Conformal Invariance in Einstein-Cartan-Weyl Space

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We consider conformally invariant form of the actions in Einstein, Weyl, Einstein-Cartan and
Einstein-Cartan-Weyl space in general dimensions (\(>2\)) and investigate the relations among them.
In Weyl space, the observational consistency condition for the vector field determining non-metricity
of the connection can be obtained from the equation of motion. In Einstein-Cartan space a similar
role is played by the vector part of the torsion tensor. We consider the case where the trace part
of the torsion is the Kalb-Ramond type of field. In this case, we express conformally invariant action
in terms of two scalar fields of conformal weight \(-1\), which can be cast into some interesting form.
We discuss some applications of the result.

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I. INTRODUCTION

The issues related to conformal transformation and conformal symmetry in Einstein’s general relativity have been
studied for a long time. Einstein’s general relativity formulated in 1916 was successful in all known experiments and it
describes the reality very well. In spite of these successes, there were some attempts to impose conformal symmetry on
Einstein’s general relativity. One of the remarkable attempts of these was made by Herman Weyl \[1–4\]. He proposed
Einstein’s metricity condition can be generalized to incorporate conformal invariance (\(\nabla_\mu g_{\alpha\beta} \sim C_{\mu}g_{\alpha\beta}\), see next
page for possible space-time geometry). These attempts, however, had a serious problem \[5–7\]. It is that quantum
mechanics provides absolute standards of length and Weyl’s theory leads to observational inconsistency, even though
some authors suggested how to overcome this inconsistency \[8\]. Other attempts \[9–17\] at incorporating conformal
invariance in the theory of general relativity have been studied in the context of particle physics and mathematical
physics. Recently, conformal invariant gravity was suggested as the model of dark energy in \[18, 19\] and conformal
quintessence model was introduced in \[20–23\].

On the other hand, one of the most famous theory for a generalizing Einstein’s theory is the Einstein-Cartan
theory \[24–26\]. In 1920s, Élie Cartan suggested that space-time with torsion can be related to the intrinsic angular
momentum, before the concept of spin was introduced. In the early 1960s, Sciama \[27, 28\] and Kibble \[29\] reinterpreted
Cartan’s theory as the theory of gravitation with spin and torsion. According to them, in order to incorporate spinors
into the theory of general relativity, vierbein(or tetrad) must be introduced. This theory describes General Relativity
in terms of gauge theory with some local gauge transformations, such as the local Poincare group \[27–34\]. Since then,
torsion have been widely studied in general relativity \[33–38\]. Some author showed that torsion tensor corresponds
to Kalb-Ramond field \[39\] in Einstein-Cartan space with Weyl’s non-metricity condition (hereafter referred to as
Einstein-Cartan-Weyl space) \[40\] and the traceless part of contorsion tensor to Kalb-Ramond field in Einstein-Cartan
space \[41\]. In cosmology, some of the applications include torsion quintessence \[42\], possible role of torsion in current
accelerating universe \[43\] and early inflation \[44\].

In particular, the relation between torsion and conformal symmetry was studied by several authors. It was shown
that the torsion could play an important role in conformal invariance of the action and behave like an effective gauge
field in \[45, 46\]. Also, in the non-minimally coupled metric-scalar-torsion theory, for some special choice of the action,
torsion acts as a compensating field and the full theory can be conformally equivalent to general relativity on classical
level \[47, 48\].

In this paper, we consider the local conformal invariance in the Einstein-Cartan-Weyl space and explicitly construct
an action with local conformal invariance. We only pay attention to conformal invariance at classical action level

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and we do not discuss a view point of frame (Einstein frame or Jordan frame)\(^1\). In order to achieve the conformal invariance, we introduce scalar fields which are obtained through some special ansatz of the Weyl gauge field and the trace part of the torsion. For Weyl gauge field, this ansatz is natural because it can solve the problem of observational inconsistency.\(^2\) For the torsion vector field, it is consistent with the equations of motion in Einstein-Cartan space. The role of these scalar fields are some kind of gauge field which give conformal symmetry in the theory. We first construct the conformal-invariant actions in Einstein, Weyl, Einstein-Cartan space in general dimensions and search for relations among them. It can be shown that these actions are all equivalent to each other through the aforementioned ansatz. Then, we extend the analysis to Einstein-Cartan-Weyl space and explicitly construct conformally invariant action. Here, we end up with two scalar fields coming from the Weyl’s gauge field and torsion vector field. One of the motivation is that these two fields could constitute the two scalar fields of the quintom model \(^{54–57}\) and provide a possible geometric origin of the dark energy \(^{58–60}\). We will examine the conformal symmetry in each of the four spaces with the following types of connection:

A. metric compatible and torsionless (Einstein space): \(\nabla_\mu g_{\alpha \beta} = 0\), \(\Gamma^\rho_{\mu \nu} = \{\rho_{\mu \nu}\}\).

B. Weyl’s type and torsionless (Weyl space): \(\nabla_\mu g_{\alpha \beta} \sim \mathcal{C}_\mu g_{\alpha \beta}\), \(\bar{\Gamma}^\rho_{\mu \nu} = \{\rho_{(\mu \nu)}\}\).

C. metric compatible with torsion (Einstein–Cartan space): \(\bar{\nabla}_\mu g_{\alpha \beta} = 0\), \(\bar{\Gamma}^\rho_{\mu \nu} = \{\rho_{(\mu \nu)}\} - K^\rho_{\mu \nu}\).

