Non-imprisonment conditions on spacetime

E. Minguzzi

Abstract

The non-imprisonment conditions on spacetimes are studied. It is proved that the non-partial imprisonment property implies the distinction property. Moreover, it is proved that feeble distinction, a property which stays between weak distinction and causality, implies non-total imprisonment. As a result the non-imprisonment conditions can be included in the causal ladder of spacetimes. Finally, totally imprisoned causal curves are studied in detail, and results concerning the existence and properties of minimal invariant sets are obtained.

1 Introduction

Given a spacetime, i.e. a time oriented Lorentzian manifold, the non-total future imprisonment condition is satisfied if no future-inextendible causal curve can enter and remain in a compact set. Analogously, the non-partial future imprisonment condition requires that no future-inextendible causal curve reenters a compact set indefinitely in the future. Since the formulation of these properties uses only the causal structure of spacetime, it is expected that they could be related with other conformal invariant properties such as strong causality or distinction [2].

The classic book by Hawking and Ellis [6, Sect. 6.4] devotes to this issue several interesting propositions. Unfortunately, some of them prove to be too weak and as today the relationship between non-partial (total) imprisonment and the other causality properties has yet to be clarified. The aim of this work is to show that the non-imprisonment conditions can be included in the causal ladder of spacetimes [7, 11], a hierarchy of conformal invariant properties which is very useful in order to establish at first sight the relationship between the most common conformal invariant properties that have appeared in the literature.

Note that in general the generic conformal invariant property does not find a place in the causal ladder. For instance, the condition of reflectivity [5, 11], defined by the property $I^+(q) \subseteq I^+(p) \Leftrightarrow I^-(p) \subseteq I^-(q)$, despite being conformal invariant, is not present by itself in the ladder (one has to add to it some distinguishing condition so as to obtain the level of causal continuity [7]). Thus the
Figure 1: The portion of the causal ladder which summarizes the results of this work and [10] concerning the causal ladder. An arrow $A \Rightarrow B$ means that $A$ implies $B$ and there are examples which show that $A$ differs from $B$. Note that feeble distinction and reflectivity imply causal continuity, for a proof of this result see [10].

fact that the non-partial and non-total imprisonment conditions can find a place in the causal ladder is a rather fortunate circumstance given the importance of these conditions for the study of the causal structure of spacetime.

Figure 1 displays the relevant part of the causal ladder and summarizes the final picture as will be given by this work. In it I included some results recently obtained in [10].

I denote with $(M, g)$ a $C^r$ spacetime (connected, time-oriented Lorentzian manifold), $r \in \{3, \ldots, \infty\}$ of arbitrary dimension $n \geq 2$ and signature $(-,+,\ldots,+)$. The subset symbol $\subset$ is reflexive, $X \subset X$. Several versions of the limit curve theorem will be repeatedly used. The reader is referred to [9] for a sufficiently strong formulation.

## 2 Non-partial imprisonment

In this section I shall introduce some definitions and basic results. I will then consider the property of non-partial imprisonment leaving the property of non-total imprisonment to the next section.

**Definition 2.1.** A future inextendible causal curve $\gamma : I \to M$, is totally future imprisoned in the compact set $C$ if there is $t \in I$, such that for every $t' > t$, $t' \in I$, $\gamma(t') \in C$, i.e. if it enters and remains in $C$. It is partially future imprisoned if for every $t \in I$, there is $t' > t$, $t' \in I$, such that $\gamma(t') \in C$, i.e. if
it does not remain in the compact set it continually returns to it. The curve
*escapes to infinity in the future* if it is not partially future imprisoned in any
compact set. Analogous definitions hold in the past case.

**Remark 2.2.** In Hawking and Ellis [6, p. 194] it is stated that a future in-
extendible causal curve $\gamma : [0, b) \to M$ which is not partially imprisoned in a
compact set, intersects every compact set only a finite number of times. How-
ever, this statement is incorrect unless very strong differentiability conditions
are imposed on the curve. The point is that the causal curve may escape and
reenter the same compact set infinitely often while the parameter does not
go to $b$. Consider for instance 1+1 Minkowski spacetime $ds^2 = -dt^2 + dx^2$
and the $C^{k-2}$, $k > 2$, timelike curve $\gamma = (t, x(t))$ with $x(t) = 0$ for $t \leq 0,$
$x(t) = \frac{1}{k+2}(\tanh t)^k \sin(1/\tanh t)$ for $t > 0$. It enters and escapes the compact
set $[-1, 1] \times [-1, 0]$ an infinite number of times on any neighborhood of $t = 0$.

