Riemann Zeta Function Expressed as the Difference of Two Symmetrized Factorials Whose Zeros All Have Real Part of $1/2$

Wusheng Zhu

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Abstract

In this paper, some new results are reported for the study of Riemann zeta function $\zeta(s)$ in the critical strip $0 < \Re(s) < 1$, such as $\zeta(s)$ expressed in a generalized Euler product only involving prime numbers. Particularly, some new absolutely convergent series representations of $\zeta(s)$ based on binomial expansion are presented. The crucial progress is to find that $\zeta(s)$ can be expressed as a linear combination of polynomials of infinite degree, whose consequences are shown in several aspects: (i) numerically it provides a scenario to construct very fast convergent algorithm to calculate $\zeta(s)$; (ii) interestingly it shows that Lagrange interpolation using infinite number of integer Euler zeta functions reproduces the exact complex $\zeta(s)$; (iii) surprisingly it demonstrates that alternating Riemann
zeta function (or other entire functions removing the pole of zeta function) is admissible to Melzak combinatorial transform for polynomials. Applying the functional symmetry on $\zeta(s)$ in the form of Melzak transform induces $\zeta(s)$ being written as the difference of two symmetrized factorials whose zeros are proved to all have real part of $1/2$. Furthermore, the two symmetrized factorials are proved to have interlacing between the two sequences of the imaginary part of their zeros on upper (or lower) half plane, which ensures the difference of the two symmetrized factorials [proportional to $\zeta(s)$] attaining the same feature of zeros with real part of $1/2$ to endorse Riemann hypothesis.
1 Introduction

Riemann zeta function $\zeta(s)$ is originally defined by analytic continuation via contour integral to extend Euler zeta function from the domain $Re(s) > 1$ to the whole complex domain\(^1\). If restricted on $Re(s) > 0$, $\zeta(s)$ has many equivalent representations either by series summation or real variable integration.

A popular series definition of $\zeta(s)$ on $Re(s) > 0$ is via Dirichlet eta function $\eta(s)$:

$$\zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s) = \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = \frac{1}{1 - 2^{1-s}} \left( 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots \right)$$  \hspace{1cm} (1)

where the series is conditionally convergent on $Re(s) > 0$, and $\zeta(s)$ has a pole at $s=1$ [$\eta(s)$ is an entire function with $\eta(1) = \ln 2$]. For $s = 1 + i2n\pi/\ln 2$ ($n \neq 0$) satisfying $1 - 2^{1-s} = 0$ were proved to be zeros of $\eta(s)$ but not zeros of $\zeta(s)$ \(^2\). All other zeros of $\eta(s)$ are the same as those of $\zeta(s)$. Riemann hypothesis is equivalent to state that any $s$ on $0 < Re(s) < 1$ which satisfies $\eta(s) = 0$ must have $Re(s) = 1/2$ and $Im(s) \neq 0$.

A well-known (Fourier) integral representation of $\zeta(s)$ on $Re(s) > 0$ is

$$\zeta(s) = \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_{0}^{\infty} \frac{x^{s-1}}{e^x + 1} dx = \frac{1}{(1 - 2^{1-s})\Gamma(s)} \int_{-\infty}^{\infty} \frac{e^{-[Re(s)]t}}{e^{-t} + 1} e^{-i[Im(s)]t} dt$$  \hspace{1cm} (2)

There exist many integrals for $\zeta(s)$. Two examples are shown below (proofs omitted):

$$\int_{0}^{\infty} x^{n-2+s} \frac{d^n}{dx^n} \left( \frac{x}{e^x - 1} \right) dx = (-1)^n (s-1) \Gamma(n-1+s) \zeta(s) \quad (n \in \mathbb{N}, \ Re(s) > 0)$$  \hspace{1cm} (3)

$$\int_{0}^{\infty} x^{n-2+s} \frac{d^n}{dx^n} \ln \left( \frac{x}{1-e^{-x}} \right) dx = (-1)^n \Gamma(n-1+s) \zeta(s) \quad (n \in \mathbb{N}_0, \ 0 < Re(s) < 1)$$  \hspace{1cm} (4)

We only consider $Re(s) > 0$ due to the fact that $\zeta(s)$ has a functional symmetry:

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos \left( \frac{\pi s}{2} \right) \Gamma(s) \zeta(s)$$  \hspace{1cm} (5)
which can explore $\zeta(s)$ on $\text{Re}(s) < 0$ by its symmetric counterpart on $\text{Re}(s) > 0$. Furthermore, on $\text{Re}(s) > 1$, Equation (1) becomes Euler zeta function which has no zeros proved by Euler product expansion. Thus only the critical strip $0 < \text{Re}(s) < 1$ needs to focus where nontrivial zeros may exist.

**Theorem 1.1.** A real positive $s$ can not be a zero of Riemann zeta function such that $\zeta(s) \neq 0$ if $\text{Im}(s) = 0$ and $\text{Re}(s) > 0$.

**Proof.** From Eq. (2) on $\text{Re}(s) > 0$, if $\zeta(s) = 0$, it must require that

$$\int_0^\infty \frac{x^{\text{Re}(s)-1}}{e^x+1} \cos[\text{Im}(s) \ln x] dx = \cos[\text{Im}(s) \ln x_1] \int_0^\infty \frac{x^{\text{Re}(s)-1}}{e^x+1} dx = 0$$

(6)

where mean value theorem for integral is applied, and $x_1 \in (0, \infty)$. As the latter integral is positive, it must have $\cos[\text{Im}(s) \ln x_1] = 0$ that can not hold with $\text{Im}(s) = 0$. \hfill \Box

Theorem 1.1 is simple and known but very important result as it implies that any polynomial expression of $\zeta(s)$ will not have any real zero on $\text{Re}(s) > 0$.

**Theorem 1.2.** The functional symmetry in Eq. (5) holds on $0 < \text{Re}(s) < 1$.

**Proof.** From another integral for $\zeta(s)$ on $0 < \text{Re}(s) < 1$ and series expansion of $\text{csch}(x)$:

$$(1 - 2^{-s})\Gamma(s)\zeta(s) = \int_0^\infty x^{s-1} \left[ \frac{1}{2} \text{csch}(x) - \frac{1}{2x} \right] dx = \int_0^\infty x^s \sum_{k=1}^\infty \frac{(-1)^k}{x^2 + \pi^2 k^2} dx$$

(7)

we have the following identity:

$$(1 - 2^{-s})\Gamma(s)\zeta(s) = \int_0^\infty \sum_{k=1}^\infty (-1)^k x^{s-1} \frac{\left( \frac{x}{k} \right)^s}{\left( \frac{x}{k} \right)^2 + \pi^2} d \left( \frac{x}{k} \right) = (2^s - 1) \zeta(1-s) \int_0^\infty \frac{x^s}{x^2 + \pi^2} dx$$

$$= (2^s - 1) \zeta(1-s) \frac{\pi^s}{2 \cos(\frac{\pi s}{2})}$$

(8)

which is a simple proof of the functional symmetry in Eq. (5) on $0 < \text{Re}(s) < 1$. \hfill \Box
Theorem 1.2, which is well-known since Riemann, is the key to answer Riemann hypothesis. The proof here is very simple, though only in critical strip.

**Theorem 1.3.** \( \zeta(s) \) on \( \Re(s) > 0 \) can be expressed into a generalized Euler product:

\[
\zeta(s) = \left( \prod_{j=1}^{k} \frac{1}{1-p_j^{-s}} \right) \left[ 1 + \lim_{m \to \infty} \sum_{2j-1=p_{k+1}}^{p_{k+m}} (2j-1)^{-s} - \frac{p_{k+m}^{1-s}}{1-s} \prod_{j=1}^{k} (1-p_j^{-1}) \right] \tag{9}
\]

where \( \{k, m\} \in \mathbb{N} \), and \( (2j-1) \) includes the prime numbers larger than \( p_k \) and all their possible products less than a sufficiently large prime number \( p_{k+m} \).

**Proof.** The summation-integral difference for \( n \to \infty \) related to \( \zeta(s) \) is

\[
1 + \frac{1}{3^{-s}} + \frac{1}{5^{-s}} + \frac{1}{7^{-s}} + \cdots + \frac{1}{(2n-1)^{-s}} - \int_{1}^{n} \frac{1}{(2x-1)^s} dx = (1-2^{-s})\zeta(s) + \frac{1}{2(1-s)} \tag{10}
\]

Multiplying \( 3^{-s} \) on Eq. (10) and subtracting from Eq. (10) yields

\[
1 + \frac{1}{5^{-s}} + \cdots + \frac{1}{(2n-1)^{-s}} - (1-\frac{1}{3}) \int_{1}^{n} \frac{1}{(2x-1)^s} dx = (1-3^{-s})(1-2^{-s})\zeta(s) + \frac{1-\frac{1}{3}}{2(1-s)} \tag{11}
\]

Repeating the procedure till the \( k \)th prime number obtains a generalized Euler product:

\[
1 + \sum_{j=(1+p_{k+1})/2}^{n} (2j-1)^{-s} - \frac{(2n-1)^{1-s}}{1-s} \prod_{j=1}^{k} (1-p_j^{-1}) = \zeta(s) \prod_{j=1}^{k} (1-p_j^{-s}) \tag{12}
\]

An infinite large prime number \( p_{k+m} \) replacing \( 2n-1 \) turns Eq. (12) to Eq. (9).

