TIME EVOLUTION
OF THE EXTERNAL FIELD PROBLEM
IN QED

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Abstract

We construct the time-evolution for the second quantized Dirac equation subject to a smooth, compactly supported, time dependent electromagnetic potential and identify the degrees of freedom involved. Earlier works on this (e.g. Ruijsenaars) observed the Shale-Stinespring condition and showed that the one-particle time-evolution can be lifted to Fock space if and only if the external field had zero magnetic components. We scrutinize the idea, observed earlier by Fierz and Scharf, that the time-evolution can be implemented between time varying Fock spaces. In order to define these Fock spaces we are led to consider classes of reference vacua and polarizations. We show that this implementation is up to a phase independent of the chosen reference vacuum or polarization and that all induced transition probabilities are well-defined and unique.

Keywords: Second Quantized Dirac Equation; External Field; Polarization Classes; Time-Varying Fock Spaces.

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1 Introduction

1.1 The Problem and State of the Art

This paper purports the the second-quantized time-evolution of Dirac fermions subject to a classical time-dependent external electromagnetic potential $A : \mathbb{R}^4 \to \mathbb{R}^4$. In the early works [SS65, Rui77a, Rui77b, Bel75, Bel76] it was recognized that the construction of such a time-evolution in the presence of an external potential which has non-zero magnetic components turns out to be impossible on one fixed Fock space. Let us briefly explain the nature of the problem and state the classical results.

It is well known that the spectrum of the free Dirac operator $H^0 = -i\alpha \cdot \nabla + \beta m$ is $(-\infty, -m] \cup [+m, +\infty)$ and, thus, allows for wave functions associated with “negative energy”. Throughout this work we use Planck units $\hbar = c = 1$. The two components of the spectrum give rise to a splitting of the one-particle Hilbert space $L_2(\mathbb{R}^3, \mathbb{C}^4)$, i.e. the space of square integrable $\mathbb{C}^4$ valued functions on $\mathbb{R}^3$, into two spectral subspaces $L_2(\mathbb{R}^3, \mathbb{C}^4) = H_- \oplus H_+$. While a wave function $\psi \in H_+$ can be interpreted to describe the dynamics of electrons with positive kinetic energy the interpretation of the negative energy wave functions is not straightforward as we do not seem to see particles of negative kinetic energy in nature. Moreover, there is no mechanism in quantum mechanics to prevent transitions from the positive to the negative spectral subspace so we can not simply regard those negative energy wave functions as unphysical. In order to solve this problem Dirac very early came up with a physical theory for his equation, the so-called Dirac Sea or Hole Theory:

Admettons que dans l’Univers tel que nous le connaissons, les états d’énergie négative soient presque tous occupés par des électrons, et que la distribution
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1.1 The Problem and State of the Art

ainsi obtenue ne soit pas accessible à notre observation à cause de son uniformité dans toute l’étendue de l’espace. Dans ces conditions, tout état d’énergie négative non occupé représentant une rupture de cette uniformité, doit se révéler à observation comme une sorte de lacune. Il est possible d’admettre que ces lacunes constituent les positrons.

P.A.M. Dirac, Théorie du Positron (1934), in Selected Papers on Quantum Electrodynamics, Ed. J. Schwinger, Dover Pub. (1958)

It is assumed that all negative energy states are occupied by electrons which then constitute the Dirac sea. Due to its uniformity the Dirac sea is hidden from our observation and, thus, physically inaccessible. What can be observed are the holes in the Dirac sea created by Dirac sea electrons which made transitions to the positive energy spectrum. The holes are called positrons.

The exclusion principle will operate to prevent a positive-energy electron ordinarily from making transitions to states of negative energy. It will be possible, however, for such an electron to drop into an unoccupied state of negative energy. In this case we should have an electron and positron disappearing simultaneously, their energy being emitted in the form of radiation. The converse process would consist in the creation of an electron and a positron from electromagnetic radiation.

P.A.M. Dirac, Theory of the positron, in: The Principles of Quantum Mechanics, Oxford (1930)

Dirac’s theory predicted the existence and properties of positrons, pair creation and pair annihilation, which shortly later were verified by Anderson [And33].