The previous definitions extend to the spacetime

**Definition 2.3.** A spacetime is *non-total future imprisoning* if no future inex-
extendible causal curve is totally future imprisoned in a compact set. A spacetime
is *non-partial future imprisoning* if no future inextendible causal curve is par-
tially future imprisoned in a compact set. Analogous definitions hold in the past
case.

Actually, Beem proved [1, theorem 4] that a spacetime is non-total future
imprisoning if and only if it is non-total past imprisoning, thus in the non-total
case one can simply speak of the *non-total imprisonment* property (condition $N$,
in Beem’s terminology [1]). This result will also be obtained in the next section.
A spacetime which is both non-partial future imprisoning and non-partial past
imprisoning is simply said to be *non-partial imprisoning*.

The following result is immediate

**Proposition 2.4.** The non-total imprisonment condition implies causality.

*Proof.* Assume causality does not hold. A closed causal curve can be made
inextendible while keeping the same compact image, simply by extending the
parametrization so as to make many rounds over the original curve. The result
is a inextendible curve whose image is contained in a compact set (since it is a
compact set itself), in contradiction with the non-total imprisonment condition.

Carter constructed an example of causal but total imprisoning spacetime.
This classical example can be found in figure 39 of [6]. Thus causality and
non-total imprisonment do not coincide.

The next result is well known [6, Prop. 6.4.7], [2, Prop. 3.13].

**Proposition 2.5.** If the spacetime is strongly causal then it is non-partial im-
prisoning.
Strong causality differs from non-partial imprisonment, see for instance figure 38 of [8].

The next observation is important for the placement of the non-partial imprisonment condition in the causal ladder

**Proposition 2.6.** If a spacetime is non-partial future (resp. past) imprisoning then it is past (resp. future) distinguishing. In particular non-partial imprisoning spacetimes are distinguishing.

**Proof.** I give the proof in the “non partial future imprisoning - past distinguishing” case, the other case being analogous.

Assume \((M, g)\) is non-partial future imprisoning. If \((M, g)\) is non-past distinguishing there are \(x \neq z\) such that \(I^-(x) = I^-(z)\). Let \(U \ni x\) be a relatively compact set such that \(z \notin \bar{U}\), and let \(V \ni z\) relatively compact set such that \(x \notin \bar{V}, U \cap V = \emptyset\). Take \(x_1 \in I^-(x) \cap \bar{U}\), then there is a timelike curve \(\sigma^z_1\) which connects \(x_1\) to \(z\). Let \(z_1 \in \sigma^z_1 \cap \bar{V} \subset I^-(z)\), and parametrize \(\sigma^z_1\) so that \(x_1 = \sigma^z_1(0)\) and \(z_1 = \sigma^z_1(1)\). There is a timelike curve \(\sigma^z_2\) which connects \(z_1\) to \(x\). Let \(x_2 \in \sigma^z_2 \cap \bar{U} \subset I^-(x)\), and parametrize \(\sigma^z_2\) so that \(z_1 = \sigma^z_2(1)\) and \(x_2 = \sigma^z_2(2)\). Continue in this way and obtain sequences \(x_n \in \bar{U}, z_n \in \bar{V}, \sigma_n^z, \sigma_n^x\). The timelike curve

\[
\sigma = \ldots \circ \sigma^z_2[3,4] \circ \sigma^z_1[2,3] \circ \sigma^z_1[1,2] \circ \sigma^z_1[0,1]
\]

is future inextendible and is partially future imprisoned in both \(\bar{U}\) and \(\bar{V}\). The contradiction proves that \((M, g)\) is past distinguishing. \(\square\)

The next example proves that past distinction differs from non-partial future imprisonment (analogously future distinction differs from non-partial past imprisonment).

