Sharp multiplicative inequalities with BMO I

Dmitriy Stolyarov  Vasily Vasyunin  Pavel Zatitskiy

January 28, 2020

Abstract

We find the best possible constant $C$ in the inequality $\|\varphi\|_{L^r} \leqslant C \|\varphi\|_{L^p} \|\varphi\|_{BMO}^{1-p/r}$, where $2 \leqslant r$ and $p < r$. We employ the Bellman function technique to solve this problem in the case of an interval and then transfer our results to the circle and the line.

2010 MSC subject classification: 42B35, 60G45.
Keywords: bounded mean oscillation, Bellman function, interpolation.

1 Introduction

The space $BMO$ plays an important role in analysis. It serves as a good substitute for $L^\infty$ in the endpoint estimates that fail for the latter space. On the other hand, some estimates that are trivial for $L^\infty$ become more interesting for the case of $BMO$. Consider the classical multiplicative inequality

$$\|\varphi\|_{L^r} \leqslant \|\varphi\|_{L^p}^{\frac{p}{r}} \|\varphi\|_{BMO}^{1-\frac{p}{r}}, \quad 0 < p < r < \infty. \quad (1.1)$$

This inequality holds true for any function $\varphi$ on a measurable space, and follows from the simple estimate $|\varphi(x)| \leqslant \|\varphi\|_{L^\infty}$. The bound (1.1) is sharp.

The inequality

$$\|\varphi\|_{L^r} \leqslant C \|\varphi\|_{L^p}^{\frac{p}{r}} \|\varphi\|_{BMO}^{1-\frac{p}{r}}, \quad 1 \leqslant p < r < \infty, \quad (1.2)$$

is less trivial. Here $C$ is a constant that should depend neither on $\varphi$, nor on $p$, nor on $r$. For this inequality, one needs to specify the domain of $\varphi$. In the case of the Euclidean space $\mathbb{R}^d$, the inequality

$$\|\varphi\|_{L^r(\mathbb{R}^d)} \leqslant C_d \|\varphi\|_{L^p(\mathbb{R}^d)}^{\frac{p}{r}} \|\varphi\|_{BMO(\mathbb{R}^d)}^{1-\frac{p}{r}}, \quad 1 \leqslant p < r < \infty,$$

was first obtained in [1]. It plays an important role in interpolation and extrapolation theory for BMO, see [7]. We note that the exponents $\frac{p}{r}$ and $1 - \frac{p}{r}$ in the inequalities above are also dictated by interpolation theory, see [7]. Alternatively, one may restore the exponents from the dilation invariance of (1.1) and (1.2).

Our aim is to obtain sharp versions of (1.2). For that we need to specify the choice of the BMO norm.

The progress in computation of sharp constants in the John–Nirenberg type inequalities for BMO in higher dimensions is scant. In fact, even the asymptotical behavior of these constants is unknown, see [2]. So, we limit ourselves to the case $d = 1$. Let us consider the case of the BMO space on an interval $I$. A real-valued function $\varphi \in L^1(I)$ belongs to $BMO(I)$ provided the quantity

$$\|\varphi\|_{BMO(I)}^2 = \sup_{\text{subinterval of } I} \langle |\varphi - \langle \varphi \rangle_I|^2 \rangle_I$$

(1.3)

*Support by the Russian Science Foundation grant 19-71-10023.
is finite. Here and in what follows we use the notation \( \langle \psi \rangle_E \) to denote the average of a function \( \psi \) over a set \( E \) of positive measure, that is

\[
\langle \psi \rangle_E = \frac{1}{|E|} \int_E \psi.
\]

The choice of the exponent 2 in the definition (1.3) does not affect the validity of (1.2) since, by the John–Nirenberg inequality, one obtains an equivalent norm for other values of \( p \). However, this choice is important for sharp constants. Most of the work related to sharp constants for BMO functions was done with the quadratic norm. However, see \[5\], \[6\], \[11\], and \[13\] for the results concerning the classical 1-norm and arbitrary \( p \)-norm.

We warn the reader that the inequality (1.2) cannot be true for functions on the interval since the seminorm (1.3) vanishes on constant functions. So, this inequality needs a slight modification, which is our first main result.

**Theorem 1.1.** The inequality

\[
\|\varphi\|_{L^r(I)} \leq \left( \frac{\Gamma(r + 1)}{\Gamma(p + 1)} \right)^{\frac{2}{p}} \|\varphi\|_{L^p(I)}^{\frac{2}{p}} \|\varphi\|_{\text{BMO}(I)}^{1 - \frac{2}{p}}, \quad \langle \varphi \rangle_I = 0,
\]

holds true and is sharp when \( p \geq 1 \) and \( \max(2, p) \leq r < \infty \).

We must say a couple of words about our tools. We will be using the Bellman function method. It allows to derive sharp inequalities for non-compact infinite dimensional objects (such as the unit ball of the BMO space) from certain finite dimensional boundary value problems. The papers \[8\] and \[9\] laid the foundation of the method. We refer the reader to \[12\] and \[19\] for the basics of the theory and to \[10\] for the probabilistic point of view (in the probability theory, this technique is usually called the Burkholder method).

The Bellman functions are convenient for BMO problems. Their successful application lead to the computation of sharp constants in various forms of the John–Nirenberg inequalities and related problems, see \[14\], \[15\], and \[18\]. Later the branch of the Bellman function method that works with BMO problems was converted into a theory in \[3\] and \[16\].

The Bellman function appearing in our proof of Theorem 1.1 is interesting in itself. All the Bellman functions in the papers cited in the previous paragraph are two-dimensional, whereas our function is three-dimensional. The two-dimensional optimization problems related to BMO are well-understood (see \[3\] and \[4\]), which is not quite true for higher dimensional ones. The difficulty increases dramatically. Luckily, the Bellman function appearing in the proof of Theorem 1.1 is tractable.

Using the technique developed in \[17\], we will transfer our results to the circle and the line.

**Theorem 1.2.** The inequality

\[
\|\varphi\|_{L^r(T)} \leq \left( \frac{\Gamma(r + 1)}{\Gamma(p + 1)} \right)^{\frac{2}{p}} \|\varphi\|_{L^p(T)}^{\frac{2}{p}} \|\varphi\|_{\text{BMO}(T)}^{1 - \frac{2}{p}}, \quad \varphi \in \text{BMO}(T), \quad \int_T \varphi = 0,
\]

holds true and is sharp when \( p \geq 1 \) and \( \max(2, p) \leq r < \infty \).

**Theorem 1.3.** The inequality

\[
\|\varphi\|_{L^r(R)} \leq \left( \frac{\Gamma(r + 1)}{\Gamma(p + 1)} \right)^{\frac{2}{p}} \|\varphi\|_{L^p(R)}^{\frac{2}{p}} \|\varphi\|_{\text{BMO}(R)}^{1 - \frac{2}{p}}, \quad \varphi \in L^p(R),
\]

holds true and is sharp when \( p \geq 1 \) and \( \max(2, p) \leq r < \infty \).
The main purpose of this paper is to find an explicit formula for formulas (3.4), (3.14), \( \phi \). In other words, the set of functions we introduce the main character. This is the Bellman function.

Section 4. Section 5 contains the proof of Theorem 1.1. Section 6 provides the derivation of Theorems 1.2 and 1.3 from Theorem 1.1.

2 Optimization problem

We introduce the main character. This is the Bellman function \( B_{p,r}: \mathbb{R}^3 \to \mathbb{R} \cup \{\pm \infty\} \). It is defined as

\[
B_{p,r}(x_1, x_2, x_3) = \sup \left\{ \langle |\varphi|^p \rangle : \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle = x_1, \langle \varphi^2 \rangle = x_2, \langle |\varphi|^p \rangle = x_3 \right\}. \tag{2.1}
\]

We say that \( \varphi \) is a test function for the point \( x \in \mathbb{R}^3 \) if

\[\|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle = x_1, \langle \varphi^2 \rangle = x_2, \langle |\varphi|^p \rangle = x_3.\]

The main purpose of this paper is to find an explicit formula for \( B_{p,r} \). We state this result by referring to the formulas appearing in the forthcoming sections.

**Theorem 2.1.** For \( (r - 2)(p - r) < 0 \) the function \( B_{p,r} \) coincides with the function \( G \) given by formulas (3.14), (3.1), and (3.16) on the domains (3.1). The functions \( m_r \) and \( k_r \) are defined by (2.3) and (2.5) respectively.