D. Weyl’s type with torsion (Einstein–Cartan–Weyl space): \(\bar{\nabla}_\mu g_{\alpha \beta} \sim \mathcal{C}_\mu g_{\alpha \beta}\), \(\bar{\Gamma}^\rho_{\mu \nu} = \{\rho_{(\mu \nu)}\} - K^\rho_{\mu \nu}\).

This paper is organized as follows. In Sec. II, we review Weyl’s theory and discuss the conditions for Weyl’s vector field \(\mathcal{C}_\mu\) to avoid the observational consistency problem. In Sec. III, we introduce Einstein-Hilbert action with conformal symmetry and discuss relations with Weyl action. In Sec. IV, Einstein-Cartan action with conformal symmetry is considered and we will show that this action link to other actions. In the course of constructing the conformal-invariant action, we consider two cases. One is to take the trace part of torsion as a physical field itself, the other is as an anti-symmetric tensor \(B_{\mu \nu}\) of Kalb-Ramond type. In Sec V, we consider Einstein-Cartan-Weyl action with conformal symmetry. We briefly summarize the results and discuss them in Sec VI.

Our notations.

The metric signature is \(-,+,+...+\). The Riemann, Ricci tensor and curvature scalar are given by the Christoffel symbols \(\Gamma^\rho_{\mu \nu}\) and the inverse metric \(g^{\mu \nu}\)

\[
\begin{align*}
R^\rho_{\sigma \mu \nu} &= \partial_\mu \Gamma^\rho_{\sigma \nu} - \partial_\nu \Gamma^\rho_{\sigma \mu} + \Gamma^\rho_{\mu \lambda} \Gamma^\lambda_{\sigma \nu} - \Gamma^\rho_{\nu \lambda} \Gamma^\lambda_{\sigma \mu},
R_{\mu \nu} &= \partial_\mu \Gamma^\rho_{\nu \rho} - \partial_\nu \Gamma^\rho_{\mu \rho} + \Gamma^\rho_{\nu \lambda} \Gamma^\lambda_{\rho \mu} - \Gamma^\rho_{\rho \lambda} \Gamma^\lambda_{\mu \nu},
R &= g^{\mu \nu} R_{\mu \nu}.
\end{align*}
\]

The affine connection for A\(\sim\)D is \(\Gamma^\rho_{\mu \nu}\), \(\bar{\Gamma}^\rho_{\mu \nu}\), \(\bar{\Gamma}^\rho_{\mu \nu}\) respectively, i.e.,

\[
\begin{align*}
\nabla_\mu V_\nu &= \partial_\mu V_\nu - \Gamma^\rho_{\mu \nu} V_\rho, \\
\bar{\nabla}_\mu V_\nu &= \partial_\mu V_\nu - \bar{\Gamma}^\rho_{\mu \nu} V_\rho, \\
\bar{\nabla}_\mu V_\nu &= \partial_\mu V_\nu - \bar{\Gamma}^\rho_{\mu \nu} V_\rho.
\end{align*}
\]

\(^1\) In a conformally symmetric theory all choices of frames are equivalent. The change of the conformal frame corresponds to the invariant choice of the dynamical variables at the classical level. However, when matters are coupled in such a way that the conformal invariance is no longer valid, a choice of frame becomes very important. As discussed in Ref. \(^{49}\), a different choice of frame leads to a different physical interpretation of the theory. In this work we do not deal with such issues because we are only dealing with conformally invariant theories.

\(^2\) There also exist literatures \(^{51–53}\) considering conformal invariance in Einstein-Cartan-Weyl space. However, in these papers, this problem was not considered. In our approach, the Weyl gauge field is assumed to have the form of gradient scalar (see eq.\(^{11}\)) from the beginning.
II. A REVIEW OF THE WEYL’S THEORY: INTRODUCTION AND OBSERVATIONAL PROBLEM

In 1918, Herman Weyl suggested an intuitive notion of gauge invariance according to natural generalization of metricity condition used in Einstein’s general relativity. He assumed that Einstein’s metricity condition can be replaced by explicitly imposing conformal symmetry.

\[ \nabla_\mu g_{\alpha\beta} = -2fC_\mu g_{\alpha\beta}. \]

(1)

where \( \nabla \) is the covariant derivative in Weyl space, \( f \) is nonzero constant and \( C_\mu \) is some vector field. This condition is invariant with respect to conformal transformations

\[ g_{\alpha\beta} \rightarrow e^{2\omega}g_{\alpha\beta}, \]

(2)

\[ C_\mu \rightarrow C_\mu - \frac{1}{f}\partial_\mu \omega. \]

(3)

Consequently, for a vector transported around a closed loop by parallel displacement both the direction and length can change, but the angle between two transported vectors must be conserved. And his initial intention is to identify conformal objects as follows

\[ \text{because in Weyl space, Riemann curvature tensor has no the antisymmetric property for first two indices any more,} \]

\[ \text{This is just Einstein-Maxwell action with cosmological constant!} \]

It is interesting to note that there exists 2nd Ricci tensor \( \tilde{R}_{\mu\nu} \), (some author refer it as “homothetic curvature” because in Weyl space, Riemann curvature tensor has no the antisymmetric property for first two indices any more, i.e., \( \tilde{R}_{\mu\nu\rho\sigma} \neq -\tilde{R}_{\nu\mu\rho\sigma} \)). Defining the homothetic curvature as \( \tilde{R}_{\mu\nu} \), it can be written as

\[ \tilde{R}_{\mu\nu} = g^{\rho\sigma}\tilde{R}_{\rho\mu\nu} = \partial_\mu\tilde{\Gamma}_{\nu\rho}^\sigma - \partial_\nu\tilde{\Gamma}_{\mu\rho}^\sigma = nf(\partial_\mu C_\nu - \partial_\nu C_\mu) \]

