Let $R$ be the $hh$-curvature associated with the Chern connection or the Cartan connection. Adopting the pulled-back tangent bundle approach to the Finslerian Geometry, an intrinsic characterization of $R$-Einstein metrics is given. Finslerian metrics which are locally conformally $R$-Einstein are classified.

1. Introduction

Finslerian metrics are of considerable interest due to their rich structure including Riemann, Randers, Landsberg and Berwald type metrics. Some areas in which they have significant impacts are Differential Geometry, Einstein’s theory of General Relativity and Biology [1,2]. A natural and important problem is the classification of metrics conformally Einstein. In 1923, Brinkmann obtained in [4] the necessary and sufficient conditions for an $n$-dimensional Riemannian manifold to be conformally Einstein. Later, Szekeres [14] in 1963, Kozameh-Newmann-Tod [5] in 1985, Listing [6] in 2001, Gover-Nurowski [11] in 2005, as well as Kühnel-Rademacher [12] in 2016 studied this problem from different points of view, both for (pseudo-)Riemannian metrics. This motivates us to study the above problem for a general Finslerian metric.

In the present paper, we study and characterize Finslerian metrics which are locally conformal to $R$-Einstein metrics. Unfortunately, the specificity of the Finslerian metric and his associated fundamental tensor do not allow us to use the same technics and tools as in the Riemannian case to obtain general classifications of (locally or globally) conformally Finslerian $R$-Einstein metrics. Hence, we exploit the pulled-back bundle approach and introduce a globally theory on conformal Finslerian $R$-Einstein geometry. Let $M$ be an $n$-dimensional $C^\infty$ connected manifold and $\tilde{TM} := TM\setminus\{0\}$ its slit tangent bundle. The submersion $\pi : \tilde{TM} \longrightarrow M$ pulls back the tangent bundle $TM$ to a vector bundle $\pi^*TM$ over $TM$. Given a Finslerian metric $F$ on $M$ and $g$ its fundamental tensor, we have introduced in [9], the following tensor. The trace-free horizontal Ricci tensor of a Finslerian manifold $(M, F)$ is the application

$$E^H_F : \Gamma(\pi^*TM) \times \chi(\tilde{TM}) \rightarrow C^\infty(\tilde{TM}, \mathbb{R})$$

$$\quad \rightarrow \quad (\text{Ric}^H_F - \frac{1}{n} \text{Scal}^H_F)(\xi, X)$$

where $\text{Ric}^H_F$ is the horizontal Ricci tensor, $\text{Scal}^H_F$ is the horizontal scalar curvature and $g := \pi^*g$ is the pullback of $g$ by the submersion $\pi : \tilde{TM} \longrightarrow M$. One of advantage of
the tensor $E^H_F$, it vanishes when $F$ is an $R$-Einstein metric. Furthermore, it is insensitive to whether we use the Chern connection or the Cartan connection. Our main results in this work are given by the following.

**Proposition 1.** Let $F$ be a Finslerian metric on an $n$-dimensional manifold. $F$ is locally conformal to an $R$-Einstein metric $\tilde{F}$, with $\tilde{F} = e^u F$, if and only if the conformal factor $e^u$ is a solution of the equation

\begin{align}
E_F(\partial_i, \hat{\partial}_j) - (n-2) \left( \nabla_j \nabla_i u - \nabla_i u \nabla_j u \right) \\
+ \frac{(n-2)}{n} \left( \nabla^d \nabla_d u - \nabla^d u \nabla_d u \right) g_{ij} \\
+ \frac{(n-1)}{2nF} \left( \nabla_r u \nabla^q u \right) \frac{\partial (F^2 g^{rs} - 2y^r y^s)}{\partial y^q} g^{kl} A_{skl} g_{ij} = 0.
\end{align}

(1.1)

To determine the solution(s) of the equation (1.1), we consider it as a system of partial differential equations in the conformal factor $e^u$ and curvatures associated with $F$ on a neighborhood of the given manifold. The explicit solution $u$ can tell us how $F$ is constructed. Hence, we prove the following.

**Theorem 1.** A Finslerian metric $F$ on a 2-dimensional manifold is locally conformally $R$-Einstein if and only if one of the following two cases holds:

(i) the conformal factor is constant and $F$ is $R$-Einstein.
(ii) $F$ is a Riemannian metric.

Note that, the warped product of two $R$-Einstein metrics with different horizontal scalar curvatures is not $R$-Einstein. It is studied in [3] the special case where the conformal factor only depends on the base of a warped product Riemannian manifold. Thus we have the following.

**Theorem 2.** Let $F$ be a Finslerian metric on a cylinder $\mathbb{R} \times \mathbb{M}$ of dimension $n \geq 3$ and $F$ a Finslerian metric on $\mathbb{M}$. Let $u$ be a $C^\infty$ function on $\mathbb{R} \times \mathbb{M}$ such that $u(t, x) = u(t)$ for every $t \in \mathbb{R}$ and $x \in \mathbb{M}$. Then $F$ is locally horizontally conformal to an Einstein metric $\tilde{F}$, with $\tilde{F} = e^u F$, if and only if one of the following cases occurs:

(i) $u$, in the conformal factor $e^u$, is a constant function.
(ii) $e^{u(t, x)} = \alpha e^{s^+ t} + \beta e^{-s^+ t}$, where $s^+ = \sqrt{-\text{Scal}_{\tilde{F}}^H / (n-1)(n-2)}$, for some real constants $\alpha$ and $\beta$, and $\tilde{F}$ is horizontally Ricci-constant with positive horizontal scalar curvature $\text{Scal}_{\tilde{F}}^H$.
(iii) $e^{u(t, x)} = \mu \cos\left(\sqrt{-\text{Scal}_{\tilde{F}}^H / (n-1)(n-2)} t\right) + \gamma \sin\left(\sqrt{-\text{Scal}_{\tilde{F}}^H / (n-1)(n-2)} t\right)$, for some real constants $\mu$ and $\gamma$, and $\tilde{F}$ is horizontally Ricci-constant with negative horizontal scalar curvature $\text{Scal}_{\tilde{F}}^H$.

For non-warped product Finslerian metrics, we obtain the following.
Theorem 3. A Finslerian metric $F$ on a 3-dimensional (respectively 4-dimensional) manifold is locally conformally $R$-Einstein if and only if the conformal factor is constant and the Finslerian analogous of Cotton-York (respectively of Bach) tensor vanishes.

The rest of this paper is organised as follows. In Section 2, we give some basic notions on Finslerian manifolds. The Section 3 is devoted to study of Finslerian $R$-Einstein metrics. In the Section 4, we derive Finslerian locally conformal $R$-Einstein equation. The Theorem 1 is proved in Section 5. An intrinsic theory on Finslerian warped product metrics is developed in Section 6 and the Theorem 2. Finally the Theorem 3 is proved in Section 7.

2. Preliminaries

Throughout this paper, all manifolds are assumed to be connected and, all manifolds and mappings are supposed to be differentiable of class $C^\infty$. However, our results presented hold under the differentiability of class $C^4$. Let $M$ be an $n$-dimensional manifold. We denote by $T_xM$ the tangent space at $x \in M$ and by $TM := \bigcup_{x \in M} T_x M$ the tangent bundle of $M$. Set $\hat{T}M = TM \setminus \{0\}$ and $\pi : TM \longrightarrow M, \pi(x, y) \mapsto x$ the natural projection. Let $(x^1, \ldots, x^n)$ be a local coordinate on an open subset $U$ of $M$ and $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ be the local coordinate on $\pi^{-1}(U) \subset TM$. The local coordinate system $(x^i)_{i=1,\ldots,n}$ produces the local coordinate bases $\{\frac{\partial}{\partial x^i}\}_{i=1,\ldots,n}$ and $\{dx^i\}_{i=1,\ldots,n}$ respectively, for $TM$ and cotangent bundle $T^*M$. We use Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise will be noted.

Definition 1. A function $F : TM \longrightarrow [0, \infty)$ is called a Finsler metric on $M$ if:

1. $F$ is $C^\infty$ on the entire slit tangent bundle $\hat{T}M$,
2. $F$ is positively 1-homogeneous on the fibers of $TM$, that is
   \[ \forall c > 0, \ F(x, cy) = cF(x, y), \]
3. the Hessian matrix $(g_{ij}(x, y))_{1 \leq i, j \leq n}$ with elements
   \[ g_{ij}(x, y) := \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \] (2.1)
   is positive definite at every point $(x, y)$ of $\hat{T}M$.

Remark 1. $F(x, y) \neq 0$ for all $x \in M$ and for every $y \in T_x M \setminus \{0\}$.

Consider the differential map $\pi_* : \hat{T}M \longrightarrow TM$. The vertical subspace of $TTM$ is defined by $V := ker(\pi_*)$ and is locally spanned by the set $\{F \frac{\partial}{\partial y^i}, 1 \leq i \leq n\}$, on each $\pi^{-1}(U) \subset \hat{T}M$.