For the case \( (r - 2)(p - r) > 0 \) the function \( G \) constructed by formulas (3.14) and (3.11) will coincide with the minimal Bellman function, see Section 4.

To describe the function \( B_{p,r} \), we will need two auxiliary Bellman functions \( B^\pm_{p,r}: \mathbb{R}^2 \to \mathbb{R} \cup \{\pm \infty\} \) defined by the rule

\[
B^+_{p,r}(x_1, x_2) = \sup \left\{ \langle |\varphi|^p \rangle : \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle = x_1, \langle \varphi^2 \rangle = x_2 \right\}, \tag{2.2}
\]
\[
B^-_{p,r}(x_1, x_2) = \inf \left\{ \langle |\varphi|^p \rangle : \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle = x_1, \langle \varphi^2 \rangle = x_2 \right\}. \tag{2.3}
\]

The latter two functions were studied in detail in [14]. We survey these results since they will play an important role in our study.

2.1 Description of \( B^\pm_{p,r} \)

The domain of both functions \( B^\pm_{p,r} \) is

\[\Omega^2_\varepsilon = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1^2 \leq x_2 \leq x_1^2 + \varepsilon^2 \right\}.\]

By the domain of a Bellman function we mean the set of \( x \) where the function is not equal to \(-\infty\). In other words, the set of functions \( \varphi \) over which we optimize in formulas (2.2) and (2.3) is non-empty for these \( x \) (there exists at least one \( \varphi \) such that \( \langle \varphi \rangle = x_1, \langle \varphi^2 \rangle = x_2, \) and \( \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon \)). Both functions also satisfy the boundary condition \( B^\pm_{p,r}(t, t^2) = |t|^p, t \in \mathbb{R} \). From now on we omit the index \( \varepsilon \) in the notation of domains and functions.
To describe $B^\pm$, we need some auxilliary functions. For $p > 1$ and $u \geq 0$ define

$$m_p(u) = \frac{p}{\varepsilon} \int_u^{+\infty} e^{(u-t)/\varepsilon} t^{p-1} \, dt,$$  \hspace{1cm} (2.4)

$$k_p(u) = \frac{p}{\varepsilon} \int_u^0 e^{(t-u)/\varepsilon} t^{p-1} \, dt.$$  \hspace{1cm} (2.5)

For any $u \in \mathbb{R}$ we denote the segment connecting the points $(u, u^2)$ with $(u + \varepsilon, (u + \varepsilon)^2 + \varepsilon^2)$ by $S_+(u)$ and the segment connecting $(u, u^2)$ with $(u - \varepsilon, (u - \varepsilon)^2 + \varepsilon^2)$ by $S_-(u)$. Note that these segments touch upon the upper boundary of $\Omega^2$, that is the parabola $x_2 = x_1^2 + \varepsilon^2$. For any $(x_1, x_2) \in \Omega^2$ there exist unique $u_+ = u_+(x_1, x_2) \in \mathbb{R}$ such that $(x_1, x_2) \in S_+(u_+), u_+ \leq u_-.$

Define the function $A_{m_p}$ on $\Omega^2$ in the following way. We put

$$A_{m_p}(x) = u^p + m_p(u)(x_1 - u), \quad x \in S_p(u), \quad u \geq 0,$$

$$A_{m_p}(x) = |u|^p - m_p(|u|)(x_1 - u), \quad x \in S_p(u), \quad u \leq 0.$$  \hspace{1cm} (2.6)

In the triangle between the tangents $S_-(0)$ and $S_+(0)$, we set

$$A_{m_p}(x) = \frac{m_p(0)}{2\varepsilon} x_2, \quad |x_1| \leq \varepsilon, \quad 2\varepsilon|x_1| \leq x_2 \leq x_1^2 + \varepsilon^2.$$  \hspace{1cm} (2.7)

Formulas (2.4) and (2.5) define the function $A_{m_p}$ on the entire domain $\Omega^2$. Note that $A_{m_p}$ is $C^1$-smooth and even with respect to $x_1$.

Define the function $A_{k_p}$ on $\Omega^2$ as follows. We put

$$A_{k_p}(x) = u^p + k_p(u)(x_1 - u), \quad x \in S_p(u), \quad u \leq \varepsilon,$$

$$A_{k_p}(x) = |u|^p - k_p(|u|)(x_1 - u), \quad x \in S_p(u), \quad u \leq -\varepsilon.$$  \hspace{1cm} (2.8)

In the domain $x_2 \leq \varepsilon^2$, we set

$$A_{k_p}(x_1, x_2) = x_2^{p/2}, \quad x_1^2 \leq x_2 \leq x_2^2.$$  \hspace{1cm} (2.9)

Formulas (2.8) and (2.9) define the function $A_{k_p}$ on the entire domain $\Omega^2$. This function is also $C^1$-smooth and even with respect to $x_1$.

Now we are ready to describe the functions $B^\pm$:

$$B^+ = \begin{cases} A_{m_p}, & \text{if} \quad 2 \leq p < \infty \\ A_{k_p}, & \text{if} \quad 1 < p \leq 2 \end{cases} \quad \text{and} \quad B^- = \begin{cases} A_{k_p}, & \text{if} \quad 2 \leq p < \infty \\ A_{m_p}, & \text{if} \quad 1 < p \leq 2. \end{cases}$$  \hspace{1cm} (2.10)

Here we collect some useful relations for derivatives of the functions $m_p$ and $k_p$:

$$m_p''(u) = \frac{p(p-1)(p-2)}{\varepsilon} \int_u^{+\infty} e^{(u-t)/\varepsilon} t^{p-3} \, dt,$$  \hspace{1cm} (2.11)

$$k_p''(u) = \frac{p(p-2)}{\varepsilon} e^{u-\varepsilon/\varepsilon} t^{p-3} \varepsilon + \frac{p(p-1)(p-2)}{\varepsilon} \int_u^0 e^{(t-u)/\varepsilon} t^{p-3} \, dt,$$  \hspace{1cm} (2.12)

$$- \varepsilon m_p'(u) + m_p(u) = pu^{p-1}, \quad \varepsilon k_p'(u) + k_p(u) = pu^{p-1},$$  \hspace{1cm} (2.13)

$$\varepsilon (m_p^{(\ell+1)} + k_p^{(\ell+1)}) = m_p^{(\ell)} - k_p^{(\ell)}, \quad \ell \geq 0,$$  \hspace{1cm} (2.14)

where the notation $g^{(k)}$ means the $k$-th derivative of $g$.  \hspace{1cm}
2.2 Simple properties of the optimization problem

The domain of the function $B_{p,r;x}$ introduced in (2.1) is described in terms of the functions $B_{p,r}^\pm$ from (2.2).

Proposition 2.2. The set

$$\Omega^3_p = \left\{ x \in \mathbb{R}^3 : (x_1, x_2) \in \Omega^2_p, x_3 \in [B_{p,r}(x_1, x_2), B_{p,r}^+(x_1, x_2)] \right\},$$

is the domain of the Bellman function $B_{p,r;x}$.

At this point we note that for $x \notin \Omega^2_p$ there are no test functions and we definitely have $B_{p,r}(x) = -\infty$ in this case. On the other hand, a test function $\varphi$ for any $x \in \Omega^2_p$ with $x_3 = B_{p,r}^+(x_1, x_2)$ was constructed in [14]. In other words, we have proved that the points on the boundary of $\Omega^2_p$ belong to the domain of $B_{p,r}$. We will complete the proof of Proposition 2.2 after formulating Proposition 2.3.

A function $G : \omega \to \mathbb{R} \cup \{\pm \infty\}$, where $\omega \subset \mathbb{R}^d$ is an arbitrary set, is called locally concave if for any segment $\ell \subset \omega$, the restricted function $G|_{\ell}$ is concave.

We collect standard facts concerning Bellman functions of such kind.

Proposition 2.3. 1. The function $B_{p,r}$ satisfies the boundary conditions on the skeleton of $\Omega^3_p$:

$$B_{p,r}(t, t^2, |t|^r) = |t|^r, \quad t \in \mathbb{R}. \quad (2.15)$$

2. The function $B_{p,r}$ is locally concave on $\Omega^3_p$.

3. The function $B_{p,r}$ is the pointwise minimal among all locally concave on $\Omega^3_p$ functions $G$ that satisfy the boundary condition (2.15).

