Relationships Between Characteristic Path Length, Efficiency, Clustering Coefficients, and Density

Alexander Strang  
Case Western Reserve University  
Cleveland, OH 44106

Oliver Haynes  
Rochester Institute of Technology  
Rochester, NY 14623

Nathan D. Cahill  
Rochester Institute of Technology  
Rochester, NY 14623

Darren A. Narayan  
Rochester Institute of Technology  
Rochester, NY 14623-5604

Abstract

The graph theoretic properties of the clustering coefficient, characteristic (or average) path length, global and local efficiency, provide valuable information regarding the structure of a graph. These four properties have applications to biological and social networks and have dominated much of the literature in these fields. While much work has done in applied settings, there has yet to be a mathematical comparison of these metrics from a theoretical standpoint. Motivated by networks appearing in neuroscience, we show in this paper that these properties can be linked together using a single property - graph density.

1 Introduction

Graph theory provides an abundance of valuable tools for analyzing social and biological networks. There are many well known distance metrics which are used to analyze networks including diameter, density, characteristic path length, clustering coefficient, global and local efficiency, as well as centrality metrics which include betweenness centrality and closeness centrality. As these properties involve shortest paths there would seem to be some relation between them. However it is surprising to see that some of these properties are inextricably linked.

In 2016, Brandes, Borgatti, and Freeman, showed that betweenness centrality and closeness centrality are dual properties [2]. In this paper we establish relationships among distance metrics in graphs (i) global efficiency is linked to the characteristic path length and (ii) local efficiency is linked to the clustering coefficient. Furthermore we show that both of these relationships are linked through a single property - graph density.

We begin by reviewing some well known properties of graphs. Let $G$ be a graph with $n$ vertices and let $d(i,j)$ represent the distance (the number of edges in a shortest path between vertices $i$ and $j$ in $G$). If there is no path connecting $i$ and $j$, then $d(i,j) = \infty$. The diameter of $G$ is denoted $diam(G)$ and equals $\max_{i,j} d(i,j)$. The characteristic path length $L$ is the average distance over all pairs of vertices, $L = \frac{1}{n(n-1)} \sum_{i \neq j} d(i,j)$. In 2001, Latora and Marchiori [8] introduced the concept of efficiency and defined the global efficiency to be $E_{glob}(G) = \frac{1}{n(n-1)} \sum_{i \neq j} \frac{1}{d(i,j)}$. They also defined a local version of efficiency. First, let $G_i$ be the subgraphs of $G$ which are induced by the
neighbors of \(i\). The \textit{local efficiency} is defined to be \(E_{\text{loc}}(G) = \frac{1}{n} \sum_{i \in G} E_{\text{glob}}(G_i)\) and is the average of the global efficiencies over the subgraphs \(G_i\). The \textit{clustering coefficient} was defined by Watts and Strogatz [13] to be \(CC(G) = \frac{1}{n} \sum_{i} \frac{|E(G_i)|}{\binom{|V(G_i)|}{2}}\) where \(G_i\) is the subgraph induced by the neighbors of \(i\) (the open neighborhood of \(i\)) and \(V(G_i)\) is the vertex set of the graph \(G_i\), and \(E(G_i)\) is the edge set of the graph \(G_i\). Many papers have since followed, some recent works include: [7] and [11]. A "closed" variant of the clustering coefficient can be defined by \(CC(G) = \frac{1}{n} \sum_{i} \frac{|E(G_i)|}{\binom{|V(G_i)|}{2}}\) where \(G_i\) is the subgraph induced by vertices in the closed neighborhood of \(i\) (which includes the vertex \(i\) in addition to all of its neighbors). The \textit{density of a graph} \(D(G) = \frac{|E(G)|}{\binom{|V(G)|}{2}}\). Throughout the paper, we will use \(n\) to denote the number of vertices in a graph and \(m\) to denote the number of edges. For any undefined notation, see the textbook by D. B. West [14].

Our results were initially motivated by research studies in neuroscience involving functional connectivity of the human brain. This led to a series of results involving local and global network properties of graphs. In Section 2, we establish relationships among four local properties: \(E_{\text{loc}}(G) \leq \frac{1}{2}(1 + CC(G))\) and \(E_{\text{loc}}(G) = \frac{1}{2}(1 + CC(G))\). In Section 3, we present the following connection between two global properties, \(E_{\text{glob}}(G) \leq \frac{3}{2} - L(G)\).

