CANCELLABLE ELEMENTS OF THE LATTICE OF MONOID VARIETIES

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Abstract. The set of all cancellable elements of the lattice of semigroup varieties has recently been shown to be countably infinite. But the description of all cancellable elements of the lattice $\text{MON}$ of monoid varieties remains unknown. This problem is addressed in the present article. The first example of a monoid variety with modular but non-distributive subvariety lattice is first exhibited. Then a necessary condition of the modularity of an element in $\text{MON}$ is established. These results play a crucial role in the complete description of all cancellable elements of the lattice $\text{MON}$. It turns out that there are precisely five such elements.

1. Introduction and summary

The present article is concerned with the lattice $\text{MON}$ of all monoid varieties, where monoids are considered as semigroups with an identity element that is fixed by a 0-ary operation. For many years, results on the lattice $\text{MON}$ were scarce. But recently, interest in this lattice has grown significantly; in particular, the study of its special elements was initiated in the articles \cite{5,6}. In the present work, we continue these investigations.

Special elements play an important role in general lattice theory; see \cite[Section III.2]{3}, for instance. We recall definitions of those types of special elements that are relevant here. An element $x$ of a lattice $L$ is

\begin{itemize}
  \item \textit{cancellable} if $\forall y, z \in L$: $x \lor y = x \lor z$ \& $x \land y = x \land z \implies y = z$;
\end{itemize}
modular if $\forall y, z \in L: y \leq z \rightarrow (x \land z) \lor y = (x \lor y) \land z$.

It is easy to see that every cancellable element is modular.

Our main goal is to describe all cancellable elements of the lattice $\text{MON}$. To formulate our main result, we need some definitions and notation. Let $\mathcal{X}^+$ (respectively, $\mathcal{X}^*$) denote the free semigroup (respectively, monoid) over a countably infinite alphabet $\mathcal{X}$. Elements of $\mathcal{X}$ are called letters and elements of $\mathcal{X}^*$ are called words. Words unlike letters are written in bold. An identity is written as $u \approx v$, where $u, v \in \mathcal{X}^*$.

Let $T$, $SL$, and $\text{MON}$ denote the variety of trivial monoids, the variety of semilattice monoids, and the variety of all monoids, respectively. For any identity system $\Sigma$, let $\text{var}\Sigma$ denote the variety of monoids given by $\Sigma$. Put

$$C_2 = \text{var}\{x^2 \approx x^3, xy \approx yx\} \quad \text{and} \quad D = \text{var}\{x^2 \approx x^3, x^2y \approx yxy \approx yx^2\}.$$ 

Then the following is our main result.

**Theorem 1.1.** A monoid variety is a cancellable element of the lattice $\text{MON}$ if and only if it coincides with one of the varieties $T$, $SL$, $C_2$, $D$ or $\text{MON}$.

Many articles were devoted to special elements of different types in the lattice $\text{SEM}$ of all semigroup varieties; an overview of results published before 2015 can be found in the survey [16].\(^1\) It is natural to compare Theorem 1.1 with the description of cancellable elements of the lattice $\text{SEM}$ that was found in 2019 [15]. Theorem 1.1 shows that there are only five cancellable elements in the lattice $\text{MON}$. In contrast, the set of all cancellable elements of the lattice $\text{SEM}$ is countably infinite.

In general, the set of cancellable elements in a lattice need not form a sublattice. For example, the elements $x$ and $y$ of the lattice in Fig. 1 are cancellable but their join $x \lor y$ is not. However, the class of all cancellable elements of $\text{SEM}$ forms a distributive sublattice of $\text{SEM}$; see Corollary 3.14 in the extended version of the survey [16].\(^2\) Theorem 1.1 shows that the same is true for monoid varieties too; in fact, the five cancellable elements in $\text{MON}$ constitute a chain.

For a variety $V$ of monoids, we denote its subvariety lattice by $\mathcal{L}(V)$. Now since the chain $T \subseteq SL \subseteq C_2 \subseteq D$ coincides with the lattice $\mathcal{L}(D)$ of subvarieties of $D$ (see Fig. 2), a monoid variety $V$ is a cancellable element of the lattice $\text{MON}$ if and only if either $V \subseteq D$ or $V = \text{MON}$. It is routinely verified that the variety $D$ can be given by the single identity $x^3yz \approx yxzx$.

\(^1\) An extended version of this survey, which is periodically updated as new results are found and/or new articles are published, is available at http://arxiv.org/abs/1309.0228v20.

\(^2\) Actually, this fact easily follows from Theorem 1.1 in [15] but it is not mentioned in [15] explicitly.
Therefore it is easy to check the cancellability of proper elements of the lattice $\text{MON}$; a monoid variety is proper if it is different from $\text{MON}$.

