Decoupling in QED and QCD*

Andrey Grozin
Budker Institute of Nuclear Physics SB RAS, Novosibirsk
and Novosibirsk State University

Abstract
Decoupling of a heavy flavour in QCD is discussed in a pedagogical way. First we consider a simpler case: decoupling of muons in QED. All calculations are done up to 2 loops.

1 Introduction
QCD with all 6 flavours is rarely used. If characteristic momenta \( p_i \ll M_Q \) (where \( M_Q \) is the mass of a heavy flavour \( Q \)), it is better to use a low-energy effective theory without \( Q \). Its Lagrangian has the QCD form plus \( 1/M_Q^\alpha \) corrections. Operators in the full QCD are expanded in \( 1/M_Q \) via appropriate operators in the effective theory.

The pioneering paper [1] discussed decoupling effects in the \( \overline{\text{MS}} \) scheme at two loops; however, it contained a calculational error. The correct two-loop result was obtained in [2] as a by-product of a three-loop calculation. A simple and efficient method to find decoupling effects was developed in [3]; the two-loop result [2] was confirmed, and new three-loop results were derived. After that, the erratum to [1] appeared, in which the authors fixed their error, and confirmed the results of [2, 3]. A few years ago, four-loop decoupling coefficients have been calculated [4, 5].

In this lecture, we shall discuss decoupling effects in QCD at two-loop level. First we shall discuss the QED case which is very similar to QCD but simpler (Sect. 2). We consider QED with electrons and muons; when characteristic momenta \( p_i \ll M \) (where \( M \) is the muon mass), the effective low-energy theory containing only electrons and photons can be used instead. After that we discuss QCD with light flavours \( q_i \) and a heavy flavour \( Q \) (Sect. 3).

---

* Lecture at 5-th Helmholtz international summer school Calculations for modern and future colliders, Dubna, July 23 – August 2, 2012.

1 The result of [5] contains one master integral which was not known analytically, only numerically, with 37-digits precision. An analytical expression for this integral has been published later [6].


2 Decoupling in QED

2.1 Full theory and effective low-energy theory

Let’s consider QED with \( n_f = 2 \) lepton flavours — massless electron \( \psi \) and heavy muon \( \Psi \):

\[
L = \bar\psi_0 i \gamma_\mu \gamma_5 \psi_0 + \bar\Psi_0 (i \gamma_\mu - M_0) \Psi_0 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}_0 - \frac{1}{2a_0} (\partial_\mu A_0^\mu)^2
\]

(2.1)

(the covariant gauge is used; 0 means bare quantities). When characteristic momenta \( p_i \ll M \) (characteristic distances \( \gg 1/M \)), the low-energy effective theory containing only light fields can be used instead. Its Lagrangian contains all possible operators allowed by symmetries; those with dimensionalities \( > 4 \) are suppressed by powers of \( 1/M \):

\[
L' = \bar\psi'_0 i \gamma_\mu \gamma_5 \psi'_0 - \frac{1}{4} F'_{\mu\nu} F'^{\mu\nu}_0 - \frac{1}{2a'_0} (\partial_\mu A'^\mu_0)^2 + \frac{1}{M_0^2} \sum_i C^0_i O^0_i + \cdots
\]

(2.2)

(quantities in the effective theory are denoted by primes). Muons only exist in loops of size \( \sim 1/M \) producing local interactions of light fields. Coefficients in the effective Lagrangian are tuned to reproduce scattering amplitudes of the full theory expanded in \( p_i/M \) up to some order. We shall not discuss \( 1/M^n \) corrections here.

Operators of the full theory can be expressed via all operators of the effective theory which are allowed by symmetries. Contributions of higher-dimensionality operators are suppressed by powers of \( 1/M \). Coefficients are tuned to reproduce on-shell matrix elements of the full-theory operators expanded in \( p_i/M \) up to some order. In particular, the light fields of the full QED can be written as

\[
A_0 = \left[ \zeta^0_A \right]^{-1/2} A'_0 + \frac{1}{M_0^2} \sum_i C^0_{A_i} O^0_{A_i} + \cdots
\]

\[
\psi_0 = \left[ \zeta^0_\psi \right]^{-1/2} \psi'_0 + \frac{1}{M_0^2} \sum_i C^0_{\psi_i} O^0_{\psi_i} + \cdots
\]

(2.3)

We shall not discuss \( 1/M^n \) corrections.

Similarly, the parameters of the full-theory Lagrangian are related to those of the effective-theory one:

\[
e_0 = \left[ \zeta^0_\alpha \right]^{-1/2} e'_0, \quad a_0 = \left[ \zeta^0_A \right]^{-1} a'_0.
\]

(2.4)

We shall soon see why decoupling of the gauge fixing parameter \( a \) is determined by the same coefficient \( \zeta_A \) as that of the photon field.

Now we shall recall some information about renormalization of QED which will be used later. For more details, see textbooks, e.g., [7].

The photon propagator has the structure

\[
-iD_{\mu\nu}(p) = -iD^0_{\mu\nu}(p) + (-i)D^0_{\mu\alpha}(p)i\Pi_{\alpha\beta}(p)(-i)D^0_{\beta\nu}(p) + \cdots
\]

(2.5)

\[
+ (-i)D^0_{\mu\alpha}(p)i\Pi_{0\beta}(p)(-i)D^0_{\beta\gamma}(p)i\Pi_{0\gamma}(p)(-i)D^0_{0\nu}(p) + \cdots
\]
where the photon self energy \( i\Pi_{\mu\nu}(p) \) is the sum of all one particle irreducible diagrams (which cannot be cut into two disconnected pieces by cutting a single photon line). Due to the Ward identity, it is transverse:

\[
\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi(p^2),
\]

and we obtain

\[
D_{\mu\nu}(p) = \frac{1}{p^2 [1 - \Pi(p^2)]} \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) + a_0 \frac{p_\mu p_\nu}{(p^2)^2}.
\]

The electron propagator has the structure

\[
iS(p) = iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p) + \cdots
\]

where the electron self energy \(-i\Sigma(p)\) is the sum of all one particle irreducible diagrams (which cannot be cut into two disconnected pieces by cutting a single electron line). For the massless electron, it has one Dirac structure

\[
\Sigma(p) = \slashed{p} \Sigma_V(p^2),
\]

due to chirality conservation. We obtain

\[
S(p) = \frac{1}{\slashed{p} [1 - \Sigma_V(p^2)]}.
\]

We shall use two renormalization schemes, \( \overline{\text{MS}} \) and on-shell. In the \( \overline{\text{MS}} \) scheme, bare fields and parameters are related to renormalized ones as

\[
A_0 = Z_A^{1/2}(\alpha(\mu)) A(\mu), \quad \psi_0 = Z_\psi^{1/2}(\alpha(\mu), a(\mu)) \psi(\mu),
\]

\[
e_0 = Z_e^{1/2}(\alpha(\mu)) e(\mu), \quad a_0 = Z_A(\alpha(\mu)) a(\mu),
\]

where all renormalization constants have the minimal structure

\[
Z_i(\alpha) = 1 + \frac{z_1}{\varepsilon} + \frac{z_2}{\varepsilon^2} + \frac{z_3}{\varepsilon^3} + \cdots
\]

(zeroth and positive powers of \( \varepsilon \) are not allowed). We shall see in a moment why renormalization of \( a \) is determined by the same constant \( Z_A \) as that of the photon field. In \( d = 4 - 2\varepsilon \) dimensions, dimensionality of \( e \) is \( \varepsilon \); \( \alpha \) should be dimensionless, therefore, we have to introduce renormalization scale \( \mu \) to construct

\[
\frac{\alpha(\mu)}{4\pi} = \mu^{-2\varepsilon} \frac{e^2(\mu)}{(4\pi)^{d/2}} e^{-\gamma \varepsilon}.
\]
Multiplying the bare photon propagator (2.7) by $Z^{-1}_A$ converts $D_\perp(p^2)$ to $D^r_\perp(p^2;\mu)$ which is finite at $\varepsilon \to 0$; at the same time it converts $a_0$ to $a(\mu)$ which is also finite. This is the reason why renormalization of the photon field $A$ and the gauge parameter $a$ is given by a single renormalization constant $Z_A$.

