INCOMPLETE PADÉ APPROXIMATION AND CONVERGENCE OF ROW SEQUENCES OF HERMITE-PADÉ APPROXIMANTS

J. CACOQ, B. DE LA CALLE YSERN, AND G. LÓPEZ LAGOMASINO

Abstract. We give a Montessus de Ballore type theorem for row sequences of Hermite-Padé approximations of vector valued analytic functions refining some results in this direction due to P.R. Graves-Morris and E.B. Saff. We do this introducing the notion of incomplete Padé approximation which contains, in particular, simultaneous Padé approximation and may be applied in the study of other systems of approximants as well.

1. Introduction

Let

\[ f(z) = \sum_{n=0}^{\infty} \phi_n z^n, \quad \phi_n \in \mathbb{C}, \]

(1)

denote a formal or convergent Taylor expansion about the origin. By \( D_0(f) \) and \( R_0(f) \) we denote the disk and radius of convergence, respectively, of the series (1). In [5], Jacques Hadamard introduced the notion of \( m \)th disk of meromorphy \( D_m(f) \) of \( f \). When \( R_0(f) = 0 \) this disk is defined to be the empty set. If \( R_0(f) > 0 \) then \( D_m(f) \) is the largest disk centered at the origin to which the analytic element \( (f, D_0(f)) \) can be extended as a meromorphic function having no more than \( m \) poles. Let \( R_m(f) \) denote the radius of \( D_m(f) \). In the cited paper, Hadamard proves a beautiful formula which gives the values of the numbers \( R_m(f) \) for all \( m \in \mathbb{Z}_+ \) using exclusively the data provided by the Taylor coefficients \( \phi_n \). For \( m = 0 \), it reduces to Cauchy’s formula for the radius of convergence of a Taylor series. Hadamard’s finding is intimately connected with the convergence theory of Padé approximations.

Definition 1. Let \( f \) be the formal expansion (1). Let \( n, m \in \mathbb{Z}_+, n \geq m \), be given. Then, there exist polynomials \( Q, P \), satisfying

2010 Mathematics Subject Classification. Primary 41A21, 41A28; Secondary 41A25.

Key words and phrases. Montessus de Ballore Theorem, simultaneous approximation, Hermite-Padé approximation.

The work of B. de la Calle received support from MINCINN under grant MTM2009-14668-C02-02 and from UPM through Research Group “Constructive Approximation Theory and Applications”. The work of J. Cacoq and G. López was supported by MINCINN under grant MTM2009-14668-C01-02.

1
a.1) \( \deg P \leq n - m, \quad \deg Q \leq m, \quad Q \neq 0, \)
a.2) \( [Qf - P](z) = Az^{n+1} + \cdots. \)

Any pair \((Q, P)\) which satisfies a.1) \(-\) a.2) defines a unique rational function
\(\pi_{n,m} = P/Q\) which is called the \(\text{Padé approximation of type} (n,m)\) of \(f\).

We have slightly modified (in an equivalent form) the usual definition
of an \((n,m)\) \(\text{Padé approximation having in mind the aims of this paper.} \)
Let \(\pi_{n,m} = P_{n,m}/Q_{n,m}\) where \(Q_{n,m}\) and \(P_{n,m}\) are polynomials obtained
 cancelling all common factors and, unless otherwise stated, normalizing \(Q_{n,m}\)
so that
\[
(2) \quad Q_{n,m}(z) = \prod_{|\zeta_{n,k}| \leq 1} (z - \zeta_{n,k}) \prod_{|\zeta_{n,k}| > 1} \left(1 - \frac{z}{\zeta_{n,k}}\right).
\]

Robert de Montessus de Ballore, using Hadamard’s work, proved the
 following result (see [7]). Let \(Q_m(f)\) stand for the polynomial (properly nor-
 malized as in (2)) whose zeros are the poles of \(f\) in \(D_m(f)\) with multiplicity
equal to the order of the corresponding pole. By \(P_m(f)\) we denote this set
of poles. Given a compact set \(K \subset \mathbb{C}\), \(\| \cdot \|_K\) denotes the sup norm on \(K\).

**Montessus de Ballore Theorem.** Assume that \(R_0(f) > 0\) and that \(f\) has
 exactly \(m\) poles in \(D_m(f)\) (counting multiplicities), then
\[
(3) \quad \limsup_{n \to \infty} \|f - \pi_{n,m}\|_K^{1/n} = \frac{\|z\|_K}{R_m(f)},
\]
where \(K\) is any compact subset of \(D_m(f) \setminus P_m(f)\). Additionally
\[
(4) \quad \limsup_{n \to \infty} \|Q_m(f) - Q_{n,m}\|^{1/n} = \frac{\max\{|\zeta| : \zeta \in P_m(f)\}}{R_m(f)},
\]
where \(\| \cdot \|\) denotes the coefficient norm in the space of polynomials.

From this result it follows that if \(\zeta\) is a pole of \(f\) in \(D_m(f)\) of order \(\tau\),
 then for each \(\varepsilon > 0\), there exists \(n_0\) such that for \(n \geq n_0\), \(Q_{n,m}\) has exactly
\(\tau\) zeros in \(\{z : |z - \zeta| < \varepsilon\}\). We say that each pole of \(f\) in \(D_m(f)\) attracts
as many zeros of \(Q_{n,m}\) as its order when \(n\) tends to infinity. In Montessus’
paper the geometric rate expressed in (3) and (4) was not explicitly given.

The simultaneous Hermite-Padé approximation of systems of functions
has been a subject of major interest in the recent past. Though most results
deal with what could be called diagonal sequences of simultaneous approxi-
mants, there are some interesting results due to P. R. Graves-Morris and E.
B. Saff for row sequences which extend the Montessus Theorem, see [8]-[10].

**Definition 2.** Let \(f = (f_1, \ldots, f_d)\) be a system of \(d\) formal Taylor expansions
as in (1). Fix a multi-index \(\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d \setminus \{0\}\) where \(0\) denotes
the zero vector in \(\mathbb{Z}_+^d\). Set \(|\mathbf{m}| = m_1 + \cdots + m_d\). Then, for each \(n \geq \max \{m_1, \ldots, m_d\}\),
there exist polynomials \(Q, P_j, j = 1, \ldots, d\), such that
b.1) \(\deg P_j \leq n - m_j, j = 1, \ldots, d, \quad \deg Q \leq |\mathbf{m}|, \quad Q \neq 0, \)
b.2) \([Qf_j - P_j](z) = A_jz^{n+1} + \cdots. \)
The vector rational function $R_{n,m} = (P_1/Q, \ldots, P_d/Q)$ is called an \((n,m)\) Hermite-Padé approximation of \(f\).

Unlike the scalar case, in general, this vector rational approximation is not uniquely determined and in the sequel we assume that given \((n,m)\) one particular solution is taken. For that solution we write
\begin{equation}
R_{n,m} = (R_{n,m,1}, \ldots, R_{n,m,d}) = (P_{n,m,1}, \ldots, P_{n,m,d})/Q_{n,m},
\end{equation}
where \(Q_{n,m}\) has no common factor simultaneously with all the \(P_{n,m,j}\) and is normalized the same way as \(Q_{n,m}\) above.

**Definition 3.** A vector \(f = (f_1, \ldots, f_d)\) of functions meromorphic in some domain \(D\) is said to be polewise independent with respect to the multi-index \(m = (m_1, \ldots, m_d)\) in \(D\) if there do not exist polynomials \(p_1, \ldots, p_d\), at least one of which is non-null, satisfying
\begin{itemize}
  \item[c.1)] \(\deg p_j \leq m_j - 1, j = 1, \ldots, d\), if \(m_j \geq 1\),
  \item[c.2)] \(p_j \equiv 0\) if \(m_j = 0\),
  \item[c.3)] \(\sum_{j=0}^d p_j f_j \in \mathcal{H}(D)\),
\end{itemize}
where \(\mathcal{H}(D)\) denotes the space of analytic functions in \(D\).

This notion was introduced in [8]. When \(d = 1\) polewise independence merely expresses that the function has at least \(m\) poles in \(D\).

**Definition 4.** Let \(f = (f_1, \ldots, f_d)\) be a system of formal Taylor expansions about the origin and \(D = (D_1, \ldots, D_d)\) a system of domains such that, for each \(k = 1, \ldots, d\), \(f_k\) is meromorphic in \(D_k\). We say that a point \(a\) is a pole of \(f\) in \(D\) of order \(\tau\) if there exists an index \(k \in \{1, \ldots, d\}\) such that \(a \in D_k\) and it is a pole of \(f_k\) of order \(\tau\), and for the rest of the indices \(j \neq k\) either \(a\) is a pole of \(f_j\) of order less than or equal to \(\tau\) or \(a \notin D_j\).

Polewise independence of \(f\) with respect to \(m\) in \(D\) implies that \(f\) has at least \(|m|\) poles in \(D = (D_1, \ldots, D_d)\) counting multiplicities, see Lemma 1 in [8]. In those cases when \(D = (D_1, \ldots, D_d)\) we say that \(a\) is a pole of \(f\) in \(D\).

