Approaching the isoperimetric problem in $H^m_C$ via the hyperbolic log-convex density conjecture

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Received: 10 October 2022 / Accepted: 2 November 2023
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Abstract

We prove that geodesic balls centered at some base point are uniquely isoperimetric sets in the real hyperbolic space $H^m_C$ endowed with a smooth, radial, strictly log-convex density on the volume and perimeter. This is an analogue of the result by G. R. Chambers for log-convex densities on $\mathbb{R}^n$. As an application we prove that in any rank one symmetric space of non-compact type, geodesic balls are uniquely isoperimetric in a class of sets enjoying a suitable notion of radial symmetry.

Mathematics Subject Classification 49Q20 · 49Q05 · 53A10

1 Introduction

We denote by $(H^n_R, g_H)$ the real hyperbolic space of dimension $n \in \mathbb{N}$ with constant sectional curvature equal to $-1$. Call $d_H$ the induced Riemannian distance. Choose an arbitrary base point $o \in H^n_R$. We say that a function $f : H^n_R \to \mathbb{R}_{>0}$ is (strictly) radially log-convex if

$\ln(f(x)) = h(d_H(o, x))$,

for a smooth, (strictly) convex and even function $h : \mathbb{R} \to \mathbb{R}$. We define the weighted perimeter and volume of a set with finite perimeter $E \subset H^n_R$ as

$V_f(E) := \int_E f \, d\mathcal{H}^n$, and $P_f(E) = \int_{\partial^* E} f \, d\mathcal{H}^{n-1}$.

Here, following the notation in [14], $\partial^* E$ denotes the reduced boundary of $E$. A set of finite perimeter $E$ with volume $V_f(E) = v > 0$ is called isoperimetric if it solves the minimization problem

$\mathcal{J}(v) := \inf \left\{ P_f(F) : V_f(F) = v, F \subset H^n_R \text{ of finite perimeter} \right\}$.

(1.1)
The first goal of this paper is to show the following characterization of the isoperimetric sets, which will be developed in Sect. 3.

**Theorem 1.1** For any strictly radially log-convex density $f$, geodesic balls centered at $o \in H^d_{\mathbb{R}}$ uniquely minimize the weighted perimeter for any given weighted volume with respect to $P_f$ and $V_f$.

Our main motivation in proving such result is the tight relation of this problem with the (unweighted) isoperimetric problem in the complex hyperbolic spaces $H^m_{\mathbb{C}}$, the quaternionic spaces $H^m_{\mathbb{H}}$ and the Cayley plane $H^2_{\mathbb{O}}$ restricted to a family of sets sharing a particular symmetry that we define as follows.

**Definition 1.2** (Hopf-symmetric sets) Let $K \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $d = \dim(K) \in \{2, 4, 8\}$ and $(M, g) = (H^m_{K}, g)$ be the associated rank one symmetric space of non-compact type of real dimension $n = dm$, $m = 2$ if $K = \mathbb{O}$. Fix an arbitrary point $o \in M$ and let $N$ be the unit length radial vector field emanating from $o$. Then, up to renormalization of the metric, the Jacobi operator $R(\cdot, N)N$ arising from the Riemannian curvature tensor $R$ is a self-adjoint operator of $TM$, and has exactly three eigenvalues: $\{0, -1, -4\}$. The $(-4)$-eigenspace defines at every point $x \neq o$ a distribution $H_x$ of real dimension $d - 1$. A $C^1$-set $E \subset M$ with normal vector field $\nu$ is said to be $Hopf$-symmetric if $\nu(x)$ is orthogonal to $H_x$ at each point $x \in \partial E$, $o \notin \partial E$.

**Remark 1.3** Let $h : S^{n-1} \rightarrow \mathbb{C}P^{m-1}$ be the celebrated Hopf fibration. Then, for any $C^1$-profile $\rho : S^{n-1} \rightarrow (0, +\infty)$ so that $\rho$ is constant along the fibres of $h$, the set with boundary
\[
\partial E := \{\exp_o(\rho(x)x) : x \in S^{n-1} \subset T_oM\},
\]
is Hopf-symmetric, where $\exp_o$ is the exponential map of $M$ at an arbitrary point $o \in M$.

**Remark 1.4** Being $Hopf$-symmetric has not to be confused with the standard notion of being $Hopf$ in $H^m_{\mathbb{C}}$, that is a set with principal curvature along the characteristic directions $J\nu$, where $J$ denotes the associated complex structure. It is worth saying that spheres are the only Hopf, compact, embedded constant mean curvature surfaces in $H^m_{\mathbb{C}}$, as it is proven by A. A. Borisenko in in [3]. The natural generalization of this concept when $K \in \{\mathbb{H}, \mathbb{O}\}$ is being a curvature-adapted hypersurface, that is, the normal Jacobi operator $R(\cdot, \nu)\nu$ commutes with the shape operator.

We adopt the notation of Definition 1.2 for the rest of the paper. Let $P$ and $V$ be the perimeter and volume functionals induced by $g$ in $H^m_{K}$. Consider the (unweighted) isoperimetric problem
\[
\inf\{P(F) : V(F) = v, F \subset H^m_{K} \text{ Hopf-symmetric}\}. \tag{1.2}
\]
We dedicate Sect. 4 to the proof of the following theorem.

**Theorem 1.5** If geodesic balls centered at $o \in H^d_{\mathbb{R}}$ are isoperimetric with respect to Problem (1.1) for the strictly radial log-convex density
\[
f(x) = \cosh(d_H(o, x))^d - 1, \quad d = \dim(K),
\]
then geodesic balls in $H^m_{K}$ are optimal with respect to the isoperimetric Problem (1.2).

The explicit expression of the perimeter for Hopf-symmetric sets that we will develop in the proof of Theorem 1.5, and Theorem 1.1 will lead to the following consequence.
Corollary 1.6 In the class of Hopf-symmetric sets, geodesic balls are the unique isoperimetric sets in $H_{mg}^m$. 

In the past two decades, numerous researchers have shown great interest in studying the isoperimetric problem within manifolds with positive densities on the perimeter and volume. In the context of radial weights on $\mathbb{R}^n$, C. Rosales, A. Cañete, V. Bayle and F. Morgan established the existence of isoperimetric sets by imposing certain growth conditions on the weight. Furthermore, they proved that spheres centered at the origin are stable if and only if the weight is log-convex, see [20, Theorem 3.10]. Under this last assumption, K. Brakke conjectured that balls centered at the origin are not only stable, but in fact global minimizers of the weighted perimeter. In the aforementioned paper, the authors completely proved this conjecture in the one dimensional case, and in all dimensions for the particular weight $\exp(|x|^2)$ via a symmetrization argument, see [20, Corollary 4.11, Theorem 5.2]. In [19] A. Pratelli and F. Morgan provided several important new results and examples related to this topic. In particular, existence, boundedness and mean convexity of the isoperimetric sets are ensured in all dimensions if the density is log-convex. The conjecture was proved in the large volume regime by A. Kolesnikov and R. Zhdanov in [12] through an ingenious application of the divergence theorem, proving that large isoperimetric sets have to be level sets of the given radial weight. The small volume regime was then proved by A. Figalli and F. Maggi in [8] via a rescaling argument taking advantage of deep quantitative stability estimates on the spheres. Moreover, the authors established the general result when the weight is close enough in norm to the before mentioned exponential function $\exp(|x|^2)$ via an interpolation argument. The conjecture was finally proven true by the remarkable article by G. R. Chambers [5], who obtained a complete characterization of the isoperimetric sets. The analysis relies on a meticulous examination of the generating profile of spherical symmetrized sets. In fact, the first and main part of this paper is an adaptation of the method to our negatively curved case. It is worth saying that this strategy was moreover successfully employed by W. Boyer, B. Brown, G. R. Chambers, A. Loving and S. Tammen in [4] to show the surprising fact that for all volumes balls whose boundary passes through the origin are isoperimetric with respect to the radial polynomial weight $|x|^p$, for all $p > 0$.

For what concerns curved ambient spaces, various results have been obtained. In [18] F. Morgan, M. Hutchings, and H. Howards focused their attention on the plane endowed with a smooth, rotationally symmetric metric with radially increasing Gauss curvature, proving that an isoperimetric set in this case is either a circle, a complement of a circle or an annulus. In warped product manifolds a significant result is due to S. Howe, who in addition of generalizing the aforementioned result by A. Kolesnikov and R. Zhdanov, established several situations in which the fibres minimize the vertical volume, see [11]. For what concerns the hyperbolic setting, Brakke’s conjecture in the two dimensional case was proved according to I. McGuillivray in [15]. L. Di Giosia, J. Habib, L. Kenigsberg, D. Pittman, and W. Zhu proved in [6] via a direct application of Chamber’s theorem, that balls centered at the origin are isoperimetric for smooth log-convex radially symmetric densities in $H^n_{mg}$ introducing an additional weight on the perimeter to counteract the impact arising from the negatively curved metric. The work of E. Bongiovanni, A. Diaz, A. Kakkar, and N. Sothanaphan in [2] provide an affirmative answer to Brakke’s conjecture for large volume sets containing the origin, in the general setting of two dimensional surfaces of revolution, in which the product of the metric factor with the given volume density is eventually log-convex. This applies for instance to the hyperbolic plane with density equal to $\exp(d_H(x, o)^2)$ for some fixed base point $o \in H^2_{mg}$, see [2, Corollary 5.10]. Finally, very recently in [13] H. Li and B. Xu showed sharp isoperimetric inequalities in $H^n_{mg}$ endowed with radial density of the form
\[ \phi(\sinh(d_H(x, o)) \cosh(d_H(x, 0))), \]

for \( \phi \) even, log-convex, and \( o \in H^n_R \) any base point. The proof, that generalizes the result by J. Scheuer and C. Xia in [21], cleverly applies the result of G. R. Chambers by projecting the hyperbolic space onto \( \mathbb{R}^n \) and employing a comparison argument. This result proves Theorem 1.5 in the case of the complex hyperbolic space by simply taking \( \phi \equiv 1 \). Our contribution consists in a further generalization: observe that the density

\[ f(x) := \phi(\sinh(d_H(x, o)) \cosh(d_H(x, 0))) \]

is always strictly log-convex, but the converse is not true: for instance when \( f(x) = \cosh(d_H(x, o))^{d-1} \) for \( d > 2 \) (like in Theorem 1.5), the associated function

\[ \phi(R) := \frac{\cosh(\text{arsinh}(R))^{d-1}}{\cosh(\text{arsinh}(R))} = \cosh(\text{arsinh}(R))^{d-2} \]

is not log-convex.

**Remark 1.7** In extending the proof of Brakke’s conjecture from the Euclidean space to the hyperbolic space, we decided for simplicity to assume the weight to be strictly log-convex rather than simply log-convex. This choice was motivated by the technical difficulties arising from the presence of regions with constant weight. It is worth noting that this restriction has no bearing on the application being studied.

## 2 Preliminaries

In what follows, we will always assume \( E \subset H^n_R \) to be an isoperimetric set with respect to the weighted problem (1.1).

