A NOTE ON STRONG-FORM STABILITY FOR THE SOBOLEV INEQUALITY

ROBIN NEUMAYER

Abstract. In this note, we establish a strong form of the quantitative Sobolev inequality in Euclidean space for $p \in (1, n)$. Given any function $u \in \dot{W}^{1,p}(\mathbb{R}^n)$, the gap in the Sobolev inequality controls $\|\nabla u - \nabla v\|_p$, where $v$ is an extremal function for the Sobolev inequality.

1. Introduction

Sobolev inequalities, broadly speaking, establish integrability or regularity properties of a function in terms of the integrability of its gradient. A fundamental example is the classical Sobolev inequality on Euclidean space, which states the following. Given $n \geq 2$ and $p \in (1, n)$, there exists a constant $S = S(n, p)$ such that

$$\|\nabla u\|_p \geq S\|u\|_{p^*}. \quad (1.1)$$

for any function $u \in \dot{W}^{1,p}(\mathbb{R}^n)$. Here, $p^* = np/(n - p)$, and $\dot{W}^{1,p}(\mathbb{R}^n)$ is the space of functions such that $u \in L^{p^*}(\mathbb{R}^n)$ and $|\nabla u| \in L^p(\mathbb{R}^n)$. Let us take $S$ to be the largest possible constant for which (1.1) holds. Aubin [Aub76] and Talenti [Tal76] determined that equality is achieved in (1.1) for the function

$$\bar{v}(x) = \left(1 + |x|^{p'}\right)^{(p-n)/p},$$

as well as its translations, dilations, and constant multiples. Here and in the sequel, we let $p' = p/(p - 1)$ denote the Hölder conjugate of $p$. In fact, these functions are the only such extremal functions for (1.1), and we will let

$$\mathcal{M} = \left\{ v \mid v(x) = c\bar{v}(\lambda(x - y)) \text{ for some } c \in \mathbb{R}, \lambda \in \mathbb{R}_+, y \in \mathbb{R}^n \right\}$$

denote this $(n + 2)$-dimensional space of extremal functions.

Brezis and Lieb raised the question of quantitative stability for the Sobolev inequality in [BL85], asking whether the deviation of a given function from attaining equality in (1.1) controls its distance to the family of extremal functions $\mathcal{M}$. The strongest notion of distance that one expects to control is the $L^p$ norm between gradients. With this in mind, let us define the asymmetry of a function $u \in \dot{W}^{1,p}(\mathbb{R}^n)$ by

$$A(u) = \inf \left\{ \frac{\|\nabla u - \nabla v\|_p}{\|u\|_{p^*}} : v \in \mathcal{M} \right\}.$$
Note that \( A(u) \) is invariant under the symmetries of the Sobolev inequality (translations, dilations, and constant multiples) and is equal to zero if and only if \( u \in \mathcal{M} \). To quantify the deviation from equality in (1.1), we define the deficit of a function \( u \in \dot{W}^{1,p}(\mathbb{R}^n) \) to be

\[
\delta(u) = \frac{\|\nabla u\|_p^{p'} - S^p \|u\|_p^{p'}}{\|u\|_p^{p'}} \quad \text{if } p < 2
\]

and

\[
\delta(u) = \frac{\|\nabla u\|_p^p - S^p \|u\|_p^p}{\|u\|_p^p} \quad \text{if } p \geq 2.
\]

Like the asymmetry, the deficit is a non-negative functional that is invariant under translations, dilations, and constant multiples, and is equal to zero if and only if \( u \in \mathcal{M} \).

By way of a concentration compactness argument as in [Lio85], one readily establishes the qualitative stability of (1.1). That is, if \( \{u_i\} \) is a sequence of functions with \( \delta(u_i) \to 0 \), then \( A(u_i) \to 0 \). The first quantitative result was established in the case \( p = 2 \) in [BE91], where Bianchi and Egnell showed that there is a dimensional constant \( C \) such that

\[ A(u)^2 \leq C \delta(u). \]

This result, in addition to being optimal in the strength of the distance controlled, is sharp in the sense that the exponent 2 cannot be replaced by a smaller one. The proof relies strongly on the fact that \( W^{1,2}(\mathbb{R}^n) \) is a Hilbert space, and in the absence of this structure, the case when \( p \neq 2 \) has proven much more difficult to treat. Nevertheless, in [CFMP09], Cianchi, Fusco, Maggi, and Pratelli established a quantitative stability result in which the deficit controls the distance of a function to \( \mathcal{M} \) in terms of the \( L^{p^*} \) norm; see Theorem 2.1 below for a precise statement. The argument combines symmetrization arguments in the spirit of [FMP08] with a mass transportation argument in one dimension. More recently, in [FN19], Figalli and the author strengthened this result in the case \( p \geq 2 \) by showing that the deficit of a function controls a power of \( A(u) \). The main idea there was to view \( W^{1,p}(\mathbb{R}^n) \) as a weighted Hilbert space and to establish a spectral gap for the linearized operator in the second variation as in [BE91]. However, bounding the difference between the deficit and the second variation required the use of the main result of [CFMP09].

