A Fock space approach to the theory of strictly positive kernels

MICHIRO SETO
National Defense Academy, Yokosuka 239-8686, Japan
E-mail address: mseto@nda.ac.jp

Abstract
In this paper, we give a new approach to the theory of strictly positive kernels. Our method is based on the structure of Fock spaces. As its applications, various examples of strictly positive kernels are given. Moreover, we give a new proof of the universal approximation theorem for the Gauss kernel.

2020 Mathematical Subject Classification: Primary 46E22; Secondary 30C40
keywords: kernel function, pseudo-hyperbolic distance, reproducing kernel Hilbert space

1 Introduction
The purpose of this paper is to introduce a new method for dealing with strictly positive kernel functions. Our method is based on the structure of Fock spaces. We will apply the tensor algebra structure of the full Fock space to the theory of strictly positive kernel functions. This is the main idea of this paper. As consequences, we obtain not only various examples of strictly positive kernel functions, but also a new proof of the universal approximation theorem for the Gauss kernel.

This paper is organized as follows. In Section 2, we introduce huge reproducing kernel Hilbert spaces from power series whose coefficients are strictly positive. Although all materials of Section 2 are well known to specialists in Hilbert space operator theory, we will give the details for the sake of general readers. In Section 3, we investigate relations between the strict positivity of kernel functions and reproducing kernel Hilbert spaces constructed in Section 2. In Section 4, various examples of strictly positive kernels are given as applications of results obtained in Section 3. In Section 5, we prove the universal approximation theorem for the Gauss kernel with our method.

2 Preliminaries
Let $k$ be a kernel function on a set $X$, and let $H_k$ be the reproducing kernel Hilbert space generated by $k$. We will use the notation $k_x(y) = k(y, x)$ throughout the whole of this
paper. We fix a sequence \( \{ a_n \}_{n \geq 0} \) such that \( a_n > 0 \) for any \( n \geq 0 \) and
\[
\sum_{n=0}^{\infty} a_n \| k_x \|_{\mathcal{H}_k}^{2n} = \sum_{n=0}^{\infty} a_n k(x,x)^n < \infty
\]
for any \( x \) in \( X \). Moreover, we set \( \varphi(z) = \sum_{n=0}^{\infty} a_n z^n \).

In this section, we construct a huge reproducing kernel Hilbert space from \( \mathcal{H}_k \) and \( \varphi \). The contents of this section are well known to specialists. For example, see Exercise (k) in p. 320 of Nikolski [6] and Chapters 5 and 7 in Paulsen-Raghupathi [7]. However, we give the details for the sake of general readers.

Let \( \mathcal{H}_k^n \) denote the reproducing kernel Hilbert space obtained by the pull-back construction with the \( n \)-fold tensor product space
\[
\mathcal{H}_k^n = \mathcal{H}_k \otimes \cdots \otimes \mathcal{H}_k
\]
and the \( n \)-dimensional diagonal map
\[
\Delta_n : X \to X^n, \; \lambda \mapsto (x, \ldots, x).
\]
More precisely, \( \mathcal{H}_k^n \) is equal to \( \{ F \circ \Delta_n : F \in \mathcal{H}_k^{\otimes n} \} \) as vector spaces and its inner product is defined by
\[
\langle f,g \rangle_{\mathcal{H}_k^n} = \langle P(\ker \Delta_n)^\perp F, P(\ker \Delta_n)^\perp G \rangle_{\mathcal{H}^{\otimes n}} \quad (f = F \circ \Delta_n, \; g = G \circ \Delta_n),
\]
where we define \( \ker \Delta_n = \{ F \in \mathcal{D}^{\otimes n} : F \circ \Delta_n = 0 \} \) and \( P(\ker \Delta_n)^\perp \) denotes the orthogonal projection onto the orthogonal complement of \( \ker \Delta_n \). Then, it is easy to see that \( k_x^{\otimes n} \) belongs to \( (\ker \Delta_n)^\perp \). Hence, for any function \( f = F \circ \Delta_n \) in \( \mathcal{H}_k^n \), we have
\[
\langle f, k_x^{\otimes n} \rangle_{\mathcal{H}_k^n} = \langle P(\ker \Delta_n)^\perp F, P(\ker \Delta_n)^\perp k_x^{\otimes n} \rangle_{\mathcal{H}^{\otimes n}}
\]
\[
= \langle f, k_x^{\otimes n} \rangle_{\mathcal{H}^{\otimes n}}
\]
\[
= F(x, \ldots, x)
\]
\[
= f(x).
\]
This concludes that \( k_x^{n} \) is the reproducing kernel of \( \mathcal{H}_k^n \) (for the further details of this construction, see Theorems 5.7 and 5.16 in [7]).

