$E_{10}$ Symmetry

in One-dimensional Supergravity

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Abstract

We consider dimensional reduction of the eleven-dimensional supergravity to less than four dimensions. The three-dimensional $E_{8(+8)}/SO(16)$ nonlinear sigma model is derived by direct dimensional reduction from eleven dimensions. In two dimensions we explicitly check that the Matzner-Misner-type $SL(2, \mathbb{R})$ symmetry, together with the $E_8$, satisfies the generating relations of $E_9$ under the generalized Geroch compatibility (hypersurface-orthogonality) condition. We further show that an extra $SL(2, \mathbb{R})$ symmetry, which is newly present upon reduction to one dimension, extends the symmetry algebra to a real form of $E_{10}$. The new $SL(2, \mathbb{R})$ acts on certain plane wave solutions propagating at the speed of light. To show that this $SL(2, \mathbb{R})$ cannot be expressed in terms of the old $E_9$ but truly enlarges the symmetry, we compactify the final two dimensions on a two-torus and confirm that it changes the conformal structure of this two-torus.

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1 Introduction

It is now widely believed that there exists a fundamental eleven-dimensional quantum theory which incorporates all five superstring theories. Various duality symmetries of compactified superstrings \cite{1, 2, 3} are discrete subgroups of “hidden symmetries” of their effective supergravity theories. An interesting picture is that there is some huge discrete symmetry of the most symmetric vacuum of M theory, and other dualities arise according to the variety of symmetry breaking. This fundamental symmetry would then include all known duality groups as subgroups.

The eleven-dimensional supergravity \cite{4} possesses (continuous) $E_{7(+7)}$ global (with $SU(8)$ local) symmetry \cite{5} when it is compactified to four dimensions on a seven torus $T^7$. Its discrete subgroup $E_{7(+7)}(\mathbb{Z})$ is known as U-duality of the compactified typeII superstring \cite{3}. Below four dimensions the eleven-dimensional supergravity exhibits $E_8$ and $E_9$ in three and two dimensions \cite{6, 7, 8, 9, 10} with the compactification spaces being $T^8$ and $T^9$, respectively. One is then curious about what happens if one further goes to one dimension. In 1982 Julia conjectured that the symmetry group will be enlarged to $E_{10}$ \cite{11}, whose Lie algebra belongs to a certain class of Kac-Moody algebra, “hyperbolic Kac-Moody algebra”. This paper is an attempt to answer to this question.

The hyperbolic Kac-Moody algebra $E_{10}$ is defined by the Cartan matrix

\[
K_{ij} = \begin{bmatrix}
2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{bmatrix}, \tag{1.1}
\]

whose Dynkin diagram is shown in Fig.1. In general one can define a Kac-Moody algebra associated with any $N \times N$ Cartan matrix $K_{ij}$ that satisfies (i) $K_{ii} = 2$, (ii) $K_{ij}$ ($i \neq j$) is non-positive integer and (iii) if $K_{ij} = 0$ then $K_{ji} = 0$. Given such a matrix $K_{ij}$, the algebra is defined as arbitrary number of multiple commutators with the relations

\[
[h_i, e_j] = K_{ij} e_j, \quad [h_i, f_j] = -K_{ij} f_j, \quad [e_i, f_j] = \delta_{ij} h_j, \quad [h_i, h_j] = 0 \tag{1.2}
\]
Figure 1: Dynkin diagram of $E_{10}$.

$$i, j = 1, \ldots, N$$

and the Serre relations

$$(\text{ad} e_i)^{1-K_{ij}} e_j = 0, \quad (\text{ad} f_i)^{1-K_{ij}} f_j = 0. \quad (1.3)$$

The set of generators $e_i, f_i, h_i (i = 1, \ldots, N)$ are called the Chevalley generators. A Kac-Moody algebra is said hyperbolic if a removal of any vertices from its Dynkin diagram leaves diagrams of either finite-dimensional simple Lie algebras, affine Kac-Moody algebras or their direct sum. A well-known intriguing feature of hyperbolic Kac-Moody algebras, in particular in relation to supergravity and string theory, is that hyperbolic Kac-Moody algebras can exist only up to rank = 10. $E_{10}$ is one of three hyperbolic Kac-Moody algebras with the possible highest rank (See [13] for a review.). One is tempted to suspect [11] that this restriction might be linked in some way to the fact that the allowed highest dimensions for supergravity is eleven.

Toward the proof of the $E_{10}$ symmetry, an important progress was brought about in 1991 by Nicolai, who pointed out a special feature of dimensional reduction to one dimension [15]. He showed in $D = 4$, $N = 1$ supergravity that one must take a null Killing vector at the stage of the reduction from two to one dimension if one wants to keep the duality relations non-trivial. After this novel type of reduction [1], he wrote out for the first time the set of the action of the Chevalley generators on the fields of the $SL(2, \mathbb{R})$ symmetry which newly emerged in one dimension, confirming that this together with the Lie algebra of the Geroch group generates the hyperbolic “$SL(2, \mathbb{R})$ double-hat” algebra.

What was subtle in his argument was the independence of the new $SL(2, \mathbb{R})$. Suppose that one starts with a conformal-gauge two-metric (i.e. a zweibein proportional to the identity) as in ref.[16] by invoking the degrees of freedom of diffeomorphisms. Introducing a constant Killing vector then leads it to a flat metric, which is essen-

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1See [16] for other aspects of null reduction.
tially unique on a plane \( \mathbb{R}^2 \). Hence the new \( SL(2, \mathbb{R}) \) can do nothing more than the old Geroch group can on the two-metric because a global scale transformation can be realized by the central charge of the Geroch group. For this reason, although the \( E_9 \) symmetry has been known for some time, the straightforward-looking generalization to \( E_{10} \) has never been accomplished.

In this paper, to achieve the non-trivial realization of “Nicolai’s \( SL(2, \mathbb{R}) \)”, we parameterize the two-metric by Lorentzian analogue of Beltrami differentials and wrap the final two dimensions on a two-torus. In fact, even when a non-conformally-flat metric has been chosen, it necessarily becomes flat again once a constant null Killing vector is introduced. The escape in our case is that the new \( SL(2, \mathbb{R}) \) can act on the conformal structure of the two-torus. One may view these solutions as certain plane wave solutions propagating at the speed of light. We will show that the new \( SL(2, \mathbb{R}) \) does cause a change of a “Beltrami differential” of the Lorentzian metric that cannot be absorbed by any diffeomorphisms.

Another issue in establishing the \( E_{10} \) symmetry is how the fields of the eleven-dimensional supergravity parameterize the symmetric space \( E_{8(+8)}/SO(16) \) after the reduction to three dimensions. Although the three-dimensional locally supersymmetric \( E_{8(+8)}/SO(16) \) nonlinear sigma model is well-known, the direct construction from eleven dimensions by dimensional reduction seems to have never appeared in print. Of course one could also start from \( E_{7(+7)}/SU(8) \) in four dimensions, but the way \( E_{7(+7)}/SU(8) \) is embedded into \( E_{8(+8)}/SO(16) \) is a bit complicated. In this paper we obtain the \( E_{8(+8)}/SO(16) \) nonlinear sigma model by direct dimensional reduction from eleven to three dimensions. We use Freudenthal’s classical realization of \( E_8 \) It turns out that this realization clearly reflects the (bosonic) field content in three dimensions after duality transformations, showing how the eleven-dimensional fields fit into the \( E_{8(+8)}/SO(16) \) scalar manifold.

Our strategy for the construction of \( E_{10} \) is as follows. We proceed step by step. We first read off the set of transformations that correspond to the Chevalley generators of \( E_{8(+8)} \) in the \( E_{8(+8)}/SO(16) \) sigma model. Next we reduce the dimension to two,
write out the Chevalley generators of the extra $SL(2, \mathbb{R})$ in two dimensions, and confirm that this $SL(2, \mathbb{R})$ has correct commutation relations with the $E_8$ so that the symmetry is extended to $E_9$. Finally we reduce one more dimension, find the new $SL(2, \mathbb{R})$ transformation, and check that it successfully enlarges the symmetry algebra to $E_{10}$.

This paper is organized as follows. In sect.2 we derive the $E_{8(+8)}/SO(16)$ nonlinear sigma model by dimensional reduction from eleven dimensions and read off the transformations corresponding to the Chevalley generators. In sect.3 we deal with the reduced two-dimensional model and check that the symmetry is enlarged to $E_9$. In sect.4 we reduce the dimension to one and show that the symmetry is $E_{10}$. The non-triviality of Nicolai’s $SL(2, \mathbb{R})$ is also proved. Finally we give conclusions in sect.5.

2 $E_8$ in three dimensions

2.1 $E_8/SO(16)$ nonlinear sigma model from the eleven-dimensional supergravity

We start with dimensional reduction of the eleven-dimensional supergravity \[^4\] to three dimensions. The bosonic Lagrangian with all fermionic fields set to zero is

$$\mathcal{L} = E^{(11)} \left[ R^{(11)} - \frac{1}{12} F_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} F^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}} \right] + \frac{8}{12^4} e^{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4\hat{\mu}_5\hat{\mu}_6\hat{\mu}_7\hat{\mu}_8\hat{\mu}_9\hat{\mu}_{10}\hat{\mu}_{11}} \hat{F}_{\hat{\mu}_1\hat{\mu}_2\hat{\mu}_3\hat{\mu}_4} \hat{F}_{\hat{\mu}_5\hat{\mu}_6\hat{\mu}_7\hat{\mu}_8} A_{\hat{\mu}_9\hat{\mu}_{10}\hat{\mu}_{11}}. \quad (2.1)$$

$\hat{\mu}, \hat{\nu}, \ldots$ are the eleven-dimensional spacetime indices. Decomposing $x^{\hat{\mu}}$ into uncompactified coordinates $x^{\hat{\mu}} = t, x, y$ and compactified coordinates $x^i, i = 1, \ldots, 8$ and dropping the $x^i$-dependence, we get the three-dimensional Lagrangian

$$\mathcal{L} = E^{(3)} \left[ R^{(3)} - G^{(3)}_{\mu\nu} \partial_{\mu} \ln e \partial_{\nu} \ln e + \frac{1}{4} G^{(3)}_{\mu\nu} g^{ij} \partial_{\mu} g_{ij} \right]$$

[^4]: Note that if the relations \((1.2)\) and \((1.3)\) are regarded as those of a real Lie algebra, with $K_{ij}$ defined by \((2.43)\), they define a real form $E_{8(+8)}$ of the complex Lie algebra $E_8$. (The number $+8$ in parenthesis denotes the difference between the numbers of positive and negative generators in the invariant bilinear form.) For any simple Lie algebra $X_N$, the real Lie algebra defined in this way is always $X_N(+N)$, since positive and negative roots can be paired either symmetrically or anti-symmetrically, giving the same number of generators with opposite signs. This leaves $N$ excess coming from the Cartan subalgebra. In that sense the symmetries should be called $E_{9(+9)}$ and $E_{10(+10)}$, respectively, although both are infinite-dimensional. They are called ‘normal’ (or ‘split’) real forms.
To complete squares of $F_{\mu
u}$ where the eleven-dimensional Lorentz indices $\hat{\alpha}$ and $B_{\mu
u}$ is the Kaluza-Klein vector fields associated with the decomposition of the elfbein $E^{(11)}_\hat{\mu} = \begin{bmatrix} e^{-1}E_3^{(3)} & B_\mu^i e_i^a \\ 0 & e_i^a \end{bmatrix}$. The local Lorentz flat metric is

$$\eta_{\hat{\alpha}\hat{\beta}} = \begin{bmatrix} \eta_{\alpha\beta}^{(3)} \\ \delta_{ab} \end{bmatrix},$$

where the eleven-dimensional Lorentz indices $\hat{\alpha}$ are also decomposed into $\alpha = 0, \ldots, 3$ and $a = 1, \ldots, 8$ similarly. $\eta_{\alpha\beta}^{(3)}$ has signature $(-+++)$. $A'_\mu$ is an invariant combination under Kaluza-Klein gauge transformations.

