BILINEAR FORMS ON WEIGHTED BESOV SPACES

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ABSTRACT. We compute the norm of some bilinear forms on products of weighted Besov spaces in terms of the norm of their symbol in a space of pointwise multipliers defined in terms of Carleson measures.

1. Introduction

The object of this paper is the study of some bilinear forms on products of weighted holomorphic Besov spaces, and their relationship with Hankel operators and weak products.

If \( \varphi \) and \( \psi \) are measurable functions on \( \mathbb{D} \) (on \( \mathbb{T} \) if \( t = 0 \)) such that \( \varphi \psi \in L^1(d\nu_t) \), let

\[
\langle \langle \varphi, \psi \rangle \rangle_t := \int_{\mathbb{D}} \overline{\varphi} \psi d\nu_t, \quad t > 0, \quad \langle \langle \varphi, \psi \rangle \rangle_0 := \int_{\mathbb{T}} \varphi \overline{\psi} d\sigma.
\]

Here, for \( t > 0 \), we write \( d\nu_t(z) := t(1 - |z|^2)^{t-1}d\nu(z) \), where \( d\nu \) is the normalized Lebesgue measure on the unit disk \( \mathbb{D} \), and \( d\sigma \) denotes the normalized Lebesgue measure on the circle \( \mathbb{T} \).

We also consider the pairings

\[
\langle h, b \rangle_t := \lim_{r \to 1^-} \langle \langle h(rz), b(rz) \rangle \rangle_t,
\]

whose domain is the subset of \( H \times H \) for which the limit exists. In particular, if either \( b \in H \cap L^1(d\nu_t) \), \( t > 0 \), or \( b \in H^1 \), \( t = 0 \), then we have that for any \( h \in H(\mathbb{D}) \),

\[
\langle h, b \rangle_t = \langle \langle h, b \rangle \rangle_t.
\]

In this paper we compute the norm of the bilinear form \( \Lambda_b(f, g) := \langle fg, b \rangle_t \), defined on products of weighted Besov spaces with weights of Békollé type, in terms of the norm of \( b \) in a space of pointwise multipliers related to these Besov spaces.

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Let us precise these results. Throughout the paper we denote by $H := H(\mathbb{D})$ (resp. $H(\mathbb{D})$) the space of holomorphic functions on the unit disk $\mathbb{D}$ (resp. on a neighborhood of $\mathbb{D}$).

If $1 < p < \infty$ and $t > 0$, the Békollé class $B_{p,t}$ consists of non-negative functions $\theta \in L^1(d\nu_t)$ such that the measures $d\mu_t := \theta d\nu_t$ and $d\mu'_t := \theta^{-p'/p} d\nu_t$ satisfy the so called $B_p$ condition

\[ B_{p,t}(\theta) := \sup_{z \in \mathbb{D}} \left( \frac{\mu_t(T_z)}{\nu_t(T_z)} \right)^{1/p} \left( \frac{\mu'_t(T_z)}{\nu_t(T_z)} \right)^{1/p'} < \infty, \]

where $p'$ is the conjugate exponent of $p$,

\[ T_z := \{ w \in \mathbb{D} : |1 - w\overline{z}/|z|| < 2(1 - |z|^2) \}, \quad z \neq 0, \quad \text{and} \quad T_0 := \mathbb{D}. \]

If $1 \leq p < \infty$, $s \in \mathbb{R}$, $\theta \in B_{p,t}$ and $d\mu_t = \theta d\nu_t$, then the Besov space $B^p_s(\mu_t)$ consists of holomorphic functions $f$ on $\mathbb{D}$ satisfying

\[ \|f\|_{B^p_s(\mu_t)} := \int_{\mathbb{D}} \left| (1 + R)^{k_s} f(z) \right|^p (1 - |z|^2)^{(k_s - s)p} d\mu_t(z) < \infty. \]

Here, $k_s := \min\{k \in \mathbb{N} : k > s\}$ and $R$ denotes the radial derivative.

As it happens for the unweighted case, if we replace $k_s$ by another non-negative integer $k > s$ we obtain equivalent norms (see for instance [7, Section 3]). In particular, if $s < 0$, then we can take $k = 0$, and thus we have that $B^p_s(\mu_t) = H \cap L^p(\mu_{t-\kappa p})$.

The classical unweighted Besov space $B^p_s$ corresponds to $B^p_s(\mu_0)$, where $d\mu_0(z) = \frac{d\nu(z)}{1 - |z|^2}$. Observe that this space is already included in the scale of weighted Besov spaces we have considered, simply because $B^p_s(\mu_0) = B^p_{s+\kappa p}(\nu_t)$ for any $t > 0$. In order to recover some well-known results for the unweighted case and the pairing $\langle \cdot, \cdot \rangle_0$, we write $B_{p,0} = \{1\}$.

**Definition 1.1.** The space $CB^p_s(\mu_t)$ consists of the functions $g \in B^p_s(\mu_t)$ for which

\[ \|g\|_{CB^p_s(\mu_t)} := \sup_{0 \neq f \in B^p_s(\mu_t)} \frac{\|f(1 + R)^{k_s} g\|_{B^p_{s-k_s}(\mu_t)}}{\|f\|_{B^p_s(\mu_t)}} \]

is finite.

The space $CB^p_s(\mu_t)$ can be described either in terms of Carleson measures or in terms of pointwise multipliers. Indeed,

(i) $b \in CB^p_s(\mu_t)$ if and only if the measure

\[ d\mu_b(z) := |(1 + R)^{k_s} b(z)|^p (1 - |z|^2)^{(k_s - s)p} d\mu_t(z), \]

satisfying

\[ \int_{\mathbb{D}} |b(z)|^p d\mu_t(z) < \infty. \]
is a Carleson measure for $B^p_s(d\mu_t)$, that is, if and only if the embedding $B^p_s(\mu_t) \subset L^p(d\mu_t)$ is continuous.

(ii) $b \in CB^p_s(\mu_t)$ if and only if $(1 + R)^{k_s} b \in Mult(B^p_s(\mu_t) \to B^p_{s-k_s}(\mu_t))$, where $Mult(B^p_s(\mu_t) \to B^p_{s-k_s}(\mu_t))$ denotes the space of pointwise multipliers from $B^p_s(\mu_t)$ to $B^p_{s-k_s}(\mu_t)$.

When $t = 0$, that is for the unweighted case, we simply denote the space $CB^p_s(\mu_0)$ by $CB^p_s$.

The spaces $CB^p_s$ appear naturally when dealing with some problems on operators on $B^p_s$. For instance, it is well known that $Mult(B^p_s) = H^\infty \cap CB^p_s$. In some special cases it is not difficult to give a full description of the space $CB^p_s$. If $s > 1/p$, then $B^p_s$ is a multiplicative algebra and $CB^p_s = B^p_s$. If $s < 0$, then it is easy to check that $CB^p_s$ coincides with the Bloch space $B^\infty_s$. Different type of characterizations of the spaces $CB^p_s$, for $0 \leq s \leq 1/p$, have been obtained by several authors (see for instance [14], [1], [13], [3], [7], [8] and the references therein).

One of the main results of this paper is the following theorem.

**Theorem 1.2.** Let $1 < p < \infty$, $0 < s < 1$, $t \geq 0$ and $\theta \in B_{p,t}$. For $b \in H(\mathbb{D})$ the following assertions are equivalent:

(i) $b \in CB^p_s(\mu_t)$.

(ii) $\Gamma_1(b) := \sup_{0 \neq f, g \in H(\mathbb{D})} \frac{|\langle |f|, (1 + R)b \rangle|}{\| f \|_{B^p_s(\mu_t)} \| g \|_{B^{p'}_{s'}(\mu_t)}} < \infty$.

(iii) $\Gamma_2(b) := \sup_{0 \neq f, g \in H(\mathbb{D})} \frac{|\langle f, g b \rangle|}{\| f \|_{B^p_s(\mu_t)} \| g \|_{B^{p'}_{s'}(\mu_t)}} < \infty$.

Moreover, $\| b \|_{CB^p_s(\mu_t)} \approx \Gamma_1(b) \approx \Gamma_2(b)$.

The symbol $\approx$ means here that each term is bounded by constant times the other term, with constants which do not depend on the function $b$.

