On the third logarithmic coefficient in some subclasses of close-to-convex functions

Nak Eun Cho · Bogumiła Kowalczyk · Oh Sang Kwon · Adam Lecko · Young Jae Sim

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Abstract
For analytic functions $f$ in the unit disk $\mathbb{D}$ normalized by $f(0) = 0$ and $f'(0) = 1$ satisfying in $\mathbb{D}$ respectively the conditions $\text{Re}((1 - z)f'(z)) > 0$, $\text{Re}((1 - z^2)f'(z)) > 0$, $\text{Re}((1 - z + z^2)f'(z)) > 0$, $\text{Re}((1 - z)^2f'(z)) > 0$, the sharp upper bound of the third logarithmic coefficient in case when $f''(0)$ is real was computed.

Keywords Univalent functions · Close-to-convex functions · Functions convex in the direction of the imaginary axis · Logarithmic coefficients · Carathéodory class

Mathematics Subject Classification 30C45

1 Introduction
Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$ and $\mathbb{T} := \partial \mathbb{D}$. Let $\mathcal{H}$ be the class of all analytic functions in $\mathbb{D}$, $\mathcal{A}$ be its subclass of $f$ normalized by $f(0) := 0$ and $f'(0) := 1$, 

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Young Jae Sim
yjsim@ks.ac.kr

Nak Eun Cho
necho@pknu.ac.kr

Bogumiła Kowalczyk
b.kowalczyk@matman.uwm.edu.pl

Oh Sang Kwon
oskwon@ks.ac.kr

Adam Lecko
alecko@matman.uwm.edu.pl

1 Department of Applied Mathematics, Pukyong National University, Pusan 48513, Korea
2 Department of Complex Analysis, University of Warmia and Mazury in Olsztyn, ul. Słoneczna 54, 10-710 Olsztyn, Poland
3 Department of Mathematics, Kyungsung University, Pusan 48434, Korea
i.e., of the form
\[ f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1, \quad z \in \mathbb{D}. \]  
and \( S \) be the subclass of \( A \) of all univalent functions.

Given \( f \in S \) let
\[ \log f(z) = 2 \sum_{n=1}^{\infty} \gamma_n z^n, \quad z \in \mathbb{D} \setminus \{0\}, \quad \log 1 := 0. \]  
The numbers \( \gamma_n \) are called logarithmic coefficients of \( f \). As is well known, the logarithmic coefficients play a crucial role in Milin conjecture ([23], see also [10, p. 155]), namely that for \( f \in S \),
\[ \sum_{m=1}^{n} \sum_{k=1}^{m} \left( k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0. \]
De Branges [8] showing Milin conjecture confirmed the famous Bieberbach conjecture (e.g., [10, p. 37]). It is surprising that for the class \( S \) the sharp estimates of single logarithmic coefficients \( S \) are known only for \( \gamma_1 \) and \( \gamma_2 \), namely,
\[ |\gamma_1| \leq 1, \quad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e} = 0.635 \ldots \]
and are unknown for \( n \geq 3 \).

As usual, instead of the whole class \( S \) one can take into account their subclasses for which the problem of finding sharp estimates of logarithmic coefficients can be studied. When \( f \in S^* \), the class of starlike functions, the inequality \( |\gamma_n| \leq 1/n \) holds for \( n \in \mathbb{N} \) (see e.g. [30, p. 42]). Moreover, for \( f \in SS^*(\beta) \), the class of strongly starlike function of order \( \beta \) (0 < \( \beta \leq 1 \)), it holds that \( |\gamma_n| \leq \beta/n \) (see [28]). Also, the bounds of \( \gamma_n \) for functions in the class of gamma-starlike functions, close-to-convex functions and Bazilevič functions were examined in [30, p. 116], [9,27,29], respectively. In two recent papers, namely, in [15] the bounds of early logarithmic coefficients of the subclasses \( F_1, F_2, F_3 \) of \( S \) and in [1] of the subclass \( F_4 \) of \( S \) of functions \( f \) satisfying respectively the condition
\[ \text{Re} \left\{ (1-z) f'(z) \right\} > 0, \quad z \in \mathbb{D}, \]  
\[ \text{Re} \left\{ (1-z^2) f'(z) \right\} > 0, \quad z \in \mathbb{D}, \]  
\[ \text{Re} \left\{ (1-z+z^2) f'(z) \right\} > 0, \quad z \in \mathbb{D}, \]  
\[ \text{Re} \left\{ (1-z)^2 f'(z) \right\} > 0, \quad z \in \mathbb{D}, \]
were computed. Let us note that each class defined above is the subclass of the well known class of close-to-convex functions, so therefore families \( F_i \), \( i = 1, \ldots, 4 \), contain only univalent functions (e.g., [12, Vol. II, p. 2]). Both cited paper contains sharp bounds of \( \gamma_1 \) and \( \gamma_2 \) and partial results for \( \gamma_3 \) only. The first three results in theorem below were shown in [15], and the last one in [1].