**Corollary 1.2.** Suppose that $M$ is any monoid that generates a proper subvariety $V$ of $\text{MON}$. Then $V$ is a cancellable element of the lattice $\text{MON}$ if and only if $M$ satisfies the identity $x^3yz \approx yzx$.

The article consists of five sections. Section 2 contains definitions, notation, certain known results and their simple corollaries. In Section 3, the first example of a monoid variety with modular but non-distributive subvariety lattice is given. In Section 4, a necessary condition of the modularity of an element in $\text{MON}$ is established in Proposition 4.3. Results from Sections 3 and 4 will then be used in Section 5 to prove Theorem 1.1.

2. Preliminaries

Acquaintance with rudiments of universal algebra is assumed of the reader. Refer to the monograph [2] for more information.

Recall that a variety is periodic if it consists of periodic monoids. Equivalently, a variety is periodic if and only if it satisfies the identity $x^n \approx x^{n+m}$ for some $n, m \geq 1$. For any word $w$ and any set $X$ of letters, the word obtained from $w$ by deleting all the letters of $X$ is denoted by $w_X$. The content of a word $w$, denoted by $\text{con}(w)$, is the set of letters occurring in $w$. The partition lattice over a set $X$ is denoted by $\text{Part}(X)$. Let $\mathcal{L}_{\text{FIC}}(\mathfrak{X}_*)$ denote the lattice of all fully invariant congruences on the monoid $\mathfrak{X}_*$, and for any variety $V$ of monoids, let $\text{FIC}(V)$ denote the fully invariant congruence on $\mathfrak{X}_*$ corresponding to $V$. It is well known that the mapping $\text{FIC}: \text{MON} \longrightarrow \mathcal{L}_{\text{FIC}}(\mathfrak{X}_*)$ is an anti-isomorphism of lattices; see [2, Theorem 11.9 and Corollary 14.10], for instance. For any $u, v \in \mathfrak{X}_+$, we put $u \preceq v$ if $v = a\xi(u)b$ for some words $a, b \in \mathfrak{X}_+$ and some endomorphism $\xi$ of $\mathfrak{X}_+$. It is easily seen that the relation $\preceq$ on $\mathfrak{X}_+$ is a quasi-order. For an arbitrary anti-chain $A \subseteq \mathfrak{X}_+$ under the relation $\preceq$, let $L_A$ denote the set of all monoid varieties $V$ for which $A$ is a union of $\text{FIC}(V)$-classes. Define the map $\varphi_A: L_A \longrightarrow \text{Part}(A)$ by the rule $\varphi_A(V) = \text{FIC}(V)|_A$ for any $V \in L_A$. 

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Lemma 2.1 [4, Lemma 3]. Let $A$ be any anti-chain under the quasi-order $\preceq$. Suppose that for any words $u, v \in A$ and any nonempty set $X \subseteq \text{con}(u)$, the equalities $\text{con}(u) = \text{con}(v)$ and $u_X = v_X$ hold. Then

(i) the set $L_A$ is a sublattice of the lattice $\text{MON}$;
(ii) the map $\varphi_A$ is a surjective anti-homomorphism of the lattice $L_A$ onto the lattice $\text{Part}(A)$;
(iii) for any partition $\beta \in \text{Part}(A)$, there exists a non-periodic monoid variety $V \in L_A$ such that $\varphi_A(V) = \beta$.

Recall that a band is left regular if it is a semilattice of left zero bands. It is well known that the class of left regular band monoids coincides with the variety

$$\text{LRB} = \text{var}\{xy \approx xyx\}.$$  

The initial part of a word $w$, denoted by $\text{ini}(w)$, is the word obtained from $w$ by retaining the first occurrence of each letter. The following assertion is well known and easily verified.

Lemma 2.2. An identity $u \approx v$ holds in $\text{LRB}$ if and only if $\text{ini}(u) = \text{ini}(v)$.

For any $n \geq 2$, the variety

$$C_n = \text{var}\{x^n \approx x^{n+1}, xy \approx yx\}$$

is generated by the monoid $\langle a, 1 \mid a^n = 0 \rangle$ [1, Corollary 6.1.5]. Note that the variety $C_2$ has already been introduced in Section 1. A word $w$ is an isoterm for a variety $V$ if the $\text{FIC}(V)$-class of $w$ is singleton. The following result is easily deduced from [10, Lemma 3.3].

Lemma 2.3. Let $n \geq 1$. For any monoid variety $V$, the following are equivalent:

a) $x^n$ is not an isoterm for $V$;

b) $V$ satisfies the identity $x^n \approx x^{n+m}$ for some $m \geq 1$;

c) $C_{n+1} \not\subseteq V$.

