Research Article

Estimation procedures on Type-II censored data from a scaled Muth distribution

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ABSTRACT

In the present paper, we consider the estimation problem for the scaled Muth distribution under Type-II censoring scheme. In order to estimate the model parameters $\alpha$ and $\beta$, the maximum likelihood, the least-squares, and the maximum spacing estimators are derived. To show estimation efficiencies of the estimators obtained with this paper, we present an extensive Monte-Carlo simulation study in which the estimators are compared according to bias and mean squared error criteria. Furthermore, we evaluate the applicability of the scaled Muth distribution by taking into account both full and Type-II censored data situations by an analysis conducted on a real-life dataset.

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INTRODUCTION

The Muth distribution (MD) was firstly introduced on the positive real interval by Muth [1] as the statistical lifetime model with increasing, decreasing, and bathtub failure rates. Moreover, Jodra et al. [2] comprehensively studied some statistical features of the MD. In a wide variety of lifetime data observed in many fields from engineering and natural sciences to health and social sciences, the MD has got the potential of a good option for analyzing datasets with such failure rates. In this regard, it is a powerful alternative to popular lifetime models such as exponential, gamma, Weibull, log-normal, Rayleigh, and inverse Gaussian, with a weak probability mass property in the tail [2]. The probability density function of the MD is

$$f(x, \alpha) = (e^{\alpha x} - \alpha) \exp \left( \alpha x - \frac{1}{\alpha} (e^{\alpha x} - 1) \right), \quad x > 0,$$

and the corresponding cumulative distribution function is

$$F(x, \alpha) = 1 - \exp \left( \alpha x - \frac{1}{\alpha} (e^{\alpha x} - 1) \right), \quad x > 0,$$

where $\alpha \in (0,1)$ is a parameter that plays a vital role in the behavior of the distribution. The expected value of the

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MD with the single parameter is $E(X) = 1$. This is a strong constraint at the real data modeling stage. Jodra et al. [2] derived a new two-parameter form of MD called scaled Muth distribution (SMD) by adding a scale parameter to the distribution in their study. By this new form, a strong restriction on the expected value of the MD is removed. The pdf and cdf of the SMD are, respectively,

$$f(x, \alpha, \beta) = \frac{1}{\beta} \left( \frac{e^{\alpha x} - \alpha}{e^{\beta x} - \alpha} \right) \exp \left( \frac{\alpha x}{\beta} - \frac{1}{\alpha} \left( \frac{e^{\alpha x} - \alpha}{e^{\beta x} - \alpha} - 1 \right) \right), \; x > 0,$$

and

$$F(x, \alpha, \beta) = 1 - \exp \left( \frac{\alpha x}{\beta} - \frac{1}{\alpha} \left( \frac{e^{\alpha x} - \alpha}{e^{\beta x} - \alpha} - 1 \right) \right), \; x > 0,$$

where $\beta > 0$ is the scale parameter of the SMD. We present Figure 1 to illustrate the pdf behavior of the SMD for different values of the parameters $\alpha$ and $\beta$.

In the reliability and lifetime data analysis, it is well known that a complete dataset should be used to optimally analyze a phenomenon. However, sometimes the dataset may not be completely obtained or time cost constraints may be encountered. In such cases, the researcher may consider the censoring of the data. Censoring of the data will mostly enable significant cost savings in reliability engineering while time-saving in the modeling of lifetimes. There are various censoring schemes in the literature and commonly used of these are Type-I and Type-II. The main opinion in the Type-I censoring scheme, the experiment continues till a predetermined time $T$. The base idea in the Type-II censoring scheme, an experiment ends when records a predetermined number of data (failure times). So far, many researchers have made valuable studies on the solution of the statistical inference problem under the Type-II censoring scheme for specific probability distribution models, see [3–9]. This paper mainly motivated to examine different estimation procedures under the Type-II censoring scheme for the SMD. As far as we know, no attempt has yet been made to discuss different estimation procedures under the Type-II censoring scheme of SMD. This is quite an important task for areas where censored data is encountered, such as reliability engineering and lifetime analyzing, since optimally estimating model parameters have significant effects on determining a suitable model to data and modeling performance.

The rest of the paper is organized as follows. In the section Inference on SMD Parameters, we investigate the parameter estimators of the SMD under the Type-II censoring scheme by considering the different estimation methods, such as maximum likelihood (ML), least-squares (LS), and maximum spacing (MSP). The section Simulation Experiments includes a comprehensive Monte-Carlo simulation study in which compares the efficiencies of the estimators obtained in the section Inference on SMD Parameters according to bias and mean-squared error criteria. In the section Data Analysis, we present an illustrative data example to show the usefulness of SMD in modeling the Type-II censored data. Finally, the section Conclusion concludes the study.

**INFEERENCE ON THE SMD PARAMETERS**

Assume that $X_1, X_2, \ldots X_n$ is a random sample from the SMD with parameters $\alpha$ and $\beta$, where $X_1, X_2, \ldots X_n$ random variables imply the failure times for the $n$ independent unit. We also denote the order statistics of the random sample $X_1, X_2, \ldots X_n$ by $X_{(1)}, X_{(2)}, \ldots X_{(n)}$. Note that we only observe the first $r$ (before pre-decided $r < n$) order statistics of sample $X_1, X_2, \ldots X_n$ under the Type-II censoring scheme.

Now, we obtain the ML, LS, and MSP estimators of SMD parameters for Type-II censored data, which are commonly used estimators in the literature.

