Well-posedness and long time behavior for the electron inertial Hall-MHD system in Besov and Kato-Herz spaces

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Abstract

In this paper, we study the wellposedness of the Hall-magneto-hydrodynamic system augmented by the effect of electron inertia. Our main result consists of generalising the wellposedness one in [13] from the Sobolev context to the general Besov spaces and Kato-Herz space, then we show that we can reduce the required regularity of the magnetic field in the first result modulo an additional condition on the maximal time of existence. Finally, we show that the $\hat{L}^p$ (and eventually the $L^p$) norm of the solution $(u, B, \nabla \times B)$ associated to an initial data in $\hat{B}^{3-1}_{p, \infty} (\mathbb{R}^3)$, is controled by $t^{-\frac{2}{p}(1-\frac{2}{p})}$, for all $p \in (3, \infty)$, which provides a polynomial decay to zero of the $\hat{L}^p$ norm of the solution.

Keywords: MHD equation, Littlewood-Paley, critical spaces.

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1 Introduction

In this work we consider the incompressible 3D electron inertia Hall-MHD equations derived from the two fluid model of ion and electron

\begin{equation}
\begin{aligned}
\partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P &= j \times [B - \delta(1-\delta)\lambda^2 \Delta B]
\partial_t B' - \eta \Delta B' + (1-\delta)\lambda^2 \mu_e \Delta^2 B = \nabla \times \left\{ [u - (1-\delta)\lambda j] \times B' \right\} - \lambda \mu_e \Delta(\nabla \times u)
\end{aligned}
\end{equation}

\begin{align*}
j &= \frac{4\pi}{\varepsilon} \nabla \times B
B' &= B - \delta(1-\delta)\lambda^2 \Delta B - \delta \lambda(\nabla \times u)
div u = div B = 0
(u_{|t=0}, B_{|t=0}) = (u_0, B_0)
\end{align*}

Where $u$ is the hydrodynamc velocity, $B$ the magnetic field, the scalar function $P$ denotes the pressure which can be recovered, due the incompressibility condition, from the relation $-\Delta P = \nabla \cdot (u \cdot \nabla u - j \times [B - \delta(1-\delta)\lambda^2 \Delta B])$. $c$ is the speed of light and $j$ denotes the electric current density. $\mu_e$ is the kinematic viscosity of the electron fluid, and if we denote $\mu_i$ that of the ion fluid, then the kinematic viscosity $\mu$ will be $\mu_i + \mu_e$. the very small parameter $\delta$ is given by $\delta = m_e/M$, where $M = m_e + m_i$, which is the sum of masses of ion and electron. If we denote $e, n, L_0$ the charge, number density and the length scale, respectively, and if we denote $w_M \overset{\text{def}}{=} (4\pi e^2 n/M)^{\frac{1}{2}}$, then $\lambda$ will be $\lambda = c/w_M L_0$.

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As pointed out in the introduction of [13], the system above can be seen as full two-fluid MHD description of a completely ionized hydrogen plasma, retaining the effects of the Hall current, electron pressure and electron inertia. For more details about the derivation of the system, we refer the reader to [1], [13].

As shown in [13], the system above can be simplified into the following one

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \Delta \mathbf{u} + \nabla P &= (\nabla \times \mathbf{B}) \times \mathbf{H} \\
\frac{\partial \mathbf{H}}{\partial t} - \Delta \mathbf{H} + 2\nabla \times \left((\nabla \times \mathbf{B}) \times \mathbf{H}\right) &= \nabla \times (\mathbf{u} \times \mathbf{H}) + \nabla \times ((\nabla \times \mathbf{B}) \times (\nabla \times \mathbf{u})) \\
\mathbf{H} &= (\mathbb{I} - \Delta) \mathbf{B} \\
\text{div} \mathbf{u} &= \text{div} \mathbf{B} = 0 \\
(u|_{t=0}, B|_{t=0}) &= (u_0, B_0),
\end{align*}
\]

where we omit the constants which will play no significant role in the mathematical study of the system.

If we try to deal with (H.MHD) as it is, the structure of the force terms will prevent us from establishing the wellposedness for all \( p \in [1, \infty[ \). the issue in fact will be at the level of estimating the remainder terms in Bony’s decomposition. Also, it is worth noting that the system above does not have a scaling invariance structure as the classical Navier Stokes equations does, let us first rewrite it in an appropriate form. To do so, we recall some vectorial concepts.

For \( U, V \) two divergence free vector fields \( \mathbb{R}^3 \), we have

\[
\nabla \times (U \times V) = V \cdot \nabla U - U \cdot \nabla V
\]

\[(\nabla \times U) \times U = U \cdot \nabla U - \frac{1}{2}|V|^2\]

\[V \times (\nabla \times U) + U \times (\nabla \times V) = -\nabla \times (U \times V) - 2U \cdot \nabla V + \nabla(U \cdot V)\]

\[\nabla \times (\nabla \times U) = -\Delta U\]

If we denote \( J \overset{\text{def}}{=} \nabla \times B \), then according to [2] and [4] we obtain

\[(\nabla \times B) \times (B - \Delta B) = B \cdot \nabla B - J \cdot \nabla J + \nabla \left(\frac{|B|^2 - |J|^2}{2}\right),\]

in the other hand, we have

\[u \times H + (\nabla \times B) \times (\nabla \times u) = u \times B + u \times (\nabla \times J) + J \times (\nabla \times u),\]

thus according to [3], we infer that

\[u \times H + (\nabla \times B) \times (\nabla \times u) = u \times B - \nabla \times (J \times u) - 2J \cdot \nabla u + \nabla(J \cdot u),\]

therefore, (H.MHD) can be written as follows

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} - \Delta \mathbf{u} &= \mathbf{B} \cdot \nabla \mathbf{B} - J \cdot \nabla J - \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \hat{P} \\
\frac{\partial \mathbf{B}}{\partial t} - \Delta \mathbf{B} &= (\mathbb{I} - \Delta)^{-1} \nabla \times \Theta \\
\frac{\partial J}{\partial t} - \Delta J &= -\Delta (\mathbb{I} - \Delta)^{-1} \Theta + \nabla (\mathbb{I} - \Delta)^{-1} (\nabla \cdot \Theta) \\
\text{div} \mathbf{u} = \text{div} \mathbf{B} = \text{div} J &= 0 \\
(u, B, J)|_{t=0} &= (u_0, B_0, \nabla \times B_0),
\end{align*}
\]
where,
\[ \tilde{P} \stackrel{\text{def}}{=} -p + \frac{|B|^2 - |J|^2}{2} \]
\[ \Theta \stackrel{\text{def}}{=} u \times B - 2B \cdot \nabla B - J \cdot \nabla J - \nabla \times (J \times u) - 2J \cdot \nabla u + \nabla (J \cdot u). \]

Unlike \((H,MHD)\), system \((S)\) has the same scaling as the classical Navier Stokes equations, that is \((S)\) is invariant under the following transformation:

If \((u, B, J)\) is the solution associated to \((u_0, B_0, J_0)\), then \((u_\lambda, B_\lambda, J_\lambda)\) is the one associated to \((u_0\lambda, B_0\lambda, J_0\lambda)\), with

\[ f_{0,\lambda}(x) \stackrel{\text{def}}{=} \lambda f_0(\lambda x), \quad f_\lambda(t, x) \stackrel{\text{def}}{=} \lambda f(\lambda^2 t, \lambda x), \]

and all the following spaces are then critical with respect to this transformation (the definition of the functional spaces is in Appendix)

\[ B^{\frac{3}{m-1}}_{m,n}(\mathbb{R}^3) \hookrightarrow H_0^1(\mathbb{R}^3) \hookrightarrow B^{\frac{3}{p}-1}_{p,r}(\mathbb{R}^3) \hookrightarrow B^{\frac{3}{q,r'}-1}_{q,r'}(\mathbb{R}^3) \hookrightarrow B^{-1}_{\infty,\infty}(\mathbb{R}^3) \]

for all \(m, n, p, q, r, r'\) satisfying

\[ (m \leq 2 \leq p < \infty, \quad n \leq 2 \leq r < r' < \infty). \]

