THE "QUANTUM" TURÁN PROBLEM FOR OPERATOR SYSTEMS

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Abstract. Let $V$ be a linear subspace of $M_n(\mathbb{C})$ which contains the identity matrix and is stable under Hermitian transpose. A “quantum $k$-clique” for $V$ is a rank $k$ orthogonal projection $P \in M_n(\mathbb{C})$ for which $\dim(PVP) = k^2$, and a “quantum $k$-anticlique” is a rank $k$ orthogonal projection for which $\dim(PVP) = 1$. We give upper and lower bounds both for the largest dimension of $V$ which would ensure the existence of a quantum $k$-anticlique, and for the smallest dimension of $V$ which would ensure the existence of a quantum $k$-clique.

1. Background

In finite dimensions, an operator system is a linear subspace $V$ of $M_n(\mathbb{C})$ with the properties

- $I_n \in V$
- $A \in V \Rightarrow A^* \in V$

where $I_n$ is the $n \times n$ identity matrix and $A^*$ is the Hermitian transpose of $A$.

A natural class of examples arises from graphs with vertex set $\{1, \ldots, n\}$. Given such a graph $G$, we can define an operator system

$$V_G = \text{span}\{E_{ij} : i = j \text{ or } i \text{ is adjacent to } j\}$$

where $E_{ij}$ is the $n \times n$ matrix with a 1 in the $(i,j)$ entry and 0’s elsewhere. Note that the symmetry of the edge set of $G$ is reflected in the stability of $V_G$ under Hermitian transpose. (These are precisely the operator systems which are bimodules over the diagonal subalgebra of $M_n(\mathbb{C})$.)

Operator systems have been studied by C*-algebraists for decades, but only recently have they begun to be thought of as being, in some way, a matrix or “quantum” analog of graphs. More generally, we can regard the notion of a linear subspace of $M_n(\mathbb{C})$ as a linearization of the notion of a subset of $\{1, \ldots, n\}^2$, i.e., a relation on the set $\{1, \ldots, n\}$. The two conditions which define operator systems are then matrix versions of reflexivity and symmetry, so that an operator system becomes a matrix version of a reflexive, symmetric relation on a set — which is effectively the same as a graph on that set. This point of view was developed in [9, 10].
The term “quantum” is supported by the fact that operator systems appear in
the theory of quantum error correction, playing a role exactly analogous to the
role played by ordinary graphs in classical error correction [3]. In the classical case
we have a confusability graph which tells us when two transmitted signals could
be received as the same signal, and in the quantum case we have a confusability
operator system which tells us when two transmitted states could be received as the
same state. The two settings even have a natural common generalization; see [10].

The first paper to demonstrate that there could be a “quantum graph theory” for
operator systems was [3], where, driven by the needs of quantum error correction, a
“quantum Lovász number” was defined for an arbitrary operator system, in analogy
to the classical Lovász number of a graph. The error correction perspective on
quantum graphs was developed further in [6].

The present paper is a sequel to [11], where an operator system version of Ram-
sey’s theorem was proven. This result involves quantum versions of graph-theoretic
cliques and anticliques. The theory of error correction tells us what a quantum
anticlique should be, because in classical error correction a “code” is realized as an
anticlique in the confusability graph, whereas in quantum error correction a “code”
is realized as an orthogonal projection \( P \in M_n(\mathbb{C}) \) satisfying \( PAP = \lambda P \) for all \( A \)
belonging to the confusability operator system \( \mathcal{V} \). Equivalently, this condition can
be stated as \( \dim(PVP) = 1 \) where \( PVP = \{PAP : A \in \mathcal{V}\} \).

Observe that if \( P \in M_n(\mathbb{C}) \) is any orthogonal projection (i.e., \( P = P^2 = P^* \))
and \( \mathcal{V} \subseteq M_n(\mathbb{C}) \) is any operator system, then \( PVP \) is effectively a set of linear
transformations from \( \text{ran}(P) \) to itself, and the condition that \( P \) should be a code is
that this set should be minimal, consisting only of the scalar multiples of the identity
operator on \( \text{ran}(P) \). If these are the anticliques of \( \mathcal{V} \), then it is natural to take the
cliues of \( \mathcal{V} \) to be the orthogonal projections \( P \) for which \( PVP \) is maximal, i.e., it
consists of all linear operators from \( \text{ran}(P) \) to itself. This can also be expressed by
saying that \( \dim(PVP) = k^2 \). We therefore make the following definition.

**Definition 1.1.** Let \( \mathcal{V} \subseteq M_n(\mathbb{C}) \) be an operator system. A rank \( k \) orthogonal
projection \( P \in M_n(\mathbb{C}) \) is a quantum \( k \)-anticlique for \( \mathcal{V} \) if \( \dim(PVP) = 1 \), and a
quantum \( k \)-clique for \( \mathcal{V} \) if \( \dim(PVP) = k^2 \).

In general, if we identify \( PM_n(\mathbb{C})P \) with \( M_k(\mathbb{C}) \), where \( k = \text{rank}(P) \), then \( PVP \)
becomes an operator system in \( M_k(\mathbb{C}) \). This is the induced operator system which
is analogous to a subgraph induced on a subset of the vertex set of a graph. (Some
intuition for this analogy is given in [10], again with the natural common general-
ization mentioned earlier.) Thus \( P \) is a quantum clique if the induced operator
system is a full matrix algebra and it is a quantum anticlique if the induced operator
system is trivial.

The classical theorem of Ramsey states that for any \( k \) there exists \( n \) such that
every graph with \( n \) vertices has either a \( k \)-clique or a \( k \)-anticlique. The quantum
Ramsey theorem proven in [11] states that for any \( k \) there exists \( n \) such that every
operator system in \( M_n(\mathbb{C}) \) has either a quantum \( k \)-clique or a quantum \( k \)-anticlique.
The most surprising aspect of this result is that in the quantum setting \( n \) grows
polynomially in \( k \), not exponentially as in the classical case. (The specific value
given in [11] is \( n = 8k^{11} \), but this is surely not optimal. An easy lower bound
is \( n = (k - 1)(k^2 - 1) = k^3 - k^2 - k + 1 \), obtained by taking \( r = k^2 - 1 \) in the
construction described in Proposition [2.1] below.) A quantum Ramsey theorem for
infinite-dimensional operator systems was proven in [4].
Michael Jury suggested to me the problem of finding a version of Turán’s theorem for operator systems. The classical theorem of Turán gives the maximum number of edges a graph with \( n \) vertices can have without having any \((k+1)\)-cliques; by taking edge complements, we see that \( \binom{n}{2} \) minus this number is the minimum number of edges a graph with \( n \) vertices can have without having any \((k+1)\)-anticliques. The analogous questions for operator systems are: what is the maximum dimension \( T^{↑}(n, k) \) of an operator system in \( \mathbb{M}_n(\mathbb{C}) \) having no quantum \((k+1)\)-cliques, and what is the minimum dimension \( T^{↓}(n, k) \) of an operator system in \( \mathbb{M}_n(\mathbb{C}) \) having no quantum \((k+1)\)-anticliques. These two questions constitute a “quantum Turán problem”. The goal of this paper is not to give exact answers to the m, but merely to provide upper and lower bounds for both values. Specifically, we prove

\[
\sqrt{\frac{n}{k}} < T^{↓}(n, k) \leq \left\lceil \frac{n}{k} \right\rceil \quad \text{and} \quad 2(k-1)n - (k-1)^2 + 3 \leq T^{↑}(n, k) < 16(k+1)^8n.
\]

