The noncommutative family
Atiyah-Patodi-Singer index theorem

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Abstract

In this paper, we define the eta cochain form and prove its regularity. We decompose the eta form as a pair of the eta cochain form with the Chern character of an idempotent matrix and we also decompose the Chern character of the index bundle for a fibration with boundary as a pair of the family Chern-Connes character for a manifold with boundary with the Chern character of an idempotent matrix. We define the family $b$-Chern-Connes character and then we prove that it is entire and give its variation formula. By this variation formula, we prove another noncommutative family Atiyah-Patodi-Singer index theorem. Thus, we extend the results of Gezler and Wu to the family case. Finally, we state an equivariant generalization of our results.

Keywords: Eta cochain form; family Chern-Connes character for manifolds with boundary; family $b$-Chern-Connes character; variation formula.

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1 Introduction

In [APS], Atiyah-Patodi-Singer introduced the eta invariant and proved their famous Atiyah-Patodi-Singer index theorem for manifolds with boundary. In [BC], using Cheeger’s cone method, Bismut and Cheeger defined the eta form which is a family version of the eta invariant and extended the APS index formula to the family case under the condition that all boundary Dirac operators are invertible. In [MP1,2], using the Melrose’s $b$-calculus, Melrose and Piazza extended the Bismut-Cheeger family index theorem to the case that boundary Dirac operators are not invertible. In [Do], Donnelly extended the APS index theorem to the equivariant case by modifying the Atiyah-Patodi-Singer original method. In [Zh], Zhang got this equivariant Atiyah-Patodi-Singer index theorem by using a direct geometric method in [LYZ].

On the other hand, in [Wu], Wu proved the Atiyah-Patodi-Singer index theorem in the framework of noncommutative geometry. To do so, he introduced the eta cochain (called the higher eta invariant in [Wu]) which is a generalization of the classical Atiyah-Patodi-Singer eta invariant in [APS], then proved its regularity by using the Getzler symbol calculus [Ge1] as adopted in [BF] and computed its radius of convergence. Subsequently, he proved the variation formula of eta cochains, using which he
got the noncommutative Atiyah-Patodi-Singer index theorem. In [Ge2], using super-
connection, Getzler gave another proof of the noncommutative Atiyah-Patodi-Singer
index theorem, which was more difficult, but avoided mention of the operators $b$ and
$B$ of cyclic cohomology. In [Wa1], we defined the equivariant eta cochain and proved
its regularity using the method in [CH], [Fe] and [Zh]. Then we proved an equiv-
ariant noncommutative Atiyah-Patodi-Singer index theorem. In [Wa2], we defined
infinitesimal equivariant eta cochains and proved their regularity. In [LMP], Lesch,
Moscovici and Pflaum presented the Chern-Connes character of the Dirac operator
associated to a $b$-metric on a manifold with boundary in terms of a retracted cocy-
cle in relative cyclic cohomology. Blowing-up the metric one recovered the pair of
characteristic currents that represent the corresponding de Rham relative homology
class, while the blowdown yielded a relative cocycle whose expression involves higher
ta cochains and their $b$-analogues. The corresponding pairing formula with relative
$K$-theory classes captured information about the boundary and allowed to derive ge-
ometric consequences. In [Xi], Xie proved an analogue for odd dimensional manifolds
with boundary, in the $b$-calculus setting, of the higher Atiyah-Patodi-Singer index
theorem by Getzler and by Wu. Xie also obtained a natural counterpart of the eta
invariant for even dimensional closed manifolds.

The purpose of this paper is to extend the theorems due to Getzler and Wu to
the family case.

This paper is organized as follows: in Section 2, we define the eta cochain form
and prove its regularity. In Section 3, we decompose the eta form as a pair of the
eta cochain form with the Chern character of an idempotent matrix and we also
decompose the Chern character of the index bundle for a fibration with boundary
as a pair of the family Chern-Connes character for manifolds with boundary with
the Chern character of an idempotent matrix. In Section 4, We define the family
$b$-Chern-Connes character and then we prove that it is entire and give its variation
formula. In Section 5, by this variation formula, we prove another noncommutative
family Atiyah-Patodi-Singer index theorem. Thus, we extend the results of Gezler
and Wu to the family case. Then, we state an equivariant generalization of our results.

2 The eta cochain form

In this Section, we define the eta cochain form and prove its regularity.

Firstly, we recall the Bismut superconnection. Let $M$ be a $n + q$ dimensional
compact connected manifold without boundary and $X$ be a $q$ dimensional compact
connected manifold without boundary. We assume that $\pi : M \to X$ is a submersion
of $M$ onto $X$, which defines a fibration of $M$ with the fibre $Z$. For $y \in X$, $\pi^{-1}(y)$ is a
submanifold $M_y$ of $M$. Denote by $TZ$ the $n$-dimensional vector bundle on $M$ whose
fibre $T_xM_{\pi(x)}$ is the tangent space at $x$ to the fibre $M_{\pi(x)}$. We assume that $M$ and $X$
are oriented. We take a smooth horizontal subbundle $T^H M$ of $TM$. A vector field
$X \in \Gamma(X, TX)$ will be identified with its horizontal lift $X^H \in \Gamma(M, T^H M)$. Moreover
$T^H_xM$ is isomorphic to $T_{\pi(x)}X$ via $\pi_*$. We take a Riemannian metric on $X$ and then
lift the Euclidean scalar product $g_X$ of $TX$ to $T^HM$. We further assume that $TZ$ is endowed with a scalar product $g_Z$. Thus we can introduce on $TM$ a new scalar product $g_X \oplus g_Z$, and denote by $\nabla^L$ the Levi-Civita connection on $TM$ with respect to this metric. Set $\nabla^X$ denote the Levi-Civita connection on $TX$ and we still denote by $\nabla^X$ the pullback connection on $T^HM$. Let $\nabla^Z = P_Z(\nabla^L)$ where $P_Z$ denotes the orthogonal projection to $TZ$. Set $\nabla^\oplus = \nabla^X \oplus \nabla^Z$ and $S = \nabla^L - \nabla^\oplus$ and $T$ be the torsion tensor of $\nabla^\oplus$. Denote by $SO(TZ)$ the $SO(n)$ bundle of oriented orthonormal frames in $TZ$. Now we assume that the bundle $TZ$ is spin. Denote by $S(TZ)$ the associated spinor bundle and $\nabla^S$ can be lifted to a connection on $S(TZ)$. Let $D$ be the Dirac operator in the tangent direction defined by $D = \sum_{j=1}^n c(e_j^*) \nabla^S(TZ)$ where $\nabla^S(TZ)$ is a spin connection on $S(TZ)$. Set $E$ be the vector bundle $\pi^*(\wedge T^*X) \otimes S(TZ)$. Then the Bismut superconnection acting on $E$ is defined by

$$B = D + \sum_{\alpha=1}^q f^\alpha (\nabla^{S(TZ)})_{\alpha} + \frac{1}{2} k(f_{\alpha}) - \frac{1}{4} c(T), \quad (2.1)$$

where

$$k(f_{\alpha}) = \sum_{j=1}^n \langle \nabla^{T_Z} f_{\alpha} e_j - [f_{\alpha}, e_j], e^j \rangle, \quad c(T) = - \sum_{\alpha < \beta} \sum_j f^\alpha f^\beta c(e_j) \langle [f^H_\alpha, f^H_\beta], e_j \rangle. \quad (2.2)$$

Let $\psi_t : dy_{\alpha} \rightarrow \frac{dy_{\alpha}}{\sqrt{t}}$ be the rescaling operator. Let $B_t = \sqrt{t}\psi_t(B)$ and $F_t = B_t^2$. Let $\text{tr}^{even}$ denote taking the trace with value in $\Omega^{even}(X)$. When $\text{dim}Z$ is odd, for $a_0, \cdots, a_{2k} \in C^\infty(M)$, we define the family cochain $\text{ch}_{2k}(B_t, \frac{dB_t}{dt})$ by the formula:

$$\text{ch}_{2k}(B_t, \frac{dB_t}{dt})(a_0, \cdots, a_{2k}) = \sum_{j=0}^{2k} (-1)^j \langle a_0, [B_t, a_1], \cdots, [B_t, a_j], \frac{dB_t}{dt}, [B_t, a_{j+1}], \cdots, [B_t, a_{2k}] \rangle_t, \quad (2.3)$$

If $A_j$ ($0 \leq j \leq q$) are operators on $\Gamma(E)$, we define:

$$\langle A_0, \cdots, A_q \rangle_t = \int_{\Delta_q} \text{tr}^{even}[A_0 e^{-s_0 F_1} A_1 e^{-s_1 F_1} \cdots A_q e^{-s_q F_1}] d\sigma, \quad (2.4)$$

where $\Delta_q = \{(\sigma_1, \cdots, \sigma_q) | \sigma_0 + \cdots + \sigma_q = 1, \sigma_j \geq 0\}$ is a simplex in $\mathbb{R}^q$. When $\text{dim}Z$ is even, in (2.4), we use $\text{str}$ instead of $\text{tr}^{even}$ and define $\text{ch}_{2k}(B_t, \frac{dB_t}{dt})$.

We assume that the kernel of $D$ is a complex vector bundle. Formally, the eta cochain form is defined to be an even cochain sequence by the formula:

$$\tilde{\eta}_{2k}(B) = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{ch}_{2k}(B_t, \frac{dB_t}{dt}) dt, \quad \text{when } \text{dim}Z \text{ is odd}; \quad (2.5)$$

$$\tilde{\eta}_{2k}(B) = \int_0^\infty \text{ch}_{2k}(B_t, \frac{dB_t}{dt}) dt, \quad \text{when } \text{dim}Z \text{ is even}. \quad (2.6)$$
Then \( \hat{\eta}_0(B)(1) \) is the eta form defined by Bismut and Cheeger. In order to prove that the above definition is well defined, it is necessary to check the integrality near the two ends of the integration. Firstly, the regularity at infinity comes from the following lemma.