Let us now write down the Duhamel’s formula corresponding to \((S)\). Let \(\mathcal{P}\) be the Leray projector, we denote

\[ \mathcal{Q}(u, v) \stackrel{\text{def}}{=} \nabla \cdot (u \otimes v), \]
\[ \mathcal{P}(u, v) \stackrel{\text{def}}{=} u \times v, \]
\[ \mathcal{R}(u, v) \stackrel{\text{def}}{=} \nabla \times (u \times v), \]

for a three dimensional vectors \(K = (K_1, K_2, K_3)\), \(L = (L_1, L_2, L_3)\), we define

\[ \Gamma(K, L) \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{Q}(K_2, L_2) - \mathcal{Q}(K_1, L_1) - \mathcal{Q}(K_3, L_3) \\ \mathcal{P}(K_1, L_2) - \mathcal{R}(k_1, L_1) - \mathcal{Q}(K_3, L_3) - 2\mathcal{Q}(K_2, L_2) - 2\mathcal{Q}(K_3, L_1) \\ \mathcal{P}(K_1, L_2) - \mathcal{R}(k_3, L_1) - \mathcal{Q}(K_3, L_3) - 2\mathcal{Q}(K_2, L_2) - 2\mathcal{Q}(K_3, L_1) \end{pmatrix}, \]
\[ \Omega(K, L) \stackrel{\text{def}}{=} \mathcal{P} \begin{pmatrix} \Gamma_1(K, L) \\ (Id - \Delta)^{-1}\nabla \times \Gamma_2(K, L) \\ -\Delta(Id - \Delta)^{-1}\Gamma_2(K, L) \end{pmatrix}, \quad (6) \]

and

\[ \zeta(K, L) \stackrel{\text{def}}{=} \mathcal{K}\Omega(K, L), \quad (7) \]

where

\[ \mathcal{K}\varphi(t, \cdot) = \int_0^t e^{(t-s)\Delta} \varphi(s, \cdot)ds, \]

finally, if we denote \(\mathcal{U} \stackrel{\text{def}}{=} (u, B, J)\), then \((S)\) is equivalent to

\[
\left\{ \begin{array}{l}
\partial_t \mathcal{U} - \Delta \mathcal{U} = \zeta(\mathcal{U}, \mathcal{U}) \\
\text{div} \mathcal{U} = \text{div} B = \text{div} J = 0 \\
\mathcal{U}|_{t=0} = \mathcal{U}_0
\end{array} \right.
\]

\( (S_\zeta) \)
Remark 1. As mentioned for the classical Navier Stokes equations in a paper of I.Gallagher, D.Iftime and F.Planchon \[8\], the theory of weak solutions to the Navier Stokes equations in related to special structure of the equation, namely to the energy inequality, while the Kato’s approach is more general and can be applied to more general parabolic or dispersive equations, this work is an example of many. The main issue here consists at writing the equations in an appropriate form in order to be able to adapte the techniques used for the classical Navier Stokes \[2, 8, 12, 6\].

Before stating our results, let us fix some notations which will be of a constant use in this paper,

- For \(A, B\) two real quantities, \(A \lesssim B\) means \(A \leq cB\), for some \(c > 0\) independent of \(A\) and \(B\).

- \((c_{j,r})_{j \in \mathbb{Z}}\) will be a sequence satisfying \(\sum_{j \in \mathbb{Z}} c_{j,r} \leq 1\). This sequence is allowed to differ from a line to line, also let us point out that, due to the embedding \(\ell^r(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})\), we will often use the inequality \(c_{j,r}^2 \leq c_{j,r}\).

- The \(\hat{L}^p\) norm of \(u\) is given by \(\|u\|_{\hat{L}^p} \overset{def}{=} \|\hat{u}\|_{L^{p'}}\), where \(p'\) is the usual conjugate of \(p\).

- We use the notation \(\mathcal{L}(B_{p,r}^s) \overset{def}{=} \bigcap_{\rho \in [1, \infty]} \tilde{L}^\rho([0,T]; B_{p,r}^{s+\frac{2}{r}})\), and for \(T > 0\), \(\mathcal{L}_T(B_{p,r}^s) \overset{def}{=} \bigcap_{\rho \in [1, \infty]} \tilde{L}^\rho([0,T]; B_{p,r}^{s+\frac{2}{r}})\)

Let us now define what we will mean by a solution to \((S_\zeta)\) in this paper

Definition 1. Let \(T > 0\), and \(U_0\) be given in some Banach space \(\mathcal{X}\), we say that \(U\) is a solution to \((S_\zeta)\) on \((0,T)\) if \(U \in L^\infty_{loc}([0,T]; \mathcal{X})\) and satisfies, for a.e \(t \in [0,T]\)

\[
(\mathcal{L}(B_{p,r}^s) \subset \mathcal{L}_T(B_{p,r}^s) \overset{def}{=} \bigcap_{\rho \in [1, \infty]} \tilde{L}^\rho([0,T]; B_{p,r}^{s+\frac{2}{r}})\)

The authors in \[13\] proved the wellposedness of (H.MHD) under the condition of \(\|u_0\|_{H^{\frac{3}{2}}} + \|B_0\|_{H^\frac{1}{2}} + \|B_0\|_{H^\frac{3}{2}}\) small enough, our first result consists of generalising this last one to the Besov context, it reads as follows

Theorem 1. Let \(p \in [1, \infty), r \in [1, \infty]\) and \(U_0 = (u_0, B_0, \nabla \times B_0)\) be in \(B_{p,r}^{\frac{3}{2} - 1}(\mathbb{R}^3)\)

There exists \(c_0 > 0\) such that, if

\[
\|U_0\|_{B_{p,r}^{\frac{3}{2} - 1}} < c_0
\]

then \((S_\zeta)\) has a unique global solution \(U\) which is also in \(\mathcal{L}(B_{p,r}^{\frac{3}{2} - 1})\), with

\[
\|U\|_{\mathcal{L}(B_{p,r}^{\frac{3}{2} - 1})} < 2c_0
\]

Remark 2. One may show that the solution, in the case \(r < \infty\), is continuous in time with value in \(B_{p,r}^{\frac{3}{2} - 1}\), while in the case \(r = \infty\) it is just weakly-continuous in time.

Remark 3. One may prove a local in time wellposedness for large initial data, by slightly modifying the proof of theorem \[7\], we will give some details about that in corollary \[7\].
In terms of the required regularity, in Theorem 1 we ask for the initial data of $B$ to be in $B_{p,r}^{-\frac{3}{2} - 1}(\mathbb{R}^3) \cap B_{p,r}^{\frac{3}{2}}(\mathbb{R}^3)$, it is worth to note that, it is because of the two non linear terms $u \times B$ in $(S_\zeta)_2$, and $B \cdot \nabla B$ in $(S_\zeta)_1$, that we don’t know how to prove an analogue result to theorem 1 starting from initial data $B_0$ only in $B_{p,r}^{\frac{3}{2}}$. However, in the case $r = 1$, we will prove that, a small enough “compared to the maximal time of existence $T^*$” initial data of $B_0$ in $B_{p,1}^{\frac{3}{2}}$ should generate a unique solution, at least up to time $T^*$. More precisely, we will prove

**Theorem 2.** Let $T > 0$, $p \in [1, \infty)$ and $(u_0, B_0)$ be two divergence vector fields in $B_{p,1}^{\frac{3}{2} - 1} \times B_{p,1}^{\frac{3}{2}}$, there exists $c_0 > 0$ such that if

$$
\|u_0\|_{B_{p,1}^{\frac{3}{2} - 1}} + (2 + T) \|B_0\|_{B_{p,1}^{\frac{3}{2}}} < c_0
$$

Then $(S_\zeta)$ has a unique solution $(u, B)$ on $(0, T)$ with

$$
\|u\|_{\mathcal{L}_T(B_{p,1}^{\frac{3}{2} - 1})} + (2 + T) \|B\|_{\mathcal{L}_T(B_{p,1}^{\frac{3}{2}})} < 2c_0
$$

The question of the behavior, for large time, of the solution obtained in theorem 1 can be established along ”approximately” the same lines as shown for instance for the 3D Navier Stokes equations in [8], that is it should be possible to prove that $\|U(t)\|_{B_{p,r}^{\frac{3}{2} - 1}}$ tends to zero as $t$ tends to infinity.

It is also well known, for the Navier Stokes equations, that for an initial data $u_0 \in B_{p,r}^{\frac{3}{2} - 1}$, $p \in (3, \infty)$, also the $L^\infty$ norm of the velocity decays to zero at infinity, and more precisely it is controled by $Ct^{-\frac{1}{2}}$. The proof of this result relies on the fact that the bilinear operator in Duhamel’s formula acts well on the Kato’s space, together with the fact that we can iterate, sufficiently many times as much as we want, the solution $u$ in the Duhamel’s formul in order to obtain a solution of the form of a sum of some $N$ multi-bilinear terms of $e^{t\Delta} u_0$, and a more regular remainder term $r_{N+1}$ which is unique in $L^\infty_t(\mathbb{R}^3)$. A priori, this approach should work as well in our case, but we will not enter into the details of that in this paper.

