BI-LIPSCHITZ SOLUTIONS TO THE PRESCRIBED JACOBIAN INEQUALITY IN THE PLANE AND APPLICATIONS TO NONLINEAR ELASTICITY

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Abstract. We show that the prescribed Jacobian inequality in the plane admits – unlike the prescribed Jacobian equation – a bi-Lipschitz solution in case of right-hand sides of class $L^\infty$ (with identity boundary conditions). Our construction in particular utilizes a new refinement of a covering result due to Alberti, Csörgő, and Preiss, which enables us to construct bi-Lipschitz maps stretching a given measurable subset of the plane by a chosen factor. We then apply our result to a model functional in nonlinear elasticity, the integrand of which blows up as the Jacobian determinant of the map in consideration drops below a certain positive threshold. For such functionals, the derivation of the equilibrium equations for minimizers requires an additional regularization of test functions, which is provided by our newly constructed maps.

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1. Introduction

The prescribed Jacobian equation

\begin{equation}
\left\{
\begin{array}{ll}
det \nabla \phi = f & \text{in } \Omega \\
\phi = \text{id} & \text{on } \partial \Omega
\end{array}
\right.
\end{equation}

(with $\Omega \subset \mathbb{R}^d$ a connected bounded open set, $f : \Omega \to \mathbb{R}_+^+$ and $\phi : \overline{\Omega} \to \mathbb{R}^d$) is an important subject of studies in geometric analysis, as by the change of variables...
formula the equation amounts to prescribing the volume distortion of the mapping $\phi$. An obvious necessary condition for the solvability of (1) is
\[ \int_{\Omega} f = |\Omega|. \]
For $f \in L^\infty(\Omega)$, the prescribed Jacobian equation in general does not admit a bi-Lipschitz solution, even if $f$ is additionally assumed to be close to 1 and continuous (see Burago-Kleiner [6] and McMullen [14]).

In the present work, in Theorem 1 we show that on the contrary, in the plane (i.e. for $\Omega \subset \mathbb{R}^2$) somewhat surprisingly the prescribed Jacobian inequality
\[ \begin{cases} 
\det \nabla \phi \geq f & \text{in } \Omega \\
\phi = \text{id} & \text{on } \partial \Omega 
\end{cases} \tag{2} \]
adopts a bi-Lipschitz solution in case of $f \in L^\infty(\Omega)$ with $f \geq 0$ and $\int_{\Omega} f < |\Omega|$. As we shall see below, this bi-Lipschitz regularity is in general sharp. We also give an analogous result for the divergence operator (cf. Theorem 3).

Note that the equation (1) is well-understood in case of positive regular right-hand sides $f \in C^{r,\alpha}(\Omega)$ for $r \geq 0$ and $0 < \alpha < 1$; in this case it admits a solution with sharp regularity $\phi \in C^{r+1,\alpha}$ (cf. [10] and also [15, 7]), provided that the necessary condition $\int_{\Omega} f = |\Omega|$ holds. For $f \in C^0(\Omega)$ or even $f \in L^\infty(\Omega)$, the only known existence result (see [15]) yields a Hölder continuous solution (in a appropriate weak sense) of (1). One could consult e.g. [8] for a complete overview of the available theory for the prescribed Jacobian equation.

In the present work, we shall reduce the existence proof for the prescribed Jacobian inequality (2) to the existence of bi-Lipschitz maps which stretch a given measurable subset of the plane. Given a bounded open set $\Omega$ in $\mathbb{R}^2$ with $C^{1,1}$ boundary, let $\tau > 0$ and consider any measurable set $M \subset \Omega$ with small measure. Then we are able to construct a bi-Lipschitz mapping $\phi = \phi_{\tau,M}$ (with $\phi : \overline{\Omega} \to \overline{\Omega}$) preserving the boundary pointwise, stretching $M$ by a factor of at least $1 + \tau$, and compressing in a controlled way outside $M$, i.e.
\[ \begin{align*}
\det \nabla \phi & \geq 1 + \tau \quad \text{a.e. in } M \\
\det \nabla \phi & \geq 1 - C|M|^{1/2} \tau \quad \text{a.e. in } \Omega,
\end{align*} \]

\[ \text{together with uniform } W^{1,p} \text{-estimates for } 1 \leq p \leq \infty. \]

Our bi-Lipschitz stretching maps have another interesting application in nonlinear elasticity. Consider the model functional from nonlinear elasticity
\[ \mathcal{F}[v] = \int_{\Omega} |\nabla v|^2 + \frac{1}{[\det \nabla v - \mu]^\beta} \, dx \tag{3} \]
with $\mu, \beta > 0$ (here, by convention $0^{-1} = \infty$). In contrast to the classical case in nonlinear elasticity, this functional blows up as $\det \nabla v$ drops below $\mu > 0$; for the classical functionals in nonlinear elasticity, one would have $\mu = 0$. It is natural to consider functionals of the form (3) with $\mu > 0$, as most physical materials may not be compressed beyond a certain point in practice.

Making use of our bi-Lipschitz stretching maps, we are able to show that minimizers of (3) have the additional regularity
\[ \det \nabla u \cdot (\det \nabla u - \mu)_+^{\beta - 1} \in L^1(\Omega) \]
(note that the property $\det \nabla u - \mu)_+^{\beta} \in L^1(\Omega)$ comes for free).
Taking into account this additional regularity, we are then able to deduce the equilibrium equation
\[
\int_{\Omega} 2(\nabla \xi(u) \cdot \nabla u) : \nabla u - \beta(\det \nabla u - \mu)^{-\beta - 1} \cdot \det \nabla u \cdot \div \xi(u) = 0 \quad \forall \xi \in C^\infty_c(\Omega; \mathbb{R}^2)
\]
for minimizers of our functional. The notion of equilibrium equations goes back to Ball [3], who considered functionals like (3) in the case \(\mu = 0\). Note that the direct ansatz
\[
\lim_{\tau \to 0^+} \frac{F[(\text{id} + \tau \xi) \circ u] - F[u]}{\tau} \geq 0
\]
for the derivation of the equilibrium equation fails in the case \(\mu > 0\). Indeed, in the course of the derivation our stretching maps are required again, this time as a tool to deal with the lack of regularity of the functional (3) for trial functions with inf \(\det \nabla v \leq \mu\).

Before stating our main results, let us describe the construction of our bi-Lipschitz stretching maps in more detail. Our construction makes use of an improvement of a covering lemma due to Alberti, Csörnyei, and Preiss [1]. This covering lemma from [1] states that any compact subset \(K\) of the unit square may be covered (for \(\delta > 0\) small enough) by a collection of horizontal and vertical 1-Lipschitz strips with height (respectively width) \(2\delta\), the number of strips being bounded by \(\delta^{-1} \sqrt{|K|}\). Here, a horizontal (respectively vertical) 1-Lipschitz strip is defined to be the graph of a 1-Lipschitz function over the horizontal (respectively vertical) axis thickened by \(\delta\) in the vertical (respectively horizontal) direction; cf. the central picture in Figure 2 for a sketch of such a covering. In the present work, we show additionally that the overlap of the strips may be controlled; more precisely, the horizontal strips may be chosen to cover any point at most 3 times (same for the vertical strips); cf. Lemma 17. By stretching the horizontal strips in the vertical direction and the vertical strips in the horizontal direction, we obtain a mapping which stretches a given compact subset \(K\) of the unit square; cf. Proposition 11. We then correct the boundary conditions by following the flow of an appropriately constructed solenoidal vector field; cf. Proposition 14. Using inner regularity of the Lebesgue measure, our result is subsequently extended to general measurable sets (cf. Lemma 13). Finally we extend the result – which by now has only been proven for the unit square – to domains in the plane with boundary of class \(C^{1,1}\); cf. Lemma 12.

**Notation.** Throughout the paper, we use the following notation:

- By \((t)_+\) we denote \(\max\{t, 0\}\) and \((t)_-\) stands for \(\min\{t, 0\}\).
- The scalar product of two matrices \(A\) and \(B\) of the same size is denoted by \(A : B\), namely \(A : B = \sum_{i,j} A_{ij} B_{ij}\).
- We denote the Lebesgue measure of a measurable set \(M \subset \mathbb{R}^d\) by \(|M|\).
- For \(M \subset \mathbb{R}^d\), \(\chi_M\) denotes the usual characteristic function of \(M\):
  \[
  \chi_M = \begin{cases} 
  1 & \text{in } M \\
  0 & \text{in } M^c.
  \end{cases}
  \]
- The notation \(\sharp P\) refers to the number of elements of the set \(P\).
- A function \(g : \mathbb{R} \to \mathbb{R}\) is said to be 1–Lipschitz if it is Lipschitz with \(|g'| \leq 1\) a.e..
• By $C$ we denote a generic constant whose value may change from appearance to appearance.
• By $\text{id}$ we denote the identity map, while by $\text{Id}$ we denote the unit matrix.

2. Main results

Our first main result guarantees the existence of a bi-Lipschitz solution to the prescribed Jacobian inequality.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with boundary of class $C^{1,1}$. Let $f \in L^\infty(\Omega)$ be a nonnegative function with $\int_{\Omega} f \, dx < |\Omega|$. Then there exists a bi-Lipschitz map $\phi : \Omega \to \Omega$ satisfying

$$\begin{cases} \det \nabla \phi \geq f & \text{a.e. in } \Omega \\ \phi = \text{id} & \text{on } \partial \Omega. \end{cases}$$

Moreover the regularity of $\phi$ is sharp in general: there exists an $f$ for which there is no $C^1$ solution $\phi$ to (4).

**Remark 2.** (i) Recall that for $f \in C^0(\Omega)$ with $f > 0$ and $\int_{\Omega} f = |\Omega|$, in general the prescribed Jacobian equation

$$\det \nabla \phi = f \quad \text{a.e. in } \Omega \quad \text{and} \quad \phi = \text{id} \quad \text{on } \partial \Omega$$

does not admit a bi-Lipschitz solution, cf. [6, 14]. Hence the assumption $\int_{\Omega} f < |\Omega|$ is sharp in general: indeed from the change of variables formula $\int_{\Omega} \det \nabla \phi = |\Omega|$ we infer that

$$\begin{cases} (\text{4}) \text{ implies (5)} & \text{if } \int_{\Omega} f = |\Omega| \\ (\text{5}) \text{ has no solution} & \text{if } \int_{\Omega} f > |\Omega|. \end{cases}$$

(ii) By a completely symmetric proof, we can show that the prescribed Jacobian equation with the reverse inequality

$$\det \nabla \phi \leq f \quad \text{a.e. in } \Omega \quad \text{and} \quad \phi = \text{id} \quad \text{on } \partial \Omega$$

admits a bi-Lipschitz solution for any measurable $f$ with $\text{essinf } f > 0$ and $\int_{\Omega} f > |\Omega|$.

The linearized version of the previous theorem is also valid.

**Theorem 3.** Let $\Omega \subset \mathbb{R}^2$ be a bounded connected open set with boundary of class $C^{1,1}$. Let $f \in L^\infty(\Omega)$ be such that $\int_{\Omega} f < 0$ (resp. $\int_{\Omega} f > 0$). Then there exists $u \in W^{1,\infty}_0(\Omega; \mathbb{R}^2)$ satisfying

$$\text{div } u \geq f \quad \text{a.e. in } \Omega \quad \text{(resp. div } u \leq f \quad \text{a.e. in } \Omega).$$

Moreover the regularity of $u$ is sharp in general: there exists an $f$ for which there is no $C^1$ solution $u$ to (6).

**Remark 4.** For $f \in C^0(\Omega)$ and $\int_{\Omega} f = 0$ there does in general not exist $u \in W^{1,\infty}_0(\Omega; \mathbb{R}^2)$ such that $\text{div } u = f$ a.e. (cf. [14]). This implies (similarly to Remark 2 (i)) that the assertion $\int_{\Omega} f < 0$ (resp. $\int_{\Omega} f > 0$) is sharp in general.

Our second main result concerns the theory of nonlinear elasticity. We consider functionals of the form

$$\mathcal{F}[v] := \int_{\Omega} |\nabla v|^2 + h(\det \nabla v).$$
Here \( h : \mathbb{R} \to \mathbb{R}_+^+ \cup \{\infty\} \) denotes a convex monotonously decreasing function with \( h \equiv \infty \) on \((-\infty, \mu]\) and \( h(s) \to \infty \) as \( s \to \mu^+ \) for some \( \mu > 0 \). We also require \( h \) to be continuously differentiable on \((\mu, \infty)\). For example, we may take for some \( \beta > 0 \) and some \( \mu > 0 \)

\[
h(s) = \begin{cases} 
(s - \mu)^{-\beta} & \text{for } s > \mu \\
\infty & \text{otherwise.}
\end{cases}
\]

We then have the following result.

**Theorem 5.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set with boundary of class \( C^{1,1} \) and \( u_0 \in W^{1,2}(\Omega; \mathbb{R}^2) \) be such that \( u_0 \) is an homeomorphism from \( \overline{\Omega} \) to \( \overline{\Omega} \) and \( F[u_0] < \infty \). Then any minimizer \( u \in u_0 + W^{1,2}_0(\Omega; \mathbb{R}^2) \) of

\[
\inf_v \left\{ F[v] : v \in u_0 + W^{1,2}_0(\Omega; \mathbb{R}^2) \right\}
\]

has the additional regularity

\[
h'(\det \nabla u) \cdot \det \nabla u \in L^1(\Omega)
\]

and satisfies the equilibrium equation

\[
\int_{\Omega} 2(\nabla \xi(u) \cdot \nabla u) : \nabla u + h'(\det \nabla u) \cdot \det \nabla u \cdot \operatorname{div} \xi(u) = 0
\]

for any vector field \( \xi \in C^\infty_c(\Omega; \mathbb{R}^2) \).

**Remark 6.** (i) The existence of a minimizer \( u \) is well known and holds without assuming that \( u_0 \) is a homeomorphism; see e.g. [3] or [9].

(ii) Recall that the equilibrium equation is formally obtained from

\[
(8) \quad \lim_{\tau \to 0^+} \frac{F[\varphi_\tau \circ u] - F[u]}{\tau} \geq 0,
\]

where \( \varphi_\tau \) is the flow of \( \xi \), i.e.

\[
\frac{d}{d\tau} \varphi_\tau = \xi(\varphi_\tau) \quad \text{and} \quad \varphi_0 = \text{id}.
\]

However since \( h \) blows up at \( \mu > 0 \), we do not even know whether \( F[\varphi_\tau \circ u] \) is finite for any arbitrarily small \( \tau \): indeed we do not necessarily have that

\[
\det \nabla \varphi_\tau(u) \cdot \det \nabla u > \mu \quad \text{a.e.,}
\]

which is obviously required for the finiteness of \( F[\varphi_\tau \circ u] \). We will overcome this difficulty by replacing \( F[\varphi_\tau \circ u] \) by \( F[\varphi_\tau \circ \phi_{\tau,M} \circ u] \) in the above limit for some appropriately constructed map \( \phi_{\tau,M} \) (see Theorem 8 below), in particular enforcing the condition

\[
\det \nabla (\varphi_\tau \circ \phi_{\tau,M} \circ u) > \mu \quad \text{a.e.}
\]

(iii) Note that in the standard case in nonlinear elasticity, i.e.

\[
h(s) = \begin{cases} 
s^{-\beta} & \text{for } s > 0 \\
\infty & \text{otherwise}
\end{cases}
\]

(where \( h \) only blows up at \( 0 \)), the property \( h'(\det \nabla u) \cdot \det \nabla u = -\beta h(\det \nabla u) \in L^1(\Omega) \) is automatically satisfied. Furthermore the equilibrium equations follow directly using the ansatz [8].
By duality we obtain the following analogue to Lemma 2.4 in Bauman-Owen-Phillips [4].

**Corollary 7.** Assume that a minimizer $u$ from the previous theorem has the additional regularity
\[
\nabla u \in L^{2p}(\Omega)
\]
for some $p > 1$. We then have
\[
h'(\det \nabla u) \cdot \det \nabla u \in L^p(\Omega)
\]
Furthermore, the estimate
\[
||h'(\det \nabla u) \cdot \det \nabla u||_{L^p(\Omega)} \leq C||\nabla u||_{L^2_p(\Omega)}^2
\]
is satisfied for some constant $C$ depending only on $\Omega$, $p$, $h$, and $F[u_0]$.

All the above results rely on the following core theorem of our paper. The theorem essentially states that for any $\tau > 0$ and any measurable set $M \subset \Omega \subset \mathbb{R}^2$ with small enough measure, we can construct a bi-Lipschitz map mapping $\Omega$ to itself which

- stretches the set $M$ by a factor of at least $1 + \tau$,
- compresses $\Omega \setminus M$ by no more than a factor of $1 - C\sqrt{|M|}\tau$,
- has good uniform estimates for its difference from the identity,
- preserves the boundary pointwise.

In particular, for $|M|$ small enough the compression is almost negligible, as is the difference of the map from the identity in the $W^{1,p}$ norm for $p < \infty$.

**Theorem 8.** Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with boundary of class $C^{1,1}$. Then there exist constants $C, c > 0$ depending only on $\Omega$ with the following property: for any $\tau > 0$ and for any measurable set $M \subset \Omega$ with $\max\{|M|, \sqrt{|M|}\tau\} \leq c$ there exists a bi-Lipschitz mapping $\phi = \phi_{\tau,M} : \overline{\Omega} \rightarrow \overline{\Omega}$ satisfying
\[
\phi = \text{id} \quad \text{on } \partial \Omega,
\]
\[
\|
\phi - \text{id}\|_{W^{1,p}(\Omega)} \leq C|M|^{1/(2p)} \exp(C\tau^3) \quad \text{for every } 1 \leq p \leq \infty,
\]
\[
\|
\phi - \text{id}\|_{L^\infty(\Omega)} \leq C|M|^{1/2}\tau,
\]
\[
\det \nabla \phi \geq 1 + \tau \quad \text{a.e. in } M,
\]
\[
\det \nabla \phi \geq 1 - C|M|^{1/2}\tau \quad \text{a.e. in } \Omega.
\]

**Remark 9.** (i) We would like to stress that the constant $C$ only depends on $\Omega$. In particular (10) with $p = \infty$ gives the uniform Lipschitz estimate
\[
\|
\phi - \text{id}\|_{W^{1,\infty}(\Omega)} \leq C\tau \exp(C\tau^3).
\]

(ii) Note that by (3) we always have $\int_{\Omega} \det \nabla \phi = |\Omega|$. Hence a smallness assumption on $|M|$ (depending on $\tau$) is needed for (12) to hold true.

(iii) By a symmetric construction we can construct compressing maps $\varphi = \varphi_{\tau,M}$ satisfying
\[
\det \nabla \varphi \leq \frac{1}{1 + \tau} \quad \text{a.e. in } M \quad \text{and} \quad \det \nabla \varphi \leq \frac{1}{1 - C\sqrt{|M|}\tau} \quad \text{a.e. in } \Omega
\]
(instead of (13) and (14)) together with (3) and (11).
Linearizing the previous theorem we obtain, as an immediate corollary, the following result for the divergence operator.

**Corollary 10.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set with boundary of class \( C^{1,1} \). Then there are constants \( C, c > 0 \) depending only on \( \Omega \) with the following property: for any measurable set \( M \subset \Omega \) with \( |M| \leq c \) there exists \( v \in W^{1,\infty}_0(\Omega) \) such that

\[
\|v\|_{W^{1,p}(\Omega)} \leq C|M|^{1/(2p)} \quad \text{for every } 1 \leq p \leq \infty,
\]

\[
\text{div } v \geq 1 \quad \text{a.e. in } M,
\]

\[
\text{div } v \geq -C|M|^{1/2} \quad \text{a.e. in } \Omega.
\]

3. **Reduction to bi-Lipschitz stretchings**

We first show how our result on the prescribed Jacobian inequality (Theorem 1) reduces to the existence of the bi-Lipschitz stretching maps in Theorem 8.

The idea of the proof is the following: first (by convolution and classical results for the prescribed Jacobian equation in the smooth case) we exhibit a smooth map \( \varphi \) such that \( \varphi = \text{id} \) on \( \partial \Omega \) and \( \det \nabla \varphi > f \) outside a set \( M \) of small measure. Postcomposing \( \varphi \) with the stretching map \( \phi_{\tau,\varphi(M)} \) for some well chosen \( \tau \) then yields the desired solution to the prescribed Jacobian inequality.

**Proof of Theorem 1.**

**Step 1 (sharp regularity).** We first show that there exists a nonnegative function \( f \in L^\infty(\Omega) \) with \( \int_{\Omega} f < |\Omega| \) such that no solution to (4) can be \( C^1 \). Let \( M \subset \Omega \) be an open dense (in \( \Omega \)) set with \( |M| < |\Omega|/2 \) and let \( f = 2\chi_M \). We argue by contradiction and assume that there exists a solution \( \phi \in C^1 \) of (4). Then we would have

\[
\det \nabla \phi \geq 2 \quad \text{a.e. in } M.
\]

By continuity of \( \det \nabla \phi \) and the fact that \( M \) is open and dense, the previous inequality would imply

\[
\det \nabla \phi \geq 2 \quad \text{everywhere in } \Omega,
\]

which contradicts

\[
\int_{\Omega} \det \nabla \phi = |\Omega|.
\]

In the remainder of the proof, we are concerned with the existence part of our theorem.

