A PRIORI BOUND FOR NONLINEAR ELLIPTIC EQUATION AND SYSTEM INVOLVING A FRACTIONAL LAPLACIAN

WOOCHEOL CHOI

Abstract. We give a new proof of a priori estimates of Gidas-Spruck type for the nonlinear elliptic equation with square root of the Laplacian. We also use the approach to establish a priori estimates for the system of the Lane-Emden type involving a fractional Laplacian.

1. Introduction

We consider the nonlinear problem:

\[
\begin{align*}
A_{1/2}u &= f(u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega, \\
u &> 0 \quad \text{in } \Omega,
\end{align*}
\]

where \(\Omega\) is a smooth bounded domain of \(\mathbb{R}^n\) and \(A_{1/2}\) is the fractional Dirichlet Laplacian \((-\Delta)^s_{\Omega}\) on the domain \(\Omega\) with \(s = \frac{1}{2}\). The Dirichlet Laplacian with a suitable domain is selfadjoint and has a discrete spectrum with eigenvalues of finite multiplicity. Then it admits a orthonormal eigenbasis and the fractional Laplacian is then defined by functional calculus using the spectral theorem.

The study of the problem (1.1) was initiated by Cabré-Tan [CT]. They converted the nonlocal problem (1.1) to a local problem on a half-cylinder \(C := \Omega \times [0, \infty)\):

\[
\begin{align*}
\Delta v &= 0 \quad \text{in } C = \Omega \times (0, \infty), \\
v &= 0 \quad \text{on } \partial_C := \partial \Omega \times [0, \infty), \\
\frac{\partial v}{\partial \nu} &= f(v) \quad \text{on } \Omega \times \{0\}, \\
v &> 0 \quad \text{in } C,
\end{align*}
\]

where \(\nu\) is the unit outer normal vector to \(C\) at \(\Omega \times \{0\}\). The trace \(v\) of \(v\) on \(\Omega \times \{0\}\) is a solution of (1.1) when \(v\) is a solution of (1.2).

By studying the local problem (1.2) with classical local techniques, Cabré-Tan [CT] established existence of positive solutions for problem with subcritical power nonlinearities, regularity, symmetric property. They also prove a priori estimates of Gidas-Spruck type by a blow-up argument along with proving a nonlinear Liouville type result of the square root of the Laplacian on the half-space (see [CT, Theorem 1.3]).

The first aim of this paper is to extend the result of Cabré-Tan [CT] on a priori estimates to functions satisfying a general condition by giving a new proof.

**Theorem 1.1.** Let \(n \geq 2\) and \(2^* = \frac{2n}{n-1}\). Assume that \(\Omega \subset \mathbb{R}^n\) is a smooth bounded domain and \(f(u) = u^p\), \(1 < p < 2^* - 1 = \frac{n+1}{n-1}\).

\[\text{2000 Mathematics Subject Classification. Primary.}\]
Then, there exists a constant $C(p, \Omega)$ depending only on $p$ and $\Omega$ such that every weak solution of (1.1) satisfies

\[ \|u\|_{L^\infty(\Omega)} \leq C(p, \Omega). \]

Moreover, the above statement holds for locally Lipschitz continuous function $f : \mathbb{R}^+ \to \mathbb{R}$ satisfying

\[ \liminf_{u \to \infty} \frac{f(u)}{u^{1/2}} > \lambda_1^{1/2}, \quad \lim_{u \to \infty} \frac{f(u)}{u^{(n+1)/(n-1)}} = 0, \]

with one of the following assumptions

1. $\Omega$ is convex and

\[ \limsup_{n \to \infty} \frac{uf(u) - \theta F(u)}{u^2 f(u)^{2/n}} \leq 0, \quad \text{for some } \theta \in [0, 2^* = \frac{2n}{n-1}). \] (1.3)

2. Condition (1.3) holds and the function $u \to f(u)u^{-\frac{n+1}{n-1}}$ is nonincreasing on $(0, \infty)$.

The main step of the proof is to get the uniform bound $\int_{\Omega} u^{p+1}(x) \, dx \leq C(p, \Omega)$ from the Pohozaev identity of Tan [T] given for solutions of (1.2). Then, we shall use a bootstrap argument with the $L^p - L^q$ estimates for the problem

\[ \begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \] (1.4)

This estimate will be proved by using the bound of the Green function of $(-\Delta)^s$ obtained by Song [S].

Our approach is a nonlocal version of the method of Figueiredo-Lions-Nussbaum [FLN] for the problem

\[ \begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega. \end{cases} \] (1.5)

The first step of this approach is to use the moving plane argument to get the $L^\infty$ bound of $u$ near the boundary $\partial\Omega$. The moving plane argument for (1.4) is achieved in [CT]. Therefore this step is done without difficulty when $\Omega$ is strictly convex. However, we have more work to derive the bound $\int_{\Omega} u^{p+1} \, dx$ from the Pohozaev identity for a solution $v$ of (1.2). It is due to the non-local nature of the Pohozaev Identity in view of its trace function $u = \text{Tr}_{\Omega \times \{0\}} v$. More precisely, we can not control the left side of the Pohozaev identity (see formula (3.6)) by only using $L^\infty$ estimates of $u$ near the boundary in contrast with the case for the problem (1.5). We shall overcome this difficulty by a suitable partition of $v$ and an estimate using a Green function related to the equation (1.2). This approach does not need a Liouville-type result, and so the function $f$ is not required to have a precise asymptotic as $u \to \infty$. It is also useful to study the system type problems.

The second part of this paper is concerned with the following system:

\[ \begin{cases} A_{1/2}u = v^p & \text{in } \mathcal{C}, \\ A_{1/2}v = u^q & \text{in } \mathcal{C}, \\ u > 0, \ v > 0 & \text{in } \mathcal{C}, \\ u = v = 0 & \text{on } \partial\mathcal{C}. \end{cases} \] (1.6)

The analogue problem to (1.3) for the Laplacian is the Lane-Emden system, which has been investigated widely in the last decades (see [QS] and references therein). We shall prove the following theorem on a priori estimates of this system.
Theorem 1.2. Assume that \( \Omega \subset \mathbb{R}^n \) is a smooth convex bounded domain and \( p, q > 1 \) satisfy the condition,
\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-1}{n}.
\]
Then, there exists a constant \( C(p,q,\Omega) \), which depends only on \( p \) and \( \Omega \), such that every weak solution of (1.6) satisfies
\[
\|u\|_{L^\infty(\Omega)} + \|v\|_{L^\infty(\Omega)} \leq C(p,q,\Omega).
\]

This theorem is analogous to the result of Clément-Figueiredo-Mitidieri \[CFM\] for the Lane-Emden system with the usual Laplacian. As a preliminary step of the proof, we shall obtain a Pohozaev type identity and exploit the moving plane method for the system (1.6).

This paper is organized as follows. In Section 2, we first review on the function spaces related to the fractional Laplacian and their properties. We then recall some results of Cabré-Tan \[CT\] and prove the \( L^p - L^q \) estimates. In Section 3, we prove several lemmas and use them to establish the proof of Theorem 1.1. In Section 4, we obtain the Pohozaev Identity and exploit the moving plane method for the Lane-Emden system (1.6). Based on these things, we shall finally complete the proof of Theorem 1.2.

We shall use the notation \( \lesssim \) when the constant of an estimate \( A \leq CB \) depends only on the function \( f \), the domain \( \Omega \) and some parameters fixed in the proofs.

2. The Functional Frame Work

This section is devoted to study the fractional Laplacians and related function spaces. Consider \( L^2(\Omega) \) normalized eigenfunctions \( \phi_1, \phi_2, \phi_3, \ldots \), of \( -\Delta \), corresponding to positive eigenvalues \( \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \), counted with multiplicity. We then have \( \phi_1 > 0 \) and
\[
\int_\Omega \phi_i \phi_j = \delta_{ij}.
\]
For each \( r \in \mathbb{R} \) we define the fractional Laplacian in the following way:
\[
(-\Delta)^{r/2} : \sum_{k=1}^\infty \xi_k \phi_k \longrightarrow \sum_{k=1}^\infty \lambda_k^{r/2} \xi_k \phi_k,
\]
with domain
\[
H^r_0(\Omega) := \{ u = \sum_{k=1}^\infty \xi_k \phi_k : \sum_{k=1}^\infty \lambda_k^r \xi_k^2 < \infty \}.
\]
By the correspondence
\[
\sum_{k=1}^\infty \xi_k \phi_k \leftrightarrow (\xi_1, \xi_2, \cdots),
\]
the space \( H^r_0(\Omega) \) is identified with a subspace of \( l^\infty \),
\[
\omega^r = \{ \xi = (\xi_1, \xi_2, \cdots) : \sum_{k=1}^\infty \lambda_k^r \xi_k^2 < \infty \},
\]
which is a Hilbert space with the inner product
\[
(\xi, \eta)_r = \sum_{k=1}^\infty \lambda_k^r \xi_k \eta_k.
\]
We see that \((-\Delta)^s : H_0^s(\Omega) \to H_0^{s-2s}(\Omega)\) is an isomorphism. Define the bilinear operator 
\[ A_s(u, v) = \int_\Omega (-\Delta)^s u(x)v(x)\,dx. \]
Then we have for \(u = \sum_{k=1}^\infty \xi_k\phi_k\) and \(v = \sum_{k=1}^\infty \eta_k\phi_k\) that 
\[ A_s(u, v) = \sum_{k=1}^\infty \lambda_k^{s/2} \xi_k\eta_k = \sum_{k=1}^\infty \lambda_k^{s/2} \xi_k \lambda_k^{s/2-r} \eta_k. \]
By Cauchy-Schwartz inequality we see 
\[ |A_s(u, v)| \leq \left\{ \sum_{k=1}^\infty \lambda_k^{s/2} \xi_k^2 \right\}^{1/2} \left\{ \sum_{k=1}^\infty \lambda_k^{s/2-r} \eta_k^2 \right\}^{1/2} = \|u\|_{H^s_c}\|v\|_{H^{s-r}_c}.
\]
Thus the bilinear operator \(A_s\) can be defined on the space \(H^s_0 \times H^{s-r}_0\) for any \(r \in \mathbb{R}\). This fact allows us to use \(A_s\) in different sobolev spaces in the variational argument for proving the existence of nontrivial solutions to the Lane-Emden system (see \([HV]\)). The sobolev embedding is given by 
\[ H^s_0(\Omega) \to L^p(\Omega), \quad \text{if} \ 1 \leq p \leq \frac{2N}{N-2r} < \infty, \quad 2r < N. \]
This embedding is compact if \(1 \leq p < 2N/(N-2r).\) If \(2r \geq N\), these properties hold for any \(1 \leq p < \infty.\)

