A String Model for AdS Gravity and Higher Spins

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Abstract

We construct a string sigma-model which low energy limit describes the anti de Sitter gravity and spin 3 massless fields in Vasiliev’s frame-like formalism. The model is based on vertex operators generating vielbein and connection fields in the Mac Dowell - Mansoury - Stelle - West (MMSW) formulation of gravity. The structure of the vertex operators is based on the hidden symmetry generators in RNS superstring theory, realizing the isometry group of the AdS space. The beta-function equations in the sigma-model lead to equations of motion in the MMSW gravity with negative cosmological constant with the AdS geometry being the vacuum solution. Generalizations for the higher spin fields are analyzed and equations of motion for spin 3 fields in $d = 3$ in frame-like formalism are obtained in the low energy limit of string theory. These equations correspond to those of $sl(3, R)$ truncated Chern-Simons action based on higher spin algebra $hs(1, 1)$.

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1. Introduction

Describing higher spin fields in anti-de Sitter geometry is a fascinating and challenging problem \[1, 2\]. As there is no well-defined S-matrix in AdS geometry, one could hope to circumvent no-go theorems \[3, 4\] and look for consistently interacting theories of higher spins. At the same time, higher spin fields in AdS backgrounds are important ingredients of the AdS/CFT correspondences, as there are multitudes of corresponding operators appearing in dual conformal field theories (e.g. see \[5\]). Understanding relations between holography and higher spin dynamics is therefore crucial for the entire concept of AdS/CFT in general. A frame-like formalism is a particularly powerful and efficient approach to gravity and higher spin field theories in curved backgrounds \[6, 7, 8, 9, 10\]. In this approach, gravity and higher spin field theories are formulated in terms of gauge theories of vielbeins and connections. For an ordinary theory of gravity, such an approach has been first developed by Cartan and Weyl and then generalized by Mac Dowell, Mansouri, Stelle and West who proposed manifestly gauge-invariant frame-like formulation of gravity with nonvanishing cosmological constant \[11, 12\]. The frame-like approach for the higher spin fields is the generalization of the MMSW formalism for fields with spins greater than 2. It has been proposed by Vasiliev \[6, 7\] and later developed in a number of papers. In particular, equations of motion for anti-de Sitter (AdS) gravity, as well as for the higher spin fields in AdS space-time become remarkably compact and elegant in this formalism.

String theories in AdS backgrounds constitute, in turn, another crucial ingredient of the AdS/CFT correspondence. In fact, this correspondence can be most naturally understood as isomorphism between vertex operators on the string theory side in AdS space and appropriate observables in conformal field theory, so their correlation functions match exactly. It is actually the perturbative dynamics of strings in AdS space that could provide a powerful test for the AdS/CFT, in order to approach the strongly coupled regime of gauge theories. Unfortunately, string theory in AdS backgrounds, in its standard formulations, is difficult to approach beyond semiclassical limit, although even in this limit some remarkable results for anomalous dimensions of gauge theory operators have been obtained (e.g. see \[13\]).

At the same time, string theory appears to be an efficient and natural framework to describe the dynamics of interacting higher spin field theories both in flat space and in AdS \[14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30\].
In this paper we propose a sigma-model in RNS string theory, based on hidden symmetry generators \cite{31} that realise the $o(d-1,2)$ isometry algebra of $AdS_d$. Namely, the sigma model is based on the RNS superstring theory perturbed by the vertex operators which structure is determined by the AdS isometry generators. As will be demonstrated below, the vertex operators, constructed in this work, can be regarded as sources for connection gauge fields and vielbeins in space-time (which can be unified into a single $o(d-1,2)$ connection gauge field). The construction is based on certain hidden space-time symmetry also generators (reviewed in this paper) that realize AdS isometry group in $d$ dimensions. It is particularly remarkable that the commutation relations of these operators (computed in the Section 2) fix the negative sign of the cosmological constant, leading to the appearance of the AdS geometry in the sigma-model constructed in the paper. BRST nontriviality constraints on the vertex operators in the sigma-model lead to $o(d-1,2)$ gauge transformations on the unified connection, while the BRST invariance conditions produce linearized equations of motion for the connection field (including the zero torsion constraint). Beta-function equations for the sigma-model, in turn, lead to full equations of motion for AdS gravity (with cosmological constant) in the frame-like formalism. The rest of the paper is organized as follows. In the next section (section 2) we review the realization of AdS isometry by special matter-ghost mixing symmetry generators ($\alpha$-symmetries \cite{31}). These generators will be used as building blocks to construct closed string vertex operators for the unified connection gauge field. In Section 3, we analyze the BRST constraints for the connection vertex operators. We find that the BRST nontriviality conditions lead to gauge symmetry transformations for the $o(d-1,2)$ connection field in MMSW gravity, while the BRST-invariance conditions lead to linearized equations of motion for this field. In Section 4, we study the sigma-model in RNS string theory, based on the constructed vertex operators for the connection. We calculate the beta-function in this model and show that it leads to full equations of motion for MMSW gravity in the frame-like formalism with cosmological constant. The source of the cosmological constant comes from the vertices corresponding to transvections in AdS isometry transformations.

In the concluding section we discuss the physical implications of our results and their generalizations to higher spin field theories in the frame-like approach, particularly deriving frame-like equations of motion for spin 3 fields on $AdS_3$ from string theory sigma-model.

2. AdS Isometry and Space-Time $\alpha$-Symmetry

In string theory the space-time symmetry generators are typically conformal dimension 1 primary fields, integrated over the worldsheet boundary. For example, in RNS
string theory the Poincare isometries of flat space-time are realized by the operators of translations and rotations given by

\[ T_m = \oint \frac{dz}{2i\pi} \partial X^m \]

\[ L_{mn} = \oint \frac{dz}{2i\pi} \psi_m \psi_n + \ldots \]

where we have skipped the ghost dependent terms in the expression for the rotation generator (that ensure the overall BRST invariance of the generator). Here \( X^m (m = 0, \ldots, d-1) \) are the space-time coordinates, and \( \psi^m \) are their worldsheet superpartners. What is far less trivial is that, in addition the standard Poincare isometries, RNS string theory also possesses a set of additional surprising symmetries that are realized nonlinearly and mix matter and ghost degrees of freedom \[31\]. In particular, there is a subgroup of these generators that realize the \( o(d-1,2) \) isometry of \( AdS_d \). Recall that the \( AdS_d \) isometry algebra is given by:

\[
\begin{align*}
[T_{ab}, T_{cd}] &= \eta_{ac} T_{bd} - \eta_{ab} T_{cd} - \eta_{cd} T_{ab} + \eta_{bd} T_{ac} \\
[T_a, T_{bc}] &= \eta_{ab} T_c - \eta_{ac} T_b \\
[T_a, T_b] &= \Lambda T_{ab}
\end{align*}
\]

where \( \Lambda \sim -\frac{1}{\rho^2} \) is negative cosmological constant and \( R \) is the \( AdS \) radius. In other words, the main property distinguishing the \( AdS \) isometry algebra (2) from the one of the flat space is the noncommutation of the vector generators (proportional to the cosmological constant). For this reason, these generators are known as the generators of transvections in the \( AdS \) space, to distinguish them from translations in flat space-time.