Next, let \( \mathcal{F} \) denote the Hilbert space with the inner product
\[
\langle (f_0, f_1, \ldots)^\top, (g_0, g_1, \ldots)^\top \rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} a_n \langle f_n, g_n \rangle_{\mathcal{H}_k^n},
\]
where \( f_n \) and \( g_n \) are functions in \( \mathcal{H}_k^n \), and we set \( \mathcal{H}_k^0 = \mathbb{C} \). We consider the map \( \Gamma \) defined as follows:
\[
\Gamma : \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix} \mapsto \sum_{n=0}^{\infty} a_n f_n \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix} \in \mathcal{F}.
\]
Proposition 2.1. $\Gamma$ is a linear map from $\mathcal{F}$ to the vector space consisting of functions on $X$. Moreover, $\ker \Gamma$ is closed.

Proof. For any $F = (f_0, f_1, \ldots)^\top$ in $\mathcal{F}$, we have
\[
\left| \sum_{\ell=n+1}^{m} a_\ell f_\ell(x) \right| \leq \sum_{\ell=n+1}^{m} |a_\ell f_\ell(x)| \\
\leq \sum_{\ell=n+1}^{m} a_\ell \|f_\ell\|_{\mathcal{H}_k^\ell} \|k_\ell^\ell\|_{\mathcal{H}_k^\ell} \\
\leq \left( \sum_{\ell=n+1}^{m} a_\ell \|f_\ell\|_{\mathcal{H}_k^\ell}^2 \right)^{1/2} \left( \sum_{\ell=n+1}^{m} a_\ell \|k_\ell^\ell\|_{\mathcal{H}_k^\ell}^2 \right)^{1/2} \\
= \left( \sum_{\ell=n+1}^{m} a_\ell \|f_\ell\|_{\mathcal{H}_k^\ell}^2 \right)^{1/2} \left( \sum_{\ell=n+1}^{m} a_\ell \|k_\ell^\ell\|_{\mathcal{H}_k^\ell}^2 \right)^{1/2}. \quad (2.1)
\]
Hence, $\sum_{n=0}^{\infty} a_n f_n(x)$ converges. This concludes that $\Gamma$ is a linear map from $\mathcal{F}$ to the vector space consisting of functions on $X$. Moreover, by (2.1), we have
\[
\| (\Gamma F)(x) \| \leq \| F \|_{\mathcal{F}} \left( \sum_{n=0}^{\infty} a_n k(x, x)^n \right)^{1/2}. \quad (2.2)
\]
This inequality concludes that $\ker \Gamma$ is closed. \qed

By Proposition 2.1, the pull-back construction can be applied to $\Gamma$.

Definition 2.1. We define $\varphi(\mathcal{H}_k)$ as the reproducing kernel Hilbert space obtained by the pull-back construction with the linear map $\Gamma$, that is, $\varphi(\mathcal{H}_k)$ is equal to the range of $\Gamma$ as vector spaces and its inner product is defined by
\[
\langle f, g \rangle_{\varphi(\mathcal{H}_k)} = \langle P(\ker \Gamma)^{-1} F, P(\ker \Gamma)^{-1} G \rangle_{\mathcal{F}} \quad (f = \Gamma F, \ g = \Gamma G, \ F, G \in \mathcal{F}).
\]

We summarize basic properties of $\varphi(\mathcal{H}_k)$.

Proposition 2.2. $\varphi(\mathcal{H}_k)$ is a reproducing kernel Hilbert space consisting of functions on $X$. More precisely, for any $f$ in $\varphi(\mathcal{H}_k)$, there exists a vector $(f_0, f_1, \ldots)^\top$ in $\mathcal{F}$ such that
\[
f = \sum_{n=0}^{\infty} a_n f_n.
\]
Moreover, the reproducing kernel of $\varphi(\mathcal{H}_k)$ is
\[
\sum_{n=0}^{\infty} a_n k_n^x = \varphi(k_x),
\]
that is,
\[
f(x) = \langle f, \varphi(k_x) \rangle_{\varphi(\mathcal{H}_k)} \quad \text{for any } f \text{ in } \varphi(\mathcal{H}_k) \text{ and any } x \text{ in } X.
\]
Proof. We shall show that \( \varphi(k_x) \) is the reproducing kernel of \( \varphi(H_k) \). If \( (h_0, h_1, h_2, \ldots)^\top \) is a vector in ker \( \Gamma \), then

\[
\langle (h_0, h_1, h_2, \ldots)^\top, (1, k_x, k_x^2, \ldots)^\top \rangle_F = \sum_{n=0}^{\infty} a_n \langle h_n, k_x \rangle_{H_k^n} = \sum_{n=0}^{\infty} a_n h_n(x) = 0.
\]

Hence, \( (1, k_x, k_x^2, \ldots)^\top \) belongs to \( (\ker \Gamma)^\perp \). Next, for any function \( f = \Gamma(f_0, f_1, f_2, \ldots)^\top \) in \( \varphi(H_k) \), we have

\[
\langle f, \varphi(k_x) \rangle_{\varphi(H_k)} = \langle P_{(\ker \Gamma)^\perp} (f_0, f_1, f_2, \ldots)^\top, P_{(\ker \Gamma)^\perp} (1, k_x, k_x^2, \ldots)^\top \rangle_F
\]

\[
= \sum_{n=0}^{\infty} a_n \langle f_n, k_x^n \rangle_{H_k^n}
\]

\[
= \sum_{n=0}^{\infty} a_n f_n(x)
\]

\[
= f(x).
\]

This concludes the proof.

