Domains of holomorphy of generating functions of Pólya frequency sequences of finite order.

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Abstract: A domain $G \subset \mathbb{C}$ is the domain of holomorphy of the generating function of a Pólya frequency sequence of order $r$ if and only if it satisfies the following conditions: (A) $G$ contains the point $z = 0$, (B) $G$ is symmetric with respect to the real axis, (C) $T = \text{dist}(0, \partial G) \in \partial G$.

1 Introduction.

The Pólya frequency sequences, also called multiply positive sequences, were first introduced by Fekete in 1912 (see [4]). They were studied in detail by Karlin (see [5]).

The class of all Pólya frequency sequences of order $r \in \mathbb{N} \cup \{\infty\}$ ($r$-times positive) is denoted by $PF_r$ and consists of the sequences $\{c_k\}_{k=0}^{\infty}$ such that all minors of order $\leq r$ (all minors if $r = \infty$) of the infinite matrix

$$
\begin{pmatrix}
  c_0 & c_1 & c_2 & c_3 & \ldots \\
  0 & c_0 & c_1 & c_2 & \ldots \\
  0 & 0 & c_0 & c_1 & \ldots \\
  0 & 0 & 0 & c_0 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

are non-negative. The class of corresponding generating functions

$$
f(z) = \sum_{k=0}^{\infty} c_k z^k
$$
is also denoted by $PF_r$. The radius of convergence of a $PF_r$ generating function ($PF_r$ g.f.) is positive provided $r \geq 2$ ([3], p.394). Further we will suppose, without loss of generality, that $c_0 = 1$.

The class $PF_\infty$ was completely described in [1] (see also [4], p. 412):

**Theorem [1]:** The class $PF_\infty$ is formed by the functions

$$f(z) = e^{\gamma z} \prod_{k=1}^{\infty} \frac{(1 + \alpha_k z)}{(1 - \beta_k z)},$$

where $\gamma \geq 0, \alpha_k \geq 0, \beta_k \geq 0, \sum (\alpha_k + \beta_k) < \infty$.

In 1955, I.J. Schoenberg set up the problem of characterizing the classes $PF_r, r \in \mathbb{N}$. Some results were obtained that showed essential differences between the properties of $PF_\infty$ g.f. and those of $PF_r$ g.f. with $r \in \mathbb{N}$ (see [7, 6] and [2]).

**2 Statement of results.**

This paper deals with the study of $PF_r$ domains of holomorphy with $r \in \mathbb{N}$.

A domain $G \subset \mathbb{C}$ is called a $PF_r$ domain of holomorphy if there exists a $PF_r$ g.f. which is analytic in $G$ and admits no analytic continuation across the boundary of $G$.

It follows from Theorem [1] that if $G$ is a $PF_\infty$ domain of holomorphy, then $\mathbb{C}\setminus G$ is at most a countable set of points $\{p_k\}$ on the positive ray such that $\sum(1/p_k) < \infty$ (the points cannot be "too close" to each other).

The situation with the $PF_r$ domains of holomorphy for which $r \in \mathbb{N}$, is quite different. They can be much more complicated as the main result of [3] shows:

**Theorem [3]:** Let $E$ be a closed set in $\mathbb{C}$, satisfying the conditions: (i) $E$ is symmetric with respect to the real axis, (ii) $E \cap \{z : |z| \leq 1\} = \emptyset$. For any $r \in \mathbb{N}$, there exists a function $f(z)$ such that: (i) $f(z) \in PF_r$, (ii) the set of all singularities of $f(z)$ coincides with $E \cup \{1\}$.
From the theorem above, we obtain conditions, which, taken together, are sufficient for a domain $G$ to be a $PF_r$ domain of holomorphy with $r \in \mathbb{N}$:

(I) $G$ contains the point $z = 0$;

(II) $G$ is symmetric with respect to the real axis;

(III) Let $E$ be the set from Theorem 3. $\partial G = \{T\} \cup \partial E$ with $T \in \mathbb{R}$, $T > 0$ and $E \cap \{z : |z| \leq T\} = \emptyset$.