### 2.1 Qualitative properties of the isoperimetric sets

The main argument of this paper is grounded in the principles of existence and regularity of isoperimetric sets in manifolds with densities. We refer to the work of E. Milman [16, Section 2.2 and 2.3] as a very general reference. Existence, boundedness and mean-convexity of isoperimetric sets in \( \mathbb{R}^n \) endowed with a various family of densities was the focus of the article by F. Morgan and A. Pratelli [19]. For completeness, the detailed application of their arguments to our hyperbolic setting can be found in the Appendix 1, Theorems A.2, A.3, and A.4. Regularity of area minimizing surfaces has been the object of study of geometric measure theorists for many decades. The result ensuring smoothness away from a singular set of Hausdorff dimension at most \( n - 8 \) is by now a well-established and widely acknowledged fact. For a presentation of the historical background, we recommend referring to [17, Chapter 8], and [16, Section 2.2] for numerous references on the subject. In analogy with the unweighted case, the last crucial property of the isoperimetric sets \( E \) is to have constant weighted mean curvature

\[ H_f := H + \partial_n \ln(f), \]

at each regular point of \( \partial E \). Here, \( H \) denotes the unaveraged inward Riemannian mean curvature, and \( \nu \) the outward pointing unit normal. The peculiar form of \( H_f \) is obtained via a direct computation of the volume preserving first variation of the perimeter, see [20, Section 3]. The next theorem summarizes all the before mentioned properties of isoperimetric sets.
Theorem 2.1 (Existence and regularity) For any volume \( v > 0 \) there exists a set \( E \subset H^m_{\mathbb{R}} \) of finite perimeter and weighted volume \( V_f(E) = v \) solving the isoperimetric problem (1.1). Moreover, \( E \) enjoys the following properties:

- \( \partial E \) is a bounded \( C^\infty \) embedded hypersurface outside a singular set of Hausdorff dimension at most \( n-8 \).
- There exists \( \lambda \in \mathbb{R} \) such that at any regular point \( x \in \partial E \), \( H_f(x) = \lambda \). As a consequence, \( \partial E \) is mean-convex at each regular point \( y \in \partial E \), that is \( H(y) \geq (n-1) \).
- If the tangent cone at \( x \in \partial E \) lies in a halfspace, then it is a hyperplane, and therefore \( \partial E \) is regular at \( x \). In particular, \( \partial E \) is regular at points \( x^* \in \partial E \) satisfying \( d_H(x^*, 0) = \sup_{x \in \partial E} d_H(x, 0) \).

2.2 The Poincaré model of \( H^n_{\mathbb{R}} \)

Adopting the Poincaré model, \( H^n_{\mathbb{R}} \) is conformal to the open Euclidean unit ball. At a point \( x \in H^n_{\mathbb{R}} \) the metric is

\[
g_H = \frac{4}{(1 - r^2)^2} g_{\text{flat}},
\]

where \( r = |x| \) will always denote the Euclidean distance of \( x \) from the origin, and \( g_{\text{flat}} \) the usual Euclidean metric of \( \mathbb{R}^n \). The hyperbolic distance from the origin is then given by

\[
d_H(x, 0) = 2 \arctanh(r).
\]

We define the boundary at infinity \( \partial_{\infty} H^n_{\mathbb{R}} \) of \( H^n_{\mathbb{R}} \) to be the Euclidean unit sphere \( \partial B(0, 1) = S^{n-1} \). We will identify the base point \( o \in H^n_{\mathbb{R}} \) of the radial density \( f \) with the origin 0 in \( B(0, 1) \).

2.3 Isometries and special frames in \( H^2_{\mathbb{R}} \)

Denote by \( e_1 \) and \( e_2 \) the horizontal and vertical Cartesian axes in the two dimensional Poincaré disk model. Also, let \( (H^2_{\mathbb{R}})^{+} \) be the intersection of \( H^2_{\mathbb{R}} \) with the closed upper half-plane having \( e_1 \) as boundary. The isometry group of \( (H^2_{\mathbb{R}}, g_H) \) is completely determined (up to orientation) by the Möbius transformations preserving the boundary \( \partial_{\infty} H^2_{\mathbb{R}} \). Hence, geodesic circles coincide with Euclidean circles completely contained in \( H^2_{\mathbb{R}} \). Their curvature lies in \( (1, +\infty) \). Circles touching \( \partial_{\infty} H^2_{\mathbb{R}} \) in a point are called horocycles, and have curvature equal to 1. Geodesics are arcs of (possibly degenerate) circles hitting \( \partial_{\infty} H^2_{\mathbb{R}} \) perpendicularly in two points. Arcs of (possibly degenerate) circles that are not geodesics are called hypercycles, and have constant curvature in \( (-1, 1) \setminus \{0\} \). It will be convenient to work with a particular frame: define

\[
S : (H^2_{\mathbb{R}})^{+} \to \mathbb{R},
\]

to be the hyperbolic distance of a point in \( (H^2_{\mathbb{R}})^{+} \) from the horizontal axis \( e_1 \). Set \( X = \nabla S \), where we naturally extend by continuity \( X \) at \( e_1 \). Then, denoting with \( X^\perp \) the counterclockwise rotation of \( X \) by \( \pi/2 \) radians, since the level sets of \( S \) are equidistant to each other, \( \{X, X^\perp\} \) forms an orthonormal frame of \( (H^2_{\mathbb{R}})^{+} \), see Fig. 1. The integral curves of \( X \) are all geodesic rays hitting \( e_1 \) perpendicularly. For each \( l \in [0, 1] \), let \( \delta_l \) be the integral curve of \( X^\perp \) so that \( \delta_l(0) = (0, l) \). Then, \( \delta_l \) is a family of equidistant hypercycles foliating \( (H^2_{\mathbb{R}})^{+} \), crossing \( e_2 \) perpendicularly and with constant curvature which coincides with the
Fig. 1 The special frame \( \{X, X^\perp\} \)

Euclidean one: \( K_1 = \frac{2l_1}{1+l_1^2} = \frac{1}{R(l)} \), where \( R(l) \in (0, +\infty) \) is the radius of the Euclidean circle representing the curve. Similarly, set \( \{N, N^\perp\} \) to be the orthonormal frame on \( H^2_{E \setminus \{0\}} \) where \( N \) is the radial unit length vector field emanating from the origin. Then, integral curves of \( N \) are rays of geodesics, and integral curves of \( N^\perp \) are concentric geodesic cycles. Notice that the frame \( \{X, X^\perp\} \) is invariant under the one-parameter subgroup of hyperbolic isometries fixing \( e_1 \) (\( X^\perp \) is the infinitesimal generator of the action by translations) and, up to reverse the orientation, under the reflections with respect to any geodesic integral curve of \( X \). Finally, notice that on \( e_1 \) and \( e_2 \), \( \{X, X^\perp\} \) is a positive rescaling of \( \{(0, 1), (-1, 0)\} \).

For a regular curve parametrized by arc length \( \eta \) we denote with \( \kappa_\eta(t) \) the inward signed curvature of \( \eta \) at \( \eta(t) \). We recall the identity

\[
\kappa_\eta \dot{\eta}^\perp = \nabla \dot{\eta} \dot{\eta},
\]

where here \( \nabla \) denotes the standard Levi-Civita connection associated to \( g_H \).

2.4 Reduction to \( H^2_{E_\mathbb{R}} \)

From now on, let \( E \) be an isoperimetric set with arbitrary weighted volume. Since both the density \( f \) and the conformal term of \( g_H \) are radial, the coarea formula implies that spherical symmetrization pointed at the origin preserves the weighted volume and does not increase the weighted perimeter of \( E \) (see [19, Theorem 6.2]). For this reason, we will assume \( E \) spherically symmetric with respect to the \( e_1 \) axis. Now, intersecting \( E \) with the Euclidean plane spanned by \( \{e_1, e_2\} \), we obtain a spherically symmetric profile \( \Omega \subset H^2_{E_\mathbb{R}} \). Let \( x^* \) be the furthest point of \( \Omega \) lying in the positive part of the \( e_1 \) axis (this is always possible by reflecting \( \Omega \) with respect to the \( e_2 \) geodesic). Let \( \gamma : [-a, a] \to H^2_{E_\mathbb{R}} \) be a counter-clockwise, arclength parametrization of the boundary of the connected component of \( \Omega \) containing \( x^* \), so that \( \gamma(0) = x^* \), see Fig. 2. The curve \( \gamma \) enjoys the following properties:

- \( \gamma \) is smooth on \((-a, a)\). Indeed, if there exists \( a^* \in (-a, a) \) such that \( \gamma(a^*) \) is not regular, then \( \partial E \) contains a singular set of Hausdorff dimension \( n - 2 \), but this is impossible because of Theorem 2.1.
- \( \gamma \) is symmetric with respect to the axis \( e_1 \).
– The curve $\gamma$ forms a simple, closed curve.
– Writing $\gamma = (\gamma_1, \gamma_2)$ in cartesian coordinates, one has that $\text{sgn}(\gamma_2(t)) = \text{sgn}(t)$. In particular, $\gamma : [0, a) \rightarrow (H^2_\mathbb{R})_+$. 
– $\dot{\gamma}(0) = X(\gamma(0))$.

To translate Eq. (2.1) as a property of the profile $\gamma$, we need the following definition.

**Definition 2.2** For any $t \in [0, a)$, denote by

– $C_t = C_t(s)$ the (possibly degenerated) oriented circle tangent to $\gamma(t)$, with center on $e_1$, parametrized by arclength and such that $C_t(0) = \gamma(t)$. Denote by $\kappa(C_t)$ its constant curvature.
– $c_t = c_t(s)$ the (possibly degenerated) oriented circle tangent to $\gamma(t)$, parametrized by arclength, such that $c_t(0) = \gamma(t)$ and $\kappa(c_t) = \kappa(\gamma(t))$.
– $x(C_t)$ and $x(c_t)$ the hyperbolic center of $C_t$ and $c_t$ respectively. Similarly, let $x_1(C_t)$ and $x_1(c_t)$ be the first Euclidean coordinate of $x(C_t)$ and $x(c_t)$ respectively.

**Remark 2.3** Let $F \subset B(0, 1) \subset \mathbb{R}^n$. Then, at every regular point $x \in \partial F$, the mean curvature $H$ is related with the Euclidean mean curvature $H_{\text{flat}}$ by

$$H(x) = \frac{1 - r^2}{2} H_{\text{flat}}(x) + (n - 1) g_{\text{flat}}(x, \bar{v}),$$
where $\tilde{v}$ is the outward normal vector to $\partial F$ with Euclidean norm equal to one. In particular, when $n = 2$, denoting with $\kappa_{\text{flat}}$ the usual Euclidean curvature, one has that

$$\kappa_{\gamma} = \frac{1 - |\gamma'(t)|^2}{2} \kappa_{\gamma} \text{ flat} + g_{\text{flat}}(\gamma, \tilde{v}).$$

Therefore, $\kappa_{\gamma}(C_t) = \kappa_{\gamma}$, that is comparison circles $C_t$ and $C_t$ in the hyperbolic setting coincide with comparison circles with respect to the Euclidean metric. From this formula, we also deduce that for any Euclidean circle $C$,

$$\kappa_C = \frac{1}{2} \left( \frac{1 - |x_0|^2}{\tau_0} + \tau_0 \right) = \coth(\tau),$$

where $x_0$ and $\tau_0$ are the Euclidean center and radius, and $\tau$ is the hyperbolic radius.

**Lemma 2.4** On $t \in [0, a)$ it holds

$$H(t) = \kappa_{\gamma}(t) + (n - 2)\kappa(C_t).$$

In particular,

$$H_f(t) = \kappa_{\gamma}(t) + (n - 2)\kappa(C_t) + h'(d_H(o, \gamma(t))) g_H(v(t), N(\gamma(t))) = \lambda,$$

where $v = -\dot{\gamma} \perp$.

We call $H_1(t) := \partial_t \ln(f)(\gamma(t)) = h'(d_H(o, \gamma(t))) g_H(v(t), N(\gamma(t)))$ the term coming from the log-convex density.