**Step 2 (preliminaries).** Define

\[
\beta := \frac{|\Omega| - \int_{\Omega} f}{|\Omega|} \in (0, 1].
\]

Take \( \tau_0 \) big enough so that

\[
\frac{(1 + \tau_0)\beta}{2} \geq \|f\|_{L^\infty}.
\]

Next choose \( \epsilon_0 > 0 \) small enough so that

\[
(1 - C\tau_0 \sqrt{\|f\|_{L^\infty} + 1})\epsilon_0 \left( \frac{\beta}{2} + y \right) \geq y \quad \text{for every } y \in [0, \|f\|_{L^\infty}].
\]
taking $\epsilon_0$ even smaller we can moreover assume that
\begin{equation}
\left(\|f\|_{L^\infty} + 1\right)\epsilon_0 \sqrt{\left(\|f\|_{L^\infty} + 1\right)\epsilon_0 \tau_0} \leq c,
\end{equation}
where $C, c > 0$ are the constants (depending only on $\Omega$) in the statement of Theorem 8.

**Step 3 (approximation).** Extending $f$ by 1 outside of $\Omega$, mollifying the resulting function, and adding an appropriate constant, it is elementary to construct a sequence $f_\nu \in C^\infty(\overline{\Omega})$, $\nu \in \mathbb{N}$, such that $\int_\Omega f_\nu = |\Omega|$,
\begin{equation}
\frac{\beta}{2} \leq f_\nu \leq \|f\|_{L^\infty} + 1 \quad \text{in } \Omega,
\end{equation}
and
\[ f_\nu(x) \to f(x) + \beta \quad \text{for a.e. } x \in \Omega. \]
The last formula implying convergence in measure, there exist $\nu_0$ and a measurable set $A \subset \Omega$ such that $|A| \leq \epsilon_0$ and
\begin{equation}
f_{\nu_0} \geq \frac{\beta}{2} + f \quad \text{a.e. in } \Omega \setminus A.
\end{equation}

**Step 4 (conclusion).** By a classical result for the Jacobian equation (cf. e.g. Theorem 10.7 in [8]; recall that $\int_\Omega f_{\nu_0} = |\Omega|$) there exists $\varphi \in \text{Diff}^\infty(\overline{\Omega}; \overline{\Omega})$ so that $\det \nabla \varphi = f_{\nu_0}$ in $\Omega$ and $\varphi = \text{id}$ on $\partial \Omega$.

Using (19), we have
\begin{equation}
|\varphi(A)| = \int_A \det \nabla \varphi \leq (\|f\|_{L^\infty} + 1)|A| \leq (\|f\|_{L^\infty} + 1)\epsilon_0,
\end{equation}
hence, from (16), we can apply Theorem 8 with $M = \varphi(A)$ and $\tau = \tau_0$ and get a bi-Lipschitz mapping $\psi : \overline{\Omega} \to \overline{\Omega}$ such that $\psi = \text{id}$ on $\partial \Omega$ and
\begin{align}
\det \nabla \psi &\geq 1 + \tau_0 \quad \text{a.e. in } \varphi(A), \\
\det \nabla \psi &\geq 1 - C|\varphi(A)|^{1/2}\tau_0 \quad \text{a.e. in } \Omega.
\end{align}
We claim that $\phi := \psi \circ \varphi$ has all the desired properties. First we obviously have $\phi = \text{id}$ on $\partial \Omega$. It remains to show that
\[ \det \nabla \phi = \det \nabla \psi(\varphi) \cdot f_{\nu_0} \geq f \quad \text{a.e in } \Omega. \]

First, using (20), (17) and (14), we obtain that a.e. in $A$
\[ \det \nabla \psi(\varphi) \cdot f_{\nu_0} \geq (1 + \tau_0)f_{\nu_0} \geq (1 + \tau_0)\frac{\beta}{2} \geq \|f\|_{L^\infty}. \]

Finally, using (21), (19), (18), and (15), we get that a.e. in $\Omega \setminus A$
\[ \det \nabla \psi(\varphi) \cdot f_{\nu_0} \geq (1 - C\tau_0 \sqrt{|\varphi(A)|})f_{\nu_0} \geq (1 - C\tau_0 \sqrt{(\|f\|_{L^\infty} + 1)\epsilon_0})f_{\nu_0} \]
\[ \geq (1 - C\tau_0 \sqrt{(\|f\|_{L^\infty} + 1)\epsilon_0}) \left( \frac{\beta}{2} + f \right) \geq f, \]
which ends the proof.

We defer the proof of Theorem 3 to Section 4.3, as it is very similar to the above one.
4. Bi-Lipschitz stretching of a given set with small measure

In this section we prove Theorem 8. Its proof will consist of two main parts. In the first one we show that it is enough to prove the theorem when \( \Omega = (0,1)^2 \) (cf. Lemma 12), when the set \( M \) is compact (cf. Lemma 13), and without preserving the boundary values of the mapping pointwise (cf. Proposition 14). This way we will only be left with proving the simplified version of the result, cf. Proposition 11 below.

In the second part we prove Proposition 11. To this end we first prove two covering lemmas (cf. Lemma 15 and Lemma 17) based on results in [1]. The combination of our two lemmas will allow us to cover any compact set \( K \subset (0,1)^2 \) by a finite number of strips generated by 1–Lipschitz functions in both axis directions with the two following properties:

- The total width of the strips do not exceed \( 2\sqrt{|K|} \);
- no point is covered by more than three \( x \)-strips, neither do more than three \( y \)-strips cover a given point.

4.1. Simplification to compact subsets of the unit square. In this subsection, we show that it is enough to prove the following simplified version of Theorem 8:

**Proposition 11.** There exist universal constants \( C, c > 0 \) with the following property: for any \( \tau > 0 \) and any compact set \( K \subset (0,1)^2 \) with \( \max\{|K|, \sqrt{|K|}\tau\} \leq c \) there exists a bi-Lipschitz mapping \( \phi = \phi_{\tau,K} : [0,1]^2 \rightarrow [0,1]^2 \) satisfying

\[
\|\phi - \text{id}\|_{W^{1,p}([0,1]^2)} \leq C|K|^{1/(2p)} \tau \quad \text{for all } 1 \leq p \leq \infty,
\]

\[
\det \nabla \phi \geq 1 + \tau \quad \text{a.e. on } K,
\]

\[
\det \nabla \phi \geq 1 - C\sqrt{|K|} \tau \quad \text{a.e. on } [0,1]^2,
\]

\[
\|\phi - \text{id}\|_{L^\infty([0,1]^2)} \leq C\sqrt{|K|}\tau.
\]

Moreover the following properties concerning the boundary values hold true:

\[
\phi^2(s,0) = \phi^1(0,s) = 0 \quad \text{and} \quad \phi^2(s,1) = \phi^1(1,s) = 1 \quad \text{for every } s \in [0,1],
\]

\[
\|\phi - \text{id}\|_{W^{1,p}(\partial[0,1]^2)}, \|\phi^{-1} - \text{id}\|_{W^{1,p}(\partial[0,1]^2)} \leq C|K|^{1/(2p)} \tau \quad \text{for all } 1 \leq p \leq \infty.
\]

We first notice that is enough to prove Theorem 8 for \( \Omega = (0,1)^2 \). The proof will essentially follow from the fact that \( (9)-(13) \) behave well under fixed \( C^{1,1} \) mappings. As the proof is not difficult but tedious, it will be provided in Appendix A.

**Lemma 12.** It is enough to prove Theorem 8 when \( \Omega = (0,1)^2 \).

We then show that we can assume with no loss of generality that the set \( M \) is compact. The proof will mainly rely on inner regularity of the Lebesgue measure, on weak lower semicontinuity for the norms \( \| \cdot \|_{W^{1,p}} \), and on the weak continuity of the determinant.

**Lemma 13.** It is enough to prove Theorem 8 for compact sets \( M \subset \Omega \).

**Proof.** Let \( \tau > 0 \) and \( M \subset \Omega \) be a measurable set with \( |M| \) and \( \sqrt{|M|}\tau \) small enough. We assume that Theorem 8 has been proven for compact sets \( K \) and we show by a limiting process that the same holds for measurable sets \( M \). Recall that the constant \( C \) appearing in (10), (11), and (13) is independent of \( K \) (and of \( \tau \)).
Step 1. By inner regularity of the Lebesgue measure, we may choose an increasing sequence of compact sets $K_\nu \subset M$ with $\lim_{\nu \to \infty} |K_\nu| = |M|$. For any $K_\nu$ we have by assumption a bi-Lipschitz map $\phi_\nu = (\phi_\nu)_{\tau,K_\nu} : \Omega \to \overline{\Omega}$ satisfying

$$\phi_\nu = \text{id} \quad \text{on } \partial \Omega,$$

(22)

$$\|\phi_\nu - \text{id}\|_{W^{1,p}(\Omega)} \leq C|K_\nu|^{1/(2p)}\tau \exp(C\tau^3) \quad \text{for every } 1 \leq p \leq \infty,$$

(23)

$$\det \nabla \phi_\nu \geq 1 + \tau \quad \text{a.e. in } K_\nu,$$

(24)

$$\det \nabla \phi_\nu \geq 1 - C|K_\nu|^{1/2} \tau \quad \text{a.e. in } \Omega.$$

(25)

Taking $\sqrt{|M|} \tau$ smaller if necessary we infer from (25) that

$$\det \nabla \phi_\nu \geq 1/2 \quad \text{a.e. in } \Omega.$$

Step 2. By (23) with $p = \infty$ and the last inequality, the maps $\phi_\nu, \phi_\nu^{-1}$ are uniformly bounded in $W^{1,\infty}(\Omega)$, namely

$$\|\phi_\nu - \text{id}\|_{W^{1,\infty}(\Omega)} \leq C\tau \exp(C\tau^3)$$

and

$$\sup_\nu \|\phi_\nu^{-1}\|_{W^{1,\infty}} < \infty.$$

Hence, up to a subsequence we know that

$$\phi_\nu \rightharpoonup \phi \quad \text{in } W^{1,\infty}(\Omega) \quad \text{as } \nu \to \infty$$

holds for some bi-Lipschitz map $\phi$ from $\Omega$ to $\overline{\Omega}$. By compactness of the embedding of $W^{1,\infty}$ in $C^0$, we see that $\phi_\nu$ converges to $\phi$ also in $C^0(\overline{\Omega})$. Hence, using (22) we get

$$\phi = \text{id} \quad \text{on } \partial \Omega.$$

Note also that by the weak lower semicontinuity we have

$$\|\phi - \text{id}\|_{W^{1,p}(\Omega)} \leq \liminf_{\nu \to \infty} \|\phi_\nu - \text{id}\|_{W^{1,p}(\Omega)} \quad \text{for every } 1 \leq p \leq \infty.$$
which implies (by (28))
\[
\int_{\Omega} \chi_A \det \nabla \phi \, dx \geq \int_{\Omega} (1 + \tau) \chi_A \, dx,
\]
where \( \chi_A \) is the usual characteristic function of \( A \). By arbitrariness of \( A \subset K_\nu \), we obtain \( \det \nabla \phi \geq 1 + \tau \) a.e. on \( K_\nu \) for all \( \nu \). Since \(|M \setminus (\cup_{\nu \geq 1} K_\nu)| = 0\), we deduce
\[
\det \nabla \phi \geq 1 + \tau \quad \text{a.e. in } M.
\]
Similarly, using (25) and (28) we get for any measurable subset \( A \subset \Omega \)
\[
\int_{\Omega} \chi_A \det \nabla \phi \, dx = \lim_{\nu \to \infty} \int_{\Omega} \chi_A \det \nabla \phi_\nu \, dx
\]
\[
\geq \lim_{\nu \to \infty} \int_{\Omega} (1 - C \sqrt{|K_\nu|} \tau) \chi_A \, dx = \int_{\Omega} (1 - C \sqrt{|M|} \tau) \chi_A \, dx,
\]
which, again by arbitrariness of \( A \), implies
\[
\det \nabla \phi \geq 1 - C \sqrt{|M|} \tau \quad \text{a.e. in } \Omega.
\]
This ends the proof. \( \square \)

We finally notice that it is enough to prove Theorem 8 when \( \Omega = (0,1)^2 \), when \( M \subset \Omega \) is compact and without assuming the bi-Lipschitz mapping to preserve the boundary pointwise. The main idea of the proof will be to exhibit a well-chosen isotopy correcting the boundary values, leaving the determinant unchanged, and preserving the \( W^{1,p} \) estimates. As the proof is technical and not the core of the proof of Theorem 8, we chose to postpone it to Appendix B.

**Proposition 14.** In order to prove Theorem 8 it is enough to prove Proposition 11; i.e. it is enough to show that there exist constants \( C, c > 0 \) such that for any compact subset \( K \subset (0,1)^2 \), there exists a bi-Lipschitz map \( \phi = \phi_{r,K} : [0,1]^2 \to [0,1]^2 \) with the properties
\[
\| \phi - \text{id} \|_{W^{1,p}((0,1)^2)} \leq C |K|^{1/(2p)} \tau \quad \text{for every } 1 \leq p \leq \infty,
\]
\[
\| \phi - \text{id} \|_{L^\infty((0,1)^2)} \leq C |K|^{1/2} \tau,
\]
\[
\det \nabla \phi \geq 1 + \tau \quad \text{a.e. in } K,
\]
\[
\det \nabla \phi \geq 1 - C |K|^{1/2} \tau \quad \text{a.e. in } (0,1)^2.
\]
as well as
\[
\phi^2(s,0) = \phi^1(0,s) = 0 \quad \text{and} \quad \phi^2(s,1) = \phi^1(1,s) = 1 \quad \text{for every } s \in [0,1],
\]
\[
\| \phi - \text{id} \|_{W^{1,p}((0,1)^2)}, \| \phi^{-1} - \text{id} \|_{W^{1,p}((0,1)^2)} \leq C |K|^{1/(2p)} \tau \quad \text{for all } 1 \leq p \leq \infty,
\]
where we have written \( \phi = (\phi^1, \phi^2) \).

4.2. **Covering and stretching.** Having performed the desired reduction in Lemma 12, Lemma 13 and Proposition 14 we see that it is enough to show Proposition 11 in order to prove Theorem 8. As already said we will first establish two covering results to prove Proposition 14.

The first result, based on a result in [1], essentially says that it is possible to cover any set of \( \sharp P \) grid points in the plane by graphs of at most \( \sqrt{\sharp P} \) 1-Lipschitz functions over the x-axis and graphs of at most \( \sqrt{\sharp P} \) 1-Lipschitz functions over the y-axis. Furthermore, and this will be crucial for our purpose, the horizontal Lipschitz graphs will be chosen to keep a vertical distance from each other of at
least the grid spacing (and similarly for the vertical Lipschitz graphs). Note that this last property is not contained in [1].

Lemma 15. Let $l \geq 2$ be an integer and let

$$P \subset \left\{ \frac{3}{2^{l+1}}, \frac{5}{2^{l+1}}, \ldots, 1 - \frac{3}{2^{l+1}} \right\}^2$$

be a set of points in the unit square. Then there exist $N, M \in \mathbb{N}$ with $N, M \leq \sqrt{\#P}$ and 1-Lipschitz functions $f_i, g_j : [0, 1] \to \left[ \frac{3}{2^{l+1}}, 1 - \frac{3}{2^{l+1}} \right]$, $1 \leq i \leq N$, $1 \leq j \leq M$ with the following properties:

(29) \quad $P \subset \left( \bigcup_{i=1}^{N} \{ (x, f_i(x)) : x \in [0, 1] \} \right) \cup \left( \bigcup_{j=1}^{M} \{ (g_j(y), y) : y \in [0, 1] \} \right)$

and for any $1 \leq i < k \leq N$ and any $1 \leq j < m \leq M$

(30) \quad $f_i \geq f_k + \frac{1}{2^{l}}$ and $g_j \geq g_m + \frac{1}{2^{l}}$ in $[0, 1]$.

Remark 16. The previous lemma is no longer true in higher dimensions (cf. Question 8.2 in [1]).

Explanation of the accompanying pictures

In Figure 1 the strategy of the proof of the proposition is illustrated. For the notation used in the following explanations, see the proof below.

(i) The first picture illustrates Step 1 and represents the covering of the set $P$ (the black grid points) by 1-Lipschitz graphs over the horizontal and over the vertical axis. The black points on a single horizontal 1-Lipschitz graph correspond to one tuple of points $\{ (p_{i1}, q_{i1}), \ldots, (p_{iL_i}, q_{iL_i}) \}$.

(ii) The graphs of the maps $\tilde{f}_i$ (cf. Step 2) are depicted in the second picture. The graph of each of them consists only of horizontal or diagonal segments.

(iii) In Step 3, the $\hat{f}_i$ are constructed. The green line in the second picture represents the graph of $\hat{f}_1$. The third and the fourth pictures show the construction of the maps $\hat{f}_2$ (in blue) and the graph of $\hat{f}_3$ (in red). Note that the resulting graphs keep a vertical distance from each other of at least the grid spacing.

(iv) The last picture contains the construction of the final $f_i$ (cf. Step 4) for a different constellation of points. The corresponding $f_i$ are depicted in the fifth picture.

Proof. Let us first summarize briefly the idea of the proof. In Step 1 we use a covering result in [1] to find an appropriate partition of the set $P$ into subsets which may be covered by a single graph of a 1-Lipschitz function. In Step 2 we construct 1-Lipschitz maps $\tilde{f}_i$ and $\tilde{g}_j$ satisfying all the desired properties except possibly (30). In Step 3 we exhibit 1-Lipschitz maps $\hat{f}_i$ and $\hat{g}_j$ subject to the properties stated in the lemma except that those maps may take values outside of $[\frac{3}{2^{l+1}}, 1 - \frac{3}{2^{l+1}}]$. Finally in the last step we construct our final maps $f_i$ and $g_j$.

Step 1. Applying Theorem 2.1 in [1] (whose proof is very elegant, short and self-contained) to the set $P$, we obtain $N \leq \sqrt{\#P}$ tuples of points

$$\{(p_{i1}^1, q_{i1}^1), \ldots, (p_{iL_i}^i, q_{iL_i}^i)\}, \quad 1 \leq i \leq N,$$

and $M \leq \sqrt{\#P}$ tuples of points

$$\{(s_{j1}^1, t_{j1}^1), \ldots, (s_{jR_j}^j, t_{jR_j}^j)\}, \quad 1 \leq j \leq M,$$
Figure 1. The pictures above illustrate the proof of Lemma 15.
For an explanation of the pictures, see the remarks in the accompanying text.
such that
\[
P = \left( \bigcup_{n=1}^{N} \bigcup_{i=1}^{L_i} (p^n_i, q^n_i) \right) \cup \left( \bigcup_{j=1}^{M} \bigcup_{i=1}^{R_j} (s^j_i, t^j_i) \right)
\]
and
\[
|q^n_{i+1} - q^n_i| \leq p^n_{i+1} - p^n_i \quad \text{for any } 1 \leq i \leq N \text{ and any } 1 \leq n \leq L_i - 1,
\]
\[
|s^j_{i+1} - s^j_i| \leq t^j_{i+1} - t^j_i \quad \text{for any } 1 \leq j \leq M \text{ and any } 1 \leq m \leq R_j - 1.
\]

Step 2. We construct 1-Lipschitz maps
\[
\mathcal{T}_i, g_j : [0,1] \to \left[ \frac{3}{2i+1}, 1 - \frac{3}{2i+1} \right], \quad 1 \leq i \leq N, 1 \leq j \leq M
\]
satisfying (29). Define for $1 \leq i \leq N$
\[
\mathcal{T}_i (x) := \begin{cases} 
q^n_i & \text{for } 0 \leq x \leq p^n_i \\
\max \{q^n_i, q^n_{i+1} - p^n_{i+1} + x\} & \text{for } p^n_i \leq x \leq p^n_{i+1} \text{ if } q^n_i < q^n_{i+1} \\
\min \{q^n_i, q^n_{i+1} + p^n_{i+1} - x\} & \text{for } p^n_i \leq x \leq p^n_{i+1} \text{ if } q^n_i \geq q^n_{i+1} \\
q^n_{L_i} & \text{for } p^n_{L_i} \leq x \leq 1
\end{cases}
\]
and for $1 \leq j \leq M$
\[
g_j (y) := \begin{cases} 
s^j_i & \text{for } 0 \leq y \leq t^j_i \\
\max \{s^j_i, s^j_{i+1} - t^j_{i+1} + y\} & \text{for } t^j_i \leq y \leq t^j_{i+1} \text{ if } s^j_i < s^j_{i+1} \\
\min \{s^j_i, s^j_{i+1} + t^j_{i+1} - y\} & \text{for } t^j_i \leq y \leq t^j_{i+1} \text{ if } s^j_i \geq s^j_{i+1} \\
s^j_{R_j} & \text{for } t^j_{R_j} \leq y \leq 1.
\end{cases}
\]

It is then trivial to see that the $\mathcal{T}_i$ and $g_j$ are 1–Lipschitz. Since every $q^n_i$ and every $s^j_i$ as well as every $t_j$ and every $p_j$ belong to $\left\{ \frac{3}{2i+1}, \frac{5}{2i+1}, \ldots, 1 - \frac{3}{2i+1} \right\}$, it is direct that
\[
(31) \quad \mathcal{T}_i (a), g_j (a) \in \left\{ \frac{3}{2i+1}, \frac{5}{2i+1}, \ldots, 1 - \frac{3}{2i+1} \right\} \quad 1 \leq i \leq N, 1 \leq j \leq M
\]
holds for every $a \in \{ \frac{3}{2i+1}, \frac{5}{2i+1}, \ldots, 1 - \frac{3}{2i+1} \}$ and that the $\mathcal{T}_i$ and $g_j$ take values in $[\frac{3}{2i+1}, 1 - \frac{3}{2i+1}]$. Finally since
\[
\mathcal{T}_i (p^n_i) = q^n_i \quad \text{and} \quad g_j (t^j_i) = s^j_i,
\]
we get at once
\[
(32) \quad P \subset \left( \bigcup_{i=1}^{N} \{(a, \mathcal{T}_i (a))\} \right) \cup \left( \bigcup_{j=1}^{M} \{(a, g_j (a))\} : a \in \left\{ \frac{3}{2i+1}, \frac{5}{2i+1}, \ldots, 1 - \frac{3}{2i+1} \right\} \right)
\]
which trivially implies that the $\mathcal{T}_i$ and $g_j$ satisfy (29).

