Lagrangian and Hamiltonian structures for the constant astigmatism equation

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Abstract

In this paper we found a Lagrangian representation and corresponding Hamiltonian structure for the constant astigmatism equation. Utilizing this Hamiltonian structure and extra conservation law densities we construct a first evolution commuting flow of the third order. We also apply the recursion operator and present a second Hamiltonian structure. This bi-Hamiltonian structure allows us to replicate infinitely many local commuting flows and corresponding local conservation law densities.

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1. Introduction

Many integrable equations have been found in classical differential geometry. One of them is the famous Bonnet equation, also known as the sine-Gordon equation. This equation expresses angles between asymptotic directions of surfaces of negative constant Gaussian curvature. Recently, interest in the Bonnet equation was renewed due to a successful search for integrable cases of Weingarten surfaces. The equation describing surfaces of constant astigmatism

\[ u_{tt} + \left( \frac{1}{u} \right)_{xx} + 2 = 0 \]  

was considered in several papers (for details, see [2–5]). A transformation between the Bonnet equation and (1) was also found. However, this transformation is very sophisticated (see, for instance, [5]). It is not so easy to recompute solutions and Hamiltonian structures from the Bonnet equation to (1). For this reason, we independently construct the Lagrangian, the corresponding Hamiltonian structure, the first evolution commuting flow of the third order and a second Hamiltonian structure of (1) in this paper. The inverse transformation (from (1) back to the Bonnet equation) is less complicated. We believe that our results can be effectively utilized in the theory of the Bonnet equation.
2. The Lagrangian and Hamiltonian structure

The Lagrangian

\[ S = \int \left( \frac{1}{2} \Omega_{xx} \Omega_{tt} - f(\Omega, \Omega_x, \Omega_{xx}, \ldots) \right) \, dx \, dt \]  

(2)

determines the Euler–Lagrange equation

\[ \Omega_{ttx} = \frac{\delta F}{\delta \Omega}, \]

(3)

where \( F = \int f(\Omega, \Omega_x, \Omega_{xx}, \ldots) \, dx \). Obviously, two local conservation laws (of the energy and of the momentum) can be obtained (due to Noether’s Theorem) from the energy–momentum tensor. For instance, the conservation law of the momentum is

\[ (\Omega_x, \Omega_{xt})_t = \left( \frac{1}{2} \Omega_{xx}^2 + \Omega_x \Omega_{tt} - G \right)_t, \]

where \( G_x = \frac{\delta F}{\delta \Omega} \), while the conservation law of the energy is

\[ \left( \frac{1}{2} \Omega_{tt}^2 + f(\Omega, \Omega_x, \Omega_{xx}, \ldots) \right)_t = (\Omega, \Omega_{tt} + Q)_t, \]

where \( Q_x = -\frac{\delta F}{\delta \Omega} \). If, for instance, \( f(\Omega, \Omega_x, \Omega_{xx}) \), then

\[ (\Omega_{xx}, \Omega_{xt})_t = \left( \frac{1}{2} \Omega_{xx}^2 + \Omega_x \Omega_{tt} - f + \frac{\partial f}{\partial \Omega_x} \Omega_x - \left( \frac{\partial f}{\partial \Omega_{xx}} \right)_x \Omega_x + \frac{\partial f}{\partial \Omega_{xx}} \Omega_{xx} \right)_x, \]

\[ \left( \frac{1}{2} \Omega_{tt}^2 + f(\Omega, \Omega_x, \Omega_{xx}) \right)_t = (\Omega, \Omega_{tt} + \frac{\partial f}{\partial \Omega_x} \Omega_x + \left( \frac{\partial f}{\partial \Omega_{xx}} \right)_x \Omega_x + \frac{\partial f}{\partial \Omega_{xx}} \Omega_{xx} \right)_x, \]

while the Euler–Lagrange equation (3) can be written as a Hamiltonian system

\[ \Omega_t = \partial_x^{-1} \frac{\delta H}{\delta w}, \quad w = \partial_x^{-1} \frac{\delta H}{\delta \Omega} \cdot \quad H = \int \left( f(\Omega, \Omega_x, \Omega_{xx}) + \frac{1}{2} w^2 \right) \, dx, \]

where \( w = \Omega_{xx} \).