Because, unlike the classical case, there is no natural symmetry between quantum cliques and quantum anticliques, we are really dealing with two distinct questions. Broadly speaking, it is easy to find quantum cliques and hard to find quantum anticliques. This is dramatically illustrated by the fact that our upper bound on the maximum dimension of an operator system having no quantum \((k+1)\)-cliques is linear in \( n \). As there are \( n^2 \) available dimensions in \( \mathbb{M}_n(\mathbb{C}) \), this means that when \( n \) is large compared to \( k \) one needs only a comparatively small number of dimensions to guarantee that quantum \((k+1)\)-cliques exist. In contrast, the upper bound on the lower quantum Turán number is \( \left\lceil \frac{n}{k} \right\rceil \), meaning that \( \dim(V) \) has to be even smaller than this to ensure that quantum \((k+1)\)-anticliques exist.

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2. Lower quantum Turán numbers

We define the lower quantum Turán number \( T^{↓}(n, k) \) to be the smallest number \( d \) such that some operator system in \( \mathbb{M}_n(\mathbb{C}) \) whose dimension is \( d \) has no quantum \((k+1)\)-anticliques.

Every rank 1 projection is always both a quantum 1-anticlique and a quantum 1-clique for any operator system, so let us assume throughout that \( k \geq 1 \).

Classically, a graph on \( n \) vertices which lacks \((k+1)\)-anticliques, and has the minimum number of edges for doing so, looks like a disjoint union of \( k \) many cliques of equal or nearly equal size. So a natural guess for an operator system in \( \mathbb{M}_n(\mathbb{C}) \) which lacks quantum \((k+1)\)-anticliques and has the smallest possible dimension is a direct sum of \( k \) many matrix algebras of equal or nearly equal size, \( \mathcal{V} = \mathbb{M}_{n_1}(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_{n_k}(\mathbb{C}) \). This operator system indeed has no quantum \((k+1)\)-anticliques; in fact, it has no quantum 2-anticliques because it contains the diagonal operator system \( D_n \), which itself has no quantum 2-anticliques [11 Proposition 2.1]. But this shows that this \( \mathcal{V} \) is far from being minimal: its dimension is approximately \( \frac{n^2}{k} \), whereas the dimension of \( D_n \) is \( n \). Quantum \((k+1)\)-anticliques for \( k > 1 \) can be blocked using even fewer dimensions.

**Proposition 2.1.** Let \( P_1, \ldots, P_r \) be orthogonal projections in \( \mathbb{M}_n(\mathbb{C}) \), each of rank at most \( k \), satisfying \( P_1 + \cdots + P_r = I_n \). Then the operator system \( \mathcal{V} = \text{span}(P_1, \ldots, P_r) \) has no quantum \((k+1)\)-anticliques.
Proof. Let $P$ be a rank $k+1$ orthogonal projection in $M_n(\mathbb{C})$ and assume $PVP = \mathbb{C} \cdot P$. For each $i$, the matrix $PP_iP$ has rank at most rank$(P_i) \leq k$, so the only way it can be a scalar multiple of $P$ is for it to be zero. But this implies that $P = P(P_1 + \cdots + P_r)P = 0$, a contradiction. 

(If $r = \dim(\mathcal{V}) \leq k^2 - 1$ and each $P_i$ has rank $k-1$, then this operator system has neither quantum $k$-cliques nor quantum $k$-anticlques, explaining a parenthetical comment made in the introduction.)

The minimum value of $r$ for which there exist $r$ projections, each of rank at most $k$, which sum to $I_n$ is $\left[\frac{n}{k}\right]$. Thus the following corollary is immediate.

**Corollary 2.2.** $T^+(n, k) \leq \left[\frac{n}{k}\right]$.

If $r = \left[\frac{n}{k}\right]$ then the operator system described in Proposition 2.1 is minimal in the sense that every operator system properly contained in it does have a quantum $(k+1)$-anticlique. In order to prove this, it will be useful to have the following alternative characterization of quantum anticlques. (This characterization is implicit in [5].)

**Lemma 2.3.** Let $\mathcal{V} \subseteq M_n(\mathbb{C})$ be an operator system. Then $\mathcal{V}$ has a quantum $k$-anticlique if and only if there exists an orthonormal set $\{v_1, \ldots, v_k\}$ in $\mathbb{C}^n$ such that for every Hermitian $A \in \mathcal{V}$

$$\langle Av_i, v_j \rangle = 0 \quad \text{and} \quad \langle Av_i, v_i \rangle = \langle Av_j, v_j \rangle$$

whenever $i \neq j$.

**Proof.** If there is a quantum $k$-anticlique $P$ for $\mathcal{V}$ then any orthonormal basis $\{v_1, \ldots, v_k\}$ of its range is easily seen to have the stated properties, since $PAP = \lambda P$ implies $\langle Av_i, v_j \rangle = \langle PAPv_i, v_j \rangle = \lambda \langle v_i, v_j \rangle$ for all $i$ and $j$. Conversely, suppose we are given a set $\{v_1, \ldots, v_k\}$ satisfying the conditions of the lemma and let $P$ be the orthogonal projection onto its span. Since every matrix in $\mathcal{V}$ is a linear combination of two Hermitian matrices in $\mathcal{V}$, the stated equations will be true of any matrix in $\mathcal{V}$. So fix a matrix $A \in \mathcal{V}$ and let $\lambda$ be the common value of the inner products $\langle Av_i, v_i \rangle$. Then $P = \sum v_i v_i^*$ and so

$$PAP = \sum_{i,j} v_i v_i^* A v_j v_j^* = \sum \lambda v_i v_i^* = \lambda P,$$

since $v_i^* A v_j = \langle Av_j, v_i \rangle$ is 0 when $i \neq j$ and $\lambda$ when $i = j$. Thus $PAP$ is a scalar multiple of $P$ for every $A \in \mathcal{V}$, i.e., $P$ is a quantum anticlique. 