In this note, we establish a reduction theorem that, paired with [CFMP09], allows us to deduce a strong-form quantitative stability result in which the deficit of a function controls a power of \( A(u) \). For \( p \geq 2 \), this recovers the main result of [FN19] with a simpler proof, while in the case \( p \in (1,2) \), it provides the first known quantitative estimate for (1.1) at the level of gradients.

**Theorem 1.1.** Fix \( n \geq 2 \) and \( p \in (1,n) \). There exist constants \( C_1(n,p) \) and \( C_2(n,p) \) such that the following holds. For any \( u \in \dot{W}^{1,p}(\mathbb{R}^n) \) and for any \( v \in \mathcal{M} \) with \( \|u\|_{p^*} = \|v\|_{p^*} \), we have

\[
\left( \frac{\|\nabla u - \nabla v\|_p}{\|u\|_{p^*}} \right)^\alpha \leq C_1 \delta(u) + C_2 \frac{\|u - v\|_{p^*}}{\|u\|_{p^*}}. \tag{1.2}
\]

Here, \( \alpha = p' \) if \( p \in (1,2) \) and \( \alpha = p \) if \( p \in [2,n) \).
Pairing Theorem 1.1 with the main result of [CFMP09] (Theorem 2.10 below), we establish the following quantitative estimate.

**Corollary 1.2.** Fix $n \geq 2$ and $p \in (1, n)$. There exist constants $C = C(n, p)$ and $\beta = \beta(n, p)$ such that the following holds. For any $u \in \dot{W}^{1,p}(\mathbb{R}^n)$, we have

$$A(u)^\beta \leq C \delta(u). \quad (1.3)$$

The value of $\beta$ in Corollary 1.2 is given by

$$\beta = \begin{cases} 
p\left(p^*(3+4p-\frac{3p+1}{n})\right)^2 & \text{if } p \in (1, 2) \\
p^*(3+4p-\frac{3p+1}{n})^2 & \text{if } p \in [2, n). \end{cases}$$

The proof of Theorem 1.1 is elementary and at its core relies on the convexity of the function $t \mapsto t^p$. It is inspired by the recent paper [HS], in which Hynd and Seuffert give a qualitative description of extremal functions in (a certain form of) Morrey’s inequality. Interestingly, they are able to establish a quantitative stability result, even without knowing the explicit form of extremal functions.

Quantitative stability for Sobolev-type inequalities has been a topic of interest in recent years. Closely related to the main results here, a strong-form quantitative stability result was shown for the Sobolev inequality (1.1) with $p = 1$ in [FMP13], following [FMP07, Cia06]. Quantitative stability results have also been shown for (a different form of) Morrey’s inequality [Cia08], the log-Sobolev inequality [ML14, BGRS14, FIL16], the higher order Sobolev inequality [BWW03, GW10], the fractional Sobolev inequality [CFW13], Gagliardo-Nirenberg-Sobolev inequalities [CF13, DT13, DT16, Seu, Ngu], and Strichartz inequalities [Neg].

More broadly, strong-form stability estimates (in which the gap in a given inequality controls the strongest possible norm, typically involving the oscillation of a set or function) have been studied for various functional and geometric inequalities. For instance, such results have been shown for isoperimetric inequalities in Euclidean space [FJ14], on the sphere [BDF17], and in hyperbolic space [BDS15], as well as for anisotropic [Neu16] and Gaussian [Eld15, BBJ17] isoperimetric inequalities.

Apart from their innate interest from a variational perspective, quantitative stability estimates have found applications in the study of geometric problems [FM11, CS13, KM14] and PDE [CF13, DT16]. Certain applications, such as those in [FMM18, CNT], necessitate strong-form quantitative estimates of the type established here.

