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We study possibility of improving staggered fermions using various fat links in order to reduce perturbative corrections to the gauge-invariant staggered fermion operators. We prove five theorems on SU(3) projection, triviality in renormalization, multiple SU(3) projections, uniqueness and equivalence. As a result of these theorems, we show that, at one loop level, the renormalization of staggered fermion operators is identical between SU(3) projected Fat7 links and hypercubic links, as long as the action and operators are constructed by imposing the same perturbative improvement condition. In addition, we propose a new view of SU(3) projection as a tool of tadpole improvement for the staggered fermion doublers. As a conclusion, we present alternative choices of constructing fat links to improve the staggered fermion action and operators, which deserve further investigation.

I. INTRODUCTION

The staggered fermion formulation on the lattice has a number of advantages for numerical studies. In particular, it preserves part of the chiral symmetry, which is essential to calculate the weak matrix elements for $\epsilon'/\epsilon$ and to remove the unwanted additive renormalization of light quark masses. In addition, computational cost with staggered fermions is noticeably cheaper than any other fermion formulation on the lattice. The chiral symmetry insures that the discretization errors are quadratic in lattice spacing $a$. By construction, the staggered fermions carry four degenerate flavors, which in itself is not a major problem. The major problem is that the naive (i.e. unimproved) staggered fermion action allows flavor changing quark-gluon interactions, which are relatively large [1]. These flavor changing interactions are quadratic in $a$ and get weaker as the lattice approaches the continuum.

Recently, in [1–4], it was understood that the dominant flavor changing interactions originate from a gluon exchange between quarks when the gluon carries a high transverse momentum. This interaction is a pure lattice artifact of order $a^2$ and can be removed by modifying the lattice action at tree level. Numerical studies in [3] showed that the fat link improvement at tree level reduces the flavor changing effects noticeably.

In [4], the concept of the fat link action (Fat7) introduced in [3] was systematically re-interpreted in terms of Symanzik improvement programme. There are two kinds of $O(a^2)$ error at the tree level: one is the flavor changing interaction originated from one gluon exchange between quarks [3] and the other is a flavor conserving kinetic term. Based on this observation, a proposal of how to correct both was made in [4]. In this paper, we adopt the same notation as in [4] and we call the fat link introduced in [3,4] “Fat7” afterwards for notational convenience.

Recently, the hypercubic (HYP) action (a new fat link action) was introduced in [5,6]. The HYP fat link is constructed such that the HYP smearing smoothes the gauge fields within the hypercubes attached to the original thin link with SU(3) projection after each smearing. The HYP fat link improvement reduces the flavor symmetry breaking effects more efficiently than the Fat7 action.

A problem with staggered fermions is that the composite operators receive relatively large perturbative corrections even at one loop. In this paper, our improvement goal is to minimize the perturbative corrections. Recently, in Ref. [7], it was observed that the contribution from the staggered fermion doublers (it was named “doubler-tadpole”) to the renormalization is as large as that from the usual gluon tadpoles introduced in [8]. This implies that the fat links can also improve the perturbative behavior efficiently, since the origin of doubler-tadpoles are the same as that of the flavor changing interactions.

In this paper, we prove that one-loop renormalization of the staggered operators is identical between the SU(3) projected Fat7 links and the HYP link. We also propose alternative choices of fat links to improve the action and operators for better perturbative behavior, which is relatively cheap to implement in numerical simulations on the lattice. This paper is organized as follows. In Sec. II, we describe our notation for various fat links and, briefly, review previous works. Sec. III is devoted to proof of five theorems on the equivalence, uniqueness and triviality of the one-loop renormalization for the HYP fat link and the SU(3) projected Fat7 links. In Sec. IV, we interpret the meaning of these theorems, which leads to a proposal of how to construct fat links in order to reduce the perturbative corrections most efficiently. We close with some conclusions.
II. NOTATIONS AND REVIEW

First, we review the improvement of removing the flavor changing interactions in Ref. [4]. We define a covariant second derivative as

$$\Delta^{(2)}_{\mu}(x) = \frac{1}{u_0^2 a^2} \left( U_{\mu}(x)U_{\mu}(x + \hat{\rho})U^\dagger_{\mu}(x + \hat{\mu}) - 2u_0^2 U_{\mu}(x) + U^\dagger_{\mu}(x - \hat{\rho})U_{\mu}(x - \hat{\rho} + \hat{\mu}) \right)$$  (1)

Using this definition, we define the smearing operator $L_{\rho}$ (prefactor) as

$$L_{\rho}(\alpha) \cdot U_{\mu} = \left( 1 + \frac{\alpha^2 \Delta^{(2)}_{\rho}}{4} \right) U_{\mu}$$  (2)

The smeared link, $L_{\rho}(\alpha = 1) \cdot U_{\mu}$ is identical to the thin link $U_{\mu}$ up to order $a^2$ but vanishes when a single gluon carries a momentum $p_{\rho} = \pi/a$. Using $L_{\rho}$, we can rewrite the Fat7 link as

$$V_{\mu}^L = \frac{1}{6} \sum_{\text{perm}(\nu, \rho, \lambda)} L_{\nu}(\alpha_1) \cdot \left( L_{\rho}(\alpha_2) \cdot \left( L_{\lambda}(\alpha_3) \cdot U_{\mu} \right) \right)$$  (3)

Here, $\text{perm}(\nu, \rho, \lambda)$ represents all the possible permutations of $\nu, \rho, \lambda$ indices ($\nu \neq \rho \neq \lambda \neq \mu$). When $\alpha_i = 1$, the prefactors vanish at tree level when a single gluon emission carries a momentum $p_{\rho} = \pi/a$ for any $\nu \neq \mu$.

