Stochastic Maximum Principle under Probability Distortion *

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Abstract

Within the framework of Kahneman and Tversky’s cumulative prospective theory, this paper considers a continuous-time behavioral portfolio selection model, which includes both running and terminal terms in the objective functional. Despite the existence of S-shaped utility functions and probability distortions, a necessary condition for optimality is derived by stochastic maximum principle. Finally, the results are applied to various cases.

Key words: cumulative prospective theory, S-shaped utility function, probability distortion, stochastic maximum principle, behavioral portfolio optimization

1. Introduction

Expected utility theory (EUT) prevailed for a long time as the dominant preference measure under risk. Along with the theory in continuous financial portfolio selection problems, many approaches, such as dynamic programming, stochastic maximum principle, martingale and convex duality have been developed, see Merton (1969), Peng (1990), Duffie & Epstein (1992), Karatzas et al. (1991). The EUT, proposed by von Neumann & Morgenstern (1944), is premised on the tenets that the utilities of outcomes are weighted by their probabilities and decision makers are consistently risk averse. These, however, have been violated by substantial phenomena.

Allais paradox (Allais, 1953) argues that individuals evaluate (overweight or underweight the probability of) every outcome depending on the other outcomes of a prospect. Related studies in response to this fact are Fishburn (1988), Schmeidler

*This research is supported by Macao Science and Technology Development Fund FDCT 025/2016/A1.

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(1989), etc. On the other hand, risk-seeking behavior pervades decision problems, e.g., people would love to spend $x$ on the lottery with expected payoff no more than $x$. Likewise in loss situation, people usually prefer a possible large loss to a certain loss. Quite a few economists, such as Yaari (1987), have investigated the modification of EUT on these challenges.

The most notable effort to alternate EUT is Kahneman & Tversky’s prospect theory (PT) (1979), which takes investors’ psychology into account in the face of uncertainty. Later the PT was evolved into cumulative prospect theory (CPT) (Tversky & Kahneman, 1992). A significant difference between CPT and PT is that weighting is applied to the cumulative distribution functions, but not applied to the probabilities of individual outcomes; that is, the new version can be extended to the continuous distributions. The key elements of CPT are

- A benchmark (evaluated at terminal time $T$) serves as a base point to distinguish gains from losses. Without loss of generality, it is assumed to be 0 in this paper.

- Utility functions are concave for gains and convex for losses, and steeper for losses than for gains.

- Probability distortions (or weighting) are nonlinear transformation of the probability measures, which overweight small probabilities and underweight moderate and high probabilities.

There have been burgeoning research focuses on merging the CPT or PT into portfolio choice issues. Most of them are limited to the discrete-time setting, see for example Benartzi & Thaler (1995), Shefrin & Statman (2000), Levy & Levy (2003). The pioneering analytical research on continuous-time asset allocation featuring behavioral criteria is done by Jin & Zhou (2008). Since then, a few extensive works have been published, see He & Zhou (2011a, 2011b) and Jin & Zhou (2013). Jin & Zhou (2008) developed a new theory to work out the optimal terminal value in continuous-time CPT models, featuring both S-shaped utility functions and probability distortions. Their prominent idea is to change the decision variable from the random variable to its quantile function, such that the non-concave/convex objective turns to be a concave functional. The whole machinery is quite involved. To achieve the optimal control process that replicates the optimal terminal value, a further calculation is necessary. Nonetheless, their theory aims at a particular portfolio choice problem in a self-financing market (i.e. there is no consumption or income).

The main motivation of our work is to deal with probability distortion for model with consumption. In order to come closer to reality, bankruptcy is not allowed in our problem. Below are two examples which motivate our work.

Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ a filtered complete probability space on which is defined a standard $\mathcal{F}_t$-adapted $m$-dimensional Brownian motion $W_t \equiv (W^1_t, \cdots, W^m_t)^\top$ with $W_0 = 0$. It is assumed that $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$. 

Example 1.1 (Investment vs. Consumption) We illustrate a model from Pham (2009, Section 3.6.2). The financial market consists of a bond with price $S_0^0$ given by

$$dS_t^0 = r_t S_t^0 dt, \quad S_0^0 = s_0 > 0,$$

and $m$ stocks with prices per share $S_t^i$, $i = 1, \ldots, m$, satisfying the geometric Brownian motion

$$dS_t^i = S_t^i \left( b_t^i dt + \sum_{j=1}^{m} \sigma_{ij}^i dW_t^j \right), \quad S_0^i = s_i > 0.$$

The interest rate $r_t$, the vector $b_t = (b_1^1, \ldots, b_m^m)\top$ of stock appreciation rates, and the volatility matrix $\sigma_t = \{\sigma_{ij}^i\}_{1 \leq i, j \leq m}$ are taken to be $F_t$-progressively measurable stochastic processes.

In this financial market, bankruptcy is not allowed. The wealth process $X_t$ is required to be positive. Let $u_t^i$ (which may be negative, or may exceed 1) be the proportion of wealth invested in stock $i$, and $c_t$ be the consumption per unit of wealth at time $t$. The remaining proportion $1 - \sum_{i=1}^{m} u_t^i$ is invested in the bond. Then $X_t$ evolves according to the forward stochastic differential equation (SDE)

$$\begin{align*}
  dX_t &= \sum_{i=1}^{m} \frac{u_t^i X_t}{S_t^i} dS_t^i + \frac{(1 - \sum_{i=1}^{m} u_t^i) X_t}{S_t^i} dS_t^0 - c_t X_t dt \\
  &= X_t (r_t + (b_t - r_t 1_m)\top u_t - c_t) dt + X_t u_t \sigma_t dW_t, \quad t \in [0, T]; \\
  X_0 &= x_0 > 0,
\end{align*}$$

where $u_t = (u_1^t, \ldots, u_m^t)\top$ and $c_t$ together is the portfolios of the investor. It should be emphasized an important point concerning the way we specify our trading strategies. Like in most papers in the literature, the model define a trading strategy or portfolio, say $u_t$, as the proportions or fractions of wealth allocated to different assets, see Merton (1969), Karatzas et al. (1991), Karatzas & Shreve (1998).

Within the continuous-time CPT framework of Jin & Zhou (2008), the objective is to find the optimal consumption path $c_t$ and portfolio strategy on shares $u_t$ such that the prospective preference

$$J(c_t, u_t) = \int_0^T \int_0^\infty \mathbb{P}\{\zeta(c_t X_t) > y\} dy dt + \int_0^\infty w(\mathbb{P}\{I(X_T) > x\}) dx,$$

$t \in \{0, 1, \ldots, T\}$, augmented by all the null sets. Throughout this paper $A^\top$ denotes the transpose of a matrix $A$; $a^\pm$ denote the positive and negative parts of the real number $a$.

Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a copy of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any random variable $\xi$ over $(\Omega, \mathcal{F}, \mathbb{P})$ we denote by $\tilde{\xi}$ a copy of $\xi$ defined over $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. The expectation $\tilde{\mathbb{E}}[\cdot] = \int_{\tilde{\Omega}} \cdot d\tilde{\mathbb{P}}$ acts over the variable $\tilde{\omega} \in \tilde{\Omega}$ only. In what follows, we replace $F_Y(Y) = \tilde{\mathbb{P}}\{Y \leq Y\}$ by $F_Y(Y)$ for convenience.
achieves the maximum. Here \( \zeta(\cdot), l(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) are the investor’s utility functions for consumption and terminal wealth, respectively. \( w(\cdot) : [0, 1] \to [0, 1] \) represents the distortion of probability. There is no distortion on consumption. In fact, the prospective functional could be written as

\[
J(c, u) = \mathbb{E} \int_0^T \zeta(c_t X_t) dt + \mathbb{E} \left( l(X_T) w'(1 - F_{X_T}(X_T)) \right).
\]