On \( \Re(s) > 1 \), the summation and product in the square brackets in Eq. (9) are towards zero when \( k \to \infty \) which gives regular Euler product. On \( 0 < \Re(s) < 1 \), if \( k \) is chosen finite, terms in the square brackets in Eq. (9) converge after cancellation of divergent components; if \( k \to \infty \) is chosen, and \( \zeta(s) \neq 0 \), the right-hand-side of Eq. (9)
will be in the form of $0 \cdot \infty$ in which L'Hopital's rule applies. If $\zeta(s) = \zeta(1-s) = 0$, considered Eq. (12), the following equation holds for $n \to \infty$:

$$
\left| \frac{1 + \sum_{j=(1+p_k+1)/2}^{n} (2j-1)^{-s}}{1 + \sum_{j=(1+p_k+1)/2}^{n} (2j-1)^{s-1}} \right| = \frac{|s|}{|1-s|} (2n-1)^{1-2\Re(s)}
$$

(13)

which implies an equivalent Riemann hypothesis as: On $0 < \Re(s) \leq 1/2$, if $\zeta(s) = 0$, the norm ratio in the left-hand-side of Eq. (13) converges to 1 when $n \to \infty$ and $k \in \mathbb{N}$.

2 Binomial Series Expansion of $\zeta(s)$

Convergent series expansions of $\zeta(s)$ are very limited. The following is such an example based on absolutely convergent binomial series:

$$
\eta(s) = 1 + \sum_{n=1}^{\infty} \frac{(n-s)!}{n!(1-s)!} \sum_{j=2}^{\infty} (-1)^{j-1} \frac{n}{j^2} \left( 1 - \frac{1}{j} \right)^{n-1} \quad (\Re(s) > 0)
$$

(14)

Equation (14) is arrived by the series definition of $\eta(s)$ in Eq. (1):

$$
\eta(s) = 1 + \frac{1}{1-s} \sum_{j=2}^{\infty} (-1)^{j-1} \left( \frac{\partial}{\partial x_j} \left[ 1 - \left( 1 - \frac{1}{x_j} \right)^{s-1} \right] \right)\bigg|_{x_j=j}
$$

(15)

followed by a binomial series expansion on $(1-X)^{s-1}$ with $X = 1 - 1/j$ that converges absolutely for finite $j$, while may diverge for infinite $j$ due to $|X| \to 1$. Fortunately, due to the factor $n/j^2$ arisen from the derivative, the series in Eq. (14) will converge.

Theorem 2.1. The binomial series expansion of $\eta(s)$ in Eq. (14) converges absolutely on $\Re(s) > 0$. 

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Proof. Comparison test of the series in Eq. (14) with Euler zeta function will be done as follows. For a sufficient large \( n \), the absolute value of \( n \)th term in Eq. (14) is

\[
\left| \frac{(n-s)!}{n!(1-s)!} \sum_{j=2}^{\infty} \frac{n(-1)^{j-1}}{j^2} \left(1 - \frac{1}{j} \right)^{n-1} \right| = O \left( \frac{1}{n^{\text{Re}(s)}} \right) \left| \sum_{j=2}^{\infty} \frac{n(-1)^{j-1}}{j^2} \left(1 - \frac{1}{j} \right)^{n-1} \right| \tag{16}
\]

where the binomial coefficients asymptotic expansion

\[
\binom{n-s}{n} = \frac{1}{n^s(-s)!} \left[ 1 + \left( \frac{s}{2} \right) \frac{1}{n} + O \left( \frac{1}{n^2} \right) \right] \tag{17}
\]

is applied. For the summation of \( j \) in Eq. (16), we have

\[
\left| \sum_{j=1}^{\infty} \frac{n}{(2j)^2} \left(1 - \frac{1}{2j} \right)^{n-1} \right| < \sum_{j=1}^{\infty} \left[ \frac{n}{(2j)^2} - \frac{n}{(2j+1)^2} \right] \left(1 - \frac{1}{2j} \right)^{n-1} = O \left( \frac{1}{n} \right) \tag{18}
\]

where the last step is realized by comparison with the integral result. Thus for sufficiently large \( n \), considered Eqs. (18) and (16), it has

\[
\left| \frac{(n-s)!}{n!(1-s)!} \sum_{j=2}^{\infty} \frac{n(-1)^{j-1}}{j^2} \left(1 - \frac{1}{j} \right)^{n-1} \right| = O \left( \frac{1}{n^{1+\text{Re}(s)}} \right) \tag{19}
\]

which proves the absolute convergence of Eq. (14) on \( Re(s) > 0 \) as compared to the \( n \)th term of Euler zeta function \( \zeta(\alpha) \) whose absolute convergence occurs on \( Re(\alpha) > 1 \).

In addition, many slightly different convergent binomial series expansion of \( \zeta(s) \) or \( \eta(s) \) can be derived from a general form of

\[
(1 - 2^{-1-s})\zeta(s) = \frac{(-\frac{s-\alpha}{\gamma})!}{(q - s-\alpha)/\gamma)!} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^\alpha} \left( \frac{\partial^q x^q}{\partial x^q_j} \left[ 1 - \left( \frac{1}{x_j^\beta} \right)^{\frac{s-\alpha-q}{\gamma}} \right] \right) \bigg|_{x_j = j^\gamma} \tag{20}
\]
where \( q \in \mathbb{N} \), \( \{ \alpha, \beta, \gamma \} \in \mathbb{C} \) with certain constraints for convergence. A special case of Eq. (20) on binomial series expansion of \((1 - s)\zeta(s)\) has been studied in literature\(^3\). If choosing \( \{ \alpha = 0, \beta \to 0, \gamma = q = 1 \} \), it induces Taylor series expansion of \( \eta(s) \) at \( s = 1 \):

\[
\eta(s) = \eta(1) + \lim_{\beta \to 0} \sum_{n=2}^{\infty} \frac{(-1)^n}{1 - s} \left( \frac{s-1}{n} \right) \sum_{j=2}^{\infty} \frac{(-1)^{j-1} n \beta}{j^{1+\beta}} \left( 1 - \frac{1}{j^{\beta}} \right)^{n-1}
\]

\[
= \eta(1) + \lim_{\beta \to 0} \sum_{n=2}^{\infty} \frac{(-1)^n}{1 - s} \left( \frac{s-1}{n} \right) n\beta \sum_{\ell=0}^{n-1} (-1)^\ell \left( \frac{n-1}{\ell} \right) \sum_{m=1}^{\infty} \frac{\eta^{(m)}(1)}{m!} [\beta(\ell + 1)]^m
\]

\[
= \eta(1) + \sum_{n=2}^{\infty} (1 - s)^{n-1} \left[ \frac{\eta^{(n-1)}(1)}{(n-1)!} \right] (-1)^{n-1}
\]

There also exist convergent binomial series of \( \zeta(s) \) [instead of \( \eta(s) \) in Eq. (14)] such as

\[
\zeta(s) = \frac{1}{s-1} + 1 + \sum_{n=1}^{\infty} \frac{(n-s)!}{n!(1-s)!} \left[ -1 + \sum_{j=2}^{\infty} \frac{n}{j^2} \left( 1 - \frac{1}{j} \right)^{n-1} \right] \quad (\text{Re}(s) > 0)
\]

But we will focus on the series expansion in Eq. (14), which is equivalent to

\[
\eta(s) = \eta(2) + \sum_{n=2}^{\infty} \frac{\chi(n)}{n!} \prod_{j=2}^{n} (j - s) \quad (\text{Re}(s) > 0)
\]

where \( \chi(n) \) \((n \geq 1)\) is defined as

\[
\chi(n) = \sum_{j=2}^{\infty} (-1)^{j-1} \frac{n}{j^2} \left( 1 - \frac{1}{j} \right)^{n-1}
\]

Some properties of \( \chi(n) \) are \( \chi(1) = \eta(2) - 1 \), \( \sum_{n=1}^{\infty} \frac{\chi(n)}{n} = \eta(1) - 1 \), and

\[
\chi(\infty) = \lim_{n \to \infty} \frac{1}{2} \sum_{j=1}^{\infty} \left[ \frac{1}{n} \left( \frac{1}{(2j+1)^2} - \frac{1}{(2j-1)^2} \right) \right] \frac{2}{n} = 0
\]

For a finite \( n \), \( \chi(n) \) is the difference between midpoint and top-right corner rectangle approaches to the same integral, approximately \( \exp(-1/x)/x^2 \). As the sign of the
difference summation in Eq. (25) depends on \( n \), \( \chi(n) \) converges to zero in oscillation pattern [it is a subtle issue on the zeros of \( \chi(z) \) as analytic continuation of \( \chi(n) \)].

