AUSLANDER-REITEN CONJECTURE FOR NON-GORENSTEIN COHEN-MACaulay RINGS

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Abstract. Let $R$ be a Cohen-Macaulay local ring and $Q$ be a parameter ideal of $R$. Due to M. Auslander, S. Ding, and Ø. Solberg, the Auslander-Reiten conjecture holds for $R$ if and only if it holds for the residue ring $R/Q$. In the former part of this paper, we study the Auslander-Reiten conjecture for the ring $R/Q^{\ell}$ in connection with that for $R$ and prove the equivalence of them for the case where $R$ is Gorenstein and $\ell \leq \dim R$. In the latter part, we study the existence of Ulrich ideals and generalize the result of maximal embedding dimension by J. Sally. Due to these two of our results, we finally show that the Auslander-Reiten conjecture holds if there is an Ulrich ideal whose residue ring is a complete intersection. Besides them, we also explore the Auslander-Reiten conjecture for determinantal rings.

1. Introduction

The purpose of this paper is to study the vanishing of Ext modules. The vanishing of homology plays a very important role in the study of rings and modules. The Auslander-Reiten conjecture and several related conjectures are problems about the vanishing. For a guide to these conjectures, one can consult [7, Appendix A] and [6, 16, 24, 25]. These conjectures originate from the representation theory of rings. However, interesting results also have been developed from the theory of commutative rings; see, for examples, [1, 19, 20, 21]. Let us recall the Auslander-Reiten conjecture over a commutative Noetherian ring $R$.

Conjecture 1.1. [3] Let $M$ be a finitely generated $R$-module. If $\text{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$, then $M$ is a projective $R$-module.

Although a lot of partial results on the Auslander-Reiten conjecture are known, in this paper, we are especially interested in the following one; see [11 Theorem 3.1], [21 Proposition 1.9.], and [19 Theorem 0.1].

Theorem 1.2. (Araya, Auslander-Ding-Solberg, Huneke-Leuschke) Suppose that $R$ is a Gorenstein ring which is a complete intersection in codimension one. Then the Auslander-Reiten conjecture holds for $R$. In particular, the Auslander-Reiten conjecture holds for complete intersections and Gorenstein normal domains.

As is well known, non-zerodivisors preserve the Auslander-Reiten conjecture (Proposition 2.1). Hence, through Theorem 1.2 many Gorenstein rings which satisfy the Auslander-Reiten conjecture are given. Even if a given local ring is not Gorenstein, the conjecture still holds if the ring is Golod or almost Gorenstein (21 Proposition 1.4.) and
However, Golod rings and almost Gorenstein rings do not form the best possible classes of rings that satisfy the Auslander-Reiten conjecture. In fact, if a local ring \((R, \mathfrak{m})\) is a Golod ring (resp. a non-Gorenstein almost Gorenstein ring) and \(x \in \mathfrak{m}\) is a non-zerodivisor of \(R\), then the ring \(R/(x^n)\) is no longer Golod (resp. almost Gorenstein), where \(n > 1\) \([14, \text{Theorem } 3.7.]\) and \([23, \text{Proposition } 4.6.]\). Motivated by these results, in this paper, we study the Auslander-Reiten conjecture for non-Gorenstein rings.

In Section 2, we study the Auslander-Reiten conjecture for the residue ring \(R/Q\ell\) in connection with that for \(R\), where \(Q\) is an ideal of \(R\) generated by a regular sequence on \(R\) and \(\ell\) is a positive integer. As a result, we have the following which is one of the main results of this paper.

**Theorem 1.3.** (Theorem 2.2) Suppose that \(R\) is a Gorenstein local ring. Let \(Q = (x_1, x_2, \ldots, x_n)\) be an ideal of \(R\) generated by a regular sequence on \(R\). Then the following conditions are equivalent.

1. The Auslander-Reiten conjecture holds for \(R\).
2. There is an integer \(\ell > 0\) such that the Auslander-Reiten conjecture holds for \(R/Q\ell\).
3. For all integers \(1 \leq \ell \leq n\), the Auslander-Reiten conjecture holds for \(R/Q\ell\).

As is well-known, unlike localizations and dividing by non-zerodivisors, homological properties do not necessarily preserve through dividing by the powers of parameter ideals. In fact, letting \(R\) be a Gorenstein ring and \(Q = (x_1, x_2, \ldots, x_n)\) be an ideal of \(R\) generated by a regular sequence on \(R\), \(R/Q\ell\) is no longer Gorenstein if \(n \geq 2\) and \(\ell \geq 2\). Therefore Theorem 1.3 gives a new class of rings which satisfy the Auslander-Reiten conjecture.

The powers of parameter ideals are related to determinantal rings. Let \(s, t\) be positive integers and \(A[X] = A[X_{ij}]_{1 \leq i \leq s, 1 \leq j \leq t}\) be a polynomial ring over a commutative ring \(A\). Assume \(s \leq t\) and let \(\mathbb{I}_s(X)\) denote the ideal of \(A[X]\) generated by the maximal minors of the \(s \times t\) matrix \((X_{ij})\). With these assumptions and notations, we have the following.

**Theorem 1.4.** (Theorem 2.9) Suppose that \(2s \leq t + 1\) and \(A\) is a Gorenstein ring which is a complete intersection in codimension one. Then the Auslander-Reiten conjecture holds for the determinantal ring \(A[X]/\mathbb{I}_s(X)\).

In Section 3, we study a new class of rings arising from Theorem 1.3 that is, the class of rings \(R\) that there exist a parameter ideal \(q\) of \(R\), a complete intersection \(S\), and a parameter ideal \(Q\) of \(S\) such that \(R/q \cong S/Q^2\) as rings. We will see that the condition is characterized by an ideal condition and strongly related to the existence of Ulrich ideals. Here the notion of Ulrich ideals is given by \([12]\) and a generalization of maximal ideals of rings possessing maximal embedding dimension. It is known that Ulrich ideals enjoy many good properties, see \([12, 15]\) and \([11, \text{Theorem } 1.2]\). The ubiquity and existence of Ulrich ideals are also studied \([11, 13, 15]\). In the current paper, we study the existence of Ulrich ideals whose residue rings are complete intersections in connection with a new class of rings. As a goal of this paper, we have the following.

**Theorem 1.5.** (Corollary 3.8) Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring of dimension \(d\). Suppose that there exists an Ulrich ideal of \(R\) whose residue ring is a complete intersection. Let \(v\) (resp. \(r\)) denotes the embedding dimension of \(R\) (resp. the Cohen-Macaulay type of \(R\)). Then the following assertions are true.

