The Structure Constants of the Exceptional Lie Algebra $\mathfrak{g}_2$ in the Cartan-Weyl Basis

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Abstract

The purpose of this paper is to answer the question whether it is possible to realize simultaneously the relations $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$, $N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}$ and $N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = \frac{-1}{2}q(p+1)\langle\alpha, H_\alpha\rangle$ by the structure constants of the Lie algebra $\mathfrak{g}_2$. We show that if the structure constants obey the first relation, the three last ones are violated, and vice versa. Contrary to the second case, the first one uses the Cartan matrix elements to derive the structure constants in the form of $\langle\beta, H_\alpha\rangle$. The commutation relations corresponding to the first case are exactly documented in the prior literature. However, as expected, a Lie algebra isomorphism is established between the Cartan-Weyl bases obtained in both approaches.

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1 Introduction

The simple Lie algebras over the complex numbers have been classified into four infinite classes, including the special linear, odd orthogonal, symplectic and even orthogonal algebras, as well as that into five finite exceptional Lie algebras $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$ with the dimensions 14, 56, 78, 133 and 248, respectively [1, 2, 3, 4, 5, 6, 7]. Among the exceptional Lie algebras, the smallest and easiest one to consider the root system, the Dynkin diagram, fundamental representations, the structure constants, etc. is the algebra $\mathfrak{g}_2$. Its corresponding Lie group as a subgroup of the Spinor group $Spin(7)$ is obtained by fixing a point in $S^7$ [8]. The group $G_2$ has been attracted great attentions in many areas such as gauge theories in particle physics [9, 10, 11, 12, 13, 14], (super)gravity theory [15, 16, 17] and etc. For example, in order to add a new matter to the minimal $SU(3)$ model for electroweak unification, it is developed by embedding in the exceptional Lie group $G_2$ [18]. The reason to choose this is that any such group must be at least rank 2 and

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contain the subgroup $SU(3)$. $G_2$ is not only a rank 2 group that contains $SU(3)$ but also it has the minimum number of generators with respect to any other group that properly involves $SU(3)$.

Many studies have investigated different aspects of the Lie group $G_2$ and the Lie algebra $g_2$. Some areas are listed here. The complex form of $G_2$ is the isotropy group of a generic 3-form in 7-dimensional complex space [19][20]. Whilst, the compact and noncompact real forms of the group $G_2$ appear as the automorphism groups of the octonion algebra and the split octonions, respectively [6][7]. In Refs. [21][22], it has been shown that the generators of $g_2$ can be appeared in the role of the derivations of a certain non-associative Cayley algebra. The Casimir operators of $g_2$, with the generators written in terms of $A_2$ and $B_3$ bases, have been considered in Refs. [19][23]. All finite-dimensional irreducible representations of $g_2$ have been realized in spaces of complex-valued polynomials of six variables [24].

Our goal here is to derive the commutation relations corresponding to the exceptional Lie algebra $g_2$ in the Cartan-Weyl basis from two different approaches, one is regularly used and the other is unknown. We focus on the relations that should be satisfied by the structure constants corresponding to the generators of the type of raising and lowering operators. We denote that the structure constants of the two bunches of commutation relations saturate the known equations but different from each other. However, it will be shown, by using an appropriate set of the linear relations between the two bunches of the Cartan-Weyl bases, that the commutation relations of the two different viewpoints are isomorphic to each other. Let us follow the problem in details in the next section.

2 Preliminaries

Let $g$ be a semisimple, complex and finite-dimensional Lie algebra. Suppose $h$ is a maximal abelian subalgebra of $g$ and $\Delta$ is the system of all non-zero roots of $g$ with respect to it. The non trivial subalgebra $h$ is called the Cartan subalgebra and consists of linearly independent and simultaneously ad-diagonalizable elements. This yields the triangular decomposition of $g$ with respect to $h$ [25][26][27],

$$g = h \oplus \bigoplus_{\alpha \in \Delta} g_\alpha,$$

(2.1)

in which $g_\alpha$’s are eigenspaces of the operators belonging to $\text{ad} h$,

$$\text{ad} H(E_\alpha) \equiv [H, E_\alpha] = \langle \alpha, H \rangle E_\alpha, \quad H \in h, \ E_\alpha \in g_\alpha. $$

(2.2)

The Cartan subalgebra $h$ is an eigenspace with the eigenvalue 0 and so $h = g_0$. The eigenvalue equation (2.2) offers a definition for the root $\alpha$ as the linear form $\alpha(H) \equiv \langle \alpha, H \rangle$ on $h$. The main properties of roots and root subspaces are listed as following:

- If $\alpha$ is a root belonging to $\Delta$, then so is $-\alpha$. For $-1 \neq a \in \mathbb{C}$, $a\alpha$ does not belong to the root system $\Delta$. This allows us to partition root system $\Delta$ into two subsets $\Delta_+$ and $\Delta_-$, containing the positive and negative roots, respectively: $\Delta = \Delta_+ \cup \Delta_-$. 

- For $\alpha, \beta, \alpha + \beta \in \Delta$ we have $[g_\alpha, g_\beta] \subset g_{\alpha + \beta}$ and $[g_\alpha, g_{-\alpha}] \subset h$. Also, $[g_\alpha, g_\beta] = 0$ if $\alpha + \beta$ is not a root of $\Delta$.

- All root subspaces $g_\alpha$ are one-dimensional. Therefore, we can assign a basis $E_\alpha$ to any root subspaces $g_\alpha$. 


• If $\alpha + \beta \neq 0$, then $E_\alpha$ and $E_\beta$ are orthogonal with respect to the Killing form $K(E_\alpha, E_\beta) \equiv \text{Tr}(\text{ad}E_\alpha \circ \text{ad}E_\beta)$.

• The restriction of $K$ to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate. For any $\alpha$, there is a unique element $H_\alpha \in \mathfrak{h}$ such that $K(H_\alpha, H) = \langle \alpha, H \rangle$ for all $H \in \mathfrak{h}$.

• The Killing form is nondegenerate on $\mathfrak{g} \times \mathfrak{g}$. The invariance of the Killing form fixes the normalization of the $E_{\pm \alpha}$ generators to one, $K(E_\alpha, E_{-\alpha}) = 1$. Then, we have $[E_\alpha, E_{-\alpha}] = H_\alpha$.

From the above considerations for $\mathfrak{g}$, the commutation relations in the Cartan-Weyl basis are resulted as

$$[H_\alpha, H_\beta] = 0, \quad [H_\alpha, E_\beta] = \langle \beta, H_\alpha \rangle E_\beta, \quad \alpha, \beta \in \Delta,$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \quad [E_\alpha, E_\beta] = N_{\alpha, \beta} E_{\alpha + \beta}, \quad \alpha + \beta \neq 0, \quad \alpha, \beta \in \Delta,$$

in which, $N_{\alpha, \beta} = 0$ if $\alpha + \beta$ is not a root. The root $\alpha \in \Delta$ is called simple if it can not be written as a sum of two positive roots. Let us collect the properties of simple roots $[28]$:

• Simple roots are linearly independent and their number coincides with the rank of $\mathfrak{g}$, which is, the dimension $l$ of the Cartan subalgebra $\mathfrak{h}$. We denote the set of simple roots by $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_l\} \subset \Delta_+$.

• The difference of two simple roots is not a root.