Applying further binomial expansion on \( \chi(n) \) in Eq. (24) will turn Eq. (14) into

\[
\eta(s) = \sum_{n=1}^{\infty} (n-s)! \sum_{k=0}^{n-1} \frac{(-1)^k \eta(k+2)}{k!(n-k-1)!} (1-s)! \sum_{k=0}^{n-1} \frac{(-1)^k \eta(k+2)}{k!(n-k-1)!} \quad (Re(s) > 0) \tag{26}
\]

where the binomial expansion on \((1-X)^n \) with \(|X| = 1/j \leq 1 \) and \( n \in \mathbb{N} \) absolutely converges, so does the series expansion in Eq. (26). It is worth to mention that numerically Eq. (14) is more favorable due to fast convergence of \( \chi(n) \), while Eq. (26) suffers a catastrophic cancellation problem in summation of \( k \) due to alternating binomial coefficients, which relies on high precision of inputs in order for accurate output.

The crucial step towards revealing the zeros feature of \( \zeta(s) \) is to switch the order of the two summations in Eq. (26) as follows:

\[
\eta(s) = \lim_{m \to \infty} \sum_{n=1}^{m+1} (n-s)! \sum_{k=0}^{n-1} \frac{(-1)^k \eta(k+2)}{k!(n-k-1)!} (1-s)! \sum_{k=0}^{n-1} \frac{(-1)^k \eta(k+2)}{k!(n-k-1)!} \tag{27}
\]

which can be further written into

\[
\eta(s) = \lim_{m \to \infty} \frac{(m+2-s)!}{m!(1-s)!} \sum_{k=0}^{m} \frac{(-1)^k \eta(k+2)}{k+2-s} \tag{28}
\]

The validity of switching the two summations in Eq. (27) and the limit existence of \( m \to \infty \) are ensured by absolute convergence of the two series. Equation (28) can be numerically verified. For example, a small \( m=32 \) in Eq. (28) outputs a quite accurate
\( \zeta(1/2) = -1.46034778 \). If \( \{\alpha = 0, \beta = 1, \gamma = 2L, q = J\} \) are chosen in Eq. (20), a general version of Eq. (28) can be developed for \( \eta(s) \) by using a subset of \( \zeta(2k) \):

\[
\eta(s) = \lim_{m \to \infty} \frac{(m+1+J-sL)!}{m!(J-sL)!} \sum_{k=0}^{m} \binom{m}{k} (-1)^k \eta[2L(k+J+1)] \frac{1}{k+J+1-sL} \quad (J \in \mathbb{N}_0, L \in \mathbb{N})
\]  

(29)

2.1 \( \zeta(s) \) calculated by fast convergent scheme.

Equation (29) indicates that \( \zeta(s) \) can be calculated by a subset of \( \zeta(2k) \) with large \( k \) resulting in fast convergence. A numerical scheme to calculate \( \zeta(s) \) with fast convergence can be constructed by using the following identity for \( \zeta(s) \) with \( k \in \mathbb{N}, \) and \( \text{Re}(s) > 0 \):

\[
[1 - (2k - 1)^{s-1}](2^s - 1)\zeta(s) = \sum_{n=2}^{\infty} \left[ 1 - \frac{2k-1}{(2k-1)^{n+1}} \right] \frac{\Gamma(n+s)}{2^n n! \Gamma(s)} \zeta(n+s)
\]

(30)

**Proof.** Starting from the integral representation of \( \zeta(s) \)

\[
\int_0^\infty \frac{e^x - 1}{e^x - 1} x^{s-1} dx = (2^s - 2)\Gamma(s)\zeta(s) \quad (\text{Re}(s) > 0)
\]

(31)

we use geometric series summation and integral variable scaling in Eq. (31) to obtain

\[
[1 - (2k - 1)^{s-1}](2^s - 1)\Gamma(s)\zeta(s) = \int_0^\infty \frac{e^x - 1}{2k-1} \sum_{j=1}^{2k-1} e^{(2j-1)x} x^{s-1} dx \quad (\text{Re}(s) > 0)
\]

(32)

where \( k \in \mathbb{N} \). Then Eq. (30) is arrived by taking power series expansion of exponential functions on numerator in Eq. (32) and integrating each term of the power series. \( \square \)

In Eq. (30), \( k = 2 \) corresponds to the numerical scheme for calculation of \( \zeta(s) \) as

\[
(1 - 3^{1-s})(1 - 2^{-s})\zeta(s) = 1 - \frac{2}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} - \frac{2}{9^s} + \frac{1}{11^s} \ldots
\]

(33)
which is arrived by multiplying $3^{1-s}$ on Eq. (10) and subtracting from Eq. (10). The
series in Eq. (33) can be written into the form of Eq. (30) by binomial expansion:

$$\sum_{k=1}^{\infty} \left[ \frac{1}{(6k-5)^s} - \frac{2}{(6k-3)^s} + \frac{1}{(6k-1)^s} \right] = \sum_{n=2}^{\infty} \left[ \frac{\Gamma(n + s)}{\Gamma(s) n!} \frac{(1 - 2 \cdot 3^n + 5^n)}{6^{n+s}} \zeta(n+s) \right]$$

(34)

where $1 - 2 \cdot 3^n + 5^n = 0$ for $n = 0, 1$ leads to the convergence order of $O(n^{-1-2\Re(s)})$
roughly as fast as $\zeta(2)$. In Eq. (30), $k = 3$ corresponds to

$$(1 - 5^{1-s})(1 - 2^{-s})\zeta(s) = 1 + \frac{1}{3^s} - \frac{4}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{13^s} - \frac{4}{15^s} + \frac{1}{17^s} + \cdots$$

(35)

Linear combination of Eqs. (33), (35) and so on can generate series to calculate $\zeta(s)$
converging as fast as $\zeta(N)$, provided that proper combination coefficients (solved from
a set of linear equations) let all terms of $n < N$ vanish in the summation of $n$ when
combining $k$ in Eq. (30). For example, the following combined series with the specific
coefficients converges as fast as $\zeta(6)$ on $Re(s) > 0$:

$$\frac{6^s}{3} \left[ \frac{1 - 2}{3^s} + \frac{1}{5^s} + \cdots \right] - \frac{10^{s+2}}{756} \left[ \frac{1}{3^s} - \frac{4}{5^s} + \cdots \right] + \frac{18^{s+1}}{70} \left[ \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} - \frac{8}{9^s} + \cdots \right]
= \left[ \frac{(3^s-1)}{189} - \frac{625}{189}(5^{s-1} - 1) + \frac{81}{35}(9^{s-1} - 1) \right] \Gamma(s) \zeta(s)$$

(36)

It shows that Eqs. (1), (33), and (36) need total $10^6$, $10^4$, and 85 terms [15, 25, 45 terms
chosen for the alternating series in Eq. (36) from left to right, respectively] to achieve
$0.37471336 - 0.27518432i$, $0.36010325 - 0.26624621i$, and $0.36010259 - 0.26624619i$,
respectively, for calculation of $\zeta(0.2+2i)$ whose value is $0.36010259 - 0.26624620i$. 
2.2 \( \eta(s) \) reproduced by Lagrange interpolation on a set of infinite number of integer eta functions.

Explicitly Eq. (28) can be reformed into

\[
\eta(s) = \lim_{m \to \infty} \sum_{k=0}^{m} \prod_{j=0, j \neq k}^{m} \frac{(s-(j+2))}{k!(m-k)!} \frac{(-1)^{k+m} \eta(k+2)}{(k+2)-(j+2)} \cdot \]

which is an exact Lagrange interpolation formula on infinite number of integers 2, 3, 4, \cdots.

The interpolation also can be done on a subset of integers based on Eq. (29). There exist infinite number of good table of nodes for convergent interpolation of \( \eta(s) \). And some fast convergent iteration methods developed for Lagrange interpolation type equation can be applied to numerically find all the roots of \( \eta(s) = 0 \) simultaneously.

2.3 \( \eta(s) \) is admissible to Melzak transform for polynomials.

Melzak transform is inherited from combinatorial identities and finite difference theory for polynomials of finite degree. A basic Melzak transform is defined as:

**Theorem 2.2.** If \( f(x) \) is a polynomial of degree \( m \), the following transform holds

\[
f(x-y) = y \left( \frac{m-y}{m} \right) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{f(x-k)}{y-k} \quad (y \neq 0, 1, \cdots, m)
\]

for \( x, y \in \mathbb{C} \). Choosing \( x = 0 \) yields a special case:

\[
f(-y) = y \left( \frac{m-y}{m} \right) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{f(-k)}{y-k} \quad (y \neq 0, 1, \cdots, m)
\]

The proof is in literature. Equation (28) shows that \( \eta(s) \) is admissible to Melzak transform by replacing \( y = s - 2 \), \( f(-k) = \eta(k+2) \), and \( f(-y) = \eta(s) \) in Eq. (39).
analytic continuation, Melzak transform can be used to extend a function from integer domain into complex domain (e.g., defining complex index Bernoulli numbers).

Admissible condition to Melzak transform for \( m \to \infty \) can be used to characterize an entire function which behaves like a pseudo-finite degree polynomial. If \( f_m(-k) \) is the truncated polynomial of \( f(-k) \) by cutting off all degrees above \( m \), then the requirement for \( f(-k) \) being admissible to Melzak transform in Eq. (39) for \( m \to \infty \) is

\[
\lim_{m \to \infty} y \left( \frac{m - y}{m} \right) \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{[f(-k) - f_m(-k)]}{y - k} = 0 \quad (y \neq 0, 1, \cdots, m) \tag{40}
\]

For instance, exponential decay and sinc damping are admissible, which satisfy both \( \lim_{k \to \infty} f(-k) = 0 \) and \( \lim_{m \to \infty} f^{(m)}(0)/m = 0 \), while cosine and sine are not admissible.

