THE WHITTAKER-SHINTANI FUNCTIONS FOR SYMPLECTIC GROUPS

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ABSTRACT. In this note, we give a formula for the Whittaker-Shintani functions for the $p$-adic symplectic groups, which is a generalization of the Zonal spherical functions and Whittaker functions. We then use the formula to give an alternative proof of a conjecture given by T. Shintani on the unramified calculation of $L$-functions for $\text{Sp}_{2m} \times \text{GL}_1$.

1. INTRODUCTION

1.1. Let $G$ be a connected reductive linear algebraic group defined over a number field $F$ with its ring of Adeles $A$, $\Pi = \otimes_v \Pi_v$ an irreducible unitary automorphic cuspidal representation of $G(A)$, and $r$ a finite dimensional representation of the $L$-group $L_G$ of $G$. Following Langlands, one may define the partial $L$-function as

$$L_S(s, \Pi, r) = \prod_{v \notin S} L(s, \Pi_v, r_v)$$

where $S$ is a finite set of places of $F$ outside of which both $G$ and $\Pi_v$ are unramified. Langlands conjectured that this partial $L$-function continues to a meromorphic function in $\mathbb{C}$ which has only finitely many poles and satisfies a standard functional equation relating its value at $s$ to $1 - s$. One of the successful approaches to this conjecture, the Rankin-Selberg method, is by constructing a global zeta-integral plus an Euler product expansion, and equating the unramified local zeta-integrals with the “Langlands factors” $L(s, \Pi_v, r_v)$ (referred to as “unramified computation”).

One of the interesting cases of the above conjecture is the partial tensor $L$-function, where $(G, \Pi, r) = (\text{Sp}_{2n} \times \text{GL}_k, \pi \otimes \tau, 'standard').$ Here $\pi$ and $\tau$ are irreducible cuspidal automorphic representations of $\text{Sp}_{2n}$ and $\text{GL}_k$ respectively. The purpose of this paper is to give an explicit formula for the Whittaker-Shintani functions, which is one of the key steps towards the unramified computation, for the case when $\pi$ is non-generic and $k < n$, following the global construction of zeta-integral in [5]. When $\pi$ is generic the unramified calculation is completed in [6] using the

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Casselman-Shalika formula ([4]) for the Whittaker functions, and the formula for Whittaker-Shintani functions play a parallel role in the non-generic case.

The main idea of this paper actually comes from [4] for the Casselman-Shalika formula for Whittaker functions, and [7] where the Whittaker-Shintani functions for orthogonal groups are defined in a similar way and an explicit formula is given. However, the calculation in our case is more technical since the Jacobi group we are dealing with is not reductive. While the general formula we obtain is not as explicit as in the orthogonal case due to its nature, we still have an explicit formula (5) when restricted to the torus, which is enough for the unramified computation.

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1.2. Let $G$ and $M$ be symplectic groups, defined over a non-archimedean local field, of rank $n$ and $m$ respectively with $n \geq m + 1$. Let $\text{Ind}_{G}^{B} (\chi)$ be the unramified principle series of $G$. Let $M^{J}$ be the Jacobi group and $B_{M^{J}}$ its Borel subgroup as defined in (A) in Section 2 and let $\text{Ind}_{M^{J}}^{M} (\xi, \psi)$ be an unramified principle series of $M^{J}$ as defined in (C) in Section 2. Let $U$ be the unipotent radical of a parabolic subgroup $P_{n-m-1}$ of $G$ and $\psi_{U}$ be a character on $U$ which is stabilized by $M^{J}$ (see (6) and (7)). Then one can define an $M^{J}$-invariant, $(U, \psi_{U})$-covariant pairing $l_{\chi, \xi, \psi}$ between $\text{Ind}_{G}^{B} (\chi)$ and $\text{Ind}_{M^{J}}^{M} (\xi, \psi)$. Let $F_{\chi}^{0}$ and $F_{\xi, \psi}^{0}$ be the normalized spherical vectors in $\text{Ind}_{G}^{B} (\chi)$ and $\text{Ind}_{M^{J}}^{M} (\xi, \psi)$, and we define

$$W_{\chi, \xi, \psi}(g) = l_{\chi, \xi, \psi}(R(g)F_{\chi}^{0}, F_{\xi, \psi}^{0}).$$

This function is a Whittaker-Shintani function attached to $(\chi, \xi, \psi)$ (see Definition 2.3). We will show later that for given $(\chi, \xi, \psi)$ (unramified) such function is unique up to a scalar. Denote by $W_{\chi, \xi, \psi}^{0}$ the normalized Whittaker-Shintani function which is equal to 1 at the identity. In this paper we show the following two theorems (for the definition of $X^{0}, Z, K_{M^{J}}, p^{d}, \lambda, p_{G}, K_{G}$, see Section 2).

**Theorem 1.1.** Let $(\chi, \xi) \in \mathbb{C}^{n} \times \mathbb{C}^{m}$, and let $d \in \Lambda_{m}^{+}, f \in \Lambda_{n}^{+}$. Let $W_{\chi, \xi, \psi}^{0}$ be the normalized Whittaker-Shintani function attached to $(\chi, \xi, \psi)$. Then we have

$$\int_{X^{0}} dx W_{\chi, \xi, \psi}^{0} (p^{d} x p^{f}) = \zeta(1) \prod_{i=1}^{m} \zeta(2i) \sum_{w \in W_{G}, w' \in W_{M}} b(w \chi, w' \xi) d(w \chi) d'(w' \xi) (\delta^{+}(p^{f}) \delta^{-}(p^{d})).$$

(2)
where
\[
d(\chi) = \prod_{1 \leq a < b \leq n} \zeta(\chi_a + \chi_b) \prod_{i=1}^n \zeta(\chi_i), \quad d'(\xi) = \prod_{1 \leq a < b \leq m} \zeta(\xi_a + \xi_b) \prod_{j=1}^m \zeta(2\xi_j),
\]
and
\[
b(\chi, \xi) = \prod_{i < j + n - m} \zeta^{-1}(\chi_i - \xi_j + \frac{1}{2}) \cdot \prod_{i > j + n - m} \zeta^{-1}(-\chi_i + \xi_j + \frac{1}{2}) \cdot \prod_{1 \leq j \leq m} \zeta^{-1}(\frac{1}{2}) \cdot \prod_{1 \leq i \leq n} \zeta^{-1}(\chi_i + \xi_j + \frac{1}{2})
\]

**Theorem 1.2.** Under the same notation and assumption as in the previous theorem, the support of \(W_{\chi, \xi, \psi}^0\) is on
\[
\bigcup_{d \in \Lambda_m^+} \text{ZUK}_{M^?} (p^d \lambda p^f) K_G.
\]
If we let \(L(d', f') = \int_{X^0} dx W_{\chi, \xi, \psi}^0 (p^d x p^f)\), then there exists \(a(d') \geq 0\) independent of \((\chi, \xi, \psi)\) such that
\[
W_{\chi, \xi, \psi}^0 (p^d \lambda p^f) = \sum_{d'} a(d') L(d', f + d - d').
\]
where \(d'\) runs over the set \(\{d' \mid d' \in \Lambda_m^+, f + d - d' \in \Lambda_n^+, d' \leq d\}\) and \(a(d) > 0\). In particular, we have
\[
W_{\chi, \xi, \psi}^0 (p^f) = L(0, f).
\]

The paper is organized as follows. In **Section 2** we give the notation we use in this paper. In **Section 3** we use the Rankin-Selberg convolution to find an integral expression of the pairing \(l_{\chi, \xi, \psi}\) when \((\chi, \xi)\) belongs to \(Z_c \subset \mathbb{C}^n \times \mathbb{C}^m\) which contains a Hausdorff open set. In **Section 4** we show that the pairing \(l_{\chi, \xi, \psi}\) between \(\text{Ind}_{B_G}^G(\chi)\) and \(\text{Ind}_{B_M^?}^M(\xi, \psi)\) satisfying Condition A (see **Definition 2.2**) is unique up to a scalar. Then in **Section 5** we apply the Bernstein's theorem to extend this pairing defined by the integral to generic \((\chi, \xi)\). In **Section 6** we discuss the double cosets of \(G\) on which the Whittaker-Shintani function is supported. By considering the vectors invariant under certain open compact subgroups (in **Section 7**) and applying the intertwining operators (in **Section 8**) we give an explicit formula in **Section 9** for the Whittaker-Shintani function attached to generic \((\chi, \xi)\), and we obtain its value at the identity by an combinatorial argument in **Section 10**. After showing the uniqueness of the normalized Whittaker-Shintani function in **Section 11** we apply the Bernstein’ theorem again to extend the formula to all \((\chi, \xi)\) in **Section 12**. In **Section 13** we use the formula we obtained to give an alternative proof of in [9, Theorem 6.1], the unramified calculation of \(L\)-functions for \(\text{Sp}_{2n} \times \text{GL}_1\).
The application in the last section is in fact a special case of [5, Theorem 4.3], the unramified calculation of L-functions for $\text{Sp}_{2n} \times \text{GL}_k$. The proof for the general case will be shown in our [10]. We also expect parallel results for unitary groups (with respect to skew hermitian forms). This will be covered in the future.

2. Notation

In this paper, we let $F$ be a non-archimedean local field of characteristic 0. Let $\mathcal{O}$ be its maximal compact subring and $p$ the uniformizer. Suppose the order of the residue field is $q$ which is not a power of 2. All the groups are defined over $F$. Throughout the paper we fix $\psi$ to be an additive character on $F$ with conductor 0.

(A) Groups. Let $G = \text{Sp}_{2n}$, $H = \text{Sp}_{2m+2}$ and $M = \text{Sp}_{2n}$, where $m, n$ are two positive integers with $n \geq m + 1$. $M$ (or $H$) embeds to $G$ as $\text{diag}(1, g)$ (or $\text{diag}(1, g, 1)$) for $g \in M$ (or $g \in H$). Let $\mathcal{M}_{2n \times 2n}(F)$ be the $2n \times 2n$ matrix over $F$. For any subgroup $\tilde{G}$ of $G$, and any $i \geq 0$, we define $\tilde{G}^i$ as $\tilde{G}^i = \tilde{G} \cap (I_{2n} + \mathcal{M}_{2n \times 2n}(p^i\mathcal{O}))$.

Let $K_G = G^0$, and $K_M = M \cap K_G$. Let $J$ be the Heisenberg group of dimension $2m + 1$ embedding to $H$ as

$$J(x, y, z) = \begin{pmatrix} 1 & x & y & z \\ 1 & 1 & t & y \\ 1 & -t & x \\ 1 & & & \end{pmatrix}$$

where $x, y \in \mathbb{F}^m$, $z \in \mathbb{F}$. Let $M^J = M \ltimes J$, and $K_M^J = K_M \ltimes J^0$. We let $X(x) = J(x, 0, 0)$, $Y(y) = J(0, y, 0)$, $Z(z) = J(0, 0, z)$ and let $X, Y, Z$ be the group of them respectively. Let $B_G$, $B_H$ and $B_M$ be the standard Borel subgroup of $G, H, M$, and $B_M^J = B_M \ltimes (Y \ltimes Z)$, and let $N_G, N_H, N_M, N_M^J$ be their unipotent radical respectively. Let $T_G$ be the toral part of of $B_G$, and let

$$\Lambda_G^+ = \{(d_1, ..., d_k) \in \mathbb{Z}^k \mid d_1 \geq d_2 \geq ... \geq d_k \geq 0\}$$

$$T_G^+ = \{\text{diag}(t_1, ..., t_n, t_n^{-1}, ..., t_1^{-1}) \mid |t_1| \leq ... \leq |t_n| \leq 1\}$$

$$T_G^- = \{t^{-1} \mid t \in T_G^+\}$$

The definition of $T_M^+$ and $T_M^-$ are similar. Let $P_1^{n-m-1}$ be the standard parabolic subgroup of $G$ with Levi decomposition

$$P_1^{n-m-1} = \text{GL}_1^{n-m-1} \times H \ltimes U. \quad (6)$$

Let $\psi_U$ be the character on $U$ given by

$$\psi_U(u) = \psi\left(\sum_{i=1}^{n-m-1} u_{i,i+1}\right), \quad (7)$$
which is stabilized by $M^j$. We denote by $I_G$, $I_M$ the Iwahori subgroups of $G$ and $M$, and $T_M = I_M \rtimes J^0$. Let $W_G$, $W_M$ be the Weyl group of $G$ and $M$ with respect to $T_G$ and $T_M$.

(B) **Elements.** Let $w_0^G$ be the longest Weyl element in $G$. For $k \leq n$, and given $t_1, \ldots, t_k \in F^*$, we let $d_k(t_1, \ldots, t_k) = diag(I_{n-k}, t_1, \ldots, t_k, t_k^{-1}, \ldots, t_1^{-1}, I_{n-k}) \in T_G$. Let $\mathbb{Z} = \mathbb{Z} \cup \{\infty\}$, and let $\nu$ be the normalized valuation from $F$ to $\mathbb{Z}$. For $a, b \in \mathbb{Z}^+$, we define an order in $\mathbb{Z}$ such that $a \geq b$ if and only if $a - b \in (\mathbb{N} \cup \{\infty\})^k$. We define $\min(a, b) = (\min(a_1, b_1), \ldots, \min(a_k, b_k))$. When $a \in \mathbb{Z}^+$, we let $\lambda(a) = X(p^{a_1}, \ldots, p^{a_m})$. Here $p^\infty = 0$. Let $\lambda = \lambda(0)$. For $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, we let $p^a = d_k(p^{a_1}, \ldots, p^{a_k})$.

(C) **Representations.** Let $\chi$ and $\xi$ be unramified characters on $T_G$ and $T_M$. We parametrize them as $\chi = (\chi_1, \ldots, \chi_n) \in \mathbb{C}^n$ and $\xi = (\xi_1, \ldots, \xi_m) \in \mathbb{C}^m$ such that $\chi(d_n(t_1, \ldots, t_n)) = \prod_{i=1}^n |t_i|^{\chi_i}$ and $\xi(d_m(t_1, \ldots, t_m)) = \prod_{j=1}^m |t_j|^\xi_j$. Let $\text{Ind}_{B_M}^{M^j}(\xi, \psi)$ be a representation of $M^j$ consisting of smooth functions on $M^j$ such that

$$f(b_M(0, y, z)m^j) = \xi^{\frac{1}{2}} \delta_{B_M}(b_m) \psi(z)f(m^j),$$

with $M^j$ acting by right translation. Sometimes we write $\xi \psi$ as a character on $B_M^j$ such that

$$\xi \psi(b_M(0, y, z)) = \xi(b_M)\psi(z).$$

**Remark 2.1.** Although $B_{M^j} \setminus M^j$ is not compact, functions in $\text{Ind}_{B_M}^{M^j}(\xi, \psi)$ are compactly supported on $B_{M^j} \setminus M^j$ by smoothness. In fact by Iwasawa decomposition on $M$, we have

$$M^j = B_{M^j} \ltimes K_M.$$

Suppose $f \in \text{Ind}_{B_M}^{M^j}(\xi, \psi)$ which is right $K_f$ invariant for some open compact subgroup $K_f$. Note that $K_f \backslash K_M$ is finite. Let $k$ be a representative in one of the cosets and suppose $f(xk) \neq 0$. Then by the smoothness of $f$, $f(xk) = f(xyk)$ when $y$ is in a neighbourhood of $0 \in F_m$. But note that $f(xyk) = \psi(2\langle x, y \rangle)f(xk)$, so $\psi(\langle x, y \rangle) = 1$ for all such $y$, which implies that $x$ belongs to a compact subset.

In particular, if $f$ is $K_{M^j}$-invariant, then $f(x) \neq 0$ implies $x \in X^0$. So the spherical vector in $\text{Ind}_{B_M}^{M^j}(\xi, \psi)$ is supported on $B_{M^j}K_{M^j}$, and is unique up to a scalar.

(4) **Functions and Functional.** We denote by $\zeta(s)$ the local zeta function as $\zeta(s) = (1 - q^{-s})^{-1}$. For any set $\mathcal{X}$ we denote by $Ch_{\mathcal{X}}$ its characteristic function. For $\varphi_1 \in C_c^\infty(G)$, we let

$$F_\mathcal{X}(\varphi_1)(g) = \int_{B_G} \varphi_1^{-1} \delta_{B_G}^\frac{1}{2} (bg) \varphi_1(b_g) g d_b g.$$  

(8)
Then the map \( \varphi_1 \mapsto F_{\xi,\psi}(\varphi_1) \) is surjective from \( C^\infty_c(G) \) to \( \text{Ind}^{G}_{BG}(\chi) \). Similarly for \( \varphi_2 \in C^\infty_c(M^1) \), we let

\[
F_{\xi,\psi}(\varphi_2)(m^1) = \int_{B^M_{M^1}} (\xi \psi)^{-1} \delta_{B^M_{M^1}}^\perp (b_{M^1}) \varphi_2(b_{M^1} m^1) \, \mathrm{d} b_{M^1}.
\] (9)

The map \( \varphi_2 \mapsto F_{\xi,\psi}(\varphi_2) \) is surjective from \( C^\infty_c(M^1) \) to \( \text{Ind}^{M^1}_{BM^1}(\xi, \psi) \). Let \( K_{\chi,\xi,\psi} \) be a function defined on \( G \) such that

\[
K_{\chi,\xi,\psi}(b_{G} w_0 G \lambda J(0, y, z) b_{M^1} u) = \chi^{-1} \delta^{1/2}(b_{G}) \xi^{1/2} (b_{M^1}) \psi(z) \psi^{-1}(u),
\] (10)

and \( K_{\chi,\xi,\psi}(g) = 0 \) for all other \( g \). For \( \varphi_1 \in C^\infty_c(G) \) and \( \varphi_2 \in C^\infty_c(M^1) \), let

\[
I_{\chi,\xi,\psi}(\varphi_1, \varphi_2)(g) = \int_{G} \int_{M^1} \int_{M^1} \varphi_1(g') K_{\chi,\xi,\psi}(g') (g^{-1}(m^1)^{-1}) \varphi_2(m^1),
\] (11)

and let \( I^0_{\chi,\xi,\psi}(g) = I_{\chi,\xi,\psi}(Ch_{G}, Ch_{M^1})(g) \). Let \( F_{\chi}^0 = F_{\chi}(Ch_{G}) \) and \( F_{\xi,\psi}^0 = F_{\xi,\psi}(Ch_{M^1}) \) be spherical elements in \( \text{Ind}^{G}_{BG}(\chi) \) and \( \text{Ind}^{M^1}_{BM^1}(\xi, \psi) \) respectively. Let \( H_G \) be the spherical Hecke algebra of \( G \), and \( H_{M^1, \psi} \) be the spherical Hecke algebra of \( M^1 \) with respect to \( \psi \) as defined in section 4 of [8], and let them act on \( \text{Ind}^{G}_{BG}(\chi)^{K_G} \) and \( \text{Ind}^{M^1}_{BM^1}(\xi, \psi)^{K_{M^1}} \) by characters \( \omega_{\chi} \) and \( \omega_{\psi} \) respectively. For any function \( f \) on \( G \), let \( (L(g_0) f)(g) = f(g_0^{-1} g) \), and \( (R(g_0) f)(g) = f(g g_0) \).

**Definition 2.2.** A pairing \( l_{\chi,\xi,\psi} \) between \( \text{Ind}^{G}_{BG}(\chi) \) and \( \text{Ind}^{M^1}_{BM^1}(\xi, \psi) \) is called satisfying Condition A if

(i) \( l_{\chi,\xi,\psi}(F_{\chi}, F_{\xi,\psi}) = l_{\chi,\xi,\psi}(R(m^1) F_{\chi}, R(m^1) F_{\xi,\psi}) \) for any \( m^1 \in M^1 \).

(ii) \( l_{\chi,\xi,\psi}(R(u) F_{\chi}, F_{\xi,\psi}) = \psi(u) l_{\chi,\xi,\psi}(F_{\chi}, F_{\xi,\psi}) \) for any \( u \in U \).

**Definition 2.3.** For \( (\chi, \xi) \in \mathbb{C}^n \times \mathbb{C}^m \), a function \( W_{\chi,\xi,\psi} \in C^\infty(G) \) is called a Whittaker-Shintani Function attached to \( (\chi, \xi) \) if

(i) \( W_{\chi,\xi,\psi}(z u_{M^1} g k_G) = \psi^{-1}(z) \psi(u) W_{\chi,\xi,\psi}(g) \).

(ii) \( L(\varphi_{M^1}) R(\varphi_G) W_{\chi,\xi,\psi} = \omega_{\xi}(\varphi_{M^1}) \omega_{\chi}(\varphi_G) \cdot W_{\chi,\xi,\psi} \) for any \( \varphi_{M^1} \in H_{M^1, \psi} \) and \( \varphi_G \in H_G \).

