A high-order unfitted finite element method for moving interface problems

Chuwen Ma · Weiying Zheng

Abstract We propose a kth-order unfitted finite element method (2 ≤ k ≤ 4) to solve moving interface problem of the Oseen equations. Thorough error estimates for the discrete solutions are presented by considering errors from interface-tracking, time integration, and spatial discretization. In literatures on time-dependent Stokes interface problems, error estimates for the discrete pressure are usually sub-optimal, namely, (k − 1)th-order, under the $L^2$-norm. We have obtained a (k − 1)th-order error estimate for the discrete pressure under the $H^1$-norm. Numerical experiments for a severely deforming interface show that optimal convergence orders are obtained for k = 3 and 4.

Keywords: optimal control, wave equation, unbounded domain, boundary integral equation, well-posedness, convolution quadrature, stability, convergence, error estimate.

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1 introduction

Partial differential equations (PDEs) on time-varying domains are frequently encountered in various applications in biology, physics, and engineering, such as blood-flows, fluid-structure interaction, free-surface problems, etc. It is well-known that moving interface problems may cause challenges to high-order numerical simulations and rigorous error analysis. In this paper, we study numerical methods for two-dimensional Oseen equations with a time-varying interface.

Let $\Omega \subset \mathbb{R}^2$ be an open rectangle whose boundary is denoted by $\Sigma = \partial \Omega$. For any $t \geq 0$, let $\Omega_1(t)$ and $\Omega_2(t)$ be two time-varying sub-domains of $\Omega$ occupied by two immiscible fluids. We assume that $\Omega$ is fixed and denote the interface by $\Gamma(t) = \partial \Omega_1(t) \subset \Omega$. Then $\partial \Omega_2(t) = \Sigma \cup \Gamma(t)$ (see Fig. 1). The linear interface problem of two-phase incompressible fluids is given as follows

$$ \frac{\partial u_i}{\partial t} + (w \cdot \nabla) u_i - \nu_i \Delta u_i + \nabla p_i = f_i, \quad \text{div} \, u_i = 0 \quad \text{in} \quad \Omega_i(t), \quad (1.1a) $$

$$ u_i|_{t=0} = u_{i,0} \quad \text{in} \quad \Omega_i(0), \quad (1.1b) $$

$$ \nu_1 \partial_n u_1 - p_1 n = \nu_2 \partial_n u_2 - p_2 n, \quad u_1 = u_2 \quad \text{on} \quad \Gamma(t), \quad (1.1c) $$

$$ u_2 = 0 \quad \text{on} \quad \Sigma. \quad (1.1d) $$

where $u_i$, $p_i$, and $f_i$, $i = 1, 2$ stand for the flow velocity, the pressure, and the applied body force in each phase, respectively. Here $w$ is the advection velocity which drives the variation of $\Gamma(t)$, namely,

$$ \Gamma(t) = \{ X(t; 0, x) : \forall x \in \Gamma(0) \}, $$

where, for any $t \geq s \geq 0$ and any $x \in \Omega$, $X(t; s, x)$ is the solution to the initial-value problem of ordinary differential equation (ODE)

$$ \frac{d}{dt} X(t; s, x) = w(X(t; s, x), t), \quad X(s; s, x) = x. \quad (1.2) $$

Moreover, (1.1b) stands for the transmission conditions for the solution across $\Gamma(t)$, and the unit normal $n$ points to the interior of $\Omega_2(t)$. The viscosities $\nu_1, \nu_2$ are positive constants. Without loss of generality, we assume $0 < \nu_2 < \nu_1 = 1$ and define $\nu \in L^\infty(\Omega)$ by $\nu = \nu_i$ in $\Omega_i(t)$, $i = 1, 2$.

In the literature on interface problems, numerical methods on unfitted grids (referred to “unfitted methods” hereafter) are very popular during the past few decades. To mention some of them, we refer to [44,43] for immersed boundary methods, to [24,27] for immersed interface methods, to [25,26,28] for immersed finite element methods, to [3,17,19,29,42] for Nitsche extended finite element
methods, to [5,10] for cut finite element methods, and to [1] for immersed discontinuous Galerkin method. The essential idea is to double the degrees of freedom on interface elements so that interface conditions can be enforced explicitly in basis functions or weakly in discrete formulations. Similar ideas are used in fictitious-domain methods which enhance the stability of numerical solutions by penalizing face jumps of their normal derivatives [8,9,20,33].

Although unfitted methods have been well-developed for stationary problems, they may encounter a significant challenge for solving dynamic interface problems. Since the computational domain (or its sub-domains) is varying in time, numerical solutions computed in previous time steps are compounded with flow maps in the current time step. Traditional methods for time integrations cannot be applied directly to this case [13,45]. One way to deal with the issue is the space-time method, which uses discontinuous Galerkin method to the time variable and uses extended finite element method to the spatial variable [21,22]. We refer to the very recent work [15] which presents error estimates for immersed finite element method for the parabolic equation with a time-varying interface. In [23], Lehrenfeld and Olshanaskii propose an implicit Euler finite element method for the advection-diffusion equation on time-varying domains. In [41], von Wahl and Richter and Lehrenfeld extend this method to solve the Stokes equations on a time-varying domain. By extending the discrete solution to a slightly large neighbourhood at every time step, Lou and Lehrenfeld propose a high-order method for solving the advection-diffusion equation [31]. The method is based on isoparametric unfitted finite element and backward-differentiation formulas (BDF). In [32], the authors proposed a high-order finite element method for solving the advection-diffusion equation on time-varying domain. Thorough error estimates are given for third- and fourth-order methods by taking account of errors from boundary tracking, time integration, and spatial discretization.

This paper extends the numerical study for varying-boundary problem of the advection-diffusion equation to varying-interface problem of the Oseen equations. The extension is essentially nontrivial, considering the discrete LBB condition for varying-interface problems and the time integration along characteristic curves. We have overcome the difficulties by defining a modified Stokes projection onto finite element spaces and introducing adequate penalty terms.
to the pressure variable. The main contributions of the paper are summarized as follows.

1. For $2 \leq k \leq 4$, we propose a $k$th-order numerical method for the varying-interface problem (1.1). The method uses $k$th-order time integration along characteristic curves and $k$th-order finite elements on unfitted Eulerian meshes. The interface is formed dynamically with the $(k+1)$th-order interface-tracking method in each time step.

2. We present a thorough error analysis for the finite element method by taking into account all errors from spatial discretization, time integration, and interface-tracking process. Optimal error estimates are obtained for the discrete velocity under weighted $H^1$-norm.

3. By incorporating adequate residual-based penalties into the discrete formulation, we have obtained the $(k-1)$th-order error estimates for the discrete pressure under weighted $H^1$-norm, while error estimates for the discrete pressure are of $(k-1)$th-order under the $L^2$-norm in the literature [41].

4. Numerical experiment for a severely deforming interface show that optimal convergence orders are obtained for both the third-order and the fourth-order methods.

The rest of the paper is organized as follows. In section 2, we reformulate the Oseen model in Lagrangian coordinates. Discrete flow maps are introduced via the $(k+1)$th-order Runge-Kutta (RK) scheme. In section 3, we introduce the interface-tracking algorithm which generates the computational interface in each time step. Error estimates between the exact interface and the approximate one are cited from [32]. In section 4, we propose a $k$th-order unfitted finite element method for solving the interface problem in each time step. The well-posedness of the discrete problem is established. In section 5, we introduce a modified Stokes projection operator and prove error estimates for the projection. Section 6 is devoted to proving the stability of numerical solutions. In section 7, a thorough error analysis is presented by taking into account all errors from approximate interfaces, time integrations, and finite element discretization. In section 8, optimal convergence orders are verified for both the third-order and the fourth-order methods by a numerical experiment with severely deforming interface.

2 The interface-tracking algorithm

In this section, we present an algorithm for constructing an approximate interface in each time step. In practice, it is difficult to track the interface $\Gamma(t)$ with the true solution to the ODE (1.2). A realistic approach is to track a finite number of control points on $\Gamma(0)$ by solving (1.2) numerically and construct an approximate interface based on these control points.

Let $T$ be the final time of evolution and let $t_n = n\tau$, $n = 0, \ldots, N$, be a uniform partition of $[0, T]$ with time-step size $\tau = T/N$. Let $X^{n-j,n} := X(t_n; t_{n-j} \cdot)$, $0 \leq j \leq n$, denote exact flow maps at discrete time points.
The uniqueness of the solution to (1.2) implies that $X^{n-j,n}$ is one-to-one and maps $\Omega(t_{n-j})$ to $\Omega(t_n)$ for $i = 1, 2$. For convenience, we denote the inverse of $X^{n-j,n}$ by $X^{n,n-j} := (X^{n-j,n})^{-1}$.

### 2.1 Approximate flow maps

For any $x^{n-1} \in \Omega$, the image $x^n = X^{n-1,n}(x^{n-1})$ is calculated by the RK-$(k + 1)$ (the $(k + 1)^{\text{th}}$-order Runge-Kutta) scheme

$$
\begin{aligned}
&\left\{ x^{(i)} = x^{n-1} + \tau \sum_{j=1}^{i-1} \alpha_{ij} w(x^{(j)}, t^{(j)}), \quad t^{(j)} = t_{n-1} + \gamma_{j} \tau, \\
&x^n = x^{n-1} + \tau \sum_{i=1}^{n} \beta_{i} w(x^{(i)}, t^{(i)}). \tag{2.1}\right.
\end{aligned}
$$

Here $n_{k+1}$ is the number of stages and $\alpha_{ij}, \beta_{i}, \gamma_{i}$ are coefficients of the RK-$(k + 1)$ scheme, satisfying $\alpha_{jj} = 0$ for $j \geq i$.

We denote the map from $x^{n-1}$ to $x^{(i)}$ by

$$
\phi_{n-1}^{(i)}(x^{n-1}) := x^{(i)}, \quad i = 1, \ldots, n_{k+1}.
$$

Then $X^{n-1,n}$ can be represented explicitly as follows

$$
X^{n-1,n}(x) = x + \tau \sum_{i=1}^{n_{k+1}} \beta_{i} w(\phi_{n-1}^{(i)}(x), t^{(i)}). \tag{2.2}
$$

The multi-step mapping is defined by

$$
X^{n-i,n} = X^{n-1,n} \circ X^{n-2,n-1} \circ \cdots \circ X^{n-i,n-i+1}, \quad 1 \leq i \leq n. \tag{2.3}
$$

The inverse of $X^{n-i,n}$ is denoted by $X^{n,n-i} := (X^{n-i,n})^{-1}$.

**Lemma 2.1 ([32, Lemma 3.3])** There exists a constant $C > 0$ independent of $\tau$ such that for $\nu = 0, 1$,

$$
||X^{m,n} - X^{m,n}||_{W^{\nu, \infty}(\Omega)} \leq C(n - m)^{\nu k + 2 - \mu}, \quad 0 \leq m \leq n \leq N. \tag{2.4}
$$

### 2.2 Interface-tracking algorithm

Now we describe the algorithm which tracks the interface $\Gamma(t_n)$ approximately. The algorithm is adopted from [32]. Let $J$ be the number of control points on the initial boundary $\Gamma^0 \equiv \Gamma(0)$, let $L$ be the arc length of $\Gamma^0$, and let $\eta := L/J$ be the segment size for interface-tracking. Let $\mathcal{P}^0 = \{p_j^0 \in \Gamma^0 : j = 0, 1, \ldots, J\}$ be a partition of $\Gamma^0$ with $p_0 = p_J^0$. Suppose there is a parametrization $\Gamma^0 = \{\chi_0(l) : l \in [0, L]\}$ which satisfies $\chi_0 \in C^4([0, L])$ and

$$
\chi_0(L_j) = p_j^0, \quad L_j = j\eta, \quad 0 \leq j \leq J.
$$

The set of nodes on $\Gamma^0$ is defined by $\mathcal{L} = \{L_j : j = 0, 1, \ldots, J\}$. 

Algorithm 1 (32) Given \( n \geq 1 \) and \( \Gamma_0^\eta = \Gamma_0 \), the interface-tracking algorithm for constructing \( \Gamma_n^\eta \) from \( \Gamma_{n-1}^\eta \) consists of two steps.

1. Trace forward all control points in \( \mathcal{P}^{n-1} \) to obtain the set of control points at \( t = t_n \),
   \[
   \mathcal{P}^n = \{ p_j^n = X_{\tau}^{n-1,n}(p_{j-1}^{n-1}) : j = 0, \ldots, J \}.
   \]

2. Compute the cubic spline interpolation \( \chi_n \in C^2([0, L]) \) based on \( L \) and \( \mathcal{P}^n \). Define
   \[
   \Gamma^\eta_n := \{ \chi_n(l) : l \in [0, L] \}.
   \]

In each time step, we clarify three interfaces, the exact interface \( \Gamma(t_n) \), the approximate interface \( \Gamma_n \) obtained with the RK-\((k + 1)\) scheme, and the practical interface \( \Gamma^\eta_n \) used for numerical computations. Then three interfaces are, respectively, parameterized as follows

\[
\begin{align*}
\Gamma(t_n) &= \{ \hat{x}_n(l) : l \in [0, L] \}, & \hat{x}_n &= X^{0,n} \circ \chi_0, \\
\Gamma_n &= \{ \tilde{x}_n(l) : l \in [0, L] \}, & \tilde{x}_n &= X^{0,n} \circ \chi_0, & 1 \leq n \leq N, \\
\Gamma^\eta_n &= \{ \chi_n(l) : l \in [0, L] \}, & \chi_n &\in C^2([0, L]),
\end{align*}
\]

2.3 Error estimates for interface tracking

Let \( L^2(\Omega) \) denote the space of square-integrable functions on \( \Omega \), \( L_0^2(\Omega) \) the subspace of functions whose integrals on \( \Omega \) are zero, and \( H^m(\Omega) \) the subspace of functions whose partial derivatives of order up to \( m \) belong to \( L^2(\Omega) \). The inner product on \( L^2(\Omega) \) is denoted by

\[
(u, v)_\Omega = \int_\Omega uv, \quad \forall u, v \in L^2(\Omega).
\]

Moreover, \( L^\infty(\Omega) \) denotes the space of functions which are essentially bounded on \( \Omega \) and \( W^{m,\infty}(\Omega) \) the subspace of functions whose partial derivatives of order up to \( m \) belong to \( L^\infty(\Omega) \). We denote vector-valued quantities by boldface symbols, such as \( L^2(\Omega) = (L^2(\Omega))^2 \), and denote matrix-valued quantities by blackboard bold symbols, such as \( \mathbb{L}^2(\Omega) = (L^2(\Omega))^{2 \times 2} \). Hereafter, \( (\cdot, \cdot)_\Omega \) denotes the inner products on \( L^2(\Omega) \), \( L^2(\Omega) \), and \( \mathbb{L}^2(\Omega) \), in their respective circumstances.

Throughout the paper, the notation \( f \lesssim g \) means \( f \leq Cg \) where \( C \) is a generic constant and independent of sensitive quantities, such as the segment size \( \eta \), the spatial mesh size \( h \), the time-step size \( \tau \), the number of time steps \( n \), and the viscosity \( \nu \). So are the constants \( C_i, i = 0, 1, \cdots \), in the following texts.

To focus on high-order error estimates, we make the assumptions on the driving velocity \( \mathbf{w} \) and the exact interfaces \( \Gamma(t_n) \):
Assumption 2 Assume that

- \( \mathbf{w} \in C^r([0, T] \times \bar{\Omega}) \) and \( \mathbf{w}(:, t) \in C^r_0(\Omega) \), \( r \geq 4 \), for all \( t \geq 0 \), and that
- the parametrization of \( \Gamma(t_n) \) satisfies \( \| \mathbf{x}_n \|_{C^4([0, L])} \lesssim 1 \) for all \( 0 \leq n \leq N \).

Let the Jacobi matrices of \( \mathbf{X}_{n-i,n} \), \( \mathbf{X}_{r-n,i} \), and \( \mathbf{X}_{n,n} \), \( \mathbf{X}_{r,n} \) be denoted by

\[
\mathbf{j}^{n-i,n} = \frac{\partial \mathbf{X}_{n-i,n}}{\partial \mathbf{x}}, \quad \mathbf{j}_{n-i} = (\mathbf{j}^{n-i,n})^{-1},
\]
\[
\mathbf{j}_r^{n-i,n} = \frac{\partial \mathbf{X}_{r-n,i}}{\partial \mathbf{x}}, \quad \mathbf{j}^{n-i,n} = (\mathbf{j}_r^{n-i,n})^{-1}.
\]

From (2.2), it is easy to see that

\[
\mathbf{j}_r^{n-1,n}(\mathbf{x}) = 1 + \tau \sum_{i=1}^{n+1} \alpha_i \mathbf{w}(\mathbf{X}_n, t_i) \mathbf{w}(\mathbf{X}_n, t_i), \quad \forall \mathbf{x} \in \Omega, \quad (2.7)
\]

where

\[
\mathbf{w}(\mathbf{X}_n, t_i) = \mathbf{1}, \quad \mathbf{w}(\mathbf{X}_n, t_i) = \mathbf{1} + \tau \sum_{j=1}^{i-1} \alpha_j \mathbf{w}(\mathbf{X}_n, t_j) \mathbf{w}(\mathbf{X}_n, t_j), \quad i \geq 2. \quad (2.8)
\]

From [32, Lemma 3.3], we cite some useful estimates for the Jacobi matrices

\[
\|\mathbf{j}_r^{m,n}\|_{W^{1,\infty}(\Omega)} + \|\mathbf{j}_r^{m,n}\|_{W^{1,\infty}(\Omega)} \lesssim 1, \quad 0 \leq m, n \leq N, \quad (2.9)
\]
\[
\|\mathbf{j}_r^{n-1,n} - \mathbf{I}\|_{L^\infty(\Omega)} + \|\mathbf{j}_r^{n-1,n} - \mathbf{I}\|_{L^\infty(\Omega)} \lesssim \tau, \quad 0 \leq i \leq k + 1. \quad (2.10)
\]

Theorem 3 Let Assumption 2 be satisfied. Then \( \| \mathbf{x}_n \|_{C^4([0, L])} \lesssim 1 \) and

\[
\begin{align*}
\| \mathbf{x}_n - \mathbf{x}_n \|_{C^4([0, L])} & \lesssim \tau^{k-\mu}, \\
\| \mathbf{x}_n - \mathbf{x}_n \|_{C^4([0, L])} & \lesssim \tau^{k-\mu} + \tau^{k+1-\mu}, \quad \mu = 0, 1. \quad (2.11)
\end{align*}
\]

Proof From (2.3) and (2.6), we have \( \mathbf{x}_n(t) = \mathbf{x}_{n-1} \circ \mathbf{x}_{n-1} \). The chain rule implies

\[
\mathbf{x}_n' = \prod_{m=1}^{n} \mathbf{x}_m', \quad \mathbf{x}_n'' = \{ (\mathbf{x}_n' - \mathbf{x}_n) \mathbf{x}_n'' \} \mathbf{x}_n + \mathbf{x}_n'' \mathbf{x}_n - \mathbf{x}_n'.
\]

Since \( \mathbf{w} \in C^r([0, T] \times \bar{\Omega}) \) with \( r \geq 4 \), from (2.7) we infer that \( \mathbf{j}_r^{n-1,n} \in C^3(\Omega) \) and \( \| \mathbf{j}_r^{n-1,n} \|_{W^{3,\infty}(\Omega)} \lesssim \tau \) for \( j = 1, 2, 3 \). Together with (2.9)–(2.10), they show that

\[
\| \mathbf{x}_n'' \|_{C([0, L])} \lesssim (1 + C\tau) \| \mathbf{x}_n'' \|_{C([0, L])} \lesssim 1,
\]
\[
\| \mathbf{x}_n'' \|_{C([0, L])} \lesssim (1 + C\tau) (C\tau + \| \mathbf{x}_n'' \|_{C([0, L])}) \lesssim 1 + \| \mathbf{x}_n'' \|_{C([0, L])} \lesssim 1.
\]
Furthermore, \( \|X''_n\|_{C([0,L])} \) and \( \|X^{(4)}_n\|_{C([0,L])} \) can be estimated similarly.

Since \( X_n \) is the cubic spline interpolation of \( X_n \), the first inequality of (2.11) follows directly from standard error estimates for cubic spline interpolations. The second inequality can be proven by triangular inequality and standard error estimates for the RK-(\( k+1 \))-scheme.

**Theorem 4 ([32, Theorem 3.5])** Suppose \( r \geq \text{max}(k+2,4) \). For any \( 0 \leq n, m \leq N \) and \( |m-n| \leq k \), there holds for \( \mu = 0,1 \)

\[
\|X_n - X_{\tau}^{m,n} \circ X_m\|_{C^0([0,L])} \lesssim \tau^{\mu} \sum_{i=0}^{k+1} (\tau^i \eta_i^{\min(4,r-i)} + \tau^{k+1}).
\] (2.12)

2.4 A semi-discrete scheme

For any \( x_0 \in \Omega_0 \), we use the continuous flow map and write \( x \equiv x(t) = X(t; x_0) \). The material derivative of \( u_i \) is defined as

\[
\frac{d}{dt} u_i(x(t), t) = \frac{\partial u_i}{\partial t}(x, t) + w(x, t) \cdot \nabla_x u_i(x, t), \quad i = 1, 2.
\] (2.13)

The momentum equation of (1.1) can be written equivalently as follows

\[
\frac{d\mathbf{u}_i}{dt} - \nu_i \Delta \mathbf{u}_i + \nabla p_i = \mathbf{f}_i \quad \text{in} \quad \Omega_i(t), \quad i = 1, 2.
\] (2.14)

We apply the \( k \)-th order backward differentiation formula (BDF-\( k \)) to (3.2) and obtain a semi-discrete approximation to problem (1.1):

Given \((u_i^n, u_j^n)\) for \( n - k \leq m < n \), solve the coupled problems for \((u_i^n, u_j^n)\)

\[
\begin{align*}
\frac{1}{\tau} A^k \mathbb{U}^n_i - \nu_i \Delta u^n_i + \nabla p^n_i &= f^n_i, \quad \text{div} \mathbf{u}_i^n = 0 \quad \text{in} \quad \Omega^n_i, \quad i = 1, 2, \quad (2.15a) \\
\nu_i \partial_n u^n_i - p^n_i n &= \nu_2 \partial_n u_2^n - p_2^n n, \quad u^n_1 = u^n_2 \quad \text{on} \quad \Gamma^n_i, \quad (2.15b) \\
u^n_2 &= 0 \quad \text{on} \quad \Sigma, \quad (2.15c)
\end{align*}
\]

where \( \mathbb{U}_i^n = [U_i^{n-k,n}, \ldots, U_i^{n,n}] \), \( U_i^{m,n} = u_i^m \circ X_{\tau}^{n,m} \), and \( A^k \mathbb{U}_i^n = \sum_{j=0}^{k} \lambda^k_j \mathbb{U}_i^{n-j,n} \).

Here \( f^n_i = f_i(t_n) \) and \( \lambda^k_0, \ldots, \lambda^k_k \) are coefficients for the BDF-\( k \) (see [30, 32]).