If $b \in L^1(d\nu_t)$, then the small Hankel operator $h^t_b$, $t \geq 0$, is defined on $H(\mathbb{D})$ by

$$h^t_b(f)(z) := \int_{\mathbb{D}} f(w)\overline{b(w)} \frac{d\nu_t(w)}{(1 - wz)^{1+t}}, \quad t > 0,$$

$$h^0_b(f)(z) := \int_{\mathbb{T}} \frac{f(\zeta)\overline{b(\zeta)}}{1 - \zeta z} d\sigma(\zeta).$$

Notice that, by Fubini’s theorem, if $f, g \in H(\mathbb{D})$, then $\langle g, h^t_b(f) \rangle = \langle f, b \rangle_t$.

Thus, we have $\Gamma_2(b) = \| h^t_b \|_{\mathcal{L}(B^p_s(\mu_t) \to \overline{B^p_s(\mu_t)})}$.

In the above theorem we compute the norm of the bilinear forms on the product $B^p_s(\mu_t) \times B^p_{s'}(\mu_t)$. However, using that the operator $(1 + R)^{s'}$ is a bijection from $B^p_s(\mu_t)$ to $B^p_{s-s'}(\mu_t)$, that $B_{p,t} \subset B_{p,t+t_0}$, $t_0 \geq 0$ and

$$B^p_s(\mu_t) = B^p_{s+t_0/p}(\mu_t+t_0),$$

(1.3)
we can use Theorem 1.2 to compute norms of bilinear forms on products $B^p_{s_0}(\mu_{t_0}) \times B^{p'}_{s_1}(\mu'_{t_1})$ for some particular choices of the indexes $s_0$, $s_1$, $t_0$ and $t_1$. For instance, we have:

**Corollary 1.3.** Let $1 < p < \infty$, $t_0, t_1 \geq 0$, $\theta \in B_{p,t_0}$ and $s_0 \in \mathbb{R}$. For $s_1 \in \mathbb{R}$ satisfying $s_0 + s_1 < 0$ and $0 < \frac{s_0}{p'} - \frac{s_1}{p} < 1$, let $t = t_0 - s_0 - s_1$.

Then we have

$$
\|R_{t_1+t}^{t-t_1}b\|_{CB^p_{s_0/p'-s_1/p}(\mu_t)} \approx \sup_{0 \neq f, g \in H(\mathbb{D})} \frac{|\langle f, g, t \rangle|}{\|f\|_{B^p_{s_0}(\mu_{t_0})} \|g\|_{B^{p'}_{s_1}(\mu'_{t_1})}},
$$

where $R_{t_1+t}^{t-t_1}$ is a fractional differential operator of order $t - t_1$ (see (2.5)).

For $s_0, s_1 < 0$ we prove the following result:

**Theorem 1.4.** If $1 < p < \infty$, $t \geq 0$, $\theta \in B_{p,t}$ and $s_0, s_1 < 0$, then

$$
\|b\|_{B^\infty_{s_0-s_1}} \approx \sup_{0 \neq f, g \in H(\mathbb{D})} \frac{|\langle f, g, t \rangle|}{\|f\|_{B^p_{s_0}(\mu_{t_0})} \|g\|_{B^{p'}_{s_1}(\mu'_{t_1})}}.
$$

The results in Theorem 1.2 for the unweighted case are stated in a different formulation by different authors. For instance, see [13] and [15] for the case $p = 2$, and [5] for $p \neq 2$. See also the references therein. The proof of our results follow some of the ideas used in [15], modifying the Hilbert techniques valid only for the case $p = 2$ in order to cover the weighted case and $p \neq 2$. Our approach permit us to compute the norms of the bilinear form on $B^p_{s_0}(\mu_{t_0}) \times B^{p'}_{s_1}(\mu'_{t_1})$ only when $s_0 < 0$ or $s_1 < 0$. It seems more difficult to compute this norm for the cases $s_0, s_1 > 0$.

Some results for the unweighted case and $p = 2$ can be found for instance in [15] ($s_0 > s_1$) and in the recent papers [2] and [8] ($s_0 = s_1 = 1/2$).

As it happens in the unweighted case (see for instance [13], [9], [3]), from the equivalences between (i) and (ii) in Theorem 1.2, we obtain the following duality result for weak products.

**Theorem 1.5.** Let $1 < p < \infty$, $t \geq 0$ and $\theta \in B_{p,t}$. If we consider the pairing $\langle \cdot, \cdot \rangle_t$, we then have:

(i) If $0 < s < 1$, then $(B^p_{s_0}(\mu_{t_0}) \circ B^{p'}_{s_1}(\mu'_{t_1})')' \equiv CB^p_{s_0}(\mu_{t_0})$.

(ii) If $s_0, s_1 < 0$, then $(B^p_{s_0}(\mu_{t_0}) \circ B^{p'}_{s_1}(\mu'_{t_1})')' \equiv B^\infty_{-s_0-s_1}$, and consequently we have $B^p_{s_0}(\mu_{t_0}) \circ B^{p'}_{s_1}(\mu'_{t_1}) = B^1_{s_0+s_1-t}$.

The same arguments used to prove Corollary 1.3 from Theorem 1.2 combining the above theorem with (1.3), give a description of the dual of $B^p_{s_0}(\mu_{t_0}) \circ B^{p'}_{s_1}(\mu'_{t_0})$ for $s_0, s_1$ and $t_0$ satisfying the conditions in Corollary 1.3. These results cover some well-known results stated in section 5 in [9] for the unweighted case.
The paper is organized as follows. In Section 2 we give some definitions and we state some properties of the class of weights in \( B_{p,t} \) and its corresponding weighted Besov spaces. In Section 3 we obtain estimates of \( \|b\|_{CB^p_s(\mu_t)} \) which in particular give the proof of Theorem 1.4. Section 4 is devoted to the proof of Theorem 1.2 and Corollary 1.3. In Section 5 we use our previous results to prove Theorem 1.5.

2. Notations and preliminaries

Throughout this paper, the expression \( F \lesssim G \) means that there exists a positive constant \( C \) independent of the essential variables and such that \( F \leq CG \). If \( F \lesssim G \) and \( G \lesssim F \) we will write \( F \approx G \).

2.1. Differential and integral operators.

We denote the partial derivatives of first order by \( \partial := \frac{\partial}{\partial z} \) and \( \overline{\partial} := \frac{\partial}{\partial \overline{z}} \) respectively. Let \( R := z\partial \) be the radial derivative.

For \( s, t \in \mathbb{R}, t > 0 \) and \( k \) a non-negative integer, we consider the differential operator \( R^k_t \) of order \( k \) defined by
\[
R^k_t f := \left( 1 + \frac{R}{t + k - 1} \right) \cdots \left( 1 + \frac{R}{t} \right) f.
\]

If we need to specify the variable of differentiation, then we write \( \partial_z, R_z \) and \( R^k_{t,z} \), respectively.

The operators \( R^k_t \) satisfy the following formula:
\[
R^k_t \frac{1}{(1 - z\overline{w})^t} = \frac{1}{(1 - z\overline{w})^{t+k}}.
\]

Definition 2.1. For \( N > 0 \) and \( M \geq 0 \), we consider the following integral operators:
\[
\mathcal{P}^{N,M}(\varphi)(z) := \int_D \varphi(w)\mathcal{P}^{N,M}(z,w)d\nu(w), \quad \text{where} \quad \mathcal{P}^{N,M}(z,w) := N\frac{(1 - |w|^2)^{N-1}}{(1 - z\overline{w})^{1+M}}.
\]
\[
\mathcal{P}^{N,M}(\varphi)(z) := \int_D \varphi(w)\mathcal{P}^{N,M}(z,w)d\nu(w), \quad \text{where} \quad \mathcal{P}^{N,M}(z,w) := |\mathcal{P}^{N,M}(z,w)|.
\]

We extend the definition to the case \( N = 0 \) by writing
\[
\mathcal{P}^{0,M}(\varphi)(z) := \int_T \frac{\varphi(\zeta)}{(1 - z\overline{\zeta})^{1+M}}d\sigma(\zeta), \quad \mathcal{P}^{0,M}(\varphi)(z) := \int_T \frac{\varphi(\zeta)}{|1 - z\overline{\zeta}|^{1+M}}d\sigma(\zeta).
\]

If \( N = M \), then we denote \( \mathcal{P}^{N,N} \) and \( \mathcal{P}^{N,N} \) by \( \mathcal{P}^N \) and \( \mathcal{P}^N \), respectively.

