REGULAR ORBITS OF QUASISIMPLE LINEAR GROUPS II

MELISSA LEE

ABSTRACT. Let $V$ be a finite-dimensional vector space over a finite field, and suppose $G \leq \text{GL}(V)$ is a group with a unique subnormal quasisimple subgroup $E(G)$ that is absolutely irreducible on $V$. A base for $G$ is a set of vectors $B \subseteq V$ with pointwise stabiliser $G_B = 1$. If $G$ has a base of size 1, we say that it has a regular orbit on $V$. In this paper we investigate the minimal base size of groups $G$ with $E(G)/Z(E(G)) \cong \text{PSL}_n(q)$ in defining characteristic, with an aim of classifying those with a regular orbit on $V$.

1. INTRODUCTION

A permutation group $G \leq \text{Sym}(\Omega)$ is said to have a regular orbit on $\Omega$ if there exists $\omega \in \Omega$ with trivial stabiliser in $G$. The study of regular orbits arises in a number of contexts, particularly where $\Omega$ is a vector space $V$, and $G \leq \text{GL}(V)$. For instance, if $V$ is a finite vector space, $G \leq \text{GL}(V)$, and all orbits of $G$ on $V \setminus \{\emptyset\}$ are regular, then the affine group $GV$ is a Frobenius group with Frobenius complement $G$, and such $G$ were classified by Zassenhaus [36]. Regular orbits also underpin parts of the proof of the $k(GV)$-conjecture [14], asserting that the number of conjugacy classes $k(GV)$ of $GV$, with $|G|$ coprime to $|V|$, is at most $|V|$. One of the major cases was where $G$ is almost quasisimple and acts irreducibly on $V$. In this case, the existence of a regular orbit of $G$ on $V$ was sufficient to prove the $k(GV)$-conjecture. A classification of pairs $(G,V)$ where $G$ has a regular orbit on $V$ and $(|G|,|V|) = 1$ was completed by Köhler and Pahlings [21], building on work of Goodwin [15,16] and Liebeck [28].

A subset $B \subset V$ is a base for $G$ if its pointwise stabiliser in $G$ is trivial. The minimal size of a base for $G$ is called the base size and denote it by $b(G)$. For example, if $G$ has a regular orbit on $V$, then $b(G) = 1$. Each element $g \in G$ is characterised by its action on a base, so $b(G) \geq \lceil \log |G|/\log |V| \rceil$. In recent years, there has been a number of advancements towards classifying finite primitive groups $H$ with small bases. A primitive group with $b(H) = 1$ is cyclic, so the smallest case of interest is where $b(H) = 2$. There has been a number of contributions to a classification here, including a partial classification for diagonal type groups [9], a complete classification for primitive actions of $S_m$ and $\text{Alt}_m$ [3,20] and sporadic groups [7] and also substantial progress for almost simple classical groups [6].

A finite group $G$ is said to be quasisimple if it is perfect and $G/Z(G)$ is a non-abelian simple group. We further define $G$ to be almost quasisimple if $G$ has a unique quasisimple subnormal subgroup, which forms the layer $E(G)$ of $G$, and the quotient $G/F(G)$ of $G$ by its Fitting subgroup $F(G)$ is almost simple. A primitive affine group $H = GV$ has base size 2 if and only if $G \leq \text{GL}(V)$ has a regular orbit in its irreducible action on $V$. In classifying $G$ with a regular orbit, one would naturally use Aschbacher’s theorem [1] to determine the possibilities for irreducible subgroups of $\text{GL}(V)$. In this paper, we investigate the case where $G$ is a member of the $C_9$ class of Aschbacher’s theorem. Slightly more broadly than this, we consider $G \leq \text{GL}(V)$ almost quasisimple such that $E(G)$ acts absolutely irreducibly on $V$.

The pairs $(G, V)$ where $G$ has a regular orbit on $V$ have been classified for $E(G)/Z(E(G))$ a sporadic or alternating group [10,11], and for $G$ of Lie type with $(|G|,|V|) = 1$ by the aforementioned proof of the $k(GV)$-conjecture. The authors of [27] showed that $b(G) \leq 6$ for $G \leq \text{GL}(V)$ with $E(G)$ quasisimple, as long as $V$ is not the natural module for $E(G)$. Moreover, Guralnick and Lawther [18] classified $(G, V)$ where $G$ is a simple algebraic group over an algebraically closed field of characteristic $p > 0$ that has a regular orbit on the irreducible $G$-module $V$. They also determine the generic stabilisers in each case. Their methods, which provide the foundation for the techniques in this paper, rely heavily on detailed analyses of highest weight representations of these simple algebraic groups.

This paper is the second in a series of three, which analyse base sizes of pairs $(G, V)$, where $G \leq \text{GL}(V)$ is almost quasisimple and $E(G)$ is a group of Lie type that acts absolutely irreducibly on $V$, with $(|G|,|V|) > 1$. The first paper [24] dealt with the cross-characteristic representations of $G$, while the
present paper considers the case where \(E(G)/Z(E(G)) \cong \text{PSL}_n(q)\) and \(V\) is an absolutely irreducible module for \(E(G)\) in defining characteristic. The final paper in the series will consider the remaining almost quasisimple groups \(G \leq \Gamma L(V)\) of Lie type in defining characteristic.

We say that two representations \(\rho_1, \rho_2\) of \(E(G)\) (and their corresponding modules) are quasiequivalent if there exists \(g \in \text{Aut}(E(G))\) such that \(\rho_1\) is equivalent to \(g\rho_2\). For an almost quasisimple group of Lie type \(G = G(q)\), and a natural number \(i\), we define \(\epsilon_i = 1\) if \(i \mid q\), and \(\epsilon_i = 0\) otherwise.

The main theorem of this work is as follows.

**Theorem 1.1.** Let \(V = V_d(q_0)\) be a \(d\)-dimensional vector space over \(\mathbb{F}_{q_0}\) with \(q_0 = p^f\), and let \(G \leq \Gamma L(V)\) be an almost quasisimple group such that \(E(G)/Z(E(G)) \cong \text{PSL}_n(q)\) with \(n \geq 2\) and \(q = p^f\). Suppose that the restriction of \(V\) to \(E(G)\) is an absolutely irreducible module \(V(\lambda)\) of highest weight \(\lambda\). Set \(k = f/\epsilon\). Either \(G\) has a regular orbit on \(V\), or up to quasiequivalence, \(\lambda\) and \(n\) appear in Table 1.1 if \(k = 1\) and Table 1.2 otherwise.

There are several corollaries that we can deduce from Theorem 1.1.

**Corollary 1.2.** Let \(G, V\) be as in Theorem 1.1. If \(n \geq 7\) and \(\log_q |V| > n^2 + n\), then \(G\) has a regular orbit on \(V\).

The authors of [27] show that if \(G\) as in Theorem 1.1 is contained in \(\Gamma L(V)\), then \(b(G) \leq 6\), unless \(V\) is the natural module for \(E(G)\). The next corollary gives an improvement to this result.

**Corollary 1.3.** Let \(G, V\) be as in Theorem 1.1. Then either \(V\) is the natural module for \(G\), or \(b(G) \leq 5\).

If \(q = q_0\), that is, \(k = 1\), and \(G\) is quasisimple, we can say precisely when \(G\) has a regular orbit on \(V\).

**Corollary 1.4.** Let \(G, V\) be as in Theorem 1.1, and further suppose \(G\) is quasisimple. If \(q = q_0\), then either \(G\) has a regular orbit on \(V\), or \(b(G) > 1\) and \(V\) appears in Table 1.1.

We now give some remarks on the content of Tables 1.1 and 1.2.

**Remark 1.5.**
1. The highest weights in Tables 1.1 and 1.2 with coefficients including \(p^a\) have \(1 \leq a \leq e - 1\), and \(\lambda_i\) denotes the \(i\)th fundamental dominant weight of the root system associated with \(G\).
2. The groups \(\text{PSL}_2(4)\) and \(\text{PSL}_2(5)\) are isomorphic, and the proof of Theorem 1.1 for these groups is given in [20].
3. The rows in Tables 1.1 and 1.2 for \(\lambda = \lambda_1\) are asterisked because if \(G\) contains no field automorphisms, then \(b(G) = [n/k] + c\), where \(c = 1\) if \(i = (k, n) > 1\) and \(G\) contains scalars in \(\mathbb{F}_{q^i} \setminus \mathbb{F}_{q}\), and \(c = 0\) otherwise. If \(G\) contains field automorphisms, then \(b(G) \leq [n/k] + 1\), with equality if \(G\) contains scalars in \(\mathbb{F}_{q^i} \setminus \mathbb{F}_{q}\) or \(\log_q |G| > kn/[n/k]\).
4. Notice that if \(G\) has no regular orbit on \(V\), then the possibilities for the parameter \(k\) are very restricted. Indeed, by Tables 1.1 and 1.2, excluding the natural module for \(E(G)\), the parameter \(k\) is either an integer \(k \in \{1, 3\}\), or can be written as \(k = 1/c\), with \(c \in \{2, 4\}\).
5. The entry for \(\lambda = 3\lambda_1\) and \(n = 2\) is asterisked because by Proposition 5.13 \(G\) has a regular orbit on \(V\) for \(G\) a subgroup of index at least 3 in \(\Gamma L_2(q)/K\), where \(K\) is the kernel of the action of \(\text{GL}_2(q)\). Otherwise, \(b(G) \leq 2\).
6. The entries for \(\lambda = 3\lambda_1\) with \(n = 6\) in Tables 1.1 and 1.2 are asterisked by Propositions 5.14 and 8.13, 2/k \(\leq b(G) \leq (3 + \delta)/k\), where \(\delta\) is 1 if \(G\) contains a graph automorphism and zero otherwise.
7. The row for \(\lambda = \lambda_1 + \lambda_1\) in Table 1.2 is asterisked because by Proposition 8.4, there is a regular orbit of \(G\) on \(V\) if \(G\) does not contain graph automorphisms.
8. If \(\lambda = (p^n + 1)\lambda_1\) or \(\lambda_1 + p^n\lambda_{n-1}\), then by Proposition 6.6, there is no regular orbit of \(G\) on \(V\) unless \((a, e) = 1\), \((n, p) = (2, 2)\) or \((2, 3)\) and \(G = \text{SL}_2(q)/K\), where \(K\) is the kernel of the action of \(\text{SL}_2(q)\).
9. The entries in the lower section of Table 1.1 are asterisked because if \(G\) is quasisimple, then \(G\) has a regular orbit on \(V\). Otherwise, we determine that \(b(G) \leq 2\).
10. Each row in Table 1.2 with \(\lambda = \left(\sum_{i=0}^{m-1} p^{ie/m}\right)\lambda_1\) for some \(m \in \{2, 3, 4\}\) describes an \(\text{SL}_m(q)\)-module over a subfield \(\mathbb{F}_{q^{1/m}} \subset \mathbb{F}_q\). In each of these cases, \(k = 1/m\). The construction of such modules is described in Section 4.
Table 1.1. A description of modules $V(\lambda)$, with $k = 1$, where $G$ may not have a regular orbit.

| $\lambda$ | Dimension | Notes | $n$ |
|-----------|------------|-------|-----|
| $\lambda_1$ | $n$ | $b(G) \leq n + 1^*$ | $[2, \infty)$ |
| $2\lambda_1$, $p \geq 3$ | $\binom{n+1}{2}$ | $2 \leq b(G) \leq 3$ | $[2, \infty)$ |
| $\lambda_2$ | $\binom{n}{2}$ | $b(G) = 3$ | $[7, \infty)$ |
| $3\lambda_1$, $p \geq 5$ | $4$ | $b(G) = 2^*$ | $2$ |
| $\lambda_3$ | $\binom{n}{3}$ | $b(G) = 2$ | $[7, 8]$ |
| $\lambda_1 + \lambda_{n-1}$ | $n^2 - 1 - \epsilon_n$ | $b(G) = 2^*$ | $[3, \infty)$ |
| $\lambda_1 + \lambda_2$, $q = 3^e$ | $16$ | $b(G) = 2$ | $4$ |
| $(p^e + 1)\lambda_1$ | $n^2$ | $b(G) = 2^*$ | $[2, \infty)$ |
| $\lambda_1 + p^e\lambda_{n-1}$ | $n^2$ | $b(G) = 2^*$ | $[3, \infty)$ |
| $2\lambda_2$, $p \geq 3$ | $20 - \epsilon_3$ | $1 \leq b(G) \leq 2^*$ | $4$ |
| $\lambda_3$ | $84$ | $1 \leq b(G) \leq 2^*$ | $9$ |
| $3\lambda_1$, $p \geq 5$ | $10$ | $1 \leq b(G) \leq 2^*$ | $3$ |
| $\lambda_4$ | $70$ | $1 \leq b(G) \leq 2^*$ | $8$ |

Table 1.2. A description of modules $V(\lambda)$, with $k \neq 1$, where $G$ may have no regular orbit.

| $\lambda$ | Dimension | $k$ | Notes | $n$ |
|-----------|------------|-----|-------|-----|
| $\lambda_1$ | $n$ | $k \geq 2$ | $b(G) \leq \lceil n/k \rceil + 1^*$ | $[2, \infty)$ |
| $2\lambda_1$, $p \geq 3$ | $\binom{n+1}{2}$ | $2$ | $1 \leq b(G) \leq 2$ | $[2, \infty)$ |
| $\lambda_2$ | $\binom{n}{2}$ | $2$ | $b(G) = 2$ | $[5, \infty)$ |
| $\lambda_3$ | $20$ | $2, 3$ | $2/k \leq b(G) \leq \lceil 4/k \rceil^*$ | $6$ |
| $\lambda_1 + \lambda_1$ | $8 - \epsilon_3$ | $2$ | $1 \leq b(G) \leq 2^*$ | $3$ |
| $(p^{e/2} + 1)\lambda_1$ | $n^2$ | $\frac{1}{2}$ | $2 \leq b(G) \leq 3$ | $[2, \infty)$ |
| $(p^{e/2} + 1)\lambda_2$ | $\binom{n}{2}^2$ | $\frac{1}{2}$ | $1 \leq b(G) \leq 2^*$ | $4$ |
| $2(p^{e/2} + 1)\lambda_1$, $p \geq 3$ | $\binom{n+1}{2}^2$ | $\frac{1}{2}$ | $1 \leq b(G) \leq 2$ | $2$ |
| $(p^{e/2} + p^{2e/2} + 1)\lambda_1$ | $n^3$ | $\frac{1}{3}$ | $1 \leq b(G) \leq 2$ | $3$ |
| $(p^{e/2} + p^{2e/2} + 3)\lambda_1$ | $n^4$ | $\frac{1}{4}$ | $b(G) = 2$ | $2$ |

Table 1.2. A description of modules $V(\lambda)$, with $k \neq 1$, where $G$ may have no regular orbit.
(xi) The papers [10] [11] [15] [16] [24] mentioned in the introduction each assume that $V$ (as in Theorem 1.1) is a faithful $d_{0}$-dimensional $\mathbb{F}_{q}G$-module, with $r_{0}$ prime such that $E(G)$ acts irreducibly, but not necessarily absolutely irreducibly, on $V$. We can reconcile this with our study of absolutely irreducible modules for $E(G)$ over finite fields of arbitrary prime power order, following [13] §3. Define $k = \text{End}_{\mathbb{F}_{q}}G(V)$, $K = \text{End}_{\mathbb{F}_{q}E(G)}(V)$, $t = |K : k|$ and $d = \dim_{K}(V)$. Then $E(G) \leq \text{GL}_{d}(K)$ is absolutely irreducible, and $G \leq \text{GL}_{d}(K)\langle \phi \rangle \leq \Gamma \text{L}_{d}(K)$, where $\phi$ is a field automorphism of order $t$.

Our main approach for proving Theorem 1.1 relies on the simple observation that if $G$ has no regular orbit on the absolutely irreducible $E(G)$-module $V$, then every vector $v \in V$ is fixed by a non-trivial prime order element in $G$. For each $V$, we then set out to give a proof by contradiction by showing that $|V|$ exceeds sum of the sizes of the fixed point spaces $C_{V}(x)$ of prime order elements $x \in G \setminus 1$ (this is formalised in Proposition 3.1). This technique requires a method of determining reasonably precise upper bounds for the dimensions of fixed point spaces of elements of $G$.

To compute these upper bounds, we adopt the techniques pioneered by Guralnick and Lawther in [18], and also based on the work of Kenneally [22]. Their methods rely heavily on weight theory in simple algebraic groups of Lie type. They obtain upper bounds on eigenspace dimensions by defining a set of equivalence relations on the weights of the representation $\rho$ corresponding to $V$. These equivalence relations are derived from subsystems $\Psi$ of the root system $\Phi$ associated to a simple algebraic group $\mathcal{G}$. Larger subsystems $\Psi$ generally give tighter upper bounds, but apply to fewer conjugacy classes due to our underlying assumptions. The technique is described in more detail in Section 2.

The techniques implemented by Guralnick and Lawther provide a starting point for the proof of Theorem 1.1. However, our application to finite groups presents additional challenges. We are also required to sum over conjugacy classes of our group, and this often means that more delicate upper bounds on dimensions of fixed point spaces are required. In some cases, this method does not work at all, and different considerations are needed.

**Remark 1.6.** There are striking examples where Guralnick and Lawther [18] show that $\mathcal{G}$, a simple algebraic group of type $A_{l}$, has no regular orbit on an irreducible $\mathbb{F}_{p}\mathcal{G}$-module $V = V(\lambda)$, but according to Theorem 1.1 the corresponding finite group $G(q)$ has a regular orbit on $V$ realised over $\mathbb{F}_{q}$, where $q = p^{e}$. For example, when $l = 1$, $V = V((p^{e}+1)\lambda_{1})$ and $(a, e) = 1$, then [18] Proposition 3.1.8 asserts that there is a regular orbit under the action of $\text{SL}_{2}(\mathbb{F}_{q})$, but we prove in Proposition 6.6 that there is a regular orbit on $V$ under the action of $\text{SL}_{2}(\mathbb{F}_{q})$ when $p = 2$ or $3$.

The rest of this work is set out as follows. In Section 2 we present some preliminary results, which will provide the machinery for the bulk of the proofs in Sections 5 and 6 which will prove Theorem 1.1. We also include some explanation on the techniques of the proofs and a guide on how information is presented in tables preceding each calculation. The proof of Theorem 1.1 is split across four sections. In the notation of Theorem 1.1 the modules $V = V(\lambda)$ where $k = 1$ are dealt with in Section 5 for $\lambda p$-restricted and Section 6 otherwise. Section 7 deals with absolutely irreducible modules with $k < 1$ that are not realised over a proper subfield of $\mathbb{F}_{p^{e}}$. Finally, Section 8 completes the proof of Theorem 1.1 by considering field extensions of the modules discussed in Sections 5 and 6.

2. Background

Let $\mathcal{G}$ be a simple algebraic group over $\mathbb{F}_{p}$, $p$ prime. Let $T \leq \mathcal{G}$ be a fixed maximal torus of $\mathcal{G}$, and $\Phi$ be the root system of $\mathcal{G}$ with respect to $T$, with base $\Delta = \{\alpha_{1}, \ldots, \alpha_{l}\}$ of simple roots, and corresponding fundamental dominant weights $\alpha_{1}, \ldots, \alpha_{l}$. We define a partial ordering on weights $\lambda, \mu$ by saying $\mu \preceq \lambda$ if and only if $\lambda - \mu$ is a non-negative linear combination of simple roots. By a theorem of Chevalley, the irreducible modules of $\mathcal{G}$ in defining characteristic $p$ are characterised by their unique highest weights $\lambda$ under $\preceq$ and conversely, every dominant weight $\lambda$ gives rise to an irreducible $\mathcal{G}$-module, denoted $V(\lambda)$.

If the weight $\lambda = a_{1}\alpha_{1} + \ldots + a_{l}\alpha_{l}$ with $0 \leq a_{i} < p$, then we say $\lambda$ is $p$-restricted. Given a module $V = V(\lambda)$ with corresponding representation $\rho : \mathcal{G} \rightarrow \text{GL}(V)$ and a Frobenius endomorphism of $\mathcal{G}$ given by $\sigma : x \mapsto x^{p^{e}}$, we may define a new module $V(\lambda)^{(p^{e})}$ with corresponding representation $\rho^{(p^{e})}$ by setting $v\rho^{(p^{e})}(g) = v\rho(\sigma(g))$. Notice that $V(\lambda)$ and $V(\lambda)^{(p^{e})}$ are quasiequivalent. We will sometimes say that $V(\lambda)^{(p^{e})}$ is the image of $V(\lambda)$ under a field twist. The following celebrated theorem provides a
method of representing each irreducible $\mathbb{F}_pG$-module as a tensor product of field twists of $p$-restricted modules.

**Theorem 2.1 (33).** Let $\overline{G}$ be a simple algebraic group of simply connected type over $\mathbb{F}_p$. If $\omega_0, \ldots, \omega_k$ are $p$-restricted weights, then

$$V(\omega_0 + p\omega_1 + \cdots + p^k\omega_k) \cong V(\omega_0) \otimes V(\omega_1)^{(p)} \otimes \cdots \otimes V(\omega_k)^{(p^k)}$$

Let $\sigma$ be a Frobenius endomorphism of $\overline{G}$, such that $\overline{G}_\sigma$ is an untwisted group of Lie type over $\mathbb{F}_q$. Steinberg proved that the restriction of $V(\lambda)$ for $\lambda \in \{\alpha_1 \lambda_1 + \cdots + \alpha_t \lambda_t | 0 \leq \alpha_i \leq q - 1\}$ to the finite group $\overline{G}_\sigma$ gives a complete set of representatives of irreducible $\mathbb{F}_qE(\overline{G})$-modules up to equivalence.

So if $G$ is an almost quasisimple group such that $E(G) = \overline{G}_\sigma$, the finite absolutely irreducible $\mathbb{F}_qE(\overline{G})$-modules are the realisations of the $V(\lambda)$ over $\mathbb{F}_q$.

Our primary method for proving Theorem 2.1 comes from the observation that if $G$ has no regular orbit on $V$, then $V$ is the union of the fixed points spaces $C_V(g)$ for $g \in G \setminus F(G)$. Since conjugates in $G$ have conjugate fixed point spaces, we find that

$$|V| \leq \sum_{x \in A} |x^{G}|C_V(x)|,$$

where $A$ is a set of non-identity conjugacy class representatives of $G$.

We say that an element $g \in G$ is of projective prime order if the coset $gF(G)$ has prime order in $G/F(G)$. Note that this implies that there exists a scalar $\beta \in \mathbb{F}_q$ such that $\beta g$ is of prime order. We call the order of $\beta g$ the projective order of $g$. For every non-central element of $g \in G$, there exists $j \in \mathbb{N}$ such that $g^j$ is of projective prime order. Moreover, $C_V(g) \subseteq C_V(g^j)$. Therefore, we may instead sum over classes of projective prime order elements in (2.1).

### 2.1. Properties of prime order elements in $\text{Aut}(\text{PSL}_n(q))$.

From now on, let $G$ be an almost quasisimple group with $E(G)/Z(E(G)) \cong \text{PSL}_n(q)$, with $q = p^e$ for $p$ prime; let $\overline{G} = \text{SL}_n(\mathbb{F}_p)$ with $l = n - 1$ and let $\sigma$ be a Frobenius endomorphism of $\overline{G}$ such that $\overline{G}_\sigma = \text{SL}_n(q)$.

For $G_0$, a simple group, and non-identity $x \in \text{Aut}(G_0)$, define $\alpha(x)$ to be the minimal number of $G_0$-conjugates of $x$ needed to generate the group $\langle G_0, x \rangle$. For a projective prime order element $g$ of an almost quasisimple group $G$, define $\alpha(g) = \alpha(gF(G))$, and also define

$$\alpha(G) = \max \{ \alpha(x) | 1 \neq x \in G/F(G) \}.$$

The following results give upper bounds for $\alpha(G)$ when $\text{soc}(G/F(G)) \cong \text{PSL}_n(q)$.

**Proposition 2.2 (19 Lemmas 3.1, 3.2, Theorem 4.1).** Let $H$ be an almost simple group with socle $H_0 = \text{PSL}_n(q)$ for $n \geq 2$. Then for $x \in H \setminus \{1\}$, one of the following holds:

(i) $\alpha(x) \leq n$,
(ii) $H_0 = \text{PSL}_4(q)$ with $q \geq 3$, and $x$ is an involutory graph automorphism and $\alpha(x) \leq 6$,
(iii) $H_0 = \text{PSL}_3(q)$, $x$ is an involutory graph-field automorphism and $\alpha(x) \leq 4$,
(iv) $H_0 = \text{PSL}_2(q)$, with $q \neq 9$, $x$ is an involution and $\alpha(x) \leq 3$, except that $\alpha(x) \leq 4$ if $x$ is an involutory field automorphism or if $q = 5$ and $x$ is an involutory diagonal automorphism.
(v) $H_0 = \text{PSL}_2(9)$, $x$ has order 2 or 3 and $\alpha(x) \leq 3$, or $x$ is an involutory field automorphism and $\alpha(x) = 5$.
(vi) $H_0 = \text{PSL}_4(2)$, $x$ is a graph automorphism and $\alpha(x) = 7$.

For a group $G$, denote by $i_r(G)$ the number of elements of order $r$ in $G$.

**Proposition 2.3 (23 Prop. 1.3).** Let $\overline{G}$ be a simply connected simple algebraic group over $\mathbb{F}_p$ with associated root system $\Phi$. Let $\sigma$ be a Frobenius endomorphism of $\overline{G}$ such that $G_0 = \text{soc}(\overline{G}_\sigma/Z(G_\sigma))$ is a finite simple group of Lie type over $\mathbb{F}_q$. Assume that $G_0$ is not of type $2F_4$, $2G_2$ or $2B_2$. Then

(i) $i_2(\text{Aut}(G_0)) < 2(q^{N_2} + q^{N_2-1})$, where $N_2 = \dim \overline{G} - \frac{1}{2} |\Phi|$, and
(ii) $i_3(\text{Aut}(G_0)) < 2(q^{N_3} + q^{N_3-1})$, where $N_3 = \dim \overline{G} - \frac{1}{3} |\Phi|$.  

Let $V = V_\phi(q^k)$ be an absolutely irreducible module for $\text{SL}_n(q)$, where $q = p^e$ for some prime $p$. We observe that by [23] Proposition 5.4.6, $k$ can be written as $k = a/b$, where $a, b \in \mathbb{N}$, $b | e$ and $a \geq 1$.  

Proposition 5.4.6]. We can write

\[ \sum_{x \in F} |x^H||C_V(x)| \leq \log(\log_2 q + 2)q^{n^2/2 + \frac{1}{2}(d+\zeta d^1/2)}, \]

where \( \zeta = 1 \) if there exists \( x \in F \) that acts linearly on \( V \), and \( \zeta = 0 \) otherwise.

**Proof.** Suppose first that all \( x \in F \) of projective prime order \( r \) acts as a field automorphism on \( V \). For such \( x \) there exists a basis \( V \) such that \( x \) fixes all vectors with coefficients lying in \( \mathbb{F}_{q^{r/2}} \) and no other vectors. This observation, together with [17 Proposition 4.9.1(d)] and [3 Table B.3], give

\[ \sum_{x \in F} |x^H||C_V(x)| \leq \sum_{r|e} \frac{|\text{PGL}_n(q)|}{|\text{PGL}_n(q^{r/2})|} q^{kd/r} < \sum_{r|e} \frac{(q^{1/2} - 1)}{(q - 1)} q^{n^2(1-1/r)+kd/r} \]

Now, \( \frac{(q^{1/2} - 1)}{(q - 1)} < 1 \) for all \( q \) and \( r \). We also have \( n^2(1-1/r) + kd/r \leq n^2/2 + kd/2 \) if \( d \geq n^2/k \). So

\[ \sum_{x \in F} |x^H||C_V(x)| \leq \log(\log_2 q + 2)q^{n^2/2+kd/2}, \]

since the number of distinct prime divisors of \( N \in \mathbb{N} \) is at most \( \log(N+2) \).

Now suppose that there exists \( x \in F \) of projective prime order \( r \) that acts linearly on \( V \). By [23 Proposition 5.4.6], we can write \( V \otimes \mathbb{F}_q \cong M \otimes \mathbb{F}_q \otimes \cdots \otimes \mathbb{F}_q(q^{-1}) \) for some irreducible \( \text{SL}_n(q) \) module \( \mathbb{M} \). Then \( x \) permutes the tensor factors of \( V \) cyclically by one place.

Let \( B = \{ e_1 \otimes \cdots \otimes e_j | 1 \leq i_j \leq d \} \) be a basis of \( V \), and set \( d_0 = \dim M \) so that \( |B| = d_0 \). Now \( \langle x \rangle \) has \( d_0 \) orbits of size 1 on \( B \), consisting of basis vectors \( e_1 \otimes \cdots \otimes e_{i_j} \) where \( i_j = i_{j+1} \) for \( 1 \leq j \leq r-1 \). The remaining basis vectors lie in \( (d_0 - d_0)/r \) orbits of size \( r \). Therefore, \( x \) has one eigenspace of dimension \( d_0 + (d_0 - d_0)/r \leq (d+1/2)/2 \), while the other eigenspaces of \( x \) have dimension \( (d_0 - d_0)/r \leq (d-d^2)/2 \).