**Lemma 2.1** We assume that the kernel of \( D \) is a complex vector bundle. For \( a_0, \ldots, a_{2k} \in C^\infty(M) \), we have

\[
\text{ch}_{2k}(B_t, \frac{dB_t}{dt})(a_0, \ldots, a_{2k}) = O(t^{-\frac{3}{2}}), \quad \text{as } t \to \infty. \tag{2.7}
\]

**Proof.** Since the kernel of \( D \) is a complex vector bundle, our proof is very similar to the proof of Lemma 3.5 in [Wa2] (see revised version arXiv:1307.8189). We just use Lemma 9.4 in [BGV] instead of Lemma 3.4 in [Wa2]. We use \( D + \frac{c(T)}{4} \) and \( \psi_t : dy_a \to \frac{d\eta_t}{\sqrt{t}} \) instead of \( D - \frac{c(X)}{4} \) and \( \psi_t : X \to \frac{X}{t} \) in Lemma 3.5 in [Wa2] respectively where \( X \) is the Killing vector field. We note that \( \text{ch}_{2k}(B_t, \frac{dB_t}{dt}) \) corresponds to \( \frac{1}{2\sqrt{t}} \text{ch}_k(\sqrt{t}D_X, D_X) \) in [Wa2]. Comparing with the single operator case in Lemma 2 in [CM], the operator \( [B_t, a_j] = \sqrt{t}(c(dZa_j) + \frac{1}{\sqrt{t}} d_Xa_j \wedge) \) is instead of \( \sqrt{t}[D, a_j] \) and \( \delta_t(g) = 1 + O(t^{-\frac{1}{2}})S_0 \) and \( S_0 \) is a smooth operator. By these differences, in the discussions of Lemma 2 in [CM], the number of copies of \( e^{-\sigma_t tD^2}(I - H) \) may be less than \( \frac{k}{2} + 1 \). But the coefficients of \( S_0 \) and \( d_Xa_j \wedge \) are \( O(t^{-\frac{1}{2}}) \). Through careful observations, we still get (2.7). \( \Box \)

In the following, we prove the regularity at zero of the eta cochain form. We know that \( \frac{dB_t}{dt} = \frac{1}{2\sqrt{t}} \psi_t(D + \frac{c(T)}{4}) \). We introduce the Grassmann variable \( dt \) which anticommute with \( c(e_j) \) and \( dy_a \). Set \( \hat{F} = F + dt(D + \frac{c(T)}{4}) \). Let

\[
\text{ch}_{2k}(\hat{F})(a_0, \ldots, a_{2k}) = t^k \int_{\Delta_{2k}} \psi_t t^r e^{-\sigma_0} \hat{F} [B, a_1] \cdots [B, a_{2k}] e^{-\sigma_2k \hat{F}} d\sigma. \tag{2.8}
\]

By the Duhamel principle and \( (dt)^2 = 0 \), we have

\[
e^{-\sigma_0 \hat{F}} = e^{-\sigma_0 F} - t^2 \int_0^{\sigma_0} e^{-r(\sigma_0 - \xi)} F(D + \frac{c(T)}{4}) e^{-\xi F} d\xi. \tag{2.9}\]

By (2.8) and (2.9), we have

\[
\text{ch}_{2k}(\hat{F})(a_0, \ldots, a_{2k}) = \text{ch}_{2k}(F)(a_0, \ldots, a_{2k}) + t^2 \text{ch}_{2k}(B_t, \frac{dB_t}{dt})(a_0, \ldots, a_{2k}) dt. \tag{2.10}
\]

Let \( A \) be an operator and \( l \) be a positive integer. Write

\[
A[l] = [F, A[l-1]], \quad A[0] = A, \quad A[l] = [F, A[l-1]], \quad A(0) = A.
\]

Similar to Lemma 4.4 in [Wa3], we have
Lemma 2.2 Let $A$ be a finite order fibrewise differential operator with form coefficients, then for any $s > 0$, we have:

\[
e^{-sF}A = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l A^{(l)} e^{-sF} + (-1)^N s^N A^{(N)}(s); \tag{2.11}
\]

\[
e^{-\hat{s}F}A = \sum_{l=0}^{N-1} \frac{(-1)^l}{l!} s^l A^{[l]} e^{-\hat{s}F} + (-1)^N s^N A^{[N]}(s), \tag{2.12}
\]

where $A^{(N)}(s)$ and $A^{[N]}(s)$ are given by

\[
A^{(N)}(s) = \int_{\Delta_N} e^{-u_1 sF} A^{(N)} e^{-(1-u_1)sF} du_1 du_2 \cdots du_N; \tag{2.13}
\]

\[
A^{[N]}(s) = \int_{\Delta_N} e^{-u_1 \hat{s}F} A^{[N]} e^{-(1-u_1)\hat{s}F} du_1 du_2 \cdots du_N. \tag{2.14}
\]

As in [CH], [Fe], [Wa3], by Lemma 2.2, we have for a sufficient large $N$,

\[
\text{ch}_{2k}(F)(a_0, \ldots, a_{2k}) = \psi_t \sum_{\lambda_1, \ldots, \lambda_{2k}=0}^{N} \frac{(-1)^{\lambda_1+\cdots+\lambda_{2k}}}{\lambda_1! \cdots \lambda_{2k}!} C t^{||\lambda||+k} t^{\text{tr even}} \left[ a_0[B, a_1]^{(\lambda_1)} \cdots [B, a_{2k}]^{(\lambda_{2k})} e^{-tF} \right] + O(t^\frac{2}{3}), \tag{2.15}
\]

\[
\text{ch}_{2k}(\hat{F})(a_0, \ldots, a_{2k}) = \psi_t \sum_{\lambda_1, \ldots, \lambda_{2k}=0}^{N} \frac{(-1)^{\lambda_1+\cdots+\lambda_{2k}}}{\lambda_1! \cdots \lambda_{2k}!} C t^{||\lambda||+k} t^{\text{tr even}} \left[ a_0[B, a_1]^{[\lambda_1]} \cdots [B, a_{2k}]^{[\lambda_{2k}]} e^{-t\hat{F}} \right] + O(t^\frac{3}{2}), \tag{2.16}
\]

where $C$ is a constant. Recall Lemma 2.17 in [Wa3] which extends the corresponding Lemma in [Po] and [PW] (for related definitions, see [Wa3]).

Lemma 2.3 (Lemma 2.17 in [Wa3]) Let $Q \in \Psi_Y^s(\mathbb{R}^n \times \mathbb{R}, S(T(M_z)) \otimes T^*_z X)$ have Getzler order $m$ and model operator $Q_{(m)}$. Then as $t \to 0^+$ we have:

1) $\sigma[\psi_t K_Q(0,0,t)]^{(j)} = \omega^{\text{odd}} O(t^{\frac{j-n-m-2}{2}}) + O(t^{\frac{j-n-m-1}{2}})$, if $m-j$ is odd;

2) $\sigma[\psi_t K_Q(0,0,t)]^{(j)} = t^{\frac{j-n-m-2}{2}} K_{Q_{(m)}}(0,0,1)^{(j)} + \omega^{\text{odd}} O(t^{\frac{j-n-m-1}{2}}) + O(t^{\frac{j-n-m}{2}})$, if $m-j$ is even,

where $[K_Q(0,0,t)]^{(j)}$ denotes the degree $j$ form component in $M_z$ and $\omega^{\text{odd}} O(t^{\frac{j-n-m-2}{2}})$ denotes that the coefficients of $t^{\frac{j-n-m-2}{2}}$ are in $\wedge^\text{odd}(T^*_z X) \otimes (T^* (M_z))$.

Lemma 2.4 The following estimate holds

\[
\text{ch}_{2k}(B_t, \frac{dB_t}{dt}) \sim O(1) \quad \text{when } t \to 0, \tag{2.17}
\]
Proof. By (2.10), (2.15) and (2.16), in order to prove Lemma 2.4, we only prove

\[ \psi t^{[\lambda + k]_{\text{tr}} \text{even}} [a_0[B, a_1]^{[\lambda_1]} \cdots [B, a_{2k}]^{[\lambda_{2k}]} e^{-t \widetilde{F}}] \]

\[- \psi t^{[\lambda + k]_{\text{tr}} \text{even}} [a_0[B, a_1]^{[\lambda_1]} \cdots [B, a_{2k}]^{[\lambda_{2k}]} e^{-t \widetilde{F}}] = O(t^{\frac{3}{2}})dt. \]  

(2.18)

This a local problem and we fix a point \( x_0 \) in \( M_z \). Set

\[ h(x) = 1 + \frac{1}{2} dt \sum_{j=1}^{n} x_j c(e_j) \]  

(2.19)

as in [Zh]. By (5.29) in [Wa3], we have

\[ h[F + dt(D + \frac{c(T)}{4})h^{-1}] = F + dt u, \]  

(2.20)

where the Getzler order \( O_G(u) \leq 0 \) of \( u \). Write

\[ \widetilde{A}^{|l|} = [h \hat{F} h^{-1}, \widetilde{A}^{|l|-1}], \quad \widetilde{A}^{|0|} = A. \]