Second, we review the flavor conserving improvement proposed in Ref. [4]. We define a covariant first derivative as

$$\Delta^{(1)}_{\rho}(x) = \frac{1}{2u_0^2 a^2} \left( U_{\rho}(x)U_{\mu}(x + \hat{\rho})U^\dagger_{\mu}(x + \hat{\mu}) - U^\dagger_{\rho}(x - \hat{\rho})U_{\mu}(x - \hat{\rho} + \hat{\mu}) \right)$$  (4)

The $O(a^2)$ corrections introduced when $U_{\mu}$ is replaced by $V_{\mu}^L$ cancels all the tree level flavor changing interactions. There is still a remaining $O(a^2)$ error at low energy as a result of introducing the $\Delta^{(2)}_{\rho}$’s of $V_{\mu}^L$. This flavor conserving error can be removed by further modifying $V_{\mu}^L$.

$$V_{\mu}^L \to V'_{\mu} = V_{\mu}^L - \sum_{\rho \neq \mu} \frac{a^2 (\Delta^{(1)}_{\rho})^2}{4} U_{\mu}$$  (5)

Recently, Hasenfratz and Knechtli made an interesting proposal of hypercubic blocking (HYP) in [5,6]. The basic form of smearing transformation is a SU(3) projected modified APE blocking:

$$V_{\mu}^H = \text{Proj}_{SU(3)}[V_{\mu}^H]$$

$$V_{\mu}^H = \left[ (1 - \alpha_1') U_{\mu}(x) + \frac{\alpha_1'}{6} \sum_{\nu \neq \mu} \left( \overline{M}_{\nu,\mu}(x) \overline{M}_{\nu,\mu}(x + \hat{\nu}) \overline{M}_{\nu,\mu}(x + \hat{\mu}) + \overline{M}_{\nu,\mu}(x - \hat{\nu}) \overline{M}_{\nu,\mu}(x - \hat{\nu} + \hat{\mu}) \right) \right]$$  (6)

$$\overline{M}_{\mu,\nu} = \text{Proj}_{SU(3)}[M_{\mu,\nu}]$$

$$M_{\mu,\nu} = \left[ (1 - \alpha_2') U_{\mu}(x) + \frac{\alpha_2'}{4} \sum_{\rho \neq \mu, \nu} \left( \overline{W}_{\rho,\mu,\nu}(x) \overline{W}_{\mu,\rho,\nu}(x + \hat{\rho}) \overline{W}_{\rho,\nu,\mu}(x + \hat{\rho}) \right) + \overline{W}_{\rho,\mu,\nu}(x - \hat{\rho}) \overline{W}_{\mu,\rho,\nu}(x - \hat{\rho} + \hat{\mu}) \right]$$  (7)

$$\overline{W}_{\mu,\nu,\rho} = \text{Proj}_{SU(3)}[W_{\mu,\nu,\rho}]$$

$$W_{\mu,\nu,\rho} = \left[ (1 - \alpha_3') U_{\mu}(x) + \right.$$

2
\[
\frac{\alpha_3'}{2} \sum_{\lambda \neq \mu, \nu, \rho} \left( U_\lambda(x)U_\mu(x + \hat{\rho})U_\lambda^\dagger(x + \hat{\mu}) \\
+ U_\lambda^\dagger(x - \hat{\rho})U_\mu(x - \hat{\rho})U_\lambda(x - \hat{\rho} + \hat{\mu}) \right)
\]

(8)

The hypercubic blocking is composed of three smearing transformations with each accompanied by SU(3) projection. By construction, the resulting fat link \(V_H^\mu\) mixes with thin links only from the hypercubes attached to the original thin link \(U_\mu\). The corresponding gauge field \(H_\mu\) is defined as

\[
V_H^\mu(x) = \exp\left(iaH_\mu(x + \frac{1}{2}\hat{\mu})\right)
\]

(9)

For more details, please refer to Refs. [5,6].