**Theorem 1** Let \( f \in A \) be of the form (1). Then
1. if \( f \in F_1 \) and \( 1 \leq a_2 \leq 3/2 \), then
\[ |\gamma_3| \leq \frac{1}{288} (11 + 15\sqrt{30}) = 0.323466 \ldots; \]
2. if \( f \in \mathcal{F}_2 \) and \( 0 \leq a_2 \leq 1 \), then 
\[
|\gamma_3| \leq \frac{1}{972} (95 + 23\sqrt{46}) = 0.258223 \ldots ;
\]
3. if \( f \in \mathcal{F}_3 \) and \( 1/2 \leq a_2 \leq 3/2 \), then 
\[
|\gamma_3| \leq \frac{1}{7776} (743 + 131\sqrt{262}) = 0.368238 \ldots ;
\]
4. if \( f \in \mathcal{F}_4 \) and \( 1 \leq a_2 \leq 2 \), then 
\[
|\gamma_3| \leq \frac{1}{243} (28 + 19\sqrt{19}) = 0.456045 \ldots
\]

In this paper we improve all results in Theorem 1 for \( \gamma_3 \) for the general case when \( a_2 \) is real. Differentiating (2) and using (1) we get
\[
\gamma_3 = \frac{1}{2} \left( a_4 - a_2 a_3 + \frac{1}{3} a_2^3 \right),
\]
(7)

Since each class \( \mathcal{F}_i \), \( i = 1, \ldots, 4 \), has a representation by using the Carathéodory class \( \mathcal{P} \), i.e., the class of functions \( p \in \mathcal{H} \) of the form
\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D},
\]
(8)

having a positive real part in \( \mathbb{D} \), the coefficients of functions in \( \mathcal{F}_i \), so \( \gamma_3 \) has a suitable representation expressed by the coefficients of functions in \( \mathcal{P} \). Therefore to get the upper bound of \( \gamma_3 \) our computing is based on parametric formulas for the second and third coefficients in \( \mathcal{P} \). The proof of results of Theorem 1 are based on the well known formula on \( c_2 \) and on the formula \( c_3 \) due to Libera and Zlotkiewicz [21,22] with the restriction that \( c_1 \geq 0 \). Since all classes \( \mathcal{F}_i \) are not rotation invariant, to omit the assumption \( c_1 \geq 0 \), we will use a general formula for \( c_3 \), which was found in [4]. However to be self contained we present a proof for \( c_3 \) here. Moreover in our computation of the sharp bound of \( \gamma_3 \) we use a lemma due to Ohno and Sugawa [24].

Let us mention that the conditions (3), (4) and (6) were discovered by Ozaki [25] as useful criteria of univalence. Recall also that the classes \( \mathcal{F}_2 \) and \( \mathcal{F}_4 \) have nice geometrical interpretations, and therefore they play an important role in the geometric function theory. Each function \( f \in \mathcal{F}_2 \) maps univalently \( \mathbb{D} \) onto a domain \( f(\mathbb{D}) \) convex in the direction of the imaginary axis, i.e., for every \( w_1, w_2 \in f(\mathbb{D}) \) such that \( \text{Re } w_1 = \text{Re } w_2 \) the line segment \( [w_1, w_2] \) lies in \( f(\mathbb{D}) \), with the additional property that there exist two points \( \omega_1, \omega_2 \in \partial f(\mathbb{D}) \) for which \( \{\omega_1 + it : t > 0\} \subset \mathbb{C} \setminus f(\mathbb{D}) \) and \( \{\omega_2 - it : t > 0\} \subset \mathbb{C} \setminus f(\mathbb{D}) \) (see e.g., [12, p. 199]). Each function in the class \( \mathcal{F}_4 \) maps univalently \( \mathbb{D} \) onto a domain \( f(\mathbb{D}) \) called convex in the positive direction of the real axis, i.e., \( \{w + it : t \geq 0\} \subset f(\mathbb{D}) \) for every \( w \in f(\mathbb{D}) \) [2,6,7,11,18,19].

At the end, let us say that the conditions (3)–(6) were generalized by replacing polynomials standing at \( f' \) by any quadratic polynomial [16,17], and by any polynomial of any degree having their roots in \( \mathbb{C} \setminus \mathbb{D} \) [13,14].

2 Lemmas

The formula (9) is due to Carathéodory [3] (see e.g., [10, p. 41]). The formula (10) can be found in [26, p. 166]. In a recent paper [4] the formula (11) was shown and the extremal
functions (13) and (14) were computed also. When \( c_1 \geq 0 \) the formula (11) was found by Libera and Zlotkiewicz [21,22] (see also [20]).

**Lemma 1** If \( p \in \mathcal{P} \) is of the form (8), then
\[
\begin{align*}
    c_1 &= 2\zeta_1, \\
    c_2 &= 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2
\end{align*}
\] (9)
and
\[
c_3 = 2\zeta_1^3 + 4(1 - |\zeta_1|^2)\zeta_1\zeta_2 - 2(1 - |\zeta_1|^2)\overline{\zeta_1}\overline{\zeta_2}^2 + 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)\zeta_3
\] (11)
for some \( \zeta_i \in \overline{\mathbb{D}} \), \( i \in \{1, 2, 3\} \).

