Plane symmetric cosmological models

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In this work, we perform the Lie symmetry analysis on the Einstein-Maxwell field equations in plane symmetric spacetime. Here Lie point symmetries and optimal system of one dimensional subalgebras are determined. The similarity reductions and exact solutions are obtained in connection to the evolution of universe. The present study deals with the electromagnetic energy of inhomogeneous universe where $F_{12}$ is the non-vanishing component of electromagnetic field tensor. To get a deterministic solution, it is assumed that the free gravitational field is Petrov type-II non-degenerate. The electromagnetic field tensor $F_{12}$ is found to be positive and increasing function of time. As a special case, to validate the solution set, we discuss some physical and geometric properties of a specific sub-model.

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I. INTRODUCTION

The standard Friedman-Robertson-Walker (FRW) cosmological model prescribes a homogeneous and an isotropic distribution for its matter in the description of the present state of the universe. At the present state of evolution, the universe is spherically symmetric and the matter distribution in the universe is on the whole isotropic and homogeneous. But in early stages of evolution, it could have not had such a smoothed picture. Close to the big bang singularity, neither the assumption of spherical symmetry nor that of isotropy can be strictly valid. So we consider plane-symmetric, which is less restrictive than spherical symmetry and can provide an avenue to study inhomogeneities. Inhomogeneous cosmological
models play an important role in understanding some essential features of the universe such as the formation of galaxies during the early stages of evolution and process of homogenization. The early attempts at the construction of such models have been done by Tolman [1] and Bondi [2] who considered spherically symmetric models. Inhomogeneous plane-symmetric models were considered by Taub [3, 4] and later by Tomimura [5] and Szekeres [6]. Recently, Senovilla [7] obtained a new class of exact solutions of Einstein’s equation without big bang singularity, representing a cylindrically symmetric, inhomogeneous cosmological model filled with perfect fluid which is smooth and regular everywhere satisfying energy and causality conditions. Later, Ruis and Senovilla [8] have separated out a fairly large class of singularity free models through a comprehensive study of general cylindrically symmetric metric with separable function of \( r \) and \( t \) as metric coefficients. Recently Bali and Tyagi [9], Pradhan et al. [10, 11] obtained a plane-symmetric inhomogeneous cosmological models of perfect fluid distribution with electro-magnetic field. In the recent past Pradhan et al. [12], Yadav [13], Ali and Yadav [14] and Ali et al. [15] have studied inhomogeneous cosmological models with perfect fluid as source of matter; the latter were invoked to palliate the problems associated with plane-symmetric space-time.

A lesson given by the history of cosmology is that the concept of the electromagnetic field tensor revives in the days of crisis and we have more reasons than ever to believe that this term is the necessary ingredient of any cosmological model. The occurrence of magnetic fields on galactic scale is well-established fact today, and their importance for a variety of astrophysical phenomena is generally acknowledged as pointed out by Zeldovich et al. [16]. Also Harrison [17] has suggested that magnetic field could have a cosmological origin. As a natural consequence, we should include magnetic fields in the energy-momentum tensor of the early universe. The choice of anisotropic cosmological models in Einstein system of field equations leads to the cosmological models more general than Robertson-Walker model [18].

Strong magnetic fields can be created due to adiabatic compression in clusters of galaxies. Primordial asymmetry of particle (say electron) over antiparticle (say positron) have been well established as CP (charged parity) violation. Asseo and Sol [19] speculated the large-scale inter galactic magnetic field and is of primordial origin at present measure \( 10^{-8} \) G and gives rise to a density of order \( 10^{-35} \) g/cm\(^3\). The present day magnitude of magnetic energy is very small in comparison with the estimated matter density, it might not have been negligible during early stage of evolution of the universe. FRW models are approximately valid as present day magnetic field is very small. The existence of a primordial magnetic field is limited to Bianchi Types I, II, III, VI\(_0\) and VII\(_0\) as shown by Hughston and Jacobs [20]. Large-scale magnetic fields give rise to anisotropies in the universe. The anisotropic pressure created by the magnetic fields dominates the evolution of the shear anisotropy and it decays slower than if the pressure was isotropic [21, 22]. Such fields can be generated at the end of an inflationary epoch [23–25]. Anisotropic magnetic field models have significant contribution in the evolution of galaxies and stellar objects.

Many natural phenomena are described by a system of nonlinear PDEs which is often difficult to be solved analytically, as there is no general theory for completely solving the nonlinear PDEs. Symmetry group analysis, advocated by Sophus Lie during the 19th century, provides an efficient method for obtaining exact solutions of PDEs [26–29]. Lie group of transformations has been extensively applied to linear and nonlinear differential equations in the area of applied mathematics and theoretical physics such as: quantum mechanics, fluid dynamics system, relativity, particle physics and cosmology [30, 31]. The method of Lie symmetry group is one of the important techniques for finding exact solutions for the
Einstein field equations described by a system of NLPDEs \[14, 15\]. Several applications of Lie groups in the theory of differential equations were discussed in the literature, the most important ones are: reduction of order of ordinary differential equations, transform the partial differential equations to ordinary differential equations, construction of invariant solutions, mapping solutions to other solutions and generating new solutions from known ones. The classification of group-invariant solutions of differential equations by means of the so-called optimal system is one of the main applications of Lie group analysis of differential equations. The method was first conceived by Ovsianikov \[27\]. Discussion on optimal systems can be found in ref. \[28\]. Also Ibragimov \[31\], in his paper has given some examples of optimal system.