In constrast of that, we will treat the case of initial data in the Herz-space $\hat{B}_{p,r}^{\frac{3}{2} - 1}(\mathbb{R}^3)$ (see Appendix for the definition and some properties of such spaces), and we will give some details in the case $r = \infty$ as an example. More precisely, we will prove

**Theorem 3.** Let $p \in (3, \infty)$ and $u_0, B_0$ be two divergence free vector fields, there exists $c_0$ such that if

$$
\|e^{t\Delta} u_0\|_{\hat{K}_p^{1 - \frac{3}{2p}}} + \|e^{t\Delta} B_0\|_{\hat{K}_p^{1 - \frac{3}{2p}}} + \|e^{t\Delta} (\nabla \times B_0)\|_{\hat{K}_p^{1 - \frac{3}{2p}}} < c_0
$$

Then $(S_\zeta)$ has a unique global solution $\mathcal{U} = (u, B, \nabla \times B)$ in $\hat{K}_p^{1 - \frac{3}{2p}}$ satisfying

$$
\|\mathcal{U}(t, \cdot)\|_{L^p} \lesssim t^{-\frac{1}{2}(1 - \frac{1}{p})}
$$

**Remark 4.** By virtue of the caracterisation (20), we can show that the obtained solution in theorem 3 is also in $\mathcal{L}(\hat{B}_{p,\infty}^{\frac{3}{2} - 1})$, by following approximately the same steps of the proof of theorem 1 and 2. In fact if $\mathcal{U}_0 \in \hat{B}_{p,r}^{\frac{3}{2} - 1}$, small enough, we may construct a unique global solution in $\mathcal{L}(\hat{B}_{p,r}^{\frac{3}{2} - 1})$ by proceeding as in the proofs we will show in the next section, we left the details of this to the reader.
The rest of the paper is organized as follows: In section two we will prove, respectively, the wellposedeness in theorem \[1\] and theorem \[2\] then we provide some details about the proof of the wellposedeness in Kato-Herz space and the decay property described in theorem \[3\]. Finally, the Appendix is devoted to the definitions of the functional spaces used in this work, together with some useful technical results.

2 Proof of the three theorems

2.1 Proof of Theorem \[1\]

The main key to prove theorem \[1\] is the following proposition

**Proposition 1.** Let \((p,r)\) be in \([1,\infty)^2\), \(u, v\) in \(L^{\frac{3}{2}}(B_{p,r}^{-1})\), \(Q, R\) and \(P\) be giving as in the introduction, then we have

\[
\|Q(u, v)\|_{L^1(B_{p,r}^{-1})} \lesssim \|u\|_{L^2(B_{p,r}^{-1})} \|v\|_{L^2(B_{p,r}^{-1})}
\]

\[
\|R(u, v)\|_{L^1(B_{p,r}^{-1})} \lesssim \|u\|_{L^2(B_{p,r}^{-1})} \|v\|_{L^2(B_{p,r}^{-1})}
\]

\[
\|P(u, v)\|_{L^1(B_{p,r}^{-1})} \lesssim \|u\|_{L^2(B_{p,r}^{-1})} \|v\|_{L^2(B_{p,r}^{-1})}
\]

**Proof**

All we need to show is how to prove the last inequality, the two first ones then can be proved along the same lines seen that \(D^1(P) \approx Q \approx R\), where \(D^1\) one-order Fourier-multiplier.

We consider the Bony’s decomposition described in the Appendix, to write

\[
u v = T_u v + T_v u + R(u, v),
\]

for the first term, we have

\[
\|\Delta_j T_u v\|_{L^1(L^p_x)} \lesssim \sum_{k \geq j} \|S_{k-1} u\|_{L^4(L^\infty_x)} \|\Delta_k v\|_{L^4(L^p_x)}
\]

using proposition A.2.1 we infer that

\[
\|\Delta_j T_u v\|_{L^1(L^p_x)} \lesssim \sum_{k \geq j} c_{k,r} 2^{-k \frac{p}{2}} \|u\|_{L^4(B_{p,r}^{\frac{3}{2}})} \|v\|_{\dot{L}^4(B_{p,r}^{\frac{3}{2} + \frac{1}{2}})}
\]

\[
\lesssim c_{j,r} 2^{-j \frac{p}{2}} \|u\|_{L^4(B_{p,r}^{\frac{3}{2} + \frac{1}{2}})} \|v\|_{\dot{L}^4(B_{p,r}^{\frac{3}{2} + \frac{1}{2}})}
\]

\(T_v u\) enjoys the same estimate by commuting \(u\) and \(v\) in the previous one, we obtain then

\[
\|\Delta_j T_v u\|_{L^1(L^p_x)} \lesssim c_{j,r} 2^{-j \frac{p}{2}} \|v\|_{L^4(B_{p,r}^{\frac{3}{2} + \frac{1}{2}})} \|u\|_{\dot{L}^4(B_{p,r}^{\frac{3}{2} + \frac{1}{2}})}
\]

Finally, the remainder term, can be dealt with along the same lines, we infer that

\[
\|\Delta_j R(u, v)\|_{L^1(L^p_x)} \lesssim \sum_{k \geq j + N_0} \|\tilde{\Delta}_k u\|_{L^4(L^\infty_x)} \|\tilde{\Delta}_k v\|_{\dot{L}^4(L^\infty_x)}
\]

\[
\lesssim 2^{-j \frac{p}{2}} \sum_{k \geq j + N_0} 2^{(k-j)\frac{p}{2}} c_{k,r} 2^{(k-j)\frac{p}{2}} \|u\|_{L^4(B_{p,r}^{\frac{3}{2}})} \|v\|_{\dot{L}^4(B_{p,r}^{\frac{3}{2} + \frac{1}{2}})}
\]

\[
\lesssim c_{j,r} 2^{-j \frac{p}{2}} \|u\|_{L^4(B_{p,r}^{\frac{3}{2} + \frac{1}{2}})} \|v\|_{\dot{L}^4(B_{p,r}^{\frac{3}{2} + \frac{1}{2}})}
\]
using then, the embedding \( L^{\frac{3}{p} + \frac{1}{2p}}(B^{\frac{3}{p}}_{p,r}) \cap \tilde{L}^{4}(B^{\frac{3}{p} - \frac{1}{2p}}_{p,r}) \), we obtain
\[
\|uv\|_{L^{1}(B^{\frac{3}{p}}_{p,r})} \lesssim \|u\|_{L^{\frac{3}{p} - 1}(B^{\frac{3}{p} - 1}_{p,r})} \cdot \|v\|_{L^{\frac{3}{p} - 1}(B^{\frac{3}{p} - 1}_{p,r})}
\]
(8)

It follows then
\[
\|P(u, v)\|_{L^{1}(B^{\frac{3}{p}}_{p,r})} \approx \|uv\|_{L^{1}(B^{\frac{3}{p}}_{p,r})} \lesssim \|u\|_{L^{\frac{3}{p} - 1}(B^{\frac{3}{p} - 1}_{p,r})} \cdot \|v\|_{L^{\frac{3}{p} - 1}(B^{\frac{3}{p} - 1}_{p,r})}
\]
(9)

and
\[
\|Q(u, v)\|_{L^{1}(B^{\frac{3}{p} - 1}_{p,r})} \approx \|R(u, v)\|_{L^{1}(B^{\frac{3}{p} - 1}_{p,r})} \lesssim \|u\|_{L^{\frac{3}{p} - 1}(B^{\frac{3}{p} - 1}_{p,r})} \cdot \|v\|_{L^{\frac{3}{p} - 1}(B^{\frac{3}{p} - 1}_{p,r})}
\]
(10)

Proposition [1] is proved.

Remark 5. We can replace \( \tilde{L}^{1}(\cdot) \) and \( L^{(\cdot)} \) in the previous proposition, respectively, by \( \tilde{L}^{1}_{T}(\cdot) \) and \( L^{T}(\cdot) \), for \( T > 0 \).

In order to prove theorem [1] we will use the following abstract lemma of Banach fixed point theorem. The reader can see lemma 4 in [3] for more details.