**Example 1.2 (Investment vs. Gambling)** In addition to the investment in aforesaid market, an investor is allowed to buy lottery tickets. Here the wealth is required to be positive as well. For simplicity, let \( c_t \in \mathbb{R}^+ \) be the wager per unit of wealth at time \( t \) and \( K_t \) be the odds of winning. For instance, if \( K_t \) is 8 with 0.1 and –1 with 0.9, the investor will win \( 8c_t X_t \) with probability 0.1 and lose the wager \( c_t X_t \) with probability 0.9 at \( t \). The value process is governed by

\[
\begin{align*}
    dX_t &= X_t (r_t + (b_t - r_t \mathbf{1}_m)^T u_t) dt + X_t u_t^T \sigma_t dW_t + K_t c_t X_t dt, \quad t \in [0, T]; \\
    X_0 &= x_0 > 0,
\end{align*}
\]

where \( u_t = (u_{t1}, \ldots, u_{tm})^T \) and \( c_t \) are the portfolios of the investor. For this case, the portfolio selection problem is to find the most preferable portfolios to maximize the distorted expected payoff

\[
J(c, u) = \int_0^T \left( \int_0^\infty \varpi_+(\mathbb{P}\{\zeta_+(K_t^+ c_t X_t) > y\}) dy \right) dt + \int_0^\infty \mathbb{E} \left( \mathbb{P}\{l(X_T) > x\} \right) dx,
\]

where \( \zeta_+(\cdot), \zeta_-(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) are utility functions measure the gains and losses of gambling, respectively. \( \varpi_+(\cdot), \varpi_-(\cdot) : [0, 1] \to [0, 1] \) represent the distortions in probability for the gains and losses, respectively. \( w(\cdot) \) and \( l(\cdot) \) are as same as those in the last example. Straightforwardly, the distorted payoff could be written as

\[
J(c, u) = \mathbb{E} \int_0^T \left( \zeta_+(K_t^+ c_t X_t) \varpi'_+ \left(1 - F_{K_t^+ c_t X_t}(K_t^+ c_t X_t)\right) \right) dt - \zeta_-(K_t^- c_t X_t) \varpi'_- \left(1 - F_{K_t^- c_t X_t}(K_t^- c_t X_t)\right) dt + \mathbb{E} \left( l(X_T) w'(1 - F_{X_T}(X_T)) \right).
\]

The objective is to find an optimal portfolio \((u, c)\) to maximize \(J\).

In general, we will consider optimization problems with probability distortions and running utilities. Resulting from the distorted probability, time-consistency of the conditional expectation with respect to a filtration is invalid. Thus the dynamic programming approach is failed upon the underlying problem. On the other hand,
the quantile formulation introduced in Jin & Zhou (2008) is feasible to those of the control being a random variable rather than a stochastic process. It doesn’t work on the running terms. In this paper, we therefore employ the stochastic maximum principle to conquer the aforementioned difficulties, and strive to acquire the necessary condition of the optimal control process for the general optimization problems.

The rest of this article is organized as follows. Next section will formulate a general continuous-time portfolio selection model under the CPT, featuring S-shaped utility functions and probability distortions. After that, the main results of this paper are presented. The stochastic maximum principle is used to obtain the necessary condition for optimality in Section 3. Finally, we list three interesting cases.

2. Problem Formulation and Main Result

We define a positive state process

\[
\begin{aligned}
    dX_t &= b(t, u_t, X_t)dt + \sigma(t, u_t, X_t)dW_t \\
    X_0 &= x_0 > 0,
\end{aligned}
\]

and the agent’s prospective functional

\[
    J(u) = \mathbb{E} \int_0^T \left( \zeta_+(u_t^+) \omega_+'(1 - F_{u_t^+}(u_t^+)) - \zeta_-(u_t^-) \omega_-'(1 - F_{u_t^-}(u_t^-)) \right) dt \\
    &+ \mathbb{E} \left( l(X_T)w'(1 - F_{X_T}(X_T)) \right),
\]

where \( u \) is a control process taking values in a convex set \( U \subseteq \mathbb{R} \). According to CPT, the following assumptions will be in force throughout this paper, where \( x \) denotes the state variable, \( u \) denotes the control variable.

(H.1) \( b(\cdot, \cdot, \cdot) : [0, T] \times U \times \mathbb{R}^+ \rightarrow \mathbb{R}, \sigma(\cdot, \cdot, \cdot) : [0, T] \times U \times \mathbb{R}^+ \rightarrow \mathbb{R} \), are continuously differentiable with respect to \((u, x)\). The first derivatives of \( b, \sigma \) with respect to \((x, u)\) are Lipschitz continuous. We further assume \( b(t, u, 0) = \sigma(t, u, 0) = 0 \).

(H.2) \( \zeta_{\pm}(\cdot), l(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) are supposed to be differentiable, strictly increasing, strictly concave, with \( \zeta_{\pm}(0) = l(0) = 0 \) and the Inada conditions \( \zeta_{\pm}'(0+) = l'(0+) = \infty \).

(H.3) \( \omega_{\pm}(\cdot), w(\cdot) : [0, 1] \rightarrow [0, 1] \), are differentiable and strictly increasing, with \( \omega_{\pm}(0) = w(0) = 0, \omega_{\pm}(1) = w(1) = 1 \). Moreover, the first derivatives of \( \omega_{\pm}(\cdot), w(\cdot) \) are all bounded.

Normally, the utility function is taken to be \( l(x) = x^\gamma, 0 < \gamma < 1 \). We might adopt the decumulative weighting function used in Lopes’s SP/A theory (Lopes, 1987) for the distortion of probability. It takes the form: \( w(p) = \nu p^{\alpha+1} + (1 - \nu)[1 - (1 - p)^{\beta+1}] \).
where \(0 \leq \nu \leq 1\) and \(\alpha, \beta \geq 0\). Clearly, \(p^{\alpha+1}\) and \(1 - (1 - p)^{\beta+1}\) are convex and concave functions, respectively. Define

\[
U = \left\{ u : [0, T] \times \Omega \to U \mid u_t \text{ is } \mathcal{F}_t\text{-adapted and } \mathbb{E} \int_0^T |u_t|^4 dt < \infty \right\}.
\]

**Definition 2.1** A control process \(u \in U\) is said to be admissible, and \((u, X)\) is called an admissible pair, if

1. \(X\) is the unique solution of equation (2.1) under \(u\);
2. both \(u^+_t\) and \(u^-_t\) possess continuous (except at 0) distribution functions;
3. \(\mathbb{E} \int_0^T \left( |\zeta_+(u^+_t)\varpi_+(1 - F_{u^+_t}(u^+_t))|^8 + |\zeta_-(u^-_t)\varpi_-(1 - F_{u^-_t}(u^-_t))|^8 \right) dt < \infty\).
4. \(\mathbb{E} \int_0^T \left( \left| \frac{d}{du} \ln \zeta_+(u^+_t) \right|^8 + \left| \frac{d}{du} \ln \zeta_-(u^-_t) \right|^8 + \left| \zeta''_+(u^+_t) \right|^4 + \left| \zeta''_-(u^-_t) \right|^4 \right) dt < \infty\).

The set of all admissible controls is denoted by \(U_{ad}\).

**Remark 2.2** If \(\zeta_\pm(u) = \frac{u^\gamma}{\gamma}, 0 < \gamma < 1\), the condition (4) is satisfied provided that the admissible control are restrict to those with \(\mathbb{E} \int_0^T |u_t|^{-8} dt < \infty\).

Meanwhile, some technical assumptions for the terminal state are in force throughout this paper.

**Assumption 2.3** The terminal state \(X_T\) corresponding to the control process \(u \in U_{ad}\) is supposed to has continuous distribution function. Besides,

\[
\mathbb{E} |l(X_T)w'(1 - F_{X_T}(X_T))|^8 + \mathbb{E} \left| \frac{d}{dx} \ln l(X_T) \right|^8 + \mathbb{E} |l''(X_T)|^4 < \infty.
\]

The condition (3) in Definition 2.1 as well as the first term of inequality (2.3) guarantee that the prospective functional \(J(u)\) is always finite. Generally, in the case that the supremum of \(J\) is finite with bounded initial investment \(x_0\), the model is regarded as well-posed; otherwise, it is ill-posed.

