Simpler and Stronger Approaches for Non-Uniform Hypergraph Matching and the Füredi, Kahn, and Seymour Conjecture∗

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Abstract

A well-known conjecture of Füredi, Kahn, and Seymour (1993) on non-uniform hypergraph matching states that for any hypergraph with edge weights \( w \), there exists a matching \( M \) such that the inequality \( \sum_{e \in M} g(e)w(e) \geq \text{OPT}_{LP} \) holds with \( g(e) = |e| - 1 + \frac{1}{|e|} \), where \( \text{OPT}_{LP} \) denotes the optimal value of the canonical LP relaxation. While the conjecture remains open, the strongest result towards it was very recently obtained by Brubach, Sankararaman, Srinivasan, and Xu (2020)—building on and strengthening prior work by Bansal, Gupta, Li, Mestre, Nagarajan, and Rudra (2012)—showing that the aforementioned inequality holds with \( g(e) = |e| + O(|e| \exp(-|e|)) \). Actually, their method works in a more general sampling setting, where, given a point \( x \) of the canonical LP relaxation, the task is to efficiently sample a matching \( M \) containing each edge \( e \) with probability at least \( \frac{x(e)}{g(e)} \).

We present simpler and easy-to-analyze procedures leading to improved results. More precisely, for any solution \( x \) to the canonical LP, we introduce a simple algorithm based on exponential clocks for Brubach et al.’s sampling setting achieving \( g(e) = |e| - (|e| - 1)x(e) \). Apart from the slight improvement in \( g \), our technique may open up new ways to attack the original conjecture. Moreover, we provide a short and arguably elegant analysis showing that a natural greedy approach for the original setting of the conjecture shows the inequality for the same \( g(e) = |e| - (|e| - 1)x(e) \) even for the more general hypergraph \( b \)-matching problem.

1 Introduction

The maximum matching problem is one of the most heavily studied problems in Combinatorial Optimization. It has a natural and well-known generalization to hypergraphs, where the task is to find a maximum weight subset of non-overlapping (hyper-)edges. Formally, for a hypergraph \( H = (V, E) \) with \( E \subseteq 2^V \) and edge weights \( w \in \mathbb{R}_{\geq 0}^E \), the task is to solve

\[
\max\{w(M): M \subseteq E \text{ with } e_1 \cap e_2 = \emptyset \ \forall e_1, e_2 \in M, e_1 \neq e_2\},
\]

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where $w(M) := \sum_{e \in M} w(e)$. (We use the shorthand $x(U) := \sum_{e \in U} x(e)$ for any $x \in \mathbb{R}^E_{\geq 0}$ and $U \subseteq E$.) Depending on the context, the above problem is called hypergraph matching or set packing, where the former is often used when deriving results as a function of the sizes of hyperedges.

Interestingly, despite intensive work there remain large gaps in our understanding of the approximability of hypergraph matching. Depending on the context, the above problem is called hypergraph matching or set packing, where the former is often used when deriving results as a function of the sizes of hyperedges.

In general hypergraphs the problem is equivalent to maximum independent set, and thus is $n^{1-\varepsilon}$-hard to approximate as shown by Håstad [15]. Nevertheless, significant progress has been achieved for $k$-uniform hypergraphs, i.e., when all edges have size $k$. (Note that $k = 2$ corresponds to the classical matching problem.) In this setting, approximation guarantees are described as a function of $k$, and strong results have been obtained both with local search techniques [3, 9, 11] and approaches based on linear programming [8, 10, 18] (see Section 1.2 below for additional information).

When dealing with non-uniform instances, which is the focus of this work, it is harder to identify a good fine-grained way to express approximation guarantees, as there is no obvious global parameter like $k$ in the case of $k$-uniform hypergraphs. Nevertheless, one would expect the problem to be easier to approximate if there are mostly hyperedges of small size. An approach to this problem was suggested almost three decades ago by Füredi, Kahn, and Seymour [10], who consider edge-wise guarantees with respect to a solution $x$ to the canonical LP relaxation, which is also called the fractional matching LP and is defined as follows:

$$\begin{align*}
\max & \quad \sum_{e \in E} w(e)x(e) \\
\text{s.t.} & \quad x(\delta(v)) \leq 1 \quad \forall v \in V \\
& \quad x \in \mathbb{R}^E_{\geq 0},
\end{align*}$$