III. THEOREMS FOR THE FAT LINK IMPROVEMENT

Here we present a series of theorems to clarify the nature of the fat links obtained using the smearing transformations given in Eqs. (3), (5) and (6). Let us consider a general form of the fat links. The gauge fields are related to the thin links as

\[
U_\mu(x) = \exp\left(iaA_\mu(x + \frac{1}{2}\hat{\mu})\right)
\]

(10)

where the gauge coupling is absorbed into a redefinition of gauge fields without loss of generality. We will set \(a = 1\) for notational convenience. We can write the general form of the fat links using gauge fields:

\[
V_\mu(x) = 1 + i \sum_{\nu, y} \Lambda^{(1)}_{\mu \nu}(x, y)A_\nu(y) + \sum_{\nu, \rho, y, z} \Lambda^{(2)}_{\mu \nu \rho}(x, y, z)A_\nu(y)A_\rho(z) + O(A^3)
\]

(11)

Here, \(V_\mu\) represents the general form of the fat links such as \(V_L^\mu\), \(V_L'^\mu\) and \(V_H^\mu\). We also introduce a notation for SU(3) projection:

\[
\nabla_\mu = \text{Proj}_{SU(3)} [V_\mu].
\]

(12)

The SU(3) projected link is related to the effective gauge field, \(B_\mu\).

\[
\nabla_\mu(x) = \exp\left(iaB_\mu(x + \frac{1}{2}\hat{\mu})\right)
\]

(13)

Here, note that \(B_\mu = \sum_{a} B^a_\mu T_a\). We may express \(B^a_\mu\) as a perturbative expansion in powers of \(A^a\) fields.

\[
B^a_\mu = \sum_{n=1}^{\infty} B^{a(n)}_\mu = B^{a(1)}_\mu + B^{a(2)}_\mu + O(A^3)
\]

(14)

Here, \(B^{a(n)}_\mu\) represents a term of order \(A^n\).

A. SU(3) Projection

Theorem 1 (SU(3) Projection\(^1\))

1. The linear term is invariant under SU(3) projection.

\[
B^{(1)}_\mu(x) = \sum_{\nu, y} \Lambda^{(1)}_{\mu \nu}(x, y)A_\nu(y) = \Lambda^{(1)}_\mu \cdot A
\]

(15)
2. The quadratic term is antisymmetric in gauge fields.

\[ B^{(2)}_{\mu} = \frac{1}{2} \sum_{a,b} f_{abc} \sum_{\nu,\rho} \sum_{y,z} \Lambda^{(2)}_{\mu\nu\rho}(x, y, z) A^a_\nu(y) A^b_\rho(z) \]

\[ B^{(2)}_{\mu} = -\frac{i}{2} \sum_{\nu,\rho, y, z} \Lambda^{(2)}_{\mu\nu\rho}(x, y, z) [A_\nu(y), A_\rho(z)] \]  

(16)

where \( f_{abc} \) are the antisymmetric SU(3) structure constants with non-zero values defined by \([T_a, T_b] = i f_{abc} T_c\).  

**Proof 1.1** For the SU(3) projection, we define \( X \) as follows:

\[ X = \text{Tr}(\overline{V}_\mu V_\mu) \]  

(17)

The SU(3) projection means that \( V_\mu \) is determined such that it should maximize \( \text{Re}(X) \) and minimize \( (\text{Im}(X))^2 \) under the condition:

\[ \text{sign}(\det(V_\mu)) = \text{sign}(\det(\overline{V}_\mu)) \]  

(18)

Using perturbation, we may expand \( X \) in powers of gauge fields.

\[ X = \sum_{i=0}^{\infty} X^{(i)} \]  

(19)

Here, \( X^{(n)} \) is a term of order \( A^n \). It is easy to show that \( X^{(0)} = 3 \) and \( X^{(1)} = 0 \). The \( X^{(2)} \) term provides a condition to determine \( B^{(1)}_\mu \).

\[ X^{(2)} = \text{Tr}\left[ -\frac{1}{2} (B^{(1)}_\mu - \Lambda^{(1)}_\mu \cdot A)^2 + \frac{1}{2} (\Lambda^{(1)}_\mu \cdot A)^2 + \Lambda^{(2)}_\mu \cdot A^2 \right] \]  

(20)

Here, note that \( X^{(2)} \) is real: \( X^{(2)} = \text{Re}(X^{(2)}) \) and \( \text{Im}(X^{(2)}) = 0 \). It is clear that \( B^{(1)}_\mu \) should satisfy the following condition in order to maximize \( \text{Re}(X^{(2)}) \).

\[ B^{(1)}_\mu = \Lambda^{(1)}_\mu \cdot A \]  

(21)

This proves the first part of Theorem 1.

**Proof 1.2** The \( X^{(3)} \) term is

\[ X^{(3)} = \text{Tr}\left[ \frac{1}{2} \left\{ (\Lambda^{(1)}_\mu \cdot A - B^{(1)}_\mu)^2, B^{(2)}_\mu \right\} - \frac{i}{3} (B^{(1)}_\mu)^3 + i \Lambda^{(3)}_\mu \cdot A^3 - i \frac{i}{2} \left\{ B^{(1)}_\mu, \Lambda^{(2)}_\mu \cdot A^2 \right\} \right] \]

\[ = \text{Tr}\left[ -\frac{i}{3} (\Lambda^{(1)}_\mu \cdot A)^3 + i \Lambda^{(3)}_\mu \cdot A^3 - i \frac{i}{2} \left\{ \Lambda^{(1)}_\mu \cdot A, \Lambda^{(2)}_\mu \cdot A^2 \right\} \right] \]  

(22)

