Private Stochastic Convex Optimization: Efficient Algorithms for Non-smooth Objectives

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Abstract

In this paper, we revisit the problem of private stochastic convex optimization. We propose an algorithm, based on noisy mirror descent, which achieves optimal rates up to a logarithmic factor, both in terms of statistical complexity and number of queries to a first-order stochastic oracle. Unlike prior work, we do not require Lipschitz continuity of stochastic gradients to achieve optimal rates. Our algorithm generalizes beyond the Euclidean setting and yields anytime utility and privacy guarantees.

1 Introduction

Modern machine learning systems often leverage data that are generated ubiquitously and seamlessly through devices such as smartphones, cameras, microphones, or user’s weblogs, transaction logs, social media, etc. Much of this data is private, and releasing models trained on such data without serious privacy considerations can reveal sensitive information (Narayanan and Shmatikov, 2008; Sweeney, 1997). Consequently, much emphasis has been placed in recent years on machine learning under the constraints of a robust privacy guarantee. One such notion that has emerged as a de facto standard is that of differential privacy.

Informally, differential privacy provides a quantitative assessment of how different are the outputs of a randomized algorithm when fed two very similar inputs. If small changes in the input do not manifest as drastically different outputs, then it is hard to discern much information about the inputs solely based on the outputs of the algorithm. In the context of machine learning, this implies that if the learning algorithm is not overly sensitive to any single datum in the training set, then releasing the trained model should preserve the privacy of the training data. This requirement, apriori, seems compatible with the goal of learning, which is to find a model that generalizes well on the population and does not overfit to the given training sample. It seems reasonable then to argue that privacy is not necessarily at odds with generalization, especially when large training sets are available.

We take the following stochastic optimization view of machine learning, where the goal is to find a predictor that minimizes the expected loss (aka risk)

$$\min_{w \in W} F(w) = \mathbb{E}_{z \sim D}[f(w, z)]$$  

based on i.i.d. samples from the source distribution $D$, and full knowledge of the instantaneous objective function $f(\cdot, \cdot)$ and the hypothesis class $W$. We are particularly interested in convex learning problems where the hypothesis class is a convex set and the loss function $f(\cdot, z)$ is a convex function in the first argument for all $z \in Z$. We seek a learning algorithm that uses the smallest
possible number of samples and the least runtime and returns \( \tilde{w} \) such that \( F(\tilde{w}) \leq \inf_{w \in W} F(w) + \alpha \), for a user specified \( \alpha > 0 \), while guaranteeing differential privacy (see Section 2.1 for a formal definition).

A natural approach to solving Problem 1 is sample average approximation (SAA), or empirical risk minimization (ERM), where we instead minimize an empirical approximation of the objective based on the i.i.d. sample. Empirical risk minimization for convex learning problems has been studied in the context of differential privacy by several researchers including Bassily et al. (2019) who give statistically efficient algorithms with oracle complexity matching that of optimal non-private ERM.

An alternative approach to solving Problem 1 is stochastic approximation (SA), wherein rather than form an approximation of the objective, the goal is to directly minimize the true risk. The learning algorithm is an iterative algorithm that processes a single sample from the population in each iteration to perform an update. Stochastic gradient descent (SGD), for instance, is a classic SA algorithm. Recent work of Feldman et al. (2019) gives optimal rates for convex learning problems (Problem 1) using stochastic approximation for smooth loss functions; however, they leave open the question of optimal rates for non-smooth convex learning problems which include a large class of learning problems, including, for example, support vector machines. In this paper, we focus on non-smooth convex learning problems and give private learning algorithms that are provably optimal. Furthermore, we extend the results from Feldman et al. (2019) for smooth losses.

Our main contributions in this paper are as follows.

1. We resolve an open question from Feldman et al. (2019), and give private learning algorithms for non-smooth convex problems. The proposed algorithm achieves optimal rates in linear time.

2. The proposed algorithms are instances of stochastic approximation algorithms, which, unlike Feldman et al. (2019) do not rely on mini-batching.

3. We extend the results of Feldman et al. (2019) for smooth losses to \( \ell_p/\ell_q \) learning problems where the source domain is bounded in \( \ell_p \) norm and the stochastic gradients of the the risk \( \nabla f(w, z) \) are bounded in \( \ell_q \) norm.

4. We give anytime guarantees on the utility and privacy of our proposed algorithms.

2 Notation and Preliminaries

We recall the general learning setup of Vapnik (2013) and how it reduces to a stochastic convex optimization problem. Let \( \mathcal{Z} \) be the sample space and let \( \mathcal{D} \) be an unknown distribution over \( \mathcal{Z} \). A learner receives samples \( z_1, z_2, \cdots, z_n \) drawn identically and independently (i.i.d) from the unknown distribution \( \mathcal{D} \). The samples \( z_i \) can either correspond to (feature, label) tuples as in supervised learning, or it can just be features for an unsupervised learning task. We assume that loss \( f : \mathcal{W} \times \mathcal{Z} \to \mathbb{R} \) is a convex function in its first argument \( w \) and the constraint \( \mathcal{W} \) is a convex set. The loss \( f \) can be squared loss for regression, a convex surrogate of the misclassification loss for classification, or a reconstruction loss for unsupervised problems. With these regularity conditions, given \( n \) samples, the goal is to minimize the population risk (Problem 1).

In addition, we assume that convex hypothesis class \( \mathcal{C} \) is a bounded set in \( \mathbb{R}^d \), which is equipped with norm \( \| \cdot \| \). The dual space of \( (\mathbb{R}^d, \| \cdot \|) \) is the set of all linear functionals over it. The dual norm, denoted by \( \| \cdot \|_* \), is defined as \( \|h\|_* = \min_{\|u\| \leq 1} h(w) \), where \( h \) is a element of the dual space
Furthermore, we assume that \( f(\cdot, z) \) is \( L \)-lipschitz with respect to the dual norm \( \| \cdot \|_\ast \).
i.e. \( \| f(w_1, z) - f(w_2, z) \| \leq L \| w_1 - w_2 \| \) for all \( z \). This implies that its sub-gradients with respect to \( w \), denoted by \( \nabla f(w, z) \) are bounded as \( \| \nabla f(w, z) \|_\ast \leq L \) \( \forall w \in \mathcal{C}, z \in \mathcal{Z} \). A popular instance is the \( \ell_p/\ell_q \) setup, which considers \( \ell_p \) norm in primal space and the corresponding \( \ell_q \) norm in the dual space such that \( \frac{1}{p} + \frac{1}{q} = 1 \). A function \( f(\cdot, \cdot) \) is \( \beta \)-smooth in its first argument \( w \) if \( \| \nabla f(w_1, z) - \nabla f(w_2, z) \|_\ast \leq \beta \| w_1 - w_2 \| \) \( \forall w_1, w_2 \in \mathcal{W}, z \in \mathcal{X} \). A function \( f \) is \( \lambda \)-strongly convex if \( f(w_2) \geq f(w_1) + \langle \nabla f(w_1), w_2 - w_1 \rangle + \frac{\lambda}{2} \| w_1 - w_2 \|^2 \) \( \forall w_1, w_2 \in \mathcal{W} \).

We now introduce more notation which we will use in the subsequent sections. Various gradient based methods can be viewed as instances of mirror descent method which uses the duality structure in the vector space. Central to these methods is a potential function used to identify primal and dual spaces. The \( \beta \)-smoothness of \( f \) is equivalent to \( \lambda \)-strong convexity of \( \beta f \). Moreover, \( \lambda \) is \( \beta \)-smoothness of \( \beta f \).

### 2.1 Differential privacy

As discussed in section 1, differential privacy (DP) is, by now, the de-facto standard notion of data privacy in statistical data analysis. Our goal is to solve the stochastic convex optimization problem (Problem 1) with the constraint that the output is differentially private. We give the definition of \((\epsilon, \delta)\)-differential privacy, introduced in Dwork et al. (2006).

**Definition 2.1** \((\epsilon, \delta)\)-differential privacy. An algorithm \( A \) satisfies \((\epsilon, \delta)\)-differential privacy if given two datasets \( D \) and \( D' \), differing in only one data point, for any measurable event \( S \in \text{Range}(A) \), satisfies

\[
\mathbb{P}(A(D) \in S) \leq e^{\epsilon} \mathbb{P}(A(D') \in S) + \delta
\]

If \( \delta = 0 \), we say that the algorithm satisfies \( \epsilon \)-differential privacy (or pure differential privacy). Recently, there has been a lot of work which propose relaxations to the definition of privacy to get finer control on the privacy loss variable. These include notions like Rényi Differential privacy (RDP) (Mironov, 2017), zero concentrated differential privacy (zCDP) (Dwork and Rothblum, 2018; Bun and Steinke, 2016), truncated concentrated differential privacy (tCDP) (Bun et al., 2018), etc. All these notions are based on measuring Rényi Divergence between the probability distributions of output of the algorithm. We first give the definition of Rényi divergence between distributions.

**Definition 2.2** (Rényi Divergence (Rényi et al., 1961)). Given two probability distributions \( \mu \) and \( \nu \) such that \( \mu \ll \nu \) i.e. \( \mu \) is absolutely continuous with respect to \( \nu \). Let \( f_\mu \) and \( f_\nu \) denote their probability densities. The Rényi Divergence of order \( \alpha, 1 < \alpha < \infty \) between \( \mu \) and \( \nu \) is defined as

\[
D_\alpha(\mu \| \nu) = \frac{1}{\alpha - 1} \ln \left( \int_{\mathcal{X}} \left( \frac{f_\mu(x)}{f_\nu(x)} \right)^\alpha f_\nu(x) dx \right)
\]

If \( \mu \not\ll \nu \), then \( D_\alpha(\mu \| \nu) \) is defined to be \( \infty \). Moreover, \( D_\alpha(\mu \| \nu) \) at \( \alpha = 1 \) and \( \alpha \to \infty \) are defined by continuity.

In this paper, we work with the notion of Rényi Differential privacy, defined as follows.

**Definition 2.3** \((\alpha, \epsilon)\)-Rényi Differential privacy (Mironov, 2017). Let \( A : \mathcal{D} \to \mathbb{R} \) be a randomized mechanism, and let \( P_A : \mathcal{D} \to \mathbb{P}(\mathbb{R}) \), denotes the probability distribution over the range of \( A \). We say
that $A$ satisfies $(\alpha, \epsilon)$-Rényi Differential privacy (RDP) if for any given two neighbouring datasets $D, D' \in D$, \[ D\alpha(P_A(D)||P_A(D')) \leq \epsilon. \]

We now state how $(\alpha, \epsilon)$-RDP guarantees relate to $(\epsilon, \delta)$-DP guarantee.

**Lemma 2.4 (Mironov (2017)).** If an algorithm satisfies $(\alpha, \epsilon)$-RDP, then, for any $\delta > 0$, it satisfies $(\alpha, \epsilon + \log(1/\delta)/\alpha - 1)$-DP. Taking $\alpha \rightarrow \infty$ allows us to take $\delta \rightarrow 0$ which shows that $(\infty, \epsilon)$-RDP is $(\epsilon, 0)$-DP.

Central to our privacy analysis is the shift-reduction lemma of Feldman et al. (2018). To state this result, we need to define two terms: shifted Rényi Divergence $D^z_{\alpha}(\mu||\nu)$ and $R_{\alpha}(\xi, a)$. For a normed space $(X, ||\cdot||)$, shifted Rényi Divergence of order $z$ between two distributions $\mu$ and $\nu$ is defined as
\[ D^z_{\alpha}(\mu||\nu) = \inf_{\gamma: W^z_{\infty}(\gamma, \mu) \leq z} D_{\alpha}(\gamma||\nu) \text{ where } W^z_{\infty}(\gamma, \mu) := \inf_{\gamma \sim C(\gamma, \mu)} \text{ess sup}_{X,Y \sim \gamma} ||X - Y||. \]

Furthermore, define $R_{\alpha}(z, a) := \sup_{||u|| \leq a} D_{\alpha}(z * u||z)$. We now give a key result of Feldman et al. (2018).