Let \(R_0(f)\) be the largest disk in which all the expansions \(f_j, j = 1, \ldots, d\) correspond to analytic functions. If \(R_0(f) = 0\), we take \(D_m(f) = \emptyset, m \in \mathbb{Z}_+\); otherwise, \(R_m(f)\) is the radius of the largest disk \(D_m(f)\) centered at the origin to which all the analytic elements \((f_j, D_0(f_j))\) can be extended so that \(f\) has at most \(m\) poles counting multiplicities. By \(Q_m(f)\) we denote the polynomial whose zeros are the poles of \(f\) in \(D_m(f)\) counting multiplicities and normalized as \(Q_{n,m}\). This set of poles is denoted by \(P_m(f)\).

In [8], Graves-Morris and Saff proved the following analog of the Montes-sus de Baliore Theorem for simultaneous approximation.

**Graves-Morris/Saff Theorem.** Assume that \(R_0(f) > 0\). Fix a multi-index \(m \in \mathbb{Z}_+^d \setminus \{0\}\) and suppose that \(f\) is polewise independent with respect to \(m\) in \(D_m(f)\), then
\begin{equation}
\limsup_{n \to \infty} \|f_k - R_{n,m,k}\|_K^{1/n} \leq \frac{\|z\|_K}{R_{|m|}(f)}, \quad k = 1, \ldots, d,
\end{equation}
where $K$ is any compact subset of $D_{[m]}(f) \setminus P_{[m]}(f)$. Additionally

$$\limsup_{n \to \infty} \|Q_{[m]}(f) - Q_{n,m}\|^{{1/n}} \leq \frac{\max\{|\zeta| : \zeta \in P_{[m]}(f)\}}{R_{[m]}(f)}.$$ 

It also follows from this result that each pole of $f$ in $D_{[m]}(f)$ attracts exactly as many zeros of $Q_{n,m}$ as its order when $n$ tends to infinity.

The aim of this paper is to complement and refine some of the statements of the Graves-Morris/Saff Theorem. For this purpose, in Section 3 we introduce the notion of incomplete Padé approximation and study some of its properties. They are used in Section 4 to obtain our results for row sequences of Hermite-Padé approximation. Section 5 contains examples that illustrate to what extent our main Theorem 11 improves the one due to Graves-Morris/Saff. In passing, we mention that incomplete Padé approximants may be used to study the convergence of other systems of approximating rational (scalar or vector) functions. Section 2 contains some auxiliary results. Our approach is strongly influenced by the viewpoint of A.A. Gonchar as presented in [3] for studying row sequences of Padé approximants.

2. Convergence in $\sigma$-content

Let $B$ be a subset of the complex plane $\mathbb{C}$. By $U(B)$ we denote the class of all coverings of $B$ by at most a numerable set of disks. Set

$$\sigma(B) = \inf \left\{ \sum_{i=1}^{\infty} |U_i| : \{U_i\} \in U(B) \right\},$$

where $|U_i|$ stands for the radius of the disk $U_i$. The quantity $\sigma(B)$ is called the 1-dimensional Hausdorff content of the set $B$. This set function is not a measure but it is semi-additive and monotonic, properties which will be used later. Clearly, if $B$ is a disk then $\sigma(B) = |B|$.

**Definition 5.** Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence of functions defined on a domain $D \subset \mathbb{C}$ and $\varphi$ another function defined on $D$. We say that $\{\varphi_n\}_{n \in \mathbb{N}}$ converges in $\sigma$-content to the function $\varphi$ in compact subsets of $D$ if for each compact subset $K$ of $D$ and for each $\varepsilon > 0$, we have

$$\lim_{n \to \infty} \sigma\{z \in K : |\varphi_n(z) - \varphi(z)| > \varepsilon\} = 0.$$

Such a convergence will be denoted by $\sigma\text{-}\lim_{n \to \infty} \varphi_n = \varphi$ in $D$.

The next lemma was proved by A.A. Gonchar in [2].

**Gonchar’s Lemma.** Suppose that $\sigma\text{-}\lim_{n \to \infty} \varphi_n = \varphi$ in $D$. Then the following assertions hold true:

i) If the functions $\varphi_n$, $n \in \mathbb{N}$, are holomorphic in $D$, then the sequence $\{\varphi_n\}$ converges uniformly on compact subsets of $D$ and $\varphi$ is holomorphic in $D$ (more precisely, it is equal to a holomorphic function in $D$ except on a set of $\sigma$-content zero).
ii) If each of the functions $\varphi_n$ is meromorphic in $D$ and has no more than $k<+\infty$ poles in this domain, then the limit function $\varphi$ is (again except on a set of $\sigma$-content zero) also meromorphic and has no more than $k$ poles in $D$.

iii) If each function $\varphi_n$ is meromorphic and has no more than $k<+\infty$ poles in $D$ and the function $\varphi$ is meromorphic and has exactly $k$ poles in $D$, then all $\varphi_n$, $n\geq N$, also have $k$ poles in $D$; the poles of $\varphi_n$ tend to the poles $z_1,\ldots,z_k$ of $\varphi$ (taking account of their orders) and the sequence $\{\varphi_n\}$ tends to $\varphi$ uniformly on compact subsets of the domain $D' = D \setminus \{z_1,\ldots,z_k\}$.

3. Incomplete Padé approximants

**Definition 6.** Let $f$ denote a formal Taylor expansion about the origin. Fix $m^* \leq m$. Let $n \geq m$. We say that the rational function $R_{n,m}$ is an incomplete Padé approximation of type $(n,m,m^*)$ corresponding to $f$ if $R_{n,m}$ is the quotient of any two polynomials $P$ and $Q$ that verify

\begin{align*}
d.1) \ deg P & \leq n-m^*, \ deg Q \leq m, \ Q \neq 0, \\
d.2) \ [Qf - P](z) & = A z^{n+1} + \cdots.
\end{align*}

Notice that given $(n,m,m^*)$, $n \geq m \geq m^*$, any one of the Padé approximants $\pi_{n,m^*},\ldots,\pi_{n,m}$ is an incomplete Padé approximation of type $(n,m,m^*)$ of $f$. The so-called Padé-type approximants (see [1]) where $m-m^*$ zeros of $Q$ are fixed and $m^*$ are left free are also incomplete Padé approximants. Moreover, from Definition 2 and 5 it follows that $R_{n,m,k}$, $k = 1,\ldots,d$, is an incomplete Padé approximation of type $(n,[m],m_k)$ with respect to $f_k$.

Given $n \geq m \geq m^*$, $R_{n,m}$ is not unique so we choose one candidate. As before, after canceling out common factors between $Q$ and $P$, we write

$$R_{n,m} = P_{n,m}/Q_{n,m},$$

where, additionally, $Q_{n,m}$ is normalized as in 2. Suppose that $Q$ and $P$ have a common zero at $z = 0$ of order $\lambda_n$. From d.1)-d.2) readily follows that

\begin{align*}
d.3) \ deg P_{n,m} & \leq n-m^* - \lambda_n, \ deg Q_{n,m} \leq m - \lambda_n, \ Q_{n,m} \neq 0, \\
d.4) \ [Q_{n,m} f - P_{n,m}](z) & = A z^{n+1-\lambda_n} + \cdots.
\end{align*}

where $A$ is, in general, a different constant from the one in d.2).

When $f$ denotes a convergent series, it is well known by the specialists that any row sequence $\{\pi_{n,m}\}_{n\geq m}$, where $m \geq m^*$ is fixed, converges to $f$ in $\sigma$-content in compact subsets of $D_{m^*}(f)$. This is also true for any sequence of incomplete Padé approximations when $m \geq m^*$ is fixed. Before giving a formal statement of that result, let us introduce some additional definitions.

Take an arbitrary $\varepsilon > 0$ and define the open set $J_\varepsilon$ as follows. For $n \geq m$, let $J_{n,\varepsilon}$ denote the $\varepsilon/6m n^2$-neighborhood of the set $P_{n,m}^2 = \{\zeta_{n,1},\ldots,\zeta_{n,m_n}\}$ of finite zeros of $Q_{n,m}$. If $R_0(f) > 0$, let $J_{m-1,\varepsilon}$ denote the $\varepsilon/6m$-neighborhood of the set of poles of $f$ in $D_m(f)$. Otherwise, $J_{m-1,\varepsilon} = \emptyset$. Set $J_\varepsilon =$
\[ \bigcup_{n \geq m} J_{n, \varepsilon} \]. We have \( \sigma(J_{\varepsilon}) < \varepsilon \) and \( J_{\varepsilon_1} \subset J_{\varepsilon_2} \) for \( \varepsilon_1 < \varepsilon_2 \). For any set \( B \subset \mathbb{C} \) we put \( B(\varepsilon) = B \setminus J_{\varepsilon} \).

Let \( \{ \varphi_n \}_{n \in \mathbb{N}} \) be a sequence of functions defined on a domain \( D \) and \( \varphi \) another function also defined on \( D \). Clearly, if \( \{ \varphi_n \}_{n \in \mathbb{N}} \) converges uniformly to \( \varphi \) on \( K(\varepsilon) \) for every compact \( K \subset D \) and each \( \varepsilon > 0 \), then \( \sigma \)-\( \lim_{n \to \infty} \varphi_n = \varphi \) in \( D \).