The sum of one-particle-irreducible vertex diagrams (not including the external propagators) is the vertex function

$$\Gamma^\mu(p,p') = i e_0 \Gamma^\mu(p,p').$$

(2.14)

It should be equal to $\Gamma^\mu(p,p') = Z_\Gamma \Gamma^\mu(p,p';\mu)$ where $Z_\Gamma$ is a minimal renormalization constant and $\Gamma^\mu(p,p';\mu)$ is finite at $\varepsilon \to 0$. The scattering amplitude $e_0 \Gamma^\mu Z_\psi Z^{1/2}_A$ must be finite; this means that the product $Z^1_A Z_\Gamma Z_\psi Z^{1/2}_A$ is finite. But the only minimal (2.12) renormalization constant finite at $\varepsilon \to 0$ is 1. Therefore, $Z^1_A Z_\Gamma Z_\psi Z^{1/2}_A = 1$, and

$$Z_\alpha = [Z_\Gamma Z_\psi]^{-2} Z^{-1}_A.$$

(2.15)

The Ward identity says that

$$\Gamma^\mu(p,p') q_\mu = S^{-1}(p') - S^{-1}(p).$$

(2.16)

Multiplying this equation by $Z_\psi$ make its right-hand side finite; hence its left-hand side is finite, too, and $Z_\Gamma = Z_\psi^{-1}$. Thus the Ward identity makes situation in QED simple: $Z_\alpha = Z^{-1}_A$.

In the on-shell renormalization scheme

$$A_0 = [Z^{os}_A(e_0)]^{1/2} A_{os}, \quad \psi_0 = [Z^{os}_\psi(e_0,a_0)]^{1/2} \psi_{os},$$

$$e_0 = [Z^{os}_\alpha(e_0)]^{1/2} e_{os}, \quad a_0 = Z^{os}_A(e_0) a_{os}$$

(2.17)

($Z^{os}_i$ are not minimal). By definition, the renormalized propagators in this scheme tend to the free ones near their mass shells:

$$D^{os}_\perp(p^2) \to D^{0}_\perp(p^2) = \frac{1}{p^2}, \quad S^{os}_\perp(p) \to S_0(p) = \frac{1}{p}$$

(2.18)

at $p^2 \to 0$. This means

$$Z^{os}_A(e_0) = \frac{1}{1 - \Pi(0)}, \quad Z^{os}_\psi(e_0,a_0) = \frac{1}{1 - \Sigma(0)}.$$

(2.19)

When $p$ and $p'$ are on the mass shell, and the initial electron and the final one have physical polarizations, a single-photon scattering of an electron is described by 2 form factors; only one survives at $q \to 0$. The scattering amplitude in this limit is $e_{os} \gamma^\mu =
\[ e_0 \Gamma^\mu Z_\psi^{\text{os}} [Z_\psi^{\text{os}}(e_0)]^{1/2}, \] where \( e_{\text{os}} \) is, by definition, the on-shell electron charge. The vertex is \( \Gamma^\mu = Z_\Gamma^{\text{os} \cdot \mu} \), and therefore
\[ Z_\alpha^{\text{os}} = [Z_\Gamma^{\text{os}} Z_\psi^{\text{os}}]^{-2} [Z_A^{\text{os}}]^{-1}. \tag{2.20} \]
The Ward identity (2.16) at \( q \to 0 \)
\[ \Gamma^\mu(p, p) = \frac{\partial S^{-1}(p)}{\partial p_\mu} \tag{2.21} \]
near the mass shell \( (p^2 \to 0) \) gives
\[ \Gamma^\mu(p, p) = \frac{\partial}{\partial p_\mu} \left[ \frac{Z_\psi^{\text{os}}}{p} \right]^{-1} = [Z_\psi^{\text{os}}]^{-1} \gamma^\mu, \tag{2.22} \]
and we obtain \( Z_\Gamma^{\text{os}} = [Z_\psi^{\text{os}}]^{-1} \); hence \( Z_\alpha^{\text{os}} = [Z_A^{\text{os}}]^{-1} \).

The \( \overline{\text{MS}} \) renormalized fields and parameters in the full theory and the effective one are related by
\[
\begin{align*}
A(\mu) &= \zeta_A^{-1/2}(\mu) A'(\mu), & \psi(\mu) &= \zeta_\psi^{-1/2}(\mu) \psi'(\mu), \\
e(\mu) &= \zeta_\alpha^{-1/2}(\mu) e'(\mu), & a(\mu) &= \zeta_A^{-1}(\mu) a'(\mu),
\end{align*} \tag{2.23}
\]
where the renormalized decoupling coefficients are
\[
\begin{align*}
\zeta_A(\mu) &= \frac{Z_A(\alpha(\mu))}{Z_A'(\alpha'(\mu))} \zeta_0, & \zeta_\psi(\mu) &= \frac{Z_\psi(\alpha(\mu), a(\mu))}{Z_\psi'(\alpha'(\mu), a'(\mu))} \zeta_0, & \zeta_\alpha(\mu) &= \frac{Z_\alpha(\alpha(\mu))}{Z_\alpha'(\alpha'(\mu))} \zeta_0 .
\end{align*} \tag{2.24}
\]
They satisfy the renormalization group equations
\[
\begin{align*}
\frac{d}{d \log \mu} \log \zeta_A(\mu) &= \gamma_A(\alpha(\mu)) - \gamma_A'(\alpha'(\mu)), \\
\frac{d}{d \log \mu} \log \zeta_\psi(\mu) &= \gamma_\psi(\alpha(\mu), a(\mu)) - \gamma_\psi'(\alpha'(\mu), a'(\mu)), \tag{2.25} \\
\frac{d}{d \log \mu} \log \zeta_\alpha(\mu) &= 2 [\beta(\alpha(\mu)) - \beta'(\alpha'(\mu))],
\end{align*}
\]
where the anomalous dimensions and the \( \beta \) functions are defined as
\[
\begin{align*}
\frac{d}{d \log \mu} \log Z_A(\alpha(\mu)) &= \gamma_A(\alpha(\mu)), \\
\frac{d}{d \log \mu} \log Z_\psi(\alpha(\mu), a(\mu)) &= \gamma_\psi(\alpha(\mu), a(\mu)), \tag{2.26} \\
\frac{d}{d \log \mu} \log Z_\alpha(\alpha(\mu)) &= 2 \beta(\alpha(\mu)).
\end{align*}
\]
Note that the \( \overline{\text{MS}} \) charge \( \alpha(\mu) \) and the gauge parameter \( a(\mu) \) satisfy the RG equations
\[
\begin{align*}
\frac{d}{d \log \mu} \log \alpha(\mu) &= -2\varepsilon - 2\beta(\alpha(\mu)), & \frac{d}{d \log \mu} \log a(\mu) &= -\gamma_A(\alpha(\mu)). \tag{2.27}
\end{align*}
\]
2.2 Photon field and electron charge