In this paper we consider just the case \( f(\Omega, \Omega_x, \Omega_{xx}) = -2\Omega - \ln \Omega_{xx} \). The corresponding Euler–Lagrange equation (3) is nothing but the constant astigmatism equation (1), where \( u = \Omega_{xx} \). Thus, the constant astigmatism equation (1) possesses the local Lagrangian representation

\[ S = \int \left( \frac{1}{2} \Omega_{xx} \Omega_{tt} + \ln \Omega_{xx} + 2\Omega \right) \, dx \, dt, \]

(4)

two local conservation laws (the momentum and the energy, respectively):

\[ (\Omega_x, \Omega_{xt})_t = \left( \frac{1}{2} \Omega_{xx}^2 + \Omega_x \Omega_{tt} + 2\Omega - \frac{\Omega_{xxx}}{\Omega_{xx}^2} \ln \Omega_{xx} \right)_x, \]

\[ \left( \frac{1}{2} \Omega_{tt}^2 - 2\Omega - \ln \Omega_{xx} \right)_t = (\Omega, \Omega_{tt} - \frac{\Omega_{xxx}}{\Omega_{xx}^2} \Omega_x - \frac{\Omega_{tt}}{\Omega_{xx}})_x \]

and non-local Hamiltonian structure

\[ \Omega_t = \partial_x^{-1} \frac{\delta H}{\delta w}, \quad w = \partial_x^{-1} \frac{\delta H}{\delta \Omega}, \]

(5)

where the Hamiltonian \( H = \int \left( \frac{1}{2} w^2 - 2\Omega - \ln \Omega_{xx} \right) \, dx \) and the momentum \( P = \int \Omega_{xx} w \, dx \).
Remark. Under the substitution $u = \Omega_{xx}$, the above non-local Hamiltonian structure assumes a local form

$$u_y = \frac{\delta H}{\delta w}, \quad w_y = \frac{\delta H}{\delta u},$$

where the momentum $P = \int u w \, dx$ is still local, but the Hamiltonian $H = \int \left( \frac{1}{2} u^2 - \ln u - 2\Omega \right) \, dx$ is essentially non-local.

Also, the constant astigmatism equation (1) has two extra conservation laws (see [5])

$$\partial_t \sqrt{4u + \left( \frac{u_x}{u} \pm \frac{w_x}{u} \right)^2} = \pm \partial_x \sqrt{4u + \left( \frac{u_x}{u} \pm \frac{w_x}{u} \right)^2}.$$

Thus, one can construct a third order symmetry (see (6))

$$u_y = \frac{\delta \tilde{H}}{\delta w}, \quad w_y = \frac{\delta \tilde{H}}{\delta u}, \quad \tilde{H} = \int \sqrt{4u + \left( \frac{u_x}{u} \pm \frac{w_x}{u} \right)^2} \, dx.$$

Remark. Any higher commuting flow to the constant astigmatism equation (1) can also be written via the same function $\Omega_{xx}$ only. Indeed, taking into account that $u = \Omega_{xx}$ and $w = \Omega_{xt}$, the evolution system (7) reduces to two three-dimensional equations

$$\Omega_{xx} + 2\Omega_{xy} \sqrt{\frac{\Omega_{xx}}{1 - \Omega_{xx}^2}} \pm \Omega_{xxx} = 0, \quad \Omega_{xt} = \sqrt{\frac{1 - \Omega_{xx}^2}{\Omega_{xx}}} \pm \frac{\Omega_{xy}}{\Omega_{xx}}.$$

