Hawking Radiation, Effective Actions and Covariant Boundary Conditions

Rabin Banerjee\textsuperscript{1,2*}, Shailesh Kulkarni\textsuperscript{2†}

\textsuperscript{1}Institute of Quantum Science, College of Science and Technology, Nihon University, Tokyo 101-8308, Japan.
\textsuperscript{2}S.N. Bose National Centre for Basic Sciences, JD Block, Sector III, Salt Lake, Kolkata-700098, India

From an appropriate expression for the effective action, the Hawking radiation from charged black holes is derived, using only covariant boundary conditions at the event horizon. The connection of our approach with the Unruh vacuum and the recent analysis \cite{1,2,3} of Hawking radiation using anomalies is established.

Introduction:

Hawking radiation arises upon the quantisation of matter in a background spacetime with an event horizon. It therefore plays an important role in black hole physics. Apart from Hawking’s \cite{4} original derivation, there are other approaches \cite{5,6}, although none is completely clinching or conclusive. This has led researchers to consider alternative derivations providing new insights into the problem. Here we discuss another approach that is based solely on the structure of the effective action and boundary conditions at the event horizon. We therefore guarantee the universality of Hawking radiation which ought to be determined by properties at the event horizon only—a feature that is usually lacking in approaches based on effective actions \cite{7,8}. To put our work in a proper perspective, however, it is desirable to elaborate on the recent approaches \cite{1,2,3} to the Hawking effect which rely on the cancellation of gauge and gravitational anomalies.

An anomaly, it might be recalled, is a breakdown of some classical symmetry due to the process of quantisation. For example, a gauge anomaly is an anomaly in gauge symmetry, taking the form of nonconservation of the gauge current. Similarly, a gravitational anomaly occurs from a breaking of general covariance, taking the form of nonconservation of the energy momentum tensor (For reviews see, \cite{9,10}). The simplest manifestation of these (gauge and gravitational) anomalies, which is also relevant for the present discussion, occurs for 1+1 dimensional chiral fields.

Recently, Robinson and Wilczek \cite{1}, followed by Iso, Umetsu and Wilczek \cite{2}, gave a new derivation of the Hawking effect. They found that, by the process of dimensional reduction, effective field theories become two dimensional and chiral near the event horizon of a black hole. This leads to the occurrence of gauge and gravitational anomalies. The Hawking flux is necessary to cancel these anomalies.

An essential aspect of \cite{1,2} is that a two dimensional chiral theory admits two types of anomalous currents (and/or energy momentum tensors)- the consistent and the covariant \cite{9,10,11,12,13,14}, which are actually related by local counterterms. The covariant divergence of these currents and energy momentum tensors yields either the consistent or covariant form of the anomaly. Then the Hawking flux was derived in \cite{1,2} by a cancellation of the consistent anomaly but the boundary condition necessary to fix the parameters was obtained from a vanishing of the covariant current at the horizon \cite{2}. It was also observed \cite{15} that an incorrect result for the charge flux would be obtained if, instead, the vanishing of the consistent current at the horizon was taken as the boundary condition.

The approach of \cite{1,2} was very recently generalised by us \cite{3}. It was shown that the complete analysis was feasible in terms of covariant expressions only. The flux from a charged black hole was correctly determined by a cancellation of the covariant anomaly with the boundary condition being the vanishing of the covariant current (and energy momentum tensor) at the horizon. Apart from being conceptually clean and more natural (all expressions being covariant), it simplified the original analysis \cite{1,2} considerably. This was true not just for charged black holes, but for other black holes as well \cite{17}.

From the analysis of \cite{2,3} it appears therefore that covariant boundary conditions at the horizon play a fundamental role. We adopt the arguments of \cite{1,2} which imply that effective field theories are chiral and two dimensional near the horizon. Then, exploiting known structures of the two dimensional effective actions, the relevant expressions for the currents and the energy momentum tensors are derived by only imposing covariant boundary conditions at the horizon. The Hawking flux from charged black holes is correctly reproduced in this manner. Finally, we establish the connection of our approach with calculations based on the Unruh vacuum \cite{15,18}.