where $\delta(v) := \{ e \in E : v \in e \}$ denotes all hyperedges containing $v$. More precisely, Füredi et al. [10] address the question of finding the “smallest possible” function $g : E \to \mathbb{R}_{\geq 0}$ so that for every edge-weighted hypergraph $H = (V, E)$, there exists a matching $M$ satisfying

$$\sum_{e \in M} g(e)w(e) \geq \sum_{e \in E} w(e)x(e), \quad (1)$$

where $x \in [0, 1]^E$ is an optimal solution to the fractional matching LP. This leads to the following well-known conjecture, claiming that $g(e) = |e| - 1 + \frac{1}{|e|}$ is achievable.

**Conjecture 1** (Füredi et al. [10]). For any hypergraph $H = (V, E)$ and edge weights $w \in \mathbb{R}^E_{\geq 0}$, there is a matching $M$ such that

$$\sum_{e \in M} \left(|e| - 1 + \frac{1}{|e|}\right)w(e) \geq \sum_{e \in E} w(e)x(e),$$

where $x$ is an optimal solution to the fractional matching LP.

Even though the conjecture remains open, partial progress has been achieved in several directions. In particular, Füredi et al. [10] showed non-constructively that it holds if $H$ is either an

\footnote{A simple way to parameterize the instance would be by $\max_{e \in E} |e|$, but this is very coarse as it does not allow for exploiting the existence of small hyperedges. Moreover, this parameterization essentially falls back to the $k$-uniform case as results for $k$-uniform hypergraph matching typically extend to hypergraphs with edges of size at most $k$.}
unweighted, uniform, or intersecting hypergraph. The conjecture, if true, is best possible in the sense that it is known to be tight even for unweighted \(k\)-uniform hypergraphs for infinitely many values of \(k\), namely whenever \(k\) is one unit more than a prime power \([10]\).

A step towards the general form of Conjecture \([1]\) was made by Bansal, Gupta, Li, Mestre, Nagarajan, and Rudra \([2]\) in the context of the more general stochastic set packing problem. Although not stated explicitly in \([2]\), their algorithm implies \([1]\) for \(g(e) = |e| + 1 + o(|e|)\). (A formal proof of this is given in \([4]\).) More precisely, they present a randomized procedure returning a matching that contains each edge \(e\) with probability at least \(x(e) |e| - (|e| - 1)\). Motivated by this, Brubach, Sankararaman, Srinivasan, and Xu \([4]\) presented a strengthening of the algorithm in \([2]\) to sample a matching containing each edge \(e\) with probability at least \(x(e) |e| + O(|e| \exp(-|e|))\), which implies \([1]\) for \(g(e) = |e| + o(|e|)\). Inspired by these results, Brubach et al. \([4]\) formulated the following slightly stronger conjecture.

**Conjecture 2** (Brubach et al. \([4]\)). Let \(H = (V, E)\) be a hypergraph, \(w \in \mathbb{R}_{\geq 0}^E\), and let \(x \in [0, 1]^E\) be a fractional matching. Then, it is possible to efficiently sample a matching \(M\) from a distribution such that each edge \(e \in E\) is contained in \(M\) with probability at least \(x(e) |e| - 1 + 1/|e|\).

Such sampling settings are motivated by fairness considerations and have recently attracted significant attention in other contexts, such as in clustering (see, e.g., \([13, 14]\)).

Conjecture \([2]\) clearly implies Conjecture \([1]\). Moreover, it was observed in \([4]\) that Conjecture \([1]\) implies a non-constructive version of Conjecture \([2]\), i.e., that there exists a distribution over matchings containing each edge \(e\) with probability at least \(x(e) |e| - 1 + 1/|e|\). We highlight that by adjusting a well-known technique based on both duality and the ellipsoid method by Carr and Vempala \([7]\), one can derive that a constructive version of Conjecture \([1]\) where the matching \(M\) has to be found efficiently, also implies Conjecture \([2]\) (For completeness, we show this in Appendix \([A]\).) Hence, from a theoretical perspective, a key difference between Conjecture \([1]\) and Conjecture \([2]\) is the constructiveness. Nevertheless, procedures working directly in the sampling setting remain of interest, as they are typically significantly faster than procedures obtained by using the ellipsoid method through the duality argument.

