Comments on Newton-Schrödinger Equations

pertaining to [1] and relevant references therein. The
Newton-Schrödinger equations (NSE) play a prominent
role in alternative quantum theories (AQT) [2] and have
served as a theoretical platform for experimentalists to
investigate the interaction of quantum matter with clas-
sical gravity. The latest development is the application
of a many-body NSE to macroscopic mechanical objects
[3]. In this note we show that NSEs do not follow from
general relativity (GR) and quantum field theory (QFT).
We come to this conclusion from two considerations: 1)
Working out a model (see [3]) with matter described by
a scalar field interacting with weak gravity, solve the con-
straint, canonically quantize the system then take the
nonrelativistic (NR) limit. This procedure is the exact
analogue of deriving the NR limit of quantum electro-
dynamics (QED). 2) Taking the NR limit of the semi-
classical Einstein equation (SCE), the central equation
of relativistic semiclassical gravity (RSCG) [10], a fully
covariant theory based on GR+QFT with self-consistent
backreaction of quantum matter on the spacetime dy-
namics [3]. We first lay out two key physical differences:

A. In NSE the gravitational self-energy defines non-
linear terms in Schrödinger’s equation. In Diosi’s theory
[4], the gravitational self-energy defines a stochastic term
in the master equation. With GR+QFT gravitational
self-energy only contributes to mass renormalization, at
least in the weak field (WF) limit. The Newtonian interac-
tion term at the field level induces a divergent self-energy
contribution to the single-particle Hamiltonian. It does
not induce nonlinear terms to the Schrödinger equation
for any number of particles.

B. The one-particle NS equation appears as the Hartree
approximation for N particle states as N → ∞. Con-
ider the ansatz |Ψ⟩ = |χ⟩ ⊗ |χ⟩ ⊗ · · · ⊗ |χ⟩ for a N-particle
system. At the limit N → ∞ the generation of par-
cle correlations in time is suppressed and one gets an
equation which reduces to the NS equation for χ [3, 8].
However, in the Hartree approximation, χ(r) is not the
wave-function ψ(r) of a single particle, but a collective
variable that describes a system of N particles under a
mean field approximation.

The NSE governing the wave function of a single
particle ψ(r, t) is of the form i∂ψ/∂t = −ℏc2∇2ψ + m2V_N[ψ]|ψ⟩ Eq.(1), where V_N(r) is the (normalized) grav-
itational (Newtonian) potential given by V_N(r, t) = − ∫ dr′|ψ(r′, t)|2/|r − r′|. It satisfies the Poisson equation
∇2V_N = 4πGρ, with the mass density ρ = m|ψ(r, t)|2
being the nonrelativistic limit of energy density ε = T_00.

The semiclassical Einstein equation is of the form
G_{μν} = 8πG(Ψ|T_{μν}|Ψ) Eq.(2), where ⟨T_{μν}⟩ is the expec-
tation value of the stress energy density operator T_{μν}
with respect to some quantum state |Ψ⟩ of the field.
In the weak field limit the spacetime metric has the form
ds2 = (1− 2V)c2dt2 − dr2, and the non-relativistic limit
of the SCE Eq becomes ∇2V = 4πGε, where ε = T_00 is
the energy density operator. This can be solved to yield
V(r) = −G ∫ dr′(|ψ(r′)|2/|r − r′|) Eq.(3). Note ⟨ψ(r′)|2⟩ in
genral contains ultraviolet divergences and need be reg-
ularized via known procedures.

Two key differences between the NR limit of SCE and
NSE are: 1) the energy density ε(r) is an operator, not a
c-number. The Newtonian potential is not a dynamical
object in GR, but subject to constraint conditions. 2) the
state |Ψ⟩ of a field is a N-particle wave function. Quan-
tum matter is coupled to classical gravity as a mean-field
theory, well defined only when N is sufficiently large.

The (misplaced) procedure leading one from SCE to
a NS equation is the treatment of m|ψ(r, t)|2 as a mass
density for a single particle, while in fact the mass den-
sity is a quantum operator ε(r) = ψ†(r)ψ(r) in the QFT
Hilbert space. Not treating these quantities as operators
bears the consequences A and B.

To cross check these observations we have car-
ried out an independent calculation following the
same procedures as in obtaining the non-relativistic
limit of QED. Treating the matter degrees of free-
dom in terms of quantum fields ψ(r) and ψ†(r), we
obtain the Schrödinger equation for the fields
i∂|Ψ⟩/∂t = H|Ψ⟩, with H = −ℏ2c2 ∫ drψ†(r)∇2ψ(r) −
G ∫ dr dr′(|ψ†ψ(r)|(|ψ†ψ(r′)|/|r − r′|) Eq.(4). This equa-
tion is very different from the NSE when considering a
single particle state. For single-particle states the grav-
itational interaction leads only to a mass-renormalization
term (similar to mass renormalization in QED). This is
point A we made above. Using the Hartree approxima-
tion to Eq. (4) leads to the same result as the NR WF
limit of SCE, not NSE. (Point B). Details are in [9].

Our analysis via two routes based on GR+QFT shows
that NSEs are not derivable from them. Coupling of clas-
sical gravity with quantum matter can only be via mean
fields. There are no N-particle NSEs. Theories based
on Newton-Schrödinger equations assume unknown physics.

C. Anastopoulos, University of Patras, Greece and B.
L. Hu, University of Maryland, USA. February 12, 2014.

[1] H. Yang et al, Phys. Rev. Lett. 110, 170401 (2013).
[2] A. Bassi et al, Rev. Mod. Phys. 85, 471- 527 (2013).
[3] C. Anastopoulos and B. L. Hu, Class. Quant. Grav. 30, 165007 (2013).
[4] B. L. Hu, in EmQM2013, Vienna. J. Phys. (Conf).
[5] B. L. Hu and E. Verdaguer, Liv. Rev. Rel. 11 (2008) 3
[6] L. Diosi, Phys. Lett. A105, 199 (1984).
[7] R. Alicki and J. Messer, J. Stat. Phys. 32, 299 (1983).
[8] S. L. Adler, Class. Quant. Grav. 30, 195015 (2013).
[9] C. Anastopoulos and B. L. Hu, (in preparation)
[10] There are four levels of semiclassical gravity theories [2] and one needs be careful which level one refers to when
debating issues, better use the most developed level [3].