The degrees of freedom of vector fields in three dimensions can be traded by those of scalar fields by duality transformations. To see this, a usual trick is to introduce the following Lagrange multiplier terms

$$\mathcal{L}_{\text{Lag.mul.}} = \varphi^{ij} \varepsilon^{\mu
u\rho} \partial_\mu F'_{\nu\rho j} + \frac{1}{2} \psi_i \varepsilon^{\mu
u\rho} \partial_\mu B_{\nu\rho}^i.$$  

Up to complete squares of $F'_{\mu
u}$ and $B_{\mu
u}^i$, we find

$$\mathcal{L} + \mathcal{L}_{\text{Lag.mul.}} = E^{(3)} \left[ R^{(3)} + G^{(3)\mu\nu} \left( \frac{1}{4} \partial_\mu g^{ij} \partial_\nu g_{ij} - \partial_\mu \ln e \partial_\nu \ln e - \frac{1}{3} g^{ij} g^{jm} g^{kn} \partial_\mu A_{ijk} \partial_\nu A_{lmn} + 2 g^{ij} \partial_\mu A_{ij} \partial_\nu A_{lmn} \right) \right.$$  

$$- \frac{1}{3} \varepsilon^{\mu
u\rho} \partial_\mu g^{ij} \partial_\rho g_{ij} - \frac{1}{3} \varepsilon^{ijpqr1st1u1} \partial_\mu A_{pqr1s} A_{t1u1} \right) \cdot \left( \partial_\nu \varphi^{kl} - \frac{1}{3} \varepsilon^{klpqr2s2t2u2} \partial_\nu A_{pqr2s} A_{2t2u2} \right)$$  

$$- \frac{1}{2} e^{-2} g^{ij} \left( \partial_\mu \varphi_i + \partial_\mu A_{ik1l1} \varphi_k^{l1} - A_{ik1l1} \partial_\mu \varphi_k^{l1} + \frac{1}{54} \varepsilon^{m1n1pqr1st1u1} A_{m1n1} \partial_\mu A_{pqr1s} A_{t1u1} \right) \cdot \left( \partial_\nu \varphi_j + \partial_\nu A_{jk2l2} \varphi^{k2l2} - A_{jk2l2} \partial_\nu \varphi^{k2l2} + \frac{1}{54} \varepsilon^{m2n2pqr2s2t2u2} A_{m2n2} \partial_\nu A_{pqr2s} A_{2t2u2} \right) \right],$$

where

$$F'_{\mu
u} = 2 \partial_\mu A'_{\nu ij}, \quad A'_{\mu ij} = A_{\mu ij} - B^k_{\mu} A_{ijk},$$

$$B_{\mu
u}^i = 2 \partial_\mu B_{\nu}^i.$$
where the degrees of freedom of $28 + 8$ three-dimensional vectors $A'_{\mu ij}$ and $B^i_{\mu}$ are replaced by those of the scalars $\phi^{ij}$ and $\psi_i$. Duality relations are

$$F^{(3)\mu
u ij} = -E^{(3)-1}e^{-2\epsilon_{\mu\nu\rho}}(\partial_{\rho}\phi^{ij} - \frac{1}{36}\epsilon^{ijklmnpq}\partial_{\rho}A_{kln}A_{npq}) \quad (2.9)$$

with $F^{\mu\nu ij}_\mu = F'_{\mu
u ij} + B^k_{\mu\nu}A_{ijk}$, and

$$B^\mu_{\nu} = -E^{(3)-1}e^{-2\epsilon_{\mu\nu\rho}}(\partial_{\rho}\psi_k - A_{kij}\partial_{\rho}\phi^{ij} + \phi^{ij}\partial_{\rho}A_{kij}$$

$$+ \frac{1}{54}\epsilon^{ijklmnpq}A_{ijk}\partial_{\rho}A_{lmn}A_{pqr}). \quad (2.10)$$

The derivation of (2.8) is standard.

We will now show how this nonlinear sigma model fits into the symmetric space $E_8(\pm 8)/SO(16)$. We first observe that if all the fields originating from $A_{\hat{\mu}\hat{\nu}\hat{\rho}}$ are turned off, the remaining scalars describe $SL(9,\mathbb{R})/SO(9)$. This can be seen as follows. The sigma-model part of the Lagrangian (2.8) is then simply given by

$$E^{(3)}G^{(3)\mu\nu}(\frac{1}{4}\partial_{\mu}g^{ij}\partial_{\nu}g_{ij} - \partial_{\mu}\ln e\partial_{\nu}\ln e - \frac{1}{2}e^{2}\partial_{\mu}\psi_{i}\partial_{\nu}\psi_{j}). \quad (2.11)$$

We parameterize the coset $SL(9,\mathbb{R})/SO(9)$ in terms of a nine-by-nine symmetric matrix $M \equiv VV^T$ with, taking $e_i^a$ to be upper-triangular,

$$V'_{i' a'} = \begin{bmatrix} e_i^a & -e^{-1}\psi_i \\ 0 & e^{-1} \end{bmatrix}. \quad (2.12)$$

Here we introduced the extended curved and flat indices $i' = i, 9$ and $a' = a, 9$ in a similar way to [5] where the $SL(7,\mathbb{R})$ is enlarged to the $SL(8,\mathbb{R})$. The form of $VV^T$ ensures that the orthogonal group factor is gauged out. One may then easily see that (2.11) can be written

$$\frac{1}{4}E^{(3)}G^{(3)\mu\nu}\text{Tr}\partial_{\mu}M^{-1}\partial_{\nu}M. \quad (2.13)$$

This is an example of a general rule that $d$-dimensional pure gravity exhibits $SL(d-2,\mathbb{R})/SO(d-2)$ symmetry when it is reduced to three dimensions [1]. Another (the oldest) example is “Ehlers’ $SL(2,\mathbb{R})$” [18, 9] known for a long time in general relativity.

As a complex Lie algebra, $SL(9,\mathbb{C})$ can be extended to $E_8$ by adding two third-rank antisymmetric tensors [17]. Let $V$ be the space of third-rank antisymmetric tensor representation of $SL(9,\mathbb{C})$, and $V^*$ be the dual space of $V$, $E_8$ is decomposed as $SL(9,\mathbb{C}) \oplus V \oplus V^*$. Their dimensions are $248 = 80 \oplus 84 \oplus \overline{84}$. $E_8$ can be defined
by the following commutation relations (Lie brackets):

\[
[X, \ Y]^{j'}_{i'} = X^{l'}_{p'} Y^{j'}_{k'} - X^{j'}_{i'} Y^{l'}_{k'},
\]

\[
[X, \ v]^{j'}_{i'} = X^{l'}_{p'} v^{j'}_{k'} + X^{j'}_{i'} v^{l'}_{k'} + X^{l'}_{k'} v^{j'}_{i'},
\]

\[
[X, \ v^s]^{j'}_{i'} = -\left(X^{l'}_{p'} v^{s} v^{j'}_{k'} + X^{j'}_{i'} v^{s} v^{l'}_{k'} + X^{l'}_{k'} v^{s} v^{j'}_{i'}\right),
\]

\[
[v, \ w]^{j'}_{i'} = \frac{1}{36\sqrt{3}} \epsilon^{j'}_{j'k'p'} \epsilon^{l'}_{l'q'} v^{l'}_{i'} v^{j'}_{k'} \nu^{p'}_{q'},
\]

\[
[v^s, \ w^s]^{j'}_{i'} = \frac{1}{36\sqrt{3}} \epsilon^{s}^{j'k'p'} \epsilon^{s}^{l'q'} v^{s} v^{j'}_{k'} \nu^{s} v^{s} v^{l'}_{q'},
\]

\[
[v, \ w^s]^{j'}_{i'} = -\frac{1}{6} \left(v^{s} v^{i'} v^{j'} k' l' m' n' - \frac{1}{9} \delta^{j'}_{i'} \nu^{k'} m' v^{s} k' l' m' n'\right),
\]

(2.14)

\[X^{j'}_{i'}, Y^{j'}_{i'} \in SL(9, \mathbb{C}), \ v^{j'}_{i'}, w^{j'}_{i'} \in V \text{ and } v^{s} v^{j'}_{i'}, w^{s} v^{j'}_{i'} \in V^s.\]

If these relations are regarded as those of a real Lie algebra, then they define \(E_{8(\pm8)}\). We write down the generators in adjoint representation of \(E_{8(\pm8)}\) in the three-by-three block form

\[
\text{ad} X = \begin{bmatrix}
X^{l'}_{p'} \delta^{j'}_{m'} & X^{j'}_{i'} \delta^{l'}_{n'} & 0 \\
0 & X^{l'}_{p'} m' n' & 0 \\
0 & 0 & -X^{j'}_{i'} k' l' m' n'
\end{bmatrix},
\]

(2.15)

\[
\text{ad} v = \begin{bmatrix}
-3(v^{l'}_{m'} k' p' q' \delta^{j'}_{n'}) & 0 & \frac{1}{36\sqrt{3}} \epsilon^{j'}_{j'k'p'} \epsilon^{l'}_{l'q'} v^{l'}_{i'} v^{j'}_{k'} \\
0 & 0 & \frac{1}{6} (v^{s} v^{j'}_{i'} m' n' \delta^{l'}_{k'}) - \frac{1}{6} v^{s} v^{l'}_{k'} m' n' \delta^{j'}_{i'} \\
3(v^{s} v^{l'}_{m'} \delta^{j'}_{n'}) & 0 & 0
\end{bmatrix},
\]

with \(X^{l'}_{p'} m' n' \equiv X^{l'}_{p'} \delta^{m'}_{n'} + X^{j'}_{i'} \delta^{l'}_{n'} + X^{l'}_{k'} \delta^{j'}_{n'} \delta^{l'}_{i'} \). They map tensors \((v^{m'}_{i'}, w^{m'}_{m' n'})\) to \((v^{s} v^{l'}_{m'} n', w^{s} v^{l'}_{m' n'})\). 

In view of the structure of the algebra, \(56 + 28 = 84\ A_{i j k}\) and \(\varphi^{i j}\) should fit somehow into \(84 + 84\ v^{i j k'},\ v^{s} v^{i j k'}\). To find the correct assignment, a useful hint is that \(\text{ad}(v + v^s)\) must be a nilpotent matrix so that \(\exp(\text{ad}(v + v^s))\) becomes a polynomial.
The most natural candidate is the case that all $A_{ijk}$ and $\varphi^{ijk}$ correspond only to the elements associated with negative (or positive) roots. In this way a half of $84 + 84$ elements are selected. Moreover, it turns out that $v_{ijk}$ and $v^{*ijk}$ ($i, j, k = 1, \ldots, 8$) can be taken as the elements corresponding to negative roots. In view of this nice index structure, we assume

\[
v_{i'j'k'} \equiv A_{i'j'k'} = -2\sqrt{3}\delta^{i'j'}_{ij} \delta^{j'k'}_{jk} A_{ijk},
\]

\[
v^{*i'j'k'} \equiv \varphi^{i'j'k'} = -2\sqrt{3} \cdot 3 \delta^{[i'j']_{ij}} \delta^{i'k'}_{jk} \varphi^{ij},
\]

where $-2\sqrt{3}$ is a normalization for later convenience. One may then verify that $(\text{ad}(v + v^*))^5 = 0$. Therefore $\exp(\text{ad}(v + v^*))$ is a fourth-order polynomial of $A_{i'j'k'}$ and $\varphi^{i'j'k'}$. After some calculation we obtain

\[
\exp(\text{ad}(v + v^*)) = \begin{bmatrix}
\exp(\text{ad}(v + v^*))_{Y,Y} & \exp(\text{ad}(v + v^*))_{Y,w} & \exp(\text{ad}(v + v^*))_{Y,w^*} \\
\exp(\text{ad}(v + v^*))_{w,Y} & \exp(\text{ad}(v + v^*))_{w,w} & \exp(\text{ad}(v + v^*))_{w,w^*} \\
\exp(\text{ad}(v + v^*))_{w^*,Y} & \exp(\text{ad}(v + v^*))_{w^*,w} & \exp(\text{ad}(v + v^*))_{w^*,w^*}
\end{bmatrix},
\]

