The $AdS_3$ central charge in string theory

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Abstract

We evaluate the vacuum expectation value of the central charge operator in string theory in an $AdS_3$ vacuum. Our calculation provides a rare non-zero one-point function on a spherical worldsheet. The evaluation involves the regularization both of a worldsheet ultraviolet divergence (associated to the infinite volume of the conformal Killing group), and a space-time infrared divergence (corresponding to the infinite volume of space-time). The two divergences conspire to give a finite result, which is the classical general relativity value for the central charge, corrected in bosonic string theory by an infinite series of tree level higher derivative terms.
1 Introduction

The boundary conditions on quantum gravity in three-dimensional anti-de Sitter space can be chosen such that the asymptotic symmetry group contains two copies of the centrally extended Virasoro algebra. The value of the central charge was computed in classical general relativity \[1\]. The calculation can also be done using holographic renormalisation \[2\].

The space-time $AdS_3$ also arises as a factor in string theory vacua. In string theory, the central charge in the space-time boundary conformal field theory was computed in $AdS_3$ with Neveu-Schwarz-Neveu-Schwarz flux in \[3\][5][4][6][7] and for the case of $AdS_3$ with Ramond-Ramond flux in \[8\]. In string theory, the result is a central charge operator which can take different values in different states of the theory \[6\]. While the difference in central charge between vacua separated by a long string has been computed \[3\], the value in a given vacuum has not directly been evaluated. A natural assumption is that the evaluation of the central charge operator in the $AdS_3$ vacuum should give rise to the Brown-Henneaux central charge, at least at string tree level and in the weak curvature limit.

The problem of the evaluation of the central charge in string theory in $AdS_3$ with Neveu-Schwarz-Neveu-Schwarz flux is plagued both with conceptual and technical difficulties. An attempt to repeat the holographic Weyl anomaly calculation \[2\] in string theory would meet a basic obstacle. The difficulty is that in the derivation one uses the on-shell value of the bulk space-time gravity action – in string theory the on-shell action is zero \[9\].

Another difficulty becomes apparent when we realize that we are supposed to evaluate a one-point function in string theory. In almost all circumstances, a one-point function in string theory is zero \[10\], whether by symmetry, definition of regularization scheme, or by the fact that one has an on-shell background. One formal argument says that the one-point function will be divided by the infinite volume of the worldsheet conformal Killing group (leaving one point fixed), and will therefore vanish. Another argument says that one-point functions must vanish due to conformal symmetry. These statements are known to permit exceptions, for example for a zero-momentum dilaton operator \[11\], but not many. Every exception is worth studying.

There is another fundamental reason to understand the $AdS_3$ central charge in string theory. It is the central quantity to compute in a check on the microscopic description of the degrees of freedom responsible for the leading term in the macroscopic entropy of black holes \[12\][13]. As such, the value of the central charge has propagated throughout the literature. We believe it is important to compute it directly in string theory.

2 The one-point function of the central charge operator

Our goal is to calculate the one-point function of the central charge operator in bosonic string theory on $AdS_3$ with Neveu-Schwarz-Neveu-Schwarz flux. The calculation is universal for all bosonic string theories including an $AdS_3$ space-time that factorizes. Our computation is easily generalized to superstring theory in which the supersymmetric current algebra level is not renormalized. We discuss the central charge operator, first in the fully interacting picture, and then in the asymptotically free field variables that will be handy in our calculation.

2.1 The central charge operator

The classical string worldsheet action (with $\alpha' = 2$) is given by:

$$S_{int} = \frac{k}{2\pi} \int d^2 z (\partial \bar{\phi} \partial \phi + e^{2\tilde{\phi}} \partial \bar{\gamma} \partial \gamma), \quad (2.1)$$
where $\tilde{\phi}$ is the radial coordinate of the string, and the coordinates $\gamma, \bar{\gamma}$ parameterize the planar sections of the Poincaré patch of $AdS_3$ with Neveu-Schwarz-Neveu-Schwarz flux. The fully interacting conformal field theory can be solved [14]. It contains primary operators of the form:

$$
\Phi_h = \frac{1}{\pi}(|\gamma - x|^2 e^{\tilde{\phi}/2} + e^{-\tilde{\phi}/2})^{-2h},
$$