We now consider the extended domain 
\[ \mathcal{C} = \Omega \times (0, \infty), \]
with its later boundary 
\[ \partial_t \mathcal{C} = \partial \Omega \times [0, \infty). \]
We recall from Cabré-Tan \([CT]\) the sobolev space on \(\mathcal{C}\) with zero assumption on \(\partial_t \mathcal{C}\), 
\[ H^1_{0,L}(\mathcal{C}) = \{ v \in H^1(\mathcal{C}) : v = 0 \ \text{a.e. on} \ \partial_t \mathcal{C} \}, \]
equipped with the norm 
\[ \|v\| = \left( \int_\mathcal{C} |\nabla v|^2 \,dx\,dy \right)^{1/2}. \]
For \(n \geq 2\), it follows from the sobolev trace inequality that 
\[ \left( \int_\Omega |w(x, 0)|^{2n/(n-1)} \,dx \right)^{(n-1)/2n} \leq C \left( \int_\mathcal{C} |\nabla w(x, y)|^2 \,dx\,dy \right)^{1/2} \forall w \in H^1_{0,L}(\mathcal{C}), \]
where the constant \(C > 0\) depends on the dimension \(n.\) We set \(\text{tr}_\Omega\) be the trace operator on \(\Omega \times \{0\}\) for functions in \(H^1_{0,L}(\mathcal{C})\): 
\[ \text{tr}_\Omega v := v(x, 0) t \ \text{for} \ v \in H^1_{0,L}(\mathcal{C}). \]
It is well known that traces of \(H^1\) functions are \(H^{1/2}\) functions on the boundary and we thus have \(\text{tr}_\Omega v \in H^{1/2}(\Omega).\)

**Proposition 2.1** \([CT]\). Let \(\mathcal{V}_0(\Omega)\) be the space of all traces on \(\Omega \times \{0\}\) of functions in \(H^1_{0,L}(\mathcal{C})\). Then we have the following properties:
\[ \mathcal{V}_0(\Omega) := \{ u = \text{tr}_\Omega v \mid v \in H^1_{0,L}(\mathcal{C}) \} \]
\[ = \{ u \in L^2(\Omega) \mid u = \sum_{k=1}^\infty b_k \phi_k \ \text{satisfying} \ \sum_{k=1}^\infty b_k^2 \lambda_k^{1/2} < \infty \}, \]
where \(\lambda_k, \phi_k\) is the spectral decomposition of \(-\Delta\) in \(\Omega\) as above, with \(\{\phi_k\}\) an orthonormal basis in \(L^2(\Omega)\).
Proposition 2.2 ([CT]). If \( u \in V_0(\Omega) \), then there exists a unique harmonic extension \( v \) in \( \mathcal{C} \) of \( u \) such that \( v \in H^1_{0,L}(\mathcal{C}) \). In particular, if the expansion of \( u \) is written by \( u(x) = \sum_{k=1}^{\infty} b_k \phi_k(x) \in V_0(\Omega) \), then
\[
v(x,y) = \sum_{k=1}^{\infty} b_k \phi_k(x) \exp(-\lambda_k^{1/2} y) \in H^1_{0,L}(\mathcal{C}),
\]
where \( \lambda_k, \phi_k \) is the spectral decomposition of \( -\Delta \) in \( \Omega \) as above. Let us define the operator \( A_{1/2} : V_0(\Omega) \to V'_0(\Omega) \) by
\[
A_{1/2}u := \frac{\partial v}{\partial \nu} |_{\Omega \times \{0\}},
\]
where \( V'_0(\Omega) \) is the dual space of \( V_0(\Omega) \). Then
\[
A_{1/2}u = \sum_{k=1}^{\infty} b_k \lambda_k^{1/2} \phi_k,
\]
and \( A_{1/2}^2 \) is equal to \( -\Delta \) in \( \Omega \) with zero Dirichlet boundary value on \( \partial \Omega \). More precisely, the inverse \( B_{1/2} := A_{1/2}^{-1} \) is the unique square root of the inverse Laplacian \( (-\Delta)^{-1} \) in \( \Omega \) with zero Dirichlet boundary value on \( \partial \Omega \).

From the above proposition we deduce the following equation for a harmonic extension \( v \).
\[
\int_{\Omega \times \{y = 0\}} |\nabla_x v(x,0)|^2 dx = \int_{\Omega \times \{y = 0\}} -\Delta_x v(x,0)v(x,0) dx
= \int_{\Omega \times \{y = 0\}} A_{1/2}v(x,0)A_{1/2}v(x,0) dx
= \int_{\Omega \times \{y = 0\}} \left| \frac{\partial}{\partial \nu} v(x,0) \right|^2 dx.
\]
(2.1)

It will be useful in the proof of Lemma (3.3).

Now we recall the definition of weak solution of the problem involving the square root of the Laplacian. For the linear equation
\[
\begin{cases}
A_{1/2}u = g(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
(2.2)
the concept of weak solution is defined in different ways in Cabré-Tan [CT] and Hulshof-Vorst [HV].

Definition 2.3 ([HV]). A function \( u \in L^2(\Omega) \) is a weak solution of (2.2) if it satisfies the following equation.
\[
A_{1/2}(u, \phi) = \int_{\Omega} u(-\Delta)^{1/2} \phi \ dx = \int_{\Omega} g(x) \phi \ dx, \quad \phi \in H^2_0(\Omega).
\]
(2.3)

Definition 2.4 ([CT]). A function \( v \in H^1_{0,L}(\mathcal{C}) \) with \( v(\cdot, 0) = u \) is a weak solution of
\[
\begin{cases}
\Delta v = 0 & \text{in } \mathcal{C}, \\
v = 0 & \text{on } \partial_t \mathcal{C}, \\
\frac{\partial v}{\partial \nu} = g(x) & \text{on } \Omega \times \{0\},
\end{cases}
\]
(2.4)
when it satisfies
\[
\int_{\mathcal{C}} \nabla v \nabla \xi \ dx \ dy = (g(x), \xi(\cdot, 0)),
\]
(2.5)
for all \( \xi \in H^1_{0,L}(\mathcal{C}) \). Then we say that \( u = tr_{\Omega} v \) is a weak solution of (2.2).
Lemma 2.5. Suppose that $v \in H^1_0, L^1(C)$ is a weak solution of (2.4), then $u = tr\Omega v \in H^{1/2}(\Omega)$ is a weak solution of (2.2) in the sense of Definition 2.3.

Proof. If $v \in C^2(\overline{C})$ satisfies $\Delta v = 0$ in $C$ and $\int_{\Omega} A_{1/2}v(x,0)\xi(x,0) dx = \int_{\Omega} v(x,0)A_{1/2}\xi(x,0) dx$. Thus we get $\int_{\Omega} \nabla v \cdot \nabla \xi dx = \int_{\Omega} g(x)\phi(x)dx$.

This equality holds for any $v \in H^1_0, L^1(C)$ and $\xi(x,0) := \phi(x) \in H^1_0(\Omega)$ by an approximation argument. Then, since $v$ satisfies (2.5), we get $\int_{\Omega} v(x,0)A_{1/2}\phi(x)dx = \int_{\Omega} g(x)\phi(x)dx$.

We finish the section with proving the $L^p \rightarrow L^q$ estimates necessary for the bootstrap argument in the proofs of Theorem 1.1 and Theorem 1.2.

Lemma 2.6. For $0 < \alpha < 2$, we assume that $u$ and $f$ satisfies the equation

\[
\begin{align*}
(-\Delta)^{\alpha/2} u &= f \quad \text{in} \ \Omega, \\
u &= 0 \quad \text{on} \ \partial\Omega.
\end{align*}
\]

Suppose $1 \leq p \leq q \leq \infty$ satisfy

\[
\frac{1}{p} - \frac{1}{q} < \frac{\alpha}{n}.
\] (2.6)

If $f \in L^p(\Omega)$, then $u \in L^q(\Omega)$ and

\[
\|u\|_q \leq C(\Omega, p, q)\|f\|_p.
\]

Moreover, if $q \neq \infty$, the above inequality holds for the case $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$.

Proof. Let $G_\Omega(x, y)$ be the Green function of $(-\Delta)^{\alpha/2}$. It follows from [S] Theorem 1.1 that there exists a constant $C = C(\Omega) > 0$ such that

\[
G_\Omega(x, y) \leq C \frac{1}{|x - y|^{n-\alpha}}.
\]

Thus,

\[
|u(x)| = \left| \int_{\Omega} G_\Omega(x, y)f(y)dy \right| \lesssim \int_{\Omega} \frac{|f(y)|}{|x - y|^{n-\alpha}}dy.
\]
Then, the Hardy-Littlewood-Sobolev inequality \([S]\) yields
\[
\|u\|_{L^q(\Omega)} \lesssim \|f\|_{L^p(\Omega)}
\]
for \(1 \leq p < q < \infty\) satisfying
\[
\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}.
\]
Then Hölder’s inequality yields the asserted inequality for the case \(q \neq \infty\).

Suppose now that \(q = \infty\) and \(\frac{1}{p} < \frac{\alpha}{n}\). Since \((n - \alpha)p' < n\) we obtain
\[
\int_{\Omega} \frac{|f(y)|}{|x - y|^{d-\alpha}} dy \leq \left( \int_{\Omega} \left( \frac{1}{|x - y|^{d-\alpha}} \right)^{p'} dy \right)^{1/p'} \|f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}.
\]
It completes the proof. \(\square\)

3. The new approach

In this section we prove several lemmas and establish the proof of Theorem 1.1 using the lemmas. For \(s > 0\) we set
\[
\Omega_s = \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \leq s \}.
\]

**Lemma 3.1.** Let \(u\) be a \(C^2(\bar{\Omega})\) solution of \((1.1)\) with \(f\) satsifying
\[
\liminf_{n \to \infty} \frac{f(u)}{u} > \lambda_1^{1/2}.
\]
Suppose that \(\Omega\) is strictly convex. Then, for any \(s > 0\) there exists a number \(C = C(s, p, \Omega) > 0\) such that
\[
\int_{\Omega \setminus \Omega_s} f(u) dx \leq C,
\]
and
\[
\sup_{x \in \Omega_s} u(x) \leq C.
\]

**Proof.** We have
\[
\int_{\Omega} \sqrt{\lambda_1} \phi_1 u dx = \int_{\Omega} (A_{1/2} \phi_1) u dx = \int_{\Omega} \phi_1 A_{1/2} u dx = \int_{\Omega} \phi_1 f(u) dx.
\]
The condition \((3.1)\) gives that \(f(u) > (\lambda_1^{1/2} + \delta)u - C\) for some \(\delta > 0\) and \(C > 0\). Then, the above inequality yields
\[
\int_{\Omega} \sqrt{\lambda_1} \phi_1 u dx > \int_{\Omega} \sqrt{\lambda_1} + \delta u \phi_1 dx - \int_{\Omega} C \phi_1 dx.
\]
It gives
\[
\int_{\Omega} \phi_1 u dx \leq \frac{1}{\delta} \int_{\Omega} C \phi_1 dx \leq C(\delta, \Omega, f).
\]
Because \( \phi_1 \geq c(s, \Omega) \) on \( \Omega \setminus \Omega_s \), we have

\[
\int_{\Omega \setminus \Omega_s} u \, dx \lesssim \int_{\Omega_s} \phi_1 u \, dx \lesssim 1. \tag{3.5}
\]

From the identity (3.4), we obtain the estimate (3.2).