**ML Estimators**

Let $X_1, X_2, \ldots X_n$ be a random sample from SMD, and the $(n - r)$ of $n$ observations be censored according to the Type-II censoring scheme. In this situation, by considering

![Figure 1. Pdf of the SMD for different values of the parameters $\lambda$ and $\beta$.](image-url)
the pdf (3) and the cdf (4), the likelihood function of the SMD with parameters $\alpha$ and $\beta$ is immediately written as

$$
L(\alpha, \beta) = \left( \frac{n!}{(n-r)!} \right) \prod_{i=1}^{r} \frac{1}{\beta} \frac{\alpha x_i^{(i)}}{e^{\beta x_i^{(i)}} - \alpha} \exp \left( \frac{\alpha x_i^{(i)}}{\beta} - \frac{1}{\alpha} \left( \frac{\alpha x_i^{(i)}}{e^{\beta x_i^{(i)}} - 1} \right) \right)
$$

and the corresponding log-likelihood function is

$$
\ln L(\alpha, \beta) = \ln \left( \frac{n!}{(n-r)!} \right) - r \ln \beta - \sum_{i=1}^{r} \ln \left( \frac{\alpha x_i^{(i)}}{e^{\beta x_i^{(i)}} - 1} \right)
$$

Equations (7) and (8). Unfortunately, this nonlinear system cannot be solved with respect to the parameters $\alpha$ and $\beta$ analytically. But, we can use a numerical method to obtain ML estimates of the parameters. Newton-Raphson is an iterative approach to derive the root(s) of a real-valued function using its derivative and is widely used in the literature to obtain the numerical solution of likelihood equations. The main iterative formula of the Newton-Raphson is

$$
\hat{\theta}_{j+1} = \hat{\theta}_j - H(\hat{\theta}_j)^{-1} \nabla(\hat{\theta}_j)
$$

where $j$ shows the iteration number, $\hat{\theta}_j$ shows the estimates of parameter vector at step $j$, $\nabla(\hat{\theta}_j)$ and $H(\hat{\theta}_j)$ imply the first and second derivatives of the likelihood equations with respect to parameters, respectively. By using these notations, in here, we can easily write the statements needed to run the Newton-Raphson iterative formula as follows:

$$
\hat{\theta}_j = \left[ \hat{\theta}_j \right]
$$

Derivating the log-likelihood function given by (6) with respect to the parameters $\alpha$ and $\beta$, we have the following score functions:

$$
\frac{\partial \ln L(\alpha, \beta)}{\partial \alpha} = (n-r) \left[ \frac{\alpha x_i^{(i)}}{e^{\beta x_i^{(i)}} - 1} - \frac{x_i^{(i)} e^{\beta x_i^{(i)}}}{\alpha^2} + \frac{x_i^{(i)}}{\beta} \right] - r \frac{\alpha x_i^{(i)}}{\alpha} + \sum_{i=1}^{r} \frac{\alpha x_i^{(i)}}{\alpha^2} + \sum_{i=1}^{r} \alpha x_i^{(i)} e^{\beta x_i^{(i)} - 1} - \sum_{i=1}^{r} \alpha x_i^{(i)} e^{\beta x_i^{(i)} - 1} = 0
$$

and

$$
\frac{\partial \ln L(\alpha, \beta)}{\partial \beta} = \left[ \frac{\alpha x_i^{(i)}}{e^{\beta x_i^{(i)}} - 1} - \frac{x_i^{(i)} e^{\beta x_i^{(i)}}}{\alpha} \right] - r \frac{\alpha x_i^{(i)}}{\beta} + \sum_{i=1}^{r} \frac{\alpha x_i^{(i)}}{\beta} e^{\beta x_i^{(i)} - 1} - \sum_{i=1}^{r} \frac{\alpha x_i^{(i)}}{\beta} e^{\beta x_i^{(i)} - 1} = 0
$$

The ML estimators of the parameters $\alpha$ and $\beta$ are obtained from solution of the nonlinear system given by
\[ h_{12} = h_{21} = \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \sum_{i=1}^{r} \frac{ax_{(i)}^{\alpha x_{(i)}}}{\beta^2} \left( \frac{x_{(i)} e^{\frac{\alpha x_{(i)}}{\beta}} - \frac{2ax_{(i)}^{\alpha x_{(i)}}}{\beta^2}}{\beta^4} + \frac{2ax_{(i)}^{\alpha x_{(i)}}}{\beta^3} \right) + \frac{x_{(i)} e^{\frac{\alpha x_{(i)}}{\beta}} - \frac{2ax_{(i)}^{\alpha x_{(i)}}}{\beta^2}}{\beta^3} \]

\[ = \sum_{i=1}^{r} \alpha x_{(i)}^{\alpha x_{(i)}} e^{\frac{\alpha x_{(i)}}{\beta}} - \frac{x_{(i)} e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^2} \left( 2ax_{(i)}^{\alpha x_{(i)}} \right) + \frac{2ax_{(i)}^{\alpha x_{(i)}}}{\beta^3} \]

\[ + \frac{2ax_{(i)}^{\alpha x_{(i)}}}{\beta^3} + (n-r) \left( \frac{x_{(i)} e^{\frac{\alpha x_{(i)}}{\beta}}}{\beta^3} - \frac{2ax_{(i)}^{\alpha x_{(i)}}}{\beta^2} \right) \]

\[ + \frac{r}{\beta^2} \]

Thus, by starting with an initial estimation \( \hat{\theta}_0 \) of the parameter vector \( \theta \), the method is repeated until the root(s) is obtained according to a predetermined convergence criterion. Then, we have the ML estimates of the parameters \( \alpha \) and \( \beta \), say \( \hat{\alpha} \) and \( \hat{\beta} \), from the corresponding elements of the \( \hat{\theta} \) vector obtained at the last stage of the iteration.