Step 3. We now construct 1–Lipschitz functions $\hat{f}_i, \hat{g}_j$ satisfying (29) and (50), which however may take values in $[-1, 1 - \frac{3}{2i+1}]$ instead of $[\frac{3}{2i+1}, 1 - \frac{3}{2i+1}]$. The procedure for obtaining the $\hat{g}_j$ is analogous to the procedure for the $\hat{f}_i$, so we only state our construction for the $\hat{f}_i$.

The $\hat{f}_i, 1 \leq i \leq N$, are defined by induction as follows. Let $x \in [0,1]$ and let $\nu_1, \ldots, \nu_N$ be such that $\{\nu_1, \ldots, \nu_N\} = \{1, \ldots, N\}$ and
\[
\mathcal{T}_{\nu_1} (x) \geq \cdots \geq \mathcal{T}_{\nu_N} (x)
\]
hold. Note that the \( \nu_k \) are not necessarily uniquely defined and may depend on \( x \). However, ambiguities in the definition of the \( \nu_k \) may only arise if \( \mathcal{F}_{\nu_i}(x) = \mathcal{F}_{\nu_{i+1}}(x) \) for some \( i \); note that the ambiguities will not affect the definition of the \( \hat{f}_i \) below.

We now define

\[
\hat{f}_1(x) := \mathcal{F}_{\nu_1}(x),
\]

\[
\hat{f}_{i+1}(x) := \min \left\{ \hat{f}_i(x) - \frac{1}{2^i}, \mathcal{F}_{\nu_{i+1}}(x) \right\} \quad \text{for } 1 \leq i \leq N - 1.
\]

Recalling that the \( \mathcal{F}_i \) take value in \( \left[ \frac{3}{2^{r+1}}, 1 - \frac{3}{2^{r+1}} \right] \) we get by a simple induction that the \( \hat{f}_i \) take values in

\[
\left[ \frac{3}{2^{r+1}} - \frac{(i-1)}{2^i}, 1 - \frac{3}{2^{i+1}} - \frac{(i-1)}{2^i} \right].
\]

Moreover, (31) gives for every \( 1 \leq i \leq N \) and every \( a \in \left\{ \frac{3}{2^{r+1}}, \ldots, 1 - \frac{3}{2^{r+1}} \right\} \)

\[
(33) \quad \hat{f}_i(a) \in \left\{ \frac{3 - 2(i-1)}{2^i}, \frac{5 - 2(i-1)}{2^i+1}, \ldots, 1 - \frac{3}{2^{i+1}} \right\}.
\]

By construction the \( \hat{f}_i \) satisfy (30). Moreover, recalling that \( |\mathcal{F}_i| \leq 1 \) we easily deduce that the same holds for the maps \( \hat{f}_i \). Using (31) and (33) it is not difficult to see that for every \( a \in \left\{ \frac{3}{2^{r+1}}, \frac{5}{2^{r+1}}, \ldots, 1 - \frac{3}{2^{r+1}} \right\} \) it holds that

\[
(34) \quad \{ \mathcal{F}_1(a), \ldots, \mathcal{F}_N(a) \} \subset \{ \hat{f}_1(a), \ldots, \hat{f}_N(a) \}.
\]

To show this, we argue by contradiction and suppose that there were some \( a \) and some \( j \) with \( \mathcal{F}_j(a) \neq \hat{f}_k(a) \) for all \( k \). Let \( i \) be such that \( j = \nu_i \). If \( i = 1 \), we have a contradiction, so assume that \( i > 1 \). Then obviously we would have \( \hat{f}_i(a) < \mathcal{F}_{\nu_i}(a) = \mathcal{F}_j(a) \), implying \( \hat{f}_i(a) \leq \mathcal{F}_j(a) - \frac{1}{2^i} \). The definition of \( \hat{f}_i \) then yields \( \hat{f}_{i-1}(a) \leq \mathcal{F}_j(a) \). By assumption this implies \( \hat{f}_{i-1}(a) < \mathcal{F}_j(a) \). Proceeding by induction, we arrive at a contradiction, as \( \hat{f}_i(a) = \mathcal{F}_{\nu_i}(a) \geq \mathcal{F}_j(a) \).

Proceeding exactly in the same way we construct 1-\( \nu \)-Lipschitz maps \( \hat{g}_j, 1 \leq j \leq M \) taking values in \( \left[ \frac{3}{2^{r+1}} - \frac{(j-1)}{2^r}, 1 - \frac{2(j-1)}{2^{r+1}} - \frac{(j-1)}{2^r} \right] \) and satisfying for every \( a \in \left\{ \frac{3}{2^{r+1}}, \frac{5}{2^{r+1}}, \ldots, 1 - \frac{3}{2^{r+1}} \right\} \)

\[
(35) \quad \hat{g}_j(a) \in \left\{ \frac{3 - 2(j-1)}{2^j}, \frac{5 - 2(j-1)}{2^{j+1}}, \ldots, 1 - \frac{3}{2^{j+1}} \right\}
\]

as well as

\[
(36) \quad \{ \mathcal{G}_1(a), \ldots, \mathcal{G}_M(a) \} \subset \{ \hat{g}_1(a), \ldots, \hat{g}_M(a) \}.
\]

Combining (32), (35), and (36), we see that the \( \hat{f}_i \) and \( \hat{g}_j \) satisfy (30). The \( \hat{f}_i, \hat{g}_i \) therefore satisfy all the conditions of the lemma, except perhaps for the condition \( \hat{f}_i, \hat{g}_i \geq \frac{3}{2^{r+1}} \).

**Step 4.** We finally define our functions \( f_i \) and \( g_j \) by pushing the \( \hat{f}_i \) and the \( \hat{g}_j \) back into the domain \( \left[ \frac{3}{2^{r+1}}, 1 - \frac{3}{2^{r+1}} \right] \) if necessary. Again we only make explicit the construction for the \( f_i \), the one for the \( g_j \) being completely symmetric.
We define the \( f_i \) inductively by
\[
  f_N := \max \left( f_N, \frac{3}{2^i+1} \right),
\]
\[
  f_{i-1} := \max \left( f_{i-1}, f_i + \frac{1}{2^i} \right) \quad \text{for } N \geq i \geq 2.
\]

First it is obvious to see that the \( f_i \) are \( 1 \)-Lipschitz maps which satisfy (30). Furthermore, we have \( f_i \geq \frac{3}{2^i+1} \) for any \( 1 \leq i \leq N \). Recalling that
\[
  \hat{f}_i \leq 1 - \frac{3}{2^i+1} - \frac{i-1}{2^i}
\]
we notice by induction, starting with \( i = N \), that
\[
  (37) \quad \hat{f}_i \leq 1 - \frac{3}{2^i+1} - \frac{i-1}{2^i} \leq 1 - \frac{3}{2^i+1} \quad \text{for every } 1 \leq i \leq N.
\]

Here, to start the induction (i.e. to treat the case \( i = N \)), we have used
\[
  N \leq \sqrt{2P} \leq \sqrt{(2^i - 2)^2} = 2^i - 2
\]
which implies
\[
  \frac{3}{2^i+1} \leq 1 - \frac{3}{2^i+1} - \frac{N-1}{2^i}.
\]

Moreover, using (33) and (37) we directly get by induction that for every \( 1 \leq i \leq N \) and every \( a \in \{\frac{3}{2^i+1}, \frac{5}{2^i+1}, \ldots, 1 - \frac{3}{2^i+1}\} \) we have
\[
  f_i(a) \in \left\{ \frac{3}{2^i+1}, \frac{5}{2^i+1}, \ldots, 1 - \frac{3}{2^i+1} \right\}.
\]

Using (33) and the previous formula, proceeding as in the previous step but now in the reverse direction it is not difficult to see that for every \( a \in \{\frac{3}{2^i+1}, \frac{5}{2^i+1}, \ldots, 1 - \frac{3}{2^i+1}\} \)
\[
  (38) \quad \{\hat{f}_1(a), \ldots, \hat{f}_N(a)\} \cap \left\{ \frac{3}{2^i+1}, \frac{5}{2^i+1}, \ldots, 1 - \frac{3}{2^i+1} \right\} \subset \{f_1(a), \ldots, f_N(a)\}.
\]

Proceeding similarly we get \( 1 \)-Lipschitz functions \( g_j \), \( 1 \leq j \leq M \), satisfying (30), taking values in \( \left\{ \frac{3}{2^i+1}, 1 - \frac{3}{2^i+1}\right\} \), and satisfying for every \( a \in \{\frac{3}{2^i+1}, \frac{5}{2^i+1}, \ldots, 1 - \frac{3}{2^i+1}\} \) the inclusion
\[
  (39) \quad \{\hat{g}_1(a), \ldots, \hat{g}_M(a)\} \cap \left\{ \frac{3}{2^i+1}, \frac{5}{2^i+1}, \ldots, 1 - \frac{3}{2^i+1} \right\} \subset \{g_1(a), \ldots, g_M(a)\}.
\]

Combining (32), (34), (36), (38) and (39), we directly get that (29) is satisfied. This concludes the proof. \( \square \)

We now state our second covering lemma needed to prove Proposition 11. This result essentially says that for any compact subset \( K \) of the unit square and any \( \zeta > 0 \) there is an integer \( l \) such that we can cover \( K \) by at most \( 2^l \sqrt{|K| + \zeta} \) strips of height \( 1/2^{l-1} \) generated by horizontal graphs of \( 1 \)-Lipschitz functions and at most \( 2^l \sqrt{|K| + \zeta} \) strips of height \( 1/2^{l-1} \) generated by vertical graphs of \( 1 \)-Lipschitz functions. Furthermore, the horizontal strips do not intersect more than three times, as do the vertical strips. This result is an improvement of a result in [1] (the latter result does not contain a uniform control on the number of intersections of the strips).
Lemma 17. For any compact set $K \subset (0, 1)^2$ and any $\zeta > 0$ there exist $l, N, M \in \mathbb{N}$ and 1-Lipschitz maps $f_i, g_j : [0, 1] \to [\frac{1}{2^l}, 1 - \frac{1}{2^l}], 1 \leq i \leq N, 1 \leq j \leq M$, with the following properties:

$$
\frac{N}{2^l} \leq \sqrt{|K|} + \zeta \quad \text{and} \quad \frac{M}{2^l} \leq \sqrt{|K|} + \zeta,
$$

$$
K \subset \bigcup_{i=1}^{N} \{ |f_i(x) - y| \leq \frac{1}{2^l} \} \cup \bigcup_{j=1}^{M} \{ |g_j(y) - x| \leq \frac{1}{2^l} \} \subset [0, 1]^2,
$$

$$
\sum_{i=1}^{N} \chi_{\{ |f_i(x) - y| \leq \frac{1}{2^l} \}}(x, y) \leq 3 \quad \text{and} \quad \sum_{j=1}^{M} \chi_{\{ |g_j(y) - x| \leq \frac{1}{2^l} \}}(x, y) \leq 3,
$$

where $\{ |f_i(x) - y| \leq \frac{1}{2^l} \}$ is a compact notation for

$$
\{(x, y) : 0 \leq x \leq 1, |f_i(x) - y| \leq \frac{1}{2^l} \},
$$

and where $\{ |g_j(y) - x| \leq \frac{1}{2^l} \}$ is a compact notation for

$$
\{(x, y) : 0 \leq y \leq 1, |g_j(y) - x| \leq \frac{1}{2^l} \}.
$$

Proof. The proof is based on a fairly easy application of Lemma 15. Let $K \subset (0, 1)^2$ be a compact and $\zeta > 0$.

**Step 1.** For every integer $l$, define $P_l$ to be the set of points

$$
(a, b) \in \left\{ \frac{1}{2^l+1}, \frac{3}{2^l+1}, \ldots, 1 - \frac{1}{2^l+1} \right\}^2
$$

such that

$$
\left[ a - \frac{1}{2^l+1}, a + \frac{1}{2^l+1} \right] \times \left[ b - \frac{1}{2^l+1}, b + \frac{1}{2^l+1} \right] \cap K \neq \emptyset.
$$

Note that the $2^{2l}$ squares

$$
\left[ a - \frac{1}{2^l+1}, a + \frac{1}{2^l+1} \right] \times \left[ b - \frac{1}{2^l+1}, b + \frac{1}{2^l+1} \right]
$$

cover $[0, 1]^2$ and their interiors are pairwise disjoint. Hence, by compactness of $K$ and monotone convergence, we obtain

$$
\lim_{l \to \infty} \int_{[0, 1]^2} \sum_{(a, b) \in P_l} \chi_{\left[ a - \frac{1}{2^l+1}, a + \frac{1}{2^l+1} \right] \times \left[ b - \frac{1}{2^l+1}, b + \frac{1}{2^l+1} \right]}(x, y) \, d(x, y)
$$

$$
= \int_{[0, 1]^2} \chi_K(x, y) \, d(x, y)
$$

or equivalently

$$
\lim_{l \to \infty} \frac{\#P_l}{2^{2l}} = |K|.
$$

Hence for every $l$ large enough we have that

$$
\frac{\sqrt{\#P_l}}{2^l} \leq \sqrt{|K|} + \zeta.
$$
Moreover, by the very definition of $P_l$ we have that
\begin{equation}
K \subset \bigcup_{(a,b) \in P_l} \left[ a - \frac{1}{2l+1}, a + \frac{1}{2l+1} \right] \times \left[ b - \frac{1}{2l+1}, b + \frac{1}{2l+1} \right].
\end{equation}

Finally, since $K$ is compact in $(0, 1)^2$ (and hence away from the boundary), taking $l$ bigger if necessary, we may assume that
\[ P_l \subset \left\{ \frac{3}{2l+1}, \frac{5}{2l+1}, \ldots, 1 - \frac{3}{2l+1} \right\}^2. \]

**Step 2.** We fix $l$ large enough so that the last inclusion and \[10\] hold. Applying Lemma \[13\] to $P_l$ provides us with $1-$Lipschitz maps
\[ f_i, g_j : [0, 1] \to \left[ \frac{3}{2l+1}, 1 - \frac{3}{2l+1} \right], \quad 1 \leq i \leq N, \ 1 \leq j \leq M, \]
such that
\begin{equation}
N, M \leq \sqrt{2l},
\end{equation}
\begin{equation}
P_l \subset \left\{ \bigcup_{i=1}^N \{ (x, f_i(x)) : x \in [0, 1] \} \right\} \cup \left\{ \bigcup_{j=1}^M \{ (g_j(y), y) : y \in [0, 1] \} \right\},
\end{equation}
\begin{equation}
f_i(x) \geq f_k(x) + \frac{1}{2l} \quad \text{for every } 1 \leq i < k \leq N \text{ and every } x \in [0, 1],
\end{equation}
\begin{equation}
g_j(y) \geq g_m(y) + \frac{1}{2l} \quad \text{for every } 1 \leq j < m \leq M \text{ and every } y \in [0, 1].
\end{equation}

**Step 3.** We now claim that the maps $f_i$ and $g_j$ satisfy all the desired properties. First, combining \[10\] and \[12\] immediately gives
\[ \frac{N}{2l}, \frac{M}{2l} \leq \sqrt{|K| + \frac{\zeta}{2l}}. \]

Next \[14\] and \[15\] directly imply that
\[ \sum_{i=1}^N \chi(\{|f_i(x) - y| \leq \frac{1}{2l}\}) \leq 3 \quad \text{and} \quad \sum_{j=1}^M \chi(\{|g_j(y) - x| \leq \frac{1}{2l}\}) \leq 3. \]

Recalling that the $f_i$ and the $g_j$ take their values in $[\frac{3}{2l+1}, 1 - \frac{3}{2l+1}]$, we trivially get
\[ \left( \bigcup_{i=1}^N \left\{ |f_i(x) - y| \leq \frac{1}{2l} \right\} \right) \cup \left( \bigcup_{j=1}^M \left\{ |g_j(y) - x| \leq \frac{1}{2l} \right\} \right) \subset [0, 1]^2. \]

We finally prove that $K$ is included in the union of the strips. Let $(x, y) \in K$. By \[11\] there exists $(a, b) \in P_l$ such that
\[ (x, y) \in \left[ a - \frac{1}{2l+1}, a + \frac{1}{2l+1} \right] \times \left[ b - \frac{1}{2l+1}, b + \frac{1}{2l+1} \right]. \]

By \[13\] we know that either $(a, b) = (a, f_i(a))$ for some $1 \leq i \leq N$ or $(a, b) = (g_j(b), b)$ for some $1 \leq j \leq M$, which easily implies that (recalling that $|f'_i|, |g'_j| \leq 1$) either, in the first case,
\[ \left[ a - \frac{1}{2l+1}, a + \frac{1}{2l+1} \right] \times \left[ b - \frac{1}{2l+1}, b + \frac{1}{2l+1} \right] \subset \left\{ |f_i(x) - y| \leq \frac{1}{2l} \right\} \]

or second case, etc.etc.,
or, in the second case,

\[
\left[a - \frac{1}{2^j+1}, a + \frac{1}{2^j+1}\right] \times \left[b - \frac{1}{2^j+1}, b + \frac{1}{2^j+1}\right] \subseteq \left\{\{g_j(y) - x| \leq \frac{1}{2^j}\right\}
\]

which proves

\[
K \subseteq \left(\bigcup_{i=1}^{N}\left\{f_i(x) - y| \leq \frac{1}{2^i}\right\}\right) \cup \left(\bigcup_{j=1}^{M}\left\{g_j(y) - x| \leq \frac{1}{2^j}\right\}\right)
\]

and ends the proof. \(\square\)

With the help of the two previous lemmas we are now in position to prove Proposition 11 which we recall here for the convenience of the reader.

**Proposition.** There exist universal constants \(C, c > 0\) with the following property: for any \(\tau > 0\) and any compact set \(K \subset (0, 1)^2\) with \(\max\{|K|, \sqrt{|K|}\tau}\) \leq c there exists a bi-Lipschitz mapping \(\phi = \phi_{\tau, K} : [0, 1]^2 \to [0, 1]^2\) satisfying

\[
\|\phi - \text{id}\|_{W^{1,p}([0, 1]^2)} \leq C|K|^{1/(2p)}\tau \quad \text{for all } 1 \leq p \leq \infty,
\]

\[
\det \nabla\phi \geq 1 + \tau \quad \text{a.e. on } K,
\]

\[
\det \nabla\phi \geq 1 - C\sqrt{|K|\tau} \quad \text{a.e. on } [0, 1]^2,
\]

\[
\|\phi - \text{id}\|_{L^{\infty}([0, 1]^2)} \leq C\sqrt{|K|\tau}.
\]

Moreover the following properties concerning the boundary values hold true:

\[
\phi^2(s, 0) = \phi^1(0, s) = 0 \quad \text{and} \quad \phi^2(s, 1) = \phi^1(1, s) = 1 \quad \text{for every } s \in [0, 1],
\]

\[
\|\phi - \text{id}\|_{W^{1,p}(\partial [0, 1]^2)}, \|\phi^{-1} - \text{id}\|_{W^{1,p}(\partial [0, 1]^2)} \leq C|K|^{1/(2p)}\tau \quad \text{for all } 1 \leq p \leq \infty.
\]

**Remark 18.** (i) The constants \(C\) and \(c\) can be computed explicitly in the following proof.

(ii) In fact the proof will provide a map \(\phi\) of the form \(\phi = \text{id} + \tau\xi\), where \(\xi\) is independent of \(\tau\). Moreover \(\tau \to \phi\) and \(\tau \to \phi^{-1}\) are Lipschitz.

In the accompanying figure, the strategy of the proof of the proposition is illustrated.