3 The finite element method

The purpose of this section is to propose a fully discrete finite element scheme for solving (1.1). Suppose \( \Gamma_{\eta,1}^n \subset \Omega \) is the approximate interface obtained with Algorithm 1. Let \( \Omega_{\eta,1}^n \) denote the open domain surrounded by \( \Gamma_{\eta}^n \), namely, \( \Gamma_{\eta}^n = \partial \Omega_{\eta,1}^n \). Define

\[
\Omega_{\eta,2}^n = \Omega \setminus \bar{\Omega}_{\eta,1}^n, \quad \Omega_{\eta}^n = \Omega \setminus \Gamma_{\eta}^n = \Omega_{\eta,1}^n \cup \Omega_{\eta,2}^n.
\]
3.1 A semi-discrete scheme

For any \( x_0 \in \Omega_0 \), we use the continuous flow map and write \( x(t) = X(t; 0, x_0) \). The material derivative of \( u_i \) is defined as

\[
\frac{d}{dt} u_i(x(t), t) = \frac{\partial u_i}{\partial t}(x(t)) + w(x, t) \cdot \nabla x u_i(x, t), \quad i = 1, 2.
\]  
(3.1)

The momentum equation of (1.1) can be written equivalently as follows

\[
\frac{du_i}{dt} - \nu_i \Delta u_i + \nabla p_i = f_i \text{ in } \Omega_i(t), \quad i = 1, 2.
\]  
(3.2)

We apply the \( k \)th-order backward differentiation formula (BDF-\( k \)) to (3.2) and obtain a semi-discrete approximation to problem (1.1):

Given \((u_1^n, u_2^n)\) for \( n - k \leq m < n \), solve the coupled problems for \((u_1^n, u_2^n)\)

\[
\frac{1}{\tau} A^k U_i^n - \nu_i \Delta u_i^n + \nabla p_i^n = f_i^n, \quad \text{div} u_i^n = 0 \text{ in } \Omega_{n,i}^n, \quad i = 1, 2, \quad (3.3a)
\]

\[
\nu_1 \partial_n u_1^n - p_1^n n = \nu_2 \partial_n u_2^n - p_2^n n, \quad u_1^n = u_2^n \text{ on } \Gamma_n^n, \quad (3.3b)
\]

\[
u_1 u_1^n = 0 \text{ on } \Sigma, \quad (3.3c)
\]

where \( U_i^n = [U_i^{n-k,i}, \ldots, U_i^{n,m}] \), \( U_i^{m,n} = u_i^n \circ X_{n,m} \), and \( A^k U_i^n = \sum_{j=0}^k \lambda_j^k U_i^{n-j,n} \).

Here \( f_i^n = f_i(t_n) \) and \( \lambda_0^k, \ldots, \lambda_k^k \) are coefficients for the BDF-\( k \) (see [30,32]).

3.2 Finite element meshes

Let \( T_h \) be the uniform partition of \( \Omega \) into closed squares of side-length \( h \). It generates the covers of \( \Omega_{n,1}^n, \Omega_{n,2}^n, \) and \( \Gamma^n_i \), which are, respectively, denoted by

\[
T_{h,i} := \left\{ K \in T_h : K \cap \Omega_{n,i}^n \neq \emptyset \right\}, \quad \text{ and } \quad T_{h,B} := \left\{ K \in T_h : \text{length}(K \cap \Gamma_n^i) > 0 \right\}.
\]

For each \( i \), we define a domain containing \( \Omega_{n,i}^n \) and a domain contained in \( \Omega_{n,i}^n \)

\[
\Omega_{h,i}^n := \text{interior}(\cup \{ K : K \in T_{h,i}^n \}), \quad \omega_{h,i}^n := \text{interior}(\cup \{ K : K \in T_{h,i}^n \} \setminus T_{h,B}^n).
\]  
(3.4)

Let \( \mathcal{E}_h \) denote the set of all interior edges of \( T_h \) and let \( \mathcal{E}_{h,B} \) denote the set of edges of \( T_{h,B}^n \) which are not on the boundary of \( \Omega_{h,i}^n \), namely,

\[
\mathcal{E}_{h,B} = \left\{ E \in \mathcal{E}_h : E \not\subset \partial \Omega_{h,i}^n \text{ and } \exists K \in T_{h,B}^n \text{ s.t. } E \subset \partial K \right\}.
\]

The sets are illustrated in Fig 2. Now we make two mild assumptions on the mesh.

**Assumption 5** We assume that
each element in $T_{h,i}^n \setminus T_{h,B}^n$ has at most two edges on $\partial \omega_{h,i}^n$ for $i = 1, 2$, and that

- there exists an integer $I > 0$ such that, for $i = 1, 2$ and any $K \in T_{h,B}^n$, one can find at most $I$ elements $\{K_j^{(i)}\}_{j=1}^I \subset T_{h,i}^n$ (see Fig. 3), which satisfy

\[
K_1^{(i)} = K, \quad K_I^{(i)} \subset \omega_{h,i}^n, \quad K_j^{(i)} \cap K_{j+1}^{(i)} \in \mathcal{E}^n_{i,B}.
\]

Fig. 2 (I) $T_{h,B}^n$: the set of red squares. (II) $\mathcal{E}^n_{1,B}$ (blue edges in the left figure) and $\mathcal{E}^n_{2,B}$ (blue edges in the right figure). (III) Left figure: $\Omega_{h,1}^n$ (red and yellow squares) and $\omega_{h,1}^n$ (yellow squares). (IV) Right figure: $\Omega_{h,2}^n$ (red and yellow squares) and $\omega_{h,2}^n$ (yellow squares).

3.3 Finite element spaces

Let $Q_k$ be the space of polynomials whose degrees are no more than $k$ for each variable. The finite element spaces with and without homogeneous boundary conditions are defined over $T_h$ as

\[
V_h(k, \Omega) = \{ q \in H^1(\Omega) : q|_K \in Q(K), \forall K \in T_h \},
\]

\[
V_{h,0}(k, \Omega) = V_h(k, \Omega) \cap H^1_0(\Omega).
\]

Fig. 3 An illustration of Assumption 5 with $I = 5$. 
Correspondingly, $V_h(k, \Omega)$ and $V_{h,0}(k, \Omega)$ denote finite element spaces of vector-valued functions. For a subdomain $D \subset \Omega$, we write
\[
V_h(k, D) = V_h(k, \Omega)|_D, \quad V_{h,0}(k, D) = V_{h,0}(k, \Omega)|_D.
\]
The spaces for the discrete velocity and the discrete pressure are, respectively, defined as
\[
V^n_h := \{ (v_{h,1}, v_{h,2}) : v_{h,1} \in V_h(k, \Omega^n_{h,1}), v_{h,2} \in V_{h,0}(k, \Omega^n_{h,2}) \},
\]
\[
Q^n_h := \{ (q_{h,1}, q_{h,2}) : q_{h,i} \in V_h(k-1, \Omega^n_{h,i}), i = 1, 2, \sum_{i=1,2} (q_{h,i}, \nu^{-1}v_{h,i})_{\Omega^n_{h,i}} = 0 \}.
\]
As a convention, we extend each $v_{h,i} \in V_h(k, \Omega^n_{h,i})$ to the exterior of $\Omega^n_{h,i}$ such that the extension, denoted still by $v_{h,i}$, belongs to $V_h(k, \Omega)$ and vanishes at all degrees of freedom outside $\Omega^n_{h,i}$. It is easy to see that
\[
\|v_{h,i}\|_{H^m(\Omega)} \lesssim \|v_{h,i}\|_{H^m(\Omega^n_{h,i})}, \quad m = 0, 1, \quad i = 1, 2. \tag{3.5}
\]

3.4 The discrete problem

Following Hansbo, Larson, and Zahedi [18], we define the “harmonic weights”
\[
\kappa_1 = \nu_2/(\nu_1 + \nu_2), \quad \kappa_2 = \nu_1/(\nu_1 + \nu_2).
\]
Let $a = (a_1, a_2)$ be a pair of vectoral (or scalar) functions satisfying $\Omega^n_{h,i} \subset \text{Dom}(a_i)$ for $i = 1, 2$. The magic formula shows
\[
[a \ b] = [a] \ [b] + [b] \ [a],
\]
where the jump and averages of $a$ across $\Gamma^n_n$ are, respectively, defined by
\[
[a] = a_1 - a_2, \quad [a] = \kappa_1 a_1 + \kappa_2 a_2, \quad [a] = \kappa_2 a_1 + \kappa_1 a_2 \quad \text{on \ } \Gamma^n_n.
\]
We abuse the notation and define $\nu = \nu_i$ in $\Omega^n_{h,i}$, $i = 1, 2$, for all $0 \leq n \leq N$. It is easy to see that
\[
\kappa_1 \nu_1 = \kappa_2 \nu_2 = \nu_1 \nu_2/(\nu_1 + \nu_2) = [\nu]/2. \tag{3.6}
\]
For any edge $E \in \mathcal{E}_h$, let $n_E$ be the unit normal on $E$ and define the jump of $v$ across $E$ by
\[
[v] (x) = \lim_{\varepsilon \to 0^+} [v(x - \varepsilon n_E) - v(x + \varepsilon n_E)], \quad \forall x \in E.
\]
For a function $\alpha \in L^\infty(\Omega^n_n)$, we define the weighted inner product of $a$ and $b$ as
\[
\langle \alpha a, b \rangle_{\Omega^n_n} = (\alpha a_1, b_1)_{\Omega^n_{h,1}} + (\alpha a_2, b_2)_{\Omega^n_{h,2}}.
\]
Let $\Delta_h$ denote the discrete Laplacian operator on piecewise regular functions,

$$\Delta_h v_h \in L^2(\Omega) \text{ satisfying } (\Delta_h v_h) |_{K} = \Delta(v_h) |_{K}, \quad \forall K \in \mathcal{T}_h.$$ 

It is clear that $\Delta_h v = \Delta v$ if $v \in H^2(\Omega)$. We also use the symbols and write

$$\text{div } v_h = (\text{div } v_{h,1}, \text{div } v_{h,2}), \quad \nabla v_h = (\nabla v_{h,1}, \nabla v_{h,2}), \quad \Delta_h v_h = (\Delta_h v_{h,1}, \Delta_h v_{h,2}).$$

Now we introduce the fully discrete approximation to the semi-discrete problem (3.3): Suppose that the discrete solutions $u^n_h \in \mathcal{V}_h^n$ are obtained for $n - k \leq m < n$. Define

$$U^{m,n}_h := u^m_h \times X^{n,m}_h, \quad U^n_h := [U^{n-k,n}_h, \ldots, U^{n,n}_h], \quad A^k U^n_h = \sum_{j=0}^k \lambda_j U^{n-j,n}_h.$$

The fully discrete problem reads: Find $(u^n_h, p^n_h) \in \mathcal{V}_h^n \times \mathcal{Q}_h^n$ such that for any $(v_h, q_h) \in \mathcal{V}_h^n \times \mathcal{Q}_h^n$,

$$\begin{align*}
&\frac{1}{\tau} \langle A^k U^n_h, v_h \rangle_{\Omega^n} + B^n_0(u^n_h, v_h) + B^n_0(v_h, p^n_h) - \langle f^n, v_h \rangle_{\Omega^n} = 0, \\
&\mathcal{R}_h^n(u^n_h, q_h) - J^n(p^n_h, q_h) - \mathcal{R}_h^n(U^n_h, p^n_h; q_h) = 0.
\end{align*}
$$

Here $f^n = (f^n_l, f^n_0)$ and $\mathcal{R}_h^n$ is the residual functional from the semi-discrete momentum equation

$$\mathcal{R}_h^n(U^n_h, p^n_h; q_h) = \gamma_1 \nu^2 h^2 \langle \tau^{-1} A^k u^n_h - \nu \Delta_h u^n_h + \nabla p^n_h - f^n, \nu^{-1} \nabla q_h \rangle_{\Omega^n},$$

where $\gamma_1 \in (0, 1)$. The bilinear forms in (3.7) are defined as follows

$$\begin{align*}
\mathcal{B}_n(u^n_h, v_h) &= \langle \nu \nabla u^n_h, \nabla v_h \rangle_{\Omega^n} - \mathcal{F}_n(v_h, v_h) + \mathcal{J}_n(u^n_h, v_h) + \mathcal{J}_n(v_h, v_h), \\
\mathcal{J}_n(u^n_h, v_h) &= \int_{\Gamma^0_n} \{(\nu \partial_n u^n_h) \cdot [v_h] + (\nu \partial_n v_h) \cdot [u_h]\}, \\
\mathcal{J}_n(v_h, v_h) &= \gamma_0 \nu h^{-1} \int_{\Gamma^0_n} [u_h] : [v_h], \\
\mathcal{B}_n(v_h, q_h) &= -\langle \text{div } v_h, q_h \rangle_{\Omega^n} + \int_{\Gamma^0_n} \{[v_h] \cdot n\} \{q_h\}, \\
\mathcal{F}_n(u^n_h, v_h) &= \sum_{l=1}^2 \sum_{E \in \mathcal{E}_n^l, i=1}^k h^{2l-1} / ((l-1)!)^2 \int_{E} \nu_i \left[\partial_n^l u^n_{h,i}\right] : \left[\partial_n^l v_{h,i}\right], \\
\mathcal{F}_n(v_h, q_h) &= \sum_{l=1}^2 \sum_{E \in \mathcal{E}_n^l, i=1}^k h^{2l+1} / ((l)!)^2 \int_{E} \nu_i^{-1} \left[\partial_n^l p_{h,i}\right] \left[\partial_n^l q_{h,i}\right],
\end{align*}$$

where $\partial_n v_{h,i} := (\partial_n v_{h,1,i}, \partial_n v_{h,2,i})$ and $\partial_n^l v_{h,i}$ stands for the $l$th-order normal derivative of $v_{h,i}$. The parameters $\gamma_0, \gamma_1$ are positive and independent of $\tau, h$, and $\eta$. 
Remark 1 The interface-zone penalties $J^u_n$ and $J^p_n$ are used to enhance the stability of the discrete velocity and the discrete pressure, respectively. The residual-based bilinear form $J^p_n$ is favorable to proving optimal $H^1$-error estimate for the discrete velocity.

Remark 2 The discrete scheme (3.7) is consistent with the semi-discrete problem (3.3). In fact, let $u^n = (u^n_1, u^n_2)$, $p^n = (p^n_1, p^n_2)$ be the solution to (3.3) and write

$$U^{m,n} := (u^m_1 \circ X^{n,m}_1, u^m_2 \circ X^{n,m}_2), \quad U^n = [U^{n-k,n}, \ldots, U^{n,n}].$$

Then (3.7) still holds if we replace $(u^n_1, p^n_1, U^n_1)$ with $(u^n, p^n, U^n)$.

4 The well-posedness of discrete problems

The purpose of this section is to prove the well-posedness of (3.7) in each time step. First we introduce some product spaces and mesh-dependent norms.

4.1 Mesh-dependent norms

For a positive function $\alpha \in L^\infty(\Omega)$ and an integer $m \geq 0$, we define the norms on $H^m(\Omega_{n,1}) \times H^m(\Omega_{n,2})$ and $L^2(\Omega_{n,1}) \times L^2(\Omega_{n,2})$, respectively, as

$$||v||_{m,\Omega^n} = \left( \sum_{i=1,2} ||v_i||_{H^m(\Omega_{n,i})}^2 + \alpha v \right)^{\frac{1}{2}}, \quad ||\alpha v||_{0,\Omega^n} = \left( \sum_{i=1,2} ||\alpha v_i||_{L^2(\Omega_{n,i})}^2 \right)^{\frac{1}{2}}.$$

For convenience, we also introduce two product spaces

$$\mathbf{V}^n = \{ v \in H^1(\Omega_{n,1}) \times H^1(\Omega_{n,2}) : v_2|_\Sigma = 0, \text{ and } \partial_n v_1, \partial_n v_2 \in L^2(\Gamma^n) \},$$

$$\mathbf{Q}^n = \{ q \in L^2(\Omega_{n,1}) \times L^2(\Omega_{n,2}) : \sum_{i=1}^2 (q_i, \nu^{-1}_i)_{\Omega_{n,i}} = 0 \text{, and } q_1, q_2 \in L^2(\Gamma^n) \}.$$

Clearly $\mathbf{V}^n \subset \mathbf{V}^n$ and $\mathbf{Q}^n \subset \mathbf{Q}^n$. The mesh-dependent norms on $\mathbf{V}^n$ and $\mathbf{Q}^n$ are defined as

$$||v||_{\mathbf{V}} = \left\{ \| \nu^{\frac{1}{2}} \nabla v \|_{0,\Omega^n}^2 + J^0_n(v, v) + h \| \nu \|^{-1} \| \nu \partial_n v \|_{L^2(\Gamma^n)}^2 \right\}^{\frac{1}{2}},$$

$$||q||_{\mathbf{Q}} = \left\{ \| \nu^{-\frac{1}{2}} q \|_{0,\Omega^n}^2 + h \| \nu \|^{-1} \| q \|_{L^2(\Gamma^n)}^2 \right\}^{\frac{1}{2}}.$$

We define three more norms for piecewise regular functions

$$||v||_{\mathbf{V}} = \left\{ \| \nu^{\frac{1}{2}} \nabla v \|_{0,\Omega^n}^2 + J^0_n(v, v) + J^u_n(v, v) \right\}^{\frac{1}{2}}, \quad \text{(4.1)}$$

$$||q||_{\mathbf{Q}} = \left\{ \| \nu^{-\frac{1}{2}} q \|_{0,\Omega^n}^2 + J^p_n(q, q) \right\}^{\frac{1}{2}}, \quad \text{(4.2)}$$
\[ \| q_h \|_{1, Q} = \left\{ \| q_h \|_Q^2 + \sum_{i=1,2} \| \nu \frac{1}{2} \nabla q_{h,i} \|_{L^2(\Omega_{h,i}^n)}^2 \right\}^{\frac{1}{2}}. \] (4.3)

They induce the product norm
\[ \|(u, q)\|_{V, Q} = \left( \| u \|_V^2 + \| q \|_Q^2 \right)^{\frac{1}{2}}. \]

**Lemma 4.1** Let Assumption 5 be satisfied. For any \( v_h \in V_h^n \) and \( q_h \in Q_h^n \), there hold
\[ h \| \nu \|^{-1} \| \nu \partial_n v_h \|_{L^2(\Gamma_h^n)} \lesssim \| \nu \frac{1}{2} \nabla v_h \|_{0, \Omega_h^n} + \mathcal{J}_u^n(v_h, v_h), \] (4.4)
\[ h \| \nu \|^{-1} \| q_h \|_{L^2(\Gamma_h^n)} \lesssim \| q_h \|_{Q_h^n}. \] (4.5)

**Proof** Thanks to Assumption 5, we can cite [32, Lemma 5.2] for the inequalities
\[ \| \nabla^\mu v_{h,i} \|_{L^2(\Omega_{h,i}^n)} \lesssim \| \nabla^\mu v_{h,i} \|_{L^2(\omega_{h,i}^n)} + \sum_{E \in E_{h,B}^n} \sum_{i=1}^k \frac{h^{2l+1-2\mu}}{(l-1)!} \int_E \| \partial_n^i v_{h,i} \|^2, \] (4.6)
\[ \| q_{h,i} \|_{L^2(\Omega_{h,i}^n)} \lesssim q_{h,i} \|_{L^2(\omega_{h,i}^n)} + \sum_{E \in E_{h,B}^n} \sum_{i=1}^{k-1} \frac{h^{2l+1}}{(l-1)!} \int_E \| \partial_n^i q_{h,i} \|^2, \] (4.7)

where \( \mu = 0, 1 \) and \( i = 1, 2 \). The definitions of \( \mathcal{J}_u^n \) and \( \mathcal{J}_p^n \) show that
\[ \sum_{i=1,2} \nu_i \| \nabla^\mu v_{h,i} \|_{L^2(\Omega_{h,i}^n)} \lesssim \sum_{i=1,2} \nu_i \| \nabla^\mu v_{h,i} \|_{L^2(\omega_{h,i}^n)} + h^{2-2\mu} \mathcal{J}_u^n(v_h, v_h), \] (4.8)
\[ \sum_{i=1,2} \nu_i^{-1} \| q_{h,i} \|_{L^2(\Omega_{h,i}^n)} \lesssim \sum_{i=1,2} \nu_i^{-1} \| q_{h,i} \|_{L^2(\omega_{h,i}^n)} + \mathcal{J}_p^n(q_h, q_h). \] (4.9)

For any \( K \in T_{h,B}^n \) and \( \Gamma_K = K \cap \Gamma_h^n \), we cite [16] for the trace inequality
\[ \| v \|_{L^2(\partial K)} + \| v \|_{L^2(K)} \lesssim h^{-\frac{1}{2}} \| v \|_{H^1(K)} + h^\frac{1}{2} \| v \|_{H^1(K)}, \quad \forall v \in H^1(K). \] (4.10)

It follows from (3.6) and the triangular inequality that
\[ h \| \nu \|^{-1} \| \nu \partial_n v_h \|_{L^2(\Gamma_K)} \lesssim h \sum_{i=1,2} \kappa_i \nu_i \| v_{h,i} \|_{H^1(\Gamma_K)} \lesssim \sum_{i=1,2} \nu_i \| v_{h,i} \|_{H^1(K)}, \] (4.11)
\[ h \| \nu \|^{-1} \| q_h \|_{L^2(\Gamma_K)} \lesssim h \sum_{i=1,2} \kappa_i \nu_i^{-1} \| q_{h,i} \|_{L^2(\Gamma_K)} \lesssim \sum_{i=1,2} \nu_i^{-1} \| q_{h,i} \|_{L^2(K)}. \] (4.12)

Taking the sum of (4.11) over all \( K \in T_{h,B}^n \) and using (4.8) yield (4.4). Taking the sum of (4.12) over all \( K \in T_{h,B}^n \) and use (4.9) yield (4.5).
Lemma 4.2 For any \( \mathbf{v}_h \in \mathbf{V}_h^n \) and \( q_h \in Q_h^n \), there hold

\[
h^2 \sum_{i=1,2} \nu_i \| \Delta_h \mathbf{v}_{h,i} \|^2_{L^2(\Omega_{n,i}^h)} \lesssim \left\| \nu \frac{1}{2} \nabla \mathbf{v}_h \right\|^2_{0, \Omega_n^h} + \mathcal{J}_n^0(\mathbf{v}_h, \mathbf{v}_h),
\]

\[
h^2 \sum_{i=1,2} \nu_i^{-1} \| \nabla q_{h,i} \|^2_{L^2(\Omega_{n,i}^h)} \lesssim \| q_h \|^2_{Q^n}.
\]

Proof The lemma is a direct result of (4.8)–(4.9) and inverse estimates.