For \( \zeta_1 \in \mathbb{T} \), there is a unique function \( p \in \mathcal{P} \) with \( c_1 \) as in (9), namely,
\[
p(z) = \frac{1 + \zeta_1 z}{1 - \zeta_1 \overline{z}}, \quad z \in \mathbb{D}.
\] (12)

For \( \zeta_1 \in \mathbb{D} \) and \( \zeta_2 \in \mathbb{T} \), there is a unique function \( p \in \mathcal{P} \) with \( c_1 \) and \( c_2 \) as in (9)-(10), namely,
\[
p(z) = \frac{1 + (\overline{\zeta_1}\zeta_2 + \zeta_1) z + \zeta_2 z^2}{1 + (\overline{\zeta_1}\zeta_2 - \zeta_1) z - \zeta_2 z^2}, \quad z \in \mathbb{D}.
\] (13)

For \( \zeta_1, \zeta_2 \in \mathbb{D} \) and \( \zeta_3 \in \mathbb{T} \), there is a unique function \( p \in \mathcal{P} \) with \( c_1, c_2 \) and \( c_3 \) as in (9)-(11), namely,
\[
p(z) = \frac{1 + (\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 + \zeta_1) z + (\overline{\zeta_1}\zeta_3 + \zeta_1 \overline{\zeta_2}\zeta_3 + \overline{\zeta_2}\zeta_3) z^2 + \zeta_3 z^3}{1 + (\overline{\zeta_2}\zeta_3 + \overline{\zeta_1}\zeta_2 - \zeta_1) z + (\overline{\zeta_1}\zeta_3 - \zeta_1 \overline{\zeta_2}\zeta_3 - \overline{\zeta_2}\zeta_3) z^2 - \zeta_3 z^3}, \quad z \in \mathbb{D}.
\] (14)

The next lemma is a special case of more general results due to Choi, Kim and Sugawa [5] (see also [24]). Define
\[
Y(a, b, c) := \max_{z \in \mathbb{D}} \left( |a + bz + cz^2| + 1 - |z|^2 \right), \quad a, b, c \in \mathbb{R}.
\]

**Lemma 2** [5] If \( ac \geq 0 \), then
\[
Y(a, b, c) = \begin{cases}
|a| + |b| + |c|, & \text{if } |b| \geq 2(1 - |c|), \\
1 + |a| + \frac{b^2}{4(1 - |c|)}, & \text{if } |b| < 2(1 - |c|).
\end{cases}
\]

If \( ac < 0 \), then
\[
Y(a, b, c) = \begin{cases}
1 - |a| + \frac{b^2}{4(1 - |c|)}, & \text{if } -4ac(c^{-2} - 1) \leq b^2 \wedge |b| < 2(1 - |c|), \\
1 + |a| + \frac{b^2}{4(1 + |c|)}, & \text{if } b^2 < \min \left\{ 4(1 + |c|)^2, -4ac(c^{-2} - 1) \right\}, \\
R(a, b, c), & \text{otherwise},
\end{cases}
\]
where
\[
R(a, b, c) = \begin{cases}
|a| + |b| - |c|, & \text{if } |c|(|b| + 4|a|) \leq |ab|, \\
-|a| + |b| + |c|, & \text{if } |ab| \leq |c|(|b| - 4|a|), \\
(|c| + |a|)\sqrt{1 - \frac{b^2}{4ac}}, & \text{otherwise}.
\end{cases}
\]
3 Logarithmic coefficients

Now we will prove the main results of this paper.

3.1 The class $\mathcal{F}_1$

Recall that $f \in \mathcal{F}_1$ if $f \in A$ and

$$\text{Re}((1-z)f'(z)) > 0, \quad z \in \mathbb{D}.\]

**Theorem 2** If $f \in \mathcal{F}_1$ is of the form (1) with $a_2 \in \mathbb{R}$, then

$$|\gamma_3| \leq \frac{1}{288} (11 + 15\sqrt{30}) = 0.323466 \ldots$$

(15)

The inequality is sharp with the extremal function

$$f(z) = \int_0^z \frac{p(t)}{1-t} \, dt, \quad z \in \mathbb{D},$$

(16)

where

$$p(z) = \frac{(1+z)(6+(7-2\sqrt{30})z+6z^2)}{(1-z)(6+(1+\sqrt{30})z+6z^2)}, \quad z \in \mathbb{D}.\]

(17)

**Proof** Let $f \in \mathcal{F}_1$ be of the form (1) with $a_2 \in \mathbb{R}$. Then there exists $p \in \mathcal{P}$ of the form (8) such that

$$(1-z)f'(z) = p(z), \quad z \in \mathbb{D}.\]

(18)

Substituting the series (1) and (8) into (18) and equating the coefficients we get

$$a_2 = \frac{1}{2}(1+c_1), \quad a_3 = \frac{1}{3}(1+c_1+c_2), \quad a_4 = \frac{1}{4}(1+c_1+c_2+c_3).$$

(19)