**Proposition 2.4.** Let $P_1, \ldots, P_r$ be orthogonal projections in $M_n(\mathbb{C})$ satisfying $P_1 + \cdots + P_r = I_n$. Then any operator system properly contained in $\mathcal{V} = \text{span}(P_1, \ldots, P_r)$ has a quantum $k$-anticlique where $k$ is the sum of the two smallest ranks of the $P_i$'s.

**Proof.** Let $\mathcal{V}_0$ be an operator system properly contained in $\mathcal{V}$. Its Hermitian part $\mathcal{V}_0^h$ has the form

$$\mathcal{V}_0^h = \left\{ \sum a_i P_i : \vec{a} = (a_1, \ldots, a_r) \in E \right\}$$

where $E$ is some proper subspace of $\mathbb{R}^r$ which includes the vector $(1, \ldots, 1)$ (since we require $I_n \in \mathcal{V}_0$). So we can find a nonzero $\vec{b} \in \mathbb{R}^r$ such that $\vec{a} \cdot \vec{b} = 0$ for all $\vec{a} \in E$. Since $(1, \ldots, 1) \in E$, it follows that $\vec{b}$ contains both strictly positive and strictly negative components; by rearranging, we can assume that $b_1, \ldots, b_j > 0$ and $b_{j+1}, \ldots, b_r \leq 0$. We can also assume that $b_1 + \cdots + b_j = -b_{j+1} - \cdots - b_r = 1$. 


For each $i$ let $e_{i,1}, \ldots, e_{i,\text{rank}(P_i)}$ be an orthonormal basis of $\text{ran}(P_i)$. Let $k_1$ be the smallest rank among $P_1, \ldots, P_j$ and let $k_2$ be the smallest rank among $P_{j+1}, \ldots, P_r$, so that $k \leq k_1 + k_2$. Then for $1 \leq l \leq k_1$ set $v_l = \sqrt{b_l}e_{1,l} + \cdots + \sqrt{b_{k_1}}e_{j_1, l}$, and for $1 \leq l \leq k_2$ set $v_{k_1+l} = \sqrt{-b_{j+1}}e_{j+1,l} + \cdots + \sqrt{-b_{k_1+k_2}}e_{r,l}$. The vectors $v_l$ form an orthonormal set of size $k_1 + k_2$. For any $A = a_1P_1 + \cdots + a_rP_r \in V_0^n$ we then have $\langle Av_l, v'_l \rangle = 0$ whenever $l \neq l'$, and for any $1 \leq l \leq k_1$ and $k_1 + 1 \leq l' \leq k_1 + k_2$ we also have

$$\langle Av_l, v_l \rangle = a_1b_1 + \cdots + a_jb_j = -a_{j+1}b_{j+1} - \cdots - a_rb_r = \langle Av'_l, v'_l \rangle.$$ 

So Lemma 2.3 implies that $V_0$ has a quantum $(k_1 + k_2)$-anticlique.

**Corollary 2.5.** Let $V$ be the operator system from Proposition 2.7 and assume that $r = \lceil \frac{n}{3} \rceil$. Then every operator system properly contained in $V$ has a quantum $(k + 1)$-anticlique.

**Proof.** Any family of projections, each of rank at most $k$, which sums to $I_n$ must contain at least $r = \lceil \frac{n}{3} \rceil$ members. Thus if it contains exactly this many members then the sum of the two smallest ranks of the $P_i$’s must be at least $k+1$, as otherwise these two projections could be replaced by a single projection of rank at most $k$.

The conclusion now follows from Proposition 2.4.

It is not to be expected that the kind of minimality expressed in Corollary 2.5 can only happen at dimension $\lceil \frac{n}{3} \rceil$. The following is an easy counterexample.

**Example 2.6.** Take $n = 6$ and suppose $P_1 + P_2 = I_6$ where $\text{rank}(P_1) = \text{rank}(P_2) = 3$. Then by Propositions 2.7 and 2.3, $\text{span}(P_1, P_2)$ is a two-dimensional operator system with no quantum $4$-anticliques, but every operator system properly contained in it (there is only one, namely $C \cdot I_6$) has a quantum $4$-anticlique.

Alternatively, suppose $Q_1 + Q_2 + Q_3 = I_6$ where $\text{rank}(Q_1) = \text{rank}(Q_2) = \text{rank}(Q_3) = 2$. Then $\text{span}(Q_1, Q_2, Q_3)$ is a three-dimensional operator system which has no quantum $4$-anticliques (Proposition 2.7), but I claim that any operator system properly contained in it does have a quantum $4$-anticlique. To see this, note first that any two-dimensional operator system in $M_6(\mathbb{C})$ equals $\text{span}(I_6, A)$ for some Hermitian matrix $A$, and if it is contained in $\text{span}(Q_1, Q_2, Q_3)$ then we can write $A = aQ_1 + bQ_2 + cQ_3$ for some $a, b, c \in \mathbb{R}$. Without loss of generality assume $a \leq b \leq c$. If either $a = 0$ or $b = c$ then the existence of a quantum $4$-anticlique is immediate. Otherwise let $\{e_{i,1}, e_{i,2}\}$ be an orthonormal basis for $\text{ran}(Q_i)$ ($i = 1, 2, 3$) and set $\alpha = \frac{c-b}{c-a}$ and $\gamma = \frac{b-a}{c-a}$ so that $\alpha + \gamma = 1$ and $\alpha C + \gamma = b$. Then the set $S = \{\sqrt{\alpha}e_{1,1}, \sqrt{\gamma}e_{3,1}, \sqrt{\alpha}e_{1,2}, \sqrt{\gamma}e_{3,2}, e_{2,1}, e_{2,2}\}$ satisfies the conditions given in Lemma 2.3, so $\text{span}(I_6, A)$ has a quantum $4$-anticlique.

In general, for any Hermitian $A \in M_n(\mathbb{C})$, a straightforward modification of the argument used in this example shows that we can always find a quantum $\lceil \frac{n}{3} \rceil$-anticlique for the two-dimensional operator system $V = \text{span}(I_n, A)$. Let us record this fact:

**Proposition 2.7.** Let $A \in M_n(\mathbb{C})$ be Hermitian. Then $\text{span}(I_n, A)$ has a quantum $\lceil \frac{n}{3} \rceil$-anticlique.