\*E-mail: rabin@bose.res.in

\†E-mail: shailesh@bose.res.in
General Setting and Effective Actions:

We are interested in discussing the Hawking effect from a charged black defined by the Reissner-Nordstrom metric given by,

\[ ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\Omega^2_{(d-2)} \]  

(1)

where \( d\Omega^2_{(d-2)} \) is the line element on the \((d-2)\) sphere. The function \( f(r) \) admits an event horizon at \( r_+ \) so that \( f(r_+) = 0 \). The gauge potential is defined by \( A = -\frac{Q}{2}dt \).

As already mentioned by using a dimensional reduction technique, the effective field theory near the horizon becomes a two dimensional chiral theory. The metric of this two dimensional theory is identical to the \( r-t \) component of the full metric \( \Omega \). Hence the determinant of the metric simplifies to unity \( (\sqrt{-g} = 1) \) and many expressions mimic their flat space counterparts. The theory away from the horizon is not chiral and hence is anomaly free.

We now summarise, step by step, our methodology. For a two dimensional theory the expressions for the effective actions, whether anomalous (chiral) or normal, are known \[18, 19\]. Both these are required in our analysis. For deriving the Hawking flux, only the form of the anomalous (chiral) effective action, which describes the theory near the horizon, is required. The currents and energy momentum tensors are computed by taking appropriate functional derivatives of this effective action. Next, the parameters appearing in these solutions are fixed by imposing the vanishing of covariant currents (energy momentum tensors) at the horizon. Once these are fixed, the Hawking fluxes are obtained from the asymptotic \(+\infty\) limits of the currents and energy momentum tensors. To show the connection with the Unruh vacuum the form of the usual effective action, which describes the theory away from the horizon, is necessary. The currents and energy momentum tensors, obtained from this effective action, are solved by using the knowledge of the corresponding chiral expressions. The results reproduce the expectation values of the currents and energy momentum tensors for the Unruh vacuum.

\begin{align*}
\Gamma &= \frac{1}{96\pi} \int d^2x d^2y \sqrt{-g}R(x) \frac{1}{\Delta_g}(x, y)\sqrt{-g}R(y) + \\
&\quad \frac{e^2}{2\pi} \int d^2x d^2y \epsilon^{\mu\nu} \delta \sigma A_\sigma(x) \frac{1}{\Delta_g}(x, y)\epsilon^{\rho\sigma} \delta \rho A_\sigma(y). \\
T_{\mu\nu(o)} &= \frac{2}{\sqrt{-g}} \frac{\delta \Gamma}{\delta g^{\mu\nu}}.
\end{align*}

Here \( R \) is the two-dimensional Ricci scalar given by \( R = f'' \), and and \( \Delta_g = \nabla^\mu \nabla_\mu \) is the laplacian in this background.

The energy-momentum tensor \( T_{\mu\nu(o)} \) in the region outside the horizon is defined as,

\[ T_{\mu\nu(o)} = \frac{2}{\sqrt{-g}} \frac{\delta \Gamma}{\delta g^{\mu\nu}}. \]

(3)

The explicit form for \( T_{\mu\nu(o)} \) is thus given by

\[ T_{\mu\nu(o)} = \frac{1}{48\pi} \left( 2g_{\mu\nu}R - 2\nabla_\mu \nabla_\nu G + \nabla_\mu G \nabla_\nu G - \frac{1}{2}g_{\mu\nu} \nabla^\sigma \nabla_\sigma G \right) + \frac{e^2}{\pi} \left( \nabla_\mu B \nabla_\nu B - \frac{1}{2}g_{\mu\nu} \nabla^\rho B \nabla_\rho B \right) \]

(4)