Therefore, repeating the same computation as for the previous case, we see that

\[ \sum_{x \in F} |x^H||C_V(x)| \leq \sum_{r|e} \frac{1}{r - 1} \frac{|\text{PGL}_n(q)|}{|\text{PGL}_n(q^{r/2})|} (q^{\frac{1}{2}(d-d^2)/2} + (r-1)q^{\frac{1}{2}(d-d^2)/2}) < \sum_{r|e} \frac{2(q^{1/2} - 1)}{(q - 1)} q^{\frac{1}{2}n^2+\frac{1}{2}(d+1/2)} \]

\[ < \log(\log_2 q + 2)q^{n^2/2+k(d+1/2)/2}, \]

as required. \( \square \)

Let \( \tau \) be a graph automorphism of \( \text{SL}_n(q) \), and let \( V = V_0(q^k) \) be an absolutely irreducible \( \mathbb{F}_q \text{SL}_n(q) \)-module for some \( k \in \mathbb{N} \). We say that \( \tau \) preserves \( V \) if it acts as a linear transformation on \( V \). Moreover, if \( V = V(\lambda) \) is an absolutely irreducible highest weight module for \( \text{SL}_n(q) \), then \( \tau \) preserves \( V \) if and only if the Dynkin diagram automorphism induced by \( \tau \) preserves \( \lambda \).

**Proposition 2.5.** Let \( \tau \in \text{Aut}(\text{SL}_n(q)) \), with \( q = p^k \), be an involutory graph automorphism, preserving the d-dimensional irreducible \( \text{SL}_n(q) \) module \( V = V(\lambda) \) over a finite field \( \mathbb{F}_q \) of characteristic \( p \). Define \( S \) to be the set of all weights \( \mu \) in \( V(\lambda) \) fixed by the Dynkin diagram automorphism induced by \( \tau \), and let \( d = \dim V(\lambda) \). Then

\[ \dim C_V(\tau) \leq \frac{d}{2} + \frac{1}{2} \sum_{\mu \in S} \dim V_\mu. \]

Furthermore, if \( \tau x \) is an involutory graph-field automorphism, then \( |C_V(\tau x)| \leq q^{d/2} \) if \( \tau x \) acts linearly on \( V \), and \( |C_V(\tau x)| \leq q^{d/2} \) otherwise.

**Proof.** Let \( V = V \otimes \mathbb{F}_q \). Now, \( V \) can be written as a direct sum of weight spaces with respect to some maximal torus of \( \mathbb{G} \). If \( \tau \) swaps two weight spaces \( V_\mu, V_{\tau(\mu)} \) with \( \mu \notin S \), then \( \tau \) has a fixed point space...
The result for graph-field automorphisms follows from the proof of Proposition 2.4.

Proposition 2.6 (Lemma 2.8). The number of graph automorphisms of order 2 in Aut(PSLₙ(𝑞)) for 𝑛 ≥ 3 is at most 2𝑞²−𝑛−1, and the number of graph-field automorphisms of order 2 in Aut(PSLₙ(𝑞)) is less than 𝑞(𝑛²−1)/2.

3. Techniques

The following proposition gives the main results used to prove Theorem 1.1. For an almost quasisimple group 𝐺, let 𝐺ₓ denote the set of elements of 𝐺 of projective prime order 𝑠, and 𝐺ₓ′ denote the set of elements of 𝐺 with projective prime order coprime to 𝑠.

Proposition 3.1 (Proposition 3.1). Let 𝐺 ≤ GL𝑉(𝑉) be an almost quasisimple group, acting irreducibly on the 𝑑-dimensional module 𝑉 = 𝑉d(𝑟) over 𝐹. Set 𝐻 = 𝐺/𝐹(𝐺) and let 𝑃 be a set of conjugacy class representatives of elements of projective prime order in 𝐻. For 𝑥 ∈ 𝐺, let 𝑥 = 𝑥(𝐹) ∈ 𝐻, and denote the order of 𝑥 by 𝑥. Also let 𝑉 = 𝑉 ⊗ 𝐹. Then the following statements hold.

(i) For 𝑛 ∈ 𝐹, the 𝑛-eigenspace 𝐸𝑛(𝑉) of 𝑥 in 𝑉 satisfies

\[ \dim_{𝐹}(𝐸_{𝑛}(𝑉)) \leq \left( \dim_{𝐹}(𝑉) \left( 1 - \frac{1}{\alpha(𝑛)} \right) \right) \]

Further, if 𝐺 has no regular orbit on 𝑉 = 𝑉d(𝑟) with 𝑟 = 𝑠, for 𝑠 prime, then:

(ii) \[ |𝑉| ≤ \sum_{𝑥 ∈ 𝑃∩𝐺_{ suger}} \frac{1}{\alpha(𝑥)} |𝑥| \max\{|C_{𝑉}(𝑥)| : 𝑥 ∈ 𝐹\} \]

(iii) \[ |𝑉| ≤ \sum_{𝑥 ∈ 𝑃∩𝐺_{ suger}} \frac{1}{\alpha(𝑥)} |𝑥| \max\{|C_{𝑉}(𝑥)| : 𝑥 ∈ 𝐹\} \]

(iv) \[ |𝑉| ≤ 2 \sum_{𝑥 ∈ 𝑃∩𝐺_{ suger}} |𝑥| \max\{|C_{𝑉}(𝑥)| : 𝑥 ∈ 𝐹\} \]

(v) \[ |𝑉| = r^d \leq 2 \sum_{𝑥 ∈ 𝑃∩𝐺_{ suger}} |𝑥| r^{(1-1/\alpha(𝑥))d} \]

and for fixed 𝑥, if this inequality fails for a given 𝑑, then it fails for all 𝑑 ≥ 𝑑.

Lemma 3.2. Let 𝐺, 𝐻 and 𝑉 = 𝑉d(𝑞) be as in Proposition 3.1. If one of Proposition 3.1(ii)–(v) fails for 𝐺 ≤ Γ𝐿𝑉(𝑉), then 𝐹 ∩ 𝐺 has a regular orbit on 𝑉 = 𝑉 ⊗ 𝐹 for each integer 𝑘 ≥ 1.

Proof. If one of Proposition 3.1(ii)–(v) fails for 𝐺 acting on 𝑉 (proving that 𝐺 has a regular orbit on 𝑉), then in particular Proposition 3.1(ii) fails, since the right hand side of the inequality is at most the right hand sides of parts (iii), (iv) and (v). Therefore

\[ q_0^d \geq \sum_{𝑥 ∈ 𝑃∩𝐾} \frac{1}{\alpha(𝑥)} |𝑥| F(𝐺)^H \left| q_0^{\dim_{𝑉}(𝑟)(𝑥)} \right| \]

and multiplying both sides of the inequality by \( q_0^{(k-1)d} \), we have

\[ q_0^{kd} \geq \sum_{𝑥 ∈ 𝑃∩𝐾} \frac{1}{\alpha(𝑥)} |𝑥| F(𝐺)^H \left| q_0^{\dim_{𝑉}(𝑟)(𝑥)+k(d-1)} \right| \]

and the result follows.
The next result presents an additional method of bounding $|C_V(g)|$ for $g$ of projective prime order in $G$. For $g \in G \leq GL(V)$ with $V = V_2(q_0)$, let $ε_V^g$ (or just $ε_V$ and $ε(g)$) denote the dimension of the largest eigenspace of $g$ on $V = V \otimes F_{q_0}$.

**Proposition 3.3 (29 Lemma 3.7).** Let $V_1$ and $V_2$ be vector spaces over $\mathbb{F}_{q_0}$ of dimension $d_1$ and $d_2$ respectively. Assume that $g = g_1 \otimes g_2 \in GL(V_1) \otimes GL(V_2)$ is an element of projective prime order that acts on $V = V_1 \otimes V_2$. Then

$$ε_V^g \leq \min \left\{ d_2 ε_{V_1}^g(g_1), d_1 ε_{V_2}^g(g_2) \right\}.$$  

where $V_i = V_i \otimes F_{q_0}$.

The following lemma shows that computing the base size of $V \otimes F_{q_0}$ can enable us to bound (and sometimes determine) the base size of $V$.

**Lemma 3.4.** Suppose $V$ is a $d$-dimensional vector space over $\mathbb{F}_r$, and let $G \leq GL(V)$. If $G$ has a base of size $c$ on $V \otimes F_{q_0}$, then $G$ has a base of size at most $ci$ on $V$.

**Proof.** Let $\{v_1, \ldots, v_ε\}$ be a base for $G$ acting on $V \otimes F_{q_0}$. Let $η_1, \ldots, η_ε$ be a basis for $F_{q_0}$ over $F_r$. Write each $v_j$ as $v_j = \sum_{k=1}^{ε} η_k v_{j,k}$, where $v_{j,k} \in V$. Then $\{v_{j,k} \mid 1 \leq j \leq c, 1 \leq k \leq ε\}$ is a base for $G$, since any non-trivial $g \in G$ stabilising the set must also stabilise $\{v_1, \ldots, v_ε\}$. □

**Proposition 3.5.** Let $\{a_1, a_2, \ldots, a_t\}$ be a non-decreasing sequence of natural numbers. The permutation $ν \in S_t$ that maximises $\sum_{i=1}^{t} a_i ν(i)$ is the identity permutation.

**Proof.** We proceed by induction on $t$. The results holds true for $t = 2$ since $(a_1 - a_2)^2 \geq 0$. Suppose the result holds for $t = k$, and suppose $t = k + 1$. If $ν$ can be written as a product of disjoint cycles of length less than $k + 1$, then the result follows by the inductive hypothesis. Otherwise, $ν$ is a $(k + 1)$-cycle. Let $η$ be the permutation defined by $η(i) = ν(i)$, except that $η(ν^{-1}(k + 1)) = ν(k + 1)$, and $η(k + 1) = k + 1$. Then, by the inductive hypothesis and recalling that the sequence of $a_i$s is increasing, we have

$$\sum_{i=1}^{k} a_i a_{η(i)} \leq \left( \sum_{i=1}^{k} a_i^2 \right) + (a_{k+1} - a_{ν^{-1}(k+1)})(a_{k+1} - a_{ν(k+1)}),$$

so that

$$\sum_{i=1}^{k} a_i a_{η(i)} + a_{ν^{-1}(k+1)}a_{k+1} + a_{k+1}a_{ν(k+1)} - a_{ν^{-1}(k+1)}a_{ν(k+1)} \leq \sum_{i=1}^{k+1} a_i^2.$$  

The left-hand side of the final inequality is equal to $\sum_{i=1}^{k+1} a_i a_{ν(i)}$, so the result follows. □

We use Proposition 3.5 to bound the dimension of fixed point spaces of semisimple elements acting on certain modules. For a finite dimensional vector space $V$, let $V_t(s)$ denote the $t$-eigenspace of an element $s \in GL(V)$.

**Proposition 3.6.** Let $G$ be almost quasisimple with $E(G)/Z(E(G)) \cong PSL_n(q)$, let $s \in G$ be semisimple of projective prime order with eigenvalues $t_1, t_2, \ldots, t_m$ on the natural module $W$ for $E(G)$ over $\mathbb{F}_q$, arranged so that the multiplicities $a_i$ of the $t_i$ are weakly decreasing. Suppose $t \in \mathbb{F}_q$. Then:

(i) if $V$ is the symmetric square of $W$, then $\dim V_t(s) \leq a_1 + \frac{1}{2} \sum a_i^2$;

(ii) if $V$ is the exterior square of $W$, then $\dim V_t(s) \leq \frac{1}{2} \sum a_i^2$, and

(iii) if $V$ is the exterior cube of $W$, then $\dim V_t(s) \leq \frac{1}{3} n \sum a_i^2$.

**Proof.** We will prove the proposition by considering the action of $s$ on $V = V \otimes F_{q_0}$ for the various $V$. Throughout, let $\overline{W} = W \otimes F_{q_0}$, and denote by $\overline{W}^\otimes n$ the tensor product of $a$ copies of $\overline{W}$. Also let $\{e_i \mid 1 \leq i \leq n\}$ be a basis of $\overline{W}$ consisting of eigenvectors of $s$.

First suppose that $V$ is the symmetric square of $W$. Notice that $\{e_i \otimes e_j \mid 1 \leq i \leq j \leq n\}$ is a basis of $\overline{W}^{\otimes 2}$ comprising eigenvectors of $s$, and let $B_1 = \{e_i \otimes e_j + e_j \otimes e_i \mid 1 \leq i < j \leq n\}$ and $B_2 = \{e_i \otimes e_i \mid 1 \leq i \leq n\}$. Notice that $B_1 \cup B_2$ is a basis of $\overline{V}$ consisting of eigenvectors for $s$. The number of elements of $B_1$ lying in the $t$-eigenspace $\overline{V}_t(s)$ of $s$ on $\overline{V}$ is at most $\frac{1}{2} \sum a_i^2$ by Proposition 3.5. Moreover, the number of vectors in $B_2$ contained in $\overline{V}_t(s)$ is at most $a_1$ if $s$ has odd projective prime order, and $2a_1$ if $s$ has projective prime order 2. Therefore, the result follows for elements of odd
projective prime order. If $s$ is of projective prime order 2, it has two eigenvalues $\pm \zeta$ of multiplicities $a_1, a_2$. Therefore, the eigenspaces of $s$ on $V$ have dimensions $a_1 a_2$ and $\frac{1}{2}(a_1^2 + a_2^2 + a_1 + a_2)$. Both of these quantities are less than $\dim V(s) \leq a_1 + \frac{1}{2} \sum a_i^2$, as required.

Now suppose $V$ is the exterior cube of $W$. Note that $\{e_i \otimes e_j \otimes e_k \mid 1 \leq i \leq j \leq k \leq n\}$ is a basis of $W^\tens 3$ comprising eigenvectors of $s$. We can write $V = W^\tens 3 / U$, where $U$ is the subspace of $W^\tens 3$ generated by simple tensors with a repeated factor. Notice that $e_i \otimes e_j \otimes e_k \otimes U = e_{\sigma(i)} \otimes e_{\sigma(j)} \otimes e_{\sigma(k)} + U$ for all $\sigma \in S_3$, and that vectors of the form $e_i \otimes e_j \otimes e_k + U$ form a complete set of eigenvectors for $s$ on $V$. Therefore, by Propositions 3.3 and 3.5, $\dim V(s) \leq \frac{1}{6} \dim W^\tens 3(s) \leq \frac{1}{6} n \sum a_i^2$ as required. The proof for the exterior square of $W$ is similar.

4. Proof of Theorem 1.1 I: First steps

In this section, we focus on the case where $G$ and $V$ have the same underlying field, $\mathbb{F}_q$. We reduce the proof of Theorem 1.1 here to a finite list of cases given in Tables 4.1 and 4.2. Here, as well as in future sections, we will consider the realisation of absolutely irreducible $\mathbb{F}_q \text{SL}_n(q)$-modules of $V(\lambda)$ up to quasi-equivalence, i.e., up to duality and images under field automorphisms of $\text{SL}_n(q)$.

**Proposition 4.1.** Let $V = V_d(q)$ be a $d$-dimensional vector space over $\mathbb{F}_q$. Suppose $G \leq \Gamma\Gamma(V)$ is an almost quasisimple group with $E(G)/Z(E(G)) \cong \text{PSL}_n(q)$, for $n \geq 2$ and $q = p^r$, such that the restriction of $V$ to $E(G)$ is absolutely irreducible. Also assume that $E(G)/Z(E(G))$ is not isomorphic to an alternating group i.e., $(n, q) \neq (2, 4)$ or $(2, 5)$. If $d \geq n^3$, then $G$ has a regular orbit on $V$.

**Proof.** Throughout, we will assume that $d \geq n^3$. First suppose $n \geq 5$. For $x \in G \setminus F(G)$ we have, by Propositions 2.2 and 3.3, $\dim C_V(x) \leq \left\lfloor \frac{n-1}{n} \dim V \right\rfloor$. If $G$ has no regular orbit on $V$, then by Propositions 2.3 and 2.6 as well as inequality (2.1),

$$q^d \leq (|\text{PGL}_n(q)| + q^{(n^2+n)/2-1} + q^{(n^2-1)/2})(q^{\left\lfloor \frac{n-1}{n} \right\rfloor d} + q^{\left\lfloor \frac{1}{n} \right\rfloor d}) + 2 \log \log q + 2)q^{n^2/2+d/2}$$

$$\leq q^{d-1}(q^{\left\lfloor \frac{2n-1}{n} \right\rfloor d} + q^{\left\lfloor \frac{1}{n} \right\rfloor d}) + 2 \log \log q + 2)q^{\left\lfloor \frac{1}{n} \right\rfloor d} + 2 \log \log q + 2)q^{n^2/2+d/2} < \frac{3}{2} q^{d-1}.$$ 

So this is a contradiction and $G$ has a regular orbit on $V$ when $d \geq n^3$. Now suppose $n = 3$ or 4 with $(n, q) \neq (4, 2)$. Then for $x \in G \setminus F(G)$ of projective prime order, $\dim C_V(x) \leq \left\lfloor \frac{n-1}{n} \dim V \right\rfloor$, except that $\dim C_V(x) \leq \left\lfloor \frac{3}{2} d \right\rfloor$ when $n = 3$ and $x$ is a graph automorphism, or $\dim C_V(x) \leq \left\lfloor \frac{1}{2} d \right\rfloor$ when $n = 4$ and $x$ is a graph-field automorphism. Therefore, if $V$ is not preserved by graph automorphisms, then we may use the argument above to prove the result. If $V$ is preserved by graph automorphisms and $G$ has no regular orbit on $V$ then for $n = 3$ we have, by Proposition 2.6,

$$q^d \leq |\text{PGL}_3(q)|q^{\left\lfloor \frac{2}{3} d \right\rfloor} + q^{\left\lfloor \frac{1}{3} d \right\rfloor} + (2 + q^2 + q^4)(q^{\left\lfloor \frac{1}{3} d \right\rfloor} + q^{\left\lfloor \frac{1}{3} d \right\rfloor}) + 2 \log \log q + 2)q^{n^2/2+d/2}.$$ 

So $G$ has a regular orbit on $V$. On the other hand, if $n = 4, q \geq 3$ and $G$ has no regular orbit on $V$ then by Proposition 2.6

$$q^d \leq (|\text{PGL}_4(q)| + q^{10})(q^{\left\lfloor \frac{2}{3} d \right\rfloor} + q^{\left\lfloor \frac{1}{3} d \right\rfloor}) + (2q^8)(q^{\left\lfloor \frac{1}{3} d \right\rfloor} + q^{\left\lfloor \frac{1}{3} d \right\rfloor}) + 2 \log \log q + 2)q^{n^2/2+d/2} < 2q^{d-1},$$

and again this implies that $G$ has a regular orbit on all of $V$ at dimension at least $n^3$. If instead $(n, q) = (4, 2)$, then the result follows by 11. Finally, if $n = 2$, then by Proposition 2.2 provided $q \geq 7$ and $q \neq 9$, $\dim C_V(x) \leq \left\lfloor \frac{1}{2} \dim V \right\rfloor$ for $x$ of odd prime order, and $\dim C_V(x) \leq \left\lfloor \frac{1}{2} \dim V \right\rfloor$ for involutions, except that $\dim C_V(x) \leq \left\lfloor \frac{1}{2} \dim V \right\rfloor$ for involutory field automorphisms. Therefore, by Proposition 2.3 if $G$ has no regular orbit on $V$, then

$$q^d \leq 4(q^2 + q)(q^{\left\lfloor 2d/3 \right\rfloor} + q^{\left\lfloor d/3 \right\rfloor}) + 2|\text{PGL}_2(q)|q^{\left\lfloor d/2 \right\rfloor} + \frac{|\text{PGL}_2(q)|}{|\text{PGL}_2(q)^{1/2}|}q^{\left\lfloor 3d/4 \right\rfloor} + 2 \log \log q + 2)q^{n^2/2+d/2}.$$ 

This gives a contradiction for $d \geq 9$ and $q \geq 7$, as well as $d = 8$ and $q \geq 11$. When $(d, q) = (8, 7)$, deleting the final two terms in the inequality (since they account for field automorphisms) gives the result. If $(n, q) = (2, 9)$, then the result follows by 11.

By Proposition 1.1 we only need consider those irreducible $\mathbb{F}_q G$-modules of dimension less than $n^3$. The following result gives a characterisation of all such modules.
Proposition 4.2. Let $G = \text{SL}_n(q)$ with $q = p^c$ and $p$ prime, and suppose $V = V(\lambda)$ is an absolutely irreducible $d$-dimensional $F(q)G$-module with $d < n^3$. If $\lambda$ is $p$-restricted, then it appears in Table 4.1. Otherwise, we can write $V(\lambda) = V(\mu_1)^{(p^a)} \otimes V(\mu_2)^{(p^b)}$ for $0 \leq a < b < e$ and $\mu_1, \mu_2$ $p$-restricted. In this case, $\lambda_1$ and $\lambda_2$ appear in Table 4.2.

Proof. First let $\lambda$ be $p$-restricted. For $\lambda \geq 19$, Martinez [31] provides a complete list of irreducible $G$-modules of dimension at most $(l + 1)^3$, and all of these are included in Table 4.1. For $2 \leq l \leq 18$, Lübeck [A.6–A.21] [30] provides a complete list of such modules. If instead $\lambda$ is not $p$-restricted, then for $l \geq 12$, the result follows from inspection of [30] Table 2, and for $l \leq 11$, we instead inspect [30] Appendices A.6–A.15.

We will aim to prove Theorem 1.1 for the modules $V(\lambda) = V(\mu_1)^{(p^a)} \otimes V(\mu_2)^{(p^b)}$ to consider.

$$
\begin{array}{c|c}
\lambda & l \\
\hline
\lambda_1 & [1, \infty] \\
\lambda_2 & [2, \infty] \\
2\lambda_1 & [1, \infty] \\
\lambda_1 + \lambda_1 & [5, \infty] \\
\lambda_3 & [5, \infty] \\
3\lambda_1 & [1, \infty] \\
\lambda_1 + \lambda_3 & [2, \infty] \\
\lambda_1 + \lambda_1 & [4, \infty] \\
2\lambda_1 + \lambda_1 & [2, \infty] \\
\lambda_4 & [7, 28] \\
2\lambda_2 & [3, 17] \\
\lambda_5 & [9, 14] \\
4\lambda_1 & [1, 13] \\
\lambda_6 & [11, 12] \\
\end{array}
$$

**Table 4.1.** List of $p$-restricted modules $V(\lambda)$ to consider.

$$
\begin{array}{c|c|c|c}
\lambda_1 & \lambda_2 & l \\
\hline
\lambda_1 & \lambda_1 & [1, \infty] \\
\lambda_1 & \lambda_2 & [2, \infty] \\
\lambda_1 & \lambda_{l-1} & [2, \infty] \\
\lambda_1 & \lambda_1 & [5, 7] \\
\lambda_2 & \lambda_2 & [3, 4] \\
\lambda_2 & \lambda_{l-1} & [4, 5] \\
\lambda_2 & \lambda_1 & [3] \\
\end{array}
$$

**Table 4.2.** The list of non-$p$-restricted modules $V(\lambda) = V(\mu_1)^{(p^a)} \otimes V(\mu_2)^{(p^b)}$ to consider.

We will aim to prove Theorem 1.1 for the modules $V = V(\lambda)$ in Tables 4.1 and 4.2 using Proposition 3.1. In order to do this, we need a method of computing tighter upper bounds on the sizes of fixed point spaces for projective prime order elements than those afforded by Proposition 3.1(i). We now describe this method, which has been pioneered by Guralnick and Lawther in [18].

Let $G \leq \text{TL}(V)$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_n(q)$ such that $E(G)$ is absolutely irreducible on $V$. Let $\overline{G}$ be a simple algebraic group and $\sigma$ a Frobenius endomorphism of $\overline{G}$ such that $E(G) \leq \overline{G}_\sigma$ and $E(G) \leq \overline{G} \leq G\ell(V)$.

Let $s \in G$ be a semisimple element of projective prime order $r \neq p$. Then there exists $\nu \in F_q^*$ such that $s = \nu s \in \overline{G}$. Note that $s$ is of projective prime order $r$, and that $s$ and $\hat{s}$ have eigenspaces of the same dimension on $V$. In addition, $s$ lies in a fixed maximal torus $T \leq \overline{G}$, and we define $\Phi$ to be the root system of $\overline{G}$ with respect to $T$. Let $\Phi(s) = \{ \alpha \in \Phi \mid \alpha(s) = 1 \}$ so that $C_{\overline{G}}(s) = \langle T, U_\alpha \mid \alpha \in \Phi(s) \rangle$, where $U_\alpha$ is the root subgroup of $\overline{G}$ corresponding to $\alpha$. For a closed subsystem $\Psi \subseteq \Phi$, we define the subsystem subgroup $\overline{G}_\Psi = \langle U_\alpha \mid \alpha \in \Psi \rangle$.

Let $\Psi$ be a standard subsystem of $\Phi$. That is, $\Psi = \langle S \rangle$ for some $S \subseteq \Delta$. We define an equivalence relation $\sim$ on the set of weights of $V = V(\lambda)$ of highest weight $\lambda$ by setting $\mu_1 \sim \mu_2$ if and only if $\mu_1 - \mu_2$ is a linear combination of roots in $\Psi$. We call the corresponding equivalence classes of $\Psi$-nets. If $\Psi$ is generated by a single simple root, the resulting $\Psi$-nets are called weight strings.

The fixed point space $C_V(g)$ of $g \in G$ acting on $V$ is its 1-eigenspace. We use $\Psi$-nets to compute lower bounds for codim$C_V(s)$ (and indeed the codimension of any eigenspace) for semisimple $s \in \overline{G}_\sigma$ of projective prime order $r$ as follows. Assume that $\Psi \cap \Phi(s)$ is empty; then in any $\Psi$-net, any pair of weights that differ by a multiple $M \alpha$ of a root $\alpha$ can only correspond to the same eigenvalue if $M \alpha(s) = 1$, so unless $r \mid M$, the two weight spaces must lie in different eigenspaces for $s$. We can use this to compute a lower bound for the contribution of the $\Psi$-net to codim$C_V(s)$, and we denote this lower bound by $c(s)$.

We may also use these $\Psi$-nets to compute a lower bound $c(u_{\Psi})$ for regular unipotent elements $u_{\Psi} \in \overline{G}_\Psi$. The sum of weight spaces of a given $\Psi$-net forms a (not necessarily irreducible) $\overline{G}_\Psi$-module, and so if we assume $u_{\Psi}$ is of prime order $p$, we can compute the contribution of the $\Psi$-net
to \( \text{codim}_V(u_\Psi) \). We do this using the Jordan canonical form of \( u_\Psi \) in its action on the composition factors of the \( \Psi \)-net module. We denote the lower bound on \( \text{codim}_V(u_\Psi) \) achieved by analysing these \( \Psi \)-nets by \( c(u_\Psi) \). Once we have computed lower bounds for \( \text{codim}_V(s) \) and \( \text{codim}_V(u_\Psi) \), we then apply Proposition 4.3 to attempt to prove that there is a regular orbit of \( G \) on \( V \). For this to be successful, we need to calculate lower bounds on the numbers of semisimple elements \( s \) with \( \Phi(s) \cap \Psi = \emptyset \) and unipotent elements \( u \in G \sigma \) conjugate to \( u_\Psi \) for a given \( \Psi \).

**Proposition 4.3.** Let \( \Psi \) and \( N_s(\Psi) \) be as in Table 4.3. The number of prime order semisimple elements \( g \in \text{PGL}_n(q) \) that are \( \text{PGL}_n(F_q) \)-conjugate to \( s \in T \) such that \( \Phi(s) \) intersects every subsystem of type \( \Psi \subseteq \Phi \) is at most \( N_s(\Psi) \).

| \( \Psi \) | \( N_s(\Psi) \) |
|---|---|
| \( A_1^2 \) | \( q^{2n-1} \) |
| \( A_1^3 \) | \( 2q^{4n-4} \) |
| \( A_2 \) | \( \begin{cases} 2q^{n^2/2+3/2} & n \text{ odd} \\ 4q^{n^2/2+2} & n \text{ even} \end{cases} \) |
| \( A_3 \) | \( 4q^{[n^2/2]+2} + q^{2n^2/3+2} + 2q^{2n^2/3+1} \) |

**Table 4.3.** Upper bounds on the number of prime order semisimple elements \( s \) such that \( \Phi(s) \cap \Psi \neq \emptyset \).