The compatibility condition $(\Omega_{xx})_y = (\Omega_{xt})_{xx}$ leads to the single equation

$$\left( \Omega_{xxx} \pm 2\Omega_{xy} \sqrt{\frac{\Omega_{xx}}{1 - \Omega_{xx}^2}} \right)_y + \left( \Omega_{xy} \pm \sqrt{\frac{1 - \Omega_{xx}^2}{\Omega_{xx}}} \right)_{xx} = 0,$$

which is nothing but an Euler–Lagrange equation associated with the local Lagrangian representation (see (4))

$$\tilde{S} = \int \left[ \Omega_{xy} \ln \Omega_{xx} \pm 2 \sqrt{\Omega_{xx} (1 - \Omega_{xx}^2)} \right] \, dx \, dy.$$

Meanwhile, one can express $\Omega_t$ from the first equation in (8) and substitute it into the second equation in (8). This again gives the constant astigmatism equation.

3. Bi-Hamiltonian structure

In this section we present a second Hamiltonian structure for the constant astigmatism equation (1) and its hierarchy.

Infinitely many symmetries (here $t^k$ are group parameters)

$$u_{t^k} = \left( -u_t \partial_t^{-1} + u_x \partial_x^{-2} \partial_t + 2u \partial_x^{-1} \partial_y \right) u_{t^k}, \quad k = 1, 2, \ldots$$

of the constant astigmatism equation (1) are connected to each other by the recursion operator

$$R = -u_t \partial_t^{-1} + u_x \partial_x^{-2} \partial_t + 2u \partial_x^{-1} \partial_y.$$

However, for the construction of a second Hamiltonian structure, this recursion operator should at first be rewritten in a matrix form, because the constant astigmatism equation (1) is

\[\text{Found by Artur Sergyeyev, see [5].}\]
a two component system. Indeed, introducing the field variable \( q \) such that \( u = q \), (1) takes an evolution form

\[
\begin{align*}
u_t &= q, \quad q_t = -\left( \frac{1}{u} \right)_{xx} - 2. \tag{10}
\end{align*}
\]

Thus, differentiating (9) with respect to \( t \), one can obtain

\[
q_{t+1} = -q \partial_x^{-1} u_t + q x \partial_x^{-2} q_t + q \partial_x^{-1} q_t + u_t \partial_x^{-2}(q_t)_{xx} + 2 u \partial_x^{-1} (q_t)_{xx}.
\]

Finally, taking into account (10), we eliminate derivatives with respect to \( t \). This yields a desirable relationship

\[
u_{t+1} = -q \partial_x^{-1} u_t + (2 u \partial_x^{-1} + u_x \partial_x^{-2}) q_t,
\]

\[
q_{t+1} = \left( \frac{2}{u} \partial_x - 3 \frac{u_x}{u^2} + \left[ \left( \frac{1}{u} \right)_{xx} + 2 \partial_x^{-1} \right] u \partial_x + (q \partial_x^{-1} + q x \partial_x^{-2}) q_t.
\]

Thus, under the potential substitution \( q = w \), the above transformation of symmetries can be written in the matrix form

\[
\left( \begin{array}{c} u \\ w \end{array} \right)_{t+1} = \mathcal{R} \left( \begin{array}{c} u \\ w \end{array} \right)_{t} = \left( \begin{array}{cc} 2 & -w_x \partial_x^{-1} \\ 2 & u + u_x \partial_x^{-1} \\ 2 & u_x \partial_x^{-1} \end{array} \right) \left( \begin{array}{c} u \\ w \end{array} \right)_{t}.
\]

