RANKS OF OVERPARTITIONS: ASYMPTOTICS AND INEQUALITIES

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Abstract. In this paper we compute asymptotics for \( N(a, c, n) \), the number of overpartitions of \( n \) with rank congruent to \( a \) modulo \( c \). As an application we prove, among others, some inequalities conjectured by Ji, Zhang and Zhao (2018), and Wei and Zhang (2018) on ranks of overpartitions for \( n = 6 \) and \( n = 10 \).

1. Introduction and statement of results

1.1. Motivation. A partition of a positive integer \( n \) is a non-increasing sequence of positive integers (called its parts), usually written as a sum, which add up to \( n \). The number of partitions of \( n \) is denoted by \( p(n) \). For example, \( p(4) = 5 \) as the partitions of 4 are 4, 3+1, 2+2, 2+1+1, 1+1+1+1. Extending the definition, we set by convention \( p(0) = 1 \) and \( p(n) = 0 \) for \( n < 0 \).

Among many other famous achievements, Ramanujan [22] proved that if \( n \geq 0 \), then

\[
\begin{align*}
p(5n + 4) \equiv 0 \pmod{5}, \\
p(7n + 5) \equiv 0 \pmod{7}, \\
p(11n + 6) \equiv 0 \pmod{11}.
\end{align*}
\]

In order to give a combinatorial proof of these congruences, Dyson [11] introduced the rank of a partition, often known also as the Dyson rank, which is defined to be the largest part of the partition minus the number of its parts. Dyson conjectured that the partitions of \( 5n + 4 \) form 5 groups of equal size when sorted by their ranks modulo 5 and that the same is true for the partitions of \( 7n + 5 \) when working modulo 7, conjecture which was proven by Atkin and Swinnerton-Dyer [4].

An overpartition of \( n \) is a partition of \( n \) in which the first occurrence of a part may be overlined. We denote by \( \overline{p}(n) \) the number of overpartitions of \( n \). For example, \( \overline{p}(4) = 14 \) as the overpartitions of 4 are \( \overline{4}, \overline{4}, \overline{3+1}, \overline{3+1}, \overline{3}, \overline{2+2}, \overline{2+1+1}, \overline{2+1+1}, \overline{2+1+1}, \overline{2+1+1}, \overline{1+1+1+1}, \overline{1+1+1+1} \).

Overpartitions are natural combinatorial structures associated with the \( q \)-binomial theorem, Heine’s transformation or Lebesgue’s identity. For an overview and further motivation, the reader is referred to [8] and [20].

Both the rank of partitions and overpartitions have been studied extensively. By proving that some generating functions associated to the rank are holomorphic parts of harmonic Maass forms, Bringmann and Ono [7] showed that the rank partition function satisfies some other congruences of Ramanujan type. In the same spirit, Bringmann and Lovejoy [16] proved that the overpartition rank generating function is the holomorphic part of a harmonic Maass form of weight 1/2, while Dewar [10] made certain refinements.

It is customary to denote by \( N(m, n) \) the number of partitions of \( n \) with rank \( m \) and by \( \overline{N}(a, m, n) \) the number of partitions of \( n \) with rank congruent to \( a \) modulo \( m \). The corresponding quantities for overpartitions, \( \overline{N}(m, n) \) and \( \overline{N}(a, m, n) \), are denoted by an overline.

By means of generalized Lambert series, Lovejoy and Osburn [19] gave formulas for the rank differences \( \overline{N}(s, \ell, \ell n + d) - \overline{N}(s, \ell, \ell n + d) \) for \( \ell = 3 \) and \( 5 \) and \( 0 \leq d, s < \ell \), while rank differences for \( \ell = 7 \) were determined by Jennings-Shaffer [13]. Recently, by using \( q \)-series manipulations and the 3 and 5-dissection of the overpartition rank generation function, Ji, Zhang and Zhao [15] proved some identities and inequalities for the rank difference generating functions of overpartitions modulo 6 and 10, and conjectured a few others. Some further inequalities were conjectured by Wei and Zhang [24].

It is one goal of this paper to prove these conjectures. The other, more general, goal is to compute asymptotics for the ranks of overpartitions and this is what we will start with, the inequalities mentioned...
above following then as a consequence. In doing so we rely on the Hardy-Ramanujan circle method and the modular transformations for overpartitions established by Bringmann and Lovejoy [9]. While the main ideas are essentially those used by Bringmann [5] in computing asymptotics for partition ranks, several complications arise and some modifications need to be carried out.

The structure of the paper is as follows. The rest of this section is dedicated to introducing some notation that is needed in the sequel and to formulating our results. Theorem [11] is one of them and an outline of its proof is given in Section 2. The proof is given in detail in Section 3. In the final section we show how to use Theorem [11] in order to prove the inequalities conjectured by Ji, Zhang and Zhao [15] and Wei and Zhang [24], which are stated in Theorems 2–4 together with some new inequalities.

1.2. Notation and preliminaries. The overpartition generating function (see, e.g., [8]) is given by

\[ P(q) := \sum_{n \geq 0} p(n)q^n = \frac{\eta(2z)}{\eta^2(z)} = 1 + 2q + 4q^2 + 8q^3 + 14q^4 + \cdots, \]  

(1.1)

where

\[ \eta(z) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \]

denotes, as usual, Dedekind’s eta function and \( q = e^{2\pi i z}, z \in \mathbb{C} \) with \( \text{Im}(z) > 0 \). If we use the standard \( q \)-series notation

\[ (a)_n := \prod_{r=0}^{n-1} (1 - au^r), \]

\[ (a, b)_n := \prod_{r=0}^{n-1} (1 - aq^r)(1 - bq^r), \]

for \( a, b \in \mathbb{C} \) and \( n \in \mathbb{N} \cup \{\infty\} \), then we know from [16] that

\[ \mathcal{O}(u; q) := \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} \mathcal{N}(m, n)u^mq^n = \sum_{n=0}^{\infty} \sum_{u=0}^{\infty} \frac{(-1)^nu^q\frac{1}{2}n(n+1)}{(uq, q/u)_{n+1}} \]

\[ = \frac{(-q)_{\infty}}{(q)_{\infty}} \left( 1 + 2 \sum_{n \geq 1} \frac{(1-u)(1-u^{-1})(-1)^{n}q^{n^2+n}}{(1-uq^n)(1-u^{-1}q^n)} \right). \]  

(1.2)

If \( 0 < a < c \) are coprime positive integers with \( c > 2 \), and if by \( \zeta_n = e^{\frac{2\pi i n}{c}} \) we denote the primitive \( n \)-th root of unity, we set

\[ \mathcal{O} \left( \frac{a}{c}; q \right) := \mathcal{O} \left( \zeta_n^a; q \right) = 1 + \sum_{n=1}^{\infty} A \left( \frac{a}{c}; n \right) q^n. \]  

(1.3)

Let \( k \) be a positive integer. Set \( \tilde{k} = 0 \) if \( k \) is even and \( \tilde{k} = 1 \) if \( k \) is odd. Moreover, put \( k_1 = \frac{k}{(c, k)}, \)
\( c_1 = \frac{c}{(c, k)}, \) and let the integer \( 0 \leq \ell < c_1 \) be defined by the congruence \( \ell \equiv ak_1 \pmod{c_1} \). If \( \frac{b}{c} \in (0, 1) \), let

\[ s(b, c) := \begin{cases} 0 & \text{if } 0 < \frac{b}{c} \leq \frac{1}{4}, \\ 1 & \text{if } \frac{1}{4} < \frac{b}{c} \leq \frac{3}{4}, \\ 2 & \text{if } \frac{3}{4} < \frac{b}{c} < 1, \end{cases} \quad \text{and} \quad t(b, c) := \begin{cases} 1 & \text{if } 0 < \frac{b}{c} \leq \frac{1}{2}, \\ 3 & \text{if } \frac{1}{2} < \frac{b}{c} < 1. \end{cases} \]

Throughout we will often use the shorthand notation \( s = s(b, c) \) and \( t = t(b, c) \). In what follows, \( 0 \leq h < k \) are integers such that \( (h, k) = 1 \) (in case \( k = 1 \), we set \( h = 0 \) and this is the only case when \( h = 0 \) is allowed). Let \( h' \) be an integer defined by \( hh' \equiv -1 \pmod{k} \). Moreover, let

\[ \omega_{h,k} := \exp \left( \pi i \sum_{\mu=0}^{k-1} \left( \frac{\mu}{k} \right) \left( \frac{h\mu}{k} \right) \right) \]
be the multiplier occuring in the partition function transformation law, where

\[
((x)) := \begin{cases} x - \lfloor x \rfloor - \frac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}
\]

We next define several Kloosterman sums. Here and throughout we use the notation \( \sum_h \) to denote summation over the integers \( 0 \leq h < k \) that are coprime to \( k \).

If \( c \mid k \), let

\[
A_{a,c,k}(n,m) := (-1)^{k+1} \tan \left( \frac{\pi a}{c} \right) \sum_{h} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} \cot \left( \frac{\pi ah'}{c} \right) \cdot e^{-\frac{2\pi h' a^2 k_1}{c}} \cdot e^{\frac{2\pi i}{k} (nh + mh')}
\]

and

\[
B_{a,c,k}(n,m) := -\frac{1}{\sqrt{2}} \tan \left( \frac{\pi a}{c} \right) \sum_{h} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot \frac{1}{\sin \left( \frac{2\pi h'}{c} \right)} \cdot e^{-\frac{2\pi h' a^2 k_1}{c}} \cdot e^{\frac{2\pi i}{k} (nh + mh')}.
\]

If \( c \nmid k \) and \( 0 < \frac{\ell}{c_1} \leq \frac{1}{4} \), let

\[
D_{a,c,k}(n,m) := \frac{1}{\sqrt{2}} \tan \left( \frac{\pi a}{c} \right) \sum_{h} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{\frac{2\pi i}{k} (nh + mh')},
\]

and if \( c \nmid k \) and \( \frac{3}{4} < \frac{\ell}{c_1} < 1 \), let

\[
D_{a,c,k}(n,m) := -\frac{1}{\sqrt{2}} \tan \left( \frac{\pi a}{c} \right) \sum_{h} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{\frac{2\pi i}{k} (nh + mh')}.
\]

To state our results, we need at last the following quantities. The motivation behind their expressions becomes clear if one writes down explicitly the computations done in Section 3. If \( c \nmid k \), let

\[
\delta_{c,k,r} := \begin{cases} \frac{1}{16} - \frac{\ell}{2c_1} + \frac{\ell^2}{c_1^2} - \frac{r}{c_1} & \text{if } 0 < \frac{\ell}{c_1} \leq \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} < \frac{\ell}{c_1} \leq \frac{3}{4}, \\ \frac{1}{16} - \frac{3\ell}{2c_1} + \frac{\ell^2}{c_1^2} + \frac{1}{2} - r \left( 1 - \frac{\ell}{c_1} \right) & \text{if } \frac{3}{4} < \frac{\ell}{c_1} < 1, \end{cases}
\]

and

\[
m_{a,c,k,r} := \begin{cases} -\frac{1}{2c_1^2} (2(ak_1 - \ell)^2 + c_1(ak_1 - \ell) + 2rc_1(ak_1 - \ell)) & \text{if } 0 < \frac{\ell}{c_1} \leq \frac{1}{4}, \\ 0 & \text{if } \frac{1}{4} < \frac{\ell}{c_1} \leq \frac{3}{4}, \\ -\frac{1}{2c_1^2} (2(ak_1 - \ell)^2 + 3c_1(ak_1 - \ell) - 2rc_1(ak_1 - \ell) - c_1^2(2r - 1)) & \text{if } \frac{3}{4} < \frac{\ell}{c_1} < 1. \end{cases}
\]

1.3. **Statement of results.** We are now in shape to state our main results.