On the other hand, it is not difficult to see that if $G$ is the domain of holomorphy of the $PF_r$ g.f. $f(z) = \sum_{k=0}^{\infty} c_k z^k, 2 \leq r < \infty$, then the following conditions must be satisfied:

(A) $G$ contains the point $z = 0$ (this condition is assured by $f(z) \in PF_r \subset PF_2$);

(B) $G$ is symmetric with respect to the real axis (since $c_k \in \mathbb{R}, k = 0, 1, 2, \ldots$);

(C) $T = \text{dist}(0, \partial G) \in \partial G$ (by the well-known Pringsheim’s theorem on singularities of power series with non-negative coefficients).

The class of domains satisfying (A)-(C) is much larger than the class satisfying (I)-(III). The question of the exact description of all $PF_r$ domains of holomorphy arises.

The following result shows that condition (III) is not necessary, since $\partial G$ can contain other points of $\{z : |z| = T\}$ besides $T$.

**Theorem 1.** Let $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be a function with real and bounded Taylor coefficients, i.e.

$$|b_k| < C, \ k = 0, 1, 2, \ldots,$$  \hspace{1cm} (1)

for some constant $C = C(g), C > 0$. Then for any $r \in \mathbb{N}$ there exists a $\varepsilon > 0$ such that the function

$$f_\varepsilon(z) = \frac{1}{(1 - z)^{r^2}} + \varepsilon g(z)$$
is a PF_r g.f.

The function \( h(z) = \sum_{k=0}^{\infty} z^k \) satisfies (1) so for \( g = h \) the set \( \{ z : |z| = 1 \} \) coincides with the singularity set of \( f_{\varepsilon} \), i.e. \( \mathbb{D} = \{ z : |z| < 1 \} \) is a PF_r domain of holomorphy (from now on we denote the unit disc by \( \mathbb{D} \)).

**Theorem 2:** The domain \( \Omega \) is the domain of holomorphy of a function with bounded Taylor coefficients if and only if \( \Omega \) contains the unit disc.

The necessity of the condition \( \mathbb{D} \subset \Omega \) is evident.

With the help of Theorem 2 we arrive at the main result of our paper:

**Theorem 3:** The domain \( G \) is a PF_r domain of holomorphy if and only if \( G \) satisfies conditions (A)-(C).

It follows from our theorems and the proof of Theorem 2 that among the \( PF_r, r \in \mathbb{N} \), there are functions that have no limit near the points of the boundary of their domains. Corollary 1 of the following theorem shows that a PF_r g.f. with \( r \geq 2 \) cannot be bounded in its domain of holomorphy.

**Theorem 4:** Let \( f(z) \) be a PF_r g.f., \( r \geq 2 \), and \( T \) be its radius of convergence, \( T < \infty \). Then \( (1 - z/T)f(z) \) is a PF_{r-1} g.f..

**Corollary 1:** Let \( f(z) \) satisfy the conditions of Theorem 4 for \( r = 2 \). Then

\[
\lim_{x \to T^-}(1 - x/T)f(x) = a > 0,
\]

where \( a \) can be equal to \(+\infty\).

**Corollary 2:** Let \( f(z) \) satisfy the conditions of Theorem 4 and \( T \) be an essential singularity of the function \( f \). Then

\[
\lim_{x \to T^-}(1 - x/T)^{r-1}f(x) = a > 0,
\]

where \( a \) can be equal to \(+\infty\).
3 Proofs of the results:

For proving Theorem 1 we will need two lemmas whose proofs can be found in [3].

**Lemma 1:** Let \( r, n \in \mathbb{N}, r \geq 1 \) and \( 1 \leq n \leq r \). Let

\[
\frac{1}{(1 - z)^r} = \sum_{k=0}^{\infty} \frac{(k + 1) \cdots (k + r - 1)}{(r - 1)!} z^k = \sum_{k=0}^{\infty} a_k(r) z^k
\]

and

\[
a_k(r) = 0 \text{ for } k < 0.
\]

Then for \( k \geq 0 \),

\[
\det \|a_{k+j-i}\|_{i,j=1,n} = \prod_{i=1}^{n} \frac{(i-1)!}{(r-i)!} (k + i) \cdots (k + i + r - n - 1),
\]

where we consider \((k + i) \cdots (k + i + r - n - 1) = 1\) for \( n = r \).