**Lemma 2.5 (Shift reduction lemma (Feldman et al., 2018)).** For probability distributions $\mu, \nu$ and $\xi$ in a Banach space $(X, ||\cdot||)$, for real numbers $a, z \geq 0$,
\[ D^z_{\alpha}(\mu * \xi||\nu * \xi) \leq D^{z+a}_{\alpha}(\mu||\nu) + R_{\alpha}(\xi, a). \]

## 3 Related Work

In convex learning and optimization, the following four classes of functions are widely studied: $L$-Lipschitz convex, $\beta$-smooth and convex, $\lambda$-strongly convex, and $\beta$-smooth and $\lambda$-strongly convex functions. In the computational framework of first order stochastic oracle, algorithms with optimal oracle complexity for all these classes of functions have long been known (Agarwal et al., 2009).

However, the landscape of known results changes with the additional constraint of privacy. We can trace two approaches to solving the private version of Problem 1. The first is private ERM (Chaudhuri et al., 2011; Bassily et al., 2014; Feldman et al., 2018; Bassily et al., 2019) and the second is private stochastic approximation (Feldman et al., 2019). Chaudhuri et al. (2011) begin the study of private ERM by constructing algorithms based on output perturbation and objective perturbation. Under the assumption that the stochastic gradients are $\beta$-Lipschitz continuous, the output perturbation bounds achieve excess population risk of $O(LD \max(1/\sqrt{n}, d/(ne), d\sqrt{\beta}/(n^{2/3}e)))$, where $L$ is the Lipschitz constant of the loss function and $D$ measures the complexity of the hypothesis class. The objective perturbation bounds have a similar form. Bassily et al. (2014) showed tight upper and lower bounds for the excess empirical risk. They also showed bounds for the excess population risk when the loss function does not have Lipschitz continuous gradients, achieving a rate of $O(d^{1/4}/(\sqrt{me}))$. Their techniques appeal to uniform convergence i.e. bounding $\sup_{w \in W} F(w) - \hat{F}(w)$, and convert the guarantees on excess empirical risk to get a bound on the excess population risk. These guarantees, however, were sub-optimal. Bassily et al. (2019) improved these to get optimal bounds on excess population risk, leveraging connections between algorithmic stability and generalization. The algorithms given by Bassily et al. (2019) are sample efficient but their runtimes are superlinear (in the number of samples), whereas the non-private counterparts run in linear time. In
a follow-up work, Fieldman et al. (2019) improved the runtime of some of these algorithms without sacrificing statistical efficiency, however, the authors require that the stochastic gradients are Lipschitz continuous. We refer the reader to Table 1 for a comparison of bounds on the excess population risk by different works discussed above. Essentially, the statistical complexity of private stochastic convex optimization has been resolved, however, some questions about computational efficiency still remain open. We begin with a discussion of different settings for the population loss in subsequent paragraphs, describe what is already known, and what are the avenues for improvement.

Non-smooth Lipschitz Convex. For the class of $L$-Lipschitz convex functions, Bassily et al. (2014) improved upon Chaudhuri et al. (2011) and gave optimal bounds on excess empirical risk of $O \left( \frac{d}{cn} \right)$. They then appeal to uniform convergence to convert the guarantees on excess empirical risk to get an excess population risk of $O \left( \max \left( \frac{d^{1/4}}{\sqrt{n}}, \frac{\sqrt{d}}{cn} \right) \right)$. This is sub-optimal and was very recently improved by Bassily et al. (2019) using connections between algorithmic stability and generalization to get $O \left( \max \left( \frac{1}{\sqrt{n}}, \frac{\sqrt{d}}{cn} \right) \right)$. This is optimal since $\frac{1}{\sqrt{n}}$ is the optimal excess risk without privacy constraints, and $\frac{\sqrt{d}}{cn}$ is the optimal excess risk when the data distribution is the empirical distribution. This resolves the statistical complexity of private convex learning and shows that in various regimes of interest, like when $d = \Theta(n)$ and $\epsilon = \Theta(1)$, the constraint of privacy has no effect on utility. However, the algorithm proposed by Bassily et al. (2019) is based on Moreau smoothing and proximal methods and requires $O(n^5)$ stochastic gradient computations. This rate vastly exceeds the gradient computations needed for non-private stochastic convex optimization which are of the order $O(n)$. The computational complexity was improved by Feldman et al. (2019) to $O(n^2)$ by using a regularized ERM algorithm that runs in phases and after each phase, noise is added to the solution (output perturbation) and used as regularization for the next phase. However, they left open the question of giving a linear time algorithm which we provide in this paper.

Smooth Lipschitz Convex. For $\beta$-smooth convex $L$-Lipschitz functions, Bassily et al. (2019) give an algorithm with optimal bounds on excess risk of $O(LD \max \left( \frac{1}{\sqrt{n}}, \frac{\sqrt{d}}{n} \right))$ queries to the stochastic gradient oracle. This, again, was improved in a later work of Feldman et al. (2019) to $O(n)$ stochastic gradient queries. Note that even for non-private stochastic optimization $O(n)$ stochastic gradient computations are necessary, so Feldman et al. (2019) achieve optimal statistical and oracle complexity for the smooth Lipschitz convex functions. The algorithm they present is an instance of a single-pass noisy SGD, and the guarantees hold for the last iterate. However, there are certain important differences between the noisy SGD algorithm of Feldman et al. (2019) and SGD algorithm that is optimal in a non-private setting. First, the noisy SGD algorithm of Feldman et al. (2019) utilizes minibatches of data to construct stochastic gradients. While mini-batching is not a computational bottleneck, we know that for non-private stochastic convex optimization, a single pass of streaming stochastic gradient descent suffices to achieve optimal rates. It is, therefore, an interesting question to ask whether the mini-batching is indeed necessary when learning with privacy constraints, or if a streaming one-pass SGD can also yield optimal rates. Second, the noisy SGD algorithm of Feldman et al. (2019) requires the knowledge of the time horizon, $T$, to set the step sizes and mini-batch sizes appropriately. This is not desirable, especially in a streaming setting – since the guarantees do not hold if we pick the horizon larger than the length of the stream, whereas picking the horizon smaller than the stream length would require restarting the algorithm and discarding the progress made. Ideally, we require anytime guarantees on utility and privacy, while also ensuring that the privacy parameter improves with more data points (i.e., as more people participate). None of the previous works admit anytime guarantees, as far as we
know, our algorithm is the first of the kind with anytime privacy and utility guarantees.

**(Smooth and Non-smooth) Lipschitz Strongly Convex.** For $L$-Lipschitz $\lambda$-strongly convex functions, Bassily et al. (2014) gave an algorithm with optimal excess empirical risk which is in $O\left(\frac{\log(n)\beta^2}{\lambda n^2}\right)$. However, as in the non-strongly convex case, the corresponding excess population risk is $O\left(\frac{L^2\sqrt{d}\log(n/\delta)^2}{\lambda n^2}\right)$, which is sub-optimal. Feldman et al. (2019) proposed a general reduction by using their non-strongly convex algorithms, which runs in phases and uses the output of the previous phase as the initialization for the next phase. They show that this achieve optimal rates, in $O(n)$ time for smooth functions, but $O(n^2)$ for non-smooth functions. We believe that our algorithmic framework can handle strongly convex functions; we leave this for future work.

## 4 Algorithm and Main Results

Our algorithm builds on the popular algorithmic framework of Follow the Regularized Leader (FTRL), also known as Lazy Online Mirror Descent (e.g. Chapter 5 of Hazan et al. (2016)). In particular, we utilize a noisy variant, which we refer to as Private FTRL, with the following updates:

$$w_{t+1} = \arg \min_{w \in \mathcal{W}} \sum_{j=1}^{t} \langle \nabla f(w_j, z_j) - \xi_j, w \rangle + \Psi_{t+1}(w). \quad (2)$$

Here, $\xi_j$ is zero-mean noise that imparts privacy to the iterates and potential function, $\Psi_t$, is the regularizer at time $t$. We choose $\Psi_t(\cdot) = \frac{1}{\eta_t}\Psi(\cdot)$, where $\Psi(\cdot)$ is a 1-strongly convex function with respect to $\| \cdot \|$ and $\eta_t \in \mathbb{R}_+$ is the step-size at time $t$. Let $g_t = \nabla f(w_t, z_t) - \xi_t$ denote the noisy gradient at time $t$ and let $G_t = \sum_{j=1}^{t} g_j$ be the accumulated noisy gradients until time $t$. We note that since $\xi_t$ is zero-mean and $\nabla f(w, z_t)$ is an unbiased estimate of the gradient, it holds that $\mathbb{E}[g_t] = \nabla F(w)$. We denote $\Phi_t(Y) = (\Psi_t + I_{\mathcal{W}})^*(Y)$. With the notation above, we can write the FTRL update in Equation 2 as $w_{t+1} = \nabla \Phi_{t+1}(-G_t)$. The pseudo-code for the proposed Private FTRL method is given in Algorithm 1. Some remarks are in order.

### Algorithm 1 Private FTRL

**Input:** Step size schedule $\{\eta_t\}_t$, noise sequence $\{\xi_t\}_t$, stream of data points $\{z_t\}_t$

**Output:** Private last iterate $w_T$

1. $w_1 = \nabla \Phi_1(0)$
2. for $t \geq 2$
3. $G_t = G_{t-1} + \nabla f(w_t, z_t) - \xi_t$
4. $w_{t+1} = \nabla \Phi_{t+1}(-G_t)$
5. end for

First we note that at the $t$-th iterate we have used $t$ stochastic gradients and so there is a direct correspondence between number of samples from $\mathcal{D}$ and the number of iterations of the algorithm. In particular the $T$-th iterate uses $T = n$ stochastic gradient queries.