(2.2)

and holomorphic and anti-holomorphic currents $J(x; z)$ and $\bar{J}(\bar{x}; \bar{z})$ which are linear combinations of the three holomorphic and anti-holomorphic affine currents [5]. There exists a scheme [15] in which the primary operators satisfy the worldsheet operator product expansion [5]:

$$
\Phi_1(x; z)\Phi_h(y; w) = \delta^2(x - y)\Phi_h(y; w) + \ldots
$$

(2.3)

String theory on $AdS_3$ contains worldsheet vertex operators corresponding to diffeomorphisms that fall off slowly at the boundary of space-time. These are graviton vertex operators. It can be argued that these operators are only formally BRST exact [5]. Using the worldsheet operator product expansions of these vertex operators, one can prove that they form two space-time Virasoro algebras in space-time, as they do in the semi-classical limit [1]. A result of the calculation is that the space-time Virasoro algebras contain a central charge operator $C$ [5] which can indeed be proven to commute with the full algebra. The central charge operator contains one right and one left oscillator excitation, and a bulk-boundary propagator $\Phi_1$. In the scheme defined above, the central charge operator has the expression [5][7]:

$$
C = -\frac{6}{k-2}\int d^2 z : \Phi_1 J \bar{J} : (z).
$$

(2.4)

Although seemingly dependent on the boundary insertion point $(x, \bar{x})$, it can be shown that the derivative of the operator is BRST exact [5].

### 2.2 Free field variables

The vacuum expectation value of the central charge operator is fixed by the asymptotic symmetry group, and therefore it is sufficient for our purposes to compute near the boundary of the $AdS_3$ space. In this region, there is a set of free field variables that can be used to calculate quantities that depend on a perturbative asymptotic interaction. They are a real bosonic field $\phi$ and a holomorphic $(\beta, \gamma)$ system of dimensions $(1, 0)$ (as well as an anti-holomorphic $(\bar{\beta}, \bar{\gamma})$ system). The free field action is given by:

$$
S_0 = \frac{1}{4\pi} \int d^2 z \partial \phi \bar{\partial} \phi
+ \frac{1}{2\pi} \int d^2 z (\beta \bar{\partial} \gamma + \bar{\beta} \partial \bar{\gamma}),
$$

(2.5)

supplemented with a linear coupling of the scalar $\phi$ to the worldsheet curvature such that the central charge of the theory is equal to $c = 1 + \frac{6}{k-2} + 2 = \frac{3k}{k-2}$. The operator product expansions of the free fields are:

$$
\phi(z)\phi(w) \approx -\log |z - w|^2
$$

$$
\beta(z)\gamma(w) \approx \frac{1}{z - w}.
$$

(2.6)
We introduce the constant \( Q = \sqrt{\frac{2}{k-2}} \). The operator \( \beta \bar{\beta} e^{-Q \phi} \) is exactly marginal. We can therefore perturb the free conformal field theory with this operator. We will study the perturbed theory with interaction term:

\[
S = S_0 - \frac{\mu}{2\pi} \int d^2 z \beta \bar{\beta} e^{-Q \phi}. \tag{2.7}
\]

Classically, the fields \( \beta, \bar{\beta} \) can be integrated out to yield the theory with action:

\[
S_{cl} = \frac{1}{4\pi} \int d^2 z \partial \bar{\phi} \partial \phi 
+ \frac{1}{2\pi} \int d^2 z \frac{4}{Q^2} \partial \bar{\phi} \partial \phi 
+ \frac{1}{2\pi} \int d^2 z \frac{1}{\mu} e^{2 \phi} \partial \bar{\gamma} \partial \gamma. \tag{2.8}
\]

which we can rewrite as:

\[
S_{cl} = \frac{1}{4\pi} \int d^2 z \frac{4}{Q^2} \partial \bar{\phi} \partial \phi 
+ \frac{1}{2\pi} \int d^2 z \frac{1}{\mu} e^{2 \phi} \partial \bar{\gamma} \partial \gamma. \tag{2.9}
\]

At leading order in \( 1/k \), we can choose:

\[
\mu \approx 1/k, \tag{2.10}
\]

and we recognize the string worldsheet action (2.1) in an \( AdS_3 \) background with purely Neveu-Schwarz-Neveu-Schwarz flux.