**LS Estimators**

We assign this section of the paper to investigate the LS estimators of the SMD under the Type-II censored scheme. The LS estimation method was first introduced in 1988 by Swain et al. [10] for estimating the parameters of the Beta distribution at the complete data situations. Suppose \( X_1, X_2, \ldots, X_n \) be a random sample from any continuous distribution with cdf \( F(x_1) \), and also \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) be the ordered observations. In the complete data case, the LS estimators of the distributional parameters are obtained by minimizing the given below quadratic function with respect to distribution parameters.

\[ Q_{\text{complete}} = \sum_{i=1}^{n} \left( F(x_{(i)}) - P_i \right)^2, \quad (16) \]

where, \( P_i = \frac{i}{n+1} \) is the value of the empirical cumulative distribution function corresponding to \( i \)-th observation. Thus, considering the cdf of SMD given by the equation (4) the LS estimators of the parameters \( \alpha \) and \( \beta \) are obtained numerically minimizing the quadratic function \( Q_{\text{complete}} \) given in the equation (16) with respect to the parameters \( \alpha \) and \( \beta \).

Now, we investigate the LS estimators of the SMD parameters for the case of Type-II censored data. Let \( X_{(1)}, X_{(2)}, \ldots, X_{(n)} \) be an ordered random sample from SMD, and the last \((n-r)\) of \( n \) observations be censored, namely, \( X_{(1)}, X_{(2)}, \ldots, X_{(n-r)} \) can be observed as \( X_{(1)}, X_{(2)}, \ldots, X_{(n-r)} \) and \( X_{(n+1)}, X_{(n+2)}, \ldots, X_{(n)} \) cannot be observed. By considering this assumption and Kaplan-Meier estimator of the empirical distribution function, we can obtain the LS estimators of the parameters \( \alpha \) and \( \beta \) of SMD under the Type-II censoring scheme by numerically minimizing the quadratic function \( Q_{\text{censored}} \) given as below with respect to the \( \alpha \) and \( \beta \).

\[ Q_{\text{censored}} = \sum_{i=1}^{n} \left( 1 - \exp \left( \frac{\alpha x_{(i)}}{\beta} - \frac{2ax_{(i)}^{\alpha x_{(i)}}}{\beta^2} \right) - P_i^* \right)^2, \quad (17) \]

where \( P_i^* \) is the Kaplan-Meier estimator of the empirical distribution function. For further information about Kaplan-Meier estimator, we refer the readers to [11]. Under the Type-II censoring scheme, \( P_i^* \) can be easily calculate as follow:

\[ P_i^* = 1 - \prod_{j=i}^{r} \left( 1 - \frac{1}{n - j + 1} \right), \quad i = 1, 2, \ldots, r. \quad (18) \]

**MSP Estimators**

In this subsection of the paper, we investigate the MSP estimator of the SMD parameters for Type-II censored data. The MSP estimators were originally studied by [9]. It is a strong alternative to the ML estimators and has got useful
features such as consistency and asymptotically unbiasedness. For advanced information about MSP estimation method, we refer the readers to [12, 13, 14].

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be ordered sample from the SMD with parameters $\alpha$ and $\beta$. By considering the notations of the [13], in the case of complete data, the MSP estimators of the SMD can be easily obtained by maximizing the utility function $S$ given as

$$S = \sum_{j=1}^{n} \ln \left[ F(X_{(j)}, \alpha, \beta) - F(X_{(i)}, \alpha, \beta) \right]$$

with respect to parameters $\alpha$ and $\beta$, where $F(\cdot, \alpha, \beta)$ is the cdf of SMD given by equation (4), $F(X_{(j)}, \alpha, \beta) \equiv 0$, and $F(X_{(r)}, \alpha, \beta) \equiv 1$.

Now, we consider the Type-II censored data case. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be ordered sample taken from the SMD with parameters $\alpha$ and $\beta$, and among the last $(n-r)$ of $n$ observations be censored according to Type-II censoring scheme. For obtaining the MSP estimators under censoring case, Ng et al. [15] modified the utility function $S$ given in [13] as follow

$$S_{\text{censored}} = \sum_{j=1}^{n} \ln \left[ F(X_{(j)}, \alpha, \beta) - F(X_{(i)}, \alpha, \beta) \right] + (n-r) \ln \left[ 1 - F(X_{(r)}, \alpha, \beta) \right].$$

Hence, by considering the $S_{\text{censored}}$, utility function and cdf of the SMD, the MSP estimators of the SMD parameters for the Type-II censored data are easily obtained by maximizing

$$S_{\text{censored}} = \sum_{j=1}^{n} \ln \left[ \frac{\alpha x_{(j)}}{\beta} - \frac{1}{\alpha} \left( \frac{\alpha x_{(j)}}{\beta} - 1 \right) \right]$$

with respect to parameters $\alpha$ and $\beta$.

**SIMULATION EXPERIMENTS**

In this section, we conduct Monte-Carlo simulation studies to investigate the estimation performances of the ML, LS and MSP estimators obtained in the previous section for both complete and Type-II censored data situations. Throughout the Monte-Carlo simulation study, we set the values of the SMD parameters to $\alpha = (0.25, 0.50, 0.75)$ and $\beta = (0.5, 2.0)$. These randomly selected values of the $\alpha$ and $\beta$ exemplify the various formal behaviors of SMD pdf. In each combination of the parameter values, we generate random samples of different sizes $n = 30, 60, 100, 200$ considering various censoring proportions, $p = 0, 0.1, 0.2, 0.3$, (where $p = 1 - \frac{r}{n}$, $r \leq n$), from SMD distribution, and estimate the $\alpha$ and $\beta$ parameters using the ML, LS and MSP estimators. In addition, the biases and the mean square error (MSE) values of these estimators are also calculated to clarify the estimation performances of them. The simulated results are given by Tables 1–6.

As can be seen from the simulated results given by Tables 1–6, all estimators produce quite gratifying estimations in all the combinations of the parameter values, sample sizes, and censoring proportions. One can also see from Tables 1–6 that both the biases and MSE values of all estimators gradually increase to an acceptable level as the censoring proportion $p$ increases for all sample sizes $n$. Furthermore, we can conclude from the simulated results that all estimators are asymptotically unbiased and consistent because both biases and MSE values decrease when the sample of size $n$ increases, and that the ML and MSP estimators outperform the LS estimators with smaller biases and MSE values.