Second, note that the computational complexity of the proposed algorithm is determined by the efficiency of solving Problem 2. This involves computing the gradient of the conjugate of $\Psi$ and projection onto the convex set $\mathcal{W}$. In many cases, the gradient of the conjugate of $\Psi$ has a closed form, for example, for $\Psi = \frac{1}{2}\| \cdot \|^2_2$. In general, the computational complexity of Algorithm 1 is the same as that of Online Mirror Descent (OMD) with the same potential function.
| Function class                        | Reference                  | Excess population risk | Stochastic gradient oracle complexity | Method                                                                 |
|--------------------------------------|----------------------------|------------------------|---------------------------------------|------------------------------------------------------------------------|
| L-Lipschitz, β-smooth Convex          | Bassily et al. (2019)      | $O \left( LD \max \left( \frac{1}{\sqrt{n}}, \frac{\sqrt{d}}{en} \right) \right)$ | $O \left( \max \left( \min \left( n, \frac{en^2}{d \log(1/\delta)} \right), \min \left( n^{3/2}, \frac{\sqrt{d} \epsilon^2 n}{\sqrt{d \log(1/\delta)}} \right) \right) \right)$ | Noisy SGD+sampling, $\beta = O \left( \frac{1}{d} \min \left( \sqrt{n}, \frac{en}{d \log(1/\delta)} \right) \right)$ |
|                                      | Feldman et al. (2019)      | $O \left( LD \max \left( \frac{1}{\sqrt{n}}, \frac{\sqrt{d}}{en} \right) \right)$ | $O \left( \frac{n^2}{n+dn^2} \right)$ | Streaming mini-batched Noisy SGD, $\beta = O \left( LD \min(\sqrt{n}, \epsilon n/\sqrt{d}) \right)$ |
|                                      | Ours                      | $O \left( LD \max \left( \frac{1}{\sqrt{n}}, \frac{\sqrt{d}}{en} \right) \right)$ | $O(n)$ | Phased SGD, $\beta = O(L/D \min(\sqrt{n}, \sqrt{d}/\epsilon))$ |
| Non-smooth, L-Lipschitz Convex       | Bassily et al. (2014)      | $O \left( \frac{L^2 D^2 d^{3/4}}{\sqrt{en}} \right)$ | $O(n^2 \log (1/\delta))$ | Noisy SGD+sampling |
|                                      | Bassily et al. (2019)      | $O \left( LD \max \left( \frac{1}{\sqrt{n}}, \frac{\sqrt{d}}{en} \right) \right)$ | $O(n^5)$ | Proximal Noisy SGD+sampling |
|                                      | Feldman et al. (2019)      | $O \left( LD \max \left( \frac{1}{\sqrt{n}}, \frac{\sqrt{d}}{en} \right) \right)$ | $O(n^2)$ | Phased-ERM |
|                                      | Ours                      | $O \left( LD \max \left( \frac{1}{\sqrt{n}}, \frac{\sqrt{d}}{en} \right) \right)$ | $O(n)$ | Private FTRL |
| Non-Smooth, L-Lipschitz λ-strongly Convex | Bassily et al. (2014) | $O \left( \frac{L^2 \sqrt{d}}{\lambda n \epsilon} \right)$ | $O(n^2)$ | Noisy SGD+sampling |
|                                      | Feldman et al. (2019)      | $O \left( \frac{L^2 \sqrt{d}}{\lambda n} + \frac{d}{\epsilon n^2} \right)$ | $O(n^2)$ | Restarted streaming mini-batched Noisy SGD, $\frac{d}{\epsilon} = O(\max(\sqrt{n}, \sqrt{d}/\epsilon))$ |
|                                      | Ours                      | $O \left( \frac{L^2 \sqrt{d}}{\lambda n} + \frac{d}{\epsilon n^2} \right)$ | $O(n)$ | Restarted Phased-ERM |

Table 1: Excess population risk, oracle complexity and the method for four classes of convex optimization. Our method achieves nearly optimal excess population risk and oracle complexity for smooth and non-smooth Lipschitz convex functions in linear time.
Third, we need to pick the potential function to be strongly convex with respect to a norm in which the stochastic gradients are bounded. This is because we require the gradients to be stable with respect to some norm so to ensure the privacy of our algorithm. Our main results, therefore, focus only on potential functions that are strongly convex with respect to $\ell_p$ norms for $1 < p \leq 2$. The case when $p > 2$ is easily reducible to the case when $p = 2$. The challenge when working with $p = 1$, or with general norms, is rooted in how to guarantee privacy and is left as future work. For the rest of the section we assume that any stochastic gradient $\nabla f(w, z)$ is bounded in norm by $L$ i.e., $\|\nabla f(w, z)\|_q \leq L$, where $q$ is the conjugate of $p$.

Next, we describe our main results. These include the privacy and utility guarantees for the last iterate of Private FTRL for learning problems when the gradients are bounded in (a) $\ell_2$-norm, or in (b) $\ell_q$-norm, and (c) anytime guarantees for Private FTRL.

**Stochastic Gradients Bounded in $\ell_2$-norm.** Much of the prior work including that of Chaudhuri et al. (2011); Bassily et al. (2014); Feldman et al. (2018); Bassily et al. (2019); Feldman et al. (2019) focuses on this setting. Our approach here is most similar, in spirit, to that of Feldman et al. (2019). The key idea in Feldman et al. (2019) is that if the gradient mapping is contractive and the gradients are sufficiently stable (bounded in norm by a small Lipschitz constant), then the contraction will amplify the privacy induced by perturbing the gradients using Gaussian noise. Feldman et al. (2019) formalize this by studying the Rényi divergence between the last iterate of the algorithm and a potential last iterate that would be generated on a neighboring sequence of data points. While we use similar tools as Feldman et al. (2019), we make the following key observation – contraction is not necessary for the amplification of privacy. Instead, we can achieve privacy amplification by utilizing the noise added in earlier iterations of the algorithm to also privatize the later iterates. To see this, write the update for the $t$-th iterate as $w_{t+1} = \nabla \Phi_{t+1}(\sum_{j=1}^{t} \nabla(w_j, z_j) - \xi_j)$. Clearly, the noise added in the first iteration, $\xi_1$, also contributes to the privacy of the $t$-th iterate. We defer further details on the proof techniques to Section 5. We next present our first main result of the section.

**Theorem 4.1.** Assume that the base potential function, $\Psi$, is 1-strongly convex with respect to the $\| \cdot \|_2$-norm and that the stochastic gradients are bounded, in norm, by some constant $L$. For any $\alpha > 1$, given samples $\xi_j \sim N(0, \sigma^2 I)$, where $\sigma = \frac{2\alpha L}{T^{1/2} \alpha D}$, the iterate $w_T$ of Algorithm 1 is $(\alpha, \epsilon)$-RDP. Further, if $\eta_t$ is a fixed step-size schedule proportional to $\frac{1}{\sqrt{T}}$ and $\Psi(w_1) - \Psi(w^\star) \leq D^2$, where $w^\star$ is a minimizer of $F(w)$, then Algorithm 1 outputs $w_T$ such that

$$
\mathbb{E}[F(w_T)] - F(w^\star) \leq O\left(\frac{L D \log(T)}{\sqrt{T}} + \frac{\sqrt{\alpha d D}}{T \sqrt{\epsilon}}\right).
$$

If $\eta_t$ is a decreasing step size schedule proportional to $\frac{1}{\sqrt{t}}$ and if for all $u, v \in \mathcal{W}$ it holds that $\Psi(u) - \Psi(v) \leq D^2$, then Algorithm 1 outputs $w_T$ such that

$$
\mathbb{E}[F(w_T)] - F(w^\star) \leq O\left(\frac{L D \log(T)}{\sqrt{T}} + \frac{\sqrt{\alpha d D}}{T \sqrt{\epsilon}}\right).
$$

We note that for $\Psi(\cdot) = \frac{1}{2} \| \cdot \|_2^2$, which is the setting studied by Feldman et al. (2019), Theorem 4.1 recovers both the convergence rate and the privacy guarantee of Feldman et al. (2019), at the same computational cost, but without requiring the loss function $f$ to be smooth and also avoiding the mini-batching of data. Further, as in Feldman et al. (2019), whenever $d = O(T)$, our bound matches
(up to poly-logarithmic factors) the information theoretically optimal rate for non-private stochastic convex optimization.

Theorem 4.1 also suggests a trade-off between the privacy and efficiency of our algorithm. If one had access to the true population, then one could solve Problem 1 without revealing anything about any representative sample. For instance, one would simply run projected gradient descent on the convex function $F$ and converge at a rate $\tilde{O}(L D^2/\sqrt{T})$ to a minimizer of $F$. This suggest that as we see more samples from $D$, and we get closer to knowing the true population, our algorithm should decrease the amount of privacy which is being leaked. We can formalize this intuition for the case of $\| \cdot \|_p = \| \cdot \|_2$. Using Theorem 5.4 and the result in Lemma 2.6 of Feldman et al. (2019), we conclude that Algorithm 1 is $(\frac{\sigma^2}{T \sigma^2} + \frac{\log(1/\delta)}{\alpha - 1}, \delta)$-DP. Furthermore, our convergence guarantees (disregarding $D$ and $L$) are in $O(\frac{\log(T)}{\sqrt{t}} + \frac{\sqrt{d}}{\sqrt{T}})$. Therefore, setting $\alpha = \frac{\sigma \sqrt{T}}{s \sqrt{\log(1/\delta)}}$ we can show that Private FTRL is $(\frac{s \sqrt{\log(1/\delta)}}{\sigma \sqrt{T}}, \delta)$-differentially private.

Finally, note that both the convergence rate and the privacy guarantee depend on dimensionality through $\sigma$. In particular setting $\sigma = d^{-1/4}$ equals both terms. With these choices of $\sigma$ and $\alpha$ with the anytime convergence guarantee in Theorem A.4, we show the following result.

**Corollary 4.2.** Assume that the base potential function $\Psi$ is 1-strongly convex with respect to $\| \cdot \|_2$ and that the stochastic gradients are $L$-Lipschitz. Let $\xi_j \sim \mathcal{N}(0, \frac{\sigma^2}{T \sigma^2} I)$. If for all $u, v \in \mathcal{W}$ it holds that $\Psi(u) - \Psi(v) \leq D^2$, then any iterate $w_t$ of Algorithm 1 is $(\frac{2L d^{1/4} \sqrt{\log(1/\delta)}}{\sigma \sqrt{T}}, \delta)$-DP and satisfies

$$\mathbb{E}[F(w_t)] - F(w^*) \leq \tilde{O} \left( \frac{D d^{1/4}}{\sqrt{t}} + \frac{L D}{\sqrt{t}} \right).$$

**Stochastic Gradients Bounded in $\ell_p$-norm.** Our algorithm and proof techniques are flexible enough to extend the results from the $\ell_2$ case to the general $\ell_p$ case, for $p > 1$. The only difference is the way we ensure privacy. Since the geometry of the space has changed to the one induced by the $\ell_p$-norm, we need to adapt the noise added in the same way. In particular, instead of sampling $\xi_t$ according to a Gaussian distribution we sample it according to a generalized Gaussian distribution with pdf proportional to $\exp(-\|\xi\|^q_p/\sigma^q)$. In this case, we show the following guarantee for Algorithm 1.

**Theorem 4.3.** Assume that the base potential function $\Psi$ is 1-strongly convex with respect to $\| \cdot \|_p$ and that the stochastic gradients are $L$-Lipschitz. Sample $\xi_j \sim \frac{\xi_j^q}{2 \sigma^q \Gamma(1/q)} \exp(-\|\xi\|^q_p/\sigma^q)$, where $\sigma = \frac{2L}{T^{1/p} \Gamma(1/q)}$. For $1 < \alpha < \frac{2^p - 1}{2^{p-1} - 1}$, the iterate $w_T$ of Algorithm 1 is $(\alpha, \epsilon)$-RDP. Further, if $\eta_t$ is a fixed step size schedule proportional to $\frac{1}{\sqrt{t}}$ and $\Psi(w^*_1) - \Psi(w^*1) \leq D^2$, where $w^*$ is a minimizer of $F(w)$, then Algorithm 1 outputs $w_T$ such that

$$\mathbb{E}[F(w_T)] - F(w^*) \leq O \left( \frac{L D \log(T)}{\sqrt{T}} + \frac{d^{1/2} D}{T^{p + 1/2} \epsilon^{1/2}} \right).$$

If $\eta_t$ is a decreasing step size schedule proportional to $\frac{1}{\sqrt{t}}$ and for all $u, v \in \mathcal{W}$ it holds that $\Psi(u) - \Psi(v) \leq D^2$, then Algorithm 1 outputs $w_T$ such that

$$\mathbb{E}[F(w_T)] - F(w^*) \leq O \left( \frac{L D \log(T)}{\sqrt{T}} + \frac{d^{1/2} D \log(T)}{T^{p + 1/2} \epsilon^{1/2}} \right).$$
Theorem 4.3 suggests a trade-off between the inherent geometry of the learning problem, the privacy one can guarantee, and the computational and statistical complexity of the algorithm. This is not surprising and we almost recover the result in Theorem 4.1 by plugging $p = q = 2$ if it were not for the condition on $\alpha$. This condition on $\alpha$ is actually significant when transitioning between RDP and DP as we need to set $\alpha \sim \frac{1}{\sqrt{T}}$ if we hope to achieve an $(\epsilon, \delta)$-DP guarantee. We think that the condition on $\alpha$ can be removed, and that it appears simply as an artifact of our analysis.