3 Strictly positive kernels

In this section, we will investigate relations between the strict positivity of \( \varphi(k) \) and the structure of \( \varphi(H_k) \). Some of results obtained in Kuwahara-S [5] are generalized into the setting of this paper.

Lemma 3.1 ([5]). Let \( \psi \) be a function in \( H_k \). Then, the multiplication operator \( M_\psi \) with symbol \( \psi \) is a densely defined closable linear operator in \( \varphi(H_k) \). In particular, the adjoint operator \( M_\psi^* \) is a densely defined closed linear operator in \( \varphi(H_k) \), and every \( \varphi(k_x) \) is an eigenfunction of \( M_\psi^* \). More precisely,

\[
M_\psi^* \varphi(k_x) = \overline{\psi(x)} \varphi(k_x).
\]

Proof. We define the bounded linear operator \( \tau_\psi \) as follows:

\[
\tau_\psi : \mathcal{H}_k^\otimes n \to \mathcal{H}_k^\otimes n+1, \quad F \mapsto \psi \otimes F.
\]

Then, the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{H}_k^\otimes n & \xrightarrow{\tau_\psi} & \mathcal{H}_k^\otimes n+1 \\
\Delta_n \downarrow & & \downarrow \Delta_{n+1} \\
\mathcal{H}_k^n & \xrightarrow{M_\psi|_{\mathcal{H}_k^n}} & \mathcal{H}_k^{n+1},
\end{array}
\]
where $\Delta_n$ is identified with the linear map $F \mapsto F \circ \Delta_n$. Hence, for any function $f_n$ in $\mathcal{H}_k^n$, $\psi f_n$ belongs to $\mathcal{H}_k^{n+1}$. Let $F = (f_0, f_1, \ldots, f_N, 0, \ldots)^\top$ be a vector with finite support in $\mathcal{F}$. We set $f = \Gamma F$. Then,

$$
\psi f = \psi \sum_{n=0}^{N} a_n f_n = \sum_{n=0}^{N} a_n \psi f_n = \sum_{n=0}^{N} a_{n+1} \frac{a_n}{a_{n+1}} \psi f_n = \sum_{n=1}^{N+1} a_{n-1} \frac{a_n}{a_{n-1}} \psi f_{n-1},
$$

where we note that $\psi f_{n-1}$ belongs to $\mathcal{H}_k^n$. Hence, setting

$$
G = \left( 0, \frac{a_0}{a_1} \psi f_0, \frac{a_1}{a_2} \psi f_1, \ldots, \frac{a_N}{a_{N+1}} \psi f_N, 0, \ldots \right)^\top,
$$

$G$ belongs to $\mathcal{F}$ and $\Gamma G = \psi f$, that is, $\psi f$ belongs to $\varphi(\mathcal{H}_k)$. Therefore, $M_\psi$ is a densely defined linear operator in $\varphi(\mathcal{H}_k)$. Moreover, it is easy to see that $M_\psi$ is closable and $M_\psi^* \varphi(k_x) = \overline{\psi(x) \varphi(k_x)}$. \hfill \Box

We extract the next definition and theorem from the proof of the main theorem in [5] where exponentials of de Branges-Rovnyak kernels were discussed.

**Definition 3.1** ([5]). Let $\mathcal{H}_k$ be a reproducing kernel Hilbert space on a set $X$. Then, $X$ is said to be finitely separated by $\mathcal{H}_k$ if, for any positive integer $n$ and for any distinct points $x_1, \ldots, x_n$ in $X$, there exists a function $\psi$ in $\mathcal{H}_k$ such that $\psi(x_i) \neq \psi(x_j)$ whenever $i \neq j$.

**Theorem 3.1** ([5]). Let $\mathcal{H}_k$ be a reproducing kernel Hilbert space on a set $X$. If $X$ is finitely separated by $\mathcal{H}_k$, then $\varphi(k)$ is strictly positive definite.