Due to the normalization \( \mathbb{C} \), for any compact set \( K \subset \mathbb{C} \) and for every \( \varepsilon > 0 \), there exist constants \( C_1, C_2 \), independent of \( n \), such that

\[
\| Q_{n,m} \|_K < C_1, \quad \min_{z \in K(\varepsilon)} |Q_{n,m}(z)| > C_2 n^{-2m},
\]

where the second inequality is meaningful when \( K(\varepsilon) \) is a non-empty set.

In the sequel, \( C \) will denote positive constants, generally different, that are independent of \( n \) but may depend on all the other parameters involved in each formula where they appear.

**Proposition 1.** Let \( R_0(f) > 0 \). Fix \( m \) and \( m^* \) nonnegative integers, \( m \geq m^* \). For each \( n \geq m \), let \( R_{n,m} \) be an incomplete Padé approximant of type \((n, m, m^*)\) for \( f \). Then

\[ \sigma- \lim_{n \to \infty} R_{n,m} = f \text{ in } D_{m^*}(f). \]

**Proof.** Let \( Q_{m^*} \) denote the polynomial \( Q_{m^*}(f) \) normalized to be monic. Using d.3), we have

\[
[Q_{m^*} Q_{n,m} f - Q_{m^*} P_{n,m}] (z) = Az^{n+1+\lambda} + \cdots,
\]

which implies that

\[
\frac{[Q_{m^*} Q_{n,m} f - Q_{m^*} P_{n,m}](z)}{z^{n+1-\lambda}} \in \mathcal{H}(D_{m^*}(f)).
\]

Set \( |z| < r < R_{m^*}(f) \) with \( r \) arbitrarily close to \( R_{m^*}(f) \) and let \( \Gamma_r = \{ z \in \mathbb{C} : |z| = r \} \). By Cauchy’s integral formula we obtain

\[
\int_{\Gamma_r} \frac{Q_{m^*}(\zeta) P_{n,m}(\zeta)}{\zeta^{n+1-\lambda} - z} \frac{d\zeta}{\zeta} = \frac{1}{2\pi i} \int_{\Gamma_r} \frac{[Q_{m^*} Q_{n,m} f](\zeta)}{\zeta^{n+1-\lambda}} \frac{d\zeta}{\zeta - z} + \frac{[Q_{m^*} Q_{n,m} f - Q_{m^*} P_{n,m}](z)}{z^{n+1-\lambda}}.
\]

(9)

where the second integral after the first equality is zero due to the fact that the integrand is an analytic function outside \( \Gamma_r \) with a zero of multiplicity at least two at infinity (see d.3)).

Fix an arbitrary compact set \( K \subset D_{m^*}(f) \) and take \( 0 < r < R_{m^*}(f) \) such that \( K \) and all of the poles of \( f \) are contained in the disk \( \{ z \in \mathbb{C} : |z| < r \} \).

We also select an arbitrarily small \( \varepsilon > 0 \). From (9) it follows that

\[
[Q_{m^*}(f - R_{n,m})](z) = \frac{z^{n+1-\lambda}}{2\pi i} \int_{\Gamma_r} \frac{[Q_{m^*} Q_{n,m} f](\zeta)}{Q_{n,m}(z) \zeta^{n+1-\lambda}} \frac{d\zeta}{\zeta - z}.
\]
for all \( z \in K(\varepsilon) \). Using this last formula, Equation (8), and the continuity of \( Q_{m^*}f \) on \( \Gamma_r \), we obtain
\[
\|Q_{m^*}(f - R_{n,m})\|_{K(\varepsilon)} \leq C \frac{\|z\|_K^n}{r^n} \min_{\zeta \in K(\varepsilon)} \|Q_{n,m}(\zeta)\| \leq C \frac{\|z\|_K^n}{r^n} n^{2m}.
\]
Taking \( n \)-th root, making \( n \) tend to infinity, and letting \( r \) approach \( R_{m^*}(f) \), we arrive at
\[
\limsup_{n \to \infty} \|Q_{m^*}(f - R_{n,m})\|_{1/n}^{1/n} \leq \frac{\|z\|_K}{R_{m^*}(f)} < 1.
\]
As \( \varepsilon > 0 \) is arbitrary, we have proved \( \sigma\text{-}\lim_{n \to \infty} Q_{m^*}R_{n,m} = Q_{m^*}f \) in \( D_{m^*}(f) \), which is equivalent to the statement we wanted to prove. \( \square \)

Let us find the radius of the largest disk centered at the origin in compact subsets of which the sequence \( \{R_{n,m}\}_{n \geq m} \) converges to \( f \) in \( \sigma \)-content. This number, which depends on the specific sequence of incomplete Padé approximants considered, lies between \( R_{m^*}(f) \) and \( R_m(f) \) (see Section 5.1 below). We need some formulas.

**Lemma 1.** Let a formal power series \( \langle \rangle \) be given. Fix \( m \geq m^* \) two positive integers. Consider a corresponding sequence of incomplete Padé approximations. For each \( n \geq m \), we have
\[
R_{n+1,m}(z) - R_{n,m}(z) = \frac{A_{n,m}z^{n+1} - \lambda_{n,m}q_{n,m-m^*}(z)}{Q_{n,m}(z)Q_{n+1,m}(z)},
\]
where \( A_{n,m} \) is some constant and \( q_{n,m-m^*} \) is a polynomial of degree less than or equal to \( m - m^* \) normalized as in \( 2 \).

**Proof.** Using d.4) we have
\[
z^{\lambda_n}[Q_{n,m}f - P_{n,m}](z) = Az^{n+1} + \cdots
\]
and
\[
z^{\lambda_{n+1}}[Q_{n+1,m}f - P_{n+1,m}](z) = A'z^{n+2} + \cdots.
\]
Multiplying the first equation by \( z^{\lambda_{n+1}}Q_{n+1,m} \), the second by \( z^{\lambda_n}Q_{n,m} \), and deleting one of the equations so obtained from the other, it follows that
\[
z^{\lambda_n+\lambda_{n+1}}[Q_{n,m}P_{n+1,m} - Q_{n+1,m}P_{n,m}](z) = Bz^{n+1} + \cdots.
\]
Taking into consideration d.3) we see that on the left-hand side we have a polynomial of degree \( \leq n + 1 + m - m^* \). Consequently,
\[
z^{\lambda_n+\lambda_{n+1}}[Q_{n,m}P_{n+1,m} - Q_{n+1,m}P_{n,m}](z) = z^{n+1}q_{n,m-m^*},
\]
where \( \deg q_{n,m-m^*} \leq m - m^* \). Dividing by \( z^{\lambda_n+\lambda_{n+1}}Q_{n,m}Q_{n+1,m} \) and normalizing \( q_{n,m-m^*} \) as in \( 2 \) we obtain the desired formula. \( \square \)

Take an arbitrary \( \varepsilon > 0 \) and define the open set \( J'_\varepsilon \) as follows. For \( n \geq m \), let \( J'_{n,\varepsilon} \) denote the \( \varepsilon/6nm^2 \)-neighborhood of the set of zeros of \( q_{n,m-m^*} \). Set \( J'_\varepsilon = \bigcup_{n \geq m} J'_{n,\varepsilon} \). For any compact set \( K \subset \mathbb{C} \) we put \( K'(\varepsilon) = K \setminus J'_\varepsilon \).
Due to the fact that the polynomial $q_{n,m-m^*}$ is normalized as in (2), for any compact set $K$ of $\mathbb{C}$ and for every $\varepsilon > 0$, there exist constants $M_1, M_2$, independent of $n$, such that
\begin{equation}
\|q_{n,m-m^*}\|_K < M_1, \quad \min_{z \in K(\varepsilon)} |q_{n,m-m^*}(z)| > M_2 n^{-2m},
\end{equation}
where the second inequality is meaningful when $K'(\varepsilon)$ is a non-empty set.

Define
\begin{equation}
R_m^*(f) = \limsup_{n \to \infty} \frac{1}{\|A_{n,m}\|^{1/n}}, \quad D_m^*(f) = \{z : |z| < R_m^*(f)\}.
\end{equation}

**Theorem 7.** Let $f$ be a formal power series as in (1). Fix $m$ and $m^*$ nonnegative integers, $m \geq m^*$. Let $\{R_{n,m}\}_{n \geq m}$ be a sequence of incomplete Padé approximants of type $(n,m,m^*)$ for $f$. If $R_m^*(f) > 0$ then $R_0^*(f) > 0$. Moreover,
\[D_m^*(f) \subset D_m^*(f) \subset D_m(f)\]
and $D_m^*(f)$ is the largest disk in compact subsets of which $\sigma\lim_{n \to \infty} R_{n,m} = f$. Moreover, the sequence $\{R_{n,m}\}_{n \geq m}$ is pointwise divergent in $\{z : |z| > R_m^*(f)\}$ except on a set of $\sigma$-content zero.

**Proof.** According to Lemma 1
\begin{equation}
R_{n+1,m}(z) - R_{n,m}(z) = \frac{A_{n,m}z^{n+1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}(z)}{Q_{n,m}(z) Q_{n+1,m}(z)}.
\end{equation}

Considering telescopic sums, it follows that the sequence $\{R_{n,m}\}_{n \geq m}$ converges or diverges with the series
\[\sum_{n \geq n_0} \frac{A_{n,m}z^{n+1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}(z)}{Q_{n,m}(z) Q_{n+1,m}(z)},\]
where $n_0$ is chosen conveniently so that $Q_{n_0,m}(z) \neq 0$ at the specific point under consideration.