Then

\[ \psi t^{[\lambda + k]_{\text{tr}} \text{even}} [a_0[B, a_1]^{[\lambda_1]} \cdots [B, a_{2k}]^{[\lambda_{2k}]} e^{-t \widetilde{F}}] \]

\[ = \psi t^{[\lambda + k]_{\text{tr}} \text{even}} [a_0[B, a_1]^{[\lambda_1]} \cdots [B, a_{2k}]^{[\lambda_{2k}]} e^{-\theta h h^{-1}}]. \]  

(2.21)

By the Volterra calculus, we have

\[ \left( \frac{\partial}{\partial t} + F + dt u \right)^{-1} = \left( \frac{\partial}{\partial t} + F \right)^{-1} - dt \left( \frac{\partial}{\partial t} + F \right)^{-1} u \left( \frac{\partial}{\partial t} + F \right)^{-1}. \]  

(2.22)

Let

\[ a_0[B, a_1]^{[\lambda_1]} \cdots [B, a_{2k}]^{[\lambda_{2k}]} = A_0 + dt A_1, \]  

(2.23)

where

\[ A_0 = a_0[B, a_1]^{(\lambda_1)} \cdots [B, a_{2k}]^{(\lambda_{2k})}. \]

Then

\[ a_0[B, a_1]^{[\lambda_1]} \cdots [B, a_{2k}]^{[\lambda_{2k}]} \left( \frac{\partial}{\partial t} + F + dt u \right)^{-1} - A_0 \left( \frac{\partial}{\partial t} + F \right)^{-1} \]

\[ = - A_0 dt \left( \frac{\partial}{\partial t} + F \right)^{-1} u \left( \frac{\partial}{\partial t} + F \right)^{-1} + dt A_1 \left( \frac{\partial}{\partial t} + F \right)^{-1}. \]  

(2.24)

By (2.24) in [Wa3], in order to prove (2.18), we only need to prove

\[ t^{k + [\lambda]} \psi t^{\text{tr even}} [A_0 \left( \frac{\partial}{\partial t} + F \right)^{-1} u \left( \frac{\partial}{\partial t} + F \right)^{-1}] = O(t^{\frac{3}{2}}), \]  

(2.25)

\[ t^{k + [\lambda]} \psi t^{\text{tr even}} [A_1 \left( \frac{\partial}{\partial t} + F \right)^{-1}] = O(t^{\frac{3}{2}}). \]  

(2.26)
We note that \( A_0 \frac{\partial}{\partial t} + F \) is an \( \text{End}^{-}(\wedge^{*}(TX) \otimes S(TZ)) \) endomorphism. So when we take \( \text{tr}^{\text{even}} \), only the coefficient of \( c(e_1) \cdots c(e_n) \) is left and other terms are zero. Note that

\[
O_G(t^{k+\lambda} A_0 \frac{\partial}{\partial t} + F)^{-1} u \left( \frac{\partial}{\partial t} + F \right)^{-1} \leq -4, \tag{2.27}
\]

so by Lemma 2.3 (1) for \( j = n \) odd and \( m = -4 \) and taking \( \text{tr}^{\text{even}} \), we get (2.25). By \( O_G(u) \leq 0 \) and (2.23), we get \( O_G(t^{k+\lambda} A_1 \frac{\partial}{\partial t} + F)^{-1} \leq -4 \). Again \( j = n \), so we get (2.26). Thus we prove Lemma 2.4.

\[\blacksquare\]

**Remark.** We also introduce a new Bismut superconnection on \( \tilde{M} = M \times \mathbb{R}_+ \rightarrow X \times \mathbb{R}_+ \) as in [BGV, Thm. 10.32] and prove a formula which is similar to (2.10). Then we can give a new proof of Lemma 2.4 as in [BGV, p. 347].

By Lemma 2.1 and Lemma 2.4, the eta cochain form is well defined.

For the idempotent \( p \in M_r(C^\infty(M)) \), its Chern character \( \text{Ch}(p) \) in entire cyclic homology is defined by the formula (for more details see [GS]):

\[
\text{Ch}(p) = \text{Tr}(p) + \sum_{k \geq 1} \frac{(-1)^k (2k)!}{k!} \text{Tr}_{2k} \left( (p - \frac{1}{2}) \otimes p^{\otimes 2k} \right) \tag{2.28}
\]

where

\[
\text{Tr}_{2k} : M_r(C^\infty(M)) \otimes (M_r(C^\infty(M))/M_r(C))^{\otimes 2k} \rightarrow C^\infty(M) \otimes (C^\infty(M)/C)^{\otimes 2k}
\]

is the generalized trace map. Let

\[
||dp|| = ||[B,p]|| = \sum_{i,j} ||d_M p_{i,j}|| \tag{2.29}
\]

where \( p_{i,j} \) is the entry of \( p \). Similar to Proposition 2.17 in [Wa1], we have

**Proposition 2.5** Suppose that all \( D_z \) are invertible with \( \lambda \) the smallest positive eigenvalue of all \( |D_z| \). We assume that \( ||dp|| < \lambda \), then the pairing \( \langle \hat{\eta}^*(B), \text{Ch}_*(p) \rangle \) is well-defined.

## 3 The family index pairing for manifolds with boundary

In this section, we decompose the eta form as a pair of the eta cochain form with the Chern character of an idempotent matrix and we also decompose the Chern character of the index bundle for a fibration with boundary as a pair of the family Chern-Connes character for a manifold with boundary with the Chern character of an idempotent matrix.

Suppose that all \( D_z \) are invertible with \( \lambda \) the smallest positive eigenvalue of all \( |D_z| \) and \( ||dp|| < \lambda \). Let \( H = \Gamma(M, \wedge^*(TX) \otimes S(TZ)) \) and

\[
p(B \otimes I_r)p : p(H \otimes C^r) = L^2(M, \wedge^*(TX) \otimes S(TZ) \otimes p(C^r))
\]


\[ \rightarrow L^2(M, \wedge^* (TX) \otimes S(TZ) \otimes p(C^r)) \]

be the Bismut superconnection with the coefficient from \( F = p(C^r) \). Then we have

**Theorem 3.1** Under the assumption as above, we have up to an exact form on \( X \)
\[
\tilde{\eta}(p(B \otimes I_r)p) = \langle \hat{\eta}^*(B), \text{Ch}_*(p) \rangle,
\]
(3.1)

where the left term is the Bismut-Cheeger eta form.

Let
\[
B = \begin{bmatrix} 0 & -B \otimes I_r \\ B \otimes I_r & 0 \end{bmatrix} \quad p = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} \quad \sigma = \sqrt{-1} \begin{bmatrix} 0 & I_r \\ I_r & 0 \end{bmatrix}
\]
be operators from \( H \otimes C^r \oplus H \otimes C^r \) to itself, then
\[
B \sigma = -\sigma B; \quad \sigma p = p \sigma.
\]

Moreover \( Be^{tB^2} \) and \( e^{tB^2} \) (\( t > 0 \)) are traceclass. For \( u \in [0, 1] \), let
\[
B_u = (1 - u)B + u[pBp + (1 - p)B(1 - p)] = B + u(2p - 1)[B, p],
\]
then
\[
B_u = \begin{bmatrix} 0 & -B_u \\ B_u & 0 \end{bmatrix} = B + u(2p - 1)[B, p].
\]

We consider the infinite dimensional bundle \( H \otimes C^r \oplus H \otimes C^r \) on \( X \times [0, 1] \times \mathbb{R} \times [0, \infty) \), parameterized by \( (b, u, s, t) \). Let
\[
\tilde{B} = t^2 \psi t B_u + s \sigma(p - \frac{1}{2}),
\]
then \( A = d_{(u,s,t)} + \tilde{B} \) be a superconnection on \( H \otimes C^r \oplus H \otimes C^r \). Direct computations show that
\[
(d + \tilde{B})^2 = t \psi t B_u^2 - s^2/4 - (1 - u)t^{\frac{1}{2}} s \sigma[p \psi t B, p] + ds \sigma(p - \frac{1}{2})
\]
\[
+ t^2 du(2p - 1)[p \psi t B, p] + \frac{1}{2} t^{\frac{1}{2}} dt \psi t [D_u + \frac{\sigma(T)}{4}].
\]

We also consider \( A \) as \( A_t \), which is a family superconnection parameterized by \( t \) on the superbundle with the base \( X \times [0, 1] \times \mathbb{R} \) and the fibre \( H \otimes C^r \oplus H \otimes C^r \). Let \( \Gamma_u = \{ u \} \times \mathbb{R} \subset [0, 1] \times \mathbb{R} \) be a contour oriented in the direction of increasing \( s \) and \( \gamma_s = [0, 1] \times \{ s \} \) be a contour oriented in the direction of increasing \( u \). By the Duhamel principle and the Stokes theorem as in page 225 in [Wa1], then
\[
dX \omega = \int_{[0,1] \times \mathbb{R}} d \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) = \left( \int_{\Gamma_1} - \int_{\Gamma_0} - \int_{\gamma_{-\infty}} + \int_{\gamma_{+\infty}} \right) \left[ \int_{0}^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) \right],
\]
(3.7)
where $\text{Str}^{\text{even}}$ denotes taking the supertrace with value in $\Omega^{\text{even}}(X) \otimes \Omega([0,1] \times \mathbb{R})$. So in the cohomology of $X$, we have

$$
\int_{\Gamma_0} \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) = \int_{\Gamma_1} \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}). \quad (3.8)
$$

Similar to (3.8) in [Wa1], we have

$$
\int_{\Gamma_0} \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) = -4\sqrt{-1}\pi [(\widehat{\eta}^*(B), \text{Ch}(p)) - (\widehat{\eta}^*(B), \text{rk}(p)\text{Ch}_*(1))]. \quad (3.9)
$$

Similar to (3.8) in [Wa1], we have

$$
\int_{\Gamma_1} \int_0^{+\infty} \text{Str}^{\text{even}}(e^{A^2}) = -2\sqrt{-1} \int_{-\infty}^{+\infty} e^{-s^2/4} ds
$$

$$
\cdot \int_0^{+\infty} \psi_t \text{Tr}^{\text{even}}[(p - 1/2)(D_1 + \frac{c(T)}{4})e^{-tB_s^2}] d\sqrt{t}. \quad (3.10)
$$

By the following lemma 3.2 and (3.8)-(3.10), similar to (3.12) in [Wa1], we can prove Theorem 3.1. □

**Lemma 3.2** Let $B_s = B + s(2p - 1)[B,p]$ for $s \in [0,1]$. We assume that all $D_s$ be invertible and $||d_{MP}|| < \lambda$, then we have $\widehat{\eta}(B_0) = \widehat{\eta}(B_1)$.