Since \( \Omega \) is strictly convex, the moving plane method in [CT, p. 2091] yields that the solution increases along an arbitrary line toward inside of \( \Omega \) starting from any point on \( \partial \Omega \). Then, it is well-known that the estimate (3.5) gives the uniform bound near the boundary (see e.g. [QS, Lemma 13.2]). It completes the proof. \( \square \)

Remark 3.2. In fact Lemma 3.1 holds without the convexity condition. In the proof of Theorem 1.1, we shall prove it by using the Kelvin transformation in the extended domain.

In the case of the classical problem (1.5), if one has the \( L^\infty \) bound (3.3) near the boundary, one can use \( W^{1,p} \) estimate for the equation (1.3) on a localized domain near the boundary \( \partial \Omega \) to get a uniform \( L^\infty \) estimates of \( |\nabla v| \) on the boundary \( \partial \Omega \). Then, for \( f(u) = u^p, \ p < \frac{n+2}{n-2} \), the Pohozaev identity

\[
\frac{1}{2} \int_{\partial \Omega} |\nabla v|^2 (z, \nu) d\sigma = \left( \frac{n}{p+1} - \frac{n-1}{2} \right) \int_{\Omega} v^{p+1} dx
\]

(3.6)

where \( v \) is the harmonic extension of \( u \). In this formula, the left-hand side would not be bounded by using only the \( L^\infty \) estimate of \( u(x) = v(x, 0) \) near \( \partial \Omega \) since the harmonic extension \( v(z) \) consists of all values of \( u(x) \) on \( \Omega \). Nevertheless, we shall get the bound of the left-hand side by using both the \( L^\infty \) bound (3.2) near the boundary and the \( L^p \) bound (3.3) away from the boundary. For this we split the source of the equation

\[
\begin{cases}
\Delta v = 0 & \text{in } C, \\
v = 0 & \text{on } \partial_L C, \\
\frac{\partial v}{\partial n} = f(u) & \text{on } \Omega \times \{0\}
\end{cases}
\]

in the following way:

\[
f(u)(x) = f(u)\chi(x) + f(u)(1 - \chi(x)),
\]

where \( \chi \) is a smooth bump function supported in \( \Omega \setminus \Omega_\delta \) and \( \chi = 1 \) on \( \Omega \setminus \Omega_{2\delta} \) with a small number \( \delta > 0 \). We recall that the linear problem

\[
\begin{cases}
\Delta v = 0 & \text{in } C, \\
v = 0 & \text{on } \partial_L C, \\
\frac{\partial v}{\partial n} = g(x) & \text{on } \Omega \times \{0\}
\end{cases}
\]

(3.7)

has a unique solution \( v \in H^1_0(L) \) for each \( g \in H^{1/2}(\Omega) \)(see [CT, p. 2069]). Thus, there exist solutions \( v_1 \) and \( v_2 \) such that

\[
\begin{cases}
\Delta v_1 = 0 & \text{in } \Omega \times (0, \infty), \\
v_1 = 0 & \text{on } \partial \Omega \times [0, \infty), \\
\frac{\partial v_1}{\partial n} = f(u)(x)\chi(x) & \text{on } \Omega \times \{0\},
\end{cases}
\]

and

\[
\begin{cases}
\Delta v_2 = 0 & \text{in } C, \\
v_2 = 0 & \text{on } \partial_L C, \\
\frac{\partial v_2}{\partial n} = g(x) & \text{on } \Omega \times \{0\},
\end{cases}
\]

(3.8)
and
\[
\begin{cases}
\Delta v_2 = 0 & \text{in } \Omega \times (0, \infty), \\
v_2 = 0 & \text{on } \partial \Omega \times [0, \infty), \\
\frac{\partial v_2}{\partial n} = f(u)(1 - \chi(x)) & \text{on } \Omega \times \{0\}.
\end{cases}
\]

Then the function \( v_1 + v_2 \) satisfy
\[
\begin{cases}
\Delta (v_1 + v_2) = \Delta v_1 + \Delta v_2 = 0 & \text{in } C, \\
v_1 + v_2 = 0 & \text{on } \partial L C, \\
\frac{\partial (v_1 + v_2)}{\partial n} = f(u)\chi(x) + f(u)(1 - \chi(x)) = f(u)(x) & \text{on } \Omega \times \{0\}.
\end{cases}
\]

It follows from the uniqueness that \( v = v_1 + v_2 \). We thus have
\[
\int_{\partial_L C} \left| \frac{\partial v}{\partial \nu} \right|^2 (z, \nu) d\sigma \leq 2 \left( \int_{\partial_L C} \left| \frac{\partial v_1}{\partial \nu} \right|^2 (z, \nu) d\sigma + \int_{\partial_L C} \left| \frac{\partial v_2}{\partial \nu} \right|^2 (z, \nu) d\sigma \right). \tag{3.8}
\]

In order to estimate each term in the right-hand side, we shall prove the following two lemmas.

**Lemma 3.3.** Let \( g \in L^2(\Omega) \). Suppose that \( u \in H^1_{\partial L}(C) \cap L^\infty(C) \) is a weak solution of the following linear problem:
\[
\begin{cases}
\Delta u = 0 & \text{in } C, \\
u = 0 & \text{on } \partial L C, \\
\frac{\partial u}{\partial n} = g & \text{on } \Omega \times \{0\}.
\end{cases} \tag{3.9}
\]

Then, there exists a constant \( C = C(\Omega) \) such that
\[
\int_{\partial_L C} \left| \frac{\partial u}{\partial \nu}(z) \right|^2 (z, \nu) d\sigma \leq C \int_\Omega |g(x)|^2 dx. \tag{3.10}
\]

**Proof.** We recall the well-known identity:
\[
\text{div} \left\{ (z, \nabla u) \nabla u - z \cdot \frac{\nabla u^2}{2} \right\} + \left( \frac{n + 1}{2} - 1 \right) |\nabla u|^2 = 0.
\]

Integrating this over \( \Omega \times (0, R) \), then by the divergence theorem we have
\[
\frac{1}{2} \int_{\partial \Omega \times (0, R)} |\nabla u|^2 (z, \nu) d\sigma + \int_{\Omega \times \{y = 0\}} (z, \nabla_x u)(\nabla u, \nu) dx = 0.
\]

Integrating this over \( \Omega \times (0, R) \), then by the divergence theorem we have
\[
\frac{1}{2} \int_{\partial \Omega \times (0, R)} |\nabla u|^2 (z, \nu) d\sigma + \int_{\Omega \times \{y = 0\}} (z, \nabla_x u)(\nabla u, \nu) dx = 0.
\]

To show that the last term in the left-hand side goes to zero as \( R \to \infty \) in a sequence, we take for \( k \in \mathbb{N} \) a number \( R_k \in [2^k, 2^{k+1}] \) such that
\[
\int_{\Omega \times \{y = R_k\}} |\nabla u|^2 dx = \inf_{r \in [2^k, 2^{k+1}]} \int_{\Omega \times \{y = r\}} |\nabla u|^2 dx.
\]

We then see from \( \int_{\Omega \times (0, \infty)} |\nabla u|^2 dx \leq C \) that
\[
R_k \int_{\Omega \times \{y = R_k\}} |\nabla u|^2 dx \leq \int_{\Omega \times \{2^k \leq y \leq 2^{k+1}\}} |\nabla u|^2 dz \to 0 \quad \text{as } k \to \infty.
\]
Observe that the last term of (3.11) is bounded by a constant multiple of $R \int_{\Omega \times \{y = R\}} |\nabla u|^2 dx$. Therefore, by taking $R = R_k$ in (3.11) and letting $k \to \infty$ we get the following formula

$$\frac{1}{2} \int_{\partial \Omega \times (0, \infty)} |\nabla u|^2(z, \nu)d\sigma$$

$$= - \int_{\Omega \times \{0\}} (x, \nabla_x u)(\nabla u, \nu)dx - \left(\frac{n - 1}{2}\right) \int_{\Omega \times (0, \infty)} |\nabla u|^2 dxdy$$

$$= - \int_{\Omega \times \{0\}} (x, \nabla_x u)(g(x))dx + \left(\frac{n - 1}{2}\right) \int_{\Omega \times \{0\}} u(g(x))dx.$$  

In order to prove (3.10) it is enough to bound the right-hand side by a constant times $\int_{\Omega \times \{0\}} |f|^2 dx$. For the first term, we use the formula (2.1) and (3.9) to obtain

$$\left| \int_{\Omega \times \{0\}} (x, \nabla_x u)(g(x))dx \right|$$

$$\lesssim \left( \int_{\Omega \times \{0\}} |\nabla_x u(x, 0)|^2 dx \right)^{1/2} \cdot \left( \int_{\Omega \times \{0\}} |g(x)|^2 dx \right)^{1/2}$$

$$\lesssim \left( \int_{\Omega \times \{0\}} \frac{\partial u}{\partial \nu}(x, 0)^2 dx \right)^{1/2} \cdot \left( \int_{\Omega \times \{0\}} |g(x)|^2 dx \right)^{1/2}$$

$$= \left( \int_{\Omega \times \{0\}} |g(x)|^2 dx \right) .$$

To bound the second term, we use the Sobolev inequality $\|u\|_{L^2(\Omega)} \lesssim \|\nabla u\|_{L^2(\Omega)}$ and (2.1) to deduce

$$\left| \int_{\Omega \times \{0\}} u(x)g(x)dx \right|$$

$$\lesssim \left( \int_{\Omega \times \{0\}} u(x)^2 dx \right)^{1/2} \cdot \left( \int_{\Omega \times \{0\}} (g(x))^2 dx \right)^{1/2}$$

$$\lesssim \left( \int_{\Omega \times \{0\}} |\nabla_x u(x)|^2 dx \right)^{1/2} \cdot \left( \int_{\Omega \times \{0\}} (g(x))^2 dx \right)^{1/2}$$

$$\lesssim \left( \int_{\Omega \times \{0\}} (g(x))^2 dx \right) .$$

The lemma is proved. \hfill \square

**Lemma 3.4.** Let $g \in L^1(\Omega)$ and suppose that $g$ has support in $\Omega \setminus \Omega_s$ for some $s > 0$. Then there exists $C = C(s, \Omega) > 0$ such that a weak solution $u \in H^1_{0,\nu}(C) \cap L^\infty(\Omega)$ of the problem (3.9) satisfies the following estimate

$$\int_{\partial_\nu C} \left| \frac{\partial u}{\partial \nu}(z) \right|^2 (z, \nu)d\sigma \leq C(s, \Omega) \left( \int_{\Omega} |g|dy \right)^2 .$$

**Proof.** Consider the Dirichlet-Nuemann problem

$$\begin{cases}
\Delta u = 0 & \text{in } \mathcal{C}, \\
u u = 0 & \text{on } \partial_\nu \mathcal{C}, \\
\frac{\partial u}{\partial \nu} = g & \text{on } \Omega \times \{0\}.
\end{cases} \tag{3.12}$$