1. The Auslander-Reiten conjecture holds for \(R\).
(2) \( r + d \leq v \).
(3) There exist a parameter ideal \( q \) of \( R \), a local complete intersection \( S \) of dimension \( r \), and a parameter ideal \( Q \) of \( S \) such that \( R/q \cong S/Q^2 \) as rings.
(4) Assume that there are a regular local ring \( T \) of dimension \( v \) and a surjective ring homomorphism \( T \to R \). Let \( 0 \to F_{v-d} \to \cdots \to F_1 \to F_0 \to R \to 0 \) be a minimal \( T \)-free resolution of \( R \). Then

\[
\text{rank}_T F_0 = 1 \quad \text{and} \quad \text{rank}_T F_i = \sum_{j=0}^{v-r-d} \beta_{i-j} \left( v - r - d \right) \quad \text{for } 1 \leq i \leq v - d,
\]

where \( \beta_k = \begin{cases} 1 & \text{if } k = 0 \\ k \cdot \binom{r+1}{k+1} & \text{if } 1 \leq k \leq r \\ 0 & \text{otherwise.} \end{cases} \)

Theorem 1.5 (3) claims that an Ulrich ideal determines the structure of the ring. Furthermore, Theorem 1.5 (4) recovers the result of J. Sally [22, Theorem 1.] by taking the maximal ideal \( m \) as an Ulrich ideal.

Let us fix our notations throughout this paper. In what follows, unless otherwise specified, let \( R \) denote a Cohen-Macaulay local ring with the maximal ideal \( m \). For each finitely generated \( R \)-module \( M \), let \( \mu_R(M) \) (resp. \( \ell_R(M) \)) denote the number of elements in a minimal system of generators of \( M \) (resp. the length of \( M \)). If \( M \) is a Cohen-Macaulay \( R \)-module, \( t_R(M) \) denotes the Cohen-Macaulay type of \( M \). Let \( v(R) \) (resp. \( r(R) \)) denote the embedding dimension of \( R \) (resp. the Cohen-Macaulay type of \( R \)). For convenience, letting \( M \) and \( N \) be \( R \)-modules, \( \text{Ext}^i_R(M, N) = 0 \) (resp. \( \text{Tor}^{R>0}_i(M, N) = 0 \)) denotes \( \text{Ext}^i_R(M, N) = 0 \) for all \( i > 0 \) (resp. \( \text{Tor}^{R}_i(M, N) = 0 \) for all \( i > 0 \)).

2. POWERS OF PARAMETER IDEALS AND DETERMINANTAL RINGS

The purpose of this section is to prove Theorem 1.3. First of all, let us sketch a brief proof that non-zero-divisors preserve the Auslander-Reiten conjecture since Theorem 1.3 is based on the fact.

**Proposition 2.1.** Let \((R, m)\) be a Noetherian local ring and \(a \in m\) be a non-zero-divisor of \(R\). Then the Auslander-Reiten conjecture holds for \(R\) if and only if it holds for the residue ring \(R/(a)\).

**Proof.** (if part) Let \( M \) be a finitely generated \( R \)-module such that \( \text{Ext}^{>0}_R(M, M \oplus R) = 0 \). Take an exact sequence \( 0 \to X \to F \to M \to 0 \), where \( F \) is a free \( R \)-module of rank \( \mu_R(M) \). By applying the functors

\[
\text{Hom}_R(-, R), \text{Hom}_R(M, -), \text{and} \text{Hom}_R(-, X)
\]

to the above short exact sequence, we have \( \text{Ext}^{>0}_R(X, X \oplus R) = 0 \). Thus \( \text{Ext}^{>0}_R(X, X \oplus R) = 0 \) since \( a \in m \) is a non-zero-divisor of \( X \) and \( R \), where \( X \) denotes \( \bar{R}((a) \otimes R \ast) \). Therefore the \( \bar{R} \)-module \( \bar{X} \) is free and so is the \( R \)-module \( X \). Hence the \( R \)-module \( \text{Hom}_R(M, R) \) is free and so is the \( R \)-module \( M \cong \text{Hom}_R(\text{Hom}_R(M, R), R) \).

(only if part) Let \( N \) be a finitely generated \( \bar{R} \)-module such that \( \text{Ext}^{>0}_R(N, N \oplus \bar{R}) = 0 \). Then there exists a finitely generated \( R \)-module \( M \) such that \( M/aM \cong N \) and \( \text{Tor}^{R>0}_i(M, \bar{R}) = 0 \) by [2] Proposition 1.7. Hence, by the exact sequence \( 0 \to M \to
Theorem 2.2. Let \((S, n)\) be a Gorenstein local ring and \(x_1, x_2, \ldots, x_n\) be a regular sequence on \(S\). Set \(Q = (x_1, x_2, \ldots, x_n)\). Then the following conditions are equivalent.

(1) The Auslander-Reiten conjecture holds for \(S\).

(2) The Auslander-Reiten conjecture holds for \(S/Q\).

(3) There is an integer \(\ell > 0\) such that the Auslander-Reiten conjecture holds for \(S/Q^\ell\).

(4) For all integers \(1 \leq \ell \leq n\), the Auslander-Reiten conjecture holds for \(S/Q^\ell\).

Proof. The implications (1) \(\iff\) (2) follow from Proposition 2.1 and the implication (4) \(\Rightarrow\) (3) is trivial. Hence we have only to show that (1) \(\Rightarrow\) (4) and (3) \(\Rightarrow\) (1). First of all, we reduce our assertions to the case where \(Q\) is a parameter ideal. Set \(R = S/Q^\ell\). Note that \(R\) is a Cohen-Macaulay local ring with \(\dim R = \dim S - n\) since \(Q^\ell\) is perfect. In fact, \(Q^\ell\) is generated by \(\ell \times \ell\)-minors of the \(\ell \times (n + \ell - 1)\) matrix

\[
\begin{pmatrix}
 x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\
 x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 x_1 & x_2 & x_3 & \cdots & x_{n-1} & x_n \\
\end{pmatrix},
\]

whence the projective dimension of \(S/Q^\ell\) over \(S\) is \(n\), see [8] or [3] (2.14) Proposition.

Suppose \(\dim R > 0\). Then we can take \(a \in S\) so that \(a\) is a non-zerodivisor of \(R\) and \(S/Q\). By replacing \(R\) and \(S\) by \(R/aR\) and \(S/aS\), we finally may assume that \(R\) is an Artinian local ring, that is, \(Q\) is a parameter ideal of \(S\). We may also assume that \(n \geq 2\) and \(\ell \geq 2\) by Proposition 2.1.

(1) \(\Rightarrow\) (4) Assume \(1 \leq \ell \leq n\). Suppose that \(M\) is a finitely generated \(R\)-module such that \(\Ext^0_R(M, M \oplus R) = 0\). We will show that \(\Ext^0_{S/Q}(M/QM, M/QM \oplus S/Q) = 0\) in several steps. Note that we have an exact sequence

\[
0 \to Q^{i-1}/Q^i \to S/Q^i \to S/Q^{i-1} \to 0 \quad (*)
\]

of \(R\)-modules for all \(2 \leq i \leq \ell\) and \(Q^i/Q^{i+1}\) is an \(S/Q\)-free module of rank \(i+n-1\) for all \(i > 0\).