For \( N \geq 0 \), we also define
\[
\mathcal{K}^{N}(\overline{\partial}\varphi)(z) := \int_D \overline{\partial}\varphi(w)\mathcal{K}^{N}(w,z)d\nu(w), \quad \text{where} \quad \mathcal{K}^{N}(w,z) := \frac{(1 - |w|^2)^N}{(1 - z\overline{w})^N} \frac{1}{w - z}.
\]
The weighted Cauchy-Pompeiu representation formula is given by:

**Theorem 2.2.** Let \( N \geq 0 \) and \( \varphi \in C^1(\overline{D}) \). Then \( \varphi(z) = \mathcal{P}^N(\varphi)(z) + \mathcal{K}^N(\overline{\partial}\varphi)(z) \).

Since \( R_{1+N}^k f = R_{1+N}^k \mathcal{P}^N(f) = \mathcal{P}^{N,N+k}(f) \), it is natural to extend the definition of \( R_{1+N}^k \) for a noninteger order by considering

\[
R_{1+N}^s := \mathcal{P}^{N,N+s}(f), \quad s, N > 0.
\]

Note that by Theorem 2.2 we have

\[
\int_D \mathcal{P}^{N+s,N}(w,z)\mathcal{P}^{N,N+s}(u,w)dv(w) = \mathcal{P}^N(u,z).
\]

Therefore, for \( s > 0 \) we can define the inverse of \( R_{1+N}^s \) by \( R_{1+N}^{-s} := \mathcal{P}^{N+s,N}(f) \).

Let us recall the following estimate.

**Lemma 2.3.** If \( q < 2 \), \( N > 0 \), \( M \neq N - q \) and \( z \in D \), then

\[
\int_D \frac{\mathcal{P}^{N,M}(w,z)}{|w-z|^q}dv(w) \lesssim (1 + (1 - |z|^2)^{N-M-q}).
\]

**Proof.** The case \( q = 0 \) is well known (see for instance [16, Lemma 4.2.2]). The case \( q \neq 0 \) can be reduced to the case \( q = 0 \) using the change of variables \( w = \varphi_z(u) := \frac{z-u}{1-uz} \). Indeed, we have

\[
\int_D \frac{\mathcal{P}^{N,M}(w,z)}{|w-z|^q}dv(w) = (1 - |z|^2)^{N-M-q} \int_D \frac{(1 - |u|^2)^{N-1}}{|1-uz|^{1+2N-M-q} |u|^q} dv(u),
\]

which ends the proof.

### 2.2. Bézouté weights.

In this section we recall some properties of the Bézouté weights \( B_{p,t} \). We refer to [4] for more details. Recall that if \( t > 0 \) and \( \theta \in B_{p,t} \), then \( d\mu_t = \theta dv_t \) and \( d\mu_t' = \theta^{-t'/t} dv_t \).

Since, for any \( w \in T_z \), \( 1 - |w|^2 \leq 4(1 - |z|^2) \), we have:

**Lemma 2.4.** If \( 1 < p < \infty \), \( 0 < t_0 < t_1 \) and \( \theta \in B_{p,t_0} \), then \( B_{p,t_1}(\theta) \subseteq B_{p,t_0}(\theta) \).

Thus, \( B_{p,t_0} \subset B_{p,t_1} \).

The next result was proved in [4, Theorem 1 and Propositions 3, 5]

**Theorem 2.5.** Let \( 1 < p < \infty \), \( t > 0 \) and let \( \theta \) be a positive locally integrable function \( \theta \) on \( D \). Then, the following assertions are equivalent:

(i) \( \theta \in B_{p,t} \).

(ii) The integral operator \( \mathbb{P}^t \) is bounded on \( L^p(d\mu_t) \).

(iii) The integral operator \( \mathbb{P}^t \) is bounded on \( L^p(d\mu_t) \).
It is well known that any weight in the Muckenhoupt class $A_p$ satisfies a doubling condition. Similarly to what happens for these classes of weights, any weight in $B_{p,t}$ satisfies a doubling type condition with respect to tents.

We also have a characterization of weights in $B_{p,t}$ in terms of the kernels $\mathbb{P}^{t,M}$, which is analogous to the one satisfied for the weights in $A_p$ (see\cite{12}, \cite{6}).

**Proposition 2.6.** Let $1 < p < \infty,$ $t > 0$ and $\theta \in B_{p,t}$. We then have:

(i) The measure $\mu_t$ satisfies the following doubling type measure condition:

if $0 < r_1 < r_2 < 1$ and $\zeta \in \mathbb{T}$, then

$$\frac{\mu_t(T_{r_2}\zeta)}{\mu_t(T_{r_1}\zeta)} \leq B_{p,t}(\theta)^p \left( \nu_t(T_{r_1}\zeta) \right)^p \approx B_{p,t}(\theta)^p \left( \frac{1 - r_1}{1 - r_2} \right)^{(1+t)p}.$$

(ii) If $M > (1 + t)(\max\{p, p'\} - 1)$, the following equivalence holds:

$$B_{p,t}(\theta) \lesssim \sup_{z \in \mathbb{D}} (1 - |z|^2)^M \mathbb{P}^{t,M}(\theta)(z) \lesssim B_{p,t}(\theta)^p \mathbb{P}^{t,M}(\theta^{-p'/p})(z)^{1/p'} \lesssim B_{p,t}(\theta).$$

**Proof.** Part (i) follows easily from Hölder’s inequality and the fact that $\theta \in B_{p,t}$. Indeed, the embedding $T_{r_2}\zeta \subset T_{r_1}\zeta$ gives

$$\nu_t(T_{r_2}\zeta) \leq \left( \int_{T_{r_2}\zeta} d\mu_t \right)^{1/p} \left( \int_{T_{r_1}\zeta} d\mu_t \right)^{1/p'} \leq \mu_t(T_{r_2}\zeta)^{1/p} B_{p,t}(\theta) \frac{\nu_t(T_{r_1}\zeta)}{(\mu_t(T_{r_1}\zeta))^{1/p}}.$$

Since $\nu_t(T_{r_2}\zeta) \approx (1 - r)^{1+t}$, we conclude the proof.

In order to prove (ii) it is enough to prove the following estimates, valid for $z \in \mathbb{D}$:

(2.6) $$\frac{\mu_t(T_z)}{\nu_t(T_z)} \lesssim (1 - |z|^2)^M \mathbb{P}^{t,M}(\theta)(z) \lesssim B_{p,t}(\theta)^p \frac{\mu_t(T_z)}{\nu_t(T_z)},$$

(2.7) $$\frac{\mu_t'(T_z)}{\nu_t(T_z)} \lesssim (1 - |z|^2)^M \mathbb{P}^{t,M}(\theta^{-p'/p})(z) \lesssim B_{p,t}(\theta^{-p'/p})^p \frac{\mu_t'(T_z)}{\nu_t(T_z)}.$$

Observe that (2.7) follows from (2.6) since $\theta \in B_{p,t}$ if and only if $\theta^{-p'/p} \in B_{p',t}$.

The estimate on the left hand side of (2.6) is valid for any $M > 0$ and $t > 0$, and follows from

$$\frac{\mu_t(T_z)}{\nu_t(T_z)} = \frac{1}{\nu_t(T_z)} \int_{T_z} \theta d\nu_t \lesssim (1 - |z|^2)^M \int_{T_z} \frac{\theta(w)}{|1 - wz|^{1+t+M}} d\nu_t(w) = (1 - |z|^2)^M \mathbb{P}^{t,M}(\theta)(z).$$

Let us prove the estimate on the right hand side of (2.6). If $z = 0$ then $T_0 = \mathbb{D}$ and thus the result is clear. If $z \neq 0$ then let $\zeta = z/|z|$ and $J_z$ the integer part of $-\log_2(1 - |z|)$. Consider the sequence $\{z_k\} \subset \mathbb{D}$ defined by

$$z_k = (1 - 2^k(1 - |z|))\zeta \text{ if } k = 0, 1, \ldots, J_z, \text{ and } z_k = 0 \text{ if } k > J_z.$$
Observe that \(z_0 = z\) and that \(1 - |z_k|^2 \approx |1 - wz|\) for \(w \in T_{z_k} \setminus T_{z_{k-1}}\). Therefore,

\[
(1 - |z|^2)^M \mathbb{P}_{t+M}(\theta)(z) = (1 - |z|^2)^M \sum_{k=0}^{J_z+1} \int_{T_{z_k} \setminus T_{z_{k-1}}} \frac{\theta(w) \, dw}{|1 - wz|^{1+M}} \lesssim \sum_{k=0}^{J_z+1} (1 - |z|^2)^M (2k(1 - |z|^2))^{1+M} \mu_t(T_{z_k}).
\]

