ASSOCIATED VARIETIES OF MODULES OVER KAC-MOODY ALGEBRAS AND $C_2$-COFINITENESS OF $W$-ALGEBRAS

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Abstract. First, we establish the relation between the associated varieties of modules over Kac-Moody algebras $\hat{\mathfrak{g}}$ and those over affine $W$-algebras. Second, we prove the Feigin-Frenkel conjecture on the singular supports of $G$-integrable admissible representations for the degenerate cases in the sense of Frenkel-Kac-Wakimoto [FKW92]. In fact we show that the associated varieties of $G$-integrable degenerate admissible representations are irreducible $\text{Ad}G$-invariant subvarieties of the nullcone of $\mathfrak{g}$, by determining them explicitly. Third, we prove the $C_2$-cofiniteness of a large number of simple $W$-algebras, including all the (non-principal) exceptional $W$-algebras recently discovered by Kac-Wakimoto [KW08].

1. Introduction

This article addresses some basic problems concerning the representation theory of Kac-Moody algebras, that of (affine) $W$-algebras and the interrelation between them. It has three aims.

Let $\mathfrak{g}$ be a complex simple Lie algebra, $\hat{\mathfrak{g}}$ the non-twisted Kac-Moody algebra associated with $\mathfrak{g}$. For a nilpotent element $f$ of $\mathfrak{g}$ and $k \in \mathbb{C}$, both the $W$-algebra $W^k(\mathfrak{g}, f)$ at level $k$ and representations of $W^k(\mathfrak{g}, f)$ are constructed by means of the BRST cohomology functor $H^\infty_{f+0}(?)$ associated with the generalized Drinfeld-Sokolov reduction [KRW03].

The first aim of this article is to establish the relation between the associated varieties ([Ara10], see (7)) of modules over $\hat{\mathfrak{g}}$ and those over $W^k(\mathfrak{g}, f)$. More precisely, we show that

$$X_{H^\infty_{f+0}(M)} \cong X_M \cap \mathcal{S}$$

for a finitely generated graded Harish-Chandra $G[[t]]$-module $M$ of level $k$, where $X_M$ is the associated variety of $M$ and $\mathcal{S}$ is the Slodowy slice (Theorem 4.5.2). From (1) we deduce that the $W^k(\mathfrak{g}, f)$-module $H^\infty_{f+0}(M)$ is $C_2$-cofinite [Zhu96] if and only if (i) $X_M$ is contained in the nullcone $\mathcal{N}$ of $\mathfrak{g}$ and (ii) the closure of the orbit $\text{Ad}G.f$ appears as an irreducible component of $X_M$ (Theorem 4.6.2). This result may be viewed as a chiralization of a theorem [Pre07, Theorem 3.1] of Premet on finite $W$-algebras.

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$C_2$-cofinite representations of vertex algebras may be regarded as analogues of finite-dimensional representations ([Ara10]).
The second is to determine the associated varieties of (Kac-Wakimoto) admissible representations of \( \hat{\mathfrak{g}} \). Based on the above and the related results we determine the associated varieties of all the degenerate (in the sense of [FKW92]) \( \mathfrak{g} \)-integrable irreducible admissible representations and show that they are Ad\( \mathfrak{g} \)-invariant irreducible subvarieties of \( \mathcal{N} \), that is, closures of some nilpotent orbits in \( \mathfrak{g} \). This in particular proves the Feigin-Frenkel conjecture (see Conjecture 1) on the singular supports of admissible representations for the degenerate cases (Theorems 5.5.1, 5.6.1). These nilpotent orbits depend only on the level \( k \) of the representations, that is, for each degenerate admissible number \( k \), there exists a unique nilpotent orbit \( \mathcal{O}[k] \) in \( \mathfrak{g} \) such that

\[ X_{\mathfrak{L}_\lambda} = \mathcal{O}[k] \]

for any \( \mathfrak{g} \)-integrable irreducible admissible representation \( \mathfrak{L}_\lambda \) of level \( k \). The orbits \( \mathcal{O}[k] \) are explicitly determined and listed in Tables 2–10 (cf. (46)).

The third is to apply the above results to the \( C_2 \)-cofiniteness problem of \( \mathcal{W} \)-algebras. Recently, a remarkable family of \( \mathcal{W} \)-algebras, called the exceptional \( \mathcal{W} \)-algebras, was discovered by Kac and Wakimoto [KW08]. They conjectured that exceptional \( \mathcal{W} \)-algebras are rational and \( C_2 \)-cofinite. We prove the \( C_2 \)-cofiniteness of all the (non-principal) exceptional \( \mathcal{W} \)-algebras (Theorem 5.7.2). More precisely we prove that, for each degenerate admissible number \( k \), the simple quotient of \( \mathcal{W}_k(\mathfrak{g}, f) \) with \( f \in \mathcal{O}[k] \) is \( C_2 \)-cofinite (Theorem 5.7.2), and that each (non-principal) exceptional \( \mathcal{W} \)-algebra is isomorphic to \( \mathcal{W}_k(\mathfrak{g}, f) \) for some degenerate admissible number \( k \) and \( f \in \mathcal{O}[k] \). We note that there are also a considerable number of \( C_2 \)-cofinite \( \mathcal{W} \)-algebras which are not exceptional, see Tables 2, 4, 6, 8, 9, 10. At this moment we do not know whether these not-exceptional \( C_2 \)-cofinite \( \mathcal{W} \)-algebras are non-rational.

Our strategy to prove (1) is based on Ginzburg’s reproof [Gin09] of Premet’s conjecture [Pre07, Conjecture 3.2] (proved by Losev [Los10]) on finite \( \mathcal{W} \)-algebras. A “chiralization” of the argument of [Gin09] proves the vanishing of the BRST cohomology of the associated graded spaces (Theorem 4.3.3). The difficult part is the proof of the convergency of the corresponding spectral sequence because our algebras are not Noetherian. We overcome this problem by using the right exactness of the functor \( H^\infty \mathcal{W}_k(\mathfrak{g}, f) \) (Theorem 4.2.2, cf. [AM09]), see §4.4 for the detail. As a result we obtain the strong vanishing assertion (Theorem 4.4.6) of the BRST cohomology, which gives (1) as desired.

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2 Admissible representations of vertex operators algebras have nothing to do with (Kac-Wakimoto) admissible representations of \( \hat{\mathfrak{g}} \).

3 Degenerate \( \mathfrak{g} \)-integrable admissible representations may be considered as those \( \mathfrak{g} \)-integrable admissible representation whose associated varieties are strictly contained in \( \mathcal{N} \), while non-degenerate \( \mathfrak{g} \)-integrable admissible representations are those \( \mathfrak{g} \)-integrable admissible representation whose associated varieties should equals to \( \mathcal{N} \), see Theorem 5.5.1.

4 A complex number \( k \) is called a degenerate admissible number if the simple affine vertex algebra \( \mathfrak{L}_k \) at level \( k \) is a degenerate admissible representations.

5 In the principal nilpotent case the exceptional \( \mathcal{W} \)-algebras are exactly the minimal series \( \mathcal{W} \)-algebras discovered by Frenkel, Kac and Wakimoto [FKW92].

6 Conjecturally [FKW92, KRW03], \( \mathcal{W}_k(\mathfrak{g}, f) = H^\infty f(L_k) \). We have proved this in [Ara05] for a minimal nilpotent element \( f \) and in [Ara07] for a principal nilpotent element \( f \) under some regularity condition on \( k \). In type \( A \) one can show this for any nilpotent element using the results of [Ara08] under some regularity condition on \( k \). The detail will appear elsewhere.
Note that the vanishing of the BRST cohomology proves the exactness of the functor

\[ H_f^{\infty+0}(?) : KL_k \to \mathcal{W}^k(g,f)\)-Mod

as well (Theorem 4.4.5), where KL_k is the category of graded Harish-Chandra (\(\widehat{g}, G[[t]]\))-modules of level k, or equivalently, the full subcategory of the category O of \(\widehat{g}\) consisting G-integrable representations of level k. This generalizes some of the exactness results obtained in [Ara04, Ara05, Ara07, FG07].

For a finitely generated object M of KL_k, the associated variety \(X_M\) is an \(\text{Ad}_G\)-invariant, conic, Poisson subvariety of \(g^*\), while the variety \(X_{H_f^{\infty+0}(M)}\) of the \(\mathcal{W}^k(g,f)\)-module \(H_f^{\infty+0}(M)\) is a \(C^*\)-invariant, Poisson subvariety of the Slodowy slice \(S\). Because \(f\) is the unique fixed point of the \(C^*\)-action of \(S\), from (1) it follows that

\[ H_f^{\infty+0}(M) \neq 0 \iff \text{Ad}_G.f \subset X_M \]

(compare [Mat87, Theorem 2], [Gin09, Corollary 4.1.6]). Although \(X_M\) is not necessarily contained in \(N\), we show that

\[ H_f^{\infty+0}(M) \text{ is zero or } C_2\text{-cofinite} \iff X_M \subset N \]

(Proposition 4.6.1). In particular for a principal nilpotent element \(f_{pri}\) we have

\[ H_{f_{pri}}^{\infty+0}(M) \neq 0 \iff X_M \supset N, \]

\[ H_{f_{pri}}^{\infty+0}(M) = 0 \iff X_M \subset N. \]

In the case that \(L_{\lambda} \in KL_k\) is an admissible representation with highest weight \(\lambda\) its character is given by the Weyl-Kac type character formula [KW88], and therefore the character of \(H_f^{\infty+0}(L_{\lambda})\) is computable [FKW92, KRW03] by the vanishing of the BRST cohomology and the Euler-Poincaré principle. If \(L_{\lambda}\) is degenerate, or equivalently [FKW92] \(H_f^{\infty+0}(L_{\lambda}) = 0\), then (5) proves the Feigin-Frenkel conjecture which says \(X_{L_{\lambda}} \subset N\). Furthermore, by (2), we see that the variety \(X_{L_{\lambda}}\) can be determined by knowing for which nilpotent \(f\) the cohomology \(H_f^{\infty+0}(L_{\lambda})\) is zero.

If \(g\) is simply-laced then all the G-integrable admissible representations are principal admissible [KW89]. If \(g\) is not simply-laced then there are also non-principal G-integrable admissible representations. In the principal admissible cases, i.e., if \(L_{\lambda} \in KL_k\) is a degenerate principal admissible representation of level k, then we show that

\[ X_{L_{\lambda}} \cong N_q, \]

where

\[ N_q := \{ x \in g; (\text{ad } x)^{2q} = 0 \}, \]

and \(q\) is the denominator of \(k\) (Theorem 5.5.1). The irreducibility of \(N_q\) is known\(^7\) by [oGVAG04]. Note that \(N_q = \{ f \in N; \text{ht}(f) < 2q \} \cup \{ 0 \}\), where \(\text{ht}(f)\) is the height [Pan94] of \(f\). In the non-principal admissible cases we show that

\[ X_{L_{\lambda}} \cong L N_q, \]

\(^7\)The irreducibility of \(\{ x \in g; (\text{ad } x)^n = 0 \}\) is not true in general if \(n\) is odd.
where
\[ \mathcal{L}_q := \{ f \in \mathcal{N}; \text{ht}^\vee (f) < 2q \} \cup \{0\}, \]
\( \text{ht}^\vee (f) \) is the coheight of \( f \), \( q = (\text{the denominator of } k) / r^\vee \) and \( r^\vee \) is the lacety of \( \mathfrak{g} \). The irreducibility of \( \mathcal{L}_q \) follows from the classification of nilpotent elements and their closure relations (Theorem 5.6.1).

An exceptional \( W \)-algebra is by definition the simple \( W \)-algebra \( W_k(\mathfrak{g}, f) \) at a principal admissible level \( k \) such that \( (q, f) \) is an exceptional pair \( \text{[KW08]} \), where \( q \) is the denominator of \( k \). The exceptional pairs were classified in \( \text{[KW08][EKV09]} \). From the classification it follows that \( (q, f) \) is an exceptional pair if and only if (i) \( \text{Ad } G, f = N_q \) and (ii) \( f \) is of principal type (Theorem 5.8.1). This fact together the above explained results proves the \( C_2 \)-cofiniteness of the (non-principal) exceptional \( W \)-algebras.

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Notation. Throughout this paper the ground field is the complex number \( \mathbb{C} \) and tensor products and dimensions are always meant to be as vector spaces over \( \mathbb{C} \) if not otherwise stated.

2. Preliminaries

In \( \text{[Ara10]} \) we recall some definitions and results from \( \text{[Ara10]} \). In \( \text{[2.2]} \) we recall some fundamental facts on Kac-Moody algebras. In \( \text{[2.4]} \) we recall some basic facts on affine vertex algebras and their associated graded vertex Poisson algebras. In \( \text{[2.5]} \) we define a category \( \mathcal{C} \) of modules over the associated graded vertex Poisson algebras of affine vertex algebras.

2.1. Functions on jet schemes of affine Poisson varieties as vertex Poisson algebras. For a scheme \( X \) of finite type, let \( X_m \) the \( m \)-th jet scheme of \( X \), \( X_\infty \) the infinite jet scheme of \( X \) (or the arc space of \( X \)). In the case that \( X \) is an affine scheme \( \text{Spec } R \),
\[ R_\infty := \mathbb{C}[X_\infty] \]
is a differential algebra generated by \( R \). We denote by \( T \) the derivation of \( R_\infty \), and we set \( a_{(n)} = T^{(n-1)}(a)/(n-1)! \) for \( a \in R_\infty, n \geq 1 \).

If \( R \) is a Poisson algebra, then \( R_\infty \) is equipped with the level 0 vertex Poisson algebra structure \( \text{[Ara10]} \) Proposition 2.3.1). It is a unique vertex Poisson algebra structure on \( R_\infty \) such that
\[ a_{(n)} b = \begin{cases} \{a, b\} & \text{if } n = 0, \\ 0 & \text{if } n > 0, \end{cases} \text{ for } a, b \in R. \]

That is, \( L_{2A_0} \) is principal admissible.
Here \(a_{(n)}\) with \(n \geq 0\) is the coefficient of \(z^{-n-1}\) in the field \(Y_{-}(a, z)\) in the notation of [FBZ04 16.2]. Throughout the paper by a \(R_{\infty}\)-module \(M\) we mean a module over the vertex Poisson algebra in the sense of [Li04] unless otherwise stated. In particular \(M\) is equipped with the action of \(a_{(n)}\) with \(a \in R_{\infty}\), \(n \geq 0\), such that

\[
[a_{(m)}, b_{(n)}] = \sum_{i \geq 0} \binom{m}{i} (a_{(i)}b)_{m+n-i},
\]

\[
a_{(n)}(b_{(-1)}v) = (a_{(n)}b)_{(-1)}v + b_{(-1)}(a_{(n)}v)
\]

for all \(a, b \in R_{\infty}\), \(m, n \geq 0\), \(v \in M\).

Let \(\mathfrak{g}\) be a simple Lie algebra as in Introduction, \(\{x^{i}; i \in I\}\) a basis of \(\mathfrak{g}\). Then \(\mathbb{C}[\mathfrak{g}_{\infty}] = \mathbb{C}[\mathfrak{g}]_{\infty} = \mathbb{C}[x^{i}_{-\infty}; i \in I, n \geq 1]\). Below we regard \(\mathbb{C}[\mathfrak{g}_{\infty}]\) as a vertex Poisson algebra with the level zero vertex Poisson algebra structure induced from the Kirillov-Kostant Poisson algebra structure on \(\mathbb{C}[\mathfrak{g}]^*\).

Let \(G\) be the adjoint group of \(\mathfrak{g}\). The \(m\)-the jet scheme \(G_{m}\) of \(G\) is an algebraic group, which is isomorphic to a semi-direct product of \(G\) and a unipotent group. The infinite jet scheme \(G_{\infty}\) of \(G\) is a proalgebraic group \(G[[t]]\), which is isomorphic to a semi-direct product of \(G\) and a pronipotent group. We have \(\text{Lie}(G_{m}) = \mathfrak{g}_{m} = \mathfrak{g}[t]/(t^{m+1})\), and \(\text{Lie}(G_{\infty}) = \mathfrak{g}_{\infty} = \mathfrak{g}[t]\). The group \(G_{\infty}\) acts on \(\mathfrak{g}_{\infty}\) by adjoint.

For a \(\mathbb{C}[\mathfrak{g}_{\infty}]\)-module \(M\), the assignment \(x \otimes t^{n} \mapsto x_{(n)}(t)\) with \(x \in \mathfrak{g}\), \(n \geq 0\), defines a \(\mathfrak{g}[t]\)-module structure on \(M\). In fact a \(\mathbb{C}[\mathfrak{g}_{\infty}]\)-module is the same as a \(\mathbb{C}[\mathfrak{g}_{\infty}^*]\)-module \(M\) as a ring equipped with an action of \(\mathfrak{g}[t]\) such that \(x_{(n)}(am) = (x_{(n)}a)m + ax_{(n)}m\) for \(x \in \mathfrak{g}\), \(n \geq 0\), \(a \in \mathbb{C}[\mathfrak{g}_{\infty}]\), \(m \in M\), where \(x_{(n)} = x \otimes t^{n}\).

\section*{2.2. Singular supports and associated varieties of modules over vertex algebras.}

For a vertex algebra \(V\), there is a canonical decreasing filtration \(\{F_{p}V\}\), which we refer to as the \(Li\) filtration.\footnote{The \(Li\) filtration is defined independent of the grading of \(V\). Hence it unifies the notion of the standard filtration and the Kazhdan filtration in our case (compare [Kos78, Lyn79, GG02]).} The associated graded space \(\text{gr}F V = \bigoplus F_{p}V/F_{p+1}V\) is naturally a vertex Poisson algebra. We have \(F_{0}V = V\) and \(F_{1}V = C_{2}(V)\), where \(C_{2}(V)\) is the linear span of the vectors \(a_{(-2)}b\) with \(a, b \in V\).

In particular \(\text{gr}F V\) contains \(Zhu’s\ Poisson\ algebra\ \(R_{\infty}\)\footnote{The \(Li\) filtration is defined independent of the grading of \(V\). Hence it unifies the notion of the standard filtration and the Kazhdan filtration in our case (compare [Kos78, Lyn79, GG02]).}\) as its subspace, where

\[R_{\infty} := V/C_{2}(V)\]

In fact \(R_{\infty}\) is a subring of \(\text{gr}F V\) and its Poisson algebra structure can be obtained [Li05] by restricting the vertex Poisson algebra structure of \(\text{gr}F V\).