The propagators of both $A_{os}$ and $A'_{os}$ are equal to the free propagator at $p^2 \to 0$:

$$D_{\perp}^{os}(p^2) = D_{\perp}^{os}(p^2) \left[ 1 + \mathcal{O}(p^2) \right],$$

and therefore

$$A_{os} = A'_{os} + \mathcal{O} \left( \frac{1}{M^2} \right).$$

Hence the bare decoupling coefficient is

$$\zeta^0_A = \frac{Z_{A}^{os}(e'_0)}{Z_{A}^{os}(e_0)}.$$  \hspace{1cm} (2.30)

We have

$$Z_{A}^{os}(e_0) = \frac{1}{1 - \Pi(0)} \quad \text{and} \quad Z_{A}^{os}(e'_0) = \frac{1}{1 - \Pi'(0)}. \hspace{1cm} (2.31)$$

But $\Pi'(0) = 0$ because all loop integrals are scale-free. Hence $Z_{A}^{us} = 1$:

$$\zeta^0_A = 1 - \Pi(0). \hspace{1cm} (2.32)$$

At one loop, the photon self energy (2.6) is

$$\Pi(p^2) = i \left( p^2 g_{\mu\nu} - p_{\mu} p_{\nu} \right) \Pi(p^2).$$

Contraction in $\mu$ and $\nu$ we get

$$\Pi(p^2) = \frac{4i e^2_0}{(d-1)p^2} \int \frac{d^d k}{(2\pi)^d} \frac{N}{D_1 D_2},$$

$$D_1 = M_0^2 - (k+p)^2, \quad D_2 = M_0^2 - k^2,$$

$$N = 1 \frac{1}{4} \text{Tr} \gamma_{\mu}(k + \not{p} + M_0) \gamma^{\mu}(k + M_0).$$

Now we expand the integrand in $p$ up to $p^2$; the problem reduces to trivial 1-loop vacuum integrals:

$$\Pi(0) = - \frac{4 e^2_0 M_0^{-2\epsilon}}{3 (4\pi)^{d/2}} \Gamma(\epsilon). \hspace{1cm} (2.33)$$

The bare decoupling coefficient of the photon field with the 1-loop accuracy is

$$\zeta^0_A = 1 + \frac{4 e^2_0 M_0^{-2\epsilon}}{3 (4\pi)^{d/2}} \Gamma(\epsilon) + \cdots \hspace{1cm} (2.34)$$
Re-expressing it via renormalized quantities in the full theory, we obtain

\[ \zeta_A^0 = 1 + \frac{4}{3} \frac{\alpha(\mu)}{4\pi \varepsilon} Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu)) \left( \frac{\mu}{M(\mu)} \right)^{2\varepsilon} \Gamma(1+\varepsilon) e^{\gamma\varepsilon} + \cdots \]  

(2.35)

where the \( \overline{\text{MS}} \) renormalized muon mass is defined by

\[ M_0 = Z_m(\alpha(\mu)) M(\mu). \]  

(2.36)

The \( \overline{\text{MS}} \) renormalization constants of the photon fields in the full theory and the effective one with the 1-loop accuracy are

\[ Z_A^{(\gamma)}(\alpha) = 1 - \frac{4}{3} n_f \frac{\alpha}{4\pi \varepsilon} + \cdots \]  

(2.37)

(see, e.g., the textbook [7]). There are 2 lepton flavours, electron and muon, in the full theory \( (n_f = 2) \) and only 1 (electron) in the low-energy effective theory \( (n_f = 1) \). We may neglect the difference between \( \alpha'(\mu) \) and \( \alpha(\mu) \) in corrections. Combining these pieces together, we arrive at the renormalized decoupling coefficient of the photon field with the 1-loop accuracy

\[ \zeta_A(\mu) = 1 + \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \cdots \]  

(2.38)

where

\[ L = 2 \log \frac{\mu}{M(\mu)}. \]  

(2.39)

Note that \( L \) depends on \( \mu \) in a complicated way because the \( \overline{\text{MS}} \) renormalized muon mass \( M(\mu) \) also depends on \( \mu \). It is convenient to do decoupling at \( \mu = \bar{\mu} \) where \( \bar{\mu} \) is defined as the solution of the equation

\[ M(\bar{\mu}) = \bar{\mu}; \]  

(2.40)

then \( L = 0 \). The decoupling coefficient \( \zeta_A(\mu) \) for other \( \mu \) can be obtained by solving the RG equation \([2,25]\) with this initial condition.

The scattering amplitude of an on-shell electron with a physical polarization in an electromagnetic field at \( q \to 0 \) should be the same in both theories:

\[ e_0 \Gamma^\mu Z_\psi^{\text{os}} [Z_A^{\text{os}}]^{1/2} = e'_0 \Gamma'^\mu Z_\psi^{\text{os}} [Z_A'^{\text{os}}]^{1/2}. \]  

(2.41)

The vertex functions in the two theories are

\[ \Gamma^\mu = Z_\Gamma^{\text{os} \gamma^\mu}, \quad \Gamma'^\mu = Z_\Gamma'^{\text{os} \gamma^\mu}, \]  

(2.42)

and hence

\[ \zeta_\alpha^0 = \frac{[Z_\Gamma^{\text{os}} Z_\psi^{\text{os}}]^{2} Z_\alpha^{\text{os}}}{[Z_\Gamma'^{\text{os}} Z_\psi^{\text{os}}]^{2} Z_A'^{\text{os}}} = \frac{Z_\alpha^{\text{os}}}{Z_A^{\text{os}}} . \]  

(2.43)
As we already discussed, \( Z_\Gamma^{\text{os}} Z_\psi^{\text{os}} = 1 \) due to the Ward identity; the same is true in the effective theory. Moreover, in the effective theory \( Z_\Gamma^{\text{os}} = Z_\psi^{\text{os}} = Z_A^{\text{os}} = 1 \) because all loop integrals are scale-free. Therefore

\[
\zeta_\alpha = \left[ \zeta_\alpha \right]^{-1}.
\]  

(2.44)

In other words, the on-shell electron charge (measured at large distances in smooth macroscopic fields, i.e., at \( q \to 0 \), in Millikan-like experiments) must be the same in both theories

\[
\alpha_{\text{os}} = \alpha'_{\text{os}}.
\]  

(2.45)

Thus we immediately obtain (2.43).