**Theorem 1.** If \( 0 < a < c \) are coprime positive integers and \( \varepsilon > 0 \) is arbitrary, then

\[
A \left( \frac{a}{c} ; n \right) = i \sqrt{\frac{2}{n}} \sum_{1 \leq k \leq \sqrt{n} \atop c \mid k, 2 \mid k} \frac{B_{a,c,k}(-n,0)}{\sqrt{k}} \cdot \sinh \left( \frac{\pi \sqrt{n}}{k} \right) \\
+ 2 \sqrt{\frac{2}{n}} \sum_{1 \leq k \leq \sqrt{n} \atop c \mid k, 2 \mid k, c_1 \neq 0} \frac{D_{a,c,k}(-n,m_{a,c,k,r})}{\sqrt{k}} \cdot \sinh \left( \frac{4\pi \delta_{c,k,r} \sqrt{n}}{k} \right) + O(n^\varepsilon).
\]

On using Theorem 1 together with the identity

\[
\mathcal{N}(a,c,n)q^n = \frac{1}{c} \sum_{n=0}^{\infty} p(n)q^n + \frac{1}{c} \sum_{j=1}^{c-1} \zeta_c^{-aj} \cdot O(\zeta_c^j; q),
\]

(1.6)
which follows by the orthogonality of roots of unity, and the Rademacher-type convergent series expansion
\[
\overline{p}(n) = \frac{1}{2\pi} \sum_{2|k} \sqrt{k} \sum_{h} \frac{\omega_{h,k}^2}{w_{2h,k}} \cdot e^{-2\sinh_k \cdot \frac{d}{dn}} \left( \frac{1}{\sqrt{n}} \sinh \left( \frac{\pi \sqrt{n}}{k} \right) \right)
\]
found by Zuckerman [25 p. 321, eq. (8.53)], we obtain the following.

**Corollary 1.** If \(0 \leq a < c\) are integers and \(\varepsilon > 0\) is arbitrary, then
\[
\overline{N}(a,c,n) = \frac{1}{2\pi \cdot c} \sum_{2|k} \sqrt{k} \sum_{h} \frac{\omega_{h,k}^2}{w_{2h,k}} \cdot e^{-2\sinh_k \cdot \frac{d}{dn}} \left( \frac{1}{\sqrt{n}} \sinh \left( \frac{\pi \sqrt{n}}{k} \right) \right)
\]

\[+ \frac{1}{c} \sum_{j=1}^{c-1} \zeta^{-aj} \left( i \sqrt{\frac{2}{n}} \sum_{c_j \mid k, 2|k} B_{j',c_j,k}(-n,0) \right) \cdot \sinh \left( \frac{4\pi \sqrt{\delta_{c_j,k,r(n)}}}{k} \right) + O_c(n^\varepsilon),
\]

where \(c_j = \frac{x}{(c_j)}, \overline{c}_j = \frac{x}{(c_j)}\) and \(j' = \frac{x}{(c_j)}\).

As applications, we prove the following inequalities conjectured by Ji, Zhang and Zhao [15 Conjecture 1.6 and Conjecture 1.7], and Wei and Zhang [24 Conjecture 5.10].

**Conjecture 1** (Ji–Zhang–Zhao, 2018).

(i) For \(n \geq 0\) and \(1 \leq i < 4\) we have
\[
\overline{N}(0,10,5n+i) + \overline{N}(1,10,5n+i) \geq \overline{N}(4,10,5n+i) + \overline{N}(5,10,5n+i).
\]

(ii) For \(n \geq 0\) we have
\[
\overline{N}(1,10,n) + \overline{N}(2,10,n) \geq \overline{N}(3,10,n) + \overline{N}(4,10,n).
\]

**Conjecture 2** (Wei–Zhang, 2018). For \(n \geq 11\) we have
\[
\overline{N}(0,6,3n) \geq \overline{N}(1,6,3n) = \overline{N}(3,6,3n) \geq \overline{N}(2,6,3n),
\]

\[
\overline{N}(0,6,3n+1) \geq \overline{N}(1,6,3n+1) = \overline{N}(3,6,3n+1) \geq \overline{N}(2,6,3n+1),
\]

\[
\overline{N}(1,6,3n+2) \geq \overline{N}(2,6,3n+2) \geq \overline{N}(0,6,3n+2) \geq \overline{N}(3,6,3n+2).
\]

More precisely, we prove the following inequalities.

**Theorem 2.** For \(n \geq 2\) we have
\[
\overline{N}(1,10,n) + \overline{N}(2,10,n) > \overline{N}(3,10,n) + \overline{N}(4,10,n),
\]

\[
\overline{N}(0,10,n) + \overline{N}(3,10,n) > \overline{N}(2,10,n) + \overline{N}(5,10,n),
\]

\[
\overline{N}(0,10,n) + \overline{N}(1,10,n) > \overline{N}(4,10,n) + \overline{N}(5,10,n),
\]

\[
\overline{N}(0,6,n) + \overline{N}(1,6,n) > \overline{N}(2,6,n) + \overline{N}(3,6,n),
\]

\[
\overline{N}(0,6,3n) + \overline{N}(3,6,3n) > \overline{N}(1,6,3n) + \overline{N}(2,6,3n),
\]

\[
\overline{N}(0,6,3n+1) + \overline{N}(3,6,3n+1) > \overline{N}(1,6,3n+1) + \overline{N}(2,6,3n+1),
\]

\[
\overline{N}(0,6,3n+2) + \overline{N}(3,6,3n+2) < \overline{N}(1,6,3n+2) + \overline{N}(2,6,3n+2).
\]

**Theorem 3.** For \(n \geq 11\) we have
\[
\overline{N}(0,6,3n) > \overline{N}(1,6,3n) > \overline{N}(2,6,3n),
\]

\[
\overline{N}(0,6,3n+1) > \overline{N}(1,6,3n+1) > \overline{N}(2,6,3n+1),
\]

\[
\overline{N}(1,6,3n+2) > \overline{N}(2,6,3n+2) > \overline{N}(0,6,3n+2) > \overline{N}(3,6,3n+2).
\]
Remark 1. The inequality (1.12) was proven by Ji, Zhang and Zhao [15] for \( n \equiv 0 \pmod{5} \).

Remark 2. The inequality (1.13) is new. The identities from (1.7) and (1.8) were proven by Ji, Zhao and Zhang [15], who further proved that \( N(0, 6, 3n) > N(2, 6, 3n) \) for \( n \geq 1 \), and \( N(0, 6, 3n + 1) > N(2, 6, 3n + 1) \) for \( n \geq 0 \). While (1.13) follows easily now for \( n \equiv 0, 1 \pmod{3} \), the inequality is not at all clear for \( n \equiv 2 \pmod{3} \), as the same authors also showed that \( N(0, 6, 3n + 2) < N(2, 6, 3n + 2) \) for \( n \geq 1 \) and \( N(1, 6, 3n + 2) > N(3, 6, 3n + 2) \) for \( n \geq 0 \). For a list of the identities and inequalities already proven, see [15, Theorem 1.4].

Remark 3. The identity and inequalities from (1.7) were also proven by Wei and Zhang [24, p. 25].

### 2. Strategy of the proof of Theorem 1

For the benefit of the reader, we outline the main steps in proving Theorem 1 along with several other estimates that will be used in the sequel.

#### 2.1. Circle method.

The main idea of our approach is the Hardy-Ramanujan circle method. By Cauchy’s Theorem we have, for \( n > 0 \),

\[
A \left( \frac{a}{c}; n \right) = \frac{1}{2\pi i} \int_{C} \frac{O \left( \frac{a}{c}; q \right)}{q^{n+1}} dq,
\]

where \( C \) may be taken to be the circle of radius \( e^{-\frac{2\pi}{n}} \) parametrized by \( q = e^{-\frac{2\pi}{n} + 2\pi it} \) with \( t \in [0, 1] \), in which case we obtain

\[
A \left( \frac{a}{c}; n \right) = \int_{0}^{1} \frac{O \left( \frac{a}{c}; e^{-\frac{2\pi}{n} + 2\pi it} \right)}{e^{2\pi - 2\pi int}} dt.
\]

If \( \frac{bn}{k_1} < \frac{b}{k} < \frac{bn}{k_2} \) are adjacent Farey fractions in the Farey sequence of order \( N := \lfloor n^{1/2} \rfloor \), we put

\[
\vartheta'_{h,k} := \frac{1}{k(k_1 + k)} \quad \text{and} \quad \vartheta''_{h,k} := \frac{1}{k(k_2 + k)}.
\]

Splitting the path of integration along the Farey arcs \( -\vartheta'_{h,k} \leq \Phi \leq \vartheta''_{h,k} \), where \( \Phi := t - \frac{b}{k} \) and \( 0 < h < k \leq N \) with \( (h, k) = 1 \), we have

\[
A \left( \frac{a}{c}; n \right) = \sum_{h,k} e^{-\frac{2\pi in}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} \frac{O \left( \frac{a}{c}; e^{2\pi i(h+iz)} \right)}{e^{2\pi i/k}} d\Phi,
\]

where \( z = \frac{k}{n} - k\Phi i \).

The reader familiar with some basics from Farey theory will recognize immediately the inequality

\[
\frac{1}{k+k_j} \leq \frac{1}{N+1}
\]

for \( j \in \{1, 2\} \), together with several other known facts (which are otherwise very easy to prove) such as

\[
\text{Re}(z) = \frac{k}{n}, \quad \text{Re} \left( \frac{1}{z} \right) > \frac{k}{2}, \quad |z|^{-\frac{1}{2}} \leq n^{-\frac{1}{2}} \cdot k^{-\frac{1}{2}} \quad \text{and} \quad \vartheta'_{h,k} + \vartheta''_{h,k} \leq \frac{2}{k(N+1)}.
\]

For a nice introduction to Farey fractions, one can consult Apostol [3, Chapter 5.4].
2.2. Modular transformation laws. Our next step in the proof of Theorem 1 requires the modular transformation laws. For \( O(\frac{a}{c}; q) \) established by Bringmann and Lovejoy [6], the proof of which can be found in [6] pp. 11–17. For \( 0 < a < c \) coprime with \( c > 2 \), let

\[
\mathcal{U}\left(\frac{a}{c}; q\right) = \mathcal{U}\left(\frac{a}{c}; z\right) := \eta\left(\frac{z}{2}\right) \sin\left(\frac{\pi a}{c}\right) \sum_{n \in \mathbb{Z}} \frac{(1 + q^n)q^{n^2 + \frac{4}{c}}}{1 - 2 \cos\left(\frac{2\pi a}{c}\right) q^n + q^{2n}}.
\]

\[
\mathcal{U}(a, b, c; q) = \mathcal{U}(a, b, c; z) := \eta\left(\frac{z}{2}\right) e^{\frac{\pi a(4b - c - 2s(b,c))}{c}} q^{\frac{2s(b,c) + b}{c} + \frac{b^2}{c^2}} \sum_{m \in \mathbb{Z}} \frac{q^{m(2m+1) + ms(b,c)}}{1 - e^{-2\pi i m/c} q^{m+\frac{b}{c}}},
\]

\[
\mathcal{V}(a, b, c; q) = \mathcal{V}(a, b, c; z) := \eta\left(\frac{z}{2}\right) e^{\frac{\pi a(4b - c - 2s(b,c))}{c}} q^{\frac{2s(b,c) + b}{c} + \frac{b^2}{c^2}} \sum_{m \in \mathbb{Z}} (-1)^m q^{m(2m+1) + ms(b,c)} \frac{(1 + e^{-2\pi i m/c} q^{m+\frac{b}{c}})}{1 - e^{-2\pi i m/c} q^{m+\frac{b}{c}}},
\]

\[
O(a, b, c; q) = O(a, b, c; z) := \eta\left(\frac{z}{2}\right) e^{\frac{\pi a(4b - c - 2s(b,c))}{c}} q^{\frac{2s(b,c) + b}{c} + \frac{b^2}{c^2}} \sum_{m \in \mathbb{Z}} q^{m^2 + m} \frac{(1 + e^{-2\pi i m/c} q^{m+\frac{b}{c}})}{1 - e^{-2\pi i m/c} q^{m+\frac{b}{c}}}.
\]

Furthermore, if

\[
H_{a,c}(x) := \frac{e^x}{1 - 2 \cos\left(\frac{2\pi a}{c}\right) e^x + e^{2x}},
\]

we define the Mordell-type integral

\[
I_{a,c,k,\nu} := \int_{\mathbb{R}} e^{-\frac{2\pi ax^2}{k}} H_{a,c} \left( \frac{2\pi i \nu}{k} - \frac{2\pi zx}{k} - \frac{\nu}{2} \right) dx.
\]

If \( k \) is even and \( c|k \), or if \( k \) is odd, \( a = 1 \) and \( c = 4k \), there might be a pole at \( x = 0 \). In these cases we need to take the Cauchy principal value of the integral. We will make this precise at a later stage.

