HYSTERESIS-DRIVEN PATTERN FORMATION IN
REACTION-DIFFUSION-ODE SYSTEMS

ALEXANDRA KÖTHE
Institute of Applied Mathematics and Bioquant
Heidelberg University
Heidelberg, 69120, Germany

ANNA MARCINIAK-CZOCHRA*
Institute of Applied Mathematics, Bioquant and
Interdisciplinary Center for Scientific Computing (IWR)
Heidelberg University
Heidelberg, 69120, Germany

IZUMI TAKAGI
Institute for Mathematical Sciences, Renmin University of China
Beijing 100872, China
and
Mathematical Institute, Tohoku University
Sendai, 980-8578, Japan

ABSTRACT. The paper is devoted to analysis of far-from-equilibrium pattern formation in a system of a reaction-diffusion equation and an ordinary differential equation (ODE). Such systems arise in modeling of interactions between cellular processes and diffusing growth factors. Pattern formation results from hysteresis in the dependence of the quasi-stationary solution of the ODE on the diffusive component. Bistability alone, without hysteresis, does not result in stable patterns. We provide a systematic description of the hysteresis-driven stationary solutions, which may be monotone, periodic or irregular. We prove existence of infinitely many stationary solutions with jump discontinuity and their asymptotic stability for a certain class of reaction-diffusion-ODE systems. Nonlinear stability is proved using direct estimates of the model nonlinearities and properties of the strongly continuous diffusion semigroup.

1. Introduction. This paper is devoted to mathematical analysis of hysteresis-driven pattern formation. Mathematically, a pattern is a stable spatially inhomogeneous stationary solution of an evolution equation describing the development of a system.

2010 Mathematics Subject Classification. Primary: 35K57, 35B36; Secondary: 35J25, 35K20.
Key words and phrases. Reaction-diffusion-ODE system, pattern formation, bistable kinetics, hysteresis, stationary solution with jump discontinuity, asymptotic stability.

This work is supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Collaborative Research Center 1324 (SFB1324, project B6). It has been supported in part by JSPS Kakenhi, Grant Numbers 16KT0128 and 19K03557.

* Corresponding author: Anna Marciniak-Czochra.
Most mathematical models of biological or ecological patterns are based on reaction-diffusion equations with the Turing mechanism (diffusion-driven instability) [30] which can explain de novo pattern formation [11]. Diffusion-driven instability is related to a local behavior of solutions in the neighborhood of a constant stationary solution that is destabilized by diffusion. It may lead to emergence of close to the equilibrium patterns that are stable continuous and spatially periodic structures around the destabilized constant equilibrium. The Turing concept became a paradigm for biological pattern formation and led to development of numerous theoretical models, though its biological validation has remained elusive. Another type of patterns are far-from-equilibrium patterns that arise in systems with multiple constant stationary solutions that usually exhibit bistability and hysteresis. Bistability (or more generally multistability) is a phenomenon of systems toggling among two (or more) stable equilibrium points. Usually, between the stable equilibria, there is an unstable intermediate equilibrium working as a threshold. Hysteresis refers to a system where the output does not only depend on the input, but also on the history of the system that may lead to different responses to the same input. Bistability and hysteresis are mechanisms important for biological systems, which generate oscillations and switching between discrete states, and integrate transient stimuli [1]. They can create an all-or-none response and transform a graded input signal into a discontinuous output. Moreover, hysteresis may be responsible for irreversibility, which is of particular importance in developmental processes [4].

To understand the role of bistability and hysteresis in pattern formation, we propose a prototype mathematical model based on a reaction-diffusion equation coupled to an ordinary differential equation (ODE) with hysteresis. Such models have recently been employed in biology [8, 10, 16, 21, 26, 31] and in ecology [15]. They allow describing cell-to-cell communication coupled to intracellular dynamics as well as binding of a diffusing molecule to an immobile substrate [12] or receptor [16, 17]. Reaction-diffusion-ODE can be obtained as a homogenization limit of the models describing coupling of cell-localized processes with cell-to-cell communication via diffusion in a cell assembly [23, 18]. Furthermore, they may occur as reduced problems in the analysis of reaction-diffusion systems with small diffusion coefficients. For example, in [24] certain solutions of the reduced system have been used to construct large-amplitude solutions of a system of two reaction-diffusion equations having in interior transition layer, with one diffusion coefficient being sufficiently small and the other sufficiently large.

Reaction-diffusion-ODE models with hysteresis may additionally exhibit diffusion-driven instability what leads to an interesting dynamics as shown recently in Ref. [6, 13]. In this paper, to streamline the presented analysis and explore the role of bistability and hysteresis, we focus on a model that does not exhibit the Turing mechanism. Instead, it has two spatially homogenous stationary solutions that are always locally asymptotically stable. Diffusion acts to average different states and is the cause of spatio-temporal patterns. We prove that bistability without the hysteresis effect is not sufficient for existence of stable spatially heterogenous patterns. Moreover, we provide a systematic description of stationary solutions that may differ from those of the usual reaction-diffusion systems [20, 19, 7]. In particular, we show a co-existence of infinite number of stationary solutions that exhibit jump-discontinuities in non-diffusing variables. In applications, discontinuities are useful for defining sharp boundaries of gene expression pattern [8].
This paper is organized as follows: In Section 2, we set up our problem and show that there is no spatially heterogeneous stable stationary solution if the nonlinearity does not exhibit hysteresis (Theorem 2.2). In the remainder of the paper we consider only nonlinearities with hysteresis. Section 3 is devoted to the existence (Theorem 3.2) and the asymptotic stability (Theorem 3.7 and Corollary 1) of monotone increasing (or decreasing) stationary solutions with jump discontinuity. In Section 4 we consider the location of the point of discontinuity (Theorem 4.3). Finally in Section 5 we consider the existence and stability of spatially periodic stationary solutions (Corollary 2) and then proceed to spatially irregular stationary solutions. We prove the uniqueness (Proposition 4) and the stability (Theorem 5.4) of spatially irregular stationary solutions, and discuss the existence (Example 5.3) of such solutions.

2. Problem setting. In this section we introduce a prototype reaction-diffusion-ODE model with hysteresis and discuss its basic properties. We consider a one-dimensional problem
\begin{align}
  u_t &= \frac{1}{\gamma} u_{xx} + f(u, v) \quad \text{for} \quad x \in (0, 1), \ t > 0, \\
  v_t &= g(u, v) \quad \text{for} \quad x \in [0, 1], \ t > 0,
\end{align}
supplemented with the homogeneous Neumann boundary condition for \( u \)
\[ u_x(t, 0) = u_x(t, 1) = 0 \]
and initial conditions
\[ u(0, x) = u_0(x), \quad v(0, x) = v_0(x). \]
To focus on a simple structure of nonlinearities exhibiting hysteresis, we consider a specific choice of the kinetic functions
\[ f(u, v) = \alpha v - \beta u, \]
\[ g(u, v) = u - p(v), \]
where \( \alpha, \beta \) are positive constants and \( p(v) \) is a polynomial of degree three such that \( p(v) \to +\infty \) as \( v \to +\infty \). The parameters are chosen such that there exist three intersection points of \( f = 0 \) and \( g = 0 \) with non-negative coordinates.

The above assumptions yield a bistable system with or without the hysteresis effect, see Fig. 1. We define:

Case 1 (Bistability without hysteresis): Polynomial \( p(v) \) is assumed to be monotone increasing (cf. Fig. 1a).

Case 2 (Hysteresis): Polynomial \( p(v) \) is assumed to be non-monotone. In this case, we denote by \( H = (u_H, v_H) = (p(v_H), v_H) \) the local maximum of \( v \mapsto p(v) \) and by \( T = (u_T, v_T) = (p(v_T), v_T) \) the local minimum of \( v \mapsto p(v) \). Moreover, we assume that the coordinates of \( H \) and \( T \) are positive and that \( \lim_{v \to +\infty} p(v) = +\infty \) holds (cf. Fig. 1b).

Here, by bistability it is meant that the kinetic system
\begin{align}
  \frac{du}{dt} = f(u, v), \quad \frac{dv}{dt} = g(u, v)
\end{align}
has two asymptotically stable equilibria.

We check that the system is bistable in both cases but only a system with hysteresis allows emergence of stable patterns (spatially heterogeneous stationary solutions).
Lemma 2.1. Assume that there exist three spatially homogeneous stationary solutions \( S_0, S_1 \) and \( S_2 \), \( 0 < u_1 < u_2, 0 < v_1 < v_2 \), of system (1)-(3) with nonlinearities given by (5). Then, \( S_0 \) and \( S_2 \) are asymptotically stable, while \( S_1 \) is unstable.

Proof. First, we check bistability of the kinetic system (6), i.e., asymptotic stability of the spatially homogeneous stationary solutions \( S_0 \) and \( S_2 \) of system (1)-(3) under constant perturbations.

Let \( J(\bar{u}, \bar{v}) \) be Jacobi-matrix of function \( (f(u, v), g(u, v)) \) at a stationary solution \( (\bar{u}, \bar{v}) \). As \( \alpha \beta \) is the slope of \( f(u, v) = 0 \) solved with respect to \( v \), we obtain \( p'(0) > \frac{\alpha}{\beta} > 0, p'(v_2) > \frac{\alpha}{\beta} > 0 \). Thus, calculating

\[
\det J(\bar{u}, \bar{v}) = \beta \left(p'(\bar{v}) - \frac{\alpha}{\beta}\right) \quad \text{and} \quad \text{Tr} J(\bar{u}, \bar{v}) = -\left(\beta + p'(\bar{v})\right)
\]

at the solutions \( S_0 \) and \( S_2 \), we conclude about their asymptotic stability.

For \( S_1 \) we have to distinguish between Case 1 and Case 2. In Case 2 it holds \( p'(v_1) < 0 \), hence \( \det J(S_1) < 0 \). In Case 1, since \( p'(v_1) < \frac{\alpha}{\beta} \), it also holds that \( \det J(S_1) < 0 \). Therefore, linearization at \( S_1 \) has one positive and one negative eigenvalue.

The stability properties do not change for spatially heterogeneous perturbations, since diffusion causes a destabilization of stable spatially constant stationary solution (diffusion-driven instability) in a reaction-diffusion-ODE system if and only if the ODE component exhibits autocatalysis, i.e., \( g_v(\bar{u}, \bar{v}) > 0 \), see Theorem 2.11 in Ref. [19]. It is not the case for model (6), since \( p'(0) > 0 \) and \( p'(v_2) > 0 \) and hence \( g_v(\bar{u}, \bar{v}) < 0 \), independently of the choice of admissible model parameters. Consequently, \( S_0 \) and \( S_2 \) are asymptotically stable as spatially homogenous stationary solutions of system (1)-(3).
Next, we observe that bistability is not sufficient for emergence of stable patterns. Let \((U, V)\) be a stationary solution of system (1)-(3),

\[
\frac{1}{\gamma} U_{xx} + f(U, V) = 0, \quad x \in (0, 1) \tag{7}
\]

\[
g(U, V) = 0, \tag{8}
\]

\[
U_x(0) = U_x(1) = 0. \tag{9}
\]

Since, in Case 1, the polynomial \(p\) is monotone increasing, equation (8) can be uniquely solved with respect to \(V\). We obtain an elliptic boundary value problem

\[
0 = \frac{1}{\gamma} U_{xx} + f(U, h(U)) \text{ for } x \in [0, 1], \text{ and } U_x(0) = U_x(1) = 0, \tag{10}
\]

where \(h(U)\) satisfies \(g(U, h(U)) = 0\). Since \(f(U, h(U))\) is a smooth function, equation (10) has classical solutions \(U \in C^2([0, 1])\), and hence also \(V = h(U) \in C^2([0, 1])\). Such solutions can be explicitly constructed by the phase plane analysis (cf. [2]). Their instability follows from analysis of the linearized problem using a similar argument as in case of a scalar reaction-diffusion equation. It results in

**Theorem 2.2.** There exists no stable non-homogeneous stationary solution \((U, V)\) of system (1)-(4) in Case 1.

Details of the proof are provided in Appendix for the sake of completeness.

3. Patterns with jump-discontinuity in the model with hysteresis. In this section, we show that all stationary spatially heterogeneous solutions of problem (7)-(9) with parameters satisfying Case 2 have a jump-discontinuity in \(V(x)\). Then, we focus on construction and analysis of monotone solutions.

3.1. Construction of discontinuous steady-states. In Case 2, polynomial \(p\) is non-monotone and the stationary algebraic equation (8) cannot be solved uniquely with respect to \(V\). We denote by \(V = h_H(U)\) the lower solution branch and by \(V = h_T(U)\) the upper one (cf. Fig. 1b). There is a third solution branch \(h_0\) in the middle, but since the spatially constant steady state \(S_1\) located on this branch is unstable, we do not expect to obtain stable stationary solutions using this branch. Indeed, we show in Section 3.2 that the stability condition is not fulfilled for \(V = h_0(U)\).

The stationary problem reads

\[
0 = \frac{1}{\gamma} U_{xx} + f(U, h_i(U)) \text{ for } x \in [0, 1], \text{ and } U_x(0) = U_x(1) = 0, \tag{11}
\]

where \(i = H\) or \(i = T\). Using phase plane analysis, we conclude that there is no solution of this problem neither for \(i = H\) nor \(i = T\), which means that there exists no stationary spatially heterogenous \(C^2\)-solutions.

In the next step, we observe that the phase planes associated to the equation (11) for \(i = H\) and \(i = T\) overlap for \(u \in (u_T, u_H)\) and search for solutions with jump discontinuity (cf. Fig. 2A). Heuristically, to construct a solution, we select a value \(\bar{u} \in (u_T, u_H)\) and “glue” the phase planes together at \(\bar{u}\).

It is convenient to define the following functions:

\[
q_H(u) = f(u, h_H(u)) \text{ for } 0 \leq u < u_H, \quad \text{and} \quad q_T(u) = f(u, h_T(u)) \text{ for } u > u_T. \tag{12}
\]
Let $q_u$ denote the function with discontinuity at $\bar{u}$ defined by
$$
q_u(u) = \begin{cases} 
q_H(u) & \text{when } u \leq \bar{u} \\
q_T(u) & \text{when } u > \bar{u}.
\end{cases}
$$
We search for a weak solution of the boundary value problem
$$
\frac{1}{\gamma} U_{xx} + q_u(U) = 0 \quad \text{for } x \in (0,1) \quad \text{and} \quad U_x(0) = U_x(1) = 0,
$$
i.e. a function $U \in H^1(0,1)$ satisfying
$$
-\frac{1}{\gamma} \int_0^1 U_x(x) \varphi_x(x) dx + \int_0^1 q_u(U(x)) \varphi(x) dx = 0
$$
for all $\varphi \in C^\infty([0,1])$. Notice that weak solutions $U \in H^1(0,1)$ belong to $H_2^N(0,1) = \{ v \in H^2(0,1) \mid v'(0) = v'(1) = 0 \} \subset C^1([0,1])$ by virtue of the standard elliptic regularity theory.

