Energy-Momentum Tensor of Field Fluctuations in Massive Chaotic Inflation

F. Finelli *, 1,2, G. Marozzi † 2, G. P. Vacca ‡ 3,2 and G. Venturi § 2

1 Department of Physics, Purdue University, West Lafayette, IN 47907, USA
2 Dipartimento di Fisica, Università degli Studi di Bologna and I.N.F.N., via Irnerio, 46 – 40126 Bologna – Italy
3 H. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D-33761 Hamburg, Germany

Abstract

We study the renormalized energy-momentum tensor (EMT) of the inflaton fluctuations in rigid space-times during the slow-rollover regime for chaotic inflation with a mass term. We use dimensional regularization with adiabatic subtraction and introduce a novel analytic approximation for the inflaton fluctuations which is valid during the slow-rollover regime. Using this approximation we find a scale invariant spectrum for the inflaton fluctuations in a rigid space-time, and we confirm this result by numerical methods. The resulting renormalized EMT is covariantly conserved and agrees with the Allen-Folacci result in the de Sitter limit, when the expansion is exactly linearly exponential in time. We analytically show that the EMT tensor of the inflaton fluctuations grows initially in time, but saturates to the value $H^2H_0^2$, where $H$ is the Hubble parameter and $H_0$ is its value when inflation has started. This result also implies that the quantum production of light scalar fields (with mass smaller or equal to the inflaton mass) in this model of chaotic inflation depends on the duration of inflation and is larger than the usual result extrapolated from the de Sitter result.

I. INTRODUCTION

Particle production in expanding universe, pioneered by L. Parker [1], is an essential ingredient of inflationary cosmology [2]. The nearly scale invariant spectrum of density perturbations predicted by inflationary models [3] is at present inextricably related to the
concept of amplification of vacuum fluctuations by the geometry. The scale invariant spectrum for massless minimally coupled fields during a de Sitter era was indeed computed before inflation was suggested.

The calculation of the energy carried by these amplified fluctuations is then the natural question. To answer this question a renormalization scheme is necessary, as in ordinary Minkowski space-time. Ultraviolet divergences due to fluctuations on arbitrary short scales are common in field theory. In Minkowski space-time, infinities in a free theory are removed by the subtraction of the vacuum expectation value of the energy, also called normal ordering, the physical justification being that these vacuum contributions are unobservable.

A similar prescription is used in order to regularize infinities in cosmological space-times. However, one additional problem is the absence of an unambiguous choice of vacuum, because of the absence of a class of privileged observers, which are the inertial observers in the Minkowski space-times. The idea is then to subtract the energy associated with a vacuum for which the effects of particle production by the time-dependence of the metric are minimized. This vacuum is determined by the assumption of an adiabatic expansion of the metric. This procedure is therefore called adiabatic subtraction \cite{4,5}. In this way the infinities of field theory are removed.

Even with a minimal prescription such as the above, there are several surprising effects accompanying the renormalized energy-momentum tensor (henceforth EMT) of a test field in cosmological space-times. To name a few, there could be the avoidance of singularities due to quantum effects \cite{6}, conformal anomalies which break classical conformal symmetries at the quantum level \cite{7}, violation of the various energy conditions \cite{8}. One of the first models of inflation proposed by Starobinsky \cite{9} was indeed based on the role of the conformal anomaly, which both avoids the singularity and produces an inflationary phase.

While the adiabatic vacuum for a test field - and its associated EMT - can be computed for generic cosmological space-times, the unrenormalized EMT can be calculated analytically only if exact analytic solutions for the field Fourier modes are available. Because of this, space-times such as de Sitter have been thoroughly investigated \cite{10,11}, since analytic solutions for a scalar field with generic mass and coupling are available. In the absence of analytic solutions, numerical schemes are implemented \cite{12}. In a de Sitter space-time, the back-reaction of a test field seems important only if $m^2 + \xi R = 0$, with $m$ and $\xi$ separately different from zero, or $m^2 + \xi R < 0$ ($m$ is the mass of the test field and $\xi$ its coupling to the curvature $R$) \cite{13,14}. In the former case, the EMT of the test field grows linearly in time, while in the latter case it grows exponentially. The EMT of a massless minimally coupled field is constant in de Sitter space-time \cite{15}.

Inflationary models based on the use of scalar fields, have an accelerated stage, usually called slow rollover, in which the Hubble parameter is almost frozen. During this stage it is rare to have exact solutions for the fluctuations. However, these inflationary models are more attractive than the de Sitter space-time one in furnishing phenomenological models whose predictions can be tested against observations. Chaotic models \cite{15} are the simplest among these. In this paper we study analytically the problem of back-reaction of inflaton fluctuations during the regime of slow-rollover for the case of a massive inflaton. We consider inflaton fluctuations in rigid space-times, i.e. we neglect metric perturbations coupled to them, as pioneeringly investigated by Abramo, Brandenberger and Mukhanov \cite{16}. We plan to come back to this issue in a future work \cite{17}. 
The plan of the paper is as follows: in section II we describe the background classical dynamics for a massive inflaton and the novel analytical approximation for its fluctuations during the slow-rollover regime. In section III we discuss the normalization of quantum fluctuations and in section IV we compare the numerical evaluation of the spectrum with the analytic approximation. We discuss the EMT of inflaton fluctuations, the adiabatic subtraction and the renormalization in sections V-VII, respectively. In section VIII the problem of the back-reaction of the EMT of inflaton fluctuations is addressed and in section IX our novel analytic approximation is compared with the slow-rollover technique [18]. In section X we analyze the production of a secondary massive field $\chi$ lighter than the inflaton and we show that its production depends on the duration of inflation. In section XI we conclude and in the two appendices we relegate useful formulae for the adiabatic expansion and the dimensional regularization with cut-off.

II. ANALYTIC APPROXIMATION

We consider inflation driven by a classical minimally coupled massive scalar field. The action is:

$$S \equiv \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right]$$

where $\mathcal{L}$ is the lagrangian density and $m$ is the mass of the field $\phi$. Further we consider the Robertson-Walker line element with flat spatial section:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2(t) d\vec{x}^2$$

The scalar field is separated in its homogeneous component and the fluctuations around it, $\phi(t, x) = \phi(t) + \varphi(t, x)$. During slow rollover the potential energy dominates and therefore

$$H^2 = \frac{\kappa^2}{3} \left[ \frac{\dot{\phi}^2}{2} + m^2 \phi^2 \right] \simeq \kappa^2 m^2 \phi^2$$

where $\kappa^2 = 8\pi G = 8\pi/M^2_{\text{pl}}$. The Hubble parameter evolves as:

$$\dot{H} = -\frac{\kappa^2}{2} \dot{\phi}^2$$

On using the equation of motion for the scalar field:

$$\ddot{\phi} + 3H \dot{\phi} + m^2 \phi = 0$$

and neglecting the second derivative with respect to time we have $3H \dot{\phi} \simeq -m^2 \phi$, and from Eq. (4) one obtains:

$$\dot{H} \simeq -\frac{m^2}{3} \equiv \dot{H}_0$$

which leads to a linearly decreasing Hubble parameter and correspondingly to an evolution for the scale factor which is not exponentially linear in time, i.e.:

$$H(t) \simeq H_0 + \dot{H}_0 t \quad a(t) \simeq \exp[H_0 t + \dot{H}_0 t^2/2]$$

In Fig. [1] the comparison of the analytic approximation with the numerical evolution for the Hubble parameter $H$ is displayed.
FIG. 1. We plot the numerical evolution of $H$ (solid line) and the analytic approximation (dashed line). The initial condition corresponds to $H(t = 0) \approx 9.2m$ and the time is in $1/m$ units.

We now consider the equations of motion for the inflaton fluctuations $\varphi$ in a rigid space-time (i.e. without metric perturbations):

$$\ddot{\varphi}_k + 3H \dot{\varphi}_k + \omega_k^2 \varphi_k = 0,$$

where $\omega_k^2 = k^2/a^2 + m^2$ and the $\varphi_k$ are the Fourier modes of the inflaton fluctuations,

$$\varphi(t, x) = \frac{1}{\sqrt{V}} \sum_k \left[ e^{ik \cdot x} \varphi_k(t) + e^{-ik \cdot x} \varphi_k^*(t) \right].$$

As already emphasized in the introduction, exact solutions for scalar fields in an expanding universe are rare, and indeed we do not have exact solutions for Eq. (8) with the time evolution given by Eq. (7). We therefore introduce an approximation scheme based on an analogy with de Sitter space-time, where exact solution for scalar fields with arbitrary mass and coupling to the curvature do exist. We introduce $\psi_k = a^{3/2} \varphi_k$ and we split the time dependence in $\psi_k$ as follows:

$$\psi_{k} = \psi_{k}(\zeta, H) \quad \text{with} \quad \zeta = \frac{k}{aH}.$$ 

The equation for $\psi_k$ is

$$\ddot{\psi}_k + \left[ \frac{k^2}{a^2} + m^2 - \frac{3}{2} \dot{H} - \frac{9}{4} H^2 \right] \psi_k = 0. \quad (11)$$