The \( X^{(3)} \) term may include an imaginary part. In other word, the leading contribution to the \( \text{Im}(X) \) could be of order \( A^3 \). The \( X^{(3)} \) term, however, does not provide any clue to determine \( B^{(2)}_\mu \) because its coefficient vanishes. Therefore, it is necessary to study the next higher order term, \( X^{(4)} \):

\[ X^{(4)} = \text{Tr}\left[ -\frac{1}{2} (B^{(2)}_\mu^2 + \frac{i}{2} (B^{(1)}_\mu)^2 + i \Lambda^{(2)}_\mu \cdot A^2)^2 + Y \right] \]  

(23)

where \( Y \) is defined as

---

1Theorems 1 and 2 were known to Patel and Sharpe and used in their perturbative calculations [9], although they did not present their derivation and details [10]. Theorem 1 is also mentioned in [11], although their derivation and details are not presented [12].
\[
Y = \text{Tr} \left[- \frac{1}{2} \left( \frac{1}{2} (B^{(1)}_\mu)^2 + \Lambda^{(2)}_\mu \cdot A^2 \right)^2 + \Lambda^{(4)}_\mu \cdot A^4 + \frac{1}{2} \left\{ B^{(1)}_\mu, \Lambda^{(3)}_\mu \cdot A^3 \right\} - \frac{1}{4} \left\{ (B^{(1)}_\mu)^2, \Lambda^{(2)}_\mu \cdot A^2 \right\} - \frac{1}{8} (B^{(1)}_\mu)^4 \right] \tag{24}
\]

Here, note that \( Y \) is a known constant term, because \( B^{(1)}_\mu \) is fixed by the \( X^{(2)} \) term. The coefficient of the \( B^{(3)}_\mu \) term vanishes when \( B^{(1)}_\mu = \Lambda^{(1)}_\mu \cdot A \). The \( B^{(4)}_\mu \) term vanishes because it is traceless. Therefore, it would be best if \( B^{(2)}_\mu \) could satisfy the following condition:

\[
B^{(2)}_\mu = -i \frac{1}{2} (B^{(1)}_\mu)^2 - i \Lambda^{(2)}_\mu \cdot A^2 = J + K \tag{25}
\]

However, Eq. (25) can not be satisfied with real \( B^{(2)}_\mu \) gauge fields. From the following relation,

\[
J = -i \frac{1}{2} (B^{(1)}_\mu)^2
= -i \frac{1}{4} \sum_{a,b} B^{a(1)}_\mu B^{b(1)}_\mu \left( \frac{1}{3} \delta_{ab} + d_{abc} T_c \right)
\tag{26}
\]

note that the coefficient of the \( J \) term in the adjoint representation is purely imaginary, whereas \( B^{a(2)}_\mu \) is real. Hence, there is no way that \( B^{(2)}_\mu \) can cancel off any part of the \( J \) term. How about the \( K \) term?

\[
K = -i \Lambda^{(2)}_\mu \cdot A^2
= -i \sum_{a,b} \sum_{\nu,\rho} \sum_{y,z} \Lambda^{(2)}_{\mu \nu \rho} (x,y,z) A^a_\nu (y) A^b_\rho (z) \frac{1}{2} \left( \frac{1}{3} \delta_{ab} + (d_{abc} + i f_{abc}) T_c \right)
\tag{27}
\]

The \( K \) term contains a non-trivial real antisymmetric term in the adjoint representation. Thus, in order to maximize \( \text{Re} X^{(4)} \), \( B^{(2)}_\mu \) must satisfy the following:

\[
B^{c(2)}_\mu = \frac{1}{2} \sum_{a,b} f_{abc} \sum_{\nu,\rho,\nu,\rho} \Lambda^{(2)}_{\mu \nu \rho} (x,y,z) A^a_\nu (y) A^b_\rho (z)
\tag{28}
\]

This completes a proof of the second part of Theorem 1.

**B. Renormalization at one loop**

**Theorem 2 (Triviality of one-loop renormalization)**

1. At one loop level, only the \( B^{(1)}_\mu \) term contributes to the renormalization of the gauge-invariant staggered fermion operators.

2. At one loop level, the contribution from \( B^{(n)}_\mu \) for any \( n \geq 2 \) vanishes.

3. At one loop level, the renormalization of the gauge-invariant staggered operators can be done by simply replacing the propagator of the \( A_\mu \) field by that of the \( B^{(1)}_\mu \) field.

This theorem is true, regardless of details of the smearing transformation.