**Definition 3.1.** A pair of functions $(U, V)$ is called a solution with jump discontinuity at $\bar{u}$ of the stationary problem (7)-(9), if $U \in H^1(0,1)$ is a weak solution of equation (13). Function $V \in L^\infty(0,1)$ is given for almost all $x \in [0,1]$ by
$$
V(x) = \begin{cases} 
h_H(U(x)) & \text{if } U(x) < \bar{u} \\
h_T(U(x)) & \text{if } U(x) > \bar{u}.
\end{cases}
$$
The value $0 < \bar{x} < 1$ such that $U(\bar{x}) = \bar{u}$ is called the layer position of the solution $(U, V)$.

**Theorem 3.2.** For all diffusion coefficients $\frac{1}{\gamma}$ with $\gamma > 0$, there exists a unique monotone increasing solution $(U, V)$ with jump at $\bar{u}$ of the stationary problem (7)-(9) of system (7)-(9) with nonlinearities satisfying Case 2.

**Proof.** We apply the classical method of phase plane analysis [2] and the analysis of time-maps [28]. The first integral of equation (13) reads
$$
\frac{U^2(x)}{2\gamma} + Q_u(U(x)) = E,
$$
where the function
$$
Q_u(u) = \int_0^u q_u(\tilde{u}) \, d\tilde{u}
$$
is called the potential and corresponds in a physical context to the potential energy of the system. The constant $E \in \mathbb{R}$ is arbitrary and corresponds to the total energy of the system.

Problem (13) can be written as a boundary value problem for the system of first order equations
$$
U_x = W, \quad W_x = -\gamma q_u(U) \quad \text{and} \quad W(0) = W(1) = 0.
$$

A monotone increasing solution of problem (13) is given by a solution of system (16) satisfying
$$
(U(0), W(0)) = (u_0, 0) \quad \text{and} \quad (U(1), W(1)) = (u_e, 0),
$$
where $u_0, u_e \in \mathbb{R}$ are such that $u_0 < u_e$ and $W(x) > 0$ for all $x \in (0,1)$, see Fig. 2A. It follows from the first integral (14)
$$
W = U_x = \sqrt{2\gamma(E - Q_u(U))},
$$
where the constant

$$E = Q_{\bar{u}}(u_0) = Q_{\bar{u}}(u_e)$$  

is determined by the Neumann boundary condition at $x = 0$ and $x = 1$. Hence, because of (18), a necessary condition for the solution $U(x)$ requires that $Q_{\bar{u}}(U(x)) \leq E$ holds for $x \in [0, 1]$.

By definition $Q_{\bar{u}}$ fulfils $Q_{\bar{u}}(u) = q_{\bar{u}}(u)$ and $Q_{\bar{u}}(0) = 0$. It is continuous for all $u$ but not differentiable in $\bar{u}$. Assuming that $\bar{u} < u_2$, we conclude that the potential has local maxima at $u = 0$ and at $u = u_2$, because $q_{\bar{u}}(0) = f(0, h_H(0)) = f(0, 0) = 0$, and $q_{\bar{u}}(u_2) = f(u_2, h_H(u_2)) = f(u_2, v_2) = 0$, respectively. Furthermore, it has a local minimum at $u = \bar{u}$. Therefore, for $\bar{u} \in (u_T, \min(u_H, u_2))$, it is possible to find values $0 < u_0 < \bar{u} < u_e < u_2$ fulfilling condition (19) and such that $Q_{\bar{u}}(u) < Q_{\bar{u}}(u_0)$ for all $u \in (u_0, u_e)$.

Equation (18) yields

$$1 = \frac{1}{\sqrt{2\gamma}} \int_0^1 \frac{U_x(x)dx}{\sqrt{E - Q_{\bar{u}}(U(x))}} = \frac{1}{\sqrt{2\gamma}} \int_{u_0}^{u_e} \frac{du}{\sqrt{E - Q_{\bar{u}}(u)}}$$  

We split this integral at the minimum of the potential and denote

$$T_{u_0}^1(0) = \frac{1}{\sqrt{2\gamma}} \int_{u_0}^{\bar{u}} \frac{du}{\sqrt{Q_{\bar{u}}(u_0) - Q_{\bar{u}}(u)}} , \quad T_{u_e}^2(0) = \frac{1}{\sqrt{2\gamma}} \int_{\bar{u}}^{u_e} \frac{du}{\sqrt{Q_{\bar{u}}(u_e) - Q_{\bar{u}}(u)}} .$$  

(21)

$T_{u_0}^1(0)$ and $T_{u_e}^2(0)$ are called time-maps, since our variable $x$ corresponds to time in classical models from mechanics. For a forward orbit in the phase plane representation of the system (16), with $q_{\bar{u}}(u) = f(u, h_H(u))$ and starting at $(u_0, 0)$, $T_{u_0}^1(0)$ is the “time” $x$ required to reach the $u = \bar{u}$ axis for the first time. In distinction to this, $T_{u_e}^2(0)$ is the “time” $x$, required for a backward orbit in the phase plane representation system (16), with $q_{\bar{u}}(u) = f(u, h_T(u))$ and starting at $(u_e, 0)$, to reach the $u = \bar{u}$ axis for the first time.
Thus, the total “time” to connect \((u_0, 0)\) with \((u_c, 0)\), where \(u_c\) depends on \(u_0\) through equation (19), is given by

\[
T_u(u_0) := T^1_u(u_0) + T^2_u(u_c(u_0))
\]

(22)

and the problem of finding a monotone increasing solution of (13) is reduced to that of finding a value \(u_0 \in (0, \bar{u})\) such that

\[
T_u(u_0) = 1.
\]

(23)

Lemma 3.3. The mappings \(T^1_u : (0, \bar{u}) \to (0, \infty)\) and \(T^2_u : (\bar{u}, u_2) \to (0, \infty)\) are well-defined, continuous and surjective.

Proof. The proofs of well-definedness and continuity are standard and can be found in [28]. We sketch the proof of surjectivity for completeness.

The integrand of \(T^1_u\) has a singularity at \(U = u_0\). Since \(Q_u(u_0) = q_H(u_0) \neq 0\), using the Taylor expansion we conclude that the integral is convergent.

Moreover, the Taylor expansion at 0 and the fact that \(Q_u\) has a local maximum at 0 (since \(q_H(0) = 0\) and \(q_H'(0) < 0\)) allow to estimate

\[
Q_u(u_0) - Q_u(u) \leq Q_u(0) - (Q_u(0) + q_H(0)u + \frac{q_H'(0)}{2}u^2) \leq Cu^2,
\]

where \(C = \max_{[0, u]} -\frac{1}{2}q_H'(\eta)\) (note that \(C > 0\) since \(q_H'(0) < 0\)) and therefore we obtain

\[
\lim_{u_0 \to 0} T^1_u(u_0) \geq \lim_{u_0 \to 0} \frac{1}{\sqrt{2\gamma C}} \int_{u_0}^{\bar{u}} \frac{du}{\sqrt{u^2}} = \lim_{u_0 \to 0} \frac{1}{\sqrt{2\gamma C}} \left( \ln(\bar{u}) - \ln(u_0) \right) = \infty.
\]

For \(u_0\) and \(u\) close to \(\bar{u}\), we use the Taylor expansion of \(Q_u\) at \(u_0\) and obtain

\[
Q_u(u_0) - Q_u(u) = Q_u(u_0) - (Q_u(u_0) + q_H(u)(u - u_0)) = -q_H(u)(u - u_0).
\]

with \(u \in (u_0, u)\). Thus, we calculate the limit

\[
\lim_{u_0 \to \bar{u}} T^1_u(u_0) = \lim_{u_0 \to \bar{u}} \frac{1}{\sqrt{2\gamma}} \int_{u_0}^{\bar{u}} \frac{du}{\sqrt{-q_H(\eta)(u - u_0)}} \leq \lim_{u_0 \to \bar{u}} \left( \max_{\eta \in [u_0, \bar{u}]} \frac{1}{\sqrt{2\gamma |q_H(\eta)|}} \right) \int_{u_0}^{\bar{u}} \frac{du}{\sqrt{\bar{u} - u_0}}
\]

\[
= \frac{1}{\sqrt{2\gamma |q_H(\bar{u})|}} \lim_{u_0 \to \bar{u}} 2\sqrt{\bar{u} - u_0} = 0.
\]

The corresponding results for \(T^2_u(u_c)\) are obtained in a similar fashion. \(\Box\)

The maps \(T^1_u\) and \(T^2_u\), resp., are defined for all \(u_0 \in (0, \bar{u})\) and \(u_c \in (\bar{u}, u_2)\). However, \(T^1_u(u_0)\) is defined only for \(u_0\) such that there exists a \(u_c\) satisfying (19). Thus, the domain of definition of \(T^1_u(u_0)\) depends on the local maxima of the potential. Therefore, if \(Q_u(u_2) \geq Q_u(0) = 0\), we denote \(u_{\min} = 0\) and \(\bar{u} < u_{\max} \leq u_2\) is the solution of \(Q_u(u_{\max}) = 0\). If \(Q_u(u_2) \leq 0\), we denote \(u_{\max} = u_2\) and \(0 \leq u_{\min} < \bar{u}\) is the solution of \(Q_u(u_{\min}) = Q_u(u_2)\).

Lemma 3.4. The mapping \(T_u : (u_{\min}, \bar{u}) \to (0, \infty)\) is well-defined, continuous and surjective.
Proof. Clearly $T_{\bar{u}}$ is continuous as a sum and the composition of continuous functions. Because of the continuity of $Q_{\bar{u}}$ we conclude that $u_0 \to \bar{u}$ implies $u_c \to \bar{u}$ and, therefore,
\[
\lim_{u_0 \to u} T_{\bar{u}}(u_0) = \lim_{u_0 \to u} T_{\bar{u}}^1(u_0) + \lim_{u_c \to \bar{u}} T_{\bar{u}}^2(u_c) = 0.
\]
Finally, we know that that either $u_{\min} = 0$ or $u_{\max} = u_2$ holds true and therefore
\[
\lim_{u_0 \to u_{\min}} T_{\bar{u}}(u_0) = \lim_{u_0 \to u_{\min}} T_{\bar{u}}^1(u_0) + \lim_{u_c \to u_{\max}} T_{\bar{u}}^2(u_c) = \infty,
\]
because either the first or the second limit is infinite. \qed

The previous lemma provides existence of a monotone increasing solution of problem (13) for any diffusion coefficient $\frac{1}{\gamma} > 0$. Next, we show uniqueness of this solution, which is a consequence of the monotonicity of the time-map $T_{\bar{u}}$. To prove this, we need the following representation of the derivatives of the time-maps.

**Lemma 3.5.** The time-maps are differentiable and the derivatives have the following form
\[
\frac{dT_{\bar{u}}^1}{du_0}(u_0) = -\frac{1}{\sqrt{\gamma}} \frac{q_H(u_0)}{Q_{\bar{u}}(u_0) - Q_{\bar{u}}(\bar{u})} \int_{u_0}^{\bar{u}} \left( \frac{(Q_{\bar{u}}(u) - Q_{\bar{u}}(\bar{u}))q'_H(u)}{q_H(u)^2} - \frac{1}{2} \right) \frac{du}{\sqrt{Q_{\bar{u}}(u_0) - Q_{\bar{u}}(u)}}
\]
and
\[
\frac{dT_{\bar{u}}^2}{du_c}(u_c) = -\frac{1}{\sqrt{\gamma}} \frac{q_T(u_c)}{Q_{\bar{u}}(u_c) - Q_{\bar{u}}(\bar{u})} \int_{\bar{u}}^{u_c} \left( \frac{(Q_{\bar{u}}(u) - Q_{\bar{u}}(\bar{u}))q'_T(u)}{q_T(u)^2} - \frac{1}{2} \right) \frac{du}{\sqrt{Q_{\bar{u}}(u_c) - Q_{\bar{u}}(u)}}.
\]

**Proof.** This formula has been shown in (2.18) and (2.19) of Ref. [14] under the assumption $q_{\bar{u}} \in C^0$ with piecewise continuous $q'_{\bar{u}}$ and $q_{\bar{u}}(\bar{u}) = 0$. We adapt that proof to requirements satisfied by our problem.

First, we notice that the derivative of $T_{\bar{u}}^2(u_c)$ can be written in the form
\[
\frac{dT_{\bar{u}}^2}{du_c}(u_c) = \frac{1}{\sqrt{2\gamma} \sqrt{\Delta Q_{\bar{u}}(u_c, \bar{u})}} - \frac{1}{\sqrt{2\gamma}} \int_{\bar{u}}^{u_c} \frac{q_T(u_c) - q_T(u)}{(\Delta Q_{\bar{u}}(u_c, u)))^{3/2}} \frac{du}{\Delta Q_{\bar{u}}(u_c, u))^{3/2}},
\]
where
\[
\Delta Q_{\bar{u}}(u_1, u_2) = Q_{\bar{u}}(u_1) - Q_{\bar{u}}(u_2),
\]
since the respective proof provided in Theorem 1 of [14] requires only existence of $q'_T$ for $u > \bar{u}$. Now, we observe that it holds, for any $\bar{u} < a < b < u_c$,
\[
\int_a^b \frac{q_T(b) \, du}{(\Delta Q_{\bar{u}}(u_c, u)))^{3/2}} = \frac{1}{\Delta Q_{\bar{u}}(u_c, \bar{u})} \left( \int_a^b \frac{q_T(b) \, du}{\sqrt{\Delta Q_{\bar{u}}(u_c, u))^{3/2}}} + \int_a^b \frac{q_T(b) \Delta Q_{\bar{u}}(u_c, u) \, du}{(\Delta Q_{\bar{u}}(u_c, u)))^{3/2}} \right).
\]
Using integration by parts of the second integral with
\[
v(u) = \frac{q_T(b) \Delta Q_{\bar{u}}(u_c, u)}{q_T(u)} \quad \text{and} \quad w(u) = \frac{q_T(u)}{(\Delta Q_{\bar{u}}(u_c, u)))^{3/2}}
\]
which imply
\[
v'(u) = q_T(b) \left( 1 - \Delta Q_{\bar{u}}(u_c, \bar{u})q'_T(u) \right) \quad \text{and} \quad w'(u) = \frac{2}{\sqrt{\Delta Q_{\bar{u}}(u_c, u))^{3/2}}},
\]
we obtain
\[
\int_a^b \frac{q_T(b) \, du}{(\Delta Q_\bar{u}(u, a))^{3/2}} = \frac{1}{\Delta Q_\bar{u}(u_e, \bar{u})} \left( \frac{2(\Delta Q_\bar{u}(b, \bar{u})}{\sqrt{\Delta Q_\bar{u}(u_e, b)}} - \frac{q_T(b)}{\sqrt{\Delta Q_\bar{u}(u_e, a)}} \right) \left( \frac{2(\Delta Q_\bar{u}(a, \bar{u})}{\sqrt{\Delta Q_\bar{u}(u_e, a)}} \right)
\]
\[+ \frac{2q_T(b)}{\Delta Q_\bar{u}(u_e, \bar{u})} \int_a^b \left( \frac{\Delta Q_\bar{u}(u, \bar{u})}{q_T(u)^2} - \frac{1}{2} \right) \frac{du}{\sqrt{\Delta Q_\bar{u}(u_e, u)}} \]