**Proof.** First let \( \Psi \) be of type \( A_1^2 \). The semisimple elements \( s \) with \( \Phi(s) \) intersecting every conjugate of \( \Psi \) are those with centraliser type \( A_{l-1} \). Let \( r \) be the order of \( s \). By \([3, \text{Table B.3}]\), \( r \mid q - 1 \) and each element \( s \) has eigenvalues \( (\gamma_1^{n-1}, \gamma_2) \) for \( \gamma_1, \gamma_2 \) both \( r \)th roots of unity. The total number of these elements in \( \text{PGL}_n(q) \) is less than

\[
(q - 1) \frac{|\text{GL}_n(q)|}{|\text{GL}_{n-1}(q)||\text{GL}_1(q)|} < q^{2n-1},
\]
as required. Now suppose \( \Psi \) is of type \( A_1^3 \). The closed subsystems of \( \Phi \) that intersect every conjugate of \( \Psi \) are those of type \( A_{l-1}, A_1A_{l-2}, \text{ or } A_{l-2} \). We have already dealt with the first case, so suppose \( \Phi(s) \) is of type \( A_1A_{l-2} \). Then by \([3, \text{Table B.3}]\), \( s \) has eigenvalues \( (\gamma_1^2, \gamma_2^{n-2}) \) for \( \gamma_1 \) some \( r \)th roots of unity in \( F_q \) (so in particular, \( q > 2 \)). The total number of elements of this form in \( \text{PGL}_n(q) \) is less than

\[
(q - 1) \frac{|\text{GL}_n(q)|}{|\text{GL}_2(q)||\text{GL}_{n-2}(q)|} < 2q^{4n-7}
\]

Now suppose \( \Phi(s) \) is of type \( A_{l-2} \). Then \( s \) has centraliser type either \( \text{GL}_1(q)^2\text{GL}_{n-2}(q) \) or \( \text{GL}_1(q)^2 \times \text{GL}_{n-2}(q) \). In the former case, \( r \mid q - 1 \) and \( s \) has two eigenvalues of multiplicity one, and a third of multiplicity \( n - 2 \). The total number of elements of this type in \( \text{PGL}_n(q) \) is at most

\[
(q - 1)^2 \frac{|\text{GL}_n(q)|}{|\text{GL}_1(q)^2||\text{GL}_{n-2}(q)|} < q^{4n-4}
\]

Finally, if \( s \) has centraliser type \( \text{GL}_1(q^2) \times \text{GL}_{n-2}(q) \), then \( r > 2, r \mid q + 1 \) and \( s \) has eigenvalues \( (\gamma, \gamma^q, 1^{n-2}) \) for \( \gamma \) a primitive \( r \)th root of unity in \( F_{q^2} \setminus F_q \). The total number of these elements in \( \text{PGL}_n(q) \) is at most

\[
\frac{q + 1}{2} \frac{|\text{GL}_n(q)|}{|\text{GL}_1(q^2)||\text{GL}_{n-2}(q)|} < \frac{3}{2} q^{4n-5}
\]

The result follows from taking the sum of bounds computed in each case. The proofs for \( \Psi \) of type \( A_2 \) or \( A_3 \) are similar. \( \square \)

Suppose \( u_1, u_2 \in \text{PGL}_n(q) \) are unipotent of prime order \( p \). Define \( \pi_i \) to be the partition of \( n \) given by the Jordan blocks in the Jordan canonical form of \( u_i \) on the natural module for \( \text{GL}_n(q) \), and arrange \( \pi_i \) so that the parts are in weakly decreasing order. Now, the closure of \( u_1^\sigma \) with respect to the Zariski topology contains \( u_2 \) if and only if \( \pi_1 \) dominates \( \pi_2 \) in the usual partial dominance ordering of partitions \([32, \text{§4}]\).
Proposition 4.4. Let Ψ and \( N_u(Ψ) \) be as in Table 4.4, and define \( \overline{G} = \text{PGL}_n(\mathbb{F}_q) \). The number of unipotent elements \( u \in \text{PGL}_n(q) \) of prime order such that the closure of \( u^r \) does not contain a regular unipotent element in \( \overline{G}_\Psi \) is at most \( N_u(Ψ) \).

| Ψ   | \( N_u(Ψ) \)                        |
|-----|-----------------------------------|
| \( A_1^2 \) | \( q^{2n-1} \)                     |
| \( A_2 \) | \( 4 \left( \frac{q^{n^2/2+2}-1}{q^2-1} \right) \) |
| \( A_3 \) | \( 8 \left( \frac{q^{2n^2/3+7/2}-1}{(q^2-1)(q^{3/2}-1)} \right) \) |

Table 4.4. Upper bounds on the number of unipotent elements \( u \) of prime order for certain subsystems \( Ψ \).

Proof. Let Ψ be of type \( A_1^2 \). The unipotent elements \( u \) satisfying the hypothesis are conjugates of a root element. The number of conjugates of a root element is the index of a root parabolic in \( \text{PGL}_n(q) \):

\[
\frac{|\text{GL}_n(q)|}{q^{2n-3}|\text{GL}_1(q)||\text{GL}_{n-2}(q)|} = \frac{(q^n - 1)(q^{n-1} - 1)}{q - 1} < q^{2n-1}.
\]

Now suppose \( Ψ \) is of type \( A_2 \). Then \( u \) must have associated partition \((2^l, 1^{n-2j})\). Therefore, by Table B.3 the total number of such elements is equal to

\[
\sum_{j=1}^{n/2} q^{2(n^2-2j)+j^2}|\text{GL}(j, q)||\text{GL}(n-2j, q)| \leq \sum_{j=1}^{n/2} \frac{q^{4j^2}}{q^{2(n-2j)^2}+2j^2+(n-2j)^2} = \sum_{j=1}^{n/2} 4q^{2j(n-j)}
\]

Let \( f(j) = 2j(n-j) \). Then \( f \) is increasing with respect to \( j \), achieving a maximum of \( |n^2/2| \) at \( j = [n/2] \). Moreover, \( |f(j+1) - f(j)| = 2(n-2j) - 2 \geq 2 \), so

\[
\sum_{j=1}^{n/2} 4q^{2j(2n-j)} < 4(1 + q^2 + \ldots + q^{n^2/2}) = 4 \left( \frac{q^{n^2/2+2}-1}{q^2-1} \right).
\]

The proof for \( Ψ \) of type \( A_3 \) is similar. \( \square \)

Summary of table organisation. Let \( V(λ) \) be a highest weight module with highest weight \( λ \) in Table 4.1 or 4.2. In Sections 5 and 6 we use the technique outlined in the discussion preceding Proposition 4.3 to find upper bounds for the eigenspace dimensions of projective prime order elements in \( G \) acting on \( V(λ) \). We will summarise information about the weights and \( Ψ \)-nets of \( V(λ) \), as well as the lower bounds for eigenspace dimensions obtained from these \( Ψ \)-nets in a collection of tables. We now give a general description of these tables and then give a worked example.

Weyl orbit tables. We use a table to summarise information about the orbits of the Weyl group \( W \) on the set of weights of the module. The first column of such a table assigns an index \( i \) to each of the Weyl group orbits. The second column lists the unique dominant weight \( μ \) lying in each Weyl orbit, while the third column gives the size of \( W_μ \). Finally, the fourth column gives the multiplicity of each of the weights in the orbit.

Weight string tables. After the Weyl orbit table, we usually proceed by analysing the weight strings of the module i.e., the equivalence classes of the set of weights under addition or subtraction by a fixed simple root \( α \). Unless explicitly stated, we assume that \( α = α_1 \). Let \( s \in \overline{G} \) be a semisimple element of prime order, and for a subset \( Ψ \) of the root system \( Φ \), let \( u_Ψ \) denote a regular unipotent element in \( \overline{G}_Ψ \).

The first column of a weight string table describes the form \( μ_{i_1} μ_{i_2} \ldots μ_{i_k} \) of weight strings, with each \( μ_i \) a weight lying in the \( i \)th Weyl orbit (as designated in the Weyl orbit table). The second column gives the number of weight strings of this form in the module. The next (possibly several) columns labelled \( c(s) \) give the minimum contribution from each type of weight string to the codimension of the largest eigenspace of a semisimple element \( s \in \overline{G} \) of prime order \( r \). The final collection of columns
labelled $c(u_\Psi)$ give the minimum contribution from each type of weight string to the codimension of the fixed point space of a regular unipotent element $u_\Psi = u_\alpha(1)$. The final row of the table totals the contributions of each type of weight string to give the values of $c(s)$ and $c(u_\Psi)$ arising from the analysis of weight strings.

$\Psi$-net tables. If the lower bounds on $\text{codim} C_V(s)$ and $\text{codim} C_V(u)$ computed from the weight strings of $V(\lambda)$ are insufficient to prove that $G$ has a regular orbit on $V$ by Proposition 3.1, we proceed by computing the $\Psi$-nets for larger standard subsystems $\Psi \subseteq \Phi$. We usually do this using GAP [13]. As in the case of the weight strings, we summarise this information in a table. We will first explain our notation for $\Psi$-nets, which originates from [18]. We may write a standard subsystem $\Psi = \langle \alpha_i \mid i \in S \rangle$ as a product $\Psi = \Psi_1 \ldots \Psi_l$ where each $\Psi_j$ is an irreducible root system. For each $\alpha_i \in \Psi$, there exists $\Psi_j$ with $\alpha_i \in \Psi_j$ and we write $\omega_i$ for the fundamental dominant weight of $\overline{G}_\Psi$, corresponding to $\alpha_i$.

We may then write the highest weight of any $\overline{G}_\Psi$-module as a non-negative linear combination of the $\omega_i$. Each $\Psi$-net forms a $\overline{G}_\Psi$-module, and the highest weight $\nu$ of this module is used to denote the $\Psi$-net in the first column of a $\Psi$-net table. The next column, labelled $n_i$ gives the number of weights in each $\nu$ from the $i$th orbit under the Weyl group. The next column, labelled "Mult" gives the number of $\Psi$-nets of this form in the module. The columns labelled $c(s)$ give the minimum contribution from each type of $\Psi$-net to the codimension of the largest eigenspace of a semisimple element $s \in \overline{G}$ of prime order $r$. The columns labelled $c(u_\Psi)$ give the minimum contribution from each type of weight string to the codimension of the fixed point space of a regular unipotent element $u_\Psi$ in the subsystem subgroup $\overline{G}_\Psi$.

Throughout, we make extensive use of GAP [13] to determine the $\Psi$-nets of each module and compute the contributions of some $\Psi$-nets to $c(s)$. We also use GAP to give information about the number of prime order elements in some small linear groups and to explicitly construct some small modules to determine whether there is a regular orbit. We also use Mathematica [35] to confirm that the inequalities arising from Proposition 3.11(i)(ii)[(v)] fail, so as to prove the existence of a regular orbit. We now demonstrate this technique with a worked example. In this example and throughout this paper, for $G = G(q)$ acting on $V$, we define $c_k = 1$ if the characteristic of $\mathbb{F}_q$ divides $k$, and $c_k = 0$ otherwise.

**Worked Example 4.5.** Let $G$ be an almost quasisimple group with $E(G)/Z(E(G)) \cong \text{PSL}_{d+1}(q)$, $l \in [4, \infty)$ such that $E(G)$ acts absolutely irreducibly on the realisation $V$ of $V(\lambda_1 + \lambda_{l-1})$ over $\mathbb{F}_q$. By [33] ($l \leq 17$) and [31] ($l \geq 18$), we have $d = \dim V = 3 \left[ \frac{l+1}{2} \right] - \left[ \frac{l+2}{2} \right] - \epsilon_l(l+1)$.

Using GAP [13], we compute the weights of the representation, and see that they lie in two orbits under the Weyl group, each with a unique dominant weight $\mu$. This information, along with the size of each Weyl orbit and the multiplicity of weights in each of the Weyl orbits is given in Table 4.5. Let

| $i$ | $\mu$ | $|W,\mu|$ | Multiplicity |
|-----|-------|----------|--------------|
| 1   | $\lambda_1 + \lambda_{l-1}$ | $3 \left[ \frac{l+1}{2} \right]$ | 1 |
| 2   | $\lambda_l$ | $l + 1$ | $l - 1 - \epsilon_l$ |

**Table 4.5.** The Weyl orbit table of $V(\lambda_1 + \lambda_{l-1})$.

$\Psi = \langle \alpha_1 \rangle$. After computing the weight strings with respect to $\Psi$ using GAP, we write each of them as a list of $\mu_i$, based on the Weyl orbit of each weight in the string. We write these in the "String" column of the weight string table (Table 4.10), and record their multiplicities in the "Mult" column. For each weight string, we then determine its minimum contribution to $c(s)$, a lower bound on $\text{codim} C_V(s)$ for a semisimple element $s$ of prime order $r$, using our assumption that $\Psi \cap \Phi(s) = \emptyset$. We do this by trying to find the maximal dimension of a union of weight spaces that could correspond to the same eigenvalue of $s$. For example, the contribution of $\mu_1$ to $c(s)$ is zero, because there is no restriction on the eigenvalue corresponding to the weight based on our assumption. If we instead consider the weight string $\mu_1 \mu_2 \mu_1$, then each consecutive pair of weights differ by $\alpha_1$, so if they correspond to the same eigenvalue, it would imply that $\gamma := \alpha_1(s) = 1$, violating our assumption that $\Psi \cap \Phi(s) = \emptyset$. On the other hand, the first and third weights in the string may lie in the same eigenspace if and only if $2\alpha_1(s) = \gamma^2 = 1$. By our assumption, this can only occur if $r = 2$. However, since $\mu_1$ has multiplicity 1 and $\mu_2$ has multiplicity $l - 1 - \epsilon_l$, the largest contribution to an eigenspace of $s$ from this weight string is $l - 1 - \epsilon_l$, so the contribution of each of the $l - 1$ weight strings of this form to $c(s)$ is 2.
We subsequently compute the contributions of the weight strings to \( c(u_\Psi) \). Define the \( A_1 \) type subgroup \( A = \langle U_{2,l_\alpha} \rangle \). We take the sum of the weight spaces in each weight string and consider it as an \( A \)-module. For example, the string \( \mu_2 \mu_2 \) corresponds to an \( \overline{\mathcal{G}}_\Psi A \)-module with \( l - 1 - \epsilon_l \) composition factors isomorphic to the natural module. Now \( u_\Psi \) has a 1-dimensional fixed point space on each natural module, so the total contribution to \( c(u_\Psi) \) is \( l - 1 - \epsilon_l \). On the other hand, consider the weight string \( \mu_1 \mu_2 \mu_1 \). In characteristic \( p = 2 \), the composition factors of this \( A \)-module are one copy of the twisted natural module \( V(\omega_1)^{(2)} \) and \( l - 2 - \epsilon_l \) copies of the trivial module. Now, \( u_\Psi \) has a one-dimensional fixed point space on each of these, so the total contribution to \( c(u_\Psi) \) by the \( l - 1 \) strings of this type is \( l - 1 \). If instead \( p \geq 3 \), then the weight string has the following composition factors: one copy of the symmetric square \( V(2\omega_1) \), and \( l - 2 - \epsilon_l \) copies of the trivial module. Again, \( u_\Psi \) has a one-dimensional fixed point space on each of these, so the total contribution to \( c(u_\Psi) \) by the \( l - 1 \) strings of this type is \( 2l - 2 \). We continue in this way for the remaining weight strings, and the results are summarised in Table 4.6. Now let \( \Psi = \langle \alpha_1, \alpha_2 \rangle \) be a subsystem of \( \Phi \) of type \( A_2 \). The \( \Psi \)-net table is given in Table 4.7, where \( \delta_{l,A} \) is the Kronecker delta function. We compute the \( \Psi \)-nets of the weights of \( V \) using GAP [13], consider each \( \Psi \)-net as a \( \overline{\mathcal{G}}_\Psi \)-module and determine its highest weight. We then perform a similar analysis to that for the weight string table. For instance, take the \( \overline{\mathcal{G}}_\Psi \) module of highest weight \( 2\omega_1 \). The weights lying in the first Weyl orbit are \( s_1 = \{ 2\omega_1, 2\omega_1 - 2\alpha_1, 2\omega_1 - 2\alpha_1 - 2\omega_2 \} \), while those lying in the second are \( s_2 = \{ 2\omega_1 - \alpha_1, 2\omega_1 - \alpha_1 - \alpha_2, 2\omega_1 - 2\alpha_1 - \alpha_2 \} \). Examining Table 4.3 we see that each weight in \( s_1 \) has multiplicity 1, while each weight in \( s_2 \) has multiplicity \( l - 1 - \epsilon_l \). The collection of weights that we can take with maximal weight space dimension, which also do not pairwise differ by an element of \( \Psi \) is obtained by taking one element each of \( s_1 \) and \( s_2 \), for example \( \{ 2\omega_1, 2\omega_1 - 2\alpha_1 - \alpha_2 \} \). Therefore, the contribution to \( c(s) \) for this \( \Psi \)-net is the sum of multiplicities of the remaining four weights (two from each of \( s_1 \) and \( s_2 \), which is \( 2(l - 1 - \epsilon_l + 1) = 2(l - \epsilon_l) \). We now consider the contribution of this \( \Psi \)-net to \( c(u_\Psi) \). For \( p \geq 3 \), the composition factors of the \( \Psi \)-net with highest weight \( 2\omega_1 \) are: one copy of \( V(2\omega_1) \), and \( l - 2 - \epsilon_l \) copies of \( V(\omega_2) \). A regular unipotent element \( u_\Psi \) has a fixed point space of dimension two on the former and 1 on the latter, giving a total contribution to \( c(u_\Psi) \) of \( 4 + 2(l - 2 - \epsilon_l) = 2l - 2\epsilon_l \). We continue in this manner with the other \( \Psi \)-nets to complete the \( \Psi \)-net table.

| String | Multiplicity | \( c(s) \) | \( c(u_\Psi) \) |
|--------|--------------|-------------|---------------|
| \( \mu_1 \) | \((l - 4)(l - 1) + \binom{l}{2}\) | \( r = 2 \) | \( r \geq 3 \) |
| \( \mu_1 \mu_1 \) | \( 3\binom{l}{2} \) | \( 3\binom{l}{2} \) | \( 3\binom{l}{2} \) |
| \( \mu_1 \mu_2 \mu_1 \) | \( l - 1 \) | \( 2l - 2 \) | \( l - 1 \) |
| \( \mu_2 \mu_2 \) | 1 | \( l - 1 - \epsilon_l \) | \( l - 1 - \epsilon_l \) |
| Total | | \( 3\binom{l}{2} - \epsilon_l \) | \( 3\binom{l}{2} - \epsilon_l \) |

Table 4.6. Weight string table for \( V(\lambda_1 + \lambda_{l-1}) \).

| \( \nu \) | \( n_1 \) | \( n_2 \) | Mult | \( r \geq 3 \) | \( p = 3 \) | \( p = 5 \) |
|----------|----------|----------|------|-------------|-------------|-------------|
| \( \omega_1 + \omega_2 \) | 6 | 1 | \( l - 2 \) | \( 6 - \epsilon_l \delta_{l,A}(l - 2) \) | \( 4(l - 2) \) | \( 6(l - 2) \) |
| \( 2\omega_1 \) | 3 | 3 | 1 | \( 2(l - \epsilon_l) \) | \( 2l - 2\epsilon_l \) | \( 2l - 2\epsilon_l \) |
| \( \omega_2 \) | 3 | 0 | \( (l - 2)(l - 3) \) | \( 2(l - 2)(l - 3) \) | \( 2(l - 2)(l - 3) \) | \( 2(l - 2)(l - 3) \) |
| \( \omega_1 \) | 3 | 0 | \( \binom{l}{2} \) | \( (l - 1)(l - 2) \) | \( (l - 1)(l - 2) \) | \( (l - 1)(l - 2) \) |
| 0 | 1 | 0 | \( \binom{l}{3} \) | 0 | 0 | 0 |
| Total | | | \( 3l^2 - 5l + 2 - 2\epsilon_l(1 + \epsilon_2) \) | \( 3l^2 - 7l + 6 - 2\epsilon_l \) | \( 3l^2 - 5l + 2 - 2\epsilon_l \) |

Table 4.7. The \( A_2 \)-net table for \( V(\lambda_1 + \lambda_{l-1}) \).

We are now ready to prove that \( G \) has a regular orbit on \( V \), where in this example \( V = V(\lambda_1 + \lambda_{l-1}) \). We will use Proposition 3.4\( (4) \), and also Proposition 2.4 for field automorphisms, and Propositions
4.3 and 4.4 to determine an upper bound on the number of elements of prime order in \( G/F(G) \) that we cannot use bounds computed in Table 4.7 for. If \( G \) has no regular orbit on \( V \), then

\[
q^d \leq 2|\text{PGL}_n(q)| \left| q^{d - \left(3q^2 - 5q + 2 - 2r(1 + \varepsilon)\right)} + (8q^{n^2/2 + 2} + 4 \left( \frac{q^{n^2/2 + 2} - 1}{q^2 - 1}\right)) q^{d - 3(\frac{d}{2}) + \varepsilon}\right|
\]

\[+ 2q^{2n - 1} q^{d - 3(\frac{d}{2}) - 1 + \varepsilon} + 2 \log_2(q + 2) q^{n^2/2 + d/2}.\]

This is a contradiction for \( l \geq 4 \) and \( q \geq 2 \), so \( G \) has a regular orbit on \( V \).

We now analyse the modules in Table 4.1 individually. We will start by fixing some notation. Firstly, \( V = V(\lambda) \) will denote an absolutely irreducible \( \mathbb{F}_q \text{SL}_n(q) \) module of highest weight \( \lambda \). Moreover, \( G \leq \text{GL}(V) \) will be an almost quasisimple group with \( E(G)/Z(E(G)) \cong \text{PSL}_n(q) \) such that the restriction of \( V \) to \( E(G) \) is absolutely irreducible. We will also write \( d = \dim V \), \( q = p^e \) and let \( r \neq p \) be a prime. We will also use both \( l \) and \( n = l + 1 \) throughout for convenience in notation.

5. Proof of Theorem 1.1 II : \( p \)-restricted modules

The main result of this section is as follows.

**Theorem 5.1.** Let \( V = V_d(q) \) be a \( d \)-dimensional vector space over \( \mathbb{F}_q \) with \( q = p^e \), and let \( G \leq \text{GL}(V) \) be almost quasisimple with \( E(G)/Z(E(G)) \cong \text{PSL}_n(q) \). Further suppose that the restriction of \( V = V(\lambda) \) to \( E(G) \) is an absolutely irreducible module of \( p \)-restricted highest weight \( \lambda \). Either \( G \) has a regular orbit on \( V \), or \( b(G) > 1 \) and \( n, b(G) \) and \( \lambda \) (up to quasisequivalence) appear in Table 1.1.

By Propositions 4.1 and 4.2, the proof of Theorem 5.1 reduces to an analysis of the modules \( V(\lambda) \) with highest weight \( \lambda \) in Table 1.1. We begin this analysis with some modules where there is no regular orbit under the action of \( G \).

**Proposition 5.2.** Theorem 5.1 holds for \( G \) with \( E(G)/Z(E(G)) \cong \text{PSL}_n(q) \) acting on \( V = V(\lambda_1) \) and \( V(2\lambda_1) \) with \( l \in \{1, \infty\} \), and \( V = V(\lambda_2) \) for \( l \in \{3, \infty\} \). Namely:

(i) if \( V = V(\lambda_1) \) then \( b(G) \leq n + 1 \) if \( G \) contains field automorphisms and \( b(G) = n \) otherwise,

(ii) if \( V = V(2\lambda_1) \) then \( 2 \leq b(G) \leq 3 \), and

(iii) if \( V = V(\lambda_2) \) then \( b(G) = 3 \) if \( n \geq 7 \), \( 3 \leq b(G) \leq 4 \) for \( n \in \{5, 6\} \) and \( 3 \leq b(G) \leq 5 \) for \( n = 4 \).

**Proof.** We have \( \log |G|/\log |V| > n - 1, 2 \) and 1 for \( V(\lambda_1) \), \( V(\lambda_2) \) and \( V(2\lambda_1) \) respectively, except that \( \log |G|/\log |V| < 1 \) for some \( G \) with \( l = 1 \) acting on \( V(2\lambda_1) \). In this case, \( q \) is odd and since \( L_2(q) \cong \Omega_3(q) \), there is no regular orbit of \( G \) on \( V \) by [23, Lemma 2.10.5(iv)]. We will prove the upper bounds on base size for these three modules later in Propositions 5.2 and 5.13. \( \square \)

As before, for \( G \) acting on \( V \), we define \( \epsilon_k = 1 \) if the characteristic of \( \mathbb{F}_q \) divides \( k \), and \( \epsilon_k = 0 \) otherwise.

5.1. \( \lambda = 4\lambda_1 \), with \( l \in \{2, 13\} \).

**Proposition 5.3.** Theorem 5.1 holds for \( G \) with \( E(G)/Z(E(G)) \cong \text{PSL}_{l+1}(q) \) acting on \( V = V(4\lambda_1) \) for \( l \in \{1, 13\} \). Namely, \( G \) has a regular orbit on \( V \).

**Proof.** Here \( d = \binom{l+4}{4} \) and \( p \geq 5 \). The Weyl orbit and weight string tables are given in Tables 5.1 and 5.2 respectively.

| \( i \) | \( \mu \) | \( \text{Mult} \) |
|---|---|---|
| 1 | \( 4\lambda_1 \) | \( l + 1 \) |
| 2 | \( 2\lambda_1 + \lambda_2 \) | \( \binom{l+1}{2} \) |
| 3 | \( 2\lambda_2 \) | \( \binom{l+1}{3} \) |
| 4 | \( \lambda_1 + \lambda_3 \) | \( \binom{l+1}{4} \) |
| 5 | \( \lambda_4 \) | \( \binom{l+1}{5} \) |

**Table 5.1.** The Weyl orbit table of \( V(4\lambda_1) \).
This gives a contradiction for all \( l \geq 2 \) and \( q \geq 5 \). When \( l = 1 \), we use more accurate upper bounds on the number of involutions, elements of order 3 and semisimple elements in \( \text{PGL}_2(q) \), and this gives the result for \( q \geq 180 \). For the remaining values of \( q \), we use GAP [13] to construct each module \( V \) and find regular orbits explicitly. \( \square \)

5.2. \( \lambda = \lambda_1 + \lambda_3 \), with \( l \in [5, 11] \).

Proposition 5.4. Theorem 3.1 holds for \( G \) with \( E(G)/Z(E(G)) \cong \text{PSL}_{l+1}(q) \) acting on \( V = V(\lambda_1 + \lambda_3) \) for \( l \in [5, 11] \). Namely, \( G \) has a regular orbit on \( V \).

Proof. Here \( d = 3^{(l+2)/4} - \epsilon_2^{(l+1)/4} \). The Weyl orbit and weight string tables are found in Tables 5.3 and 5.4 respectively.

| \( i \) | \( \mu = \lambda_1 + \lambda_3 \) | \( |W, \mu| \) | Mult |
|---|---|---|---|
| 1 | \( \lambda_1 + \lambda_3 \) | \( 3^{(l+1)/4} \) | 1 |
| 2 | \( \lambda_3 \) | \( 3^{(l+1)/4} \) | \( 3 - \epsilon_2 \) |

Table 5.3. Weyl orbit table for \( V(\lambda_1 + \lambda_3) \).

| String | \( c(s) \) | \( c(u_\Psi) \) |
|---|---|---|
| \( \mu_1 \) | \( l-1 \) | 0 | 0 |
| \( \mu_1 \mu_2 \mu_3 \mu_2 \mu_1 \) | 1 | 2 | 3 | 4 | 5 |
| \( \mu_2 \) | \( 2^{l(2)} \) | 0 | 0 | 0 | 0 |
| \( \mu_4 \mu_2 \) | \( l-1 \) | \( l-1 \) | \( l-1 \) | \( l-1 \) | \( l-1 \) |
| \( \mu_2 \mu_4 \mu_4 \mu_2 \) | \( l-1 \) | \( 2(l-1) \) | \( 2(l-1) \) | \( 3(l-1) \) | \( 3(l-1) \) |
| \( \mu_3 \) | \( \lambda \) | 0 | 0 | 0 | 0 |
| \( \mu_3 \mu_4 \mu_3 \) | \( l-1 \) | \( l-1 \) | \( 2(l-1) \) | \( 2(l-1) \) | \( 2(l-1) \) |
| \( \mu_4 \) | \( 3^{l(2)} \) | 0 | 0 | 0 | 0 |
| \( \mu_2 \mu_4 \mu_4 \mu_4 \) | \( 2^{l(2)} \) | \( 2^{l(2)} \) | \( 2^{l(2)} \) | \( 2^{l(2)} \) | \( 2^{l(2)} \) |
| \( \mu_4 \mu_4 \mu_4 \mu_4 \mu_4 \) | \( \lambda \) | 0 | 0 | 0 | 0 |
| \( \mu_5 \) | \( \lambda \) | 0 | 0 | 0 | 0 |
| \( \mu_5 \mu_5 \) | \( \lambda \) | 0 | 0 | 0 | 0 |
| \( \mu_5 \mu_5 \mu_5 \mu_5 \mu_5 \) | \( \lambda \) | 0 | 0 | 0 | 0 |
| \( \mu_5 \mu_5 \mu_5 \mu_5 \mu_5 \) | \( \lambda \) | 0 | 0 | 0 | 0 |
| \( \mu_5 \mu_5 \mu_5 \mu_5 \mu_5 \) | \( \lambda \) | 0 | 0 | 0 | 0 |
| \( \mu_5 \mu_5 \mu_5 \mu_5 \mu_5 \) | \( \lambda \) | 0 | 0 | 0 | 0 |
| \( \mu_5 \mu_5 \mu_5 \mu_5 \mu_5 \) | \( \lambda \) | 0 | 0 | 0 | 0 |

Table 5.2. The weight string table of \( V(4\lambda_1) \).

