Neutron spin polarization in strong magnetic fields

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Abstract

The effects of strong magnetic fields on the inner crust of neutron stars are investigated after taking into account the anomalous magnetic moments of nucleons. Energy spectra and wave functions for protons and neutrons in a uniform magnetic field are provided. The particle spin polarizations and the yields of protons and neutrons are calculated in a free Fermi gas model. Obvious spin polarization occurs when $B \geq 10^{14}$G for protons and $B \geq 10^{17}$G for neutrons, respectively. It is shown that the neutron spin polarization depends solely on the magnetic field strength.

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The observed magnetic field strengths of neutron stars are in the range of $10^{12} \sim 10^{14}$ G [1]. For certain magnetars, which are newly born neutron stars, it can be up to $10^{15}$ G [2-5]. An extra strong magnetic field up to $10^{18}$ G [6] may exist in the interior of neutron stars according to the estimation based on the scalar virial theorem (for a neutron star with radius equaling 10km and mass equaling a sun’s). Such strong field has vast effects on the charged particles as well as uncharged particles when the anomalous magnetic moment (AMM) terms are taken into account. Thus, it influences the structure of neutron stars. Furthermore, the linear polarization in prompt $\gamma$-ray emission was discovered for GRB021206 [7], which can be attributed to the existence of a strong magnetic field in fireball. After simple calculation, for a fireball of $\gamma$-ray burst with radius 100km and total energy $2 \times 10^{51}$ erg one can infer it having a nuclear matter density about $10^6$ g/cm$^3$ and a magnetic field around $3.5 \times 10^{15}$ G. The origin of the magnetic field, though still ambiguous, might be involved with the surrounding of fireball sources. The above two situations contain the similar characteristic, i.e., the presence of strong magnetic fields in the low density matter.

The effects of the magnetic field enter in two aspects. Firstly, charged particles in a strong field can be Landau quantized. This quantization leads to polarization of particles and a consequent softening of the equation of state (EOS) [8]. Secondly, when the AMM term is taken into account, it can cause further split of energy levels, and the neutral particles can be spin polarized too. The effects of very strong magnetic fields on the equation of state for high density matter in neutron stars with the incorporation of the nuclear AMM have been studied [9, 10, 11]. An ideal neutron-proton-electron ($npe$) gas was applied to investigate the properties of low-density matter in strong magnetic fields, with the incorporation of muon degree of freedom at high density [12]. Nevertheless, the spin polarization of particles induced by the constant magnetic field has not yet been discussed thoroughly. It would be especially interesting for neutral particles since it is simply the effect of the anomalous magnetic moment terms. In this paper we investigate the particle spin polarization in a $npe$ system possessing chemical
equilibrium and charge neutrality. Calculations for pure neutron matter will also be carried out since illustratively it approaches the situation of inner crust of neutron stars.

We consider the problem of low-density nuclear matter in an external magnetic field where the strong interactions are negligible. The Lagrangian density is written as

\[ \mathcal{L} = \overline{\psi} \left( i \gamma_\mu \partial^\mu - e \frac{1 + \tau_0}{2} \gamma_\mu A^\mu - \frac{1}{4} g_b \mu_N \sigma_{\mu\nu} F^{\mu\nu} - m_N \right) \psi + \overline{\psi_e} \left( i \gamma_\mu \partial^\mu - e \gamma_\mu A^\mu - \frac{1}{4} g_e \mu_B \sigma_{\mu\nu} F^{\mu\nu} - m_e \right) \psi_e , \]

where \( A^\mu = (0, 0, Bx, 0) \) refers to a constant external magnetic field along \( z \) direction, \( \sigma_{\mu\nu} = i/2 [\gamma_\mu, \gamma_\nu] \), \( F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), and \( \tau_0 \) is the third component of the isospin operator of the nucleon. \( m_N \) and \( m_e \) are the free nucleon mass and electron mass, \( \mu_N \) and \( \mu_B \) are the nuclear magneton of nucleons and Bohr magneton of electrons, \( g_p = 3.58569470156 \), \( g_n = -3.8260854690 \), and \( g_e = \alpha/\pi \) are the coefficients of the AMM terms for protons, neutrons and electrons, respectively. Since we will consider the magnetic field up to \( 10^{18} \) G, which is stronger than the critical field of electrons, the higher order terms of the electron self-energy beyond the AMM term will play a role. The effects of those high-order terms are sophisticated and largely cancel the effects of the electron AMM term [13]. Thus, following Ref. [9], for electrons only the electromagnetic interactions are taken into account.