Furthermore, since \((\partial / \partial u) \Delta Q_\bar{u}(u_e, u) = -q_T(u)\), we have
\[
\int_a^b q_T(u) \, du (Q_\bar{u}(u_e) - Q_\bar{u}(u))^{3/2} = \frac{2}{\sqrt{Q_\bar{u}(u_e) - Q_\bar{u}(b)}} - \frac{2}{\sqrt{Q_\bar{u}(u_e) - Q_\bar{u}(a)}}
\]
which yields
\[
\int_a^b \frac{q_T(b) - q_T(u)}{\Delta Q_\bar{u}(u_e, u))^{3/2}} \, du = \frac{1}{\Delta Q_\bar{u}(u_e, \bar{u})} \left( -2\sqrt{\Delta Q_\bar{u}(u_e, b)} - \frac{q_T(b)}{\sqrt{\Delta Q_\bar{u}(u_e, a)}} \right) \left( \frac{2\Delta Q_\bar{u}(a, \bar{u})}{\sqrt{\Delta Q_\bar{u}(u_e, a)}} \right)
\]
\[+ \frac{2q_T(b)}{\Delta Q_\bar{u}(u_e, \bar{u})} \int_a^b \left( \frac{\Delta Q_\bar{u}(u, \bar{u})}{q_T(u)^2} - \frac{1}{2} \right) \frac{du}{\sqrt{\Delta Q_\bar{u}(u_e, u)}}
\]

Finally, letting \(a \to \bar{u}\) and \(b \to u_e\), we obtain
\[
\int_{\bar{u}}^{u_e} \frac{q_T(u_e) - q_T(u)}{\Delta Q_\bar{u}(u_e, u))^{3/2}} \, du = \frac{2}{\sqrt{\Delta Q_\bar{u}(u_e, \bar{u})}}
\]
\[+ \frac{2q_T(u_e)}{\Delta Q_\bar{u}(u_e, \bar{u})} \int_{\bar{u}}^{u_e} \left( \frac{\Delta Q_\bar{u}(u, \bar{u})}{q_T(u)^2} - \frac{1}{2} \right) \frac{du}{\sqrt{\Delta Q_\bar{u}(u_e, u)}}
\]
Together with representation (26), this integral yields the formula in the lemma. In
a similar fashion we deduce the result for \(\frac{dT_\bar{u}}{du_0}(u_0)\).

\[\square\]

**Proposition 1.** Consider the stationary problem (7)-(9) in Case 2. Then the
derivatives of \(T_\bar{u}\) and \(T_{\bar{u}}^2\) have the following signs
\[
\frac{dT_\bar{u}}{du_0}(u_0) < 0 \quad \text{and} \quad \frac{dT_{\bar{u}}^2}{du_e}(u_e) > 0.
\]

Therefore, for all jumps \(\bar{u} \in (u_T, \min(u_H, u_2))\), the derivative of the
time-map \(T_\bar{u}\) is negative, i.e.,
\[
\frac{dT_\bar{u}}{du_0}(u_0) < 0.
\]

**Proof.** To show the negativity of \(\frac{dT_\bar{u}}{du_0}(u_0)\), we rewrite the integral representation
(24) of the derivative of the time-map in the form
\[
\frac{dT_\bar{u}}{du_0}(u_0) = -\frac{q_H(u_0)}{\sqrt{T_F^2 \Delta Q_\bar{u}(u_0, \bar{u})}} \int_{\bar{u}}^{u_e} \frac{l_H(u) \, du}{\sqrt{\Delta Q_\bar{u}(u_0, u)}}
\]
with a function \(l_H\) defined by
\[
l_H(u) = \frac{\Delta Q_\bar{u}(u, \bar{u}) \cdot q_H(u)}{q_H(u)^2} - \frac{1}{2}.
\]
We observe that $-q_H(u_0)/\sqrt{2\Delta Q_\bar{u}(u_0, \bar{u})}$ is positive, as $q_H(u_0) < 0$ and $Q_\bar{u}$ is decreasing in the interval $(0, \bar{u})$. Thus, it remains to show that $l_H$ is negative for $u \in [u_0, \bar{u}]$.

For this purpose, we multiply $l_H$ by the square of $q_H$ and calculate the derivative

$$\frac{d}{du} (q_H(u)^2 l_H(u)) = \Delta Q_\bar{u}(u, \bar{u}) q_H''(u)$$

which leads together with

$$q_H(u)^2 l_H(u)|_{u=\bar{u}} = -\frac{1}{2} q_H(\bar{u})^2 =: -C_H < 0$$

to the representation

$$q_H(u)^2 l_H(u) = -C_H + \int_{\bar{u}}^{u} \Delta Q_\bar{u}(\bar{u}, \bar{u}) q_H''(\bar{u}) d\bar{u} = -C_H - \int_{u}^{\bar{u}} \Delta Q_\bar{u}(\bar{u}, \bar{u}) q_H''(\bar{u}) d\bar{u}.$$  

To see that this expression is negative, we observe that $q_H''(u) = \alpha h_H''(u) = -\alpha p''(h_H(u))/[p'(h_H(u))]^3$ is of positive sign. Indeed, for all $u \in (u_0, \bar{u})$ it holds $p'(h_H(u)) > 0$ as well as $p''(h_H(u)) < 0$. Hence, we obtain $l_H(u) < 0$, which proves $\frac{d}{du_0} T_{\bar{u}}(u_0) < 0$. In a similar fashion, we can show that $\frac{d}{du_0} T_{\bar{u}}^2(u_e) > 0$.

To accomplish the proof, we deduce

$$\frac{d u_e}{du_0}(u_0) = \frac{q_H(u_0)}{q_H(u_e)} < 0$$ \quad \text{(27)}$$

by differentiating equation (19) with respect to $u_0$. Therefore, we obtain

$$\frac{dT_{\bar{u}}}{du_0}(u_0) = \frac{dT_{\bar{u}}}{du_0}(u_0) + \frac{dT_{\bar{u}}^2}{du_e}(u_e) \cdot \frac{du_e}{du_0}(u_0) < 0.$$

\hfill \Box
3.2. **Stability analysis.** A standard approach to showing stability of stationary solutions is based on linear stability analysis which in case of discontinuous solutions requires careful treatment, e.g., considering a special topology that allows excluding the discontinuity points \([3, 6]\). In this section, we show the stability of stationary solutions by applying direct estimates.

Numerical simulations of the basic model with hysteresis considered here suggest that the selected pattern depends on the initial conditions. More precisely, the ultimate stationary solution depends on the position of the initial functions relative to the separatrix of the saddle point of the kinetic system. For this reason it is suitable to consider the stability in \(L^\infty(0,1)\) sense. A perturbation which is small in \(L^\infty(0,1)\) cannot move a point on the stationary solution \((U,V)\) lying on one side of the separatrix to the other side. In contrast, a perturbation which is small in \(L^2(0,1)\) may have high values on some small interval leading to another stationary solution. Thus, a stationary solutions which is stable in \(L^\infty(0,1)\) sense might be unstable with respect to \(L^2(0,1)\)-perturbations, as illustrated in the following example.

**Example 3.6.** Consider a model with hysteresis for the kinetic functions 
\[
 f(u,v) = 1.4 v - u
\]
and
\[
 p(v) = v^3 - 6.3 v^2 + 10v
\]
Let \((U(x), V(x))\) be a monotone increasing stationary solution with layer position at \(\bar{x} = 0.4\) (see Figure 4a). We consider the different perturbations \((\varphi_0(x), \psi_0(x))\):

(i) In case of a random perturbation satisfying 
\[
 \|\varphi_0\|_{L^\infty(0,1)} = \|\psi_0\|_{L^\infty(0,1)} = 0.4,
\]
solutions of system (1)-(2) with initial condition \((U(x) + \varphi_0(x), V(x) + \psi_0(x))\)
approach \((U(x), V(x))\) (cf. Fig. 4b).

(ii) A perturbation with a step function
\[
 \varphi_0(x) = \psi_0(x) = \begin{cases} 
 5 & \text{for } x \in [0.39, 0.4] \\
 0 & \text{else}
\end{cases}
\]
which is small in \(L^2(0,1)\) (\(\|\varphi_0\|_{L^2(0,1)} = 0.05\), but large in \(L^\infty(0,1)\) (\(\|\varphi_0\|_{L^\infty(0,1)} = 5\)), leads to a stationary solution with layer position \(\bar{x} = 0.39\) (cf. Fig. 4c).

(iii) Finally, a perturbation
\[
 \varphi_0(x) = \psi_0(x) = \begin{cases} 
 5 & \text{for } x \in [0.19, 0.2] \\
 0 & \text{else}
\end{cases}
\]
with norm \(\|\varphi_0\|_{L^2(0,1)} = 0.05\) and \(\|\varphi_0\|_{L^\infty(0,1)} = 5\), leads to a stationary solution which is not monotone.

We remark here that such strong dependence of the pattern on initial data is related to the specific simple nonlinearities we use to model hysteresis, i.e., the existence of two spatially homogeneous stationary solutions that are always stable. As shown recently in Ref. [6], hysteresis can be also exhibited by models with diffusion-driven instability of one of the spatially constant stationary solutions and, in such a case, pattern selection is more robust and in case of small perturbations depends on the diffusive scaling of the system (a phenomenon similar to the selection of Turing patterns).

Let \((U(x), V(x))\) be a steady-state solution constructed in Section 3.1. In the remainder of this section, we derive conditions for its local asymptotic stability. Since \(V(x)\) is discontinuous, we need to make the meaning of “small perturbations”
A monotone increasing stationary solution \((U(x), V(x))\) with layer position at \(x = 0.4\).

A perturbation which is small in the \(L^\infty (0, 1)\) norm does not change the stationary solution.

A perturbation which is small in the \(L^2 (0, 1)\), but not the \(L^\infty (0, 1)\) norm that shifts the layer position.

A perturbation which is small in the \(L^2 (0, 1)\), but not the \(L^\infty (0, 1)\) norm that leads to a non-monotone stationary solution.

Figure 4. Simulations of the generic model with hysteresis for different types of perturbations of a stationary solution. The plots show initial conditions (dotted lines) and the approached stationary solution (continuous lines) after a sufficiently large time \(t_{end}\).

Let
\[
    u(t, x) = U(x) + \varphi(t, x), \quad v(t, x) = V(x) + \psi(t, x)
\]
be a solution of the initial-boundary value problem (1)–(4). Let \((u_0(x), v_0(x)) = (U(x) + \varphi_0(x), V(x) + \psi_0(x))\) be the initial functions. The perturbation \((\varphi(t, x), \psi(t, x))\) satisfies the following equations:

\[
    \varphi_t(t, x) = \frac{1}{\gamma} \varphi_{xx}(t, x) + \alpha \psi(t, x) - \beta \varphi(t, x), \quad t > 0, \ 0 < x < 1, \\
    \psi_t(t, x) = \varphi(t, x) - p'(V(x))\psi(t, x) + R(t, x)\psi(t, x)^2, \quad t > 0, \ 0 \leq x \leq 1,
\]
\[ \varphi_x(t, 0) = \varphi_x(t, 1) = 0, \quad t \geq 0. \]  

Here, \( R(t, x) \) is a bounded function. In fact, by the linearity of (1) we obtain the first equation immediately. From the equation (2) for \( v = V(x) + \psi(t, x) \) we have by the Taylor expansion of \( p(v) = p(V(x) + \psi(t, x)) \) around \( V(x) \), for each \( x \) fixed,

\[
\begin{align*}
\psi_t(x) &= \varphi(t, x) - p(V(x) + \psi(t, x)) + p(V(x)) \\
&= \varphi(t, x) - p'(V(x))\psi(t, x) - \frac{1}{2} p''(V(x) + \theta\psi(t, x))\psi(t, x)^2,
\end{align*}
\]

where \( \theta = \theta(t, x) \) satisfies \( 0 < \theta < 1 \). Hence, we see that

\[
R(t, x) = -\frac{1}{2} p''(V(x) + \theta\psi(t, x)).
\]

In the following we assume that \( \sup_{x \in [0, 1]} |\psi(t, x)| \leq 1 \) for all \( t \geq 0 \); and later we verify that the solution we are considering satisfies this assumption. Set

\[
R_\ast = \max \left\{ \frac{1}{2} |p''(v)| \left| \min_{0 \leq x \leq 1} V(x) - 2 \leq v \leq \max_{0 \leq x \leq 1} V(x) + 2 \right. \right\}.
\]

We can regard \((\varphi(t, x), \psi(t, x))\) as a solution of the following integral equations:

\[
\begin{align*}
\varphi(t, x) &= (S(t)\varphi_0)(x) + \alpha \int_0^t (S(t-s)\psi(s, \cdot))(x) \, ds, \\
\psi(t, x) &= e^{-\gamma t} V(x)^t \psi_0(x) + \int_0^t e^{-(t-s)} p'(V(x)) \varphi(s, x) \, ds \\
&+ \int_0^t e^{-(t-s)} p'(V(x)) R(s, x)\psi(s, x)^2 \, ds,
\end{align*}
\]

where \( S(t) \) is the analytic semigroup on \( L^2(0, 1) \) generated by the elliptic operator \( \varphi \mapsto \frac{1}{2} \varphi_{xx} - \beta \varphi \) under the homogeneous Neumann boundary condition. We assume that the initial data \((\varphi_0, \psi_0)\) is taken from \( H_0^2(0, 1) \times L^\infty(0, 1) \) so that (32)–(33) has a unique solution \((\varphi, \psi) \in C^0([0, T); H_0^2(0, 1) \times L^\infty(0, 1))\), where \( 0 < T \leq \infty \) denotes the maximum existence time. It is known (see, e.g., Lemma 3 in p. 25 of [27]) that

\[
\|S(t)\varphi\| \leq e^{-\beta t} \|\varphi\|_\infty \quad \text{for all} \quad \varphi \in L^\infty(0, 1).
\]

We shall prove that \((U(x), V(x))\) is exponentially asymptotically stable under the assumption

\[
K := \text{ess inf}_{x \in [0, 1]} p'(V(x)) > \frac{\alpha}{\beta},
\]

which we call the stability condition. To state the result precisely, we define a positive constant \( R_0 \) by

\[
R_0 = \max \left\{ \frac{1}{2} \left( R_\ast + \frac{1}{2} \left( K - \frac{\alpha}{\beta} \right) \right) \right\}.
\]

