On the Bifurcation of Limit Cycles Due to Polynomial Perturbations of Hamiltonian Centers

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Abstract. We study the number of limit cycles bifurcating from the period annulus of a real planar polynomial Hamiltonian ordinary differential system with a center at the origin when it is perturbed in the class of polynomial vector fields of a given degree.

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1. Introduction and Statement of the Main Results

In the qualitative theory of real planar polynomial differential systems, one of the main problems is the determination of limit cycles of a given vector field. The notion of limit cycle goes back to Poincaré, see [14–17]. He defined a limit cycle for a vector field in the plane as a periodic orbit of the differential system isolated in the set of all periodic orbits. The first works in determining the number of limit cycles of a given vector field can be traced back to Liénard [11] and Andronov [1]. After these works, the detection of the number of limit cycles of a polynomial differential system, intrinsically related to the so-called 16th Hilbert problem [7–9], has been extensively studied in the mathematical community, see, for instance, the books [3,19] and the papers [5,6,12,13].

One of the main tools of producing limit cycles is perturbing a system having a center. The notion of center goes back to Poincaré, see [14–17], who defined a center for a vector field on the real plane as a singular point having a neighborhood filled with periodic orbits with the exception of the singular point. If a system has a center, then when we perturb it, we may have a limit cycle that bifurcates in the perturbed system from some of the periodic orbits forming a center. This tool is one of the most effective ways of producing limit cycles, but it requires the knowledge of the first integral of the unperturbed system (the one having a center). It is well known that the determination of first integrals is also a very hard problem. This is why in this paper, we will
focus on an unperturbed planar differential system from which we know a first integral of it.

More precisely, in this paper, we consider the planar polynomial Hamiltonian system:

\[
\dot{x} = -F_y, \quad \dot{y} = F_x,
\]

with Hamiltonian \( F = F(x, y) \) of the form:

\[
F(x, y) = \sum_{i=1}^{n} c_i H(x, y)^i,
\]

where \( H(x, y) = (x^2 + y^2)/2 \) and the first \( c_i \neq 0 \) is positive. We assume \( c_n \neq 0 \) for convenience so that the degree of system (1) is \( 2n - 1 \). Note that \( H \) is a first integral of system (1), and since \( H \) has a local minimum at the origin of system, (1) has a center at the origin.

To simplify the notation we will write system (1) as

\[
\dot{x} = -y \cdot G(x, y), \quad \dot{y} = x \cdot G(x, y),
\]

where

\[
G(x, y) = \sum_{i=1}^{n} i c_i \left( \frac{x^2 + y^2}{2} \right)^{i-1} := \Gamma(H(x, y)).
\]

We note that the circles \( x^2 + y^2 = \text{constant} \) on which \( G(x, y) \neq 0 \) are periodic orbits and that the ones on which \( G(x, y) = 0 \) are filled of singular points of the differential system (2), and consequently of the differential system (1). Let \( \eta > 0 \) be the smallest real number, such that the circle \( x^2 + y^2 = \eta^2 \) is filled of singular points if it exists, otherwise \( \eta = +\infty \). We also note that all the singular points of the differential system (2) except the origin are on circles \( x^2 + y^2 = \text{constant} \), where \( G(x, y) = 0 \). Therefore, the period annulus of the center at the origin (i.e., the connected set formed by the union of all the periodic orbits surrounding the origin and having the origin in its inner boundary) is the annulus \( \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \eta^2\} \).

First, we will study the number of limit cycles that appear when system (1) [or (2)] is perturbed in the class of all polynomial differential systems in the form:

\[
\dot{x} = -yG(x, y) + \varepsilon A(x, y), \quad \dot{y} = xG(x, y) + \varepsilon B(x, y),
\]

where \( A \) and \( B \) are arbitrary real polynomials, such that

\[
m = \max\{\deg(A), \deg(B)\}.
\]

