{(m|j,-j)} = M^{(m|i+j,-i-j)} + M^{(m|i-j,-j-i)} \). From this and (A.4a),(A.4b) we obtain \( \mathcal{N}_r = M^{(m|i,-i)} \). Finally, we need the matrix \( \mathcal{N}_r = \mathcal{N}_{\Lambda_r} \). (A.4c) says \( \mathcal{N}_1 \mathcal{N}_r = 2 \mathcal{N}_r \), so each column of \( \mathcal{N}_r \) is an eigenvector of \( \mathcal{N}_1 \) with eigenvalue 2. This eigenspace is 1-dimensional, spanned by \( (1, 1, \ldots, 1)^t \), so each column of \( \mathcal{N}_r \) is constant. Since also \( \mathcal{N}_r = \mathcal{N}_r \), we get that \( \mathcal{N}_r = a \cdot 1_{mm} \). The constant \( a \) can be determined by quantum-dimension calculations. The result is \( \mathcal{N} = \mathcal{N}(m) \), given in §5.2.
Next, turn to Case 1(b). Here we can put $N_1^J = \tilde{M}^{(m)}$. There is only one order-2 symmetry of the graph $D^0_m$: $N_J$ must interchange the two degree-1 vertices (i.e. the nodes $m - 1$ and $m$).

All that remains is to determine the matrix $N_r$. From (A.4c) we get that its columns lie in the nullspace $\text{Null}(N_1^J - I - N_J)$, and so are of the form $(x, x, \ldots, x, y, z)^t$ where $y + z = x$. Now use $N_r^t = N_r$ and $N_J^{-1} N_r N_J = N_r$ to get

$$N_r = \begin{pmatrix} 2a & \cdots & 2a & a & a \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2a & \cdots & 2a & a & a \\ a & \cdots & a & b & c \\ a & \cdots & a & c & b \end{pmatrix}$$

where $a = b + c$. Its Perron-Frobenius eigenvector is $(2, 2, \ldots, 2, 1, 1)^t$, with eigenvalue $a(2m - 3)/2$, and this must equal the quantum-dimension $S_{\Lambda_r,0}/S_{00} = \sqrt{n}$. This fixes $a$.

The trace (3.8) tells us $b = a + 1/2$ (if $\sigma = \Lambda_r$) or $b = a - 1/2$ (if $\sigma = J\Lambda_r$). Then $N_{\Lambda_1 + \Lambda_r} = N_J N_r$ will be the same as $N_r$, except with $b$ and $c$ interchanged. The result is $N'(m, \sigma)$.

Incidentally, the fact that case 1 requires $n$ to be a perfect square follows from Galois (Thm.3(iv)): when $\sqrt{n} \notin \mathbb{Z}$, the Galois orbit of 0 is $\{0, J\}$.

Case 2(a) is similar to Case 1(a), so the details won’t be repeated. To get that the upper-left $m \times m$ and lower-right $m' \times m'$ blocks in $N_r$ are 0, use Thm.3(x). The result is $N'(m, m')$.

Case 2(b) essentially reduces to two copies of the Case 1(b) argument. To determine $N_J$, use the nonnegativity of $N_2 = (N_1^J)^2 - I - N_J$, as well as the fact that $N_J$ must be a symmetry of the graph $D^0_m \cup D^0_{m'}$. Again we get $\text{Tr}(N_r) = 0$. The result is $N'(m, m')$.

A.3. The $\mathfrak{so}(even)$ level 2 proof.

Recall the weights $\lambda^i$ of $\mathfrak{so}(n)_2 = D_{r,2}$, where $n = 2r$. In [38] we find that the $S$ entries are

$$S_{00} = \frac{1}{\sqrt{r}} S_{0\Lambda_r} = \frac{1}{2} S_{0\lambda^a} = \frac{1}{2\sqrt{n}} \quad (A.5a)$$

$$S_{\lambda^a\lambda^b} = \frac{2}{\sqrt{n}} \cos\left(\pi \frac{ab}{r}\right) \quad (A.5b)$$

$$S_{\lambda^a\Lambda_r} = S_{\lambda^a\Lambda_{r-1}} = 0 \quad (A.5c)$$

$$S_{\Lambda_r\Lambda_r} = S_{\Lambda_{r-1}\Lambda_{r-1}} = \frac{1}{4}(1 + (i)^r) \quad (A.5d)$$

$$S_{\Lambda_r\Lambda_{r-1}} = \frac{1}{4}(1 - (i)^r) \quad (A.5e)$$

for $a, b \in \{1, 2, \ldots, r - 1\}$. The remaining entries of $S$ are given by (2.4) and $S = S^t$.

The only fusion products we need are

$$\Lambda_1 \boxtimes \Lambda_1 = 0 \boxtimes (2\Lambda_1) \boxtimes \Lambda_2 \quad (A.6a)$$
where $1 < i < r - 1$. Hence the obvious fusion-generator consists of $\Lambda_1$, the spinors $\Lambda_r$ and $\Lambda_{r-1}$, and the simple-currents $J_v = 2\Lambda_1$ and $J_s = 2\Lambda_r$.