In RNS string theory, the \( AdS \) isometry algebra (2) can be realized by using the the generators of the \( \alpha \)-symmetries \[31\] inducing nonlinear global symmetries in space-time. Namely, consider the RNS superstring theory in flat space with the action given by:

\[
S_{RNS} = S_{matter} + S_{bc} + S_{\beta\gamma} + S_{Liouville}
\]

\[
S_{matter} = -\frac{1}{4\pi} \int d^2z (\partial X_m \bar{\partial} X^m + \psi_m \bar{\partial} \psi^m + \bar{\psi}_m \partial \bar{\psi}^m)
\]

\[
S_{bc} = \frac{1}{2\pi} \int d^2z (b \bar{\partial} c + \bar{b} \partial \bar{c})
\]

\[
S_{\beta\gamma} = \frac{1}{2\pi} \int d^2z (\beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma})
\]

\[
S_{Liouville} = -\frac{1}{4\pi} \int d^2z (\partial \varphi \bar{\partial} \varphi + \bar{\partial} \lambda \lambda + \partial \lambda \bar{\lambda} + \mu_0 e^B \varphi (\lambda \bar{\lambda} + F))
\]
where $\varphi, \lambda, F$ are components of super Liouville field and the Liouville background charge is

$$q = B + B^{-1} = \sqrt{\frac{9-d}{2}}$$

(4)

The ghost fields $b, c, \beta, \gamma$ bosonized according to

$$b = e^{-\sigma}, c = e^{\sigma}$$

$$\gamma = e^{\phi - \chi} \equiv e^{\phi} \eta$$

$$\beta = e^{\chi - \phi} \partial \chi \equiv \partial \xi e^{-\phi}$$

(5)

and the BRST charge is

$$Q = Q_1 + Q_2 + Q_3$$

$$Q_1 = \oint \frac{dz}{2i\pi} (cT - bc\partial c)$$

$$Q_2 = -\frac{1}{2} \oint \frac{dz}{2i\pi} (\gamma \psi_m \partial X_m - q \partial \lambda)$$

$$Q_3 = -\frac{1}{4} \oint \frac{dz}{2i\pi} b \gamma^2$$

(6)

Then, in the limit $\mu_0 \to 0$ the action (3) is particularly symmetric under the following global space-time transformations (for complete list of symmetries e.g see [31])

$$\delta X^m = e^m \{ \partial (e^\phi \lambda) + 2e^\phi \partial \lambda \}$$

$$\delta \lambda = -e^m \{ e^\phi \partial^2 X_m + 2 \partial (e^\phi \partial X_m) \}$$

$$\delta \gamma = e^m e^{2\phi - \chi} (\lambda \partial^2 X_m - 2 \partial \lambda \partial X_m)$$

$$\delta \beta = \delta b = \delta c = 0$$

(7)

with the generator of (7) given by

$$T_m = \frac{1}{\rho} \oint \frac{dz}{2i\pi} e^\phi (\lambda \partial^2 X_m - 2 \partial \lambda \partial X_m)$$

(8)

where $\rho$ is some constant (which we shall relate to AdS radius and cosmological constant, while relating $T_m$ to generator of transvections). This generator is not BRST-invariant and therefore the symmetry transformations generated by (8) are incomplete (similarly, the rotation generator $T_{mn} = \oint \frac{dz}{2i\pi} \psi_m \psi_n$ is not BRST invariant and therefore only induces rotations for the $\psi$-fields but not for bosons). To make both $T_m$ and $T_{mn}$ complete one has to restore their BRST invariance by adding ghost dependent correction terms. These
terms can be obtained by the homotopy $K$-transformation described in [31], [29], which we shall briefly review below. Let $Q$ be the BRST operator given by (6) and let

$$T = \oint \frac{dz}{2i\pi} V(z)$$  

be some global symmetry generator, incomplete (in the sense described above) and not BRST invariant, satisfying

$$[Q_{brst}, V(z)] = \partial U(z) + W(z)$$  

and therefore

$$[Q_{brst}, T] = \oint \frac{dz}{2i\pi} W(z)$$  

Introduce the homotopy operator

$$K(z) = -4ce^{2\chi-2\phi}(z) \equiv \xi \Gamma^{-1}(z)$$  

satisfying

$$\{Q_{brst}, K(z)\} = 1$$  

In general, the homotopy operator has a non-singular OPE with $W$. Suppose this OPE is given by

$$K(z_1)W(z_2) \sim (z_1 - z_2)^N Y(z_2) + O((z_1 - z_2)^{N+1})$$  

where $N \geq 0$ and $Y$ is some operator of dimension $N + 1$.

Then the complete BRST-invariant symmetry generator $L$ can be obtained from the incomplete non-invariant symmetry generator $T$ by the following homotopy transformation:

$$T \to L(w) = K \circ T = T + \frac{(-1)^N}{N!} \oint \frac{dz}{2i\pi} (z - w)^N : K \partial^N W : (z) + \frac{1}{N!} \oint \frac{dz}{2i\pi} \partial^{N+1} [(z - w)^N K(z)] K \{Q_{brst}, U\}$$  

where $w$ is some arbitrary point on the worldsheet and $K \circ$ represents the transformation (15) using the $K(z)$ operator (12). It is straightforward to check the invariance of $L$ by using some partial integration along with the relation (13) as well as the obvious identity

$$\{Q_{brst}, W(z)\} = -\partial(\{Q_{brst}, U(z)\})$$
that follows directly from (10). The homotopy transformed BRST-invariant $L$-generators are then typically of the form

$$L(w) = \oint \frac{dz}{2i\pi} (z - w)^N \tilde{V}_{N+1}(z)$$

with the conformal dimension $N + 1$ operator $\tilde{V}_{N+1}(z)$ in the integrand satisfying

$$[Q_{brst}, \tilde{V}_{N+1}(z)] = \partial^{N+1}\tilde{U}_0(z)$$

where $\tilde{U}_0$ is some operator of conformal dimension zero. Although for $N > 0$ the $L$-operator depends on an arbitrary point on the worldsheet, such a dependence is irrelevant in correlation functions since it can be shown [31] that all the $w$-derivatives of $L$ are BRST exact in small Hilbert space. We shall refer to $L$ as homotopy image of $K$. For our purposes, it will be also convenient to generalize the definitions (10)-(15) as follows. Namely, we shall refer to operator $L = K\Upsilon \circ T$ as a partial homotopy transform of $T$ based on $\Upsilon$, if the operator $T = \oint V$ satisfies $[Q_1, V] = \partial(cU) + W$, $\Upsilon$ is some dimension 1 operator, the OPE of $K$ and $\Upsilon$ is non-singular with the leading order $N > 0$ and $L$ is related to $K$ according to the transformation (15) with $W$ replaced by $\Upsilon$, i.e.

$$L(w) = K\Upsilon \circ T = T + \frac{(-1)^N}{N!} \oint \frac{dz}{2i\pi} (z - w)^N : K\partial^N \Upsilon : (z)$$

$$+ \frac{1}{N!} \oint \frac{dz}{2i\pi} \partial^{N+1} [(z - w)^N K(z)] K\{Q_{brst}, U\}$$

Particularly, if $[Q, T] = \oint \Upsilon$, the partial homotopy transform obviously coincides with the usual homotopy transform (15). In the following sections, we shall particularly use the partial homotopy transforms in order to construct operators with necessary on-shell conditions.