Second, we will study the number of limit cycles that appear when system (1) [or (2)] is perturbed in the class of all polynomial differential systems in the form:

\[
\dot{x} = -yG(x, y) + \sum_{i=1}^{\infty} \varepsilon^i A_i(x, y),
\]

\[
\dot{y} = xG(x, y) + \sum_{i=1}^{\infty} \varepsilon^i B_i(x, y),
\]

where

\[
A_i(x, y) = \sum_{j=1}^{n} i c_i \left( \frac{x^2 + y^2}{2} \right)^{i-1}.
\]
where $A_i$ and $B_i$ are arbitrary real polynomials, such that
\[ m = \max \{ \deg(A_i), \deg(B_i) \} = m \quad \text{for all} \quad i = 1, 2, \ldots. \]

Let $\eta_0 = \eta$ if $\eta < +\infty$, and let $\eta_0 < +\infty$ if $\eta = +\infty$. Then, we can parameterize the set of periodic orbits surrounding the origin and intersecting the interval $(0, \eta_0)$ by $h$, such that $h = x^2/2 = H(x, 0)$ with $x \in (0, \eta_0)$. We define the Poincaré map $P_\varepsilon(h)$ for system (4) on $(0, x_0)$, and the displacement map $\Delta(h, \varepsilon) = P_\varepsilon(h) - h$ which has a power series representation
\[
\Delta(h, \varepsilon) = \varepsilon M_1(h) + \varepsilon^2 M_2(h) + \cdots
\]
that converges for sufficiently small $\varepsilon$. The functions $M_i(h)$, defined for $h \geq 0$, are called the $i$th Melnikov function, and each positive simple zero of the first non-vanishing Melnikov function corresponds to a limit cycle of system (4).

To study the limit cycles that bifurcate from an unperturbed system when we perturb it, the vast majority of the papers study the simple zeros of $M_1(h)$, assuming that it is the first non-vanishing Melnikov function. There are much fewer papers studying the simple zeros of $M_2(h)$ assuming that it is the first non-vanishing Melnikov function, and there very few papers which study the simple zeros of $M_2(h)$ assuming that it is the first non-vanishing Melnikov function. In this paper, we will study the simple zeroes of all the Melnikov functions $M_k$ for an arbitrary $k$, assuming that it is the first non-vanishing Melnikov function.

As far as we know, there are only two papers that provide a similar result working with Melnikov functions at any order and perturbing the linear center $\dot{x} = -y$, $\dot{y} = -x$. The first one goes back to Iliev [10] where he was the first one in doing so. Due to the fact that this is an extremely hard problem involving very difficult computations, he started with the linear center, i.e., system (2) with $G = 1$. Following the ideas of Iliev, in [2], the authors study the number of zeros of the Melnikov function at any order for system (2) in the case in which $G = (x^2 + y^2)^{m-1}$. Our system (2) generalizes the systems studied in [2,10], because the unperturbed part is taken to be more general and we extend their results to this more general situation.

Our first main result is the following.

**Theorem 1.** The first non-vanishing Melnikov function $M_k(h)$ for system (4) has at most $[k(m - 1)/2] + (k - 1)(n - 1)$ positive zeros counting their multiplicities.

Here, $[p]$ denotes the integer part of the real number $p$.

The second main result of the paper is the following.

**Theorem 2.** The first non-vanishing Melnikov function $M_k(h)$ for system (5) has at most

(i) $[(m - 1)/2] + 2(k - 1)(n - 1)$ positive zeros counting their multiplicities if $m \leq 2n - 1$;

(ii) $[k(m - 1)/2] + (k - 1)(n - 1)$ positive zeros counting their multiplicities if $m \geq 2n - 1$. 
When \( G = 1 \), the bounds obtained in Theorem 2 coincide with the upper bounds obtained in [10]. When \( n = m - 1 \) and \( c_i = 0 \) for \( i = 1, \ldots, m - 1 \), system (2) coincides with the one studied in [2]. However, the upper bounds provided by Theorem 2 are larger than those obtained in [2] due to the fact that our \( G \) is full. Since a particular case of our system is the system studied in [2], where the authors obtain a better bound and prove that the bounds are not reached, and since the bounds in Theorem 1 are the same as those in statement (ii) of Theorem 2, we believe that the bounds in Theorem 2 are not going to be reached.

The paper is divided as follows. In Sect. 2, we introduce three lemmas that will be used in the proofs of Theorems 1 and 2. The proof of Theorem 1 is given in Sect. 3, and the proof of Theorem 2 is given in Sect. 4.