**DATA ANALYSIS**

In this section, we give an illustrative application on a practical dataset called the Air-condition system dataset to show data modeling with SMD for both complete and Type-II censored data situations, considering various censoring plans. The Air-condition system dataset contains 27 observations deal with times between successive failures (in hours) of the air-conditioning system of an airplane (airplane number 7913) [16]. The sorted complete data are as follows: 1, 4, 11, 16, 18, 18, 24, 31, 39, 46, 51, 54, 63, 68, 77, 80, 82, 97, 106, 111, 141, 142, 163, 191, 206, 216.

Before analysis, we investigate the underlying distribution of the data. We first draw the Total Time on Test (TTT) plot of data, see [17], to decide the suitable distribution families with consistent hazard rate function. Figure 2 shows the TTT plot of the data.

From Figure 2, it is lucid that the underlying distribution of the Air-condition system dataset has an increasing hazard rate function. Thus we can propose the SMD, Weibull, gamma and log-normal distributions for modeling the data.

Table 7 shows the calculated Kolmogorov-Smirnov (K-S) statistics and the corresponding p-values considering probability distribution models SMD, Weibull, gamma, and log-normal.

According to K-S test results given by Table 7, the proposed distributions are suitable for fitting the dataset.
Table 1. Simulated results for $\alpha = 0.25$ and $\beta = 0.5$

| $p$ | $n$ | Method | $\alpha = 0.25$ | $\beta = 0.5$ | $p$ | $n$ | Method | $\alpha = 0.25$ | $\beta = 0.5$ |
|-----|-----|--------|-----------------|--------------|-----|-----|--------|-----------------|--------------|
|     |     |        | Bias | MSE |        |     |        | Bias | MSE |        |     |        | Bias | MSE |        |     |        | Bias | MSE |        |
| 0   | 30  | ML     | 0.0832 | 0.0401 | 0.0004 | 0.0051 | 0.2 | 30  | ML     | 0.0709 | 0.0544 | -0.0040 | 0.0081 |
|     |     | LS     | 0.0410 | 0.0542 | 0.0020 | 0.0057 |     |     | LS     | 0.0036 | 0.0503 | -0.0122 | 0.0084 |
|     |     | MSP    | 0.0761 | 0.0429 | 0.0009 | 0.0052 |     |     | MSP    | 0.0037 | 0.0491 | 0.0179 | 0.0091 |
| 60  |     | ML     | 0.0252 | 0.0133 | -0.0012 | 0.0027 |     |     | ML     | 0.0240 | 0.0332 | 0.0002 | 0.0025 |
|     |     | LS     | 0.0036 | 0.0286 | -0.0108 | 0.0029 |     |     | LS     | -0.0331 | 0.0405 | 0.0041 | 0.0030 |
|     |     | MSP    | 0.0240 | 0.0155 | -0.0013 | 0.0028 |     |     | MSP    | -0.0202 | 0.0350 | 0.0137 | 0.0028 |
| 100 |     | ML     | 0.0129 | 0.0091 | -0.0040 | 0.0015 |     |     | ML     | 0.0204 | 0.0195 | -0.0024 | 0.0023 |
|     |     | LS     | -0.0090 | 0.0183 | -0.0072 | 0.0019 |     |     | LS     | -0.0135 | 0.0234 | -0.0010 | 0.0027 |
|     |     | MSP    | 0.0117 | 0.0097 | -0.0040 | 0.0015 |     |     | MSP    | -0.0096 | 0.0205 | 0.0061 | 0.0025 |
|     | 30  | ML     | 0.0641 | 0.0481 | 0.0047 | 0.0056 |     |     | ML     | 0.0831 | 0.0690 | -0.0081 | 0.0109 |
|     |     | LS     | 0.0171 | 0.0590 | -0.0099 | 0.0065 |     |     | LS     | -0.0230 | 0.0675 | 0.0004 | 0.0130 |
|     |     | MSP    | 0.0100 | 0.0457 | 0.0181 | 0.0063 |     |     | MSP    | 0.0061 | 0.0652 | 0.0246 | 0.0135 |
|     | 60  | ML     | 0.0484 | 0.0231 | 0.0045 | 0.0030 |     |     | ML     | 0.0500 | 0.0405 | -0.0024 | 0.0050 |
|     |     | LS     | -0.0044 | 0.0300 | 0.0040 | 0.0033 |     |     | LS     | -0.0369 | 0.0425 | 0.0110 | 0.0053 |
|     |     | MSP    | 0.0121 | 0.0228 | 0.0123 | 0.0032 |     |     | MSP    | -0.0008 | 0.0398 | 0.0165 | 0.0058 |
|     | 100 | ML     | 0.0275 | 0.0164 | -0.0044 | 0.0014 |     |     | ML     | 0.0474 | 0.0276 | -0.0042 | 0.0028 |
|     |     | LS     | 0.0048 | 0.0204 | -0.0071 | 0.0017 |     |     | LS     | 0.0033 | 0.0325 | 0.0032 | 0.0037 |
|     |     | MSP    | 0.0052 | 0.0161 | 0.0002 | 0.0014 |     |     | MSP    | 0.0108 | 0.0281 | 0.0083 | 0.0032 |
|     | 200 | ML     | 0.0065 | 0.0098 | 0.0002 | 0.0011 |     |     | ML     | 0.0230 | 0.0212 | 0.0037 | 0.0019 |
|     |     | LS     | -0.0135 | 0.0123 | 0.0005 | 0.0013 |     |     | LS     | -0.0120 | 0.0240 | 0.0101 | 0.0024 |
|     |     | MSP    | -0.0106 | 0.0104 | 0.0036 | 0.0011 |     |     | MSP    | -0.0020 | 0.0217 | 0.0123 | 0.0022 |