For the mixed problem like the above equation, Taylor-Ott-Brown [TOB] proved existence of the Green function if the whole domain $\mathcal{C}$ is bounded. One can see that, the proof of [TOB, Theorem 3.6] use the boundedness assumption of $\mathcal{C}$ only to have the coercivity estimate $\|h\|_{L^2(\mathcal{C})} \leq$
$C\|\nabla h\|_{L^2(C)}$ for any function $h$ vanishing on the boundary with Dirichlet condition. On the other hand, for the domain $C = \Omega \times (0, \infty)$, we can use the Sobolev inequality in each slide $\Omega \times \{s\}$ for $0 < s < \infty$ to deduce that

$$\|h\|_{L^2(C)} \leq C\|\nabla_x h\|_{L^2(C)} \leq C\|\nabla h\|_{L^2(C)}$$

for function $h$ vanishing on $\partial_L C = \partial \Omega \times (0, \infty)$. Thus the coercivity estimate holds and it guarantees the existence of a unique Green function $G_1$ for the problem (3.12). We then have the following expression

$$u(z) = \int_{\Omega} G_1(z, y) f(y) dy. \quad (3.13)$$

In order to find an upper bound of $G_1$ we shall use the Green function $G$ on $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n, x_n > 0\}$ with the Neumann condition on $\partial \Omega = \{x : x_n = 0\}$ which is given by

$$G(z, w) = \frac{1}{(n-2)\sigma_n|z-w|^{n-1}} + \frac{1}{(n-2)\sigma_n|z-w^*|^{n-1}},$$

where $y^* = (y_1, \ldots, y_n, -y_{n+1})$ for $y = (y_1, \ldots, y_{n+1})$. That is, it satisfies

$$\begin{cases}
\Delta G(z, w) = \delta_w(z) & \text{in } C, \\
\frac{\partial}{\partial \nu} G(z, w) = 0 & \text{on } \Omega \times \{0\}, \\
G(z, w) > 0 & \text{on } \partial_L C.
\end{cases}$$

On hand, the Green function $G_1$ satisfies

$$\begin{cases}
\Delta G_1(z, w) = \delta_w(z) & \text{in } C, \\
\frac{\partial}{\partial \nu} G_1(z, w) = 0 & \text{on } \Omega \times \{0\}, \\
G_1(z, w) = 0 & \text{on } \partial_L C.
\end{cases}$$

Then, applying the maximum principle [CT, Proposition 4.4] to $G - G_1$, we conclude that

$$G_1(z, w) \leq G(z, w).$$

In order to prove the lemma, we need to estimate $\frac{\partial}{\partial \nu} G_1(z, y) f(y) dy$ which involves the derivatives of $G_1(z, y)$ for $z \in \partial_L C$. To use the regularity property $\Delta_z G_1 \equiv 0$ on an open domain, we shall use the identity (3.14) of $u$ on $\partial_L C_r$ for $0 < r < s$:

$$\begin{align*}
\frac{1}{2} \int_{\partial L} \frac{\partial}{\partial \nu} u(z)^2 (z, \nu) d\sigma_z + \frac{1}{2} \int_{\partial \Omega \setminus \partial \Omega \times (0, \infty)} |\nabla u|^2 (z, \nu) d\sigma_z \\
+ \int_{\Omega_r \times \{y = 0\}} (x, \nabla x u)(\nabla u, \nu) dx + \frac{n-1}{2} \int_{\Omega_r \times (0, \infty)} |\nabla u|^2 dxdy = 0. \quad (3.14)
\end{align*}$$

First we note that the third term is zero because $\frac{\partial}{\partial \nu} = 0$ on $\Omega_r \times \{y = 0\}$. As for the last term, we use an integration by part with the fact that $u = 0$ on $\partial \Omega \times [0, \infty)$ to get

$$\int_{\Omega_r \times (0, \infty)} |\nabla u|^2 dxdy = \int_{\Omega_r \times \{y\}} u \frac{\partial u}{\partial \nu} dx + \int_{\partial \Omega_r \setminus \partial \Omega \times [0, \infty)} u \frac{\partial u}{\partial \nu} d\sigma.$$ 

Then, the identity (3.14) gives the following identity.

$$\begin{align*}
\frac{1}{2} \int_{\partial \Omega \times (0, \infty)} |\nabla u|^2 (z, \nu) d\sigma = \frac{1}{2} \int_{\partial \Omega \setminus \partial \Omega \times (0, \infty)} |\nabla u|^2 (z, \nu) d\sigma + \frac{n-1}{2} \int_{\partial \Omega \setminus \partial \Omega \times (0, \infty)} u \frac{\partial u}{\partial \nu} d\sigma.
\end{align*}$$
We now set $C_r := (\Omega \setminus \Omega_r) \times [0, \infty)$ and integrate the above formula over $s/4 < r < s/2$, then we can deduce the following inequality

$$
\frac{1}{2} \int_{\partial_r C} |\nabla u|^2 (z, \nu) d\sigma = \frac{2}{s} \int_{\Omega_{s/2} \setminus \Omega_{s/4} \times (0, \infty)} \left\{ |\nabla u|^2 (z, \nu) + (n-1)u \frac{\partial u}{\partial \nu} \right\} d\sigma 
\lesssim \int_{c_{s/4} \setminus c_{s/4}} \left[ |\nabla u|^2 + u^2 \right] dz.
$$

Using the formula (3.15) and Cauchy-Schwartz inequality, we get

$$
|\nabla u(z)|^2 = \left[ \int_{\Omega} \nabla \chi G_1 (z, y) g(y) dy \right]^2 
\leq \left( \int_{\Omega} |\nabla \chi G_1 (z, y)|^2 |g(y)| dy \right) \left( \int_{\Omega} |g(y)| dy \right).
$$

Therefore,

$$
\int_{c_{s/2} \setminus c_{s/4}} \left[ |\nabla u|^2 + u^2 \right] dz 
\leq \left( \int_{\Omega} \left[ \int_{c_{s/2} \setminus c_{s/4}} |\nabla G_1 (z, y)|^2 + |G_1 (z, y)|^2 dz \right] |g(y)| dy \right) \left( \int_{\Omega} |g(y)| dy \right).
$$

In order to bound the term involving $\nabla G_1$, we shall use the following result.

**Sublemma 3.5.** Let $A$ be a smooth bounded domain. Suppose that $v$ satisfy the equation.

$$
\begin{cases}
\Delta v = 0 & \text{on } A \times [0, \infty), \\
\frac{\partial}{\partial \nu} v = 0 & \text{on } A \times \{0\}.
\end{cases}
$$

Then, for each $\delta > 0$, there exists a constant $C = C(A, \delta) > 0$ such that

$$
\int_0^\infty \int_{A \setminus A_\delta} |\nabla_x v(x, s)|^2 dx ds \leq C \int_0^\infty \int_A |v(x, s)|^2 dx ds.
$$

**Proof.** Let $\phi$ be a positive bump function compactly supported on $A$ which equals 1 on $A_\delta$ and satisfies

$$
|\nabla_x \phi(x)|^2 \leq C \phi(x) \quad \forall x \in A,
$$

where the constant $C > 0$ can be chosen depending only on $\delta$ and $A$. Using $\Delta v = 0$, an integration by parts gives

$$
0 = \int_0^\infty \int_A \Delta v \cdot v(x, s) \phi(x) dx ds 
= \int_A \frac{\partial}{\partial s} v(x, s) v(x, s) \phi(x) dx 
- \int_0^\infty \int_A \nabla_x v \cdot \nabla_x v(x, s) \phi(x) dx ds 
- \int_0^\infty \int_A \nabla_x v \cdot v(x, s) \nabla_x \phi(x) dx ds.
$$

Note that $\frac{\partial}{\partial \nu} v \big|_A = 0$ in this identity and use Young’s inequality to get

$$
\int_0^\infty \int_A |\nabla_x v(x, s)|^2 \phi(x) dx ds 
\leq \int_0^\infty \int_A C |v(x, s)|^2 dx ds + \int_0^\infty \int_A \frac{1}{4C} |\nabla_x v(x, s)|^2 |\nabla_x \phi(x)|^2 dx ds.
$$
Using (3.18) we obtain
\[ \int_0^\infty \int_A |\nabla v(x,s)|^2 \phi(x) dx ds \leq C(\delta, A) \int_0^\infty \int_A |v(x,s)|^2 dx ds. \]
\[ \square \]

Now we choose \( A = \Omega_{s/2} \setminus \Omega_{s/16} \) and take a bump function \( \phi \) such that \( \phi = 1 \) on \( \Omega_{s/4} \setminus \Omega_{s/8} \) satisfying (3.18). Then we get
\[ \int_{s/8}^{s/2} \int_{\partial L_c \setminus \partial L_c} \left| \nabla G_1(z,y) \right|^2 d\sigma_z ds \leq s \int_{s/8}^{s/2} \int_{\partial L_c \setminus \partial L_c} \frac{1}{(s/2 + |z_{n+1}|)^2} d\sigma_z ds \]
\[ \leq 1, \]
where the third inequality holds because \( y \in \Omega \setminus \Omega_s \) and \( z \in \Omega_{s/2} \times [0, \infty) \). Using this with (3.15) and (3.16) we deduces that
\[ \int_{\partial L_c} \left| \frac{\partial u}{\partial \nu} (z,\nu) \right|^2 (z,\nu) d\sigma \leq C(s,\Omega) \left( \int_{\Omega} |g(y)| dy \right)^2, \]
which is the asserted inequality.
\[ \square \]

**Proposition 3.6.** Suppose \( 1 < p < \frac{n+1}{n-1} \) and \( u \in C^2(\bar{\Omega}) \) is a solution of the equation (1.1) with \( f(u) = u^p \). Then there exists a constant \( C = C(p,\Omega) > 0 \) such that
\[ \int_{\Omega} u^{p+1}(x) dx \leq C. \]
In the general case of Theorem 1.1, there exists a constant \( C = C(f,\Omega) > 0 \) such that
\[ \int_{\Omega \times \{0\}} \left\{ nF(v) - \frac{n-1}{2} v f(v) \right\} dx \leq C, \]
where \( F(v) := \int_0^v f(s) ds). \)

We are now ready to prove the bound of \( L^{p+1}(\Omega) \) norm of the solution \( u \).

**Proof of Proposition 3.6.** In this proof, we assume that \( \Omega \) is strictly convex. The general case will be proved in the last part of the proof of Theorem 1.1.

We first make use of Lemma 3.1 to get a number \( s > 0 \) and a constant \( C = C(s,\Omega) > 0 \) such that
\[ \sup_{x \in \Omega_s} u(x) \leq C \]
\[ \text{(3.20)} \]
and
\[ \int_{\Omega \setminus \Omega_s} f(u)(x) dx \leq C. \]
\[ \text{(3.21)} \]
We find a bump function \( \chi \) supported in \( \Omega \setminus \Omega_s \) and \( \chi = 1 \) on \( \Omega \setminus \Omega_{2s} \). Set \( v \) be the harmonic extension of \( u \). Then \( v \) satisfies the equation (1.2). We split the function \( v \) as \( v = v_1 + v_2 \), where
$v_1$ and $v_2$ satisfy the following equations

$$
\begin{cases}
\Delta v_1 = 0 & \text{in } C, \\
v_1 = 0 & \text{on } \partial L C, \\
\frac{\partial v_1}{\partial \nu} = (1 - \chi(x))f(u)(x) & \text{on } \Omega \times \{0\}
\end{cases}
$$

and

$$
\begin{cases}
\Delta v_2 = 0 & \text{in } C, \\
v_2 = 0 & \text{on } \partial L C, \\
\frac{\partial v_2}{\partial \nu} = \chi(x)f(u)(x) & \text{on } \Omega \times \{0\}.
\end{cases}
$$

Using Lemma 3.3 and (3.20) we get

$$
\int_{\partial L C} \left| \frac{\partial v_1}{\partial \nu} \right|^2 (x, \nu) d\sigma \lesssim \int_{\Omega} |(1 - \chi(x))f(u)(x)|^2 dx \lesssim 1.
$$

On hand, we use (3.21) and Lemma 3.4 to have

$$
\int_{\partial L C} \left| \frac{\partial v_2}{\partial \nu} \right|^2 (x, \nu) d\sigma \lesssim \left( \int_{\Omega} |\chi(x)f(u)(x)| dx \right)^2 \lesssim \left( \int_{\Omega \backslash \Omega_\epsilon} |f(u)(x)| dx \right)^2 \lesssim 1.
$$

From (3.8) and the above two inequalities, we get

$$
\int_{\partial L C} \left| \frac{\partial u}{\partial \nu} \right|^2 (x, \nu) d\sigma \lesssim 1.
$$

Then the Pohozaev identity [5.6] concludes the proof.

