Some relations between left (right) semi-uninorms and coimplications on a complete lattice

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Uninorms are important generalizations of triangular norms and conorms, with the neutral elements lying anywhere in the unit interval, left (right) semi-uninorms are non-commutative and non-associative extensions of uninorms, and coimplications are extensions of the Boolean implication. In this paper, we study the relationships between left (right) semi-uninorms and coimplications on a complete lattice. We first discuss the residual coimplicators of left and right semi-uninorms and show that the right (left) residual coimplicator of a disjunctive right (left) semi-uninorm is a right infinitely \(\lor\)-distributive coimplication which satisfies the neutrality principle. Then, we investigate the left and right semi-uninorms induced by a coimplication and demonstrate that the operations induced by right infinitely \(\lor\)-distributive coimplications, which satisfy the order property or neutrality principle, are left (right) infinitely \(\land\)-distributive left (right) semi-uninorms or right (left) semi-uninorms. Finally, we prove that the meet-semilattice of all disjunctive right (left) infinitely \(\land\)-distributive left (right) semi-uninorms is order-reversing isomorphic to the join-semilattice of all right infinitely \(\lor\)-distributive coimplications that satisfy the neutrality principle.

**Keywords:** fuzzy connective; uninorm; semi-uninorm; left (right) semi-uninorm; coimplication

1. Introduction

Uninorms, introduced by Yager & Rybalov (1996) and studied by Fodor, Yager, & Rybalov (1997), are special aggregation operators that have been proven useful in many fields such as fuzzy logic, expert systems, neural networks, aggregation, and fuzzy system modelling (see Gabbay & Metcalfe, 2007; Tsadiras & Margaritis, 1998; Yager, 2001, 2002). This kind of operation is an important generalization of both triangular norms (t-norms for short) and triangular conorms (t-conorms for short) and a special combination of t-norms and t-conorms (see Fodor et al., 1997). But, there are real-life situations when truth functions cannot be associative or commutative (see Flondor, Georgescu, & Iorgulescu, 2001; Fodor & Keresztfalvi, 1995). By throwing away the commutativity from the axioms of uninorms, Mas, Monserrat, & Torrens (2001) introduced the concepts of left and right uninorms on \([0, 1]\) and later on a finite chain (Mas, Monserrat, & Torrens, 2004), Wang and Fang (2009a, 2009b) studied the left and right uninorms on a complete lattice. By removing the associativity and commutativity from the axioms of uninorms, Liu (2012) introduced the concept of semi-uninorms on a complete lattice and Su, Wang, & Tang (2013) discussed the notion of left and right semi-uninorms on a complete lattice. On the other hand, it is well known that a uninorm (semi-uninorm, left and right uninorms) \(U\) can be conjunctive or disjunctive whenever \(U(0, 1) = 0\) or \(1\), respectively. This fact allows us to use uninorms (semi-uninorm, left and right uninorms) in defining fuzzy implications and coimplications (see De Baets & Fodor, 1999; Liu, 2012; Mas, Monserrat, & Torrens, 2007; Ruiz & Torrens, 2004; Wang & Fang, 2009a, 2009b).

In this paper, based on De Baets & Fodor (1999), Liu (2012), Ruiz & Torrens (2004), Su & Wang (2013) and Wang & Fang (2009b), we study left (right) semi-uninorms and coimplications on a complete lattice. After recalling some necessary definitions and examples about the left and right semi-uninorms in the third section and show that the right (left) residual coimplicator of a disjunctive right (left) infinitely \(\land\)-distributive left (right) semi-uninorm is a right infinitely \(\lor\)-distributive coimplication that satisfies the neutrality principle. In Section 4, we investigate the left and right semi-uninorms induced by a coimplication and give some conditions such that the operations induced by a coimplication constitute left or right semi-uninorms. Finally, we prove that the meet-semilattice of all disjunctive right (left) infinitely \(\land\)-distributive left (right) semi-uninorms is order-reversing isomorphic to the join-semilattice of all right infinitely \(\lor\)-distributive coimplications that satisfy the neutrality principle.

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semi-uninorms and right infinitely $\lor$-distributive coimplications which satisfy the neutrality principle.

The knowledge about lattices required in this paper can be found in Birkhoff (1967).

Throughout this paper, unless otherwise stated, $L$ always represents any given complete lattice with maximal element $1$ and minimal element $0$; $J$ stands for any index set.

2. Left and right semi-uninorms

Noting that the commutativity and associativity are not desired for aggregation operators in a lot of cases, Liu (2012) introduced the concept of semi-uninorms on a complete lattice. Here, we recall some necessary concepts about the left and right semi-uninorms and illustrate these notions by means of two examples.

**Definition 2.1 (Su et al., 2013).** A binary operation $U$ on $L$ is called a left (right) semi-uninorm if it satisfies the following two conditions:

1. (U1) there exists a left (right) neutral element, that is, an element $e_L \in L (e_R \in L)$ satisfying $U(e_L, x) = x (U(x, e_R) = x)$ for all $x \in L$,
2. (U2) $U$ is non-decreasing in each variable.

For any left (right) semi-uninorm $U$ on $L$, $U$ is said to be left-conjunctive and right-conjunctive if $U(0, 1) = 0$ and $U(1, 0) = 0$, respectively. $U$ is called conjunctive if both $U(0, 1) = 0$ and $U(1, 0) = 0$ since it satisfies the classical boundary conditions of AND. $U$ is said to be left-disjunctive and right-disjunctive if $U(1, 0) = 1$ and $U(0, 1) = 1$, respectively. We call $U$ disjunctive if both $U(1, 0) = 1$ and $U(0, 1) = 1$ by a similar reason.

If a left (right) semi-uninorm $U$ is associative, then $U$ is the left (right) uninorm on $L$ (see Wang & Fang, 2009a, 2009b).

If a left (right) semi-uninorm $U$ with the left (right) neutral element $e_L$ ($e_R$) has a right (left) neutral element $e_R$ ($e_L$), then $e_L = U(e_L, e_R) = e_R$. Let $e = e_L = e_R$. Here, $U$ is the semi-uninorm (see Liu, 2012). In particular, if the neutral element $e = 1$, then the semi-uninorm $U$ becomes a $t$-semiprobability (see Suárez García & Gil Álvarez, 1986) or a semi-copula (see Bassan & Spizzichino, 2005; Durante, Klement, Mesiar, & Sempi, 2007); if the neutral element $e = 0$, then the semi-uninorm $U$ becomes a $t$-semiconorm (see De Cooman & Kerre, 1994).