(continued)
### Table 2. Simulated results for \( \alpha = 0.25 \) and \( \beta = 2 \) (cont.)

| \( p \) | \( n \) | Method | \( \alpha = 0.25 \) | \( \beta = 2 \) | \( \alpha = 0.25 \) | \( \beta = 2 \) |
|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE |
| 0.1 | 30 | ML | 0.0624 | 0.0405 | 0.0013 | 0.0903 | 0.3 | 30 | ML | 0.1136 | 0.0645 | -0.0695 | 0.1499 |
| | | LS | -0.0174 | 0.0445 | -0.0292 | 0.1142 | LS | 0.0074 | 0.0489 | -0.0400 | 0.1655 |
| | | MSP | 0.0047 | 0.0378 | 0.0570 | 0.0971 | MSP | 0.0300 | 0.0547 | 0.0578 | 0.1794 |
| 60 | ML | 0.0361 | 0.0216 | 0.0003 | 0.0423 | 60 | ML | 0.0419 | 0.0353 | -0.0393 | 0.0981 |
| | LS | 0.0008 | 0.0265 | -0.0159 | 0.0458 | LS | -0.0105 | 0.0346 | -0.0315 | 0.1003 |
| | MSP | -0.0008 | 0.0221 | 0.0317 | 0.0447 | MSP | -0.0106 | 0.0343 | 0.0365 | 0.1090 |
| 100 | ML | 0.0309 | 0.0167 | 0.0119 | 0.0260 | 100 | ML | 0.0007 | 0.0316 | 0.0157 | 0.0616 |
| | LS | 0.0126 | 0.0192 | -0.0015 | 0.0296 | LS | -0.0440 | 0.0327 | 0.0356 | 0.0610 |
| | MSP | 0.0086 | 0.0162 | 0.0316 | 0.0266 | MSP | -0.0308 | 0.0322 | 0.0624 | 0.0692 |
| 200 | ML | 0.0039 | 0.0101 | 0.0007 | 0.0179 | 200 | ML | 0.0207 | 0.0209 | 0.0153 | 0.0366 |
| | LS | -0.0208 | 0.0147 | 0.0043 | 0.0197 | LS | -0.0204 | 0.0258 | 0.0489 | 0.0496 |
| | MSP | -0.0133 | 0.0108 | 0.0156 | 0.0177 | MSP | -0.0036 | 0.0211 | 0.0489 | 0.0409 |

### Table 3. Simulated results for \( \alpha = 0.5 \) and \( \beta = 0.5 \)

| \( p \) | \( n \) | Method | \( \alpha = 0.5 \) | \( \beta = 0.5 \) | \( \alpha = 0.5 \) | \( \beta = 0.5 \) |
|---|---|---|---|---|---|---|
| | | Bias | MSE | Bias | MSE | Bias | MSE |
| 0 | 30 | ML | 0.0380 | 0.0233 | -0.0034 | 0.0035 | 0.2 | 30 | ML | 0.0570 | 0.0458 | -0.0032 | 0.0053 |
| | | LS | -0.0041 | 0.0361 | -0.0147 | 0.0039 | LS | -0.0282 | 0.0630 | -0.0017 | 0.0059 |
| | | MSP | 0.0036 | 0.0247 | -0.0037 | 0.0035 | MSP | 0.0007 | 0.0552 | 0.0128 | 0.0060 |
| 60 | ML | 0.0099 | 0.0124 | 0.0003 | 0.0018 | 60 | ML | 0.0155 | 0.0240 | -0.0039 | 0.0019 |
| | LS | -0.0147 | 0.0200 | -0.0055 | 0.0021 | LS | -0.0437 | 0.0391 | -0.0010 | 0.0024 |
| | MSP | 0.0082 | 0.0133 | 0.0004 | 0.0018 | MSP | 0.0171 | 0.0283 | 0.0042 | 0.0019 |
| 100 | ML | 0.0181 | 0.0114 | -0.0003 | 0.0011 | 100 | ML | 0.0105 | 0.0113 | 0.0001 | 0.0011 |
| | LS | 0.0062 | 0.0147 | -0.0050 | 0.0012 | LS | -0.0160 | 0.0141 | -0.0013 | 0.0012 |
| | MSP | 0.0198 | 0.0112 | -0.0004 | 0.0012 | MSP | 0.0089 | 0.0125 | 0.0047 | 0.0012 |
| 200 | ML | -0.0049 | 0.0051 | -0.0023 | 0.0007 | 200 | ML | 0.0160 | 0.0096 | -0.0010 | 0.0008 |
| | LS | -0.0081 | 0.0073 | -0.0060 | 0.0008 | LS | 0.0001 | 0.0120 | -0.0007 | 0.0009 |
| | MSP | -0.0058 | 0.0051 | -0.0022 | 0.0007 | MSP | 0.0034 | 0.0102 | 0.0020 | 0.0008 |
| 0.1 | 30 | ML | 0.0169 | 0.0353 | -0.0039 | 0.0034 | 0.3 | 30 | ML | 0.0547 | 0.0638 | 0.0012 | 0.0053 |
| | LS | -0.0474 | 0.0561 | -0.0090 | 0.0043 | LS | -0.0482 | 0.0805 | 0.0082 | 0.0057 |
| | MSP | 0.0030 | 0.0148 | 0.0052 | 0.0035 | MSP | -0.0150 | 0.0839 | 0.0280 | 0.0075 |
| 60 | ML | 0.0251 | 0.0206 | 0.0053 | 0.0018 | 60 | ML | 0.0084 | 0.0324 | 0.0008 | 0.0024 |
| | LS | -0.0114 | 0.0262 | 0.0022 | 0.0019 | LS | -0.0626 | 0.0490 | 0.0099 | 0.0031 |
| | MSP | 0.0005 | 0.0219 | 0.0099 | 0.0019 | MSP | -0.0318 | 0.0400 | 0.0141 | 0.0028 |
| 100 | ML | 0.0232 | 0.0110 | 0.0015 | 0.0010 | 100 | ML | 0.0286 | 0.0190 | 0.0008 | 0.0014 |
| | LS | 0.0090 | 0.0135 | -0.0025 | 0.0011 | LS | -0.0212 | 0.0289 | 0.0092 | 0.0020 |
| | MSP | 0.0080 | 0.0112 | 0.0041 | 0.0010 | MSP | 0.0057 | 0.0212 | 0.0080 | 0.0015 |
| 200 | ML | 0.0078 | 0.0083 | 0.0024 | 0.0009 | 200 | ML | 0.0012 | 0.0125 | -0.0004 | 0.0010 |
| | LS | -0.0053 | 0.0099 | 0.0014 | 0.0001 | LS | -0.0202 | 0.0163 | 0.0008 | 0.0011 |
| | MSP | -0.0028 | 0.0087 | 0.0043 | 0.0009 | MSP | -0.0165 | 0.0149 | 0.0048 | 0.0011 |
### Table 4. Simulated results for $\alpha = 0.5$ and $\beta = 2$