We now make the ansatz $\psi_k = \zeta^\mu Z_\nu(\lambda \zeta)$ with $\mu, \nu$ and $\lambda$ functions of $H$. On expressing the first and second time derivatives as derivatives with respect to $(\zeta, H)$ and using $\dot{H} \approx 0$, as follows from Eq. (3), we obtain from Eq. (11) after a little algebra:

$$\zeta^\mu \left[ \zeta^2 \frac{\partial^2}{\partial \zeta^2} + \zeta \frac{\partial}{\partial \zeta} + (\lambda^2 \zeta^2 - \nu^2) \right] Z_\nu + \text{Res}_1 \zeta^{\mu+1} \frac{\partial Z_\nu}{\partial \zeta} + \text{Res}_2 \zeta^{\mu+2} Z_\nu + \text{Res}_3 \zeta^\mu Z_\nu + \mathcal{O}\left(\frac{\dot{H}^2}{H^4}\right) = 0 \quad (12)$$
where we have neglected quadratic and higher order terms in $\dot{H}/H^2$. Indeed, in order to have a value of density perturbations compatible with observations, $m$ is constrained to be $O(10^{-5} - 10^{-6} M_{pl})$: from Eqs. (13) one can see that working to first order in $\dot{H}/H^2$ during slow-rollover ($\phi \sim \text{few} M_{pl}$) is a good approximation. On considering $H$ and $\zeta$ as independent variables, the first term vanishes if $Z_\nu$ is a Bessel function of argument $\lambda \zeta$ and index $\nu$. On requiring that the residual functions $\text{Res}_i, i = 1 - 3$ vanish individually, the parameters $\lambda, \mu$ and $\nu$ are determined to be:

$$\lambda = 1 - \frac{\dot{H}}{H^2}, \quad \mu = \frac{\dot{H}}{2H^2} \quad (13)$$

$$\nu^2 = \frac{9}{4} - \frac{m^2}{H^2} - 3\frac{\dot{H}}{H^2}. \quad (14)$$

Hence the general solution to Eq. (8) is:

$$\varphi_k = \frac{1}{a^{3/2}} \zeta^\mu \left[ A H^{(1)}_\nu(\lambda \zeta) + B H^{(2)}_\nu(\lambda \zeta) \right] \quad (15)$$

where $H^{(1,2)}_\nu$ are the Hankel functions of first and second kind respectively, and $A, B$ are time-independent coefficients to the order of our approximation.

We note that in the de Sitter limit ($\dot{H}/H^2 \to 0$) the solution in Eq. (15) tends to the de Sitter solution [10,19,20], since $\lambda = 1, \mu = 0$.

On using Eq. (6) the value for the index $\nu$ in Eq. (14) corresponds to an exact scale invariant spectrum for the inflaton fluctuations $\varphi$, i.e. $\nu = 3/2$. We shall show numerically in section IV that this analytic approximation is very good for the relevant spectrum range. This numerical analysis agrees with a previous numerical estimate of the same spectral index [21]. This scale invariant spectral index could seem a little surprising, since in de Sitter space-time, a mass term would lead to a spectrum, which is slightly blue shifted with respect to scale invariance ($\nu < 3/2$). To see this, it is sufficient to put $\dot{H} = 0$ in Eqs. (13,14). On considering $\dot{H} \neq 0$ (and negative), it appears to give a positive contribution to the mass term in Eq. (11), instead it compensates the mass term in Eq. (14). The interpretation is the following: on considering a de Sitter stage in which $H$ slowly decreases, a fluctuation freezes when it crosses the Hubble radius, with an amplitude determined by the value of the Hubble radius at the horizon crossing. However $H$ decreases, therefore if $k_1 > k_2$, this effect implies that the amplitude for the mode $k_1$ is smaller than the one for the mode $k_2$, since the latter crosses the Hubble radius first. This effect is a red tilt of the de Sitter scale invariant spectrum. For the case of slow-rollover in a chaotic inflationary model with a massive inflaton, these red and blue shifts exactly compensate, leading to a scale-invariant spectrum for the inflaton fluctuations $\varphi$ in rigid space-time.

### III. QUANTIZED FLUCTUATIONS

We now consider quantized fluctuations of the inflaton. This means that Eq. (9) is promoted to an operator form:
\[
\dot{\varphi}(t, x) = \frac{1}{\sqrt{V}} \sum_k \left[ \varphi_k(t) e^{ik \cdot \hat{b}_k} + e^{-ik \cdot \varphi^*_k(t)} \right]
\] (16)

where the \( \hat{b}_k \) are time-independent Heisenberg operators (also called time independent invariants in \[22,23\]). In order to have the usual commutation relations among the \( \hat{b}_k \):

\[
[\hat{b}_k, \hat{b}_{k'}] = [\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger] = 0 \quad [\hat{b}_k, \hat{b}_{k'}] = \delta^{(3)}(k - k')
\] (17)

one must normalize the solution to the equations of motion through the Wronskian condition:

\[
\dot{\varphi}_k \dot{\varphi}^*_k - \dot{\varphi}_k \dot{\varphi}^*_k = \frac{i}{a^3}.
\] (18)

This normalization condition yields to the following relation among the coefficients \( A, B \) of Eq. (15):

\[
|A|^2 - |B|^2 = \frac{\pi}{4H} \lambda \zeta^{-2\mu}.
\] (19)

The fact that \( A, B \) depend on time should not surprise. In time-dependent perturbation theory, these coefficients, which would be time independent for exact solutions, acquire a time dependence \[23\], just as the Wronskian of the solutions. In our case, this time dependence is consistent with the approximation, i.e. self-consistent to (including) order \( \dot{H}/H^2 \).

The solution corresponding to the Bunch-Davies vacuum \[20\] in de Sitter space-time, that is the adiabatic vacuum for \( k \to \infty \) during the slow-rollover phase, corresponds to choosing \( A = (\pi \lambda/4H)^{1/2} \zeta^{-\mu} \), \( B = 0 \). With this choice, for \( \lambda \zeta \to \infty \), the solution (15) becomes:

\[
\varphi_k \simeq -\frac{1}{a \sqrt{2k}} e^{+i\lambda \zeta}
\] (20)

Let us now discuss the behaviour for \( \lambda \zeta \ll 1 \). On using Eqs. (10,13) one sees that \( \lambda \zeta \ll 1 \) implies \( k \ll aH \), i.e. wavelengths which are much larger than the Hubble radius. In this limit, the solution is \[26\]:

\[
\varphi_k \simeq -i \frac{\Gamma(\nu)}{\pi a^{3/2}} \left( \frac{\pi \lambda}{4H} \right)^{1/2} \left( \frac{\lambda \zeta}{2} \right)^{-\nu}.
\] (21)

In order to compute expectation values of operators with respect to states in the time-independent invariant \( b \) \[22,23\] basis it is useful to introduce the modulus of mode functions \( x_k = |\varphi_k/\sqrt{2}| \). The variable \( x_k \) satisfies the following Pinney equation:

\[
\ddot{x}_k + 3H \dot{x}_k + \omega_k^2 x_k = \frac{1}{a^3 x_k^3}.
\] (22)

We now rescale \( x_k = \rho_k/a^{3/2} \) to eliminate the damping term and obtain:

\[
\ddot{\rho}_k + \left[ \omega_k^2 - \frac{9}{4} H^2 - \frac{3}{2} \dot{H} \right] \rho_k = \frac{1}{\rho_k^{3}}.
\] (23)
The general solution to Eq. (23) is given as a nonlinear combination of two independent solutions \( y_1, y_2 \) to the linear part of Eq. (23). From Eq. (11) we can use the following solutions:

\[
\begin{align*}
y_1 &= \zeta \mu J_\nu(\lambda \zeta) \\
y_2 &= \zeta \mu N_\nu(\lambda \zeta)
\end{align*}
\]

(24)

Therefore the solution to Eq. (23) is

\[
\rho_k = \left( L y_1^2 + M y_2^2 + 2 N y_1 y_2 \right)^{\frac{1}{2}}
\]

(25)

where the coefficients satisfy \( LM - N^2 = 1/\bar{W}^2 \), with \( \bar{W} \) the (time-dependent) Wronskian of \( y_1, y_2 \). The choice of initial conditions for \( \rho_k \) which corresponds to the adiabatic vacuum for \( k \to \infty \) is \( N = 0 \) and \( L = M = \lambda \pi / (2H \zeta^{2\nu}) \). The solution for \( x_k \) is:

\[
x_k = \frac{1}{a^{3/2}} \left( \frac{\pi}{2H} \right)^{1/2} \lambda^{1/2} \left[ J_\nu^2(\lambda \zeta) + N^2_\nu(\lambda \zeta) \right]^{1/2}
\]

(26)

which coincides with the Bunch-Davies choice in the de Sitter limit [22].