Let $R_m^*(f) > 0$ and $K \subset D_m^*(f)$. Fix $\varepsilon > 0$. Using (3) and (10), we have
\begin{equation}
\limsup_{n \to \infty} \left\| \frac{A_{n,m}z^{n+1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}(z)}{Q_{n,m}(z) Q_{n+1,m}(z)} \right\|^{1/n}_{K(\varepsilon)} \leq \left\| \frac{z}{R_m^*(f)} \right\| < 1.
\end{equation}

Therefore, the series converges uniformly on $K(\varepsilon)$ for every $K \subset D_m^*(f)$ and every $\varepsilon > 0$. Thus $\sigma\lim_{n \to \infty} R_{n,m} = \varphi$ in $D_m^*(f)$, where, according to Gonchar’s Lemma, $\varphi$ is (except on a set of $\sigma$-content zero) a meromorphic function with at most $m$ poles in $D_m^*(f)$. On the other hand, if $|z| > R_m^*(f)$ and $z \notin J'_\varepsilon$ from (3) and (10) it follows that
\[\limsup_{n \to \infty} \left\| \frac{A_{n,m}z^{n+1-\lambda_n-\lambda_{n+1}} q_{n,m-m^*}(z)}{Q_{n,m}(z) Q_{n+1,m}(z)} \right\|^{1/n}_{K(\varepsilon)} \geq \left\| \frac{z}{R_m^*(f)} \right\| > 1,
\]
and the series diverges. Thus, the sequence $\{R_{n,m}\}_{n \geq m^*}$ pointwise diverges in $\{z : |z| > R_m^*(f)\}$ except on a set of $\sigma$-content zero (namely, $\cap_{\varepsilon > 0} J'_\varepsilon$).
We conclude the proof of the theorem if we show that \( R^*_m(f) > 0 \) implies that \( R_0(f) > 0 \). Indeed, if this is true, then necessarily \( \varphi = f \) in \( D^*_m(f) \) since by Proposition \([\text{1}]\) \( f \) is the \( \sigma \)-limit of \( \{R_{n,m}\}_{n \geq m} \) at least in compact subsets of \( D^*_m(f) \). Since \( D^*_m(f) \) is the largest disk centered at the origin in compact subsets of \( \{R_{n,m}\}_{n \geq m} \) converges to \( f \) in \( \sigma \)-content, we get that \( D_m(f) \subset D^*_m(f) \). On the other hand, \( D_m(f) \) is the largest disk centered at the origin in which \( f \) admits a meromorphic extension with no more than \( m \) poles, therefore \( D^*_m(f) \subset D_m(f) \).

Let \( R^*_m(f) > 0 \), then \( \sigma_{\text{lim}}_{n \to \infty} R_{n,m} = \varphi \) in \( D^*_m(f) \), where \( \varphi \) has at most \( m \) poles in this disk. Choose a subsequence of indices \( \Lambda \subset \mathbb{N} \) such that for all \( n \in \Lambda \) the number of poles of \( R_{n,m} \) is exactly equal to \( m_0, m_0 \leq m \), and \( \lim_{n \in \Lambda} \zeta_{n,j} = z_j, j = 1, \ldots, m_0 \). Suppose that \( \ell \) of the points \( z_j \) equal zero and let \( U \) be a neighborhood of \( z = 0 \) that does not contain any \( z_j \) other than zero and is contained in \( D^*_m(f) \). From Gouchar’s Lemma it follows that \( \lim_{n \in \Lambda} R_{n,m} = \varphi \) uniformly on each compact subset of \( U^* = U \setminus \{0\} \), where \( \varphi \) is holomorphic in \( U^* \), and its Laurent expansion in \( U^* \) has the form

\[
\varphi(z) = \sum_{k=-\ell}^{\infty} \varphi_k z^k.
\]

If we show that \( \varphi_k = 0, k = -\ell, \ldots, -1, \) and \( \varphi_k = \phi_k, k \geq 0, \) we obtain that \( \varphi \) is analytic in \( U \) and coincides with \( f \) in that set. In consequence, \( R_0(f) > 0 \).

Choose \( r > 0 \) such that \( \Gamma = \{z : |z| = r\} \) belongs to \( U^* \). For all sufficiently large \( n \in \Lambda \) the points \( \zeta_{n,j}, j = 1, \ldots, \ell \), are inside \( \Gamma \) and the points \( \zeta_{n,j}, j = \ell + 1, \ldots, m^* \), are outside this curve. From now on we only consider such \( n \)'s. Let us compare the Taylor expansion of \( R_{n,m} \) about \( z = 0 \)

\[
R_{n,m}(z) = \sum_{k=0}^{\infty} \alpha_{n,k} z^k,
\]

with its Laurent expansion on \( \Gamma \),

\[
R_{n,m}(z) = \sum_{k=-\infty}^{\infty} \beta_{n,k} z^k.
\]

For notational convenience we set \( \phi_k = 0 \) and \( \alpha_{n,k} = 0 \) for \( k = -1, -2, \ldots \) and \( \varphi_k = 0 \) for \( k = -\ell - 1, -\ell - 2, \ldots \) We restrict our attention to the case when all \( \zeta_{n,k}, k = 1, \ldots, \ell \), are distinct. The general case is proved analogously with some additional technical difficulties.

Let \( c_{n,j}, j = 1, \ldots, \ell \), be the residue of \( R_{n,m} \) at \( \zeta_{n,j} \). The Taylor expansion of \( R_{n,m} \) about \( z = 0 \) and its Laurent expansion on \( \Gamma \) differ only because of the expansion of the fractions \( c_{n,j}/(z - \zeta_{n,j}), j = 1, \ldots, \ell \). Therefore, it is easy to verify that

\[
\beta_{n,k} - \alpha_{n,k} = \sum_{j=1}^{\ell} \frac{c_{n,j}}{s_{n,j}} \frac{1}{z^k}, \quad k \in \mathbb{Z}.
\]

(14)
By the definition of $R_{n,m}$ (in particular, see (d.4)), $\alpha_{n,k} = \phi_k$ for $k < n + m - \lambda_n$; therefore, $\lim_{n \to \Lambda} \alpha_{n,k} = \phi_k$, $k \in \mathbb{Z}$. On the other hand, from the uniform convergence of $R_{n,m}$ to $\varphi$ on $\Gamma$ we also have that $\lim_{n \to \Lambda} \beta_{n,k} = \varphi_k$, $k \in \mathbb{Z}$. We obtain
\begin{equation}
\lim_{n \to \Lambda} (\beta_{n,k} - \alpha_{n,k}) = \varphi_k - \phi_k, \quad k \in \mathbb{Z}.
\end{equation}

Set $\varepsilon_{n,k} = \beta_{n,k} - \alpha_{n,k}$ and
\begin{equation}
L_n(z) = \prod_{j=1}^{\ell}(1 - \zeta_{n,j} z) = 1 + \gamma_{n,1} z + \cdots + \gamma_{n,\ell} z^\ell.
\end{equation}

Using (14), for arbitrary $k \in \mathbb{Z}$, we obtain
\begin{equation}
\varepsilon_{n,k} + \gamma_{n,1} \varepsilon_{n,k+1} + \cdots + \gamma_{n,\ell} \varepsilon_{n,k+\ell} = \sum_{j=1}^{\ell} \frac{c_{n,j}}{s_{n,j}} L_n(\zeta_{n,j}^{-1}) = 0.
\end{equation}

Since $\lim_{n \to \Lambda} \gamma_{n,j} = 0$, $j = 1, \ldots, \ell$, and $\lim_{n \to \Lambda} \varepsilon_{n,k+j} = \varphi_{k+j} - \phi_{k+j}$, $j = 1, \ldots, \ell$, from (10) it follows that $\lim_{n \to \infty} \varepsilon_{n,k} = 0$. Using (15) we obtain that $\varphi_k = \phi_k$, $k \in \mathbb{Z}$, as we wanted to prove.

Next, we will prove that each pole of the function $f$ in $D_m^*(f)$ attracts, with geometric rate, at least as many zeros of $Q_{n,m}$ as its order. For this purpose, let us define two indicators of the asymptotic behavior of the poles of the incomplete Padé approximants. These indicators were first introduced by A.A. Gonchar in [3] for the study of inverse type theorems for row sequences of Padé approximants. Let
\begin{equation}
P_{n,m} = \{\zeta_{n,1}, \ldots, \zeta_{n,\nu_n}\}, \quad n \in \mathbb{N}, \quad \nu_n \leq m,
\end{equation}
denote the collection of zeros of $Q_{n,m}$ (repeated according to their multiplicity). It is easy to verify that $| \cdot |_1 : \mathbb{C}^2 \to \mathbb{R}_+$ given by
\begin{equation}
|z - \omega|_1 = \min\{1, |z - \omega|\}, \quad z, \omega \in \mathbb{C},
\end{equation}
defines a distance in $\mathbb{C}$ (although $| \cdot |_1$ does not define a norm in $\mathbb{C}$).

Choose a point $a \in \mathbb{C}$. The first indicator is defined by
\begin{equation}
\Delta(a) = \lim_{n \to \infty} \sup_{\nu_n} \prod_{j=1}^{\nu_n} |\zeta_{n,j} - a|_1^{1/n} = \lim_{n \to \infty} \prod_{|\zeta_{n,j} - a|_1 < 1} |\zeta_{n,j} - a|_1^{1/n}.
\end{equation}

Obviously, $0 \leq \Delta(a) \leq 1$ (when $\nu_n = 0$ the product is taken to be 1).