(6)

\[ \equiv nfF_{\mu\nu}. \]

(7)

Since conformal weight of \( R \) is \(-2\), we can construct the most general conformal action by introducing some scalar field \( \varphi \) (conformal weight is \(-1\)) in 4D as follows

\[ S = \int d^4x\sqrt{-g}\left\{ \frac{\alpha_1}{12}\varphi^2\tilde{R} - \frac{\alpha_2}{2}(D_\mu\varphi)(D_\nu\varphi)g^\mu\nu - \frac{\alpha_3}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4!}\varphi^4 \right\}. \]

(8)

where \( D_\mu = \nabla_\mu - fC_\mu \) and \( \alpha_1, \alpha_2, \alpha_3, \) are constants. We have added a \( \lambda\varphi^4 \) term which is also conformally invariant in 4D. We can easily check the conformal invariance of the action with respect to \( \varphi \) and \( \varphi \rightarrow e^{-\omega}\varphi. \) But, we should note that \( F_{\mu\nu} \) term in this action or \( \tilde{R}_{\mu\nu} \) term in (8) pauses some problems of observational inconsistencies. We will come back to this shortly after. From (8), we can rewrite (8) as follows,

\[ S = \int d^4x\sqrt{-g}\left\{ \frac{\alpha_1}{12}\varphi^2R - \frac{\alpha_2}{2}(\nabla_\mu\varphi)(\nabla_\nu\varphi) - \frac{1}{2}f^2(\alpha_1 + \alpha_2)\varphi^2C^2 + (\alpha_1 + \alpha_2)\varphi_c\nabla^c\varphi - \frac{\alpha_3}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4!}\varphi^4 \right\}. \]

(9)

Using the freedom of conformal invariance and particular choices of arbitrary constants \( \alpha_1, \alpha_3 \), we can set \( \varphi = \sqrt{3/4\pi G}, \alpha_1 = -\alpha_2 = 1 \) and \( \alpha_3 = 1 \). Then the above action (8) becomes

\[ S = \int d^4x\sqrt{-g}\left\{ \frac{1}{16\pi G}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{4!}\left(\frac{3}{4\pi G}\right)^2 \right\}. \]

(10)

This is just Einstein-Maxwell action with cosmological constant!

But, in 1918, Einstein rejected Weyl’s theory. Einstein pointed out that according to Weyl’s theory, the reading of an atomic clock would depend not only on space-time geometry but also on the unit length of the measurement. Consequently, Weyl’s theory would disagree with well-known observations. Now, consider the length in Weyl space, \( L = g_{\mu\nu}l^\mu l^\nu \). Then, the change of the length under an infinitesimal parallel transport \( dx^\sigma \) is

\[ dL = \nabla_\sigma g_{\mu\nu}dx^\sigma l^\mu l^\nu \]

(11)

\[ = -2fLC_\sigma dx^\sigma. \]

(12)
We have used the result of (11) in the second step. But, the above result causes observational problems. For example, set two clocks at a given space-time point P. If these two clocks travel to another point Q through different paths C_1 and C_2, these two clocks are not synchronized according to gravitational effect in the theory of general relativity. This is the “First clock effect”. In Weyl space, we have an additional synchronization loss (”Second clock effect”) due to variation of the unit length of measurements by different paths (conservation of the unit length implies ∇_σ g_{μν} = 0). To avoid “Second clock effect” problem, we have to impose the coincidence of the unit length of measurements for both observers at P without reference to any path. This implies that

\[ \int_{C_1} dL = \int_{C_2} dL \]

(13)

\[ \Leftrightarrow \int dL = 0 = -2fL \oint C_\mu dx^\mu. \]

(14)

Using the Stokes theorem we obtain the following condition,

\[ \nabla_\mu C_\nu - \nabla_\nu C_\mu = 0. \]

(15)

Consequently, to keep the observational consistency, \( F_\mu^\nu \) term should be vanishing (\( F_\mu^\nu \) of (11) and (8)~(11)) and \( C_\mu \) must be a pure gauge. So the homothetic curvature in Weyl’s original action cannot describe the electromagnetic interaction.

Now, the solution of (15) can be written as

\[ C_\mu = \frac{2}{(n-2)f} \partial_\mu \phi, \]

(16)

by introducing a scalar field \( \phi \) which transforms as

\[ \phi \rightarrow e^{\frac{2n}{n-2}} \phi. \]

(17)

### III. CONFORMAL EINSTEIN-HILBERT ACTION AND WEYL’S ACTION

**Case A.** \( \nabla_\mu g_{\alpha\beta} = 0, \Gamma^\rho_{\mu\nu} = \{^\rho_{\mu\nu}\} \) with \( S_{CEH} = S_{\text{conformal Einstein–Hilbert}}. \)

The Einstein-Hilbert action of general relativity has the form

\[ S_{EH} = \int d^n x \sqrt{-g} R. \]

(18)

This action, of course, is not invariant under conformal transformation of the metric, i.e., \( g_{\mu\nu} \rightarrow e^{2\omega} g_{\mu\nu} \), because the volume element and the curvature scalar transform as [64]

\[
\begin{align*}
    d^n x \sqrt{-g} &\rightarrow d^n x e^{n\omega} \sqrt{-g}, \\
    R &\rightarrow e^{-2\omega}(R - 2(n-1)\nabla_\gamma \nabla^\gamma \omega - (n-2) \nabla_\gamma \omega \nabla^\gamma \omega),
\end{align*}
\]

(19)

(20)

where \( n \) is the space-time dimensions.