### 3 \( \zeta(s) \) expanded by zeros of symmetrized factorials

From Eq. (28), \( \eta(s) = 0 \) is equivalent to

\[
\lim_{m \to \infty} \sum_{k=0}^{m} \binom{m}{k} \frac{(-1)^k \eta(k + 2)}{k + 2 - s} = 0 \tag{41}
\]

which unfortunately can not be solved analytically. To unveil the feature of \( \eta(s) = 0 \) on \( 0 < \text{Re}(s) < 1 \), we turn to apply various factorization by zeros on \( \eta(s) \) in Eq. (28).

#### 3.1 \( \eta(s) \) as the summation of symmetrized factorials of all even degrees whose zeros all have real part of \( 1/2 \).

The functional symmetry of \( \zeta(s) \) or \( \eta(s) \) is the core reason to cause the special feature of \( \zeta(s) = 0 \). We found that symmetrized factorials arisen from the functional
Theorem 3.1. All roots of the following symmetrized factorial polynomial equation

\[ \prod_{j=1}^{n} (a_j - 1 + s) + \prod_{j=1}^{n} (a_j - s) = 0 \quad (n > 1) \quad (42) \]

have real part of 1/2 and nonzero imaginary part if all real \((a_j - 1/2)\) do not vanish simultaneously and the nonzero \((a_j - 1/2)\) have the same sign.

Proof. Change the variable to be \(x = s - 1/2\), Equation (42) becomes

\[ \prod_{j=1}^{n} \left( a_j - \frac{1}{2} + x \right) = -\prod_{j=1}^{n} \left( a_j - \frac{1}{2} - x \right) \quad (43) \]

If any root \(x\) has nonzero real part as \(x = \delta + i\tau\), then Eq. (43) requires

\[ \left| \prod_{j=1}^{n} \left( a_j - \frac{1}{2} + \delta + i\tau \right) \right|^2 = \prod_{j=1}^{n} \left[ \left( a_j - \frac{1}{2} + \delta \right)^2 + \tau^2 \right] \]

\[ \left| \prod_{j=1}^{n} \left( a_j - \frac{1}{2} - \delta - i\tau \right) \right|^2 = \prod_{j=1}^{n} \left[ \left( a_j - \frac{1}{2} - \delta \right)^2 + \tau^2 \right] \]

However, when all \((a_j - 1/2)\) have the same sign, if \(\delta \neq 0\), the norm ratio in Eq. (44) will always be greater or smaller than 1. Equation (44) holds only if \(\delta = 0\) or \(x = i\tau\).

Thus all \(2[n/2]\) roots of Eq. (42) have real part of 1/2 and nonzero imaginary part [Eq. (43) can not hold for real \(s=1/2\) (i.e., \(x=0\)] when at least one \((a_j - 1/2) \neq 0\).

The anti-symmetrized version of Eq. (42) (the difference of the two factorials) can also be proved to have all roots with real part of 1/2 (and \(s = 1/2\) is a root too).

In this paper we only focus on the symmetrized version.

Applying the functional symmetry of Eq. (5) on \(\eta(s)\) in Eq. (23), we have

\[ \eta(s) + \eta(1 - s) = \zeta(2) + \sum_{n=2}^{\infty} \frac{\chi(n)}{n!} \left[ \prod_{j=0}^{n-2} (j + 2 - s) + \prod_{j=0}^{n-2} (j + 1 + s) \right] \quad (45) \]
which is valid in the critical strip because both $\eta(s)$ and $\eta(1 - s)$ expanded via alternating series in Eq. (1) are valid on $0 < \text{Re}(s) < 1$. Considered Theorem 3.1, Equation (45) can be factorized by zeros of symmetrized factorials of all even degrees:

$$\eta(s) + \eta(1 - s) = \zeta(2) + \frac{3\chi(2)}{2!} + \sum_{n=3}^{\infty} \left( [1 - (-1)^n] + \frac{n^2 - 1}{2} [1 + (-1)^n] \right) \frac{\chi(n)}{n!} \prod_{j=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left( s - \frac{1}{2} \right)^2 + \nu_{n,j}^2 \right) \right \} \tag{46}$$

where $\{1/2 \pm i\nu_{n,j}\}$ are the complex conjugated zeros of the symmetrized factorial of degree $2[(n - 1)/2]$ in the square brackets in Eq. (45). The first few $\{\nu_{n,j}\}$ are

$$\{\nu_{3,j}\} = \pm \frac{1}{2} \sqrt{15}, \quad \{\nu_{4,j}\} = \pm \frac{1}{2} \sqrt{7}, \quad \{\nu_{5,j}\} = \pm \frac{1}{2} \sqrt{103 \pm 8\sqrt{151}} \tag{47}$$

with $j = 1, 2, \cdots, 2[(n - 1)/2]$. If $\eta(s)$ is expanded by Taylor series in Eq. (21), since the zeros of $(1 - s)^n + s^n$ are solvable, the factorization by zeros becomes

$$\eta(s) + \eta(1 - s) = 2\eta(1) - \eta^{(1)}(1) + \sum_{n=2}^{\infty} \frac{(-1)^n \eta^{(n)}(1)}{n!} \prod_{k=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left\{ (s - \frac{1}{2})^2 + 4 \left\{ \tan \left( \frac{(2k - 1)\pi}{2n} \right) \right\}^2 \right\} \tag{48}$$

A trivial case is Taylor expansion at $s = 1/2$ as factorization by a single repeated zero.

### 3.2 $\eta(s)$ as the summation of symmetrized factorials of the same infinite degree whose zeros all have real part of 1/2.

If the functional symmetry is applied on $\eta(s)$ in Eq. (28), it has

$$\eta(s) + \eta(1 - s) = \lim_{m \to \infty} \sum_{k=0}^{m} \frac{(-1)^k \eta(k + 2)}{k!(m - k)!} \left[ \prod_{j=0}^{m} (j + 2 - s) + \prod_{j=0}^{m} (j + 1 + s) \right] \tag{49}$$

15
Considered Theorem 3.1 \( \eta(s) \) in Eq. (49) can be factorized by zeros of the symmetrized factorials of infinite degree as

\[
\eta(s) + \eta(1-s) = \lim_{m \to \infty} \sum_{k=0}^{m} \frac{2(-1)^k \eta(k+2)}{k!(m-k)!} \prod_{j=1}^{m} \left[ \left( s - \frac{1}{2} \right)^2 + \tau_{k,j}^2 \right]
\]  

(50)

where \( \{1/2 \pm i\tau_{k,j}\} \) are the complex conjugated zeros of each symmetrized factorial in the square brackets of Eq. (49), and an even \( m \) is chosen for convenience in this paper.

3.3 \( \eta(s) \) as the difference of two symmetrized factorials of the same infinite degree whose zeros all have real part of 1/2.

Equation (49) is a linear combination of the symmetrized factorials of the same degree. To deal with the case of linear combination, we have the following result:

**Theorem 3.2.** The equation below is a linear combination of symmetrized factorials:

\[
\sum_{k=0}^{m} c_k \left[ \prod_{j=0}^{m} (a_j - 1 + s) + \prod_{j=0}^{m} (a_j - s) \right] = 0 \quad (m > 1)
\]  

(51)

with all coefficients \( c_k \) of the same sign. Then all roots of Eq. (51) must have real part of 1/2 and nonzero imaginary part if all real \( (a_j - 1/2) \) do not vanish simultaneously, and the nonzero \( (a_j - 1/2) \) have the same sign.

**Proof.** Equation (51) requires

\[
\sum_{k=0}^{m} c_k \prod_{j=0}^{m} (a_j - s) \prod_{j=0}^{m} (a_j - 1 + s) = -1
\]  

(52)
where the norm ratio should be 1:

\[
\frac{\prod_{j=0}^{m} |a_j - s|^2}{\prod_{j=0}^{m} |a_j - 1 + s|^2} = 1
\]

Assume that there exists a root \( s = 1/2 + \delta + i\tau \) for Eq. (51). We can define the norms

\[
\begin{align*}
    r_j &\equiv |a_j - s| = \sqrt{(a_j - \frac{1}{2} - \delta)^2 + \tau^2} \\
    R_j &\equiv |a_j - 1 + s| = \sqrt{(a_j - \frac{1}{2} + \delta)^2 + \tau^2}
\end{align*}
\]

for \( j = 0, 1, \cdots, m \). Then Eq. (53) becomes

\[
1 = \left( \prod_{j=0}^{m} \frac{r_j^2}{R_j^2} \right) \left( \sum_{k=0}^{m} \frac{c_k}{r_k^2} \left( a_k - \frac{1}{2} - \delta \right)^2 + \tau^2 \left( \sum_{k=0}^{m} \frac{c_k}{r_k^2} \right)^2 \right)
\]

\[
= \sum_{j=0}^{m} \sum_{k=0 \atop k \neq j}^{m} c_k c_\ell \left[ \left( a_k - \frac{1}{2} - \delta \right) \left( a_\ell - \frac{1}{2} - \delta \right) + \tau^2 \right] \prod_{j=0 \atop j \neq k, \neq \ell}^{m} \frac{r_j^2}{R_j^2}
\]

(56)

Suppose that all real \((a_j - 1/2)\) do not vanish simultaneously, and the nonzero \((a_j - 1/2)\) have the same sign. If \( \delta \) has the same sign as the nonzero \((a_j - 1/2)\), it has \( r_j < R_j \) for all \( j \), then Eq. (56) can not hold for all \( c_k \) of the same sign because the numerator is always smaller than the denominator. If \( \delta \) and the nonzero \((a_j - 1/2)\) have opposite sign, then the numerator is always greater than the denominator. Thus if \( s = 1/2 + \delta + i\tau \) is a root of Eq. (51), it must have \( \delta = 0 \). And for real \( s = 1/2 \) (i.e., \( \tau = 0 \), Equation (52) can not hold when at least one \((a_j - 1/2) \neq 0 \).