The first statement of Proposition 2.3 follows from the fact that if $\langle \varphi^2 \rangle = \langle \varphi \rangle^2$, then $\varphi$ is constant function and $\| \varphi \| = \| \langle \varphi \rangle \|$. The second one is not so trivial. It is a consequence of the following fact (see Corollary 3.13 in [10]): if $\varphi_i \in \text{BMO}(I)$ with $\| \varphi_i \|_{\text{BMO}(I)} \leq \varepsilon$, $i = 1, 2$, and the segment with the endpoints $(\varphi_i, \varphi_i^2)$ lies in $\Omega^3_p$, then for any $\theta \in (0, 1)$ there exists a function $\varphi \in \text{BMO}(I)$ such that

$$\| s \in I : \varphi(s) > \lambda | = \theta, | s \in I : \varphi_1(s) > \lambda | + (1 - \theta) | s \in I : \varphi_2(s) > \lambda |, \lambda \in \mathbb{R}, \quad \| \varphi \|_{\text{BMO}(I)} \leq \varepsilon.$$ 

This fact also leads to the existence of a test function for any $x \in \Omega^3_p$. Indeed, any $x \in \Omega^3_p$ may be represented as a convex combination $x = \theta y + (1 - \theta) z$, where $y$ and $z$ lie on the boundary of $\Omega^3_p$ and $x$, $y$ and $z$ have one and the same first two coordinates. By the results of [14], we know that there exist test functions $\varphi_1$ and $\varphi_2$ for $y$ and $z$ correspondingly. Application of Corollary 3.13 from [10] cited in the previous paragraph to $\varphi_1$ and $\varphi_2$ produces a test function for $x$ and completes the proof of Proposition 2.2.

The third statement of Proposition 2.3 is usual for the Bellman function technique and is proved by the so-called Bellman induction, see e.g. [10].

In view of Proposition 2.3 it suffices to construct a $C^1$-smooth function $G : \Omega^3_p \to \mathbb{R}$ such that

1) the function $G$ is locally concave on $\Omega^3_p$;

2) the function $G$ fulfills the boundary conditions (2.15);

3) for any point $x \in \Omega^3_p$ with $x_3 = B_{p,r}^+(x_1, x_2)$, there is a function $\varphi_x \in \text{BMO}(I)$ such that

$$\| \varphi_x \|_{\text{BMO}(I)} \leq \varepsilon, \quad \langle \varphi_x \rangle_1 = x_1, \quad \langle \varphi_x^2 \rangle_1 = x_2, \quad \langle |\varphi_x|^r \rangle_1 = x_3, \quad \langle |\varphi_x|^r \rangle_1 = G(x); \quad (2.16)$$

4) for any point $x \in \Omega^3_p$, there exists a two-dimensional plane $L[x] \subset \mathbb{R}^3, x \in L[x]$, such that $G$ is linear on the connected component of $L[x] \cap \Omega^3_p$ containing $x$. 

5
If all of the above requirements hold, then, \( G = B_{p,r} \). Indeed, the inequality \( G(x) \geq B_{p,r}(x), x \in \Omega^3_x \) follows from conditions 1), 2) and the third statement of Proposition 2.3. The reverse inequality \( G(x) \leq B_{p,r}(x) \) for \( x \in \Omega^3_x \) with \( x_3 = B_{p,r}(x_1, x_2) \) follows from condition 3) and the definition of the Bellman function \( B_{p,r} \). For other \( x \in \Omega^3_x \), the inequality \( G(x) \leq B_{p,r}(x) \) is implied by condition 4): \( G \) is linear on \( L[x] \) while \( B_{p,r} \) is concave there. We will provide more details in Subsection 3.4.

A function \( \varphi_x \) satisfying (2.10) is called an optimizer for \( G \) at \( x \).

## 3 Solution to the optimization problem

Our aim is to construct the function \( G \) on \( \Omega^2_x \) described at the end of the previous section. We split \( \Omega^2_x \) into three subdomains \( \Xi_+, \Xi_0, \Xi_- \):

\[
\begin{align*}
\Xi_0 &= \left\{ x \in \Omega^2_x : |x_1| \leq 2\varepsilon, x_2 \geq 4\varepsilon|x_1| - 3\varepsilon^2, (p - 2)(x_3 - \varepsilon^2 - x_2 - \frac{4\varepsilon}{3}m_p(\varepsilon)) \geq 0 \right\}, \\
\Xi_+ &= \left\{ x \in \Omega^2_x \setminus \Xi_0 : x_1 > 0 \right\}, \\
\Xi_- &= \left\{ x \in \Omega^2_x \setminus \Xi_0 : x_1 < 0 \right\}.
\end{align*}
\]

(3.1)

The latter condition defining \( \Xi_0 \) may look strange. However, it is needed to distinguish the cases \( p \geq 2 \) and \( p < 2 \).

We will construct \( G \) on each of these domains by an individual formula, verify the local concavity on each of the domains, and also prove that the three parts provide a \( C^1 \)-smooth function on the union of the domains. This will lead to a locally concave \( C^1 \)-smooth function \( G \) on \( \Omega^2_x \).

### 3.1 Construction on \( \Xi_+ \)

Let \( u \geq \varepsilon \). Consider the two-dimensional plane \( L_u \) that passes through \( U = (u, u^2, w^p) \) and the points

\[
\begin{align*}
U_+ &= \left(u + \varepsilon, (u + \varepsilon)^2 + \varepsilon^2, A_m(u + \varepsilon, (u + \varepsilon)^2 + \varepsilon^2) \right) = \left(u + \varepsilon, (u + \varepsilon)^2 + \varepsilon^2, w^p + \varepsilon m_p(u) \right), \\
U_- &= \left(u - \varepsilon, (u - \varepsilon)^2 + \varepsilon^2, A_k(u - \varepsilon, (u - \varepsilon)^2 + \varepsilon^2) \right) = \left(u - \varepsilon, (u - \varepsilon)^2 + \varepsilon^2, w^p - \varepsilon k_p(u) \right).
\end{align*}
\]

(3.2)

The equation of \( L_u \) is

\[
\begin{align*}
x_3 = w^p + \frac{k_p}{4\varepsilon} \cdot (x_2 - x_1^2 + (x_1 - u)^2) + \frac{m_p + k_p}{2} \cdot (x_1 - u).
\end{align*}
\]

(3.3)

Here and in what follows, we omit the argument of \( m_p \) and \( k_p \) if this does not lead to ambiguity.

Let \( T_u \) be the intersection of \( \Omega^2_x \) with the triangle with the vertices \( U, U_-, U_+ \). So, \( T_u \) is a curvilinear triangle. We define the function \( G \) on \( T_u \) by linearity:

\[
G(x_1, x_2, x_3) = w^p + \frac{m_p - k_p}{4\varepsilon} \cdot (x_2 - x_1^2 + (x_1 - u)^2) + \frac{m_p + k_p}{2} \cdot (x_1 - u).
\]

(3.4)

Note that equations (3.3) and (3.4) completely define \( G \) on \( \Xi_+ \) since the latter domain is foliated by the triangles \( T_u \), when \( u \) runs through \((\varepsilon, +\infty)\). Let us prove this. The triangle \( T_u \) is simply the boundary between \( \Xi_0 \) and \( \Xi_+ \). Recall that for any \( x \in \Xi_+ \) there exist unique \( u_ \in \mathbb{R} \) such that \( (x_1, x_2) \in S_\pm(u_ \pm) \). For \( x_1, x_2 \) fixed we will verify that \( x_3 \) defined by (3.3) is a monotone function of \( u \in [\max(\varepsilon, u_+), u_-] \) (see (3.9) further). If \( u_+ \geq \varepsilon \), then \( x_3 \) as a function of \( u \) runs from \( A_m(x_1, x_2) \) to \( A_k(u_+, x_1, x_2) \) when \( u \in [u_+, u_-] \). If \( u_+ \leq \varepsilon \), then it runs from \( w^p + \frac{x_2 - \varepsilon^2}{4\varepsilon}m_p(\varepsilon) \) to \( A_k(u_+, x_1, x_2) \) when \( u \in [\varepsilon, u_-] \). We have proved that the \( T_u \), \( u \in (\varepsilon, +\infty) \), foliate \( \Xi_+ \).
In order to show the local concavity of $G$, let us verify that the Hessian of $G$ is either non-positive or non-negative on the entire domain $\Xi_+$. The restrictions of this function to the planes $L_u$, which are always transversal to the $x_3$ axis, are linear. Thus, it suffices to show that the second derivative of $G$ with respect to $x_3$ does not change its sign in $\Xi_+$.