The space of Whittaker-Shintani functions attached to \( (\chi, \xi, \psi) \) is denoted by \( \mathcal{W}S_{\chi,\xi,\psi} \). Sometimes we omit \( \psi \) because it is fixed in this paper. A Whittaker-Shintani function is called a Normalized Whittaker-Shintani function if it equals 1 at the identity.

**3. Integral expression for the pairing**

We first use the function \( K_{\chi,\xi,\psi} \), as defined in [10], to construct a pairing between \( \text{Ind}^{G}_{BG}(\chi) \) and \( \text{Ind}^{M^1}_{BM^1}(\xi, \psi) \) satisfying Condition A. For any element \( g \in B_{G} w_0^G N_{G} \), the way to express \( g = b w_0^G n \) with \( b \in B_{G} \) and \( n \in N_{G} \) is unique. From this it is
not hard to see that the set $B_G u_{G}^G \lambda Y Z B_M U$ has the same property. So the function $K_{\chi, \xi, \psi}$ is well-defined. Moreover $B_G u_{G}^G \lambda Y Z B_M U$ is zarisky open in $G$ by lemma (3.3) below. Let $\varphi_1 \in C_c^\infty(G)$ and $\varphi_2 \in C_c^\infty(M^j)$, and let $F_{\chi}(\varphi_1)$ and $F_{\xi, \psi}(\varphi_2)$ be defined as in (8) and (9). Then we let

$$E(F_{\chi}(\varphi_1))(g) = \int_{B_G^G \backslash G} F_{\chi}(\varphi_1)(gg)K_{\chi, \xi, \psi}(g) \, dg,$$

where $dg$ is the right $G$-invariant functional on $\text{Ind}_{\text{BG}}^G(\delta_{BG}^{1/2})$ determined by the Haar measure of $G$. By direct calculation we have

$$E(F_{\chi}(\varphi_1))(g) = \int_G \varphi_1(g'g)K_{\chi, \xi, \psi}(g') \, dg,$$

which is convergent when $K_{\chi, \xi, \psi}$ is continuous on $G$. $E(F_{\chi}(\varphi_1))$ satisfies

1. $E(R(u)F_{\chi}(\varphi_1))(g) = \psi_U(u)E(F_{\chi}(\varphi_1))(g)$ when $u \in U$.
2. When restricted to $M^j$, $E(F_{\chi}(\varphi_1)) \in \text{Ind}_{\text{BM}^j}^\text{MJ}(\xi^{-1}, \psi)$.

So $E$ actually gives an $M^j$-homomorphism from the twisted Jacquet-module $(\text{Ind}_{\text{BG}}^G(\chi))_{U, \psi_U}$ to $\text{Ind}_{\text{BM}^j}^\text{MJ}(\xi^{-1}, \psi)$. Now we consider the integral

$$l_{\chi, \xi, \psi}(F_{\chi}(\varphi_1), F_{\xi, \psi}(\varphi_2)) = \int_{B_M^j \backslash M^j} E(F_{\chi}(\varphi_1))(\hat{m}^j)F_{\xi, \psi}(\varphi_2)(\hat{m}^j) \, dm^j$$

where $d\hat{m}^j$ is the right $M^j$-invariant functional on $\text{Ind}_{\text{BM}^j}^\text{MJ}(\delta_{BM}^{1/2})$ determined by the Haar measure of $M^j$. Substituting $E$ and $F_{\xi, \psi}$ by definition, we have

$$l_{\chi, \xi, \psi}(F_{\chi}(\varphi_1), F_{\xi, \psi}(\varphi_2)) = \int_G \int_{M^j} \varphi_1(g')K_{\chi, \xi, \psi}(g'(m^j)^{-1})\varphi_2(m^j) \, dm^j \, dg'. \quad (12)$$

Note that the right hand side is actually $I_{\chi, \xi, \psi}(\varphi_1, \varphi_2)(e)$ as defined in (11).

It is easy to see that the pairing $l_{\chi, \xi, \psi}(F_{\chi}(\varphi_1), F_{\xi, \psi}(\varphi_2))$ satisfies Condition A if the integral is convergent, and the integral is convergent if $K_{\chi, \xi, \psi}$ is continuous on $G$. In the rest of this section we will prove the following proposition.

**Proposition 3.1.** Let $Z_c$ be the set of unramified characters $(\chi, \xi)$ satisfying

$$\begin{align*}
\text{Re}(\chi_i - \chi_{i+1}) &\geq 1 & \text{for } 1 \leq i \leq n - m - 1 \\
\text{Re}(\chi_{n-m-1+j} - \xi_j) &\geq \frac{1}{2} & \text{for } 1 \leq j \leq m \\
\text{Re}(-\chi_{n-m+j} + \xi_j) &\geq \frac{1}{2} & \text{for } 1 \leq j \leq m \\
\text{Re}(\chi_n) &\geq 1
\end{align*}$$

then when $(\chi, \xi) \in Z_c$, the function $K_{\chi, \xi, \psi}$ is continuous on $G$, and as a consequence, the integral (12) is convergent.
Since $K_{\chi,\xi,\psi}$ is defined continuously on an Zariski open subset of $G$ (we will see this soon) and extend by 0 to $G$, we only need to show the continuity outside the Zariski open set, for the function $|K_{\chi,\xi,\psi}|$. The method we use here is similar to that in [7].

First by the Bruhat decomposition we have

$$G = \bigcup_{w \in W_G} B_G w N_G.$$  

And we know that $B_G w_0^G N_G$ is zariski open in $G$. In fact we have

**Lemma 3.2.** There exists $\alpha_k \in \mathfrak{o}[G]$ for $1 \leq k \leq n$ such that

$$B_G w_0^G N_G = \{ g \mid \alpha_k(g) \neq 0 \text{ for all } k \}.$$  

**Proof.** For $g \in G$, let its matrix be $g = (g_{ij})_{1 \leq i, j \leq 2n}$. Let $N_{2n} = \{1, 2, \ldots, 2n\}$. For $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ both belonging to $N_{2n}^k$, we let $g_{IJ} = (g_{is, jt})_{1 \leq s, t \leq k}$. We define

$$\Delta_{I,J}(g) = \det g_{IJ}.$$  

(14)

For $1 \leq k \leq n$, let $I_k = \{2n+1-k, 2n+1-(k-1), \ldots, 2n\}$, and $J_k = \{1, 2, \ldots, k\}$, and we take

$$\alpha_k(g) = \Delta_{I_k, J_k}(g).$$  

(15)

Then one can check that

1. For any $n_1, n_2 \in N_G$, $\alpha_k(n_1 g n_2) = \alpha_k(g)$.
2. $\alpha_k(d_n(t_1, \ldots, t_n) g d_n(s_1, \ldots, s_n)) = \prod_{i=1}^{k} t_i^{-1} s_i \cdot \alpha_k(g)$.
3. Let $w \in W_G$. If $\alpha_k(w) \neq 0$ for all $1 \leq k \leq n$, then $w = w_0^G$.

Combining these properties with the Bruhat decomposition of $G$, we have our lemma. \hfill \Box

Next we have

**Lemma 3.3.** There exists $\beta_l \in \mathfrak{o}[G]$ for $1 \leq l \leq m$ such that

$$B_G w_0^G \lambda YZB_M U = \{ g \in G \mid \alpha_k(g) \neq 0, \beta_l(g) \neq 0 \text{ for all } 1 \leq k \leq n, 1 \leq l \leq m \},$$  

where $\alpha_k$ is as defined in lemma 3.2.

**Proof.** Note that for any $w \in W$, $B_G w N_G = B_G w X UN_{M^1}$. For any $X(x_1, \ldots, x_m) \in X$, we have $X(x_1, \ldots, x_m) = s^{-1} X(r_1, \ldots, r_m) s$, where $s = d_m(s_1, \ldots, s_m) \in T_M$ such that

$$(s_i, r_i) = \begin{cases} (x_i, 1) & \text{if } x_i \neq 0; \\ (1, 0) & \text{if } x_i = 0 \end{cases}$$
From this we can see that
\[ B_G w N_G = \bigcup_{r \in \{0,1\}^m} B_G w X(r) B_{M^l} U, \]  
and when \( w = w_0^G \), the union is disjoint. For \( 1 \leq l \leq m \), we let
\[ J'_l = \{ 1, 2, \ldots, (n - m), \ldots, n - m + l \}, \]
and we define
\[ \beta_l(g) = \Delta_{J_{n-m+l-1}, J'_l}(g). \]  
Then \( \beta_l \) satisfies
\[
(1) \quad \beta_l(n_1 g) = \beta_l(g) \text{ for any } n_1 \in N_G.
\]
\[
(2) \quad \beta_l(g n_2 u) = \beta_l(g) \text{ for any } n_2 \in N_M, u \in U.
\]
\[
(3) \quad \beta_l(d_n(t_1, \ldots, t_n) g d_m(s_1, \ldots, s_m)) = \prod_{i=1}^{n-m} t_{i-1}^{1} \cdot \prod_{j=1}^{l-1} t_{n-m+j}^{1} \cdot \prod_{j=1}^{l'} s_j \cdot \beta_l(g).
\]
\[
(4) \quad \beta_l(w_0^G X(r)) = \pm r_l. \text{ The sign in front of } r_l \text{ depends on } l, \text{ which is not important}
\]
since we are only interested in \(|\beta_l|\).

In fact, (1) is by the definition of \( I_k \) while (3) and (4) are by direct calculation. For (2), note that if we only consider the first \( n \) column of \((g_{ij})\), multiplying elements in \( N_M U \) from the right corresponds to column operations adding multiples of column \( k_1 \) to column \( k_2 \) where \( 1 \leq k_1 < k_2 \leq n \) with \( k_1 \neq n - m \). On the other hand elements in \( J'_l \) are consecutive from 1 to \( n - m + l \) with \( n - m \) missing, so \( \Delta_{J_{n-m+l-1}, J'_l}(g) \) is invariant under such column operations.

So for \( g \in B_G w X(r) B_{M^l} U \), \( \alpha_k(g) \neq 0 \) and \( \beta_l(g) \neq 0 \) for all \( 1 \leq k \leq n \) and \( 1 \leq l \leq m \) if and only if \( w = w_0^G \) (by lemma 3.2) and \( X(r) = \lambda \) (by the property of \( \beta_l \)'s), completing our proof. \( \square \)

**Remark 3.4.** If we let \( \varpi_i = \epsilon_1 + \ldots + \epsilon_i \in \text{Hom}(T_G, GL_1) \) and \( \varpi'_j = \epsilon'_1 + \ldots + \epsilon'_j \in \text{Hom}(T_M, GL_1) \) be the dominant weights of \( G \) and \( M \) with respect to \( B_G \) and \( B_M \), then the properties of \( \alpha_k \) and \( \beta_l \) actually shows that under the \( B_G \times B_M \) action, \( \alpha_k \) has the highest weight \( (\varpi_k, 0) \) when \( 1 \leq k \leq n - m \), and \( (\varpi_k, \varpi'_{k-(n-m)}) \) when \( n - m + 1 \leq k \leq n \), and \( \beta_l \) has the highest weight \( (\varpi_{n-m+l-1}, \varpi'_l) \).

Now we can expressed \( K_{\chi, \xi, w} \) by \( \alpha_k \) and \( \beta_l \). First we have

**Lemma 3.5.** Let \( g = d_n(t_1, \ldots, t_n) n_G w_0^G \lambda d_m(s_1, \ldots, s_m) n_M U \in B_G w_0^G \lambda B_{M^l} U \), we have
\[
|t_i| = \begin{cases} \alpha_1^{-1}(g) & \text{if } i = 1; \\ \alpha_{i-1}^{-1}(g) & \text{if } 2 \leq i \leq n - m; \\ \beta_{l-(n-m)}^{-1}(g) & \text{if } i > n - m \end{cases}
\]
and
\[
|s_j| = |\beta_j^{-1}(g)| \quad \text{for } 1 \leq j \leq m
\]
By this we have

**Lemma 3.6.** For \( g \in B_G u_0^G \lambda B_{M^J} U \), we have

\[
|K_{\chi, \xi, \psi}(g)| = \prod_{i=1}^{n-m-1} |(\chi_i \chi_{i+1}^{-1} \cdot |^{-1})(\alpha_i(g))| \cdot \prod_{j=1}^{m} |(\chi_{n-m-1+j} \xi_{j}^{-1} \cdot |^{-1})(\alpha_{n-m+j}(g))|
\]

\[
\cdot |(\chi_{n} \cdot |^{-1})(\alpha_{n}(g))| \cdot \prod_{j=1}^{m} |(\chi_{n-m-j} \xi_{j} \cdot |^{-1})(\beta_{j}(g))|.
\]

The proof of these are by direct calculation. Note that \( \alpha_k, \beta_l \) are continuous functions on \( G \), so when the assumptions in proposition 3.1 are satisfied, the extension of \( |K_{\chi, \xi, \psi}(g)| \) by 0 to outside the set \( B_G u_0^G \lambda B_{M^J} U \) is continuous, and so \( K_{\chi, \xi, \psi}(g) \) is continuous.

4. **Uniqueness for the pairing for generic \((\chi, \xi)\)**

The pairing \( l_{\chi, \xi, \psi} \) satisfying Condition A corresponds to the homomorphism

\[
\text{Hom}_{B_{M^J} U}(\text{Ind}_{B_G}(\chi), \xi^{-1} \psi^{-1} \delta_{B_{M^J}} \otimes \psi_U).
\]

In this section we prove that

**Proposition 4.1.** For generic \((\chi, \xi)\),

\[
\text{dim Hom}_{B_{M^J} U}(\text{Ind}_{B_G}^G(\chi), \xi^{-1} \psi^{-1} \delta_{B_{M^J}} \otimes \psi_U) \leq 1.
\]

Let \( U_d \) be the union of double cosets of \( B_G \backslash G / B_{M^J} U \) with codimension \( \leq d \). Then \( U_0 = B_G u_0^G \lambda B_{M^J} U \) is open in \( G \), and for any \( d \geq 1 \) we have the exact sequence

\[
0 \to S(U_{d-1}) \to S(U_d) \to \sum_{\text{codim } U = d} S(U) \to 0.
\]

Obviously we have

\[
\text{dim Hom}_{B_{M^J} U}(\text{Ind}_{B_G}^G(\chi, U_0), \xi^{-1} \psi^{-1} \delta_{B_{M^J}} \otimes \psi_U) \leq 1,
\]

so we only need to show that

**Lemma 4.2.** Suppose \((\chi, \xi)\) is generic, and let \( U = B_G g B_{M^J} U \) be a double coset different from \( U_0 \), then

\[
\text{dim Hom}_{B_{M^J} U}(\text{Ind}_{B_G}^G(\chi, U), \xi^{-1} \psi^{-1} \delta_{B_{M^J}} \otimes \psi_U) = 0.
\]

**Proof.** When \( U = B_G g B_{M^J} U \), we let \( G_g = B_{M^J} U \cap g^{-1} B_G g \), then

\[
\text{Hom}_{B_{M^J} U}(\text{Ind}_{B_G}^G(\chi, U), \xi^{-1} \psi^{-1} \delta_{B_{M^J}} \otimes \psi_U) = \text{Hom}_G(g^{-1}(\chi \delta_{B_G}) \otimes \xi \psi \delta_{B_{M^J}} \otimes \psi_U^{-1}, \delta_g)
\]
where $\delta_g$ is the modulus character of $G_g$. So we need to show that
\[ g^{-1}(\chi \delta_{BG}^{\frac{1}{2}}) \cdot \xi \psi \delta_{BMU}^{\frac{1}{2}} \cdot \psi_U^{-1} \cdot \delta_g^{-1} \neq 1 \] (19)
when restricted to $G_g$ for any generic $(\chi, \xi)$. Recall from (16) that
\[ G = \bigcup_{w \in W_G, r \in \{0,1\}^m} B_G w X(r) B_{Mj}. \]
So we can assume $g = w X(r)$ with either $w \neq w_0^G$ or $r \neq (1,1,\ldots,1)$. What we need to find is $b_1 \in B_G$ and $b_2 \in B_{Mj} U$ such that
\[ b_1 g = gb_2 \]
\[(\chi \delta_{BG}^{\frac{1}{2}})(b_1) \neq \xi \psi \delta_{BMU}^{\frac{1}{2}} \psi_U \cdot \delta_g(b_2). \] (20)

First suppose $r \neq (1,1,\ldots,1)$. In this case we claim that $T_G \cap (g T_{M} g^{-1})$ contains a nontrivial torus. Let $t = d_m(t_1, \ldots, t_m) \in T_M$. Note that $T_G$ is stabilized by the adjoint action of $W_G$, so it suffices to show that there exists a nontrivial torus $T_s$ of $T_M$ such that when $t \in T_s$, $X(r)^{-1} t X(r) \in T_G$. Note that
\[ X(r)^{-1} t X(r) = t \cdot X((1-t_1)r_1, (1-t_2)r_2, \ldots, (1-t_m)r_m), \]
so when $r_j = 0$ for some $j$, we can let $T_s = \{ t = d_m(1, \ldots, \hat{t}_j, \ldots, 1) \}$ be the torus we claimed. Then since $(\chi, \xi)$ is generic, one can find some $b_2 \in T_s$ and $b_1 = gb_2 g^{-1} \in T_M$ so that (20) is satisfied, completing the proof for this case.

Now suppose $r = (1, \ldots, 1) \in F^m$ and $w \neq w_0^G$, so $X(r) = \lambda$ by our notation. In this case there is a simple root $\alpha$ in $G$ such that $w N_\alpha w^{-1} \in N_G$.

When $\alpha = e_i - e_{i+1}$ with $1 \leq i \leq n - m - 1$, note that $\lambda \in M^3$ stabilizes $\psi_U$, so $\psi_U(\lambda^{-1} n_\alpha(t) \lambda) = \psi_U(n_\alpha(t)) \neq 1$ for some $t \in F$. On the other hand, $w n_\alpha(t) w^{-1} \in N_G$ by our assumption. So let $b_1 = w n_\alpha(t) w^{-1}$ and $b_2 = \lambda^{-1} n_\alpha(t) \lambda$ we have (20).

When $\alpha = e_i - e_{i+1}$ with $i \geq n - m$, we let $r(t, i) = X(1, \ldots, 1, i + t, \ldots, 1)$, where $i' = i - (n - m)$. Then for $t \neq -1$ we have
\[ w n_\alpha(t) \lambda = w X(r(t, i)) n_\alpha(t) \]
\[ = (d_m^{-1}(r(t, i)))^w w \lambda d_m(r(t, i)) n_\alpha(t). \]
So let $b_1 = (d_m^{-1}(r(t, i))) n_\alpha(t) \lambda$ and $b_2 = d_m(r(t, i)) n_\alpha(t)$. For generic $(\chi, \xi)$, we can always find some $t \in F$ so that (20) is satisfied.

When $\alpha = 2e_n$, we have
\[ w n_\alpha(t) \lambda = w \lambda Y(t) Z(-t) n_\alpha(t). \]
So let $b_1 = (n_\alpha(t)) \lambda$ and $b_2 = Y(t) Z(-t) n_\alpha(t)$, we have $(\chi \delta_{BG}^{\frac{1}{2}})(b_1) = 1$, and we can find some $t \in F$ so that $\psi(Z(-t)) \neq 1$, so (20) is satisfied. \( \Box \)
5. Rationality(I)

In this section we show that the pairing $l_{\chi,\xi,\psi}$ defined in (12) can be extended to all generic $(\chi,\xi)$.

**Lemma 5.1.** Given $\varphi_1$ and $\varphi_2$, the pairing $l_{\chi,\xi,\psi}(F_{\chi}(\varphi_1), F_{\xi,\psi}(\varphi_2))$ as defined in (12) can be extended as a rational function on $(\chi,\xi)$, and for generic $(\chi,\xi)$ it is the pairing between $\text{Ind}_{BG}^G(\chi)$ and $\text{Ind}_{BM}^M(\xi,\psi)$ satisfying Condition A.

**Proof.** By lemma (8.4), which does not depend on the rationality of $l_{\chi,\xi,\psi}$, we know that for $(\chi,\xi) \in \mathcal{Z}_c$, $l_{\chi,\xi,\psi}(F_{\chi}(\text{Ch}_G), F_{\xi,\psi}(\text{Ch}_M)) = \text{Vol}(I_G)\text{Vol}(I_M)$. (21)

Now we apply the Bernstein theorem. Note that by the last section, the pairing $l_{\chi,\xi,\psi}$ satisfying Condition A and (21) is unique for generic $(\chi,\xi)$. When $(\chi,\xi) \in \mathcal{Z}_c$, the pairing $l_{\chi,\xi,\psi}$ defined by $l_{\chi,\xi,\psi}(F_{\chi}(\varphi_1), F_{\xi,\psi}(\varphi_2)) = I_{\chi,\xi,\psi}(\varphi_1, \varphi_2)$ satisfies these conditions. So it extends to a rational function in $(\chi,\xi)$ when $\varphi_1$ and $\varphi_2$ are given. Moreover, it is regular at generic $(\chi,\xi)$ where the uniqueness is valid. $\square$

From now on we still denote by $l_{\chi,\xi,\psi}$ its own meromorphic continuation to all generic $(\chi,\xi)$, and similarly for $I_{\chi,\xi,\psi}(g)$ and $I_{0,\chi,\xi,\psi}(g)$.