This is proven by ordering the eigenvalues of $A$ as $\lambda_1 \leq \cdots \leq \lambda_n$, then letting $r = \lceil \frac{n}{3} \rceil$ and for $1 \leq i \leq r-1$ finding a convex combination $\alpha_i \lambda_i + \alpha_{r+i} \lambda_{r+i} = \lambda_r$. 


and then applying Lemma 2.3 to the vectors $\sqrt{\alpha_i}v_i + \sqrt{\alpha_j}v_{r+1}$ plus the one additional vector $v_r$, where $v_i$ is the eigenvector belonging to $\lambda_i$.

In Example 2.6 this number is improved to $\lceil \frac{n}{2} \rceil$ because the two middle eigenvalues of $A$ are equal and their corresponding eigenvectors can both be used separately.

Actually, Turán’s theorem does not just give the minimum number of edges in a $(k+1)$-anticliqueless graph on $n$ vertices; it explicitly describes the structure of such a graph with that minimum number of edges — and there is only one up to isomorphism. I do not know whether $\lceil \frac{n}{2} \rceil$ is the minimum dimension of a quantum $(k+1)$-anticliqueless operator system in $M_n(\mathbb{C})$, but the operator system described in Proposition 2.4 with $r = \lceil \frac{n}{2} \rceil$ is not the only quantum $(k+1)$-anticliqueless operator system of that dimension. We can see this from the following extension of Proposition 2.1.

**Proposition 2.8.** Let $A_1, \ldots, A_r$ be positive matrices in $M_n(\mathbb{C})$, each of rank at most $k$, and suppose that the dimension of $\ker(\sum A_i)$ is also at most $k$. Then the operator system $\mathcal{V} = \text{span}(I_n, A_1, \ldots, A_r)$ has no quantum $(k+1)$-antiquites.

**Proof.** As in the proof of Proposition 2.1 if we assume that $P$ is a quantum $(k+1)$-antiquite for $\mathcal{V}$ then comparing ranks shows that $PA_iP = 0$ for all $i$. Thus $P(\sum A_i)P = 0$, which implies that $(\sum A_i)^{1/2}P = 0$ and hence that $(\sum A_i)P = (\sum A_i)^{1/2}(\sum A_i)^{1/2}P = 0$. This shows that $\text{ran}(P)$ is contained in $\ker(\sum A_i)$, which contradicts the hypothesis that $\dim(\ker(\sum A_i)) \leq k$.

Thus there are many operator systems of dimension $\lceil \frac{n}{2} \rceil$ which have no quantum $(k+1)$-antiquites. Indeed, if $A_1, \ldots, A_r$ are positive matrices of rank $k$, where $r = \lceil \frac{n}{2} \rceil - 1$, then generically the kernel of their sum will have dimension at most $k$ and Proposition 2.8 will apply.

Now let us turn to lower bounds for $T^c(n, k)$. The next pair of results are basically [5, Theorems 3 and 4], with two small improvements. For the reader’s convenience I include the full proofs.

**Lemma 2.9.** Let $\mathcal{V}$ be an operator system in $M_n(\mathbb{C})$ and let $d = \dim(\mathcal{V})$. Assume every matrix in $\mathcal{V}$ is diagonal. If $(k-1)d+1 \leq n$ then $\mathcal{V}$ has a quantum $k$-anticlique.

**Proof.** Write $\mathcal{V} = \text{span}(A_1, \ldots, A_d)$ with each $A_i$ Hermitian and $A_1 = I_n$. Then for each $1 \leq j \leq n$ let $b_j \in \mathbb{R}^{d-1}$ be the vector whose components are the $(j, j)$ entries of $A_2, \ldots, A_d$. That is, $b_j$ is the sequence of eigenvalues of the $A_i$, excepting $A_1 = I_n$, belonging to the $j$th standard basis vector $e_j$. By a theorem of Tverberg [7, 8], if $n \geq kd-(d-1) = (k-1)d+1$, then the index set $\{1, \ldots, n\}$ can be partitioned into $k$ blocks $S_1, \ldots, S_k$ such that the convex hulls of the sets $\{b_j : j \in S_l\} \subset \mathbb{R}^{d-1}$, for $1 \leq l \leq k$, have nonempty intersection. That is, we can find a single point $\bar{b} \in \mathbb{R}^{d-1}$ such that for each $1 \leq l \leq k$ some convex combination $\sum_{j \in S_l} \mu_j b_j$ equals $\bar{b}$. Letting $v_l = \sum_{j \in S_l} \sqrt{\mu_j} e_j$, we then have that $\langle A_i v_l, v_l \rangle = 0$ whenever $l \neq l'$, for any $i$ (even $i = 1$), and if $i \neq 1$ then $\langle A_i v_l, v_l \rangle$ equals the $i$th component of $\bar{b}$, while $\langle A_1 v_l, v_l \rangle = 1$ for any $l$. So $\mathcal{V}$ has a quantum $k$-anticlique by Lemma 2.3.

**Theorem 2.10.** Let $\mathcal{V}$ be an operator system in $M_n(\mathbb{C})$ and let $d = \dim(\mathcal{V})$. If $(k-1)d+1 \leq \lceil \frac{n}{k} \rceil$ then $\mathcal{V}$ has a quantum $k$-anticlique.

**Proof.** We reduce to Lemma 2.9 by compressing $\mathcal{V}$ to an operator system which contains only diagonal matrices. To do this, write $\mathcal{V} = \text{span}(A_1, \ldots, A_d)$ with
each $A_i$ Hermitian and $A_1 = I_n$. Start the construction by letting $v_1$ be a norm 1 eigenvector of $A_2$. Then let $E_1 = \{ Bv_1 : B \in \mathcal{V} \}$ and let $P_1$ be the orthogonal projection onto $E_1^\perp$. Having constructed $v_j$, $E_j$, and $P_j$, let $v_{j+1}$ be a norm 1 eigenvector for $P_jA_2P_j$, let $E_{j+1} = P_jVv_{j+1}$, and let $P_{j+1}$ be the orthogonal projection onto $(E_1 + \cdots + E_j)^\perp$. Continue until all of $\mathbb{C}^n$ is exhausted.

Since $v_j$ is an eigenvector for $P_{j-1}A_2P_{j-1}$ (setting $P_0 = I_n$), and also $P_{j-1}A_1P_{j-1}v_j = v_j$, it follows that the dimension of $E_j$ is at most $d - 1$. Thus we have a sequence $(v_1, \ldots, v_r)$ with $r \geq \lceil \frac{n}{2k} \rceil$. Also, by construction $A_i v_j$ is orthogonal to $v_{j'}$ when $j < j'$, for any $i$. Thus if $P$ is the orthogonal projection onto the span of the $v_j$‘s, then the matrices $PA_iP$ are diagonal with respect to the $v_j$ basis. In other words, $PVP$ satisfies the hypotheses of Lemma 2.9 with $r \geq \lceil \frac{n}{2k} \rceil$ in place of $n$. So $(k - 1)d + 1 \leq \lceil \frac{n}{2k} \rceil$ implies that $PVP$ has a quantum $k$-anticlique, and hence that $\mathcal{V}$ does as well.