Let us now apply the above prescription to the symmetry generators (1), (8). The homotopy transformed full BRST-invariant rotation generator is then given by

$$L_{mn} = \oint \frac{dz}{2i\pi} [\psi_m \psi_n + 2 ce^{\chi} - \phi \psi_m \partial X_n] - 4 \partial c e^{2\phi - 2\chi}$$

$$= -4\{Q, \xi \Gamma^{-1} \psi_m \psi_n\}$$

Note that the generator (20) can be written as a BRST commutator in the large Hilbert space. It is straightforward to check that the generator (20) induces (up to the terms, BRST exact in small Hilbert space) Lorenz rotations for all the matter fields (both $X$ and
Similarly, the homotopy transformation of the generator (8) gives full BRST-invariant symmetry generator given by

\[
L^m(w) = \oint \frac{dz}{2i\pi} (z - w)^2 \left\{ \frac{1}{2} P^{(2)}_{2\phi - 2\chi - \sigma} e^{\phi} F^m_{\frac{4}{3}} - 12 \partial c e^{2\chi - \phi} F^m \right. \\
+ c e^\chi \left[ -\frac{2}{3} \partial^3 \psi^m \lambda + \frac{4}{3} \partial^3 \varphi \partial X^m + 2 \partial^2 \psi^m \partial \lambda \\
+ P^{(1)}_{\phi - \chi} (-2 \partial \varphi \partial^2 X^m + 4 \partial^2 \varphi \partial X^m - 2 \partial^2 \psi^m \lambda + 4 \partial \psi^m \partial \lambda) \\
\left. + P^{(2)}_{\phi - \chi} (2 \partial \varphi \partial X^m + 2 \partial^2 \psi^m \partial \lambda - 2 \partial \psi^m \lambda - \frac{2}{3} \partial^3 \varphi \partial^2 X^m) + P^{(3)}_{\phi - \chi} (-\frac{2}{3} \psi^m \lambda + \frac{4}{3} \partial X^m) \right] \right\}
\tag{21}
\]

so the full vector symmetry generator is again the BRST commutator in the large Hilbert space. Here \( F^m_\frac{4}{3} = \lambda \partial^2 X_m - 2 \partial \lambda \partial X_m \) and the conformal weight \( n \) polynomials \( P^{(n)}_{a\phi + b\chi + c\sigma} \) (where \( a, b, c \) are some constants) are defined according to

\[
P^{(n)}_{a\phi + b\chi + c\sigma} = e^{-a\phi(z) - b\chi(z) - c\sigma(z)} \frac{d^n}{dz^n} e^{a\phi(z) + b\chi(z) + c\sigma(z)}
\tag{22}
\]

(with the product taken in algebraic rather than OPE sense). The BRST-invariant symmetry generator can also be constructed at dual \(-3\) picture (as well as the pictures below; but not above minimal negative picture \(-3\) at which it is annihilated by the picture changing). At picture \(-3\) the symmetry generator is given by

\[
L^m = \oint \frac{dz}{2i\pi} e^{-3\phi} F^m \tag{23}
\]

The symmetry generators (21), (23) at pictures \(+1\) and \(-3\) are related by the sequence of \(Z\)-transformations and the picture-changing \([31]\) according to

\[
L^m_{(+1)} = Z : \Gamma^2 : Z : \Gamma^2 : L^m_{(-3)}
\]

where \( \Gamma = \{Q, e^\chi\} \) is the picture-changing operator for the \(\beta - \gamma\) system while \( Z = b\delta(T) \) is the operator of picture-changing for the \(b - c\) system (particularly, it maps unintegrated vertex operators to integrated). The manifest integral form of \( Z \) is given e.g. in \([29]\).

With some effort, it can now be shown that the operators \( L^{mn} \) and \( L_m \) realize the \(AdS_d\) isometry algebra (2) with the cosmological constant \( \Lambda = -\frac{1}{\rho^2} \).
To demonstrate this, we start with the OPE of the primary fields \( F_m^n(z) \) (related to the matter ingredient of \( L^m \)) Straightforward calculation gives

\[
\begin{align*}
F_m^n(z)F_m^n(w) &= -\frac{6\eta_{mn}}{(z-w)^5} + \frac{14\partial^\lambda\partial\eta_{mn}(w) + 8\partial X^m\partial X^n(w)}{(z-w)^3} \\
&\quad + \frac{10\partial^2 X^m\partial X^n(w) - 2\partial X^m\partial^2 X^n(w) + 7\eta_{mn}\partial^2 \lambda\lambda(w)}{(z-w)^2} \\
&\quad + \frac{6\partial^3 X^m\partial X^n(w) - 3\partial^2 X^m\partial^2 X^n(w) + 3\eta_{mn}\partial^3 \lambda\lambda(w) + 2\eta_{mn}\partial^2 \lambda\partial\lambda(w)}{z-w} \\
&\quad + (z-w)^0 \left[ \frac{7}{3}\partial^4 X^m\partial X^n(w) - 2\partial^3 X^m\partial^2 X^n(w) \right. \\
&\quad + \frac{11}{12}\eta_{mn}\partial^4 \lambda\lambda(w) + \frac{4}{3}\eta_{mn}\partial^3 \lambda\partial\lambda(w) \left. \right] + : F_m^n F_m^n : (w) \\
&\quad + (z-w)^1 \left[ \frac{13}{60}\eta_{mn}\partial^5 \lambda\lambda(w) - \frac{1}{2}\eta_{mn}\partial^4 \lambda\partial\lambda(w) + \frac{2}{3}\partial^5 X^m\partial X^n(w) \right. \\
&\quad - \frac{5}{6}\partial^4 X^m\partial^2 X^n(w) + \partial F_m^n F_m^n : (w) \left. \right] \\
&\quad + (z-w)^2 \left[ \frac{1}{24}\eta_{mn}\partial^6 \lambda\lambda(w) - \frac{2}{15}\eta_{mn}\partial^5 \lambda\partial\lambda(w) \right. \\
&\quad \left. \right] + \frac{3}{20}\partial^6 X^m\partial X^n(w) - \frac{1}{4}\partial^5 X^m\partial^2 X^n(w) + \frac{1}{2}\partial^2 F_m^n F_m^n : (w) + \ldots \\
\end{align*}
\]