**Proof** In [5, Proposition 3.1] it is shown that the Euclidean mean curvature of the spherically symmetric set $E$ can be computed as

$$H_{\text{flat}} = \kappa_{\gamma} \text{ flat} + (n - 2)\kappa_{\text{flat}}(C_t).$$

Thanks to Remark 2.3 we have that

$$H(\gamma(t)) = \frac{1 - |\gamma'(t)|^2}{2} H_{\text{flat}}(\gamma(t)) + (n - 1) g_{\text{flat}}(\gamma(t), v)$$

$$= \frac{1 - |\gamma'(t)|^2}{2} \left( \kappa_{\gamma} \text{ flat} + (n - 2)\kappa_{\text{flat}}(C_t) \right) + (n - 1) g_{\text{flat}}(\gamma(t), v)$$

$$= \kappa_{\gamma} + (n - 1)\kappa(C_t).$$

\[\square\]

### 3 The proof

We have seen that existence, boundedness, and regularity of isoperimetric sets (Theorems 2.1, A.2, and A.3) together with the radial nature of the density $f$ allows us to assume the optimal set $E$ to be bounded and spherically symmetric with generating curve smooth away from the axis of symmetrization. Consequently, the problem is reduced to a planar situation, in which the profile curve $\gamma$ solves the ordinary differential equation induced by the constant weighted mean curvature $H_f$ of the original isoperimetric set, as stated in Lemma 2.4. Adapting Chamber’s analysis to our specific situation presents difficulties as the nonexistence of a natural choice of a frame on the tangent space as in the Euclidean plane. Consequently, to carry out a rigorous curvature-comparison analysis (for instance Lemma
3.15), it is crucial to carefully select a frame that appropriately accommodates the geometry of our particular case, as we did in Sect. 2.3. The proof of Theorem 1.1 relies on showing that $\gamma$ represents a circumference centered at the origin. The argument presented shows that refuting this possibility leads to a surprising consequence: the curve $\gamma$ must make a curl, as represented in Fig. 3, contradicting the fact that $\gamma$ is the parameterization of a spherically symmetric set. More rigorously, the contradiction arises as the combination of the next two lemmas.

**Lemma 3.1** For every $t \in (0, a)$

$$g_H(N, \dot{\gamma}) \leq 0.$$  

**Proof** The fact that the set $\Omega$ is spherically symmetric implies that $t \mapsto g_{\flat}(\gamma(t), \gamma(t))$ is non increasing. Differentiating in $t$ gives the desired sign of the angle between $N$ and $\dot{\gamma}$. □

Sect. 3.1 is devoted to the proof of the next lemma.

**Lemma 3.2** (The tangent lemma) If $\gamma$ is not a circle centered in the origin, there exist $0 < a_0 < a_1 < a_2 < a$ such that $\dot{\gamma}(a_0) = X_{\perp}(\gamma(a_0))$, $\dot{\gamma}(a_1) = -X(\gamma(a_1))$ and $\dot{\gamma}(a_2) = X(\gamma(a_2))$.

Assuming now that Lemma 3.2 holds true, the proof of the main result goes as follows.

**Proof of Theorem 1.1** If $\gamma$ is a circle centered at the origin we are done. Otherwise, Lemma 3.2 ensures the existence of $0 < a_2 < a$ such that $\dot{\gamma}(a_2) = X(\gamma(a_2))$. This violates the inequality of Lemma 3.1, because at $a_2$

$$g_H(N, \dot{\gamma}) = g_H(N, X) > 0.$$  

Therefore, the profile curve $\gamma$ has to be a circumference centered in the origin. Uniqueness is established by observing that up to measure zero the only set which, when spherically symmetrized, results in a centered ball, is a centered ball itself. □
3.1 Proof of the tangent lemma

The proof is made by following the behaviour of \( \gamma \) step by step: first we show that \( \gamma \) has to arch upwards with curvature strictly greater than one. The endpoint of this arc will be \( \gamma(a_0) \), where \( \dot{\gamma}(a_0) = X^\perp(\gamma(a_0)) \). Then, it goes down curving strictly faster than before, and this result about curvature is the tricky point to generalize in the hyperbolic setting. It turns out that the special frame given by the hypercyclical foliation \( (\delta_l)_{l \in [0,1]} \) is the good one. Then, arguing by contradiction, we will show that this behaviour must end at a point \( 0 < a_0 < a_1 < a \), where \( \dot{\gamma}(a_1) = -X(\gamma(a_1)) \). Finally, we prove the existence of \( a_2 \) so that \( \dot{\gamma}(a_2) = X(\gamma(a_2)) \) taking advantage of the mean-curvature convexity of \( \Omega \). We start by looking at what happens at the starting point.

**Lemma 3.3** One has that \( \dot{\gamma}(0) = X(\gamma(0)), \dot{\kappa}(0) = 0 \) and \( \kappa(0) \geq \kappa(0) > 1 \).

**Proof** This is a consequence of the symmetry of \( \gamma \) with respect to the \( e_1 \) axis, and of the fact that \( \gamma(0) \) represent the furthest point from the origin of \( \Omega \).

**Lemma 3.4** If there exists \( t^* \in [0,a) \) such that \( x_1(C_{t^*}) = 0 \) and \( \kappa(t^*) = \kappa(C_{t^*}) \), then \( \gamma \) has to be a centered circle.

**Proof** In this case \( \gamma(t) \) and \( C_{t^*}(s) \) solve the same ODE, with same initial data. Therefore, they have to coincide locally, and hence globally.

**Definition 3.5** Call \( \alpha : [0,a) \to [\pi, -\pi) \) the oriented angle made by \( \dot{\gamma} \) with \( X^\perp \). We say that \( \dot{\gamma}(t) \) is in the I, II, III and IV quadrant if \( \alpha(t) \) belongs to \([\pi/2, \pi], [0, \pi/2], [0, -\pi/2] \) and \([ -\pi/2, -\pi] \) respectively. We add strictly if \( \dot{\gamma} \) is not collinear to \( X \) and \( X^\perp \).

**Lemma 3.6** If for some \( t \in [0,a) \), \( \dot{\gamma}(t) \) belongs to the II quadrant, then \( x_1(C_t) \geq 0 \).

**Proof** We first treat the case \( t \in (0,a) \). Expressing \( N(\gamma(t)) \) in the \( \{X, X^\perp\} \) frame, we have thanks to Lemma 3.1 that

\[
0 \geq g_H(N, \dot{\gamma}) = g_H(X, N)g_H(X, \dot{\gamma}) + g_H(X^\perp, N)g_H(X^\perp, \dot{\gamma}) = g_H(X, N) \sin(\alpha) + g_H(X^\perp, N) \cos(\alpha).
\]

(3.1)

If \( \alpha = \pi/2 \), then

\[
0 \geq g_H(X, N),
\]

which is possible only when \( \gamma_2(t) = 0 \), that is \( t \in [0,a) \). If \( \alpha \in [0, \pi/2] \) then \( \cos(\alpha) > 0 \), implying that \( g_H(X^\perp, N) \leq -g(X, N) \tan(\alpha) \leq 0 \). Notice that this is possible only if \( \gamma_1(t) \geq 0 \). Calling \( -\vartheta < 0 \) the angle that \( N \) makes with \( X \), we get by Eq. (3.1) that

\[
\tan(\alpha) \leq \tan(\vartheta).
\]

(3.2)

Now, observe that the two geodesic rays \( \sigma_\gamma, \sigma_N \) starting at \( \gamma(t) \) with initial velocities \( \dot{\gamma}_N(0) = \dot{\gamma}_N(t) \) and \( \dot{\sigma}_N(0) = N^\perp(\gamma(t)) \), together with the axis \( e_1 \) and the geodesic orthogonal to \( e_1 \) passing from \( \gamma(t) \) bound two geodesic triangles \( \Delta_\gamma \) and \( \Delta_N \). Call \( d \) the distance between \( \gamma(t) \) and \( e_1 \). Then, the length of the sides \( \ell_\gamma \) and \( \ell_N \) of \( \Delta_\gamma \) and \( \Delta_N \) respectively, lying on \( e_1 \) are given via hyperbolic trigonometric laws by

\[
\tanh(\ell_\gamma) = \tan(\alpha) \sinh(d) \leq \tanh(\vartheta) \sinh(d) = \tanh(\ell_N).
\]

But this implies that \( x(C_t) \), which is the intersection of \( \sigma_\gamma \) with \( e_1 \), has first coordinate positive, as claimed. If \( t = 0 \), then \( C_0 = c_0 \) and approximates \( \gamma(t) \) up to the fourth order.
Therefore, if \( x_1(C_t) < 0 \), then there exists \( \varepsilon > 0 \) such that \( \gamma|_{(\varepsilon, 2\varepsilon)} \) lies outside the ball of centered in the origin and with radius \( d_H(\gamma(0), o) \). This is a contradiction because by construction \( \gamma(0) \) is the furthest point of \( \Omega \) from \( o \).

Our next goal is to show four important properties of the curve \( \gamma \). The proof is made by comparison with the circles \( c_t \) and \( C_t \), and the preservation of the weighted mean curvature \( H_f \). For this reason, we need the following preliminary lemma.

**Lemma 3.7** Let \( \eta = \eta(s) \) be an arc-length, counter-clockwise parametrization of a circle centered at \((0, y)\) such that \( \eta(0) = (r, y) \) and \( \eta(L) = (0, y + \tau) \). Let \( O = (-\tilde{\eta}, 0) \) be an arbitrary point lying on \( e_1 \) with \( \tilde{\eta} \in [0, 1) \), and \( \nu(s) \) the outward pointing normal to \( \eta(s) \). Then, setting

\[
\tilde{H}_1(s) := \partial_v(h(d_H(O, x)))|_{x = \eta(s)},
\]

if \( y = 0 \), then

\[
\tilde{H}_1(s) \leq 0, \text{ in } (0, L],
\]

and

\[
\tilde{H}_1''(0) \leq 0.
\]

Both inequalities are strict if \( \tilde{\eta} \neq 0 \). If \( y \in (0, 1) \) and \( \tilde{\eta} \neq 0 \), then

\[
\tilde{H}_1'(L) < 0.
\]

**Proof** Let \( T : H^2_{\mathbb{R}} \to H^2_{\mathbb{R}} \) be the unique isometry fixing \( e_1 \) and sending the origin to \( O \). Then,

\[
\tilde{H}_1(s) = \partial_v(h(d_H(O, x)))|_{x = \eta(s)} = h'(d_H(O, \eta(s)))g_H(\nu(s), T_s N(\eta(s))).
\]

For the sake of exposition, we will omit the arguments in the following computations. Differentiating one time in \( s \) we have that

\[
\tilde{H}_1'(s) = h''g_H(T_s N, \tilde{\eta})g_H(\nu, T_s N) + h'\frac{d}{ds}g_H(\nu, T_s N)
\]

\[
= h''g_H(T_s N, \tilde{\eta})g_H(\nu, T_s N) - h'\left(g_H(\nabla_\tilde{\eta} 1, T_s N) + g_H(\tilde{\eta} 1, \nabla_\tilde{\eta} T_s N)\right)
\]

\[
= h''g_H(T_s N, \tilde{\eta})g_H(\nu, T_s N) - h'\left(-g_H(T_s N, \tilde{\eta})\kappa_\nu + g_H(T_s N, \tilde{\eta})g_H(T_s N 1, \tilde{\eta})\kappa_1\right),
\]

where \( \kappa_1 \) is the curvature of the integral curve of \( T_s N 1 \) passing through \( \eta(s) \), which is a geodesic sphere centered at \( O \). Suppose first that \( y = 0 \) and \( \tilde{\eta} \neq 0 \). Then, \( \tilde{\eta} = N 1 \), and

\[
\tilde{H}_1'(s) = h''g_H(T_s N, N 1)g_H(N, T_s N) - h'g_H(T_s N, N 1)(-\kappa_\nu + g_H(T_s N, N)\kappa_1) < 0,
\]

because \( h'' > 0, h' > 0, g_H(T_s N, N 1) < 0, g_H(T_s N, N) > 0 \) and \( \kappa_\nu > \kappa_1 \) since \( \tilde{\eta} \neq 0 \). This proves Equation (3.3) when \( \tilde{\eta} \neq 0 \). The same holds in the context of Eq. (3.5) since \( \dot{\eta}(L) = N 1 \). Up to relaxing the inequalities, the proof when \( \tilde{\eta} = 0 \) is exactly the same. To prove Eq. (3.4), we differentiate \( \tilde{H}_1 \) one more time, obtaining

\[
\tilde{H}_1''(s) = h''g_H(T_s N, \tilde{\eta})^2g_H(\nu, T_s N) + h''\frac{d}{ds}g_H(T_s N, \tilde{\eta})g_H(\nu, T_s N)
\]

\[
+ h''g_H(T_s N, \tilde{\eta})\frac{d}{ds}g_H(\nu, T_s N) + h''g_H(T_s N, \tilde{\eta})\frac{d}{ds}g_H(\nu, T_s N)
\]
Observe that in zero \(g_H(T_sN, \dot{\gamma}) = 0\), hence only the second and last term survive
\[
\tilde{H}'(0) = \int_0^v dh \left| \frac{d^2}{ds^2} g_H(v, T_sN) \right|.
\]