**Proof.** It suffices to show that $\left\{ \varphi(k_x) \right\}_{j=1}^n$ is linearly independent for any $n$ in $\mathbb{N}$ and any $n$ distinct points $x_1, \ldots, x_n$ in $X$. Suppose that

$$
\sum_{j=1}^{n} c_j \varphi(k_{x_j}) = 0
$$

for some $n$ in $\mathbb{N}$, some $n$ distinct points $x_1, \ldots, x_n$ in $X$, and some $c_1, \ldots, c_n$ in $\mathbb{C}$. Then, for any function $\psi$ in $\mathcal{H}_k$, by Lemma 3.1, we have

$$
\begin{pmatrix}
\frac{1}{\psi(x_1)} & \cdots & \frac{1}{\psi(x_n)} \\
\vdots & \ddots & \vdots \\
\frac{1}{\psi(x_1)^{-1}} & \cdots & \frac{1}{\psi(x_n)^{-1}}
\end{pmatrix}
\begin{pmatrix}
c_1 \varphi(k_{x_1}) \\
c_2 \varphi(k_{x_2}) \\
\vdots \\
c_n \varphi(k_{x_n})
\end{pmatrix}
= \begin{pmatrix}
\sum_{j=1}^{n} c_j \varphi(k_{x_j}) \\
\sum_{j=1}^{n} c_j \psi(x_j) \varphi(k_{x_j}) \\
\vdots \\
\sum_{j=1}^{n} c_j \psi(x_j)^{-1} \varphi(k_{x_j})
\end{pmatrix}
= \begin{pmatrix}
\sum_{j=1}^{n} c_j \varphi(k_{x_j}) \\
M_\psi \sum_{j=1}^{n} c_j \varphi(k_{x_j}) \\
\vdots \\
(M_\psi^*)^{n-1} \sum_{j=1}^{n} c_j \varphi(k_{x_j})
\end{pmatrix}
= 0.
$$
However, by the assumption, there exists a function $\psi$ in $\mathcal{H}_k$ such that

$$
\prod_{1 \leq i < j \leq n} (\psi(x_i) - \psi(x_j)) \neq 0.
$$

Then, the Vandermonde matrix

$$
\begin{pmatrix}
1 & \cdots & 1 \\
\psi(x_1) & \cdots & \psi(x_n) \\
\vdots & \vdots & \vdots \\
\psi(x_1)^{n-1} & \cdots & \psi(x_n)^{n-1}
\end{pmatrix}
$$

is nonsingular. Therefore, we have that

$$
\begin{pmatrix}
c_1 \varphi(k_{x_1}) \\
c_2 \varphi(k_{x_2}) \\
\vdots \\
c_n \varphi(k_{x_n})
\end{pmatrix} = 0.
$$

This concludes that $c_1 = \cdots = c_n = 0$. Indeed, if $\varphi(k_{x_j}) = 0$ as a function for some $1 \leq j \leq n$, then we would have $f(x_j) = 0$ for any $f$ in $\varphi(\mathcal{H}_k)$. However, $\varphi(\mathcal{H}_k)$ includes $\mathbb{C}$ by definition. Hence, $\varphi(k_{x_j}) \neq 0$ for any $1 \leq j \leq n$.

Further, in the case where $\varphi(z) = e^z$, we have the following useful results (cf. Theorem 4.1 in Guella [2]).

**Lemma 3.2.** Let $\mathcal{H}_k$ be a reproducing kernel Hilbert space on a set $X$. If $\exp tk$ is strictly positive definite for some $t > 0$, then so is $\exp(-t\|k_x - k_y\|_{\mathcal{H}_k}^2)$.

**Proof.** Suppose that $\exp tk$ is strictly positive definite. Then, since

$$
\exp(2t \Re \langle k_x, k_y \rangle_{\mathcal{H}_k}) = \exp(t \langle k_x, k_y \rangle_{\mathcal{H}_k}) \exp(t \langle k_x, k_y \rangle_{\mathcal{H}_k}) = \exp tk(x, y) \exp tk(x, y),
$$

$\exp(2t \Re \langle k_x, k_y \rangle_{\mathcal{H}_k})$ is strictly positive definite by the Schur product theorem (Theorem 7.5.3 in Horn-Johnson [4]). Hence, for any distinct points $x_1, \ldots, x_n$ in $X$ and any $(c_1, \ldots, c_n)$ in $\mathbb{C}^n \setminus \{0\}$, we have

$$
\sum_{i,j=1}^n c_i c_j e^{-t\|k_{x_i} - k_{x_j}\|_{\mathcal{H}_k}^2} = \sum_{i,j=1}^n c_i c_j e^{-t\|k_{x_i} - k_{x_j}\|_{\mathcal{H}_k}^2} \exp(2t \Re \langle k_{x_i}, k_{x_j} \rangle_{\mathcal{H}_k}) > 0.
$$

This concludes the proof.