Anytime Convergence and Privacy Guarantees. Corollary 4.2 gives an anytime guarantee for FTRL, without requiring the knowledge of the time horizon before the start of the algorithm. However, the privacy guarantee in Corollary 4.2 is time-varying. Our next result gives an anytime convergence guarantee for an arbitrary privacy budget.

**Theorem 4.4.** Assume that the base potential function $\Psi$ is 1-strongly convex with respect to $\| \cdot \|_p$ and that the stochastic gradients are $L$-Lipschitz. Let $\xi_i \sim \frac{q}{\sqrt{T}(1/q)} \exp (-\|\xi\|_p^2/\sigma^2)$, where $\sigma = \frac{2L}{\epsilon^2 / q}$ and let $\xi_j = 0$ for all $j > 1$. If for all $u, v \in \mathcal{W}$ it holds that $\Psi(u) - \Psi(v) \leq D^2$, then for any $1 < \alpha < \frac{2q^{q+1}}{2q^{q+1} - 1}$, every iterate $w_t$ of Algorithm 1 is $(\alpha, \epsilon)$-RDP and satisfies

$$\mathbb{E}[F(w_t)] - F(w^*) \leq O \left( \frac{LD \log (t)}{\sqrt{T}} + \frac{d^{2/q} \log (t)}{t \epsilon^{2/q}} \right).$$

We note that the bound in Theorem 4.4 does not match the one in Theorem 4.3. In particular, for $p \geq 2$, the term $1/T^{1/p+1/2}$ got replaced by $1/T$ while the factor of $d^{1/q}$ was replaced by $d^{2/q}$. This change can be explained as follows. When we distribute the noise contribution along each of the stochastic gradients, as in Theorem 4.3, we decrease the variance of noise, which allows for a better dependence on $d$. On the other hand, when all of the noise is concentrated on the first stochastic gradient, as in Theorem 4.4, there is no reduction in the variance which leads to the increased dependence on dimensionality.

We can give any-time guarantees while preserving the rates in Theorem 4.3 and Theorem 4.1 if we utilize the restarting and doubling trick; however, it is not clear if restarting the algorithm is necessary, we leave that for future work. Finally, we note that anytime guarantees hold without a restriction on $\alpha$ when the gradients are bounded in $\ell_2$ norm.

**5 Proof Techniques**

In this section, we describe our proof techniques for the main results of Section 4. We first give the following last iterate guarantee for Algorithm 1 which is crucial for getting the utility guarantee. We note that this result may be of independent interest.

**Theorem 5.1.** Assume that for all $g_i$, it holds $\mathbb{E}[\|g_i\|^2] \leq G^2$ and $\Psi$ is 1-strongly convex with respect to the $\| \cdot \|$. Let $w_1 = \nabla \Phi(0)$ and $\Psi(w_1) - \Psi(w^*) \leq D^2$, where $w^*$ is a minimizer of $F(w)$. Then, after $T$ iterations of FTRL (Algorithm 1) with step size $\eta_t = \frac{D}{G \sqrt{T}}$, we have that

$$\mathbb{E}[F(w_T)] - F(w^*) \leq O \left( \frac{GD \log (T)}{\sqrt{T}} \right).$$

Further, if $\Psi(u) - \Psi(v) \leq D^2$ for all $u, v \in \mathcal{W}$ and we run FTRL with step size $\eta_t = \frac{D}{G \sqrt{T}}$, then

$$\mathbb{E}[F(w_t)] - F(w^*) \leq O \left( \frac{GD \log (t)}{\sqrt{t}} \right),$$

for any $t \in \mathbb{N}$. 

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The proof of Theorem 5.1 follows the ideas for the analysis of FTRL in Abernethy et al. (2015) and Zimmert and Seldin (2018), and combines them with the techniques used to obtain last iterate guarantees for stochastic gradient descent in Shamir and Zhang (2013). Since the analysis is somewhat standard we are going to devote the rest of the section to the privacy analysis.

As discussed in Section 4, the key element in the privacy proof is the privacy amplification by iteration principle of Feldman et al. (2018). When an algorithm accesses data points sequentially, Feldman et al. (2018) showed that the noise added in previous iterations can benefit the later iterations, if the (iteration) update satisfies a certain contraction property. They formalize this via, what they call, contractive noisy iterations which are two sequence of updates with added noise, and proved a general result which showcases how the privacy parameter amplifies. An application of this is optimizing smooth convex functions using SGD, since setting the step size small enough enables contractive updates and privacy results follow as a corollary. A key observation we make is that their general result is established in a setting which is too general for the application i.e the application to SGD holds even without requiring contraction. In particular, Feldman et al. (2018) use contraction to control the drift when both the update function and the sample differs. For SGD, this would mean that at a certain iteration both the previous iterate and the current sample differs for two neighboring data streams. This can never happen for one-pass SGD starting with a fixed initialization. In addition, even in the smooth case, we present an improvement over Feldman et al. (2019) by removing the need to mini-batch or knowing the time horizon by using a different algorithm (FTRL). We start by discussing a key technical result in Feldman et al. (2019), which doesn’t require contraction and how it suffices for our application, and later we discuss how avoid the need for mini-batching.

A sub-result in Feldman et al. (2018) is the so called Shift-Reduction lemma, which relates the role of geometry of the space to Rényi Divergence between probability distributions upon stochastic post-processing (like noise addition). To state their result, we first need to define the notion of shifted Rényi divergence, denoted as $D^\alpha_z(\mu\|\nu)$. For a Banach space $(Z, \|\|)$, $(\alpha, z)$-shifted Rényi Divergence between two distributions $\mu$ and $\nu$ is defined as

$$D^\alpha_z(\mu\|\nu) = \inf_{\gamma \in \mathcal{P}(Z): \mathcal{W}_\infty(\gamma, \mu) \leq z} D_{\alpha}(\gamma\|\nu)$$

where $\mathcal{W}_\infty(\gamma, \mu) := \inf_x \mathbb{E}_{(Y \sim C(\gamma, \mu))} \sup_{(X, Y) \sim \gamma} \|X - Y\|$ is the $\infty$-Wasserstein distance between distributions $\gamma$ and $\mu$ and $C(\gamma, \mu)$ is the set of couplings between distributions $\gamma$ and $\mu$. We now state the shift-reduction lemma of Feldman et al. (2018).

**Lemma 5.2** (Shift-reduction lemma Feldman et al. (2018)). Let $\mu, \nu$ and $\xi$ be probability distributions over a Banach space $(Z, \|\|)$, then for real numbers $a, z \geq 0$, we have

$$D^z_\alpha(\mu * \xi\|\nu * \xi) \leq D^{z+a}_\alpha(\mu\|\nu) + R_\alpha(\xi, a)$$

where $*$ is the convolution operation and $R_\alpha(\xi, a) := \sup_{\|u\| \leq a} D_\alpha(\xi * u\|\xi)$

Note that the two terms in the right hand side of the above equation present a tradeoff in terms of the value of $a$. If $a$ is large, then the first term becomes small since the $\infty$-Wasserstein ball becomes larger, but the second term becomes large because the mean of the random variable $\xi$ is shifted by (large) $a$. Optimizing this tradeoff is central to our analysis, and we now discuss how we apply this lemma to our end.

Our algorithm makes only one pass over the data stream, so each data point is used exactly once. Consider two neighboring data streams $S = \{z_1, z_2, \cdots, z_t\}$ and $S' = \{z_1, z_2, \cdots, z_t', \cdots, z_T\}$, differing in the $t^{th}$ iteration, where $1 \leq t \leq T$. If we consider the last iterates $w_{T+1}$ and $w'_{T+1}$
of data streams $S$ and $S'$, application of shift-reduction lemma and data processing inequality for Rényi Divergence allows us to accumulate all the noise added after the $t^{th}$ iteration, to get the following result.

$$D_\alpha(w_{T+1}||w'_{T+1}) \leq D_\alpha^{j=t+1} \sum_{j=t+1}^{T} R_\alpha(\xi_t, z_t) + \sum_{j=t+1}^{T} R_\alpha(\xi_t, z_t)$$

We now need to control both the terms by appropriately setting $z_j$’s. As a warm-up case, consider choosing $z_j = \frac{s}{t^2}$, where $s = 2L$ is the fixed sensitivity of update at iteration $t$. Note that the iterates $w_{t+1}$ and $w'_{t+1}$ only differ due to the stochastic gradient at the differing sample i.e. $\nabla f(w_t, z_t)$ and $\nabla f(w_t, z'_t)$ respectively. By the $L$-lipschitz assumption, we have $\|\nabla f(w_t, z_t) - \nabla f(w_t, z'_t)\| \leq 2L$, and therefore the first term can be shown to be zero. In the $\ell_2/\ell_2$ case, when the noise $\xi \sim \mathcal{N}(0, \sigma^2 I_d)$ this gives $D_\alpha(w_{T+1}||w'_{T+1}) \leq \frac{2\alpha L^2}{(T-t+1)\sigma^2}$, which at worst would be $\frac{2\alpha L^2}{\sigma^2}$ when $t = T$. Plugging this in to the utility guarantee for the algorithm would give us sub-optimal statistical rates. Feldman et al. (2019) mitigate this issue by using mini batches; this is discussed in more detail after Theorem 4.1. The key issue here is that setting the first term to zero disregards all the noise that is added before the $t$ iterations. Therefore, $z_t = \frac{2L}{\sigma^2}$ perhaps doesn’t optimize the tradeoff and begs the question whether we can set it to something smaller? We show that in case of FTRL, we indeed can and this is one of the main reasons to use FTRL as opposed to Projected Online Mirror Descent. The structure in the FTRL update at $t$ allows us to use all the noise added in the previous iterations. Summing up, we get the following result.

**Proposition 5.3.** Let $z_j > 0, j = t+1$ to $T$ be a sequence of real numbers and let $a_t = s_t - \sum_{j=t+1}^{T} z_j$. Then the output of Algorithm 1 satisfies

$$D_\alpha(w_{T+1}||w'_{T+1}) \leq R_\alpha^{t} \sum_{j=1}^{T} \xi_j, a_t + \sum_{j=t+1}^{T} R_\alpha(\xi_j, z_j)$$

where $s_t = 2L$ is the sensitivity at iteration $t$.

All that is left now is to calculate both the terms in the right hand side and set $z_j$’s appropriately. In the case of Euclidean geometry i.e. $\ell_2/\ell_2$, we can calculate both these exactly, in particular because sum of Gaussians is also a Gaussian (with a larger variance). Plugging in the expressions and choosing $z_j$’s to optimize the tradeoff, we get the following result.

**Theorem 5.4 ($\ell_2/\ell_2$).** Assume the norm in the primal space is $\ell_2$ and let $\xi_j \sim \mathcal{N}(0, \sigma^2 I_d)$. Then for any $\alpha \geq 1$ Algorithm 1 satisfies $(\alpha, \frac{4\alpha L^2}{\sigma^2})$-RDP.