Assume that \(V\) is \textit{finitely strongly generated}, or equivalently, the ring \(R_{\infty}\) finitely generated.

The spectrum \(\text{Spec} R_{\infty}\) is called the \textit{associated variety} of \(V\) and denoted by \(X_{V}\).

The embedding \(R_{\infty} \hookrightarrow \text{gr}F V\) induces [Li05, Ara10] the surjective homomorphism

\[(R_{\infty})_{\infty} \twoheadrightarrow \text{gr}F V\]

of vertex Poisson algebras. The closed subscheme

\[SS(V) := \text{supp}(R_{\infty})_{\infty}(\text{gr}F V)\]

of \((X_{V})_{\infty}\) is called \textit{singular support} of \(V\). We have

\[\dim SS(V) = 0 \iff \dim X_{V} = 0 \iff V\text{ is }C_{2}\text{-cofinite.}\]
Here the first equivalence follows from the fact that
\[ X_V = \pi_{\infty,0}(SS(V)), \]
where \( \pi_{\infty,0} : (X_V)_\infty \to X_V \) is the natural projection.

Assume that \( V \) is \( Q_{\geq 0} \)-graded by a Hamiltonian \( H \), and let \( M = \bigoplus_{d \in \mathbb{C}} M_d \) be a lower truncated graded \( V \)-module, where \( M_d = \{ m \in M ; Hm = dm \} \). A compatible filtration of \( M \) is a decreasing filtration \( M = F_0 M \supset F_1 M \supset \cdots \) such that
\[
\begin{align*}
    a_{(n)} \Gamma^q M &\subset \Gamma^{p+q-n-1} M \quad \text{for } a \in F^p V, \quad \forall n \in \mathbb{Z}, \\
    a_{(n)} \Gamma^q M &\subset \Gamma^{p+q-n} M \quad \text{for } a \in F^p V, \quad n \geq 0, \\
    H.\Gamma^p M &\subset \Gamma^p M \quad \text{for all } p \geq 0, \\
    \Gamma^p M_d & = 0 \quad \text{for } p \gg 0,
\end{align*}
\]
where \( \Gamma^p M_d = M_d \cap \Gamma^p M \). The associated graded space \( \text{gr}^F M = \bigoplus F^p M / F^{p+1} M \) is a graded module over the graded vertex Poisson algebra \( \text{gr}^F V \).

We denote by \( \{ F^p M \} \) the Li filtration of a \( V \)-module \( M \), which is a compatible filtration. We have \( F^0 M / F^1 M = M / C_2(M) \), where \( C_2(M) \) is the linear span of the vectors \( a_{(-2)} m \) with \( a \in V, \ m \in M \). The \( \text{gr}^F V \)-module structure of \( \text{gr}^F M \) gives a \( R_V \)-module structure on \( M / C_2(M) \).

Assume that \( M \) is \textit{finitely strongly generated} over \( V \); that is, \( M / C_2(M) \) is finitely generated over \( R_V \). The \textit{associated variety} of \( M \) is by definition
\[ X_M = \text{supp}_{R_V} (M / C_2(M)), \]
which is a Poisson subvariety of \( X_V \). Clearly,
\[ M \text{ is } C_2\text{-cofinite } \iff \dim X_M = 0. \]

The \textit{singular support} of \( M \) is by definition
\[ SS(M) = \text{supp}_{(R_V)_\infty} (\text{gr}^F M). \]

Because
\[ X_M = \pi_{\infty,0}(SS(M)), \]
\( \dim SS(M) = 0 \) implies that \( \dim X_M = 0 \). The converse is also true if \( V \) is conformal.

The singular support \( SS(M) \) may be computed by using any \textit{good filtration} of \( M \), that is, a compatible filtration \( \{ F^p M \} \) of \( M \) such that \( \text{gr}^F M \) is finitely generated over \( \text{gr}^F M \).

Throughout the paper \( \{ F^p M \} \) denotes the Li filtration and a general compatible filtration will be denoted by \( \{ F^p M \} \).

### 2.3. Kac-Moody algebras and the category \( KL_k \)

Let \( b \) be a Borel subalgebra of \( g \), \( h \subset b \) the Cartan subalgebra, \( \Delta_+ \) the corresponding set of positive roots, \( \Delta = \Delta_+ \sqcup -\Delta_+ \). Let \( g_\alpha \) the root space of root \( \alpha \in \Delta \), \( \theta \) the highest root, \( \theta_s \) the highest short root, \( \rho = \sum_{\alpha \in \Delta_+} \alpha / 2, \rho^\vee = \sum_{\alpha \in \Delta_+} \alpha^\vee / 2 \), where \( \alpha^\vee = 2\alpha / (\alpha | \alpha) \). Let \( \Pi = \{ \alpha_1, \ldots, \alpha_\ell \} \subset \Delta_+ \) be the set of simple roots of \( \Delta \), where \( \ell \) is the rank of \( g \).

Let \( P^+ = \{ \lambda \in h^*: \lambda(\alpha^\vee) \in \mathbb{Z}_{\geq 0} \text{ for } \alpha \in \Delta_+ \} \), the set of the integral dominant weights of \( g \). Denote by \( E_\lambda \) the irreducible finite-dimensional representation of \( g \) with highest weight \( \lambda \in P^+ \).
Let $\mathfrak{g}$ be the non-twisted affine Kac-Moody algebra associated with $\mathfrak{g}$:
\[
\tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus CK \oplus CD,
\]
whose commutation relations are given by
\[
[x_{(m)}, y_{(n)}] = [x, y]_{(m+n)} + m(x|y)\delta_{m+n,0}K,
\]
\[
[D, x_{(m)}] = mx_{(m)}, \quad [K, \tilde{\mathfrak{g}}] = 0
\]
with $x, y \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$, where $x_{(m)} = x \otimes t^m$ and $(\cdot | \cdot)$ is a normalized invariant bilinear form on $\mathfrak{g}$. Set
\[
\tilde{\mathfrak{h}} := \{[h, \tilde{\mathfrak{g}}] = \mathfrak{g}[t, t^{-1}] \oplus CK.
\]
Let $\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ be the standard Cartan subalgebra of $\tilde{\mathfrak{g}}$, $\tilde{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0$ the dual of $\tilde{\mathfrak{g}}$. Here $\delta(D) = \Lambda_0(K) = 1$, $\delta(h) = \Lambda_0(h) = \delta(K) = \Lambda_0(D) = 0$.

Set $\tilde{\mathfrak{g}}_+ = [b_+, b_+] + \mathfrak{g}[t] \subset \tilde{\mathfrak{g}}$, $\tilde{\mathfrak{g}}_- = [b_-, b_-] + \mathfrak{g}[t^{-1}]t^{-1} \subset \tilde{\mathfrak{g}}$, where $b_-$ be the opposite Borel subalgebra of $\mathfrak{g}$. Then
\[
\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{g}}_+
\]
gives the standard triangular decomposition of $\tilde{\mathfrak{g}}$. Denote by $\tilde{\Delta}_+$ the corresponding set of positive roots of $\tilde{\mathfrak{g}}$, by $\tilde{\Delta}_+^\circ$ the set of positive real roots of $\tilde{\mathfrak{g}}$, $\tilde{Q}_+ = \sum_{\alpha \in \tilde{\Delta}_+} \mathbb{Z}_{\geq 0} \alpha \subset \tilde{\mathfrak{h}}^*$. Let $\tilde{W}$ be the subgroup of $GL(\tilde{\mathfrak{h}}^*)$ generated by the reflection $s_\alpha$ with $\alpha \in \tilde{\Delta}_+^\circ$, where $s_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha$. The dot action of $\tilde{W}$ on $\tilde{\mathfrak{h}}^*$ is given by $w \cdot \lambda = w(\lambda + \tilde{\rho}) - \tilde{\rho}$, where $\tilde{\rho} = \rho + h^\vee \Lambda_0$ and $h^\vee$ is the dual Coxeter number of $\tilde{\mathfrak{g}}$.

For $\lambda \in \tilde{\mathfrak{h}}^*$, let $L_\lambda$ be the irreducible highest weight representation of $\tilde{\mathfrak{g}}$ with highest weight $\lambda$.

For $k \in \mathbb{C}$, let $\text{KL}_k$ be as in Introduction. The category $\text{KL}_k$ is the full subcategory of the category of $\tilde{\mathfrak{g}}$-modules consisting of objects $M$ satisfying
- $M$ has level $k$, that is, $K$ acts on $M$ as the multiplication by $k$,
- $M = \bigoplus_{d \in \mathbb{C}} M_d$ and $\text{dim} M_d < \infty$ for all $d \in \mathbb{C}$, where $M_d = \{m \in M; Dm = -dm\}$,
- there exists a finite subset $\{d_1, \ldots, d_s\}$ of $\mathbb{C}$ such that $M_{d_i} = 0$ unless $d \in \bigcup_{i=1}^{s} (d_i + \mathbb{Z}_{\geq 0})$.

The action of $\mathfrak{g}[t] \subset \tilde{\mathfrak{g}}$ on $M \in \text{KL}_k$ integrates to the action of $G_{\infty} = G[[t]]$.

The irreducible representation $L_\lambda$ is an object of $\text{KL}_k$ if and only if $\lambda \in \tilde{P}_k^+$, where
\[
\tilde{P}_k^+ := \{\lambda \in \tilde{\mathfrak{h}}^*_k; \lambda \in P^+\}.
\]
Here $\tilde{\mathfrak{h}}^*_k := \{\lambda \in \tilde{\mathfrak{h}}^*_k; \lambda(K) = k\}$ and $\tilde{\mathfrak{h}}^*_k \rightarrow \tilde{\mathfrak{h}}^*$, $\lambda \mapsto \tilde{\lambda}$, is the restriction map.

For $\lambda \in \tilde{P}_k^+$, let $V_\lambda \in \text{KL}_k$ be the Weyl module with highest weight $\lambda$:
\[
V_\lambda = U(\tilde{\mathfrak{g}})U(\mathfrak{g}[t] \oplus CK \oplus CD)E_\lambda,
\]
where $E_\lambda$ is the $\mathfrak{g}$-module $E_\lambda$ considered as a $\mathfrak{g}[t] \oplus CK \oplus CD$-module on which $\mathfrak{g}[t]$ acts trivially, $K = kid_{E_\lambda}$ and $D = \lambda(D) id_{E_\lambda}$.

Let $\text{KL}_k^A$ be the full subcategory of $\text{KL}_k$ consisting of objects $M$ that admits a Weyl flag, that is a finite filtration $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ such that each successive quotient $M_i/M_{i+1}$ is isomorphic to $V_{\lambda_i}$ for some $\lambda_i$. Any object of $\text{KL}_k^A$
is finitely generated and any finitely generated object of $\text{KL}_k$ is a quotient of some object of $\text{KL}_k^\infty$.

2.4. Universal affine vertex algebras and their associated graded vertex Poisson algebras. For $k \in \mathbb{C}$, the Weyl module $V_{k,A_0}$ is equipped with a unique vertex algebra structure such that

$$1 = 1 \otimes 1 \text{ is the vacuum},$$

$$Y(x_{(-1)}1, z) = x(z) := \sum_{n \in \mathbb{Z}} x(n) z^{-n-1} \quad \text{for } x \in \mathfrak{g}.$$ 

The vertex algebra $V_{k,A_0}$ is called universal affine vertex algebra associated with $\mathfrak{g}$ and denoted by $V^k(\mathfrak{g})$. The vertex algebra $V^k(\mathfrak{g})$ is graded by the Hamiltonian $H = -D$.

We denote by $V_k(\mathfrak{g})$ be the unique simple graded quotient of $V^k(\mathfrak{g})$.

The category $V_k(\mathfrak{g})\text{-Mod}$ of $V_k(\mathfrak{g})$-modules can be identified with the full subcategory of the category of $\hat{\mathfrak{g}}$-modules consisting of $\mathfrak{g}$-modules $M$ of level $k$ such that $x(z)$ is a field on $M$ for any $x \in \mathfrak{g}$. In particular $\text{KL}_k$ may be thought as a full subcategory of $V_k(\mathfrak{g})\text{-Mod}$. We have $V_k(\mathfrak{g}) \cong \text{KL}_{k,A_0}$ as $\mathfrak{g}$-modules.

We have

$$R_{V^k(\mathfrak{g})} \cong \mathbb{C}[\mathfrak{g}^*], \quad \text{gr}^F V^k(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}_\infty^*]$$

as Poisson algebras and vertex Poisson algebras, respectively (see e.g., [Ara10 Proposition 2.7.1]). In particular

$$X_{V^k(\mathfrak{g})} = \mathfrak{g}^*, \quad \text{SS}(V^k(\mathfrak{g})) = \mathfrak{g}_\infty^*.$$ 

Because the action of $H$ on $V^k(\mathfrak{g})$ stabilizes the Li filtration, the vertex Poisson algebra $\mathbb{C}[\mathfrak{g}_\infty^*]$ is graded by the Hamiltonian $H$. If $M$ is a graded $V^k(\mathfrak{g})$-module and $\{T^p M\}$ is a compatible filtration then $\text{gr}^T M$ is a graded $\mathbb{C}[\mathfrak{g}_\infty^*]$-module.

2.5. The category $\mathcal{C}$. Let $\mathcal{C}$ be the full subcategory of the category of graded modules over the vertex Poisson algebra $\mathbb{C}[\mathfrak{g}_\infty^*]$ consisting of modules $M$ such that

1) $\dim M_d < \infty$ for all $d \in \mathbb{C}$, where $M_d = \{m \in M; Hm = dm\}$,
2) there exists a finite subset $\{d_1, \ldots, d_r\}$ of $\mathbb{C}$ such that $M_d = 0$ unless $d \in \bigcup_{i=1}^r (d_i + \mathbb{Z}_{\geq 0})$.

For $M \in \mathcal{C}$ each homogeneous subspace $M_d$ is a direct sum of finite-dimensional representations of $\mathfrak{g} \subset \mathfrak{g}[t]$. The action of $\mathfrak{g}[t]$ on $M$ integrates to the action of $G_\infty$.

Let $M \in \text{KL}_k$, $\{T^p M\}$ a compatible filtration. Then $\text{gr}^T M$ is an object of $\mathcal{C}$.

We have $\text{gr}^F V_{A} \cong M(E_{A}, -\lambda(D))$, where

$$M(E, d) = \mathbb{C}[\mathfrak{g}_\infty^*] \otimes E$$

for a finite-dimensional representation $E$ of $\mathfrak{g}$ and $d \in \mathbb{C}$, whose graded $\mathbb{C}[\mathfrak{g}_\infty^*]$-module structure is given by

$$x_{(n)}(m \otimes u) = (x_{(n)}m) \otimes u \quad \text{for } n \neq 0,$$

$$x_{(0)}(m \otimes u) = (x_{(0)}m) \otimes u + m \otimes (x_{(0)}u),$$

$$H(m \otimes u) = (Hm \otimes u) + d(m \otimes u)$$

for $x \in \mathfrak{g}$, $m \in \mathbb{C}[\mathfrak{g}_\infty^*]$, $u \in E$. 

Lemma 2.5.1. Let $M$ be an object of $C$. Then there exists a filtration $0 = M_0 \subset M_1 \subset \ldots$ such that (1) $M = \bigcup M_i$, (2) each $M_i/M_{i-1}$ is a quotient of $M(E_i, d_i)$ for some finite dimensional simple $g$-module $E_i$ and $d_i \in \mathbb{C}$ and (3) $d_i \neq d_j$ for $i < j$. Moreover if $M$ is finitely generated then there exists a finite filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with this property.

3. Jet schemes of Slodowy slices and their modules

In this section we collect some fundamental facts about jet schemes of Slodowy slices and prove two (co)homology vanishing assertions Propositions 3.6.1, 3.7.1.

3.1. Jet schemes of Slodowy slices. Let $f$ be a nilpotent element of $g$. By the Jacobson-Morozov theorem, $f$ can be embedded into an $sl_2$-subalgebra $s = \text{span}_\mathbb{C}\{e, h, f\}$, so that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

We assume that $(e, f) = 1$. The form $(\ | \ )$ gives an isomorphism

$$\nu : g \xrightarrow{\sim} g^*$$

of $G$-modules. The induced isomorphisms $g_m \xrightarrow{\sim} g^*_m$ and $g_\infty \xrightarrow{\sim} g^*_\infty$ are denoted by $\nu_\infty$ and $\nu_m$, respectively.

Let $\chi_\infty, \chi_m$ and $\chi$ be the images of $f \in g_\infty, f + g[t]t^{m+1} \in g_m$ and $f \in g$ by $\nu_\infty$, $\nu_m$ and $\nu$, respectively. Then $\chi_m = \pi_{\infty, m}(\chi_\infty), \chi = \pi_{\infty, 0}(\chi_\infty) = \pi_{m, 0}(\chi_m)$, where $\pi_{\infty, m} : X_\infty \to X_m, \pi_{\infty, 0} : X_\infty \to X$ and $\pi_{m, 0} : X_m \to X$ are the projections.

Let $N \subset g^*$ be the image of the set of nilpotent elements of $g$, and let $S \subset g^*$ be the image of $f + g^*$, where $g^*$ is the centralizer of $e$ in $g$. The affine space $S$ is transversal at $\chi$ to $\text{Ad} G \chi$, and is often called the Slodowy slice.

We have

$$S_m = \chi_m + \nu_m(g^*_m) \subset g^*_m, \quad S_\infty = \chi_\infty + \nu_\infty(g^*_\infty) \subset g^*_\infty.$$

Because $g_m = [g_m, f] \oplus g^*_m$, one has the following assertion.

Lemma 3.1.1. The affine space $S_m$ is transverse to the orbit $\text{Ad} G_m \chi_m$ at $\chi_m$ for any $m \geq 0$.

It is known [GG02, §3] that the Poisson structure of $g^*$ restricts to $S$. Here $\mathbb{C}[S_\infty]$ is equipped with the level zero vertex Poisson algebra induced from the Poisson structure of $\mathbb{C}[S]$.