By the doubling property \((10)\), we have

\[
\mu_t(T_{z_k}) \lesssim \mathcal{B}_{p,t}(\theta)^p \frac{(1 - |z_k|)^{(1+t)p}}{(1 - |z|)^{(1+t)p}} \mu_t(T_z) \approx \mathcal{B}_{p,t}(\theta)^p 2^{k(1+t)p} \mu_t(T_z).
\]

Since \(M > (1 + t)(p - 1)\) and \(\nu_t(T_z) \approx (1 - |z|^2)^{1+t}\) we obtain

\[
(1 - |z|^2)^M \mathbb{P}_{t+M}(\theta)(z) \lesssim \mathcal{B}_{p,t}(\theta)^p \nu_t(T_z),
\]

which concludes the proof of the right hand side estimate in \((2.6)\). \(\square\)

As a consequence of the above proposition and the estimate \(1 - |w|^2 \leq 2|1 - wz|\), we obtain:

**Corollary 2.7.** If \(1 < p < \infty, t \geq 0, N > 0, M > (1 + t + N)(\max\{p, p'\} - 1)\) and \(\theta \in \mathcal{B}_{p,t}\), then

\[
\sup_{z \in D} (1 - |z|^2)^M \left( \mathbb{P}_{t+N,t+N+M}(\theta)(z) \right)^{1/p} \left( \mathbb{P}_{t+N,t+N+M}(\theta^{t-1}/p)(z) \right)^{1/p'} \lesssim \mathcal{B}_{p,t}(\theta)^2.
\]

### 2.3. Weighted Besov spaces.

In this section we recall some properties of the weighted Besov spaces \(B^p_s(\mu_t)\) introduced in Section \([11]\).

The next result is well known for the unweighted case (see for instance \([17\) Chapters 2, 6]). The proof for the weighted Besov spaces can be done following the same arguments used to prove Theorem 3.1 in \([7\).

**Proposition 2.8.** Let \(1 < p < \infty, s \in \mathbb{R}, t \geq 0\) and \(\theta \in \mathcal{B}_{p,t}\). If \(k > s\) is a nonnegative integer, then

\[
\int_D |D^k f(z)|^p (1 - |z|^2)^{(k-s)p} \, d\mu_t(z) \quad \text{and} \quad \sum_{m=0}^{k} \int_D |\partial^m f(z)|^p (1 - |z|^2)^{(k-s)p} \, d\mu_t(z)
\]

provide equivalent norms on \(B^p_s(\mu_t)\), where \(D^k\) is either \((1 + R)^k\) or \(R^k_L\).

The next embedding relates weighted and unweighted Besov spaces.

**Lemma 2.9.** If \(1 < p < \infty, s \in \mathbb{R}, t \geq 0\) and \(\theta \in \mathcal{B}_{p,t}\), then \(B^p_s(\mu_t) \subset B^1_{s-t}\).
Proof. Since for any positive integer \( k \) we have \( B^p_k(\mu_t) = (1 + R)^{-k}B^p_{\delta-k}(\mu_t) \) and \( B^1_{s-t} = (1 + R)^{-k}B^1_{s-t-k} \), it is sufficient to prove the above embedding for \( s < 0 \).

In this case, Hölder’s inequality gives

\[
\|f\|_{B^1_{s-t}} \leq \left( \int_D |f|^p d\mu_{t-sp} \right)^{1/p} \left( \int_D d\mu'_t \right)^{1/p'},
\]

which proves the result. \( \square \)

In order to state a duality relation between weighted Besov spaces, we need the next lemma.

Lemma 2.10. The pairing \( \langle \cdot, \cdot \rangle_\delta \) defined in (1.2) satisfies that for \( f, g \in H(\Omega) \):

(i) \( \langle f, g \rangle_\delta = \langle f, R^k_{\delta+1}g \rangle_{\delta+k} = \langle R^k_{\delta+1}f, g \rangle_{\delta+k} \).

(ii) If \( \tau \in \mathbb{R} \) then we have \( \langle f, g \rangle_\delta = \langle (1 + R)^\tau f, (1 + R)^{-\tau}g \rangle_\delta \).

Proof. Let us prove (i) for \( k = 1 \), that is

\[
\langle f, g \rangle_\delta = \left( f, \left( 1 + \frac{R}{\delta + 1} \right) g \right)_{\delta+1} = \left( \left( 1 + \frac{R}{\delta + 1} \right) f, g \right)_{\delta+1}.
\]

Observe that the second equality can be deduced from the first one by conjugation.

If \( \delta = 0 \), then Stokes’ theorem gives

\[
\langle f, g \rangle_0 = \frac{1}{2\pi i} \lim_{r \to 1^-} \int_T f(r\zeta)\overline{g(r\zeta)}\zeta d\zeta = \lim_{r \to 1^-} \int_D \overline{\partial(\overline{z}f(rz)\overline{g(rz)})} \ d\nu(z)
\]

\[
= \lim_{r \to 1^-} \int_D f(rz)((1 + R)g(rz)) \ d\nu(z) = \langle f, (1 + R)g \rangle_1.
\]

The case \( \delta > 0 \) follows from the identity

\[
\delta(1 - |z|^2)\delta^{-1} = (\delta + 1)(1 - |z|^2)^\delta - \overline{\partial(\overline{z}(1 - |z|^2)\delta)}
\]

and integration by parts.

A simple iteration of these identities gives \( \square \).

Assertion (ii) follows from the facts that \( (1 + R)^\tau z^m = (1 + m)^\tau z^m \) and that \( \langle z^k, z^m \rangle_\delta = 0, k \neq m \). \( \square \)

The next result extends the well known duality \( (B^p_\theta)' \equiv B^{p'}_{-\theta} \) for the case \( t = 0 \) (see [10]).

Proposition 2.11. Let \( 1 < p < \infty, t \geq 0 \) and \( \theta \in B_{p,t} \). If \( s \in \mathbb{R} \), then, the dual of \( B^p_s(\mu_t) \) with respect to the pairing \( \langle \cdot, \cdot \rangle_t \) is the Besov space \( B^{p'}_{-s}(\mu'_t) \).
Proof. As in the unweighted case, from the duality \((L^p(\mu_t))' \equiv L^{p'}(\mu'_t)\), with respect to the pairing \(\langle \cdot, \cdot \rangle_{t+1}\), Theorem 2.2 and the Hahn-Banach theorem, we obtain
\[
\left( B^p_{-1/p}(\mu_t) \right)' = (H \cap L^p(\mu_t))' \equiv H \cap L^{p'}(\mu'_t) = B^{p'}_{-1/p'}(\mu'_t),
\]
with respect to the pairing \(\langle \cdot, \cdot \rangle_{t+1}\), and consequently with respect to the pairing \(\langle \cdot, \cdot \rangle_{t+1}\).

Next, we use the above result and Lemma 2.10 to prove the general case.

If \(g \in B^p_{-s}(\mu'_t)\) and \(f \in B^p_s(\mu_t)\), then
\[
|\langle f, g \rangle_t| = |\langle R^1_t f, g \rangle_{t+1}| \leq \|g\|_{L^{p'}(\mu_{-s'})} \|R^1_t f\|_{L^p(\mu_{(1-s)p+t})} \approx \|g\|_{B^p_{-s}(\mu_t)} \|f\|_{B^p_s(\mu_t)}.
\]
Thus, the map \(g \mapsto \langle \cdot, g \rangle_t\) is an injective map from \(B^p_{-s}(\mu'_t)\) to \(B^p_s(\mu_t)\).

Let us prove that this map is surjective. If \(\Lambda\) is a linear form on \(B^p_s(\mu_t)\), then \(\Lambda \circ (1+R)^{-s-1/p}\) is also a linear form on \(B^p_{-1/p}(\mu_t)\). Thus, there exists \(g \in B^p_{-1/p}(\mu'_t)\) such that for any \(h \in B^p_{-1/p}(\mu_t)\),
\[
\Lambda \circ (1+R)^{-s-1/p}(h) = \langle h, g \rangle_{t+1} = \langle (1+R)^{-s-1/p}h, (1+R)^{s+1/p}g \rangle_{t+1} = \langle (1+R)^{-s-1/p}h, R^1_{1+t}(1+R)^{s+1/p}g \rangle_t,
\]
where in the second identity we have used \((ii)\) in Lemma 2.10 and in the last one \((i)\) in the same lemma.

Since for any \(f \in B^p_s(\mu_t)\), we have that \(h = (1+R)^{s+1/p}(f) \in B^p_{-1/p}(\mu_t)\), we deduce that \(\Lambda(f) = \langle f, g \rangle_t\) with \(G := R^1_{1+t}(1+R)^{s+1/p}g \in B^p_{-s}(\mu_t)\).