The second indicator, a nonnegative integer $\mu(a)$, is defined as follows. We suppose that for each $n$ the points in $P_{n,m}$ are enumerated in nondecreasing distance to the point $a$. We put
\begin{equation}
\delta_j(a) = \lim_{n \to \infty} \sup_{|\zeta_{n,j} - a|_1 < 1} |\zeta_{n,j} - a|_1^{1/n}.
\end{equation}

These numbers are defined by (17) for $j = 1, \ldots, m'$, $m' = \liminf_{n \to \infty} \nu_n$; for $j = m' + 1, \ldots, m$ we define $\delta_j(a) = 1$. We have $0 \leq \delta_j(a) \leq 1$. If $\Delta(a) = 1$ (in that case all $\delta_j(a) = 1$), then $\mu(a) = 0$. If $\Delta(a) < 1$, then for...
some $\mu, 1 \leq \mu \leq m$, we have that $\delta_1(a) \leq \cdots \leq \delta_\mu(a) < 1$ and $\delta_{\mu+1}(a) = 1$

Clearly, $\Delta(a) < 1 \Leftrightarrow \mu(a) \geq 1$ and $\sum_{a \in \mathbb{C}} \mu(a) \leq m$. We shall need $\Delta(a)$

and $\mu(a)$ only for points $a \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. It is easy to verify that

\begin{equation}
\Delta(a) = \limsup_{n \to \infty} |Q_{n,m}(a)|^{1/n}.
\end{equation}

**Theorem 8.** Let $R_0(f) > 0$. Fix $m$ and $m^*$ nonnegative integers, $m \geq m^*$. For each $n \geq m$, let $R_{n,m}$ be an incomplete Padé approximant of type $(n,m,m^*)$ for $f$. Let $a$ be a pole of $f$ in $D_m^*(f)$ of order $\tau$. Then

\[\Delta(a) \leq \frac{|a|}{R_{m}^*(f)} \quad \text{and} \quad \mu(a) \geq \tau.\]

**Proof.** Let $a$ be a pole of $f$ in $D_m^*(f)$ of order $\tau$ and take $r > 0$ sufficiently small so that the disk of center $a$ and radius $r$, denoted by $D_{a,r}$, contains no other pole of $f$. It follows from Gonchar’s Lemma that the approximants $R_{n,m}$ have at least $\tau$ poles in $D_{a,r}$ for sufficiently large $n \in \mathbb{N}$. If this was not so, from Theorem 7 we have that there exists a subsequence $\{R_{n,m}\}_{n \in \Lambda}$ converging in $\sigma$-content to $f$ in compact subsets of $D_{a,r}$ with each approximant having less than $\tau$ poles in $D_{a,r}$ and part ii) of Gonchar’s Lemma would imply that $f$ has less than $\tau$ poles in $D_{a,r}$, which is absurd. As $r > 0$

is arbitrarily small, we have proved that each pole of $f$ in $D_m^*(f)$ attracts at least as many zeros of $Q_{n,m}$ as its order.

Fix $\varepsilon > 0$ arbitrarily small and take again $r > 0$ sufficiently small so that $D_{a,r}$ contains no other pole of $f$. Since $\sigma(J_{\varepsilon}) < \varepsilon$, we can choose $r$ such that $\Gamma_{a,r} = \{z : |z - a| = r\} \subset D_{m^*}^*(f) \setminus J_{\varepsilon}$. Let $\zeta_{n,1}, \ldots, \zeta_{n,m_n}$ be the zeros of $Q_{n,m}$ in $D_{a,r}$ indexed in non-decreasing distance from $a$. That is,

\[|a - \zeta_{n,1}| \leq |a - \zeta_{n,2}| \leq \cdots \leq |a - \zeta_{n,m_n}|.\]

For all sufficiently large $n$ we know that $\zeta_{n,\tau} \in D_{a,r}$. We will only consider such $n$’s. Consequently, we have $\tau \leq m_n \leq m$. Set

\[Q_{n,a}(z) = \prod_{j=1}^{m_n}(z - \zeta_{n,j}).\]

For any $\rho$ with $|a| + \tau < \rho < R_m^*(f)$, it follows from (12) and (13) that

\begin{equation}
\|f - R_{n,m}\|_{\Gamma_{a,r}} < C q^n, \quad q = \frac{|a| + \tau}{\rho} < 1,
\end{equation}

for sufficiently large $n$.

Let $p(z)/(z - a)^{\tau}$ be the principal part of the function $f$ at the point $a$ and $p_n/Q_{n,a}$ the sum of the principal parts of $R_{n,m}$ corresponding to its poles in $D_{a,r}$. We have $\deg p < \tau$, $p(a) \neq 0$, and $\deg p_n < \mu_p$. It is known that the norm of the holomorphic component of a meromorphic function may be bounded in terms of the norm of the function and the number of poles (see
plying the maximum principle, we have
for sufficiently large \( n \). Therefore, getting rid of the denominators and applying the maximum principle, we have
\[
\left\| \frac{p(z)}{(z-a)^r} - \frac{p_n(z)}{Q_{n,a}(z)} \right\|_{\Gamma_{a,r}} < C q^n,
\]
for sufficiently large \( n \). Thus, using (19), we obtain
\[
\|p(z) Q_{n,a}(z) - (z-a)^r p_n(z)\|_{\mathcal{P}_{a,r}} < C q^n,
\]
for sufficiently large \( n \). All the factors in \( Q_{n,m} \) that contribute to the limit value \( \Delta(a) \) are present in \( Q_{n,a} \), see (18) and (2). So, making \( z = a \) in (20) and taking limits as \( n \) tends to infinity gives the inequality \( \Delta(a) \leq q \). As \( r, \varepsilon, \) and \( \rho \) are arbitrary we have proved that \( \Delta(a) \leq |a|/R_n^\ast(f) \). To conclude the proof we must show that \( \mu(a) \geq \tau \). We will prove it by induction.

Since \( \Delta(a) < 1 \), we have \( \delta_1(a) < 1 \). Let \( \delta_1(a) \leq \cdots \leq \delta_k(a) < 1 \) and \( k < \tau \). We differentiate the polynomial inside the norm in (20) \( k \) times. As this polynomial has degree bounded by \( 2m - 1 \), its \( k \)th derivative satisfies an inequality similar to (20) by virtue of Bernstein’s inequality (see, for instance, Section 4.4.2 in [11]). If we put \( z = a \) in the corresponding inequality, we obtain
\[
\left| \left( p(z) \prod_{j=1}^{\mu_n} (z - \xi_{n,j}) \right)^{(k)} (a) \right| < C q^n,
\]
for sufficiently large \( n \). Now
\[
\left( p(z) \prod_{j=1}^{\mu_n} (z - \xi_{n,j}) \right)^{(k)} (a) = \sum_{|\alpha| = k} \frac{k!}{\alpha!} p^{(\beta)}(a) \prod_{j=1}^{\mu_n} (z - \xi_{n,j})^{(\alpha_j)} (a),
\]
where \( \alpha = (\beta, \alpha_1, \ldots, \alpha_{\mu_n}) \in \mathbb{Z}_{+}^{\mu_n+1} \), \( \alpha! = \beta! \cdot \alpha_1! \cdot \cdots \cdot \alpha_{\mu_n}! \), and \( |\alpha| = \beta + \alpha_1 + \cdots + \alpha_{\mu_n} \). By \( \sum_{|\alpha| = k} \) we mean that the sum is taken over all the multi-indices \( \alpha \) such that \( |\alpha| = k \). The total amount of such multi-indices is bounded independently of \( n \). One of them is \((0,1,\ldots,1,0,\ldots,0)\) corresponding to the term
\[
k! p(a) \prod_{j=k+1}^{\mu_n} (z - \xi_{n,j}).
\]
Each of the remaining terms must necessarily contain one factor \((z - \xi_{n,j})\), \( j \in \{1,2,\ldots,k\} \). Since we have assumed that \( \delta_j(a) < 1 \) for \( j = 1,\ldots,k \), it follows from (21) and (22) that
\[
\limsup_{n \to \infty} \prod_{j=k+1}^{\mu_n} |z - \xi_{n,j}|^{1/n} < 1,
\]
which in turn implies \( \limsup_{n \to \infty} |z - \xi_{n,k+1}|^{1/n} < 1 \), that is, \( \delta_{k+1}(a) < 1 \). Therefore it holds \( \mu(a) \geq \tau \) and we are done. \( \square \)
The estimate $\Delta(a) \leq |a|/R_m^*(f)$ can be sharpened if one knows that a
given pole attracts exactly as many zeros of $Q_{n,m}$ as its order.

**Theorem 9.** Let $R_0(f) > 0$ and let $a$ be a pole of $f$ in $D_m^*(f)$ of order $\tau$.
Assume that $\liminf_{n \to \infty} |a - z_{n,\tau+1}| > 0$. Then

$$\delta_1(a) \leq \cdots \leq \delta_{\tau}(a) \leq \left( \frac{|a|}{R_m^*(f)} \right)^{1/\tau}.$$  

In particular, $\delta_1(a) = \cdots = \delta_{\tau}(a) = (|a|/R_m^*(f))^{1/\tau}$ if and only if $\Delta(a) = |a|/R_m^*(f)$.