**Example 2.7.** Consider the spacetime \(N = \mathbb{R} \times S^1 \times \mathbb{R}\) of coordinates \((t, \theta, y)\), \(\theta \in [0, 2\pi]\), and metric

\[
g = -dt \otimes d\theta - d\theta \otimes dt + t^2(dy - yd\theta)^2 + (dy + yd\theta)^2
\]

The vector field \(\partial/\partial \theta\) is Killing and the field \(\partial/\partial t - y\partial/\partial y\) is lightlike on the null surface \(t = 0\), while \(\partial/\partial \theta\) is lightlike and future directed everywhere. If \(V\) is the tangent vector to a future directed causal curve then \(g(V, \partial/\partial t) \leq 0\) which reads \(d\theta[V] \geq 0\). The equality sign holds only if \(V \propto \partial/\partial t\), while in all the other cases \(d\theta[V] > 0\). But it is also \(g(V, V) \leq 0\) which leads to \(dt[V]d\theta[V] \geq 0\), from which it follows that \(t\) is a quasi-time function, that is, it is non-decreasing over every causal curve. The curve \(\gamma = (t(\lambda), \theta(\lambda), y(\lambda))\) with \(t(\lambda) = 0, \theta(\lambda) = \lambda, y = -\exp(-\lambda)\) is a lightlike line partially imprisoned in the compact set \([-1, 1] \times [\pi/2, \pi] \times [-1, 1]\). Call \(\Gamma\) the set of events on the surface \(\theta = 0\) which can be connected to \(q = (0, 0, 0)\) through a causal curve which intersects the surface \(\theta = 0\) only at the endpoints. Note that no point of \(\gamma\) can be connected with a causal curve to \(q\), indeed the causal curve would have to be lightlike in order to keep itself in the surface \(t = 0\), which would imply the coincidence with \(\gamma\) which, however, does not pass through \(q\). Thus \(\gamma \cap \Gamma = \emptyset\).
Identify

Remove $\Gamma$ from the spacetime, then $\gamma$ is still partially imprisoned in the new spacetime $(M, g|_M)$ (see figure 2) but while $(N, g)$ was not past distinguishing, $(M, g|_M)$ is in fact past distinguishing. Indeed, the only set of points at which past distinction can be violated is the $\theta$ axis, however, thanks to the removal of $\Gamma$ the points on it have all distinct chronological pasts. Indeed, if $r_1 < r_2$, $r_1, r_2$ belong to the $\theta$ axis, then it can’t be $r_2 \in \mathcal{I}^-(r_1)$, for there would be a sequence of timelike curves starting from a neighborhood of $r_2$ and reaching $r_1$ which is impossible since they would intersect $\Gamma$.

If from the spacetime of example 2.7 one also removes a set analogous to $\Gamma$, that is the set of events on the surface $\theta = 0$ which can be reached from $q$ with a causal curve which intersects the surface $\theta = 0$ only at the endpoints, then one gets a spacetime that is distinguishing but not non-partial imprisoning, thus distinction and non-partial imprisonment differ.

3 Non-total imprisonment

Definition 3.1. Let $\gamma : \mathbb{R} \rightarrow M$ be a causal curve. Denote with $\Omega_f(\gamma)$ and $\Omega_p(\gamma)$ the following sets

\[
\Omega_f(\gamma) = \bigcap_{t \in \mathbb{R}} \overline{\gamma[t, +\infty)}
\]

\[
\Omega_p(\gamma) = \bigcap_{t \in \mathbb{R}} \overline{\gamma(-\infty, t]}
\]

They give, respectively, the set of accumulation points in the future of $\gamma$ and the set of accumulation points in the past of $\gamma$. The sets $\Omega_f$ and $\Omega_p$ are well
known from the study of dynamical systems [12 Sect. 3.2]. The points of \( \Omega_f(\gamma) \) are called \( \omega \)-limit points of \( \gamma \), while the points of \( \Omega_p(\gamma) \) are called \( \alpha \)-limit points of \( \gamma \). Note, however, that the analogy with dynamical systems is not complete because so far no flow has been defined on \( M \).

**Proposition 3.2.** Let \( \gamma : \mathbb{R} \to M \) be a inextendible causal curve. The set \( \Omega_f(\gamma) \) is closed. The curve \( \gamma \) is partially future imprisoned in a compact set iff \( \Omega_f(\gamma) \neq \emptyset \). The curve \( \gamma \) is totally future imprisoned in a compact set \( C \) iff \( \Omega_f(\gamma) \neq \emptyset \) and \( \Omega_f(\gamma) \) is compact. In this case \( \Omega_f(\gamma) \) is the intersection of all the compact sets in which \( \gamma \) is totally future imprisoned, moreover, \( \Omega_f(\gamma) \) is connected. Analogous statements hold in the past case. Finally, for every causal curve \( \gamma \), the closure of its image is given by \( \overline{\gamma} = \Omega_p(\gamma) \cup \gamma \cup \Omega_p(\gamma) \).