**Lemma 1.** Let \( \mathcal{X} \) be an abstract Banach space with norm \( \|\cdot\| \), let \( \zeta \) be a bilinear operator mapping \( \mathcal{X} \times \mathcal{X} \) into \( \mathcal{X} \) satisfying
\[
\forall x_1, x_2 \in \mathcal{X}, \quad \|\zeta(x_1, x_2)\| \leq \eta \|x_1\| \cdot \|x_2\|, \quad \text{for some } \eta > 0,
\]
then for all \( y \in \mathcal{X} \) such that
\[
\|y\| < \frac{1}{4\eta}
\]
the equation
\[
x = y + \zeta(x, x)
\]
has a solution \( x \in \mathcal{B}_{\mathcal{X}}(0, \frac{1}{2\eta}) \).
This solution is the unique one in the ball \( \mathcal{B}_{\mathcal{X}}(0, \frac{1}{2\eta}) \).

**Proof of theorem [1]** In order to apply lemma [1] all we need to show is that
\[
\|\zeta(U, V)\|_{L^{(\frac{3}{p})}(B^{\frac{3}{p}}_{p,r})} \lesssim \|U\|_{L^{\frac{3}{p} - 1}(B^{\frac{3}{p} - 1}_{p,r})} \cdot \|V\|_{L^{\frac{3}{p} - 1}(B^{\frac{3}{p} - 1}_{p,r})}
\]
\[
\|e^{t \Delta U_{0}}\|_{L^{(\frac{3}{p})}(B^{\frac{3}{p}}_{p,r})} \lesssim \|U_{0}\|_{B^{\frac{3}{p}}_{p,r}}
\]
the second inequality follows directly from inequality (23) from lemma A.2.4 in Appendix, and the first one follows by combining proposition [1] proposition A.2.2 and inequality (24) from lemma A.2.4 indeed, \( \zeta = (\zeta_{1}, \zeta_{2}, \zeta_{3})^{T} \) contains in each component the bilinear operators \( Q, P, R \):

- For \( Q \) and \( R \):
  - in \( \zeta_{1} \) we apply directly proposition [1]
  - in \( \zeta_{2} \) we apply proposition [1] and inequality (21) for \( \rho = 1 \) and \( k = 1 \)
  - in \( \zeta_{2} \) we apply proposition [1] and inequality (22) for \( \rho = 1 \) and \( k = 2 \)

- For \( P \)
  - in \( \zeta_{2} \) we apply proposition [1] and inequality (21) for \( \rho = 1 \) and \( k = 0 \)
  - in \( \zeta_{2} \) we apply proposition [1] and inequality (22) for \( \rho = 1 \) and \( k = 1 \)
Remark 6. As a corollary, one may replace the smallness condition on the initial data in theorem 1, by another one on the maximal time of existence, namely we can prove

Corollary 1. Let \( p \in [1, \infty), \ r \in [1, \infty), \) and \( U_0 = (u_0, B_0, \nabla \times B_0) \), be in \( B^{\frac{3}{p}-1}_{p,r}(\mathbb{R}^3). \)
There exists \( T^* > 0 \) and a unique solution \( \mathcal{U} \) to \((S_{\xi})\) on \([0,T]\), for all \( T < T^* \).

This solution is also in \( \mathcal{L}_T(B^{\frac{3}{p}-1}_{p,r}). \)

Proof

We split the solution \( \mathcal{U} \) into a sum
\[
\mathcal{U} = V + W,
\]
where \( V \) is given by
\[
V(t, \cdot) \overset{\text{def}}{=} e^{t\Delta} U_0,
\]
It remains then to solve, by fixed point argument, the equation on \( W \)
\[
W = \zeta(V, V) + \zeta(W, W) + \zeta(V, W),
\]
let us point out that, according to the previous calculations, we have
\[
\|\Omega(W, W)\|_{\mathcal{L}^{\frac{3}{2}-1}_{T}(B^{\frac{3}{2}-1}_{p,r})} \leq \gamma \|W\|_{\mathcal{L}^{\frac{3}{2}-1}_{T}(B^{\frac{3}{2}-1}_{p,r})}^2
\]
for some \( \gamma > 0 \), where \( \Omega \) is given by \((6)\), and also we have
\[
\|\zeta(V, V)\|_{\mathcal{L}^{\frac{3}{2}-1}_{T}(B^{\frac{3}{2}-1}_{p,r})} = \|K\Omega(V, V)\|_{\mathcal{L}^{\frac{3}{2}-1}_{T}(B^{\frac{3}{2}-1}_{p,r})} \lesssim \|\Omega(V, V)\|_{\mathcal{L}^{\frac{3}{2}}_{T}(B^{\frac{3}{2}}_{p,r})}
\]

On the other hand we know that \( \Omega(V, V) \in \mathcal{L}^{\frac{3}{2}}_{T}(B^{\frac{3}{2}}_{p,r}) \) from the estimates of proposition 1 and lemma [A.2.4], which gives in particular
\[
\|\Omega(V, V)\|_{\mathcal{L}^{\frac{3}{2}}_{T}(B^{\frac{3}{2}}_{p,r})} \lesssim \|V\|_{\mathcal{L}^{\frac{3}{2}}_{T}(B^{\frac{3}{2}}_{p,r})}^2 \lesssim \|V_0\|_{B^{\frac{3}{2}}_{p,r}}^2 < \infty
\]

For the linear term on \( W \), by virtue of lemma [A.2.4] and the proof of proposition 1, where we showed that we can obtain the estimates of \( Q, P \) and \( R \), by suing only the norm of \( V \) in \( \mathcal{Z}_T \overset{\text{def}}{=} \mathcal{L}^{\frac{3}{2}}_{T}(B^{\frac{3}{2}}_{p,r}) \cap \mathcal{L}^{\frac{3}{2}}_{T}(B^{\frac{3}{2}+\frac{1}{2}}_{p,r}) \), we infer that
\[
\|\zeta(V, W)\|_{\mathcal{L}^{\frac{3}{2}-1}_{T}(B^{\frac{3}{2}-1}_{p,r})} \leq C \|V\|_{\mathcal{Z}_T} \|W\|_{\mathcal{L}^{\frac{3}{2}-1}_{T}(B^{\frac{3}{2}-1}_{p,r})}
\]
for some universal constant \( C > 0 \), we chose \( T_1 \) such that, for all \( T < T_1 \)
\[
C \|V\|_{\mathcal{Z}_T} < \lambda < 1,
\]
then, we can chose \( T^* \leq T_1 \) small as much as we want such that, for all \( T < T^* \leq T_1 \), we have
\[
\|\Omega(V, V)\|_{\mathcal{L}^{\frac{3}{2}-1}_{T}(B^{\frac{3}{2}-1}_{p,r})} < \varepsilon_1 \leq \frac{(1 - \lambda)^2}{4\gamma}
\]
The result then can be reached by a direct application of lemma [A.2.4].
2.2 proof of Theorem 2

The key estimates to prove theorem 2 is shown in the following proposition

**Proposition 2.** Let $p$ be in $[1, \infty)$, $T > 0$, $u$ in $L_T(B^{\frac{3}{p}-1}_{p,1})$ and $w, z$ be in $L_T(B^{\frac{3}{p}}_{p,1})$, then we have

$$
\|Q(w, z)\|_{L^1_{T}(B^{\frac{3}{p}-1}_{p,1})} \leq T \|w\|_{L^\infty_{T}(B^{\frac{3}{p}}_{p,1})} \|z\|_{L^\infty_{T}(B^{\frac{3}{p}}_{p,1})}
$$

(11)

$$
\|Q(w, z)\|_{L^1_{T}(B^{\frac{3}{p}+1}_{p,1})} \leq \|w\|_{L^\infty_{T}(B^{\frac{3}{p}}_{p,1})} \|z\|_{L^\infty_{T}(B^{\frac{3}{p}}_{p,1})}
$$

(12)

$$
\|P(u, w)\|_{L^1_{T}(B^{\frac{3}{p}+1}_{p,1})} \leq \|u\|_{L^\infty_{T}(B^{\frac{3}{p}}_{p,1})} \|w\|_{L^\infty_{T}(B^{\frac{3}{p}}_{p,1})}
$$

(13)

$$
\|R(u, w)\|_{L^1_{T}(B^{\frac{3}{p}}_{p,1})} \leq \|u\|_{L^\infty_{T}(B^{\frac{3}{p}}_{p,1})} \|w\|_{L^\infty_{T}(B^{\frac{3}{p}}_{p,1})}
$$

(14)

**Proof**

The proof does not work for $r > 1$, where the embedding $B^{\frac{3}{p}}_{p,r}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ fails to be true unless when $r = 1$. In this part of the paper, we will denote $d_j \overset{\text{def}}{=} c_{j,1}$

**Proof of (11):** Inequality (11) follows directly from the fact that $\tilde{L}^\infty(B^{\frac{3}{p}}_{p,1})$ is an algebra and the (local in time) embedding

$$
\|a\|_{L^1(B^{\frac{3}{p}}_{p,1})} \leq T \|a\|_{L^\infty(B^{\frac{3}{p}}_{p,1})}
$$