Now we are in a position to prove our main theorem.

**proof of Theorem 1.1.** For simplicity, we shall first prove the theorem for $f(u) = u^p$ with the convexity assumption on $\Omega$. Since $p < \frac{n+1}{n-1}$ we get $q_1 > p$ for $\frac{p}{p+1} - \frac{1}{q_1} = \frac{1}{n} - \epsilon$ with sufficiently small $\epsilon > 0$. Using Lemma 2.6 we get

$$
\|u\|_{q_1} \lesssim \|(-\Delta)^{1/2} u\|_{\frac{p+1}{p}} \lesssim \|u^p\|_{\frac{p+1}{p}} \lesssim 1.
$$

For $k \geq 1$, we define $q_k$ by the relation $\frac{p}{q_k} - \frac{1}{q_{k+1}} = \frac{1}{n} - \epsilon$ and stop the sequence when we have $\frac{p}{q_N} < \frac{1}{n} - \epsilon$. Then, using Lemma 2.6 we have

$$
\|u\|_{q_{k+1}} \lesssim \|(-\Delta)^{1/2} u\|_{\frac{q_k}{p}} \lesssim \|u^p\|_{\frac{q_k}{p}} \lesssim 1,
$$

for $k = 1, \cdots, N - 1$. We then have $\|u\|_{q_N} \lesssim 1$, and use Lemma 2.6 again to deduce $\|u\|_{L^\infty} \lesssim 1$. It completes the proof when $\Omega$ is convex and $f(u) = u^p$, $p < \frac{n+1}{n-1}$.
In order to prove this for general \( f \) satisfying the condition (1), we first see from Proposition 3.6 that
\[
\int_{\Omega \times \{0\}} \left\{ nF(v) - \frac{n - 1}{2} vf(v) \right\} dx \leq C. \tag{3.23}
\]

From the condition (1.3), for any \( \epsilon > 0 \), we can find \( C_\epsilon > 0 \) such that
\[
uf(u) \leq \theta F(u) + \epsilon u^2 f(u)^{1/n} + C_\epsilon, \tag{3.24}
\]

In the below, \( C_\epsilon \) may be chosen differently in each line. We note that
\[
\int_\Omega u^2 |f(u)|^{\frac{1}{n-1}} dx \leq \|u\|_2^2 \|f(u)\|_1^{1/n} \leq C\|A_1^{1/2} u\|_2^2, \tag{3.25}
\]

and
\[
\int_\Omega uf(u)dx = \int_\Omega u A_{1/2} u dx = \int_\Omega A_{1/2}^2 u \cdot A_{1/2}^{1/2} u dx = \|A_{1/2}^{1/2} u\|_2^2. \tag{3.26}
\]

From (3.23) and (3.24), we can deduce that
\[
\left( \frac{n}{\theta} - \frac{n - 1}{2} \right) \int_\Omega uf(u)dx \leq \frac{\epsilon}{\theta} \int_\Omega u^2 f(u)^{1/n} dx + C_\epsilon.
\]

Choose \( \epsilon = \epsilon(\theta, n) > 0 \) small enough so that \( \left( \frac{n}{\theta} - \frac{n - 1}{2} \right) > \frac{\theta}{\theta} \). Then combining (3.26) and (3.24) with the above inequality yields that
\[
\|A_{1/2}^{1/2} u\|_2^2 \leq C,
\]

for a constant \( C = C(\theta, n) > 0 \).

Let \( p > 1 \) and \( q = (p+1) \frac{n}{n-1} \). Then,
\[
\left( \int_\Omega u^q dx \right)^{\frac{n-1}{n}} = \|u^{(p+1)/2}\|_{\frac{n}{n-1}}^2 \leq C \int_{\Omega \times (0, \infty)} |\nabla u^{\frac{p+1}{2}}|^2 dx \]
\[
= C_p \int_{\Omega \times (0, \infty)} \nabla u \cdot \nabla (u^p) dx \]
\[
= C_p \int_{\Omega} \frac{\partial u}{\partial \nu} \cdot u^p dx \]
\[
\leq \epsilon C_p \int_{\Omega} u^{\frac{n+1}{n-1}} u^p dx + C_\epsilon. \tag{3.27}
\]
Since \( p + 1 = \frac{2}{n} q \) we have

\[
\int_{\Omega} u^{\frac{n+1}{n-1}} dx = \int_{\Omega} u^{\frac{n}{n-1}}\left(\int_{\Omega} u^{\frac{2}{n-1}} dx\right)^{\frac{n-1}{n}} \leq C \left( \int_{\Omega} u^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \int_{\Omega} u^{\frac{2}{n-1}} dx \leq C \left( \int_{\Omega} u^{\frac{n}{n-1}} dx \right)^{\frac{n-1}{n}} \int_{\Omega} u^{\frac{2}{n-1}} dx.
\]

(3.28)

For a sufficiently small \( \epsilon > 0 \), the above inequality and (3.27) yields

\[
\left( \int_{\Omega} u^{\frac{n}{n-1}} dx \right)^{\frac{1}{q}} \leq C \epsilon,
\]

Since \( p \) is an arbitrary number, we can use Lemma 2.6 to conclude that \( \|u\|_{L^\infty} \leq C \epsilon \).

It completes the proof for general \( f \) with the convexity assumption on \( \Omega \).

Finally, we shall remove the convexity assumption by using the Kelvin transform in \( \mathbb{R}^{n+1} \) space to \( v \). Since \( \Omega \) is smooth, for a point \( x_0 \) we can find a ball which contact \( x_0 \) from the exterior of \( \Omega \).

We may assume \( x_0 = 1 \) and the ball is \( B(0,1) \) without loss of generality. Set

\[
w(z) = |z|^{1-n} v \left( \frac{z}{|z|^2} \right).
\]

Then, \( w \) satisfies

\[
\begin{cases}
\Delta w = 0 & \text{in } \kappa(C), \\
w > 0 & \text{in } \kappa(C), \\
w = 0 & \text{on } \kappa(\partial \Omega \times [0, \infty)), \\
\frac{\partial w}{\partial \nu} = g(y, w) & \text{on } \kappa(\Omega \times \{0\}),
\end{cases}
\]

where \( g(y, w) := f(|y|^{n-1} w)/|y|^{n+1} \). For \( \lambda > 0 \) we set

- \( D_\lambda = \kappa(C) \cap \{ z \in \mathbb{R}^{n+1}_+: |z| \leq 1, \ z_1 > 1 - \lambda \} \),
- \( \partial D_\lambda = D_\lambda \cap \partial \mathbb{R}^{n+1}_+ \),
- \( T_\lambda(y) = (2 - 2\lambda - y_1, y_2, \ldots, y_{n+1}) \).

Let \( \zeta_\lambda = w_\lambda - w \) defined on \( D_\lambda \). We claim that \( v_\lambda \geq 0 \) if \( \lambda > 0 \) is sufficiently small. Set \( v_\lambda = \max\{0, -v_\lambda\} \). Then,

\[
0 = \int_{D_\lambda} \zeta_\lambda^- \Delta \zeta_\lambda^- dxdy = \int_{\partial D_\lambda} \zeta_\lambda^- \frac{\partial \zeta_\lambda^-}{\partial \nu} d\nu + \int_{D_\lambda} |\nabla \zeta_\lambda^-|^2 dxdy.
\]

(3.29)

We have

\[
\int_{\partial D_\lambda} \left(-\zeta_\lambda^-\right) \frac{\partial \zeta_\lambda^-}{\partial \nu} dx = \int_{\partial D_\lambda} \left(-\zeta_\lambda^-\right) (g(T_\lambda x, w_\lambda) - g(x, w)) dx
\]

\[
= \int_{\partial D_\lambda \cap \{w_\lambda < w\}} (w - w_\lambda)(g(x, w) - g(T_\lambda x, w_\lambda)) dx
\]

(3.30)
Since \( u \to f(u)u^{-\frac{n+1}{2}} \) is nonincreasing, we see that \( g(x, w) \leq g(T_\lambda x, w_\lambda) \) because \(|x| \geq |T_\lambda(x)|\).
Using this we deduces that
\[
\int_{D_\lambda} |\nabla \zeta_\lambda^-|^2 \, dx \, dy \leq \int_{\partial D_\lambda \cap \{w_\lambda \leq w\}} (w - w_\lambda)(g(x, w) - g(x, w_\lambda)) \, dx
\]
\[
\leq \int_{\partial D_\lambda \cap \{w_\lambda \leq w\}} (w - w_\lambda)^2 h(x, w, w_\lambda) \, dx
\]
\[
= \int_{\partial D_\lambda \cap \{w_\lambda \leq w\}} (\zeta_\lambda^-)^2 h(x, w, w_\lambda) \, dx,
\]
where \( h(x, w, w_\lambda) = \frac{g(x, w) - g(x, w_\lambda)}{w - w_\lambda} \). Since \( f \) is locally Lipschitz it is bounded by using \( \sup_{\partial D_\lambda} \|u\| + |w_\lambda| \). By Hölder’s inequality we deduce that
\[
\int_{D_\lambda} |\nabla \zeta_\lambda^-|^2 \, dx \, dy \leq C \int_{\partial D_\lambda \cap \{w_\lambda \leq w\}} (\zeta_\lambda^-)^2 \, dx \leq C \delta D_\lambda \cap \{w_\lambda \leq w\}\]}
\[
\leq C |\partial D_\lambda \cap \{w_\lambda \leq w\}|^{1/n}\|\zeta_\lambda^- (-, 0)\|_{L_{2n/(n-1)}(\Omega)}^2.
\]
Using the trace inequality, we get
\[
\|\zeta_\lambda^- (-, 0)\|_{L_{2n/(n-1)}(\Omega)} \leq C |\partial D_\lambda \cap \{w_\lambda \leq w\}|^{1/n}\|\zeta_\lambda^- (-, 0)\|_{L_{2n/(n-1)}(\Omega)},
\]
which yields that \( \zeta_\lambda^- \equiv 0 \) for small \( \lambda \).