![Figure 1: The Green function of \( \bar{\psi}_0, \psi_0, A_0 \).](image)

We can look at the charge decoupling from a slightly different point of view. Let’s consider the Green function of \( \bar{\psi}_0, \psi_0, A_0 \). It is the vertex with the full propagators attached (Fig. 1). It differs from the similar Green function in the effective theory by the bare decoupling constants of the three fields:

\[
e_0 \Gamma_{SSD} = \left[ \zeta_\psi \right]^{-1} \left[ \zeta_A \right]^{-1/2} e_0' \Gamma' S' S' D'.
\]  

(2.46)

On the other hand, \( S = \left[ \zeta_\psi \right]^{-1} S', D = \left[ \zeta_A \right]^{-1} D' \), and hence

\[
e_0 \Gamma^\mu = \zeta_\psi \left[ \zeta_A \right]^{1/2} e_0' \Gamma'^\mu.
\]  

(2.47)

Writing down \( \Gamma^\mu = \left[ \zeta_\Gamma \right]^{-1} \Gamma'^\mu \) where \( \zeta_\Gamma = Z_\Gamma^{\text{os}} / Z_\Gamma^{\text{os}} \), we obtain

\[
\zeta_\alpha = \left[ \zeta_\Gamma \zeta_\psi \right]^{-2} \left[ \zeta_A \right]^{-1},
\]  

(2.48)

and this is equivalent to (2.43).

The MS renormalized charge decoupling is

\[
\alpha(\mu) = \zeta_\alpha^{-1}(\mu) \alpha'(\mu),
\]  

(2.49)

where

\[
\zeta_\alpha(\mu) = \frac{Z_\alpha(\alpha(\mu)) \zeta_\alpha^0}{Z_\alpha'(\alpha'(\mu))}.
\]  

(2.50)

Taking into account \( Z_\alpha^{(\mu)} = \left[ Z_\alpha^{(\mu)} \right]^{-1} \), we obtain

\[
\zeta_\alpha(\mu) = \zeta_A^{-1}(\mu).
\]  

(2.51)
With the 1-loop accuracy $\zeta_A(\mu)$ is given by (2.38).

The photon self energy with the 2-loop accuracy (Fig. 2) is

$$\Pi(0) = -\frac{4}{3} \frac{e^2 \bar{M}_0^{2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)$$

$$- \frac{2}{3} \frac{(d-4)(5d^2 - 33d + 34)}{d(d-5)} \left( \frac{e^2 \bar{M}_0^{2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

(2.52)

It reduces to the integrals (Fig. 3)

$$\int \frac{d^d k_1 d^d k_2}{D_1^{n_1} D_2^{n_2} D_3^{n_3}} = -\pi^d M^{2(d-n_1-n_2-n_3)} V(n_1, n_2, n_3),$$

$$D_1 = M^2 - k_1^2, \quad D_2 = M^2 - k_2^2, \quad D_3 = -(k_1 - k_2)^2,$$

(2.53)

which can be expressed via $\Gamma$ functions

$$V(n_1, n_2, n_3) = \frac{\Gamma \left( \frac{d}{2} - n_3 \right) \Gamma \left( n_1 + n_3 - \frac{d}{2} \right) \Gamma \left( n_2 + n_3 - \frac{d}{2} \right) \Gamma \left( n_1 + n_2 + n_3 - d \right)}{\Gamma \left( \frac{d}{2} \right) \Gamma(n_1) \Gamma(n_2) \Gamma(n_1 + n_2 + 2n_3 - d)}.$$  

(2.54)

The bare decoupling coefficient of the photon field with the 2-loop accuracy is

$$\zeta_A^0 = 1 + \frac{4}{3} \frac{e^2 \bar{M}_0^{2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)$$

$$+ \frac{2}{3} \frac{(d-4)(5d^2 - 33d + 34)}{d(d-5)} \left( \frac{e^2 \bar{M}_0^{2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots$$

(2.55)
Re-expressing in via renormalized quantities in the full theory, we obtain

\[
\zeta_A^0 = 1 + \frac{4}{3} e^{L \varepsilon} \frac{\alpha(\mu)}{4\pi \varepsilon} Z_\alpha(\alpha(\mu)) Z_m^{-2\varepsilon}(\alpha(\mu)) \\
- \varepsilon \left(6 - \frac{13}{3} \varepsilon + \cdots\right) e^{2L \varepsilon} \left(\frac{\alpha(\mu)}{4\pi \varepsilon}\right)^2 + \cdots
\] (2.56)

We need 1-loop corrections to \(Z_\alpha\) and \(Z_m\) in the full theory:

\[
Z_\alpha = Z_A^{-1} = 1 + 2 \cdot \frac{4}{3} \frac{\alpha(\mu)}{4\pi \varepsilon} + \cdots \quad Z_m = 1 - 3 \frac{\alpha(\mu)}{4\pi \varepsilon} + \cdots
\] (2.57)

The \(\overline{\text{MS}}\) renormalization constants of the photon fields in the full theory and the effective one with the 2-loop accuracy are

\[
Z_A^{(\prime)}(\alpha) = 1 - \frac{4}{3} n_f \frac{\alpha}{4\pi \varepsilon} - 2\varepsilon n_f \left(\frac{\alpha}{4\pi \varepsilon}\right)^2 + \cdots
\] (2.58)

(see, e.g., the textbook [7]). Substituting everything into \(\zeta_A(\mu)\) (2.24), we arrive at

\[
\zeta_A^{-1}(\mu) = \zeta_A(\mu) = 1 + \frac{4}{3} L \frac{\alpha(\mu)}{4\pi} + \left(-4L + \frac{13}{3}\right) \left(\frac{\alpha(\mu)}{4\pi}\right)^2 + \cdots
\] (2.59)

where \(L\) is defined in (2.39).

In particular, at \(\mu = \overline{M}\) (2.40) \((L = 0)\)

\[
\zeta_A(\overline{M}) = 1 - \frac{13}{3} \left(\frac{\alpha(\overline{M})}{4\pi}\right)^2 + \cdots
\] (2.60)

If we use \(\mu = M_{\text{os}}\) instead, where the on-shell muon mass is related to the \(\overline{\text{MS}}\) one by

\[
\frac{M(\mu)}{M_{\text{os}}} = 1 - 6 \left(\log \frac{\mu}{M_{\text{os}}} + \frac{2}{3}\right) \frac{\alpha}{4\pi} + \cdots
\] (2.61)

then \(L = \frac{8\alpha}{4\pi}\) and

\[
\zeta_A(M_{\text{os}}) = 1 - 15 \left(\frac{\alpha(M_{\text{os}})}{4\pi}\right)^2 + \cdots
\] (2.62)

In general, for any \(\mu = \overline{M}(1 + \mathcal{O}(\alpha))\) we have \(\zeta_A(\mu) = 1 + \mathcal{O}(\alpha^2)\) (with different coefficients of \(\alpha^2\)); however, for, say, \(\mu = 2\overline{M}\) or \(\overline{M}/2\) the 1-loop term appears. It is most convenient to use some \(\mu_0 = \overline{M}(1 + \mathcal{O}(\alpha))\) as an initial condition, and obtain \(\zeta_A(\mu)\) for other \(\mu\) from the RG equation (2.25).
2.3 Electron field and mass