(2.16)

\[
\exp(\text{ad}(v + v^*))_{Y,Y} = \delta^{t'}_{t} \delta^{l'}_{l} - \frac{1}{12} \delta^{t'}_{t} \delta^{l'}_{l} + \frac{1}{12} (A_{ijp'q'} \varphi^{i'j'}_{pq'} \delta^{l'}_{l} + A_{im'p'q'} \varphi^{i'j'}_{pq'} \delta^{j'}_{j}) + 4 A_{ijp'q'} \varphi^{i'j'}_{pq'} \delta^{l'}_{l} + \frac{1}{432} \delta^{t'}_{t} \delta^{l'}_{l} \delta^{i'}_{i} \delta^{j'}_{j} \delta^{k'}_{k} + \frac{1}{3} A_{ijp'q'} \varphi^{i'j'}_{pq'} \delta^{l'}_{l} \delta^{i'}_{i} \delta^{j'}_{j} \delta^{k'}_{k} - \frac{1}{3} A_{im'p'q'} \varphi^{i'j'}_{pq'} \delta^{l'}_{l} \delta^{i'}_{i} \delta^{j'}_{j} \delta^{k'}_{k},
\]

(2.17)

\[
\exp(\text{ad}(v + v^*))_{Y,w} = \frac{1}{6} \varphi_{[m'n']_{ij}} - \frac{1}{9} \delta^{t'}_{t} \varphi_{t'm'n'} - \frac{1}{432} \delta^{t'}_{t} \varphi_{t'm'n'} A_{ijp'q'} A_{p'q'r'} + \frac{1}{36} A_{ijp'q'} \varphi^{i'j'}_{pq'} \varphi^{t'}_{t'} A_{p'q'r'} \varphi^{t'}_{t'} A_{p'q'} \varphi^{t'}_{t'} - \frac{1}{3} A_{ijp'q'} \varphi^{i'j'}_{pq'} \varphi^{t'}_{t'} A_{p'q'r'} \varphi^{t'}_{t'} A_{p'q'} \varphi^{t'}_{t'} A_{p'q'r'} A_{p'q'q'},
\]

(2.18)

\[
\exp(\text{ad}(v + v^*))_{Y,w^*} = \frac{1}{6} \varphi_{[m'n']_{ij}} - \frac{1}{9} \delta^{t'}_{t} \varphi_{t'm'n'} - \frac{1}{432} \delta^{t'}_{t} \varphi_{t'm'n'} A_{ijp'q'} A_{p'q'r'} + \frac{1}{36} A_{ijp'q'} \varphi^{i'j'}_{pq'} \varphi^{t'}_{t'} A_{p'q'r'} \varphi^{t'}_{t'} A_{p'q'} \varphi^{t'}_{t'} - \frac{1}{3} A_{ijp'q'} \varphi^{i'j'}_{pq'} \varphi^{t'}_{t'} A_{p'q'r'} \varphi^{t'}_{t'} A_{p'q'} \varphi^{t'}_{t'} A_{p'q'r'} A_{p'q'q'},
\]

(2.19)
\[
= -\frac{1}{6}(A_{\nu'[m'n']\delta_{\mu}} - \frac{1}{9} \delta_{\mu} A_{\nu'[m'n']}) + \frac{1}{432\sqrt{3}} \varepsilon_{\nu'[w'q'] \nu'[w']\nu'[w']} A_{\nu'[m'n'] A_{\nu'[q'r']} A_{s't'u'}} \\
- \frac{1}{36} \left( \frac{1}{6} \varphi_j^{p'q'} A_{\nu'[p'q'] A_{\nu'[m'n']}} - \varphi_j^{p'q'} A_{\nu'[p'[v]} A_{\nu'[q']]} \right) \\
+ \frac{1}{432\sqrt{3}} \varepsilon_{\nu'[w'q'] \nu'[w']\nu'[w']} A_{\nu'[m'n'] A_{\nu'[q'r']} A_{s't'u'}},
\] (2.20)

\[
\exp(\text{ad}(v + v^*))_{w,Y} = -3(A_{\nu'[j'k'] \delta_{\mu}} - \frac{1}{9} A_{\nu'[j'k'] \delta_{\mu}}) + \frac{1}{24\sqrt{3}} \varepsilon_{\nu'[w'q'q'] \nu'[w']\nu'[w']} \varphi_i^{p'q'} \varphi_j^{q'r'} \\
+ \frac{1}{6} \left( -\frac{1}{2} \varphi_j^{p'q'} A_{\nu'[p'q'] A_{\nu'[j'k']}} + 3 \varphi_j^{p'q'} A_{\nu'[p'[v]} A_{\nu'[j'k']} \right) \\
- \frac{1}{24\sqrt{3}} \varepsilon_{\nu'[w'q'q'] \nu'[w']\nu'[w']} A_{\nu'[j'k'] A_{\nu'[q'r']} A_{s't'u'}},
\] (2.21)

\[
\exp(\text{ad}(v + v^*))_{w,w} = \delta_{[\nu'[j'k'] \delta_{\mu}} + \frac{1}{2} \left( \frac{1}{9} A_{\nu'[j'k'] \delta_{\mu}} A_{\nu'[j'k']} \right) - (A_{\nu'[j'k'] \delta_{\mu}} A_{\nu'[j'k'] A_{\nu'[j'k'] \delta_{\mu}}} + \frac{1}{9} A_{\nu'[j'k'] \delta_{\mu}} A_{\nu'[j'k'] \delta_{\mu}}) \\
+ \frac{1}{72\sqrt{3}} \varepsilon_{\nu'[w'q'q'] \nu'[w']\nu'[w']} A_{\nu'[j'k'] A_{\nu'[q'r']} A_{s't'u'}},
\] (2.22)

\[
\exp(\text{ad}(v + v^*))_{w,w^*} = \frac{1}{36\sqrt{3}} \varepsilon_{\nu'[j'k'] \nu'[w'q'] \nu'[w']} \varphi_i^{p'q'} \\
+ \frac{1}{4} \left( A_{\nu'[j'k'] A_{\nu'[j'k']}} - \frac{1}{9} A_{\nu'[j'k'] A_{\nu'[j'k']}} \right),
\] (2.23)

\[
\exp(\text{ad}(v + v^*))_{w^*,Y} = +3(\varphi_j^{[j'k']} \delta_{\mu}} - \frac{1}{9} \varphi_j^{[j'k']} \delta_{\mu}}) - \frac{1}{24\sqrt{3}} \varepsilon_{\nu'[w'q'] \nu'[w']\nu'[w']} A_{\nu'[w'q'] A_{\nu'[w']}},
\] (2.24)

\[
\exp(\text{ad}(v + v^*))_{w^*,w} = \frac{1}{36\sqrt{3}} \varepsilon_{\nu'[j'k'] \nu'[w'q'] \nu'[w']} A_{\nu'[w'q']} \\
+ \frac{1}{4} \left( \varphi_j^{[j'k']} \varphi_i^{m'n'} - \frac{1}{9} \varphi_j^{[j'k']} \varphi_i^{m'n'} \right),
\] (2.25)
Here the bracket means that the indices inside are totally anti-symmetrized. We have employed the equations $A_{\nu'j'k'} \varphi_{\nu'j'k'} = 0$, $A_{\nu'j'k'}A_{\nu''k''l''} \varphi_{\nu''k''l''} = 0$, etc. We further calculate $\exp(-\text{ad}(v + v^*)) \cdot \partial_{\mu} \exp(\text{ad}(v + v^*))$ to find

$$
\begin{align*}
\exp(-\text{ad}(v + v^*)) \partial_{\mu} \exp(\text{ad}(v + v^*)) &= \\
&= \left[ Y_{\nu'}^{\nu} \delta_{m'}^{\nu} - Y_{m'}^{\nu} \delta_{\nu'}^{\nu} - \frac{1}{6} \left( w^{*j'[m'n']} \delta_{i'}^{\nu} - \frac{1}{9} w^{*l'm'n'} \delta_{i'}^{\nu} \right) - \frac{1}{6} \left( w_{i'}^{[m'n']} \delta_{i'}^{\nu} - \frac{1}{9} w_{l'm'n'} \delta_{i'}^{\nu} \right) \right] \\
&= -3 \left( w_{m'j'k'} \delta_{i'}^{\nu} - \frac{1}{9} w_{i'j'k'} \delta_{m'}^{\nu} \right) Y_{m'j'k'}^{\nu} - \frac{1}{36} \epsilon^{i'j'k'} \epsilon^{r's't'u'} \partial_{\mu} A_{\nu'j'k'}^{r's't'u'} \\
&= -3 \left( w^{*i'j'k'} \delta_{m'}^{\nu} - \frac{1}{9} w^{*i'j'k'} \delta_{m'}^{\nu} \right) \epsilon^{i'j'k'} \epsilon^{r's't'u'} \partial_{\mu} A_{\nu'j'k'}^{r's't'u'} - Y_{m'j'k'}^{\nu} \\
&= \left( Y_{\nu'}^{\nu} - \frac{1}{12} \left( A_{\nu'j'k'} \partial_{\mu} \varphi_{\nu'j'k'} - \varphi_{\nu'j'k'} \partial_{\mu} A_{\nu'j'k'} \right) \right) \\
&- \frac{1}{1296 \sqrt{3}} \epsilon^{\nu'w'q'r's't'u'} A_{\nu'w'} A_{q'r's't'u'} \partial_{\mu} A_{s't'u'} \partial_{\mu} A_{s't'u'},
\end{align*}
$$

with

$$
\begin{align*}
Y_{\nu'}^{\nu} &= \frac{1}{12} \left( A_{\nu'j'k'} \partial_{\mu} \varphi_{\nu'j'k'} - \varphi_{\nu'j'k'} \partial_{\mu} A_{\nu'j'k'} \right) \\
&- \frac{1}{1296 \sqrt{3}} \epsilon^{\nu'w'q'r's't'u'} A_{\nu'w'} A_{q'r's't'u'} \partial_{\mu} A_{s't'u'} \partial_{\mu} A_{s't'u'}, \\
w_{\nu'j'k'} &= \partial_{\mu} A_{\nu'j'k'}, \\
w^{*i'j'k'} &= \partial_{\mu} \varphi_{\nu'j'k'} - \frac{1}{72 \sqrt{3}} \epsilon^{i'j'k'} \epsilon^{r's't'u'} \partial_{\mu} A_{s't'u'}. \quad (2.28)
\end{align*}
$$

All terms higher than fourth-order vanish. This is exactly what we want because the Lagrangian (2.8) contains bilinear of cubic or lower-order terms only. Since a triple $[Y_{\nu'}^{\nu}, w_{\nu'j'k'}, w^{*i'j'k'}]$ completely specifies an element of $E_{8(8)}$ in adjoint representation, we write

$$
(2.27) \equiv \text{ad}[Y_{\nu'}^{\nu}, w_{\nu'j'k'}, w^{*i'j'k'}] \quad (2.29)
$$

for simple notation.