In order to impose conformal symmetry on the action (18), we need something to cancel overall weight \( e^{(n-2)\omega} \). Now, one can introduce the most general conformal-invariant action as follows

\[ S_{CEH} = \int d^n x \sqrt{-g} \left\{ \phi^2 R + \frac{4(n-1)}{n-2} \nabla_\gamma \phi \nabla^\gamma \phi - \lambda \phi \frac{2n}{n-2} \right\}, \]

(21)

which is invariant w.r.t \( g_{\mu\nu} \rightarrow e^{2\omega(x)} g_{\mu\nu}, \phi \rightarrow e^{\frac{2n}{n-2}\omega(x)} \phi \) and \( \lambda \) is some constant. This is a well known action which is written by many authors [8, 10, 15, 22, 63, 65]. Some authors sometimes refer this to scale invariant gravity or conformal gravity. Through variation of (21), we can obtain the field equations for \( g_{\mu\nu}, \phi \) as

\[ \delta g_{\mu\nu} S = \phi^2(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \frac{2n}{n-2} \nabla_\mu \phi \nabla_\nu \phi - 2\phi \nabla_\mu \nabla_\nu \phi + 2g_{\mu\nu} \phi \nabla_\gamma \nabla^\gamma \phi - \frac{2g_{\mu\nu}}{n-2} \nabla_\gamma \phi \nabla^\gamma \phi + \frac{2\lambda \phi}{2} g_{\mu\nu} \phi \frac{2n}{n-2}, \]

(22)

\[ \delta \phi S = \phi R - \frac{4(n-1)}{n-2} \nabla_\gamma \nabla^\gamma \phi - \frac{n}{n-2} \lambda \phi \frac{2n}{n-2} = 0. \]

(23)

3. Do not confuse with conformal gravity included \( \int d^4 x C_{\mu\nu\rho\sigma} C_{\mu\nu\rho\sigma} \) where \( C_{\mu\nu\rho\sigma} \) is the Weyl tensor which has only fourth order derivative terms. We do not consider this action in this paper.
Here, if we contract eq. (22) with $g^{\mu \nu}$, then eq. (22) is equivalent to eq. (23). Thus, we have only one independent equation (22). This can be viewed as a result of $\phi$ being a pure gauge. Nevertheless, $\phi$ has still gauge degree of freedom given by $\phi \to e^{\frac{2n-\omega(x)}{2\pi}} \phi$. If we gauge fix $\phi$ and choose $\phi = \phi_0 = \sqrt{1/16\pi G}$, the action (21) reduces to the Einstein-Hilbert action with a cosmological constant. In this sense, the usual Einstein gravity can be thought of as a gauge fixed version of the conformally invariant action.

**case B.** $\nabla_{\mu} g_{\alpha\beta} = -2fC_{\mu} g_{\alpha\beta}$. $\Gamma^\lambda_{\alpha\beta} = \{\alpha\beta\} - f(C^\lambda_{\alpha\beta} - C_{\alpha\beta\lambda} - C_{\beta\alpha\lambda}).$

As explained in Section II, in order to avoid observational inconsistency we choose $F_{\mu\nu} = 0$. Then, the most general conformal invariant action can be written as

$$S_{\text{WEYL}} = \int d^n x \sqrt{-g} \left\{ \phi^2 \mathcal{R} - \alpha g^{\mu\nu}(D_{\mu}\phi)(D_{\nu}\phi) - \lambda \phi \right\},$$

(24)

by introducing the scalar $\phi$ with conformal weight $(2 - n)/2$ as before, where

$$D_{\mu}\phi = \nabla_{\mu}\phi - \frac{(n-2)f}{2}C_{\mu}\phi.$$  

(25)

If $C_{\mu}$ field is being treated as auxiliary, it can be eliminated through equations of motion which yields exactly eq. (10). Substituting back into the above equation, we obtain

$$S_{\text{WEYL}} = \int d^n x \sqrt{-g} \left\{ \phi^2 \mathcal{R} + \frac{4(n-1)}{n-2} \nabla_\gamma \phi \nabla_\gamma \phi - \lambda \phi \right\}.$$  

(26)

This action, as anticipated, is exactly equivalent to the action (21) up to a total derivative. Consequently, starting from the condition (15) in Weyl space, one can obtain conformal gravity action.