The propagators of both $\psi_{os}$ and $\psi'_{os}$ are equal to the free propagator at $p^2 \to 0$:

$$\hat{p}S_{os}(p) = \hat{p}S'_{os}(p) \left[ 1 + O(p^2) \right],$$

and therefore

$$\psi_{os} = \psi'_{os} + O \left( \frac{1}{M^2} \right). \quad (2.64)$$

Hence the bare decoupling coefficient is

$$\zeta^0_\psi = \frac{Z_{\psi}(e'_0)}{Z_{\psi}(e_0)}, \quad (2.65)$$

where

$$Z_{\psi}(e_0) = \frac{1}{1 - \Sigma_{\psi}(0)} , \quad Z_{\psi}(e'_0) = \frac{1}{1 - \Sigma'_{\psi}(0)}. \quad (2.66)$$

Only diagrams with muon loops contribute to $\Sigma_{\psi}(0)$; $\Sigma'_{\psi}(0) = 0$ because all loop integrals are scale-free. Hence $Z_{\psi}^{\text{os}} = 1$:

$$\zeta^0_\psi = 1 - \Sigma_{\psi}(0). \quad (2.67)$$

The calculation reduces to the integrals (2.54). The result is

$$\Sigma_{\psi}(0) = -i e^2_0 \frac{(d-1)(d-4)}{d} \int \frac{d^d k}{(2\pi)^d} \frac{i k^\mu i k^\nu}{(k + \hat{p})^2} \left( \frac{-i}{k^2} \right)^2 i (k^2 g_{\mu \nu} - k_\mu k_\nu) \Pi(k^2),$$

where $\Pi(k^2)$ is the muon-loop contribution to the photon self energy. Expanding the right-hand side up to linear terms in $p$, we obtain

$$\Sigma_{\psi}(0) = -i e^2_0 \frac{(d-1)(d-4)}{d} \int \frac{d^d k}{(2\pi)^d} \frac{\Pi(k^2)}{(-k^2)^2}. \quad (2.68)$$

The calculation reduces to the integrals (2.54). The result is

$$\Sigma_{\psi}(0) = \frac{2(d-1)(d-4)(d-6)}{d(d-2)(d-5)(d-7)} \left( \frac{e^2_0 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \ldots \quad (2.69)$$

Figure 4: Electron self energy.
The bare decoupling coefficient is

\[ \zeta_0 = 1 - \frac{2(d - 1)(d - 4)(d - 6)}{d(d - 2)(d - 5)(d - 7)} \left( \frac{e^2 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \right)^2 + \cdots = 1 - \varepsilon \left( 1 - \frac{5}{6} \varepsilon + \cdots \right) \left( \frac{\alpha}{4\pi \varepsilon} \right)^2 + \cdots \]  

(2.70)

To find the renormalized decoupling coefficient \( \zeta_\psi(\mu) \) we need the \( \overline{\text{MS}} \) renormalization constants \( Z_\psi^{(\prime)} \). They are determined by the anomalous dimensions

\[ \gamma_\psi^{(\prime)}(\alpha, a) = 2a \frac{\alpha}{4\pi} - (4n_f + 3) \left( \frac{\alpha}{4\pi} \right)^2 + \cdots \]  

(2.71)

(see, e.g., the textbook [7]). Taking into account \( \alpha'(\bar{M}) = \alpha(\bar{M}) [1 + \mathcal{O}(\alpha^2)] \), \( \alpha'(\bar{M}) = \alpha(\bar{M}) [1 + \mathcal{O}(\alpha^2)] \), we see that

\[ \frac{Z_\psi(\alpha(\bar{M}), a(\bar{M}))}{Z_\psi^{(\prime)}(\alpha'(\bar{M}), a'(\bar{M}))} = 1 + \varepsilon \left( \frac{\alpha(\bar{M})}{4\pi \varepsilon} \right)^2, \]

and hence

\[ \zeta_\psi(\bar{M}) = 1 + \frac{5}{6} \left( \frac{\alpha(\bar{M})}{4\pi} \right)^2 + \cdots \]  

(2.72)

(\( \zeta_\psi(\mu) \) for any \( \mu \sim \bar{M} \) has the same form, because there is no \( \mathcal{O}(\alpha_s) \) term). To find \( \zeta_\psi(\mu) \) for other values of \( \mu \), the RG equation (2.25) can be used.

Until now we considered QED with massless electrons and heavy muons. Now let’s take the small electron mass \( m \) into account. The electron self energy has two Dirac structures

\[ \Sigma(p) = \frac{1}{p - m_0 - \Sigma(p)} = \frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\hat{p} - 1 + \Sigma_S(p^2)/m_0}. \]  

(2.74)

and the propagator is

\[ S(p) = \frac{1}{p - m_0 - \Sigma(p)} = \frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\hat{p} - 1 + \Sigma_S(p^2)/m_0}. \]

Near the mass shell

\[ \frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\hat{p} - 1 + \Sigma_S(p^2)/m_0} = \left[ \zeta_0^{(\prime)} \right]^{-1} \frac{1}{1 - \Sigma_V'(p^2)} \hat{p} - \frac{1}{1 - \Sigma_V'(p^2)/m_0}. \]  

(2.75)

We shall work in the linear approximation in \( m \). Comparing the overall factors, we reproduce (2.65); comparing the denominators, we obtain

\[ \frac{1 + \Sigma_S(0)}{1 - \Sigma_V(0)/m_0} = \frac{1 + \Sigma_S'(0)}{1 - \Sigma_V'(0)/m_0}'. \]  

(2.76)
(we may set \(m_0 = 0\) in \(\Sigma_{V,S}(0)\)). The bare masses in the two theories are related by

\[
m_0 = [\zeta_m^0]^{-1} m'_0;
\]

(2.77)

we obtain

\[
\zeta_m^0 = [\zeta_q^0]^{-1} \frac{1 + \Sigma_S(0)}{1 + \Sigma_S'(0)} = \frac{1 + \Sigma_S(0)}{1 - \Sigma_V(0)}
\]

(2.78)

(because \(\Sigma_V'(0) = \Sigma_S'(0) = 0\)).

In other words, the on-shell electron mass is the same in both theories:

\[
m_{os} = m'_{os},
\]

(2.79)

because it can be directly measured in experiment. In view of \(m_0 = Z_{m_{os}}^0 m_{os}\) and a similar relation in the effective theory, this leads to

\[
\zeta_m^0 = \frac{Z_{m_{os}}(e'_0)}{Z_{m_{os}}(e_0)}.
\]

(2.80)

The on-shell electron mass renormalization constant in the full theory \(Z_{m_{os}}^0\) depends on two masses, \(m_{os}\) and \(M_{os}\); if we neglect corrections suppressed by powers of \(m_{os}^2/M_{os}^2\), this (more physical) definition of \(\zeta_m^0\) coincides with our previous definition based on expansion in \(m\) up to linear terms.