**Acknowledgments:** The author is supported by Grant No. DMS-1638352 at the Institute for Advanced Study.

### 2. Proofs of Theorem 1.1 and Corollary 1.2

In the proof of Theorem 1.1, we will make use of the following version of Clarkson’s inequalities for vector-valued functions, which state the following. Let $F, G : \mathbb{R}^n \to \mathbb{R}^n$ with
If $|F|, |G| \in L^p(\mathbb{R}^n)$. Then
\[
\left\| \frac{F + G}{2} \right\|_p^{p'} + \left\| \frac{F - G}{2} \right\|_p^{p'} \leq \left( \frac{1}{2} \|F\|_p^p + \frac{1}{2} \|G\|_p^p \right)^{p'/p}
\] (2.1)
if $p \in (1, 2)$, and
\[
\left\| \frac{F + G}{2} \right\|_p^p + \left\| \frac{F - G}{2} \right\|_p^p \leq \frac{1}{2} \|F\|_p^p + \frac{1}{2} \|G\|_p^p.
\] (2.2)
if $p \geq 2$. These inequalities were shown for scalar- and complex-valued functions in [Cla36], and were extended to functions mapping from $\mathbb{R}$ to $\mathbb{R}^n$ in [Boa40]. Though Clarkson’s inequalities have been generalized in a number of directions, we could not locate a reference for the precise form of (2.1) and (2.2), so in Section 3 we prove (2.2) and show how to deduce (2.1) from its scalar-valued analogue.

Proof of Theorem 1.1. We first consider the case $p \in (1, 2)$. Applying (2.1) with $F = \nabla u$ and $G = \nabla v$, we find that
\[
\left\| \frac{\nabla u - \nabla v}{2} \right\|_p^{p'} \leq \left( \frac{1}{2} \|\nabla u\|_p^p + \frac{1}{2} \|\nabla v\|_p^p \right)^{p'/p} - \left\| \frac{\nabla u + \nabla v}{2} \right\|_p^{p'}
\] (2.3)
Next, the Sobolev inequality (1.1) implies that
\[
\|\nabla v\|_p^p \leq \|\nabla u\|_p^p,
\] (2.4)
and
\[
\|\nabla u + \nabla v\|_p^{p'} \geq S^{p'} \|u + v\|_p^{p'}.
\] (2.5)
In (2.4) we have used the assumption that $\|u\|_{p^*} = \|v\|_{p^*}$. Together (2.3), (2.4), and (2.5) imply that
\[
\left\| \frac{\nabla u - \nabla v}{2} \right\|_p^{p'} \leq \|\nabla u\|_p^{p'} - S^{p'} \left\| \frac{u + v}{2} \right\|_p^{p'}.
\] (2.6)
Finally, we claim that
\[
\left\| \frac{u + v}{2} \right\|_{p^*}^{p'} \geq \|u\|_{p^*}^{p'} - p' \|u\|_{p^*}^{p'-1} \left\| \frac{u - v}{2} \right\|_{p^*}^{p'}.
\] (2.7)
Indeed, Minkowski’s inequality implies that
\[
\left\| \frac{u + v}{2} \right\|_{p^*}^{p'} \geq \left( \|u\|_{p^*}^{p'} - \left\| \frac{u - v}{2} \right\|_{p^*}^{p'} \right)^{p'}.
\] (2.8)
Then, convexity of the function $t \mapsto t^{p'}$ implies that
\[
\left( \|u\|_{p^*}^{p'} - \left\| \frac{u - v}{2} \right\|_{p^*}^{p'} \right)^{p'} \geq \|u\|_{p^*}^{p'} - p' \|u\|_{p^*}^{p'-1} \left\| \frac{u - v}{2} \right\|_{p^*}^{p'}.
\] (2.9)
Together (2.8) and (2.9) imply (2.7). Finally, combining (2.6) and (2.7) and dividing through by \( \|u\|_{p'} \) establishes the proof of (1.2) with \( C_1 = 2^p \) and \( C_2 = p'2^{p'-1} \).

