Engineering many-body quantum Hamiltonians with non-ergodic properties using quantum Monte Carlo

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We present a computational framework to identify Hamiltonians of interacting quantum many-body systems that host non-ergodic excited states. We combine quantum Monte Carlo simulations with the recently proposed eigenstate-to-Hamiltonian construction, which maps the ground state of a specified parent Hamiltonian to a single non-ergodic excited state of a new derived Hamiltonian. This engineered Hamiltonian contains non-trivial, systematically-obtained, and emergent features that are responsible for its non-ergodic properties. We demonstrate this approach by applying it to quantum many-body scar states where we discover a previously unreported family of Hamiltonians with spatially oscillating spin exchange couplings that host scar-like properties, including revivals in the quantum dynamics, and towers in the inverse participation ratio; and to many-body localization, where we find a two-dimensional Hamiltonian with correlated disorder that exhibits non-ergodic scaling of the participation entropy and inverse participation ratios of order unity. The method can be applied to other known ground states to discover new quantum many-body systems with non-ergodic excited states.

Quantum Monte Carlo | Eigenstate-to-Hamiltonian construction approach | Many-body localization | Many-body scars

1. Introduction

Statistical mechanics is foundational to classical many-body theory, underpinning our understanding of the natural world and responsible for virtually all industrial progress before the advent of the transistor. Concepts from statistical mechanics have attained such familiarity that we describe other areas of physics using the same language. For example, path integral formulations of quantum mechanics invoke the unphysical concept of imaginary time to recast the propagator as a Boltzmann factor, and the normalization constant as a partition function. This success at transplanting the ideas, formalism and language of statistical mechanics to quantum theory has led to the assumption that the postulates of statistical mechanics continue to hold. This is the eigenstate thermalization hypothesis (1–3) that can be understood as follows. For an ergodic many-body quantum system, a state evolves under unitary time evolution, remaining pure and thus, non-thermal. However, after a partial trace is done over a designated subsystem, the remaining subsystem density matrix is mixed, thermal, and for sufficiently long time and large system size independent of initial conditions. In other words, an ergodic quantum many-body system will act as its own bath and thermalize in the usual statistical physics sense.

Recently, several counter-examples to this ergodic scenario have been found in both theory and experiment that show non-ergodic properties including for integrable systems (3), quantum many-body scars (QMBS) (4–9), many-body localization (MBL) (10–14), and systems with fragmented Hilbert spaces (15–18). Non-ergodicity can be characterized by the scaling of the entanglement entropy $S_R = -Tr \rho_A \log \rho_A$, with $\rho_A$ the density matrix for subsystem A. Thermal states are ergodic and have $S_R$ proportional to the volume of A, while non-ergodic states have $S_R$ growing more slowly, for example with the area of A (19) or the log of the volume of A (20). Non-ergodic systems can preserve quantum information and present the same richness to quantum systems that non-fully chaotic classical dynamics like the solar system give to classical systems. Without tools from statistical mechanics, we lack a theoretical framework to understand the emergence of these non-ergodic properties.

From another perspective, non-ergodic states are as familiar as thermal states. Most ground states of quantum many-body systems are generically non-ergodic and follow an area-law scaling of their entanglement entropy (21). The description of ground state properties is much better developed than for

Significance Statement

The choice of numerical technique is often dictated by trade-offs between factors such as speed, accuracy, attainable system size, and the accessible energy regime. While quantum Monte Carlo is a fast and highly accurate method able to handle large systems, it is generally limited to low energy states or thermal ensembles. We demonstrate a method that uses quantum Monte Carlo to engineer Hamiltonians with new emergent properties that host interesting non-ergodic excited states. For example, we find quantum many-body scars in a spin chain with alternating couplings, and many-body localization in a two-dimensional Heisenberg model with strongly correlated random magnetic field.

FFA, GL, PS, and SA conceived the project. HKT and BJJK did the initial proof of concept under the supervision of FFA, PS, and SA. NS, HKT and DCWF developed the codes and implemented the research under the guidance of PS, GL, and SA. NS, HKT, DCWF and GL analyzed the results. All authors discussed the results. NS, DCWF, GL, PS, and SA wrote the paper. The authors declare that they have no competing financial interests.

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excited states, and in particular, numerical approaches such as quantum Monte Carlo (QMC) can handle large Hilbert space sizes $\gtrsim 2^{1000}$, and can be applied to different types of many-body ground states. In this work we demonstrate how to apply the QMC approach developed for low-energy properties of quantum many-body systems to understand non-ergodic properties at high energy. We find new Hamiltonians with emergent symmetries that support non-ergodic states including Hamiltonians with oscillating spin exchange couplings that have QMBS states and two-dimensional correlated disorder Hamiltonians with many-body localization. These symmetry properties emerge non-trivially and systematically from the method and were not known previously.

One may wonder how QMC, whose sampling is ergodic in nature, traversing the full Hilbert space could possibly give results for a non-ergodic system. While it is true that QMC sampling generates any possible state in the Hilbert space, states are then accepted or rejected according to some criteria, such as energy in the Metropolis-Hastings algorithm. Rejected states no longer contribute to the expectation value computation. Since a finite sized system has a finite gap between the ground state and first excited state, an appropriate choice of “temperature” in the Boltzmann weighting function then ensures the exponentially vanishing contribution of any excited state to the expectation value. This is what allows QMC to accurately quantify the observables for non-ergodic ground states. While QMC uses the Boltzmann factor as a weighting function, this does not imply that the method implicitly assumes the eigenstate thermalization hypothesis. Here the Boltzmann factor is used merely as a weighting function to target the ground state properties at sufficiently small temperature. In principle, one could use a different weighting function that is unrelated to thermalization and statistical mechanics. One could use a small negative temperature to target only the highest energy eigenstate, or in particular, one could use the shift-invert method (22) with the weighting function $(E - H)^{-1}$, where $H$ is the Hamiltonian, to specifically target the highly excited state with energy closest to $E$. While the use of the shift-invert operator might be the first choice, at least conceptually, for targeting excited states using QMC, in practice this does not work for our purpose. This is because the form of the shift-invert function introduces competing frustrated interactions resulting in a sign problem for the QMC. Instead, in this work, we show that combining the eigenstate-to-Hamiltonian construction (EHC) (23) with QMC successfully obtains Hamiltonians with non-ergodic excited state properties.