**Proof.** The closure is a consequence of the definition as intersection of closed sets.

If \( \gamma \) is partially future imprisoned in the compact set \( C \) then for every \( n \in \mathbb{N} \) there is \( t_n \in \mathbb{R} \) such that \( x_n = \gamma(t_n) \in C \) and \( t_n > n \). If \( x \in C \) is an accumulation point for \( x_n \), there is a subsequence \( x_{n_k} = \gamma(t_{n_k}) \) such that \( x_{n_k} \to x \). Choose \( t \in \mathbb{R} \) then every neighborhood \( U \ni x \) contains \( x_{n_k} \) for large \( k \), thus \( x \in \gamma([t, +\infty)) \) and since \( t \) is arbitrary \( x \in \Omega_f(\gamma) \). For the converse, assume \( \Omega_f(\gamma) \) is not empty, and take \( x \in \Omega_f(\gamma) \). Let \( V \ni x \) be a neighborhood of compact closure, then \( \gamma \) is partially future imprisoned in the compact set \( C = \overline{V} \).

Assume \( \gamma \) is totally future imprisoned in a compact set \( C \). Since \( T \) can be chosen so large that \( \gamma([T, +\infty)) \subset C \), it is \( \Omega_f(\gamma) \subset C \), and in particular \( \Omega_f(\gamma) \) is compact. Call \( I \) the intersection of all the compact sets totally future imprisoning \( \gamma \), then since \( \Omega_f(\gamma) \subset C \) holds for any such compact set \( C \), \( \Omega_f(\gamma) \subset I \).

Now, assume only that \( \Omega_f(\gamma) \) is non-empty and compact. Let \( h \) be an auxiliary complete Riemannian metric on \( M \), and let \( \rho \) be the corresponding continuous distance function. By the Hopf-Rinow (Heine-Borel) theorem any closed and bounded set of \( (M, h) \) is compact. Thus \( \Gamma_{\epsilon} = \{ y \in M : \rho(y, \Omega_f(\gamma)) \leq \epsilon \} \) is compact and \( \bigcap_{\epsilon > 0} \Gamma_{\epsilon} = \Omega_f(\gamma) \). But \( \gamma \) is totally future imprisoned in each \( \Gamma_{\epsilon} \), \( \epsilon > 0 \). Indeed, if not \( \gamma \) intersects indefinitely the set \( S_{\epsilon/2} = \{ y \in M : \rho(y, \Omega_f(\gamma)) = \epsilon/2 \} \), which is compact and thus there would be a accumulation point \( z \in S_{\epsilon/2} \cap \Omega_f(\gamma) \) a contradiction since \( S_{\epsilon/2} \cap \Omega_f(\gamma) = \emptyset \). Thus \( \gamma \) is totally future imprisoned in a compact set iff \( \Omega_f(\gamma) \) is non-empty and compact. From \( \bigcap_{\epsilon > 0} \Gamma_{\epsilon} = \Omega_f(\gamma) \), it follows that \( I \subset \Omega_f(\gamma) \), and using the other inclusion, \( I = \Omega_f(\gamma) \).

As for the connectedness of \( \Omega_f(\gamma) \), without loss of generality we can assume \( \gamma \) entirely contained in the compact set \( C \), and we already know that \( \Omega_f(\gamma) \subset C \). If there are two disjoint non-empty closed sets \( A \) and \( B \) such that \( \Omega_f = A \cup B \), then there are two open sets \( A' \supset A \), \( B' \supset B \), such that \( A \cap B = \emptyset \). Since, by definition of \( \Omega_f \), \( \gamma \) is partially imprisoned in \( A \) and \( B \) it crosses infinitely often both sets and there is a sequence of points \( z_r = \gamma(t_r) \in \gamma \subset C \), \( t_r \to +\infty \), \( z_r \notin B \cup A \). Thus there is an accumulation point \( z \in C \setminus (A \cup B) \) a contradiction since \( z \in \Omega_f(\gamma) \). The proof in the past case is analogous.