Clearly, $U(0, 0) = 0$ and $U(1, 1) = 1$ hold for any left (right) semi-uninorm $U$ on $L$. Moreover, the left (right) neutral elements need not to be unique. In fact, the projection operator given by $U(x, y) = x$ for all $x, y \in L$ is such that any element in $L$ is a right neutral element. But, left (right) neutral elements are all idempotent (see De Baets, 1999) because $U(e_L, e_L) = e_L (U(e_R, e_R) = e_R)$ for any left (right) neutral element $e_L$ ($e_R$) of $U$.

**Definition 2.2 (Wang & Fang, 2009a, 2009b).** A binary operation $U$ on $L$ is called left (right) infinitely $\lor$-distributive if

\[
U \left( \bigvee_{j \in J} x_j, y \right) = \bigvee_{j \in J} U(x_j, y)
\]

\[
U \left( x, \bigvee_{j \in J} y_j \right) = \bigvee_{j \in J} U(x, y_j)
\]

\[
\forall x, y, x_j, y_j \in L;
\]

left (right) infinitely $\land$-distributive if

\[
U \left( \bigwedge_{j \in J} x_j, y \right) = \bigwedge_{j \in J} U(x_j, y)
\]

\[
U \left( x, \bigwedge_{j \in J} y_j \right) = \bigwedge_{j \in J} U(x, y_j)
\]

\[
\forall x, y, x_j, y_j \in L.
\]

If a binary operation $U$ is left infinitely $\lor$-distributive ($\land$-distributive) and also right infinitely $\lor$-distributive ($\land$-distributive), then $U$ is said to be infinitely $\lor$-distributive ($\land$-distributive).

Noting that the least upper bound of the empty set is $0$ and the greatest lower bound of the empty set is $1$ (see Birkhoff, 1967), we have that

\[
U(0, y) = U \left( \bigvee_{j \in \emptyset} x_j, y \right) = \bigvee_{j \in \emptyset} U(x_j, y) = 0
\]

\[
U(x, 0) = U \left( x, \bigvee_{j \in \emptyset} y_j \right) = \bigvee_{j \in \emptyset} U(x, y_j) = 0
\]

for any $x, y \in L$ when $U$ is left (right) infinitely $\lor$-distributive and

\[
U(1, y) = U \left( \bigwedge_{j \in \emptyset} x_j, y \right) = \bigwedge_{j \in \emptyset} U(x_j, y) = 1
\]

\[
U(x, 1) = U \left( x, \bigwedge_{j \in \emptyset} y_j \right) = \bigwedge_{j \in \emptyset} U(x, y_j) = 1
\]

for any $x, y \in L$ when $U$ is left (right) infinitely $\land$-distributive.

For the sake of convenience, we introduce the following symbols:
\( \mathcal{U}^e_L(L) \): the set of all left semi-uninorms with the left neutral element \( e_L \) on \( L \);

\( \mathcal{U}^e_R(L) \): the set of all right semi-uninorms with the right neutral element \( e_R \) on \( L \);

\( \mathcal{U}^\infty_L(L) \): the set of all right infinitely ∨-distributive left semi-uninorms with the left neutral element \( e_L \) on \( L \);

\( \mathcal{U}^\infty_R(L) \): the set of all left infinitely ∨-distributive right semi-uninorms with the right neutral element \( e_R \) on \( L \);

\( \mathcal{U}^\infty_{\leftrightarrow}^{e_L}(L) \): the set of all left infinitely ∨-distributive right semi-uninorms with the right neutral element \( e_R \) on \( L \);

\( \mathcal{U}^\infty_{\leftrightarrow}^{e_R}(L) \): the set of all right infinitely ∨-distributive left semi-uninorms with the left neutral element \( e_L \) on \( L \).

Now, we present two examples of left and right semi-uninorms on \( L \).

**Example 2.1 (see Su et al., 2013).** Let \( e_L \in L \),

\[
\begin{align*}
U^e_{sh}(x,y) &= \begin{cases} 
y & \text{if } x \geq e_L, \\
0 & \text{otherwise},
\end{cases} \\
U^e_{sm}(x,y) &= \begin{cases} 
y & \text{if } x \leq e_L, \\
1 & \text{otherwise},
\end{cases} \\
U^{*e}_{sh}(x,y) &= \begin{cases} 
1 & \text{if } y = 1, \\
y & \text{if } x \geq e_L, \\
0 & \text{otherwise},
\end{cases} \\
U^{*e}_{sm}(x,y) &= \begin{cases} 
1 & \text{if } x = 1 \text{ or } y = 1, \\
y & \text{if } e_L \leq x < 1, \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]

where \( x \) and \( y \) are elements of \( L \). Then \( U^e_{sh} \) and \( U^e_{sm} \) are, respectively, the smallest and greatest elements of \( \mathcal{U}^e_L(L) \); and \( U^{*e}_{sh} \) and \( U^{*e}_{sm} \) are, respectively, the smallest and greatest elements of \( \mathcal{U}^\infty_{\leftrightarrow}^{e_L}(L) \). Moreover, it is easy to verify that \( U^{*e}_{sh} \) is the smallest disjunctive right infinitely ∨-distributive right semi-uninorm with the right neutral element \( e_R \).

**Example 2.2 (see Su et al., 2013).** Let \( e_R \in L \),

\[
\begin{align*}
U^{*e}_{sh}(x,y) &= \begin{cases} 
x & \text{if } y \geq e_R, \\
0 & \text{otherwise},
\end{cases} \\
U^{*e}_{sm}(x,y) &= \begin{cases} 
x & \text{if } y \leq e_R, \\
1 & \text{otherwise},
\end{cases} \\
U^{e}_{sh}(x,y) &= \begin{cases} 
1 & \text{if } x = 1 \text{ or } y = 1, \\
x & \text{if } e_R \leq y < 1, \\
0 & \text{otherwise},
\end{cases} \\
U^{e}_{sm}(x,y) &= \begin{cases} 
0 & \text{if } x = 1 \text{ or } y = 0, \\
1 & \text{otherwise},
\end{cases}
\end{align*}
\]

where \( x \) and \( y \) are elements of \( L \). Then \( U^{*e}_{sh} \) and \( U^{e}_{sm} \) are, respectively, the smallest and greatest elements of \( \mathcal{U}^{*e}_R(L) \); and \( U^{*e}_{sm} \) and \( U^{e}_{sm} \) are, respectively, the smallest and greatest elements of \( \mathcal{U}^\infty_{\leftrightarrow}^{e_R}(L) \). Moreover, it is easy to verify that \( U^{e}_{sm} \) is the smallest disjunctive left infinitely ∨-distributive left semi-uninorm with the left neutral element \( e_L \).