2. EHC-QMC: Eigenstate to Hamiltonian Construction with Quantum Monte Carlo

While studies of quantum systems typically commence with Hamiltonians from which eigenstates or various observables are derived, the EHC (23, 24) offers an alternative by addressing the reverse question: given a particular eigenstate, what Hamiltonian hosts it? Figure 1 illustrates our application of the EHC. It involves first defining an input state $|\Psi\rangle$, assumed here as a ground state of a parent Hamiltonian $H$, and a set of local operators $\{O_i\}_{i=1,N_O}$, such that $H = \sum_{i=1}^{N_O} h_i O_i$. The next step is to calculate the covariance matrix $C$ whose
elements are

\[ C_{ij} = \langle \Psi | \mathcal{O}_i \mathcal{O}_j | \Psi \rangle - \langle \Psi | \mathcal{O}_i | \Psi \rangle \langle \Psi | \mathcal{O}_j | \Psi \rangle \]  

An eigenvector of the covariance matrix \( C \) provides the coefficients \( \hat{h}_i \) defining a target Hamiltonian \( \hat{H} = \sum_{i=1}^{N_0} \hat{h}_i \). Its associated eigenvalue represents the energy variance of \( |\Psi\rangle \) with respect to the target Hamiltonian \( \hat{H} \). If this eigenvalue is zero, then \( |\Psi\rangle \) is an exact eigenstate of \( \hat{H} \). Since \( |\Psi\rangle \) is the ground state of a local Hamiltonian, it satisfies an area law. But \( |\Psi\rangle \) is also an excited eigenstate of \( \hat{H} \) and therefore violates the ETH by construction.

The EHC was originally conceived to identify new Hamiltonians sharing the same ground state as the parent Hamiltonian and considered only zero eigenvalues of the covariance matrix where the mapping between the ground states of the parent and target Hamiltonians is exact (23). By contrast, in this work we use this approach as an approximate method to find a target Hamiltonian \( \hat{H} \) with \( |\Psi\rangle \) as a highly excited state. Here, the eigenvalue of the covariance matrix considered is non-zero, but vanishingly small in the thermodynamic limit. Our approach is similar to Ref. (24) where the authors have used the density matrix renormalization group (DMRG) numerical method in 1D supplemented by the EHC to approximately map many-body localization to a class of localized ground states known as the Bose-glass.

The effectiveness of the EHC relies on the judicious choice of the set of operators \( \{\mathcal{O}_i\}_{i=1, N_0} \). The calculation of the elements \( C_{ij} \) can be more involved depending on this choice, but the size of the covariance matrix \( C \) depends on the number of these operators \( N_0 \) (rather than on the dimension of the Hilbert space which is exponential in system size). With a proper choice, the method can be highly efficient, enabling exploration of large system sizes and overcoming the limitations of exact approaches. Further details of our EHC procedure is described in the Supplemental Material.

Developing approximate but reliable methods to describe the high energy properties of quantum many body systems, that are challenging to access by exact approaches, is a crucial goal in this field (25–28). Our approach enables us to harness QMC’s ability to access larger system sizes to explore non-ergodic excited states. We note that nearly all previously known models with non-ergodic properties were constructed through inspiration, fine-tuning, or serendipity (29–35). Our method circumvents this by providing a systematic approach to extend non-ergodic ground state properties to the high-energy sector. In addition, we have developed metrics to quantify the accuracy of the approximation providing diagnostics to determine when the EHC-QMC construction works.

Before we describe our results, we note some drawbacks of our approach. First, QMC is unable to obtain the full spectrum of the target Hamiltonian. It is therefore necessary to complement our approach with exact diagonalization (ED) at smaller system sizes to fully characterize the properties of the non-ergodic states we have identified. For example, we use ED to confirm that the ground state maps to a single eigenstate as opposed to a superposition of excited states. Second, the QMC-EHC approach is non-exhaustive by construction. We are unable to access non-ergodic excited states with a large average sign \( \sim 1 \) as these will suffer from a sign problem (36). Similarly, we are unable to access ergodic excited states, and therefore unable to characterize the full ergodic-to-non-ergodic transition of excited states as would be necessary to describe a many-body localization transition. Despite these drawbacks, our approach provides significant advantages over existing tools for studying non-ergodic excited states including accessing substantially larger system sizes, and revealing new emergent symmetries and other essential features of Hamiltonians that support non-ergodic properties.

While we illustrate the method by focusing on two main classes of non-ergodic excited states, namely, QMBS and MBL, there are some key differences between them: For MBL, all the excited states (at least within some energy window) are non-ergodic, while the non-ergodic QBMS states differ from the other excited eigenstates that are ergodic. Moreover, our MBL non-ergodic states occurs in a disordered system that necessitates appropriately averaging over hundreds of disorder configurations, while the QBMS states are for a single realization of a clean system. Despite these differences we are still able to use the same EHC-QMC framework to discover new examples in both these cases.

### 3. Quantum Many Body Scar States

The usual behavior for most Hamiltonians is that all highly excited states are thermal/ergodic, with, for example, an entanglement entropy scaling with volume. Yet Hamiltonians have been discovered both theoretically and experimentally (4, 5, 8, 9) in which a vanishing fraction of states are the so-called “scar states”, with anomalous non-ergodic properties, e.g. sub-volume entropy scaling. If an initial state has a strong overlap with those scar states, the time-evolved state will exhibit periodic revivals (5), in strong contrast with the systematic relaxation to an effective thermal equilibrium for systems with ergodic properties (1–3). These scar states are named for an analogy to classical scars, unstable periodic orbits in systems that do not generally host periodic orbits, and to our knowledge, the engineering of systems hosting such scar states to date has required fine-tuned kinetic constraints that emulate the formation of these classical closed orbits (8, 37). The QMC-EHC approach therefore provides an untapped niche for the systematic development of scar Hamiltonians. We show here that starting from the ground state of a clean, translationally invariant, spin chain model, the EHC provides a target Hamiltonian with quantum many-body scarring. We confine our search of target Hamiltonians to the possibly inhomogeneous spin-1/2 Heisenberg model, on a 1D chain with periodic boundary conditions

\[
\hat{H} = \sum_i [\hat{J}_{\perp, i} \frac{1}{2} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + \hat{J}_{z, i} S_i^z S_{i+1}^z] \]  

where \( S_i^+(-) \) is the raising (lowering) operator for a spin at site \( i \), \( S_i^z \) is the projection of that spin on the \( z \)-axis, \( \hat{J}_{\perp, i} \) are the couplings for swaps of nearest neighbour singlets and \( \hat{J}_{z, i} \) is the site-resolved nearest neighbour Ising interaction. The family of Hamiltonians of Eqn. 2 commutes with the total spin operator, with the largest sector of net zero spin having a Hilbert space size of \( \frac{N}{2} \), exponential in the system size \( N \). We use the ground state of the homogeneous case, \( \hat{J}_{\perp, i} = \hat{J}_{z, i} = J = 1 \), as our input state \( |\Psi\rangle \). This state displays antiferromagnetic quasi-long-range order and the usual non-ergodic properties associated with ground states (38). We choose the EHC operator basis as \( \mathcal{O}_i \equiv \frac{1}{2} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) \).
4. Many-Body Localization

For MBL Hamiltonians all excited states in a finite energy range exhibit nonergodic properties. First raised in the context of the (vanishing) conductivity of disordered wires (40), these systems, where disorder plays a crucial role, are the subject of intense theoretical and experimental interest. While the experimental signatures of MBL appear identical in one and two dimensions (41–44), the very existence of MBL remains theoretically controversial due to an instability mechanism called the “thermal avalanche” (45, 46) that makes MBL impossible in two and higher dimensions, and possibly pushing the critical disorder strength in one dimension to large or even infinite values (47–52).