For the last statement let \( \gamma \) have domain \( (a, b) \) (finiteness irrelevant), and let \( x \in \overline{\gamma} \). For some sequence \( t_n \in (a, b) \), \( \gamma(t_n) \to x \). Either \( t_n \) admits a subsequence
which converges to \( t_0 \in (a, b) \), in which case by continuity \( x = \gamma(t_0) \in \gamma \), or there is a subsequence which converges to \( b \), in which case \( x \in \Omega_f(\gamma) \) or finally, there is a subsequence which converges to \( a \), in which case \( x \in \Omega_p(\gamma) \).

\[ \square \]

**Definition 3.3.** A lightlike line is an achronal inextendible causal curve.

Since every causal curve with endpoints which is not a lightlike geodesic can be varied to give a timelike curve joining the same points [6 Prop. 4.5.10], a lightlike line is necessarily a lightlike geodesic which, again by achronality, maximizes the Lorentzian distance between any of its points. Conversely, a inextendible lightlike geodesic which maximizes the Lorentzian distance between any of its points is clearly a line since the Lorentzian length calculated along the curve vanishes (indeed this is the definition given by [2 Prop. 8.12]).

**Proposition 3.4.** Let \( C \) be the chronology violating set of \((M, g)\) and let \( \gamma : \mathbb{R} \to M \) be a causal curve

(i) Let \( \alpha \) be a lightlike line such that \( \alpha \subset \Omega_f(\gamma) \) then \( \Omega_f(\alpha) \cup \Omega_p(\alpha) \subset \Omega_f(\gamma) \).

(ii) If \( \gamma \cap C = \emptyset \) then \( \Omega_f(\gamma) \) is achronal (and thus \( \Omega_f(\gamma) \cap C = \emptyset \)). Moreover, given \( y \in \Omega_f(\gamma) \) there passes through \( y \) one and only one lightlike line \( \alpha \).

This line is such that \( \alpha \subset \Omega_f(\gamma) \).

(iii) If \( \gamma \cap C = \emptyset \) and \( \gamma \) is a lightlike line then either \( \gamma \subset \Omega_f(\gamma) \) or \( \gamma \cap \Omega_f(\gamma) = \emptyset \). More generally, if \( \gamma \cap C = \emptyset \) but \( \gamma \) is only a causal curve then \( \gamma \) cannot leave \( \Omega_f(\gamma) \) if it ever enters it.

Analogous statements hold in the past case.

**Proof.** Let \( x \in \Omega_f(\alpha) \cup \Omega_p(\alpha) \) and take \( T \in \mathbb{R} \), and \( U \ni x \). There is \( t' \) such that \( \alpha(t') \in U \). But \( \alpha(t') \in \Omega_f(\gamma) \) and \( U \) is a neighborhood also for \( \alpha(t') \) thus there is \( t > T \), such that \( \gamma(t) \in U \). Since \( T \) and \( U \) are arbitrary, \( x \in \Omega_f(\gamma) \).

Assume \( \Omega_f(\gamma) \) is not achronal then there are \( x_1, x_2 \in \Omega_f(\gamma) \) such that \( (x_1, x_2) \in I^+ \). Using the fact that \( I^+ \) is open there are neighborhoods \( U_1 \ni x_1 \) and \( U_2 \ni x_2 \) such that \( U_1 \times U_2 \subset I^+ \). Since \( x_1, x_2 \in \Omega_f(\gamma) \) it is possible to find \( t_2 < t_1 \) such that \( \gamma(t_2) \in U_2 \) and \( \gamma(t_1) \in U_1 \), thus \( \gamma(t_2) \leq \gamma(t_1) \) and hence \( \gamma \cap C \neq \emptyset \), a contradiction.

Take \( y \in \Omega_f(\gamma) \), and assume without loss of generality that \( \gamma \) is parametrized with respect to \( h \)-length where \( h \) is a complete Riemannian metric. It is possible to find a sequence \( t_k, t_{k+1} > t_k + k \), such that \( \gamma(t_k) \to y \). By the limit curve theorem case (2) [9] the segments \( \gamma_k = \gamma|_{[t_k, t_{k+1}]} \) have both endpoints that converge to \( y \) and since their \( h \)-length goes to infinity and \( \gamma \cap C = \emptyset \), there is a lightlike line \( \alpha \) passing through \( y \) which is a limit (cluster) curve for \( \gamma_k \).