An horizontal subspace $\mathcal{H}$ of $TTM$ is by definition any complementary to $V$. The bundles $\mathcal{H}$ and $V$ give a smooth splitting

\[ TTM = \mathcal{H} \oplus V. \] (2.2)

An Ehresmann connection is a selection of a horizontal subspace $\mathcal{H}$ of $TTM$. As explain in [?], $\mathcal{H}$ can be canonically defined from the geodesics equation.
Definition 2. Let \( \pi : \tilde{T}M \longrightarrow M \) be the restricted projection.

1. An Ehresmann-Finsler connection of \( \pi \) is the subbundle \( \mathcal{H} \) of \( T\tilde{T}M \) given by

\[
\mathcal{H} := \ker \theta,
\]

where \( \theta : T\tilde{T}M \longrightarrow \pi^*TM \) is the bundle morphism defined by

\[
\theta = \frac{\partial}{\partial x^i} \otimes 1 (dy^i + N^i_j dx^j),
\]

where \( N^i_j(x, y) := \frac{\partial G^i(x, y)}{\partial y^j} \) with \( G^i(x, y) := \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial x^r} + \frac{\partial g_{jr}}{\partial x^l} - \frac{\partial g_{lr}}{\partial x^j} \right) y^j y^k. \)

2. The form \( \theta : T\tilde{T}M \longrightarrow \pi^*TM \) induces a linear map

\[
\theta_{(x, y)} : T(x, y)\tilde{T}M \longrightarrow T_x M,
\]

for each point \((x, y) \in \tilde{T}M\); where \( x = \pi(x, y) \).

The vertical lift of a section \( \xi \) of \( \pi^*TM \) is a unique section \( \nu(\xi) \) of \( T\tilde{T}M \) such that for every \((x, y) \in \tilde{T}M\),

\[
\pi_* (\nu(\xi))_{(x, y)} = 0_{(x, y)} \text{ and } \theta_{(x, y)}(\nu(\xi))_{(x, y)} = \xi_{(x, y)},
\]

3. The differential projection \( \pi_* : T\tilde{T}M \longrightarrow \pi^*TM \) induces a linear map

\[
\pi_*_{(x, y)} : T(x, y)\tilde{T}M \longrightarrow T_x M,
\]

for each point \((x, y) \in \tilde{T}M\); where \( x = \pi(x, y) \).

The horizontal lift of a section \( \xi \) of \( \pi^*TM \) is a unique section \( \nu(\xi) \) of \( T\tilde{T}M \) such that for every \((x, y) \in \tilde{T}M\),

\[
\pi_* (\nu(\xi))_{(x, y)} = \xi_{(x, y)} \text{ and } \theta_{(x, y)}(\nu(\xi))_{(x, y)} = 0_{(x, y)}.
\]

We have the following.

Definition 3. A Finslerian tensor field \( T \) of type \((q, 0; p_1, p_2)\) on \( \tilde{T}M \) is a \( C^\infty \) section of the tensor bundle

\[
\left( \pi^*TM \otimes \ldots \otimes \pi^*TM \otimes T^*\tilde{T}M \otimes \ldots \otimes T^*\tilde{T}M \otimes \bigotimes_{p_1-\text{times}} \pi^*TM \right) \otimes \bigotimes_{p_2-\text{times}} \pi^*TM.
\]

where \((p_1, p_2 \text{ and } q \in \mathbb{N})\) which is \( C^\infty(\tilde{T}M, \mathbb{R}) \)-linear in each argument.

Remark 2. In a local chart,

\[
T = T_{k_1 \ldots k_q}^{i_1 \ldots i_{p_1} j_1 \ldots j_{p_2}} \partial_{k_1} \otimes \ldots \otimes \partial_{k_q} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_{p_1}} \otimes \epsilon^{j_1} \otimes \ldots \otimes \epsilon^{j_{p_2}}
\]

where \((\partial_{k_1} \otimes \ldots \otimes \partial_{k_q} \otimes dx^{i_1} \otimes \ldots \otimes dx^{i_{p_1}} \otimes \epsilon^{j_1} \otimes \ldots \otimes \epsilon^{j_{p_2}})_{k \in \{1, \ldots, n\}, i \in \{1, \ldots, n\}, j \in \{1, \ldots, n\}}\) is a basis section of this tensor and, the \( \partial_{k_i} := \frac{\partial}{\partial x^{k_i}} \) as well as \( \epsilon^{j_s} \) are respectively the basis sections for \( \pi^*TM \) and \( T^*\tilde{T}M \) dual of \( T\tilde{T}M \).

Examples 1.

1. A vector field \( X \) on \( \tilde{T}M \) is of type \((0, 0; 0, 1)\).
2. A section \( \xi \) of \( \pi^*TM \) is of type \((0, 0; 1, 0)\).
3. The fundamental tensor \( g \) is of type \((0, 0; 2, 0)\).
The following lemma defines the Chern connection on $\pi^*TM$.

**Lemma 1.** Let $(M, F)$ be a Finslerian manifold and $g$ its fundamental tensor. There exists a unique linear connection $\nabla$ on the vector bundle $\pi^*TM$ such that, for all $X, Y \in \chi(TM)$ and for every $\xi, \eta \in \Gamma(\pi^*TM)$, one has the following properties:

(i) $\nabla_X \pi^*_s Y - \nabla_Y \pi^*_s X = \pi^*_s [X, Y]$, 
(ii) $X(g(\xi, \eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta) + 2\mathcal{A}(\theta(X), \xi, \eta)$

where $\mathcal{A} := \frac{1}{2} \frac{\partial g}{\partial y^i} dx^i \otimes dx^j \otimes dx^k$ is the Cartan tensor.

One has

$$\nabla \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i} \quad \Gamma^i_{jk} := \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{jk}}{\partial x^l} - \frac{\partial g_{lk}}{\partial x^j} \right)$$

(2.10)

where

$$\left\{ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^i_j \frac{\partial}{\partial y^j} = h(\frac{\partial}{\partial x^i}) \right\}_{i=1,...,n}$$

(2.11)

The generalized Cartan connection on $\pi^*TM$ is given as follows.

**Lemma 2.** Let $(M, F)$ be a Finslerian manifold and $g$ its fundamental tensor. There exists a unique linear connection $\nabla$ on the vector bundle $\pi^*TM$ such that, for all $X, Y \in \chi(TM)$ and for every $\xi, \eta, \nu \in \Gamma(\pi^*TM)$, one has the following properties:

(i) $\nabla_X \pi^*_s Y - \nabla_Y \pi^*_s X = \pi^*_s [X, Y] + (A(\theta(X), \pi^*_s Y, \nu))^2 - (A(\pi^*_s X, \theta(Y), \nu))^2$,
(ii) $X(g(\xi, \eta)) = g(\nabla_X \xi, \eta) + g(\xi, \nabla_X \eta)$

where $A$ is the Cartan tensor and $\mathcal{A}$ the section of $\pi^*TM$ dual to $A$ defined by $g(A(\xi, \eta, \nu)^2, \nu) = A(\xi, \eta, \nu)$.

3. **Finslerian $R$-Einstein metrics**

3.1. **First curvature** $R$ associated with the Chern connection or the Cartan connection.

**Definition 4.** The full curvature of a linear connection $\nabla$ on the vector bundle $\pi^*TM$ over the manifold $TM$ is the application

$$\phi : \chi(TM) \times \chi(TM) \times \Gamma(\pi^*TM) \to \Gamma(\pi^*TM)$$

$$(X, Y, \xi) \mapsto \phi(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]} \xi.$$

By the relation (2.2), we have

$$\nabla_X = \nabla_{\hat{X}} + \nabla_{\tilde{X}},$$

where $X = \hat{X} + \tilde{X}$ with $\hat{X} \in \Gamma(H)$ and $\tilde{X} \in \Gamma(V)$.

Using the metric $F$, one can define the full curvature of $\nabla$ as:

$$\Phi(\xi, \eta, X, Y) = g(\phi(X, Y)\xi, \eta)$$

$$= g(\phi(\hat{X}, \hat{Y})\xi + \phi(\tilde{X}, \tilde{Y})\xi + \phi(\hat{X}, \tilde{Y})\xi + \phi(\tilde{X}, \hat{Y})\xi, \eta)$$

$$= R(\xi, \eta, X, Y) + P(\xi, \eta, X, Y) + Q(\xi, \eta, X, Y),$$

where $R(\xi, \eta, X, Y) = g(\phi(\hat{X}, \hat{Y})\xi, \eta), P(\xi, \eta, X, Y) = g(\phi(\tilde{X}, \tilde{Y})\xi + g(\phi(\hat{X}, \tilde{Y})\xi + \phi(\tilde{X}, \hat{Y})\xi, \eta)$ and $Q(\xi, \eta, X, Y) = g(\phi(\tilde{X}, \tilde{Y})\xi, \eta)$ are respectively the first (horizontal) curvature, mixed curvature and vertical curvature.
In particular, if $\nabla$ is the Chern connection, the $Q$-curvature vanishes.

**Proposition 2.** Let $\Phi$ be the full curvature tensor associated with the Cartan connection and $\Phi$ be the full curvature tensor associated with the Chern connection. Then in the horizontal direction, $\Phi = \Phi$.