Note first that since $a_2$ is real, so is $c_1$, and (9) holds with some $\zeta_1 \in [-1,1]$. Moreover, from (19) it follows that $a_2 \in [-1/2,3/2].$

By (7) and (19) we get

$$48\gamma_3 = 3 + c_1 - c_1^2 + 4c_1c_2 + 2c_2 + 6c_3.$$  

Using now (9)–(11) we have

$$48\gamma_3 = 3 + 2\zeta_1 + 4\zeta_1^3 + 4(1-\zeta_1^2)\zeta_2 + 8(1-\zeta_1^2)\zeta_1\zeta_2$$

$$- 12(1-\zeta_1^2)\zeta_1\zeta_2^2 + 12(1-\zeta_1^2)(1-|\zeta_2|^2)\zeta_3,$$

(20)

with $\zeta_1 \in [-1,1]$ and $\zeta_2, \zeta_3 \in \mathbb{D}.$

Hence for $\zeta_1 = 1$ and $\zeta_1 = -1$ we respectively have

$$\gamma_3 = \frac{3}{16} = 0.1875, \quad \gamma_3 = -\frac{1}{16} = -0.0625.$$  

(21)

Let now $\zeta_1 \in (-1,1).$ Then from (20) we obtain

$$48|\gamma_3| \leq 12(1-\zeta_1^2)\Psi(A, B, C),$$

(22)

where

$$\Psi(A, B, C) := |A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2,$$

(23)
with
\[A := \frac{3 + 2\xi_1 + 4\xi_1^3}{12(1 - \xi_1^2)}, \quad B := \frac{1}{3}(1 + 2\xi_1), \quad C := -\xi_1.\]

Note that
\[AC < 0, \quad \xi_1 \in [-1, \xi') \cup (0, 1),\]
and
\[AC \geq 0, \quad \xi_1 \in [\xi', 0].\]

where \(\xi' = -0.72808\ldots\) is the zero of the equation \(3 + 2x + 4x^3 = 0, \ x \in (-1, 0)\).

**A.** Let \(\xi_1 \in [\xi', 0].\) Then the inequality \(|B| < 2(1 - |C|)\) holds, so by (22) and Lemma 2 we have
\[48|\gamma_3| \leq 12(1 - \xi_1^2) \left(1 + |A| + \frac{B^2}{4(1 - |C|)}\right) = \varphi(\xi_1),\] (24)
where
\[\varphi(x) := \frac{1}{3}(46 + 9x - 36x^2 + 8x^3), \quad x \in [-1, 1].\] (25)

Since \(\varphi\) is increasing on \([\xi', 0]\), so \(\varphi(x) \leq \varphi(0) = 46/3\) for all \(x \in [\xi', 0]\). Therefore from (24) we get
\[|\gamma_3| \leq \frac{23}{72} = 0.319444\ldots\]

**B.** Let \(\xi_1 \in (0, 1).\) Then the following inequalities hold:
\[B^2 + 4AC(C^{-2} - 1) = \frac{1}{9\xi_1}(-9 - 5\xi_1 + 4\xi_1^2 - 8\xi_1^3) < 0\]
and
\[B^2 - 4(1 + |C|)^2 = -\frac{1}{9}(35 + 68\xi_1 + 32\xi_1^2) < 0.\]

Therefore from (22) and Lemma 2 it follows that
\[48|\gamma_3| \leq 12(1 - \xi_1^2) \left(1 + |A| + \frac{B^2}{4(1 + |C|)}\right) = \varphi(\xi_1),\] (26)
where \(\varphi\) is the function defined by (25). Since \(\varphi'(x) = 0\) occurs only at \(x = (6 - \sqrt{30})/4 =: x_0\) in \((0, 1)\) and \(\varphi''(x_0) = -4\sqrt{30} < 0\), it follows that
\[\varphi(x) \leq \varphi(x_0) = \frac{1}{6}(11 + 15\sqrt{30}), \quad x \in (0, 1).\]

Thus by (26) we get
\[|\gamma_3| \leq \frac{1}{288}(11 + 15\sqrt{30}) = 0.323466\ldots\]

**C.** Let \(\xi_1 \in (-1, \xi').\) Note that then \(B^2 < 4(1 + |C|)^2.\) Furthermore, \(B^2 + 4AC(C^{-2} - 1) < 0\) holds if and only if \(\xi_1 \in [-1, \xi''],\) where \(\xi'' = -0.73448\ldots\) is the zero of the
equation $9 + 5x - 4x^2 + 8x^3 = 0$, $x \in (-1, 1)$. Therefore, when $\zeta_1 \in (-1, \zeta'')$, by (22) and Lemma 2 we have

$$|\gamma_3| \leq \frac{1}{4}(1 - \zeta_1^2) \left(1 + |A| + \frac{B^2}{4(1 + |C|)}\right)$$

$$= \frac{1}{144}(28 - \zeta_1 - 28\zeta_1^2 - 8\zeta_1^3) < \frac{1}{8} = 0.125.$$  

(27)

For $\zeta_1 \in (\zeta''', \zeta')$ it holds $B^2 + 4AC(C^{-2} - 1) > 0$ and $|B| < 2(1 - |C|)$. Hence by (22) and Lemma 2 we get

$$|\gamma_3| \leq \frac{1}{4}(1 - \zeta_1^2) \left(1 - |A| + \frac{B^2}{4(1 - |C|)}\right)$$

$$= \frac{1}{144}(46 + 9\zeta_1 - 36\zeta_1^2 + 8\zeta_1^3) < \frac{7}{48} = 0.145833 \ldots$$  

(28)

Summarizing, from (21) and parts A-C it follows that the inequality (15) is true.