IV. NUMERICAL ANALYSIS

In this section we present the numerical analysis of the time evolution of the \( \phi \) modes. Besides checking the validity of the analytical approximation introduced in section II, this analysis is useful in order to understand how a natural infrared cut-off emerges in the problem, on assuming that inflation is not eternal in the past, but starts at some finite time. This infrared cut-off becomes relevant when the \( \phi \) fluctuations are generated in an infrared state [27], as occurs for \( \nu \geq 3/2 \) (see Eq. (14)).

First we want to analyze the properties of the spectrum of the inflaton fluctuations. We numerically evolve Eqs. (3, 5) and Eq. (22). We present numerical data for an interval of comoving wavenumbers for which \( 1 \leq k/m \leq 10^5 \) at the initial time \( t_0 \) (\( a(t_0) = 1 \)). The initial conditions are \( \phi(t_0) = 4.5M_{pl}, \dot{\phi}(t_0) = 0 \) for the inflaton. If we consider the vacuum state for each mode of the field fluctuations to be the initial condition, one has

\[
\begin{align*}
x_k(t_0) &= \frac{1}{a^{3/2}(t_0) \omega^{1/2}_k(t_0)} \\
\dot{x}_k(t_0) &= 0
\end{align*}
\]

(27)

This fact can be easily seen in terms of the invariant operators introduced to quantize time dependent harmonic oscillators [22–24].

As a second initial condition, we consider the limit for large \( k \) of the conditions in (27), which correspond to setting the mass equal to zero. A third set of initial conditions is related to the adiabatic expansion in conformal time, (see later Eq. (39)):

\[
\begin{align*}
x_k(t_0) &= \frac{1}{a(t_0) \Omega^{1/2}_k(t_0)} \\
\dot{x}_k(t_0) &= -H(t_0)x_k(t_0)
\end{align*}
\]

(28)
where $\Omega_k$ will be defined in Eq. (40). Let us note that for the last case the frequency becomes imaginary below a certain threshold and so we shall consider the region above it. In Fig. 2 we exhibit the three cases. The first two initial conditions lead to spectra practically equal for $k$ of order $m$ and above, the third set instead has a spectrum which joins the others at values of $k$ of the order of $H_0$.

![FIG. 2.](image)

FIG. 2. We show on a logarithmic scale the r.m.s. of the inflaton fluctuations $x_k$ obtained numerically for the three different sets of initial conditions. The spectrum is shown as function of $\log(k/m)$.

The spectrum of the fluctuations, related to the initial conditions in (27), are displayed, over a broader range, in Fig. 3.

![FIG. 3.](image)

FIG. 3. We show on a logarithmic scale the r.m.s. of the inflaton fluctuations $x_k$ obtained numerically (solid line) and an asymptotic linear fit (dotted line). The spectrum is shown as function of $\log(k/m)$ and the linear fit is given by $0.7708 - 1.5 \log(k/m)$, in agreement with [21].

Figs. 2 and 3 display the spectrum at $t = 10/m$ (for this case inflation lasts a period of time $\sim 27/m$). We note that this scale invariant spectrum extends only up to a certain scale, $\ell$, which is of the same order as the initial Hubble radius $H(t_0)$ (we have checked this by changing the initial conditions for the homogeneous mode of the inflaton). For comov-
ing modes smaller than $\ell$ the spectrum oscillates and bends towards the initial conformal adiabatic vacuum state, as shown in Fig. (3). For all practical purposes we can therefore safely consider a scale invariant spectrum for $k > \ell$, where $\ell = CH(t_0)$, with $C$ a numerical coefficient $O(1)$. This numerical evidence favours the picture in which the amplification of the modes occurs mainly at the crossing of Hubble radius. Since $m << H$ during inflation, this means that all the modes for which $m \lesssim H(t_0)$, are not stretched by the geometry.

Secondly, we wish to show how accurate the approximation (24) is mode by mode. In order to do this we numerically solve Eq. (23) and compare it to the approximation employed. The agreement is very good up to times very close to the end of inflation. In Fig. 4 we show for the mode with $k = 10 m$ the time evolution of the relative error $(\rho^{num}_k - \rho^{approx}_k)/\rho^{num}_k$. The larger $k$ is the better the agreement. We are therefore allowed to use the approximation for the inflaton dynamics almost till the end of inflation.

![Graph](image)

**FIG. 4.** Relative error $(\rho^{num}_k - \rho^{approx}_k)/\rho^{num}_k$ (for $k = 10 m$) as function of time (in unit of $1/m$).

Let us now consider the correlator $\langle \phi^2 \rangle$ (similar considerations are valid for other quantities bilinear in the field). We note from the spectral analysis that the integral over the modes can be split into two parts

$$\langle \phi^2 \rangle = \frac{\hbar}{(2\pi)^3} \int d^3k |\phi_k|^2 = \frac{\hbar}{2(2\pi)^3} \int d^3k \ k^2 [ \int_0^\ell dk \ k^2 x_k^2 + \int_\ell^\infty dk \ k^2 x_k^2 ]$$

(29)

Below the scale $\ell$ both the interval with the oscillations and the tilt around the scale invariant spectrum shown in Fig. 1 are included. The analytic treatment of the far infrared modes which contribute to the first integral would need an analytic approximation for the modes which at the beginning of inflation are outside the Hubble radius. This would amount to knowing the phase and the initial quantum states which precede the inflationary phase. Therefore we shall proceed by considering only the second integral and we shall neglect the first one in the far infrared. Even on neglecting the first integral which would require extra assumptions, the correct leading behaviour of the renormalized quantities is obtained [28].

We conclude this section by noting that other methods to deal with infrared states are present in the literature. Infrared states were studied by Ford and Parker [27] for massless fields in Robertson-Walker space-times with a power-law expansion of a particular kind. By
matching an earlier static space-time with a space-time with a scale factor which expands in time with a power law, they noted that an infrared finite state cannot evolve to an infrared divergent state [27]. The same scheme was also used by Vilenkin and Ford [29] for the problem of massless minimally coupled scalar fields in de Sitter space-time. An earlier radiation dominated phase was matched to the de Sitter metric. Through this matching the infrared tail becomes suppressed leading to an infrared finite state. As we shall see both the calculations performed by eliminating the infrared tail (i.e. working with the cut-off) and suppressing the infrared tail (through the Bogoliubov coefficients obtained by the matching prescription) lead to the same result to leading order. Indeed, their physical motivation is the same: inflation is not eternal, but starts at a finite time. However, the two methods treat the infrared tail in a different way: the agreement to leading order implies that the relevant contribution to the correlator $\langle \varphi^2 \rangle$ - and to the EMT - comes from intermediate modes, and not from the furthest infrared modes.

V. THE ENERGY-MOMENTUM TENSOR

The classical energy-momentum tensor (henceforth EMT) of inflaton fluctuations is:

$$T_{\mu\nu} = \partial_{\mu}\varphi\partial_{\nu}\varphi + g_{\mu\nu} \left[ -\frac{1}{2} g^{\alpha\beta} \partial_{\alpha}\varphi \partial_{\beta}\varphi - \frac{m^2}{2} \varphi^2 \right]$$  \hspace{1cm} (30)

and its operator form is simply obtained by promoting $\varphi$ to an operator as in Eq. (16).

When averaged over the vacuum state annihilated by $\hat{b}_k|0\rangle = 0$ (31)

the energy-momentum tensor assumes a perfect fluid form because of the symmetries of the RW background [30]:

$$\langle T_{\mu\nu} \rangle = \text{diag}(\epsilon, a^2 p \delta_{ij}),$$  \hspace{1cm} (32)

where $\epsilon$ and $p$ are the energy density and the pressure density respectively.

In the following we consider, according to the previous sections, $\nu = 3/2$, employing the dimensional regularization [3] to treat the UV behaviour. Therefore the integrands will be in 3 dimensions and the integration measure analytically continued in $d$ dimension.