**Lemma 2:** (see [9]) If \( \sum_{k=0}^{\infty} c_k \lambda^k < \infty \) for certain \( \lambda, 0 < \lambda < 1 \), \( c_k = 0 \) for \( k < 0 \) and \( \det \|c_{k+j-i}\|_{i,j=1,n} > 0 \) for all \( k \geq 0 \) and \( n, 1 \leq n \leq r \), then \( \{c_k\}_{k=0}^{\infty} \in PF_r \).

Now, we are going to prove the following fact, which is slightly more general than Theorem 1.

**Theorem 1’:** Let the function \( g \) satisfy the hypothesis of Theorem 1. Then for any \( r \in \mathbb{N} \) and any \( \alpha \in \mathbb{N} \), there exists a \( \varepsilon > 0 \) such that the functions \( f_\varepsilon(z), f_\varepsilon'(z), \ldots, f_\varepsilon^{(\alpha)}(z) \) are \( PF_r \) g.f.

**Proof of Theorem 1’:** Let \( r \in \mathbb{N} \) be a fixed number. Let \( f_\varepsilon^{(p)}(z) = \sum_{k=0}^{\infty} c_k^{(p)}(r^2) z^k \), where \( 0 \leq p \leq \alpha \). Then \( c_k^{(p)}(r^2) = a_k^{(p)}(r^2) + \varepsilon b_k^p \) where \( a_k^{(p)}(r^2) \) and \( b_k^p \) are the Taylor coefficients of the functions

\[
\frac{r^2 \cdots (r^2 + p - 1)}{(1 - z)^{r^2 + p}} \left( \frac{1}{(1 - z)^{r^2}} \text{ for } p = 0 \right)
\]
and \( g^{(p)}(z) \) respectively (consider \( a_k^0(r^2) = a_k(r^2) \) and \( b_k^0 = b_k \)). More explicitly,

\[
a_k^p(r^2) = \frac{(k+1) \cdots (k+r^2+p-1)}{(r^2-1)!}, \quad \text{for } r - 1 + p > 0, k = 0, 1, 2, \ldots;
\]
\[
a_k^0(1) = 1, \quad \text{for } k = 0, 1, 2, \ldots; \tag{4}
\]
\[
a_k^p(r^2) = 0, \quad \text{for } k < 0;
\]

and

\[
b_k^p = (k + p) \cdots (k + 1)b_{k+p}, \quad \text{for } k = 0, 1, 2, \ldots, p \neq 0;
\]
\[
b_k^0 = b_k, \quad \text{for } k = 0, 1, 2, \ldots; \tag{5}
\]
\[
b_k^p = 0, \quad \text{for } k < 0.
\]

Therefore, for each \( p, 0 \leq p \leq \alpha \),

\[
a_k^p(r^2) = O(k^{r^2+p-1}) \quad \text{and} \quad b_k^p = O(k^p), \quad k \to \infty. \tag{6}
\]

For \( n, 1 \leq n \leq r \), we have

\[
\begin{bmatrix}
c_k^p & c_{k+1}^p & \cdots & c_{k+n-1}^p \\
c_{k-1}^p & c_k^p & \cdots & c_{k+n-2}^p \\
\vdots & \vdots & \ddots & \vdots \\
c_{k-n+1}^p & c_{k-n+2}^p & \cdots & c_k^p
\end{bmatrix} = \begin{bmatrix}
a_k^p + \varepsilon b_k^p & a_{k+1}^p + \varepsilon b_{k+1}^p & \cdots & a_{k+n-1}^p + \varepsilon b_{k+n-1}^p \\
a_k^p + \varepsilon b_{k-1}^p & a_k^p + \varepsilon b_k^p & \cdots & a_{k+n-2}^p + \varepsilon b_{k+n-2}^p \\
\vdots & \vdots & \ddots & \vdots \\
a_k^p + \varepsilon b_{k-n+1}^p & a_{k-n+2}^p + \varepsilon b_{k-n+2}^p & \cdots & a_k^p + \varepsilon b_k^p
\end{bmatrix}
\]