3.2. Good gradings. Let

$$g_j = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} g_j$$

be a good grading [KRW03] for $s$, that is, a $\frac{1}{2}\mathbb{Z}$-grading of $g$ such that

$$e \in g_1, \quad h \in g_0, \quad f \in g_{-1}, \quad \text{(11)}$$

$$\text{ad} f : g_j \to g_{j-1} \text{ is injective for } j \geq \frac{1}{2}, \quad \text{(12)}$$

$$\text{ad} f : g_j \to g_{j-1} \text{ is surjective for } j \leq \frac{1}{2}. \quad \text{(13)}$$

Let $x_0$ be the element in $g_0$ which defines the grading, i.e.,

$$g_j = \{ x \in g ; [x_0, x] = jx \}. \quad \text{(14)}$$
Good gradings were classified in [EK05]. The grading defined by \( x_0 = h/2 \) is obviously good and is called the Dynkin grading.

We assume that the grading (11) is compatible with the triangular decomposition of \( \mathfrak{g} \), that is, \( \mathfrak{b}^+ \subset \bigoplus_{j \geq 0} \mathfrak{g}_j \), \( \mathfrak{h} \subset \mathfrak{g}_0 \) and \( h, x_0 \in \mathfrak{b} \).

Put

\[
\Delta_j = \{ \alpha \in \Delta; \mathfrak{g}_\alpha \subset \mathfrak{g}_j \}, \quad \Delta_{0,+} = \Delta_+ \cap \Delta_0,
\]

so that \( \Delta = \bigcup_{j \in \frac{1}{2} \mathbb{Z}} \Delta_j, \Delta_+ = \Delta_{0,+} \cup \bigcup_{j > 0} \Delta_j \).

### 3.3. The fundamental isomorphisms.

Set

\[
m = \bigoplus_{j \geq 1} \mathfrak{g}_j, \quad n = \mathfrak{g}_+ \oplus \bigoplus_{j \geq 1} \mathfrak{g}_j.
\]

Then \( m \subset n \), and they are both nilpotent subalgebras of \( \mathfrak{g} \).

Denote by \( N \) the unipotent subgroup of \( G \) with \( \text{Lie}(N) = n \). It is known from [Kos78] [GG02] the coadjoint action map gives the isomorphism of affine varieties

\[
(15) \quad N \times S \rightarrow \chi + m^\perp,
\]

where \( m^\perp \subset \mathfrak{g}^* \) is the annihilator of \( m \). Since \( (X \times Y)_m \cong X_m \times Y_m \), we immediately get the following assertion.

**Proposition 3.3.1.** The coadjoint action maps give the following isomorphisms:

\[
\begin{align*}
N_m \times S_m & \rightarrow \chi_m + (m^\perp)_m \quad \text{for all } m \geq 0, \\
N_\infty \times S_\infty & \rightarrow \chi_\infty + (m^\perp)_\infty.
\end{align*}
\]

For a Lie algebra \( \mathfrak{a} \) and \( \mathfrak{a} \)-module \( M \), we write \( H^\bullet_{\text{Lie}}(\mathfrak{a}, M) \) (respectively \( H^\bullet_{\text{Lie}}(\mathfrak{a}, M) \)) for the Lie algebra \( \mathfrak{a} \)-cohomology (respectively \( \mathfrak{a} \)-homology) with the coefficient in a \( \mathfrak{a} \)-module \( M \).

**Corollary 3.3.2.**

\[
H^i_{\text{Lie}}(n_m, \mathbb{C}[\chi_r + (m^\perp)_m]) \cong \begin{cases} \mathbb{C}[S_m] & \text{for } i = 0, \\
0 & \text{for } i > 0,
\end{cases}
\]

\[
H^i_{\text{Lie}}(n_\infty, \mathbb{C}[\chi_\infty + (m^\perp)_\infty]) \cong \begin{cases} \mathbb{C}[S_\infty] & \text{for } i = 0, \\
0 & \text{for } i > 0.
\end{cases}
\]

### 3.4. The \( \mathbb{C}^* \)-action.

Let \( I_\infty \) be the ideal of \( \mathbb{C}[\mathfrak{g}_\infty^*] \) generated by \( x - \chi_\infty(x) \) with \( x \in m[t^{-1}]t^{-1}, I_m = I_\infty \cap \mathbb{C}[\mathfrak{g}_m^*] \). Then

\[
\mathbb{C}[\chi_\infty + (m^\perp)_\infty] = \mathbb{C}[\mathfrak{g}_\infty^*/I_\infty], \quad \mathbb{C}[\chi_m + (m^\perp)_m] = \mathbb{C}[\mathfrak{g}_m^*/I_m].
\]

Set

\[
H_{\text{new}} = H - (x_0)_{(0)} \in \text{End} \mathbb{C}[\mathfrak{g}_\infty^*].
\]

This defines a new \( \frac{1}{2} \mathbb{Z} \)-grading on the vertex Poisson algebra \( \mathbb{C}[\mathfrak{g}_\infty^*] \):

\[
\mathbb{C}[\mathfrak{g}_\infty^*] = \bigoplus_{\Delta \in \frac{1}{2} \mathbb{Z}} \mathbb{C}[\mathfrak{g}_\infty^*]_{\Delta_{\text{new}}, \Delta_{\text{new}} = \{ \alpha \in \mathbb{C}[\mathfrak{g}_\infty^*]; H_{\text{new}} \alpha = \Delta \alpha \}.
\]

The operator \( H_{\text{new}} \) gives a new grading on \( M \in \mathcal{C} \) as well.
As easily seen $H_{\text{new}}$ stabilizes $I_\infty$ and $I_\infty$. Therefore it acts on $\mathbb{C}[\chi_\infty + (m^+)_\infty]$ and $\mathbb{C}[\chi_m + (m^+)_m]$. It gives non-negative gradings

\begin{equation}
\mathbb{C}[\chi_\infty + (m^+)_\infty] = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} \mathbb{C}[\chi_\infty + (m^+)_\infty]_\Delta, \quad \dim \mathbb{C}[\chi_\infty + (m^+)_\infty]_0 = 1,
\end{equation}

\begin{equation}
\mathbb{C}[\chi_m + (m^+)_m] = \bigoplus_{\Delta \in \mathbb{Z}_{\geq 0}} \mathbb{C}[\chi_m + (m^+)_m]_\Delta, \quad \dim \mathbb{C}[\chi_m + (m^+)_m]_0 = 1,
\end{equation}

where $\mathbb{C}[\chi_\infty + (m^+)_\infty]_\Delta$ and $\mathbb{C}[\chi_m + (m^+)_m]_\Delta$ are eigenspaces of $H_{\text{new}}$ with eigenvalue $\Delta$. This gives $C^*$-actions on $\chi_\infty + (m^+)_\infty$ and $\chi_m + (m^+)_m$, which contract to the unique fixed points $\chi_\infty$ and $\chi_m$, respectively. Therefore by Lemma 3.1.1 and [Gin09] 1.4 we have the following assertion.

**Proposition 3.4.1.** For each $m \geq 0$, any $G_m$ orbit in $\mathfrak{g}_m$ meets the affine space $\chi + (m^+)_m$ transversely.

As $\mathfrak{n}[t]$ is stable under the adjoint action of $H_{\text{new}}$, it follows from Corollary 3.3.2 that there are induced contracting $C^*$-actions on $S_\infty$ and $S_m$ as well.

### 3.5. The flatness

Let $\mathbb{C}[m^*_\chi]_{\chi_\infty}$ be the localization of the polynomial algebra $\mathbb{C}[m^*_\chi]$ at the point $\chi_\infty$.

**Proposition 3.5.1.** Let $M$ be a finitely generated object of $\mathcal{C}$. Then, for any $x \in \chi_\infty + (m^+)_\infty$, the localization $M_x$ of $M$ at $x$ is flat over $\mathbb{C}[m^*_\chi]$.

**Proof.** By Lemma 2.5.1 it suffice to prove the assertion in the case that $M$ is a quotient of $M(E,d)$ for some finite-dimensional simple $\mathfrak{g}$-module $E$ and $d \in \mathbb{C}$.

Let $U$ be the image of $\mathbb{C} \otimes E \subset M(E,d)$ under the quotient map $M(E,d) \to M$. Set $M_r = \mathbb{C}[\mathfrak{g}^*_r] U$. Then $M = \bigcup_r M_r$. Note that the $G_\infty$-module structure of $M$ induces the $G_r$-module structure on $M_r$.

We have $\mathbb{C}[m^*_\chi]_{\chi_\infty} \subset \mathbb{C}[m^*_r]_{\chi_\infty}$ and $\mathbb{C}[m^*_\chi]_{\chi_\infty} = \bigcup_r \mathbb{C}[m^*_r]_{\chi_r}$. Let $x_r = \pi_{\infty,r}(x)$. Then $M_x = \bigcup_r (M_r)_{x_r}$. By [Eis95] Corollary 6.5 it is sufficient to show that each $(M_r)_{x_r}$ is flat over $\mathbb{C}[m^*_r]_{\chi_r}$.

Let $\tilde{M}_r$ be the sheaf on the affine space $\mathfrak{g}^*_r$ corresponding to the $\mathbb{C}[\mathfrak{g}^*_r]$-module $M_r$. Because $\tilde{M}_r$ is $G_r$-equivalent and coherent, we can apply by Proposition 3.4.1 the argument of [Gin09] Corollary 1.3.8 to get that $(M_r)_{x_r}$ is flat over $\mathbb{C}[m^*_r]_{\chi_r}$. This completes the proof. \qed

### 3.6. A homology vanishing

Let $\{y_i\}$ be a basis of $\mathfrak{m}[t^{-1}]t^{-1}$. Set

\[ t_i = y_i - \chi(y_i), \]

so that

\begin{equation}
I_\infty = \sum_i t_i \mathbb{C}[\mathfrak{g}^*_\chi].
\end{equation}

For a module $M$ over $\mathbb{C}[m^*_\chi]$ as a ring, let $H^0_\text{Kos}(\mathfrak{m}[t^{-1}]t^{-1}, M \otimes \mathbb{C}_-\chi)$ denote the homology of the Koszul complex associated with $M$ and the sequence $t_1, t_2, \ldots$. By definition

\[ H^0_\text{Kos}(\mathfrak{m}[t^{-1}]t^{-1}, M \otimes \mathbb{C}_-\chi) = M/I_\infty M. \]

**Proposition 3.6.1.** We have $H^i_\text{Kos}(\mathfrak{m}[t^{-1}]t^{-1}, M \otimes \mathbb{C}_-\chi) = 0$ for $i > 0$ and any object $M$ of $\mathcal{C}$.
Let

\[ M_{\text{tor}} = \{ m \in M ; m_x = 0 \text{ for all } x \in \chi_\infty + (m^+)_{\infty} \} . \]

Then \( M_{\text{tor}} \) is a submodule of \( M \) over \( \mathbb{C}[m^*_\infty] \) as a ring.

The following assertion follows from [Eis95] Proposition 17.14b.

**Lemma 3.6.2.** \( H^*_C(m[t^{-1}]t^{-1}, M_{\text{tor}} \otimes \mathbb{C}_{-\chi}) = 0 \).

**Proof of Proposition 3.6.1.** We may assume that \( M \) is finitely generated as the homology functor commutates with the inductive limits. Set \( \bar{M} = M/M_{\text{tor}} \). By Lemma 3.6.2 it suffices to show that

\[ H^*_C(m[t^{-1}]t^{-1}, \bar{M} \otimes \mathbb{C}_{-\chi}) = 0 \text{ for } i > 0 . \]

Clearly, we may assume that \( \bar{M} \neq 0 \). It is enough to show that \( \{ t_1, t_2, \ldots \} \) is a regular sequence on \( \bar{M} \), that is, \( t_r \) is not a zero divisor of \( \bar{M} / \sum_{i=1}^{r-1} t_i \bar{M} \) for all \( r \geq 0 \).

Let \( a \in \bar{M} \) such that \( t_r a \in \sum_{i=1}^{r-1} t_i \bar{M} \). By localizing it at \( x \in \chi_\infty + (m^+)_{\infty} \), we get that \( t_r a_x \in \sum_{i=1}^{r-1} t_i M_x \). Because \( M_x = M \) for \( x \in \chi + (m^+)_{\infty} \), \( M_x \) is flat over \( \mathbb{C}[m^*_\infty]_{\chi} \) by Proposition 3.6.1. In particular \( \{ t_1, t_2, \ldots \} \) is a regular sequence on \( M_x \) ([Eis95] Exercise 18.18). This forces \( a_x = 0 \) for any \( x \in \chi_\infty + (m^+)_{\infty} \). Hence we get that \( a = 0 \). This completes the proof. \( \square \)

### 3.7. A cohomology vanishing

Let \( M \) be an object of \( \mathcal{C} \). Then \( M/I_\infty \) is naturally a module over \( \mathbb{C}[m^*_\infty] \) as a ring. Because the action of \( n[t] \subset \mathfrak{g}[t] \) on \( M \) stabilizes \( I_\infty M \), \( M/I_\infty M \) is a \( n[t] \)-module. We have

\[
(\mathbb{C}[\mathfrak{g}_\infty]/I_\infty)^{n[t]} \cong \mathbb{C}[S_\infty]
\]

as differential algebras by Corollary 3.3.2. Thus \( (M/I_\infty M)^{n[t]} \) is a module over \( \mathbb{C}[S_\infty] \) as a ring\(^{10}\).

The proof of the following assertion is based on [GG02] 6.2.

**Proposition 3.7.1.** Let \( M \in \mathcal{C} \). Then we have \( H^*_\text{Lie}(n[t], M/I_\infty M) = 0 \) for \( i > 0 \).

**Proof.** We show that the multiplication map

\[ \varphi : \mathbb{C}[\chi_\infty + (m^+)_{\infty}] \otimes \mathbb{C}[S_\infty] (M/I_\infty M)^{n[t]} \to M/I_\infty M \]

is an isomorphism of \( n[t] \)-modules. This proves the assertion because

\[
\begin{align*}
H^*_\text{Lie}(n[t], \mathbb{C}[\chi_\infty + (m^+)_{\infty}] \otimes \mathbb{C}[S_\infty]) (M/I_\infty M)^{n[t]} \\
= H^*_\text{Lie}(n[t], \mathbb{C}[N_\infty] \otimes (M/I_\infty M)^{n[t]}) \quad \text{(by Proposition 3.3.1)} \\
= H^*_\text{Lie}(n[t], \mathbb{C}[N_\infty]) \otimes (M/I_\infty M)^{n[t]} \\
= \left\{ \begin{array}{ll}
(M/I_\infty M)^{n[t]} & \text{for } i = 0, \\
0 & \text{for } i > 0.
\end{array} \right\
\end{align*}
\]

To prove \( \varphi \) is an isomorphism we have only to show that \( (\ker \varphi)^{n[t]} = 0 \) and \( (\operatorname{coker} \varphi)^{n[t]} = 0 \) because \( n[t] \) acts locally nilpotently on \( \ker \varphi \) and \( \operatorname{coker} \varphi \).

Applying the left exact functor \( H^0_\text{Lie}(n[t], \?) \) to the exact sequence \( 0 \to \ker \varphi \to \mathbb{C}[\chi_\infty + (m^+)_{\infty}] \otimes \mathbb{C}[S_\infty] (M/I_\infty M)^{n[t]} \to M/I_\infty M \), we obtain the exact sequence \( 0 \to \)?

\(^{10}\)We will see in (the proof of) Theorem 3.3.3 that \( (M/I_\infty M)^{n[t]} \) is actually a \( \mathbb{C}[S_\infty] \)-module as a vertex Poisson algebra.
(ker \varphi)^{n[i]} \to (M/I_{\infty}M)^{n[i]} \to (M/I_{\infty}M)^{n[i]}. This gives that (ker \varphi)^{n[i]} = 0. Similarly, the exact sequence 0 \to \mathbb{C}[\chi_{\infty} + (m^{\perp})_{\infty}] \otimes_{\mathbb{C}[S_{\infty}]} (M/I_{\infty}M)^{n[i]} \to M/I_{\infty}M \to \coker \varphi \to 0 gives the exact sequence

0 \to (M/I_{\infty}M)^{n[i]} \to (M/I_{\infty}M)^{n[i]} \to (\coker \varphi)^{n[i]} \to 0

by the vanishing of \( H^1 \) in (20). This gives that (coker \varphi)^{n[i]} = 0, completing the proof. 

By Propositions 3.6.1 and 3.7.1 the assignment \( M \mapsto (M/I_{\infty}M)^{n[i]} \) yields an exact functor from \( \mathcal{C} \) to the category of \( \mathbb{C}[S_{\infty}] \)-modules.

**Proposition 3.7.2.** Let \( M \) be a finitely generated object of \( \mathcal{C} \). Then \( (M/I_{\infty}M)^{n[i]} \) is finitely generated over \( \mathbb{C}[S_{\infty}] \) as a ring.

**Proof.** By Lemma 2.5.1 and the exactness of the functor it suffice to prove the assertion for \( M = M(E,d) \) with some finite-dimensional simple \( g \)-module \( E \) and \( d \in \mathbb{C} \). But in this case the assertion is obvious because

\( (M/I_{\infty}M)^{n[i]} = ([\mathbb{C}[\mathfrak{g}^*]/I_{\infty}) \otimes E)^{n[i]} \cong \mathbb{C}[S_{\infty}] \otimes E. \)

\[ \square \]

4. Associated varieties of module over affine vertex algebras and affine \( W \)-algebras

In this section we establish the relation between the associated varieties of modules over \( \hat{\mathfrak{g}} \) and those over \( \mathcal{W}^k(\mathfrak{g},f) \) by proving Theorems 4.4.5, 4.4.6, 4.5.2 and 4.6.2. As an application in 4.4.7 we give a proof of the characterization of integrable representations of \( \hat{\mathfrak{g}} \) in terms of vertex algebra theory. In 4.9 we comment on the critical level case, which is of another interest.