### 1.1 Our contribution

One main contribution of this work is to improve on Brubach et al.’s \([4]\) sampling result, through a significantly simpler procedure that allows for a very short and clean analysis, and leads to an improved factor as highlighted below. The algorithm and its analysis are presented in Section \([2]\).

**Theorem 1.** Let \(H = (V, E)\) be a hypergraph, \(w \in \mathbb{R}_{\geq 0}^E\), and let \(x \in [0, 1]^E\) be a fractional matching. Then, there is an efficient sampling procedure returning a matching \(M\) with

\[
\Pr[e \in M] \geq \frac{x(e)}{|e| - (|e| - 1)x(e)} \quad \forall e \in E.
\]

Note that while Theorem \([1]\) does not prove Conjecture \([2]\) it is not implied by it either, i.e., the two statements are incomparable. Nevertheless, it strengthens Brubach et al.’s result and, as we discuss in the conclusion, may open up new ways to attack Conjecture \([1]\). However, the arguably most important point is the very clean and simple underlying algorithm and analysis.
Inspired by the result of Parekh and Pritchard [18] on $k$-uniform hypergraph $b$-matching, we then study Conjectures 1 and 2 in the more general non-uniform hypergraph $b$-matching setting.

We recall that a hypergraph $b$-matching problem consists of a hypergraph $H = (V, E)$, edge weights $w \in \mathbb{R}^E_{\geq 0}$, and vertex capacities $b \in \mathbb{Z}^V_{\geq 1}$, and the task is to find a maximum weight $b$-matching $M$, which is a subset of edges such that, for any $v \in V$, at most $b(v)$ edges in $M$ contain $v$. Moreover, a fractional $b$-matching is a point in the polytope $\{ x \in [0,1]^E : x(\delta(v)) \leq b(v) \ \forall v \in V \}$. We show that a natural greedy algorithm, which picks edges in order of decreasing weights, allows for deriving the following result. This part is discussed in Section 3.

**Theorem 2.** Given a hypergraph $H = (V, E)$, $w \in \mathbb{R}^E_{\geq 0}$, $b \in \mathbb{Z}^V_{\geq 1}$, and an optimal fractional $b$-matching $x \in [0,1]^E$, the greedy algorithm returns a $b$-matching $M$ that satisfies

$$\sum_{e \in M} (|e| - (|e| - 1)x(e)) w(e) \geq \sum_{e \in E} w(e)x(e).$$

Again, we think that a key contribution of this result is the simplicity of the procedure and the way we analyze the greedy algorithm, which leads to a concise proof.

We finally highlight that, by extending a result of Carr and Vempala [7] based on LP duality and the ellipsoid method, we can use Theorem 2 to obtain an efficient sampling algorithm returning a $b$-matching that contains each edge $e \in E$ with probability at least $\frac{x(e)}{e - (e - 1)x(e)}$. Because the employed arguments are quite standard, we formalize this in Appendix A as Theorem 3. Even though this allows our results for $b$-matchings to be used to obtain an alternative proof of Theorem 1, the resulting sampling algorithm is significantly more involved and slower (due to the use of the ellipsoid method) than our much shorter and elegant approach that we use to prove Theorem 1.

### 1.2 Further discussion on prior work

We expand on the progress for $k$-uniform hypergraph matching that we briefly mentioned above. This special case remains APX-hard for any $k \geq 3$ as shown by Kann [17]. Nevertheless, local search techniques led to an exciting sequence of strong results, culminating in an approximation factor of $\frac{k+1}{3} + \varepsilon$ for unweighted graphs by Cygan [9] (see also Fürer and Yu [11]) and $\frac{k+1}{2} + \varepsilon$ for weighted graphs by Berman [3]. This is contrasted by an approximation hardness of $\Omega(\frac{k}{\log k})$ by Hazan, Safra, and Schwartz [16].