Let $\mathcal{N}$ be any NIM-rep, with exponent $\mathcal{E}$. Write $\mathcal{N}_i := \mathcal{N}_{\Lambda_i}$, $\mathcal{N}_v := \mathcal{N}_{J_v}$, $\mathcal{N}_s := \mathcal{N}_{J_s}$ and $\mathcal{N}_{c} := \mathcal{N}_{J_c}$. Consider the matrix $\mathcal{N}_1$: since $r(\mathcal{N}_1) = 2$ and $\mathcal{N}_1^2 = \mathcal{N}_1$, its graph is a disjoint union of the graphs of Figure 2. Their eigenvalues are given in Table 2.

The graph $\mathcal{N}_1$ has $S_{\Lambda_1,\mu}/S_{0\mu} = 2$ only for the simple-currents $0$ and $J_v$, so we’ll have 1 or 2 indecomposable components, as in §A.2. Likewise, $S_{\Lambda_1,\mu}/S_{0\mu} = -2$ iff $\mu = J_s, J_c$, so a component of $\mathcal{N}_1$ will be bipartite iff either $J_s$ or $J_c$ are exponents.

$\mathcal{N}_v$ will be an order 1 or 2 symmetry of the fusion graph of $\Lambda_1$: $\mathcal{N}_1 \mathcal{N}_v = \mathcal{N}_v \mathcal{N}_1 = \mathcal{N}_1$. It stabilises each component. If the graph has a degree-1 vertex $i$, then this symmetry must move that vertex to a different degree-1 vertex (otherwise the $(i, i)$ entry of $\mathcal{N}_2 = \mathcal{N}_1^2 - I - \mathcal{N}_v$ will equal $-1$). This eliminates the possibility of having components $E_6(1), E_7(1), E_8(1)$, and determines the permutation $\mathcal{N}_v$ restricted to any of the other possible components from Table 2 (except for $0A_2^1$ and $A_3^1$). For later convenience, we’ll write $D_0^2 := 0A_2^1$, $D_3^1 := A_3^1$ and $E_4 := A_1^1$, and give both $D_3^1$ and $E_4$ the adjacency matrix displayed in §5.3. We take $\mathcal{N}_v$ to act trivially on both $0A_2^1$ and $A_3^1$, but to switch the vertices of $D_0^2$, switch the last two vertices of $E_4$, and to switch vertices $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ of $D_3^1$.

The Galois orbits of the spinors are: for $r$ odd, $\{\Lambda_r, \Lambda_{r-1}, \Lambda_1 + \Lambda_r, \Lambda_1 + \Lambda_{r-1}\}$; for $r$ even, $\{\Lambda_r, \Lambda_1 + \Lambda_{r-1}\}$ and $\{\Lambda_{r-1}, \Lambda_1 + \Lambda_r\}$. The Galois orbit of $0$ is: $\{0, J_c\}$ unless $r$ is a perfect square, in which case it’s only $\{0\}$. Finding the Galois orbit of some primary $\mu$ is easy once you know the $S$ entries: just apply (2.6) to the ratios $S_{v\mu}/S_{0\mu}$.

Note that the components $A_i^m$ and $0A_0^m$ contribute no spinors to $\mathcal{E}$, while the components $D_0^m$ ($m \geq 2$) and $E_4$ contribute exactly one spinor to $\mathcal{E}$, and $D_3^m$ ($m \geq 3$) exactly two. The reason is that the number of spinors in $\mathcal{E}$ is precisely the dimension of the common nullspace of $\mathcal{N}_1$ and $\mathcal{N}_2 = \mathcal{N}_1^2 - I - \mathcal{N}_v$.

Consider first when the graph $\mathcal{G}$ of $\mathcal{N}_{\Lambda_1}$ is connected. Then we know $J_v \notin \mathcal{E}$. By the Galois symmetry Thm. 3(iv), $r$ must then be a perfect square. Likewise, we know the graph cannot be $D_0^m$ ($m \geq 2$) or $E_4$, because then there would only be one spinor in $\mathcal{E}$.

The following matrices will be useful: Write $1_k^\top$ for the $2 \times k$ matrix whose $(i, j)$th entry is $(-1)^{i+j}$. Write $(\pm 1)_{k\ell}$ for the $k \times \ell$ matrix whose $(i, j)$th entry is $(-1)^{i+j}$.

Case 1(a): The graph $\mathcal{G}$ is the circle $A_{p-1}^1$. Then from Table 2, $m$ must divide $2r$. No spinors are in $\mathcal{E}$. The set $\mathcal{E}$ of exponents is now determined (up to the choice of $J'$ when $m$ is even), and by (3.6) we see that all simple-currents must map to $I_m$. Also, for $m$ even, $r$ must be even because otherwise $CJ' = J_vJ'$ would also be in $\mathcal{E}$.