In this paper, we have attempted to find the exact solution for accelerating universe in plane symmetric space-time with electromagnetic fluid distribution under the background of general relativity. We organize the paper as follows: In Sec. II we introduce the mathematical modeling of the accelerating universe in the plane symmetric space-time. In Sec. III, solution of the field equations have been obtained. Sec. IV is developed to study some physical and geometrical properties of the obtained model. The paper ends with a short discussion in Sec. V.

II. MATHEMATICAL FRAMWORK

A. The Einstein-Maxwell spacetime geometry

We consider the metric in the form
\[
ds^2 = A^2 \left(dx^2 - dt^2\right) + B^2 \, dy^2 + C^2 \, dz^2,
\]
where the metric potentials \(A, B \) and \(C\) are functions of the spatial and temporal coordinates \(x, t\) both.

The energy-momentum tensor is taken as
\[
T^j_i = (\rho + p)v_i v^j + \rho g^j_i + E^j_i,
\]
where \(E^j_i\) is the electromagnetic field given by
\[
E^j_i = \bar{\mu} \left[ h_i h^j (v_i v^j + \frac{1}{2} g^j_i) - h_i h^j \right].
\]

Here \(\rho\) and \(p\) are the energy density and isotropic pressure respectively and \(v^i\) is the flow vector satisfying the relation
\[
g_{ij} v^i v^j = -1,
\]
whereas \(\bar{\mu}\) is the magnetic permeability and \(h_i\) the magnetic flux vector defined by
\[
h_i = \frac{1}{\bar{\mu}} * F_{ij} v^j,
\]
where \(* F_{ij}\) is the dual electro-magnetic field tensor defined by Synge \[34\]
\[
* F_{ij} = \frac{\sqrt{-g}}{2} \epsilon_{ijkl} F^{kl}.
\]
Here $F_{ij}$ is the electro-magnetic field tensor and $\epsilon_{ijkl}$ is the Levi-Civita tensor density. The coordinates are considered to be comoving so that $v^1 = 0 = v^2 = v^3$ and $v^4 = \frac{1}{A}$. We consider that the current is flowing along the $z$-axis so that $h_3 \neq 0$, $h_1 = 0 = h_2 = h_4$. The only non-vanishing component of $F_{ij}$ is $F_{12}$. The Maxwell equations

$$F_{ij;k} + F_{jk;i} + F_{ki;j} = 0,$$

and

$$\left[ \frac{1}{\mu} F^{ij} \right]_{;j} = 0,$$

require that $F_{12}$ be function of $x$ alone. We assume that the magnetic permeability as a function of $x$ and $t$ both. Here the semicolon represents a covariant differentiation.

The Einstein field equations (in gravitational units $G = c = 1$) read as

$$R^i_j - \frac{1}{2} R g^i_j = -8\pi T^i_j,$$

which for the line element (1) has been set up as follows:

$$E_1 = \frac{B_{st}}{B} + \frac{C_{st}}{C} - \frac{A_t}{A} \left( \frac{B_x}{B} + \frac{C_x}{C} \right) - \frac{A_x}{A} \left( \frac{B_t}{B} + \frac{C_t}{C} \right) = 0,$$

$$E_2 = \frac{B_t C_t - B_x C_x}{B C} + \frac{B_{tt}}{B} + \frac{C_{xx}}{C} + \frac{A_{xx} - A_{tt}}{A} + \frac{A_t}{A} \left( \frac{A_t}{A} - \frac{B_x}{B} - \frac{C_x}{C} \right) - \frac{A_x}{A} \left( \frac{A_x}{A} + \frac{B_x}{B} + \frac{C_x}{C} \right) = 0,$$

$$\chi A^2 \mu(x, t) = \frac{B_{xx} - 2 B_{tt}}{2 B} - \frac{C_{tt}}{2 C} + \frac{A_{xx} - A_{tt}}{2 A} + \frac{B_x C_x - B_t C_t}{2 B C} + \frac{A_t}{2 A} \left( \frac{A_t}{A} + \frac{B_t}{B} + \frac{C_t}{C} \right) - \frac{A_x}{2 A} \left( \frac{A_x}{A} - \frac{B_x}{B} - \frac{C_x}{C} \right),$$

$$\chi A^2 \rho(x, t) = \frac{C_{tt} - 2 C_{xx}}{2 C} - \frac{B_{xx}}{2 B} + \frac{A_{xx} - A_{tt}}{2 A} + \frac{3 (B_t C_t - B_x C_x)}{2 B C} + \frac{A_t}{2 A} \left( \frac{A_t}{A} + \frac{B_t}{B} + \frac{C_t}{C} \right) + \frac{A_x}{2 A} \left( \frac{A_x}{A} + \frac{B_x}{B} + \frac{C_x}{C} \right),$$