Corollary 1 For any \( \mathbf{v}_h \in \mathbf{V}_h^n \) and \( q_h \in Q_h^n \), there hold

\[
\| \mathbf{v}_h \|^2_{\mathbf{V}} \lesssim \| \mathbf{v}_h \|^2_{\mathbf{V}} \approx \sum_{i=1,2} \nu_i |v_{h,i}|^2_{H^1(\Omega_{n,i}^h)} + \mathcal{J}_n^0(\mathbf{v}_h, \mathbf{v}_h),
\]

(4.13)

\[
\| q_h \|^2_{Q} \lesssim \| q_h \|^2_{Q} \approx \sum_{i=1}^2 \nu_i^{-1} \| q_{h,i} \|^2_{L^2(\Omega_{n,i}^h)}.
\]

(4.14)

Proof The inequalities \( \| q_h \|^2_{Q} \lesssim \| q_h \|^2_{Q} \) and \( \| \mathbf{v}_h \|^2_{\mathbf{V}} \lesssim \| \mathbf{v}_h \|^2_{\mathbf{V}} \) follow directly from Lemma 4.1. For \( i = 1, 2 \) and \( 1 \leq l \leq k \), using inverse estimates, we have

\[
h^{2l-1} \int_E \left[ |\partial_{x_l} \mathbf{v}_{h,i} |^2 \right] \lesssim h^{2l-2} \| \nabla \mathbf{v}_{h,i} \|^2_{L^2(K_1 \cup K_2)} \lesssim \| \nabla \mathbf{v}_{h,i} \|^2_{L^2(K_1 \cup K_2)},
\]

where \( E \in \mathcal{E}_{i,B}^n \), \( K_1, K_2 \in \mathcal{T}_{h,i}^n \) are the two elements sharing \( E \). Together with (4.8), it yields

\[
\sum_{i=1,2} \nu_i \| \nabla \mathbf{v}_{h,i} \|^2_{L^2(\Omega_{n,i}^h)} \approx \left\| \nu \frac{1}{2} \nabla \mathbf{v}_h \right\|^2_{0, \Omega_n^h} + \mathcal{J}_n^0(\mathbf{v}_h, \mathbf{v}_h),
\]

which shows the equivalence \( \| \mathbf{v}_h \|^2_{\mathbf{V}} \approx \sum_{i=1,2} \nu_i |v_{h,i}|^2_{H^1(\Omega_{n,i}^h)} + \mathcal{J}_n^0(\mathbf{v}_h, \mathbf{v}_h). \)

Similarly, we can prove

\[
\sum_{i=1,2} \nu_i^{-1} \| q_{h,i} \|^2_{L^2(\Omega_{n,i}^h)} \approx \| q_h \|^2_{Q}.
\]

The proof is finished.

4.2 Inf-sup conditions

Inf-sup conditions play the key role in proving the well-posedness of mixed problems. In [18, Lemma 3.11], Hansbo, Larson, and Azhedi proved the inf-sup condition for mixed finite element approximation to Stokes interface problem, where the velocity is discretized with linear finite elements on a triangular mesh of size \( h/2 \), while the pressure is discretized with linear finite elements on a triangular mesh of size \( h \). Here we follow them to establish the inf-sup condition on \( \mathbf{V}_h^n \times Q_h^n \). We also refer to [36] which studies mixed finite element method for Stokes interface problem on body-fitted meshes.
Lemma 4.3 Let Assumption 3 be satisfied and let \( h \) be small enough. For any \( q_h \in Q_h^n \), there exists a function \( \mathbf{v}_h \equiv (v_h|_{\partial n_1}, v_h|_{\partial n_2}) \), \( v_h \in V_{h,0}(k, \Omega) \) satisfies such that
\[
\| \mathbf{v}_h \|_V^2 \lesssim \| q_h \|_Q^2 \lesssim \langle \text{div } v_h, q_h \rangle_{Q_h^n} + \mathcal{J}_p^n(q_h, q_h).
\]

Proof In [7] Theorem 3.2, Brezzi and Falk proved the inf-sup condition for the \( Q_k-Q_{k-1} \) Hood-Taylor triangular finite elements. A careful inspection shows that the theorem holds for any rectangular mesh in which each element has at most two edges on the boundary of domain. Therefore, we have, for any \( \xi_h \in V_h(k-1, \omega_{h,i}^n) \cap L_2^2(\omega_{h,i}^n) \),
\[
\sup_{\nu_h,0 \in V_h(k, \omega_{h,i}^n) \cap H_0^1(\omega_{h,i}^n)} \frac{\langle \text{div } \nu_h, \xi_h \rangle_{\omega_{h,i}^n}}{\| \nu_h \|_{H_0^1(\omega_{h,i}^n)}} \gtrsim \| \xi_h \|_{L_2^2(\omega_{h,i}^n)} \quad i = 1, 2, (4.15)
\]
The rest of the proof uses (4.15) and is parallel to the proof of [18, Lemma 3.11] where the inf-sup condition is proven for linear finite elements on triangular meshes. Here we only sketch the proof and omit the details.

Step 1. Define \( \sigma_1 \equiv \nu_1 \| \Omega_{h,1} \|^{-1} \) and \( \sigma_2 \equiv -\nu_2 \| \Omega_{h,2} \|^{-1} \) and write \( \sigma = (\sigma_1, \sigma_2) \). Let \( \hat{q}_0 \) be the projection of \( q_h \) onto the one-dimensional space \( \text{Span} \{\sigma\} \) under the inner product \( \langle \nu^{-1} \cdot \rangle_{\Omega_h^n} \). Following the proof of [18] Lemma 3.9, we can find a function \( \mathbf{v}_0 \in V_{h,0}(k, \Omega) \) such that \( \mathbf{v}_0 \equiv (v_0|_{\Omega_{h,1}^n}, v_0|_{\Omega_{h,2}^n}) \) satisfies
\[
\| \mathbf{v}_0 \|_V^2 \lesssim \| \nu^{-1} q_0 \|_{0, \Omega_h^n}^2 \leq \| q_0 \|_Q^2 \lesssim \langle \text{div } \mathbf{v}_0, q_0 \rangle_{\Omega_h^n}.
\]

Step 2. For \( \hat{q}_h = q_h - q_0 \), following the proof of [18] Lemma 3.10 and using (4.14), we can find a \( \hat{v}_h \in V_h \) which satisfies \( \hat{v}_{h,i} = 0 \) in \( \Omega_{h,i} \setminus \omega_{h,i} \), \( i = 1, 2 \), and
\[
\| \hat{v}_h \|_V \lesssim \| \hat{q}_h \|_Q^2 \gtrsim \sum_{i=1,2} \nu_i^{-1} \| \hat{q}_h \|_{L_2^2(\omega_{h,i}^n)} \lesssim \langle \text{div } \hat{v}_h, \hat{q}_h \rangle_{\Omega_h^n} + \mathcal{J}_p^n(\hat{q}_h, \hat{q}_h),
\]
Define \( \hat{v}_h = \hat{v}_{h,1} \) in \( \Omega_{h,1} \) and \( \hat{v}_h = \hat{v}_{h,2} \) elsewhere. It is clear that \( \hat{v}_h \in V_{h,0}(k, \Omega) \).

Step 3. Define \( v_h = v_0 + \alpha \hat{v}_h \) and \( \hat{q}_h = q_0 + \hat{q}_h \) for some \( \alpha > 0 \). By the Cauchy-Schwarz inequality and the equality \( \langle \text{div } \hat{v}_h, q_0 \rangle_{\Omega_h^n} = 0 \), we have
\[
\langle \text{div } v_h, q_h \rangle_{\Omega_h^n} = \langle \text{div } v_0, q_0 \rangle_{\Omega_h^n} + \alpha \langle \text{div } \hat{v}_h, \hat{q}_h \rangle_{\Omega_h^n} + \langle \text{div } v_h, \hat{q}_h \rangle_{\Omega_h^n} \gtrsim C_0 \| q_0 \|_Q^2 + C_1 \alpha \| \hat{q}_h \|_Q^2 - \alpha \mathcal{J}_p^n(\hat{q}_h, \hat{q}_h) - C_2 \| q_0 \|_Q \| \hat{q}_h \|_Q \gtrsim \langle \text{div } \hat{v}_h, \hat{q}_h \rangle_{\Omega_h^n} + \mathcal{J}_p^n(\hat{q}_h, \hat{q}_h),
\]
where \( C_0, C_1, C_2 \) are positive constants independent of \( h, \tau, \) and \( \nu \). Since \( \mathcal{J}_p^n(\hat{q}_h, \hat{q}_h) = \mathcal{J}_p^n(\hat{q}_h, q_h) \), choosing \( \alpha = 4C_2^2/(C_0C_1) \) and using (4.9), we find that
\[
\| q_h \|_Q^2 \leq 2 \| q_0 \|_Q^2 + 2 \| \hat{q}_h \|_Q^2 \lesssim \langle \text{div } v_h, q_h \rangle_{\Omega_h^n} + \mathcal{J}_p^n(q_h, q_h).
\]
Then $v_h := v_0 + \alpha v_h \in V_h, 0, k, \Omega)$ and satisfies $v_h = v_{h,i}$ in $\Omega_h$ for $i = 1, 2$.

**Lemma 4.4** Suppose $\gamma_0$ is large enough. For any $u_h, v_h \in \mathcal{V}_h^n$ and $q_h \in \mathcal{Q}_h^n$,

$$|B^n_0(u_h, p_h)| \lesssim \|v_h\|_\mathcal{V} \|q_h\|_Q,$$

$$|\mathcal{A}_h^n(u_h, v_h)| \lesssim \|u_h\|_\mathcal{V} \|v_h\|_\mathcal{V}, \quad \mathcal{A}_h^n(v_h, q_h) \geq 0.9 \|v_h\|_\mathcal{V}^2.$$

**Proof** The continuity of $B^n_0$ is obvious. By Lemma 4.1, it is clear that, for any $\varepsilon > 0$,

$$|\mathcal{F}(v_h, v_h)| \leq (\varepsilon h)^{-\frac{1}{2}} \eta \nu \|\nu\|_{L^2(I_h^n)} + \varepsilon h \|\nu\|_{L^2(I_h^n)}^{-1} \|\nu\|_{L^2(I_h^n)}^2 \leq (\varepsilon^{-1} \gamma_0^{-1} + C_0 \varepsilon) \|v_h\|_\mathcal{V}^2.$$

Choosing $\varepsilon$ small enough and $\gamma_0$ large enough such that $\varepsilon^{-1} \gamma_0^{-1} + C_0 \varepsilon \leq 0.1$, we have $\mathcal{A}_h^n(v_h, q_h) = \|v_h\|_\mathcal{V}^2 - \mathcal{F}(v_h, v_h) \geq 0.9 \|v_h\|_\mathcal{V}^2$. The proof for the continuity of $\mathcal{A}_h^n$ is similar.

### 4.3 The well-posedness

To favor the analysis, we rewrite (3.7) into an equivalent variational form: find $(u_h, p_h) \in \mathcal{V}_h^n \times \mathcal{Q}_h^n$ such that

$$\mathcal{K}_1((u_h, p_h), (v_h, q_h)) = \mathcal{G}(v_h, q_h), \quad \forall (v_h, q_h) \in \mathcal{V}_h^n \times \mathcal{Q}_h^n$$

(4.16)

where

$$\mathcal{K}_1((u_h, p_h), (v_h, q_h)) = \tau^{-1} \lambda_0^k (u_h, v_h)_{\mathcal{V}_h^n} + \mathcal{A}_h^n(u_h, v_h) + \mathcal{B}_h^n(v_h, p_h)$$

$$- \mathcal{B}_h^n(u_h, q_h) + \mathcal{F}(p_h, q_h) + \gamma_1 \nu_2 h^2 \mathcal{A}_h^n(u_h, q_h)$$

$$+ \gamma_1 \nu_2 h^2 (\nu^{-1} \nabla p_h, \nabla q_h)_{\Omega_h^n},$$

$$\mathcal{B}_h^n(u_h, q_h) = \sum_{i=1,2} \sum_{K \in \mathcal{T}_h} \nu_i^{-1} \int_{K \cap \Omega_h^n} (\tau^{-1} \lambda_0^k u_{h,i} - \nu_i \Delta u_{h,i}) \cdot \nabla q_{h,i},$$

$$\mathcal{G}(v_h, q_h) = \sum_{i=1,2} \int_{\Omega_h^n} \left( f_i^n - \frac{1}{\tau} \sum_{j=1}^{k} \lambda_j^k U_{h,i}^{n-j,i} \right) \cdot (v_{h,i} - \gamma_1 \nu_2 h^2 \nu_i^{-1} \nabla q_{h,i}) .$$

**Theorem 6** Suppose that $\gamma_0, \gamma_1^{-1}$ are large enough and $h = O(\tau)$. Then problem (4.16) has a unique solution.
Proof Since (4.16) is a linear problem on finite-dimensional space, it suffices to prove that there exits an $\alpha_0 > 0$ such that for any $(u_h, p_h) \in V_h^n \times Q_h^n$
\[
\sup_{(w_h, q_h) \in V_h^n \times Q_h^n} \mathcal{K}_1((u_h, p_h), (v_h, q_h)) \| (v_h, q_h) \|_{V, Q} > \alpha_0 \| (u_h, p_h) \|_{V, Q}. \tag{4.17}
\]
From Lemma 4.3, there exists a $w_h \in V_{h,0}(k, \Omega)$ and a $u_h \in V_h^n$ such that $\| w_{h,i} \|_{H_{h,i}^i, i = 1,2}$, and
\[
C_0 \| p_h \|_{Q}^2 \leq \langle \text{div} w_h, p_h \rangle_{\Omega} + \mathbb{B}_n(p_h, p_h), \quad \| w_h \|_V \leq C_1 \| p_h \|_{Q}, \tag{4.18}
\]
where $C_0, C_1$ are positive constants independent of $h, \tau$, and $\nu$. Choosing $(v_h, q_h) = (u_h + \alpha w_h, p_h)$ for some $\alpha \in (0, 1)$ to be specified later, we have
\[
\mathcal{K}_1((u_h, p_h), (v_h, p_h)) \geq \lambda_h^k \tau^{-1} \| u_h \|_{0, \Omega}^2 + \mathbb{A}_n(u_h, u_h) + C_0 \| p_h \|_{Q}^2 \nonumber
\]
\[
+ \lambda_h^k \alpha \tau^{-1} \| u_h \|_{0, \Omega}^2 + C_0 \| u_h \|_{0, \Omega}^2 \quad \text{and} \quad \| u_h \|_{H^1(\Omega)} \leq C \| u_h \|_{H^1(\Omega)} \| w_h \|_V. \tag{4.19}
\]
Using Lemma 4.2 and the assumption $\nu = O(\tau)$, we have
\[
\mathbb{B}_n(u_h, p_h) \leq \| \nu^{-\frac{1}{2}} (\tau^{-1}\lambda_h^k u_h, i - \nu \Delta_h u_h) \|_{0, \Omega}^2 \| \nu^{-\frac{1}{2}} \nabla p_h \|_{0, \Omega}^2 \nonumber
\]
\[
\leq C_2 h^{-2} \left( \| \nu^{-\frac{1}{2}} u_h \|_{0, \Omega} \| \nu^{-\frac{1}{2}} \nabla p_h \|_{0, \Omega} + \frac{1}{2} \| \nu^{-\frac{1}{2}} \nabla p_h \|_{0, \Omega}^2 \right), \tag{4.20}
\]
where $C_2 > 0$ is a constant independent of $h, \tau$, and $\nu$. Moreover, by the relation between $w_h$ and $w_h$ and Poincaré's inequality, there exists a constant $C > 0$ independent of $h, \tau$ such that
\[
\left| \langle u_h, w_h \rangle_{\Omega} \right| \leq \| u_h \|_{0, \Omega} \| w_h \|_{L^2(\Omega)} \leq C \| u_h \|_{0, \Omega} \| w_h \|_V. \nonumber
\]
Together with (4.18) and Lemma 4.4, this implies
\[
\left| \langle u_h, w_h \rangle_{\Omega} \right| \leq C \| u_h \|_{0, \Omega} \| w_h \|_{Q} \nonumber \leq C \| u_h \|_{0, \Omega} \| w_h \|_{Q} + \frac{\tau}{4 \lambda_h^k} C_0 \| p_h \|_{Q}^2, \tag{4.21}
\]
\[
\mathbb{A}_n(u_h, w_h) \leq C \| u_h \|_V \| p_h \|_{Q} \nonumber \leq C_4 \| u_h \|_V \| p_h \|_{Q} + \frac{1}{4} C_0 \| p_h \|_{Q}^2. \tag{4.22}
\]
Since $\mathbb{A}_n(u_h, u_h) \geq 0.9 \| u_h \|_V^2$ by Lemma 4.4 inserting (4.20)–(4.22) into (4.19) leads to
\[
\mathcal{K}_1((u_h, p_h), (v_h, q_h)) \geq \{ \lambda_h^k \tau^{-1} - C_3 \alpha (\lambda_h^k / \tau)^2 - C_2 \gamma_1 \} \| u_h \|_{0, \Omega}^2 + (0.9 - C_4 \alpha - C_2 \gamma_1 \nu_2) \| u_h \|_{V}^2 + \frac{1}{2} C_0 \alpha \| p_h \|_{Q}^2. \nonumber
\]
Suppose $\gamma_1$ is small enough such that $4 C_2 \gamma_1 \leq \min(\nu_2^{-1}, \lambda_h^k \tau^{-1})$. Then (4.17) holds with $\alpha_0 = 0.5 \min(1, C_0 \alpha)$ if we take $\alpha = 0.1 \min \{ 1, C_4^{-1} (C_2 \lambda_h^{-1})^{-1} \}$.
5 Modified Stokes-projection operator

In this section we define a modified Stokes-projection operator from $\mathbf{V}^n \times \mathbf{Q}^n$ to $\mathbf{V}^n_h \times \mathbf{Q}^n_h$. It is a powerful tool for proving the stability and error estimates of numerical solutions. To begin with, we make the assumption on the regularity of the solution to Stokes interface problem:

**Assumption 7** For any $\xi \in L^2(\Omega^n_0)$, the Stokes interface problem

\[
\begin{aligned}
&-\nu \Delta z + \nabla r = \xi, & &\text{div } z = 0 \quad \text{in } \Omega^n_0, \\
&\| (\nu \nabla z + r) \cdot n \| = \| z \| = 0 \quad \text{on } \Gamma^n_0, \\
&z = 0 \quad \text{on } \Sigma,
\end{aligned}
\]

has a unique solution which satisfies

\[
\sum_{i=1,2} \left( \nu_i^{\frac{3}{2}} \| z \|_{H^2(\Omega^n_{0,i})} + \nu_i \| r \|_{H^1(\Omega^n_{1,i})} \right) \lesssim \| \xi \|_{L^2(\Omega^n_0)}.
\]

For the case that $\Omega^n_{0,1}$ and $\Omega^n_{0,2}$ have smooth boundaries, the solution to elliptic interface problem has the $H^2$-regularity in each sub-domain (cf. [2]). Here we do not intend to prove the result for Stokes interface problem and just treat it as an assumption. Another issue involves the outer boundary $\Sigma = \partial \Omega$ which is not smooth, but Lipschitz continuous. Our method and theory can be extended straightforwardly to the case that $\Sigma$ is smooth. Since the paper is focused on dealing with the moving interface $\Gamma(t)$, we only consider the case that $\Sigma$ is fitted with $\Gamma_h$ for simplicity.

5.1 The definition

For any $u, p \in \mathbf{V}^n \times \mathbf{Q}^n$ and $f \in (\mathbf{Q}^n_h)'$, the modified Stokes projection $S^n(u, p, f) = (u_h, p_h) \in \mathbf{V}^n_h \times \mathbf{Q}^n_h$ is defined by the solution to the mixed problem

\[
\begin{aligned}
\mathcal{A}^n_h(u_h, v_h) + \mathcal{B}^n_0(v_h, p_h) &= a^n_h(u, v_h) + \mathcal{B}^n_0(v_h, p), & &\forall v_h \in \mathbf{V}^n_h, \quad (5.1a) \\
\mathcal{B}^n_0(u_h, q_h) - \mathcal{J}^n_p(u_h, q_h) &= f(q_h), & &\forall q_h \in \mathbf{Q}^n_h, \quad (5.1b)
\end{aligned}
\]

where $a^n_h(u, v) = \langle \nu \nabla u, \nabla v \rangle_{\Omega^n_0} - \mathcal{F}^n(u, v) + \mathcal{J}^n_p(u, v)$. We define two norms on $(\mathbf{Q}^n_h)'$ as

\[
\| f \|_{(\mathbf{Q}^n_h)'} := \sup_{q_h \neq 0} \frac{f(q_h)}{\| q_h \|_{\mathbf{Q}}}, \quad \| f \|_{1,(\mathbf{Q}^n_h)'} := \sup_{q_h \neq 0} \frac{f(q_h)}{\| q_h \|_{1,\mathbf{Q}}}.
\]

**Theorem 8** Let the assumptions in Lemma 4.3 be satisfied. Define

\[
\mathcal{K}_0((u_h, p_h), (v_h, q_h)) = \mathcal{A}^n_h(u_h, v_h) + \mathcal{B}^n_0(v_h, p_h) - \mathcal{B}^n_0(u_h, q_h) + \mathcal{J}^n_p(p_h, q_h).
\]

\[
\mathcal{K}_0((u_h, p_h), (v_h, q_h)) = S^n(u, p, f) + \mathcal{J}^n_p(p_h, q_h).
\]
For any \((u_h, p_h) \in \mathcal{V}_h \times \mathcal{Q}_h^n\), there exists a \(v_h \in \mathcal{V}_h^n\) such that
\[
\left\|v_h\right\|_{\mathcal{V}}^2 \leq \left\|(u_h, p_h)\right\|_{\mathcal{V}, \mathcal{Q}}^2 \lesssim \mathcal{K}_0((u_h, p_h), (v_h, p_h)).
\] (5.4)

Proof The proof is similar to that of Theorem 6. By Lemma 4.3, there exist a \(w_h \in \mathcal{V}_h^n\) satisfying \(\left\|w_h\right\| = 0\) on \(\Gamma^\tau_n\) and two positive constants \(C_0, C_1\) independent of \(\tau, h\) such that
\[
C_0\left\|w_h\right\|_{\mathcal{V}}^2 \leq C_1\left\|p_h\right\|_{\mathcal{Q}}^2 \lesssim \langle \text{div} w_h, p_h \rangle_{\Omega^\tau_n} + J^n(p_h, p_h).
\]

Set \(v_h = u_h - \alpha w_h\) and \(q_h = p_h\) for some \(\alpha \in (0, 1)\) to be specified. By Lemma 4.4, there exists a constant \(C_2 > 0\) such that
\[
\mathcal{K}_0((u_h, p_h), (v_h, q_h)) = \mathcal{K}^n_u((u_h, u_h - \alpha w_h) + \alpha \langle \text{div} w_h, p_h \rangle_{\Omega^\tau_n} + J^n(p_h, p_h)
\]
\[
\geq 0.9\left\|u_h\right\|_{\mathcal{V}}^2 - C_2\alpha\left\|u_h\right\|_{\mathcal{V}}\left\|w_h\right\|_{\mathcal{V}} + C_1\alpha\left\|p_h\right\|_{\mathcal{Q}}^2
\]
\[
\geq (0.9 - 0.5C_2^2C_1^{-1}\alpha)\left\|u_h\right\|_{\mathcal{V}}^2 + 0.5C_1\alpha\left\|p_h\right\|_{\mathcal{Q}}^2.
\]

Taking \(\alpha = C_2^{-2}C_1^{-1}\) leads to (5.4).