There can’t be another lightlike line \( \alpha' \) passing through \( y \) indeed it is possible to show that it cannot be distance maximizing. Indeed, let \( w_1, w_2 \in \alpha \), \( w_1 < y < w_2 \), and analogously let \( w'_1, w'_2 \in \alpha' \), \( w'_1 < y < w'_2 \). The limit curve theorem

\[ ^1 \]I am indebted to an anonymous referee for this statement and its proof.
which has allowed to construct α states also that \((w_2, w_1) \in \tilde{J}^+\). Now, \(\alpha'\) and \(\alpha\) have different tangent vectors at \(y\), otherwise, being geodesics they would coincide, thus rounding off the corners at \(y\) it follows \(w'_1 \ll w_2\) and \(w_1 \ll w'_2\) which together with \((w_2, w_1) \in \tilde{J}^+\) give, since \(J^+\) is open, \(w'_1 \ll w'_2\), thus \(\alpha'\) is not a line, a contradiction.

Now, \(\alpha \in \Omega_f(\gamma)\), indeed, take \(w \in \alpha\), and let \(U \ni w\) and \(T \in \mathbb{R}\). Since \(\alpha\) is a limit curve, \(U\) intersects all but a finite number of \(\gamma_k\), in particular it is possible to find \(s \in \mathbb{N}\) such that \(t_s > T\) and \(\gamma_k\) intersects \(U\). Thus there is \(t'_s \in [t_s, t_{s+1}]\) such that \(\gamma(t'_s) \in U\) and \(t'_s \geq t_s > T\). From the arbitrariness of \(U\) and \(T, w \in \Omega_f(\gamma)\).

Finally, assume that \(z \in \gamma \cap \Omega_f(\gamma)\) and that \(\gamma\) is a lightlike line. By point (ii) through \(z\) there passes a unique lightlike line \(\alpha\), and moreover such line has the property \(\alpha \subset \Omega_f(\gamma)\). But through \(z\) there passes already \(\gamma\) thus \(\alpha = \gamma\). More generally, if \(\gamma\) is only a causal curve, let \(z \in \gamma \cap \Omega_f(\gamma), z = \gamma(t)\). Split \(\gamma\) in the two curves \(\gamma_p = \gamma|_{[-\infty, t]}\) and \(\gamma_f = \gamma|_{[t, +\infty)}\) and do the same with the lightlike line \(\alpha\) passing through \(z\). The causal curve \(\gamma_f \circ \alpha_p\) is achronal because if not there are points \(z_p \in \alpha_p, z_f \in \gamma_f\), with \(z_p \ll z_f\). But since \(z_p \in \Omega_f(\gamma) = \Omega_f(\gamma_f)\), it would be \(\gamma_f \cap C \neq \emptyset\) a contradiction. Thus since \(\gamma_f \circ \alpha_p\) is achronal it is a lightlike line and hence it coincides with \(\alpha\), in particular \(\gamma_f\) is contained in \(\Omega_f(\gamma)\).

The proofs in the past case are analogous.

**Corollary 3.5.** Non-total past imprisonment is equivalent to non-total future imprisonment.

**Proof.** Assume non-total past imprisonment holds, then the spacetime is causal. If non-total future imprisonment does not hold then there is a future imprisoned causal curve \(\eta\), and \(\Omega_f(\eta)\) is compact and non-empty (Prop. 3.2). Take \(p \in \Omega_f(\eta)\), since the spacetime is chronological, there is a lightlike line passing through \(p\) (Prop. 3.3 (ii)) contained and hence totally past imprisoned in \(\Omega_f(\eta)\). The contradiction proves non-total past imprisonment implies non-total future imprisonment, the other direction being analogous.

Propositions 3.2 and 3.3 imply that if \(\gamma \cap C = \emptyset\) and the inextendible causal curve \(\gamma\) is partially future imprisoned in a compact set then \(\Omega_f(\gamma)\) is non-empty, generated by lightlike lines, and that there is a privileged field of future directed null directions over \(\Omega_f(\gamma)\). Introduce a complete Riemannian metric on \(M\), and normalize the field of directions so as to obtain a field of future directed lightlike vectors \(n : \Omega_f(\gamma) \to T\Omega_f(\gamma)\) which satisfies \(\nabla_n n = h(x)n\), for some scalar field \(h\). It is then possible to define over \(\Omega_f(\gamma)\) the integral flow \(\phi : \mathbb{R} \times \Omega_f(\gamma) \to \Omega_f(\gamma)\) of \(n\),

\[
\frac{d\phi_t(x_0)}{dt} = n(\phi_t(x_0)).
\]

which has the peculiarity of not having fixed points since \(n \neq 0\). By proposition 3.3 (ii), \(\Omega_f(\gamma)\) is invariant under the flow \(\phi_t\). Since the concept of invariance under the flow \(\phi\), is actually independent of the Riemannian metric chosen to define \(\phi\), it is convenient to state it in an equivalent but clearer way as follows
Definition 3.6. A non-empty closed subset \( \Omega \) of \( M \) is said to be invariant, if (i) through each one of its points there passes one and only one lightlike line and (ii) the entire line is contained in \( \Omega \). An invariant set is minimal if it has no invariant proper subset.