A monoid is completely regular if it is a union of its maximal subgroups. A variety is completely regular if it consists of completely regular monoids. It is well known that a monoid variety is completely regular if and only if it satisfies the identity $x \approx x^{n+1}$ for some $n \geq 1$.

Lemma 2.4 [7, Lemma 2.14]. If a monoid variety $V$ is non-completely regular and noncommutative, then $D \subseteq V$.

Lemma 2.5. Let $V$ be any monoid variety such that $C_2 \subseteq V$. Suppose that $V$ does not contain the variety

$$E = \text{var}\{x^2 \approx x^3, x^2y \approx xyx, x^2y^2 \approx y^2x^2\}.$$  

Then $V$ satisfies the identity $x^py^q \approx xy^r$ for some $p, q \geq 1$ and $r \geq 2$.
Proof. If $D \subseteq V$, then the result follows from [7, Lemma 4.1 and Proposition 4.2]. Therefore suppose that $D \not\subseteq V$, so that by Lemma 2.4, the variety $V$ is either completely regular or commutative. But $V$ cannot be completely regular because $C_2 \subseteq V$. Hence $V$ is commutative and satisfies the identity $yx x \approx y x^2$. □

3. Monoid variety with modular but non-distributive subvariety lattice

There are many examples of monoid varieties with non-distributive subvariety lattice; see [5,6,13], for instance. However, all these varieties have non-modular subvariety lattice as well. In this section, we present the first example of a monoid variety whose subvariety lattice is modular but non-distributive. To this end, the following varieties are required: the variety $D_2$ generated by the monoid

$$\langle a, b, 1 \mid a^2 = b^2 = bab = 0 \rangle = \{a, b, ab, ba, aba, 1, 0\},$$

the variety $R$ generated by the monoid

$$\langle a, b, 1 \mid a^3 = b^2 = ba = 0 \rangle = \{a, b, a^2, ab, a^2b, 1, 0\}$$

and the variety $R^\delta$ dual to $R$. It is proved in [9, Lemmas 2.2.8 and 2.2.9] that

$$D_2 = \text{var}\{x^3 \approx x^2, x^3yzt \approx yxztx, xyzty \approx yxzty, xzxyty \approx xzytx, xtyzxy \approx xtyzyx\},$$

$$R \vee R^\delta = \text{var}\{x^4 \approx x^3, x^3yzt \approx yxztx, xyzty \approx yxzty, xzxyty \approx xzytx, xtyzxy \approx xtyzyx\}.$$ 

It is easily seen that $D_2 = (R \vee R^\delta) \land \text{var}\{x^3 \approx x^2\}$.

Proposition 3.1. The lattice $\mathcal{L}(R \vee R^\delta)$ of subvarieties of $R \vee R^\delta$ is given in Fig. 2. In particular, this lattice is modular but not distributive.

Proof. It is easily shown that $C_3 \subseteq R \vee R^\delta$. According to Lemma 2.3, any subvariety $V$ of $R \vee R^\delta$ such that $C_3 \not\subseteq V$ satisfies the identity $x^3 \approx x^2$, whence $V \subseteq D_2$. Therefore, the lattice $\mathcal{L}(R \vee R^\delta)$ is the disjoint union of the lattice $\mathcal{L}(D_2)$ and the interval $[C_3, R \vee R^\delta]$. It is proved in [10, Lemmas 4.4 and 4.5] that the lattice $\mathcal{L}(D_2)$ coincides with the 5-element chain in Fig. 2. Thus it remains to describe the interval $[C_3, R \vee R^\delta]$. It follows from [12, Proposition 4.1] that every noncommutative variety in this interval is defined within $R \vee R^\delta$ by some of the identities $xy x \approx x^2 y$, $yx x \approx y x^2$ or $x^2 y \approx y x^2$.

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It is then routinely shown that the interval \([C_3, R \lor R^\delta]\) is as described in Fig. 2, where

\[
R = (R \lor R^\delta) \land \text{var}\{xyx \approx yx^2\},
\]

\[
D_2 \lor C_3 = (R \lor R^\delta) \land \text{var}\{x^2y \approx yx^2\},
\]

and \(D \lor C_3 = (D_2 \lor C_3) \land R = (D_2 \lor C_3) \land R^\delta = R \lor R^\delta\). The proof of this proposition is thus complete. □

4. Necessary condition of the modularity of an element in \(\text{MON}\)

Given any word \(w\) and letter \(x\), let \(\text{occ}_x(w)\) denote the number of occurrences of \(x\) in \(w\). Let \(\lambda\) denote the empty word. Let \(W = W_1 \cup W_2\), where

\[
W_1 = \{y^{r_1}xt^{r_2}z^{r_3}y^{r_4}t^{r_5}x^{r_6} | r_1, r_2, r_3, r_4, r_5, r_6 \geq 2\},
\]

\[
W_2 = \{y^{r_1}xt^{r_2}z^{r_3}xy^{r_4}t^{r_5}x^{r_6} | r_1, r_2, r_3, r_4, r_5, r_6 \geq 2\}.
\]

Let us fix the following two words:

\[p = y^2xt^2z^2y^2t^2xz^2\] and \[q = y^2xt^2z^2x^2y^2t^2xz^2\].