In the language of quantum field theory the Dirac sea is represented in the so-called second quantization procedure by the “vacuum vector” on which two types of creation operators act. Those creating electrons and those creating positrons. This way one implements Dirac’s idea that one only considers the “net description of particles: electrons and positrons” and neglects what is going on “deep down in the sea”, assuming that nothing physically relevant happens in there. The Hilbert space for this many particle system is the Fock space built by successive applications of creation operators on the vacuum.

As Dirac already pointed out in [Dir34b], the Dirac sea, however, is inaccessible and the choice of $\mathcal{H}_-$ in the presence of an external field is not obvious at all. In addition, Dirac’s invention and likewise quantum field theory are plagued by a serious problem: As soon as an electromagnetic field $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, \vec{A})$ enters the Dirac equation, i.e. as soon as “interaction is turned on”, one has generically transitions of negative energy wave functions to positive energy wave functions, i.e. pair creation and pair annihilation. For a mathematical proof of pair creation in the adiabatic regime see [DP07, DP08]. While pair
creation and annihilation is an observed phenomenon it nevertheless has mathematically a devastating side effect. Pictorially speaking, the negative energy states are “rotated” by the external field and thus develop components in the positive energy subspace. Thus the Dirac sea containing infinitely many particles generically produces under the influence of an external field infinitely many electrons as soon as the field acts. Such a state does not anymore belong to the Fock space and there is no reason to hope that in general a lift of the one-particle Dirac time-evolution to this Fock space exists.

This problem manifests itself in the classical results as follows. Under reasonable assumptions on the external potential $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec{A})$ the one-particle Dirac Hamiltonian $H^A(t) = -i\alpha \cdot (\nabla - \vec{A}(t)) + \beta m + A(t)^0$ generates a unitary time-evolution $U^A : \mathcal{H} \to \mathcal{H}$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ of square integrable, $\mathbb{C}^4$ valued functions on $\mathbb{R}^3$. Having Dirac’s sea idea \cite{Dir34b, Dir34a} in mind one introduces a splitting of this Hilbert space $\mathcal{H}$ into a Hilbert direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm$ are the spectral subspaces of the free Dirac Hamiltonian $H^0$ and $P_\pm$ the corresponding orthogonal projectors. The “states” of $\mathcal{H}_-$ are assumed to be filled with electrons, historically referred to as Dirac sea. In modern quantum field theory the notion of the Dirac sea is replaced by the so called vacuum. In order to extract finite and physical meaningful expressions from this infinite particle picture Dirac’s idea is to focus only on the “net balance” between the initial Dirac sea and the time-evolved Dirac sea while neglecting what is going on “deep down in the seas”. Transitions between $\mathcal{H}_-$ and $\mathcal{H}_+$ are thought to describe pair creation and annihilation \cite{Dir34b, Dir34a}; for a mathematically rigorous proof of the creation of a pair of asymptotically free moving electron and positron in an adiabatically changing strong field see \cite{DP08, DP07}. The Fock space $\mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) = \Lambda \mathcal{H}_+ \otimes \overline{\Lambda \mathcal{H}_-}$ serves as a mathematical setup for this infinite particle picture. One intends to lift the one-particle Dirac time-evolution $U^A$ on $\mathcal{H}$ to this Fock space. The Shale-Stinespring Theorem \cite{SS65} gives the following necessary and sufficient condition for the existence of such a lift. For times $t_0, t_1 \in \mathbb{R}$ the one-particle Dirac time-evolution $U^A(t_1, t_0)$ can be lifted to a second-quantized time-evolution on $\mathcal{F}(\mathcal{H}_+, \mathcal{H}_-)$ if and only if the off-diagonal terms $U^A_{+, -}(t_1, t_0) := P_+ U^A(t_1, t_0) P_-$ and $U^A_{-, +}(t_1, t_0) := P_- U^A(t_1, t_0) P_+$ are both Hilbert-Schmidt operators. Such a lift, if it exists, is unique up to a phase. Ruijsenaars \cite{Rui77a, Rui77b} supplied the physical implications of the Shale-Stinespring Theorem: the operators $U^A_{+, -}(t_1, t_0)$ and $U^A_{-, +}(t_1, t_0)$ are Hilbert-Schmidt operators for all times $t_0, t_1$ if and only if $\vec{A} = 0$, a somehow devastating result. Not only it means there are no lifts of the one-particle Dirac time-evolution for external potentials with non-zero magnetic components, it also means that gauge transformations which add non-zero spatial components to the external potential cannot be implemented.

