Global adiabaticity and non-Gaussianity consistency condition

Antonio Enea Romano,1,3 Sander Mooij2 and Misao Sasaki3

1Instituto de Física, Universidad de Antioquia, A.A.1226, Medellín, Colombia
2Grupo de Cosmología y Astrofísica Teórica, Departamento de Física, FCFM, Universidad de Chile, Blanco Encalada 2008, Santiago, Chile
3Center for Gravitational Physics, Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan

In the context of single-field inflation, the conservation of the curvature perturbation on comoving slices, \( R_c \), on super-horizon scales is one of the assumptions necessary to derive the consistency condition between the squeezed limit of the bispectrum and the spectrum of the primordial curvature perturbation. However, the conservation of \( R_c \) holds only after the perturbation has reached the adiabatic limit where the constant mode of \( R_c \) dominates over the other (usually decaying) mode. In this case, the non-adiabatic pressure perturbation defined in the thermodynamic sense, \( \delta P_{\text{nad}} \equiv \delta P - c_w^2 \delta \rho \), becomes negligible on superhorizon scales. Therefore one might think that the adiabatic limit is the same as thermodynamic adiabaticity. This is in fact not true. In other words, thermodynamic adiabaticity is not a sufficient condition for the conservation of \( R_c \) on super-horizon scales. In this paper, we consider models that satisfy \( \delta P_{\text{nad}} = 0 \) on all scales, which we call global adiabaticity (GA), which is guaranteed if \( c_w^2 = c_s^2 \), where \( c_s \) is the phase velocity of the propagation of the perturbation. A known example is the case of ultra-slow-roll (USR) inflation in which \( c_w^2 = c_s^2 = 1 \). In order to generalize USR we develop a method to find the Lagrangian of GA K-inflation models from the behavior of background quantities as functions of the scale factor. Applying this method we show that there indeed exists a wide class of GA models with \( c_w^2 = c_s^2 \), which allows \( R_c \) to grow on superhorizon scales, and hence violates the non-Gaussianity consistency condition.

I. INTRODUCTION

A period of accelerated expansion during the early stages of the evolution of the Universe, called inflation \[1,2\], is able to account for several otherwise difficult to explain features of the observed Universe such as the high level of isotropy of the cosmic microwave background (CMB) \[3\] radiation and the small value of the curvature. Some of the simplest inflationary models are based on a single slowly-rolling scalar field, and they are in good agreement with observations. It is commonly assumed in slow-roll models that adiabaticity in the thermodynamic sense, \( \delta P_{\text{nad}} \equiv \delta P - c_w^2 \delta \rho = 0 \) where \( c_w^2 = \dot{P}/\dot{\rho} \), implies the conservation of the curvature perturbation on uniform density slices \( \zeta \), and hence the conservation of the curvature perturbation on comoving slices \( R_c \), on super-horizon scales.

In \[3\] it was shown that there can be important exceptions, i.e., in some cases thermodynamic adiabaticity does not necessarily imply the super-horizon conservation of \( R_c \) and \( \zeta \), and that they can differ from each other. This can happen even for models in which \( c_w^2 = c_s^2 \). An example is ultra-slow-roll (USR) inflation \[4,5\], which has exact adiabaticity \( \delta P_{\text{nad}} = 0 \) on all scales. In USR inflation, both \( R_c \) and \( \zeta \) exhibit super-horizon growth but their behavior is very different from each other. As has been stressed in \[8\], the non-freezing of \( R_c \) has important phenomenological consequences. Since the freezing of \( R_c \) on superhorizon scales is a necessary ingredient \[9\] for Maldacena’s consistency relation \[10\] to hold, models that do not conserve \( R_c \) can actually violate that consistency condition. In this paper focusing on K-inflation, i.e., Einstein-scalar models with a general kinetic term, we explore in a general way other single field models which have \( c_w^2 = c_s^2 \), hence satisfy \( \delta P_{\text{nad}} = 0 \) on all scales which we call globally adiabatic (GA), but which may not conserve \( R_c \). We find a generalization of the USR model. A different generalization without imposing the condition \( c_w^2 = c_s^2 \) was discussed in \[11,12\].

The method we adopt is based on establishing a general condition for the non-conservation of \( R_c \) in terms of the dependence of the background quantities, in particular the slow-roll parameter \( \epsilon \equiv -\dot{H}/H^2 \) and the sound velocity \( c_s \), on the scale factor \( a \).