Put \(K = \text{var}\{p \approx q\}\).

**Lemma 4.1.** The set \(W\) is a \(\text{FIC}(K)\)-class.

**Proof.** Let \(u \approx v\) be any identity of \(K\) with \(u \in W\). We need to verify that \(v \in W\). By assumption, there is a deduction of the identity \(u \approx v\)
from the identity $p \approx q$, that is, a sequence $w_0, w_1, \ldots, w_m$ of words such that $w_0 = u$, $w_m = v$ and, for each $i = 0, 1, \ldots, m - 1$, there are words $a_i, b_i \in \mathcal{X}^*$ and an endomorphism $\xi_i$ of $\mathcal{X}^*$ such that $w_i = a_i \xi_i(s_i) b_i$ and $w_{i+1} = a_i \xi_i(t_i) b_i$, where $\{s_i, t_i\} = \{p, q\}$. By trivial induction on $m$, it suffices to only consider the case when $u = a \xi(s) b$ and $v = a \xi(x) b$ for some words $a, b \in \mathcal{X}^*$, an endomorphism $\xi$ of $\mathcal{X}^*$ and words $s$ and $t$ such that $\{s, t\} = \{p, q\}$.

Since any subword of $u$ of the form $ab$, where $a$ and $b$ are distinct letters, occurs only once in $u$ and all letters occurring in $s$ are multiple, the following holds:

(I) For any $a \in \text{con}(s)$, either $\xi(a) = \lambda$ or $\xi(a)$ is a power of some letter. Further, since $\text{occ}_x(u) \leq 3$ and $\text{occ}_y(s) = \text{occ}_z(s) = \text{occ}_t(s) = 4$, we have

$$(\Pi) \ x \not\in \text{con}(\xi(yzt)).$$

We note that if $\xi(s) = \lambda$ or $\xi(s)$ is a power of some letter, then the required statement is evident. So, we may assume that

$$(\PiII) |\text{con}(\xi(s))| \geq 2.$$ Let $u = y_{\ell_1} x_{\ell_2} z_{\ell_3} x_{\ell_4} y_{\ell_5} x_{\ell_6}$, where $c \in \{0, 1\}$ and $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6 \geq 2$, and let

$$d = \begin{cases} 
0 & \text{if } s = p, \\
1 & \text{if } s = q.
\end{cases}$$

If $\xi(x) = \lambda$, then $\xi(s) = \xi(t)$, whence $v = u \in W$. So, it remains to consider the case when $\xi(x) \neq \lambda$. Then (I) implies that $\xi(x)$ is a power of some letter.

Suppose that $\xi(x)$ is a power of $y$. Then (II) implies that

$$\text{con}(\xi(t^2 z x^d y^2 t^2))$$

contains one of the letters $x, z$ and $t$. This is only possible when $\xi(t^2 z x^d y^2 t^2) = y^p x_{\ell_2} z_{\ell_3} x_{\ell_4} y^q$ for some $0 \leq p \leq \ell_1$ and $0 \leq q \leq \ell_4$. But since $x \not\in \{y\} = \text{con}(\xi(x))$ by assumption and $x \not\in \text{con}(\xi(yzt))$ by (II), the contradiction $x \not\in \text{con}(\xi(t^2 z x^d y^2 t^2))$ is deduced. Therefore, $\xi(x)$ cannot be a power of $y$. Similarly, $\xi(x)$ cannot be a power of $z$ as well.

Suppose now that $\xi(x)$ is a power of $t$. Then (III) implies that $\text{con}(\xi(t^2 z x^d y^2 t^2))$ contains one of the letters $x, y$, and $z$. This is only possible when $\xi(t^2 z x^d y^2 t^2) = t^p z_{\ell_3} x_{\ell_4} y^q$ for some $0 \leq p \leq \ell_2$ and $0 \leq q \leq \ell_5$. Then by (I), either $\xi(t) = \lambda$ or $\xi(t)$ is a power of $t$. This implies that $\xi(z_{\ell_3} x_{\ell_4} y^q) = z_{\ell_3} x_{\ell_4} y^q$. Taking into account that $\xi(x)$ is a power of $t$, we apply (I) again and obtain that $\xi(z_{\ell_3} y^q) = z_{\ell_3} x_{\ell_4}$, $\xi(y^q) = y^{q_0}$ and $c = d = 0$. This is only possible when $\xi(x t^2 z x^d) = y^r x_{\ell_2} z_{\ell_3} x_{\ell_4}$ for some $0 \leq r \leq \ell_1$. But since $x \not\in \{t\} = \text{con}(\xi(x))$ by assumption and $x \not\in \text{con}(\xi(yzt))$ by (II), the contradiction $x \not\in \text{con}(\xi(x t^2 z x^d))$ is deduced. Therefore, $\xi(x)$ cannot be a power of $t$.