**Proof of (12):**

According to Bony’s decomposition, we have

$$
wz = Twz + Tz w + R(w, z),
$$

we show then how to estimate the first and the third term, we have

$$
\|Q_j w z\|_{L_j^1 L_p} \lesssim \|Q_{j-1} w\|_{L_j^\infty L^\infty} \|Q_j z\|_{L_j^1 L_p}
$$

$$
\lesssim d_j 2^{-j(\frac{3}{p}+2)} \|w\|_{L_j^\infty L^\infty} \|w\|_{L_j^1(B^{\frac{3}{p}+2}_{p,1})}
$$

the embedding $B^{\frac{3}{p}+1}_{p,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, together with Minkoski inequality, give

$$
\|Q_j w z\|_{L_j^1 L_p} \lesssim d_j 2^{-j(\frac{3}{p}+2)} \|w\|_{L_j^\infty(B^{\frac{3}{p}+2}_{p,1})} \|w\|_{L_j^1(B^{\frac{3}{p}+2}_{p,1})}.
$$

For the remainder term, we proceed as follows

$$
\|\Delta_j R(w, z)\|_{L_j^1 L_p} \lesssim \sum_{k \geq j + N_0} \|\Delta_k w\|_{L_j^1 L_p} \|\Delta_k z\|_{L_j^\infty L^\infty}
$$

$$
\lesssim 2^{-j(\frac{3}{p}+2)} \sum_{k \geq j + N_0} d_k 2^{(j-k)(\frac{3}{p}+2)} \|w\|_{L_j^1(B^{\frac{3}{p}+2}_{p,1})} \|z\|_{L_j^\infty(B^{\frac{3}{p}}_{p,1})}
$$

$$
\lesssim 2^{-j(\frac{3}{p}+2)} d_j \|w\|_{L_j^1(B^{\frac{3}{p}+2}_{p,1})} \|z\|_{L_j^\infty(B^{\frac{3}{p}}_{p,1})}
$$

Inequality (12) follows.

**Proof of (13):** Let us point out again that, due to $\mathcal{D}^1(\mathcal{P}) \approx R$, (13) and (14) can be proved
Minkoski inequality gives then

\[ uw = T_u w + T_w u + R(u, w) \]

For \( T_u w \), we have

\[
\| \Delta_j (T_u w) \|_{L^1_T L^p} \lesssim \| S_{j-1} u \|_{L^\infty_T L^\infty} \| \Delta_j w \|_{L^1_T L^p} \\
\lesssim d_j 2^{-j(\frac{4}{p}+1)} \| u \|_{L^\infty_T (B^{\frac{4}{p}+1}_{p,1})} \| w \|_{L^1_T (B^{\frac{4}{p}+2}_{p,1})} \\
\lesssim d_j 2^{-j(\frac{4}{p}+1)} \| u \|_{L^\infty_T (B^{\frac{4}{p}+1}_{p,1})} \| w \|_{L^1_T (B^{\frac{4}{p}+2}_{p,1})}.
\]

For \( T_w u \), by using the embedding \( B^{\frac{4}{p}+1}_{p,1}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3) \), we infer that

\[
\| \Delta_j (T_w u) \|_{L^1_T L^p} \lesssim \| S_{j-1} w \|_{L^\infty_T L^\infty} \| \Delta_j u \|_{L^1_T L^p} \\
\lesssim d_j 2^{-j(\frac{4}{p}+1)} \| w \|_{L^\infty_T (L^\infty)} \| u \|_{L^1_T (B^{\frac{4}{p}+1}_{p,1})} \\
\lesssim d_j 2^{-j(\frac{4}{p}+1)} \| w \|_{L^\infty_T (B^{\frac{4}{p}+1}_{p,1})} \| u \|_{L^1_T (B^{\frac{4}{p}+1}_{p,1})}.
\]

Minkoski inequality gives then

\[
\| \Delta_j (T_w u) \|_{L^1_T L^p} \lesssim d_j 2^{-j(\frac{4}{p}+1)} \| w \|_{L^\infty_T (B^{\frac{4}{p}+1}_{p,1})} \| u \|_{L^1_T (B^{\frac{4}{p}+1}_{p,1})}.
\]

For the remainder term, we have

\[
\| \Delta_j R(u, w) \|_{L^1_T L^p} \lesssim \sum_{k \geq j + N_0} \| \Delta_k u \|_{L^1_T L^p} \| \Delta_k w \|_{L^\infty_T L^\infty} \\
\lesssim 2^{-j(\frac{4}{p}+1)} \sum_{k \geq j + N_0} d_k 2^{(j-k)(\frac{4}{p}+1)} \| u \|_{L^1_T (B^{\frac{4}{p}+1}_{p,1})} \| w \|_{L^\infty_T (B^{\frac{4}{p}+1}_{p,1})} \\
\lesssim 2^{-j(\frac{4}{p}+1)} d_j \| u \|_{L^1_T (B^{\frac{4}{p}+1}_{p,1})} \| w \|_{L^\infty_T (B^{\frac{4}{p}+1}_{p,1})}.
\]

This ends the proof of inequality (14), and eventually (15).

Lemma 2 is then proved. \( \square \)

The proof of theorem 2 is based on the following variation of lemma 1.

**Lemma 2.** Let \( \{A_i\}_{i \in \{1, 2, 3, 4\}} \) be a set of bilinear operators with

\[
\| A_1(x_1, x_2) \|_\chi \leq \eta_1 \| x_1 \|_\chi \| x_2 \|_\chi \\
\| A_2(y_1, y_2) \|_\chi \leq (1 + T) \| y_1 \|_\gamma \| y_2 \|_\gamma \\
\| A_3(x_1, y_2) \|_\gamma \leq \eta_3 \| x_1 \|_\chi \| y_2 \|_\gamma \\
\| A_4(y_1, y_2) \|_\gamma \leq \eta_4 \| y_1 \|_\gamma \| y_2 \|_\gamma
\]

for some non negative \( T, (\eta_i)_{i \in \{1, 2, 3, 4\}} \).

Let \( \eta \overset{\text{def}}{=} \max_{i \in \{1, 2, 3, 4\}} \eta_i \). Then for all \( (x_0, y_0) \in (\chi \times \gamma) \) such that

\[
\| x_0 \|_\chi + (2 + T) \| y_0 \|_\gamma \leq \frac{1}{24 \eta}
\]

the system

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\[
\begin{aligned}
\begin{cases}
x = x_0 + A_1(x, x) + A_2(y, y) \\
y = y_0 + A_3(x, y) + A_4(y, y)
\end{cases}
\end{aligned}
\]

has a unique solution \((x, y)\) in \(\mathcal{X} \times \mathcal{Y}\), which also satisfies
\[
\|x\|_\mathcal{X} + (2 + T) \|y\|_\mathcal{Y} < \frac{1}{12\eta}
\]

**Proof**

Let us present brevly the outlines of the proof, the idea is as usual:
Defining the sequence \((x^n, y^n)\) by
\[
\begin{aligned}
\begin{cases}
(x^0, y^0) = (x_0, y_0) \\
x^{n+1} = x_0 + A_1(x^n, x^n) + A_2(y^n, z^n) \\
y^{n+1} = y_0 + A_3(x^n, y^n) + A_4(y^n, y^n)
\end{cases}
\end{aligned}
\]

If we denote \(z^n \equiv (1 + T)y^n\), then the system above is equivalent to
\[
\begin{aligned}
\begin{cases}
(x^0, y^0, z^0) = (x_0, y_0, (1 + T)y_0) \\
x^{n+1} = x_0 + A_1(x^n, x^n) + A_2(y^n, z^n) \\
y^{n+1} = y_0 + A_3(x^n, y^n) + A_4(y^n, y^n) \\
z^{n+1} = z_0 + A_3(x^n, z^n) + A_4(y^n, z^n)
\end{cases}
\end{aligned}
\]

with \(\tilde{A}_2 = \frac{1}{1+T}A_2\) whose norm is less than \(\eta\).

Let \(\alpha \equiv \|x_0\|_\mathcal{X} + \|y_0\|_\mathcal{Y} + \|z_0\|_\mathcal{Y} < \frac{1}{24\eta}\), we claim then \((x^n, y^n, z^n)\) to be a Cauchy bounded sequence in \(\mathcal{B}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Y}}(0, 2\alpha)\).