4.1. The BRST complex of the generalized quantized Drinfeld-Sokolov reduction. ([KRW03], see also [Ara05]). For a \( V^k(\mathfrak{g}) \)-module \( M \), let \( (C(M),Q_{(0)}) \) be the BRST complex of the generalized quantized Drinfeld-Sokolov reduction associated with \( (\mathfrak{g},f,x_0) \). We have

\[ C(M) = V^k(\mathfrak{g}) \otimes \mathcal{F} \hat{x} \otimes \bigwedge^{\mathfrak{g}+} = \bigoplus_{i \in \mathbb{Z}} C^i(M), \quad C^i(M) = V^k(\mathfrak{g}) \otimes \mathcal{F} \hat{x} \otimes \bigwedge^{\mathfrak{g}+i}, \]

where \( \mathcal{F} \hat{x} \) is the vertex algebra of neutral free superfermions, \( \bigwedge^{\mathfrak{g}+} = \bigoplus_{i \in \mathbb{Z}} \bigwedge^{\mathfrak{g}+i} \) is the vertex algebra of the semi-infinite forms associated with \( \mathfrak{n}[t,t^{-1}] \), and \( Q_{(0)} \) is a zero mode of a certain odd element \( Q \in C^1(M) \) satisfying \( Q(z)Q(w) \sim 0 \). We set

\[ H^+_i(M) := H^+_i(C(M)) = H^+_i(C(M),Q_{(0)}), \]

\[ \mathcal{W}^k(\mathfrak{g},f) := H^+_i(0)(V^k(\mathfrak{g})). \]

Being a tensor product of vertex (super)algebras, \( C(V^k(\mathfrak{g})) \) is a vertex (super)algebra. It follows that \( \mathcal{W}^k(\mathfrak{g},f) \) inherits the vertex algebra structure from \( C(V^k(\mathfrak{g})) \). The vertex algebra \( \mathcal{W}^k(\mathfrak{g},f) \) is called the (universal) \( W \)-algebra associated with \( (\mathfrak{g},f) \) at level \( k \). It is a conformal vertex algebra provided that \( k \neq -h^\vee \). For a \( V^k(\mathfrak{g}) \)-module \( M \), the \( C(V^k(\mathfrak{g})) \)-module structure on \( C(M) \)
induces the $W^k(g,f)$-module structure on $H_f^\neq\bullet(M)$. It follows that the assignment $M \mapsto H^0(C(M))$ gives a functor $V^k(g)$-Mod $\rightarrow W^k(g,f)$-Mod, where $W^k(g,f)$-Mod denotes the category of $W^k(g,f)$-modules.

By abuse of notation we set

$$H_{\text{new}} := -D - x_0 \in \hat{g}.$$  

The operator $H_{\text{new}}$ gives a $\frac{1}{2}\mathbb{Z}$-grading on $C(V^k(g))$, where $\hat{h}$ acts on $C(V^k(g))$ diagonally (see [Ara05, §3]). Because $H_{\text{new}}$ commutes with $Q(0)$, the vertex algebra $W^k(g,f)$ is $\frac{1}{2}\mathbb{Z}$-graded by the Hamiltonian $H_{\text{new}}$. Similarly, for $M \in \text{KL}_k$, the $W^k(g,f)$-module $H_f^\neq\bullet(M)$ is graded by $H_{\text{new}}$. Set

$$H_f^\neq\bullet(M)_d := \{m \in H_f^\neq\bullet(M); H_{\text{new}}m = dm\}.$$ 

Then $H_f^\neq\bullet(M)_d$ is the cohomology of the subcomplex

$$C(M)_d := \{m \in C(M); H_{\text{new}}m = dm\},$$

and we have the decomposition

$$H_f^\neq\bullet(M) = \bigoplus_{d \in \mathbb{C}} H_f^\neq\bullet(M)_d.$$ 

We denote by $W_k(g,f)$ the unique simple graded quotient of $W^k(g,f)$.

**Remark 4.1.1.** The character of all highest weight representations of $W^k(g,f)$ was determined for a principal nilpotent element $f_{\text{pri}}$ in [Ara07]. It was determined for a minimal nilpotent element $f_{\text{min}}$ in [Ara05] provided that $k \neq -h^\vee$. For $g = sl_t$, one can choose a good even grading for any nilpotent element $f$, so that $W^k(sl_t, f)$ is $\mathbb{Z}_{\geq 0}$-graded. In this case the character of all ordinary representations of $W^k(sl_t, f)$ was determined in [Ara08] provided that $k \neq -h^\vee$.

### 4.2. The right exactness of the functor $H_f^\neq0$.

**Theorem 4.2.1.** ([KW04 Theorem 6.3]), (i) Let $M \in \text{KL}_k^\Delta$. Then $H_f^\neq0(M) = 0$ for all $i \neq 0$. In particular $H_f^\neq0(V^k(g)) = 0$ for all $i \neq 0$.

(ii) For $\lambda \in P_+^k$, we have $H_f^\neq0(V_\lambda) = \bigoplus_{d \in -\lambda(D+\mathfrak{x}_0) + \frac{1}{2}\mathbb{Z}_{\geq 0}} H_f^\neq0(V_\lambda)_d$ and\n
$$\dim H_f^\neq0(V_\lambda)_d < \infty \text{ for all } d \in \mathbb{C}. \text{ In particular } H_f^\neq0(M)_d \text{ is finite-dimensional for all } d \in \mathbb{C}, M \in \text{KL}_k^\Delta.$$

**Theorem 4.2.2.** Let $M \in \text{KL}_k$. Then $H_f^\neq0(M) = 0$ for all $i > 0$. Therefore the functor $H_f^\neq0(\cdot) : \text{KL}_k \rightarrow W^k(g,f)$-Mod is right exact.

**Proof.** By Theorem 4.2.1, the assertion can be proven in the similar manner as [AM09 Theorem 3.4]. \hfill \Box

**Corollary 4.2.3.** Let $M$ be a finitely generated object of $\text{KL}_k$. Then $H_f^\neq0(M)_d$ is finite-dimensional for all $d \in \mathbb{C}$. 

4.3. Cohomology vanishing of associated graded BRST complexes. The following two assertions are easily seen from the definition \[\text{[KRW03].}\]

**Lemma 4.3.1.**  
(i) We have \(R_{\mathfrak{g}_1/2} \cong C[H_{1/2}^*]\) as Poisson algebras, where the Poisson structure of \(C[H_{1/2}^*]\) is given by the symplectic form \(\mathfrak{g}_{1/2} \times \mathfrak{g}_{1/2} \rightarrow \mathbb{C}, (x, y) \mapsto (f, [x, y]).\)

(ii) The homomorphism \(C[H_{1/2}^*] = (R_{\mathfrak{g}_1/2})_{\infty} \rightarrow \text{gr} F \mathfrak{g}^\times\) is an isomorphism of vertex Poisson algebras.

**Lemma 4.3.2.**  
(i) \(R_{\mathfrak{g}_1^*} = \mathfrak{g}_1^* \otimes \mathfrak{g}_1^*\) as Poisson superalgebras, where the Poisson superalgebra structure on the right-hand side is given by \(\{\psi_f, \psi_x\} = f(x), \{\psi_x, \psi_y\} = 0\) for \(x, x' \in \mathfrak{n}, f, f' \in \mathfrak{n}^*\). Here \(\psi_x\) (respectively \(\psi_f\)) denotes the element of \(\Lambda^1(\mathfrak{n})\) (respectively \(\Lambda^1(\mathfrak{n}^*)\)) corresponding to \(x \in \mathfrak{n}\) (respectively to \(f \in \mathfrak{n}^*\)).

(ii) \(\text{gr}^F R_{\mathfrak{g}_1^*} = \Lambda^* (\mathfrak{n}^* [t^{-1}]^r) \otimes \Lambda^* (\mathfrak{n}^* [t^{-1}])\) as vertex Poisson superalgebras, where the vertex Poisson superalgebra structure of the right-hand side is the level 0 vertex Poisson superalgebra induced from the Poisson superalgebra structure of \(\Lambda^* (\mathfrak{n}) \otimes \Lambda^* (\mathfrak{n}^*)\).

Let \(M\) be a \(V^k(\mathfrak{g})\)-module. The Li filtration of \(C(M)\) is given by

\[
F^p C(M) = \sum_{r+s+t=p} F^r M \otimes F^s \mathfrak{g}^x \otimes F^t \mathfrak{g}^\times .
\]

More generally, for a compatible filtration \(\{\Gamma^p M\}\) of \(M\),

\[
\Gamma^p C(M) := \sum_{r+s+t=p} \Gamma^r M \otimes \Gamma^s \mathfrak{g}^x \otimes \Gamma^t \mathfrak{g}^\times \mathfrak{g}^\times
\]

defines a compatible filtration of \(C(V^k(\mathfrak{g}))\)-module \(C(M)\). We have:

\[
\text{gr}^F C(M) = \text{gr}^F M \otimes C([\mathfrak{g}^*_{1/2})_{\infty}] \otimes \Lambda^* (\mathfrak{n}^* [t^{-1}]) \otimes \Lambda^* (\mathfrak{n}^* [t^{-1}]).
\]

If \(\{\Gamma^p M\}\) is a good filtration then so is \(\{\Gamma^p C(M)\}\).

We have \(Q(0) \Gamma^p C(M) \subset \Gamma^p C(M)\). Thus, \((\text{gr}^F C(M), Q(0))\) is a cochain complex.

It follows that the zeroth cohomology \(H^0((\text{gr}^F C(V^k(\mathfrak{g})))\) is a vertex Poisson algebra, and \(H^*((\text{gr}^F C(M))\) is a module over the vertex Poisson algebra \(H^0((\text{gr}^F C(V^k(\mathfrak{g})))\).

**Theorem 4.3.3.**  
(i) We have \(H^i((\text{gr}^F C(V^k(\mathfrak{g}))) = 0\) for all \(i \neq 0\), and \(H^0((\text{gr}^F C(V^k(\mathfrak{g}))) \cong C[S_{\infty}]\) as vertex Poisson algebras.

(ii) Let \(M \in \mathcal{K}_L, \{\Gamma^p M\}\) a compatible filtration of \(M\). Then

\[
H^i((\text{gr}^F C(M)) \cong \begin{cases} (\text{gr}^F M/I_{\infty} \text{gr}^F M)^{n[t]} & \text{for } i = 0, \\ 0 & \text{for } i \neq 0 \end{cases}
\]

as modules over \(H^0((\text{gr}^F C(V^k(\mathfrak{g}))) = C[S_{\infty}]\).

**Proof.** (ii) Set

\[
\tilde{C} = \text{gr}^F C(M) = \bigoplus_{i \leq 0, j \geq 0} \tilde{C}_{i,j},
\]

where

\[
\tilde{C}_{i,j} = \text{gr}^F M \otimes C([\mathfrak{g}^*_{1/2})_{\infty}] \otimes \Lambda^{-i} (\mathfrak{n}^* [t^{-1}]) \otimes \Lambda^j (\mathfrak{n}^* [t^{-1}]).
\]
From the explicit form $\text{[KRW03]}$ of $Q$, we see that the operator $Q_{(0)}$ decomposes as

$$Q_{(0)} = \bar{Q}^- + \bar{Q}^+,$$

where $\bar{Q}^- \mathcal{C}_{i,j} \subset \mathcal{C}_{i+1,j}$ and $\bar{Q}^+ \mathcal{C}_{i,j} \subset \mathcal{C}_{i,j+1}$. Because $Q^2_{(0)} = 0$, we get that

$$\bar{Q}^- \mathcal{C}_{i,j} = \{ \bar{Q}^-, \bar{Q}^+ \} = 0.$$

Consider the spectral sequence $E_r \Rightarrow H^*(C)$ whose zeroth differential is $\bar{Q}^-$ and first differential is $\bar{Q}^+$. This is a converging spectral sequence because the complex $\mathcal{C}$ is a direct sum of subcomplexes $\Gamma^p C(M)/\Gamma^{p+1} C(M)$ with $p \in \mathbb{Z}$, and $(\Gamma^p C(M)/\Gamma^{p+1} C(M)) \cap C_{i,j} = 0$ for $j > 0$ by (21).

By definition the $E_1$-term is the cohomology of the complex $(\mathcal{C}, \bar{Q}^-)$. Consider the decomposition

$$\bar{C} \cong \bar{C}^{(1)} \otimes \bar{C}^{(2)} \otimes \bar{C}^{(3)},$$

where

$$\bar{C}^{(1)} = \mathbb{C}[[\mathfrak{g}_{1/2}]] \otimes \bigwedge^n (\mathfrak{g}_{1/2}[t^{-1}]t^{-1}),$$

$$\bar{C}^{(2)} = \text{gr}^F M \otimes \bigwedge^n (m[t^{-1}]t^{-1}),$$

and $\bar{C}^{(3)} = \bigwedge^n ([t^{-1}])$. From the explicit form of $\bar{Q}^-$ we find that (22) is a tensor product decomposition of the complex $(\mathcal{C}, \bar{Q}^-)$, where $\bar{C}^{(1)}$ is viewed as the Koszul complex with the $\mathbb{C}[[\mathfrak{g}_{1/2}]]$-module $\mathbb{C}[[\mathfrak{g}_{1/2}]]$ associated with a sequence consisting of a basis of $\mathfrak{g}_{1/2}[[t^{-1}]t^{-1}]$, $\bar{C}^{(2)}$ is viewed as a Koszul complex of $\text{gr}^F M$ considered in (3.0) and $\bar{C}^{(3)}$ is regarded as a trivial complex. Therefore Proposition 3.6.1 gives that

$$H^i(\bar{C}, \bar{Q}^-) = \begin{cases} (\text{gr}^F M/I_{\infty} \text{gr}^F M) \otimes \Lambda^n [t^{-1}] & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

Next let us compute the $E_2$-term. We find from the explicit form of $\bar{Q}^-$ and (23) that the complex $(H^0(\bar{C}, \bar{Q}^-), \bar{Q}^+)$ is identical to the the Chevalley complex for computing Lie algebra cohomology $H^*_{\text{lie}}(\mathfrak{g}, \text{gr}^F M/I_{\infty} \text{gr}^F M)$ considered in (3.7). Therefore Proposition 3.7.1 gives that

$$H^i(\bar{C}, \bar{Q}^-), \bar{Q}^+) \cong \begin{cases} (\text{gr}^F M/I_{\infty} \text{gr}^F M)^{n[t]} & \text{for } i = j = 0, \\ 0 & \text{otherwise}. \end{cases}$$

Therefore the spectral sequence collapse at $E_2 = E_{\infty}$, proving the assertion (ii). Note that $E_{\infty}^0 = \mathbb{C}$ lies entirely in $E_{\infty}^{0,0}$, and therefore the corresponding filtration of $H^0(\text{gr}^F C(M))$ is trivial.

Applying (ii) to $M = V^k(\mathfrak{g})$ and $\Gamma = F$, we obtain the cohomology vanishing assertion of (i) and an isomorphism

$$\Phi : H^0(\text{gr}^F C(V^k(\mathfrak{g}))) \cong \mathbb{C}[S_{\infty}].$$

The triviality of the filtration associated with the spectral sequence considered above implies that $\Phi$ is an isomorphism of differential algebras. It remains to show that $\Phi$ is an isomorphism of vertex Poisson algebras.

Consider the subcomplex $R_{C(V^k(\mathfrak{g}))}$ of $\text{gr}^F C(V^k(\mathfrak{g}))$. We have

$$R_{C(V^k(\mathfrak{g}))} \cong R_{V^k(\mathfrak{g})} \otimes R_{\mathfrak{g}} \otimes R_{\Lambda} \otimes \cdots.$$
as Poisson algebras. It follows from the explicit form of $Q_{(0)}$ that the assignment $R_{C(V^k(g))} \rightarrow H^0(R_{C(V^k(g))})$ is exactly the BRST realization [KSS7] of Hamiltonian reduction described in [CG02 §3.2]. Hence the restriction of $\Phi$ gives the isomorphism

$$\Phi|_{H^0(R_{C(V^k(g))})} : H^0(R_{C(V^k(g))}) \cong C[S]$$

of Poisson algebras.

Since $C[S_\infty]$ is generated by its subring $C[S]$, $H^0(\text{gr}^F C(V^k(g)))$ is also generated by $H^0(R_{C(V^k(g))})$ as a differential algebra. By the property of the Li filtration we have $a(n)b = 0$ for $n > 0$, $a, b \in H^0(R_{C(V^k(g))})$. This means that the vertex Poisson algebra structure of $H^0(\text{gr}^F C(V^k(g)))$ coincides with that of $C[S_\infty]$ on the generating subspaces $H^0(R_{C(V^k(g))}) \cong C[S]$. But we have shown in [Ara10 Proposition 2.3.1] that this uniquely determines the whole vertex Poisson algebra structure. This completes the proof. □

Consider the decomposition $H(\text{gr}^F C(M)) = \bigoplus_{d \in \mathbb{C}} H(\text{gr}^F C(M))_d$, where

$$H(\text{gr}^F C(M))_d = \{m \in \text{gr}^F C(M); H_{\text{new}}m = dm\}.$$

**Proposition 4.3.4.**

(i) Let $M \in \text{KL}_k^\lambda, \{\Gamma^pM\}$ a compatible filtration. Then $H^0(\text{gr}^F C(M))_d$ is finite-dimensional for all $d \in \mathbb{C}$.

(ii) Let $M$ be a finitely generated object of $\text{KL}_k, \{\Gamma^pM\}$ a compatible filtration. Then $H^0(\text{gr}^F C(M))_d$ is finite-dimensional for all $d \in \mathbb{C}$.

**Proof.** (i) By Theorems 4.2.1, 4.3.3, and the Euler-Poincaré principle it follows that $\dim H^0(\text{gr}^F C(M))_d = \dim H^0\text{gr}^F(M)_d$ for all $d$. Hence the assertion follows from Theorem 4.2.1 (ii). (ii) Because $M$ is finitely generated, there is a surjection $\phi : P \rightarrow M$ with some $P \in \text{KL}_k^\lambda$. Let $\Gamma^pP = \phi^{-1}(\Gamma^pM)$. Then $\{\Gamma^pP\}$ gives a compatible filtration of $P$. By Theorem 4.3.3, the surjection $\text{gr}^F P \rightarrow \text{gr}^F M$ gives rise to a surjection $H^0(\text{gr}^F P) \rightarrow H^0(\text{gr}^F M)$. Therefore (ii) follows from (i). □

4.4. **Strong vanishing of BRST cohomology.** Below we denote by $M_d$ the eigenspace of $H_{\text{new}}$ with eigenvalue $d$ for a semisimple $\text{CH}_{\text{new}}$-module $M$.