The Dirac equation for nucleons in a homogeneous magnetic field can be written as

\[ [i \gamma_\mu \partial^\mu - e \frac{1 + \tau_0}{2} \gamma_\mu A^\mu - \frac{1}{4} g_b \mu_N \sigma_{\mu\nu} F^{\mu\nu} - m_N ]\psi = 0 . \]

Solving the above equation in the chiral representation we obtain the energy spectra and corresponding eigenfunctions. For positive- and negative-energy protons,

\[ E_p(S, sn, \nu) = sn \left[ \left( m_p^2 + 2eB\nu + S\Delta_p \right)^2 + p_z^2 \right]^{1/2} , \]
\[ \Psi_p(S, sn, \nu) = \sqrt{\frac{E_p(S, sn, \nu) + p_z}{2E_p(S, sn, \nu)}} \left( \frac{2eB\nu e^{-iE_p(S, sn, \nu)t + ip_yy + ip_zz}}{(m_p + S\sqrt{m_p^2 + 2eB\nu})^2 + 2eB\nu} \right)^{1/2} \]

\[ \times \left( \begin{array}{c}
\frac{i}{\sqrt{2eB\nu}} (m_p + S\sqrt{m_p^2 + 2eB\nu}) I_{\nu;p_y}(x) \\
\frac{E_p(S, sn, \nu) - p_z}{\Delta_p + S\sqrt{m_p^2 + 2eB\nu}} I_{\nu-1;p_y}(x) \\
\frac{i}{\sqrt{2eB\nu}} \Delta_p + S\sqrt{m_p^2 + 2eB\nu} (p_z - E_p(S, sn, \nu)) I_{\nu;p_y}(x) \\
I_{\nu-1;p_y}(x)
\end{array} \right), \quad (4) \]

where the Hermite polynomials are defined as given in Ref. [14]

\[ I_{\nu;p_y}(x) = \left( \frac{eB}{\pi} \right)^{1/4} \exp \left[ -\frac{1}{2} \frac{eB}{eB} \left( x - \frac{p_y}{eB} \right)^2 \right] \frac{1}{\sqrt{\nu!}} H_{\nu} \left( \sqrt{2eB} \left( x - \frac{p_y}{eB} \right) \right), \]

\[ H_{\nu}(x) = (-1)^\nu \exp \left( \frac{x^2}{2} \right) \frac{d^\nu}{dx^\nu} \exp \left( -\frac{x^2}{2} \right), \quad \text{for } \nu = 0, 1, 2, \ldots . \quad (5) \]

In the above formulae, \( \Delta_p = -\frac{1}{2} g_p\mu_N B \) is the AMM term of protons, \( sn = \pm 1 \) denotes the positive-energy and negative-energy solutions and \( S = \pm 1 \) indicates the spin-up and spin-down particles, respectively. Here the spin-up and spin-down are just relative notions because now the spin operators of considered particles do not commute with the corresponding Hamiltonian. The set of functions \( I_{\nu;p_y}(x) \) is complete and orthonormal [14], and \( \nu \) is the quantum number of Landau levels for charged particles [15, 16].
\( \nu = 0 \), the eigenfunctions turn out to be
\[
\Psi_p(\pm 1, sn, 0) = \sqrt{\frac{E_p(\pm 1, sn, 0) + p_z}{2E_p(\pm 1, sn, 0)}} e^{-iE_p(\pm 1, sn, 0)t + ip_yy + ip_zz} \times iI_{\nu p_y}(x) \begin{pmatrix} 1 \\ 0 \\ \frac{p_z - E_p(\pm 1, sn, 0)}{\Delta_p + m_p} \\ 0 \end{pmatrix},
\]
where \( \Delta_p = -\frac{1}{2}g_n\mu_N B \) is the AMM term of neutrons. The solutions of Eq. (2) has been given in Ref. [9] in the Dirac representation. The above formulae obtained in the chiral representation are concise and compact. One can see that after including the AMM
terms the energy levels of nucleons become spin non-degenerate. Consequently, in the derivation the wave functions are solely determined without the need of introducing additional physical quantity to form common eigenstates. If the AMM term of electrons is taken into account, one gets the same energy spectra and eigenfunctions as those of protons. The solution of electrons without the AMM is available in Ref. [14]. We will directly use their results in the present investigation. The electron energy spectra read

\[ E_e(sn, \nu) = sn \sqrt{m_e^2 + p_e^2 + 2eB\nu} . \]  

(10)

In a free Fermi gas model the chemical equilibrium is realized between nucleons and electrons

\[ \mu_n = \mu_p + \mu_e , \]  

