Scalar-Graviton Scattering in Noncommutative Space and Deformed Newton Gravity

A. Jahan*, A. Parvishi**
Research Institute for Astronomy and Astrophysics of Maragha (RIAAM) Maragha, IRAN, P. O. Box: 55134 - 441*
Islamic Azad University, Urmia Branch, Urmia, Iran**
jahan@riaam.ac.ir

Abstract

The isotropy of Newton potential in noncommutative space disappears as it deforms to a momentum dependent one. We generalize the earlier derivation of such a deformed potential to the relativistic regime by calculating the 2 scalar-1 graviton scattering amplitude by taking into account the noncommutativity of space.

PACS: 11.10.Nx, 04.60.-m

1. Introduction

There are several serious motivations, arising from the different arenas of the planck scale physics [1-3], to consider a noncommutative (NC) algebra for the coordinates $x^\mu$ spanning the space-time manifold, via

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu},$$

(1)

where $\theta^{\mu\nu}$ assumed to be a constant antisymmetric matrix. As a result of ”$\theta$-deformation” of the algebra of space-time coordinates one must replace the usual product among the fields with Weyl-Moyal product or $\star$-product [4], i.e.

$$\phi_1(\vec{x}) \star \phi_2(\vec{x}) \equiv \lim_{x\rightarrow y} e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial x^\mu}{\partial \phi_1} \frac{\partial x^\nu}{\partial \phi_2}} \phi_1(x)\phi_2(y).$$

(2)

From the Feynman’s rules point of view the only effect of the $\star$-product is to modify the $n$-point interaction vertices ($3 \leq n$) by the phase factor [4]

$$\tau(p_1, \ldots, p_n) = e^{-\frac{1}{2} \sum_{a<b} (p_a \wedge p_b)},$$

(3)

where $p_a \wedge p_b = \theta^{\mu\nu} p_a^\mu p_b^\nu$. Here the momentum flow of the a-th field into the vertex is denoted by $p_a$. In the case of noncommutative QED (QED living on a NC space-time), by analyzing the electron-photon
interaction vertex one can demonstrate that at the quantum mechanical level the \( \theta \)-deformation of spatial coordinates gives rise to a deformed Coulomb potential \([5]\)

\[
V_{\theta} = -\frac{Ze^2}{|\vec{x}|} - \frac{Ze^2}{4|\vec{x}|^3} \vec{L} \cdot \vec{\theta} + O(\theta^2),
\]

(4)

with \( \theta_{ij} = \frac{1}{2} \epsilon_{ijk} \theta_k \) and \( \vec{L} = \vec{x} \times \vec{p} \). However one may defer the field theoretic considerations and follow a more economical way to achieve (4) by redefining the coordinates

\[
\vec{x}_i \rightarrow x_i - \frac{\theta_{ij}}{2} p_j.
\]

(5)

in the Coulomb potential \( \hat{V} = -\frac{Ze^2}{\sqrt{\vec{x}_i \vec{x}_i}} \) defined over a noncommutative space. The prescription which has been also followed to consider the \( \theta \)-deformation of the Newton potential and its possible classical phenomenological consequences \([6, 7]\).

So, as the aim of this letter, it seems to be a logical step to deduce the deformed Newton potential by looking at the matter-graviton interaction vertex in a NC background space. In the next section we write down the explicit form of the momentum space vertex factor for a Klein-Gordon field scattering from a graviton in NC space. In section 3, the two-body scattering problem in NC flat background is considered and the deformed Newton potential is re-derived by calculating the Fourier transform of the scattering amplitude. The scalar particle scattering off a static source of the graviton is also considered in section 4. Our result is the relativistic generalization of the previous calculations \([6, 7]\). In this work we assume \( c = \hbar = 1 \) and \( \eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1) \).

2. Matter-Graviton Coupling

The Klein-Gordon lagrangian density in curved space-times is given by

\[
\mathcal{L}_{KG} = \frac{\sqrt{-g}}{2} (\partial_{\alpha} \phi \partial^{\alpha} \phi - m^2 \phi^2),
\]

(6)

By expanding the metric tensor and its determinants up to the first order in parameter \( \kappa = \sqrt{32 \pi G} \) as \([8, 9, 10]\)

\[
g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} + O(\kappa^2),
\]

(7)

\[
g = 1 + \kappa^2 + O(\kappa^2),
\]