6. Double coset decomposition

Let $W_{\chi,\xi,\psi}(g)$ be a Whittaker-Shintani function. In this section we discuss the support of $W_{\chi,\xi,\psi}(g)$ on the double cosets $ZUKM \backslash G/KG$. We denote by $g_1$ $g_2$ when $g_1$ and $g_2$ belong to the same double coset. We are going to show that

**Theorem 6.1.** The support of $W_{\chi,\xi,\psi}(g)$ is contained in the double coset $\bigcup_{d \in \Lambda^+_m, f \in \Lambda^+_n} \text{ZUK}_{M^d}(p^d \lambda p^f)K_G$.

First, by the Iwasawa decomposition on $G$ and the Cartan decomposition on $M$, we have $G = \text{ZUK}_{M^d}(\text{XYT}_{1}^{n-m}T_{1}^{n-m})K_G$.

Here $T_{1}^{n-m}$ is the embedding of $\text{GL}_{n-m}^{1}$ to $G$ as $T_{1}^{n-m}(t_1, \ldots, t_{n-m}) = d_n(t_1, \ldots, t_{n-m}, 1, 1, \ldots, 1)$.

So we only need to consider the support of $W_{\chi,\xi,\psi}$ on the set $XYp^{(a,b)}$ where $a \in \mathbb{Z}^{n-m}$ and $b \in \Lambda^+_m$. We have
Lemma 6.2. Suppose $W_{\chi,\xi,\psi}(xy^{p(a,b)}) \neq 0$. Then $y \in Y^0$ and $a \in \Lambda_{n-m}$.

Proof. The proof is given in lemma 2.1 in [9]. First for any $z \in Z^0$, we have

$$W_{\chi,\xi,\psi}(xy^{p(a,b)}) = W_{\chi,\xi,\psi}(xy^{p(a,b)}z) = \psi(p^{2a_{n-m}z})W_{\chi,\xi,\psi}(xy^{p(a,b)}).$$

So $a_{n-m} \geq 0$.

To show $y \in Y^0$, we argue by contradiction. Let $y = Y(y_1, ..., y_m)$ with $|y_j| = |p|^{-r}$, $r > 0$. We let $E_{\alpha}$ be the root subgroup in $G$ of the root $\alpha$. Let $E_{\alpha}(t)$ be the canonical embedding of $F$ to $E_{\alpha}$. Define $E'_{\beta}(t)$ similarly on $M$. Then for any $t \in O^*$,

$$W_{\chi,\xi,\psi}(xy^{p(a,b)}) = W_{\chi,\xi,\psi}(xy^{p(a,b)}E'_{-2e_j}^t(p^{2b_j+r}t))$$

But note that $p^{rty_j^2} \not\in w^{-r}O^*$, one can choose $t \in O^*$ so that $\psi(p^{rty_j^2}) \neq 1$, which is a contradiction. So $y \in Y^0$.

To show $a \in \Lambda_{n-m}^+$, we also argue by contradiction. Since we already have $a_{n-m} \geq 0$, we assume that $a_i < a_{i+1}$ for some $i \leq n-m-1$. Note that when $i \leq n-m-1$, $E_{e_i-e_{i+1}} \subset U$. For any $t \in O$, we have

$$W_{\chi,\xi,\psi}(xy^{p(a,b)}) = W_{\chi,\xi,\psi}(xy^{p(a,b)}E_{e_i-e_{i+1}}t)$$

But when $a_i < a_{i+1}$, we can always find some $t \in O$ such that $\psi(p^{a_i-a_{i+1}t}) \neq 1$, contradicting the assumption. So we have $a \in \Lambda_{n-m}^+$. \[\square\]

By this lemma the support of $W_{\chi,\xi,\psi}$ is on $ZUK_M(XT_{n-m}^+T_m^+)K_G$. Next we narrow the support further on the $X$-part. Note that if $v(x_i) = c_i$ for $1 \leq i \leq m$, then $X((x_1, ..., x_m)) t \lambda(c)t$ for any $t \in T_G$. So we only need to consider for which $c$ that $\lambda(c)T_{n-m}^+T_m^+$ is contained in the support.

Lemma 6.3. Suppose $x = \lambda(c)$, then for any $t \in T_n$,

$$xt \lambda(min(c,0))t$$

Proof. Note that $\lambda \in K_M$, so $xt \lambda xt \in T_M^0 \lambda(min(c,0))tT_M^0$. \[\square\]

By this lemma, we just need to consider the support of $W_{\chi,\xi,\psi}$ on $xp(a,b)$ with $x = \lambda(-d)$ for some $d \geq 0$.

Lemma 6.4. Let $x = \lambda(-d)$ with $d \geq 0$. Let $a \in \Lambda_{n-m}^+$ and $b \in \Lambda_m^+$. Suppose $W_{\chi,\xi,\psi}(xp^{(a,b)}) \neq 0$, then $d \leq b$. 

Proof. Suppose not, so we let $d_j > b_j$ for some $j$. Let $t \in \mathcal{O}^*$, then
\[
W_{\chi, \xi, \psi}(x p^{(a,b)}(a)) = W_{\chi, \xi, \psi}(x p^{(a,b)} E_{2e_j}(t)) = W_{\chi, \xi, \psi}(x E_{2e_j}(p^{2b_j} t) p^{(a,b)})
\]
\[
= \psi(p^{2b_j - 2d_j} t) W_{\chi, \xi, \psi}(Y(t, \ldots, 0, p^{2b_j - d_j}, 0, \ldots, 0) x p^{(a,b)})
\]
By lemma [5.4], when $W_{\chi, \xi, \psi}(x p^{(a,b)}(a)) \neq 0$, we have $p^{2b_j - d_j} \in \mathcal{O}$, so the last formula equals $\psi(p^{2b_j - 2d_j} t) W_{\chi, \xi, \psi}(x p^{(a,b)})$. But when $d_j > b_j$, we can choose some $t \in \mathcal{O}^*$ such that $\psi(p^{2b_j - 2d_j} t) \neq 1$, contradicting the assumption.

Note that when $x = \lambda(-d)$ with $d \geq 0$, $x p^{(a,b)} = p^d \lambda p^{(a,b-d)}$. Let $r = b - d$. So by the previous lemmas, the support of $W_{\chi, \xi, \psi}$ is on the union of $ZUK_M(p^d \lambda p^{(a,r)}) K_G$ for all $a \in \Lambda_{n-m}^+$, $d \geq 0$, $r \geq 0$ and $d + r \in \Lambda_m^+$. The following two lemmas help us to narrow our choice of $a, b, d$ so that we get theorem (6.1). We also need to use them in the later calculations for the Whittaker-Shintani function.

**Lemma 6.5.** Suppose $g = p^d \lambda p^{(a,r)}$ with $a \in \Lambda_{n-m}^+$, $d \geq 0$, $r \geq 0$, and $d + r \in \Lambda_m^+$, then

1. Suppose $a_{n-m} < r_1$. Let $\overline{r} = (a_{n-m}, r_2, \ldots, r_m)$. Then

\[
p^d \lambda p^{(a,r)} = p^d \lambda p^{(a,\overline{r})}.
\]

Moreover, if $d \in \Lambda_m^+$, then so is $d + r - \overline{r}$. We call the process from $(d; a, r)$ to $(d + r - \overline{r}; a, \overline{r})$ **Operation 1**.

2. Fix $i$, let $\overline{r} = (\overline{r}_1, \ldots, \overline{r}_m)$ where

\[
\overline{r}_j = \begin{cases} 
    \overline{r}_j & \text{if } j > i \text{ and } r_j > r_i \\
    r_j & \text{otherwise},
\end{cases}
\]

then

\[
p^d \lambda p^{(a,r)} = p^{d+r-\overline{r}} \lambda p^{(a,\overline{r})}.
\]

Moreover, if $a_{n-m} \geq r_1$, then $a_{n-m} \geq \overline{r}_1$; if $d \in \Lambda_m^+$, then so is $d + r - \overline{r}$. For given $i$, we call the process from $(d; a, r)$ to $(d + r - \overline{r}; a, \overline{r})$ **Operation (2, i)**.

3. Given $i$, let $\overline{d} = (d_1, \ldots, d_m)$ where

\[
\overline{d}_j = \begin{cases} 
    d_i & \text{if } j < i \text{ and } d_j < d_i \\
    d_j & \text{otherwise},
\end{cases}
\]

then

\[
W_{\chi, \xi, \psi}(p^d \lambda p^{(a,r)}) = W_{\chi, \xi, \psi}(p^{\overline{d}} \lambda p^{(a,r+d-\overline{d})}).
\]

Moreover, if $(a, r) \in \Lambda_m^+$, then so is $(a, r + d - \overline{d})$. For given $i$, we call the process from $(d; a, r)$ to $(\overline{d}; a, r + d - \overline{d})$ **Operation (3, i)**.
Proof. For part 1, note that when \( a < r_1 \),
\[
 p^d \lambda p(a,r) \ p^d \lambda p(a,r) \\
 p^d \lambda((a_{n-m} - r_1, 0, \ldots, 0))p^{(a,r)} \ p^{d+r-\tau} \lambda p(a,\tau),
\]
so we have part 1. For part 2, we fix \( i \), then we have
\[
 p^d \lambda p(a,r) \ p^d \lambda p(a,r) \prod_{j \mid j > i, r_j > r_i} E'_{e_i - e_j}(1) \\
 \prod_{j \mid j > i, r_j > r_i} E'_{e_i - e_j}(p^{d_i + r_i - d_j - r_j}) p^d \lambda(\tilde{\tau} - r) p^{(a,r)} \ p^{d+r-\tilde{\tau}} \lambda p(a,\tilde{\tau})
\]
the rest of part 2 is correct since \( d + r \in \Lambda_m^+ \). Part 3 is similar to part 2. For fix \( i \), we have
\[
 p^d \lambda p(a,r) \prod_{j \mid j < i, d_j < d_i} E'_{-e_j + e_i}(1) p^d \lambda p(a,r) \\
 p^d \lambda(d - \tilde{d}) p^{(a,r)} \prod_{j \mid j < i, d_j < d_i} E'_{e_i - e_j}(p^{d_i + r_i - d_j - r_j}) p^d \lambda p(a,r + d - \tilde{d}).
\]
The rest of part 3 is correct since \( d + r \in \Lambda_m^+ \) and \( \tilde{r}_1 \leq r_1 \).

Proposition 6.6. Suppose \( a \in \Lambda_{n-m}^+, d \geq 0, r \geq 0 \) and \( d + r \in \Lambda_m^+ \). Starting from \((d; a, r)\), if we take the Operation 1, and then Operation (2,i), for \( i \) from 1 to \( m \), and then Operation (3,i), for \( i \) from \( m \) to 1, we will end the process with \((\tilde{d}; a, \tilde{\tau})\) such that

1. \( \tilde{d} + \tilde{\tau} = d + r \).
2. \( \tilde{d} \geq d \), so equivalently, \( \tilde{\tau} \leq r \).
3. \( \tilde{d} \in \Lambda_m^+ \), and \( (a, \tilde{\tau}) \in \Lambda_m^+ \).
4. For any triple \((d'; a, r')\) satisfying (1), (2) and (3) above, \( \tilde{d} \leq d' \).

In other words, for \((d; a, r)\) such that \( a \in \Lambda_{n-m}^+, d \geq 0, r \geq 0 \) and \( d + r \in \Lambda_m^+ \), there exists \((\tilde{d}; a, \tilde{\tau})\) satisfying (1), (2) and (3) above such that \( p^d \lambda p(a,r) \ p^d \lambda p(a,\tilde{\tau}). \)
Moreover, among all the triples satisfying (1), (2) and (3), \((\tilde{d}; a, \tilde{\tau})\) is the one with the smallest \( \tilde{d} \).

Proof. Part (1) is obvious. Part (2) is true because in all the operations, either \( d \) increases or \( r \) decreases, and by (1) they are equivalent. Part (3) is true because by operation 1, \( a_{n-m} \geq \tau_1 \), and after operation (2,i), \( \tau_i \geq \tau_j \) for all \( j > i \), and after operation (3,i), \( \tilde{d}_i \leq \tilde{d}_j \) for all \( j < i \).

To prove part (4), we let \((d'; a, r')\) satisfy (1), (2) and (3). Note that \((\tilde{d}; a, \tilde{\tau})\) is the result of \( 2i + 1 \) operations on \((d; a, r)\). We show below that for any triple \((D; a, R)\) satisfying \( D + R = d' + r' \) and \( D \leq d' \), it still satisfies the same conditions after one
of the operations in lemma (6.5). Repeating it for \(2i + 1\) times we prove (4) since initially we have \(d \leq d'\).

To be precise, let \((D; a, R)\) be such a triple, and suppose after one of the operations in lemma (6.5) it becomes \((D'; a, R')\). We need to show \(D' + R' = d' + r'\) and \(D' \leq d'\). The former one is obvious by the definition of the operations. For the latter first suppose the operation we took is \((3,i)\), then for any \(j\), either \(D'_j = D_j \leq d'_j\), or \(D'_j = D_i\), in which case \(j < i\), implying \(D'_j \leq d'_j\). So \(D'_j \leq d'_j\) anyway and hence \(D' \leq d'\). If the operation we took is \((1)\) or \((2,i)\), we can similarly prove that \(R' \geq r'\), which implies, by \(D' + R' = d' + r'\), that \(D' \leq d'\). So in any case \(D' \leq d'\).

□

Following lemma (6.3) and (6.6), theorem (6.1) is implied.

7. Vectors invariant under certain open compact subgroups

7.1. The \(I_G\)-invariant vectors in \(\text{Ind}_{B_G}^G(\chi)\).

Let \(I_G\) be the Iwahori subgroup of \(G\). For \(w \in W_G\), let \(\Phi^w_{\chi}\) be the element in \(\text{Ind}_{B_G}^G(w\chi)\) defined as

\[
\Phi^w_{\chi}(g) = \int_{B_G} \, \text{d} b \, \text{Ch}_{I_G}(bg)(w\chi)^{-1} \delta_{B_G}^\dagger(b).
\]

Then \(\Phi^w_{\chi}\) is \(I_G\)-invariant. And we have

\[
\Phi^w_{\chi}(1) = \text{Vol}(B_G \cap I_G) = 1.
\]

For a Weyl element \(w\) and a character \(\chi\) on the \(T_G\), let \(T_{w,\chi}\) be the Intertwining operator from \(\text{Ind}_{B_G}^G(\chi)\) to \(\text{Ind}_{B_G}^G(w\chi)\) defined as

\[
T_{w,\chi} f(g) = \int_{N \cap wNw^{-1}} \, \text{d} n \, f(w^{-1}ng).
\]

The integral is convergent when \(\text{Re}(\chi)\) is sufficiently large, and by [3] it continues holomorphically to generic \(\chi\). We write \(T_{w,\chi}\) as \(T_w\) when there is no risk of confusion. For generic \(\chi\), the \(G\)-intertwining operator from \(\text{Ind}_{B_G}^G(\chi)\) to \(\text{Ind}_{B_G}^G(w\chi)\) is unique for every \(w \in W_G\). So

\[
T_{w_1} \circ T_{w_2} = c \cdot T_{w_1w_2}
\]

for some constant \(c\). Moreover, if we let

\[
c_\alpha(\chi) = \frac{\zeta((\chi, \bar{\alpha}))}{\zeta((\chi, \bar{\alpha}) + 1)} \quad \text{and} \quad c_w(\chi) = \prod_{\alpha > 0, \omega \alpha < 0} c_\alpha(\chi),
\]

then

\[
T_{w_1} \circ T_{w_2} = c \cdot T_{w_1w_2}.
\]
then by theorem 3.1 in \[2\]
\[T_wF_0^\chi = c_w(\chi)F_0^{\phi w\chi}.\]
So if we let
\[T_{w,\chi} = c_w(\chi)^{-1}T_{w,\chi},\]
then
\[T_{w_1w_2,\chi} = T_{w_1w_2,\chi}.\]

Following section 5 in \[4\] we state the following results without proof.

**Lemma 7.1.** Elements in \(\{T_{w^{-1}}\phi_1^{w\chi}\}_{w\in W_G}\) form a basis of \(\text{Ind}_{B_G}(\chi)^{I_G}\).

By this lemma, we have

**Lemma 7.2.** one can express \(F_0^{\chi}\) as a linear combination of \(\{T_{w^{-1}}\phi_1^{w\chi}\}_{w\in W_G}\). In fact we have
\[F_0^{\chi} = \text{Vol}(I_G)^{-1}\sum_{w\in W_G} c_w(\chi)T_{w^{-1}}\phi_1^{w\chi}.\]

Next we have

**Proposition 7.3.** For \(a \in T^-_G\),
\[R(\text{Ch}_{I_G}^aI_G)\phi_1^\chi = \text{Vol}(I_G)^{-1}\delta^{-\frac{1}{2}}(a)\phi_1^\chi.\] (22)
So as a consequence,
\[R(\text{Ch}_{I_G}^aI_G)F_0^{\chi} = \sum_{w\in W_G} c_w(\chi)(w\chi)\delta^{-\frac{1}{2}}(a)T_{w^{-1}}\phi_1^{w\chi}.\] (23)

**7.2. The \(I_M\)-Invariant vectors in \(\text{Ind}_{B_M^J}(\xi,\psi)\).**

A similar discussion can be applied to \(M^J\) with some modification. Let \(I_M\) be the Iwahori subgroup of \(M\), and let \(\tilde{I}_M = I_M \ltimes J^0\). Consider the space
\[\text{Ind}_{B_M^J}(\xi,\psi)^{\tilde{I}_M}.\]
We have the following lemma.

**Lemma 7.4.** Any \(f \in \text{Ind}_{B_M^J}(\xi,\psi)^{\tilde{I}_M}\) is supported on \(B_M^JW_M^{\tilde{I}_M}\).

**Proof.** Since
\[M = B_MW_MI_M,\]
we have
\[M^J = B_M^JXW_MI_M.\]
Suppose $f \in \text{Ind}_{B_{M^J}}^{M^J}(\xi, \psi)_{|M}$ with $f(X(x)w) \neq 0$, then since $W_M$ normalizes $J^0$, we have

$$f(X(x)w) = f(X(x)Y(y)w)$$

for any $y \in O^m$. But then

$$f(X(x)Y(y)w) = f(Y(y)X(x)Z(2\langle x, y \rangle)w) = \psi(\langle x, y \rangle)f(X(x)w).$$

So $\psi(\langle x, y \rangle) = 1$ for any $y \in O^m$, which means $x \in O^m$, completing our proof since $B_{M^J}X^0W_MI_M = B_{M^J}W_MI_M$. □

By this lemma, we have

$$\dim \text{Ind}_{B_{M^J}}^{M^J}(\xi, \psi)_{|M} \leq \text{Card}(W_M).$$

For any character $\xi$ on $T_M$, let $T_{w,\psi}^\xi$ be the intertwining operator from $\text{Ind}_{B_{M^J}}^{M^J}(\xi, \psi)$ to $\text{Ind}_{B_{M^J}}^{M^J}(w\xi, \psi)$ defined as

$$T_{w,\psi}^\xi(f)(g) = \int_{N^J \cap wN^Jw^{-1} \setminus N^J} f(w^{-1}ng) \, dn.$$  

Similar to the previous subsection, the integral is convergent when $\text{Re}(\xi)$ is sufficiently large, and continues holomorphically to generic $\xi$. In fact the only difference is that we are integrating on part of $X$, on which smooth elements in $\text{Ind}_{B_{M^J}}^{M^J}(\xi, \psi)$ is compactly supported. For generic $\xi$, the $M^J$-intertwining operator from $\text{Ind}_{B_{M^J}}^{M^J}(\xi, \psi)$ to $\text{Ind}_{B_{M^J}}^{M^J}(w\xi, \psi)$ is unique, so we have

$$T_{w_1,\psi}^\xi \circ T_{w_2,\psi}^\xi = c \cdot T_{w_1w_2,\psi}^\xi$$

for some constant $c$. By a similar method as in theorem 3.1 in [2] we have the following result.