Using this OPE it is straightforward to compute the commutator \([L^m, L^n]\). It is convenient to choose one of the vector at picture +1 representation (21) and another at negative picture −3 representation (23). Because of the isomorphism between positive and negative picture representations, ensured by the appropriate \( Z, \Gamma \) transformations (see below equation (23)), the final result will be picture-independent. Then, using (21) and the BRST invariance of \( L^m \) at negative picture (23), we get

\[
[L^m, L^n] = \frac{1}{\rho^2} \{ Q, \int \frac{dz_1}{2i\pi} (z_1 - w)^2 e^{2\phi - \chi} F_m^n(z_1), \int \frac{dz_2}{2i\pi} e^{-3\phi} F_m^n(z_2) \} = \{ Q, U(z_2) \}
\]
where

\[ U(z_1) \equiv \oint \frac{dz_2}{2i\pi} U_1(z_2) + \oint \frac{dz_2}{2i\pi} (z_1 - z_2) U_2(z_2) + \oint \frac{dz_2}{2i\pi} (z_1 - z_2)^2 U_3(z_2) \]

\[ = \oint \frac{dz_2}{2i\pi} c e^{2\chi - 4\phi} \left[ \frac{7}{3} \partial^4 X^{[m} \partial X^{n]} - 2 \partial^3 X^{[m} \partial^2 X^{n]} + 6 P^{(1)}_{2\chi-\phi+\sigma} \partial^3 X^{[m} \partial X^{n]} \right. \]

\[ + 4 P^{(2)}_{2\chi-\phi+\sigma} \partial^3 X^{[m} \partial X^{n]} + : F^{m} F^{n} : \]

\[ + \oint \frac{dz_2}{2i\pi} (z_1 - z_2) c e^{2\chi - 4\phi} \left[ \frac{4}{3} P^{(3)}_{2\chi-\phi+\sigma} \partial^2 X^{[m} \partial X^{n]} + 3 P^{(2)}_{2\chi-\phi+\sigma} \partial^3 X^{[m} \partial X^{n]} \right. \]

\[ + P^{(1)}_{2\chi-\phi+\sigma} \left( \frac{7}{3} \partial^4 X^{[m} \partial X^{n]} - 2 \partial^3 X^{[m} \partial X^{n]} \right) \]

\[ + \frac{2}{3} \partial^5 X^{[m} \partial X^{n]} - \frac{5}{6} \partial^4 X^{[m} \partial X^{n]} \]

\[ + \oint \frac{dz_2}{2i\pi} (z_1 - z_2)^2 c e^{2\chi - 4\phi} \left[ \frac{1}{3} P^{(4)}_{2\chi-\phi+\sigma} \partial^2 X^{[m} \partial X^{n]} + P^{(3)}_{2\chi-\phi+\sigma} \partial^3 X^{[m} \partial X^{n]} + \right. \]

\[ P^{(2)}_{2\chi-\phi+\sigma} \left( \frac{7}{6} \partial^4 X^{[m} \partial X^{n]} - \partial^3 X^{[m} \partial X^{n]} \right) \]

\[ + P^{(1)}_{2\chi-\phi+\sigma} \left( \frac{2}{3} \partial^5 X^{[m} \partial X^{n]} - \frac{5}{6} \partial^4 X^{[m} \partial X^{n]} \right) + \frac{3}{20} \partial^3 X^{[m} \partial X^{n]} - \frac{1}{4} \partial^5 X^{[m} \partial X^{n]} \]

where, for convenience, we split the overall integral into 3 parts, with the integrands proportional to \( U_1(z_2), \ (z_1 - z_2) U_2(z_2) \) and \((z_1 - z_2)^2 U(z_2)\) accordingly.

To relate the right hand side of the commutator (26) to the rotation generator (20) one has to perform double picture changing transform of \{Q, U(z_1)\} in order to bring it to picture zero. We shall demonstrate the procedure explicitly for the \( U_1 \) integral, with the other two integrals treated similarly. For that, we first of all need a manifest expression for the commutator of the BRST charge with the \( U_1(z_1) \) operator (26). Straightforward calculation gives:

\[ \{Q, U_1(z)\} = -2 \partial c e^{2\chi - 4\phi} (P^{(2)}_{2\chi-\phi+2\sigma} + P^{(2)}_{2\chi-\phi+\sigma}) \partial^2 X^{[m} \partial X^{n]} \]

\[ + 9 \partial^2 c e^{2\chi - 4\phi} P^{(1)}_{2\chi-\phi+\sigma} \partial^2 X^{[m} \partial X^{n]} \]

\[ - \frac{3}{2} \partial^2 c e^{2\chi - 4\phi} \partial^3 X^{[m} \partial X^{n]} + \partial c e^{2\chi - 4\phi} \left( \frac{7}{3} \partial^4 X^{[m} \partial X^{n]} - 2 \partial^3 X^{[m} \partial X^{n]} \right) \]

\[ + 34 \partial^3 c e^{2\chi - 4\phi} \partial^2 X^{[m} \partial X^{n]} + 12 \partial^2 c e^{2\chi - 4\phi} \partial^3 X^{[m} \partial X^{n]} \]

\[ + 4 c e^{2\chi - 3\phi} [\partial^2 \psi^{|m} \partial X^{|n]} + \partial \psi^{|m} \partial^2 X^{|n]} - P^{(1)}_{\phi-\chi} (\psi^{|m} \partial^2 X^{|n]} - 2 \partial \psi^{|m} \partial X^{|n]} \]

\[ + \psi^{|m} \partial X^{|n]} (P^{(2)}_{\phi-\chi} + P^{(2)}_{2\chi-\phi+\sigma}) \]

The next step is to perform the normal ordering of the integrand of this expression with \( \xi = e^\chi \) around the midpoint. We get

\[ \xi \{Q, U_1(z)\} := -8 \partial c e^{2\chi - 4\phi} \partial^2 X^{[m} \partial X^{n]} + 4 c e^{2\chi - 3\phi} [\psi^{|m} \partial^2 X^{|n]} \]

\[ + 4 \partial \psi^{|m} \partial X^{|n]} - 2 \psi^{|m} \partial X^{|n]} P^{(1)}_{3\phi-\chi-2\sigma} \]

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The next step is to perform the commutation of this expression with $Q$ which, by definition, gives us $\{Q, U(z)\}$ at picture $-1$. Straightforward calculation gives:

$$\{Q, \xi \{Q, U(z)\}\} = c e^{x-2\phi} (4P^{(1)}_{\phi + \chi - 2\psi m \psi n} + 2\psi^m \partial \psi^n) \tag{29}$$

Next, the normal ordering of this expression with $\xi$ around the midpoint gives:

$$\xi \{Q, \xi \{Q, U(z)\}\} := 4c e^{x-2\phi} \psi^m \psi^n \tag{30}$$

Finally, the commutator of this expression with $Q$ by definition gives us $U_1$ at picture zero:

$$U_{1(0)}(z) \equiv \{Q, \xi \{Q, \xi \{Q, U_1(z)\}\}\} = 4\{Q, e^{2x-2\phi} \psi^m \psi^n\} \tag{31}$$

which, according to (20) is nothing but the integrand of the full rotation generator with the inverse sign. The picture transform of $U_2$ and $U_3$ in the remaining terms of (20) is performed similarly. Applying picture changing transformation twice and integrating out total derivatives we find the contributions from the second and the third integrals cancel each other and the commutator remains unchanged. This concludes the proof that the commutator of two operators $[L^m, L^n] = -\frac{1}{\rho^2} L^{mn}$ reproduces the commutation of two transvections in the $AdS$ isometry algebra (2).