**Proof.** By $||d_{MP}|| < \lambda$, then $D_s = D + s(2p - 1)[D,p]$ is invertible for $s \in [0,1]$. Similar to the discussions of Proposition 4.4 in [Wu], the eta form of $B_s$ is well defined. Therefore $\eta(B_s)$ is smooth. Let $B_s = D_s + A_{[1]} - \frac{c(T)}{2}$ and $A_0 = (2p - 1)[D,p]$. Then by the definition of the eta form and the Duhamel principle, we have

$$
\frac{d}{ds} \widehat{\eta}(B_s) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} \psi_t \text{tr}^{\text{even}}[A_0 e^{-tB_s^2}] d\sqrt{t} + L, \quad (3.11)
$$

where

$$
L = -\frac{1}{\sqrt{\pi}} \int_0^{+\infty} \psi_t \text{tr}^{\text{even}} \left\{ t(B_s + \frac{c(T)}{2} - A_{[1]}) \right\}
$$

$$
\cdot \int_0^1 e^{-\sigma B_s^2} [(2p - 1)[B,p], B_s]_+ e^{-(1-\sigma)tB_s^2} d\sigma \right\} d\sqrt{t}. \quad (3.12)
$$

By $\text{tr}^{\text{even}}(AB) = \text{tr}^{\text{even}}(BA)$ and $B_s e^{-\sigma B_s^2} = e^{-\sigma B_s^2} B_s$, we have

$$
\text{tr}^{\text{even}} \left\{ B_s \int_0^1 e^{-\sigma B_s^2} [(2p - 1)[B,p], B_s]_+ e^{-(1-\sigma)tB_s^2} d\sigma \right\}
$$

$$
= \int_0^1 \text{tr}^{\text{even}} \left\{ (2p - 1)[B,p] e^{-\sigma B_s^2} [B_s, B_s]_+ e^{-(1-\sigma)tB_s^2} d\sigma \right\}, \quad (3.13)
$$

and

$$
\text{tr}^{\text{even}} \left\{ \left( \frac{c(T)}{2} - A_{[1]} \right) \int_0^1 e^{-\sigma B_s^2} [(2p - 1)[B,p], B_s]_+ e^{-(1-\sigma)tB_s^2} d\sigma \right\} \right\}$$

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\[ = \int_0^1 \text{tr}^{\text{even}} \left\{ (2p - 1)[B, p]e^{-\sigma t B^2} \left[ \frac{c(T)}{2} - A_{[1]} + B_s + e^{-(1-\sigma) t B^2} d\sigma \right] \right\}. \tag{3.14} \]

By (3.12)-(3.14)

\[ L = -\int_0^{+\infty} \frac{\sqrt{t}}{2\sqrt{\pi}} \psi_t \int_0^1 \text{tr}^{\text{even}} \left\{ (2p - 1)[B, p]e^{-\sigma t B^2} [D_s + \frac{c(T)}{4}, B_s] + e^{-(1-\sigma) t B^2} d\sigma \right\} dt, \tag{3.15} \]

By

\[ \frac{d(t\psi_t B^2)}{dt} = \frac{1}{2} \psi_t [D_s + \frac{c(T)}{4}, B_s] +, \tag{3.16} \]

(3.11) and (3.15), using the Duhamel principle and the Leibniz rule, then we get

\[ \frac{\partial}{\partial t} \left( \frac{\sqrt{t}}{\sqrt{\pi}} \psi_t \text{tr}^{\text{even}} \left[(2p - 1)[D, p]e^{-t B^2}\right] \right) = \frac{\partial}{\partial s} \left( \frac{1}{2\sqrt{\pi t}} \psi_t \text{tr}^{\text{even}} \left[(D_s + \frac{c(T)}{4})e^{-t B^2}\right] \right). \tag{3.17} \]

So

\[ \frac{d}{ds} \tilde{\eta}(B_s) = \frac{\sqrt{t}}{\sqrt{\pi}} \psi_t \text{tr}^{\text{even}} \left[(2p - 1)[D, p]e^{-t B^2}\right] |_0^{+\infty}. \tag{3.18} \]

By \(D_s\) being invertible, \(\text{tr}^{\text{even}} \left[(2p - 1)[D, p]e^{-t B^2}\right]\) exponentially decays, so

\[ \lim_{t \to +\infty} \frac{\sqrt{t}}{\sqrt{\pi}} \psi_t \text{tr}^{\text{even}} \left[(2p - 1)[D, p]e^{-t B^2}\right] = 0. \tag{3.19} \]

By Lemma 2.3, similar to the discussions on page 164 in [Wu], we have

\[ \lim_{t \to 0} \frac{\sqrt{t}}{\sqrt{\pi}} \psi_t \text{tr}^{\text{even}} \left[(2p - 1)[D, p]e^{-t B^2}\right] = c_0 \int_Z \hat{A}(TZ) \text{tr} \left\{ (2p - 1)(d_Z p) \exp \left[ \frac{-1}{2\pi} (A' \wedge A' + dA') \right] \right\} = 0, \tag{3.20} \]

where \(A' = s(2p - 1)d_M p\). Then by (3.18)-(3.20), we prove Lemma 3.2. \(\square\)

Let \(N\) be a fibration with the even-dimensional compact spin fibre. Let \(M\) be the boundary of \(N\). We endow \(N\) with a metric which is a product in a collar neighborhood of \(M\). Denote by \(B\) (\(B_M\)) the Bismut superconnection on \(N\) (\(M\)). Let \(C_*^\infty(N) = \{ f \in C^\infty(N) | f \) is independent of the normal coordinate \(x_n\) near the boundary \}. \(\}

**Definition 3.3** The family Chern-Connes character on \(N\), \(\tau = \{ \tau_0, \tau_2, \cdots, \tau_{2q}, \cdots \}\) is defined by

\[ \tau_{2q}(B)(f^0, f^1, \cdots, f^{2q}) := -\tilde{\eta}_{2q}(B_M)(f^0|_M, f^1|_M, \cdots, f^{2q}|_M) \]

\[ + \frac{1}{(2q)!(2\pi \sqrt{-1})^q} \int_Z \hat{A}(TZ) f^0 df^1 \wedge \cdots \wedge df^{2q}, \tag{3.21} \]
where \( f^0, f^1, \ldots, f^{2q} \in C_\infty^\infty(N) \).

Similar to Proposition 4.2 in [Wa1], we have

**Proposition 3.4** The family Chern-Connes character is \( b - B \) closed (for the definitions of \( b, B \), see [FGV]). That is, in the cohomology of \( X \), we have

\[
b\tau_{2q-2} + B\tau_{2q} = 0. \tag{3.22}
\]

By Proposition 2.5, we have

**Proposition 3.5** Suppose that all \( D_{M,z} \) are invertible with \( \lambda \) the smallest positive eigenvalue of all \( |D_{M,z}| \). We assume that \( ||d(p|_M)|| < \lambda \), then the pairing \( \langle \tau, \operatorname{Ch}(p) \rangle \) is well-defined.

We let \( C_1(M) = M \times (0,1], \tilde{N} = N \cup_{M \times \{1\}} C_1(M) \) and \( \mathcal{U} \) be a collar neighborhood of \( M \) in \( N \). For \( \varepsilon > 0 \), we take a metric \( g^\varepsilon \) of \( \tilde{N} \) such that on \( \mathcal{U} \cup M \times \{1\} \)

\[
g^\varepsilon = \frac{dr^2}{\varepsilon} + r^2 g^M.
\]

Let \( S = S^+ \oplus S^- \) be spinors bundle associated to \((\tilde{N}_z, g^\varepsilon)\) and \( H^\infty \) be the set \( \{ \xi \in \Gamma(\tilde{N}_z, S) \mid \xi \text{ and its derivatives are zero near the vertex of cone} \} \). Denote by \( L^2_c(\tilde{N}_z, S) \) the \( L^2 \)-completion of \( H^\infty \) (similar define \( L^2_c(\tilde{N}_z, S^+) \) and \( L^2_c(\tilde{N}_z, S^-) \)). Let

\[
D_{z,\varepsilon} : H^\infty \rightarrow H^\infty ; \quad D_{z,+\varepsilon} : H^\infty_+ \rightarrow H^\infty_-,\n\]

be the Dirac operators associated to \((\tilde{N}_z, g^\varepsilon)\) which are Fredholm operators for the sufficient small \( \varepsilon \). When \( D_{M_z} \) is invertible, the index bundle of \( \{D_z\} \) is well defined by [BC]. We recall the Bismut-Cheeger family index theorem for the twisting bundle \( \text{Imp} \) with the connection \( pd \) in [BC]

\[
\text{ch}[\text{Ind}(pD_{z,+\varepsilon,p})] = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!(2\pi \sqrt{-1})^r} \int_Z \widehat{A}(TZ) \text{Tr}[p(dp)^{2r}] - \widehat{\eta}(pB_M p). \tag{3.22}
\]

By the Stokes theorem and the trace property and \( p(dp)^2 = (dp)^2 p \), we have

\[
\int_M ^{\widehat{A}(TM)} \text{Tr}[p_M(dMp_M)^{2k-1}] = 0. \tag{3.23}
\]

By Theorem 3.1 and Definition 3.3 and (3.23), using similar discussions on page 232 in [Wa1], we get

**Theorem 3.6** Suppose that all \( D_{M,z} \) are invertible with \( \lambda \) the smallest positive eigenvalue of all \( |D_{M,z}| \). We assume that \( ||d(p|_M)|| < \lambda \) and \( p \in M_{r \times r}(C_\infty^\infty(N)) \), then in the cohomology of \( X \)

\[
\text{ch}[\text{Ind}(pD_{z,+\varepsilon,p})] = \langle \tau(B), \text{Ch}(p) \rangle. \tag{3.24}
\]
Remark. When $N$ has the odd-dimensional compact spin fibre with boundary, we have a similar theorem as in [MP2].