Taking advantage of the explicit expression for \(\frac{d}{ds} g(T_sN, N)\) we obtained before, we get
\[
\frac{d^2}{ds^2} g_H(T_sN, N) = -\frac{d}{ds} g \left( g_H(T_sN, N^\perp) \left( -\kappa_\gamma + g_H(T_sN, N) \kappa_1 \right) \right)
= (\kappa_\gamma - g_H(T_sN, N) \kappa_1) \frac{d}{ds} g_H(T_sN, N^\perp),
\]
which implies that
\[
\tilde{H}'(0) = \left( h'' g_H(T_sN, N) + h' \kappa_\gamma - h' g_H(T_sN, N) \kappa_1 \right) \frac{d}{ds} g_H(T_sN, N^\perp).
\]

Hence, we are left to show that \(\frac{d}{ds} g_H(T_sN, N^\perp) < 0\). Developing again we get
\[
\frac{d}{ds} g_H(T_sN, N^\perp) = \frac{d}{ds} g_H(\nabla N, N^\perp) |_{s=0} + \frac{d}{ds} g_H(T_sN, \nabla N^\perp) |_{s=0}
\]
\[
= g_H(T_sN, N)^2 \kappa_1 |_{s=0} - g_H(TN, N) \kappa_\gamma |_{s=0}
\]
\[
= \kappa_1 - \kappa_\gamma < 0.
\]

We are now ready to prove the next result.

**Lemma 3.8** The following four points hold.

\(i\) If for \(t \in (0, a)\) one has that \(\kappa_{\gamma}(t) \geq \kappa(C_t) > 1\), then \(t \mapsto x_1(C_t)\) is smooth and \(\frac{d}{dt} x_1(C_t) \geq 0\).

\(ii\) If \(\gamma\) is not a centered circle, then \(\kappa_{\gamma}(0) > 0\).

\(iii\) If for \(t \in (0, a)\), \(\dot{\gamma}(t)\) is in the II quadrant and \(\kappa_{\gamma}(t) = \kappa(C_t) > 1\), then \(\dot{\kappa}_{\gamma}(t) \geq 0\).

Moreover, if \(\dot{\gamma}(t) \neq X_{\gamma}(\gamma(t))\) and \(C_t\) is not centered in the origin, then \(\dot{\kappa}_{\gamma}(t) > 0\).

\(iv\) If for \(t \in (0, a)\) one has that \(\dot{\gamma}(t) = X_{\gamma}(\gamma(t))\), \(\gamma_1(t) > 0\) and \(\kappa_{\gamma}(t) \geq \kappa(C_t) > 1\), then \(\kappa_{\gamma}(t) > 0\).

**Proof** We start with point \(i\). Observe that since \(C_t\) approximates \(\gamma\) up to the third order around \(\gamma(t)\), it suffices to prove \(\frac{d}{dt} x_1(C_t) \geq 0\) replacing \(\gamma\) with \(C_t\). Also, we can suppose \(x(C_t)\) on \(e_2\) by composing with the unique hyperbolic isometry translating \(x(C_t)\) on \(e_2\) and fixing \(e_1\). The curvature condition \(\kappa_{\gamma}(t) \geq \kappa(C_t) > 1\) ensures that \(x(C_t) \in (H^2_+)\). By monotonicity of the function \(\tanh(\cdot/2)\), it suffices to prove the claim for the Euclidean center of \(C_t\). Thus, we have reduced the problem to an explicit computation in the Euclidean plane, that can be found in [5, Lemma 5.3]. Thanks to Lemma 3.7 the proofs of the other points go exactly as in [5, Lemma 3.4, Lemma 3.5 and Lemma 3.7]. We show point \(ii\). Differentiating \(H_f\) twice, we get that
\[
\ddot{\kappa}_{\gamma}(0) = -(n - 2)\ddot{\kappa}(C_0) - H''_1.
\]
By symmetry, $c_0 = C_0$. Moreover, since $\dot{k}_\gamma(0) = 0$, we have that both $c_0$ and $C_0$ approximate $\gamma$ up to the fourth order near zero. Hence, $\dot{k}(C_0) = 0$. Therefore, it suffices to determine the sign of $H_1'$ replacing $\gamma$ with $C_0$. Let $T : H^2_\mathcal{CE} \to H^2_\mathcal{CE}$ be the unique isometry fixing $e_1$ that moves $x(C_0)$ to the origin. The result follows by Equation (3.4) of Lemma 3.7 setting $O = T(0)$, and noticing that $T(0) \neq 0$ by Lemma 3.4. The proofs of points iii. and iv. are similar: in the first case the condition $\kappa_\gamma(t) = \kappa(C_t) > 1$ implies that $C_t$ approximates $\gamma$ near $t$ up to the third order, the same holds if $\dot{\gamma}(t) = X^\perp(\gamma(t))$ by symmetry. Hence, substituting $\gamma$ with $C_t$ and differentiating one time $H_f$, we have to determine the sign of $H_1'$ in the case of a circle, via Lemma 3.7.

We are now ready to analyse the first behaviour of $\gamma$.

**Definition 3.9** (Upper curve) The upper curve is the (possibly empty) maximal connected interval $I_U \subset [0, a)$ such that $0 \in I_U$ and for all $t \in I_U$

a. $\dot{\gamma}(t)$ is in the II quadrant,

b. $\kappa_\gamma(t) \geq \kappa(C_t) > 1$,

c. $t \mapsto x_1(C_t)$ is smooth and $\frac{d}{dt} x_1(C_t) \geq 0$.

We set

$$a_0 := \sup I_U.$$

In the discussion, we will sometimes identify the upper curve with its image through $\gamma$.

**Definition 3.10** We say that a curve $\eta$ is graphical with respect to the hypercyclic foliation $(\delta_t)_{t \in [0, 1]}$ if $\eta$ meets each $\delta_t$ at most once.

Notice that the upper curve (if non-empty) is graphical with respect to the hypercyclical foliation because $\dot{\gamma}$ is in the II quadrant.

**Proposition 3.11** The upper curve is non empty and enjoys the following properties

i. $0 < a_0 < a$,

ii. $a_0 \in I_U$,

iii. $\gamma_1(a_0) > 0$,

iv. $\dot{\gamma}(a_0) = X^\perp(\gamma(a_0))$.

**Proof** Thanks to Lemma 3.8, the proof goes exactly as [5, Lemma 3.11 and Proposition 3.12]. We sketch for completeness the idea behind each point. We start by showing that the upper curve is non empty.

By Lemma 3.3 we know that $\dot{\gamma}(0) = X(\gamma(0))$, $\dot{\kappa}_\gamma(0) = 0$, and $\kappa_\gamma(0) \geq \kappa(C_0) > 1$. Moreover, by Lemma 3.8 point ii. since by assumption $\gamma$ is not a centered circle, we get that $\dot{\kappa}_\gamma(0) > 0$. By continuity, since $c_0 = C_0$ approximates $\gamma$ up to the fourth order near zero, we have that there exists $\varepsilon > 0$ such that for all $t \in [0, \varepsilon)$ points a. and b. in Definition 3.9 are satisfied. Finally, point c. of Definition 3.9 follows from Lemma 3.8 point i. which asserts that $\kappa_\gamma(t) \geq \kappa(C_t) > 1$ implies that $x_1(C_t)$ is smooth and $\frac{d}{dt} x_1(C_t) \geq 0$. Hence, $[0, \varepsilon) \subset I_U$, proving that the upper curve cannot be empty.

Notice that $0 < a_0$ cannot be equal to $a$ since otherwise the curve $\gamma$ does not close itself on $e_1$, simply because $\dot{\gamma}$ belongs to the II quadrant by definition of $I_U$. By the regularity of $\gamma$ and that $I_U$ is defined by three close conditions, we have that $a_0 \in I_U$. By composing with the unique hyperbolic isometry sending $\gamma(a_0)$ on $e_2$ fixing $e_1$, we can see that $x_1(C_{a_0}) \leq 0$ because $\dot{\gamma}(a_0)$ belongs to the II quadrant. Lemma 3.4 and Lemma 3.6 imply that $x_1(C_0) \geq 0$.
and since $\frac{d}{dt} x_1(C_t) \geq 0$ in $I_U$, one must have that $\gamma_1(a_0) > 0$. The last point is proved by contradiction: if $\dot{\gamma}(a_0) \neq X^\perp(\gamma(a_0))$, then $a_0 \in I_U$ implies that $\dot{\gamma}(a_0)$ is strictly in the II quadrant. If $\kappa_\gamma(a_0) = \kappa(C_{a_0}) > 1$, then $c_{a_0} = C_{a_0}$ approximates $\gamma$ to the third order and Lemma 3.8 point iii. implies that there exists some $\delta > 0$ such that $\kappa_\gamma(t) \geq \kappa(C_t) > 1$ for $t \in [a_0, a_0 + \delta]$. The same holds if $\kappa_\gamma(a_0) > \kappa(C_{a_0}) > 1$ by continuity. This means that $[a_0, a_0 + \delta] \subset I_U$, which is not possible by the very definition of $a_0$. Hence, $\dot{\gamma}(a_0) = X^\perp(\gamma(a_0))$.

**Definition 3.12** (Lower curve) The lower curve is the maximal connected interval $I_L \subset [a_0, a)$ such that for all $t \in I_L$

a. $a_0 \in I_L$,

b. $\dot{\gamma}(t)$ is in the III quadrant,

c. calling $\tilde{t} \in I_U$ the unique time such that $S(\gamma(t)) = S(\gamma(\tilde{t}))$ we have that $\kappa_\gamma(t) \geq \kappa_\gamma(\tilde{t})$.

We set

$$a_1 := \sup I_L.$$ 

Notice that $a_0$ truly belongs to $I_L$, so $I_L \neq \emptyset$. Also, the lower curve is graphical with respect to the hypercyclical foliation because $\dot{\gamma}$ is in the III quadrant. Our next goal is to prove that $a_1 < a$. Again, we proceed by contradiction, and the intuition is the following: suppose that $a_1 = a$. If $\kappa_\gamma(t) = \kappa_\gamma(\tilde{t})$ for all $t \in I_L$, then the lower curve is nothing else than the upper curve reflected with respect to the geodesic orthogonal to $e_1$ and passing through $\gamma(a_0)$. Hence, $\lim_{t \to a^+} \alpha(t) = -\frac{\pi}{2}$. Otherwise, if the $\gamma|_{I_L}$ curves strictly faster than the upper curve at some point, then the angle of incidence $\lim_{t \to a^+} \alpha(t) < -\frac{\pi}{2}$ (see Fig. 4). But this cannot be true, because it contradicts the regularity of $\partial E$ pointed out in Theorem 2.1. To prove that the lower curve curves strictly faster than the upper curve we need first to express the curvature with respect to the $\{X, X^\perp\}$ frame, and next prove three comparison lemmas.