• Every positive root is a linear combination of the simple roots with non-negative integer coefficients.

For every simple root $\alpha_i$ one can consider the standard triple $\{H_{\alpha_i}, E_{\alpha_i}, F_{\alpha_i}\}$ in $\mathfrak{g}$ with $H_{\alpha_i} \in \mathfrak{h}$, $E_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$, and $F_{\alpha_i} \in \mathfrak{g}_{-\alpha_i}$. Also, a basis for the Cartan subalgebra $\mathfrak{h}$ can be proposed as $\{H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_l}\}$ such that $\langle \alpha, H_{\alpha_\beta} \rangle \in \mathbb{R}$, $i = 1, 2, \ldots, l$, for all $\alpha \in \Delta$. The structure constants in form of $\langle \beta, H_\alpha \rangle$ constitute the nonsingular Cartan matrix for when $\alpha$ and $\beta$ are chosen to be the simple roots, i.e. $a_{ij} = \langle \alpha_j, H_{\alpha_i} \rangle$, where $a_{ii} = 2$ and $a_{ij} \in \{0, -1, -2, -3\}$ for $i \neq j$. One can now construct the full algebra starting from the Cartan-Chevalley generators $H_{\alpha_i}$, $E_{\alpha_i}$, and $F_{\alpha_i}$, $i = 1, 2, \ldots, l$, subject to the following generating relations

$$[H_{\alpha_i}, H_{\alpha_j}] = 0, \quad [H_{\alpha_i}, E_{\alpha_j}] = a_{ij} E_{\alpha_j}, \quad [H_{\alpha_i}, F_{\alpha_j}] = -a_{ij} F_{\alpha_j}, \quad [E_{\alpha_i}, F_{\alpha_j}] = \delta_{ij} H_{\alpha_i}.$$

This presentation of the Lie algebra is completed by the so-called Serre relations

$$(\text{ad}E_{\alpha_i})^{1-a_{ij}} E_{\alpha_j} = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_{\alpha_i}^{1-a_{ij}-k} E_{\alpha_i} E_{\alpha_j}^k = 0, \quad i \neq j,$$

$$(\text{ad}F_{\alpha_i})^{1-a_{ij}} F_{\alpha_j} = \sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_{\alpha_i}^{1-a_{ij}-k} F_{\alpha_i} F_{\alpha_j}^k = 0, \quad i \neq j.$$

The structure constants in the Cartan-Chevalley basis are all integers and they are actually the elements of the Cartan matrix. The Lie algebra $\mathfrak{g}$ is spanned by the Cartan-Chevalley generators and the set of multiple commutators

$$[\cdots [E_{\alpha_{i_1}}, E_{\alpha_{i_2}}], E_{\alpha_{i_3}}], \cdots, E_{\alpha_{i_k}}], \quad [\cdots [F_{\alpha_{i_1}}, F_{\alpha_{i_2}}], F_{\alpha_{i_3}}], \cdots, F_{\alpha_{i_k}}],$$

restricted by the Serre relations $[270]$. Therefore, it can be resulted that the algebra $\mathfrak{g}$ is generated by the Chevalley generators $E_{\alpha_i}$ and $F_{\alpha_i}$ with $i = 1, 2, \ldots, l$.

Now, let $\alpha$, $\beta \neq \pm \alpha$ and $\alpha + \beta$ be roots. We remind the facts presented in the literature concerning the nonzero structure constants $N_{\alpha, \beta}$ (see, e.g., Refs. $[29]$ $[30]$):
• The constants $N_{\alpha,\beta}$ are real and skew symmetric.
• Jacobi identity: $N_{\alpha,\beta}N_{\gamma,\eta} + N_{\beta,\gamma}N_{\alpha,\eta} + N_{\gamma,\alpha}N_{\beta,\eta} = 0$, where $\eta = -\alpha - \beta - \gamma$, and $\eta$ is not one of $-\alpha, -\beta$ and $-\gamma$.
• If we assume the Cartan generators to be self-adjoint and $E_\alpha$ and $F_\alpha$ to be Hermitian conjugate of each other as well as the structure constants $\langle \beta, H_\alpha \rangle$ and $N_{\alpha,\beta}$ to be real then, we will obtain
  \[ N_{\alpha,\beta} = -N_{-\alpha,-\beta}. \] (2.8)
• By using the properties of Killing form and the normalization condition $K(E_\alpha, E_{-\alpha}) = 1$, it is shown that $H_\alpha + H_\beta + H_{-\alpha-\beta} = 0$. This relation, in turn, leads to
  \[ N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}. \] (2.9)
• Considering the $\alpha$-chain through $\beta$,
  \[ \beta - p\alpha, \cdots , \beta - \alpha, \beta + \alpha, \cdots , \beta + q\alpha, \] (2.10)
in which, the nonnegative integer numbers $p$ and $q$ are defined by $p - q = \langle \beta, H_\alpha \rangle$, we have
  \[ N_{\alpha,\beta}N_{-\alpha,-\beta} = -\frac{1}{2}q(p + 1)\langle \alpha, H_\alpha \rangle. \] (2.11)

In what follows we would like to respond to the question “whether the structure constants in the well known commutation relations of the exceptional Lie algebra $\mathfrak{g}_2$ saturate simultaneously the relations (2.8), (2.9) and (2.11)”. For this, in Sections 2 and 3, we consider two different approaches to the extraction of the commutation relations of $\mathfrak{g}_2$, where one obeys (2.8) and rejects (2.9) and (2.11), and the other, vice versa. It is found that the structure constants of the type of $\langle \beta, H_\alpha \rangle$ are computed directly from the Cartan matrix in the former case, while in the latter case it does not. In any case, the Cartan subalgebra generators are defined as the first relation of (2.4).

3 The commutation relations of $\mathfrak{g}_2$ based on (2.8)

The starting point is the Dynkin diagram of the rank-two exceptional Lie algebra $\mathfrak{g}_2$: $\circ \equiv \circ$. The number of lines shows that the allowed angle between two roots is $5\pi/6$. There are only two simple roots, say $\alpha_1$ and $\alpha_2$, and we may choose them such that $|\alpha_1/\alpha_2| = \sqrt{3}$. The Dynkin diagram determines the Cartan matrix as $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ (see, e.g., [26, 30, 31, 32]). It means
  \[ \langle \alpha_1, H_{\alpha_1} \rangle = \langle \alpha_2, H_{\alpha_2} \rangle = 2, \quad \langle \alpha_1, H_{\alpha_2} \rangle = -1, \quad \langle \alpha_2, H_{\alpha_1} \rangle = -3. \] (3.1)