4 The family $b$-Chern-Connes character

In this section, we define the family $b$-Chern-Connes character which is the family version of the Getzler’s $b$-Chern-Connes character in [Ge2] and then we prove that it is entire and give its variation formula.

Let us recall the exact $b$-geometry (see [LMP],[Xi]). Let $N$ be a compact fibration with boundary $M$ and denote by $N^0$ its interior of $N$. We take the $b$-metric $g_b = \frac{1}{r} dr \otimes dr + g_M$ near the $M$ where $r$ is the normal coordinate near the boundary. Let $x = ln r$ which gives an isometry between the infinite cylinder $(-\infty, c_0] \times M, g_{cyl} = dx \otimes dx + g_M)$ and the collar neighborhood $U$ with the exact $b$-metric. Now we consider the complete Riemannian manifold $\tilde{N} = (-\infty, c_0] \times M \cup_M (N \setminus N^0)$ instead of $N^0$ with the exact $b$-metric. Let $C^\infty_{exp}(\tilde{N})$ be the space of smooth functions on $\tilde{N}$ which expands exponentially on the infinite cylinder $(-\infty, c_0] \times M$. A smooth function $f \in C^\infty(\tilde{N})$ expands exponentially on $(-\infty, c_0] \times M$ if $f(x, y) \sim \sum_{k=0}^{\infty} e^{kx} f_k(y)$ for $(x, y) \in (-\infty, c_0] \times M$, where $f_k(y) \in C^\infty(M)$ for each $k$. That is

$$f(x, y) - \sum_{k=0}^{N-1} e^{kx} f_k(y) = e^{Nz} R_N(x, y),$$

where all derivative of $R_N(x, y)$ in $x$ and $y$ are bounded.

On $(-\infty, c_0] \times M$, we write $a = a_c + e^x a_\infty$ for $a \in C^\infty_{exp}(\tilde{N})$ with $a_c, a_\infty \in C^\infty(\tilde{N})$ and $a_c$ constant with respect to $x$. Following [Xi], define the $b$-norm of $a$ by $b||a|| := ||a_c|| + 2||a_\infty||$. The $b$-integral of $a$ along the fibre is defined by

$$\int^b a d\text{vol} := \int_{N_x \setminus U_x^2} a|_{N_x \setminus U_x^2} d\text{vol} + \int_{(-\infty, c_0] \times M} e^x a_\infty d\text{vol}. \quad (4.2)$$

Following [LMP, A.1] and [MP1], we can define the $b$-pseudodifferential operator with coefficients in $\Lambda^*(TX)$ and the pointwise trace of the Schwartz kernel of smooth $b$-pseudodifferential operators is a $b$-function. We define the $b$-trace is the $b$-integral of this $b$-function. That is, for $A \in \Psi^{-\infty}_b(\tilde{N}_z, \Lambda^*(TX) \otimes S(T\tilde{N}_z))$ and its Schwartz kernel $k_A$, define the $b$-trace which is in $\Omega(X)$ by

$${b}\text{Str}(A) = \int^b \text{Str}(k_A(x, x)) d\text{vol}.$$

(4.3)

Let $B$ be the Bismut superconnection on $\tilde{N}$ and $B_t = t \psi t B$ and $F_t = B_t^2$. By [MP1], $e^{-F_t} \in \Psi^{-\infty}_b(\tilde{N}_z, \Lambda^*(TX) \otimes S(T\tilde{N}_z))$. For $A_0, \cdots, A_q \in \Psi^{-\infty}_b(\tilde{N}, \Lambda^*(TX) \times S(T\tilde{N}_z))$, we define

$$\langle (A_0, \cdots, A_q) \rangle_b = \int_{\Delta_q} {b}\text{Str}[A_0 e^{-\sigma_0 F} A_1 e^{-\sigma_1 F} \cdots A_q e^{-\sigma_q F}] d\sigma,$$

(4.4)
\[ \langle A_0, \cdots, A_q \rangle_{b,t} = \int_{\triangle_q} b\text{Str}[A_0e^{-\sigma_0F_t} A_1e^{-\sigma_1F_t} \cdots A_qe^{-\sigma_qF_t}]d\sigma. \]  

(4.5)

For \( f_0, \cdots, f_k \in C_0^\infty(\tilde{\mathcal{N}}) \), we define the family b-Chern-Connes character by

\[ b_{\text{ch}}^k(B)(f_0, \cdots, f_k) := \langle \langle f_0, [B, f_1], \cdots, [B, f_k] \rangle \rangle_b; \quad (4.6) \]

and

\[ b_{\text{ch}}^k(B_t)(f_0, \cdots, f_k) := \langle \langle f_0, [B_t, f_1], \cdots, [B_t, f_k] \rangle \rangle_{b,t}. \quad (4.7) \]

Define

\[ b_{\text{ch}}^k(B, V) := \sum_{0 \leq j \leq k} (-1)^j \deg V \langle \langle f_0, [B, f_1], \cdots, [B, f_j], V, [B, f_{j+1}], \cdots, [B, f_k] \rangle \rangle_b. \quad (4.8) \]

Similarly we may define \( b_{\text{ch}}^k(B_t, V) \). The family b-Chern-Connes character is well defined by the following Proposition 4.7. We recall the following lemma

\begin{lemma} [\cite{MP1, Proposition 9}] For \( A \in \Psi^\infty_b(\tilde{\mathcal{N}}, \wedge^*(TX) \times S(T\tilde{N}_z)) \) and \( L \in \Psi^{-\infty}_b(\tilde{\mathcal{N}}, \wedge^*(TX) \times S(T\tilde{N}_z)) \), we have

\[ b_{\text{Str}}[A, L] = \frac{\sqrt{-1}}{2\pi} \int_{-\infty}^{+\infty} \text{Str}_M \left( \frac{\partial I(A, \lambda)}{\partial \lambda} \cdot I(L, \lambda) \right) d\lambda, \quad (4.9) \]

where \( I(L, \lambda) \) is the indicial family of \( L \) (for the definition, see \cite{LMP} or \cite{MP1}). \end{lemma}

Let \( D \) be the Dirac operator on the cylinder \((-\infty, +\infty) \times M\), then \( D = c(dx) \frac{d}{dx} + D_\theta \). On the boundary, \( c(dx) \) gives a natural identification of the even and odd half spinor bundle, then with respect to this splitting

\[ D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & D_\theta \\ D_\theta & 0 \end{pmatrix}. \quad (4.10) \]

By \cite{MP1, p.139}, we have

\begin{lemma} The following equality holds

\[ I(B, \lambda) = \sqrt{-1}c(dx)\lambda + B^M; \quad I(F, \lambda) = \lambda^2 + (B^M)^2 \quad (4.11) \]

where with respect the above splitting

\[ B^M = D_\theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sum_{\alpha=1}^q f_\alpha^* \wedge (\nabla_{f_\alpha}^S(TM/X) + \frac{1}{2} k^M(f_\alpha)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

\[ - \frac{1}{4} c(T^M) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (4.12) \]

\end{lemma}
By Lemma 4.2, we have

\[ F^M := (B^M)^2 \in \Omega^{even}(X) \left( \begin{array}{cc} L_1 & 0 \\ 0 & L_1 \end{array} \right) + \Omega^{odd}(X) \left( \begin{array}{cc} 0 & L_2 \\ L_2 & 0 \end{array} \right), \tag{4.13} \]

where \( L_1, L_2 \in \text{End}(S(TM_z)) \). Similarly, we have

\[ I([B, a], \lambda) = \left( \begin{array}{cc} 0 & [D_\partial, a_\partial] \\ [D_\partial, a_\partial] & 0 \end{array} \right) + d_X a_\partial \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right). \tag{4.14} \]

By Lemma 4.1 and Lemma 4.2, we have

**Lemma 4.3** For \( K \in \Psi_{b, cl}^{\infty}(\hat{N}, \wedge^*(TX) \times S(T\hat{N}_z)) \), we have

\[ b\text{Str}[B, K] = d_X b\text{Str}(K) - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Str}_M [c(dx)I(K, \lambda)] d\lambda. \tag{4.15} \]

By Lemmas 4.1-4.3, similar to Lemma 6.3 in [Ge2], we have

**Lemma 4.4** Let \( A_j \in \Psi_{b, cl}^{\infty}(\hat{N}, \wedge^*(TX) \times S(T\hat{N}_z)) \) which indicial family is independent of \( \lambda \).