**Proof.** Let us maintain the notation used in the proof of Theorem 8. We
may assume that $Q_{n,a}(z) = \prod_{j=1}^{\tau} (z - \zeta_{n,j})$.

Recall that $p(a) \neq 0$. So, taking $z = a$ in (20), we obtain $|Q_{n,a}(a)| < C q^n$, for sufficiently large $n$. From this, (21), and the formula

$$(p Q_{n,a}(k))(a) = p(a) Q_{n,a}^{(k)}(a) + \sum_{j=0}^{k-1} \binom{k}{j} p^{(k-j)}(a) Q_{n,a}^{(j)}(a)$$

it readily follows by induction that

$$\left| Q_{n,a}^{(k)}(a) \right| \leq C q^n, \quad k = 0, 1, \ldots, \tau - 1,$$

for sufficiently large $n$. These inequalities and the expression

$$Q_{n,a}(z) = (z - a)^\tau + \sum_{k=0}^{\tau-1} \frac{Q_{n,a}^{(k)}(a)}{k!}(z - a)^k$$

give $\| (z - a)^\tau - Q_{n,a}(z) \|_{D_a,r} < C q^n$, for $n \geq N \in \mathbb{N}$. If we put here $z = \zeta_{n,\tau}$
we obtain

$$|\zeta_{n,\tau} - a|^{\tau} < C q^n, \quad n \geq N,$$

which implies $\delta_{\tau}(a) \leq q$. As $q = (|a| + r)/\rho$ and $r > 0$ and $\rho < R_m^*(f)$ are
arbitrary, we have

$$\delta_{\tau}(a) \leq \left( \frac{|a|}{R_m^*(f)} \right)^{1/\tau},$$

which is all we need to show since $\delta_1(a) \leq \cdots \leq \delta_{\tau}(a)$ is trivial.

On the other hand, according to Theorem 8, $\Delta(a) \leq |a|/R_m^*(f)$ is always true
and the last statement readily follows. \qed
4. Simultaneous approximation

Throughout this section, \( f = (f_1, \ldots, f_d) \) denotes a system of formal power expansions about the origin; that is,

\[
f_k(z) = \sum_{j=0}^{\infty} \phi_{k,j} z^j, \quad k = 1, \ldots, d,
\]

and \( \mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}^d_+ \setminus \{0\} \) is a fixed multi-index. We are concerned with the simultaneous approximation of \( f \) by sequences of vector rational functions defined according to Definition 2 taking account of (5). That is, for each \( n \in \mathbb{N}, n \geq |\mathbf{m}| \), let \( (R_{n,m_1}, \ldots, R_{n,m_d}) \) be a Hermite-Padé approximation of type \((n, \mathbf{m})\) corresponding to \( f \). As we mentioned earlier, \( R_{n,m,k} \) is an incomplete Padé approximant of type \((n, |\mathbf{m}|, m_k)\) with respect to \( f_k, k = 1, \ldots, d \). In the sequel, we consider \( \Delta \) and \( \mu \) defined as in Section \( \S \) taking \( \mathcal{P}_{n,m} \) to be the collection of zeros of the common denominator \( Q_{n,m} \).

The number \( R_{|\mathbf{m}|}^*(f_k) \) determines the radius of the largest disk, denoted by \( D_{|\mathbf{m}|}^*(f_k) \), centered at the origin in compact subsets of which we have \( \sigma\text{-}\lim_{n \to \infty} R_{n,m,k} = f_k \). Theorem 7 gives \( R_{m_k}(f_k) \leq R_{|\mathbf{m}|}^*(f_k) \leq R_{|\mathbf{m}|}(f_k) \) and a formula for finding \( R_{|\mathbf{m}|}^*(f_k) \). The following result is a rather straightforward consequence of Theorem 8.

**Corollary 1.** Suppose that \( R_0(f) > 0 \). For each \( k = 1, \ldots, d \), if \( a \) is a pole of \( f_k \) in \( D_{|\mathbf{m}|}^*(f_k) \) of order \( \tau \), then \( \Delta(a) \leq |a|/R_{|\mathbf{m}|}^*(f_k) \) and \( \mu(a) \geq \tau \).

**Proof.** Fix \( k = 1, \ldots, d \). Denote the denominator of \( R_{n,m,k} \), considered as an incomplete Padé approximant of \( f_k \), by \( Q_{n,m,k} \). This polynomial is either \( Q_{n,m} \) or a divisor of it, since there may be some additional cancelations of common factors with the numerator of \( R_{n,m,k} \). Let \( \Delta_k \) and \( \mu_k \) stand for the indicators \( \Delta \) and \( \mu \), respectively, when using \( Q_{n,m,k} \) instead of \( Q_{n,m} \). Let \( a \) be a pole of \( f_k \) in \( D_{|\mathbf{m}|}^*(f_k) \) of order \( \tau \). Then, Theorem 8 gives

\[
\Delta_k(a) \leq |a|/R_{|\mathbf{m}|}^*(f_k), \quad \mu_k(a) \geq \tau.
\]

It is clear that \( \mu(a) \geq \mu_k(a) \) whereas

\[
\Delta(a) = \limsup_{n \to \infty} |Q_{n,m}(a)|^{1/n} \leq \limsup_{n \to \infty} |Q_{n,m}(a)|^{1/n} = \Delta_k(a),
\]

which proves the result. \( \square \)

To each pole \( a \) of \( f \) in a system of domains \( \mathbf{D} = (D_1, \ldots, D_d) \) (see Definition 4) we associate an index \( k(a) \) \( \in \{1, \ldots, d\} \) as follows. The index \( k(a) \) verifies that \( a \in D_{k(a)} \) and \( a \) is a pole of \( f_{k(a)} \) of the same order as it is as a pole of \( f \) in \( \mathbf{D} \). If there are several indices \( k \) satisfying that condition we choose one among those with greatest \( R_{|\mathbf{m}|}^*(f_k) \).

Given a system \( f = (f_1, \ldots, f_d) \) and a multi-index \( \mathbf{m} \in \mathbb{Z}^d_+ \setminus \{0\} \), put

\[
D_{\mathbf{m}}^*(f) = \left(D_{|\mathbf{m}|}^*(f_1), \ldots, D_{|\mathbf{m}|}^*(f_d)\right), \quad D_{\mathbf{m}}^*(f) = \bigcap_{k=1}^{d} D_{|\mathbf{m}|}^*(f_k),
\]
and let $R^*_m(f)$ stand for the radius of $D^*_m(f)$. By $Q_m(f)$ we denote the polynomial whose zeros are the poles of $f$ in $D^*_m(f)$ counting multiplicities and normalized as $Q_{n,m}$ in (2). This set of poles is denoted by $P_m(f)$. For $k = 1, \ldots, d$, set $P_{m,k}(f) = P_m(f) \cap D^*_m(f_k)$.

**Lemma 2.** The following assertions hold:

a) If $R_0(f) > 0$ then $f$ has at most $|m|$ poles in $D^*_m(f)$.

b) If $R^*_m(f) > 0$ then $R_0(f) > 0$.

c) $R^*_m(f) \leq R_{|m|}(f)$.

d) Suppose that $f$ is polewise independent with respect to $m$ in $D_{|m|}(f)$, then $R^*_m(f) = R_{|m|}(f)$.

**Proof.** Suppose that $R_0(f) > 0$ and $f$ has more than $|m|$ poles in $D^*_m(f)$. Due to Corollary 11 each of those poles attracts as many zeros of $Q_{n,m}$ as its order. Then, deg $Q_{n,m} > |m|$, which is absurd. Therefore, a) takes place.

If $R^*_m(f) > 0$ then, for each $k = 1, \ldots, d$, $R^*_m(f_k) > 0$. By virtue of Theorem 7 taking $m = |m|$ and $m^* = m_k$, this implies $R_0(f_k) > 0, k = 1, \ldots, d$. This proves assertion b). As for c), if $R_0(f) = 0$ then the result is trivial due to part b). Suppose that $R_0(f) > 0$ and $R_{|m|}(f) < R^*_m(f)$. Then $f$ has at least $|m| + 1$ poles in $D^*_m(f)$, which contradicts part a).

Regarding d) we can assume that $R_0(f) > 0$ since the case $R_0(f) = 0$ is trivial. The Graves-Morris/Saff Theorem says in particular that, for each $k = 1, \ldots, d$, we have $\sigma$-lim$_{n \to \infty} R_{n,m,k} = f_k$ in $D_{|m|}(f)$. From the definition of $D^*_{|m|}(f_k)$ it follows that $D_{|m|}(f) \subset D^*_{|m|}(f_k), k = 1, \ldots, d$. Hence $D_{|m|}(f) \subset D^*_m(f)$ and by c) the equality follows. With this, we conclude the proof.

We will use the following concept in the next theorem.

**Definition 10.** We say that a compact set $K \subset \mathbb{C}$ is $\sigma$-regular if for each $z_0 \in K$ and for each $\delta > 0$, it holds

$$\sigma\{z \in K : |z - z_0| < \delta\} > 0.$$  

We are ready to prove our main result.