The energy density is

$$\epsilon = \langle T_{00} \rangle = \frac{\hbar}{2(2\pi)^d} \int_{|k| > \ell} d^d k \left[ |\dot{\varphi}_k|^2 + \frac{k^2}{a^2} |\varphi_k|^2 + m^2 |\varphi_k|^2 \right]$$

$$= \frac{\hbar}{4(2\pi)^d} \int_{|k| > \ell} d^d k \left[ \dot{x}_k^2 + \frac{1}{a^2 x_k^2} + \left( \frac{k^2}{a^2} + m^2 \right) x_k^2 \right],$$  \hspace{1cm} (33)

and the pressure density $p$, related to the space-space component of the EMT, is:

$$p = \frac{\langle T_{ii} \rangle}{a^2} = \frac{\hbar}{4(2\pi)^d} \int_{|k| > \ell} d^d k \left[ \dot{x}_k^2 + \frac{1}{a^2 x_k^2} + \left( \frac{2 - d}{d} \frac{k^2}{a^2} - m^2 \right) x_k^2 \right].$$  \hspace{1cm} (34)
On using Eqs. (26) and (73) in Appendix B with $\alpha = d - 1$, we may now compute the second part of the integral (29):

$$
\langle \varphi^2 \rangle = \frac{\hbar}{2(2\pi)^d} \int_{|k| > \ell} d^d k \, x_k^2
$$

(35)

$$
= \frac{\hbar}{16\pi^2} H^2 \left(1 - \frac{2m^2}{3H^2}\right) \left\{2 - 4\ln 2 - 2 \left( \frac{\ell}{2\pi^{1/2}} \right)^{d-3} \Gamma\left(\frac{1}{2} - \frac{d}{2}\right) + \mathcal{O}\left(\frac{1}{a^2}\right) \right\} + \mathcal{O}(d - 3)
$$

(36)

Analogously, the energy and pressure densities in Eqs. (33,34), with the help of the formulae in Appendix B, are:

$$
\epsilon = \langle T_{00} \rangle = \frac{\hbar}{16\pi^2} H^4 \left\{4 \left(\frac{1}{2\pi^{1/2}}\right)^{d-3} \Gamma(1 - d) + \frac{m^2}{H^2} \left[-1 + \gamma - 2\ln 2\right] + \mathcal{O}\left(\frac{1}{a^2}\right) \right\} + \mathcal{O}(d - 3)
$$

(37)

$$
p = \frac{\langle T_{ii} \rangle}{a^2} = \frac{\hbar}{16\pi^2} H^4 \left\{-4 \left(\frac{1}{2\pi^{1/2}}\right)^{d-3} \ell^{d-3} \Gamma(1 - d) + \frac{m^2}{H^2} \left[+1 - \gamma + 2\ln 2\right] + \mathcal{O}\left(\frac{1}{a^2}\right) \right\} + \mathcal{O}(d - 3)
$$

(38)

The poles given by the negative values of the argument of the $\Gamma$ function in Eqs. (36,37,38) represent part of the ultraviolet infinities of field theory.

VI. THE ADIABATIC SUBTRACTION

In order to remove the divergent quantities which appear in the integrated quantities as poles in the $\Gamma$ functions, we shall employ the method of adiabatic subtraction [4]. Such a method consists in replacing $x_k$ with an expansion in powers of derivatives of the logarithm of the scale factor in Eqs. (33-35). This expansion coincides with the adiabatic expansion introduced by Lewis in [31] for a time dependent oscillator.

Usually it is more convenient to formulate the adiabatic expansion by using the conformal time $\eta$ [4] ($d\eta = dt/a$). We follow this procedure and write an expansion in derivatives with respect to the conformal time (denoted by $'$) for $x_k$. Then go back to the cosmic time and we insert the expansion in the expectation values we wish to compute. Adiabatic expansion in cosmic time and conformal time lead to equivalent results, because of the explicit covariance under time reparametrization [12].

We rewrite Eq. (24) in conformal time in the following way:

$$(ax_k)'' + \Omega_k^2 (ax_k) = \frac{1}{(ax_k)^3}$$

(39)

where
\[ \Omega_k^2 = k^2 + m^2 a^2 - \frac{1}{6} a^2 R \]  

(40)

and \( R \) is the Ricci curvature:

\[ R = 6 \frac{a''}{a^3}. \]  

(41)

The fourth order expansion for \( \langle \phi^2 \rangle \), the energy and pressure densities are therefore (as before \( \nu = 3/2 \), integrands in 3 space dimensions and the measure is analytically continued in \( d \) dimensions), using the expression in (76) and the results of appendix A and B,

\[
\langle \phi^2 \rangle^{(4)} = \frac{\hbar}{16\pi^2} H^2 \left\{ -2 + \frac{4m^2}{3H^2} \right\} \left( \frac{am}{2\pi^{1/2}} \right)^{d-3} \Gamma \left( \frac{1}{2} - \frac{d}{2} \right) + \frac{2m^2}{9H^2} - \frac{4}{3} + \\
+ \frac{1}{m^2} \left[ \frac{7}{45} m^2 + \frac{29}{15} H^2 \right] + \mathcal{O}\left( \frac{1}{a^3} \right) \right\} + \mathcal{O}(d - 3) 
\]  

(42)

\[
\epsilon^{(4)} = \langle T_{00} \rangle^{(4)} = \frac{\hbar}{16\pi^2} H^4 \left\{ \frac{4m^2}{H^2} \left( \frac{1}{2\pi^{1/2}} \right)^{d-3} a^{d-3} m^{d-3} \Gamma(1 - d) \right. \\
+ \frac{119}{60} + \frac{m^2}{H^2} \left[ \frac{33}{10} + \gamma \right] + \mathcal{O}\left( \frac{1}{a^3} \right) \right\} + \mathcal{O}(d - 3) 
\]  

(43)

\[
p^{(4)} = \frac{\langle T_{ii} \rangle^{(4)}}{a^2} = \frac{\hbar}{16\pi^2} H^4 \left\{ -4 \frac{m^2}{H^2} \left( \frac{1}{2\pi^{1/2}} \right)^{d-3} a^{d-3} m^{d-3} \Gamma(1 - d) \right. \\
- \frac{119}{60} + \frac{m^2}{H^2} \left[ \frac{1309}{270} - \gamma \right] + \mathcal{O}\left( \frac{1}{a^3} \right) \right\} + \mathcal{O}(d - 3) 
\]  

(44)

VII. THE CONSERVED RENORMALIZED EMT

On subtracting the adiabatic part given in Section VI from the bare integrated contribution given in Section V and taking the limit \( d \to 3 \) one obtains the finite renormalized expectation value for the correlator and for the energy-momentum tensor.

The renormalized expectation value of \( \langle \phi^2 \rangle \) is therefore, neglecting terms of order \( 1/a^3 \) and for \( a > H/m \) \[28\],

\[
\langle \phi^2 \rangle^{\text{REN}} = \langle \phi^2 \rangle - \langle \phi^2 \rangle^{(4)} = \frac{\hbar}{16\pi^2} H^2 \left\{ 4 - \frac{8m^2}{3H^2} \right\} \left( \ln a - \ln \frac{CH(t_0)}{m} \right) - \left( 1 - \frac{2m^2}{3H^2} \right) 4 \ln 2 + \\
- \frac{14m^2}{9H^2} + \frac{10}{3} - \frac{1}{m^2} \left[ \frac{7}{45} m^2 + \frac{29}{15} H^2 \right] + \mathcal{O}\left( \frac{1}{a^3} \right) \right\}. 
\]  

(45)

Considering (45), we note that for late times it resembles more a massless, than a massive, field in de Sitter space-time. This is a consequence of the scale invariant spectrum (114) of inflaton fluctuations (the same spectrum occurs for massless minimally coupled fields in de
Sitter space-time). The leading behaviour for $\langle \varphi^2 \rangle_{\text{REN}}$ agrees for late times with the de Sitter result \[14,29,32\] when $\dot{H} = 0$ and $a(t) = a_0 e^{H_{\text{DS}} t}$:

$$
\langle \varphi^2 \rangle_{\text{DS \, REN}} \sim \frac{\hbar}{4\pi^2} H_{\text{DS}}^3 t.
$$

At earlier times, $\langle \varphi^2 \rangle_{\text{REN}}$ is dominated by a contribution $O(\ln a)$ takes over. We warn the reader about the massless limit taken at face value of Eq. (45). In the massless limit, as discussed in the appendix B, one has to use a different analytic continuation for the adiabatic part, related to the expression in (77) and the result for $\langle \varphi^2 \rangle_{\text{REN}}$ will be finite, different from Eq. (45), but with the same leading contribution (46). However, this massless limit is just of academic interest, since for $m = 0$ inflation would not happen.

Even if both inflaton fluctuations for a massive inflaton and a massless minimally coupled scalar fields in de Sitter share the same scale invariant spectrum, the energy and pressure carried by fluctuations for these two cases are very different. For the latter case, a linear growth in time is present only for the correlator; the EMT does not contain the correlator, but only bilinear quantities less infrared than $\varphi^2$ (for a nice explanation of this difference see [12]). For the case of inflaton fluctuations, the correlator appears directly in the EMT because of the nonvanishing mass. The kinetic and gradient terms should be smaller than the potential term, as for the massless minimally coupled case. Therefore for the case of inflation driven by a massive inflaton, the EMT of inflaton fluctuations should grow in time. This is what we shall show in the following.