By using Poisson summation and proceeding similarly to [2] and [5], Bringmann and Lovejoy [6] proved the following transformation laws.

**Theorem 5 ([6] Theorem 2.1).** Assume the notation above and let \( q = e^{\frac{2\pi i}{k}(h + iz)} \) and \( q_1 = e^{\frac{2\pi i}{k}(h'+\nu z)} \), with \( z \in \mathbb{C} \) and \( \text{Re}(z) > 0 \).

1. If \( c|k \) and \( 2|k \), then

\[
O\left(\frac{a}{c}; q\right) = (-1)^{k_1+1}\cdot e^{-\frac{2\pi a^2 h h' k}{k}} \cdot \tan\left(\frac{\pi a}{c}\right) \cdot \cot\left(\frac{\pi a h}{c}\right) \cdot \omega_{h,k/2}^2 \cdot \omega_{h,k/2}^2 z^{-\frac{1}{2}} \cdot O\left(\frac{a h}{c}; q_1\right) + \sum_{\nu=0}^{k-1} (-1)^{\nu} e^{-\frac{2\pi a h'}{k} z^2} \cdot I_{a,c,k,\nu}(z).
\]

2. If \( c|k \) and \( 2 \nmid k \), then

\[
O\left(\frac{a}{c}; q\right) = -\sqrt{2} \cdot e^{\frac{\pi i h'}{k}} \cdot \tan\left(\frac{\pi a}{c}\right) \cdot \omega_{h,k}^2 \cdot \omega_{h,k}^2 z^{-\frac{1}{2}} \cdot \mathcal{U}\left(\frac{a h'}{c}; q_1\right) + \sum_{\nu=0}^{k-1} e^{-\frac{\pi i h'}{k} (2\nu^2 - \nu)} \cdot I_{a,c,k,\nu}(z).
\]

\[1\text{In passing we correct the definitions of } \mathcal{U}(a, b, c; q) \text{ and } \mathcal{V}(a, b, c; q). \text{ Some misprints occured in their original expressions from [6] p. 8 and the necessary changes become clear on consulting the proof, see [6] pp. 11–17.}

\[2\text{Some further corrections are in order; namely, the "-" sign in front of the expressions from (3)–(6) in their original formulation [6] Theorem 2.1 should be a "+", and the other way around for (1)–(2), the reason being that the "+" sign from the expression of the residues } \lambda_{a,m}^2 \text{ (see [6] p. 13)} \text{ is meant to be a "-". All necessary modifications are made here.}\]
(3) If \( c \nmid k, 2|k, \) and \( c_1 \neq 2, \) then
\[
\mathcal{O} \left( \frac{a}{c}; q \right) = 2e^{-2\pi ia^2 k'/k_{1c}} \cdot \tan \left( \frac{\pi a}{c} \right) \frac{\omega_{h,k}^2}{\omega_{k/2}} z^{-\frac{1}{2}} \cdot (-1)^{c_1(k+k_{1})} \cdot \mathcal{O} \left( \frac{ah'}{c_1}, \ell c, c; q_1 \right) + \frac{4 \sin^2 \left( \frac{\pi a}{c} \right) \cdot \omega_{h,k}^2}{\omega_{k/2} \cdot k} \sum_{\nu=0}^{k-1} (-1)^\nu e^{-2\pi i h'_{\nu}} \cdot I_{a,c,k,\nu}(z).
\]

(4) If \( c \nmid k, 2|k, \) and \( c_1 = 2, \) then
\[
\mathcal{O} \left( \frac{a}{c}; q \right) = e^{2\pi ia^2 k'/k_{1c}} \cdot \tan \left( \frac{\pi a}{c} \right) \frac{\omega_{h,k}^2}{\omega_{k/2}} z^{-\frac{1}{2}} \cdot \mathcal{V} \left( \frac{ah'}{c}; q_1 \right) + \frac{4 \sin^2 \left( \frac{\pi a}{c} \right) \cdot \omega_{h,k}^2}{\omega_{k/2} \cdot k} \sum_{\nu=0}^{k-1} (-1)^\nu e^{-2\pi i h'_{\nu}} \cdot I_{a,c,k,\nu}(z).
\]

(5) If \( c \nmid k, 2 \nmid k, \) and \( c_1 \neq 4, \) then
\[
\mathcal{O} \left( \frac{a}{c}; q \right) = \sqrt{2} \cdot e^{-2\pi ia^2 k'/k_{1c}} \cdot \tan \left( \frac{\pi a}{c} \right) \frac{\omega_{h,k}^2}{\omega_{2h,k}} z^{-\frac{1}{2}} \cdot \mathcal{U} \left( \frac{ah'}{c_1}, c; q_1 \right) + \frac{4 \sqrt{2} \sin^2 \left( \frac{\pi a}{c} \right) \cdot \omega_{h,k}^2}{\omega_{2h,k} \cdot k} \sum_{\nu=0}^{k-1} e^{-2\pi i h'_{\nu}} \cdot I_{a,c,k,\nu}(z).
\]

(6) If \( c \nmid k, 2 \nmid k, \) and \( c_1 = 4, \) then
\[
\mathcal{O} \left( \frac{a}{c}; q \right) = e^{-2\pi ia^2 k'/k_{1c}} \cdot \tan \left( \frac{\pi a}{c} \right) \frac{\omega_{h,k}^2}{\sqrt{2} \cdot \omega_{2h,k}} z^{-\frac{1}{2}} \cdot \mathcal{V} \left( \frac{ah'}{c_1}, c; q_1 \right) + \frac{4 \sqrt{2} \sin^2 \left( \frac{\pi a}{c} \right) \cdot \omega_{h,k}^2}{\omega_{2h,k} \cdot k} \sum_{\nu=0}^{k-1} e^{-2\pi i h'_{\nu}} \cdot I_{a,c,k,\nu}(z).
\]

In addition to these modular transformations, we need some further estimates.

2.3. The Mordell integral \( I_{a,c,k,\nu}. \) In the previous subsection we introduced
\[
I_{a,c,k,\nu} = \int_{\mathbb{R}} e^{-2\pi ix^2 k^2/k} H_{a,c} \left( \frac{2\pi i v}{k} - \frac{2\pi x}{k} - \frac{\tilde{k} \pi i}{2k} \right) dx.
\]
Recalling the definition of \( H_{a,c}(x), \) it is easy to see that
\[
H_{a,c}(x) = \frac{e^x}{1 - 2 \cos \left( \frac{2\pi a}{c} \right) e^x + e^{2x}} = \frac{1}{4 \sinh \left( \frac{x}{2} + \frac{\pi a}{c} \right) \sinh \left( \frac{x}{2} - \frac{\pi a}{c} \right)},
\]
and so \( H_{a,c}(x) \) can only have poles in points of the form
\[
x = 2\pi i \left( s \pm \frac{a}{c} \right)
\]
with \( s \in \mathbb{Z}. \)

For \( 2|k, c|k \) and \( \nu = \frac{k_1 a}{c} \) or \( \nu = k \left( 1 - \frac{a}{c} \right) \) there may be a pole at \( x = 0. \) The same is true if \( 2 \nmid k, \nu = 0, a = 1 \) and \( c = 4k. \) In both cases we must consider the Cauchy principal value of the integral \( I_{a,c,k,\nu}, \) that is, instead of \( \mathbb{R} \) we choose as path of integration the real line indented below 0.

The following\(^3\) is adapted after Bringmann [5 Lemma 3.1].

\(^3\)Note that there are a few typos in the formulation of the original result from which this lemma is inspired, namely Bringmann [5 Lemma 3.1]. More precisely, \( n \frac{2}{3} \) should read \( n^\frac{2}{3}, k \) should read \( k^{-\frac{1}{2}} \) and the 6kc factor from the definition of \( g_{a,c,k,\nu} \) should be removed. These changes however do not affect the proof.
Lemma 1. Let \( n \in \mathbb{N}, N = \lfloor n^{1/2} \rfloor \) and \( z = \frac{k}{n} - k \Phi i \), where \(-\frac{1}{k(k+k_1)} \leq \Phi \leq \frac{1}{k(k+k_2)}\) and \( \frac{b_1}{k} < \frac{b_2}{k} \) are adjacent Farey fractions in the Farey sequence of order \( N \). Then

\[
z^{\frac{1}{2}} \cdot I_{a,c,k,\nu} \ll_c \ k^{-\frac{3}{2}} \cdot g_{a,c,k,\nu},
\]

where

\[
g_{a,c,k,\nu} := \begin{cases} 
\left( \min \left\{ \left| \nu - \frac{1}{k} + \frac{a}{c} \right|, \left| \nu - \frac{1}{k} - \frac{a}{c} \right| \right\} \right)^{-1} & \text{if } k \text{ is odd, } \nu \neq 0 \text{ and } \frac{a}{c} \neq \frac{1}{k}, \\
\left( \min \left\{ \left| \nu + \frac{a}{c} \right|, \left| \nu - \frac{1}{k} \right| \right\} \right)^{-1} & \text{if } k \text{ is even and } \nu \notin \left\{ \frac{ka}{c}, k \left(1 - \frac{a}{c} \right) \right\}, \\
\left( \frac{a}{k} \right)^{-1} & \text{otherwise},
\end{cases}
\]

and \( \{x\} = x - \lfloor x \rfloor \) is the fractional part of \( x \in \mathbb{R} \).