**Theorem 11.** Let $P_m(f) = \{a_1, \ldots, a_\nu\}$. Suppose that $R_0(f) > 0$ and that $f$ has exactly $|m|$ poles in $D^*_m(f)$. Then,

$$\limsup_{n \to \infty} \|f_k - R_{n,m,k}\|_{K}^{1/n} \leq \|z\|_K \frac{1}{R^*_m(f_k)}, \quad k = 1, \ldots, d,$$

where $K$ is any compact subset of $D^*_{|m|}(f_k) \setminus P_{m,k}(f)$. Also, we have

$$\limsup_{n \to \infty} \|Q_m(f) - Q_{n,m}\|^{1/n} \leq \max_{i=1,\ldots,\nu} \left\{ \frac{|a_i|}{R^*_m(f_k(a_i))} \right\}.$$

If, additionally, $K$ is $\sigma$-regular, then we have equality in (25).
Proof. Let \( a \) be an arbitrary pole of \( f \) in \( D^*_m(f) \) and let \( \tau \) be its order. Then, \( a \) is a pole of \( f_{k(a)} \) in \( D^*_{[m]}(f_{k(a)}) \) of order \( \tau \). According to Corollary 1 we have \( \mu(a) \geq \tau \). As this is true for any other pole of \( f \) in \( D^*_m(f) \) and \( \deg Q_{n,m} \leq |m| \), we have \( \deg Q_{n,m} = |m| \) for sufficiently large \( n \), \( \mu(a) = \tau \), and

\[
(27) \quad \lim_{n \to \infty} \|Q_m(f) - Q_{n,m}\| = 0.
\]

Take \( r > 0 \) sufficiently small so that \( D_{a,r} \) contains no other pole of \( f \). Let \( \zeta_{n,1}, \ldots, \zeta_{n,\mu_n} \) be the zeros of \( Q_{n,m} \) in \( D_{a,r} \) indexed in increasing distance from \( a \). That is,

\[
|a - \zeta_{n,1}| \leq |a - \zeta_{n,2}| \leq \cdots \leq |a - \zeta_{n,\mu_n}|.
\]

We know that \( \mu_n \geq \tau \) and \( \liminf_{n \to \infty} |a - z_{n,\tau+1}| > 0 \), so we can use the arguments employed in Theorem 9. In particular, formulas (23) and (24) prove that

\[
(28) \quad \limsup_{n \to \infty} \|(z - a)^\tau - Q_{n,a}\|^{1/\tau} \leq \frac{|a|}{R^*_m(f_{k(a)})},
\]

where

\[
Q_{n,a}(z) = \prod_{j=1}^\tau (z - \zeta_{n,j}).
\]

Formula (28) holds true for each of the poles of \( f \), so it may be rewritten as

\[
(29) \quad \limsup_{n \to \infty} \|(z - a_i)^{\tau_i} - Q_{n,a_i}\|^{1/\tau_i} \leq \frac{|a_i|}{R^*_m(f_{k(a_i)})},
\]

where \( \tau_i \) is the order of \( a_i \) as a pole of \( f \) in \( D^*_m(f) \), \( i = 1, \ldots, \nu \).

Let \( Q_{m} \) and \( Q_{n,m} \) be the polynomials \( Q_m \) and \( Q_{n,m} \) respectively normalized to be monic. We can write

\[
(Q_m - Q_{n,m})(z) = Q_m(z) - \frac{(Q_m Q_{n,a_1})(z)}{(z - a_1)^{\tau_1}} + \frac{(Q_m Q_{n,a_2})(z)}{(z - a_1)^{\tau_1}} - \cdots
\]

\[
+ \frac{(Q_m Q_{n,a_1} \cdots Q_{n,a_{\nu-1}})(z)}{(z - a_1)^{\tau_1} \cdots (z - a_{\nu-1})^{\tau_{\nu-1}}} - Q_{n,m}(z).
\]

Therefore

\[
|Q_m - Q_{n,m}|(z) \leq \sum_{i=1}^{\nu} \left| \frac{(Q_m Q_{n,a_1} \cdots Q_{n,a_{\nu-1}})(z)}{(z - a_1)^{\tau_1} \cdots (z - a_i)^{\tau_i}} \cdot [(z - a_i)^{\tau_i} - Q_{n,a_i}(z)] \right|.
\]

Since

\[
\lim_{n \to \infty} \frac{(Q_m Q_{n,a_1} \cdots Q_{n,a_{\nu-1}})(z)}{(z - a_1)^{\tau_1} \cdots (z - a_i)^{\tau_i}} = \frac{Q_m(z)}{(z - a_i)^{\tau_i}}, \quad i = 1, \ldots, \nu,
\]

uniformly on compact subsets of \( \mathbb{C} \), with the aid of (29), we obtain the inequality (26).
Now, fix $k = 1, \ldots, d$. Let $K$ be an arbitrary compact subset of $D_{\lvert m \rvert}^*(f_k) \setminus \mathcal{P}_{m,k}(f)$. Due to (27), and reasoning only for sufficiently large values of $n$, we have that $K = K(\varepsilon)$ for all $\varepsilon > 0$ sufficiently small, where the definition of $J_0$ is given for $Q_{n,m}$. Then, the inequality (25) follows from the formulas (12) and (13) when applied to the incomplete Padé approximant $R_{n,m,k}$.

Suppose now that the compact set $K \subset D_{\lvert m \rvert}^*(f_k) \setminus \mathcal{P}_{m,k}(f)$ is $\sigma$-regular, see Definition 10. Let us consider the constants $A_{n,m,k}$ and the polynomials $q_{n,m-m*,k}$ defined according to Lemma 1 for the incomplete Padé approximant $R_{n,m,k}$, where $m = \lvert m \rvert$ and $m* = m_k$. Denote the denominator of $R_{n,m,k}$ by $Q_{n,m,k}$. Put $J_0 = \cap_{\varepsilon>0} J_\varepsilon^*$ and take $z_0 \in K$ such that $\|z\|_K = |z_0| > 0$. As $J_0'$ is a set of $\sigma$-content zero and the compact set $K$ is $\sigma$-regular, there exists a sequence $\{z_j\}_{j \in \mathbb{N}} \subset K \setminus J_0'$ verifying $\lim_{j \to \infty} z_j = z_0$.

We may assume that $|z_j| > 0$ for all $j \in \mathbb{N}$.

From Lemma 1 it follows that

$$|A_{n,m,k}| = \frac{|Q_{n+1,m,k} Q_{n,m,k}(z_j)| |R_{n+1,m,k}(z_j) - R_{n,m,k}(z_j)|}{|z_j|^m |n+1 - \lambda_n - \lambda_{n+1}| |q_{n,m-m*,k}(z_j)|}.$$ 

We may write

$$|R_{n+1,m,k}(z_j) - R_{n,m,k}(z_j)| \leq \|f_k - R_{n+1,m,k}\|_K + \|f_k - R_{n,m,k}\|_K.$$ 

So, taking into account the formulas (8) and (10), we arrive at

$$\frac{1}{R_{\lvert m \rvert}^*(f_k)} = \limsup_{n \to \infty} |A_{n,m,k}|^{1/n} \leq \frac{1}{|z_j|} \limsup_{n \to \infty} \|f_k - R_{n,m,k}\|_K^{1/n}.$$ 

Taking limits in the above expression as $j$ tends to infinity, we obtain that the inequality (25) is actually an equality when $K$ is a $\sigma$-regular compact set, as we wanted to prove.

As was mentioned earlier, if $f$ is polewise independent in $D_{\lvert m \rvert}(f_k)$ it follows from Lemma 1 in [8] that $f$ has exactly $\lvert m \rvert$ poles in $D_{\lvert m \rvert}(f)$ and, due to part d) of Lemma 2, it has at least $\lvert m \rvert$ poles in $D_{\lvert m \rvert}^*(f)$. Now, part a) of Lemma 2 proves that Theorem 11 includes the Graves-Morris/Saff Theorem as a particular case although we have used the latter in establishing this fact.

Theorem 11 improves the Graves-Morris/Saff Theorem in several aspects. First of all, (25) gives the correct bound since we have shown that it is exact for $\sigma$-regular compact sets. The applicability of Theorem 11 is greater since there are systems $f$ that are not polewise independent and still have exactly $\lvert m \rvert$ poles in $D_{\lvert m \rvert}^*(f)$. Even when the system $f$ is polewise independent the region of convergence of the approximants given by Theorem 11 is in general larger than that of the Graves-Morris/Saff Theorem. Finally, the bounds (25) and (26) are less than or equal to (6) and (7), respectively. Several examples in Section 5.3 illustrate these improvements. In Section 5.2 we show that, in general, the bound (26) is still not exact.
5. Examples

5.1. On the values of $R_m^*(f)$. The purpose of this example is to show that $R_m^*(f)$ may take any value between $R_m^*(f)$ and $R_m(f)$ depending on the sequence of incomplete Padé approximants considered. Take $m^* = 1$, $m = 2$, and

$$f(z) = \frac{1}{1 - z^2}.$$  

Then, $R_1(f) = 1$, $R_2(f) = +\infty$. Consider

$$g(z) = \frac{z}{1 + z^2}, \quad h(z) = \frac{1}{1 + z}, \quad w_p(z) = \frac{1}{1 + z} + \frac{1}{1 - z/p}, \quad p > 1.$$  

Fix $m = (1, 1)$ and set $f = (f, g)$. It is clear that $R_2(f) = 1$ and the system $f$ is not polewise independent with respect to $m$ in $D_2(f)$. On the other hand, $R_1(f) = R_1(g) = 1$ and $R_2(f) = R_2(g) = +\infty$. It is very easy to see that $Q_{n,m} = 1 - z^2$ if $n$ is even and $Q_{n,m} = 1 + z^2$ when $n$ is odd. So, $R_2^*(f) = 1$ since $R_2^*(f) \geq R_1(f) = 1$ and $R_2^*(f)$ cannot be greater than 1. Otherwise, from part iii) of Gonchar’s Lemma, it follows that the polynomial $Q_{n,m}$ tends to $1 - z^2$, which is not true. An analogous argument proves that $R_2^*(g) = 1$. This example also shows that the reciprocal of the statement d) in Lemma 2 does not hold in general.