$$\chi \frac{F_{12}^2(x)}{B^2 \mu(x, t)} = \frac{B_x C_x - B_t C_t}{B C} - \frac{C_{tt}}{B C} + \frac{B_{xx}}{B} + \frac{A_{tt} - A_{xx}}{A} + \frac{A_t}{A} \left( \frac{C_t}{C} + \frac{B_t}{B} - \frac{A_t}{A} \right) + \frac{A_x}{A} \left( \frac{A_x}{A} + \frac{B_x}{B} + \frac{C_x}{C} \right).$$
The four-acceleration vector $\dot{u}_i$, the rotation $\omega_{ij}$, the scalar expansion $\Theta$, shear scalar $\sigma^2$ and proper volume $V$ are respectively found to have the following expressions \cite{35, 36}:

\begin{align}
\dot{u}_i &= u_{i:j} u^j = \frac{A_x}{A} \left( 1, 0, 0, 0 \right), \\
\omega_{ij} &= u_{[i:j]} + \dot{u}_{[i} u_{j]} = 0 \Rightarrow \omega_{41} = -\omega_{14} = A_x, \\
\Theta &= u^i_{;i} = \frac{1}{A} \left( \frac{A_t}{A} + \frac{B_t}{B} + \frac{C_t}{C} \right), \\
\sigma^2 &= \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{\Theta^2}{3} - \frac{1}{A^2} \left( \frac{A_t B_t}{A B} + \frac{A_t C_t}{A C} + \frac{B_t C_t}{B C} \right), \\
V &= \sqrt{-g} = A^2 B C,
\end{align}

where $g$ is the determinant of the metric $\Box$. The shear tensor is

$$
\sigma_{ij} = u_{(i:j)} + \dot{u}_{(i} u_{j)} - \frac{1}{3} \Theta (g_{ij} + u_{i} u_{j}),
$$

and the non-vanishing components of the $\sigma^j_i$ are

$$
\begin{align*}
\sigma^1_1 &= \frac{1}{3} A \left( \frac{2 A_t}{A} - \frac{B_t}{B} - \frac{C_t}{C} \right), \\
\sigma^2_2 &= \frac{1}{3} A \left( \frac{2 B_t}{B} - \frac{C_t}{C} + \frac{A_t}{A} \right), \\
\sigma^3_3 &= \frac{1}{3} A \left( \frac{2 C_t}{C} - \frac{B_t}{B} + \frac{A_t}{A} \right), \\
\sigma^4_4 &= 0.
\end{align*}
$$

The Einstein field equations (10)-(14) constitute a system of five highly non-linear differential equations with six unknowns variables, $A$, $B$, $C$, $p$, $\rho$ and $\frac{F^2}{B}$. Therefore, one physically reasonable conditions amongst these parameters are required to obtain explicit solutions of the field equations. Let us assume that the expansion scalar $\Theta$ in the model (11) is proportional to the eigenvalue $\sigma^1_1$ of the shear tensor $\sigma^k_j$. Then from (17) and (21), we get

$$
\frac{2 A_t}{A} - \frac{B_t}{B} - \frac{C_t}{C} = 3 \gamma \left( \frac{A_t}{A} + \frac{B_t}{B} + \frac{C_t}{C} \right),
$$

where $\gamma$ is a constant of proportionality.

The above equation can be written in the form

$$
\frac{A_t}{A} = n \left( \frac{B_t}{B} + \frac{C_t}{C} \right),
$$

where $n = \frac{1+3\gamma}{2-3\gamma}$.

If we integrate the above equation with respect to $t$, we can get the following relation

$$
A(x, t) = f(x) B^n(x, t) C^n(x, t),
$$

where $f(x)$ is a constant of integration which is an arbitrary function of $x$. 

5
If we substitute the metric function $A$ from (24) in the Einstein field equations, the equations (10) and (11) transform to the nonlinear partial differential equations of the coefficients $B$ and $C$ only, as the following new form:

$$E_1 = \frac{B_{xt}}{B} + \frac{C_{xt}}{C} - 2n \left( \frac{B_x B_t + B_x C_t + B_t C_x}{B C} + \frac{C_x C_t}{C^2} \right) - \frac{f'}{f} \left( \frac{B_t}{B} + \frac{C_t}{C} \right) = 0,$$

$$E_2 = n \left( \frac{B_{xx}}{B} - \frac{C_{tt}}{C} \right) + (1 - n) \frac{B_{tt}}{B} + (1 + n) \frac{C_{xx}}{C} + 2n \left( \frac{B_x^2}{B^2} - \frac{C_x^2}{C^2} \right) + (1 - 2n) \frac{B_t C_t}{B C} - (1 + 2n) \frac{B_x C_x}{B C} - \frac{f'}{f} \left( \frac{B_x}{B} + \frac{C_x}{C} \right) + \frac{f f'' - f'^2}{f^2} = 0,$$

where the prime indicates derivative with respect to the coordinate $x$.

If we solve the system of second order nonlinear PDEs (25)-(26), we shall obtain the exact solution of the consideration problem. The classical method for finding this solution is a separation method by taking $B^* = B_1(x) B_2(t)$ and $C^* = C_1(x) C_2(t)$ [37-39]. The symmetry analysis method is a powerful method which gives an invariant solutions. Here, we shall use the Lie group analysis method and explain it in details for this work. In the last century, the application of this method has been developed by a number of mathematicians. Ovsyannikov [27], Olver [28], Baumann [30], and Bluman and Anco [40] are some of the mathematicians who have enormous amount of studies in this field.