Corollary 2.12. Let \(1 < p < \infty, t' > t \geq 0\) and \(\theta \in \mathcal{B}_{p,t}\). If \(s \in \mathbb{R}\), then
\[
(B^p_s(\mu_t))' = B^{p'}_{-s-t'}(\mu'_t)\text{ with respect to the pairing } \langle \cdot, \cdot \rangle_{t'}.
\]
In particular, if \(t = 0\), then \((B^p_s)\' = B^{p'}_{-s-t'}\) with respect to the pairing \(\langle \cdot, \cdot \rangle_0\).

Proof. By the above proposition, we have
\[
(B^p_s(\mu_t))' = \left( B^p_{s+(t-t')/p}(\mu_{t'}) \right)' \equiv \left( B^{p'}_{-s-(t-t')/p}(\mu_{t'}) \right) = \left( B^{p'}_{-s+t-t'}(\mu'_t) \right)
\]
which ends the proof. \(\square\)

3. Estimates of \(\|b\|_{CB^p_s(\mu_t)}\) and proof of Theorem 1.3.4

We introduce a variation in the definition of the constants \(\Gamma_1(b)\) and \(\Gamma_2(b)\) in Theorem 1.2 which allow us to cover some general situations.

Definition 3.1. If \(1 < p < \infty, s_0, s_1 \in \mathbb{R}, t \geq 0, \theta \in \mathcal{B}_{p,t}\) and \(b \in H\), then
\[
\Gamma_3(b) = \Gamma(b, p, s_0, s_1, t) := \sup_{0 \neq f, g \in H(\mathcal{F})} \frac{|\langle f, g \rangle_t|}{\|f\|_{B^p_{s_0}(\mu_t)} \|g\|_{B^p_{s_1}(\mu'_t)}}.
\]
We will start proving the following theorem.

**Theorem 3.2.** Let $1 < p < \infty$, $s_0, s_1 \in \mathbb{R}$, $t \geq 0$ and $\theta \in B_{p,t}$. Then $\|b\|_{B_{s_0-s_1}^{\infty}} \lesssim \Gamma_3(b)$.

If $s_0, s_1 < 0$, then the converse inequality holds.

The proof of this result will be a consequence of Lemmas 3.4 and 3.6.

**Lemma 3.3.** Let $1 < p < \infty$, $s_0, s_1 \in \mathbb{R}$, $t \geq 0$ and $\theta \in B_{p,t}$. Let

$$\tau > \lambda := (1 + t)(\max\{p, p'\} - 1) + \max\{0, -s_0p, -s_1p'\}.$$  

For $z \in \mathbb{D}$, we consider the functions

$$f_z(w) = \frac{1}{(1 - w\bar{z})^{(1 + t + \tau)/p}} \quad \text{and} \quad g_z(w) = \frac{1}{(1 - w\bar{z})^{(1 + t + \tau)/p'}},$$

Then

$$\|f_z\|_{B^p_{s_0}(\mu_t)} \|g_z\|_{B^{p'}_{s_1}(\mu_t')} \lesssim B_{p,t}(\theta)^2 (1 - |z|^2)^{\tau - s_0 - s_1}.$$  

**Proof.** If $m > s_0$ is a non-negative integer, then

$$\|f_z\|_{B^p_{s_0}(\mu_t)} \approx \int_{\mathbb{D}} \frac{(1 - |w|^2)^{t + (m - s_0)p - 1}}{|1 - wz|^{1 + t + \tau + mp}} \theta(w) d\nu(w).$$

Analogously, if $m > s_1$, then

$$\|g_z\|_{B^{p'}_{s_1}(\mu_t')} \approx \int_{\mathbb{D}} \frac{(1 - |w|^2)^{t + (m - s_1)p' - 1}}{|1 - wz|^{1 + t + \tau + mp'}} \theta^{p'/p}(w) d\nu(w).$$

Therefore, if $N, M$ satisfy $0 < N < \min\{(m - s_0)p, (m - s_1)p'\}$ and $(1 + t + N)(\max\{p, p'\} - 1) < M < \min\{k\tau + s_0p, \tau + s_1p'\}$, then the estimate $1 - |z|^2 \leq 2(1 - w\bar{z})$ and Corollary 2.7 give

$$\|f_z\|_{B^p_{s_0}(\mu_t)} \|g_z\|_{B^{p'}_{s_1}(\mu_t')} \lesssim (1 - |z|^2)^{M - \tau - s_0 - s_1} \left( \frac{1}{M^{t+N,t+N+M}} \left( \frac{1}{M^{t+N,t+N+M}} \right) \right)^{1/p} \lesssim B_{p,t}(\theta)^2 (1 - |z|^2)^{\tau - s_0 - s_1},$$

which ends the proof. \(\square\)

**Lemma 3.4.** Let $1 < p < \infty$, $s_0, s_1 \in \mathbb{R}$, $t \geq 0$, $\theta \in B_{p,t}$ and $b \in H$. Then $\|b\|_{B_{s_0-s_1}^{\infty}} \lesssim \Gamma_3(b)$.

**Proof.** We want to prove that for some positive integer $k$, we have

$$\|b\|_{B_{s_0-s_1}^{\infty}} \approx \sup_{z \in \mathbb{D}} (1 - |z|^2)^{k + s_0 + s_1} |R_{1+t}^k b(z)| \lesssim \Gamma_3(b).$$
By Cauchy formula, we have
\[
R^k_{1+t}b(z) = t \lim_{r \to 1} R^k_{1+t} \int_D b(rw) \frac{(1 - |w|^2)^{t-1}}{(1 - rzw)^{1+t+k}} d\nu(w) \\
= t \lim_{r \to 1} \int_D b(rw) \frac{(1 - |w|^2)^{t-1}}{(1 - rzw)^{1+t+k}} d\nu(w).
\]

Assume that \( k \) is a positive integer satisfying (3.8), and let
\[
f_z(w) = \frac{1}{(1 - wz)^{(1+t+k)/p}} \quad \text{and} \quad g_z(w) = \frac{1}{(1 - wz)^{(1+t+k)/p'}}.
\]

Since \( |R^k_{1+t}g(z)| = |⟨f_z, g_b⟩| \), Lemma 3.3 gives
\[
|R^k_{1+t}b(z)| \leq \Gamma_3(b) \|f_z\|_{B^p_{s_0}(\mu_t)} \|g_z\|_{B^p'_{s_1}(\mu_t')} \lesssim \Gamma_3(b)(1 - |z|^2)^{-k-s_0-s_1},
\]
which concludes the proof.

\[\square\]

**Corollary 3.5.** Let \( 1 < p < \infty \) and \( 0 < s < 1 \). If \( b \) satisfies condition (iii) in Theorem 1.2, that is \( \Gamma_3(b, p, s, -s, t) < \infty \), then \( b \in B^p_s(\mu_t) \cap B^\infty_0 \).

**Proof.** The above lemma gives \( b \in B^\infty_0 \). The fact that \( b \in B^p_s(\mu_t) \) follows from the estimate \( |⟨g, b⟩| \leq C_b\|1\|_{B^r_p(\mu_t)} \|g\|_{B^p'_{s_1}(\mu_t')} \) and the duality result in Proposition 2.11. \(\square\)

**Lemma 3.6.** If \( 1 < p < \infty \) and \( s_0, s_1 < 0 \), then \( \Gamma_3(b) \lesssim \|b\|_{B^\infty_{-s_0-s_1}} \).