If \( G \) has no regular orbit on \( V \), then by Propositions 2.3, 2.4 and 3.1 (v),
\[
q^{l(4)} \leq 2|\text{PGL}_{l+1}(q)|q^{d-l(1/2) + (l+1)/2)} - 2l + 4(q^{l(1/2)}-1 - q)q^{d-1}(q^{l(1/2)}-1) + 2 (l+1)/2) + 2 \log (\log_2 q + 2) q^{n/2} + d/2
\]
This gives a contradiction for all \( l \geq 2 \) and \( q \geq 5 \). When \( l = 1 \), we use more accurate upper bounds on the number of involutions, elements of order 3 and semisimple elements in \( \text{PGL}_2(q) \), and this gives the result for \( q \geq 180 \). For the remaining values of \( q \), we use GAP [13] to construct each module \( V \) and find regular orbits explicitly.
If $G$ has no regular orbit on $V$ then by Propositions 2.3, 2.4 and 3.1(v),
$$q^d \leq 2|\text{PGL}_n(q)|q^{d-(l(l-1)/2-\epsilon_l)} + 4(q^{(n+1)/2})^{-1} q^{d-(l(l-1)/2)}q^{d-(l(l+1)/2)} + 2\log(\log_2 q + 2)q^{n^2/2+d/2}$$
This is a contradiction for $l \geq 5$ and $q \geq 2$, so $G$ has a regular orbit on $V$ here. □

5.3. $\lambda = \lambda_2 + \lambda_3$, $l \in [4,7]$.

**Proposition 5.5.** Theorem 5.1 holds for $G$ with $E(G)/Z(E(G)) \cong \text{PSL}_{d+1}(q)$ acting on $V = V(\lambda_2 + \lambda_3)$ for $l \in [4,7]$. Namely, $G$ has a regular orbit on $V$.

**Proof.** First suppose $l = 4$. Here $d = 75 - \epsilon_2 - 24\epsilon_3$. The Weyl orbit and weight string tables are given in Table 5.5.

| $i$ | $\mu$ | $|W.\mu|$ | $\text{Mult}$ |
|-----|-------|------------|-------------|
| 1   | $\lambda_2 + \lambda_3$ | 30 | 1 |
| 2   | $\lambda_1 + \lambda_4$ | 20 | 2 - $\epsilon_3$ |
| 3   | 0     | 1 | 5 - $\epsilon_2 - 4\epsilon_3$ |

**Table 5.5.** Weyl orbit and weight string tables for $V(\lambda_2 + \lambda_3)$, with $l = 4$.

Here we must also consider graph and graph field automorphisms, since they preserve the set of weights of the module. If $\tau$ is a graph automorphism, then we determine using Proposition 2.5 that $\text{dim} C_V(\tau) \leq 46 - 16\epsilon_3 - \epsilon_2$. Therefore, if $G$ has no regular orbit on $V$, then by Propositions 2.3, 2.4 and 3.1(v)
$$q^{75-\epsilon_2-24\epsilon_3} \leq 2|\text{PGL}_5(q)|q^{d-34+8\epsilon_3} + 4(q^{14} + q^{13})q^{d-34+15\epsilon_3} + 2\log(\log_2 q + 2)q^{25/2+d/2}$$

which gives a contradiction for all $q \geq 2$. Therefore, $G$ has a regular orbit on $V$. Now suppose $l = 5$. Here $d = 210 - 6\epsilon_2 - 84\epsilon_3$. The Weyl orbit and weight string tables are given in Table 5.6.

| $i$ | $\mu$ | $|W.\mu|$ | $\text{Mult}$ |
|-----|-------|------------|-------------|
| 1   | $\lambda_2 + \lambda_3$ | 60 | 1 |
| 2   | $\lambda_1 + \lambda_4$ | 60 | 2 - $\epsilon_3$ |
| 3   | $\lambda_5$ | 6 | 5 - $\epsilon_2 - 4\epsilon_3$ |

**Table 5.6.** Weyl orbit and weight string tables for $V(\lambda_2 + \lambda_3)$ with $l = 5$.

If $G$ has no regular orbit on $V$, then by Propositions 2.3 and 3.1(v),
$$q^d \leq 2|\text{PGL}_6(q)|q^{d-50} + 2\log(\log_2 q + 2)q^{36/2+d/2}$$
This gives a contradiction for $q \geq 2$. The arguments for $l = 6,7$ are similar. □
5.4. \( \lambda = \lambda_2 + \lambda_4 \), with \( l = 5 \).

**Proposition 5.6.** Theorem 5.1 holds for \( G \) with \( E(G)/Z(E(G)) \cong \text{PSL}_6(q) \) acting on \( V = V(\lambda_2 + \lambda_4) \). Namely, \( G \) has a regular orbit on \( V \).

**Proof.** Here \( d = 189 - \epsilon_5 - 35\epsilon_2 \). The Weyl orbit and weight string tables are given in Table 5.7.

| \( i \) | \( \mu \) | \( |W,\mu| \) | Mult |
|-------|-------|---------|------|
| 1     | \( \lambda_2 + \lambda_4 \) | 90      | 1    |
| 2     | \( \lambda_1 + \lambda_5 \) | 30      | 3 - \( \epsilon_2 \) |
| 3     | 0     | 1       | 9 - \( \epsilon_5 - 5\epsilon_2 \) |

**Table 5.7.** The Weyl orbit and weight string tables for \( V(\lambda_2 + \lambda_4) \) with \( l = 5 \).

Using Proposition 2.3, we also determine that for a graph automorphism \( \tau \), we have \( \dim C_V(\tau) \leq 114 - 23\epsilon_2 - \epsilon_5 \).

Therefore, if \( G \) has no regular orbit on \( V \), then by Propositions 2.4, 2.6, 3.1 (v) and 3.1 (v)

\[
q^{189 - \epsilon_5 - 35\epsilon_2} \leq 2|\text{PGL}_6(q)|q^{d-54} + 2\log(\log_2 q + 2)q^{36/2 + d/2} + (2q^{20} + 2q^{35/2})q^{114-23\epsilon_2-\epsilon_5}.
\]

This gives a contradiction for all \( q \geq 2 \), so \( G \) has a regular orbit on \( V \). \( \square \)

5.5. \( \lambda = 3\lambda_1 + \lambda_2 \), \( l \in [2, 4] \).

**Proposition 5.7.** Theorem 5.1 holds for \( G \) with \( E(G)/Z(E(G)) \cong \text{PSL}_{d+1}(q) \) acting on \( V = V(3\lambda_1 + \lambda_2) \) for \( l \in [2, 4] \). Namely, \( G \) has a regular orbit on \( V \).

**Proof.** Note here that \( p \geq 5 \). First suppose that \( l = 2 \). Here \( d = 24 - 6\epsilon_5 \). The Weyl orbit and weight string tables are given in Tables 5.8 and 5.9. If \( G \) has no regular orbit on \( V \), then

| \( i \) | \( \mu \) | \( |W,\mu| \) | Mult |
|-------|-------|---------|------|
| 1     | \( 3\lambda_1 + \lambda_2 \) | 6       | 1    |
| 2     | \( \lambda_1 + 2\lambda_2 \) | 6       | 1    |
| 3     | \( 2\lambda_1 \)           | 3       | 2 - \( \epsilon_5 \) |
| 4     | \( \lambda_2 \)            | 3       | 2 - \( \epsilon_5 \) |

**Table 5.8.** The Weyl orbit table of \( V(3\lambda_1 + \lambda_2), l = 2 \).

| String | Mult | \( c(s) \) | \( c(\psi) \) |
|--------|------|-----------|-----------|
| \( \mu_1 \) | 1    | 1         | 1         |
| \( \mu_1 \mu_1 \) | 1    | 2         | 2         |
| \( \mu_1 \mu_1 \mu_2 \) | 1    | 2 - \( \epsilon_5 \) | 2 - \( \epsilon_5 \) |
| \( \mu_2 \mu_2 \) | 1    | 2 - \( \epsilon_5 \) | 2 - \( \epsilon_5 \) |
| \( \mu_2 \mu_2 \mu_2 \) | 1    | 3 - \( \epsilon_5 \) | 3 - \( \epsilon_5 \) |

**Table 5.9.** The weight string table of \( V(3\lambda_1 + \lambda_2), l = 2 \).

\[
q^d \leq 2|\text{PGL}_3(q)|q^{d-13} + 4(q^5 + q^4)q^{d-(12-4\epsilon_5)} + 2(q^6 + q^5)q^{d-(14-4\epsilon_5)} + 2\log(\log_2 q + 2)q^{9/2+d/2}
\]

This gives a contradiction for all \( q \geq 5 \). The proofs for \( l = 3, 4 \) are similar. \( \square \)

5.6. \( \lambda = \lambda_1 + \lambda_2 + \lambda_3 \), \( l = 3 \).

**Proposition 5.8.** Theorem 5.1 holds for \( G \) with \( E(G)/Z(E(G)) \cong \text{PSL}_4(q) \) acting on \( V = V(\lambda_1 + \lambda_2 + \lambda_3) \). Namely, \( G \) has a regular orbit on \( V \).
Proposition 5.9. In the following proposition.

Proof. Here $\dim V = 64 - 20\lambda_3 - 6\epsilon_3$ and $V$ is preserved by graph automorphisms. We determine that a graph automorphism $\tau$ has $\dim C_V(\tau) \leq 44 - 5\epsilon_5 - 14\epsilon_3$.

If $G$ has no regular orbit on $V$, then
\[
q^d \leq 2|\text{PGL}_4(q)|q^{d-32+12\lambda_3+4\epsilon_5} + 2\log(\log_2 q + 2)q^{16/2+6/2} + (2q^{15/2} + 2^q)q^{44-5\epsilon_5-14\epsilon_3}
\]
and this gives a contradiction for all $q \geq 2$.

There are a number of highest weight modules $V(\lambda)$ with $\lambda$ in Table 4.1 for which the proof of Theorem 3.4 is straightforward application of Proposition 3.1(v) with the lower bounds on codimension computed from the corresponding weight string tables. For conciseness, we treat these cases together in the following proposition.

Proposition 5.9. Theorem 5.4 holds for $G$ with $E(G)/Z(E(G)) \cong \text{PSL}_{l+1}(q)$ acting on $V = V(\lambda)$ for $l$ and $\lambda$ given in Table 5.12. Namely, $G$ has a regular orbit on $V$ in each case.

Proof. Using weight string tables for each case (omitted here), we compute lower bounds on $c(s)$ for prime order semisimple elements $s \in G$ and $c(u_\Psi)$ for unipotent elements $u_\Psi \in G$. We also, if applicable, bound the size of fixed point space of an involutory graph automorphism using Proposition 2.5. For a given $\lambda$ and $l$, let $m$ be the minimum value of the $c(s)$ and $c(u_\Psi)$ listed in the row corresponding the $\lambda$ and $l$ in Table 5.12. Also let $m_\Psi$ be equal to the entry listed in the $\dim C_V(\tau)$ column if it exists, otherwise set $m_\Psi = \infty$. If $G$ has no regular orbit on $V$, then applying Propositions 2.4, 2.6, and 3.1(iv) we see that $q^{\dim V(\lambda)}$ is less than
\[
2|\text{PGL}_{l+1}(q)|q^{\dim V(\lambda)-m} + 2(2q^{(n^2+n)/2-1} + 2q^{(n^2-1)/2})q^{\dim V(\lambda)-m_\Psi} + 2\log(\log_2 q + 2)q^{(l+1)^2/2+\dim V(\lambda)/2}.
\]
This gives a contradiction for every choice of $\lambda$ and $l$, so the result follows.

5.7. $\lambda = 2\lambda_2$, with $l \in [3, 17]$.

Proposition 5.10. Theorem 5.4 holds for $G$ with $E(G)/Z(E(G)) \cong \text{PSL}_{l+1}(q)$ acting on $V = V(2\lambda_2)$ for $l \in [3, 17]$. Namely, $G$ has a regular orbit on $V$ if $l \geq 4$ or $l = 3$ and $G$ is quasisimple, and $b(G) \leq 2$ otherwise.

Proof. Here $d = (l+1)^2 - (l+1)(l+1) - \epsilon_3(l+1)$ and $p \geq 3$. The Weyl orbit and weight string tables are given in Tables 5.13 and 5.14 respectively.
\[ \lambda \quad l \quad \text{dim} V(\lambda) \quad c(s) \quad c(\nu \Psi) \quad \text{dim} C_V(r) \\
\hline
\lambda_6 \quad 11 \quad 924 \quad 252 \quad 252 \quad 252 \quad 252 \quad 430 \\
\lambda_{1} + \lambda_{4} \quad 7 \quad 504 - 56\epsilon_5 \quad 175 - 15\epsilon_5 \quad 175 - 15\epsilon_5 \quad 175 - 15\epsilon_5 \quad 175 - 15\epsilon_5 \quad - \\
2\lambda_1 + \lambda_2 \quad 2 \quad 15 \quad 7 \quad 9 \quad 9 \quad - \quad 8 \quad 9 \quad - \\
3 \quad 45 \quad 20 \quad 24 \quad 24 \quad - \quad 22 \quad 24 \quad - \\
4 \quad 105 \quad 43 \quad 50 \quad 50 \quad - \quad 47 \quad 50 \quad - \\
5 \quad 210 \quad 79 \quad 90 \quad 90 \quad - \quad 86 \quad 90 \quad - \\
\lambda_1 + 2\lambda_2 \quad 3 \quad 60 \quad 27 \quad 34 \quad 34 \quad - \quad 31 \quad 34 \quad - \\
\lambda_{1} + \lambda_{3} \quad 5 \quad 175 - 34\epsilon_3 \quad 74 - 23\epsilon_3 \quad 80 - 10\epsilon_3 \quad 80 - 10\epsilon_3 \quad - \quad 70 \quad 80 \quad 69 - 12\epsilon_3 \\
(p = 5) \quad 4 \quad 103 \quad 45 \quad 52 \quad 52 \quad - \quad - \quad 52 \quad - \\
5\lambda_1 \quad 1 \quad 6 \quad 3 \quad 4 \quad 4 \quad - \quad - \quad 3 \quad (p \geq 7) \quad - \\
2 \quad 21 \quad 9 \quad 12 \quad 14 \quad - \quad - \quad 12 \quad (p \geq 7) \quad - \\
3 \quad 56 \quad 22 \quad 29 \quad 34 \quad - \quad - \quad 31 \quad (p \geq 7) \quad - \\
3\lambda_2 \quad 3 \quad 50 \quad 22 \quad 28 \quad 30 \quad - \quad - \quad 30 \quad 15 \quad - \\
\lambda_1 + \lambda_4 \quad 6 \quad 371 - 84\epsilon_{1,-1} \quad 80 - 5\epsilon_{1,-1} \quad 80 - 5\epsilon_{1,-1} \quad 80 - 5\epsilon_{1,-1} \quad 70 - 5\epsilon_{1,-1} \quad 80 - 5\epsilon_{1,-1} \quad 80 - 5\epsilon_{1,-1} \\
\lambda_1 + \lambda_5 \quad 7 \quad 630 - 112\epsilon_{1,-1} \quad 140 - 6\epsilon_{1,-1} \quad 140 - 6\epsilon_{1,-1} \quad 140 - 6\epsilon_{1,-1} \quad 125 - 6\epsilon_{1,-1} \quad 140 - 6\epsilon_{1,-1} \quad 140 - 6\epsilon_{1,-1} \\
\lambda_1 + \lambda_6 \quad 8 \quad 990 - 144\epsilon_{1,-1} \quad 224 - 7\epsilon_{1,-1} \quad 224 - 7\epsilon_{1,-1} \quad 224 - 7\epsilon_{1,-1} \quad 263 - 7\epsilon_{1,-1} \quad 224 - 7\epsilon_{1,-1} \quad 224 - 7\epsilon_{1,-1} \\
\hline
\text{Table 5.12. Lower bounds for codimensions of eigenspaces in some $p$-restricted modules $V(V(\lambda))$.} \\
\hline
i \quad \mu \quad |W(\mu)| \quad \text{Mult} \\
\hline
1 \quad 2\lambda_2 \quad \frac{l+1}{2} \quad 1 \\
2 \quad \lambda_1 + \lambda_3 \quad 3 \frac{l+1}{3} \quad 1 \\
3 \quad \lambda_4 \quad \frac{l+1}{4} \quad 2 - \epsilon_3 \\
\hline
\text{Table 5.13. Weyl orbit table of $V(2\lambda_2)$.)} \\
\hline
| String | Mult | $r = 2$ | $r \geq 3$ | $p \geq 3$ \\
\hline
\mu_1 \quad \frac{l-1}{2} + 1 \quad 0 \quad 0 \quad 0 \\
\mu_1 \mu_2 \mu_1 \quad l - 1 \quad l - 1 \quad 2(l - 1) \quad 2(l - 1) \\
\mu_2 \quad 3 \frac{l-1}{3} \quad 0 \quad 0 \quad 0 \\
\mu_2 \mu_2 \quad \frac{l-1}{2} \quad (l-1)^2 \quad (l-1)^2 \quad (l-1)^2 \\
\mu_2 \mu_3 \mu_2 \quad \frac{l+1}{2} \quad 2 - \epsilon_3 \frac{l-1}{2} \quad 2 \frac{l+1}{2} \quad 2 \frac{l+1}{2} \\
\mu_3 \quad \frac{l-1}{4} \quad 0 \quad 0 \quad 0 \\
\mu_3 \mu_3 \quad \frac{l+1}{3} \quad 2 - \epsilon_3 \frac{l-1}{3} \quad 2 - \epsilon_3 \frac{l-1}{3} \quad 2 - \epsilon_3 \frac{l-1}{3} \\
\hline
\text{Total} \quad 2 \frac{l+1}{3} - \epsilon_3 \frac{l+1}{3} \quad 2 \frac{l+1}{3} + l - 1 - \epsilon_3 \frac{l-1}{3} \quad 2 \frac{l+1}{3} + l - 1 - \epsilon_3 \frac{l-1}{3} \\
\hline
\text{Table 5.14. Weight string table of $V(2\lambda_2)$.)} \\
\hline
If $l \geq 4$ and $G$ has no regular orbit on $V$, then
\[ q^d \leq 2|\text{PGL}_n(q)|q^{d-2(l+1)/3-l+1+\epsilon_3(l+1)/3} + 4q^{(l+1)/2-1} + q^{(l+1)/2-2}q^{d-2(l+1)/3+\epsilon_3(l+1)/3} + 2\log_2(q + 2)q^{2d/2+d/2} \]
which gives a contradiction for $l \geq 5$ and all $q \geq 3$. It remains to consider $l = 3, 4$.

Suppose $l = 4$, and let $\Psi = \{\alpha_1, \alpha_2\}$. The $W$-net table is given in Table 5.13.

If $G$ has no regular orbit on $V$, then by Propositions 2.3, 3.1.11, 4.13 and 4.3,
\[ q^d \leq 2|\text{PGL}_5(q)|q^{d-28+2\epsilon_3} + 4(q^{14} + q^{13})q^{d-22+2\epsilon_3} + (2q^9 + q^8)q^{d-20+4\epsilon_3} + 2\log_2(q + 2)q^{25/2+d/2} \]
This gives a contradiction for all $q \geq 3$.

Finally, suppose $l = 3$. This case is examined in [18, Proposition 3.1.3] for $H$ a simple algebraic group of type $A_3$. The authors consider the equivalent action of a group of type $D_3$ acting on the
symmetric square of its natural module. They present a vector which they subsequently prove is a regular orbit representative. Applying the same argument to the realisation of this module over $\mathbb{F}_q$, we see that if $G$ is quasisimple, then it has a regular orbit on $V$. Otherwise, $G$ has $b(G) \leq 2$ by Proposition 8.13.

5.8. $\lambda = \lambda_5$, with $l \in [9, 14]$.

**Proposition 5.11.** Theorem 5.7 holds for $G$ with $E(G)/Z(E(G)) \cong \text{PSL}_{l+1}(q)$ acting on $V = V(\lambda_5)$ for $l \in [9, 14]$. Namely, $G$ has a regular orbit on $V$.

**Proof.** Here $d = \binom{l+1}{5}$. The Weyl orbit and weight string tables are given in Table 5.16.

![Table 5.15. The $\Psi$-net table for $V(2\lambda_2)$ and $\Psi = \langle \alpha_1, \alpha_3 \rangle$.](image)

| $\nu$ | $n_1$ | $n_2$ | $n_3$ | Mult | $r = 2$ | $r \geq 3$ | $p \geq 3$ |
|-------|-------|-------|-------|-------|--------|--------|--------|
| 0     | 1     | 0     | 0     | 2     | 0      | 0      | 0      |
| $2\omega_1$ | 2 | 1 | 0 | 1 | 1 | 2 | 2 |
| $2\omega_3$ | 2 | 1 | 0 | 1 | 1 | 2 | 2 |
| $2\omega_1 + 2\omega_3$ | 4 | 4 | 1 | 1 | 4 | 6 | 6 |
| $\omega_1$ | 0 | 2 | 0 | 1 | 1 | 1 | 1 |
| $\omega_3$ | 0 | 2 | 0 | 1 | 1 | 1 | 1 |
| $\omega_1 + \omega_3$ | 0 | 4 | 0 | 3 | 6 | 6 | 6 |
| $2\omega_1 + \omega_3$ | 0 | 4 | 2 | 1 | $4 - \epsilon_3$ | 5 - $\epsilon_3$ | $-\epsilon_3$ |
| $\omega_1 + 2\omega_3$ | 0 | 4 | 2 | 1 | $4 - \epsilon_3$ | 5 - $\epsilon_3$ | $4 - \epsilon_3$ |

| Total | 22 - 2$\epsilon_3$ | 28 - 2$\epsilon_3$ | 26 - 2$\epsilon_3$ |

Table 5.15. The $\Psi$-net table for $V(2\lambda_2)$ and $\Psi = \langle \alpha_1, \alpha_3 \rangle$.

If $l > 9$ and $G$ has no regular orbit on $V$ then

$$q^{\binom{l+1}{5}} \leq 2|\text{PGL}_{l+1}(q)|q^{\binom{l+1}{5} - \binom{l-1}{4}} + 2\log(\log_q(2q + 2)q^{a/2} + q^{\binom{l+1}{5}})$$

This gives a contradiction for $l \geq 10$ and all $q \geq 2$, so it remains to consider $l = 9$. When $l = 9$, we have $d = 252$, and $V = V(\lambda_5)$ is preserved by graph automorphisms in $\text{Aut}(\text{PSL}_{10}(q))$. Using Proposition 2.3, we determine that a graph automorphism $\tau$ has dim $C_V(\tau) \leq 142$. We give the $\Psi$-net table for $\Psi$ of type $A_2^2$ in Tables 5.17.

![Table 5.16. The Weyl orbit and weight string table for $V(\lambda_5)$.](image)

| $i$ | $\mu$ | $|W, \mu|$ | Mult | $c(s)$ | $c(u_\Psi)$ |
|-----|-------|-----------|------|--------|--------|
| 1   | $\lambda_5$ | $\binom{l+1}{5}$ | 1    | $\mu_1$ | $\binom{l+1}{5} - 2\binom{l-1}{4}$ |
|     |       |           |      | $\mu_1\mu_1$ | $\binom{l-1}{4}$ |
|     |       |           |      | $\mu_1\mu_1$ | $\binom{l-1}{4}$ |
| Total |       |           |      | $\binom{l-1}{4}$ | $\binom{l-1}{4}$ |

Table 5.16. The Weyl orbit and weight string table for $V(\lambda_5)$.

Therefore, if $G$ has no regular orbit on $V$, then by Propositions 5.11(iv) 4.3 and 4.4

$$q^{252} \leq 2|\text{PGL}_{10}(q)|q^{152} + 4q^{19}q^{d-70} + 2\log(\log_q(2q + 2)q^{50+126} + 2(2q^{54} + 2q^{99/2})q^{142})$$

This gives a contradiction for all $q \geq 2$, so $G$ has a regular orbit on $V$. □

5.9. $\lambda = 2\lambda_1 + \lambda_1$.

**Proposition 5.12.** Theorem 5.7 holds for $G$ with $E(G)/Z(E(G)) \cong \text{PSL}_{l+1}(q)$ acting on $V = V(2\lambda_1 + \lambda_1)$ for $l \in [2, \infty)$. Namely, $G$ has a regular orbit on $V$.
Table 5.18. Weyl orbit table of $V(2\lambda_1 + \lambda_l)$.

| String | Multiplicity |
|--------|--------------|
| $\mu_1$ | $(l - 1)(l - 2)$ |
| $\mu_1\mu_1$ | $l - 1$ |
| $\mu_1\mu_2\mu_1$ | $l - 1$ |
| $\mu_2\mu_2\mu_1$ | $l - 1$ |
| $\mu_2\mu_2\mu_2$ | $0$ |

Table 5.19. Weight string table for $V(2\lambda_1 + \lambda_l)$.

| $\nu$ | $m_1$ | Mult | $r \geq 2$ | $p \geq 2$ |
|-------|-------|------|------------|------------|
| 0     | 1     | 52   | 0          | 0          |
| $\omega_1$ | 2     | 30   | 30         | 30         |
| $\omega_3$ | 2     | 30   | 30         | 30         |
| $\omega_1 + \omega_3$ | 4     | 20   | 40         | 40         |

Table 5.17. $\Psi = (\alpha_1, \alpha_3)$.

Proof. Here $d = 3(l^2 + 2) + (\frac{l+1}{2}) - \epsilon_{l+2}(l + 1)$ . The Weyl group has three orbits on the weights of the module, which are summarised in Table 5.18. The weight string table is given in Table 5.19. For conciseness, define $\xi = \epsilon_{l+2}$.

If $G$ has no regular orbit on $V$, then by Propositions 4.3, 4.4 and 2.3

$$q^d \leq |\text{PGL}_n(q)|q^{d - \frac{1}{2}(3l^2 + 3l - 2\xi)} + 4(q^{(n+1)/2} - 1 + q^{(n+1)/2} - 2)q^{d - \frac{1}{2}(3l^2 + l - 2\xi)} + q^{(n-1)}q^{d - \frac{1}{2}(3l^2 + l - 2\xi - 2)} + 2q\log(2q + 2)q^{n^2/2+d/2}$$

which gives a contradiction for $l \geq 3$ and $q \geq 3$, and also $l = 2$, $q \geq 5$. Using more accurate counts of unipotent elements gives the result for $q = 3$. So $G$ has a regular orbit on $V$. \hfill \square

5.10. $\lambda = \lambda_1 + \lambda_l$.

Proposition 5.13. Theorem 5.7 holds for $G$ with $E(G)/Z(E(G)) \cong \text{PSL}_{4+1}(q)$ acting on $V = V(\lambda_1 + \lambda_l)$ with $l \in [2, \infty)$. Namely, $b(G) = 2$.

Proof. Here $d = (l + 1)^2 - 1 - \epsilon_{l+1}$. If $p \nmid l + 1$, then $|V| < |G|$ and there is no regular orbit. So suppose $p \mid l + 1$. To prove that $G$ has no regular orbit on $V$, it is sufficient to show it for $H = \text{PSL}_{l+1}(q)$. Now, $V$ is the adjoint module for $H$ and the action of $H$ equivalent to conjugation on $n \times n$ matrices with trace 0. This action is rank preserving. The number of matrices of rank $k$ for $0 \leq k \leq n - 1$ is less than $|H|$, so there cannot be a regular orbit here. Moreover, there is no regular orbit of $H$ on the rank $n$ matrices, since $\text{GL}_n(q)$ has no conjugacy classes of size $|H|$ [12, Theorem 6.4]. So $H$ has no regular orbit on $V$, so neither does $G$.\hfill \square
We now show that \( b(G) = 2 \). There are \( q^{n - 1} - 1 \) elements of \( \mathbb{F}_q^n \) with relative field trace 0 over \( \mathbb{F}_q \). Since \( n \geq 3 \), we can always find \( \xi \in \mathbb{F}_q^n \) not contained in any proper subfield of \( \mathbb{F}_q^n \) with relative field trace 0. Therefore, let \( A \) be a regular semisimple element of \( \text{GL}_n(q) \) with eigenvalues \( \{\xi, \xi^q, \ldots, \xi^{q^{n - 1}}\} \) over \( \mathbb{F}_q^n \). Then \( A \) has matrix trace 0, and \( |G_A| = |C_G(A)| < q^n \). We claim that \( C_G(A) \) has a regular orbit on \( V \). Take \( \Psi \) to be of type \( A_1 \). The relevant Weyl orbit and \( \Psi \)-net tables are given below.

| \( i \) | \( \mu \) | \( |W_\mu| \) | Multiplicity |
|-------|-------|----------|-------------|
| 1     | \( \lambda_1 + \lambda_1^t \) | \( t^2 + l \) | 1           |
| 2     | 0     | 1        | \( l - \epsilon_{l+1} \) |

Table 5.20. The Weyl orbit and weight string tables of \( V(\lambda_1 + \lambda_1^t) \).

If \( C_G(A) \) has no regular orbit on \( V \), then

\[
q^l \leq |C_G(A)|q^{d - 2l} < q^{d - l + 1}
\]

which gives a contradiction for all \( l > 1 \). Therefore, \( b(G) = 2 \) on the adjoint module. \( \square \)

5.11. \( \lambda = \lambda_3 \). For \( s \in \text{GL}(V) \), we denote the \( t \)-eigenspace of \( s \) on \( V \) by \( V_t(s) \).