### B. The method of symmetry analysis

Let us consider a one-parameter Lie group of transformations:

$$\begin{cases}
    x^* = x + \epsilon \xi_1(x, t, B, C), & t^* = t + \epsilon \xi_2(x, t, B, C), \\
    B^* = B + \epsilon \eta_1(x, t, B, C), & C^* = C + \epsilon \eta_2(x, t, B, C),
\end{cases}$$

with a small parameter $\epsilon < 1$ where the coefficients $\xi_1$, $\xi_2$, $\eta_1$ and $\eta_2$ are functions of corresponding variables. We have assumed that the system (25)-(26) is invariant under the transformations given in the above equation. The corresponding infinitesimal generator of Lie algebra (symmetries) is of the form

$$X = \sum_{i=1}^{2} \xi_i \frac{\partial}{\partial x_i} + \sum_{a=1}^{2} \eta_a \frac{\partial}{\partial u_a},$$

where $x_1 = x$, $x_2 = t$, $u_1 = B$ and $u_2 = C$. The components $\xi_1$, $\xi_2$, $\eta_1$ and $\eta_2$ of the infinitesimals symmetries corresponding to $x$, $t$, $B$ and $C$ respectively, are determined from the invariance conditions:

$$\begin{cases}
    P_{r(2)}^{} X \bigg|_{(E_1=0, E_2=0)} = 0, \\
    P_{r(2)}^{} X \bigg|_{(E_1=0, E_2=0)} = 0,
\end{cases}$$

where $P_{r(2)}$ is the operator of the system (25)-(26). The method of finding the symmetry algebra is to find the conditions of invariance which are determined by the divergence of the infinitesimal generator $X$. The corresponding Lie derivative of the system is defined as

$$\mathcal{L}_X = \{ P_{r(2)}^{} X \} = 0,$$

where $\{ \}$ denotes the Lie derivative. The conditions of invariance for the system are given by

$$\begin{cases}
    P_{r(2)}^{} X \bigg|_{(E_1=0, E_2=0)} = 0, \\
    P_{r(2)}^{} X \bigg|_{(E_1=0, E_2=0)} = 0,
\end{cases}$$

where $P_{r(2)}$ is the operator of the system (25)-(26).
where $E_1 = 0$, $E_2 = 0$ are the system (25)-(26) under study and $P^{(2)}_r$ is the second prolongation of the symmetries $X$.

Since our equations (25)-(26) are at most of order two, therefore, we need second order prolongation of the infinitesimal generator in Eq. (29). It is worth noting that, the 2-th order prolongation is given by

$$P^{(2)}_r X = \sum_{i=1}^{2} \xi_i \frac{\partial}{\partial x_i} + \sum_{\alpha=1}^{2} \eta_{\alpha} \frac{\partial}{\partial u_{\alpha}} + \sum_{i=1}^{2} \sum_{\alpha=1}^{2} \eta_{\alpha i} \frac{\partial}{\partial u_{\alpha,i}} + \sum_{j=1}^{2} \sum_{i=1}^{2} \sum_{\alpha=1}^{2} \eta_{\alpha ij} \frac{\partial}{\partial u_{\alpha,ij}},$$

(30)

where

$$\begin{cases}
\eta_{\alpha i} = D_i (\eta_\alpha) - \sum_{j=1}^{2} u_{\alpha,j} D_j (\xi_j), \\
\eta_{\alpha ij} = D_j (\eta_{\alpha i}) - \sum_{k=1}^{2} u_{\alpha,k i} D_j (\xi_k).
\end{cases}$$

(31)

The Hach operators $D_x$ and $D_t$ are the total derivatives with respect to $x$ and $t$ respectively where $u_{\alpha,i} = \frac{\partial u_{\alpha}}{\partial x_i}$ and $u_{\alpha,i,j} = \frac{\partial^2 u_{\alpha}}{\partial x_i \partial x_j}$. Expanding the system of Eqs. (29) along with the original system of Eqs. (25)-(26) to eliminate $B_{xx}$ and $B_{xt}$ while we set the coefficients involving $C_x$, $C_t$, $C_{xx}$, $C_{xt}$, $B_x$, $B_t$, $B_{tt}$ and various products to zero give rise to the essential set of over-determined equations. Solving the set of these determining equations, the components of symmetries takes the following form:

$$\xi_1 = c_1 x + c_2, \quad \xi_2 = c_1 t + c_3, \quad \eta_1 = c_4 B, \quad \eta_2 = c_5 C,$$

(32)

where the function $f(x)$ must be taken the following forms

$$\begin{cases}
f(x) = c_6 \exp [c_7 x], \quad \text{if} \quad c_1 = 0, \\
f(x) = c_8 (c_1 x + c_2)^{c_9}, \quad \text{if} \quad c_1 \neq 0,
\end{cases}$$

(33)

where $c_i$, $i = 1, 2, ..., 9$ are an arbitrary constants.