Now, take $f = (f, h)$ with the same multi-index $m$. Obviously, $R_2^*(h) = +\infty$ since $R_1(h) = +\infty$. The system $f$ is polewise independent with respect to $m$ in $D_2(f) = \mathbb{C}$. Using part d) of Lemma 2 we obtain $R_2^*(f) = +\infty$.

Finally, consider $f = (f, w_p)$ and fix $m = (1, 1)$. We have $R_2(f) = p$ and the system $f$ is polewise dependent with respect to $m$ in $D_2(f)$. As $R_2^*(w_p) \geq R_1(w_p) = p$, necessarily $R_2^*(f) \geq p$ due to Lemma 2 again. Then $Q_{n,m}$ tends to $1 - z^2$ and $R_2^*(w_p) = p$. An easy calculation shows that

$$Q_{n,m}(z) = \begin{cases} \lambda_n \left( z^2 + \frac{p^2 - 1}{p^n - p} z - 1 \right), & \text{if } n \text{ is even,} \\ z^2 - \frac{p^n - p^2}{p^n - 1}, & \text{if } n \text{ is odd,} \end{cases}$$

with $\lim_{n \to \infty} \lambda_n = 1$. Now, $R_2^*(f)$ may be worked out by means of formula (III) according to Lemma I. Keeping in mind the notation adopted there and using the expression of $Q_{n,m}$ calculated before, it turns out that

$$|A_{n,2}| = \lambda_n \frac{p(p^2 - 1)}{p^n - 1}, \quad n \text{ even.}$$

Then, $\lim_{n \to \infty} |A_{n,2}|^{1/n} = 1/p$, which implies

$$p \leq R_2^*(f) = \frac{1}{\lim \sup_{n \to \infty} |A_{n,2}|^{1/n}} \leq p.$$  

Thus, we have proved that $R_2^*(f) = p$ may take any value between $R_1(f) = 1$ and $R_2(f) = \infty$, both ends included.
5.2. Comparison between the Graves-Morris/Saff Theorem and Theorem [11]. First, let us see that there are very simple systems $f$ that are not polewise independent in $D_{|m|}(f)$ and still they have exactly $|m|$ poles in $D_m^*(f)$. Set

$$f_1(z) = \frac{1}{1 - z} + \frac{1}{2 - z}, \quad f_2(z) = \frac{1}{3 - z},$$

and fix the multi-index $m = (1, 1)$. Put $f = (f_1, f_2)$. It is clear that $R_2(f) = 3$ and, as $0f_1 + f_2$ is analytic in $D_2(f)$, the system $f$ is not polewise independent in $D_2(f)$. Also, as $R_1(f_2) = \infty$, we have $R_m^*(f_2) = \infty$ and one of the poles of $Q_{n,m}$ is attracted by the point $z = 3$ on account of Corollary [11].

On the other hand, $R_1(f_1) = 2$, so $R_m^*(f_1) \geq 2$ but $R_2^*(f_1)$ cannot be greater than 2 since in that case two other poles of $Q_{n,m}$ would be attracted by the points $z = 1$ and $z = 2$, which is absurd. Then $R_m^*(f_1) = 2$ and the system $f$ has exactly two poles, $z = 1$ and $z = 3$, in $D_m^*(f)$. This example also shows that the inequality appearing in part c) of Lemma [2] may be strict.

Now, fix again $m = (1, 1)$ and take $g = (g_1, g_2)$, where

$$g_1(z) = \frac{1}{1 - z} + \log(3 - z), \quad g_2(z) = \frac{1}{2 - z} + \log(10 - z).$$

Obviously, $R_1(g_1) = R_2^*(g_1) = R_1(g_2) = 3$ and $R_1(g_2) = R_2^*(g_2) = R_2(g_2) = 10$. The system $g$ is polewise independent in $D_2(g)$ with $R_2(g) = 3$. The Morris-Graves/Saff Theorem gives

$$\limsup_{n \to \infty} \|g_2 - R_{n,m,2}\|_K^{1/n} \leq \frac{\|z\|_K}{3},$$

for any compact subset $K$ of $\{z : |z| < 3\}$ and

$$\limsup_{n \to \infty} \|Q_m(g) - Q_{n,m}\|^{1/n} \leq 2/3,$$

where $Q_m(g)(z) = (1 - z)(1 - z/2)$. On the other hand, Theorem [11] gives

$$\limsup_{n \to \infty} \|g_2 - R_{n,m,2}\|_K^{1/n} \leq \frac{\|z\|_K}{10},$$

for any compact subset $K$ of $\{z : |z| < 10\}$ and

$$\limsup_{n \to \infty} \|Q_m(g) - Q_{n,m}\|^{1/n} \leq \max\{1/3, 1/5\} = 1/3.$$

5.3. On the rate of convergence of $\{Q_{n,m}\}$. Let us show that the rate of convergence of the sequence of polynomials $Q_{n,m}$ given by the inequality (26) is not exact in general. Fix $m = (1, 1)$ and consider the system $h = (h_1, h_2)$, where

$$h_1(z) = \frac{1}{1 - z} + \frac{1}{2 - z} + \log(3 - z), \quad h_2(z) = \frac{1}{1 - z} + \log(3 - z) + \log(4 - z).$$

Obviously $R_2(h) = 3$ and the system $h$ is polewise independent in $D_2(h)$. As $R_1(h_2) = R_2(h_2) = 3$, we have $R_m^*(h_2) = 3$. On the other hand, we have
that $R_2(h_1) = 3$, from which it follows that $R_2^*(h_1) \leq 3$. Using part d) of Lemma 2, we obtain $R_2^*(h_1) = 3$. Therefore, Theorem 11 gives
\[
\limsup_{n \to \infty} \left\| Q_m(h) - Q_{n,m} \right\|^{1/n} \leq \max\{1/3, 2/3\} = 2/3,
\]
where $Q_m(h)(z) = (1 - z)(1 - z/2)$.

Consider now the system $\hat{h} = (\hat{h}_1, \hat{h}_2)$, where $\hat{h}_1 = h_1 - h_2$ and $\hat{h}_2 = h_2$. We have $R_1(\hat{h}_1) = 4 = R_2(\hat{h}_1)$, hence $R_2^*(\hat{h}_1) = 4$. As before, $R_2^*(\hat{h}_2) = 3$. Obviously, the $(n,m)$ Hermite-Padé approximants of the systems $h$ and $\hat{h}$ have the same common denominator $Q_{n,m}$. Using again Theorem 11 for the new auxiliary system, we obtain a better estimate
\[
\limsup_{n \to \infty} \left\| Q_m(h) - Q_{n,m} \right\|^{1/n} \leq \max\{1/3, 2/4\} = 1/2.
\]

REFERENCES

[1] C. Brezinski, Padé-Type Approximation and General Orthogonal Polynomials, Birkhäuser, Basel, 1980.
[2] A.A. Gonchar, On the convergence of generalized Padé approximants of meromorphic functions, Sb. Math. 27 (1975) 503–514.
[3] A.A. Gonchar, Poles of rows of the Padé table and meromorphic continuation of functions, Sb. Math. 43 (1982) 527–546.
[4] A.A. Gonchar, L.D. Grigorjan, On estimates of the norm of the holomorphic component of a meromorphic function, Sb. Math. 28 (1976) 571–575.
[5] J. Hadamard, Essai sur l’étude des fonctions données par leur développement de Taylor, J. Math. Pures Appl. 8 (1892) 101–186.
[6] M. Marden, Geometry of Polynomials, Amer. Math. Soc., Providence, Rhode Island, 1949.
[7] R. de Montessus de Ballore, Sur les fractions continues algébriques, Bull. Soc. Math. France 30 (1902) 28–36.
[8] P.R. Graves-Morris, E.B. Saff, A de Montessus theorem for vector-valued rational interpolants, Lecture Notes in Math. 1105, Springer, Berlin, 1984, pp. 227–242.
[9] P.R. Graves-Morris, E.B. Saff, Row convergence theorems for generalized inverse vector-valued Padé approximants, J. Comp. Appl. Math. 23 (1988) 63–85.
[10] P.R. Graves-Morris, E.B. Saff, An extension of a row convergence theorem for vector Padé approximants, J. Comp. Appl. Math. 34 (1991) 315–324.
[11] T. Sheil-Small, Complex Polynomials, Cambridge University Press, Cambridge, 2002.