Claim 1. \(\Ext^0_R(M, S/Q) = 0\).

Proof of Claim 1. By applying the functor \(\Hom_R(M, -)\) to the exact sequence (*) for \(i > 0\), we have the following exact sequence and isomorphism

\[
\Ext^j_R(M, S/Q) \oplus (i+n-1) \to \Ext^j_R(M, S/Q^i) \to \Ext^j_R(M, S/Q^{i+1}) \to \Ext^{j+1}_R(M, S/Q) \oplus (i+n-1)
\]

\(\Ext^j_R(M, S/Q^\ell) \cong \Ext^{j+1}_R(M, S/Q) \oplus (\ell+n-1)\) \(\text{(i)}\)

of \(R\)-modules for all \(2 \leq i \leq \ell - 1\) and \(j > 0\). Set \(E_j = \ell_R(\Ext^j_R(M, S/Q))\) for \(j > 0\). Then, by (i),

\[
E_{j+1} \left( \ell - 1 + n - 1 \right) = \ell_R(\Ext^j_R(M, S/Q^{\ell-1})) \leq E_j \left( \ell - 2 + n - 1 \right) + \ell_R(\Ext^j_R(M, S/Q^{\ell-2}))
\]

\[
\leq E_j \left( \ell - 2 + n - 1 \right) + \left( \ell - 3 + n - 1 \right) + \ell_R(\Ext^j_R(M, S/Q^{\ell-3})) \leq \cdots
\]

\[
\leq E_j \cdot \sum_{i=0}^{\ell-2} \left( i + n - 1 \right) = E_j \left( \ell - 2 + n \right),
\]
that is,
\[ E_{j+1} \leq \left(\frac{\ell + n - 2}{n - 1}\right) E_j = \frac{\ell - 1}{n} E_j \]
for all \( j > 0 \). Hence, for enough large integer \( m \geq 0 \),
\[ E_{m+1} \leq \left(\frac{\ell - 1}{n}\right)^m E_1 < 1 \]

since \( \ell \leq n \). Hence \( \text{Ext}^j_R(M, S/Q) = 0 \) for all \( j > m \). On the other hand, for \( j > 0 \), \( \text{Ext}^{j+1}_R(M, S/Q) = 0 \) implies that \( \text{Ext}^j_R(M, S/Q^i) = 0 \) for all \( 1 \leq i \leq \ell - 1 \) by above isomorphism and exact sequence (i). Hence, by using descending induction, \( \text{Ext}^j_R(M, S/Q) = 0 \) for all \( j > 0 \).

Let \( \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \) be a minimal \( R \)-free resolution of \( M \). Then, by applying the functor \( \text{Hom}_R(-, S/Q) \) to the minimal free resolution, we have the following commutative diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \text{Hom}_R(M, S/Q) & \longrightarrow & \text{Hom}_R(F_0, S/Q) & \longrightarrow & \text{Hom}_R(F_1, S/Q) & \longrightarrow & \cdots \\
& & \cong & & \cong & & \cong & & \\
0 & \longrightarrow & \text{Hom}_{S/Q}(M/QM, S/Q) & \longrightarrow & \text{Hom}_{S/Q}(F_0/QF_0, S/Q) & \longrightarrow & \text{Hom}_{S/Q}(F_1/QF_1, S/Q) & \longrightarrow & \cdots .
\end{array}
\]

The upper row is exact by Claim 1 and so is the lower row. Since \( S/Q \) is self-injective, the sequence \( \cdots \rightarrow F_1/QF_1 \rightarrow F_0/QF_0 \rightarrow M/QM \rightarrow 0 \) is a minimal \( S/Q \)-free resolution of \( M/QM \). Hence \( \text{Tor}^R_{>0}(M, S/Q) = 0 \). Moreover, by applying the functor \( M \otimes_R - \) to the sequence (*),
\[ \text{Tor}^R_{>0}(M, S/Q^i) = 0 \quad \text{for all} \ 1 \leq i \leq \ell - 1. \] (ii)

Apply the functor \( M \otimes_R - \) to (*) again. Then we get the exact sequence
\[ 0 \rightarrow (M/QM)^{\oplus (i+1)} \rightarrow M/Q^i M \rightarrow M/Q^{i-1} M \rightarrow 0 \] (**)

of \( R \)-modules for all \( 2 \leq i \leq \ell \) by (ii). Therefore, by applying the functor \( \text{Hom}_R(M, -) \) to (**), we get the following exact sequence and isomorphism
\[ \text{Ext}_R^j(M, M/QM)^{\oplus (i+1)} \rightarrow \text{Ext}_R^j(M, M/Q^i M) \rightarrow \text{Ext}_R^j(M, M/Q^{i-1} M) \rightarrow \text{Ext}_R^{j+1}(M, M/QM)^{\oplus (i+1)} \]
\[ \text{Ext}_R^j(M, M/Q^{\ell-1} M) \cong \text{Ext}_R^{j+1}(M, M/QM)^{\oplus (i+1)} \]

of \( R \)-modules for all \( 2 \leq i \leq \ell - 1 \) and \( j > 0 \). By setting \( E'_j = \ell_R(\text{Ext}_R^j(M, M/QM)) \) for all \( j > 0 \) and calculation as the proof of Claim 1 we have \( \text{Ext}^R_{>0}(M, M/QM) = 0 \). This induces that \( \text{Ext}^0_{S/Q}(M/QM, M/QM) = 0 \). In fact, we have the commutative diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & \text{Hom}_R(M, M/QM) & \longrightarrow & \text{Hom}_R(F_0, M/QM) & \longrightarrow & \text{Hom}_R(F_1, M/QM) & \longrightarrow & \cdots \\
& & \cong & & \cong & & \cong & & \\
0 & \longrightarrow & \text{Hom}_{S/Q}(M/QM, M/QM) & \longrightarrow & \text{Hom}_{S/Q}(F_0/QF_0, M/QM) & \longrightarrow & \text{Hom}_{S/Q}(F_1/QF_1, M/QM) & \longrightarrow & \cdots 
\end{array}
\]

and both of rows are exact. Since \( \cdots \rightarrow F_1/QF_1 \rightarrow F_0/QF_0 \rightarrow M/QM \rightarrow 0 \) is a minimal \( S/Q \)-free resolution of \( M/QM \) by (ii), \( \text{Ext}^0_{S/Q}(M/QM, M/QM) = 0 \). Thus we have
\[ \text{Ext}^0_{S/Q}(M/QM, M/QM \oplus S/Q) = 0 \]
since \( S/Q \) is self-injective, whence \( M/QM \) is \( S/Q \)-free by Proposition 2.1. This shows that \( M \) is \( R \)-free because \( \text{Tor}_1^R(M, S/Q) = 0 \) by (ii).