**Example 2.1** Let us consider gauge-invariant staggered bilinear operators as an example. The Feynman diagrams at one loop are given in Refs. [9,13,14]. Note that the \( B^{(1)}_\mu \) contribution can be obtained by simply replacing the gauge field propagator of \( \langle A_\mu (x) A_\nu (y) \rangle \) by \( \langle B^{(1)}_\mu (x) B^{(1)}_\nu (y) \rangle \). The \( B^{(2)}_\mu \) terms, in principle, may contribute to the tadpole diagrams (c) and (e) in Figure 1 of Ref. [13]. However, this tadpole contribution from the \( B^{(2)}_\mu \) term vanishes because the gauge field propagator \( \langle A^{a}_\mu (x) A^{a}_\nu (y) \rangle \) is symmetric in color indices (proportional to \( \delta_{bc} \)) and \( f_{abc} \cdot \delta_{bc} = 0 \). It is easy to show that the \( B^{(n>2)}_\mu \) terms can not contribute to any one-loop diagrams. This proves Theorem 2 for the gauge-invariant bilinear operators.
Proof 2.1 At one loop level, the \( B^{(1)}_\mu \) terms can contribute to the renormalization of the gauge invariant operators exactly in the same way as the \( A_\mu \) terms. In other words, the one-loop contribution from the \( B^{(1)}_\mu \) terms can be calculated by simply replacing the propagator of the \( A_\mu \) field by that of the \( B^{(1)}_\mu \) field. At one loop, the \( B^{(2)}_\mu \) terms contributes only to the tadpole diagrams, whereas this is not true for the higher loop corrections. Here, the tadpole diagrams mean that the two \( A_\mu \) gauge fields in the \( B^{(2)}_\mu \) term are contracted with each other. The gauge field propagator \( \langle A_\mu^6(x)A_\mu^6(y) \rangle \) is proportional to \( \delta_{bc} \) and so symmetric with respect to the color index exchange: \( b \leftrightarrow c \). However, the \( B^{(2)}_\mu \) term is antisymmetric with respect to \( b \leftrightarrow c \), whereas the gauge field propagator is symmetric. Therefore, the \( B^{(2)}_\mu \) contribution to the one-loop renormalization vanishes, since \( f_{abc} \cdot \delta_{bc} = 0 \), regardless of details of the smearing transformations. The contribution from the \( B^{(n>2)}_\mu \) terms to the one-loop renormalization vanishes because they are at least proportional to

\[
\langle A_\lambda \rangle^{n-2} = \langle A_\lambda(1) \rangle \langle A_\lambda(2) \rangle \cdots \langle A_\lambda(n-2) \rangle (x_{n-2})
\]

and \( \langle A_\lambda(x) \rangle = 0 \) (the vacuum can not break Lorentz symmetry or rotational symmetry in QCD). Note that the information on the smearing transformation is contained only in \( \Lambda^{(n)} \) and this proof is independent of \( \Lambda^{(n)} \). In other words, this theorem is valid, regardless of details of the smearing transformation. This completes a proof of Theorem 2.

C. Multiple SU(3) Projections

As an example, let us consider \( V^L_\mu \), the Fat7 link for the flavor symmetry improvement. One can perform a single SU(3) projection, or alternatively apply the SU(3) projections in front of any \( L_\mu(\alpha_i) \) operators used to construct \( V_\mu^L \), which we call “multiple SU(3) projections”. As in Eq. (12), the single SU(3) projected Fat7 link is defined as

\[
\nabla^L_\mu = \text{Proj}_{SU(3)} [V^L_\mu] \\
\nabla^L_\mu(x) = \exp \left( iaC_\mu(x + \frac{1}{2}\hat{\mu}) \right)
\]

The multiple SU(3) projection is not unique. As an example, we may define a multi-SU(3) projected Fat7 link as

\[
\nabla^M_\mu = \frac{1}{6} \sum_{\text{perm}(\nu,\rho,\lambda)} \text{Proj}_{SU(3)} \cdot L_\nu(\alpha_1) \cdot \left( \text{Proj}_{SU(3)} \cdot L_\rho(\alpha_2) \cdot \left( \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3) \cdot U_\mu \right) \right)
\]

\[
\nabla^M_\mu(x) = \exp \left( iaD_\mu(x + \frac{1}{2}\hat{\mu}) \right)
\]

The other example is

\[
\nabla^{M'}_\mu = \frac{1}{6} \sum_{\text{perm}(\nu,\rho,\lambda)} \text{Proj}_{SU(3)} \cdot L_\nu(\alpha_1) \cdot \left( \text{Proj}_{SU(3)} \cdot L_\rho(\alpha_2) \cdot \left( \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3) \cdot U_\mu \right) \right)
\]

\[
\nabla^{M'}_\mu(x) = \exp \left( iaD'_\mu(x + \frac{1}{2}\hat{\mu}) \right)
\]

Another possibility is

\[
\nabla^{M''}_\mu = \frac{1}{6} \sum_{\text{perm}(\nu,\rho,\lambda)} \text{Proj}_{SU(3)} \cdot L_\nu(\alpha_1) \cdot \left( L_\rho(\alpha_2) \cdot \left( \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3) \cdot U_\mu \right) \right)
\]

\[
\nabla^{M''}_\mu(x) = \exp \left( iaD''_\mu(x + \frac{1}{2}\hat{\mu}) \right)
\]

Theorem 3 (Role of multiple SU(3) projections)

1. The linear gauge field term in the perturbative expansion is universal.

\[
C^{(1)}_\mu = D^{(1)}_\mu = D'^{(1)}_\mu = D''^{(1)}_\mu
\]
2. In general, the quadratic terms may be different from one another. But all of them are antisymmetric in gauge fields.