**Explanation of Figure 2**

For the notation used in the following explanations, see the proof below.

In all pictures, the unit square is divided into \(2^{2l}\) squares of size \(1/2^l\) (the division is sketched by gray lines). In the first three pictures, the compact set \(K\) is sketched; it corresponds to the gray regions.

(i) The black points in the first and second pictures represent the set \(P_l\). It consists of the centers of all grid squares which have nonempty intersection with \(K\). The centers of the grid squares which have empty intersection with \(K\) are colored in gray.

(ii) The colored lines in the second picture represent the graphs of the \(1\)-Lipschitz maps \(f_i\) and \(g_j\) which cover the set \(P_1\), as provided by Lemma 15.

(iii) The third picture represents the covering of \(K\) by the strips of width \(2\delta = 1/2^{l-1}\) generated by the \(f_i\) and the \(g_j\). This covering is provided by Lemma 17.

(iv) The fourth and the last picture illustrate the construction of our final stretching map \(\phi = (\phi_1, \phi_2)\) in Proposition 11 to define \(\phi_2\) we stretch all the horizontal strips by a factor of \(1 + 2\tau\) (cf. the penultimate picture); similarly, to define \(\phi_1\) we stretch all the vertical strips by a factor of \(1 + 2\tau\) (cf. the last picture).
Figure 2. The above sketches illustrate our second covering lemma as well as Proposition 11. For an explanation of the pictures, see the accompanying text.
**Proof.** First note that if $|K| = 0$ then $\phi = \text{id}$ trivially has the properties stated in the proposition; hence we can assume that $|K| > 0$.

**Step 1 (preliminaries).** Applying Lemma 17 to $K$ and $\zeta = |K|$, there exist $\delta = 1/2^2 > 0$, $N, M \in \mathbb{N}$ and $1$–Lipschitz functions $f_1, g_2 : [0, 1] \to [0, 1]$, $1 \leq i \leq N$, $1 \leq j \leq M$ such that

\[
\delta N \leq 2\sqrt{|K|} \quad \text{and} \quad \delta M \leq 2\sqrt{|K|},
\]

\[
K \subset \left( \bigcup_{i=1}^{N} \{|f_i(x) - y| \leq \delta\} \right) \cup \left( \bigcup_{j=1}^{M} \{|g_j(y) - x| \leq \delta\} \right) \subset [0, 1]^2,
\]

\[
\sum_{i=1}^{N} \chi_{\{|f_i(x) - y| \leq \delta\}} \leq 3 \quad \text{and} \quad \sum_{j=1}^{M} \chi_{\{|g_j(y) - x| \leq \delta\}} \leq 3.
\]

**Step 2 (definition of $\phi$).** Define our map $\phi = (\phi_1, \phi_2)$ by

\[
\phi_2(x, y) = (1 - 4\delta N\tau)y + 2\tau \sum_{i=1}^{N} \left( \min(2\delta, y - f_i(x) + \delta) - \min(0, y - f_i(x) + \delta) \right)
\]

\[
= (1 - 4\delta N\tau)y + 2\tau \sum_{i=1}^{N} \left\{ \begin{array}{ll}
2\delta & \text{if } y \geq f_i(x) + \delta \\
0 & \text{if } y \leq f_i(x) - \delta
\end{array} \right.
\]

and, symmetrically,

\[
\phi_1(x, y) = (1 - 4\delta M\tau)x + 2\tau \sum_{j=1}^{M} \left( \min(2\delta, x - g_j(y) + \delta) - \min(0, x - g_j(y) + \delta) \right)
\]

\[
= (1 - 4\delta M\tau)x + 2\tau \sum_{j=1}^{M} \left\{ \begin{array}{ll}
2\delta & \text{if } x \geq g_j(y) + \delta \\
0 & \text{if } x \leq g_j(y) - \delta
\end{array} \right.
\]

**Step 3 (properties of $\phi_2$).** First, since (by 46) $\delta N \leq 2\sqrt{|K|}$, we get for every $(x, y) \in [0, 1]^2$

\[
|\phi_2(x, y) - y| \leq 4\tau\delta Ny + 4\tau\delta N \leq 16\tau\sqrt{|K|}.
\]

Moreover, since all the strips are contained in $[0, 1]^2$ (cf. the second inclusion in 17) we get that for $1 \leq i \leq N$ and $x \in [0, 1]$

\[
1 \geq f_i(x) + \delta \quad \text{and} \quad 0 \leq f_i(x) - \delta
\]

which directly implies

\[
\phi_2(x, 0) = 0 \quad \text{and} \quad \phi_2(x, 1) = 1 \quad \text{for all } x \in [0, 1].
\]

We now investigate the derivatives of $\phi_2$. It is elementary to see that $\phi_2$ is Lipschitz (and thus a.e differentiable) and that for a.e. $(x, y)$ we have

\[
\partial_x \phi_2(x, y) = -2\tau \sum_{i=1}^{N} f_i'(x) \chi_{\{|f_i(x) - y| \leq \delta\}},
\]

\[
\partial_y \phi_2(x, y) = 1 - 4\delta N\tau + 2\tau \sum_{i=1}^{N} \chi_{\{|f_i(x) - y| \leq \delta\}}.
\]


We first investigate $\partial_x \phi_2$. Recalling that $|f'_{i}| \leq 1$ and (43) we immediately obtain
\[\partial_x \phi_2 = 0 \quad \text{a.e. outside } \cup_{i=1}^{N} \{ |f_i(x) - y| \leq \delta \},\]
\[|\partial_x \phi_2| \leq 6\tau \quad \text{a.e. in } \cup_{i=1}^{N} \{ |f_i(x) - y| \leq \delta \}.
\]
Since by (46) we infer
\[| \cup_{i=1}^{N} \{ |f_i(x) - y| \} \| \leq 2\delta N \leq 4\sqrt{|K|},\]
we directly obtain from the last three formulas that
\[\int_{[0,1]^2} |\partial_x \phi_2|^p = \int_{\cup_{i=1}^{N} \{ |f_i(x) - y| \leq \delta \}} |\partial_x \phi_2|^p \leq 4\sqrt{|K|}(6\tau)^p\]
holds, which yields for every $1 \leq p \leq \infty$
\[(51) \quad \|\partial_x \phi_2\|_{L^p([0,1]^2)} \leq 24\tau |K|^{1/(2p)}.
\]
We now deal with $\partial_y \phi_2$. A closer look at the definition of $\phi_2$ tells us that for every $x \in [0,1]$ the function $y \to \phi_2(x,y)$ is differentiable outside the finite set \{ $y \in [0,1] : |f_i(x) - y| = \delta$ for some $i$ \} (with the same formula for $\partial_y \phi_2$ as above). Using (46) and (48) we get for every fixed $x \in [0,1]$
\[(52) \quad \partial_y \phi_2(x,y) = 1 - 4\tau \delta N \geq 1 - 8\tau \sqrt{|K|}
\]
for every $y \in [0,1] \setminus \cup_{i=1}^{N} \{ y \in [0,1] : |f_i(x) - y| \leq \delta \}$,
\[(53) \quad 1 - 8\tau \sqrt{|K|} \leq 1 - 4\tau \delta N \leq \partial_y \phi_2(x,y) \leq 1 + 6\tau
\]
for a.e. $y \in \cup_{i=1}^{N} \{ y \in [0,1] : |f_i(x) - y| \leq \delta \}$. Hence, taking $\tau \sqrt{|K|}$ small enough, we obtain from (52) and (53) for any $x \in [0,1]$
\[(54) \quad 1/2 \leq \partial_y \phi_2(x,y) \leq 1 + 6\tau \quad \text{for a.e. } y.
\]
Combining the previous equation with (51) we get that for every $x \in [0,1]$
\[(55) \quad y \to \phi_2(x,y) \quad \text{is a bi-Lipschitz function from } [0,1] \to [0,1].
\]
Next, using (52) and (53) and noticing as before that
\[| \{ y \in [0,1] : |f_i(x) - y| \leq \delta \text{ for some } i \} | \leq 2\delta N \leq 4\sqrt{|K|},
\]
we get for every fixed $x \in [0,1]$
\[
\int_{[0,1]} |\partial_y \phi_2(x,y) - 1|^p dy = \int_{\{ y \in [0,1] : |f_i(x) - y| > \delta \ \forall i \}} |\partial_y \phi_2(x,y) - 1|^p dy
\]
\[
+ \int_{\{ y \in [0,1] : |f_i(x) - y| \leq \delta \text{ for some } i \}} |\partial_y \phi_2(x,y) - 1|^p dy \leq (6\tau \sqrt{|K|})^p + 4\sqrt{|K|}(6\tau)^p \leq ((6\tau)^p + (24\tau)^p)\sqrt{|K|} \leq (30\tau)^p \sqrt{|K|}.
\]
Thus, for every $x \in [0,1]$ and every $1 \leq p \leq \infty$ we have
\[(56) \quad \|\partial_y \phi_2(x,\cdot) - 1\|_{L^p([0,1])} \leq 30\tau |K|^{1/(2p)}.\]
Let us denote the inverse of the bi-Lipschitz map \( y \to \phi_2(x, y) \) by \( l_x : [0, 1] \to [0, 1] \).
Then, using (54) and the usual change of variables, we deduce
\[
\int_{[0,1]} |l'_x(y) - 1|^p dy = \int_{[0,1]} \left| \frac{1 - \partial_y \phi_2(x, l_x(y))}{\partial_y \phi_2(x, l_x(y))} \right|^p dy
\]
\[
= \int_{[0,1]} |\partial_y \phi_2(x, y) - 1|^p |\partial_y \phi_2(x, y)|^{1-p} dy \leq 2^{p-1} \int_{[0,1]} |\partial_y \phi_2(x, y) - 1|^p dy.
\]
Therefore using (56) we get for every \( 1 \leq p \leq \infty \)
\[
\|l'_x - 1\|_{L^p([0,1])} \leq 60\tau|K|^{1/(2p)}.
\]
Finally integrating (56) with respect to \( x \) gives for every \( 1 \leq p \leq \infty \)
\[
\|\partial_y \phi_2 - 1\|_{L^p([0,1]^2)} \leq 30\tau|K|^{1/(2p)}.
\]

**Step 4.** In the rest of proof \( C \) will denote a generic universal constant that may change from appearance to appearance. Switching the role of \( x \) and \( y \) we get, proceeding exactly as in the previous step, the analogous properties for \( \phi_1 \). First using the counterparts of (51) and (55) for \( \phi_1 \) we have that for every \( y \in [0, 1] \)
\[
x \to \phi_1(x, y) \quad \text{is a bi-Lipschitz function from } [0, 1] \text{ to } [0, 1]
\]
and
\[
\phi_1(0, y) = 0 \quad \text{and} \quad \phi_1(1, y) = 1 \quad \text{for all } y \in [0, 1].
\]
This directly implies that \( \phi : \partial(0,1)^2 \to \partial(0,1)^2 \) is bi-Lipschitz. Moreover by (49), (50), (57), and their counterpart for \( \phi_1 \), we get that
\[
\|\phi - \text{id}\|_{W^{1,p}(\partial(0,1)^2)}, \|\phi^{-1} - \text{id}\|_{W^{1,p}(\partial(0,1)^2)} \leq C\tau|K|^{1/(2p)} \quad \text{for every } 1 \leq p \leq \infty.
\]
Finally using (49), (51), (58) and their counterpart for \( \phi_1 \) we get
\[
\|\phi - \text{id}\|_{L^\infty([0,1]^2)} \leq C\tau \sqrt{|K|}
\]
and
\[
\|\phi - \text{id}\|_{W^{1,p}([0,1]^2)} \leq C\tau|K|^{1/(2p)} \quad \text{for every } 1 \leq p \leq \infty.
\]

**Step 5.** It remains to check the two assertions concerning the determinant and to show that \( \phi \) is bi-Lipschitz from \([0, 1] \) to \([0, 1] \). Recall that for a.e. \( (x, y) \in [0, 1]^2 \)
\[
\nabla \phi(x, y) = \left( 1 - 4\tau \delta M + 2\tau \sum_{j=1}^M g_j(y)|x| \leq \delta \right) \right) \right) \right) \right) \right) \right) \right)
\]
\[
\text{Denote by } S \text{ the union of all the strips, i.e.}
\]
\[
S := \left( \bigcup_{i=1}^N \{ |f_i(x) - y| \leq \delta \} \right) \cup \left( \bigcup_{j=1}^M \{ |g_j(y) - x| \leq \delta \} \right).
\]
Then, obviously, a.e. outside \( S \) we have
\[
\nabla \phi = \left( 1 - 4\delta M \tau \right) \begin{pmatrix} 0 & 0 \\ 0 & 1 - 4\delta N \tau \end{pmatrix}.
\]
Thus, recalling that \( M\delta, N\delta \leq 2\sqrt{|K|} \) (cf. (46)),
\[
\det \nabla \phi = 1 - 4\tau(\delta M + \delta N) + 16\delta^2 M N \tau^2 \geq 1 - 16\tau \sqrt{|K|} \quad \text{a.e. in } [0, 1]^2 \setminus S.
\]
Trivially for all \((x, y) \in S\) we have

\[
1 \leq \sum_{i=1}^{N} \chi(|f_i(x) - y| \leq \delta) + \sum_{j=1}^{M} \chi(|g_j(y) - x| \leq \delta).
\]

Thus we get for a.e. \((x, y) \in S\), recalling that \(|f'_i|, |g'_j| \leq 1\) and using (48),

\[
\begin{align*}
\det \nabla \phi(x, y) &= \left[1 - 4\tau \delta M + 2\tau \sum_{j=1}^{M} \chi(|g_j(y) - x| \leq \delta)\right] \left[1 - 4\tau \delta N + 2\tau \sum_{i=1}^{N} \chi(|f_i(x) - y| \leq \delta)\right] \\
&\quad - 4\tau^2 \left[\sum_{j=1}^{M} g'_j(y) \chi(|g_j(y) - x| \leq \delta)\right] \left[\sum_{i=1}^{N} f'_i(x) \chi(|f_i(x) - y| \leq \delta)\right] \\
&\quad \geq 1 - 4\tau \delta (M + N) - 8\tau \delta N \sum_{j=1}^{M} \chi(|g_j(y) - x| \leq \delta) - 8\tau \delta M \sum_{i=1}^{N} \chi(|f_i(x) - y| \leq \delta) \\
&\quad + 2\tau \left(\sum_{j=1}^{M} \chi(|g_j(y) - x| \leq \delta) + \sum_{i=1}^{N} \chi(|f_i(x) - y| \leq \delta)\right) \\
&\quad + 4\tau^2 \left(\sum_{j=1}^{M} \chi(|g_j(y) - x| \leq \delta)\right) \left[\sum_{i=1}^{N} f'_i(x) \chi(|f_i(x) - y| \leq \delta)\right] \\
&\quad \geq 1 - 4\tau \delta (M + N) - 8\tau \delta N \sum_{j=1}^{M} \chi(|g_j(y) - x| \leq \delta) - 8\tau \delta M \sum_{i=1}^{N} \chi(|f_i(x) - y| \leq \delta) \\
&\quad + 2\tau \left(\sum_{j=1}^{M} \chi(|g_j(y) - x| \leq \delta) + \sum_{i=1}^{N} \chi(|f_i(x) - y| \leq \delta)\right) \\
&\quad \geq 1 + 2\tau [1 - 2\delta(N + M) - 12\tau \delta(N + M)].
\end{align*}
\]

Recalling (49), we can choose \(|K|\) and \(\sqrt{|K|} \tau\) small enough so that

\[
1 + 2\tau (1 - 2\delta(N + M) - 12\tau \delta(N + M)) \geq 1 + \tau.
\]

Combining the last two estimates gives

\[
(60) \quad \det \nabla \phi \geq 1 + \tau \quad \text{a.e. in } S.
\]

Since \(K \subset S\) (cf. (17)), the last inequality obviously holds a.e. in \(K\). Combining (60) and the last estimate gives

\[
\det \nabla \phi \geq 1 - 16\sqrt{|K|} \tau \quad \text{a.e. on } [0, 1]^2.
\]

Taking \(\sqrt{|K|} \tau\) small enough, the last equation implies

\[
\det \nabla \phi \geq 1/2 > 0 \quad \text{a.e. in } \Omega.
\]
Hence, recalling from Step 4 that \( \phi|_{\partial(0,1)^2} : \partial(0,1)^2 \to \partial(0,1)^2 \) is bi-Lipschitz, classical results using the degree theory show that \( \phi : [0,1]^2 \to [0,1]^2 \) is also bi-Lipschitz (cf. e.g. Theorem 2 in [2] noting that \( \phi|_{\partial(0,1)^2} \) can be easily extended to a homeomorphism from \([0,1]^2 \) to \([0,1]^2 \)).

4.3. Linearization. We prove the linearized version of Theorem 8.

**Proof of Corollary 10.** We apply Theorem 8 for any \( \tau \leq 1 \) and for a set \( M \) with small enough measure. In particular we have that

\[
\left\| \phi - \text{id} \right\|_{W^{1,\infty}(\Omega)} \leq C
\]

for some constant \( C \) depending only on \( \Omega \). Hence there exists \( v \in W^{1,\infty}(\Omega) \) such that, up to a subsequence,

\[
\frac{\phi - \text{id}}{\tau} \rightharpoonup v \quad \text{in} \quad W^{1,\infty} \text{ as } \tau \to 0^+.
\]

Using the weak lower semicontinuity of the norms \( \| \cdot \|_{W^{1,p}} \) and the expansion

\[
\det \nabla \phi = \text{div} \phi + \det(\nabla \phi - \text{Id}) \quad \text{(the latter term being bounded by } C\tau^2 \text{)}
\]

we easily conclude (similarly to Steps 2 and 3 of the proof of Lemma 12) that \( v \) has all the desired properties. \( \square \)

We prove our main result on the divergence operator.

**Proof of Theorem 3.** By linearity, it is trivially enough to treat the case \( \int_{\Omega} f < 0 \).

**Step 1 (sharp regularity).** We first show that there exists \( f \in L^\infty(\Omega) \) with \( \int_{\Omega} f < 0 \) such that no solution of (6) can be \( C^1 \). Let \( M \subset \Omega \) be an open dense (in \( \Omega \)) set with \( |M| < |\Omega|/2 \) and let \( f = -\chi_{\Omega \setminus M} + \chi_M \). We argue by contradiction and assume that there exists a solution \( u \) of (6) in \( C^1 \). Then we would have

\[
\text{div } u \geq 1 \quad \text{a.e. in } M.
\]

By continuity of \( \text{div } u \) and the fact that \( M \) is open and dense this previous inequality would imply

\[
\text{div } u \geq 1 \quad \text{everywhere in } \Omega,
\]

which contradicts

\[
\int_{\Omega} \text{div } u = 0.
\]

In the remainder we deal with the existence part.

**Step 2 (preliminaries).** Let \( f_+ := \max\{f, 0\}, f_- := \min\{f, 0\} \) and \( \beta := -\int_{\Omega} f/|\Omega| > 0 \). Take \( \delta_0 > 0 \) big enough so that

\[
\delta_0 - \|f_-\|_{L^\infty} + \beta/2 \geq \|f_+\|_{L^\infty}.
\]

Next choose \( 0 < \epsilon_0 < c \) small enough so that

\[
\delta_0 C \sqrt{\epsilon_0} \leq \beta/2
\]

where \( C, c > 0 \) are the constants (depending only on \( \Omega \)) in the statement of Corollary 11.

**Step 2 (approximation).** Proceeding by convolution (applied to \( f + \beta \)) it is elementary to construct a sequence \( f_\nu \in C^\infty(\overline{\Omega}) \) such that \( \int_{\Omega} f_\nu = 0, \)

\[
-\|f_-\|_{L^\infty} + \beta/2 \leq f_\nu \leq \|f_+\|_{L^\infty} + 2\beta \quad \text{in } \Omega
\]
and

\[ f_\nu(x) \to f(x) + \beta \quad \text{for a.e. } x \in \Omega. \]

The last formula implying convergence in measure, there exist \( \nu_0 \) and a measurable set \( M \subset \Omega \) such that \( |M| \leq \epsilon_0 \) and

\[ f_{\nu_0} \geq \frac{\beta}{2} + f \quad \text{a.e. in } \Omega \setminus M. \]

\( \textbf{Step 3 (conclusion).} \) By a classical result for the divergence equation (cf. e.g. Theorem 9.2 in [8]) there exists \( v \in C^\infty(\Omega; \mathbb{R}^n) \) so that (recall that \( \int_\Omega f_{\nu_0} = 0 \))

\[ \text{div } v = f_{\nu_0} \quad \text{in } \Omega \quad \text{and } v = 0 \quad \text{on } \partial \Omega. \]

Next we apply Corollary 10 to \( M \) and get \( w \in W^{1,\infty}_0(\Omega; \mathbb{R}^2) \) such that

\[ \text{div } w \geq 1 \quad \text{a.e. in } M, \]

\[ \text{div } w \geq -C|M|^{1/2} \geq -C\sqrt{\epsilon_0} \quad \text{a.e. in } \Omega. \]

We claim that \( u = v + \delta_0 w \) has all the wished properties. First we obviously have \( u = 0 \) on \( \partial \Omega \). It remains to show that

\[ \text{div } u = f_{\nu_0} + \delta_0 \text{div } w \geq f \quad \text{a.e in } \Omega. \]

First, using (61), (63) and (65), we obtain, that a.e. in \( M \)

\[ f_{\nu_0} + \delta_0 \text{div } w \geq -\|f_-\|_{L^\infty} + \beta/2 + \delta_0 \geq \|f_+\|_{L^\infty} \geq f. \]

Finally, using (21), (64) and (62), we get that a.e. in \( \Omega \setminus M \)

\[ f_{\nu_0} + \delta_0 \text{div } w \geq f + \beta/2 - \delta_0 C\sqrt{\epsilon_0} \geq f, \]

which ends the proof. \( \square \)

5. Applications to nonlinear elasticity

Theorem 8 has applications e.g. in the theory of nonlinear elasticity, as already mentioned in Section 2. We consider functionals of the form

\[ \mathcal{F}[v] := \int_\Omega |\nabla v|^2 + h(\det \nabla v) \, dx \]

where \( \Omega \) is a bounded open set in \( \mathbb{R}^2 \) with boundary of class \( C^{1,1} \), \( h : \mathbb{R} \to \mathbb{R}_0^+ \cup \{\infty\} \) denotes a convex monotonously decreasing function with \( h \equiv \infty \) on \( (-\infty, \mu] \) and \( h(s) \to \infty \) as \( s \to \mu^+ \) for some fixed \( \mu > 0 \). We moreover require \( h \) to be continuously differentiable on \( (\mu, \infty) \).