By tracking back the above proof, we see that equality in (15) holds when it is satisfied that

$$\zeta_1 = \frac{1}{4}(6 - \sqrt{30}), \quad \zeta_3 = 1$$

(29)

and

$$|A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2 = 1 + |A| + \frac{B^2}{4(1 + |C|)},$$

(30)

where

$$A = \frac{-2490 + 731\sqrt{30}}{5460}, \quad B = \frac{8 - \sqrt{30}}{6}, \quad C = \frac{1}{4}(6 - \sqrt{30}).$$

Indeed we can easily check that one of the solutions of Eq. (30) is

$$\zeta_2 = \frac{1}{105}(25 - \sqrt{30}).$$

(31)

By Lemma 1 a function $p$ of the form (14) with $\zeta_i (i \in \{1, 2, 3\})$ given by (29) and (31), i.e., the function (17) belongs to $\mathcal{P}$. Thus the function (16) belongs to $\mathcal{F}_1$. Substituting (29) and (31) into (20) we get equality in (15). This ends the proof of the theorem.  

$\square$

### 3.2 The class $\mathcal{F}_2$

Recall that $f \in \mathcal{F}_2$ if $f \in \mathcal{A}$ and

$$\text{Re}\{(1 - z^2)f'(z)\} > 0, \quad z \in \mathbb{D}.$$  

Theorem 3 If $f \in \mathcal{F}_2$ is of the form (1) with $a_2 \in \mathbb{R}$, then

$$|\gamma_3| \leq \frac{1}{972}(95 + 23\sqrt{46}) = 0.258223 \ldots$$

(32)

The inequality is sharp with the extremal function

$$f(z) = \int_0^z \frac{p(t)}{1 - t^2}dt, \quad z \in \mathbb{D},$$

(33)
where
\[ p(z) = \frac{(1 + z)(9 + (7 - 2\sqrt{46})z + 9z^2)}{(1 - z)(9 + (1 + \sqrt{46})z + 9z^2)}, \quad z \in \mathbb{D}. \tag{34} \]

**Proof** Let \( f \in \mathcal{F}_2 \) be of the form (1). Then there exists \( p \in \mathcal{P} \) of the form (8) such that
\[ (1 - z^2)f'(z) = p(z), \quad z \in \mathbb{D}. \tag{35} \]
Substituting the series (1) and (8) into (35) by equating the coefficients we get
\[ a_2 = \frac{1}{2}c_1, \quad a_3 = \frac{1}{3}(1 + c_2), \quad a_4 = \frac{1}{4}(c_1 + c_3). \tag{36} \]

Note first that since \( a_2 \) is real, so is \( c_1 \) and (9) holds with some \( \zeta_1 \in [-1, 1] \). Moreover, from (36) it follows that \( a_2 \in [-1, 1] \).

By (7) and (36) we get
\[ 48\gamma_3 = 2c_1 + c_1^3 - 4c_1c_2 + 6c_3. \]
Using now (9)–(11) we have
\[ 12\gamma_3 = \zeta_1^3 + \zeta_1 + 2(1 - \zeta_1^2)\zeta_1\zeta_2 - 3(1 - \zeta_1^2)\zeta_1\zeta_2^2 + 3(1 - \zeta_1^2)(1 - |\zeta_2|^2)\zeta_3. \tag{37} \]
with \( \zeta_1 \in [-1, 1] \) and \( \zeta_2, \zeta_3 \in \mathbb{D} \).