From the two last relations of (3.1), we find that $\alpha_1$-chain through $\alpha_2$ and the $\alpha_2$-chain through $\alpha_1$ involve the positive roots $\alpha_2$ and $\alpha_2 + \alpha_1$ as well as $\alpha_1$, $\alpha_1 + \alpha_2$, $\alpha_1 + 2\alpha_2$ and $\alpha_1 + 3\alpha_2$, respectively. For the last root of the second chain we have $\langle \alpha_1 + 3\alpha_2, H_{\alpha_1} \rangle = -1$. This leads us to the $\alpha_1$-chain through $\alpha_1 + 3\alpha_2$ with the positive roots $\alpha_1 + 3\alpha_2$ and $2\alpha_1 + 3\alpha_2$. Then, for $\mathfrak{g}_2$, there exist six positive roots: $\Delta^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$. Therefore, based on the triangular decomposition, the exceptional Lie algebra $\mathfrak{g}_2$ involves the fourteen Cartan-Weyl generators as
  \[ H_{\alpha_1}, H_{\alpha_2}, E_{\alpha_1}, E_{\alpha_2}, E_{\alpha_1+\alpha_2}, E_{\alpha_1+2\alpha_2}, E_{\alpha_1+3\alpha_2}, E_{2\alpha_1+3\alpha_2}, F_{\alpha_1}, F_{\alpha_2}, F_{\alpha_1+\alpha_2}, F_{\alpha_1+2\alpha_2}, F_{\alpha_1+3\alpha_2}, F_{2\alpha_1+3\alpha_2}. \] (3.2)
Therefore, it is necessary to present the well known two- and four-dimensional representations of the standard triples of the exceptional Lie algebra of spin representations of the special unitary group $SU$. In what follows, we use the procedure proposed in Ref. [31], which is based on the use of spin representations of the special unitary group $SU(2)$, to obtain the rest of the commutation relations of the exceptional Lie algebra $\mathfrak{g}_2$. According to the relations (3.3) and (3.4), every one of the standard triples $\{H_\alpha, E_\alpha, F_\alpha\}$ and $\{H_{\alpha_1}, E_{\alpha_1}, F_{\alpha_1}\}$ constitutes a copy of $su(2)$ commutation relations

$$[E_{\alpha_1}, F_{\alpha_1}] = H_{\alpha_1}, \quad [E_{\alpha_2}, F_{\alpha_2}] = H_{\alpha_2}, \quad [H_{\alpha_1}, E_{\alpha_1}] = 2E_{\alpha_1}, \quad [H_{\alpha_1}, F_{\alpha_1}] = -2F_{\alpha_1} \quad (3.3)$$

$$[H_{\alpha_1}, E_{\alpha_2}] = -E_{\alpha_2}, \quad [H_{\alpha_1}, F_{\alpha_2}] = -3E_{\alpha_1}, \quad [H_{\alpha_2}, E_{\alpha_1}] = F_{\alpha_2}, \quad [H_{\alpha_2}, F_{\alpha_1}] = 3F_{\alpha_1} \quad (3.4)$$

respectively. In what follows, we use the procedure proposed in Ref. [31], which is based on the use of spin representations of the special unitary group $SU(2)$, to obtain the rest of the commutation relations of the exceptional Lie algebra $\mathfrak{g}_2$. According to the relations (3.3) and (3.4), every one of the standard triples $\{H_\alpha, E_\alpha, F_\alpha\}$ and $\{H_{\alpha_1}, E_{\alpha_1}, F_{\alpha_1}\}$ constitutes a copy of $su(2)$ commutation relations

$$[J_z, J_+] = J_+, \quad [J_z, J_-] = -J_-, \quad [J_+, J_-] = 2J_z \quad (3.10)$$

Therefore, it is necessary to present the well known two- and four-dimensional representations of $su(2)$ Lie algebra as below

$$J_\pm \left| \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right> = \left| \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right>, \quad J_\pm \left| \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right> = 0 \quad (3.11)$$

$$J_\pm \left| \begin{array}{c} \frac{3}{2} \\ \frac{3}{2} \end{array} \right> = \sqrt{3} \left| \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \end{array} \right>, \quad J_\pm \left| \begin{array}{c} \frac{3}{2} \\ \frac{3}{2} \end{array} \right> = 2 \left| \begin{array}{c} \frac{3}{2} \\ \frac{1}{2} \end{array} \right>, \quad J_\pm \left| \begin{array}{c} \frac{3}{2} \\ \frac{3}{2} \end{array} \right> = 0 \quad (3.12)$$

Comparing the relations (3.3) and (3.4) with (3.10), it is found that the two-, four- and two-dimensional chains (3.7), (3.8) and (3.9) for the generators can be corresponded to the following definitions

$$E_{\alpha_2} = \left| \begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{array} \right>, \quad E_{\alpha_1+\alpha_2} = \left| \begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ 2 \end{array} \right>, \quad \text{ad}H_{\alpha_1} = 2J_z, \quad \text{ad}E_{\alpha_1} = -\frac{3}{2}J_+, \quad \text{ad}F_{\alpha_1} = -\sqrt{3}J_- \quad (3.13)$$

$$E_{\alpha_1} = \left| \begin{array}{c} \frac{3}{2} \\ -\frac{3}{2} \\ 2 \end{array} \right>, \quad E_{\alpha_1+\alpha_2} = \left| \begin{array}{c} \frac{3}{2} \\ -\frac{3}{2} \\ 3 \end{array} \right>, \quad E_{\alpha_1+2\alpha_2} = \left| \begin{array}{c} \frac{3}{2} \\ -\frac{3}{2} \\ 4 \end{array} \right>, \quad E_{\alpha_1+3\alpha_2} = \left| \begin{array}{c} \frac{3}{2} \\ -\frac{3}{2} \\ 5 \end{array} \right>, \quad \text{ad}H_{\alpha_2} = 2J_z, \quad \text{ad}E_{\alpha_2} = \frac{J_+}{\sqrt{2}}, \quad \text{ad}F_{\alpha_2} = \sqrt{2}J_- \quad (3.14)$$

$$E_{\alpha_1+3\alpha_2} = \left| \begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ 3 \end{array} \right>, \quad E_{2\alpha_1+3\alpha_2} = \left| \begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \\ 4 \end{array} \right>, \quad \text{ad}H_{\alpha_1} = 2J_z, \quad \text{ad}E_{\alpha_1} = \sqrt{\frac{3}{2}}J_+, \quad \text{ad}F_{\alpha_1} = \sqrt{\frac{3}{2}}J_- \quad (3.15)$$
respectively. Here, the labels 1, 2 and 3 on the right end of the kets denote the chain issue. From now on, for simplicity, the skew symmetric property is implicitly used to determine the structure constants. From (3.14) and (3.15) we immediately get

$$\text{ad}E_\alpha E_\alpha = \frac{J_+}{\sqrt{2}} \frac{1}{2} \begin{bmatrix} -2 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \equiv \sqrt{\frac{3}{2}} E_{\alpha+2} \Rightarrow \{E_\alpha, E_\alpha\} = \sqrt{\frac{3}{2}} E_{\alpha+2}, \quad (3.16)$$

$$\text{ad}E_\alpha E_{\alpha+2} = \frac{J_+}{\sqrt{2}} \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \equiv \sqrt{\frac{3}{2}} E_{\alpha+3} \Rightarrow \{E_\alpha, E_{\alpha+2}\} = \sqrt{\frac{3}{2}} E_{\alpha+3}, \quad (3.17)$$

$$\text{ad}E_\alpha E_{\alpha+3} = \frac{J_+}{\sqrt{2}} \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \equiv \sqrt{\frac{3}{2}} E_{\alpha+3} \Rightarrow \{E_\alpha, E_{\alpha+3}\} = \sqrt{\frac{3}{2}} E_{\alpha+3}, \quad (3.18)$$