The only novel aspects of these two proofs are (1) elimination of the first coordinates of the vectors $b_j$ in Lemma 2.9 and (2) our choice of $v_j$ to be an eigenvector of $P_{j-1}A_2P_{j-1}$. Both yield small improvements on the inequality that has to be assumed, meaning that in both cases the inequality is slightly weakened.

Replacing $\lceil \frac{n}{2k} \rceil$ with $\frac{n}{2k}$ yields, if anything, a stronger condition on $k$. So $(k - 1)d + 1 \leq \frac{n}{2k} \Rightarrow \mathcal{V}$ has a quantum $k$-anticlique. Substituting $k + 1$ for $k$ and solving for $d$ yields the condition $d \leq \frac{k - 1 + \sqrt{(k + 1)^2 + 4kn}}{2k}$ and thus any operator system whose dimension is at most this value must have a quantum $(k+1)$-anticlique. As

$$\sqrt{\frac{n}{k}} = \frac{\sqrt{4kn}}{2k} \leq \frac{k - 1 + \sqrt{(k + 1)^2 + 4kn}}{2k},$$

$T_\downarrow(n, k)$ must be larger than this value. Together with Corollary 2.2 this yields the following estimate.

**Theorem 2.11.** $\sqrt{\frac{n}{k}} < T_\downarrow(n, k) \leq \left\lceil \frac{n}{k} \right\rceil$.

The more precise lower bound $\frac{k - 1 + \sqrt{(k + 1)^2 + 4kn}}{2k}$ is only marginally better than $\sqrt{\frac{n}{k}}$. But when $k = 1$ it does improve $T_\downarrow(n, 1) > \sqrt{n}$ to $T_\downarrow(n, 1) > \sqrt{n} + 1$.

The obvious inefficiency in the proof of Theorem 2.10, where we start by compressing to a diagonal operator system, plus the minimality demonstrated in Corollary 2.3, make it natural to conjecture that the lower quantum Turán number $T_\downarrow(n, k)$ exactly equals $\left\lceil \frac{n}{k} \right\rceil$. When $n = 3$ and $k = 1$, Proposition 2.7 and Corollary 2.2 yield $T_\downarrow(3, 1) = 3$, so the first interesting case is $n = 4$, $k = 1$, when Theorem 2.10 yields $3 \leq T_\downarrow(4, 1) \leq 4$ and the natural conjecture is $T_\downarrow(4, 1) = 4$, i.e., that every three-dimensional operator system in $M_4(\mathbb{C})$ has a quantum 2-anticlique. But even this special case seems hard. I have only been able to prove two partial positive results. The first is an immediate consequence of either Corollary 2.3 or Lemma 2.4. (It can also be inferred from Theorem 2.10 below.)

**Proposition 2.12.** Let $\mathcal{V}$ be an operator system in $M_4(\mathbb{C})$ consisting of diagonal matrices, and whose dimension is at most 3. Then $\mathcal{V}$ has a quantum 2-anticlique.

The other partial result is more substantive. Its content resides almost entirely in the next lemma, which is a slightly modified version of a theorem of Bryant [1].
Lemma 2.13. Let $B_1, B_2, B_3 \in M_2(\mathbb{C})$ with $B_1$ and $B_3$ Hermitian and let $a \geq 1$. Then there exist $\lambda \in [0, 1]$ and $U \in SU(2)$ such that

$$SB_1 S + CUB_2S + SB_1^2 U^* C + CUB_3 U^* C$$

is a scalar multiple of $I_2$, where $S = \text{diag}(\sqrt{\lambda}, \sqrt{1-\lambda})$ and $C = \text{diag}(\sqrt{1-\lambda}, \sqrt{1-\lambda}/a)$.

Proof. For any unit vector $\vec{z} = (z_0, z_1) \in \mathbb{C}^2$ we have a special unitary matrix $U_{\vec{z}} = \begin{bmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{bmatrix}$, and this identifies the 3-sphere $S^3$ with $SU(2)$. Define $f_\lambda : SU(2) \to M_2(\mathbb{C})$ by $f_\lambda(U) = SB_1 S + CUB_2 S + SB_2^2 U^* C + CUB_3 U^* C$, with $S$ and $C$ defined as given above. (Recall that $M_2(\mathbb{C})$ is the Hermitian part of $M_2(\mathbb{C})$.) Also define $g : M_2(\mathbb{C}) \to \mathbb{R} \oplus \mathbb{C}$ by

$$g(A) = (a_{11} - a_{22}, 2a_{12})$$

where $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Note that $g$ is real-linear and $g(A) = 0$ if and only if $A$ is a scalar multiple of $I_2$. Finally, let $F_\lambda : SU(2) \to \mathbb{R} \oplus \mathbb{C}$ be the map $F_\lambda = g \circ f_\lambda$.

If $B_3$ is a scalar multiple of $I_2$ then $f_0(U) = U B_3 U^*$ is a scalar multiple of $I_2$ for any $U \in SU(2)$, and we are done. So assume this is not the case. Since $S^2 + C^2 = I_2$, adding a scalar multiple of $I_2$ to both $B_1$ and $B_3$ does not change the problem, so we can assume one of the eigenvalues of $B_3$ is 0. Multiplying $B_1$, $B_2$, and $B_3$ by a nonzero scalar, we can assume the other eigenvalue is 1. We can then find $V \in SU(2)$ such that $VB_3 V^* = \text{diag}(1, 0)$, and if $U$ solves the problem for $B_1$, $VB_2$, and $V B_3 V^*$ then $UV$ solves the problem for $B_1$, $B_2$, and $B_3$. So we may assume $B_3 = \text{diag}(1, 0)$.

To reach a contradiction, suppose $F_\lambda(U) \neq 0$ for all $\lambda \in [0, 1]$ and $U \in SU(2)$. Then we can define $ar{F}_\lambda : SU(2) \to S^2 \subset \mathbb{R} \oplus \mathbb{C} \cong \mathbb{R}^3$ by $\bar{F}_\lambda(U) = \frac{F_\lambda(U)}{\sqrt[3]{\|F_\lambda(U)\|}}$.