We will now construct the $E_{8(8)}/SO(16)$ nonlinear sigma-model Lagrangian. Let $\mathcal{V}$ be an element of the Lie group of $E_{8(8)}$ defined by

$$
\mathcal{V} \equiv \mathcal{V}_- \mathcal{V}_+ \quad (2.30)
$$

10
with
\[ V_+ = \begin{bmatrix} V_{ij}' V_{ij}' & V_{ij}' V_{ij}' V_{ij}' c' \\ V_{ij}' V_{ij}' & V_{ij}' V_{ij}' V_{ij}' c' \\ V_{ij}' V_{ij}' & V_{ij}' V_{ij}' V_{ij}' c' \end{bmatrix} \] (2.31)

and
\[ V_- = \exp(\text{ad}(v + v^*)). \] (2.32)

The invariant tangent vector field is
\[ V^{-1} \partial_\mu V = \text{ad}[Z_{a'b'}, w_{a'b'} w_{a'b'}], \] (2.33)

where
\[ Z_{a'b'} = \begin{bmatrix} e_i^d \partial_i e^d_i & -e^{-1} e_i^d (\partial_i \psi_i - Y_i^9) \\ 0 & -e^{-1} \partial_i e \end{bmatrix}, \] (2.34)
\[ Y_i^9 = A_{ipq} \partial_i \phi_{pq} - \phi_{pq} \partial_i A_{ipq} - \frac{1}{54} e^{ipqrst} A_{ipqrst}, \] (2.35)
\[ w_{a'b'} = -2\sqrt{3} \delta_{[a'} \delta_{b']} \epsilon_{c} e_{i}^a e_{j}^b \partial_{ij}, \] (2.36)
\[ w^{a'b'} = -6\sqrt{3} \delta_{[a'} \delta_{b']} e_i^c \epsilon_{j}^e \partial_{ij} \phi_{ij} - \frac{1}{36} \epsilon_{ijpqsr} \partial_i A_{pqrst}. \] (2.37)

To construct a coset nonlinear sigma model, we introduce the symmetric space involution \( \tau \) such that for \( E_{8(\pm8)} = H(= SO(16)) \oplus K \),
\[ \tau(H) = +H, \quad \tau(K) = -K. \] (2.38)

Defining the “symmetric matrix”
\[ \mathcal{M} \equiv V \cdot (V)^{-1}, \] (2.39)

one finds
\[ \text{Tr} \partial_\mu \mathcal{M}^{-1} \partial^\mu \mathcal{M} = -\text{Tr} \left( V^{-1} \partial_\mu V - \tau(V^{-1} \partial_\mu V) \right)^2. \] (2.40)

Thus the \( H \) components of \( V^{-1} \partial_\mu V \) are projected out. Since \( \tau \) acts on a triple as
\[ \tau(\text{ad}[Z_{a'b'}, w_{a'b'} c', w^{a'b'} c']) = \text{ad}[-Z_{a'b'}, w_{a'b'}^* c', w^{a'b'} c'], \] (2.41)

we obtain
\[ \text{Tr} \partial_\mu \mathcal{M}^{-1} \partial^\mu \mathcal{M} \]
This completes the direct derivation of the E₈ model by dimensional reduction from eleven dimensions. For later convenience we write out the transformation formulas of the scalar fields corresponding to the Chevalley generators of E₈ with the Cartan matrix

\[
K_{ij} = \begin{bmatrix}
2 & -1 \\
-1 & 2 & -1 \\
 & -1 & 2 & -1 \\
 & & -1 & 2 & -1 & -1 \\
 & & & -1 & 2 & -1 \\
 & & & & -1 & 2 \\
 & & & & & -1 & 2
\end{bmatrix}.
\]  

(2.43)

Labeling the vertices as in Fig. 2, we denote the kth SL(2, ℝ) subalgebra by SL(2, ℝ)ₙ (n = 1, ..., 8). They generate E₈ as already remarked. Then SL(2, ℝ)ₖ (k = 1, ..., 7) act on \( \mathcal{V} \) as (before compensating SO(16) gauge transformations) \(^6\)

\[
\delta_{εₖ} \mathcal{V} = \text{ad}[E_{(k)} ε', 0, 0] \mathcal{V},
\]

\(^6\)In fact it is \(-δ_{εₙ}\) that satisfy the algebra of \( xₙ \) (\( x = e, f, h \)) because \([δₓ, δₓ'] = δ[ₓ',ₓ] \).
Figure 2: Dynkin diagram of $E_8$.

\[
\begin{align*}
\delta_{f_k} \mathcal{V} &= \text{ad}[F_{(k)\nu'}', 0, 0] \mathcal{V}, \\
\delta_{h_k} \mathcal{V} &= \text{ad}[H_{(k)\nu'}', 0, 0] \mathcal{V}, \\
\end{align*}
\]

with

\[
\begin{align*}
E_{(k)\nu'}'' &= \delta_{\nu'}^k \delta_{k+1}'', \\
F_{(k)\nu'}'' &= \delta_{k}^{k+1} \delta_{\nu'}^k, \\
H_{(k)\nu'}'' &= \delta_{\nu'}^k \delta_{k}'' - \delta_{k+1}^{k+1} \delta_{\nu'}^k, \\
\end{align*}
\]

whereas for $SL(2, \mathbb{R})_8$,

\[
\begin{align*}
\delta_{e_8} \mathcal{V} &= \text{ad}[0, 6\sqrt{3}\delta_{[\nu'}^0 \delta_{j']^k]}], 0] \mathcal{V}, \\
\delta_{f_8} \mathcal{V} &= \text{ad}[0, 0, -6\sqrt{3}\delta_{[\nu'}^0 \delta_{j']^k} \delta_{k}^8]] \mathcal{V}, \\
\delta_{h_8} \mathcal{V} &= \text{ad}[\tilde{H}_{(8)\nu'}', 0, 0] \mathcal{V}, \\
\end{align*}
\]

where

\[
\tilde{H}_{(8)} \equiv \frac{1}{3}(H_{(1)} + 2H_{(2)} + 3H_{(3)} + 4H_{(4)} + 5H_{(5)} + 3H_{(6)} + H_{(7)} - H_{(8)}), \\
H_{(8)\nu'}' \equiv \delta_{\nu'}^8 \delta_{8}'' - \delta_{\nu'}^0 \delta_{g}''.
\]

Using the equation

\[
\mathcal{V}_-^-1 \delta \mathcal{V}_- + \delta \mathcal{V}_+ \mathcal{V}_-^-1 = \mathcal{V}_-^-1 \mathcal{X} \mathcal{V}_-
\]

for $\delta \mathcal{V} = \mathcal{X} \mathcal{V}$, we may express these formulas as those for $A_{[\nu'j']^k'}$, $\varphi_{[\nu'j']^k'}$ and $V_{\nu'}^a'$. The result is as follows:

\[
\begin{align*}
\delta_{e_k} A_{[\nu'j']^k'} &= 3A_{[k+1j']^0}^{k'}, \\
\delta_{e_k} \varphi_{[\nu'j']^k'} &= -3\varphi_{[k'j']^0}^{k'} \delta_{k+1}, \\
\delta_{e_k} V_{\nu'}^a' &= \delta_{\nu'}^k \delta_{k+1}^{k'} V_{\nu'}^a', \\
\end{align*}
\]
\[\begin{align*}
\delta_{f_k} A_{i'j'k'} &= 3A_{k[i'j'k']} \delta_{i'j'}^{k+1}, \\
\delta_{f_k} \varphi^{i'j'k'} &= -3\varphi^{k+1[i'j'k']} \delta_k, \\
\delta_{f_k} V_{i'} &= \delta_{i'}^{k+1} \delta_{k} V_{i'},
\end{align*}\]

\[\begin{align*}
\delta_{h_k} A_{i'j'k'} &= 3(A_{k[i'j'k']} \delta_{i'}^{k+1} - A_{k+1[i'j'k']} \delta_{i'}^{k+1}), \\
\delta_{h_k} \varphi^{i'j'k'} &= -3(\varphi^{k+1[i'j'k']} - \varphi^{k+1[i'j'k']}), \\
\delta_{h_k} V_{i'} &= (\delta_{i'}^{k} \delta_{k} - \delta_{i'}^{k+1} \delta_{k+1}) V_{i'},
\end{align*}\]

\[(k = 1, \ldots, 7),\]

\[\begin{align*}
\delta_{es} A_{i'j'k'} &= 6\sqrt{3} \delta_{i'j'k'}^{678}, \\
\delta_{es} \varphi^{i'j'k'} &= \frac{1}{12} \epsilon^{i'j'k'678} q' q'' A_{i'q'q''}, \\
\delta_{es} V_{i'} &= \left( -\frac{\sqrt{3}}{2} \varphi^{[78] \delta_{i'}^{678}} + \frac{1}{432} \epsilon^{i'q'q''678} s't' u' A_{i'q'q''} A_{s't'u'} \right) V_{i'}, \\
\delta_{fs} A_{i'j'k'} &= -\frac{1}{6} \epsilon^{i'j'k'678} q' q'' \varphi^{i'j'k'} - \frac{3\sqrt{3}}{2} \left( A_{i'q'78 A_{k'k'}} - \frac{1}{9} A_{i'q'78 A_{678}} \right), \\
\delta_{fs} \varphi^{i'j'k'} &= -6\sqrt{3} \delta_{i'j'k'}^{678} + \frac{3\sqrt{3}}{2} \left( A_{i'q'78 \varphi^{i'j'k'} \delta_{i'}^{678}} - \frac{1}{9} A_{i'q'78 \varphi^{i'j'k'}} \right) \\
&\quad + \frac{1}{144} \epsilon^{i'j'k'p'q'q's't'u'} A_{i'q'78 A_{678}} A_{s't'u'}, \\
\delta_{fs} V_{i'} &= -\sqrt{3} \left( \frac{1}{2} A_{i'q'78 \delta_{i'}^{678}} - \frac{1}{9} \delta_{i'}^{678} A_{678} \right) V_{i'} \\
&\quad - \frac{\sqrt{3}}{18} \left( A_{i'q'78 A_{678}} + \frac{1}{4} A_{i'q'78 A_{678}} A_{i'q'78 A_{678}} \right) \varphi^{i'q'q''}, \\
\delta_{hs} A_{i'j'k'} &= 3A_{i'j'k'} \tilde{H}_{(8)i'j'}, \\
\delta_{hs} \varphi^{i'j'k'} &= -3\varphi^{i'j'k'} \tilde{H}_{(8)i'}, \\
\delta_{hs} V_{i'} &= \tilde{H}_{(8)i'} V_{i'}. \tag{2.55}
\end{align*}\]

Since \(\delta_{fs}\) makes \(\varphi^{i'j'k'}\) deviate from the condition \(\varphi^{i'j'k'} = 0 (i', j', k' \neq 9)\), and consequently breaks the traceless condition for \(V_{i'}\), one needs a compensating local \(SO(16)\) transformation. Such a transformation is given by

\[\begin{align*}
\delta_{fs}^{\text{(comp)}} Y &= YY, \\
Y &= \text{ad}[0, w_{a'b'c'}, w_{a'b'c'}], \\
w_{a'b'c'} &= 6\sqrt{3} \delta_{a'c'} \delta_{b'c'} \epsilon_{a}^{b} \epsilon_{b}^{c}. \tag{2.56}
\end{align*}\]
The relation
\[ \mathcal{V}_-^{-1} \delta_{fs}^{(comp)} \mathcal{V}_- + \delta_{fs}^{(comp)} \mathcal{V}_+ \mathcal{V}_+^{-1} = \mathcal{V}_+ \mathcal{V}_+^{-1} \] (2.57)

enables us to find
\[ \delta_{fs}^{(comp)} A_i'j'k' = 6\sqrt{3}g_{i0}g_{j7}g_{k8}\delta_{ij}^0\delta_{j'k'}^0, \]
\[ \delta_{fs}^{(comp)} \varphi^{i'j'k'} = 6\sqrt{3}(\delta_{i0}^{1'}\delta_{j7}^0 + 3\psi_{[i0'}\delta_{j7'}^0\delta_{k0']}) - \frac{1}{12}\epsilon^{i'j'k'}pqrs't'u'v'g_{p6}g_{q7}g_{r8}A_{a't'u'v'}, \]
\[ \delta_{fs}^{(comp)} \psi^{'} = \left[ \frac{\sqrt{3}}{2}g_{i0}g_{j7}g_{k8}\delta_{i}^{[1'}\varphi_{j7k']} + \frac{\sqrt{3}}{2} (\delta_{[i8}A_{78]}^{'} + \psi_{[6}A_{78]}^{'}\delta_{9}^{t'}) \right. \]
\[ + \frac{1}{432} \epsilon^{i'm'n'pqrstu} A_{i'm'n'}A_{s't'u'v'}g_{p6}g_{q7}g_{r8} \] \[ \left. \right] \psi^{'} \]. (2.58)