Similarly, the form for the gauge current can be obtained,

\[ J_\mu(o) = \frac{\delta \Gamma}{\delta A_\mu} = \frac{e^2}{\pi} \epsilon^{\mu\nu} \delta \rho B. \]

(5)

Here

\[ G(x) = \int d^2y \Delta_g^{-1}(x, y)\sqrt{-g}R(y), \]

\[ B(x) = \int d^2y \Delta_g^{-1}(x, y)\epsilon^{\mu\nu} \delta \rho A_\nu(y). \]

(6)

(7)

From now on we would omit the \( \sqrt{-g} = 1 \) factor from all the expressions. Hence we work with the antisymmetric numerical tensor \( \epsilon^{\mu\nu} \) defined by \( \epsilon^{tt} = 1 \). \( B(x) \) and \( G(x) \) satisfy,

\[ \nabla_\mu \nabla_\mu B = -\partial_\tau A_\tau; \quad \nabla_\mu \nabla_\mu G = R = f'' \]

(8)
respectively. The solutions for $B$ and $G$ are now given by

$$B = B_o(r) - at + b; \partial_r B_o = \frac{1}{f}(A_t + c),$$

(9)

$$G = G_o(r) - 4pt + q; \partial_r G_o = -\frac{1}{f}(f' + z),$$

(10)

where $a, b, c, p, q$ and $z$ are constants. Also note that $B_o$ and $G_o$ are functions of $r$ only.

The current (5) and the energy momentum tensor (4) satisfy the normal Ward identities,

$$\nabla^{\mu} J_{\mu}^{(o)} = 0; \quad \nabla^{\mu} T_{\mu \nu}^{(o)} = F_{\mu \nu} J_{\mu}^{(o)}$$

(11)

Note that in the presence of an external gauge field the energy momentum tensor is not conserved; rather the Lorentz force term is obtained.

In the region near the horizon we have gravitational as well as gauge anomaly so that the effective theory is described by an anomalous (chiral) effective action which is given by (12),

$$\Gamma_{(H)} = -\frac{1}{3}z(\omega) + z(A)$$

(12)

where $\omega$ and $A$ are the gauge field and the spin connection, respectively, and,

$$z(v) = \frac{1}{4\pi} \int d^2 x d^2 y e^{\mu \nu} \partial_{\mu} v_{\nu}(x) \nabla^{-1}(x, y) \partial_{\rho} \epsilon^{\rho \sigma} v_{\sigma}(y)$$

(13)

From a variation of this effective action the energy momentum tensor and the gauge current are computed. To get their covariant forms in which we are interested, however, appropriate local polynomials have to be added. This is possible since energy momentum tensors and currents are only defined modulo local polynomials. The final results for the covariant energy momentum tensor and the covariant current are given by (14),

$$\delta \Gamma_{H} = \int d^2 x \left( \frac{1}{2} \delta g_{\mu \nu} T^{\mu \nu} + \delta A_{\mu} J^{\mu} \right) + l$$

(14)

where the local polynomial is given by,

$$l = \frac{1}{4\pi} \int d^2 x \epsilon^{\mu \nu} (A_{\mu} \delta A_{\nu} - \frac{1}{3} w_{\mu} \delta w_{\nu} - \frac{1}{24} Re^a_{\mu} \delta e^a_{\nu})$$

(15)

The covariant energy momentum tensor $T^{\mu \nu}$ and the covariant gauge current $J^{\mu}$ are obtained from the above relations as,

$$T^{\mu \nu} = e^2 \frac{4\pi}{24} (D^{\mu} D_{\nu} B)$$

$$+ \frac{1}{4\pi} \left( \frac{1}{48} D^{\mu} G D_{\nu} G - \frac{1}{24} D^{\mu} D_{\nu} G + \frac{1}{24} \delta^{\mu \nu} R \right)$$

(16)

$$J^{\mu} = -\frac{e^2}{2\pi} D^{\mu} B.$$  

(17)

Note the presence of the chiral covariant derivative,

$$D_{\mu} = \nabla_{\mu} - \epsilon_{\mu \nu} \nabla^{\nu} = -\epsilon_{\mu \nu} D^{\nu},$$

(18)

instead of the usual one that occurred previously in (4). The definitions of $B$ and $G$ are provided in (6), (7).