(8)

where \( g = \det g_{\mu\nu} \) and \( h = \eta^{\mu\nu} h_{\mu\nu} \), and then substituting them in (6) we find the Lagrangian density

\[
\mathcal{L}_{KG} = \frac{1}{2} (\partial_{\alpha} \phi \partial^{\alpha} \phi - m^2 \phi^2) - \frac{\kappa}{2} h^{\mu\nu} \left[ \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} (\partial_{\alpha} \phi \partial^{\alpha} \phi - m^2 \phi^2) \right] + O(\kappa^2).
\]

(9)

Now the second term of (9), first order in \( \kappa \), clearly describes the 2 scalar-1 graviton interaction in a flat background. Thus we have the interaction term as

\[
\mathcal{L}_{\text{int}} = - \frac{\kappa}{2} h^{\mu\nu} \left[ \partial_{\mu} \phi \partial_{\nu} \phi - \frac{1}{2} (\partial_{\alpha} \phi \partial^{\alpha} \phi - m^2 \phi^2) \right].
\]

(10)
from which one obtains the corresponding vertex factor [8, 9, 10]

$$\tau_{\alpha\beta}(p, p') = -\frac{i\kappa}{2}(p_\alpha p'_\beta + p'_\alpha p_\beta - \eta_{\alpha\beta}p \cdot p')$$  \hspace{1cm} (11)

In de Donder gauge (harmonic gauge) $\partial^\alpha h_{\alpha\beta} - \frac{1}{2}\partial_\beta h = 0$ the Einstein-Hilbert action takes the form

$$S_{EH} = \frac{1}{2}\int d^4x h_{\mu\nu}Q^{\mu\nu,\alpha\beta}h_{\alpha\beta}$$  \hspace{1cm} (12)

with

$$Q^{\mu\nu,\alpha\beta} = \frac{1}{2}(\eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} - \eta^{\mu\nu}\eta^{\alpha\beta})\partial_2$$  \hspace{1cm} (13)

So the momentum-space graviton propagator is found to be

$$D_{\mu\nu,\alpha\beta}(q) = -\frac{i}{2q^2}(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta})$$  \hspace{1cm} (14)

### 3. Two Body Scattering in NC Space

In a NC flat background the interaction term must be changed to [11]

$$L_{int} = \frac{k}{2}h^{\mu\nu} \left[ \partial_\mu \phi \ast \partial_\nu \phi - \frac{1}{2}(\partial_\alpha \phi \ast \partial_\alpha \phi - m^2\phi \ast \phi) \right].$$  \hspace{1cm} (15)

Therefore, with the aid of formula (3) we obtain the deformed momentum space 2 scalar-1 graviton vertex factor as

$$\tau^\theta_{\alpha\beta}(p, p') = -\frac{i\kappa}{2}(p_\alpha p'_\beta + p'_\alpha p_\beta - \eta_{\alpha\beta}p \cdot p')e^{i\vec{p} \cdot \vec{p}'}.$$  \hspace{1cm} (16)

where we have assumed $\theta_{\mu0} = 0$ that implies $\vec{p} \wedge \vec{q} \rightarrow \theta_{ij}p_ip_j$ to avoid the problematic features of the NC models [4]. Now let us look at a typical two-body scattering mediated by a graviton. For the scalar particles with masses $m_1$ and $m_2$ the scattering amplitude is

$$M_\theta = \tau^{\mu\nu}_\theta(p_1, p'_1)D_{\mu\nu,\alpha\beta}(p_1 - p'_1)\tau^{\alpha\beta}_\theta(p_2, p'_2)$$  \hspace{1cm} (17)

$$= \frac{4\pi G}{(p_1 - p'_1)^2} \left\{ \left[(p_1 + p_2)^2 - m_1^2 - m_2^2\right]^2 + \left[(p_1 - p'_2)^2 - m_1^2 - m_2^2\right]^2 - \left[(p'_1 - p_1)^2 + 4m_1^2m_2^2\right]^2 \right\} e^{i\vec{p} \wedge (\vec{p}' - \vec{p}')}$$

In the non-relativistic limit we have

$$(p'_1 - p_1)^2 \approx -\vec{q}^2$$  \hspace{1cm} (18)

$$(p_1 + p_2)^2 \approx (m_1 + m_2)^2$$  \hspace{1cm} (19)

$$(p_1 - p'_2)^2 \approx (m_1 - m_2)^2 + \vec{q}^2$$  \hspace{1cm} (20)

By substituting (18)-(20) in (17) we find the deformed gravitational potential

$$U_\theta(\vec{x}) = -\frac{1}{4m_1m_2} \int \frac{d^3q}{(2\pi)^3} e^{i\vec{x} \cdot \vec{q}} M_\theta(\vec{q})$$  \hspace{1cm} (21)

$$= -G\frac{m_1m_2}{\sqrt{(x_1 - \frac{2}{3}\theta_{ij}p_i)(x_1 - \frac{4}{3}\theta_{ij}p_j)}} + 4\pi G\delta(\vec{x})$$
where
\[ \tilde{x}_i = x_i - \frac{\theta_{ij} p_j}{2} \] (22)

One must note that the second term of (21) is repulsive and can only be measured for bound s-states and thus does not appear for gravitational field as a spin zero massless field [10].