Thus the total transformation \( \delta_{fs}^{(tot)} = \delta_{fs} + \delta_{fs}^{(comp)} \) reads (taking the factor \(-2\sqrt{3}\) in eqs.\(2.16\) into account)
\[ \delta_{fs}^{(tot)} A_{ijk} = -3(g_{i0}g_{j7}g_{k8})_{[678]} - \frac{1}{2}\epsilon_{ijk678pq} \varphi^{pq} \]
\[ + 9(A_{78}A_{ijk})_{[678]} - A_{ijk}A_{678}, \]
\[ \delta_{fs}^{(tot)} \varphi^{ij} = -3\psi_{[i0}\delta_{j7}] - \frac{1}{12} \epsilon^{ijpqstu} g_{p6}g_{q7}g_{r8}A_{stu} \]
\[ - (6A_{p[78}\varphi_{j7k]}^{[i} - A_{678}\varphi^{ij}) + \frac{1}{12} \epsilon^{ijpqstu} A_{p[q78}A_{6]qr}A_{stu}, \]
\[ \delta_{fs}^{(tot)} \psi_{i} = 6(A_{i[78}\psi]^{[j} - A_{i78}\psi_{j}) - \frac{1}{36} \epsilon^{jkpqstu} A_{ijk}A_{stu}g_{p6}g_{q7}g_{r8} \]
\[ - (4A_{ij[6\psi_{k}]) + A_{i[78}A_{6]}jk - \frac{2}{3} A_{ijk}A_{678})\varphi_{j} \]. (2.59)

We also write down \( \delta_{es} \) on these fields:
\[ \delta_{es} A_{ijk} = -3\delta_{[i}^{a}g_{j7}^{a}g_{k8]}, \]
\[ \delta_{es} \varphi^{ij} = \frac{1}{12} \epsilon^{i678pq} A_{pq}, \]
\[ \delta_{es} \psi_{i} = 0, \]
\[ \delta_{es} \psi_{i} = -3\varphi_{78}^{[i}g_{78]^{i} - \frac{1}{36} \epsilon^{pq678stu} A_{ipq}A_{stu} \]. (2.60)

As a final comment we note that \( \delta_{fs}, \ k = 1, \ldots, 7 \) break the upper-triangular gauge condition for \( V_{i'}^{'} \). In checking the \( E_9 \) relations in the next section this is harmless since \( \delta_{fs} V_{i'}^{'} \) is well-defined whether or not \( V_{i'}^{'} \) is in upper-triangular form.

In contrast, we have to compensate nonzero \( \varphi_{678}^{i} \) as we did above because by definition one of \( i', j', k' \) must be 9 for \( \varphi^{i'j'k'} \) and cannot be expressed in terms of \( \varphi^{ij} \).
3 $E_9$ in two dimensions

3.1 “Matzner-Misner’s” $SL(2, \mathbb{R})$

In this subsection we will analyze the extra $SL(2, \mathbb{R})$ symmetry that is newly present in two dimensions. As it mixes third and fourth rows of the elfbein, it is not manifest in the dualized Lagrangian \( \mathcal{L}' \) in which the coordinates $y$ and $x^1$ are differently treated. Hence we start with Cremmer-Julia’s four-dimensional Lagrangian \[5\] and then reduce the dimension to two.

The bosonic part of the four-dimensional Lagrangian obtained by dimensional reduction of the eleven-dimensional supergravity can be written in the form \[5\]

\[
\mathcal{L}' = E^{(4)} \left[ R^{(4)} - \frac{1}{2} \left( 1^{MNPQ}_{\mu IJ} + j^{MNPQ}_{\mu IJ} a_{IJ} \right) F^I_{MN} F^J_{PQ} + \frac{1}{48} \text{Tr} G^{(4)M} \partial_M \mathcal{R}^{-1} \partial_N \mathcal{R} \right]
\]

with

\[
1^{MNPQ} \equiv G^{(4)P}[M G^{(4)N}]Q, \quad j^{MNPQ} \equiv \frac{1}{2} E^{(4)} - \epsilon^{MNPQ}.
\]

$m_{IJ}, a_{IJ}$ are some symmetric functions of scalar fields. $\mathcal{R}$ is a symmetric matrix given in terms of $m$ and $a$ as

\[
\mathcal{R} = \begin{bmatrix}
m + am^{-1}a & am^{-1} \\\m^{-1}a & m^{-1}
\end{bmatrix}.
\]

$I, J$ are the internal indices labeling 28 abelian vector fields. $M, N, \ldots$ are the four-dimensional spacetime indices, which we split into $(\mu, \bar{\mu})$ with $\mu = t, x$ and $\bar{\mu} = y, x^1$.

Assuming $\partial_{\bar{\mu}} = 0$, we get the two-dimensional Lagrangian

\[
\mathcal{L}' = \tau E^{(2)} \left[ \tilde{R}^{(2)} + \tilde{G}^{(2)\, \bar{\mu} \bar{\nu}} \left( \frac{1}{4} \partial_{\bar{\mu}} \bar{\tau} \partial_{\bar{\nu}} \partial_{\bar{\mu}} \bar{\tau} + \partial_{\bar{\nu}} \ln \bar{\tau} \partial_{\bar{\mu}} \ln \bar{\tau} \right) 
\right.
\]

\[
- \frac{1}{4} \tilde{G}^{(2)\, \bar{\mu} \bar{\nu}} \tilde{G}^{(2)\, \bar{\mu} \bar{\sigma}} \tilde{G}^{(2)\, \bar{\sigma} \bar{\nu}} \tilde{B}^{\bar{\mu} \bar{\nu}} \tilde{B}^{\bar{\mu} \bar{\sigma}}
\]

\[
- \frac{1}{2} m_{IJ} \left( \tilde{G}^{(2)\, \bar{\mu} \bar{\nu}} \tilde{G}^{(2)\, \bar{\mu} \bar{\sigma}} (F^I_{\bar{\mu} \bar{\nu}} - 4 F^I_{\bar{\mu} \bar{\sigma}} B_{\bar{\nu} \bar{\sigma}}) F^J_{\bar{\mu} \bar{\nu}} 
\right)
\]

\[
+ 2 \tilde{G}^{(2)\, \bar{\mu} \bar{\sigma}} \tilde{G}^{(2)\, \bar{\mu} \bar{\nu}} \tilde{G}^{(2)\, \bar{\sigma} \bar{\nu}} \tilde{B}^{\bar{\mu} \bar{\nu}} \tilde{B}^{\bar{\mu} \bar{\sigma}} F^I_{\bar{\mu} \bar{\nu}} F^J_{\bar{\mu} \bar{\nu}}
\]

\[
+ \bar{\epsilon}^{-1} (\tilde{E}^{(2)})^{-1} a_{IJ} \epsilon^{\bar{\mu} \bar{\nu}} \epsilon^{\bar{\mu} \bar{\sigma}} F^I_{\bar{\mu} \bar{\nu}} F^J_{\bar{\mu} \bar{\nu}}
\]

\[
+ \frac{1}{48} \text{Tr} \tilde{G}^{(2)\, \bar{\mu} \bar{\nu}} \partial_{\bar{\mu}} \mathcal{R}^{-1} \partial_{\bar{\nu}} \mathcal{R} \right],
\]

The name “Matzner-Misner” appeared in ref.\[20\], in which the first evidence for $A_1^{(1)}$ including central charge was presented.
where the vierbein is decomposed as
\[
E^{(4)A}_M = \begin{bmatrix}
\tilde{E}^{(2)\pi}_{\pi} & \tilde{E}^{(2)\pi}_{\pi} \\
0 & \tilde{E}^{(2)\pi}_{\pi}
\end{bmatrix}.
\] (3.5)

The zweibein without tilde is reserved for the symbol used in the next section. \(E^{(4)A}_M\) is related to the four-by-four block of the elfbein by
\[
E^{(11)A}_M = \left(\frac{e}{\epsilon_{a=1}}\right)^{-\frac{1}{2}} E^{(4)A}_M.
\] (3.6)

The Lagrangian \(\mathcal{L}'\) is manifestly invariant under the \(SL(2, \mathbb{R})\) group transformation
\[
\begin{align*}
e^{\pi}_{\pi} & \mapsto \Omega^{\pi}_{\pi} e^{\pi}_{\pi}, \\
F^I_{\mu\nu} & \mapsto \Omega^{\pi}_{\pi} F^I_{\mu\nu}, \\
B^\pi_{\nu} & \mapsto (\Omega^{-1})^\mu_{\nu} B^\pi_{\nu}
\end{align*}
\] (3.7)

for \(\Omega^{\pi}_{\pi} \in SL(2, \mathbb{R})\) group. This is a generalization of Matzner-Misner’s \(SL(2, \mathbb{R})\) symmetry \([19, 20, 9]\) in general relativity.

**3.2 From \(E_8\) to \(E_9\)**

The Lagrangian \(\mathcal{L}'\) differs from \(\mathcal{L}\) by some other Lagrange multiplier terms, but their equations of motion are equivalent if the duality relations are used \([5]\). Hence the equations of motion of \(\mathcal{L}'\) are also equivalent to those derived from \(\mathcal{L} + \mathcal{L}_{\text{Lag.mul.}}\) through the relations (2.9) and (2.10). Thus the equations of motion of the eleven-dimensional supergravity possess \(SL(2, \mathbb{R}) \times E_8\) symmetry if they are reduced to two dimensions. We will show that the infinitesimal form of this \(SL(2, \mathbb{R})\) enlarges the symmetry algebra from \(E_8\) to \(E_9\).

Let \(SL(2, \mathbb{R})_0\) be the infinitesimal form of (3.7), generated by
\[
\begin{align*}
\delta_X e^{\pi}_{\pi} &= X^{\mu\nu} e^{\pi}_{\mu\nu}, \\
\delta_X F^I_{\mu\nu} &= X^{\mu\nu} F^I_{\mu\nu}, \\
\delta_X B^\pi_{\nu} &= -X^\mu_{\nu} B^\pi_{\nu}.
\end{align*}
\] (3.8)

We denote the generators for
\[
X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (\equiv E), \quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} (\equiv F) \text{ and } \begin{bmatrix} 1 & -1 \end{bmatrix} (\equiv H) \tag{3.9}
\]
by $\delta_{e_0}$, $\delta_{f_0}$ and $\delta_{h_0}$, respectively.

We first note that the action of $SL(2, \mathbb{R})_0$ is expressed in terms of eleven-dimensional fields as

$$
\delta_X E^{(11)\tilde{\alpha}}_{\mu} = 
\begin{bmatrix}
0 & 0 & X & 0 & \cdots \\
0 & \cdots & 0 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
\begin{bmatrix}
\vdots \\
E^{(11)\tilde{\alpha}}_{\nu} \\
E^{(11)\tilde{\alpha}}_{\nu} \\
\vdots \\
\end{bmatrix}
$$

$$
\equiv X(0)_{\mu} \hat{\nu} E^{(11)\tilde{\alpha}}_{\nu}
$$

(3.10)

and

$$
\delta_X A_{\mu\nu\hat{\rho}} = 3(X(0)_{\mu} \hat{\nu} A_{\sigma\sigma\hat{\rho}})_{[\mu\nu\hat{\rho}]},
$$

(3.11)

In fact, if we define

$$
X_{(k)\mu} \hat{\nu} = 
\begin{bmatrix}
0 & 0 & 0 & \cdots & X & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix},
$$

(3.12)

the action of $SL(2, \mathbb{R})_k$ ($k = 1, \ldots, 7$) can also be written as

$$
\delta_{x_k} E^{(11)\tilde{\alpha}}_{\mu} = X_{(k)\mu} \hat{\nu} E^{(11)\tilde{\alpha}}_{\nu},
$$

(3.13)

$$
\delta_{x_k} A_{\mu\nu\hat{\rho}} = 3(X_{(k)\mu} \hat{\nu} A_{\sigma\sigma\hat{\rho}})_{[\mu\nu\hat{\rho}]},
$$

(3.14)

where $x = e, f, h$ corresponding to $X = E, F, H$. Hence $SL(2, \mathbb{R})_0$ commutes with $SL(2, \mathbb{R})_k$ ($k = 2, \ldots, 7$). It is also easy to see that $SL(2, \mathbb{R})_0$ and $SL(2, \mathbb{R})_1$ do not commute but generate $SL(3, \mathbb{R})$. As we already remarked, (3.13) is well-defined irrespective of the decomposition of $E^{(11)\tilde{\alpha}}_{\mu}$ or whether or not $E^{(3)\tilde{\alpha}}_{\mu}$ or $e_i^a$ is in upper-triangular form. Thus the matter is reduced to a trivial algebra of matrices. If the set
\(-\delta e_0, -\delta f_0, -\delta h_0\) is regarded as a part of the Chevalley generators, the corresponding elements of the “extended Cartan matrix” are

\[
K_{0k} = K_{k0} = 2\delta_{k,0} - \delta_{k,1} \quad (k = 0, \ldots, 7).
\tag{3.15}
\]