We first derive the necessary condition for the comoving curvature perturbation \( R_c \) to grow on superhorizon scales. Next we determine \( \rho(a) \) and \( P(a) \) by solving the continuity equation. Then using the equivalence between barotropic fluids and K-inflationary models which satisfy the condition \( c_w^2 = c_s^2 \) \[13,14\], we determine the corresponding Lagrangian for the equivalent scalar field.
model. Using this method we obtain a new class of GA scalar field models which do not conserve $R_c$.

Throughout the paper we denote the proper-time derivative by a dot ($\dot{\rho} = d\rho/d\tau$), the conformal-time derivative by a prime ($\rho = d\rho/d\eta = a(d\tau/d\eta)$) and the Hubble expansion rates in proper and conformal times by $H = \dot{a}/a$ and $H = \dot{a}/a$, respectively. We also use the terminology “adiabaticity” for thermodynamic adiabaticity $\delta P_{nad} = 0$ throughout the paper.

II. CONSERVATION OF $R_c$ AND GLOBAL ADIABATICITY

We set the perturbed metric as

$$ds^2 = a^2 \left[ -(1 + 2\Lambda)d\eta^2 + 2\partial_j Bdx^j d\eta + \{\delta_{ij} (1 + 2 R) + 2\partial_i \partial_j E\} dx^i dx^j \right].$$

(1)

In [3] it was shown that independently of the gravity theory and for generic matter the energy-momentum conservation equations imply

$$\delta P_{nad} = \left[ \left( \frac{c_w}{c_s} \right)^2 - 1 \right] (\rho + P) \frac{A_c}{R_c},$$

(2)

where the subscript $c$ means a quantity evaluated on co-moving slices defined by $\delta T^0_0 = 0$ (or equivalently on slices on which the scalar field is homogeneous). In the case of general relativity, the additional relation $A_c = R_c/H$ gives an important relation for the time derivative of $R_c$

$$\delta P_{nad} = \left[ \left( \frac{c_w}{c_s} \right)^2 - 1 \right] (\rho + P) \frac{R_c}{H},$$

(3)

The non-adiabatic pressure perturbation is given according to its thermodynamics definition

$$\delta P_{nad} \equiv \delta P - \frac{c_w^2}{2} \delta P.$$  

(4)

This definition of $\delta P_{nad}$ is important because it is gauge invariant and $\delta P_{nad} = \delta P_{ud}$, where $\delta P_{ud}$ is the pressure perturbation on uniform density slices. It appears in the equation for the curvature perturbation on uniform density slices $\zeta \equiv R_{ud}$ obtained from the energy conservation law [13],

$$\zeta' = -\frac{\mathcal{H}}{(\rho + P)} \frac{1}{3} \frac{\delta P_{nad}}{\delta (v - E')_{ud}}$$

(5)

where $v$ is the 3-velocity potential ($v = \delta \phi/\phi'$ for a scalar field). In general, the curvature perturbations on uniform density and comoving slices are related as

$$\zeta = R_c + \frac{\delta P_{nad}}{3(\rho + P)(c_s^2 - c_w^2)}.$$  

(6)

A common interpretation of these equations (see for example [16, 17]) is that when $\delta P_{nad} = 0$ with $c_w^2 \neq c_s^2$, $\zeta$ and $R_c$ are equal because of eq. (3), and they are both conserved on super-horizon scales because of eq. (3).

The equation (3) is the key relation to understand how $R_c$ depends on the non-adiabatic pressure $\delta P_{nad}$. First of all let us note that this equation is valid on any scale. The advantage of it with respect to eq. (5) is that it does not involve gradient terms, so it allows us to directly relate $\delta P_{nad}$ to $R_c$, if $c_w^2 \neq c_s^2$, while in eq. (5) $\zeta$ depends on spatial gradients, which in the case of USR are not negligible on super-horizon scales [3]. This explains while in USR in which $c_s^2 = c_w^2 = 1$, both $R_c$ and $\zeta$ are not conserved despite $\delta P_{nad} = 0$.

It should be noted here that for slow-roll attractor models $c_w^2 \neq c_s^2$ in general, and $R_c$ is time-varying on sub-horizon scales. This implies that the non-adiabatic pressure perturbation $\delta P_{nad}$ on sub-horizon scales is not zero. In other words, the attractor models are adiabatic only on super-horizon scales, and we call these models super-horizon adiabatic (SHA).