1. If \( \varepsilon_j = (|A_0| + \cdots + |A_j|) (|A_j| + \cdots + |A_k|) \), then

\[ \langle \langle A_0, \cdots, A_k \rangle \rangle_{b,t} = (-1)^{\varepsilon_j} \langle \langle A_j, \cdots, A_k, A_0, \cdots, A_{j-1} \rangle \rangle_{b,t}. \tag{4.16} \]

2. \( \langle \langle A_0, \cdots, A_k \rangle \rangle_{b,t} = \sum_{j=0}^{k} (-1)^{\varepsilon_j} \langle \langle A_j, \cdots, A_k, A_0, \cdots, A_{j-1} \rangle \rangle_{b,t}. \tag{4.17} \)

3. \( -d_X \langle \langle A_0, \cdots, A_k \rangle \rangle_{b,t} + \sum_{j=0}^{k} (-1)^{|A_0|+\cdots+|A_j-1|} \langle \langle A_0, \cdots, [B_j, A_j], \cdots, A_k \rangle \rangle_{b,t} \]

\[ = \langle \langle (A_0), \cdots, (A_k) \rangle \rangle_{\partial,t}, \tag{4.18} \]

where when \( \dim M_z \) is odd,

\[ \langle \langle (A_0), \cdots, (A_k) \rangle \rangle_{\partial,t} := \frac{-1}{2\sqrt{\pi}} \int_{\Delta_k} \text{Str}_M \left[ c(dx)A_{0, \partial} e^{-\sigma_0 F_t^M} \cdots A_{k, \partial} e^{-\sigma_k F_t^M} \right] d\sigma. \tag{4.19} \]

4. For \( 0 \leq j < k \),

\[ \langle \langle A_0, \cdots, [F_t, A_j], \cdots, A_k \rangle \rangle_{b,t} \]

\[ = -\langle \langle A_0, \cdots, A_j A_{j+1}, \cdots, A_k \rangle \rangle_{b,t} + \langle \langle A_0, \cdots, A_{j-1} A_j, \cdots, A_k \rangle \rangle_{b,t}. \tag{4.20} \]

For \( j = k \),

\[ \langle \langle A_0, \cdots, A_{k-1}, [F_t, A_k] \rangle \rangle_{b,t} \]
= \langle \langle A_0, \cdots, A_{k-2}, A_{k-1} A_k \rangle \rangle_{b,t} - (-1)^k \langle \langle A_k A_0, A_1, \cdots, A_{k-1} \rangle \rangle_{b,t}. \quad (4.21)

By Lemma 4.4, similar to the proof of Theorem 6.2 in [Ge2], we get

**Theorem 4.5** When \( \dim M_z \) is odd, for any \( k \geq 0 \), the following equality holds

\[
 b^b \text{ch}^{k-2}(B_t) + B^b \text{ch}^k(B_t) - d_X b^b \text{ch}^{k-1}(B_t) = C h^{k-1}(B^M_t) \circ i_M^*, \quad (4.22)
\]

where

\[
 C h^{k-1}(B^M_t) \circ i_M^*(a_0, \cdots, a_k) = \langle \langle (a_0)_{\partial}, [B^M_t, a_1, \partial], \cdots, [B^M_t, a_k, \partial] \rangle \rangle_{\partial, t}. \quad (4.23)
\]

By Theorem 4.5, we have

**Theorem 4.6** When \( \dim N_z \) is even and \( k - 1 \) is even, the following equality holds

\[
 \frac{d b^b \text{ch}^{k-1}(B_t)}{dt} + b^b \text{ch}^{k-2}(B_t, \frac{dB_t}{dt}) + B^b \text{ch}^k(B_t, \frac{dB_t}{dt})
 + d_X b^b \text{ch}^{k-1}(B_t, \frac{dB_t}{dt}) = - \frac{1}{\sqrt{\pi}} \text{ch}^{k-1}(B^M_t, \frac{dB^M_t}{dt}). \quad (4.24)
\]

**Proof.** We know that \( B_t \) is a superconnection on the infinite dimensional bundle \( C^\infty(N, E) \to X \) which we write \( E \to X \). Let \( \bar{X} = X \times \mathbb{R}_+ \), and \( \bar{E} \) be the superbundle \( \pi^* \mathcal{E} \) over \( \bar{X} \), which is the pull-back to \( \bar{X} \) of \( \mathcal{E} \). Define a superconnection \( \hat{B} \) on \( \bar{E} \) by the formula

\[
 (\hat{B} \beta)(y, t) = (B_t \beta(\cdot, t))(y) + dt \wedge \frac{\partial \beta(y, t)}{\partial t}. \quad (4.25)
\]

The curvature \( \hat{F} \) of \( \hat{B} \) is

\[
 \hat{F} = F_t - \frac{dB_t}{dt} \wedge dt, \quad (4.26)
\]

where \( F_t = B^2_t \) is the curvature of \( B_t \). By the Duhamel principle, then

\[
 e^{-\hat{F}} = e^{-F_t} - dt \left( \int_0^1 e^{-uF_t} \frac{dB_t}{dt} e^{-(1-u)F_t} du \right). \quad (4.27)
\]

Then for any \( l \geq 0 \), we have

\[
 b^b \text{ch}^l(\hat{B}) = b^b \text{ch}^l(B_t) - dt b^b \text{ch}^l(B_t, \frac{dB_t}{dt}). \quad (4.28)
\]

By Theorem 4.5, we have

\[
 b^b \text{ch}^{k-2}(\hat{B}) + B^b \text{ch}^k(\hat{B}) - d_X b^b \text{ch}^{k-1}(\hat{B}) = C h^{k-1}(B^M_t). \quad (4.29)
\]

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By Theorem 4.5 and (4.29), (4.28) and \( d_X = d_X + dt \frac{d}{dt} \), we have

\[
\begin{align*}
    dt \left[ d^{b \text{ch}} k^{-1}(B_t) + b^{b \text{ch}} k^{-2}(B_t, \frac{dB_t}{dt}) + B^{b \text{ch}} k(B_t, \frac{dB_t}{dt}) \right] + d_X^{b \text{ch}} k^{-1}(B_t, \frac{dB_t}{dt}) = \widetilde{Ch}^{-1}(B_t^M) - \widetilde{Ch}^{-1}(\widetilde{B}^M). \\
    \text{(4.30)}
\end{align*}
\]

By (4.19), (4.28), (4.12) and (4.13), we get

\[
\widetilde{Ch}^{-1}(B_t^M) - \widetilde{Ch}^{-1}(\widetilde{B}^M) = -\frac{1}{\sqrt{\pi}} dt \text{ch}^{-1}(B_t^M, \frac{dB_t^M}{dt}). \quad \text{(4.31)}
\]

By (4.30) and (4.31), we get (4.24). \( \square \)

Now we prove that the family \( b \)-Chern-Connes character is entire. For \( A \in \Psi_b^{\infty}(\hat{N}_z, \Lambda^*(TX) \otimes S(T\hat{N}_z)) \) and its Schwartz kernel \( k_A \), we define

\[
\text{Str}^N U(A) = \int_{N_u \setminus U_z} \text{Str}(k_A(x,x))d\text{vol}; \quad b\text{Str}^\text{end}(A) = \int_{(-\infty,c_0) \times M_z} \text{Str}(k_A(x,x))d\text{vol}. \quad \text{(4.32)}
\]

So for \( a_0, \ldots, a_q \in C^\infty_{\exp}(\hat{N}) \),

\[
\begin{align*}
    \int_{\Delta_q} b\text{Str} \left[ a_0 e^{-\sigma_0 F} [B, a_1] e^{-\sigma_1 F} \cdots [B, a_q] e^{-\sigma_q F} \right] d\sigma \\
    = \int_{\Delta_q} \text{Str}^N U \left[ a_0 e^{-\sigma_0 F} [B, a_1] e^{-\sigma_1 F} \cdots [B, a_q] e^{-\sigma_q F} \right] d\sigma \\
    + \int_{\Delta_q} b\text{Str}^\text{end} \left[ a_0 e^{-\sigma_0 F} [B, a_1] e^{-\sigma_1 F} \cdots [B, a_q] e^{-\sigma_q F} \right] d\sigma. \quad \text{(4.33)}
\end{align*}
\]

By the discussions on the compact fibration as in [BeC], we have

\[
\left| \int_{\Delta_q} \text{Str}^N U \left[ a_0 e^{-\sigma_0 F} [B, a_1] e^{-\sigma_1 F} \cdots [B, a_q] e^{-\sigma_q F} \right] d\sigma \right| \leq \text{Tr}(e^{-\frac{H^2}{2}})^b ||a_0||^b ||a_1|| \cdots b ||a_q||. \quad \text{(4.34)}
\]

On the cylinder, we get

\[
[B, a_j] = C_j + e^x B_j; \quad a_0 = C_0 + e^x B_0, \quad \text{(4.35)}
\]

where

\[
C_j = c(d_{M_z}(a_j)_c) + d_X(a_j)_c; \quad B_j = c(d_{N_z}(a_j)_\infty) + c((a_j)_\infty dx) + d_X(a_j)_\infty,
\]

and \( C_j \) is constant along the normal direction \( x \). The second term in (4.33) can be written as a sum of terms of the following two types:

I) \( \int_{\Delta_q} b\text{Str}^\text{end} \left[ C_0 e^{-\sigma_0 F} C_1 e^{-\sigma_1 F} \cdots C_q e^{-\sigma_q F} \right] d\sigma, \)
II) \( \int_{\Delta_{m}} \text{Str}^end \left[ A_{0}e^{-\sigma_{0}F} \ldots e^{-\sigma_{j}F}e^{xB}e^{-\sigma_{j+1}F} \ldots A_{q}e^{-\sigma_{q}F} \right] d\sigma \), where \( A_{j} = C_{j} \) or \( e^{xB} \).