C. Determination of the optimal system

The general Lie point symmetries (28) becomes

$$X = (c_1 x + c_2) \frac{\partial}{\partial x} + (c_1 t + c_3) \frac{\partial}{\partial t} + c_4 B \frac{\partial}{\partial B} + c_5 C \frac{\partial}{\partial C}.$$  

(34)

Consequently, the non-linear field equations (25)-(26) admits the 5-dimensional Lie algebra spanned by the independent symmetries shown below

$$X_1 = x \frac{\partial}{\partial x} + t \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial t}, \quad X_4 = B \frac{\partial}{\partial B}, \quad X_5 = C \frac{\partial}{\partial C}.$$  

(35)
The forms of the symmetries $X_i$, $i = 1, \ldots, 5$ suggest their significations: $X_2$ and $X_3$ generate the symmetry of space translation, $X_1$, $X_4$ and $X_5$ are associated with the scaling transformations. When the Lie algebra of these symmetries is computed, the only non-vanishing relations are

$$[X_1, X_2] = -X_2, \quad [X_1, X_3] = -X_3.$$  \hspace{1cm} (36)

It is well known that reduction of the independent variables by one is possible using any linear combinations of the generators of symmetries (35). We will construct a set of minimal combinations known as optimal system [27, 28]. An optimal system of a Lie algebra is a set of $l$-dimensional subalgebra such that every $l$-dimensional is equivalent to a unique element of the set under some element of the adjoint representation. The adjoint representation of a Lie algebra $\{X_i, i = 1, \ldots, 5\}$ is constructed using the formula

$$\text{Ad}(\exp[\varepsilon X_i])X_j = \sum_{k=0}^{\infty} \frac{\varepsilon^k}{k!} (\text{Ad}(X_i))^k X_j = X_j - \varepsilon [X_i, X_j] + \frac{\varepsilon^2}{2} [X_i, [X_i, X_j]] - \ldots.$$  \hspace{1cm} (37)

In order to find the optimal system of the field equations (25)-(26), first the following adjoint table is constituted as the following

| Ad | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
|----|-------|-------|-------|-------|-------|
| $X_1$ | $X_1$ | $e^\varepsilon X_2$ | $e^\varepsilon X_3$ | $X_4$ | $X_5$ |
| $X_2$ | $X_1 - \varepsilon X_2$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
| $X_3$ | $X_1 - \varepsilon X_3$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
| $X_4$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |
| $X_5$ | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ |

Using simplification procedure in refs. [27, 28], we acquire an optimal system of one-dimensional subalgebras to be those spanned by

$$\{X^{(1)} = X_1 + a_4 X_4 + a_5 X_5, \quad X^{(2)} = a_2 X_2 + X_3 + a_4 X_4 + a_5 X_5, \quad X^{(3)} = X_2 + a_4 X_4 + a_5 X_5, \quad X^{(4)} = X_4 + a_5 X_5, \quad X^{(5)} = X_5\}.$$  \hspace{1cm} (38)

### III. SIMILARITY SOLUTIONS OF THE EINSTEIN-MAXWELL FIELD EQUATIONS

The characteristic equations corresponding to the symmetries (32) are given by:

$$\frac{dx}{c_1 x + c_2} = \frac{dt}{c_1 t + c_3} = \frac{dB}{c_4 B} = \frac{dC}{c_5 C}.$$  \hspace{1cm} (39)

In the case of symmetries $X^{(3)}$ or $X^{(4)}$ or $X^{(5)}$, we have $c_1 = c_3 = 0$. From the characteristic equations in (39) this leads to the similarity variable as $\xi = t$ where similarity functions
$B$ and $C$ are functions of $t$ only. Thus, we shall consider the invariant solutions associated with the optimal systems of symmetries $X^{(1)}$ and $X^{(2)}$ only as in the following two cases:

**Case I:** The symmetries $X^{(2)}$ has the characteristic equation in the form (39) such that $c_1 = 0$ and $c_3 = 1$. Therefore, the similarity variable and similarity functions can be written as follows:

$$
\xi = x + b t, \quad B(x, t) = \Psi(\xi) \exp[cx], \\
C(x, t) = \Phi(\xi) \exp[dx],
$$

where $b = -c_2$, $c = \frac{c_4}{c_2}$ and $d = \frac{c_5}{c_2}$ are an arbitrary constants.

**Case II:** The symmetries $X^{(1)}$ has the characteristic equation in the form (39) such that $c_1 = 1$ and $c_2 = c_3 = 0$. Therefore, the similarity variable and similarity functions can be written as the following:

$$
\xi = \frac{t}{x}, \quad B(x, t) = x^c \Psi(\xi), \quad C(x, t) = x^d \Phi(\xi),
$$

where $c = c_4$ and $d = c_5$ are an arbitrary constants.

However, one can perform mathematical and physical analysis by considering several subcases under the above two cases and conclude that Case I and some of its subcases lead us to the physically interesting and viable solutions. Therefore, to save time as well as space, we shall consider only Case I and following subcases in our calculations.