The first diagram contributing to \(\Sigma_S(0)\) appears at 2 loops (Fig. 4). In the linear approximation, we retain \(m_0\) in the numerator of the electron propagator, and set \(m_0 = 0\) in other places. The result is

\[
\Sigma_S(0) = -\frac{2(d-1)(d-6)}{(d-2)(d-5)(d-7)} \left( \frac{\epsilon_0^2 M_0^{-2\epsilon}}{(4\pi)^{d/2}} \Gamma(\epsilon) \right)^2 + \cdots
\]

(2.81)

Substituting it and (2.69) into (2.78), we obtain

\[
\zeta_m^0 = 1 - \frac{8(d-1)(d-6)}{d(d-2)(d-5)(d-7)} \left( \frac{\epsilon_0^2 M_0^{-2\epsilon}}{(4\pi)^{d/2}} \Gamma(\epsilon) \right)^2 + \cdots = 1 + \left( 2 - \frac{5}{3} \epsilon + \frac{89}{18} \epsilon^2 + \cdots \right) \left( \frac{\alpha}{4\pi\epsilon} \right)^2 + \cdots
\]

(2.82)

To find the renormalized decoupling coefficient

\[
\zeta_m(\mu) = \frac{Z_m(\alpha(\mu))}{Z'_m(\alpha'(\mu))} \zeta_m^0
\]

(2.83)

we need the \(\overline{\text{MS}}\) renormalization constants \(Z_m^{(\ell)}\). They are determined by the anomalous dimensions

\[
\gamma_m^{(\ell)}(\alpha) = 6 \frac{\alpha}{4\pi} + \left( 3 - \frac{20}{3} n_f \right) \left( \frac{\alpha}{4\pi} \right)^2 + \cdots
\]

(2.84)
(see, e.g., the textbook \[7\]); we obtain
\[
\frac{Z_m(\alpha(M))}{Z'_m(\alpha'(M))} = 1 - \left(2 - \frac{5}{3}\varepsilon\right)\left(\frac{\alpha}{4\pi\varepsilon}\right)^2 + \ldots
\]
and hence
\[
\zeta_m(M) = 1 + \frac{89}{18} \left(\frac{\alpha(M)}{4\pi}\right)^2 + \ldots
\]
(2.85)

To find \(\zeta_m(\mu)\) for other values of \(\mu\), the RG equation
\[
\frac{d\log \zeta_m(\mu)}{d\log \mu} + \gamma_m(\alpha(\mu)) - \gamma'_m(\alpha'(\mu)) = 0
\]
(2.86)
can be used.

### 3 Decoupling in QCD

#### 3.1 Full theory and effective low-energy theory

Now we shall consider QCD with \(n_l\) massless flavours \(q_i\) and a single heavy flavour \(Q\):
\[
L = \sum_{i=1}^{n_l} \bar{q}_i i \not{D} q_i + \bar{Q} (i \not{D} - M_0) Q_0 - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{2a_0} (\partial_\mu A^a_0)^2 + (\partial_\mu \bar{c}_0^a) (D^\mu c_0^a) .
\]
(3.1)

When characteristic momenta \(p_i \ll M\), the low-energy effective theory containing only light fields can be used instead:
\[
L' = \sum_{i=1}^{n_l} \bar{q}_i' i \not{D} q_i' - \frac{1}{4} G'^a_{\mu\nu} G'^{a\mu\nu} - \frac{1}{2a'_0} (\partial_\mu A'^a_0)^2 + (\partial_\mu \bar{c}'_0^a) (D'^\mu c'_0^a) + \mathcal{O} \left(\frac{1}{M^2}\right) .
\]
(3.2)

The fields and the parameters of the full theory are related to those of the effective theory:
\[
A_0 = \left[\zeta_A^0\right]^{-1/2} A_0', \quad q_0 = \left[\zeta_q^0\right]^{-1/2} q_0', \quad c_0 = \left[\zeta_c^0\right]^{-1/2} c_0', \quad g_0 = \left[\zeta_g^0\right]^{-1/2} g_0', \quad a_0 = \left[\zeta_A^0\right]^{-1} a_0'.
\]
(3.3)

The gluon self energy up to 2 loops (Fig. 5) is
\[
\Pi(0) = -\frac{4}{3} T_F g^2_0 M_0^{-2\varepsilon} \Gamma(\varepsilon)
- \frac{1}{d(d - 5)} \left[\frac{2}{3} (d - 4)(5d^2 - 33d + 34) C_F - \frac{d^5 - 20d^4 + 145d^3 - 458d^2 + 588d - 232}{(d - 2)(d - 7)} C_A \right]
\times T_F \left(\frac{g^2_0 M_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)\right)^2 + \ldots
\]
(3.4)
the $C_F$ term trivially follows from QED; the calculation reduces to the vacuum integrals (2.54). The bare gluon decoupling coefficient is $\zeta^A_0 = 1 - \Pi(0)$.

The quark self energy up to 2 loops (Fig. 4) can be trivially obtained from the QCD results (2.69), (2.81):

$$\Sigma_V(0) = \frac{2(d - 1)(d - 4)(d - 6)}{d(d - 2)(d - 5)(d - 7)} C_F T_F \left( \frac{g_0^2 M_0^{2\varepsilon}}{(4\pi)^{d/2} \Gamma(\varepsilon)} \right)^2 + \cdots$$

$$\Sigma_S(0) = -\frac{2(d - 1)(d - 6)}{(d - 2)(d - 5)(d - 7)} C_F T_F \left( \frac{g_0^2 M_0^{2\varepsilon}}{(4\pi)^{d/2} \Gamma(\varepsilon)} \right)^2 + \cdots$$

Therefore, the bare decoupling coefficients for the light-quark fields (2.67) and masses (2.78) are

$$\zeta^A_q = 1 - \frac{2(d - 1)(d - 4)(d - 6)}{d(d - 2)(d - 5)(d - 7)} C_F T_F \left( \frac{g_0^2 M_0^{2\varepsilon}}{(4\pi)^{d/2} \Gamma(\varepsilon)} \right)^2 + \cdots$$

$$\zeta^A_m = 1 - \frac{8(d - 1)(d - 6)}{d(d - 2)(d - 5)(d - 7)} C_F T_F \left( \frac{g_0^2 M_0^{2\varepsilon}}{(4\pi)^{d/2} \Gamma(\varepsilon)} \right)^2 + \cdots$$

The ghost propagator is

$$G(p) = \frac{1}{p^2 - \Sigma_c(p^2)};$$

therefore, the on-shell renormalization constant of the ghost field is

$$Z^{\text{os}}_c = \frac{1}{1 - \frac{d\Sigma_c}{dp^2}(0)}.$$
3.2 Decoupling of $\alpha_s$

All elementary vertices of QCD are determined by a single coupling constant $g_0$. Therefore, any vertex (quark–gluon, 3–guon, ghost–gluon, 4–gluon) can be used to obtain the decoupling relation for $g_0$. In the full theory, we expand these vertex functions in their external momenta up to the first non-vanishing term, and compare them to the corresponding elementary vertices in the effective theory (there are no loop corrections in it, because there is no scale). This first non-vanishing term of the expansion ought to have the structure of the elementary QCD vertex, otherwise the Lagrangian of the effective theory would not have the QCD form (higher terms of expansions in external momenta lead to higher-dimensional operators in the effective low-energy Lagrangian, suppressed by powers of $1/M^2$).