Finally, suppose that $\xi(x)$ is a power of $x$. Then since $x^2$ is not a subword of $u$, we have $\xi(x) = x$.

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Suppose that \( c = 0 \). Then \( d = 0 \) because otherwise, \( \text{occ}_x(u) < \text{occ}_x(\xi(s)) \). Then \( \xi(t^2 z^2 y^2 t^2) = t^{\ell_2} z^{\ell_3} y^{\ell_4} t^{\ell_5} \). It follows from (I) that \( \xi(z^2) = z^{\ell_3}, \xi(y^2) = y^{\ell_4} \) and \( \xi(t^2) = t^{\ell_2} = t^{\ell_5} \). Then

\[
\xi(s) = y^{\ell_4} x t^{\ell_2} z^{\ell_3} y^{\ell_4} t^{\ell_5} x z^{\ell_3}, \quad a = y^{\ell_1 - \ell_4} \quad \text{and} \quad b = z^{\ell_6 - \ell_3}.
\]

Therefore, \( \xi(t) = y^{\ell_4} x t^{\ell_2} z^{\ell_3} y^{\ell_4} t^{\ell_5} x z^{\ell_3} \), whence \( v = y^{\ell_1} x t^{\ell_2} z^{\ell_3} y^{\ell_4} t^{\ell_5} x z^{\ell_6} \in W \), and we are done.

Suppose now that \( c = 1 \). If \( x \in \text{con}(b) \), then \( d = 0 \) because otherwise, \( \text{occ}_x(u) < \text{occ}_x(\xi(s)b) \). This is only possible when

\[
\begin{align*}
a \xi(y^2) &= y^{\ell_1}, \quad x \xi(t^2 z^2 y^2 t^2)x = x t^{\ell_2} z^{\ell_3} x \quad \text{and} \quad \xi(z^2)b = y^{\ell_4} t^{\ell_5} x z^{\ell_6}.
\end{align*}
\]

The second equality implies that \( \xi(t^2 z^2 y^2 t^2) = t^{\ell_2} z^{\ell_3} \). Clearly, \( \xi(t^2) = \lambda \), whence \( \xi(z^2 y^2) = t^{\ell_2} z^{\ell_3} \). In view of (I), we have \( \xi(z^2) = t^{\ell_2} \) and \( \xi(y^2) = z^{\ell_3} \). But this contradicts the fact that \( \xi(z^2)b = z^{\ell_4} t^{\ell_5} x z^{\ell_6} \). Therefore, \( x \notin \text{con}(b) \).

Analogously, one can verify that \( x \notin \text{con}(a) \). It follows that \( d = 1 \). Then

\[
\begin{align*}
a \xi(y^2) &= y^{\ell_1}, \quad x \xi(t^2 z^2 y^2 t^2)x = x t^{\ell_2} z^{\ell_3} x y^{\ell_4} t^{\ell_5} x \quad \text{and} \quad \xi(z^2)b = z^{\ell_6}.
\end{align*}
\]

It follows from (I) that \( \xi(z^2) = z^{\ell_3}, \xi(y^2) = y^{\ell_4} \) and \( \xi(t^2) = t^{\ell_2} = t^{\ell_5} \). Then

\[
\begin{align*}
\xi(s) &= y^{\ell_3} x t^{\ell_2} z^{\ell_3} x y^{\ell_4} t^{\ell_5} x z^{\ell_4}, \quad a = y^{\ell_1 - \ell_3} \quad \text{and} \quad b = z^{\ell_6 - \ell_4}.
\end{align*}
\]

Therefore, \( \xi(t) = y^{\ell_3} x t^{\ell_2} z^{\ell_3} x y^{\ell_4} t^{\ell_5} x z^{\ell_4} \), whence \( v = y^{\ell_1} x t^{\ell_2} z^{\ell_3} y^{\ell_4} t^{\ell_5} x z^{\ell_6} \in W \), and we are done.

For any \( n \geq 1 \), put

\[
B_n = \text{var}\{x^n \approx x^{n+1}\}.
\]

**Lemma 4.2.** Suppose that \( V \) is any proper monoid variety that is a modular element of the lattice \( \text{MON} \). Then \( V \) is periodic.