We note that these works assume uncorrelated disorder. Including disorder correlations can significantly change these conclusions. For example, in a recent work (53), we have shown that confining potentials that are ubiquitous in experimental realizations of MBL (and a form of correlated disorder) can shift the critical dimension of MBL from $d=1$ to $d=2$. Similarly, Stark MBL has also been shown to arise in the absence of disorder (32–35), and quasi-periodic potentials also show...
signatures of MBL properties (54–57). These potentials may generically be thought of as the correlated part of an otherwise uncorrelated disorder potential, if any. Below we will find that disorder correlations emerge from our EHC procedure even when we start with a parent Hamiltonian with uncorrelated disorder. Similar correlations were observed by Ref. (24) when using EHC in conjunction with DMRG to investigate MBL in 2D which is made feasible by the QMC.

We consider the spin-1/2 Heisenberg model in a random, but possibly correlated field, which acts as quenched disorder. In two dimensions, it is described by the Hamiltonian

$$\hat{H} = J \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j + \sum_i \hat{h}_i \mathbf{S}_i^z,$$

where $\hat{J}$ is the coupling strength which we set to unity, $\mathbf{S}_i$ is the spin operator at site $i$, $\langle i,j \rangle$ indicates nearest-neighbour sites, and $\hat{h}_i$ is the local disordered magnetic field. Our input state is the ground state of $H$ with uncorrelated disorder $\hat{h}_i \in [-h, \Delta h]$. Beyond a critical disorder strength $h > h_c \approx 2.35$, $H$ has a Bose glass ground state (see e.g. Ref. (58) and Supplementary Material). The Bose glass ground state is insulating and characterized by correlators $\langle \mathbf{S}_i \cdot \mathbf{S}_j \rangle$ that decay exponentially with distance (58). This is different from the clean and homogeneous Heisenberg Hamiltonian considered in the previous section whose ground state has quasi long-range antiferromagnetic order (38). As with the QMBS Hamiltonian, the Hilbert space size of the zero net spin sector is $N^2 \sum_{\langle i,j \rangle} \sim N^{-1/2} 2^N$, growing exponentially with system size.

We proceed to construct the target Hamiltonian hosting the ground state of $H$ as an excited state. We select our EHC basis operators as $\mathbf{O}_0 = \sum_{\langle i,j \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$ and $\mathbf{O}_l = \mathbf{S}_i^l$, $i = 1, \ldots , N$ so that the number of operators $N_C = N + 1$ and the only possible outcome of EHC is to change the disorder configuration. We calculate the $(N+1) \times (N+1)$ covariance matrix $C$ using QMC. Details of the calculation and calibration checks performed with ED is discussed in the Supplementary Material. As described earlier, we look for (near) zero eigenvalues of $C$. Two of these are trivially zero, corresponding to the original Hamiltonian and the total spin operator, $S_i^z = \sum_i S_i^z$, respectively. So in practice, we look for the third smallest eigenvalue to get our target Hamiltonian $\hat{H}$. This eigenvalue scales to zero in the thermodynamic limit (see Sect. 5).

Figure 3 illustrates the main properties of the obtained target Hamiltonian that is consistent with many-body localization. We plot the IPR of all the eigenstates (similar to Fig. 2 (a),(b)). In dark-red squares, we show the target Hamiltonian $H$ obtained from a 1D version of $H$ at $h = 10$, and in blue squares, the parent Hamiltonian $H$ at $h = 1$ (see Eq. (3)). For this illustration, we consider the 1D case because we are restricted to ED for this characterization of the spectrum, and the ED has too strong finite-size effects in 2D. We note that the parent Hamiltonian $H$ is the paradigmatic model of MBL, which, at $h = 1$, is in the ETH phase where all states have ergodic/thermal properties. This is also manifest in the low values of IPR proportional to the inverse of the Hilbert space size $N$. In contrast, the eigenstates of the target Hamiltonian $H$ have large values of the IPR – orders of magnitude larger than those of ergodic states for the same Hilbert space size. This is a clear signature of the MBL nature of the target Hamiltonian, albeit in 1D and for small system sizes.

Next we look at the properties in 2D using QMC that enables us to describe the properties of $\Psi$ up to very large Hilbert space sizes $\sim \mathcal{O}(2^{500})$. In Fig. 3 (b), we represent the participation entropy $S_\infty = \lim_{q \to \infty} \frac{1}{q} \ln \sum_{|\Psi|} \langle |\Psi| (|\Psi|)^2 \rangle$ of the state $|\Psi\rangle$, where $|i\rangle$ are many-body states in $S_2$ basis, as a function of the size of the Hilbert space $N$. The dashed lines show the scaling of $S_\infty$ for $q \to \infty$ with $N$ where $S_\infty = D \ln N + c$. $D$ is the multifractal dimension and $c$ is a constant. The benchmark (shown in blue) is ETH ergodic regime where $D = 1$. The data for our target Hamiltonians are obtained for very large 2D samples (up to $10 \times 10$) using QMC and all have $D < 1$. This indicates a vanishing fraction of the states in configuration space contribute to $|\Psi\rangle$ demonstrating its non-ergodic properties (59). Additional characterization of the MBL properties of the target Hamiltonian is shown in the Supplementary Material.

Finally, in Fig. 3 (c), we show that, despite $H$ having uncorrelated disorder, the new Hamiltonian $H$ always has correlated disorder. The new disorder can be expressed as $\hat{h}_i = h_i + \Delta h_i$, with $\Delta h_i$ used to define a correlation function $C(r) = (\sum_{d_{ij}=r} \Delta h_i \Delta h_j) / (\sum_i \Delta h_i^2)$, where $d_{ij}$ is the distance between sites $i$ and $j$. This correlation function is plotted as a
function of $r/r_{\text{max}}$, where $r_{\text{max}}$ is the maximum distance in our 2D sample. We further show how the emergent correlations scale with system size. We clearly observe self-similar disorder correlations as the scaled distance between sites increases, with one prominent maximum and minimum. The strength of the correlation, and thus of the square of the confining potential, scales with the linear dimension $L$ of the system ($N = L^2$), mirroring previous findings (53) about MBL stabilized by a confining potential in two dimensions.