From eq. (4) we can immediately deduce that in general relativity there are two possible scenarios for the non-conservation of $R_c$,

$$\begin{align*}
(1) \quad & c_s^2 = c_w^2, \quad \delta P_{nad} = 0, \\
(2) \quad & c_s^2 \neq c_w^2, \quad \delta P_{nad} \neq 0.
\end{align*}$$

(7)

The second case was studied in [11, 12]. Here we focus on the first case. It is trivial to see that because of the gauge invariance of $\delta P_{nad}$ the condition $c_s^2 = c_w^2$ automatically implies $\delta P_{nad} = 0$. The models satisfying the condition $c_s^2 - c_w^2 = \delta P_{nad} = 0$ are adiabatic on any scale, and because of this we call them globally adiabatic (GA). In GA models an explicit calculation can reveal the super-horizon behavior of $R_c$, and $\zeta$, as was shown in [3] in the case of USR. Below, we develop an inversion method to find a new class of models that violate the conservation of $R_c$ without solving the perturbations equations.

III. GLOBALLY ADIABATIC K-ESSENCE MODELS

The condition $c_s^2 = c_w^2$ has been studied in the context of K-inflation [13] described by the action

$$S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ M_{Pl}^2 R + 2 \mathcal{P}(X, \phi) \right] ,$$

(8)

and it was shown that it is satisfied by scalar field models with the Lagrangian of the form,

$$\mathcal{P}(X, \phi) = u(X g(\phi)) = u(Y).$$

(9)

These models are equivalent to a barotropic perfect fluid, i.e. a fluid with equation of state $P(\rho)$. See also [18–21]. We note again that these models are adiabatic on any scale (GA), contrary to the slow-roll attractor models, which are adiabatic only on super-horizon scales (SHA). The fact that they are mutually exclusive can be readily
seen by considering the hypothetical case of $\delta P_{nad} = 0$ and $c_w^2 \neq c_s^2$. In this case Eq. (3) which is valid on any scale would mean $R_c$ should be frozen on all scales. In contrast, the condition $c_w^2 = c_s^2$ allows for the curvature perturbation to evolve both on sub-horizon and super-horizon scales.

In [13] it was shown that it is possible to associate any barotropic perfect fluid with an equivalent K-inflation model according to

$$2 \int F(u) = \log(Y),$$

where $F(P) = \rho(P) + P$ and $Y = g(\phi) X$ with $X = -g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi/2$. These models are the ones which could violate the conservation of $R_c$ for adiabatic perturbations, since they satisfy $c_w^2 = c_s^2$. It is noted of course that the global adiabaticity is not the sufficient condition for the non-conservation of $R_c$. Not all GA models violate the conservation of $R_c$ on super-horizon scales.

IV. GENERAL CONDITIONS FOR SUPER-HORIZON GROWTH OF $R_c$

From the equation for the curvature perturbation on comoving slices,

$$\frac{\partial}{\partial t} \left( \frac{a^3 \epsilon}{\rho} \frac{\partial}{\partial t} R_c \right) - a \epsilon \Delta R_c = 0, \quad (11)$$

we can deduce, after re-expressing the time derivative in terms of the derivative respect to the scale factor $a$, that on superhorizon scales there is (apart from a constant solution) a solution of the form,

$$R_c \propto \int a^\delta f(a); \quad f(a) = \frac{c_w^2(a)}{H a^\epsilon(a)}, \quad (12)$$

where we have introduced the function $f(a)$ for later convenience. In conventional slow-roll inflation $c_w^2$ and $\epsilon$ are both slowly varying, hence the integral rapidly approaches a constant, rendering $R_c$ conserved. The time dependent part of the above solution corresponds to the decaying mode.

The necessary and sufficient condition for super-horizon freezing is that there exists some $\delta > 0$ for which

$$\lim_{a \to \infty} a^\delta f(a) = 0. \quad (13)$$

By definition of inflation, $H$ must be sufficiently slowly varying; $\epsilon = -H/H^2 \ll 1$. So we may neglect the time dependence of $H$ in (12) at leading order, while $\epsilon$ and $c_w^2$ may vary rapidly in time. For models for which $\epsilon \approx a^{-n}$ and $c_w^2 \approx a^\alpha$ we get

$$f \propto a^{\alpha+n^2}, \quad (14)$$

hence the condition for freezing is

$$q + n - 3 < 0. \quad (15)$$

If this condition is violated, i.e. $q + n - 3 \geq 0$, then the solution (12) will grow on super-horizon scales. This happens for example in USR., which corresponds to $c_w^2 = 1$ and $\epsilon \propto a^{-6}$, i.e. $q = 0$, and $n = 6$. (The super-horizon growth of $R_c$ in USR can also be understood as a direct consequence of the non-attractor nature of USR [22].) In general, we expect that $q$ would not become very large. This implies $\epsilon$ should decrease sufficiently rapidly. Conversely, if $\epsilon$ decreases sufficiently rapidly, then the growth of $R_c$ on superhorizon scales will follow.