**Theorem 3.7.** Suppose that the steady-state solution \((U(x), V(x))\) satisfies the stability condition (35). If the initial data \((\varphi_0, \psi_0) \in H_0^2(0, 1) \times L^\infty(0, 1)\) satisfy

\[
\|\varphi_0\|_\infty + K\|\psi_0\|_\infty < \frac{1}{4R_0} \left( K - \frac{\alpha}{\beta} \right)^2,
\]

then there exist positive constants \( C_\ast \) and \( \delta \), depending on \( \|\varphi_0\|_\infty \) and \( \|\psi_0\|_\infty \), such that

\[
\|\varphi(t, \cdot)\|_\infty \leq C_\ast e^{-\delta t} \quad \text{and} \quad \|\psi(t, \cdot)\|_\infty \leq C_\ast e^{-\delta t} \quad \text{for all} \quad t \geq 0.
\]
Proof. First we choose a positive constant $\delta$ so small that the following inequalities are satisfied:

$$K - 2\delta > 0, \quad \beta - \delta > 0, \quad K - \delta > \frac{\alpha}{\beta - \delta},$$

$$\|\varphi_0\|_\infty + (K - \delta)\|\psi_0\|_\infty < \frac{K - 2\delta}{4R_0(K - \delta)} \left( K - \delta - \frac{\alpha}{\beta - \delta} \right)^2. \quad (38)$$

Put

$$\Phi(t) = \sup_{0<s<t} e^{\delta s}\|\varphi(s, \cdot)\|_\infty \quad \text{and} \quad \Psi(t) = \sup_{0<s<t} e^{\delta s}\|\psi(s, \cdot)\|_\infty. \quad (39)$$

By applying (34) to (32), we obtain

$$\|\varphi(t, \cdot)\|_\infty \leq \|\varphi_0\|_\infty e^{-\beta t} + \alpha \int_0^t e^{-K(t-s)-\delta s} (e^{\delta s}\|\psi(s, \cdot)\|_\infty) ds$$

$$\leq \|\varphi_0\|_\infty e^{-\beta t} + \alpha \Psi(t) \int_0^t e^{-\beta(t-s)-\delta s} ds$$

$$\leq \|\varphi_0\|_\infty e^{-\beta t} + \frac{\alpha e^{-\delta t}}{\beta - \delta} \Psi(t).$$

Hence,

$$e^{\delta t}\|\varphi(t, \cdot)\|_\infty \leq \|\varphi_0\|_\infty + \frac{\alpha}{\beta - \delta} \Psi(t). \quad (40)$$

Since $p'(V(x)) \geq K$ in $[0, 1]$ we obtain from (33) with the help of (31) that

$$\|\psi(t, \cdot)\|_\infty \leq e^{-Kt}\|\psi_0\|_\infty + \int_0^t e^{-K(t-s)-\delta s} (e^{\delta s}\|\psi(s, \cdot)\|_\infty) ds$$

$$+ \int_0^t e^{-K(t-s)} R_s e^{-\delta s} \left( e^{\delta s}\|\psi(s, \cdot)\|_\infty \right)^2 ds$$

$$\leq e^{-Kt}\|\psi_0\|_\infty + \int_0^t e^{-K(t-s)-\delta s} \left( \|\varphi_0\|_\infty + \frac{\alpha \Psi(s)}{\beta - \delta} \right) ds$$

$$+ R_s \int_0^t e^{-K(t-s)-\delta s} \Psi(s)^2 ds.$$
for any $0 \leq t \leq \tau < T$. Taking the supremum of the left-hand side over $0 \leq t \leq \tau$, and then changing $\tau$ to $t$, we conclude that

$$
\Psi(t) \leq \|\psi_0\|_\infty + \frac{1}{K - \delta} \|\varphi_0\|_\infty + \frac{\alpha}{(\beta - \delta)(K - \delta)} \Psi(t) + \frac{R_0}{K - 2\delta} \Psi(t)^2. \tag{41}
$$

Now consider the quadratic function

$$
\nu_\delta(y) = \frac{R_0}{K - 2\delta} y^2 - \left(1 - \frac{\alpha}{(\beta - \delta)(K - \delta)}\right)y + \|\psi_0\|_\infty + \frac{\|\varphi_0\|_\infty}{K - \delta}.
$$

Let $D_\delta$ denote the discriminant of $\nu_\delta(y)$:

$$
D_\delta = \left(1 - \frac{\alpha}{(\beta - \delta)(K - \delta)}\right)^2 - \frac{4R_0}{(K - \delta)(K - 2\delta)} \left(\|\varphi_0\|_\infty + (K - \delta)\|\psi_0\|_\infty\right).
$$

By (38) we know that $D_\delta > 0$. Note also that the coefficient of the linear term in $\nu_\delta(y)$ is negative. Hence, the equation $\nu_\delta(y) = 0$ has two distinct positive roots $\rho_{1,\delta} < \rho_{2,\delta}$.

We observe that $\nu_\delta(\Psi(0)) > 0$ whenever $\|\varphi_0\|_\infty + \|\psi_0\|_\infty > 0$, since $\nu_\delta(\Psi(0)) = \nu_\delta(\|\psi_0\|_\infty) = R_0(K - 2\delta)^{-1} \|\varphi_0\|_\infty + \alpha(\beta - \delta)^{-1} (K - \delta)^{-1} \|\psi_0\|_\infty + (K - \delta)\|\varphi_0\|_\infty$. Moreover, we claim that $\Psi(0) < \rho_{1,\delta}$. To see this, we note that $1 - \alpha/[(\beta - \delta)(K - \delta)] < 2$. Hence from (38) we obtain

$$
\|\psi_0\|_\infty < \frac{K - 2\delta}{4R_0(K - \delta)^2} \left(K - \delta - \frac{\alpha}{\beta - \delta}\right)^2 = \frac{K - 2\delta}{4R_0} \left(1 - \frac{\alpha}{(\beta - \delta)(K - \delta)}\right)^2 = \frac{\rho_{1,\delta} + \rho_{2,\delta}}{2}.
$$

On the other hand, since $\nu_\delta(y) > 0$ if and only if $y < \rho_{1,\delta}$ or $y > \rho_{2,\delta}$, we see that $\Psi(0) = \|\psi_0\|_\infty < \rho_{1,\delta}$ or $\Psi(0) > \rho_{2,\delta}$, and the latter possibility is ruled out by the estimate above.

From (41) we have $\nu_\delta(\Psi(t)) \geq 0$ for all $0 \leq t < T$, where $T$ is the maximum existence time. The continuity of $\Psi(t)$ and $\Psi(0) < \rho_{1,\delta}$ imply that $\Psi(t) < \rho_{1,\delta}$ for $t \in [0, T)$, hence

$$
e^{\delta t} \|\psi(t, \cdot)\|_\infty \leq \Psi(t) < \frac{K - 2\delta}{2R_0} \left(1 - \frac{\alpha}{(\beta - \delta)(K - \delta)}\right) =: C_1 \quad \text{for all } t \in [0, T). \tag{42}
$$

By virtue of (40) we obtain also that

$$
\|\psi(t, \cdot)\|_\infty \leq \|\varphi_0\|_\infty + \frac{C_1 \alpha}{\beta - \delta} =: C_2 \quad \text{for all } t \in [0, T). \tag{43}
$$

In view of $(K - 2\delta)(1 - \alpha/[(\beta - \delta)(K - \delta)]) < K - \alpha/\beta$, we obtain

$$
C_1 < \frac{1}{2R_0} \left(K - \frac{\alpha}{\beta}\right) \leq 1
$$
due to (36). Hence, (42) implies, in particular, $\|\psi(t, \cdot)\|_\infty < 1$ for all $t \in [0, T)$. It is now easy to verify that $T = +\infty$, and this completes the proof of the theorem by choosing $C_* = \max\{C_1, C_2\}$.

The stability condition seems to be difficult to check, because it depends on the stationary solution. However, we can translate it into a condition for the jump. For this, we denote by $u_H^0$ the zero of $q_H'$ in $[0, u_H]$ and by $u_L^0$ the zero of $q_H'$ in $[u_L, u_2]$. These values exist uniquely and their relative position depends on the kinetic functions.
Corollary 1. Consider the model (1)-(4) for kinetic functions such that \( u^0_T < u^0_H \). Let \((U(x), V(x))\) be a stationary solution with jump \( u^0_T < \bar{u} < u^0_H \), then \((U(x), V(x))\) is asymptotically stable.

Proof. By definition of a stationary solution with jump \( \bar{u} \), the function \( V(x) \) is given by \( h_H(U(x)) \) for \( x \in [0, 1] \) fulfilling \( U(x) \leq \bar{u} \). Therefore, for those \( x \) the requirement \( p'(V(x)) > \frac{\alpha}{\beta} \) is equivalent to

\[
q_H'(U(x)) = \alpha \frac{1}{p'(h_H(U(x)))} - \beta = \alpha \left( \frac{1}{p'(h_H(U(x)))} - \frac{\beta}{\alpha} \right) < 0.
\]

From this we see that \( q_H'(U(x)) < 0 \) holds exactly when \( U(x) < u^0_H \). Therefore, \( \bar{u} < u^0_H \) is necessary and sufficient for fulfilling the stability condition.

Similarly, for \( x \in [0, 1] \) fulfilling \( U(x) > \bar{u} \) it holds \( V(x) = h_T(U(x)) \) and the stability condition requires

\[
q_T'(U(x)) = \alpha \frac{1}{p'(h_T(U(x)))} - \beta = \alpha \left( \frac{1}{p'(h_T(U(x)))} - \frac{\beta}{\alpha} \right) < 0.
\]

It holds \( q_T'(U(x)) < 0 \) if \( u^0_T < U(x) \), hence we deduce \( u^0_T < \bar{u} \). Thus, the stability condition is fulfilled. \( \square \)

Definition 3.8. Kinetic functions such that \( u^0_T < u^0_H \) holds are called admissible. In this situation, values \( \bar{u} \in (u^0_T, u^0_H) \) are called admissible jumps.

Consequently, Corollary 1 says that for an admissible kinetic function all monotone increasing stationary solutions having an admissible jump \( \bar{u} \in (u^0_T, u^0_H) \) are stable.

The disadvantage of the above construction of patterns is the fact that we cannot directly set up \( \bar{u} \) in a simulation of the time-dependent system. It is rather the choice of the initial condition by which we may influence the final pattern.

4. Dependence on the jump. To investigate pattern selection, we analyze the dependence of the layer position \( \bar{x} \) on the jump \( \bar{u} \). First, we state all relevant functions and values as depending on \( \bar{u} \). Then, we show negativity of the derivative of \( \bar{x}(\bar{u}) \), which we use for the investigation of the range of this function. This plays an important role in pattern selection and depends on the parameters of the kinetic functions as well as on the diffusion coefficient.

4.1. The potential and time-maps.

Lemma 4.1. The derivative of potential \( Q(\bar{u}, u) := Q_{\bar{u}}(u) \) defined in equation (15) with respect to \( \bar{u} \) is given by

\[
\frac{\partial Q}{\partial \bar{u}}(\bar{u}, u) = \begin{cases} 
0 & \text{for } \bar{u} > u \\
q_H(\bar{u}) - q_T(\bar{u}) & \text{for } \bar{u} < u.
\end{cases}
\]

For \( \bar{u} = u \) it is not differentiable.

Proof. For \( \bar{u} > u \), it holds \( Q(\bar{u}, u) = \int_{u}^{\bar{u}} q_H(\bar{u}) d\bar{u} \) which is independent of \( \bar{u} \), whereas for \( \bar{u} < u \) it holds \( Q(\bar{u}, u) = \int_{\bar{u}}^{u} q_H(\bar{u}) d\bar{u} - \int_{\bar{u}}^{u} q_T(\bar{u}) d\bar{u} \). By calculating the derivative of these representations with respect to \( \bar{u} \) the result follows immediately. \( \square \)

We recall that \( u_0 \in (0, \bar{u}) \), respectively \( u_e = u_e(u_0, \bar{u}) \in (\bar{u}, u_2) \), are values for which there exists an \( L > 0 \), such that \((U(0), U_x(0)) = (u_0, 0)\) and \((U(L), U_x(L)) = (u_e, 0)\), where \( U \) is a monotone increasing solution of equation (13).
Proposition 2. Functions $u_e(\bar{u}, u_0), T_1(\bar{u}, u_0), T_2(\bar{u}, u_e), T(\bar{u}, u_0), u_0(\bar{u})$ and $u_e(\bar{u})$, defined in Section 3.1 are continuously differentiable as functions of $\bar{u}$. The derivatives of these functions with respect to $\bar{u}$ have the following form and sign.

i) $\frac{\partial T_1}{\partial \bar{u}}(\bar{u}, u_0) = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, \bar{u})}} \right) > 0,$

ii) $\frac{\partial T_2}{\partial \bar{u}}(\bar{u}, u_0) = \frac{-1}{\sqrt{2\pi}} \left( \frac{1}{\sqrt{Q(\bar{u}, u_e) - Q(\bar{u}, \bar{u})}} \right) < 0,$

iii) $\frac{\partial u_e}{\partial \bar{u}}(\bar{u}, u_0) = \frac{q_T(\bar{u}) - q_H(\bar{u})}{q_T(u_e(\bar{u}, u_0))} > 0,$

iv) $\frac{\partial T}{\partial \bar{u}}(\bar{u}, u_0) = \frac{q_T(\bar{u}) - q_H(\bar{u})}{q_T(u_e(\bar{u}, u_0))} \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e(\bar{u}, u_0)) > 0,$

v) $\frac{du_0}{\partial \bar{u}}(\bar{u}) = \frac{q_H(\bar{u}) - q_T(\bar{u})}{q_T(u_e(\bar{u}, u_0))} \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e(\bar{u}, u_0)) > 0,$

vi) $\frac{du_e}{\partial \bar{u}}(\bar{u}) = \frac{q_T(\bar{u}) - q_H(\bar{u})}{q_T(u_e(\bar{u}, u_0))} \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e(\bar{u}, u_0)) > 0.$

Proof. We observe that $u_0, u_e \neq \bar{u}$ and, therefore, $Q(\bar{u}, u_0)$, respectively $Q(\bar{u}, u_e)$ are differentiable, which yields differentiability and continuity of $u_e(\bar{u}, u_0, \bar{u})$, the time-maps and, consequently, of $u_0(\bar{u})$ and $u_e(\bar{u})$. The sign of the derivatives follows from $q_H(u) < 0$ for $u \in (0, u_H)$, $q_T(u) > 0$ for $u \in (u_T, u_2)$ (cf. Fig. 1B). Moreover, we use Proposition 1.