### 2.3 A note on normalization

The precise normalization of the central charge operator in equation (2.4) was computed in the fully interacting picture [7]. We want to translate the operator, including its overall normalization into the free field picture we will use in the following. This translation was analyzed in detail in [16][17]. We need to choose a coefficient for the free field interaction term and normalize the operators accordingly.

Suppose that we normalize our operators in the free field approach naively as:

\[
\Phi_{h}^{free} = (|\gamma - x|^2 e^{Q \phi/2} + e^{-Q \phi/2})^{-2h}, \tag{2.11}
\]

and we pick the interaction term in equation (2.7) as in [16]:

\[
\mu \equiv \frac{1}{k}. \tag{2.12}
\]

Given these choices, the relation between the free field operators and the operators in the fully interacting formalism is [15][16][17]:

\[
\Phi_{h}^{free} = E(h) \Phi_{h} \tag{2.13}
\]

where

\[
E(h) = -b^4 \frac{b^2}{\pi k} e^{-h \Delta(b^2)} e^{-2h+1} \Delta((2h - 1)b^2). \tag{2.14}
\]
We used the notations:
\[
\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)}
\]
\[
b^2 = \frac{1}{k-2}.
\]
Thus, the interacting operator \( \Phi_1 \) that we wish to use is given by the naive expression in the free field approach times a factor of \( E(1)^{-1} = -(k-2)/(\pi k) \). Thus, in the free field variable path integral, we must map the operator \( \Phi_1 \) to the expression:
\[
\Phi_1 = -\frac{k-2}{\pi k} \frac{1}{(|\gamma - x|e^{Q\phi/2} + e^{-Q\phi/2})^2}.
\]
\[
(2.15)
\]

2.4 The central charge operator in the free field formalism

We are finally ready to write down the normalized central charge operator in the free field formalism. We need the currents in the free field approach:
\[
J(x; z) = -\beta(x - \gamma)^2 + 2(x - \gamma) \frac{1}{Q} \partial \phi - k \partial \gamma,
\]
and can then easily compute the asymptotic expression for the central charge operator:
\[
C = 6k \int d^2 z \delta^{(2)}(\gamma - x) \partial \gamma \partial \bar{\gamma} + O(e^{-Q\phi}).
\]
\[
(2.17)
\]

It will become manifest that only the leading term in the large radius limit can contribute to the final result.

2.5 The conformal field theory one-point function

We first evaluate the central charge one-point function in the conformal field theory, and then move on to embed the calculation in string theory. When calculating the one-point function of this operator using the free field operator products, we must descend from the exponential of the action the appropriate number of interaction terms in order to find a non-zero result. In the case at hand, we find a non-zero result only when we descend a single interaction term:
\[
\langle C \rangle_{\text{CFT}} = 6k \int d^2 w : \delta^{(2)}(\gamma - x) \partial \gamma \partial \bar{\gamma}(w) : \frac{\mu}{2\pi} \int d^2 z : \beta \bar{\beta} e^{-Q\phi}(z) : 
\]
\[
(2.19)
\]
We have the chiral correlator equality:
\[
\langle : \delta(\gamma - x) \partial \gamma(w) :: \beta(z) : \rangle = \langle \partial_w \left( \frac{1}{z-w} \delta(\gamma - x) \right) \rangle. 
\]
\[
(2.20)
\]
Using this result, we can partially integrate and obtain:
\[
\langle C \rangle_{\text{CFT}} = \frac{3}{\pi} \int d^2 z \int d^2 w (2\pi \delta^{(2)}(z-w))^2 \delta^{(2)}(\gamma_0 - x) e^{-Q\phi_0},
\]
\[
(2.21)
\]
before integrating over the zero-modes. The result is ultraviolet divergent on the worldsheet. We want to represent it in a form that is more easily regularized. To that end, we rewrite:
\[
\int d^2 z \int d^2 w (2\pi \delta^{(2)}(z-w))^2 = \int d^2 z \int d^2 w \frac{1}{|z-w|^4}.
\]
\[
(2.22)
\]
2.6 The string theory one-point function

Let’s now move to the calculation of the string theory one-point function. We need to take into account various modifications. Firstly, we need to add the overall normalization of the path integral [18]. We further need to divide out the conformal field theory correlator by the volume of the conformal Killing group of the sphere. We must also still perform the integral over zero modes. And, we need to regularize the divergences.