**Remark 2.4** If \(l(x) = \frac{x^\gamma}{\gamma}, 0 < \gamma < 1\), \(\hat{b}(t, u, x) = x^{-1}b(t, u, x)\) and \(\hat{\sigma}(t, u, x) = x^{-1}\sigma(t, u, x)\) are bounded on \([0, T] \times U \times \mathbb{R}^+\), then \(\mathbb{E} \left| \frac{d}{dx} \ln l(X_T) \right|^8\) and \(\mathbb{E} |l''(X_T)|^4\) are finite. In fact, applying Itô’s formula to \(X_t^{4\gamma-8}\), we finally get

\[
\mathbb{E} \left( l''(X_T)^4 \right) = (\gamma - 1)^4 X_0^{4\gamma-8} \cdot \mathbb{E} \exp \left\{ (4\gamma - 8) \left( \int_0^T (\hat{b}(t, u_t, X_t) - \frac{1}{2} \hat{\sigma}^2(t, u_t, X_t)) dt + \int_0^T \hat{\sigma}(t, u_t, X_t) dW_t \right) \right\},
\]

which is bounded. So is \(\mathbb{E} \left| \frac{d}{dx} \ln l(X_T) \right|^8\).
**Problem.** Our optimal control problem is to find \( \bar{u} \in \mathcal{U}_{ad} \) such that

\[
J(\bar{u}) = \max_{u \in \mathcal{U}_{ad}} J(u).
\]

Let \((\bar{u}, \bar{X})\) be an optimal pair of the problem (2.4). Before stating the main result of this paper, we formulate the adjoint equation

\[
dp_t = -\left(b_x(t, \bar{u}_t, \bar{X}_t) \sigma_x(t, \bar{u}_t, \bar{X}_t) q_t dt + q_t dW_t, \right.
\]

\[
p_T = l'(\bar{X}_T)w'(1 - F_{\bar{X}_T}(\bar{X}_T)).
\]

**Theorem 2.5** If \( \bar{u} \) is the optimal control with the state trajectory \( \bar{X} \), then there exists a pair \((p, q)\) of adapted processes which satisfies (2.5) such that a.e. \( t \in [0, T] \),

\[
p_t b_u(t, \bar{u}_t, \bar{X}_t) + \sigma_u(t, \bar{u}_t, \bar{X}_t) q_t = \begin{cases} 
-\zeta^+_{\cdot} (\bar{u}_t^+) \varpi^+_{\cdot} (1 - F_{\bar{u}_t^+}(\bar{u}_t^+)) & \text{if } \bar{u}_t > 0, \\
-\zeta^-_{\cdot} (\bar{u}_t^-) \varpi^-_{\cdot} (1 - F_{\bar{u}_t^-}(\bar{u}_t^-)) & \text{if } \bar{u}_t < 0,
\end{cases}
\]

**Remark 2.6** Theorem 2.5 remains true when \( \zeta_{\pm}(\cdot) \), \( l(\cdot) \) are replaced by functions which are twice continuously differentiable and get zero value at zero. Additionally, the model can be generalized. For instance, we may use \( u_t^\pm X_t \) instead of \( u_t^\pm \) in the objective functional. Again, the risk preference can be defined as

\[
J(u) = \mathbb{E} \int_0^T \left( f(t, u_t, X_t) + \zeta^+_{\cdot} (u_t^+) \varpi^+_{\cdot} (1 - F_{u_t^+}(u_t^+)) - \zeta^-_{\cdot} (u_t^-) \varpi^-_{\cdot} (1 - F_{u_t^-}(u_t^-)) \right) dt + \mathbb{E} \left( l(X_T)w'(1 - F_{X_T}(X_T)) \right),
\]

where \( f(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}^m \times \mathbb{R}^+ \rightarrow \mathbb{R} \) is supposed to be twice continuously differentiable with respect to \( u \) and \( x \). The necessity of optimality for such a problem is

\[
p_t b_u(t, \bar{u}_t, \bar{X}_t) + \sigma_u(t, \bar{u}_t, \bar{X}_t) q_t + \partial_u f(t, \bar{u}_t, \bar{X}_t)
\]

\[
= \begin{cases} 
-\zeta^+_{\cdot} (\bar{u}_t^+) \varpi^+_{\cdot} (1 - F_{\bar{u}_t^+}(\bar{u}_t^+)) & \text{if } \bar{u}_t > 0, \\
-\zeta^-_{\cdot} (\bar{u}_t^-) \varpi^-_{\cdot} (1 - F_{\bar{u}_t^-}(\bar{u}_t^-)) & \text{if } \bar{u}_t < 0,
\end{cases}
\]

\( a.e.t \in [0, T], \ a.s., \) where

\[
dp_t = -\left(b_x(t, \bar{u}_t, \bar{X}_t) \sigma_x(t, \bar{u}_t, \bar{X}_t) q_t + \partial_x f(t, \bar{u}_t, \bar{X}_t) \right) dt + q_t dW_t,
\]

\[
p_T = l'(\bar{X}_T)w'(1 - F_{\bar{X}_T}(\bar{X}_T)).
\]

We refer to Peng (1990) for the classical optimal control problem of which cost function is \( f(t, u_t, X_t) \). The conclusion can be proved by the same argument to be given in the next section, combining with those for classical stochastic maximum principle (Yong & Zhou 1999).
3. Proof of the Main Result

Suppose \( \varepsilon \in [0, 1) \). Take \( u, \in \mathcal{U} \) such that \( u_t \) has the same sign as \( \bar{u}_t \) \( (u_t = 0 \text{ if } \bar{u}_t = 0) \). Define

\[
u^\varepsilon = \bar{u} + \varepsilon (u - \bar{u}).\]

The convexity of \( U \) guarantees that \( u^\varepsilon \in \mathcal{U} \), and obviously,

\[
J(\bar{u}) - J(u^\varepsilon) \geq 0.
\]

Denote the state trajectory corresponding to the perturbation \( u^\varepsilon \) of \( \bar{u} \) by \( X^\varepsilon \).

In the rest of this paper, we adopt the short-hand notations

\[
v_t = u_t - \bar{u}_t, \quad \tilde{\phi}(t) = \phi(t, \bar{u}_t, \bar{X}_t).
\]

Now we proceed to proving Theorem 3.5 by a few lemmas.

**Lemma 3.1** Under condition (H.1), we have

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} |X^\varepsilon_t - \bar{X}_t|^4 \right) = 0.
\]

**Proof:** From the state equation, one has

\[
d(X^\varepsilon_t - \bar{X}_t) = \left( b(t, \bar{u}_t + \varepsilon v_t, X^\varepsilon_t) - b(t, \bar{u}_t, \bar{X}_t) \right) dt
+ \left( \sigma(t, \bar{u}_t + \varepsilon v_t, X^\varepsilon_t) - \sigma(t, \bar{u}_t, \bar{X}_t) \right) dW_t.
\]

By Condition (H.1), Cauchy-Schwarz inequality as well as Burkholder-Davis-Gundy inequality, we obtain

\[
\mathbb{E} \sup_{0 \leq t \leq T} |X^\varepsilon_t - \bar{X}_t|^4 \leq 8 \mathbb{E} \sup_{0 \leq t \leq T} \left\{ \int_0^t \left( b(s, \bar{u}_s + \varepsilon v_s, X^\varepsilon_s) - b(s, \bar{u}_s, \bar{X}_s) \right) ds \right\}^4
+ 8 \mathbb{E} \sup_{0 \leq t \leq T} \left\{ \int_0^t \left( \sigma(s, \bar{u}_s + \varepsilon v_s, X^\varepsilon_s) - \sigma(s, \bar{u}_s, \bar{X}_s) \right) dW_s \right\}^4
\]

\[
\leq 8 \mathbb{E} \sup_{0 \leq t \leq T} \left\{ \int_0^t \left( b(s, \bar{u}_s + \varepsilon v_s, X^\varepsilon_s) - b(s, \bar{u}_s, \bar{X}_s) \right) ds \right\}^4
+ 8 \mathbb{E} \sup_{0 \leq t \leq T} \left\{ \int_0^t \left( \sigma(s, \bar{u}_s + \varepsilon v_s, X^\varepsilon_s) - \sigma(s, \bar{u}_s, \bar{X}_s) \right) dW_s \right\}^4
\]

\[
\leq 8 \mathbb{E} \sup_{0 \leq t \leq T} \left\{ \int_0^t \left( b(s, \bar{u}_s + \varepsilon v_s, X^\varepsilon_s) - b(s, \bar{u}_s, \bar{X}_s) \right)^2 ds \cdot \int_0^t 1^2 ds \right\}^2
+ 8 K_T \mathbb{E} \left\{ \int_0^T \left( \sigma(s, \bar{u}_s + \varepsilon v_s, X^\varepsilon_s) - \sigma(s, \bar{u}_s, \bar{X}_s) \right)^2 ds \right\}^2.
\]
The result follows by Gronwall’s inequality.