**Lemma 3.13** Let $\eta$ any regular curve parametrized by arclength such that $\dot{\eta}(t)$ is not collinear to $X(\eta(t))$. Then,

$$-\kappa_\eta(t) = \dot{\beta}(t) - K_1(\eta(t)) \cos(\beta(t)),$$
where $\beta(t)$ denotes the angle between $\dot{\eta}$ and $X^\bot$, and $K_1$ is the curvature of the leaf $\delta_1$ passing through $\eta(t)$.

**Proof** Decompose $\dot{\eta} = AX + BX^\bot$. Then, since $\kappa_\eta \dot{\eta}^\bot = \nabla_\dot{\eta} \dot{\eta}$, we get that

$$- \cos(\beta)\kappa_\eta = g_H\left(\nabla_\dot{\eta} \dot{\eta}, X\right) = \partial_t (\sin(\beta)) - g_H\left(\dot{\eta}, \nabla_\dot{\eta} X\right).$$

Now, keeping in mind that $\nabla_X X = 0$ and $g_H(\nabla_X X^\bot, X) = -K_1(\eta(t))$, we get

$$g_H\left(AX + BX^\bot, \nabla_{AX+BX^\bot} X\right) = B^2 g_H\left(X^\bot, \nabla_X X\right) = \cos(\beta)^2 K_1(\eta(t)).$$

Dividing both sides by $\cos(\beta)$ we get the desired identity. □

We can prove our first curvature comparison lemma.

**Lemma 3.14** (κ comparison lemma) Let $\eta_1 : (0, A_1] \to H^2_{R}$ and $\eta_2 : (0, A_2] \to H^2_{R}$ be two hypercyclical graphical curves parametrized by arclength and with velocity vectors in the II quadrant. Suppose that there exists $l_0 \in (0, 1)$ such that

$$\lim_{t \to 0^+} \eta_1(t) and \lim_{t \to 0^+} \eta_2(t),$$

exist and belong to the same leaf $\delta_{l_0}$. Also, suppose that $\eta_1(A_1) = \eta_2(A_2)$, $\dot{\eta}_1(A_1) = \dot{\eta}_2(A_2)$, and that if $S(\eta_1(t)) = S(\eta_2(\tau))$ then $\kappa_{\eta_1}(t) \geq \kappa_{\eta_2}(\tau)$. Then, calling $\alpha_1$ and $\alpha_2$ the angle made by $\dot{\eta}_1$ and $\dot{\eta}_2$ with $X^\bot$ we have that

$$\lim_{t \to 0^+} \alpha_1(t) \geq \lim_{t \to 0^+} \alpha_2(t).$$

Moreover, if for some $t$ and $\tau$ such that $S(\eta_1(t)) = S(\eta_2(\tau))$, one has that $\kappa_1(t) > \kappa_2(\tau)$, then

$$\lim_{t \to 0^+} \alpha_1(t) > \lim_{t \to 0^+} \alpha_2(t).$$

**Proof** Since the curves are graphical with respect to the hypercyclical foliation we can operate a change of variable: we observe that the two height functions $s_1(t) := S(\eta_1(t))$ and $s_2(\tau) = S(\eta_2(\tau))$ are bijections with same image of the form $(l_0, L]$. By hypothesis $\kappa_{\eta_1}(s_1^{-1}(l)) \geq \kappa_{\eta_2}(s_2^{-1}(l))$ for every $l \in (l_0, L]$. Comparing the two curves in the $l \in (l_0, L]$ variable, since $s_1 = g_H(\nabla X, \dot{\eta}_1) = g_H(X, \dot{\eta}_1) = \sin(\alpha_i), i = 1, 2$, we get by Lemma 3.13 that

$$0 \leq \kappa_{\eta_1}(l) - \kappa_{\eta_2}(l) = \dot{\alpha}_2(l) \sin(\alpha_2(l)) - \dot{\alpha}_1(l) \sin(\alpha_1(l)) - \frac{2l}{1 + l^2}(\cos(\alpha_2(l)) - \cos(\alpha_1(l))).$$

Multiplying by $(1 + l^2)$ and integrating we finally get that

$$0 \leq \int_{l_0}^{L} (1 + l^2)(\cos(\alpha_1) - \cos(\alpha_2)) dl' + 2l(\cos(\alpha_1) - \cos(\alpha_2)) dl = \int_{l_0}^{L} \frac{dl'}{dl} \left((1 + l^2)(\cos(\alpha_1) - \cos(\alpha_2))\right) dl = \lim_{l \to l_0^+} (1 + l^2)(\cos(\alpha_2(l)) - \cos(\alpha_1(l))).$$

If the two curvatures are different somewhere, then the inequality between the two angles is strict. □
Lemma 3.15 \((\kappa(C_t) \text{ comparison lemma})\) Let \(\eta_1, \eta_2\) be as in Lemma 3.14. Then, for any two points \(\eta_1(t_1)\) and \(\eta_2(t_2)\) on the same leaf \(\delta_1\), calling \(C^1\) and \(C^2\) the comparison circles at \(\eta_1(t_1)\) and \(\eta_2(t_2)\) as in Definition 2.2, we have that

\[ \kappa(C^1) \leq \kappa(C^2). \]

**Proof** For \(i = 1, 2\), the hyperbolic radius of \(C_i\) together with \(e_1\) and the geodesic starting from \(\eta_i(t_i)\) and hitting \(e_1\) perpendicularly bound a geodesic triangle \(\Delta_i\). Let \(d_{i1}^1\) be the hyperbolic radius, \(d_{i2}^1\) be the side touching \(e_1\) and \(d_{i3}^1\) the the remaining side of \(\Delta_i\). Similarly, for \(i = 1, 2\) and \(j = 1, 2, 3\), call \(\beta_{ij}^1\) the angle opposite to \(d_{ij}^1\), and \(\ell_{ij}^1\) the length of \(d_{ij}^1\). We refer to Fig. 5. By construction \(\beta_{11}^1 = \beta_{12}^1 = \pi / 2\), \(\beta_{21}^1 = \alpha_1\), and since \(\eta_1(t_1)\) and \(\eta_2(t_2)\) are in the same hypercycle by hypothesis, we get \(\ell_{11}^1 = \ell_{12}^1\). Then, by the hyperbolic law of cosines and by Lemma 3.14 we get

\[ \kappa(C^1) = \coth(\ell_{11}^1) = \frac{\cos(\alpha_1)}{\tanh(\ell_{11}^1)} \leq \frac{\cos(\alpha_2)}{\tanh(\ell_{12}^3)} = \coth(\ell_{12}^3) = \kappa(C^2). \]  \(\text{(3.6)}\)

\(\square\)

Lemma 3.16 \((H_1 \text{ comparison lemma})\) Let \(\eta_1, \eta_2\) be as in Lemma 3.14 and let \(\tilde{\eta}_1\) be the reflection of \(\eta_1\) with respect to the geodesic passing through \(\eta_1(A_1)\) and crossing \(e_1\) perpendicularly. Reverse its parametrization, so that the angle that \(\tilde{\eta}_1\) makes with \(X^\perp\) is equal to \(\tilde{\alpha}_1 = -\alpha_1\). Moreover, suppose that

\[ g_H(N, \tilde{\eta}_2) \leq 0. \]

Denote the unitary outward pointing normals to \(\eta_1\) and \(\eta_2\) by \(\tilde{v}_1\) and \(v_2\). Then, for any two points \(\tilde{\eta}_1(t_1)\) and \(\eta_2(t_2)\) on the same leaf \(\delta_1\) we have that

\[ g_H(N(\tilde{\eta}_1(t_1)), \tilde{v}_1(t_1)) \leq g_H(N(\eta_2(t_2)), v_2(t_2)). \]
with equality if and only if \( \hat{\eta}_2(t_2) \) and \( \hat{\eta}_1(t_1) \) are tangent to the same circle centered in the origin and \( \hat{\eta}_1(t_1) = -\hat{\eta}_2(t_2) \).

**Proof** Parametrize \( \delta_l : \mathbb{R} \to H^2_0 \) by arclength in the \( X^\perp \) direction, so that \( \delta_l \) intersects \( e_2 \) at \( \delta_l(0) \). Let \( \hat{\vartheta}(s) \) be the angle that \( N \) makes with \( X^\perp \) at \( \delta_l(s) \) and \( s_2 < s_1 \) be such that \( \delta_l(s_1) = \tilde{\eta}_1(t_1) \) and \( \delta_l(s_2) = \eta_2(t_2) \). Let \( \Theta_1, \Theta_2 \in [0, \frac{\pi}{2}] \) be the angles that \( N \) makes with \( \varphi_1(t_1) \) and \( \varphi_2(t_2) \) respectively. Then, \( \Theta_1 := \vartheta(s_1) - \tilde{\alpha}_1 - \frac{\pi}{2}, \Theta_2 := \vartheta(s_2) - \alpha_2 - \frac{\pi}{2} \),

\[
g_H(N(\tilde{\eta}_1(t_1))), \tilde{\varphi}_1(t_1)) = \cos(\Theta_1),
\]

and

\[
g_H(N(\eta_2(t_1))), \varphi_2(t_2)) = \cos(\Theta_2).
\]

We need to investigate if \( \Theta_2 \leq \Theta_1 \), and when \( \Theta_2 < \Theta_1 \). Let \( s_2^* \in \mathbb{R} \) be such that the unit vector at \( \delta_l(s_2^*) \) that forms an angle of \( \alpha_2 \) with \( X^\perp \) is tangent to a circle centered in the origin. The value \( s_2^* \) exists in the interval \( [s_2, 0] \) because by Lemma 3.6, Eq. (3.2), we have that \( \vartheta(s_2) - \frac{\pi}{2} \geq \alpha_2 \) and, in the intersection of \( \delta_l \) with \( e_2 \) we have that \( \vartheta(0) - \frac{\pi}{2} = 0 \leq \alpha_2 \). By continuity there must be a point \( s_2 \leq s_2^* < 0 \) such that \( \vartheta(s_2^*) - \frac{\pi}{2} = \alpha_2 \). Then

\[
\Theta_2 = -\vartheta(s_2^*) + \vartheta(s_2) = -\int_{s_2}^{s_2^*} \hat{\vartheta}(s) \, ds.
\]

Set \( s_1^* := -s_2^* \), and notice that \( \hat{\vartheta}(-s) = \pi - \vartheta(s) \) for every \( s \geq 0 \). Then,

\[
\Theta_1 = \vartheta(s_1) - \tilde{\alpha}_1 - \frac{\pi}{2} = \vartheta(s_1) + \alpha_2 - (\tilde{\alpha}_1 + \alpha_2) - \frac{\pi}{2} = \vartheta(s_1) + \vartheta(s_2^*) - \pi - (\tilde{\alpha}_1 + \alpha_2) = \vartheta(s_1) - \vartheta(s_1^*) - (\tilde{\alpha}_1 + \alpha_2) = -\int_{s_1}^{s_1^*} \hat{\vartheta}(s) \, ds - (\tilde{\alpha}_1 + \alpha_2).
\]

Hence,

\[
\Theta_2 - \Theta_1 = (\tilde{\alpha}_1 + \alpha_2) - \int_{s_2}^{s_2^*} \hat{\vartheta}(s) \, ds + \int_{s_1^*}^{s_1} \hat{\vartheta}(s) \, ds = -\int_{s_2}^{s_2^*} \hat{\vartheta}(s) \, ds + \int_{s_1^*}^{s_1} \hat{\vartheta}(s) \, ds,
\]

since \( \tilde{\alpha}_1 + \alpha_2 \leq 0 \) by Lemma 3.14. Let \( \ell = 2 \arctan(h) \) be the hyperbolic distance of \( \delta_l \) from \( e_1 \). We claim that

\[
\hat{\vartheta}(s) = -(\sinh(\ell) \cosh(\ell) \cosh^2(\sech(\ell)s) + \coth(\ell) \sinh^2(\sech(\ell)s))^{-1} < 0.
\]