Moreover, for $k$-uniform hypergraph matching, the integrality gap of the fractional matching LP is essentially settled. More precisely, Füredi et al. [10] showed, non-constructively, that the integrality gap is at most $k - 1 + \frac{1}{k}$, and almost 20 years later, Chan and Lau [8] gave an elegant algorithmic version of that result. We recall that a matching lower bound on the integrality gap was given for all $k$ that are one unit more than a prime power [10]. Furthermore, Parekh and Pritchard [18] showed that one can obtain an LP-relative $(k - 1 + \frac{1}{k})$-approximation for the more general $b$-matching problem on $k$-uniform hypergraphs. This is a non-trivial generalization, which may be surprising in view of the fact that many results on matchings easily generalize to $b$-matchings.

### 1.3 Brief overview of techniques

The main tool for obtaining Theorem 1 is the so-called exponential clocks technique. It was previously used by Buchbinder, Naor, and Schwartz [6] in the context of the Multiway Cut problem,
and a further application in the context of Dynamic Facility Location was found by An, Norouzi-Fard, and Svensson [1]. At a high level, it refers to competing independent exponential random variables, where an exponential clock wins a competition if it has the smallest value among all participating exponential clocks. Similar in spirit to the randomized rounding algorithms of Bansal et al. [2] and Brubach et al. [4], where the first step is to independently sample a random variable for each edge based on its $x$-value, and then extract a matching based on these random variables, the exponential clocks algorithm first independently samples an exponential random variable for each edge and then picks an edge if it is the “winner” among all edges in its neighborhood (i.e., the realization of its exponential variable is the smallest among its neighborhood). The first use of the technique on matching problems was by Bruggmann and Zenklusen [5], where they used it as a tool to demonstrate the existence of certain distributions over matchings on graphs (not hypergraphs) in the context of designing contention resolution schemes for matchings. (We provide more details on this in Section 2.)

In this work, we generalize the arguments of [5] and obtain the improved result stated in Theorem 1. Besides being a very elegant and easy-to-analyze algorithm, we hope that it may open up new ways to attack Conjectures 1 and 2 as our analysis already shows that, loosely speaking, when applied to an extreme point of the fractional matching $LP$, it implies an “average version” of these conjectures. We expand on this in the conclusion.

Regarding Theorem 2, we do a careful analysis of the natural greedy algorithm for the more general hypergraph $b$-matching problem. The analysis is based on a concise charging scheme that charges the $LP$-contribution of non-matching edges first to tight vertices and then to edges in the $b$-matching to achieve the same factor (function $g$) as the one in Theorem 1. Thus, it addresses a generalization of Conjecture 1 in the $b$-matching setting.

2 The exponential clocks rounding scheme

In this section, we present our algorithm based on exponential clocks that proves Theorem 1. For $\lambda > 0$, we denote by $Exp(\lambda)$ an exponential random variable with parameter $\lambda$. Our algorithm is highlighted below as Algorithm 1. We use the notation $N(e) := \{f \in E : f \neq e \text{ and } f \cap e \neq \emptyset\}$ for the edges overlapping with an edge $e$ and, for a fractional matching $x$, we denote its support by $\text{supp}(x) := \{e \in E : x(e) > 0\}$.

**Algorithm 1: Exponential clocks rounding**

- Let $Z_e \sim \text{Exp}(x(e))$ for $e \in \text{supp}(x)$ be indep. random variables and $z_e$ a realization of $Z_e$;
- return $M = \{e \in \text{supp}(x) : z_e < z_f \ \forall f \in N(e) \cap \text{supp}(x)\};$

In words, our algorithm realizes for each edge an exponential random variable with parameter equal to its $x$-value, and picks each edge whose exponential random variable realized to a value smaller than any of its overlapping edges. As already noted, our algorithm is a generalization of a procedure used by Bruggmann and Zenklusen (see proof of Lemma 15 in [5]) for a different purpose in the context of contention resolution schemes for matchings on graphs.²

²More precisely, [5] uses Algorithm 1 for classical graphs $G = (V, E)$ to show that, for any $y \in \mathbb{R}_{\geq 0}^E$, the vector $(y(e)/y(e) + \sum_{f \in N(e)} y(f))_{e \in E}$ is in the matching polytope, by proving that Algorithm 1 returns a matching $M \subseteq E$ with $\Pr[e \in M] = y(e)/y(e) + \sum_{f \in N(e)} y(f)$ for all $e \in E$. 


To show that Algorithm 1 proves Theorem 1, we recall the following basic facts about exponential random variables, which lie at the heart of procedures based on exponential clocks.