\[ \leq 8 T^3 \mathbb{E} \sup_{0 \leq s \leq T} \int_0^t \left( b(s, \bar{u}_s + \varepsilon v_s, X^\varepsilon_s) - b(s, \bar{u}_s, \bar{X}_s) \right)^4 ds \\
+ 8 T K_T \mathbb{E} \int_0^T \left( \sigma(s, \bar{u}_s + \varepsilon v_s, X^\varepsilon_s) - \sigma(s, \bar{u}_s, \bar{X}_s) \right)^4 ds \\
\leq K \mathbb{E} \int_0^T (|X^\varepsilon_s - \bar{X}_s| + \varepsilon |v_s|)^4 ds \\
\leq K \int_0^T \mathbb{E} \sup_{0 \leq s \leq t} |X^\varepsilon_s - \bar{X}_s|^4 dt + K \mathbb{E} \int_0^T \varepsilon^4 |v_s|^4 ds \]

The result follows by Gronwall’s inequality.

Lemma 3.2 Suppose that \( \bar{X}_T \) possess continuous distribution function. Then,
\[ \lim_{\varepsilon \to 0} \mathbb{E} |F_{X^\varepsilon_T} - F_{\bar{X}_T}|^4 = 0. \]

Proof: Notice that there is a subsequence \( \varepsilon^* \subset \varepsilon \) such that
\[ \lim_{\varepsilon^* \to 0} \mathbb{E} |F_{X^\varepsilon^*_T} - F_{\bar{X}_T}|^4 = \lim_{\varepsilon \to 0} \mathbb{E} |F_{X^\varepsilon_T} - F_{\bar{X}_T}|^4, \]
which always exists. \( X^\varepsilon_T \overset{L^4}{\to} \bar{X}_T \) implies that \( X^\varepsilon^*_T \overset{L^4}{\to} \bar{X}_T \). Moreover, there is a subsequence (of \( X^\varepsilon^*_T \)) \( \bar{X}^\varepsilon^*_T \) converging to \( \bar{X}_T \) almost surely. We then have
\[ \lim_{\varepsilon^* \to 0} \mathbb{E} |F_{X^\varepsilon^*_T} - F_{\bar{X}_T}|^4 = \lim_{\varepsilon^* \to 0} \mathbb{E} |F_{X^\varepsilon^*_T} - F_{\bar{X}_T}|^4. \]

As a consequence, the problem is turned to demonstrate
\[ (3.1) \quad \lim_{\varepsilon^* \to 0} \mathbb{E} |F_{X^\varepsilon^*_T} - F_{\bar{X}_T}|^4 = 0, \]
given \( X^\varepsilon^*_T \overset{a.s.}{\to} \bar{X}_T \). Note that
\[ |F_{X^\varepsilon^*_T} - F_{\bar{X}_T}| \leq |F_{X^\varepsilon^*_T} - F_{\bar{X}_T}|^4 + |F_{X^\varepsilon^*_T} - F_{\bar{X}_T}|^4 \leq \sup_x |F_{X^\varepsilon^*_T} - F_{\bar{X}_T}|^4 \to 0. \]
Equality (3.1) then follows by the dominated convergence theorem.

Remark 3.3 One can also verify that for all \( \lambda, \mu \in [0, 1] \),
\[ \lim_{\varepsilon \to 0} \mathbb{E} \left| \lambda F_{X^\varepsilon_T} (\mu X^\varepsilon_T + (1 - \mu) \bar{X}_T) + (1 - \lambda) F_{\bar{X}_T} (\mu X^\varepsilon_T + (1 - \mu) \bar{X}_T) - F_{\bar{X}_T} (\bar{X}_T) \right|^4 = 0. \]
Moreover, if \( \bar{u}_t^\pm \) have continuous (except at 0) distribution functions, then
\[ \lim_{\varepsilon \to 0} \mathbb{E} |F_{u^+_t} - F_{\bar{u}^+_t}|^4 = 0, \quad \forall t \in [0, T]. \]
Lemma 3.4 Let $Z_t$ be such that

\begin{equation}
\begin{aligned}
    dZ_t &= (\tilde{b}_x(t)Z_t + \tilde{b}_u(t)v_t)dt + (\tilde{\sigma}_x(t)Z_t + \tilde{\sigma}_u(t)v_t)dW_t \\
    Z_0 &= 0.
\end{aligned}
\end{equation}

Then under condition (H.1), it holds that

\begin{equation}
\lim_{\varepsilon \to 0} \mathbb{E} \left( \sup_{0 \leq t \leq T} \left| \frac{X^\varepsilon_t - \bar{X}_t}{\varepsilon} - Z_t \right|^2 \right) = 0.
\end{equation}

Proof: Let $y^\varepsilon_t = \frac{X^\varepsilon_t - \bar{X}_t}{\varepsilon} - Z_t$, then

\begin{align*}
    dy^\varepsilon_t &= \left\{ \frac{1}{\varepsilon} \left( b(t, \bar{u}_t + \varepsilon v_t, \bar{X}_t + \varepsilon(Z_t + y^\varepsilon_t)) - b(t, \bar{u}_t, \bar{X}_t) \right) - b_x(t, \bar{u}_t, \bar{X}_t)Z_t \\
    & \quad - b_u(t, \bar{u}_t, \bar{X}_t)v_t \right\} dt + \left\{ \frac{1}{\varepsilon} \left( \sigma(t, \bar{u}_t + \varepsilon v_t, \bar{X}_t + \varepsilon(Z_t + y^\varepsilon_t)) - \sigma(t, \bar{u}_t, \bar{X}_t) \right) \\
    & \quad - \sigma_x(t, \bar{u}_t, \bar{X}_t)Z_t - \sigma_u(t, \bar{u}_t, \bar{X}_t)v_t \right\} dW_t.
\end{align*}

One can easily show that $\mathbb{E} \int_0^T Z_t^4 dt + \mathbb{E} \int_0^T |y^\varepsilon_t|^4 dt < \infty$. Since the drift and diffusion coefficients of $y^\varepsilon_t$ are similar, we focus on the drift one only.

\begin{align*}
    \frac{1}{\varepsilon} \left( b(t, \bar{u}_t + \varepsilon v_t, \bar{X}_t + \varepsilon(Z_t + y^\varepsilon_t)) - b(t, \bar{u}_t, \bar{X}_t) \right) - b_x(t, \bar{u}_t, \bar{X}_t)Z_t - b_u(t, \bar{u}_t, \bar{X}_t)v_t & \\
    &= \int_0^1 b_x(t, \bar{u}_t + \lambda \varepsilon v_t, \bar{X}_t + \lambda \varepsilon(Z_t + y^\varepsilon_t))(Z_t + y^\varepsilon_t) d\lambda \\
    & \quad + \int_0^1 b_u(t, \bar{u}_t + \lambda \varepsilon v_t, \bar{X}_t + \lambda \varepsilon(Z_t + y^\varepsilon_t))v_t d\lambda - b_x(t, \bar{u}_t, \bar{X}_t)Z_t - b_u(t, \bar{u}_t, \bar{X}_t)v_t \\
    &= \int_0^1 \left( b_x(t, \bar{u}_t + \lambda \varepsilon v_t, \bar{X}_t + \lambda \varepsilon(Z_t + y^\varepsilon_t)) - b_x(t, \bar{u}_t, \bar{X}_t) \right)Z_t d\lambda \\
    & \quad + \int_0^1 \left( b_u(t, \bar{u}_t + \lambda \varepsilon v_t, \bar{X}_t + \lambda \varepsilon(Z_t + y^\varepsilon_t)) - b_u(t, \bar{u}_t, \bar{X}_t) \right)v_t d\lambda \\
    & \quad + \int_0^1 b_x(t, \bar{u}_t + \lambda \varepsilon v_t, \bar{X}_t + \lambda \varepsilon(Z_t + y^\varepsilon_t))y^\varepsilon_t d\lambda.
\end{align*}