Remark 3.7. The definition of invariant set must embody the requirement of non-emptiness, otherwise the empty set would be an invariant set and no invariant set but the empty set would be minimal. Unfortunately, in many references about dynamical systems, where similar definitions are introduced, the non-emptiness condition is incorrectly omitted (for instance [4, p. 184]). Note that according to the terminology of this work a minimal invariant set is an invariant set and hence it is non-empty.

Lemma 3.8. The union of a finite family of invariant sets is an invariant set. The intersection of an arbitrary family of invariant sets, if non-empty, is an invariant set.

Proof. They are a trivial consequence of the definitions.

There may be other invariant sets inside \( \Omega_f(\gamma) \), indeed for every \( y \in \Omega_f(\gamma) \) there is a unique lightlike line \( \alpha \subset \Omega_f(\gamma) \) passing through \( y \), and \( \Omega_f(\alpha) \cap \Omega_p(\alpha) \subset \Omega_f(\gamma) \). Using again proposition 3.4(ii), it follows that both \( \Omega_f(\alpha) \) and \( \Omega_p(\alpha) \) satisfy conditions (i) and (ii) of invariant sets (definition 3.6). However, they could be empty. They are certainly non-empty if \( \gamma \) is not only partially imprisoned but also totally imprisoned. Indeed, in this case since \( \alpha \subset \Omega_f(\gamma) \), and \( \Omega_f(\gamma) \) is compact, \( \alpha \) is totally imprisoned both in the past and in the future and hence both \( \Omega_p(\alpha) \) and \( \Omega_f(\alpha) \) are non-empty and compact (proposition 3.2).

Thus a totally future imprisoned curve leads to a partial order of invariant sets (where the order is the usual inclusion) inside \( \Omega_f(\gamma) \). Actually, it is possible to prove that there exists a minimal invariant set as the next proposition shows.

Theorem 3.9. Let \( \eta \) be an inextendible causal curve totally future imprisoned in a compact set \( C \), and let \( \eta \cap C = \emptyset \) with \( C \) the chronology violating set of \((M, g)\). Then there is a minimal invariant set \( \Omega \subset \Omega_f(\eta) \subset C \). Through every point of \( \Omega \) there passes one and only one lightlike line, this lightlike line is entirely contained in \( \Omega \) and for every lightlike line \( \alpha \subset \Omega \) it is \( \overline{\alpha} = \Omega_f(\alpha) = \Omega_p(\alpha) = \Omega \). Another consequence is that all the points belonging to \( \Omega \) share the same chronological past and future. An analogous version holds with \( \eta \) past imprisoned.

Proof. Since \( \eta \) is totally future imprisoned, \( \Omega_f(\eta) \) is non-empty and compact. Consider the set of all the invariant subsets of \( \Omega_f(\eta) \) ordered by inclusion. This set is non-empty since \( \Omega_f(\eta) \) itself is invariant. By Hausdorff’s maximum principle (equivalent to Zorn’s lemma and the axiom of choice) there is a maximal chain of invariant sets, \( Z \), and the intersection of all the elements of the chain gives a set \( \Omega \) which, if non-empty, is an invariant set (lemma 3.8 which has to be minimal otherwise the chain would not be maximal. Since \( \Omega_f(\eta) \) is compact all the elements of \( Z \) are non-empty and compact, thus \( \Omega \) is non-empty being the intersection of a nested family of non-empty compact sets. This last result is
standard in topology ([3, theorem 3.1.1], in some references it is called Cantor’s intersection lemma), I include here the proof.

If \( \bigcap_{U \in Z} U = \emptyset \) then taking complements, \( \bigcup_{U \in Z} U^C = M \), thus the sets \( U^C \) with \( U \in Z \) would give an open covering of \( M \) and hence of \( \Omega_f(\eta) \). Extract a finite subcovering of this last set, and label the elements of the covering \( U_1^C, \ldots, U_n^C \) with \( U_{j+1} \subset U_j \). Thus \( U_n^C \supset \Omega_f(\gamma) \) which is impossible since \( U_n \subset \Omega_f(\gamma) \) is non-empty. The contradiction proves that \( \Omega \) is non-empty and hence minimal invariant.