Next, we calculate the derivatives:

i) & ii) It holds $Q(\bar{u}, u_0) - Q(\bar{u}, u) = \int_{u_0}^{u} q_H(\bar{u})d\bar{u}$ for $u \in (u_0, \bar{u})$ and $Q(\bar{u}, u_e) - Q(\bar{u}, u) = \int_{u_0}^{u} q_T(\bar{u})d\bar{u}$ for $u \in (\bar{u}, u_e)$, respectively, therefore, the integrand of both integrals is independent of $\bar{u}$. Thus, $\bar{u}$ appears only as integration limit and these statements follow directly.

iii) We consider $u_0$ as fixed and calculate the derivative with respect to $\bar{u}$ of both sides of equation (19) Because of $u_0 < \bar{u}$, we obtain

$$\frac{d}{d\bar{u}} Q(\bar{u}, u_0) = \frac{\partial Q}{\partial \bar{u}}(\bar{u}, u_0) = 0.$$ 

Whereas $u_e > \bar{u}$ yields

$$0 = \frac{d}{d\bar{u}} Q(\bar{u}, u_e(\bar{u}, u_0)) = \frac{\partial Q}{\partial \bar{u}}(\bar{u}, u_e(\bar{u}, u_0)) + \frac{\partial u_e}{\partial \bar{u}}(\bar{u}, u_0) \cdot \frac{\partial Q}{\partial u_e}(\bar{u}, u_e(\bar{u}, u_0))$$

$$= q_H(\bar{u}) - q_T(\bar{u}) + \frac{\partial u_e}{\partial \bar{u}}(\bar{u}, u_0) \cdot q_T(u_e(\bar{u}, u_0)).$$

Solving the last equation with respect to $\frac{\partial u_e}{\partial \bar{u}}(\bar{u}, u_0)$ yields the assertion.

iv) We calculate the derivative of equation (22) using the chain rule

$$\frac{\partial T}{\partial \bar{u}}(\bar{u}, u_0) = \frac{\partial T_1}{\partial \bar{u}}(\bar{u}, u_0) + \frac{\partial T_2}{\partial \bar{u}}(\bar{u}, u_e(\bar{u}, u_0)) + \frac{\partial u_e}{\partial \bar{u}}(\bar{u}, u_0) \cdot \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e(\bar{u}, u_0))$$

$$= 0 + \frac{q_T(\bar{u}) - q_H(\bar{u})}{q_T(u_e(\bar{u}, u_0))} \cdot \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e(\bar{u}, u_0)),$$

where we used the fact that $\frac{\partial T_1}{\partial \bar{u}}(\bar{u}, u_0)$ and $\frac{\partial T_2}{\partial \bar{u}}(\bar{u}, u_e(\bar{u}, u_0))$ add up to zero by formulas i) and ii) combined with relation (19)

v) We differentiate formula (23), which implicitly determines $u_0(\bar{u})$, and obtain

$$\frac{\partial T}{\partial \bar{u}}(\bar{u}, u_0(\bar{u})) + \frac{du_0}{d\bar{u}}(\bar{u}) \frac{\partial T}{\partial u_0}(\bar{u}, u_0(\bar{u})) = 0.$$
This can be solved with respect to \( \frac{d\bar{u}}{du}(\bar{u}) \) and leads together with iv) of this proposition to the assertion.

vi) We apply the chain rule to the defining equation
\[
\frac{du}{d\bar{u}}(\bar{u}) = u_e(\bar{u}, u_0(\bar{u})).
\]
(44)

We skip the details because we do not use them later on.

4.2. Dependence of the layer position on the jump.

**Definition 4.2.** The layer position \( \bar{x}(\bar{u}) \in (0, 1) \) of a monotone increasing solution of equation (10) is defined by
\[
\bar{x}(\bar{u}) = T_1(\bar{u}, u_0(\bar{u})).
\]

This is the position where the monotone increasing solution of (13) switches from the phase plane corresponding to \( \frac{1}{\gamma} U_{xx}(x) + q_H(U(x)) = 0 \) to that corresponding to \( \frac{1}{\gamma} U_{xx}(x) + q_T(U(x)) = 0 \).

**Proposition 3.** The layer position \( \bar{x}(\bar{u}) \) is continuously differentiable and its first derivative with respect to \( \bar{u} \) is given by
\[
\frac{d\bar{x}}{d\bar{u}}(\bar{u}) = \frac{1}{\sqrt{2\gamma}} \frac{1}{\sqrt{Q(\bar{u}, u_0(\bar{u}))-Q(\bar{u}, u)}} + \frac{q_H(\bar{u}) - q_T(\bar{u})}{q_T(u_e(\bar{u}))} \frac{\partial T_1}{\partial u_0}(\bar{u}, u_0(\bar{u})) \frac{\partial T_1}{\partial u}(\bar{u}, u_0(\bar{u})).
\]
(45)

For admissible kinetic functions, the relation \( \frac{d\bar{x}}{d\bar{u}}(\bar{u}) < 0 \) holds for all admissible jumps \( \bar{u} \in (u^T_0, u^T_H) \).

**Proof.** The function \( \bar{x}(\bar{u}) \) is continuously differentiable as a composition of such functions. Applying the chain rule to \( T_1(\bar{u}, u_0(\bar{u})) \) yields
\[
\frac{d\bar{x}}{d\bar{u}}(\bar{u}) = \frac{\partial T_1}{\partial \bar{u}}(\bar{u}, u_0(\bar{u})) + \frac{d\bar{u}}{d\bar{u}}(\bar{u}) \frac{\partial T_1}{\partial u_0}(\bar{u}, u_0(\bar{u}))
\]
\[
= \frac{1}{\sqrt{2\gamma} E_*} + \frac{q_H(\bar{u}) - q_T(\bar{u})}{q_T(u_e(\bar{u}))} \frac{\partial T_1}{\partial u_0}(\bar{u}, u_0(\bar{u})) \frac{\partial T_1}{\partial u}(\bar{u}, u_0(\bar{u}))
\]
\[
= \frac{1}{\sqrt{2\gamma} E_*} + \frac{q_H(u_e(\bar{u})) - q_T(u_e(\bar{u}))}{q_T(u_e(\bar{u}))} \frac{\partial T_1}{\partial u_0}(\bar{u}, u_0(\bar{u})) + \frac{q_H(u_0(\bar{u}))}{q_T(u_e(\bar{u}))} \frac{\partial T_1}{\partial u_0}(\bar{u}, u_0(\bar{u}))
\]
where we used Proposition 2 i) & v) and equation (28).

For better readability, we write from now on only \( u_0 \) and \( u_e \) and omit the dependencies on \( \bar{u} \).

We multiply equation (45) by the denominator of the right-hand side \( q_T(u_e) \frac{\partial T_1}{\partial u_0}(\bar{u}, u_0) + q_H(u_0) \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e) \) which is negative (cf. equation (28)). Therefore, showing the negativity of \( \frac{d\bar{x}}{d\bar{u}}(\bar{u}) \) is equivalent to showing the positivity of
\[
\frac{1}{\sqrt{2\gamma} E_*} \left( q_T(u_e) \frac{\partial T_1}{\partial u_0}(\bar{u}, u_0) + q_H(u_0) \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e) + (q_H(u_0) - q_T(\bar{u})) \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e) \right).
\]
(46)

In Lemma 3.5, it is shown that the derivatives of the time-maps with respect to \( u_0 \) and \( u_e \), resp., have the form
\[
\frac{\partial T_1}{\partial u_0}(\bar{u}, u_0) = -q_H(u_0) \sqrt{2\gamma E_*} \cdot \text{int}_H, \quad \frac{\partial T_2}{\partial u_e}(\bar{u}, u_e) = -q_T(u_e) \sqrt{2\gamma E_*} \cdot \text{int}_T,
\]
where we denote by $\text{int}_H$ and $\text{int}_T$ the functions

$$\text{int}_H = \text{int}_H(u_0) = \int_{u_0}^{\bar{u}} \left( \frac{(Q(\bar{u}, u) - Q(\bar{u}, \bar{u}))}{q_H(u)} - \frac{1}{2} \right) \frac{du}{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}} < 0,$$

$$\text{int}_T = \text{int}_T(u_c) = \int_u^{u_c} \left( \frac{(Q(\bar{u}, u) - Q(\bar{u}, \bar{u}))}{q_T(u)} - \frac{1}{2} \right) \frac{du}{\sqrt{Q(\bar{u}, u_c) - Q(\bar{u}, u)}} < 0.$$

Using these representations, we reformulate expression (46)

$$\frac{q_H(u_0)q_T(u_c)}{2\gamma E_1^2} \left( (q_H(\bar{u}) - q_T(\bar{u})) \text{int}_T - E_1 \text{int}_H - E_1 \text{int}_T \right) = \frac{q_H(u_0)q_T(u_c)}{2\gamma E_1^2} \left( \text{int}_T (q_H(\bar{u}) \text{int}_H - E_1) - \text{int}_H (q_T(\bar{u}) \text{int}_T + E_1) \right).$$

Remark that $q_H(u_0)q_T(u_c)$ is negative, the whole expression is positive if the term in the parentheses is negative. This is the case if it holds

$$q_H(\bar{u}) \text{int}_H > E_1 \quad \text{and} \quad -q_T(\bar{u}) \text{int}_T > E_1.$$  \hfill (47)  

For showing the first estimate in (47), we investigate in detail the integral $\text{int}_H$. Therefore, we split the integrand into a suitable sum of two summands.

$$\text{int}_H = \int_{u_0}^{\bar{u}} \left( \frac{(Q(\bar{u}, u) - Q(\bar{u}, u_0) + Q(\bar{u}, u_0) - Q(\bar{u}, \bar{u}))}{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, \bar{u})}} - \frac{1}{2\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}} \right) \frac{du}{q_H(u)}$$

$$= \int_{u_0}^{\bar{u}} \left( \frac{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}q_H'(u)}{q_H(u)} + \frac{1}{2\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}} \right) \frac{du}{\text{int}_H}$$

$$+ \int_{u_0}^{\bar{u}} \frac{E^2q_H''(u)}{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}q_H(u)} \frac{du}{\text{int}_H}.$$

We calculate the integral $\text{int}_H^1$ using the fundamental theorem of calculus

$$\text{int}_H^1 = \int_{u_0}^{\bar{u}} \frac{\partial}{\partial u} \left( \frac{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}}{q_H(u)} \right) du = \frac{E_1}{q_H(\bar{u})}. \hfill (48)$$

For the calculation of $\text{int}_H^2$, we recall that $\bar{u} < u_H^0$ implies that $q_H'(u)$ is negative for all $u < \bar{u}$ (cf. Fig. 1B). Thus, the integrand is negative and we estimate

$$\text{int}_H^2 = E_1^2 \int_{u_0}^{\bar{u}} \frac{q_H''(u)}{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}q_H'(u)} du \leq E_1^2 \int_{u_0}^{\bar{u}} \frac{q_H''(u)}{E_1^2q_H''(u)} du$$

$$= E_1 \int_{u_0}^{\bar{u}} \frac{du}{q_H(u)} \left( -1 + \frac{1}{q_H(u_0)} \right) du = E_1 \left( 1 - \frac{1}{q_H(u_0)} \right). \hfill (49)$$

We use here the estimate

$$\frac{1}{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}} > \frac{1}{\sqrt{Q(\bar{u}, u_0) - Q(\bar{u}, u)}} = \frac{1}{E_1},$$

which holds true because $Q(\bar{u}, u)$ is monotone decreasing in $u$ for $u \in (0, \bar{u})$.

Furthermore, it holds $0 > q_H(u_0) > q_H(\bar{u})$, because $q_H$ is decreasing for $u \leq \bar{u}$. This yields $\frac{q_H''(u_0)}{q_H(u_0)} > 1$. Hence, we obtain the estimate (47) by putting the results
(48) and (49) together as follows
\[ q_H(\bar{u}) \int_H = q_H(\bar{u})(\int^1_H + \int^2_H) \geq q_H(\bar{u})E_\ast \left( \frac{1}{q_H(\bar{u})} + \frac{1}{q_H(u_0)} - \frac{1}{q_H(\bar{u})} \right) \]
\[ = E_\ast \frac{q_H(\bar{u})}{q_H(u_0)} > E_\ast. \]

To show the second estimate in equation (47), we argue in the same way, which accomplishes the proof of this proposition.

A direct consequence of the preceding proposition is the next theorem.

**Theorem 4.3.** We consider model (1)-(3) with admissible kinetic functions. Then, for every \( \bar{x} \in I^0 := (\bar{x}(u^0_H), \bar{x}(u^0_H)) \) there is a unique monotone increasing stationary solution with layer position \( \bar{x} \). Moreover, this solution is stable.

**Proof.** As the continuous function \( \bar{x} : (u^0_T, u^0_H) \rightarrow [0, 1], \bar{u} \mapsto \bar{x}(\bar{u}) \) is monotone decreasing, its range is given by \( I^0 \). Thus, for every \( \bar{x} \in I^0 \), there is a unique \( \bar{u} \in (u^0_T, u^0_H) \) such that the monotone increasing stationary solution \((U, V)\) with jump \( \bar{u} \) has the layer position \( \bar{x} \). Moreover, this stationary solution is unique (cf. Theorem 3.2) and stable (cf. Corollary 1).

This result can be applied directly to monotone decreasing stationary solutions. Given a monotone increasing solution \((U, V)\) of problem (13), we obtain a monotone decreasing one by setting \( U(x) = \bar{U}(1-x) \) and \( V(x) = \bar{V}(1-x) \) for \( x \in [0, 1] \). Thus, for every \( \bar{x} \in (1-\bar{x}(u^0_H), 1-\bar{x}(u^0_H)) \) there is a unique monotone decreasing stationary solution with layer position \( \bar{x} \) which is stable.

### 4.3. The range of layer positions and the role of diffusion coefficient.

A consequence of Theorem 4.3 is that if the interval \( I^0 = (\bar{x}(u^0_H), \bar{x}(u^0_H)) \subseteq (0, 1) \) is large enough, the model (1)-(4) admits stable patterns for a large number of initial functions. Thus, in the following section, we investigate how the parameters of the kinetic functions influence the length of the interval \( I^0 \).

**Theorem 4.4.** We consider the model (1)-(3) and fix a jump \( \bar{u} \in (u_T, \min(u_H, u_2)) \). If \( Q(\bar{u}, u_2) > 0 \) then the layer position \( \bar{x}(\bar{u}) \) of the monotone increasing solution \((U, V)\) tends to 1 as \( \gamma \rightarrow \infty \), whereas if \( Q(\bar{u}, u_2) < 0 \) then \( \bar{x}(\bar{u}) \rightarrow 0 \) as \( \gamma \rightarrow \infty \).