On subtracting Eq. (43) from Eq. (37), the renormalized energy density $\epsilon_{\text{REN}}$ is:

$$
\epsilon_{\text{REN}} = \langle T_{00} \rangle_{\text{REN}} = \langle T_{00} \rangle - \langle T_{00} \rangle_{(4)} = \frac{\hbar}{16\pi^2} H^4 \left\{-2 \frac{m^2}{H^2} \left[ \ln \frac{CH(t_0)}{m} - \ln a(t) \right] - \frac{119}{60} + \frac{m^2}{H^2} \left[ \frac{23}{10} - 2 \ln 2 \right] + O \left( \frac{1}{a^2} \right) \right\}.
$$

Similarly, by subtracting Eq. (44) from Eq. (38), the renormalized pressure density $p_{\text{REN}}$ is:

$$
p_{\text{REN}} = \frac{\langle T_{ii} \rangle_{\text{REN}}}{a^2} = \frac{\langle T_{ii} \rangle - \langle T_{ii} \rangle_{(4)}}{a^2} = \frac{\hbar}{16\pi^2} H^4 \left\{2 \frac{m^2}{H^2} \left[ \ln \frac{CH(t_0)}{m} - \ln a(t) \right] + \frac{119}{60} + \frac{m^2}{H^2} \left[ \frac{1039}{270} + 2 \ln 2 \right] + O \left( \frac{1}{a^2} \right) \right\}
$$

We note that this result does not agree for $\dot{H} = 0$ (and therefore for $m = 0$ because of Eq. (8)) and $\xi = 0$ with the de Sitter result \[3,30\] obtained with the Bunch-Davies vacuum:

$$
T_{\mu\nu}^{\text{DS \, BD}} = \frac{g_{\mu\nu}}{64\pi^2} \left[ m^2 \left[ m^2 + \left( \xi - \frac{1}{6} \right) R \right] \left[ \psi \left( \frac{3}{2} + \nu \right) + \psi \left( \frac{3}{2} - \nu \right) - \ln \frac{12m^2}{R} \right] - m^2 \left( \xi - \frac{1}{6} \right) R - \frac{1}{18} m^2 R - \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2 R^2 + \frac{1}{2160} R^2 \right]
$$

where $R$ is the curvature in de Sitter ($R = 12H_{\text{DS}}^2$ with $H_{\text{DS}}$ as the Hubble parameter in de Sitter space-time) \[13,34\]. Indeed, the limit of Eq. (49) for vanishing mass and coupling is finite and is \[4\]:

13
Instead, the result (47,48) agrees with the Allen-Folacci [14] result for $m = 0$:

$$T_{\mu\nu}^{\text{DS AF}} = \frac{119 H^4_{\text{DS}}}{960\pi^2} g_{\mu\nu}$$

(51)

However, the renormalized EMT of inflaton fluctuations in Eqs. (47,48) grows in time, as the logarithm of the scale factor. This feature is due both to the fact that the inflaton is massive and to the infrared state in which its fluctuations are generated. For a test field in de Sitter space-time a linear growth in time of the EMT is possible, only for $m^2 + \xi R = 0$, with $m$ and $\xi$ both different from zero.

The renormalized EMT in de Sitter space-time in the Bunch-Davies vacuum (49) and in the Allen-Folacci vacuum corresponds to a perfect fluid with an equation of state $w = p/\epsilon = -1$, which is identical to the background driven by a cosmological constant. The conservation of the renormalized EMT (49,51) is direct consequence of its symmetries:

$$T_{\mu\nu}^{\text{DS} \, \text{REN}} \propto g_{\mu\nu} \rightarrow \nabla^\mu T_{\mu\nu}^{\text{DS} \, \text{REN}} = 0$$

(52)

since $\nabla^\mu g_{\mu\nu} = 0$.

The renormalized EMT in an inflationary stage driven by a massive inflaton corresponds to a perfect fluid, but with an equation of state which differs from $-1$ by terms $\mathcal{O}(m^2/H^2)$, as one can see from Eqs. (47) and (48).

The derivation of the conservation of the renormalized EMT in chaotic inflation is also straightforward. The renormalized EMT is conserved consistently with the approximation used, i.e. to the order $\mathcal{O}(m^2/H^2)$. The conservation can be easily checked mode by mode, i.e. by considering the covariant derivative inside the integrals in $\mathbf{k}$ in the difference between the bare value and the fourth order adiabatic value and using the equations of motion for the field modes (8). The conservation of the final renormalized value EMT can also be checked by inserting the expressions given by Eqs. (47) and (48) in

$$\frac{d\epsilon_{\text{REN}}}{dt} + 3H(\epsilon_{\text{REN}} + p_{\text{REN}}) = 0.$$  

(53)

and retaining only the terms up to and including $\mathcal{O}(m^2/H^2)$.

VIII. BACK-REACTION ON THE GEOMETRY

We now discuss the back-reaction of the amplified $\varphi$ fluctuations on the geometry.

We consider the back-reaction equations perturbatively: we evaluate the higher order geometrical terms [4] generated by renormalization as their background value and we do not use them to generate higher order differential equations. We note that, in accord with the approximations used, many higher order derivative terms are implicitly absent in Eqs. (17,18), since they would be of higher order in powers of $\dot{H}/H^2$ and because $\dot{H} \simeq 0$. We estimate the back-reaction effects without changing the structure of the left hand side of Einstein equations. Hence the back-reaction equations we consider are:
\[ H^2 = \frac{8\pi G}{3} \left[ \frac{\dot{\phi}^2}{2} + \frac{m^2}{2} \phi^2 + \epsilon_{\text{REN}} \right] \]  \( (54) \)

\[ \dot{H} = -4\pi G (\dot{\phi}^2 + \epsilon_{\text{REN}} + p_{\text{REN}}). \]  \( (55) \)

The main point is that the energy and pressure of inflaton fluctuations grows in time as the logarithm of the scale factor, while the Hubble parameter driven by the background energy density decreases linearly in time. However, the approximation we use, i.e. Eqs. (44), is valid for a certain time interval, \( \Delta t \) given by:

\[ \Delta t \sim \frac{H_0}{m^2} \]  \( (56) \)

Therefore the term which grows in the renormalized EMT will saturate at the value:

\[ \epsilon_{\text{REN}}(\Delta t) \sim -p_{\text{REN}}(\Delta t) \sim \frac{\hbar}{8\pi^2} H^2 H_0^2. \]  \( (57) \)

The term (57) is larger than the Allen-Folacci value \( \sim H^4 \) in Eq. (51) and has the opposite sign. Therefore, the energy density of fluctuations starts with a negative value and changes sign to a positive value when the logarithm takes over. This behaviour is shown in Figs. 5 and 3.

Such a value leads to a contribution to the Einstein equations of the order \( H^2 H_0^2/M_{\text{pl}}^2 \). The importance of back-reaction is therefore related to the ratio \( H_0^2/M_{\text{pl}}^2 \). If inflation starts at a Planckian energy density then back-reaction during inflation cannot be neglected.

A slightly different conclusion can be reached on considering the variation in time of the Hubble parameter, i.e. Eq. (52). The important point to note is that the leading contribution in \( \epsilon_{\text{REN}}, p_{\text{REN}} \), i.e. the terms \( \sim H^4 \) and \( \sim m^2 H^2 \log a \), do not contribute to \( \dot{H} \) since these terms have an equation of state \( p_{\text{REN}}/\epsilon_{\text{REN}} = -1 \). The contribution of the inflaton fluctuations to \( \dot{H} \) is therefore of order \( m^2 H^2/M_{\text{pl}}^2 \), which is suppressed with respect to the classical value by the factor \( H^2/M_{\text{pl}}^2 \).

\[ \text{FIG. 5. Time evolution of the renormalized energy, where the cosmological time is in units of 1/m.} \]
FIG. 6. Time evolution of the magnitude of the back-reaction $8\pi G \epsilon_{\text{REN}}/(3H^2|_{\epsilon_{\text{REN}}=0})$, where the cosmological time is in unit of $1/m$.

IX. COMPARISON WITH THE SLOW-ROLLOVER CALCULATION

We now compare the approximation which lead us to Eq. (15) with the slow-rollover technique introduced by Stewart and Lyth [18]. The latter technique was developed directly for scalar and tensor metric perturbations [18], and not for field perturbations in rigid space-times, as treated here. However, the equation for gravitational waves differ from Eq. (8) only by the presence of the mass term $m$. Therefore, as a first check we note from Eq. (14) that

$$\nu_{\text{GW}}^2 = \frac{9}{4} - \frac{3}{2} \frac{\dot{H}}{H^2} \nu_{\text{GW}} \simeq \frac{3}{2} - \frac{\dot{H}}{H^2}$$

(58)

where the second relation holds when $\dot{H}/H^2$ is small. The value $\nu_{\text{GW}}$ in Eq. (58) coincides with the index $\mu$ of Eq. (41) of [18]. On the other hand, if one applies the Stewart-Lyth procedure to Eq. (8) one would obtain an index for Hankel functions which is precisely $\nu$ in Eq. (14).

A natural question is to ask whether inflaton fluctuations are generated in infrared states also for other models of chaotic inflation. Analytic approximations, such as the one presented in Section 2, are very difficult to obtain. However, since our calculation agrees with the slow-rollover result [18] for inflaton fluctuations in rigid space-times and gravitational waves in the case of a massive inflaton, one can use the latter technique to estimate the spectral index of fluctuations in a generic chaotic model with potential $V(\phi) = \lambda \phi^n/n$ (here $\lambda$ has the dimensions of a mass elevated to the power $4-n$). We follow Stewart and Lyth and we study the equation:

$$(a\varphi_k)'' + \left( k^2 + m_{\text{eff}}^2 a^2 - \frac{a''}{a} \right) (a\varphi_k) = 0,$$

(59)

where $m_{\text{eff}}^2 = V_{,\phi\phi} = \lambda (n-1) \dot{\phi}^{n-2}$. On using $a(\eta) = -1/|H \eta (1 - \epsilon)|$, after some algebra one obtains the following spectral index for the constant mode:
\[ \nu = \frac{1}{2} + \frac{1 - 2\epsilon + 2\epsilon/n}{1 - \epsilon} \]  

(60)

where the slow-rollover parameter \( \epsilon \) is taken as constant and is defined as:

\[ \epsilon = -\frac{\dot{H}}{H^2} = \frac{n^2}{16\pi G\phi^2} \]  

(61)

The slow-rollover approximation is better for large values of the inflaton. The result (60) agrees for \( n = 2 \) with the quadratic case. However, for \( n > 2 \) the spectral index \( \nu \) is smaller than the critical value \( 3/2 \), leading to inflaton fluctuations which are infrared finite.