Now we set
\[
\eta = \sup\{\lambda > 0 : T_\lambda(D_\lambda) \subset \kappa(C)\},
\]
and
\[
S := \{0 < \lambda \leq \frac{\eta}{2} : \zeta_\lambda \geq 0 \text{ on } D_\lambda \} \cup \{0\}.
\]
We shall prove that \( S = [0, \eta/2] \). Since \( \zeta_\lambda \) is a continuous function of \( \lambda \), \( S \) is closed. Thus, It suffices to show that \( S \) is also open in \([0, \eta/2] \). Note that the constant \( C \) in the inequality \( 3.32 \) can be chosen uniformly for \( \lambda \in [0, \eta/2] \) since \( \sup_{0 < \lambda < \eta/2 \sup_{\partial D_\lambda} \|u\| + |w_\lambda|} \) is bounded.

Choose any \( 0 < \lambda_0 < \eta/2 \) contained in \( S \). Then we have \( \zeta_{\lambda_0} \geq 0 \). Since \( \zeta_{\lambda_0} \geq 0 \) on \( \kappa(\partial \Omega \times [0, \infty)) \cap D_{\lambda_0} \) and \( \Delta \zeta_{\lambda_0} \equiv 0 \) in \( D_{\lambda_0} \), we see that \( \zeta_{\lambda_0} > 0 \) in \( D_{\lambda_0} \) by the maximum principle. Thus we can find \( c > 0 \) such that
\[
|D_{\lambda_0, c} := \{x \in D_{\lambda_0} : \zeta_{\lambda_0} > c\}| \geq |D_{\lambda_0}| - \delta/2.
\]
By continuity, there is \( \epsilon > 0 \) such that \( \zeta_\lambda > \frac{\epsilon}{2} \) and \( |D_\lambda \setminus D_{\lambda_0}| < \frac{\delta}{2} \) for \( \lambda \in [\lambda_0, \lambda_0 + \epsilon) \). For such \( \lambda \) we then see that
\[
|\{x \in D_\lambda : \zeta_\lambda > \frac{\epsilon}{2}\}| \geq |D_\lambda| - \delta - \frac{\delta}{2} = |D_\lambda| - \delta.
\]
This yields that
\[
|\{x \in D_\lambda : \zeta_\lambda \leq 0\}| \leq \delta.
\]
Then the inequality \( 3.32 \) implies that \( \zeta_\lambda \geq 0 \) for \( \lambda \in [\lambda_0, \lambda_0 + \epsilon) \). Therefore we have that \( w \) increase in any line in \( \Omega \) starting from a boundary point. Since \( w(x) \geq w(y) \) we deduce that \( u(x/|x|^2) \geq cu(y/|y|^2) \) holds with some \( c \in (0, 1) \) uniformly for \( (x, y) \) satisfying \( \min(|x|, |y|) > 1/2 \).

Then we can obtain the \( L^\infty \) bound near the boundary \( \partial \Omega \). The rest of the proof is same with it of the restricted case. The proof is completes. \( \square \)
4. The Lane-Emden System

In this section, we study the Lane-Emden system involving the square root of the Laplacian

\[
\begin{align*}
A_{1/2}u & = v^p \quad \text{in } C, \\
A_{1/2}v & = u^q \quad \text{in } C, \\
u > 0, \ v > 0 & \quad \text{in } C, \\
u = v = 0 & \quad \text{on } \partial C.
\end{align*}
\]

(4.1)

We shall use the same letter \((u, v)\) to denote the harmonic extension of a solution \((u, v)\) of (4.1).

Then, we have

\[
\begin{align*}
\Delta u & = \Delta v = 0 \quad \text{in } C, \\
u = v = 0 & \quad \text{on } \partial \Sigma, \\
\frac{\partial u}{\partial \nu} & = v^p, \ \frac{\partial v}{\partial \nu} = u^q \quad \text{on } \Omega \times \{0\}, \\
u > 0, \ v > 0 & \quad \text{in } C.
\end{align*}
\]

(4.2)

We say that \((p, q)\) is

1. subcritical if \(\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-1}{n}\),
2. critical if \(\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-1}{n}\),
3. supercritical if \(\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-1}{n}\).

First, the existence of weak solution and Brezis-Kato type estimate will follow from the same proof of [HV]. We shall obtain a Pohozaev type identity, which proves nonexistence of nontrivial solutions for the system (4.1) in critical and supercritical cases. Next, we shall establish a moving plane argument. Then, we shall finally prove a priori estimate for subcritical cases by applying the method which of the proof of Theorem 1.1.

The following two theorems are analogous to the result on the existence and Brezis-Kato estimate for the Lane-Emden system (see [HV, Theorem 1]), where the subcritical range is given by

\[
\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}.
\]

The difference of the range of \((p, q)\) comes from the different ranges of sobolev embeddings. With only changing the index of the embeddings we can apply the proofs of [HV] straightforwardly to obtain the following theorems.

**Theorem 4.1.** If \((p, q)\) is sub-critical, one can find \(\alpha, \beta > 0\) satisfying

\[
\frac{1}{2} - \frac{1}{p+1} < \frac{\alpha}{n}, \ \frac{1}{2} - \frac{1}{q+1} < \frac{\beta}{n}, \ \text{and } \alpha + \beta = 1.
\]

Then, there exists a (nontrivial) weak solution \((u, v)\) \(H_0^2(\Omega) \times H_0^2(\Omega)\) of the system (4.1).

**Proof.** See the proof of [HV, Theorem 1]. The only difference of the range of \((p, q)\) is due to the different ranges of the Sobolev inequalities.

**Theorem 4.2.** Assume \((p, q)\) is sub-critical or critical. Let \((u, v)\) be a weak solution of the system (4.1). Then, \(u \in C^{2,\alpha}\) and \(v \in C^{2,\alpha}\) for some \(0 < \alpha < 1\).

**Proof.** One may see that the proof of [HV, Theorem 3.1] can be adapted to our problem with using the embedding in Lemma 2.6 thus we have \(u \in L^\infty(\Omega)\) and \(v \in L^\infty(\Omega)\) for subcritical case. Then we can apply the regularity result [CT, Proposition 3.1] to get \(u \in C^{2,\alpha}(\Omega)\) and \(v \in C^{2,\alpha}(\Omega)\).
We shall obtain a Pohozaev type identity for the system. It will gives the nonexistence result for the critical and supercritical cases.

**Theorem 4.3.** Suppose that \((u, v) \in C^2(\bar{C}) \times C^2(\bar{C})\) satisfies

\[
\begin{cases}
\Delta u = \Delta v = 0 & \text{in } C, \\
u = 0 & \text{on } \partial L C.
\end{cases}
\]

Then we have

\[
\int_{\partial L C} (x \cdot \nu) \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\sigma = -\int_{\Omega \times \{y = 0\}} [(x, \nabla_x v)(\nabla, \nu) + (x, \nabla_x u)(\nabla v, \nu)] dx - (n - 1) \int_C \nabla u \cdot \nabla v dx. \tag{4.4}
\]

**Proof.** We have

\[
div[(x, \nabla v)\Delta u + (x, \nabla u)\Delta v]
\]

\[
(x, \nabla v)\Delta u + (x, \nabla u)\Delta v + x \cdot \nabla (\nabla u \cdot \nabla v) + 2 \nabla u \cdot \nabla v. \tag{4.5}
\]

Therefore, from (4.3) we have in \(C\) that

\[
div[(x, \nabla v)\nabla u + (x, \nabla u)\nabla v] = x \cdot \nabla (\nabla u \cdot \nabla v) + 2 \nabla u \cdot \nabla v. \tag{4.6}
\]

We have

\[
div[(x)(\nabla u \cdot \nabla v)] = (\text{div} x)(\nabla u \cdot \nabla v) + x \cdot \nabla (\nabla u \cdot \nabla v) = (n + 1)(\nabla u \cdot \nabla v) + x \cdot \nabla (\nabla u \cdot \nabla v).
\]

The above two formulas gives the following equality.

\[
div[(x, \nabla v)\nabla u + (x, \nabla u)\nabla v] - \text{div}(x(\nabla u \cdot \nabla v)) + (n - 1) \nabla u \cdot \nabla v = 0. \tag{4.7}
\]

The divergence theorem with that \(u = v = 0\) on \(\partial \Omega \times [0, \infty)\), we get

\[
\int_{\Omega \times (0, R)} \text{div}(x, \nabla v)\nabla u + (x, \nabla u)\nabla v) dx
\]

\[
= \int_{\Omega \times (0, R)} 2(x \cdot \nu) \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\sigma + \int_{\Omega \times \{y = 0\}} [(x, \nabla_x v)(\nabla u, \nu) + (x, \nabla_x u)(\nabla v, \nu)] dx \tag{4.8}
\]

\[
+ \int_{\Omega \times \{y = R\}} [(x, \nabla_x v)(\nabla u, \nu) + (x, \nabla_x u)(\nabla v, \nu)] dx
\]

and

\[
\int_{\Omega \times (0, R)} \text{div}(x(\nabla u \cdot \nabla v)) dx = \int_{\partial \Omega \times (0, R)} (x \cdot \nu)(\frac{\partial u}{\partial \nu} \cdot \frac{\partial v}{\partial \nu}) d\sigma + \int_{\Omega \times \{y = R\}} R(\nabla u \cdot \nabla v) dx. \tag{4.9}
\]

By a limiting argument used in the proof of Lemma 3.3 we obtain

\[
\int_{\mathcal{C}} \text{div}(x, \nabla v)\nabla u + (x, \nabla u)\nabla v) dx
\]

\[
= \int_{\mathcal{C}} 2(x \cdot \nu) \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\sigma + \int_{\Omega \times \{y = 0\}} [(x, \nabla_x v)(\nabla u, \nu) + (x, \nabla_x u)(\nabla v, \nu)] dx \tag{4.10}
\]

and

\[
\int_{\mathcal{C}} \text{div}(x(\nabla u \cdot \nabla v)) dx = \int_{\mathcal{C}} (x \cdot \nu)(\frac{\partial u}{\partial \nu} \cdot \frac{\partial v}{\partial \nu}) d\sigma. \tag{4.11}
\]
We integrate (4.7) over $C$, and use the above two formulas to have
\[\int_C (x \cdot \nu) \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\sigma + \int_{\Omega \times \{y = 0\}} [(x, \nabla_x v)(\nabla u, \nu) + (x, \nabla_x u)(\nabla v, \nu)] dx = (n - 1) \int_C \nabla u \nabla v dx,\]
which is the desired identity (4.4).

**Theorem 4.4.** Assume that the domain $\Omega$ is bounded and starshaped, and $p, q > 1$ satisfy
\[\frac{1}{p + 1} + \frac{1}{q + 1} \leq \frac{n - 1}{n}.\]
Then (4.6) has no positive solution.

**Proof.** We may assume that $\Omega$ is starshaped with respect to the origin, that is, $(x \cdot \nu) > 0$ for any $x \in \partial L \Omega$. It easily implies that $(x \cdot \nu) > 0$ holds also for $x \in \partial L C$.