Note that $\mathcal{N}_r + \mathcal{N}_{r-1}$ is symmetric, and by (A.6d),(A.6e) obeys $\mathcal{N}_1(\mathcal{N}_r + \mathcal{N}_{r-1}) = 2(\mathcal{N}_r + \mathcal{N}_{r-1})$. So we get $\mathcal{N}_r + \mathcal{N}_{r-1} = 2\sqrt{m}1_{mm}$. For $m$ odd, all spinors must map to the same matrix, by (3.6). For $m$ even, define the spinor $\sigma$ as in (i) in §5.3 and consider
\(N_\sigma - N_{C_1,\sigma}\): it also must be symmetric (since \(C\) is trivial) and by (A.6d),(A.6e) its columns will be eigenvectors of \(N_1\) with eigenvalue \(-2\). The rest follows.

Case 1(b): When the graph is \(0A_m^0\), the argument is similar to but simpler than that of Case 1(a). \(N_s\) is determined as follows: it is nontrivial iff \(r/m\) is odd, i.e. iff \(r\) is odd (since \(m|\sqrt{T}\)) ; it also must be a symmetry of \(0A_m^0\) (since \(N_1 = N_sN_1N_s^{-1}\)).

Case 1(c): Suppose the fusion graph is \(D_m^{(1)}\), \(m \geq 4\). Then we know \(\mathcal{E}\) has precisely two spinors, so by Galois \(r\) must be even (hence a multiple of 4) and the spinors are \(\sigma, J_v\) for some \(\sigma = \Lambda_r, \Lambda_{r-1}\). This fixes the exponents, apart from some choice of \(J' = J_s, J_c\).

By (A.6d),(A.6e), the columns of \(N_r + N_{r-1}\) will be 0-eigenvectors of \(N_1 - I - N_v\), and any \(N_{\text{spinor}}\) must commute with \(N_v\). Hence we get

\[
N_r + N_{r-1} = \begin{pmatrix}
U & a_{1, m-4, 2} & V \\
2a_{m-4, m' - 4} & a_{1, m-4, 2} \\
W & a_{1, m' - 4} & W
\end{pmatrix}
\]

where \(U = \begin{pmatrix} u & a - u \\ a - u & u \end{pmatrix}\), etc. By rearranging appropriately the row/column indices, we may suppose \(u \geq a - u\) and \(v \geq a - v\). By computing maximal eigenvalues, we obtain 
\(a = \sqrt{r}/(m - 3)\).

By (3.6) we get \(N_{C_1,\sigma} = N_vN_{C_1,\sigma}\) and hence \((N_r + N_{r-1}) - N_v (N_r + N_{r-1}) = N_\sigma - N_vN_\sigma\) has eigenvalues \(\pm 2\sqrt{2}\) (multiplicity 1) and 0. So the nonzero eigenvalues of

\[
\begin{pmatrix}
\Delta u & -\Delta u & \Delta v & -\Delta v \\
-\Delta u & \Delta u & -\Delta v & \Delta v \\
\Delta v & -\Delta v & \Delta w & -\Delta w \\
-\Delta v & \Delta v & -\Delta w & \Delta w
\end{pmatrix}
\]

must be \(\pm 2\sqrt{2}\), where \(\Delta u = 2u - a\), etc. Its trace should be 0, so \(\Delta w = -\Delta u\). We obtain \(\Delta u = \Delta v = 1\), so \(a\) and \(m\) are odd and \(u, v, w\) are all determined.

Note from \(\Delta u > 0\) that \(N_\sigma\) must have nonzero diagonal entries and hence a nonzero trace, so \(S_{\sigma, J'}/S_{0, J'} = +\sqrt{r}\) – i.e. \(\sigma\) and \(J'\) are related as in (i) in §5.3. Thus \(\text{Tr}(N_{C_1,\sigma}) = \text{Tr}(N_{J_vC_1,\sigma}) = 0\), so the upper-left and lower-right \(2 \times 2\) blocks of \(N_{C_1,\sigma}\) are 0_{22}. This also implies \(N_s = I_m\), by (3.6).

Arguing as above, we find

\[
N_\sigma - N_{C_1,\sigma} = \begin{pmatrix}
U & a_{1, m' - 4} \\
2a_{(\pm 1)m-4, m' - 4} & a_{(\pm 1)}^{1, m-4} \\
V & a_{1, m' - 4} \\
a_{1, m' - 4} & V
\end{pmatrix}
\]

where \((\pm 1)_{k\ell}\) and \(1_{k}^{m}\) were defined earlier. This determines everything.

That completes the discussion of the fusion graph of \(N_1\) being connected. The other possibility is that the fusion graph possesses two connected components \(G_1\) and \(G_2\), and that \(J_v \in \mathcal{E}\). Because \(J_v \in \mathcal{E}\), the matrices \(N_{\text{spinor}}\) must all be traceless. Also, note that
Case 2(d): Now consider easy to find by Tr($D_m^0$, $m \geq 2$). The total number of spinors in $E$ must be even, by Galois, so $G_2$ must be some $D_m^0$ ($E_4$, unlike $D_m^0$, is bipartite). Then $r$ must be even (since there are only two spinors in $E$), and the spinors must be $\sigma, J_v \sigma$ for either $\sigma = \Lambda_{r-1}$ or $\sigma = \Lambda_r$. The matrix $N_v = I_{m-2} \oplus I_2^r \oplus I_{r-2} \oplus I_2^r$, and the exponents $E = \{0, J_v, \langle \lambda 2^r/(2m-3) \rangle, \langle \lambda 2^r/(2m'-3) \rangle, \sigma, J_v \sigma\}$, are now determined.