Proof. Let us first treat the case when \( k \) is odd and we encounter no poles. We have \( \widetilde{k} = 1 \) and

\[
I_{a,c,k,\nu} = \int_{\mathbb{R}} e^{-\frac{2kz^2}{\pi}} H_{a,c}\left( \frac{2\pi i\nu}{k} - \frac{2\pi z x}{k} - \frac{\pi i}{2k} \right) dx.
\]

If we write \( \frac{z}{k} = re^{i\phi} \) with \( r > 0 \), then \( |\phi| < \frac{\tilde{k}}{2} \) since \( \text{Re}(z) > 0 \). The substitution \( \tau = \frac{z}{k} \) yields

\[
z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) = \frac{k}{\pi z} \int_{L} e^{-\frac{2kz^2}{\pi}} H_{a,c}\left( \frac{2\pi i\nu}{k} - \frac{\pi i}{2k} - 2\tau \right) d\tau,
\]

where \( L \) is the line passing through 0 at an angle of argument \( \pm \phi \). One easily sees that, for \( 0 \leq t \leq \phi \),

\[
e^{-\frac{2kR^2}{\pi} z t} H_{a,c}\left( \frac{2\pi i\nu}{k} - \frac{\pi i}{2k} \pm 2Re^{it} \right) dx \to 0 \quad \text{as } R \to \infty.
\]

As the integrand from (2.2) has no poles, we can shift the path of integration \( L \) to the real line and obtain

\[
z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) = \frac{k}{\pi z} \int_{\mathbb{R}} e^{-\frac{2kz^2}{\pi}} H_{a,c}\left( \frac{2\pi i\nu}{k} - \frac{\pi i}{2k} - 2t \right) dt.
\]

The inequality

\[
\left| \sinh \left( \frac{\pi i\nu}{k} - \frac{\pi i}{4k} - t \pm \frac{\pi ia}{c} \right) \right| \geq \left| \sin \left( \frac{\pi \nu}{k} - \frac{\pi}{4k} \pm \frac{\pi a}{c} \right) \right|
\]

follows immediately for \( t \in \mathbb{R} \) from the definition of sinh and some easy manipulations, while the estimate

\[
\left| \sin \left( \frac{\pi \nu}{k} - \frac{\pi}{4k} - \frac{\pi a}{c} \right) \right| \geq \min \left\{ \left| \nu - \frac{1}{k} + \frac{a}{c} \right|, \left| \nu - \frac{1}{k} - \frac{a}{c} \right| \right\} \gg_c \min \left\{ \left| \nu - \frac{1}{k} + \frac{a}{c} \right|, \left| \nu - \frac{1}{k} - \frac{a}{c} \right| \right\}
\]

is clear. Therefore we have

\[
z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) \ll_c \frac{k}{\min \left\{ \left| \nu - \frac{1}{k} + \frac{a}{c} \right|, \left| \nu - \frac{1}{k} - \frac{a}{c} \right| \right\} } \int_{\mathbb{R}} e^{-\frac{2kRe(\frac{1}{z})^2}{\pi}} dt.
\]

Noting that

\[
e^{-\frac{2kR^2}{\pi} z t} = e^{-\frac{2k \pi}{\pi} Re(\frac{1}{z})^2}, \quad Re\left( \frac{1}{z} \right) = \frac{1}{|z|}, \quad |z|^{-\frac{1}{2}} \leq \sqrt{2} \cdot \sqrt{n} \cdot k^{-1},
\]

and making the substitution \( t \mapsto \sqrt{\frac{2kRe(\frac{1}{z})}{\pi}} \cdot t \), we obtain the claim.

If \( k \) is even and \( c \nmid k \), then we proceed similarly as above. If however the integrand in (2.1) has a pole at \( x = 0 \), in both of the cases \( c \nmid k \) and \( c \nmid k \), instead of \( \mathbb{R} \), we must consider the path of integration to be the real line indented below 0.

For simplicity, let us present the case when \( 2 \nmid k \), as the case \( 2 \nmid k \) is completely analogous. After doing the same change of variables as before and (if needed) shifting the path of integration (which will now consist of a straight line passing through 0 at an angle \( \pm \phi \) with a small segment centered at 0 removed and replaced by a semicircle inclined also at an angle \( \pm \phi \)), the new path of integration will be given by
\[ \gamma_{R,\varepsilon} = [-R, -\varepsilon] \cup C_{\varepsilon} \cup [R, \varepsilon] \] where \( C_{\varepsilon} \) is the positively oriented semicircle of radius \( \varepsilon \) around 0 below the real line and
\[
I_{a,c,k,\nu} = \frac{k}{\pi z} \int_{\gamma_{R,\varepsilon}} e^{-\frac{2k}{\pi z} t} H_{a,c} \left( \frac{2\pi i \nu}{k} - 2t \right) dt = \frac{k}{4\pi z} \int_{\gamma_{R,\varepsilon}} e^{-\frac{2k}{\pi z} t} \sinh(t) \sinh \left( t - \frac{2\pi ia}{c} \right) dt.
\]

If we let \( D_{R,\varepsilon} \) be the enclosed path of integration \( \gamma_{R,\varepsilon} \cup [R, R + \pi i a/c] \cup [R + \pi i a/c, -R + \pi i a/c] \cup [-R + \pi i a/c, -R] \) and set
\[ f(w) := \frac{e^{-\frac{2k}{\pi z} w}}{\sinh(w) \sinh \left( w - \frac{2\pi i a}{c} \right)}, \]
then by the Residue Theorem we obtain
\[
\frac{4\pi z}{k} \cdot I_{a,c,k,\nu} = -\frac{2\pi a}{c} + \left( \int_{-R}^{R} + \int_{-R + \pi i a/c}^{R + \pi i a/c} + \int_{R + \pi i a/c}^{-R} \right) \frac{e^{-\frac{2k}{\pi z} w} \sinh(w) \sinh \left( w - \frac{2\pi i a}{c} \right)}{\sinh(w) \sinh \left( w - \frac{2\pi i a}{c} \right)} dw,
\]

since inside and on \( D_{R,\varepsilon} \) the only pole of \( f \) is at \( w = 0 \), with residue
\[ \text{Res}_{w=0} f(w) = \frac{i}{\sin \left( \frac{2\pi a}{c} \right)}. \]

On \([-R + \pi i a/c, -R]\) and \([R + \pi i a/c, R]\) we have \( |\sinh(w) \sinh \left( w - \frac{2\pi i a}{c} \right)| \geq \sin^2 w R \) and \( |e^{-\frac{2k}{\pi z} w}| = e^{-\frac{2k}{\pi z} \Re \left( \frac{1}{2} \right) R^2} \), thus the two corresponding integrals tend to 0 as \( R \to 0 \), whereas on \([-R + \pi i a/c, R + \pi i a/c]\) we have, after a change of variables,
\[
\int_{-R + \pi i a/c}^{R + \pi i a/c} \frac{e^{-\frac{2k}{\pi z} w} \sinh(w) \sinh \left( w - \frac{2\pi i a}{c} \right)}{\sinh(w) \sinh \left( w - \frac{2\pi i a}{c} \right)} dw = \int_{-R}^{R} \frac{e^{\frac{2k}{\pi z} \left( t + \frac{2\pi i a}{c} \right)^2}}{\sinh(t + \frac{2\pi i a}{c}) \sinh(t - \frac{2\pi i a}{c})} dt.
\]

Proceeding now along the same lines as in the case when \( k \) is odd we obtain
\[ z^{\frac{1}{2}} \cdot I_{a,c,k,\nu}(z) \ll \left( \frac{\pi a}{c} \right)^{-1} \cdot k \frac{1}{|z|^2} \int_{\mathbb{R}} e^{-\frac{2k}{\pi z} \Re \left( \frac{1}{2} \right) t^2} dt, \]
and the proof is complete. \( \square \)

2.4. Kloosterman sums. The following is a variation of Bringmann [5, Lemma 3.2], cf. Andrews [2, Lemma 4.1].

**Lemma 2.** Let \( m, n \in \mathbb{Z}, 0 \leq \sigma_1 < \sigma_2 \leq k \) and \( D \in \mathbb{Z} \) with \( (D, k) = 1 \).

(i) We have
\[
\sum_{h_1 \leq D h' \leq \sigma_2} \frac{\omega_{h_1,k}^2}{\omega_{2h,k}^2} \cdot e^{\frac{2\pi i}{k} (hn + h' m)} \ll (24n + 1, k)^{\frac{1}{2}} \cdot k^\frac{1}{2} + \varepsilon. \quad (2.3)
\]

(ii) If \( c | k \), we have
\[
\tan \left( \frac{\pi a}{c} \right) \sum_{h_1 \leq D h' \leq \sigma_2} \frac{\omega_{h_1,k}^2}{\omega_{2h,k} \sin \left( \frac{2\pi a}{c} \right)} \cdot e^{-\frac{2\pi i h'^2 k_1}{c}} \cdot e^{\frac{2\pi i}{k} (nh + mh')} \ll (24n + 1, k)^{\frac{1}{2}} \cdot k^\frac{1}{2} + \varepsilon. \quad (2.4)
\]

(iii) If \( c | k \), we have
\[
\tan \left( \frac{\pi a}{c} \right) \sum_{h_1 \leq D h' \leq \sigma_2} \frac{\omega_{h_1,k}^2}{\omega_{2h,k} (-1)^{k_1 + 1} \cot \left( \frac{\pi a h'}{c} \right)} \cdot e^{-\frac{2\pi i h'^2 k_1}{c}} \cdot e^{\frac{2\pi i}{k} (nh + mh')} \ll (24n + 1, k)^{\frac{1}{2}} \cdot k^\frac{1}{2} + \varepsilon. \quad (2.5)
\]
The implied constants are independent of \( a \) and \( k \), and \( \varepsilon > 0 \) can be taken arbitrarily.
Proof. Part (i) follows simply by replacing $\omega_{h,k}$ by $\frac{\omega_{h,k}^2}{\omega_{2h,k}}$ in the proof of Andrews [2, Lemma 4.1]. As the proof of (2.4) is completely analogous to that of (2.4), we deal only with part (ii). We set $\tilde{c} = c$ if $k$ is odd and $\tilde{c} = 2c$ if $k$ is even. Since $e^{-2\pi h'/a^2 k}$ depends only on the residue class of $h'$ modulo $\tilde{c}$, the left-hand side of (2.4) can be rewritten as

$$\tan \left( \frac{\pi a}{c} \right) \sum_{c_j} e^{-\frac{2\pi h'_a}{c} c_j} \sum_{\sigma_1 \leq Dh' \leq \sigma_2} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{\frac{2\pi i}{c}(nh+mh')}$$

where $c_j$ runs over a set of primitive residues modulo $\tilde{c}$. Furthermore, we have

$$\sum_{\sigma_1 \leq Dh' \leq \sigma_2} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{\frac{2\pi i}{c}(nh+mh')} = \frac{1}{c} \sum_{r \pmod{\tilde{c}}} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \cdot e^{\frac{2\pi i}{c}(r'(h'-c_j))}$$

and the proof is concluded on invoking part (i) and noting that $\frac{kr}{c} \in \mathbb{Z}$. \qed

3. PROOF OF THEOREM 1

We turn our focus now to the proof of Theorem 1, the whole section being dedicated to this purpose. We proceed as described in the strategy presented in Section 2.4. Invoking Theorem 5, we obtain

$$A \left( \frac{a}{c} ; n \right) = i \tan \left( \frac{\pi a}{c} \right) \sum_{h,k \neq 2} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} (-1)^{k+1} \cot \left( \frac{\pi ah'}{c} \right) e^{-\frac{2\pi h'/a^2}{c} k_1} \int_{-\theta_{h,k}'}^{\theta_{h,k}'} z^{-\frac{1}{2}} e^{\frac{2\pi n z}{k}} \cdot O \left( \frac{ah'}{c} ; q_1 \right) d\Phi$$

$$- \sqrt{2} \tan \left( \frac{\pi a}{c} \right) \sum_{h,k \neq 2} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \int_{\epsilon(h+k)} \frac{2\pi i a^2 h'/k}{c} \int_{-\theta_{h,k}'}^{\theta_{h,k}'} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} \cdot U \left( \frac{ah'}{c} ; q_1 \right) d\Phi$$

$$+ 2 \tan \left( \frac{\pi a}{c} \right) \sum_{h,k \neq 2} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} \epsilon(h+k) e^{-\frac{2\pi i a^2 h'/k}{c} c_1} \cdot \int_{-\theta_{h,k}'}^{\theta_{h,k}'} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} \cdot O \left( \frac{ah'}{c} ; q_1 \right) d\Phi$$

$$+ \tan \left( \frac{\pi a}{c} \right) \sum_{h,k \neq 2} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} e^{-\frac{2\pi i a^2 h'/k}{c} c_1} \cdot \int_{-\theta_{h,k}'}^{\theta_{h,k}'} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} \cdot V \left( \frac{ah'}{c} ; q_1 \right) d\Phi$$

$$+ \frac{\sqrt{2}}{2} \tan \left( \frac{\pi a}{c} \right) \sum_{h,k \neq 2} \frac{\omega_{h,k}^2}{\omega_{2h,k}} \epsilon(h+k) e^{-\frac{2\pi i a^2 h'/k}{c} c_1} \cdot \int_{-\theta_{h,k}'}^{\theta_{h,k}'} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} \cdot U \left( \frac{ah'}{c} ; q_1 \right) d\Phi$$

$$+ \frac{1}{2} \tan \left( \frac{\pi a}{c} \right) \sum_{h,k \neq 2} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} e^{-\frac{2\pi i a^2 h'/k}{c} c_1} \cdot \int_{-\theta_{h,k}'}^{\theta_{h,k}'} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} \cdot V \left( \frac{ah'}{c} ; q_1 \right) d\Phi$$

$$+ 4 \sin^2 \left( \frac{\pi a}{c} \right) \sum_{h,k \neq 2} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} e^{-\frac{2\pi i h'/k}{c}} \epsilon(h+k) e^{-\frac{2\pi i h'/k}{c} k} \cdot \sum_{\nu=0}^{k-1} (-1)^\nu \cdot e^{-\frac{2\pi i h'/k}{c} \nu} \cdot \int_{-\theta_{h,k}'}^{\theta_{h,k}'} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} \cdot I_{a,c,k,\nu}(z) d\Phi$$
For the convenience of the reader, we divide our proof into several steps. We start by estimating the sums where we write

For the convenience of the reader, we divide our proof into several steps. We start by estimating the sums where we write

3.1. Estimates for the sums \( \Sigma_2, \Sigma_5 \) and \( \Sigma_6 \). To estimate \( \Sigma_2 \), notice that we can write

where \( m_r, n_r \in \mathbb{Z} \) and the coefficients \( a_2(r) \) and \( b_2(r) \) are independent of \( a, c, k \) and \( h \).