4. Scattering off a Static Graviton Source

The linearized Einstein equation in de Donder gauge reads
\[ \Box \left[ h_{\mu\nu}(x) - \frac{1}{2} \eta_{\mu\nu} h(x) \right] = -\kappa T_{\mu\nu}(x). \] (23)

On the other hand for a static source of graviton with mass \( M \) we have the energy-momentum tensor as \( T_{\mu\nu}(x) = M \delta_{\mu0} \delta_{\nu0} \delta(\vec{x}) \). Hence in momentum space one finds
\[ h_{\mu\nu}(\vec{q}) = \frac{\kappa M}{q^2} \delta(\vec{q}) \left[ \eta_{\mu\nu} + 2 \theta_{ij} p_i q_j \right], \] (24)

Therefore, having at hand the ingredients needed for calculation of the scattering amplitude \( M \) we get [10]
\[ iM_{\theta} = \frac{4\pi GM}{q^2} (2m^2 + 4\vec{p}^2) e^{i\vec{p} \cdot \vec{q}}. \] (25)

The Fourier transform of scattering amplitude leaves us with the Newton potential as
\[ U_{\theta} = -\frac{4\pi GM m^2 + 2\vec{p}^2}{E} \int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec{x} \cdot \vec{q} + i\frac{1}{2} \theta_{ij} p_i p_j}}{q^2}, \] (26)
\[ = -\frac{m^2 + 2\vec{p}^2}{E} \frac{GM}{\sqrt{(x_i - \frac{1}{2} \theta_{ij} p_j)(x_i - \frac{1}{2} \theta_{ik} p_k)}}, \]
\[ = -\frac{m^2 + 2\vec{p}^2}{E} \left( \frac{GM}{4|x|^3} \vec{L} \cdot \vec{\theta} + O(\theta^2) \right). \]

In non-relativistic limit, i.e. \( |\vec{p}| << m \) and \( E \approx m \) the above result coincides with that of [6, 7]
\[ U_{\theta}^{\text{non-rel}} = -\frac{GmM}{|x|} \frac{GmM}{4|x|^3} \vec{L} \cdot \vec{\theta} + O(\theta^2) \] (27)
which was gained by implementing (5) in \( \hat{U} = -G\frac{mM}{\sqrt{\lambda} |x|} \). For a relativistic particle with \( E \simeq |\vec{p}| \) we have
\[ U_{\theta}^{\text{rel}} = -\frac{2GME}{|x|} \frac{GME}{2|x|^3} \vec{L} \cdot \vec{\theta} + O(\theta^2). \] (28)
where the first term of (17) is the well-known result arising from the scalar-graviton scattering in absence of the noncommutativity [10].

5. Conclusion

We followed a field theoretic approach to re-derived the \( \theta \)-deformed Newton potential by calculating the matter-graviton scattering amplitude in a NC flat background. In the case of static graviton source our result confirms the earlier derivation and generalizes it to the relativistic regime.
Acknowledgments

This work has been supported financially by Research Institute for Astronomy & Astrophysics of Maragha.

References

[1] N. Seiberg, E. Witten, JHEP 9909 (1999) 032.
[2] H. S. Snyder, Phys. Rev. 71 (1947) 38.
[3] S. Doplicher, K. Fredenhagen, and J. E. Roberts, Commun. Math. Phys. 172 (1995) 187.
[4] R. Szabo, Phys. Rept. 378 (2003) 207.
[5] M. Chaichian, M. M. Sheikh-Jabbari, and A. Tureanu, Phys. Rev. Lett. 86 (2001) 2716.
[6] J. M. Romero, J. A. Santiago, J. D. Vergara, Phys. Lett. A310 (2003) 9.
[7] B. Mirza, M. Dehghani, Commun. Theor. Phys. 42 (2004) 183.
[8] N. E. J. Bjerrum-Bohr, Ph.D. thesis, The Niels Bohr Institute, University of Copenhagen, July 2004.
[9] B. Holstein, Am. J. Phys. 74 (2006) 1002.
[10] M. Scadron, Advanced Quantum Theory and its Applications through Feynman Diagrams, Springer-Verlag, New York (1979).
[11] J. W. Moffat, Phys. Lett. B493 (2000) 142.