Let us give an intuition for the Shale-Stinespring condition. We regard the time-evolution $U^A(t_1, t_0) : \mathcal{H}_- \oplus \mathcal{H}_+ \to \mathcal{H}_- \oplus \mathcal{H}_+$ in matrix form:

$$U^A(t_1, t_0) = \begin{pmatrix} U^A_{+, +}(t_1, t_0) & U^A_{-, +}(t_1, t_0) \\ U^A_{-, -}(t_1, t_0) & U^A_{+, -}(t_1, t_0) \end{pmatrix}. \quad (1.1)$$
The non-diagonal terms describe pair creation and annihilation. In leading order, for $U^A(t_1,t_0)$ close to the identity and neglecting multiple pair creation, the squared Hilbert-Schmidt norm

$$\|U^A_{+-}(t_1,t_0)\|_{L_2}^2 = \sum_{n\in\mathbb{N}} \|U^A_{+-}(t_1,t_0)\varphi_n\|_{\mathcal{H}}^2$$

for any orthonormal basis $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathcal{H}_-$ may be interpreted as the probability of creating one pair from the Dirac sea. In this sense the Shale-Stinespring condition ensures that the pair creation probabilities are well-defined.

Metaphorically speaking: The negative energy states in $\mathcal{H}_-$ are “rotated” by the interaction term $-i\alpha \cdot \vec{A} + A^0$ of the Dirac Hamiltonian $H^A$ and develop components in the positive energy subspace $\mathcal{H}_+$ as soon as the field acts. While the term $A^0$ induces only a mild rotation, the rotation induced by $-i\alpha \cdot \vec{A}$ is strong due to the presence of the $\alpha$ matrix. The catastrophe of ill-defined pair creation probabilities happens as long as the field is acting. When the field is switched off, most of spinors are however rotated back into the “free Dirac sea”. Therefore focusing on the scattering matrix only one expects that the off-diagonal of the $S$-matrix $P_+ S^A P_-$ consists of Hilbert-Schmidt operators. Hence, a lift of the $S$-matrix to Fock space exists [Bel75, Bel76]. This lift is, as we said, only unique up to a phase. In the scattering situation the initial Dirac sea is “more or less” restored, so that ingoing states in Fock space are transformed to outgoing states in the same Fock space.

Since the $S$-matrix captures the asymptotic time-evolution it is desirable to interpolate the asymptotic free time-evolutions of scattering by a full time-evolution when the external field acts. Due to the catastrophic pair creation discussed above one must adjust the sea, i.e. the vacuum, so that the most of the spinors remain “sea-vectors”. Therefore, one considers a second quantized Dirac time-evolution in the presence of an external field with respect to time-dependent reference polarizations in contrast to one fixed polarization $\mathcal{H}_-$; we shall call all closed subspaces of $\mathcal{H}$ with infinite dimension and codimension polarizations. One implements the second quantized time-evolution on time-varying Fock spaces instead of only one fixed Fock space. Such an implementation was already described in [FS79], and this idea was further developed by Mickelsson [Mic98]. Mickelsson gives a time-dependent unitary transformation of the Dirac Hamiltonian $H^A$ such that its off-diagonal parts become Hilbert-Schmidt operators. Furthermore, he identifies the missing phase of the second quantized time-evolution up to a remaining freedom. A related but different approach to this phase is described in [SW86].