Hence for \( \zeta_1 = 1, \zeta_1 = -1 \) and \( \zeta_1 = 0 \) we respectively have
\[ \gamma_3 = \frac{1}{6}, \quad \gamma_3 = -\frac{1}{6}, \quad |\gamma_3| \leq \frac{1}{4}(1 - |\zeta_2|^2) \leq \frac{1}{4}. \tag{38} \]
Now let \( \zeta_1 \in (-1, 1) \setminus \{0\} =: I \). Then from (37) we obtain
\[ 12|\gamma_3| \leq 3(1 - \zeta_1^2)\Psi(A, B, C), \tag{39} \]
where \( \Psi \) is defined by (23) with
\[ A := \frac{\zeta_1(1 + \zeta_1^2)}{3(1 - \zeta_1^2)}, \quad B := \frac{2}{3}\zeta_1, \quad C := -\zeta_1. \]
Note now that \( AC < 0 \) for \( \zeta_1 \in I \). Moreover,
\[ B^2 \leq -4AC(C^{-2} - 1), \quad \zeta_1 \in I, \]
since
\[ B^2 + 4AC(C^{-2} - 1) = -\frac{4}{9}(3 + 2\zeta_1^2) < 0, \quad \zeta_1 \in I, \]
and
\[ B^2 < 4(1 + |C|)^2, \quad \zeta_1 \in I, \]
since
\[ B^2 - 4(1 + |C|)^2 = -\frac{4}{9}(8\zeta_1^2 + 18|\zeta_1| + 9) < 0, \quad \zeta_1 \in I. \]
Therefore by Lemma 2 we get
\[
\Psi(A, B, C) \leq 1 + |A| + \frac{B^2}{4(1 + |C|)} = 1 + \frac{|\xi_1|(1 + \xi_1^2)}{3(1 - \xi_1^2)} + \frac{\xi_1^2}{9(1 + |\xi_1|)}.
\]
Hence and from (39) it follows that
\[
12|\gamma_3| \leq \varphi(\xi_1),
\]
where
\[
\varphi(x) := \frac{1}{3}(9 + 3|x| - 8x^2 + 2|x|^3), \quad x \in I.
\]
We note that the function \( \varphi \) is even in I. As easy to verify
\[
\varphi(x) \leq \varphi(x_0) = \frac{1}{81}(95 + 23\sqrt{46}), \quad x \in I,
\]
where \( x_0 := (8 - \sqrt{46})/6 = 0.202945 \ldots \). Hence and by (40) we obtain
\[
|\gamma_3| \leq \frac{1}{12}\varphi(x_0) \leq \frac{1}{972}(95 + 23\sqrt{46}).
\]
This and (38) show that the inequality (32) is true.

By tracking back the above proof, we see that equality in (32) holds when it is satisfied that
\[
\xi_1 = \frac{1}{6}(8 - \sqrt{46}), \quad \xi_3 = 1
\]
and
\[
|A + B\xi_2 + C\xi_2^2| + 1 - |\xi_2|^2 = 1 + |A| + \frac{B^2}{4(1 + |C|)},
\]
where
\[
A = \frac{-1688 + 283\sqrt{46}}{3150}, \quad B = \frac{1}{9}(8 - \sqrt{46}), \quad C = -\frac{1}{6}(8 - \sqrt{46}).
\]
Indeed we can easily check that one of the solutions of the equation (42) is
\[
\xi_2 = \frac{1}{75}(11 - \sqrt{46}).
\]

By Lemma 1 a function \( p \) of the form (14) with \( \zeta_i \ (i \in \{1, 2, 3\}) \) given by (41) and (43), i.e., the function (34) belongs to \( \mathcal{P} \). Thus the function (33) belongs to \( \mathcal{F}_2 \). Substituting (41) and (43) into (37) we get equality in (32). This ends the proof of the theorem. \( \square \)

### 3.3 The class \( \mathcal{F}_3 \)

Recall that \( f \in \mathcal{F}_3 \) if \( f \in \mathcal{A} \) and
\[
\text{Re}((1 - z + z^2)f'(z)) > 0, \quad z \in \mathbb{D}.
\]
Theorem 4 If \( f \in \mathcal{F}_3 \) is of the form (1) with \( a_2 \in \mathbb{R} \), then
\[
|\gamma_3| \leq \frac{1}{7776}(743 + 131\sqrt{262}) = 0.368238\ldots
\]
This result is sharp.

Proof
Let \( f \in \mathcal{F}_3 \) be of the form (1) with \( a_2 \in \mathbb{R} \). Then there exists \( p \in \mathcal{P} \) of the form (8) such that
\[
(1 - z + z^2)f'(z) = p(z), \quad z \in \mathbb{D}.
\]
Substituting the series (1) and (8) into (45) by equating the coefficients we get
\[
a_2 = \frac{1}{2}(1 + c_1), \quad a_3 = \frac{1}{3}(c_1 + c_2), \quad a_4 = \frac{1}{4}(-1 + c_2 + c_3).
\]

Note first that since \( a_2 \) is real, so is \( c_1 \) and (9) holds with some \( \gamma_1 \in [-1, 1] \). Moreover, from (46) it follows that \( a_2 \in [-1/2, 3/2] \).

By (7) and (46) we get
\[
48\gamma_3 = -5 - c_1 - c_1^2 + c_1^3 - 4c_1c_2 + 2c_2 + 6c_3.
\]
Using now (9)–(11) we have
\[
48\gamma_3 = -5 - 2\gamma_1 + 4\gamma_1^3 + 4(1 - \gamma_1^2)\gamma_2 + 8(1 - \gamma_1^2)\gamma_1 \gamma_2
\]
\[\quad - 12(1 - \gamma_1^2)\gamma_1 \gamma_2^2 + 12(1 - \gamma_1^2)(1 - |\gamma_2|^2)\gamma_3,
\]
with \( \gamma_1 \in [-1, 1] \) and \( \gamma_2, \gamma_3 \in \mathbb{D} \).