Note that the result (3.16) can be also obtained by using (3.13). The condition (2.8) using the latter four relations will enable us to determine the following four commutation relations

$$\{F_\alpha, F_\alpha\} = -\sqrt{\frac{3}{2}} F_{\alpha+2}, \quad \{F_\alpha, F_{\alpha+2}\} = -\sqrt{\frac{3}{2}} F_{\alpha+3}, \quad (3.20)$$

The next commutation relations follow easily from the fact that the number of the positive roots of $$g_2$$ is exactly six:

$$\{E_\alpha, E_{\alpha+2}\} = \{E_\alpha, E_{\alpha+3}\} = 0$$

$$\{E_\alpha, E_{\alpha+2}\} = \{E_\alpha, E_{\alpha+3}\} = 0$$

$$\{E_\alpha, E_{\alpha+2}\} = \{E_\alpha, E_{\alpha+3}\} = 0$$

$$\{E_\alpha, E_{\alpha+2}\} = \{E_\alpha, E_{\alpha+3}\} = 0$$

Similar relations are also held for the F-generators corresponding to the negative roots. The above relations, together with the Jacobi identity, give rise to

$$\{E_\alpha, E_{\alpha+2}\} = -\sqrt{\frac{3}{2}} \{E_\alpha, E_{\alpha+2}\}$$

$$\{E_\alpha, E_{\alpha+3}\} = \sqrt{\frac{3}{2}} \{E_\alpha, E_{\alpha+3}\}$$

In the same way, from the Jacobi identity and earlier commutation relations we find

$$\{H_\alpha, E_{\alpha+2}\} = 2H_\alpha + \frac{2}{3} H_\alpha$$

$$\{H_\alpha, E_{\alpha+3}\} = 4H_\alpha + \frac{2}{3} H_\alpha$$

$$\{H_\alpha, E_{\alpha+3}\} = 8H_\alpha + 8H_\alpha$$

In the same way, from the Jacobi identity and earlier commutation relations we find

$$\{H_\alpha, E_{\alpha+2}\} = 2H_\alpha + \frac{2}{3} H_\alpha,$$

$$\{H_\alpha, E_{\alpha+3}\} = 4H_\alpha + \frac{2}{3} H_\alpha,$$

$$\{H_\alpha, E_{\alpha+3}\} = 8H_\alpha + 8H_\alpha.$$
Now, with the help of (3.24) we can directly calculate the following commutation relations without the use of the Jacobi identity

\[
\begin{align*}
[F, H_1] & = -\frac{2}{3} E_1, \\
[H_1, E_1] & = 2 E_1, \\
[H_1, E_2] & = \frac{4}{3} F_2, \\
[H_1, F_2] & = 0,
\end{align*}
\]

(3.25)

\[
\begin{align*}
[F_0, E_{01+2}] & = -\sqrt{\frac{2}{3}} E_{02}, \\
[F_0, E_{01+3}] & = \sqrt{\frac{2}{3}} E_{02}, \\
[F_0, E_{01+4}] & = \sqrt{\frac{2}{3}} E_{02}, \\
[F_0, E_{01+5}] & = 0.
\end{align*}
\]

(3.26)

Now, with the help of (3.24) we can directly calculate the following commutation relations without the use of the Jacobi identity

\[
\begin{align*}
[H_1, E_1] & = 2 E_1, \\
[H_1, E_2] & = \frac{4}{3} F_2, \\
[H_1, F_2] & = 0,
\end{align*}
\]

(3.27)

The condition (2.8) together with the Jacobi identity, leads to

\[
\begin{align*}
[E_1, F_1] & = \frac{\sqrt{2}}{3} F_2, \\
[E_1, F_2] & = \frac{\sqrt{2}}{3} F_2, \\
[E_1, F_3] & = 0, \\
[E_1, F_4] & = \frac{\sqrt{2}}{3} F_2,
\end{align*}
\]

(3.28)

Now, we can calculate the commutator of elements of the Cartan subalgebra with the \(F\)-generators corresponding to a positive and non-simple root. For example,

\[
\begin{align*}
[H_2, F_1] & = \sqrt{\frac{2}{3}} [H_2, F_1], \\
[H_2, F_2] & = -\sqrt{\frac{2}{3}} [H_2, F_2], \\
[H_2, F_3] & = -\sqrt{\frac{2}{3}} [H_2, F_3], \\
[H_2, F_4] & = \sqrt{\frac{2}{3}} [H_2, F_4],
\end{align*}
\]

(3.29)
By rescaling the generators corresponding to the positive non-simple roots as

\[
\begin{align*}
X_{\alpha_1} &\equiv E_{\alpha_1}, & X_{\alpha_2} &\equiv E_{\alpha_2}, & X_{\alpha_1+\alpha_2} &\equiv -\sqrt{\frac{3}{2}}E_{\alpha_1+\alpha_2}, \\
X_{\alpha_1+2\alpha_2} &\equiv -\frac{\sqrt{3}}{2}E_{\alpha_1+2\alpha_2}, & X_{\alpha_1+3\alpha_2} &\equiv -\sqrt{\frac{3}{2}}E_{\alpha_1+3\alpha_2}, & X_{2\alpha_1+3\alpha_2} &\equiv -\sqrt{\frac{3}{4}}E_{2\alpha_1+3\alpha_2}, \\
X_{\alpha_1+2\alpha_2} &\equiv -\sqrt{\frac{3}{2}}E_{\alpha_1+2\alpha_2}, & X_{\alpha_1+3\alpha_2} &\equiv -\sqrt{\frac{3}{2}}E_{\alpha_1+3\alpha_2}, & X_{2\alpha_1+3\alpha_2} &\equiv -\sqrt{\frac{3}{4}}E_{2\alpha_1+3\alpha_2}, \\
\end{align*}
\]

(3.30)

and likewise for the negative roots denoted by \(Y\), we obtain the familiar commutation relations of the exceptional Lie algebra \(g_2\), as shown in Table 1 (see e.g., Refs. [24, 32, 33, 34]).

4 The commutation relations of \(g_2\) based on (2.9) and (2.11)

In this section we will denote the Cartan-Chevalley generators by \(H'_{\alpha_i}\), \(X'_{\alpha_i}\) and \(Y'_{\alpha_i}\), which correspond respectively to the zero and the simple (positive and negative) roots. The conditions (2.9) for the root system of the exceptional Lie algebra \(g_2\) can be explicitly described as follows