2. Preliminary Results

We first present a lemma, proved in [10], which will be a key factor in calculating the Melnikov functions.

**Lemma 3.** Any polynomial one-form \( \tau \) of degree \( s \) can be expressed as

\[
\tau = dQ(x, y) + q(x, y)\,dH + \alpha(H)y\,dx
\]

where \( Q(x, y), q(x, y), \) and \( \alpha(h) \) are polynomials of degree \( s + 1, s - 1 \) and \([ (s - 1)/2 ]\), respectively.

**Corollary 4.** For a polynomial one form \( \tau \) of degree \( s \), we have \( \int_{H=h} \tau = -\alpha(h)2\pi h \), where \( \alpha(h) \) is a polynomial of degree \([ (s - 1)/2 ]\).

**Proof.** The proof follows directly from Lemma 3. \( \square \)

**Corollary 5.** Let \( \tau \) be a polynomial one form of degree \( s \). Then, the one form \( \tau/G^l \) can be expressed as

\[
\frac{\tau}{G^l} = dS + r\,dH + \frac{\alpha(H)}{G^{l+1}}y\,dx,
\]

where

\[
S = \frac{Q}{G^l}, \quad r = \frac{qG + lQJ}{G^{l+1}}, \quad J = \sum_{i=2}^{n} i(i - 1)H^{i-2},
\]

such that \( Q(x, y), q(x, y), \) and \( \alpha(h) \) are polynomials of degree \( s + 1, s - 1, \) and \([ (s - 1)/2 ]\), respectively.

**Proof.** Choose the polynomials \( Q, q \) and \( \alpha \) as in Lemma 3. Then, the proof follows by substitution. \( \square \)

We rewrite system (4) as

\[
dH - \varepsilon \frac{\omega}{G} = 0,
\]

where \( \omega = A(x, y)dy - B(x, y)dx \) is a polynomial one form of degree \( m \). We will calculate the Melnikov functions for system (4) using the following well-known result due to Françoise [4] and Roussarie [18].
**Lemma 6.** For system (4), we have

\[ M_1(h) = \int_{H=h} \Omega_1, \]

where \( \Omega_1 = \omega / G \). In addition, if for some \( k \geq 2 \), we have \( M_1(h) = \cdots = M_{k-1}(h) \equiv 0 \), then

\[ M_k(h) = \int_{H=h} \Omega_k, \]

where \( \Omega_k = r_{k-1} \omega / G \), and \( r_{k-1} \) is determined successively by \( \Omega_i = dS_i + r_i dH \) for \( i = 1, \ldots, k-1 \).

We rewrite system (5) as

\[ dH - \frac{\varepsilon \omega_1}{G} - \varepsilon^2 \frac{\omega_2}{G} - \cdots = 0, \]

where \( \omega_i = A_i(x, y)dy - B_i(x, y)dx \) is a polynomial one form of degree \( m \) for each \( i = 1, 2, \ldots \). We will calculate the Melnikov functions for system (4) using the following well-known result due to Françoise [4] and Roussarie [18].

**Lemma 7.** For system (5), we have

\[ M_1(h) = \int_{H=h} \Omega_1, \]

where \( \Omega_1 = \omega / G \). In addition, if for some \( k \geq 2 \), we have \( M_1(h) = \cdots = M_{k-1}(h) \equiv 0 \), then

\[ M_k(h) = \int_{H=h} \Omega_k, \]

where

\[ \Omega_k = \frac{\omega_k}{G} + \sum_{i=1}^{k-1} r_{k-i} \frac{\omega_i}{G}, \]

and \( r_{k-i} \) is determined successively by \( \Omega_i = dS_i + r_i dH \) for \( i = 1, \ldots, k-1 \).

### 3. Proof of Theorem 1

To proof Theorem 1, we first prove the following lemma, where we look at the \( r_i \) in Lemma 6 in more detail.

**Lemma 8.** Assume \( M_1(h) = \cdots = M_{k-1}(h) \equiv 0 \) for some \( k \geq 2 \), and define \( p_0 = 1 \). For \( i = 1, \ldots, k-1 \) let \( p_i \) be the polynomial such that the function \( r_i \) of Lemma 6 is \( r_i = p_i / G^{2i} \), and let \( Q_i \) and \( q_i \) defined satisfying \( p_{i-1} \omega = dQ_i + q_i dH \), see Lemma 3. Then, \( p_i \) is a polynomial of degree \( i(2n - 3) + im \) given by \( p_i = q_i G + (2i - 1)Q_i J \).