**Lemma 7.5.** For $\alpha$ being a simple root in $M$, if we let

$$\tilde{c}_\alpha(\xi) = \frac{\zeta(\langle \xi, \alpha \rangle)}{\zeta(\langle \xi, \alpha \rangle + 1)},$$

and let

$$\tilde{c}_w(\xi) = \prod_{\alpha > 0, w\alpha < 0} \tilde{c}_\alpha(\xi),$$

then,

$$T_{w,\psi}^\xi(F_0^0(\xi,\psi))(e) = \tilde{c}_w(\xi)$$
So if we define
\[ \overline{T}_{w}^{\xi, \psi} = (\tilde{c}_{w}(\xi))^{-1}T_{w}^{\xi, \psi}, \]
then
\[ \overline{T}_{w_{1}}^{\xi, \psi} \circ \overline{T}_{w_{2}}^{\xi, \psi} = \overline{T}_{w_{1}w_{2}}^{\xi, \psi} \]
Let
\[ \Psi_{1}^{\xi, \psi} = F_{\xi, \psi}(\text{Ch}_{M}), \]
then we have
\[ \Psi_{1}^{\xi, \psi}(g) = \begin{cases} \xi_{\psi}x_{B_{M}J}(b) & \text{if } g = bn_{-}x, b \in B_{M}^{0}, n_{-} \in N_{M}^{-1}, x \in X^{0}. \\ 0 & \text{if } g \notin B_{M}^{0}N_{M}^{-1}X^{0}. \end{cases} \]

**Lemma 7.6.** The set \( \{ \overline{T}_{w_{-1}}^{w, \xi, \psi} \}_{w \in W_{M}} \) forms a basis of \( \text{Ind}_{B_{M}J}^{M}(\xi, \psi)T_{M}^{M} \).

**Proof.** Since \( \dim \text{Ind}_{B_{M}J}^{M}(\xi, \psi)T_{M}^{M} \leq \text{Card}(W_{M}) \), it suffices to prove that elements in \( \{ \overline{T}_{w_{-1}}^{w, \xi, \psi} \}_{w \in W_{M}} \) are linear independent. Consider
\[ \overline{T}_{w}(\Psi_{1}^{\xi, \psi})(w_{0}^{M}) = (\tilde{c}_{w}(\xi))^{-1}\int_{N^{J} \cap wN^{J} \setminus N^{J}} dn^{J}n^{J}w_{0}^{M} \]
By the definition of \( \Psi_{1}^{\xi, \psi} \), the integral is non-zero implies
\[ w^{-1}N^{J}w_{0}^{M} \cap B_{M}JN_{M}^{-1}X^{0} \neq \emptyset, \]
which implies
\[ w^{-1}N^{J} \cap B_{M}JN_{M}^{-1}X^{0}(w_{0}^{M})^{-1} \neq \emptyset. \]
Note that \( w^{-1}N^{J} \subseteq B_{M}w^{-1}N_{M} \rtimes J \), and \( B_{M}JN_{M}^{-1}X^{0}(w_{0}^{M})^{-1} \subseteq B_{M}(w_{0}^{M})^{-1}N_{M} \rtimes J \), so by the Bruhat-decomposition on \( M \) we have
\[ w = w_{0}^{M} \]
if \( \overline{T}_{w}(\Psi_{1}^{\xi, \psi})(w_{0}^{M}) \neq 0. \) On the other hand, when \( w = w_{0}^{M} \),
\[ \overline{T}_{w_{0}^{M}}(\Psi_{1}^{\xi, \psi})(w_{0}^{M}) = (\tilde{c}_{w_{0}^{M}}(\xi))^{-1}\int_{N^{-}X} dn dx \Psi_{1}^{\xi, \psi}(nx) \]
which equals
\[ (\tilde{c}_{w_{0}^{M}}(\xi))^{-1} \text{Vol}(I_{M}) \]
by the definition of \( \Psi_{1}^{\xi, \psi} \). Now let \( \overline{T}_{w_{0}^{M}}w \) acts on all elements in \( \{ \overline{T}_{w_{-1}}^{w, \xi, \psi} \}_{w \in W_{M}} \) and evaluate them at \( w_{0}^{M} \). Only \( \overline{T}_{w_{0}^{M}}w_{0}^{M} \circ \overline{T}_{w_{-1}}^{w, \xi, \psi} \) is non-zero. So they are linear independent, and hence form a basis of \( \text{Ind}_{B_{M}J}^{M}(\xi, \psi)T_{M}^{M} \). \( \square \)

So then we have
Lemma 7.7. We can write $F^0_{\chi,\psi}$ as a linear combination of $\{T_{w^{-1}}\xi,\psi\}_{w \in W_M}$ as

$$ F^0_{\chi,\psi} = \text{Vol}(I_M)^{-1} \sum_{w \in W_M} \tilde{c}_{w_0^{-1}}(w\chi)T_{w^{-1}}\xi,\psi. \quad (24) $$

Proof. Since $\{T_{w^{-1}}\xi,\psi\}_{w \in W_M}$ is the basis of $\text{Ind}_{B_{Mj}}^{Mj}(\xi,\psi)^{T_M}$, we assume

$$ F^0_{\chi,\psi} = \sum_{w \in W_M} b_w T_{w^{-1}}\xi,\psi. $$

for some $b_w \in \mathbb{C}$. Let $T_{w_0^{-1}}$ acts on both sides of the equation and take the value at $w_0^M$, we obtain that

$$ 1 = b_w(\tilde{c}_{w_0^{-1}}(w\xi))^{-1}\text{Vol}(I_M), $$

completing our proof. \qed

Next we consider the action $R(\text{Ch}_{T_M}a\text{Ch}_{T_M})$ on $F^0_{\chi,\psi}$ for $a \in T_M^-$. We have

Proposition 7.8.

$$ R(\text{Ch}_{T_M}a\text{Ch}_{T_M})F^0_{\chi,\psi} = \sum_{w \in W_M} \tilde{c}_{w_0^{-1}}(w\xi) \cdot (w\xi)\delta_{B_{Mj}}^j(a) \cdot T_{w^{-1}}\xi,\psi $$

Proof. Consider $R(\text{Ch}_{T_M}a\text{Ch}_{T_M})\Psi^\xi,\psi_1$. Note that it belongs to $\text{Ind}_{B_{Mj}}^{Mj}(\xi,\psi)^{T_M}$, so by lemma (7.4), we only need to consider its value on $W_M$. Note that when $a \in T_M^-$, we have the decomposition

$$ I_M a I_M = N_{M_j}^{-1} X^0 \cdot a B_{Mj}^0. $$

Since $\Psi^\xi,\psi_1$ is $I_M$-invariant, and $\text{Vol}(X^0) = \text{Vol}(B_{Mj}^0) = 1$, so

$$ R(I_M a I_M)\Psi^\xi,\psi_1(w) = \int_{N_{M_j}^{-1} X^0} dn \Psi^\xi,\psi_1(wnxa) $$

Suppose it is non-zero, then by considering the support of $\Psi^\xi,\psi_1$, we have

$$ wN_{1,M}^- a X \cap B_{Mj}^0 N_{1,0}^- X^0 \neq \emptyset $$

so

$$ wN_{1,M}^- a X w_0^M \cap B_{Mj}^0 w_0^M N_{1,0}^- Y^0 \neq \emptyset $$

Note that

$$ wN_{1,M}^- a X w_0^M \subseteq B_{Mj} w_0^M N_{1,0}^- J $$

and

$$ B_{Mj}^0 w_0^M N_{1,0}^- Y^0 \subseteq B_{Mj} w_0^M N_{1,0}^- J. $$

So by Bruhat decomposition of $M$ we have

$$ w = e. $$
This implies that \( R(I_M a I_M) \Psi_1^{\xi,\psi} \) is propotional to \( \Psi_1^{\xi,\psi} \). Consider \( R(I_M a I_M) \Psi_1^{\xi,\psi}(e) \), which is equal to
\[
\int_{N^*_M X^0} dn \ dx \Psi_1^{\xi,\psi}(nxa).
\]
Note that \( \Psi_1^{\xi,\psi} \in \text{Ind}_{B_{M^1}}^{M^1} (\xi, \psi) \), the integral is equal to
\[
\xi \delta_{B_{M^1}}^{1/2} (a) \int_{N^*_M X^0} dn \ dx \Psi_1^{\xi,\psi}(a^{-1}nxa).
\]
Considering the support of \( \Psi_1^{\xi,\psi} \),
\[
\xi \delta_{B_{M^1}}^{1/2} (a) \int_{N^*_M X^0} dn \ \Psi_1^{\xi,\psi}(a^{-1}nxa) = \xi \delta_{B_{M^1}}^{1/2} (a) \text{Vol}(N^*_M X^0 \cap aN^*_M X^0 a^{-1})
\]
When \( a \in T_M, aN^*_M X^0 a^{-1} \in N^*_M X^0 \), so
\[
\text{Vol}(N^*_M X^0 \cap aN^*_M X^0 a^{-1}) = \text{Vol}(aN^*_M X^0 a^{-1}) = \delta_{B_{M^1}}^{-1} (a) \text{Vol}(I_M).
\]
So we have
\[
R(I_M a I_M) \Psi_1^{\xi,\psi} = \xi \delta_{B_{M^1}}^{-1} (a) \text{Vol}(I_M) \Psi_1^{\xi,\psi}.
\]
Applying this to both sides of equation (24), our proposition is proved. \( \square \)

8. \( \gamma \)-factor

In this section we assume \((\chi, \xi)\) is generic. We are going to show that

**Theorem 8.1.** Let \( \chi = (\chi_1, ..., \chi_n) \) and \( \xi = (\xi_1, ..., \xi_m) \). Let \( \Gamma(\chi, \xi) \) be a function on \((\chi, \xi)\) given by
\[
\Gamma(\chi, \xi) = \prod_{1 \leq a < b \leq n} \zeta^{-1}(\chi_a - \chi_b + 1) \zeta^{-1}(\chi_a + \chi_b + 1) \cdot \prod_{i=1}^{n} \zeta^{-1}(\chi_i + 1)
\]
\[
\cdot \prod_{1 \leq a < b \leq n} \zeta^{-1}(\xi_a - \xi_b + 1) \zeta^{-1}(\xi_a + \xi_b + 1) \cdot \prod_{j=1}^{m} (1 + \xi_j(p) |p|^{1/2})
\]
\[
\cdot \prod_{j=1}^{m} \prod_{i=1}^{(n-m)+j-1} \zeta(\chi_i - \xi_j + \frac{1}{2}) \cdot \prod_{i=1}^{n} \prod_{j=1}^{m} \zeta(\chi_i + \xi_j + \frac{1}{2}) \zeta(-\chi_i + \xi_j + \frac{1}{2})
\]

Then for any fixed \( g \in G \),
\[
\frac{l_{\chi, \xi, \psi}(R(g) F_0^{\chi, \psi})}{\Gamma(\chi, \xi)}
\]
is \( W_G \times W_M \)-invariant.
If we can prove that for \((w, w') = (w_\alpha, 1)\) and \((w, w') = (1, w_\beta)\) where \(\alpha\) is a simple root in \(G\) and \(\beta\) is a simple root in \(M\),

\[
\frac{\Gamma(w_\chi, w'_\xi, \xi)}{\Gamma(w_\chi, w'_\xi, \xi)} = \frac{l_{w_\chi, w'_\xi, \xi}(R(g)F_{w_\chi}^0, F_{w'_\xi}^0)}{l_{\xi, \xi}(R(g)F_{\chi}^0, F_{\xi, \xi}^0)}
\]

then theorem (8.1) is implied. Since we can calculate the left hand side above directly, we only need to consider the ratio

\[
\frac{l_{w_\chi, w'_\xi, \xi}(R(g)F_{w_\chi}^0, F_{w'_\xi}^0)}{l_{\xi, \xi}(R(g)F_{\chi}^0, F_{\xi, \xi}^0)}.
\]

(25)

We obtain its value by the uniqueness of the pair \(l_{\chi, \xi, w, w'}\). To be precise, for \((w, w') \in W_G \times W_M\), let

\[
\tilde{l}_{\chi, \xi, w, w'}(F_{\chi, \xi}, F_{\xi, \xi}) = l_{w_\chi, w'_\xi, \xi}(T_{w'}F_{\chi, \xi}, F_{w'_\xi}).
\]

Then \(\tilde{l}_{\chi, \xi, w, w'}\) is also a pairing on \(\text{Ind}^G \otimes \text{Ind}^M\) satisfying Condition A. By the uniqueness of such pairing, there exists a constant, which we denote by \(\gamma(\chi, \xi, w, w')\), such that

\[
\tilde{l}_{\chi, \xi, w, w'} = \gamma(\chi, \xi, w, w')l_{\chi, \xi, w, w'}.
\]

Then, since \(T_{w'}F_{\chi} = c(\chi)F_{w_\chi}^0\) and \(T_{w'}F_{\xi, \xi} = c(\xi)F_{w'_\xi}^0\), we have

\[
\frac{l_{w_\chi, w'_\xi, \xi}(R(g)F_{w_\chi}^0, F_{w'_\xi}^0)}{l_{\xi, \xi}(R(g)F_{\chi}^0, F_{\xi, \xi}^0)} = c(\chi)\cdot \frac{\tilde{l}_{\chi, \xi, w, w'}(R(g)F_{\chi}^0, F_{\xi, \xi}^0)}{l_{\chi, \xi, \xi}(R(g)F_{\chi}^0, F_{\xi, \xi}^0)} = c(\chi)\gamma(\chi, \xi, w, w')
\]

So in the rest of this section we calculate \(\gamma(\chi, \xi, w_\alpha, 1)\) and \(\gamma(\chi, \xi, 1, w_\beta)\). The result will be stated in theorem (8.3) and (8.7), which implies the theorem (8.1).

8.1. The calculation of \(\gamma(\chi, \xi, w_\alpha, 1)\).

Let \(I_{\text{M}^l} = I_{\text{M}^l} \times (X^1Y^0Z^0)\), and let

\[
F_{\xi, \psi}^1 = F_{\xi, \psi}(\text{Ch}_{\text{M}^l}).
\]

For \(w \in W_G\), let \(\Phi_{w_\chi} = F_{\chi}(\text{Ch}_{G,w_1G})\). Then by theorem 3.4 in [2]

\[
T_{w_\alpha}(\Phi_{w_\chi} + \Phi_{w_\chi}) = c(\chi)(\Phi_{w_\alpha, w_1} + \Phi_{w_\alpha, w_1, w_1}).
\]

So we have

\[
\gamma(\chi, \xi, w_\alpha, 1) = c(\chi) \cdot \frac{l_{w_\alpha, w_\xi, \xi, \psi}(R(\lambda_{w_\alpha}) \circ (\Phi_{w_\alpha, w_1} \odot \Phi_{w_\alpha, w_1}), F_{\xi, \psi})}{l_{\xi, \xi}(R(\lambda_{w_\alpha}), F_{\xi, \psi})}.
\]

(26)
Let \( i' = i - (n - m) \). The result of the calculation is stated as the proposition below:

**Proposition 8.2.** For generic \((\chi, \xi)\), the value of \( l_{x, \xi, \psi}(R(\lambda w_0^G) \circ (\Phi_{x, 1} + \Phi_{x, \omega}), F_{x, \psi}^1) \) equals

\[
\frac{\text{Vol}(I_G) \cdot \text{Vol}(I_M)}{|p|^{-1} \cdot \frac{1 - |p|^2}{(1 - (\chi_i \chi_{i+1})(p)) V \cdot V}}
\]

for \( \alpha = e_i - e_{i+1}, 1 \leq i \leq n - m - 1, \) and equals

\[
\frac{\text{Vol}(I_G) \cdot \text{Vol}(I_M)}{|p|^{-1} \cdot \frac{1 - \chi_n(p)}{|p|}}
\]

for \( \alpha = 2e_n \).

Substituting this in (26), we have

**Theorem 8.3.** For generic \((\chi, \xi)\), the \( \gamma \)-factor \( \gamma(\chi, \xi, w_\alpha, 1) \) is equal to

\[
c_{\alpha}(\chi) \cdot \frac{\zeta(\chi_i - \chi_{i+1} + 1)}{\zeta(\chi_{i+1} - \chi_i + 1)}
\]

for \( \alpha = e_i - e_{i+1}, 1 \leq i \leq n - m - 1, \) and

\[
c_{\alpha}(\chi) \cdot \frac{\zeta(\chi_i - \chi_{i+1} + 1) \zeta(-\chi_{i+1} + \xi + 1)}{\zeta(\chi_i - \chi_{i+1} + 1) \zeta(-\chi_{i+1} + \xi + 1)}
\]

for \( \alpha = e_i - e_{i+1}, n - m \leq i \leq n - 1, \) and

\[
c_{\alpha}(\chi) \cdot \frac{\zeta(\chi_n + 1)}{\zeta(-\chi_n + 1)}
\]

for \( \alpha = 2e_n \).

Recall that for \((\chi, \xi) \in Z_c\), we have

\[
l_{x, \xi, \psi}(R(g)F_{x}(\varphi_1), F_{x, \psi}(\varphi_2)) = l_{x, \xi, \psi}(\varphi_1, \varphi_2)(g).
\]

First we calculate \( l_{x, \xi, \psi}(\text{Ch}_{I_G}, \text{Ch}_{I_M}) (\lambda w_0^G) \), in fact,

**Lemma 8.4.** For \((\chi, \xi) \in Z_c\), we have

\[
l_{x, \xi, \psi}(\text{Ch}_{I_G}, \text{Ch}_{I_M}) (\lambda w_0^G) = \text{Vol}(I_G) \text{Vol}(I_M).
\]

**Proof.** By definition,

\[
l_{x, \xi, \psi}(\text{Ch}_{I_G}, \text{Ch}_{I_M}) (\lambda w_0^G) = \int_{I_G \times I_M} K_{x, \xi, \psi}(xw_0 \lambda^{-1} x') dx dx'
\]
From now on we write $\mathcal{K}_{\chi, \xi, \psi}$ by $\mathcal{K}$ for simplicity. Note that
\[ \lambda^{-1} = d_m(-1, -1, \ldots, -1)\lambda d(-1, -1, \ldots, -1) \in T_M^0 \cdot T_M^0, \]
so we have
\[ \int_{I_G \times I_{M^d}} dx dx' \mathcal{K}(xw_0 \lambda^{-1}x') = \int_{I_G \times I_{M^d}} dx dx' \mathcal{K}(xw_0 \lambda x'). \]
The proposition is implied if $\mathcal{K}(xw_0 \lambda x') = 1$ for all $x \in I_G$ and $x' \in I_{M^d}$. To show this we note that $I_G = B_G^0 N_G^{-1}$ and $I_{M^d} = X^1 N_M^{-1} B_M^0 Y^0 Z^0$. So by the definition of $\mathcal{K}$, we only need to show
\[ \mathcal{K}(n'w_0 \lambda xn^-) = 1 \]
for $n' \in N_G^{-1}$, $x \in X^1$ and $n^- \in N_M^{-1}$. Note that when $x \in X^1$, we have $\lambda x \in T_M^0 \cdot T_M^0$, and note that $T_M^0$ normalizes both $N_G^{-1}$ and $N_M^{-1}$, so it reduces to show that
\[ \mathcal{K}(n'w_0 \lambda n^-) = 1 \]
for $n' \in N_G^{-1}$, and $n^- \in N_M^{-1}$. The proof for this is similar to that in lemma [8.10], so we just skip it here.

Now we consider the calculation of $I_{\chi, \xi, \psi}(\text{Ch}_{I_G w_0 I_G}, \text{Ch}_{I_{M^d}})(\lambda w_0)$. First, by a similar method as in lemma [8.10], we have
\[ I_{\chi, \xi, \psi}(\text{Ch}_{I_G w_0 I_G}, \text{Ch}_{I_{M^d}})(\lambda w_0) = |p|^{-1} \text{Vol}(I_G) \text{Vol}(I_M) \int_{N_G^0} dn_\alpha \mathcal{K}(w_\alpha n_\alpha w_0 \lambda). \]
Combining this with [27], we have
\[ I_{\chi, \xi, \psi}(\text{Ch}_{I_G} + \text{Ch}_{I_G w_0 I_G}, \text{Ch}_{I_{M^d}})(\lambda w_0) = \text{Vol}(I_G) \text{Vol}(I_M) (1 + |p|^{-1} \int_{N_G^0} dn_\alpha \mathcal{K}(w_\alpha n_\alpha w_0 \lambda)) \]
Note that $w_\alpha n_\alpha(t) \in T_G^0 T_\alpha(t^{-1}) N_\alpha(-t) N_{-\alpha}(t^{-1})$, so it is equal to
\[ \text{Vol}(I_G) \text{Vol}(I_M) \cdot \left(1 + |p|^{-1} \int_{|t| \leq 1} dt \mathcal{K}(T_\alpha(t^{-1}) w_0 n_\alpha(t^{-1}) \lambda) \right). \]
We consider the calculation of this case by case.