Firstly we estimate the type I) integral. Without generality, we set \( q = 1 \). Let \( B^{2} = D^{2} + A_{[+]} \) and \( D_{R} \) be the Dirac operator on the cylinder \( (-\infty, c_{0}) \times M_{z} \). By the Duhamel principle, we have

\[
C_{0}e^{-\sigma_{0}F}C_{1}e^{-\sigma_{1}F} = C_{0} \sum_{m \geq 0} (-\sigma_{0})^{m} \int_{\Delta_{m}} e^{-\sigma_{0}v_{0}D_{2}}A_{[+]} \ldots A_{[+]e^{-\sigma_{0}v_{m}D_{2}}dv}
\]

\[
\times C_{1} \sum_{l \geq 0} (-\sigma_{1})^{l} \int_{\Delta_{l}} e^{-\sigma_{1}v_{l}D_{2}}A_{[+] \ldots A_{[+]e^{-\sigma_{1}v_{l}D_{2}}dv'}
\]

\[
= C_{0} \sum_{m \geq 0} (-\sigma_{0})^{m} \int_{\Delta_{m}} e^{-\sigma_{0}v_{0}D_{2}} - e^{-\sigma_{0}v_{0}D_{2}'R]}A_{[+] \ldots A_{[+]e^{-\sigma_{0}v_{m}D_{2}}dv}
\]

\[
\times C_{1} \sum_{l \geq 0} (-\sigma_{1})^{l} \int_{\Delta_{l}} e^{-\sigma_{1}v_{l}D_{2}'R]}A_{[+] \ldots A_{[+]e^{-\sigma_{1}v_{l}D_{2}}dv'}
\]

\[
+ \ldots + C_{0} \sum_{m \geq 0} (-\sigma_{0})^{m} \int_{\Delta_{m}} e^{-\sigma_{0}v_{0}D_{2}'R]}A_{[+] \ldots A_{[+]e^{-\sigma_{0}v_{m}D_{2}'R}dv}
\]

\[
\times C_{1} \sum_{l \geq 0} (-\sigma_{1})^{l} \int_{\Delta_{l}} e^{-\sigma_{1}v_{l}D_{2}'R]}A_{[+] \ldots A_{[+]e^{-\sigma_{1}v_{l}D_{2}}dv'}. \tag{4.36}
\]

We know that \( A_{[+]} \) is independent of \( x \) on the cylinder and \( D_{R}^{2} = \Delta_{R} + D_{M_{z}}^{2} \), so

\[
\text{Str}^end \left[ C_{0} \sum_{m \geq 0} (-\sigma_{0})^{m} \int_{\Delta_{m}} e^{-\sigma_{0}v_{0}D_{2}'R]}A_{[+] \ldots A_{[+]e^{-\sigma_{0}v_{m}D_{2}'R}dv}
\]

\[
\times C_{1} \sum_{l \geq 0} (-\sigma_{1})^{l} \int_{\Delta_{l}} e^{-\sigma_{1}v_{l}D_{2}'R]}A_{[+] \ldots A_{[+]e^{-\sigma_{1}v_{l}D_{2}}dv'} \right|_{(-\infty, c_{0}) \times M_{z}} = 0. \tag{4.37}
\]

We estimate the first term \( K_{1} \) in (4.36) and the estimate of other terms is similar.

Since \( D \) and \( D_{R} \) are self adjoint, we can apply the functional calculus to these two operators. Then \( ||e^{-uD^{2}}|| \leq 1 \) and \( ||e^{-uD_{R}^{2}}|| \leq 1 \) for \( u \geq 0 \). By Theorem 3.2 (1) in [LMP], similar to the proof of Lemma 2.2 in [Wa2], then \( ||K_{1}||_{1} \) is bounded. By the measure of the boundary of the simplex being zero, we can estimate \( K_{1} \) in the interior of the simplex, that is \( \sigma_{0} > 0, \sigma_{1} > 0, v_{j} > 0, v_{j}' > 0 \). We note that the zero order \( b \)-pseudodifferential operator is bounded and

\[
||(1 + D^{2})^{-\frac{1}{2}}e^{-uD^{2}}|| \leq L_{0}u^{-\frac{1}{2}}; \ ||e^{-uD^{2}} - e^{-uD_{R}^{2}}||_{1} \leq L_{0}'u^{r}, \tag{4.38}
\]

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where $L_0, L'_0$ are constant and $r$ is any integer. So

$$
||K_1|| \leq ||C_0|||C_1|| \int_{\Delta_1} (\sigma_0)^m (\sigma_1)^l \int_{\Delta_m} ||e^{-\sigma_0 v_0 D^2} - e^{-\sigma_0 v_0 D^2}||_1
$$

$$
\cdot ||A_{[+]}(1 + D^2)^{r-\frac{1}{2}}|| (1 + D^2)^{\frac{1}{2}}e^{-\sigma_0 v_1 D^2}||
\cdot \int_{\Delta_l} ||e^{-\sigma_1 v_l D^2}|| \cdot ||A_{[+]}(1 + D^2)^{r-\frac{1}{2}}|| (1 + D^2)^{\frac{1}{2}}e^{-\sigma_1 v_l' D^2}||
\cdot \int_{\Delta_l} \cdot ||A_{[+]}(1 + D^2)^{r-\frac{1}{2}}|| (1 + D^2)^{\frac{1}{2}}e^{-\sigma_1 v_l' D^2}||
\leq \delta_0 ||C_0|||C_1|| \int_{\Delta_1} (\sigma_0)^m (\sigma_1)^l \int_{\Delta_m} \int_{\Delta_l}
\cdot ||\delta_1 A_{[+]}(1 + D^2)^{r-\frac{1}{2}}|| (1 + D^2)^{\frac{1}{2}}e^{-\sigma_1 v_l' D^2}||
\leq \delta_0 \frac{1}{q!} (q + 1 + \dim X) \left( \prod_{j=0}^q ||C_j|| \right) \left( \delta_1 e^{\delta_2 ||A_{[+]}(1 + D^2)^{r-\frac{1}{2}}||} \right)^{q+1}. \tag{4.39}
$$

In order to estimate the type II integral, we decompose the type II integral as (4.36). Up to the last term, other terms have the same estimate with corresponding terms in (4.36). Using the same trick as in [Xi], we get that the bound of the 1-norm of the last term is $\delta_0 \frac{1}{q!} (q + 1 + \dim X) \prod_{j=0}^q ||A_j||$.

By the above estimate, $\text{bch}(B)$ is well-defined. Similarly, for a fixed $t > 0$, $\text{bch}(B_t)$ and $\text{bch}(B_t, \frac{d\text{bch}}{dt})$ are well-defined. We recall that an even cochain $\{\Phi_{2 n}\}$ is called entire if $\sum_n ||\Phi_{2 n}|| n! z^n$ is entire, where $||\Phi|| := \sup_p ||\Phi|| \leq 1 \{||\Phi(f^0, f^1, \ldots, f^{2k})||\}$ for $f_j \in C^\infty(\hat{N})$. By the above estimate, then we have

**Proposition 4.7** $\text{bch}(B)$ is an entire cochain and $\langle \text{bch}(B), \text{ch}(p) \rangle$ is well defined.

### 5 The family Atiyah-Patodi-Singer index theorem for twisted Dirac operators

In this section, we extend the Getzler’s index theorem to the family case. Let

$$
\hat{A}(R^{\hat{N}/X}) = \det^\frac{1}{2} \left( \frac{R^{\hat{N}/X}}{4\pi \sqrt{-1}} \right) \left( \frac{R^{\hat{N}/X}}{4\pi \sqrt{-1}} \right).
$$

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**Theorem 5.1** Suppose that all $D_{M,z}$ are invertible with $\lambda$ the smallest positive eigenvalue of all $|D_{M,z}|$. We assume that $|d(p_M)| < \lambda$ and $p \in M_{r \times r}(C^\infty_\text{exp}(\tilde{N}))$, then in the cohomology of $X$

$$\text{ch[Ind}(pD_z+p)] = \int_{\tilde{N}/X} \hat{A}(R^{\tilde{N}/X}) \text{ch(Imp)} - \langle \hat{\eta}^*(B^M), \text{ch}_*(p_M) \rangle.$$  

(5.2)

**Proof.** By Theorem 4.6 and $(B + b)(\text{ch}(p)) = 0$, for fixed $t_1, t_2 > 0$, we have in the cohomology of $X$,

$$\langle \hat{\eta}^*(B_{t_2}), \text{ch}_*(p) \rangle - \langle \hat{\eta}^*(B_{t_1}), \text{ch}_*(p) \rangle = -\frac{1}{\sqrt{\pi}} \langle \int_{t_1}^{t_2} \text{ch}^*(B^M_t, \frac{dB^M_t}{dt}) dt, \text{ch}_*(p_M) \rangle.$$  

(5.3)

Let $t_1$ go to zero and $t_2$ go to $+\infty$. By Proposition 5.2 and Theorem 5.3 in [LMP], similar to the computations in Section 4 in [Wa3], we get

$$\lim_{t_1 \to 0} \langle \hat{\eta}^*(B_{t_1}), \text{ch}_*(p) \rangle = \int_{\tilde{N}/X} \hat{A}(R^{\tilde{N}/X}) \text{ch(Imp)}.$$  

(5.4)

Then we have

$$\lim_{t_2 \to +\infty} \langle \hat{\eta}^*(B_{t_2}), \text{ch}_*(p) \rangle = \int_{\tilde{N}/X} \hat{A}(R^{\tilde{N}/X}) \text{ch(Imp)}.$$  