\[
C^{(2)}_{\mu} = -\frac{1}{2} \sum_{\nu,\rho, y, z} \Omega_{\mu \nu \rho}(x, y, z) [A_\nu(y), A_\rho(z)]
\]

\[
D^{(2)}_{\mu} = -\frac{1}{2} \sum_{\nu,\rho, y, z} \Gamma_{\mu \nu \rho}(x, y, z) [A_\nu(y), A_\rho(z)]
\]

\[
D'^{(2)}_{\mu} = -\frac{1}{2} \sum_{\nu,\rho, y, z} \Gamma'_{\mu \nu \rho}(x, y, z) [A_\nu(y), A_\rho(z)]
\]

\[
D''^{(2)}_{\mu} = -\frac{1}{2} \sum_{\nu,\rho, y, z} \Gamma''_{\mu \nu \rho}(x, y, z) [A_\nu(y), A_\rho(z)]
\]  
\[ (35) \]

This theorem is true, regardless of the details of the smearing operator $L_\mu(\alpha)$ and its parameter $\alpha$.

**Proof 3.1** By Theorem 1, the linear term of the smearing operators are the same under SU(3) projection. In other words, Theorem 1 tells us that the linear term of the $L_\lambda(\alpha_3)$ operator is identical to that of the $\text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3)$ operator. Then the repeated application of Theorem 1 tells us that the linear term of the $L_\rho(\alpha_2) \cdot L_\lambda(\alpha_3)$ is identical to that of the following operators:

1. $L_\rho(\alpha_2) \cdot \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3)$
2. $\text{Proj}_{SU(3)} \cdot L_\rho(\alpha_2) \cdot L_\lambda(\alpha_3)$
3. $\text{Proj}_{SU(3)} \cdot L_\rho(\alpha_2) \cdot \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3)$

Applying Theorem 1 once more, we obtain the final results: the linear terms of the following 8 operators are the same.

1. $L_\mu(\alpha_1) \cdot L_\rho(\alpha_2) \cdot L_\lambda(\alpha_3)$
2. $L_\mu(\alpha_1) \cdot L_\rho(\alpha_2) \cdot \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3)$
3. $L_\mu(\alpha_1) \cdot \text{Proj}_{SU(3)} \cdot L_\rho(\alpha_2) \cdot L_\lambda(\alpha_3)$
4. $L_\mu(\alpha_1) \cdot \text{Proj}_{SU(3)} \cdot L_\rho(\alpha_2) \cdot \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3)$
5. $\text{Proj}_{SU(3)} \cdot L_\mu(\alpha_1) \cdot L_\rho(\alpha_2) \cdot L_\lambda(\alpha_3)$
6. $\text{Proj}_{SU(3)} \cdot L_\mu(\alpha_1) \cdot L_\rho(\alpha_2) \cdot \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3)$
7. $\text{Proj}_{SU(3)} \cdot L_\mu(\alpha_1) \cdot \text{Proj}_{SU(3)} \cdot L_\rho(\alpha_2) \cdot L_\lambda(\alpha_3)$
8. $\text{Proj}_{SU(3)} \cdot L_\mu(\alpha_1) \cdot \text{Proj}_{SU(3)} \cdot L_\rho(\alpha_2) \cdot \text{Proj}_{SU(3)} \cdot L_\lambda(\alpha_3)$

This proves the first part of Theorem 3.

**Proof 3.2** As long as the final operation is the SU(3) projection, we can apply Theorem 1, which directly proves the second part of Theorem 3. Note that Theorem 1 is independent of details of the smearing operator and so is this proof.

**D. Uniqueness of the Perturbative Improvement Program**

The perturbative improvement program for the flavor symmetry restoration is a procedure of removing the flavor changing interactions. Both Fat7 and HYP programs are based upon the same philosophy of maximally reducing the flavor changing interactions. Since the technical details of how to construct the fat links are different, they look different superficially. Here, we address the question of whether the perturbative improvement programs can be different from each other even though they share the same underlying philosophy. First, let us present the conclusion and explain the details later: the answer is NO and the perturbative improvement program should be unique at least at one loop level. The following two theorems will prove this in two steps.
Theorem 4 (Uniqueness)
If we impose the perturbative improvement condition of removing the flavor changing interactions on the HYP action, the HYP gauge field, $H_\mu$, defined in Eq. (9) satisfies the following:

1. The linear gauge field term in perturbative expansion is identical to that of the SU(3) projected Fat7 links.

$$H^{(1)}_\mu = C^{(1)}_\mu = D^{(1)}_\mu = D'^{(1)}_\mu = D''^{(1)}_\mu$$

where $C_\mu, D_\mu, D'_\mu$ and $D''_\mu$ are defined in Eqs. (30-33).

