GLOBAL COMPARISON PRINCIPLES FOR THE 
p-LAPLACE OPERATOR ON RIEMANNIAN MANIFOLDS

STEFANO PIGOLA AND GIONA VERONELLI

ABSTRACT. We prove global comparison results for the $p$-Laplacian on a $p$-parabolic manifold. These involve both real-valued and vector-valued maps with finite $p$-energy.

1. INTRODUCTION

Let $(M, (\cdot, \cdot))$ be a connected, $m$-dimensional manifold and let $p > 1$. Recall that the $p$-Laplacian of a real valued function $u : M \to \mathbb{R}$ is defined by $\Delta_p u = \text{div} \left( |\nabla u|^{p-2} \nabla u \right)$. The function $u$ is said to be $p$-subharmonic if $\Delta_p u \geq 0$. In case any bounded above, $p$-subharmonic function is necessarily constant we say that the manifold $M$ is $p$-parabolic. It is known that $p$-parabolicity of a complete manifold $M$ is related to volume growth properties of the underlying manifold. Namely, $M$ is $p$-parabolic provided, for some $x \in M$,

$$\left( \frac{1}{\text{vol}_{m-1} \partial B_r (x)} \right)^{\frac{1}{p-1}} \notin L^1 (\mathbb{R}) ,$$

where $\partial B_r (x)$ denotes the metric sphere centered at $x$, of radius $r > 0$, and $\text{vol}_{m-1}$ is the $(m-1)$-dimensional Hausdorff measure; [9]. Thus, for instance, the standard Euclidean space $\mathbb{R}^m$ is $p$-parabolic if $m \leq p$.

Now, suppose that $M$ is $p$-parabolic, with $p \geq 2$. It is known, [7], that a $p$-subharmonic function $u : M \to \mathbb{R}$ with finite $p$-energy $|\nabla u| \in L^p (M)$ must be constant. We shall show that this is nothing but a very special case of a genuine comparison principle for the $p$-Laplace operator.

Theorem 1. Let $(M, (\cdot, \cdot))$ be a connected, $p$-parabolic manifold, $p \geq 2$. Assume that the smooth functions $u, v : M \to \mathbb{R}$ satisfy

$$\Delta_p u \geq \Delta_p v \text{ on } M,$$

and

$$|\nabla u|, |\nabla v| \in L^p (M).$$

Then, $u = v + A$ on $M$, for some constant $A \in \mathbb{R}$. 

Date: April 3, 2009.

2000 Mathematics Subject Classification. 35B05, 31C45.

Key words and phrases. Nonlinear comparison, finite $p$-energy, $p$-parabolicity.
Besides real-valued functions one is naturally led to consider manifold-valued maps. Several topological questions are related to the $p$-Laplacian of maps; [11]. Recall that the $p$-Laplacian (or the $p$-tension field) of a map $u : M \to N$ between Riemannian manifolds is defined by

$$\Delta_p u = \text{div} \left( |du|^{p-2} du \right),$$

Here, $du \in T^*M \otimes u^{-1}TN$ denotes the differential of $u$ and the bundle $T^*M \otimes u^{-1}TN$ is endowed with its Hilbert-Schmidt scalar product $\langle \cdot, \cdot \rangle$. Moreover, $-\text{div}$ stands for the formal adjoint of the exterior differential $d$ with respect to the standard $L^2$ inner product on vector-valued 1-forms. Say that $u$ is $p$-harmonic if $\Delta_p u = 0$. In [10], Schoen and Yau prove a general comparison principle for homotopic (2-)harmonic maps with finite (2-)energy into non-positively curved targets. They assume that the complete, non compact manifold $M$ has finite volume but the request that $M$ is (2-)parabolic suffices, [7]. In this direction, comparisons for homotopic $p$-harmonic maps with finite $p$-energy into non-positively curved manifolds are far from being completely understood. Some progress in the special situation of a single map homotopic to a constant has been made in [7]. In this note, we focus our attention on the case $N = \mathbb{R}^n$. According to [7], it is clear that, if $M$ is $p$-parabolic, then every $p$-harmonic map $u : M \to \mathbb{R}^n$ with finite $p$-energy $|du| \in L^p (M)$ must be constant. However, using the very special structure of $\mathbb{R}^n$, we are able to extend this conclusion, thus establishing a comparison principle for maps $u, v : M \to \mathbb{R}^n$ having the same $p$-Laplacian. This can be considered as a further step towards the comprehension of the general comparison problem alluded to above.