(5.5)

By Lemma 5.2 in the following, we have

$$\lim_{t_2 \to +\infty} \langle \hat{\eta}^*(B_{t_2}), \text{ch}_*(p) \rangle = \lim_{t_1 \to +\infty} \langle \hat{\eta}^*(B_{t_1}), \text{ch}_*(p) \rangle = \lim_{t_1 \to +\infty} \langle \hat{\eta}^*(B_{t_1}), \text{ch}_*(p) \rangle = \lim_{t_2 \to +\infty} \langle \hat{\eta}^*(B_{t_2}), \text{ch}_*(p) \rangle.$$  

(5.6)

By all $D_{M,z}$ being invertible and Proposition 15 in [MP1], we have

$$\text{ch[Ind}(pD_z+p)] = \lim_{t_1 \to +\infty} \langle \hat{\eta}^*(B_{t_1}), \text{ch}_*(p) \rangle.$$  

(5.7)

By (5.3) and (5.5)-(5.7) and the definition of the eta cochain form, we get Theorem 5.1. $\blacksquare$

**Lemma 5.2** The formula (5.6) holds.

**Proof.** Let $B_{t,u} = \sqrt{t}\psi_1(B + u(2p - 1)[B_t,p])$. Using the same discussions with Theorem 4.6, we get in the cohomology of $X$

$$\langle \frac{\partial \hat{\eta}^*(B_{t,u})}{\partial u}, \text{ch}_*(p) \rangle = -\frac{1}{\sqrt{\pi}} \langle \hat{\eta}^*(B^M_{t,u}, \frac{\partial B^M_{t,u}}{\partial u}), \text{ch}_*(p_M) \rangle.$$  

(5.8)

Then

$$\langle \hat{\eta}^*(B_{t,1}), \text{ch}_*(p) \rangle - \langle \hat{\eta}^*(B_t), \text{ch}_*(p) \rangle = -\frac{1}{\sqrt{\pi}} \langle \int_{0}^{1} \hat{\eta}^*(B^M_{t,u}, \frac{\partial B^M_{t,u}}{\partial u}) du, \text{ch}_*(p_M) \rangle.$$  

(5.9)
By \([B_{t,1}, p] = 0\), it holds that
\[
\langle b^* (B_{t,1}), ch_p \rangle = b^* (pB_t p).
\] (5.10)

By (5.9) and (5.10) and the following lemma, we know that Lemma 5.2 is correct. □

**Lemma 5.3** The following equality holds
\[
\lim_{t \to +\infty} \left( \int_0^1 \text{ch}^* (B_{t,u}^M, \frac{\partial B_{t,u}^M}{\partial u}) du, \text{ch}^* (p_{B_M}) \right) = 0.
\] (5.11)

**Proof.** By \([B_{t,u}^M, p_M] = (1 - u)[B_{t,u}^M, p_M]\) and \(\frac{\partial B_{t,u}^M}{\partial u} = (2p - 1)[B_t, p]\), we have
\[
\langle \int_0^1 \text{ch}^* (B_{t,u}^M, \frac{\partial B_{t,u}^M}{\partial u}) du, \text{ch}^* (p_{B_M}) \rangle = \sum_{l=0}^{+\infty} \frac{(2l)!}{l!} \sum_{j=0}^{l+1} (-1)^j \cdot \int_{u \in [0,1]} (1 - u)^2 \psi_t (p_M - \frac{1}{2}) [B_{t,u}^M, p_M], \cdots, (2p_M - 1)[B_{t,u}^M, p_M], \cdots, [B_{t,u}^M, p_M] \rangle_{t,u}.
\] (5.12)

For the large \(t\), we have
\[
||\text{tr} e^{-tB_{t,u}^M} || \leq c_0 e^{-t(\lambda - u|\text{dp}|)^2}.
\] (5.13)

In the following, we drop off the index \(M\). Using the same trick in Lemma 4.2 in [Wa3] and (5.13), we get the following estimate. For any \(1 \geq \sigma > 0\), \(t > 0\) and \(t\) is large and any order \(l\) fibrewise differential operator \(A\) with form coefficients, we have
\[
\|e^{-\sigma t B^2} A\|_{\sigma - 1} \leq C_0 (\sigma t)^{-\frac{1}{2} + \frac{\dim X}{2}} e^{-[(1-\varepsilon)(\lambda - u|\text{dp}|)^2 - \varepsilon]t},
\] (5.14)

where \(C_0\) is a constant and \(\varepsilon\) is any small positive constant. By (5.14) and the Hölder inequality, we have
\[
\langle p - \frac{1}{2}, [B, p], \cdots, (2p - 1)[B, p], \cdots, [B, p] \rangle_{t,u} \leq C_0 \|B_{t,u}^M\|^{2l+1} t^{\frac{\dim X}{2}} e^{-[(1-\varepsilon)(\lambda - u|\text{dp}|)^2 - \varepsilon]t},
\] (5.15)

By (5.12) and (5.15), we get
\[
\langle \int_0^1 \text{ch}^* (B_{t,u}^M, \frac{\partial B_{t,u}^M}{\partial u}) du, \text{ch}^* (p_{B_M}) \rangle = O(e^{c_0(|\text{dp}| - \lambda)t}),
\] (5.16)

where \(c_0\) is a positive constant, so Lemma 5.3 holds. □

In the last, we announce an equivariant generalization of Theorem 5.1 and more details and further generalizations will appear elsewhere.

Let \(G\) a compact Lie group acting isometrically on \(N\) and preserving the fibre and
the spin structure. We assume that $G$ acts on $(-\infty, c_0)$ trivially near the boundary. We define the equivariant eta cochain form on $M$ as follows. When $\dim M_z$ is odd, for $a_0, \cdots, a_{2k} \in C^\infty(M)$ and $a_j(x) = a_j(h \cdot x)$ for any $h \in G$, we define the family cochain $\text{ch}_{2k}(B_t, \frac{dB_t}{dt})(h)$ by the formula:

$$
\text{ch}_{2k}(B_t, \frac{dB_t}{dt})(a_0, \cdots, a_{2k})(h) = \sum_{j=0}^{2k} (-1)^j \langle a_0, [B_t, a_1], \cdots, [B_t, a_j], \frac{dB_t}{dt}, [B_t, a_{j+1}], \cdots, [B_t, a_{2k}] \rangle_t(h).
$$

(5.17)

If $A_j$ $(0 \leq j \leq q)$ are operators on $\Gamma(E)$, we define:

$$
\langle A_0, \cdots, A_q \rangle_t(h) = \int_{\Delta_q} \text{tr}^{\text{even}}[h^* A_0 e^{-\sigma_0 F_1} A_1 e^{-\sigma_1 F_1} \cdots A_q e^{-\sigma_q F_1}] dt, 
$$

(5.18)

where $h^*$ is the lift on the spinors bundle of $h$. Formally, the **equivariant eta cochain form** is defined to be an even cochain sequence by the formula:

$$
\tilde{\eta}_{2k}(B)(h) = \frac{1}{\sqrt{\pi}} \int_0^\infty \text{ch}_{2k}(B_t, \frac{dB_t}{dt})(h) dt, \quad \text{when } \dim M_z \text{ is odd};
$$

(5.19)

$$
\tilde{\eta}_{2k}(B)(h) = \int_0^\infty \text{ch}_{2k}(B_t, \frac{dB_t}{dt})(h) dt, \quad \text{when } \dim M_z \text{ is even}. 
$$

(5.20)

Then $\tilde{\eta}(B)(1)(h)$ is the equivariant eta form defined in [Wa3]. Applying Lemma 3.6 in [Wa3] instead of Lemma 2.3, similar to Lemma 2.4, we can prove the regularity of the equivariant eta cochain form. The equivariant case of Theorem 3.1 is correct without changing the proof. For $A \in \Psi^\infty_c(\check{N}, \wedge^* TX \otimes S(T \check{N}))$ and its Schwartz kernel $k_A$, define the equivariant $b$-trace which is in $\Omega(X)$ by

$$
b\text{Str}(A)(h) = \int b \text{Str}(h^* k_A(x, h \cdot x)) d\text{vol}.
$$

(5.21)

Then the equivariant case of Lemma 4.1, Lemma 4.4, Theorem 4.5 and Theorem 4.6 holds since $a_j(x) = a_j(h \cdot x)$ and $h^* B = B h^*$. So we get the equivariant generalization of Theorem 5.1. Define character forms on the fixed point submanifold $\check{N}^h$ of $h$ and the normal bundle $\check{N}$:

$$
\hat{A}(R^T \check{N}^h/X) = \text{det}^\frac{1}{2} \left( \frac{R^T \check{N}^h/X \sqrt{-1}}{\sinh(R^T \check{N}^h/X \sqrt{-1})} \right); \quad \nu(h)(R^N) := \text{det}^\frac{1}{2}(1 - h^N e^{-2\pi R^N}).
$$

(5.22)

**Theorem 5.4** Suppose that all $D_{M,z}$ are invertible with $\lambda$ the smallest positive eigenvalue of all $|D_{M,z}|$. We assume that $|d(p)| < \lambda$ and $p \in M_{r \times r}(C^\infty_{\exp}(\check{N}))$ and
\( p(h \cdot x) = p(x), \) then in the cohomology of \( X \)

\[
\text{ch}_h[\text{Ind}(pD_z + p)] = \int_{N^h/X} \hat{A}(R^{T^h/N}) \nu_h(R^N) \text{ch}(\text{Imp}) - \langle \hat{\eta}^*(B^M)(h), \text{ch}_*(p_M) \rangle.
\]

(5.23)

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