The EHC-QMC method allows us to obtain new Hamiltonians with MBL properties in two dimensions which was thought to be impossible due to thermal avalanches. The difference, of course, is the strong disorder correlations that emerge from the method. Starting from the localized ground state of a Hamiltonian with weak uncorrelated disorder, we obtain a target Hamiltonian with strong correlated disorder whose entire spectrum is non-ergodic. In addition, Hamiltonians corresponding to higher eigenvalues of the covariance matrix also exhibited MBL properties (not shown).

5. Reliability of the Method

Since the EHC is an approximate method, it is necessary to assess its reliability. The eigenvalue $e_n$ of the covariance matrix determines the variance of the target Hamiltonian $\hat{H}$ in the state $|\Psi\rangle$. We have $\sigma^2(\hat{H}) = \langle \Psi | (\hat{H}^2) | \Psi \rangle - \langle \Psi | (\hat{H}) | \Psi \rangle^2$, where $\sigma^2(\hat{H}) = e_n / (\bar{J}^2)$ for the QMBS case, and $\sigma^2(\hat{H}) = e_n / (\bar{J}^2)$ for the MBL case. Here $n$ is the index of the considered eigenvalue. In Fig. 4(a), we display the energy density variance $\sigma^2(\hat{H}/N)$ as a function of system size for both cases, where the data is obtained using ED at small system sizes and using QMC at large sizes. The MBL data is disordered averaged. We clearly observe a power-law decay and very good agreement between ED and QMC data. Since a vanishing energy variance signals that $|\Psi\rangle$ is an exact eigenstate of $\hat{H}$, this decrease and consistency between ED and QMC is a first indication that our EHC approach works reliably even at large system sizes.

However, this does not guarantee that $|\Psi\rangle$ maps to a single eigenstate of the new Hamiltonian. In fact, as the excited states of a many-body system have an exponentially large density, $|\Psi\rangle$ could correspond to a superposition of eigenstates. This limitation is common to all such approximate methods (25, 26, 60). To address this question, we use exact diagonalization, keeping in mind the limited applicability to small system sizes. We determine the eigenstates $|\psi_n\rangle$ of $\hat{H}$ close in energy to that of $|\Psi\rangle$, and calculate their corresponding overlap $O_{\alpha} = \langle \Psi | \psi_n \rangle^2$. In particular, we focus on the maximum value of these overlaps, max(overlap), since max(overlap) close to 1 indicate that EHC works i.e. $|\Psi\rangle$ maps mainly to a single eigenstate of $\hat{H}$.

An alternate figure of merit that is accessible to QMC is the relative residual $R = \sigma(\hat{H})/\Delta_N$, where $\Delta_N$ the mean level spacing of the many-body target Hamiltonian $\hat{H}$. Fig. 4(b) compares $R$ with max(overlap) for the MBL case. We present a 2D histogram of the $R$ and max(overlap) values for over 2000 disorder realizations (with the color indicating the frequency). We clearly observe that the EHC does not work at weak disorder (e.g. $h = 1$) where most disorder configurations have large $R$ and small max(overlap) values. By contrast, EHC works for most (but not all) disorder configurations at large disorder ($h > 20$) despite all disorder configurations having similar $R$ values. It is not surprising that the EHC approach works when $R < 1$, i.e. when the error in the energy is small compared to $\Delta_N$ for $R < 1$, we always find max(overlap) $\approx 1$. However, it might be counter intuitive that EHC also works at large disorder when $R > 1$. This is because even if the energy resolution is not sufficient (as is the case in QMC), the non-ergodic properties associated with MBL nevertheless allow EHC to work (see Supplemental Material).

For the QMBS case, we have also verified that system sizes up to $L = 16$ have large values of max(overlap) $\geq 0.8$ indicating that the EHC method works. At even larger system sizes only accessible through QMC such a characterization is not possible. Instead, we compare directly the QMC-obtained target Hamiltonian at large sizes with the ED results at small sizes. In Fig. 4(c), we show that the coupling parameters $J_{zz}$ and $J_{zz}$, which define the target Hamiltonian $\hat{H}$, obtained
by EHC-QMC align perfectly with those derived from ED. This is significant since the considered eigenvalue of the covariance matrix for QMC corresponding to the lowest non-trivial, non-degenerate eigenvalue, is the 8th eigenvalue for \( L = 48 \), whereas it is the 4th eigenvalue for ED. This is shown in Fig. 4(d). We interpret this perfect correspondence as further evidence of the reliability of the EHC-QMC approach.

6. Conclusion and Perspectives

We have developed a novel method, the EHC combined with QMC to systematically build quantum many-body Hamiltonians with non-ergodic excited states. Systems exhibiting non-ergodic properties have garnered significant interest recently because they can evade standard thermalization, enabling the preservation of quantum information over extended or even infinite times, even at infinite temperature. Previously, such systems were typically discovered through serendipity or arguments based on elaborate intuition and symmetry.

Our EHC-QMC approach takes a different path by systematically constructing such Hamiltonians through a procedure where the specific symmetries or correlations of the parameters defining the system emerge naturally. The key idea is to start from ground states that inherently possess non-ergodic properties and then search for target Hamiltonians that host such ground states as highly excited eigenstates. The EHC construction engineers the Hamiltonian supporting non-ergodic excited states, while the QMC provides an accurate description of ground state properties harnessing its capacity for large system sizes in higher dimensions. Our two examples illustrate that our approach provides a useful method to systematically construct Hamiltonians with non-ergodic states. While our focus here has been on the non-ergodic properties of highly excited states, our method allows us to systematically construct new Hamiltonians with desired excited state properties inherited from a specific ground state. In condensed matter systems, ground states are known to exhibit unique and interesting properties, such as the quasi-long-range ordering seen in the QMBS case we have described, but also superconductivity, Wigner crystalization, the quantum Hall effect, and topological order, among others. With our EHC-QMC method, these ground state properties can be promoted to highly excited states. This opens up the possibility for symmetry breaking and long-range order to occur even at infinite temperature. For example, we speculate that this algorithm could engineer a system with an "infinite temperature" superconducting state, as provocative as that prospect may seem.