Hence for \( \gamma_1 = 1 \) and \( \gamma_1 = -1 \) we respectively have
\[
\gamma_3 = -\frac{1}{16}, \quad \gamma_3 = -\frac{7}{48}.
\]

Now let \( \gamma_1 \in (-1, 1) \). Then from (47) we get
\[
48|\gamma_3| \leq 12(1 - \gamma_1^2)\Psi(A, B, C),
\]
where \( \Psi \) is defined by (23) with
\[
A := \frac{-5 - 2\gamma_1 + 4\gamma_1^3}{12(1 - \gamma_1^2)}, \quad B := \frac{1}{3}(1 + 2\gamma_1), \quad C := -\gamma_1.
\]

Note that \( A < 0 \) for \( \gamma_1 \in (-1, 1) \).

Let \( \gamma_1 \in (-1, 0) \). Then \( AC < 0 \) and it can be easily checked that the following inequalities are true:
\[
B^2 + 4AC(C^{-2} - 1) = \frac{1}{9\gamma_1}(15 + 7\gamma_1 + 4\gamma_1^2 - 8\gamma_1^3) < 0
\]
and
\[
B^2 - 4(1 + |C|)^2 = -\frac{1}{9}(35 - 76\gamma_1 + 32\gamma_1^2) < 0.
\]

Hence from (49) and Lemma 2 and it follows that
\[
|\gamma_3| \leq \frac{1}{4}(1 - \gamma_1^2) \left(1 + |A| + \frac{B^2}{4(1 + |C|)}\right)
\]
\[
= \frac{1}{144}(52 + 11\gamma_1 - 28\gamma_1^2 - 8\gamma_1^3) \leq \frac{13}{36} = 0.361111\ldots
\]
Let now $\zeta_1 \in [0, 1)$. Then $AC \geq 0$ and we consider two subcases, i.e., $\zeta_1 \in [5/8, 1)$ and $\zeta_1 \in [0, 5/8)$. For $\zeta_1 \in [5/8, 1)$, it holds $|B| \geq 2(1-|C|)$. Thus by (49) and Lemma 2 we have

$$|\gamma_3| \leq \frac{1}{4}(1-\zeta_1^2)(|A|+|B|+|C|) = \frac{1}{48}(9 + 22\zeta_1 - 4\zeta_1^2 - 24\zeta_1^3) \leq \frac{327}{1024} = 0.319335\ldots$$

(51)

For $\zeta_1 \in [0, 5/8)$ it holds $|B| < 2(1-|C|)$. Thus (49) and Lemma 2 lead to

$$48|\gamma_3| \leq 12(1-\zeta_1^2)\left(1 + |A| + \frac{B^2}{4(1-|C|)}\right) = \varphi(\zeta_1),$$

(52)

where

$$\varphi(x) := \frac{1}{3}(52 + 11x - 28x^2 - 8x^3), \quad x \in [0, 5/8).$$

As easy to verify, for $x \in [0, 5/8)$,

$$\varphi(x) \leq \varphi(x_0) = \frac{1}{162}(743 + 131\sqrt{262}) = 17.675433\ldots,$$

where $x_0 = (-14 + \sqrt{262})/12 = 0.182201\ldots \in [0, 5/8)$. Hence and by (52) it follows that

$$|\gamma_3| \leq \frac{1}{7776}(743 + 131\sqrt{262}).$$

(53)

Summarizing (48), (50), (51) and (53) show that the inequality (44) is true.

By tracking back the above proof, we see that equality in (44) holds when it is satisfied that

$$\zeta_1 = \frac{1}{12}(-14 + \sqrt{262}), \quad \zeta_3 = 1$$

(54)

and

$$|A + B\zeta_2 + C\zeta_2^2| + 1 - |\zeta_2|^2 = 1 + |A| + \frac{B^2}{4(1-|C|)},$$

(55)

where

$$A = \frac{9526 - 1601\sqrt{262}}{35604}, \quad B = -\frac{1}{18}(8 - \sqrt{262}), \quad C = \frac{1}{12}(14 - \sqrt{262}).$$

Indeed we can check that $\zeta_2$ defined by

$$\zeta_2 = \frac{4924 - 269\sqrt{262} \pm 2i\sqrt{57399872 - 3438382\sqrt{262}}}{11946 - 555\sqrt{262}},$$

(56)

satisfies the equation (55).

By Lemma 1 a function $p$ of the form (14) with $\zeta_i (i \in \{1, 2, 3\})$ given by (54) and (56) belongs to $\mathcal{P}$. Note that $|\zeta_2| = 0.912\ldots$ Thus the corresponding function function

$$f(z) = \int_0^z p(t) \frac{1}{1 - t^2} \, dt, \quad z \in \mathbb{D},$$

belongs to $\mathcal{F}_3$. Substituting such chosen $\zeta_1$, $\zeta_2$ and $\zeta_3$ into (47) we get equality in (44). This ends the proof of the theorem. $\square$
3.4 The class $\mathcal{F}_4$

Recall that $f \in \mathcal{F}_4$ if $f \in \mathcal{A}$ and

$$\text{Re}\{(1 - z)^2 f'(z)\} > 0, \ z \in \mathbb{D}.$$  

**Theorem 5** If $f \in \mathcal{F}_4$ is of the form (1) with $a_2 \in \mathbb{R}$, then

$$|\gamma_3| \leq \frac{1}{243} (28 + 19\sqrt{19}) = 0.456045 \ldots$$  

(57)