Let $M$ be a finitely generated object of $\text{KL}_k, \{\Gamma^pM\}$ a compatible filtration. We want to show that $\{\Gamma^pC(M)_d\}$ is a regular filtration of the complex $C(M)_d$ in the sense of [CE50, p.324], that is, $H^*(\Gamma^pC(M)_d) = 0$ for a sufficiently large $p$.

**Proposition 4.4.1.** Let $M \in \text{KL}_k, \{\Gamma^pM\}$ a compatible filtration.

(i) We have $H^1(C(M)/\Gamma^pC(M)) = 0$ for all $i \neq 0$ and $p \geq 1$.

(ii) Let $d \in \mathbb{C}$ and suppose that $H^0(\text{gr}^F C(M))_d$ is finite-dimensional. Then there exists $p_0 \in \mathbb{N}$ such that

$$\dim H^0(C(M)/\Gamma^pC(M))_d = \dim H^0(\text{gr}^F C(M))_d$$

for all $p \geq p_0$.

**Proof.** (i) Set $C = C(M)$, $\Gamma^p(C/\Gamma^pC) = \Gamma^pC/(\Gamma^pC \cap \Gamma^pC)$. Then, $\{\Gamma^p(C/\Gamma^pC)\}$ is a regular filtration of the complex $C/\Gamma^pC$, and we have the corresponding converging spectral sequence. Because the complex $\text{gr}^F(C/\Gamma^pC)$ is a direct summand of $\text{gr}^F C$, Theorem 4.3.3 gives that $H^i(\text{gr}^F(C/\Gamma^pC)) = 0$ for $i \neq 0$. Therefore the
spectral sequence collapses at $E_1 = E_{\infty}$, and we get that
\begin{equation}
H^i(C/\Gamma^pC) = 0 \quad \text{for } i \neq 0, \quad (26)
\end{equation}
\begin{equation}
\text{gr}^p H^0(C/\Gamma^pC) = H^0(\text{gr}^p(C/\Gamma^pC)) = \bigoplus_{i=0}^{p-1} H^0(\Gamma^iC/\Gamma^{i+1}C). \quad (27)
\end{equation}
The assertion (ii) follows from (27). \hfill \Box

Let $M \in \text{KL}_k$, $\{\Gamma^p M\}$ a compatible filtration. Let $\{\Gamma^p H_f^{\text{ss}}(M)\}$ be the induced filtration of $H_f^{\text{ss}}(M)$, i.e.,

$$
\Gamma^p H_f^{\text{ss}}(M) = \text{Im}(H^\bullet(\Gamma^p C(M))) \to H^\bullet(C(M)) \subset H_f^{\text{ss}}(M).
$$

**Proposition 4.4.2.** Let $M \in \text{KL}_k$, $\{\Gamma^p M\}$ a compatible filtration. Then for each $d \in \mathbb{C}$, $H_f^{\text{ss}}(M)_d = \Gamma^p H_f^{\text{ss}}(M)_d$ for all $i \neq 0$ and $p \geq 1$.

**Proof.** We have the injection
\begin{equation}
H_f^{\text{ss}}(M)/\Gamma^p H_f^{\text{ss}}(M) \hookrightarrow H^i(C(M)/\Gamma^p C(M)). \quad (28)
\end{equation}
But $H^i(C(M)/\Gamma^p C(M)) = 0$ for all $i \neq 0$ by Proposition 4.4.1. \hfill \Box

**Proposition 4.4.3.** Let $M$ be a finitely generated object of $\text{KL}_k$. Then the following conditions are equivalent.

(i) $H_f^{\text{ss}}(M) = 0$ for all $i \neq 0$.

(ii) For any compatible filtration $\{\Gamma^p M\}$, we have $\Gamma^p H_f^{\text{ss}}(M) = 0$ for $i \neq 0$, $p \geq 0$ and $\Gamma^p H_f^{\text{ss}}(M)_d = 0$ for $p \gg 0$, $d \in \mathbb{C}$.

(iii) For any compatible filtration $\{\Gamma^p M\}$, we have $H^i(\Gamma^p C(M)) = 0$ for $i \neq 0$, $p \geq 0$ and $H^0(\Gamma^p C(M))_d = 0$ for $p \gg 0$, $d \in \mathbb{C}$.

(iv) There exists a compatible filtration $\{\Gamma^p M\}$ such that $H^\bullet(\Gamma^p C(M))_d = 0$ for $p \gg 0$, $d \in \mathbb{C}$.

**Proof.** Put $C = C(M)$. The direction (iii) $\Rightarrow$ (iv) is obvious. The condition (iv) means that the filtration $\{\Gamma^p C(M)_d\}$ is regular. Therefore, the associated spectral sequence converges to $H_f^{\text{ss}}(M)$. By Theorem 4.3.3 this collapses at $E_1 = E_{\infty}$, and hence (iv) implies (i). Next let us show that (i) implies (ii). By Proposition 4.4.2 $\Gamma^p H_f^{\text{ss}}(M) = 0$ for all $i \neq 0$. It remains to show that $\Gamma^p H_f^{\text{ss}}(M)_d = 0$ for $p \gg 0$. Consider the composition
\begin{equation}
H_f^{\text{ss}}(M) \to H_f^{\text{ss}}(M)/\Gamma^p H_f^{\text{ss}}(M) \to H^0(C/\Gamma^p C). \quad (29)
\end{equation}
By Propositions 4.3.3, 4.4.1 $H^0(C/\Gamma^p C)_d = H^0(\text{gr}^p C)_d$ for $p \gg 0$. On the other hand, from the assumption, Theorem 4.3.3 and the Euler-Poincaré principle it follows that $\dim H_f^{\text{ss}}(M)_d = \dim H^0(\text{gr}^p C)_d$. Hence the two maps in (29) must be isomorphisms for $p \gg 0$, proving (ii). Finally, let us show (ii) implies (iii). By the assumption and the injectivity of (29), $H_f^{\text{ss}}(M)_d$ is embedded into $H_f^{\text{ss}}(M)$ for $p \gg 0$. Therefore, Proposition 4.4.1 gives that $H_f^{\text{ss}}(M) = 0$ for $i \neq 0$. \hfill \Box
By considering the long exact sequence corresponding to the exact sequence \(0 \to \Gamma^p C \to C \to C/\Gamma^p C \to 0\), we get that
\[
H^i(\Gamma^p C) = 0 \quad \text{for} \quad i \neq 0, 1,
\]
\[
0 \to H^0(\Gamma^p C) \to H^0_I(0(M) \to H^0(\Gamma C/\Gamma^p C) \to H^1(\Gamma^p C) \to 0 \quad \text{(exact)}.
\]
But from the cohomology vanishing we know that the maps in (29) are isomorphisms for \(p \gg 0\). Thus the middle map of the above exact sequence is an isomorphism. This show that \(H^0(\Gamma^p C)_d = H^1(\Gamma^p C)_d = 0\) for \(p \gg 0\), completing the proof. \(\square\)

**Proposition 4.4.4.** Let \(M\) be an object of \(\mathcal{KL}_k\), \(\{\Gamma^p M\}\) a compatible filtration.

(i) We have \(H^i(\Gamma^p C(M)) = 0\) for \(i > 0\).

(ii) Let \(d \in \mathbb{C}\) and suppose that \(H^0(\text{gr}^\Gamma C(M))_d\) is finite-dimensional. Then and \(H^0(\Gamma^p C(M))_d = 0\) for a sufficiently large \(p\).

**Proof.** (i) The fact that \(H^i(\Gamma^p C(M))_d = 0\) for \(i \geq 2\) follows from Theorem 4.2.1 and Proposition 4.4.1 by considering the long exact sequence associated with the short exact sequence
\[
0 \to \Gamma^p C(M) \to C(M) \to C(M)/\Gamma^p C(M) \to 0.
\]
To prove \(H^1(\Gamma^p C(M)) = 0\), let us first assume that \(M\) is finitely generated, so that there is an exact sequence
\[
0 \to N \to P \xrightarrow{\phi} M \to 0
\]
in \(\mathcal{KL}_k\) with \(P \in \mathcal{KL}_k^P\). Put \(\Gamma^p P = \phi^{-1}(\Gamma^p M)\), \(\Gamma^p N = N \cap \Gamma^p P\). Then \(\{\Gamma^p P\}\) and \(\{\Gamma^p N\}\) are compatible as well. By Theorem 4.2.1 and Proposition 4.4.1 we have \(H^i(\Gamma^p C(P)) = 0\) for \(i \neq 0\). Thus the exact sequence \(0 \to \Gamma^p N \to \Gamma^p P \to \Gamma^p M \to 0\) gives \(H^1(\Gamma^p C(M)) = 0\) as \(H^2(\Gamma^p C(N)) = 0\).

Next let \(M\) be arbitrary. There is an increasing series \(0 = M_0 \subset M_1 \subset \ldots\) of finitely generated submodules of \(M\) such that \(M = \bigcup_i M_i\). Let \(\Gamma^p M_i = M_i \cap \Gamma^p M\) be the induced filtration. Because the cohomology functor commutes with the injective limits we get that \(H^1(\Gamma^p C(M)) = \lim H^1(\Gamma^p C(M_i)) = 0\).

(ii) First, assume that \(M\) is finitely generated, so that (31) exists. By (i), we get the surjection \(H^0(\Gamma^p C(P))_d \to H^0(\Gamma^p C(M))_d\) by considering the long exact sequence associated with the short exact sequence \(0 \to \Gamma^p C(N) \to \Gamma^p C(P) \to \Gamma^p C(M) \to 0\). As \(H^0(\Gamma^p C(P))_d = 0\) for \(p \gg 0\) by Theorem 4.2.1 and Proposition 4.4.3 we get that \(H^0(\Gamma^p C(M))_d = 0\) for \(p \gg 0\). Note that we then have
\[
\text{gr}^\Gamma H^0_I(0(M) \cong H^0(\text{gr}^\Gamma C(M)).
\]
Next let \(M\) be arbitrary. By Proposition 4.4.1 and (i) the long exact sequence associated with (30) gives the exact sequence
\[
0 \to H^0(\Gamma^p C(M))_d \to H^0(\Gamma^p C(M)/\Gamma^p C(M))_d \to 0.
\]
Therefore, by Proposition 4.4.1 it is sufficient to show that
\[
\dim H^0_I(0(M)_d = \dim H^0(\text{gr}^\Gamma C(M))_d.
\]
Consider the increasing series \(0 = M_0 \subset M_1 \subset \ldots\) of finitely generated submodules of \(M\) such that \(M = \bigcup_i M_i\) as above, and let \(\Gamma^p M_i\) be the induced filtration. By Theorem 4.3.3 \(H^0(\text{gr}^\Gamma C(M_i)) \subset H^0(\text{gr}^\Gamma C(M))\) and we have \(H^0(\text{gr}^\Gamma C(M)) = \)
\[
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\]
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\[ \bigcup_j H^0(\text{gr}^F C(M_i)) \]. Because \( H^0(\text{gr}^F C(M))_d \) is finite-dimensional there exists \( i_0 \in \mathbb{N} \) such that
\[ H^0(\text{gr}^F C(M_i))_d = H^0(\text{gr}^F C(M))_d \quad \text{for } i \geq i_0, \]
or equivalently, \( H^0(\text{gr}^F C(M_i/M_{i_0}))_d = 0 \) for \( i \geq i_0 \) by Proposition 4.4.1. Because 
\[ \text{gr}^F H^i_f(\text{gr}^F C(M_i)) \cong H^0(\text{gr}^F C(M_i)) \quad \text{and} \quad \text{gr}^F H^i_f(\text{gr}^F C(M_i/M_{i_0})) \cong H^0(\text{gr}^F C(M_i/M_{i_0})), \]
\[ H^i_f(\text{gr}^F C(M_i)) \cong H^0(\text{gr}^F C(M_i)) \] it follows that the embedding \( M_{i_0} \hookrightarrow M_i \) induces an isomorphism
\[ H^i_f(\text{gr}^F C(M_i))_d \to H^i_f(\text{gr}^F C(M))_d \quad \text{for } i \geq i_0. \]
Hence \( H^i_f(\text{gr}^F C(M))_d = H^i_f(\text{gr}^F C(M_i))_d \), proving (32). This completes the proof. \( \square \)

**Theorem 4.4.5.** We have \( H^i_f(\text{gr}^F C(M)) = 0 \) for \( i \neq 0 \) and for any object \( M \) in \( \mathcal{V}_k \). In particular the functor \( \mathcal{V}_k \to \mathcal{W}^k(\mathfrak{g}, f)\text{-Mod}, M \mapsto H^i_f(\text{gr}^F C(M)) \), is exact.

**Proof.** By Proposition 4.4.1 it remains to show that \( H^i_f(\text{gr}^F C(M)) = 0 \) for all \( i \leq -1 \). We proceed by induction on \( i \). Because the cohomology functor commutes with the injective limits we may assume that \( M \) is finitely generated. Thus, there is \( P \in \mathcal{V}_k \) and an exact sequence
\[ 0 \to N \to P \to M \to 0 \]
in \( \mathcal{V}_k \). Let \( \{ \Gamma^p P \} \) be a compatible filtration of \( P \), and let \( \{ \Gamma^p N \} \) be induced filtration. We have the short exact sequence \( 0 \to \text{gr}^F N \to \text{gr}^F P \to \text{gr}^F M \to 0 \), that induces an exact sequence \( 0 \to H^0(\text{gr}^F C(N)) \to H^0(\text{gr}^F C(P)) \to H^0(\text{gr}^F C(M)) \) \( \to 0 \) by Theorem 4.3.3. It follows from Proposition 4.3.4 that \( H^0(\text{gr}^F C(N))_d, H^0(\text{gr}^F C(P))_d \) and \( H^0(\text{gr}^F C(M))_d \) are finite-dimensional for all \( d \in \mathbb{C} \).

We have the commutative diagram
\[ \begin{array}{cccccc}
0 & \to & H^i_f(\text{gr}^F C(M))_d & \to & H^i_f(\text{gr}^F C(P))_d & \to & H^i_f(\text{gr}^F C(M))_d & \to & 0 \\
0 & \to & H^0(\text{gr}^F C(1))_d & \to & H^0(\text{gr}^F C(2))_d & \to & H^0(\text{gr}^F C(3))_d & \to & 0,
\end{array} \]
where \( C_1 = C(N) \), \( C_2 = C(P) \), \( C_3 = C(M) \). But the vertical arrows are all isomorphisms by Proposition 4.4.4 for a sufficiently large \( p \). Therefore we get that \( H^i_f(\text{gr}^F C(M)) = 0 \).

Next suppose that we have shown that \( H^i_f(\text{gr}^F C(M)) = 0 \) for all objects \( M \in \mathcal{V}_k \). Since \( H^i_f(\text{gr}^F C(M)) = 0 \) by Theorem 4.2.1 the long exact sequence associated with (31) gives that \( H^i_f(\text{gr}^F C(M)) = 0 \) as desired. \( \square \)

**Theorem 4.4.6.**
(i) The Li filtration of \( \mathcal{W}_k(\mathfrak{g}) \) induces the Li filtration of the vertex algebra \( \mathcal{V}_k(\mathfrak{g}, f) \), that is, \( F^p \mathcal{W}_k(\mathfrak{g}, f) = \text{Im}(\mathcal{H}^0(F^p C(\mathcal{W}_k(\mathfrak{g})))) \to \mathcal{W}_k(\mathfrak{g}, f) \). We have \( \text{gr}^F \mathcal{W}_k(\mathfrak{g}, f) \cong \mathbb{C}[S] \) as vertex Poisson algebras.
(ii) Let \( M \) be a finitely generated object of \( \mathcal{V}_k \), \( \{ \Gamma^p M \} \) be a compatible filtration of \( M \). Then
\[ \text{gr}^F H^i_f(\text{gr}^F C(M)) \cong H^0(\text{gr}^F C(M)) \cong (\text{gr}^F M/I_\infty \text{gr}^F M)^{n[i]} \]
as modules over the vertex Poisson algebras \( \mathbb{C}[S] \).
Corollary 4.4.7 (DSK06). We have $R_{W^k(g, f)} \cong C[S]$ as Poisson algebras.

Corollary 4.4.8. The vertex algebra $H_f^\to(V_k(g))$ is $C_2$-cofinite if and only if it is $C_2$-cofinite as a module over $W^k(g, f)$. In this case the simple $W$-algebra $W_k(g, f)$ is also $C_2$-cofinite.

Proof. The first assertion follows from the fact that $H_f^\to(V_k(g))$ is a quotient vertex algebra of $W^k(g, f)$ by Theorem 4.4.5. The same fact shows that $W_k(g, f)$ is a quotient of $H_f^\to(V_k(g))$, proving the second assertion. □

4.5. BRST Reduction of associated varieties.

Proposition 4.5.1. Let $M$ be a finitely generated object of $KL$, $\{G^p M\}$ a good filtration of $M$. Then $\{G^p H_f^\to(M)\}$ is also a good filtration of $H_f^\to(M)$. In particular, $H_f^\to(M)$ is finitely strongly generated over $W^k(g, f)$.

Proof. The assertion follows from Proposition 3.7.2 and Theorem 4.4.6. □

We identify $\text{gr}^F W^k(g, f)$ and $R_{W^k(g, f)}$ with $C[S_\infty]$ and $C[S]$, respectively, through Theorem 4.4.6 and Corollary 4.4.7. Thus, for a finitely generated object $M$ of $KL$, $SS(H_f^\to(M))$ is a $C^*$-invariant subscheme of $S_\infty$, and $X_{H_f^\to(M)}$ a $C^*$-invariant Poisson subvariety of $S$.

Theorem 4.5.2. Let $M$ be a finitely generated object of $KL$.

(i) The singular support $SS(H_f^\to(M))$ is isomorphic to the scheme theoretic intersection $SS(M) \cap S_\infty$.

(ii) The associated variety $X_{H_f^\to(M)}$ is isomorphic to the scheme theoretic intersection $X_M \cap S$. 

Proof. (i) follows from the fact that $SS(M) \cap S_\infty$ is $C^*$-invariant. (ii) follows from the fact that $X/M \cap S$. □
(i) follows from Theorem 4.4.6 and Proposition 4.5.1. (ii) follows (i) and (9). □

Proposition 4.5.3. Let $M$ be a finitely generated object of $\mathcal{KL}_k$. Then $H_{f}^{\infty,0}(M) \neq 0$ if and only if $X_M$ contains the closure over $G \cdot X$ of the coadjoint orbit $\text{Ad}^{-1}G \cdot X$.