**IV. EINSTEIN-CARTAN WITH CONFORMAL SYMMETRY**

**case C.** $\nabla_{\mu} g_{\alpha\beta} = 0$, $\bar{\Gamma}^\rho_{\mu\nu} = \{\rho\mu\nu\} - K^\rho_{\mu\nu\rho}$. Let us first consider metric compatibility condition,

$$\nabla_{\mu} g_{\alpha\beta} = \partial_{\mu} g_{\alpha\beta} - \bar{\Gamma}^\lambda_{\mu\alpha\beta} - \bar{\Gamma}^\lambda_{\mu\beta\alpha} = 0.$$  

(27)

From the above condition (27), one can easily find the connection as $\bar{\Gamma}^\gamma_{\alpha\beta} = \{\alpha\beta\} - K^\gamma_{\alpha\beta}$, where $K^\gamma_{\alpha\beta}$ is the contorsion tensor, which is given in terms of the torsion tensor by

$$K^\gamma_{\alpha\beta} = -S^\gamma_{\alpha\beta} + S^\gamma_{\beta\alpha} - S^\gamma_{\alpha\beta},$$  

(28)

with the torsion tensor $S^\gamma_{\mu\alpha} = \bar{\Gamma}^\gamma_{\mu\alpha}$. It is important to note that $K^\gamma_{\alpha\beta}$ is anti-symmetric for last two indices and $S^\gamma_{\alpha\beta}$ anti-symmetric for first two indices. Now, we can generally decompose the contorsion tensor (28) into traceless and traceful parts as follows

$$K^\gamma_{\alpha\beta} = \tilde{K}^\gamma_{\alpha\beta} - \frac{2}{n-1}(\delta^\gamma_{\alpha} S_{\beta} - g_{\alpha\beta} S^\gamma),$$  

(29)

where $\tilde{K}^\alpha_{\beta\gamma} = 0$ and $S_\beta$ is the trace of the torsion tensor, $S_\beta = S^\gamma_{\alpha\beta}$. This decomposition, of course, means that we decompose torsion tensor as

$$S^\gamma_{\alpha\beta} = \tilde{S}^\gamma_{\alpha\beta} - \frac{1}{n-1}(\delta^\gamma_{\alpha} S_\beta - \delta^\gamma_{\beta} S_\alpha),$$  

(30)

where $\tilde{S}^\alpha_{\beta\gamma} = 0$. Making use of the connection $\tilde{\Gamma}^\gamma_{\alpha\beta}$ with (29), we can write curvature scalar as follows

$$\tilde{R} = R - 4\nabla_\mu S^\mu - \frac{4(n-2)}{n-1} S_\mu S^\mu - \tilde{K}_{\nu\rho\sigma} \tilde{K}^{\nu\rho\sigma},$$  

(31)

Note that $F(\phi, R)$ gravity without conformal invariance in general is equivalent to a system described by the Einstein-Hilbert action plus scalar fields via conformal transformation [63, 66].
where $R$ is the Riemann curvature scalar calculated from the usual Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$. Here, using scalar field $\phi$, we can construct the following conformal-invariant action
\[
\int d^n x \sqrt{-g} \left\{ R - 4\nabla_\mu S^\mu - \frac{4(n-2)}{n-1} S_\mu S^\mu - \tilde{K}_{\nu\rho\alpha} \tilde{K}^{\nu\rho\alpha} - \lambda \phi^{\frac{4(n-2)}{n-1}} \right\} \tag{32}
\]
is invariant under the following conformal transformations:
\[
g_{\mu\nu} \rightarrow e^{2\omega} g_{\mu\nu}, \quad \phi \rightarrow e^{\frac{2}{n-2}\omega} \phi, \tag{33}
\]
\[
S_\mu \rightarrow S_\mu + \frac{1-n}{2} \nabla_\mu \omega, \tag{34}
\]
\[
\tilde{K}_{\mu\alpha}^\lambda \rightarrow \tilde{K}_{\mu\alpha}^\lambda. \tag{35}
\]
Taking the variation with respect to $S_\mu$ and $\tilde{K}_{\nu\rho\alpha}$ then we obtain the following equation \[48\]
\[
S_\nu = \frac{n-1}{n-2} \frac{\nabla_\nu \phi}{\phi}, \quad \tilde{K}_{\nu\rho\alpha} = 0, \tag{36}
\]
which is consistent with the transformation law. From the above equation, the action \[22\] can be expressed simply as
\[
\int d^n x \sqrt{-g} \left\{ \phi^2 R + \frac{4(n-1)}{n-2} \nabla_\gamma \phi \nabla^\gamma \phi - \lambda \phi^{\frac{4(n-2)}{n-1}} \right\} \tag{37}
\]
up to a surface term. This action is also exactly same with the action \[21\] and \[20\]. As a result, the torsion vector $S_\mu$ \[30\] is equivalent to $C_\mu$ \[16\] in Weyl space, when $f = 2/(n-1)$.

More generally, in 4D if we extend to two scalar fields as follows
\[
S = \int d^4 x \sqrt{-g} (a_1 \phi_1^2 + a_2 \phi_1 \phi_2 + a_3 \phi_2^2) \tilde{R}
\]
(39)
then can find $S_\mu$ through the variation of it as
\[
S_\mu = \frac{3}{4} \left( \frac{2a_1 \phi_1 \nabla_\mu \phi_1 + 2a_3 \phi_2 \nabla_\mu \phi_2 + a_2 \phi_1 \nabla_\mu \phi_2 + a_2 \phi_2 \nabla_\mu \phi_1}{a_1 \phi_1^2 + a_2 \phi_1 \phi_2 + a_3 \phi_2^2} \right). \tag{39}
\]