**Proof.** Seeking a contradiction, suppose that \( V \) is not periodic, so that \( V \) contains the variety \( \text{COM} \) of all commutative monoids. Since \( V \) is proper and non-periodic, it satisfies some non-trivial identity \( u \approx v \) such that every letter from \( \text{con}(uv) \) occurs \( n \) times on both sides for some \( n \geq 1 \), that is, \( n = \text{occ}_a(u) = \text{occ}_a(v) \) for all \( a \in \text{con}(uv) \). Then by [14, Lemma 3.2], there exist two distinct letters \( x \) and \( y \) such that the identity obtained from \( u \approx v \) by retaining \( x \) and \( y \) is non-trivial. Therefore we may assume that \( \text{con}(u) = \text{con}(v) = \{x, y\} \) with \( n = \text{occ}_x(u) = \text{occ}_x(v) = \text{occ}_y(u) = \text{occ}_y(v) \).

Suppose that \( \text{LRB} \subseteq V \). In view of Lemma 2.2, we may assume without loss of generality that \( \text{ini}(u) = \text{ini}(v) = xy \). Let \( u' \) and \( v' \) be words that obtain from \( u \) and \( v \), respectively, by making the substitution \( (x, y) \mapsto (y, x) \). Then \( \text{ini}(u') = \text{ini}(v') = yx \). Put

\[
A = \{w \in \{x, y\}^+ \mid \text{occ}_x(w) = n + 1, \text{occ}_y(w) = n\}.
\]
Let \( w, w' \in A \) and \( w \preceq w' \). This means that \( w' = aw(\xi)bw \) for some words \( a, b \in \mathcal{X}^+ \) and some endomorphism \( \xi \) of \( \mathcal{X}^+ \). Since the length of \( w \) equals to the length of \( w' \), we have \( a = b = \lambda \). Then \( w' = \xi(w) \). But this is only possible when \( \xi(x) = x \) and \( \xi(y) = y \) because \( \text{occ}_y(w) < \text{occ}_x(w) \). Hence \( w = w' \). So, \( A \) is an anti-chain under the quasi-order \( \preceq \). Then \( L_A \) is a sublattice of \( \text{MON} \) by Lemma 2.1(i) and \( V \in L_A \).

Clearly, \( ux, u'x, vx, v'x \in A \). Evidently, \( \text{ini}(ux) = \text{ini}(vx) = xy \) and \( \text{ini}(u'x) = \text{ini}(v'x) = yx \). In view of Lemma 2.2, the words \( ux \) and \( u'x \) lie in distinct \( \text{FIC}(V) \)-classes. Then, since \( V \) satisfies the nontrivial identities \( ux = vx \) and \( u'x = v'x \), the equivalence \( \gamma = \varphi(V) \) contains at least two non-singleton classes. It is verified in [11, Proposition 2.2] that a partition \( \rho \in \text{Part}(X) \) is a modular element in \( \text{Part}(X) \) if and only if \( \rho \) has at most one non-singleton class. This result implies that \( \gamma \) is not a modular element of the lattice \( \text{Part}(A) \). Then there are \( \alpha, \beta \in \text{Part}(A) \) such that \( \alpha \subset \beta \) and

\[
(\gamma \wedge \beta) \vee \alpha \subset (\gamma \vee \alpha) \wedge \beta.
\]

According to Lemma 2.1, we can find a non-periodic variety \( X \in L_A \) such that \( \varphi(X) = \alpha \). Put

\[
Y = X \wedge \text{var}\{w \approx w' \mid (w, w') \in \beta\}.
\]

Clearly, \( Y \in L_A \) and \( \varphi(Y) = \beta \). Then

\[
(V \wedge X) \vee Y \subset (V \vee Y) \wedge X
\]

because otherwise, the inclusion (4.1) does not hold. We see that \( V \) is not a modular element of the lattice \( \text{MON} \), which is a contradiction.

Suppose now that \( LRB \not\subseteq V \). Then Lemma 2.2 allows us to assume that \( u \) starts with the letter \( x \) but \( v \) starts with the letter \( y \). Let

\[
Z = \text{var}\{x^{n+1} \approx x^{n+2}, x^n v \approx x^{n+1} v\}.
\]

We note that \( V \wedge B_{n+1} \subseteq Z \). Indeed, \( V \wedge B_{n+1} \) satisfies the identities

\[
x^n v \approx x^n u \approx x^{n+1} u \approx x^{n+1} v
\]

and so the identity \( x^n v \approx x^{n+1} v \). Clearly, the word \( x^n v \) is an isoterms for both \( V \vee Z \) and \( B_{n+1} \). It follows that \( x^n v \) is an isoterms for \( (V \vee Z) \wedge B_{n+1} \) as well. However, \( x^n v \) is not an isoterms for \( Z \) because \( Z \) satisfies \( x^n v \approx x^{n+1} v \). Therefore,

\[
(V \wedge B_{n+1}) \vee Z = Z \subset (V \vee Z) \wedge B_{n+1}.
\]

This means that \( V \) is not a modular element of the lattice \( \text{MON} \), which again is a contraction. \( \square \)

The following is the main result of this section.