Proof. Because $X_{H_{f}^{\infty,0}(M)}$ is stable under the $C^*$-action, it follows that $H_{f}^{\infty,0}(M) \neq 0$ if and only if $X_M$ contains the unique $C^*$-fixed point $X$ of $\mathcal{S}$. The assertion follows because $X_M$ is $G$-invariant and closed. □

4.6. The $C_2$-cofiniteness condition of $W$-algebras.

Proposition 4.6.1. Let $M$ be an finitely generated object of $\mathcal{KL}_k$. Suppose that $H_{f}^{\infty,0}(M)$ is either 0 or $C_2$-cofinite. Then $X_M$ is contained in the nullcone $\mathcal{N}$.

Proof. Let $p_1, \ldots, p_\ell$ be the homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$, so that

$$\mathcal{N} = \{ x \in \mathfrak{g}; p_i(x) = 0 \text{ for all } i \}.$$  

We identify $p_i$ with its image in $\mathbb{C}[\mathfrak{g}_s]$. Thus, $p_1, \ldots, p_\ell \in \mathbb{C}[\mathfrak{g}_s]^G$. (In fact, $p_1, \ldots, p_\ell$ generate $\mathbb{C}[\mathfrak{g}_s]^G$ as a differential algebra $[EF01]$.)

By Proposition 3.3.1 we have

$$\mathbb{C}[\mathcal{S}_\infty] \cong \mathbb{C}[\chi + (m^\perp)_\infty]^N.$$  

Let $\tilde{p}_i$ denote the restriction of $p_i$ to $\chi + m_\infty$. Then $\tilde{p}_i$ belongs to $\mathbb{C}[\chi + (m^\perp)_\infty]^N = \mathbb{C}[\mathcal{S}_\infty]$.  

Let $\{ \Gamma \}$ be a good filtration of $M$. By the assumption, each $\tilde{p}_i$ acts locally nilpotently on $\text{gr}^\Gamma H_{f}^{\infty,0}(M)$. Since we have

$$\text{gr}^\Gamma H_{f}^{\infty,0}(M) = (\text{gr}^\Gamma M/I_\infty \text{gr}^\Gamma M)^{\mathbb{N}[t]}$$  

by Theorem 4.4.6, it follows that

$$\text{gr}^\Gamma M/I_\infty \text{gr}^\Gamma M \cong \mathbb{C}[\chi + (m^\perp)_\infty] \otimes_{\mathbb{C}[\mathcal{S}_\infty]} \text{gr}^\Gamma H_{f}^{\infty,0}(M)$$  

from the proof of Proposition 3.7.1. This shows that each $\tilde{p}_i$ acts locally nilpotently on $\text{gr}^\Gamma M/I_\infty \text{gr}^\Gamma M$, i.e., there exists $r_i \in \mathbb{N}$ such that

$$p_i^{r_i} = u_i + g_i$$  

for some $u_i \in \text{Ann}_{\mathbb{C}[\mathfrak{g}_s]}(\text{gr}^\Gamma M)$ and $g_i \in I_\infty$. Because $p_i$ is homogeneous and $\text{Ann}_{\mathbb{C}[\mathfrak{g}_s]}(\text{gr}^\Gamma M)$ is graded we may assume that $u_i$ and $g_i$ are also homogeneous. This forces that

$$g_i \in \sum_{x \in \mathfrak{h}^*; \langle x, \nu \rangle \neq 0} x \mathbb{C}[\mathfrak{g}]^*.$$  

It follows that $p_i^{r_i}(u) = u_i(u)$ for all $u \in \mathfrak{h}^*$, and thus,

$$X_M \cap \mathfrak{h}^* = 0. \tag{38}$$  

Now let $x \in X_M$, $x_s$ the semisimple part of $x$ (we are identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ via $\nu$). Because $X_M$ is $G$-invariant, the standard argument shows that $x_s \in X_M$. But any semisimple element is conjugate to some element of $\mathfrak{h}$. Therefore $x_s$ must be zero by (38). This completes the proof. □
Theorem 4.6.2. Let $M$ be a finitely generated object of $\mathcal{K}L_k$. Then the following conditions are equivalent.

(i) $H^+_{H^G}f(M)$ is $C_2$-cofinite.

(ii) $X_M \subset \mathcal{N}$ and $\text{Ad}G \cdot \chi$ appears as an irreducible component of $X_M$.

Proof. Because $X_{H^G}^0_{f(M)} \subset \mathcal{S}$ is $C^*$-invariant, $\dim X_{H^G}^0_{f(M)} = 0$ if and only if $X_{H^G}^0_{f(M)} = \{\chi\}$. Therefore by Theorem 4.5.2 $H^+_{H^G}f(M)$ is $C_2$-cofinite if and only if $X_M \cap \mathcal{S} = \{\chi\}$.

Suppose that $H^+_{H^G}f(M)$ is $C_2$-cofinite. Then by Proposition 4.6.1, $X_M$ is contained in $\mathcal{N}$. Thus the irreducible components of $X_M$ are the closures of some nilpotent coadjoint orbits, say, $\mathcal{O}_1, \ldots, \mathcal{O}_r$. But the transversality of $\mathcal{S}$ with any $\text{Ad}G$-orbit implies that

$$\overline{\mathcal{O}}_i \cap \mathcal{S} = \{\chi\} \iff \mathcal{O}_i = \text{Ad}G \cdot \chi.$$  

Hence, (i) implies (ii). The direction (ii) $\Rightarrow$ (i) follows also from (39).

Corollary 4.6.3. The vertex algebra $H^+_{H^G}f(V_k(\mathfrak{g}))$ is $C_2$-cofinite if and only if $X_{V_k(\mathfrak{g})} \subset \mathcal{N}$ and $\text{Ad}G \cdot \chi$ appears as an irreducible component of $X_{V_k(\mathfrak{g})}$. In this case $\mathcal{W}_k(\mathfrak{g}, f)$ is $C_2$-cofinite.

4.7. The minimal nilpotent element case. Let $f_{\text{min}}$ be a minimal nilpotent element of $\mathfrak{g}$, $\chi_{\text{min}} = \nu(f_{\text{min}})$, $\mathcal{O}_{\text{min}} = \text{Ad}G \cdot \chi_{\text{min}}$. Then $\mathcal{O}_{\text{min}} \subset \mathcal{N}$ is the unique nonzero nilpotent orbit of minimal dimension (see [CM93]).

The following assertion is a special case of [Ara05 Main Theorem].

Theorem 4.7.1. Let $\lambda \in \mathcal{P}^+_k$. Then $H^+_{f_{\text{min}}}f(\mathfrak{L}_\lambda)$ is non-zero if and only if $\mathfrak{L}_\lambda$ is not an integrable representation of $\mathfrak{g}$.

From Proposition 4.5.3 and Theorem 4.7.1 we immediately get the following assertion.

Proposition 4.7.2. Let $\lambda \in \mathcal{P}^+_k$. Then $\mathcal{O}_{\text{min}} \subset X_{\mathfrak{L}_\lambda}$ if and only if $\mathfrak{L}_\lambda$ is not an integrable representation.

Proposition 4.7.3 (cf. [DM06]). For $\lambda \in \mathcal{P}^+_k$, $\mathfrak{L}_\lambda$ is an integrable representation if and only if it is $C_2$-cofinite as a module over $V^k(\mathfrak{g})$.

Proof. The assertion for the “if” part is well-known (see e.g. [Ara10] for a proof). Let us show the “only if” part. Suppose that $\mathfrak{L}_\lambda$ is $C_2$-cofinite. Then $\mathfrak{L}_\lambda$ is in particular finitely strongly generated. But $\mathfrak{L}_\lambda$ is finitely strongly generated and if and only if $\lambda \in \mathcal{P}^+_k$ ([Ara10 Example 3.2.1]). Because $X_{\mathfrak{L}_\lambda} = \{0\}$, $\mathfrak{L}_\lambda$ must be integrable by Proposition 4.7.2.

4.8. The principal nilpotent element case. Let $f_{\text{pri}}$ be a principal nilpotent element, $\chi_{\text{pri}} = \nu(f_{\text{pri}})$, so that $\mathcal{N} = \text{Ad}G \cdot \chi_{\text{pri}}$. The following is an immediate consequence of Proposition 4.5.3 and Theorem 4.6.2.

Proposition 4.8.1. Let $M$ be a finitely generated object of $\mathcal{K}L_k$.

(i) $X_M \subset \mathcal{N}$ if and only if $H^+_{f_{\text{pri}}}f(M) \neq 0$.

(ii) $X_M = \mathcal{N}$ if and only if $H^+_{f_{\text{pri}}}f(M)$ is (non-zero and) $C_2$-cofinite.

(iii) $X_M \not\subset \mathcal{N}$ if and only if $H^+_{f_{\text{pri}}}f(M) = 0$. 

The assertion follows from Theorem 4.4.6, (40) and (41).

Proof. 4.9.2

Remark 5.1.

Kac-Wakimoto admissible representations. In this section we first prove the Feign-Frenkel conjecture for the degenerate cases and determine the associated varieties of G-integrable degenerate admissible representations (Theorems 5.5.1 5.6.1). Second we prove the $C_2$-cofiniteness of a large number of $W$-algebras including all the (non-principal) exceptional $W$-algebras (Theorems 5.7.2 5.8.2). In [5.9] we comment on the trivial representations of principal $W$-algebras. In this section $\mathfrak{g}^*$ is often identified with $\mathfrak{g}$.

5. Associated varieties of Kac-Wakimoto admissible representations and $C_2$-cofiniteness of $W$-algebras

4.9. The critical level case. By [Pre02 Theorem 5.1], the scheme $S \cap N$ is reduced, irreducible, Gorenstein, and complete intersection of dimension $\dim N - \dim \text{Ad} G \chi$. By [Gin09 Proposition 1.3.3], the coadjoint action gives the isomorphism $N \times (S \cap N) \cong (S \cap N) \cap (\chi + m^+)$. It follows that

$$\text{(40)} \quad (\mathbb{C}[N_{\infty}]/I_{\infty} \mathbb{C}[(N)_{\infty}])^{t[1]} \cong \mathbb{C}[(S \cap N)_{\infty}]$$

in the same manner as [119].

On the other hand, by a theorem of Frenkel and Gaitsgory [FG04], the maximal graded submodule of $V^{-h^\vee}(\mathfrak{g})$ is generated by the Feigin-Frenkel center $j(\mathfrak{g}) \subset V^{-h^\vee}(\mathfrak{g})$. Because $\text{gr}^F j(\mathfrak{g}) \cong \mathbb{C}[\mathfrak{g}]_{\text{co}},$ from [EF01] it follows that

$$\text{(41)} \quad \text{gr}^F V_{-h^\vee}(\mathfrak{g}) \cong \mathbb{C}[N_{\infty}]$$

as vertex Poisson algebras.

Theorem 4.9.1. We have $\text{gr}^F H^{\pm +0}_{f}(V_{-h^\vee}(\mathfrak{g})) \cong \mathbb{C}[(S \cap N)_{\infty}]$ as vertex Poisson algebras. In particular $R_{V_{-h^\vee}(\mathfrak{g})} \cong \mathbb{C}[S \cap N]$. 

Proof. The assertion follows from Theorem 4.4.6 (40) and (41). 

Remark 4.9.2. From Theorem 4.9.1 it follows that $H^{\pm +0}_{f}(V_{-h^\vee}(\mathfrak{g}))$ is $C_2$-cofinite if and only if $f$ is a principal nilpotent element, and that $H^{\pm +0}_{\text{prin}}(V_{-h^\vee}(\mathfrak{g}))$ is one-dimensional [FG07].

5. Associated varieties of Kac-Wakimoto admissible representations and $C_2$-cofiniteness of $W$-algebras

In this section we first prove the Feign-Frenkel conjecture for the degenerate cases and determine the associated varieties of G-integrable degenerate admissible representations (Theorems 5.5.1 5.6.1). Second we prove the $C_2$-cofiniteness of a large number of $W$-algebras including all the (non-principal) exceptional $W$-algebras (Theorems 5.7.2 5.8.2). In [5.9] we comment on the trivial representations of principal $W$-algebras. In this section $\mathfrak{g}^*$ is often identified with $\mathfrak{g}$.

5.1. Kac-Wakimoto admissible representations. A subset $\Delta'$ of $\hat{\Delta}$ is called a subroot system if $s_\alpha(\beta) \in \Delta'$ for any $\alpha, \beta \in \Delta'$. For a subroot system $\Delta'$, $\Delta'_+ = \Delta' \cap \Delta^\vee$ is a set of positive root and $\Pi' = \{ \alpha \in \Delta'_+ ; s_\alpha(\Delta'_+ \setminus \{ \alpha \}) \subset \Delta'_+ \}$ is the set of simple roots (MP95 KT98).

For $\lambda \in \mathfrak{h}^*$, let $\Delta(\lambda)$ be the associated integral root system defined by

$$\Delta(\lambda) = \{ \alpha \in \hat{\Delta}^\vee ; \langle \lambda + \hat{\rho}, \alpha^\vee \rangle \in \mathbb{Z} \}.$$ 

Then $\Delta(\lambda)$ is a subroot system of $\hat{\Delta}$. Let $\Delta(\lambda)_+$ and $\Pi(\lambda)$ be the sets of positive roots and simple roots of $\Delta(\lambda)$ as above. The subgroup $\tilde{W}(\lambda) = \langle s_\alpha ; \alpha \in \Delta(\lambda) \rangle$ of $\tilde{W}$ is called the integral Weyl group of $\lambda$.

A weight $\lambda \in \mathfrak{h}^*$ is called admissible [KW89] if

(i) $\lambda$ is regular dominant, that is, $\langle \lambda + \hat{\rho}, \alpha^\vee \rangle \not\in \{0, -1, -2, \ldots \}$ for any $\alpha \in \Delta^\vee$, 

(ii) $Q\Delta(\lambda) = Q\Delta^\vee$, where $\Delta(\lambda) = \{ \alpha \in \hat{\Delta}^\vee ; \langle \lambda + \hat{\rho}, \alpha^\vee \rangle \in \mathbb{Z} \}$. 


For $\lambda \in \hat{h}^*$, $L_\lambda$ is called an admissible representation if $\lambda$ is admissible. The condition (i) implies [KWS88] that the formal character $\text{ch} L_\lambda$ of $L_\lambda$ is given by

$$\text{ch} L_\lambda = \sum_{w \in \hat{W}(\lambda)} (-1)^{\ell(w)} e^{w \lambda} \prod_{\alpha \in \hat{\Delta}^+} (1 - e^{-\alpha})^{-\dim \hat{g}_\alpha},$$

where $\ell : \hat{W}(\lambda) \to \mathbb{Z}_{\geq 0}$ is the length function.

5.2. $G$-integrable admissible representations and the Feigin-Frenkel conjecture. A complex number $k$ is called admissible if $k \Lambda_0$ is admissible. Set

$$\text{Adm}^k_+ = \{ \lambda \in \hat{P}^k_+; \lambda \text{ is admissible} \}, \quad \text{Adm}_+ = \bigcup_{k \in \mathbb{C}} \text{Adm}^k_+.$$

The following assertion is easy to see.

Lemma 5.2.1. The set $\text{Adm}^k_+$ is non-empty if and only if $k$ is admissible.

The admissible numbers of $\hat{g}$ are classified in [KW08]. Set

$$A = A_\hat{g} = \bigcup_{q \in \mathbb{N}, (q,r^\vee) = 1} A[q], \quad L A = LA_\hat{g} = \bigcup_{q \in \mathbb{N}} L A[q],$$

where

$$A[q] := \{-h^\vee + \frac{p}{q}; p,q \in \mathbb{N}, (p,q) = 1, \ p \geq h^\vee \},$$

$$LA[q] := \{-h^\vee + \frac{p}{r^\vee q}; p,q \in \mathbb{N}, (p,q) = 1, (p,r^\vee) = 1, \ p \geq h_\hat{g} \}.$$

Here $h_\hat{g}$ is the Coxeter number of $\hat{g}$.

Proposition 5.2.2 ([KWS9] [KW08]).

(i) A complex number $k$ is admissible if and only if $k \in A \cup L A$.

(ii) Let $k \in A[q]$ with $q \in \mathbb{N}$, $(q,r^\vee) = 1$. Then, for $\lambda \in \text{Adm}^k_+$, $\hat{\Delta}(\lambda) = \hat{\Delta}(k\Lambda_0) = \{ \alpha + nq\delta; \alpha \in \Delta, n \in \mathbb{Z} \}$, $\hat{\Pi}(\lambda) = \{ -\theta + q\delta, \alpha_1, \ldots, \alpha_\ell \}$.

(iii) Let $k \in L A[q]$ with $q \in \mathbb{N}$. Then, for $\lambda \in \text{Adm}^k_+$, we have $\hat{\Delta}(\lambda) = \hat{\Delta}(k\Lambda_0) = \{ \alpha + r^\vee n\delta; \alpha \in \Delta_{\text{long}}, n \in \mathbb{Z} \} \cup \{ \alpha + nd\delta; \alpha \in \Delta_{\text{short}}, n \in \mathbb{Z} \}$, $\hat{\Pi}(\lambda) = \{ -\theta + q\delta, \alpha_1, \ldots, \alpha_\ell \}$. Here $\Delta_{\text{long}}$ and $\Delta_{\text{short}}$ are the sets of long roots and short roots of $\hat{g}$, respectively.

| $\hat{g}$ | $h_\hat{g}$ | $h^\vee$ | $L h^\vee$ | $r^\vee$ |
|---|---|---|---|---|
| $A_l$ | $l + 1$ | $l + 1$ | $l + 1$ | 1 |
| $B_l$ | $2l$ | $2l - 1$ | $l + 1$ | 2 |
| $C_l$ | $2l$ | $l + 1$ | $2l - 1$ | 2 |
| $D_l$ | $2l - 2$ | $2l - 2$ | $2l - 2$ | 1 |
| $E_6$ | 12 | 12 | 12 | 1 |
| $E_7$ | 18 | 18 | 18 | 1 |
| $E_8$ | 30 | 30 | 30 | 1 |
| $F_4$ | 12 | 9 | 9 | 2 |
| $G_2$ | 6 | 4 | 4 | 3 |

Table 1.
Conjecture 1 (Feigin and Frenkel). If \( X \) is of type \( A \), \( B \), \( C \), or \( D \) then \( \tilde{\Delta}(\lambda) \cong \hat{\Delta}_{\text{re}} \) as root systems, that is, \( \lambda \) is a principal admissible weight \([KW89]\).