Assuming that this identity holds, letting \( \delta := s_2^* - s_2 \), we have that

\[
\Theta_2 - \Theta_1 \leq -\int_{s_2}^{s_2^*} \hat{\vartheta}(s) \, ds + \int_{s_1^*}^{s_1} \hat{\vartheta}(s) \, ds + \int_{s_1^*}^{s_1} \hat{\vartheta}(s) \, ds \\
= \int_0^{\delta} \hat{\vartheta}(s_2^* - \tau) - \hat{\vartheta}(s_2^* + \tau) \, d\tau + \int_{s_1}^{s_1^*} \hat{\vartheta}(s) \, ds \\
\leq \int_{s_1}^{s_1^*} \hat{\vartheta}(s) \, ds \leq 0,
\]

where we used that \( s_1^* = -s_2^* \) and \( \hat{\vartheta}(s) \) is a strictly negative, even function increasing in \([0, +\infty) \). This implies that \( \Theta_2 \leq \Theta_1 \), with equality if and only if \( \delta = 0 \) and \( \alpha_1 = \alpha_2 \), that
is when \( \tilde{\eta}_2 \) and \( \tilde{\eta}_1 \) are tangent to the same circle centered in the origin. We are left to prove Equation (3.7). Let \( \beta(t) \) be the angle that \( X \) makes with \( N \) in \( \delta_t(s) \). Since \( \beta(s) + \frac{\pi}{2} = \vartheta(s) \), it suffices to compute \( \dot{\beta}(s) \). The hypercycle \( \delta_t(s) \) has curvature \( \frac{2l}{1+l^2} = \tanh(t) \). The circle centered in the origin passing through \( \delta_t(s) \) has curvature \( \coth(d_H(0, \delta_t(s))) = \frac{\cos(\beta(s))}{\tanh(t)} \), by the hyperbolic trigonometric laws (as in Equation (3.6)). Now, we obtain an ODE for \( \beta(s) \) arguing as in Lemma 3.13: at any time \( s \in \mathbb{R} \) we have that

\[
- \tanh(t) \cos(\beta) = - \tanh(t) g_H(\dot{\delta}_t, N^\perp) = g_H(\nabla_{\dot{\delta}_t} \dot{\delta}_t, N)
\]

\[
= \frac{d}{ds} (g_H(\dot{\delta}_t, N)) - g_H(\dot{\delta}_t, \nabla N)
\]

\[
= - \cos(\beta) \dot{\beta} + g_H(\dot{\delta}_t, N^\perp)^2 g_H(\nabla N N^\perp, N)
\]

\[
= - \cos(\beta) \dot{\beta} - \frac{\cos(\beta)^3}{\tanh(t)}.
\]

Dividing both sides by \( \cos(\beta) \neq 0 \) it follows that

\[
\begin{align*}
\dot{\beta}(s) &= \tanh(t) - \frac{\cos(\beta(s))^2}{\tanh(t)}, \quad s \in \mathbb{R}, \\
\beta(0) &= 0.
\end{align*}
\]

By integration, one can compute the explicit solution

\[
\beta(s) = - \arctan(\text{csch}(t) \tanh(\sech(t)s)),
\]

that by differentiation gives

\[
\dot{\beta}(s) = - \left( \sinh(t) \cosh(t) \cosh^2(\text{sech}(t)s) + \coth(t) \sinh^2(\text{sech}(t)s) \right)^{-1},
\]

proving Eq. (3.7). \( \square \)

We can prove the main result about the lower curve.

**Proposition 3.17** It \( \gamma \) is not a circle centered in the origin, the lower curve is contained in \([a_0, a], \) that is \( 0 < a_1 < a. \) Furthermore, \( a_1 \in I_L \) and \( \gamma(a_1) = -X(\gamma(a_1)) \).

**Proof** By property iv. of Lemma 3.8, \( a_1 > a > 0. \) Suppose by contradiction that \( a_1 = a. \) Set \( \tilde{\eta}_1 \) to be the (reparametrized) lower curve and \( \eta_2 \) the upper curve. Choose any point \( t \in I_L \) with corresponding \( \tilde{t} \in I_U. \) Applying Lemma 3.15 and Lemma 3.16 to \( \tilde{\eta}_1(t) \) and \( \eta_2(\tilde{t}) \), and taking advantage of the expression for \( H_f \) given in Lemma 2.4, we get that

\[
H_f(\gamma(\tilde{t})) = H_f(\gamma(t)) = \kappa(\gamma(t) + (n - 2) \kappa(C_{\tilde{t}}) + h'(d_H(0, \gamma(t))) g_H(\nu(t), N)
\]

\[
< \kappa(\gamma(t) + (n - 2) \kappa(C_{\tilde{t}}) + h'(d_H(0, \gamma(t))) g_H(\nu(t), N).
\]

We have that

\[
d_H(0, \gamma(t)) \leq d_H(0, \gamma(\tilde{t})).
\]

This can be verified again via the trigonometric rules for hyperbolic triangles: fix \( t \in I_U, \) and call \( \beta \) and \( \tilde{\beta} \) the angle that \( N \) makes with \( X(\gamma(t)) \) and \( X(\gamma(\tilde{t})) \) respectively. Notice that \( 0 \leq \beta \leq \tilde{\beta}. \) Then, calling \( d \) the distance of \( \gamma(t) \) and \( \gamma(\tilde{t}) \) from \( e_1, \) we get that

\[
\tanh(d_H(0, \gamma(t))) = \frac{\tanh(d)}{\cos(\beta)} \leq \frac{\tanh(d)}{\cos(\tilde{\beta})} = \tanh(d_H(0, \gamma(\tilde{t}))).
\]
Hence
\[ H_f(\gamma(\bar{t})) < \kappa_\gamma(t) + (n - 2)\kappa(C_\bar{t}) + h'(d_H(0, \gamma(\bar{t})))g_H(v(\bar{t}), N), \]
implying
\[ \kappa_\gamma(\bar{t}) < \kappa_\gamma(t), \]
since \( h \) is strictly convex. This is a contradiction because Lemma 3.14 tells us that the lower curve hits the \( e_1 \) axis with an angle strictly smaller than \( -\frac{\pi}{2} \). Therefore, \( a_1 < a \). Since \( \gamma \) is smooth in \( a_1 < a \), and the conditions on \( I_L \) are closed, we deduce that \( a_1 \in I_L \). Suppose now that \( \alpha(a_1) \) is strictly in the III quadrant. Since \( a_1 \in I_L \), we can apply again the comparison lemmas to \( \gamma(a_1) \) and \( \gamma(\bar{a}_1) \) to infer
\[ \kappa_\gamma(\bar{a}_1) < \kappa_\gamma(a_1). \]
By continuity of \( \kappa_\gamma \) and \( \dot{\gamma} \) around \( a_1 \), we get that there exists a neighbourhood of \( a_1 \) in which \( \dot{\gamma} \) is in the III quadrant and the above inequality holds in the not strict sense. But this implies that \( a_1 \) is not the supremum of \( I_L \). Therefore, the velocity vector of \( \gamma \) at \( a_1 \) has to be equal to \(-X\).

We prove the last part of the tangent lemma.

**Proposition 3.18** If \( \gamma \) is not a centered circle, then there exists \( 0 < a_1 < a_2 \) such that \( \dot{\gamma}(a_2) = X(\gamma(a_2)) \).

**Proof** If \( a_2 \) exists we are done. Otherwise, we show that the non existence contradicts the mean-curvature convexity of \( \Omega \). Let
\[ I_c := \{ t \in [a_1, a) : \dot{\gamma} \text{ is in the I or IV quadrant} \}. \]
Here the index stands for \textit{curl curve}. Set \( \bar{a}_2 := \sup I_c \). Since \( \kappa(a_1) > 1 \) we have that \( a_1 < \bar{a}_2 \). If \( \bar{a}_2 < a \), then the mean convexity of \( \Omega \) implies that
\[ \kappa_\gamma(\bar{a}_2) \geq (n - 1) - (n - 2)\kappa(C_{\bar{a}_2}) > 0. \]
To see this, move \( \gamma(\bar{a}_2) \) on \( e_2 \) as in Lemma 3.8. Then, \( C_{\bar{a}_2} \) is oriented clockwise, and hence has negative curvature. But this implies that we can extend \( I_c \) after \( \bar{a}_2 \), contradicting the definition of \( \bar{a}_2 \). So, we need to rule out the situation in which \( \bar{a}_2 = a \). If it is the case, then again for mean-convexity one has that in \( I_c \)
\[ \kappa_\gamma(t) > 1. \]
Moreover, for \( t \in I_c \setminus \{ a_1 \} \) we have that \( \dot{\gamma} \) lies in the IV quadrant, because otherwise this implies together with \( \kappa_\gamma(t) > 0 \) that \( \gamma \) cannot close at \( e_1 \). Therefore \( \alpha(t) \) lies in the IV quadrant and it is strictly increasing, implying that
\[ \lim_{t \to a^+} \alpha(t) < -\frac{\pi}{2}. \]
This cannot happen because of the before mentioned regularity properties of isoperimetric sets.

The proof of the tangent lemma is then the collection of the results we showed in this section.

**Proof of Lemma 3.2** The existence of the chain \( 0 < a_0 < a_1 < a_2 < a \) is ensured by Proposition 3.11, Proposition 3.17 and Proposition 3.18.
4 Symmetric sets in $H^m_\mathbb{K}$

Consider any rank one symmetric space of non-compact type $(M^n, g) = (H^m_\mathbb{K}, g)$, $\mathbb{K} \in \{\mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Set $d = \dim(\mathbb{K}) \in \{2, 4, 8\}$ so that the real dimension of $M$ is $n = md$. Recall that if $\mathbb{K} = \mathbb{O}$, we only have the Cayley plane $H^2_\mathbb{O}$. As classical references on symmetric spaces we cite the books of Eberlein [7] and Helgason [10]. Fix an arbitrary base point $o$, and let $N$ be the unit-length, radial vector field emanating from it. As in Definition 1.2, let $\mathcal{H}$ be the distribution on $M \setminus \{o\}$ induced by the $(-4)$-eigenspace of the Jacobi operator $R(\cdot, N)N$. Denote with $\mathcal{V}$ the orthogonal complement of $\mathcal{H}$ with respect to $g$. For every $x \in M \setminus \{o\}$, we have the orthogonal splitting

$$T_x M = \mathcal{H}_x \oplus \mathcal{V}_x,$$

with orthogonal projections $(\cdot)^\mathcal{H}$ and $(\cdot)^\mathcal{V}$. Let now $(\tilde{M}^n, g_H) = (H^m, g_H)$, and choose an arbitrary base point $\bar{o}$ in it. The isometric identification of $T_o M$ with $T_o \tilde{M}$ according to the flat metrics $(\exp_o^M)^*g|_0$ and $(\exp_{\bar{o}}^{\tilde{M}})^*g_H|_0$, induces a well defined diffeomorphism

$$\Psi = \exp_{\bar{o}}^{\tilde{M}} \circ (\exp_o^M)^{-1} : M \to \tilde{M}.$$

With a slight abuse of notation, we still denote with $g_H$ the metric $\Psi^*g_H$, that makes $M$ isometric to $\tilde{M}$. The following lemma allows us to compare $g$ with $g_H$.

**Lemma 4.1** For every $x \in M \setminus \{o\}$, the splitting

$$T_x M = \mathcal{H}_x \oplus \mathcal{V}_x,$$

is orthogonal with respect to $g_H$. In particular, letting $d_H$ be the Riemannian distance induced by $g_H$ on $M$, one has that

$$g(v, w) = \cosh^2(d_H(o, x))g_H(v^\mathcal{H}, w^\mathcal{H}) + g_H(v^\mathcal{V}, w^\mathcal{V}),$$

(4.1)

for all $v, w \in T_x M$.