3. The residual complicators of left and right semi-uninorms

Recently, Mas et al. (2007) and Ruiz & Torrens (2004) studied the implications and coimplications derived from uninorms on \([0,1]\), Wang & Fang (2009b) discussed the residual complicators of left and right uninorms on a complete lattice, Su & Wang (2013) researched pseudo-uninorms and coimplications on a complete lattice. In this section, based on Mas et al. (2007), Ruiz & Torrens (2004), Su & Wang (2013) and Wang & Fang (2009b), we consider the residual complicators of left and right semi-uninorms on a complete lattice.

First of all, we recall the definitions of implications and coimplications.

**Definition 3.1** (Baczynski & Jayaram, 2008; De Baets, 1997; De Baets & Fodor, 1999). An implication \( I \) on \( L \) is a hybrid monotonous (with decreasing first and increasing second partial mappings) binary operation that satisfies the corner conditions \( I(0,0) = I(1,1) = 1 \) and \( I(1,0) = 0 \).

A coimplication \( C \) on \( L \) is a hybrid monotonous binary operation that satisfies the corner conditions \( C(0,0) = C(1,1) = 0 \) and \( C(0,1) = 1 \).

Implications are extensions of the Boolean implication \( \Rightarrow (P \Rightarrow Q \text{ meaning that } P \text{ is sufficient for } Q) \). Coimplications are extensions of the Boolean coimplication \( \Leftarrow (P \Leftarrow Q \text{ meaning that } P \text{ is not necessary for } Q) \) (see De Baets, 1997; De Baets, Tsiporkova, & Mesar, 1999).

Note that for any coimplication \( C \) on \( L \), due to the monotonicity, the absorption principle holds, that is, \( C(x,0) = C(1,x) = 0 \) for any \( x \in L \).

We denote the set of all implications and the set of all right infinitely ∨-distributive coimplications on \( L \) by \( \mathcal{C}(L) \) and \( \mathcal{C}_r(L) \), respectively.

**Example 3.1** (Su & Wang, 2014). Let

\[
\begin{align*}
C_{h}(x,y) &= \begin{cases} 
1 & \text{if } (x,y) = (0,1), \\
0 & \text{otherwise},
\end{cases} \\
C_{m}(x,y) &= \begin{cases} 
0 & \text{if } x = 1 \text{ or } y = 0, \\
1 & \text{otherwise},
\end{cases}
\end{align*}
\]
where \( x \) and \( y \) are elements of \( L \). It is easy to see that \( C_w \) and \( C_M \) are, respectively, the smallest and greatest elements of \( C(L) \) and \( C_M \) is also the largest element of \( C_v(L) \).

Example 3.2 (Su & Wang, 2014). Let \( L = \{0, a, b, 1\} \) be a lattice, where \( 0 < a < 1, 0 < b < 1, a \land b = 0 \) and \( a \lor b = 1 \). Define two coimplications \( C_1 \) and \( C_2 \) as follows:

\[
\begin{array}{c|ccc|c|ccc|}
 & a & b & 1 & & a & b & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
a & 0 & 0 & 0 & a & 0 & 0 & 0 \\
b & 0 & 0 & 0 & b & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{array}
\]

It is easy to see that \( C_1 \) and \( C_2 \) are two right infinitely \( \lor \)-distributive coimplications, and \( C_1 \land C_2 = C_w \). But \( C_w \) is not right infinitely \( \lor \)-distributive. Suppose \( C \) is the smallest right infinitely \( \lor \)-distributive coimplication, then \( C_1 \geq C \) and \( C_2 \geq C \). Thus, \( C_w = C_1 \land C_2 \geq C \) and \( C_w = C \) since \( C_w \) is the smallest coimplication. It leads to the contradiction that \( C_w \) is right infinitely \( \lor \)-distributive. Therefore, there is no the smallest right infinitely \( \lor \)-distributive coimplication.

This illustrates that \( C_v(L) \) is not a \( \land \)-semilattice.

**Definition 3.2 (Wang & Fang, 2009b).** Let \( U \) be a binary operation on \( L \). Define \( C^L_U, C^R_U \in L^{L \times L} \) as follows:

\[
\begin{align*}
C^L_U(x,y) &= \land \{z \in L | y \leq U(z,x)\} \quad \forall x, y \in L, \\
C^R_U(x,y) &= \land \{z \in L | y \leq U(x,z)\} \quad \forall x, y \in L.
\end{align*}
\]

Here, \( C^L_U \) and \( C^R_U \) are, respectively, called the left and right residual coimplicators of \( U \).

For any operation \( U \) on \( L \), it is straightforward to verify that (see Wang & Fang, 2009b, Theorems 3.1 and 3.2)

1. \( C^L_U(x,0) = C^R_U(x,0) = 0 \) for any \( x \in L \).
2. For any \( x, y \in L \), \( C^L_U(y, U(x,y)) \leq x \) and \( C^R_U(x, U(x,y)) \leq y \).
3. If \( U \) is right-disjunctive, then \( C^L_U(1,y) = 0 \) and if \( U \) is left-disjunctive, then \( C^R_U(1,y) = 0 \).

By virtue of Definition 3.2, it is easy to see that \( C^L_U \) and \( C^R_U \) are all decreasing in the first variable and increasing in the second one when \( U \) is a left (right) semi-uninorm; \( C^L_U(e,x) = C^R_U(e,x) = x \) for any \( x \in L \) when \( U \) is a semi-uninorm with the neutral element \( e \) on \( L \).