**Proof.** If $X, Y \in \mathcal{H}$ then $X = \hat{X} = \hat{X}^k \delta_{x^k}$ and $Y = \hat{Y} = \hat{Y}^r \delta_{x^r}$. By the relation (2.4), we get

$$\theta(\hat{X}) = \left[ \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N_{ij}^k dx^j) \right] (\hat{X}^k \delta_{x^k})$$

$$= -\frac{\hat{X}^k}{F} N^k_s \delta^s_i \frac{\partial}{\partial x^i} + \frac{\hat{X}^k}{F} N^j_l \delta^l_k \frac{\partial}{\partial x^i} = 0.$$  

(3.1)

Using the relation (3.1), the both connections ($'\nabla$ and $\nabla$) verify the equations (i) and (ii) in the Lemma 1 and the Lemma 2. That is, for horizontal vectors fields on $\hat{T}M$, $'\nabla$ and $\nabla$ are torsion-free and are compatible with respect to the Finslerian metric $F$. Thus, from the Lemma 1 and Lemma 2, $\Phi = \Phi$. $\square$

### 3.2. $R$-Einstein metric.

With respect to the Chern connection or the Cartan connection, we have the following.

**Definition 5.** The horizontal Ricci tensor $\text{Ric}_F^H$ and the horizontal scalar curvature $\text{Scal}_F^H$ of $(M, F)$ are respectively defined by

$$\text{Ric}_F^H(\xi, X) := \Phi(\xi, \frac{\partial}{\partial x^i}, X, h(\frac{\partial}{\partial x^i})) g^{ij},$$  

(3.2)

$$\text{Scal}_F^H := \text{trace}_{\pi^* g} \left( \text{Ric}_F^H \right), \quad g := \pi^* g.$$  

(3.3)

**Remark 3.** Let $l := \frac{y^i}{F} \frac{\partial}{\partial x^i}$ be the distinguish section for $\pi^* TM$. The tensor $\text{Ric}_F^H$ can be expressed in terms of the classical Akbar-Zadeh Ricci curvatures $[13]$ $\text{Ric}$ and $\text{Ric}_{ij}$ as follows.

$$\text{Ric}_F^H(l, h(l)) = \left[ \frac{\partial}{\partial x^i} \right] \text{Ric}(l, \frac{\partial}{\partial x^i}, h(l), \frac{\partial}{\partial x^j})$$

$$= \frac{\partial}{\partial x^i} \text{Ric} + \hat{l}^i \text{Ric}_{ij}.$$  

It is known [?], $F$ is Einstein if there exists a $C^\infty$ function $k$ on $M$ such that

$$\text{Ric} = (n - 1)k.$$  

(3.4)

**Remark 4.** If $F$ is a Finslerian Einstein metric on an $n$-dimensional manifold $M$ then its associated horizontal scalar curvature is a function on $M$. That is, for any $(x, y)$ of $T M$,

$$\text{Scal}_F^H(x, y) = n(n - 1)k(x).$$

Now, we introduce the following.
Definition 6. [10] A Finslerian metric $F$ on an $n$-dimensional manifold is $R$-Einstein if
\[ \text{Ric}^H_F = \frac{1}{n} \text{Scal}^H_F g. \quad (3.5) \]

Remark 5. If $F$ satisfies (3.5) for a constant function $\text{Scal}^H_F$ (respectively for $\text{Scal}^H_F \equiv 0$) then $F$ is said to be horizontally Ricci-constant (respectively, $F$ is called horizontally Ricci-flat metric).

3.3. Schur’s type lemma.

Definition 7. Let $T$ be a $(0, 0; p_1, p_2)$-tensor on $(M, F)$ and $X \in T\hat{T}M$. The covariant derivative of $T$ in the direction of $X$ is given by the following formula:
\[
(\nabla_X T) (\xi_1, \ldots, \xi_{p_1}, X_1, \ldots, X_{p_2}) := X (T (\xi_1, \ldots, \xi_{p_1}, X_1, \ldots, X_{p_2}))
- \sum_{i=1}^{p_1} [T (\xi_1, \ldots, \nabla_X \xi_i, \ldots, \xi_{p_1}, X_1, \ldots, X_{p_2})]
- \sum_{j=1}^{p_2} [T(\xi_1, \ldots, \xi_{p_1}, X_1, \ldots, h(\nabla_X \pi_s X_j), \ldots, X_{p_2})]
- \sum_{j=1}^{p_2} [T(\xi_1, \ldots, \xi_{p_1}, X_1, \ldots, h(\nabla_X \theta (X_j)), \ldots, X_{p_2})]
- \sum_{j=1}^{p_2} [T(\xi_1, \ldots, \xi_{p_1}, X_1, \ldots, v(\nabla_X \pi_s X_j), \ldots, X_{p_2})]
- \sum_{j=1}^{p_2} [T(\xi_1, \ldots, \xi_{p_1}, X_1, \ldots, v(\nabla_X \theta (X_j)), \ldots, X_{p_2})].
\]

We obtain the Finslerian horizontal Bianchi identity given in the following.

Lemma 3. If $\xi, \eta \in \Gamma(\pi_s T^*M)$ and $X, Y, Z \in \chi(T^*M)$ then
\[
(\nabla_Z R) (\xi, \eta, X, Y) + (\nabla_X R) (\xi, \eta, Y, Z) + (\nabla_Y R) (\xi, \eta, Z, X) = 0.
\]

Proof. The Lemma is obtained from the symmetry of $\nabla$ and the Jacobi identity and by the Definition applied to the first curvature $R$. \hfill \square

We prove a Schur lemma for $\text{Scal}^H_F$.

Lemma 4. If $F$ is horizontally an Einstein metric on a connected manifold of dimension $n \geq 3$ then its horizontal scalar curvature is constant.

Proof. If $F$ is horizontally an Einstein metric then the relation (3.5) holds.

Applying the horizontal covariant derivative on each side of the relation (3.5), we obtain
\[
\nabla_k \text{Ric}^H_F (\partial_i, \partial_j) = \frac{1}{n} (\nabla_k \text{Scal}^H_F) g_{ij}.
\]
Multiplying this last equation by \( g^{ik} \) we get
\[
\nabla^i \text{Ric}^H_F (\partial_i, \partial_j) = \frac{1}{n} \nabla_j \text{Scal}^H_F.
\]
(3.6)
where \( \nabla^i := g^{ik} \nabla_k \).

By contracting twice on equation (3) written in a local coordinate, we have
\[
\frac{1}{2} \nabla_j \text{Scal}^H_F = \nabla^i \text{Ric}^H_F (\partial_i, \partial_j)
\]
(3.6)
\[
= \frac{1}{n} \nabla_j \text{Scal}^H_F.
\]
(3.7)

When \( n > 2 \), the equations (3.6) and (3.7) together with the Lemma 4 imply
\[
0 = \nabla_j \text{Scal}^H_F
= \frac{\partial \text{Scal}^H_F}{\partial x^j}.
\]

Hence, \( \text{Scal}^H_F \) must be constant.

\[\square\]

4. Finslerian locally conformal \( R \)-Einstein equation

**Definition 8.** A Finslerian metric \( F \) on a manifold \( M \) is locally conformally \( R \)-Einstein if each point \( x \in M \) has a neighborhood \( U \) on which there exists a \( C^\infty \)-function \( u \) such that the conformal deformation \( \tilde{F} \) of \( F \) with \( \tilde{F} = e^u F \), is an \( R \)-Einstein metric on \( U \).

**Lemma 5.** \([9]\) Let \( F \) and \( \tilde{F} \) be two Finslerian metrics on an \( n \)-dimensional manifold \( M \). If \( F \) is conformal to \( \tilde{F} \), with \( \tilde{F} = e^u F \), then the trace-free horizontal Ricci tensors \( E^H_F \) and \( \tilde{E}^H_F \) are related by
\[
\tilde{E}^H_F = E^H_F - (n - 2) (H_u - du \circ du) - \frac{(n - 2)}{n} (\Delta^H u + ||\nabla u||^2) g + \Psi^{E^H_F}_u \quad (4.1)
\]
where \( \Psi^{E^H_F}_u \) is the \((0, 0; 1, 1)\)-tensor on \((M, F)\) given by
\[
\Psi^{E^H_F}_u (\xi, X) := (2 - n) [A(\nabla u, B(X), \xi) + A(\nabla u, \pi_x X, B(h(\xi)))]
+ (n - 4) A(B(h(\nabla u), \pi_x X, \xi))
+ \frac{1}{n} g^{ij} \left[ 2(n - 2) A(\nabla u, \partial_i, B(\partial_j)) \right]
- 3 A(B(h(\nabla u), \partial_j, \partial_i)) g(\xi, \pi_x X).
\]
(4.2)
for every $\xi \in \Gamma(\pi^*TM)$ and $X \in \chi(\tilde{T}M)$ with $\Theta_{ij} = \Theta(\partial_i, \partial_j)$ and $\mathcal{B}$ is the application which maps $\pi^*TM$ to $\pi^*TM$ defined by

$$\mathcal{B} = B^i_j \partial_i \otimes dx^j$$

with

$$B^i_j = \frac{1}{2F}(\nabla^i u) \frac{\partial(F^i g^{ij} - 2g^i g^j)}{\partial y^j}.$$  \hspace{1cm} (4.3)