Since \( \Omega \) is invariant through every point of it there passes a unique lightlike line entirely contained in \( \Omega \). Taken \( \alpha \subset \Omega \), it is \( \Omega_f(\alpha) \cap \alpha \cup \Omega_f(\alpha) \subset \Omega \), but since \( \Omega \) is actually the minimal invariant set the equality must hold. Since \( \overline{\Omega} = \Omega_p(\alpha) \cup \alpha \cup \Omega_f(\alpha) \), we have also \( \overline{\Omega} = \Omega \).

Finally, let \( x, z \in \Omega \), \( x \neq z \), and let \( z' \in I^+(z) \). Consider the lightlike line \( \gamma_x \) passing through \( x \). Since \( \Omega_f(\gamma_x) = \Omega \), \( z \in \Omega_f(\gamma_x) \), and in particular since there is a whole neighborhood of \( z \) in the past of \( z' \), \( z' \in I^+(x) \), and hence \( I^+(z) \subset I^+(x) \). Exchanging the roles of \( x \) and \( z \), and of pasts and futures, we get \( I^+(x) = I^+(z) \) and \( I^-(x) = I^-(z) \).

Recall that a feeibly distinguishing spacetime is a spacetime for which no two causally related events can have the same chronological past and chronological future. It is a weaker condition than weak distinction in which the causal relation of the events is not mentioned. Moreover, it is stronger than causality and differs from both causality and weak distinction [10].

Actually, there is not very much difference between weak distinction and feeble distinction, as it is quite difficult to produce examples of spacetimes which are feeibly distinguishing but non-weakly distinguishing. Nevertheless, this difference is mentioned and emphasized here because the next theorem stated with feeble distinction is slightly stronger than with weak distinction.

The next result improves Hawking and Ellis’ [6] who assume past or future distinction. Note that without this result the relative strength of non-total imprisonment and weak distinction would have been left open, a fact that has so far forbidden the placement of non-total imprisonment into the causal ladder.

**Corollary 3.10.** If a spacetime is feeably distinguishing then it is non-total imprisoning.

**Proof.** It follows trivially from theorem [8,4], because if \( \eta \) is a causal curve totally future imprisoned in a compact set, then there is a minimal invariant set \( \Omega \subset \Omega_f(\eta) \) (it is easy to see that feeble distinction implies chronology and hence \( \eta \cap C = \emptyset \)). Let \( \gamma \subset \Omega \) be a lightlike line, then the points of \( \gamma \) are causally related but share the same chronological past and future in contradiction with feeble distinction.

The fact that feeble distinction differs from non-total imprisonment follows, for instance, from figure 1(A) of [10].
4 Conclusions

The properties of non-total imprisonment and non-partial imprisonment have been placed into the causal ladder. The placement of non-partial imprisonment has required the proof that non-partial imprisonment implies distinction (Prop. 2.3) and the production of an example which shows that the two levels do indeed differ (figure 3).

More interesting and rich has proved the study of total imprisonment. To start with I considered a partial future imprisoned causal curve $\gamma$. I proved that on every event of the limit set $\Omega_f(\gamma)$ there passes one and only one lightlike line, and that this line is contained in $\Omega_f(\gamma)$. This result allows to define a flow on the same set and makes it possible to study this situation in analogy with dynamical systems. If $\gamma$ is actually totally imprisoned then it is possible to infer the existence of a minimal invariant set on which feeble distinction is violated. Thus feeble distinction implies non-total imprisonment, and the non-imprisonment conditions can thus be placed in the causal ladder.

There remains some interesting work to be done along the lines followed here in the study of totally imprisoned curves. For instance, in 2+1 dimensions one could perhaps apply the Poincaré-Bendixon-Schwartz theorem to the invariant set to infer more results on the connection between imprisonment and manifold topology. However, in order to follow this path it is necessary to prove that the invariant set is actually smooth or has a higher degree of differentiability than Lipschitz continuity (which follows from its achronality, see proposition 3.4(ii)). Similar ideas have indeed been applied in the study of chronology violating spacetimes [5, 13] but often without addressing the differentiability issues which indeed arise because the Poincaré-Bendixon-Schwartz theorem, as formulated by Schwartz, would require the invariant set to be a $C^2$ manifold.

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