**Proposition 5.14.** Theorem 7.1 holds for \( G \) with \( E(G)/Z(E(G)) \cong \text{PSL}_{l+1}(q) \) acting on \( V = V(\lambda_3) \) with \( l \in [5, \infty) \). Namely, one of the following holds.

(i) \( l \geq 9 \), or \( l = 8 \) with \( G \) quasisimple, and \( G \) has a regular orbit on \( V \).

(ii) \( l = 8 \), \( E(G) \neq G \) and \( b(G) \leq 2 \),

(iii) \( l \in [6,7] \) and \( b(G) = 2 \), or

(iv) \( l = 5 \) and we have \( 2 \leq b(G) \leq 3 + \delta \), where \( \delta = 1 \) if \( G \) contains a graph automorphism, and \( \delta = 0 \) otherwise.

**Proof.** Here \( d = \binom{l+1}{3} \), and we note that \( |V| < |G| \) for \( 5 \leq l \leq 7 \), so \( b(G) \geq 2 \) here. We give the Weyl and \( \Psi \)-net tables in Table 5.26. If \( \Psi \) is of type \( A_1, A_2, A_3 \) or \( A_2A_1^2 \), then the \( \Psi \)-net tables are computed in [13] Proposition 2.6.3, so in these cases, we give the lower bounds on \( c(s) \) and \( c(\psi) \) computed there. We also give the \( \Psi \)-net tables for \( \Psi \) of type \( A_2, A_3 \) and \( A_1^2 \) as well as the \( A_1^2, A_2A_2 \) and \( A_6 \) tables for semisimple elements with \( n = 10 \) in Table 5.30. For conciseness, we give a condensed version of the \( A_1^2 \) table and the \( A_6^2 \) table for \( n = 10 \). In each row of one of these tables, the indices \( i, j \) and \( k \) are distinct, and \( i, j, k \in \{1,3,5,7\} \) or \( \{1,3,5,7,9\} \) for \( \Psi = A_1^2 \) and \( A_6^2 \) respectively. The entries in the \( c(s) \) and \( c(\psi) \) columns give the total contribution of the \( \Psi \)-nets with highest weight of the given type.

Suppose \( s \in G \) is semisimple of projective prime order \( r \), and write the eigenvalues of \( s \) on the \( n \)-dimensional natural module \( W \) for \( E(G) \) over \( \mathbb{F}_q \) as \( t_1, t_2, \ldots, t_m \), ordered so that the multiplicities \( a_i \) of the \( t_i \) are weakly decreasing. Since \( V \) is isomorphic to the exterior cube of \( W \), \( \dim C_V(s) \leq \frac{1}{2}n \sum a_i^2 \) by Proposition 3.10 and by Table B.3. \(|s^{\text{PGL}_n(q)}| < 2^m q^n - \sum a_i^2 \), where \( m \) is the number of distinct \( t_i \). Therefore,

\[
\text{codim}_{\mathbb{F}_q} C_V(s) - \log_q(|s^{\text{PGL}_n(q)}|) \geq \left( \binom{n}{3} - \frac{1}{6}n \sum a_i^2 \right) - \left( n^2 - \sum a_i^2 + m \right) > \left( \frac{n}{6} - 1 \right) \left( n^2 - \sum a_i^2 \right) - \frac{n^2}{2} + \frac{n}{3} - m
\]

(5.1)

If \( a_1 \leq n - 4 \), we have \( n^2 - \sum a_i^2 \geq 8n - 32 \), so the right hand side of (5.1) is at least \( \frac{1}{6}(5n^2 - 84n + 192) \). If \( a_1 \geq n - 3 \), we instead note that \( \dim C_V(s) \leq \frac{1}{2}n \sum a_i^2 \) and treat them separately in the inequality to come. The number of semisimple elements of \( \text{PGL}_n(q) \) with an \( (n-3) \)-dimensional eigenspace is at most \( 4q^{6n-9} \), while the number of elements with an eigenspace of dimension \( n - 1 \) or \( n - 2 \) is counted
in the proof of Proposition \[\text{4.3}\] By \[\text{12}\] Lemma 3.7, the number of conjugacy classes in \(\text{PGL}_n(q)\) is at most \(q^{n-1} + 5q^{n-2}\). Therefore, if \(n \geq 7\) and \(G\) has no regular orbit on \(V\), then
\[
q^d \leq 2(2^{q^{n-1} + 5q^{n-2}}d^{\frac{1}{2}(5n^2 - 8n + 192)} + 4q^{6n-9}d^{-(3l^2 - 21l + 50)/2} + 4q^{4n-4}d^{-(l-1)}/2 + q^{2n-1}d^{-(l-1)/2})
+ q^{(l+1)}d^{-(3l^2 - 13l/2 + 9)} + 8 \left( \frac{q^{2n^2/3 + 7/2 - 1}}{(q^2 - 1)(q^{3l/2} - 1)} \right) d^{-(2l-3l+2)} + 4 \left( \frac{q^{n^2/2 + 2 - 1}}{q^2 - 1} \right) d^{-(l-5l+8)}
+ 2 \log(\log_2 q + 2)q^{n^2/2 + d/2}
\]
This is a contradiction for \(n \geq 16\) and \(q \geq 2\). Using the \(\Psi\)-net tables given in Table \[\text{5.26}\] we determine that every unipotent element \(u \in G\) of prime order has \(\dim C_V(u) + \log_q(a_{\text{PGL}_n(q)}) \leq d - 3\) for \(10 \leq n \leq 14\), with the exception of \(u \in \text{PGL}_{10}(q)\) with associated partition \(\left(3^2, 1^4\right)\). Here, we compute that \(\dim C_V(u) = 44\). Moreover, if \(n \geq 11\), we compute from Table \[\text{5.26}\] that every semisimple \(s\) element of \(G\) of projective prime order has \(\dim V_s + \log_q(a_{\text{PGL}_n(q)}) \leq d\). Therefore, if \(G\) has no regular orbit on \(V\) with \(11 \leq n \leq 14\),
\[
q^d \leq 2(2^{q^{n-1} + 5q^{n-2}}d^{n - 4q^{6n-9}d^{-(3l^2 - 21l + 50)/2} + 4q^{4n-4}d^{-(l-1)}/2 + q^{2n-1}d^{-(l-1)/2}})
+ c_p(n)q^{d-3} + 2 \log(\log_2 q + 2)q^{n^2/2 + d/2}
\]
where \(c_p(n)\) is the number of partitions of \(n\) with parts of size at most \(p\). This gives a contradiction for \(n \geq 11\) and \(q \geq 2\).

When \(n = 10\), we conduct a detailed analysis of conjugacy classes of semisimple elements of prime order in \(G/F(G)\) including finding tighter upper bounds for the number of classes with given eigenvalue multiplicities. Using this technique and applying Proposition \[\text{5.4}\]\(\text{[iii]}\), we see that \(G\) has a regular orbit on \(V\) for \(q \geq 2\).

The case where \(n = 9\) is treated in \[\text{18}\] Propositions 3.1.1, 3.1.4, where the authors construct a connected simple algebraic group \(\hat{G}\) of type \(A_8\) acting on \(V(\lambda_3)\) as a subgroup of a simply connected group of type \(E_8\) acting on its adjoint module \(\mathcal{L}\). They give an explicit vector \(v \in \mathcal{L}\) and show that it is a representative of a regular orbit under the action of \(\hat{G}\). We can apply the same argument to the realisation over \(\mathbb{F}_q\), and find if \(G\) is quasisimple, then there is a regular orbit of \(G\) on \(V\). If \(l = 8\) and \(G\) is not quasisimple, then we show \(b(G) \leq 2\) in Proposition \[\text{8.13}\].

Finally, for \(6 \leq n \leq 8\), we apply Proposition \[\text{5.\text{[iv]}\[\text{v}\}\] with the information in Table \[\text{5.26}\] and Lemma \[\text{3.3}\] and find that \(b(G) = 2\) for \(n = 7, 8\) and \(b(G) \leq 3\) for \(n = 6\) if \(G\) contains no graph automorphisms, and \(b(G) \leq 4\) if it does. \(\square\)

5.12. \(\lambda = 3\lambda_1\).

**Proposition 5.15.** Theorem \[\text{5.7}\] holds for \(G\) with \(E(G)/Z(E(G)) \cong \text{PSL}_{q+l}(q)\) acting on \(V = V(3\lambda_1)\) for \(l \in [1, \infty)\). Namely, if \(l \geq 3\) or \(l = 2\) and \(G\) is quasisimple, then \(G\) has a regular orbit on \(V\). If \(l = 2\) and \(G\) is not quasisimple, then \(b(G) \leq 2\). If \(l = 1\) then, letting \(K\) denote the kernel of the action of \(\text{GL}_2(q)\), \(G\) has a regular orbit on \(V\) for \(G\) a subgroup of \(\text{GL}_2(q)/K\) of index at least 3, and has \(b(G) \leq 2\) otherwise, with equality if \(G\) is a subgroup of index at most 2 in \(\text{GL}_2(q)/K\).

**Proof.** Here \(d = \left(\frac{l+4}{3}\right)\) and \(p \geq 5\). The Weyl orbit table is given in Table \[\text{5.31}\]. The \(\Psi\)-nets for \(\Psi\) of types \(A_1\) and \(A_2\) are given in \[\text{18}\] Proposition 2.6.2, and so in Table \[\text{5.31}\] we give the bounds on \(c(s)\) and \(c(u_3)\) from these tables.

Let \(\Psi = \langle \alpha_1, \alpha_2, \alpha_3 \rangle\), a subsystem of type \(A_3\). Then the \(\Psi\)-net table for semisimple elements is given in Table \[\text{5.32}\].

So if \(G\) has no regular orbit on \(V\), then
\[
q^d \leq \left(\left|\text{PGL}_n(q)\right| - i_2(G) - i_3(G)\right) q^{d-\frac{1}{2}(3l^2 + l + 2)} + 2q^{2n-1}q^{d-\frac{1}{2}(l^2 + l + 2)} + 2(q^{2n} + q^{2n-1})q^{d-\frac{1}{2}(l^2 + 3l)}
+ 4\left(\theta(n+1) - n\theta(n) - 1\right)q^{d-(l^2 + l + 4)} + 4\left(\frac{1}{3}q^{n(2n+1)-1} + \frac{1}{3}q^{n(2n+1)-2}\right)q^{d-l(l+1)}
+ 2 \log(\log_2 q + 2)q^{n^2/2 + d/2} + 2(4q^{n^2/2 + 2} + 43\left(\frac{1}{3}q^{n^2/3 + 2} + 2q^{n^2/3 + 1}\right)q^{d-(l^2 + l + 2)}
+ q^{n-1}q^{d-l(l+1)}}.
\]
| $i\cdot\mu$ | $|W,\mu|$ | Multiplicity |
|---|---|---|
| 1 $\lambda_3$ | $(l+1)/3$ | 1 |

**Table 5.21.** Weyl orbit table.

| $\nu$ | $n_1$ | Mult | $r \geq 2$ | $p \geq 2$ |
|---|---|---|---|---|
| 0 | 1 | $(l-2)/3 + 1$ | 0 | 0 |
| $\omega_2$ | 3 | $l-2$ | $2(l-2)$ | $2(l-2)$ |
| $\omega_1$ | 3 | $(l-2)/2$ | $2(l-2)/2$ | $2(l-2)/2$ |
| **Total** | | $2(l-1)/2$ | | $2(l-1)/2$ |

**Table 5.23.** $\Psi$-net table for $\Psi = \langle \alpha_1, \alpha_2 \rangle$.

| $\nu$ | $n_1$ | Mult | $r \geq 2$ | $p \geq 2$ |
|---|---|---|---|---|
| 0 | 1 | $(l-7)/2 + 4(l-7)$ | 0 | 0 |
| $\omega_1$ | 2 | $(l-7)/2 + 3$ | $4(l-7)/3 + 3$ | $4(l-7)/3 + 3$ |
| $\omega_1 + \omega_j$ | 4 | $l-7$ | $12(l-7)$ | $12(l-7)$ |
| $\omega_1 + \omega_j + \omega_k$ | 8 | 1 | 16 | 16 |
| **Total** | | $4(l-4)/2 + 16$ | | $4(l-4)/2 + 16$ |

**Table 5.24.** $\Psi$-net table for $\Psi = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$.

| $\nu$ | $n_1$ | Mult | $r \geq 2$ | $p \geq 2$ |
|---|---|---|---|---|
| 0 | 1 | 6 | 0 |
| $\omega_1$ | 3 | 4 | 8 |
| $\omega_2$ | 3 | 6 | 12 |
| $\omega_4$ | 3 | 4 | 8 |
| $\omega_5$ | 3 | 6 | 12 |
| $\omega_2 + \omega_4$ | 9 | 1 | 6 |
| $\omega_1 + \omega_5$ | 9 | 1 | 6 |
| $\omega_2 + \omega_5$ | 9 | 4 | 24 |
| **Total** | | 76 |

**Table 5.25.** $\Psi$-net table for $\Psi = \langle \alpha_1, \alpha_3, \alpha_5, \alpha_7 \rangle$.

This gives a contradiction for $l \geq 4$ and $q \geq 5$. Now suppose $l = 3$. The number of non-regular semisimple elements in $\text{PGL}(4,q)$ is at most

$$
(q-1)\frac{|\text{GL}(1,q)|}{|\text{GL}(1,q)||\text{GL}(3,q)|} + \binom{q-1}{2} \frac{|\text{GL}(q)|}{|\text{GL}(2,q)|^2} + (q-1)\frac{|\text{GL}(4,q)|}{|\text{GL}(2,q)||\text{GL}(1,q)|^2} + \frac{q+1}{2} \frac{|\text{GL}(4,q)|}{|\text{GL}(2,q)|} < \frac{1}{2}(q^{12} + 3q^9)
$$
If $G$ has no regular orbit on $V$, then
\[
q^{20} \leq 2|\text{PGL}_4(q)|q^{20-16} + 2(\frac{1}{2}q^{12} + \frac{3}{2}q^9)q^{20-14} + q^{12}q^{20-14} + 2(q^7 + q^7)q^{20-9} + \frac{\text{GL}_4(q)}{|\text{GL}(1,q)||\text{GL}_3(q)|}q^{20-7} + 2\log(\log_2 q + 2)q^{16/2 + 20/2} + 4q^{9} + 4q^{8}q^{20-10} + 4(q^{11} + q^{10})q^{20-12}.
\]
This gives a contradiction for $q \geq 13$. We use GAP [13] to compute the number of elements of $\text{PGL}_4(q)$ of each prime order for $q \leq 11$, and substituting these into the inequality we have a contradiction in each case, so $G$ has a regular orbit on $V$.

Suppose now that $l = 2$. This case is treated in [18, Proposition 3.1.2], where the authors construct a group of type $A_2$ acting on $V(3\lambda_1)$ as a subgroup of a simply connected group of type $D_4$ acting on its adjoint module. They prove the existence of a regular orbit by providing an explicit vector with trivial stabiliser. Applying the same argument to the reduction of the module over $\mathbb{F}_q$, we see that if $G$ is quasisimple, then $G$ has a regular orbit on $V$. Otherwise, we show in Proposition 8.13 that $b(G) \leq 2$.

Finally, we consider $l = 1$. Let $K$ be the kernel of the action of $\text{GL}_2(q)$ on $V$. Note that $|K| = (3, q - 1)$, and let $H = \mathbb{F}_q^* \circ (\text{GL}_2(q)/K)$. Let $v_j = e_1^2 e_2 + je_2^3$ for $j \in \mathbb{F}_q$ and $z$ be a generator of $\mathbb{F}_q^*$. Define $S_1 = \{v_2, v_{-2}, v_3\}$, and $S_2 = \{v_1, v_{-1}, v_{-3}\}$. Assume that $q \equiv 1 \mod 4$. Then $e_1^2 e_2$ is a regular orbit representative for $\text{SL}_2(q)$. Moreover, if $G \leq H$ does not contain the image of $\text{Diag}(-1,1)$ in $\text{GL}(V)$, then $S_1$ contains representatives of three distinct regular orbits of $G$. That is, each vector in $S_1$ has a stabiliser of order 2 in $H$. Moreover, each element of $S_2$ lies in a distinct orbit of $H$ on $V$, and has a stabiliser of order 6 in $H$. Now suppose $q \equiv 3 \mod 4$. We again find that $e_1^2 e_2$ is a regular orbit representative for $\text{SL}_2(q)$. Moreover, each vector in $S_2$ lies in a distinct orbit of $H$ of size $|H|/2$, while each element of $S_1$ lies in a distinct orbit of $H$ of size $|H|/6$.

We now show that there is no regular orbit of $\text{GL}_2(q)/K$ (and therefore $H$) on $V$. If $K$ is trivial, then the number of vectors of $V$ lying in orbits not already mentioned is less than $|\text{GL}_2(q)|$, so there can be no regular orbit.

If instead $|K| = 3$, we also determine that $e_1^2 + ze_3^3$ and $e_1 + z^2 e_2^3$ lie in distinct $\text{GL}_2(q)/K$ orbits and both have a stabiliser of size 3 in $\text{GL}_2(q)/K$. The number of vectors in orbits not mentioned is then less than $|\text{GL}_2(q)|/3$ for $q \geq 7$, so $\text{GL}_2(q)/K$ has no regular orbit on $V$. We determine that the same is true for $q = 5$ using an explicit construct of the module in GAP [13]. In the cases where we have not shown that there is a regular orbit, we prove that $b(G) \leq 2$ in Proposition 8.13.

\[
\begin{array}{lllllll}
  c(s) & c(u\Psi) \\
  \Psi & r = 2 & r = 3 & r \geq 5 & p \geq 5 \\
  A_1 & \frac{1}{2}(l^2 + l + 2) & \frac{1}{2}(l + 3) & \frac{1}{2}(l^2 + 3l + 2) & \frac{1}{2}(l^2 + 3l + 2) \\
  A_{2,3} & l^2 - l + 4 & \frac{1}{2}(l + 1) & \frac{1}{2}(l^2 + l + 2) & \frac{1}{2}(l^2 + l + 2) \\
\end{array}
\]

Table 5.31. The Weyl orbit table and some $\Psi$-net bounds for $V(3\lambda_1)$

| $i$ | $\lambda_i$ | $|W_{\mu_i}|$ | Multiplicity |
|-----|-------------|---------------|--------------|
| 1   | $3\lambda_1$ | $l + 1$       | 1            |
| 2   | $\lambda_1 + \lambda_2$ | $l(l + 1)$ | 1            |
| 3   | $\lambda_3$ | $(l+1)^3$     | 1            |

Table 5.32. The $A_3$-net table of $V(3\lambda_1)$.
5.13. \(\lambda = \lambda_4\).

**Proposition 5.16.** Theorem 5.1 holds for \(G\) with \(E(G)/Z(E(G)) \cong \text{PSL}_{d+1}(q)\) acting on \(V = V(\lambda_4)\) for \(l \in [7, 28]\). Namely, if either \(l \geq 8\) or \(l = 7\) and \(G\) is quasisimple, then \(G\) has a regular orbit on \(V\). If \(l = 7\) and \(G\) is not quasisimple, then \(b(G) \leq 2\).

**Proof.** The Weyl group has one orbit on the weights of the module, and the Weyl orbit table is given in Table 5.33. The analysis of \(\Psi\)-nets for \(\Psi\) of type \(A_1, A_2^1\) and \(A_2\) is conducted in [18, Proposition 2.6.4]. Set \(c = 2(l^{-2} + l - 3)\). The values of \(c(s)\) and \(c(u_{\Psi})\) computed in [18, Proposition 2.6.4] are given in Table 5.34.

| \(i\) | \(\mu\) | \(|W_\mu|\) | Multiplicity |
|-------|-------|--------|-------------|
| 1     | \(\lambda_4\) | \(l^{-1+1}\) | 1           |

**Table 5.34.** The Weyl orbit table of \(V(\lambda_4)\).

If \(G\) has no regular orbit on \(V\), then

\[
q^{l^{-1+1}} \leq 2|\text{PGL}_n(q)|q^{d-2(l^{-1+1})} + 2(4q^{n/2+2})q^d-c + 2(q^{2n-1} + q^{2n-1})q^{d-(n-2)} + 4(q^{n+1})^{-1} + q^{(n+1)-2})q^{d-c} + 2\log(\log_2 q + 2)q^{n/2+d/2}.
\]

This gives a contradiction for \(l = 9\) and \(q \geq 2\). Now let \(l = 8\), and \(\Psi = \langle \alpha_1, \alpha_2, \alpha_3 \rangle\). The \(\Psi\)-net table is given in Table 5.35.

| \(\nu\) | \(n_1\) | Mult | \(r \geq 2\) | \(p \geq 2\) |
|-------|-------|------|-------------|-------------|
| \(\omega_1\) | 4     | 10   | 30          | 30          |
| \(\omega_2\) | 6     | 10   | 40          | 40          |
| \(\omega_3\) | 4     | 5    | 15          | 15          |
| 0      | 1     | 6    | 0           | 0           |

**Table 5.35.** \(A_3\)-net table for \(V(\lambda_4)\) and \(l = 8\).

So if \(l = 8\) and \(G\) has no regular orbit on \(V\) then

\[
q^{126} \leq 2|\text{PGL}_9(q)|q^{126-85} + 2(4q^{n/2+2}) + \frac{43}{3}q^{2n/3+2} + 2(2\log_2 q + 2)q^{d-70} + 2(2q^{n/2+3/2})q^{d-50} + 2(q^{2n-1} + q^{2n-1})q^{d-35} + 4(q^{n+1})^{-1} + q^{(n+1)-2})q^{d-50} + 2\log(\log_2 q + 2)q^{n/2+d/2}.
\]

This gives a contradiction for \(l = 8\) and all \(q \geq 2\).

The case where \(l = 7\) is treated in [18, Propositions 3.1.1, 3.1.4], where the authors construct a connected simple algebraic group \(\tilde{G}\) of type \(A_7\) acting on \(V(\lambda_4)\) as a subgroup of a simply connected group of type \(E_7\) acting on its adjoint module \(\mathfrak{g}\). They give an explicit vector \(v + Z(\mathfrak{g}) \in \mathfrak{g}/Z(\mathfrak{g})\) and show that it is a representative of a regular orbit under the action of \(\tilde{G}\). We can apply the same argument to the realisation over \(\mathbb{F}_q\), and find that if \(G\) is quasisimple, then there is a regular orbit of \(G\) on \(V\). If instead \(G\) is not quasisimple, then we show \(b(G) \leq 2\) in Proposition 5.13.

5.14. \(\lambda = \lambda_1 + \lambda_2\).

**Proposition 5.17.** Theorem 5.1 holds for \(G\) with \(E(G)/Z(E(G)) \cong \text{PSL}_{d+1}(q)\) acting on \(V = V(\lambda_1 + \lambda_2)\) for \(l \in [2, \infty)\). Namely, \(G\) has a regular orbit on \(V\) unless \(l = 3, q = 3^e\), where \(b(G) = 2\).
This gives a contradiction for $l \leq 2(l^2-l+1)$, where $PGL_5(2)$ has no regular orbit on $V$. We have completed the proof.

Proof. This module is of dimension $d = 2(l^2-l+1) - c_3(l+1)$. The Weyl orbit table can be found in Table 5.36.

First suppose $p \neq 3$. We compute using [3, Table B.3] that $\frac{\mathcal{O}_{-}^{(l+1)}}{\mathcal{O}_{-}^{(l+1)}(2)^2} = 3$. We summarise the bounds obtained there in Table 5.37.

| $c(s)$ | $c(u_\Psi)$ |
|-------|----------|
| $\Psi$ | $r = 2$ | $r \geq 3$ | $p = 2$ | $p = 5$ |
| $A_1$ | $l^2$ | $l^2$ | $l^2-l+1$ | $l^2$ |
| $A_2$ | $-2l^2-2l+1$ | $-2l^2-2l+2$ |

Table 5.37. Values of $c(s)$ and $c(u_\Psi)$ on $V(\lambda_1+\lambda_2)$ with $p \neq 3$ for different choices of $\Psi$.

Now let $\Psi = \langle \alpha_1, \alpha_3 \rangle$ of type $A_2^2$. The corresponding $\Psi$-net table is given in Table 5.38.

| $\nu$ | $n_1$ | $n_2$ | Mult | $r \geq 2$ | $p = 2$ | $p \geq 5$ |
|-------|------|------|-------|-----------|--------|--------|
| $0$   | $2$  | $0$  | $2(l^3-3)$ | $0$      | $0$    | $0$    |
| $\omega_1$ | $2$  | $0$  | $l-2$ | $l-2$      | $l-2$  | $l-2$  |
| $\omega_2$ | $2$  | $0$  | $l-2$ | $l-2$      | $l-2$  | $l-2$  |
| $2\omega_1 + \omega_3$ | $4$  | $2$  | $1$   | $4$        | $4$    | $4$    |
| $2\omega_1$ | $2$  | $1$  | $l-3$ | $2(l-3)$   | $l-3$  | $2(l-3)$ |
| $2\omega_2$ | $2$  | $1$  | $l-3$ | $2(l-3)$   | $l-3$  | $2(l-3)$ |
| $\omega_1 + \omega_3$ | $4$  | $2$  | $1$   | $4$        | $4$    | $4$    |
| $0$   | $0$  | $1$  | $l^3-3$ | $0$      | $0$    | $0$    |
| $\omega_1$ | $2$  | $2$  | $2(l^3-3)$ | $2(l^3-3)$ | $2(l^3-3)$ |
| $\omega_2$ | $2$  | $2$  | $2(l^3-3)$ | $2(l^3-3)$ | $2(l^3-3)$ |
| $\omega_1 + \omega_3$ | $4$  | $2$  | $l-3$ | $4(l-3)$   | $4(l-3)$ | $4(l-3)$ |
| Total | $2l^2-2l+2$ | $2l^2-2l+2$ | $2l^2-2l+2$ |

Table 5.38. $A_2^2$-net table for $V(\lambda_1+\lambda_2)$ with $p \neq 3$.

If $G$ has no regular orbit on $V$ then

$$q^d \leq 2|PGL_{n}(q)||q^{d-(2l^2-2l+1)} + 2(4q^{n^2/2+2}q^{d-2l^2-2l+2}) + 4q^{2n-1}q^{d-l^2} + 4 \left( \frac{q^{n^2/2+2}-1}{q^2-1} \right) q^{d-2l^2-2l+2}$$

$$4(q^{(n+1)} - q^{(n+1)-1} + q^{(n+1)-2}) q^{d-l^2} + 2 \log(\log_2 q + 2) q^{n^2/2+d/2}.$$ 

This gives a contradiction for $l \geq 5$ and $q \geq 2$, as well as $l = 4$ with $q \geq 4$. When $(l, q) = (4, 2)$, we replace $|PGL_5(2)|$ with the number of elements of prime order in $PGL_5(2)$ in the inequality above, and $4(q^{(n+1)} - q^{(n+1)-1} + q^{(n+1)-2})$ with twice the number of involutions in $PGL_5(2)$. This gives a contradiction and the result follows. Now suppose that $l = 3$ and $p \neq 3$. We compute (see [3, Table B.3]) that the number of semisimple elements of prime order in $PGL_4(q)$ with centralisers of type $A_2$, $A_2^2$ and $A_1$ is less than $q^4$, $\frac{q^4}{2}$ and $\frac{q^4}{2}$ respectively. Using GAP, we also compute that a regular semisimple element has all eigenspaces of dimension at most 4. Therefore, if $G$ has no regular orbit on $V$, then by
Propositions 3.1(iv), 2.3 and 4.4

\[ q^{20} \leq 2|\text{PGL}_4(q)|q^{20-16} + \frac{4}{3}q^{12}q^{20-13} + 3q^9q^{20-10} + 2q^7q^{20-9} + 4(q^9 + q^8)q^{20-10} + 4 \left( \frac{q^{10} - 1}{q^2 - 1} \right) q^{20-10} + q^{12}q^{20-14} + 2\log(q)q^2 + 2q^8 + 10 + \frac{q^4(q-1)(q^3-1)}{q-1}q^{20-7}. \]

This gives a contradiction for \( q \geq 13 \), so \( G \) has a regular orbit on \( V \). For the remaining \( q \), an explicit construction of \( G \leq \Gamma L(V) \) in GAP [13] shows that \( G \) has a regular orbit on \( V \) here as well.