Theorem 9 Problem (5.1) has a unique solution which satisfies
\[
\left\|(u_h, p_h)\right\|_{\mathcal{V}, \mathcal{Q}} \lesssim \left\|(u, p)\right\|_{\mathcal{V}, \mathcal{Q}} + \|f\|_{\mathcal{Q}_h^n}^\ast.
\]

Proof Note that (5.1) is equivalent to the variational problem
\[
\mathcal{K}_0((u_h, p_h), (v_h, q_h)) = a^n_h(u, u_h) + B^n_0(v_h, p) - f(q_h), \quad \forall (v_h, q_h) \in \mathcal{V}_h^n \times \mathcal{Q}_h^n.
\] (5.5)

By Theorem 8 we have the inf-sup condition
\[
\left\|(u_h, p_h)\right\|_{\mathcal{V}, \mathcal{Q}} \lesssim \sup_{(v_h, q_h) \in \mathcal{V}_h^n \times \mathcal{Q}_h^n} \frac{\mathcal{K}_0((u_h, p_h), (v_h, q_h))}{\left\|(v_h, q_h)\right\|_{\mathcal{V}, \mathcal{Q}}}.\] (5.6)

This implies that (5.5) has a unique solution. Moreover, combining (5.5) and (5.6) yields
\[
\left\|(u_h, p_h)\right\|_{\mathcal{V}, \mathcal{Q}} \lesssim \sup_{(v_h, q_h) \in \mathcal{V}_h^n \times \mathcal{Q}_h^n} \frac{a^n_h(u, u_h) + B^n_0(v_h, p) - f(q_h)}{\left\|(v_h, q_h)\right\|_{\mathcal{V}, \mathcal{Q}}}
\]
\[
\lesssim \left\|(u, p)\right\|_{\mathcal{V}, \mathcal{Q}} + \|f\|_{\mathcal{Q}_h^n}^\ast.
\]

The proof is finished.
5.2 Quasi-interpolation operators

For $1 \leq m \leq k$ and $D \subseteq \Omega$, let $\pi_{m,D} : H^1(D) \to V_h(m,D)$ be the quasi-interpolation (or Scott-Zhang) operator which respects homogeneous Dirichlet boundary conditions \cite{Scott-Zhang}. We have the well-known results: for any $K \in T_h$ and $E \in \mathcal{E}_h$,

$$\|\pi_{m,E}v - v\|_{H^l(K)} \lesssim h^{\min\{m-1,l\}} |v|_{H^{l+1}(E)}, \quad (5.7)$$

$$\|\pi_{m,E}v - v\|_{H^l(E)} \lesssim h^{\min\{m-1/2,l-1/2\}} |v|_{H^{l+1}(E)}, \quad 0 \leq l \leq s, \quad (5.8)$$

where $D_A$ is the union of all elements having non-empty intersection with $A$, for $A = K$ or $E$. Similarly, let $\pi_{m,D}$ denote the interpolation operator on vector-valued functions. For any $v \in H^1(\Omega_{h,1}^n) \times H^1(\Omega_{h,2}^n)$ and any $q \in H^1(\Omega_{h,1}^n) \times H^1(\Omega_{h,2}^n)$, we define

$$\pi_m(v) = (\pi_{m,\Omega_{h,1}^n} v_1, \pi_{m,\Omega_{h,2}^n} v_2) \in \mathcal{V}_h^n,$$

$$\pi_m(q) = (\pi_{m,\Omega_{h,1}^n} q_1, \pi_{m,\Omega_{h,2}^n} q_2) \in \mathcal{Q}_h^n.$$

**Lemma 5.1** Suppose that the components of $v$ and $q$ satisfy $v_i \in H^{k+1}(\Omega_{h,i}^n)$, $q_i \in H^k(\Omega_{h,i}^n)$ for $i = 1,2$. Let $v_h = \pi_h(v)$ and $q_h = \pi_h(q)$. Then

$$\|v - v_h\|_{\mathcal{V}}^2 + \|v - v_h\|_{2}^2 + h^2 \|\nabla v - \nabla v_h\|_{0,\Omega_h^n}^2 \lesssim h^{2k} \sum_{i=1,2} \nu_i |v_i|_{H^{k+1}(\Omega_{h,i}^n)}^2, \quad (5.9)$$

$$\|q - q_h\|_{\mathcal{Q}}^2 + \|q - q_h\|_{2}^2 + h^2 \sum_{i=1,2} \nu_i^{-1} |q_i|_{H^k(\Omega_{h,i}^n)}^2 \lesssim h^{2k} \sum_{i=1,2} \nu_i^{-1} |q_i|_{H^{k+1}(\Omega_{h,i}^n)}^2. \quad (5.10)$$

**Proof** Since $q_i \in H^k(\Omega_{h,i}^n)$ and $v_i \in H^{k+1}(\Omega_{h,i}^n)$, from \cite{Scott-Zhang} we have

$$\mathcal{J}_p^n(q - q_h, q - q_h) \lesssim \sum_{\kappa \in \mathcal{T}_{h,B}^n} h^{2k} \nu_i^{-1} |q_i|_{H^{k}}(\Omega_{h,i}^n) \lesssim h^{2k} \sum_{i=1}^2 |q_i|_{H^k(\Omega_{h,i}^n)}^2,$$

$$\mathcal{J}_v^n(v - v_h, v - v_h) \lesssim \sum_{\kappa \in \mathcal{T}_{h,B}^n} h^{2k} \nu_i |v_i|_{H^{k+1}}(\Omega_{h,i}^n) \lesssim h^{2k} \sum_{i=1}^2 \nu_i |v_i|_{H^{k+1}(\Omega_{h,i}^n)}.$$
Now we define the operator $B^n_0: \mathbf{V}^n \rightarrow (Q^n_h)^\prime$ as follows: for any $u \in \mathbf{V}^n$, $B^n_0u \in (Q^n_h)^\prime$ satisfies

$$(B^n_0u)(q_h) = B^n_0(u, q_h), \quad \forall q_h \in Q^n_h.$$ 

For convenience in notation, we define

$$M_k(u, p) = 2 \sum_{i=1}^{2} \left( \nu_i |u_i|^2_{H^{k+1}(\Omega^n_{h,i})} + \nu_i^{-1} |p_i|^2_{H^k(\Omega^n_{h,i})} \right).$$

**Theorem 10** Let Assumption 7 be satisfied and let $(u_h, p_h) \equiv S^n(u, p, f)$ be the solution to problem (5.1). Suppose $M_k(u, p) < \infty$. Then

$$\|\nu u - u_h, p - p_h\|_{\mathbf{V}, Q} + h\|\nu^\frac{1}{2} \Delta h(u - u_h)\|_{0, \Omega^n_h} + h\|\nu^{-\frac{1}{2}} \nabla (p - p_h)\|_{0, \Omega^n} \lesssim h^k M^2_k(u, p) + \|f - B^n_0u\|_{(Q^n_h)^\prime}.$$ 

**Proof** For convenience, we write

$$e_u = u - u_h, \quad e_p = p - p_h, \quad \xi_h = u - \pi_k(u), \quad \eta_h = \pi_k(u) - u_h,$$

$$\nu_k = \pi_k(p) - p_h, s_h = \pi_{k-1}(p) - p_h.$$ 

Since $u_i \in H^{k+1}(\Omega^n_{h,i})$ and $p_i \in H^k(\Omega^n_{h,i})$, we have $J^n_p(u, p_h) = J^n_p(p, q_h) = 0$ for all $q_h \in \mathbf{V}^n_h$ and $p_h \in Q^n_h$. From (5.5), we have

$$J^n_0((e_u, e_p), (\nu_h, q_h)) = B^n_0(u, q_h) - f(q_h), \quad \forall (\nu_h, q_h) \in \mathbf{V}^n_h \times Q^n_h.$$
By Theorem 8, there is a \( \mathbf{w}_h \in \mathbf{V}_h^n \) which satisfies \( \| \mathbf{w}_h \|_{\mathbf{V},Q} \lesssim \| (\mathbf{w}_h, s_h) \|_{\mathbf{V},Q} \) and
\[
\| (\mathbf{w}_h, s_h) \|_{\mathbf{V},Q} \lesssim \mathcal{K}_0((\mathbf{w}_h, s_h), (\mathbf{w}_h, s_h))
\]
\[
= \mathcal{K}_0(\mathbf{e}_u, s_h), (\mathbf{w}_h, s_h)) - \mathcal{K}_0((\mathbf{w}_h, s_h), (\mathbf{w}_h, s_h))
\]
\[
\lesssim (\| f - B_0^n u \|_{(Q^n')}) + \| (\mathbf{w}_h, s_h) \|_{\mathbf{V},Q} \| (\mathbf{w}_h, s_h) \|_{\mathbf{V},Q}.
\]
where we have used Lemma 4.4 in the second inequality. It follows that
\[
\| (\mathbf{w}_h, s_h) \|_{\mathbf{V},Q} \lesssim \| f - B_0^n u \|_{(Q^n')} + \| (\mathbf{w}_h, s_h) \|_{\mathbf{V},Q}.
\]
Together with Lemma 5.1, this yields the estimate for \( \| (\mathbf{u} - \mathbf{u}_h, p - p_h) \|_{\mathbf{V},Q} \).

Finally, from Lemma 4.2 and Lemma 5.1, we have
\[
\| \nu^\frac{1}{2} \Delta_h \mathbf{e}_u \|_{0,\Omega^n} + \| \nu^{-\frac{1}{2}} \nabla \mathbf{e}_u \|_{0,\Omega^n} \lesssim \| \nu^\frac{1}{2} \Delta_h \mathbf{e}_u \|_{0,\Omega^n} + \| \nu^{-\frac{1}{2}} \nabla \mathbf{e}_u \|_{0,\Omega^n}
\]
\[
+ h^{-1} \| (\mathbf{w}_h, s_h) \|_{\mathbf{V},Q} \lesssim h^{-1} \left( h^k M^1_k(\mathbf{u}, p) + \| f - B_0^n u \|_{(Q^n')} \right).
\]
The proof is finished.

**Theorem 11** Let Assumption 7 and Assumption 8 be satisfied. Then upon a hidden constant depending only \( \Omega^n_{h,1} \) and \( \Omega^n_{h,2} \), \( (\mathbf{u}_h, p_h) \equiv S^p(\mathbf{u}, p, f) \) satisfies
\[
\| \mathbf{u} - \mathbf{u}_h \|_{0,\Omega^n} \lesssim h \| (\mathbf{u} - \mathbf{u}_h, p - p_h) \|_{\mathbf{V},Q} + \| f - B_0^n u \|_{1,(Q^n')}.
\]

**Proof** We shall use the duality technique to prove the lemma. Let \( \mathbf{e}_u \in L^2(\Omega^n) \) be defined as \( \mathbf{e}_u := \mathbf{u}_i - \mathbf{u}_{h,i} \) in \( \Omega^n_{h,i} \) for \( i = 1, 2 \). Consider the auxiliary problem
\[
\begin{cases}
-\nu \Delta \mathbf{z} + \nabla r = \mathbf{e}_u, & \text{div } \mathbf{z} = 0 \quad \text{in } \Omega^n,

\langle (\nu \nabla \mathbf{z} + r) \cdot \mathbf{n} \rangle = [\mathbf{z}] = 0 \quad \text{on } \Gamma^n,

\mathbf{z} = 0 \quad \text{on } \Sigma.
\end{cases}
\]
(5.13)

By Assumption 7 we have \( \| \nu^\frac{1}{2} \mathbf{z} \|_{H^2(\Omega^n)} + \| \nu^{-\frac{1}{2}} r \|_{H^1(\Omega^n)} \lesssim \| \mathbf{e}_u \|_{L^2(\Omega^n)}. \) By Stein’s extension theory (see 38 Chapter 6), there exist \( \mathbf{z}_i \in H^2(\Omega^n_{h,i}) \) and \( r_i \in H^1(\Omega^n_{h,i}) \), \( i = 1, 2 \), which satisfy \( \mathbf{z}_i = \mathbf{z} \) and \( r_i = r \) in \( \Omega^n_{h,i} \), and
\[
\nu^\frac{1}{2}_i \| \mathbf{z}_i \|_{H^2(\Omega^n_{h,i})} + \nu^{-\frac{1}{2}}_i \| r_i \|_{H^1(\Omega^n_{h,i})}.
\]
\[
\lesssim \|\nu_{\gamma}^{\frac{1}{2}} z\|_{H^2(\Omega_h^{\ast})} + \|\nu_{\gamma}^{-\frac{1}{2}} r\|_{H^1(\Omega_h^{\ast})} \lesssim \|e_u\|_{L^2(\Omega_h^{\ast})}. \tag{5.14}
\]

Define \( \bar{z} = (z_1, z_2) \), \( r = (r_1, r_2) \), \( z_h = \pi_{1}(\bar{z}) \), \( r_h = \pi_{1}(r) \), \( e_\gamma = \bar{z} - z_h \), and \( e_r = r - r_h \). From Lemma 5.1 and (5.14), we obtain
\[
\|\langle e_\gamma, e_r \rangle\|_{V, Q} + h^\frac{1}{2} \|\nabla e_\gamma\|_{0, \Omega_h^{\ast}} \lesssim \|e_u\|_{L^2(\Omega_h^{\ast})}. \tag{5.15}
\]

Write \( e_u = u - u_h \) and \( e_r = p - p_h \). The weak form of (5.13) yields
\[
a^n_h(\bar{z}, e_u) + B^0_h(\bar{e}_u, e_r) = \langle e_u, e_u \rangle_{\Omega_h^{\ast}}, \quad B^0_h(\bar{z}, e_r) = 0.
\]

Write \( f_u := f - B^0_h u \) for convenience. From (5.1), we also have
\[
a^n_h(e_u, z_h) + B^0_h(z_h, e_p) = J^0_u(u_h, z_h),
\]
\[
B^n_h(e_u, r_h) + J^n_p(p_h, r_h) + f_u(r_h) = 0.
\]

Combining the four equalities above and noting that \( a^n_h \) is symmetric, we have
\[
\|e_u\|^2_{L^2(\Omega_h^{\ast})} = a^n_h(e_\gamma, e_u) + B^n_h(e_\gamma, e_p) + B^n_h(e_u, e_r) + J^n_p(u_h, z_h) - J^n_p(p_h, r_h) - f_u(r_h)
\]
\[
= a^n_h(e_\gamma, e_u) + B^n_h(e_\gamma, e_p) + B^n_h(e_u, e_r) - f_u(r_h)
\]
\[
+ h \sum_{i=1,2} \sum_{E \in \mathcal{E}_h^{i, b}} \int_E \nu_i \|\partial_n (u_i - u_{h, i})\| \|\partial_n (z_i - z_{h, i})\| \|
\]
\[
- h^3 \sum_{i=1,2} \sum_{E \in \mathcal{E}_h^{i, b}} \int_E \nu_i^{-1} \|\partial_n (p_i - p_{h, i})\| \|\partial_n r_{h, i}\| , \tag{5.16}
\]

where in the second equality, we have used the fact that \( z_{h, i} \) and \( r_{h, i} \) are piecewise linear.

From (5.7) and (5.14), we know that \( \nu_{\gamma}^{-\frac{1}{2}} \|r_{h, i}\|_{H^1(\Omega_{h, i}^{\ast})} \lesssim \nu_{\gamma}^{-\frac{1}{2}} \|r\|_{H^1(\Omega_h^{\ast})} \lesssim \|e_u\|_{L^2(\Omega_h^{\ast})} \) for \( h \) small enough. By arguments similar to the proof of Lemma 4.4 and using Theorem 10 and inequality (5.15), we have
\[
\left| a^n_h(e_\gamma, e_u) + B^n_h(e_\gamma, e_p) + B^n_h(e_u, e_r) \right| \lesssim h\|\langle e_u, e_p \rangle\|_{V, Q}\|e_u\|_{L^2(\Omega_h^{\ast})}, \tag{5.17}
\]
\[
|f_u(r_h)| \lesssim \|f_u\|_{1, (Q_h^{\ast})} \|r_h\|_{1, Q} \lesssim \|f_u\|_{1, (Q_h^{\ast})} \|e_u\|_{L^2(\Omega_h^{\ast})}. \tag{5.18}
\]

By the trace inequality (4.10) and the Cauchy-Schwarz inequality, we also have
\[
h \sum_{i=1,2} \sum_{E \in \mathcal{E}_h^{i, b}} \int_E \nu_i \|\partial_n (u_i - u_{h, i})\| \|\partial_n (z_i - z_{h, i})\| \lesssim J^n_u(e_u, e_u)^\frac{1}{2} \sum_{i=1,2} \nu_i^\frac{1}{2} (|z_i - z_{h, i}|_{H^1(\Omega_{h, i}^{\ast})} + h |z_i|_{H^2(\Omega_{h, i})}),
\]
\[ \lesssim h \mathcal{J}_u^n(e_u, e_u) \lesssim h \|e_u\|_{L^2(\Omega_n^\tau)}. \]  

(5.19)

Similarly, by norm equivalence and the estimates in (5.14)–(5.15), we get

\[ h^3 \sum_{i=1,2} \sum_{E \in E_n^h} \int_E \nu_i^{-1} \|\partial_n (p_i - p_{h,i})\| \|\partial_n r_{h,i}\| \]

\[ \lesssim h \mathcal{J}_p^n(e_p, e_p) \lesssim h \|e_p\|_{L^2(\Omega_n^\tau)}. \]

(5.20)

The proof is finished by inserting (5.17)–(5.20) into (5.16).

**Remark 3** If \( f = B_0^h u \) and \( M_k(u, p) < \infty \), Theorems 10 and 11 indicate that

\[ \|u - u_h\|_{0,\Omega_n^\tau} \lesssim h\|\{u - u_h, p - p_h\}\|_{\mathcal{V},\mathcal{Q}} \lesssim h^{k+1} M_k^2(u, p). \]

(5.21)

### 5.4 Modified Stokes projections of discrete solutions

Note from (3.7) that the discrete solutions are coupled with flow maps. Numerical solutions in previous time steps are not piecewise polynomials any more in the current time step. In the subsequent analysis, the modified Stokes projections will be applied to them.

Following same lines in [32, Appendix A], we have the preliminary but useful estimates for the pull-back map \( v_h \rightarrow v_h \circ X^{n,n-l} \) for a finite element function \( v_h \in V_h(k, \Omega) \).

**Lemma 5.2** Suppose \( \eta = O(\tau^{\max(k/3,1)}) \). Write \( v_h^{n-l,n} = v_h \circ X^{n,n-l}_\tau \) for \( v_h \in V_h(k, \Omega) \), \( 0 \leq l \leq k \leq n \). There is a constant \( C > 0 \) independent of \( \tau, h \), and \( n \) such that, for \( \mu = 0,1, i = 1, 2 \), and \( l \leq j \leq k \),

\[ \|v_h^{n-l,n}\|_{H^1(\Omega_n^\tau)}^2 \leq C h^{-1} \|v_h\|_{H^1(\Omega)}^2, \]

(5.22)

\[ \|v_h^{n-l,n}\|_{L^2(\Gamma_n^\tau)}^2 \leq (1 + C \tau) \|v_h\|_{L^2(\Gamma_{n-l}^\tau)}^2 + C \tau^{k+2} h^{-2} \|v_h\|_{H^1(\Omega)}^2, \]

(5.23)

\[ \|\nabla \nu v_h^{n-l,n}\|_{L^2(\Omega_{n-l}^\tau)}^2 \leq (1 + C \tau) \|\nabla \nu v_h^{n-j,n-l}\|_{L^2(\Omega_{n-l}^\tau)}^2 \]

\[ + C \tau^{k+2} h^{-1} \|\nabla \nu v_h\|_{L^2(\Omega)}^2. \]

(5.24)

Note that \( \Omega_{n,i}^\tau \neq \Omega_i(t_n) \) for \( i = 1, 2 \). We also need to estimate functions on \( \Omega_{n,i}^\tau \oplus \Omega_i(t_n) \), where \( A \oplus B = (A \backslash B) \cup (B \backslash A) \) for two sets \( A \) and \( B \).
Using (3.5), (3.6), and Lemma 5.2, we easily get
\[ \|v\|^2_{L^2(\Omega^n \cap \Gamma(t_n(u)))} \lesssim \tau^{k+1} \|v\|^2_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega), \]  
(5.25)
\[ \|v\|^2_{L^2(\Omega^n \cap \Omega(X^n, u^n))} \lesssim \tau^{k+2} \|v\|^2_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega), \]  
(5.26)
\[ \|\nabla v_h\|^2_{L^2(\Omega^n \setminus \Omega(t_n(u)))} \lesssim \tau^{k+1} h^{-1} \|\nabla v_h\|^2_{L^2(\Omega^n)}, \quad \forall v_h \in V_h(k, \Omega), \]  
(5.27)
\[ \|\nabla v_h\|^2_{L^2(\Omega^n \cap \Omega(X^n, u^n))} \lesssim \tau^{k+2} h^{-1} \|\nabla v_h\|^2_{L^2(\Omega^n)}, \quad \forall v_h \in V_h(k, \Omega). \]  
(5.28)

**Proof.** The proof of (5.28) can be found in [32, Appendix A]. It is left to prove (5.25)–(5.27). First we cite an important result from [35, Lemma 10 and (17)]:

*Lemma 5.3* Suppose \( \eta = O(\tau^{\max{k/3,1}}) \). Then for \( i = 1, 2 \) and \( |m-n| \leq k \),
\[ \|v\|^2_{L^2(\Omega^n \cap \Omega(t_n(u)))} \lesssim \tau^{k+1} \|v\|^2_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega), \]
(5.29)
\[ \|v\|^2_{L^2(\Omega^n \cap \Omega(X^n, u^n))} \lesssim \tau^{k+2} \|v\|^2_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega), \]
(5.30)

where \( \eta \) is a constant, and \( C \) is a constant independent of \( \tau \). Using \( \eta = O(\tau^{\max{k/3,1}}) \) and Theorem 3, we infer that
\[ \|\nabla v_h\|^2_{L^2(\Omega^n \setminus \Omega(t_n(u)))} \lesssim \tau^{k+1} h^{-1} \|\nabla v_h\|^2_{L^2(\Omega^n)}. \]
(5.31)

Moreover, inequality (5.26) is a consequence of (5.29) and Theorem 3.

Finally, from (5.30) we find that
\[ \|\nabla v_h\|^2_{L^2(\Omega^n \cap \Omega(t_n(u)))} \lesssim \tau^{k+1} \sum_{K \in \mathcal{T}_h^n} h \|\nabla v_h\|^2_{L^\infty(K)} \lesssim \tau^{k+1} h^{-1} \|\nabla v_h\|^2_{L^2(\Omega^n)}. \]

This is inequality (5.27). The proof is finished.