Hence, substitution of the transformations (40) in the Eqs. (25) - (26) lead to the following system of ordinary differential equations:

$$
\left[ c_7 - c + 2n (d - c) \right] \frac{\Psi'}{\Psi} + \left[ c_7 - d + 2n(c - d) \right] \frac{\Phi'}{\Phi} + 2n \left( \frac{\Psi'}{\Psi} + \frac{\Phi'}{\Phi} \right)^2 - \frac{\Psi''}{\Psi} - \frac{\Phi''}{\Phi} = 0,
$$

$$
\left[ (n - 1) b^2 - n \right] \frac{\Psi''}{\Psi} - \left[ n + 1 - nb^2 \right] \frac{\Phi''}{\Phi} + 2n \left( \frac{\Psi''}{\Psi^2} + \frac{\Phi''}{\Phi^2} \right) + \left[ 2n + 1 + (2n - 1)b^2 \right] \frac{\Psi'}{\Psi} + \frac{\Phi'}{\Phi} + \\
\left[ c_7 + d + 2n(c + d) \right] \frac{\Psi'}{\Psi} + \left[ c_7 + d + 2n(c + d) - 2a_4 \right] \frac{\Phi'}{\Phi} + \frac{d(c - d) + c_7(c + d) + n(c + d)^2}{\Phi} = 0.
$$

The equations (42) and (43) are non-linear ordinary differential equations which is very difficult to solve. However, it is worth noting that, these equations are easy to solve in a special case with the consideration $b = -1$.

Subtracting between equation (42) and (43) we get the following equation:

$$
\frac{\Psi'}{\Psi} + \frac{(c+d)\Psi'}{\Psi} = \alpha_0,
$$

where $\alpha_0 = \frac{d(c_7-d)+c(c_7+d)+n(c+d)^2}{d-c}$.

After integration the above equation with respect to $\xi$, we get:

$$
\Phi(\xi) = r_1 \Psi^{\alpha_1}(\xi) \exp[\alpha_0 \xi],
$$
where \( \alpha_1 = \frac{d+c}{d-c} \) while \( r_1 \) is an arbitrary constant of integration.

The equation (42), after using the transformation

\[
\Psi(\xi) = r_2 \exp \left[ \alpha_2 \int \Omega(\xi) \, d\xi \right]
\]

and (45), becomes:

\[
\Omega' = \alpha_3 \Omega^2 + \alpha_4 \Omega + \alpha_5,
\]

where

\[
\begin{aligned}
\alpha_3 &= \frac{\alpha_2 \left( (c^2+3d^2)(c^2+2cd-d^2)+4d^2 \right) \alpha_0 + (c+d) c_7}{d(c-d)(c+d)^2}, \\
\alpha_4 &= \frac{c^4-8c^2d^2-4c^2d+3d^4-2}{2d(c+d)^2} \left[ c^2 + 5c^2d^2 - 5d^4 \right] \alpha_0 + 8d \alpha_0 \left[ (d-c) \alpha_0 - (c+d) c_7 \right] - c_7, \\
\alpha_5 &= \frac{\alpha_0 (c-d) \left[ 2c(d+\alpha_0)(d+c_7+\alpha_0) - d(d-c_7+\alpha_0)(d+2\alpha_0) + c^2(3d+c_7+3\alpha_0) \right]}{2d\alpha_2 (c+d)^2},
\end{aligned}
\]

and \( r_2 \) is constant while \( \Omega(\xi) \) is a new function of \( \xi \).

To get the solution of the above ordinary differential equation we consider the following special case of \( \alpha_3 \neq 0, \alpha_4 = 0 \) and \( \alpha_5 \neq 0 \).

Now the general solution of the equation (17) becomes

\[
\Omega(\theta) = \sqrt{\frac{\alpha_5}{\alpha_3}} \tan \left[ \sqrt{\frac{\alpha_5}{\alpha_3}} \xi \right].
\]

The above solution is very complicated because the values of \( \alpha_3 \) and \( \alpha_5 \) are very complicated. Therefore, we shall study the simple case as follows: \( \alpha_0 = \frac{d^2 - 3d^2}{2} - c \).

Now, using (19), (46), (45) and (40), and after some calculation, we can obtain the solutions of the metric functions in this case as the following:

**Case (b - 1):** \( m > 3 + \frac{2}{m} \)

\[
\begin{aligned}
A(x, t) &= q_1 \exp \left[ K \left( \frac{(2m^3-5m^2+m-2)x-(m^2-3m-2)t}{\sqrt{2}(m-1)^2} \right) K_0 \right] \\
&\times \cos^{\gamma_0} \theta, \\
B(x, t) &= q_2 \exp \left[ \frac{2\sqrt{2} \gamma_0 K_x}{K_0} \cos^{(m-1)} \gamma_0 \theta \right], \\
C(x, t) &= q_3 \exp \left[ \frac{\sqrt{2} \gamma_0 K}{K_0} \left( \frac{(m^2-m-2)x-(m^2-3m-2)t}{K_0} \right) \right] \\
&\times \cos^{(m+1)} \gamma_0 \theta,
\end{aligned}
\]

where \( \gamma_0 = \frac{m}{(m-1)^2} \), \( K_0^2 = m - 3 - \frac{2}{m} \), \( f(x) = c_0 \exp \left[ \frac{\sqrt{2} K (m^3-3m^2+m-1)x}{K_0 (m-1)^2} \right] \) and \( \theta = K(x-t) \), the symbols \( K, m, q_1, q_2 \) and \( q_3 \) all are being arbitrary constants, however, here \( m \) will never be 0 or 1.
Case (b - 2) : \( m < 3 + \frac{2}{m} \)