V. BAROTROPIC MODEL

We have shown that GA models could violate the super-horizon conservation of $R_c$, so now we will look for GA K-essence models which do indeed violate it, based on the freezing condition in eq. (13). Inspired by the equivalence between barotropic fluids and GA K-essence models [13] we will first look for barotropic fluids that can give the growing curvature perturbation on superhorizon scales. From the very beginning we will set $c_w^2 = c_s^2$.

Using the Friedmann equation we can write the slow-roll parameter $\epsilon$ as

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{3 \rho + P}{2 \rho}. \quad (16)$$

In terms of the scale factor and $\epsilon$ the energy conservation equation reads

$$\frac{d\rho}{da} + \frac{3}{a} (\rho + p) = \frac{dp}{da} + \frac{2 \epsilon \rho}{a} = 0. \quad (17)$$

We may now define the quantity $b(a) = 2 \epsilon \rho$. It appears naturally in the continuity equation and plays a crucial role in regards to the super-horizon behavior of curvature perturbations because the function $f(a)$ can be re-written in terms of it as

$$f(a) \propto \frac{H c_w^2}{a^\alpha b(a)}. \quad (18)$$

Integrating the energy conservation equation we get

$$\rho(a) = \rho_0 \exp \left[ -2 \int a_0^a a \right] = \int \frac{b(a)}{a} da. \quad (19)$$

Using eq. (10), we then obtain

$$P(a) = \left( \frac{2}{3} \epsilon - 1 \right) \rho. \quad (20)$$

The sound velocity is given by

$$c_w^2 = c_s^2 = \frac{dP}{d\rho} = -1 + \frac{1}{3} \frac{db(a)}{da}$$

$$= -1 + \frac{1}{3} \frac{db(a)}{da} / \left( \frac{dp}{da} \right)$$

$$= -1 - \frac{a}{3 b(a)} \frac{db(a)}{da}. \quad (21)$$
We now consider the behavior of $f(a)$ introduced in \[12\]. As mentioned before, we consider the case when $\epsilon$ decreases sufficiently rapidly. In this case, $\rho = 3H^2M_p^2$ approaches a constant rapidly. Hence the time dependence of $\rho$ may be neglected compared to that of other quantities that vary far more rapidly. With this approximation, assuming $\epsilon \propto a^{-n}$, we find
\[ c_s^2 \approx \frac{n - 3}{3}, \]
which means $q \approx 0$, and
\[ f(a) = \frac{c_s^2(a)}{H_0^3(a)} \propto a^{n-3}, \]
which satisfies the condition for the growth if $n > 3$, in accordance with the original anticipation. In passing, it is interesting to note that the condition $n > 3$ implies $c_s^2 > 0$, a necessary condition to avoid the gradient instability of the perturbation. Thus virtually all GA models that are free from the gradient instability exhibit superhorizon growth of the comoving curvature perturbation $\mathcal{R}_c$.

VI. SCALAR FIELD MODEL

Let us now find a scalar field model that corresponds to the barotropic model discussed in the previous section. As a warm-up, let us consider the USR case, whose fluid interpretation has already been studied in [23]. In this case, we exactly have $c_s^2 = 1$. From eq. (21), this implies $b/2 = c\rho(= 3(\rho + P)/2) \propto a^{-6}$. Also $c_s^2 = 1$ implies $\rho = P + \text{const.}$ Inserting this into eq. (10) gives
\[ \frac{2dP}{2P + \text{const.}} = \frac{dY}{Y}. \]
Thus up to a constant term $P$ and $Y$ are the same,
\[ P = Y + \text{const.}. \]
Absorbing $g(\phi)$ in $Y$ into the definition of the scalar field by $g^{1/2}d\phi \rightarrow d\phi$, this is indeed the Lagrangian for a minimally coupled massless scalar with a cosmological constant:
\[ L = P(\phi, X) = X - V_0. \]
This is consistent with $\rho + P = 2X \propto c\rho \propto a^{-6}$.