**Theorem 2.** Suppose that $(M, \langle \cdot, \cdot \rangle)$ is $p$-parabolic, $p \geq 2$. Let $u, v : M \to \mathbb{R}^n$ be smooth maps satisfying

(1) \hspace{1cm} \Delta_p u = \Delta_p v \text{ on } M,

and

$$|du|, |dv| \in L^p (M).$$

If $(M, \langle \cdot, \cdot \rangle)$ is $p$-parabolic then $u = v + A$, for some constant $A \in \mathbb{R}$.

**Acknowledgement.** The authors express their gratitude to A.G. Setti for useful discussions during the preparation of this paper.

2. Main tools

In the proofs of Theorems 1 and 2 we will make an essential use of two main ingredients: (a) a version for the $p$-Laplacian of a classical inequality for the mean-curvature operator; (b) a parabolicity criterion which, in a sense, can be considered as a global form of the divergence theorem in non-compact settings.
2.1. A key inequality. The following basic inequality was discovered by Lindqvist, [4].

**Lemma 3.** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional, real vector space endowed with a positive definite scalar product. Let $p \geq 2$. Then, for every $x, y \in V$ it holds

$$|x|^p + (p-1)|y|^p \geq p |y|^{p-2} \langle x, y \rangle + \frac{1}{2p-1-1} |x-y|^p.$$ 

As consequence, we deduce the validity of the next

**Corollary 4.** In the above assumptions, for every $x, y \in V$, it holds

$$\langle |x|^{p-2} x - |y|^{p-2} y, x-y \rangle \geq \frac{2}{p(2p-1-1)} |x-y|^p.$$ 

**Proof.** We start computing

$$\langle |x|^{p-2} x - |y|^{p-2} y, x-y \rangle = |x|^p + |y|^p - \langle x, y \rangle \left( |x|^{p-2} + |y|^{p-2} \right).$$

On the other hand, applying twice Lindqvist inequality with the role of $x$ and $y$ interchanged we get

$$p (|x|^p + |y|^p) \geq p \left( |x|^{p-2} + |y|^{p-2} \right) \langle x, y \rangle + \frac{2}{(2p-1-1)} |x-y|^p.$$ 

Inserting into the above completes the proof. $\square$

**Remark 5.** Inequality (2) can be considered as a version for the $p$-Laplacian of the classical Mikljukov-Hwang-Collin-Krust inequality; [6], [3], [1]. This latter states that, for every $x, y \in V$,

$$\left\langle \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}}, x-y \right\rangle \geq \frac{\sqrt{1+|x|^2} + \sqrt{1+|y|^2}}{2} \left| \frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}} \right|^2,$$

equality holding if and only if $x = y$.

2.2. The Kelvin-Nevanlinna-Royden criterion. There is a very useful characterization of (non) $p$-parabolicity in terms of special vector fields on the underlying manifold. It goes under the name of Kelvin-Nevanlinna-Royden criterion. In the linear setting $p = 2$ it was proved in a paper by Lyons and Sullivan, [5]. See also [8]. The following non-linear extension is due to Gol’dshtein and Troyanov, [2].

**Theorem 6.** The manifold $M$ is not $p$-parabolic if and only if there exists a vector field $X$ on $M$ such that:

(a) $|X| \in L^{\frac{p}{p-1}}(M)$
(b) $\text{div} \ X \in L^1_{\text{loc}}(M)$ and $\min (\text{div} \ X, 0) = (\text{div} \ X)_- \in L^1(M)$
(c) $0 < \int_M \text{div} \ X \leq +\infty$.

In particular, if $M$ is $p$-parabolic and $X$ is a vector field satisfying (a) $|X| \in L^{\frac{p}{p-1}}(M)$, (b) $\text{div} \ X \in L^1_{\text{loc}}(M)$, and (c) $\text{div} \ X \geq 0$ on $M$, then we must necessarily conclude that $\text{div} \ X = 0$ on $M$. 
3. Proofs of the comparison principles

We are now in the position to prove the main results.