By virtue of the definition of \((x^n, y^n, z^n)\) and the continuity of \(A_i\), we proceed by induction to obtain
\[
\begin{aligned}
\begin{cases}
\|x^{n+1}\|_\mathcal{X} \leq \|x_0\|_\mathcal{X} + 8\alpha \eta^2 \\
\|y^{n+1}\|_\mathcal{Y} \leq \|y_0\|_\mathcal{Y} + 8\alpha \eta^2 \\
\|z^{n+1}\|_\mathcal{Y} \leq \|z_0\|_\mathcal{X} + 8\alpha \eta^2
\end{cases}
\end{aligned}
\]

which gives
\[
\|x^{n+1}\|_\mathcal{X} + \|y^{n+1}\|_\mathcal{Y} + \|z^{n+1}\|_\mathcal{Y} < 2\alpha
\]

In order to prove that \((x^n, y^n, z^n)\) is a Cauchy sequence, similar calculus lead to
\[
I_n \equiv \|x^{n+1} - x^n\|_\mathcal{X} + \|y^{n+1} - y^n\|_\mathcal{Y} + \|z^{n+1} - z^n\|_\mathcal{Y} \leq (24\eta\alpha)I_{n-1}
\]

This should be enough to conclude the proof. \(\square\)

**Proof of Theorem** 2 In order to apply lemma 2 let us rewrite the system \((S_\zeta)\) as follows, we define
\[
\varphi \left( \begin{array}{c}
\psi_1(x_1, x_2) \\
\psi_2(y_1, y_2) \\
\psi_3(x_1, y_1) \\
\psi_4(y_1, y_2)
\end{array} \right) \equiv \left( \begin{array}{c}
-Q(x_1, x_2) \\
-Q(y_1, y_2) \\
\mathcal{P}(x_1, y_1) - \mathcal{R} \left( \nabla \times y_1, \nabla \times x_1 \right) - 2Q \left( \nabla \times y_1, x_1 \right) \\
-2Q(y_1, y_2) - Q \left( \nabla \times y_1, \nabla \times y_2 \right)
\end{array} \right),
\]
then
\[
\varphi \left( \begin{array}{c}
x_1 \\
y_1 \\
x_2 \\
y_2
\end{array} \right) = \left( \begin{array}{c}
\varphi_1(x_1, x_2) \\
\varphi_2(y_1, y_2) \\
\varphi_3(x_1, y_1) \\
\varphi_4(y_1, y_2)
\end{array} \right) \equiv \left( \begin{array}{c}
\psi_1(x_1, x_2) \\
\psi_2(y_1, y_2) \\
(Id - \Delta)^{-1} \nabla \times \psi_3(x_1, y_1) \\
(Id - \Delta)^{-1} \nabla \times \psi_4(y_1, y_2)
\end{array} \right),
\]
and then, we define
\[
\begin{pmatrix}
A_1(x_1, x_2) \\
A_2(y_1, y_2) \\
A_3(x_1, y_1) \\
A_4(y_1, y_2)
\end{pmatrix}
\defeq K_P \varphi \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix},
\]
with
\[
K_P(t, \cdot) = \int_0^t e^{(t-s)\Delta} P \varphi(s, \cdot) ds,
\]
therefore, the system \((S_\zeta)\) is equivalent to the following one
\[
\begin{aligned}
u(t, \cdot) &= e^{t\Delta} u_0 + A_1(u, u) + A_2(B, B) \\
B(t, \cdot) &= e^{t\Delta} B_0 + A_3(u, B) + A_4(B, B)
\end{aligned}
\]
The proof of theorem 2 can be reduced to a direct application of lemma 2, thus all we need to show then is that \(\{A_i\}_{i \in \{1, 2, 3, 4\}}\) satisfies the hypothesis of lemma 2 with \(X = L(B_{p,1}^{\frac{2}{p}-1})\) and \(Y = L(B_{p,1}^{\frac{2}{p}})\), to do so, according to lemma A.2.4 we should estimate \(\varphi\) in
\[
\hat{L}_T^1(B_{p,1}^{\frac{2}{p}-1} \times B_{p,1}^{\frac{2}{p}-1} \times B_{p,1}^{\frac{2}{p}} \times B_{p,1}^{\frac{2}{p}})
\]
Now, each component of \(\varphi\) contains a combination of \(Q, P,\) and \(R,\) we will thus show how to use proposition 1, proposition 2 and lemma A.2.2 to deal with each one

- For \(Q\)
  - in \(\varphi_1,\) we apply proposition 1
  - in \(\varphi_2,\) for \(Q(y_1, y_2)\) we apply inequality (11) from proposition 2, and for \(Q(\nabla \times y_1, \nabla \times y_2),\) we apply proposition 1
  - in \(\varphi_3,\) we apply respectively proposition 1 then inequality (21) from proposition A.2.2 for \(k = 2.\)
  - in \(\varphi_4,\)
    * for \(Q(y_1, y_2),\) we apply inequality (12) from proposition 2, then inequality (21) from proposition A.2.2 for \(k = 0\)
    * for \(Q(\nabla \times y_1, \nabla \times y_2),\) we apply respectively inequality proposition 1 then inequality (21) from proposition A.2.2 for \(k = 2.\)
- For \(P,\) we apply inequality (13) from proposition 2, then inequality (21) from proposition A.2.2 for \(k = 0.\)
- For \(R,\) we apply inequality (14) from proposition 2, then inequality (21) from proposition A.2.2 for \(k = 1.\)

This ends the proof of theorem 2.

2.3 Proof of Theorem 3

In this section, we shall give some details about the wellposedeness in the hat-Kato space \(\hat{K}_{p}^{1-\frac{2}{p}},\) and then we establish the decay property described in theorem 3. It is all based on the following proposition
Proposition 3. Let $\zeta$ be giving by (7), then $\zeta$ maps $\hat{K}_p^{1-\frac{2}{p}} \times \hat{K}_p^{1-\frac{2}{p}}$ into $\hat{K}_p^{1-\frac{2}{p}}$, that is there exists $\kappa > 0$ such that

$$\|\zeta(L,M)\|_{\hat{K}_p^{1-\frac{2}{p}}} \leq \kappa \|L\|_{\hat{K}_p^{1-\frac{2}{p}}} \|M\|_{\hat{K}_p^{1-\frac{2}{p}}},$$

for all $L, M \in \hat{K}_p^{1-\frac{2}{p}}$.

Proof

By taking into account the inequality, for all $m \in [0,2]$,

$$\frac{1}{1+|\xi|^2} \lesssim \frac{1}{2^m}$$

in order to prove the continuity property of $\zeta$ on $\hat{K}_p^{1-\frac{2}{p}}$, we only need to show that, for all $t > 0$

$$t^{\frac{2}{p}(1-\frac{2}{p})}\int_0^t \|e^{-(t-s)}|\cdot((\hat{L} \ast \hat{M})(s,\cdot))\|_{L^{p'}} \, ds \lesssim \|L\|_{\hat{K}_p^{1-\frac{2}{p}}} \|M\|_{\hat{K}_p^{1-\frac{2}{p}}},$$

that is, due to (16), all the components of $\zeta$ can be dominated by a Gaussian multiplied by order one Fourier-multiplier, as in the proof of theorem 2.

Let us then gives some details about the proof of (17). By setting, for $p > 3$,

$$\frac{1}{r} \equiv 1 - \frac{2}{p} \iff \frac{1}{r'} = \frac{1}{p'} - \frac{1}{r} = 1,$$

by virtue of Holder inequality, we infer that

$$\int_0^t \|e^{-(t-s)}|\cdot((\hat{L} \ast \hat{M})(s,\cdot))\|_{L^{p'}} \, ds \lesssim \int_0^t \|G(t-s,\cdot)\|_{L^p} \|\hat{L}(s,\cdot)\|_{L^{p'}} \|\hat{M}(s,\cdot)\|_{L^{p'}} \, ds,$$

where

$$G(t,\xi) \equiv e^{-|\xi|^2} \xi,$$

by a change of variable in the $L^p$ norm of $G(t-s,\cdot)$, we obtain

$$\|G(t-s,\cdot)\|_{L^p} \lesssim \frac{1}{(t-s)^{\frac{1}{2} + \frac{2}{p}} \|G(1,\cdot)\|_{L^{p'}}},$$

this gives

$$\int_0^t \|e^{-(t-s)}|\cdot((\hat{L} \ast \hat{M})(s,\cdot))\|_{L^{p'}} \, ds \lesssim \|L\|_{\hat{K}_p^{1-\frac{2}{p}}} \|M\|_{\hat{K}_p^{1-\frac{2}{p}}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2} + \frac{2}{p}} \|G(1,\cdot)\|_{L^{p'}}} \, ds,$$

$$\lesssim t^{-\frac{1}{2}(1-\frac{2}{p})} \|L\|_{\hat{K}_p^{1-\frac{2}{p}}} \|M\|_{\hat{K}_p^{1-\frac{2}{p}}},$$

inequality (17) follows, and then proposition 3 is then proved.