Let us generalize the USR case. As in the previous section, we consider models that have the behavior of $c\rho$ as
\[ 2c\rho = b(a), \]
where $b(a)$ should decrease faster than $a^{-3}$ asymptotically at $a \rightarrow \infty$ but otherwise is an arbitrary function. Then we have
\[ F(P) \equiv \rho + P = 2H^2c = \frac{2c\rho}{3} = \frac{b(a)}{3}, \]
which gives
\[ \frac{dY}{Y} = 2\frac{dP}{F(P)} = 6\frac{dP}{2c\rho} = \frac{6}{b(a)}\frac{dP}{P}. \]

For $dP$, using the energy conservation law, we may rewrite it as
\[ dP = d(-\rho + F(P)) = -d\rho + \frac{db(a)}{3} = 3\frac{da}{a}(P + \rho) + \frac{db(a)}{3} = b(a)\frac{da}{a} + \frac{db(a)}{3}. \]
Therefore we have
\[ \frac{dY}{Y} = 6\frac{dP}{b(a)} = 6\frac{da}{a} + 2\frac{db}{b}. \]

Hence
\[ Y \propto a^6b^2. \]
This is consistent with the USR case in which $b(a) \propto a^{-6}$ and $Y = X \propto a^{-6}$.

This relation is quite useful since it allows to rewrite the freezing function $f(a)$ as
\[ f(a) \propto \frac{Hc^2}{\sqrt{Y}}, \]
from which we can deduce that $Y(a)$ determines the super-horizon behavior of $\mathcal{R}_c$. In particular, for the models we are considering in which $c_s$ is constant, we infer that super-horizon growth can happen in the limit $Y \rightarrow 0$.

For a given choice of $b(a)$, eq. (32) can be inverted to give the scale factor as a function of $Y$, $a = a(Y)$. Also eq. (30) can be integrated to give $P = P(a)$. Combining these two, one can obtain the Lagrangian for the scalar field, $L = P = P(Y)$.

Note that in GA models there is a one-to-one correspondence between the scale factor and state variables such as $P(a)$ and $\rho(a)$, which is the reason why we can also write a barotropic equation of state $P(\rho) = P(a(\rho))$. Once any of the functions $P(a), \rho(a), b(a), c(a), Y(a)$ is specified, all the others are specified too, as well as the equation of state $P(\rho)$ or its scalar field equivalent Lagrangian $P(Y)$, which is in fact the basis of the inversion method that we are developing in this paper.

VII. EXAMPLES

Here we give a couple of specific K-inflation models that are globally adiabatic and violate the conservation of $\mathcal{R}_c$. Given the parametric behaviour of $b \equiv 2\epsilon\rho$, our inversion method allows us to deduce the Lagrangian.
A. Ex 1: Generalized USR

Let us consider a specific case where \( b(a) \) is a power-law function,

\[
2\epsilon \rho = b(a) = ca^{-n}.
\]

(34)

where \( c \) is a constant. We assume \( n > 3 \) in order to have the growth on superhorizon scales.

From eq. (32) we have

\[
a \propto Y^{1/(6-2n)}.
\]

(35)

Now eq. (30) gives

\[
P = \int_{a}^{\infty} \left( \frac{b(a) da}{a} + \frac{db(a)}{3} \right)
\]

\[
= -\frac{c}{n} a^{-n} + \frac{c}{3} a^{-n} + \text{const.}
\]

\[
= \frac{n-3}{3n} b(a) + \text{const.}
\]

(36)

Plugging eq. (35) into this, we finally obtain

\[
L = P(Y) = Y^{n/(2n-6)} - V_0.
\]

(37)

Since this may be regarded as a natural generalization of the USR case, which corresponds to the case \( n = 6 \), we call it the generalized USR (GUSR) model. Lagrangians involving \( Y^n \) terms have already been studied in [11, 24, 25], but those models are either not exactly globally adiabatic because of the presence of a not constant potential or they satisfy the relation \( \epsilon \propto a^{-n} \) only approximately and during a limited time range, while for GUSR \( \epsilon \propto a^{-n} \) is an exact relation and is valid at any time. As the Lagrangian is of the type described in eqs. (9) and (20) (remember that after a field transformation \( Y \) can be made equal to \( X \), we understand that this scalar field model is indeed equivalent to a barotropic fluid. Hence we have \( c_w^2 = c_s^2 \) and therefore \( \delta P_{nad} = 0 \). Indeed the second condition for super-horizon growth of \( R_c \) given in eq. (17) is satisfied. More precisely, we note that for the GUSR model, the sound velocity is exactly constant,

\[
c_w^2 = c_s^2 = \frac{n-3}{3}.
\]

(38)

The power spectrum of the comoving curvature perturbation can be explicitly computed for this model. One finds [24] that the spectral index is a function of \( n \): \( n_s - 1 = 6 - n \), in agreement with the scale invariant spectrum of the original ultra slow-roll inflation in which one has \( n = 6 \). Hence, the model can be constrained by the observational value. Note as well, from eq. (35), that to have a slightly red-tilted spectrum, we need a slightly superluminal speed of sound.