**Fact 1.** Let $X_i \sim \text{Exp}(\lambda_i)$ for $i \in \{1, \ldots, n\}$ be indep. random variables. Then $\min\{X_1, \ldots, X_n\} \sim \text{Exp}(\lambda_1 + \ldots + \lambda_n)$.

**Fact 2.** Let $X_1 \sim \text{Exp}(\lambda_1), X_2 \sim \text{Exp}(\lambda_2)$ be indep. random variables. Then $\Pr[X_1 < X_2] = \frac{\lambda_1}{\lambda_1 + \lambda_2}$.

**Proof of Theorem 1.** Algorithm 1 clearly returns a matching $M$, and for each $e \in E$ with $x(e) = 0$ we trivially have $\Pr[e \in M] = 0 = x(e)/(|e| - (|e| - 1)x(e))$. Hence, it remains to check the probability of $M$ containing an edge $e \in \text{supp}(x)$, which is equal to

$$
\Pr[e \in M] = \Pr[Z_e < \min\{Z_f: f \in N(e) \cap \text{supp}(x)\}] \quad \text{(by construction)}
$$

$$
= \Pr[Z_e < \text{Exp}(x(N(e)))] \quad \text{(by Fact 1)}
$$

$$
= \frac{x(e)}{x(e) + x(N(e))} \quad \text{(by Fact 2)}.
$$

The result now follows by observing that

$$
x(N(e)) \leq \sum_{v \in e} \sum_{f \in \delta(v): f \neq e} x(f) \leq \sum_{v \in e} (1 - x(e)) = |e|(1 - x(e)),
$$

where the second inequality holds as $x$ is a feasible fractional matching, and thus, $x(\delta(v)) \leq 1$.

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**3 The greedy algorithm for non-uniform hypergraph $b$-matching**

In this section, we switch to the more general hypergraph $b$-matching problem, and prove Theorem 2 by analyzing a natural greedy algorithm with respect to the fractional hypergraph $b$-matching LP. A description of the greedy procedure is given in Algorithm 2 below.

**Algorithm 2:** Greedy algorithm for hypergraph $b$-matching

1. Order the edges $E = \{e_1, \ldots, e_m\}$ such that $w(e_1) \geq w(e_2) \geq \ldots \geq w(e_m)$;
2. $M = \emptyset$;
3. for $j = 1$ to $m$ do
   - if $M \cup \{e_j\}$ is a $b$-matching then
     - $M = M \cup \{e_j\}$;
4. return $M$;

To analyze Algorithm 2, we now present a charging scheme showing that the returned matching satisfies the properties of Theorem 2 and thus implies the theorem. For each edge $e \in E \setminus M$, the charging scheme assigns its LP-weight $x(e)w(e)$ first to a vertex $v \in V$ that is saturated by $M$, which means that the number of edges in $M$ containing $v$ is equal to $b(v)$. In a second step, the charge of saturated vertices is assigned to edges $f \in M$ in a way that $w(f)(|f| - (|f| - 1)x(f))$ is large enough to pay for all charges assigned to it, including its own charge $x(f)w(f)$.
Proof of Theorem 2. Algorithm[2] clearly returns a $b$-matching. Let $S = \{v \in V : |\delta(v) \cap M| = b(v)\}$ be all vertices saturated by $M$. Note that the only reason for an edge $f$ not to be added to $M$, i.e., $f \in E \setminus M$, is that $f$ was “blocked” by a vertex $v_f \in f \cap S$ that was saturated when $f$ was considered by the algorithm. If $f$ contains more than one saturated vertex at the moment $f$ is considered by the algorithm, we denote by $v_f$ an arbitrary vertex among those. Notice that because $v_f$ was saturated when Algorithm[2] considered $f$ and edges are considered in non-increasing order of weight, every edge of $M$ containing $v_f$ is at least as heavy as $f$, i.e.,

$$w(f) \leq w(e) \quad \forall e \in M \cap \delta(v_f) \ .$$

The relation below shows that the total LP-contribution of edges in $E \setminus M$ can be bounded by an expression involving the $x$-load on saturated vertices and the weights of edges in $M$ containing them:

$$\sum_{f \in E \setminus M} w(f)x(f) = \sum_{v \in S} \sum_{f \in E \setminus M: v_f = v} w(f)x(f)$$