Suppose that $(u, v) \in C^2(\bar{\Omega}) \times C^2(\bar{\Omega})$ satisfies (4.6) and denote also by $(u, v)$ the harmonic extension of $(u, v)$. Let $f(v) = v^p$ and $g(u) = u^p$ and set
\[F(v) = \int_0^v f(s) ds \quad \text{and} \quad G(u) = \int_0^u g(s) dx.\]
Because $F(0) = 0$ and $u = 0$ on $\partial \Omega \times \{0\}$, we get
\[\int_{\Omega \times \{0\}} (x, \nabla_x v)(\nabla u, \nu) dx = \int_{\Omega \times \{0\}} (x, \nabla_x v)f(v) dx = \int_{\Omega \times \{0\}} (x, \nabla_x F(v)) dx = - \int_{\Omega \times \{0\}} nF(v) dx. \tag{4.12}\]
Likewise, we have
\[\int_{\Omega \times \{0\}} (x, \nabla_x u)(\nabla v, \nu) dx = - \int_{\Omega \times \{0\}} nG(u) dx.\]

For a solution $(u, v)$ of the system (4.1), we get
\[\frac{1}{2} \int_{\partial L \mathcal{C}} (x \cdot \nu) \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\sigma = \int_{\Omega \times \{0\}} \left(\frac{n}{p + 1} - (n - 1)\theta\right) u^{p+1} + \left(\frac{n}{q + 1} - (n - 1)(1 - \theta)\right) v^{q+1} dx. \tag{4.13}\]
Since $u = v = 0$ on $\partial L \mathcal{C}$ we have $\frac{\partial u}{\partial \nu} \geq 0$ and $\frac{\partial v}{\partial \nu} \geq 0$ on $\partial L \mathcal{C}$. If $(p, q)$ is super-critical we can find $\theta \in (0, 1)$ such that
\[\frac{n}{p + 1} - (n - 1)\theta < 0 \quad \text{and} \quad \frac{n}{q + 1} - (n - 1)(1 - \theta) < 0.\]
It implies that $u \equiv v \equiv 0$ on $\Omega \times \{0\}$. In the critical case, we have
\[\frac{1}{2} \int_{\partial L \mathcal{C}} (x \cdot \nu) \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\sigma, \]
which implies that $\frac{\partial u}{\partial \nu}(x_0) = 0$ or $\frac{\partial v}{\partial \nu}(x_0) = 0$. Since $\Delta u = \Delta v = 0, u, v \geq 0$ on $\mathcal{C}$, it follows from Hopf’s lemma that $u \equiv 0$ or $v \equiv 0$, which yields that $u \equiv v \equiv 0$. It completes the proof. \hfill \Box

Next, we shall establish the moving plaine argument, which will gives the symmetry result and the $L^\infty$ bound near the boundary of positive solutions to (4.1). As a preliminary step of it, we need the following lemma.
Lemma 4.5. Assume that $c \leq 0$, $d \leq 0$ and $\Omega$ is a bounded (not necessarily smooth) domain of $\mathbb{R}^n$ and $c,d \in L^\infty(\Omega)$. Let $u,v \in C^2(\bar{\Omega}) \cap L^\infty(\Omega)$, where $\mathcal{C} = \Omega \times (0,\infty)$, satisfy

\[
\begin{cases}
\Delta u = \Delta v = 0 & \text{in } \mathcal{C}, \\
u \geq 0, \ v \geq 0 & \text{on } \partial_t \mathcal{C}, \\
\frac{\partial u}{\partial \nu} + c(x)v \geq 0 & \text{on } \Omega \times \{0\}, \\
\frac{\partial v}{\partial \nu} + d(x)u \geq 0 & \text{on } \Omega \times \{0\}.
\end{cases}
\] (4.14)

Moreover, we assume $u(x_0) = v(x_0) = 0$ holds for some point $x_0 \in \bar{\mathcal{C}}$. Then, there exists $\delta > 0$ depending only on $\|c\|_{L^\infty(\Omega)}$, $\|d\|_{L^\infty}$ and $n$ such that if

\[
|\Omega \cap \{u(\cdot,0) < 0\}| \cdot |\Omega \cap \{v(\cdot,0) < 0\}| \leq \delta,
\]

then $u \geq 0$ and $v \geq 0$ in $\mathcal{C}$.

Proof. Let $u^- = \max\{0,-u\}$ and $v^- = \max\{0,-v\}$. Since $u^- = v^- = 0$ on $\partial \Omega \times [0,\infty)$, we see

\[
0 = \int_{\mathcal{C}} u^- \Delta u dxdy = \int_{\Omega \times \{0\}} u^- \frac{\partial u}{\partial \nu} dxdy + \int_{\mathcal{C}} |\nabla u^-|^2 dxdy.
\]

Then, using $c \leq 0$ we deduce that

\[
\int_{\mathcal{C}} |\nabla u^-|^2 dxdy = - \int_{\Omega \times \{0\}} v^- \frac{\partial u}{\partial \nu} dxdy = \int_{\Omega \times \{0\}} u^- cvdx
\]

\[
\leq \int_{\Omega \times \{0\}} u^- (-c)v^- dxdy \leq |\Omega \cap \{u^-(\cdot,0) > 0\}|^{1/n} |\Omega \cap \{v^-(\cdot,0) > 0\}|^{1/n} \|c\|_{L^\infty(\mathcal{C})} \|u^-\|_{L^{2n/(n-1)}(\Omega)} \|v^-\|_{L^{2n/(n-1)}(\Omega)}.
\] (4.15)

By the same argument for $v^-$, we get

\[
\int_{\mathcal{C}} |\nabla v^-|^2 dxdy \leq |\Omega \cap \{u^-(\cdot,0) > 0\}|^{1/n} |\Omega \cap \{v^-(\cdot,0) > 0\}|^{1/n} \|c\|_{L^\infty(\mathcal{C})} \|d\|_{L^\infty} \|u^-\|_{L^{2n/(n-1)}(\Omega)} \|v^-\|_{L^{2n/(n-1)}(\Omega)}.
\] (4.16)

Multiplying the above two inequalities, we obtain

\[
\left(\int_{\mathcal{C}} |\nabla u^-|^2 dxdy\right) \left(\int_{\mathcal{C}} |\nabla v^-|^2 dxdy\right)
\]

\[
\leq |\Omega \cap \{u^-(\cdot,0) > 0\}|^{1/n} |\Omega \cap \{v^-(\cdot,0) > 0\}|^{1/n} \|c\|_{L^\infty(\mathcal{C})} \|d\|_{L^\infty} \|u^-\|_{L^{2n/(n-1)}(\Omega)} \|v^-\|_{L^{2n/(n-1)}(\Omega)}.
\] (4.17)

We now use the Sobolev trace inequality

\[
S_0 \|u^-(\cdot,0)\|_{L^{2n/(n-1)}(\Omega)}^2 \leq \int_{\bar{\mathcal{C}}} |\nabla u^-|^2 dxdy
\]

and

\[
S_0 \|v^-(\cdot,0)\|_{L^{2n/(n-1)}(\Omega)}^2 \leq \int_{\bar{\mathcal{C}}} |\nabla v^-|^2 dxdy.
\]

Then it follows that

\[
S_0^2 \|u^-(\cdot,0)\|^2_{L^{2n/(n-1)}(\Omega)} \|v^-(\cdot,0)\|^2_{L^{2n/(n-1)}(\Omega)}
\]

\[
\leq |\Omega \cap \{u^-(\cdot,0) > 0\}|^{1/n} |\Omega \cap \{v^-(\cdot,0) > 0\}|^{1/n} \|c\|_{L^\infty(\mathcal{C})} \|d\|_{L^\infty} \|u^-\|_{L^{2n/(n-1)}(\Omega)} \|v^-\|_{L^{2n/(n-1)}(\Omega)}.
\]

If we choose $\delta$ so that $S_0^2 > \delta^{1/n} \|c\|_{L^\infty(\mathcal{C})} \|d\|_{L^\infty}$, the above inequality yields that $u^- \equiv 0$ or $v^- \equiv 0$. Say $u^- \equiv 0$, then we have $\int_{\mathcal{C}} |\nabla u^-|^2 dxdy = 0$ from (4.17). Thus we have $\nabla v^- \equiv 0$, and since $v(x_0) = 0$, we conclude that $v^- \equiv 0$. It completes the proof. \qed
For \( y \in \partial \Omega \) and \( \lambda > 0 \) we set

\[
T(y, \lambda) := \{ x \in \mathbb{R}^n : \langle y - x, \nu(y) \rangle = \lambda \},
\]

\[
\Sigma(y, \lambda) := \{ x \in \Omega : \langle y - x, \nu(y) \rangle \leq \lambda \},
\]

and define \( R(y, \lambda) \) be the reflection with respect to the hyperplane \( T(y, \lambda) \). We also set \( \Sigma'(y, \lambda) := R(y, \lambda)\Sigma(y, \lambda) \) and

\[
\lambda_y := \sup \{ \lambda > 0 : \Sigma(y, \lambda) \subset \Omega \}. \tag{4.18}
\]

**Lemma 4.6.** Suppose that \((u, v) \in C^2(\Omega)\) is a solution of (1.6). Then, for any \( y \in \partial \Omega \) and \( x \in \Sigma(y, \lambda) \) the following inequalities hold for any \( \lambda \in (0, \lambda_y) \).

\[
u(R(y, \lambda)x) \geq u(x), \quad v(R(y, \lambda)x) \geq v(x)
\]

Proof. We may assume that \( 0 \in \partial \Omega \) and \( \nu = (1, 0) \) is a normal direction to \( \partial \Omega \) at this point. It is sufficient to prove the lemma at this point. For \( \lambda > 0 \) we set

\[\Sigma_\lambda = \{ (x_1, x') \in \Omega : x_1 > \lambda \} \quad \text{and} \quad T_\lambda = \{ (x_1, x') \in \Omega : x_1 = \lambda \} .\]

For \( x \in \Sigma_\lambda \), define \( x_\lambda = (2\lambda - x_1, x') \). From the definition (4.18) we see

\[\{ x_\lambda : x \in \Sigma_\lambda \} \subset \Omega \quad \forall \lambda < \lambda_0 .\]

We denote also by \((u, v)\) the harmonic extension of \((u, v)\) in \( C \). Then, \((u, v) \in C^2(\Omega)\) satisfies

\[
\begin{cases}
\Delta u = \Delta v = 0 & \text{in } C,
\quad u = v = 0 & \text{on } \partial C,
\quad \frac{\partial u}{\partial \nu} = v^p, \quad \frac{\partial v}{\partial \nu} = u^q & \text{on } \Omega \times \{0\},
\quad u > 0, \quad v > 0 & \text{in } C .
\end{cases} \tag{4.19}
\]

For \((x, y) \in \Sigma_\lambda \times [0, \infty)\), we set

\[u_\lambda(x, y) = u(x_\lambda, y) = u(2\lambda - x_1, x', y)\]

and

\[\alpha_\lambda(x, y) = (u_\lambda - u)(x, y) , \quad \beta_\lambda(x, y) = (v_\lambda - v)(x, y) .\]