And we introduce the following form
\[
\tilde{K}_{\nu\rho\alpha} = F(\phi_1, \phi_2) H_{\nu\rho\alpha}, \tag{40}
\]
where $F(\phi_1, \phi_2) = (c_1 \phi_1^2 + c_2 \phi_1 \phi_2 + c_3 \phi_2^2)^{-1}$, $H_{\nu\rho\alpha} = \nabla_\nu B_{\rho\alpha} + \nabla_\rho B_{\alpha\nu} + \nabla_\alpha B_{\nu\rho}$ and $B_{\nu\rho} = -B_{\rho\nu}$. Varying for the field $B_{\mu\nu}$ in the action \[38\] then the equation suggests a solution with scalar field $T$ \[70\] as
\[
H_{\nu\rho\alpha} = \frac{1}{W_1 F^2} k \epsilon_{\nu\rho\alpha\beta} \nabla^\beta T, \tag{41}
\]
where $k$ is a constant, $W_1 = a_1 \phi_1^4 + a_2 \phi_1 \phi_2 + a_3 \phi_2^4$, $\epsilon_{\nu\rho\alpha\beta}$ is the Levi-Civita tensor and
\[
T = \left\{ \frac{1}{2} \ln \left( b_1 \phi_1^2 + b_2 \phi_1 \phi_2 + b_3 \phi_2^2 \right) - G \ln \phi_1 - H \ln \phi_2 \right\}, \tag{42}
\]
Here, $b_{1,2,3}$ are constants, $G + H = 1$. It is important to note that $\tilde{K}_{\nu\rho}^\alpha$ \[41\] is invariant for conformal transformation, i.e., $g_{\mu\nu} \rightarrow e^{2\omega} g_{\mu\nu}$, $\phi_{1,2} \rightarrow e^{-\omega} \phi_{1,2}$. For simplicity we consider only $W_1 = F^{-1}$. In this case, substituting \[39\]~\[42\] into \[38\] and rearranging terms then we obtain
\[ \int d^3x \sqrt{-g} \left\{ (a_1 \phi_1^2 + a_2 \phi_1 \phi_2 + a_3 \phi_2^2) R - A_1 (\nabla_\mu \phi_1)^2 - A_2 (\nabla_\mu \phi_2)^2 - A_3 \nabla_\gamma \phi_1 \nabla^\gamma \phi_2 - A_4 \frac{\phi_1}{\phi_2} \nabla_\gamma \phi_1 \nabla^\gamma \phi_2 \\
- A_5 \frac{\phi_1}{\phi_2} \nabla_\gamma \phi_1 \nabla^\gamma \phi_2 - A_6 \frac{\phi_2}{\phi_1} (\nabla_\mu \phi_2)^2 - A_7 \frac{\phi_2}{\phi_1} (\nabla_\mu \phi_1)^2 - A_8 \frac{\phi_1}{\phi_2} (\nabla_\mu \phi_2)^2 - A_9 \frac{\phi_1}{\phi_2} (\nabla_\mu \phi_1)^2 \right. \\
+ 12k^2 (a_1 \phi_1^2 + a_2 \phi_1 \phi_2 + a_3 \phi_2^2) \frac{(b_1 \phi_1 \nabla_\mu \phi_1 + b_2 \phi_2 \nabla_\mu \phi_2 + \frac{1}{2} b_3 \phi_1 \nabla_\mu \phi_2 + \frac{1}{2} b_3 \phi_2 \nabla_\mu \phi_1)}{(b_1 \phi_1^2 + b_2 \phi_1 \phi_2 + b_3 \phi_2^2)} \left( G \frac{\nabla_\mu \phi_1}{\phi_1} + H \frac{\nabla_\mu \phi_2}{\phi_2} \right) \\
+ \frac{3}{2} \frac{3 (2a_1 \phi_1 \nabla_\mu \phi_1 + 2a_3 \phi_2 \nabla_\mu \phi_2 + a_2 \phi_1 \nabla_\mu \phi_2 + a_2 \phi_2 \nabla_\mu \phi_1)^2}{a_1 \phi_1^2 + a_2 \phi_1 \phi_2 + a_3 \phi_2^2} \\
- 6k^2 (a_1 \phi_1^2 + a_2 \phi_1 \phi_2 + a_3 \phi_2^2) \frac{(b_1 \phi_1 \nabla_\mu \phi_1 + b_2 \phi_2 \nabla_\mu \phi_2 + \frac{1}{2} b_3 \phi_1 \nabla_\mu \phi_2 + \frac{1}{2} b_3 \phi_2 \nabla_\mu \phi_1)}{(b_1 \phi_1^2 + b_2 \phi_1 \phi_2 + b_3 \phi_2^2)^2} \right\}, \quad (43) \]

where

\[
A_1 = a_1 (6k^2 G^2), \quad A_2 = a_3 (6k^2 H^2), \quad A_3 = a_2 (12k^2 G H), \\
A_4 = a_1 (12k^2 G H), \quad A_5 = a_3 (12k^2 G H), \quad A_6 = a_1 (6k^2 H^2), \\
A_7 = a_3 (6k^2 G^2), \quad A_8 = a_2 (6k^2 G^2), \quad A_9 = a_2 (6k^2 H^2).
\]

In the case of \( k = 0 \), the coefficients \( A_{1-9} \) are all zero. Then, the action (43) becomes

\[
S = \int d^4x \sqrt{-g} \left\{ (a_1 \phi_1^2 + a_2 \phi_1 \phi_2 + a_3 \phi_2^2) R + \frac{3}{2} \frac{3 (2a_1 \phi_1 \nabla_\mu \phi_1 + 2a_3 \phi_2 \nabla_\mu \phi_2 + a_2 \phi_1 \nabla_\mu \phi_2 + a_2 \phi_2 \nabla_\mu \phi_1)^2}{a_1 \phi_1^2 + a_2 \phi_1 \phi_2 + a_3 \phi_2^2} \right\}. \quad (44)
\]

When \( a_1 = a_2 = 0, \ a_3 = 1 \) or \( a_3 = a_2 = 0, \ a_1 = 1 \), the above action (44) reduces to that of only one scalar field \( \phi \) in (20), (37). In this case, \( S_\mu \) (39) is equivalent to the one in (46).