\[
\begin{align*}
N_{\alpha_1,\alpha_2} &= N_{\alpha_2,-\alpha_1-\alpha_2} = N_{-\alpha_1-\alpha_2,\alpha_1}, \\
N_{-\alpha_1,-\alpha_2} &= N_{-\alpha_2,\alpha_1+\alpha_2} = N_{\alpha_1+\alpha_2,-\alpha_1}, \\
N_{\alpha_1,\alpha_1+3\alpha_2} &= N_{\alpha_1+3\alpha_2,-2\alpha_1-3\alpha_2} = N_{-2\alpha_1-3\alpha_2,\alpha_1}, \\
N_{-\alpha_1,-\alpha_1+3\alpha_2} &= N_{-\alpha_1-3\alpha_2,2\alpha_1+3\alpha_2} = N_{2\alpha_1+3\alpha_2,-\alpha_1}, \\
N_{\alpha_2,\alpha_1+\alpha_2} &= N_{\alpha_1+\alpha_2,-\alpha_1-2\alpha_2} = N_{-\alpha_1-2\alpha_2,\alpha_2}, \\
N_{-\alpha_2,-\alpha_1-\alpha_2} &= N_{-\alpha_1-2\alpha_2,\alpha_1+2\alpha_2} = N_{\alpha_1+2\alpha_2,-\alpha_2}, \\
N_{\alpha_2,\alpha_1+2\alpha_2} &= N_{\alpha_1+2\alpha_2,-\alpha_1-3\alpha_2} = N_{-\alpha_1-3\alpha_2,\alpha_2}, \\
N_{-\alpha_2,-\alpha_1-2\alpha_2} &= N_{-\alpha_1-2\alpha_2,\alpha_1+3\alpha_2} = N_{\alpha_1+3\alpha_2,-\alpha_2}, \\
N_{\alpha_1+2\alpha_2,\alpha_1+2\alpha_2} &= N_{\alpha_1+2\alpha_2,-2\alpha_1-3\alpha_2} = N_{-2\alpha_1-3\alpha_2,\alpha_1+\alpha_2}, \\
N_{-\alpha_1-\alpha_2,-\alpha_1-2\alpha_2} &= N_{-\alpha_1-2\alpha_2,\alpha_1+3\alpha_2} = N_{2\alpha_1+3\alpha_2,-\alpha_1-\alpha_2}. \\
\end{align*}
\]

(4.1)
One can investigate and see that there exist fifteen various root chains of the type (2.10), each of them corresponds to one relation in the form of (2.11):

\[
\begin{align*}
\{ & \beta = \alpha_2, \alpha = \alpha_1, p = 0, q = 1 \quad & \beta = \alpha_1, \alpha = \alpha_2, p = 0, q = 3, \\
& N_{\alpha_1,\alpha_2} N_{-\alpha_1,-\alpha_2} = \frac{-1}{2} \langle \alpha_1, H_{\alpha_1}^\prime \rangle = \frac{-3}{2} \langle \alpha_2, H_{\alpha_2}^\prime \rangle, \\
\{ & \beta = \alpha_1 + 3 \alpha_2, \alpha = \alpha_1, p = 0, q = 1 \quad & \beta = \alpha_1, \alpha = \alpha_1 + 3 \alpha_2, p = 0, q = 1, \\
& N_{\alpha_1,\alpha_1+3\alpha_2} N_{-\alpha_1,-\alpha_1-3\alpha_2} = \frac{-1}{2} \langle \alpha_1, H_{\alpha_1}^\prime \rangle = \frac{-1}{2} \langle \alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle, \\
\{ & \beta = -\alpha_1 - \alpha_2, \alpha = \alpha_1, p = 0, q = 1 \quad & \beta = \alpha_1, \alpha = -\alpha_1 - \alpha_2, p = 0, q = 3, \\
& N_{\alpha_1,-\alpha_1-\alpha_2} N_{-\alpha_1,\alpha_1+\alpha_2} = \frac{-1}{2} \langle \alpha_1, H_{\alpha_1}^\prime \rangle = \frac{-3}{2} \langle \alpha_1 + \alpha_2, H_{\alpha_1+\alpha_2}^\prime \rangle, \\
\{ & \beta = -2\alpha_1 - 3 \alpha_2, \alpha = \alpha_1, p = 0, q = 1 \quad & \beta = \alpha_1, \alpha = -2\alpha_1 - 3 \alpha_2, p = 0, q = 1, \\
& N_{\alpha_1,-2\alpha_1-3\alpha_2} N_{-\alpha_1,2\alpha_1+3\alpha_2} = \frac{-1}{2} \langle \alpha_1, H_{\alpha_1}^\prime \rangle = \frac{-1}{2} \langle 2\alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle, \\
\{ & \beta = \alpha_1 + \alpha_2, \alpha = \alpha_2, p = 1, q = 2 \quad & \beta = \alpha_2, \alpha = \alpha_1 + \alpha_2, p = 1, q = 2, \\
& N_{\alpha_2,\alpha_1+\alpha_2} N_{-\alpha_2,-\alpha_1-\alpha_2} = -2 \langle \alpha_2, H_{\alpha_2}^\prime \rangle = -2 \langle \alpha_1 + \alpha_2, H_{\alpha_1+\alpha_2}^\prime \rangle, \\
\{ & \beta = \alpha_1 + 2 \alpha_2, \alpha = \alpha_2, p = 2, q = 1 \quad & \beta = \alpha_2, \alpha = \alpha_1 + 2 \alpha_2, p = 2, q = 1, \\
& N_{\alpha_2,\alpha_1+2\alpha_2} N_{-\alpha_2,-\alpha_1-2\alpha_2} = \frac{-3}{2} \langle \alpha_2, H_{\alpha_2}^\prime \rangle = \frac{-3}{2} \langle \alpha_1 + \alpha_2, H_{\alpha_1+2\alpha_2}^\prime \rangle, \\
\{ & \beta = -\alpha_1 - 3 \alpha_2, \alpha = \alpha_2, p = 0, q = 3 \quad & \beta = \alpha_2, \alpha = -\alpha_1 - 3 \alpha_2, p = 0, q = 1, \\
& N_{\alpha_2,-\alpha_1-3\alpha_2} N_{-\alpha_2,\alpha_1+3\alpha_2} = \frac{-3}{2} \langle \alpha_2, H_{\alpha_2}^\prime \rangle = \frac{-3}{2} \langle \alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle, \\
\{ & \beta = \alpha_1 + 2 \alpha_2, \alpha = \alpha_1 + \alpha_2, p = 2, q = 1 \quad & \beta = \alpha_1 + \alpha_2, \alpha = \alpha_1 + 2 \alpha_2, p = 2, q = 1, \\
& N_{\alpha_1,\alpha_1+2\alpha_2} N_{-\alpha_1,-\alpha_1-2\alpha_2} = \frac{-3}{2} \langle \alpha_1 + \alpha_2, H_{\alpha_1+2\alpha_2}^\prime \rangle = \frac{-3}{2} \langle 2\alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle, \\
\{ & \beta = -\alpha_1 - 3 \alpha_2, \alpha = \alpha_1 + 2 \alpha_2, p = 0, q = 3 \quad & \beta = \alpha_1 + 2 \alpha_2, \alpha = -\alpha_1 - 3 \alpha_2, p = 0, q = 1, \\
& N_{\alpha_1+2\alpha_2,-\alpha_1-3\alpha_2} N_{-\alpha_1-2\alpha_2,\alpha_1+3\alpha_2} = \frac{-3}{2} \langle \alpha_1 + 2 \alpha_2, H_{\alpha_1+2\alpha_2}^\prime \rangle = \frac{-3}{2} \langle \alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle, \\
\{ & \beta = -2\alpha_1 - 3 \alpha_2, \alpha = \alpha_1 + 3 \alpha_2, p = 0, q = 1 \quad & \beta = \alpha_1 + 3 \alpha_2, \alpha = -2\alpha_1 - 3 \alpha_2, p = 0, q = 1, \\
& N_{\alpha_1+3\alpha_2,-2\alpha_1-3\alpha_2} N_{-\alpha_1-3\alpha_2,\alpha_1+3\alpha_2} = \frac{-3}{2} \langle \alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle = \frac{-3}{2} \langle 2\alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle, \\
\{ & \beta = \alpha_1 + 3 \alpha_2, \alpha = -\alpha_1 - 2 \alpha_2, p = 0, q = 3 \quad & \beta = -\alpha_1 - 2 \alpha_2, \alpha = \alpha_1 + 3 \alpha_2, p = 0, q = 1, \\
& N_{\alpha_1,-2\alpha_1-3\alpha_2} N_{-\alpha_1,3\alpha_2,\alpha_1+3\alpha_2} = \frac{-3}{2} \langle \alpha_1 + 2 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle = \frac{-3}{2} \langle \alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle, \\
\{ & \beta = 2\alpha_1 + 3 \alpha_2, \alpha = -\alpha_1 - 3 \alpha_2, p = 0, q = 3 \quad & \beta = -\alpha_1 - 3 \alpha_2, \alpha = 2\alpha_1 + 3 \alpha_2, p = 0, q = 1, \\
& N_{-\alpha_1-3\alpha_2,2\alpha_1+3\alpha_2} N_{-\alpha_1,3\alpha_2,\alpha_1+3\alpha_2} = \frac{-3}{2} \langle \alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle = \frac{-3}{2} \langle 2\alpha_1 + 3 \alpha_2, H_{\alpha_1+3\alpha_2}^\prime \rangle, \\
\{ & \beta = -\alpha_1 - \alpha_2, \alpha = \alpha_2, p = 2, q = 1 \quad & \beta = \alpha_2, \alpha = -\alpha_1 - \alpha_2, p = 2, q = 1, \\
& N_{\alpha_2,-\alpha_1-\alpha_2} N_{-\alpha_2,\alpha_1+\alpha_2} = \frac{-3}{2} \langle \alpha_2, H_{\alpha_2}^\prime \rangle = \frac{-3}{2} \langle \alpha_1 + \alpha_2, H_{\alpha_1+\alpha_2}^\prime \rangle, \\
\{ & \beta = -\alpha_1 - 2 \alpha_2, \alpha = \alpha_2, p = 1, q = 2 \quad & \beta = \alpha_2, \alpha = -\alpha_1 - 2 \alpha_2, p = 1, q = 2, \\
& N_{\alpha_2,-\alpha_1-2\alpha_2} N_{-\alpha_2,\alpha_1+2\alpha_2} = -2 \langle \alpha_2, H_{\alpha_2}^\prime \rangle = -2 \langle \alpha_1 + 2 \alpha_2, H_{\alpha_1+2\alpha_2}^\prime \rangle,
\end{align*}
\]
Similarly, we get realization of the constraints (2.9) and (2.11).