**Proof of Proposition** Let $F$ and $\bar{F}$ be two conformal Finslerian metrics on a manifold of dimension $n$. If $F$ is conformally $R$-Einstein then $\bar{F}_{\bar{F}}$ vanishes. By the Lemma in a local chart we have

$$0 = \left[ \left( \frac{\partial^2 u}{\partial y^i} \right)^2 + \frac{n-2}{n} \left( \frac{\partial^2 u}{\partial y^j} \right) \frac{\partial^2 u}{\partial y^k} \right] (\partial_t, \partial_j)$$

where

$$\Psi^{EH}_{\bar{F}}(\partial_t, \partial_j) = (2-n) \left[ A(\nabla u, B(\partial_j), \partial_t) + A(\nabla u, \pi, \partial_j, B_h(\partial_i)) \right]$$

$$+ (n-4) A(B(h(\nabla u), \pi, \partial_j, \partial_t))$$

$$+ \frac{1}{n} g^{kl} \left[ 2(n-2) A(\nabla u, \partial_k, B_h(\partial_l)) - 3A(B_h(\nabla u), \partial_l, \partial_k) \right] g(\partial_l, \pi, \partial_j).$$

$$+ g^{ij} \left[ \left( \Theta(\partial_{ij}, \Theta(\partial_{1j}, \Theta(\partial_{1j}, \partial_{2j}))) \right)(\partial_t, \partial_k) - g \left( \Theta(\partial_{ij}, \Theta(\partial_{1j}, \partial_{1j})), \partial_k \right) \right]$$

$$+ g^{kl} \left[ \left( \nabla l \Theta \right)(\partial_t, \partial_k), \partial_k \right] - g \left( \left( \nabla l \Theta \right)(\partial_t, \partial_j), \partial_k \right) \right]$$

$$- \frac{1}{n} g^{rs} g^{kl} \left[ A(B_h(\Theta(\partial_{sk}), \partial_t, \partial_r)), A(\nabla u, B_h(\Theta(\partial_{sk}), \partial_t, \partial_r)) \right] g_{ij}$$

$$- \frac{1}{n} g^{rs} g^{kl} \left[ g \left( \left( \nabla s \Theta \right)(\partial_t, \partial_r), \partial_r \right) - g \left( \left( \nabla s \Theta \right)(\partial_t, \partial_r), \partial_r \right) \right] g_{ij}.$$  \hspace{1cm} (4.5)

Using the relation (4.3), we have $B(\partial_j) = B_{s1}^s \delta^s_2 \partial_s = B_{s2}^s \partial_s$ and $B_h(\nabla u) = B_{s2}^s \partial_s \otimes dx^s(\nabla^i u \partial_i) = \nabla^i u B_{i}^s \partial_s$. Thus, from (4.6), we have

$$I_1 = (2-n) \left[ A(\nabla u, B(\partial_j), \partial_t) + A(\nabla u, \pi, \partial_j, B_h(\partial_i)) \right]$$

$$+ (n-4) A(B(h(\nabla u), \pi, \partial_j, \partial_t))$$

$$= (n-4) \nabla^s u B_{s2}^s A_{s1ij} - (n-2) \left( \nabla^s u B_{s2}^s A_{s1js2} + \nabla^s u B_{s2}^s A_{s1is2} \right),$$

$$I_2 = \frac{1}{n} g^{kl} \left[ 2(n-2) A(\nabla u, \partial_k, B(\partial_l)) - 3A(B_h(\nabla u), \partial_l, \partial_k) \right] g_{ij}$$

$$= - \frac{1}{n} g^{kl} \nabla^s u \left[ - 3B_{s2}^s A_{s1kl} + 3B_{s1} A_{s1ls2} - (2n-1) B_{s1} A_{s1ls2} \right] g_{ij},$$

$$I_3 = - \frac{1}{n} g^{rs} g^{kl} \left[ A(B_h(\Theta(\partial_{sk}), \partial_t, \partial_r)), A(\nabla u, B_h(\Theta(\partial_{sk}), \partial_t, \partial_r)) \right] g_{ij}. $$
\[ I_4 = g^{kl} \left[ g \left( \Theta(\partial_j, h(\Theta(\partial_t, h(\partial_i))), \partial_k) \right) - g \left( \Theta(\partial_t, h(\Theta(\partial_j, h(\partial_i))), \partial_k) \right) \right] \]
\[ = g^{kl} \delta_i^j \delta_j^l \left[ g \left( \Theta(\partial_s, h(\Theta(\partial_t, h(\partial_r))), \partial_k) \right) - g \left( \Theta(\partial_t, h(\Theta(\partial_s, h(\partial_r))), \partial_k) \right) \right] \]
\[ = \frac{1}{n} g^{kl} g^{rs} g_{ij} \left[ g \left( \Theta(\partial_s, h(\Theta(\partial_t, h(\partial_r))), \partial_k) \right) - g \left( \Theta(\partial_t, h(\Theta(\partial_s, h(\partial_r))), \partial_k) \right) \right] \]
\[ = -I_3, \]
\[ I_5 = -\frac{1}{n} g^{rs} g^{kl} \left[ g \left( (\nabla_x \Theta)(\partial_j, h(\xi)), \partial_i \right) - g \left( (\nabla \Theta)(h(\xi), X, \partial_t) \right) \right] \]
\[ = -I_5. \]

Hence, putting the expressions of \( I_1, I_2, I_3, I_4, I_5 \) and \( I_6 \) in the right-hand side of (4.5) we obtain the equation (4.1). \( \square \)

**Remark 6.** The equation (4.1) is called Finslerian locally conformal \( R \)-Einstein equation.

5. Locally Conformally \( R \)-Einstein Metrics in Dimensions 1 and 2

5.1. For \( n = 1 \). Every Finslerian metric is conformally \( R \)-Einstein.

**Theorem 4.** Let \((M, F)\) be a Finslerian manifold of dimension one. Then \((M, F)\) is always \( R \)-flat.

**Proof.** This follows from the Lemma 1 and the skewsymmetry of the curvature \( R \). \( \square \)

5.2. For \( n = 2 \): Proof of the Theorem

**Proof.** When \( n = 2 \), the equation (1.1) reduces to
\[ E_F(\partial_i, \hat{\partial}_j) + \frac{1}{4F} (\nabla_r u \nabla_y u) \frac{\partial(F^2 g^{rs} - 2y^r y^s)}{\partial y^l} g^{kl} A_{ijkl} = 0. \] (5.1)

Contracting (5.1) by \( g^{ij} \) yields
\[ \frac{1}{2F} (\nabla_r u \nabla_y u) \frac{\partial(F^2 g^{rs} - 2y^r y^s)}{\partial y^l} g^{kl} A_{ijkl} = 0. \] (5.2)

Since \( F \) is a Finslerian metric, \( F(x, y) \neq 0 \) for every \((x, y) \in \hat{T}M\) and since \( g \) is positive-definite the \( g^{kl} \) functions do not vanish for any \( k, l \in \{1, 2\} \). Hence, the only solution of the equation (5.2) are \( \nabla_r u = 0 \) or \( A \equiv 0 \).

(i) If \( \nabla_r u = 0 \), the conformal factor \( e^u \) is constant. Further, if \( u \) is constant then by equation (5.1) \( E_F^{B/2} \) vanishes.

(ii) If \( A \equiv 0 \), by Deicke’s theorem (13), \( F \) is Riemannian. Hence, the result follows by the fact that any Riemannian metric on a 2-dimensional manifold is Einstein (see [28]).

Conversely, if the conformal deformation is homothetic and \( F \) is horizontally locally Einstein then the relation (5.1) is satisfied. Thus, if \( A \) vanishes it is known that \( F \) is Riemannian and, when \( n = 2 \), every Riemannian metric in conformally Einstein. \( \square \)
Examples 2. For a Finsler-Minkowskian metric on $\mathbb{R}^2$, $F(x, y) = F(y)$. It is known [7] the conformal deformations of $F$ are of the form $\tilde{F} = cF$ for all $c > 0$. Since, the $R$-curvature on $\mathbb{R}^2$ vanishes, the tensor $E^H_F$ vanishes. Then $F$ is globally (and automatically locally) conformally $R$-Einstein.

6. Locally conformally $R$-Einstein metrics on a cylinder of dimension $n \geq 3$

6.1. Warped product of Finsler metrics. Let $\overset{1}{M}$ and $\overset{2}{M}$ be two manifolds. The set of all product coordinate systems in $\overset{1}{M} \times \overset{2}{M}$ is an atlas on $M = \overset{1}{M} \times \overset{2}{M}$ called product manifold of $\overset{1}{M}$ and $\overset{2}{M}$.

Examples 3. The product $\mathbb{R} \times \overset{1}{M}$ is called an infinite cylinder over $\overset{1}{M}$.

Examples 4. In the Example 3, if we replace $\mathbb{R}$ by an open interval $(1, \varepsilon)$, we obtain a finite cylinder $(1, \varepsilon) \times \overset{1}{M}$ over $\overset{1}{M}$.

Remark 7. In general, the product manifold of $k$ manifolds $\overset{1}{M}$, ..., $\overset{k-1}{M}$ and $\overset{k}{M}$ is the cartesian product $M = \overset{1}{M} \times \ldots \times \overset{k}{M}$.