Now, for the remainder of the proof, suppose that \( p = 3 \). Here \( d = 2(t_5^3) - (t_3^3) \). The \( \Psi \)-net tables for \( \Psi \) of type \( A_1 \), \( A_2^2 \) and \( A_3 \) are given in the proof of [18] Proposition 2.6.6 and we summarise the resulting lower bounds on \( c(s) \) and \( c(u_\Psi) \) in Table 5.39. If \( G \) has no regular orbit on \( V \),

| \( \Psi \) | \( c(s) \) | \( c(u_\Psi) \) |
|---|---|---|
| \( A_1 \) | \( \frac{1}{2}(l+1) \) | \( \frac{1}{2}(l^2 + 3l - 2) \) |
| \( A_1^2 \) | \( l^2 - l + 2 \) | \( l^2 + l - 2 \) |
| \( A_3 \) | \( - \) | \( \frac{1}{2}(3l^2 + l - 6) \) |

Table 5.39. Values of \( c(s) \) and \( c(u_\Psi) \) on \( V(\lambda_1 + \lambda_2) \) for different choices of \( \Psi \).

\[ q^d \leq 2|\text{PGL}_n(q)|q^{d-\frac{1}{2}(32^2+4)+2(q^{2n-1} + q^{2n-1})q^{d-\frac{1}{2}(l+1)} + 4(q^{(n+1)/2}) - 1 + q^{(n+1)/2} - 2)q^{d-(l^2-l+2)} + 4\log(q^2 + 2)q^{n^2/2+d/2} \]

This gives a contradiction for \( l \geq 6 \) and \( q \geq 3 \), and also for \( l = 5 \) when \( q \geq 9 \), so \( G \) has a regular orbit here. When \( l = 5 \) and \( q = 3, 9 \), we explicitly compute the number of prime order elements of \( \text{PGL}_6(q) \), as well as the number of unipotent elements of prime order. These, along with the co-dimension bounds for \( \Psi = A_1 \), \( A_2^2 \) and \( A_3 \) show that \( G \) has a regular orbit on \( V \) by an application of Proposition 3.1. When \( l = 4 \), we use [3] Tables B.2, B.3 to more precisely count elements of prime order in \( \text{PGL}_5(q) \), including those enumerated in Propositions 2.3 and 4.4. Substituting this information into the inequality gives the result for \( l = 4 \) and \( p = 3 \). Finally, if \( l = 3 \), then there is no regular orbit of \( G \) on \( V \) by [3], so \( b(G) \geq 2 \). Now, if \( G \) has no base of size 2 in its action on \( V \), then by Lemma 3.4

\[ q^{2\times16} \leq 2|\text{PGL}_4(q)|q^{2d-(l^2+l+2)} + 2(q^{2n-1} + q^{2n-1})q^{2d-(l^2+l+1)} + 4(q^{(n+1)/2} - 1 + q^{(n+1)/2} - 2)q^{2d-(l^2-l+2)} + 2\log(q^2 + 2)q^{n^2/2+d/2} \]

This gives a contradiction for \( q \geq 3 \), so \( b(G) = 2 \).

This concludes our analysis of highest weight modules \( V = V(\lambda) \) over \( \mathbb{F}_q \) with \( p \)-restricted highest weight \( \lambda \).

6. Proof of Theorem 1.1: Tensor product modules

In this section, we aim to prove the following result which will complete the proof of Theorem 1.1 for \( G, V \) having the same underlying field \( \mathbb{F}_q \).

**Theorem 6.1.** Let \( V = V(\nu_1) \otimes V(\nu_2)^{(p^a)} \) be a highest weight module defined over \( \mathbb{F}_q \), \( q = p^e \), with \( \nu_1, \nu_2 \) in Table 2.2 and \( 1 \leq a < e \). Let \( G \leq \Gamma L(V) \) be almost quasisimple with \( E(G)/Z(E(G)) \cong \text{PSL}_{l+1}(q) \) such that the restriction of \( V \) to \( E(G) \) is absolutely irreducible. Then either \( G \) has a regular orbit on \( V \), or \( \nu_1 = \lambda_1 \) and \( \nu_2 = \lambda_1 \) or \( \nu_1 \) up to quasi-equivalence. In the latter case, \( b(G) = 2 \) unless \( l = 1, (a, e) = 1, p = 2, 3 \) and \( G = \text{SL}_2(q) \), in which case there is a regular orbit.

We begin by examining the modules with \( \nu_1 = \lambda_2 \).
6.1. Tensor product modules with $V(\lambda_2)$ as a factor. In this subsection, we investigate the modules in Table 4.2 with $\iota_1 = \lambda_2$. We assume that $\iota_2 \neq \lambda_1$ or $\iota_1$ here; we treat tensor product modules with a factor quasiequivalent to $\lambda_1$ or $\lambda_1$ in Section 6.2. We also suppose in this section that $l > 2$, because if $l = 2$, then $V(\lambda_2) = V(\lambda_1)^*$ and $V(\lambda)$ is quasiequivalent to a module with $\iota_1 = \lambda_1$.

We begin by giving the Weyl orbit table and some $\Psi$-net tables for $V(\lambda_2)$ in Table 6.2.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|l|l|}
\hline
\multicolumn{2}{|c|}{\nu} & \multicolumn{2}{|c|}{c(s)} \cr
\hline
0 & 1 & (l-2) & 0 \cr
\hline
\omega_1 & 3 & l-2 & 2(l-2) \cr
\hline
\omega_2 & 3 & 1 & 2 \cr
\hline
\hline
\textbf{Total} & & 2l-2 & 2l-2 \cr
\hline
\end{tabular}
\caption{$\Psi = \langle \alpha_1 \rangle$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|l|l|}
\hline
\multicolumn{2}{|c|}{\nu} & \multicolumn{2}{|c|}{c(u_{\Psi})} \cr
\hline
0 & 1 & (l+1)-2(l-1) & 0 \cr
\hline
\lambda_1 & 2 & l-1 & l-1 \cr
\hline
\omega_1 + \omega_3 & 4 & 1 & 2 \cr
\hline
\hline
\textbf{Total} & & 2l-4 & 2l-4 \cr
\hline
\end{tabular}
\caption{$\Psi = \langle \alpha_1, \alpha_2 \rangle$.}
\end{table}

Table 6.5. The Weyl orbit table and some $\Psi$-net tables for $V(\lambda_2)$.

We are now ready to prove the following result.

**Proposition 6.2.** Let $G$ be as in Theorem 5.1 with $l \geq 3$ and $V = V(\lambda_2) \otimes V(\iota_2)^*(\Psi)$ with $\iota_2$ appearing in Table 4.2. Then $G$ has a regular orbit on $V$.

**Proof.** Let $d_2 = \dim V(\iota_2)$. From Table 4.2 we see that $d_2 \geq \binom{l+1}{2}$. If $G$ has no regular orbit on $V$, then from the $\Psi$-net tables for $V(\lambda_2)$ given above, as well as Propositions 3.3 and 3.4

\[
q^{(l+1)2d_2} \leq 2|PGL_n(q)|q^{d_2(\binom{l+1}{2})-(2d-4)} + 2\log(\log_2 q + 1)q^{n^2/2+d/2} + (q^{2n-1} + 2q^{2n-1})q^{d_2(\binom{l+1}{2})-(l-1)}
\]

This gives a contradiction for $d_2 \geq \binom{l+1}{2}$, when $l \geq 4$ and $q \geq 2$ as required. The inequality is also false for $l = 3$ and $d \geq 8$, leaving $\iota_2 = \lambda_2$ to consider. So suppose $\iota_2 = \lambda_2$.

Here we must take graph automorphisms $\tau$ into consideration, since they preserve the module. We compute that $\dim C_V(\tau) \leq 25$. Therefore, if $G$ has no regular orbit on $V$, then by Propositions 3.3, 4.3, 4.4 and 2.6

\[
q^{36} \leq 2|PGL_4(q)|q^{36-4 \times 6} + 8q^{10}q^{36-12} + 12q^{36-24} + 4\left(\frac{q^{10} - 1}{q^{2}-1}\right)q^{36-12} + 2\log(\log_2 q + 2)q^{26} + (2^{15}/2 + 2^9)q^{25}
\]

This gives a contradiction for all non-prime $q$, so $G$ has a regular orbit on $V$. \qed

6.2. Tensor product modules with $V(\lambda_1)$ as a factor. We begin with some discussion about the Weyl orbits and $\Psi$-nets of $V(\lambda_1)$.

Let $V = V(\lambda_1)$. The weights of $V$ are of the form $\{\lambda_1 - \alpha_i \mid \alpha_i \in \Delta\} \cup \{\lambda_1\}$, where $\Delta$ is a fixed base for $\Phi$. All of these weights lie in one orbit under the Weyl group.

Suppose $\Psi = \Psi_1 \Psi_2 \ldots \Psi_k$ is a standard subsystem of $\Phi$, where each of the $\Psi_i$ are irreducible. There is one $\Psi$-net for each $\Psi_i$ whose weight space sum is a copy of the natural module for $\mathcal{G}_{\Psi_i}$. Each of the remaining weights individually forms a $\Psi$-net.
Therefore, if $|\Delta_i|$ is the size of a base of $\Psi_i$, we find that for a semisimple element $s \in G$ of prime order with $\Psi \cap \Phi(s) = \emptyset$, we have

$$c(s) \geq \sum_{i=1}^{k} |\Delta_i|,$$

and the same bound holds for $c(\omega \Psi)$.

**Proposition 6.3.** Suppose $G \leq \Gamma L(V)$ is almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_n(q)$, and that $E(G)$ acts absolutely irreducibly on $V = V(\lambda_1) \otimes V(t_2)^{(p^s)}$, where $1 \leq a < e$ and $t_2$ appears in Table 7.3. If either $n \geq 4$ and $\dim V(t_2) \geq \frac{a^2+1}{2}$, or $n = 3$ and $\dim V(t_2) \geq 6$, then $G$ has a regular orbit on $V$.

**Proof.** Let $d_2 = \dim V(t_2)$. Note that for $n \geq 3$, $V$ is never fixed by a graph automorphism, and that $q$ is not prime. If $G$ has no regular orbit on $V$ then by Propositions 4, 3 and 2.3, as well as the discussion preceding this proposition,

$$q^{nd_2} \leq 2|\text{PGL}_n(q)|q^{d_2(n-2)} + 2 \log(q + 2)q^{n^2/2 + nd_2/2} + (2q^{2n-1} + q^{2n-1})q^{(n-1)d_2}$$

This gives a contradiction for $d \geq (n^2 + 1)/2$ and $n \geq 4$ and $q \geq 4$, and also $n = 3$ with $d \geq 6$ and $q \geq 4$.

The remaining possibilities for $V(t_2)$ are given in the table below.

| $t_2$ | $\lambda_1$ | $\lambda_1$ | $\lambda_2$ | $\lambda_{l-1}$ | $2\lambda_1$ |
|-------|-------------|-------------|-------------|----------------|-------------|
| $l$   | $[1, \infty)$ | $[1, \infty)$ | $[3, \infty)$ | $[3, \infty)$ | $1$          |

**Table 6.6.** Remaining possibilities for $V(t_2)$.

**Proposition 6.4.** Theorem 6.3 holds for $V = V(\lambda_1) \otimes V(2\lambda_1)^{(p^s)}$ and $l = 1$. Namely, $G$ has a regular orbit on $V$.

**Proof.** Here $\dim V = 6$. By the proof of [13] Proposition 2.7.3, the codimension of the largest eigenspace of an element $x$ of prime order on $V(\lambda_1)$ is at least 2 if $x$ has odd order, and 1 otherwise. Therefore, if $G$ has no regular orbit on $V$, then by Propositions 3.3, 3.3 and 2.3

$$q^6 \leq \frac{3}{2}|\text{PGL}_2(q)|q^2 + 2(q^2 + q)(2q^3) + 2 \log(q + 2)q^{2+6/2}$$

and this is a contradiction for $q \geq 9$. □

**Proposition 6.5.** Theorem 6.3 holds for $V = V(\lambda_1) \otimes V(\lambda_2)^{(p^s)}$ and $V = V(\lambda_1) \otimes V(\lambda_{l-1})^{(p^s)}$ for $l \geq 3$. Namely, $G$ has a regular orbit on $V$.

**Proof.** We will make use of the $\Psi$-net tables for $V(\lambda_2)$ given at the beginning of Section 6.1. Suppose $l \geq 4$, and set $d = \dim V = (l + 1)(l+1)$. If $G$ has no regular orbit on $V$, then by Propositions 6.3, 1.1 and 3.3

$$q^d \leq 2|\text{PGL}_n(q)|q^{d-(2l-2)(l+1)} + 8q^{n^2/2+2}q^{d-(2l-4)(l+1)} + 4\left(\frac{q^{n^2/2+2} - 1}{q^2 - 1}\right)q^{d-(2l-4)(l+1)}$$

$$+ 4q^{2n-1}q^{d-(l-1)(l+1)} + 2 \log(q + 2)q^{n^2/2+d/2}$$

This gives a contradiction for all $q \geq 4$, so $G$ has a regular orbit on $V$. Now suppose that $l = 3$. If $G$ has no regular orbit on $V$ then by Proposition 6.5

$$q^{24} \leq 2|\text{PGL}_4(q)|q^{24-16} + (8q^{10} + 4 \left(\frac{q^{10} - 1}{q^2 - 1}\right))q^{24-12} + 4q^7q^{24-8} + 2 \log(q + 2)q^{20}$$

This gives a contradiction for all $q \geq 8$, so $G$ has a regular orbit on $V$. For $q \leq 8$, we replace $|\text{PGL}_4(q)|$ with the number of elements of prime order in $\text{PGL}_4(q)$, and also use more precise versions of Proposition 4.3 and 4.4 in order to give a contradiction in the above inequality. □
Proposition 6.6. Theorem 6.4 holds for $V = V(\lambda_1) \otimes V(\lambda_1)^{(p^a)}$ and $V = V(\lambda_1) \otimes V(\lambda_1)^{(p^a)}$ for $l \geq 3$. Namely, $b(G) = 2$, unless $(a, \log_p q) = 1$, $p = 2$ or 3 and $G = \text{SL}_2(q)/K$, where $K$ is the kernel of the action of $\text{SL}_2(q)$ on $V$. In these cases, $b(G) = 1$.

Proof. Let $q = p^c$. To show that there is no regular orbit of $G$ on $V$, it is sufficient to consider $G = \text{SL}_n(q)/Z$, where $Z$ is the kernel of the action of $\text{SL}_n(q)$. Let $b = \gcd(a, c)$, then $|Z| = \gcd(p^b+1, n)$ and $\gcd(p^b-1, n)$ for $\lambda = (p^a+1)\lambda_1$ and $p^b\lambda_1 + \lambda_1$ respectively. The action of $g \in G$ on $V(\lambda_1) \otimes V(\lambda_1)^{(p^a)}$ or $V(\lambda_1) \otimes V(\lambda_1)^{(p^a)}$ is equivalent to the action of $G$ on $n \times n$ matrices as

$$A \mapsto g^T Ag^{(p^a)} \quad \text{or} \quad A \mapsto g^{-1}Ag^{(p^a)}$$

respectively. These maps are rank and determinant preserving. The number of matrices of rank $k$ ($1 \leq k \leq n$) is (see [24] p. 338) for example):

$$q(t) \prod_{j=1}^k \frac{(q^n-k+1)^2}{(q^j-1)}.$$ 

This is less than $|G|$ for $1 \leq k \leq n - 2$. So if $G$ has a regular orbit on $V$, then it is composed of matrices of rank $n$ or rank $n-1$. We show that $G$ has no regular orbit on matrices of rank $n$ using Shintani descent (see [3] §2.6 for an overview).

Suppose $\lambda = \lambda_1 + p^a \lambda_2$. Let $\sigma : x \mapsto x^{(p^b)}$ be a field automorphism of $\text{GL}_n(q)$, let $\varphi = \sigma^c : x \mapsto x^{(p^a)}$, where $c = a/b$, and define the semidirect product $K = \text{GL}_n(q) \rtimes \langle \sigma \rangle$. The stabiliser in $H = \text{SL}_n(q) \leq K$ of $A \in \text{GL}_n(q)$ under the action $A \mapsto g^{-1}Ag^{(p^a)}$ is equal to $C_H(A\varphi^{-1})$. Let $\hat{c}$ be an integer such that $\hat{c} \equiv -c \mod \mathbb{Z}/b$ and such that $\gcd(\hat{c}, |K|) = 1$. Then the map $f : K \to K$ defined by $x \mapsto x^{\hat{c}}$ is a bijection. Therefore, $A\varphi^{-1}$ has a preimage of the form $y\varphi \in K$. Moreover, we see that $C_H(A\varphi^{-1}) = C_H(y\varphi)$. Applying Shintani descent (see [4] Lemma 2.13), for instance), $C_H(y\varphi) = C_{\text{SL}_n(p^c)}(h)$ for some $h \in \text{GL}_n(p^c)$. Since every element of $\text{GL}_n(p^c)$ is centralised by a non-scalar in $\text{SL}_n(p^c)$, $C_H(y\varphi)$ is not contained in the centre of $H$, and there is no regular orbit of $G$ on $V$. The argument for $\lambda = (p^a+1)\lambda_1$ is similar; there we define $\sigma$ to be a graph-field automorphism $x \mapsto (x^{(p^a)})^{-1}$.

It remains to consider whether there exists a regular orbit on the matrices of rank $n - 1$. Suppose $n \geq 3$. We first deal with $V(\lambda_1) \otimes V(\lambda_1)^{(p^a)}$. Let

$$S = \left\{ \left[ \begin{array}{cc} N & 0 \\ 0 & 0 \end{array} \right] \mid N \in \text{GL}_{n-1}(q) \right\}.$$ 

Note that every element of $S$ has non-trivial stabiliser in $\text{SL}_n(q)/Z$. The setwise stabiliser of $S$ in $\text{SL}_n(q)$ is

$$K = \left\{ \left[ \begin{array}{cc} A & 0 \\ 0 & b \end{array} \right] \mid A \in \text{GL}_{n-1}(q), b = \det A^{-1} \right\}.$$ 

Let $S^G = \{ g^{-1}Ag^{(p^a)} \mid g \in \text{SL}_n(q), A \in S \}$, and note that $\text{SL}_n(q)$ has no regular orbit that includes an element of $S^G$. We now count the number of elements in $S^G$. Let $g_1, g_2, \ldots, g_t$ be a set of right coset representatives of $\text{SL}_n(q)/Z$; then the set of $g_i^{-1}Sg_i^{(p^a)}$ for $1 \leq i \leq t$ partition $S^G$. We have $|g_i^{-1}Sg_i^{(p^a)}| = |S|$, so $|S^G| = |\text{SL}_n(q)||S|/|Z| = |\text{SL}_n(q)|$. The number of remaining matrices of rank $n - 1$ is less than $2q^{2d-2}$, which is less than $|\text{SL}_n(q)|/|Z|$, so there can be no regular orbit. For $V = V(\lambda_1) \otimes V(\lambda_1)^{(p^a)}$, we repeat the same argument, this time setting

$$S = \left\{ \left[ \begin{array}{ccc} N & v_1 & 0 \\ v_2 & 0 & c \\ 0 & 0 & 0 \end{array} \right] \mid N \in \text{GL}_{n-2}(q), v_1, v_2 \in \mathbb{F}_q^{n-2}, c \in \mathbb{F}_q \right\},$$ 

and find that there is no regular orbit under $\text{SL}_n(q)/Z$ here either.

Finally, if $n = 2$, then the stabiliser of an element of $S$ contains $\text{Diag}(\nu, \nu^{-1})$ for $\nu \in \mathbb{F}_q^\times$. All such elements lie in the kernel of the action if and only if $b = 1$ and $p = 2$ or 3. In both cases, $\text{Diag}(1, 0)$ is a regular orbit representative of $G$ on $V$. When $p = 3$, $\text{Diag}(j, 0)$, with $j \in \mathbb{F}_q^\times$ a non-square is a representative of a second regular orbit of $G$ on $V$. The two regular orbits are not interchanged by a graph or diagonal automorphism. No group properly containing $G$ has a regular orbit on $V$, since there is no regular orbit of $G$ containing rank 2 matrices, and the number of remaining rank 1 matrices is $q^2 - 1$, which is less than the order of $G$. Where we show that is no regular orbit, we have $b(G) = 2$ by Proposition 6.13. \qed
This completes the proof of Theorem 6.1 and therefore Theorem 1.1 for modules with $k = 1$.

7. Absolutely Irreducible Representations over Subfields

In this section, we consider the following embedding of $\text{SL}_n(q^c) \leq \text{SL}_m(q)$ for $c \geq 1$. Let $W$ be the $n$-dimensional natural module for $\text{SL}_n(q^c)$ over $\mathbb{F}_q$, with standard basis $\{e_1, \ldots, e_n\}$, and let

$$V = W \otimes W(q) \otimes W(q^2) \otimes \cdots \otimes W(q^{c-1}),$$

so that $\dim V = n^c$ and $B = \{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_c} \mid 1 \leq i_j \leq n\}$ is a basis of $V$.

The semilinear map on $V$ sending $\nu e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_c}$ to $\nu^a e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_c}$ for $\nu \in \mathbb{F}_q^*$ induces an automorphism of $\text{SL}_n(q^c)$ defined by $\varphi : (a_{ij}) \mapsto (a_{ij}^\nu)$. As discussed in the proof of Proposition 2.4, $\langle \varphi \rangle$ fixes $e_1 \otimes e_1 \otimes \cdots \otimes e_1$ for $1 \leq i \leq c$ and has orbits of length $c$ on the remaining elements of $B$.

Let $\theta$ be a set of $\langle \varphi \rangle$-orbit representatives and let $\{\nu_1, \nu_2, \ldots, \nu_c\}$ be a basis of $\mathbb{F}_q^*$ over $\mathbb{F}_q$. We define a new basis of $V$ as follows:

$$B_2 = \{\sum_{i=0}^{c-1} \nu_j^i \varphi^i(v) \mid v \in \theta, 1 \leq j \leq c\}.$$ 

Notice that every element of $B_2$ is preserved under the action of $\varphi$. An element $g \in \text{SL}_n(q^c)$ preserves the $\mathbb{F}_q$ span of $B_2$ since

$$(\sum_{i=0}^{c-1} \nu_j^i \varphi^i(v))g = \sum_{i=0}^{c-1} \nu_j^i \varphi^i(vg) = \sum_{i=0}^{c-1} \varphi^i(v_jvg)$$

which is fixed by the action of $\varphi$. We can therefore realise $V$ as a $\mathbb{F}_q\text{SL}_n(q^c)$-module, and Proposition 5.4.6 shows that all absolutely irreducible $\mathbb{F}_q\text{SL}_n(q^c)$-modules arise in this way, replacing $W$ with some arbitrary irreducible $\text{SL}_n(q^c)$-module.

In the notation of Theorem 1.1, the absolutely irreducible $\mathbb{F}_q\text{SL}_n(q^c)$-modules $V$ constructed in this way correspond to cases where $k < 1$ (indeed, $k = 1/c$), and we further assume in this section that $V$ cannot be realised over any proper subfield of $\mathbb{F}_q$. The remaining cases with $k < 1$ will be dealt with in Section 8.

The main theorem of this section is as follows.

**Theorem 7.1.** Let $V = V_{mc}(q)$ be a $m^c$-dimensional vector space over $\mathbb{F}_q$, with $c > 1$. Let $G \leq \GammaL(V)$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_n(q^c)$, such that $V = V(\lambda)$ is an absolutely irreducible $\mathbb{F}_qE(G)$-module and $V$ cannot be realised over a proper subfield of $\mathbb{F}_q$. Then one of the following holds.

(i) $G$ has a regular orbit on $V$;

(ii) $\lambda, k = 1/c$ and $n$ appear in Table 1.2.

We begin by reducing the proof of Theorem 7.1 to a finite list of modules that require further consideration.

**Proposition 7.2.** Suppose $G \leq \GammaL(V)$ is as in Theorem 7.1, with $V = V_{mc}(q)$, whose restriction to $E(G)$ is absolutely irreducible. Then one of the following holds.

(i) $G$ has a regular orbit on $V$;

(ii) $(n, m, c) = (n, n, 2)$ or $(2, 2, 3)$ and $G$ has no regular orbit on $V$.

(iii) The tuple $(n, m, c)$ appear in Table 7.1.

**Proof.** Notice that if $(n, m, c)$ appears in part (iii), then $|V| < |G|$ and there cannot be a regular orbit of $G$ on $V$. Suppose $n \geq 5$. If $G$ has no regular orbit on $V$, then $q^{mc} \leq 2|\text{Aut}(\text{PSL}_n(q^c))||q^{\frac{n-1}{2}m^c}|$. Since $m \geq n$, this contradicts for $c \geq 4$, for $c = 3$ if $m > n$, and for $c = 2$ if $m > \left(\frac{n}{2}\right)$ or $m = \left(\frac{n}{2}\right)$ and $n \geq 7$. By [30], a full list of the remaining modules not satisfying the inequality (excluding $(n, m, c) = (n, n, 2)$) is given in Table 7.1. Similarly, if $n = 4$ and $G$ has no regular orbit on $V$, then by Propositions 2.2, 2.3 and 2.6

$$q^{mc} \leq 2(|\text{PGL}_4(q^c)| + 2q^{15c/2})q^{4m^c} + 4\log(q^{15c} + 2)q^{8c + (m^c + m^{c/2})/2} + 2q^{6c}q^{6m^c/7}.$$ 

This gives a contradiction for $c \geq 5$, for $c = 3, 4$ if $m > 4$ and for $c = 2$ if $m \geq 14$ and the remaining modules are given in Table 7.1. If instead $n = 3$ and $G$ has no regular orbit on $V$ we have

$$q^{mc} \leq 2|\text{PGL}_3(q^c)|(q^{5m^c} + q^{4m^c})4\log(q^{15c} + 2)q^{10c + (m^c + m^{c/2})/2} + 2q^{5c}q^{3m^c/4}.$$
This gives a contradiction for $c \geq 5$ for $m \geq 3$, for $c = 3, 4$ if $m > 3$ and for $c = 2$ if $m \geq 7$. Finally let $n = 2$ and suppose $q \geq 7$ and $q \neq 9$. If $G$ has no regular orbit on $V$, then by Propositions 2.2 and 2.3

$$q^{mc} \leq 2|\text{PGL}_2(q^c)|q^{(mc/2)} + 2(q^2 + q)(q^{2m^2/3} + q^{mc/3}) + 4\log(\log_2 q^c + 2)q^{2c+(mc+m^2/3)/2}.$$ 

This gives a contradiction for $c \geq 5$ and $m \geq 2$, for $r = 3, 4$ if $m > 2$ and for $r = 2$ if $m \geq 4$. Lastly, if $q = 9$, then examining [21] p.4, we see that the only absolutely irreducible $\text{SL}_2(9)$ modules realised over $\mathbb{F}_3$ have $(m, c) = (2, 2)$ or $(3, 2)$, and so the result follows.

Suppose $(n, m, c) = (2, 2, 3)$ and let $G$ and $V$ be as in Theorem 7.1. In Proposition 8.12 we show that $b(G) = 1$ on $V \otimes \mathbb{F}_p$, so by Proposition 8.14 $b(G) = 2$ on $V$. Similarly, if $(n, m, c) = (n, n, 2)$, we show that $G$ has a regular orbit on $V \otimes \mathbb{F}_p$, in Proposition 8.3 so for the action of $G$ on $V$, we have $2 \leq b(G) \leq 3$ by Proposition 8.12. We will use similar arguments in this section for other $G$ and $V$ where we cannot show there is a regular orbit.

| $n$  | $m$ | $c$ |
|------|-----|-----|
| $\geq 3$ | $n$ | 3 |
| $4 \leq n \leq 6$ | $(\binom{n}{2})$ | 2 |
| 4 | 10 | 2 |
| 4 | 4 |
| 3 | 6 | 2 |
| 3 | 4 |
| 2 | 3 | 2 |
| 2 | 4 |

Table 7.1. Representations of $\text{SL}_n(q^c) < \text{GL}_{mc}(q)$ left to consider.

**Proposition 7.3.** Let $V = V_n(a)$ be an $n^3$-dimensional vector space over $\mathbb{F}_q$, where $n \geq 3$. Suppose that $G \leq \text{GL}(V)$ is almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_n(q^3)$ and that the restriction of $V$ to $E(G)$ is absolutely irreducible. If $n \geq 4$, then $G$ has a regular orbit on $V$ and if $n = 3$, then $b(G) \leq 2$.