2. The quadratic terms are antisymmetric in gauge fields.

$$H^{(2)}_\mu = -\frac{1}{2} \sum_{\nu,\rho,y,z} \Xi_{\mu\nu\rho}(x,y,z) [A_{\nu}(y), A_{\rho}(z)]$$

Proof 4.1 The perturbative improvement condition of removing the flavor changing interactions imposes the three restrictions: one gluon emission vertex vanishes at the momentum $q_\nu = \pi/a$ for $\forall \nu \neq \mu$ (any direction transverse to the original link direction, $\mu$). These three restrictions fix the coefficients of Fat7 links as $\alpha_1 = \alpha_2 = \alpha_3 = 1$ at tree level, regardless of the SU(3) projection. The same restrictions determine the coefficients of the HYP blocking: $\alpha'_1 = 7/8$, $\alpha'_2 = 4/7$ and $\alpha'_3 = 1/4$. The key point is that the improvement condition is necessary and sufficient to determine all the coefficients so that the linear terms satisfy the universal relation given in Eq. (36). One may ask why there is no ambiguity in the flavor conserving terms. By construction, the HYP link incorporates the gauge degrees of freedom within the hypercubes attached to the original link. As in Eq. (5), the flavor conserving kinetic terms, however, goes beyond the hypercubes attached to the original link. Hence, there was no ambiguity originated from the flavor conserving interactions. This proves the first part of Theorem 4.

Proof 4.2 By construction, the HYP fat links are SU(3)-projected after each smearing transformation as shown in Eq. (6–8). By Theorem 1, the quadratic terms of any SU(3) projected fat links are antisymmetric in gauge fields. This proves the second part of Theorem 4.

Theorem 5 (Equivalence at one loop)
If we impose the perturbative improvement condition to remove the flavor changing interactions, at one loop level,

1. the renormalization of the gauge invariant staggered operators is identical between the HYP staggered action and those improved staggered actions made of the SU(3) projected Fat7 links,

2. and the contribution to the one-loop renormalization can be obtained by simply replacing the propagator of $A_\mu$ field by that of the $H^{(1)}_\mu = C^{(1)}_\mu = D^{(1)}_\mu = D'^{(1)}_\mu = D''^{(1)}_\mu$ field.

Here, the SU(3) projected Fat7 links collectively represent the $\nabla^L_\mu$, $\nabla^M_\mu$, $\nabla^{M'}_\mu$ and $\nabla^{M''}_\mu$ gauge links defined in Eqs. (30-33).

Proof 5.1 The quadratic terms in the HYP fat links and the SU(3) projected Fat7 links make no contribution to one loop renormalization by Theorem 2. Since the quadratic and higher order terms can not contribute, only the linear terms contribute to the renormalization at one loop by Theorem 2. The linear terms are identical between the HYP fat links and the SU(3) projected Fat7 links by Theorem 4. Therefore, the one-loop renormalization of the gauge invariant staggered fermion operators are the same between the HYP and SU(3) projected Fat7 links. Since only the linear terms contribute, the renormalization constants can be calculated by simple replacement of the gauge propagator, $\langle A_\mu(x)A_\nu(y) \rangle$ with $\langle H^{(1)}_\mu(x)H^{(1)}_\nu(y) \rangle$ by Theorem 2. This completes a proof of Theorem 5.

\[2\text{Note that } u_0 = 1 \text{ at tree level.}\]
IV. INTERPRETATION OF THE THEOREMS

In the original definition of the Fat7 links, the $SU(3)$ projection was not included in Ref. [4], because the improvement aims at removing the $O(a^2)$ terms based upon the Symanzik improvement programme. Hence, the flavor symmetry restoration was only a part of the improvement goal in Ref. [4]. However, in this paper we have a somewhat different goal to minimize the perturbative corrections to staggered fermion operators by constructing the action and operators using fat links.

It has been a long-standing problem that the naive (i.e. unimproved) staggered fermion operators receive a large perturbative corrections. Recently, in [7], it was pointed out that the conventional tadpole improvement program suggested in [8] works well for the Wilson fermions but it does not work for the staggered fermions. It was observed that the contribution from the staggered fermion doublers (it is called “doubler-tadpole” in [7]) are very much like the usual gluon tadpoles originally introduced in [8]. Therefore, subtracting only the usual gluon tadpoles was not enough to improve the perturbative behavior. Note that the origin of the doubler-tadpoles are the same as that of the flavor changing interactions at the tree level. This observation of the doubler-tadpole problem guided us to an improvement program of removing the doubler-tadpoles systematically, which is identical to the idea of removing the flavor changing quark-gluon vertex suggested in [3,4,15]. Therefore, in this section, we will focus on the improvement programs for the flavor symmetry restoration: the Fat7 improvement and the HYP improvement, and interpret the meaning of the theorems given in Sec. III.

A. SU(3) Projection and Renormalization

Theorems 1 and 2 tell us that the SU(3) projection makes the one-loop renormalization of the staggered operators so simple that it can be calculated simply by substituting the fat link gauge propagator $\langle B_\mu^{(1)}(x)B_\nu^{(1)}(y) \rangle$ for the original gauge propagator $\langle A_\mu(x)A_\nu(y) \rangle$ in calculating each Feynman diagram. By construction, the smearing prefactor of the fat link is designed to suppress the high momentum gluon exchange. Therefore, the SU(3) projected fat links guarantees that the perturbative correction will be smaller than that of the original thin links.