Although the approximation presented here gives the same result as the slow-rollover technique to the lowest order [18] for gravitational waves and inflaton fluctuations in rigid space-times for a massive chaotic model, it will not be so when one includes metric perturbations [17].

**X. MODULI PRODUCTION**

Let us now discuss the quantum production of a light scalar field \( \chi \) in this model of massive chaotic inflation. By the term light, we mean a scalar field with a mass \( M \), which is smaller than (or equal to) the inflaton one, \( m \). We assume a vanishing homogeneous component for \( \chi \). Therefore, for \( \chi \) the approximation of a rigid space-time is correct.

With \( M < m \), we see from Eqs. (34) that the index \( \nu_\chi \) for the Hankel functions, which are involved in the solutions for the \( \chi \) modes, is larger than \( 3/2 \). The leading contribution for the renormalized value of \( \langle \chi^2 \rangle \) is:

\[ \langle \chi^2 \rangle_{\text{REN}} \sim \frac{1}{16\pi^2} \beta H^2 \frac{a^{2\nu_\chi - 3} - 1}{2\nu_\chi - 3} \]  

(62)

where \( \beta \) is a numerical coefficient. This formula generalizes Eq. (45) to the case of a mass \( M < m \). The case \( M = m \) represents a limiting case, for which a logarithm appears. Analogously, in de Sitter space-time, a limiting case is also present for \( \nu = 3/2 \) [12], and a logarithm of the scale factor appears instead of a power.

For a massive \( \chi \) the following relation holds:

\[ \nu_{GW} > \nu_\chi > \nu = \frac{3}{2} \]  

(63)

where \( \nu_{GW} \) is defined in Eq. (58) and, owing to the smallness of the ratio \( m^2/H^2 \), \( \nu_\chi \) is very close to \( 3/2 \). However, the growth in time of \( \langle \chi^2 \rangle_{\text{REN}} \) is more rapid than the growth of \( \langle \phi^2 \rangle_{\text{REN}} \).

This fact can be checked numerically: for example, on taking the initial condition for inflation already used in the previous sections for the other numerical checks, one finds for \( M = 0, \nu_\chi \simeq 1.5043 \) and for \( M = 0.5m, \nu_\chi \simeq 1.5032 \). Such a value is time independent after a very short transient phase needed for the modes to freeze and corresponds to the one given in (58) for \( H \) computed for a time very close to the beginning of the inflation.

We can therefore substitute \( H(t \simeq 0) \simeq H_0 \) in place of \( H(t) \) in \( \nu \). We have checked this
relation for different initial $H_0$. This fact is in agreement with the fact that the approximate solution, which we employed for the inflaton field, can be used for other fields with different masses, but only at the beginning of the inflation period. Thus we can control the spectral behaviour of the moduli field, but not its normalization.

To study the behaviour in (62) it is convenient to rewrite the evolution of the scale factor $a$ and its exponent $(2\nu - 3)$ as

$$a(t) = \exp \left( \frac{3}{2} \frac{H_0^2 - H^2}{m^2} \right) , \quad 2\nu - 3 \simeq \frac{2m^2 - M^2}{H_0^2}.$$  \hspace{1cm} (64)

Therefore the main result (let us only write the dominant contribution) is that for $M < m$ and at the end of inflation

$$\langle \chi^2 \rangle_{\text{REN}} \sim H^2 H_0^2 \left( \frac{\exp \left[ \frac{H_0^2 - H^2}{m^2} \left( 1 - \frac{M^2}{m^2} \right) \right] - 1}{m^2 - M^2} \right) > \langle \varphi^2 \rangle_{\text{REN}} \sim H^2 \frac{H_0^2}{m^2}$$  \hspace{1cm} (65)

and for the renormalized EMT associated with the $\chi$ field

$$\epsilon^{(\chi)}_{\text{REN}} \sim \frac{3}{16\pi^2} \frac{M^2}{m^2} H^2 H_0^2 \left\{ \exp \left[ \frac{H_0^2 - H^2}{H_0^2} \left( 1 - \frac{M^2}{m^2} \right) \right] - 1 \right\}.$$  \hspace{1cm} (66)

The results in Eqs. (57, 65, 66) are very interesting. They show that the production of a scalar field $\chi$ with mass $M$ smaller or equal to the inflaton mass $m$ depends on the duration of inflation and is larger than the usual extrapolation of the de Sitter result ($\langle \chi^2 \rangle \sim H^4/m^2$ and $\epsilon_\chi \sim (M^2/m^2)H^4$). This result implies that the quantum production of light scalar fields in chaotic inflation with a mass term is even greater than expected on extrapolating the de Sitter result, and depends on the duration of inflation, as is also stated in [21,35]. Again, if inflation starts at a Planckian energy density, the back-reaction of light scalar fields cannot be neglected during inflation.

We note that the factor in curly brackets in (65, 66) is larger than the logarithmic term one has for the $\nu = 3/2$ case, by a factor of 2.

We see that on computing the exact numerical solution of the Pinney equation for the modes of the inflaton with mass $m$ and for a moduli field with $M = 0.5m$, one obtains for the same momenta amplitudes more than one order of magnitude larger for the latter at the end of inflation, again in agreement with [21]. We therefore expect an enhancement of 2-3 orders of magnitude for its backreaction with respect to the inflaton case.

**XI. DISCUSSION AND CONCLUSIONS**

We have computed the renormalized conserved EMT of the inflaton fluctuations $\varphi(t,x)$ in rigid space-times during the inflationary stage driven by a mass term. The method of dimensional regularization has been applied by using an analytic approximation valid during the slow-rollover regime. All the results agree with the Allen-Folacci results for $T_{\mu\nu}$ of a test field in de Sitter space-time [14], in the limit for which the Hubble parameter is constant (which is also the massless limit because of Eq. (3)).
We find that the EMT of inflaton fluctuations grows in time. The reason for this behavior is that chaotic inflation driven by a massive scalar field produces a scale invariant spectrum of fluctuations even if the field is massive. This effect is due to the decrease of the Hubble parameter during the slow-rollover regime.

In de Sitter space-time, the renormalized EMT of a quantum field grows linearly in time only if \( m^2 + \xi R_{DS} = 0 \) with \( m \) and \( \xi \) different from zero \(^{12}\). A massless minimally coupled scalar field in de Sitter space-time, characterized by a scale invariant spectrum of fluctuations, leads to a correlator which grows in time. However, only bilinear quantities less infrared than the correlator appear in the EMT, and therefore the expectation value of the EMT of a massless minimally coupled scalar field is constant in time \(^{12}\). In massive chaotic inflation, inflaton fluctuations are generated with a scale invariant spectrum. Since the correlator appears directly in the EMT because of the nonvanishing mass, then the renormalized EMT grows in time just as the correlator does.

We find that the growth of the EMT of inflaton fluctuations during slow-rollover leads to a positive energy density which reaches a maximum value \( O(H^2 H_0^2) \), where \( H_0 \) is the Hubble radius at the beginning of inflation. This value exceeds the usual value \( O(H^4) \), which is of the same order of magnitude as the conformal anomaly. These values also show that back-reaction effects cannot be neglected if inflation starts at Planckian energies, i.e. at \( H_0 \sim M_{Pl} \). If inflation started at Planckian energies, although the contribution of the terms \( O(H^2 H_0^2) \) and of the conformal anomaly would be of the same order of magnitude, we think that the two contributions could be different because of the different signs and of the different behaviours in time.

In this model of chaotic inflation, we have also analyzed the geometric production of an additional field \( \chi \) with mass \( M \) smaller than the inflaton mass \( m \). Of course, on considering the normalization, we have found that \( \epsilon_\chi > \epsilon_\phi \sim H^2 H_0^2 \). This result implies that the quantum production of light fields depends on the duration of inflation and it is greater than expected on extrapolating the de Sitter result (in de Sitter \( \epsilon_\chi \sim H^4 \)). As in the case of inflaton fluctuations, the energy density of light scalar fields could be comparable to the background one at the end of inflation, if inflation started at Planckian energy densities. Also in this case, the back-reaction of \( \chi \) fluctuations does not appear to be negligible during inflation.