\[
= \sum_{l=0}^{n} \varepsilon^l S_l^p(k, n), \quad \text{where} \quad S_l^p(k, n) = \begin{bmatrix}
a_k^p & a_{k+1}^p & \cdots & a_{k+n-1}^p \\
a_{k-1}^p & a_k^p & \cdots & a_{k+n-2}^p \\
\vdots & \vdots & \ddots & \vdots \\
a_{k-n+1}^p & a_{k-n+2}^p & \cdots & a_k^p
\end{bmatrix},
\]
\[ S^p_{n}(k, n) = \begin{vmatrix} b^p_k & b^p_{k+1} & \cdots & b^p_{k+n-1} \\ b^p_{k-1} & b^p_k & \cdots & b^p_{k+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ b^p_{k-n+1} & b^p_{k-n+2} & \cdots & b^p_k \end{vmatrix} \]

and

\[ S^p_{l}(k, n), 1 \leq l \leq n - 1, \text{ is the sum of determinants of all possible matrices formed by } n - l \text{ rows from } S^0_p(k, n) \text{ and } l \text{ complementary rows from } S^p_n(k, n). \]

The sum \( S^p_{l}(k, n) \) has \( \binom{n}{l} \) summands. Each summand is a determinant of order \( n \). Each determinant is the sum of \( n! \) summands which are the products of \( n - l \) entries of the matrix of \( S^0_p(k, n) \) and \( l \) entries of the matrix \( S^p_n(k, n) \). It follows from (6) that the modulus of each entry of \( S^p_n(k, n) \) does not exceed \( C_p(k + 1)^p \), where \( C_p \) is a constant, and each element of \( S^0_p(k, n) \) is non-negative and does not exceed \( a^{p}_{k+r-1} \), since \( \{a^p_k\} \) is a non-decreasing sequence. Hence, we have for \( l, 1 \leq l \leq n, \)

\[ |S^p_{l}(k, n)| < \binom{n}{l} n!(C_p)^l(k + 1)^{pl}(a^{p}_{k+r-1})^{n-l} \leq B(k + 1)^{pl}(a^{p}_{k+r-1})^{n-l}, \]

where \( B = 2^r r! \max_0 \leq r \leq a(C_p)^r \). Obviously \( B \) does not depend on \( k \).

As a result, we have

\[ |S^p_{l}(k, n)| \geq |S^p_{0}(k, n)| - \sum_{l=1}^{n} |S^p_{l}(k, n)| \geq \]

\[ \geq S^p_{0}(k, n) - \varepsilon B \sum_{l=1}^{n} (k + 1)^{pl}(a^{p}_{k+r-1})^{n-l} \]

for \( \varepsilon < 1 \).
From (3) we have

\[(k + 1)^{pl}(a_{k+r-1}^p)^{n-l} = O(k^{(r^2+p-1)(n-l)+pl}) = O(k^{(r^2-1)(n-l)+np}), k \to \infty.\]

By (2), (3) and (4) we have

\[a_{k}^p(r^2) = \frac{r^2 + p - 1}{(r^2 - 1)!}a_k(r^2 + p), k \in \mathbb{Z}.\]

Hence, using Lemma 1 we obtain

\[S_{p}^0(k, n) = \prod_{j=0}^{p-1}(r^2 + j)^n \prod_{i=1}^{n} \frac{(i - 1)!}{(r^2 + p - i)!}(k + i) \cdots (k + i + r^2 + p - n - 1) \geq M(k + 1)^{n(r^2+p-n)}, n = 1, 2, \ldots, r, k = 0, 1, 2, \ldots, (8)\]

where \(M\) is a positive number.