By using Condition (H.1) as well as Cauchy-Schwarz inequality, we conclude that the first two terms on the right hand side of the above equality tend to zero in
same treatment, one has

The proof for the second term is similar. Dealing with the diffusion part of the Davis-Gundy inequality, in addition to the boundedness condition of $b_x$, we have

\[
\mathbb{E} \int_0^T \left\{ \int_0^1 \left( b_x(t, \bar{u}_t + \lambda v_t, \bar{X}_t + \lambda (Z_t + y^\varepsilon_t)) - b_x(t, \bar{u}_t, \bar{X}_t) \right) \mathcal{Z}_t \frac{d\lambda}{\lambda^2} \right\}^2 dt \\
\leq \mathbb{E} \int_0^T \left\{ \int_0^1 K \lambda \varepsilon(|Z_t + y^\varepsilon_t| + |v_t|) \mathcal{Z}_t \frac{d\lambda}{\lambda^2} \right\}^2 dt \\
\leq \mathbb{E} \int_0^T K \left\{ \int_0^1 \lambda \varepsilon(|Z_t + y^\varepsilon_t| + |v_t|) \frac{d\lambda}{\lambda^2} \right\} \cdot Z_t^2 dt \\
\leq \mathbb{E} \int_0^T K \left\{ \int_0^1 \lambda \varepsilon(|Z_t + y^\varepsilon_t| + |v_t|) \frac{d\lambda}{\lambda^2} \right\} \cdot \int_0^1 \mathcal{Z}_t \mathcal{Z}_t^2 dt \\
\leq \mathbb{E} \int_0^T \left\{ K \int_0^1 \left( \lambda \varepsilon(|Z_t + y^\varepsilon_t|) \right)^2 \frac{d\lambda}{\lambda^2} + K \int_0^1 \left( \lambda \varepsilon(v_t) \right)^2 \mathcal{Z}_t \mathcal{Z}_t^2 dt \right\} \\
\leq K \int_0^T \mathbb{E} \left\{ \int_0^1 \left( \lambda \varepsilon(|Z_t + y^\varepsilon_t|) \right)^2 \frac{d\lambda}{\lambda^2} \right\} \cdot \mathbb{E} \mathcal{Z}_t \mathcal{Z}_t^2 dt \\
\leq K \left\{ \int_0^T \mathbb{E} \left\{ \int_0^1 \left( \lambda \varepsilon(|Z_t + y^\varepsilon_t|) \right)^2 \frac{d\lambda}{\lambda^2} \right\} \cdot \mathbb{E} \mathcal{Z}_t \mathcal{Z}_t^2 dt \right\}^{\frac{1}{2}} \\
+ K \left\{ \int_0^T \mathbb{E} \left\{ \int_0^1 \left( \lambda \varepsilon(v_t) \right)^2 \mathcal{Z}_t \mathcal{Z}_t^2 \right\} \cdot \mathbb{E} \mathcal{Z}_t \mathcal{Z}_t^2 dt \right\}^{\frac{1}{2}} \\
\to 0 \text{ as } \varepsilon \to 0.
\]

The proof for the second term is similar. Dealing with the diffusion part of $y^\varepsilon_t$ by the same treatment, one has

\[
y_t^\varepsilon = \int_0^t \int_0^1 b_x(s, \bar{u}_s + \lambda v_s, \bar{X}_s + \lambda (Z_s + y^\varepsilon_s)) y^\varepsilon_s d\lambda ds + \int_0^t \rho^\varepsilon_s ds \\
+ \int_0^t \int_0^1 \sigma_x(s, \bar{u}_s + \lambda v_s, \bar{X}_s + \lambda (Z_s + y^\varepsilon_s)) y^\varepsilon_s d\lambda dW_s + \int_0^t \tau^\varepsilon_s dW_s,
\]

where $\mathbb{E} \int_0^T |\rho^\varepsilon_t|^2 dt, \mathbb{E} \int_0^T |\tau^\varepsilon_t|^2 dt$ go to zero as $\varepsilon$ goes to zero. Using the Burkholder-Davis-Gundy inequality, in addition to the boundedness condition of $b_x, \sigma_x$, finally we have

\[
\mathbb{E} \sup_{0 \leq t \leq T} |y^\varepsilon_t|^2 \leq K \int_0^T \sup_{0 \leq s \leq t} |y^\varepsilon_s|^2 dt + \mathbb{E} \int_0^T |\rho^\varepsilon_s|^2 ds + \mathbb{E} \int_0^T |\tau^\varepsilon_s|^2 ds.
\]

Applying Gronwall’s inequality, the result follows. □
Lemma 3.5  The Gateaux derivative of the objective functional $J$ is given by

$$\frac{d}{d\varepsilon} J(\bar{u} + \varepsilon v) \big|_{\varepsilon=0} = \mathbb{E} \int_0^T \left( \zeta'_+ (\bar{u}_t^+) \varphi'_+ (1 - F_{\bar{u}_t^+} (\bar{u}_t^+)) v_t 1_{\bar{u}_t^+ > 0} \right.$$

$$\left. + \zeta'_- (\bar{u}_t^-) \varphi'_- (1 - F_{\bar{u}_t^-} (\bar{u}_t^-)) v_t 1_{\bar{u}_t^- < 0} \right) + \mathbb{E}\left( l'(\bar{X}_T) w'(1 - F_{\bar{X}_T}(\bar{X}_T)) Z_T \right).$$

Proof: Recall the objective functional (2.2). Those three integrals are alike in structure. We discuss the last term in details. Rewrite the last integral in (2.2) as below,

$$\mathbb{E}(l(X_T)w'(1 - F_{X_T}(X_T))) = \int_0^\infty l(x) w' (1 - F_{X_T}(x)) dF_{X_T}(x)$$

$$= \int_0^\infty l'(x) w(1 - F_{X_T}(x)) dx.$$

We have its Gateaux derivative

$$I \triangleq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \int_0^\infty l'(x) w(1 - F_{X_T^\varepsilon}(x)) dx - \int_0^\infty l'(x) w(1 - F_{\bar{X}_T}(x)) dx \right)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\infty l'(x) \int_0^1 w'(1 - \lambda F_{X_T^\varepsilon}(x) - (1 - \lambda) F_{\bar{X}_T}(x)) d\lambda (F_{X_T^\varepsilon}(x) - F_{\bar{X}_T}(x)) dx,$$