**Remark 4** Using (3.5), (3.6), and Lemma 5.2 we easily get
\[ \|\nu \frac{1}{\sqrt{\tau}} \nabla U_h^{n-l,n} \|^2_{L^2(\Omega^n)} \leq (1 + C\tau) \|\nu \frac{1}{\sqrt{\tau}} \nabla U_h^{n-l,n} \|^2_{L^2(\Omega^n)} + C\tau^{k+2} h^{-1-\mu} \sum_{n=1}^{l,\Omega^n} \|\nu \|_{L^2(\Omega^n)} \|U_h^{n-l,n} \|^2_{L^2(\Omega^n)}, \]
(5.32)
\[ \|\nu \|_{L^2(I^{n-l,n}_\Omega)} \leq (1 + C\tau) \|\nu \|_{L^2(I^{n-l,n}_\Omega)} + C\tau^{k+2} h^{-1} \sum_{n=1}^{l,\Omega^n} \|\nu \|_{L^2(I^{n-l,n}_\Omega)} \|U_h^{n-l,n} \|^2_{L^2(\Omega^n)}, \]
(5.33)
where the jump and average are defined as
\[ U_h^{n-l,n} = u_h^{n-l} \circ X^{n-l,n} - u_h^{n-l} \circ X^{n-l,n}, \]
\[ \nu \partial_n U_h^{n-l,n} = \nu \partial_n (u_h^{n-l} \circ X^{n-l,n}) + \kappa_2 \nu \partial_n (u_h^{n-l} \circ X^{n-l,n}). \]

Together with (4.13), they show \( \|U_h^{n-l,n}\|_{\mathcal{V}} \leq \|U_h^{n-1}\|_{\mathcal{V}}. \)
Lemma 5.4 Let $u_h^{n-1}$ be the discrete velocity at $t_{n-1}$ and $q_h \in Q_h^n$. Extend $u_h^{n-1}$ and $q_{h,i}$ from $\Omega_h^n$ to $\Omega$ according to the convention in (3.5) and denote their extensions still by $u_h^{n-1} \in V_h(k, \Omega)$ and $q_{h,i} \in V_h(k-1, \Omega)$. Moreover, let $\hat{q}_{h,i} = \pi_{1,\Omega}(q_{h,i} \circ X_h^{n-1,n}) \in V_h(1, \Omega)$ be the Scott-Zhang interpolations of $q_{h,i} \circ X_h^{n-1,n}$ for $i = 1, 2$. Then $\hat{q}_h = (\hat{q}_{h,1}, \hat{q}_{h,2})$ satisfies

$$\left| B_0(u_h^{n-1,n}, q_h) - B_0^{-1}(u_h^{n-1}, \hat{q}_h) \right| \lesssim h \left\| u_h^{n-1} \right\| \left\| \nabla \right\| q_h \left\| \right\| _{1, \Omega}. $$

Proof For convenience, we denote the unit normals on $\Gamma^n_q$, $\Gamma^{n-1}_q$ by $n_n$ and $n_{n-1}$, respectively. Using the parametric representations of $\Gamma^n_q$ and $\Gamma^{n-1}_q$, we find that

$$\int_{\Gamma_q^n} \left[ U_h^{n-1,n} \cdot n \right] \{ q_h \} = \int_{\Gamma_q^{n-1}} \left[ U_h^{n-1,n} \cdot n \right] \circ \chi_n \{ q_h \} \circ \chi_n \left| \chi'_n \right| = \sum_{j=0}^3 I_j,$$

where

$$I_0 = \int_{\Gamma_q^n} \left( \left[ U_h^{n-1,n} \cdot n_{n-1} \right] \{ \hat{q}_h \} \circ \chi_n \left| \chi'_{n-1} \right| = \int_{\Gamma_q^{n-1}} \left[ U_h^{n-1,n} \cdot n_{n-1} \right] \{ \hat{q}_h \},
$$

$$I_1 = \int_{\Gamma_q^n} \left( \left[ U_h^{n-1,n} \cdot n \right] \circ \chi_n \left| \chi'_n \right| - \left[ U_h^{n-1,n} \cdot n_{n-1} \right] \circ \chi_n \left| \chi'_{n-1} \right| \} \{ \hat{q}_h \} \circ \chi_n,$n

$$I_2 = \int_{\Gamma_q^n} \left[ U_h^{n-1,n} \cdot n_{n-1} \right] \circ \chi_n \left( \{ q_h \} \circ \chi_n - \{ q_h \circ X_h^{n-1,n} \} \circ \chi_n \left| \chi'_{n-1} \right| \right),
$$

$$I_3 = \int_{\Gamma_q^n} \left[ U_h^{n-1,n} \cdot n_{n-1} \right] \circ \chi_n \left( \{ q_h \circ X_h^{n-1,n} \} - \{ \hat{q}_h \} \right) \circ \chi_n \left| \chi'_{n-1} \right| .$$

This shows that $B_0(u_h^{n-1,n}, q_h) = B_0^{-1}(u_h^{n-1}, \hat{q}_h) + \sum_{j=1}^5 I_j$, where

$$I_4 = - \sum_{i=1,2} \int_{\Omega_{i,q,i}} (\nabla u_h^{n-1}) \circ X_h^{n-1} \circ (\mathbb{J}_h^{n-1,n} - I) q_{h,i},
$$

$$I_5 = - \sum_{i=1,2} \int_{\Omega_{i,q,i}} \text{div} u_h^{n-1} q_{h,i} (\det \mathbb{J}_h^{n-1,n} - 1)
$$

$$- \sum_{i=1,2} \int_{\Omega_{i,q,i}} \text{div} u_h^{n-1}(q_{h,i} \circ X_h^{n-1,n} - \hat{q}_{h,i}) \det \mathbb{J}_h^{n-1,n}
$$

$$- \sum_{i=1,2} \int_{\Omega_{i,q,i} \setminus \Omega_{i,q,i}} \text{div} u_h^{n-1} q_{h,i} \circ X_h^{n-1,n} \det \mathbb{J}_h^{n-1,n}
$$

$$+ \sum_{i=1,2} \int_{\Omega_{i,q,i} \setminus \Omega_{i,q,i}} \text{div} u_h^{n-1} q_{h,i} \circ X_h^{n-1,n} \det \mathbb{J}_h^{n-1,n}. $$

Here $A : B = \sum_{i,j} A_{i,j} B_{i,j}$ stands for the Hadamard product of matrices $A$ and $B$. It suffices to estimate $I_0, \cdots, I_5$ term by term.
By (2.10) and (3.5), the estimation for $I_4$ is easy and gives

$$|I_4| \lesssim \tau \sum_{i=1,2} |u_{h,i}^{n-1}|_{H^1(\Omega)} \|q_{h,i}\|_{L^2(\Omega^m_{n,i})} \lesssim h \|u_{h,i}^{n-1}\|_\mathcal{V} \|\nu^{-\frac{1}{2}} q_h\|_{0,\Omega^m_n}.$$  

Thanks to (5.7), (2.10), and (3.5), we have the error estimates

$$\|q_{h,i} \circ X_{\tau}^{n-1,n} - \hat{q}_{h,i}\|_{L^2(\Omega)} \lesssim h \|q_{h,i} \circ X_{\tau}^{n-1,n}\|_{H^1(\Omega)} \lesssim h \|q_{h,i}\|_{H^1(\Omega^m_{n,i})}. \quad (5.35)$$

Using (2.10), (4.7), (5.7), and Lemma 5.3, we obtain

$$|I_5| \lesssim h \sum_{i=1,2} |u_{h,i}^{n-1}|_{H^1(\Omega)} \|q_{h,i}\|_{H^1(\Omega)} \lesssim h \|u_{h,i}^{n-1}\|_\mathcal{V} \|q_h\|_{1,\mathcal{Q}}.$$  

Note that $n_n = (-\chi_n',\chi_n')/|\chi_n'|$. From Theorem 4 and (2.10), we have

$$\|X_n - X_{\tau}^{n-1,n} \circ \chi_n\|_{C^1([0,L])} \lesssim \|X_n - X_{\tau}^{n-1,n} \circ \chi_n\|_{C^1([0,L])} + \|X_{\tau}^{n-1,n} \circ \chi_n - \chi_n\|_{C^1([0,L])} \lesssim \tau.$$  

This implies $\|n_n \circ \chi_n - n_{n-1} \circ \chi_{n-1} |\chi_n'\|_{L^\infty([0,L])} \lesssim \tau$. By Theorem 4, Lemma 4.1 and norm equivalence, $I_1$ can be estimated as follows

$$|I_1| \lesssim \int_0^L \left[ \left| u_{h,i}^{n-1} (X_{\tau}^{n-1,n} \circ \chi_n) \right| - \left| u_{h,i}^{n-1} (X_{\tau}^{n-1} \circ \chi_n') \right| \right] \|q_h\|_{0,\Omega^m_n} + \tau \int_0^L \left[ \left| u_{h,i}^{n-1} \cdot n_{n-1} \right| \circ \chi_{n-1} \right] \|q_h\|_{0,\Omega^m_n} \lesssim \tau^{k+2}h^{-\frac{1}{2}} \sum_{i=1,2} |u_{h,i}^{n-1}|_{H^1(\Omega^m_{n,i})} \|q_h\|_{L^2(\Omega^m_{n,i})} + \tau \|u_{h,i}^{n-1}\|_{L^2(\Omega^m_{n,i})} \|q_h\|_{L^2(\Omega^m_{n,i})} \lesssim h \|u_{h,i}^{n-1}\|_\mathcal{V} \|q_h\|_{\mathcal{Q}}.$$  

Similarly, by Theorem 4 and inverse estimate, $I_2$ is estimated as follows

$$|I_2| \lesssim \int_0^L \left[ \left| u_{h,i}^{n-1} \cdot n_{n-1} \right| \circ \chi_{n-1} \right] \|q_h\|_{0,\Omega^m_n} \lesssim \tau^{k+2}h^{-3/2} \|u_{h,i}^{n-1}\|_{L^2(\Omega^m_{n,i})} \|q_{h,i}\|_{L^2(\Omega^m_{n,i})} \lesssim h^{k+1} \|u_{h,i}^{n-1}\|_\mathcal{V} \|q_h\|_{\mathcal{Q}}.$$  

Finally, from (5.12) and the interpolation error estimate (5.7), we have

$$|I_3| \lesssim \left( \left\| u_{h,i}^{n-1} \right\|_{L^2(\Omega^m_{n,i})} \right) \|q_h \circ X_{\tau}^{n-1,n} - \hat{q}_h \|_{L^2(\Omega^m_{n,i})} \lesssim h \|u_{h,i}^{n-1}\|_\mathcal{V} \|q_h\|_{1,\mathcal{Q}}.$$  

The proof is finished.
Corollary 2 Let Assumption[4] Assumption[7] and the assumptions in Lemma[5.2] be satisfied. Suppose \( k \geq 2 \) and \( h = O(\tau) \). Then \((U^{n-1}_{0}, P^{n-1}_{0}, \hat{U}^{n-1}_{0}, \hat{P}^{n-1}_{0}) = S^n(U^{n-1}_{h}, \hat{U}^{n-1}_{h}, 0, 0)\) satisfies

\[
\left\| \left( \begin{array}{c} U^{n-1}_{h} \\ \hat{U}^{n-1}_{h} 
\end{array} \right) \right\|_{\mathcal{V}, \mathcal{Q}} \lesssim \left\| \left( \begin{array}{c} U^{n-1}_{h} \\ \hat{U}^{n-1}_{h} 
\end{array} \right) \right\|_{\mathcal{V}}.
\]

(5.36)

\[
\left\| f^{n-1}_{h} - f^{n-1}_{h} \right\|_{0, \Omega_{h}^n} \lesssim \gamma_{1} \nu_{2} h \left\{ \nu_{2}^{1/2} \left\| A^{k} U^{n-1}_{h} - \tau f^{n-1}_{h} \right\|_{0, \Omega_{h}^n} \right. \\
+ \left. h \| \nu^{-1/2} \nabla P^{n-1}_{h} \|_{0, \Omega_{h}^n} \right\} \\
+ h \left\{ \left\| u^{n-1}_{h} \right\|_{\mathcal{V}} + J^{n-1}_{p} \left( P^{n-1}_{h}, P^{n-1}_{h} \right) \right\}.
\]

(5.37)

Proof The stability (5.36) comes directly from Theorem[9] and Remark[4]. It is left to prove (5.37). From Theorem[11] Remark[4] and inequality (5.36), we have

\[
\left\| U^{n-1}_{h} - \hat{U}^{n-1}_{h} \right\|_{0, \Omega_{h}^n} \lesssim h \left\| U^{n-1}_{h} - \hat{U}^{n-1}_{h} \right\|_{\mathcal{V}} + \left\| J_{0}^{n-1} U^{n-1}_{h} \right\|_{1, (\Omega_{h}^n)^{\prime}} \\
\lesssim h \left\| \left( \begin{array}{c} u^{n-1}_{h} \\ \hat{u}^{n-1}_{h} 
\end{array} \right) \right\|_{\mathcal{V}} + \left\| J_{0}^{n-1} U^{n-1}_{h} \right\|_{1, (\Omega_{h}^n)^{\prime}}.
\]

(5.38)

For any \( 0 \neq q_{h} \in \mathcal{Q}_{h}^{n} \), we extend \( q_{h,i} \) to \( \Omega \) according to the convention in (3.5) and let \( \hat{q}_{h,i} = \pi_{1, \Omega} \left( q_{h,i} \circ X^{-n-1}_{h} \right) \in V_{h}(1, T_{h}) \) be the Scott-Zhang interpolation for \( i = 1, 2 \). Setting \( \hat{q}_{h} = \left( \hat{q}_{h,1}, \hat{q}_{h,2} \right) \) and using Lemma[5.4] we find that

\[
\left| \left( \begin{array}{c} B_{0}^{0} U^{n-1}_{h} \end{array} \right) (q_{h}) \right| \leq \left| B_{0}^{0} \left( u^{n-1}_{h}, \hat{q}_{h} \right) \right| + C h \left\| u^{n-1}_{h} \right\|_{\mathcal{V}} \left\| q_{h} \right\|_{1, \mathcal{Q}}.
\]

(5.39)

It suffices to estimate \( B_{0}^{0} \left( u^{n-1}_{h}, \hat{q}_{h} \right) \).

From equation (3.7b), we have

\[
B_{0}^{0} \left( u^{n-1}_{h}, \hat{q}_{h} \right) = J_{p}^{n-1} \left( p^{n-1}_{h}, \hat{q}_{h} \right) + J_{h}^{n-1} \left( U^{n-1}_{h}, \hat{P}^{n-1}_{h}, \hat{q}_{h} \right).
\]

By norm equivalence and (5.7), the first term on the right-hand side satisfies

\[
\left| J_{p}^{n-1} \left( p^{n-1}_{h}, \hat{q}_{h} \right) \right| \lesssim h \left\| J_{p}^{n-1} \left( p^{n-1}_{h}, p^{n-1}_{h} \right) \right\|^{1/2} \sum_{i=1,2} \nu_{i}^{-1/2} \left\| q_{h,i} \right\|_{H^{1}(\Omega_{h,i})},
\]

(5.37)

By inverse estimates and (5.7), the second term satisfies

\[
\left| B_{h}^{n-1} \left( U^{n-1}_{h}, \hat{P}^{n-1}_{h}, \hat{q}_{h} \right) \right| \lesssim \gamma_{1} \nu_{2} h \left\{ \nu_{2}^{1/2} \left\| A^{k} U^{n-1}_{h} - \tau f^{n-1}_{h} \right\|_{0, \Omega_{h}^n} + \left\| u^{n-1}_{h} \right\|_{\mathcal{V}} \\
+ h \| \nu^{-1/2} \nabla P^{n-1}_{h} \|_{0, \Omega_{h}^n} \right\} \left\| q_{h} \right\|_{1, \mathcal{Q}}.
\]

Together with (5.39) and (5.2), they yield the desired result

\[
\left| B_{0}^{0} U^{n-1}_{h} \right|_{1, (\Omega_{h}^n)^{\prime}} \lesssim \gamma_{1} \nu_{2} h \left\{ \nu_{2}^{1/2} \left\| A^{k} U^{n-1}_{h} - \tau f^{n-1}_{h} \right\|_{0, \Omega_{h}^n} \\
+ h \| \nu^{-1/2} \nabla P^{n-1}_{h} \|_{0, \Omega_{h}^n} \right\} \left\| q_{h} \right\|_{1, \mathcal{Q}}.
\]

The proof is finished.
6 Stability of numerical solutions

The purpose of this section is to prove the stability of numerical solutions. It also paves the way to a priori finite element error estimates. First we cite the telescope formulas for BDF schemes from [30, Section 2 and Appendix A].

Lemma 6.1 Suppose \(1 \leq k \leq 4\) and let \(a_0 = 1\) and \(a_1 = \delta_{k,3} + \delta_{k,4}\), where \(\delta_{ij}\) is the Kronecker delta. Define \(\Lambda^n_h = u^n_h\) and

\[
\Psi_l^n(U^n_h) = \sum_{j=1}^l c_{l,j}^k U_h^{n+1-j,n}, \quad \Phi_l^n(U^n_h) = \sum_{j=1}^l c_{l,j}^k U_h^{n-j,n},
\]

where the parameters \(c_{l,j}^k, 1 \leq j \leq l \leq k+1\), are given in [30, Table 2.2]. Then

\[
\left(A^n_h\right)_{l=0}^{k+1} a_l A^n_l(U^n_h) = \sum_{l=1}^{k+1} \left[\Psi_l^n(U^n_h)\right]^2 - \sum_{l=1}^k \left[\Phi_l^n(U^n_h)\right]^2.
\]

Lemma 6.2 Assume that the penalty parameter \(\gamma_0\) in \(\mathcal{J}_0^n\) is large enough. Let \(p_h^n\) be the discrete pressure at \(t_n\). Then

\[
\|p_h^n\|_Q^2 \lesssim \mathcal{J}_p^n(p_h^n, p_h^n) + \frac{1}{\nu_2^2} \sum_{j=0}^k \left\{ \|u_h^{n-j}\|_{0, \Omega^n_{n-j}}^2 + h^{k+3} \nu_2^{-1} \|u_h^{n-j}\|_{V}^2 \right\}
+ \|u_h^n\|_V^2 + \nu^{-\frac{1}{2}} \mathcal{J}_t^n(u_h^n).
\]

Proof By Lemma 4.3 there is a \(v_h \equiv (v_h|_{\Omega_h}, v_h|_{\Omega_{h-2}})\), \(v_h \in V_{0,0}(k, \Omega)\), satisfies

\[
\|\nu^{-\frac{1}{2}} v_h\|_{0, \Omega^n_0} \leq \|v_h\|_V^2 \leq \|p_h^n\|_Q^2 \leq \langle \text{div } v_h, p_h^n \rangle_{\Omega^n_0} + \mathcal{J}_p^n(p_h^n, p_h^n),
\]

where we have used Poincaré’s inequality in the first inequality due to \(v_h \in H_0^1(\Omega)\). From (5.32) and (4.6), we have

\[
\|A^n_h\|_{h, 0, \Omega^n_0}^2 \lesssim \sum_{j=0}^k \left\{ \|u_h^{n-j}\|_{0, \Omega^n_{n-j}}^2 + \nu_2^{-1} h^{k+3} \mathcal{J}_u^n(u_h^{n-j}, u_h^{n-j}) \right\}.
\]

Using (3.7a) and (4.5), we find that

\[
\langle \text{div } v_h, p_h^n \rangle_{\Omega^n_0} = \tau^{-1} \langle A^n_h v_h, p_h^n \rangle_{\Omega^n_0} + \mathcal{J}_u^n(u_h^n, v_h)
+ \int_{\Gamma^n_0} [v_h] \cdot n \left\{ p_h^n \right\} - \langle f^n, v_h \rangle_{\Omega^n_0}
\lesssim \tau^{-1} \|\nu^{-\frac{1}{2}} v_h\|_{0, \Omega^n_0} \sum_{j=0}^k \nu^{-\frac{1}{2}} u_h^{n-j} \|u_h^{n-j}\|_{0, \Omega^n_{n-j}}.
\]
\[ + \sum_{j=0}^{k} \nu_2^{-1} h^{(k+3)/2} f^n(u_{n-j}^n, u_{h}^{n-j})^{1/2} \]
\[ + \left( \|u_h^n\|_{V} + \gamma_0^{-1} \|p_h^n\|_{Q} \right) \|v_h\|_{V} \]
\[ + \|\nu^{1/2} v_h\|_{0, \Omega_h^n} \|\nu^{-1/2} f^n\|_{0, \Omega_h^n}. \]

The proof is finished by using (6.1) and assuming that \(\gamma_0\) is large enough.

**Theorem 12** Suppose \(2 \leq k \leq 4\), \(\tau \leq h = O(\tau)\), and \(\gamma_0, \gamma_1^{-1}\) are large enough. Let \(u_h^n\) be the solution to the discrete problem (3.7). There is an \(h_0 > 0\) small enough such that, for any \(h \in (0, h_0)\) and \(m \geq k\),

\[
\|u_h^m\|_{0, \Omega_h^n}^2 + \sum_{n=k}^{m} \tau \left( \|u_h^n\|_{V}^2 + \gamma_1 \nu_2 h^2 \|\nu p_h^n\|_{1, Q}^2 \right)
\leq \sum_{n=0}^{m} \tau \|f^n\|_{L^2(\Omega_h^n)}^2 + \sum_{n=0}^{k-1} \left( \|u_h^n\|_{0, \Omega_h^n}^2 + \tau \|u_h^n\|_{V}^2 \right). 
\]

**Proof** Without loss of generality, we only prove the lemma for \(k = 4\). The proofs for \(k < 4\) are similar. The discrete problem can be written as follows

\[ \langle A^4 U_h^n, \psi_h \rangle_{\Omega_h^n} + \tau \mathcal{K}_0(u_h^n, p_h^n, (\psi_h, q_h)) \]
\[ + \gamma_1 \nu_2 h^2 \mathcal{B}_1(u_h^n, q_h) + \gamma_1 \nu_2 h^2 \langle \nu \nabla p_h^n, \nabla q_h \rangle_{\Omega_h^n} \]
\[ = \tau \langle f^n, \psi_h \rangle_{\Omega_h^n} - \gamma_1 \nu_2 h^2 \sum_{i=1,2} \nu_i^{-1} \left( f_i^n - \sum_{j=1}^{k} \lambda_j^{i} U_h^{n-j,i} \nabla q_{h,i} \right)_{\Omega_h^n}, \]  \(6.3\)

For \(k = 4\), the coefficients \(a_0 = a_1 = 1\) in Lemma 6.1. From [30], the coefficients \(\lambda_0, \ldots, \lambda_k\) for the BDF-4 scheme also imply that \(A^0 U_h^n = u_h^n\) and \(A^1 U_h^n = u_h^n - U_h^{n-1,n}\). Then

\[
\sum_{l=0,1} a_l A^l U_h^n = 2u_h^n - U_h^{n-1,n}. 
\]

Define \((\hat{U}_h^{n-1,n}, p_h^{n-1,n}) = S^n((U_h^{n-1,n}, 0), 0)\), take \((\psi_h, q_h) = (2u_h^n - \hat{U}_h^{n-1,n}, 0)\) in (6.3). Using Lemma 6.1 we have

\[
\sum_{l=1}^{5} \|\psi_l(U_h^n)\|_{0, \Omega_h^n}^2 - \sum_{l=1}^{4} \|\Phi_l(U_h^n)\|_{0, \Omega_h^n}^2 + \tau \mathcal{K}_0((u_h^n, p_h^n), (\psi_h, 0)) = \tau(A_1 + A_2), 
\]

\(6.4\)
where $A_1 = \langle f^n, v_h \rangle_{\Omega_n^0}$ and $A_2 = \tau^{-1} \langle A^4 y_h, \bar{U}_{h}^{n-1,n} - \bar{U}_h^{n-1,0} \rangle_{\Omega_n^0}$. We are going to estimate each term in $[6.4]$ separately.