Next consider the relation between \(SL(2, \mathbb{R})_0\) and \(SL(2, \mathbb{R})_8\). \(\delta_{h_8}\)-charges of the scalar fields in three dimensions can be read off from (2.55). Let \(q(i)\) be a function defined by

\[
q(i) = \begin{cases} 
-\frac{1}{3} & i = 1, \ldots, 5, \\
\frac{2}{3} & i = 6, 7, 8,
\end{cases}
\tag{3.16}
\]

then \(\delta_{h_8}\)-charges \(q_8\) are

\[
q_8(e_i^a) = q(i), \\
q_8(\psi_i) = \frac{1}{3} + q(i), \\
q_8(A_{ijk}) = q(i) + q(j) + q(k), \\
q_8(\phi^{ij}) = \frac{1}{3} - q(i) - q(j).
\tag{3.17}
\]

\(E^{(3)\alpha}_\mu\) is of course \(\delta_{h_8}\)-neutral. Using the duality relation, \(\delta_{h_8}\)-charges of \(E^{(11)\alpha}_y\) and \(A_{yjk}\) turn out to be

\[
q_8(E^{(11)\alpha}_y) = -\frac{1}{3}
\tag{3.18}
\]

and

\[
q_8(A_{yjk}) = -\frac{1}{3} + q(i) + q(j),
\tag{3.19}
\]

which are equal to \(q(e_1^a)\) and \(q(A_{1jk})\), respectively. Thus \(SL(2, \mathbb{R})_0\) does not change \(\delta_{h_8}\)-charge.

Similarly, \(\delta_{h_0}\)-charges \(q_0\) of the eleven-dimensional fields are

\[
q_0(E^{(11)\alpha}_{\mu\hat{\nu}\hat{\rho}}) = +\delta^\mu_{\hat{\mu}} - \delta^1_{\hat{\mu}}, \quad q_0(A_{\mu\hat{\nu}\hat{\rho}}) = (\delta^\nu_{\hat{\nu}} + \delta^\rho_{\hat{\rho}}) - (\delta^1_{\hat{\nu}} + \delta^1_{\hat{\rho}}).
\tag{3.20}
\]

The duality relations translate these equations into those in terms the three-dimensional fields as follows:

\[
q_0(e_i^a) = -\delta^1_i, \\
q_0(\psi_i) = -1 - \delta^1_i, \\
q_0(A_{ijk}) = -(\delta^1_i + \delta^1_j + \delta^1_k),
\tag{3.21}
\]

\[
q_0(\phi^{ij}) = -1 + \delta^1_i + \delta^1_j, \\
q_0(E^{(3)\alpha}_\mu) = -1 + \delta^\nu_{\mu}.
\]
In view of (2.53)(2.54), we also find that $SL(2, \mathbb{R})_8$ does not change $\delta_{h_0}$-charge.

### 3.3 Serre relation

What remains to check is the Serre relation between $SL(2, \mathbb{R})_0$ and $SL(2, \mathbb{R})_8$. Since $\delta_{f_0}$ gives rise to a non-zero $E^{(11)\alpha}_{\mu} \gamma^i - B_\mu \gamma^i$, one needs a local Lorentz transformation $\delta^{(\text{comp})}_{f_0}$ to restore the “block-upper-triangular” form of the elfbein (2.5). Including this contribution, the total variation $\delta_{f_0}^{(\text{tot})} \equiv \delta_{f_0} + \delta^{(\text{comp})}_{f_0}$ is given as

\[
\begin{align*}
\delta_{f_0}^{(\text{tot})} E^{3\alpha}_{\mu} &= 2B_y \gamma^i E^{3\alpha}_{\mu}, \\
\delta_{f_0}^{(\text{tot})} B^i_{\mu} &= e^{-2}G_{\mu y}^{i1} - B_\mu B^i_y, \\
\delta_{f_0}^{(\text{tot})} e^a_i &= \delta^1_i B_\mu e^a_{\mu}, \\
\delta_{f_0}^{(\text{tot})} A_{ijk} &= 3\delta^1_i A_{jk}, \\
\delta_{f_0}^{(\text{tot})} A_{\mu ij} &= 2\delta^1_{\mu} A_{ij}, \\
\delta_{f_0}^{(\text{tot})} A_{\mu \nu i} &= \delta^1_{\mu} A_{\nu i}, \\
\delta_{f_0}^{(\text{tot})} A_{\mu \nu \rho} &= 0.
\end{align*}
\] (3.22)

The rule of $\delta_{e_0}$ is much simpler:

\[
\begin{align*}
\delta_{e_0} E^{3\alpha}_{\mu} &= \delta_{e_0} e^a_i = \delta_{e_0} A_{ijk} = 0, \\
\delta_{e_0} B^i_{\mu} &= \delta^y_{\mu} \delta^1_i, \\
\delta_{e_0} A_{\mu ij} &= \delta^y_{\mu} A_{ij}, \\
\delta_{e_0} A_{\mu \nu i} &= 2\delta^y_{\mu} A_{\nu i}, \\
\delta_{e_0} A_{\mu \nu \rho} &= 3\delta^y_{\mu} A_{\nu \rho}.
\end{align*}
\] (3.23)

To prove that the symmetry algebra is $E_9$ one needs to show $[\delta_{e_0}, \delta_{e_8}] = [\delta_{f_0}, \delta_{f_8}] = 0$ on all the fields that appear in the theory. What makes the situation difficult is the fact that $SL(2, \mathbb{R})_8$ is originally defined as a variation on the three-dimensional dualized fields $\phi^{ij}$ and $\psi_i$ while $SL(2, \mathbb{R})_0$ naturally acts on the eleven-dimensional fields, or “un-dualized fields” $A'_{\mu ij}$ and $B_\mu ^i$. Thus all one can do is to verify the commutator to vanish only on their derivatives using the eqs. (2.9)(2.10) [15]. Since there appear only the field strengths in the original Lagrangian, the gauge potentials $A'_{\mu ij}$ and $B_\mu ^i$ themselves do not affect physics classically, but they can quantum mechanically. We will restrict ourselves to the classical aspect of $E_9$ in this paper.
Let us examine $[\delta_{f_0}^{(\text{tot})}, \delta_{f_8}^{(\text{tot})}]$. It is easy to find that $[\delta_{f_0}^{(\text{tot})}, \delta_{f_8}^{(\text{tot})}] e_i^a = [\delta_{f_0}^{(\text{tot})}, \delta_{f_8}^{(\text{tot})}] B_{\mu\nu} = 0$ if and only if
\[
\begin{align*}
\delta_{f_8}^{(\text{tot})} B_y^i &= 6 A'_{[78} \delta_i^{[6]}, \\
\delta_{f_8}^{(\text{tot})} B_1^1 &= 0.
\end{align*}
\] (3.24)

They are consistent with the equation
\[
\delta_{f_8}^{(\text{tot})} A_{\mu\nu}^k = 6 F'_{\mu\nu[78} \delta_i^{k]},
\] (3.25)

which is derived from (2.59) and (2.10). Then $[\delta_{f_0}^{(\text{tot})}, \delta_{f_8}^{(\text{tot})}] E^{(3)\alpha}_\mu = 0$ follows.

Since $\delta_{f_0}^{(\text{tot})} \delta_{f_8}^{(\text{tot})} A_{ijk}$ contains a term $-\frac{1}{2} \epsilon_{ijk678pq} \delta_{f_0}^{(\text{tot})} \varphi^{pq}$, we calculate the commutator on $\partial_\mu A_{ijk}$ as we noted above. Useful formulas are
\[
\begin{align*}
\delta_{f_8}^{(\text{tot})} F_{\mu\nu}^{(3)ij} &= -3 B_{[\mu} \delta^{ij}_{\nu]} - 12 (A_{p78} F_{\mu\nu}^{(3)p[j} \delta_i^{]678} + \frac{2}{3} A_{678} F_{\mu\nu}^{(3)ij}) \\
&\quad + \frac{1}{6} E^{(3)1} e^{\mu\rho\phi} \epsilon^{ijpqrsu} \partial_\rho A_{stu} g_{\phi6} g_{\theta7} g_{\gamma8},
\end{align*}
\] (3.26)
\[
\delta_{f_0}^{(\text{tot})} \left( E^{(3)} \epsilon^2 F^{(3)\bar{\mu}\bar{\nu}ij} \right) = E^{(3)} \epsilon^2 \left[ 2 F^{\mu\bar{\nu}} y_j \delta^{1j} + B_y^1 F^{(3)\bar{\mu}\bar{\nu}ij} + (2 B_y^1 F^{(3)\bar{\mu}\bar{\nu}1j})^{[ij]} \\
&\quad - \epsilon^{ik} g^{jl} g^{1m} \phi^{1m} A_{klm} \right],
\] (3.27)

where $F''_{\mu\nu\rho\delta} = F_{\mu\nu\rho} - 3 B_{[\mu} \delta^{ij}_{\nu]} - 3 B_{[\mu} B_{\nu]} F_{\rho]ki}$ is the Kaluza-Klein invariant non-dynamical three-form field strength, which was ignored in the three-dimensional reduced Lagrangian (2.2). Using these equations, one finds
\[
[\delta_{f_0}^{(\text{tot})}, \delta_{f_8}^{(\text{tot})}] \partial_\mu A_{ijk} = -\frac{3}{2} \delta_{[j}^{\mu} \epsilon_{ijk]678pq} \epsilon^{\mu\nu\rho} E^{(3)} \epsilon^2 F''_{\mu\nu\rho} y_j \delta^{1j}.
\] (3.28)

Thus the commutator can be consistently set to zero.

The commutator on $F_{\mu\nu\rho\delta}^{(3)}$ is the most complicated but can be done straightforwardly. In practice much labor can be saved if one calculates $[\delta_{f_0}^{(\text{tot})}, \delta_{f_8}^{(\text{tot})}] (E^{(3)} \epsilon^2 F^{(3)\bar{\mu}\bar{\nu}ij})$ and uses the above formulas. It is worth noting that $F^{(3)\bar{\mu}\bar{\nu}ij} = B_{k}^{\mu\rho} = 0$ since $\delta_y = 0$ for any field in rhs of the duality relations (2.9) (2.10). In fact, this is a part of Geroch’s compatibility condition considered in the next subsection. After some calculation one may confirm that $[\delta_{f_0}^{(\text{tot})}, \delta_{f_8}^{(\text{tot})}] (E^{(3)} \epsilon^2 F^{(3)\bar{\mu}\bar{\nu}ij})$ vanishes up to terms proportional to $F''_{\mu\nu\rho\delta}$.

Let us now turn to the examination of $[\delta_{e_0}, \delta_{e_8}]$. This vanishes on $E^{(3)\alpha}_\mu$, $\epsilon_i^a$ and $A_{ijk}$ trivially. It is also easy to see that $\delta_{e_0} B_{\mu}^{(1)} = \delta_{e_0} F_{\mu\nu\rho\delta}^{(3)} = 0$, so that
\[
\begin{align*}
\delta_{e_0} \varphi^{ij} &= e_i^j, \\
\delta_{e_0} (\partial_\mu \psi_i) &= -c^{jk} A_{ijk} + d_i,
\end{align*}
\] (3.29)
where \( c^{ij}, d_i \) are constants. Thus \([\delta_{e_0}, \delta_{e_8}]\) vanishes also on \( \partial_\mu \varphi^{ij} \) and \( \partial_\mu \psi_i \). (One may even check that it vanishes on \( \varphi^{ij} \), and does on \( \psi_i \) provided that \( c^{67} = c^{78} = c^{86} = 0 \).)