By the usual argument (see Case 1(c)) and using the fact that $\text{Tr}(N_{\text{spinor}}) = 0$, we get $N_r + N_{r-1} = \begin{pmatrix} 0_{mm} & B \\ B^t & 0_{m'm'} \end{pmatrix}$ where $B$ is the $m \times m'$ matrix

$\begin{pmatrix} 2a & \cdots & 2a & a & a \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 2a & \cdots & 2a & a & a \\ a & \cdots & a & b & c \\ a & \cdots & a & c & b \end{pmatrix}$

where $b+c = a = 2\sqrt{r/(2m-3)/(2m'-3)}$. Hence 4 divides $r$. We find that the eigenvalues of that $(m + m') \times (m + m')$ matrix are $\pm 2\sqrt{r}, \pm (b-c)$, and 0 (multiplicity $m + m' - 4$). However, the eigenvalue $S_{\lambda \mu} / S_{0\mu} + S_{\lambda \mu} / S_{0\mu}$ corresponding to exponent $\mu = \sigma$ will equal $\sqrt{2}$. This forces $b-c = \pm \sqrt{2}$, which contradicts integrality.

Case 2(b): Suppose $G_1$ is the graph $E_4$; then so must be $G_2$. By the usual arguments we get that $r$ is even and $E = \{0, J_v, J_s, J_c, \Lambda_{r/2}, \Lambda_{r/2}, \sigma, J_v \sigma\}$ for some $\sigma \in \{\Lambda_{r-1}, \Lambda_r\}$. Now proceed as in Case 2(a).

Case 2(c): Consider next $G_1$ being nonbipartite (i.e. of type $A_{even}^{(1)}$ or $A_{m}^{0}$). Then so must be $G_2$. There are no spinors in $E$, and neither $J_s, J_c$ are in $E$. The exponents $E$ are thus determined, and we find from (3.6) that all spinors map to the identical matrix, which is easy to find by $\text{Tr}(N_{\text{spinor}}) = 0$ and the method of Case 1(a).

Case 2(d): Now consider $G_1 = A_{m-1}^{(1)}$, when $m$ is even. Suppose for contradiction that $G_2 = D_{m-1}^{(1)}$. Then there are only two spinors in $E$ — say $\sigma, J_v \sigma$ for $\sigma \in \{\Lambda_{r-1}, \Lambda_r\}$ — so $r$ must be even. We know $N_v$ is as in Case 2(a). In the usual way (eigenvectors of $N_1 - I - N_v$, etc), we find that $N_r + N_{r-1} = \begin{pmatrix} 0_{mm} & D \\ D^t & 0_{m'm'} \end{pmatrix}$ where $D$ is the $m \times m'$ matrix

$D = \begin{pmatrix} a & a & 2a & \cdots & 2a & a & a \\ a & a & 2a & \cdots & 2a & a & a \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a & a & 2a & \cdots & 2a & a & a \end{pmatrix}$

Choose $\sigma'$ to be the spinor $\Lambda_r, \Lambda_{r-1}$ for which $S_{\sigma, \sigma'} \neq 0$ (so $\sigma' = C_1 \Lambda_r$). Then we find by (3.6) that $N_{\sigma'} \neq N_{J_v \sigma'}$ while $N_{C_1 \sigma'} = N_{J_v C_1 \sigma'}$. So $0 \neq (I - N_v) (N_r + N_{r-1}) = 0$. 

$G_1$ is bipartite iff either $J_s$ or $J_c$ is in $E$; but by Thm.3(iii), that’s true iff both $J_s$ and $J_c = J_s J_v$ are in $E$. Hence $G_1$ is bipartite iff $G_2$ is bipartite. We will first eliminate the possibility that $G_1$ is $D_m^0$ or $E_4$. 

\[
\begin{pmatrix} 2a & \cdots & 2a & a & a \\ \vdots & \cdots & \vdots & \vdots & \vdots \\ 2a & \cdots & 2a & a & a \\ a & \cdots & a & b & c \\ a & \cdots & a & c & b \end{pmatrix}
\]
That contradiction means $G_2$ must be $A_{m'}^{(1)}$ for some even $m'$. Then we know the exponents, and we get that $N_r = N_{J_a} \Lambda_r \neq N_{r-1} = N_{J_a} \Lambda_{r-1}$. We find $N_r \pm N_{r-1}$ as in Case 1(c); flipping if necessary the order of the vertices $1, 2, \ldots, m$ yields the precise placement of $C$’s and $C''$’s given in (iv).

Case 2(e): Finally, consider the case where $N_1$ is $D_{m-1}^{(1)} \cup D_{m'}^{(1)}$. The exponents consist of $0, J_1, J_2, \langle \lambda r/(m-3) \rangle, \langle \lambda r/(m' -3) \rangle$, and four spinors. These spinors must be closed under Galois, so when $r$ is odd all four distinct spinors must appear, each with multiplicity 1. When $r$ is even, this can also happen, but another possibility is that the spinors are $\sigma, J_a \sigma$, for $\sigma \in \{\Lambda_r, \Lambda_{r-1}\}$, each appearing with multiplicity 2.