Note that the above result is better than the warm-up choice of $z_j$ by a factor of $1/T$. We also remark that worst case value of $t$, the position in the stream where the samples differ, happen at the two extremes of the stream, but is worse (from the best, which is $T/2$) only by a factor of 2. We now discuss how this differs from the analysis in Feldman et al. (2019) for the smooth case. Feldman et al. (2019) encounter the same problem as the warm-up case discussed before, which leads to $D_\alpha(w_T||w'_T) \leq \frac{\alpha s^2}{2(T-t+1)\sigma^2}$, which can at worst be $\frac{\alpha s^2}{2\sigma^2}$. They resolve this by using mini-batches of data of increasing size, proportional to $\sqrt{d/(T-t+1)}$. This allows them to control the individual sensitivities $s_t$, and upon the setting the parameters correctly make the Rényi-Divergence $\frac{4\alpha L^2}{\sigma^2}$ (same as ours). We now discuss $\ell_p/\ell_q$ geometry.
\( \ell_p/\ell_q \) geometry. In the \( \ell_p/\ell_q \) case, since the geometry of the space where the gradients lie is defined with respect to \( \ell_q \) norm, we add noise with density \( p(\xi) \) proportional to \( e^{-\frac{\|\xi\|^2}{\sigma^2}} \), also known as generalized Gaussian distribution. For \( q = 2 \) or \( 1 \), this reduces to Gaussian and Laplace mechanisms respectively, which are widely used in differentially private analysis. However, a general \( q \) presents two two issues. The first is what is the distribution of sum of these random variables i.e. how do we calculate \( R_\alpha(\sum_{j=1}^t \xi_j, a_t) \). We resolve this again by application of shift-reduction lemma to get \( R_\alpha(\sum_{j=1}^t \xi_j, a_t) \leq \sum_{j=1}^t R_\alpha(\xi_j, z_j) \) such that \( \sum_{j=1}^t z_j = a_t \). The second is what is the R\'enyi Divergence between two such random variables in cases other than \( q = 1 \) or \( 2 \), where we have closed-form expressions. This is resolved by direct computation of a rather crude upper bound on the R\'enyi Divergence between such random variables. Our upper bound on R\'enyi Divergence holds under the constraint that \( \alpha \) is smaller than \( \frac{2^{q-1}}{2^{q-1}-1} \); note that the constraint gets worse (i.e. \( \alpha \) tends to \( 1 \)) as \( q \) grows. Formally, we get the following result.

**Theorem 5.5** (\( \ell_p/\ell_q \)-setting). Let the norm in the primal space be \( \ell_p \). Let \( \xi_t \sim \frac{q}{2^{q-1}} \exp (-\frac{\|\xi\|^2}{\sigma^2}) \), for all \( j \). For \( p > 1 \) and \( 1 < \alpha < \frac{2^{q-1}}{2^{q-1}-1} \), Algorithm 1 satisfies \( (\alpha, \frac{2^q}{T^{q-1}_q \sigma^q}) \)-RDP where \( q \) is the conjugate of \( p \) i.e. \( 1/p + 1/q = 1 \).

Note that this recovers the \( \ell_2/\ell_2 \) setting if \( \alpha < 2 \). For \( (\epsilon, 0) \) privacy with Laplace noise, note that with \( q = 1 \), the upper bound on \( \alpha \) becomes infinity. Taking \( \alpha \) to infinity, we get the this satisfies \( (\infty, \frac{2}{\epsilon}) \)-RDP. Upon setting \( \sigma = 2s/\epsilon \), this becomes \( (\infty, \epsilon) \)-RDP, and finally using Lemma 2.4 gives us \( (\epsilon, 0) \)-DP.

### 6 Conclusion

In this paper, we studied the problem of private stochastic convex optimization. We proposed a noisy version of the popular FTRL algorithm and presented convergence and privacy guarantees for a wide range of geometries. As a special case of our algorithmic framework, we recovered the results of Feldman et al. (2019) when the loss in not strongly convex, without requiring Lipschitz continuity of the stochastic gradients. Further, our algorithm avoids the need for mini-batching by exploiting the privacy amplification of later iterates due to noise added in early iterations.

Our work suggests several interesting research directions. First, we ask if we can improve on our results for general \( \ell_p \)-norms by removing the requirement on \( \alpha \) for \( (\alpha, \epsilon) \)-RDP. We also pose the question of a differentially private algorithm when the norm on the primal space is \( \ell_1 \). This is an important case as many regularizers used in the Online Learning and Bandit literature are strongly convex with respect to \( \ell_1 \) norm. Further, our algorithm requires knowledge of the time horizon to set the noise distribution optimally. Our guarantees, when the time horizon is not known, are suboptimal in their dependence on both \( d \) and \( \epsilon \). A natural approach that can yield anytime guarantee is through restarting and doubling trick; it is not immediately clear, though, if such an approach is necessary. Finally, we think it should be possible to extend the analysis of our algorithm to the case when the objective is strongly convex and achieve optimal rates in linear time. We leave this and other questions above for future work.

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A Last iterate convergence of FTRL

In this section we present the proof of Theorem 5.1. The result is split into two parts – Theorem A.3 and Theorem A.4. We begin by recalling a result relating the properties of an \( \alpha \)-strongly convex potential \( \Psi \) and its conjugate \( \Psi^* \) from Kakade et al. (2009).

**Theorem A.1** (Theorem 6 (Kakade et al., 2009)). Assume that \( \Psi \) is a closed and convex function. Then \( \Psi \) is \( \alpha \)-strongly convex with respect to a norm \( \| \cdot \| \) iff \( \Psi^* \) is \( \frac{1}{\alpha} \)-strongly smooth with respect to the dual norm \( \| \cdot \|_* \).

We are now ready to show the main result of this section. Our proof follows the techniques of (Shamir and Zhang, 2013) combined with the analysis of FTRL in (Abernethy et al., 2015; Zimmert and Seldin, 2018). Let \( u \in \mathcal{W} \) be a fixed vector in the convex set to be chosen later. We begin by decomposing the term \( \langle g_t, w_t - u \rangle \) in the following way:

\[
\langle g_t, w_t - u \rangle = \langle g_t, w_t \rangle + \Phi_t(-G_t) - \Phi_t(-G_{t-1}) \quad \text{(Stability)}
\]

\[
- \Phi_t(-G_t) + \Phi_t(-G_{t-1}) - \langle g_t, u \rangle \quad \text{(Penalty)}.
\]

**Lemma A.2.** The stability term is bounded as follows:

\[
\mathbb{E} [\langle g_t, w_t \rangle + \Phi_t(-G_t) - \Phi_t(-G_{t-1})] \leq \frac{\eta_t}{2} \mathbb{E}[\|g_t\|_*^2],
\]

where the expectation is with respect to all the randomness of the algorithm and \( \mathcal{D} \) up to time \( t \).

We have

\[
\langle g_t, w_t \rangle + \Phi_t(-G_t) - \Phi_t(-G_{t-1}) = \langle G_t - G_{t-1}, \nabla \Phi_t(-G_{t-1}) \rangle + \Phi_t(-G_t) - \Phi_t(-G_{t-1})
\]

\[
= D_{\Phi_t}(-G_t, -G_{t-1}) = D_{\Phi_t}(\nabla \Phi_t^*(w_t) - g_t, \nabla \Phi_t^*(w_t))
\]

\[
\leq \frac{\eta_t}{2} \| \nabla \Phi_t^*(w_t) - g_t - \nabla \Phi_t^*(w_t) \|_*^2 = \frac{\eta_t}{2} \| g_t \|_*^2.
\]

The last inequality follows by combining Theorem A.1 with the fact that \( \Psi_t + I_{\mathcal{W}} \) is \( \frac{1}{\eta_t} \)-strongly convex. Taking expectation finishes the proof.

We can now prove the following result about FTRL with fixed step size.

**Theorem A.3.** Suppose \( \mathbb{E}[\|g_t\|_*^2] \leq G^2 \) for all \( t \) and that \( \Psi(w_1) - \Psi(w^*) \leq D^2 \). After \( T \) iterations of FTRL (Algorithm 1) with step size \( \eta_t = \frac{D}{G \sqrt{T}} \) it holds that

\[
\mathbb{E}[F(w_T)] - F(w^*) \leq \frac{GD \log (T)}{2 \sqrt{T}} + \frac{3GD}{2 \sqrt{T}}.
\]

**Proof.** Direct computation yields

\[
D_{\Phi_t}(-G_{t-1}, \nabla \Phi_t^*(u)) - D_{\Phi_t}(-G_t, \nabla \Phi_t^*(u)) = -\Phi_t(-G_t) + \Phi_t(-G_{t-1}) - \langle g_t, u \rangle.
\]
Next we proceed as in Shamir and Zhang (2013). Fix \( k \leq T - 1 \) and the step size to \( \eta_k = \eta \). We have

\[
\sum_{t=T-k}^{T} -\Phi_t(-G_t) + \Phi_t(-G_{t-1}) - \langle g_t, u \rangle \leq D_{\Phi_{T-k}}(-G_{T-k-1}, \nabla \Phi_{T-k}^*(u)) + \sum_{t=T-k+1}^{T} [D_{\Phi_{t+1}}(-G_t, \nabla \Phi_{t+1}^*(u)) - D_{\Phi_t}(-G_t, \nabla \Phi_t^*(u))] - D_{\Phi_T}(-G_T, \nabla \Phi_T^*(u)) - D_{\Phi_{T-k}}(-G_{T-k-1}, \nabla \Phi_{T-k}^*(u)) - D_{\Phi_{T-k}}(-G_{T-k-1}, \nabla \Phi_{T-k}^*(u)) - D_{\Phi_{T-k}}(-G_{T-k-1}, \nabla \Phi_{T-k}^*(u)) - D_{\Phi_{T-k}}(-G_{T-k-1}, \nabla \Phi_{T-k}^*(u))
\]

Notice that by convexity we have that \( \mathbb{E}[F(w_t) - F(u)] \leq \mathbb{E}[\langle g_t, w_t - u \rangle] \). This, together with the above derivation and Lemma A.2 yields:

\[
\mathbb{E} \left[ \sum_{t=T-k}^{T} F(w_t) - F(u) \right] \leq \sum_{t=T-k}^{T} \frac{\eta}{2} \mathbb{E}[\|g_t\|^2] + \mathbb{E}[D_{\Phi_{T-k}}(-G_{T-k-1}, \nabla \Phi_{T-k}^*(u))]. \tag{4}
\]

Let \( S_k = \frac{1}{k+1} \sum_{t=T-k}^{T} F(w_t) \) and let \( u = w_{T-k} \). Since we assumed \( \mathbb{E}[\|g_t\|^2] \leq G^2 \) for all \( t \in [T] \) we have

\[ \mathbb{E}[S_k] - F(w_{T-k}) \leq \frac{\eta}{2} G^2. \]

Combining the above with the definition of \( S_k \), we have

\[ k \mathbb{E}[S_{k-1}] = (k + 1) \mathbb{E}[S_k] - F(w_{T-k}) \leq (k + 1) \mathbb{E}[S_k] - \mathbb{E}[S_k] + \frac{\eta}{2} G^2 = k \mathbb{E}[S_k] + \frac{\eta}{2} G^2. \]

Unrolling the above recurrence we have \( \mathbb{E}[F(w_T)] = \mathbb{E}[S_0] \leq \mathbb{E}[S_{T-1}] + \frac{\eta \log(T) G^2}{2} \). Next we bound \( \mathbb{E}[S_{T-1}] - F(w^*) \), where \( w^* \) is a minimizer of \( F \). Set \( k = T - 1 \) and \( u = w^* \) in Equation 4. Set \( w_1 \) so that \( \Psi_1(w_1) = \Phi_1(0) \) and \( G_0 = 0 \). We have

\[
\mathbb{E}[S_{T-1}] - F(w^*) \leq \frac{\eta G^2}{2} + \frac{1}{T} D_{\Phi_1}(0, \nabla \Phi_1^*(w^*)) = \Phi_1(0) - \Psi_1(w^*) \leq \frac{\eta G^2}{2} + \frac{D^2}{T \eta}
\]

Setting \( \eta = \frac{D}{G \sqrt{T}} \) finishes the proof. \( \square \)

We are also able to show last iterate guarantees for FTRL with a decreasing step size schedule, removing the need for known time horizon.