Our bi-Lipschitz mappings from Theorem 8 then enable us to prove Theorem 5 which we restate for the convenience of the reader.

**Theorem.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set with boundary of class \( C^{1,1} \) and \( u_0 \in W^{1,2}(\Omega; \mathbb{R}^2) \) be such that \( u_0 \) is an homeomorphism from \( \overline{\Omega} \) to \( \overline{\Omega} \) and \( \mathcal{F}[u_0] < \infty \). Then any minimizer \( u \in u_0 + W^{1,2}_0(\Omega; \mathbb{R}^2) \) of

\[ \inf_v \left\{ \mathcal{F}[v] : v \in u_0 + W^{1,2}_0(\Omega; \mathbb{R}^2) \right\} \]

has the additional regularity

\[ h'(\det \nabla u) \cdot \det \nabla u \in L^1(\Omega). \]
and satisfies the equilibrium equation

\[ \int_\Omega 2(\nabla \xi(u) \cdot \nabla u) : \nabla u + h'(\det \nabla u) \cdot \det \nabla u \cdot \text{div} \xi(u) = 0 \]

for any vector field \( \xi \in C^\infty_c(\Omega; \mathbb{R}^2) \).

Since the proof of Theorem 5 is rather long, we split into several parts.

We start with a lemma stating the existence of a minimizer and some of its well-known properties.

**Lemma 19.** Let \( u_0 \in W^{1,2}(\Omega; \mathbb{R}^2) \) be a homeomorphism from \( \Omega \) to \( \Omega \) with \( F[u_0] < \infty \). There exists at least one minimizer \( u \in u_0 + W^{1,2}(\Omega; \mathbb{R}^2) \) of the functional \( F \).

Moreover \( u \) (in fact any \( v \in u_0 + W^{1,2}(\Omega; \mathbb{R}^2) \) with \( F[v] < \infty \)) has the following properties:

1) \( u : \Omega \to \Omega \) is continuous, sends zero Lebesgue measure sets to zero Lebesgue measure sets, is a.e. one-to-one and satisfies

\[ \det \nabla u > \mu > 0 \quad \text{a.e. in } \Omega. \]

2) For any \( f \in L^1(\Omega) \), the usual change of variables formula holds true, namely

\[ \int_{\Omega} f = \int_{\Omega} f \circ u \cdot \det \nabla u. \]

3) Define

\[ N_\delta := \{ x \in \Omega : \det \nabla u(x) < \mu + \delta \} \quad \text{and} \quad M_\delta := u(N_\delta). \]

Then \( M_\delta \) is measurable and

\[ \lim_{\delta \to 0^+} |N_\delta| = 0 \quad \text{and} \quad \lim_{\delta \to 0^+} |M_\delta| = 0. \]

4) For any \( \mu < s \leq t \) it holds that

\[ |h'(t)| \leq |h'(s)| \quad \text{and} \quad |h(t) - h(s)| \leq |h'(s)|(t - s). \]

Furthermore for every \( \delta > 0 \) we get that

\[ h'(f(\tau) \cdot \det \nabla u) \to h'(\det \nabla u) \quad \text{in } L^\infty(\Omega \setminus N_\delta) \quad \text{as } \tau \to 0^+ \]

for every \( f \in C^0([0, \infty)) \) with \( f(0) = 1 \).

The proof of this lemma is provided in Appendix D.

We now exhibit the stretching mappings and their properties needed in order to prove Theorem 5.

**Lemma 20.** There exists a constant \( C > 0 \) depending only on \( \Omega \) and \( \mu \) such that the following assertion holds: for every \( \tau, \delta > 0 \) small enough there exists a bi-Lipschitz mapping \( \phi_{\tau, \delta} : \Omega \to \Omega \) with the properties

\[ \phi_{\tau, \delta} = \text{id} \quad \text{on} \quad \partial \Omega, \]

\[ \|\phi_{\tau, \delta} - \text{id}\|_{W^{1,p}(\Omega)} \leq C\tau |M_\delta|^{1/(2p)} \quad \text{for every } 1 \leq p \leq \infty, \]

\[ \|\phi_{\tau, \delta} - \text{id}\|_{L^\infty(\Omega)} \leq C\tau |M_\delta|^{1/2}, \]

\[ \det \nabla \phi_{\tau, \delta}(u) \geq 1 + \tau \quad \text{a.e. in } N_\delta, \]

\[ \det \nabla \phi_{\tau, \delta} \geq 1 - C\tau |M_\delta|^{1/2} \quad \text{a.e. in } \Omega. \]
Furthermore
\[ (80) \quad \| \nabla \phi_{\tau, \delta}(u) - \text{Id} \|_{L^p(\Omega)} \leq C \tau |M_\delta|^{1/(2p)} \text{ for every } 1 \leq p \leq \infty. \]

Finally there exists \( g_\delta \in L^\infty(\Omega) \) such that we have for a subsequence
\[ (81) \quad \frac{1}{\tau} \left| \det \nabla \phi_{\tau, \delta}(u) - 1 \right| \rightharpoonup g_\delta \text{ in } L^\infty(\Omega) \text{ as } \tau \to 0^+ \]

and
\[ (82) \quad \| g_\delta \|_{L^\infty(\Omega)} \leq C \text{ and } \| g_\delta \|_{L^1(\Omega)} \leq C |M_\delta|^{1/2}. \]

**Proof.** Step 1. Using (72), we can apply Theorem 5 to \( M = M_\delta \) for every \( \delta \) and \( \tau \) small enough and find a bi-Lipschitz mapping \( \phi_{\tau, \delta} \) satisfying (9)-(13). Recalling that \( u(N_\delta) = M_\delta \), (75)-(79) follow directly from (9)-(13). Combining (69), (70) and (76) we get
\[ \int_\Omega |\nabla \phi_{\tau, \delta}(u) - \text{Id}|^p \leq \frac{1}{\mu} \int_\Omega \left| \det \nabla \phi_{\tau, \delta} - 1 \right|^p \leq \frac{1}{\mu} C \tau^p |M_\delta|^{1/2}, \]
which proves (80).

Step 2. We prove the last assertion. In what follows \( C \) will denote a generic constant depending only on \( \Omega \) and \( \mu \) that may change from appearance to appearance. Using (80) with \( p = \infty \) we get that
\[ \left\| \frac{1}{\tau} \left| \det \nabla \phi_{\tau, \delta}(u) - 1 \right| \right\|_{L^\infty(\Omega)} \leq C. \]

Hence, after passing to a subsequence we obtain for every \( \delta \) small enough some \( g_\delta \in L^\infty(\Omega) \) satisfying \( \| g_\delta \|_{L^\infty(\Omega)} \leq C \) and (81). Moreover, using (69) and (70) we get
\[ \int_\Omega \left| \frac{\det \nabla \phi_{\tau, \delta}(u) - 1}{\tau} \right| \leq \int_\Omega \frac{\det \nabla u}{\mu} \left| \frac{\det \nabla \phi_{\tau, \delta}(u) - 1}{\tau} \right| \leq \frac{1}{\mu} \int_\Omega \left| \frac{\det \nabla \phi_{\tau, \delta} - \text{Id}}{\tau} \right| \leq C \tau + C \int_\Omega \left| \frac{\nabla \phi_{\tau, \delta} - \text{Id}}{\tau} \right| \leq C \tau + C |M_\delta|^{1/2}, \]
where we have used (76) with \( p = 1 \) for the last inequality. Combining the previous equation with (81) we get at once the second estimate in (82) which ends the proof.  

5.1. Higher integrability. We are now in position to prove Theorem 5. We start with (67).

**Proof of (67).** We use the previous mappings \( \phi_{\tau, \delta} \) varying only \( \tau \) and we fix once and for all \( \delta \) to be small enough.

Step 1. Using (72) and (71) we obviously have
\[ \int_{\Omega \backslash N_\delta} |h'(\det \nabla u)| \, dx \leq |h'(|\mu + \delta|)\int_{\Omega \backslash N_\delta} 1 \, dx < \infty; \]
therefore it is enough to show that
\[ (83) \quad - \int_{N_\delta} h'(\det \nabla u) \cdot \det \nabla u \, dx < \infty \]
to prove (67). Since $u$ is a minimizer of $\mathcal{F}$ we have using (75)
\[
\frac{\mathcal{F}(\phi_{\tau, \delta} \circ u) - \mathcal{F}(u)}{\tau} \geq 0
\]
or equivalently
\[
\int_{\Omega} \frac{\nabla \phi_{\tau, \delta}(u) \cdot \nabla u}{\tau^2} - \left| \nabla \phi_{\tau, \delta}(u) \right|^2 d\tau + \int_{\Omega} h[\det \nabla \phi_{\tau, \delta}(u) \cdot \det \nabla u] - h[\det \nabla u] d\tau \geq 0.
\]
First using (76) with $p = \infty$, we directly get that
\[
\int_{\Omega} \frac{\nabla \phi_{\tau, \delta}(u) \cdot \nabla u}{\tau^2} - \left| \nabla \phi_{\tau, \delta}(u) \right|^2 d\tau \leq C
\]
holds for some constant $C$ independent of $\tau$.

Step 2. We now handle the second integral in (84). Using (79), (78), and the fact that $h$ is decreasing, we obtain
\[
\int_{\Omega} h[\det \nabla \phi_{\tau, \delta}(u) \cdot \det \nabla u] - h[\det \nabla u] d\tau \leq \int_{\Omega \setminus N_\delta} h[(1 - C|M_\delta|^{1/2}) \cdot |\det \nabla u|] d\tau + \int_{N_\delta} h[(1 + \tau) \cdot |\det \nabla u|] d\tau.
\]
Combining (84), (85), and (86), we deduce
\[
\int_{\Omega \setminus N_\delta} h[(1 - C|M_\delta|^{1/2}) \cdot |\det \nabla u|] d\tau - h[(1 + \tau) \cdot |\det \nabla u|] d\tau \leq C + \int_{N_\delta} h[(1 + \tau) \cdot |\det \nabla u|] d\tau.
\]
Then, for every $\tau$ small enough with respect to $\delta$, we have that
\[
(1 - C|M_\delta|^{1/2}) \cdot |\det \nabla u| \geq \mu + \delta/2 \quad \text{a.e. in } \Omega \setminus N_\delta.
\]
Hence, by (73) we have
\[
|h[(1 - C|M_\delta|^{1/2}) \cdot |\det \nabla u|] - h[\det \nabla u]| \leq C\tau|M_\delta|^{1/2} \cdot |\det \nabla u| \cdot |h'(\mu + \delta/2)| \quad \text{a.e. in } \Omega \setminus N_\delta.
\]
Thus we can apply the dominated convergence theorem to get that
\[
\lim_{\tau \to 0^+} \int_{\Omega \setminus N_\delta} h[(1 - C|M_\delta|^{1/2}) \cdot |\det \nabla u|] d\tau - h[(1 + \tau) \cdot |\det \nabla u|] d\tau = C|M_\delta|^{1/2} \int_{\Omega \setminus N_\delta} -h'(\det \nabla u) \cdot \det \nabla u \; dx < \infty.
\]
Now Fatou’s lemma (which is applicable since $h$ is decreasing) implies that
\[
- \int_{N_\delta} h'(\det \nabla u) \cdot \det \nabla u \; dx \leq \liminf_{\tau \to 0} \int_{N_\delta} h[(1 - C|M_\delta|^{1/2}) \cdot |\det \nabla u|] d\tau - h[(1 + \tau) \cdot |\det \nabla u|] d\tau.
\]
Hence, combining \(87\), \(88\) with the previous estimate we get that
\[
-\int_{N_k} h'(\det \nabla u) \cdot \det \nabla u < \infty,
\]
which is precisely \(83\). \hfill \Box

5.2. **The equilibrium equations.** We finally prove \(68\). To accomplish this, we need to vary \(\delta\) as well in the mapping \(\phi_{\tau,\delta}\) and not only \(\tau\) as in the last proof.

*Proof of \(68\).* It is obviously enough to show the inequality
\[
(89) \quad \int_\Omega 2(\nabla \xi(u) \cdot \nabla u + h'(\det \nabla u) \cdot \det \nabla u \cdot \div \xi(u)) \, dx \geq 0,
\]
for every \(\xi \in C^\infty_0(\Omega; \mathbb{R}^2)\) for which \(\|\xi\|_{W^{1,\infty}}\) is small enough. We denote by \(C\) a generic constant depending only on \(\Omega\) and \(\mu\) which may change from appearance to appearance.

**Step 1 (preliminaries).** Let \(\xi \in C^\infty_0(\Omega; \mathbb{R}^2)\) and define \(\psi_\tau \in C^\infty(\Omega; \Omega)\) to be the flow of \(\xi\), i.e.
\[
\frac{d}{d\tau} \psi_\tau = \xi(\psi_\tau) \quad \text{and} \quad \psi_0 = \text{id}.
\]
We now know that \(\psi_\tau = \text{id}\) holds near \(\partial \Omega\), that we have
\[
(90) \quad \frac{\nabla \psi_\tau - \text{Id}}{\tau} \to \nabla \xi \quad \text{and} \quad \frac{\det \nabla \psi_\tau - 1}{\tau} \to \div \xi \quad \text{in } L^\infty(\Omega) \text{ as } \tau \to 0,
\]
and, choosing \(\|\xi\|_{W^{1,\infty}}\) small enough, we may assume that for \(\tau \in (0,1)\)
\[
(91) \quad \|\nabla \psi_\tau - \text{Id}\|_{L^\infty(\Omega)}, \|\det \nabla \psi_\tau - 1\|_{L^\infty(\Omega)} \leq \tau/2.
\]
Moreover since the \(C^2\) norm of \(\psi_\tau\) is uniformly bounded we trivially have
\[
(92) \quad |\nabla \psi_\tau(x) - \nabla \psi_\tau(y)||, \det \nabla \psi_\tau(x) - \det \nabla \psi_\tau(y)| \leq C|x - y| \quad \text{for all } x, y \in \Omega.
\]

**Step 2.** As \(u\) is a minimizer of \(\mathcal{F}\) and as we have \(\psi_\tau = \phi_{\tau,\delta} = \text{id}\) on \(\partial \Omega\), we know that
\[
\frac{\mathcal{F}(\psi_\tau \circ \phi_{\tau,\delta} \circ u) - \mathcal{F}(u)}{\tau} \geq 0
\]
or equivalently
\[
(93) \quad \int_\Omega \frac{|\nabla \psi_\tau(\phi_{\tau,\delta}(u)) \cdot \nabla \phi_{\tau,\delta}(u) \cdot \nabla u|_2^2 - |\nabla u|_2^2}{\tau} \, dx + \int_\Omega \frac{h[\det \nabla \psi_\tau(\phi_{\tau,\delta}(u)) \cdot \det \nabla \phi_{\tau,\delta}(u) \cdot \det \nabla u] - h[\det \nabla u]}{\tau} \, dx \geq 0.
\]
We are going to handle to two integrals in the last expression separately. We will first show (cf. Part 1) that
\[
(94) \quad \lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_\Omega \frac{|\nabla \psi_\tau(\phi_{\tau,\delta}(u)) \cdot \nabla \phi_{\tau,\delta}(u) \cdot \nabla u|_2^2 - |\nabla u|_2^2}{\tau} \, dx \right\} = 2 \int_\Omega (\nabla \xi(u) \cdot \nabla u) : \nabla u \, dx
\]
and then (cf. Part 2) that
\[
\lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{\Omega} h[\det \nabla \psi_{\tau,\delta}(u)] \cdot \det \nabla \phi_{\tau,\delta}(u) \cdot \det \nabla u - h[\det \nabla u] \, dx \right\} \\
= \int_{\Omega} h'(\det \nabla u) \cdot \det \nabla u \cdot \text{div} \, \xi(u) \, dx.
\]
(95)

The combination of (93), (94) and (95) will then directly give (89) which will finish the proof.

**Part 1.** We show (94). To this end, it is enough to prove that
\[
\lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{\Omega} |A_{\tau,\delta}| \, dx \right\} = 0,
\]
(96)
\[
\lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{\Omega} B_{\tau,\delta} \, dx \right\} = \int_{\Omega} 2(\nabla \xi(u) \cdot \nabla u) : \nabla u \, dx,
\]
(97)
\[
\lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{\Omega} |C_{\tau,\delta}| \, dx \right\} = 0,
\]
(98)
where
\[
A_{\tau,\delta} := \frac{\nabla \psi_{\tau}(\phi_{\tau,\delta}(u)) \cdot \nabla \phi_{\tau,\delta}(u) \cdot \nabla u^2 - |\nabla \psi_{\tau}(u) \cdot \nabla \phi_{\tau,\delta}(u) \cdot \nabla u|^2}{\tau},
\]
\[
B_{\tau,\delta} := \frac{\nabla \psi_{\tau}(u) \cdot \nabla \phi_{\tau,\delta}(u) \cdot \nabla u^2 - |\nabla \phi_{\tau,\delta}(u) \cdot \nabla u|^2}{\tau},
\]
\[
C_{\tau,\delta} := \frac{\nabla \phi_{\tau,\delta}(u) \cdot \nabla u^2 - |\nabla u|^2}{\tau}.
\]

**Step 3.** We first show (96). Using (76) with \(p = \infty\) and (92) we easily obtain that,
\[
|A_{\tau,\delta}| \leq C|\nabla u|^2 \left\| \frac{\phi_{\tau,\delta} - \text{id}}{\tau} \right\|_{L^\infty} \text{ a.e. in } \Omega.
\]
The previous equation and (77) readily imply
\[
\int_{\Omega} |A_{\tau,\delta}| \, dx \leq C|M_\delta|^{1/2}
\]
and hence (96) follows from (72).

**Step 4.** We now show (97). Using (76) with \(p = \infty\) and (91) we get that
\[
|B_{\tau,\delta}| \leq C|\nabla u|^2 \left| \frac{\nabla \psi_{\tau}(u) - \text{Id}}{\tau} \right| \leq C|\nabla u|^2 \text{ a.e. in } \Omega.
\]
Hence by the dominated convergence theorem we get using (76) and (91) that
\[
\lim_{\tau \to 0^+} \int_{\Omega} B_{\tau,\delta} \, dx = \lim_{\tau \to 0^+} \int_{\Omega} B_{\tau,\delta} \, dx = 2 \int_{\Omega} (\nabla \xi(u) \cdot \nabla u) : \nabla u \, dx,
\]
which proves (97).

**Step 5.** We finally show (98). Using (76) with \(p = \infty\) it follows that we have
\[
|C_{\tau,\delta}| \leq C\left| \frac{\nabla \phi_{\tau,\delta}(u) - \text{Id}}{\tau} \right| |\nabla u|^2 \text{ a.e. in } \Omega.
\]
Recall (cf. (80))
\[
\left| \frac{\nabla \phi_{\tau, \delta}(u) - \text{Id}}{\tau} \right|_{L^\infty} \leq C \quad \text{and} \quad \int_{\Omega} |\nabla \phi_{\tau, \delta}(u) - \text{Id}| \ dx \leq C|\mathcal{M}_\delta|^{1/2},
\]
hence, combining the last two equations, we obtain (98) from Lemma 21 and (72).

