WHEN THE ROBIN INEQUALITY DOES NOT HOLD

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Abstract. In mathematics, the Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It is one of the seven Millennium Prize Problems selected by the Clay Mathematics Institute to carry a US 1,000,000 prize for the first correct solution. In 1915, Ramanujan proved that under the assumption of the Riemann Hypothesis, the inequality $\sigma(n) < e^{\gamma} \times n \times \log \log n$ holds for all sufficiently large $n$, where $\sigma(n)$ is the sum-of-divisors function and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. In 1984, Guy Robin proved that the inequality is true for all $n > 5040$ if and only if the Riemann Hypothesis is true. Let $n > 5040$ be $n = r \times q$, where $q$ denotes the largest prime factor of $n$. If $n > 5040$ is the smallest number such that Robin inequality does not hold, then we show the following inequality is also satisfied:

\[ \sqrt{e} + \frac{\log \log r}{\log \log n} > 2. \]

1. Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of $n$ [Cho+07]:

\[ \sum_{d | n} d. \]

Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\text{Robins}(n)$ holds provided

\[ f(n) < e^{\gamma} \times \log \log n. \]

The constant $\gamma$ is the Euler-Mascheroni constant, and log is the natural logarithm. The importance of this property is:

**Theorem 1.1.** [RH] If $\text{Robins}(n)$ holds for all $n > 5040$, then the Riemann Hypothesis is true [Rob84].

There are several known results about the possible counterexamples of $\text{Robins}(n)$ when $n > 5040$ [Cho+07]. In addition, we show that

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Theorem 1.2. [counterexample] Let $n > 5040$ be $n = r \times q$, where $q$ denotes the largest prime factor of $n$. If $n > 5040$ is the smallest number such that Robins$(n)$ does not hold, then
\[
\sqrt{e} + \frac{\log \log r}{\log \log n} > 2.
\]

2. Some Useful Lemmas

The following lemma is a very helpful inequality:

Lemma 2.1. [ineq] We have
\[
\frac{x}{1 - x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}
\]
where $y = 1 - x$.

Proof. We know $1 + x \leq e^x$ [Koz21]. Therefore,
\[
\frac{x}{1 - x} \leq \frac{e^{x-1}}{1 - x} = \frac{1}{(1 - x) \times e^{1-x}} = \frac{1}{y \times e^{y}}.
\]
However, for every real number $y \in \mathbb{R}$ [Koz21]:
\[
y \times e^{y} \geq y + y^2 + \frac{y^3}{2}
\]
and this can be transformed into
\[
\frac{1}{y \times e^{y}} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.
\]
Consequently, we show
\[
\frac{x}{1 - x} \leq \frac{1}{y + y^2 + \frac{y^3}{2}}.
\]
\[\square\]

Here, it is another practical result:

Lemma 2.2. [prop] Suppose that $n > 5040$ and let $n = r \times q$, where $q$ denotes the largest prime factor of $n$. We have
\[f(n) \leq (1 + \frac{1}{q}) \times f(r).
\]

Proof. Suppose that $n$ is the form of $m \times q^k$ where $q \nmid m$ and $m$ and $k$ are natural numbers. We have that
\[f(n) = f(m \times q^k) = f(m) \times f(q^k)
\]
since $f$ is multiplicative and $m$ and $q$ are coprimes \cite{Voj20}. However, we know that 
\[ f(q^k) \leq f(q^{k-1}) \times f(q) \]
because of we notice that $f(a \times b) \leq f(a) \times f(b)$ when $a, b \geq 2$ \cite{Voj20}. In this way, we obtain that 
\[ f(q^{k-1}) \times f(q) = f(q^{k-1}) \times (1 + \frac{1}{q}) \]
according to the value of $f(q)$ \cite{Voj20}. In addition, we analyze that 
\[ f(m) \times f(q^{k-1}) = f(m \times q^{k-1}) = f(r) \]
because $f$ is multiplicative and $m$ and $q$ are coprimes \cite{Voj20}. Finally, we obtain that 
\[ f(n) = f(m) \times f(q^k) \leq f(m) \times f(q^{k-1}) \times f(q) = f(r) \times (1 + \frac{1}{q}) \]
and as a consequence, the proof is completed.