**Proof.** We know by Lemma 6 that

\[ M_1(h) = \int_{H=h} \Omega_1 = \int_{H=h} \frac{\omega}{G} = \frac{1}{\Gamma(h)} \int_{H=h} \omega, \]
and by Lemma 3 that
\[ \omega = dQ_1 + q_1 dH + \alpha(H)y \, dx, \]
where \( Q_1, q_1, \) and \( \alpha \) are polynomials of degree \( m + 1, m - 1, \) and \( [(m-1)/2], \) respectively. Using this information and induction on \( k, \) we shall prove that \( p_i \) is a polynomial of degree \( i(2n - 3) + im \) given by \( p_i = q_iG + (2i - 1)Q_iJ \) for \( i = 1, \ldots, k - 1. \)

Let \( k = 2. \) Then, we have \( M_1(h) \equiv 0, \) which means \( \omega = dQ_1 + q_1 dH \) due to Corollary 4. By Lemma 6, we have \( \Omega_2 = r_1 \omega/G, \) where \( \Omega_1 = dS_1 + r_1 dH. \) Then, by Corollary 5, we obtain
\[ r_1 = \frac{q_1G + Q_1J}{G^2} = \frac{p_1}{G^2}, \]
and consequently, \( \deg(p_1) = 2n - 3 + m. \) Hence, the statement is true for \( k = 2. \)

Now, assume that the statement holds for some arbitrary \( k > 2. \) Then, we have \( M_1(h) = \cdots M_{k-1}(h) \equiv 0 \) and
\[ r_i = \frac{p_i}{G^{2i}} = \frac{q_iG + (2i - 1)Q_iJ}{G^{2i}}, \quad \text{(7)} \]
for \( i = 1, \ldots, k - 1, \) where \( \deg(p_i) = i(2n - 3) + im. \) To show that the statement holds for \( k + 1, \) we further assume that \( M_k(h) = \int_{H=h} \Omega_k \equiv 0. \) Then, by Lemma 6, we get
\[ \Omega_k = r_{k-1} \frac{\omega}{G} = dS_k + r_k \, dH, \]
for some functions \( S_k \) and \( r_k. \) We know from (7) that
\[ \Omega_k = \frac{p_{k-1} \omega}{G^{2k-1}}, \]
where \( p_{k-1} \omega \) is a polynomial one form of degree \( (k - 1)(2n - 3) + km, \) which can be expressed as \( dQ_k + q_k dH, \) where \( \deg(Q_k) = (k - 1)(2n - 3) + km + 1 \) and \( \deg(q_k) = (k - 1)(2n - 3) + km - 1, \) by Lemma 3, because \( M_k(h) \equiv 0. \) Hence, by Lemma 6 and Corollary 5, we obtain
\[ \Omega_k = d \left( \frac{Q_k}{G^{2k-1}} \right) + \frac{q_kG + (2k - 1)Q_kJ}{G^{2k}} \, dH. \]
Therefore, we get
\[ r_k = \frac{p_k}{G^{2k}} = \frac{q_kG + (2k - 1)Q_kJ}{G^{2k}} \]
where \( \deg(p_k) = k(2n - 3) + km. \)

**Proof of Theorem 2.** As a result of Lemma 8, we have that if for some \( k \geq 2 \)
\( M_1(h) = \cdots = M_{k-1}(h) \equiv 0, \) then
\[ M_k(h) = \frac{1}{(\Gamma(h))^{2k-1}} \int_{H=h} (q_{k-1}G + (2k - 3)Q_{k-1}J) \omega \]
has at most \([k(m - 1)/2] + (k - 1)(n - 1)\) positive zeros taking into account their multiplicities due to Corollary 4, concluding the proof of Theorem 1. \( \square \)
4. Proof of Theorem 2

To proof Theorem 2, we first prove the following lemma, where we look at the $r_i$ in Lemma 7 in more detail.