(i) Let \( k \in A, \lambda \in \text{Adm}_k \). Then \( \tilde{\Delta}(\lambda) \cong \hat{\Delta}_{\text{re}} \) as root systems, that is, \( \lambda \) is a principal admissible weight \([KW89]\).

(ii) Let \( k \in L A, \lambda \in \text{Adm}_k \). Then \( \tilde{\Delta}(\lambda) \) is isomorphic to the root system of Langlands dual Lie algebra \( L \hat{g} \) of \( \hat{g} \), i.e., if \( g \) is of type \( B_\ell, C_\ell, G_2 \) and \( F_4 \), then \( L \hat{g} \) is of type \( D_{\ell+1}, A_{2\ell-1}^{(2)}, D_4^{(3)} \), and \( E_6^{(2)} \), respectively.

(iii) The assignment

\[
\lambda \mapsto Lk \coloneqq \frac{1}{r^\vee(k + h^\vee)} - Lh^\vee
\]

gives bijections \( A_g \to L_{A_{\tilde{g}}} \) and \( L_{A_{\tilde{g}}} \to \tilde{A}_g \), where \( L_g \) is the Langlands dual Lie algebra of \( g \) and \( Lh^\vee \) is the dual Coxeter number of \( L_g \).

Conjecture 1 holds for \( g = \mathfrak{sl}_2 \).

For \( g = \mathfrak{sl}_2 \), Conjecture 1 is a theorem of Feigin and Malikov.

**Theorem 5.2.4** (Feigin and Malikov \([FM97]\)). Conjecture 1 holds for \( g = \mathfrak{sl}_2 \).

**Proof.** Suppose that \( L_\lambda \) is admissible. If \( L_\lambda \) is integrable then \( X_{L_\lambda} = \{0\} \) (see Proposition 4.7.3). If \( L_\lambda \) is not integrable then \( H_f^{+0}(L_\lambda) \) with \( f \neq 0 \) is a minimal series representation of the Virasoro algebra \([FKW92] \), see also \([Ara10]\). Therefore \( X_{L_\lambda} = \mathcal{N} \) by Proposition 1.8.1 and a theorem of Beilinson, Feigin and Mazur \([BFM] \) (see \([Ara10] \). Theorem 3.4.3). \( \square \)

5.3. Height and coheight of nilpotent elements. In this subsection we assume that the grading (11) is Dynkin, so that \( x_0 = h/2 \). The largest integer \( j \) such that \( g_{j/2} \neq 0 \) is called the height \([Pan99]\) of the nilpotent element \( f \) and denoted by \( \text{ht}(f) \). Because \( g_\theta \subset g_{\text{ht}(f)/2} \), it follows that

\[
\text{ht}(f) = 2\theta(x_0).
\]

Define the coheight \( \text{ht}^\vee(f) \) of \( f \) by

\[
\text{ht}^\vee(f) = 2\theta_\ast(x_0).
\]

If \( g \) is a classical Lie algebra then the nilpotent orbits are parameterized by certain partitions of \( N \), where \( N = \ell + 1 \) (respectively \( 2\ell + 1, 2\ell, 2\ell \)) when \( g \) is of type \( A_\ell \) (respectively \( B_\ell, C_\ell, D_\ell \)). If \( X = A \) (respectively \( B, C, D \)), let \( \mathcal{P}_X(N) \) be the set of partitions of \( N \) parameterizing the set of nilpotent orbits for \( A_\ell \) (respectively \( B_\ell, C_\ell, D_\ell \)). A precise description of \( \mathcal{P}_X(N) \) \([CM93] \) Theorems 5.1.2-5.1.4] is given as follows.

- \( \mathcal{P}_A(N) \): all partition of \( N \).
- \( \mathcal{P}_B(N) \): partitions of \( N \) such that even parts occur with even multiplicity.
- \( \mathcal{P}_C(N) \): partitions of \( N \) such that odd parts occur with even multiplicity.
- \( \mathcal{P}_D(N) \): partitions of \( N \) such that even parts occur with even multiplicity.

Let \( \mathcal{O}_d \) be the nilpotent orbit corresponding to a partition \( d \) (when \( g \) is of type \( D \) and \( d \) is a very even partition there are two orbits \( \mathcal{O}_d^I \) and \( \mathcal{O}_d^II \) corresponding to \( d \)). The following assertion can be seen from the formula for the weighted Dyndin diagram (see \([CM93] \). 5.3)).

**Proposition 5.3.1.** Let \( f \) be a nilpotent element corresponding to a partition \( (d_1, d_2, \ldots, d_n) \) in a classical Lie algebra \( g \).
Proof. Let $Ch$ be the formal character of the $\hat{h}$-module $C^i(L_\lambda)$. We have

$$\sum_{i \in \mathbb{Z}} (-1)^i z^i Ch^i(L_\lambda) = Ch L_\lambda \prod_{n \geq 0} \prod_{\alpha \in \Delta_{i+1}} (1 + z e^{-\alpha + n \delta}) \prod_{\alpha \in \Delta_{i+1}} (1 + z^{-1} e^{\alpha + n \delta})$$

where $\Delta_{i+1} = \bigcup_{i > 0} \Delta_i$, see [KRW03] [Ara05].
Let $\rho^\vee = \frac{1}{2} \sum_{\alpha \in \Delta_f(\lambda)} \alpha^\vee$. By Theorem 4.10 (iv), the Euler-Poincaré principle, we have

$$\chi_q H_f^{\mathfrak{sl}(n)}(L_{\lambda}) = \lim_{t \to 0} \left( \sum_{w \in \mathcal{W}(\lambda)} (-1)^{\ell(w)} q^{-\langle w \cdot \lambda, D + x_0 + t\rho^\vee \rangle} \prod_{\alpha \in \Delta_{0,+}} (1 - q^{\ell(\alpha, \rho^\vee)})^{-1} \right) \times \prod_{j \geq 1} (1 - q^j)^{-\#\Delta_{0,+}} \prod_{j \geq 0} (1 - q^{\frac{j}{k} + j})^{-\#\Delta_{0,+}}.$$

Choose a representative $\hat{W}_f(\lambda)$ of the coset $\hat{W}_f(\lambda) \backslash W(\lambda)$. Then

$$\sum_{w \in \hat{W}(\lambda)} (-1)^{\ell(w)} q^{-\langle w \cdot \lambda, D + x_0 + t\rho^\vee \rangle} = \sum_{y \in \hat{W}_f(\lambda)} (-1)^{\ell(y)} q^{-\langle y \cdot \lambda, D + x_0 \rangle} \sum_{u \in \hat{W}_f(\lambda)} (-1)^{\ell(u)} q^{\ell(u \cdot \lambda, \rho^\vee)}.$$

Because $(-1)^{\ell(u)} = (-1)^{\ell_1(u)}$, where $\ell_1 : \hat{W}_f(\lambda) \to \mathbb{Z}_{\geq 0}$ is the length function, we get that

$$\sum_{u \in \hat{W}_f(\lambda)} (-1)^{\ell(u)} q^{\ell(u \cdot \lambda, \rho^\vee)} = q^{\langle \theta_0 \lambda, \rho^\vee \rangle} \prod_{\alpha \in \Delta_f(\lambda) \cap \Delta_{0,+}} (1 - q^{\ell(\theta(\lambda + \rho), \alpha^\vee)}).$$

This gives the desired results. \(\square\)

**Proposition 5.4.2.** Let $q \in \mathbb{N}$.

(i) Suppose that $\text{ht}(f) \geq 2q$. Then there exists a root $\alpha$ such that $\alpha(x_0) = q$.

(ii) Suppose that $\text{ht}(f) \geq 2q$. Then either of the following holds:

(a) There exists a short root $\alpha$ such that $\alpha(x_0) = q$.

(b) There exists a long root $\alpha$ such that $\alpha(x_0) = r q^2$.

**Proof.** (i) If $\text{ht}(f)$ is even, that is, $\theta(x_0) \in \mathbb{Z}$, then the assertion follows from the $\mathfrak{sl}_2$-representation theory. If $\text{ht}(f)$ is odd, then it follows from the fact [Pan99, Proposition 2.4] that the weighted Dynkin diagram of $f$ contains no $2$’s.

(ii) We may assume that $\hat{g}$ is not simply laced.

Let $\hat{g} = \mathfrak{so}_{2\ell+1}$. We assume that the simple roots of $\hat{g}$ are labeled as in [CM93], so that $\theta = \alpha_1 + \cdots + \alpha_\ell$ and the positive short roots are of the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_\ell$ with $1 \leq i \leq \ell$. Suppose that there is no simple positive root $\alpha$ such that $\alpha(x_0) = q$. Then the condition $\theta(x_0) \geq 2q$ implies that there exists $i$ such that $(\alpha_i + \cdots + \alpha_j)(x_0) = q + 1/2$ and $\alpha_i(x_0) = 1$. Then $\beta = \alpha_i + 2(\alpha_{i+1} + \cdots + \alpha_\ell)$ is a long root satisfying $\beta(x_0) = 2q$.

Let $\hat{g} = \mathfrak{sp}_{2\ell}$. We show that the condition (a) holds. Let $(d_1, \ldots, d_n) \in \mathcal{P}(N)$ be the partition corresponding to $f$. First, suppose that $d_2 \leq d_1 - 2$. (Then $d_1$ must be even.) By Proposition 5.3.1, $\theta_s(x_0) = d_1 - 2$. Let $h_i = d_1 - 1 - 2i$ for $i = 1, \ldots, (d_1 - 2)/2$. The formula for the weighted Dynkin diagram implies that the numbers $(h_i - h_j)/2$ and $(h_i + h_j)/2$ with $i < j$ appear as the values of some positive short roots at $x_0$. Because $(h_1 + h_2)/2 = d_1 - 2$, the assertion (a) holds. Next, suppose that $d_2 \geq d_1 - 2$. By Proposition 5.3.1, $\theta_s(x_0) = (d_1 + d_2)/2 - 1$. If $d_2 = d_1 - 1$ or $d_2 = d_1 - 2$ then $q \geq \theta_s(x_0)$ implies that $q \geq d_1 - 2$. Thus,
the assertion can be proven in the similar manner as above. If $d_1 = d_2$ then the assertion can be proven in the similar manner as well.

For types $G_2$ and $F_4$ one can consult Dynkin’s tables of the weighted Dynkin diagrams (see [CM93, ch. 8]). \hfill \square

**Theorem 5.4.3.**

(i) Let $k \in \mathcal{A}[q]$ with $(q, r^\vee) = 1$, $\lambda \in \text{Adm}_+^k$. Then $H_f^\infty(L_\lambda) \neq 0$ if and only if $\text{ht}(f) < 2q$.

(ii) Let $k \in L\mathcal{A}^\vee$, $\lambda \in \text{Adm}_+^k$. Then $H_f^\infty(L_\lambda) \neq 0$ if and only if $\text{ht}(f) < 2q$.

**Proof.** (i) Suppose that $\text{ht}(f) \geq 2q$. By Proposition 5.4.2 there exists $\alpha \in \Delta_+$ such that $\alpha(x_0) = q$. It follows that $-\alpha + q\delta \in \widehat{\Delta}(\lambda) \setminus \Delta_0$. Hence $H_f^\infty(L_\lambda) = 0$ by Proposition 5.4.1.

Conversely, suppose that $\text{ht}(f) < 2q$. Then $\langle \alpha, D + x_0 \rangle \geq 0$ for any $\alpha \in \widehat{\Delta}(\lambda) \cap \Delta_+$, and the equality holds if and only if $\alpha \in \Delta_0$. Let $\overline{W}_0$ be a set of representative of the coset $W_0 \setminus \overline{W}(\lambda)$. It follows that we have $\langle y \circ \alpha, D + x_0 \rangle \leq \langle \lambda, D + x_0 \rangle$ for $y \in \overline{W}_0$ and the equality holds if and only if $y \in W_0$. Therefore, by Proposition 5.4.1 the dimension of $H_f^\infty(L_\lambda) - (\langle \lambda, D + x_0 \rangle)$ is $\prod_{\alpha \in \Delta_0, +} (\frac{\langle \lambda + 2, 0, 0 \rangle}{\langle \rho, \alpha \rangle}) \neq 0$. In particular, $H_f^\infty(L_\lambda) \neq 0$. (ii) can be proven in the similar way as in (i). \hfill \square

5.5. Associated varieties of $G$-integrable degenerate admissible representations and the $C_2$-cofiniteness of $W$-algebras. For $q \in \mathbb{N}$, set

$$N_q := \{ f \in N; \text{ht}(f) < 2q \} \cup \{0\} \subset N,$$

$$L_n N_q := \{ f \in N; \text{ht}^\vee(f) < 2q \} \cup \{0\} \subset N.$$ 

Note that

$$N_q = \{ x \in \mathfrak{g}; (ad x)^2q = 0 \}.$$ 

by the $\mathfrak{sl}_2$-representation theory. Because

$$\text{ht}(f_{\text{prin}}) = 2(h_{\mathfrak{g}} - 1), \quad \text{ht}^\vee(f_{\text{prin}}) = 2(Lh^\vee - 1),$$

we have

$$N_q = N \iff q \geq h_{\mathfrak{g}}, \text{ and } L N_q = N \iff q \geq L h^\vee.$$ 

An admissible number $k$ is called non-degenerate if

$$k \in \mathcal{A}[q] \text{ with } q \geq h_{\mathfrak{g}}, (q, r^\vee) = 1, \text{ or } k \in L\mathcal{A}[q] \text{ with } q \geq L h^\vee,$$

and otherwise called degenerate. An admissible representation $L_\lambda \in KL_k$ is called ("+-") non-degenerate if $k$ is non-degenerate, otherwise called ("+-") degenerate ([FWK92]).

**Theorem 5.5.1.** Let $k$ be an admissible number, $\lambda \in \text{Adm}_+^k$.

(i) We have $N \subset X_{L_\lambda}$ if and only if $k$ is non-degenerate.

(ii) We have $X_{L_\lambda} \subset \overline{N}$ if and only if $k$ is degenerate. In this case we have

$$X_{L_\lambda} = \begin{cases} N_q & \text{if } k \in \mathcal{A}[q] \text{ with } (q, r^\vee) = 1, \\ L N_q & \text{if } k \in L\mathcal{A}[q]. \end{cases}$$
| \( g \) | \( q \) | \( \mathcal{O}_q \) | exceptional |
|-----|-----|-----|-----|
| \( \mathfrak{sl}_n \) | any | \((q, \ldots, q, s), 0 \leq s \leq q - 2\) | yes |
| \( \mathfrak{sp}_{2n} \) | odd | \((q, \ldots, q, s), 0 \leq s \leq q - 1, s \text{ even}\) | yes |
| | | \((q, \ldots, q, q - 1, s), 1 \leq s \leq q - 1, s \text{ even}\) | yes if \( s = q - 1 \) |
| | | \((q, \ldots, q, s), 0 \leq s \leq q - 1, s \text{ even}\) | yes |
| | even | \((q, \ldots, q, 1), 0 \leq s \leq q - 1, s \text{ odd}\) | yes if \( s = 1 \) |
| | odd | \((q + 1, q, \ldots, q)\) | yes |
| | even | \((q + 1, q, \ldots, q, s, 1), 1 \leq s \leq q - 1, s \text{ odd}\) | yes if \( s = 1 \) |
| | even | \((q + 1, q, \ldots, q, q - 1, s), 1 \leq s \leq q - 1, s \text{ odd}\) | yes if \( s = q - 1 \) |
| \( \mathfrak{so}_{2n+1} \) | odd | \((q, \ldots, q, s), 0 \leq s \leq q, s \text{ odd}\) | yes if \( s = q \) |
| | even | \((q, \ldots, q, 1), 0 \leq s \leq q - 1, s \text{ odd}\) | yes if \( s = 1 \) |
| | odd | \((q + 1, q, \ldots, q, s), 0 \leq s \leq q - 1, s \text{ odd}\) | yes |
| | even | \((q + 1, q, \ldots, q, q - 1, s, 1), 0 \leq s \leq q - 1, s \text{ odd}\) | yes if \( s = q - 1 \) |

Table 2. Orbits \( \mathcal{O}_q \) in classical Lie algebras

**Proof.** Let \( k \in \mathbb{A}[q] \) (respectively \( k \in L\mathbb{A}[q] \)). Let us assume that (45) is Dynkin. By Theorem 5.4.3 and (45), \( H_{\infty}^{2+0}(L\lambda) \neq 0 \) if and only if \( q \geq h_g \) (respectively \( q \geq Lh^\vee \)). Thus (i) and the first assertion of (ii) follow from Proposition 4.8.1. The second assertion of (ii) follows from the first assertion, Proposition 4.5.3 and Theorem 5.4.3. □

Theorem 5.5.1 (ii) proves Conjecture 1 for degenerate admissible representations. Note that, for a non-degenerate admissible representation \( L\lambda \), Theorem 5.5.1 (i) imply that \( X_{L\lambda} \) should equal to \( N \) according to Conjecture 1.

**5.6. Irreducibility of \( N_q \) and \( L^\nu N_q \).**

**Theorem 5.6.1.** Let \( q \in \mathbb{N} \).

(i) \([\text{GVAG04}], \text{not necessarily} (q, r^\vee) = 1\) There exists a unique nilpotent orbit \( \mathcal{O}_q \) such that \( N_q = \overline{\mathcal{O}_q} \). If \( q \geq h_q \) then \( \mathcal{O}_q = \mathcal{O}_{\text{prin}} \). The orbits \( \mathcal{O}_q \) for \( q < h_q \) are listed in Tables 2, 3, 4, 5, 6, 8 and 10.