For convenience, let

$$g_\varepsilon(x) = l'(x) \int_0^1 w'(1 - \lambda F_{X_T^\varepsilon}(x) - (1 - \lambda) F_{\bar{X}_T}(x)) d\lambda, \ x > 0,$$

and define

$$G_\varepsilon(x) = \int_0^x g_\varepsilon(y) dy, \ G_\varepsilon(0+) = 0.$$

Then it becomes

$$I = \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^\infty g_\varepsilon(x) (F_{X_T^\varepsilon}(x) - F_{\bar{X}_T^\varepsilon}(x)) dx$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^\infty \left( F_{\bar{X}_T}(x) - F_{X_T^\varepsilon}(x) \right) dG_\varepsilon(x)$$

$$= - \lim_{\varepsilon \to 0} \varepsilon^{-1} \int_0^\infty G_\varepsilon(x) d(F_{X_T^\varepsilon}(x) - F_{\bar{X}_T^\varepsilon}(x))$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{-1} \mathbb{E}(G_\varepsilon(X_T^\varepsilon) - G_\varepsilon(\bar{X}_T))$$

$$= \lim_{\varepsilon \to 0} \varepsilon^{-1} \mathbb{E} \int_0^1 g_\varepsilon(\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T) (X_T^\varepsilon - \bar{X}_T) d\mu.$$
The other terms can be studied similarly. Hence the Gateaux derivative of $J$ is translated to be
\[
\frac{d}{d\varepsilon} J(\bar{u} + \varepsilon v) \big|_{\varepsilon = 0} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \int_0^T \int_0^1 g_1^\varepsilon (\tau u_t^\varepsilon + (1 - \tau) \bar{u}_t)(u_t^\varepsilon - \bar{u}_t) \mathbb{I}_{\bar{u}_t > 0} d\tau dt \\
+ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \int_0^T \int_0^1 g_2^\varepsilon (-\tau u_t^\varepsilon - (1 - \tau) \bar{u}_t)(u_t^\varepsilon - \bar{u}_t) \mathbb{I}_{\bar{u}_t < 0} d\tau dt \\
(3.4) + \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{E} \int_0^1 g_\varepsilon (\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T)(X_T^\varepsilon - \bar{X}_T) d\mu,
\]
where
\[
g_1^\varepsilon(x) = \zeta_+^\varepsilon(x) \int_0^1 \varphi_+(1 - \lambda F_{(u_t^\varepsilon)}^\varepsilon)(x) - (1 - \lambda) F_{u_t^\varepsilon}(x) d\lambda, \quad x > 0,
\]
\[
g_2^\varepsilon(x) = \zeta_-^\varepsilon(x) \int_0^1 \varphi_-(1 - \lambda F_{(u_t^\varepsilon)}^-)(x) - (1 - \lambda) F_{u_t^\varepsilon}(x) d\lambda, \quad x > 0.
\]
Next, we go back to the calculation of $I$. To prove
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left| \int_0^1 g_\varepsilon (\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T) d\mu - l'(\bar{X}_T)w\left(1 - F_{X_T}(\bar{X}_T)\right) \right|^2 = 0,
\]
we adopt the shorthand notation for simplicity,
\[
J^{\varepsilon;\lambda;\mu} = \lambda F_{X_T^\varepsilon}(\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T) + (1 - \lambda) F_{X_T}(\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T),
\]
\[
J_1 = w'(1 - J^{\varepsilon;\lambda;\mu}) - w'(1 - F_{X_T}(\bar{X}_T)).
\]
Condition (H.3) implies that $J_1$ is bounded. Therefore, by Cauchy-Schwarz inequality,
\[
\lim_{\varepsilon \to 0} \mathbb{E} \left| \int_0^1 g_\varepsilon (\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T) d\mu - l'(\bar{X}_T)w\left(1 - F_{X_T}(\bar{X}_T)\right) \right|^2 \\
\leq \lim_{\varepsilon \to 0} \mathbb{E} \left| \int_0^1 l'(\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T) \int_0^1 w\left(1 - \lambda F_{X_T^\varepsilon}(\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T) - (1 - \lambda) F_{X_T}(\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T) \right) d\lambda d\mu - l'(\bar{X}_T)w\left(1 - F_{X_T}(\bar{X}_T)\right) \right|^2 \\
\leq \lim_{\varepsilon \to 0} K \mathbb{E} \left| l'(\bar{X}_T) \right|^2 \cdot \left| \int_0^1 \int_0^1 w\left(1 - J^{\varepsilon;\lambda;\mu}\right) d\lambda d\mu - w'(1 - F_{X_T}(\bar{X}_T)) \right|^2 \\
\leq \lim_{\varepsilon \to 0} K \mathbb{E} \int_0^1 \int_0^1 l''(\tau \mu X_T^\varepsilon + (1 - \tau \mu) \bar{X}_T) \left( X_T^\varepsilon - \bar{X}_T \right) \mu^2 d\tau d\mu \\
+ K \left( \mathbb{E} \left| l'(\bar{X}_T) \right|^4 \right)^{\frac{1}{2}} \cdot \lim_{\varepsilon \to 0} \mathbb{E} \left| \int_0^1 \int_0^1 w'\left(1 - J^{\varepsilon;\lambda;\mu}\right) - w'(1 - F_{X_T}(\bar{X}_T)) \right|^4 d\lambda d\mu \right)^{\frac{1}{2}}.
\]
Thanks to Lemma 3.1,

\[
\lim_{\varepsilon \to 0} K \mathbb{E} \int_0^1 \int_0^1 l''(\tau \mu X_T^\varepsilon + (1 - \tau \mu) \bar{X}_T) \left( X_T^\varepsilon - \bar{X}_T \right)^2 \mu^2 d\tau d\mu \\
\leq \lim_{\varepsilon \to 0} K \mathbb{E} \left( l''(X_T^\varepsilon) + l''(\bar{X}_T) \right)^2 \left( X_T^\varepsilon - \bar{X}_T \right)^2 \\
\leq \lim_{\varepsilon \to 0} K \left( \mathbb{E}(l''(X_T^\varepsilon) + l''(\bar{X}_T))^4 \mathbb{E}(X_T^\varepsilon - \bar{X}_T)^4 \right)^{\frac{1}{2}} = 0.
\]

Meanwhile, according to the Remark of Lemma 3.2, we have

\[
I_2 \triangleq \lim_{\varepsilon \to 0} \mathbb{E} \int_0^1 \int_0^1 \left| w'(1 - J^{\varepsilon, \lambda, \mu}) - w'(1 - F_{X_T}(\bar{X}_T)) \right|^4 d\lambda d\mu \\
\leq \lim_{\varepsilon \to 0} \mathbb{E} \int_0^1 \int_0^1 \left| w''(1 - \tau J^{\varepsilon, \lambda, \mu} - (1 - \tau) F_{X_T}(\bar{X}_T)) \right| d\tau \left( F_{X_T}(\bar{X}_T) - J^{\varepsilon, \lambda, \mu} \right) \\
\cdot \mathbb{1}_{\delta \leq J^{\varepsilon, \lambda, \mu}, F_{X_T}(\bar{X}_T) \leq 1 - \delta} d\lambda d\mu \\
+ \lim_{\varepsilon \to 0} \mathbb{E} \int_0^1 \int_0^1 |J_1|^4 \cdot \mathbb{1}_{J^{\varepsilon, \lambda, \mu} < \delta} d\lambda d\mu + \lim_{\varepsilon \to 0} \mathbb{E} \int_0^1 \int_0^1 |J_1|^4 \cdot \mathbb{1}_{F_{X_T}(\bar{X}_T) < \delta} d\lambda d\mu \\
+ \lim_{\varepsilon \to 0} \mathbb{E} \int_0^1 \int_0^1 |J_1|^4 \cdot \mathbb{1}_{J^{\varepsilon, \lambda, \mu} > 1 - \delta} d\lambda d\mu + \lim_{\varepsilon \to 0} \mathbb{E} \int_0^1 \int_0^1 |J_1|^4 \cdot \mathbb{1}_{F_{X_T}(\bar{X}_T) > 1 - \delta} d\lambda d\mu \\
\leq \lim_{\varepsilon \to 0} K_\delta \mathbb{E} \int_0^1 \int_0^1 |F_{X_T}(\bar{X}_T) - J^{\varepsilon, \lambda, \mu}|^4 d\lambda d\mu + \lim_{\varepsilon \to 0} K \mathbb{E} \int_0^1 \int_0^1 \mathbb{1}_{J^{\varepsilon, \lambda, \mu} < \delta} d\lambda d\mu \\
+ \lim_{\varepsilon \to 0} \mathbb{E} \int_0^1 \int_0^1 \mathbb{1}_{J^{\varepsilon, \lambda, \mu} > 1 - \delta} d\lambda d\mu + K \mathbb{E}(\mathbb{1}_{F_{X_T}(\bar{X}_T) < \delta}) + K \mathbb{E}(\mathbb{1}_{F_{X_T}(\bar{X}_T) > 1 - \delta}) \\
= K \int_0^1 \int_0^1 \lim_{\varepsilon \to 0} P\{J^{\varepsilon, \lambda, \mu} < \delta\} d\lambda d\mu + K P\{F_{X_T}(\bar{X}_T) < \delta\} \\
+ K \int_0^1 \int_0^1 \lim_{\varepsilon \to 0} P\{J^{\varepsilon, \lambda, \mu} > 1 - \delta\} d\lambda d\mu + K P\{F_{X_T}(\bar{X}_T) > 1 - \delta\} \\
\leq 2 K P\{F_{X_T}(\bar{X}_T) \leq \delta\} + 2 K P\{F_{X_T}(\bar{X}_T) \geq 1 - \delta\}.
\]