On replacing \( z \) by \( z_1 = z/2 \) in the generating function \( \mathcal{P}(q_1) \), we have

where we write \( q_1 = e^{2 \pi i z_1} \). It follows that

where we can write

We treat the sum coming from the constant term and the two sums coming from the case \( r \geq 1 \) separately. The former will contribute to the main term, while the latter two sums will contribute to the error term. We denote the associated sums by \( S_1, S_2 \) and \( S_3 \) and we first estimate \( S_2 \).
We write
\[ \int_{-\theta''_{h,k}}^{\theta''_{h,k}} = \int \frac{1}{k(N+k)} + \int \frac{1}{k(k_1+k)} + \int \frac{1}{k(k(N+k))} \]  
and denote the associated sums by \( S_{21}, S_{22} \) and \( S_{23} \). This way of splitting the integral is motivated by the Farey dissection used by Rademacher [21, pp. 504–509]. It allows us to interchange summation with the integral and yields a so-called complete Kloosterman sum and two incomplete Kloosterman sums. Lemma 2 applies to both types of sums.

We know (see, e.g., [12]) that
\[ \overline{p}(n) \sim \frac{1}{8n} e^{\pi \sqrt{n}}, \]
thus
\[ \overline{p}(n) < e^{\pi \sqrt{n}} \text{ as } n \to \infty. \] (3.3)

We first consider \( S_{21} \). As the integral does not depend on \( h \), we can perform summation with respect to \( h \). Using in turn the bound \( (3.3) \) applied to \( a_2(r) \), Lemma \( 2 \) the estimates from \( (3.1) \), and the well-known bound \( \sigma_0(n) = o(n^\varepsilon) \) for all \( \varepsilon > 0 \), we obtain
\[
S_{21} \ll \sum_{r=1}^{\infty} a_2(r) \cdot \sum_{\ell \mid k} \tan \left( \frac{\pi a}{c} \right) \sum_{h} \frac{\omega_{h,k}^2}{\omega_{2h,k} \sin \left( \frac{\pi a h'}{c} \right)} \cdot e^{-\frac{2\pi i h' k_1}{c}} - \frac{2\pi i h n}{k} + \frac{2\pi i h k'}{k} \cdot \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} z^{-\frac{1}{2}} \cdot e^{-\frac{2\pi i k}{k}(r - \frac{1}{16}) + \frac{2\pi i n}{k}} d\Phi
\]
\[
\ll \sum_{r=1}^{\infty} a_2(r) \cdot \sum_{\ell \mid k} \frac{1}{d} \sum_{k \leq N} (dk)^{-1+\varepsilon}
\ll \sum_{d \mid 24n+1} d^{-\frac{1}{2}+\varepsilon} \int_{1}^{N/d} x^{-1+\varepsilon} dx = \sum_{d \mid 24n+1} d^{-\frac{1}{2}} \cdot d^\varepsilon \cdot \left( \frac{N}{d} \right)^\varepsilon \ll \sum_{d \mid 24n+1} d^{-\frac{1}{2}} \cdot n^\varepsilon
\ll n^{\varepsilon}. \]

For a proof of the fact that \( \sigma_0(n) = o(n^\varepsilon) \) see, e.g., Apostol [3, p. 296]. Here we bound trivially
\[
\sum_{d \mid 24n+1} d^{-\frac{1}{2}} < \sum_{d \mid 24n+1} 1 = \sigma_0(24n + 1) = o(n^\varepsilon)
\]
and choose \( 0 < \varepsilon < \varepsilon/2 \), where \( \sigma_0(n) \) denotes, as usual, the number of divisors of \( n \).

Since \( S_{22} \) and \( S_{23} \) are treated in the exact same way, we only consider \( S_{22} \). Writing
\[
\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} = \sum_{\ell = k_1 + k}^{N+k-1} \int_{-\frac{1}{k\ell}}^{\frac{1}{k\ell}} \]
we see that
\[
S_{22} \ll \sum_{r=1}^{\infty} a_2(r) \sum_{\ell \mid k} \sum_{\ell = k_1 + k}^{N+k-1} \cdot \tan \left( \frac{\pi a}{c} \right) \sum_{h \leq \ell} \frac{\omega_{h,k}^2}{\omega_{2h,k} \sin \left( \frac{\pi a h}{c} \right)} \cdot e^{-\frac{2\pi i h' k_1}{c}} - \frac{2\pi i h n}{k} + \frac{2\pi i h k'}{k} \cdot \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} z^{-\frac{1}{2}} \cdot e^{-\frac{2\pi i k}{k}(r - \frac{1}{16}) + \frac{2\pi i n}{k}} d\Phi
\]
Again from basic facts of Farey theory it follows that
\[
N - k < k_1, k_2 \leq N \quad \text{and} \quad k_1 \equiv -k_2 \equiv -h' (\text{mod } k),
\]
conditions which imply the restriction of \( h' \) to one or to two intervals in the range \( 0 \leq h' < k \). Therefore we can use Lemma 2 to estimate the above expression just as in the case of \( S_{21} \).

As for the estimation of \( S_1 \), we can split the integration path into

\[
\int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \int_{-\frac{1}{k(k_1+k)}}^{\frac{1}{k(k_1+k)}} - \int_{-\frac{1}{k(k_2+k)}}^{\frac{1}{k(k_2+k)}}
\]

gives

\[
S_{12} \approx \left| \sum_{c|k} \sum_{\ell=N}^{k_1+k-1} \int_{-\frac{1}{k\ell}}^{\frac{1}{k\ell}} - \frac{1}{k\ell} \cdot z^{-\frac{1}{2}} \cdot e^{\frac{q}{8kz} + \frac{2\pi i n}{k}} d\Phi \cdot \tan \left( \frac{\pi a}{c} \right) \sum_{\ell<k_1+k-1 \leq N+k-1} \frac{\omega h,k}{\omega 2h,k \sin (\pi ah/c)} \cdot e^{-2\pi i h'/c^2 k_1} - 2\pi i h/c} \right|.
\]

Using the fact that

\[
\text{Re}(z) = \frac{k}{n}, \quad \text{Re} \left( \frac{1}{z} \right) < k \quad \text{and} \quad |z|^2 \geq \frac{k^2}{n^2},
\]

this sum can be estimated as before against \( O(n^\varepsilon) \). Thus,

\[
\sum_2 = i \sum_{c|k} B_{a,c,k}(-n,0) \int_{-\frac{1}{kN}}^{\frac{1}{kN}} - \frac{1}{kN} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi i n}{k} + \frac{q}{8kz}} d\Phi + O(n^\varepsilon). \tag{3.4}
\]

We stop here for the moment with the estimation of \( \sum_2 \) and turn our attention to \( \sum_5 \). This sum is treated in a similar manner, but some comments regarding necessary modifications are in order. On noting that

\[
U(a, b, c; q) = \frac{\eta(\frac{z}{n})}{\eta^2(\frac{z}{n})} \left( \frac{\pi i a}{c} e^{\frac{2\pi i b}{c} - 2\pi b(c)} \right) q^{\frac{1}{c}} \left( \sum_{m \geq 0} e^{-\frac{\pi i a}{c} q} \frac{m(2m+1)}{2} + ms(b,c) + \frac{b}{2c} \right) - \sum_{m \geq 1} e^{-\frac{\pi i a}{c} q} \frac{m(2m+1)}{2} - ms(b,c) - \frac{b}{2c} \right)
\]

we can rewrite

\[
e^{\frac{\pi i h}{8k}} \cdot \frac{2\pi i h'}{c_1} \cdot U \left( a b', \frac{\ell c}{c_1}, c; q \right) = \sum_{r \geq r_0} a_5(r) e^{\frac{2\pi i m h'}{k}} e^{\frac{\pi r}{c_1}} + \sum_{r \geq r_1} b_5(r) e^{\frac{2\pi i m h'}{k}} e^{\frac{\pi r}{c_1}}, \tag{3.5}
\]

where \( m_r, n_r \) and \( r_0, r_1 \in \mathbb{Z} \). By the same argument as for \( S_{21} \), one sees immediately that the part which might contribute to the main term can come only from those terms with \( r < 0 \). A straightforward but rather tedious computation shows that such terms can arise only for \( s = 0, m = 0 \) (from the first sum) and for \( s = 2, m = 1 \) (from the second sum). In the former case the contribution is given by

\[
e^{-\frac{2\pi i m h'}{c_1} + \frac{4\pi i ah'}{c_1} - \frac{\pi i h'}{c_1}} e^{-\frac{1}{16} \frac{c_1^2}{c_1} + \frac{\ell c_1^2}{2r_1}} \sum_{\delta_{c,k,r} > 0} e^{-\frac{2\pi i h'}{c_1}} q_1^{\frac{\ell c}{c_1}},
\]

and in the latter by

\[
-e^{-\frac{2\pi i m h'}{c_1} + \frac{4\pi i ah'}{c_1} - \frac{3\pi i h'}{c_1}} e^{-\frac{1}{16} \frac{c_1^2}{c_1} + \frac{\ell c_1^2}{2r_1}} \sum_{\delta_{c,k,r} > 0} e^{-\frac{2\pi i h'}{c_1}} q_1^{\left( 1 - \frac{\ell}{c_1} \right) r}.
\]