The quark–gluon vertex function at 0-th order in its external momenta obviously has a single Dirac and colour structure $\gamma^\mu t^a$.

The 3–gluon vertex function should be expanded up to linear terms in its external momenta. The Bose symmetry allows two structures:

\begin{equation}
\Gamma_{a_1a_2a_3}^{\mu_1\mu_2\mu_3} = \left( g^{\mu_1\mu_2} (k_1^\mu_3 - k_2^\mu_3) + \text{cycle} \right) \text{ and } \delta_{a_1a_2a_3}^{\mu_1\mu_2\mu_3} = \left( g^{\mu_1\mu_2} k_3^\mu_3 + \text{cycle} \right).
\end{equation}

The Slavnov–Taylor identity $\langle T\{\partial^{\mu}A_{\mu}(x), \partial^{\nu}A_{\nu}(y), \partial^{\lambda}A_{\lambda}(z)\} \rangle = 0$ leads to $\Gamma_{a_1a_2a_3}^{\mu_1\mu_2\mu_3} k_1^{\mu_1} k_2^{\mu_2} k_3^{\mu_3} = 0$, thus excluding the second possibility.

The ghost–gluon vertex has a single structure: $p^\mu f^{abc}$, where $p^\mu$ is the outgoing ghost momentum. We shall prove this statement in a moment.

As a result, each vertex function of the full theory, expanded in its external momenta up to the first non-vanishing term, is equal to the corresponding elementary vertex times a scalar quantity $\Gamma_i$. Any one of these vertices can be used to find the bare decoupling coefficient $\zeta_0^\alpha$:

\begin{equation}
\zeta_0^\alpha(g_0) = \Gamma_{A^c \bar{c}}^2 \left[ Z_c^{\alpha} \right]^2 Z_A^{\alpha} = \Gamma_{Aq \bar{q}}^2 \left[ Z_q^{\alpha} \right]^2 Z_A^{\alpha} = \Gamma_{A \bar{A}A}^2 \left[ Z_A^{\alpha} \right]^3.
\end{equation}

We shall use the ghost–gluon vertex here, because the calculation is simplest in this case.

Now we shall discuss the ghost–gluon vertex function in the full theory expanded in its external momenta up to linear terms. Let’s consider the right-most vertex on the ghost line:

\begin{equation}
\mu \quad = \quad A^{\mu \nu} p^\nu.
\end{equation}

The tensor $A^{\mu \nu}$ may be calculated at zero external momenta, hence $A^{\mu \nu} = Ag^{\mu \nu}$. Therefore all loop diagrams have the Lorentz structure of the tree vertex, as expected.
Now let’s consider the left-most vertex:

\[
\begin{array}{c}
k \\
0 \\
k
\end{array}
\]

It gives \( k^\lambda \), thus singling out the longitudinal part of the gluon propagator. Therefore, all loop corrections vanish in Landau gauge. Furthermore, diagrams with self-energy insertions into the left-most gluon propagator vanish in any covariant gauge:

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram1.png}} \\
= 0
\end{array}
\]

In the diagrams including a quark triangle, the contraction of \( k^\lambda \) transforms the gluon propagator to a spin 0 propagator and a factor \( k^\rho \) which contracts the quark-gluon vertex. After decomposing \( k \) into a difference of the involved fermion denominators one obtains in graphical form

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram2.png}} \\
= a_0
\end{array}
\]

The diagrams with a massless triangle vanish. The non-vanishing diagrams contain the same Feynman integral, but differ by the order of the colour matrices along the quark line, thus leading to a commutator of two colour matrices.

The remaining diagram contains a three-gluon vertex with a self energy inserted in the right-most gluon propagator. The contraction of \( k^\lambda \) with the three-gluon vertex cancels the gluon propagator to the right of the three-gluon vertex:

\[
\begin{array}{c}
\text{\includegraphics[width=0.5\textwidth]{diagram3.png}} \\
= a_0
\end{array}
\]
The colour structure of the three-gluon vertex is identical to the commutator above, however with opposite sign. Therefore, after summing all contributions the complete 2-loop result is zero.

Thus the bare decoupling coefficient $\zeta_0^0$ with the 2-loop accuracy can be obtained (3.10) from the on-shell field renormalization constants $Z^{\text{os}}_A$ (3.4) and $Z^{\text{os}}_c$ (3.9):\

$$
\left[\zeta_0^0\right]^{-1} = 1 + \frac{4}{3} T_F \frac{g_0^2 M_0^{-2\epsilon}}{(4 \pi)^{d/2}} \Gamma(\epsilon) + \frac{d - 4}{d(d - 5)} \left[\frac{2}{3} (5d^2 - 33d + 34) C_F - \frac{d^3 - 14d^2 + 53d - 32}{d - 7} C_A\right] T_F \left(\frac{g_0^2 M_0^{-2\epsilon}}{(4 \pi)^{d/2}} \Gamma(\epsilon)\right)^2 + \cdots
$$

(3.11)

It is gauge invariant, as expected. At 2 loops, $Z^{\text{os}}_A$ and $Z^{\text{os}}_c$ don't depend on $a_0$; this means that $\Gamma_{A\bar{c}c}$ does not depend on $a_0$ at this order, and its 2-loop term trivially vanishes in Landau gauge $a_0 = 0$. Therefore non-vanishing 2-loop diagrams for $\Gamma_{A\bar{c}c}$ must cancel, as discussed above. At 3 loops this is no longer so.

We have the expression for $g_0^2 = \zeta_0^0 (g_0) g_0^2$ via the bare quantities of the full theory. We re-express it via the renormalized quantities using

$$
\frac{g_0^2}{(4 \pi)^{d/2}} \Gamma(\epsilon) = \mu^{2\epsilon} \frac{\alpha_s(\mu)}{4 \pi \epsilon} Z_\alpha(\alpha_s(\mu)) \Gamma(1 + \epsilon) e^{-\gamma\epsilon},
$$

(3.12)

where

$$
Z_\alpha(\alpha) = 1 - \beta_0 \frac{\alpha}{4 \pi \epsilon} + \left(\beta_0^2 - \frac{1}{2} \beta_1 \epsilon\right) \left(\frac{\alpha}{4 \pi \epsilon}\right)^2 + \cdots
$$

(3.13)

and (2.36).

Inverting the series

$$
\frac{g_0^2}{(4 \pi)^{d/2}} \Gamma(\epsilon) = \mu^{2\epsilon} \frac{\alpha_s'(\mu')}{4 \pi \epsilon} Z_\alpha'(\alpha_s'(\mu')) \Gamma(1 + \epsilon) e^{-\gamma\epsilon}
$$

(3.14)

we obtain

$$
\frac{\alpha_s'(\mu')}{4 \pi} = \frac{g_0^2}{(4 \pi)^{d/2}} \frac{\mu'^{-2\epsilon}}{\epsilon} e^{-\gamma\epsilon} \left[1 + \beta_0 \frac{g_0^2}{(4 \pi)^{d/2}} \frac{\mu'^{-2\epsilon}}{\epsilon} e^{-\gamma\epsilon} + \left(\beta_0^2 + \frac{1}{2} \beta_1 \epsilon\right) \left(\frac{g_0^2}{(4 \pi)^{d/2}} \frac{\mu'^{-2\epsilon}}{\epsilon} e^{-\gamma\epsilon}\right)^2 + \cdots\right].
$$

(3.15)

Finally, we substitute the expression for $g_0^2$ via $\alpha_s(\mu)$ obtained above, and arrive at $\alpha_s'(\mu')$ expressed via $\alpha_s(\mu)$.