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SUPPORTING INFORMATION

A. Computation of the Covariance Matrix with the Stochastic Series Expansion (SSE) QMC

The eigenstate-to-Hamiltonian construction (EHC) approach crucially needs the evaluation of a quantum covariance matrix, which mainly involves the computation of a collection of expectation values of correlators with respect to the ground state wavefunction. In SSE QMC (61, 62), ground state expectation values for finite size systems is obtained by choosing a sufficiently large inverse temperature \( \beta \) (that depends on the system size). The spectrum of any finite-size system is discrete and for simulations performed at temperatures smaller than the finite-size gap (between the ground state and the first excited state), contributions from higher energy states are exponentially suppressed, yielding ground state expectation values for the finite size system. Estimates for thermodynamic quantities are then obtained through a simultaneous finite-size and finite-temperature scaling (the temperature for each simulation is adjusted carefully to ensure that it is smaller than the finite size gap). In the literature (63), this approach has been successfully applied by all finite-temperature QMC algorithms (SSE, determinantal QMC, world line QMC, path integral QMC) to investigate the ground state phases of interacting spins, bosons and fermions, both with and without disorder. We perform our SSE-QMC simulations at low finite temperature, with \( \beta \) values changing from \( \beta = 2L \) to \( \beta = 4L \) and further to \( \beta = 8L \), where \( L \) is the linear dimension of our system size under study. We observe that the ground state energy converges readily with increasing \( \beta \), and hence we set \( \beta = 8L \) to ensure we are in the ground state for computation of various observables of interest.

Having obtained the ground state (say \( |\psi\rangle \)) of a parent Hamiltonian (say \( H \)), we choose a set of local operators \( \{O_i\}_{i=1,N_O} \), as our EHC basis operators, such that \( H = \sum_{i=1}^{N_O} h_i O_i \). We emphasize that the choice of this set of local operators is not unique, and mainly dictated by the problem at hand and the ease of numerical evaluation.

The next step is to compute the covariance matrix \( C \) whose elements are given by:

\[
C_{ij} = \langle \psi | O_i O_j | \psi \rangle - \langle \psi | O_i | \psi \rangle \langle \psi | O_j | \psi \rangle \quad :[4]
\]

It should be noted that unlike the Hamiltonian, whose dimension is exponential in system size, the dimension of a well chosen covariance matrix is linear in system size. Thus the covariance matrix can be diagonalized numerically by well established computational packages, and its eigenvalues and eigenvectors can be obtained precisely.

An eigenvector of the covariance matrix \( C \) provides the coefficients \( h_i \) defining a target Hamiltonian \( \tilde{H} = \sum_{i=1}^{N_O} \tilde{h}_i O_i \). Its associated eigenvalue represents the energy variance of \( |\psi\rangle \) with respect to the target Hamiltonian \( \tilde{H} \). That is, if this eigenvalue is zero, then \( |\psi\rangle \) is an exact eigenstate of \( \tilde{H} \). On the other hand, if this eigenvalue is non-zero (which typically happen on finite system sizes), then it’s scaling behavior with increased system size needs to be studied. If the eigenvalue decays at least as a power-law fashion, then it is argued that \( |\psi\rangle \) is an approximate eigenstate of \( \tilde{H} \) (24, 64).

A. Scar model. The Hamiltonian of interest is the \( \hat{S} = 1/2 \) antiferromagnetic Heisenberg model on a 1D chain with periodic boundary,

\[
H = \sum_i \left[ (J/2)(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + JS_i^z S_{i+1}^z \right]. \quad :[5]
\]

We write the Hamiltonian as above to distinguish between its diagonal part \( (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) \) and off-diagonal part \( (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) \) on each bond in the standard basis of diagonal spin components. In SSE QMC, we use the Taylor expansion to expand the exponential part of the partition function. Hence, the partition function can be written as a sum of the products (strings) of different Hamiltonian operators with the inverse temperature \( \beta \) as its order, in which its sequence is usually referred to as an operator string (see chapter-5 of Ref (63) for further details).
We choose $2N$ EHC basis operators to construct the covariance matrix, $\mathbf{C}$, such that $\mathcal{O}_i = (1/2)(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) \text{ and } \mathcal{O}_{i\neq N} = S_i^+ S_{i+1}^-$ with $i = 1, \ldots, N$. Thus the dimension of the covariance matrix in this case is $2N \times 2N$.

As the parent Hamiltonian follows translation invariance, the covariance matrix constructed as above also requires to follow translation invariance. Thus, the correlation, $\langle \mathcal{O}_i \mathcal{O}_j \rangle$, should depend only on $i-j$. With QMC we compute,

$$\langle \mathcal{O}_i \mathcal{O}_j \rangle = \frac{1}{(\beta)} \langle (n-1)N(i,j) \rangle \quad [6]$$

where, $N(i,j)$ denotes the number of times the indices $i, j$ appear next to each other in the operator string (65). QMC provides statistical estimations of these correlators and thus may not fulfill this invariance exactly. Therefore, we impose it such that $\langle \mathcal{O}_i \mathcal{O}_{i+r} \rangle = \sum_i \langle \mathcal{O}_i \mathcal{O}_{i+r} \rangle / N$, during the evaluation of the matrix elements of the covariance matrix.

While using the EHC method, we are mainly focused on the lowest eigenvalues of the covariance matrix to build our target Hamiltonians. For small system sizes ($L \leq 24$), we can compare the covariance matrix computed with ED and QMC methods. In figure 5(a-b), we show the covariance matrix for $L = 16$. It can be seen that the matrix elements of the covariance matrix computed with ED and QMC agree perfectly. Furthermore, the covariance matrix exhibits translational invariance. While the covariance matrices evaluated with ED and QMC methods agree pretty well, we are interested in the eigenvalues and eigenvectors of this for our EHC approach. With ED, we find that the target Hamiltonian defined by the fourth eigenvector (non-degenerate eigenvector) of the covariance matrix exhibits quantum many-body scars. We systematically increase the QMC measurement parameters, $N_{\text{meas}}$, and $N_{\text{meas}}$ and observe that for sufficiently large QMC averaging ($N_{\text{meas}} \approx 10^3$, $N_{\text{meas}} \approx 10^4$) of the EHC correlators, the eigenvalues of the covariance matrix computed via QMC converges to the ED values. In figure 5(c), we show the behavior of fourth eigenvalue of the covariance matrix, $\epsilon_4$ with increasing $N_{\text{meas}}$ for a fixed $N_{\text{meas}} = 10^3$. In our QMC simulations, we typically use $N_{\text{meas}} \approx 1 \times 10^3$ and $N_{\text{meas}} = 10^4$ for measurement of our observables.

With increasing system sizes, we have to rely on the QMC method only. We find that not all target Hamiltonians defined by the fourth eigenvector of the covariance matrix exhibit quantum many-body scars. In stead, the target Hamiltonian we select corresponds to the smallest nontrivial and nondegenerate eigenvalue of the covariance matrix. We have systematically observed that this choice yields a target Hamiltonian with scarring properties.

B. MBL model. The Hamiltonian of interest is the $S = 1/2$ antiferromagnetic Heisenberg model with a random disorder in 2D,

$$H = J \sum_{\langle i,j \rangle} S_i \cdot S_j + \sum_i h_i S_i^z \quad [7]$$

We choose $N+1$ EHC basis operators such that $\mathcal{O}_0 = \sum_{\langle i,j \rangle} S_i \cdot S_j$ and $\mathcal{O}_i = S_i^z$, $i = 1, \ldots, N$. Thus the dimension of the covariance matrix is $(N+1) \times (N+1)$.