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Proposition 4.3. Suppose that $V$ is any proper monoid variety that is a modular element of the lattice $\text{MON}$. Then $V$ satisfies the identities

\begin{align}
(4.2) & \quad x^2 \approx x^3, \\
(4.3) & \quad x^2 y \approx y x^2.
\end{align}

Proof. By Lemma 4.2, the variety $V$ is periodic and so it satisfies the identity $x^n \approx x^{n+m}$ for some $n, m \geq 1$; we may assume $n$ and $m$ to be the least possible.

First, suppose that $n = 1$, so that $V$ is completely regular. If $X$ is a non-commutative completely regular variety, then it is verified in [5, Lemma 3.1] that

$$(X \wedge D) \lor C_2 \subset (X \lor C_2) \land D,$$

whence $X$ is not a modular element of the lattice $\text{MON}$. If $X$ is a commutative variety containing a nontrivial group, then it is proved in [5, Lemma 3.2] that

$$(X \wedge B_2) \lor Q \subset (X \lor Q) \land B_2,$$

where $Q = \text{var}\{y x y z x y \approx y x z x y z\}$, whence $X$ is again not a modular element of the lattice $\text{MON}$. In view of these two facts, the variety $V$ is commutative and does not contain any nontrivial group. Since $V$ is also completely regular, it is idempotent and so is contained in $\text{SL}$. Obviously, $\text{SL}$ satisfies (4.2) and (4.3).

So, it remains to consider the case when $n > 1$. Then $C_n \subseteq V$ and $C_{n+1} \not\subseteq V$ by Lemma 2.3. It follows from [6, Lemma 2] that $E \not\subseteq V$. Then by Lemma 2.5, $V$ satisfies the identity $x^{p_1} y x^{q_1} \approx y x^{r_1}$ for some $p_1, q_1 \geq 1$ and $r_1 \geq 2$. The dual arguments imply that $V$ also satisfies the identity $x^{p_2} y x^{q_2} \approx x^{r_2} y$ for some $p_2, q_2 \geq 1$ and $r_2 \geq 2$. Since one can substitute $x^n$ for $x$ in these identities and $V$ satisfies $x^n \approx x^{n+m}$, we may assume without loss of generality that

$$p_1, p_2, q_1, q_2, r_1, r_2 \in \{n, n+1, \ldots, n+m-1\}.$$

Evidently, there exist $\ell_1$ and $\ell_2$ such that the identities

$$x^{p_1} y x^{q_1+\ell_1} \approx y x^{r_1+\ell_1} \quad \text{and} \quad x^{p_2+\ell_2} y x^{q_2} \approx x^{r_2+\ell_2} y$$

are equivalent modulo $x^n \approx x^{n+m}$ to the identities

$$x^{p_1} y x^{q_2} \approx y x^{r_1+\ell_1} \quad \text{and} \quad x^{p_1} y x^{q_2} \approx x^{r_2+\ell_2} y,$$

respectively. Therefore $V$ satisfies $x^{r_2+\ell_2} y \approx y x^{r_1+\ell_1}$, whence it satisfies

\begin{align}
(4.4) & \quad x^k y \approx y x^k.
\end{align}
for some \( k \geq n \). It follows that the meet \( V \land B_2 \) satisfies the identities (4.2) and (4.4); it also satisfies the identity \( p \approx q \) because

\[
p = y^2xt^2z^2y^2t^2xz^2 \approx y^kxt^kz^k y^k x^k z^k \approx y^2xt^2z^2t^2z^2 = q.
\]

Therefore \( V \land B_2 \subseteq K \), so that \((V \land B_2) \lor K = K\).

Suppose that \( n > 2 \) or \( m > 1 \). Recall from the beginning of the section that

\[
W_1 = \{ y^{r_1}xt^{r_2}z^{r_3}y^{t_4}t^{t_5}x^rz^r \mid r_1, r_2, r_3, r_4, r_5, r_6 \geq 2 \}.
\]