The final result for $\mu' = \mu$ is $\alpha_s'(\mu) = \zeta_\alpha(\mu) \alpha_s(\mu)$, where the renormalized decoupling coefficient is

$$
\zeta_\alpha(\mu) = 1 - \frac{4}{3} LT_F \frac{\alpha_s(\mu)}{4 \pi} + \left[\frac{16}{9} T_F L^2 + 4 \left(C_F - \frac{5}{3} C_A\right) L - \left(\frac{13}{3} C_F - \frac{32}{9} C_A\right)\right] T_F \left(\frac{\alpha_s(\mu)}{4 \pi}\right)^2 + \cdots
$$

(3.16)
(\(L\) is defined by (2.39)). It is convenient to use \(\mu = \bar{M}\) defined by (2.40):

\[
\zeta_\alpha(\bar{M}) = 1 - \left( \frac{13}{3} C_F - \frac{32}{9} C_A \right) T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots \tag{3.17}
\]

(here the \(C_F\) term trivially follows from QED (2.60)). For other values of \(\mu\) the RG equation (2.25) can be used. For example, Fig. 7 (produced using RunDec [9]) shows \(\alpha_s^{(5)}(\mu)\) and \(\alpha_s^{(4)}(\mu)\) near \(\mu = M_b\).

![Graph showing \(\alpha_s^{(5)}(\mu)\) and \(\alpha_s^{(4)}(\mu)\) near \(\mu = M_b\).]

Figure 7: Crossing the b-quark threshold: \(\alpha_s^{(5)}(\mu)\) and \(\alpha_s^{(4)}(\mu)\).

The decoupling relation for the light-quark masses \(m'(\bar{M}) = \zeta_m(\bar{M}) m(\bar{M})\) can be trivially obtained from the QED result (2.85):

\[
\zeta_m(\bar{M}) = 1 - \frac{89}{18} C_F T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots \tag{3.18}
\]

For other values of \(\mu\) the RG equation (2.86) can be used.

### 3.3 QCD fields

Decoupling of the gluon field and the gauge parameter are given by the same quantity \(\zeta_0^\Lambda\):

\[
a'_0 = \zeta_0^\Lambda(g_0, a_0, M_0) a_0. \tag{3.19}
\]

At the first step we express the bare quantities in the right-hand side via the renormalized ones using (3.12), (2.36), and

\[
a_0 = Z_\Lambda(\alpha_s(\mu), a(\mu)) a(\mu), \tag{3.20}
\]
and thus obtain an expression for $a'_0$ via the renormalized parameters of the full theory $a(\mu), \alpha_s(\mu), M(\mu)$. Next we find $a'(\mu')$ in terms of $a'_0$ by solving the equation

$$a'_0 = Z'_A(a'_s(\mu'), a'(\mu')) a'(\mu')$$

(3.21)

iteratively. Substituting the expression for $a'_0$ obtained earlier, we arrive at an expression for $a'(\mu')$ via $a(\mu), \alpha_s(\mu), M(\mu)$.

It is convenient to use $\mu' = \mu = \bar{M}$: $a'(\bar{M}) = \zeta_A(\bar{M}) a(\bar{M})$,

$$\zeta_A(\bar{M}) = 1 + \frac{13}{12} (4C_F - C_A) T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots$$

(3.22)

(the $C_F$ term trivially follows from QED (2.59)). Coefficients of this expansion are gauge-dependent starting from $\alpha_s^3$. For other values of $\mu, \mu'$ the RG equation (2.25) can be used.

For the quark field decoupling we can trivially obtain from the QED result (2.72)

$$\zeta_q(\bar{M}) = 1 + \frac{5}{6} C_F T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots$$

(3.23)

Coefficients of this expansion are gauge-dependent starting from $\alpha_s^3$.

For the ghost field we find

$$\zeta_c(\bar{M}) = 1 - \frac{89}{72} C_A T_F \left( \frac{\alpha_s(\bar{M})}{4\pi} \right)^2 + \cdots$$

(3.24)

### 4 Conclusion

Decoupling effects in QED are tiny; they are discussed in Sect. 2 for pedagogical reasons: QED calculations are similar to QCD but simpler.

In QCD decoupling effects are essential for any quantity if we want to study it in a wide range of $\mu$. The most important application is $\alpha_s(\mu)$: if we want, e.g., to relate $\alpha_s(m_\tau)$ to $\alpha_s(m_Z)$, we need to take into account both RG running (in each interval it is controlled by the corresponding $\beta$ function) and decoupling. The same is true for, say, $m_s(\mu)$. Parton distribution functions are also extracted from experiment at very different values of $\mu$, and in addition to RG running (given by the evolution equations) we need to take into account also decoupling effects. These effects are more difficult to calculate than, e.g., for $\alpha_s(\mu)$; however, the methods used are similar.

QCD without one or more heaviest flavours is one of the simplest examples of an effective low-energy theory. There are other similar examples, e.g., the effective Lagrangian of the Higgs–gluon interaction. In the Standard Model, it is produced by the $t$-quark loop; however, characteristic momenta are $\ll M_t$, and this loop can be replaced by a local interaction.

I am grateful to K. G. Chetyrkin, M. Höchele, J. Hoff, and M. Steinhauer for discussions of decoupling, and to D. I. Kazakov for inviting me to give a talk at Calc-2012. The work was supported by RFBR (grant 12-02-00106-a) and by Russian Ministry of Education and Science.
References

[1] W. Bernreuther, W. Wetzel, Nucl. Phys. B 197 (1982) 228; Erratum: B 513 (1998) 758.

[2] S. A. Larin, T. van Ritbergen, J. A. M. Vermaseren, Nucl. Phys. B 438 (1995) 278 [hep-ph/9411260].

[3] K. G. Chetyrkin, B. A. Kniehl, M. Steinhauser, Nucl. Phys. B 510 (1998) 61 [hep-ph/9708255].

[4] K. G. Chetyrkin, J. H. Kühn, C. Sturm, Nucl. Phys. B 744 (2006) 121 [hep-ph/0512060].

[5] Y. Schröder, M. Steinhauser, JHEP 01 (2006) 051 [hep-ph/0512058].

[6] B. A. Kniehl, A. V. Kotikov, A. I. Onishchenko, O. L. Veretin, Phys. Rev. Lett. 97 (2006) 042001 [hep-ph/0607202].

[7] A. G. Grozin, Lectures on QED and QCD: Practical calculation and renormalization of one- and multi-loop Feynman diagrams, World Scientific (2007).

[8] A. A. Vladimirov, Teor. Mat. Fiz. 43 (1980) 210 [Theor. Math. Phys. 43 (1980) 417].

[9] K. G. Chetyrkin, J. H. Kühn, M. Steinhauser, Comput. Phys. Commun. 133 (2000) 43 [hep-ph/0004189].