The family of maps $\bar{F}_\lambda$ constitutes a homotopy from $\bar{F}_0$ to $\bar{F}_1$. Now $S = 0$ and $C = I_2$ when $\lambda = 0$, so that $f_0(U) = U B_3 U^*$. Recalling that we have reduced to the case where $B_3 = \text{diag}(1, 0)$, a short computation shows that $\bar{F}_0(U_{\vec{z}}) = \bar{F}_0(U_{\bar{z}}) = (|z_0|^2 - |z_1|^2, 2z_0z_1)$, i.e., it is the Hopf map from $S^3$ to $S^2$.

This map is homotopically nontrivial, so to generate a contradiction we need only to show that $\bar{F}_1$ is null homotopic. When $\lambda = 1$ we have $C = \text{diag}(0, a')$ with $a' = \sqrt{1 - 1/a}$. So $F_1(U) \in \mathbb{R} \oplus \mathbb{C}$ is a constant (namely $g(S B_1 S)$) plus something real-linear in the entries of $U$ (namely $g(C UB_2 S + SB_2 U^* C)$) plus something in $\mathbb{R} \oplus 0$ (namely $g(C UB_3 U^* C)$). Letting $X = \{U \in SU(2) : F_1(U) \in \mathbb{R} \oplus 0\}$, it follows that $X$ is the intersection of $SU(2) \cong S^3$ with an affine real-linear subspace of $\left\{ \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} : \alpha, \beta \in \mathbb{C} \right\} \cong \mathbb{R}^4$ whose real dimension is at least 2. Thus $X$ is connected, and therefore its image under $F_1$ in $\mathbb{R} \oplus 0$ is connected. Since this image does not contain 0, it must therefore lie entirely in $(0, \infty) \oplus 0$ or $(-\infty, 0) \oplus 0$; in either case, the image of $F_1$ cannot be all of $S^2$ and so $F_1$ must be null homotopic. This contradicts the homotopic nontriviality of $\bar{F}_0$, and we conclude that $g(f_\lambda(U)) = F_\lambda(U) \in \mathbb{R} \oplus \mathbb{C}$ must be 0 for some $\lambda \in [0, 1]$ and $U \in SU(2)$. So $f_\lambda(U)$ is a scalar multiple of $I_2$ for this $\lambda$ and $U$. \qed

Theorem 2.14. Let $A, B \in M_4(\mathbb{C})$ be Hermitian and assume $A$ has a repeated eigenvalue. Then $V = \text{span}(I_4, A, B)$ has a quantum 2-anticlique.
Proof. If $A$ has a triple eigenvalue then there is a rank 3 orthogonal projection $P$ such that $PAP$ is a scalar multiple of $P$. We can then identify $PM_4(\mathbb{C})P$ with $M_3(\mathbb{C})$ and invoke Proposition [2.7] to infer that $\text{span}(I_3, PB P)$ has a quantum 2-anticlique $Q$. This $Q$ will then be a quantum 2-anticlique for $V$.

So assume $A$ has an eigenvalue of multiplicity exactly 2. By adding a scalar multiple of $I_4$ to $A$, we can assume that this eigenvalue is 0. There are now two cases to consider. First, suppose the two nonzero eigenvalues of $A$ have opposite sign. Without loss of generality say 

$$A = \begin{bmatrix} a & -b \\ b & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with $a, b > 0$. Multiplying $A$ by a nonzero scalar, we can also assume that $\frac{1}{a} + \frac{1}{b} = 1$. Then let 

$$W = \begin{bmatrix} \sqrt{a} & 0 & 0 \\ 0 & \sqrt{b} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so that $W^*W = I_3$ and $W^*AW = 0$. Again by Proposition [2.7], $\text{span}(I_3, W^*BW) \subset M_3(\mathbb{C})$ has a quantum 2-anticlique $Q$, and $P = WQW^*$ is then a quantum 2-anticlique for $V$.

In the other case, the two nonzero eigenvalues of $A$ have the same sign. Multiplying by a scalar and diagonalizing, we can assume that 

$$A = \begin{bmatrix} 1 & a \\ a & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

with $a \geq 1$. In this basis write 

$$B = \begin{bmatrix} B_1 & B_2^* \\ B_2 & B_3 \end{bmatrix},$$

with $B_1, B_2, B_3 \in M_2(\mathbb{C})$ and $B_1$ and $B_3$ Hermitian. Then find $\lambda$ and $U$ as in Lemma [2.13] and define 

$$P = \begin{bmatrix} S^2 & SCU \\ U^*SC & U^*C^2U \end{bmatrix},$$

with $S$ and $C$ as in the statement of that lemma. A computation now shows that both $PAP$ and $PBP$ are scalar multiples of $P$. To see that $\text{rank}(P) = 2$, observe that $P$ is unitarily conjugate to 

$$\begin{bmatrix} S^2 & SC \\ SC & C^2 \end{bmatrix},$$

which after interchanging the middle two basis vectors is the direct sum of 

$$\begin{bmatrix} \sqrt{\lambda} & \sqrt{1-\lambda} \\ \sqrt{1-\lambda} & \sqrt{\lambda} \end{bmatrix} \begin{bmatrix} \sqrt{\lambda/a} \\ \sqrt{1-\lambda/a} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \sqrt{\lambda/a} \\ \sqrt{1-\lambda/a} \end{bmatrix}.$$

□

In other words, any three-dimensional operator system in $M_4(\mathbb{C})$ has a quantum 2-anticlique provided it contains a nonscalar matrix that has a repeated eigenvalue. Unfortunately, for generic Hermitian $A, B \in M_4(\mathbb{C})$ the operator system $\text{span}(I_4, A, B)$ does not have this property [2].

3. Upper quantum Turán numbers

We define the upper quantum Turán number $T^\uparrow(n, k)$ to be the largest number $d$ such that some operator system in $M_n(\mathbb{C})$ whose dimension is $d$ has no quantum $(k + 1)$-cliques. As before, we restrict attention to the case $k \geq 1$.

Evaluating $T^\uparrow(n, k)$ and $T^\downarrow(n, k)$ are very different problems. In general there is no natural "quantum" analog of edge complementation which would interchange quantum cliques and anticliques. In finite dimensions we can consider the orthogonal complement $V^\perp$ of an operator system $V \subseteq M_n(\mathbb{C})$ relative to the Hilbert-Schmidt inner product $\langle A, B \rangle = \text{tr}(AB^*)$, but it will not contain $I_n$. In order to produce a "complementary" operator system we could define $V^\dagger = V^\perp + \mathbb{C} \cdot I_n$, and this is a genuine complementation operation in the sense that $V^{\dagger\dagger} = V$. This operation
transforms quantum anticliques into quantum cliques, but not vice versa (incidentally making precise the idea that anticliques are more special than cliques). We can infer from this fact that $T^\uparrow(n, k) \leq n^2 + 1 - T^\downarrow(n, k)$, but this upper bound is terrible compared to the one proven below\(^2\).