**Proof.** Multiplying the equation \( T_1(\bar{u}, u_0(\bar{u})) + T_2(\bar{u}, u_\ast(\bar{u})) = 1 \) by \( \sqrt{\gamma} \) yields
\[ \frac{1}{\sqrt{2}} \int_{u_0(\bar{u})}^{\bar{u}} \frac{du}{\sqrt{Q(\bar{u}, u_0(\bar{u})) - Q(\bar{u}, u)}} + \frac{1}{\sqrt{2}} \int_{\bar{u}}^{u_\ast(\bar{u})} \frac{du}{\sqrt{Q(\bar{u}, u_\ast(\bar{u})) - Q(\bar{u}, u)}} = \sqrt{\gamma}. \]

Hence, as \( \gamma \rightarrow \infty \), at least one of the integrals on the left-hand side must diverge. This is possible only when \( u_\ast(\bar{u}) \rightarrow 0 \), since \( Q(\bar{u}, u_2) > 0 \) implies \( \bar{u} < u_\ast(\bar{u}) < u_{\max}(\bar{u}) \) and hence
\[ \frac{1}{\sqrt{2}} \int_{\bar{u}}^{u_{\max}(\bar{u})} \frac{du}{\sqrt{Q(\bar{u}, u_\ast(\bar{u})) - Q(\bar{u}, u)}} < \frac{1}{\sqrt{2}} \int_{\bar{u}}^{u_{\max}(\bar{u})} \frac{du}{\sqrt{Q(\bar{u}, u_{\max}(\bar{u})) - Q(\bar{u}, u)}} < \infty \]
because \( T^2(\bar{u}, u_\ast) \) is monotone increasing in \( u_\ast \). Therefore, the first integral on the left-hand side of (51) diverges to \( \infty \) as \( \gamma \rightarrow \infty \). This yields that the layer position \( \bar{x}(\bar{u}) = T_1(\bar{u}, u_0(\bar{u})) \) tends to 1 as \( \gamma \rightarrow \infty \).

For \( Q(\bar{u}, u_2) < 0 \), we conclude similarly. Under this assumption \( u_0(\bar{u}) \in (u_{\min}(\bar{u}), \bar{u}) \) with \( 0 < u_{\min}(\bar{u}) \). In this situation, the first integral on the left-hand side of
(51) is bounded by \( \frac{1}{\sqrt{2}} \int_{u_{\text{min}}}^{u_{\text{max}}} \frac{du}{\sqrt{Q(u,H)}} < \infty \), whereas the second integral tends to infinity as \( \gamma \to \infty \). Hence, \( 1 - \bar{x} = T_2(\bar{u}, u_\epsilon) \to 1 \) and the layer position \( \bar{x} \) tends to zero.

A similar result can be found in [22], Theorem 9.

**Definition 4.5.** We denote by \( u^* \in (u_T, \min(u_H, u_2)) \) a value fulfilling \( Q(u^*, u_2) = 0 \).

**Theorem 4.6.** We consider model (1)-(3) with admissible kinetic functions. Suppose that there exists a \( u^* \) in the interval \( (u_T^0, u_H^0) \) such that \( Q(u^*, u_2) = 0 \). Then, for each \( \delta > 0 \) there is a positive constant \( \Gamma \) such that for any

\[ \bar{x} \in (\delta, 1 - \delta) \]

there exists a unique monotone increasing and a unique monotone decreasing stationary solution with jump at \( \bar{x} \), provided that the diffusion coefficient satisfies \( 1/\gamma > \Gamma \). Moreover, these solutions are stable.

**Proof.** As \( \frac{\partial}{\partial u} Q(\bar{u}, u_2) = q_H(\bar{u}) - q_T(\bar{u}) < 0 \) (cf. 4.1) it holds \( Q(u_T^0, u_2) > 0 \) and \( Q(u_H^0, u_2) < 0 \), because of \( u_T^0 < u^* < u_H^0 \). Applying Theorem 4.4 yields that \( \bar{x}(u_T^0) \) tends to 1, whereas \( \bar{x}(u_H^0) \) tends to 0 for \( \gamma \) tending to infinity. Thus, for every \( \delta > 0 \) there is \( \gamma_1 \) such that \( \bar{x}(u_T^0) > 1 - \delta \) for all \( \gamma > \gamma_1 \). Similarly, there is \( \gamma_2 \) such that \( \bar{x}(u_H^0) < \delta \) for all \( \gamma > \gamma_2 \). For \( \gamma > \max\{\gamma_1, \gamma_2\} \), we obtain that \( (\delta, 1 - \delta) \subset I^0 = I^0(\gamma) \), which leads to the result by virtue of Theorem 4.3.

We illustrate the dependence of \( \bar{x}(\bar{u}) \) on the kinetic function and the diffusion coefficient in Fig. 5. For the plots in Fig. 5A and B, we used the kinetic functions (60) which admit the value \( u^* \in (u_T^0, u_H^0) \) (cf. Appendix) and two different diffusion coefficients \( \frac{1}{\gamma} \). We remark that already for \( \gamma = 50 \) the interval \( I^0 \) is given by \( (0.05, 0.98) \). For \( \gamma = 200 \) this interval is only slightly larger. However, for \( \bar{u} < u^* \) the layer positions \( \bar{x}(\bar{u}) \) approach 1 for growing \( \gamma \), whereas for \( \bar{u} > u^* \) they approach zero, this yields a steep slope of \( \bar{x}(\bar{u}) \) at \( \bar{u} = u^* \). This effect is stronger for larger values of \( \gamma \) and increases the range of \( \bar{x}(\bar{u}) \) for \( \bar{u} \in (u^* - \epsilon, u^* + \epsilon) \) significantly.

In Fig. 5C, we used the kinetic functions (61), where \( Q(\bar{u}, u_2) > 0 \) for all \( \bar{u} \in (u_T, u_H) \). We observe that the range of \( \bar{x}(\bar{u}) \) is significantly smaller than in Fig. 5B, although we used the same diffusion coefficient.

To sum up, we showed in this section that model (1)-(3) admits stable monotone stationary solutions with jump at any point in the open interval \( (0, 1) \) if the kinetic functions fulfil the following conditions:

1. \( u_H^0 < u_T^0 \), this is necessary for the stability of the solutions.
2. There is \( u^* \in (u_H^0, u_T^0) \), this is necessary to make the interval \( I^0 \) large.

In this situation for \( \gamma \) large enough every layer position \( \bar{x} \in (0, 1) \) is possible.

Heuristically, the nullclines of kinetic functions meeting the conditions have the following qualitative behaviour. The nullcline \( g = 0 \) has to be curved and not only slightly bend. Moreover, the straight line \( f(u, v) = \alpha v - \beta u \) has to cut the “S” described by \( p(v) \) in the middle close to \( S_1 \).

**5. Spatially irregular stationary solutions.** In this section, we use the monotone solutions to construct all stationary solutions starting with spatially periodic ones and, finally, spatially irregular solutions.
**Definition 5.1.** Let $k \in \mathbb{N}$ and $k \geq 2$. We call a function $U \in C([0,1])$ a periodic function on $[0,1]$ of mode $k$ if $U(x)$ is monotone on $[0, \frac{1}{k}]$ and

$$U(x) = \begin{cases} U\left(x - \frac{2j}{k}\right) & \text{for } x \in \left[\frac{2j}{k}, \frac{2j+1}{k}\right] \\ U\left(\frac{2j+2}{k} - x\right) & \text{for } x \in \left[\frac{2j+1}{k}, \frac{2j+2}{k}\right] \end{cases}$$

(52)

for every $j \in \{0, 1, 2, \ldots\}$ such that $2j + 2 \leq k$.

**Corollary 2.** For every jump $\bar{u} \in (u_T, \min(u_H, u_2))$ and every diffusion coefficient $\frac{1}{\gamma} > 0$, there are exactly two solutions $U$ of problem (13), which are periodic of mode $k$. Restricted to the interval $[0, \frac{1}{k}]$ one of them is monotone increasing, whereas the other one is monotone decreasing. If the kinetic functions are admissible and the jump $\bar{u}$ is admissible, then these periodic stationary solutions are asymptotically stable.

**Proof.** The proof is standard and can be found, e.g., in [20]. It basically consists of using a monotone solution $\tilde{U}$ of problem (13) with jump $\bar{u}$ for the diffusion coefficient $\frac{k^2}{\gamma}$. By setting

$$U(x) = \tilde{U}(kx) \quad \text{for } x \in \left[0, \frac{1}{k}\right]$$

which we continue periodically for $x \in [0,1]$ by formula (52), the function $U$ is periodic of mode $k$ by construction and a solution of (13) with jump $\bar{u}$ for the diffusion coefficient $\frac{1}{\gamma}$.

Additionally to periodic patterns, the model (1)-(3) admits another class of stationary solutions. This is a consequence from the observation that the phase planes of the equations $\frac{1}{\gamma} U_{xx} + q_H(U) = 0$ and $\frac{1}{\gamma} U_{xx} + q_T(U) = 0$ are overlapping. For a periodic solution the switch between these phase planes takes always place at the same value $\bar{u}$, as one can see for the blue trajectory in Fig. 6. However, a similar construction of discontinuous patterns can be performed with switches at different values $\bar{u}^1, \bar{u}^2, \ldots$ as one can see for the red trajectory in Fig. 6. Then, there exist sub-intervals of $[0, 1]$ such that the solution restricted to each of them is a monotone stationary solution with jump at $\bar{u}^1, \bar{u}^2, \ldots$, respectively.
Figure 6. The phase planes of $\frac{1}{\gamma}U_{xx} + q_H(U) = 0$ and $\frac{1}{\gamma}U_{xx} + q_T(U) = 0$ are overlapping. In blue we see a periodic solution with jump at $\bar{u}$. We cannot determine the mode of a periodic solution in the phase plane. It corresponds to how often the trajectory has been traveled through. In red we see a irregular solution with three different jumps.

Definition 5.2. A pair of functions $(U, V) \in H_N^2(0,1) \times L^\infty(0,1)$ is called irregular stationary solution with jumps at $\bar{u}_1^1, \bar{u}_2^1, \ldots, \bar{u}_k^1$ of system (1)-(3), if there are values $x^0 = 0 < x^1 < \cdots < x^i < x^{i+1} < \cdots < x^{k-1} < 1 = x^k$

and jumps $\bar{u}_j \in (u_T, \min(u_H, u_2))$, for $i = 1, \ldots, k$ such that the restriction of $U$ to the intervals $[x^i, x^{i+1}]$ are alternately monotone increasing and monotone decreasing. Thereby, for $x \in [x^i, x^{i+1}]$ the function $U$ is given by $U(x) = \tilde{U}_i(x)$, where $\tilde{U}_i(x)$ is a monotone increasing (resp. decreasing) solution of the equation

$$\frac{1}{\gamma} \tilde{U}_i''(\tilde{x}) + q_{\bar{u}_i}(\tilde{U}_i(\tilde{x})) = 0,$$

for $\tilde{x} \in [x^i, x^{i+1}]$ with boundary condition $\tilde{U}_i'(x^{i-1}) = \tilde{U}_i'(x^{i+1}) = 0$. The $V$-component of an irregular solution is given by

$$V(x) = \begin{cases} h_H(U(x)) & \text{for } x \in [x^i, x^{i+1}] \quad \text{if } U(x) \leq \bar{u}_i \\ h_T(U(x)) & \text{for } x \in [x^i, x^{i+1}] \quad \text{if } U(x) > \bar{u}_i. \end{cases}$$

We emphasise here that the function $U$ is continuous, in particular at the values $x^i$ for $i = 1, \ldots, k - 1$. Hence, this means that the ending value of $\tilde{U}_i$ and the starting value of $\tilde{U}_{i+1}$ coincide. Using the notation

$$\tilde{U}_i(x^{i-1}) =: u_0^i \quad \text{and} \quad \tilde{U}_i(x^{i+1}) =: u_e^i,$$
We observe that in this sum all values $u^i$ sum of the time-maps as between two consecutive values that connect $\bar{u}$ to $u$. Moreover, we have to assure that $U$ is a solution on the interval $[0, 1]$. This can be expressed in terms of the time-maps by

$$\sum_{i=1}^{k} T(\bar{u}^i, u^i_0) = 1. \quad (54)$$

We recall that the time-map $T(\bar{u}, u_0)$ is initially defined as the “time” $x$ a monotone increasing stationary solution with jump at $\bar{u}$ needs to connect $0 < u_0 < \bar{u}$ with $\bar{u} < u_0(0, \bar{u}) < u_2$. However, this can be extended for monotone decreasing solutions that connect $\bar{u} < u_0 < u_2$ with $0 < u_0(\bar{u}, u_0) < \bar{u}$. Here, it holds $T(\bar{u}, u_0) = T_2(\bar{u}, u_0) + T_1(\bar{u}, u_0(\bar{u}, u_0))$, because $T_1(\bar{u}, \cdot)$ is only defined for values smaller than $\bar{u}$ and $T_2(\bar{u}, \cdot)$ for values larger than $\bar{u}$.

**Proposition 4.** There are at most two irregular stationary solutions $(U, V)$ with jumps at $\bar{u}^1, \ldots, \bar{u}^k$. Restricted to the first sub-interval $[0, x^1]$ one of them is monotone increasing and the other one is monotone decreasing.

**Proof.** First, we remark that an irregular solution with jumps at $\bar{u}^1, \ldots, \bar{u}^k$ is unique for a given partition $x^1, \ldots, x^{k-1}$ of the interval $[0, 1]$ and fixed monotonicity of the sub-interval $[0, x^1]$. This is clear as $U|_{[0, x^1]}$ is given by formula $U(x) = \hat{U}^i(x)$ and the function $\hat{U}^i$ is the unique monotone increasing (resp. decreasing) solution of equation (53) (cf. Theorem 3.2).