**Proof.** Let \( k \) be a positive integer such that \( k > -s_0 - s_1 \). Then
\[
|⟨f, b⟩| = |⟨f, R^k_{1+t}b⟩| \lesssim \|b\|_{B^\infty_{-s_0-s_1}} \|f\|_{L^1(\theta dw_{-s_0-s_1})} \\
\leq \|b\|_{B^\infty_{-s_0-s_1}} \|f\|_{L^p(\theta dw_{-s_0-s_1})} \|g\|_{L^{p'}(\theta^{-1} dw_{1-p'})} \\
\approx \|b\|_{B^\infty_{-s_0-s_1}} \|f\|_{B^p_{s_0}(\mu_t)} \|g\|_{B^p'_{s_1}(\mu_t')},
\]
which ends the proof. \(\square\)

**Proof of Theorem 1.4.** The proof is an immediate consequence of Lemmas 3.4 and 3.6. \(\square\)

**Theorem 3.7.** Let \( 1 < p < \infty \), \( s < 1 \), \( t \geq 0 \) and \( \theta \in \mathcal{B}_{p,t} \). Then, \( CB^p_s(\mu_t) \subset B^p_s(\mu_t) \cap B^\infty_0 \). If \( s < 0 \), then \( CB^p_s(\mu_t) = B^\infty_0 \).
Proof. The first inclusion follows from the same arguments used to prove Lemma 3.4. For a non-negative integer $k > s$ which we precise later, we have

\[
|R_{1+t+(1-s)p}^k (I + R)b(z)| = |R_{1+t+(1-s)p}^k D^{t+(1-s)p}((I + R)b)(z)| = |D^{t+(1-s)p,t+(1-s)p+k}((I + R)b)(z)|
\]

\[
\lesssim \left( \int_D \frac{(1 - |w|^2)^{(1-s)p+t-1} |(I + R)b(w)|^p}{|1 - w|^1 + t + (1-s)p + k} \theta(w) d\nu(w) \right)^{1/p} 
\]

\[
\cdot \left( \int_D \frac{(1 - |w|^2)^{(1-s)p+t-1}}{|1 - w|^1 + t + (1-s)p + k} \theta^{-p'/p}(w) d\nu(w) \right)^{1/p'}
\]

\[
\lesssim \|b\|_{CB^p_s(\mu_t)} \left( \int_D \frac{(1 - |w|^2)^{(1-s)p+t-1} \theta}{|1 - w|^1 + t + (1-s)p + k} \right)^{1/p} \cdot \left( \int_D \frac{(1 - |w|^2)^{(1-s)p+t-1}}{|1 - w|^1 + t + (1-s)p + k} \theta^{-p'/p}(w) d\nu(w) \right)^{1/p'}
\]

\[
\leq \|b\|_{CB^p_s(\mu_t)} (1 - |z|^2)^{-1} \left( \frac{\theta}{1 + t + (1-s)p + k} \theta^{-p'/p}(z) \right)^{1/p'}
\]

If $k > (1 + t + (1-s)p)(\max\{p, p'\} - 1)$, then Corollary 2.7 with $N = (1-s)p$ and $M = k$, gives $|R_{1+t+(1-s)p}^k (I + R)b(z)| \lesssim \|b\|_{CB^p_s(\mu_t)} (1 - |z|^2)^{-1-k}$ which proves that $b \in B^\infty_{B_0}$.

Next, if $s < 0$, then we have $k_s = 1$ and the inequality $\|b\|_{CB^p_s(\mu_t)} \lesssim \|b\|_{B^\infty_{B_0}}$ follows from

\[
\int_D |f(z)|^p |(1 + R)b(z)|^p (1 - |z|^2)^{(1-s)p} d\mu_t(z) \lesssim \|b\|_{B^\infty_{B_0}}^p \|f\|_{B^p_s(\mu_t)}^p,
\]

which concludes the proof.

\[\square\]

Remark 3.8. Observe that if $0 < s < 1$, $0 < \varepsilon < 1 - s$ and $\|g\|_{B^\infty_{s+\varepsilon-1}} < \infty$, then

\[
\|gf\|_{B^p_{s-\varepsilon-1}(\mu_t)} \lesssim \|g\|_{B^\infty_{s+\varepsilon-1}} \|f\|_{B^p_{s-1}(\mu_t)} \lesssim \|g\|_{B^\infty_{s+\varepsilon-1}} \|f\|_{B^p_{s-1}(\mu_t)}.
\]

Therefore, $g \in \text{Mult}(B^p_s(\mu_t) \rightarrow B^p_{s-1}(\mu_t))$. In particular,

\[
B^\infty_{B_0} \subset B^\infty_{s+\varepsilon-1} \subset \text{Mult}(B^p_s(\mu_t) \rightarrow B^p_{s-1}(\mu_t)).
\]

This gives that $g \in CB^p_s(\mu_t)$ if and only if for some (any) $l > 0$, $(l + R)g \in \text{Mult}(B^p_s(\mu_t) \rightarrow B^p_{s-1}(\mu_t))$. 

4. Proof of Theorem 1.2 and Corollary 1.3

4.1. Proof of (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) in Theorem 1.2.

The fact that (i) $\Rightarrow$ (ii) is a consequence of Hölder’s inequality. Indeed, since $0 < s < 1$, we have

$$\langle |fg|, |1 + Rb| \rangle_{t+1} \leq \|g\|_{L_p^s(\partial^s D_{D}, d\nu_{1-s})} \|f(1 + R)b\|_{L_p^t(\partial^t D_{D}, d\nu_{1-t})}$$

$$\leq \|g\|_{B_s^p(\mu_t)} \|f\|_{B_t^p(\mu_t)} \|b\|_{CB^p(\mu_t)}.$$

Clearly (ii) $\Rightarrow$ (iii) is a consequence of Lemma 2.10 (i). Indeed, if $\langle |fg|, |1 + Rb| \rangle_{t+1} < \infty$ for any $f, g \in H(D)$, then by Corollary 3.5 (see also Remark 3.8) we have $\langle |fg|, |R_{t+1}^1 b| \rangle_{t+1} < \infty$. Thus

$$\langle fg, b \rangle_t = \langle fg, R_{t+1}^1 b \rangle_{t+1} \leq \langle |fg|, |R_{t+1}^1 b| \rangle_{t+1},$$

which concludes the proof.

Observe that if $b \in CB^p_s(\mu_t)$, the above estimates give

$$(4.9) \quad \langle fg, b \rangle_t \leq \|b\|_{CB^p_s(\mu_t)} \|f\|_{B^p_t(\mu_t)} \|g\|_{B^p_s(\mu_t)}.$$

Thus we have $\Gamma_2(b) \leq \Gamma_1(b) \leq \|b\|_{CB^p_s(\mu_t)}$.

4.2. Proof of (iii) $\Rightarrow$ (i) in Theorem 1.2 for the unweighted case $t = 0$.

In the next proposition we use Corollary 3.5 and the weighted Cauchy-Pompeiu’s formula, to give an easy proof of (iii) $\Rightarrow$ (i) in Theorem 1.2 for the unweighted case $t = 0$. This last case has been proved using different methods in [13] for $p = 2$ and in [5] for any $p > 1$. Our approach follows the techniques in [15].

**Proposition 4.1.** Let $1 < p < \infty$ and $0 < s < 1$. Assume that $b \in H$ satisfies $\langle fg, b \rangle_0 \leq C_b \|f\|_{B^p_s} \|g\|_{B^p_s}$ for any $f, g \in H(D)$. Then $b \in CB^p_s$.

**Proof.** By Lemma 2.9 we have $b \in B^p_s \subset B^p_0$. Therefore, for $f \in H(D)$, the weighted Cauchy-Pompeiu’s representation formula in Theorem 2.2 gives

$$(4.10) \quad (1 + R)b(z) = \mathcal{P}^1((1 + R)b\overline{f}) + \mathcal{K}^1((1 + R)b\overline{f}).$$

In order to prove this proposition it is enough to show that the $L^p(d\nu_{1-s})$-norms of the two terms in the right hand side in (4.10) are bounded by a constant times $\|f\|_{B^p_s}$.

The first term $h = \mathcal{P}^1((1 + R)b\overline{f})$ is a holomorphic function on $D$. Thus, by Corollary 2.12 it suffices to prove that $\langle h, g \rangle_1 \leq C \|f\|_{B^p_s} \|g\|_{B^p_s}$ for any $g \in H(D)$.