One may then ask whether this behaviour of the back-reaction due to the fluctuations in rigid space-time is common to other inflationary chaotic models. Analytic approximations, such as the one presented in Section 2, are very difficult to obtain. However, since our calculation agrees with the slow-rollover result \(^{18}\) for the massive case, we have used the latter technique to estimate the spectrum of inflaton fluctuations in rigid space-time for a generic inflaton potential \( V(\phi) = \lambda \phi^n / n \). We have found that for \( n > 2 \) the inflaton fluctuations are generated in an infrared finite state, leading to a back-reaction which does not increase in time. However, we think that we must address the problem while including metric perturbations, in order to fully understand this issue. It is known that chaotic inflationary models predict a spectrum of curvature perturbations which is red tilted \(^{18}\) - i.e. with a spectrum more infrared than the scale invariant one - , a result which does not hold for field perturbations in rigid space-time, as we have shown. Since infrared states could lead to a correlator which grows in time, the possibility exists that a back-reaction growing in time is common to all the chaotic inflationary models once metric perturbations...
are included.

Other important issues are whether an eventual self-consistent scheme to include the back-reaction would prevent the development of infrared states. The effect of the self-consistent inclusion of back-reaction effects on the spectrum of fluctuations during inflation is, to our knowledge, an issue still to be fully explored. It would be interesting also to investigate the effect of the inclusion of the higher order terms [36] in the back-reaction equations [34,53]. Obviously, the Starobinsky model [9] is a surprising example of the importance of higher order terms.

Acknowledgments

We would like to thank Raul Abramo, Robert Brandenberger, Sergei Khlebnikov and Igor Tkachev for discussions and comments on the manuscript. One of us (F. F.) would like to thank Salman Habib and Katrin Heitmann for many important discussions on renormalization in curved space-times and for warm hospitality at Los Alamos Laboratories, where part of this work was written.

XII. APPENDIX A: THE ADIABATIC FOURTH ORDER EXPANSION

From Eqs. (39,40,41) one obtains the expansion for $x_k$ up to the fourth adiabatic order:

$$x_k^{(4)} = \frac{1}{a} \frac{1}{\Omega_k^{1/2}} \left( 1 + \frac{1}{4} \epsilon_2 + \frac{5}{32} \epsilon_2^2 - \frac{1}{4} \epsilon_4 \right)$$

(67)

where $\Omega_k$ is defined in Eq. (40) and $\epsilon_2, \epsilon_4$ are given by:

$$\epsilon_2 = -\frac{1}{2} \frac{\Omega_k''}{\Omega_k} + \frac{3}{4} \frac{\Omega_k'^2}{\Omega_k^3}$$

$$\epsilon_4 = \frac{1}{4} \frac{\Omega_k''}{\Omega_k} - \frac{1}{4} \frac{1}{\Omega_k^2} \epsilon_2''$$

(68)

The solution in Eq. (67) must be expanded again since the Ricci curvature is of adiabatic order 2. Therefore $x_k^{(4)}$ is:

$$x_k^{(4)} = \frac{1}{c^{1/2} \Sigma_k^{1/2}} \left\{ \frac{1}{4} \frac{1}{c} \frac{R}{\Sigma_k^2} + \frac{5}{32} \frac{c^2 R^2}{36 \Sigma_k^4} + \frac{1}{16} \frac{1}{c^2 \Sigma_k^4} \left[ c'' \left( m^2 - \frac{R}{6} \right) - 2c' \frac{R'}{6} - c \frac{R''}{6} \right] - \frac{5}{64} \frac{1}{\Sigma_k^6} \left[ c'' m^4 - 2c' m^2 \frac{R}{6} - 2c' m^2 c \frac{R'}{6} \right] + \frac{9}{64} \frac{1}{\Sigma_k^6} \frac{R}{c} c'' m^2 - \frac{65}{256} \frac{1}{\Sigma_k^6} c' m^4 + \frac{5}{32} \epsilon_2^{2s} - \frac{1}{4} \epsilon_4^s \right\}$$

(69)

where $c = a^2$ and
\[ \Sigma_k = (k^2 + a^2 m^2)^{1/2} \]
\[ \epsilon_{2*} = -\frac{1}{2} \Sigma_k' + \frac{3}{4} \Sigma_k^2 \]
\[ \epsilon_{4*} = \frac{1}{4} \Sigma_k' \epsilon_k - \frac{1}{4} \Sigma_k^2 \epsilon_k \tag{70} \]

**XIII. APPENDIX B: DIMENSIONAL REGULARIZATION WITH CUT-OFF**

We start with \( \varphi^2 \) as an example. According to the discussion following Eq. (29), we neglect the infrared piece of the integral since it gives a small finite part. Therefore the relevant integral

\[ \langle \varphi^2 \rangle = \frac{1}{\pi^3} \frac{\hbar}{a^3} \frac{\lambda}{4H} \Gamma(3/2) \int_0^{\infty} dk k^2 x^2 \]

\[ = \frac{\hbar}{(2\pi)^3} \frac{\lambda}{a^3} \frac{\pi}{4H} \Gamma(3/2) \int_0^{\infty} dk k^2 \left[ J^2_\nu \left( \frac{\lambda k}{aH} \right) + N^2_\nu \left( \frac{\lambda k}{aH} \right) \right] \tag{71} \]

on extending it to \( d \)-dimensions (integrands in 3 space dimensions and analytic continuation of the measure to \( d \) dimensions):

\[ \langle \varphi^2 \rangle = \frac{\hbar}{(2\pi)^d} \frac{\lambda}{a^3} \frac{\pi}{4H} \Gamma(d/2) \int_0^{\infty} dk k^{d-1} \left[ J^2_\nu \left( \frac{\lambda k}{aH} \right) + N^2_\nu \left( \frac{\lambda k}{aH} \right) \right]. \tag{72} \]

The two-point function can be computed by using the following integral \[37\]:

\[
\int_0^{\infty} dx \, x^\alpha (J^2_\nu (x) + N^2_\nu (x)) = \frac{1}{\pi^2} \left[ \cos \left( \frac{\pi}{2} (\alpha + 1 - 2\nu) \right) \frac{\Gamma((\alpha + 1)/2) \Gamma((\alpha + 1)/2 - \nu)}{\Gamma(1 + \alpha/2) \Gamma((1 - \alpha)/2)} + \sin \left( \frac{\pi}{2} (\alpha + 1 - 2\nu) \right) \frac{\Gamma(-\alpha/2) \Gamma((1 - \alpha)/2 + \nu)}{\Gamma((1 - \alpha)/2) \Gamma((1 - \alpha)/2 + \nu)} \right] \\
\times \Gamma(\alpha/2 + \nu) + \frac{1}{(\alpha + 1) \nu} \left[ 2\pi \left( \frac{\lambda \ell}{aH} \right)^d \right] \Gamma(\pi \nu) _2 F_3 \left( \frac{1}{2}, \frac{\alpha + 1}{2}, \frac{\alpha + 3}{2}, 1 - \nu, 1 + \nu; - \left( \frac{\lambda \ell}{aH} \right)^2 \right) \\
- \frac{1}{(\alpha + 1 + 2\nu) \Gamma(1 + \nu)^2} \left[ 4^{-\nu} \left( \frac{\lambda \ell}{aH} \right)^{\alpha + 1 + 2\nu} (2\pi^2 + \cos(2\nu \pi) \Gamma(-\nu)^2 \Gamma(1 + \nu)^2) \\
\times _2 F_3 \left( \frac{1}{2} + \nu, \frac{\alpha + 1}{2} + \nu; 1 + \nu, \frac{\alpha + 3}{2} + \nu, 1 + 2\nu; - \left( \frac{\lambda \ell}{aH} \right)^2 \right) \right]. \tag{73} \]

With \( \nu = 3/2 \) and \( \alpha = d - 1 \), and on using:

\[ \]
the expectation value for $\langle \varphi^2 \rangle$ is:

$$
\langle \varphi^2 \rangle = \frac{\hbar}{16\pi^2} H^2 \left( 1 - \frac{2m^2}{3H^2} \right) \left\{ 2 - 4 \ln 2 - 2 \left( \frac{\ell}{2} \right)^{d-3} \Gamma \left( \frac{1}{2} - \frac{d}{2} \right) + O \left( \frac{1}{a^2} \right) \right\} + O(d - 3)
$$

Similarly, the integral used for the fourth order adiabatic quantities in $d$-dimensions is (integrands in 3 space dimensions and analytic continuation of the measure to $d$ dimensions):

$$
\frac{2\pi^{\alpha/2+1/2}}{\Gamma((\alpha + 1)/2)} \int_0^{\ell} dk k^{\alpha} \frac{1}{(k^2 + a^2 m^2)^{n/2}} = \frac{2\pi^{\alpha/2+1/2}}{\Gamma((\alpha + 1)/2)} \int_0^{\ell} dk k^{\alpha} \frac{1}{(k^2 + a^2 m^2)^{n/2}} - \frac{2\pi^{\alpha/2+1/2}}{\Gamma((\alpha + 1)/2)} \int_0^\ell dk k^{\alpha} \frac{1}{(k^2 + a^2 m^2)^{n/2}}
$$

$$
= \pi^{\alpha/2+1/2} \left( a^2 m^2 \right)^{\alpha/2+1/2-n/2} \Gamma(n/2 - \alpha/2 - 1/2) \Gamma(n/2) - \frac{2\pi^{\alpha/2+1/2}}{\Gamma((\alpha + 1)/2)} \frac{1}{1 + \alpha} (a^2 m^2)^{-n/2} \ell^{1+\alpha}
$$

$$
\begin{aligned}
2F_1 \left( \frac{n}{2}, \frac{1 + \alpha}{2}; \frac{3 + \alpha}{2}; - \left( \frac{\ell^2}{a^2 m^2} \right) \right)
\end{aligned}
$$

Let us note that on taking the massless limit one can analytically continue the hypergeometric function and after straightforward calculations one gets

$$
- \frac{2\pi^{\alpha/2+1/2}}{\Gamma((\alpha + 1)/2)} \frac{1}{1 + \alpha - n} \ell^{1+\alpha-n},
$$

which is of course the result one would obtain setting $m = 0$ from the beginning. Therefore all the massless singularities in (76) correctly cancel. We note that in the massless limit the analytic continuation misses the UV divergencies stronger than the logarithmic ones.