Also, for each \(n, 1 \leq n \leq r\), and \(l, 1 \leq l \leq n\),

\[n(r^2 + p - n) = n(r^2 - 1) - (n^2 - n) + np \geq n(r^2 - 1) - (n^2 - 1) + np \geq n(r^2 - 1) - l(r^2 - 1) + np = (r^2 - 1)(n - l) + np.\]

Hence, using (8), for each \(p, 0 \leq p \leq \alpha\), and for any \(n, 1 \leq n \leq r\),

\[B \sum_{l=1}^{n}(k + 1)^{pl}(a_{k+r-1}^p)^{n-l} = O(k^{n(r^2+p-n)}), k \to \infty\]

and by (8) there is a \(\varepsilon_n^p > 0\) such that the inequality

\[\varepsilon_n^p B \sum_{l=1}^{n}(k + 1)^{pl}(a_{k+r-1}^p)^{n-l} < \frac{1}{2}S_{0}^p(k, n)\]

holds for any \(k \geq 0\).
Let \( \varepsilon = \min \{ \varepsilon_n \} \). Since Lemma 1 \( S^0_\varepsilon(k, n) > 0 \), then, using (\( 7 \)), for any \( k \geq 0 \), any \( n, 1 \leq n \leq r \), and any \( p, 0 \leq p \leq \alpha \) we have

\[
\begin{vmatrix}
 c^p_k & c^p_{k+1} & \cdots & c^p_{k+n-1} \\
c^p_{k-1} & c^p_k & \cdots & c^p_{k+n-2} \\
\vdots & \vdots & \ddots & \vdots \\
c^p_{k-n+1} & c^p_{k-n+2} & \cdots & c^p_k
\end{vmatrix} > \frac{1}{2} S^p_0(k, n) > 0.
\]

Relying on Lemma 2 we conclude that \( \{ c^p_k \}_{k=0}^\infty \in PF_r \). \( \square \)

**Proof of Theorem 2:**

The necessity of the condition \( \mathbb{D} \subset \Omega \) is evident.

The sufficiency will be proved by constructing a function \( g(z) \) with bounded Taylor coefficients, analytic in \( \Omega \), that cannot be analytically continued through \( \partial \Omega \).

Let \( \Omega \) be a domain with \( \mathbb{D} \subset \Omega \) and \( \{ \zeta_k \}_{k=1}^\infty \subset \partial \Omega \) a countable set of points dense in \( \partial \Omega \).

In the neighborhood of radius \( 1/2n, n \in \mathbb{N} \), of the point \( \zeta_k \) there exists a point \( z(n, k) \in \Omega \). Let us denote by \( \lambda(n, k) \) the closest to \( z(n, k) \) point of \( \partial \Omega \), i.e. \( |\lambda(n, k) - z(n, k)| = dist(z(n, k), \partial \Omega) \). It is obvious that \( |\zeta_k - \lambda(n, k)| < 1/n, n \in \mathbb{N} \). Hence, the countable set \( \{ \lambda(n, k) \}_{n,k=1}^\infty \) is dense in \( \partial \Omega \). Let us number the sets \( \{ z(n, k) \} \) and \( \{ \lambda(n, k) \} \) with one parameter \( k \in \mathbb{N} \) preserving the correspondence \( |z_k - \lambda_k| = dist(z_k, \partial \Omega) \). Now we choose a sequence \( \{ d_k \}_{k=1}^\infty \subset \mathbb{R}_+ \) such that \( \sum_{k=1}^\infty d_k < \infty \) and put

\[
g(z) = \sum_{k=1}^\infty \frac{d_k}{\lambda_k - z}.
\]

If \( z \in K, K \) is a compact in \( \Omega \), and \( \delta = dist(K, \partial \Omega) \), then

\[
\left| \frac{d_k}{\lambda_k - z} \right| < \frac{d_k}{\delta}.
\]

Thus, the series defining \( g \) converges uniformly on each compact \( K \subset \Omega \) and, therefore, \( g \) is analytic in \( \Omega \).
Now, we prove that $g$ cannot be analytically continued through $\partial \Omega$ applying an idea that can be found in Levin’s [8], page 117. We fix a point $\lambda_p$ and will show that $g(z)$ tends to infinity for certain $z$ approaching $\lambda_p$. Let $N_p$ be a number such that
\[
\sum_{k=N_p+1}^{\infty} d_k < \frac{d_p}{2}.
\]
Then
\[
|g(z)| \geq \frac{d_p}{|\lambda_p - z|} - \left| \sum_{k=1,k\neq p}^{N_p} \frac{d_k}{\lambda_k - z} \right| - \sum_{k=N_p+1}^{\infty} \frac{d_k}{|\lambda_k - z|}.
\]
Note that for $z = \alpha \lambda_p + (1 - \alpha) \lambda_p$, $0 < \alpha < 1$, and $k \neq p$ the inequalities
\[
|\lambda_k - z| > |\lambda_k - \lambda_p| - |z - \lambda_p| \geq |\lambda_p - \lambda_p| - |z - \lambda_p| = |\lambda_p - z|
\]
hold.