Proof of theorem 3

According to proposition 3 we can apply the fixed point lemma 1 that is, if

$$\|e^{t\Delta} u_0\|_{\hat{K}_p^{\frac{1}{2}-1}} + \|e^{t\Delta} B_0\|_{\hat{K}_p^{\frac{1}{2}-1}} + \|e^{t\Delta}(\nabla \times B_0)\|_{\hat{K}_p^{\frac{1}{2}-1}} \leq \frac{1}{4\kappa},$$

then we can construct a unique solution $\mathcal{U}$ of $(S_\zeta)$ in $\hat{K}_p^{\frac{1}{2}-1}$ with

$$\|\mathcal{U}\|_{\hat{K}_p^{\frac{1}{2}-1}} < \frac{1}{2\kappa}.$$
By virtue of the continuity of the Fourier transform, from $L^q$ into $L^{q'}$, for $q \in [1, 2]$, we infer that, for $p \in (3, \infty)$,
$$
\|U(t, \cdot)\|_{L^p} = \|\widehat{U}(t, \cdot)\|_{L^p} \lesssim \|\widehat{U}(t, \cdot)\|_{L^{q'}}
$$
it follows then, for all $t > 0$
$$
\|U(t, \cdot)\|_{L^p} \lesssim \|\widehat{U}(t, \cdot)\|_{L^{q'}} \lesssim t^{-\frac{1}{2}(1 - \frac{2}{p})}.
$$
Theorem 3 is then proved. \hfill \Box

A Appendix

In this section we recall some basic tools of a constant use in the analysis of our paper, we begin by recalling some definitions and functional spaces, then we shall provide some properties of these spaces in the next subsection.

A.1 Functional spaces

Let us begin by recalling the Littlewood-Paley decomposition and the associated Besov spaces. Let $(\psi, \varphi)$ be a couple of smooth functions with value in $[0, 1]$ satisfying:

$$
\text{Supp } \psi \subset \{\xi \in \mathbb{R} : |\xi| \leq \frac{4}{3}\}, \quad \text{Supp } \varphi \subset \{\xi \in \mathbb{R} : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}
$$

$$
\psi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}, \quad \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \forall \xi \in \mathbb{R}\setminus\{0\}.
$$

Let $a$ be a tempered distribution, $\widehat{a} = \mathcal{F}(a)$ its Fourier transform and $\mathcal{F}^{-1}$ denotes the inverse of $\mathcal{F}$. We define the homogeneous dyadic blocks $\Delta_q$ by setting:

$$
\Delta_q a \overset{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-q}|\xi|\widehat{a})), \quad S_q \overset{\text{def}}{=} \sum_{j<q} \Delta_j \forall q \in \mathbb{Z}.
$$

Although the previous sections, we used the Bony’s decomposition which reads as follows, for tempered distributions $u$ and $v$, we have

$$
uv = T_u v + T_v u + R(u, v),
$$
with

$$
T_u v \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} S_{j-1} u \Delta_j u, \quad R(u, v) \overset{\text{def}}{=} \sum_{j \in \mathbb{Z}} \Delta_j u \Delta_j v,
$$
where $\Delta_j \overset{\text{def}}{=} \sum_{i \in \{-1, 0, 1\}} \Delta_{j+i}$.

According to the support properties above, we have

$$
\Delta_q T_u v = \Delta_q \sum_{j<q} S_{j-1} u \Delta_j u
$$
$$
\Delta_q R(u, v) = \Delta_q \sum_{j \geq q+N_0} \Delta_j u \Delta_j v,
$$
for some $N_0 \in \mathbb{Z}$.

Based on the dyadic decomposition presented above, we recall the definition of the usual Besov spaces on $\mathbb{R}^d$ and the Chemin-Lerner spaces defined on $\mathbb{R}^+ \times \mathbb{R}^d$. 

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Definition 2. Let $s$ be a real number and $p, r$ be in $[1, +\infty]$, we define the space $B^{s}_{p, r}(\mathbb{R}^{d})$ as the space of tempered distributions $u$ in $\mathcal{S}(\mathbb{R}^{d})$ such that

$$\|u\|_{B^{s}_{p, r}} := \|2^{js} \|\Delta_{j} u\|_{L^{p}(\mathbb{Z})}\|_{L^{r}(\mathbb{Z})} < \infty$$

And for $\rho \in [1, \infty]$, the space $\tilde{L}^{\rho}(B^{s}_{p, r})$ is the space of the tempered distributions $f$ in $\mathcal{S}(\mathbb{R}^{+} \times \mathbb{R}^{d})$ such that

$$\|f\|_{\tilde{L}^{\rho}(B^{s}_{p, r})} := \|2^{js} \|\Delta_{j} u\|_{L^{p}(\mathbb{Z})}\|_{L^{\rho}(\mathbb{Z})} < \infty$$

Next, we recall the definition of Kato spaces, then we introduce the Kato-Herz and the Fourier-Herz spaces used in theorem 3 for more details about the Fourier-Herz spaces the reader can see for instance [4].

Definition A.1.1 (Kato spaces). Let $p$ be in $[1, \infty]$, $\sigma \in \mathbb{R}^{+}$, we define the space $K^{\sigma}_{p, r}(T)$ (or simply $K^{\sigma}_{p, r}$ when $T = \infty$), as the space of functions $u$ on $\mathbb{R}^{+} \times \mathbb{R}^{d}$, such that

$$\|u\|_{K^{\sigma}_{p, r}(T)} \overset{\text{def}}{=} \sup_{t \in (0, T)} \left\{ \|u(t, \cdot)\|_{L^{\rho}(\mathbb{R}^{d})} \right\} < \infty$$

the case $r = \infty$, we simply denote $K^{\sigma}_{p, \infty} = K^{\sigma}_{p}$, such that

$$\|u\|_{K^{\sigma}_{p}(T)} \overset{\text{def}}{=} \sup_{t \in (0, T)} \left\{ \|u(t, \cdot)\|_{L^{\rho}(\mathbb{R}^{d})} \right\} < \infty$$

Definition A.1.2 (Fourier-Herz and Kato-Herz spaces). Let $p, r$ be in $[1, \infty]$, $(s, \sigma) \in \mathbb{R}^{+} \times \mathbb{R}$, we define the Fourier-Herz space $\tilde{B}^{s}_{p, r}(\mathbb{R}^{d})$ as the space of tempered distribution $w$ on $\mathbb{R}^{d}$ such that

$$\|u\|_{\tilde{B}^{s}_{p, r}} := \|2^{js} \|\Delta_{j} u\|_{L^{p}(\mathbb{Z})}\|_{L^{r}(\mathbb{Z})} < \infty,$$

and we define $\tilde{K}^{\sigma}_{p, r}(T)$ (or simply $\tilde{K}^{\sigma}_{p, r}$ when $T = \infty$), as the space of functions $u$ on $\mathbb{R}^{+} \times \mathbb{R}^{d}$, such that

$$\|u\|_{\tilde{K}^{\sigma}_{p, r}(T)} \overset{\text{def}}{=} \sup_{t \in (0, T)} \left\{ \|\hat{u}(t, \cdot)\|_{L^{\rho}(\mathbb{R}^{d})} \right\} < \infty$$

the case $r = \infty$, we simply denote $\tilde{K}^{\sigma}_{p, \infty} = \tilde{K}^{\sigma}_{p}$, such that

$$\|u\|_{\tilde{K}^{\sigma}_{p}(T)} \overset{\text{def}}{=} \sup_{t \in (0, T)} \left\{ \|\hat{u}(t, \cdot)\|_{L^{\rho}(\mathbb{R}^{d})} \right\} < \infty$$

Remark 7. In terms of the scaling, the Fourier-Herz space $\tilde{B}^{s}_{p, r}$ (resp. the Kato-Herz space $\tilde{K}^{s}_{p, r}$) has the same scale as the usual Besov $B^{s}_{p, r}$ (resp. the usual Kato $K^{s}_{p, r}$).

A.2 Some technical results

We begin this subsection by recalling the Bernstein lemma from [2].