**Example 3.3** For those left and right semi-uninorms in Examples 2.1 and 2.2, a simple computation shows that

\[
\begin{align*}
C^L_U(x,y) &= C^L_{e_{L}}(x,y) = \begin{cases} 
0 & \text{if } y = 0, \\
y & \text{if } x \geq e_R,
\end{cases} \\
C^R_U(x,y) &= C^R_{e_{R}}(x,y) = \begin{cases} 
0 & \text{if } y = 0, \\
1 & \text{otherwise,}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
C^L_{e_{L}}(x,y) &= C^L_{e_{L}}(x,y) = \begin{cases} 
0 & \text{if } x = 1 \text{ or } y = 0, \\
1 & \text{otherwise,}
\end{cases} \\
C^R_{e_{R}}(x,y) &= C^R_{e_{R}}(x,y) = \begin{cases} 
0 & \text{if } y = 0, \\
y & \text{if } x \geq e_L, \\
1 & \text{otherwise,}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
C^R_{e_{R}}(x,y) &= C^R_{e_{R}}(x,y) = \begin{cases} 
0 & \text{if } y = 0, \\
1 & \text{otherwise,}
\end{cases} \\
C^R_{e_{L}}(x,y) &= C^R_{e_{L}}(x,y) = \begin{cases} 
0 & \text{if } x = 1 \text{ or } y = 0, \\
y & \text{if } e_R \leq x < 1, \\
1 & \text{otherwise,}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
C^L_{e_{L}}(x,y) &= C^L_{e_{L}}(x,y) = \begin{cases} 
0 & \text{if } x = 1 \text{ or } y = 0, \\
y & \text{if } e_L \leq x < 1, \\
1 & \text{otherwise,}
\end{cases}
\end{align*}
\]

When \( e_L, e_R \in L \setminus \{0, 1\} \), we see that \( C^L_{e_{L}}, C^L_{e_{R}}, C^R_{e_{L}}, C^R_{e_{R}} \) and \( C^R_{e_{L}}, C^R_{e_{R}} \) are four coimplications, \( C^L_{e_{L}}, C^L_{e_{R}}, C^R_{e_{L}}, C^R_{e_{R}} \) and \( C^R_{e_{L}}, C^R_{e_{R}} \) are four right infinitely \( \lor \)-distributive coimplications, but \( C^L_{e_{L}}, C^L_{e_{R}}, C^L_{e_{L}}, C^L_{e_{R}} \) and \( C^R_{e_{L}}, C^R_{e_{R}} \) are not coimplications.
THEOREM 3.1 Let $U \in \mathcal{U}^R_{0}$ ($L$).

(1) For any $x, y \in L, y \leq x \Rightarrow C^R_U(x, y) \leq e_l$.
(2) $C^R_U$ satisfies the neutrality principle with respect to $e_l$ (w.r.t. $e_l$ for short), that is, $C^R_U(e_l, y) = y$ for any $y \in L$.
(3) If $U$ is left-disjunctive, then $C^R_U \in \mathcal{C}(L)$.
(4) If $U \in \mathcal{U}^R_{0}$ ($L$) is left-disjunctive, then $C^R_U \in \mathcal{C}_L$ and

$$C^R_U(x, y) = \min\{z \in L \mid y \leq U(x, z)\}.$$ 

Here, $C^R_U$ is called the right residual coimpliation of the left semi-uninorm $U$.

Proof Clearly, statements (1) and (2) hold.

(3) If $U$ is a left-disjunctive left semi-uninorm with the left neutral element $e_l$, then it follows from the statements before Example 3.3 that $C^R_U$ is non-increasing in its first and non-decreasing in its second variable and $C^R_U(1, 1) = 1$. Moreover,

$$C^R_U(0, 0) = \bigwedge\{z \in L \mid 0 \leq U(0, z)\} = 0.$$ 

By the non-decreasingness of $U$, we see that

$$C^R_U(0, 1) = \bigwedge\{z \in L \mid U(0, z) = 1\} \geq \bigwedge\{z \in L \mid z = U(e_l, z) \geq U(0, z) = 1\} = 1.$$ 

Thus, $C^R_U$ is a coimpliation on $L$.

(4) Refer to the Proofs of Theorem 3.1 in Su & Wang (2013) and Theorem 3.5 in Wang & Fang (2009b).

When $e_l < 1$, for the right infinitely $\vee$-distributive left semi-uninorm $\mathcal{U}^R_{\Omega^R_{\vee}}$, we see that $C^R_{\mathcal{U}^R_{\Omega^R_{\vee}}}$ is in $\mathcal{C}(L)$ by Example 3.3, but $C^L_{\mathcal{U}^L_{\Omega^L_{\wedge}}}(e_l, y) = e_l \neq y$ when $0 < y < e_l$, that is, $C^L_{\mathcal{U}^L_{\Omega^L_{\wedge}}}$ does not satisfy the neutrality principle. This illustrates Theorem 3.1 does not hold for the left residual coimpliation of a left semi-uninorm.

If $P$ and $Q$ are two propositions, then the property $U(x, C^L_{\mathcal{U}^L_{\Omega^L_{\wedge}}}(x, y)) \geq y$ is a generalization of the following tautology $Q \Rightarrow (P \vee (P \Leftrightarrow Q))$ in classical logic and is in some sense dual to the modulus ponens (see De Baets, 1997). By Theorem 3.2 in Su & Wang (2013), we know that $U$ and $C^R_U$ satisfy the generalized dual modus ponens rule and the following right residual principle:

$$y \leq U(x, z) \Leftrightarrow C^R_U(x, y) \leq z \forall x, y, z \in L,$$

when $U$ is a right infinitely $\wedge$-distributive left semi-uninorm on $L$. Similarly, $U$ and $C^R_U$ satisfy the generalized dual modus ponens rule in the form: $U(C^L_U(x, y), x) \geq y$ and the following left residual principle (see Su & Wang, 2013, Theorem 3.1):

$$y \leq U(z, x) \Leftrightarrow C^L_U(x, y) \leq y \forall x, y, z \in L,$$

when $U$ is left infinitely $\wedge$-distributive right semi-uninorm on $L$. Thus, for right semi-uninorms on $L$, we have a similar result.

THEOREM 3.2 Let $U \in \mathcal{U}^R_{0}$ ($L$).

(1) For any $x, y \in L, y \leq x \Rightarrow C^L_U(x, y) \leq e_R$.
(2) $C^L_U$ satisfies the neutrality principle with respect to $e_R$ (w.r.t. $e_R$, for short), that is, $C^L_U(e_R, y) = y$ for any $y \in L$.
(3) If $U$ is right-disjunctive, then $C^L_U \in \mathcal{C}(L)$.
(4) If $U \in \mathcal{U}^L_{0}$ ($L$) is right-disjunctive, then $C^L_U \in \mathcal{C}_L$ and

$$C^L_U(x, y) = \min\{z \in L \mid y \leq U(z, x)\}.$$ 

Here, $C^L_U$ is called the left residual coimpliation of the right semi-uninorm $U$.