In particular, setting \( a_1 = a_3 = b_1 = b_3 = 0, \ a_2 = b_2 = 1, \ G = -1/2, \ H = 3/2 \) and \( k^2 = 1/4 \) then the action (43) can be expressed simply like

\[
S = \int d^4x \sqrt{-g} (\phi_1 \phi_2 R + 6 \nabla_\mu \phi_1 \nabla^\mu \phi_2) \quad (45)
\]

\[
= \int d^4x \sqrt{-g} \left\{ (\Phi_1^2 - \Phi_2^2) R + 6 \nabla_\mu \Phi_1 \nabla^\mu \Phi_1 - 6 \nabla_\mu \Phi_2 \nabla^\mu \Phi_2 \right\}, \quad (46)
\]

where \( \Phi_1 = (\phi_1 + \phi_2)/2 \) and \( \Phi_2 = (\phi_1 - \phi_2)/2 \). This action is invariant for \( g_{\mu \nu} \rightarrow e^{2\omega} g_{\mu \nu}, \ \Phi_1 \rightarrow e^{-\omega} \Phi_1 \). As we have shown in Section III, conformally invariant action i.e., (21), (23) and (37) is reduced to Einstein-Hilbert action when \( \phi = \phi_0 = \sqrt{1/16\pi G} \). Similarly, it is pointed out that the above action (46) can be reduced to superquintessence or conformal quintessence (20, 23) when \( \Phi_1 = \sqrt{1/16\pi G} \). In the case of \( \Phi_2 = 0 \), of course, it is exactly same with the action (21), (20) and (37).

More generally, in (44) if we add extra terms according to conformal transformation rule of \( \hat{R} \), i.e., \( \hat{R} \rightarrow e^{-2\omega} \hat{R} \) then

\[
S = \int d^4x \sqrt{-g} \left\{ (\phi_1 \phi_2 R + 6 \nabla_\mu \phi_1 \nabla^\mu \phi_2 - V(\phi_1, \phi_2)) \right\} \quad (47)
\]

\[
= \int d^4x \sqrt{-g} \left\{ (\Phi_1^2 - \Phi_2^2) R + 6 \nabla_\mu \Phi_1 \nabla^\mu \Phi_1 - 6 \nabla_\mu \Phi_2 \nabla^\mu \Phi_2 - V(\Phi_1, \Phi_2)) \right\}, \quad (48)
\]

where

\[
V(\phi_1, \phi_2) = a_1 \phi_1^4 + a_2 \phi_1^3 + a_3 \phi_2^3 + a_4 \phi_1 \phi_2 + a_5 \phi_1 \phi_2^2, \\
V(\Phi_1, \Phi_2) = \beta_1 \Phi_1^4 + \beta_2 \Phi_1^3 + \beta_3 \Phi_1^2 \Phi_2 + \beta_4 \Phi_1^2 \Phi_2 + \beta_5 \Phi_1 \Phi_2^2, \\
\beta_1 = a_1 + a_2 + a_3 + a_4 + a_5, \\
\beta_2 = a_1 + a_2 + a_3 - a_4 - a_5, \\
\beta_3 = 6a_1 + 6a_2 - 2a_3, \\
\beta_4 = 4a_1 - 4a_2 + 2a_4 - 2a_5, \\
\beta_5 = 4a_1 - 4a_2 - 4a_4 + 4a_5.
\]
In the case of $\Phi_1 = \sqrt{1/16\pi G}$, the action (16) can be reduced to conformal quintessence model with a power-law potential for $\Phi_2$ [22, 23]. In particular, regarding the above action (18) as matter in Einstein space one can construct the action of conformal quintom model as

$$S = \int d^4x \sqrt{-g} \left\{ R + (\Phi_1^2 - \Phi_2^2)R + 6\nabla_\mu \Phi_1 \nabla^\mu \Phi_1 - 6\nabla_\mu \Phi_2 \nabla^\mu \Phi_2 - V(\Phi_1, \Phi_2) \right\}. \quad (49)$$

V. EINSTEIN-CARTAN-WEYL THEORY WITH CONFORMAL SYMMETRY

case D. $\hat{\nabla}_\mu g_{\alpha\beta} \sim C_\mu g_{\alpha\beta}$, $\hat{\Gamma}^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} - K^\rho_{\mu\nu}$. Let us consider the extended conformal condition

$$\hat{\nabla}_\mu g_{\alpha\beta} = -2fC_\mu g_{\alpha\beta}$$

$$= \partial_\mu g_{\alpha\beta} - \hat{\Gamma}^\lambda_{\mu\alpha} g_{\lambda\beta} - \hat{\Gamma}^\lambda_{\mu\beta} g_{\alpha\lambda}. \quad (50)$$

From the above condition (50) one can find the connection as follows

$$\hat{\Gamma}^{\gamma}_{\alpha\beta} = \left\{ \frac{\gamma_{\alpha\beta}}{f} \right\} - f(C^{\gamma} g_{\alpha\beta} - C_\alpha \delta^{\gamma}_\beta - C_\beta \delta^{\gamma}_\alpha) - K^{\gamma}_{\alpha\beta}. \quad (51)$$