**Lemma 9.** Assume that $M_1(h) = \cdots = M_{k-1}(h) \equiv 0$ for some $k \geq 2$. Then, $\Omega_k$ as in Lemma 7 can be written as $\Omega_k = \tau_k/G^{2k-1}$, where $\tau_k$ is a polynomial of degree

$$\deg(\tau_k) = \begin{cases} m + 4(k - 1)(n - 1), & \text{if } m \leq 2n - 1, \\ km + (k - 1)(2n - 3), & \text{if } m \geq 2n - 1. \end{cases}$$

*Proof.* First, we will prove by induction on $k$ that for $k \geq 2$, we have that the $r_i$ for $i = 1, \ldots, k - 1$ given in Lemma 6 satisfies

$$r_i = \frac{q_i G + (2i - 1)Q_i J}{G^{2i}} := \frac{p_i}{G^{2i}}, \quad (8)$$

where $q_i$ and $Q_i$ are polynomials, such that

$$\deg(q_i) = \begin{cases} m + 4(i - 1)(n - 1) - 1, & \text{if } m \leq 2n - 1, \\ im + (i - 1)(2n - 3) - 1, & \text{if } m \geq 2n - 1, \end{cases}$$

and $\deg(Q_i) = \deg(q_i) + 2$.

Let $k = 2$. Then, we have

$$M_1(h) = \int_{H = h} \Omega_1 = \frac{1}{\Gamma(h)} \int_{H = h} \omega_1 \equiv 0,$$

which means

$$\omega_1 = dQ_1 + q_1 \, dH$$

where $Q_1$ and $q_1$ are polynomials of degree $m + 1$ and $m - 1$, respectively, due to Lemma 3 and Corollary 4. Then, by Corollary 5, we get

$$\Omega_1 = \frac{\omega_1}{G} = d \left( \frac{Q_1}{G} \right) + \frac{q_1 G + Q_1 J}{G^2} \, dH,$$

and thus

$$r_1 = \frac{q_1 G + Q_1 J}{G^2},$$

which prove the induction hypothesis for $k = 2$.

Now, assume that the lemma holds for some arbitrary $k > 2$. Then, by (8), we have

$$\Omega_k = \frac{\omega_k}{G} + \sum_{i=1}^{k-1} \frac{r_{k-i} \omega_i}{G} = \frac{\omega_k}{G} + \sum_{i=1}^{k-1} \frac{p_{k-i} \omega_i}{G^{2k-2i+1}}$$

$$= \frac{\omega_k G^{2k-2} + \sum_{i=1}^{k-1} p_{k-i} \omega_i G^{2i-2}}{G^{2k-1}} := \frac{\tau_k}{G^{2k-1}}. \quad (10)$$
We see that $\deg(\omega_k G^{2k-2}) = m + 4(k - 1)(n - 1)$ and by the induction hypothesis [see (8) and (9)], we obtain

\[
\deg(p_{k-i}\omega G^{2i-2}) = \begin{cases} 
2m + 4(k - 1)(n - 1) - (2n - 1), & \text{if } m \leq 2n - 1, \\
km + (k-1)(2n-3) + (1-i)(m-(2n-1)), & \text{if } m \geq 2n - 1.
\end{cases}
\]

In particular, the degree of $\tau_k$ is

\[
\deg(\tau_k) = \begin{cases} 
m + 4(k - 1)(n - 1), & \text{if } m \leq 2n - 1, \\
km + (k-1)(2n-3), & \text{if } m \geq 2n - 1.
\end{cases}
\tag{11}
\]

and the lemma is proved if we end the induction process.

To end the induction process, i.e., that (8) and (9) hold for $k + 1$, we further assume that $M_k(h) \equiv 0$. Then, we have $\tau_k = dQ_k + q_k dH$. This together with (11) completes the induction process. \qed

**Proof of Theorem 2.** As a result of Lemmas 7 and 9, we have that if $M_1(h) = \cdots = M_{k-1}(h) \equiv 0$ for some $k \geq 2$, then

\[
M_k(h) = \frac{1}{(\Gamma(h))^{2k-1}} \int_{H=h} \tau_k.
\]

Now, the proof of Theorem 2 follows from Lemma 9 and Corollary 4. \qed

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