We can compute $\Lambda_r \pm \Lambda_{r-1}$ in the usual way, and thus obtain $N_r$ and $N_{r-1}$ (the answer depends on $r$ being odd or even, and also depends on some parameters). The exponents tell us the eigenvalues of $N_{r'} - N_{J_a} \sigma'$ for either choice of $\sigma' = \Lambda_r, \Lambda_{r-1}$, and this then fixes the values of the various parameters.

Incidentally, in reading off eigenvalues from the matrices arising here, it is helpful to recall facts such as the sum of the squares of the eigenvalues of a symmetric matrix $D$, equals the sum of the squares of the entries of $D$.

A.4. The matrix classifications.

Let $G$ be any multi-digraph. Write $r(G)$ for its largest eigenvalue. We will prove first that the only connected multigraphs (multiple edges and loops are allowed, but no directed edges) with $r(G) = 2$, are listed in Figure 2.

Incidentally, to find the eigenvalues and eigenvectors of the graphs $G = A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, use the fact that they’re the McKay graphs for cyclic, dihedral, and $S_4, A_5, S_5$ groups. To find the eigenvalues and eigenvectors of $A_n^0$ and $D_n^0$, use the fact that they’re $Z_2$-foldings of $A_n^{(1)}$ and $D_n^{(1)}$.

Let $G$ be any connected multigraph with $r = 2$. To prove that it must lie in Figure 2, we simply use the following fact over and over:

(PF1) [30] If $A$, $B$ are nonnegative matrices, and entry-wise $A_{ab} \leq B_{ab} \forall a, b$, then $r(A) \leq r(B)$. If in addition $A, B$ are symmetric and indecomposable, then $r(A) < r(B)$ unless $A = B$.

For example, the Fact tells us that if $G$ has any multiple edges, then it must be $A_n^{(1)}$. If $G$ has at least 2 loops, then it must be one of the $A_n^0$. If $G$ has a vertex with at least 4 edges leaving it, then it must be $D_n^{(1)}$. Etc.

The other matrix classification we need is much more difficult: finding all indecomposable $Z_2$-matrices $A$ with largest eigenvalue $< 2$, and which obey $A^t = \Pi A = A \Pi$ for some permutation matrix $\Pi = (\delta_{b, \pi a})$. By replacing $A$ with some $\Pi^k A$, we can (and in §A.1 we do) assume $\Pi$ has order a power of 2. We will write $n_a$ for the smallest positive power of this permutation $\pi$ which fixes $a$ — so each $n_a$ will likewise be a power of 2.

Lemma A. Let $A$ be an indecomposable matrix with nonnegative integer entries and with $r(A) < 2$, and obeying the relations $A^t = \Pi A = A \Pi$ for some permutation matrix $\Pi$ whose order is a power of 2. Then up to equivalence, $(A, \Pi)$ corresponds to a diagram in Figure 3.
In §5.1 we explain how to obtain the digraph $A$ and permutation $\Pi$, given a diagram in Figure 3. Note e.g. that both $D_n(1)$ and $C_{n-1}(1)$ have the same digraph $A$, but different permutations $\Pi$.

The equations $A = \Pi A \Pi^t$ and $A^t = \Pi A$ tell us, for all $a, b, \ell$,

$$A_{ab} = A_{\pi^t a, \pi^t b} = A_{\pi^t b, \pi^t + 1 a} \quad (A.7)$$

Note also (from Perron-Frobenius theory [30]) that $(A^t A)_{aa} \leq r(A^t A) = r(A)^2 < 4$, so:

(PF2) all entries of $A$ are 0’s and 1’s;
(PF3) there are at most three 1’s in any row or column.

Claim 1. Let $A_{ab} \neq 0$. Then either:

(i) $n_a = n_b$, and $A_{\pi^+ a, \pi^+ b} \neq 0$ iff $i \equiv j \pmod{n_a}$, in which case $A_{\pi^+ a, \pi^+ b} = A_{ab} = 1$.
(ii) $n_a = 2n_b$, and $A_{\pi^+ a, \pi^+ b} \neq 0$ iff $i \equiv j \pmod{n_b}$, in which case $A_{\pi^+ a, \pi^+ b} = A_{ab} = 1$.
(iii) $n_b = 2n_a$, and $A_{\pi^+ a, \pi^+ b} \neq 0$ iff $i \equiv j \pmod{n_a}$, in which case $A_{\pi^+ a, \pi^+ b} = A_{ab} = 1$.

To see this, first write $n$ for the maximum of $n_a$ and $n_b$; then (A.7) and (PF3) tell us that $n/n_a$ and $n/n_b$ either equal 1 or 2 (since $n$ must be a power of 2). The more refined statements in (i)–(iii) arise by restricting to the submatrix of $A$ consisting of the rows and columns lying in the $\pi$-orbits of $a$ and $b$, and using (PF1). QED

It is convenient to assign to $A$ a diagram in which each $\pi$-orbit $\langle \pi \rangle a$ is associated a node. Each node is labeled with its size $n_a$, and we connect nodes $\langle \pi \rangle a$ and $\langle \pi \rangle b$ if $A_{\pi^+ a, \pi^+ b} \neq 0$ for some $i, j$. We want to show that this diagram lies in Figure 3.