**Case a.** When $\alpha = e_i - e_{i+1}$ with $1 \leq i \leq n - m - 1$ we have $n_\alpha(t^{-1}) \lambda = \lambda n_\alpha(t^{-1})$ if $i < n - m - 1$, and $n_\alpha(t^{-1}) \lambda = \lambda n_\alpha(t^{-1}) u$ for some $u \in U$ with $\psi_U(u) = 1$ if $i = n - m - 1$. So
\[ \mathcal{K}(T_\alpha(t^{-1}) w_0 n_\alpha(t^{-1}) \lambda) = \mathcal{K}(T_\alpha(t^{-1}) w_0 \lambda n_\alpha(t^{-1})), \]
and so
\[ \int_{|t| \leq 1} dt \mathcal{K}(T_\alpha(t^{-1}) w_0 n_\alpha(t^{-1}) \lambda) = \int_{|t| \leq 1} dt (\chi_i \chi_{i+1}^{-1})(t) |t|^{-1} \psi(t^{-1}). \]
We let
\[ I(n) = \int_{|t| = |p|^{n}} dt (\chi_i \chi_{i+1}^{-1})(t)|t|^{-1} \psi(t^{-1}) \]
\[ = (\chi_i \chi_{i+1}^{-1}(p))^n |p|^{-n} \int_{|t| = |p|^{n}} dt \psi(t^{-1}). \]

Using lemma (8.11) below we have \( I(0) = 1 - |p| \), \( I(1) = (-1)(\chi_i \chi_{i+1}^{-1})(p)|p| \), and \( I(n) = 0 \) if \( n \geq 2 \). Combining these we have
\[ I(\alpha) = \chi_{\alpha}(1 + 1) \chi_{\alpha}^{-1}, \]
which is the first part of proposition (8.2).

**Case b.** When \( \alpha = e_i - e_{i+1} \) with \( n - m \leq i \leq n - 1 \), we let \( t_j = (1, 1, \ldots, 1 + t^{-1}, 1, \ldots, 1) \in F^m \). Then \( n\alpha(t^{-1}) = X(t_1) \) if \( i = n - m \), and \( n\alpha(t^{-1}) = X(t_i t_{i+1}^{-1})n\alpha(t^{-1}) \) if \( i \geq n - m + 1 \). Here \( i = i - (n - m) \). Note that
\[ w_0 X(t_j) = d'(t_j) w_0 \lambda d'(t_j), \]
where \( d' \) is the \( j \)-th unramified character on \( F^* \).

To calculate this we apply the lemma 8.6 in [7].

**Lemma 8.5.** Suppose \( \chi \) and \( \chi' \) are two unramified character on \( F^* \), then
\[ 1 + |p|^{-1} \int_O dt \chi(t) \chi'(1 + t) = (|p|^{-1} - 1) \frac{1 - |p|^2(\chi \chi')(p)}{(1 - |p| \chi(p))(1 - |p| \chi'(p))}. \]

Applying the lemma for \( \chi = \chi_i \chi_{i+1}^{-1} \cdot |^{-1} \) and \( \chi' = \chi_{i+1}^{-1} \chi_i |^{-1} \), we have, when \( \alpha = e_i - e_{i+1} \) with \( n - m \leq i \leq n - 1 \),
\[ I(\chi \chi'(1 + t)) = \chi(\chi_i \chi_{i+1}^{-1})(p) \]
\[ = \chi_{i+1}^{-1} \chi_i (|p|^{-1} - 1) \frac{1 - |p|^{2}}{(1 - (\chi_i \chi_{i+1}^{-1})(p)|p|^{2})}. \]

which is the second part of proposition (8.2).

**Case c.** When \( \alpha = 2e_n \), we have
\[ n\alpha(t^{-1}) = \chi(t)^{-1} \chi(t)^{-1} \]
\[ \int_{|t| \leq 1} dt \chi(t) = \int_{|t| \leq 1} dt \chi(t) t^{-1} \psi(t^{-1}). \]
Similar to Case a, if we let
\[
\tilde{I}(i) = \int_{|t|=|p|^i} dt \chi_n(t) |t|^{-1} \psi(t^{-1})
\]
\[
= \chi_n(p)^i |p|^{-i} \int_{|t|=|p|^i} dt \psi(t^{-1}),
\]
then by lemma (8.11),
\[
\tilde{I}(i) = \begin{cases} 
1 - |p| & \text{if } i = 0; \\
- \chi_n(p) |p| & \text{if } i = 1; \\
0 & \text{if } i \geq 2.
\end{cases}
\]
So
\[
I_{\chi,\xi,\psi}(\text{Ch}_{I_G} + \text{Ch}_{w_{I_G}} \text{Ch}_{I_M})(\lambda w_0) = \text{Vol}(I_G) \text{Vol}(I_M) |p|^{-1} \cdot (1 - \chi_n(p) |p|),
\]
which is part 3 of proposition (8.2).
So by the calculation in the Case a, b and c, we have proved proposition (8.2), which implies theorem (8.3).

8.2. Calculation of \(\gamma(\chi, \xi, 1, w_\beta)\).
Let \(\bar{\chi}_i = \chi_{n-m+i}\) for \(1 \leq i \leq m\). Let \(\Phi_{\chi,1} = F_\chi(\text{Ch}_{I_G})\) as before. Recall that \(T_M = I_M \ltimes J^0\). Let
\[
\Psi_{\xi,\psi,w} = F_{\xi,\psi}(\text{Ch}_{I_M w_{I_M}}).
\]
Similar to theorem 3.4 in [2], we have
\[
T_{w_\beta}(\Psi_{\xi,\psi,1} + \Psi_{\xi,\psi, w_\beta}) = \tilde{c}_\beta(\xi)(\Psi_{w_\beta \xi,\psi,1} + \Psi_{w_\beta \xi,\psi, w_\beta}). \tag{28}
\]
So we have
\[
\gamma(\chi, \xi, 1, w_\beta) = \tilde{c}_\beta(\xi) \cdot \frac{l_{\chi, w_\beta \xi, \psi}(R(\lambda w_0^G) \Phi_{\chi,1}, \Psi_{w_\beta \xi, \psi,1} + \Psi_{w_\beta \xi, \psi, w_\beta})}{l_{\chi, \xi, \psi}(R(\lambda w_0^G) \Phi_{\chi,1}, \Psi_{\xi, \psi,1} + \Psi_{\xi, \psi, w_\beta})} \tag{29}
\]
What we are going to show is

**Proposition 8.6.** For generic \((\chi, \xi)\), the value of \(l_{\chi, \xi, \psi}(R(\lambda w_0^G) \Phi_{\chi,1}, \Psi_{\xi, \psi,1} + \Psi_{\xi, \psi, w_\beta})\) equals
\[
|p|^{-1}(1 - |p|)^m \text{Vol}(I_G) \text{Vol}(I_M) \prod_{j \neq i, i+1} \zeta \left( \frac{1}{2} - \bar{\chi}_j + \xi_j \right) \cdot \frac{\zeta(\xi_i - \bar{\chi}_i + 1) \zeta(\xi_{i+1} - \bar{\chi}_{i+1} + 1) \zeta(\xi_i - \bar{\chi}_{i+1} + 1) \zeta(-\xi_{i+1} + \bar{\chi}_i + \frac{1}{2}) \zeta(\xi_i - \bar{\chi}_i + \frac{1}{2})}{\zeta(\chi_i - \bar{\chi}_{i+1} + 1) \zeta(\xi_i - \xi_{i+1} + 1)}
\]
when $\beta = e_i - e_{i+1}$, and equals

$$|p|^{-1}(1 - |p|)^m \text{Vol}(I_G) \text{Vol}(I_M) \prod_{j=1}^{m-1} \zeta(-\bar{x}_j + \xi_j + \frac{1}{2})$$

$$\cdot \frac{(1 - \bar{x}_m \cdot |i\rangle(p)(1 + \xi_m \cdot |\frac{i}{2}\rangle(p))}{(1 - \bar{x}_m^{-1} \xi_m \cdot |\frac{i}{2}\rangle(p))(1 - \bar{x}_m \xi_m \cdot |\frac{i}{2}\rangle(p))}$$

when $\beta = 2e_m$.

Substituting this in (29) we have

**Theorem 8.7.** For generic $(\chi, \xi)$, the $\gamma$-factor $\gamma(\chi, \xi, 1, w_\beta)$ equals

$$\tilde{c}_\beta(\xi) \frac{\zeta(\xi - \xi_{i+1} + 1)\zeta(-\bar{x}_i + \xi_{i+1} + \frac{1}{2})\zeta(\bar{x}_i - \xi_i + \frac{1}{2})}{\zeta(-\xi + \xi_{i+1} + 1)\zeta(\bar{x}_i - \xi_{i+1} + \frac{1}{2})\zeta(-\bar{x}_i + \xi_i + \frac{1}{2})}$$

when $\beta = e_i - e_{i+1}$, and equals

$$\tilde{c}_\beta(\xi) \frac{\zeta(-\xi_m - \bar{x}_m + \frac{1}{2})\zeta(-\xi_m + \bar{x}_m + \frac{1}{2})\zeta(2\xi_m + 1)\zeta(-\xi_m + \frac{1}{2})}{\zeta(\xi_m - \bar{x}_m + \frac{1}{2})\zeta(\xi_m + \bar{x}_m + \frac{1}{2})\zeta(-2\xi_m + 1)\zeta(\xi_m + \frac{1}{2})}$$

when $\beta = 2e_m$.

To prove (8.6), first we have

**Proposition 8.8.** For $(\chi, \xi) \in Z_C$,

$$I_{\chi, \xi, \psi}(\text{Ch}_{I_G}, \text{Ch}_{T_M})(\lambda w_0^G) = \text{Vol}(I_G) \text{Vol}(I_M) \prod_{j=1}^{m} \frac{1 - |p|}{1 - \bar{x}_j^{-1} \xi_j(p)|p|^{\frac{j}{2}}}$$

**Proof.** By definition, we have

$$I_{\chi, \xi, \psi}(\text{Ch}_{I_G}, \text{Ch}_{T_M})(\lambda w_0^G) = \int_{I_G} \int_{T_M} dg \, dm \mathcal{K}(gw_0^G \lambda^{-1} m).$$

Since $\lambda \in T_M$, and $\mathcal{K}$ is left $B_0^0$-invariant and right $B_M^0$-invariant, and $\text{Vol}(B_0^0) = \text{Vol}(B_M^0) = 1$, we have

$$\int_{I_G} \int_{T_M} dg \, dm \mathcal{K}(gw_0^G \lambda^{-1} m) = \int_{N^{-1}_G} dn \int_{N^{-1}_M \times_0} dn' dx' \mathcal{K}(nw_0^G n' x').$$

By a similar discussion as in lemma (8.10), which is given later, we have, for all $n \in N^{-1}_G$ and $n' \in N^{-1}_M$,

$$\int_{\times_0} dx' \mathcal{K}(nw_0^G n' x') = \int_{\times_0} dx' \mathcal{K}(w_0^G x').$$

So

$$I_{\chi, \xi, \psi}(\text{Ch}_{I_G}, \text{Ch}_{T_M})(\lambda w_0^G) = \text{Vol}(I_G) \text{Vol}(I_M) \int_{\times_0} dx' \mathcal{K}(w_0^G x').$$
Parametrize $X^0$ as $X^0 = \{X(x) = X(x_1, \ldots, x_m) | x_i \in O \text{ for all } i\}$, then, $w_N^G X(x_1, \ldots, x_m) = d_m(x_1, \ldots, x_m) w_N^G \lambda d_m(x_1, \ldots, x_m)$ if all $x_i$ are non-zero, and $K(w_N^G x') = 0$ otherwise. So

$$\int_{X^0} dx' K(w_N^G x') = \prod_{j=1}^m \int_{x_j \in O} dx_j (\hat{\chi}_j^{-1} \xi_j | \cdot |^{-\frac{1}{2}})(x_j) = \prod_{j=1}^m \frac{1 - |p|}{1 - \hat{\chi}_j^{-1} \xi_j(p)|p|^{\frac{1}{2}}},$$

where the last equality follows from the following lemma. \hfill \Box

**Lemma 8.9.** Let $\chi$ be an unramified character on $F^*$. Then

$$\int_{x \in O^*} \chi(x) \, dx = \frac{1 - |p|}{1 - \chi(p)|p|}.$$

We skip its proof.

Next we consider $I_{\chi, \xi, \psi}(\text{Ch}_{I_G}, \text{Ch}_{I_M^0 \overline{I}_M})(\lambda w_N^G)$. By definition, it is equal to

$$\int_{I_G} dx \int_{I_M^0 \overline{I}_M} dx' K(xw_N^G \lambda^{-1} x').$$

Note that $\lambda \in \overline{I}_m$, and $I_M w_\beta \overline{I}_M = N_{\overline{M}}^{-1} w_\beta N_{-\overline{M}}^0 J_0 B_{\overline{M}}$. So the above integral is equal to

$$\int_{N_G^{-1}} dn \int_{N_{\overline{M}}^{-1}} dn' \int_{N_{-\overline{M}}^{-1}} dn_{-\beta} \int_{J_0} du K(nw_N^G n' w_\beta n_{-\beta} u).$$

We claim that

**Lemma 8.10.** For any $n \in N_G^{-1}$ and $n' \in N_M^{-1}$,

$$\int_{N_{-\beta}^{-1}} \int_{J_0} du K(nw_N^G n' w_\beta n_{-\beta} u) = \int_{N_{-\beta}^{-1}} \int_{J_0} du K(w_N^G n_{-\beta} u) = \int_{N_{-\beta}^{-1}} \int_{J_0} du K(w_N^G \bar{n} w_\beta n_{-\beta} u).$$

**Proof.** First we have

$$\int_{N_{-\beta}^{-1}} \int_{J_0} du K(nw_N^G n' w_\beta n_{-\beta} u) = \int_{N_{-\beta}^{-1}} \int_{J_0} du K(\bar{n} w_N^G w_\beta n_{-\beta} u). \ (30)$$

for some $\bar{n} \in N_G^{-1}$. This is true because

1. $w_N^G N_{\overline{M}}^{-1} w_0^G = N_M^1$,
2. $N_{\overline{M}}^{-1} N_M \subset B_{\overline{M}}^0 N_{\overline{M}}^{-1}$,
3. $K$ is left $B_{\overline{M}}^0$-invariant.

Continue the calculation, the right hand side of equation (30) is equal to

$$\int_{N_{-\beta}^{-1}} \int_{J_0} du K(w_N^G \bar{n} w_\beta n_{-\beta} u) \ (31)$$
for some $\tilde{n} \in N^1_G$. We write $\tilde{n}$ as $\tilde{n} = n_1 n_2 J(x, y, z) n_3$ where $n_1 \in N^1_{1, \beta}$, $n_2 \in N^1_{M, \beta}$, $J(x, y, z) \in J^1$, and $n_3 \in U^1$. Here $N_{M, \beta}$ is the subgroup of $N_M$ generated by all positive roots in $M$ except $\beta$. Then (31) is equal to

$$\int_{N^0_{-\beta}} dn_{-\beta} \int_{J^0} du \mathcal{K}(w_0^G n_1 n_2 J(x, y, z) n_3 w_{-\beta} u)$$

Our lemma will be proved if we remove $n_1 n_2 J(x, y, z) n_3$ in the integral, which is by the following steps

1. First, $n_3$ can be removed because $U^1$ is normalized by $w_{-\beta}$, $N^0_{-\beta}$, and $J^0$, and $\mathcal{K}$ is right $U^1$-invariant.
2. Note that $J^1$ is normalized by $w_{\beta} N^0_{-\beta}$, we can remove $J^1(x, y, z)$ by a change of variable in the integral of $J^0$.
3. We can remove $n_2$ because $N^1_{M, \beta}$ is normalized by $w_{\beta} N^0_{-\beta}$, and $N^1_{M, \beta}$ normalizes $J^0$, and $\mathcal{K}$ is right $N^1_{M, \beta}$-invariant.
4. Finally, note that $N^1_{\beta} w_{\beta} = w_{\beta} N^1_{-\beta}$, we can remove $n_1$ by a change of variable in the integral of $N^0_{-\beta}$

So the lemma is proved.  

By the lemma we have

$$I_{\chi, \xi, \psi}(\text{Ch}_{\text{IC}}; \text{Ch}_{M w_{-\beta} I_M})(\lambda w_0^G) = |p|^{-1} \text{Vol}(I_G) \text{Vol}(I_M) \int_{|t| \leq 1} dt \int_{X^0} dx \mathcal{K}(w_0^G w_{-\beta} n_{-\beta} (t) x)$$

Combining this with proposition (8.8), we have

$$I_{\chi, \xi, \psi}(\text{Ch}_{\text{IC}}; \text{Ch}_{M} + \text{Ch}_{M w_{-\beta} I_M})(\lambda w_0^G) = \text{Vol}(I_G) \text{Vol}(I_M) \left( \prod_{j=1}^{m} \frac{1 - |p|}{1 - \chi_j^{-1} \xi_{j}(p) |p|^2} + |p|^{-1} \int_{|t| \leq 1} dt \int_{X^0} dx \mathcal{K}(w_0^G w_{-\beta} n_{-\beta} (t) x) \right).$$

(32)

Now we calculate

$$\int_{|t| \leq 1} dt \int_{X^0} dx \mathcal{K}(w_0^G w_{-\beta} n_{-\beta} (t) x).$$

Note that $w_{\beta} n_{-\beta} (t) = n_{-\beta} (-t^{-1}) N_{\beta} (t) T_{\beta} (t)$, so

$$\int_{|t| \leq 1} dt \int_{X^0} dx \mathcal{K}(w_0^G w_{-\beta} n_{-\beta} (t) x)$$

$$= \int_{|t| \leq 1} dt \int_{X^0} dx \mathcal{K}(w_0^G n_{-\beta} (-t^{-1}) n_{\beta} (t) T_{\beta} (t) x)$$

$$= \int_{|t| \leq 1} dt \int_{X^0} dx \mathcal{K}(w_0^G x n_{-\beta} (t) T_{\beta} (t) (\xi_{B_M}^{-2})(T_{\beta} (t))).$$
Below we discuss this integral case by case.

**Case a.** When \( \beta = \epsilon_i - \epsilon_{i+1} \).

We parametrize \( X^0 \) as \( X(x) = X(x_1, \ldots, x_m) \) with \( x_j \in \mathcal{O} \). Then we have

\[
X(x_1, \ldots, x_m)^{n_\beta(t)T_\beta(t)} = X(x_1, \ldots, x_{i-1}, t^{-1}x_i, -x_i + tx_{i+1}, x_{i+2}, \ldots, x_m)
\]

and

\[
(\xi^\beta_{Bm})^{1/2}(T_\beta(t)) = (\xi_i \xi_{i+1}^{-1} | \cdot |^{-1})(t).
\]

Let \( x(i, t) = (x_1, \ldots, x_{i-1}, t^{-1}x_i, -x_i + tx_{i+1}, x_{i+2}, \ldots, x_m) \), then

\[
\int_{|t| \leq 1} dt \int_{X^0} dx \mathcal{K}(w_0 X^{n_\beta(t)T_\beta(t)})(|\xi^\beta_{Bm})^{1/2}|(T_\beta(t))
\]

\[
= \int_{|t| \leq 1} dt \int_{|x_i| \leq 1, i=1,2,\ldots,m} dx_i \ (\xi_i \xi_{i+1}^{-1})(t) |t|^{-1} \mathcal{K}(w_0 X(x(i, t))).
\]

Note that if none of the components of \( x(i, t) \) is zero, then

\[
w_0 X(x(i, t)) = d_m(x(i, t)) w_0 \lambda d_m(x(i, t))
\]

and \( \mathcal{K}(w_0 X(x(i, t))) = 0 \) if otherwise. So the integral above is equal to

\[
\prod_{j \in \{1,2,\ldots,i-1,i+2,\ldots,m\}} \int_{x_j \in \mathcal{O}} (\tilde{\chi}_j^{1/2} | \xi_j| | \cdot |^{-1})(x_j) dx_j.
\]

\[
\int_{|t| \leq 1, |x_i| \leq 1, |x_{i+1}| \leq 1} dt \ dx_i \ dx_{i+1} (\xi_i \xi_{i+1}^{-1} | \cdot |^{-1})(t)
\]

\[
(\tilde{\chi}_i^{1/2} | \xi_i| | \cdot |^{-1})(t^{-1}x_i) (\tilde{\chi}_{i+1}^{1/2} \xi_{i+1}^{-1} | \cdot |^{-1})(-x_i + tx_{i+1})
\]

By lemma \((8.9)\), it is equal to

\[
\prod_{j \in \{1,2,\ldots,i-1,i+2,\ldots,m\}} \frac{1 - |p|}{1 - \tilde{\chi}_j^{1/2} | \xi_j| |p|^{1/2}}
\]

\[
\int_{|t| \leq 1, |x_i| \leq 1, |x_{i+1}| \leq 1} dt \ dx_i \ dx_{i+1} (\xi_i \xi_{i+1}^{-1} | \cdot |^{-1})(t)
\]

\[
(\tilde{\chi}_i^{1/2} | \xi_i| | \cdot |^{-1})(t^{-1}x_i) (\tilde{\chi}_{i+1}^{1/2} \xi_{i+1}^{-1} | \cdot |^{-1})(-x_i + tx_{i+1})
\]

(33)

Let \( I(a, b) \) be part of the integral above for \( |t| = |p|^a \) and \( |x_i| = |p|^b \) with \( a, b \geq 0 \), then the integral is equal to \( \sum_{a, b \geq 0} I(a, b) \). When \( a > b \), i.e., \( |x_i| > |t| \), we have \( | -x_i + tx_{i+1}| = |x_i| \). So

\[
I(a, b) = (1 - |p|)^2 |p|^{1/2} \chi_i^{a-b} \xi_{i+1}^{-b} \xi_{i+1}^{b} x_i b x_{i+1}^{a+b}.
\]
When \( a \leq b \), i.e., \( |x_i| \leq |t| \) we can change the variable \( x_{i+1} \rightarrow x_{i+1} + \frac{\varpi}{t} \), we have

\[
I(a, b) = \sum_{c \geq 0} (1 - |p|^c)|p|^{\frac{b+c}{2}} |x|^a \tilde{x}_{i+1}^{a-c} \xi^b \xi_{i+1},
\]

where the summand for \( c \) corresponds to \( |x_{i+1}| = |p|^c \) after the change of variable.