The remaining commutators of (2) are computed similarly. Note the highly nontrivial appearance of the minus sign on the right hand side of the commutator as a result of the ghost structure of the operators, that indicates that the effective cosmological constant in the symmetry algebra is negative, so the effective geometry generated is of the $AdS$ type.

The combination of operators $(L^m, L^{mn})$ (20), (21) that we considered so far, is not the only possible realization of the $AdS$ symmetry algebra (2) in RNS theory. In particular, it is easy to check that $so(d-1,2)$ isometry of the $AdS$ space is realized by $S^m, L^{mn}$ where

$$L^{mn} = K \circ T^{mn} \equiv K \circ \int \frac{dz}{2i\pi} \psi^m \psi^n \tag{32}$$

is the same full rotation operator (20) (where the $K$ represents the homotopy transformation to ensure the BRST-invariance) while $S^m$ is the homotopy transformation of the operator $\int \frac{dz}{2i\pi} \lambda \psi^m$, representing the rotation in the Liouville-matter plane:

$$S^m = K \circ \rho^{-1} \int \frac{dz}{2i\pi} \lambda \psi^m = \rho^{-1} \int \frac{dz}{2i\pi} [\lambda \psi^m + 2c e^{x-\phi} (\partial \varphi \psi^m - \partial X^m \lambda - q P^{(1)}_{\phi - \chi \psi^m} - 4\partial c e^{2x-2\phi} \lambda \psi^m] \tag{33}$$

$$= -4\{Q, \rho^{-1} \int \frac{dz}{2i\pi} c e^{2x-2\phi} \lambda \psi^m\}$$
Again, using (33) it and the procedure identical to the one explained above, it is straightforward to show that $S^m$ satisfy the commutation relation for transvections: $[S^m, S^n] = -\frac{L^{mn}}{\rho^2}$ and the rest of $so(d - 1, 2)$ relations (2) with $L^{mn}$. In order to construct vertex operators for spin connection in $AdS$ space, we will actually need the realization using the linear combination of the transvections (21),(33), given by

$$P^m = \frac{1}{\sqrt{2}}(L^m + S^m)$$

(34)

One can show that these generators realize the transvections on $AdS$ space provided that the two-form (20) is shifted according to

$$L^{mn} \to P^{mn}(w) = L^{mn} + K \circ \int \frac{dz}{2i\pi} e^{-3\phi}(\psi^{[m} \partial^2 X^{n]} - 2\partial\psi^{[m} \partial X^{n]}) + qK \circ \int \frac{dz}{2i\pi} ce^{-\phi} \lambda \psi^m \psi^n$$

(35)

where $K$ is again the homotopy transformation. The new term, proportional to the background charge, appears as a result of the Liouville terms of $Q$ entering the game of picture changing. The AdS isometry algebra (2) is then realized by the combination of $P^m$ and $P^{mn}$. In the next section, we will use these $AdS$ isometry generators as building blocks to construct vertex operators for $AdS$ frame fields in closed string theory.

### 3. Vertex operators for frame fields and on-shell conditions

In this section we construct vertex operators for connection gauge fields for MMSW gravity with cosmological constant, using the generators (20),(21), (33)-(35) and study their BRST properties. We find that the BRST invariance constraints leads to linearized equations of motion for MMSW gravity around the AdS vacuum, while the nontriviality constraints entail the gauge transformations for the frame fields. To start with, let us recall the basic facts about MMSW formulation of gravity. In the frame-like approach, the description of the dynamics of the theory in terms of metric tensor $g_{mn}$ is replaced by introducing two dynamical fields - the frame field $e^a_m$ and the connection gauge field $\omega^a_m$ with the indices $a$ and $b$ living in the tangent space. Using these fields one constructs one-forms $e^a = e^a_m dx^m$ and $\omega^{ab} = \omega^{ab}_m dx^m$ and unifies them into a single one-form $\omega = e^a T_a + \frac{1}{2} \omega^{am} T_{ab}$ where $T_a$ and $T_{ab}$ are isometry generators of $AdS_d$. The curvature is then the two-form defined according to

$$R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} - \rho^{-2} e^a \wedge e^b$$

(36)
while the two-form of torsion is given by

\[ T^a = de^a + \omega^a_c \wedge e^c \] (37)

It is convenient to unify the connection and the frame fields into a single \( o(d-1,2) \) gauge field

\[ \omega^{AB} \equiv (\omega^{ab}, \omega^{ad}) \] (38)

where by definition \( \omega^{ad} = \rho^{-1} e^a \) and the \( o(d-1,2) \) index is split in the \((d,1)\) way as \( A \equiv (a,d) \). The curvature and the torsion are then unified into a single tensor \( R^{AB} = (R^{ab}, R^{ad}) \) with \( R^{ad} = \rho^{-1} T^a \). The AdS\(_d\) geometry is then the solution of the vacuum equations

\[ R^{AB}(\omega) = 0 \] (39)

which combine the constant curvature and the zero torsion constraints. The gauge symmetry transformations for the 1-form \( \omega^{AB} = \omega^{AB}_m dx^m \) are given by

\[ \delta^\text{gauge} \omega^A_m = D^m \rho^{AB} \]
\[ \delta^\text{diff} \omega^A_m = \partial_n \epsilon^m \omega^A_n + \epsilon^m \partial_n \omega^A_m \] (40)

where \( \rho^{AB} \) and \( \epsilon^m \) are the parameters of the gauge and diffeomorphism transformations accordingly.

Our goal now will be to construct a sigma-model based on vertex operators for connection and frame gauge fields, which beta-functions reproduce (39). Since AdS geometry would appear as the vacuum solution of (39), the low-energy limit of the string theory sigma-model we are looking for would describe the MMSW gravity on anti-de Sitter space in the frame-like formulation. Just as a standard graviton operator (describing fluctuations of metric around flat vacuum) is given by the structure bilinear in translation operators (multiplied by \( e^{ipX} \)), we shall look for vertex operators for the MMSW gauge fields as closed string bilinears based on generators (20),(21),(33)-(35) realizing the AdS\(_d\) isometry (2). The operator that we propose is given by

\[ G(p) = e^a_m(p) F_a \bar{L}^m + + \omega^{ab}_m(p) (F^b_m \bar{L}_a - \frac{1}{2} F_{ab} \bar{L}^m) + \text{c.c.} \] (41)

where

\[ F_m = -2K_{U_1} \circ \int dz \lambda \psi^m \bar{e}^{ipX}(z) \]
\[ U_1 = \lambda \psi^m e^{ipX} + \frac{i}{2} \gamma \lambda ((\bar{p} \gamma \psi^m - p_m P^{(1)}_{\phi-\chi}) e^{ipX} \] (42)
or manifestly

\[ F_m = -2 \int dz \{ \lambda \psi_m (1 - 4 c c e^2 x - 2 \phi) + 2 c e^x - \phi (\lambda \partial x_m - \partial \phi \psi_m + q \psi_m P^{(1)}_{\phi - x}) \} e^{i p X} (z) \]  

\[ F_{m a} = -4 q K U_2 \circ \int dz c e^x - \phi \lambda \psi_m \psi_a \]  

\[ U_2 = [Q - Q_3, c e^x - \phi \lambda \psi_m \psi_a e^{i p X}] = -\frac{i}{2} c \lambda (\bar{\partial} \bar{\psi}) \psi_m \psi_a - p_m \psi_a P^{(1)}_{\phi - x} e^{i p X} (z) \]  