**Part 2.** We show (93). To this end, it is obviously enough to prove that (99)
\[
\lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{N_\delta} |D_{\tau, \delta}| \ dx \right\} = \lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{N_\delta} |D_{\tau, \delta}| \ dx \right\} = \lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{N_\delta} |E_{\tau, \delta}| \ dx \right\} = 0,
\]
and
\[
\lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{\Omega \setminus N_\delta} |E_{\tau, \delta}| \ dx \right\} = \int_{\Omega} h'(\det \nabla u) \cdot \det \nabla u \cdot \text{div}(u) \ dx,
\]
and
\[
\lim_{\delta \to 0^+} \left\{ \lim_{\tau \to 0^+} \int_{\Omega \setminus N_\delta} |F_{\tau, \delta}| \ dx \right\} = 0,
\]
where
\[
D_{\tau, \delta} := \frac{h[\det \nabla \psi_{\tau}(\phi_{\tau, \delta}(u)) \cdot \det \nabla \phi_{\tau, \delta}(u) \cdot \det \nabla u] - h[\det \nabla \psi_{\tau}(u) \cdot \det \nabla \phi_{\tau, \delta}(u) \cdot \det \nabla u]}{\tau},
\]
\[
E_{\tau, \delta} := \frac{h[\det \nabla \psi_{\tau}(u) \cdot \det \nabla \phi_{\tau, \delta}(u) \cdot \det \nabla u] - h[\det \nabla \phi_{\tau, \delta}(u) \cdot \det \nabla u]}{\tau},
\]
\[
F_{\tau, \delta} := \frac{h[\det \nabla \phi_{\tau, \delta}(u) \cdot \det \nabla u] - h[\det \nabla u]}{\tau}.
\]

**Step 6.** We first show (99). The claim will follow once we prove that (103)
\[
|D_{\tau, \delta}|, |E_{\tau, \delta}|, |F_{\tau, \delta}| \leq C|h'(\det \nabla u) \cdot \det \nabla u| \quad \text{a.e. in } N_\delta.
\]
Indeed, assuming the last estimate and recalling that $h'(\det \nabla u) \cdot \det \nabla u \in L^1$ (cf. (97)) and $\lim_{\delta \to 0^+} |N_\delta| = 0$ (cf. (72)), we directly get (99) since
\[
\lim_{\delta \to 0^+} \int_{N_\delta} |h'(\det \nabla u) \cdot \det \nabla u| \ dx = 0.
\]
We now show (103). Recall that for every any $\mu < s \leq t$ (cf. (73))
\[
|h'(t)| \leq |h'(s)| \quad \text{and} \quad |h(t) - h(s)| \leq |h'(s)|(t - s).
\]
Using (78) and (71) we get for a.e $x \in N_\delta$ and a.e. $y \in \Omega$
\[
\det \nabla \phi_{\tau, \delta}(u(x)) \geq 1 \quad \text{and} \quad \det \nabla \phi_{\tau, \delta}(u(x)) \cdot \det \nabla \psi_{\tau}(y) \geq 1.
\]
Hence, using the last two estimates, we easily deduce that, a.e. in $N_\delta$,
\[
|D_{\tau, \delta}| \leq |h'(\det \nabla u) \cdot \det \nabla u \cdot \det \nabla \phi_{\tau, \delta}(u)| \cdot |\det \nabla \psi_{\tau}(\phi_{\tau, \delta}(u)) - \det \nabla \psi_{\tau}(u)|, \\
|E_{\tau, \delta}| \leq |h'(\det \nabla u) \cdot \det \nabla u \cdot \det \nabla \phi_{\tau, \delta}(u)| \cdot \frac{|\det \nabla \psi_{\tau}(u) - 1|}{\tau}, \\
|F_{\tau, \delta}| \leq |h'(\det \nabla u) \cdot \det \nabla u| \cdot |\det \nabla \phi_{\tau, \delta}(u) - 1|.
By \((76)\) and \((91)\) we obtain
\[
\left\| \det \nabla \phi_{\tau, \delta} \right\|_{L^\infty} \leq C.
\]
Using \((92)\) and \((76)\), we infer
\[
\left\| \det \nabla \psi_{\tau} \phi_{\tau, \delta}(u) - \det \nabla \psi_{\tau}(u) \right\|_{L^\infty} \leq C \left\| \phi_{\tau, \delta} - \frac{u}{\tau} \right\|_{L^\infty} \leq C.
\]
We now trivially get \((103)\) from the last five estimates.

**Step 8.** We show \((100)\). As before, combining \((76)\) and \((91)\) we have for a.e. \(x, y \in \Omega\)
\[
\det \nabla \psi_{\tau}(x) \cdot \det \nabla \phi_{\tau, \delta}(y) \geq 1 - C\tau \quad \text{for a.e. } x, y \in \Omega.
\]
Recall that (cf. \((71)\))
\[
det \nabla u \geq \mu + \delta \quad \text{a.e. in } \Omega \setminus N_{\delta}.
\]
Combining the last two equations we can then apply \((73)\) and get (for every \(\tau\) small enough with respect to \(\delta\))
\[
|D_{\tau, \delta}| \leq \left| h'((1 - C\tau) \det \nabla u \cdot \det \nabla \phi_{\tau, \delta}(u) \cdot \det \nabla u \right| \cdot \frac{\left| \det \nabla \psi_{\tau}(\phi_{\tau, \delta}(u)) - \det \nabla \psi_{\tau}(u) \right|}{\tau}.
\]
Now, by \((74)\) we have that
\[
h'((1 - C\tau) \det \nabla u) \rightarrow h'(\det \nabla u) \quad \text{in } L^\infty(\Omega \setminus N_{\delta}) \text{ as } \tau \rightarrow 0^+.
\]
Using \((77)\) and \((92)\) we get
\[
\left\| \det \nabla \psi_{\tau}(\phi_{\tau, \delta}(u)) - \det \nabla \psi_{\tau}(u) \right\|_{L^\infty} \leq C \left\| \frac{\phi_{\tau, \delta} - \text{id}}{\tau} \right\|_{L^\infty(\Omega)} \leq C|M_{\delta}|^{1/2}.
\]
Combining the last three estimates and \((76)\) we get that
\[
\limsup_{\tau \to 0^+} \int_{\Omega \setminus N_{\delta}} |D_{\tau, \delta}| \, dx \leq C|M_{\delta}|^{1/2} \int_{\Omega \setminus N_{\delta}} |h'(\det \nabla u) \cdot \det \nabla u| \, dx.
\]
Recalling \((77)\) and \((72)\), this immediately implies \((101)\).

**Step 7.** We now show \((100)\). First, using \((76)\) and \((91)\) we immediately obtain
\[
\det \nabla \psi_{\tau}(x) \cdot \det \nabla \phi_{\tau, \delta}(y) \geq 1 - C\tau \quad \text{for a.e. } x, y \in \Omega.
\]
We show \((101)\). As before, combining \((76)\) and \((91)\) we have for a.e. \(x, y \in \Omega\)
\[
\det \nabla \psi_{\tau}(x) \cdot \det \nabla \phi_{\tau, \delta}(y) \leq 1 + C\tau.
\]
By the previous estimate, we can apply the mean value theorem and get for a.e. \(x \in \Omega \setminus N_{\delta}\)
\[
E_{\tau, \delta}(x) = h'[f_{\tau, \delta}(x) \cdot \det \nabla u(x)] \cdot \det \nabla \phi_{\tau, \delta}(u(x)) \cdot \det \nabla u(x) \cdot \frac{\det \nabla \psi_{\tau}(u(x)) - 1}{\tau}
\]
for some \(f_{\tau, \delta}(x) \in [1 - C\tau, 1 + C\tau]\). Now since \(h'\) is increasing we get that a.e. in \(\Omega \setminus N_{\delta}\)
\[
h'((1 - C\tau) \det \nabla u) \leq h'(f_{\tau, \delta} \cdot \det \nabla u) \leq h'((1 + C\tau) \det \nabla u)
\]
and hence, a direct application of \((74)\) gives
\[
h'(f_{\tau, \delta} \cdot \det \nabla u) \rightarrow h'(\det \nabla u) \quad \text{in } L^\infty(\Omega \setminus N_{\delta}) \text{ as } \tau \rightarrow 0^+.
\]
Using (76) and (90) we have that
\[ \det \nabla \phi_{\tau,\delta}(u) \to 1 \quad \text{and} \quad \frac{\det \nabla \psi_{\tau}(u) - 1}{\tau} \to \text{div} \xi(u) \quad \text{in} \ L^\infty(\Omega) \quad \text{as} \quad \tau \to 0^+. \]

Combining (105) with the last two assertions, we directly obtain
\[ \lim_{\tau \to 0^+} \int_{\Omega \setminus N_\delta} E_{\tau,\delta} \, dx = \int_{\Omega \setminus N_\delta} h'(\det \nabla u) \cdot \det \nabla u \cdot \text{div} \xi(\tau) \, dx. \]

Combining the last equation with (72) gives (101).

Step 9. We finally prove (102). By (76) we have
\[ \det \nabla \phi_{\tau,\delta}(x) \geq 1 - C\tau \quad \text{for a.e.} \quad x \in \Omega, \]
therefore we deduce using (73) that a.e. in \( \Omega \setminus N_\delta \)
\[ |F_{\tau,\delta}| \leq |h'(1 - C\tau) \det \nabla u| \cdot |\det \nabla \phi_{\tau,\delta}(u) - 1| \cdot \frac{\tau}{\tau}. \]

By (71) we know that
\[ h'(1 - C\tau) \det \nabla u \to h'(\det \nabla u) \quad \text{in} \ L^\infty(\Omega \setminus N_\delta) \quad \text{as} \quad \tau \to 0^+. \]

Recall that by (81)
\[ \frac{|\det \nabla \phi_{\tau,\delta}(u) - 1|}{\tau} \xrightarrow{\tau} g_\delta \quad \text{in} \ L^\infty(\Omega) \quad \text{as} \quad \tau \to 0^+ \]
for some \( g_\delta \in L^\infty(\Omega) \). Combining the last three assertions we get that
\[ \limsup_{\tau \to 0^+} \int_{\Omega \setminus N_\delta} |F_{\tau,\delta}| \, dx \leq \int_{\Omega \setminus N_\delta} |h'(\det \nabla u) \cdot \det \nabla u \cdot g_\delta| \, dx. \]

Since (cf. 67) \( h'(\det \nabla u) \cdot \det \nabla u \in L^1(\Omega) \) and since \( g_\delta \) moreover satisfies (cf. 82)
\[ \|g_\delta\|_{L^\infty(\Omega)} \leq C \quad \text{and} \quad \lim_{\delta \to 0^+} \|g_\delta\|_{L^1(\Omega)} = 0, \]
we can apply Lemma 21 to the right-hand side of (106) and obtain (102). This finishes the proof of Theorem 5.

In the previous proof we have used the following very elementary lemma.

**Lemma 21.** Let \( \Omega \subset \mathbb{R}^n \) be a bounded set and \( f \in L^1(\Omega) \). Let also \( g_\delta \in L^\infty(\Omega) \) be a sequence functions satisfying \( \|g_\delta\|_{L^\infty} \leq C \) for some constant \( C \) independent of \( \delta \) and

\[ \lim_{\delta \to 0} \int_{\Omega} |g_\delta| \, dx = 0. \]

Then
\[ \lim_{\delta \to 0} \int_{\Omega} |f \cdot g_\delta| \, dx = 0. \]

**Proof.** We have
\[ \limsup_{\delta \to 0} \int_{\Omega} |f \cdot g_\delta| \, dx \]
\[ \leq \limsup_{\nu \to \infty} \left\{ \limsup_{\delta \to 0} \int_{\{f \leq \nu\}} |f \cdot g_\delta| \, dx \right\} + \limsup_{\nu \to \infty} \left\{ \limsup_{\delta \to 0} \int_{\{f > \nu\}} |f \cdot g_\delta| \, dx \right\} \]
\[ \leq C \limsup_{\nu \to \infty} \int_{\{f > \nu\}} |f| \, dx = 0, \]
the last equality holding since \( f \in L^1(\Omega) \).

\( \square \)

Proof of Corollary \([7]\) In what follows \( C \) will denote a constant depending only on \( h, p, \Omega \) and \( F[u_0] \). The precise value of \( C \) may change from appearance to appearance.

Step 1. As in Lemma \([19]\) let

\[ N_\delta = \{ x \in \Omega : \det \nabla u < \mu + \delta \} \]

We claim that we can select \( \delta > 0 \) small enough depending only on \( \mu, h, p, \Omega \) and \( F[u_0] \) such that

\( |u(N_\delta)| \leq |\Omega|/2 \), \( \text{(107)} \)

To see this, recall that \( h \) is decreasing and that \( F[u] \leq F[u_0] \), which implies

\[ |u(N_\delta)| \leq (\mu + \delta)\, |N_\delta| \leq (\mu + \delta) \int_{N_\delta} \frac{h(\det \nabla u)}{h(\mu + \delta)} \leq \frac{\mu + \delta}{h(\mu + \delta)} F[u_0]. \]

Noting that \( h(\mu + \delta) \) converges to \( \infty \) as \( \delta \to 0 \), the claim is established. From now on, we fix a \( \delta > 0 \) satisfying \( \text{(107)} \).

By \( (73) \) we have that

\[ \int_{\Omega \setminus N_\delta} |h'(\det \nabla u) \cdot \det \nabla u|^p \leq |h'(\mu + \delta)|^{2p} \int_{\Omega \setminus N_\delta} |\det \nabla u|^p \leq C|\nabla u|_{L^2 p(\Omega)}^{2p}. \]

To prove the corollary it is therefore enough to establish that

\[ \int_{N_\delta} |h'(\det \nabla u) \cdot \det \nabla u|^p \leq C|\nabla u|_{L^2_p(\Omega)}^{2p} \]

or, equivalently by duality (and density),

\( (108) \)

\[ \int_{N_\delta} g \cdot h'(\det \nabla u) \cdot \det \nabla u \leq C|\nabla u|_{L^2 p(\Omega)}^{2p} \]

for every \( g \in L^\infty(N_\delta) \) with \( \|g\|_{L^{p/(p-1)}(N_\delta)} \leq 1 \). So we fix such a \( g \) and show \( \text{(108)} \) in the two remaining steps.

Step 2. Define \( f : u(N_\delta) \to \mathbb{R} \) by

\[ g = f \circ u. \]

Recalling (cf. Lemma \([19]\) that \( u : \overline{\Omega} \to \overline{\Omega} \) is continuous, sends zero Lebesgue measure sets to zero Lebesgue measure sets, and is almost everywhere one-to-one, we get that \( f \) is well defined, bounded and measurable. Moreover since

\[ \det \nabla u < \mu + \delta \quad \text{in } N_\delta \]

we deduce using \( \text{(70)} \) that

\( \text{(109)} \)

\[ \|f\|_{L^{p/(p-1)}(u(N_\delta))} \leq C. \]

Define \( \mathcal{F} \in L^\infty(\Omega) \) by

\[ \mathcal{F} := \begin{cases} f & \text{in } u(N_\delta), \\ -\int_{u(N_\delta)} f \, dx/|u(N_\delta)^c| & \text{in } u(N_\delta)^c. \end{cases} \]

Using \( \text{(107)} \) and \( \text{(109)} \) we immediately get

\( \text{(110)} \)

\[ \|\mathcal{F}\|_{L^{p/(p-1)}(\Omega)} \leq C \quad \text{and} \quad \|\mathcal{F}\|_{L^\infty(u(N_\delta)^c)} \leq C. \]
Note also that

\[ \int_{\Omega} f = 0. \]

By density there exists a sequence \( f_\nu \in C^\infty_c(\Omega) \) such that \( \int_{\Omega} f_\nu = 0 \),

(111) \[ \lim_{\nu \to \infty} \| f_\nu - \overline{f} \|_{L^{p/(p-1)}(\Omega)} = 0 \quad \text{and} \quad \sup_{\nu} \| f_\nu \|_{L^\infty(\Omega)} < \infty. \]

By a classical result for the divergence (see e.g. [5]) there exists \( \xi_\nu \in C^\infty_c(\Omega; \mathbb{R}^2) \) such that

\[ \text{div} \xi_\nu = f_\nu \quad \text{in} \ \Omega \quad \text{and} \quad \| \xi_\nu \|_{W^{1,p/(p-1)}(\Omega)} \leq C \| f_\nu \|_{L^{p/(p-1)}(\Omega)}. \]

**Step 3.** By (68) we have that

(112) \[ \int_{\Omega} f_\nu(u) \cdot h'(\det \nabla u) \cdot \det \nabla u = -\int_{\Omega} 2(\nabla \xi_\nu(u) \cdot \nabla u) : \nabla u. \]

Since \( \det \nabla u > \mu \) a.e. in \( \Omega \) (cf. (69)) we get using (70) that

\[ \| \nabla \xi_\nu \circ u \|_{L^{p/(p-1)}(\Omega)} \leq \frac{1}{\mu^{1/p}} \| \nabla \xi_\nu \|_{L^{p/(p-1)}(\Omega)}. \]

and hence using (111)

\[ \| \nabla \xi_\nu \circ u \|_{L^{p/(p-1)}(\Omega)} \leq C. \]

From (112) and the previous inequality we get

(113) \[ \int_{\Omega} f_\nu(u) \cdot h'(\det \nabla u) \cdot \det \nabla u \leq C \| \nabla u \|_{L^{2p}(\Omega)}^2. \]

Note that using the left assertion in (111) as well as (69) and (60), we get

\[ \| f_\nu \circ u - \overline{f} \circ u \|_{L^{p/(p-1)}(\Omega)} \to 0. \]

Furthermore, we notice that \( \| f_\nu \circ u \|_{L^\infty(\Omega)} = \| f_\nu \|_{L^\infty(\Omega)} \) as the preimage of any null set under \( u \) is a null set. Using the previous assertion, the right assertion of (111), and the fact that \( h(\det \nabla u) \cdot \det \nabla u \in L^1(\Omega) \) (cf. (67)), we can apply Lemma 21 and deduce from (113) that

\[ \int_{\Omega} f_\nu(u) \cdot h'(\det \nabla u) \cdot \det \nabla u \leq C \| \nabla u \|_{L^{2p}(\Omega)}^2. \]

or equivalently

\[ \int_{\Omega \setminus N_\delta} f_\nu(u) \cdot h'(\det \nabla u) \cdot \det \nabla u \leq -\int_{\Omega \setminus N_\delta} \overline{f}(u) \cdot h'(\det \nabla u) \cdot \det \nabla u + C \| \nabla u \|_{L^{2p}(\Omega)}^2. \]

Since \( u \) is a.e. one-to-one we have that \( u(x) \in \Omega \setminus u(N_\delta) \) for a.e. \( x \in \Omega \setminus N_\delta \). Hence using (73) and the right assertion in (110) we directly get

\[ \int_{\Omega \setminus N_\delta} \overline{f}(u) \cdot h'(\det \nabla u) \cdot \det \nabla u \leq C |h'(\mu + \delta)| \int_{\Omega \setminus N_\delta} \det \nabla u \]

\[ \leq C \| \nabla u \|_{L^{2p}(\Omega)}^2. \]

Combining the last two inequalities we get (108) which ends the proof.
APPENDIX A. REDUCTION TO THE CASE OF THE UNIT SQUARE

Proof of Lemma 12. So we assume that Theorem 8 has been proven for $\Omega = (0, 1)^2$ and show how to deduce it for any bounded open set $\Omega \subset \mathbb{R}^2$ with $C^{1,1}$ boundary. To this end we recall that any bi-Lipschitz mapping in the plane $\psi$ satisfies

$$ \frac{1}{L^2} |M| \leq |\psi(M)| \leq L^2 |M| $$

whenever $|x - y|/L \leq |\psi(x) - \psi(y)| \leq L|x - y|$ holds for every $x, y$.

Step 1. We assume first that for $\Omega$ there exists a bijection $\psi : [0, 1]^2 \rightarrow \mathbb{R}^2$ such that $\psi \in C^{1,1}([0, 1]^2; \Omega)$ and $\psi^{-1} \in C^{1,1}(\Omega; [0, 1]^2)$ hold. We call such a set $\Omega$ $C^{1,1}$-equivalent to the unit square.