Hence,
\[
|g(z)| \geq \frac{d_p}{2|\lambda_p - z|} - \left| \sum_{k=1,k\neq p}^{N_p} \frac{d_k}{\lambda_k - z} \right|,
\]
which shows that $g(z) \to \infty$ when $z \to \lambda_p$, $z = \alpha \lambda_p + (1 - \alpha) \lambda_p$, $0 < \alpha < 1$.

We still have to show that $g$ is a function with bounded Taylor coefficients. Denoting
\[
g(z) = \sum_{n=0}^{\infty} b_n z^n
\]
we have
\[
b_n = \frac{g^{(n)}(0)}{n!} = \sum_{k=1}^{\infty} \frac{d_k}{\lambda_k^{n+1}}, \quad \text{for} \quad n = 0, 1, 2, \ldots.
\]
But, $|\lambda_k| \geq 1$ and $\sum_{k=1}^{\infty} d_k < \infty$, so the sequence $\{b_n\}_{n=0}^{\infty}$ is bounded. \(\square\)

Theorem 3 follows at once from Theorem 1 and Theorem 2.

Proof of Theorem 4: Since a $PF_2$ g.f. is either a polynomial or a transcendental function with positive Taylor coefficients (see [8], page 393), we only consider the case $c_k > 0$ for $k = 0, 1, 2, \ldots$, and $c_k = 0$ for $k < 0$. 

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Following \( [1] \) (see also \( [4] \), page 407), we put

\[
f_1(z) = (1 - z/T) f(z) = \sum_{k=0}^{\infty} c_k^{(1)} z^k,
\]

where \( c_k^{(1)} = c_k - c_{k-1}/T \).

Every minor

\[
\begin{vmatrix}
c_{k_1} & c_{k_2} & \ldots & c_{k_n} & \frac{c_m}{c_{m-n}} \\
c_{k_1-1} & c_{k_2-1} & \ldots & c_{k_n-1} & \frac{c_{m-1}}{c_{m-n}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{k_1-n} & c_{k_2-n} & \ldots & c_{k_n-n} & \frac{c_{m-n}}{c_{m-n}}
\end{vmatrix}
= \frac{1}{c_{m-n}}
\begin{vmatrix}
c_{k_1} & c_{k_2} & \ldots & c_{k_n} & c_m \\
c_{k_1-1} & c_{k_2-1} & \ldots & c_{k_n-1} & c_{m-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{k_1-n} & c_{k_2-n} & \ldots & c_{k_n-n} & c_{m-n}
\end{vmatrix}
\]

for \( n = 1, 2, \ldots, r - 1; k_1 < k_2 < \ldots < k_n < m \), is nonnegative by hypothesis. Since \( \lim_{m \to \infty} c_m/c_{m-1} = 1/T \), we have that \( \lim_{m \to \infty} c_m/c_{m-l} = 1/T^l \).

Letting \( m \) tend to \( \infty \) in the considered minors, we obtain

\[
\begin{vmatrix}
c_{k_1} & c_{k_2} & \ldots & c_{k_n} & 1/T^n \\
c_{k_1-1} & c_{k_2-1} & \ldots & c_{k_n-1} & 1/T^{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{k_1-n} & c_{k_2-n} & \ldots & c_{k_n-n} & 1
\end{vmatrix} \geq 0.
\]