The inequality is sharp with the extremal function

$$f(z) = \int_0^z \frac{p(t)}{(1 - t)^2} dt, \ z \in \mathbb{D},$$  

(58)

where

$$p(z) = \frac{(1 + z) \left( 9 + (14 - 4\sqrt{19})z + 9z^2 \right)}{(1 - z) \left( 9 + 2(1 + \sqrt{19})z + 9z^2 \right)}, \ z \in \mathbb{D}.$$  

(59)

**Proof** Let $f \in \mathcal{F}_4$ be of the form (1) with $a_2 \in \mathbb{R}$. Then there exists $p \in \mathcal{P}$ of the form (8) such that

$$(1 - z)^2 f'(z) = p(z), \ z \in \mathbb{D}.$$  

(60)

Putting the series (1) and (8) into (60) by equating the coefficients we get

$$a_2 = \frac{1}{2} (2 + c_1), \quad a_3 = \frac{1}{3} (3 + 2c_1 + c_2),$$
$$a_4 = \frac{1}{4} (4 + 3c_1 + 2c_2 + c_3).$$  

(61)

As in earlier consideration, $\xi_1 \in [-1, 1]$ and from (61) it follows that $a_2 \in [0, 1]$.

By (7) and (61) we get

$$12\gamma_3 = \frac{1}{4} (8 + 2c_1 - 2c_1^2 + c_1^3 - 4c_1c_2 + 4c_2 + 6c_3).$$

Using now (9)–(11) we have

$$12\gamma_3 = \xi_1^3 + \xi_1 + 2 + 2(1 - \xi_1^2)\xi_2 + 2(1 - \xi_1^2)\xi_1\xi_2$$
$$- 3(1 - \xi_1^2)\xi_1\xi_2^2 + 3(1 - \xi_1^2)(1 - |\xi_2|^2)\xi_3,$$  

(62)

with $\xi_1 \in [-1, 1]$ and $\xi_2, \xi_3 \in \overline{\mathbb{D}}$.

Hence for $\xi_1 = 1$ and $\xi_1 = -1$ we respectively have

$$\gamma_3 = 1/3, \quad \gamma_3 = 0.$$  

(63)

Let now $\xi_1 \in (-1, 1)$. Then

$$12|\gamma_3| \leq 3(1 - \xi_1^2)\Psi(A, B, C),$$  

(64)

where $\Psi$ is defined by (23) with

$$A := \frac{2 - \xi_1 + \xi_1^2}{3(1 - \xi_1)}, \quad B := \frac{2}{3} (1 + \xi_1), \quad C := -\xi_1.$$  

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For $\zeta_1 \in (-1, 0]$ it holds $AC \geq 0$ and $|B| < 2(1 - |C|)$. Thus by (64) and Lemma 2 we have
\[
|\gamma_3| \leq \frac{1}{4}(1 - \zeta_1^2) \left(1 + |A| + \frac{B^2}{4(1 - |C|)}\right)
\]
\[
= \frac{1}{18}(8 + 2\zeta_1 - 5\zeta_1^2 + \zeta_1^3) \leq \frac{4}{9}.
\]
\hspace*{1cm}(65)

For $\zeta_1 \in (0, 1)$ it can be easily checked that
\[
AC < 0, \quad B^2 < -4AC(C^{-2} - 1), \quad B^2 < 4(1 + |C|)^2.
\]

Therefore by (64) and Lemma 2 we get
\[
12|\gamma_3| \leq 3(1 - \zeta_1^2) \left(1 + |A| + \frac{B^2}{4(1 + |C|)}\right) = \varphi(\zeta_1),
\]
\hspace*{1cm}(66)

where
\[
\varphi(x) := \frac{2}{3}(8 + 2x - 5x^2 + x^3), \quad x \in (0, 1).
\]

As easy to verify
\[
\varphi(x) \leq \varphi(x_0) = \frac{4}{81}(28 + 19\sqrt{19}), \quad x \in (0, 1),
\]
where $x_0 := (5 - \sqrt{19})/3$. Hence and by (66) it follows that
\[
|\gamma_3| \leq \frac{1}{243}(28 + 19\sqrt{19}).
\]
\hspace*{1cm}(67)

Summarizing, (63), (65) and (67) show that the inequality (57) is true.

A similar method used for the proof of Theorem 2, the equality in (57) when it is satisfied that
\[
\zeta_1 = \frac{1}{3}(5 - \sqrt{19}), \quad \zeta_2 = \frac{1}{3}, \quad \zeta_3 = 1.
\]
\hspace*{1cm}(68)

By Lemma 1 a function $p$ of the form (14) with $\zeta_i (i \in \{1, 2, 3\})$ given by (68), i.e., the function (59) belongs to $\mathcal{P}$. Thus the function (58) belongs to $\mathcal{F}_4$. Substituting (68) into (62) we get equality in (57). This ends the proof of the theorem. \qed

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