Next, the proof for the case \( p \geq 2 \) is completely analogous. Indeed, applying Clarkson’s inequality (2.2) followed by the Sobolev inequality (1.1), and then (2.7) (with \( p \) replacing \( p' \)), we find that

\[
\left\| \frac{\nabla u - \nabla v}{2} \right\|_{p}^{p} \leq \frac{1}{2} \left\| \nabla u \right\|_{p}^{p} + \frac{1}{2} \left\| \nabla v \right\|_{p}^{p} - \left\| \frac{\nabla u + \nabla v}{2} \right\|_{p}^{p} \\
\leq \left\| \nabla u \right\|_{p}^{p} - S_p \left\| \frac{u + v}{2} \right\|_{p^*}^{p} \\
\leq \left\| \nabla u \right\|_{p}^{p} - S_p \left\| u \right\|_{p^*}^{p} + p \left\| \frac{u}{p'} \right\|_{p^*}^{p-1} \left\| \frac{u - v}{2} \right\|_{p^*}^{p}.
\]

Dividing by \( \|u\|_{p'} \) establishes (1.2) with \( C_1 = 2^p \) and \( C_2 = p'2^{p'-1} \).

Now, let us recall the main result from [CFMP09]. The notion of \( L^{p^*} \) asymmetry considered there is

\[
\lambda(u) = \inf \left\{ \frac{\|u - v\|_{p^*}}{\|u\|_{p^*}} : v \in \mathcal{M}, \|v\|_{p^*} = \|u\|_{p^*} \right\}
\]

**Theorem 2.1** (Cianchi, Fusco, Maggi, Pratelli). Fix \( n \geq 2 \) and \( p \in (1, n) \). There exists a constant \( C = C(n, p) \) such that the following holds. For any \( u \in \dot{W}^{1,p}(\mathbb{R}^n) \),

\[
\lambda(u)^\beta \leq C \frac{\left\| \nabla u \right\|_{p} - S_{p,n} \|u\|_{p^*}}{\|u\|_{p^*}^{\beta}},
\]

(2.10)

Here \( \beta = \left( p^* \left( 3 + 4p - \frac{3p+1}{n} \right) \right)^2 \).

We now prove Corollary 1.2 by combining Theorems 1.1 and 2.1.

**Proof of Corollary 1.2**. The only point to check is that

\[
\frac{\left\| \nabla u \right\|_{p} - S_{p,n} \|u\|_{p^*}}{\|u\|_{p^*}^{\beta}} \leq \delta(u). \tag{2.11}
\]

To see this, note that for any \( q \geq 1 \), the function \( t \mapsto t^q - t \) is increasing for \( t \geq 1 \). In particular, if \( a \geq b \geq 1 \), we have

\[
a^q - b^q \geq a - b. \tag{2.12}
\]

Let \( a = \|\nabla u\|_{p}/S_{p,n} \|u\|_{p^*} \) and \( b = 1 \). Then applying (2.12) with \( q = p' \) for \( p \in (1, 2) \) and \( q = p \) for \( p \in [2, n) \) establishes (2.11). With this in hand, Corollary 1.2 follows immediately from (1.2) and (2.10). \( \square \)
3. Clarkson’s inequalities for vector valued functions on \( \mathbb{R}^n \)

For \( p \in (1, 2) \), Clarkson [Cla36] established the following inequality for, in particular, real numbers \( a \) and \( b \):

\[
|a + b|^{p'} + |a - b|^{p'} \leq 2(|a|^p + |b|^p)^{p'/p}.
\] (3.1)

Let us see how to deduce (2.1) from (3.1). We make use of the reverse Minkowski inequality: if \( s \in (0, 1) \), then for \( (a_1, \ldots, a_n) \subset \mathbb{R}^n \) and \( (b_1, \ldots, b_n) \subset \mathbb{R}^n \) we have

\[
\left( \sum a_i^s \right)^{1/s} + \left( \sum b_i^s \right)^{1/s} \leq \left( \sum (a_i + b_i)^s \right)^{1/s}.
\] (3.2)

This inequality follows from the concavity of the function \( t \mapsto t^s \). We take \( s = 2/p' \) and let \( a_i = |F_i + G_i|^{p'} \) and \( b_i = |F_i - G_i|^{p'} \) for \( i = 1, \ldots, n \). Here \( F_i \) denotes the \( i \)th component of \( F \) in some fixed basis. Then, applying (3.2) followed by (3.1), we find that

\[
|F - G|^{p'} + |F + G|^{p'} \leq \left( \sum (|F_i + G_i|^{p'} + |F_i - G_i|^{2p'})^{p'/2} \right)^{p'/2}
\leq 2 \left( \sum (|F_i|^{p'} + |G_i|^{2p'})^{p'/2} \right).
\] (3.3)