For $k = 1$, evaluation of $T^\uparrow(n, 1)$ is not trivial, but it is completely solved:

**Theorem 3.1.** ([11] Theorem 3.3) For any $n \geq 2$, $T^\uparrow(n, 1) = 3$.

(The cited result only states that $T^\uparrow(n, 1) < 4$, but the reverse inequality follows from the trivial lower bound $T^\uparrow(n, k) \geq (k+1)^2 - 1$. If $\dim(V) < (k+1)^2$ then $V$ obviously cannot have any quantum $(k+1)$-cliques.)

In contrast, it follows from [11, Proposition 2.3] that $T^\uparrow(n, 2) \to \infty$ as $n \to \infty$.

The example which shows this can be described more abstractly, in a way that generalizes to larger values of $k$.

**Proposition 3.2.** Let $Q$ be an orthogonal projection in $M_n(\mathbb{C})$ of rank $n-k+1$. Then the operator system

\[ V_Q = \{ A \in M_n(\mathbb{C}) : QAQ \text{ is a scalar multiple of } Q \} \]

has no quantum $(k+1)$-cliques. Indeed, no two-dimensional extension of $V_Q$ has any quantum $(k+1)$-cliques, but every three-dimensional extension of $V_Q$ does have a quantum $(k+1)$-clique.

**Proof.** Let $P$ be a rank $k+1$ orthogonal projection. Then since $\operatorname{rank}(P) + \operatorname{rank}(Q) = n+2$ there is a rank 2 orthogonal projection $P_0$ which lies below both $P$ and $Q$. Since $Q$ is a quantum anticlique for $V_Q$, so is $P_0$, i.e., $\dim(P_0V_QP_0) = 1$. Thus any two-dimensional extension $V_Q''$ of $V_Q$ must satisfy $\dim(P_0V_Q''P_0) \leq 3$, so that $P_0$ cannot be a quantum 2-clique for $V_Q''$. This implies that $P$ cannot be a quantum $(k+1)$-clique for $V_Q''$.

Now let $V_Q^n$ be a three-dimensional extension of $V_Q$. Then $\dim(QV_Q^nQ) = 4$. (Consider the map $F : A \mapsto QAQ$ from $M_n(\mathbb{C})$ to $QM_n(\mathbb{C})Q$. We have $V_Q = \ker(F) + \mathbb{C} \cdot I_n$, so if $V$ is a $d$-dimensional extension of $V_Q$ then $\dim(F(V)) = d + 1$.) So by Theorem [11] $QV_Q^nQ$ has a quantum 2-clique $Q_0$, and I claim that the projection $P = (I - Q) + Q_0$ is then a quantum $(k+1)$-clique for $V_Q^n$. To see this, let $A \in M_n(\mathbb{C})$ be any matrix which satisfies $PAP = A$; we must show that $A \in PV_Q^nP$. Since $Q_0$ is a quantum 2-clique for $V_Q^n$, we can find $B_0 \in V_Q^n$ such that $Q_0B_0Q_0 = Q_0A_0Q_0$. Let $B_1 = Q_0B_0Q$; then $Q(B_1 - B_0)Q = 0$ and so $B_1 - B_0 \in V_Q$, which implies that $B_1 \in V_Q^n$. Similarly, $Q(A - Q_0AQ_0)Q = Q(PAP - Q_0A_0Q_0)Q = 0$ so $A - Q_0AQ_0 \in V_Q$, and finally $B = A - Q_0AQ_0 + B_1$ belongs to $V_Q^n$ and satisfies $PBP = A$. Thus we have shown that $PV_Q^nP$ contains $A$, as desired. \(\square\)

The last part of Proposition 3.2 shows that the operator systems $V_Q^n$ are maximal for not having any quantum $(k + 1)$-cliques. Of course, this does not rule out

\(^2\)Maybe quantum clique for $V$ should be redefined to simply mean quantum anticlique for $V^\dagger$? This would automatically introduce a symmetry between quantum cliques and quantum anticliques, but it suffers from two drawbacks: first, it does not generalize to the infinite-dimensional setting, and second, the quantum Ramsey theorem from [11] would fail. According to [11] Proposition 2.1] the diagonal operator system $D_n$ has no quantum 2-anticliques, but $D_n^\dagger$ also has no quantum 2-anticliques. (Suppose $P$ is a quantum 2-anticlique for $D_n^\dagger$. Let $v_1$ and $v_2$ be orthonormal vectors in $\operatorname{ran}(P)$ and consider the operator $A : v \mapsto \langle v, v_1 \rangle v_2$. Then $A = PAP$ and $\operatorname{tr}(A) = 0$, so that $\operatorname{tr}(AP^*) = \operatorname{tr}(APB^*P) = 0$ for all $B \in D_n^\dagger$, which implies that $A \in D_n$. But $A$ cannot belong to $D_n$ because it does not commute with $A^*$, contradiction.)
the possibility that other operator systems whose dimensions are larger could lack quantum \((k + 1)\)-cliques.

If \(Q\) is diagonalized as \(Q = \text{diag}(0, \ldots, 0, 1, \ldots, 1)\) (with \(k - 1\) zeros and \(n - k + 1\) ones) then \(\mathcal{V}_Q\) appears as the set of matrices whose restriction to the bottom right \((n - k + 1) \times (n - k + 1)\) corner is a scalar multiple of the \((n - k + 1) \times (n - k + 1)\) identity matrix, and which can be anything on the top and left \((k - 1) \times n\) and \(n \times (k - 1)\) strips. Thus \(\dim(\mathcal{V}_Q) = 2(k - 1)n - (k - 1)^2 + 1\) and we infer the following corollary.

**Corollary 3.3.** \(T^\uparrow(n, k) \geq 2(k - 1)n - (k - 1)^2 + 3\).

The classical analog of the operator system \(\mathcal{V}_Q\) is the graph on \(n\) vertices which is the edge complement of a single \((n - k + 1)\)-clique. In other words, the only missing edges are those both of whose endpoints lie within a fixed set of \(n - k + 1\) vertices. Such a graph contains no \((k + 1)\)-cliques, but the number of edges is has is linear in \(n\), whereas the classical Turán numbers go like \(n^2\).