Let \( a \approx b \) be any identity of \( V \lor K \) with \( a \in W_1 \). If \( n > 2 \), then \( b \in W_1 \) by Lemmas 2.3 and 4.1. Clearly, \( V \) contains the variety \( A_m \) of all Abelian groups of exponent \( m \). It is well known and easily verified that an identity \( w \approx w' \) holds in \( A_m \) if and only if \( \text{occ}_a(w) \equiv \text{occ}_a(w') \pmod{m} \) for all \( a \in \mathfrak{X} \). This fact and Lemma 4.1 imply that if \( m > 1 \), then \( b \in W_1 \). We see that if \( n > 2 \) or \( m > 1 \), then \( b \in W_1 \) in either case. Evidently, if \( B_2 \) satisfies an identity \( c \approx d \) with \( c \in W_1 \), then \( d \in W_1 \). This implies that if an identity of the form \( p \approx w \) holds in \((V \lor K) \land B_2\), then \( w \in W_1 \). In particular, \((V \lor K) \land B_2 \) violates \( p \approx q \). Therefore,

\[
(V \land B_2) \lor K = K \subset (V \lor K) \land B_2.
\]

This means that \( V \) is not a modular element in \( \text{MON} \). It follows that \( n = 2 \) and \( m = 1 \). Then \( V \) satisfies (4.2). Besides that, since (4.4) holds in the variety \( V \), this variety satisfies (4.3).

Proposition 4.3 is thus proved. \( \Box \)

5. Proof of Theorem 1.1

**Necessity.** Let \( V \) be any proper monoid variety that is a cancellable element of the lattice \( \text{MON} \). Since any cancellable element is modular, Proposition 4.3 implies that \( V \) satisfies the identities (4.2) and (4.3). If \( V \) does not coincide with any of the varieties \( T, \ S_L, C_2 \) and \( D \), then \( V \) contains the variety \( D_2 \) by [8, Lemma 3.3(i)]. Proposition 3.1 and the fact that \( C_3 \not\subseteq V \) imply that \( V \lor R = V \lor R^\delta \) and \( V \land R = V \land R^\delta = D \), contradicting the assumption that \( V \) is a cancellable element of \( \text{MON} \). Hence \( V \) coincides with one of the varieties \( T, \ S_L, C_2 \) and \( D \).

**Sufficiency.** Obviously, \( T \) and \( \text{MON} \) are cancellable elements of \( \text{MON} \). An element \( x \) of a lattice \( L \) is costandard if

\[
\forall y, z \in L: \quad (x \land z) \lor y = (x \lor y) \land (z \lor y).
\]

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It is easily seen that any costandard element is cancellable. It is shown in [5, Theorem 1.2] that the varieties $SL$ and $C_2$ are costandard elements of the lattice $MON$. Therefore, these varieties are cancellable elements of this lattice.

So, it remains to establish that $D$ is a cancellable element in $MON$. Let $X$ and $Y$ be monoid varieties such that $D \vee X = D \vee Y$ and $D \wedge X = D \wedge Y$. If $D \subseteq X$, then $D = D \wedge X = D \wedge Y$, so that $D \subseteq Y$, whence $X = D \vee X = D \vee Y = Y$ and we are done. Therefore by symmetry, we may assume that $D \not\subseteq X$ and $D \not\subseteq Y$.

Now the subvariety lattice $L(D)$ is the chain $T \subseteq SL \subseteq C_2 \subseteq D$; see Fig. 2. It follows that $D \wedge X = D \wedge Y \in \{T, SL, C_2\}$. If $D \wedge X = D \wedge Y = T$, then $X$ and $Y$ are varieties of groups by [7, Lemma 2.1]. Then $X \vee Y$ is a variety of groups too and so $SL \not\subseteq X \vee Y$, whence

\[(5.1) \quad D \wedge (X \vee Y) = D \wedge X = D \wedge Y.\]

If $D \wedge X = D \wedge Y = SL$, then $X$ and $Y$ are completely regular varieties by [7, Corollary 2.6]. Then $X \vee Y$ is completely regular and so $C_2 \not\subseteq X \vee Y$, whence the equality (5.1) is true. Finally, if $D \wedge X = D \wedge Y = C_2$, then $X$ and $Y$ are commutative by Lemma 2.4. Then $X \vee Y$ is commutative and so $D \not\subseteq X \vee Y$, whence the equality (5.1) is true again. We see that the equality (5.1) holds in any case.

Clearly,

\[(5.2) \quad D \vee (X \vee Y) = D \vee X = D \vee Y.\]

Then

\[
X = (D \wedge X) \vee X \quad \text{because } D \wedge X \subseteq X \\
= (D \wedge (X \vee Y)) \vee X \quad \text{by (5.1)} \\
= (D \vee X) \wedge (X \vee Y) \quad \text{by [6, Proposition 4]} \\
= (D \vee (X \vee Y)) \wedge (X \vee Y) \quad \text{by (5.2)} \\
= (X \vee Y) \quad \text{because } X \vee Y \subseteq D \vee (X \vee Y). 
\]

We see that $X = X \vee Y$. By symmetry, $Y = X \vee Y$, whence $X = Y$. Therefore, $D$ is a cancellable element in $MON$. □

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