We differentiate (3.3) with respect to $x_3$ and get

$$ 1 = u_{x_3} \left( p u^{p-1} + \frac{m'_p - k'_p}{4\varepsilon} \cdot (x_2 - x_1^2 + (x_1 - u)^2) - \frac{m_p - k_p}{2\varepsilon} \cdot (x_1 - u) + \frac{m' + k'_p}{2} \cdot (x_1 - u) - \frac{m_p + k_p}{2} \right), $$

which, with the help of (2.13), may be restated as

$$ 1 = u_{x_3} \cdot \frac{m'_p - k'_p}{4\varepsilon} \cdot (x_2 - x_1^2 + (x_1 - u)^2 - 2\varepsilon^2) $$

$$ = u_{x_3} \cdot \frac{m'' + k''_p}{4} \cdot (x_2 - x_1^2 + (x_1 - u)^2 - 2\varepsilon^2). $$

Similarly, we differentiate (3.3) with respect to $x_3$ and get

$$ G_{x_3}(x_1, x_2, x_3) = u_{x_3} \cdot \frac{m'' + k''_p}{4} \cdot (x_2 - x_1^2 + (x_1 - u)^2 - 2\varepsilon^2). $$

Equations (3.5) and (3.6) imply

$$ G_{x_3}(x_1, x_2, x_3) = \frac{m'' + k''_p}{m''_p + k''_p}. $$

It follows from (2.11) and (2.12) that

$$ \text{sign} \left( m''_p(u) \right) = \text{sign} \left( k''_p(u) \right) = \text{sign} (p - 2) $$

when $u \geq \varepsilon$. We have $x_2 - x_1^2 + (x_1 - u)^2 - 2\varepsilon^2 \leq 0$ for $x \in T_u$. Thus, (3.5) implies

$$ \text{sign} (u_{x_3}) = -\text{sign} (p - 2), \quad x \in T_u. $$

Therefore,

$$ \text{sign} \left( G_{x_3 u} \right) = -\text{sign} \left( (p - 2) G_{x_3 u} \right). $$

Let us compute that latter sign, using formulas (3.7) and (2.14):

$$ G_{x_3 u}(x) = \frac{(m''_p - k''_p)(m''_p + k''_p) - (m''_p - k''_p)(m''_p + k''_p)}{\varepsilon (m''_p + k''_p)^2} $$

$$ = \frac{2(k''_p m''_p - m''_p k''_p)}{\varepsilon (m''_p + k''_p)^2} = \frac{2k''_p m''_p}{\varepsilon (m''_p + k''_p)^2}. $$

Thus, we need to investigate the sign of the expression $\frac{m''_p}{k''_p} - \frac{m''_p}{k''_p}$. Without loss of generality, we may assume $\varepsilon = 1$ (we may always substitute $u = u/\varepsilon$). The following notation is convenient:

$$ I_1 = \int_0^{e^{-t}} e^{-t} u^{p-3} dt, \quad \bar{I}_1 = \frac{\partial}{\partial p} I_1 = \int_0^{e^{-t}} e^{-t} u^{p-3} \log t dt $$

$$ I_2 = \int_1^{e^{-t}} u^{e^{-t} p-3} dt, \quad \bar{I}_2 = \frac{\partial}{\partial p} I_2 = \int_1^{e^{-t}} e^{-t} u^{p-3} \log t dt. $$
Consider the derivative
\[
\frac{\partial}{\partial p} \frac{m''_p}{k''_p} = \frac{\partial}{\partial p} \left( \frac{(p-1)I_1}{e^{1-u} + (p-1)I_2} \right) = \\
= \frac{(I_1 + (p-1)I_1)(e^{1-u} + (p-1)I_2) - (p-1)I_1(I_2 + (p-1)I_2)}{(e^{1-u} + (p-1)I_2)^2} = \\
= \frac{(I_1 + (p-1)I_1)e^{1-u} + (p-1)^2(I_2 - I_2 I_1)}{(e^{1-u} + (p-1)I_2)^2}.
\]

Note that the quantities \(I_1, I_2, \tilde{I}_1, \tilde{I}_2\) are non-negative. Moreover, the inequalities
\[
\tilde{I}_1 > I_1 \log u, \quad \tilde{I}_2 \leq I_2 \log u
\]
hold true. Therefore, the expression in (3.12) is non-negative. Consequently,
\[
\text{sign} \left( \frac{m''_p}{k''_p} - \frac{m''_p}{k''_p} \right) = \text{sign}(r-p).
\]

Finally, we investigate the sign of \(G_{x_1x_3}\):
\[
\text{sign} \left( G_{x_1x_3} \right) = -\text{sign} \left( (p-2)G_{x_1u} \right) = -\text{sign}(p-2) \cdot \text{sign} \left( k''_r \right) \cdot \text{sign}(k''_p) \cdot \text{sign} \left( \frac{m''_r}{k''_r} - \frac{m''_p}{k''_p} \right) \cdot \text{sign}(r-2) \cdot \text{sign}(p-r).
\]

Thus, the constructed function \(G\) is locally concave on \(\Xi_+\) provided \((r-2)(p-r) < 0\) and locally convex if \((r-2)(p-r) > 0\).

By symmetry, we define the function \(G\) on \(\Xi_-\):
\[
G(x_1, x_2, x_3) = G(-x_1, x_2, x_3), \quad x \in \Xi_-.
\]

Thus, the concavity (or convexity) of this symmetrized function on \(\Xi_-\) is the same as on \(\Xi_+\).

### 3.2 Construction on \(\Xi_0\)

The point \(U = (u, u^2, v^p)\) lies on the skeleton of \(\Omega_2^3\) for any \(u \in [0, \varepsilon]\). Let \(\bar{L}_u\) be the two dimensional plane that passes through \(U\), \(U_+ = (u + \varepsilon, (u + \varepsilon)^2 + \varepsilon^2, u^p + \varepsilon m_p(u))\), and \(\bar{U} = (-u, u^2, v^p)\). Note that the segments connecting \(U\) with \(U_+\) lie on the boundary of \(\Omega_2^3\).

The plane \(\bar{L}_u\) is defined by the equation
\[
x_3 = u^p + \frac{x_2 - u^2}{2(u + \varepsilon)} m_p(u).
\]

It contains the point \(U_+\) that is symmetric to \(U_+\) with respect to the coordinate plane \(x_1 = 0\). Let \(\bar{T}_u\) be the intersection of \(\Omega_2^3\) with the quadrilateral with the vertices \(U, U_+, \bar{U}\) and \(U_+\). So, \(\bar{T}_u\) is a curvilinear quadrilateral for any \(u \in (0, \varepsilon)\). We define the function \(G\) by linearity on \(\bar{T}_u\):
\[
G(x_1, x_2, x_3) = u^p + \frac{x_2 - u^2}{2(u + \varepsilon)} m_p(u), \quad x \in \bar{T}_u.
\]
We state that the domain $\Xi_0$ is foliated by $\hat{T}_u$, $u \in [0, \varepsilon]$. Let us show this. We first note that $\hat{T}_u$ is the common boundary of $\Xi_0$ and $\Xi \cup \Xi_+$ Recall that for any $x \in \Xi_0$ there exist unique $u_\pm \in \mathbb{R}$ such that $(x_1, x_2) \in S_u(\pm)$. For $x_1, x_2$ fixed we will verify that $x_3$ defined by (3.15) is a monotone function of $u \in [\max(0, u_+), \min(\sqrt{x_2}, \varepsilon)]$ (see 3.20 further). If $x_2 \leq \varepsilon^2$, then $x_3$ as a function of $u$ runs from $A_{m_p}(x_1, x_2)$ to $A_{\varepsilon}(x_1, x_2)$ when $u \in [\max(0, u_+), \sqrt{x_2}]$. If $x_2 \geq \varepsilon^2$, then it runs from $A_{m_p}(x_1, x_2)$ to $\varepsilon^p + \frac{\varepsilon^2 - \varepsilon}{\varepsilon} m_p(\varepsilon)$ when $u \in [\max(0, u_+), \varepsilon]$.

Let us verify that $G$ is either locally concave or locally convex on the entire domain $\Xi_0$ (depending on $p$ and $r$). Similar to the previous subsection, it suffices to investigate the sign of $G_{x_3x_3}$.