| Method | $\alpha = 0.5$ | $\beta = 2$ | Method | $\alpha = 0.5$ | $\beta = 2$ |
|--------|----------------|-------------|--------|----------------|-------------|
|        | Bias | MSE  | Bias | MSE  | Bias | MSE  | Bias | MSE  | Bias | MSE  | Bias | MSE  |
| 0      | 0.0394 | 0.0309 | 0.0068 | 0.0419 | 0.2     | 0.0571 | 0.0444 | -0.0211 | 0.0390 |   |
| LS     | 0.0054 | 0.0380 | -0.0451 | 0.0448 | LS     | -0.0306 | 0.0645 | -0.0166 | 0.0468 |   |
| MSP    | 0.0367 | 0.0327 | 0.0065 | 0.0425 | MSP    | -0.0018 | 0.0556 | 0.0271 | 0.0410 |   |
| 60     | 0.0210 | 0.0154 | 0.0249 | 0.0200 | 60 ML  | 0.0176 | 0.0222 | -0.0197 | 0.0196 |   |
| LS     | 0.0022 | 0.0233 | 0.0033 | 0.0207 | LS     | -0.0323 | 0.0272 | -0.0180 | 0.0223 |   |
| MSP    | 0.0186 | 0.0162 | 0.0258 | 0.0205 | MSP    | -0.0145 | 0.0266 | 0.0046 | 0.0200 |   |
| 100    | 0.0148 | 0.0087 | -0.0004 | 0.0105 | 100 ML | 0.0105 | 0.0128 | 0.0012 | 0.0124 |   |
| LS     | 0.0041 | 0.0127 | -0.0119 | 0.0116 | LS     | -0.0179 | 0.0156 | 0.0046 | 0.0140 |   |
| MSP    | 0.0141 | 0.0094 | 0.0003 | 0.0108 | MSP    | -0.0083 | 0.0150 | 0.0159 | 0.0131 |   |
| 200    | 0.0053 | 0.0051 | 0.0026 | 0.0066 | 200 ML | 0.0046 | 0.0107 | 0.0050 | 0.0074 |   |
| LS     | -0.0040 | 0.0072 | -0.0050 | 0.0075 | LS     | -0.0102 | 0.0153 | 0.0050 | 0.0086 |   |
| MSP    | 0.0047 | 0.0055 | 0.0031 | 0.0067 | MSP    | -0.0077 | 0.0122 | 0.0151 | 0.0077 |   |
| 0.1    | 0.0463 | 0.0379 | -0.0165 | 0.0323 | 0.3     | 0.0431 | 0.0530 | -0.0043 | 0.0489 |   |
| LS     | -0.0123 | 0.0423 | -0.0453 | 0.0383 | LS     | -0.0750 | 0.0698 | 0.0306 | 0.0650 |   |
| MSP    | 0.0012 | 0.0414 | 0.0089 | 0.0323 | MSP    | -0.0304 | 0.0683 | 0.0738 | 0.0624 |   |
| 60     | 0.0098 | 0.0143 | -0.0014 | 0.0186 | 60 ML  | 0.0414 | 0.0287 | -0.0013 | 0.0259 |   |
| LS     | -0.0158 | 0.0198 | -0.0166 | 0.0214 | LS     | -0.0173 | 0.0406 | 0.0176 | 0.0361 |   |
| MSP    | -0.0155 | 0.0169 | 0.0128 | 0.0189 | MSP    | 0.0032 | 0.0349 | 0.0372 | 0.0310 |   |
| 100    | 0.0236 | 0.0104 | -0.0004 | 0.0102 | 100 ML | 0.0234 | 0.0155 | -0.0261 | 0.0153 |   |
| LS     | 0.0142 | 0.0124 | -0.0171 | 0.0122 | LS     | -0.0115 | 0.0232 | -0.0105 | 0.0184 |   |
| MSP    | 0.0086 | 0.0111 | 0.0035 | 0.0103 | MSP    | 0.0002 | 0.0185 | -0.0038 | 0.0153 |   |
| 200    | 0.0088 | 0.0075 | -0.0115 | 0.0072 | 200 ML | 0.0129 | 0.0125 | -0.0053 | 0.0074 |   |
| LS     | -0.0029 | 0.0105 | -0.0184 | 0.0080 | LS     | -0.0139 | 0.0195 | 0.0054 | 0.0109 |   |
| MSP    | -0.0019 | 0.0084 | -0.0058 | 0.0072 | MSP    | -0.0038 | 0.0153 | 0.0107 | 0.0078 |   |

(continued)
Table 5. Simulated results for $\alpha = 0.75$ and $\beta = 0.5$ (cont.)