We could try to get a better lower bound by considering the matrix analog of a \((k + 1)\)-cliqueless graph with the maximal number of edges. This is the edge complement of a disjoint union of \(k\) many cliques of equal or nearly equal size. The matrix analog would be the operator system \(\{A \in M_n(\mathbb{C}) : P_iA^r P_s \text{ is a scalar multiple of } P_{i'}\} \text{ for } 1 \leq i \leq k\) where \(P_1, \ldots, P_k\) are orthogonal projections of equal or nearly equal rank which sum to \(I_n\). But this idea does not work because this operator system typically does have quantum \((k + 1)\)-cliques. This is most simply illustrated in the case \(k = 2\) when we are dealing with a “complete bipartite” operator system which might be expected to have no quantum 3-cliques. This expectation fails badly, however:

**Proposition 3.4.** Let \(\mathcal{V}_0 \subset M_{2k}(\mathbb{C})\) be the set of matrices of the form \[
\begin{bmatrix}
0 & A \\
B & 0
\end{bmatrix}
\]
with \(A, B \in M_k(\mathbb{C})\), and let \(\mathcal{V}\) be the operator system \(\mathcal{V} = \mathcal{V}_0 + \mathbb{C} \cdot I_{2k}\). Then \(\mathcal{V}\) has a quantum \(k\)-clique.

**Proof.** Let \(E = \{v + v' : v' \in \mathbb{C}^k\} \subset \mathbb{C}^{2k}\) and let \(P\) be the orthogonal projection onto \(E\). Any linear operator from \(E\) to itself has the form \((v + v') \mapsto (Av + Av')\) for some \(A \in M_k(\mathbb{C})\). But for any \(A \in M_k(\mathbb{C})\) the matrix \(A' = \begin{bmatrix} 0 & A \\
A & 0 \end{bmatrix}\) satisfies

\[
(PA'P)(v + v') = A'(v + v') = Av + Av,
\]
so that \(PVP\) contains every linear operator from \(E\) to itself. That is, \(P\) is a quantum \(k\)-clique.

In fact, linearity in \(n\) is the most we can ask for in a lower bound on \(T^\uparrow(n, k)\), because — incredibly — we can give an upper bound on \(T^\uparrow(n, k)\) which is also linear in \(n\). The argument uses the following result from [11]. Let \((e_i)\) be the standard basis of \(\mathbb{C}^n\).

**Lemma 3.5.** ([11] Lemma 4.4) Let \(n = k^4 + k^3 + k - 1\) and let \(\mathcal{V}\) be an operator system contained in \(M_n(\mathbb{C})\). Suppose \(\mathcal{V}\) contains matrices \(A_1, \ldots, A_{k+1,k}\) such that for each \(i\) we have \(\langle A_i e_i, e_{i+1} \rangle \neq 0\), and also \(\langle A_r e_r, e_s \rangle = 0\) whenever \(\max(r, s) > i + 1\) and \(r \neq s\). Then \(\mathcal{V}\) has a quantum \(k\)-clique.

We need this lemma to prove the next result, which is extracted from the proof of [11] Theorem 4.5]. For the reader’s convenience I include the proof here.
Lemma 3.6. Let $\mathcal{V}$ be an operator system in $M_n(\mathbb{C})$ and suppose that for each nonzero $v \in \mathbb{C}^n$ we have $\dim(\mathcal{V}v) \geq 8k^8$. Then $\mathcal{V}$ has a quantum $k$-clique.

Proof. Let $v_1$ be any nonzero vector in $\mathbb{C}^n$ and find $A_1 \in \mathcal{V}$ such that $v_2 = A_1v_1$ is nonzero and orthogonal to $v_1$. Then find $A_2 \in \mathcal{V}$ such that $v_3 = A_2v_2$ is nonzero and orthogonal to each of $v_1$, $A_1v_1$, $A_1^*v_1$, $A_1v_2$, and $A_1^*v_2$. Continue in this way, at the $r$th step finding $A_r \in \mathcal{V}$ such that $v_{r+1} = A_rv_r$ is nonzero and orthogonal to the span of the vectors $v_1$ and $A_iv_j$ and $A_i^*v_j$ for $i < r$ and $j \leq r$. The dimension of this span is at most $2r^2 - 2r + 1$, so as long as $r \leq 2k^4$ its dimension is less than $8k^8$ and a suitable matrix $A_r$ can be found. Compressing to the span of the $v_i$ for $1 \leq i \leq k^4 + k^3 + k - 1$ then puts us in the situation of Lemma 3.5, so there is a quantum $k$-clique by that result. \hfill \Box

Theorem 3.7. Let $\mathcal{V}$ be an operator system in $M_n(\mathbb{C})$ of dimension at least $16k^8n$. Then $\mathcal{V}$ has a quantum $k$-clique.

Proof. Fix $k$; the proof goes by induction on $n$. The smallest sensible value of $n$ is $n = 16k^8$, as for smaller values of $n$ the dimension of $\mathcal{V}$ is at most $n^2 < 16k^8n$.

When $n$ exactly equals $16k^8$, the only way to have $\dim(\mathcal{V}) \geq 16k^8n$ is if $\mathcal{V} = M_n(\mathbb{C})$, so it certainly has a quantum $k$-clique. In the induction step, first suppose that there exists a nonzero vector $v \in \mathbb{C}^n$ such that $\dim(\mathcal{V}v) < 8k^8$. Let $P$ be the rank $n - 1$ orthogonal projection onto the orthocomplement of $\mathbb{C} \cdot v$ in $\mathbb{C}^n$. If $A \in \mathcal{V}$ satisfies $PAP = 0$ then, with respect to an orthonormal basis of which $v$ is the first element, $A$ is the sum of a matrix which is zero except on the leftmost column and a matrix which is zero except on the topmost row. Since $\dim(\mathcal{V}v) < 8k^8$, it follows that the set $\{A \in \mathcal{V} : PAP = 0\}$ has dimension at most $16k^8$. Thus

$$\dim(PVP) = \dim(\mathcal{V}) - 16k^8 \geq 16k^8(n - 1),$$

and the induction hypothesis tells us that $PVP$ has a quantum $k$-clique, so $\mathcal{V}$ does as well.

Otherwise, for every nonzero vector $v \in \mathbb{C}^n$ we have $\dim(\mathcal{V}v) \geq 8k^8$, and then $\mathcal{V}$ has a quantum $k$-clique by Lemma 3.6. \hfill \Box

Putting this together with Corollary 3.3 yields the promised bounds on $T^\dag(n, k)$.

Corollary 3.8. $2(k - 1)n - (k - 1)^2 + 3 \leq T^\dag(n, k) < 16(k + 1)^8n$.

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