We differentiate (3.15) with respect to $x_3$ and obtain

$$
1 = u_{x_3} \left( pu^{p-1} + \frac{-2(u + \varepsilon) - (x_2 - u^2)}{2(u + \varepsilon)^2} \cdot m_p + \frac{x_2 - u^2}{2(u + \varepsilon)} \cdot m'_p \right) \quad \text{(3.17)}
$$

$$
= u_{x_3} \left( pu^{p-1} + \frac{-2(u + \varepsilon) - (x_2 - u^2)}{2(u + \varepsilon)^2} \cdot (\varepsilon m'_p + pu^{p-1}) + \frac{x_2 - u^2}{2(u + \varepsilon)} \cdot m'_p \right) = \varepsilon m'_p + pu^{p-2} \quad \text{(3.18)}
$$

Similarly, (3.16) leads to

$$
G_{x_3} = u_{\varepsilon} \frac{-2(u + \varepsilon) + (x_2 - u^2)}{2(u + \varepsilon)^2} \cdot u \cdot \left( m'_r - ru^{r-2} \right).
$$

Formulas (3.17) and (3.18) imply

$$
G_{x_3} = \frac{m'_r - ru^{r-2}}{m'_p - pu^{p-2}} \quad \text{(3.19)}
$$

For $x$ in $\hat{T}_u$ we have $x_2 - (u + \varepsilon)^2 - \varepsilon^2 \leq 0$. By (2.11), $\text{sign}(m''_p) = \text{sign}(p - 2)$. Therefore, formula (3.17) implies

$$
\text{sign}(u_{x_3}) = \text{sign}(2 - p). \quad \text{(3.20)}
$$

We obtain

$$
\text{sign}(G_{x_3x_3}) = \text{sign}(2 - p) \text{sign}(G_{x_3u}). \quad \text{(3.21)}
$$

Formula (3.19) implies

$$
G_{x_3u}(x) = \frac{(m''_p - r(r - 2)u^{r-3})(m'_p - pu^{p-2}) - (m''_p - p(p - 2)u^{p-3})(m'_p - ru^{r-2})}{(m'_p - pu^{p-2})^2} \quad \text{(3.22)}
$$

$$
= \frac{(m''_p - r(r - 2)u^{r-3})(\varepsilon m'_p + p(p - 2)u^{p-2}) - (m''_p - p(p - 2)u^{p-3})(\varepsilon m'_p + r(r - 2)u^{r-2})}{(m'_p - pu^{p-2})^2} \quad \text{(3.23)}
$$

$$
= \frac{(u + \varepsilon)(p(p - 2)u^{p-3}m'_p - r(r - 2)u^{r-3}m'_p)}{(m'_p - pu^{p-2})^2} \quad \text{(3.24)}
$$

$$
= \frac{(u + \varepsilon)(pu^{p-2})u^{p+5}}{(m'_p - pu^{p-2})^2} \left( \frac{m''_p}{r(r - 2)u^{r-2}} - \frac{m''_p}{p(p - 2)u^{p-2}} \right). \quad \text{(3.25)}
$$
We compute the derivative of the latter expression to investigate its sign:
\[ \frac{\partial}{\partial p} \left( \frac{e^{m_p''}}{p(p-2)u^{p-2}} \right) = \frac{\partial}{\partial p} \left( \frac{p-1}{u^{p-2}} \int_1^{+\infty} e^{-t} t p^{-3} \, dt \right) = \int_1^{+\infty} e^{-t} t p^{-3} \, dt + (p-1) \int_1^{+\infty} e^{-t} t p^{-3} \log t \, ds > 0. \]

Consequently, \( \text{sign}(G_{x,u}) = \text{sign}((r-2)(p-2)(r-p)) \) and by (3.21) \( \text{sign}(G_{x,x,u}) = \text{sign}((r-2)(p-r)) \).

Thus, the constructed function \( G \) is locally concave on \( \Xi_0 \) provided \( (r-2)(p-r) < 0 \) and locally convex if \( (r-2)(p-r) > 0 \).

### 3.3 Concatenations

We have defined \( G \) on three subsets of \( \Omega_3 \), namely on \( \Xi_+ \), \( \Xi_0 \), and \( \Xi_- \). Now we verify that the constructed function is defined on the entire domain \( \Omega_3 \) and is \( C^1 \)-smooth. Due to symmetry, we may study the part of \( \Omega_3 \) where \( x_1 > 0 \) only.

Note that the planes \( L_u \) (see (3.3)) and \( \tilde{L}_u \) (see (3.15)) coincide when \( u = \varepsilon \) since \( k_p(\varepsilon) = 0 \) by (2.5). Similarly, the values of \( G \) on that common plane delivered by formulas (3.3) and (3.16) coincide since \( k_\varepsilon(\varepsilon) = 0 \). Thus, we have shown that \( G \) is correctly defined on \( \Omega_3 \) and is continuous on this domain.

To show that \( G \) is \( C^1 \) smooth, it suffices to verify that the limits of \( G_{x,e}(x) \) as \( x \) approaches \( L_\varepsilon \) from different sides coincide. Indeed, once we proved this, the other directional derivatives will glue continuously since \( G \) is linear on \( L_\varepsilon \). By virtue of (3.7) and (3.19), we need to prove
\[ \frac{m''_p(\varepsilon) + k''_p(\varepsilon)}{m''_p(\varepsilon) + k''_p(\varepsilon)} = \frac{m'_p(\varepsilon) - r \varepsilon^{-2}}{m'_p(\varepsilon) - r \varepsilon^{-2}} = \frac{m'_p(\varepsilon) - e \varepsilon^{-2}}{m'_p(\varepsilon) - e \varepsilon^{-2}}. \]

This may be done as follows:
\[ \frac{m''_p(\varepsilon) + k''_p(\varepsilon)}{m''_p(\varepsilon) + k''_p(\varepsilon)} = \frac{m'_p(\varepsilon) - k'_p(\varepsilon)}{m'_p(\varepsilon) - k'_p(\varepsilon)} = \frac{m'_p(\varepsilon) - r \varepsilon^{-2}}{m'_p(\varepsilon) - e \varepsilon^{-2}}, \]

where in the last identity we have used that \( k'_p(\varepsilon) = e \varepsilon^{-2} - \frac{1}{r} k_p(\varepsilon) = e \varepsilon^{-2} \), which is true by (2.13) and (2.24).

To summarize, we have proved that \( G \) is \( C^1 \) smooth. Therefore, its local concavity/convexity on the parts \( \Xi_0, \Xi_\pm \) of \( \Omega_3 \) implies its local concavity/convexity on the entire domain \( \Omega_3 \).

### 3.4 Optimizers

In the previous section, we have constructed a locally concave on \( \Omega_3 \) function \( G \). Let us verify that it coincides with \( B_{p,r} \). For that, it suffices, given arbitrary \( x \in \Omega_3 \), to construct a function \( \varphi_x \) satisfying (2.10).

Such functions are usually called optimizers. We will reason in a slightly different way. Namely, we will construct the optimizers for some specific points on the boundary of \( \Omega_3 \), and then, using a concavity argument show that \( G = B_{p,r} \) on the entire domain \( \Omega_3 \).

First, we construct the optimizers for the vertices of the curvilinear triangle \( T_u, u \in [\varepsilon, +\infty) \), (see (3.22)). The constant function \( \varphi \equiv u \) is an optimizer for the point \( U = (u, u^2, |u|^p) \). The functions
\[ \varphi_{U_+}(t) = -\varepsilon \ln t + u, \quad t \in I = (0, 1]; \]
\[ \varphi_{U_-}(t) = -\varepsilon \chi_{[0,1/2)}(t) + \varepsilon \chi_{[1/2,1)}(t) + (1 + \ln t) \chi_{[1,e^{\frac{u}{\varepsilon}}]}(t), \quad t \in I = [0,e^{\frac{u}{\varepsilon}}^\pm]. \]
are the optimizers for the points $U_\pm$ (see [14]). One may verify that $\|\varphi_{U_\pm}\|_{\text{BMO}(I)} = \varepsilon$ and

\[
\langle \varphi_{U_\pm} \rangle_1 = u \pm \varepsilon, \quad \langle \varphi_{U_\pm}^2 \rangle_1 = (u \pm \varepsilon)^2 + \varepsilon^2, \quad \langle |\varphi_{U_\pm}| \rangle_1 = A_{m_p}(u + \varepsilon, (u + \varepsilon)^2 + \varepsilon^2), \quad \langle |\varphi_{U_\pm}|^2 \rangle_1 = A_{k_p}(u - \varepsilon, (u - \varepsilon)^2 + \varepsilon^2).
\]