So applying this to equation (32) and by some calculation,

\[
I_{\chi, \xi, \psi}(Ch_{I_1}, Ch_{I_2}, Ch_{I_3})(\lambda w_0^G) = \text{Vol}(I_G)\text{Vol}(I_M)
\]

\[
\cdot \left( \prod_{j=1}^{m} \frac{1 - |p|}{1 - \tilde{x}_j^{-1} \xi_j(p)}|p|^{\frac{1}{2}} + |p|^{-1} \prod_{j \neq i, i+1} \frac{1 - |p|}{1 - \tilde{x}_j^{-1} \xi_j(p)}|p|^{\frac{1}{2}} \sum_{a,b \geq 0} I(a, b) \right)
\]

\[
= |p|^{-1}(1 - |p|)^m \text{Vol}(I_G)\text{Vol}(I_M) \prod_{j \neq i, i+1} \zeta(\frac{1}{2} - \tilde{x}_j + \xi_j)
\]

\[
\cdot \zeta(\xi_i - \tilde{x}_i + \frac{1}{2}) \zeta(\xi_{i+1} - \tilde{x}_{i+1} + \frac{1}{2}) \zeta(\xi_i - \tilde{x}_{i+1} + \frac{1}{2}) \zeta(-\xi_{i+1} + \tilde{x}_i + \frac{1}{2})
\]

\[
\zeta(\tilde{x}_i - \tilde{x}_{i+1} + 1) \zeta(\xi_i - \xi_{i+1} + 1)
\]

which is the first part of proposition (8.6).

**Case b.** When \( \beta = 2e_m \), we have

\[
(\xi_d^{-\frac{2}{3}})(T_\beta(t)) = \xi_m(t)|t|^{-\frac{2}{3}},
\]

and

\[
X(x_1, ..., x_m)^{n_\beta(t)T_\beta(t)} = X(x_1, ..., x_{m-1}, t^{-1}x_m)Y_1(-x_m)Z(t^{-1}x_m^2).
\]

So

\[
\int_{|t| \leq 1} \int_{x^0} dx \mathcal{K}(w_0^G x^{n_\beta(t)T_\beta(t)})(\xi_d^{-\frac{2}{3}})(T_\beta(t))
\]

\[
= \int_{|t| \leq 1} \int_{x^0} dx \psi(t^{-1}x_m^2) \xi_m(t)|t|^{-\frac{2}{3}} \mathcal{K}(w_0^G X(x_1, ..., x_{m-1}, t^{-1}x_m)) \quad (34)
\]

\[
= \prod_{j=1}^{m-1} \frac{1 - |p|}{1 - \tilde{x}_j^{-1} \xi_j(p)}|p|^{\frac{1}{2}} \cdot \int_{t,x_m \in O} dt dx_m (\tilde{x}_m^{-1} \xi_m) |t|^{-\frac{2}{3}}(t^{-1}x_m)(\xi_m)|t|^{-\frac{2}{3}}(t)|t^{-1}x_m^2| \quad (35)
\]

Let \( I(a, b) \) be part of the integral above when \( |t| = |p|^a \) and \( |x_m| = |p|^b \). To calculate \( I(a, b) \), we apply the following lemma which can be proved by direct calculation

**Lemma 8.11.** Suppose \( |x| = |p|^i \), then

\[
\int_{|t| = |p|^j} \psi(t^{-1}x) dt = \begin{cases} |p|^j(1 - |p|) & \text{if } j \leq i \\ -|p|^{i+2} & \text{if } j = i + 1 \\ 0 & \text{if } j \geq i + 2 \end{cases}
\]
So \( I(a, b) = (1 - |p|^2) \tilde{\chi}_m \eta^b_\nu |p|^{\frac{1}{2} b} \) when \( 0 \leq a \leq 2b \), and \( I(a, b) = -(1 - |p|) \tilde{\chi}_m \eta^b_\nu |p|^{\frac{1}{2} b + 1} \) when \( a = 2b + 1 \), and \( I(a, b) = 0 \) when \( a \geq 2b + 2 \). Applying these to equation (32) and by some calculation, we have

\[
I(\chi, \xi, \psi)(Ch_{I_G}, Ch_{I_M I_{W_0} I_M})/(\lambda w_0 G) = |p|^{-1} (1 - |p|)^m \text{Vol}(I_G) \text{Vol}(I_M) \prod_{j=1}^{m-1} \zeta(-\tilde{\chi}_m + \xi_j + \frac{1}{2}) \cdot \frac{(1 - \tilde{\chi}_m \cdot |\tilde{\gamma}(p)| (1 + \xi_m |\tilde{\tau}(p)|) (1 - \tilde{\chi}_m \xi_m |\tilde{\tau}(p)|)}{(1 - \tilde{\chi}_m \cdot |\tilde{\tau}(p)|) (1 - \tilde{\chi}_m \xi_m |\tilde{\tau}(p)|)}
\]

which is the second part of the proposition (8.6). Combining Case a and b, we have proposition 8.6, which implies theorem 8.7.

9. Formula for generic \((\chi, \xi)\)

In this section we discuss the value of \( I^0_{\chi, \xi, \psi}(p^d \lambda p^f) \) for generic \((\chi, \xi)\) where \( d \in \Lambda^+_m \) and \( f \in \Lambda^+_n \). First we claim that

**Lemma 9.1.** For any \( g \in I_G \) and \( m \in I_M \), we have

\[ p^d \lambda w_0^G g^{-f} p^d m \lambda w_0^G g^{-f} \]

in the double coset \( U K M^+ \backslash G / K G \).

**Proof.** When \( d \in \Lambda^+_m \) and \( f \in \Lambda^+_n \), we have \( p^d B_0^G p^{-d} \in B^G_M \) and \( p^f B_0^G p^{-f} \in B^G_0 \), so it suffices to prove the lemma when \( g \in N^1_G \) and \( m \in N^1_M \), which is true by the method in lemma (8.10). \( \square \)

Now we consider

\[ \Gamma^{-1}(\chi, \xi) \cdot l_{\chi, \xi, \psi}(R(\lambda w_0^G)R(I_G p^{-f} I_G)F^0_{\chi}, R(T_M p^{-d} T_M)F^0_{\xi, \psi}). \]

By lemma (9.1) it is equal to

\[ \Gamma^{-1}(\chi, \xi) \text{Vol}(I_G) \text{Vol}(I_M) \int_{X^0} \text{dx} I^0_{\chi, \xi, \psi}(p^d x p^f). \]

On the other hand, by lemma (7.3) and (7.8), it is equal to

\[ \Gamma^{-1}(\chi, \xi) \sum_{w \in W_G, w' \in W_M} c_{w_G} (w \chi)(w \chi \delta_{B_0^G}) \tilde{c}_{w_M} (w' \xi)(w' \xi \delta_{B_0^M}) \cdot l_{\chi, \xi, \psi}(R(\lambda w_0^G)T_{w^{-1}} \Phi_1^{w \chi}, T_{w'^{-1}} \Psi_1^{w' \xi, \psi}). \]
So we have
\[
\Gamma^{-1}(\chi, \xi) \int_{X_0}^0 I_{X, \xi, \psi}(p^d x p^f) = \text{Vol}(I_G)^{-1} \text{Vol}(I_M)^{-1} \\
\cdot \sum_{w \in W_G, w' \in W_M} \left( c_{w_w^G}(w\chi)(w\chi\delta_{B_G}^{-\frac{1}{2}})(p^{-f})\tilde{c}_{w'_w^M}(w'\xi)(w'\xi\delta_{B_M}^{-\frac{1}{2}})(p^{-d}) \right)
\cdot I_{X, \xi, \psi}(R(\lambda w_0)T_{w-1}\Phi^w_1, T_{w'-1}\Psi_{w'}^1) \cdot \Gamma^{-1}(\chi, \xi).
\]

To calculate the right hand side of the equation above, we use the following lemma without proof.

**Lemma 9.2.** For generic \((\chi, \xi)\), and \((d, f) \in \Lambda_m^+ \times \Lambda_n^+\), let
\[
B(\chi, \xi, d, f) = \sum_{w \in W_G, w' \in W_M} A(\chi, \xi, w, w')(w\chi)(p^f)(w'\xi)(p^d).
\]

Suppose \(B(\chi, \xi, d, f)\) is \(W_G \times W_M\)-invariant for all \((d, f) \in \Lambda_m^+ \times \Lambda_n^+\), then we have
\[
A(\chi, \xi, w, w') = A(w\chi, w'\xi, e, e).
\]

By this lemma, the summation on the right hand side of (37) is determined by its summand at \((w, w') = (e, e)\), which, by proposition 8.8, is equal to
\[
(1 - |p|)^m b(\chi, \xi) d(\chi) d'(\xi) (\chi\delta_{B_G}^{-\frac{1}{2}})(p^{-f}) (\xi\delta_{B_M}^{-\frac{1}{2}})(p^{-d}) \text{Vol}(I_G) \text{Vol}(I_M),
\]
where
\[
d(\chi) = \prod_{1 \leq a < b \leq n} \zeta(\chi_a - \chi_b) \zeta(\chi_a + \chi_b) \prod_{i=1}^n \zeta(\chi_i),
\]
\[
d'(\xi) = \prod_{1 \leq a < b \leq m} \zeta(\xi_a - \xi_b) \zeta(\xi_a + \xi_b) \prod_{j=1}^m \zeta(2\xi_j),
\]
and
\[
b(\chi, \xi) = \prod_{i < j + n - m} \zeta^{-1}(\chi_i - \xi_j + \frac{1}{2}) \cdot \prod_{i > j + n - m} \zeta^{-1}(-\chi_i + \xi_j + \frac{1}{2}) \cdot \prod_{1 \leq j \leq m} \zeta^{-1}(\xi_j + \frac{1}{2}) \cdot \prod_{1 \leq j \leq m} \zeta^{-1}(\chi_i + \xi_j + \frac{1}{2}).
\]

Applying the lemma 9.2 we have
\[
\int_{X_0} dx I_{X, \xi, \psi}(p^d x p^f) = (1 - |p|)^m \Gamma(\chi, \xi) \cdot \sum_{w \in W_G, w' \in W_M} b(w\chi, w'\xi) d(w\chi) d'(w'\xi) (w\chi\delta_{B_G}^{-\frac{1}{2}})(p^{-f}) (w'\xi\delta_{B_M}^{-\frac{1}{2}})(p^{-d}).
\]
Though this is not a direct formula for $I^0_{\chi, \xi, \psi}(p^d \lambda p^f)$, we have the following theorem.

**Theorem 9.3.** Let $l(d, f) = I^0_{\chi, \xi, \psi}(p^d \lambda p^f)$, and let

$$L(d, f) = \int_{\mathcal{X}_0} dx I^0_{\chi, \xi, \psi}(p^d x p^f),$$

then $l(0, f) = L(0, f)$. Moreover, there exists $a(d') \geq 0$ such that

$$l(d, f) = \sum_{d'} a(d') L(d', f + d - d')$$

where the sum is over the set $\{d' \mid d' \leq d, d' \in \Lambda_m^+, f + d - d' \in \Lambda_n^+\}$, and we have $a(d) > 0$.

**Proof.** We have $l(0, f) = L(0, f)$ by the $K_{M^1}$-invariance on the left. Because of this, if we can prove

$$L(d, f) = \sum_{d'} b(d') l(d', f + d - d')$$

where $d'$ runs over $\{d' \mid d' \leq d, d' \in \Lambda_m^+, f + d - d' \in \Lambda_n^+\}$ with $b(d') \geq 0$ and $b(d) > 0$, then the theorem is implied. Expand the right hand side of equation (39), and then by lemma (6.3), $L(d, f)$ is a linear combination of $I^0_{\chi, \xi, \psi}(p^d X(c) p^f)$ with non-negative coefficients where $0 \leq c \leq d$. Note that $I^0_{\chi, \xi, \psi}(p^d X(c) p^f) = I^0_{\chi, \xi, \psi}(p^{d-c} \lambda p^{f+c})$, and that $(d - c; f + c)$ satisfies the assumption in lemma (6.6). So by lemma (6.6), $I^0_{\chi, \xi, \psi}(p^{d-c} \lambda p^{f+c}) = I^0_{\chi, \xi, \psi}(p^{d'} \lambda p^{f+d-d'})$ where $d'$ satisfies the conditions in our theorem, so we have (40). To show that $b(d) \neq 0$, note that when $x \in (O^*)^m$, $I^0_{\chi, \xi, \psi}(p^d X(x) p^f) = l(d, f)$, so $b(d) \geq (1 - |p|)^m > 0$, completing our proof. \qed

10. Normalization

By equation (38), we have

$$I^0_{\chi, \xi, \psi}(e) = (1 - |p|)^m \Gamma(\chi, \xi) \cdot \sum_{w \in W_G, w' \in W_M} b(w \chi, w' \xi) d(w \chi) d'(w' \xi).$$

Now we calculate the value of $I^0_{\chi, \xi, \psi}(e)$. The method is similar to that in section 11 in [7]. Our conclusion is

**Theorem 10.1.** The value of $I^0_{\chi, \xi, \psi}(e)$ at identity is given by

$$I^0_{\chi, \xi, \psi}(e) = (1 - |p|)^m \Gamma(\chi, \xi) \zeta(1) m \prod_{i=1}^{m} \zeta^{-1}(2i).$$

Let $C = \sum_{w \in W_G, w' \in W_M} b(w \chi, w' \xi) d(w \chi) d'(w' \xi)$, then what we need to show is

$$C = \zeta(1) m \prod_{i=1}^{m} \zeta^{-1}(2i).$$
In this section we simply denote $\chi_i(p)$ (resp. $\xi_j(p)$) as $\chi_i$ (resp. $\xi_j$) for convenience. Let $b_1(\chi)$ be a function defined on $\chi \in \mathbb{C}^n$ and $b_2(\xi)$ a function on $\xi \in \mathbb{C}^m$, we define

$$\mathcal{A}_{W_G}(b_1(\chi)) = \sum_{w \in W_G} sgn(w)b_1(w\chi).$$

and

$$\mathcal{A}_{W_M}(b_2(\xi)) = \sum_{w' \in W_M} sgn(w')b_2(w'\xi).$$

Then it is not hard to see that

**Lemma 10.2.** For $w \in W_G$ and $w' \in W_M$, we have

$$\mathcal{A}_{W_G}(b_1(w\chi)) = sgn(w)\mathcal{A}_{W_G}(b_1(\chi)),$$

$$\mathcal{A}_{W_M}(b_2(w'\xi)) = sgn(w')\mathcal{A}_{W_M}(b_2(\xi)).$$

For $\epsilon \in \mathbb{Z}^n$ and $\mu \in \mathbb{Z}^m$, let $\chi^\epsilon = \prod_i \chi_i^{\epsilon_i}$ and $\xi^\mu = \prod_j \xi_j^{\mu_j}$. We say $\epsilon$ (resp. $\mu$) is regular if $\epsilon = \mu$ (resp. $w'\mu = \mu$) implies $w = \epsilon$ (resp. $w' = \epsilon$). Then we have

**Lemma 10.3.** For non-regular $\epsilon$ and $\mu$,

$$\mathcal{A}_{W_G}(\chi^\epsilon) = 0; \quad \mathcal{A}_{W_M}(\xi^\mu) = 0.$$

Let $\rho_1 = (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2})$ and let $\rho_2 = (m, m - 1, \ldots, 1)$. Let

$$P(\chi) = \chi^{\rho_1}; \quad Q(\xi) = \xi^{\rho_2}.$$

Note that $\rho_1$ is the half sum of positive roots in $\hat{G} = SO_{2n+1}(\mathbb{C})$, and $\rho_2$ is that of $M = SP_{2m}$, and note that $W_G = W_{\hat{G}}$, so, by the Weyl character formula on $\hat{G}$ and on $M$, we have

$$d(\chi) = \frac{(-1)^n}{P(\chi)\mathcal{A}_{W_G}(P(\chi))}, \quad d'(\xi) = \frac{(-1)^m}{Q(\xi)\mathcal{A}_{W_M}(Q(\xi))}.$$

From which we know that

$$d(w\chi) = \frac{(-1)^n sgn(w)}{P(w\chi)\mathcal{A}_{W_G}(P(\chi))}, \quad d'(w'\xi) = \frac{(-1)^m sgn(w')}{Q(w'\xi)\mathcal{A}_{W_M}(Q(\xi))}.$$

So we have

$$C = (-1)^{m+n} \frac{\mathcal{A}_{W_G} \circ \mathcal{A}_{W_M})(b(\chi, \xi)P(\chi^{-1})Q(\xi^{-1}))}{\mathcal{A}_{W_G}(P(\chi))\mathcal{A}_{W_M}(Q(\xi))}.$$

Note that $sgn(w_0^G)sgn(w_0^M) = (-1)^{m+n}$, $w_0^G(\chi) = \chi^{-1}$ and $w_0^M(\xi) = \xi^{-1}$. So

$$C = \frac{\mathcal{A}_{W_G} \circ \mathcal{A}_{W_M})(b(\chi^{-1}, \xi^{-1})P(\chi)Q(\xi))}{\mathcal{A}_{W_G}(P(\chi))\mathcal{A}_{W_M}(Q(\xi))}.$$

To calculate $C$, we need to simplify $\mathcal{A}_{W_G} \circ \mathcal{A}_{W_M})(b(\chi^{-1}, \xi^{-1})P(\chi)Q(\xi))$. 

Lemma 10.4. Let
\[ A(\chi, \xi) = \frac{b(\chi^{-1}, \xi^{-1})P(\chi)Q(\xi)}{\prod_{i=1}^{n-m} \prod_{j=1}^{m} \zeta^{-1}(-\chi_i + \xi_j + \frac{1}{2}) \zeta^{-1}(-\chi_i - \xi_j + \frac{1}{2})} \]
then
\[ C = \frac{(A_{W_G} \circ A_{W_M})(A(\chi, \xi))}{A_{W_G}(P(\chi))A_{W_M}(Q(\xi))}. \]

Proof. Consider \( b(\chi^{-1}, \xi^{-1})P(\chi)Q(\xi) \), which is equal to
\[
\prod_{i<j+n-m} \zeta^{-1}(-\chi_i + \xi_j + \frac{1}{2}) \cdot \prod_{i>j+n-m} \zeta^{-1}(+\chi_i - \xi_j + \frac{1}{2}) \prod_{1 \leq j \leq m} \zeta^{-1}(-\xi_j + \frac{1}{2}) \cdot \chi^{\rho_1 \xi_{\rho_2}}.
\]
Let \( b(\chi^{-1}, \xi^{-1})P(\chi)Q(\xi) = \sum_{\epsilon, \mu} c_{\epsilon, \mu} \chi^\epsilon \xi^\mu \), then
\[
(A_{W_G} \circ A_{W_M})(b(\chi^{-1}, \xi^{-1})P(\chi)Q(\xi)) = \sum_{\epsilon, \mu \text{ regular}} c_{\epsilon, \mu} \cdot (A_{W_G} \circ A_{W_M})(\chi^\epsilon \xi^\mu).
\]
By considering the power of \( \chi \), for those \((\epsilon, \mu)\) with \( c_{\epsilon, \mu} \neq 0 \), we have
\[
\epsilon_i \in \left\{ \frac{1}{2} + (n-i) - 2m, \frac{1}{2} + (n-i) \right\} \quad \text{when } 1 \leq i \leq n-m
\]
\[
\epsilon_i \in \left\{ \frac{1}{2} - m, m - \frac{1}{2} \right\} \quad \text{when } n-m + 1 \leq i \leq n
\]
Suppose \( \epsilon \) is regular, then for \( n-m+1 \leq i \leq n \), \( \{\epsilon_i\} \) must be a permutation of \( \{\frac{1}{2}, \frac{3}{2}, \ldots, m - \frac{1}{2}\} \), which implies \( \epsilon_{n-m} = \frac{1}{2} + m \), which then implies \( \epsilon_{n-m-1} = \frac{1}{2} + (m+1) \), and so on. Eventually we have, \( \epsilon_i = \frac{1}{2} + n - i \) for \( 1 \leq i \leq n - m \). In other words, for \( 1 \leq i \leq n-m \), \( \epsilon_i \) should attain its upper bound. This implies that \( (A_{W_G} \circ A_{W_M})(b(\chi^{-1}, \xi^{-1})P(\chi)Q(\xi)) \) remains the same if we divides \( \prod_{i=1}^{n-m} \prod_{j=1}^{m} \zeta^{-1}(-\chi_i + \xi_j + \frac{1}{2}) \zeta^{-1}(-\chi_i - \xi_j + \frac{1}{2}) \) from \( b(\chi^{-1}, \xi^{-1})P(\chi)Q(\xi) \), which equals \( A(\chi, \xi) \).

