NESTED ARTIN APPROXIMATION

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Abstract. A short proof of the linear nested Artin approximation property of the algebraic power series rings is given here.

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Introduction

The solution of an old problem (see for instance [3]), the so-called nested Artin approximation property, is given in the following theorem.

Theorem 1 ([4], [5, Theorem 3.6]). Let \( k \) be a field, \( A = k\langle x \rangle, \ x = (x_1, \ldots, x_n) \) the algebraic power series over \( k \), \( f = (f_1, \ldots, f_s) \in A[Y]^s, Y = (Y_1, \ldots, Y_m) \) and \( 0 < r_1 \leq \ldots \leq r_m \leq n, c \) be some non-negative integers. Suppose that \( f \) has a solution \( \hat{y} = (\hat{y}_1, \ldots, \hat{y}_m) \) in \( k[[x]] \) such that \( \hat{y}_i \in k[[x_1, \ldots, x_r]] \) for all \( 1 \leq i \leq m \) (that is a so-called nested formal solution). Then there exists a solution \( y = (y_1, \ldots, y_m) \) of \( f \) in \( A \) such that \( y_i \in k(x_1, \ldots, x_{r_i}) \) for all \( 1 \leq i \leq m \) and \( y \equiv \hat{y} \mod (x)^c k[[x]] \).

The proof relies on an idea of Kurke from 1972 and the Artin approximation property of rings of type \( k[[u]]\langle x \rangle \) (see [4], [9], [5]). When \( f \) is linear, interesting relations with other problems and a description of many results on this topic are nicely explained in [2]. Also a proof of the above theorem in the linear case can be found in [2] using just the classical Artin approximation property of \( A \) (see [1]). Unfortunately, we had some difficulties in reading [2], but finally we noticed a shorter proof using mainly the same ideas. This proof is the content of the present note.

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1. Linear nested Artin approximation property

We start recalling [8, Lemma 9.2] with a simplified proof.

Lemma 2. Let \((A, m)\) be a complete normal local domain, \( u = (u_1, \ldots, u_n), v = (v_1, \ldots, v_m), B = A[[u]][v]\) be the algebraic closure of \( A[[u]][v] \) in \( A[[u, v]] \) and \( f \in B \). Then there exists \( g \) in the algebraic closure \( A(v, Z) \) of \( A[[v, Z]] \), \( Z = (Z_1, \ldots, Z_s) \) in \( A[[v, Z]] \) for some \( s \in \mathbb{N} \) and \( \hat{z} \in A[[u]]^s \) such that \( f = g(\hat{z}) \).

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Proof. Changing $f$ by $f - f(u = 0)$ we may assume that $f \in (u)$. Note that $B$ is the Henselization of $C = A[[u]][v]_{(m, u, v)}$ by \textcolor{blue}{[2]} and so there exists some etale neighborhood of $C$ containing $f$. Using for example \textcolor{blue}{[9]} Theorem 2.5 there exists a monic polynomial $F$ in $X$ over $A[[u]][v]$ such that $F(f) = 0$ and $F'(f) \not\equiv (m, u, v)$, let us say $F = \sum_{i,j} F_{ij} v^i X^j$ for some $F_{ij} \in A[[u]]$.

Set $\hat{z}_{ij} = F_{ij}(u = 0) \in (u)A[[u]]$, $\hat{z} = (\hat{z}_{ij})$ and $G = \sum_{ij}(F_{ij}(u = 0) + Z_{ij})v^iX^j$ for some new variables $Z = (Z_{ij})$. We have $G(\hat{z}) = F$. Set $G' = \partial G/\partial X$. As
\[
G(Z = 0) = G(\hat{z}(u = 0)) \equiv F(f) = 0 \text{ modulo } (u),
\]
\[
G'(Z = 0) = G'(\hat{z}(u = 0)) \equiv F'(f) \not\equiv 0 \text{ modulo } (m, u, v)
\]
we get $G(X = 0) \equiv 0, G'(X = 0) \not\equiv 0$ modulo $(m, v, Z)$. By the Implicit Function Theorem there exists $g \in (m, v, Z)A(v, Z)$ such that $G(g) = 0$. It follows that $G(g(\hat{z})) = 0$. But $F = G(\hat{z}) = 0$ has just a solution $X = f$ in $(m, u, v)B$ by the Implicit Function Theorem and so $f = g(\hat{z})$. \hfill $\square$

Lemma 3. Let $(A, m)$ be a Noetherian local ring, $f \in A[U]$, $U = (U_1, \ldots, U_s)$ a linear system of polynomial equations, $c \in N$ and $\hat{u}$ a solution of $f$ in the completion $\hat{A}$ of $A$. Then there exists a solution $u \in A^c$ of $f$ such that $u \equiv \hat{u}$ modulo $m^c \hat{A}$.