Let $\overset{1}{M}$ and $\overset{2}{M}$ be two $C^\infty$ manifolds. For every $(x_1, x_2) \in \overset{1}{M} \times \overset{2}{M}$, we have the following properties derived from $\overset{1}{M}$ and $\overset{2}{M}$.

(1) The projections
\[
\overset{1}{p} : \overset{1}{M} \times \overset{2}{M} \to \overset{1}{M} \quad \text{such that} \quad \overset{1}{p}(x_1, x_2) = x_1
\]
\[
\overset{2}{p} : \overset{1}{M} \times \overset{2}{M} \to \overset{1}{M} \quad \text{such that} \quad \overset{2}{p}(x_1, x_2) = x_2
\]
are $C^\infty$ submersions.

(2) $\dim(\overset{1}{M} \times \overset{2}{M}) = \dim \overset{1}{M} + \dim \overset{2}{M}$.

The warped product manifold of two Finslerian manifolds is defined as follows.

Definition 9. Let $(\overset{1}{M}, \overset{1}{F})$ and $(\overset{2}{M}, \overset{2}{F})$ be two Finslerian manifolds. Let $f$ be a positive $C^\infty$ function on $\overset{1}{M}$. The warped product of $(\overset{1}{M}, \overset{1}{F})$ and $(\overset{2}{M}, \overset{2}{F})$ is a manifold $M = \overset{1}{M} \times \overset{1}{f} \overset{2}{M}$ equipped with the Finslerian metric
\[
F : T \overset{1}{M} \times \overset{2}{M} \to \mathbb{R}^+
\]
such that for any vector tangent $y \in T_x M$, with $x = (x_1, x_2) \in M$ and $y = (y_1, y_2)$,
\[
F(x, y) = \sqrt{\overset{1}{F}^2(\overset{1}{p}_* y) + f^2(\overset{1}{p}(x_1, x_2)) \overset{2}{F}^2(x_2, \overset{2}{p}_* y)}
\]
(6.2)
where $\overset{1}{p}$ and $\overset{2}{p}$ are respectively the projections of $\overset{1}{M} \times \overset{2}{M}$ onto $\overset{1}{M}$ and $\overset{2}{M}$.

Remark 8. Let $F$ be a Finsler metric on a warped product manifold $\overset{1}{M} \times \overset{1}{f} \overset{2}{M}$. 


(1) $F$ is not $C^\infty$ on the tangent vectors of the form $(y_1, 0)$ nor $(0, y_2)$ at a point $(x_1, x_2) \in \mathring{1} M \times f \mathring{2} M$.

(2) $\mathring{1} M$ is called the base manifold while $\mathring{2} M$ is the fiber manifold and $f$ is called the warping function.

If $f \equiv 1$ then $\left( \mathring{1} M \times f \mathring{2} M, \sqrt{F^2 (x, P_y)} + f^2 (x, P_y) \right)$ reduces to a Finslerian product manifold $\left( \mathring{1} M \times \mathring{2} M, \sqrt{F^2 (x, P_y)} + f^2 (x, P_y) \right)$.

The function $F$ defined in (6.1) and (6.2) is a Finslerian manifold. More precisely,

(i) $F$ is $C^\infty$ on $\mathring{T} \mathring{1} M \times \mathring{T} \mathring{2} M$ since $F$ and $\mathring{2} F$ are respectively $C^\infty$ on $\mathring{T} \mathring{1} M$ and $\mathring{T} \mathring{2} M$.

(ii) $F$ is homogeneous of degree 1 in $y = (y_1, y_2) \in T_x M$. Namely, for any $c > 0$,

\[
F(x, cy) = \sqrt{F^2 (x, (cy_1)) + f^2 (x, (cy_2))} = c \sqrt{F^2 (x, y_1) + f^2 (x, y_2)} = c F(x, y).
\]

(iii) If $n_1$ and $n_2$ are respectively the dimensions of $(\mathring{1} M, \mathring{1} F)$ and $(\mathring{2} M, \mathring{2} F)$, each element of the Hessian matrix $(g_{ij} (x, y))_{1 \leq i, j \leq n_1 + n_2}$ of $\mathring{2} F^2$, has the form:

\[
g_{ij} (x, y) := \frac{\partial^2 \left[ \frac{1}{2} F^2 (x, y) \right]}{\partial y_i \partial y_j} = \frac{1}{2} \frac{\partial^2 \left[ F^2 (x, y_1) + f^2 (x_1) F^2 (x, y_2) \right]}{\partial y_i \partial y_j} = \frac{1}{2} \frac{\partial^2 F^2 (x_1, y_1)}{\partial y'_i \partial y'_j} + f^2 (x_1) \frac{\partial^2 F^2 (x_2, y_2)}{\partial y'_2 \partial y'_2},
\]

for every point $(x, y) = (x_1, x_2, y_1, y_2) \in \mathring{T} \mathring{1} M \times \mathring{T} \mathring{2} M$. Thus,

\[
(g_{ij} (x, y)) = \begin{pmatrix}
0 & \hat{g}_{ij} (x_1, y_1) \\
\hat{g}_{ij} (x_2, y_2) & 0
\end{pmatrix}
\]

(6.3)

where $\hat{g}_{ij} (x_1, y_1) := \frac{1}{2} \frac{\partial^2 F^2 (x_1, y_1)}{\partial y'_i \partial y'_j}$ and $\hat{g}_{ij} (x_2, y_2) := \frac{1}{2} f^2 (x_1) \frac{\partial^2 F^2 (x_2, y_2)}{\partial y'_2 \partial y'_2}$.

So the fundamental tensor $g$ of $F$ is positive definite at every point $(x_1, x_2, y_1, y_2) \in \mathring{T} \mathring{1} M \times \mathring{T} \mathring{2} M$ since $\mathring{1} g$ and $\mathring{2} g$ are.

6.2. Curvatures associated with warped product Finslerian metrics. Given the submersions $\pi: \mathring{T} \mathring{1} M \rightarrow \mathring{1} M$ and $\mathring{2} \pi: \mathring{T} \mathring{2} M \rightarrow \mathring{2} M$, the fundamental tensors $\mathring{1} g$ and $\mathring{2} g$ associated with
\( F \) and \( F' \) are Riemannian metrics on the respective pulled-back tangent bundles \( \pi^* T^1 M \) and \( \pi^* T^2 M \). Thus, \( \pi \) gives rise to the Ehresmann-Finsler connection
\[
\mathcal{H} = \ker \theta_1 \quad \text{where} \quad \theta_1 : T\tilde{T}^1 M \to \pi^* T^1 M
\]
while \( \pi' \) gives rise to the Ehresmann-Finsler connection
\[
\mathcal{H}' = \ker \theta_2 \quad \text{where} \quad \theta_2 : T\tilde{T}^2 M \to \pi^* T^2 M.
\] The Ehresmann-Finslerian product connection \( \mathcal{H} \) is given by the product form \( \theta \) of \( \theta_1 \) and \( \theta_2 \), that is
\[
\theta = \theta_1 \times \theta_2 : T\tilde{T}^1 M \times T\tilde{T}^2 M \to T(\tilde{T}^1 M \times \tilde{T}^2 M) \to \pi^* T^1 M \times \pi^* T^2 M
\]
such that
\[
\ker \theta = \ker (\theta_1 \times \theta_2) = \ker \theta_1 \oplus \ker \theta_2.
\]

Now, let \( \mathcal{V} \) and \( \mathcal{V}' \) be the vertical subbundle of \( T\tilde{T}^1 M \) and \( T\tilde{T}^2 M \), respectively. We obtain the following decomposition
\[
T\tilde{T}(\pi^* T^1 M) = \mathcal{H} \oplus \mathcal{V} \oplus \mathcal{H}' \oplus \mathcal{V}'
\]

**Proposition 3.** Let \((\pi^* T^1 M, F)\) and \((\pi^* T^2 M, F')\) be two Finslerian manifolds. On a warped product manifold \( M = M_1 \times_f M_2 \) if \( \xi \in \Gamma(\pi^* T^1 M), \xi \in \Gamma(\pi^* T^2 M) \) and \( X \in \chi(\tilde{T}^1 M) \) then

(i) \( \nabla^1_X \xi = \nabla^1_X \xi \) where \( \nabla \) is the Chern connection associated with \((M, F)\).

(ii) \( \nabla^2_X \xi = \frac{1}{2} X (f^2) \xi \).