Thus, for proving the proposition, it will be enough to show that for a fixed set of jumps $\bar{u}^1, \ldots, \bar{u}^k$ and prescribed monotonicity on the first sub-interval, there is only one possible partition of the interval $[0, 1]$. Looking at the phase plane (cf. Fig. 6), we see that an irregular solution which is increasing on $[x^{i-1}, x^i]$ has to be decreasing on $[x^i, x^{i+1}]$ and vice versa. Therefore, if $u_0^i < \bar{u}^i$, then $u_0^{i+1} > \bar{u}^{i+1}$ and vice versa. Moreover, using the relation $u^i_c = u^i_0$ and equation (44), there is the connection

$$Q(\bar{u}^i, u^i_0) = Q(\bar{u}^i, u_0^{i+1}) \quad (55)$$

between two consecutive values $u^i_0$. Fixing the value $u^i_0$, this successively determines $u^i_0$ for $i = 2, \ldots, k$. Hence, the following sum of time-maps is well-defined

$$T(\bar{u}^1, \ldots, \bar{u}^k, u^1_0) := T(\bar{u}^1, u^1_0) + T(\bar{u}^2, u^2_0(u^1_0)) + \cdots + T(\bar{u}^k, u^k_0(u^1_0)).$$

We observe that in this sum all values $u^i_0$ which are larger than the corresponding $\bar{u}^i$ can be replaced by $u_0^{i+1} < \bar{u}^i$. This follows from the condition $u^i_c = u_0^{i+1}$ and equation (55). Therefore, it holds

$$T(\bar{u}^i, u^i_0) = T(\bar{u}^i, u_0^{i+1})$$

if $u^i_0 > \bar{u}^i$. In the situation where $U|_{[0, x^1]}$ is monotone increasing, we rewrite the sum of the time-maps as

$$T(\bar{u}^1, \ldots, \bar{u}^k, u^1_0) = T(\bar{u}^1, u^1_0) + T(\bar{u}^2, u^2_0) + T(\bar{u}^3, u^3_0) + T(\bar{u}^4, u^4_0) + T(\bar{u}^5, u^5_0) + \cdots.$$

Now, we calculate the derivative of equation (55) with respect to $u^i_0$ and obtain the relations

$$\frac{du^i_0}{du^i_0} \cdot q_H(u^i_0) = \frac{du^{i+1}_0}{du^{i+1}_0} \cdot q_R(u^{i+1}_0) \quad \text{and} \quad \frac{du_0^{i+1}}{du_0^i} \cdot q_R(u_0^{i+1}) = \frac{du_0^{i+2}}{du_0^i} \cdot q_R(u_0^{i+2})$$

this implies

$$u^i_c = u_0^{i+1}$$

for all $i = 1, \ldots, k - 1$. Theorem 3.2)
Here, \( i = 1, 3, 5, \ldots \leq k \) runs through all odd numbers. We start at \( i = 1 \) and use these relations together, which successively yields
\[
\frac{d}{du_0} u_{0}^{\bar{i}(u_0^{1})} = \frac{q_H(u_0^{1})}{q_H(u_0^{1})} > 0. \tag{56}
\]

Finally, we calculate the derivative of the time-map
\[
\frac{\partial}{\partial u_0^{1}} T(\bar{u}^1, \ldots, \bar{u}^k, u_0^k) = \frac{\partial T}{\partial u_0^{1}} (\bar{u}^1, u_0^1) + \frac{\partial T}{\partial u_0^{2}} (\bar{u}^2, u_0^2) + \frac{\partial T}{\partial u_0^{3}} (\bar{u}^3, u_0^3) + \cdots
\]
\[
= \frac{\partial T}{\partial u_0^{1}} (\bar{u}^1, u_0^1) + \frac{q_H(u_0^1)}{q_H(u_0^1)} \frac{\partial T}{\partial u_0^{2}} (u_0^2, u_0^3) + \frac{q_H(u_0^1)}{q_H(u_0^1)} \frac{\partial T}{\partial u_0^{3}} (u_0^3, u_0^4) + \cdots < 0,
\]
by virtue of Proposition 2 and the positivity of relation (56).

This result yields the uniqueness of a value \( u_0^1 \) such that
\[
T(\bar{u}^1, \ldots, \bar{u}^k, u_0^k) = 1 \tag{57}
\]
holds, which in turn determines uniquely the partition of the interval \([0, 1]\). Indeed the sub-intervals \([x^{i-1}, x^i]\) are given by
\[
x^i = \sum_{j=1}^{i} T(\bar{u}^j, u_0^j). \tag{58}
\]
This proves the uniqueness of an irregular solution having jumps at \( \bar{u}^1, \bar{u}^2, \ldots, \bar{u}^k \) which is monotone increasing on \([0, x^1]\). The second case is treated in a similar fashion by calculating the derivative of the time-map with respect to \( u_0^k \).

**Corollary 3.** An irregular solution with jumps at \( \bar{u}^1, \bar{u}^2, \ldots, \bar{u}^k \) has \( k \) layer positions \( \bar{x}^1, \bar{x}^2, \ldots, \bar{x}^k \). They are given by the formula
\[
\bar{x}^i = \begin{cases} 
  x^{i-1} + T_1(\bar{u}^i, u_0^i) & \text{if } u_0^1 < \bar{u}^i, \\
  x^i - T_1(\bar{u}^i, u_0^{i+1}) & \text{if } u_0^1 > \bar{u}^i.
\end{cases}
\]

**Proof.** There is exactly one layer position \( \bar{x}^i \) in every subinterval \([x^{i-1}, x^i]\). Depending on the monotonicity of \( U\vert_{[x^{i-1}, x^i]} \) the layer position is given by \( x^{i-1} + T_1(\bar{u}^i, u_0^i) \) and \( x^{i-1} + T_2(\bar{u}^i, u_0^i) \), respectively. We observe that by definition and because of formula (58)
\[
x^i = x^{i-1} + T(\bar{u}^i, u_0^i) = x^{i-1} + T_1(\bar{u}^i, u_0^i) + T_2(\bar{u}^i, u_0^i) = x^{i-1} + T_1(\bar{u}^i, u_0^{i+1}) + T_2(\bar{u}^i, u_0^i)
\]
holds. Thus, if \( U\vert_{[x^{i-1}, x^i]} \) is monotone decreasing the layer condition is given by
\[
\bar{x}^i = x^{i-1} + T_2(\bar{u}^i, u_0^i) + T_1(\bar{u}^i, u_0^{i+1}) - T_1(\bar{u}^i, u_0^{i+1}) = x^i - T_1(\bar{u}^i, u_0^{i+1}).
\]

Having shown the uniqueness of irregular solutions, we now turn our attention to their existence. The proof is not so obvious as it might seem. Actually there is not necessarily an irregular solution for every set of jumps and every diffusion coefficient \( \frac{1}{\gamma} \). We investigate an example to see the problem that occurs.

**Example 5.3.** We consider kinetic functions having jumps \( \bar{u}^1 < \bar{u}^2 \), such that it holds \( Q(\bar{u}^1, u_2^1) > 0 \) and \( Q(\bar{u}^2, u_2^2) < 0 \). We recall that to construct an irregular solution with jumps \( \bar{u}^1 \) and \( \bar{u}^2 \), we need to find values \( u_0^1, u_0^1, u_0^2 \) and \( u_0^2 \) fulfilling
\[
\bar{U}^1(0) = u_0^1, \quad \bar{U}^1(x^1) = u_0^1, \quad \bar{U}^2(x^1) = u_0^2 \quad \text{and} \quad \bar{U}^2(1) = u_0^2,
\]
where \( \bar{U}^1 \) and \( \bar{U}^2 \) are monotone solutions of equation (53). We require \( U(x) \) to be continuous and therefore the condition

\[
u^1_e = u^2_0
\]

has to be fulfilled, because both values correspond to \( U(x^1) \).

Here, we require \( U|_{[0,x^1]} \) to be monotone increasing, which yields that \( \bar{U}^1 \) has to be increasing and \( \bar{U}^2 \) decreasing. Hence, the possible ranges for \( u^1_e \) and \( u^2_0 \) are given by

\[
\bar{u}^1 < u^1_e < u_{\max}(\bar{u}^1) < u_2 \quad \text{and} \quad \bar{u}^2 < u^2_0 < u_2
\]

where \( u_{\max}(\bar{u}^1) < u_2 \), because of the assumption \( Q(\bar{u}^1, u_2) > 0 \).

For \( \gamma \) small, it is possible to find these values. However, as \( \gamma \) increases, the value \( u^2_0 \) tends towards \( u_2 \), whereas \( u^1_e \) is bounded by \( u_{\max}(\bar{u}^1) \). Hence, there will be a value \( \gamma_{\max}(\bar{u}^1, \bar{u}^2) \) such that an irregular solution with jumps at \( \bar{u}^1, \bar{u}^2 \) which is monotone increasing on \([0,x^1]\) does not exist for diffusion coefficients \( \frac{1}{\gamma} \), with \( \gamma > \gamma_{\max}(\bar{u}^1, \bar{u}^2) \).
Generalizing this, the existence of an irregular solutions for a prescribed set of jumps \( \bar{u}^1, \ldots, \bar{u}^k \) depends essentially on the existence of values \( u^1_0, \ldots, u^k_0 \) related to each other by equation (55) and such that equation (57) holds. Depending on the signs of \( Q(\bar{u}^i, u^i_2) \) and \( Q(\bar{u}^{i+1}, u^i_2) \) there might be no solutions to equation (55) considered for \( i \) and \( i+1 \) at the same time which fulfills \( u^{i+1}_0 = u^i_2 \), when \( \gamma \) is larger than some \( \gamma_{\text{max}} \).

The value \( \gamma_{\text{max}} \) depends on the relative position of the jumps. When all \( \bar{u}^i \) are equal, the solution is periodic and exists for all \( \gamma \). Thus, the closer the \( \bar{u}_i \) are to each other, the larger is \( \gamma_{\text{max}} \). In particular, when all jumps are close to \( u^* \), the value \( \gamma_{\text{max}} \) gets larger, because then \( Q(\bar{u}^i, u^i_2) \approx 0 \) for all \( i \) and the possible range for \( u^i_0 \) is large.

**Theorem 5.4.** Consider the model (1)-(3) with admissible kinetic functions. Let \((U, V)\) be an irregular solution with jumps \( \bar{u}^1, \ldots, \bar{u}^k \) which are all admissible, then \((U, V)\) is asymptotically stable.

**Proof.** This theorem follows from Theorem 3.7, we only have to show that the stability condition (35) is fulfilled. As it holds

\[
\text{essinf}_{x \in [0,1]} p'(V(x)) = \min_{i \in \{1, \ldots, k\}} \text{essinf}_{x \in [x^{i-1}, x^i]} p'(V(x)),
\]

it suffices to prove the stability condition \( \text{essinf}_{x \in [x^{i-1}, x^i]} p'(V(x)) > \frac{2}{3} \) for every subinterval \([x^{i-1}, x^i]\). Using that \( \bar{u}^i \in (u^i_0, u^i_2) \), it holds \( q'_H(U(x)) < 0 \) and \( q'_T(U(x)) < 0 \) for those \( x \) such that \( U(x) \leq \bar{u} \) and \( U(x) > 0 \), respectively (cf. Fig. 1). Remembering that \( h_H \) and \( h_T \) are local inverses of the polynomial \( p \) and using the chain rule yields \( q_i(U(x)) = \alpha (1/p'(h_i(U(x)))) - \frac{2}{3} \) for \( i = H, T \). This yields the desired stability condition.

Finally, we would like to address the question if a certain initial function will lead to a stable irregular pattern. When an initial function is crossing the separatrix at several points, then these points set up the layer positions of the final pattern, provided that there is a stable pattern with such layer positions. So the problem can be reformulated as follows: Is there an irregular solution having a prescribed set of layer positions? Unfortunately, we are not able to deduce a formula for the partition \( x^1, \ldots, x^{k-1} \), when we only know the layer positions, but not the jumps. Thus, we cannot apply Theorem 4.3, because the interval \( I^0 \) depends on the diffusion coefficient and it changes if we consider it on a subinterval \([x^{i-1}, x^i]\) instead of \([0, 1]\).

However, under the conditions which we have encountered at the end of section 4.3, the existence of a stable irregular solution having a prescribed set of layer positions is most likely to exists: the kinetics have to be admissible and there is a value \( u^* \in (u^*_T, u^*_H) \). In this situation, the range of layer positions in every subinterval \([x^{i-1}, x^i]\) is large and most layer positions correspond to a jump \( \bar{u}^i \) close to \( u^* \). This makes it more likely that a stable irregular solution exists.

To examine this, we perform simulations of model (1)-(3) with discontinuous initial conditions. To do so, we choose a number \( k \in \mathbb{N}, k \geq 2 \) and positions \( 0 < \tilde{x}_1 < \cdots < \tilde{x}_{k-1} < 1 \) for the discontinuities, such that \( \tilde{x}_i - \tilde{x}_{i-1} \geq 0.05 \) holds. Moreover, we select points \( P_1, \ldots, P_k \in \mathbb{R}^2 \) with positive coordinates, such that \( P_i \) with even index \( i \) are attracted by the steady state \( S_0 \), whereas those with odd index \( i \) are attracted by \( S_2 \) (or the other way around). Now we define the initial
conditions by

\[
(u_0(x), v_0(x)) = \begin{cases} 
  P_1 & \text{for } 0 \leq x \leq \tilde{x}_1 \\
  P_i & \text{for } \tilde{x}_{i-1} < x \leq \tilde{x}_i \text{ when } 2 \leq i \leq k - 1 \\
  P_k & \text{for } \tilde{x}_{k-1} < x \leq 1
\end{cases} \quad (59)
\]

We expect the formation of irregular stationary solution with \(k\) jumps. The diffusion coefficient equals \(\frac{1}{\gamma} = \frac{1}{1000}\) for all simulations.

For the admissible kinetic functions (60) in Appendix 1 which fulfill \(u^* \in (u_0^*, u_H^*)\), we always observed the formation of a stable nonhomogeneous stationary solution. These stationary solutions are irregular and have layer positions exactly at the discontinuities of the initial function. (Cf. Fig. 8A for \((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (0.1, 0.4, 0.6, 0.95)\), Fig. 8B for \((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (0.1, 0.15, 0.25, 0.3)\) and Fig. 8C for \((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (0.2, 0.3, 0.4, 0.8)\).

For the admissible kinetic functions (62) in Appendix 1 we observed that the broader region of high concentration persists, whereas the narrower region of high concentration disappears (Fig. 8C for \((\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = (0.2, 0.3, 0.4, 0.8)\)) the narrow peak disappears and the broad peak persists). However, for the kinetic functions (61) in Appendix 1 we observed the opposite behaviour.

To sum up, the simulations show that a solution of the model strongly depends on the initial data. When there exists a stable stationary solution with layer positions where it has been prescribed by the initial function, we observe that the time-dependent solution is quickly approaching this stationary solution. When there is no such stable stationary solution, we observe a moving front finally leading to a constant solution or a stationary solution with fewer layer positions than prescribed by the initial function. We never observed a moving front leading to a stable stationary solution with layer positions at positions different from those prescribed by the initial function.

**Figure 8.** Simulations of model (1)-(3) for admissible kinetic functions, diffusion coefficient \(1/\gamma = 1/1000\) and initial conditions of type (59) having four discontinuities. The \(u\)-component is plotted in blue, whereas the \(v\)-component is red. The initial condition \((u(0, x), v(0, x)) = (u_0(x), v_0(x))\) is indicated by dotted lines and the stationary solution \((u(t_{end}, x), v(t_{end}, x))\) is indicated by continuous bold lines. Here, \(t_{end}\) is a sufficiently large timepoint, such that the solution \((u(t, x), v(t, x))\) does not change in time anymore. **A-C** the kinetic functions are given by (60) **D** the kinetic functions are given by (62).
Appendix A. Parameter values. Here, we state the parameter values which we have used for generating the figures.