By taking the covariant divergence of (16) and (17) we get the anomalous Ward identities,

$$\nabla_{\mu} J^{\mu} = -\frac{e^2}{2\pi} \epsilon^{\rho \sigma} \partial_{\mu} A_{\sigma} = \frac{e^2}{2\pi} \partial_{\nu} A_{\nu}$$

(19)

$$\nabla_{\mu} T^{\mu \nu}_{\nu} = F_{\mu \nu} J^{\mu} + \frac{1}{96\pi} \epsilon_{\nu \rho \sigma} \partial^{\mu} R.$$  

(20)

The anomalous terms are the covariant gauge anomaly and the covariant gravitational anomaly, respectively. These Ward identities were also obtained from different considerations in [3].
**Charge and Energy Flux:**

In this subsection we calculate the charge and energy flux by using, respectively, the expressions for the covariant current (17) and the covariant energy momentum tensor (16). We will see that the results are the same as that obtained by the anomaly cancellation (consistent or covariant) method [2, 3].

First, we derive the charge flux. Using (9) and (18) we have from (17),

\[ J^r = \frac{e^2}{2\pi} (A_t(r) + c + a) \]  

We now impose the boundary condition that the covariant current \( J^r \) vanishes at the horizon, implying \( J^r(r_+) = 0 \). This leads to a relation,

\[ c + a = -A_t(r_+) \]  

Hence the expression for \( J^r \) takes the form,

\[ J^r = \frac{e^2}{2\pi} (A_t(r) - A_t(r_+)) \]  

Now the charge flux is given by the asymptotic \( (r \to \infty) \) limit of the anomaly free current [1, 2, 3]. As observed from (19) the anomaly vanishes in this limit and hence we directly compute the flux from (23) by taking the \( (r \to \infty) \) limit. This yields,

\[ J^r(r \to \infty) = -\frac{e^2}{2\pi} (A_t(r_+)) \]  

This is the desired Hawking flux and agrees with previous findings [1, 2, 3].

We next consider the energy momentum flux by adopting the same technique. After using the solutions for \( B(x) \) and \( G(x) \), the \( r-t \) component of the covariant energy momentum tensor (16) becomes,

\[ T^r_t = \frac{e^2}{4\pi} (A_t(r) - A_t(r_+))^2 + \frac{1}{12\pi}(p - \frac{1}{4}(f' + z))^2 
+ \frac{1}{24\pi}(pf' + \frac{1}{4}ff'' - \frac{1}{4}f'(f' + z)). \]  

Now we implement the boundary condition; namely the vanishing of the covariant energy momentum tensor at the horizon, \( T^r_t(r_+) = 0 \). This condition yields,

\[ p = \frac{1}{4}(z \pm f'_+); \quad f'_+ \equiv f'(r = r_+). \]  

Using either of the above solutions in (26) we get,

\[ T^r_t = \frac{e^2}{4\pi} (A_t(r) - A_t(r_+))^2 
+ \frac{1}{192\pi} \left[f'^2 - f'^2 + 2ff''\right]. \]  

This expression is in agreement with that given in [3].

To obtain the energy flux, we recall that it is given by the asymptotic expression for the anomaly free energy momentum tensor. As happened for the charge case, here also it is found from [4] that the anomaly vanishes in this limit. Hence the energy flux is abstracted by taking the asymptotic infinity limit of (27). This yields,

\[ T^r_t(r \to \infty) = \frac{e^2}{4\pi} A_t^2(r_+) + \frac{1}{192\pi} f'^2. \]  

which correctly reproduces the Hawking flux.