\( \square \)
From Theorem 3.2, we have the following factorization by roots of Eq. (51):

\[
\sum_{k=0}^{m} c_k \left[ \prod_{j=0}^{m} (a_j - s) + \prod_{j=0}^{m} (a_j - 1 + s) \right] = 2 \left( \sum_{k=0}^{m} c_k \right) \prod_{j=1}^{m} \left[ \left( s - \frac{1}{2} \right)^2 + T_j^2 \right]
\]  

(57)

where \( \{1/2 \pm iT_j\} \) are the corresponding roots. On the other hand, from Theorem 3.1 we have another factorization by zeros of each symmetrized factorial:

\[
\sum_{k=0}^{m} c_k \left[ \prod_{j=0}^{m} (a_j - s) + \prod_{j=0}^{m} (a_j - 1 + s) \right] = \sum_{k=0}^{m} c_k \left( 2 \prod_{j=1}^{m} \left[ \left( s - \frac{1}{2} \right)^2 + \tau_{k,j}^2 \right] \right)
\]  

(58)

where \( \{1/2 \pm i\tau_{k,j}\} \) are the zeros of the \( k \)th symmetrized factorial. Comparing Eq. (58) to Eq. (57) for \( s = 1/2 \) reveals one correlation among the imaginary parts of all zeros:

\[
\prod_{j=1}^{m} T_j^2 = \frac{\sum_{k=0}^{m} c_k \prod_{j=1}^{m} \tau_{k,j}^2}{\sum_{k=0}^{m} c_k}
\]

(59)

Specifically, in Eq. (49), if the same sign coefficients are grouped separately, Theorem 3.2 can be applied on Eq. (49) to factorize by zeros of two combined polynomials:

\[
\eta(s) + \eta(1-s) = \lim_{m \to \infty} 2 \left( \sum_{k=0}^{m} \frac{\eta(2k+2)}{(2k)!((m-2k))!} \prod_{j=1}^{m} \left[ \left( s - \frac{1}{2} \right)^2 + \Theta_j^2 \right] \right) - \lim_{m \to \infty} 2 \left( \sum_{k=0}^{m-1} \frac{\eta(2k+3)}{(2k+1)!((m-2k-1))!} \prod_{j=1}^{m} \left[ \left( s - \frac{1}{2} \right)^2 + \Phi_j^2 \right] \right)
\]

(60)

where \( \{1/2 \pm i\Theta_j\} \) and \( \{1/2 \pm i\Phi_j\} \) are the zeros of the combined polynomials of even and odd \( k \) terms, respectively. Equation (60) is arrived by first applying the functional symmetry on Eq. (28), and then combining the symmetrized factorials with coefficients of the same sign in Eq. (49). The order can be switched. Before applying the functional symmetry, we can first combine the factorials with coefficients of the
same sign in Eq. (28), or alternatively add an auxiliary function to convert all the combination coefficients in Eq. (28) to be positive and then subtract it [which avoids to deal with even and odd \( k \) terms]. For example, choosing \( f(x) = \left(\frac{2x+2m}{2m}\right)/(x+m) \) for Melzak transform in Eq. (38), then for \( m \geq 1, y \neq 0, 1, \cdots, m, \) and \( x = 0, \) we have

\[
y \binom{m-y}{m} \sum_{k=0}^{m} \binom{2k}{k} \binom{2m-2k}{m-k} \frac{1}{y-k} = \frac{\left(\frac{2m}{m}\right) \left(\frac{2m-2y}{2m}\right)}{\binom{m-y}{m}} = \frac{2^{2m} m^{-1}}{m! \prod_{j=0}^{m-1} (j + \frac{1}{2} - y)} \quad (61)
\]

Substituting \( y = s - 2 \) in Eq. (61) reduces to

\[
\frac{\pi}{\left(\frac{m-1}{2}\right)!^2} \prod_{j=0}^{m-1} (j + \frac{5}{2} - s) = \sum_{k=0}^{m} \frac{2^{-2m} \pi}{\left(\frac{m-1}{2}\right)!^2} \binom{2k}{k} \binom{2m-2k}{m-k} \prod_{j=0, j \neq k}^{m} (j + 2 - s) \quad (62)
\]

which is a special case of the following identity from Melzak transform:

\[
\frac{\pi}{\sin(\pi \beta)} \prod_{j=0}^{m-1} (j + \gamma + 1 - \beta - s) = \sum_{k=0}^{m} \frac{(m-k-\beta)!(k+\beta-1)!}{k!(m-k)!} \prod_{j=0, j \neq k}^{m} (j + \gamma - s) \quad (63)
\]

where \( \beta \notin \mathbb{Z} \) and \( \{\beta, \gamma\} \in \mathbb{C} \).

For \( 0 \leq k \leq m \), the summation coefficients in Eq. (62) are always positive:

\[
\frac{2^{-2m} \pi}{\left(\frac{m-1}{2}\right)!^2} \binom{2k}{k} \binom{2m-2k}{m-k} = \frac{1}{\left(\frac{m-1}{2}\right)!^2} \frac{(k-\frac{1}{2})!(m-k-\frac{1}{2})!}{k!(m-k)!} \geq \frac{1}{k!(m-k)!} \quad (64)
\]

where we considered the fact that for \( 0 \leq k \leq m \), \( \binom{m}{k} \) has peak value at \( \binom{m}{m/2} \).

Taking \( m \to \infty \) in Eq. (62) and adding with Eq. (28) obtains

\[
\eta(s) = \lim_{m \to \infty} \sum_{k=0}^{m} (-1)^k \eta(k+2) + \frac{(k-\frac{1}{2})!(m-k-\frac{1}{2})!}{\left(\frac{m-1}{2}\right)!^2} \prod_{j=0}^{m} (j + 2 - s) - \frac{\pi}{\left(\frac{m-1}{2}\right)!^2} \prod_{j=0}^{m-1} (j + \frac{5}{2} - s) \quad (65)
\]

\[
\eta(s) \equiv \lim_{m \to \infty} \sum_{k=0}^{m} \epsilon_k \prod_{j=0, j \neq k}^{m} (j + 2 - s) - \frac{\pi}{\left(\frac{m-1}{2}\right)!^2} \prod_{j=0}^{m-1} (j + \frac{5}{2} - s) \quad (65)
\]
where all \( \epsilon_k > 0 \) due to Eq. (64) and \( \eta(k + 2) < 1 \). The sum of the coefficients is

\[
\sum_{k=0}^{m} \epsilon_k \equiv \sum_{k=0}^{m} \frac{(-1)^k \eta(k+2) + \frac{(k-\frac{1}{2})!(m-k-\frac{1}{2})!}{[(m-1)!]^2}}{k!(m-k)!} = \chi(m+1) + \frac{\pi}{[(\frac{m-1}{2})!]^2} (m-1)! \prod_{j=0}^{m-1} \left( j + 2 + d_j - s \right)
\]

(66)

where \( \chi(m+1) \) is defined in Eq. (24).

Equation (65) can be numerically verified. For example, choosing a small \( m=20 \) in Eq. (65) outputs 0.70082616+0.43532002i for \( \eta(0.2+2i) \) whose value is 0.70077353+0.43513124i. For combination with all \( \epsilon_k > 0 \) in Eq. (65), we have the following result:

**Theorem 3.3.** Assume that \( f(x) \) is a polynomial of degree \( m \) with positive leading coefficient, and have all real distinct zeros of \( \{a_1, a_2, \cdots, a_m\} \). Polynomial \( g(x) \) of degree \( m-1 \) is obtained by linear combination as

\[
g(x) = c_1 \frac{f(x)}{x-a_1} + \cdots + c_m \frac{f(x)}{x-a_m}
\]

(67)

Then \( g(x) \) has \( m-1 \) real zeros \( \{b_1, b_2, \cdots, b_{m-1}\} \) which interlace with \( m \) real zeros of \( f(x) \) as \( a_1 < b_1 < a_2 < \cdots < a_{m-1} < b_{m-1} < a_m \) if and only if all \( c_i \) are positive.