**Proof.** We have $V \otimes \mathbb{F}_p \cong V(\lambda_1) \otimes V(\lambda_1)(\varphi) \otimes V(\lambda_1)(\varphi^2)$, where $V(\lambda_1)$ is the natural module for $E(G)$. Suppose $s \in G$ is semisimple of projective prime order, and suppose the eigenvalues of $s$ on $V(\lambda_1) \otimes \mathbb{F}_p$ are $t_1, t_2, \ldots, t_m$, ordered so that the multiplicities $a_i$ of the $t_i$ are weakly decreasing. By Propositions 3.3 and 3.3, the codimension of $C_V(s)$ is at least $n(n^2 - \sum a_i^2)$, while $\log_2(|sF(G)^{\text{PGL}_n(q)}|) < 3(n^2 - \sum a_i^2) + m \log_2 2$. Therefore, noting that $m \leq n$, we have

$$(7.1) \quad \text{codim}_{F_s} C_V(s) - \log_2(|sF(G)^{\text{PGL}_n(q)}|) \geq (n - 3)(n^2 - \sum a_i^2) - n \log_2 2$$

Provided that $a_1 < n - 1$, the right hand side of (7.1) is at least $(n-3)(4n-8)$. If instead $a_1 = n-1$, then $\dim C_V(s) \leq n^3 - 2n^2 + 2n$ and we will treat such classes separately. Now suppose $uF(G) \in G/F(G)$ is of prime order with Jordan blocks $(J_1^{a_1}, \ldots, J_2^{a_2}, J_3^{a_3})$ on the natural module for $G$. By [18] Lemma 1.3.3, and Proposition 3.3, the codimension of $C_V(u)$ is at least $n(n^2 - \sum_{i<j} i a_i a_j - \sum i a_i^2)$, and $\log_2(|uF(G)^{\text{PGL}_n(q)}|) < 3(n^2 - 2 \sum_{i<j} i a_i a_j - \sum i a_i^2) + m$, where $m$ is the number of non-zero $a_i$. Therefore, since $m \leq 1 + \sum_{i<j} i a_i a_j$, we have

$$(7.2) \quad \text{codim}_{F_s} C_V(u) - \log_2(|uF(G)^{\text{PGL}_n(q)}|) \geq (n - 3)(n^2 - 2 \sum_{i<j} i a_i a_j - \sum a_i^2) - 1.$$ 

Provided that $u$ does not have Jordan canonical form $(J_2, J_1^{a_1-2})$ on $V(\lambda_1)$, the right hand side of (7.2) is at least $(n-3)(4n-8) - 1$. If $u$ does have Jordan canonical form $(J_2, J_1^{a_1-2})$, then $\dim C_V(s) \leq n^3 - 3n^2 + 4n$. By [12] Lemma 3.7, the number of conjugacy classes in $\text{PGL}_n(q^3)$ is at most $q^{3(n-1)} + 5q^{3(n-2)}$. Therefore, if $G$ has no regular orbit on $V$, 

$$q^{n^3} \leq 2(q^{3(n-1)} + 5q^{3(n-2)})(q^{n^3-(n-3)(4n-8)} - n \log_2 2 + q^{n^3-(n-3)(4n-8)+1} + 2q^{3(n-1)}q^{3n^2-2n^2+2n}$$

$$+ q^{3(2n-1)}q^{n^3-3n^2+4n} + 2\log(\log_2 q^3 + 2)q^{3n^2/2+(n^3+n^2)/2}.$$
This gives a contradiction for \( n \geq 5 \) and \( q \geq 2 \). Replacing \((q^{3(n-1)} + 5q^{3(n-2)})\) with a more accurate count of the number of conjugacy classes of prime order elements in \( \text{PGL}_4(q^3) \) gives the result for \( n = 4 \). Finally, if \( n = 3 \), then \( b(G) \leq 2 \) by Propositions 3.3 and 8.12.

**Proposition 7.4.** Let \( m = \binom{n+1}{2} \) with \( n \in [2, 4] \), and define \( V = V_{m^2}(q) \). Suppose that \( G \leq \Gamma L(V) \) is an almost quasisimple group with \( E(G)/Z(E(G)) \cong \text{PSL}_n(q^2) \) such that the restriction of \( V \) to \( E(G) \) is absolutely irreducible. If \( n \in [3, 4] \), then \( G \) has a regular orbit on \( V \). If instead \( n = 2 \), then \( b(G) \leq 2 \).

**Proof.** We have \( V \otimes \mathbb{F}_p \cong V(2\lambda_1) \otimes V(2\lambda_1)^{(q)} \), and so \( p \geq 3 \). The Weyl orbit and weight string tables for \( V(2\lambda_1) \) are given below.

| \( i \) | \( \mu \) | \( |W(\mu)| \) | \( \text{Mult} \) |
|---|---|---|---|
| 1 | \( 2\lambda_1 \) | \( l + 1 \) | 1 |
| 2 | \( \lambda_2 \) | \( \frac{l+1}{2} \) | 1 |

By Propositions 3.3 and 8.12, if \( G \) has no regular orbit on \( V \), then

\[
q^{m^2} \leq 2|\text{PGL}_n(q^2)|q^{m(m-n)} + 4(q^{n^2+n-2} + q^{n^2+n-3})q^{m(m-n+1)} + 2\log(q) + 2 q^{n^2+(m^2+m)/2}.
\]

This gives a contradiction for \( n = 4 \) and \( q \geq 3 \), as well as \( n = 3 \), \( q \geq 4 \). If \( E(G)/Z(E(G)) \cong \text{PSL}_3(q) \), then substituting the precise number of involutions and prime order elements (computed using GAP) into the equation, we also obtain a contradiction. Finally, if \( n = 2 \), then \( b(G) \leq 2 \) by Propositions 3.3 and 8.12.

**Proposition 7.5.** Let \( V = V_{n^2}(q) \), with \( n \in [2, 4] \), and let \( G \leq \Gamma L(V) \) be an almost quasisimple group with \( E(G)/Z(E(G)) \cong \text{PSL}_n(q^4) \). Further suppose that \( E(G) \) acts absolutely irreducibly on \( V \). If \( n \in [3, 4] \), then \( G \) has a regular orbit on \( V \), and if \( n = 2 \), then \( b(G) \leq 2 \).

**Proof.** We have \( V \otimes \mathbb{F}_p \cong V(\lambda_1) \otimes V(\lambda_1)^{(q)} \otimes V(\lambda_1)^{(q)} \otimes V(\lambda_1)^{(q)} \), where \( V(\lambda_1) \) is the natural module for \( E(G) \). Now, any projective prime order element has an eigenspace of dimension at most \( n-1 \) on the natural module. Therefore, by Propositions 3.3 and 3.3 for projective prime order \( x \in G \), \( \dim C_V(g) \leq n^2((n-1)^2 + 1^2) \) and so if \( G \) has no regular orbit on \( V \), then

\[
q^{n^4} \leq 2|\text{PGL}_n(q^4)|q^{n^4((n-1)^2+1)} + 2\log(q) + 2 q^{n^2+(n^2+n)/2}.
\]

This gives a contradiction for \( q \geq 2 \) and \( n = 3, 4 \), so \( G \) has a regular orbit on \( V \).

If \( n = 2 \), then \( b(G) \leq 2 \) by Propositions 3.4 and 8.12.

**Proposition 7.6.** Suppose \( V = V_{m^2}(q) \), where \( m = \binom{n}{2} \) and \( n \in [4, 6] \). Let \( G \) be an almost quasisimple group with \( E(G)/Z(E(G)) \cong \text{PSL}_n(q^2) \) such that \( E(G) \) acts absolutely irreducibly on \( V \). If \( n \in [5, 6] \), then \( G \) has a regular orbit on \( V \), and if \( n = 4 \), then \( b(G) \leq 2 \).

**Proof.** We have \( V \otimes \mathbb{F}_p \cong V(\lambda_2) \otimes V(\lambda_2)^{(q)} \). From the \( \Psi \)-net tables for \( V(\lambda_2) \) given in Section 6.1 and also Proposition 6.3, we deduce that if \( G \) has no regular orbit on \( V \) then

\[
q(\binom{n}{2})^2 \leq 2(q^{2(2n-1)} + q^{2(2n-1)}q(\binom{n}{2})^2) + 2(2q^{2(2n-1)} + 4 q^{2(2n/2+2)}) + 4 \left( \frac{q^{2(2n/2+2)} - 1}{q^4 - 1} \right) q(\binom{n}{2})^2(q(\binom{n}{2})-2n+6) + 2|\text{PGL}_n(q^2)|q(\binom{n}{2})^2(q(\binom{n}{2})-2n+4) + 2\log(q) + 2 q^{n^2+(n^2+n)/2}.
\]

This inequality is false for \( n = 5, 6 \) and \( q \geq 2 \). Finally, if \( n = 4 \), then \( b(G) \leq 2 \) by Propositions 3.3 and 8.2.

We have now completed the proof of Theorem 7.1.
be realised over a proper subfield of $F_{36}$. We also assume that quasiequivalence) appear in Table 1.2.

Proof. First note that $\lambda$ be a generator of $F$. Maintaining the notation in Theorem 8.1, let $W$ be a field extension of the modules $E$. We now consider each of the modules in Table 8.1 individually. We have already investigated these modules of definition $F$. Therefore, considering the results contained in Sections 5, 6 and 7, we see that it remains to consider field extensions of the modules $V$, or $n, b(G)$ and $\lambda$ (up to quasiequivalence) appear in Table 1.2.

Table 8.1. Remaining absolutely irreducible $\text{SL}_n(q^k)$ modules to consider, with field of definition $F_{q^k}$.

| $\lambda$ | Dimension | $c$ | $n$ |
|-----------|------------|-----|-----|
| $\lambda_1$ | $n$ | 1 | $[2, \infty)$ |
| $\lambda_2$ | $(n \choose 2)$ | 1 | $[3, \infty)$ |
| $2\lambda_1$ | $(n+1 \choose 2)$ | 1 | $[2, \infty)$ |
| $3\lambda_1$ | 4 | 1 | 2 |
| $\lambda_3$ | 6 | 1 | 3 |
| $2\lambda_2$ | 20 | 1 | 6 |
| $4\lambda_1$ | 84 | 1 | 9 |
| $\lambda_4$ | 70 | 1 | 8 |
| $2\lambda_2 + b$ | $20 - \epsilon_3$ | 1 | 4 |
| $\lambda_1 + \lambda_{n-1}$ | $n^2 - 1 - \epsilon_n$ | 1 | $[4, \infty)$ |
| $\lambda_1 + p^c\lambda_{n-1}$ | $n^2$ | 1 | $[3, \infty)$ |
| $(p^c + 1)\lambda_1$ | $n^2$ | 2 | $[2, \infty)$ |
| $2(p^{c/2} + 1)\lambda_1$ | 9 | 2 | 2 |
| $(p^{c/2} + 1)\lambda_2$ | 36 | 2 | 4 |
| $(p^{c/3} + p^{2c/3} + 1)\lambda_1$ | $n^3$ | 3 | $[2, 3]$ |
| $(p^{c/4} + p^{2c/4} + p^{3c/4} + 1)\lambda_1$ | 16 | 4 | 2 |

8. Absolutely irreducible representations over extension fields

In this section we consider absolutely irreducible $\text{SL}_n(q)$-modules $V$ defined over a field $F$ that can be realised over a proper subfield of $F$. We have already analysed such modules $V$ over $F_q$ in Sections 4 and 5 so we also assume that $k \neq 1$ (with $k$ as in Theorem 1.1) in this section. The main result of this section is as follows.

Theorem 8.1. Let $V = V_d(q^k)$ be a $d$-dimensional vector space over $F_{q^k}$ with $k \neq 1$, and let $G \leq \Gamma L(V)$ be almost quasismple with $E(G)/Z(E(G)) \cong \text{PSL}_n(q)$ and $n \geq 2$. Further suppose that the restriction of $V = V(\lambda)$ to $E(G)$ is an absolutely irreducible $F_{q^k}$-module of highest weight $\lambda$, and that $V$ can be realised over a proper subfield of $F_{q^k}$. Then either $G$ has a regular orbit on $V$, or $n, b(G)$ and $\lambda$ (up to quasiequivalence) appear in Table 1.2.

Maintaining the notation in Theorem 8.1, let $F_{q^k}$ be field of definition of $V$ i.e., the smallest subfield of $F_{q^k}$ that $V$ is realised over. By our previous work, $q_0 = q^{1/c}$ for some integer $c \geq 1$, and so we set $q = q_0^c$. Let $V_0$ be the realisation of $V$ over $F_{q_0}$. We can then write $V = V_0 \otimes F_{q^k}$, where $q^k = q_0^c$ i.e., $t = ck$.

We have already investigated these modules $V_0$ in Sections 5, 6 and 7. By Proposition 8.2. if one of the inequalities given in Proposition 3.1 (ii) (v) fails for $G$ acting on $V_0$, then $F_{q_0}^\times \circ G$ has a regular orbit on $V$. Therefore, considering the results contained in Sections 5, 6 and 7, we see that it remains to consider field extensions of the modules $V_0 = V(\lambda)$ listed in Table 8.1 where there is either no regular orbit of $G$ on $V_0$, or a different method of proof was used.

We now consider each of the modules in Table 8.1 individually.

Proposition 8.2. Let $G$ be almost quasismple with $E(G) \cong \text{SL}_n(q)$. Denote by $V_0$ the natural module for $G$ over $F_q$ and let $V = V_0 \otimes F_{q^k}$ with $k \geq 1$. If $G \leq \Gamma L(V)$ contains no field automorphisms, then $b(G) = [n/k] + c$, where $c = 1$ if $i = (k, n) > 1$ and $G$ contains scalars in $F_{q^k} \setminus F_q^\times$, and $c = 0$ otherwise. If $G$ contains field automorphisms, then $b(G) \leq [n/k] + 1$, with equality if $G$ contains scalars in $F_q^\times \setminus F_{q^k}^\times$ or $\log_q |G| > kn[n/k]$.

Proof. First note that $b(G) \geq \log |G| / \log |V|$, so $b(G) \geq [n/k]$ if $k \geq 2$, since $|G| \geq |\text{SL}_n(q)| > q^{n^2-2}$. Moreover, if $|G| > kn[n/k]$, then $b(G) \geq [n/k] + 1$. Let $e_1, \ldots, e_n$ denote the standard basis of $V_0$ and let $\omega_k$ be a generator of $F_{q^k}^\times$. If $k = 1$, then $\{e_1, \ldots, e_n\}$ is a base for $\text{GL}_n(q)$, and adding $\omega_1 e_1$ to the
does not have eigenvalues of multiplicity \( \ell \). Suppose \( g \in GB \) has projective prime order \( r \). The elements of \( B \) lie in the \( F_q^k \)-span of vectors in the fixed point space \( C_V(g) \) of \( g \). If \( g \) is unipotent, or is semisimple with \( r \mid q-1 \), then this holds if and only if \( g \) fixes \( e_1, \ldots, e_n \), i.e., \( g \) is the identity, contradicting our assumption that \( g \) is non-trivial. So either \( g \) is semisimple and \( r \) divides \( q^k-1 \) but not \( q-1 \), or \( g \) is a field automorphism.

If \( g \) is semisimple, then \( g \) has the form \( ah \), for some \( a \in F_q^k \setminus F_q \) and \( h \in SL_n(q) \). Let \( i \) be the least natural number such that \( r \mid q^i-1 \), and note that \( i \mid k \). If \( g \) fixes \( B \), then each vector contained in \( B \), excluding that where \( j = \lceil n/k \rceil - 1 \), can be written uniquely as a \( F_q^k \) linear combination of \( k/i \) linearly independent vectors in \( V_0 \otimes F_q^\ell \), while the vector with \( j = \lceil n/k \rceil - 1 \) can be written as the \( F_q^k \) linear combination of \( \min\{b,k/i\} \) such vectors, where \( b = n \mod k \). Therefore, the fixed point space \( C_V(g) \) of \( g \) has dimension at least \( \frac{\ell}{n} + \min\{b,k/i\} \). Since \( h \) is defined over \( F_q \), we must have \( b = 0 \), so \( i \mid n \), the fixed point space of \( g \) has dimension \( n/i \), and \( G \) must contain scalars \( aI \) of order \( r \). Moreover, a representative of \( gF(G) \in G/F(G) \) must have eigenvalues \( \{\alpha^{q^i} \mid 0 \leq j \leq k_i-1\} \) for some \( a \in F_q \setminus F_q \) of order \( r \), each of multiplicity \( n/i \). We see that \( e_i \notin C_V(g) \) for all \( 1 \leq i \leq n \), since otherwise \( e_ih \notin V_0 \). Now suppose that \( g \) is a field automorphism, and write \( g = p^\ell \). Then we can write \( g = h\phi \), where \( \phi \) is a standard field automorphism sending the coefficients of an element of \( B \) (with respect to the basis \( e_1,e_2,\ldots,e_n \)) to their \( p^\ell \)-th powers, and \( h \in GL_n(q) \) is the unique matrix acting on each element of \( B \) by:

\[
\begin{align*}
\min\{k,n-jk\} & \quad \sum_{i=1}^{\min\{k,n-jk\}} \omega_k^{i-1}e_{i+jk} \quad \mapsto \quad \sum_{i=1}^{\min\{k,n-jk\}} \omega_k^{\ell-i(i-1)}e_{i+jk}.
\end{align*}
\]

Now, there exists \( e_i \) not fixed by \( h\phi \), since otherwise \( h \) is the identity, and \( g = \phi \) does not fix \( B \) pointwise. So \( e_i \) is not fixed by any element in \( GB \), and therefore, \( B \cup \{e_i\} \) forms a base for \( G \) on \( V \).

Therefore, the base size of \( G \) on \( V \) is less than or equal to \( \lceil n/k \rceil + c \), where \( c = 1 \) if either \( (k,n) > 1 \) and \( G \) contains scalars outside of \( F_q \), or \( G \) contains field automorphisms, and \( c = 0 \) otherwise. It remains to show that if \( (k,n) > 1 \) and \( G \) contains scalars outside of \( F_q \), then \( b(G) = \lceil n/k \rceil + 1 \), i.e., there is no base of size \( n/k \). If there exists a base \( \mathcal{B} \) of size \( n/k \), then we can write each element of the base as a linear combination of a set of vectors \( B \subseteq V \) (over \( F_q \)) with coefficients \( \{\omega_k^i \mid 0 \leq i \leq k - 1\} \), since this is a basis of \( F_q^k \) over \( F_q \). By the proof of Proposition \( 3.4 \), \( B \) forms a base for \( G \) acting on \( V \). But a base for \( G \) acting on \( V \) must be a spanning set, and since \( |B| = n \), it follows that \( B \) is a basis of \( V \) and therefore \( \mathcal{B} \) is conjugate to \( B \) and cannot form a base of \( G \).

**Proposition 8.3.** Let \( G \) be almost quasisimple with \( E(G)/Z(E(G)) \cong PSL_n(q) \), and let \( V = V((p^a+1)\lambda_1) \) or \( V(p^a\lambda_1 + \lambda_1) \) over \( F_q^k \) for \( k > 1 \). Then \( G \) has a regular orbit on \( V \).

**Proof.** Let \( V = V((p^a+1)\lambda_1) \) over \( F_q^k \). We will complete the proof by considering the action of \( G \) on \( \overline{V} = V \otimes F_q \). Note that here \( k \) is an integer, except when \( a = c/2 \). In this case, since \( V \) can be realised over \( F_q^2 \), we instead must also consider \( k = b/2 \) for \( b \geq 3 \). Let \( \overline{W} \) be the natural module for \( G \) over \( F_q \), and suppose \( s \in G \) is semisimple of prime order \( r \). Denote the eigenvalues of \( s \) on \( \overline{W} \) by \( t_1,t_2,\ldots,t_n \), ordered so that the multiplicities \( a_j \) of the \( t_i \) are weakly decreasing. Suppose that \( \{v_i \mid 1 \leq i \leq n\} \) is a basis of eigenvectors of \( s \) on \( \overline{W} \) so that \( v_i \otimes v_j \mid 1 \leq i,j \leq n \) forms a basis of \( \overline{V} \) consisting of eigenvectors for \( s \). Fix \( t \in F_q \). The action of \( s \) on \( \overline{V} \) has eigenvalues of the form \( |s|a_j^2 \) of fixed \( i \), there is at most one choice of \( j \) so that \( t = t_ja_j \). Therefore, by Proposition \( 3.5 \), the codimension of \( C_W(s) \) is at least \( n^2 - \sum a_j^2 \). Now, by \( 3 \) Table B.3], \(|sF(G)|^{GL_n(0)}| < 2^nq^{n^2-\sum a_j^2} \), and this at most \( q^{n^2+2m-2} - \sum a_j^2 \), since \( q \geq 4 \). Therefore,

\[
\frac{k \cdot \text{codim}_{C_V(s)} - \log_q(|sF(G)|^{GL_n(0)})}{(8.1)} \geq k(n^2 - \sum a_j^2) - (n^2 + m/2 - \sum a_j^2) = (k-1)(n^2 - \sum a_j^2) - m/2
\]

If \( k \geq 2 \), then the right hand side of \( (8.1) \) is at least \((k-1)(2n-2) - n/2 \), while if \( k = 3/2 \), and \( s \) does not have eigenvalues of multiplicity \((n-1,1) \) then the right hand side of \( (8.1) \) is at least \( 2n - 5 \).
Now suppose that \( u \in G \) is unipotent of prime order \( p \), and with associated canonical Jordan form \((J_{p^k}, \ldots, J_{p^{k^2}}, J_{p^1})\) on \( \mathbb{W} \). By [13] Lemma 1.3.3], the codimension of \( C_V(u) \) is \( n^2 - \sum_{i<j} i a_{ij} - \sum_i a_i^2 \), while by [3] Table B.3], \( |uF(G)^{PGL_n(q)}| < 2^m q^{n^2-c} \), where \( c = 2 \sum_{i<j} i a_{ij} - \sum_i a_i^2 \) and \( m \) is the number of distinct Jordan block sizes. So

\[
(8.2) \quad k \text{codim}C_V(u) - \log_q(|uF(G)^{PGL_n(q)}|) \geq (k-2)(n^2 - \sum_{i<j} i a_{ij} - \sum_i a_i^2) + n^2 - \frac{m}{2} \sum_i a_i^2
\]

If \( k \geq 2 \), then the right hand side of (8.2) is at least \( n^2 - n/2 - (n-2)^2 \geq 2 \). If \( k = 3/2 \), then \( m \leq \sum_{i<j} i a_{ij} + 1 \) and if \( u \) is not a root element, then the right hand side of (8.2) is at least \( 2n-4 \).

By [12] Lemma 3.7], the number of conjugacy classes in \( PGL_n(q) \) is at most \( q^{n^2} + 5q^{n^2-2} \). Therefore, if \( k \geq 2 \) and \( G \) has no regular orbit on \( \mathbb{V} \),

\[
q^{kn^2} \leq 2(q^{n-1} + 5q^{n-2})(q^{kn^2-(7n/2-6)} + q^{kn^2-(k-1)(2n-2)-n/2}) + 2\log(\log_2 q + 2)q^{kn^2/2+n^2/2}
\]

This gives a contradiction for \( n \geq 3 \) and \( q \geq 4 \), except when \( (n, k, q) = (4, 2, 4) \), where \( E(G)/Z(E(G)) \cong \text{Alt}_8 \), or \((3, 2, q_1)\), where \( q_1 \leq 9 \). For the remaining cases where \( n = 3 \), we compute that no element of projective prime order has an eigenspace of dimension greater than 5, and substituting this in gives a contradiction for the remaining \( q \). When \( n = 2 \), then each element of prime order has eigenspaces of dimension at most 2 on \( \mathbb{V} \), so if \( G \) has no regular orbit on \( \mathbb{V} \) and \( k \geq 2 \),

\[
q^{4k} \leq 2|PGL_2(q)|q^{k(4-2)} + 2\log(\log_2 q + 2)q^{2k+2}
\]

This gives a contradiction for all \( q \geq 4 \), completing the proof. If now \( k = 3/2 \) and \( G \) has no regular orbit on \( \mathbb{V} \), then

\[
q^{kn^2} \leq 2(q^{n-1} + 5q^{n-2})(q^{kn^2-(2n-4)} + q^{kn^2-(2n-5)}) + 2q^{2n-1}q^{3/2(n^2-2n-2)} + q^{2n-1}q^{3/2(n^2-3n+4)} + 2\log(\log_2 q + 2)q^{kn^2/2+n^2/2}
\]

This gives a contradiction for all \( n, q \) except \((n, q) = (5, 4) \) and \( n \leq 4 \) for all \( q \). For these cases, we perform a more detailed analysis of the classes of elements of prime order and again apply Proposition [3, 4] to give the result. The proof for \( V = V(p^l \lambda_1 + \lambda) \) with \( k \geq 2 \) is similar. \( \square \)

**Proposition 8.4.** Let \( G \leq GL(V) \) be almost quasisimple with \( E(G)/Z(E(G)) \cong PSL_n(q) \), and let \( V = V(\lambda_1 + \lambda) \) be the adjoint module for \( E(G) \) over \( F_q \), for \( k > 1 \). Then either \( G \) has a regular orbit on \( V \), or \((n, k) = (3, 2) \), \( G \) contains graph automorphisms and \( b(G) \leq 2 \).

**Proof.** Recall that \( \dim(V) = n^2 - 1 - \epsilon_n \), where \( \epsilon_n = 1 \) if \((n, q) = 1 \) and \( \epsilon_n = 0 \) otherwise. We can consider \( V \) as a composition factor of \( \mathbb{V} = V(\lambda_1) \otimes V(\lambda) \). Now \( \mathbb{V} \) has an \((n^2-1)-\text{dimensional} \) \( F_q^\mathbb{G}_{2}(q) \)-submodule \( V_0 \). When \( \epsilon_n = 0 \), then \( V_0 \) is irreducible, and we set \( V = V_0 \) and have \( \text{dim} C_V(g) \leq \text{dim} C_V^g(g) \) for all \( g \in G \). We will aim to show the same when \( \epsilon_n = 1 \). So assume \( \epsilon_n = 1 \), then \( V_0 \) has a one-dimensional \( F_q \)-vector \( q_{GL_n(q)} \)-submodule \( V_0 = \langle w \rangle \), which is fixed by every element of \( GL_n(q) \), and \( V = V_0/V_0 \). Every element \( g \in GL(V_0) \) preserving \( V_0 \) can be considered as an element of \( GL(V) \) by setting \((v + V_1)g = vg + V_1 \). Conversely, every element of \( GL(V) \) arises this way. Suppose \( g \in GL(V_0) \) is unipotent of prime order \( p \) and let \( v + V_1 \in C_V(g) \). Clearly if \( v \in C_V(g) \), then \( v + V_1 \in C_V^g(g) \), so \( \text{dim} C_V(g) - 1 \leq \text{dim} C_V^g(g) \), since \( g \) fixes \( V_1 \) pointwise. Now suppose \( v + V_1 \in C_V^g(g) \), but \( v \notin C_V(g) \). Then \( v + V_1 \in C_V^g(g) \). So \( v + V_1 \) is fixed by every element of \( GL(V) \) and \( v + V_1 \) is fixed by every element of \( GL(V) \). In particular, \( C_V^g(g) \) is fixed by every element of \( GL(V) \) and \( v + V_1 \) is fixed by every element of \( GL(V) \).

Therefore, recalling that \( g \) fixes \( V_1 \) pointwise, we have \( \text{dim} C_V(g) \leq \text{dim} C_V^g(g) \leq \text{dim} C_V^g(g) \) as required. We now consider \( g \in G \) semisimple of prime order. Recall that we can think of the action of \( G \cap GL_n(q) \) on \( V \) as conjugation on the vector space of \( n \times n \) trace 0 matrices over \( F_q^k \), modulo the subspace \( Z \) generated by the identity matrix. Suppose \( g \in G \) is semisimple of prime order. If \( h + Z \in C_V^g(g) \), then as before \( g^{-1}h + Z = h + Z \), for some \( j \in F_q^k \). Therefore, \( g^{-p}h + Z = h + Z \), since \( g \) has projective prime order coprime to \( p \). So \( \text{dim} C_V(g) \leq \text{dim} C_V^g(g) \leq \text{dim} C_V^g(g) \) as required. Let \( \mathbb{W} \) denote the natural module for \( G \) over \( F_q \). By the proof of Proposition [8, 3] if \( s \in G \) is semisimple, then as long as \( s \) does not have eigenvalue multiplicities \((n-1, 1) \) or \((n-2, 2) \) on \( \mathbb{W} \), then

\[
k \text{codim}C_V(s) - \log_q(|sF(G)^{PGL_n(q)}|) \geq (k-1)(4n-6) - n/2 - k(1 + \epsilon_n)
\]

If \( s \in G \) has eigenvalue multiplicities \((n-1, 1) \) or \((n-2, 2) \) on \( \mathbb{W} \), then the lower bound is replaced with \((k-1)(2n-2) - 1 - k(1 + \epsilon_n) \) or \((k-1)(4n-8) - 1 - k(1 + \epsilon_n) \) respectively. The number of
conjugacy classes of elements of prime order in $\text{PGL}_n(q)$ that have eigenvalue multiplicities $(n-1,1)$ or $(n-2,2)$ on $\overline{W}$ is at most $q-1$ in each case, as long as $n \geq 5$. Moreover, if $u \in G$ is unipotent, then by the proof of Proposition 5.3

$$k \text{ codim} C_V(u) - \log_q (|uF(G)^{\text{PGL}_n(q)}|) \geq (k-2)(3n-4) + \frac{7}{2}n - 6 - k(1 + \epsilon_n).$$

By Lemma 3.7, the number of conjugacy classes in $\text{PGL}_n(q)$ is at most $q^n - 1 + 5q^{n-2}$. We must also consider graph and graph-field automorphisms here. We find that a graph automorphism $\tau$ has dim $C_V(\tau) \leq \frac{1}{2}(n^2 - 1 + \epsilon_n) + \frac{1}{2}(2[n/2] - 1 - \epsilon_n)$.