**Proof** Fix an arbitrary unit direction $N_o \in T_o M$, and let $V_o \in T_o M$ be any vector orthogonal to it with respect to $g|_o = g_H|_o$. Since the radial geodesics emanating from $o$ are the same for $g$ and $g_H$, the Jacobi field $Y(t)$ along the geodesic $t \mapsto \exp(tN_o)$, determined by the initial conditions $Y(0) = 0$, $\dot{Y}(0) = V_o$ is the same for both metrics. Let $V(t)$ and $V_H(t)$ be the parallel transport of $V_o$ along $\sigma$ with respect to $g$ and $g_H$, respectively. By the very definition of symmetric spaces, the curvature tensor $R$ is itself parallel along geodesics. This implies that

$$\sinh(t)V_H(t) = Y(t) = \frac{\sinh(\sqrt{-\kappa}t)}{\sqrt{-\kappa}} V(t),$$

provided $V_o$ belongs to the $\kappa$-eigenspace of the Jacobi operator $R(\cdot, N_o)N_o$. Therefore, parallel vector fields in the eigenspaces are collinear for the two metrics. Hence, for $t > 0$ the linear subspaces $\mathcal{H}_{\sigma(t)}$ and $\mathcal{V}_{\sigma(t)}$ are nothing else than the parallel transport of the corresponding eigenspaces of $R(\cdot, N_o)N_o$ along $\sigma$. It follows that the splitting $T_x M = \mathcal{H}_x \oplus \mathcal{V}_x$ is orthogonal not only with respect to $g$, but also with respect to the hyperbolic metric $g_H$. Equation (4.1) is a direct consequence of this fact and the definition of the distribution $\mathcal{H}$. □

We can now prove Theorem 1.5.
Proof of Theorem 1.5 Let $E \subset M$ be a Hopf-symmetric set with outward pointing normal vector field $\nu$ with respect to $g$. By the very definition of Hopf-symmetry, $\nu_H \equiv 0$. Therefore, thanks to Lemma 4.1, $\nu$ is orthonormal to $\partial E$ also with respect to $g_H$. Let $\text{vol}$ and $\text{vol}_H$ the volume forms associated to $g$ and $g_H$. We have that

$$P(E) = \int_{\partial E} \iota_{\nu} \text{vol} = \int_{\partial E} \cosh^{d-1}(d_H(o, x)) \iota_{\nu} \text{vol}_H(x),$$

where $\iota : \Omega(M)^p \to \Omega(M)^{p-1}$ denotes the interior product $\iota_X \alpha(\cdot) = \alpha(X, \cdot)$. The volume of $E$ is given by the formula

$$V(E) = \int_{E} \text{vol} = \int_{E} \cosh^{d-1}(d_H(o, x)) \text{vol}_H(x).$$

Hence, the volume and perimeter of Hopf-symmetric sets in $M$ correspond to the volume and perimeter of $\Psi(E)$ in $H^n_{\mathbb{R}}$ with density equal to $f(x) = \cosh^{d-1}(d_H(o, x))$, concluding the proof. \hfill \Box

Acknowledgements The author would like to thank Urs Lang and Alessio Figalli for their precious guidance and constant support. A special thanks to Raphael Appenzeller for the very enjoyable and instructive exchanges about the hyperbolic plane. The author would like to thank Miguel Domínguez Vázquez for suggesting an efficient way to include the octonionic case in the definition of Hopf-symmetric, and Frank Morgan, for the useful insight about the earlier contributions to the problem. Finally, the author would like to thank the anonymous referees for their time, diligent review of the manuscript, and insightful comments. The author has received funding from the European Research Council under the Grant Agreement No. 721675 “Regularity and Stability in Partial Differential Equations (RSPDE)”.

Funding Open access funding provided by Swiss Federal Institute of Technology Zurich.

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Appendix A: Existence, boundedness and mean convexity

The objective of this section is to establish the existence, boundedness, and mean convexity of the isoperimetric sets in the hyperbolic space $H^n_{\mathbb{R}}$ when equipped with a radial density function $f : H^n_{\mathbb{R}} \to \mathbb{R}_{>0}$. Expressing

$$\ln(f(x)) = h(d_H(o, x))$$

for some $h : \mathbb{R} \to \mathbb{R}$, it will be sufficient to assume $h$ lower-semicontinuous and divergent to infinity to ensure existence, and non-decreasing to ensure boundedness. We will take advantage of the log-convexity to establish the mean-convexity of the isoperimetric sets. The proof is a direct application of the arguments employed by Morgan and Pratelli in the flat case [19, Theorem 3.3, Theorem 4.3, Theorem 5.9, Theorem 6.5]. We recall that we work in the Poincaré model, that makes $H^n_{\mathbb{R}}$ conformal to the unit ball in $\mathbb{R}^n$. The metric at a point
\[ x \in H^n_R \text{ is given by} \]
\[ g_H = \frac{4}{(1 - r^2)^2} g_{\text{flat}}, \]
where \( r = |x| \) will always denote the Euclidean distance of \( x \) from the origin. In this coordinate system, the hyperbolic distance from the origin is given by
\[ d_H(x, 0) = 2 \text{artanh}(r). \]

We will denote with \( \tilde{f}(r) := \exp(h(2 \text{artanh}(r))) \) the profile of the radial weight in Poincaré coordinates. Then, one can check that for \( k \in \{n - 1, n\} \) the \( k \)-dimensional weighted Hausdorff measures associated to \( g_H \) and \( f \) can be expressed as
\[ dH^k f := f dH^k = \tilde{f}(r) \left( \frac{2}{1 - r^2} \right)^k dH^k_{\text{flat}}, \tag{A.1} \]
where we denote with the flat index the Hausdorff measures associated to \( g_{\text{flat}} \) in the Poincaré model. To simplify the exposition, let us define the function
\[ \omega(r) := \frac{2}{1 - r^2}. \]
Finally we will denote with \( B(r) \) the ball centered at the origin in \( H^n_R \) with Euclidean radius \( r \in (0, 1) \), and with \( S^{n-1}(r) \) its boundary.

We start by proving that the isoperimetric profile is monotone. This step will be important to show boundedness later on.

**Theorem A.1** (Monotonicity of the isoperimetric profile) Let \( h : \mathbb{R}_{\geq 0} \to \mathbb{R} \) be a lower-semicontinuous non-decreasing function and let \( f : H^n_R \to \mathbb{R}_{>0} \) be defined through \( f(x) = \exp(h(d_H(x, o))) \) for some base point \( o \in H^n_R \). Then, the isoperimetric profile \( \mathcal{J} \) defined in (1.1) as
\[ \mathcal{J}(v) := \inf \left\{ P_f(F) : V_f(F) = v, F \subset H^n_R \text{ of finite perimeter} \right\}, \]
is non-decreasing in \( v \in [0, +\infty) \). Moreover, \( \mathcal{J} \) is strictly increasing if there exist isoperimetric sets for all volumes.

**Proof** Let \( E \) be any set of finite perimeter with finite volume \( V_f(E) = v \). We claim that for all \( r > 0 \) such that \( E(r) := E \cap B(r) \subsetneq E \) one has that
\[ P_f(E(r)) < P_f(E). \tag{A.2} \]
If Eq. (A.2) holds, then it suffices to notice that for every \( 0 < v' < v \) there exists \( r' \in (0, 1) \) such that
\[ V_f(E(r')) = v', \]
which implies that
\[ \mathcal{J}(v') \leq P_f(E(r')) < P_f(E). \]
If \( E \) is isoperimetric, then we have immediately that \( \mathcal{J}(v') < \mathcal{J}(v) \) for all \( 0 < v < v' \). Otherwise, for every \( \varepsilon > 0 \) let \( E_\varepsilon \) be a set of finite perimeter such that \( V_f(E) = v \) and \( P_f(E_\varepsilon) \leq \mathcal{J}(v) + \varepsilon \). From the inequality
\[ \mathcal{J}(v') < P_f(E) \leq \mathcal{J}(v) + \varepsilon, \]
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we infer that \( \tilde{\mathcal{J}}(v) \leq \mathcal{J}(v') \) for all \( 0 < v < v' \). We are left to prove Eq. (A.2). Let \( \pi : \partial E \setminus B(r) \to S^{n-1}(r) \) be the normal projection on the sphere of radius \( r \). Notice that \( \pi \) is strictly 1-Lipschitz with respect to the Euclidean distance. Then,

\[
\pi(\partial E \setminus B(r)) \supseteq \partial E(r) \setminus \partial E.
\]  

(A.3)

In fact, the set \( E \) contains the (possibly empty) cone

\[
C = \{ \lambda x : \lambda \in [1, r^{-1}], x \in H \},
\]

where \( H = (\partial E(r) \setminus \partial E) \setminus \pi(\partial E \setminus B(r)) \), and the dilation \( \lambda x \) is to be understood with respect to the Euclidean structure in the Poincaré model. Since the density is non-decreasing, it follows that \( V_f(C) = +\infty \) unless \( \mathcal{H}^{n-1}_\text{flat}(H) = 0 \). By assumption, the volume of \( E \) is finite, and therefore Equation (A.3) must hold up to a set of measure zero. By the coarea formula (see for instance [14, Chapter 13]) we finally get that

\[
P_f(E(r)) = \int_{\partial E \cap B(r)} \tilde{f}(r) \omega(r)^{n-1} d \mathcal{H}^{n-1}_\text{flat} + \int_{\partial E(r) \setminus \partial E} \tilde{f}(r) \omega(r)^{n-1} d \mathcal{H}^{n-1}_\text{flat}
\]

\[
\leq \int_{\partial E \cap B(r)} \tilde{f}(r) \omega(r)^{n-1} d \mathcal{H}^{n-1}_\text{flat} + \int_{\pi(\partial E \setminus B(r))} \tilde{f}(r) \omega(r)^{n-1} d \mathcal{H}^{n-1}_\text{flat}
\]

\[
< \int_{\partial E \cap B(r)} \tilde{f}(r) \omega(r)^{n-1} d \mathcal{H}^{n-1}_\text{flat} + \int_{\partial E \setminus B(r)} \tilde{f}(|\pi(x)|) \omega(|\pi(x)|)^{n-1} d \mathcal{H}^{n-1}_\text{flat}
\]

\[
\leq P_f(E),
\]

where \( \tilde{f}(r) = \exp(h(2 \text{artanh}(r))) \). \( \square \)

We are now ready to establish existence.

**Theorem A.2** (Existence of isoperimetric sets) Let \( h : \mathbb{R} \to \mathbb{R} \) be a lower-semicontinuous function that diverges to infinity and let \( f : H^0_{\mathbb{R}} \to \mathbb{R}_{>0} \) be defined through \( f(x) = \exp(h(d_H(x, o))) \) for some base point \( o \in H^0_{\mathbb{R}} \). Then, for all volumes there exists a set attaining the isoperimetric infimum in Eq. (1.1).