Due to symmetry, $\langle \varphi \rangle_i$ has the same optimizers, therefore $B_+(\varepsilon) = B_-(\varepsilon)$. Without loss of generality, we may assume $B_{p,r}(\varepsilon) \leq G$ pointwise. For $U_+$ and $U_-$ we have $G(U_\pm) = \langle |\varphi_{U_\pm}|^2 \rangle = B_{p,r}(\varepsilon)$. Thus, $B_{p,r}(U_\pm) = G(U_\pm)$. The function $G$ is linear on $T_u$, while $B_{p,r}$ is locally concave on it. Therefore, $G \leq B_{p,r}$ on $T_u$. So, we have proved that $B_{p,r} = G$ on $T_u$ for $u \in [\varepsilon, +\infty)$. Due to symmetry, $B_{p,r} = G$ on $\Xi$. In the similar way, one may verify that $B_{p,r} = G$ on $T_u$ for $u \in [0, \varepsilon]$. Indeed, for the vertices of $T_u$ we have the same optimisers, therefore $B_{p,r} = G$ at all the vertices. Similar to the reasoning in the previous paragraph, $B_{p,r}$ is locally concave on $T_u$, while $G$ is linear on it. Consequently, $G \leq B_{p,r}$ on $T_u$, and so $B_{p,r} = G$ there. So, we have proved that $B_{p,r} = G$ on $\Omega_3^1$ entirely and have finally proved Theorem 2.1.

4 Lower Bellman function

One may also consider the lower Bellman function

$$B_{p,r,c}^{\text{min}}(x_1, x_2, x_3) = \inf \left\{ \langle |\varphi| \rangle_i : \|\varphi\|_{\text{BMO}(I)} \leq \varepsilon, \langle \varphi \rangle_i = x_1, \langle \varphi^2 \rangle_i = x_2, \langle |\varphi|^2 \rangle_i = x_3 \right\}. \quad (4.1)$$

Proposition 4.1. 1. The function $B_{p,r,c}^{\text{min}}$ satisfies boundary condition (2.15) on the skeleton.

2. The function $B_{p,r}^{\text{min}}$ is locally convex on $\Omega_3^3$.

3. The function $B_{p,r}^{\text{min}}$ is the pointwise maximal among all locally convex on $\Omega_3^1$ functions $G$ that satisfy the boundary condition (2.15).

If $(r - 2)(p - r) > 0$, then the function $G$ constructed in Section 3 is locally convex, and it can be proved by literally the same arguments that $B_{p,r}^{\text{min}} = G$ in this case.

Theorem 4.2. If $(r - 2)(p - r) > 0$, then $B_{p,r}^{\text{min}} = G$, where the function $G$ is given by formulas (3.4) and (3.10).

5 Extracting the constant

We are going to compute the best possible constant $c_{p,r}$ in the inequality

$$\|\varphi\|_{L^r(I)} \leq c_{p,r} \|\varphi\|_{L^p(I)} \|\varphi\|_{\text{BMO}(I)}, \quad \langle \varphi \rangle_i = 0. \quad (5.1)$$

Without loss of generality, we may assume $\|\varphi\|_{\text{BMO}} = 1$, so we set $\varepsilon = 1$ throughout this section. We raise the inequality to the power $r$:

$$\int |\varphi|^r \leq c_{p,r}^r \int |\varphi|^p, \quad \|\varphi\|_{\text{BMO}} = 1.$$

Let us search for the best possible constant $d_{p,r}$ in the inequality

$$B_{p,r}(0, x_2, x_3) \leq d_{p,r}^r : x_3, \quad (0, x_2, x_3) \in \Omega_3^1.$$
Note that $c_{p,r} \leq d_{p,r}$ by their definitions. On the other hand, it follows from homogeneity that the constant $d_{p,r}$ is attained at some $\varphi$ with $\|\varphi\|_{BMO} = 1$ (i.e., the optimizer for $B_{p,r}$ at $(0, x_2, x_3)$ indeed has BMO-norm equal to one). Therefore, $c_{p,r} = d_{p,r}$.

Since $x_1 = 0$, we will be investigating the values of $B_{p,r}$ on $\Xi_0$. The $x_3$ coordinate is then given by (3.13), whereas the value of $B_{p,r}$ is provided by (3.14). We need to maximize

$$\frac{\mathcal{B}_{p,r}(0, x_2, x_3)}{x_3} = \frac{2(u + 1)u^r + (x_2 - u^2)m_p(u)}{2(u + 1)u^p + (x_2 - u^2)m_p(u)} \quad x_2 \in [u^2, 1], \quad u \in [0, 1].$$

For any $u \in (0, 1]$ fixed, the latter expression is a fraction of two linear functions of $x_2$ whose denominator does not vanish on $[u^2, 1]$. Thus, the function in question attains its maximum at one of the endpoints. At the endpoint $x_2 = u^2$, the value is $u^{r-p}$, which does not exceed one since $r > p$. At $x_2 = 1$, we obtain

$$\frac{g(u)}{u} = \frac{2u^r + (1 - u)m_p(u)}{2u^p + (1 - u)m_p(u)}.$$

We claim that $g$ is decreasing on $[0, 1]$. The sign of $g'(u)$ coincides with the sign of

$$(2ru^{r-1} - mr + (1 - u)m_p)(2u^p + (1 - u)m_p) - (2pu^{p-1} - m_p + (1 - u)m_p')(2u^r + (1 - u)m_r)$$

$$(r + 1(u)^{r-1} - um_p)(2u^p + (1 - u)m_p) - (p(1 + u)u^{p-1} - um_p)(2u^r + (1 - u)m_r)$$

$$= u^2 - m_p(u) - m_p(u) - 2(r + 1(1 + u)u^{p-r-2})$$

$$= u^{r-p-1}(r(u^2 - 1) - 2u^2)(m_p(u) - m_p(u) - 2(r + 1(1 + u)u^{p-r-2})).$$

Note that

$$\frac{m_p(u)}{u} \int_1^\infty e^{u(1-t)}t^{r-1} dt.$$ 

Therefore, the conditions $r > p$ and $r > 2$ imply

$$\frac{m_p(u)}{ru^r} > \frac{m_p(u)}{ru^r} > \frac{m_2(u)}{ru^2} = \int_1^\infty e^{u(1-t)}t dt = \frac{1 + u}{u^2}.$$ 

So, the expression in (5.2) is negative, proving our claim that $g$ is decreasing. This implies that $g$ attains its maximum at 0, which is

$$g(0) = \frac{x^r}{p}(p + 1) = \frac{x^r}{p}.$$ 

Returning to 3D coordinates, we see that the extremal value is attained at the point $0, 1, \frac{x^r}{p+1} \in \Omega_2^3$:

$$\mathcal{B}_{p,r}(0, 1, \frac{x^r}{p+1}) = \frac{x^r}{p}.$$ 

Let us provide an optimiser $\varphi_0$ at the point $0, 1, \frac{x^r}{p+1}$ for the function $\mathcal{B}_{p,r}$:

$$\varphi_0(t) = \begin{cases} -\ln(2 - t), & t \in [1, 2), \\ 0, & t \in [-1, 1], \\ \ln(t + 2), & t \in (-2, -1]\end{cases}$$

on the interval $I = (-2, 2)$. It satisfies the following relations:

$$\langle \varphi_0 \rangle_I = 0, \quad \langle \varphi_0^2 \rangle_I = 1, \quad \langle |\varphi_0|^p \rangle_I = \frac{\Gamma(p + 1)}{2}, \quad \langle |\varphi_0|^r \rangle_I = \frac{\Gamma(r + 1)}{2}, \quad \|\varphi_0\|_{BMO(I)} = 1.$$ 

So, (5.1) turns into equality with $c_{p,r} = \left(\frac{\Gamma(r + 1)}{\Gamma(p + 1)}\right)^{1/r}$ for this function $\varphi_0$. 