Firstly, we compute the ratio of the divergent conformal field theory correlator and the volume of the conformal Killing group. We must calculate the (inverse of the) following regularized ratio of integrals:

\[
\int \frac{d^2 z_1 d^2 z_2 d^2 z_3}{|z_1 - z_2|^2 |z_2 - z_3|^2 |z_3 - z_1|^2} / \int \frac{d^2 z d^2 w}{|z - w|^4}
\]

\[
= |z - w|^2 \int d^2 \xi \frac{1}{|\xi - z|^2 |\xi - w|^2}
\]

\[
= 4\pi \log(1/\epsilon).
\]

The points \(z\) and \(w\) are arbitrary points that fix part of the \(SL(2, \mathbb{C})\) symmetry of the formal triple integral. In the last equality, we used a short distance cut-off \(\epsilon\) on the integral over the \(\xi\)-plane. The coefficient of the logarithmic divergence is independent of the choice of points \(z\) and \(w\). We note that the same regularization enters the calculation of the one loop mass renormalization of string states in flat space [19]. It is also reminiscent of the calculation of the space-time two-point functions in AdS3 [20].

Next, we note that the overall normalization of the bosonic string path integral including ghosts is such that it provides us with an extra factor:

\[
\frac{8\pi}{\alpha' g_s^2}.
\]

This constant is fixed by unitarity [18][21]. Putting all these results together, the full expression for the one-point function including the zero mode integral (and at \(\alpha' = 2\)) becomes:

\[
\langle C \rangle_{\text{string}} = \frac{3 V^\perp}{g_s^2 \pi \log(1/\epsilon)} \int d\phi_0 \int d\gamma_0 \int d\bar{\gamma}_0 e^{Q\phi_0} \delta^2(x - \gamma_0) e^{-Q\phi_0}.
\]

The transverse volume factor \(V^\perp\) arises from the integration over the zero modes orthogonal to AdS3. Note that we also took into account the linear dilaton contribution to the \(\phi_0\) zero-mode integral on the sphere, and that we introduced a bulk infrared cut-off \(\phi_0^{IR}\) on the radial integration region. We take the lower limit on the integration region to be close enough to the boundary for our approximation to the central charge operator to be valid (e.g. \(\phi_0 >> 1/Q\)). This is sufficient to isolate the bulk infrared divergent contribution of interest.

In bosonic string theory, we have the following relation between the string coupling and the dimensionally reduced Newton coupling constant [21]:

\[
\frac{1}{g_s^2} = \frac{\pi}{2 G_N^3 V^\perp}.
\]

Using this relation, as well as \(Q = \sqrt{2/(k - 2)}\) and \(\alpha' = 2\), we finally find:

\[
\langle C \rangle_{\text{string}} = \frac{3 \sqrt{(k - 2)\alpha'}}{2 G_N^3} \frac{\log e^{Q\phi_0^{IR}}}{\log 1/\epsilon}.
\]
We wrote the argument of the logarithm in the numerator as a function of the space-time infrared length scale (as is clear from the expression for the space-time metric). We can now justify the regularized expression:

\[ \langle C \rangle_{\text{string}} = \frac{3 \sqrt{(k - 2) \alpha'}}{2 G_N^3} \]  

(2.28)

in various ways. First of all, the logarithmic divergences we identified are universal, and so is their coefficient. Moreover, a definition of closed string one-point functions can be given in terms of the logarithmic derivative with respect to a worldsheet ultraviolet cut-off \[22\]. It is also natural to identify the logarithmic dependence on the space-time infrared cut-off as the conformal anomaly. More powerfully, by holography, we can kill two birds with one stone. A holographic interpretation for our regularized expression is to view the logarithmic divergences as cancelling one another directly. Indeed, the worldsheet ultraviolet cut-off is akin to a boundary ultraviolet cut-off, which in turn is holographically dual to a bulk infrared cut-off \[23\]. As a final argument for our final equation we note that our identification reproduces the small curvature general relativity limit \[1\]. Our result moreover contains an infinite set of tree level higher derivative corrections which are present in bosonic string theory.

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