Proof (of Theorem 1). Fix any \( x_0 \in M \), let \( A = u(x_0) - v(x_0) \) and define \( \Omega_A \) to be the connected component of the open set
\[
\{ x \in M : A - 1 < u(x_0) - v(x_0) < A + 1 \}
\]
which contains \( x_0 \). By standard topological arguments, \( \Omega_A \neq \emptyset \) is a (connected) open set. Let \( \alpha : \mathbb{R} \to \mathbb{R}_{\geq 0} \) be the piece-wise linear function defined by
\[
\alpha(t) = \begin{cases} 
0 & t \leq A - 1 \\
(t - A + 1)/2 & A - 1 \leq t \leq A + 1 \\
1 & t \geq A + 1.
\end{cases}
\]
Consider the vector field
\[
X = \alpha(u - v) \left\{ |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v \right\}.
\]
A direct computation gives
\[
\text{div } X = \alpha' (u - v) \left( |\nabla u|^{p-2} \nabla u - |\nabla v|^{p-2} \nabla v, \nabla u - \nabla v \right) + \alpha (u - v) (\Delta_p u - \Delta_p v)
\geq \frac{2}{2p-1 - 1} \alpha' (u - v) |\nabla u - \nabla v|^p,
\]
where in the last inequality we have used Corollary 4, the fact that \( \alpha, \alpha' \geq 0 \) and the assumption \( \Delta_p u - \Delta_p v \geq 0 \). Therefore
\[
\text{div } X \geq 0, \text{ on } M,
\]
the equality holding if and only if
\[
\alpha' (u - v) |\nabla u - \nabla v| = 0.
\]
On the other hand, for a suitable constant \( C > 0 \),
\[
|X|^{\frac{p}{p-1}} \leq C (|\nabla u|^p + |\nabla v|^p) \in L^1(M).
\]
Therefore, Theorem 6 yields
\[
\text{div } X = 0 \text{ on } M.
\]
Since \( \alpha' (u - v) \neq 0 \) on \( \Omega_A \), we deduce
\[
u - v \equiv A, \text{ on } \Omega_A.
\]
It follows that the open set \( \Omega_A \) is also closed. Since \( M \) is connected we must conclude that \( \Omega_A = M \) and \( u - v = A \) on \( M \).

□

Proof (of Theorem 2). We suppose that either \( u \) or \( v \) is non-constant, for otherwise there’s nothing to prove. Fix \( q_0 \in M \). Set \( C := u(q_0) - v(q_0) \in \mathbb{R}^n \)
and introduce the radial function $r: \mathbb{R}^n \rightarrow \mathbb{R}$ defined as $r(x) = |x - C|$. For $T > 0$, consider the piecewise differentiable vector field $X_T$ defined as

$$X_T := \left[ dh_T |_{(u-v)} \circ \left( |du|^{p-2}du - |dv|^{p-2}dv \right) \right]^2$$

where $h_T \in C^1(\mathbb{R}^n, \mathbb{R})$ is the function

$$h_T(x) := \begin{cases} \frac{r^2(x)}{2} & \text{if } r(x) < T \\ T r(x) - \frac{T^2}{2} & \text{if } r(x) \geq T. \end{cases}$$

We observe that $h_T \in C^2$ where $r(x) \neq T$ and that $X_T$ is well defined since there exists a canonical identification

$$T_{|u-v(q)\rangle \mathbb{R}^n} \cong T_{u(q)\rangle \mathbb{R}^n} \cong T_{v(q)\rangle \mathbb{R}^n} \cong \mathbb{R}^n.$$ 