\[
\langle 2\alpha_1 + 3\alpha_2, H'_{\alpha_1+3\alpha_2} \rangle = \langle \alpha_1 + 3\alpha_2, H'_{\alpha_1+3\alpha_2} \rangle = \langle \alpha_1, H'_{\alpha_1} \rangle = 3\langle \alpha_1 + \alpha_2, H'_{\alpha_1+\alpha_2} \rangle = 3(\alpha_1 + 2\alpha_2, H'_{\alpha_1+2\alpha_2}) = 3\langle \alpha_2, H'_{\alpha_2} \rangle = 1.
\]

(4.3)

Substituting the results (4.3) into equations (4.1) and (4.2), we obtain the following values for the structure constants in the form of \(N_{\alpha,\beta}\),

\[
N_{\alpha_1,\alpha_2} = N_{\alpha_1,\alpha_1+3\alpha_2} = -N_{\alpha_1,-\alpha_1-\alpha_2} = -N_{\alpha_1,-2\alpha_1-3\alpha_2} = N_{\alpha_2,\alpha_1+2\alpha_2} = N_{\alpha_2,-\alpha_1-2\alpha_2} = -N_{\alpha_2,-\alpha_1-3\alpha_2} = N_{\alpha_1+2\alpha_2,-\alpha_1-3\alpha_2} = 1,
\]

(4.4)

\[
N_{\alpha_1+\alpha_2,-\alpha_2} = -N_{\alpha_1+\alpha_2,-\alpha_1} = -N_{\alpha_1+3\alpha_2,-\alpha_2} = N_{\alpha_1+3\alpha_2,-\alpha_1-2\alpha_2} = N_{\alpha_1+3\alpha_2,-\alpha_1-3\alpha_2} = -N_{\alpha_1,-\alpha_1-3\alpha_2} = -N_{\alpha_1,-\alpha_2-\alpha_1-3\alpha_2} = -N_{\alpha_1,-\alpha_2-\alpha_1-3\alpha_2} = 1,
\]

(4.5)

\[
-N_{\alpha_1+2\alpha_2,-\alpha_2} = N_{\alpha_1+2\alpha_2,-\alpha_1-\alpha_2} = -N_{\alpha_1+2\alpha_2,-\alpha_1-2\alpha_2} = -N_{\alpha_1+2\alpha_2,-\alpha_1-3\alpha_2} = 2/3.
\]

(4.6)

The first relation of (2.4) for the root \(\alpha_1 + \alpha_2\), as an example, follows from the last results as

\[
H'_{\alpha_1+\alpha_2} = [X'_{\alpha_1+\alpha_2}, Y'_{\alpha_1+\alpha_2}] = [[X'_{\alpha_1}, Y'_{\alpha_2}], Y'_{\alpha_1+\alpha_2}]

= -[[Y'_{\alpha_1+\alpha_2}, X'_{\alpha_1}], X'_{\alpha_2}] - [[X'_{\alpha_2}, Y'_{\alpha_1+\alpha_2}], Y'_{\alpha_1}]

= -[Y'_{\alpha_2}, X'_{\alpha_2}] - [Y'_{\alpha_1}, X'_{\alpha_1}] = H'_{\alpha_2} + H'_{\alpha_1}.
\]

(4.7)

Similarly, we get

\[
H'_{\alpha_1+2\alpha_2} = H'_{\alpha_1} + 2H'_{\alpha_2},
\]

\[
H'_{\alpha_1+3\alpha_2} = H'_{\alpha_1} + 3H'_{\alpha_2},
\]

\[
H'_{2\alpha_1+3\alpha_2} = 2H'_{\alpha_1} + 3H'_{\alpha_2}.
\]

(4.8)

Again, as an example, for the roots \(\alpha = \alpha_1\) and \(\beta = \alpha_2\), the second relation of (2.3) is considered in the following method

\[
[H'_{\alpha_1}, X'_{\alpha_2}] = [[X'_{\alpha_1}, Y'_{\alpha_2}], X'_{\alpha_2}] = -[[X'_{\alpha_2}, X'_{\alpha_1}], Y'_{\alpha_1}] - [[Y'_{\alpha_1}, X'_{\alpha_2}], X'_{\alpha_1}]

= [X'_{\alpha_1+\alpha_2}, Y'_{\alpha_1}] = -\frac{1}{2}X'_{\alpha_2}.
\]

(4.9)

This implies \(\langle \alpha_2, H'_{\alpha_1} \rangle = -\frac{1}{2}\) which is incompatible with (3.1). Therefore, realization of the constraints (2.9) and (2.11) violates the negative integer values for the Cartan numbers. All other structure constants in the form of \(\langle \beta, H'_{\alpha} \rangle\) can be calculated in the same way as in (4.9). We recall in Table 2 the unfamiliar commutation relations of the exceptional Lie algebra \(g_2\) based on the realization of the constraints (2.9) and (2.11).
5 Conclusions