Next,

\[ \bar{L}^a = \int d\bar{z} e^{-3 \phi} \{ \bar{\lambda} \bar{\partial}^2 X^a - 2 \bar{\partial} \bar{\lambda} X^a \]  

\[ + i p^a \left( \frac{1}{2} \bar{\partial}^2 \bar{\lambda} + \frac{1}{q} \partial \bar{\phi} \bar{\partial} \bar{\lambda} - \frac{1}{2} \bar{\lambda} (\partial \bar{\phi})^2 + (1 + 3 q^2) \bar{\lambda} (3 \bar{\partial} \bar{\psi} \bar{\psi}^b - \frac{1}{2 q} \bar{\partial}^2 \bar{\phi}) \right) \} e^{i p X} (z) \]  

(similarly for its holomorphic counterpart \( L^a \)) and

\[ F_{m a} = F_{m a}^{(1)} + F_{m a}^{(2)} + F_{m a}^{(3)} \]  

where

\[ F_{m a}^{(1)} = -4 q K U_2 \circ \int dz c e^x - \phi \lambda \psi_m \psi_a \]  

\[ F_{m a}^{(2)} = K \circ \int dz \psi_m \psi_a e^{i p X} = -4 \{ Q, \int dz c e^x - 2 \phi \} e^{i p X} (z) \]  

and

\[ F_{m a}^{(3)} = \int d\bar{z} e^{-3 \phi} \left( \psi_m \partial^2 X_a - 2 \partial \psi_m \partial X_a \right) e^{i p X} (z) \]  

In the limit of zero momentum the holomorphic and the antiholomorphic components of the operator (41) correspond to AdS isometry generators (20), (21), (33)-(35) in different realizations, described above. More precisely, while the antiholomorphic part of (41) is based on the L- operators (20),(21) related to the L-realization of the symmetry algebra, the holomorphic part of (41) involves the F-operators (such as \( F^a \) and \( F^{ab} \)) which, although different from the operators of the P-representation, become related to those after one imposes the on-shell constraints on the space-time fields (see below).

We start with analyzing the BRST invariance constraints on the operator (41). The BRST commutators are given by:
The BRST invariance therefore imposes the following constraints on vielbein and connection fields:

\[ [Q, G(p)] = 0 \]

\[ [Q, G(p)] = ie^b_m(p)\bar{L}_b \int dz \gamma \lambda((\bar{p}\bar{\psi})\psi_m - P^{(1)}_{x})e^{ipX}(z) \]

\[ \omega^{ab}_m L_b \int dz \{ \gamma \lambda \psi_a \psi_m + 2i c \lambda(\bar{p}\bar{\psi})\psi_a \psi_m - P^{(1)}_{x}\psi_a \} e^{ipX}(z) \]

The BRST invariance therefore imposes the following constraints on vielbein and connection fields:

\[ p^{[n}e^{b]}_m(p) - \omega^{b[n}_m(p) = 0 \]

\[ p^{[n}\omega^{ab]}_m(p) = 0 \]

\[ p^m e^b_m(p) = 0 \]

\[ p^m \omega^{ab}_m(p) = 0 \]

The first two constraints represent the linearized equations \( R^{AB} = 0 \) (the first one being the zero torsion constraint \( T^a = R^{a\hat{a}} = 0 \) while the second reproducing vanishing Lorenz curvature \( R^{ab} = 0 \)). The last two constraints represent the gauge fixing conditions related to the diffeomorphism symmetries (40). The fact that the BRST invariance leads to space-time equations in a certain gauge is not surprising if we recall that similar constraints on a standard vertex operator of a photon also lead to Maxwell’s equations in the Lorenz gauge. Provided that the constraints (50) are satisfied the vertex operator \( G(p) \) can be written as a BRST commutator in the large Hilbert space plus terms that are manifestly in the small Hilbert space, according to

\[ G(p) = \{Q, W(p)\} + \frac{1}{q} \omega^{ab}_m \int dz e^{3\phi}(\psi^{[m}\partial^2 X_a] - 2\partial \psi^{[m}\partial^X a] e^{ipX}(z) \bar{L}_b + c.c. \]

\[ W(p) = 8e^a_m(p)\bar{L}_a \int dz \omega \lambda \psi^m e^{ipX} \]

\[ = \omega^{ab}_m L_b[ - \frac{4}{q} \int dz \omega \lambda \psi^m e^{ipX} \]

\[ + 4 \int dz (z - w) \partial cc \partial \partial^2 \xi \partial^X \lambda \psi^m e^{ipX} \]

This particularly implies that, modulo gauge transformations, the vertex operator \( G(p) \) is the element of the small Hilbert space. Let us now turn to the question of BRST nontriviality and related gauge symmetries (40). The linearized gauge symmetry transformations (40) are given by

\[ \delta e^a_m = \partial_m \rho^a + \rho^a_m \]

\[ \delta \omega^{ab}_m = \partial_m \rho^{ab} + \rho^{[a}_m \rho^{b]}_m \]
where we write $\rho^{AB} = (\rho^{ab}, \rho^{ad}) = (\rho^{ab}, \rho^a)$ The variation of $G(p)$ under (52) in the momentum space is

$$
\delta G(p) = p^m F_m \bar{L}_a \rho^a + p^m F_{ma} \bar{L}_b \rho^{ab}
$$

(53)

The two terms of the variation (53) are BRST exact in the small Hilbert space (and therefore are irrelevant in correlators) since

$$
p^m F_m = \{Q, : \Gamma : (w)\} [Q, \xi A]\}

A = \int dze^{x-3\phi} \partial \chi ((\bar{p}\bar{\partial}X)^a \lambda - (\bar{p}\bar{\psi}) \partial \varphi + (\bar{p}\bar{\psi}) P_{\phi-1+q} e^{ipX}

$$

(54)

and

$$
p^m F_{ma}^{(1)} = 4q [Q, \Gamma (w) \int dze^{-3\phi} \partial \xi \partial^2 \xi \psi_a (\bar{p}\bar{\psi}) e^{ipX}] 

p^m F_{ma}^{(2)} = \{Q, : \Gamma : (w) \int dz \partial \xi \partial (\bar{p}\bar{\psi}) \partial X_a - (\bar{p}\bar{\partial}X)_a \psi_a e^{ipX}\}

\}

p^m F_{ma}^{(3)} = \{Q, [K \circ \int dz \lambda \psi_a e^{ipX}, B]\}

B = \int dz \partial \xi e^{-4\phi} [\lambda (\partial \bar{\psi} \partial^2 \bar{X}) - 2 \partial \lambda ((\bar{p}\bar{\psi}) \partial^2 \bar{X}) - 2(\partial \bar{\psi} \partial \bar{X} - 2(\partial \bar{\psi} \partial \bar{X}__))]

$$

(55)

Therefore gauge transformations of $e$ and $\omega$ shift $G(p)$ by terms not contributing to correlators. This concludes the BRST analysis of the vertex operator for vielbein and connection fields in the frame-like description of MMSW gravity. In the next section we shall investigate the conformal beta-function of $G(p)$ in the sigma-model, showing that it reproduces the equations of motion of MMSW gravity with negative cosmological constant in the low energy limit.