(3) \( \Rightarrow \) (1) Let \( N \) be a finitely generated \( S \)-module and suppose that \( \text{Ext}_S^0(N, N + S) = 0 \). Then, by applying the functor \( \text{Hom}_S(N, -) \) to the exact sequence \( 0 \to S \xrightarrow{x} S \to S/x_1S \to 0 \) of \( S \)-modules, \( \text{Ext}_S^0(N, S/x_1S) = 0 \). Hence

\[
\text{Ext}_S^0(N, S/Q) = 0 \quad \text{(iii)}
\]

by induction on \( n \). Similarly, \( \text{Ext}_S^0(N, N/QN) = 0 \) since \( N \) is a maximal Cohen-Macaulay \( S \)-module by \( \text{Ext}_S^0(N, S) = 0 \).

Let \( \cdots \to G_1 \to G_0 \to N \to 0 \) be a minimal \( S \)-free resolution of \( N \). Then, by applying the functor \( \text{Hom}_S(-, S/Q) \) to the minimal free resolution, we see that the sequence \( \cdots \to G_1/QG_1 \to G_0/QG_0 \to N/QN \to 0 \) is a minimal \( S/Q \)-free resolution of \( N/QN \) since (iii) and \( S/Q \) is self-injective. Hence \( \text{Tor}^S_0(N, S/Q) = 0 \). Moreover, by applying the functor \( N \otimes_S - \) to the sequence (*),

\[
\text{Tor}^S_0(N, S/Q^i) = 0 \quad \text{for all } 1 \leq i \leq \ell. \quad \text{(iv)}
\]

Therefore, for all \( 1 \leq i \leq \ell \), the sequence \( \cdots \to G_1/Q^iG_1 \to G_0/Q^iG_0 \to N/Q^iN \to 0 \) is a minimal \( S/Q^i \)-free resolution of \( N/Q^iN \) and

\[
0 \to (N/QN)^{\oplus (i-1+n-1)} \to N/Q^iN \to N/Q^{i-1}N \to 0 \quad \text{(v)}
\]

is exact as \( S \)-modules. Hence, by applying the functor \( \text{Hom}_S(N, -) \) to (v),

\[
\text{Ext}_S^0(N, N/Q^iN) = 0 \quad \text{for all } 1 \leq i \leq \ell \quad \text{since } \text{Ext}_S^0(N, N/QN) = 0. \quad \text{Thus } \text{Ext}_S^0(N/Q^iN, N/Q^iN) = 0. \quad \text{Similarly, } \text{Ext}_S^0(N, S/Q^\ell) = 0, \text{ whence } \text{Ext}_S^0(N/Q^\ell N, S/Q^\ell \oplus N/Q^\ell N) = 0. \quad \text{Hence } N/Q^\ell N \text{ is } S/Q^\ell \text{-free, whence } N \text{ is } S\text{-free by (iv).} \square
\]

The following assertions are direct consequences of Theorem 2.2.

**Corollary 2.3.** Let \( S \) be a Gorenstein local ring and \( Q \) be a parameter ideal of \( S \) generated by a regular sequence on \( S \). Then the Auslander-Reiten conjecture holds for \( S \) if and only if it holds for \( S/Q^2 \).

**Corollary 2.4.** Let \( S \) be either a complete intersection or a Gorenstein normal domain. Let \( x_1, x_2, \ldots, x_n \) be regular sequence on \( S \) and set \( Q = (x_1, x_2, \ldots, x_n) \). Then the Auslander-Reiten conjecture holds for \( S/Q^\ell \) for all \( 1 \leq \ell \leq n \).

**Corollary 2.5.** Let \( R \) be a Cohen-Macaulay local ring. Suppose that there exist a parameter ideal \( q \) of \( R \), a local complete intersection \( S \), and a parameter ideal \( Q \) of \( S \) such that \( R/q \cong S/Q^2 \) as rings. Then the Auslander-Reiten conjecture holds for \( R \).

In Section 3, we will characterize rings obtained in Corollary 2.5 by the existence of ideals in \( R \). In the remainder of this section, we explore the Auslander-Reiten conjecture for determinantal rings. We start with the following.

**Proposition 2.6.** Let \( s, t \) be positive integers and assume that \( 2s \leq t + 1 \). Suppose that \( S \) is a Gorenstein local ring and \( \{x_{ij}\}_{1 \leq i \leq s, 1 \leq j \leq t} \) forms a regular sequence on \( S \). Let \( I \) be an ideal of \( S \) generated by \( s \times s \) minors of the \( s \times t \) matrix \( (x_{ij}^a) \), where \( a_{ij} \) is a positive integer for all \( 1 \leq i \leq s \) and \( 1 \leq j \leq t \). Set \( R = S/I \). Then the Auslander-Reiten conjecture holds for \( S \) if and only if it holds for \( R \).
Proof. First of all, we show the case where $\alpha_{ij} = 1$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$. Set

$A = \{(i, j) \in \mathbb{Z} \oplus \mathbb{Z} \mid 1 \leq i \leq s, 1 \leq j \leq t\}$,

$B = \bigcup_{1 \leq i \leq s-1} \{(i, i + k) \in A \mid 0 \leq k \leq t - s\}$, and

$C = B \cup \{(s, s + k) \in A \mid 0 \leq k \leq t - s\}$.

Then

$\{x_{ij} - x_{i+1,j+1} \mid (i, j) \in B\} \cup \{x_{ij} \mid (i, j) \in A \setminus C\}$

forms a regular sequence on $S/(x_{11}, x_{12}, \ldots, x_{1t-s+1})$ and $R$, whence our assertion reduces to the case of Theorem 2.2. In fact, letting $Q$ be an ideal of $S$ generated by the above regular sequence,

$R/QR \cong S/(I + Q) = S/((x_{11}, x_{12}, \ldots, x_{1t-s+1})^s + Q)$ and $s \leq t - s + 1$.

Set $D = \{(i, j) \in A \mid \alpha_{ij} > 1\}$. We prove our assertion by induction on $N = \#D$. Assume that $N > 0$ and our assertion holds for $N - 1$. Take $(i, j) \in D$. We may assume that $(i, j) = (1, 1)$. Let $J$ be an ideal of $S$ generated by $s \times s$ minors of the $s \times t$ matrix $(x_{ij}^{\beta_{ij}})$, where $\beta_{ij} = \alpha_{ij}$ for all $(i, j) \in A \setminus \{(1, 1)\}$ and $\beta_{11} = 1$. Noting that $x_{11}$ is a non-zerodivisor of $R$, $S/J$, and $S$, we have the conclusion since $R/x_{11}R \cong S/(x_{11}S + J) = S/(x_{11}S + J)$. □

Let $H \subseteq \mathbb{Z}$ be a numerical semigroup and $k$ be a field. Then the numerical semigroup ring $k[[H]]$ often have the form obtained in Proposition 2.6 see, for examples, [10, 17]. In particular, the Auslander-Reiten conjecture holds for all three generated numerical semigroup rings. Let us note another concrete example.