It is recognized that the random variable $F_{X_T}(\bar{X}_T) \sim U(0, 1)$. Taking $\delta \to 0$, we achieve $I_2$ is 0. Consequently,

\[
\lim_{\varepsilon \to 0} \mathbb{E} \left| \int_0^1 g_\varepsilon (\mu X_T^\varepsilon + (1 - \mu) \bar{X}_T) d\mu - l'(\bar{X}_T) w'(1 - F_{X_T}(\bar{X}_T)) \right|^2 = 0.
\]

In addition to Lemma 3.4, we give

\[
I = \mathbb{E} \left( l'(\bar{X}_T) w'(1 - F_{X_T}(\bar{X}_T)) Z_T \right).
\]

Other terms on the RHS of (3.4) can be treated by the same way. \qed
Lemma 3.6
\[ \mathbb{E}(p_T Z_T) = \mathbb{E} \int_0^T v_t(p_t \bar{b}_u(t) + q_t \bar{s}_u(t))dt. \]

**Proof:** In view of (3.2) and (2.3), applying Itô’s formula to \( p_t Z_t \) yeilds
\[
d(p_t Z_t) = p_t dZ_t + Z_t dp_t + d\langle p, Z \rangle_t
\]
\[
= (p_t \bar{b}_u(t)Z_t + p_t \bar{b}_u(t)q_t)dt + p_t(\bar{s}_u(t)Z_t + \bar{s}_u(t)v_t)dW_t
\]
\[
- Z_t(\bar{b}_u(t)p_t + \bar{s}_u(t)q_t)dt + Z_t q_t dW_t + (\bar{s}_u(t)Z_t + \bar{s}_u(t)v_t)q_t dt
\]
\[
= (p_t \bar{b}_u(t)v_t + \bar{s}_u(t)v_t q_t)dt + (p_t \bar{s}_u(t)Z_t + p_t \bar{s}_u(t)v_t + Z_t q_t) dW_t.
\]

Then, taking the integration over \( t \) and taking the expectation on both side, the result follows. \( \blacksquare \)

Observe that
\[ \mathbb{E}(p_T Z_T) = \mathbb{E}(l'(\bar{X}_T)w'(1 - F_{\bar{X}_T}(\bar{X}_T))Z_T). \]

We are ready to finish
**Proof of Theorem 2.5** Combining Lemma 3.5 and 3.6, the Gateaux derivative of the prospective functional is expressed in this way.
\[
\frac{d}{d\varepsilon} J(\bar{u}, + \varepsilon v) \bigg|_{\varepsilon = 0} = \mathbb{E} \int_0^T v_t(p_t \bar{b}_u(t) + \bar{s}_u(t)q_t + \zeta'_+(\bar{u}_t^+) \varpi'_+(1 - F_{\bar{u}_t^+}(\bar{u}_t^+)) \mathbb{1}_{\bar{u}_t > 0}
\]
\[
+ \zeta'_-(\bar{u}_t^-) \varpi'_-(1 - F_{\bar{u}_t^-}(\bar{u}_t^-)) \mathbb{1}_{\bar{u}_t < 0}) dt.
\]

In fact, \( v = u - \bar{u} \) and \( \bar{u} \) is the optimal control process. Thus, we arrive at
\[
\mathbb{E} \int_0^T (u_t - \bar{u}_t)(p_t \bar{b}_u(t) + \bar{s}_u(t)q_t + \zeta'_+(\bar{u}_t^+) \varpi'_+(1 - F_{\bar{u}_t^+}(\bar{u}_t^+)) \mathbb{1}_{\bar{u}_t > 0}
\]
\[
+ \zeta'_-(\bar{u}_t^-) \varpi'_-(1 - F_{\bar{u}_t^-}(\bar{u}_t^-)) \mathbb{1}_{\bar{u}_t < 0}) dt = 0.
\]

Note that \( u_t - \bar{u}_t \) is arbitrary when \( \bar{u}_t \neq 0 \). When \( \bar{u}_t = 0 \), we have
\[
p_t \bar{b}_u(t) + \bar{s}_u(t)q_t + \zeta'_+(\bar{u}_t^+) \varpi'_+(1 - F_{\bar{u}_t^+}(\bar{u}_t^+)) \mathbb{1}_{\bar{u}_t > 0}
\]
\[
+ \zeta'_-(\bar{u}_t^-) \varpi'_-(1 - F_{\bar{u}_t^-}(\bar{u}_t^-)) \mathbb{1}_{\bar{u}_t < 0} = 0,
\]
a.e.t \( \in [0, T] \), \( \mathbb{P} - a.s. \). We finish the proof of Theorem 2.5. The optimal solution for Problem (2.3) is eventually derived.
4. Application

Recalling the state equation (2.1) and the adjoint equation (2.5), we have

\[
\begin{align*}
\frac{d\hat{X}_t}{dt} &= b(t, \bar{u}_t, \hat{X}_t)dt + \sigma(t, \bar{u}_t, \hat{X}_t)dW_t, \\
\frac{dp_t}{dt} &= -(b_x(t, \bar{u}_t, \hat{X}_t)p_t + \sigma_x(t, \bar{u}_t, \hat{X}_t)q_t)dt + q_t dW_t, \\
\hat{X}_0 &= x_0, \\
p_T &= l'(\hat{X}_T)w'(1 - F_{\hat{X}_T}(\hat{X}_T)).
\end{align*}
\]

Given an optimal control \( \bar{u} \), there exists a unique solution \( \hat{X}(\bar{u}) \) to the state equation. As \( p_T \) is known, the unique solution \( (p(\bar{u}), q(\bar{u})) \) for the above backward SDE is obtained. Plugging \( \hat{X}(\bar{u}) \) and \( (p(\bar{u}), q(\bar{u})) \) into (2.6), the optimal control \( \bar{u} \) is determined.

In this section, we apply this procedure to three interesting cases. The first example will show that the result in Jin & Zhou (2008) coincides with ours without running cost.

Example 4.1 Consider Example 1.1 without consumption. The state process is modeled by

\[
\begin{align*}
\frac{dX_t}{dt} &= X_t(r_t + (b_t - r_1 1_m^\top u_t)dt + X_t u_t^\top \sigma_t dW_t, \quad t \in [0, T]; \\
X_0 &= x_0 > 0,
\end{align*}
\]

and the agent’s objective functional under the CPT becomes

\[ V(X_T) = \int_0^\infty w(\mathbb{P}\{l(X_T) > x\})dx. \]

**Hypothesis.** There exists an \( \mathbb{R}^m \)-valued, uniformly bounded, \( \mathcal{F}_t \)-progressively measurable process \( \theta \) such that \( \sigma_t \theta_t = b_t - r_1 1_m, \ a.e.t \in [0, T], \ a.s. \). Besides, \( \text{rank}(\sigma_t) = m, \ a.e.t \in [0, T], \ a.s. \).