The neutrality principle w.r.t. $e_L : C(e_l, y) = y$ and neutrality principle w.r.t. $e_R : C(e_R, y) = y$ are generalizations of the neutrality principle $C(0, y) = y$ for any $y \in L$ in De Baets (1997).

Combining Theorems 3.1 and 3.2, we know that both $C^L_U$ and $C^R_U$ are all right infinitely $\vee$-distributive coimpliations when $U$ is, respectively, an infinitely $\wedge$-distributive right and left semi-uninorm on $L$.

THEOREM 3.3 (1) If $U \in \mathcal{U}^R_{0}$ ($L$), then $C^R_U$ is right infinitely $\vee$-distributive and satisfies the left residual coimpliation and order property with respect to $e_l$ (w.r.t. $e_l$, for short):

$$y \leq x \Leftrightarrow C^L_U(x, y) \leq e_l \forall x, y \in L.$$ 

Moreover, if $U \in \mathcal{U}^R_{0}$ ($L$) is strict left-disjunctive, that is, it is disjunctive and satisfies the condition:

$$U(x, 0) = 1 \Leftrightarrow x = 1 \forall x \in L,$$

then $C^L_U$ is a right infinitely $\vee$-distributive coimpliation which satisfies the order property.

(2) If $U \in \mathcal{U}^R_{0}$ ($L$), then $C^R_U$ is right infinitely $\vee$-distributive and satisfies the right residual principle and order property with respect to $e_R$ (w.r.t. $e_R$, for short):

$$y \leq x \Leftrightarrow C^R_U(x, y) \leq e_R \forall x, y \in L.$$ 

Moreover, if $U \in \mathcal{U}^R_{0}$ ($L$) is strict right-disjunctive, that is, it is disjunctive and satisfies the condition:

$$U(0, x) = 1 \Leftrightarrow x = 1 \forall x \in L,$$

then $C^L_U$ is a right infinitely $\vee$-distributive coimpliation which satisfies the order property.
Assume that $U$ is a left infinitely $\land$-distributive left semi-uninorm with the left neutral element $e_L$. By Theorem 3.2(4), we can see that $C_U^L$ is right infinitely $\lor$-distributive and satisfies the left residual principle. If $x, y \in L$ and $y \leq x$, then it follows from Theorem 3.1(1) that $C_U^L(x, y) \leq C_U^L(x, y)$ if $C_U^L(x, y) \leq e_L$, then
\[ x = U(e_L, x) \geq U(C_U^L(x, y), x) \geq y. \]

Thus, $C_U^L$ satisfies the order property. Moreover, if $U \in \mathcal{U}^{e_L}_L(L)$ is strict left-disjunctive, then
\[ C_U^L(0, 1) = \land \{ z \in L \mid U(z, 0) = 1 \} = 1, \]
\[ C_U^L(1, 1) = \land \{ z \in L \mid U(z, 1) = 1 \} = 0. \]

Therefore, $C_U^L$ is a right infinitely $\lor$-distributive coimplication which satisfies the order property.

Similarly, we can show that $C_U^R$ is a right infinitely $\lor$-distributive coimplication, which satisfies the right residual principle and order property, when $U$ is a strict right-disjunctive right infinitely $\land$-distributive right semi-uninorm.

In particular, if $U$ is an infinitely $\land$-distributive semi-uninorm with the neutral element $e$ (see Liu, 2012), then $C_U^L$ and $C_U^R$ satisfy the residual principle (RP), neutrality principle (NP) and order property (OP), and are all right infinitely $\lor$-distributive coimplications by Theorems 3.1–3.3.

4. The left and right semi-uninorms induced by coimplications

Liu (2012) discussed the semi-uninorms induced by implications and Su & Wang (2013) studied the pseudouninorms induced by coimplications. In this section, based on these works, we investigate the left and right semi-uninorms induced by coimplications on a complete lattice.

**Definition 4.1 (Su & Wang, 2013).** Let $C$ be a binary operation on $L$. Define two induced operators $U_C^L$ and $U_C^R$ of $C$ as follows:
\[ U_C^L(x, y) = \lor \{ z \in L \mid C(z, y) \leq x \} \quad \forall x, y \in L, \]
\[ U_C^R(x, y) = \lor \{ z \in L \mid C(x, z) \leq y \} \quad \forall x, y \in L. \]

Clearly, $U_C^L(1, x) = U_C^R(x, 1) = 1$; $U_C^L(0, x) = U_C^R(x, 0) = \lor \{ z \in L \mid C(z, x) = 0 \}$ for any $x \in L$; and $U_C^L = U_C^R$ if $I$ satisfies the condition:
\[ C(y, z) \leq x \iff C(x, z) \leq y \quad \forall x, y, z \in L. \]

When $C$ is hybrid monotonic, it is easy to see that $U_C^L$ and $U_C^R$ are all non-decreasing in its each variable.

Moreover, for any binary operation $C$, it follows from Definition 4.1 that
\[ y \leq U_C^L(C(x, y), x), \quad y \leq U_C^R(x, C(x, y)) \quad \forall x, y \in L. \]

These explain that $U_C^L$ and $C$, $U_C^R$ and $C$ satisfy the generalized dual modus ponens rule.

**Example 4.1** For two coimplications $C_W$ and $C_M$ in Example 3.1, we have that
\[ U_{C_W}^L(x, y) = U_{C_M}^R(x, y) = \begin{cases} 1 & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise}, \end{cases} \]
\[ U_{C_W}^R(x, y) = \lor \{ \forall e \in L \mid 1 \} \quad \forall x, y \in L. \]

Thus, four operations induced by coimplications $C_W$ and $C_M$ are neither left semi-uninorms nor right semi-uninorms on $L$.

We know that the right residuum of a disjunctive right infinitely $\land$-distributive left semi-uninorm and the left residuum of a disjunctive left infinitely $\land$-distributive right semi-uninorm are all right infinitely $\lor$-distributive coimplications. Below, we find some conditions such that these operations induced by coimplications are left or right semi-uninorms.

**Theorem 4.1** Let $C \in \mathcal{C}(L)$.

1. If $C$ satisfies the order property w.r.t. $e_L$, then $U_C^L \in \mathcal{U}^{e_L}_L(L)$; if $C$ satisfies the neutrality principle w.r.t. $e_L$, then $U_C^R \in \mathcal{U}^{e_L}_R(L)$. Here, $U_C^L$ and $U_C^R$ are called the left semi-uninorms induced by the coimplication $C$.