| $p$ | $n$ | Method | $\alpha = 0.75$ | $\beta = 0.5$ | $p$ | $n$ | Method | $\alpha = 0.75$ | $\beta = 0.5$ |
|-----|-----|--------|----------------|--------------|-----|-----|--------|----------------|--------------|
|     |     |        | Bias MSE       |              |     |     |        | Bias MSE       |              |
| 0.1 | 30  | ML     | 0.0227 0.0292  | -0.0034 0.0027 | 0.3 | 30  | ML     | -0.0053 0.0315 | -0.0067 0.0037 |
|     |     | LS     | -0.0233 0.0362 | -0.0119 0.0030 |     |     | LS     | -0.0862 0.0542 | -0.0041 0.0040 |
|     |     | MSP    | -0.0015 0.0340 | 0.0030 0.0027 |     |     | MSP    | -0.0481 0.0453 | 0.0093 0.0042 |
| 60  | ML  | 0.0030 0.0142 | -0.0012 0.0016 | 60  | ML  | -0.0213 0.0169 | -0.0041 0.0018 | 0.0002 0.0014 |
|     |     | LS     | -0.0210 0.0152 | 0.0021 0.0017 |     |     | LS     | -0.0213 0.0282 | 0.0001 0.0016 |
|     |     | MSP    | -0.0036 0.0108 | 0.0011 0.0008 |     |     | MSP    | -0.0091 0.0255 | 0.0076 0.0015 |
| 100 | ML  | 0.0110 0.0086 | 0.0018 0.0007 | 100 | ML  | -0.0036 0.0089 | 0.0038 0.0007 | 0.0057 0.0009 |
|     |     | LS     | 0.0053 0.0108  | 0.0011 0.0008 |     |     | LS     | -0.0148 0.0129 | 0.0024 0.0011 |
|     |     | MSP    | 0.0019 0.0089 | 0.0038 0.0007 |     |     | MSP    | -0.0068 0.0112 | 0.0057 0.0009 |
| 200 | ML  | 0.0035 0.0061 | 0.0002 0.0005 | 200 | ML  | -0.0058 0.0071 | -0.0020 0.0005 | 0.0000 0.0007 |
|     |     | LS     | -0.0030 0.0063 | 0.0015 0.0005 |     |     | LS     | -0.0090 0.0086 | 0.0000 0.0007 |
|     |     | MSP    | -0.0030 0.0063 | 0.0015 0.0005 |     |     | MSP    | -0.0074 0.0075 | 0.0028 0.0007 |

Table 6. Simulated results for $\alpha = 0.75$ and $\beta = 2$

| $p$ | $n$ | Method | $\alpha = 0.75$ | $\beta = 2$ | $p$ | $n$ | Method | $\alpha = 0.75$ | $\beta = 2$ |
|-----|-----|--------|----------------|-----------|-----|-----|--------|----------------|-----------|
|     |     |        | Bias MSE       |            |     |     |        | Bias MSE       |            |
| 0   | 30  | ML     | 0.0313 0.0201  | -0.0050 0.0360 | 0.2 | 30  | ML     | 0.0007 0.0256 | -0.0224 0.0439 |
|     |     | LS     | -0.0052 0.0253 | -0.0494 0.0456 |     |     | LS     | -0.0650 0.0453 | -0.0349 0.0480 |
|     |     | MSP    | 0.0323 0.0223 | -0.0300 0.0371 |     |     | MSP    | -0.0323 0.0326 | 0.0177 0.0447 |
| 60  | ML  | 0.0241 0.0120 | 0.0096 0.0224 | 60  | ML  | -0.0027 0.0151 | -0.0058 0.0257 | -0.0251 0.0274 |
|     |     | LS     | 0.0231 0.0123 | 0.0106 0.0225 |     |     | LS     | -0.0013 0.0186 | 0.0087 0.0245 |
|     |     | MSP    | 0.0181 0.0082 | 0.0044 0.0109 |     |     | MSP    | 0.0094 0.0093 | -0.0081 0.0139 |
| 100 | ML  | 0.0129 0.0108 | -0.0127 0.0136 | 100 | ML  | 0.0194 0.0081 | 0.0041 0.0111 | -0.0119 0.0152 |
|     |     | LS     | 0.0012 0.0044 | 0.0057 0.0068 |     |     | LS     | -0.0111 0.0118 | -0.0119 0.0152 |
|     |     | MSP    | 0.0008 0.0045 | -0.0054 0.0068 |     |     | MSP    | -0.0000 0.0059 | 0.0015 0.0097 |
| 200 | ML  | 0.0288 0.0295 | -0.0121 0.0393 | 200 | ML  | 0.0012 0.0044 | 0.0057 0.0068 | -0.0063 0.0097 |
|     |     | LS     | -0.0186 0.0365 | -0.0496 0.0465 |     |     | LS     | 0.0016 0.0034 | 0.0189 0.0335 |
|     |     | MSP    | 0.0228 0.0168 | 0.0043 0.0209 |     |     | MSP    | -0.0043 0.0146 | 0.0225 0.0216 |
| 0.1 | 30  | ML     | 0.0016 0.0324 | 0.0118 0.0397 | 0.3 | 30  | ML     | -0.0172 0.0094 | -0.0034 0.0097 | 0.0058 0.0208 |
|     |     | LS     | -0.0063 0.0208 | -0.0134 0.0198 |     |     | LS     | -0.0073 0.0112 | -0.0119 0.0111 | 0.0049 0.0215 |
|     |     | MSP    | 0.0060 0.0168 | 0.0164 0.0211 |     |     | MSP    | -0.0079 0.0095 | 0.0047 0.0097 | 0.0022 0.0065 |
| 100 | ML  | 0.0172 0.0094 | -0.0034 0.0097 | 100 | ML  | 0.0011 0.0054 | 0.0014 0.0074 | 0.0099 0.0105 |
|     |     | LS     | -0.0073 0.0112 | -0.0119 0.0111 |     |     | LS     | 0.0027 0.0079 | 0.0095 0.0081 | 0.0104 0.0123 |
|     |     | MSP    | 0.0048 0.0055 | 0.0067 0.0075 |     |     | MSP    | 0.0045 0.0089 | 0.0208 0.0109 |
Figure 2. TTT plot of the air-conditioning system data.

