The continuous–time description of financial markets frequently uses the (multiplicative, driftless) Ito stochastic differential equations (SDE). The market is considered to be fair in a sense that, independently on the strategy, the expected future value of an investment is equal, if discounted prices are used, to the value of the capital used. An attempt to make the price movements partially predictable is to replace the driving white noise by a correlated process. Because the direct white noise limit procedure corresponds to the Stratonovich interpretation of the resulting SDE, the initial equation should be first properly rewritten. The paper is organized as follows. In Sec. 2 some basic results of SDE theory are collected, in Sec. 3 the Black and Scholes equation is introduced. Two examples of color noises are presented in Sec. 4, which are then used, in Secs. 5 and 6, to generalize the Black and Scholes model. The last section contains a short summary and conclusions.

II. SDE. ITO VS STRATONOVICH

The usual assumption concerning ideal market is that the evolution of appropriately discounted prices represents a fair game, i.e., that it is described by a certain martingale. Such processes, by the definition, have the property that the (future) conditional mean value $\langle x_t | x_{t_0}, \ldots, x_{t_n} \rangle = x_{t_0}$, $t \geq t_0 \geq \ldots \geq t_1$, is equal to the last value specified by the condition [1,2]. An important class of martingales are driftless Ito processes [1-4]

$$dx_t = f(x)dt + g(x)dW_t,$$ (1)

where $f(x) = 0$ and the $\circ$ sign is to indicate that the equation is interpreted according to the Ito definition that $\langle g(x) d \circ W_t \rangle = 0$ (nonanticipating property). $W_t$ is the Wiener process normalized by the condition

$$\langle \exp(yW_t) \rangle = \exp(Dty^2),$$ (2)

or, equivalently,

$$\langle \xi_t \xi_0 \rangle = 2D\delta(t),$$ (3)

where $\xi_t \equiv dW_t/dt$ is a Gaussian white noise (GWN).

The well known consequence of the nonanticipating property is that the ordinary rules of differentiation and integration are no longer valid [1-4], being replaced by the specific Ito calculus. Particularly, it turns out that $x_t \equiv x(t,W_t)$, considered as a function of two variables, represents the solution to the Ito Eq. (1) only if the usual condition $\partial x/\partial W = g(x)$ and the unusual one $\partial x/\partial t = f(x) - Dg(x)g'(x)$ are satisfied. This means that using the ordinary calculus the same process $x(t,W_t)$ is considered to be the solution of the Stratonovich equation [2,4,5]

$$dx_t = [f(x) - Dg(x)g'(x)]dt + g(x)dW_t$$ (4)

(in our notation without $\circ$ sign) and in such sense both Eqs. (1) and (4) are equivalent. The term $Dgg'$ is called “spurious drift.” Note that the Stratonovich interpretation is more popular in a physical literature because the well recognized (ordinary) methods of transforming the variables and solving the differential equations can be used.

III. BLACK AND SCHOLES MODEL

In his pioneering work [6] Bachelier adopts arithmetical Brownian motion

$$\dot{x}_t = rx + \xi_t$$ (5)

to describe the evolution of stock prices. Here $r > 0$ is an intrinsic growth rate often identified simply with the interest rate. Except for the sign $r = -\gamma < 0$, where $\gamma$ is the friction coefficient, it is the famous Langevin equation, for the Brownian particle’s velocity, of the Einstein–Smoluchowski theory of Brownian motion [7]. Because the solutions of Eq. (5) are Gaussian (Ornstein–Uhlenbeck processes [8]), they are in fact not well suited for modeling prices, which are the nonnegative quantities.

Assuming an independent and Gaussian character of the relative changes one obtains the (Samuelson) Black and Scholes equation (B&S) [9-11]

$$dS_t = rSdt + SdW_t$$ (6)

or, equivalently,

$$\dot{S}_t = [r - D]S + S\dot{\xi}_t$$ (7)

if the Stratonovich interpretation is used. The stochastic solution

$$S_t = S_0 e^{(r-D)t} \exp(W_t)$$ (8)

for modeling stock prices. Here $\xi_t \equiv dW_t/dt$ is a Gaussian white noise (GWN).
immediately follows from Eq. (7). Using Eq. (2) one verifies that
\[ \langle S_t \rangle = S_0 e^{(r-D)t} \langle \exp(W_t) \rangle = S_0 e^{rt} \] (9)
in agreement with Eq. (6). Eq. (9) shows that the average return, related to the passive investment “buy and hold,” is determined by the interest rate \( r \). Because the discounted price \( \tilde{S}_t = e^{-rt} S_t \) is a martingale the consequence of the games theory is that no other strategy can lead to a better result.

**IV. COLOR NOISE**

The Langevin random force (GWN) may be considered as a “singular” limit of certain “regular” correlated processes. Two important examples are the following.