2. If $C$ satisfies the order property w.r.t. $e_R$, then $U_C^R \in \mathcal{U}^{e_R}_R(L)$; if $C$ satisfies the neutrality principle w.r.t. $e_R$, then $U_C^L \in \mathcal{U}^{e_R}_L(L)$. Here, $U_C^L$ and $U_C^R$ are called the right semi-uninorms induced by the coimplication $C$.

3. If $C$ satisfies the order property w.r.t. $e_L$ and neutrality principle w.r.t. $e_R$, then $U_C^L$ is a semi-uninorm (see Liu, 2012) on $L$.

4. If $C$ satisfies the order property w.r.t. $e_R$ and neutrality principle w.r.t. $e_L$, then $U_C^R$ is also a semi-uninorm on $L$.

**Proof** Assume that $C \in \mathcal{C}(L)$. Then $U_C^L$ is non-decreasing in each variable. If $C$ satisfies the order property w.r.t. $e_L$,
then
\[ U_C^{\ell}(e_L, y) = \vee \{ z \in L \mid C(y, z) \leq e_L \} = \vee \{ z \in L \mid z \leq y \} = y \quad \forall y \in L. \]

Thus, \( U_C^R \in U_{\alpha}^R(L) \). If \( C \) satisfies the neutrality principle w.r.t. \( e_L \), then
\[ U_C^{R}(e_L, y) = \vee \{ z \in L \mid C(e_L, z) \leq y \} = \vee \{ z \in L \mid z \leq y \} = y \quad \forall y \in L. \]

So, \( U_C^R \in U_{\alpha}^R(L) \).

Similarly, we can show that \( U_C^R \in U_{\alpha}^R(L) \) when the coimplication \( C \) satisfies the order property w.r.t. \( e_R \) and \( U_C^R \in U_{\alpha}^R(L) \) when \( C \) satisfies the neutral element principle w.r.t. \( e_R \).

If \( C \) satisfies the order property w.r.t. \( e_L \), let \( U_C^R(x, e_R) = x \) for any \( x \in L \). Thus, \( e_L = e_R \). Then \( U_C^R \) is a semi-uninorm with the neutral element \( e \) on \( L \).

In a similar way, we can see that \( U_C^R \) is also a semi-uninorm on \( L \) when \( C \) satisfies the order property w.r.t. \( e_R \) and neutrality principle w.r.t. \( e_L \).

When \( C \in C(L) \), \( C(1, x) = 0 \) for any \( x \in L \) and hence it follows from Definition 4.1 that \( U_C^R(0, 1) = U_C^R(1, 0) = 1 \). Thus, \( U_C^R \) and \( U_C^R \) in Theorem 4.1 are all disjunctive left or right semi-uninorms induced by the coimplication \( C \).

**Theorem 4.2** Let \( C \in C_\vee(L) \).

1. If \( C \) satisfies the order property w.r.t. \( e_L \), then \( U_C^R \in U_{\alpha}^R(L) \); if \( C \) satisfies the neutrality principle w.r.t. \( e_L \), then \( U_C^R \in U_{\alpha}^R(L) \).

2. If \( C \) satisfies the order property w.r.t. \( e_R \), then \( U_C^L \in U_{\alpha}^L(L) \); if \( C \) satisfies the neutrality principle w.r.t. \( e_R \), then \( U_C^L \in U_{\alpha}^L(L) \).

3. If \( C \) satisfies the order property w.r.t. \( e_L \) and neutrality principle w.r.t. \( e_R \), then \( U_C^L \) is a left infinitely \( \wedge \)-distributive semi-uninorm on \( L \).

4. If \( C \) satisfies the order property w.r.t. \( e_R \) and neutrality principle w.r.t. \( e_L \), then \( U_C^R \) is a right infinitely \( \wedge \)-distributive semi-uninorm on \( L \).

**Proof** Assume that \( C \) is a right infinitely \( \vee \)-distributive coimplication. By the proof of Theorem 5.1 in Su & Wang (2013), we have that
\[ C(y, z) \leq x \iff z \leq U_C^L(x, y), \forall x, y, z \in L. \]

Noting that \( x \geq C(y, U_C^L(x, y)) \), we know that
\[ U_C^L(x, y) = \max \{ z \in L \mid C(y, z) \leq x \}. \]

Moreover, when the index set \( J \neq \emptyset \), for any \( x_j, y \in L \) (\( j \in J \)), we have that
\[ U_C^L(\bigvee_{j \in J} x_j, y) = \vee \{ z \in L \mid C(y, z) \leq \bigvee_{j \in J} x_j \} = \vee \{ z \in L \mid C(y, z) \leq x_j \}, \forall y \in J \]
\[ = \vee \{ z \in L \mid z \leq U_C^L(x_j, y) \} \forall j \in J \]
\[ = \vee \{ z \in L \mid z \leq \bigvee_{j \in J} U_C^L(x_j, y) \} = \bigvee_{j \in J} U_C^L(x_j, y). \]

When \( J = \emptyset \), we see that \( U_C^L(\bigvee_{j \in J} x_j, y) = U_C^L(1, y) = 1 = \bigvee_{j \in J} U_C^L(x_j, y) \). Thus, \( U_C^L \) is left infinitely \( \wedge \)-distributive.

Therefore, by virtue of Theorem 4.1, \( U_C^L \in U_{\alpha}^L(L) \) when \( C \) satisfies the order property w.r.t. \( e_L \) and \( U_C^L \in U_{\alpha}^R(L) \) when \( C \) satisfies the neutrality principle w.r.t. \( e_R \).

Similarly, we can show that \( U_C^L \) is a right infinitely \( \wedge \)-distributive right semi-uninorm and left semi-uninorm when \( C \) satisfies the order property w.r.t. \( e_R \) and the neutrality principle w.r.t. \( e_L \), respectively.

Statements (3) and (4) are the direct consequences of statements (1) and (2) and Theorem 4.1.

In Theorem 4.2, if \( C \in C_\vee(L) \) satisfies the order property w.r.t. \( e_L \) and \( U_C^L(0, x, 0) = 1 \), then
\[ 1 = C(0, U_C^L(0, x)) = C(0, \vee \{ z \in L \mid C(0, z) \leq x \}) = \vee \{ C(0, z) \mid z \in L, C(0, z) \leq x \} \leq x, \]

which is a strict left-disjunctive. Similarly, if \( C \in C_\vee(L) \) satisfies the order property w.r.t. \( e_R \), then \( U_C^L \) is strict right-disjunctive.