\section{Connections between local properties}

\subsection{Motivation from biological networks}

Results in this section were motivated by a research study of McCarthy, Benusko, and Franz, where the clustering coefficient and local efficiency properties were applied to functional MRI data from subjects with posterior-anterior shift in aging (PASA) [9] and [10]. The graph in Figure 1 appears in both [9] and [10], and shows that the clustering coefficient and local efficiency data for both task based resting state functional networks approaches the line \(E_{\text{loc}}(G) = \frac{1}{2}(1 + CC(G))\) from below, which is consistent with our theoretical findings. In fact, we will show later that Theorem 4 gives a decent approximation of the equation of this line segment when \(0.2 \leq CC(G) \leq 1\) which can be obtained using the points \((0.2, 0.6)\) and \((1, 1)\). The result is \(E_{\text{loc}}(G) \approx \frac{1}{2}(1 + CC(G))\) and we show later in the paper that this approximation improves as the density and clustering coefficient increase.

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2.2 General results

Lemma 1 Let \( v \) be a vertex in \( G \) and let \( G_v \) be the subgraph induced by the vertices in the closed neighborhood of \( v \). Then \( \text{diam}(G_v) = 2 \).

Proof. By definition of \( G_v \), \( v \) is adjacent to all other vertices. For any pair of vertices \( i \) and \( j \) not equal to \( v \) they are either adjacent or connected by a path through \( v \).

Theorem 2 \( E_{\text{loc}}(G) = \frac{1}{2} (1 + CC(G)) \)

Proof. Let \( v \) vertex in \( G \) and let \( G_v \) be the subgraph induced by the vertices in the closed neighborhood of \( v \). Let \( \alpha \) be the number of pairs vertices in \( G_v \) that are distance two apart. Then
\[
E_{\text{loc}}(G) = \frac{1}{n(n-1)} (m + \frac{\alpha}{2}) = \frac{1}{n(n-1)} \left( m + \frac{1}{2} (n(n-1) - m) \right) = \frac{1}{2} \left( 1 + \frac{m}{n(n-1)} \right) = \frac{1}{2} (1 + CC(G)).
\]
Summing over all vertices \( v \) gives \( E_{\text{loc}}(G) = \frac{1}{2} (1 + CC(G)) \).

In the next theorem we show that for the open neighborhood versions of local efficiency and the clustering coefficient we have an inequality.

Theorem 3 \( E_{\text{loc}}(G) \leq \frac{1}{2} (1 + CC(G)) \)

Proof. Let \( v \) be a vertex of \( G \). Then
\[
E_{\text{loc}}(G) = E_{\text{glob}}(G_v) \leq \frac{1}{n(n-1)} (m + \frac{\alpha}{2}) = (1 + CC(G_v)).
\]
Summing over all vertices \( v \) gives \( E_{\text{loc}}(G) \leq \frac{1}{2} (1 + CC(G)) \).

We will refer to graphs where most of the distances are either 1 or 2 as effectively closed. In our next theorem we note that for effectively closed graphs there is a linear approximation between \( E_{\text{loc}}(G) \) and \( CC(G) \).

Theorem 4 If \( d(u,v) \leq 2 \) for a large fraction of vertex pairs \( u,v \) then \( E_{\text{loc}}(G) \approx \frac{1}{2} (1 + CC(G)) \).
Proof. The proof is similar to the approach found in Theorem 2, noting that the deviation between $E_{loc}(G)$ and $\frac{1}{2}(1 + CC(G))$ is tied to the number of pairs of vertices that have a distance of more than 2. As this quantity decreases our approximation becomes closer. ■

3 Connections between global properties

3.1 Motivation from analysis of functional MRI data

In this section we investigate connections between the two global properties, characteristic path length $L(G)$, and global efficiency $E_{glob}$, motivated by a study of resting state functional MRI scans conducted by the Rochester Center for Brain Imaging at the University of Rochester in 2013. The data from the scans reflected correlations in blood oxygenated level dependent (BOLD) signals between each pair of 92 selected regions of the brain. The Pearson correlations were all between 1 (perfect correlation) and -1 (perfect anti-correlation). The correlations were then binarized with a threshold of 0 to form an adjacency matrix for each scan.