Lemma A.2.1 (Bernstein). Let $B$ be a ball of $\mathbb{R}^{d}$, and $C$ be a ring of $\mathbb{R}^{d}$. Let also $a$ be a tempered distribution and $\hat{a}$ its Fourier transform. Then for $1 \leq p_{2} \leq p_{1} \leq \infty$ we have:

$$\text{Supp} \, \hat{a} \subset 2^{k}B \implies \|\partial^{\alpha}_{a}a\|_{L^{p_{1}}} \lesssim 2^{|\alpha|/2} \|a\|_{L^{p_{2}}}$$

$$\text{Supp} \, \hat{a} \subset 2^{k}C \implies \|a\|_{L^{p_{1}}} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial^{\alpha}_{a}a\|_{L^{p_{1}}}$$
In the following proposition, we collect some useful properties and results related to the spaces defined above, the reader can see \([2, 3, 10, 9]\) for more details.

**Proposition A.2.1.** Let \((\delta, s)\) be in \(\mathbb{R} \times \mathbb{R}^+\), and \(p, r, \rho, m\) be in \([1, \infty]\),

- for \(u \in B^s_{p,r}\), there exists some sequence \((c_{j,r})_{r \in \mathbb{Z}}\) such that
  \[\|\Delta_j u\|_{L^p} \leq c_{j,r} 2^{-j\delta} \|f\|_{B^s_{p,r}}, \text{ and } \sum_{j \in \mathbb{Z}} c_{j,r}^r \leq 1\]

- In the case of non positive regularity, one may replace, equivalently, \(\Delta_j\) in the definitions of the Besov space by \(S_j\), that is we have
  \[\|u\|_{B^s_{p,r}} \approx \|2^{js} \|S_j u\|_{L^p}\]

- According to Minkowski’s inequality, we have
  \[L^p(B^s_{p,r}) \hookrightarrow \tilde{L}^p(\mathbb{R}^d) \quad \text{if} \quad \rho \leq r, \quad \tilde{L}^p(B^s_{p,r}) \hookrightarrow L^p(B^s_{p,r}) \quad \text{if} \quad r \leq \rho\]

- For \(1 \leq p \leq q \leq \infty\), and \(1 \leq r \leq m \leq \infty\), we have
  \[B^s_{p,r}(\mathbb{R}^d) \hookrightarrow B^s_{q,m}(\mathbb{R}^d)\]

- In terms of Kato spaces (resp. Kato-Herz spaces), we have the following characterization of Besov spaces (resp. Fourier-Hezr spaces) of negative regularity \(s < 0\)
  \[\|f\|_{B^s_{p,r}} \approx \|e^{\Delta f}\|_{K^s_{p,r}}\]
  \[\|f\|_{\tilde{B}^s_{p,r}} \approx \|e^{\Delta f}\|_{\tilde{K}^s_{p,r}}\]

The following proposition, has been used in the previous section, describes the continuity in Chemin-Lerner spaces of some Fourier multipliers

**Proposition A.2.2.** Let \(s\) be a real number, \((p, r)\) be in \([1, \infty]_2\), then we have, for all \(f \in B^s_{p,r}(\mathbb{R}^d)\), for all \(k \in [0, 2]\)

\[\|(I_d - \Delta)^{-1} \nabla \times f\|_{\tilde{L}^p(B^{s+k-1}_{p,r})} \lesssim \|f\|_{\tilde{L}^p(B^s_{p,r})}\]

\[\|(I_d - \Delta)^{-1} \Delta f\|_{L^p(B^{s+m-2}_{p,r})} \lesssim \|f\|_{L^p(B^s_{p,r})}\]

The proof of proposition [A.2.2] is based on the Bernstein lemma and following one

**Lemma A.2.2.** Let \(C\) be an annulus in \(\mathbb{R}^d\), \(m \in \mathbb{R}\), and \(k\) be the integer part of \(1 + \frac{d}{2}\) \((k \overset{\triangle}{=} [1 + \frac{d}{2}]\). Let \(\sigma\) be \(k\)-times differentiable function on \(\mathbb{R}^s\) such that for all \(\alpha \in \mathbb{N}^d\) with \(|\alpha| \leq k\), there exists \(C_\alpha\) satisfying

\[\forall \xi \in \mathbb{R}^d, \quad |\partial^\alpha \sigma(\xi)| \leq C_\alpha (1 + |\xi|^2)^m |\xi|^{-\alpha}\]

There exists \(C\) depends only on \(C_\alpha\) such that for any \(p \in [1, \infty]\) and \(\lambda > 0\), we have, for any \(u \in L^p\) satisfying \(\text{supp}(\hat{u}) \subset \mathbb{C}_\lambda\),

\[\|\sigma(D) u\|_{L^p} \lesssim C(1 + \lambda^2)^m \|u\|_{L^p} \quad \text{with} \quad \sigma(D) u \overset{\triangle}{=} \mathcal{F}^{-1}(\sigma \hat{u})\]
Proof
Following the proof of lemma 2.2 from [2], seen that \(\text{supp}(\tilde{u}) \subset \lambda \mathcal{C}\), we can write
\[
\sigma(D)u = \lambda dK_{\lambda}(\lambda \cdot) \ast u \quad \text{with}
\]
\[
K_{\lambda}(x) \overset{\text{def}}{=} (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x|\xi)} \tilde{\varphi}(\xi) \sigma(\lambda \xi) d\xi
\]
for some smooth function \(\tilde{\varphi}\) supported in an annulus and having value 1 in \(\mathcal{C}\).

Let \(M\) be the integer part of \((1 + |x|^2)^M\). We have
\[
(1 + |x|^2)^M |K_{\lambda}(x)| = \left| \int_{\mathbb{R}^d} e^{i(x|\xi)} (Id - \Delta)^M \tilde{\varphi}(\xi) \sigma(\lambda \xi) d\xi \right|
\]
\[
= \left| \sum_{|\alpha| + |\beta| \leq 2M} c_{\alpha,\beta} \lambda^{|eta|} \int_{\text{supp}(\tilde{\varphi})} e^{i(x|\xi)} \partial^\alpha \tilde{\varphi}(\xi) \partial^\beta \sigma(\lambda \xi) d\xi \right|
\]
\[
\lesssim C(1 + \lambda^2)^m
\]

As \(2M > d\), we deduce that
\[
\|K_{\lambda}\|_{L^1} \leq C(1 + \lambda^2)^m
\]
Thus Young’s inequality conclude the proof of the desired inequality. \(\square\)

Proof of proposition A.2.2
According to lemma A.2.2 and Bernstein lemma, we have
\[
\|(Id - \Delta)^{-1} \nabla \times (\Delta_j f)\|_{L^1 L^p_x} \lesssim \frac{2^j}{1 + 2^{2j}} \|\Delta_j f\|_{L^1 L^p_x}
\]
\[
\|(Id - \Delta)^{-1} \Delta(\Delta_j f)\|_{L^1 L^p_x} \lesssim \frac{2^{2j}}{1 + 2^{2j}} \|\Delta_j f\|_{L^1 L^p_x}
\]
the result follows from the fact that, for all \(k \in [0, 2]\) we have
\[
2^{jk} \lesssim 1 + 2^{2j}
\]
Proposition A.2.2 is then proved. \(\square\)

The following fixed point argument has been used to prove corollary 1, the proof of which can be found for instance in [3].

Lemma A.2.3. Let \(X\) be a Banach space, \(L\) a linear operator from \(X\) to \(X\), with norm equals to \(\lambda < 1\), and let \(B\) be a bilinear operator mapping from \(X \times X\) in \(X\), with norm \(\|B\| = \gamma\), then for all \(y \in X\) such that
\[
\|y\|_X < \frac{(1 - \gamma)^2}{4\gamma}
\]
the equation
\[
x = y + L(x) + B(x, x)
\]
has a unique solution in the ball \(B_X(0, \frac{1-\lambda}{2\gamma})\).

Finally we recall a result concerning the smoothing effect of the Heat Kernel, one may see [2, 7] for more details.

Lemma A.2.4. Let \((s, p, r) \in \mathbb{R} \times [1, \infty]^2\) and the operators \(\mathcal{T}\) and \(\mathcal{T}_0\) be given by
\[
(T_0 K_0)(t, x) \overset{\text{def}}{=} e^{t \Delta} K_0(x)
\]
\[ TK(t, x) \overset{\text{def}}{=} \int_0^t e^{\Delta(t-s)} K(s, x) ds \]

then,
\[ \| T_0 K_0 \|_{L^p(B_{p,r}^{s})} \lesssim \| K_0 \|_{B_{p,r}^{s}} \tag{23} \]

and
\[ \| T_0 K_0 \|_{L^p(B_{p,r}^{s})} \lesssim \| K \|_{L^1_t(B_{p,r}^{s})} \tag{24} \]

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