Therefore, if $G$ has no regular orbit on $V$, letting $d = k(n^2 - 1 - \epsilon_n)$, we have

$$q^d \leq 2(q^{n-1} + 5q^{n-2})(q^{d-(k-2)(3n-4)-(7n/2-6)+k(1+\epsilon_n)} + q^{d-(k-1)(4n-6)+n/2+k(1+\epsilon_n)})$$

$$2(q-1)(q^{d-(k-1)(2n-2)+1+k(1+\epsilon_n)} + q^{d-(k-1)(4n-8)+1+k(1+\epsilon_n)})$$

$$+ 2 \log(q^{d/2} + 2)q^{n^2/2+d/2} + 4q^{(n^2+n)/2-1}q^{d/2} + \frac{k(2[n/2]-1-\epsilon_n)}{2} + 2(q^{n^2-1}q^{d/2})$$

This inequality is false except for some cases with $k = 3$ and $n = 3$, as well as $k = 2$ and $n = 3, 4, 5$. For the remaining cases with $n = 4, 5$, we either replace $q^n - 1 + 5q^{n-2}$ with a tighter upper bound for the number of $\text{PGL}_n(q)$-classes of elements of prime order to obtain a contradiction, or we use the precise counts of elements with eigenvalue multiplicities $(n-1,1)$ or $(n-2,2)$ on $\overline{W}$. So suppose $n = 3$. By Proposition 5.3 and the proof of Proposition 5.13, if $G$ has no regular orbit on $V$, then letting $d = k(8-\varepsilon_3)$, we have

$$q^d \leq (q-1)(\frac{|GL_3(q)|}{|GL_2(q)||GL_1(q)|}(q^{4k} + q^{4k-4k}) + \frac{|GL_3(q)|}{q^3|GL_2(q)||GL_1(q)|}q^{4k} + \frac{|GL_3(q)|}{q^3|GL_1(q)|}q^{3k})$$

$$+ \frac{1}{2} \left( \frac{q-1}{2} + \frac{q+1}{2} + \frac{q^2+q+1}{3} \right)(\frac{|GL_3(q)|}{|GL_1(q)|}q^{2k} + q^{(2-\varepsilon_3)k} + 2 \log(q^{d/2} + 2)q^{n^2/2+d/2}$$

$$+ 4q^{n^2+n-1}q^{d/2} + \frac{k(2[n/2]-1-\epsilon_n)}{2} + 4q^{n^2-1}q^{d/2}$$

This gives a contradiction for the remaining cases with $k = 3$. When $k = 2$, removing the terms that account for graph automorphisms gives a contradiction for all $q$ except $q = 2$ when $\varepsilon_3 = 0$, and $q \leq 13$ when $\varepsilon_3 = 1$. For these remaining cases where $G$ does not contain graph automorphisms, we confirm the existence of a regular orbit of $G$ on $V$ using GAP. When $G$ contains graph automorphisms, we do not achieve a contradiction for $k = 2$, so $b(G) \leq 2$ here by Lemma 5.4.

**Proposition 8.5.** Let $G$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_n(q)$, and let $V = V(2\lambda_1)$ be the symmetric square for $E(G)$ over $\mathbb{F}_q$. If $k \geq 3$, then $G$ has a regular orbit on $V$. If instead $k = 2$, then $b(G) \leq 2$.

**Proof.** We will aim to apply Proposition 6.11. Let $d = k(n+1)$). We will again proceed by considering the action of $G$ on $\overline{V} = V \otimes \mathbb{F}_q$. Suppose $s \in G$ is semisimple of prime order $r$ in $G$, and write the eigenvalues of $s$ on the natural module $W$ for $G$ as $t_1, t_2, \ldots, t_m$, ordered so that the multiplicities $a_i$ of the $t_i$ are weakly decreasing. Suppose that $\{v_i : 1 \leq i \leq n\}$ is a basis of eigenvectors of $s$ on the natural module $\overline{V}$ for $G$ over $\mathbb{F}_q$ so that $\{v_i v_j : 1 \leq i \leq j \leq n\}$ forms a basis of eigenvectors for $s$. Fix $t \in \mathbb{F}_q$. Now, $V$ is isomorphic to the symmetric square of $W$, so by Proposition 3.8

$$\text{dim} V_i(s) \leq \frac{1}{2} \sum a_i^2 + a_1.$$ Therefore, we have

$$k \text{ codim} V_i(s) - \log_q (|s^{\text{PGL}_n(q)}|) \geq k \left( \frac{n+1}{2} - \frac{1}{2} \sum a_i^2 - a_1 \right) - \left( n^2 - \sum a_i^2 + m/2 + \delta_q, 3 \frac{m}{6} \right)$$

$$= \left( \frac{k}{2} - 1 \right) \left( n^2 - \sum a_i^2 \right) + \frac{kn}{2} - a_1 - m/2 - \delta_q, 3 \frac{m}{6}$$

This is at least $(k/2 - 1)(3n - 2) - \delta_q, 3 \frac{m}{6}$, since $m \leq n - a_1 + 1$ and $n \geq a_1 + 1$.

Now let $u \in G$ be unipotent of prime order $p$, and with associated canonical Jordan form $J_p^{v_p}, \ldots, J_1^{v_1}$ on the natural module of $G$. Suppose that $u$ does not have Jordan blocks $(J_2, J_{1}^{v_{1} - 2})$ or $(J_2, J_{1}^{v_{1} - 4})$; we will deal with these elements separately. Now,

$$S^2(J_2^{v_2}, \ldots, J_1^{v_1}) = \sum a_i S^2(J_i) + \sum_{i<j} a_i a_j J_i \otimes J_j + \sum \left( \frac{a_i}{2} \right) J_i \otimes J_i.$$
By \([\text{[18]}\text{ Lemmas 1.3.3, 1.3.4}]\), \(\dim C_V(u) \leq \sum a_i [i/2] + \sum_{i<j} ia_i a_j + \sum \binom{a_i}{2} i\). Therefore, for any \(u\) and \(q\),

\[
k \cdot \text{codim} C_V(u) - \log_q(|u|^{\text{PGL}_n(q)}) \geq \kappa \left( \binom{n+1}{2} - \sum_{i \text{ odd}} \frac{a_i}{2} - \sum_{i<j} \frac{a_i a_j - \frac{1}{2} \sum i a_i^2}{2} \right) - \left( n^2 - 2 \sum_{i<j} ia_i a_j - \sum \binom{a_i}{2} + m \right)
\]

\[
\geq \left( \frac{k}{2} - 1 \right) \left( n^2 - 2 \sum_{i<j} ia_i a_j - \sum \binom{a_i}{2} \right) + k \left( \frac{n}{2} - \sum_{i \text{ odd}} \frac{a_i}{2} \right) - m,
\]

which is at least \((k/2 - 1)(4n - 6)\) since \(m < 3\sqrt{n}/2\). Also note that if \(u\) has Jordan form \((J_2, J_1^{n-2})\) or \((J_2^2, J_1^{-1})\), then \(\dim C_V(u) \leq d - n\) and \(d - (n - 2)\) respectively. The number of such elements is given in the proof of Proposition \([\text{3.4}]\). If \(G\) has no regular orbit on \(V\), then

\[
q^d \leq 2(q^{n-1} + 5q^{n-2})(q^{d - \frac{d}{2} - \frac{1}{2}} - \frac{1}{2} \sum \binom{a_i}{2}) + 2 \log(\log_q q + 2) q^{n^2/2 + d/2} + q^{2n - 1} q^{d - k - n} + 4q^{4n - 8} q^{d - (k - 2)}.
\]

This is a contradiction for all \((k, n, q)\) with \(k \geq 3\), \(n \geq 3\) and \(q \geq 3\), except for \((n, q) = (3, 3), (3, 5), (4, 3)\) and \((5, 3)\). We confirm using GAP \([\text{13}]\) that \(G\) has a regular orbit on \(V\) in each of the remaining cases. When \(n = 2\), we remove the \(4q^{4n - 8} q^{d - (k - 2)}\) term, and find that there is a regular orbit of \(G\) on \(V\) for \(k \geq 3\) and \(q \geq 7\). Finally, if \(k = 2\), then \(b(G) \leq 2\) by Proposition \([\text{3.3}]\). \(\square\)

**Proposition 8.6.** Let \(G\) be almost quasisimple with \(E(G)/Z(E(G)) \cong \text{PSL}_n(q)\) with \(n \geq 4\), and let \(V = V(\lambda_2)\) be the exterior square for \(E(G)\) over \(\mathbb{F}_q^\perp\). Then \(G\) has a regular orbit on \(V\) unless one of the following holds.

(i) \((n, k) = (4, 4)\) and \(b(G) \leq 2\),

(ii) \(k = 3\) with \(n \in [4, 6]\) and \(b(G) \leq 2\),

(iii) \(k = 2\) with \(n \geq 5\) and \(b(G) \leq 2\), or

(iv) \((n, k) = (4, 2)\) and \(b(G) \leq 3\).

**Proof.** We again consider the action of \(G\) on \(V = V \otimes \mathbb{F}_q^\perp\). Assume that \(k \geq 3\). Suppose \(s \in G\) is semisimple of prime order \(r \in \mathbb{F}_q\), with eigenvalues \(t_1, t_2, \ldots, t_m\) on the natural module \(V\) of \(G\) over \(\mathbb{F}_q\), ordered so that their multiplicities \(a_i\) are weakly decreasing. Also suppose that \((t_1, t_2) \neq (n - 1, 1)\) or \((n - 2, 2)\); we will deal with these cases separately. Now, \(V\) is isomorphic to the exterior square of \(W\), so by Proposition \([\text{3.6}]\) for \(t \in \mathbb{F}_q\), we have \(\dim V_t(s) = \frac{1}{2} \sum a_i^2\). Therefore,

\[
k \cdot \text{codim} V_t(s) - \log_q(|s|^{\text{PGL}_n(q)}) \geq k \left( \binom{n}{2} - \frac{1}{2} \sum a_i^2 \right) - \left( n^2 - \sum \binom{a_i}{2} + m \right)
\]

\[
> \left( \frac{k}{2} - 1 \right) \left( n^2 - \sum \binom{a_i}{2} \right) \geq \left( \frac{k}{2} - 1 \right) \left( 3n - 6 \right),
\]

since by assumption, \(n^2 - \sum a_i^2 \geq n^2 - (n - 2)^2 - 2 = 4n - 6\). If \((t_1, t_2) = (n - 1, 1)\) or \((n - 2, 2)\), then we note that by the \(\Psi\)-net tables for \(V(\lambda_2)\) in Table \([\text{6.3}]\) we have \(V_t(s) \leq n - 2\) and \(2n - 4\) respectively. Now suppose \(u \in G\) is unipotent of prime order \(p\), and with associated canonical Jordan form \((J_p^{a_p}, \ldots, J_2^{a_2}, J_1^{a_1})\) on the natural module of \(G\). Now,

\[
\wedge^2((J_p^{a_p}, \ldots, J_2^{a_2}, J_1^{a_1})) = \sum a_i \wedge^2 (J_i) + \sum a_i a_j J_i \otimes J_j + \sum \binom{a_j}{2} J_i \otimes J_i.
\]

By \([\text{18]}\text{ Lemmas 1.3.3, 1.3.4}]\), \(\dim C_V(u) \leq \sum a_i [i/2] + \sum_{i<j} ia_i a_j + \sum \binom{a_i}{2} i\). Therefore,

\[
k \cdot \text{codim} C_V(u) - \log_q(|u|^{\text{PGL}_n(q)}) \geq \kappa \left( \binom{n}{2} + \sum_{i \text{ odd}} \frac{a_i}{2} - \sum_{i<j} \frac{a_i a_j - \frac{1}{2} \sum i a_i^2}{2} \right) - \left( n^2 - 2 \sum_{i<j} ia_i a_j - \sum \binom{a_i}{2} + m \right)
\]

\[
\geq \left( \frac{k}{2} - 1 \right) \left( n^2 - 2 \sum_{i<j} ia_i a_j - \sum \binom{a_i}{2} \right) + k \left( \sum_{i \text{ odd}} \frac{a_i}{2} - \frac{kn}{2} \right) - m.
\]

Now, since \(k \geq 3\), we have \(m \leq \sqrt{n} + k \sum_{i \text{ odd}} \frac{a_i}{2}\). Moreover, \(n^2 - 2 \sum_{i<j} ia_i a_j - \sum \binom{a_i}{2} \geq 6n\), unless \(u\) has an associated partition given in Table \([\text{8.2}]\) which we treat separately. Let \(S\) denote a set of
unipotent conjugacy class representatives of prime order with corresponding partition in Table 8.2.
Excluding classes with representatives in $S$, we have

$$k \text{ codim}C_V(u) - \log_q(|u^{PGL_n(q)}|) \geq (\frac{5k}{2} - 6)n - \sqrt{n}.$$ 

| Partition | $n^2 - 2 \sum_{i<j} ia_i a_j - \sum i a_i^2$ | codim$C_V(u)$ |
|-----------|---------------------------------|----------------|
| $(6, 1^{n-6})$ | $10n - 30$ | $5n - 18$ |
| $(5, 2, 1^{n-7})$ | $10n - 32$ | $5n - 19$ |
| $(5, 1^{n-5})$ | $8n - 20$ | $4n - 12$ |
| $(4, 3, 1^{n-7})$ | $10n - 34$ | $5n - 20$ |
| $(4, 2^2, 1^{n-8})$ | $10n - 36$ | $5n - 22$ |
| $(4, 2, 1^{n-6})$ | $8n - 22$ | $4n - 14$ |
| $(4, 1^8 - 4)$ | $6n - 12$ | $3n - 8$ |
| $(3^2, 2, 1^{n-8})$ | $10n - 38$ | $5n - 22$ |
| $(3^2, 1^{n-6})$ | $8n - 24$ | $4n - 14$ |
| $(3, 2^3, 1^{n-9})$ | $10n - 42$ | $5n - 25$ |
| $(3, 2^2, 1^{n-7})$ | $8n - 26$ | $4n - 16$ |
| $(3, 2, 1^{n-5})$ | $6n - 14$ | $3n - 9$ |
| $(3, 1^{n-3})$ | $4n - 6$ | $2n - 4$ |
| $(2^3, 1^{n-10})$ | $10n - 50$ | $5n - 30$ |
| $(2^4, 1^{n-8})$ | $8n - 32$ | $4n - 20$ |
| $(2^3, 1^{n-6})$ | $6n - 18$ | $3n - 12$ |
| $(2^2, 1^{n-4})$ | $4n - 8$ | $2n - 6$ |
| $(2, 1^{n-2})$ | $2n - 2$ | $n - 2$ |

Table 8.2. The centraliser dimensions and fixed point space codimensions of some unipotent elements $u$ acting on $V = V(\lambda_2)$.  

Therefore, if $G$ has no regular orbit on $V$,

$$q^d \leq 2(q^{n-1} + 5q^{n-2})(q^{d-(k/2)-1}(3n-6) + q^{d-(\frac{5k}{2}-6)n-\sqrt{n}) + 2 \log(\log_2 q + 2) q^{n^2/2+d/2}$$

$$+ 2q^{2n-1} q^{d-k(n-2)} + 2q^{n-7} q^{d-2k(n-2)} + \sum_{u \in S} q^{n^2-2i a_i a_j - \sum i a_i^2} q^{d-k \text{codim}C_V(u)}$$

where $d = \binom{n}{2}$ and $t_u$ is the number of parts of distinct sizes in the partition corresponding to $u$.

This gives a contradiction for all $n, q$ except for $4 \leq n \leq 6$, as well as $k = 4$ with $n = 7$ and $q \geq 2$, and also $k = 3$ with $n = 7, 8$ with $q \geq 2$, and $(n, 2)$ for $n \leq 12$. For the remaining cases, with $k = 4$ and $n \geq 5$, or $k = 3$ and $n \geq 7$, we remove terms in the inequality based on elements of $S$ that do not exist for a given $n$. This leaves us with a smaller list of $(n, q)$ that satisfy the inequality. All of these have $q \leq 5$, and by either computing the number of elements of each prime order in $G/F(G)$ precisely or constructing the module explicitly using GAP [13], we determine that there is a regular orbit. Finally, if $k = 2$, then $b(G) = 2$ by Lemma [3,4] for $n \geq 5$, and $b(G) \leq 3$ if $n = 4$.  

$\square$

**Proposition 8.7.** Let $G$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_2(q)$, and let $V = V(3\lambda_1)$ or $V(4\lambda_1)$ over $\mathbb{F}_{q^k}$ for $k > 1$. Then $G$ has a regular orbit on $V$.

**Proof.** First let $V = V(3\lambda_1)$ over $\mathbb{F}_{q^k}$ for $k > 1$. Then by Proposition [2.2] if $G$ has no regular orbit on $V$,

$$q^{4k} \leq 2|\text{PGL}_2(q)|q^{2k} + 2 \log(\log_2 q + 2) q^{2+2k}$$

This gives a contradiction for $k \geq 2$ and $q \geq 7$. Now let $V = V(4\lambda_1)$ over $\mathbb{F}_{q^k}$. If $G$ has no regular orbit on $V$, then

$$q^{5k} \leq 2|\text{PGL}_2(q)|q^{2k} + 4(q^2 + q)q^{3k} + 2 \log(\log_2 q + 2) q^{2+5k/2}$$

This gives a contradiction for $q \geq 7$ and the result follows.  

$\square$
Proposition 8.8. Let $G$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_\lambda(q)$ for $n \in \{6, 9\}$, and let $V = V(\lambda_\lambda)$ over $\mathbb{F}_q$, for $k > 1$. Then $G$ has a regular orbit on $V$, unless possibly when either $(n, k) = (6, 3)$ and $G$ contains a graph automorphism, or $(n, k) = (6, 2)$, in which cases $b(G) \leq 2$.

Proof. Suppose $n = 9$. If $G$ has no regular orbit on $V$, then by the proof of Proposition 5.14
\[
q^{84k} \leq 2|\text{PGL}_9(q)|q^{84k-42k} + 2 \times 2q^{81/2+3/2}q^{84k-32k} + (2q^{2n-1} + 4 \left(\frac{q^{81/2+2} - 1}{q^2 - 1}\right)^2)q^{84-21k} + 2 \log(\log_2 q + 2)q^{81/2+42k}
\]
This gives a contradiction except for $q = 2$, where replacing $4 \left(\frac{q^{81/2+2} - 1}{q^2 - 1}\right)$ with the precise number of unipotent elements of order 2 in $\text{PGL}_9(2)$ gives a contradiction. Now suppose that $n = 6$. Here there are graph automorphisms $\tau$ to consider. By Proposition 2.3, $\dim V(\tau) \leq 14$. Therefore, if $G$ contains a graph automorphism and has no regular orbit on $V$, then
\[
q^{20k} \leq 2|\text{PGL}_6(q)|q^{k(20-12)} + 2(4q^{20} + 4 \left(\frac{q^{20} - 1}{q^2 - 1}\right))q^{k(20-8)} + 3q^{11}q^{k(20-6)} + 2 \log(\log_2 q + 2)q^{18+10k} + (2q^{20} + 2q^{35/2})q^{14k}.
\]
This is a contradiction for $k \geq 4$ and $q \geq 2$. So $G$ has a regular orbit on $V$, and moreover $b(G) \leq 2$ for $k = 2, 3$ by Proposition 3.4 since $b(G) = 1$ for $k = 4, 6$ respectively. A similar argument shows that if $G$ does not contain a graph automorphism, then $G$ has a regular orbit on $V$ when $k \geq 3$ and $q \geq 2$, and $b(G) = 2$ when $k = 2$. \hfill $\square$

Proposition 8.9. Let $G$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_3(q)$, and let $V = V(3\lambda_\lambda)$ over $\mathbb{F}_q$, for $k > 1$. Then $G$ has a regular orbit on $V$.

Proof. By the proof of Proposition 5.15 if $G$ has no regular orbit on $V$, then
\[
q^{20k} \leq 2|\text{PGL}_4(q)|q^{k(20-8)} + 4(q^5 + q^4)q^{20k-6k} + 2(q^5 + q^4)q^{20k-4k} + 2 \log(\log_2 q + 2)q^{9/2+10k}
\]
This is a contradiction for $k \geq 2$ and $q \geq 5$. \hfill $\square$

Proposition 8.10. Let $G$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_8(q)$, and let $V = V(2\lambda_\lambda)$ over $\mathbb{F}_q$, for $k > 1$. Then $G$ has a regular orbit on $V$.

Proof. By the proof of Proposition 5.16 if $G$ has no regular orbit on $V$ over $\mathbb{F}_q$, for $k > 1$, then
\[
q^d \leq 2|\text{PGL}_n(q)|q^{d-2k(n-1)} + 2(4q^{n^2/2+2})q^{d-28k} + 2(q^{2n-1} + q^{2n-1})q^{d-k(n-2)} + 4(q^{n+1} - 1)q^{n+1} + 2 \log(\log_2 q + 2)q^{n^2/2+d/2},
\]
where $d = k(n^2)$. This gives a contradiction for $k \geq 2$ and $q \geq 2$ so the result follows. \hfill $\square$

Proposition 8.11. Let $G$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_4(q)$, and let $V = V(2\lambda_\lambda)$ over $\mathbb{F}_q$, for $k > 1$. Then $G$ has a regular orbit on $V$.

Proof. This module is preserved by graph automorphisms, and there are 9 weights fixed by a graph automorphism. By the proof of Proposition 5.10 if $G$ has no regular orbit on $V = V(2\lambda_\lambda)$ over $\mathbb{F}_q$, then letting $d = k(20 - \epsilon_3)$, we have
\[
q^d \leq 2|\text{PGL}_4(q)|q^{d-10k} + 4(q^9 + q^8)q^{d-(8-\epsilon_3)k} + 2 \log(\log_2 q + 2)q^{8+k/2} + (2q^9 + 2q^{15/2})q^{9k}
\]
This gives a contradiction for $k \geq 2$ and $q \geq 5$. \hfill $\square$

We next take care of some absolutely irreducible modules from Section 7 over extension fields. Here we replace $q_0$ with $q$ in our notation to be consistent with Section 7.

Proposition 8.12. Let $V = V_\lambda(q)$ be a finite dimensional vector space over $\mathbb{F}_q$. Let $G \leq \Gamma L(V)$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_n(q^\ell)$ for $n, \ell$ in Table 8.3. Also suppose that the restriction of $V$ to $E(G)$ is an absolutely irreducible module of highest weight $\lambda$ corresponding to $n, \ell$ in Table 8.3 Then $G$ has a regular orbit on $V \otimes \mathbb{F}_q$ for all integers $t > 1$. 


Therefore, if $n \neq 2.4$, then $q^t \geq 1$. This inequality is false for $n = 3$. We have $d = 27$. By Proposition 8.3, a semisimple element $s \in G$ has $\dim C_V(s) \leq 15$ or $9$ if $s$ has eigenspaces of dimensions $2, 1$ and $1, 1$ respectively on the natural module of $G$. The number of prime order semisimple elements in $\text{PGL}_3(q)$ whose preimages in $\text{GL}_3(q^3)$ have eigenspace dimensions $2, 1$ on the natural module for $G$ is at most $(q^3 - 1)q^6(q^4 + q^3 + 1)$. The number of prime order unipotent elements in $\text{PGL}_3(q^3)$ is at most $q^{18}$, and by [2, Corollary 1], we have $\dim C_V(u) \leq 14$ for all unipotent projective prime order $u \in G$. Therefore, if $G$ has no regular orbit on $V \otimes \mathbb{F}_{q^t}$, then $q^{27t} \leq 2|\text{PGL}_3(q^3)|q^{9t} + 2(q^3 - 1)q^6(q^6 + q^3 + 1)q^{15t} + q^{18}q^{14t} + 2\log(\log_2 q + 2)q^{27t} + \frac{1}{2}(27t + 27^t)t$.

This inequality is false for $t \geq 2$ and $q \geq 2$, so $G$ has a regular orbit on $V \otimes \mathbb{F}_{q^t}$. Now suppose that $n = 2$. We compute that all semisimple and unipotent elements $x$ of projective prime order $d$ in $G$ have $\dim C_V(x) \leq 4$, with $\dim C_V(x) \leq 3$ if the projective prime order of $x$ is at least $5$. We also compute that the number of odd prime order elements in $\text{PGL}_2(q^3)$ is at most $q^4(q^3 + 1)(q^3 + 2q - 1)$, that the number of involutions of $\text{PGL}_2(q^3)$ is $q^6$, and the number of unipotent elements is $q^9 - 1$. Therefore, if $G$ has no regular orbit on $V \otimes \mathbb{F}_{q^2}$, then by Propositions 2.3 and the proof of Proposition 2.4, $q^{8t}$ is less than or equal to $2q^3(q^3 + 1)(q^3 + q^2 + 2q - 1)q^{2t} + 4(q^7 + q^4)q^{14t} + 2q^4q^{14t} + 2q^6q^{14t} + 2\log(\log_2 q^3 + 2)q^{14t} + 2\log(\log_2 q^3 + 2)q^{14t}$, which is false, except for $t = 2$ and $q \leq 7$. When $t = 2$ and $3 \leq q \leq 7$, we delete terms from the inequality as appropriate and substitute in the precise number of elements of order $3$ and elements of order at least $5$ in $\text{PGL}_2(q^3)$ into the inequality. This gives us a contradiction in each case, so $G$ has a regular orbit on $V \otimes \mathbb{F}_{q^2}$. When $t = 2$, we determine that there is a regular orbit of $G$ on $V \otimes \mathbb{F}_{q^2}$ by explicit construction in GAP 13.

Now let $\lambda = (p^e/4 + p^e/4 + p^e/4 + 1)\lambda_1$, $c = 4$ and $n = 2$. We have $d = 16$ and by the proof of Proposition 7.5 if $G$ has no regular orbit on $V \otimes \mathbb{F}_{q^t}$, then $q^{16t} \leq 2|\text{PGL}_2(q^4)|q^{8t} + 2\log(\log_2 q^4 + 2)q^{8t}$, which gives a contradiction for $t \geq 2$ and $q \geq 2$, and so the result follows.

Now suppose that $\lambda = 2(p^e/2 + 1)\lambda_1$, with $c = 2$ and $n = 2$. Here $d = 9$ and $q$ is odd. By Propositions 2.3 and 3.4, if $G$ has no regular orbit on $V \otimes \mathbb{F}_{q^t}$, then $q^{9t} \leq 2|\text{PGL}_2(q^2)|q^{\frac{2}{3}t} + 4(q^4 + q^2)q^{\frac{2}{3}t}$, which gives a contradiction for $t \geq 3$ and $q \geq 3$. Lastly, suppose that $n = 4$, $c = 2$ and $\lambda = (p^e/2 + 1)\lambda_2$, so $d = 36$. We must consider graph automorphisms $\tau$ here, and determine that $\dim C_V(\tau) \leq 24$. Therefore if $G$ has no regular orbit on $V \otimes \mathbb{F}_{q^t}$, then by the proof of Proposition 7.4, $q^{36t} \leq 2\left[2q^{14} + 2q^{30} + 4\left(\frac{4}{q^6 - 1}\right)\right]q^{24t} + 2|\text{PGL}_4(q^2)|q^{12t} + 2\log(\log_2 q^2 + 2)q^{16t} + 4(q^{18} + q^{14})q^{24t}$.

This gives a contradiction for $t \geq 2$ and $q \geq 2$, so $G$ has a regular orbit in all cases.

We have now completed the proof of Theorem 8.1. Finally, many of the base size results in this section for $V = V_0 \otimes \mathbb{F}_{q^k}$ give upper bounds on the base size of $V_0$ by Proposition 8.1. The next proposition finishes the proof of Theorem 1.1 by completing Table 1.1.

**Proposition 8.13.** Let $V = V_d(q)$ be a $d$-dimensional vector space over $\mathbb{F}_q$, and let $G \leq \Gamma L(V)$ be almost quasisimple with $E(G)/Z(E(G)) \cong \text{PSL}_n(q)$. Further suppose that the restriction of $V = V(\lambda)$ to
to $E(G)$ is an absolutely irreducible module of highest weight $\lambda$, with $\lambda$ (up to quasiequivalence) in Table 8.4. Let $\delta = 1$ if $G$ contains a graph automorphism, and zero otherwise. Then $b(G)$ is given in Table 8.4.

$$
\begin{array}{|c|c|c|c|}
\hline
\lambda & n & b(G) & \text{Reference} \\
\hline
2\lambda_1 & [2, \infty) & 2 \leq b(G) \leq 3 & \text{Proposition 8.5} \\
\lambda_2 & [7, \infty) & 3 \leq b(G) \leq 4 & \text{Proposition 8.6} \\
 & [5, 6] & 3 \leq b(G) \leq 5 & \text{Proposition 8.6} \\
3\lambda_1, p \geq 5 & 2 & 2 & \text{Proposition 8.7} \\
 & 3 & 1 \leq b(G) \leq 2 & \text{Proposition 8.7} \\
\lambda_3 & 6 & 2 \leq b(G) \leq 3 + \delta & \text{Proposition 8.8} \\
 & 9 & 1 \leq b(G) \leq 2 & \text{Proposition 8.8} \\
\lambda_4 & 8 & 1 \leq b(G) \leq 2 & \text{Proposition 8.10} \\
2\lambda_2, p \geq 3 & 4 & 1 \leq b(G) \leq 2 & \text{Proposition 8.11} \\
(p^a + 1)\lambda_1 & [2, \infty) & 2 & \text{Proposition 8.3} \\
\lambda_1 + p^a\lambda_{n-1} & [3, \infty) & 2 & \text{Proposition 8.3} \\
\hline
\end{array}
$$

Table 8.4. Some base sizes of $F_q G$-modules $V(\lambda)$.

Proof. This follows from Proposition 8.4 and each of the references given in Table 8.4.

This concludes the proof of Theorem 1.1.

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DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE, LONDON SW7 2BZ, UK
E-mail address: m.lee16@imperial.ac.uk