Now, starting from the second row, we divide each row by \( T \) and subtract it from the previous, altering each row but the last. We obtain

\[
0 \leq \begin{vmatrix}
c_{k_1}^{(1)} & c_{k_2}^{(1)} & \ldots & c_{k_n}^{(1)} & 0 \\
c_{k_1-1}^{(1)} & c_{k_2-1}^{(1)} & \ldots & c_{k_n-1}^{(1)} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
c_{k_1-n+1}^{(1)} & c_{k_2-n+1}^{(1)} & \ldots & c_{k_n-n+1}^{(1)} & 0 \\
c_{k_1-n}^{(1)} & c_{k_2-n}^{(1)} & \ldots & c_{k_n-n}^{(1)} & 1
\end{vmatrix}
\]
For proving that \( \{c_k^{(1)}\} \in PF_{r-1} \) we introduce the function \( e^{xz} = \sum_{k=0}^{\infty} b_k z^k \), and consider \( b_k = 0 \) for \( k < 0 \). Note that (see [4], page 428)

\[
\begin{vmatrix}
b_k & b_{k+1} & \cdots & b_{k+n} \\
b_{k-1} & b_k & \cdots & b_{k+n-1} \\
\vdots & \vdots & \ddots & \vdots \\
b_{k-n} & b_{k-n+1} & \cdots & b_k
\end{vmatrix} > 0
\]

for \( k \geq 0 \) and \( n = 0, 1, 2, \ldots \).

Let \( g(z) = f_1(z) e^{xz} = \sum_{k=0}^{\infty} a_k z^k \), where \( a_k = \sum_{m=0}^{k} c_m^{(1)} b_{k-m} = \sum_{m=0}^{k} c_m^{(1)} z^{k-m}/(k-m)! \), and let \( a_k = 0 \) for \( k < 0 \).

By the Cauchy-Binet formula, we can write

\[
\begin{vmatrix}
a_k & a_{k+1} & \cdots & a_{k+n} \\
a_{k-1} & a_k & \cdots & a_{k+n-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{k-n+1} & a_{k-n+2} & \cdots & a_k
\end{vmatrix} =
\sum_{m_1<\ldots<m_n} \begin{vmatrix}
c_{m_1}^{(1)} & c_{m_2}^{(1)} & \cdots & c_{m_n}^{(1)} \\
c_{m_1-1}^{(1)} & c_{m_2-1}^{(1)} & \cdots & c_{m_n-1}^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
c_{m_1-n+1}^{(1)} & c_{m_2-n+1}^{(1)} & \cdots & c_{m_n-n+1}^{(1)}
\end{vmatrix} \begin{vmatrix}
b_{k-m_1} & b_{k+1-m_1} & \cdots & b_{k+n-1-m_1} \\
b_{k-m_2} & b_{k+1-m_2} & \cdots & b_{k+n-1-m_2} \\
\vdots & \vdots & \ddots & \vdots \\
b_{k-m_n} & b_{k+1-m_n} & \cdots & b_{k+n-1-m_n}
\end{vmatrix}
\]

In the expression above each summand is nonnegative, and the summand corresponding to \( m_1 = 0, m_2 = 1, \ldots, m_n = n - 1 \) is
for $k \geq 0$ and $n = 0, 1, 2, \ldots$.

The series $\sum_{k=0}^{\infty} a_k z^k$ converges for certain $\lambda, 0 < \lambda < 1$, and the minors
\[ \det \begin{vmatrix} a_{k+j-i} \end{vmatrix}_{i,j=1,n}, k \geq 0, 1 \leq n \leq r - 1, \] are strictly positive. By Lemma 2, \{a_k\} is a PF$_{r-1}$ sequence. Also, $a_k \to c^{(1)}_k$, when $\varepsilon \to 0$. Thus, \{c^{(1)}_k\} $\in$ PF$_{r-1}$ as well.$\Box$

**Proof of Corollary 1:** By Theorem 4, the function $f_1(z) = (1 - z/T)f(z)$ is a PF$_1$ g.f. and its radius of convergence is not less than $T$. Thus, the limit in question is either positive or $+\infty$. $\Box$

**Proof of Corollary 2:** Applying $r - 1$ times Theorem 4 (each time the obtained function has radius of convergence equal to $T$), we have that $(1 - z/T)^{r-1}f(z) \in$ PF$_1$ and the limit in question is either positive or $+\infty$. $\Box$

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