By Corollary 2 and inequality $[6.2]$, we have

$$ |A_1| \leq \| f^n \|_{0,\Omega_n^0} \left\{ 2 \| u_h^{n-1} \|_{0,\Omega_n^0} + \| U_h^{n-1,n} \|_{0,\Omega_n^0} + \| U_h^{n-1,n} - \bar{U}_h^{n-1,n} \|_{0,\Omega_n^0} \right\} $$

$$ \leq C \| f^n \|_{0,\Omega_n^0} \left\{ \| u_h^{n-1} \|_{0,\Omega_n^0} + \| u_h^{n-1} \|_{0,\Omega_n^0} + h \| u_h^{n-1} \|_{\mathbf{V}} + \gamma_1 \nu_2 h \| A^4 y_h^{n-1} - \tau f^n \|_{0,\Omega_n^0} + \gamma_1 \nu_2 h^2 \| \nabla p_h^{n-1} \|_{0,\Omega_n^0} + h \gamma_1 \nu_2 (p_h^{n-1}, p_h^{n-1})^{1/2} \right\} $$

$$ \leq C \left( 1 + \gamma_1 \nu_2^2 h \right) \sum_{j=0}^5 \left\{ \| u_h^{n-j} \|_{0,\Omega_n^0}^2 + \nu_2^{-1} h^4 \| u_h^{n-j} \|_{\mathbf{V}}^2 + \| f^{n-j} \|_{0,\Omega_n^0}^2 \right\} $$

$$ + h^2 \| u_h^{n-1} \|_{\mathbf{V}}^2 + (\gamma_1 \nu_2) h^2 \| \nabla p_h^{n-1} \|_{0,\Omega_n^0}^2 + \nu_2^{-1} h^4 \| \nabla p_h^{n-1} \|_{0,\Omega_n^0}^2 + h^2 \gamma_1 \nu_2 (p_h^{n-1}, p_h^{n-1}). $$

For a parameter $\varepsilon \in (0, 1)$ to be specified later, $A_2$ can be estimated similarly as follows

$$ |A_2| \leq C \| A^4 y_h^{n-1} \|_{0,\Omega_n^0} \left\{ \gamma_1 \nu_2 \| A^4 y_h^{n-1} - \tau f^n \|_{0,\Omega_n^0} + \| u_h^{n-1} \|_{\mathbf{V}} \right\} $$

$$ + \gamma_1 \nu_2 h \| \nabla p_h^{n-1} \|_{0,\Omega_n^0} + \gamma_1 \nu_2 (p_h^{n-1}, p_h^{n-1})^{1/2} \right\} $$

$$ \leq \gamma_1 \nu_2 h^2 \| f^n \|_{0,\Omega_n^0}^2 + C \gamma_1 \nu_2 \sum_{j=0}^5 \left\{ \| u_h^{n-j} \|_{0,\Omega_n^0}^2 + \frac{h^4}{\nu_2} \| u_h^{n-j} \|_{\mathbf{V}}^2 \right\} $$

$$ + \varepsilon \left\{ \| u_h^{n-1} \|_{\mathbf{V}}^2 + (\gamma_1 \nu_2) h^2 \| \nabla p_h^{n-1} \|_{0,\Omega_n^0}^2 + \gamma_1 \nu_2 (p_h^{n-1}, p_h^{n-1}) \right\}. \tag{6.6} $$

Note that $\Phi_t^j(U_h^{n-1}) = \Psi_t^j(U_h^{n-1}) \circ X_t^{n-1}$. By $[5.24]$ and $[5.32]$, we have

$$ \| \Phi_t^j(U_h^{n-1}) \|_{0,\Omega_n^0}^2 \leq \| \Psi_t^j(U_h^{n-1}) \|_{0,\Omega_n^0}^2 + C \gamma \sum_{j=1}^k \left\{ \| u_h^{n-j} \|_{0,\Omega_n^0}^2 + \frac{h^4}{\nu_2} \| u_h^{n-j} \|_{\mathbf{V}}^2 \right\}. \tag{6.7} $$

Now it is left to estimate $\mathcal{E}_0^n((u_h^n, p_h^n), (w_h, 0))$. Since $\mathcal{E}_h^n$ is symmetric, from $[5.1]$ and $[3.7b]$, we know that

$$ \mathcal{E}_0^n((u_h^n, p_h^n), (w_h, 0)) = 2\mathcal{E}_h^n(u_h^n, u_h^n) + 2\mathcal{E}_0^n(u_h^n, p_h^n) - \mathcal{E}_h^n(U_h^{n-1,n}, u_h^n) $$

$$ - \mathcal{E}_0^n(U_h^{n-1,n}, p_h^n) $$

$$ - 2\mathcal{E}_h^n(u_h^n, u_h^n) + 2\mathcal{E}_p^n(p_h^n, p_h^n) + 2\mathcal{E}_h^n(U_h^n, p_h^n; p_h^n) $$

$$ - a_h^n(U_h^{n-1,n}, u_h^n) + \mathcal{B}_h^n(U_h^n, p_h^n; p_h^{n-1,n}). \tag{6.8} $$
By Lemma 4.2, 5.32, and 6.2, we deduce that

\[
\mathcal{R}_h^n(\mathbf{u}_h^n, p^n_h; p^n_h) = \gamma_1 \nu_2 h^2 \langle \nu^{-1} A^4 U_h^n - \nu \Delta_h u^n_h + \nabla p_h^n - f^n, \nu^{-1} \nabla p_h^n \rangle_{\Omega_n}
\]
\[
\geq \frac{3}{4} \gamma_1 \nu_2 h^2 \|\nu^{-\frac{1}{2}} \nabla p_h^n\|^2 - C \gamma_1 h^2 \|f^n\|^2 - C \gamma_1 \nu_2 \|u^n_h\|^2
\]
\[
- C \gamma_1 \sum_{j=0}^4 \left( \|u_h^{n-j}\|^2_{0, \Omega_n} + \nu_2 h^4 \|u_h^{n-j}\|^2_{\mathcal{V}} \right).
\]

(6.9)

Similarly, using inverse estimates, 5.36, and 6.2, we have

\[
\mathcal{R}_h^n(\mathbf{u}_h^n, p^n_h; p^n_h) = \gamma_1 \nu_2 h^2 \langle \frac{1}{r} A^4 U_h^n - \nu \Delta_h u^n_h + \nabla p_h^n - f^n, \frac{1}{r} \nabla p_h^{n-1,n} \rangle_{\Omega_n}
\]
\[
\leq C \gamma_1 \|p_h^{n-1,n}\|_\mathcal{Q} \left( \|A^4 U_h^n\|_{0, \Omega_n} + h \|f^n\|_{0, \Omega_n} \right)
\]
\[
+ C \gamma_1 \nu_2 \|p_h^{n-1,n}\|_\mathcal{Q} \left( \|u_h^n\|_\mathcal{V} + h \|\nabla p_h^{n-1,n}\|_{0, \Omega_n} \right)
\]
\[
\leq C \gamma_1 \sum_{j=0}^4 \left( \|u_h^{n-j}\|^2_{0, \Omega_n} + \nu_2 h^4 \|u_h^{n-j}\|^2_{\mathcal{V}} \right) + C \gamma_1 \nu_2 \|u^n_h\|^2
\]
\[
+ C \gamma_1 h^2 \|f^n\|^2_{0, \Omega_n} + \frac{1}{2} \gamma_1 \nu_2 h^2 \|\nu^{-\frac{1}{2}} \nabla p_h^n\|^2_{0, \Omega_n}.
\]

(6.10)

By 4.4 and 5.32–5.34, we have

\[
\left| a_h^n(\mathbf{u}_h^{n-1,n}, u^n_h) \right| \leq \|\nu^{-\frac{1}{2}} \nabla u_h^n\|_{0, \Omega_n} \left\{ \|\nu^{-\frac{1}{2}} \nabla u_h^{n-1}\|_{0, \Omega_n} + C h \|u_h^{n-1}\|_{\mathcal{V}} \right\}
\]
\[
+ C \gamma_0^{-\frac{1}{2}} \left\{ \gamma_{0}^{-\frac{1}{2}} \gamma_{0}^{-\frac{1}{2}} \right\} \|u_h^{n-1}\|_{\mathcal{V}}
\]
\[
+ C \|u_h^n\|_{\mathcal{V}} \left\{ \gamma_{0}^{-\frac{1}{2}} \gamma_{0}^{-\frac{1}{2}} \right\} \|u_h^{n-1}\|_{\mathcal{V}}
\]
\[
+ (1 + C \tau) \left\{ \left( \gamma_{0}^{-\frac{1}{2}} \gamma_{0}^{-\frac{1}{2}} \right) \|u_h^{n-1}\|_{\mathcal{V}} + C \sqrt{h \gamma_{0}} \|u_h^{n-1}\|_{\mathcal{V}} \right\}
\]
\[
\leq \left\{ 0.5 + \varepsilon + C \gamma_0 \right\} \left\{ \|u_h^n\|_{\mathcal{V}}^2 + \|u_h^{n-1}\|_{\mathcal{V}}^2 \right\}.
\]

Choosing \( \gamma_0 \) large enough such that \( 2C \gamma_0^{-1} \leq \varepsilon^2 \) and choosing \( h \) small enough such that \( 2C(1 + \gamma_0) h \leq \varepsilon^2 \), we find that

\[
\left| a_h^n(\mathbf{u}_h^{n-1,n}, u^n_h) \right| \leq \left\{ 0.5 + 2 \varepsilon \right\} \left( \|u_h^n\|_{\mathcal{V}}^2 + \|u_h^{n-1}\|_{\mathcal{V}}^2 \right).
\]

(6.11)

Now assume that \( h \) and \( \gamma_1 \) are small enough such that \( \nu_2 h^4 \leq \varepsilon^2 \ll 1 \) and \( \gamma_1 \leq 1 \). Then inserting (6.9), (6.11) into (6.8) and using Lemma 4.4 and Lemma 6.1, we get

\[
\chi_0^n((\mathbf{u}_h^n, p^n_h), (\mathbf{v}_h, 0)) \geq (1.3 - 2 \varepsilon - C \gamma_1 \varepsilon_2) \|u_h^n\|_{\mathcal{V}}^2 - (0.5 + 2 \varepsilon) \|u_h^{n-1}\|_{\mathcal{V}}^2
\]
\[
- C \left\{ \sum_{j=0}^4 \|u_h^{n-j}\|_{0, \Omega_n}^2 + \varepsilon \|u_h^{n-j}\|_{\mathcal{V}}^2 + h^2 \|f^n\|^2_{0, \Omega_n} \right\}.
\]
+ 2 \mathcal{J}_p^n(p_n^h, p_n^h) + \gamma_1 \nu_2 h^2 \| \nu^{-\frac{1}{2}} \nabla p_n^h \|^2_{0, \Omega_n^2}. \quad (6.12)

By norm equivalence, we have
\[ \| \nu^{\frac{1}{2}} \nabla p_n^h \|^2_{0, \Omega_n^2} \geq \| p_n^h \|^2_{1, Q} - \| p_n^h \|^2_\mathcal{Q} - C h^{-2} \mathcal{J}_p^n(p_n^h, p_n^h). \quad (6.13) \]

Combining the above inequality with Lemma 6.2 leads to
\[ \mathcal{X}_0^n((u_n^h, p_n^h), (u_0, \Omega)) \geq (1.3 - 2 \varepsilon - C \gamma_1 \nu_2) \| u_n^h \|^2_{\mathcal{L}} - (0.5 + 2 \varepsilon) \| u_n^{h-1} \|^2_{\mathcal{L}} \]
\[ - C \{ \sum_{j=0}^4 (\| u_n^{h-j} \|^2_{0, \Omega_n^2} + \varepsilon \| u_n^{h-j} \|^2_{\mathcal{L}} + h^2 \| f_n \|^2_{0, \Omega_n^2} \} \]
\[ + \gamma_1 \nu_2 h^2 \| p_n^h \|^2_{1, Q} + (2 - C \gamma_1 \nu_2) \mathcal{J}_p^n(p_n^h, p_n^h). \quad (6.14) \]

Now substitute (6.5), (6.7), and (6.8) into (6.4) and take the sum of both sides over \( n = 4, \ldots, m \). After proper arrangements, we arrive at
\[ \sum_{l=1}^4 \| \mathcal{L}_l(\bar{U}_n^m) \|^2_{0, \Omega_n^2} + (0.8 - C \varepsilon - C \gamma_1 \nu_2 - C h^2) \sum_{n=4}^m \tau \| u_n^h \|^2_{\mathcal{L}} \]
\[ + (1 - \gamma_1 \nu_2 h^2 - \varepsilon \gamma_1 \nu_2) \gamma_1 \nu_2 h^2 \sum_{n=4}^m \tau \| p_n^h \|^2_{1, Q} \]
\[ + (2 - \varepsilon - C \gamma_1 \nu_2) \sum_{n=4}^m \tau \mathcal{J}_p^n(p_n^h, p_n^h) \]
\[ \leq C \sum_{n=4}^m \tau \| u_n^h \|^2_{0, \Omega_n^2} + C \sum_{n=0}^m \tau \| f_n \|^2_{0, \Omega_n} + C \sum_{n=0}^m \left( \| u_n^h \|^2_{0, \Omega_n^2} + \tau \| u_n^h \|^2_{\mathcal{L}} \right) \]

From [30, Table 2.2], we have \( \mathcal{L}_l(\bar{U}_n^m) = 0.06 u_n^m \). In the above inequality, we let \( \varepsilon, \gamma_1 \), and \( h \) to be small enough such that
\[ C(\varepsilon + \gamma_1 \nu_2) < 0.3, \quad \varepsilon \gamma_1 \nu_2 < 0.5, \quad Ch^2 < 0.1. \]

It results in
\[ \| u_n^h \|^2_{0, \Omega_n^2} + \sum_{n=4}^m \tau \| u_n^h \|^2_{\mathcal{L}} + \gamma_1 \nu_2 h^2 \sum_{n=4}^m \tau \| p_n^h \|^2_{1, Q} \]
\[ \leq C \sum_{n=4}^m \tau \| u_n^h \|^2_{0, \Omega_n^2} + C \sum_{n=0}^m \tau \| f_n \|^2_{0, \Omega_n} + C \sum_{n=0}^m \left( \| u_n^h \|^2_{0, \Omega_n^2} + \tau \| u_n^h \|^2_{\mathcal{L}} \right). \]

The proof is finished by using Gronwall’s inequality.

### 7 Finite element error estimates

The purpose of this section is to prove the error estimates between the exact solution and the finite element solution. Throughout this section we assume the the index \( r \) in Assumption 2 satisfies \( r \geq \max(4, k + 2) \).
7.1 Extension operator

First we write \( \Omega(t) = \Omega_1(t) \cup \Omega_2(t) \) and define

\[
Q_T = \{(x, t) : x \in \Omega(t), \; t \in [0, T]\},
\]

\[
Q_i,T = \{(x, t) : x \in \Omega_i(t), \; t \in [0, T]\}, \quad i = 1, 2.
\]

Given an integer \( 0 \leq m \leq k + 1 \), we define the Bochner space

\[
L^\infty(0, T; H^m(Q_i(t))) = \left\{ v_i \in L^2(Q_i,T) : \sup_{t \in [0, T]} \|v_i(X(t; 0, \cdot), t)\|_{H^m(Q_i(0))} < \infty \right\}.
\]

For any Lipschitz domain \( D \subset \Omega \), by [38, Chapter 6], there is an extension operator \( E_D : H^{k+1}(D) \to H^{k+1}(\mathbb{R}^2) \) depending only on \( D \) and \( k \) such that

\[
(E_Dw)|_D = w, \quad \|E_Dw\|_{H^{k+1}(\mathbb{R}^2)} \lesssim \|w\|_{H^{k+1}(D)} \quad \forall \; w \in H^{k+1}(D).
\]

Since \( X(t; 0, \cdot) : \Omega_i(t) \to \Omega_i(t) \) is one-to-one, we have \( w \circ X(t; 0, \cdot) \in H^{k+1}(\Omega_i(0)) \) for any \( w \in H^{k+1}(\Omega_i(t)) \). Let the extension operator from \( H^{k+1}(\Omega_i(t)) \) to \( H^{k+1}(\mathbb{R}^2) \) be defined as

\[
E^i_t w := \left[ E^i_0(w \circ X(t; 0, \cdot)) \right] \circ X(0; t, \cdot), \quad \text{where} \quad E^i_0 := E_{\Omega_i(0)}.
\]

The global extension operator \( E^i : L^\infty(0, T; H^{k+1}(\Omega_i(t))) \to L^\infty(0, T; H^{k+1}(\mathbb{R}^2)) \) is defined as

\[
\forall \; v \in L^\infty(0, T; H^{k+1}(\Omega_i(t))), \quad (E^i v)(\cdot, t) = E^i_t v(\cdot, t) \forall \; t \in [0, T]. \quad (7.1)
\]

Following [32, Lemma 6.1] and arguments similar to [23], we have the following lemma.

**Lemma 7.1** There exists a constant \( C > 0 \) depending only on \( \Omega_i(0) \) and the exact flow map \( X(t; 0, \Omega_0) \) such that, for any \( v \in L^\infty(0, T; H^{k+1}(\Omega_i(t))) \cap H^{k+1}(Q_i,T) \),

\[
\|E^i v\|_{H^{k+1}(\mathbb{R}^2 \times (0, T))} \leq C \|v\|_{H^{k+1}(Q_i,T)},
\]

\[
\|E^i v\|_{L^\infty(0, T; H^m(\mathbb{R}^2))} \leq C \|v\|_{L^\infty(0, T; H^m(\Omega_i(t)))}, \quad 1 \leq m \leq k + 1.
\]

Furthermore, for \( v \in L^\infty(0, T; H^m(\Omega_i(t))) \) and \( \partial_t v \in L^\infty(0, T; H^{m-1}(\Omega_i(t))) \), for any \( t \in [0, T] \), it holds

\[
\|\partial_t (E^i v)\|_{H^{m-1}(\mathbb{R}^2)} \leq C (\|v\|_{H^m(\Omega_i(t))} + \|\partial_t v\|_{H^{m-1}(\Omega_i(t))}), \quad 1 \leq m \leq k + 1.
\]
7.2 The extended solution

Let \( \mathbf{u} = (u_1, u_2), \ p = (p_1, p_2) \) be the true solution to (1.1) and assume

\[
\begin{align*}
\mathbf{u}_i \in L^\infty(0, T; H^{k+1}(\Omega_i(t))) \cap H^{k+1}(Q_i, T), & \quad \text{div } v_i = 0, \\
p_i \in L^\infty(0, T; \mathcal{H}^k(\Omega_i(t))) \cap \mathcal{H}^k(Q_i, T), & \quad (p_1/\nu_1, 1)_{\Omega_i(t)} + (p_2/\nu_2, 1)_{\Omega_i(t)} = 0.
\end{align*}
\]

Let \( \tilde{\mathbf{u}}_i = \mathcal{E}' u_i, \ \tilde{p}_i = \mathcal{E}' p_i \) be the extensions of \( u_i \) and \( p_i \), respectively. Define

\[
\dot{\mathbf{f}}_i := \frac{d\tilde{\mathbf{u}}_i}{dt} - \nu_i \Delta \tilde{\mathbf{u}}_i + \nabla \tilde{p}_i, \quad \frac{d\tilde{\mathbf{u}}_i}{dt} := \frac{\partial \tilde{\mathbf{u}}_i}{\partial t} + (\mathbf{w} \cdot \nabla)\tilde{\mathbf{u}}_i.
\]

It is easy to see that \( \dot{\mathbf{f}}_i = \mathbf{f}_i \) in \( \Omega_i(t), \ i = 1, 2 \). From Lemma 7.1, we have

\[
\begin{align*}
\|\tilde{\mathbf{u}}_i\|_{L^\infty(0, T; H^{k+1}(\Omega_i(t)))} & \lesssim \|u_i\|_{L^\infty(0, T; H^{k+1}(\Omega_i(t)))}, \\
\|\tilde{p}_i\|_{L^\infty(0, T; \mathcal{H}^k(\Omega_i(t)))} & \lesssim \|u_i\|_{L^\infty(0, T; \mathcal{H}^k(\Omega_i(t)))}.
\end{align*}
\]

For convenience, we introduce some symbols for quantities at discrete time steps

\[
\begin{align*}
\mathbf{u}^n_i &= (\tilde{\mathbf{u}}^n_1, \tilde{\mathbf{u}}^n_2), \quad \mathbf{p}^n_i = (\tilde{\mathbf{p}}^n_1, \tilde{\mathbf{p}}^n_2), \quad \tilde{\mathbf{u}}^n_i = \tilde{\mathbf{u}}_i(\cdot, t_n), \quad \tilde{p}^n_i = \tilde{p}_i(\cdot, t_n), \quad \dot{\mathbf{f}}^n_i = \dot{\mathbf{f}}_i(\cdot, t_n).
\end{align*}
\]

Define \( \tilde{\mathbf{U}}^{n,m}_i := \tilde{\mathbf{u}}^{n,m}_i \circ \mathbf{X}^{n,m}_i \) and \( \tilde{\mathbf{W}}^{n}_i := [\tilde{\mathbf{U}}^{n-k,n}_i, \ldots, \tilde{\mathbf{U}}^{n,n}_i] \). Then the extended solutions \( \tilde{\mathbf{u}}^n_i \) and \( \tilde{\mathbf{p}}^n_i \) satisfy the semi-discrete equations

\[
\begin{align*}
\frac{1}{\tau} A^k \tilde{\mathbf{U}}^n_i - \nu_i \Delta \tilde{\mathbf{u}}^n_i + \nabla \tilde{p}^n_i = \dot{\mathbf{f}}^n_i & \quad \text{in } \Omega_i^n, \quad \text{(7.3)} \\
\text{div } \tilde{\mathbf{u}}^n_i = 0 & \quad \text{in } \Omega_i(t_n) \backslash \partial \Omega_i^n, \quad \text{(7.4)}
\end{align*}
\]

where

\[
\dot{\mathbf{f}}^n_i = \dot{\mathbf{f}}_i + \mathbf{R}^n_i, \quad \mathbf{R}^n_i = \frac{1}{\tau} A^k \tilde{\mathbf{U}}^n_i - \frac{d\tilde{\mathbf{u}}_i}{dt} \big|_{t=t_n}.
\]

However, the extended solution does not satisfy the continuity conditions on the approximate interface \( \Gamma^n_\eta \), namely,

\[
\nu_1 \partial_n \tilde{\mathbf{u}}^n_1 - \tilde{\mathbf{p}}^n_1 \mathbf{n} \neq \nu_2 \partial_n \tilde{\mathbf{u}}^n_2 - \tilde{\mathbf{p}}^n_2 \mathbf{n}, \quad \tilde{\mathbf{u}}^n_1 \neq \tilde{\mathbf{u}}^n_2 \quad \text{on } \Gamma^n_\eta.
\]

From (7.3)–(7.4), it is easy to see that the extended solution satisfies the discrete equations with modified right-hand sides

\[
\begin{align*}
\tau^{-1} \langle A^k \tilde{\mathbf{u}}^n_i, \mathbf{v}_h \rangle_{\Omega^n_i} + \mathcal{S}^n_k(\tilde{\mathbf{u}}^n_i, \mathbf{v}_h) + \mathcal{B}^n_0(\mathbf{v}_h, \tilde{\mathbf{p}}^n_i) \\
= (\dot{\mathbf{f}}^n_i, \mathbf{v}_h)_{\Omega^n_i} + \mathcal{F}^n(\mathbf{v}_h), & \quad \forall \mathbf{v}_h \in \mathcal{V}^n_h, \quad \text{(7.5a)} \\
\mathcal{B}^n_0(\tilde{\mathbf{u}}^n_i, q_h) - \mathcal{I}^n_p(\tilde{\mathbf{p}}^n_i, q_h) - \mathcal{E}^n_k(\tilde{\mathbf{U}}^n_i, \tilde{\mathbf{p}}^n_i; q_h) \\
= (\mathcal{B}^n_0 \tilde{\mathbf{u}}^n_i)(q_h) - \gamma_1 \nu_2 h^2 (\dot{\mathbf{f}}^n_i, \nu^{-1} \nabla q_h)_{\Omega^n_i}, & \quad \forall q_h \in \mathcal{Q}^n_h. \quad \text{(7.5b)}
\end{align*}
\]
where \( \mathbf{U} = (\hat{U}_1, \hat{U}_2) \), \( \mathbf{f} = (\hat{f}_1, \hat{f}_2) \), and
\[
\mathcal{F}(\nu_h) = \int_{\Omega} \left[ \nu \partial_n \mathbf{U} - \hat{p}_h \right] \cdot \nu_h \, d\Omega
+ \int_{\Omega} \left[ \mathbf{u}_h \cdot \nabla \nu_h \right] \cdot \nu_h \, d\Omega,
\]
where \( \nu \) is the projection coefficient and \( \nu_h \) is the \( h \)-projection.