**Connection with Unruh vacuum**

Here we compute the anomaly free current and the energy momentum tensor, which describe the theory away from the horizon, and show that these agree with the expectation values of these observables for the Unruh vacuum.

We consider the expression for the current \( J^\mu_{(o)} \) in the region outside the horizon. From [5] and [6] we obtain,

\[ J^r_{(o)} = \frac{e^2}{\pi} a, \quad J^t_{(o)} = \frac{e^2}{\pi f} (A_t(r) + c). \]  

(29)
At asymptotic infinity the result for $J^r_{(o)}$ must agree with (24). Taken together with (22) this implies $a = c = -\frac{A_t(r_+)}{2}$ and hence the currents outside the horizon are given by,

$$J^r_{(o)} = -\frac{e^2}{2\pi} A_t(r_+); \quad J^t_{(o)} = \frac{e^2}{\pi f} \left( A_t(r) - \frac{1}{2} A_t(r_+) \right).$$  \hfill (30)

This is also the expectation value of the current for the Unruh vacuum in the $d = 2$ RN black hole [18, 15].

Now we consider components of the anomaly free energy momentum tensor defined in (4). The $r - t$ component of $T^\mu_{\nu(o)}$ is given by

$$T^r_{t(o)} = \frac{e^2}{4\pi} A^2_t(r_+) - \frac{e^2}{2\pi} A_t(r)A_t(r_+),$$

while the $t - t$ component becomes,

$$T^t_{t(o)} = \frac{e^2}{2\pi f} \left( A^2_t(r) - A_t(r_+)A_t(r) + \frac{1}{2} A^2_t(r_+) \right) + \frac{1}{48\pi f} \left[ 2f f'' - f'(f' + z) + 8p^2 + \frac{(f' + z)^2}{2} \right].$$

The asymptotic form of (31) must agree with that of (28). A simple inspection shows that $zp = -\frac{1}{16} f'^2$. Substituting this in (20) yields two solutions $p = \frac{1}{8} f'_+; \quad z = -\frac{1}{2} f'_+$ and $p = -\frac{1}{8} f'_+; \quad z = \frac{1}{2} f'_+$. Using either of these solutions in (31) and (32) we obtain,

$$T^r_{t(o)} = \frac{e^2}{4\pi} A^2_t(r_+) - \frac{e^2}{2\pi} A_t(r)A_t(r_+) + \frac{1}{192\pi f} f'^2,$$

while the $t - t$ component becomes,

$$T^t_{t(o)} = \frac{e^2}{2\pi f} \left( A^2_t(r) - A_t(r_+)A_t(r) + \frac{1}{2} A^2_t(r_+) \right) + \frac{1}{96\pi f} \left[ 4f f'' - f'^2 + \frac{1}{2} f'^2 \right].$$

Likewise $T^r_{r(o)}$ can be computed either directly or from noting the trace $T^\mu_{\nu(o)} = \frac{R}{24\pi}$ that follows from (4) and then using (31). These are also the expressions for the expectation values of the various components of the energy momentum tensor found for the Unruh vacuum [15, 18].

**Discussions:**

We have given a derivation of the Hawking flux from charged black holes, based on the effective action approach, which only employs the boundary conditions at the event horizon. It might be mentioned that generally such approaches require, apart from conditions at the horizon, some other boundary condition, as for example, the vanishing of ingoing modes at infinity [7, 8, 15]. The latter obviously goes against the universality of the Hawking effect which should be determined from conditions at the horizon only. In this we have succeeded. Also, the specific structure of the effective action from which the Hawking radiation is computed is valid only at the event horizon. This is the anomalous (chiral) effective action. Other effective action based techniques do not categorically specify the structure of the effective action at the horizon. Rather, they use the usual (anomaly free) form for the effective action and are restricted to two dimensions only [8].