In the above Hamiltonian, the Ising term $(S_i^z S_j^z)$ and the magnetic field term $(h_i S_i^z)$ are the diagonal terms $\mathcal{O}^d$, while the exchange term $(S_i^+ S_j^- + S_i^- S_j^+)$ is the off-diagonal term $\mathcal{O}^{od}$ (63). To compute the elements of the covariance matrix, we carry out the measurement of following types of terms.

$\mathcal{O}^{d2}$ terms: As both of the terms belong to diagonal type operation, we can do direct measurement for every spin state along the non-empty operator string.

$$\langle \mathcal{O}^{d1} \mathcal{O}^{d2} \rangle = \left( \frac{1}{N_H} \sum_{p=1}^{N_H} \mathcal{O}^{d1}_p \mathcal{O}^{d2}_p \right) \quad [8]$$

where $N_H$ is the total number of the non-empty operator string in each measuring step, $p$ is the slice index, and $\langle \cdots \rangle$ is the average of Monte Carlo steps (see chapter-5 of Ref (63) for further details). As the spin state only changes during the off-diagonal operation, we can boost the efficiency by bookkeeping spins on most of the sites. $\mathcal{O}^{od}$ terms: Only the exchange-exchange term in $\mathcal{C}$ belongs to this category. We cannot directly measure the off-diagonal term from the spin state (61). Instead, we use the number of appearance of the consecutive operators along the operator string to estimate its value.

$$\langle \mathcal{O}^{od1} \mathcal{O}^{od2} \rangle = \frac{1}{\beta} \langle N_{\text{num}}(\mathcal{O}^{od1}, \mathcal{O}^{od2}) \rangle \quad [9]$$

where $N_{\text{num}}$ is the number of consecutive appearances of $\mathcal{O}^{od1}$ and $\mathcal{O}^{od2}$ along the operator string in each Monte Carlo step.

$\mathcal{O}^{d1} \mathcal{O}^{od}$ terms: To calculate the combination of both diagonal and off-diagonal terms, we can combine both mentioned technique. At the occasion that $\mathcal{O}^{d1}$ appears, we measure the $\mathcal{O}^{d1}$ using direct measurement on the spin state.

$$\langle \mathcal{O}^{d1} \mathcal{O}^{od2} \rangle = \frac{1}{\beta} \sum_{p, \text{od}^{d1}} \mathcal{O}^{d1}_p \quad [10]$$

where $\mathcal{O} = \mathcal{O}^{od2}$ is the slice that the operator is off-diagonal.

B. Comparison with known models

In this section we show the behavior of relevant physical observables obtained with our method and compare them with previously known models exhibiting MBL, and scar properties respectively.

A key observable to study the non-ergodic properties of the eigenstates is the inverse participation ratio (IPR). For a single particle eigenstates, $\text{IPR} \sim 1/\text{Hilbert space size}$, indicates a fully delocalised nature. For many-body states this behavior is not straightforward. We perform Exact Diagonalization (ED) calculations in a 1D chain with periodic boundary, using the state-of-the-art Quspin package (66) to access all the eigenstates of our many-body Hamiltonians.

A. Scar model. A well known model describing the physics of Rydberg atom chain and hosting scar states is the PXP model (39). We compare the properties of our EHC built scar model with that of the PXP model.

In fig.6, we compare the inverse participation ratio (IPR) of the eigenstates of our EHC built scar model, $\tilde{H}$ and that of the PXP model as a function of energy. As expected, we see that majority of the eigenstates have vanishing IPR values, typical of any thermal eigenstate. In addition, we clearly observe several high-energy excited states exhibiting markedly higher IPR than the typical thermal eigenstates at given energies. These are the quantum many-body scar states. It should be noted that the scar-states in the PXP model appear in the eigen-spectrum with equally spaced energies. However, this is not the case for the scar states in our EHC built model.

A key characteristic of scar states is the visibly lower value of entanglement entropy in the bipartite entanglement entropy spectrum. In fig.6, we compare the bipartite entanglement entropy spectrum of our EHC built scar model $\tilde{H}$, and the PXP model as a function of the energy. Like the PXP model, our EHC built model hosts several high-energy excited states exhibiting markedly lower entanglement entropy than the typical thermal eigenstates at given energies, identified as scar states. Furthermore, these scar-states appear over the entire energy band.
Inverse Participation Ratio

(a) (b)

(c) (d)

Fig. 6. (a-b) We show the behavior of the inverse participation ratio (IPR) of the eigenstates of the PXP model and our EHC built scar model on a 1D chain of \( L = 16 \). It can be clearly seen the IPR values of the scar eigenstates are clear outliers from the typical thermal eigenstates. (c-d) We show the behavior of half-chain entanglement entropy of the eigenstates of the PXP model and our EHC built scar model on a chain of \( L = 16 \). It can be seen that the scar states (highlighted in blue) have lower entanglement entropy than the thermal states in the same energy density.

D. Computation of the variance of energy within the EHC approach

We begin with the Hamiltonian,

\[
H = J \sum_{\langle i,j \rangle} S_i^z S_j^z + h_i S_i^3
\]

and obtain the ground state,

\[
H \Psi_f = E_0 | \Psi_f \rangle
\]

We define the matrix elements of the covariance matrix,

\[
C_{ij} = \langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle,
\]

where \( O_0 = \sum_{\langle i,j \rangle} S_i^z S_j^z \) and \( O_i = S_i^z, \ i = 1, \ldots, N \).

We diagonalize the covariance matrix, and compute its eigenvalues and eigenvectors. We find \( e_1 = e_2 = 0 \), and \( e_3 \neq 0 \). The eigenvector corresponding to eigenvalue \( e_3 \), \( \Psi_3 = \{ J, h_1, \ldots, h_N \} \) defines a target Hamiltonian, \( \tilde{H} \) such that

\[
\tilde{H} = J \sum_{\langle i,j \rangle} S_i^z S_j^z + \tilde{h}_i S_i^3
\]

We rescale \( J \rightarrow J, \ h_i \rightarrow (J/J_{\tilde{h}}) \tilde{h}_i \), necessary for our QMC simulation, such that

\[
\tilde{H} = J \sum_{\langle i,j \rangle} S_i^z S_j^z + \tilde{h}_i S_i^3
\]

To compute the energy variance, we proceed as follows,

\[
\tilde{H} | \Psi \rangle = H | \Psi \rangle + (\tilde{H} - H) | \Psi \rangle
\]

\[
\sigma^2(\tilde{H}) = \langle \Psi | \tilde{H}^2 | \Psi \rangle - \langle (\Psi | \tilde{H} | \Psi \rangle)^2
\]

\[
\sigma^2(\tilde{H} - H) = \langle (\Psi | (\tilde{H} - H)^2 | \Psi \rangle - \langle (\Psi | (\tilde{H} - H) | \Psi \rangle^2
\]

We find that in the strong disorder case, the overlap is maximum \( \langle \Psi_0 | \Psi_0 \rangle | \Psi \rangle^2 \) and built the new Hamiltonian, \( H \), from this, the mean level spacing is computed as

\[
E = \frac{E_{max} - E_{min}}{N}
\]

where \( N \) is the number of states in the Hilbert space of \( H \). Further averaging over disorder realisations is needed to obtain the average mean level-spacing.