\[
\begin{align*}
A(x, t) &= q_1 \exp \left[ K \frac{(2m^4 - 5m^2 + m - 2)x - (m^2 - 3m - 2)t}{\sqrt{2}K_0(m-1)^2} \right] \\
B(x, t) &= q_2 \exp \left[ \frac{2\sqrt{2}K_\gamma x}{K_0} \cosh (m-1)\gamma \right] \\
C(x, t) &= q_3 \exp \left[ \frac{\sqrt{2}K_\gamma (m^2 - m - 2)x - (m^2 - 3m - 2)t}{K_0} \right] \\
&\times \cosh \gamma_0[\theta],
\end{align*}
\]

where \( \gamma_0 = \frac{m}{(m-1)^2} \), \( K_0^2 = \frac{2}{m} + 3 - m \), \( f(x) = c_6 \exp \left[ \frac{\sqrt{2}K(m^3 - 3m^2 + m - 1)x}{K_0(m-1)^2} \right] \), \( \theta = K(x - t) \), the symbols \( K, m, q_1, q_2 \) and \( q_3 \) all are being arbitrary constants as above, however, here also \( m \) must never be 0 or 1.

IV. VALIDITY OF THE COSMOLOGICAL MODELS: A SPECIAL CASE STUDY

As mentioned in the previous Sec. 3, we are now performing a study regarding physical and geometrical properties of the model in Eq. (50) under subcase (b-1). One can observe that if we take \( m = 0 \) or 1, the values of the constants diverse to infinity. For this reason we have purposely skipped the Case (b-2) as this prescription represents a non-realistic model.

The expressions for energy density \( \rho \), the pressure \( p \) and magnetic permeability \( \mu \) for this model as provided in the equation set (50), are given by

\[
\rho(x, t) = \frac{2\sqrt{2}\gamma_0^2 K^2}{xq_1^2 K_0^2} \exp \left[ \frac{\sqrt{2}K(m^2 - 3m - 2)t - (2m^2 - 5m^2 + m - 2)x}{K_0(m-1)^2} \right] \times \left( \sqrt{2}(3 + 4m - m^3) + K_0(1 + 2m + 3m^2) \tan[\theta] \right) \cos^{-2}\gamma_0[\theta],
\]

\[
p(x, t) = \frac{2\sqrt{2}\gamma_0^2 K^2}{xq_1^2 K_0^2} \exp \left[ \frac{\sqrt{2}K(m^2 - 3m - 2)t - (2m^2 - 5m^2 + m - 2)x}{K_0(m-1)^2} \right] \times \left( \sqrt{2}(m^3 - 2m^2 - 2m + 1) + K_0(1 - 2m - m^2) \tan[\theta] \right) \times \cos^{-2}\gamma_0[\theta],
\]

\[
\frac{F_{12}(x)}{\mu(x,t)} = \frac{4\sqrt{2}m^2 q_1^2 K^2}{x(m-1)^4 K_0^2 \cos^2(2m-2m)\gamma_0[\theta]} \times \left( \sqrt{2}(m^3 - 2m^2 - 2m - 1) - K_0(m^2 + 1) \tan[\theta] \right),
\]

where \( \gamma_0 = \frac{m}{(m-1)^2} \), \( \theta = K(x - t) \) and \( F_{12}(x) \) is an arbitrary function of the variable \( x \).
In Fig. 1 we have drawn the behaviour for $p$ and $\rho$ which show the expected evolutionary features of the universe.

By using the expressions of density (Eq. 52) and pressure (Eq. 53) we also draw plot for $p + \rho$ in Fig. 2. This figure indicates that the null energy condition (i.e. $\rho + p \geq 0$) is obeyed by the system in the early time, however violates at the later stage which supports a deceleration to acceleration feature of the universe.
The volume element is
\[ V = q_1^2 q_2 q_3 \times \exp \left[ \sqrt{2} K \left( \frac{(2 + 5 m + 2 m^2 - m^3) t - (2 - m + 6 m^2 - 3 m^3) x}{(m - 1)^2 K_0} \right) \right] \times \cos^{(2m-2)\gamma_0}[\theta]. \] (55)

The expansion scalar, which determines the volume behavior of the fluid content, is given by
\[ \Theta = (2m+1)\gamma_0 K \left( \frac{\sqrt{2} \tan[\theta]-K_0}{\sqrt{2} q_1 \cos^{\gamma_0}[\theta]} \right) \exp \left[ K \left( \frac{(m^2-3m-2)t-(2m^3-5m^2+6m-2)x}{\sqrt{2} K_0 (m-1)^2} \right) \right]. \] (56)

The non-vanishing components of the shear tensor, \( \sigma^j_i \), are
\[ \begin{cases}
 \sigma^1_1 = \left( \frac{1}{1+2m} - \frac{1}{3} \right) \Theta, \\
 \sigma^2_2 = \frac{1}{3+6m} \left( \frac{(2+7m+5m^2-2m^3)-\sqrt{2}(m^2-4m)K_0 \tan[\theta]}{(m^2-3m-2)-\sqrt{2}mK_0 \tan[\theta]} \right) \Theta, \\
 \sigma^3_3 = -\left( \sigma^1_1 + \sigma^2_2 \right).
\end{cases} \] (57)