By Lemma 2.10 this follows from $\langle h, g \rangle_1 = \langle (1 + R)b, fg \rangle_1 = \langle b, fg \rangle_0$ and the hypotheses.
In order to estimate the $L^p(dν(1−s)p)$-norm of $K^1((1 + R)b\partial f)$, note that by Corollary 3.5 we have $b \in B^\infty_0$. This fact, Hölder’s inequality and the estimates of Lemma 2.3, with $\varepsilon > 0$ small enough to be chosen later on, we have

\[ |K^1((1 + R)b\partial f)(z)|^p \leq \|b\|_{B^\infty_0}^p \left( \int_D \frac{|\partial f(w)|}{|1 -zw||w - z|} d\nu(w) \right)^p \]

\[ \leq \|b\|_{B^\infty_0}^p \int_D \frac{|\partial f(w)|^p(1 - |w|^2)^{(1-s)p-1}}{|1 -zw|^{(1-2s)p}|w - z|^1} d\nu(w) \left( \int_D \frac{(1 - |w|^2)^{sp'-1}}{|1 -zw|^{2sp'}|w - z|^1} d\nu(w) \right)^{p/p'} \]

\[ \lesssim \|b\|_{B^\infty_0}^p \int_D |\partial f(w)|^p(1 - |w|^2)^{(1-s)p-1} |w - z|^{-sp}. \]

Therefore, if $0 < \varepsilon < \min\{s, 1 - s\}$, then the above estimate, Fubini’s theorem and Lemma 2.3 give

\[ \|K^1((1 + R)b\partial f)\|_{L^p(d\nu(1-s)p)} \lesssim \|b\|_{B^\infty_0} \|\partial f\|_{L^p(d\nu(1-s)p)} \lesssim \|b\|_{B^\infty_0} \|f\|_{B^{s,p}}, \]

which ends the proof.

\[ \square \]

### 4.3. Proof of (iii) $\Rightarrow$ (i) in Theorem 1.2 for the general case.

Observe that if we use the same arguments of the above section to prove the unweighted case, then in the estimate of $K^{t+1}((1 + R)b\partial f)$ we will end up with integrals of the type

\[ \int_D \frac{(1 - |w|^2)^{N-1}}{|1 -zw|^{1+M}|w - z|} \theta(w)d\nu(w), \]

which are difficult to estimate because we do not have precise information on $\theta$ near the diagonal $z = w$. One method to avoid this difficulty is based in the use of the following modification of the Cauchy-Pompeiu’s formula, which on one hand avoid the singularity on the diagonal and in other hand increases the power of $(1 - |w|^2)$.

**Lemma 4.2.** Let $t > 0$, $b \in B^\infty_0$ and $f \in H(D)$. For any integer $m \geq 2$, we have

\[ K^{t+1}((1 + R)b\partial f) = K^{t+m}_0((1 + R)b\partial^2 f) + K^{t+m-1}_1((1 + R)b\partial f) + \sum_{j=1}^{m-1} Q^{t+j}((1 + R)b\partial f), \]
where
\[ \mathcal{K}^{t+m}_0((1+R)b\overline{\partial f})(z) := -\int_D \frac{((1+R)b\overline{\partial f})(w)\overline{w-z}}{(1-z\overline{w})^{t+m}} \, d\nu_{t+m+1}(w), \]
\[ \mathcal{K}^{t+m-1}_1((1+R)b\overline{\partial f})(z) := (t+m) \int_D \frac{((1+R)b\overline{\partial f})(w)(w-z)}{(1-z\overline{w})^{t+m+1}} \, d\nu_{t+m}(w), \]
\[ Q^{t+j}((1+R)b\overline{Rf})(z) := \int_D ((1+R)b\overline{Rf})(w) \frac{d\nu_{t+j+1}(w)}{(1-z\overline{w})^{t+j+1}}. \]

**Proof.** Recall that
\[ \mathcal{K}^t(w, z) = \frac{(1-|w|^2)^t}{(1-z\overline{w})^t} \frac{1}{w-z}. \]
Since \( 1 = \frac{1-|w|^2}{1-z\overline{w}} + \frac{m(w-z)}{1-z\overline{w}}, \) we have
\[ \mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z) = \mathcal{K}^{t+2}((1+R)b\overline{\partial f})(z) + Q^{t+1}((1+R)b\overline{Rf})(z). \]
Iterating this formula, we obtain
\[ \mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z) = \mathcal{K}^{t+m}((1+R)b\overline{\partial f})(z) + \sum_{j=1}^{m-1} Q^{t+j}((1+R)b\overline{Rf})(z). \]

An easy computation shows that
\[ \mathcal{K}^{t+m}(w, z) = \frac{(1-|w|^2)^{t+m}}{(1-z\overline{w})^{t+m}} \frac{1}{w-z} \]
\[ = \partial_w \left( \frac{(1-|w|^2)^{t+m}}{(1-z\overline{w})^{t+m}} \frac{w-z}{w-z} \right) + (t+m) \frac{(1-|w|^2)^{t+m-1}}{(1-z\overline{w})^{t+m+1}} (w-z). \]

Fixed \( z \in \mathbb{D} \) and \( 0 < \varepsilon < 1 - |z|, \) let \( \Omega_{z, \varepsilon} := \mathbb{D} \setminus \{ w \in \mathbb{D} : |w-z| < \varepsilon \}. \) If we apply Stokes’ theorem to the region \( \Omega_{z, \varepsilon} \) and let \( \varepsilon \to 0, \) we obtain
\[ \mathcal{K}^{t+m}((1+R)b\overline{\partial f})(z) \]
\[ = -\int_D ((1+R)b\overline{\partial f})(w) \frac{(1-|w|^2)^{t+m}}{(1-z\overline{w})^{t+m}} \frac{w-z}{w-z} \, d\nu(w) \]
\[ + (t+m) \int_D ((1+R)b\overline{\partial f})(w) \frac{(1-|w|^2)^{t+m-1}}{(1-z\overline{w})^{t+m+1}} (w-z) \, d\nu(w), \]
which concludes the proof. \( \square \)

**Proposition 4.3.** Let \( 1 < p < \infty, \) \( 0 < s < 1, \) \( t > 0, \) \( b \in B_0^\infty, \) \( f \in H(\mathbb{D}) \) and \( \varphi_f(w) := |\partial^2 f(w)|(1-|w|^2)^{2-s} + |\partial f(w)|(1-|w|^2)^{1-s}. \)

Then we have
\[ |\mathcal{K}^{t+1}((1+R)b\overline{\partial f})(z)| \lesssim \| b \|_{B_0^\infty} P^{t+s,t+1}(\varphi_f)(z). \]
Therefore, if \( \theta \in B_{p,t} \), then
\[
(4.12) \quad \|(1 - |z|^2)^{1-s}K^{t+1}((1 + R)b\overline{\partial f})(z)\|_{L^p(\mu_t)} \lesssim \|b\|_{B^p_{\infty}} \|f\|_{B^p_{\infty}(\mu_t)}.
\]

**Proof.** The pointwise estimate (4.11) follows from Lemma 4.2. Since \( 1 - |w|^2 \leq 2|1 - zw| \) and \( |z - w| \leq |1 - zw| \), then for \( m \geq 3 \), we have
\[
|K^{t+1}((1 + R)b\overline{\partial f})(z)| \lesssim \|b\|_{B^p_{\infty}} \left( \mathbb{P}^{t+m-2+s,t+m-1}(\varphi_f)(z) + \sum_{j=1}^{m-1} \mathbb{P}^{t+j+s-1,t+j}(\varphi_f)(z) \right)
\]
and thus
\[
\|\mathbb{P}^{t+s}(\varphi_f)\|_{L^p(\mu_t)} \lesssim \|\varphi_f\|_{L^p(\mu_t)} \lesssim \|f\|_{B^p_{\infty}(\mu_t)},
\]
which is a consequence of Theorem 2.5 and Proposition 2.8.

Now we can prove \( \text{(iii)} \implies \text{[i]} \) in Theorem 1.2.

**Proposition 4.4.** If \( b \) satisfies condition \( \text{(iii)} \) in Theorem 1.2, then \( b \in CB_s^p(\mu_t) \).

**Proof.** We want to prove that
\[
\int_{\mathbb{D}} |f(z)|^p |R^1_{t+1} b(z)|^p (1 - |z|^2)^{(1-s)p} d\mu_t(z) \lesssim C_b \|f\|_{B^p_{\infty}(\mu_t)}^p.
\]
To do so, by the Cauchy-Pompeiu’s formula in Theorem 2.2
\[
(4.13) \quad R^1_{t+1} b(z)\overline{f} = \mathcal{P}^{t+1}(R^1_{t+1} b\overline{f}) + K^{t+1}(R^1_{t+1} b\overline{f}),
\]
we will show that the two terms in the right hand side in (4.13) are both in \( L^p(\theta d\nu_{(1-s)p+t}) \) and that these norms are bounded up to a constant by \( \|f\|_{B^p_{\infty}(\mu_t)} \).

Since \( h = \mathcal{P}^{t+1}(\overline{f} R^1_{t+1} b) \) is a holomorphic function on \( \mathbb{D} \), the norm estimate of \( h \) is similar to the one for the unweighted case. Indeed, for \( g \in H(\mathbb{D}) \) Lemma 2.10 gives
\[
|\langle h, g \rangle_{t+1}| = |\langle R^1_{t+1} b, fg \rangle_{t+1}| = |\langle b, fg \rangle_t| \leq \Gamma_2(b) \|f\|_{B^p_{\infty}(\mu_t)} \|g\|_{B^p_{\infty}(\mu'_t)}
\]
which, by Corollary 2.12, proves that \( \|h\|_{L^p(\theta d\nu_{(1-s)p+t})} \leq \Gamma_2(b) \|f\|_{B^p_{\infty}(\mu_t)} \).