Let \( \tilde{\chi} = (\chi_1, \ldots, \chi_m) \in \mathbb{C}^m \) such that \( \tilde{\chi}_i = \chi_{n-m+i} \). Let
\[
\tilde{A}(\tilde{\chi}, \xi) = \frac{A(\chi, \xi)}{\prod_{i=1}^{n-m} \tilde{\chi}_i^{\frac{1}{2}+(n-i)}} = \prod_{i<j} \zeta^{-1}(-\tilde{\chi}_i + \xi_j + \frac{1}{2}) \prod_{i>j} \zeta^{-1}(\tilde{\chi}_i - \xi_j + \frac{1}{2}) \prod_{1 \leq j \leq m} \zeta^{-1}(-\xi_j + \frac{1}{2}) \prod_{1 \leq i,j \leq m} \zeta^{-1}(-\tilde{\chi}_i - \xi_j + \frac{1}{2}) \tilde{\rho}_1 \xi_{\tilde{\rho}_2},
\]
where \( \tilde{\rho}_1 = (m - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}) \). Then we have the following lemma.
Lemma 10.5. If we let $P(\tilde{\chi}) = \tilde{\chi}^{\tilde{\rho}}$, and let

$$(\mathcal{A}_{W_M} \circ \mathcal{A}_{W_M})(\tilde{\mathcal{A}}(\tilde{\chi}, \xi)) = \sum_{w_1, w_2 \in W_M} \text{sgn}(w_1) \text{sgn}(w_2) \tilde{\mathcal{A}}(w_1 \tilde{\chi}, w_2 \xi),$$

then

$$C = \frac{(\mathcal{A}_{W_M} \circ \mathcal{A}_{W_M})(\tilde{\mathcal{A}}(\tilde{\chi}, \xi))}{(\mathcal{A}_{W_M}(P(\tilde{\chi})) \mathcal{A}_{W_M}(Q(\xi)))}.$$ 

Moreover, the value of $C$ is independent of $(\chi, \xi)$.

Proof. What we actually need to prove is

$$\frac{(\mathcal{A}_{W_G} \circ \mathcal{A}_{W_M})(A(\chi, \xi))}{\mathcal{A}_{W_G}(P(\chi)) \mathcal{A}_{W_M}(Q(\xi))} = \frac{(\mathcal{A}_{W_M} \circ \mathcal{A}_{W_M})(\tilde{\mathcal{A}}(\tilde{\chi}, \xi))}{(\mathcal{A}_{W_M}(P(\tilde{\chi})) \mathcal{A}_{W_M}(Q(\xi)))}$$

and that it is independent of $(\chi, \xi)$. Let $A(\chi, \xi) = \sum_{\epsilon, \mu} d_{\epsilon, \mu} \chi^\epsilon \xi^\mu$. Then by the discussion in the previous lemma,

$$\epsilon_i = \frac{1}{2} + n - i, \text{ for } 1 \leq i \leq n - m$$
$$\epsilon_i \in \left[-(m - \frac{1}{2}), m - \frac{1}{2}\right], \text{ for } i > n - m.$$ 

So for each regular $\epsilon$, if we let $\epsilon' = (\epsilon_{n-m+1}, \ldots, \epsilon_n) \in \mathbb{Z}^m$, then

(1) $\epsilon = (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, m + \frac{1}{2}; \epsilon')$.

(2) Note that $\chi^\epsilon = \prod_{i=1}^{n-m} \chi_i^{\frac{1}{2} + (n-i)} \cdot \tilde{\chi}^{\epsilon'}$, so

$$\tilde{\mathcal{A}}(\tilde{\chi}, \xi) = \sum_{\epsilon, \mu} d_{\epsilon, \mu} \tilde{\chi}^{\epsilon'} \xi^\mu$$

(3) $\epsilon$ is regular if and only if $\epsilon'$ is regular with respect to the action of $W_M$. So when $\epsilon$ is regular, $\{|\epsilon_{n-m+1}|, \ldots, |\epsilon_n|\}$ is a permutation of $\{\frac{1}{2}, \ldots, m - \frac{1}{2}\}$, and hence there exists $w_1 \in W_M$, such that $\epsilon = w_1 \rho_1$ and $\epsilon' = w_1 \rho_1$.

By a similar consideration on the power of $\xi$, if $\mu$ is regular, we have

$$\{|\mu_1|, \ldots, |\mu_m|\}$$

is a permutation of $\{1, \ldots, m\}$.

So for $(\epsilon, \mu)$ being regular, we let $\epsilon = w_1 \rho_1$ (so $\epsilon' = w_1 \rho_1$), and $\mu = w_2 \rho_2$, where $w_1, w_2 \in W_M$, and we let $d_{w_1, w_2} = d_{\epsilon, \mu}$. Then

$$(\mathcal{A}_{W_G} \circ \mathcal{A}_{W_M})(A(\chi, \xi)) = \sum_{\epsilon, \mu \text{ regular}} d_{\epsilon, \mu} (\mathcal{A}_{W_G} \circ \mathcal{A}_{W_M})(\chi^\epsilon \xi^\mu)$$

$$= \sum_{w_1, w_2 \in M} d_{w_1, w_2} (\mathcal{A}_{W_G} \circ \mathcal{A}_{W_M})(\chi^{w_1 \rho_1} \xi^{w_2 \rho_2})$$

$$= \mathcal{A}_{W_G}(P(\chi)) \mathcal{A}_{W_M}(Q(\xi)) \sum_{w_1, w_2 \in M} d_{w_1, w_2} \text{sgn}(w_1) \text{sgn}(w_2).$$
Lemma 10.6.

If \( \tilde{A}(w \tilde{x}, w' \xi) \neq 0 \), then \( w = w' = e \).

Proof. We denote \( \tilde{x}_i = -\tilde{x}_i \) and \( \xi_j = \xi_j \). For any weyl element \( w, w' \in W_M \), there exists \( \sigma, \sigma' \), permutations of the set \( \{\pm 1, \pm 2, \ldots, \pm m\} \) such that

1. \( \sigma(-i) = -\sigma(i), \sigma'(j) = -\sigma'(j) \).
2. \( (w \tilde{x})_i = \tilde{x}_{\sigma(i)}, (w' \xi)_j = \xi'_{\sigma(j)} \).

So \( w = w' = e \) if and only if \( \sigma = \sigma' = \text{id} \). Now suppose \( \tilde{A}(w \tilde{x}, w' \xi) \neq 0 \), since \( \zeta^{-1}(0) = 0 \), we \( \sigma \) and \( \sigma' \) should satisfy the following properties:

1. For any \( i < j \), \( \tilde{x}_{\sigma(i)} = \xi_{\sigma'(j)} \neq \frac{\xi_j}{2} \).
2. For any \( i > j \), \( \tilde{x}_{\sigma(i)} = \xi_{\sigma'(j)} \neq -\frac{\xi_j}{2} \).
3. For any \( 1 \leq j \leq m \), \( \xi_{\sigma'(j)} \neq \frac{1}{2} \).
4. For any \( 1 \leq i, j \leq m \), \( \tilde{x}_{\sigma(i)} + \xi_{\sigma'(j)} \neq \frac{1}{2} \).

These four conditions actually imply \( \sigma = \sigma' = 1 \). To see this we consider \( A = \{\tilde{x}_{\sigma(i)}, 1 \leq i \leq m\} \) and \( B = \{\xi_{\sigma'(j)}, 1 \leq j \leq m\} \). Note that \( \{\tilde{x}_{\sigma(1)}, \ldots, \tilde{x}_{\sigma(m)}\} = \{1, \ldots, m\} \) and \( \{\xi_{\sigma(1)}, \ldots, \xi_{\sigma(m)}\} = \{\frac{1}{2}, \ldots, (m - \frac{1}{2})\} \). By property (3), \( \frac{1}{2} \notin B \).
which implies \(-\frac{1}{2} \in B\). Then by property (4), \(1 \notin A\). So then \(-1 \in A\). Then again by property (4), \(-\frac{3}{2} \notin B\), so then \(-\frac{3}{2} \in B\). And then by property (4), \(2 \notin A\), so \(-2 \in A\). Continuing this process, we will eventually have \(A = \{-m, -(m - 1), ..., -1\}\) and \(B = \{-m + \frac{1}{2}, -m + \frac{3}{2}, ..., -\frac{1}{2}\}\). So \(\sigma\) and \(\sigma'\) are actually permutations of \(\{1, 2, ..., m\}\).

Note that \(\tilde{\chi}_k - \xi_k = -\frac{1}{2}\), so by property (2), \(\sigma(i) \neq \sigma'(j)\) for any \(i > j\), which implies that

\[
\sigma^{-1}(k) \leq (\sigma')^{-1}(k)
\]

when \(1 \leq k \leq m\). On the other hand, since \(\tilde{\chi}_{k+1} - \xi_k = \frac{1}{2}\), we have

\[
\sigma^{-1}(k + 1) \geq (\sigma')^{-1}(k).
\]

by property (1). Combining equation (42) and (43), we have

\[
(\sigma')^{-1}(m) \geq \sigma^{-1}(m) \geq (\sigma')^{-1}(m - 1) \geq \sigma^{-1}(m - 1) \geq ... \geq (\sigma')^{-1}(1) \geq \sigma^{-1}(1),
\]

which implies that \(\sigma = \sigma' = \text{id}\).

By the lemma,

\[
C = \frac{\tilde{A}(\tilde{\chi}, \xi)}{A_{W_m}(P(\tilde{\chi}))A_{W_m}(Q(\xi))}
\]

where \(\tilde{\chi}_i = -(n + 1 - i)\) and \(\xi_j = -(m + \frac{1}{2} - j)\). Note that by the Weyl character formula,

\[
A_{W_m}(P(\tilde{\chi})) = \prod_{1 \leq a < b \leq m} \zeta^{-1}(-\tilde{\chi}_a + \tilde{\chi}_b)\zeta^{-1}(-\tilde{\chi}_a - \tilde{\chi}_b) \prod_{i=1}^n \zeta^{-1}(-\tilde{\chi}_i)P(\tilde{\chi});
\]

\[
A_{W_m}(Q(\xi)) = \prod_{1 \leq a < b \leq m} \zeta^{-1}(-\xi_a + \xi_b)\zeta^{-1}(-\xi_a - \xi_b) \prod_{j=1}^m \zeta^{-1}(-2\xi_j)Q(\xi).
\]

By direct calculation we have

\[
C = \zeta(1)^m \prod_{i=1}^m \zeta^{-1}(2i).
\]

11. Uniqueness of the Whittaker Shintani function

In this section we show the following theorem.

**Theorem 11.1.** Let \(W_{\chi, \xi, \psi}\) be a Whittaker Shintani function on \(G\). Let \(\mathcal{W}(d, f) = W_{\chi, \xi, \psi}(p^d \lambda p^f)\) with \(d \in \Lambda^+_m\) and \(f \in \Lambda^+_n\). If \(\mathcal{W}(0, 0) = 0\), then \(\mathcal{W}(d, f) = 0\) for every \(d \in \Lambda^+_m\) and \(f \in \Lambda^+_n\).

Combine this with theorem (6.1) we know that for all \((\chi, \xi)\), the space of Whittaker Shintani function \(\mathcal{W}S_{\chi, \xi, \psi}\) is of at most one dimensional. The method we use in the proof is from \(\cite{7}\) and \(\cite{9}\). First we define an order on \(\Lambda^+_n \times \Lambda^+_m\) as
Definition 11.2. Let $\varpi_k, \varpi_l'$ be the dominant weights of $G$ and $M$. For any $(d', f'), (d, f) \in \Lambda_m^+ \times \Lambda_n^+$, we write $(d, f) \succeq_{WS} (d', f')$ if

1. $\langle \varpi_l, f \rangle \geq \langle \varpi_l, f' \rangle$ for $1 \leq k \leq n - m$
2. $\langle \varpi_l, f \rangle + \langle \varpi_l', -d \rangle \geq \langle \varpi_l, f' \rangle + \langle \varpi_l', -d' \rangle$ for $n - m + 1 \leq l \leq n$
3. $\langle \varpi_{n+m+l-1}, f \rangle + \langle \varpi_l, d \rangle \geq \langle \varpi_{n+m+l-1}, f' \rangle + \langle \varpi_l, d' \rangle$ for $1 \leq l \leq m$

Then we have the following lemma

Lemma 11.3. Suppose $(d, f) \in \Lambda_m^+ \times \Lambda_n^+$.

1. If $(d', f') \in \Lambda_m^+ \times \Lambda_n^+$ satisfies
   
   \[ K_{M^d} p^d K_G p^f K_G \cap ZUK_{M^d} p^{d'} \lambda p^{f'} K_G \neq \emptyset, \]

   then $(d, f) \succeq_{WS} (d', f')$.

2. If $u \in U$ and $z \in Z$ satisfies
   
   \[ K_{M} p^d K_G p^f K_G \cap zUK_{M} p^{d} \lambda p^{f} K_G \neq \emptyset, \]

   then $\psi_U(u) = 1$ and $\psi(z) = 1$.

Before proving the lemma (11.3), we first show it implies theorem (11.1).

Proof of theorem (11.1). Consider $\int_{K_{M} p^d K_G p^f K_G} dg I_{\lambda, \xi, \psi}(g)$. On the one hand it is equal to

\[ \omega_{\lambda} (Ch_{K_G p^f K_G} \omega_{\xi} (Ch_{K_{M^d} p^{d} K_{M^d}})) W(0, 0). \]

On the other hand, by lemma (11.3) and theorem (6.1) it is equal to

\[ \sum_{(d', f') \leq_{WS} (d, f), (d', f') \in \Lambda_m^+ \times \Lambda_n^+} C_{d', f'} W(d', f'). \]

Where $C_{d, f}$ is positive by the second part of the lemma (11.3). So if $W(0, 0) = 0$, we have

\[ \sum_{(d', f') \leq_{WS} (d, f), (d', f') \in \Lambda_m^+ \times \Lambda_n^+} C_{d', f'} W(d', f') = 0 \]

for all $(d, f) \in \Lambda_m^+ \times \Lambda_n^+$. Taking the induction on $(d, f)$ by the order $\leq_{WS}$ we have

\[ W(d, f) \neq 0 \]

for all $(d, f) \in \Lambda_m^+ \times \Lambda_n^+$, completing our proof. \(\square\)

So in the rest of this section we only need to prove lemma (11.3). To prove the first part of lemma (11.3), we need the following lemma.
Lemma 11.4. Let \( \mathcal{N}_{2n} = \{1, 2, \ldots, 2n\} \), and let \( g, g^1, g^2, g^3 \in G \). For \( I = (i_1, \ldots, i_k) \), \( J = (j_1, \ldots, j_k) \in (\mathcal{N}_{2n})^k \), we denote \( f_{I,J}(g) = \prod_{s=1}^{k} g_{i_s,j_s} \), and

\[
\Delta_{I,J}(g) = \det(g_{i,j}) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{s=1}^{k} g_{i_{\sigma(s)},j_{\sigma(s)}}.
\]

If \( g = g^1 g^2 g^3 \), then we have

\[
\Delta_{I,J}(g) = \sum_{A,C \in \mathcal{N}^k_{2n}} f_{I,A}(g^1) \Delta_{A,C}(g^2) f_{C,J}(g^3).
\]

(44)

Proof. Since \( g = g^1 g^2 g^3 \), we have

\[
g_{i,j} = \sum_{a,b} g_{a,b}^1 g_{a,b}^2 g_{a,b}^3,
\]

(45)

where \( a, b \) runs over \( \mathcal{N}^k_{2n} \). So

\[
\Delta_{I,J}(g) = \sum_{\sigma \in S_k} \text{sgn}(\sigma) \sum_{(a_1,\ldots,a_k),(b_1,\ldots,b_k) \in (\mathcal{N}_{2n})^k} \prod_{s=1}^{k} g_{i_{s,a},g_{a,b}^1 g_{a,b}^2 g_{a,b}^3}.
\]

(46)

Note that \( S_k \) acts on \((\mathcal{N}_{2n})^k\). If we define \( c_s = b_{\sigma^{-1}(s)} \), then

\[
\sum_{(a_1,\ldots,a_k),(b_1,\ldots,b_k) \in (\mathcal{N}_{2n})^k} \prod_{s=1}^{k} g_{i_{s,a},g_{a,b}^1 g_{a,b}^2 g_{a,b}^3} = \sum_{(a_1,\ldots,a_k),(c_1,\ldots,c_k) \in (\mathcal{N}_{2n})^k} \prod_{s=1}^{k} g_{i_{s,a},g_{a,c}^1 g_{a,c}^2 g_{a,c}^3}.
\]

Note that for any \( \sigma \in S_k \),

\[
\prod_{s=1}^{k} g_{c_{\sigma(s)},j_{\sigma(s)}} = \prod_{s=1}^{k} g_{c_s,j_s}.
\]

So (46) is equal to

\[
\sum_{(a_1,\ldots,a_k),(c_1,\ldots,c_k) \in (\mathcal{N}_{2n})^k} \prod_{s=1}^{k} \left( g_{i_{s,a},g_{c,s}^3} \right) \cdot \sum_{\sigma \in S_k} \text{sgn}(\sigma) \prod_{s=1}^{k} g_{a,c}.
\]

(46)

which is the formula we want to prove.