The Gaussian color noise (GCN) (or the stationary Ornstein–Uhlenbeck process) is defined as the Gaussian process of a zero mean and an exponentially decaying autocorrelation function [4,8]
\[ K(t) \equiv \langle \xi_t \xi_0 \rangle = D \tau^{-1} \exp(-t/\tau), \] (10)
where \( \tau > 0 \) is the correlation-time. Because (as a generalized function) \( K(t) \to 2D \delta(t) \) for \( \tau \to 0 \) the GCN (10) approaches GWN (3) if the correlation-time goes to zero.

Another exponentially correlated process, of a different origin, is the dichotomous Markov process (DM) \( \xi_t = \sigma(-1)^{N_t} \), where \( \sigma \) is a binary variable equal \( \pm |\sigma| \) with probability 1/2 and \( N_t \) is a Poisson counting process with parameter \( \lambda \) [12-14],
\[ \langle \xi_t \xi_0 \rangle = \sigma^2 \exp(-2\lambda t). \] (11)
It may be shown that at the limit \( \sigma^2 \to \infty, \lambda \to \infty, \sigma^2/2\lambda = D = \text{const} \) the GWN (3) is also recovered [12]. The correlation-time of DM is \( 1/2\lambda = \tau \).

Note that the above mentioned GWN-limit procedures are consistent with the Stratonovich interpretation. Thus we will study Eq. (7) with color noise (10) or (11). Without loss of generality we assume that \( r = 0 \), which corresponds to the use of discounted prices.

**V. B&S MODEL WITH GCN**

Let \( r = 0 \) and
\[ \dot{S}_t = -DS + S \xi_t, \] (12)
where \( \xi_t \) is GCN (10). Then
\[ S_t = S_0 e^{-Dt} \exp \left[ \int_0^t \xi_s \, ds \right]. \] (13)
Using the general formula for stationary Gaussian processes and after that Eq. (10)
\[ \langle \exp \left[ \int_0^t \xi_s \, ds \right] \rangle = \exp \left[ \int_0^t ds_1 \int_0^{s_1} ds_2 K(s_1 - s_2) \right] = \exp \left[ Dt - D\tau(1 - e^{-t/\tau}) \right], \] (14)
one obtains
\[ \langle S_t \rangle = S_0 \exp \left[ -D\tau(1 - e^{-t/\tau}) \right] \approx S_0 e^{-D\tau}. \] (15)
The result (15) may be summarized as follows. The presence of a color noise in Eq. (12) introduces certain correlations between successive changes of prices. In contrast to the ideal market model the historical information about prices can be in principle useful to improve the strategy of investment. The cost to be payed is that the passive long-time investment leads to the partial loss of the initial capital \( S_0 \), by a constant factor \( e^{-D\tau} \approx 1 - D\tau \) in discounted prices.

**VI. B&S MODEL WITH DM**

Assume again that \( r = 0 \) and consider
\[ \dot{S}_t = -(\sigma^2/2\lambda) S + S \xi_t, \] (16)
where \( \xi_t \) is the asynchronic binary noise (11). Because
\[ S_t = S_0 e^{-\sigma^2 t/2\lambda} \exp \left[ \int_0^t \xi_s \, ds \right] \] (17)
we need
\[ \Phi(t) = \langle \exp \left[ \int_0^t \xi_s \, ds \right] \rangle \] (18)
in order to compute certain averages. Let
\[ \Psi(t) = \langle \xi_t \exp \left[ \int_0^t \xi_s \, ds \right] \rangle. \] (19)
Then \( \dot{\Phi} = \Psi + \sigma^2 \phi \) where the latter equation follows from Shapiro–Loginov formula [15]. The solution of
\[ \dot{\Phi} + 2\lambda \Phi - \sigma^2 \Phi = 0 \] (20)
satisfying \( \Phi(0) = 1 \), \( \dot{\Phi}(0) = 0 \) is
\[ \Phi(t) = \left( \frac{1}{2} + \frac{1}{2q} \right) e^{\lambda(q-1)t} + \left( \frac{1}{2} - \frac{1}{2q} \right) e^{-\lambda(q+1)t}, \] (21)
where \( q = \sqrt{1 + \sigma^2/\lambda^2} \). For sufficiently long time and (2D/\lambda =) \( \sigma^2/\lambda^2 \ll 1 \)
\[ \Phi(t) \approx \left( 1 - \frac{\sigma^2}{4\lambda^2} \right) e^{\frac{\sigma^2 t}{2\lambda} - \frac{\sigma^4 t}{8\lambda^3} + \ldots} \] (22)
and thus
\[ \langle S_t \rangle \approx S_0 (1 - D\tau) e^{-D^2 \tau t}, \] (23)
where \( \tau = 1/2\lambda \) and \( D = \sigma^2/2\lambda \). Comparing to Eq. (15) the case (23) seems even worse generating the still increasing (with time) losses in discounted prices (or decreasing the intrinsic growth rate from \( r \) to \( r - D^2 r \) in
real prices). On the other hand the “buy and hold” strategy, for selfevident reasons, is quite inappropriate for this case. In fact Eq. (17) shows that the realization of $S_t$ consists of the periods of exponential decay $S \sim e^{-(|\sigma|+D)\Delta t}$ separated by the periods of exponential growth $S \sim e^{(|\sigma|-D)\Delta t}$ (if $|\sigma| < 2\lambda$; otherwise the price always falls). The length of the periods is random with the average equal to $1/\lambda$. Moreover the trajectory of $S_t$ is continuous. “Playing with trend” one buys the stock at the beginning of a growth period and sells immediately when the move changes the direction. The distribution of waiting times for DM is given by $p(t) = \lambda e^{-\lambda t}$ and the corresponding price $S(t) = S_0 e^{(|\sigma|-\sigma^2/2\lambda)t}$, so the expected return per one cycle of an investment is