4. $\beta$-Function of $G(p)$ and AdS Gravity

The leading order contribution to the beta-function of the $G(p)$ operator (giving the equations of motion for $e$ and $\omega$ in the low energy limit of string theory) is determined by the structure constants stemming from three-point correlators on the worldsheet. Computing these structure constants will be our goal in this section. Manifest expressions for the operators (41)-(48) look quite lengthy and complicated. The computations, however, can be simplified significantly due to important property of the homotopy transformations (15): That is, consider two operators $V_1(z)$ and $V_2(w)$ (of dimension 1) that are, in general, not BRST-invariant and are the elements of the small space (i.e. independent on zero mode of
Suppose their operator products with the homotopy operator $K$ are nonsingular while their full OPE between themselves is given by

$$V_1(p_1; z)V_2(p_2; w) = \sum_{k=-\infty}^{\infty} (z-w)^k C_k(p_1, p_2)V_k(p_1 + p_2; \frac{z+w}{2})$$  \hspace{1cm} (56)$$

where $C^k$ are the OPE coefficients and $V_k$ are some operators. Then the operator product of their BRST-invariant homotopy transforms is given by

$$K_U \circ V_1(p_1; z)K_U \circ V_2(p_2; w) = \sum_{k=-\infty}^{\infty} (z-w)^k D_k(p_1, p_2)K_U \circ W_k(p_1 + p_2; \frac{z+w}{2})$$  \hspace{1cm} (57)$$

with the coefficients $D_k$ and operators $W_k$ defined as follows.

Let $K_U \circ V_1$ and $K_u \circ V_2$ are the transforms of $V_1$ and $V_2$ that are BRST-invariant (given the appropriate on-shell conditions on space-time fields) Then they can be represented as BRST commutators in the large space: $K_U \circ V_1 = \{Q, KW_1\}$ and $K_U \circ V_2 = \{Q, KW_2\}$ where $W_1$ and $W_2$ are (generally) some new operators in the small space (in many important cases $W_1$ and $W_2$ may actually coincide with $V_1$ and $V_2$). Let the full OPE of $W_1$ and $W_2$ be given by

$$W_1(p_1; z)W_2(p_2; w) = \sum_{k=-\infty}^{\infty} (z-w)^k D_k(p_1, p_2)W_k(p_1 + p_2; \frac{z+w}{2})$$  \hspace{1cm} (58)$$

with certain operators and coefficients $W_k$ and $D_k$. Then the OPE of the homotopy transforms $K_U \circ V_1(z)$ and $K_U \circ V_1(w)$ is given by the formula (57). Indeed,

$$K_U \circ V_1(z)K_U \circ V_2(w) = \{Q, KW_1\}(z)\{Q, LW_2\}(w) = \{Q, V - K[Q, V](z) : LV : (w) \}
= \{Q, \sum_{k=-\infty}^{\infty} K(w)D_k(p_1, p_2)W_k(p_1 + p_2; \frac{z+w}{2}) \} - \{Q, K[Q, W_1](z)KW_2(w) \}$$  \hspace{1cm} (59)$$

where we used the BRST invariance of $\{Q, KW_1\}$ and the OPE (58) of $W_1$ and $W_2$. The OPE (59) is then given by

$$K_U \circ V_1(z)K_U \circ V_2(w) = K_U \circ (W_1(z)W_2(w)) - \{Q, L[Q, V](z)LV(w) \}$$  \hspace{1cm} (60)$$
The first term in this OPE coincides with the right hand side of (57). The second term is the BRST commutator in the small Hilbert space. Indeed, if the OPEs of $K$ with $W_1$ and $W_2$ are nonsingular, one can cast the second term in (60) as

$$\{Q, K[Q, W_1](z)KW_2(w)\} = \{Q, C(z, w)\}$$

$$C(z, w) = \sum_{m=0}^{\infty} (z-w)^m W_1(z)[Q, \partial^m LLW_2](w)$$

Since $C(z, w)$ is the product of $W_1$ (operator in the small Hilbert space) and the BRST commutator in the large Hilbert space, it is the element of the small Hilbert space. This concludes the proof of the formula (57), up to BRST exact terms in the small space, irrelevant for the beta-function. The relation (57) is remarkably useful, since it allows us to replace the computation of the products of homotopy-transformed operators (which manifest expressions are cumbersome and complicated) with the products of operators which structure is far simpler. The sigma-model we consider is given by:

$$Z(e, \omega) = \int D[X, \psi, \bar{\psi}, ghosts] e^{-S_{RNS}} + \int d^4p G(p)$$

(62)

The leading order contributions to the $\beta$-function are given by terms quadratic in $G(p)$ and are proportional to $e^2$, $\omega^2$ and $e\omega$. Consider the contribution proportional to $e^2$ first. It is given by

$$\frac{1}{2} \int_p \int_q e^a_p(p)e^b_n(q)(F^m \bar{L}_a(p)F^n \bar{L}_b(q) + c.c.)$$

$$= \int_p \int_q e^a_m(p)e^b_n(q)((L^m + K \circ \int dz\lambda \psi^m) \bar{L}_a(L^n + K \circ \int dw\lambda \psi^n) \bar{L}_b + c.c.)$$

$$= -\frac{1}{\rho^2} \int_p \int_q \int \frac{d^2\xi_1}{|\xi_1|^2} e^a_m(p)e^b_n(q)(F^m n \bar{L}_{ab}(p + q) + c.c.)$$

$$= -\frac{1}{2\rho^2} \log \Lambda \int_p \int_q e(p) \wedge e(q)(F \bar{L}(p + q) + c.c.)$$

(63)

where $\xi_1 = z_1 - z_2$, $\Lambda$ is worldsheet cutoff and we used (25)-(35) and the homotopy OPE property (57), as well as the fact that the operator in front of the exponent in the expression for $L^a$ (similarly for $\bar{L}^a$ has no OPE singularities with $e^{ipX}$ or $e^{iqX}$, up to BRST-exact terms. One can easily recognize this logarithmic divergence contributing the
cosmological term to the low energy effective equations of motion. Similarly, the term quadratic in $\omega$ contributes to the beta-function as

$$\frac{1}{2} \int_p \int_q \omega^a_m(p) \omega^c_n(q)(F^m_a \bar{L}_b - \frac{1}{2} F^a_n \bar{L}_m + c.c.)(p)(F^c_n \bar{L}_a - \frac{1}{2} F^c_d \bar{L}_n + c.c.)(q)$$

$$= \int_p \int_q \int d^2 \xi \left\{ \omega^a_m(p) \omega^c_n(q)(F^m_n \bar{L}_b + c.c.)(p + q) \right\}$$

$$= \log \Lambda \int_p \int_q [\omega(p) \wedge \omega(q)(F \bar{L} + c.c.)(p + q)]$$

Thus the right-hand side (64) accounts for $\omega \wedge \omega$ contribution to the $\beta$-function.