Finally let us consider the non-dynamical fields (three- and four-form field strengths). One might worry that the \( SL(2, \mathbb{R})_0 \) may give rise to a dynamical field out of non-dynamical one, so that the latter may not be consistently ignored. Since \( \delta_{f_0} \) effectively replaces \( i, j, \ldots \) indices with \( y \), only \( \delta_{e_0} \) is dangerous. Happily, the Kaluza-Klein invariant combinations

\[
A'_\mu \nu \iota \equiv A_{\mu \nu k} - 2B^j_\nu A_{\iota jk} + B^i_\mu B^j_\nu A_{ijk}
\]

and

\[
A'_\mu \nu \rho \iota \equiv A_{\mu \nu \rho k} - 3B^j_\rho A_{\iota jk} + 3B^j_\nu B^k_\rho A_{\iota jk} - B^i_\mu B^j_\nu B^k_\rho A_{ijk}
\]

are \( \delta_{e_0} \)-invariant, and so are the bilinear of \( F''_\mu \nu \rho \iota \kappa \) (omitted in the Lagrangian (2.2)). Hence \( \delta_{e_0} \) commutes with \( \delta_{e_8} \) on these fields. This completes the proof of the commutativity of \( SL(2, \mathbb{R})_0 \) and \( SL(2, \mathbb{R})_8 \) and allows us to define

\[
K_{08} = K_{80} = 0.
\]

(3.30)

(3.15) and (3.30) extend the Cartan matrix (2.43) to that of \( E_9 \) (Fig.3).

### 3.4 Compatibility of the symmetry with the Killing vector fields

Let us conclude this section with a remark on Geroch’s condition concerning the compatibility of the symmetry with the Killing vector fields [14]. We have shown in this section that the new Killing vector field \( \partial_y \) gives rise to an extra symmetry \( SL(2, \mathbb{R})_0 \) which together with the \( E_8 \) forms the \( E_9 \) commutation relations. For this \( E_9 \) to be really a symmetry of the reduced system, any Killing vector field assumed to exist must be mapped to another Killing vector field again by any action of the symmetry for consistency. In other word all the new fields after the transformation must be again independent of \( y \) or \( x^i, i = 1, \ldots, 8 \). This is a trivial requirement if the infinitesimal transformation is given in terms of a function of eleven-dimensional fields only, but, if the variation includes dualized fields, it gives the constraints \( \delta_{y} \psi_k = \cdots \)
\[ \partial_y \varphi^{ij} = 0, \] which imply
\[ B^{\overline{m} k} = 0, \quad F^{\overline{m}ij} = 0. \tag{3.31} \]

Since the coordinates \( y \) and \( x^i, i = 1, \ldots, 8 \) are now on the same footing, one may also take \( y \) and seven of eight \( x^i \) as the coordinates of an eight-torus, dualize the three-dimensional vectors and impose the independence of the remaining last one that was not taken as the coordinate of the eight-torus. One would then obtain similar conditions on the vector fields which may also have \( y \) index. But the conditions (3.31) are based on a specific splitting of the three-dimensional space and do not allow a simple replacement of indices. To derive the fully \( y-x^i \) symmetric conditions we rewrite (3.31) into equivalent expressions
\[ \epsilon^{\overline{\mu}_0 \cdots \overline{\mu}_8}_{\overline{\mu}_0 (0)} \xi_{(i_1)} \cdots \xi_{(i_8)} \nabla_{\overline{\nu}} \xi_{(i)}\phi = 0, \]
\[ \epsilon^{\overline{\mu}_0 \cdots \overline{\mu}_8}_{\overline{\mu}_0 (0)} \xi_{(i_1)} \cdots \xi_{(i_8)} \xi_{(i_l)} \phi \nabla^{[\overline{\nu}} A_{\overline{\rho} \overline{\tau} \overline{\sigma}]} = 0, \tag{3.32} \]
where \( \xi_{(i)} = \delta_{(i)}^\mu \) and \( \xi_{(0)} = \delta_{(0)}^\mu \) are the Killing vector fields assumed to exist. They are manifestly \( y-x^i \) symmetric if one allows \( i \) to take 0 and regards \( x^0 \) as \( y \). With this extension (3.32) obviously generalize Geroch’s compatibility condition derived in four-dimensional pure gravity [14] and we assume them to hold for consistency. Note that the first set of equations are satisfied if all the Killing vector fields are hypersurface-orthogonal, while the second impose some conditions on the field strength of \( A_{\mu ij} \) and \( A_{\mu \nu i} \).

4 \quad \textit{E}_{10} \text{ in one dimension}

4.1 Null Killing vector in two dimensions

So far we have discussed the \( E_9 \) symmetry in two dimensions. We would now like to study the enlargement of the symmetry to \( E_{10} \). We first describe a special feature of dimensional reduction from two to one dimension [13].

We parameterize the dreibein as
\[ E^{(3)\alpha}_{\mu} = \begin{bmatrix} E^{(2)\overline{\mu} \overline{\nu}}_{\overline{\mu} \overline{\nu}} & \rho A_{\overline{\mu} \overline{\nu}} \\ 0 & \rho \end{bmatrix}, \tag{4.1} \]
where \( \mu = \overline{\mu}, \overline{\nu} \) and \( \overline{\mu} = \overline{+}, \overline{-} \) (They are dotted in order to distinguish them from Lorentz indices.) are three- and two-dimensional curved indices, whereas \( \alpha = \overline{\nu}, 2 \)
and $\overline{\alpha} = +, -$ are corresponding Lorentz indices, respectively. (We switch the notation from $t, x$ to $\dot{+}, \dot{-}$.) Conventions for the flat metrics are

$$\eta^{(3)}_{\alpha \beta} = \begin{bmatrix} \eta^{(2)}_{\overline{\alpha} \overline{\beta}} & 1 \\ 1 & 1 \end{bmatrix}, \quad \eta^{(2)}_{\alpha \beta} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (4.2)$$

Introducing the ninth Killing vector $\partial_y$, the three-dimensional nonlinear sigma model

$$\mathcal{L} = E^{(3)}(R^{(3)} + \frac{1}{240} G^{\mu \nu} \text{Tr} \partial_{\mu} \mathcal{M}^{-1} \partial_{\nu} \mathcal{M}) \quad (4.3)$$

is then reduced to

$$\mathcal{L} = \rho E^{(2)}(R^{(2)} - \frac{1}{4} \rho^2 H_{\mu \nu} H^{\mu \nu} + \frac{1}{240} \text{Tr} \partial_{\mu} \mathcal{M}^{-1} \partial^F \mathcal{M}), \quad (4.4)$$

where $H_{\mu \nu} = \partial_{[\mu} A_{\nu]} - \partial_{\nu} A_{\mu}$. (4.4) is on-shell equivalent to (3.4).

Next we take $\partial_{\dot{-}}$ as the tenth Killing vector. The duality relation (2.10) then becomes

$$B_{i \dot{-}} = 0 = E^{(3)} e^{-2} G^{(3) \dot{+} \dot{+}} g^{ij} (\partial_{\dot{+}} \psi_j + \cdots), \quad (4.5)$$
$$B_{i \dot{+}} = \partial_{\dot{+}} B_{i \dot{-}} = E^{(3)} e^{-2} G^{(3) \dot{+} \dot{+}} g^{ij} (\partial_{\dot{+}} \psi_j + \cdots), \quad (4.6)$$
$$B_{i \dot{+} \dot{+}} = \partial_{\dot{+}} B_{i \dot{+} \dot{-}} = - E^{(3)} e^{-2} G^{(3) \dot{+} \dot{+}} g^{ij} (\partial_{\dot{+}} \psi_j + \cdots). \quad (4.7)$$

The precise form of the terms represented by dots can be found in (2.10). Suppose that $G^{(3) \dot{+} \dot{+}} \neq 0$. These equations require that $B_{i \dot{-}}$ and $B_{i \dot{+}}$ are constants, and $\psi_i$, $\varphi^{ij}$ and $A_{ijk}$ are constrained by some relations before equations of motion are imposed. Then there is no degree of freedom for the dual field $\psi_i$ to carry. The duality relation ceases to relate the fields dual with each other and falls into trivial. A similar thing is true for the relation (2.9). Therefore, one needs to take the Killing vector $\partial_{\dot{-}}$ to be null-like if one wants to have non-trivial duality relations [15].

We adopt the following parameterization for the zweibein [21, 22, 23]:

$$E^{(2)\pi}_{\pi} = \begin{bmatrix} e^+_{\pi} & \mu^+ e^-_{\pi} \\ \mu^+ e^+_{\pi} & e^-_{\pi} \end{bmatrix}. \quad (4.8)$$

$\mu^+_{\pi}$ and $\mu^-_{\pi}$ are the Lorentzian analogue of Beltrami differentials. That is, for given $\mu^+_{\pi}$ and $\mu^-_{\pi}$, the line element

$$ds^2 = e^+_{\pi} e^-_{\pi} (dx^+ + \mu^+_{\pi} dx^-) (dx^- + \mu^-_{\pi} dx^+) \quad (4.9)$$
can be cast into the form $2\lambda dudv$ for some conformal factor $\lambda$ and for some $u$, $v$ such that
\begin{equation}
\mu_+^- = \frac{\partial_- u}{\partial_+ u}, \quad \mu_+^+ = \frac{\partial_- v}{\partial_+ v}.
\end{equation}

Hence $\mu_+^\pm$ and $\mu_+^\pm$ parameterize the “conformal structure” of two-dimensional Lorentzian metrics. Note that they are real and independent each other.

The condition $G^{(3)}_+^+ = 0$ implies that $\mu_+^\pm$ vanishes. The reduced Lagrangian reads
\begin{equation}
\mathcal{L} = E^{(2)-1}\left(-2\partial_+ \rho \partial_- (\mu_+^+ e_+^- e_+^-) + \frac{1}{2} \rho^3 (\partial_+ A_-)^2\right)
+ \frac{1}{240} \rho^2 \frac{2\mu_+^+}{1 - \mu_+^- \mu_+^+} \text{Tr}(\mathcal{M}^{-1} \partial_+ \mathcal{M})^2.
\end{equation}

$\mu_+^\pm$ is set to be zero after deriving the equations of motion. The independent ones are
\begin{equation}
e_+^- e_+^- \partial_+ \rho e_+^- e_+^- + \frac{1}{240} \rho \text{Tr}(\mathcal{M}^{-1} \partial_+ \mathcal{M})^2 = 0
\end{equation}
and
\begin{equation}
\partial_+ A_- = 0.
\end{equation}

### 4.2 New $SL(2, \mathbb{R})$ symmetry

We will now describe a new $SL(2, \mathbb{R})$ symmetry in one dimension. This $SL(2, \mathbb{R})$ is defined to act on the second and third rows of the elfbein. For $E^{(3)}_\mu$ in the decomposition (2.9) it only affects the lower-right two-by-two part (before compensating gauge transformations). $e_i^a$ is left unchanged. Denoting the infinitesimal transformations corresponding to the Chevalley generators by $\{-\delta_{e_-}, -\delta_{f_-}, -\delta_{h_-}\}$, the first and the last are given by
\begin{equation}
\delta_{e_-} \left[ \begin{array}{c} e_-^- \\ 0 \\ \rho \end{array} \right] = \left[ \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & \rho \\ 0 & 0 & 0 \end{array} \right] \left[ \begin{array}{c} e_-^- \\ \rho A_-^- \\ \rho \end{array} \right],
\end{equation}
\begin{equation}
\delta_{h_-} \left[ \begin{array}{c} e_-^- \\ 0 \\ \rho \end{array} \right] = \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\rho \end{array} \right] \left[ \begin{array}{c} e_-^- \\ \rho A_-^- \\ \rho \end{array} \right],
\end{equation}
while the multiplication of $\left[ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right]$ does not preserve the upper-triangular gauge, so a suitable compensating gauge transformation is needed for $\delta_{f_-}$:
\begin{equation}
\delta_{f_-} \left[ \begin{array}{c} E^{(2)}_{\mu} \\ 0 \\ \rho \end{array} \right] = \left[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array} \right] \left[ \begin{array}{c} E^{(2)}_{\mu} \\ 0 \\ \rho A_{\mu} \end{array} \right].
\end{equation}
Table 1: $SL(2,\mathbb{R})_{-1}$ transformations.