We also observe that, by Sard theorem, for a.e. $T > 0$, the level set \{ |u - v - C| = T \} is a smooth (possibly empty) hypersurface, hence a set of measure zero. Thus, the vector field $X_T$ is weakly differentiable and, for a.e. $T > 0$, the weak divergence of $X_T$ is given by

$$\text{div } X_T = d \left( \frac{r^2}{T} \right) |_{(u-v)} \circ (\Delta_p u - \Delta_p v)$$

$$+ M \text{ tr} \left( \text{Hess} \left( \frac{r^2}{T} \right) |_{(u-v)} \left( du - dv, \left| du \right|^{p-2}du - \left| dv \right|^{p-2}dv \right) \right),$$

if $r(x) < T$, and

$$\text{div } X_T = d (Tr) |_{(u-v)} \circ (\Delta_p u - \Delta_p v)$$

$$+ M \text{ tr} \left( \text{Hess} (Tr) |_{(u-v)} \left( du - dv, \left| du \right|^{p-2}du - \left| dv \right|^{p-2}dv \right) \right);$$

if $r(x) \geq T$. The first term on the RHS vanishes by assumption. Moreover, by standard computations, we have $\text{Hess}(r) = r^{-1}(\langle , \rangle_{\mathbb{R}^n} - dr \otimes dr)$ on $\mathbb{R}^n \setminus \{C\}$. Thus,

$$\text{Hess} \left( \frac{r^2}{T} \right) = dr \otimes dr + r \text{ Hess}(r) = \langle , \rangle_{\mathbb{R}^n} \quad \text{if } r(x) < T,$$

$$\text{Hess}(Tr) = T \text{ Hess}(r) = T \left( \frac{r^2}{T} \right) \left( \langle , \rangle_{\mathbb{R}^n} - dr \otimes dr \right) \quad \text{if } r(x) \geq T.$$ 

As a consequence, for $q \in M$ such that $r((u - v)(q)) < T$, by Corollary 4 we get

$$\text{div } X_T = \langle du - dv, \left| du \right|^{p-2}du - \left| dv \right|^{p-2}dv \rangle \geq \frac{|du - dv|^p}{p(2p - 1)}.$$ 

(3)
while, for \( q \in M \) such that \( r((u - v)(q)) \geq T \), it holds

\[
(4) \quad \text{div} X_T = \frac{T}{r(u-v)} \langle du - dv, |du|^{p-2} du - |dv|^{p-2} dv \rangle \\
- \frac{T}{r(u-v)} \langle dr|_{(u-v)}(du - dv), dr|_{(u-v)}(|du|^{p-2} du - |dv|^{p-2} dv) \rangle \\
\geq \frac{T}{r(u-v) p(2p-1)} |du - dv|^p - (|du| + |dv|)(|du|^{p-1} + |dv|^{p-1}) \\
\geq \frac{T}{r(u-v) p(2p-1)} - (|du|^p + |dv|^p + |du|^{p-1} |dv| + |dv|^{p-1} |du|) \\
\geq \frac{T}{r(u-v) p(2p-1)} - 2(|du|^p + |dv|^p),
\]

where we have used again Corollary 4 for the first term and Cauchy-Schwarz inequality, Young’s inequality and the facts that \( |dr| = 1 \) and \( r(u - v) \geq T \) for the second one. Let us now compute the \( L^{\frac{p}{p-1}} \)-norm of \( X_T \). Since

\[
|du|^{p-2} du - |dv|^{p-2} dv |^{\frac{p}{p-1}} \leq \left( |du|^{p-1} + |dv|^{p-1} \right)^{\frac{p}{p-1}} \leq 2^\frac{1}{p-1} (|du|^p + |dv|^p),
\]

we have

\[
\int_{\{|u-v-C|<T\}} |X_T|^{\frac{p}{p-1}} \leq \int_{\{|u-v-C|<T\}} |u-v-C|^{\frac{p}{p-1}} |du|^{p-2} du - |dv|^{p-2} dv |^{\frac{p}{p-1}} \\
\leq T^{\frac{p}{p-1}} 2^\frac{1}{p-1} (\|du\|_p^p + \|dv\|_p^p) < +\infty
\]

and

\[
\int_{\{|u-v-C|>T\}} |X_T|^{\frac{p}{p-1}} \leq \int_{\{|u-v-C|>T\}} T^{\frac{p}{p-1}} |du|^{p-2} du - |dv|^{p-2} dv |^{\frac{p}{p-1}} \\
\leq T^\frac{p}{p-1} 2^\frac{1}{p-1} \left( \|du\|_p^p + \|dv\|_p^p \right) < +\infty.
\]