**Theorem 3.2.** Let $\Phi$ be any map from a set $X$ to a Hilbert space $\mathcal{H}$. If $\Phi$ is injective, then $\exp t \langle \Phi(y), \Phi(x) \rangle_{\mathcal{H}}$ is strictly positive definite for any $t > 0$. Moreover, then, so is $\exp(-t\|\Phi(x) - \Phi(y)\|_{\mathcal{H}}^2)$.
Proof. We set $k(x, y) = \langle \Phi(y), \Phi(x) \rangle_{\mathcal{H}}$. Then, $k$ is a kernel function. First, we shall show that $\exp tk$ is strictly positive definite. By virtue of Theorem 3.1 and Lemma 3.2, it suffices to show that $X$ is finitely separated by $\mathcal{H}_{tk}$. Let $x_1, \ldots, x_n$ be any distinct points in $X$, Let $\mathcal{M}$ be the subspace generated by $\Phi(x_1), \ldots, \Phi(x_n)$. Since $\Phi$ is injective, these are $n$ distinct vectors in $\mathcal{H}$. We shall show that there exists some vector $z$ in $\mathcal{H}$ such that $\langle \Phi(x_i), z \rangle_{\mathcal{H}} \neq \langle \Phi(x_j), z \rangle_{\mathcal{H}}$ whenever $i \neq j$. Without loss of generality, we may assume that $z$ is a vector in $\mathcal{M}$. For any vector $y$ in $\mathcal{M}$, if there existed $1 \leq i(y) < j(y) \leq n$ such that $\langle \Phi(x_{i(y)}), y \rangle_{\mathcal{H}} = \langle \Phi(x_{j(y)}), y \rangle_{\mathcal{H}}$, then we would have

$$\mathcal{M} \subset \bigcup_{1 \leq i < j \leq n} \{ \Phi(x_i) - \Phi(x_j) \}^\perp,$$

where the above orthogonal complements are taken in $\mathcal{M}$. However, comparing the volumes of $\mathcal{M}$ and $\{ \Phi(x_i) - \Phi(x_j) \}^\perp$ in $\mathcal{M}$, we have a contradiction. Hence, there exists a vector $z$ in $\mathcal{M}$ such that $\langle \Phi(x_i), z \rangle_{\mathcal{H}} \neq \langle \Phi(x_j), z \rangle_{\mathcal{H}}$ $(i \neq j)$.

Moreover, setting $\psi(x) = \langle z, \Phi(x) \rangle_{\mathcal{H}}$ and $z = \sum_{j=1}^n c_j \Phi(x_j)$, we have,

$$\psi(x) = \langle z, \Phi(x) \rangle_{\mathcal{H}} = \sum_{j=1}^n c_j \langle \Phi(x_j), \Phi(x) \rangle_{\mathcal{H}} = \sum_{j=1}^n c_j k(x, x_j) = \sum_{j=1}^n c_j k_{x_j}(x).$$

Hence, $\psi$ belongs to $\mathcal{H}_{tk}$. Thus, we have the first half of the theorem. Moreover, since

$$\|k_x - k_y\|^2_{\mathcal{H}_{tk}} = k(x, x) - 2 \text{Re} k(x, y) + k(y, y) = \langle \Phi(x), \Phi(x) \rangle_{\mathcal{H}}^2 - 2 \text{Re} \langle \Phi(x), \Phi(y) \rangle_{\mathcal{H}} + \langle \Phi(y), \Phi(y) \rangle_{\mathcal{H}}^2 = \|\Phi(x) - \Phi(y)\|^2_{\mathcal{H}},$$

we have the second half by Lemma 3.2. 

\[\square\]

4 Examples

4.1 Gauss kernel

Let $\mathcal{H}$ be any Hilbert space. Then, it is well known that $\exp (-t \|x - y\|^2_{\mathcal{H}})$ is strictly positive definite on $\mathcal{H} \times \mathcal{H}$ for any $t > 0$. Indeed, this is a direct consequence of Theorem 3.2.
4.2 Drury-Arveson kernel

Let $\mathcal{H}$ be a Hilbert space, and let $B_{\mathcal{H}}$ be the open unit ball in $\mathcal{H}$. Then, it is well known that

$$\frac{1}{1 - \langle x, y \rangle_{\mathcal{H}}}$$

is strictly positive definite on $B_{\mathcal{H}} \times B_{\mathcal{H}}$ (see Subsection 7.3.1 in [7]). Indeed, consider the case where $k(x, y) = \langle x, y \rangle_{\mathcal{H}}$ and $\varphi(z) = \sum_{n=0}^{\infty} z^n$ in Theorem 3.1.