B. MBL model. With our EHC approach, we build Hamiltonians with correlated disorder that exhibit non-ergodic properties in the excited states. We compare the IPR behavior of all eigenstates of our EHC built target Hamiltonian, \( \tilde{H} \), vis-a-vis the prototypical MBL model, that is the disordered Heisenberg model, \( H \), Eq. (7). We use disorder values \( h = 1.0 \) and \( 10.0 \) for studying the ergodic and the localised regimes respectively. In figure 7, we show the IPR of all the eigenstates of the disordered Heisenberg model, Eq. (7), as a function of the energy.

C. Computation of the mean level spacing with QMC

Working with a given system size (say \( L \times L \)), for a given disorder realization in \( H \), we obtain the ground state \( | \Psi \rangle \) and built the new Hamiltonian \( \tilde{H} \) via the eigenstate-to-Hamiltonian approach. Then we run our QMC simulation on \( \tilde{H} \) to compute the ground state energy of \( \tilde{H} \) denoted by \( E_{\text{min}} \). We also run QMC simulation on \( -\tilde{H} \) to compute the corresponding ground state energy, denoted as \( -E_{\text{max}} \). From this, the mean level spacing is computed as

\[
\Delta_N = \frac{E_{\text{max}} - E_{\text{min}}}{N}
\]

E. Mapping of the ground state to excited state

The accuracy of the EHC mapping is inferred from a decaying behavior of the variance of the energy density with increased system size, and thus vanishing in the thermodynamic limit. But due to the exponentially large degeneracy of excited- eigenstates close to energy, \( E \), the question remains, whether \( \Psi_0 \) maps to a single eigenstate of \( \tilde{H} \) or a superposition of eigenstates. To address this, we perform Exact Diagonalization (ED) calculations using the state-of-the-art Quspin package[66] and compute the overlap of the ground state of the parent Hamiltonian, \( H \), and the eigenstates of the EHC built Hamiltonian, \( \tilde{H} \).

In Fig. 8(a), we show the overlap, \( O_m = | \langle \Psi_0 | \Psi_m \rangle |^2 \) of the actual ground state, \( \Psi_0 \) and the eigenstates of the target Hamiltonian \( \tilde{H} \), close to the energy, \( E = | \langle \Psi_0 | \tilde{H} | \Psi_0 \rangle | \) for a single disorder realisation of strong and weak disorder values on a 2D lattice of size \( 4 \times 4 \). We find that in the strong disorder case, the overlap is maximum \( O_m \approx 1 \) for a single eigenstate and vanishing for the remaining eigenstates. This indicates the EHC mapping to only one eigenstate.
These phases may be characterised by measuring the spin stiffness, with only a single eigenstate of the EHC built model, when the weak disorder (vanishing overlaps with all the eigenstates of the EHC built model), indicating the failure of the EHC mapping. On the other hand, when the input eigenvector is a different one (say \( |\Psi_0\rangle\)), we see that the ground state has maximum overlap with an exact eigenstate of the EHC model. In 1D Eq. (5), we observe that the ground state maps to a superposition of eigenstates. This is indicative of the fact that the ground state maps to a superposition of eigenstates.

In a similar fashion, in Fig. 8(b), we show the overlap, \( O_h = |\langle \Psi_0 | H | \Psi_0 \rangle|^2\) of the ground state, \( |\Psi_0\rangle\) and the eigenstates of the target Hamiltonian \( H\), for two different input eigenvectors of the covariance matrix (say 4th and 10th eigenvectors) of our scar model in 1D Eq. (5). We see that the ground state has maximum overlap with only a single eigenstate of the EHC built model, when the input eigenvector is the \( |\Psi_0\rangle\), the one hosting the couplings of our scar model. On the other hand, when the input eigenvector is a different one (say \( |\Psi_{10}\rangle\)), we observe that the ground state shows vanishing overlaps with all the eigenstate of the EHC built model, an indication of the failure of the EHC mapping.

**F. MBL : Ground state phase transition and physical observables**

The ground state of \( H\), Eq. (7), has two distinct phases as disorder strength \( h\) varies, with a quantum phase transition at a critical \( h_{c}\). These phases may be characterised by measuring the spin stiffness, \( \rho_s = \frac{1}{2} \frac{\partial^2 E}{\partial \phi^2}\), defined as the response of the total energy, \( E\), to a twist by angle \( \phi\). The delocalized superfluid (SF) phase (for \( h < h_c\)) has finite spin stiffness, whereas the localized Bose glass (BG) phase (for \( h > h_c\)) has vanishing spin stiffness, and \( h_c\) can be determined from the scaling of \( \rho_s\).

In SSE, the stiffness is measured by the fluctuation in winding number \((W)\) of the world lines as \( \rho_s = \langle W^2 \rangle / 2\beta\), where \( \beta\) is the inverse temperature (63). Close to the critical point, the stiffness obeys the scaling relation

\[
\rho_s(L, h) = L^{-\nu} f([h - h_c] L^{1/\nu}),
\]

where the correlation length exponent is \( \nu = 1 \) (67), and the dynamical critical exponent is found to be \( z = 2\). Plotting the scaled stiffness \( L^z \rho_s\) against \( h\) for different system sizes provides an accurate estimate of the critical disorder strength, \( h_{c}\) (58). The results are shown in Fig. 9(a), which suggest \( h_{c} \approx 2.35\). The interacting ground state changes from a delocalized hypercritical state to a localized Bose glass state for \( h > h_c\).

**Participation Entropy** : The \( q\)-th order Rényi participation entropy of a state \( |\psi\rangle\) is given by

\[
S_q = \frac{1}{1-q} \ln \sum_i p_i^q,
\]

where \( p_i = |\langle \psi | \phi_i \rangle|^2\) and the \( |\phi_i\rangle\) are some set of orthonormal basis states. In particular, we focus on \( q = 2\) and \( q \to \infty\). These two quantities provide the measure of how many states of a configuration space contribute to a wave function.