Example 2.7. Let $n$ be a positive integer. Let $k[[t]]$ and $S = k[[X, Y, Z, W]]$ be formal power series rings over a field $k$. Set $R = k[[t^{10}, t^{14}, t^{16}, t^{2n+1}]]$ and assume that $n \geq 6$. Then there exists an element $f \in (X)$ such that

$R \cong S/[[2](X^2Y^2Z^2) + (W^2 - f)]$, where $\mathbb{Z}((M))$ denote the ideal of $S$ generated by $2 \times 2$-minors of the matrix $M$. In particular, the Auslander-Reiten conjecture holds for $R$.

Proof. Let $\varphi : S \to R$ be a ring homomorphism such that

$X \mapsto t^{10}, \quad Y \mapsto t^{14}, \quad Z \mapsto t^{16}, \quad W \mapsto t^{2n+1}$.

Then, by a standard argument, $\text{Ker}\varphi = \mathbb{Z}((2)(X^2Y^2Z^2) + (W^2 - f))$, where

$f = \begin{cases} 
X^mY^{m-1}Z & \text{if } n = 6m \\
X^{m+2}Y^{m-1}Z & \text{if } n = 6m + 1 \\
X^mY^{m-1}Z & \text{if } n = 6m + 2 \\
X^{m+1}Y^{m-1}Z & \text{if } n = 6m + 3 \\
X^mY^{m-1}Z & \text{if } n = 6m + 4 \\
X^{m+2}Y^{m-1}Z & \text{if } n = 6m + 5 
\end{cases}$

for some positive integer $m$. □

Let us consider determinantal rings, which are not local rings. From now on until the end of this section, let $s$, $t$ be positive integers. Let $A$ be a commutative ring and $A[X] = A[X_{ij}]_{1 \leq i \leq s, 1 \leq j \leq t}$ be a polynomial ring over $A$. Suppose that $s \leq t$ and $\mathbb{Z}(X)$ is
an ideal of $A[X]$ generated by $s \times s$ minors of the $s \times t$ matrix $X = (X_{ij})$. The following lemma is well-known.

**Lemma 2.8.** With the above assumptions and notations, suppose that $A$ is a Gorenstein ring which is a complete intersection in codimension one. Then $A[X]$ is also a Gorenstein ring which is a complete intersection in codimension one.

**Theorem 2.9.** Suppose that $2s \leq t + 1$ and $A$ is a Gorenstein ring which is a complete intersection in codimension one. Then the Auslander-Reiten conjecture holds for the determinantal ring $A[X]/\mathbb{I}_s(X)$.

**Proof.** The case where $s = 1$ is trivial. Suppose that $s > 1$. Let $\mathfrak{N}$ be a maximal ideal of $A[X]$ such that $\mathfrak{N} \supseteq \mathbb{I}_s(X)$. It is sufficient to show that the Auslander-Reiten conjecture holds for $(A[X]/\mathbb{I}_s(X))_{\mathfrak{N}}$. For integers $1 \leq p \leq s$ and $1 \leq q \leq t$, let

$$\mathfrak{M}_{pq} = (X_{ij} \mid 1 \leq i \leq p, 1 \leq j \leq q)$$

denote a monomial ideal of $A[X]$. The case where $\mathfrak{M}_{st} \subseteq \mathfrak{N}$ follows from Proposition 2.6 and Lemma 2.8. Suppose that $\mathfrak{M}_{st} \not\subseteq \mathfrak{N}$ and take a variable $X_{ij}$ so that $X_{ij} \not\in \mathfrak{N}$. We may assume that $X_{ij} = X_{st}$. Then the matrix $X = (X_{ij})$ is transformed to

$$
\begin{pmatrix}
X_{ij} - \frac{X_{st} \cdot X_{sj}}{X_{st}} & \cdots & 0 \\
0 & \ddots & 0 \\
0 & \cdots & 1
\end{pmatrix}
$$

by elementary transformation in $A[X]_{\mathfrak{N}}$. By [5, (2.4) Proposition], we have the isomorphism $\varphi : A[X][X_{st}^{-1}] \to A[X][X_{st}^{-1}]$ of $A$-algebras, where

$$\varphi(X_{ij}) = \begin{cases} X_{ij} - \frac{X_{st} \cdot X_{sj}}{X_{st}} & \text{if } X_{ij} \in \mathfrak{M}_{s-1,t-1} \\ X_{ij} & \text{otherwise.} \end{cases}$$

Therefore we have the commutative diagram

$$
\begin{array}{ccc}
A[X]_{\mathfrak{N}} & \xrightarrow{\varphi_{\mathfrak{N}}} & A[X]_{\mathfrak{N}} \\
\subseteq & \circ & \subseteq \\
\mathbb{I}_s(X)_{\mathfrak{N}} & \xrightarrow{\varphi} & \mathbb{I}_{s-1}(X_{st})_{\mathfrak{N}},
\end{array}
$$

where $X_{st}$ is the $(s-1) \times (t-1)$ matrix that results from deleting the $s$-th row and the $t$-th column of $X$ and $\mathbb{I}_{s-1}(X_{st})$ is an ideal of $A[X]$ generated by $(s-1) \times (s-1)$ minors of $X_{st}$. Hence $(A[X]/\mathbb{I}_s(X))_{\mathfrak{N}} \cong (A[X]/\mathbb{I}_{s-1}(X_{st}))_{\mathfrak{N}}$ as rings. If $\mathfrak{M}_{s-1,t-1} \subseteq \mathfrak{N}$, the Auslander-Reiten conjecture holds for $(A[X]/\mathbb{I}_{s-1}(X_{st}))_{\mathfrak{N}}$ by Proposition 2.6 and Lemma 2.8 since $2(s-1) \leq (t-1)+1$. If $\mathfrak{M}_{s-1,t-1} \not\subseteq \mathfrak{N}$, repeat the above argument. Then, after finite steps, we finally see that the Auslander-Reiten conjecture holds for $(A[X]/\mathbb{I}_{s-1}(X_{st}))_{\mathfrak{N}}$. □

3. Ulrich ideals whose residue rings are complete intersections

In this section, we study rings obtained in Corollary 2.5 in connection with the existence of ideals. Throughout this section, let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d$. 
Lemma 3.1. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$ and $\mathfrak{q} = (x_1, x_2, \ldots, x_d)$ be a parameter ideal of $R$. Set $n = \mu_R(I)$. Suppose the following two conditions.

1. $\mathfrak{q} \subseteq I$ and $x_1, x_2, \ldots, x_d$ is a part of minimal generators of $I$.
2. $I^2 \subseteq \mathfrak{q}$ and $I/\mathfrak{q}$ is $R/I$-free.

Then $r(R) = (n - d)\cdot r(R/I)$. In particular, $n = d + r(R)$ if $R/I$ is a Gorenstein ring.