$$\leq \sum_{v \in S} \min\{w(e) : e \in \delta(v) \cap M\} \sum_{f \in E \setminus M: v_f = v} x(f)$$

$$\leq \sum_{v \in S} \min\{w(e) : e \in \delta(v) \cap M\} \sum_{f \in \delta(v) \setminus M} x(f) \ ,$$

where the equality can be interpreted as assigning, for each $f \in E \setminus M$, the LP-weight $w(f)x(f)$ to the vertex $v_f$ and the first inequality follows from [2]. Moreover, the $x$-load of a vertex $v \in S$ can be bounded as follows:

$$\sum_{f \in \delta(v) \setminus M} x(f) \leq b(v) - \sum_{e \in \delta(v) \cap M} x(e) = \sum_{e \in \delta(v) \cap M} (1 - x(e)) \ ,$$

where the inequality is due to $x$ being a fractional $b$-matching and the equality is a consequence of $v \in S$ being saturated by $M$.

Finally, by combining (3) and (4) we obtain

$$\sum_{f \in E \setminus M} w(f)x(f) \leq \sum_{v \in S} \min\{w(e) : e \in \delta(v) \cap M\} \sum_{e \in \delta(v) \cap M} (1 - x(e))$$

$$\leq \sum_{v \in S} \sum_{e \in \delta(v) \cap M} w(e)(1 - x(e))$$

$$= \sum_{e \in M} |S \cap e| w(e)(1 - x(e))$$

$$\leq \sum_{e \in M} |e| w(e)(1 - x(e)) \ .$$

By adding the term $\sum_{e \in M} w(e)x(e)$ to both sides of the above relation, the desired result is obtained. 

\[\square\]

4 Conclusion

The main goal of this work was the development of concise algorithms and analysis techniques for the Füredi, Kahn, and Seymour Conjecture and the related sampling problem, i.e., Conjectures [1]
and \(^2\) Although our techniques fall short of proving the conjectures, they lead to the currently best bounds of \(g(e) = |e| - (|e| - 1)x(e) \leq |e|\) while being simpler than prior approaches.

We now briefly discuss why our factor \(g\), and in particular the term \((|e| - 1)x(e)\), may open up new ways to attack both conjectures. For simplicity, let us focus on the original conjecture of Füredi, Kahn, and Seymour, i.e., Conjecture \(^1\). More precisely, notice that for any edge \(e\) with \(x(e) \geq 1/|e|\), our bound is at least as good as the one conjectured by Füredi et al., because in this case \(g(e) = |e| - (|e| - 1)x(e) \leq |e| - 1 + 1/|e|\). Hence, the contribution from these edges corresponds to the factor in the conjecture. Thus, the only issue is edges with small \(x\)-value. Of course, there are points in the matching polytope with only small \(x\)-values. However, this is not the case when \(x\) is an extreme point of the fractional matching polytope. (Note that an optimal solution to the fractional matching LP, as stipulated by Conjecture \(^1\), can of course always be chosen to be an extreme point.)

Indeed, if \(x\) is an extreme point of the fractional matching polytope, then the average \(x(e)\)-value of edges \(e \in \text{supp}(x)\) is at least \(1/|e|\). This can be derived by standard sparsity arguments as follows. Let \(Q := \{v \in V : x(\delta(v)) = 1\}\) be all vertices for which the corresponding constraint in the fractional matching polytope is tight with respect to \(x\). The remaining constraints of the fractional matching polytope are non-negativity constraints, which implies by classical sparsity arguments that \(|\text{supp}(x)| \leq |Q|\). Moreover, as all vertices in \(Q\) are covered by one unit of \(x\)-value we have \(|Q| \leq \sum_{e \in \text{supp}(x)} |e|x(e)\). Hence, \(1 \leq \frac{1}{|\text{supp}(x)|} \sum_{e \in \text{supp}(x)} |e|x(e)\), which can be interpreted as follows: “on average” the term \(|e|x(e)\) is at least 1, i.e., the “average load” of edges \(e \in \text{supp}(x)\) is at least \(1/|e|\).