**Proof** Fix \( v > 0 \) and let \( (E_j)_{j \geq 1} \subset H^n_{\mathbb{R}} \) be a sequence of smooth sets of weighted volume \( v \) whose perimeter converges to the infimum of Eq. (1.1). Without loss of generality, we can suppose \( P_f(E_j) < \mathcal{J}(v) + 1 \). Intersecting this sequence with balls of growing radii \( r_j \to 1 \), the sequence splits into

\[
E_j = (E_j \cap B(r_j)) \cup (E_j \setminus B(r_j)) = E_j^C \cup E_j^D.
\]

Up to taking a subsequence, a standard argument of compactness (see [9, Theorem 1.19] and [17, Theorem 13.4]) shows that \( E_j^C \) converges to an isoperimetric set, whose volume is equal to \( v \) if and only if there is no volume escaping to infinity, that is

\[
\lim_{R \to 1} \limsup_{j \to +\infty} V_f(E_j \setminus B(R)) = 0.
\]

To establish our argument, we will proceed by contradiction. Let us assume that, after selecting a subsequence if necessary, there exists a positive value \( \varepsilon > 0 \) such that for every \( R > 0 \), there exists an index \( j = j(R) \) satisfying the inequality

\[
V_f(E_j \setminus B(R)) \geq \varepsilon.
\]  

(A.4)
Fix $0 < R < 1$ a number very close to 1 yet to define, and $j = j(R)$. Thanks to (A.1) we can rewrite Equation (A.4) as
\[ \int_{R}^{1} \omega(r)^{n} \tilde{f}(r)S_j(r) \, dr \geq \varepsilon, \tag{A.5} \]
where $S_j(r) := H_{\text{flat}}^{n-1}(\partial E \cap S^{n-1}(r))$ and $\tilde{f}(r) = \exp(h(2 \text{artanh}(r)))$. Then, denoting $M_j(R) := \sup_{r \in [R, 1)} \omega(r)^{n}S_j(r)$ and $m(R) = \inf_{r \in [R, 1)} \tilde{f}(r)$ we gave that
\[ P_f(E_j) \geq M_j(R)m(R). \tag{A.6} \]
In particular, since $P_f(E_j)$ is uniformly bounded, up to taking $R$ close enough to 1, we can suppose $m(R)$ large enough, so that
\[ S_j(r) \leq \frac{H_{\text{flat}}^{n-1}(S^{n-1}(r))}{2} \]
for all $r \in [R, 1)$. By the classical isoperimetric inequality on the sphere, there exists a dimensional constant $c_n > 0$ such that
\[ H_{\text{flat}}^{n-2}(\partial (E_j \cap S^{n-1}(r))) \geq c_n S_j(r)^{\frac{n-2}{n-1}}, \]
for all $r \in [R, 1)$. By Vol’pert theorem (see [1, Theorem 3.108]), for almost every $r \in (0, 1)$ one has that
\[ \partial (E_j \cap S^{n-1}(r))) = \partial E_j \cap S^{n-1}(r). \]
Therefore, the coarea formula (see for instance [14, Chapter 13]) allows us to obtain the following estimate on the weighted perimeter:
\[ P_f(E_j) \geq \int_{R}^{1} \omega(r)^{n-1} \tilde{f}(r)H_{\text{flat}}^{n-2}(\partial (E_j \cap S^{n-1}(r))) \, dr \]
\[ = \int_{R}^{1} \omega(r)^{n-1} \tilde{f}(r)H_{\text{flat}}^{n-2}(\partial (E_j \cap S^{n-1}(r))) \, dr \]
\[ \geq c_n \int_{R}^{1} \omega(r)^{n-1} \tilde{f}(r)S_j(r)^{\frac{n-2}{n-1}} \, dr \]
\[ \geq c_n \int_{R}^{1} \omega(r)^{n} \tilde{f}(r)S_j(r)(S_j(R)\omega(r)^{n-1})^{-\frac{1}{n-1}} \, dr \]
\[ \geq c_n M_j^{-\frac{1}{n-1}} \varepsilon, \]
where in the last line we used assumption (A.5). On the other hand, thanks to (A.6) we get that
\[ (\mathcal{J}(v) + 1)^{\frac{n}{n-1}} \geq P_f(E_j)P_f(E_j)^{\frac{1}{n-1}} \geq c_n \varepsilon m(R)^{\frac{1}{n-1}}. \]
But this is impossible because $m(R)$ diverges to infinity as $R \to 1$. \qedafter

After existence, we prove boundedness of the isoperimetric set.

**Theorem A.3** (Boundedness of the isoperimetric sets) Let $h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be a lower-semicontinuous non-decreasing function and let $f : H^{n}_{\mathbb{R}} \rightarrow \mathbb{R}_{\geq 0}$ be defined through $f(x) = \exp(h(d_H(x, o)))$ for some base point $o \in H^{n}_{\mathbb{R}}$. Then, every set attaining the isoperimetric infimum in Equation (1.1) is bounded.

\[ \text{ Springer} \]
We proceed by contradiction. Let $E$ be an unbounded isoperimetric set. Let $r \in (0, 1)$ be close enough to 1 so that

$$E(r) := E \cap B(r) \subseteq E.$$ 

Define $E_r := E \cap S^{n-1}(r)$, and the two functions

$$V_f(r) := V_f(E \setminus B(r)), \quad P_f(r) := \mathcal{H}_f^{n-1}(\partial E \setminus B(r)).$$

Notice that $V_f(r)$ and $P_f(r)$ tend to zero as $r$ tends to 1. Thanks to Theorems A.1 and A.2, we have that

$$P_f(E) > P_f(E(r)) = P_f(E) - P_f(r) + \mathcal{H}_f^{n-1}(E_r),$$

implying that

$$P_f(r) > \mathcal{H}_f^{n-1}(E_r). \quad (A.7)$$

Up to taking $r$ closer to 1, we can assume that

$$\mathcal{H}_f^{n-1}(E_r) \leq \frac{1}{2} \mathcal{H}_f^{n-2}(\partial E_r),$$

where the boundary of $E_r$ is taken inside the sphere $S^{n-1}(r)$. Therefore, by classic isoperimetric inequality on the sphere there exists a dimensional constant $c_n > 0$ such that

$$\mathcal{H}_f^{n-2}(\partial E_r) \geq c_n \mathcal{H}_f^{n-1}(E_r)^{\frac{n-2}{n-1}}, \quad (A.8)$$

which by Eq. (A.7) leads to

$$\tilde{f}(r)\omega(r)^{n-2} \mathcal{H}_f^{n-2}(\partial E_r) \geq c_n \tilde{f}(r) \frac{1}{n-1} \left(\omega(r)^{n-1} \tilde{f}(r) \mathcal{H}_f^{n-1}(E_r)\right)^{\frac{n-2}{n-1}}$$

$$\geq c_n \tilde{f}(0) \frac{1}{n-1} \mathcal{H}_f^{n-1}(E_r)^{\frac{n-2}{n-1}}$$

$$> c_n \tilde{f}(0) \frac{1}{n-1} P_f(r)^{\frac{1}{n-1}} \mathcal{H}_f^{n-1}(E_r),$$

where we used that $\tilde{f}(r) = \exp(h(2 \text{artanh}(r)))$ is non-decreasing. Since

$$-P_f'(r) = \tilde{f}(r)\omega(r)^{n-1} \mathcal{H}_f^{n-2}(\partial E_r), \quad -V_f'(r) = \omega(r) \mathcal{H}_f^{n-1}(E_r),$$

we obtain that

$$-(P_f(r)^{\frac{n}{n-1}})' > -c_n \tilde{f}(0) \frac{1}{n-1} V_f'(r),$$

which integrated from $r$ to 1 leads to

$$P_f(r)^{\frac{n}{n-1}} > 2c V_f(r), \quad (A.9)$$

with $2c = c_n \tilde{f}(0)^{\frac{1}{n-1}}$. The last thing we need to do is to take advantage of the optimality of the set $E$ by operating a small perturbation of its boundary. Let $K \subset H^n_{\mathbb{R}}$ be a compact subset of $H^n_{\mathbb{R}}$ and $\Gamma : (-\varepsilon_0, \varepsilon_0) \times H^n_{\mathbb{R}} \to H^n_{\mathbb{R}}$ be any variation inside $K$, that is $\Gamma(0, \cdot) = \text{id}$ and for all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ the map $\Gamma(\varepsilon, \cdot)$ is a smooth diffeomorphism equal to the identity outside $K$. Then the first order expansion of the volume and perimeter operators can be computed as

$$V_f(E_\varepsilon) = V_f(E) + \varepsilon \int_{\partial E} g_H(v, X) \, d\mathcal{H}_f + o(\varepsilon),$$
\[ P_f(E_\varepsilon) = P_f(E) + \varepsilon \int_{\partial E} \mathbf{H}_f g_H(v, X) \, d\mathcal{H} + o(\varepsilon), \]

where \( X(x) := \frac{\partial}{\partial x} (0, x) \), \( E_\varepsilon := \Gamma(\varepsilon, E) \), and \( \mathbf{H}_f = H + \partial_v \ln(f) \) stands for the weighted mean curvature of \( E \), which is constant at every regular point if \( E \) is isoperimetric. We refer to [14, Chapter 17.3] and [20, Section 3] for the careful proof of this fact. This allows us to perturb the set \( E \) inside \( K = B(r_0) \) for some \( r_0 \) close enough to 1 according to a small parameter \( \varepsilon \in (0, \varepsilon_0) \), such that the resulting perturbed sets \( (E_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)} \) satisfy

\[ V_f(E_\varepsilon) = V_f(E) + \varepsilon, \]

and

\[ P_f(E_\varepsilon) \leq P_f(E) + \varepsilon(\mathbf{H}_f + 1), \]

Choose now \( \varepsilon \frac{1}{n} < \frac{c}{1 + \varepsilon} \) and \( R_0 > r_0 \) so that \( \varepsilon = V_f(r) < \varepsilon_0 \). Then, \( F = E_\varepsilon \cap B(R_0) \) has weighted volume equal to \( E \), and from

\[ P_f(F) = P_f(E_\varepsilon) - P_f(r) + \mathcal{H}_f^{n-1}(E_\varepsilon) \leq P_f(E) + \varepsilon(\mathbf{H}_f + 1) - 2c\varepsilon \frac{n-1}{n} + \mathcal{H}_f^{n-1}(E_\varepsilon) \]

\[ < P_f(E) - c\varepsilon \frac{n-1}{n} + \mathcal{H}_f^{n-1}(E_\varepsilon), \]

and the optimality of \( E \), we infer that

\[ \mathcal{H}_f^{n-1}(E_\varepsilon) > c\varepsilon \frac{n-1}{n}. \]

Hence

\[-V_f'(r) = \omega(r)\mathcal{H}_f^{n-1}(E_\varepsilon) > c\omega(r)\varepsilon \frac{n-1}{n} = c\omega(r)V_f(r)\varepsilon \frac{n-1}{n}, \]

implies that

\[ -(V_f(r)\varepsilon^{\frac{1}{n}})' > c\omega(r) = c(2\artanh(r))'. \]

Integrating both sides, since \( \artanh(r) \) tends to \(+\infty\) as \( r \) tends to 1, we obtain a contradiction with the assumption that \( V_f(r) > 0 \) for every \( r \in (0, 1) \).

We complete the section by proving the Riemannian mean-convexity of the isoperimetric sets.

**Theorem A.4** (Mean-convexity) Let \( h : \mathbb{R} \to \mathbb{R} \) be a convex and even function, and let \( f : H^n_{\mathbb{R}} \to \mathbb{R}_{>0} \) be defined through \( f(x) = \exp(h(d_H(x, o))) \) for some base point \( o \in H^n_{\mathbb{R}} \). Then, every set attaining the isoperimetric infimum in Eq. (1.1) is mean-convex.

**Proof** Let \( E \) be an isoperimetric set. Thanks to Theorem A.3, \( E \) is bounded, and therefore there exists \( z \in \partial E \) maximizing the distance from the base point \( o \). By the regularity properties summarized in Theorem 2.1, \( z \) is a regular point, and

\[ H(z) \geq (n - 1), \]

where \( H(z) \) denotes the unweighted mean curvature of \( \partial E \) at \( z \). Let now \( x \in \partial E \) be another regular point. Since the weighted mean curvature \( \mathbf{H}_f = H + \partial_v \ln(f) \) is constant, we have in particular that

\[ H(x) = H(z) + \partial_v \ln(f)(z) - \partial_v \ln(f)(x) \geq (n - 1) + h'(d_H(o, z)) - h'(d_H(o, x)) \geq (n - 1), \]

where in the last inequality we used the convexity of the exponent \( h \). \( \Box \)
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