| $e^+_i$ | $\delta_{e^{-1}}$ | $h^{-1}$ | $f^{-1}$ |
|---------|-------------------|---------|---------|
| $e^-_i$ | 0                 | 0       | 0       |
| $e^-_i$ | 0                 | $e^-_i$ | $-A^-e^-_i$ |
| $\mu^+_i$ | 0     | $\mu^-_i$ | $-A^-\mu^-_i$ |
| $\mu^-_i$ | 0     | 0       | 0       |
| $A^+_i$ | 0     | $A^+_i$ | $-A^-A^+ + \rho^{-1}e^+_i e^-_i$ |
| $A^-_i$ | 1     | 2$A^-_i$ | $-A^2^-_i$ |
| $\rho$ | 0     | $-\rho$ | $\rho A^-_i$ |
| $e^0_i$ | 0     | 0       | 0       |
| $\psi_i$ | 0     | 0       | 0       |
| $A_{ijk}$ | 0   | 0       | 0       |
| $\varphi^{ij}$ | 0   | 0       | 0       |

$$+ \left[ E^{(2)\pi}_{\pi} \begin{array}{cc} \rho A^\pi \end{array} \right] \cdot \rho^{-1}e^-_i \left[ \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{array} \right]$$

$$= \left[ \begin{array}{ccc} 0 & -e^-_i A^+_i & \rho^{-1}e^+_i e^-_i \\ 0 & -e^-_i A^-_i & 0 \\ 0 & 0 & \rho A^-_i \end{array} \right]. \quad (4.16)$$

These equations determine the transformation law of the components of the dreibein. We also define that the three-dimensional sigma-model scalars are invariant. These new $SL(2,\mathbb{R})$ transformations are summarized in Table 1. Note that $\mu^+_i$ is kept unchanged under these transformations, which is consistent with our assumption. It is easy to see that the variations of the equations of motion (4.12)(4.13) vanish if they are used themselves. Hence this $SL(2,\mathbb{R})$ is a symmetry.

For the fields “before the dualization” $B^i_\mu$ and $A_{\mu ij}$, the transformaions are defined as

$$\delta_{e^{-1}} B^i_\mu = B^i_y, \quad \delta_{h^{-1}} B^i_\mu = B^i_y, \quad \delta_{f^{-1}} B^i_\mu = 0,$$

$$\delta_{e^{-1}} B^i_y = 0, \quad \delta_{h^{-1}} B^i_y = -B^i_y, \quad \delta_{f^{-1}} B^i_y = B^i_y,$$

$$\delta_{e^{-1}} B^i_+ = \delta_{h^{-1}} B^i_+ = \delta_{f^{-1}} B^i_+ = 0,$$  

and

$$\delta_{e^{-1}} A^-_{\mu j} = A_{y j k}, \quad \delta_{h^{-1}} A^-_{\mu j} = A^-_{y j k}, \quad \delta_{f^{-1}} A^-_{\mu j} = 0,$$

$$\delta_{e^{-1}} A_{y j k} = 0, \quad \delta_{h^{-1}} A_{y j k} = -A_{y j k}, \quad \delta_{f^{-1}} A_{y j k} = A^-_{y j k},$$

$$\delta_{e^{-1}} A^+_{\mu j} = \delta_{h^{-1}} A^+_{\mu j} = \delta_{f^{-1}} A^+_{\mu j} = 0.$$  

(4.22)
The Killing vector ∂_− gives rise to additional Geroch compatibility conditions \( B^k_{\dot{y}} = F^{\ell+\dot{y}ij} = 0 \). They are similarly expressed in the form (3.32) with one of 0, . . . , 8 being replaced by −1, where \( \xi^{\dot{\mu}}_{(-1)} \partial_{\dot{\mu}} = \partial_{-} \).

### 4.3 Non-trivial realization of the new \( SL(2, \mathbb{R}) \)

In the previous subsection we saw an extra \( SL(2, \mathbb{R}) \) symmetry in one dimension with keeping the degrees of freedom of Beltrami differentials. In this subsection we show that this transformation certainly enlarges the symmetry which already exists in higher dimensions. For this purpose we further compactify the \( x^+ \) direction on a circle and verify that the new \( SL(2, \mathbb{R}) \) includes transformations that change the conformal structure of the \( (x^+, x^-) \)-torus. The transformation formula obtained in the last subsection shows that \( \mu_{\dot{+}}^- \) does change under \( \delta_{f,-1} \). We will show that this variation cannot be generated by a reparameterization.

Before and after the action of \( \delta_{f,-1} \), the Killing vector \( \partial_{-} \) is preserved. Such diffeomorphisms are caused by the vector fields of the form

\[
Y = \epsilon^+ (x^+) \partial_{+} + \epsilon^- (x^+) \partial_{-},
\]

(4.23)

where \( \epsilon^\pm (x^+) \) is periodic functions of \( x^+ \) only.

A variation of a differential form under a diffeomorphism is given by its Lie derivative, that is

\[
L_Y E_{\overline{\rho}}^{(2)\overline{\sigma}} = \epsilon^\overline{\tau} \partial_{\overline{\tau}} E_{\overline{\rho}}^{(2)\overline{\sigma}} + E_{\overline{\rho}}^{(2)\overline{\sigma}} \partial_{\overline{\tau}} \epsilon^\overline{\tau} = \epsilon^+ \partial_{+} E_{\overline{\rho}}^{(2)\overline{\sigma}} + E_{\overline{\rho}}^{(2)\overline{\sigma}} \partial_{+} \epsilon^\overline{\tau},
\]

(4.24)

where we used \( \partial_{-} E_{\overline{\rho}}^{(2)\overline{\sigma}} = \partial_{-} \epsilon^\overline{\tau} = 0 \). Since

\[
L_Y E_{\overline{\rho}}^{(2)-} = \mu_{\dot{+}}^- L_Y E_{\overline{\rho}}^{(2)-} + E_{\overline{\rho}}^{(2)-} L_Y \mu_{\dot{+}}^-,
\]

(4.25)

the Lie derivative of \( \mu_{\dot{+}}^- \) reads

\[
L_Y \mu_{\dot{+}}^- = \partial_{+} (\epsilon^+ \mu_{\dot{+}}^- + \epsilon^-).
\]

(4.26)

Thus the change of \( \mu_{\dot{+}}^- \) under the diffeomorphism is a total derivative, and in particular

\[
\int_{\text{period}} dx^+ L_Y \mu_{\dot{+}}^- = 0.
\]

(4.27)

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On the other hand, one can obviously take $A_+$ and $\mu_+^-$ such that they satisfy

$$\int_{\text{period}} dx^+(A_+ - \mu_+^- A_-) \neq 0.$$  (4.28)

Hence $\delta f_{-1}, \mu_+^-$ cannot be generated by any diffeomorphism. This proves that the new $SL(2, \mathbb{R})$ is indeed an enlargement of the symmetry because all the rest do not affect the conformal structure of the $(x^+, x^-)$-torus.

### 4.4 From $E_9$ to $E_{10}$

We will now focus on the question of the $E_{10}$ symmetry. Since $SL(2, \mathbb{R})_{-1}$ is so constructed that it acts on the eleven-dimensional fields as

$$\delta_{x_{-1}} E^{(11)\tilde{\mu}}_{\tilde{\nu}} = X_{(-1)\tilde{\mu}} E^{(11)\tilde{\nu}},$$  (4.29)

$$\delta_{x_{-1}} A_{\tilde{\mu}\tilde{\nu}} = 3(X_{(-1)\tilde{\mu}} A_{\tilde{\nu}\tilde{\rho}})_{[\tilde{\mu}\tilde{\nu}\tilde{\rho}]},$$  (4.30)

$(x = e, f, h$ and $X = E, F, H$ defined by (3.9)), with

$$X_{(-1)\tilde{\mu}} = \begin{pmatrix} + & - & y & 1 & \cdots & 8 \\ 0 & X & & & & \\ 0 & & \ddots & & & \\ & & & 0 & \end{pmatrix},$$  (4.31)

it commutes with $SL(2, \mathbb{R})_k$ ($k = 1, \ldots, 7$) and generates $SL(3, \mathbb{R})$ with $SL(2, \mathbb{R})_0$. It is also obvious that $SL(2, \mathbb{R})_{-1}$ and $SL(2, \mathbb{R})_8$ commute. Thus, regarding $\{-\delta_{e_{-1}}, -\delta_{f_{-1}}, -\delta_{h_{-1}}\}$ as a new set of the Chevalley generators, one may extend the Cartan matrix as

$$K_{-1 \ j} = K_{j \ -1} = 2\delta_{j, -1} - \delta_{j, 0}$$  (4.32)

($j = -1, 0, 1, \ldots, 8$), giving the Cartan matrix of $E_{10}$. This completes the proof of the main assertion of this paper.

Finally we would like to emphasize that the greatest difficulty in establishing the $E_{10}$ symmetry was (apart from the issue of non-trivial realization), after all, the check of the commutativity of $SL(2, \mathbb{R})_0$ and $SL(2, \mathbb{R})_8$ in the previous section, and

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8 Being on a torus, the constant mode of $A_+$ cannot be gauged away by the Kaluza-Klein gauge transformation.

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once we proved $E_9$, it was rather easy to confirm $E_{10}$. This was because $SL(2, \mathbb{R})_{-1}$ manifestly commutes with the $E_8$ and hence one has only to examine the relation with $SL(2, \mathbb{R})_{0}$ to show that they generate $SL(3, \mathbb{R})$. Realized linearly, the latter was shown as a consequence of a trivial algebra of matrices.

5 Conclusions

We have considered dimensional reduction of the eleven-dimensional supergravity to three, two and one dimension(s). We derived the three-dimensional $E_{8(+8)}/SO(16)$ nonlinear sigma model by direct dimensional reduction from eleven dimensions. Freudenthal’s classical construction of $E_8$ turned out to reflect its “supergravity structure” very clearly. In two dimensions we found a Matzner-Misner-type $SL(2, \mathbb{R})$ symmetry. The transformation rules corresponding to the Chevalley generators of $E_9$ were explicitly written down. We gave a complete check of the generating relations of $E_9$ on all the fields including the field strengths of $U(1)$ gauge fields, but the gauge potentials. This provides a proof of the $E_9$ (“$E_9(+9)$”) hidden symmetry of the bosonic part of the eleven-dimensional supergravity upon reduction to two dimensions. The generalized Geroch compatibility (hypersurface-orthogonality) condition was derived. The check on the gauge potentials is basically impossible in our scheme since the duality relations are only defined for their field strengths. Therefore our proof holds only classically. It is naturally expected that in the full quantum M theory only its discrete subgroup of $E_9$ should survive as an exact symmetry.

Upon further reduction to one dimension we found a new $SL(2, \mathbb{R})$ symmetry, the transformation rule of which is defined similarly to that of $[15]$ found in $D = 4$ pure gravity. We had to take the Killing vector to be null so as for the duality relations to be non-trivial. Consequently, this $SL(2, \mathbb{R})$ acts on the space of certain plane wave solutions propagating at the speed of light. The $E_9$ being established in two dimensions, it was not difficult to see that the full symmetry algebra is a real form of $E_{10}$. To show that this $SL(2, \mathbb{R})$ cannot be expressed in terms of the old $E_9$ but truly enlarges the symmetry, we compactified the final two dimensions on a two-torus and confirmed that it changes the conformal structure of this two-torus.

There remain many things to be done for a better understanding of the $E_{10}$ symmetry. We conclude this paper by listing some of them:

(i) It is a bit awkward to compactify the final two dimensions because this means that
all the dimensions are compactified in typeII string theory. As already mentioned in
the introduction, a symmetry truly larger than $E_9$ could be realized on the fields of
three-form origin without compactifying the last dimension.

(ii) One is tempted to think that $E_{10}$, which is a consequence of a null reduction,
might be related to Matrix theory [24] proposed as a description of M theory in inﬁ-
nite momentum frame. A deeper understanding of null reduction [16, 25] is desirable.
(iii) U duality of typeII string and M theory below three dimensions must be inves-
tigated.
(iv) Obviously, fermions must also be included.

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