Then we have \( u_\lambda = v_\lambda = 0 \) on \( T_\lambda \times [0, \infty) \) and obtain from (4.19) that \( u_\lambda > 0 \) and \( v_\lambda > 0 \) on \( (\partial \Omega \cap \Sigma_\lambda) \times [0, \infty) \). Since \( \partial \Sigma_\lambda = T_\lambda \cup (\partial \Omega \cap \Sigma_\lambda) \) we see that \((\alpha_\lambda, \beta_\lambda)\) satisfies

\[
\begin{cases}
\Delta \alpha_\lambda = \Delta \beta_\lambda = 0 & \text{in } \Sigma_\lambda \times (0, \infty),
\alpha_\lambda \geq 0, \quad \beta_\lambda \geq 0 & \text{on } (\partial \Sigma_\lambda) \times (0, \infty),
\frac{\partial \alpha_\lambda}{\partial \nu} + c_\lambda(x) \beta_\lambda = 0 & \text{on } \Sigma_\lambda \times \{0\},
\frac{\partial \beta_\lambda}{\partial \nu} + d_\lambda(x) \alpha_\lambda = 0 & \text{on } \Sigma_\lambda \times \{0\},
\end{cases}
\]

where

\[c_\lambda(x, 0) = \frac{v^p - v^p}{v_\lambda - v} \quad \text{and} \quad d_\lambda(x, 0) = -\frac{u^p - u^p}{u_\lambda - u} .\]

Note that \( c_\lambda \leq 0 \) and \( d_\lambda \leq 0 \). Now we choose a small number \( \kappa > 0 \) so that the set \( \Sigma_\lambda \) has small measure for \( 0 < \lambda < \kappa \). We then deduce from Lemma 4.14 that

\[\alpha_\lambda \geq 0 \quad \text{and} \quad \beta_\lambda \geq 0 \quad \text{in } \Sigma_\lambda \times (0, \infty), \quad \forall \lambda \in (0, \kappa) .\]
The strong maximum principle induces that $\alpha_\lambda$ and $\beta_\lambda$ are identically equal to zero or strictly positive in $\Sigma_\lambda \times (0, \infty)$. Since $\lambda > 0$, we have $\alpha_\lambda > 0$ and $\beta_\lambda > 0$ in $(\partial \Omega \cap \partial \Sigma_\lambda) \times (0, \infty)$, and thus we deduce that $\alpha_\lambda > 0$ and $\beta_\lambda > 0$ in $\Sigma_\lambda \times (0, \infty)$.

We let $\lambda_1 = \sup \{ \lambda > 0 | \alpha_\lambda \geq 0 \text{ and } \beta_\lambda \geq 0 \text{ in } \Sigma_\lambda \times (0, \infty) \}$. We claim that $\lambda_1 = \lambda_0$. With a view to a contradiction, we suppose that $\lambda_1 < \lambda_0$. From the continuity, we have $\alpha_{\lambda_1} \geq 0$ and $\beta_{\lambda_1} \geq 0$ in $\Sigma_{\lambda_1} \times (0, \infty)$. As before, from the strong maximum principle, we have that $\alpha_{\lambda_1} > 0$ and $\beta_{\lambda_1} > 0$ in $\Sigma_{\lambda_1} \times (0, \infty)$. Next, let $\delta > 0$ be a constant and find a compact set $K \subset \Sigma_{\lambda_1}$ such that $| \Sigma_{\lambda_1} \setminus K | \leq \delta / 2$. We have $\alpha_{\lambda_1} \geq \mu > 0$ and $\beta_{\lambda_1} \geq \eta > 0$ in $K$ for some constant $\eta$, since $K$ is compact. Thus, we obtain that $\alpha_{\lambda_1} + \epsilon \geq 0$ and $\beta_{\lambda_1} + \epsilon \geq 0$ in $K$ and that $| \Sigma_{\lambda_1} \setminus K | \leq 0$.

By using Lemma 4.14 in $\Sigma_{\lambda_1 + \epsilon} \times (0, \infty)$ to the function $(\alpha_{\lambda_1 + \epsilon}, \beta_{\lambda_1 + \epsilon})$, we have that $\alpha_{\lambda_1 + \epsilon} \geq 0$ and $\beta_{\lambda_1 + \epsilon} \geq 0$ in $K$. Thus $\{ \alpha_{\lambda_1 + \epsilon} < 0 \}, \{ \beta_{\lambda_1 + \epsilon} < 0 \} \subset \Sigma_{\lambda_1 + \epsilon} \setminus K$, which have measure at most $\delta$. We take $\delta$ to be the constant of Lemma 4.14. Then we deduce that

$$\alpha_{\lambda_1 + \epsilon} \geq 0 \text{ and } \beta_{\lambda_1 + \epsilon} \geq 0 \text{ in } \Sigma_{\lambda_1 + \epsilon} \times (0, \infty).$$

This is a contradiction to the definition of $\lambda_1$. Thus, we have that $\lambda_1 = \lambda_0$, which proves the lemma.

This lemma gives the following symmetry result.

**Theorem 4.7.** Suppose that a bounded smooth domain $\Omega \subset \mathbb{R}^n$ is convex in the $x_1$ direction and symmetric with respect to the hyperplane $\{ x_1 = 0 \}$. Let $(u, v)$ be a $C^2(\Omega)$ solution of (4.11).

Then, $u$ is symmetric in $x_1$ direction, that is, $u(-x_1, x') = u(x_1, x')$ for all $(x_1, x') \in \Omega$. Moreover we have $\frac{\partial}{\partial x_1} < 0$ for $x_1 > 0$.

We are now ready to prove the main theorem.

**Proof of Theorem 4.7.** Since $\Omega$ is convex and smooth, there exist constants $\lambda_0 > 0$ and $c_0 > 0$ such that

$$\Sigma'(y, \lambda) \subset \Omega, \quad \lambda \leq \lambda_0, \quad \text{and} \quad (\nu(x), \nu(y)) > c_0, \quad x \in \partial \Sigma(y, \lambda_0) \cap \partial \Omega. \quad (4.20)$$

If $(p, q)$ is subcritical, we may choose $\theta \in (0, 1)$ so that

$$\frac{n}{p + 1} - (n - 1)\theta > 0 \quad \text{and} \quad \frac{n}{q + 1} - (n - 1)(1 - \theta) > 0.$$

Then we get from the identity (4.13) that

$$\int_{\Omega \times \{y = 0\}} \left( u^{p+1} + u^{q+1} \right) dx \leq C(p, q, n) \int_{\partial \Sigma} (x \cdot \nu) \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\sigma.$$

Multiplying $\phi_1$ in the equations of (1.6) we have

$$\int_{\Omega} \nu \phi_1 dx = \int_{\Omega} \lambda_1 u \phi_1 dx \quad \text{and} \quad \int_{\Omega} u \phi_1 dx = \int_{\Omega} \lambda_1 v \phi_1 dx.$$

We use a convex inequality to get

$$\int_{\Omega} \lambda_1 u \phi_1 dx \geq (\int_{\Omega} \nu \phi_1 dx)^p$$

and

$$\int_{\Omega} \lambda_1 v \phi_1 dx \geq (\int_{\Omega} u \phi_1 dx)^q,$$
which yield that
\[ \int_\Omega v \phi_1 dx \lesssim C \quad \text{and} \quad \int_\Omega u \phi_1 dx \lesssim C. \]

Also we have
\[ \int_\Omega (v^p + u^q) \phi_1 dx \lesssim C. \]

From (4.20), Lemma 4.6 and the above inequality, we can obtain \( L^\infty \) bound for \( (u, v) \) near the boundary \( \partial \Omega \). Then, using Lemma 3.3 and Lemma 3.4 we get
\[ \int_{\partial \Omega} \left| \frac{\partial u}{\partial \nu} \right|^2 + \left| \frac{\partial v}{\partial \nu} \right|^2 d\sigma \lesssim C. \]

Using Cauchy-Schwartz inequality, we get
\[ \int_{\partial \Omega} (x \cdot \nu) \frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu} d\sigma \lesssim C. \]

Thus we obtain
\[ \int_\Omega (v^{p+1} + u^{q+1}) dx \lesssim C. \]

We now use the bootstrap argument to improve the integrability of \( v \) and \( u \). For this, we need
\[ \frac{p}{(p+1)p} - \frac{1}{(q+1)(p+1)} < \frac{1}{n} \quad \text{and} \quad \frac{q}{(q+1)q} - \frac{1}{(p+1)(p+1)} < \frac{1}{n}. \]

It is enough to get
\[ \frac{p}{p+1} - \frac{1}{(q+1)\rho} < \frac{1}{n} \quad \text{and} \quad \frac{q}{q+1} - \frac{1}{(p+1)\rho} < \frac{1}{n}. \]

We need to choose \( \rho \) so that
\[ \frac{1}{\rho} > \max \left[ (q+1) \left( \frac{n-1}{n} - \frac{1}{p+1} \right), (p+1) \left( \frac{n-1}{n} - \frac{1}{q+1} \right) \right]. \]

Because \( \frac{1}{p+1} + \frac{1}{q+1} = \frac{n-1}{n} + \epsilon \), we may choose \( \rho \) so that
\[ \frac{1}{\rho} > 1 - \epsilon \min(p+1, q+1). \]

It enables us the iteration, and so completes the proof. \( \square \)

ACKNOWLEDGEMENTS

I thanks to my advisor, Professor Raphaël Ponge for his support and valuable suggestions. I also thanks to Professor Panki Kim for teaching me the difference between the concept of \( ((-\Delta) |_{\Omega})^s \) and that of \( (-\Delta)^s|_{\Omega} \), and drawing me to the paper [S].

REFERENCES

[CFM] Ph. Clément, D.G. de Figueiredo and E. Mitidieri, Positive solutions of semilinear elliptic systems, Comm. Partial Differential Equations 17 (1992), 923-940.
[CT] X. Cabré, J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian. Adv. Math. 224 (2010), no 5, 2052-2093.
[CKS] Z. Chen, P. Kim, R. Song, Heat kernel estimates for the Dirichlet fractional Laplacian. J. Eur. Math. Soc. 12 (2010), no. 5, 1307-1329.
[FLN] D.G. de Figueiredo, P.-L. Lions and R.D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J.Math. Pures Appl. 61 (1982), 41-63.
[HV] J. Hulshof, R. Vorst, Differential systems with strongly indefinite variational structure. (English summary) J. Funct. Anal. 114 (1993), no. 1, 3258.

[LM] J. L. Lions, E. Magenes, ”Non-homogeneous Boundary Value Problems and Applications, I,” Springer-Verlag, New york/Berlin, 1972.

[QS] P. Quittner, P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states. Birkhuser Advanced Texts: Basler Lehrbcher. [Birkhuser Advanced Texts: Basel Textbooks] Birkhuser Verlag, Basel, 2007. xii+584 pp.

[S] R. Song, Sharp bounds on the density, Green function and jumping function of subordinate killed BM, Probab. Theory Relat. Fields 128, 606628 (2004)

[T] J. Tan, The Brezis-Nirenberg type problem involving the square root of the Laplacian. Calc. Var. Partial Differential Equations 42 (2011), no. 1-2, 2141.

[TOB] J.L. Taylor, K.A. Ott, R.M. Brown, The mixed problem in lipschitz domains with general decompositions of the boundary, to appear in Transactions of AMS.

School of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea
E-mail address: chwc1987@snu.ac.kr