Fig. 1, 5A, 5B, 6: The kinetic functions (5) are given by
\[ f(u, v) = 1.4v - u, \quad g(u, v) = u - (v^3 - 6.3v^2 + 10v). \] (60)
Here, it holds \( u_T = 0.244, u_H = 4.712 \) and the intersection points are given by \( S_0 = (0, 0), S_1 = (2, 2.8) \) and \( S_2 = (4.3, 6.02) \). The kinetics are admissible, because \( u^0_T = 0.382 < u^0_H = 4.5734 \). Moreover, there is the value \( u^* = 3.0316 \in (u^0_T, u^0_H) \).

Fig. 5C: The kinetic functions (5) are given by
\[ f(u, v) = 2.5v - u, \quad g(u, v) = u - (v^3 - 6v^2 + 10v) \] (61)
Here, it holds \( u_T = 2.911 u_H = 5.089 \) and the intersection points are given by \( S_0 = (0, 0), S_1 = (1.775, 4.438) \) and \( S_2 = (4.225, 10.512) \). The kinetics are admissible, because \( u^0_T = 3.3876 < u^0_H = 4.6124 \). But there is no value \( u^* \) and it holds \( Q(\tilde{u}, u_2) > 0 \) for all \( \tilde{u} \in (u_T, u_H) \).

Fig. 8D: The kinetic functions (5) are given by
\[ f(u, v) = 1.6v - u, \quad g(u, v) = u - (v^3 - 6v^2 + 10v) \] (62)
Here, it holds \( u_T = 2.911 u_H = 5.089 \) and the intersection points are given by \( S_0 = (0, 0), S_1 = (2.225, 3.561) \) and \( S_2 = (3.775, 6.039) \). The kinetics are admissible, because \( u^0_T = 3.1236 < u^0_H = 4.8764 \). But there is no value \( u^* \) and it holds \( Q(\tilde{u}, u_2) < 0 \) for all \( \tilde{u} \in (u_T, u_H) \).

Appendix B. Instability of patterns in the case without hysteresis.

Proof of Theorem 2.2. We consider the system (1)-(2) linearized at \((U(x), V(x))\)
\[
\begin{pmatrix}
\ddot{u} \\
\ddot{v}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{\gamma} \dddot{u} \\
0
\end{pmatrix} + \begin{pmatrix}
-\beta & \alpha \\
1 & -p'(V(x))
\end{pmatrix} \begin{pmatrix}
\ddot{u} \\
\ddot{v}
\end{pmatrix} =: \mathcal{L} \begin{pmatrix}
\ddot{u} \\
\ddot{v}
\end{pmatrix}
\]
with boundary conditions \( \dddot{u}_x(0) = \dddot{u}_x(1) = 0 \).

The eigenvalue problem reads
\[
\frac{1}{\gamma} \varphi_{xx} - (\beta + \lambda) \varphi + \alpha \psi = 0, \quad \varphi - (p'(V(x)) + \lambda) \psi = 0.
\] (63) (64)

We introduce the function
\[ r(\lambda, x) := \frac{\alpha}{\lambda + p'(V(x))} - \beta \]
and define the operator \( A(\lambda) \) depending on \( \lambda \) by
\[
\mathcal{D}(A(\lambda)) = \{ \varphi \in C^2([0, 1]) \mid \varphi(0) = \varphi_x(0) = 0 \},
\]
\[
A(\lambda) : \varphi \mapsto \frac{1}{\gamma} \varphi_{xx} + r(\lambda, x) \varphi.
\]
By assumption for Case 1, the polynomial \( p \) is monotone increasing. We notice also that \( K = \min_{x \in [0, 1]} p'(V(x)) > 0 \) and observe that for \( \lambda > -K \) the eigenvalue problem (63)-(64) is equivalent to the equation
\[ A(\lambda) \varphi = \lambda \varphi. \]
Let us first show that there exists $C > 0$ independent of $\lambda \geq 0$ such that

$$|r(\lambda, x)| \leq C$$

for all $x \in [0,1]$. Indeed, the denominator $\lambda + p'(V(x))$ is positive and linearly increasing in $\lambda$. Therefore, $r(\lambda, x)$ is decreasing in $\lambda$ and it holds

$$\frac{\alpha}{\lambda + p'(V(x))} - \beta \leq \frac{\alpha}{p'(V(x))} - \beta = r(0, x) \leq \frac{\alpha}{K} - \beta =: C.$$ 

Note that $C$ is positive, because if $\frac{\alpha}{\beta} \leq K$ the graph of $p$ would be entirely on one side of $f = 0$, which contradicts the assumption of the existence of three intersection points.

Denoting by $\mu_0(\lambda)$ the largest eigenvalue of the Neumann problem

$$A(\lambda)\varphi = \mu \varphi \quad \text{and} \quad \varphi_x(0) = \varphi_x(1) = 0,$$

and by $\nu_0(\lambda)$ the largest eigenvalue of the Dirichlet problem

$$A(\lambda)\varphi = \nu \varphi \quad \text{and} \quad \varphi(0) = \varphi(1) = 0.$$

The Sturm comparison principle (see below) yields that

$$\mu_0(\lambda) > \nu_0(\lambda)$$

holds. Indeed, $\mu_0(\lambda) \leq \nu_0(\lambda)$ implies

$$q_2(x) := r(\lambda, x) - \mu_0(\lambda) \geq r(\lambda, x) - \nu_0(\lambda) =: q_1(x).$$

By definition, the principal eigenfunction of the Dirichlet problem is of one sign and has its only zeros at $x = 0$ and $x = 1$. Hence, from the Sturm comparison principle (see Theorem B.1 below), we obtain that the principal eigenfunction corresponding to the Neumann problem has a zero in $(0,1)$, but this is impossible because the principal eigenfunction is of one sign also for the Neumann problem [29].

Next, we remark that $U_\lambda(x)$ is an eigenfunction for the eigenvalue 0 of the Dirichlet problem for $\lambda = 0$ by calculating the derivative of equation (10)

$$0 = \frac{1}{\gamma}(U_\lambda)_{xx} + \alpha h'(U_\lambda)U_x - \beta U_\lambda = \frac{1}{\gamma}(U_\lambda)_{xx} + (\alpha \frac{1}{p'(V)}) - \beta)U_\lambda = A(0)U_\lambda.$$

This yields $\nu_0(0) \geq 0$ and, therefore, $\mu_0(0) > 0$. Furthermore, $\mu_0(\lambda)$ depends continuously on $\lambda$ and can be calculated by

$$\mu_0(\lambda) = \sup_{\varphi \in W^{1,2}(0,1), ||\varphi||_2 = 1} \left[ -\frac{1}{\gamma} \langle \varphi_x, \varphi_x \rangle + \langle r(\lambda, x)\varphi, \varphi \rangle \right].$$

We obtain

$$-\int_0^1 \frac{1}{\gamma} \varphi_x \varphi_x dx + \int_0^1 r(\lambda, x)\varphi^2 dx \leq -\int_0^1 \frac{1}{\gamma} \varphi_x \varphi_x dx + \int_0^1 C\varphi^2 dx \leq C.$$ 

Therefore, $\mu_0(\lambda)$ is bounded and, hence, there exists a value $\bar{\lambda} > 0$ fulfilling $\mu_0(\bar{\lambda}) = \bar{\lambda}$. This implies the existence of an eigenfunction $\bar{\varphi} \neq 0$ satisfying

$$A(\bar{\lambda})\bar{\varphi} = \mu_0(\bar{\lambda})\bar{\varphi} = \bar{\lambda}\bar{\varphi},$$

with $\bar{\varphi}_x(0) = \bar{\varphi}_x(1) = 0$, which proves the existence of a positive eigenvalue of problem (63)-(64).

**Theorem B.1** (Sturm comparison principle). Let $\phi_1$ and $\phi_2$ be non-trivial solutions of the equations

$$Du_{xx} + q_1(x)u = 0 \quad \text{and} \quad Du_{xx} + q_2(x)u = 0,$$
respectively, for $x \in [0, 1]$ and the diffusion coefficient $D > 0$. We assume that the functions $q_1$ and $q_2$ are continuous on $[0, 1]$ and that

$$q_1(x) \leq q_2(x)$$

holds for all $x \in [0, 1]$. Then between any two consecutive zeros $x_1$ and $x_2$ of $\phi_1$, there exists at least one zero of $\phi_2$ unless $q_1(x) \equiv q_2(x)$ on $[0, 1]$.

Proof. We refer to [9].

REFERENCES

[1] D. Angeli, J. E. Ferrell and E. D. Sontag, Detection of multistability, bifurcations, and hysteresis in a large class of biological positive-feedback systems, PNAS, 101 (2004), 1822–1827.

[2] V. I. Arnold, Mathematical Methods of Classical Mechanics, Second edition. Graduate Texts in Mathematics, 60. Springer-Verlag, New York, 1989.

[3] D. G. Aronson, A. Tesei and H. Weinberger, A density-dependent diffusion system with stable discontinuous stationary solutions, Ann. Mat. Pura Appl., 152 (1988), 259–280.

[4] J. E. Ferrell and W. Xiong, Bistability in cell signaling: How to make continuous processes discontinuous, and reversible processes irreversible, Chaos, 11 (2001), 227–236.

[5] T. Gregor, E. F. Wieschaus, A. P. McGregor, W. Bialek and D. W. Tank, Stability and nuclear dynamics of the bicoid morphogen gradient, Cell, 130 (2007), 141–152.

[6] S. Härting, A. Marciniak-Czochra and I. Takagi, Stable patterns with jump discontinuity in systems with Turing instability and hysteresis, Discrete Contin. Dyn. Syst. Ser. A., 37 (2017), 757–800.

[7] S. Härting and A. Marciniak-Czochra, Spike patterns in a reaction-diffusion-ode model with Turing instability, Math. Meth. Appl. Sci., 37 (2014), 1377–1391.

[8] S. Hock, Y. Ng, J. Hasenauer, D. Wittmann, D. Lutter, D. Trimbach, W. Wurst, N. Prakash and F. J. Theis, Sharpening of expression domains induced by transcription and microRNA regulation within a spatio-temporal model of mid-hindbrain boundary formation, BMC Syst. Biol., 7 (2013), 48.

[9] J. Jaros and T. Kusano, A picone type identity for second order half-linear differential equations, Acta Math. Univ. Comenian, 68 (1999), 137–151.

[10] V. Klíka, R. Baker, D. Headon and E. Gaffney, The influence of receptor-mediated interactions on reaction-diffusion mechanism of cellular self-organization, Bulletin of Mathematical Biology, 74 (2012), 935–957.

[11] S. Kondo and T. Miura, Reaction-diffusion model as a framework for understanding biological pattern formation, Science, 329 (2010), 1616–1620.

[12] K. Korvasová, E. A. Gaffney, P. K. Maini, M. A. Ferreira and V. Klíka, Investigating the Turing conditions for diffusion-driven instability in the presence of a binding immobile substrate, J. Theor. Biol., 367 (2015), 286–295.

[13] Y. Li, A. Marciniak-Czochra, I. Takagi and B. Wu, Bifurcation analysis of a diffusion-ODE model with Turing instability and hysteresis, Hiroshima Math. J. 47 (2017), 217–247.

[14] W. S. Loud, “Periodic solutions of $x'' + cx' + g(x) = \epsilon f(t)$”, Mem. Amer. Math. Soc., 31 (1959), 58 pp.

[15] A. Marasco, et al., Vegetation pattern formation due to interactions between water availability and toxicity in plant-soil feedback, Bull. Math. Biol., 76 (2014), 2866–2883.

[16] A. Marciniak-Czochra, Receptor-based models with diffusion-driven instability for pattern formation in hydra, J. Biol. Sys., 11 (2003), 293–324.

[17] A. Marciniak-Czochra, Receptor-based models with hysteresis for pattern formation in hydra, Math. Biosci., 199 (2006), 97–119.

[18] A. Marciniak-Czochra, Strong two-scale convergence and corrector result for the receptor-based model of the intercellular communication, IMA J. Appl. Math., 77 (2012), 855–868.

[19] A. Marciniak-Czochra, G. Karch and K. Suzuki, Instability of Turing patterns in reaction-diffusion-ODE systems, J. Math. Biol. 74 (2017), 583–618.

[20] A. Marciniak-Czochra, G. Karch and K. Suzuki, Unstable patterns in reaction-diffusion model of early carcinogenesis, J. Math. Pures Appl., 99 (2013), 599–543.

[21] A. Marciniak-Czochra and M. Kimmel, Modeling of early lung cancer progression: Influence of growth factor production and cooperation between partially transformed cells, Math. Models Methods Appl. Sci., 17 (2007), 1693–1719.
[22] A. Marciniak-Czochra, M. Nakayama and I. Takagi, Pattern formation in a diffusion-ODE model with hysteresis, *Differential Integral Equations*, 28 (2015), 655–694.

[23] A. Marciniak-Czochra and M. Ptashnyk, Derivation of a macroscopic receptor-based model using homogenization techniques, *SIAM J. Math. Anal.*, 40 (2008), 215–237.

[24] M. Mimura, M. Tabata and Y. Hosono, Multiple solutions of two-point boundary value problems of Neumann type with a small parameter, *SIAM J. Math. Anal.*, 11 (1980), 613–631.

[25] C. Niehrs, The Spemann organizer and embryonic head induction, *EMBO J.*, 20 (2001), 631–637.

[26] K. Pham, A. Chauvieire, H. Hatzikirou, X. Li, H.M. Byrne, V. Cristini and J. Lowengrub, Density-dependent quiescence in glioma invasion: instability in a simple reaction-diffusion model for the migration/proliferation dichotomy, *J. Biol. Dyn.*, 6 (2012), 54–71.

[27] F. Rothe, *Global Solutions of Reaction-Diffusion Systems*, Lecture Notes in Mathematics, 1072, Springer, 1984.

[28] R. Schaaf, *Global Solution Branches of Two Point Boundary Value Problems*, Lecture Notes in Mathematics, 1458, Springer, 1990.

[29] J. Smoller, *Shock Waves and Reaction-Diffusion Equations*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, 258, Springer, New York; Heidelberg; Berlin, 1983.

[30] A. M. Turing, The chemical basis of morphogenesis, *Philos. Trans. Roy. Soc. London Ser. B*, 237 (1952), 37–72.

[31] D. M. Umulis, M. Serpe, M. B. O’Connor and H. G. Othmer, Robust, bistable patterning of the dorsal surface of the Drosophila embryo, *Proc. Nat. Ac. Sci.*, 103 (2006), 11613–11618.

Received May 2019; revised January 2020.

E-mail address: alexandra.koethe@posteo.de
E-mail address: anna.marciniak@iwr.uni-heidelberg.de
E-mail address: i.takagi@tohoku.ac.jp