The Hypothesis ensures that the financial market is arbitrage-free and complete. Under suitable conditions, the optimal terminal wealth given by Jin & Zhou (2008) (section 6) is

\[ \hat{X}_T = (l')^{-1}\left( \frac{\lambda \rho_T}{w'(F_{\rho_T}(\rho_T))} \right), \]

where

\[ \rho_t = \exp \left\{ -\int_t^T \left( r_s + \frac{1}{2} |\theta_s|^2 \right) ds - \int_t^T \theta_s^\top dW_s \right\}. \]
is the pricing kernel, and \( \lambda > 0 \) is the unique real number such that \( \mathbb{E}(\rho_T \bar{X}_T) = x_0 \). And they proved

\[
F_{\rho_T}(\rho_T) = 1 - F_{\bar{X}_T}(\bar{X}_T).
\]

In the light of Theorem 2.3, an optimal solution \((\bar{u}, \bar{X})\) must satisfy (2.3) and (2.6). In fact, substituting (4.1) into (2.5), we are able to obtain

\[
\begin{cases}
  dp_t &= -(r_t + (b_t - r_t 1_m)^\top u_t)p_t dt - u_t^\top \sigma_t q_t dt + q_t^\top dW_t, \\
  p_T &= l'(\bar{X}_T)w'(1 - F_{\bar{X}_T}(\bar{X}_T)) = \lambda \rho_T.
\end{cases}
\]

Using the Itô’s formula to \( \lambda \rho_t \), one has

\[
d(\lambda \rho_t) = -r_t \cdot \lambda \rho_t dt - \lambda \rho_t \theta_t^\top dW_t.
\]

Comparing it with the above backward SDE, it yields

\[
p_t = \lambda \rho_t, \quad q_t = -\lambda \rho_t \theta_t.
\]

With \( \sigma_t \theta_t = b_t - r_t 1_m \), we achieve

\[
p_t (b_t - r_t 1_m) + \sigma_t q_t = \lambda \rho_t (b_t - r_t 1_m) - \sigma_t \lambda \rho_t \theta_t = 0, \forall t \in [0, T],
\]

namely, \((p, q)\) satisfies (2.6). In a word, the optimal strategy obtained in this paper consists with that of Jin & Zhou (2008).

Actually, some situation would lead to no solution when \( \bar{u}_t \neq 0 \). In other words, the unique solution to the control process is 0.

**Example 4.2** Let \( b(t, u, x) = -ux, \sigma(t, u, x) = x \). Suppose there is no terminal term in objective functional; namely,

\[
J(u) = \mathbb{E} \int_0^T \left( \zeta_+ (u_t^+) \omega_+(1 - F_{u_t^+}(u_t^+)) - \zeta_- (u_t^-) \omega_- (1 - F_{u_t^-}(u_t^-)) \right) dt.
\]

If \((\bar{u}, \bar{X})\) is an optimal solution, by reason of

\[
p_t = \begin{cases} 
  \zeta_+ (\bar{u}_t^+) \omega_+(1 - F_{\bar{u}_t^+}(\bar{u}_t^+)) & \text{if } \bar{u}_t > 0, \\
  \zeta_- (\bar{u}_t^-) \omega_- (1 - F_{\bar{u}_t^-}(\bar{u}_t^-)) & \text{if } \bar{u}_t < 0,
\end{cases}
\]

we get

\[
\begin{cases}
  dp_t &= (\bar{u}_t p_t - q_t) dt + q_t dW_t, \\
  p_T &= 0.
\end{cases}
\]

Clearly, \( p_t \equiv q_t \equiv 0 \) is the unique answer, which results in no solution if \( \bar{u}_t \neq 0 \). Accordingly, \( \bar{u}_t = 0 \), a.e. \( t \in [0, T], a.s. \).
Finally, we present a solvable example and compare the result with the one without probability distortions. The process \( u^\pm_t \) in the objective functional are replaced by \( u^\pm_t X_t \), signifying the proportion of wealth process. We study a case with compounded cost function.

**Example 4.3** Let \( u_t, X_t > 0, b(t, u, x) = -ux, \sigma(t, u, x) = x, \zeta(x) = x^\alpha/(0 < \alpha < 1), \varpi_+(p) = \nu p^{\gamma+1} + (1 - \nu)[1 - (1 - p)^{\beta+1}](\gamma, \beta \geq 0, 0 \leq \nu \leq 1) \). We have

\[
\begin{align*}
dX_t &= -u_t X_t dt + X_t dW_t, \\
X_0 &= x_0,
\end{align*}
\]

and

\[
J(\mu) = \mathbb{E} \int_0^T \left( \frac{1}{\alpha} (u_t X_t)^\alpha \varpi_+(1 - F_{u_t X_t}(u_t X_t)) + X_t \right) dt.
\]

In accordance with Theorem \( \text{2.5} \), its optimal solution \( (\bar{u}_t, \bar{X}_t) \) should satisfy

\[
p_t = \left( \bar{u}_t \bar{X}_t \right)^{\alpha-1} \varpi_+(1 - F_{\bar{u}_t \bar{X}_t}(\bar{u}_t \bar{X}_t)), \ a.e. t \in [0, T], \ a.s.,
\]

where

\[
\begin{align*}
dp_t &= \left( \bar{u}_t p_t - q_t - \left( \bar{u}_t \bar{X}_t \right)^{\alpha-1} \varpi_+(1 - F_{\bar{u}_t \bar{X}_t}(\bar{u}_t \bar{X}_t)) \bar{u}_t - 1 \right) dt + q_t dW_t, \\
p_T &= 0.
\end{align*}
\]

Combing these two equations, we have

\[
\begin{align*}
dp_t &= -(q_t + 1) dt + q_t dW_t, \\
p_T &= 0.
\end{align*}
\]

It yields

\[
p_t = T - t, \ q_t = 0, \ \forall t \in [0, T].
\]

Going back to equality (4.2), given fixed \( t \), assume that \( \bar{u}_t \bar{X}_t = h(p_t) \). If this is the case, \( \bar{u}_t \bar{X}_t \) is deterministic and hence \( F_{\bar{u}_t \bar{X}_t}(\bar{u}_t \bar{X}_t) = 1 \). As a result, we infer that

\[
\bar{u}_t \bar{X}_t = \left( \frac{T - t}{(1 - \nu)(\beta + 1)} \right)^{1/(\alpha-1)}, \ a.e. t \in [0, T], \ a.s..
\]

Substituting back to the state equation, we get

\[
\bar{X}_t = V_t \left( x_0 + \int_0^t \frac{T - s}{(1 - \nu)(\beta + 1) V_s^{1/(\alpha-1)}} ds \right), \ V_t = \exp \left\{ B_t - \frac{t}{2} \right\}.
\]

Finally, the optimal control is

\[
\bar{u}_t = \frac{(T - t)^{1/(\alpha-1)}}{V_t \left( x_0 ((1 - \nu)(\beta + 1))^{1/(\alpha-1)} + \int_0^t \frac{(T - s)^{1/(\alpha-1)}}{V_s} ds \right)}, \ a.e. t \in [0, T], \ a.s..
\]
Without the distorted probability in this example, we acquire that

$$\bar{u}_t = \frac{(T - t)^{1/(\alpha-1)}}{V_t(x_0 + \int_0^t (T-s)^{1/(\alpha-1)} \frac{V_s}{V_s} ds)}, \ a.e.t \in [0, T], a.s..$$

5. Concluding Remarks

This article develops a stochastic maximum principle for a general continuous behavioral portfolio model. The optimal solution is characterized by (2.5) and (2.6). The system (2.1) and (2.2) covers highly diversified preferences including those of the classical utility maximization, financial investment activities involving consumption (or gambling, insurance) and other behavioral patterns. Various cases are studied in last section, showing that our solution is in agreement with that of Jin & Zhou (2008), the results are used to solve optimization problems with distorted probabilities and running utilities.

Unlike the majority of models in literature, the running terms here are divided into positive and negative parts. The utility functions are ill-behaved as a result of its S-shape and its infinite derivative at 0. Further, handling of $F_Y(Y)$ on account of probability distortions poses serious mathematical challenges. To overcome these difficulties, we convert this setting to a mean-field optimal control problem, and derive a mean-field stochastic maximum principle. Due to a technical reason, we restricted our utility as a one-variable function. We pose the study of the case when the utility function depends on more than one variable as a challenging open problem.

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