(11)

where the chemical potentials are defined as \( \mu_p = \epsilon_f^p \), \( \mu_n = \epsilon_f^n \) and \( \mu_e = \epsilon_f^e \). They are related with Fermi momenta as follows

\[
\left[ k_f^p(S, \nu) \right]^2 = (\epsilon_f^p)^2 - \left( \sqrt{m_p^2 + 2eB\nu + S\Delta_p} \right)^2 ,
\]

(12)

\[
\left[ k_f^n(S) \right]^2 = (\epsilon_f^n)^2 - (m_n + S\Delta_n)^2 ,
\]

(13)

\[
\left[ k_f^e(\nu) \right]^2 = (\epsilon_f^e)^2 - (m_e^2 + 2eB\nu) ,
\]

(14)

which are determined through respective particle number densities

\[ \rho_0^p = \frac{eB}{2\pi^2} \sum_S \sum_{\nu} k_f^p(S, \nu) , \]  

(15)

\[ \rho_0^n = \frac{1}{2\pi^2} \sum_S \left\{ \frac{1}{3} \left[ k_f^n(S) \right]^3 + \frac{S\Delta_n}{2} \right. 
\]

\[ \left. \times \left[ (m_n + S\Delta_n)k_f^n(S) + (\epsilon_f^n)^2 \left( \arcsin \frac{m_n + S\Delta_n}{\epsilon_f^n} - \frac{\pi}{2} \right) \right] \right\} , \]  

(16)

\[ \rho_0^e = \frac{eB}{2\pi^2} \sum_S \sum_{\nu} k_f^e(\nu) . \]  

(17)

The index \( \nu \) runs up to the largest integer for \( [k_f^p(S, \nu)]^2 \) to be positive according to Eq. (12). \( \rho_0^p = \rho_0^e \) is required for charge neutral matter. We consider the density from neutron drip point \( (4.3 \times 10^{11} \text{g/cm}^3) \) [17] to a quarter of nuclear saturation density, i.e.,
around $10^{11} \sim 10^{14}$ g/cm$^3$. The nonlinear equations preserving the charge neutral and chemical equilibrium [18] conditions are solved numerically in an iteration procedure.

Figure 1 shows the spin ratio of protons in a $npe$ system as a function of density. Different magnetic field strengths are considered in calculations as indicated in the figure. We see that the proton spin polarization increases with the increasing of the magnetic field strength and the decreasing of baryon density. The incorporating of the AMM term benefits the spin polarization. Evident proton spin polarization is exhibited at $B \geq 10^{14}$ G. On the other hand, protons become completely spin polarized throughout the range of density when $B \geq 10^{17}$ G.

The spin ratios of neutrons in a $npe$ system with the AMM term incorporated are reckoned at different magnetic field strengths and displayed in figure 2 as a function of density. We can see that the neutron spin polarization increases with the increasing of the magnetic field strength and the decreasing of baryon density, too. However, the neutron polarization is much weaker than that of protons because it is simply caused by the AMM term. Due to the opposite sign of this term neutrons incline to occupy the spin-down direction. Complete neutron spin polarization appears at $B \sim 10^{17}$ G. We have also investigated the pure neutron matter. It is interesting to see that the resultant figure of neutron spin polarization is the same as given in Fig. 2, while in the $npe$ system the particle fraction changes substantially. The ratios of proton number to the nucleon number are depicted in Fig. 3 for chemical-equilibrated $npe$ system. At strong magnetic field the proton fraction is quite large. The same neutron spin polarization in pure neutron matter and $npe$ matter indicates that it is sensitive to the magnetic field strength only.

In summary, the Dirac equations of protons and neutrons are solved in the chiral representation for a constant magnetic field with the anomalous magnetic moment terms taken into account. The properties of low-density matter in the presence of strong magnetic fields are investigated in a relativistic Fermi gas model. Obvious spin polarizations for protons and neutrons are obtained. It turns out that the neutron spin
polarization is the same both in the pure neutron matter and the chemical-equilibrated \( npe \) system.

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Figure 1: Proton spin polarization at different magnetic field strengths as a function of density. The upper and lower panel represent the conditions of without and with the AMM term of protons. The magnetic field strengths are tagged on the lines.
Figure 2: Neutron spin polarization at different magnetic field strengths as a function of density. The figure displays the results of pure neutron matter and chemical-equilibrated $npe$ system.

Figure 3: Proton fraction calculated at different magnetic field strengths as a function of density. The cases without and with the AMM term are denoted by dashed and dotted lines, respectively.