Also the divergence due to cross-terms proportional to $\sim e \omega$ vanishes provided that the zero torsion constraint (50) is satisfied.

Altogether (50), (63), and (64) imply the vanishing of the conformal beta-function for the model (62) leads to the low-energy effective equations of motion:

$$R^{ab} = d\omega^{ab} + (\omega \wedge \omega)^{ab} - \frac{1}{\rho^2} e^a \wedge e^b = 0$$

$$de^a + \omega^{ab} \wedge e^b = 0$$

which describe the AdS gravity in MMSW formalism. The cosmological term with $\Lambda = -\frac{1}{\rho^2}$ originates from the transvection symmetry generators that serve as building blocks for the vertex operators. Thus the leading order contribution to the $\beta$-function in the sigma-model model (62) describes the AdS vacuum solution of the MMSW gravity with negative cosmological constant. As we only considered the lowest order contributions the beta-function (64), we only recovered the vacuum solution with no fluctuations. The important next step will be to consider the fluctuations of spin 2 and higher around the AdS vacuum. For that, one has to extend (62) by adding terms with vertex operators, describing the higher spin fluctuations in the frame-like approach, with some of these operators constructed in [32] (see also (67) in the concluding Discussion section). To describe the fluctuations of spins 2 and higher, around the AdS vacuum, higher order corrections to the conformal $\beta$-function of (62) and (67) need to be computed. This calculation is currently in progress and we hope to present it soon in our future work.

5. Discussion. Higher Spin Dynamics on AdS and String Theory

The sigma-model considered in this work is constructed to set up a framework for a string theory description of higher spin dynamics on AdS in Vasiliev’s frame-like approach.
The basic idea is that the dynamics of Vasiliev’s frame-like fields and generalized connections on AdS can be obtained from correlators of vertex operators for these gauge fields in the presence of the background $G(p)$-field constructed in this work, which effectively generates cosmological constant and curves space-time from flat to AdS background. That is, the generating functional for higher spin frame fields $E^{a_1...a_1}$ and connections $\Omega^{a_1...a_n}$ should be

$$Z(E, \Omega, e, \omega) = \int D(X, \psi, \bar{\psi}, \text{ghosts}) e^{-S_{RNS} + G(p) + E^{a_1...a_n}U_{a_1...a_n} + \Omega^{a_1...a_n}W_{a_1...a_n}}$$

where $U$ and $W$ are the appropriate vertex operators for the higher spin gauge fields (we shall use capital letters for higher spin connections and frame fields to distinguish them from those in the theory of gravity) The Vasiliev’s unfolded equations for higher spin fields on $AdS$ space should then follow from the worldsheet beta-function equations for the sigma-model (67). The work in this direction is currently in progress. At this point we have been able to investigate the model (67) for the $s = 3$ case in three dimensions, leading to higher spin dynamics on $AdS_3$. Namely, the dynamic gauge fields in the $s = 3$ case are given by $E_{m}^{ab}$ and $\Omega_{m}^{ab}$ (which are spin 3 generalizations of frame and connection fields of MMSW theory) while the usual connection gauge field $\omega_{m}^{ab}$ can be dualized in $d = 3$ as $\omega_{a} = \epsilon_{abc}\omega^{bc}$ The spin 3 vertex operator for $E^{ab}$ is given by

$$U(p) = E_{m}^{ab}(p) \int dz e^{-3\phi}\psi^{m}\partial X_{a}\partial X_{a}e^{ipX}(z)$$

or, in the positive picture representation,

$$U(p) = E_{m}^{ab}(p)K \circ \int dz e^{\phi}\psi^{m}\partial X_{a}\partial X_{a}e^{ipX}(z)$$

and

$$W(p) = \Omega_{m}^{ab} \{ K \circ ( \int dz e^{\phi}\lambda\partial X_{a}\partial X_{b}e^{ipX}(z))\bar{L}^{m}(\bar{z}) \}$$

$$-\frac{1}{2}K \circ ( \int dz e^{\phi}\lambda\partial X^{m}\partial X_{a}e^{ipX}(z))L_{b} + c.c.$$
contribution to beta-function of the sigma-model (67) with spin 3 operators (69), (70) stems from disc amplitudes and leads to the low-energy equations of motion:

\[
\frac{1}{2} p[m \Omega^{ab}] + \epsilon^{acd} (e_c[m \wedge \Omega^{b}_{n]d} + \omega_c[m E^{b}_{n]d}) = 0
\]

\[
\frac{1}{2} p[m E^{ab}] + \epsilon^{acd} (\omega_c[m \wedge \Omega^{b}_{n]d} - \frac{1}{\rho^2} (e_c[m E^{b}_{n]d}) = 0
\]

which are the equations of motion for the higher spin part of the Chern-Simons type theory [33], [34], [35], [36], [37], [38], [39], [40]:

\[
S = S(\Gamma_+) - S(\Gamma_-)
\]

\[
S(\Gamma) \sim \int_{M_3} Tr(\Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma)
\]

with the gauge fields \(\Gamma\) taking values in higher spin algebra \(hs(1,1)\) truncated to \(sl(3,R)\) with \(sl(3,R)\) components given by

\[
A^a_\pm = \omega^a_\pm \pm \frac{1}{\rho} e^a_\pm
\]

\[
A^{ab}_\pm = \Omega^{ab}_\pm \pm \frac{1}{\rho} E^{ab}_\pm
\]

(while the \(sl(2,R)\) truncation gives the equations (66) for the MMSW gravity on AdS derived earlier in this paper from string theory). Extending these results to include higher spin components of \(hs(1,1)\) is a challenging and important problem, which requires better understanding of vertex operators of higher ghost cohomologies [31]. In the string theory context, the higher spin algebra \(hs(1,1)\) should be realized as an operator algebra of the vertex operators living in higher order ghost cohomologies. It would be particularly interesting to relate the asymptotic \(W_\infty\) symmetry of the Chern-Simons theory based on \(hs(1,1)\), discovered in remarkable paper by Henneaux and S.-J. Rey [37], to internal symmetries of string field theory based on the action (72) with the higher spin operators realizing \(hs(1,1)\) being the components of the string field \(\Gamma\). We hope to be able to elaborate on these ideas in future works. To conclude, the sigma-model for connections and gauge fields constructed in this work provides a promising framework to approach the unfolded dynamics of higher spin fields on AdS, although understanding of vertex operator structure for higher spin fields in frame-like description beyond \(s = 3\), as well as of the underlying string field theory, still needs to be developed.

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