B. Ex 2: Lambert Inflation

As another example, let us consider the case when \( \epsilon \) is a power-law function,

\[
\epsilon(a) = \epsilon_0 a^{-n}.
\]

(39)

As before, we assume \( n > 3 \). In this case, since \( d \log \rho/d \log a = -2\epsilon \propto a^{-n} \), we find

\[
\rho(a) = \rho_0 \exp \left[ \frac{2\epsilon}{n} \right].
\]

(40)

It is clear that \( \rho \) approaches a constant \( \rho_0 \) asymptotically at \( a \to \infty \).

Inserting eq. (39) and eq. (40) into eq. (21), the sound velocity is given by

\[
c_w^2 = c_s^2 = 1 - \frac{1}{3} \left( \frac{d \log \epsilon}{d \log a} + \frac{d \log \rho}{d \log a} \right) = \frac{n - 3 + 2\epsilon}{3}.
\]

(41)

Thus \( c_w^2 \) is time dependent, but it rapidly approaches a constant as \( \epsilon \) decays out. Also from eq. (39) and eq. (40), we find

\[
b(a) = 2\epsilon \rho = 2\epsilon \rho_0 \exp \left[ \frac{2\epsilon}{n} \right].
\]

(42)

Thus we have

\[
Y \propto a^6 \rho^2 \propto a^{6-2n} \exp \left[ \frac{4\epsilon}{n} \right] \propto \epsilon^{(2n-6)/6} \exp \left[ \frac{4\epsilon}{n} \right],
\]

(43)

which implies

\[
Y^{n/(2n-6)} \propto \frac{4\epsilon}{2n-6} \exp \left[ \frac{4\epsilon}{2n-6} \right].
\]

(44)

To find the Lagrangian, we manipulate eq. (40) as

\[
dP = b \frac{da}{a} + \frac{db}{3} = - \frac{b}{n} \frac{de}{\epsilon} + \frac{db}{3}
\]

\[
= \frac{2}{n} \rho_0 \epsilon^{2\epsilon/n} \frac{d\epsilon}{3} + \frac{db}{3}.
\]

(45)

Therefore, integrating this we obtain

\[
P = \rho_0 \epsilon^{2\epsilon/n} \left( -1 + \frac{2\epsilon}{3} \right) + \text{const.}
\]

(46)

One can invert eq. (44) to find \( \epsilon \) as a function of \( Y \), and then insert it into the above to obtain the Lagrangian.

Specifically, we introduce the Lambert function \( W(x) \) defined by the inverse function of \( X(z) = ze^z \),

\[
z = X^{-1}(ze^z) \equiv W(ze^z).
\]

(47)

Setting

\[
Y^{n/(2n-6)} = ze^z; \quad z = \frac{4\epsilon}{2n-6},
\]

(48)
we have
\[ \frac{4\epsilon}{2n-6} = W(y); \quad y = Y^{n/(2n-6)}. \] (49)
Inserting this into eq. 40, we finally obtain
\[ L = P(Y) \]
\[ = \rho_0 \left( \frac{n-3}{3} W(y) - 1 \right) \exp \left[ \frac{n-3}{n} W(y) \right] - V_0, \] (50)
where \( y = y(Y) \) is given in eq. 40.

Note that this model has been derived without making any approximation, and it gives exactly \( \epsilon \propto a^{-n}. \) However, as we mentioned before, in the late time limit, there is no difference between \( \epsilon \propto a^{-n} \) and \( \rho \epsilon \propto a^{-n}. \) Thus the two models discussed above are essentially the same at late times. This can be easily checked by expanding \( W(y) \) around \( y = 0, \)
\[ W(y) = y - y^2 + \cdots. \] (51)
At leading order in \( y = Y^{n/(2n-6)}, \) this gives
\[ P(Y) = \frac{n-3}{3} \rho_0 Y^{n/(2n-6)} - \rho_0 - V_0. \] (52)
By absorbing the constant coefficient into \( g(\phi) \) in the definition of \( Y, \)
\[ Y = g(\phi) X, \]
and absorbing \( \rho_0 \) into the constant \( V_0, \) eq. (50) reduces to
\[ P = Y^{n/(2n-6)} - V_0, \] (53)
which indeed coincides with the GUSR model, see eq. (50).