Finally, we end the paper by comparing Tables 1 and 2. The commutation relations in Table 1 obey (2.8) and (3.1) and violate (2.9) and (2.11). For Table 2 it is vice versa. In both Tables, the Cartan subalgebra generators are defined as the first relation of (2.4). Furthermore, linear relations as

\[
\begin{align*}
X'_{\alpha_1} &= \sqrt{3} X_{\alpha_1}, \\
X'_{\alpha_2} &= -\frac{1}{\sqrt{3}} X_{\alpha_2}, \\
X'_{\alpha_1+\alpha_2} &= -\sqrt{\frac{2}{3}} X_{\alpha_1+\alpha_2}, \\
X'_{\alpha_1+2\alpha_2} &= 2\sqrt{\frac{2}{3}} X_{\alpha_1+2\alpha_2}, \\
Y'_{\alpha_1} &= \frac{1}{2\sqrt{2}} Y_{\alpha_1}, \\
Y'_{\alpha_2} &= \frac{1}{2\sqrt{2}} Y_{\alpha_2}, \\
Y'_{\alpha_1+2\alpha_2} &= \frac{1}{2\sqrt{2}} Y_{\alpha_1+2\alpha_2}, \\
Y'_{\alpha_1+3\alpha_2} &= \frac{1}{2\sqrt{2}} Y_{\alpha_1+3\alpha_2}, \\
H'_{\alpha_1} &= \frac{1}{2} H_{\alpha_1}, \\
H'_{\alpha_2} &= \frac{1}{2} H_{\alpha_2}, \\
H'_{\alpha_1+\alpha_2} &= \frac{1}{2} H_{\alpha_1+\alpha_2}, \\
H'_{\alpha_1+2\alpha_2} &= \frac{1}{2} H_{\alpha_1+2\alpha_2}, \\
H'_{\alpha_1+3\alpha_2} &= \frac{1}{2} H_{\alpha_1+3\alpha_2},
\end{align*}
\]

hold an isomorphism between Tables 1 and 2.

References

[1] W. Killing, Die Zusammensetzung der stetigen endlichen Transformationsgruppen I, Math. Ann. 31 (1888) 252-290.

[2] W. Killing, Die Zusammensetzung der stetigen endlichen Transformationsgruppen II, Math. Ann. 33 (1889) 1-48.

[3] W. Killing, Die Zusammensetzung der stetigen endlichen Transformationsgruppen III, Math. Ann. 34 (1889) 57-122.

[4] W. Killing, Die Zusammensetzung der stetigen endlichen Transformationsgruppen IV, Math. Ann. 36 (1890) 161-189.

[5] E. Cartan, Sur la structure des groupes de tranformations finis et continus, These, Paris, Nony, 1894.

[6] E. Cartan, Nombres complexes, in Encyclopedie des sciences mathematiques, 1, ed. J. Molk (1908) 329-468.

[7] E. Cartan, Les groupes reels simples finis et continus, Ann. Sci. Ecole Norm. Sup. 31 (1914) 255-262.

[8] F. Adams, Lectures on Exceptional Lie Groups, Chicago Lectures in Mathematics, University of Chicago Press, 1996.

[9] E. K. Loginov, Multi-instantons in higher dimensions and superstring solitons, SIGMA 1 (2005) 002.

[10] B. H. Wellegehausen, A. Wipf and C. Wozar, Effective Polyakov loop dynamics for finite temperature $G_2$ gluodynamics, Phys. Rev. D 80 (2009) 065028.

[11] B. H. Wellegehausen, A. Wipf and C. Wozar, Casimir scaling and string breaking in $G_2$ gluodynamics, Phys. Rev. D 83 (2011) 016001.
[12] E. M. Ilgenfritz and A. Maas, Topological aspects of $G_2$ Yang-Mills theory, Phys. Rev. D 86 (2012) 114508.

[13] H. N. Saearp, Generalised Chern-Simons theory and $G_2$-instantons over associative fibrations, SIGMA 10 (2014) 083.

[14] Ya. Shnir and G. Zhilin, $G_2$ monopoles, Phys. Rev. D 92 (2015) 045025.

[15] T. House and A. Lukas, $G_2$ domain walls in M theory, Phys. Rev. D 71 (2005) 046006.

[16] T. House and A. Micu, M-theory compactifications on manifolds with $G_2$ structure, Class. Quant. Grav. 22 (2005) 1709-1738.

[17] Y. Herfray, K. Krasnov, C. Scarinci and Y. Shtanov, A 4D gravity theory and $G_2$-holonomy manifolds, arXiv:1602.03428 [hep-th].

[18] C. D. Carone and A. Rastogi, An exceptional electroweak model, Phys. Rev. D 77 (2008) 035011.

[19] A. M. Bincer and K. Riesselmann, Casimir operators of the exceptional group $G_2$, J. Math. Phys. 34 (1993) 5935-5941.

[20] S. Grigorian and S. T. Yau, Local geometry of the $G_2$ moduli space, Commun. Math. Phys. 287 (2009) 459-488.

[21] N. Jacobson, Lie Algebras, J. Wiley & Sons, New York, 1962.

[22] N. Jacobson, Exceptional Lie Algebras, Lecture Notes in Pure and Applied Mathematics I, Marcel Dekker, New York, 1971.

[23] A. M. Bincer, Casimir operators of the exceptional group $G_2$ in a $B_3$ basis, Can. J. Phys. 72 (1994) 319-320.

[24] S. Saenkarun, A. Loutsouk and S. Chunrungsikul, Restriction of representations of $G_2$ to $A_2$, Southeast Asian Bull. Math. 35 (2011) 675-685.

[25] A. L. Onishchik and E. B. Vinberg, Lie Groups and Algebraic Groups, Springer-Verlag, Berlin Heidelberg, 1990.

[26] G. G. A. Bauerle and E. A. de Kerf, Lie Algebras, Part 1: Finite and Infinite Dimensional Lie Algebras and Applications in Physics, Elsevier, North Holland, 1990.

[27] A. W. Knapp, Lie Groups Beyond an Introduction, 2nd edn. Progress in Mathematics, vol. 140. Birkhäuser, Boston, 2002.

[28] N. Ja. Vilenkin and A. U. Klimyk, Representation of Lie Groups and Special Functions. Volume 3: Classical and Quantum Groups and Special Functions, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992.

[29] H. Abbaspour and M. A. Moskowitz, Basic Lie Theory, World Scientific, Singapore, 2007.

[30] P. Ramond, Group Theory, A Physicists Survey, Cambridge University Press, New York, 2010.

[31] H. Georgi, Lie Algebras and Particle Physics, Second Edition, Westview Press, Boulder, 1999.
[32] H.-J. Huang, Y.-N. Li and D. Ruan, Indecomposable representations and oscillator realizations of the exceptional Lie algebra $G_2$, Eur. Phys. J. Plus 128 (2013) 66.

[33] N. J. Wildberger, A combinatorial construction of $G_2$, J. Lie Theory 13 (2003) 155-165.