And the curvature scalar calculated by using the connection $\hat{\Gamma}^\mu_{\alpha\beta}$ is

$$\hat{R} = R + (2 - 2n)f\nabla_\mu C^{\mu} + (n - 2)(1 - n)f^2C_\mu C^{\mu} - 4\nabla_\mu S^{\mu} + 4f(2 - n)C_\mu S^{\mu} - \frac{4(n - 2)}{n - 1}S_\mu S^{\mu} - \hat{K}_{\nu\rho} \hat{K}^{\nu\rho}. \quad (52)$$

The above curvature scalar (52) is invariant with respect to $g_{\mu\nu} \rightarrow e^{2\xi} g_{\mu\nu}$, $C_\mu \rightarrow C_\mu - \frac{1}{f} \left( 1 - \frac{2\xi}{1 - n} \right) \nabla_\mu \xi$, $S_\mu \rightarrow S_\mu + \xi \nabla_\mu \xi$, $\hat{K}_{\mu\alpha} \rightarrow \hat{K}_{\mu\alpha}$. \quad (53)

where $\xi$ is a constant. It is interesting to note that (52) can be written as the following simple form

$$R - 4\nabla_\mu Y^{\mu} - \frac{4(n - 2)}{n - 1}Y_\mu Y^{\mu} - \hat{K}_{\nu\rho} \hat{K}^{\nu\rho}, \quad (54)$$

where $Y_\mu = (n - 1)fC_\mu/2 + S_\mu$. For $Y_\mu \leftrightarrow S_\mu$, the above curvature scalar is exactly same with the curvature scalar (31) in Einstein-Cartan and the conformal transformation rules for $S_\mu$ (31), $Y_\mu$ (53) are the same too, i.e.,

$$Y_\mu \rightarrow Y_\mu + \frac{1 - n}{2} \nabla_\mu \xi. \quad (55)$$

As a result, in Einstein-Cartan-Weyl space the action of curvature scalar (52) can be reduced to (21), (20) and (37). In fact, the connection in Einstein-Cartan-Weyl space can be written as $\hat{\Gamma}^{\gamma}_{\alpha\beta} = \left\{ \frac{\gamma_{\alpha\beta}}{f} \right\} - K^{\gamma}_{\alpha\beta} + 2C_\alpha \delta^\gamma_\beta/(n - 1)$, where

$$K^{\gamma}_{\alpha\beta} = \bar{K}^{\gamma}_{\alpha\beta} - 2(\delta^{\gamma}_\beta Y_\beta - g_{\alpha\beta} Y^{\gamma})/(n - 1),$$

and the curvature scalar by this connection is equivalent to the curvature scalar (31) for $S_\mu \leftrightarrow Y_\mu$. This is because there is no contribution of $2C_\alpha \delta^\gamma_\beta/(n - 1)$ term in the course of calculating curvature scalar. In Eq. (53), proper combination of $S_\mu$ and $C_\mu$ is independent of $\xi$ and the full action only depends on this combination.

The fact that in the action only $Y_\mu$ appears implies that there is a new symmetry of the form : $C_\mu \rightarrow C_\mu + \alpha_\mu$ and $S_\mu \rightarrow S_\mu - (n - 1)f\alpha_\mu/2$, where $\alpha_\mu$ is arbitrary function of the space-time. The exact nature of the symmetry requires further investigation. Note that this symmetry disappears when this theory couples to external matter because $C_\mu$ and $S_\mu$ play different geometric roles when interacting with matter.

VI. CONCLUSION AND DISCUSSION

In the present paper, we studied conformally invariant actions in Einstein, Weyl, Einstein-Cartan and Einstein-Cartan-Weyl space and showed that these actions all have the same form. In particular, in Einstein-Cartan space it
is shown that we can obtain conformally invariant action with two scalar fields. This is possible only for gradient scalar fields ansatz. It is natural that we take this ansatz in Weyl space because it solves the problem of observational inconsistency. The torsion vector field is auxiliary and can be eliminated by the equation of motion resulting in gradient scalar field.

We found that the conformally invariant action in Einstein-Cartan space can be reduced to conformal quintessence model with a power-law potential. Also, one can construct the action whose matter part can be thought of as a conformal-quintom type. Quintom model has two scalar fields, and we pointed out that their geometric origin could come from the traceless part and trace part of the torsion tensor in Einstein-Cartan space. The conformal quintom model needs further investigation.

As was pointed out at the end of previous section, the degeneracy between the torsion and Weyl gauge field in the Einstein-Cartan-Weyl space can be lifted when external matter fields are introduced. It would be interesting to elaborate on this aspect further and to study the geometrical role of these fields in the presence of matter fields.

In the conformally invariant formulation of the Einstein’s action, the scalar fields \( \phi \) play the role of the conformal gauge field, which is dynamically trivial. Therefore, such a formulation would be interesting only if exact conformal invariance is broken by some mechanism. Some of the possibilities is by quantum mechanical effect, or by introducing some potential, or by using it as conformal matter. However, it is important to remark that both of the fields \( \phi_{1,2} \) in the two scalar field case can not be gauged away. It means that if we gauge away or fix one scalar field from the theory, it becomes a gravity theory with a non-minimal coupling of a scalar field. It would be interesting to check the type of non-minimally coupled theory coming from the conformally invariant action in detail.

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