Higher order terms in the expansion give an infinite class of models of the type
\[ u(Y) = \sum_i \beta_i Y^{n_i}, \] (54)
where \( \beta_i \) are appropriate coefficients.

Finally, note that in USR and as well in the two examples considered here, the shift symmetry in the potential \( (V(\phi) = V_0) \) is a direct consequence of the demand \( c_w^2 = c_s^2, \) which in turn follows from the global adiabaticity of the model. That is in line with the general statement [27] [28] that for a k-essence theory to describe a fluid, one needs a shift symmetry (i.e., there is no physical clock, the model is of the non-attractor type).

VIII. CONCLUSIONS

By introducing the notion of global adiabaticity, namely, \( c_w^2 = c_s^2 \) and \( \delta P_{nad} = \delta P - c_w^2 \delta \rho = 0, \) where \( c_w^2 = \dot{P}/\dot{\rho} \) and \( c_s \) is the propagation (phase) speed of the perturbation, we have determined the general conditions for the non-conservation of the curvature perturbations on comoving slices \( R_c \) on super-horizon scales. We have found that globally adiabatic K-essence models can exhibit this behavior.

We have then developed a method to construct the Lagrangian of a K-essence globally adiabatic (GA) model by specifying the behavior of background quantities such as \( \epsilon \rho \) where \( \epsilon \) is the slow-roll parameter, using the equivalence between barotropic fluids and GA K-essence models. We have applied the method to find the equations of state of the fluids and derive the Lagrangian of the equivalent single scalar field models. Interestingly, we have found that the requirement to avoid the gradient instability, ie, \( c_w^2 > 0 \) is almost identical to the condition for the non-conservation on superhorizon scales.

The advantage of our approach is that we have not solved any perturbation equation explicitly, since we have proceeded in the opposite way solving the inversion problem consisting of requiring certain properties to the behavior of the perturbation equation we are interested in. In other words, instead of starting from a Lagrangian and then solve the perturbations equations we have determined the equation of state or equivalently the Lagrangian which admits a solution of the perturbation equation with the particular behavior we are interested in.

We have shown that the main difference between attractor models and GA models is that the latter are adiabatic on all scales, while attractor models are approximately adiabatic in the sense of \( \delta P_{nad} = 0 \) only on super-horizon scales and \( c_w^2 \neq c_s^2. \)

The detailed study of the new models found in this paper will be done in a separate upcoming work 20 but we can already predict that they can be compatible with observational constraints on the spectral index thanks to the extra parameter \( n \) which is not present in USR. Furthermore they can violate the Maldacena’s consistency condition and consequently produce large local shape non-Gaussianity.

In the future it will be interesting to apply the inversion method we have developed to other problems related to primordial curvature perturbations, or to develop a similar method for the adiabatic sound speed as function of the scale factor.

ACKNOWLEDGMENTS

The work of MS was supported by MEXT KAKENHI No. 15H05888. SM is funded by the Fondecyt 2015 Postdoctoral Grant 3150126. This work was supported by the Dedicacion escusica and Sostenibilidad programs at UDEA, the UDEA CODI project IN10219CE and 2015-4044, and Colciencias mobility project COSOMOLOGY AFTER BICEP.
[1] A. D. Linde, “A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems,” Phys. Lett. B 108, 389 (1982).

[2] A. Albrecht and P. J. Steinhardt, “Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking,” Phys. Rev. Lett. 48, 1220 (1982).

[3] A. A. Starobinsky, “Dynamics of Phase Transition in the New Inflationary Universe Scenario and Generation of Perturbations,” Phys. Lett. B 117, 175 (1982).

[4] P. A. R. Ade et al. [Planck Collaboration], “Planck 2015 results. XX. Constraints on inflation,” arXiv:1502.02114 [astro-ph.CO].

[5] A. E. Romano, S. Mooij and M. Sasaki, “Adiabaticity and gravity theory independent conservation laws for cosmological perturbations,” Phys. Lett. B 755, 464 (2016) [arXiv:1512.05757 [gr-qc]].

[6] N. C. Tsamis and R. P. Woodward, “Improved estimates of cosmological perturbations,” Phys. Rev. D 69, 084005 (2004) [astro-ph/0307463].