**Proof.** (i) From the relation of \( g \)-almost compatibility of \( \nabla \), we obtain
\[
2g(\nabla^1_X \xi, \xi) = X [g(\xi, \xi)] + h(\xi) [g(\pi^*_1 X, \xi)] - h(\xi) [g(\xi, \pi^*_1 X)]
\]
\[
- g(\pi^*_1 X, [\xi, \xi]) - g(\xi, [\pi^*_1 X, \xi]) + g(\xi, [\pi^*_1 X, \xi])
\]
\[
+ \mathcal{A}(\theta(X), \xi, \xi) + \mathcal{A}(\theta(h(\xi)), \pi^*_1 X, \xi) - \mathcal{A}(\theta(h^2(\xi)), \pi^*_2 X, \xi)
\]
\[
= 0.
\]

(ii) For \( \xi, \eta \in \Gamma(\pi^* T^2 M) \),
\[
2g(\nabla^2_X \xi, \eta) = \frac{1}{2} X [g(\xi, \eta)]
\]
\[
= \frac{1}{2} X [(f \circ p)^2 g(\xi, \eta)]
\]
and the relation in (ii) follows. \( \square \)

As a direct consequence, we have
Corollary 1. Let \((\hat{M}, \hat{F})\) and \((\tilde{M}, \tilde{F})\) be two Finslerian manifolds. On a warped product manifold \(M = \tilde{M} \times f M\), if \(\xi, \eta \in \Gamma(\pi^* T\tilde{M})\), \(X, Y \in \chi(T\hat{M})\) and \(\tilde{X} \in \chi(T\tilde{M})\) then

(i) \(R(\xi, \eta, X, Y) = \tilde{R}(\xi, \eta, \tilde{X}, \tilde{Y})\),

(ii) \(R(\xi, \eta, X, Y) = 0\).

Proof. The proof follows from the Proposition [3].

6.3. Proof of the Theorem. We consider the special case where the conformal factor only depends on the base manifold \(\hat{M}\) of the product \(\hat{M} \times \tilde{M}\).

Proof. A Finslerian metric \(F\) on a cylinder \(\mathbb{R} \times \hat{M}\) can be written as \(F = \sqrt{t^2 + \tilde{F}^2}\) where \(\tilde{F}\) is a Finslerian metric on \(\hat{M}\). Further, if \(F\) is locally conformal to the \(R\)-Einstein metric \(e^{u(t)}\), then by Proposition [1] we have

Case 1: if \(i = j = 1\), that is \(t = y^i = y^j\), the equation (1.1) becomes

\[
0 = E_F(\partial_t, \partial_t) - (n - 2) (\nabla_t \nabla_t u - \nabla_t u \nabla_t u) + \frac{(n - 2)}{n} (\nabla^d \nabla_d u - \nabla_d u \nabla_d u) g_{tt} + \frac{(n - 1)}{2n} (\nabla_t u \nabla_t u) \frac{\partial(F^2 g^{rs} - 2y^r y^s)}{\partial y^q} g^{kl} A_{s, kl} g_{tt}
\]

\[
= \text{Ric}_F(\partial_t, \partial_t) - \frac{1}{n} \text{Scal}_F g_{tt} - (n - 2) (\nabla_t \nabla_t u - \nabla_t u \nabla_t u) + \frac{(n - 2)}{n} g^{kd} (\nabla_t \nabla_d u - \nabla_d u \nabla_d u) g_{tt} + \frac{(n - 1)}{2nF} (\nabla_t u \nabla_t u) \frac{\partial(F^2 g^{tt} - 2t^2)}{\partial t} g^{kl} A_{t, kl}
\]

since \(u = u(t)\) and \(y^q = t\) is a coordinate on \(\mathbb{R}\). It follows that

\[
0 = -\frac{1}{n} \text{Scal}_F - (n - 2) \left(u'' - u^2\right) + \frac{(n - 2)}{n} \left(u'' - u^2\right) + \frac{(n - 1)}{2nF} \left(u^2\right) \frac{\partial(F^2 - 2t^2)}{\partial t} g^{kl} \times 0
\]

\[
= \text{Scal}_F^H + (n - 1)(n - 2)(u'' - u^2).
\] (6.9)

Case 2: if \(i = 1\) and \(j \in \{2, 3, ..., n\}\) or \(j = 1\) and \(i \in \{2, 3, ..., n\}\) that is \(t \neq y^i\) or \(t \neq y^j\), by the Proposition [1] and by the fact that \(u = u(t)\), each term in the left-hand side of the equation (1.1) vanishes.
Case 3: if $i, j \in \{2, 3, \ldots, n\}$ that is $t \neq y^i$ and $t \neq y^j$, the equation (6.12) becomes

$$0 = E_F(\partial_\alpha, \hat{\partial}_\beta) - (n - 2)(\nabla_\beta \nabla_\alpha u - \nabla_\alpha u \nabla_\beta u)$$

$$+ \frac{(n - 2)}{n} \left( \nabla^d \nabla_d u - \nabla^d u \nabla_d u \right) g_{\alpha\beta}$$

$$+ \frac{(n - 1)}{2n F} \left( \nabla_r u \nabla^q u \right) \frac{\partial (F^2 g^{rs} - 2y^r y^s)}{\partial y^q} g_{kl} A_{skl} g_{\alpha\beta}$$

$$= \text{Ric}_F(\partial_\alpha, \hat{\partial}_\beta) - \frac{1}{n} \text{Scal}_F^H \hat{g}_{\alpha\beta}.$$(6.10)

Therefore $\vec{F} = e^u F = e^u \sqrt{t^2 + \frac{2^2}{F}}$ is locally an Einstein metric if and only if

$$0 = \begin{cases} \text{Scal}_F^H + (n - 1)(n - 2)(u'' - u'^2) \\ \text{Ric}_F(\partial_\alpha, \hat{\partial}_\beta) - \frac{1}{n} \text{Scal}_F^H \hat{g}_{\alpha\beta} \\ \text{E}_F(\partial_\alpha, \hat{\partial}_\beta) \end{cases} \text{ for } \alpha, \beta \in \{2, \ldots, n\}.$$ (6.11)

From the system (6.11), $\vec{F}$ is $R$-Einstein. By the Lemma 4, $\text{Scal}_F^H = \text{constant}$. Denote this constant by $s$. The system (6.11) becomes $u'' - u'^2 + \frac{s}{(n-1)(n-2)} = 0$ or equivalently

$$u'' - u'^2 + s^* = 0$$ (6.12)

where $s^* := \frac{s}{(n-1)(n-2)}$. We set $e^u = \varphi^{-1}$. Then

$$u = -\ln \varphi, \quad u' = -\frac{\varphi'}{\varphi}, \quad u'' = \frac{\varphi''}{\varphi^2} \quad \text{and} \quad u'' = -\frac{\varphi'' + \varphi'^2}{\varphi^2}.$$ 

The equation (6.12) becomes

$$\varphi'' - \varphi s^* = 0.$$ (6.13)

We distinguish three cases:

(i) $s^* = 0$. The general solution $\varphi$ of the equation (6.13) is

$$\varphi(t) = c_1 t + c_2.$$ 

Since $\varphi(t) > 0$ for every $t \in \mathbb{R}$ we necessarily have $c_1 = 0$ and $c_2 > 0$. Thus, the conformal factor satisfies $e^u(t) = \varphi^{-1}(t) = \frac{1}{c_2} = \alpha$ with $\alpha > 0$. Hence, $u$ must be a constant function on $\mathbb{R}$.

(ii) $s^* > 0$. The general solution $\varphi$ of the equation (6.13) is

$$\varphi(t) = c_3 e^{\sqrt{s^* t}} + c_4 e^{-\sqrt{s^* t}}.$$ 

(iii) $s^* < 0$. The general solution $\varphi$ of the equation (6.13) is $\varphi(t) = c_5 \cos \left( \sqrt{-s^* t} \right) + c_6 \sin \left( \sqrt{-s^* t} \right)$. Since $\varphi$ is positive, one chooses $c_5$ and $c_6$ such that $c_5 \cos \left( \sqrt{-s^* t} \right) + c_6 \sin \left( \sqrt{-s^* t} \right) > 0$ for every $t \in \mathbb{R}$. 
Conversely, if one of the cases (i), (ii) and (iii) is holds then \( e^u F \) is R-Einstein.

**Examples 5.** Let \( \tilde{F} \) be a Finslerian metric on the sphere \( S^{n-1} \) with positive constant flag curvature \( k = 1 \). We can show \( \tilde{F} \) is of horizontal scalar curvature \( \text{Scal}^H_{\tilde{F}} = (n-1)(n-2) \).

Then the Finslerian metric \( F = \sqrt{t^2 + \tilde{F}} \) is locally conformal to the R-Einstein metric \( \tilde{F} = \cosh^{-1} t F \) for \( t \in (1, \infty) \).

**7. Non-product metrics locally conformally R-Einstein**

We define the following.

**Definition 10.** Let \((M, F)\) be a Finslerian manifold of dimension \( n \geq 3 \). The Finslerian analogous of

1. the Schouten tensor over \((M, F)\) is the \((0, 0; 1, 1)\)-tensor given by
   \[
   S^H_F := \frac{1}{n-2} \left( \text{Ric}^H_F - \frac{1}{2(n-1)} \text{Scal}^H_F g \right). \tag{7.1}
   \]

2. the Weyl tensor over \((M, F)\) is the \((0, 0; 2, 2)\)-tensor defined by
   \[
   W^H_F := R - g \otimes S^H_F. \tag{7.2}
   \]

Its components in a local coordinate are defined as follows,

\[
W_F(\partial_\ell, \partial_\theta, \hat{\partial}_j, \hat{\partial}_k) = R(\partial_\ell, \partial_\theta, \hat{\partial}_j, \hat{\partial}_k)
- g_{ij}S_F(\partial_\ell, \hat{\partial}_j) - g_{jk}S_F(\partial_\theta, \hat{\partial}_k)
+ g_{ik}S_F(\partial_\theta, \hat{\partial}_j) + g_{ij}S_F(\partial_\ell, \hat{\partial}_k). \tag{7.3}
\]

3. the Cotton-York tensor of \((M, F)\) is the \((0, 0; 1, 2)\)-tensor \( \text{C}^H_F \) defined by
   \[
   \text{C}^H_F(\xi, X, Y) := (\nabla_X S^H_F)(\xi, Y) - (\nabla_Y S^H_F)(\xi, X) \tag{7.4}
   \]

for every \( \xi \in \Gamma(\pi^*TM) \) and \( X, Y \in \chi(TM) \). In a local chart,
   \[
   \text{C}_F(\partial_\ell, \hat{\partial}_j, \hat{\partial}_k) = (\nabla_j S_F)(\partial_\ell, \hat{\partial}_k) - (\nabla_k S_F)(\partial_\ell, \hat{\partial}_j). \tag{7.5}
   \]

In dimension greater than 3, we introduce the following tensor.