By this lemma we can prove the first part of Lemma (11.3).
Proof of first part of lemma (11.3). Suppose

\[ K_{M'}p^dK_{G'}p^fK_{G} \cap \mathbb{Z}K_{M'}p^d \lambda p^fK_{G} \neq \emptyset, \]

then \( p'^r w^r G^r \lambda p'^d u z = k_1 p'^r k_2 p'^d k \) for some \( k_1, k_2 \in K_{G} \) and \( k \in K_{M'} \). Apply \( \alpha_l \) and \( \beta_l \), which are defined as (15) and (17) on page 8 to both sides of the equation. When \( l \geq n - m + 1 \), we have

\[ v(\alpha_l(p'^r w^r G^r \lambda p'^d u z)) = -\langle \omega_l, f' \rangle - \langle \omega'_{l-(n-m)}, d' \rangle. \] (47)

On the other hand, note that

\[ \alpha_l(g) = \Delta_{I_l,J_l}(g). \] (48)

where \( I_l = (2n + 1 - l, 2n + 1 - (l - 1), \ldots, 2n) \), and \( J_l = (1, 2, \ldots, l) \). By (44), we have

\[ \alpha_l(k_1 p'^r k_2 p'^d k) = \sum_{A,C \in (\mathbb{N}_2)^l} f_{l,A}(k_1) \Delta_{A,C}(p'^r k_2 p'^d) f_{C,J}(k). \]

Note that for \( f_{l,A}(k_1) \Delta_{A,C}(p'^r k_2 p'^d) f_{C,J}(k) \neq 0 \), both \( A \) and \( C \) should contain distinct coordinates. Moreover, note that \( k \in K_{M'} \), so \( f_{C,J}(k) \neq 0 \) implies \( c_j = j \) for all \( 1 \leq j \leq n - m \). Under these two restrictions it is not hard to see that

\[ v(\Delta_{A,C}(p'^r k_2 p'^d)) \geq -\langle \omega_l, f \rangle - \langle \omega'_{l-(n-m)}, d \rangle. \]

So

\[ v(\alpha_l(k_1 p'^r k_2 p'^d k)) \geq -\langle \omega_l, f \rangle - \langle \omega'_{l-(n-m)}, d \rangle. \] (49)

Comparing (47) and (49) we have

\[ \langle \omega_l, f' \rangle + \langle \omega'_{l-(n-m)}, d' \rangle \leq \langle \omega_l, f \rangle + \langle \omega'_{l-(n-m)}, d \rangle \]

for all \( n - m + 1 \leq l \leq n \). Similarly, if we apply \( \alpha_l \) for \( 1 \leq l \leq n - m \) or \( \beta_l \) for \( 1 \leq l \leq m \), we will have

\[ \langle \omega_l, f' \rangle \leq \langle \omega_l, f \rangle \]

for all \( 1 \leq l \leq n - m \) and

\[ \langle \omega_{n-m+l-1}, f' \rangle + \langle \omega'_{l}, d' \rangle \leq \langle \omega_{n-m+l-1}, f \rangle + \langle \omega'_{l}, d \rangle \]

for all \( 1 \leq l \leq m \). So we have the first part of the lemma (11.3). \( \square \)

Next we prove the second part of the lemma (11.3). Suppose

\[ p^d k_1 p^f = \zuk p^d \lambda p^f k_2 \] (50)
for some $u \in U$, $z \in \mathbb{Z}$, $k \in K_{M_j}$ and $k_1, k_2 \in K_G$. We need to show that $\psi(z) = \psi_U(u) = 1$. Consider the element $r_{2n} = (0, 0, \ldots, 0, 1) \in F^{2n}$. Multiplying $r_{2n}$ from the left to both sides of (50), we have

$$k_{2n}^1 p^f = p^{-f_1} k_{2n}^2.$$  

Here $k_{2n}^1$, $k_{2n}^2$ are the 2n-th row of $k^1$ and $k^2$. Suppose $k_{2n}^1 = (k_{2n,1}^1, \ldots, k_{2n,2n}^1)$, then

$$k_{2n}^2 = \left( p^{f_1+f_1} k_{2n,1}^1, p^{f_1+f_2} k_{2n,2}^1, \ldots, p^{f_1+n} k_{2n,n}^1, p^{f_1-f_2} k_{2n,n+1}^1, \ldots, p^{f_1-f_n} k_{2n,2n-1}^1, k_{2n,2n}^1 \right).$$  

(51)

Note that when $k^2 \in K_G$, each row of it is primitive, that is, it belongs to $\mathcal{O}^{2n}$, but not $(p \mathcal{O})^{2n}$. So suppose $f_1 = f_2 = \cdots = f_k > f_{k+1} \geq \cdots \geq f_n$ for some $k$, then by (51), at least one element in $\{ k_{2n,2n-k+1}^1, \ldots, k_{2n,2n}^2 \}$ belongs to $\mathcal{O}^*$. Let’s say it is $k_{2n,2n-i+1}^1$. Let $w$ be an weyl element of $G$ transposing 1 and $i$, then $k^1 w^{-1}$ has element in $\mathcal{O}^*$ at the $(2n, 2n)$ position. Then, by the Bruhat decomposition of $K \,(mod \, p)$, there exists $x_1, x_2, y_1, y_2 \in \mathcal{O}^{n-1}$, and $z_1, z_2 \in \mathcal{O}$, such that

$$k^1 w^{-1} = E_1(x_1, y_1, z_1) \left( \begin{array}{c} \epsilon \\ k' \\ \epsilon^{-1} \end{array} \right) E_1(x_2, y_2, z_2) w_G^L,$$

where $\epsilon \in \mathcal{O}^*$, $k' \in K_{Sp_{2n-2}}$, and

$$E_1(x, y, z) = \left( \begin{array}{ccc} 1 & x & y \\ I_{2n-2} & t_y & t_x \\ -i & x & 1 \end{array} \right)$$

So (30) becomes

$$E_1(x_1, y_1, z_1)^p d \left( p^d \left( \begin{array}{c} \epsilon \\ k' \\ \epsilon^{-1} \end{array} \right) p^f \right) p^{-f} E_1(x_2, y_2, z_2) w_G^L w p^f = uz k^p d \lambda p^f k^2.$$

By the definition of $w$, it commutes with $p^f$, so $p^{-f} E_1(x_2, y_2, z_2) w_G^L w p^f \in K_G$. So we just need to show that

$$E_1(x_1, y_1, z_1)^p d \left( p^d \left( \begin{array}{c} 1 \\ k' \\ 1 \end{array} \right) p^f \right) = uz k^p d \lambda p^f k^2$$

implies $\psi(z) = \psi_U(u) = 1$. We prove this by induction on $n - m$. 


When $n - m = 1$, $U$ is trivial, and $E_1(x, y, z) = J(x, y, z)$. By (52), we have $p^f k^2 p^{-f} \in M^1$. So $k^2 \in K_{M^1}$. Suppose $k^2 = n_1 k''$ where $n_1 \in J^0$ and $k'' \in K_M$, and suppose $\tilde{f} = (f_2, \ldots, f_n)$, then we have

$$J(x_1, y_1, z_1)^{p^d} \left( p^d \begin{pmatrix} 1 & k' \\ k' & 1 \end{pmatrix} p^f \right) = z k^d p^d \lambda n_1^f p^f k''$$

Note that now both sides belongs to $M^1$, we write both sides in the form of $J \times M$. Then by comparing the $J$-part of both sides we have

$$J(x_1, y_1, z_1)^{p^d} = z (\lambda n_1^f)^{k^d}.$$ When $f \in \Lambda_n^+$, $n_1^f \in J^0$. So we assume $\lambda n_1^f = J(x_3, y_3, z_3)$ with $x_3, y_3 \in O_n^-$ and $z_3 \in \mathcal{O}$. Then we have

$$z \cdot Z(z_3 - z_1) = J(x_1, y_1, 0)^{p^d} J(x_1, y_1, 0)_{k^d}^{J(-x_3, -y_3, 0)}$$

When $J(x_1, y_1, 0)^{p^d} J(-x_3, -y_3, 0)_{k^d} \in Z$, we have

$$J(-x_3, -y_3, 0)_{k^d} = J(-x_1, -y_1, 0)^{p^d}.$$ So $z \cdot Z(z_3 - z_1) = x_1 \cdot y_1 \in Z^0$. Since $z_1, z_3 \in \mathcal{O}$, so $z \in Z^0$, and hence $\psi(z) = 1$, completing the proof for $n - m = 1$.

Assume the lemma is true for $n - m = r - 1$, and suppose now $n - m = r > 1$. Then $E_1(x_1, y_1, z_1)^{p^d} \in U$. Since $E_1(x_1, y_1, z_1) \in U^0$ and $p^d$ stabilizes $\psi_U$, we have $\psi_U(E_1(x_1, y_1, z_1)^{p^d}) = 1$. So from (52) reduces to show that

$$p^d \left( \begin{pmatrix} 1 & k' \\ k' & 1 \end{pmatrix} \right) p^f = uz k^d \lambda p^f k_2$$

implies $\psi(z) = \psi_U(u) = 1$. Let

$$G' = \{ g \in G \mid g = \begin{pmatrix} 1 & * & * \\ g' & * \\ 1 \end{pmatrix} \} \cong Sp_{2n-2} \times H_{2n-1}.$$ Then by (53), $p^f k_2 p^{-f} \in G'$, so $k_2 \in K_{G'}$. Let $k_2 = \tilde{u} k''$ where $\tilde{u} \in H_{2n-1}$ and $k'' \in K_{Sp_{2n-2}}$. Then we have

$$p^d \left( \begin{pmatrix} 1 & k' \\ k' & 1 \end{pmatrix} \right) p^f = uz k^d \lambda \cdot (\tilde{u})^p k''.$$
Suppose $u = u^1 u^2$ where $u^1 \in \mathcal{H}_{2n-1}$ and $u^2 \in \text{Sp}_{2n-2}$. Now both sides belongs to $G'$. Write them in the form $\mathcal{H}_{2n-1}\text{Sp}_{2n-2}$ we have

$$u^1(\tilde{u})z k^d p^f = 1$$

and

$$u^2 z k^d \lambda p^f l'' = p^d \begin{pmatrix} 1 & k' \\ & 1 \end{pmatrix} p^f.$$

Note that $\tilde{u} \in U^0$, $f \in \Lambda_n^+$, and $zk^d \in \mathcal{M}^J$ which stabilizes $\psi_U$, we have

$$\psi_U(u_1) = \psi_U^{-1}(\tilde{u} z k^d p^f) = \psi_U^{-1}(\tilde{u} p^f) = 1.$$

On the other hand, by assumption of the induction, when

$$u^2 z k^d \lambda p^f l'' = p^d \begin{pmatrix} 1 & k' \\ & 1 \end{pmatrix} p^f,$$

we have $\psi_U(u_2) = \psi(z) = 1$. So $\psi_U(u) = \psi_U(u_1 u_2) = 1$ and $\psi(z) = 1$, completing our proof for $n - m = r > 1$. \hfill \Box

12. The formula for normalized Whittaker-Shintani function

Let $\mathcal{L}_{\chi,\xi,\psi}$ be a pairing between $\text{Ind}_{B^+}^G(\chi)$ and $\text{Ind}_{B^+}^{M^J}(\xi,\psi)$ satisfying Condition A and

$$\mathcal{L}_{\chi,\xi,\psi}(\mathcal{F}_\chi^0, \mathcal{F}_\xi^0) = 1$$

Since every such pairing produces a normalized Whittaker-Shintani function, so it is unique for every $(\chi, \xi)$. By theorem [10.1], for those generic $(\chi, \xi)$ where $\Gamma(\chi, \xi)$ has no zeroes or poles, such pairing exists. So applying the Bernstein theorem, and the corollary in section 1 in [1], formula (38) can be extended to a regular function on $(\chi, \xi)$. Now we summarize our result.

**Theorem 12.1.** For every $(\chi, \xi) \in \mathbb{C}^n \times \mathbb{C}^m$, the normalized Whittaker-Shintani function is given by

$$\int_{X^0} dx W^0_{\chi, \xi, \psi}(p^d x p^f) = \zeta(1) - \zeta(2i) \prod_{i=1}^{m} \zeta(2i).$$

$$\sum_{w \in W_G, w' \in W_M} b(w\chi, w'\xi)d(w\chi)d'(w'\xi)((w\chi)^{-1}\delta^{\chi})(p^f)((w'\xi)^{-1}\delta^{\chi})(p^d)$$

for $d \in \Lambda_m^+$ and $f \in \Lambda_n^+$. If we let $L(d', f') = \int_{X^0} dx W^0_{\chi, \xi, \psi}(p^d x p^f)$, then

$$W^0_{\chi, \xi, \psi}(p^d \lambda p^f) = \sum_{d'} a(d') L(d', f + d - d').$$
with \( a(d') \geq 0 \) and \( a(d) > 0 \). Here in the summation on \( d' \) runs over the set \( \{d' \mid d' \in \Lambda^+_{m}, f + d - d' \in \Lambda^+_{m}, d' \leq d \} \).

In particular,

\[
W_{\chi,\xi,\psi}^0(p^f) = \mathcal{L}(0, f).
\]

13. Application

Using the formula for the Shintani function, we can give an alternative proof of the Theorem 6.1 in [9]. We rewrite the theorem as below.

**Theorem 13.1** (Theorem 6.1 in [9], conjectured by T. Shintani). Let \( G = \text{Sp}_{2m} \) and \( M = \text{Sp}_{2m} \) as defined in our paper, and suppose \( n = m + 1 \). Let \( \pi \) and \( \tilde{\sigma} \) be the unramified representation of \( G(F_v) \) and \( M(F_v) \) respectively. Let \( z_{\pi} = (p^\chi_1, \ldots, p^\chi_{m+1}, 1, p^{-\chi_m+1}, \ldots, p^{-\chi_1}) \) be the Satake parameters of \( \pi \) and \( z_{\tilde{\sigma}} = (p^\xi_1, \ldots, p^\xi_m, p^{-\xi_m}, \ldots, p^{-\xi_1}) \) be the Satake parameters of \( \tilde{\sigma} \) with respect to \( \psi \) so that \( \tilde{\sigma} \otimes \omega_\psi \cong \text{Ind}_{\text{B}_G}^{\text{M}_G}(\xi, \psi) \). Let \( W_{\chi,\xi,\psi}^0 \) be the Whittaker-Shintani function as we defined in this paper. Then we have

\[
\int_{\text{GL}_1} W_{\chi,\xi,\psi}^0 \left( \begin{array}{cc} t & \text{I}_{2m} \\ \text{t}^{-1} & \end{array} \right) \left| t \right|^{s-m-1} dt = \frac{L(\pi, s)}{L(\tilde{\sigma}, s+\frac{1}{2})\zeta(2s)} \tag{54}
\]

13.3. We denote by LHS and RHS the left hand side and right hand side of above respectively. First we have

**Lemma 13.2.** Recall that \( \rho_1 = (n - \frac{1}{2}, n - \frac{3}{2}, \ldots, \frac{1}{2}) \), where \( n = m + 1 \) now, corresponding to the half sum of positive roots in \( \text{SO}_{2n+1} = \text{SO}_{2m+3} \). Then

\[
\text{LHS} = \sum_{l \geq 0} A_{W_G}(\prod_{1 \leq j \leq m}(1 - p^{-\chi_1/2} + \frac{1}{2})p^{(l+\rho_1)}) A_{W_G}(p^{(\chi, \rho_1)}) \cdot |p|^{|s|}.
\]

Note that by abusing the notation the \( l \) in \( l + \rho_1 \) is regarded as \( (l, 0, \ldots, 0) \in \mathbb{C}^n \).

**Proof.** Since all the data are unramified, the integral on the left is actually a sum over \( t = p^l \) with \( l \in \mathbb{Z} \). By the discussion of section 6, the Whittaker-Shintani function vanishes unless \( l \geq 0 \). Substituting \( W_{\chi,\xi,\psi}^0 \) by the formula we developed in the previous sections, and note that \( \delta_{\text{B}_G}^{|\frac{1}{2}|} \left( \begin{array}{cc} t & \text{I}_{2m} \\ \text{t}^{-1} & \end{array} \right) = \left| t \right|^{m+1} \), we have

\[
\text{LHS} = c^{-1} \cdot \sum_{l \geq 0} |p|^{|s|} \cdot \sum_{w \in W_G, w' \in W_M} b(w\chi, w'\xi) d(w\chi) d'(w'\xi)(\chi^{-1})(p^l).
\]
Here \( c = \zeta(1) \prod_{i=1}^{m} \zeta^{-1}(2i) \). Recall that \( \rho_2 = (m, m-1, \ldots, 1) \). Then
\[
\begin{align*}
d(\chi) &= (-1)^{m+1} p^{-\langle \chi, \rho_1 \rangle} \mathcal{A}_{W_G}^{-1}(p^{\langle \chi, \rho_1 \rangle}) , \\
d'(\xi) &= (-1)^{m} p^{-\langle \xi, \rho_2 \rangle} \mathcal{A}_{W_M}^{-1}(p^{\langle \xi, \rho_2 \rangle}) .
\end{align*}
\]
So
\[
\text{LHS} = \frac{c^{-1}}{\mathcal{A}_{W_G}(p^{\langle \chi, \rho_1 \rangle}) \mathcal{A}_{W_M}(p^{\langle \xi, \rho_2 \rangle})} \cdot \sum_{l \geq 0} |p|^{ls} \sum_{w \in W_G, w' \in W_M} (-1)^{2m+1} \text{sgn}(w) \text{sgn}(w') p^{-\langle w \chi, l+\rho_1 \rangle} p^{-\langle w' \xi, \rho_2 \rangle} b(w \chi, w' \xi) .
\]
In fact the summation on \( W_G \times W_M \) is equal to, by a change of variable \( w \mapsto ww_0 \) and \( w' \mapsto w'w_0^M \),
\[
\sum_{w \in W_G, w' \in W_M} \text{sgn}(w) \text{sgn}(w') p^{-\langle w \chi, l+\rho_1 \rangle} p^{-\langle w' \xi, \rho_2 \rangle} b(-w \chi, -w' \xi) .
\]
We can further simplify this summation by a similar discussion as in Section 10.

By the definition of \( b(\chi, \xi) \), we have
\[
\begin{align*}
p^{\langle w \chi, l+\rho_1 \rangle} p^{\langle w' \xi, \rho_2 \rangle} b(-w \chi, -w' \xi) \\
&= \left[ \prod_{j=1}^{m} (1 - p^{-\chi_1 \pm \xi_j + \frac{i}{2} p^{l+m+\frac{1}{2}} \chi_1}) \right] \left[ \tilde{A}(\tilde{\chi}, \xi) \right] .
\end{align*}
\]
Here \( \tilde{A}(\tilde{\chi}, \xi) \) is defined right after Lemma 10.4. If we let \( W_1 \) be the subgroup of \( W_G \) stabilizing \( \chi_2, \ldots, \chi_{m+1} \) and \( W_2 \) the subgroup of \( W_G \) stabilizing \( \chi_1 \), and let \( W_0 \) be a set of representatives in \( (W_1 \times W_2) \setminus W_G \), then the first bracket is invariant under \( W_M \times W_2 \), and the second bracket is invariant under \( W_1 \). So the summation over \( W_G \times W_M \) is equal to
\[
\sum_{w_0, w_1} \text{sgn}(w_0w_1)[((w_1w_0) \circ (\prod_{j=1}^{m} (1 - p^{-\chi_1 \pm \xi_j + \frac{i}{2} p^{l+m+\frac{1}{2}} \chi_1)))]
\cdot \left[ \sum_{w_2, w_M} \text{sgn}(w_2w_M) \tilde{A}(w_2w_0\tilde{\chi}, w_M \xi) \right] .
\]
By Lemma 10.5 the second bracket is equal to
\[
c \cdot \sum_{w_2, w_M} \text{sgn}(w_2w_M) p^{\langle w_2w_0 \tilde{\chi}, \rho_1 \rangle} p^{\langle w_M \xi, \rho_2 \rangle} .
\]
where \( \~\rho \) is the set \( \{0, m-\frac{1}{2}, m-\frac{3}{2}, \ldots, \frac{1}{2}\} \). From this it is not hard to see that the summation over \( W_G \times W_M \) in (56) is equal to

\[
c \cdot \sum_{w \in W_G} \text{sgn}(w) \cdot w \circ \left( \prod_{j=1}^{m} (1 - p^{-\chi_1+\xi_j+\frac{1}{2}}) p^{(\chi_1+\xi_j)} \right) A_{W_M}(w^{(\xi, \rho_2)})
\]

Substituting the formula to equation (55) we obtain our lemma. \( \square \)

Now we can prove Theorem 13.1

(Proof of Theorem 13.1). By Weyl’s character formula, for the representation of \( SO_{2N+1}(\mathbb{C}) \) whose highest weight is \( \lambda = (\lambda_1, \ldots, \lambda_N) \in \Lambda^+_N \), the trace of \( x = \text{diag}(x_1, \ldots, x_N, 1, x_{N}^{-1}, \ldots, x_1^{-1}) \) is \( \tilde{T}_N(\lambda; x) = \frac{\det(x_i^{\lambda_i-N-j+\frac{1}{2}} - x_i^{-(\lambda_i+N-j+\frac{1}{2})})}{\det(x_i^{N-j+\frac{1}{2}} - x_i^{-(N-j+\frac{1}{2})})} \). The function \( \tilde{T}_N(\lambda; x) \) is in fact defined for all \( \lambda \in \mathbb{Z}^N \). For any set \( A = \{a_1, \ldots, a_N\} \), we let \( \wedge^i(A) = \sum_{S \subseteq A, |S| = i} (\prod_{s \in S} a_s) \). Using these notation, we can express LHS as

\[
\text{LHS} = \sum_{l \geq 0, r \in \{0, 1, \ldots, 2m\}} (-1)^r \wedge^r (\Gamma_{\tilde{\sigma}}) \cdot \tilde{T}_{m+1}((l - r, 0, \ldots, 0); z) |p|^{ls}.
\]

Here \( \Gamma_{\tilde{\sigma}} \) is the set \( \{\xi_1 + \frac{1}{2}, \ldots, \xi_m + \frac{1}{2}, -\xi_m + \frac{1}{2}, \ldots, -\xi_1 + \frac{1}{2}\} \). Next we consider RHS.

By the discussion in [6, Theorem 3.1], we have

\[
\frac{L(\pi, s)}{\zeta(2s)} = \sum_{a \geq 0} \tilde{T}_{m+1}(a; z|p|^{as}).
\]

So by the notation introduced above, we have

\[
\text{RHS} = \sum_{a \geq 0, r \in \{0, 1, \ldots, 2m\}} (-1)^r \wedge^r (\Gamma_{\tilde{\sigma}}) \tilde{T}_{m+1}((a, 0, \ldots, 0); z|p|^{a+r}s).
\]

To show that (56) equals (57), note that in (56) if \( l < r \), then \( l - r \in \{-1, \ldots, -2m\} \) since \( 0 \leq r \leq 2m \). Then it is not hard to see that \( \tilde{T}_{m+1}(l - r, 0, \ldots, 0) = 0 \) by its definition. So one can replace the summation from \( l = 0 \) to \( l \leq r \). Then by a change of the variable \( l = a + r \) with \( a \geq 0 \) we have (56) equals (57). \( \square \)

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