We use the approaches developed in (69, 70) to calculate the participation entropy. These approaches use the counting of occurrence for each spin configuration to calculate the participation entropy. \( S_q\) is found using the probability of having identical configurations in different replica in each Monte Carlo step, while \( S_{\alpha}\) is calculated using the probability of maximally occurring spin configuration. For strong disorders, the maximally occupied spin configuration is usually almost aligned with the local magnetic field.

In Fig. 9(b), we show the scaling of disorder-averaged \( S_q\) with the Hilbert space size, \( N\) in the localised regime. The slope of the line \( S_2 = D_2 \ln N - c\) represents the multifractal dimension \( D_2\), and we find \( D_2 \ll 1\). This indicates that only a vanishingly small fraction of basis states (among the exponentially large space of states in the configuration space) contribute to the Bose glass ground state in our simulations; highlighting its strong non-ergodic behavior. The behavior of \( S_{\alpha}\) is shown in the main text.

**Local entanglement entropy** : We measure the local entanglement entropy \( S^E = - \ln \langle \rho_{m_{\text{loc}}} \rangle\) for a bipartition of the system where the subsystem of interest is chosen to be one site only, using the SSE extended ensemble scheme (70, 71). In Fig. 9(c), the distribution of \( S^E\), \( P(S^E)\) shows a sharp peak close to \( S^E = 0\). This is a prominent feature of MBL (see Ref. (60)), where any given site is almost disentangled from other sites of the lattice and its reduced density matrix, \( \rho_{m_{\text{loc}}}\) can be approximated as that of a pure state. Furthermore, our results show a convergence of the data with different system sizes.

**Local magnetization** : We study the distribution of local magnetization \( P(m_{\text{loc}})\) in Fig. 9(d), and find a bipolar distribution with peak values at \( m_{\text{loc}} = \pm 1/2\), a signature of polarization along in the strong disorder limit. On the other hand for the weak disorder configuration, the overlap is finite for several eigenstates. This is indicative of the fact that the ground state maps to a superposition of eigenstates.

We compute the overlap of the ground state of our MBL model in 2D, \( |\Psi_0\rangle\), of the parent Hamiltonian, (Eq. (7)) and the eigenstates of the EHC built Hamiltonian, \( \tilde{H}\), close to the energy \( \tilde{E} = \langle \Psi_0 | \tilde{H} | \Psi_0 \rangle\), for a given disorder configuration with different disorder strengths. We see that for strong disorder (\( h = 10\)), the overlap is maximum (\( \approx 1\)) for a single eigenstate closest to \( \tilde{E}\), indicating that EHC has successfully discovered a \( \tilde{H}\) hosting \( |\Psi_0\rangle\) as an exact eigenstate. In contrast, for weak disorder (\( h = 1\)), the overlap \( \ll 1\), indicating the mapping to a superposition of eigenstates of \( \tilde{H}\) and a failure of the EHC. (b) We compute the overlap of the ground state of our 1D model Eq. (5), and the eigenstates of EHC built Hamiltonian, \( \tilde{H}\), for two different input eigenvectors of the covariance matrix. We see that the ground state has maximum overlap with only a single eigenstate of the EHC built model, when the input is the 4th eigenvector (that hosts couplings of the scar model). On the other hand, when the input eigenvector is a different one (say 10th eigenvector), we observe that the ground state shows vanishing overlaps with all the eigenstate of the EHC built model, an indication of the failure of the EHC mapping.

![Figure 8](image-url) Fig. 8. (a) We compute the overlap of the ground state of our MBL model in 2D, \( |\Psi_0\rangle\), of the parent Hamiltonian, \( H\), close to the energy \( \tilde{E} = \langle \Psi_0 | \tilde{H} | \Psi_0 \rangle\), for a given disorder configuration with different disorder strengths. We see that for strong disorder (\( h = 10\)), the overlap is maximum (\( \approx 1\)) for a single eigenstate closest to \( \tilde{E}\), indicating that EHC has successfully discovered a \( \tilde{H}\) hosting \(|\Psi_0\rangle\) as an exact eigenstate. In contrast, for weak disorder (\( h = 1\)), the overlap \( \ll 1\), indicating the mapping to a superposition of eigenstates of \( \tilde{H}\) and a failure of the EHC. (b) We compute the overlap of the ground state of our 1D model Eq. (5), and the eigenstates of EHC built Hamiltonian, \( \tilde{H}\), for two different input eigenvectors of the covariance matrix. We see that the ground state has maximum overlap with only a single eigenstate of the EHC built model, when the input is the 4th eigenvector (that hosts couplings of the scar model). On the other hand, when the input eigenvector is a different one (say 10th eigenvector), we observe that the ground state shows vanishing overlaps with all the eigenstate of the EHC built model, an indication of the failure of the EHC mapping.

![Figure 9](image-url) Fig. 9. (a) Behaviour of the scaled stiffness, \( L^z \rho_s\), with varying \( h\) near the transition region. The curves for different system sizes cross at \( h = h_{c}\), providing an accurate estimate of the critical disorder strength, \( h_{c} \approx 2.35\). (Inset) Finite size scaling of the spin stiffness, \( \rho_s\), with varying system sizes for different disorder strengths. In the thermodynamic limit, \( \rho_s \approx 0\), as \( h \to h_c\) increases, establishing the BG phase as the ground state. (b) Scaling of second order Rényi entropy, \( S_2\) with the Hilbert space size \( N\) in the presence of disorder, demonstrating non-ergodic behavior of the Bose glass ground state. (c) Distribution of local entanglement entropy \( P(S^E)\) in the ground state for various system sizes for \( h = 5\). \( P(S^E)\) shows a sharp peak at \( S^E \approx 0\) indicating that each site is almost disentangled from the other sites, giving rise to an exact eigenstate (58). As expected, the \( S^E\) peak moves towards \( S^E = 0\) with increasing system sizes. (d) (Main panel) Distribution of local magnetization \( P(m_{\text{loc}})\) in the ground state for different system sizes for \( h = 5\). \( P(m_{\text{loc}})\) is strongly peaked at the values \( m_{\text{loc}} = \pm 1/2\), indicative of the local moments being fully aligned with the local random magnetic field. (Inset) Power-law decay of maximum polarization \( \rho_{m_{\text{max}}}\) (see text) with system size, another characteristic signature of MBL (64, 68).
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The on-site disordered magnetic field. Following Refs. (24, 68), we further look into the maximum polarization, defined as \( \Delta_{\text{max}} = 1/2 - \max(|\langle m_1 \rangle|) \). We observe that the typical average of \( \Delta_{\text{min}} \) for \( L^{-1/2} \), with \( \gamma \sim 3.5 \) for \( h = 5 \) (see inset). This behavior is analogous to the freezing of local moments in the MBL phase (24, 68).

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