The next Claim takes care of the possibility that our diagram has a loop.

Claim 2. Suppose $A_{a, \pi^+ a} \neq 0$ for some $\ell$. Then $A$ is a tadpole $T_m$ whose loop is at $a$, and whose permutation $\Pi = I$.

Proof. Assume first that $n_a > 1$, i.e. that $\pi a \neq a$. Then (A.7) tells us $A_{a, \pi^+ - \ell a} = A_{a, \pi^+ a} = 1$. But $1 - \ell \not\equiv \ell \pmod{n_a}$, since $n_a > 1$ is a power of 2, which contradicts Claim 1(i).

So $\pi a = a$ and $A_{aa} = 1$. Suppose $A_{ab} \neq 0$ for some $b \neq a$. If there were any other entry $A_{ac} \neq 0$ (e.g. if $n_b = 2$), then $A$ would have a submatrix of type $D_0^1$, contradicting $r(A) < 2$. Continuing in this way, we obtain that $A$ is the adjacency matrix for the tadpole, and that $\pi$ fixes everything. QED

So it suffices to consider $A_{a, \pi^+ a} = 0$ for all $a$ and $i$ (i.e. no loops in the diagram). First we’ll show that the diagram must be a tree.

If instead it contains a cycle (i.e. a subdiagram of shape $A_{i}^{(1)}$ for some $m$), then every vertex corresponding to a node in that cycle will have a row-sum at least equal to 2, and so $r$ for that subdiagram will be at least 2.

To see that our diagram can consist only of weights $k$ and $2k$, suppose for contradiction that it has a subdiagram consisting of a node with weight $k$, followed by any number (say $i$) of weight-$2k$ nodes, followed by a weight-$4k$ node. The corresponding subgraph has an eigenvector consisting of $k + 2ik$ 2’s and $4k$ 1’s, with eigenvalue 2.

Thus our diagram will be of two types: either all nodes are of the same size $k$; or all the nodes are labelled by $k$ or $2k$, for some $k$. That the diagram must be one of the ones listed in Figure 3, now follows from routine arguments.
A.5. Proof of Theorem 3.

We conclude the Appendix with the remaining proofs of Thm.3.

A useful fact is that, whenever there is an integer \( i \) and primaries \( \mu, \nu \) such that

\[
S_{\lambda \mu} S_{0 \mu}^i = S_{\lambda \nu} S_{0 \nu}^i
\]  
(A.8a)

holds for all \( \lambda \in P_+ \), then \( \mu = \nu \). To see this, hit both sides with \( S_{\lambda \mu}^* \) and sum over \( \lambda \in P_+ \).

We also use the fact, evident from Verlinde’s formula (2.2), that the Verlinde ratios form a 1-dimensional representation of the fusion ring: for any \( \mu \in P_+ \),

\[
\frac{S_{\lambda \mu}}{S_{0 \mu}} \cdot \frac{S_{\lambda \nu}}{S_{0 \nu}} = \sum_{\nu \in P_+} N_{\lambda \nu}^{\nu \mu} S_{\nu \mu} / S_{0 \mu}
\]  
(A.8b)

Proof of (iii). Let \( J \) be a simple-current of order \( n \) in \( \mathcal{E} \). Write \( P_i \) for all \( \lambda \in P_+ \) with \( Q_J(\lambda) \equiv i/n \pmod{1} \). Consider any \( \mu \in \mathcal{E} \). We want to show \( J \mu \in \mathcal{E} \). To do this, we will use this fact from Perron-Frobenius theory [30]:

(PF4) When an irreducible nonnegative matrix \( A \) has eigenvalue \( e^{2\pi i/D} r(A) \), then the eigenvalues of \( A \) are invariant under rotation by \( 2\pi/D \), i.e. if \( s \) is an eigenvalue of \( A \) with multiplicity \( m \), then so is \( e^{2\pi i/D} s \).

The problem is that this only applies to individual matrices, and we need it simultaneously for all \( N_{\lambda} \). Write \( d(\mu) \) for the largest divisor \( d \) of \( n \) such that \( \mu = J^{n/d} \mu \). So \( d = 1 \) if \( \mu \) is not a fixed-point of \( J \). Note that \( S_{\lambda \mu} = 0 \) when \( \lambda \in P_i \) and \( d(\mu) \) does not divide \( i \) (Proof: apply (2.4) to \( S_{\lambda,J^{n/d} \mu} = S_{\lambda \mu} \)). In Claim 1 we establish a converse of that simple fact.

Claim 1. Choose any \( \mu \in P_+ \). Then \( S_{\lambda \mu} \neq 0 \) for some \( \lambda \in P_m \), iff \( d(\mu) \) divides \( m \).