**Definition 11.** The Finslerian analogous of Bach tensor for a Finslerian manifold \((M, F)\) is the \((1, 1; 0, 0)\)-tensor \( B^H_F \) defined by
   \[
   B_F(\partial_\ell, \hat{\partial}_j) = \nabla^k \text{C}_F(\partial_\ell, \hat{\partial}_j, \hat{\partial}_k) + S^H_F(\partial_\ell, \partial_\theta, \hat{\partial}_j, \hat{\partial}_k). \tag{7.6}
   \]

We have the following properties.

**Lemma 6.** Let \((M, F)\) be a Finslerian manifold of dimension \( n \geq 3 \). Then,

1. the Finslerian analogous of Weyl and of Cotton-York tensors are related as follows:
   \[
   \nabla^l W_F(\partial_\ell, \partial_\theta, \hat{\partial}_j, \hat{\partial}_k) = (n - 3) \text{C}_F(\partial_\ell, \hat{\partial}_j, \hat{\partial}_k).
   \]
(2) If $F$ is horizontally an Einstein metric then its Finslerian Cotton-York tensor vanishes

$$C_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) = 0.$$  

**Proof.** (1) Contracting the Finslerian second Bianchi identity given in Lemma 3 we get

$$g^{si}\left[\nabla_j R_{lik}s + \nabla_k R_{lisj} + \nabla_s R_{lijk}\right] = 0.$$  

Equivalent

$$-\nabla_j \text{Ric}_F(\partial_i, \hat{\partial}_k) + \nabla_k \text{Ric}_F(\partial_i, \hat{\partial}_j) + \nabla^l R_{lijk} = 0.$$  

Using this relation we have

$$\nabla^l W_F(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) = g^{ls} \nabla_s \left\{ R(\partial_l, \partial_i, \hat{\partial}_j, \hat{\partial}_k) - \frac{1}{n-2} \left( \text{Ric}_F(\partial_i, \hat{\partial}_k) \delta_{lk} - \text{Ric}_F(\partial_i, \hat{\partial}_j) \right) \right\}$$

$$= \frac{n-3}{n-2} \left( \nabla_j \text{Ric}_F(\partial_i, \hat{\partial}_k) - \nabla_k \text{Ric}_F(\partial_i, \hat{\partial}_j) \right) - \frac{n-3}{2(n-1)(n-2)} \left( \nabla_j \text{Scal}^H_F g_{ik} - \nabla_k \text{Scal}^H_F g_{ij} \right).$$

(2) If $F$ is $R$-Einstein then, by Lemma 4, $\text{Scal}^H_F$ is constant. Hence,

$$C_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) = (\nabla_j S_F)(\partial_i, \hat{\partial}_k) - (\nabla_k S_F)(\partial_i, \hat{\partial}_j)$$

$$= \nabla_j \left[ \frac{1}{n-2} \left( \text{Ric}^H_F - \frac{1}{2(n-1)} \text{Scal}^H_F g \right) \right] (\partial_i, \hat{\partial}_k)$$

$$- \nabla_k \left[ \frac{1}{n-2} \left( \text{Ric}^H_F - \frac{1}{2(n-1)} \text{Scal}^H_F g \right) \right] (\partial_i, \hat{\partial}_j).$$

Hence, formula (3.5) implies relation (7.7).

**Lemma 7.** Let $F$ be a Finslerian metric on a manifold of dimension $n \geq 3$. If $\tilde{F}$ is a conformal deformation of $F$, with $\tilde{F} = e^u F$, then

(1) the horizontal Schouten tensor behaves as follows:

$$\tilde{S}^H_F(\partial_i, \hat{\partial}_j) = S^H_F(\partial_i, \hat{\partial}_j) - \nabla_j \nabla_i u + \nabla_i u \nabla_j u + h_{ij}$$  

(7.7)
Then for every vector fields $X$, $Y$ and $h$

\[ h : = -\frac{1}{2} \nabla^k u \nabla_k u + \frac{\nabla^s u g^{kl}}{n(n-1)} [(n + 8) B^s_{kl} A_{skls} - 2 B^s_{kl} A_{lks}] \]

\[ + \frac{1}{2n(n-1)} g^{kl} g^{rs} \left\{ g(\Theta(\dot{\sigma}_s, h(\Theta_{lr})), \partial_k) - g(\Theta(\dot{\sigma}_l, h(\Theta_{rs})), \partial_k) \right\} \]

\[ + \left[ g((\nabla_s \Theta)_{lr}, \partial_k) - g((\nabla_l \Theta)_{rs}, \partial_k) \right]. \]

(2) if a horizontal $(1, 1, 0)$-tensor $T^H_F$ satisfies $T_F(\partial_i, \dot{\sigma}_j) = T_F(\pi^* \dot{\sigma}_j, h(\partial_i))$, for any $i, j = 1, ..., n$, then

\[ \left( \nabla^*_j T_F \right)(\partial_i, \dot{\sigma}_j) = \left( \nabla^*_j T_F \right)(\partial_i, \dot{\sigma}_k) \]

\[ + 2 \nabla_j u T_{ik} - \nabla_i u T_{jk} - \nabla_k u T_{ij} \]

\[ + g_{ij} T_F(\nabla u, \partial_k) + g_{jk} T_F(\partial_i, h(\nabla u)) \]

\[ + T_F(\partial_i, h(\Theta)_{jk}) - T_F(\Theta_{ij}, \dot{\sigma}_k). \]

(7.8)

(3) the horizontal Cotton-York tensor behaves as follows:

\[ \tilde{C}_F(\partial_i, \dot{\sigma}_j, \dot{\sigma}_k) = C_F(\partial_i, \dot{\sigma}_j, \dot{\sigma}_k) + W_F(\nabla u, \partial_i, \dot{\sigma}_j, \dot{\sigma}_k) + \Psi_{au}^C(\partial_i, \dot{\sigma}_j, \dot{\sigma}_k) \]

where

\[ \Psi_{au}^C(\partial_i, \dot{\sigma}_j, \dot{\sigma}_k) = \nabla_j \left( \nabla_i u \nabla_k u + h g_{ik} \right) - \nabla_k \left( \nabla_i u \nabla_j u + h g_{ij} \right) \]

\[ + \Gamma^l_{ik} \nabla_l u \nabla_j u - \Gamma^l_{ij} \nabla_l u \nabla_k u \]

\[ + g_{ij} \nabla_k \nabla_i u - h g_{ik} \]

\[ - g_{kl} \nabla^l (\nabla_j \nabla_i u - h g_{ij}) \]

\[ + g_{ij} \nabla_k \nabla_{\pi(u)} u + g_{lk} \nabla_j \nabla_{\pi(u)} u \]

\[ - \nabla_j \nabla_{\Theta_{ik}} u + \nabla_{\Theta_{ik}} u \nabla_j u + \nabla_k \nabla_{\Theta_{ij}} u - \nabla_{\Theta_{ij}} u \nabla_k u. \]

**Proof.** The assertion (1) in Lemma 7 is obtained by using the relation (7.1) and the lemmas 5 and 6 in [10].

To obtain the relation (7.8) in Lemma 7 we consider a $(1, 1, 0)$-tensor $T^H_F$ on $(M, F)$. Then for every vector fields $X, Y$ on $TM$ and for any section $\xi$ of the vector bundle $\pi^* TM$, we obtain

\[ \left( \nabla_X T^H_F \right)(\xi, Y) = \nabla_X (T^H_F(\xi, Y)) - T^H_F(\nabla_X \xi, Y) - T^H_F(\xi, h(\nabla_X \pi_u Y)) \]

(7.9)
where $\nabla_X$ is the covariant derivative with respect to $\tilde{F}$ in a given direction $X$. We have

$$
\left(\tilde{\nabla}_X T^H_F(\xi, Y)\right) = \nabla_X (T^H_F(\xi, Y)) - T^H_F \left(\nabla_X \xi + du(\pi_* X) \xi + du(\xi) \pi_* X - g(\pi_* X, \Theta(X, h(\xi)), Y)\right)
- \tilde{T}^H_F(\xi, h(\nabla_X \pi_* Y + du(\pi_* X) \pi_* Y + du(\pi_* Y) \pi_* X - g(\pi_* X, \pi_* Y) \nabla u + \Theta(X, Y))\right)
\n= \nabla_X (T_F(\xi, Y)) - 2(\nabla_X u) T_F(\xi, Y)
- (\nabla_h(\xi)) u T_F(\pi_* X, \pi_* Y) - (\nabla Y u) T_F(\xi, \tilde{X})
+ g(\xi, \pi_* X) T_F(\nabla u, \tilde{Y}) + g(\pi_* X, \pi_* Y) T_F(\xi, \nabla u)
- T_F(\xi, h(\Theta(X, Y))) - T_F(\Theta(h(\xi), X), Y).
$$

Setting $\xi = \partial_i$, $X = \partial_j$ and $Y = \partial_k$, we obtain the relation.