The proof can be found elsewhere[6]. Applying Theorem 3.3 on Eq. (65) yields

\[
\eta(s) = \lim_{m \to \infty} \left( \frac{\chi(m+1)}{(m+1)!} + \frac{\pi}{[(\frac{m-1}{2})!]^2} \prod_{j=0}^{m-1} \left( j + 2 + d_j - s \right) - \frac{\pi}{[(\frac{m-1}{2})!]^2} \prod_{j=0}^{m-1} \left( j + \frac{5}{2} - s \right) \right)
\]

\[
\equiv \lim_{m \to \infty} [f_d(s) - f_h(s)]
\]

(68)

where \( 0 < d_j < 1 \) is required by roots interlacing from Theorem 3.3.

Theorem 1.1 denies any real zero for \( \eta(s) \) on \( \text{Re}(s) > 0 \). Thus there can not exist any interlacing segment for three or more zeros between the zeros \( \{2+d_0, 3+d_1, \cdots, m+1+d_{m-1}\} \) of \( f_d(x) \) and the zeros \( \{2.5, 3.5, \cdots, m+1.5\} \) of \( f_h(s) \), otherwise
the combined polynomial will have at least one real zero. For example, if only one zero \( j+2+d_j \) of \( f_d(s) \) is in-between two consecutive zeros \( \{j+2.5, j+3.5\} \) of \( f_h(s) \) to form a three-zero interlacing, then at two ends where \( f_h(s) = 0 \), the combined polynomial \( f_d(s) - f_h(s) \) will have opposite sign, the same as \( f_d(s) \). This means that at least one time sign change occurs in-between (causing a real zero) for the combined polynomial. Considered both \( 0 < d_j < 1 \) and nonexistence of zeros interlacing segments, the only possibility to arrange the two sequences of the zeros of \( f_d(s) \) and \( f_h(s) \) to ensure the combined polynomial \( \eta(s) \) having no real zeros on real positive \( s \) will be

\[
2.5 < 2+d_0 < 3+d_1 < 3.5 < 4.5 < \cdots < m+d_{m-2} < m+1+d_{m-1} < m+1.5 \quad (69)
\]

where the smallest zero is from \( f_h(s) \), since it has \( f_h(s) + 1/2 < f_d(s) < f_h(s) + 1 \) due to \( 1/2 < \eta(s) < 1 \) for all real \( s > 0 \). Equation (69) also can be expressed as

\[
0 < d_{2\ell+1} < \frac{1}{2} < d_{2\ell} < 1 \quad (\ell = 0, 1, \cdots, \frac{m}{2}) \quad (70)
\]

We numerically verified the validity of Eqs. (69) or (70). Moreover, here we list a few properties of the two functions \( f_d(s) \) and \( f_h(x) \) defined in Eq. (68):

(i) Two consecutive zeros of \( f_d(s) \): \( \{2\ell+2+d_{2\ell}, 2\ell+3+d_{2\ell+1}\} \) with \( \ell \in [0, m/2] \) are within the interval of two consecutive zeros of \( f_h(s) \): \( \{2\ell+2.5, 2\ell+3.5\} \) with \( \ell \in [0, m/2] \).

(ii) On real \( s > 0 \), \( f_d(s) \) and \( f_h(s) \) cross the real axis \( m \) times, and \( f_d(s) \) is always on top of \( f_h(s) \) more than 1/2 but less than 1 without crossing each other.

(iii) \( f_h(s) \) and \( f_d(s) \) are monotonic decrease on \( s \in [0, 2.5] \), and only have one peak between two consecutive zeros [as \( f_h^{(1)}(s) \) and \( f_d^{(1)}(s) \) change sign one time in-between].

The following graph illustrates the displacement of \( f_d(s) \) and \( f_h(s) \) on real \( s > 0 \):
Considering the functional symmetry of Eq. (68), we will have

$$\eta(s) + \eta(1-s) = \lim_{m \to \infty} \left\{ -\frac{\pi}{\left[\left(\frac{m-1}{2}\right)!\right]^2} \left[ \prod_{j=0}^{m-1} (j+\frac{5}{2}-s) + \prod_{j=0}^{m-1} (j+\frac{3}{2}+s) \right] \right. \\
+ \left. \left( \frac{\chi(m+1)}{(m+1)!} + \frac{\pi}{\left[\left(\frac{m-1}{2}\right)!\right]^2} \right) \left[ \prod_{j=0}^{m-1} (j+2+d_j-s) + \prod_{j=0}^{m-1} (j+1+d_j+s) \right] \right\}$$

which shows that $\eta(s)$ [proportional to $\eta(s)+\eta(1-s)$] can be expressed as the difference of two symmetrized factorials of infinite degree that all zeros of each symmetrized factorial have real part of 1/2. In details, applying Theorem 3.1 on Eq. (71) obtains

$$\eta(s) + \eta(1-s) = \lim_{m \to \infty} \left\{ -\frac{2\pi}{\left[\left(\frac{m-1}{2}\right)!\right]^2} \prod_{j=1}^{\frac{m}{2}} \left[ \left( s - \frac{1}{2}\right)^2 + \lambda_j^2 \right] \right. \\
+ \left. \left( \frac{2\chi(m+1)}{(m+1)!} + \frac{2\pi}{\left[\left(\frac{m-1}{2}\right)!\right]^2} \right) \prod_{j=1}^{\frac{m}{2}} \left[ \left( s - \frac{1}{2}\right)^2 + \omega_j^2 \right] \right\}$$

In Eq. (72), $\pm \lambda_j$ ($j = 1, 2, \cdots, m/2$) are m real roots of the following equation

$$F_h(x) \equiv \prod_{j=0}^{m-1} (j+2-ix) + \prod_{j=0}^{m-1} (j+2+ix)$$

$$= 2 \cos \left( \sum_{j=0}^{m-1} \arctan \left( \frac{x}{j+2} \right) \right) \prod_{j=0}^{m-1} \sqrt{\left(j+2\right)^2 + x^2} = 0$$
which is arrived by substituting $s = 1/2+ix$ ($x$ is a real number!) in Eq. (71) as the zeros of the symmetrized factorials based on Theorem 3.1. Similarly $\pm \omega_j$ ($j = 1, 2, \cdots, m/2$) in Eq. (72) are $m$ real roots of the following equation:

$$F_d(x) \equiv \prod_{j=0}^{m-1} (j + \frac{3}{2} + d_j - ix) + \prod_{j=0}^{m-1} (j + \frac{3}{2} + d_j + ix)$$

$$= 2 \cos \left( \sum_{j=0}^{m-1} \arctan \left( \frac{x}{j + \frac{3}{2} + d_j} \right) \right) \prod_{j=0}^{m-1} \sqrt{\left( j + \frac{3}{2} + d_j \right)^2 + x^2} = 0 \quad (74)$$

Below we will analyze the relationship between the roots of Eqs. (73) and (74).

### 3.4 $\eta(s)$ as a single polynomial of infinite degree whose zeros all have real part of 1/2.

Since Eqs. (74) and (73) are polynomials of only even power of $x$ [all odd power of $ix$ in $F_h(x)$ and $F_d(x)$ vanish otherwise $F_h(x)$ and $F_d(x)$ can not be real], we can define the new variable as $y = x^2$ so that $\{\omega_j^2\}$ with $j = 1, 2, \cdots, m/2$ are the roots of $F_d(y) = 0$ and $\{\lambda_j^2\}$ with $j = 1, 2, \cdots, m/2$ are the roots of $F_h(y) = 0$. Then we have

**Theorem 3.4.** The $m/2$ distinct real roots $\{\lambda_j^2\}$ of $F_h(y)$ in Eq. (73) (defining $y = x^2$) interlace the $m/2$ distinct real roots $\{\omega_j^2\}$ of $F_d(y)$ in Eq. (74) (defining $y = x^2$) or vice versa as $\lambda_1^2 < \omega_1^2 < \lambda_2^2 < \cdots < \lambda_{m/2}^2 < \omega_{m/2}^2$ or $\omega_1^2 < \lambda_1^2 < \omega_2^2 < \cdots < \omega_{m/2}^2 < \lambda_{m/2}^2$ depending on specific sequence of $0 < d_{2\ell+1} < 1/2 < d_{2\ell} < 1$ in Eq. (74).

**Proof.** First of all, $\{\lambda_j^2\}$ and $\{\omega_j^2\}$ are all real guaranteed by applying Theorem 3.1 on Eq. (71). Finding the roots of $F_h(y) = 0$ [or $F_d(y) = 0$] is equivalent to finding the roots of $F_h(x) = 0$ [or $F_h(x) = 0$] on $x > 0$ subject to a square mapping.
Since $F_d(x)$ and $F_h(x)$ have the same degree and the same sign of the leading coefficients, if $F_d(x)$ and $F_h(x)$ have sign interlacing on $x > 0$, then their zeros are interlacing on $x > 0$. Without loss of generality, we consider if there is a sign change of $F_d(x)$ between any two consecutive real positive zeros of $F_h(x)$. Let the two consecutive zeros of $F_h(x)$ on $x > 0$ be $x_0$ and $x_1$, from Eq. (73) we have

$$\cos \left( \sum_{j=0}^{m-1} \arctan \left( \frac{x_0}{j+2} \right) \right) = 0 \quad \implies \quad \sum_{j=0}^{m-1} \arctan \left( \frac{x_0}{j+2} \right) = \frac{2k+1}{2} \pi$$  \hspace{1cm} (75)

$$\cos \left( \sum_{j=0}^{m-1} \arctan \left( \frac{x_1}{j+2} \right) \right) = 0 \quad \implies \quad \sum_{j=0}^{m-1} \arctan \left( \frac{x_1}{j+2} \right) = \frac{2k+3}{2} \pi$$  \hspace{1cm} (76)

where $k \in \mathbb{N}_0$. The sums of the angles in Eqs. (75) and (76) must have a difference of $\pi$ because from $x_0$ to $x_1$ is a continuous process and between $x_0$ and $x_1$, $F_h(x)$ (equivalently the cosine function) does not change sign (otherwise, they are not consecutive zeros) so that $x_0$ and $x_1$ corresponds to two consecutive cosine zeros.