Proof. Let $B = A[U]/(f)$ and $h : B \to \hat{A}$ be the map given by $U \to \hat{u}$. By \textcolor{blue}{[4]} Lemma 4.2 (or \textcolor{blue}{[6]} Proposition 36) $h$ factors through a polynomial algebra $A[Z]$, $Z = (Z_1, \ldots, Z_s)$, let us say $h$ is the composite map $B \xrightarrow{\hat{u}} A[Z] \xrightarrow{\hat{z}} \hat{A}$. Choose $z \in A^c$ such that $z \equiv g(Z)$ modulo $m^c \hat{A}$. Then $u = t(c)^s U(z)$ is a solution of $f$ in $A$ such that $u \equiv \hat{u}$ modulo $m^c \hat{A}$. \hfill $\square$

Proposition 4. Let $k(x, y)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_m)$ be the ring of algebraic power series in $x, y$ over a field $k$ and $M \subset k(x, y)^p$ a finitely generated $k(x, y)$-submodule. Then

$$k[[x]](M \cap k(x)^p) = (k[[x, y]]M) \cap k[[x]]^p,$$

equivalently $M \cap k(x)^p$ is dense in $(k[[x, y]]M) \cap k[[x]]^p$, that is for all $\hat{v} \in (k[[x, y]]M) \cap k[[x]]^p$ and $c \in N$ there exists $v_c \in (M \cap k(x)^p)$ such that $v_c \equiv \hat{v}$ modulo $(x)^c k[[x]]^p$. Moreover, if $c \in N$ and $\hat{v} = \sum_{i=1}^t \hat{u}_ia_i$ for some $a_i \in M$, $\hat{u}_i \in k[[x, y]]$ then there exist $u_{ic} \in k(x, y)$ such that $u_{ic} \equiv \hat{u}_i$ modulo $(x)^c k[[x, y]]$, $v_c = \sum_{i=1}^tu_{ic}a_i \in (M \cap k(x)^p)$ and $v$ is the limit of $(v_c)_c$ in the $(x)$-adic topology.

Proof. Let $\hat{v} \in (k[[x, y]]M) \cap k[[x]]^p$, let us say $\hat{v} = \sum_{i=1}^t \hat{u}_ia_i$ for some $a_i \in M$ and $\hat{u}_i \in k[[x, y]]^p$. By flatness of $k[[x]](\langle y \rangle) \subset k[[x, y]]$ we see that there exist $\hat{u}_i \in k[[x]](\langle y \rangle)$ such that $\hat{v} = \sum_{i=1}^t \hat{u}_ia_i$. Moreover by Lemma 3 we may choose $u_i$ such that $\hat{u}_i \equiv u_i$ modulo $(x)^c k[[x, y]]$. Then using Lemma 2 there exist $g_i \in k(y, Z)$, $i \in [t]$ for some variables $Z = (Z_1, \ldots, Z_s)$ and $\hat{z} \in k[[x]]^s$ such that $u_i = g_i(\hat{z})$. Note that $a_i = \sum_{r \in N^m} a_{ir}y^r$, $g_i = \sum_{r \in N^m} g_{ir}y^r$ with $a_{ir} \in k(x)^p$, $g_{ir} \in k(Z)$.

Clearly, $\hat{v}, \hat{v}$ is a solution in $k[[x]]$ of the system of polynomial equations $V = \sum_{i=1}^t a_{i0}g_{i0}(Z), V = (V_1, \ldots, V_p)$ if and only if it is a solution of the infinite system of polynomial equations
\[
(*) \ V = \sum_{i=1}^t a_{i0}g_{i0}(Z), \quad \sum_{i=1}^t \sum_{r + r' = e} a_{ir}g_{ir'}(Z) = 0, \quad e \in N^m, \ e \neq 0.
\]
Since $k\langle x, Z, V \rangle$ is Noetherian we see that it is enough to consider in (*) only a finite set of equations, let us say with $e \leq \omega$ for some $\omega$ high enough. Applying the Artin approximation property of $k\langle x \rangle$ (see [1]) we can find for any $c \in \mathbb{N}$ a solution $v_c \in k\langle x \rangle^p$, $z_c \in k\langle x \rangle^s$ of (*) such that $v_c \equiv \hat{v}$ modulo $(x)^c k[[x]]^p$, $z_c \equiv \hat{z}$ modulo $(x)^c k[[x]]^s$. Then $v_c = \sum_{i=1}^t a_{ci} g_i(z_c) \in M \cap k\langle x \rangle^p$, and $u_{ic} = g_i(z_{ic}) \in k(x, y)$ satisfies $u_{ic} \equiv \hat{u}_i \equiv \hat{u}_i$ modulo $(x)^c k[[x, y]]$. Clearly $\hat{v}$ is the limit of $(v_c)_c$ in the $(x)$-adic topology and belongs to $k[[x]](M \cap k\langle x \rangle^p)$. □

Remark 5. When $p = 1$ then the above $M$ is an ideal and we get the so-called (see [2]) strong elimination property of the algebraic power series.