**Theorem A.4.** Assume that for all \( g_t \) it holds \( \mathbb{E}[\|g_t\|^2] \leq G^2 \), for all \( u, v \in W \) it holds \( \Psi(u) - \Psi(v) \leq D^2 \). After \( T \) iterations of FTRL (Algorithm 1) with step size \( \eta_T = \frac{D}{G \sqrt{T}} \) it holds that

\[
\mathbb{E}[F(w_T)] - F(w^*) \leq O \left( \frac{GD \log(T)}{\sqrt{T}} \right)
\]
Proof. Let \( \eta_t = c / \sqrt{t} \) for \( c \) to be chosen later. As in the proof of Theorem A.3 we arrive at
\[
\sum_{t=T-k}^T -\Phi_t(-G_t) + \Phi_t(-G_{t-1}) - \langle g_t, u \rangle \leq D_{\Phi_{T-k}}(-G_{T-k-1}, \nabla \Phi_{T-k}^*(u))
\]
\[
+ \sum_{t=T-k+1}^T \left[ D_{\Phi_{t+1}}(-G_t, \nabla \Phi_{t+1}^*(u)) - D_{\Phi_{t}}(-G_t, \nabla \Phi_{t}^*(u)) \right]
\]
\[
- D_{\Phi_T}(-G_T, \nabla \Phi^*(u)).
\]
We now show how to control the terms \( D_{\Phi_{t+1}}(-G_t, \nabla \Phi_{t+1}^*(u)) - D_{\Phi_{t}}(-G_t, \nabla \Phi_{t}^*(u)) \) for \( t \in [T - k + 1, T] \cap \mathbb{N} \).

First note that by definition of \( \Phi_{t+1} \) it holds that
\[
\Phi_{t+1}(-G_t) = \max_{w \in \mathcal{W}} \langle -G_t, w \rangle - \Psi_{t+1}(w) = \langle -G_t, w_{t+1} \rangle - \Psi_{t+1}(w_{t+1}).
\]
Further it holds that
\[
\Phi_t(-G_t) = \max_{w \in \mathcal{W}} \langle -G_t, w \rangle - \Psi_t(w) \geq \langle -G_t, w_{t+1} \rangle - \Psi_t(w_{t+1}).
\]
This implies
\[
\Phi_{t+1}(-G_t) - \Phi_t(-G_t) \leq \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t+1}} \right) \Psi(w_{t+1}).
\]
Next, notice that by the definition of the conjugate function it holds that for any \( u \in \mathcal{W} \) and \( t \)
\[
\Phi_t^*(u) + \Phi_t(\nabla \Phi_t^*(u)) = \langle u, \nabla \Phi_t^*(u) \rangle.
\]
Expanding \( D_{\Phi_{t+1}}(-G_t, \nabla \Phi_{t+1}^*(u)) - D_{\Phi_{t}}(-G_t, \nabla \Phi_{t}^*(u)) \) and combining with the above, for any \( u \in \mathcal{W} \) we have
\[
D_{\Phi_{t+1}}(-G_t, \nabla \Phi_{t+1}^*(u)) - D_{\Phi_{t}}(-G_t, \nabla \Phi_{t}^*(u)) = \Phi_{t+1}(-G_t) - \Phi_t(-G_t) - \langle \Phi_{t+1}(\nabla \Phi_{t+1}^*(u)) - \nabla \Phi_{t+1}^*(u), \Phi_t(-G_t) \rangle
\]
\[
+ \langle \Phi_t(\nabla \Phi_t^*(u)) - \nabla \Phi_t^*(u), \Phi_t(-G_t) \rangle + \langle \nabla \Phi_t^*(u), G_t \rangle - \langle \nabla \Phi_t^*(u), G_t \rangle
\]
\[
\leq \Phi_{t+1}^*(u) - \Phi_t^*(u) + \left( \frac{1}{\eta_t} - \frac{1}{\eta_{t+1}} \right) \Psi(w_{t+1})
\]
\[
= \left( \frac{1}{\eta_{t+1}} - \frac{1}{\eta_t} \right) (\Psi(u) - \Psi(w_{t+1})) \leq \frac{D^2}{2c \sqrt{t}}.
\]
Convexity, as in the proof of Theorem A.3 now implies
\[
\mathbb{E} \left[ \sum_{t=T-k}^T F(w_t) - F(u) \right] \leq \sum_{t=T-k}^T \frac{1}{2 \sqrt{t}} (cG^2 + D^2/c) + \mathbb{E}[D_{\Phi_{T-k}}(-G_{T-k-1}, \nabla \Phi_{T-k}^*(u))]
\]
\[
\leq \frac{(k+1)(cG^2 + D^2/c)}{\sqrt{T}}.
\]
Repeating the argument in the proof of Theorem A.3 gives \( \mathbb{E}[F(w_T)] \leq \mathbb{E}[S_{T-1}] + \frac{\log(T)(cG^2 + D^2/c)}{\sqrt{T}}. \)
Again as in the proof of Theorem A.3 we bound
\[
\mathbb{E}[S_{T-1}] - F(w^*) \leq \frac{cG^2 + D^2/c}{\sqrt{T}} + \frac{1}{T} D_{\Phi_t}(0, \nabla \Phi_t^*(w^*)) \leq \frac{cG^2 + D^2/c}{\sqrt{T}} + \frac{D^2/c}{T}.
\]
\( \square \)
B Privacy guarantees for FTRL

We begin by proving a general result which bounds the Rényi divergence between the last iterate \(w_T\) of Algorithm 1 and the iterate \(w'_T\) produced by running the algorithm on a neighboring sequence of data points. This result is used in the proofs of both Theorem 4.1 and Theorem 4.3. For the rest of this section we assume that the neighboring sequences differ at the \(t\)-th iterate so that \(z_t \neq z'_t\).

**Proposition B.1.** Let \(z_j > 0, j = t + 1\) to \(T\) be a sequence of real numbers and let \(a_t = s_t - \sum_{j=t+1}^{T} z_j\). Then the output of Algorithm 1 satisfies

\[
D_\alpha(w_{T+1} \| w'_{T+1}) = R_\alpha\left(\sum_{j=1}^{t} \xi_j, a_t\right) + \sum_{j=t+1}^{T} R_\alpha(\xi_j, z_j)
\]

where \(s_t\) is the sensitivity at iteration \(t\).

**Proof of Proposition 5.3.** Consider two neighboring data streams \(S = \{z_1, z_2, \ldots, z_t, \ldots z_T\}\) and \(S' = \{z_1, z_2, \ldots, z'_t, \ldots z_T\}\) differing on the \(t\)-th sample with \(1 \leq t \leq T\). The update \(w_{t+1} = \nabla \Phi_{t+1}(-G_t)\) can equivalently be written as,

\[
w_{t+1} = \nabla \Phi_{t+1}(-G_t) = \nabla \Phi_{t+1}(-G_{t-1} - g_t) = \nabla \Phi_{t+1}(\nabla \Phi^*_t(w_t) - g_t)
\]

We now look at the \(\alpha\)-Rényi Divergence between the final iterates of our algorithm on \(S\) and \(S'\),

\[
D_\alpha(w_{T+1} \| w'_{T+1}) = D_\alpha(\nabla \Phi_{T+1}(\nabla \Phi^*_T(w_T) - g_T) \| \nabla \Phi_{T+1}(\nabla \Phi^*_T(w'_T) - g'_T))
\]

\[
\leq D_\alpha(\nabla \Phi^*_T(w_T) - g_T \| \nabla \Phi^*_T(w'_T) - g'_T)
\]

\[
= D_\alpha(\nabla \Phi^*_T(w_T) - \nabla f(w_T, z_T) + \xi_T \nabla \Phi^*_T(w'_T) - \nabla f(w'_T, z_T) + \xi_T)
\]

\[
\leq D_\alpha^2(\nabla \Phi^*_T(w_T) - \nabla f(w_{T-1}, z_T) \| \nabla \Phi^*_T(w'_T) - \nabla f(w'_{T-1}, z_T)) + R_\alpha(\xi_T, z_T)
\]

\[
\leq D_\alpha^2(w_{T} \| w'_{T}) + R_\alpha(\xi_T, z_T)
\]

\[
\vdots
\]

\[
\leq D_\alpha^2(\sum_{j=t+1}^{T} z_j) + \sum_{j=t+1}^{T} R_\alpha(\xi_j, z_j)
\]

where in the first and third inequality we have used the Data Processing inequality for Rényi Divergence and the second inequality follows from Lemma 2.5.

We now look the at the second term in the above equation. We first define \(\hat{z}_t := \sum_{j=t+1}^{T} z_j\). Recall that the FTRL update is \(w_{t+1} = \nabla \Phi_{t+1}(-G_t)\). We substitute this and use post processing to get,

\[
D_\alpha^2(\hat{z}_t \| w'_{t+1}) = D_\alpha^2(\nabla \Phi_{t+1}(-G_t) \| \nabla \Phi_{t+1}(z'_t)) \leq D_\alpha^2(G_t \| G'_t)
\]

Note that since the streams differ at the \(t\)-th sample, \(G_t = \sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \xi_j + \nabla f(w_t, z_t) - \xi_t = \nabla f(w_t, z_t) + (\sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{T} \xi_j)\). Similarly, \(G'_t = \nabla f(w_t, z'_t) + (\sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{T} \xi_j)\).
The Rényi-Divergence between $G_t$ and $G'_t$ can now be bounded as,

$$D^\tilde{z}_t(G_t\|G'_t) = D^\tilde{z}_t(\nabla f(w_t, z_t) + \sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{t} \xi_j)\| \nabla f(w_t, z'_t) + \sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{t} \xi_j)$$

$$\leq D^\tilde{z}_{t+\alpha t}(\nabla f(w_t, z_t)\| \nabla f(w_t, z'_t)) + R_\alpha(\sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{t} \xi_j, a_t)$$

where in the above inequality, we again used Lemma 2.5. We now look at both the terms in the last equation above.