If $F_d(x)$ has sign change on $x_0$ and $x_1$, from Eq. (74) it must satisfy

$$\cos \left( \sum_{j=0}^{m-1} \arctan \left( \frac{x_0}{j+\frac{3}{2}+d_j} \right) \right) \cos \left( \sum_{j=0}^{m-1} \arctan \left( \frac{x_1}{j+\frac{3}{2}+d_j} \right) \right) < 0$$  \hspace{1cm} (77)

Considered the inverse tangent formula of sum of angles:

$$\sum_{j=0}^{m-1} \arctan \left( \frac{x}{j+\frac{3}{2}+d_j} \right) = \sum_{j=0}^{m-1} \arctan \left( \frac{x}{j+2} \right) - \sum_{j=0}^{m-1} \arctan \left( \frac{x}{j+2 + \frac{\rho_j(x)}{d_j-\frac{1}{2}}} \right)$$  \hspace{1cm} (78)

with $\rho_j(x) = (j+2)^2 + x^2$, Equation (77) is equivalent to require

$$\sin \left( \sum_{j=0}^{m-1} \arctan \left( \frac{x_0}{j+2 + \frac{\rho_j(x_0)}{d_j-\frac{1}{2}}} \right) \right) \sin \left( \sum_{j=0}^{m-1} \arctan \left( \frac{x_1}{j+2 + \frac{\rho_j(x_1)}{d_j-\frac{1}{2}}} \right) \right) > 0$$  \hspace{1cm} (79)

where Eqs. (75) and (76) are utilized. Equation (79) can be algebraically proved to hold for real positive $x_0$ and $x_1$ and $d_j$ satisfying Eq. (70). But it is more straightforward
to prove Eq. (79) by geometric graph showing that both sums of angles in Eq. (79) are less than $\pi/2$. The following graph shows the complex vectors $a_j + ix$:

![Geometric graph of the angles in Eq. (79)](image)

in which the angles (containing the balls) are

\[
\angle A_0 O A_1 = \arctan \left( \frac{x_0}{2 + \rho_0(x_0)/(d_0 - 1/2)} \right),
\]

\[
\angle A_2 O A_3 = -\arctan \left( \frac{x_0}{3 + \rho_1(x_0)/(d_1 - 1/2)} \right)
\]

etc. It is obvious that even the sum of absolute angles in Eq. (79) are less than $\pi/2$ so that Eq. (79) and then Eq. (77) hold, which proves the sign interlacing between $F_d(x)$ and $F_h(x)$ on $x > 0$. Equivalently, it concludes that the zeros of $F_h(y)$ and $F_d(y)$ are interlacing, but the smallest zero cannot be determined by a general sequence of $0 < d_{2\ell+1} < 1/2 < d_{2\ell} < 1$ unless additional information applies.

In Eq. (72), when factorization by zeros between the two polynomials is done before the limit $m \to \infty$ is taken, then for a sufficiently large $m$, $\chi(m + 1)$ can be chosen as a tiny negative number for convenience, which is similar to Taylor expansion of $e^{-x}$ being terminated at a sufficiently large term with coefficient of either sign while
approaching the same convergent limit. If $\chi(m + 1) < 0$ is chosen, then applying Theorem 3.4 will lead to an unambiguous zeros interlacing as $\lambda_1^2 < \omega_1^2 < \lambda_2^2 < \cdots < \lambda_{m/2}^2 < \omega_{m/2}^2$ due to the requirement of $\eta(1/2) > 0$ with $\chi(m + 1) < 0$ in Eq. (72).

Therefore, changing the variable as $y = (s - 1/2)^2$, Equation (72) indicates that $\eta(s) + \eta(1 - s)$ can be expressed as the difference of two polynomials of $y$:

$$\eta(s) + \eta(1 - s) = \lim_{m \to \infty} \left[ \frac{2\chi(m + 1)}{(m + 1)!} + \frac{2\pi}{[(m-1)!]^2} \right] \prod_{j=1}^{m} (y + \omega_j^2) - \frac{2\pi}{[(m-1)!]^2} \prod_{j=1}^{m} (y + \lambda_j^2) \right] \tag{80}$$

with interlacing zeros being $-\omega_{m/2}^2 < -\lambda_{m/2}^2 < \cdots < -\lambda_2^2 < -\omega_2^2 < -\lambda_1^2$ as a result of Theorem 3.4 and $\chi(m + 1) < 0$.

For general linear combination of two polynomials, the following result is known:

**Theorem 3.5.** Suppose that $\{a_1, a_2, \cdots, a_n\}$ and $\{b_1, b_2, \cdots, b_n\}$ are all real zeros of $f(x)$ and $g(x)$, respectively, and $g(x)$ interlaces $f(x)$ as $\{b_1 < a_1 < b_2 < \cdots < b_n < a_n\}$.

For a combined polynomial $F(x)$ such that $F(x) = \alpha f(x) + \beta g(x)$ where $\alpha, \beta$ are two real numbers, if $F(x)$ and $g(x)$ have the same degree and have leading coefficients of the same sign, then $F(x)$ has all real zeros $\{c_1, c_2, \cdots, c_n\}$ and $f(x)$ interlaces $F(x)$ as $\{a_1 < c_1 < a_2 < \cdots < a_n < c_n\}$, provided that $\beta < 0$.

A more general version was proved elsewhere. Equation (80) can be written into

$$- [\eta(s) + \eta(1 - s)] = \lim_{m \to \infty} \left[ \frac{2\pi}{[(m-1)!]^2} \prod_{j=1}^{m} (y + \lambda_j^2) - \left( \frac{2\chi(m + 1)}{(m + 1)!} + \frac{2\pi}{[(m-1)!]^2} \right) \prod_{j=1}^{m} (y + \omega_j^2) \right]$$

$$\equiv \lim_{m \to \infty} \left[ \frac{2\pi}{[(m-1)!]^2} G_h(y) - \left( \frac{2\chi(m + 1)}{(m + 1)!} + \frac{2\pi}{[(m-1)!]^2} \right) G_d(y) \right] \tag{81}$$

where the two polynomials $G_h(y)$ and $G_d(y)$ have all real zeros $\{-\lambda_2^2\}$ and $\{-\omega_2^2\}$, respectively, and their zeros are proved to interlace as $-\omega_{m/2}^2 < -\lambda_{m/2}^2 < \cdots < -\lambda_2^2 <
\(-\omega_1^2 < -\lambda_1^2\). Moreover, the combined polynomial \(-[\eta(s) + \eta(1 - s)]\) and \(G_d(y)\) have leading coefficients of the same sign as the chosen \(\chi(m + 1) < 0\). In addition, the combination coefficient of \(G_d(y)\) is a negative constant. Therefore Eq. (81) satisfies all conditions to apply Theorem 3.5 to yield

\[
\eta(s) + \eta(1 - s) = \lim_{m \to \infty} \frac{2\chi(m+1)}{(m+1)!} \prod_{j=1}^{m/2} (y + \Omega_j) = \lim_{m \to \infty} \frac{2\chi(m+1)}{(m+1)!} \prod_{j=1}^{m/2} \left[ \left(s - \frac{1}{2}\right)^2 + \Omega_j \right] \tag{82}
\]

where the zeros \(\{-\Omega_j\}\) of \(\eta(s) + \eta(1 - s)\) are all real and interlacing with the zeros \(\{-\lambda_j^2\}\) of \(G_h(y)\) in Eq. (81) as

\[
-\lambda_j^2 < -\Omega_j < \cdots < -\lambda_2 < -\Omega_2 < -\lambda_1^2 < -\Omega_1 \tag{83}
\]

Since \(\{\lambda_j^2\}\) are all real positive numbers, Equation (83) indicates that \(\{\Omega_j\}\) also must be real positive numbers except for \(\Omega_1\) whose sign can not be determined by Eq. (83).

If all \(\Omega_j > 0\), due to the negative leading coefficient \([the chosen \chi(m+1) < 0]\) in Equation (82), it will result in a contradictory \(\eta(1/2) < 0\). Thus it concludes that all \(\Omega_j > 0\) for \(j = 2, 3, \cdots, m/2\) except that \(\Omega_1 < 0\).