Proof. Since $I/\mathfrak{q} \cong (R/I)^{\oplus(n-d)}$, $I = \mathfrak{q} :_R I$. Hence $I/\mathfrak{q} = (\mathfrak{q} :_R I)/\mathfrak{q} \cong \text{Hom}_R(R/I, R/\mathfrak{q})$. Therefore

\[ \text{Hom}_R(R/m, (R/I)^{\oplus(n-d)}) \cong \text{Hom}_R(R/m, \text{Hom}_R(R/I, R/\mathfrak{q})) \cong \text{Hom}_R(R/m \otimes_R R/I, R/\mathfrak{q}), \]

whence we have the conclusion by comparing the lengths of them.

For a moment, let $(R, \mathfrak{m})$ be an Artinian local ring. Then there are a regular local ring $(S, \mathfrak{n})$ and a surjective local ring homomorphism $\varphi : S \to R$. We can take $S$ so that the dimension of $S$ is equal to the embedding dimension of $R$. Set $v = v(R) = \dim S$ and $r = r(R)$. With these assumptions and notations, we have the following.

Proposition 3.2. Let $(R, \mathfrak{m})$ be an Artinian local ring and $S$ be as above. The following conditions are equivalent.

1. $r \leq v$ and there exists a regular sequence $X_1, X_2, \ldots, X_v \in \mathfrak{n}$ on $S$ such that $R \cong S/[(X_1, X_2, \ldots, X_r)^2 + (X_{r+1}, X_{r+2}, \ldots, X_v)]$ as rings.
2. There exists a nonzero ideal $I$ of $R$ such that
   - (i) $I^2 = 0$ and $I$ is $R/I$-free.
   - (ii) $R/I$ is a complete intersection.

Proof. (2) $\Rightarrow$ (1) Let $\overline{\varphi} : S \xrightarrow{\varphi} R \to R/I$ be a surjective local ring homomorphism. Set $\mathfrak{a} = \text{Ker}\varphi$ and $J = \text{Ker}\overline{\varphi}$. Since $R/I \cong S/J$ is a complete intersection, $J$ is generated by a regular sequence $x_1, x_2, \ldots, x_v \in \mathfrak{n}$ on $S$, see [4, Theorem 2.3.3.(c)]. Hence, after renumbering of $x_1, x_2, \ldots, x_v$,

\[ I = JR = (x_1, x_2, \ldots, x_v) = (x_1, x_2, \ldots, x_r) \]

by Lemma 3.1, where $\overline{x}$ denotes the image of $x \in S$ to $R$. Thus $J = (x_1, x_2, \ldots, x_r) + \mathfrak{a}$. For all $r + 1 \leq i \leq v$, take $y_i \in \mathfrak{a}$ and $c_{j_1}, c_{j_2}, \ldots, c_{j_r} \in R$ so that $x_i = y_i + \sum_{j=1}^{r} c_{j} x_{j}$. Then $J = (x_1, x_2, \ldots, x_r) + (y_{r+1}, y_{r+2}, \ldots, y_v)$. Set $X = (x_1, x_2, \ldots, x_r)$ and $Y = (y_{r+1}, y_{r+2}, \ldots, y_v)$, where $Y$ denotes $(0)$ if $r = v$. We then have inclusions

\[ J^2 + Y \subseteq \mathfrak{a} \subseteq J, \]

where the first inclusion follows from $I^2 = 0$. On the other hand, setting $S' = S/Y$,

\[ \ell_S(J/[J^2 + Y]) = \ell_{S'}(XS'/X^2S') = \ell_{S'}(S'/XS') - r = \ell_R(R/I) - r = \ell_R(I) = \ell_S(J/\mathfrak{a}), \]

where the forth equality follows from the fact that $I$ is an $R/I$-free module. Thus $\mathfrak{a} = J^2 + Y = (x_1, x_2, \ldots, x_r)^2 + (y_{r+1}, y_{r+2}, \ldots, y_v)$.

(1) $\Rightarrow$ (2) Let $X = (X_1, X_2, \ldots, X_r)$ and $Y = (X_{r+1}, X_{r+2}, \ldots, X_v)$ be ideals of $S$. Set $I = XR$. Then $I^2 = 0$ and $I = [X + Y]/[X^2 + Y] \cong [S/(X + Y)]^{\oplus r} \cong (R/I)^{\oplus r}$. 

We are now back to the Setting that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring. Let us generalize Proposition 3.2 to arbitrary Cohen-Macaulay local rings.
Theorem 3.3. Let \((R, m)\) be a Cohen-Macaulay local ring. Then the following conditions are equivalent.

1. There exists a parameter ideal \(q\) of \(R\), a local complete intersection \(S\) of positive dimension, and a primary ideal \(Q\) of \(S\) such that \(R/q \cong S/Q^2\) as rings.
2. There exists a parameter ideal \(q\) of \(R\), a local complete intersection \(S\) of dimension \(r(R)\), and a parameter ideal \(Q\) of \(S\) such that \(R/q \cong S/Q^2\) as rings.
3. There exists an \(m\)-primary ideal \(I\) and a parameter ideal \(q\) of \(R\) such that
   - \(I^2 \subseteq q \subseteq I\) and \(I/q\) is \(R/I\)-free.
   - \(R/I\) is a complete intersection.

Proof. \((1) \Rightarrow (3)\) Set \(\ell = \dim S\) and \(Q = (x_1, x_2, \ldots, x_\ell)\). Then, since \(R/q \cong S/Q^2\), we can choose \(y_1, y_2, \ldots, y_\ell \in R\) so that \(\overline{y_i}\) corresponds to \(\overline{x_i}\) for all \(1 \leq i \leq \ell\), where \(\overline{y_i}\) denotes the image of \(y_i\) in \(S/Q^2\) and \(\overline{x_i}\) denotes the image of \(x_i\) in \(R/q\). Set \(I = (y_1, y_2, \ldots, y_\ell) + q\).

Then \(R/I \cong S/Q^2\) is a complete intersection and \(I/q \cong Q/Q^2\) is \(R/I\)-free. Furthermore \(I^2 = qI + (y_1, y_2, \ldots, y_\ell)^2 \subseteq q\).

\((3) \Rightarrow (2)\) Because \(I/q\) is an ideal of \(R/q\) which satisfies the assumption of Proposition 3.2. \(\square\)

Theorem 3.3 is applicable to Ulrich ideals. Here the definition of Ulrich ideals is stated as follows.

Definition 3.4. ([12, Definition 2.1.]) Let \((R, m)\) be a Cohen-Macaulay local ring and \(I\) be an \(m\)-primary ideal of \(R\). Assume that \(I\) contains a parameter ideal \(q\) of \(R\) as a reduction. We say that \(I\) is an Ulrich ideal of \(R\) if the following conditions are satisfied.

1. \(I \neq q\), but \(I^2 = qI\).
2. \(I^2\) is a free \(R/I\)-module.

Note that the condition \((1)\) of Definition 3.4 is independent of the choice of \(q\); see, for example, [18, Theorem 2.1.]. The following assertions say that the notion of Ulrich ideals is closely related to the condition \((3)(i)\) of Theorem 3.3.