12
6 Transference

In this section we will prove Theorems [1.2] and [1.3] Inequality [1.5] is a direct consequence of [1.4] since the circle BMO-norm dominates the interval BMO-norm of the same function. To prove [1.6], we will use [1.4] and some standard limiting arguments in Subsection 6.1. The main difficulty here is to prove the sharpness of [1.5] and [1.6], we will do this in Subsection 6.2

6.1 Inequality

We start with proving [1.3]. Let \( I_n = [-n, n], \) \( n \in \mathbb{N}, \) and let \( \varphi \in \text{BMO}(\mathbb{R}). \) We apply [1.4] to \( \varphi - \langle \varphi \rangle_{I_n} \) on \( I_n \) and use the obvious inequality \( \| \varphi - \langle \varphi \rangle_{I_n} \|_{\text{BMO}(I_n)} = \| \varphi \|_{\text{BMO}(I_n)} \leq \| \varphi \|_{\text{BMO}(\mathbb{R})}.\) \hfill (6.1)

**Lemma 6.1.** For \( \varphi \in L^p(\mathbb{R}) \) one has

\[
\| \langle \varphi \rangle_{I_n} \chi_{I_n} \|_{L^p(\mathbb{R})} = \| \langle \varphi \rangle_{I_n} \|_{L^p(\mathbb{R})} \to 0, \quad n \to \infty. \tag{6.2}
\]

**Proof.** Let \( \delta > 0 \) be an arbitrary real. Pick \( N \) such that \( \int_{\mathbb{R} \setminus I_N} |\varphi|^p < \delta^p. \) Then, for any \( n > N, \) we have

\[
|\langle \varphi \rangle_{I_n}| \leq \left| \frac{I_N}{|I_n|} \right| (|\varphi|)_{I_n} + \frac{|I_n| - |I_N|}{|I_n|} |\langle \varphi \rangle_{I_n}|_{I_n} \leq \frac{|I_N|}{|I_n|} |\langle \varphi \rangle|_{I_n}|^{1/p} + \frac{|I_n| - |I_N|}{|I_n|} |\langle \varphi \rangle|_{I_n}|^{1/p}
\]

\[
\leq \frac{|I_N|^{1-1/p}}{|I_n|} \| \varphi \|_{L^p(\mathbb{R})} + \frac{(|I_n| - |I_N|)^{1-1/p}}{|I_n|} \delta.
\]

This proves [6.2]. \hfill \Box

**Corollary 6.2.** For \( \varphi \in L^p(\mathbb{R}) \) one has

\[
\| \varphi - \langle \varphi \rangle_{I_n} \|_{L^p(\mathbb{R})} \to \| \varphi \|_{L^p(\mathbb{R})}, \quad n \to \infty. \tag{6.3}
\]

**Proof.** From [6.2], we have

\[
\lim_{n \to \infty} \| \varphi - \langle \varphi \rangle_{I_n} \|_{L^p(\mathbb{R})} = \lim_{n \to \infty} \| \varphi \chi_{I_n} - \langle \varphi \rangle_{I_n} \chi_{I_n} \|_{L^p(\mathbb{R})} = \lim_{n \to \infty} \| \varphi \chi_{I_n} \|_{L^p(\mathbb{R})} = \| \varphi \|_{L^p(\mathbb{R})}.
\]

We finish the proof of [1.6] by using Fatou’s Lemma:

\[
\| \varphi \|_{L^p} \leq \liminf_{n \to \infty} \| \varphi - \langle \varphi \rangle_{I_n} \|_{L^p(\mathbb{R})}
\]

\[
\leq \lim_{n \to \infty} \left( \frac{(r+1)}{(p+1)} \right)^{1/p} \| \varphi \|_{L^p(\mathbb{R})} \| \varphi \|^{1-p}_{\text{BMO}(\mathbb{R})}
\]

\[
= \left( \frac{(r+1)}{(p+1)} \right)^{1/p} \| \varphi \|_{L^p(\mathbb{R})} \| \varphi \|^{1-p}_{\text{BMO}(\mathbb{R})}.
\]

6.2 Sharpness

We will first prove the sharpness of [1.5] and complete the proof of Theorem [1.2]. For that, we will construct a special function \( \psi_0 \) on the circle. After that, we will modify this function to prove the sharpness of [1.6], thus, completing the proof of Theorem [1.3].

Unfortunately, we have no simple formula for such a function \( \psi_0. \) We will rely upon the material of [15] and [17]. The following lemma is pivotal in the construction. In particular, this lemma proves the sharpness of [1.5].
Lemma 6.3. For any $\delta > 0$ there exists a function $\psi_0$ on $\mathbb{R}$ with the following properties:

1) $\psi_0$ is 1-periodic, i.e., $\psi_0(t + 1) = \psi_0(t)$ for $t \in \mathbb{R}$;

2) $\int_0^1 |\psi_0|^p = (|\varphi_0|^p)_t + O(\delta), \quad \int_0^1 |\psi_0|^r = (|\varphi_0|^r)_t + O(\delta)$;

3) $\|\psi_0\|_{\text{BMO}(\mathbb{R})} \leq 1 + \delta$;

4) $\int_0^1 \psi_0 = 0$ and $\int_0^1 \psi_0^2 = 1$.

Proof. The proof follows the lines of [17].

Fix $\delta > 0$. First, we will need the notion of an $\Omega^2$-martingale introduced in [16]. We say that a discrete time martingale $(M, S) = (\{M_n\}_n, \{S_n\}_n)$, where $S = \{S_n\}_n$ is a discrete time filtration of finite algebras, and the $M_n$ are $\mathbb{R}^2$-valued random variables, is an $\Omega^2$-martingale provided

1) $S_0$ is the trivial algebra;

2) there exists a summable random variable $M_\infty$ whose values lie on the parabola $x_2 = x_1^2$ almost surely and such that $M_n \to M_\infty$ as $n \to \infty$ in mean and almost surely;

3) for any atom $w \in S_n$, the convex hull of the set $\{M_{n+1}(x): x \in w\}$ lies inside $\Omega^2$.

We refer the reader to [16] for basic properties of such type martingales. By Theorem 3.7 in [16], there exists an $\Omega^2_{1+\frac{\delta}{2}}$-martingale $M$ such that

$$P(M_\infty^1 > \lambda) = \frac{1}{|I|} \cdot \left| \{ t \in I: \varphi_0(t) > \lambda \} \right|, \quad \lambda \in \mathbb{R};$$

by $M_\infty^1$ we denote the first coordinate of the random vector $M_\infty$ (recall that $\varphi_0$ is the optimizer constructed at the end of the previous section). In other words, the first coordinate of the terminate distribution of $M$ is equimeasurable with $\varphi_0$.

Next, by a routine stopping time argument, we may replace $M$ with a simple $\Omega^2_{1+\frac{\delta}{2}}$-martingale $N$ (i.e. a martingale that stops after a finite number of steps) such that $N_0 = M_0$ and

$$\mathbb{E} |N_\infty^1|^p = \mathbb{E} |M_\infty^1|^p + O(\delta); \quad \mathbb{E} |N_\infty^1|^r = \mathbb{E} |M_\infty^1|^r + O(\delta).$$

We apply Theorem 2.3 from [17] to $N$ and obtain a 1-periodic function $\psi_0$ on the line such that

$$\|\psi_0\|_{\text{BMO}(\mathbb{R})} \leq 1 + \frac{\delta}{2}$$

and $\psi_0$ is equimeasurable with $N_\infty^1$ in the sense that

$$\left| \left\{ t \in \left[ -\frac{1}{2}, \frac{1}{2} \right]: \psi_0(t) > \lambda \right\} \right| = P(N_\infty^1 > \lambda)$$

for any $\lambda \in \mathbb{R}$. In particular, the function $\psi_0$ satisfies requirements 2 and 3 of the lemma. Since

$$N_0 = M_0 = (0, 1),$$

we have $\langle \psi_0 \rangle_{\{ -\frac{1}{2}, \frac{1}{2} \}} = 0$ and $\langle \psi_0^2 \rangle_{\{ -\frac{1}{2}, \frac{1}{2} \}} = 1$. \qed
So, we have proved the sharpness of (1.5) and completed the proof of Theorem 1.2.

Now we are ready to prove the sharpness of (1.6), which will easily follow from the lemma below.

**Lemma 6.4.** For any \( \delta > 0 \) there exists a function \( \psi \) on \( \mathbb{R} \) with the following properties:

1) \( \psi(t) = 0 \) for \( t \not\in (0, 1) \);
2) \[
\int_0^1 |\psi|^p = \int_0^1 |\psi_0|^p = (|\varphi_0|^p)_I + O(\delta), \quad \int_0^1 |\psi|^r = \int_0^1 |\psi_0|^r = (|\varphi_0|^r)_I + O(\delta);
\]
3) \( \|\psi\|_{\text{BMO}(\mathbb{R})} \leq 1 + \delta \).