[7] W. H. Kinney, “Horizon crossing and inflation with large eta,” Phys. Rev. D 72, 023515 (2005) [gr-qc/0503017].

[8] M. H. Namjoo, H. Firouzjahi and M. Sasaki, “Violation of non-Gaussianity consistency relation in a single field inflationary model,” Europhys. Lett. 101, 39001 (2013) [arXiv:1210.3692 [astro-ph.CO]].

[9] P. Creminelli and M. Zaldarriaga, “Single field consistency relation for the 3-point function,” JCAP 0410, 006 (2004) [astro-ph/0407059].

[10] J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP 0305, 013 (2003) [astro-ph/0210603].

[11] X. Chen, H. Firouzjahi, M. H. Namjoo and M. Sasaki, “A Single Field Inflation Model with Large Local Non-Gaussianity,” Europhys. Lett. 102, 59001 (2013) [arXiv:1301.5690 [hep-th]].

[12] J. Martin, H. Motohashi and T. Suyama, “Ultra Slow-Roll Inflation and the non-Gaussianity Consistency Relation,” Phys. Rev. D 87, no. 2, 023514 (2013) [arXiv:1211.0083 [astro-ph.CO]].

[13] F. Arroja and M. Sasaki, “A note on the equivalence of a barotropic perfect fluid with a K-essence scalar field,” Phys. Rev. D 81, 107301 (2010) [arXiv:1002.1376 [astro-ph.CO]].

[14] S. Unnikrishnan and L. Sriramkumar, “A note on perfect scalar fields,” Phys. Rev. D 81, 103511 (2010) [arXiv:1002.0820 [astro-ph.CO]].

[15] D. Wands, K. A. Malik, D. H. Lyth and A. R. Liddle, “A New approach to the evolution of cosmological perturbations on large scales,” Phys. Rev. D 62, 043527 (2000) [astro-ph/0003278].

[16] D. H. Lyth, K. A. Malik and M. Sasaki, “A General proof of the conservation of the curvature perturbation,” JCAP 0505, 004 (2005) [astro-ph/0411220].

[17] A. J. Christopherson and K. A. Malik, “The non-adiabatic pressure in general scalar field systems,” Phys. Lett. B 675, 159 (2009) [arXiv:0809.3518 [astro-ph]].

[18] C. Quercellini, M. Bruni and A. Balbi, “Affine equation of state from quintessence and k-essence fields,” Class. Quant. Grav. 24, 5413 (2007) [arXiv:0706.3667 [astro-ph]].

[19] V. Faraoni, “The correspondence between a scalar field and an effective perfect fluid,” Phys. Rev. D 85, 024040 (2012) [arXiv:1201.1448 [gr-qc]].

[20] A. Diez-Tejedor, “Note on scalars, perfect fluids, constrained field theories, and all that,” Phys. Lett. B 727, 27 (2013) [arXiv:1309.4756 [gr-qc]].

[21] P. Wongjun, “A Perfect Fluid in Lagrangian Formulation due to Generalized Three-Form Field,” arXiv:1602.00682 [gr-qc].

[22] S. Mooij and G. A. Palma, “Consistently violating the non-Gaussian consistency relation,” JCAP 1511, no. 11, 025 (2015) [arXiv:1502.03458 [astro-ph.CO]].

[23] X. Chen, H. Firouzjahi, M. H. Namjoo and M. Sasaki, “Fluid Inflation,” JCAP 1309, 012 (2013) [arXiv:1306.2901 [hep-th]].

[24] X. Chen, H. Firouzjahi, E. Komatsu, M. H. Namjoo and M. Sasaki, “In-in and δN calculations of the bispectrum from non-attractor single-field inflation,” JCAP 1312, 039 (2013) [arXiv:1308.5341 [astro-ph.CO]].

[25] S. Hirano, T. Kobayashi and S. Yokoyama, “Ultra slow-roll G-inflation,” arXiv:1604.00141 [astro-ph.CO].

[26] A. E. Romano and S. Mooij, in preparation.

[27] R. Akhoury, C. S. Gauthier and A. Vikman, “Stationary Configurations Imply Shift Symmetry: No Bondi Accretion for Quintessence / k-Essence,” JHEP 0903, 082 (2009) [arXiv:0811.1620 [astro-ph]].

[28] I. Sawicki, I. D. Saltas, L. Amendola and M. Kunz, “Consistent perturbations in an imperfect fluid,” JCAP 1301, 004 (2013) [arXiv:1208.4855 [astro-ph.CO]].