$$\bar{S} = \int_0^\infty p(t)S(t)dt = S_0 \frac{2}{(1-|\sigma|/\lambda)^2+1}. \quad (24)$$

Note that at the GWN-limit, $|\sigma| = \sqrt{2\lambda D}$, $\lambda \to \infty$, the r.h.s. of Eq. (24) is (still) equal $S_0$, which reflects the fairness of the ideal market. The ratio $\bar{S}/S_0 > 1$ if $|\sigma|/\lambda < 2$ (or $D < 2\lambda$). The maximum $\bar{S}/S_0 = 2$ corresponds to $|\sigma| = \lambda = 2D$. Thus, in spite of the general decreasing tendency (23), the market described by Eq. (16) provides easy earn opportunities.

VII. REMARKS

The B&S equation, written in the Stratonovich form (7), can be easily generalized by an appropriate replacement of the driving noise. The B&S equation with a color noise remains exactly solvable. The particular cases of GCN and DM have been analyzed in Secs. 5 and 6. The general conclusion is the following. The limit of zero correlation–time corresponds to the ordinary B&S model, when the discounted price $\tilde{S}_t = e^{-rt}S_t$ is a martingale and the expected future price is $\langle S_t \rangle = S_0 e^{rt}$. On the correlated market ($\tau > 0$) the expected price is lowered: $\langle S_t \rangle \approx S_0 e^{-Dr}e^{rt} \approx S_0 (1 - Dr)e^{rt}$, Eq. (15), for GCN and $\langle S_t \rangle \approx S_0 (1 - D\tau)e^{(r-D\tau)t}$, Eq. (23), for DM, respectively. Thus the long–time investment is not particularly recommended. On the other hand, in the presence of correlations the historical prices contain a certain information which can be used to improve the investment. In the case of DM it is easy to identify and use the short–time trends to get a certain earn, as shows Eq. (24).

[1] Y.V. Prokhorov and Y.A. Rozanov, Probability Theory (Springer, Berlin, 1969); I.I. Gihman and A.V. Skorohod, Stochastic Differential Equations (Springer, Berlin, 1972).
[2] R. Zygadło, Phys. Rev. E 68, 046117 (2003).
[3] K. Ito, Mem. Am. Math. Soc. 4, 289 (1951).
[4] C.W. Gardiner, Handbook of Stochastic Methods (Springer, Berlin, 1983).
[5] R.L. Stratonovich, Topics in the Theory of Random Noise (Gordon and Breach, New York, 1967).
[6] L. Bachelier, Ann. Sci. Ecol. Norm. Sop. 17, 21 (1900).
[7] A. Einstein, Ann. Physik 17, 549 (1905); M. Smoluchowski, Ann. Physik 21, 758 (1906); P. Langevin, Comptes rendus 146, 530 (1908).
[8] G.E. Uhlenbeck, L.S. Ornstein, Phys. Rev. 36, 823 (1930).
[9] P.A. Samuelson, Industrial Management Review 6, 13 (1965).
[10] F. Black and M. Scholes, J. Polit. Economy 81, 637 (1973).
[11] R.N. Mantegna and H.E. Stanley Introduction to Econophysics: Correlations and Complexity in Finance (University, Cambridge, 1999).
[12] C. van den Broeck, J. Stat. Phys. 31, 447 (1983).
[13] W. Horsthemke and R. Lefever, Noise-Induced Transitions (Springer, Berlin, 1984).
[14] R. Zygadło, Phys. Rev. E 54, 5964 (1996).
[15] V.E. Shapiro and V.M. Loginov, Physica A 91, 563 (1978); V.M. Loginov, Acta Phys. Pol. B 27, 693, (1996).