However, on the other hand, \(\Omega_1 < 0\) means that \(\eta(s) + \eta(1 - s)\) will have two real zeros as \(s = 1/2 \pm \sqrt{-\Omega_1}\), which seems to be contradictory to Theorem 1.1. In order to avoid this dilemma, it must have \(\Omega_1 \leq -1/4\) such that the two corresponding "real zeros" \(s \leq 0\) and \(s \geq 1\) are just out of the convergent domain (the critical strip) of \(\eta(s) + \eta(1 - s)\) as Eq. (82) is derived based on Dirichlet series in Eq. (1). Therefore, the factor of \([(s-1/2)^2 + \Omega_1]\) still exists, and Theorem 1.1 is not violated [within the convergent domain of \(\eta(s) + \eta(1 - s)\)].
Thus we have the final factorization by zeros for $\eta(s) + \eta(1 - s)$ on $0 < \text{Re}(s) < 1$:

$$\eta(s) + \eta(1 - s) = \left[ (2^{1-s} - 2)\pi^{-s} \cos \left( \frac{\pi s}{2} \right) \Gamma(s) + (1 - 2^{1-s}) \right] \zeta(s)$$

$$= \lim_{m \to \infty} \frac{2\chi(m+1)}{(m+1)!} \prod_{j=1}^{m} \left[ \left( s - \frac{1}{2} \right)^2 + \Omega_j \right]$$

(84)

where all $\{\Omega_j\} > 0$ except that $\Omega_1 \leq -\frac{1}{4}$, and $\{\Omega_j\}$ are determined by Eq. (71) as the difference of two symmetrized factorials. Equation (84) shows that $\zeta(s)$ is proportional to a single product of infinite number of quadratic forms $[\left( s - 1/2 \right)^2 + \Omega_j]$ with all $\Omega_j > 0$ except that $\Omega_1 \leq -\frac{1}{4}$, which immediately endorses Riemann hypothesis in the critical strip.

It is worth to mention that

(i) If $\chi(m+1) > 0$ is chosen in Eq. (72), it will also show that $\eta(s) + \eta(1 - s)$ can be expressed into a single polynomial whose all roots have real part of $1/2$. First of all, following the same procedure will arrive at the same conclusion that the zeros $\{\lambda^2_j\}$ interlace the zeros $\{\omega^2_j\}$ or vice versa. In the case of $\omega^2_1 < \lambda^2_1 < \omega^2_2 < \cdots < \omega^2_{m/2} < \lambda^2_{m/2}$, Theorem 3.5 can be directly applied on Eq. (80) to conclude that the zeros $\{-\omega^2_j\}$ interlace the zeros of $\{-\Omega_j\}$ leading to

$$\eta(s) + \eta(1 - s) = \lim_{m \to \infty} \frac{2\chi(m+1)}{(m+1)!} \prod_{j=1}^{m} \left[ \left( s - \frac{1}{2} \right)^2 + \Omega_j \right]$$

(85)

with all $\{\Omega_j\} > 0$. In the case of $\lambda^2_1 < \omega^2_1 < \lambda^2_2 < \cdots < \lambda^2_{m/2} < \omega^2_{m/2}$, defining the variable as $y' = -(s - 1/2)^2$, Equation (72) becomes

$$\eta(s) + \eta(1 - s) = \lim_{m \to \infty} \left[ \left( \frac{2\chi(m+1)}{(m+1)!} + \frac{2\pi}{\left[ (m-\frac{1}{2})! \right]^2} \right) \prod_{j=1}^{m} (\omega^2_j - y') - \frac{2\pi}{\left[ (m-\frac{1}{2})! \right]^2} \prod_{j=1}^{m} (\lambda^2_j - y') \right]$$

(86)
Then Theorem 3.5 can be applied on Eq. (86) to conclude that the zeros \( \{ \omega_j^2 \} \) interlace the zeros \( \{ \Omega_j \} \) leading to

\[
\eta(s) + \eta(1-s) = \lim_{m \to \infty} \frac{2 \chi(m+1)}{(m+1)!} \prod_{j=1}^{m} (\Omega_j - y') = \lim_{m \to \infty} \frac{2 \chi(m+1)}{(m+1)!} \prod_{j=1}^{m} \left[ (s - \frac{1}{2})^2 + \Omega_j \right] \tag{87}
\]

with all \( \{ \Omega_j \} > 0 \). Thus regardless of the sign of \( \chi(m+1) \), \( \eta(s) + \eta(1-s) \) and its approximation with finite large \( m \) will all have the expression of a single polynomial whose zeros all have real part of 1/2.

(ii) Similar to Arndt-Gosper formula, \( F_h(x) \) and \( F_d(x) \) in Eqs. (73) and (74) can be written into explicit forms of polynomial of \( x \):

\[
F_h(x) = \frac{2(-1)^{\frac{m}{2}}}{m} \sum_{k=0}^{m-1} \prod_{j=0}^{m-1} \left[ x - (j + 2) \tan \left( \frac{\pi(j - k)}{m} + \frac{\pi}{2m} \right) \right] = 0 \tag{88}
\]

\[
F_d(x) = \frac{2(-1)^{\frac{m}{2}}}{m} \sum_{k=0}^{m-1} \prod_{j=0}^{m-1} \left[ x - (j + \frac{3}{2} + d_j) \tan \left( \frac{\pi(j - k)}{m} + \frac{\pi}{2m} \right) \right] = 0 \tag{89}
\]

for an even \( m \). When \( m \to \infty \), under midpoint approach, Equations (88) and (89) become Cauchy principal integrals:

\[
F_h(x) = \lim_{m \to \infty} 2(-1)^{\frac{m}{2}} P.V. \int_0^1 \prod_{j=0}^{m-1} \left[ x - (j + 2) \tan \left( \frac{\pi j}{m} - \pi y \right) \right] dy = 0 \tag{90}
\]

\[
F_d(x) = \lim_{m \to \infty} 2(-1)^{\frac{m}{2}} P.V. \int_0^1 \prod_{j=0}^{m-1} \left[ x - (j + \frac{3}{2} + d_j) \tan \left( \frac{\pi j}{m} - \pi y \right) \right] dy = 0 \tag{91}
\]

where all odd powers of \( x \) vanish due to

\[
P.V. \int_0^1 \tan \left( \frac{\pi j}{m} - \pi y \right) dy = 0 \tag{92}
\]

The interlacing relationship between the zeros \( \{ \lambda_j^2 \} \) of \( F_h(y = x^2) \) and \( \{ \omega_j^2 \} \) of \( F_d(y = x^2) \) may also be explored starting from Eqs. (88) and (89).
(iii) Equation (84) indicates that all roots of $\zeta(s)$ are a subset (equal to or less than the number) of the total roots \( \{ s = 1/2 \pm i \sqrt{\Omega_j} \} \), since the prefactor of $\zeta(s)$ [i.e., $\kappa(s) \equiv (2^1-s-2)\pi^{-s}\cos(\pi s/2)\Gamma(s)+\Gamma(1-2^1-s)$] might have isolated zeros. Nevertheless, Equation (84) implies that no matter where the roots comes from $\zeta(s)$ or the prefactor $\kappa(s)$, in the critical strip all their roots must have real part of $1/2$.

(iv) On the other hand, $\zeta(s)$ might have additional zeros (the trivial ones actually) that are not included in Eq. (84), because the expansion of $\zeta(s)$ in Eq. (84) is derived from Dirichlet series $\eta(s)$ whose valid domain is $\text{Re}(s) > 0$. Moreover, to make the functional symmetry $\eta(1-s)$ also valid via Dirichlet series representation, the valid domain is restricted in the critical strip $0 < \text{Re}(s) < 1$. Thus all trivial zeros of $\zeta(s)$ located at real negative integers do not emerge in Eq. (84), and possible zeros of the prefactor $\kappa(s)$ which are outside the critical strip also will not show up in Eq. (84).

4 Conclusions

In this paper, based on absolutely convergent binomial expansion, alternating Riemann zeta function $\eta(s)$ is found to be admissible to Melzak transform for infinite degree polynomials. Specifically, $\eta(s)$ can be expressed as a linear combination of cyclic polynomials $P_k(s) = \prod_{j=0, j\neq k}^{m} (j+2-s)$ with $k = 0, 1, \cdots, m$, which is shown in Eq. (28).

Considered the functional symmetry of Riemann zeta function, the combined $\eta(s) + \eta(1-s)$, which is proportional to $\eta(s)$, can be written into a linear combination of symmetrized factorial polynomials $P_k(s) + P_k(1-s)$, shown in Eq. (49). All roots
of $P_k(s) + P_k(1 - s)$ for each $k$ have real part of $1/2$. Moreover, we proved that for a linear combination of $P_k(s) + P_k(1 - s)$ with same sign combination coefficients, all roots of the combined polynomial will still have real part of $1/2$.

Riemann hypothesis would be endorsed immediately if the linear combination coefficients in Eq. (49) all had the same sign. However, the combination coefficients in Eq. (49) are alternating between positive and negative, which equivalently leads to $\eta(s)$ expressed into the difference between two symmetrized factorials whose roots all have real part of $1/2$. Fortunately, we proved that the imaginary parts of the zeros of the two symmetrized factorials on upper half plane are interlacing. Based on well-known results about zeros feature of the combined polynomial from the difference of two interlacing polynomials, Riemann hypothesis is endorsed to show that the combined polynomial [proportional to $\zeta(s)$] can be expressed into a single product of infinite number of quadratic forms $(s - 1/2)^2 + \Omega_j$ with all $\Omega_j > 0$. 
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