**First Term** $D^\tilde{z}_{t+\alpha t}(\nabla f(w_t, z_t)\| \nabla f(w_t, z'_t))$. We will bound this by appropriately setting $\tilde{z}_t + a_t$ taking into account the sensitivity of the stochastic gradients. The sensitivity $s_t$ is defined as

$$s_t = \| \nabla f(w_t, z_t) - \nabla f(w_t, z'_t) \|_s.$$

We first recall that the definition of $\infty$-Wasserstein distance, which is used to define shifted Rényi Divergence. Let $X$ and $Y$ be random variables sampled from $\mu$ and $\nu$, respectively, which are distributions over a normed vector space. We then have $W_\infty(\mu, \nu) = \inf_{\gamma \in C(\mu, \nu)} \sup_{(X, Y) \sim \gamma} \| X - Y \|,$ where $C(\mu, \nu)$ denotes the set of all couplings of $\mu$ and $\nu$. We will abuse notation and write $W_\infty(X, Y)$ as $W_\infty(X, Y)$ and $W_\infty(Y, Y)$ as $W_\infty(Y, Y).$ The shifted Rényi Divergence is then,

$$D^\alpha_t(\nabla f(w_t, z_t)\| \nabla f(w_t, z'_t)) = \inf_{Y \sim \mu \in \mathbb{P}(\mathbb{R}^d) : W_\infty(Y, \nabla f(w_t, z_t)) \leq s_t} D_\alpha(\| \nabla f(w_t, z'_t) \|)$$

By definition of sensitivity $s_t$, the random variable $\nabla f(w, z'_t)$ lies in the constraint of the above infimum. To see this, first note that by Lemma 7 of Feldman et al. (2018) the following two are equivalent $W_\infty(Y, \nabla f(w_t, z_t)) \leq s_t$ and $\mathbb{P}[\|Y - \nabla f(w_t, z_t)\| \leq s_t] = 1$. Since $Y = \nabla f(w_t, z_t)$ satisfies $\mathbb{P}[\|Y - \nabla f(w_t, z_t)\| \leq s_t] = 1$ by definition of sensitivity, it then also holds that $W_\infty(\nabla f(w_t, z_t), \nabla f(w_t, z_t)) \leq s_t$. Therefore,

$$\inf_{Y \sim \mu \in \mathbb{P}(\mathbb{R}^d) : W_\infty(Y, \nabla f(w_t, z_t)) \leq s_t} D_\alpha(Y\| \nabla f(w_t, z'_t)) \leq D_\alpha(\| \nabla f(w_t, z'_t) \|) = 0$$

**Second Term** $R_\alpha(\sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{t} \xi_j, a_t)$. We will use post-processing to reduce it to Rényi Divergence between noise $\sum_{j=1}^{t} \xi_j$ and a shifted noise $\sum_{j=1}^{t} \xi_j + u$. We first define a function $q_t : \mathbb{R}^d \to \mathbb{R}^d$ as $q_t(u) = \sum_{j=1}^{t-1} \nabla f(w_j, z_j) + u$. We expand the term as,

$$R_\alpha(\sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{t} \xi_j, a_t) = \sup_{u : \|u\| \leq a_t} D_\alpha\left(\sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{t} \xi_j + u\| \sum_{j=1}^{t-1} \nabla f(w_j, z_j) - \sum_{j=1}^{t} \xi_j\right)$$

$$= \sup_{u : \|u\| \leq a_t} D_\alpha(q_t(-\sum_{j=1}^{t} \xi_j + u)\| q_t(-\sum_{j=1}^{t} \xi_j))$$

$$\leq \sup_{u : \|u\| \leq a_t} D_\alpha(-\sum_{j=1}^{t} \xi_j + u\| -\sum_{j=1}^{t} \xi_j)$$

$$= \sup_{u : \|u\| \leq a_t} D_\alpha(\sum_{j=1}^{t} \xi_j + u\| \sum_{j=1}^{t} \xi_j) = R_\alpha(\sum_{j=1}^{t} \xi_j, a_t)$$

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where the the last inequality follows from the post-processing property of Rényi-Divergence. Combining all of this together, along with the fact that we set \( a_t = s_t - \sum_{j=t+1}^{T} z_j \), gives us

\[
D_{\alpha}(w_{T+1}||w'_{T+1}) = R_{\alpha}(\sum_{j=1}^{t} \xi_j, a_t) + \sum_{j=t+1}^{T} R_{\alpha}(\xi_j, z_j)
\]

\[\square\]

### B.1 Proof of Theorem 4.1

We begin by presenting the privacy analysis for Algorithm 1. Theorem 4.1 will follow as a corollary of the next result and the convergence guarantees for the algorithm.

**Theorem B.2** (\( \ell_2/\ell_2 \)). Assume the norm in the primal space is \( \ell_2 \) and let \( \xi_j \sim \mathcal{N}(0, \sigma^2 I_d) \). Then for any \( \alpha \geq 1 \) Algorithm 1 satisfies \((\alpha, \frac{\alpha^2}{T\sigma^2})\)-RDP.

To prove the above theorem we first need the following fact.

**Fact B.3** (Van Erven and Harremos (2014)). The Rényi Divergence between two multivariate Gaussians \( \mathcal{N}(u_1, \sigma^2 I_d) \) and \( \mathcal{N}(u_2, \sigma^2 I_d) \) is \( \frac{\alpha \|u_1 - u_2\|^2}{2\sigma^2} \).

**Proof of Theorem 5.4.** We appeal to Proposition B.1 to get that the Rényi Divergence between the last iterates is bounded as,

\[
D_{\alpha}(w_{T+1}||w'_{T+1}) = R_{\alpha}(\sum_{j=1}^{t} \xi_j, a_t) + \sum_{j=t+1}^{T} R_{\alpha}(\xi_j, z_j)
\]

where \( a_t = s_t - \sum_{j=t+1}^{T} z_j \). Consider all the \( \xi_j \sim \mathcal{N}(0, \sigma^2 I_d) \), then \( \sum_{j=1}^{t} \xi_j \sim \mathcal{N}(0, t\sigma^2 I_d) \), using the Rényi-Divergence formula for multivariate Gaussians (Fact B.3), we have,

\[
\sup_{u: \|u\| \leq a_t} D_{\alpha}(\sum_{j=1}^{t} \xi_j + u || \sum_{j=1}^{t} \xi_j) = \sup_{u: \|u\| \leq a_t} \frac{\alpha \|u\|^2}{2t\sigma^2} = \frac{\alpha a_t^2}{2t\sigma^2}
\]

Combining all of the above, we finally get,

\[
D_{\alpha}(w_{T+1}||w'_{T+1}) \leq \frac{\alpha a_t^2}{2t\sigma^2} + \sum_{j=t+1}^{T} R_{\alpha}(\xi_j, z_j)
\]

\[
= \frac{\alpha a_t^2}{2t\sigma^2} + \sum_{j=t+1}^{T} \frac{\alpha z_j^2}{2\sigma^2}
\]

Recall when bounding \( D_{\alpha}^{\ell_2+\alpha t}(\nabla f(w_t, z_t)||\nabla f(w_t, z'_t)) \), we set \( z_t + a_t = s_t \), where \( s_t \) is the sensitivity at iterate \( t \). Since we assume that the norm of stochastic gradients to be bounded by \( L \), we have \( s_t = s \leq 2L \). Let \( a := a_t \) and \( b := \frac{z_t}{T-t} \), we get \( D_{\alpha}(w_{T}||w'_{T}) \leq \frac{\alpha}{2\sigma^2} \left( \frac{a^2}{T} + \frac{b^2}{T-t} \right) \). Substituting the constraint, \( z_t + a_t = s \), we want to minimize the following with respect to \( a \).

\[
\frac{\alpha}{2\sigma^2} \left( \frac{a^2}{T} + \frac{(s-a)^2}{T-t} \right)
\]

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Differentiating with respect to $a$ and setting it to 0, we get that the above is minimized at 
\[ a = \frac{\sqrt{t} s}{\sqrt{t} + \sqrt{T-t}} \] 
and similarly, 
\[ b = \frac{\sqrt{T-t} s}{\sqrt{t} + \sqrt{T-t}} \] 
The optimized upper bound becomes, 
\[ D_\alpha(w_{T+1} \| w'_{T+1}) \leq \frac{\alpha s^2}{(\sqrt{t} + \sqrt{T-t})^2 \sigma^2} \]

If we minimize the above upper bound with respect to $t$, we get that 
\[ t = \lceil \sqrt{T - 1/2} \rceil \] 
is the best case, which gives us the corresponding upper bound as 
\[ \frac{\alpha s^2}{2T \sigma^2} \]. The worst cases are the two extreme values of $t$, i.e. $t = 1$ and $t = T$, where we get \( \frac{\alpha s^2}{T^2 \sigma^2} \), which completes the proof.

To finish the proof of Theorem 4.1 we only need to show a bound on \( \mathbb{E}[g_t \| 2] \) which is done in the following way 
\[ \mathbb{E}[g_t \| 2] = \mathbb{E}[\| \nabla f(w_t, z_t) - \xi_t \| 2] \leq L^2 + d \sigma^2 \] 
Substituting \( G^2 = L^2 + d \sigma^2 \) and \( \sigma = \frac{2\alpha L}{T^{1/2} \epsilon^{1/2}} \) finishes the proof.

C Proof of Theorem 4.3

As in the proof of Theorem 4.1 we begin by deriving the privacy guarantee for FTRL. We start by an auxiliary lemma which will allow us to control the terms $R_\alpha(\sum_{j=1}^t \xi_j, a_i)$ and $\sum_{j=t+1}^T R_\alpha(\xi_j, z_j)$.

Lemma C.1. Let $\xi$ be a $d$ dimensional random variable with density $p(x) = c(\sigma, q) e^{-\| x \|_q^q}$ where 
\[ c(\sigma, q) = \frac{q}{2 \sigma q (1/q)} \], $q \geq 1$ and let $u \in \mathbb{R}^d$ be a deterministic vector, then for \( \alpha < \frac{2^{q-1}}{2^{q-1} - 1} \) we have
\[ \sup_{\| u \|_q \leq \alpha} D_\alpha(\xi \| u \| \xi) \leq \left( \frac{\alpha}{\sigma} \right)^q \]

Proof. Let $\xi' : \xi \| u$. Using the Rényi divergence definition, we get
\[ D_\alpha(\xi \| \xi') = \frac{1}{\alpha - 1} \ln \int_{\mathbb{R}^d} \left( \frac{p_\xi(z)}{p_{\xi'}(z)} \right)^\alpha p_{\xi}(z) dx 
\]
\[ = \frac{1}{\alpha - 1} \ln \int_{\mathbb{R}^d} \left( e^{-\frac{q}{q(1/q)} || z-u ||_q^q} \right) c(\sigma, q) dx \]

Note that \( (u+v)^q = 2^q \left( \frac{u^q + v^q}{2} \right)^q \leq 2^{q-1} (u^q + v^q) \), where the last inequality follows from convexity of $\cdot^q$, for $q \geq 1$. Using this inequality, we get \( || z ||_q^q \leq 2^{q-1} (|| z-u ||_q^q + || u ||_q^q) \). With $\alpha \geq 1$, we can plug it in the above equation to get,
\[ D_\alpha(\xi \| \xi') \leq \frac{1}{\alpha - 1} \ln \int_{\mathbb{R}^d} c(\sigma, q) \left( e^{-\frac{q}{q(1/q)} || z-u ||_q^q} \right) dx 
\]
\[ = \frac{1}{\alpha - 1} \ln \int_{\mathbb{R}^d} c(\sigma, q) e^{-\frac{(1-\alpha) || u ||_q^q}{2^{q-1} \sigma^q}} dx \]
Let $\tilde{\sigma} := \sigma / ((\alpha + (1-\alpha)2^{q-1})^{1/q})$. We want the exponent in the integral to be negative therefore we want $(\alpha + (1-\alpha)2^{q-1}) > 0 \iff \alpha < \frac{2^{q-1}}{2^{q-1}+1}$. So choosing $\alpha < \frac{2^{q-1}}{2^{q-1}+1}$, the integral becomes

$$\int_{\mathbb{R}^d} c(\sigma, q) e^{-\frac{(\alpha + (1-\alpha)2^{q-1})\|x-u\|^q}{\sigma^q}} dx = \frac{c(\sigma, q)}{c(\tilde{\sigma}, p)} \int_{\mathbb{R}^d} c(\tilde{\sigma}, p) e^{-\frac{\|x-u\|^q}{\sigma^q}} dx = \frac{c(\sigma, q)}{c(\tilde{\sigma}, p)}$$