**Proof.** To prove the lemma, we will apply the homogenization procedure from [17] to \( \psi \). Let us briefly describe it. Let \( g \) be a function on the interval \( I = [i_1, i_2] \) and let \( J \) be an interval. Define the transfer \( g_J \) of \( g \) to \( J \) by the rule

\[
g_J(x) = g \left( \frac{(x - j_1)}{2} - \frac{i_1}{2}, \frac{(x - j_2)}{2} + \frac{i_2}{2} \right), \quad x \in J = [j_1, j_2].
\]

Now let \( \lambda \in (0, 1) \). Consider the splitting of \( [-\frac{1}{2}, \frac{1}{2}] \) into subintervals:

\[
I_{k, \pm} = \left[ \pm \frac{1 - \lambda^k - 1}{2}, \pm \frac{1 - \lambda^k}{2} \right], \quad k \in \mathbb{N}.
\]

Let \( g \) be a function defined on \( [-\frac{1}{2}, \frac{1}{2}] \). We call the function \( \Gamma_I[g] \) defined on the same interval by the formula

\[
\Gamma_I[g] = g_{I_{k, \pm}} \text{ on the interval } I_{k, \pm}, \quad k \in \mathbb{N},
\]

the \( \lambda \)-homogenization of \( g \).

Note that \( \Gamma_I[g] \) has the same distribution as \( g \). Lemma 2.7 in [17] says that

\[
\|\Gamma_I[\psi_0]\|_{\text{BMO}([-\frac{1}{2}, \frac{1}{2}])} \leq 1 + \delta,
\]

provided \( \lambda \) is sufficiently close to one. From now on we suppose \( \lambda \) to be sufficiently close to one to fulfill this property. Set

\[
\psi(x) = \Gamma_I[\psi_0] \left( x - \frac{1}{2} \right), \quad x \in [0, 1],
\]

and extend \( \psi \) to the whole line by zero to fulfill property [1] Since the distribution of \( \psi \) coincides with the distribution of \( \psi_0 \), we have requirement [2] satisfied. We also note that \( \|\psi\|_{\text{BMO}(I)} \leq 1 + \delta \).

It remains to verify property [3] Let us first show that for any \( \tilde{J} = [0, a] \) or \( \tilde{J} = [a, 1], \ a \in (0, 1), \) one has

\[
|\langle \psi \rangle_{\tilde{J}} - \langle \psi_0 \rangle_{[0,1]}| \leq \delta,
\]

where \( \langle \psi \rangle_{\tilde{J}} = \frac{1}{|\tilde{J}|} \int_{\tilde{J}} \psi \) is the average of \( \psi \) on \( \tilde{J} \).

To do this, we note the distribution of \( \Gamma_I[\psi_0] \) is close to the distribution of \( \psi_0 \) on \( [0, 1] \) (see the proof of Lemma 2.7 in [17] for details). So, we may choose \( \lambda \) to be sufficiently close to 1 in such a way that (6.4) holds true.

Let us verify property [3] For any interval \( J \subset \mathbb{R} \) we need to prove

\[
V(J) \overset{\text{def}}{=} \langle \psi^2 \rangle_J - \langle \psi \rangle_J^2 \leq (1 + \delta)^2.
\]

Consider several cases:

- if \( J \cap [0, 1] = \emptyset \), then \( \psi = 0 \) on \( J \) and \( V(J) = 0 \);

\footnote{We thank Fedor Nazarov for suggesting to use the homogenization procedure in this context.}

15
the case $J \subset [0, 1]$ had been already considered: $V(J) \leqslant \| \psi \|_{\text{BMO}[0,1]}^2 \leqslant (1 + \delta)^2$;

if $\tilde{J} = J \cap [0, 1] \neq \varnothing$, and $J \not\subset [0, 1]$, then we may apply (6.4) to $\tilde{J}$ and obtain:

$$V(J) \leqslant \langle \psi^2 \rangle_J \leqslant \langle \psi^2 \rangle_{\tilde{J}} \leqslant 1 + \delta.$$ 

The lemma is proved.

References

[1] J. Chen, X. Zhu, *A note on BMO and its applications*, J. Math. Anal. Appl. **303** (2005), 696–698.

[2] M. Cwickel, Y. Sagher, P. Shvartsman, *A new look at the John–Nirenberg and John–Strömberg theorems for BMO*, J. Funct. Anal. **263**:1 (2012), 129–166.

[3] P. Ivanisvili, D. M. Stolyarov, V. I. Vasyunin, P. B. Zatitskiy, *Bellman function for extremal problems in BMO II: evolution*, Mem. Amer. Math. Soc. **255**:1220, 2018.

[4] P. Ivanisvili, N. N. Osipov, D. M. Stolyarov, V. I. Vasyunin, P. B. Zatitskiy, *Sharp estimates of integral functionals on classes of functions with small mean oscillation*, C. R. Math. Acad. Sci. Paris **353**:12 (2015), 1081–1085.

[5] A. A. Korenovskii, *On the connection between mean oscillation and exact integrability classes of functions*, Mat. Sb. **181**:12 (1990), 1721–1727 (in Russian); translated in Math. of the USSR-Sbornik **71**:2 (1992), 561–567.

[6] A. Lerner, *The John–Nirenberg inequality with sharp constants*, C. R. Math. Acad. Sci. Paris **351**:11-12 (2013), 463–466.

[7] M. Milman, *BMO: Oscillations, Self-Improvement, Gagliardo Coordinate Spaces, and Reverse Hardy Inequalities*, Harmonic Analysis, Partial Differential Equations, Complex Analysis, Banach Spaces, and Operator Theory (Volume 1), Springer, 233–274, 2016.

[8] F. L. Nazarov, S. R. Treil, *Hunting the Bellman function: application to estimates of singular integrals and other classical problems of harmonic analysis*, Algebra I Analiz **8**:5 (1996), 32–162 (in Russian); translated in St. Petersburg Math. J. **8**:5 (1997), 721–824.

[9] F. Nazarov, S. Treil, A. Volberg, *Bellman function in stochastic optimal control and harmonic analysis (how our Bellman function got its name)*, Oper. Th.: Adv. and Appl. **129** (2001), 393–424, Birkhäuser Verlag.

[10] A. Osękowski, *Sharp Martingale and Semimartingale Inequalities*, Monografie Matematyczne IM-PAN **72**, Springer Basel, 2012.

[11] L. Slavin, *The John–Nirenberg constant of BMO$_p$, $1 \leqslant p \leqslant 2$*, https://arxiv.org/abs/1506.04969.

[12] L. Slavin, V. Vasyunin, *Cincinnati lectures on Bellman functions*, https://arxiv.org/abs/1508.07668.

[13] L. Slavin, V. Vasyunin, *The John–Nirenberg constant of BMO$_p$, $p > 2$*, Algebra I Analiz **28**:2 (2016), 72–96 (in Russian), translated in St. Petersburg Math. J. **28** (2017), 181–196.

[14] L. Slavin, V. Vasyunin, *Sharp results in the integral form John–Nirenberg inequality*, Trans. Amer. Math. Soc. **363**: 8 (2011), 4135–4169.
[15] L. Slavin and V. Vasyunin, *Sharp Lp estimates on BMO*, Indiana Univ. Math. J. 61:3 (2012), 1051–1110.

[16] D. M. Stolyarov, P. B. Zatitskiy, *Theory of locally concave functions and its applications to sharp estimates of integral functionals*, Adv. Math. 291 (2016), 228–273.

[17] D. Stolyarov, P. Zatitskiy, *Sharp transference principle for BMO and Ap*, https://arxiv.org/abs/1908.09497.

[18] V. Vasyunin, A. Volberg, *Sharp constants in the classical weak form of the John-Nirenberg inequality*, Proc. London Math. Soc. 108:6 (2014), 1417–1434.

[19] A. Volberg, *Bellman function technique in Harmonic Analysis*, Lectures of INRIA Summer School in Antibes, June 2011, [http://arxiv.org/abs/1106.3899](http://arxiv.org/abs/1106.3899).

Dmitriy Stolyarov
d.m.stolyarov@spbu.ru.

Vasily Vasyunin
vasyunin@pdmi.ras.ru

Pavel Zatitskiy
pavelz@pdmi.ras.ru.

St. Petersburg State University, Department of Mathematics and Computer Science, 14th line 29b, Vasilyevsky Island, St. Petersburg, Russia, 199178.