To evaluate \( \sum_5 \), note that one can split the sum over \( k \) into groups based on the value \( k_1 \), which is defined in terms of \( c_1 \) and \( \ell \). In each such group, the value of \( \delta_{c,k,r} \) (hence the condition \( \delta_{a,c,k,r} > 0 \)) is independent
of \( k \), and the number of terms satisfying \( \delta_{a,c,k,r} > 0 \) is finite and bounded in terms of \( c_1 \) (hence of \( c \)). Moreover, the coefficients \( a_5(r) \) and \( b_5(r) \) are independent of \( a \) and \( k \) in any such fixed group. Since the terms with \( r < 0 \) from (3.5) can be estimated as in the case of \( S_2 \), we obtain

\[
\sum_{5 \neq h, k, r} \omega_{h,k}^2 e^{2\pi i (-nh+m_{a,c,k,r}, h')} \int_{-\theta_{h,k}^r}^{\theta_{h,k}^r} z^{-\frac{1}{2}} \cdot e^{\frac{\pi}{k} \left( n z + \frac{n}{z} \right)} d\Phi + O(n^\varepsilon), \quad (3.6)
\]

where \( \delta_{c,k,r} \) and \( m_{a,c,k,r} \) were defined in (1.3)–(1.5) and, in a completely similar way,

\[
\sum_{5 \neq h, k, r} \omega_{h,k}^2 e^{2\pi i (-nh+m_{a,c,k,r}, h')} \int_{-\theta_{h,k}^r}^{\theta_{h,k}^r} z^{-\frac{1}{2}} \cdot e^{\frac{\pi}{k} \left( n z + \frac{n}{z} \right)} d\Phi + O(n^\varepsilon),
\]

where we define

\[
\delta'_{c,k,r} := \begin{cases} 
\frac{1}{16} - \frac{\ell}{2c_1} + \frac{\ell^2}{c_1} - \frac{r}{c_1} & \text{if } 0 < \frac{\ell}{c_1} \leq \frac{1}{4}, \\
\frac{1}{16} - \frac{3\ell}{2c_1} + \frac{\ell^2}{c_1} + \frac{1}{2} - r \left( 1 - \frac{\ell}{c_1} \right) & \text{if } \frac{1}{4} < \frac{\ell}{c_1} \leq \frac{3}{4}, \\
\frac{1}{16} - \frac{3\ell}{2c_1} + \frac{\ell^2}{c_1} + \frac{3}{2} - r \left( 1 - \frac{\ell}{c_1} \right) & \text{if } \frac{3}{4} < \frac{\ell}{c_1} < 1;
\end{cases}
\]

\[
\delta''_{c,k,r} := \begin{cases} 
\frac{1}{16} - \frac{3\ell}{2c_1} + \frac{\ell^2}{c_1} - \frac{r}{c_1} & \text{if } 0 < \frac{\ell}{c_1} \leq \frac{1}{4}, \\
0 & \text{if } \frac{1}{4} < \frac{\ell}{c_1} \leq \frac{3}{4}, \\
\frac{1}{16} - \frac{3\ell}{2c_1} + \frac{\ell^2}{c_1} + \frac{1}{2} - r \left( 1 - \frac{\ell}{c_1} \right) & \text{if } \frac{3}{4} < \frac{\ell}{c_1} < 1;
\end{cases}
\]

and

\[
m'_{a,c,k,r} := \begin{cases} 
-\frac{1}{2c_1} (2(ak_1 - \ell)^2 + c_1(ak_1 - \ell) + 2rc_1(ak_1 - \ell)) & \text{if } 0 < \frac{k}{c_1} \leq \frac{1}{4}, \\
-\frac{1}{2c_1} (2(ak_1 - \ell)^2 + 3c_1(ak_1 - \ell) - 2rc_1(ak_1 - \ell) - c_1^2(2r - 1)) & \text{if } \frac{1}{4} < \frac{k}{c_1} \leq \frac{3}{4}, \\
-\frac{1}{2c_1} (2(ak_1 - \ell)^2 + 5c_1(ak_1 - \ell) - 2rc_1(ak_1 - \ell) - c_1^2(2r - 3)) & \text{if } \frac{3}{4} < \frac{k}{c_1} < 1;
\end{cases}
\]

\[
m''_{a,c,k,r} := \begin{cases} 
\frac{1}{2c_1} (2(ak_1 - \ell)^2 + 3c_1(ak_1 - \ell) + 2rc_1(ak_1 - \ell)) & \text{if } 0 < \frac{k}{c_1} \leq \frac{1}{4}, \\
0 & \text{if } \frac{1}{4} < \frac{k}{c_1} \leq \frac{3}{4}, \\
-\frac{1}{2c_1} (2(ak_1 - \ell)^2 + 3c_1(ak_1 - \ell) - 2rc_1(ak_1 - \ell) - c_1^2(2r - 1)) & \text{if } \frac{3}{4} < \frac{k}{c_1} < 1.
\end{cases}
\]

An easy computation shows that if \( c_1 = 4 \), then \( \delta''_{a,c,k,r} \leq 0 \) for all \( r \geq 0 \) and \( \delta'_{a,c,k,r} > 0 \) if and only if \( r = 0, m = 1, s = 1 \) and \( \ell = 2 \), case which is impossible as it leads to \( ak_1 = 2 \) (mod 4), and by assumption \( k \) is odd, while the condition \( (a,c) = 1 \) implies that \( a \) is odd as well. Therefore \( \sum_6 \) will only contribute to the error term.

To finish the estimation of these sums, we are only left with computing integrals of the form

\[
I_{k,v} = \int_{-\frac{1}{2\pi k}}^{\frac{1}{2\pi k}} z^{-\frac{1}{2}} \cdot e^{\frac{\pi}{k} \left( n z + \frac{n}{z} \right)} d\Phi,
\]

which, upon substituting \( z = \frac{k}{n} - ik\Phi \), are equal to

\[
I_{k,v} = \frac{1}{ki} \int_{\frac{k}{n} - \frac{1}{\pi n}}^{\frac{k}{n} + \frac{1}{\pi n}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi}{k} \left( n z + \frac{n}{z} \right)} dz. \quad (3.7)
\]
To compute these integrals we proceed in the way described by Dragonette [9, p. 492] and made more precise by Bringmann [3, p. 3497]. In doing so, we enclose the path of integration by including the smaller arc of the circle through \( \frac{1}{k} \) and tangent to the imaginary axis at 0, which we denote by \( \Gamma \). If \( z = x + iy \), then \( \Gamma \) is given by \( x^2 + y^2 = w \), with \( w = \frac{k}{n} + \frac{n}{N \pi} \). Using the fact that \( 2 > w > \frac{1}{k} \), \( \text{Re}(z) \leq \frac{k}{n} \) and \( \text{Re}(\frac{1}{k}) < k \) on the smaller arc, the integral along this arc is seen to be of order \( O(n^{-\frac{3}{4}}) \). By Cauchy’s Theorem, the path of integration in (3.7) can be further changed into the larger arc of \( \Gamma \), hence

\[
I_{k,v} = \frac{1}{k} \int \frac{k + \frac{i}{n}}{n - i/n} z^{-\frac{1}{2}} e^{2\pi i (nz + y)} dz + O(n^{-\frac{3}{8}}).
\]

Making the substitution \( t = \frac{2\pi v}{kz} \) gives

\[
I_{k,v} = \frac{2\pi}{k} \left( \frac{2\pi v}{k} \right)^{\frac{1}{2}} \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-\frac{3}{2}} e^{t + \frac{2\pi iv}{k}} dt + O(n^{-\frac{1}{8}}),
\]

where \( \gamma \in \mathbb{R} \) and \( \alpha = \frac{4\pi v n}{k^2} \). Using the Hankel integral formula, we compute (see, e.g., [23 \& §3.7 and §6.2])

\[
\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} t^{-\frac{3}{2}} e^{t + \frac{2\pi iv}{k}} dt = \frac{1}{\sqrt{\pi\alpha}} \sinh(2\alpha),
\]

hence we obtain

\[
I_{k,v} = \sqrt{\frac{2}{kn}} \sinh\left( \frac{4\pi v n}{k} \right) + O(n^{-\frac{1}{8}}). \tag{3.8}
\]

On applying (3.8) to (3.4) and (3.6) for \( v = \frac{1}{16} \) and \( v = \delta_{c,k,r} \) respectively, we have

\[
\sum_2 + \sum_5 + \sum_6 = i \sqrt{\frac{2}{n}} \sum_{1 \leq k \leq \sqrt{n}} \frac{B_{a,c,k}(-n,0)}{\sqrt{k}} \sinh\left( \frac{\pi \sqrt{n}}{k} \right)
\]
\[
+ 2 \sqrt{\frac{2}{n}} \sum_{\substack{1 \leq k \leq \sqrt{n} \\vdots c, k, c_1 \neq 4 \\vdots r \geq 0 \\delta_{c,k,r} > 0}} \frac{D_{a,c,k}(-n,m_{a,c,k,r})}{\sqrt{k}} \sinh\left( \frac{4\pi \sqrt{\delta_{c,k,r} n}}{k} \right) + O(n^\varepsilon).
\]

3.2. Estimates for the sums \( \sum_1, \sum_3 \) and \( \sum_4 \). We show that these sums contribute only to the error term. Let us start our discussion with \( \sum_1 \), which equals

\[
\sum_1 = i \tan\left( \frac{\pi a}{c} \right) \sum_{h,k} \frac{\omega_{h,k}^2}{\omega_{h,k/2}} (-1)^{h+1} \cot\left( \frac{\pi a h'}{c} \right) e^{-2\pi a^2 n_{h,k}^2} \int_{-\theta_{h,k}^2}^{\theta_{h,k}^2} z^{-\frac{1}{2}} e^{\frac{2\pi a z}{k}} O\left( \frac{ah'}{c}; q_1 \right) d\Phi.
\]

Although not written down explicitly in [3], one can readily see, e.g., by inspecting the proof of Theorem 2.1 from Bringmann and Lovejoy [6, pp. 11–17], that

\[
O\left( \frac{ah'}{c}; q_1 \right) = 4 \sin^2\left( \frac{\pi a h'}{c} \right) \eta(2z_1) \sum_{n \in \mathbb{Z}} \frac{(-1)^n q_1^{2n} + q_1^{-2n}}{1 - 2q_1^2 \cos\left( \frac{2\pi a h'}{c} \right) + q_1^{2n}}
\]
\[
= \frac{\eta(2z_1)}{\eta(z_1^2)} \left( 1 + 8 \sin^2\left( \frac{\pi a h'}{c} \right) \sum_{n \geq 1} \frac{(-1)^n q_1^{2n} + q_1^{-2n}}{1 - 2q_1^2 \cos\left( \frac{2\pi a h'}{c} \right) + q_1^{2n}} \right)
\]
\[
= \mathcal{O}(q_1) \left( 1 + 8 \sin^2\left( \frac{\pi a h'}{c} \right) \sum_{n \geq 1} \frac{(-1)^n q_1^{2n} + q_1^{-2n}}{1 - 2q_1^2 \cos\left( \frac{2\pi a h'}{c} \right) + q_1^{2n}} \right),
\]

where
where we set \( q_1 = e^{2\pi i z_1} \). We can rewrite this as

\[
O \left( \frac{ah'}{c}; q_1 \right) = 1 + \sum_{r \geq 1} a_1(r) \cdot e^{-\frac{2\pi i m_r h'}{k}} \cdot e^{-\frac{2\pi r}{k \epsilon}},
\]

where \( m_r \) is a sequence in \( \mathbb{Z} \) and the coefficients \( a_1(r) \) are independent of \( a, c, k \) and \( h \). Now the sum coming from \( r \geq 1 \) will go, as we have seen in the case of \( S_2 \), into an error term of the form \( O(n^\epsilon) \), hence

\[
\sum_1 = i \tan \left( \frac{\pi a}{c} \right) \sum_{h,k} \frac{\omega_{h,k}^2}{\omega_{h,k}/2} (-1)^{k+1} \cot \left( \frac{\pi ah'}{c} \right) e^{-\frac{2\pi i a h' k}{k}} e^{-\frac{2\pi h n a}{k}} \int_{\gamma_{h,k}}^{\theta_{h,k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} d\Phi + O(n^\epsilon).
\]

As for the sum coming from the constant term, let us denote it simply by \( S \), by splitting the path of integration exactly as in the case of \( S_1 \) and working out the estimates in a similar manner, we obtain

\[
S = i \sum_{c[k], 2[k]} A_{a,c,k}(-n, 0) \int_\frac{kN}{2k}^{\frac{kN}{k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} d\Phi + O(n^\epsilon).
\]

By applying part (iii) of Lemma \[2\] and arguing as in the case of \( S_{21} \) (except that now \( m_r = 0 \)), we get

\[
\left| i \sum_{c[k], 2[k]} A_{a,c,k}(-n, 0) \int_\frac{kN}{2k}^{\frac{kN}{k}} z^{-\frac{1}{2}} \cdot e^{\frac{2\pi n z}{k}} d\Phi \right| \ll \sum_k k^{\frac{1}{2} + \epsilon} \cdot (24n + 1, k)^{\frac{1}{2}} \cdot \frac{1}{k(N + 1)} n^2 k^{-\frac{1}{2}}
\]

\[
\ll \sum_k k^{-1 + \epsilon} \cdot (24n + 1, k)^\frac{1}{2} \ll \sum_{d|24n+1} \sum_{d \leq N} d^{-\frac{1}{2} + \epsilon} \int_1^{N/d} x^{-1 + \epsilon} dx
\]

\[
\ll \sum_{d|24n+1} \sum_{d \leq N} d^{-\frac{1}{2}} \cdot \left( \frac{N}{d} \right)^\epsilon \ll n^\epsilon,
\]

proving the claim.