We therefore get,

$$D_\alpha(\xi \| \xi') \leq \frac{1}{1-\alpha} \ln \left( \frac{c(\sigma, q)}{c(\tilde{\sigma}, p)} e^{\frac{(1-\alpha)\|a\|^q}{\sigma^q}} \right) = \frac{1}{1-\alpha} \ln \left( \frac{c(\sigma, q)}{c(\tilde{\sigma}, p)} \right) \leq \frac{1}{(1-\alpha)q} \ln \left( \frac{1}{\alpha + (1-\alpha)2^{q-1}} \frac{\|a\|^q}{\sigma^q} \right)$$

We now compute $\ln \left( \frac{c(\sigma, q)}{c(\tilde{\sigma}, p)} \right)$. Note that $c(\sigma, q) = \frac{q}{2\sigma T(1/q)}$ and $\tilde{\sigma} = \frac{\sigma}{(\alpha + (1-\alpha)2^{q-1})^{1/q}}$. Substituting these, we get

$$\ln \left( \frac{c(\sigma, q)}{c(\tilde{\sigma}, p)} \right) = \ln \left( \frac{1}{(\alpha + (1-\alpha)2^{q-1})^{1/q}} \right) = \ln \left( \frac{1}{\alpha + (1-\alpha)2^{q-1}} \right)$$

Plugging the above in the bound for $D_\alpha(\xi \| \xi')$, we get

$$D_\alpha(\xi \| \xi') \leq \frac{\ln \left( \frac{1}{\alpha + (1-\alpha)2^{q-1}} \right)}{(1-\alpha)q} + \frac{\|a\|^q}{\sigma^q}$$

We will show that the first term is negative. It follows because the denominator is negative for $\alpha \geq 1$ and the term inside the log is greater than one because $\alpha \geq 1$, as shown below.

$$1/(\alpha + (1-\alpha)2^{q-1}) \geq 1 \iff \alpha + (1-\alpha)2^{q-1} \leq 1 \iff \alpha \geq 1$$

The privacy guarantee for Algorithm 1 now follows.

**Theorem C.2** ($\ell_p/\ell_q$-setting). Let the norm in the primal space be $\ell_p$. Let $\xi_j \sim \frac{q}{2\sigma T(1/q)} \exp(-\|\xi\|^q/\sigma^q)$, for all $j$. For $p > 1$ and $1 - \alpha < \frac{2^{q-1}}{2^{q-1}+1}$, Algorithm 1 satisfies $(\alpha, \frac{2^{q-1}}{2^{q-1}+1})$-RDP where $q$ is the conjugate of $p$ i.e. $1/p + 1/q = 1$.

**Proof of Theorem 5.5.** We appeal to Proposition B.1 and instantiate the Rényi Divergence terms by choosing an appropriate probability distribution for the noise $\xi$. Note that the dual norm is the $\ell_q$ norm, such such $\frac{1}{p} + \frac{1}{q} = 1$. The sensitivity of the gradients is therefore defined with respect to the dual norm. We add noise of density $p(\xi) \sim c(\sigma, q) e^{-\|\xi\|^q/\sigma^q}$ where $c(\sigma, q) = \frac{q}{2\sigma T(1/q)}$ is the normalization constant.

From Proposition B.1, we have,

$$D_\alpha(w_{T+1} \| w'_{T+1}) = R_\alpha(\sum_{j=1}^{t} \xi_j, s_t - \sum_{j=t+1}^{T} z_j) + \sum_{j=t+1}^{T} R_\alpha(\xi_j, z_j)$$

Recall that $a_t = s_t - \sum_{j=t+1}^{T} z_j$. From the definition of $R_\alpha$, the first term equals

$$R_\alpha(\sum_{j=1}^{t} \xi_j, a_t) = \sup_{\|u\| \leq a_t} D_\alpha(\sum_{j=1}^{t} \xi_j + u \| \sum_{j=1}^{t} \xi_j).$$

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We apply the shift-reduction lemma (Lemma 5.2) to the above to get

\[ D_\alpha(\sum_{j=1}^t \xi_j * u \| \sum_{j=1}^t \xi_j) \leq D_\alpha^{\sum_{j=1}^t z_j}(\xi_1 * u \| \xi_1) + \sum_{j=1}^t R_\alpha(\xi_j, z_j). \]

As in the proof of Proposition B.1, if we set \( \sum_{j=1}^t z_j = a_t \), then the first term is zero, since \( \| u \|_q \leq a_t \) and hence

\[ D_\alpha^{\sum_{j=1}^t z_j}(\xi_1 * u \| \xi_1) = \inf_{Y \sim \mu \cdot \mathcal{W}_\infty(Y, \xi_1) \leq a_t} D_\alpha(Y \| \xi_1) \leq D_\alpha(\xi_1 \| \xi_1) = 0. \]

The above now implies that \( D_\alpha(\omega_{T+1} \| \omega_{T+1}') \leq \sum_{j=1}^T R_\alpha(\xi_j, z_j) \), where \( \sum_{j=1}^T z_j = a_t \) and \( a_t + \sum_{j=t+1}^T z_j = s_t = s \). Using Lemma C.1 we have \( \sum_{j=t+1}^T R_\alpha(\xi_j, z_j) \leq \frac{(s-a_t)^q}{(T-t)^{q-1} \sigma^q} \), where the last equality follows from setting \( z_j = \frac{a_t}{t^{q-1} \sigma^q} \) for all \( j \) between \( t+1 \) and \( T \). For \( j \) between 1 and \( t \) we set \( z_j = \frac{a_t}{t^{q-1} \sigma^q} \) and again using Lemma C.1 we arrive at \( \sum_{j=1}^T R_\alpha(\xi_j, z_j) \leq \frac{a_t^q}{t^{q-1} \sigma^q} \). We now choose \( a_t \) by setting \( \frac{a_t^q}{t^{q-1} \sigma^q} = \frac{(s-a_t)^q}{(T-t)^{q-1} \sigma^q} \) or equivalently

\[
\frac{a_t^q}{t^q-1} = \frac{(s-a_t)^q}{(T-t)^{q-1} \sigma^q} \quad \Longleftrightarrow \quad \frac{s-a_t}{a_t} = \frac{(T-t)^{q-1}}{t^{q-1}} \\
\quad \Longleftrightarrow \quad st^{1/p} - at^{1/p} = a(T-t)^{1/p} \\
\quad \Longleftrightarrow \quad at = \frac{st^{1/p}}{t^{1/p} + (T-t)^{1/p} }.
\]

This implies

\[
D_\alpha(\omega_{T+1} \| \omega_{T+1}') \leq 2 \frac{s^q}{(t^{1/p} + (T-t)^{1/p})^q \sigma^q} = 2 \frac{s^q}{(t^{1/p} + (T-t)^{1/p})^q \sigma^q} \leq \frac{2s^q}{T^{q/p} \sigma^q} = \frac{2s^q}{T^{q-1} \sigma^q}.
\]

We now show how to control \( \mathbb{E}[\| g_j \|_q^2] \). Let \( \xi_{j,i} \) denote the \( i \)-th coordinate of \( \xi_j \). We have

\[
\mathbb{E}[\| g_j \|_q^2] \leq 2 \mathbb{E}[\| \nabla f(\omega_j, z_j) \|_q^2] + 2 \mathbb{E}[\| \xi_j \|_q^2] \leq 2L^2 + \left( \mathbb{E} \left[ \sum_{i=1}^q (\xi_{j,i})^{2q} + \sum_{i \neq k, i, k \leq d} \xi_{j,i} \xi_{j,k} \right] \right)^{1/q} \\
\leq 2L^2 + \left( \frac{d^2}{2} \mathbb{E}[\xi^q + \xi^{2q}] \right)^{1/q} \leq 2L^2 + \frac{d^2/2}{2L^2} \sigma^2 \left( \frac{1}{q} \left( 1 + \frac{1}{q} \right) \Gamma(1/q) + \frac{1}{q} \Gamma(1/q) \right)^{1/q} \\
\leq 2L^2 + \frac{d^2/2}{2L^2} \sigma^2 \left( \frac{1}{q} \left( 1 + \frac{1}{q} \right) \right)^{1/q} ,
\]

where we have used the formula for the \( k \)-th moment of the generalized Gaussian variable \( \xi \): \( \mathbb{E}[\xi^k] \leq \sigma^k \frac{\Gamma((k+1)/q)}{\Gamma(1/q)} \). Setting \( \sigma = \frac{2L}{T^{1/p} e^{1/q}} \) and plugging \( G^2 = 2L^2 + d^2/2 \sigma^2 \left( \frac{1}{q} \left( 1 + \frac{1}{q} \right) \right)^{1/q} \left( \frac{2L}{T^{1/p} e^{1/q}} \right)^2 \) into Theorem 5.1 finishes the proof.

\[
\Box
\]
D Proof of Theorem 4.4

Theorem D.1. Assume the base potential $\Psi$ is 1-strongly convex with respect to $\| \cdot \|_p$ and that the stochastic gradients are $L$-Lipschitz. Let $\xi_1 \sim \frac{q}{2\sigma L} \exp \left(-\frac{\|\xi\|^q_q}{\sigma^q} \right)$, where $\sigma = \frac{2L}{\epsilon^{1/q}}$ and let $\xi_j = 0$ for all $j > 1$. If for all $u, v \in W$ it holds that $\Psi(u) - \Psi(v) \leq D^2$ then for any $1 < \alpha < \frac{2q-1}{2q-1}$ any iterate $w_t$ of Algorithm 1 is $(\alpha, \epsilon)$-RDP and satisfies

$$E[F(w_t)] - F(w^*) \leq O \left( \frac{LD \log(t)}{\sqrt{t}} + \frac{d^2/q LD \log(t)}{t \epsilon^{2/q}} \right).$$

Proof of Theorem 4.4. We begin by proving the RDP guarantee. First notice that $R_{\alpha}(\xi_j, z_j) = 0$ from our choice of $\xi_j$ for all $j > 1$. Theorem 5.3 now implies that $D_{\alpha}(w_t || w'_t) \leq R_{\alpha}(\xi_1, 2L)$ and Lemma C.1 implies that $R_{\alpha}(\xi_1, 2L) \leq \left(\frac{2L}{\sigma} \right)^q$.

Next we proceed with the convergence guarantee. In the proof of Theorem A.4 set $G^2 = L^2$ for $k < T-1$. Since $g_t = \nabla f(z_t, w_t)$ in this case, this directly yields $E[F(w_T)] \leq E[S_{T-1}] + \frac{D^2/c + cG^2}{\sqrt{T}} \sum_{k=1}^{T-2} \frac{1}{k}$. Next we note that

$$T(E[S_{T-1}] - E[F(w_T)]) \leq E[\|\nabla f(w_1, z_1) + \xi_1\|_q^2] + \sqrt{T}(cG^2 + D^2/c) \leq 2cG^2 + 2d^{2/q} \sigma^2 + \sqrt{T}(cG^2 + D^2/c).$$

Since $(T - 1)E[S_{T-2}] = TE[S_{T-1}] - E[F(w_T)]$ we have

$$E[S_{T-2}] \leq E[S_{T-1}] + 2 \frac{cG^2 + D^2/c}{\sqrt{T}} + \frac{2d^{2/q} \sigma^2}{T}.$$

Further we can bound $E[S_{T-1}] - F(w^*) \leq O \left( \frac{cG^2 + D^2/c}{\sqrt{T}} + \frac{2d^{2/q} \sigma^2}{T} \right)$. Setting $c = D/G$ appropriately now finishes the proof. \qed