Hence \( X_T \) is a weakly differentiable vector field with \( |X_T| \in L^{\frac{p}{p-1}}(M) \) and \( \text{div} X_T \in L^1_{\text{loc}}(M) \). To apply Theorem \ref{thm:weak} it remains to show that \( (\text{div} X_T)_- \in L^1(M) \). By inequalities (3) and (4), we deduce that

\[
(5) \quad \int_M (\text{div} X_T)_- \leq 2 \int_{\{|u-v-C|>T\}} (|du|^p + |dv|^p) \leq 2(\|du\|_p^p + \|dv\|_p^p) < +\infty.
\]

Then, the assumptions of Theorem \ref{thm:weak} are satisfied and we get, for a.e. \( T > 0 \),

\[
\int_M \text{div} X_T \leq 0.
\]

According to (5) we now choose a sequence \( T_n \to +\infty \) such that

\[
\int_M (\text{div} X_{T_n})_- \leq 2 \int_{\{|u-v-C|>T_n\}} (|du|^p + |dv|^p) < \frac{1}{n}.
\]
As a consequence,

\[
\int_{\{u-v-C|<T_n\}} \frac{|du - dv|^p}{p(2p-1) - 1} \leq \int_{\{u-v-C|<T_n\}} (\text{div} \, X_{T_n})_+ \\
\leq \int_M (\text{div} \, X_{T_n})_+ \\
\leq -\int_M (\text{div} \, X_{T_n})_- < \frac{1}{n}.
\]

Therefore, letting \( n \) go to \(+\infty\), we obtain

\[
\int_M \frac{|d(u-v)|^p}{p(2p-1) - 1} = 0,
\]

that is, \( u - v \equiv u(q_0) - v(q_0) = C \) on \( M \).

\[\square\]

REFERENCES

[1] P. Collin, R. Krust. Le problème de Dirichlet pour l'équation des surfaces minimales sur des domaines non bornés. Bull. Soc. Math. France, 119 (1991), 443-458.

[2] V. Gol'dshtein, M. Troyanov, The Kelvin-Nevanlinna-Royden criterion for \( p \)-parabolicity. Math Z. 232 (1999), 607–619.

[3] J.-F. Hwang, Comparison principles and Liouville theorems for prescribed mean curvature equations in unbounded domains. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 15 (1988), 341–355.

[4] P. Lindqvist, On the equation \( \text{div} \, (\nabla |u|^{p-2} \nabla u) + \lambda |u|^{p-2} u = 0 \). Proc. Amer Math. Soc. 109 (1996), 157–164.

[5] T. Lyons, D. Sullivan, Function theory, random paths and covering spaces. J. Diff. Geom. 18 (1984), 229–323.

[6] V.M. Mikljukov. A new approach to Bernstein theorem and to related questions for equations of minimal surface type (in Russian). Mat. Sb. 108 (1979), 268-289; Engl. transl. in Math. USSR Sb., 36 (1980), 251-271.

[7] S. Pigola, M. Rigoli, A.G. Setti, Constancy of \( p \)-harmonic maps of finite \( q \)-energy into non-positively curved manifolds. Math. Z. 258 (2008), 347–362.

[8] S. Pigola, M. Rigoli, A.G. Setti, Vanishing and finiteness results in geometric analysis: a generalization of the Bochner technique. Progress in Mathematics 266 (2008), Birkhäuser.

[9] M. Rigoli, A.G. Setti, Liouville type theorems for \( \varphi \)-subharmonic functions. Rev. Mat. Iberoamericana. 17 (2001), 471–450.

[10] R. Schoen, S.T. Yau, Compact group actions and the topology of manifolds with non-positive curvature. Topology 18 (1979), 361–380.

[11] S.W. Wei, Representing homotopy groups and spaces of maps by \( p \)-harmonic maps. Indiana Math. J. 47 (1998), 625–669.

DIPARTIMENTO DI FISICA E MATEMATICA, UNIVERSITÀ DELL’INSUBRIA - COMO, VIA VALLEGGIO 11, I-22100 COMO, ITALY

E-mail address: stefano.pigola@uninsubria.it

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI MILANO, VIA SALDINI 50, I-20133 MILANO, ITALY

E-mail address: giona.veronelli@unimi.it