4.3 Hyperbolic cosine

Let $D$ be the open unit disk in the complex plane $\mathbb{C}$. Then it is well known that

$$\frac{1}{(1 - \lambda z)^2} \quad (\lambda \in D)$$

is the reproducing kernel of the Bergman space over $D$. If

$$\varphi(z) = 1 + \frac{1}{2!} z + \frac{1}{4!} z^2 + \cdots + \frac{1}{(2n)!} z^n + \cdots,$$

then, by Theorem 3.1,

$$\varphi\left(\frac{1}{(1 - \lambda z)^2}\right) = \cosh\left(\frac{1}{1 - \lambda z}\right)$$

is strictly positive definite on $D^2$. Note that $1/(1 - \lambda z)$ is the reproducing kernel of the Hardy space over $D$.

4.4 Pseudo-hyperbolic distance

Let $D$ be the open unit disk in the complex plane $\mathbb{C}$. Then, it is well known that $D$ is a metric space with the distance function

$$d(\lambda, \mu) = \left|\frac{\lambda - \mu}{1 - \overline{\lambda}\mu}\right| \quad (\lambda, \mu \in D),$$

which is called the pseudo-hyperbolic distance. The metric space structure of $D$ is very important not only in complex analysis but also in functional analysis (for example, see Garnett [1]). We shall prove that

$$K_t(\lambda, \mu) = \exp\left(-t \left|\frac{\lambda - \mu}{1 - \overline{\lambda}\mu}\right|^2\right)$$

is strictly positive definite on $D^2$ for any $t > 0$.

We set

$$d(\lambda, \mu) = \left|\frac{\lambda - \mu}{1 - \overline{\lambda}\mu}\right| \quad \text{and} \quad \psi(\lambda, \mu) = d(\lambda, \mu)^2.$$
Then, trivially, $\psi$ is real-valued and symmetric. Let $H^2$ denote the Hardy space over $\mathbb{D}$, and let $s_\lambda$ be the normalized reproducing kernel of $H^2$, that is, we set

$$s_\lambda(z) = \frac{\sqrt{1 - |\lambda|^2}}{1 - \overline{\lambda} z} \quad (\lambda \in \mathbb{D}).$$

Then, for any $n$ in $\mathbb{N}$, $\lambda_1, \ldots, \lambda_n$ in $\mathbb{D}$, and $c_1, \ldots, c_n$ in $\mathbb{C}$ such that $\sum_{j=1}^n c_j = 0$, we have

$$\sum_{i,j=1}^n \overline{c_i} c_j \psi(\lambda_i, \lambda_j) = \sum_{i,j=1}^n \overline{c_i} c_j (1 - |\langle s_{\lambda_i}, s_{\lambda_j} \rangle_{H^2}|^2)$$

$$= \sum_{i,j=1}^n \overline{c_i} c_j - \sum_{i,j=1}^n \overline{c_i} c_j |\langle s_{\lambda_i}, s_{\lambda_j} \rangle_{H^2}|^2$$

$$= - \sum_{i,j=1}^n \overline{c_i} c_j |\langle s_{\lambda_i}, s_{\lambda_j} \rangle_{H^2}|^2$$

$$= - \sum_{i,j=1}^n \overline{c_i} c_j \langle s_{\lambda_i}, s_{\lambda_j} \rangle_{H^2} \langle s_{\lambda_i}, s_{\lambda_j} \rangle_{H^2}$$

$$\leq 0$$

by the Schur product theorem. Hence, $\psi$ is conditionally negative definite. In particular, $\exp(-t\psi)$ is a kernel function for any $t > 0$ by Schoenberg’s generator theorem (Theorem 9.7 in [7]). Moreover, it follows from Proposition 9.3 in [7] that

$$k(\lambda, \mu) = -\psi(\lambda, \mu) + \psi(\lambda, 0) + \psi(0, \mu) - \psi(0, 0)$$

$$= -d(\lambda, \mu)^2 + |\lambda|^2 + |\mu|^2$$

(4.1)

is a kernel function on $\mathbb{D}^2$. Let $\mathcal{H}_k$ denote the reproducing kernel Hilbert space generated by this $k$. Then, we have

$$\frac{1}{2} \|k_\lambda - k_\mu\|^2_{\mathcal{H}_k} = \frac{1}{2} (k(\lambda, \lambda) - 2k(\lambda, \mu) + k(\mu, \mu))$$

$$= \frac{1}{2} \{2|\lambda|^2 - 2(-d(\lambda, \mu)^2 + |\lambda|^2 + |\mu|^2) + 2|\mu|^2\}$$

$$= d(\lambda, \mu)^2$$

$$= \psi(\lambda, \mu).$$

Hence, by virtue of Theorem 3.2, in order to prove that $K_t$ is strictly positive definite, it suffices to show that the map $\Phi : \lambda \mapsto k_\lambda$ is injective. However, it is trivial. Indeed, if $k_\lambda = k_\mu$, then

$$d(\lambda, \mu)^2 = \psi(\lambda, \mu) = \frac{1}{2} \|k_\lambda - k_\mu\|^2_{\mathcal{H}_k} = 0.$$

Hence, we have $\lambda = \mu$. 9
4.5 Word metric

Let $G$ be a free group with the finite generators $a_1, \ldots, a_N$. For any element $g$ in $G$, $|g|$ will denote the word length of $g$. Then, it is well known that $d(g, h) = |h^{-1}g|$ defines a metric on $G$, which is called the word metric on $G$. Furthermore, in [3], Haagerup showed that

$$K_t(g, h) = \exp(-|h^{-1}g|)$$

is positive semi-definite on $G \times G$ for any $t > 0$. In fact, $K_t$ is strictly positive definite.