We next deal with \( \sum_3 \) and \( \sum_4 \). The reader interested in writing down computations explicitly will see that the two sums can be expressed as

\[
O \left( \frac{ah'}{c}; q_1 \right) = \sum_{r \geq 0} a_3(r) \cdot e^{-\frac{2\pi i m_r h'}{k}} \cdot e^{-\frac{2\pi r}{k \epsilon}},
\]

and

\[
V \left( \frac{ah'}{c}; q_1 \right) = \sum_{r \geq 0} a_4(r) \cdot e^{-\frac{2\pi i m_r h'}{k}} \cdot e^{-\frac{2\pi (r+1) r}{4\epsilon k^2}},
\]

where \( m_r, n_r \in \mathbb{Z} \) and the coefficients \( a_3(r) \) and \( a_4(r) \) are independent of \( a, c, k \) and \( h \). Since \( r \geq 0 \), it is obvious that both sums will be of order \( O(n^\epsilon) \), the argument being the same as for \( S_2 \).

### 3.3. Estimates for the sums \( \sum_7 \) and \( \sum_8 \)

The estimation of the remaining sums \( \sum_7 \) and \( \sum_8 \) is not difficult and is based on Bringmann \([3\) p. 3497\]. Let us however elaborate a bit more here. Again, we split the path of integration as in \( \sum_2 \). The resulting sums can each be bounded on the various intervals of integration by

\[
\left( \sum_{k} k^{-1} \right) \left( \sum_{h} 1 \right) \cdot \sum_{\nu = 0}^{k-1} k^{-1} \cdot N^{-1} \cdot z^{\frac{1}{2}} \cdot I_{a,c,k,v}(z) \ll N^{-1} \cdot n^\frac{1}{2} \cdot k^{-\frac{1}{2}} \cdot g_{a,c,k,v} \ll k^\epsilon \ll n^\epsilon,
\]

for any \( \epsilon > 0 \). Here we have used, in turn, a trivial bound for the Kloosterman sum appearing in front of the integrals from \( \sum_7 \) and \( \sum_8 \), Lemma \[1\] and the easy estimate

\[
\sum_{1 \leq \nu \leq k} g_{a,c,k,v} \ll \sum_{1 \leq \nu \leq 4k} \frac{1}{\nu} \ll k^\epsilon.
\]

By this we conclude this section and the rather lengthy proof of Theorem \[1\]
4. Proof of the inequalities

In this section we prove the inequalities stated in Theorems 2–4. We will elaborate more on Theorem 2 while only sketching the main steps in the proof of Theorem 3, as the ideas are similar. Theorem 4 is equivalent to the inequalities (1.13)–(1.16), which are part of Theorem 2.

Before giving the proof of Theorem 2, we must establish some identities. The following is an easy generalization of [15, Lemma 3.1].

Lemma 3. If \( a \in \mathbb{N} \) is odd and \( 5 \nmid a \), then

\[
\mathcal{O}(\zeta_{10}^a; q) = \sum_{n=0}^{\infty} (N(0, 10, n) + \overline{N}(1, 10, n) - \overline{N}(4, 10, n) - \overline{N}(5, 10, n)) q^n
\]

\[
+ (\zeta_{10}^2 - \zeta_{10}^3) \sum_{n=0}^{\infty} (N(1, 10, n) + \overline{N}(2, 10, n) - \overline{N}(3, 10, n) - \overline{N}(4, 10, n)) q^n.
\]

Proof. Plugging \( u = \zeta_{10}^a \) into (4.1) gives

\[
\mathcal{O}(\zeta_{10}^a; q) = \sum_{n=0}^{\infty} N(m, n) c_{10}^{am} q^n = \frac{(-q; q)_\infty}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1 - \zeta_{10}^a)(1 - \zeta_{10}^{-a})(-1)^n q^{n^2+n}}{(1 - \zeta_{10}^a q^n)(1 - \zeta_{10}^{-a} q^n)}.
\]

(4.1)

Using the fact that \( \overline{N}(a, m, n) = \overline{N}(m - a, m, n) \), which can be easily deduced from \( \overline{N}(m, n) = \overline{N}(m, n) \) (see, e.g., Lovejoy [16, Proposition 1.1]), and noting that \( \zeta_{10}^a = -1 \) and \( 1 - \zeta_{10}^a + \zeta_{10}^2 - \zeta_{10}^3 + \zeta_{10}^4 = 0 \) for \( 5 \nmid a \), we can rewrite (4.1) as

\[
\mathcal{O}(\zeta_{10}^a; q) = \sum_{n=0}^{\infty} \sum_{\ell=0}^{9} N(\ell, 10, n) \zeta_{10}^{a\ell} q^n
\]

\[
= \sum_{n=0}^{\infty} (N(0, 10, n) + (\zeta_{10}^a - \zeta_{10}^2) \overline{N}(1, 10, n) + (\zeta_{10}^3 - \zeta_{10}^a) \overline{N}(2, 10, n)
\]

\[
+ (\zeta_{10}^2 - \zeta_{10}^3) N(3, 10, n) + (\zeta_{10}^4 - \zeta_{10}^a) \overline{N}(4, 10, n) - \overline{N}(5, 10, n)) q^n
\]

\[
= \sum_{n=0}^{\infty} (N(0, 10, n) + (1 + \zeta_{10}^2 - \zeta_{10}^3) \overline{N}(1, 10, n) + (\zeta_{10}^2 - \zeta_{10}^3) \overline{N}(2, 10, n)
\]

\[
+ (\zeta_{10}^3 - \zeta_{10}^a) \overline{N}(3, 10, n) + (1 + \zeta_{10}^2 - \zeta_{10}^3) \overline{N}(4, 10, n) - \overline{N}(5, 10, n)) q^n
\]

\[
= \sum_{n=0}^{\infty} (N(0, 10, n) + \overline{N}(1, 10, n) + \overline{N}(4, 10, n) - \overline{N}(5, 10, n)) q^n
\]

\[
+ (\zeta_{10}^2 - \zeta_{10}^3) \sum_{n=0}^{\infty} (N(1, 10, n) + \overline{N}(2, 10, n) - \overline{N}(3, 10, n) - \overline{N}(4, 10, n)) q^n,
\]

which concludes the proof.

In a similar fashion, we have the following result. For a proof of the case \( a = 1 \), see [15, Lemma 2.1].

Lemma 4. If \( a \in \mathbb{N} \) is odd and \( 3 \nmid a \), then

\[
\mathcal{O}(\zeta_{10}^a; q) = \sum_{n=0}^{\infty} (N(0, 6, n) + N(1, 6, n) - N(2, 6, n) - N(3, 6, n)) q^n.
\]

Proof of Theorem 2. Setting \( a = 1 \) and \( a = 3 \) in Lemma 3, we obtain

\[
\mathcal{O}(\zeta_{10}; q) = \sum_{n=0}^{\infty} (N(0, 10, n) + N(1, 10, n) - N(4, 10, n) - N(5, 10, n)) q^n
\]

\[
+ (\zeta_{10}^2 - \zeta_{10}^3) \sum_{n=0}^{\infty} (N(1, 10, n) + N(2, 10, n) - N(3, 10, n) - N(4, 10, n)) q^n,
\]

(4.2)
and
\[ O(\zeta_{10}^3; q) = \sum_{n=0}^{\infty} (N(0, 10, n) + N(1, 10, n) - N(2, 10, n) - N(5, 10, n))q^n \]
\[ + (\zeta_{10}^3 - \zeta_{10}) \sum_{n=0}^{\infty} (N(1, 10, n) + N(2, 10, n) - N(3, 10, n) - N(4, 10, n))q^n. \] (4.3)

Subtracting (4.3) from (4.2) yields
\[ \sum_{n=0}^{\infty} (N(1, 10, n) + N(2, 10, n) - N(3, 10, n) - N(4, 10, n))q^n = \frac{O(\zeta_{10}^3; q) - O(\zeta_{10}^3; q)}{\zeta_{10} + \zeta_{10}^3 - \zeta_{10} - \zeta_{10}^3} = \frac{O(\zeta_{10}; q) - O(\zeta_{10}^3; q)}{1 + 4 \cos \left( \frac{2\pi}{3} \right)}, \]
thus proving (1.10) is equivalent to showing that, for all \( n \geq 2, \)
\[ A \left( \frac{1}{10}; n \right) > A \left( \frac{3}{10}; n \right). \]

For \( a = 1, c = 10 \) we have \( m_{1,10,1,0} = 0 \) and \( \delta_{c,k,r} > 0 \) if and only if \( r = 0, \) in which case \( \delta_{c,k,r} = \frac{9}{400}, \) hence
\[ A \left( \frac{1}{10}; n \right) = 2 \sqrt{\frac{2}{n}} \sum_{\substack{1 \leq k \leq \sqrt{n} \backslash \equiv 1,9 \, (10)}} D_{a,c,k} \left( -n, m_{1,10,1,0} \right) \cdot \sinh \left( \frac{3\pi \sqrt{n}}{5k} \right) + O_c(n^{\varepsilon}), \] (4.4)
whereas for \( a = 3 \) and \( c = 10 \) we have \( \delta_{c,k,r} > 0 \) if and only if \( r = 0, \) in which case \( \delta_{c,k,r} = \frac{9}{400}, \) thus
\[ A \left( \frac{3}{10}; n \right) = 2 \sqrt{\frac{2}{n}} \sum_{\substack{1 \leq k \leq \sqrt{n} \backslash \equiv 3,7 \, (10)}} D_{a,c,k} \left( -n, m_{3,10,1,0} \right) \cdot \sinh \left( \frac{3\pi \sqrt{n}}{5k} \right) + O_c(n^{\varepsilon}). \] (4.5)

We further compute
\[ D_{1,10,1} \left( -n, 0 \right) = \frac{1}{\sqrt{2}} \tan \left( \frac{\pi}{10} \right), \]
and so the term corresponding to \( k = 1 \) in the sum from (4.2) is given by
\[ \frac{2}{\sqrt{n}} \tan \left( \frac{\pi}{10} \right) \sinh \left( \frac{3\pi \sqrt{n}}{5} \right). \]

Using a trivial bound for the Kloosterman sum from (4.5) and taking into account the contributions coming from the various error terms involved, estimates which we make explicit at the end of this section, we see that this term is dominant for \( n \geq 1030, \) hence
\[ A \left( \frac{1}{10}; n \right) > A \left( \frac{3}{10}; n \right) \]
for \( n \geq 1030. \) In Mathematica we see that the inequality is true for \( 2 \leq n < 1030 \) as well.