\section*{3. Proof of Main Theorem}

**Theorem 3.1.** Let $n > 5040$ be $n = r \times q$, where $q$ denotes the largest prime factor of $n$. If $n > 5040$ is the smallest number such that $\text{Robins}(n)$ does not hold, then \[
\sqrt{e} + \frac{\log \log r}{\log \log n} > 2.
\]

**Proof.** Suppose that $n$ is the smallest integer exceeding 5040 that does not satisfy the Robin’s inequality. Let $n = r \times q$, where $q$ denotes the largest prime factor of $n$. In this way, the following inequality 
\[ f(n) \geq e^\gamma \times \log \log n \]
should be true. We know that 
\[ (1 + \frac{1}{q}) \times f(r) \geq f(q \times r) \geq f(n) \geq e^\gamma \times \log \log n \]
due to lemma 2.2 \cite{prop}. Besides, this shows that 
\[ (1 + \frac{1}{q}) \times e^\gamma \times \log \log r > e^\gamma \times \log \log n \]
should be true as well. Certainly, if $n$ is the smallest counterexample exceeding 5040 of the Robin’s inequality, then $\text{Robins}(r)$ holds \cite{Cho+07}. That is the same as 
\[ (1 + \frac{1}{q}) \times \log \log r > \log \log n. \]
We have that
\[
(1 + \frac{1}{q}) \times \log \log r > \log (\log r + \log q)
\]
where we notice that \(\log(a + c) = \log a + \log(1 + \frac{c}{a})\). This follows
\[
(1 + \frac{1}{q}) \times \log \log r > \log \log r + \log(1 + \frac{\log q}{\log r})
\]
which is equal to
\[
(1 + q) \times \log \log r > q \times \log \log r + q \times \log(1 + \frac{\log q}{\log r})
\]
and thus,
\[
\log \log r > q \times \log(1 + \frac{\log q}{\log r}).
\]
This implies that
\[
\frac{\log \log r}{\log(1 + \frac{\log q}{\log r})} = \frac{\log \log r}{\log \log r + \log q} = \frac{\log \log r}{\log \log n} = \frac{\log \log r}{\log \log n - \log \log r} = \frac{\log \log r}{\log \log n \times (1 - \frac{\log \log r}{\log \log n})} = \frac{\log \log r}{(1 - \frac{\log \log r}{\log \log n})} > q
\]
should be true. If we assume that \(y = 1 - \frac{\log \log r}{\log \log n}\), then we analyze that
\[
\frac{1}{y + y^2 + \frac{y^3}{2}} \geq \frac{\log \log r}{(1 - \frac{\log \log r}{\log \log n})}
\]
because of lemma 2.1 [ineq]. As result, we have that
\[
\frac{1}{y + y^2 + \frac{y^3}{2}} > q
\]
and therefore,
\[
\frac{1}{1 + y + \frac{y^2}{2}} > q \times y.
\]
Since we have

\[ 1 + y + \frac{y^2}{2} > 1 \]

then

\[ \frac{1}{1 + y + \frac{y^2}{2}} < 1. \]

Consequently, we obtain that

\[ 1 > q \times y \]

which is the same as

\[ e > e^{q \times y}. \]

Because of we have that \( 1 + y \leq e^y \) \[\text{Koz21}\], then

\[ e > e^{q \times y} \geq (1 + y)^q = (2 - \frac{\log \log r}{\log \log n})^q \]

that is

\[ \sqrt[e]{e} > (2 - \frac{\log \log r}{\log \log n}) \]

and finally,

\[ \sqrt[e]{e} + \frac{\log \log r}{\log \log n} > 2. \]

\[ \square \]

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