Let $\tau > 0$ and $M \subset \Omega$ with $|M|$ and $\sqrt{|M|}$ small enough, which trivially implies (by (114)) that $|\psi^{-1}(M)|$ and $\sqrt{|\psi^{-1}(M)|}\tau$ are also small. Hence, by hypothesis, applying the theorem with $2\tau$ in place of $\tau$, there exists a bi-Lipschitz map $\tilde{\phi} : [0, 1]^2 \rightarrow [0, 1]^2$ stretching the set $\psi^{-1}(M) \subset (0, 1)^2$, such that

$$ \tilde{\phi} = \text{id} \quad \text{on } \partial[0, 1]^2, $$

$$ \|\tilde{\phi} - \text{id}\|_{W^{1,p}(\Omega)} \leq C|\psi^{-1}(M)|^{1/(2p)}\tau (1 + \exp(C\tau^3)) \quad \text{for all } 1 \leq p \leq \infty, $$

(115)

(116)

(117)

(118)

(119)

We claim that

$$ \phi := \psi \circ \tilde{\phi} \circ \psi^{-1} $$

satisfies (12)–(13). Indeed first (12) follows trivially from (116). In what follows $C_\psi$ will denote a generic constant depending only on $\|\psi\|_{C^{1,1}}$ and $\|\psi^{-1}\|_{C^{1,1}}$ (and hence only on $\Omega$) which may change from appearance to appearance. Using (114) and (117) we get

$$ \|\phi - \text{id}\|_{L^\infty(\Omega)} \leq C_\psi \|\tilde{\phi} \circ \psi^{-1} - \psi^{-1}\|_{L^\infty(\Omega)} \leq C_\psi |M|^{1/2}\tau, $$

which proves (11). Next note that

$$ \nabla \phi - \text{Id} = \nabla (\psi \circ \tilde{\phi} \circ \psi^{-1}) - \text{Id} = \nabla \psi (\tilde{\phi} \circ \psi^{-1}) \cdot (\nabla \tilde{\phi} (\psi^{-1}) - \text{Id}) \cdot \nabla \psi^{-1} + (\nabla \psi (\tilde{\phi} \circ \psi^{-1}) - \nabla \psi (\psi^{-1})) \cdot \nabla \psi^{-1} $$

which yields, using (114), (116), and (117),

$$ \|\nabla \phi - \text{Id}\|_{L^p(\Omega)} \leq C_\psi \|\tilde{\phi} - \text{Id}\|_{L^p([0, 1]^2)} + C_\psi \|\tilde{\phi} - \text{Id}\|_{L^\infty([0, 1]^2)} \leq C_\psi |M|^{1/(2p)}\tau (1 + \exp(C\tau^3)), $$

and hence the assertion.
proving (10). Then
\[
\det \nabla (\psi \circ \tilde{\phi} \circ \psi^{-1}) = \det \nabla \psi (\tilde{\phi} \circ \psi^{-1}) \det \nabla \tilde{\phi} (\psi^{-1}) \det \nabla \psi^{-1}
\]
\[
= \left( \det \nabla \psi (\tilde{\phi} \circ \psi^{-1}) - \det \nabla \psi (\psi^{-1}) \right) \det \nabla \tilde{\phi} (\psi^{-1}) \det \nabla \psi^{-1} + \det \nabla \tilde{\phi} (\psi^{-1})
\]
\[
\geq - C_\psi |\tilde{\phi} \circ \psi^{-1} - \psi^{-1}| + \det \nabla \tilde{\phi} (\psi^{-1})
\]
\[
\geq - C_\psi |M|^{1/2} \tau + \det \nabla \tilde{\phi} (\psi^{-1}),
\]
where we have used (114) and (117) for the last inequality. Hence using (118) and (119) we get from the last inequality
\[
\det \nabla \phi \geq \begin{cases} 
1 + \tau (2 - C_\psi |M|^{1/2}) & \text{a.e. in } M \\
1 - C_\psi |M|^{1/2} \tau & \text{a.e. in } \Omega \setminus M.
\end{cases}
\]
Taking $|M|$ small enough so that $C_\psi |M|^{1/2} \leq 1$ gives (12) and (13).

Step 2. We now prove the lemma in the general case.

Step 2.1. Let $\tau > 0$ and $M \subset \Omega$ with $|M|$ and $\sqrt{|M|} \tau$ small enough. Since $\Omega$ has a boundary of class $C^{1,1}$, there exist an integer $N$ and $N$ sets $\Omega_1, \ldots, \Omega_N$ which are $C^{1,1}$ equivalent to the unit square and such that
\[
\Omega = \bigcup_{i=1}^N \Omega_i \quad \text{and} \quad \partial \Omega \subset \bigcup_{i=1}^N \partial \Omega_i.
\]
Note that the $\Omega_i$ may intersect. In the rest of the proof by $C$ we denote a generic constant depending only on the $\Omega_i$ which may change from appearance to appearance.

Using Step 1, for any $1 \leq i \leq N$ there exists a bi-Lipschitz mapping $\phi_i$ from $\overline{\Omega}_i$ to $\Omega_1$ satisfying (9)-13 for
\[
\begin{cases}
M_1 := M \cap \Omega_1 & \text{if } i = 1 \\
M_i := M \cap \Omega_i \setminus (\Omega_{i-1} \cup \ldots \cup \Omega_1) & \text{if } 2 \leq i \leq N.
\end{cases}
\]
Obviously, we have
\[
|M_i| \leq |M|.
\]

We extend the $\phi_i$ by the identity outside $\Omega_i$ which obviously implies that (10)-(13) are satisfied on $\Omega$ (and not only on $\Omega_i$). Summarizing we have for every $1 \leq i \leq N$
\begin{align}
\phi_i &= \text{id} \quad \text{on } \Omega \setminus \Omega_i, \\
\|\phi_i - \text{id}\|_{W^{1, p}(\Omega)} &\leq C |M_i|^{1/(2p)} \tau \exp(C \tau^3) \quad \text{for every } 1 \leq p \leq \infty, \\
\|\phi_i - \text{id}\|_{L^\infty(\Omega)} &\leq C |M_i|^{1/2} \tau, \\
\det \nabla \phi_i &\geq 1 + \tau \quad \text{a.e. in } M_i, \\
\det \nabla \phi_i &\geq 1 - C |M_i|^{1/2} \tau \geq 1/2 \quad \text{a.e. in } \Omega.
\end{align}

Step 2.2 (conclusion). We claim that
\[
\phi := \phi_N \circ \cdots \circ \phi_1
\]
has all the properties stated in Theorem 8. First we obviously have that \( \phi \) is a bi-Lipschitz mapping from \( \mathbb{R}^n \) to \( \mathbb{R}^n \) with \( \phi = \text{id} \) on \( \partial \Omega \). Then from (122) we get
\[
\| \phi - \text{id} \|_{L^\infty(\Omega)} \leq \sum_{i=1}^{N} \| \phi_i - \text{id} \|_{L^\infty(\Omega)} \leq C|M|^{1/2} \tau,
\]
which proves (11). From
\[
\nabla \phi - \text{Id} = \nabla \phi_N (\phi_{N-1} \circ \ldots \circ \phi_1) \cdot \ldots \cdot \nabla \phi_1 - \text{Id} \\
= (\nabla \phi_N (\phi_{N-1} \circ \ldots \circ \phi_1) - \text{Id}) \cdot \nabla \phi_{N-1} (\phi_{N-2} \circ \ldots \circ \phi_1) \cdot \ldots \cdot \nabla \phi_1 + \ldots + \nabla \phi_1 - \text{Id},
\]
we easily deduce using (11), (121), (124), and the usual change of variables that
\[
\| \phi - \text{id} \|_{W^{1,p}(\Omega)} \leq C|M|^{1/(2p)} \tau \exp(C\tau^3) \quad \text{for all } 1 \leq p \leq \infty,
\]
which shows (10). We finally prove the two remaining assertions concerning the determinant. Note that
\[
\det \nabla \phi = \det \nabla \phi_N (\phi_{N-1} \circ \ldots \circ \phi_1) \times \ldots \times \det \nabla \phi_1.
\]
Hence, using (124) we get a.e. in \( \Omega \)
\[
\det \nabla \phi \geq \prod_{i=1}^{N} (1 - C|M_i|^{1/2} \tau).
\]
Taking \( \sqrt{|M|} \tau \) smaller if necessary we get
\[
\prod_{i=1}^{N} (1 - C|M_i|^{1/2} \tau) \geq 1 - C|M|^{1/2} \tau.
\]
Combining the last two inequalities gives (13). Next, combining (120), (123), and (124), we deduce that a.e on
\[
\{ \\
M \cap \Omega_1 & \quad \text{if } i = 1 \\
M \cap \Omega_i \setminus \bigcup_{j=1}^{i-1} \Omega_j & \quad \text{if } 2 \leq i \leq N
\}
\]
we have
\[
\det \nabla \phi \geq (1 + \tau) \prod_{j=1, j \neq i}^{N} (1 - C|M_j|^{1/2} \tau).
\]
Taking \( \sqrt{|M|} \tau \) smaller if necessary we deduce
\[
(1 + \tau) \prod_{j=1, j \neq i}^{N} (1 - C|M_j|^{1/2} \tau) \geq 1 + \tau/2.
\]
Combining the last two inequalities gives
\[
\det \nabla \phi \geq 1 + \tau/2 \quad \text{a.e. in } M.
\]
Rescaling \( \tau \) by a factor of 2 proves (12) and ends the proof. \( \square \)
APPENDIX B. CORRECTION OF THE BOUNDARY VALUES

Proof of Proposition \[ \text{[14]} \]

Step 1. Using Lemma \[ \text{[12]} \] and Lemma \[ \text{[13]} \] we already know that it is enough to prove Theorem \[ \text{[8]} \] on the unit square $\Omega = (0,1)^2$ and for compact sets $K \subset (0,1)^2$. Having established Proposition \[ \text{[11]} \], we know that for every $\tau > 0$ and any compact set $K \subset (0,1)^2$ with $|K|\sqrt{|K|} \tau < c$, there exists a bi-Lipschitz mapping $\phi = \phi_{\tau,K} : [0,1]^2 \rightarrow [0,1]^2$ satisfying

\[
\begin{align*}
&|\phi - \text{id}|_{L^\infty((0,1)^2)} \leq C\tau |K|^{1/(2p)} \quad \text{for all } 1 \leq p \leq \infty, \\
&|\phi - \text{id}|_{L^\infty((0,1)^2)} \leq C\tau |K|^{1/2}, \\
&\text{det } \nabla \phi \geq 1 + \tau \quad \text{a.e. on } K, \\
&\text{det } \nabla \phi \leq 1 - C|K|^{1/2}\tau \quad \text{a.e. on } (0,1)^2, \\
\end{align*}
\]

as well as

\[
\begin{align*}
&|\phi - \text{id}|_{W^{1,p}((0,1)^2)} \leq C\tau |K|^{1/(2p)} \quad \text{for all } 1 \leq p \leq \infty, \\
&\phi^2(s,0) = \phi^1(0,s) = 0 \quad \text{and } \phi^1(s,1) = \phi^1(1,s) = 1 \quad \text{for every } s \in [0,1].
\end{align*}
\]

We now show how to modify $\phi$ so that the new bi-Lipschitz mapping satisfies \[ \text{[14]} \]. In the rest of the proof $C$ will denote a generic universal constant that may change from appearance to appearance.

Step 2. For $t \in [0,1]$ define the homotopy $\varphi_t := (1-t) \text{id} + t\phi$. Note that $\varphi_0 = \text{id}$ and $\varphi_1 = \phi$. Using \[ \text{[120]} \] and \[ \text{[129]} \] we have that

\[
\begin{align*}
&|\frac{d}{dt} \varphi_t|_{W^{1,p}((0,1)^2)} \leq C\tau |K|^{1/(2p)} \quad \text{for all } 1 \leq p \leq \infty, \\
&|\frac{d}{dt} \varphi_t|_{L^\infty((0,1)^2)} \leq C\tau |K|^{1/2}.
\end{align*}
\]

Moreover, by Remark \[ \text{[13]} \] (ii) which implies that $\varphi_t = \phi_{t\tau,K}$, we have that $\varphi_t : \partial(0,1)^2 \rightarrow \partial(0,1)^2$ is bi-Lipschitz and, from \[ \text{[129]} \], we have

\[
|\varphi_t - \text{id}|_{W^{1,p}((0,1)^2)} \leq C\tau |K|^{1/(2p)} \quad \text{for all } 1 \leq p \leq \infty.
\]

For every $t \in [0,1]$ define $u_t : \partial(0,1)^2 \rightarrow \mathbb{R}^2$ by

\[
\frac{d}{dt} \varphi_t = u_t(\varphi_t) \quad \text{or, equivalently, } u_t := \frac{d}{dt} \varphi_t(\varphi_t^{-1}).
\]

From \[ \text{[132]} \] we get

\[
|u_t|_{L^\infty((0,1)^2)} \leq C\tau |K|^{1/2}.
\]

Moreover from \[ \text{[131]} \] and \[ \text{[133]} \] we get

\[
|\nabla u_t|_{L^p((0,1)^2)} \leq C\tau(1 + \tau^2)|K|^{1/(2p)} \quad \text{for all } 1 \leq p \leq \infty.
\]

Hence combining the previous two inequalities gives

\[
|u_t|_{W^{1,p}((0,1)^2)} \leq C\tau(1 + \tau^2)|K|^{1/(2p)} \quad \text{for every } 1 \leq p \leq \infty.
\]
Moreover, using (130) we deduce
\begin{equation}
(136) \quad u_t^2(s,0) = u_t^2(s,1) = u_t^1(0,s) = u_t^1(1,s) = 0 \quad \text{for every } s \in [0,1].
\end{equation}
Finally, using Remark 18 (ii), note that \((t,z) \to u_t(z)\) is Lipschitz in \([0,1]\times \partial(0,1)^2\).

**Step 3.** Applying Proposition 22, for every \(t \in [0,1]\) there exists \(v_t \in W^{1,\infty}((0,1)^2; \mathbb{R}^2)\) (depending linearly on \(u_t\)) such that
\begin{equation}
(137) \quad \text{div}(v_t) = 0 \quad \text{in } (0,1)^2,
\end{equation}
\begin{equation}
(138) \quad v_t = u_t \quad \text{on } \partial(0,1)^2,
\end{equation}
\begin{equation}
(139) \quad \|v_t\|_{W^{1,p}((0,1)^2)} \leq C\|u_t\|_{W^{1,p}(\partial(0,1)^2)} \quad \text{for every } 1 \leq p \leq \infty,
\end{equation}
\begin{equation}
(140) \quad \|v_t\|_{L^\infty((0,1)^2)} \leq C\|u_t\|_{L^\infty(\partial(0,1)^2)}.
\end{equation}
Moreover we have that \((t,z) \to v_t(z)\) is Lipschitz in \([0,1] \times [0,1]^2\). Combining (136) and (138) we get that \(v_t\) is perpendicular to the unit normal of \((0,1)^2\) on \(\partial(0,1)^2\).

Hence we can define the flow \(\psi_t\) of \(v_t\), i.e.
\[
\frac{d}{dt} \psi_t = v_t(\psi_t) \quad \text{and} \quad \psi_0 = \text{id}.
\]
By classical results, \(\psi_t : [0,1]^2 \to [0,1]^2\) is uniquely defined and bi-Lipschitz. Since (by (137)) \(\text{div} v_t = 0\) we get (cf. e.g. Theorem 12.5 in [8])
\begin{equation}
(141) \quad \det \nabla \psi_t \equiv 1.
\end{equation}
Moreover, since (by (138)) \(u_t = v_t\) on \(\partial(0,1)^2\), we get by uniqueness of the flow \(\psi_t = \varphi_t\) on \(\partial(0,1)^2\) for every \(t \in [0,1]\) and hence
\begin{equation}
(142) \quad \psi_1 = \varphi_1 = \phi \quad \text{on } \partial(0,1)^2.
\end{equation}
We now derive some estimates on \(\psi_t\). First using (134) and (140) we get that
\begin{equation}
(143) \quad \|\psi_t - \text{id}\|_{L^\infty((0,1)^2)} \leq C\tau \sqrt{|K|}.
\end{equation}
Next (using Gronwall Lemma, cf. e.g. Theorem 12.1 in [8]) we have the estimate
\[
\|\psi_t - \text{id}\|_{W^{1,\infty}((0,1)^2)} \leq C \int_0^t \|\psi_s\|_{W^{1,\infty}((0,1)^2)} ds \cdot \exp \left( C \int_0^t \|\psi_s\|_{W^{1,\infty}((0,1)^2)} ds \right)
\leq C\tau \exp(C\tau^3),
\]
where we have used (135) and (139) to deduce the last inequality. From the previous inequality, (155), (139) and (141) we get
\[
\|\nabla \psi_t - \text{Id}\|_{L^p((0,1)^2)} = \left\| \int_0^t \nabla v_s(\psi_s) \cdot \nabla \psi_s ds \right\|_{L^p((0,1)^2)}
\leq (1 + C\tau \exp(C\tau^3)) \int_0^t \|\nabla v_s(\psi_s) ds\|_{L^p((0,1)^2)}
\leq C\tau(1 + \exp(C\tau^3)|K|^{1/(2p)}).
\]
Finally taking into account the previous inequality and recalling that \(\det \nabla \psi_t = 1\), we get
\begin{equation}
(144) \quad \|\nabla \psi_t - \text{Id}\|_{L^p((0,1)^2)} \leq C\tau(1 + \exp(C\tau^3))|K|^{1/(2p)} \quad \text{for every } 1 \leq p \leq \infty.
\end{equation}

**Step 4 (conclusion).** We finally claim that
\[
\bar{\phi} := \psi_t^{-1} \circ \phi
\]
has all the desired properties. First, obviously by (142), we have $\tilde{\phi} = \text{id}$ on $\partial (0,1)^2$. Then, since by (141) we have $\det \nabla \tilde{\phi} = \det \nabla \phi$, we trivially infer from (127) and (128) that

$$\det \nabla \tilde{\phi} \geq 1 + \tau \quad \text{a.e. on } K \quad \text{and} \quad \det \nabla \tilde{\phi} \geq 1 - C|K|^{1/2}\tau \quad \text{a.e. on } (0,1)^2$$

holds. We finally show that

$$(145) \quad \|\tilde{\phi} - \text{id}\|_{L^\infty((0,1)^2)} \leq C\tau \sqrt{|K|}$$

and

$$(146) \quad \|\tilde{\phi} - \text{id}\|_{W^{1,p}((0,1)^2)} \leq C\tau (1 + \exp(C\tau^3))|K|^{1/(2p)} \quad \text{for all } 1 \leq p \leq \infty$$

which will end the proof. First (145) is a direct consequence of (126) and (143). Next, using (125) and (144) we get

$$\|\nabla \tilde{\phi} \cdot \text{id}\|_{L^p((0,1)^2)} = \|\nabla \psi_1^{-1}(\phi) \cdot \nabla \phi - \text{id}\|_{L^p((0,1)^2)}$$

$$\leq \|\nabla \psi_1^{-1}(\phi) - \text{id}\|_{L^p((0,1)^2)} + \|\nabla \phi - \text{id}\|_{L^p((0,1)^2)}$$

$$\leq C\tau (1 + \exp(C\tau^3))|K|^{1/(2p)} \quad \text{for every } 1 \leq p \leq \infty.$$

Combining the previous inequality with (145) gives (146) and ends the proof. \qed

APPENDIX C. SOLENOIDAL EXTENSION OF VECTOR FIELDS

**Proposition 22.** Let $u : \partial (0,1)^2 \to \mathbb{R}^2$ be a Lipschitz map satisfying

$$(147) \quad u^2(s,0) = u^2(s,1) = u^1(0,s) = u^1(1,s) = 0 \quad \text{for every } s \in [0,1]$$

(where we have written $u = (u^1, u^2)$). Then there exist a universal constant $C$ and $v \in W^{1,\infty}((0,1)^2; \mathbb{R}^2)$ satisfying

$$\text{div } v = 0 \quad \text{in } (0,1)^2,$$

$$v = u \quad \text{on } \partial (0,1)^2,$$

$$\|v\|_{W^{1,p}((0,1)^2)} \leq C\|u\|_{W^{1,p}(\partial (0,1)^2)} \quad \text{for every } 1 \leq p \leq \infty,$$

$$\|v\|_{L^\infty((0,1)^2)} \leq C\|u\|_{L^\infty(\partial (0,1)^2)}.$$

Furthermore, $v$ depends linearly on $u$.

**Proof.** The only difficulty which we face is the fact that the unit square does not have a smooth boundary. Therefore we shall work with smooth approximations of the unit square. The proof will proceed as follows:

In the first step we define a smooth open set $A \subset (0,1)^2$ and give elementary properties about its rescaled versions. Then in Step 2 we construct a solenoidal vector field $v_0$ in $(0,1)^2$ with the desired boundary values only in a subset of $\partial (0,1)^2$. Afterwards, in Step 3 we produce solenoidal vector fields $v_k$ in $1/4^k A$ (the rescaling of $A$ by a factor of $1/4^k$) correcting the boundary values near $(0,0)$. Then in Step 4, we exhibit a solenoidal vector field agreeing with $u$ on $[0,1/2] \times \{0\} \cup \{0\} \times [0,1/2]$. Finally in Step 5 we conclude by applying the same procedure to the three remaining corners.

**Step 1.** In the remainder of the proof we fix $\delta = 1/10$ and $\mu = 1/4$ and we denote by $C$ a generic universal constant that may change from appearance to appearance.
Let $A \subset (0, 1)^2$ be a smooth open set with the following properties (cf. Figure 3):

\[
A \setminus \left( ([0, \delta] \cup [1 - \delta, 1]) \times (0, 1) \right) = (\delta, 1 - \delta) \times (0, 1),
\]
\[
A \setminus \left( (0, 1) \times ([0, \delta] \cup [1 - \delta, 1]) \right) = (0, 1) \times (\delta, 1 - \delta),
\]
\[
A \cap \left[ \left( 0, \frac{\delta}{2} \right) \times \left( 0, \frac{\delta}{2} \right) \right] = A \cap \left[ \left( 1 - \frac{\delta}{2}, 1 \right) \times \left( 0, \frac{\delta}{2} \right) \right] = \emptyset,
\]
\[
A \cap \left[ \left( 0, \frac{\delta}{2} \right) \times \left( 1 - \frac{\delta}{2}, 1 \right) \right] = A \cap \left[ \left( 1 - \frac{\delta}{2}, 1 \right) \times \left( 1 - \frac{\delta}{2}, 1 \right) \right] = \emptyset.
\]

For every $k \geq 1$ define $A_k := \mu^k A$. It is then not difficult to see that

\[
\sum_{k=1}^{\infty} \chi_{A_k} \leq C \quad \text{on } \mathbb{R}^2
\]

holds.