Remark 1 (Tadpole improvement by the SU(3) projected fat links) Let us define $C_{fat}$ as the perturbative correction to the gauge invariant staggered fermion operators constructed using the SU(3) projected fat links (Fat7 type and HYP type). Similarly, $C_{thin}$ is defined as the perturbative correction to the gauge invariant staggered fermion operator constructed using the usual thin links. Then, at one loop level, for each Feynman diagram,

$$\| C_{fat} \| < \| C_{thin} \|$$

Here, note that this remark does not apply to the Fat7 link without SU(3) projection.\(^3\)

This is a direct consequence of Theorem 1 and Theorem 2 combined with the assumption that the smearing prefactor suppresses the high momentum gluon exchange which causes the doubler-tadpole problem. Therefore, we conclude that the SU(3) projection of the fat links (Fat7 type and HYP type) consistently decrease the contribution from the doubler tadpoles by the ratio of $\| B_\mu^{(1)} / A_\mu \|^2$. We may view the SU(3) projection of the fat links as one way of performing the tadpole improvement to remove the doubler tadpoles.

However, note that Remark 1 does not guarantee that the total summation of the one-loop corrections is smaller with the SU(3) projected fat links, because the smallness of individual terms does not mean much to the case of destructive cancellation between Feynman diagrams.

B. Uniqueness and Equivalence

From Theorems 3, and 5, we learn that the fat links with a single SU(3) projection and multiple SU(3) projections make no difference to one-loop renormalization of the gauge-invariant staggered fermion operators. For example, stag-

\(^3\)Explicit calculation of one-loop diagrams shows that the correction from the tadpole diagram is the same between the thin link and Fat7 link without SU(3) projection even after the first level of the tadpole improvement [16]. With no tadpole improvement, the correction using the Fat7 link was even larger.
nerged operators made of the various fat links such as $\overrightarrow{V}_{\mu}$, $\overrightarrow{V}_{\mu}^M$, $\overrightarrow{V}_{\mu}^M'$, $\overrightarrow{V}_{\mu}''$ receive the same one-loop renormalization. In addition, Theorems 4 and 5 tells us that as long as we impose the same perturbative improvement condition, the HYP staggered action and the improved staggered actions of the SU(3) projected Fat7 type give the same one-loop correction to the staggered operators, although the HYP fat links are constructed in such a way completely different from the SU(3) projected Fat7 links.

As a result of Theorems 1–5, at one loop level, we have five equivalent choices for the improvement using fat links: the four SU(3) projected Fat7 links and the HYP link. All of them share the same advantages of triviality in renormalization, smaller one-loop corrections for each Feynman diagram, uniqueness and equivalence, which are significantly better properties that the original Fat7 link action without SU(3) projection does not possess. However, one may still ask which of these five fat links are the best. We will address this question next.

V. THE FINAL PROPOSAL AND CONCLUSION

We know from Theorems 1-5 that one-loop perturbation can not make any distinction between the SU(3) projected Fat7 links and the HYP link. In other words, one-loop perturbation can not guide us any more.

Let us first make a list of advantages of various fat links of our concern. The single SU(3) projected Fat7 link, $\overrightarrow{V}_{\mu}^L$ defined in Eq. (30) is relatively cheap to calculate on the computer compared with the other fat links. In Ref. [5], it is shown that the $N = 3$ APE smearing [17] and HYP blocking reduce the flavor symmetry breaking in the pion spectrum more efficiently than the $N = 1$ APE smearing, which is consistent with Ref. [3]. This numerical results lead us to the conclusion that the fat link, $\overrightarrow{V}_{\mu}^M$ defined in Eq. (31) and the HYP fat link would probably be the best from the perspective of the flavor symmetry restoration. The fat link, $\overrightarrow{V}_{\mu}^M$ has an advantage that the interpretation of the improvement is relatively straight-forward, compared with the HYP fat link. In addition, the $\overrightarrow{V}_{\mu}^M$ link is computationally simpler to program, whereas the HYP link needs considerably more memory in order to make it fast enough.\footnote{In order to make the HYP blocking run fast, one needs to precompute the second level and the third level of the HYP blocking and save them in on-board memory [18]. This requires considerably more memory than the case of the triple SU(3) projected Fat7 links.} Considering all the advantages mentioned above, we make two final proposals.

- Use the single SU(3) projected Fat7 link $\overrightarrow{V}_{\mu}^L$, if the goal of improvement is to achieve, in a numerically cheaper way, the smaller one-loop correction with all the nice features mentioned in the previous section IV.

- Use the triple SU(3) projected Fat7 link $\overrightarrow{V}_{\mu}^M$, if the goal of improvement is to achieve the smaller one-loop correction and the better flavor symmetry restoration simultaneously with all the nice features.

These two alternatives certainly deserve further investigation.

In addition, the five theorems in this paper make the perturbative calculation simpler for the HYP type mainly because one can perform the calculation merely by replacing the thin link propagator with that of the HYP fat link. This simplicity is extensively used in calculating one loop renormalization constants of staggered fermion operators made of the HYP links [16,19].

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