On again considering the case $m \neq 0$ such that $a(t) > H/m$ and using the result (74), which also holds also for $2F_1$, one obtains, to the fourth adiabatic order, for $\langle \varphi^2 \rangle$:

$$
\langle \varphi^2 \rangle_{(4)} = \frac{\hbar}{16\pi^2} H^2 \left\{ -2 + \frac{4m^2}{3H^2} \right\} \left( \frac{am}{2\pi^{1/2}} \right)^{d-3} \Gamma \left( \frac{1}{2} - \frac{d}{2} \right) + \frac{2m^2}{9H^2} - \frac{4}{3} + \frac{1}{m^2} \left[ \frac{7}{45} m^2 + \frac{29}{15} H^2 \right] + O \left( \frac{1}{a^2} \right) \right\} + O(d - 3)
$$
REFERENCES

[1] L. Parker, *Phys. Rev.* **183**, 1057 (1969).

[2] A. D. Linde, *Particle Physics and Inflationary Cosmology* (Harwood, Chur, Switzerland, 1990); E. W. Kolb and M. S. Turner, *The Early Universe* (Addison-Wesley, Redwood City, California, 1990).

[3] V. F. Mukhanov and G. V. Chibisov, JETP Lett. **33**, 532 (1981); S. W. Hawking, Phys. Lett. **115B**, 295 (1982); A. A. Starobinsky, Phys. Lett. **117B**, 175 (1982); A. H. Guth and S.-Y. Pi, Phys. Rev. Lett. **49**, 1110 (1982).

[4] L. Parker and S. A. Fulling, *Phys. Rev.* **D 9**, 341 (1974).

[5] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, 1982).

[6] L. Parker and S. A. Fulling, *Phys. Rev.* **D 7**, 2357 (1974). For an analogous effect at the quantum mechanical level (where only the homogeneous mode of the inflaton is quantized) see [24].

[7] D. M. Capper and M. J. Duff, *Nuovo Cimento* **23A**, 173 (1974); for an enjoyable review see M. J. Duff, *Class. Quant. Grav.* **11**, 1387 (1994).

[8] M. Visser, *Lorentzian Wormholes: from Einstein to Hawking*, (Woodbury, New York, 1995)

[9] A. A. Starobinsky, *Phys. Lett.* **91B**, 99 (1980).

[10] J. S. Dowker and R. Critchley, *Phys. Rev.* **D 13**, 3224 (1976).

[11] P. R. Anderson and W. Eaker, *Phys. Rev.* **D 61**, 024003 (2000).

[12] S. Habib, C. Molina-Paris, and E. Mottola, *Phys. Rev.* **D 61**, 024010 (2000).

[13] P. R. Anderson, W. Eaker, S. Habib, C. Molina-Paris, and E. Mottola, *Phys. Rev.* **D 62**, 124019 (2000).

[14] B. Allen and A. Folacci, *Phys. Rev.* **D 35**, 3771 (1987).

[15] A. D. Linde, *Phys. Lett.* **129B**, 177 (1983).

[16] R. Abramo, R. Brandenberger, and V. F. Mukhanov, *Phys. Rev. Lett.* **78**, 1624 (1997); *Phys. Rev.* **D 56**, 3248 (1997).

[17] F. Finelli, G. Marozzi, G. P. Vacca, and G. Venturi, in preparation.

[18] E. Stewart and D. Lyth, *Phys. Lett.* **302B**, 171 (1993).

[19] N. A. Chernikov and E. A. Tagirov, *Ann. Inst. Henri Poincare* **A 9**, 109 (1968); E. A. Tagirov, *Ann. Phys. N. Y.* **76**, 561 (1973).

[20] T. S. Bunch and P. C. W. Davies, *Proc. R. Soc. London* **A 360**, 117 (1978).

[21] V. Kuzmin and I. Tkachev, *Phys. Rev.* **D 59**, 123006 (1999).

[22] C. Bertoni, F. Finelli, and G. Venturi, *Phys. Lett.* **A 237**, 331 (1998).

[23] F. Finelli, A. Gruppuso, and G. Venturi, *Class. Q. Grav.* **16**, 3923 (1999).

[24] F. Finelli, G. P. Vacca, and G. Venturi, *Phys. Rev.* **D 58**, 103514 (1998).

[25] J. J. Sakurai, *Modern Quantum Mechanics*, (The Benjamin/Cummings Publishing Company, 1985).

[26] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, (Dover Publ., New York, 1985).

[27] L. H. Ford and Leonard Parker, *Phys. Rev.* **D 16**, 245 (1977).

[28] Let us remember that the evaluation of the renormalized expectation value of $\varphi^2$ is limited by the approximation of neglecting the infrared part of the integral in the expectation values. This infrared part leads to a contribution $O(1/a^3)$ in the fourth order
adiabatic quantities, such as in Eq. (78). For the adiabatic counterterms this approximation is thus very good. Instead, for the bare renormalized expectation value of $\varphi^2$, we (numerically) estimate this infrared contribution to be $\sim H^2/3$. Such a term is negligible with respect to the leading terms $\sim H^2 \log a, H^4/m^2$ in the final finite renormalized value of $\langle \varphi^2 \rangle$ (45), but is of the same order of magnitude of the nonleading terms present in Eq. (45).

[29] A. Vilenkin and L. Ford, *Phys. Rev. D* 26, 1231 (1982).
[30] J. Guven, B. Lieberman, and C. T. Hill, *Phys. Rev. D* 39, 438 (1989).
[31] H. R. Lewis Jr., *Journal of Math. Phys.* 9, 1976 (1968).
[32] A. D. Linde, *Phys. Lett.* 116B, 335 (1982); A. A Starobinsky, *Phys. Lett.* 117B, 175 (1982).

[33] We note that Eq. (49) should also be correct for large $m$, $m > H$. In this case $\nu$ is imaginary, but the solutions with the Bessel functions are valid, and so is the dimensional regularization. From Eq. (49) it is evident that the leading behaviour of the EMT would be $O(m^4 g_{\mu\nu})$, which would correspond to a large renormalized vacuum energy. This result appears puzzling if one believes that there should not be significant particle production for $m \gg H$. However, this behaviour can be understood in the framework of adiabatic subtraction. Even for a large mass $m$ the Green's function of a test field in de Sitter space has a power-law decay, while in Minkowski space-time a massive scalar field has an exponential decay. Since the part one subtracts in the adiabatic subtraction is basically constructed perturbatively from the solutions for Minkowski space-time, we think that the subtraction is not sufficient to eliminate this large vacuum energy contribution.

[34] The renormalization of the EMT of inflaton fluctuations calculated without the cut-off introduced in Eq. (31) would lead to a result which coincides with the Bunch-Davies one (49) for $\dot{H} = 0$. However, this result would be singular since $\nu = 3/2$, and the singularity due to $\psi(3/2 - \nu)$ in the EMT persists since $m \neq 0$. As happens in the de Sitter case in the massless limit for minimally coupled scalar fields, the Bunch-Davies vacuum leads to singular expectations values. In the de Sitter case the two point function is singular, and for massive chaotic inflation both the two point function and $T_{\mu\nu}$ are singular. In the de Sitter case, on switching to the Allen-Folacci vacuum [14], one can have a regular two-point function, which however loses some de Sitter symmetries (it grows in time). It is interesting that we obtain a result which reduces to the Allen-Folacci one when $\dot{H} = 0$, on just requiring regular expectation values. This means that our procedure with the cut-off is consistent with the redefinition of the zero mode for massless minimally coupled scalar fields, which leads to the Allen-Folacci vacuum.

[35] G. Felder, L. Kofman, and A. D. Linde, *JHEP* 0002, 027 (2000).
[36] L. Parker and J. Simon, *Phys. Rev. D* 47, 1339 (1993).
[37] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, INTEGRALS AND SERIES, vol. 2: Special Functions.