By virtue of Theorem 4.2, we see that if \( C \in C_\vee(L) \) satisfies the order property (OP) and the neutrality principle (NP), then \( U_C^L \) and \( U_C^R \) are, respectively, a left infinitely \( \wedge \)-distributive semi-uninorm and a right infinitely \( \wedge \)-distributive semi-uninorm on \( L \).

When \( C \) is a right infinitely \( \vee \)-distributive coimplication on \( L \), by the proof of Theorem 4.2, we know that \( C \) and \( U_C^L \), \( C \) and \( U_C^R \) satisfy the following adjunction conditions (see Su & Wang, 2013, Theorems 5.1 and 5.2):
\[ C(y, z) \leq x \iff z \leq U_C^L(x, y), \]
\[ C(x, z) \leq y \iff z \leq U_C^R(x, y) \forall x, y, z \in L. \]

Moreover, we have that \( U_C^L(x, y) = \max \{ z \in L \mid C(x, z) \leq y \} \).

5. The relationships between left (right) semi-uninorms and coimplications

In the final section, we reveal the relationships between disjunctive right (left) infinitely \( \wedge \)-distributive left (right) semi-uninorms and right infinitely \( \vee \)-distributive coimplications which satisfy the neutrality principle on a complete lattice.
Theorem 5.1 (1) If \( U \in \mathcal{U}_L^n(L) \) is right-disjunctive, then \( C_L \in C_L^n(L) \) satisfies the neutrality principle w.r.t. \( e_R \) and \( U_{C_L}^L = U \).

(2) If \( U \in \mathcal{U}_R^n(L) \) is left-disjunctive, then \( C_R \in C_R^n(L) \) satisfies the neutrality principle w.r.t. \( e_L \) and \( U_{C_R}^R = U \).

(3) If \( C \in C_L^n(L) \) satisfies the neutrality principle w.r.t. \( e_L \), then \( U_{C}^{R} \in \mathcal{U}_{R}^{C}(L) \) is disjunctive and \( C_{C_L}^R = C \).

(4) If \( C \in C_R^n(L) \) satisfies the neutrality principle w.r.t. \( e_R \), then \( U_{C}^{L} \in \mathcal{U}_{L}^{C}(L) \) is disjunctive and \( C_{C_R}^L = C \).

Proof We only prove that statements (1) and (3) hold.

(1) If \( U \) is a right-disjunctive left infinitely \( \wedge \)-distributive right semi-uninorm, then \( C_L \in C_L^n(L) \) and satisfies the neutrality principle w.r.t. \( e_R \) by Theorem 3.2. Moreover, it follows from the left residual principle that

\[
U_{C_L}^L(x,y) = \vee \{z \in L \mid C_L(y,z) \leq x\}
= \vee \{z \in L \mid z \leq U(x,y)\}
= U(x,y) \quad \forall x,y \in L.
\]

Thus, \( U_{C_L}^L = U \).

(3) If \( C \in C_L^n(L) \) satisfies the neutrality principle w.r.t. \( e_L \), then \( U_{C}^{R} \) is a disjunctive right infinitely \( \wedge \)-distributive left semi-uninorm by Theorem 4.2. Moreover, it follows from the adjunction conditions that

\[
C_{C_L}^R(x,y) = \wedge \{z \in L \mid y \leq U_{C}^{R}(x,z)\}
= \wedge \{z \in L \mid C(x,y) \leq z\}
= C(x,y) \quad \forall x,y \in L.
\]

Therefore, \( C_{C_L}^R = C \).

We denote by \( \mathcal{U}_{L}^{C}(L) \) and \( \mathcal{U}_{R}^{C}(L) \), respectively, the set of all disjunctive right infinitely \( \wedge \)-distributive left semi-uninorms and the set of all disjunctive left infinitely \( \wedge \)-distributive right semi-uninorms; by \( C_{L}^{\text{comp}}(L) \) and \( C_{R}^{\text{comp}}(L) \), respectively, the set of all right infinitely \( \vee \)-distributive coimplifications which satisfy the neutrality principle w.r.t. \( e_L \) and the set of all right infinitely \( \vee \)-distributive coimplifications which satisfy the neutrality principle w.r.t. \( e_R \) on a complete lattice.

For any \( U_1, U_2, C_1, C_2 \in L^{L \times L} \), we define \( U_1 \wedge U_2, C_1 \vee C_2 \in L^{L \times L} \) as follows:

\[
(U_1 \wedge U_2)(x,y) = U_1(x,y) \wedge U_2(x,y),
(C_1 \vee C_2)(x,y) = C_1(x,y) \vee C_2(x,y) \quad \forall x,y \in L.
\]

It is straightforward to verify that the following theorem holds.

Theorem 5.2 (1) \( \mathcal{U}_{L}^{\text{comp}}(L) \) is a meet-semilattice with the smallest element \( U_{L}^{\text{comp}} \) and the greatest element \( U_{M}^{\text{comp}} \).

(2) \( \mathcal{U}_{R}^{\text{comp}}(L) \) is a meet-semilattice with the smallest element \( U_{R}^{\text{comp}} \) and the greatest element \( U_{M}^{\text{comp}} \).

(3) \( C_{L}^{\text{comp}}(L) \) is a join-semilattice with the smallest element \( C_{L}^{R} \) and the greatest element \( C_{M}^{R} \).

(4) \( C_{R}^{\text{comp}}(L) \) is a join-semilattice with the smallest element \( C_{R}^{L} \) and the greatest element \( C_{M}^{L} \).

Define two mappings \( \varphi : \mathcal{U}_{L}^{C}(L) \rightarrow C_{L}^{\text{comp}}(L) \) and \( \psi : \mathcal{U}_{R}^{C}(L) \rightarrow C_{R}^{\text{comp}}(L) \) as follows:

\[
\varphi(U) = C_{L}^{R} \quad \forall U \in \mathcal{U}_{L}^{\text{comp}}(L),
\psi(U) = C_{R}^{L} \quad \forall U \in \mathcal{U}_{R}^{\text{comp}}(L).
\]

Then it follows from Theorem 5.1 that \( \varphi \) and \( \psi \) are all invertible and

\[
\varphi^{-1}(C) = U_{C_L}^{R} \quad \forall C \in C_{L}^{\text{comp}}(L),
\psi^{-1}(C) = U_{C_R}^{L} \quad \forall C \in C_{R}^{\text{comp}}(L).
\]

Moreover, we have the following theorem.