Proposition 3.5. ([12, Lemma 2.3. and Proposition 2.3.])

1. If \(I\) is an Ulrich ideal of \(R\), then \(I^2 = qI \subseteq q\) and \(I/q\) is \(R/I\)-free for every parameter ideal \(q\) of \(R\) such that \(q\) is a reduction of \(I\).
2. Assume that \(R/m\) is infinite. If \(I^2 \subseteq q\) and \(I/q\) is \(R/I\)-free for all minimal reductions \(q\) of \(I\), then \(I\) is an Ulrich ideal of \(R\).

The following result recovers the result of J. Sally [22] by taking the maximal ideal \(m\) as an Ulrich ideal. For convenience, set \(d = \dim R\), \(r = r(R)\), and \(v = v(R)\).

Theorem 3.6. (cf. [22, Theorem 1.]) Suppose that there are a regular local ring \((T, n)\) of dimension \(v\) and a surjective local ring homomorphism \(\phi : T \rightarrow R\). If there exists an Ulrich ideal \(I\) of \(R\) such that \(R/I\) is a complete intersection, then \(\mu_R(I) = d + r \leq v\) and there exists a regular sequence \(x_1, x_2, \ldots, x_v\) on \(T\) such that

1. \(x_1, x_2, \ldots, x_d\) is a regular sequence on \(R\).
2. \(R/(x_1, x_2, \ldots, x_d)R \cong T/[(x_1, \ldots, x_d) + (x_{d+1}, \ldots, x_{d+r})^2 + (x_{d+r+1}, \ldots, x_v)]\).

Therefore, letting \(0 \rightarrow F_{v-d} \rightarrow \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow R \rightarrow 0\) be a minimal \(T\)-free resolution of \(R\),

\[
\operatorname{rank}_TF_0 = 1 \quad \text{and} \quad \operatorname{rank}_TF_i = \sum_{j=0}^{v-r-d} \beta_{i-j}(v-r-d)\]

for $1 \leq i \leq v - d$, where \( \beta_k = \begin{cases} 1 & \text{if } k = 0 \\ k \cdot \binom{r+1}{k+1} & \text{if } 1 \leq k \leq r \\ 0 & \text{otherwise.} \end{cases} \)

In particular, \( \text{rank}_T F_i = i \cdot \binom{r+1}{i+1} \) for $1 \leq i \leq r$ if \( \mu_R(I) = v \).

**Proof.** Let \( \overline{x} : T \xrightarrow{\varphi} R \to R/I \) be a surjective local ring homomorphism. Set \( a = \text{Ker} \varphi \) and \( J = \text{Ker} \overline{x} \). Since \( R/I \cong T/J \) is a complete intersection, \( J \) is generated by a regular sequence \( x_1, x_2, \ldots, x_v \in \mathfrak{n} \) on \( T \). Hence, after renumbering of \( x_1, x_2, \ldots, x_v \),

\[
I = JR = (x_1, x_2, \ldots, x_v) = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{d+r})
\]

by Lemma 3.1 where \( \overline{x} \) denotes the image of \( x \in T \) to \( R \). Let \( (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_d) \subseteq I \) be a minimal reduction of \( I \). Then, after renumbering of \( x_1, x_2, \ldots, x_{d+r} \),

\[
I = (x_1', x_2', \ldots, x_d', x_{d+1}', \ldots, x_{d+r}').
\]

Thus \( J = (x_1', \ldots, x_d') + (x_{d+1}, \ldots, x_{d+r}) + a \). Since \( \mu_T(J) = v \), we can choose \( v \) elements in \( \{x_1', \ldots, x_d'\}, x_{d+1}, \ldots, x_{d+r}\} \cup \{a \mid a \in a \} \) as a minimal system of generators. Assume that \( x'_i \) cannot be chosen as a part of minimal system of generators. Then

\[
I = (x_1', \ldots, x_{i-1}', x_i', x_{i+1}', \ldots, x_d', x_{d+1}', \ldots, x_{d+r}').
\]

This is a contradiction for \( \mu_R(I) = r + d \) by Lemma 3.1

Hence

\[
J = (x_1', \ldots, x_d') + (x_{d+1}, \ldots, x_{d+r}) + (y_{d+r+1}, \ldots, y_v)
\]

for some \( y_{d+r+1}, \ldots, y_v \in a \). Set \( X_1 = (x_1', \ldots, x_d') \), \( X_2 = (x_{d+1}, \ldots, x_{d+r}) \), and \( Y = (y_{d+r+1}, \ldots, y_v) \). Then \( X_1 + X_2 + Y \subseteq a + X_1 \subseteq J \), whence \( a + X_1 = X_1 + X_2 + Y \) since \( \ell_T(J/[a + X_1]) = \ell_T(J/[X_1 + X_2 + Y]) = r \cdot \ell_T(T/J) \).

Let \( 0 \to F_{v-d} \to \cdots \to F_1 \to F_0 \to R \to 0 \) be a minimal \( T \)-free resolution of \( R \). Then

\[
0 \to F_{v-d}/X_1 F_{v-d} \to \cdots \to F_1/X_1 F_1 \to F_0/X_1 F_0 \to R/X_1 R \to 0
\]

is a minimal \( T/X_1 \)-free resolution of \( R/X_1 R \) and \( R/X_1 R \cong T/[X_1 + X_2 + Y] \) since \( a + X_1 = X_1 + X_2 + Y \). On the other hand, the Eagon-Northcott complex \( \llbracket 8 \rrbracket \) gives the minimal \( T/X_1 \)-free resolution

\[
0 \to G_r \xrightarrow{\partial_{r-1}} G_{r-1} \xrightarrow{\partial_{r-2}} \cdots \xrightarrow{\partial_2} G_1 \xrightarrow{\partial_1} G_0 \xrightarrow{\partial_0} T/[X_1 + X_2] \to 0
\]

of \( T/[X_1 + X_2] \), thus \( \text{rank}_{T/X_1} G_k = \beta_k \) for all \( k \in \mathbb{Z} \). Therefore, as is well known,

\[
0 \to G_r \xrightarrow{\partial_{r+1}} G_{r-1} \oplus \cdots \oplus \xrightarrow{\partial_2} G_1 \xrightarrow{\partial_1} G_0 \xrightarrow{\partial_0} T/[X_1 + X_2 + (y_{d+r+1})] \to 0
\]

becomes a minimal \( T/X_1 \)-free resolution, where \( \partial_i = \begin{pmatrix} \partial_{i-1} \\ (-1)^{d_{d+r+1}} \partial_i \end{pmatrix} \). This show inductively \( \text{rank}_T F_i = \sum_{j=0}^{v-r-d} \beta_{(v-r-d)_j} \) as desired. \( \square \)

**Remark 3.7.** With the assumption of Theorem 3.6, the equality \( \mu_R(I) = v \) does not necessarily hold in general; see Example 3.11 (1). On the other hand, if \( R \) is a one-dimensional Cohen-Macaulay ring possessing maximal embedding dimension, every Ulrich ideal \( I \) satisfy that \( R/I \) is a complete intersection and \( \mu_R(I) = v \); see [11, Theorem 4.5].
Combining Theorem 2.2, 3.3 and 3.6, we have the following which is a goal of this paper.