### 7.3 Modified Stokes projections of the extended solution

Using (5.1) of the modified Stokes projection, we define the modified Stokes projection \( (\hat{u}_h^n, \hat{p}_h^n) = S^n(\mathbf{u}_h^n, \hat{p}_h^n, \mathbf{B}_h^\mathbf{u}_h^n) \). The approximation errors can be split into two parts
\[
\hat{u}_h^n - \mathbf{u}_h^n = \theta_h^n + \rho_h^n, \quad \hat{p}_h^n - \hat{p}_h^n = \rho_h^n + \rho_h^n.
\]
where the error functions are defined as
\[
\theta_h^n = \hat{u}_h^n - \mathbf{u}_h^n, \quad \rho_h^n = \hat{p}_h^n - \hat{p}_h^n, \quad \theta_h^n = \hat{u}_h^n - \mathbf{u}_h^n, \quad \rho_h^n = \hat{p}_h^n - \hat{p}_h^n.
\]

From (5.21), we know that
\[
\|\theta_h^n\|_{0, \Omega} + h\|\theta_h^n\|_V + h\|\rho_h^n\|_Q \lesssim h^{k+1} \tilde{M}_k^n, \quad (7.7)
\]
where by (7.2),
\[
\tilde{M}_k^n = \sum_{i=1,2} \nu_i |\hat{u}_i^n|_{H^{k+1}(\Omega)} + \nu_i^{-1} |\hat{p}_i^n|_{H^k(\Omega)}
\leq C \sum_{i=1,2} \left( \|\hat{u}_i^n\|_{L^\infty(0,T;H^{k+1}(\Omega_i))}^2 + \|\hat{p}_i^n\|_{L^\infty(0,T;H^k(\Omega_i))}^2 \right).
\]

Let \( \pi_{k-1}(\hat{p}_h^n) \) be the quasi-interpolation of \( \hat{p}_h^n \). Using (5.10), (7.7), and inverse estimate, we have
\[
\sum_{i=1}^2 \|\nu_i^{-\frac{1}{2}} \nabla \mathbf{u}_i^n\|_{0, \Omega_h^n}^2 \lesssim \sum_{i=1}^2 \|\nu_i^{-\frac{1}{2}} \nabla \left[ \pi_{k-1}(\hat{p}_i^n) - \hat{p}_i^n \right]\|_{0, \Omega_h^n}^2
+ \sum_{i=1}^2 \|\nu_i^{-\frac{1}{2}} \nabla \left[ \pi_{k-1}(\hat{p}_i^n) - \hat{p}_i^n \right]\|_{0, \Omega_h^n}^2
\lesssim h^{2k-2} \tilde{M}_k^n + h^2 \|\pi_{k-1}(\hat{p}_i^n) - \hat{p}_i^n\|_Q^2
\lesssim h^{2k-2} \tilde{M}_k^n + h^2 \|\theta_h^n\|_Q^2 + h^2 \|\pi_{k-1}(\hat{p}_i^n) - \hat{p}_i^n\|_Q^2
\lesssim h^{2k-2} \tilde{M}_k^n. \quad (7.8)
\]

It is left to estimate \( \theta_h^n \) and \( \rho_h^n \).
7.4 Error estimates for $\theta^n_h$ and $\rho^n_h$

For convenience in notation, we define

$$\Theta_{m,n} = \theta_{m,n} \circ X_{\tau,n}, \quad \Theta_{m} = \theta_{m,n} \circ X_{\tau,n}.$$ 

Subtracting (3.7) from (7.5) and using (5.1), we obtain the discrete formulation

$$\frac{1}{\tau} \langle A^k \Theta^n_0, \nu_h \rangle_{\Omega^n} + \mathcal{B}^n\Theta^n_h (\nu_h) + \mathcal{B}^n\Theta^n_h (\rho^n_h) = \mathcal{E}^1_1 (\nu_h), \quad \forall \nu_h \in \mathcal{V}^n_h \quad (7.9a)$$

$$\mathcal{B}^n_0 (\Theta^n_0, q_h) - \mathcal{J}^n_2 (\rho^n_h, q_h) - \mathcal{B}^n_h (\Theta^n_0, \rho^n_q; q_h) = \mathcal{E}^2_2 (q_h), \quad \forall q_h \in \mathcal{Q}^n_h, \quad (7.9b)$$

Clearly $\mathcal{E}^1_1$ provides a functional on $\mathcal{V}^n_h$ and $\mathcal{E}^2_2$ provides a functional on $\mathcal{Q}^n_h$. Their norms are defined as

$$\| \mathcal{E}^1_1 \|_{1, \mathcal{V}^n_h} = \sup_{\nu_h \in \mathcal{V}^n_h} \| \mathcal{E}^1_1 (\nu_h) \|_{\mathcal{V}}, \quad \| \mathcal{E}^2_2 \|_{1, \mathcal{Q}^n_h} = \sup_{q_h \in \mathcal{Q}^n_h} \| \mathcal{E}^2_2 (q_h) \|_{1, \mathcal{Q}}. \quad (7.12)$$

Lemma 7.2 Assume $k \geq 2$, $\eta = O(\tau^{\max(k/3,1)})$, $\gamma_0 h \leq 1$, and $h = O(\tau)$. Then

$$\| \mathcal{E}^1_1 \|_{1, \mathcal{V}^n_h} \lesssim \tau^k \sum_{i=1,2} \nu_i^{-\frac{1}{2}} \left( \tau \| \hat{f} \|_{H^1(\Omega)} + \sum_{j=0}^k \| \hat{u}^{n-j} \|_{H^{k+1}(\Omega)} \right) + \tau^{k-\frac{1}{2}} \sum_{i=1,2} \nu_i^{-\frac{1}{2}} \| \hat{u}_i \|_{H^{k+1}(\Omega \times (t_{n-k}, t_n))} \quad (7.13)$$

Proof Changing variables of integrations and using (2.10) and (5.21) yield

$$\frac{1}{\tau} \langle A^k \Theta^n_0, \nu_{\tau,n} \rangle_{0, \Omega^n} \leq \frac{1}{\tau} \sum_{j=0}^k \| (\det J^n_{\tau,j} \hat{u})^{n-j} \|_{0, \Omega^n_{\tau,j}} \lesssim \tau^k \sum_{j=0}^k \sum_{i=1}^2 \| \hat{u}_{i,j} \|_{k+1, \Omega}. \quad (7.14)$$
From (2.10) and the error estimates in (2.4), we have
\[ \|X_{r,n}^{j} - X_{m,n}^{j}\|_{L^\infty(\Omega)} \lesssim \tau^{k+2}. \]

Applying Taylor’s formula to the right-hand side of the following equality
\[ R^n_i = \frac{1}{\tau} A^k \tilde{u}_i^n - \frac{d\tilde{u}_i}{dt} + \frac{1}{\tau} \sum_{j=0}^k \lambda_j^k (\tilde{u}_i^{n-j} \circ X_{r,n}^{j} - \tilde{u}_i^{n-j} \circ X_{n,n}^{j}), \]
we have
\[
\|R^n_i\|_{L^2(\Omega_{n,i})} \lesssim \sum_{i=1}^k \int_{t_{n-i}}^{t_n} \frac{(t_n - \xi)^k d^{k+1}\xi}{\tau} \|\tilde{u}_i^{n-j}\|_{0,\Omega_{n,i}} + \tau^{k+1} \sum_{j=0}^k \|\tilde{u}_i^{n-j}\|_{H^1(\Omega)}. \tag{7.15}
\]

Write \( \hat{f}^n_i = (\hat{f}^n_1, \hat{f}^n_2) \), and note that \( \hat{f}^n_i = f_i^n \) in \( \Omega_i(t_n) \). Using Lemma 5.3 and norm equivalence, we deduce that
\[
\langle \hat{f}^n_i - f^n_i, \tilde{v}_h \rangle_{\Omega_{n,i}} = \sum_{i=1,2} (\hat{f}^n_i - f^n_{3-i}, \tilde{v}_h)_{\Omega_{n,i},\Omega_i(t_n)} 
\lesssim \tau^{k+1} \sum_{i=1,2} \|\nu_i^{-\frac{1}{2}} \hat{f}^n_i\|_{H^1(\Omega)} \|\tilde{v}_h\|_{\mathcal{V}}. \tag{7.16}
\]

Using \( \hat{f}^n_i = \tilde{f}^n_i + R^n_i \) and the triangular inequality, we obtain
\[
\langle \hat{f}^n_i - f^n_i, \tilde{v}_h \rangle_{\Omega_{n,i}} \lesssim \left\{ \tau^{k+1} \sum_{i=1,2} \nu_i^{-\frac{1}{2}} \left( \|\hat{f}^n_i\|_{H^1(\Omega)} + \|\tilde{u}_i^{n-j}\|_{H^1(\Omega)} \right) + \tau^{k+1} \sum_{i=1,2} \nu_i^{-\frac{1}{2}} \|\tilde{u}_i\|_{H^1+(\Omega \setminus \Gamma(t_{n-k}, t_n))} \right\} \left( \|\nu_i^{-\frac{1}{2}} \tilde{v}_h\|_{0,\Omega_{n,i}} + \|\tilde{v}_h\|_{\mathcal{V}} \right). \tag{7.16}
\]

Let \( n = (-\chi_{n,2}, \chi_{n,1}) / |\chi_n| \) and \( \hat{n} = (-\hat{\chi}_{n,2}, \hat{\chi}_{n,1}) / |\hat{\chi}_n| \) denote the unit normal vectors to the approximate interface \( I_n \) and the exact interface \( \Gamma(t_n) \), respectively. By Theorem 3, we have
\[ \|n \circ \chi_n' - \hat{n} \circ \hat{\chi}_n' / |\chi_n'| \|_{L^\infty(0,L)} \lesssim \tau^k. \]

Since \( \| \nu \nabla (u^n - p^n) \| \cdot \hat{n} = 0 \) on \( \Gamma(t_n) \) and \( \| \hat{\chi}_n - \chi_n \|_{L^\infty([0,L])} \lesssim \tau^{k+1} \), by (2.11) and the trace theorem, we obtain
\[
\| \nu \partial_n (u^n - \hat{p}^n) \|_{L^2(I_n')} = \int_0^L \| \langle \nu \nabla (u^n - \hat{p}^n) \circ \hat{\chi}_n \rangle \cdot (n \circ \chi_n) \|_{L^2(I_n')} 
\lesssim \int_0^L \| \langle \nu \nabla (u^n - \hat{p}^n) \circ \chi_n \rangle \cdot (n \circ \chi_n) \|_{L^2(I_n')}.
\]
Similarly, since $\tilde{\nu} \neq 0$, we have

$\tilde{\nu} = \nu_1 t_1, \tilde{\chi}_n = \chi_n(t_1)$.

Let the assumptions in Lemma 7.2 be satisfied and assume $\tau \leq \nu_2$. Then

$$
\frac{1}{\nu_2} \sum_{i=1,2} \| \tilde{u}_i^n \|_{H^2(\Omega)}^2 \geq \gamma_1 \nu_2 \tau^{k+2} \sum_{i=1,2} \nu_1^{-\frac{1}{2}} \left( \sum_{j=0}^{k} \| \tilde{u}_i^{n-j} \|_{H^{k+1}(\Omega)} + \| f_i \|_{H^1(\Omega)} \right) + \tau^{-\frac{1}{2}} \| \tilde{u}_i \|_{H^{k+1}(\Omega \times (t_{n-k}, t_n))} + \gamma_1 \nu_2 \tau^{k+1} \left( \hat{M}_k \right)^{\frac{1}{2}}
$$

Combining (7.14) and (7.18) and using the assumption that $\gamma_0 \leq 1$, we get

$$
|F^n(v_h)| \leq \tau^k \left( \hat{M}_k \right)^{\frac{1}{2}} \left( \| v_h \|_{0, \Omega_0} + \| v_h \|_{0, \Omega_0} \right).
$$

The proof is finished by inserting (7.14), (7.16), and (7.19) into (7.10) and using the Cauchy-Schwarz inequality.

**Lemma 7.3** Let the assumptions in Lemma 7.2 be satisfied and assume $\tau \leq \nu_2$. Then

$$
\left\| \sum_{i=1,2} \left( \text{div} \tilde{u}_i^n, q_{n,i} \right)_{\Omega_0, \Omega_i(t_n)} + \int_{\Gamma_n} \tilde{u}_i^n \cdot n \{ q_h \} \right\|_{H^2(\Omega)} \leq \tau^{k+1} \sum_{i=1,2} \| \tilde{u}_i^n \|_{H^2(\Omega)} \| q_h \|_{1, \Omega_0}.
$$
Using Lemma 5.3 (7.15), and arguments similar to (7.16), we obtain

\[
\langle \hat{f}^n - f^n, \nu^{-1} \nabla \bar{g}_h \rangle_{\Omega_n^0} \lesssim \sum_{i=1}^{2} \nu_i^{-1} (\tau^{k+\frac{1}{2}} \| \hat{f}^n_i \|_{1, \Omega} + \| R_i^n \|_{0, \Omega_n^0}) \| \nabla q h,i \|_{0, \Omega_n^0},
\]

\[
\lesssim \tau \sum_{i=1}^{2} \nu_i^{-1} \left( \frac{\tau}{2} \| \hat{f}^n_i \|_{H^1(\Omega)} + \tau^{-\frac{1}{2}} \| \bar{u}_i \|_{H^{k+1}(\Omega \times (t_{n-k}, t_n))} \right) \| \nabla q h,i \|_{L^2(\Omega_n^0)}.
\]

Then using Theorems 10 and 11, we arrive at

\[
\mathcal{G}_h(1^n, \rho^n, \bar{q}^n)
\]

\[
\lesssim \gamma_1 \nu_2 \tau^{k+1} \left( \sum_{j=0}^{k} \sum_{i=1}^{2} \nu_i^{-\frac{1}{2}} \tau \| \bar{u}_j \|_{k+1, \Omega} \right) + (M_k^n)^{\frac{1}{2}} \| \nu^{-\frac{1}{2}} \nabla \bar{q}_h \|_{0, \Omega_n^0}.
\]

(7.23)

The proof is finished by inserting (7.21)–(7.23) into (7.11).

**Lemma 7.4** Assume that the penalty parameter \( \gamma_0 \) in \( \mathcal{J}_0^n \) is large enough. Then

\[
\| \rho_h^n \|^2 \lesssim \mathcal{J}_p^n(\rho_h^n, \rho_h^n) + (\nu_2 \tau)^{-1} \| \Lambda_k \theta_h^n \|_{0, \Omega_n^0}^2 + \| \theta_h^n \|^2_{\mathcal{E}_1^0, \Omega_n^0} + \| \mathcal{E}_1^n \|^2_{1, (\nu_n^n)^{\frac{1}{2}}}.
\]

**Proof** In view of (7.9a), the proof is parallel to that of Lemma 6.2. We omit the details.

**Lemma 7.5** Let Assumption \( 3 \) Assumption \( 7 \) and the assumptions in Lemma 7.2 be satisfied. Then the modified Stokes projection \( (\hat{\theta}_h^{n,-1,n}, \hat{\rho}_h^{n,-1,n}) = S^n(\theta_h^{n,-1,n}, q_h^n) \) satisfies

\[
\| (\hat{\theta}_h^{n,-1,n}, \hat{\rho}_h^{n,-1,n}) \|_{\mathcal{V}, \Omega} \lesssim \| \theta_h^{n,-1,n} \|_{\mathcal{V}},
\]

\[
\| \theta_h^{n,-1,n} - \hat{\theta}_h^{n,-1,n} \|_{0, \Omega_n^0} \lesssim \gamma_1 \nu_2 h \left\{ \| \nu^{-\frac{1}{2}} \Lambda_k \theta_k^{n,-1} \|_{0, \Omega_n^0} + \| \mathcal{E}_2^{-1} \|^2_{1, (\Omega_n^0)^{1/2}} + \| \mathcal{E}_1^n \|^2_{1, (\nu_n^n)^{1/2}} \right\}.
\]

(7.24)

(7.25)

**Proof** The stability of \( \hat{\theta}_h^{n,-1,n} \) and \( \hat{\rho}_h^{n,-1,n} \) comes directly from Theorem 10 and Remark 4. Following the proof in Corollary 2 we have

\[
\| \theta_h^{n,-1,n} - \hat{\theta}_h^{n,-1,n} \|_{0, \Omega_n^0} \lesssim h \| \theta_h^{n,-1,n} \|_{\mathcal{V}} + \sup_{q_h \in \Omega_n^0} \frac{\mathcal{B}_0^{-1}(\theta_h^{n,-1}, q_h)}{\| q_h \|_{1, \Omega}},
\]
where we have extended \(q_{h,1}, q_{h,2}\) to \(\Omega\) by the convention in (3.5), and \(\tilde{q}_{h,i} = \pi_{1,\Omega}(q_{h,i} \circ \mathcal{X}_h^{-1,n}) \in V_h(1, T_h)\) is the Scott-Zhang interpolation.

From (7.9b), we have
\[
\mathcal{R}_n(\theta_{h}^{-1}, \tilde{q}_h) = \mathcal{E}_n^{-1}(\tilde{q}_h) + \mathcal{F}_p^{-1}(\rho_{h}^{-1}, \tilde{q}_h) + \mathcal{E}_n^{-1}(\theta_{h}^{n-1}, \rho_{h}^{n-1}, \tilde{q}_h).
\]

By arguments similar to the proof of Corollary 2, we have
\[
\begin{align*}
\mathcal{R}_n^{-1}(\theta_{h}^{-1}, \tilde{q}_h) &\lesssim h \mathcal{F}_p^{-1}(\rho_{h}^{-1}, \rho_{h}^{-1})^{1/2} \|\tilde{q}_h\|_{1,Q}, \\
\mathcal{E}_n^{-1}(\theta_{h}^{n-1}, \tilde{q}_h) &\lesssim \gamma_1 \nu_2 h \left( \|\theta_{h}^{n-1}\|_V + \|\nu^{-1/2}A^k \theta_{h}^{n-1}\|_{0,\Omega_n^{n-1}} \right) \\
&+ h \|\nu^{-1/2} \nabla \rho_{h}^{n-1}\|_{0,\Omega_n^{n-1}} + h \mathcal{F}_p^{-1}(\rho_{h}^{-1}, \rho_{h}^{-1})^{1/2} \\
&+ \|\mathcal{E}_n^{-1}\|_{1, (\mathcal{Q}_n^{n-1})'},
\end{align*}
\]

Together with (7.20) and (5.7), they yields the desired result
\[
\sup_{\tilde{q}_h \in \mathcal{Q}_h} \frac{\mathcal{R}_n^{-1}(\theta_{h}^{-1}, \tilde{q}_h)}{\|\tilde{q}_h\|_{1, Q}} \lesssim \gamma_1 \nu_2 h \left( \|\theta_{h}^{n-1}\|_V + \|\nu^{-1/2}A^k \theta_{h}^{n-1}\|_{0,\Omega_n^{n-1}} \right) \\
&+ h \|\nu^{-1/2} \nabla \rho_{h}^{n-1}\|_{0,\Omega_n^{n-1}} + h \mathcal{F}_p^{-1}(\rho_{h}^{-1}, \rho_{h}^{-1})^{1/2} \\
&+ \|\mathcal{E}_n^{-1}\|_{1, (\mathcal{Q}_n^{n-1})'}.
\]

The proof is finished.

**Theorem 13** Let the assumptions in Theorem 12 and Lemma 7.2 be satisfied. Assume \(\max(\tau, h) \leq \gamma_1 \nu_2\) and that the penalty parameter \(\gamma_0\) in \(\mathcal{F}_0^n\) satisfies \(\gamma_0^{-1} + \gamma_0 h \ll 1\). there holds for any \(k \leq m \leq N\),
\[
\|\theta_{h}^{n}\|_{0,\Omega_n^{n}} + \sum_{n=k}^{m} \tau \left( \|\theta_{h}^{n}\|_V^2 + \gamma_1 \nu_2 h^2 \|\rho_{h}^{n}\|_{1,\Omega_n^{n}}^2 + \mathcal{F}_p^n(\rho_{h}^{n}, \rho_{h}^{n}) \right) \\
\lesssim \sum_{n=0}^{k-1} \tau \left( \|\theta_{h}^{n}\|_V^2 + \|\rho_{h}^{n}\|_V^2 \right) \sum_{n=k}^{m} \left( \|\mathcal{E}_n^n\|_{1, (\mathcal{Q}_n^{n})'} + \tau \gamma_1 \nu_2^{-1} \|\mathcal{E}_n^n\|_{1, (\mathcal{Q}_n^{n})'} \right).
\]

**Proof** We only prove the theorem for \(k = 4\). The proofs for other cases are similar. Let \((\theta_{h}^{n-1,n}, \rho_{h}^{-1,n}) = S^n(\theta_{h}^{n-1,n}, 0, 0), \quad \psi_h = 2 \theta_{h}^{n} - \tilde{\theta}_{h}^{n-1,n}, \quad q_h = 0\).