(ii) There exists a unique nilpotent orbit \( L^\nu \mathcal{O}_q \) such that \( L^\nu N_q = \overline{L^\nu \mathcal{O}_q} \). If \( q \geq L^\nu h^\vee \) then \( L^\nu \mathcal{O}_q = \mathcal{O}_{\text{prin}} \). The orbits \( L^\nu \mathcal{O}_q \) for \( q < L^\nu h^\vee \) for non-simply laced Lie algebras are listed in Tables 3, 4 and 7.
$$g$$ | $$q$$ | $$L^N_q$$
--- | --- | ---
$$\mathfrak{sp}_{2n}$$ even | \((q, \ldots, q, s), 0 \leq s \leq q - 1, s \text{ even}\) | \(\text{even}\)
odd | \((q + 1, q, \ldots, q, s), 0 \leq s \leq q - 1, s \text{ even}\) | \(\text{even}\)
odd | \((q + 1, q, \ldots, q, q - 1, s), 2 \leq s \leq q - 1, s \text{ even}\) | \(\text{even}\)
$$\mathfrak{so}_{2n+1}$$ any | \((2q, \ldots, 2q, s), 0 \leq s \leq 2q - 1, s \text{ odd}\) | \(\text{even}\)
odd | \((2q, \ldots, 2q, 2q - 1, s, 1), 0 \leq s \leq 2q - 1, s \text{ odd}\) | \(\text{even}\)

Table 3. Orbits $$L^N_q$$ in non-simply laced classical Lie algebras

| $$q$$ | $$\mathcal{O}_q$$ | exceptional | $$c\left(\frac{p}{3}\right)$$ |
| --- | --- | --- | --- |
| \(\geq 6\) | $$G_2$$ | yes | \(-\frac{2(12p-14q)(7p-4q)}{pq}\) |
| (3), 4, 5 | $$G_2(a_1)$$ | no | \(-\frac{4(10p-8q)(p-2q)}{pq}\) |
| 2 | $$A_1$$ | yes | \(-\frac{63 - 9p - \frac{12}{p}}{p}\) |
| 1 | 0 | yes | \(-\frac{14(p-4)}{p}\) |

Table 4. Orbits $$\mathcal{O}_q$$ in type $$G_2$$

| $$q$$ | $$L^N_q$$ | $$c\left(\frac{p}{3}\right)$$ |
| --- | --- | --- |
| \(\geq 4\) | $$G_2$$ | \(-\frac{2(10p-12q)(4p-7q)}{pq}\) |
| 2, 3 | $$G_2(a_1)$$ | \(-\frac{4(2p-7q)(p-9q)}{pq}\) |
| 1 | $$A_1$$ | \(-\frac{2(2p-12)(p-7)}{p}\) |

Table 5. Orbits $$L^N_q$$ in type $$G_2$$

| $$q$$ | $$\mathcal{O}_q$$ | exceptional | $$c\left(\frac{p}{3}\right)$$ |
| --- | --- | --- | --- |
| \(\geq 12\) | $$F_4$$ | yes | \(-\frac{4(18p-13q)(13p-9q)}{pq}\) |
| (8), 9, (10), 11 | $$F_4(a_1)$$ | no | \(-\frac{1062 - 106p - 406q}{pq}\) |
| (6), 7 | $$F_4(a_2)$$ | no | \(-\frac{632 - \frac{30p}{2} - \frac{22}{p}}{p}\) |
| (4), 5 | $$F_4(a_3)$$ | no | \(-\frac{12(p-3)(6p-6q)}{3pq}\) |
| 3 | $$A_2 + A_1$$ | yes | \(-\frac{316 - 18p - \frac{1801}{p}}{p}\) |
| (2) | $$A_1 + \tilde{A}_1$$ | yes | \(-\frac{52(p-9)}{p}\) |
| 1 | 0 | yes | \(-\frac{52(p-91)}{p}\) |

Table 6. Orbits $$\mathcal{O}_q$$ in type $$F_4$$

Proof. (ii) Let $$X = B_t$$ or $$C_t$$, and let $$\mathfrak{g}$$ be a Lie algebra of type $$X$$, $$d \in \mathcal{P}_X(N)$$. From Proposition 5.3.1 it follows that $$\mathcal{O}_d \subset L^N_q$$ if and only if $$d$$ is dominated by
\[
\begin{array}{|c|c|c|c|}
\hline
q & L\mathcal{O}_q & c\left(\frac{p}{q}\right) \\
\hline
\geq 9 & F_4 & \frac{4(13p-18q)(9p-13q)}{p} \\
6, 7, 8 & F_4(a_1) & 1062 - \frac{1608}{p} - \frac{960}{q} \\
5 & F_4(a_2) & 632 - \frac{1088}{p} - \frac{4680}{q} \\
4 & B_3 & 562 - 21p - \frac{684}{p} \\
3 & F_4(a_3) & -\frac{12(p-18)(p-13)}{q} \\
2 & A_2 + A_1 & -\frac{9(p-16)(p-13)}{p} \\
1 & A_1 & -\frac{31(p-24)(p-13)}{p} \\
\hline
\end{array}
\]

Table 7. Orbits $L\mathcal{O}_q$ in type $F_4$

\[
\begin{array}{|c|c|c|c|c|}
\hline
q & \mathcal{O}_q & \text{exceptional} & c\left(\frac{p}{q}\right) \\
\hline
\geq 12 & E_6 & \text{yes} & -\frac{5(12p-13q)(13p-12q)}{pq} \\
9, 10, 11 & E_6(a_1) & \text{no} & -\frac{8(9p-13q)(9p-9q)}{pq} \\
8 & D_5 & \text{yes} & -\frac{45p + 1162 - \frac{2388}{p}}{pq} \\
6, 7 & E_6(a_3) & \text{no} & -\frac{80(6p-13q)(6p-27)}{pq} \\
5 & A_2 + A_1 & \text{yes} & -\frac{2(p-90)(9p-130)}{pq} \\
4 & D_4(a_1) & \text{no} & -\frac{18p + 524 - \frac{2388}{p}}{pq} \\
3 & 2A_2 + A_1 & \text{yes} & -\frac{18(p-18)(p-12)}{pq} \\
2 & 3A_1 & \text{yes} & -\frac{9p + 263 - \frac{4872}{p}}{pq} \\
1 & 0 & \text{yes} & -\frac{258(p-12)}{pq} \\
\hline
\end{array}
\]

Table 8. Orbits $\mathcal{O}_q$ in type $E_6$

\[
\begin{array}{|c|c|c|c|c|}
\hline
q & \mathcal{O}_q & \text{exceptional} & c\left(\frac{p}{q}\right) \\
\hline
\geq 18 & E_7 & \text{yes} & -\frac{7(18p-19q)(19p-18q)}{pq} \\
14, 15, 16, 17 & E_7(a_1) & \text{no} & -\frac{9(14p-19q)(11p-14q)}{pq} \\
12, 13 & E_7(a_2) & \text{no} & -\frac{(106p-171q)(9p-14q)}{pq} \\
10, 11 & E_7(a_3) & \text{no} & -\frac{960p + 2521 - \frac{2388}{p}}{pq} \\
9 & E_6(a_1) & \text{no} & \frac{56p + 2199 - \frac{2388}{p}}{pq} \\
8 & E_7(a_4) & \text{no} & -\frac{184p + 1901 - 1912}{pq} \\
7 & A_6 & \text{yes} & -\frac{p-18)(4p-931)}{pq} \\
6 & E_7(a_5) & \text{no} & -\frac{3p-17)(13p-262)}{pq} \\
5 & A_4 + A_2 & \text{yes} & \frac{9}{5} \left(\frac{-16p + 655 - \frac{6650}{p}}{pq}\right) \\
4 & A_3 + A_2 + A_1 & \text{yes} & -\frac{3(5p-60)(15p-114)}{2pq} \\
3 & 2A_2 + A_1 & \text{yes} & -\frac{18p + 721 - \frac{4872}{p}}{pq} \\
2 & 4A_1 & \text{yes} & -\frac{12(p-21)(p-19)}{pq} \\
1 & 0 & \text{yes} & -\frac{133(p-18)}{pq} \\
\hline
\end{array}
\]

Table 9. Orbits $\mathcal{O}_q$ in type $E_7$
\[ d, \text{ where} \]
\[ \bar{d}_q = \begin{cases} 
(q + 1, q, q, \ldots, q, s), & 0 \leq s \leq q - 1 \quad \text{for } \mathfrak{sp}_{2n}, \\
(2q, 2q, \ldots, 2q, s), & 0 \leq s \leq 2q - 1 \quad \text{for } \mathfrak{g} = \mathfrak{so}_{2n+1}. 
\end{cases} \]

It is known that there exists a unique maximal partition \( d_q \) in \( P_X(N) \) dominated by \( \bar{d}_q \), see [CM93] Lemma 6.3.3. The partition \( d_q \) is called the X-collapse of \( \bar{d}_q \). It follows that \( L\Omega_q = \Omega_{d_q} \) gives that \( LN_q = LN_{d_q} \).

For the types \( G_2 \) and \( F_4 \) one can consult the closure relations of nilpotent orbits described in [Car93 Chapter 13]. We give the closure relations and the values of \( \frac{1}{2} \operatorname{ht}(f) \) in exceptional Lie algebras in Tables 11 and 12. The nilpotent orbits \( L\Omega_q \) are framed there.

\[ \square \]

5.7. The \( C_2 \)-cofiniteness of \( W \)-algebras. For an admissible number \( k \), set

\[ \mathcal{O}[k] := \begin{cases} 
\Omega_q & \text{if } k \in \mathcal{A}[q] \text{ with } (q, r) = 1, \\
L\Omega_q & \text{if } k \in L\mathcal{A}[q]. 
\end{cases} \]

Remark 5.7.1. For type \( G_2 \) any nilpotent element belongs to \( \mathcal{O}[k] \) for some admissible number \( k \), see Tables 11 and 5.

By Theorems 5.5.1 and 5.6.1 we have \( X_{L\lambda} = \mathcal{O}[k] \) for any \( \lambda \in \text{Adm}^k \). Hence the following assertion follows immediately from Theorem 4.6.2 and Corollary 4.6.3.

**Theorem 5.7.2.** Let \( k \) be a degenerate admissible number.
Table 11. $G_2; \frac{ht^*(x)}{2}$

Table 12. $F_4; \frac{ht^*(x)}{2}$
(i) For $\lambda \in \text{Adm}_+^k$, the $W^k(\mathfrak{g}, f)$-module $H^0_{\mathfrak{k}}(\Lambda_\lambda)$ is $C_2$-cofinite if and only if $f \in \mathbb{O}[k]$.

(ii) The vertex algebra $H^0_{\mathfrak{k}}(V_k(\mathfrak{g}))$ is $C_2$-cofinite if and only if $f \in \mathbb{O}[k]$. In particular, the simple $W$-algebra $W_k(\mathfrak{g}, f)$ is $C_2$-cofinite if $f \in \mathbb{O}[k]$.

**Remark 5.7.3.** For $\kappa \neq 0$ let $c(\kappa)$ be the central charge of $W_{\kappa^{-1}}(\mathfrak{g}, f)$. Then

$$c(\kappa) = \dim \mathfrak{g}_0 - \frac{1}{2} \dim \mathfrak{g}_1 - \frac{12}{k + h^\vee} |\rho - (k + h^\vee) x_0|^2.$$

In Tables 3 and 10 for exceptional type Lie algebras we give the explicit formulas of $c(k + h^\vee)$ of $W_k(\mathfrak{g}, f)$ with an admissible number $k$ and $f \in \mathbb{O}[k]$ (with respect to the Dynkin grading).

5.8. **The $C_2$-cofiniteness of exceptional $W$-algebras.** Recall that a pair $(q, f)$ of a positive integer $q$ and a nilpotent element $f$ of $\mathfrak{g}$ is called exceptional if the following conditions are satisfied.

- $q$ is equal to or greater than the maximum of the Coxeter numbers of the simple factors of the minimal Levi subalgebra containing $f$,
- $\dim \mathfrak{g}^f = \dim \mathfrak{g}^{\mathfrak{g}^f}$, where $\sigma_q$ is the automorphism of $\mathfrak{g}$ such that $\sigma_q(x_\alpha) = e^{h^\vee} x_\alpha$ for a root vector $x_\alpha$, where $e_q$ is the primitive $q$-th root of unity.

Exceptional pairs are classified in [KW08] for $\mathfrak{g} = \mathfrak{sl}_n$ and in [EKV09] for a general $\mathfrak{g}$.

**Theorem 5.8.1.** The following are equivalent:

(i) $(q, f)$ is an exceptional pair.

(ii) $f \in \mathbb{O}^q$ and $f$ is of principal type (or of standard Levi type).

**Proof.** The assertion follows from the classification [EKV09] of exceptional pairs and that of $\mathbb{O}_q$. \hfill \Box

In Tables 2, 4, 6, 8, 9 and 10 we indicate whether the pair of $q$ and the nilpotent is exceptional or not.

The simple $W$-algebra $W_k(\mathfrak{g}, f)$ are called exceptional if $k \in \mathcal{A}[q]$ with $(q, r^\vee) = 1$ and $(q, f)$ is an exceptional pair. An exceptional $W$-algebra $W_k(\mathfrak{g}, f)$ is called non-principal if $f$ is not a principal nilpotent element.

By Theorems 5.7.3, 5.8.1 we obtain the following assertion.

**Theorem 5.8.2.** All the non-principal exceptional $W$-algebras are $C_2$-cofinite.

5.9. **Trivial representations of $W^k(\mathfrak{g}, f_{\text{prin}}).** The following assertion proves Conjecture [1] for a particular case of non-degenerate admissible representations.

**Theorem 5.9.1.** Let $k = -h^\vee + \frac{h^\vee}{h^\vee + 1} \in \mathcal{A}[h^\vee + 1]$ or $k = -h^\vee + \frac{h^\vee + 1}{h^\vee - h_1}$ in $\mathcal{A}[h^\vee]$. Then $H^0_{\mathfrak{k}}(V_k(\mathfrak{g})) = \mathbb{C}$.

In particular $X_{V_k(\mathfrak{g})} = \mathcal{N}$ and $SS(V_k(\mathfrak{g})) = \mathcal{N}_\infty$.

**Proof.** Let $\tilde{s}_0 = \begin{cases} s - \theta + (h^\vee)_\delta & \text{if } k + h^\vee = \frac{h^\vee}{h^\vee + 1}, \\ s - \theta + h^\vee \delta & \text{if } k + h^\vee = \frac{h^\vee + 1}{h^\vee - h_1}. \end{cases}$
By [KWS8] Corollary 1 and Proposition [5.2.2] the maximal submodule $N_k$ of $V^k(g)$ is generated by a singular vector of weight

$$
\mu = s_0 \cdot k\lambda_0 = \begin{cases} 
\theta - (h_g + 1)\delta + k\lambda_0 & \text{if } k + h^\vee = \frac{h^\vee}{h^\vee + 1}, \\
2(\theta_s - \ell h^\vee \delta) + k\lambda_0 & \text{if } k + h^\vee = \frac{h^\vee}{h^\vee + 1}.
\end{cases}
$$

Here we have used the fact that $(\rho | \theta) = h^\vee - 1, (\rho | \theta^* = h_g - 1$. Because $(\theta | \rho^\vee = h_g - 1, (\theta_s | \rho^\vee = \ell h^\vee - 1$ and $x_0 = \rho^\vee$ for $f = f_{\text{prim}}$, we get that

$$
\mu(D + x_0) = -2.
$$

By exactness of the functor $H^\bullet_{f_{\text{prim}}}(?)$, $H^\bullet_{f_{\text{prim}}}(N_k)$ is a submodule of $W^k(g, f) = H^\bullet_{f_{\text{prim}}}(V^k(g))$. Since $\text{ch} N_k = \text{ch} L_\mu + \sum_{\nu \in \mathbb{W}(k\lambda_0) \cap \mu} c_\nu \text{ch} L_\nu$ for some $c_\nu \in \mathbb{Z}_{\geq 0}$, we have

$$
\text{ch}_q H^\bullet_{f_{\text{prim}}}(N_k) = \text{ch}_q H^\bullet_{f_{\text{prim}}}(L_\mu) + \sum_{\nu \in \mathbb{W}(k\lambda_0) \cap \mu} c_\nu \text{ch}_q H^\bullet_{f_{\text{prim}}}(L_\nu).
$$

Because $\alpha(D + x_0) > 0$ for all $\alpha \in \tilde{\Delta}(k\lambda_0)_+$, from (47) it follows that

$$
\dim H^\bullet_{f_{\text{prim}}}(N_k)_2 = \dim H^\bullet_{f_{\text{prim}}}(L_\mu)_2,
$$

which is one by [Ara07] Theorem 9.1.4.

But we know from [FF90, FBZ04] that $W^k(g, f_{\text{prim}})_2$ is one-dimensional subspace of $W^k(g, f_{\text{prim}})$ spanned by the conformal vector $\omega$. Therefore $\omega \in H^\bullet_{f_{\text{prim}}}(N_k)$.

Hence $\omega(n)$ acts trivially on $H^\bullet_{f_{\text{prim}}}(V_k(g))$ for all $n \in \mathbb{Z}$. Since $\omega(1)$ is the Hamiltonian of $H^\bullet_{f_{\text{prim}}}(V_k(g))$ it follows that $H^\bullet_{f_{\text{prim}}}(V_k(g))_d = 0$ for all $d \neq 0$. Thus we get that $H^\bullet_{f_{\text{prim}}}(V_k(g)) = H^\bullet_{f_{\text{prim}}}(V_k(g))_0$, which is one-dimensional. Therefore Proposition [4.8.1] gives that $X_{V_k(g)} = \mathcal{N}$, and $\text{SS}(V_k(g)) \subset \mathcal{N}_\infty$. Since $SS(V_k(g))$ is a $G_\infty$-invariant subscheme containing $\mathcal{N}$ and $\text{Ad} G_\infty | \mathcal{N}$ is dense in $\mathcal{N}_\infty$ by [EF01], we get that $SS(V_k(g)) = \mathcal{N}_\infty$.

□

**Corollary 5.9.2.** Let $k$ be as in Theorem [5.9.1] Then

$$
\prod_{j \geq 1} (1 - q^j)^d = \sum_{w \in \mathbb{W}(k\lambda_0)} (-1)^{f_\lambda(w)} q^{-\langle w \cdot k\lambda_0, D + \rho^\vee \rangle}.
$$

**Remark 5.9.3.** For $f = f_{\text{prim}}$ we have

$$
e(\frac{P}{q}) = -\ell ((h_g + 1)p - h^\vee q)(\ell h^\vee p - (h_g + 1)q) pq\text{.}
$$

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