Some notations in Lemma 1.2 of [3] are needed. We set

$$\Lambda_j = \{(g, h) \in G \times G : g^{-1}h = a_j\} \quad \text{and} \quad \Lambda = \bigcup_{j=1}^N \Lambda_j.$$

Then, $H_\Lambda$ denotes a Hilbert space with an orthonormal basis $\{e_{(g, h)}\}_{(g, h) \in \Lambda}$ indexed by $\Lambda$. Moreover, we set $e_{(g, h)} = -e_{(h, g)}$ if $g^{-1}h = a_j^{-1}$ for some $1 \leq j \leq N$. Let $g$ be an element in $G$, and let $g = g_1 \cdots g_n$ be the word for $g$. Then, we set $f_0 = e$, the unit element of $G$, and $f_{\ell-1}f_\ell = g_\ell$ for $1 \leq \ell \leq n$. We define the map $\Phi$ from $\Lambda$ to $H_\Lambda$ as follows:

$$\Phi(g) = e_{(f_0, f_1)} + e_{(f_1, f_2)} + \cdots + e_{(f_{n-1}, f_n)}.$$ 

Then, it is proved that

$$\|\Phi(g) - \Phi(h)\|_{H_\Lambda}^2 = |h^{-1}g|$$

(see p. 283 of [3]). Hence, by virtue of Theorem 3.2, in order to show that $K_t$ is strictly positive definite, it suffices to show that $\Phi : G \to H_\Lambda$ is injective. However, it is trivial. Indeed, if $\Phi(g) = \Phi(h)$, then

$$|h^{-1}g| = \|\Phi(g) - \Phi(h)\|_{H_\Lambda}^2 = 0.$$ 

Hence, we have $g = h$.

5 Universality

In this section, we deal with real-valued kernel functions and real Hilbert spaces. Let $X$ be a locally compact Hausdorff space, and let $k = k(x, y)$ be a real-valued continuous kernel function on $X \times X$. Then, $\exp k$ is continuous, and so are all functions in $\exp H_k$. Let $K$ be a compact subset in $X$. Then it follows from (2.2) that

$$\|f\|_{\infty, K} := \sup_{x \in K} |f(x)| \leq \|F\|_F \sup_{x \in K} \exp \frac{k(x, x)}{2} < \infty.$$

for every $f$ in $\exp H_k$. Let $F_0$ denote the subspace of $F$ consisting of vectors with finite support. Then, by Lemma 3.1, $F_0$ is an algebra. In fact, the following diagram commutes:

$$\begin{array}{ccc}
H_k^{\otimes n} & \xrightarrow{\tau_k} & H_k^{\otimes m+n} \\
\Delta_n \downarrow & & \downarrow \Delta_{m+n} \\
H_k^n & \xrightarrow{\psi|_{H_k^n}} & H_k^{m+n}
\end{array}$$
where $\Psi$ is a function in $\mathcal{H}_k^\otimes m$, and we set $\psi = \Psi \circ \Delta_m$ and
\[
\tau_\psi : \mathcal{H}_k^\otimes n \to \mathcal{H}_k^\otimes m+n, \quad F \mapsto \Psi \otimes F.
\]
Hence, the closure of $\{f|_K : f \in \Gamma \mathcal{F}_0\}$ with respect to the norm $\| \cdot \|_{\infty,K}$ is a Banach algebra consisting of continuous functions on $K$. This observation leads us to a new proof of the following well known fact called the universal approximation theorem (see Definition 2.1 and Section 4 in [2]).