Of course, this is only an averaging reasoning and many edges \(e\) will have \(x\)-value below \(1/|e|\). Nevertheless, because \(x\) is a maximum weight fractional matching, there is hope that edges of value above average tend to be heavy edges, and the improved guarantee we get on those, compared to the guarantee needed by Conjecture \(^1\) may compensate for edges with small \(x\)-value. Another idea that may be helpful, possibly in combination with the above observation, is to perform a careful alteration of \(x\) before applying our rounding procedure with the goal to obtain stronger guarantees.

Finally, we think it is an interesting open question whether our exponential clocks algorithm can be extended to the \(b\)-matching setting to obtain a very efficient and hopefully simple sampling procedure for \(b\)-matchings.

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\(^3\)Moreover, also in the sampling setting, it suffices to prove Conjecture \(^2\) for extreme points only, as any fractional matching \(x\) can first be written as a convex combination \(\sum_{i=1}^{n} \lambda_i x^i\) of extreme points \(x^i\) of the fractional matching polytope. If we can get a sampling procedure fulfilling the requirements of Conjecture \(^2\) for extreme points, then we can simply first pick with probability \(\lambda_i\) the extreme point \(x^i\) and then return a matching using the sampling procedure for \(x^i\). One can easily check that such a sampling procedure fulfills the conditions of Conjecture \(^2\) if the sampling procedure for the extreme points \(x^i\) does so.
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A Relating Conjectures 1 and 2 via LP duality

In this section, we show how Theorem 2 implies the generalization of Theorem 1 to hypergraph b-matchings, though with a slower (but still polynomial-time) algorithm. The arguments used in this section follow the work of Carr and Vempala [7] for classical approximation algorithms, whereas the guarantees we are seeking are expressed as a function $g$ over the edges (see (1)). Even though the generalization is quite straightforward, we include it for completeness and because it seems that this strong connection has been overlooked in prior work.

The main tools used are LP duality and the ellipsoid method. Let $H = (V, E)$ be a hypergraph, $w \in \mathbb{R}^E_{\geq 0}$, $b \in \mathbb{Z}^V_{\geq 1}$, and let $x$ be a feasible fractional b-matching in $H$. We denote by $\mathcal{M} \subseteq 2^E$ the set of all feasible b-matchings of $H$ and let $g(e) = |e| - (|e| - 1)x(e)$. The reasoning that follows does not depend on the precise form of $g$ and thus shows that Theorem 2 implies Theorem 1 (for b-matchings) even if $g$ is replaced by another function. In particular, it shows that the existence of an efficient algorithm returning a matching satisfying Conjecture 1 implies Conjecture 2.

The following primal LP describes all vectors $\lambda \in \mathbb{R}_M$ that correspond to probabilities for sampling a b-matching such that the conditions of Theorem 1 are fulfilled.

$$\begin{align*}
\text{min} & \quad 0 \\
\text{s.t.} & \quad \sum_{M \in \mathcal{M}, e \in M} \lambda(M) \geq \frac{x(e)}{g(e)} \quad \forall e \in E \\
& \quad \sum_{M \in \mathcal{M}} \lambda(M) = 1 \\
& \quad \lambda \in \mathbb{R}_M^+ .
\end{align*}$$

Its dual is shown below.

$$\begin{align*}
\text{max} & \quad \sum_{e \in E} \frac{x(e)}{g(e)} y(e) - \mu \\
\text{s.t.} & \quad y(M) \leq \mu \quad \forall M \in \mathcal{M} \\
& \quad \mu \in \mathbb{R} \\
& \quad y \in \mathbb{R}_E^+ .
\end{align*}$$

We will show that Theorem 2 implies that the dual LP is feasible with optimal value 0, which leads to feasibility of the primal LP due to strong duality. This in turn implies existence of a desired distribution over b-matchings, and it remains to discuss how to efficiently sample from such a distribution. Note that as the primal LP only has linearly many constraints (in $|E|$), any primal vertex solution has support size linear in $|E|$. Hence, there exist desired distributions of small support.