[34] S. Saenkarun, A. Loutsiouk and S. Chunrungsikul, Studying solutions of a system of PDE through representations of $G_2$, Int. Math. Forum 4 (2009) 429-439.
|        | $X_{a_1}$ | $X_{a_2}$ | $X_{a_1+a_2}$ | $X_{a_1+2a_2}$ | $X_{a_1+3a_2}$ | $X_{2a_1+3a_2}$ | $Y_{a_1}$ | $Y_{a_2}$ | $Y_{a_1+a_2}$ | $Y_{a_1+2a_2}$ | $Y_{a_1+3a_2}$ | $Y_{2a_1+3a_2}$ |
|--------|-----------|-----------|----------------|------------------|------------------|------------------|---------|---------|----------------|------------------|------------------|------------------|
| $H_{a_1}$ | 2$X_{a_1}$ | $-X_{a_2}$ | $X_{a_1+a_2}$ | 0                | $-X_{a_1+3a_2}$ | $X_{2a_1+3a_2}$ | $-2Y_{a_1}$ | $Y_{a_2}$ | $-Y_{a_1+a_2}$ | 0                | $Y_{a_1+3a_2}$ | $-Y_{2a_1+3a_2}$ |
| $H_{a_2}$ | -3$X_{a_1}$ | 2$X_{a_2}$ | $-X_{a_1+a_2}$ | $X_{a_1+2a_2}$ | 3$X_{a_1+3a_2}$ | 0                | 3$Y_{a_1}$ | $-2Y_{a_2}$ | $Y_{a_1+a_2}$ | $-Y_{a_1+2a_2}$ | $-3Y_{a_1+3a_2}$ | 0                |
| $H_{a_1+a_2}$ | 3$X_{a_1}$ | $-X_{a_2}$ | 2$X_{a_1+a_2}$ | $X_{a_1+2a_2}$ | 0                | 3$X_{2a_1+3a_2}$ | $-3Y_{a_1}$ | $Y_{a_2}$ | $-2Y_{a_1+a_2}$ | $-Y_{a_1+2a_2}$ | 0                | $-3Y_{2a_1+3a_2}$ |
| $H_{a_1+2a_2}$ | 0 | $X_{a_2}$ | $X_{a_1+a_2}$ | $2X_{a_1+2a_2}$ | $3X_{a_1+3a_2}$ | $-3X_{2a_1+3a_2}$ | 0 | $-Y_{a_2}$ | $-Y_{a_1+a_2}$ | $-2Y_{a_1+2a_2}$ | $-3Y_{a_1+3a_2}$ | $-3Y_{2a_1+3a_2}$ |
| $H_{a_1+3a_2}$ | $-X_{a_1}$ | $X_{a_2}$ | 0 | $X_{a_1+a_2}$ | $2X_{a_1+2a_2}$ | $X_{2a_1+3a_2}$ | $Y_{a_1}$ | $-Y_{a_2}$ | 0 | $-Y_{a_1+2a_2}$ | $-2Y_{a_1+3a_2}$ | $-2Y_{2a_1+3a_2}$ |
| $H_{2a_1+3a_2}$ | $X_{a_1}$ | 0 | $X_{a_1+a_2}$ | $X_{a_1+2a_2}$ | $X_{a_1+3a_2}$ | $2X_{2a_1+3a_2}$ | $-Y_{a_1}$ | 0 | $-Y_{a_1+a_2}$ | $-Y_{a_1+2a_2}$ | $-Y_{a_1+3a_2}$ | $-2Y_{2a_1+3a_2}$ |

Table 1: The commutation relations of the exceptional Lie algebra $g_2$ based on (1.8).
| $X'_{a_1}$ | $X'_{a_2}$ | $X'_{a_1+a_2}$ | $X'_{a_1+2a_2}$ | $X'_{a_1+3a_2}$ | $X'_{2a_1+3a_2}$ | $Y'_{a_1}$ | $Y'_{a_2}$ | $Y'_{a_1+a_2}$ | $Y'_{a_1+2a_2}$ | $Y'_{a_1+3a_2}$ | $Y'_{2a_1+3a_2}$ |
|-------|-------|-------------|----------------|----------------|----------------|----------|----------|-------------|--------------|--------------|--------------|
| $H'_{a_1}$ | $X'_{a_1}$ | $\frac{1}{2}X'_{a_2}$ | $\frac{1}{2}X'_{a_1+a_2}$ | $0$ | $\frac{1}{2}X'_{a_1+3a_2}$ | $\frac{1}{2}X'_{2a_1+3a_2}$ | $-Y'_{a_1}$ | $\frac{1}{2}Y'_{a_2}$ | $\frac{1}{2}Y'_{a_1+a_2}$ | $0$ | $\frac{1}{2}Y'_{a_1+3a_2}$ | $\frac{1}{2}Y'_{2a_1+3a_2}$ |
| $H'_{a_2}$ | $\frac{1}{2}X'_{a_1}$ | $\frac{1}{2}X'_{a_2}$ | $-\frac{1}{2}X'_{a_1+a_2}$ | $\frac{1}{2}X'_{a_1+2a_2}$ | $\frac{1}{2}X'_{a_1+3a_2}$ | $0$ | $\frac{1}{2}Y'_{a_1}$ | $\frac{1}{2}Y'_{a_2}$ | $\frac{1}{2}Y'_{a_1+a_2}$ | $0$ | $\frac{1}{2}Y'_{a_1+3a_2}$ | $\frac{1}{2}Y'_{2a_1+3a_2}$ |
| $H'_{a_1+a_2}$ | $\frac{1}{2}X'_{a_1}$ | $\frac{1}{2}X'_{a_2}$ | $\frac{1}{2}X'_{a_1+a_2}$ | $\frac{1}{2}X'_{a_1+2a_2}$ | $0$ | $\frac{1}{2}Y'_{a_1}$ | $\frac{1}{2}Y'_{a_2}$ | $\frac{1}{2}Y'_{a_1+a_2}$ | $0$ | $\frac{1}{2}Y'_{a_1+3a_2}$ | $\frac{1}{2}Y'_{2a_1+3a_2}$ |
| $H'_{a_1+2a_2}$ | $0$ | $\frac{1}{2}X'_{a_1}$ | $\frac{1}{2}X'_{a_2}$ | $\frac{1}{2}X'_{a_1+a_2}$ | $\frac{1}{2}X'_{a_1+2a_2}$ | $\frac{1}{2}X'_{a_1+3a_2}$ | $\frac{1}{2}Y'_{a_1}$ | $\frac{1}{2}Y'_{a_2}$ | $\frac{1}{2}Y'_{a_1+a_2}$ | $0$ | $\frac{1}{2}Y'_{a_1+3a_2}$ | $\frac{1}{2}Y'_{2a_1+3a_2}$ |
| $H'_{a_1+3a_2}$ | $\frac{1}{2}X'_{a_1}$ | $0$ | $\frac{1}{2}X'_{a_1+a_2}$ | $\frac{1}{2}X'_{a_1+2a_2}$ | $\frac{1}{2}X'_{a_1+3a_2}$ | $\frac{1}{2}X'_{2a_1+3a_2}$ | $\frac{1}{2}Y'_{a_1}$ | $0$ | $\frac{1}{2}Y'_{a_1+a_2}$ | $\frac{1}{2}Y'_{a_1+2a_2}$ | $\frac{1}{2}Y'_{a_1+3a_2}$ | $-Y'_{2a_1+3a_2}$ |

Table 2: The commutation relations of the exceptional Lie algebra $g_2$ based on (2.9) and (2.11).