Using the \( L^p(\theta d\nu_{(1-s)p+t}) \)-norm estimate of \( K^{t+1}(R^1_{t+1} b\overline{f}) \) given in Proposition 4.3, we conclude the proof.
4.4. Proof of Corollary 1.3. Using $B^p_{\delta} (\mu_\delta) = B^p_{\delta + \tau/p} (\mu_{\delta + \tau})$, for $\tau > 0$, we will deduce the result from Theorem 1.2.

Let $1 < p < \infty$, $s_0, s_1 \in \mathbb{R}$, $t_0 \geq 0$ and $\theta \in B_{p,t_0}$. Then, for $t = t_0 - s_0 - s_1 > t_0$ we have

$$B^p_{s_0} (\mu_{t_0}) = B^p_{s_0 + (-s_0 - s_1)/p} (\mu_{t}) = B^p_{s_0/p - s_1/p} (\mu_{t}),$$ and

$$B^{p'}_{s_1} (\mu'_{t_0}) = B^{p'}_{s_1/p - s_0/p} (\mu'_t).$$

Moreover, since $\langle fg, b \rangle_{t_1} = \langle P^{t_1} (fg), b \rangle_{t_1} = \langle fg, P^{t_1} b \rangle_{t}$, we have

$$\frac{\|\langle fg, b \rangle_{t_1} \|}{\|f\|_{B^p_{s_0} (\mu_{t_0})} \|g\|_{B^{p'}_{s_1} (\mu'_{t})}} = \frac{\|\langle fg, P^{t_1} b \rangle_{t} \|}{\|f\|_{B^p_{s_0/p - s_1/p} (\mu_{t})} \|g\|_{B^{p'}_{s_1/p - s_0/p} (\mu'_t)}}.$$

Thus, Theorem 1.2 with $0 < s := s_0/p' - s_1/p < 1$, gives

$$\|P^{t_1} b\|_{CB^p_{s_0/p - s_1/p} (\mu_{t})} \approx \sup_{0 \neq f, g \in H(D)} \frac{\|\langle fg, b \rangle_{t_1} \|}{\|f\|_{B^p_{s_0} (\mu_{t_0})} \|g\|_{B^{p'}_{s_1} (\mu'_{t})}},$$

which concludes the proof.

5. Proof of Theorem 1.5.

We will determine the predual of $CB^p_s (\mu_t)$ generalizing some results for the unweighted case (see for instance [13], [3], [9] and the references therein).

5.1. Weak products and the predual of $CB^p_s (\mu_t)$.

Definition 5.1. Given two Banach spaces $X$ and $Y$ of holomorphic functions on $D$, let $X \odot Y$ be the completion of finite sums $h = \sum_{j=1}^{M} f_j g_j$, $f_j \in X$, $g_j \in Y$, using the norm

$$\|h\|_{X \odot Y} := \inf \left\{ \sum_{k=1}^{N} \|\tilde{f}_k\|_X \|\tilde{g}_k\|_Y : \sum_{k=1}^{N} \tilde{f}_k \tilde{g}_k = h \right\}.$$ 

The following well-known proposition, whose proof follows from the own definitions, will be used to prove our duality results.

Proposition 5.2. The norm of a linear form $\Lambda$ on $X \odot Y$ coincides with the norm of the bilinear form on $X \times Y$ on defined by $\tilde{\Lambda}(f, g) = \Lambda(f g)$. 

5.2. Proof of Theorem 1.5.

Proof. The embedding \( i : B^p_{s_0}(\mu_t) \rightarrow B^p_s(\mu_t) \cap B^p_{s_1}(\mu'_t) \), shows that any linear form \( \Lambda \in (B^p_s(\mu_t) \cap B^p_{s_1}(\mu'_t))^' \) produces a linear form \( \Lambda_i = \Lambda \circ i \) on \( B^p_{s_0}(\mu_t) \), which by Proposition 2.11 can be expressed as \( \Lambda_i(f) = \langle f, b \rangle_t \), for some \( b \in B^p_s(\mu_t) \).

Consequently, \( \Lambda(h) = \langle h, b \rangle_t \) for \( h \in H(\mathbb{D}) \). Since \( H(\mathbb{D}) \) is dense in both spaces \( B^p_s(\mu_t) \) and \( B^p_{s_0}(\mu_t) \), then it is also dense in \( B^p_s(\mu_t) \cap B^p_{s_1}(\mu'_t) \), and thus the norm of \( \Lambda \) coincides with the norm of the bilinear form \( (f, g) \rightarrow \langle f g, b \rangle_t \) on \( B^p_s(\mu_t) \times B^p_{s_1}(\mu'_t) \).

Therefore, the equivalence between (i) and (iii) in Theorem 1.2 concludes the proof.

The same arguments used in the first part show that the norm of a linear form \( \Lambda \) on \( B^p_{s_0}(\mu_t) \cap B^p_{s_1}(\mu'_t) \) is equivalent to the norm of the bilinear form \( (f, g) \rightarrow \langle f g, b \rangle_t \), where \( b \in B^p_{s_0}(\mu_t) \). By Theorem 3.3, this norm is equivalent to \( \|b\|_{\mathcal{B}^\infty_{\infty_0-s_1}} \) which proves the first statement.

The second statement follows from the computation by duality of the norms \( \|h\|_{B_{s_0+s_1-t}^1} \) and \( \|h\|_{B_{s_0}^p(\mu_t) \cap B_{s_1}^p(\mu'_t)} \). Indeed, if \( h \in H(\mathbb{D}) \), then

\[
\|h\|_{B_{s_0+s_1-t}^1} \approx \sup_{0 \neq b \in B^\infty_{s_0-s_1}} \frac{|\langle h, b \rangle_t|}{\|b\|_{B^\infty_{s_0-s_1}}} \approx \|h\|_{B_{s_0}^p(\mu_t) \cap B_{s_1}^p(\mu'_t)}.
\]

Since \( h \in H(\mathbb{D}) \) is dense in both spaces, we obtain the result. \( \square \)

5.3. Further remarks. Combining Theorem 1.5 with (1.3) we can obtain characterizations of weak products of type \( B^p_{s_0}(\mu_t) \cap B^p_{s_1}(\mu'_t) \) which generalize some of the results stated in Section 5 in [13].

For instance, if \( 0 < s < p \), then

\[
\left( B^p_{0}(\mu_t) \cap B^p_{s_1}(\mu'_t) \right)' = \left( B^p_{s_0}(\mu_{t+s}) \cap B^p_{s_1}(\mu_{t+s}) \right)' \equiv CB^p_{s_0}(\mu_{t+s}) = CB^p_0(\mu_t),
\]

with respect to the pairing \( \langle \cdot, \cdot \rangle_{t+s} \).

Observe that in the particular case \( p = 2 \) and \( t = 0 \), we have \( CB^2_0 = BMOA \equiv (H^1_2)', \) with respect to the pairing \( \langle \cdot, \cdot \rangle_s \). Therefore, the above duality result and the fact that \( B^2_1 = H^2 \) give \( H^2 \cap H^2_1 = H^1_2 \).

This unweighted weak factorization result can be generalized to the case \( 1 < p < 2 \). In this case \( B^p_0 \subset H^p \), and we have that \( CB^p_0 = F^{1,p}_0 \), where \( F^{1,p}_0 \) denotes the Triebel-Lizorkin space of holomorphic functions on \( \mathbb{D} \) such that the measure \( d\mu_g(z) = |\partial g(z)|^p(1 - |z|^2)^{p-1} \) is a Carleson measure for \( H^p \), that is \( \mu_g(T_z) \lesssim (1 - |z|^2) \) for any \( z \in \mathbb{D} \) (see [11], p.178). Since \( F^{\infty,p} \equiv (F^{1,p}_0)', \) with respect to the pairing \( \langle \cdot, \cdot \rangle_s \), we have \( B^p_0 \cap B^\infty_{-s} = F^{1,p}_1 \). Here, \( F^{1,p}_s \) is the Triebel-Lizorkin space of holomorphic
functions $g$ on $\mathbb{D}$ satisfying

$$
\int_T \left( \int_{|1-\zeta|^2 < 1-|w|^2} |g(w)|^{p'} \left( 1 - |w|^2 \right)^{sp'-2} d\nu(w) \right)^{1/p'} d\sigma(\zeta) < \infty.
$$

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