The dual LP clearly is feasible because of the all-zeros vector, which leads to an objective value of 0. Also observe that the dual LP is scale invariant, in the sense that if $(y, \mu)$ is feasible, then so is $(\gamma y, \gamma \mu)$ for any $\gamma \geq 0$. Thus, if there is any dual LP solution of strictly positive objective value, then the dual LP is unbounded. Consequently, the dual LP is unbounded if and only if there is a solution of value at least 1, and we describe all those solutions by the following polyhedron.

$$\mathcal{Q}(x, g) := \left\{ (y, \mu) \in \mathbb{R}_E^+ \times \mathbb{R} : \sum_{e \in E} \frac{x(e)}{g(e)} y(e) - \mu \geq 1, \quad y(M) \leq \mu \quad \forall M \in \mathcal{M} \right\} .$$

The above discussion implies the following.
Observation 1. The dual LP has optimal value 0 if and only if \( Q(x, g) = \emptyset \). Equivalently, the primal LP is feasible if and only if \( Q(x, g) = \emptyset \).

We will use the ellipsoid method together with the greedy algorithm and its approximation guarantee implied by Theorem 2 to certify in polynomially many steps that \( Q(x, g) = \emptyset \).

An ellipsoid iteration. Let \((y, \mu) \in \mathbb{R}_+^E \times \mathbb{R}\) be a candidate point of \( Q(x, g) \). If \((y, \mu)\) violates the first constraint in the inequality description of \( Q(x, g) \), then we can use this inequality as a separating hyperplane. We note that the encoding length of this separating hyperplane is polynomial with respect to the encoding length of the input hypergraph \( H \) and \( x \). So, suppose that \((y, \mu)\) satisfies

\[
\sum_{e \in E} x(e) y(e) / g(e) - \mu \geq 1.
\]

We now run the greedy algorithm using \( x \) as a feasible fractional \( b \)-matching and weights \( w \in \mathbb{R}_+^E \) given by \( w(e) = y(e) / g(e) \) for \( e \in E \), and, using Theorem 2, obtain a feasible \( b \)-matching \( M \) that satisfies

\[
\sum_{e \in E} g(e) w(e) \geq \sum_{e \in E} w(e) x(e).
\]

By the definition of \( w \), this implies that

\[
y(M) \geq \sum_{e \in E} \frac{x(e)}{g(e)} y(e).
\]

Because we have assumed that the first constraint of \( Q(x, g) \) is satisfied by \((y, \mu)\), we get

\[
y(M) \geq \mu + 1.
\]

As \( M \) is a \( b \)-matching, the above relations imply that \((y, \mu)\) violates the constraint \( y(M) \leq \mu \), which we can thus use as a separating hyperplane. Note again that its encoding length is polynomial.

In short, for any candidate point \((y, \mu) \in \mathbb{R}_+^E \times \mathbb{R}\), we can efficiently find a constraint of \( Q(x, g) \) that is violated by \((y, \mu)\), and so, the ellipsoid method will terminate in polynomially many steps certifying that \( Q(x, g) = \emptyset \). The number of ellipsoid iterations is indeed polynomially bounded as the separating hyperplanes that are produced by the algorithm have polynomial encoding length, as noted above (see Theorem 6.4.9 of [12]).

Computing the desired distribution. Combined with Observation 1, the above discussion shows that the primal LP is feasible, and the distribution we are looking for thus exists. Such a distribution can now be found through standard techniques. More precisely, the ellipsoid method certified emptiness of \( Q(x, g) \), and did so with the polynomially many constraints that were generated through the separation oracle. Hence, even when replacing in \( Q(x, g) \) the set of all \( b \)-matchings \( \mathcal{M} \) by the \( b \)-matchings \( \mathcal{M}' \subseteq \mathcal{M} \) generated in the calls to the separation oracle, the polyhedron remains empty. Hence, by strong duality, the primal LP is feasible even when replacing \( \mathcal{M} \) by the polynomially-sized subset \( \mathcal{M}' \). This reduced primal LP can thus be solved efficiently and leads to a distribution with the desired properties.

This reduction implies the following extension of Theorem 1 to the \( b \)-matching setting.

Theorem 3. Let \( H = (V, E) \) be a hypergraph, \( b \in \mathbb{Z}_{\geq 1}^V \), \( w \in \mathbb{R}_+^E \), and let \( x \in [0, 1]^E \) be a point in the fractional \( b \)-matching polytope. Then, there is an efficient sampling procedure returning a \( b \)-matching \( M \) with

\[
\Pr[e \in M] \geq \frac{x(e)}{|e| - (|e| - 1)x(e)} \quad \forall e \in E.
\]