1.2 What this Paper is about

The obstacle in implementing the second-quantized time-evolution on time-varying Fock spaces is that there is no canonical choice of polarization. To illustrate this consider $H^A$, the Dirac operator for a fixed, time independent four-vector potential $A$. The spectrum
is in general not anymore as simple as in the free case and there is no canonical way of
defining a polarization since a splitting into subspaces \( L_2(\mathbb{R}^3, \mathbb{C}^4) = \mathcal{H}^-_A \oplus \mathcal{H}^+_A \) is reasonably arbitrary. The choice of polarization becomes in particular interesting when
the external potential \( A \) is time-dependent. (Four-vector potentials defined on space-time \( \mathbb{R}^4 \) are denoted by the sans serif letter \( A \), while defined on the space \( \mathbb{R}^3 \) they are denoted
by the italic letter \( A \).) Suppose that at time \( t_0 \) the field is zero and at a later time the field
is switched on. To better understand the issue of the choice of polarization, observe that
choosing \( U^{A}(t,t_0)\mathcal{H}^- \) as the polarization at time \( t \) would not allow for the description of
pair creation: starting from a Dirac sea in \( \mathcal{H}^- \) at time \( t_0 \) all one-particle wave functions
will remain in the sea \( U^{A}(t,t_0)\mathcal{H}^- \) forever. Also, this choice of polarization depends not
only on the field \( A \) at time \( t \) but also on the whole history \( (A(s))_{s \leq t} \). We shall show below
that for another field \( \tilde{A} \) with \( \tilde{A}(t_0) = A(t_0) \) and \( \tilde{A}(t) = A(t) \) the orthogonal projectors onto
\( U^{A}(t,t_0)\mathcal{H}^- \) and \( U^{\tilde{A}}(t,t_0)\mathcal{H}^- \) differ by a Hilbert-Schmidt operator. Moreover, as discussed
in \([FS79]\), all apparent choices of polarizations like the negative spectral subspace of \( H^A \)
which allow for pair-creation are not Lorentz invariant.

This suggests that a particle/anti-particle picture can presumably not be based on
spectral considerations. Instead of choosing specific polarizations we consider equivalence
classes of polarizations. It turns out that an appropriate equivalence relation “\( \approx \)” between
polarizations is given by the condition that the difference of the corresponding orthogonal
projectors is a Hilbert-Schmidt operator. This is in accordance with: First, the intuition
described along with (1.2) which implies that transition amplitudes stay well-defined. And
second, if \( A(t_0) = 0 \), the equivalence class \( C(t) = \{ U^{A}(t,t_0)\mathcal{H}^- \}_{\approx} \), \( t > t_0 \), turns out to
depend only on the external potential \( A(t) \) at time \( t \) but not on the history \( (A(s))_{t_0<s< t} \).
A specific choice of polarization in this equivalence class is then mathematically a choice
of reference frame with respect to which we represent the second-quantized Dirac time-
evolution. This brings us to the content of this work:

1. We show that these polarization classes \( C(t) \) are uniquely identified by the spatial
(magnetic) components \( \vec{A}(t) \) of the field \( A \) at time \( t \); see Theorem 3.9, Subsection 3.2
This generalizes the case of \( \vec{A}(t) = 0 \) regarded in \([Rui77a]\) to general \( \vec{A}(t) \).

2. We give a simple representative \( e^{Q^{A}(t)}\mathcal{H}^- \in C(t) \) for each polarization class in terms of
a simple and explicit operator \( Q^{A}(t) \) which naturally appears as the key object in the
variant of the Born series of \( U^{A} \) that we use in Subsection 3.1.

3. We implement the Dirac time-evolution as unitary maps between between time-varying
Fock spaces, in Theorem 4, Subsection 4
This implementation is unique up to a phase.

4. We conclude with a brief discussion of gauge transformations of the external field; see
Theorem 3.12, Subsection 3.3.
A next step would be to derive the polarization charge current within this framework which also must be defined in a neat way so that it accounts only for the “net description” comparing two Dirac seas as mentioned above.

Our work in this field of QED was mainly inspired by Dirac’s original idea \cite{Dir34b, Dir34a}, the work of Fierz and Scharf \cite{FS79}, Scharf’s book \cite{Sch95} as well as Pressley and Segal’s book \cite{PS86} and also the work of Mickelson et al. \cite{LM96, Mic98}. Furthermore, we would like to call attention to an approach to QED by Finster \cite{Fin06, Fin09a, Fin08, Fin09b} known under the name: “The Fermionic Projector”. Though mathematically different, his approach also revisits Dirac’s original idea \cite{Dir34a, Dir34b} in a serious way. The main difference between the Fermionic Projector approach and ours is that we do not seek a distinguished polarization or, in other words, a unique vacuum. Moreover, we use a Fock space description which by construction allows for superposition and entanglement which at the moment seems to be elaborate using fermionic projectors.

The Setup. The purpose of this paragraph is to give a heuristic description of how we construct the second quantized time-evolution of the Dirac Hamiltonian in the presence of a time-dependent, external field. What is described in this Subsection will be rigorously introduced and proven in Sections 2 and 3. The definitions and assertions will later be formulated in a general form.