For evaluated models, Table 8 presents the Akaike information criterion (AIC), negative log-likelihood (N.Log-lik) values, and ML estimates of the parameters for both complete and censored data cases, where we assume that the largest \( n - r \), \( (n - r = 0, 2, 4, 8, 10) \) observations of the data are censored.

According to the analysis results provided by Table 8, the SMD is a more appropriate model than other models for modeling the dataset with minimum AIC and N.Log-lik.

| Model   | Number of Censored Observations \( (n - r) \) | AIC       | N.Log-lik. | ML Estimates |
|---------|---------------------------------------------|-----------|------------|--------------|
| SMD     | 0                                           | 290.89829 | 143.44914  | \( \hat{\alpha} = 0.26191 \) | \( \hat{\beta} = 76.86239 \) |
| Weibull | 291.91248                                    | 143.95624 | \( \hat{\theta} = 79.92387 \) | \( \hat{\lambda} = 1.12314 \) |
| Gamma   | 292.18017                                    | 144.09008 | \( \hat{\xi} = 1.13257 \) | \( \hat{\eta} = 67.82347 \) |
| Log_Normal | 299.24989                                 | 147.62949 | \( \hat{\mu} = 3.83887 \) | \( \hat{\sigma} = 1.25646 \) |
| SMD     | 2                                           | 273.95361 | 125.76762  | \( \hat{\alpha} = 2.832E-7 \) | \( \hat{\beta} = 82.96066 \) |
| Weibull | 273.88707                                    | 125.57470 | \( \hat{\theta} = 83.58582 \) | \( \hat{\lambda} = 1.02924 \) |
| Gamma   | 273.91031                                    | 125.64802 | \( \hat{\xi} = 1.03455 \) | \( \hat{\eta} = 80.02561 \) |
| Log_Normal | 278.42077                                 | 129.60015 | \( \hat{\mu} = 3.88775 \) | \( \hat{\sigma} = 1.31213 \) |
| SMD     | 4                                           | 252.46701 | 111.63635  | \( \hat{\alpha} = 1.016E-7 \) | \( \hat{\beta} = 90.17912 \) |
| Weibull | 252.72729                                    | 111.96940 | \( \hat{\theta} = 89.31712 \) | \( \hat{\lambda} = 0.94426 \) |
| Gamma   | 252.61474                                    | 111.85511 | \( \hat{\xi} = 0.94115 \) | \( \hat{\eta} = 96.57372 \) |
| Log_Normal | 256.07815                                 | 114.52497 | \( \hat{\mu} = 3.95097 \) | \( \hat{\sigma} = 1.40410 \) |
| SMD     | 8                                           | 211.16566 | 89.36563   | \( \hat{\alpha} = 2.262E-8 \) | \( \hat{\beta} = 109.16357 \) |
| Weibull | 212.01071                                    | 89.63544  | \( \hat{\theta} = 110.88645 \) | \( \hat{\lambda} = 0.80310 \) |
| Gamma   | 211.85849                                    | 89.77056  | \( \hat{\xi} = 0.77727 \) | \( \hat{\eta} = 151.76537 \) |
| Log_Normal | 212.76451                                 | 89.56906  | \( \hat{\mu} = 4.14427 \) | \( \hat{\sigma} = 1.63551 \) |
| SMD     | 10                                          | 212.39905 | 80.18482   | \( \hat{\alpha} = 9.619E-8 \) | \( \hat{\beta} = 122.00044 \) |
| Weibull | 212.39905                                    | 80.18482  | \( \hat{\theta} = 130.37211 \) | \( \hat{\lambda} = 0.74240 \) |
| Gamma   | 212.39905                                    | 80.18482  | \( \hat{\xi} = 0.70684 \) | \( \hat{\eta} = 199.75125 \) |
| Log_Normal | 193.03444                                 | 78.96027  | \( \hat{\mu} = 4.29017 \) | \( \hat{\sigma} = 1.78316 \) |

Table 7. K-S Test results for the air-conditioning system data

| Model       | K-S Test | p-value |
|-------------|----------|---------|
| SMD         | 0.0944   | 0.9513  |
| Weibull     | 0.0883   | 0.9721  |
| Gamma       | 0.0770   | 0.9933  |
| Log-Normal  | 0.1264   | 0.7350  |

values in all cases except one which the censored observation number is 2.

CONCLUSION

In this paper, the problem of estimating the parameters of the SMD under the Type II censoring scheme has been considered. We have obtained various estimators for the unknown parameters of the SMD based on the most frequently used estimation methodologies in the literature such as ML, LS, and MSE. We have also compared the estimation efficiencies of these estimators via a comprehensive simulation study. The results of the simulation study carried out on the different sample-sizes small, medium and large have revealed that all estimators obtained with the study are able to quite satisfactorily estimate the unknown parameters \( \alpha \) and \( \beta \), and that these estimators are asymptotically
unbiased and consistent. Furthermore, even if the censoring proportion increases, the ML and MSP estimators satisfactorily estimate the model parameters. In addition to these results, we give an illustrative example performed on actual data to exemplify the data modeling with SMD under complete and Type-II censored data cases. The application results have shown that SMD is a possible alternative to the famous lifetime models such as Weibull, Gamma, Log-Normal, etc. Hence, we can say that the SMD is a helpful probability model for modeling the lifetime datasets in both complete and censored data cases.

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The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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There are no ethical issues with the publication of this manuscript.

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