From the these two properties, we obtain the assertion (3) in the Lemma.

7.1. **Proof of the Theorem in dimension** $n = 3$. Let $F$ and $\tilde{F}$ be two conformal Finslerian metric, with $\tilde{F} = e^u F$, on a manifold of dimension $n \geq 3$. Then

$$
\tilde{C}_F(\partial_i, \partial_j, \partial_k) \overset{7.5}{=} \left(\tilde{\nabla}_j \tilde{S}_F\right)(\partial_i, \tilde{\partial}_k) - \left(\tilde{\nabla}_k \tilde{S}_F\right)(\partial_i, \tilde{\partial}_j)
- \nabla_j u \tilde{S}_F(\partial_i, \tilde{\partial}_k) + \nabla_k u \tilde{S}_F(\partial_i, \tilde{\partial}_j)
+ g_{ij} \tilde{S}_F(\nabla u, \partial_k) - g_{ik} \tilde{S}_F(\nabla u, \partial_j)
+ \tilde{S}_F(\Theta_{ik}, \tilde{\partial}_j) - \tilde{S}_F(\Theta_{ij}, \tilde{\partial}_k).
$$

$$
\overset{7.8}{=} \left(\nabla_J \tilde{S}_F\right)(\partial_i, \tilde{\partial}_j) - \left(\nabla_k \tilde{S}_F\right)(\partial_i, \tilde{\partial}_j)
= \nabla_j \left[\tilde{S}_F(\partial_i, \tilde{\partial}_k) - \tilde{S}_F(\nabla \partial_i, \tilde{\partial}_k)\right]
- \tilde{S}_F(\partial_i, h(\nabla_j \pi_* \partial_k)) - \left\{ \nabla_k \left[\tilde{S}_F(\partial_i, \tilde{\partial}_j)\right]\right\}
= \tilde{S}_F(\nabla_k \partial_i, \tilde{\partial}_j) - \tilde{S}_F(\partial_i, h(\nabla_k \pi_* \partial_j))\right\}
- \nabla_j u \tilde{S}_F(\partial_i, \tilde{\partial}_k) + \nabla_k u \tilde{S}_F(\partial_i, \tilde{\partial}_j)
+ g_{ij} \tilde{S}_F(\nabla u, \partial_k) - g_{ik} \tilde{S}_F(\nabla u, \partial_j)
+ \tilde{S}_F(\Theta_{ik}, \tilde{\partial}_j) - \tilde{S}_F(\Theta_{ij}, \tilde{\partial}_k)
\[\tilde{C}_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) = C_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) + \nabla_j (\nabla_i u \nabla_k u + h g_{ik}) - \nabla_k (\nabla_j u \nabla_i u + h g_{ij})\]

Conversely, if \( u = \text{constant} \) and \( C_F(\partial_i, \hat{\partial}_j, \hat{\partial}_k) \equiv 0 \) then \( e^u F \) is R-Einstein metric.
7.2. **Proof of the Theorem** in dimension $n = 4$. From the Lemma 6, if $\tilde{F}$ is $R$-Einstein metric then $C_{\tilde{F}}$ vanishes. Then the equation (7.10) holds.

Applying $\nabla^k$ to this equation, using the Definition 10 and the equation (7.10) again, we get

$$0 = B_F(\partial_i, \hat{\partial}_j) - S^F_k W_F(\partial_i, \partial_l, \hat{\partial}_k, \hat{\partial}_j) - \left[ \nabla^k \nabla^l u - (n - 3) \nabla^k u \nabla^l u \right] W_F(\partial_i, \partial_l, \hat{\partial}_k, \hat{\partial}_j) - \nabla^k \Psi^F_u(\partial_i, \hat{\partial}_j, \hat{\partial}_k) + \Psi^F_u \left( \partial_i, \hat{\partial}_j, \hat{\partial}_k \right).$$

(7.12)

Since $\tilde{F}$ is locally an $R$-Einstein metric, the equation (1.1) is equivalent to

$$0 = S^F_u(\partial_i, \hat{\partial}_j) - \frac{1}{n} J^F_F g_{ij} - \nabla_j \nabla_i u + \nabla_i u \nabla_j u$$

$$+ \frac{1}{n} \left[ \nabla^d \nabla_d u - \nabla^d u \nabla_d u \right] g_{ij}$$

$$+ \frac{(n - 1)}{2n(n - 2)^2} \left( \nabla_r u \nabla_q u \right) \frac{\partial(F^2 g^{rs} - 2y^r y^s)}{\partial y^q} g^{kl} A_{sklj} g_{ij}$$

where $J^F_F$ is the trace of $\text{Scal}^F_F$. Raising both indices and applying $W_F(\partial_i, \partial_l, \hat{\partial}_j, \hat{\partial}_k)$ to this equation, using the relation (7.7) in Lemma 7 and the equation (7.12) we obtain

$$0 = B_F(\partial_i, \hat{\partial}_j) + (n - 4) W_F(\nabla u, \partial_l, \hat{\partial}_j, \nabla u)$$

$$+ \left[ (n - 3) \nabla^k u - \nabla^k \right] \Psi^F_u \left( \partial_i, \hat{\partial}_j, \hat{\partial}_k \right).$$

Therefore, in dimension $n = 4$, we have $B_F(\partial_i, \hat{\partial}_j) + \left( \nabla^k u - \nabla^k \right) \Psi^F_u \left( \partial_i, \hat{\partial}_j, \hat{\partial}_k \right) = 0$.

Conversely, if $u = \text{constant}$ and $B^H_F \equiv 0$ then $e^u F$ is $R$-Einstein metric.

**REFERENCES**

[1] P. L. Antonelli, R. S. Ingarden and M. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Kluwer Academic Publishers, Dordrecht, 1993.

[2] D. Bao, S.-S. Chern and Z. Shen, *An Introduction to Riemannian-Finsler Geometry*, Springer-Verlag New York, (2000), 1-192.

[3] A. L. Besse, *Einstein Manifolds*, Springer-Verlag Berlin Heidelberg, (1987), 1-528.

[4] H. W. Brinkmann, *Riemann spaces conformal to Einstein spaces*, Proc. Nat. Acad. Sci. USA 9 (1923), 172-174.

[5] C. N. Kozameh, E. T. Newman and K. P. Tod, *Conformal Einstein Spaces*, Gen. Rel. and Grav., Vol. 17 (1985), 343-344.

[6] M. Listing, *Conformal Einstein Spaces in N-dimensions*, Annals of Global Analysis and Geometry, 20 (2001), 183-197.

[7] V. S. Matveev, H. B. Rademacher, M. Troyanov and A. Zeghib, *Finsler conformal Lichnerwicz-Obata Conjecture*, Annales de l’Institut de Fourier, Grenoble 59, 3 (2009), 937-949.

[8] J.-S. Mbatakou, *Intrinsic proofs of the existence of generalized Finsler connections*, Int. Electron. J. Geom., 8, 1 (2015), 1-13.

[9] G. Nibaruta, S. Degla and L. Todjihounde, *Prescribed Ricci tensor in Finslerian conformal class*, Balkan J. Geom. Appl., 23, 2 (2018), 41-55.
References

[10] G. Nibaruta, S. Degla and L. Todjihounde, *Finslerian Ricci Deformation and Conformal Metrics*, J. Appl. Math. Phys., 6 (2018), 1522-1536.

[11] A. R. Gover and P. Nurowski, Obstructions to conformally Einstein metrics in n dimensions, Journal of Geometry and Physics, www.elsevier.com/locate/jgp. (2004).

[12] W. Kühnel and H.-B. Rademacher, *Conformally Einstein product spaces*, Differ. Geom. Appl., 49 (2016), 65-96.

[13] Y.-B. Shen and Z. Shen, *Introduction to Modern Finsler Geometry*, Higher Education Press Limited Company and World Scientific Publishing Co. Pte. Ltd. (2016), 1-58.

[14] P. Szekeres, *Spaces Conformal to a Class of Spaces in General Relativity*, Proc. Roy. Soc. London, Ser. A., 274 (1963), 206-212.

Ecole Normale Supérieure de Natitingou, P. O. Box 72, Natitingou, Bénin
*Email address*: deglaserge@yahoo.fr

Ecole Normale Supérieure, Section de Mathématiques, P. O. Box 6983, Bujumbura-Burundi
*Email address*: gilbert.nibaruta@imsp-uac.org

Université d’Abomey-Calavi, Institut de Mathématiques et de Sciences Physiques, P. O. Box 613, Porto-Novo, Bénin
*Email address*: leonardt@imsp-uac.org