\textbf{Step 2.} Define $u_0 : \partial A \to \mathbb{R}^2$ as follows:

\[
u_0(x, y) := v(x, y) \quad \text{for } (x, y) \in [2\delta, 1 - 2\delta) \times \{0, 1\} \cup \{0, 1\} \times [2\delta, 1 - 2\delta),
\]

\[
u_0(x, y) := \frac{x - \delta}{\delta} u(x, y) \quad \text{for } (x, y) \in [\delta, 2\delta) \times \{0, 1\},
\]

\[
u_0(x, y) := \frac{(1 - x) - \delta}{\delta} u(x, y) \quad \text{for } (x, y) \in [1 - 2\delta, 1 - \delta) \times \{0, 1\},
\]

\[
u_0(x, y) := \frac{y - \delta}{\delta} u(x, y) \quad \text{for } (x, y) \in \{0, 1\} \times [\delta, 2\delta),
\]

\[
u_0(x, y) := \frac{(1 - y) - \delta}{\delta} u(x, y) \quad \text{for } (x, y) \in \{0, 1\} \times [1 - 2\delta, 1 - \delta),
\]

\[
u_0(x, y) := 0 \quad \text{elsewhere}.
\]
First note that we have $u_0 \in W^{1,\infty}(\partial A; \mathbb{R}^2)$ and that by (147) the vector field $(u_0^2, -u_0^1)$ is parallel to the outward unit normal of $A$. Hence we can apply Lemma 8.8 in \cite{8} and find $h \in W^{2,\infty}(A)$ verifying

\[
\begin{aligned}
\nabla h &= (u_0^2, -u_0^1) \quad \text{on } \partial A, \\
\|h\|_{W^{2,p}(A)} &\leq C\|u_0\|_{W^{1,p}(\partial A)} \quad \text{for every } 1 \leq p \leq \infty, \\
\|h\|_{W^{1,\infty}(A)} &\leq C\|u_0\|_{L^\infty(\partial A)}.
\end{aligned}
\]

The fact that the above constant $C$ is independent of $p$, although not written explicitly for the Sobolev spaces in the previous cited lemma, follows by exactly the same proof given for the Hölder spaces. Moreover the proof also gives that $h$ depends linearly on $u_0$. Define $v_0 \in W^{1,\infty}(A; \mathbb{R}^2)$ by

\[
v_0 := (-\partial_y h, \partial_x h).
\]

Then, obviously

\[
\nabla v_0 = 0 \quad \text{in } A, \\
v_0 = u_0 \quad \text{on } \partial A, \\
\|v_0\|_{W^{1,p}(A)} \leq C\|u_0\|_{W^{1,p}(\partial A)} \quad \text{for every } 1 \leq p \leq \infty, \\
\|v_0\|_{L^\infty(A)} \leq C\|u_0\|_{L^\infty(\partial A)}.
\]

Noticing that $u_0 = 0$ on $\partial A \setminus \partial(0,1)^2$ we can extend $v_0$ by 0 outside $\overline{A}$ to the full set $(0,1)^2$. We trivially get that this extension (still denoted by $v_0$) is Lipschitz and satisfies

\[
\begin{aligned}
(149) \quad \nabla v_0 &= 0 \quad \text{in } (0,1)^2, \\
(150) \quad \|v_0\|_{W^{1,p}((0,1)^2)} &\leq C\|u_0\|_{W^{1,p}(\partial A)} \quad \text{for every } 1 \leq p \leq \infty, \\
(151) \quad \|v_0\|_{L^\infty((0,1)^2)} &\leq C\|u_0\|_{L^\infty(\partial A)}.
\end{aligned}
\]

Note that $v_0 = u$ holds only on 

\[
[2\delta, 1-2\delta] \times \{0,1\} \cup \{0,1\} \times [2\delta, 1-2\delta],
\]

so we need to modify our $v_0$ in such a way that it agrees with $u$ on the whole of $\partial(0,1)^2$. This will be done in the last remaining steps.

**Step 3.** We now consider the correction to $v_0$ in the corner $[0,1/2]^2$, the correction on the three others corners will be exactly the same.

**Step 3.1.** For every integer $k \geq 1$, define the map $u_k : \partial A_k \to \mathbb{R}^2$ (recall that $A_k = \mu^k A$ where $\mu = 1/4$); recall also that $[\delta \mu^k, 2\delta \mu^{k-1}] \times \{0\} \cup \{0\} \times
By the definition of $u$ \((152)\) by

$$
\begin{align*}
  \text{for } x \in [\delta \mu^k, 2\delta \mu^k), \\
  u_k(x, 0) &:= \frac{x - \delta \mu^k}{\delta \mu^k} u(x, 0) \\
  \text{for } x \in (2\delta \mu^k, \delta \mu^{k-1}), \\
  u_k(x, 0) &:= u(x, 0) \\
  \text{for } x \in [\delta \mu^{k-1}, 2\delta \mu^{k-1}), \\
  u_k(x, 0) &:= \frac{2\delta \mu^{k-1} - x}{\delta \mu^{k-1}} u(x, 0) \\
  \text{for } y \in [\delta \mu^k, 2\delta \mu^k), \\
  u_k(0, y) &:= \frac{y - \delta \mu^k}{\delta \mu^k} u(0, y) \\
  \text{for } y \in (2\delta \mu^k, \delta \mu^{k-1}), \\
  u_k(0, y) &:= u(0, y) \\
  \text{for } y \in [\delta \mu^{k-1}, 2\delta \mu^{k-1}), \\
  u_k(0, y) &:= \frac{2\delta \mu^{k-1} - y}{\delta \mu^{k-1}} u(0, y) \\
  u_k(z) &:= 0 \\
\end{align*}
$$

Note that we have

$$
(152) \quad u_k = 0 \quad \text{on } \partial A_k \setminus \partial (0,1)^2,
$$

hence we can apply the Poincaré inequality to get (since the length of $\partial A_k$ is bounded by $C\mu^k$)

$$
(153) \quad ||u_k||_{L^p(\partial A_k)} \leq C\mu^k ||\nabla u_k||_{L^p(\partial A_k)}.
$$

By the definition of $u_k$ we trivially have

$$
(154) \quad ||\nabla u_k||_{L^p(\partial A_k)} \leq \chi_{\partial (0,1)^2 \cap \partial A_k} \left(||\nabla u||_{L^p(\partial (0,1)^2)} \right) \quad \text{on } \partial A_k.
$$

Finally, again by the Poincaré inequality, we get that

$$
(155) \quad \chi_{\partial (0,1)^2 \cap \partial A_k} ||\nabla u||_{L^p(\partial (0,1)^2)} \leq C\mu^k ||\nabla u||_{L^p(\partial (0,1)^2)}.
$$

Step 3.2. As the $A_k$ arise by rescaling $A$, we claim that we can construct a sequence of maps $v_k : A_k \to \mathbb{R}^2$ satisfying

$$
(156) \quad \text{div } v_k = 0 \quad \text{in } A_k,
$$

$$
(157) \quad v_k = u_k \quad \text{on } \partial A_k,
$$

$$
(158) \quad ||\nabla v_k||_{L^p(A_k)} \leq \frac{C\mu^k}{p} ||\nabla u||_{L^p(\partial (0,1)^2)} \quad \text{for every } 1 \leq p \leq \infty,
$$

$$
(159) \quad ||v_k||_{L^\infty(A_k)} \leq C ||u||_{L^\infty(\partial (0,1)^2)}.
$$

Indeed define the Lipschitz map $\overline{u} : \partial A \to \mathbb{R}^2$ by

$$
\overline{u}(z) := \frac{1}{\mu^k} u_k(\mu^k z).
$$

Proceeding exactly as in Step 2 there exists $\overline{v} \in W^{1,\infty}(A; \mathbb{R}^2)$ such that

$$
(160) \quad \text{div } \overline{v} = 0 \quad \text{in } A,
$$

$$
(161) \quad \overline{v} = \overline{u} \quad \text{on } \partial A,
$$

$$
(162) \quad \|\overline{v}\|_{W^{1,p}(A)} \leq C\|\overline{u}\|_{W^{1,p}(\partial A)} \quad \text{for every } 1 \leq p \leq \infty,
$$

$$
(163) \quad \|\overline{v}\|_{L^\infty(A)} \leq C\|\overline{u}\|_{L^\infty(\partial A)}.
$$

Finally define $v_k \in W^{1,\infty}(A_k; \mathbb{R}^2)$ by

$$
(164) \quad v_k(z) := \mu^k \overline{v} \left(\frac{1}{\mu^k} z\right).
$$
First, (159) comes directly from (163) and \( \|u_k\|_{L^\infty(\partial A_k)} \leq \|u\|_{L^\infty(\partial(0,1)^2)} \). Next, by (160) and (161), we get (162) and (164). Moreover, rescaling (162) we obtain for every \( 1 \leq p \leq \infty \)
\[
\mu^{-2k/p} \|\nabla u_k\|_{L^p(\partial A_k)} \leq C \mu^{-k/p} \|\nabla u_k\|_{L^p(\partial(0,1)^2)} + C \mu^{-k-1/p} \|u_k\|_{L^p(\partial(0,1)^2)}
\]
and hence (158) is verified. The claim is therefore proved. Note that the \( v_k \) still depends linearly on \( k \).

**Step 4.** We construct a solenoidal vector field \( v \) agreeing with \( u \) on \( \partial(0,1)^2 \cap [0,1/2]^2 \).

**Step 4.1.** Since \( v_k = u_k = 0 \) on \( \partial A_k \setminus \partial(0,1)^2 \) we can extend \( v_k \) to the full unit square by 0 outside \( \overline{A_k} \) and trivially get that this extension, still denoted \( v_k \), satisfies
\[
\begin{align*}
\text{div } v_k &= 0 \quad \text{in } (0,1)^2, \\
v_k &= u_k \quad \text{on } \partial A_k, \\
\|\nabla v_k\|_{L^p((0,1)^2)} &\leq C \mu^{k/p} \|\nabla u\|_{L^p(\partial(0,1)^2)} \quad \text{for every } 1 \leq p \leq \infty, \\
\|v_k\|_{L^\infty((0,1)^2)} &\leq C \|u\|_{L^\infty(\partial(0,1)^2)}.
\end{align*}
\]

Finally define
\[
v := \sum_{k=0}^{\infty} v_k.
\]

First note that \( v \) still depends linearly on \( u \). Since \( v_k \) is identically 0 outside \( \overline{A_k} \), using (142), (143), (159), (161), (165), (169), and (170) we immediately get that \( v \in W^{1,\infty}((0,1)^2, \mathbb{R}^2) \) and
\[
\begin{align*}
\text{div } v &= 0 \quad \text{in } (0,1)^2, \\
\|\nabla v\|_{L^p((0,1)^2)} &\leq C \|\nabla u\|_{L^p(\partial(0,1)^2)} \quad \text{for every } 1 \leq p \leq \infty, \\
\|v\|_{L^\infty((0,1)^2)} &\leq C \|u\|_{L^\infty(\partial(0,1)^2)}.
\end{align*}
\]

**Step 4.2.** We now claim that
\[
v = u \quad \text{on } \partial(0,1)^2 \cap [0,1/2]^2.
\]

We will only show that \( v = u \) on \( [0,1/2] \times \{0\} \), the proof that \( v = u \) on \( \{0\} \times [0,1/2] \) being completely analogous. First of all note that, since \( v_k(x,0) = 0 \) for \( k \geq 1 \) and \( x \in [2\delta,1/2] \),
\[
v(x,0) = u_0(x,0) = u(x,0) \quad \text{for } x \in [2\delta,1/2].
\]
Next, since \( v_k(0,0) = 0 \) for every \( k \geq 0 \), we obviously have,
\[
v(0,0) = 0 = u(0,0).
\]
Finally take \( x \in (0,2\delta) \). We can select \( k_1 \in \mathbb{N} \) so that (recall that \( \mu = 1/4 \))
\[
either \ x \in [\mu^{k_1} \delta, 2\mu^{k_1} \delta) \text{ or } x \in [2\mu^{k_1} \delta, \mu^{k_1-1} \delta).
\]
In the first case we get, since $v_k(x, 0) = 0$ for every $k \notin \{k_1, k_1 + 1\}$,

$$v(x, 0) = u_{k_1}(x, 0) + u_{k_1+1}(x, 0) = \frac{x - \mu^{k_1} \delta}{\mu^{k_1} \delta} u(x, 0) + \frac{2\mu^{k_1} \delta - x}{\mu^{k_1} \delta} u(x, 0) = u(x, 0)$$

whereas, in the second case, since $v_k(x, 0) = 0$ for every $k \neq k_1$, we have

$$v(x, 0) = u_{k_1}(x, 0) = u(x, 0),$$

which proves the claim.

**Step 5.** Proceeding exactly as in the last two steps, we can perform the corrections to $v_0$ in the remaining three corners, namely $[0, 1/2] \times [1/2, 1], [1/2, 1] \times [0, 1/2]$ and $[1/2, 1] \times [1/2, 1]$, and find $v \in W^{1, \infty}((0, 1); \mathbb{R}^2)$ such that

$$\text{div} v = 0 \quad \text{in} \ (0, 1)^2, \quad v = u \quad \text{on} \ \partial (0, 1)^2,$$

$$||\nabla v||_{L^p((0, 1)^2)} \leq C||\nabla u||_{L^p(\partial (0, 1)^2)} \quad \text{for every} \ 1 \leq p \leq \infty, \quad ||v||_{L^\infty((0, 1)^2)} \leq C ||u||_{L^\infty(\partial (0, 1)^2)}.$$

Recalling that by hypothesis $u^2(s, 0) = u^2(s, 1) = u^1(0, s) = u^1(1, s) = 0$ for every $s \in [0, 1]$, we can apply the Poincaré inequality to get that

$$||v||_{L^p((0, 1)^2)} \leq C||\nabla v||_{L^p((0, 1)^2)}$$

and hence

$$||v||_{W^{1, p}((0, 1)^2)} \leq C||\nabla u||_{L^p(\partial (0, 1)^2)} \quad \text{for every} \ 1 \leq p \leq \infty,$$

which concludes the proof. \qed

**Appendix D. Properties of minimizers of the functional $F$**

**Proof of Lemma 19.** The existence of a minimizer in well known (and holds without assuming $u_0$ to be a homeomorphism), for a proof one could consult [22] or [23].

**Step 1.** First note that any $v \in W^{1, 2}(\Omega; \mathbb{R}^2)$ with $I[v] < \infty$ trivially has the property $h(\det \nabla v) \in L^1(\Omega)$, which implies

$$\det \nabla v > \mu \quad \text{a.e. in} \ \Omega.$$ 

This proves in particular (69).

We now prove that $u$ is $C^0$ in $\overline{\Omega}$. To do so, we need to extend $u$ to a neighbourhood of $\Omega$. We shall do so using first a collar neighbourhood and a reflection to extend $u_0$ to some neighbourhood of $\Omega$; recall that $u_0$ coincides with $u$ on $\partial \Omega$. Since $\Omega$ has a boundary of class $C^{1, 1}$, by the classical collar neighborhood theorem there exist a neighborhood $U$ of $\partial \Omega$ and a $C^{1, 1}$ diffeomorphism $\psi$ from $U$ to $\partial \Omega \times (-1, 1)$ such that

$$\psi(U \cap \Omega) = \partial \Omega \times (0, 1) \quad \text{and} \quad \psi(U \setminus \Omega) = \partial \Omega \times (-1, 0).$$

Writing $\psi(x) = (y(x), t(x)) \in \partial \Omega \times (-1, 1)$, define

$$\overline{u}_0(x) := \left\{ \begin{array}{ll}
\psi^{-1}(y \{u_0(\psi^{-1}(y(x), -t(x))))\}, -t(u_0(\psi^{-1}(y(x), -t(x)))) & \text{if} \ x \in U \setminus \Omega \\
u_0(x) & \text{if} \ x \in \Omega.
\end{array} \right.$$ 

It is then easy to check that $\overline{u}_0$ is a homeomorphism from $U$ to $\overline{u}_0(U)$, belongs to $W^{1, 2}(U; \mathbb{R}^2)$ and satisfies $\det \nabla \overline{u}_0 > \gamma > 0$ a.e. in $U$ for some $\gamma > 0$. Extending $u$ by $\overline{u}_0$ outside $\Omega$ we get that this extension, which we still denote by $u$, obviously
belongs to $W^{1,2}(U,\mathbb{R}^2)$ and satisfies $\det \nabla u > \min\{\gamma, \mu\} > 0$ a.e. in $U$. Hence, by well known results, we first deduce that $u$ is continuous in $U$ (cf. e.g. [17]) and therefore in particular in $\overline{U}$, and also (cf. e.g. [13]) that $u$ sends zero Lebesgue measure sets to zero Lebesgue measure sets (i.e. $u$ satisfies the Lusin’s condition $N$).

**Step 2.** Using classical degree theory, we will show that $u(\Omega) = \Omega$. First, since $u = u_0$ on $\partial \Omega$ and since $u_0$ is a homeomorphism from $\overline{\Omega}$ to $\overline{\Omega}$ preserving the orientation we get that (cf. e.g. [11] or [16])

\begin{equation}
\text{deg}(u, p, \Omega) = \text{deg}(u_0, p, \Omega) = \begin{cases} 1 & \text{if } p \in \Omega \\ 0 & \text{if } p \notin \overline{\Omega}. \end{cases}
\end{equation}

Hence, again by classical results on degree theory (cf. [11] or [16]), we obtain that $u(\Omega) \supset \Omega$. It remains to prove the reverse inclusion. By the sake of the contradiction suppose that it is not the case. Then by continuity, there exists $x_0 \in \Omega$ and $\epsilon > 0$ so that

$$u(B_\epsilon(x_0)) \subset (\overline{\Omega})^c.$$  

First, on one hand, using (168), we get that

$$\text{deg}(u, u(x_0), \Omega) = 0.$$

On the other hand we claim that

$$\text{deg}(u, u(x_0), \Omega) = \int_{\Omega} \rho(u) \det \nabla u \, dx$$

where $\rho$ is smooth, has total mass 1 and has its support included in small enough ball centered at $x_0$; indeed the previous formula is well know for $C^1$ maps (cf. [11] or [16]) and a simple approximation argument shows that it is still valid for continuous and $W^{1,2}$ mappings. Using (69) the previous equation implies that $\text{deg}(u, u(x_0), \Omega) > 0$ which is our desired contradiction.

**Step 3.** We show (70). First, cf. e.g. Theorem 1.9 in Chapter 5 in [12], we have

$$\int_{\Omega} \det \nabla u(x) \, dx = \int_{\Omega} \text{deg}(u, y, \Omega) \, dy$$

and hence using (168) we get

\begin{equation}
\int_{\Omega} \det \nabla u = |\Omega|.
\end{equation}

We also know that the following formula holds true for any $f \in L^1(\Omega)$ (cf. e.g. Theorem 1.8 in Chapter 5 in [12]):

\begin{equation}
\int_{\Omega} f(u(x)) \det \nabla u(x) \, dx = \int_{\Omega} f(y) N(u, y, \Omega) \, dy,
\end{equation}

where $N(u, y, \Omega) := \#\{\Omega \cap u^{-1}(y)\}$. Now combining (169) and (170) with $f \equiv 1$ we obtain

$$\int_{\Omega} N(u, y, \Omega) \, dy = |\Omega|.$$  

Since $u(\Omega) = \Omega$ (cf. Step 2) we have in particular that $N(u, y, \Omega) \geq 1$ for every $y \in \Omega$, therefore we deduce from the previous equation that

$$N(u, y, \Omega) = 1 \quad \text{a.e } y \in \Omega,$$
which tells us that \( u \) is a.e one-to-one. The combination of (170) and the last equation proves at once (70).

**Step 4.** Since \( \det \nabla u > \mu \) a.e. in \( \Omega \) we deduce that
\[
\lim_{\delta \to 0^+} |N_\delta| = |\{ x \in \Omega : \det \nabla u < \mu + \delta \}| = 0.
\]

Since \( u \) is continuous and sends zero Lebesgue measure sets to zero Lebesgue sets, it is elementary to show that \( u \) sends in fact measurable sets to measurable sets. Applying (70) with \( f = \chi_{u(N_\delta)} \), we obtain
\[
|M_\delta| = |u(N_\delta)| = \int_{N_\delta} \det \nabla u \to 0 \quad \text{as} \quad \delta \to 0^+
\]
since \( \det \nabla u \in L^1 \). This proves (72).

**Step 5.** We prove the two remaining assertions. First, recalling that \( h \) is decreasing and convex we immediately obtain (73). Finally, since \( \lim_{t \to \infty} |h'(t)| = 0 \), we easily deduce (74). \( \square \)

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