Since any symmetry local to (1) has the same local Hamiltonian structure

\[
u_{t} = \partial_t \frac{\delta H_k}{\delta w}, \quad w_{t} = \partial_t \frac{\delta H_k}{\delta w},
\]

we automatically obtain a second Hamiltonian structure

\[
u_{t} = \partial_t \frac{\delta H_k}{\delta w} = (2 u \partial_x + u_x) \frac{\delta H_{k-1}}{\delta u} - w_x \frac{\delta H_{k-1}}{\delta w},
\]

\[
w_{t} = \partial_t \frac{\delta H_k}{\delta w} = w_x \frac{\delta H_{k-1}}{\delta u} + \left( \frac{2}{u} \partial_x - \frac{u_x}{u^2} + 2 \partial_x^{-1} \right) \frac{\delta H_{k-1}}{\delta w}.
\]

If, for instance, we start from the third order evolution system (1)

\[
u_{t} = \partial_t \frac{\delta H_1}{\delta w}, \quad w_{t} = \partial_t \frac{\delta H_1}{\delta w}, \quad H_1 = \int \sqrt{4 u + \left( \frac{u_x}{u} \pm w_x \right)^2} \, dx,
\]

then the next commuting flow

\[
u_{t} = \partial_t \frac{\delta H_2}{\delta w} = (2 u \partial_x + u_x) \frac{\delta H_1}{\delta u} - w_x \frac{\delta H_1}{\delta w},
\]

\[
w_{t} = \partial_t \frac{\delta H_2}{\delta w} = w_x \frac{\delta H_1}{\delta u} + \left( \frac{2}{u} \partial_x - \frac{u_x}{u^2} + 2 \partial_x^{-1} \right) \frac{\delta H_1}{\delta w}
\]

will also again be a local symmetry.

Thus, infinitely many local commuting flows constructed from this bi-Hamiltonian structure can be utilized for the description of multi-phase solutions for the constant astigmatism equation (1).

**Remark.** The second Hamiltonian structure

\[
\left( \begin{array}{c} u \\ w \end{array} \right)_{t} = \left( \begin{array}{cc} 2 & -w_x \partial_x^{-1} \\ 2 & u + u_x \partial_x^{-1} \\ 2 & u_x \partial_x^{-1} \end{array} \right) \left( \begin{array}{c} \frac{\delta H_{k-1}}{\delta u} \\ \frac{\delta H_{k-1}}{\delta w} \end{array} \right)
\]

is non-local. This is a linear combination of the local Hamiltonian structure of the Dubrovin–Novikov type and a pure non-local part. Such Hamiltonian structures were investigated in [6].
In the general $N$ component case, corresponding Hamiltonian operators have the form ($\alpha \neq 0$ is an arbitrary constant)

$$A^{ij} = g^{ij} \partial_t - g^{ik} \Gamma^l_{jk} u^l_t + \alpha f^i \partial_t^{-1} f^j,$$

where $g^{ij}(u)$ is a non-degenerate symmetric metric and $\Gamma^l_{jk}$ are Christoffel symbols of Levi-Civita connection, while $f^i$ are components of isometry $f^i \partial / \partial u^i$, which satisfies some special conditions. In this section, we presented a first example of integrable systems equipped by a pair of Hamiltonian operators

$$A^{ij} = g^{ij} \partial_t - g^{ik} \Gamma^l_{jk} u^l_t + \alpha f^i \partial_t^{-1} f^j,$$

$$\tilde{A}^{ij} = \tilde{g}^{ij} \partial_t - \tilde{g}^{ik} \tilde{\Gamma}^l_{jk} u^l_t + \alpha f^i \partial_t^{-1} f^j.$$  \hspace{1cm} (11)

For the integrable hierarchy of the constant astigmatism equation we choose $u^1 = u$, $u^2 = w$. Then $\alpha = 2$ and

$$g^{ik} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{g}^{ik} = 2 \begin{pmatrix} u & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad f^i = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

while $\Gamma^l_{jk} = 0$ and $\tilde{\Gamma}^l_{jk} = -\frac{1}{2} u \delta^{l}_{jk}$. The classification of integrable systems determined by pairs of compatible Poisson brackets (11) will be published elsewhere.