The shear scalar is
\[ \sigma^2 = \frac{m^2-3m-2}{6(1+2m)^2} \left( \frac{\delta_0 + \delta_1 \cos[\theta] - 2\sqrt{2}m(m^2+m+1)K_0 \sin[\theta]}{(m^2-3m-2) \cos[\theta] - \sqrt{2}mK_0 \sin[\theta]} \right) \Theta^2, \] (58)

where \( \delta_0 = 4m^4 - 12m^3 - 5m^2 + 9m - 2 \) and \( \delta_1 = 4m^4 - 16m^3 + 3m^2 - 7m - 2 \).

The non-vanishing acceleration components and the non-vanishing rotation components are given by
\[ \begin{cases}
 \dot{u}_1 = \frac{K}{\sqrt{2}(m-1)^2 K_0} \left( \delta_2 - \sqrt{2}mK_0 \tan[\theta] \right), \\
 \omega_{41} = -\omega_{14} = \frac{q_1 K (\delta_2 - \sqrt{2}mK_0 \tan[\theta])}{\sqrt{2}(m-1)^2 K_0} \\
 \times \exp \left[ K \left( \frac{\delta_2 x-(m^2-3m-2)t}{\sqrt{2}(m-1)^2 K_0} \right) \right] \cos^{\gamma_0}[\theta],
\end{cases} \] (59)

where \( \delta_2 = 2m^3 - 5m^2 + m - 2. \)
The deceleration parameter is given by [35, 36]

\[
q = \frac{\gamma_0 (1+2m)^3 K^4 \left( \delta_4 + \sqrt{2} m K_0 \tan[\theta] \right)^2}{\frac{4 q_1^2 (1-m)^2 K_0}{(m-1)^2}} \times \exp \left[ \frac{2 \sqrt{2} K \left( \delta_4 t - \delta_2 x \right)}{(m-1)^2 K_0} \right] \cos^{-2-4 \gamma_0 \theta} \]

\[
\times \left[ m^2 - 7 m + 4 + (m^2 - 5 m - 2) \cos[2 \theta] - 2 \sqrt{2} m K_0 \sin[2 \theta] \right],
\]

where \( \delta_4 = m^2 - 3 m - 2 \).

The deceleration parameter \( q \) is plotted in Fig. 3 which interestingly indicates a change over from positive \( q \) to negative \( q \) with evolution of the universe i.e. in physical sense from deceleration to accelerating universe. Therefore, a close observation of the figure actually reveals two particular features: (i) there is a flip-flop which indicates a slow rolling down of the phase of universe from deceleration to acceleration, and (ii) the phase of acceleration from deceleration has been started from around \( t = 0.29 \) Gyr. In the present epoch of an accelerating universe, \( q \) lies near \(-0.50 \pm 0.05\) \[41\] \[44\]. From our model, we can recover \( q = -0.5 \) for \( t = 0.244 \) Gyr when deceleration to acceleration occurs whereas we got \( q = -0.5 \) at \( t = 0.29 \) Gyr after fine tuning it. However, this data for time seems very low one as literature \[45\] \[50\] suggests a probable much higher value for \( t \) as \( \sim 6 \) Gyr.

V. CONCLUSION

In the present study, we perform the Lie symmetry analysis under the Einstein’s general relativistic background. Construction of the accelerating universe with perfect fluid and electromagnetic field has been done in plane symmetric spacetime in connection to Lie point symmetries and optimal system of one dimensional subalgebras. The similarity reductions and a class of exact solutions are obtained.
Some physically interesting and viable features of the present investigation are as follows:

1) The present study reveals the electromagnetic energy of inhomogeneous universe. $F_{12}$ is the non-vanishing component of electromagnetic field tensor. To get a deterministic solution, it is assumed that the free gravitational field is Petrov type-II non-degenerate. The electromagnetic field tensor ($F_{12}$) is found to be positive and increasing function of time.

2) Among the models presented in Sect. 3 only the case studied in Sect. 4 is found to be interesting with temporial behaviour as far as plots and data are concerned. Other models are with unrealistic physical features having either positive density and volume decreasing or volume increasing but density is negative.

3) The deceleration parameter $q$ as plotted in Fig. 3 interestingly indicates a change over from positive $q$ to negative $q$ with evolution of the universe i.e. from deceleration to accelerating universe. From our model, we obtain presently accepted numerical value of $q$ as $-0.5$ for $t = 0.24$ Gyr. However, this value of age seems very low with respect to $t \sim 6$ Gyr as available in literature.

As a final comment we would like to put our overall observations of the present study as follows: qualitatively (see Figs. 1-3) the model under plane-symmetric Einstein-Maxwell spacetime is very promising though quantitative result ($q$ from Fig. 3) seems does not fit for the observed data. This readily indicates that either the analysis under plane symmetric spacetime is not fully compatible with the observable universe or probably we have missed some of the threads in our whole consideration which are responsible to make the analysis partially compatible.

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