**Theorem 5.1.** Let $X$ be a locally compact Hausdorff space, and let $k = k(x,y)$ be a real-valued continuous kernel function on $X \times X$. If $\mathcal{H}_k$ separates any distinct two points in $X$, then, for any compact subset $K$ in $X$, any $\varepsilon > 0$ and any function $f$ in $C(K)$, there exist $c_1, \ldots, c_N$ in $\mathbb{R}$ and $a_1, \ldots, a_N$ in $X$ such that
\[
\left\| f - \sum_{j=1}^N c_j \exp k(x,a_j) \right\|_{\infty,K} < \varepsilon.
\]

**Proof.** Let $A$ denote the Banach algebra obtained by taking the closure of $\{f|_K : f \in \Gamma \mathcal{F}_0\}$ with respect to the norm $\| \cdot \|_{\infty,K}$. Then, it follows from the Stone-Weierstrass theorem that $A = C(K)$. Hence, for any $\varepsilon > 0$ and any function $f$ in $C(K)$, there exists a vector $G$ in $\mathcal{F}_0$ such that $\|f - G\|_{\infty,K} < \varepsilon$. Moreover, there exists a finite linear combination $h$ of reproducing kernels of $\exp \mathcal{H}_k$ such that $\|G - h\|_{\exp \mathcal{H}_k} < \varepsilon$. Hence, setting
\[
M_K = \sup_{x \in K} \exp \frac{k(x,x)}{2},
\]
$M_K$ is finite and we have
\[
|f(x) - h(x)| \leq |f(x) - (G)(x)| + |(G)(x) - h(x)| \\
\leq \|f - G\|_{\infty,K} + \|G - h\|_{\exp \mathcal{H}_k} \| \exp k\|_{\exp \mathcal{H}_k} \\
< (1 + M_K)\varepsilon
\]
for any $x$ in $K$. Therefore, we have the conclusion. \hfill \Box

**Corollary 5.1.** For any compact subset $K$ in $\mathbb{R}^n$, any $\varepsilon > 0$ and any function $f$ in $C(K)$, there exist $c_1, \ldots, c_N$ in $\mathbb{R}$ and $a_1, \ldots, a_N$ in $\mathbb{R}^n$ such that
\[
\left\| f - \sum_{j=1}^N c_j e^{-\|x-a_j\|^2_{\|x\|_n}} \right\|_{\infty,K} < \varepsilon.
\]

**Proof.** We set $k(x,y) = 2(x,y)_{\mathbb{R}^n}$. Then, $k$ is a real-valued continuous kernel function on $\mathbb{R}^2$. Let $f$ be any function in $C(K)$. Then, by Theorem 5.1, there exist $d_1, \ldots, d_N$ in $\mathbb{R}$ and $a_1, \ldots, a_N$ in $\mathbb{R}^n$ such that
\[
\|e^{\|x\|^2_{\|x\|_n}} f - \sum_{j=1}^N d_j e^{2(x,a_j)_{\mathbb{R}^n}} \|_{\infty,K} < \varepsilon.
\]
We set $c_j = d_j e^{||a_j||_R^n}$. Then, we have

$$\left| f(x) - \sum_{j=1}^N c_j e^{-||x-a_j||_R^n} \right| = \left| f(x) - e^{-||x||_R^n} \sum_{j=1}^N c_j e^{-||a_j||_R^n} e^{2(x,a_j)_R^n} \right|$$

$$= e^{-||x||_R^n} \left| e^{||x||_R^n} f(x) - \sum_{j=1}^N d_j e^{2(x,a_j)_R^n} \right|$$

$$\leq \left| e^{||x||_R^n} f - \sum_{j=1}^N d_j e^{2(x,a_j)_R^n} \right|_{\infty,K}$$

$$< \varepsilon.$$

This concludes the proof.

Acknowledgment. The author would like to thank Doctor J. C. Guella for his valuable comments on the previous version of this paper. This paper grew out of my three days lectures in Nagoya University. Special thanks are due to Professor Yoshimichi Ueda and Kenta Kojin. This work was supported by JSPS KAKENHI Grant Numbers JP20K03646 and JP21K03285.

References

[1] J. B. Garnett, *Bounded analytic functions*. Revised first edition. Graduate Texts in Mathematics, 236. Springer, New York, 2007.

[2] J. C. Guella, *On Gaussian kernels on Hilbert spaces and kernels on hyperbolic spaces*. J. Approx. Theory 279 (2022), Paper No. 105765, 36 pp.

[3] U. Haagerup, *An example of a nonnuclear C*-algebra, which has the metric approximation property*. Invent. Math. 50 (1978/79), no. 3, 279–293.

[4] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge University Press, Cambridge, 1985.

[5] S. Kuwahara and M. Seto, *Exponentials of de Branges-Rovnyak kernels*. Canad. Math. Bull. 65 (2022), no. 2, 447-455.

[6] N. K. Nikolski, *Operators, functions, and systems: an easy reading*. Vol. 1. Hardy, Hankel, and Toeplitz. Translated from the French by Andreas Hartmann. Mathematical Surveys and Monographs, 92. American Mathematical Society, Providence, RI, 2002.

[7] V. I. Paulsen and M. Raghupathi, *An introduction to the theory of reproducing kernel Hilbert spaces*. Cambridge Studies in Advanced Mathematics, 152. Cambridge University Press, Cambridge, 2016.