Proof. Call a number \( m \) ‘\( \mu \)-nice’ if \( S_{\lambda \mu} \neq 0 \) for some \( \lambda \in P_m \). The Galois automorphism \( \sigma = \sigma_{\ell} \) in (2.6) obeys \( \sigma_{\ell}(P_m) = P_{\ell m} \), because of the calculation

\[
\epsilon_{\ell}(\lambda) e^{2\pi i \ell m/n} S_{\sigma_{\lambda},0} = \sigma_{\ell}(e^{2\pi i m/n} S_{\lambda,0}) = \sigma_{\ell} S_{\lambda,J} = \epsilon_{\ell}(\lambda) S_{\sigma_{\lambda},J}
\]

Thus by (2.6), \( m \) is ‘\( \mu \)-nice’ iff the greatest common divisor \( \gcd(n,m) \) is ‘\( \mu \)-nice’. By considering various products of the form \( (S_{\lambda \mu}/S_{0 \mu})^a (S_{\lambda' \mu}/S_{0 \mu})^b \) for \( \lambda \in P_m, \lambda' \in P_{m'} \), repeated using (A.8b) to write them as sums of various \( S_{\lambda'' \mu}/S_{0 \mu} \) for \( \lambda'' \in P_{am+bm'} \), we get that \( m \) and \( m' \) are both ‘\( \mu \)-nice’ iff \( \gcd(n,m,m') \) is ‘\( \mu \)-nice’. Continuing in this way, we ultimately obtain some divisor \( D \) of \( n \) such that \( m \) is ‘\( \mu \)-nice’ iff \( D \) divides \( m \) (we’re just using the fact that \( \mathbb{Z} \) is a Principal Ideal Domain). The point is that then \( S_{\lambda,J^{n/D} \mu} = S_{\lambda \mu} \) \( \forall \lambda \in P_+ \), so by (A.8a) we get that \( \mu = J^{n/D} \mu \). QED to Claim 1

Claim 2. Choose any two primaries \( \mu, \mu' \), and write \( d = d(\mu) \). If \( S_{\lambda \mu}/S_{0 \mu} = S_{\lambda \mu'}/S_{0 \mu'} \) for all \( \lambda \in P_{d} \), then \( \mu = \mu' \).

Proof. Let \( d' = d(\mu') \). Then \( d' \) divides \( d \), by Claim 1. Choose any \( \lambda \in P_{d'} \) so that \( S_{\lambda \mu'} \neq 0 \). Then use (A.8b) repeatedly to obtain

\[
0 \neq \left( \frac{S_{\lambda \mu'}}{S_{0 \mu'}} \right)^{d'/d} = \sum_{\nu \in P_{d'}} n_{\nu} S_{\nu \mu'}/S_{0 \mu'} = \sum_{\nu \in P_{d}} n_{\nu} S_{\nu \mu}/S_{0 \mu} = \left( \frac{S_{\lambda \mu}}{S_{0 \mu}} \right)^{d'/d}
\]
for certain numbers \( n_\nu \). Thus by Claim 1, \( d \) must also divide \( d' \), and hence \( d = d' \).

Now choose any \( \lambda \in P_m \). We want to show that \( S_{\lambda\mu}/S_{0\mu} = S_{\lambda\mu'}/S_{0\mu'} \). Assume that \( d \) divides \( m \) (otherwise both ratios trivially vanish). By Claim 1 there exists a primary \( \nu \in P_d \) such that \( S_{\nu\mu} \neq 0 \). Then again by (A.8b)

\[
\frac{S_{\lambda\mu}}{S_{0\mu}} \left( \frac{S_{\nu\mu'}}{S_{0\mu'}} \right)^{n+1-m/d} = \sum_{\gamma \in P_d} n_\gamma \frac{S_{\gamma\mu}}{S_{0\mu}} = \sum_{\gamma \in P_d} n_\gamma \frac{S_{\gamma\mu'}}{S_{0\mu'}} = \frac{S_{\lambda\mu'}}{S_{0\mu'}} \left( \frac{S_{\nu\mu'}}{S_{0\mu'}} \right)^{n+1-m/d}
\]

Hence \( S_{\lambda\mu}/S_{0\mu} = S_{\lambda\mu'}/S_{0\mu'} \) \( \forall \lambda \in P_+ \), and so \( \mu = \mu' \) by (A.8a). QED to Claim 2

Recall that for both \( J \) and \( \mu \) are in \( \mathcal{E} \), and we want to show that the multiplicity \( m_{J\mu} \) equals \( m_\mu \). Write \( d = d(\mu) \). Then Claim 2 says that we can find a nonnegative linear combination \( N' = \sum_{\lambda \in P_d} a_\lambda N_\lambda \geq 0 \) (in fact almost every nonnegative linear combination will do) of fusion matrices for which the eigenvalue \( \sum_{\lambda \in P_d} a_\lambda S_{\lambda\mu}/S_{0\mu} \) has multiplicity 1. For such a choice of coefficients \( a_\lambda \), the fact (PF4) tells us that the eigenvalue \( e^{2\pi id/n} \sum_{\lambda \in P_d} a_\lambda S_{\lambda\mu}/S_{0\mu} \) of \( N' \) also will have multiplicity 1, which is to say that the only primary \( \mu' \) obeying

\[
e^{2\pi id/n} \sum_{\lambda \in P_d} a_\lambda \frac{S_{\lambda\mu}}{S_{0\mu}} = \sum_{\lambda \in P_d} a_\lambda \frac{S_{\lambda\mu'}}{S_{0\mu'}}
\]

is \( \mu' = J\mu \). Applying this to the eigenvalues of \( N' := \sum_{\lambda \in P_d} N_\lambda \), we get that \( J\mu \) is indeed in \( \mathcal{E} \), with multiplicity \( m_\mu \).