Corollary 3.8. Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring of dimension \(d\). Suppose that there exists an Ulrich ideal of \(R\) whose residue ring is a complete intersection. Then the following assertions are true.

1. The Auslander-Reiten conjecture holds for \(R\).
2. \(r + d \leq v\).
3. There exist a parameter ideal \(\mathfrak{q}\) of \(R\), a local complete intersection \(S\) of dimension \(r\), and a parameter ideal \(Q\) of \(S\) such that \(R/\mathfrak{q} \cong S/Q^2\) as rings.
4. Assume that there are a regular local ring \(T\) of dimension \(v\) and a surjective ring homomorphism \(T \to R\). Let \(0 \to F_{v-d} \to \cdots \to F_1 \to F_0 \to R \to 0\) be a minimal \(T\)-free resolution of \(R\). Then

\[
\text{rank}_T F_0 = 1 \quad \text{and} \quad \text{rank}_T F_i = \sum_{j=0}^{v-r-d} \beta_{i-j} \left(\frac{v - r - d}{j}\right)
\]

for \(1 \leq i \leq v - d\), where \(\beta_k = \begin{cases} 1 & \text{if } k = 0 \\ k \cdot \left(\frac{r+1}{k+1}\right) & \text{if } 1 \leq k \leq r \\ 0 & \text{otherwise}. \end{cases}\)

Proof. (1) This follows from (3) and Corollary 2.5
(2) Passing to the completion of \(R\), we may assume that there exist a regular local ring \(T\) of dimension \(v\) and a surjective ring homomorphism \(T \to R\). Then the assertion follows from Theorem 3.6
(3) This follows from Theorem 3.3 and Proposition 3.5.

Let us note that it is not necessarily unique for a given ring that an Ulrich ideal whose residue ring is a complete intersection.

Proposition 3.9. Let \((S, \mathfrak{n})\) be a local complete intersection of dimension three and \(f, g, h \in \mathfrak{n}\) be a regular sequence on \(S\). Set

\[R = S/(f^2 - gh, g^2 - hf, h^2 - fg).\]

Then \(R\) is a Cohen-Macaulay local ring of dimension one and \(I = (f, g, h)R\) is an Ulrich ideal of \(R\) such that \(R/I\) is a complete intersection. Furthermore, if \(f = f_1 f_2\) for \(f_1, f_2 \in \mathfrak{n}\), then \(I_1 = (f_1, g, h)R\) is also an Ulrich ideal of \(R\) such that \(R/I_1\) is a complete intersection.

Proof. By direct calculation,

\[I^2 = fI, \quad \ell_R(R/I) = \ell_S(S/(f, g, h)), \quad \ell_R(I/fR) = \ell_S((f, g, h)/((f) + (g, h)^2)) = 2 \cdot \ell_S(S/(f, g, h)).\]

Hence a surjection \((R/I)^{\oplus 2} \to I/fR\) must be an isomorphism, that is, \(I\) is an Ulrich ideal of \(R\) and \(R/I \cong S/(f, g, h)\) is a complete intersection.

Assume that \(f = f_1 f_2\). Then, similarly to the above,

\[I_1^2 = f_1 I_1, \quad \ell_R(R/I_1) = \ell_S(S/(f_1, g, h)), \quad \ell_R(I_1/f_1 R) = \ell_S((f_1, g, h)/((f_1) + (g, h)^2)) = 2 \cdot \ell_S(S/(f_1, g, h)).\]

Hence \(I_1\) is an Ulrich ideal of \(R\) and \(R/I_1 \cong S/(f_1, g, h)\) is a complete intersection.
Here are some examples arising from Proposition 3.9.

**Example 3.10.** With the same notations of Proposition 3.9, let \( S = k[[X, Y, Z]] \) be a formal power series ring over a field \( k \). Let \( \ell, m, n \) be positive integers such that \((\ell, m, n) \neq (0, 0, 0)\). Then we have the following examples.

1. Take \((f, g, h)\) so that \((X^\ell, Y^m, Z^n)\). Then
   \[
   (X^\ell, Y^m, Z^n)R, (X^\ell, Y^j, Z^k)R
   \]
   are Ulrich ideals for all \(0 \leq i \leq \ell, 0 \leq j \leq m, 0 \leq k \leq n\).

2. Take \((f, g, h)\) so that \((X^\ell, Y^m, Z^n, X^2 + Y^2, Y^2 + Z^2)\). Then
   \[
   (X^i, Y^j, Z^k, X^2 + Y^2, Y^2 + Z^2)R
   \]
   is an Ulrich ideal for all \(0 \leq i \leq \ell, 0 \leq j \leq m, 0 \leq k \leq n\). Furthermore
   \((X^\ell, Y^m, Z^n, X + Y, Y^2 + Z^2)R\) is also an Ulrich ideal if \(k\) is a field of characteristic
   two or an algebraically closed field.

We close this paper with several examples.

**Example 3.11.** Let \( k[[t]] \) and \( S = k[[X, Y, Z, W]] \) be formal power series rings over a field \( k \). The following assertions are true.

1. Let \( R_1 = k[[t^6, t^{11}, t^{16}, t^{26}]] \) and \( I = (t^6, t^{16}, t^{26}) \) be an ideal of \( R_1 \). Then \((t^6)\)
   is a reduction of \( I \), \( I \) is an Ulrich ideal of \( R_1 \), and \( R_1/I \) is a complete intersection.
   Therefore the minimal \( S \)-free resolution of \( R \) has the following form
   \[
   0 \to S^\oplus 2 \to S^\oplus 5 \to S^\oplus 4 \to S \to R \to 0.
   \]
   On the other hand, \( R_1 \cong S/((X^7 - ZW, Y^2 - XZ, Z^2 - XW, W^2 - X^6Z)) \) as rings,
   then \( R_1 \) does not have the form obtained in Proposition 2.6.

2. (cf. [21, Proposition 1.4.]) Set
   \[
   R_2 = S/(X^2 - YZ, Y^2 - ZX, Z^2 - XY, W^2).
   \]
   Then \( X \) is a non-zerodivisor of \( R_2 \) and \( R_2/XR_2 \cong k[[Y, Z, W]]/[(Y, Z)^2 + (W^2)]\).
   Hence the Auslander-Reiten conjecture holds for \( R_2 \). On the other hand, \( I = (X, W)R_2 \) is an Ulrich ideal,
   whence \( I \) is a non-free totally reflexive \( R_2 \)-module ([15, Theorem 2.8.]). Hence \( R_2 \) is not \( G \)-regular in the sense of [23]. In particular, \( R_2 \) is
   not a Golod ring.

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