From the telescope formula in Lemma 6.1, we have
\[
\sum_{l=1}^{5} \|\psi_l(\theta_{h}^{n})\|_{0,\Omega_n^{n}}^2 - \sum_{l=1}^{4} \|\Phi_l(\theta_{h}^{n})\|_{0,\Omega_n^{n}}^2 + \tau \mathcal{K}_0(\theta_{h}^{n}, \rho_{h}^{n}; (\psi_h, 0)) = \tau (B_1 + B_2),
\]
\[
(7.26)
\]
where \(B_1 = \mathcal{E}_n^n(\psi_h), \quad B_2 = \tau^{-1}(A^4 \theta_{h}^{n}, \tilde{\theta}_{h}^{n-1,n} - \tilde{\theta}_{h}^{n-1,n}) \Omega_n^{n}, \quad \text{and}
\]
\[
\mathcal{K}_0(\theta_{h}^{n}, \rho_{h}^{n}; (\psi_h, 0)) = 2 \mathcal{F}_p^n(\theta_{h}^{n}, \tilde{\theta}_{h}^{n-1,n}) - a_h^n(\theta_{h}^{n-1,n}, \theta_{h}^{n}) + 2 \mathcal{F}_p^n(\rho_{h}^{n}, \rho_{h}^{n}).
\]
From Lemma 6.1 and arguments similar to (6.7), we have

\[\mathcal{K}_0((\theta_h^n, \rho_h^n); (\nu, 0)) \geq (1.3 - 2\varepsilon - C_{\gamma_1}\nu_2)\|\theta_h^n\|_\mathcal{V}^2 - (0.5 + 2\varepsilon)\|\theta_h^{n-1}\|_\mathcal{V}^2 + 2 \mathcal{F}_p^n(\rho_h^n, \rho_h^n) + \gamma_1\nu_2h^2\|\nu^{-\frac{1}{2}}\nabla \rho_h^n\|_{0, \Omega_n}^2 - C\|\Lambda^4\theta_h^n\|_{0, \Omega_n}^2 + \mathcal{E}_2^n(2\rho_h^n + \rho_h^{n-1, n}), \tag{7.27}\]

where we have used the assumption that \(h\) and \(\gamma_0^{-1}\) are small enough such that

\[C(\gamma_0^{-1} + h\gamma_0) \leq \varepsilon^2 \ll 1.\]

From Lemma 6.1 and arguments similar to (6.7), we have

\[\|\Phi_1(\theta_h^n)\|_{0, \Omega_n}^2 \leq \|\Phi_1(\theta_h^{n-1})\|_{0, \Omega_n}^2 + C\tau \sum_{j=1}^4 \left\{ \|\theta_h^{n-j}\|_{0, \Omega_n}^2 + h^4\nu_2^{-1}\|\theta_h^{n-j}\|_\mathcal{V}^2 \right\}. \tag{7.28}\]

Inserting (7.27) and (7.28) into (7.26) and using Cauchy-Schwarz inequality, we obtain

\[
\sum_{i=1}^4 \|\Phi_i(\theta_h^n)\|_{0, \Omega_n}^2 + (1.3 - 2\varepsilon - C_{\gamma_1}\nu_2)\tau \|\theta_h^n\|_\mathcal{V}^2 + \gamma_1\nu_2h^2\|\nu^{-\frac{1}{2}}\nabla \rho_h^n\|_{0, \Omega_n}^2 + 2\tau \mathcal{F}_p^n(\rho_h^n, \rho_h^n)
\leq \sum_{i=1}^4 \|\Phi_i(\theta_h^{n-1})\|_{0, \Omega_n}^2 + C\tau \sum_{j=1}^4 \left\{ \|\theta_h^{n-j}\|_{0, \Omega_n}^2 + \nu_2^{-1}h^4\|\theta_h^{n-j}\|_\mathcal{V}^2 \right\} + (0.5 + 2\varepsilon)\|\theta_h^{n-1}\|_\mathcal{V}^2 + C(1 + \varepsilon^{-1})\tau \|\Lambda^4\theta_h^n\|_{0, \Omega_n}^2 + C\tau\varepsilon^{-1}\|\mathcal{E}_1^n\|_{1, (V_h^n)}^2 + C\tau\gamma_1\nu_2\|\Lambda^4\theta_h^n\|_{0, \Omega_n}^2 + C\tau\varepsilon^{-1}\|\mathcal{E}_1^n\|_{1, (V_h^n)}^2
+ \tau\varepsilon\left(\gamma_1\nu_2h^2\|2\rho_h^n + \rho_h^{n-1, n}\|_{0, \Omega_n}^2 + \|\nu^{-\frac{1}{2}}\nabla \rho_h^n\|_{0, \Omega_n}^2 + \|\rho_h^n\|_{0, \Omega_n}^2\right). \tag{7.29}\]

Next we estimate the right-hand side of (7.29). From (5.24) and (5.32), we have

\[\|\Lambda^4\theta_h^n\|_{0, \Omega_n}^2 \leq \sum_{j=0}^4 \left( \|\theta_h^{n-j}\|_{0, \Omega_n}^2 + h^5\nu_2^{-1}\|\theta_h^{n-j}\|_\mathcal{V}^2 \right), \tag{7.30}\]

\[\|\nu^{-\frac{1}{2}}\mathcal{E}_h^{n-1, n}\|_{0, \Omega_n}^2 \leq (1 + C\tau)\|\theta_h^{n-1}\|_{0, \Omega_n}^2 + C\tau^7\|\theta_h^{n-1}\|_\mathcal{V}^2. \tag{7.31}\]
Moreover, using (7.24), the fact that $\nu \leq 1$, and inverse estimates, we find that

$$
\|2\rho_n^h + \rho_n^{-1,n}\|_{1,Q} \leq 2\|\rho_n^h\|_{1,Q} + Ch^{-1}\|\theta_n^{-1}\|_\mathcal{V},
$$

(7.32)

$$
\|\omega_n\|_\mathcal{V} \leq 2\|\theta_n^h\|_\mathcal{V} + \|\hat{\omega}_n^{-1,n}\|_\mathcal{V} \leq 2\|\theta_n^h\|_\mathcal{V} + C\|\theta_n^{-1}\|_\mathcal{V},
$$

(7.33)

$$
\|\nu_{\frac{1}{2}}\omega_n\|_{0,\Omega_n^m} \leq 2\|\nu_{\frac{1}{2}}\theta_n^h\|_{0,\Omega_n^m} + \|\hat{\omega}_n^{-1,n} - \theta_n^{-1,n}\|_{0,\Omega_n^m} + \|\nu_{\frac{1}{2}}\theta_n^{-1,n}\|_{0,\Omega_n^m}.
$$

(7.34)

Substitute all estimates above into (7.29) and take the sum over $4 \leq n \leq m$. By the assumption $h \leq \gamma_1\nu_2$ and proper arrangements, we end up with

$$
\left\|\Psi_1^4(\Theta_n^m)\right\|^2_{0,\Omega_n^m} + \sum_{n=4}^m \tau\left\{(0.8 - C\varepsilon - C\gamma_1\nu_2)\left\|\theta_n^h\right\|^2_\mathcal{V}
+ \gamma_1\nu_2 h^2\|\nu_{\frac{1}{2}}\nabla\rho_n^h\|^2_{0,\Omega_n^m} + 2\mathcal{F}_n(\rho_n^h, \rho_n^h)\right\}
\leq \sum_{n=0}^3 \sum_{l=1}^4 \left\|\Psi_l(\Theta_n^m)\right\|^2_{0,\Omega_n^m} + C(1 + \varepsilon^{-1})\sum_{n=0}^m \tau\left\|\theta_n^h\right\|^2_{0,\Omega_n^m}
+ 4\varepsilon\tau_1\nu_2 h^2\|\rho_n^h\|^2_{1,\mathcal{Q}_n^1} + (\varepsilon\tau_1^{-1} + C\varepsilon\tau_1)\left\|\hat{\omega}_n^{-1,n} - \theta_n^{-1,n}\right\|^2_{0,\Omega_n^m}
+ C\tau^{-1}\left\|\mathcal{E}_1^2\right\|^2_{1,(\mathcal{Q}_n^1)^1} + C(\gamma_1\nu_2\varepsilon)^{-1}\|\mathcal{E}_2^2\|_{1,(\mathcal{Q}_n^1)^1}.
$$

(7.35)

Then combining (7.25) and (7.30) leads to

$$
\left\|\hat{\omega}_n^{-1,n} - \theta_n^{-1,n}\right\|^2_{0,\Omega_n^m} \leq C\gamma_1^2\nu_2 h^2\sum_{j=1}^5 \left\|\theta_n^{-j}\right\|^2_{0,\Omega_n^m} + h^4\left\|\theta_n^{-j}\right\|^2_\mathcal{V}
+ h^2\mathcal{F}_2^{n-1}(\rho_n^{-1}, \rho_n^{-1}) + \left\|\mathcal{E}_2^{n-1}\right\|^2_{1,(\mathcal{Q}_n^1)^1}
+ h^2\left\|\theta_n^{-j}\right\|^2_\mathcal{V} + C(\gamma_1\nu_2)^2 h^4\left\|\nu_{\frac{1}{2}}\nabla\rho_n^{-1}\right\|^2_{0,\Omega_n^m}.
$$

(7.36)

Using the estimate of $\rho_n^h$ in Lemma 7.4, we obtain

$$
\|\nu_{\frac{1}{2}}\nabla\rho_n^h\|^2_{0,\Omega_n^m} \geq \|\rho_n^h\|^2_{1,\mathcal{Q}_n^1} - \|\rho_n^h\|^2_{1,\mathcal{Q}_n^1} - C\gamma_1^2\nu_2 h^2\mathcal{F}_2^{n-1}(\rho_n^{-1}, \rho_n^{-1})
\geq \|\rho_n^h\|^2_{1,\mathcal{Q}_n^1} - C\gamma_1^2\nu_2 h^2\mathcal{F}_2^{n-1}(\rho_n^{-1}, \rho_n^{-1}) - C\left\|\theta_n^h\right\|^2_\mathcal{V} - C\left\|\mathcal{E}_2^2\right\|^2_{1,(\mathcal{Q}_n^1)^1}
- C\nu_2^{-1} h^{-2}\sum_{j=0}^4 \left\|\theta_n^{-j}\right\|^2_{0,\Omega_n^m} + \tau\left\|\theta_n^{-j}\right\|^2_\mathcal{V}.
$$

(7.37)

Inserting (7.36) and (7.37) into (7.35) yields

$$
\left\|\Psi_1^4(\Theta_n^m)\right\|^2_{0,\Omega_n^m} + \sum_{n=4}^m \tau\left\{(0.8 - C_0\varepsilon - C_0\gamma_1\nu_2)\left\|\theta_n^h\right\|^2_\mathcal{V}
+ (2 - C_0\varepsilon)\mathcal{F}_n(\rho_n^h, \rho_n^h) + (1 - C_0\varepsilon)\gamma_1\nu_2 h^2\|\nu_{\frac{1}{2}}\nabla\rho_n^{-1}\|^2_{0,\Omega_n^m}\right\}
$$
\[ \leq C_1 + \frac{1}{\varepsilon} \sum_{n=0}^{m} \| \theta_n^m \|_{0, \Omega_n^T}^2 + C_1 \sum_{n=0}^{m} \left( \tau \| \mathcal{E}_n \|_{1, (V_n^c)^\gamma}^2 + (\tau \gamma_n \nu_2)^{-1} \| \mathcal{F}_n \|_{1, (Q_n^c)^\gamma}^2 \right) \]
\[ + C_1 \sum_{n=0}^{\lfloor \frac{3}{\varepsilon} \rfloor} \tau \left( \| \theta_n^m \|_{V_n^c}^2 + \tau \mathcal{F}_p^n (\rho_n, \tilde{\rho}_n^m) + h^2 \| \nabla \tilde{\rho}_n^m \|_{0, \Omega_n^T}^2 \right) . \]

Choose \( \varepsilon = 0.2/C_0, \gamma_1 \nu_2 \leq 0.2/C_0, \) and \( h \leq 0.2/C_0 \) and note that \( \Psi_1^n (\theta_n^m) = 0.06 \theta_n^m. \) The proof is finished upon using Gronwall’s inequality and inverse estimates.

7.5 The main theorem

Now we are ready to present the main theorem of this paper.

**Theorem 14** Let the assumptions in Theorem 13 be satisfied and suppose that the pre-calculated initial solutions satisfy for \( j = 0, 1, \ldots, k - 1, \)
\[
\| \tilde{u}^j - u_k^m \|_{0, \Omega_n^T}^2 + \tau \| \tilde{u}^j - u_k^m \|_{V_n^c}^2 + \tau^2 \| \tilde{p}_j - p_k^m \|_{1, Q_n^c}^2 \leq C_0 \tau^{2k} . \tag{7.38}
\]

Then upon a hidden constant independent of \( \eta, \tau, h, \) and material parameters, such that for \( k \leq m \leq N \)
\[
\| \tilde{u}^m - u_k^m \|_{0, \Omega_n^T}^2 + \sum_{n=k}^{m} \sum_{i=1,2} \tau \left( \| \tilde{u}^i - u_k^m \|_{V_n^c}^2 + \gamma_1 \nu_2 h^2 \| \tilde{p}_i - p_k^m \|_{1, Q_n^c}^2 \right) \leq C_N \tau^{2k} ,
\]

where
\[
C_N := C_0 + \sum_{n=1}^{N} \tau \tilde{M}_n^m + \nu_2^{-1} \sum_{n=1}^{N} \sum_{i=1,2} \left( h^2 \| \tilde{f}_n \|_{H^2(\Omega)}^2 + \nu_i \| \tilde{u}^n \|_{H^2(\Omega)}^2 \right) \\
+ \sum_{i=1,2} \nu_i^{-1} \| \tilde{u}^i \|_{H^{k+1}(\Omega \times (0,T))}^2 .
\]

**Proof** From (7.6)–(7.7) and assumption (7.38), there exists a constant \( C > 0 \) independent of \( \eta, \tau, h, \) and \( k, \) such that for \( 0 \leq j \leq k - 1, \)

\[
\begin{aligned}
\| \theta_n^m \|_{0, \Omega_n^T} &\leq \| \tilde{u}^j - u_k^m \|_{0, \Omega_n^T} + \| \theta_n^m \|_{0, \Omega_n^T} \leq C_0 \tau^k + C (\tilde{M}_n^m)^{1/2} h^{k+1} , \\
\| \theta_n^m \|_{V_n^c} &\leq \| \tilde{u}^j - u_k^m \|_{V_n^c} + \| \theta_n^m \|_{V_n^c} \leq C_0 \tau^{k-1/2} + C (\tilde{M}_n^m)^{1/2} h^k , \\
\| \tilde{p}_n^m \|_{1, Q_n^c} &\leq \| \tilde{p}_j - p_k^m \|_{1, Q_n^c} + \| \rho_n^m \|_{1, Q_n^c} \leq C_0 \tau^{k-1} + C (\tilde{M}_n^m)^{1/2} h^{k-1} .
\end{aligned}
\]

From Theorem 13 and Lemmas 7.2–7.3, we conclude that
\[
\| \theta_n^m \|_{0, \Omega_n^T}^2 + \sum_{n=k}^{m} \tau \left( \| \theta_n^m \|_{V_n^c}^2 + \gamma_1 \nu_2 h^2 \| \rho_n^m \|_{1, Q_n^c}^2 + \mathcal{F}_p^n (\rho_n, \tilde{\rho}_n^m) \right) \leq C_N \tau^{2k} ,
\]

Using (7.7)–(7.8), we finish the proof.
8 Numerical experiment

Now we use a numerical experiment on severely deforming domain to verify the convergence orders of the finite element method for \( k = 3, 4 \). The segment size for interface-tracking is set by \( \eta = 0.5 \tau^{\max(1,k/3)} \) in Algorithm 1 for computing \( X^eta_{n-1,n} \). In order to capture large deformations of the domain accurately, we apply the cubic MARS (Mapping and Adjusting Regular Semialgebraic sets) algorithm in real computations [43]. The algorithm is a slight modification of Algorithm 1 by creating new markers or removing old markers adaptively.

The whole domain is taken as \( \Omega = (0, 1) \times (0, 1) \) and the time interval is \([0, T]\) with \( T = 1.5 \). The initial sub-domain \( \Omega_1(0) \) is a disk of radius 0.15 and centering at \((0.5, 0.75)\). The flow velocity which drives the interface \( \Gamma(t) \) is taken as

\[
\mathbf{w}(\mathbf{x}, t) = \cos(\pi t/3) \left( \sin^2(\pi x_1) \sin(2\pi x_2) - \sin^2(\pi x_2) \sin(2\pi x_1) \right)^\top, 
\]

At the final time, \( \Omega_1(T) \) is stretched into a snake-like domain (see Fig. 4).

To test the high-order error estimates, we set the true solution by smooth velocity and smooth pressure in each sub-domain

\[
\begin{align*}
\{ \mathbf{u}_1(x, t) = \cos t \left( \cos(\pi x_1) \sin(\pi x_2), -\sin(\pi x_1) \cos(\pi x_2) \right)^\top, \quad & x \in \Omega_1(t), \\
\{ p_1(x, t) = \cos(0.5\pi x_1) \sin(0.5\pi x_2), \quad & x \in \Omega_1(t), \\
\{ \mathbf{u}_2(x, t) = e^t \left( \cos(\pi x_1 x_2 + \pi t), \pi^{-1} \cos(\pi x_2 + \pi t) \right)^\top, \quad & x \in \Omega_2(t), \\
\{ p_2(x, t) = \sin(0.5\pi x_1) \cos(0.5\pi x_2). \quad & x \in \Omega_2(t).
\end{align*}
\]

The coefficients of viscosity are taken as \( \nu_1 = 1 \) and \( \nu_2 = 10^{-3} \). The body forces in (1.1a) are defined as

\[
f_i(t) = \partial_\mathbf{n} \mathbf{u}_i + (\mathbf{w} \cdot \nabla) \mathbf{u}_i - \nu_i \Delta \mathbf{u}_i + \nabla p_i \quad \text{in} \quad \Omega_i(t), \quad i = 1, 2.
\]

However, the jumps \([\mathbf{u}]\) and \([\nu \partial_\mathbf{n} \mathbf{u} - \mathbf{p} \mathbf{n}]\) do not match the interface conditions in (1.1c). To remedy the situation, we define

\[
g_0^n = \mathbf{u}_1 - \mathbf{u}_2, \quad g_1^n = \nu_1 \partial_\mathbf{n} (\mathbf{u}_1 - \mathbf{u}_2) - (p_1 - p_2) \mathbf{n} \quad \text{on} \quad \Gamma_n^0,
\]

and replace the right-hand sides of (3.7a) and (3.7b), respectively, with

\[
\left\langle \mathbf{f}^n, \mathbf{v}_h \right\rangle_{\Omega_n} + \int_{\Gamma_n^i} \left( g_1^n \cdot \{ \mathbf{v}_h \} - g_0^n \cdot \{ \nu \partial_\mathbf{n} \mathbf{v}_h \} \right), \quad \int_{\Gamma_n^0} g_0^n \cdot \mathbf{n} \{ \mathbf{g}_h \}.
\]

The approximation errors are measured with the following quantities

\[
\begin{align*}
e_{u,0} &= \| \mathbf{u}_h^N - \mathbf{u}_h^N \|_{0, \Omega_n^N}, \\
e_{u,1} &= \left( \sum_{n=k}^{N} \tau \| \nu^{\frac{1}{2}} \nabla (\mathbf{u}_h^N - \mathbf{u}_h^{N}) \|_{0, \Omega_n^N}^2 \right)^{\frac{1}{2}}, \\
e_{p,0} &= \left( \sum_{n=k}^{N} \tau \| \nu^{\frac{1}{2}} (\mathbf{p}_h^N - \mathbf{p}_h^{N}) \|_{0, \Omega_n^N}^2 \right)^{\frac{1}{2}}, \\
e_{p,1} &= \left( \sum_{n=k}^{N} \tau \| \nu^{\frac{1}{2}} \nabla (\mathbf{p}_h^N - \mathbf{p}_h^{N}) \|_{0, \Omega_n^N}^2 \right)^{\frac{1}{2}}.
\end{align*}
\]
To simplify the computation, we set pre-calculated initial values by the exact solution, namely, $u_j^h = u(t_j)$ for $0 \leq j \leq k - 1$. Throughout the section, we set $\gamma_0 = 10^3$, $\gamma_1 = 1$, and $h = \tau$.

Numerical results for $k = 3, 4$ are shown in Tables 1-4. They show that optimal convergence is obtained for both the third- and fourth-order methods. Moreover, Tables 2 and 3 show that the discrete pressure is of the $k$th-order convergence under both the weighted $L^2$-norm and the weighted $H^1$-norm. Finally, Fig. 4 shows the shapes of tracked interfaces at $t_n = 0.0, 0.5, 1.0, \text{ and } 1.5$, respectively. We find that the interface has a large deformation at $t_n = T$. Nevertheless, our numerical experiment shows that optimal convergence is obtained for such a challenging simulation.

| $h = \tau$ | $e_{u,0}$ | order | $e_{u,1}$ | order |
|------------|-----------|-------|-----------|-------|
| 1/16       | 6.479e-03 |       | 6.259e-03 |       |
| 1/32       | 8.180e-04 | 3.00  | 8.679e-04 | 2.85  |
| 1/64       | 9.941e-05 | 3.03  | 1.119e-04 | 2.96  |
| 1/128      | 1.288e-05 | 3.02  | 1.413e-05 | 2.99  |

Table 1 Convergence orders of the discrete velocity for $k = 3$.

| $h = \tau$ | $e_{p,0}$ | order | $e_{p,1}$ | order |
|------------|-----------|-------|-----------|-------|
| 1/16       | 1.83e-01  |       | 1.47e+00  |       |
| 1/32       | 2.41e-02  | 2.93  | 1.97e-01  | 2.90  |
| 1/64       | 3.07e-03  | 2.97  | 2.51e-02  | 2.97  |
| 1/128      | 3.87e-04  | 2.99  | 3.18e-03  | 2.98  |

Table 2 Convergence orders of the discrete pressure for $k = 3$.

| $h = \tau$ | $e_{u,0}$ | order | $e_{u,1}$ | order |
|------------|-----------|-------|-----------|-------|
| 1/16       | 1.46e-03  |       | 4.20e-03  |       |
| 1/32       | 8.70e-05  | 4.07  | 2.71e-04  | 3.95  |
| 1/64       | 5.17e-06  | 4.07  | 1.67e-05  | 4.02  |
| 1/128      | 3.16e-07  | 4.03  | 1.03e-06  | 4.02  |

Table 3 Convergence orders of the discrete velocity for $k = 4$.

| $h = \tau$ | $e_{p,0}$ | order | $e_{p,1}$ | order |
|------------|-----------|-------|-----------|-------|
| 1/16       | 6.80e-02  |       | 6.51e-01  |       |
| 1/32       | 4.73e-03  | 3.85  | 4.56e-02  | 3.84  |
| 1/64       | 3.08e-04  | 3.94  | 2.97e-03  | 3.94  |
| 1/128      | 1.96e-05  | 3.97  | 1.89e-04  | 3.97  |

Table 4 Convergence orders of the discrete pressure for $k = 4$. 
Fig. 4 Tracked interfaces $\Gamma^a_\eta$ at $t_n = 0.0$, $0.5$, $1.0$, and $1.5$, respectively ($h = 1/16$).

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