Taking \( \mu = J^{-1} \), we see that \( m_{J} = 1 \), and we also get from this that the \( J \in \mathcal{E} \) form a group. QED to (iii)

Proof of (viii). We will first show that we get an \( N_1 \)-grading associated to any simple-current \( J \in \mathcal{E} \). Fix a simple-current \( J \in \mathcal{E} \) and vertex 1 \( \in \mathcal{B} \). Define as in §3.3 \( g_\lambda = Q_J(\lambda) \), and \( g(y) = Q_J(\lambda) \) when \( N_{\lambda,1}^y \neq 0 \) (recall that for \( N_1 \) irreducible, \( \sum_{\lambda \in P_+} N_{\lambda,1}^y > 0 \)).

First note that \( g(y) \) is well-defined (mod 1). For if also \( N_{\mu,1}^y = 0 \), then \( (N_{\mu}\gamma_\mu\gamma_{C\mu})_{11} = 0 \), i.e. \( \text{Tr}(N_{\mu}) \neq 0 \) for some primary \( \nu \in P_+ \) with \( Q_J(\nu) \equiv Q_J(\lambda) - Q_J(\mu) \) (mod 1). But then Thm.3(x) requires that \( Q_J(\lambda) \equiv Q_J(\mu) \) (mod 1) as desired.

Now, suppose \( N_{\lambda,1}^y = 0 \), for some \( \lambda \in P_+ \) and \( y,z \in \mathcal{B} \). Choose \( \mu \in P_+ \) so that \( N_{\mu,1}^y = 0 \). Then \( (N_{\mu}\gamma_{\lambda})_{1z} \neq 0 \) and thus \( N_{\nu,1}^z = 0 \) for some primary \( \nu \in P_+ \) with \( Q_J(\nu) \equiv Q_J(\mu) + Q_J(\lambda) \) (mod 1), i.e. \( g(z) \equiv g(y) + g_\lambda \) (mod 1), as desired. Thus this gives us an \( N_1 \)-grading.

Suppose conversely that we have an \( N_1 \)-grading. Suppose \( \lambda, \mu, \nu \) are three primaries in \( P_+ \) with fusion coefficient \( N_{\lambda,\mu}^\nu \neq 0 \). Choose any vertices \( x, y \in \mathcal{B} \) such that \( (N_{\lambda,\nu})_{xy} \neq 0 \). Then there must exist a vertex \( w \in \mathcal{B} \) such that \( (N_{\lambda})_{xw} \neq 0 \) and \( (N_{\mu})_{wy} \neq 0 \). Comparing \( g_\lambda + g(x) \equiv g(w), g_\mu + g(z) \equiv g(y), \) and \( g_{\nu} + g(x) \equiv g(y), \) all taken (mod 1), we find that \( g_\lambda + g_\mu \equiv g_{\nu} \) (mod 1), i.e. \( \lambda \mapsto g_\lambda \) is a \( \mathbb{Q} \)-grading of the fusion ring. Note that the map \( \lambda \mapsto e^{2\pi ig_\lambda} S_{\lambda\mu}/S_{0\mu} \) defines a 1-dimensional representation of the fusion ring, and thus

\[
e^{2\pi ig_\lambda} \frac{S_{\lambda\mu}}{S_{0\mu}} = \frac{S_{\lambda\mu'}}{S_{0\mu'}} \quad \forall \lambda \in P_+
\]

for some \( 0' \in P_+ \). Taking the norm-squared of both sides and summing over \( \lambda \in P_+ \), we get \( S_{00} = S_{00'}^{-2} \), i.e. \( 0' \) is a simple-current \( J \), and \( g_\lambda = Q_J(\lambda) \). Let \( n \) be its order.
Now let $\vec{v}$ be the simultaneous Perron-Frobenius eigenvector, for which the matrix $N_\lambda$ has eigenvalue $S_\lambda/S_{00}$. Note that

$$\sum_{y \in B} N_{\lambda x}^y (e^{2\pi i g(y)} \vec{v}_y) = e^{2\pi i (g_\lambda + g(x))} \sum_{y \in B} N_{\lambda x}^y \vec{v}_y = e^{2\pi i (g_\lambda + g(x))} \frac{S_\lambda}{S_{00}} \vec{v}_x$$

and thus $J \in \mathcal{E}$. This is precisely the $N_1$-grading arising from the simple-current $J$, constructed in §3.3. QED to Thm.3(viii)

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Figure 1. Graphs with $r<2$
Figure 2. Graphs with $r=2$

Figure 3. The digraphs of Lemma A

\[ \begin{array}{c}
\text{(a)} \\
\text{(b)} \\
\text{(c)} \\
\text{(d)} \\
\text{(e)} \end{array} \]