An important factor concerning this analysis is to realise that effective field theories become two dimensional and chiral near the event horizon [1]. Yet another ingredient was the implementation of a specific boundary condition- the vanishing of the covariant form of the current and/or the energy momentum tensor [2, 3]. Not only that, the importance of the covariant forms was further emphasised by us in [3] where it was shown that the anomaly cancelling approach was simplified considerably if, instead of consistent anomalies used in [1, 2], covariant anomalies were taken as the starting point. Indeed, in the present computations, we have taken that form of the effective action which yields anomalous Ward identities having covariant gauge and gravitational anomalies. The unknown parameters in the covariant energy momentum tensor and the covariant current derived from this anomalous effective action were fixed by a boundary condition- namely the vanishing of these covariant quantities at the event horizon. Consequently we have shown that aspects like covariant anomalies and covariant boundary conditions are not merely confined to discussing the Hawking effect in the anomaly.
cancelling approach [1, 2, 3]. Rather they have a wider applicability since our effective action based approach is different from the anomaly cancelling approach.

Further, we have exploited the information from the chiral (anomalous) effective action, which describes the theory near the horizon, to completely fix the form of the normal effective action that describes the theory away from the horizon. The expressions for the currents and energy momentum tensors obtained from the latter reproduce the results obtained by using the Unruh vacuum approach [15, 18]. There is an alternative approach, discussed in the appendix of [18], that reveals the connection of the normal effective action with Unruh vacuum. However it uses the Kruskal coordinates and directly imposes, as a boundary condition, the vanishing of ingoing modes at infinity. Hence it is different from our approach.

Acknowledgements:
The authors thank Saurav Samanta for discussions. One of the authors (RB) also thanks members of the Institute of Quantum Science, Nihon University, Tokyo, where a part of this work was done, for their gracious hospitality and support, and Satoshi Iso for fruitful discussions.

References

[1] S. P. Robinson and F. Wilczek, Phys. Rev. Lett. 95, 011303 (2005) gr-qc/0502074.
[2] S. Iso, H. Umetsu and F. Wilczek, Phys. Rev. Lett. 96, 151302 (2006) hep-th/0602146.
[3] R. Banerjee, S. Kulkarni, Phys. Rev. D 77, 024018 (2008) arXiv:0707.2449 [hep-th].
[4] S. Hawking, Commun. Math. Phys. 43, 199 (1975)
[5] G. Gibbons, S. Hawking, Phys. Rev. D 15, 2752 (1977).
[6] M. Parikh, F. Wilczek, Phys. Rev. Lett. 85, 5042 (2000).
[7] S. Christensen and S. Fulling, Phys. Rev. D 15, 2088 (1977).
[8] For a review, see, A. Wipf, in Black Holes: Theory and Observation, edited by F. W. Hehl, C. Kiefer and R. Metzler (Springer, Berlin, 1998).
[9] R. Bertlmann, ”Anomalies In Quantum Field Theory,” (Oxford Sciences, Oxford, 2000).
[10] K. Fujikawa and H. Suzuki, ”Path Integrals and Quantum Anomalies,” (Oxford Sciences, Oxford, 2004).
[11] L. Alvarez-Gaume and E. Witten, Nucl. Phys. B 234, 269 (1984).
[12] W. A. Bardeen, B. Zumino, Nucl. Phys. B 244, 421 (1984).
[13] H. Banerjee and R. Banerjee, Phys. Lett. B 174, 313 (1986).
[14] H. Banerjee, R. Banerjee and P. Mitra, Z. Phys. C 32, 445-454 (1986).
[15] W. Unruh, Phys. Rev. D 14, 870 (1976).
[16] S. Iso, T. Morita and H. Umetsu, JHEP 0704, 068 (2007) hep-th/0612286.
[17] J.J. Peng, S. Q. Wu, arXiv:0709.0167 [hep-th].
[18] S. Iso, H. Umetsu and F. Wilczek, Phys. Rev. D 74, 044017 (2006) hep-th/0606018.
[19] H. Leutwyler, Phys. Lett. B 153, 65 (1985).