Analysis of the 25 resting state functional MRI data resulted in the following averages: $D(G) = 0.483$, $L(G) = 1.518$, and $E_{glob}(G) = 0.741$. We note that $D(G) \approx 0.5$, $L(G) \approx 1.5$, and $E_{glob}(G) \approx 0.75$. Clearly when $D(G) = 0$, $L(G) = E_{glob}(G) = 0$, and when $D(G) = 1$, $L(G) = E_{glob}(G) = 1$. We can use all of these values to form linear regressions connecting the characteristic path length and the global efficiency to the density, $L(G) = 2 - D(G)$ and $E_{glob}(G) = \frac{1}{4}(1 + D(G))$. It turns out that these can be used to formulate bounds which hold for all graphs and obtain close approximations when the distribution of edges is uniform.

3.2 General results

In our first two lemmas, we investigate bounds between the different properties.

Lemma 5 For any graph $G$, $E_{glob}(G) \leq \frac{1}{2}(1 + D(G))$.

Proof. We can express $E_{glob}(G)$ as $\frac{1}{n(n-1)} \left( m + \frac{1}{2} \alpha + \epsilon \beta \right)$ where $\alpha$ is the number of pairs of vertices which are separated by a distance of exactly 2, and $\beta$ is the number of pairs of vertices whose distance is greater than 2 and hence has an efficiency of $\epsilon < \frac{1}{2}$. Then $E_{glob}(G) = \frac{1}{n(n-1)} \left( m + \frac{1}{2} \alpha + \epsilon \beta \right) \leq \frac{1}{n(n-1)} \left( m + \frac{1}{2} \alpha + \frac{1}{2} \beta \right) = \frac{1}{n(n-1)} \left( m + \frac{n(n-1)-m}{2} \right) = \frac{1}{2} \left( m + \frac{n(n-1)}{2} \right) = \frac{1}{2} \left( 1 + \frac{m}{n(n-1)} \right) = \frac{1}{2}(1 + D(G))$. ■

Lemma 6 For any graph $G$, $L(G) \geq 2 - D(G)$.

Proof. $L = \frac{m + 2 \alpha + \epsilon \beta}{n(n-1)} \geq \frac{m + 2 \alpha + \epsilon \beta}{n(n-1)} = \frac{2(m + \alpha + \beta)}{n(n-1)} - \frac{m}{n(n-1)} = 2 - D$. ■

We note that the bound in the previous lemma is tight for the case where $G$ is a complete graph. The combination of Lemmas 5 and 6 yields the following theorem.

Theorem 7 For any graph $G$, $E_{glob}(G) \leq \frac{1}{2}(3 - L(G))$.

Corollary 8 The bounds in Lemmas 5 and 6 and Theorem 7 are tight for the case where $G$ is a graph where $\text{diam}(G) \leq 2$.

Proof. The proofs for these cases follow by considering $\beta = 0$. ■

Theorem 9 If $d(u, v) \leq 2$ for a large fraction of vertex pairs $u, v$ then $E_{glob}(G) \approx \frac{1}{4}(3 - L(G))$. 

Proof. The proof is obtained by combining Lemmas 5 and 6 and Theorem 7 and noting that the deviation between $E_{glob}(G)$ and $\frac{1}{2}(3 - L(G))$ is tied to the number of pairs of vertices that have a distance of more than 2. As this quantity decreases our approximation becomes closer.

4 Discussion

We have established a relationship between the characteristic path length and global efficiency. Likewise we showed a similar link between the local efficiency and the clustering coefficient (both open and closed versions). In both of these cases we showed that the relationships converge as the density of the graph approaches 1. It would be interesting to investigate these properties from an asymptotic standpoint to measure exactly how this convergence behaves.

We found that relationships characteristic path length and global efficiency, and local efficiency and the clustering coefficient (both open and closed versions) become prevalent when the graph has a density around 0.2. This was shown to be consistent with real world data findings such as the study by McCarthy, Benuskova, and Franz [10]. It would be interesting to conduct an analysis using more real world data to see how the relationships between graph properties behave in networks with low density.

We note that all of these graph properties are dependent on the structure of the network, in particular the distances between various pairs of nodes. More precisely a network’s structure is dependent upon the "distance distribution" that is how many pairs of nodes are separated by a distance $d$. This presents a problem of a probabilistic nature which could be explored using techniques from random graphs such as Erdős-Rényi models [5].

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