To prove (1.11) we set \( a = 1 \) and \( a = 3 \) in Lemma 3 and obtain
\[ O(\zeta_{10}; q) = \sum_{n=0}^{\infty} (N(0, 10, n) + N(3, 10, n) - N(2, 10, n) - N(5, 10, n))q^n \]
\[ + (1 + \zeta_{10}^3 - \zeta_{10}) \sum_{n=0}^{\infty} (N(1, 10, n) + N(2, 10, n) - N(3, 10, n) - N(4, 10, n))q^n, \] (4.6)
and
\[ O(\zeta_{10}^3; q) = \sum_{n=0}^{\infty} (N(0, 10, n) + N(3, 10, n) - N(2, 10, n) - N(5, 10, n))q^n \]
\[ + (1 - \zeta_{10} + \zeta_{10}^4) \sum_{n=0}^{\infty} (N(1, 10, n) + N(2, 10, n) - N(3, 10, n) - N(4, 10, n))q^n. \] (4.7)
Combining (4.6) and (4.7) and setting \( \alpha = \frac{1 + \zeta_{10}^2 - \zeta_{10}^3}{1 - \zeta_{10} + \zeta_{10}^2} \) we obtain
\[
O(\zeta_{10}; q) - \alpha \cdot O(\zeta_{10}^3; q) = (1 - \alpha) \sum_{n=0}^{\infty} (N(0, 10, n) + N(3, 10, n) - N(2, 10, n) - N(5, 10, n)) q^n,
\]
hence, as it is easy to see that \( \alpha = -(1 + 2 \cos(\pi/5)) \), proving the claim amounts to showing
\[
A \left( \frac{1}{10}; n \right) + \left(1 + 2 \cos \frac{\pi}{5}\right) A \left( \frac{3}{10}; n \right) > 0
\]
for all \( n \geq 2 \), which follows from the estimates used for proving (1.10). The proof of (1.12) follows simply on adding the inequalities (1.10) and (1.11).

We can also sketch now the proof of (1.13). Reasoning along the same lines and setting \( a = 1 \) in Lemma 4 and recalling (1.3), the claim is equivalent to proving
\[
A \left( \frac{1}{6}; n \right) > 0
\]
for \( n \geq 2 \). It is easy to see that for \( a = 1, c = 6 \) we have \( m_{1,6,1,0} = 0 \) and \( \delta_{c,k,r} > 0 \) if and only if \( r = 0 \), in which case \( \delta_{c,k,r} = \frac{1}{144} \), thus the dominant term of
\[
A \left( \frac{1}{6}; n \right) = 2 \sqrt{\frac{2}{n}} \sum_{1 \leq k \leq \sqrt{n}} \frac{D_{1,6,k}(-n, m_{1,6,k,0})}{\sqrt{k}} \cdot \sinh \left( \frac{3 \sqrt{n}}{5k} \right)
\]
will be given by
\[
\frac{2}{\sqrt{n}} \tan \left( \frac{\pi}{6} \right) \sinh \left( \frac{\pi \sqrt{n}}{3} \right).
\]

By working out similar bounds as in the proof of (1.10) and checking numerically for the small values of \( n \), the proof of (1.13) is concluded.

The inequalities (1.14)–(1.16) are a reformulation of Theorem 4. The proof relies on the identity
\[
O(\zeta_{6}^2; q) = \sum_{n=0}^{\infty} (N(0, 6, n) - N(1, 6, n) - N(2, 6, n) + N(3, 6, n)) q^n = \sum_{n=0}^{\infty} (N(0, 3, n) - N(1, 3, n)) q^n
\]
and details are left to the interested reader. □

**Proof of Theorem 3 (Sketch).** By using either [24] Lemma 5.1 (on identifying the notation \( \mathcal{R}(u; q) = O(u; q) \) or identity (1.16) (which, in combination with (1.2), amounts to the same result), we have
\[
\sum_{n=0}^{\infty} N(0, 6, n) q^n = \frac{1}{6} (O(1; q) + 2O(\zeta_6; q) + 2O(\zeta_6^2; q) + O(\zeta_6^3; q)), \quad (4.8)
\]
\[
\sum_{n=0}^{\infty} N(1, 6, n) q^n = \frac{1}{6} (O(1; q) + O(\zeta_6; q) - O(\zeta_6^2; q) - O(\zeta_6^3; q)), \quad (4.9)
\]
\[
\sum_{n=0}^{\infty} N(2, 6, n) q^n = \frac{1}{6} (O(1; q) - O(\zeta_6; q) - O(\zeta_6^2; q) + O(\zeta_6^3; q)), \quad (4.10)
\]
\[
\sum_{n=0}^{\infty} N(3, 6, n) q^n = \frac{1}{6} (O(1; q) - 2O(\zeta_6; q) + 2O(\zeta_6^2; q) - O(\zeta_6^3; q)). \quad (4.11)
\]

In light of Remark 2 to prove the inequalities (1.17)–(1.19) it suffices to show that, for \( n \geq 11 \),
\[
N(1, 6, n) > N(2, 6, n),
\]
\[
N(0, 6, 3n) > N(1, 6, 3n), \quad N(0, 6, 3n + 1) > N(1, 6, 3n + 1),
\]
\[
N(0, 6, 3n + 2) < N(1, 6, 3n + 2).
\]
Therefore, on combining (4.9) and (4.10), the first inequality above is equivalent to
\[ A \left( \frac{1}{6}; n \right) > 0, \] (4.12)
whereas the second and third are equivalent, on combining (4.8) and (4.9), to
\[ A \left( \frac{1}{6}; 3n + i \right) + 3A \left( \frac{1}{3}; 3n + i \right) > 0 \quad \text{and} \quad A \left( \frac{1}{6}; 3n + 2 \right) + 3A \left( \frac{1}{3}; 3n + 2 \right) < 0, \] (4.13)
for \( i \in \{0, 1\} \).

The attentive reader might wonder what happens with the term \( O(-1; q) \) (coming from the case \( j = c/2 \) in (4.6)) to which Theorem 5 does not apply, as its statement is formulated under the assumption \( c > 2 \). However, while working out the transformations found by Bringmann and Lovejoy in this case, see [6, Corollary 4.2] and doing the same estimates as in the proof of Theorem 1, one can easily infer that the sums involved are of order \( O(n^c) \). Therefore, as \( n \) grows large, we only need to prove (4.12) and (4.13), which follow immediately from Theorem 1. Again, explicit bounds can be provided just as described in the next subsection, and a numerical check for the small values of \( n \) concludes the proof.

4.1. Some explicit computations. As we have mentioned earlier, we will now fill in the missing details from the proof of (1.10) by explaining how to bound the different sums and error terms appearing in (4.3) and (4.5). The same arguments apply for all the other inequalities. We have already seen that
\[ A \left( \frac{1}{10}; n \right) = 2 \sqrt{\frac{2}{5}} \sum_{1 \leq k \leq \sqrt{n}} \frac{D_{a,c,k}(-n, m_{1,10}, k, 0)}{\sqrt{k}} \cdot \sinh \left( \frac{3\pi \sqrt{n}}{5k} \right), \]
and that the term corresponding to \( k = 1 \) in (4.4) equals
\[ \frac{2}{\pi} \tan \left( \frac{\pi}{10} \right) \sinh \left( \frac{3\pi \sqrt{n}}{5} \right). \] (4.14)
By using a trivial bound for the Kloosterman sums involved, the remaining terms can be estimated against
\[ \frac{4}{\sqrt{n}} \sum_{2 \leq k \leq \frac{N}{10}} k^{-\frac{3}{2}} \cdot \sinh \left( \frac{3\pi \sqrt{n}}{5(10k + 1)} \right) + \frac{4}{\sqrt{n}} \sum_{1 \leq k \leq \frac{N-9}{10}} k^{-\frac{3}{2}} \cdot \sinh \left( \frac{3\pi \sqrt{n}}{5(10k + 9)} \right), \] (4.15)
and the contribution coming from \( U \left( h', \frac{q}{10}, 10; q_1 \right) \) is seen to be less than
\[ \sqrt{2} \cdot e^{2\pi} \sum_{r=1}^{\infty} |a_5(r)| \cdot e^{-\frac{2\pi r^2}{10}} \sum_{1 \leq k \leq N \atop k \equiv 1, 9 \mod 10} k^{-\frac{1}{2}} + \sqrt{2} \cdot e^{2\pi} \sum_{r=1}^{\infty} |b_5(r)| \cdot e^{-\frac{2\pi r^2}{10}} \sum_{1 \leq k \leq N \atop k \equiv 1, 9 \mod 10} k^{-\frac{1}{2}}. \] (4.16)
Making the path of integration symmetric in (3.14) introduces an error that can be estimated against
\[ 2 \cdot e^{2\pi + \frac{\pi}{10}} \cdot n^{-\frac{3}{2}} \sum_{1 \leq k \leq N \atop k \equiv 1, 9 \mod 10} k^{\frac{1}{2}}, \] (4.17)
while integrating along the smaller arc of \( \Gamma \) gives an error not bigger than
\[ 8\pi \cdot e^{2\pi + \frac{\pi}{10}} \cdot n^{-\frac{3}{2}} \sum_{1 \leq k \leq N \atop k \equiv 1, 9 \mod 10} k. \] (4.18)
The sums \( \sum_{2}, \sum_{4} \) and \( \sum_{6} \) do not contribute in the case \( c = 10 \), whereas \( \sum_{1}, \sum_{3} \) can be treated simultaneously. The contribution coming from \( O \left( h'; q_1 \right) \) can be estimated against
\[ \frac{2 \cdot e^{2\pi}}{\sqrt{10}} \sum_{1 \leq k \leq \frac{N}{10}} k^{-\frac{1}{2}} + \frac{2 \cdot e^{2\pi}}{\sqrt{10}} \sum_{r=1}^{\infty} |a_1(r)| \cdot e^{-\pi r} \sum_{1 \leq k \leq \frac{N}{10}} k^{-\frac{1}{2}}, \] (4.19)
and that coming from $O\left(h', \frac{q}{7}, 10; q_1\right)$ against

$$2 \cdot e^{2\pi} \sum_{r=1}^{\infty} |a_3(r)| \cdot e^{-\frac{\pi r}{50}} \sum_{1 \leq k \leq N} k^{-\frac{1}{2}}.$$ \hspace{1cm} (4.20)

Using the bound (3.3) for $|a_3(r)|$, $|a_5(r)|$ and $|b_5(r)|$, we get $\sum_{r=1}^{\infty} |a_3(r)| \cdot e^{-\frac{\pi r}{50}} < 1.17944$ and $\sum_{r=1}^{\infty} |a_3(r)| \cdot e^{-\frac{7\pi r}{50}} < 4.01014 \cdot 10^{19}$, and similarly for $a_5(r)$ and $b_5(r)$. Finally, the estimates in Lemma 1 can be made explicit so as to give

$$\sum_{7} \leq 2 e^{2\pi} \sum_{2|k} k^{-\frac{3}{2}} \sum_{\nu=1}^{k} \left( \min \left\{ \left| \nu - \frac{1}{4k} + \frac{1}{10} \right|, \left| \nu - \frac{1}{4k} - \frac{1}{10} \right| \right\} \right)^{-1} \hspace{1cm} (4.21)$$

and

$$\sum_{8} \leq 2 e^{2\pi} \sum_{\frac{5k}{2|k}} k^{-\frac{3}{2}} \sum_{\nu=1}^{k} \left( \min \left\{ \left| \nu + \frac{1}{10} \right|, \left| \nu - \frac{1}{10} \right| \right\} \right)^{-1} + \frac{1}{10} \sum_{1 \leq k \leq N} k^{-\frac{1}{2}}.$$ \hspace{1cm} (4.22)

For $a = 3$ and $c = 10$ we proceed just like in (4.15) to get

$$\frac{4}{\sqrt{n}} \sum_{1 \leq k \leq \frac{N-3}{10}} k^{\frac{1}{2}} \cdot \sinh \left( \frac{3\pi \sqrt{n}}{5(10k+3)} \right) + \frac{4}{\sqrt{n}} \sum_{1 \leq k \leq \frac{N-7}{10}} k^{\frac{1}{2}} \cdot \sinh \left( \frac{3\pi \sqrt{n}}{5(10k+7)} \right)$$

as an overall bound for the main contribution in (4.5) and we use the same estimates from (4.16)–(4.22) on changing whatever necessary, e.g., the sums will now run over $k \equiv 3 \pmod{10}$ and $k \equiv 7 \pmod{10}$. Putting all estimates together we see that the term in (4.14) is dominant for $n \geq 1030$. The inequality (1.10) can be checked numerically in Mathematica to hold true also for $n < 1030$.

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