On the left-hand side, we have used \( |F| \) to denote the Euclidean norm. Next, applying the usual form of Minkowski’s inequality with \( r = 2/p \) to \( (|a_i|^p) \) and \( (|b_i|^p) \), we find

\[
\left( \sum (|a_i|^p + |b_i|^p)^{2p/2} \right)^{1/2} \leq \left( \left( \sum |a_i|^2 \right)^{p/2} + \left( \sum |b_i|^2 \right)^{p/2} \right)^{1/p}.
\]

Pairing this with (3.3), we find that

\[
|F + G|^{p'} + |F - G|^{p'} \leq 2 (|F|^p + |G|^p)^{p'/p}.
\] (3.4)

Finally, we make use of the integral form of (3.2): for \( s \in (0, 1) \), we have

\[
\|h_1\|_{L^s(\mathbb{R}^n)} + \|h_2\|_{L^s(\mathbb{R}^n)} \leq \|h_1 + h_2\|_{L^s(\mathbb{R}^n)}.
\] (3.5)

We apply (3.5) with \( s = p/p' \) and with \( h_1 = |F + G|^{p'} \) and \( h_2 = |F - G|^{p'} \), and then apply (3.4), in order to find that

\[
\|F + G\|_p^{p'} + \|F - G\|_p^{p'} \leq \left( \int (|F + G|^{p'} + |F - G|^{p'})^{p'/p} \right)^{p'/p}
\leq 2 \left( \int |F|^p + |G|^p \right)^{p'/p}.
\] (3.6)

This establishes (2.1). The corresponding inequality (2.2) for \( p \geq 2 \) is straightforward. Note that \( a^{p/2} + b^{p/2} \leq (a + b)^{p/2} \) for \( p \geq 2 \). Applying this property and then expanding the
squares, we have
\[
|F + G|^p + |F - G|^p = \left( \sum |F_i + G_i|^2 \right)^{p/2} + \left( \sum |F_i - G_i|^2 \right)^{p/2}
\leq \left( \sum (|F_i + G_i|^2 + |F_i - G_i|^2) \right)^{p/2}
= \left( 2 \left( |F|^2 + |G|^2 \right) \right)^{p/2}.
\] (3.8)

Finally, convexity of the function \( t \mapsto t^{p/2} \) implies that
\[
\left( 2 \left( |F|^2 + |G|^2 \right) \right)^{p/2} = 2^p \left( \frac{|F|^2}{2} + \frac{|G|^2}{2} \right)^{p/2}
\leq 2^{p-1} |F|^p + |G|^p.
\] (3.9)

We combine (3.8) and (3.9) and integrate to conclude the proof of (2.2).

REFERENCES

[Aub76] T. Aubin. Problèmes isopérimétriques et espaces de Sobolev. J. Differential Geom., 11(4):573–598, 1976.

[BBJ17] M. Barchiesi, A. Brancolini, and V. Julin. Sharp dimension free quantitative estimates for the Gaussian isoperimetric inequality. Ann. Probab., 45(2):668–697, 2017.

[BDF17] V. Bögelein, F. Duzaar, and N. Fusco. A quantitative isoperimetric inequality on the sphere. Adv. Calc. Var., 10(3):223–265, 2017.

[BDS15] V. Bögelein, F. Duzaar, and C. Scheven. A sharp quantitative isoperimetric inequality in hyperbolic n-space. Calc. Var. Partial Differential Equations, 54(4):3967–4017, 2015.

[BE91] G. Bianchi and H. Egnell. A note on the Sobolev inequality. J. Funct. Anal., 100(1):18–24, 1991.

[BGRS14] S. G. Bobkov, N. Gozlan, C. Roberto, and P.-M. Samson. Bounds on the deficit in the logarithmic Sobolev inequality. J. Funct. Anal., 267(11):4110–4138, 2014.

[BL85] H. Brezis and E. H. Lieb. Sobolev inequalities with remainder terms. J. Funct. Anal., 62(1):73–86, 1985.

[Boa40] R. P. Boas, Jr. Some uniformly convex spaces. Bull. Amer. Math. Soc., 46:304–311, 1940.

[BWW03] T. Bartsch, T. Weth, and M. Willem. A Sobolev inequality with remainder term and critical equations on domains with topology for the polyharmonic operator. Calc. Var. Partial Differential Equations, 18(3):253–268, 2003.

[CF13] E. A. Carlen and A. Figalli. Stability for a GNS inequality and the log-HLS inequality, with application to the critical mass Keller-Segel equation. Duke Math. J., 162(3):579–625, 2013.

[CFMP09] A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli. The sharp Sobolev inequality in quantitative form. J. Eur. Math. Soc., 11(5):1105–1139, 2009.