4. Conclusion

This paper is focused on a bi-Hamiltonian origin of the constant astigmatism equation. This equation is connected with the Bonnet (sine-Gordon) equation by differential substitutions and by a reciprocal transformation. The Bonnet equation has one local and infinitely many non-local Hamiltonian structures. However, unfortunately, at least at this time, recomputation of any Hamiltonian structure under an arbitrary reciprocal transformation is an open problem (except the case of local Hamiltonian structures of Dubrovin–Novikov type; see [7] for details). Thus, an independent reconstruction of Hamiltonian (or, moreover, bi-Hamiltonian) structures is a very important problem in the integrability of interesting nonlinear systems known in differential geometry or having physical applications. Our own interest in the bi-Hamiltonian structure is based on the opportunity to construct infinite series of local conservation laws and corresponding higher local symmetries (also known as commuting flows). However, even a single Hamiltonian structure can be utilized for the construction of multi-phase solutions (as indicated above). This procedure is based on the so-called Lax–Novikov equation

$$\delta (\Sigma \beta^m H_m) = 0,$$

where $\beta^m$ are arbitrary constants and $H_m = \int h_m [u, w] dx$ are functionals of higher conservation law densities $h_m(u, w, u_x, w_x, \ldots)$. Variation of this sum with respect to the field variables $u, w$ leads to two ordinary differential equations, which precisely describe the multi-phase solutions (for details see [8]).

The most interesting perspective aside from the constant astigmatism equation itself is the classification of bi-Hamiltonian systems (11). This problem is completely open, because even the Hamiltonian operators $\tilde{A}^{ij} = \tilde{g}^{ij} \partial_t - \tilde{g}^{ik} \tilde{\Gamma}^l_{jk} u^l_t + \alpha f^i \partial_t^{-1} f^j$ are not described for a number of components greater than two. Even in the two component case, the constant astigmatism equation is a first example equipped by such a pair of Hamiltonian structures. By the publication of this paper we hope to attract experts in bi-Hamiltonian structures to this very interesting and important question.

Some new solutions of (1) were found in [1]. Under the transformations described in [2] any solutions of the constant astigmatism should reduce to corresponding solutions of the Bonnet equations. For instance, solutions of (1) connected with multi-soliton solutions of the Bonnet equation were recently found in [1]. In our next paper we intend to utilize a method of
inverse scattering transform to construct new solutions of (1) and recalculate them to solutions of the Bonnet equation.

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References

[1] Hlaváč A and Marvan M 2012 Some results concerning the constant astigmatism equation arXiv:1206.0321
[2] Hlaváč A and Marvan M 2013 Another integrable case in two-dimensional plasticity J. Phys. A: Math. Theor. 46 045203
[3] Hlaváč A and Marvan M 2011 A reciprocal transformation for the constant astigmatism equation arXiv:1111.2027
[4] Baran H and Marvan M 2010 Classification of integrable Weingarten surfaces possessing an sl(2)-valued zero curvature representation Nonlinearity 23 2577–97
[5] Baran H and Marvan M 2009 On integrability of Weingarten surfaces: a forgotten class J. Phys. A: Math. Theor. 42 404007
[6] Ferapontov E V 1992 Nonlocal matrix Hamiltonian operators. Differential geometry and applications Teor. Mat. Fiz. 91 452–62 (in Russian)
Ferapontov E V 1992 Nonlocal matrix Hamiltonian operators. Differential geometry and applications Theor. Math. Phys. 91 642–9 (Engl. transl.)
[7] Ferapontov E V and Pavlov M V 2003 Reciprocal transformations of Hamiltonian operators of hydrodynamic type: nonlocal Hamiltonian formalism for linearly degenerate systems J. Math. Phys. 44 1150–72
[8] Novikov S P, Manakov S V, Fitaevski L P and Zakharov V E 1984 Theory of Solitons: the Inverse Scattering (New York: Consultants Bureau)