Theorem 5.3 (1) If \( U_1, U_2 \in \mathcal{U}_{L}^{\text{comp}}(L) \), then \( C_{L}^{U_1 \vee U_2} = C_{L}^{U_1} \vee C_{L}^{U_2} \).

(2) If \( U_1, U_2 \in \mathcal{U}_{R}^{\text{comp}}(L) \), then \( C_{R}^{U_1 \wedge U_2} = C_{R}^{U_1} \wedge C_{R}^{U_2} \).

(3) If \( C_1, C_2 \in C_{L}^{\text{comp}}(L) \), then \( U_{C_1 \wedge C_2}(L) = U_{C_1}^{R} \wedge U_{C_2}^{R} \).

(4) If \( C_1, C_2 \in C_{R}^{\text{comp}}(L) \), then \( U_{C_1 \vee C_2}(L) = U_{C_1}^{L} \vee U_{C_2}^{L} \).

Proof We only prove that statements (2) and (4) hold.

(2) If \( U_1, U_2 \in \mathcal{U}_{R}^{\text{comp}}(L) \), then it follows from the left residual principle that

\[
C_{L}^{U_1 \wedge U_2}(x,y) = \wedge \{z \in L \mid y \leq (U_1 \wedge U_2)(z,x)\}
= \wedge \{z \in L \mid y \leq U_1(z,x) \wedge U_2(z,x)\}
= \wedge \{z \in L \mid C_1(y,z) \leq z, C_2(y,z) \leq z\}
= \wedge \{z \in L \mid (C_1(y,z) \wedge C_2(y,z)) \leq z\}
= (C_1 \wedge C_2)(x,y) \quad \forall x,y \in L.
\]

that is, \( C_{L}^{U_1 \wedge U_2} = C_{L}^{C_1 \wedge C_2} \).

(4) If \( C_1, C_2 \in C_{R}^{\text{comp}}(L) \), then it follows from the adjunction conditions that

\[
U_{C_1 \vee C_2}(x,y) = \vee \{z \in L \mid (C_1 \vee C_2)(y,z) \leq x\}
= \vee \{z \in L \mid C_1(y,z) \vee C_2(y,z) \leq x\}
= \vee \{z \in L \mid C_1(y,z) \leq x, C_2(y,z) \leq x\}
= \vee \{z \in L \mid z \leq U_{C_1}(x,y) \wedge U_{C_2}(x,y)\}
= \vee \{z \in L \mid z \leq U_{C_1 \wedge C_2}(x,y)\}
= (U_{C_1 \wedge C_2})^L(x,y) \quad \forall x,y \in L.
\]

Therefore, \( U_{C_1 \vee C_2} = U_{C_1 \wedge C_2} \).
By virtue of Theorem 5.3, we know that
\[
\varphi(U_1 \land U_2) = \varphi(U_1) \lor \varphi(U_2) \quad \forall U_1, U_2 \in \mathcal{U}_{dr}(L),
\]
\[
\psi(U_1 \land U_2) = \psi(U_1) \lor \psi(U_2) \quad \forall U_1, U_2 \in \mathcal{U}_{rd}(L).
\]
Thus, \(\varphi\) and \(\psi\) are, respectively, order-reversing isomorphisms of the meet-semilattice \(\mathcal{U}_{dr}(L)\) onto the join-semilattice \(\mathcal{C}_{\lor_{\text{max}}}^{\text{apen}}(L)\) and the meet-semilattice \(\mathcal{U}_{rd}(L)\) onto the join-semilattice \(\mathcal{C}_{\lor_{\text{min}}}^{\text{apen}}(L)\).

Similarly, we see that
\[
\varphi^{-1}(C_1 \lor C_2) = \varphi^{-1}(C_1) \land \varphi^{-1}(C_2) \quad \forall C_1, C_2 \in \mathcal{C}_{\lor_{\text{max}}}^{\text{apen}}(L),
\]
\[
\psi^{-1}(C_1 \lor C_2) = \psi^{-1}(C_1) \land \psi^{-1}(C_2) \quad \forall C_1, C_2 \in \mathcal{C}_{\lor_{\text{min}}}^{\text{apen}}(L),
\]
that is, \(\varphi^{-1}\) and \(\psi^{-1}\) are, respectively, order-reversing isomorphisms of the join-semilattice \(\mathcal{C}_{\lor_{\text{max}}}^{\text{apen}}(L)\) onto the meet-semilattice \(\mathcal{U}_{dr}(L)\) and the join-semilattice \(\mathcal{C}_{\lor_{\text{min}}}^{\text{apen}}(L)\) onto the meet-semilattice \(\mathcal{U}_{rd}(L)\).

6. Conclusions and future works

Uninorms are important generalizations of triangular norms and conorms, with the neutral elements lying anywhere in the unit interval. Noting that the associative binary operators are often used to generate \(n\)-ary aggregation operators and the commutativity is not desired for these aggregation operators in a lot of cases, Mas et al. introduced the concepts of left and right uninorms on \([0, 1]\) in Mas et al. (2001) and later on a finite chain in Mas et al. (2004) by eliminating the commutativity from the axioms of uninorm and Liu (2012) discussed the concept of semi-uninorms on a complete lattice by removing the associativity and commutativity from the axioms of uninorm. On the other hand, Mas et al. (2007) and Ruiz & Torrens (2004) studied the implications and complications derived from uninorms on \([0, 1]\]. Wang & Fang (2009b) discussed the residual coimplications of left and right uninorms on a complete lattice, and Su & Wang (2013) investigated pseudo-uninorms and complications on a complete lattice.

In this paper, motivated by these works, we discuss the residual coimplicators of left and right semi-uninorms and the left and right semi-uninorms induced by coimplications, show that the right (left) residual coimplicator of a disjunctive right (left) infinitely \(\land\)-distributive left (right) semi-uninorm is a right infinitely \(\lor\)-distributive coimplication which satisfies the neutrality principle, give some conditions such that the operations induced by a coimplication constitute left or right semi-uninorms, demonstrate that the operations induced by a right infinitely \(\lor\)-distributive coimplication, which satisfies the order property or the neutrality principle, are left (right) infinitely \(\land\)-distributive left (right) semi-uninorms or right (left) semi-uninorms, and prove that the meet-semilattice of all disjunctive right (left) infinitely \(\land\)-distributive left (right) semi-uninorms is order-reversing isomorphic to the join-semilattice of all right infinitely \(\lor\)-distributive coimplications that satisfy the neutrality principle.

In forthcoming papers, we will investigate the relationships between left (right) semi-uninorms, implications and complications on a complete lattice.

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