The following proposition is partially contained in [2, Lemma 4.2].

**Proposition 6.** Let $M \subseteq k\langle x \rangle^p$ be a finitely generated $k\langle x \rangle$-submodule and $1 \leq r_1 < \ldots < r_e \leq n$, $p_1, \ldots, p_e$ be some positive integers such that $p = p_1 + \ldots + p_e$. Then

$$T = M \cap (k\langle x_1, \ldots, x_{r_1} \rangle^{p_1} \times \ldots \times k\langle x_1, \ldots, x_{r_e} \rangle^{p_e})$$

is dense in

$$\hat{T} = (k[[x]]M) \cap (k[[x_1, \ldots, x_{r_1}]]^{p_1} \times \ldots \times k[[x_1, \ldots, x_{r_e}]]^{p_e}).$$

Moreover, if $c \in \mathbb{N}$ and $\hat{v} = \sum_{i=1}^t \hat{u}_i a_i \in \hat{T}$ for some $a_i \in M$, $\hat{u}_i \in k[[x]]$ then there exist $u_{ic} \in k\langle x \rangle$ such that $u_{ic} \equiv \hat{u}_i$ modulo $(x)^c k[[x]]$, $v_c = \sum_{i=1}^t u_{ic} a_i \in T$ and $\hat{v}$ is the limit of $(v_c)_c$ in the $(x)$-adic topology.

**Proof.** Apply induction on $e$, the case $e = 1$ being done in Proposition 4. Assume that $e > 1$. We may reduce to the case when $r_e = n$ replacing $M$ by $M \cap k\langle x_1, \ldots, x_{r_e} \rangle^p$ if $r_e < n$. Let

$$q : \prod_{i=1}^{p_e} k[[x_1, \ldots, x_{r_i}]]^{p_i} \to \prod_{i=1}^{p_{e-1}} k[[x_1, \ldots, x_{r_i}]]^{p_i},$$

$$q : \prod_{i=1}^{p_e} k[[x_1, \ldots, x_{r_i}]]^{p_i} \to k[[x_1, \ldots, x_{r_e}]]^{p_e}$$

be the canonical projections, $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_p) \in \hat{T}$, and $M_1 = q(M)$. Assume that $\hat{v}_i = \sum_{i=1}^t \hat{u}_i a_i$ for some $\hat{u}_i \in k[[x]]$, $a_i \in M$. By induction hypothesis applied to $M_1$, $q(\hat{v}_i)$ given $c \in \mathbb{N}$ there exists $u_{ic} \in k\langle x \rangle$ with $u_{ic} \equiv \hat{u}_i$ modulo $(x)^c k[[x]]$ such that $v'_i = \sum_{i=1}^t u_{ic} q(a_i) \in q(T)$ and $q(\hat{v})$ is the limit of $(v'_c)_c$ in the $(x)$-adic topology.

Now, let $v''_i = \sum_{i=1}^t u_{ic} q(a_i) \in k(x_1, \ldots, x_{r_e})^{p_e}$. We have $v''_i \equiv q(\hat{v})$ modulo $(x)^c k[[x]]^{p_e}$. Then $v_c = (v'_c, v''_c) = \sum_{i=1}^t u_{ic} a_i \in T$, $v_c \equiv \hat{v}$ modulo $(x)^c k[[x]]^p$ and $\hat{v}$ is the limit of $(v_c)_c$ in the $(x)$-adic topology. □

**Corollary 7.** Theorem 7 holds when $f$ is linear.

**Proof.** If $f$ is homogeneous then it is enough to apply Proposition 6 for the module $M$ of the solutions of $f$ in $A = k\langle x \rangle$. Suppose that $f$ is not homogeneous, let us say $f$ has the form $g + a_0$ for some system of linear homogeneous polynomials $g \in A[Y]^s$ and $a_0 \in A^s$. The proof in this case follows [2, page 7] and we give it here only for the sake of completeness. Change $f$ by the homogeneous system of linear polynomials $\tilde{f} = g + a_0 Y_0$ from $A[Y_0, Y]^s$. A nested formal solution $\hat{y}$ of $\tilde{f}$ in $k[[x]]$ with $\hat{y}_i \in K[[x_1, \ldots, x_{r_i}]]$, $1 \leq i \leq m$ induces a nested formal solution $(\hat{y}_0, \hat{y})$, $\hat{y}_0 = 1$
of $\bar{f}$ with $r_0 = r_1$. As above, for all $c \in \mathbb{N}$ we get a nested algebraic solution $(y_0, y)$ of $\bar{f}$ with $y_i \in k\langle x_1, \ldots, x_{r_i} \rangle$ and $y_i \equiv \hat{y}_i$ modulo $(x)^ck[[x]]$ for all $0 \leq i \leq m$. It follows that $y_0$ is invertible and clearly $y_0^{-1}y$ is the wanted nested algebraic solution of $f$. □

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