[CFW13] S. Chen, R. L. Frank, and T. Weth. Remainder terms in the fractional Sobolev inequality. Indiana Univ. Math. J., 62(4):1381–1397, 2013.

[Cia06] A. Cianchi. A quantitative Sobolev inequality in BV. J. Funct. Anal., 237(2):466–481, 2006.

[Cia08] A. Cianchi. Sharp Morrey-Sobolev inequalities and the distance from extremals. Trans. Amer. Math. Soc., 360(8):4335–4347, 2008.

[Cla36] J. A. Clarkson. Uniformly convex spaces. Trans. Amer. Math. Soc., 40(3):396–414, 1936.

[CNT] R. Choksi, R. Neumayer, and I. Topaloglu. Anisotropic liquid drop models. Preprint available at arXiv:1810.08304.

[CS13] M. Cicalese and E. Spadaro. Droplet minimizers of an isoperimetric problem with long-range interactions. Comm. Pure Appl. Math., 66(8):1298–1333, 2013.
J. Dolbeault and G. Toscani. Improved interpolation inequalities, relative entropy and fast diffusion equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(5):917–934, 2013.

J. Dolbeault and G. Toscani. Stability results for logarithmic Sobolev and Gagliardo-Nirenberg inequalities. *Int. Math. Res. Not. IMRN*, (2):473–498, 2016.

R. Eldan. A two-sided estimate for the Gaussian noise stability deficit. *Invent. Math.*, 201(2):561–624, 2015.

M. Fathi, E. Indrei, and M. Ledoux. Quantitative logarithmic Sobolev inequalities and stability estimates. *Discrete Contin. Dyn. Syst.*, 36(12):6835–6853, 2016.

N. Fusco and V. Julin. A strong form of the quantitative isoperimetric inequality. *Calc. Var. Partial Differential Equations*, 50(3-4):925–937, 2014.

A. Figalli and F. Maggi. On the shape of liquid drops and crystals in the small mass regime. *Arch. Rational Mech. Anal.*, 201(1):143–207, 2011.

A. Figalli, F. Maggi, and C. Mooney. The sharp quantitative Euclidean concentration inequality. *Camb. J. Math.*, 6(1):59–87, 2018.

N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative Sobolev inequality for functions of bounded variation. *J. Funct. Anal.*, 244(1):315–341, 2007.

N. Fusco, F. Maggi, and A. Pratelli. The sharp quantitative isoperimetric inequality. *Ann. of Math.* (2), 168(3):941–980, 2008.

A. Figalli, F. Maggi, and A. Pratelli. Sharp stability theorems for the anisotropic Sobolev and log-Sobolev inequalities on functions of bounded variation. *Adv. Math.*, 242:80–101, 2013.

A. Figalli and R. Neumayer. Gradient stability for the Sobolev inequality: the case $p \geq 2$. *J. Eur. Math. Soc. (JEMS)*, 21(2):319–354, 2019.

F. Gazzola and T. Weth. Remainder terms in a higher order Sobolev inequality. *Arch. Math. (Basel)*, 95(4):381–388, 2010.

R. Hynd and F. Seuffert. Extremal functions for Morrey’s inequality. *Preprint available at arXiv:1810.04393*.

E. Indrei and D. Marcon. A quantitative log-Sobolev inequality for a two parameter family of functions. *Int. Math. Res. Not.*, (20):5563–5580, 2014.

H. Knüpfer and C. B. Muratov. On an isoperimetric problem with a competing nonlocal term II: The general case. *Comm. Pure Appl. Math.*, 67(12):1974–1994, 2014.

P.-L. Lions. The concentration-compactness principle in the calculus of variations. The limit case. I. *Rev. Mat. Iberoamericana*, 1(1):145–201, 1985.

G. Negro. A sharpened Strichartz inequality for the wave equation. *Preprint available at arXiv:1802.04114*.

R. Neumayer. A strong form of the quantitative Wulff inequality. *SIAM J. Math. Anal.*, 48(3):1727–1772, 2016.

V. H. Nguyen. The sharp Gagliardo–Nirenberg–Sobolev inequality in quantitative form. *Preprint available at arXiv:1702.01039*.

F. Seuffert. A stability result for a family of sharp Gagliardo-Nirenberg inequalities. *Preprint available at arXiv:1610.06869*.

G. Talenti. Best constant in Sobolev inequality. *Ann. Mat. Pura Appl. (4)*, 110:353–372, 1976.