Since we aim at a description depending only on polarization classes instead of specific polarizations we resort to a representation of the Fock space which is different to the standard one ($\mathcal{F}(\mathcal{H}_-, \mathcal{H}_+)$). We shall refer to it as the infinite wedge product spaces. Although our results can be rephrased in the standard Fock space language, the infinite wedge product formalism, in our opinion, is closer to Dirac’s original idea \cite{Dir34b, Dir34a} and opens up a more transparent view on the nature of the second-quantized Dirac time-evolution and on the role of the Dirac sea (i.e. Fock vacuum).

We construct Dirac seas concretely as infinite wedge products like in \cite{Dir34b}: given a polarization $V \subset \mathcal{H}$ and for that an orthonormal basis $\varphi = (\varphi_n)_{n \in \mathbb{N}}$ that spans $V$ the alternating product of all $\varphi_n$, $n \in \mathbb{N}$, is supposed to represent a Dirac sea belonging to polarization $V$. We introduce an equivalence class $S = S(\varphi)$ of other representatives, namely of all sequences $\psi = (\psi_n)_{n \in \mathbb{N}}$ in $\mathcal{H}$ such that the $\mathbb{N} \times \mathbb{N}$-matrices

$$
\langle \psi_n, \varphi_m \rangle_{n,m \in \mathbb{N}}
$$

and

$$
\langle \psi_n, \varphi_m \rangle_{n,m \in \mathbb{N}},
$$

(1.3)

\langle \cdot, \cdot \rangle denoting the inner product on $\mathcal{H}$, differ from the unity matrix only by a matrix in the trace class and thus have a determinant. In this case we write $\psi \sim \phi$. This is our notion of Dirac seas being asymptotically equal “deep down in the sea”. We define the following bracket:

$$
\langle \psi, \chi \rangle := \det(\langle \psi_n, \chi_m \rangle)_{n,m \in \mathbb{N}} = \lim_{k \to \infty} \det(\langle \psi_n, \chi_m \rangle)_{n,m=1,...,k}, \quad \psi, \chi \in S.
$$

(1.4)
With this at hand one constructs a Hilbert space $\mathcal{F}_S = \mathcal{F}_{S(\varphi)}$, where the bracket gives rise to the inner product. We refer to $\mathcal{F}_S$ as the \textit{infinite wedge space}. By this construction, see Definition 2.17, a sequence $\psi \in S$ is mapped to the wedge product $\Lambda \psi = \psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \ldots \in \mathcal{F}_S$. The rigorous construction of wedge spaces is carried out in Subsection 2.1. In Section 2.4 we also discuss the relationship of $\mathcal{F}_{S(\varphi)}$ to the standard Fock space. It is important to note that $\Lambda \varphi$ has no meaning as “the one and only” Dirac sea since $\mathcal{F}_S$ depends only on the equivalence class $S = S(\varphi)$. In fact, changing the reference $\varphi$ within the same equivalence class $S$ can be viewed as a Bogolyubov transformation.

The equivalence relation $\approx$ between two polarizations will be refined as follows: For two polarizations $V, W$ we define $V \approx_0 W$ to mean $V \approx W$ and that $V$ and $W$ have the same “relative charge”. Intuitively the “relative charge” has the following meaning: Consider two states $\Lambda \varphi$ and $\Lambda \psi$ where $\varphi$ and $\psi$ are orthonormal bases of $V$ and $W$, respectively. Then the relative charge is the difference of the electric charges of the physical states represented by $\Lambda \varphi$ and $\Lambda \psi$, respectively. Mathematically the relative charge is defined in terms of Fredholm indices in Definition 2.29. The use of the Fredholm index to describe the relative charge is quite frequent in the literature; see e.g. PS85, LM96 as well as in the work of Hainzl et al. HLS05. The relation $\approx_0$ is also an equivalence relation on the set of polarizations, and one finds an intimate connection between this equivalence relation $\approx_0$ on the set of polarizations and the equivalence relation $\sim$ on the set Dirac seas: Two equivalent Dirac seas span two equivalent polarizations and for every two polarizations $W \approx_0 V$ such that $\varphi$ spans $V$ there is a Dirac sea $\Lambda \psi \in \mathcal{F}_{S(\varphi)}$ such that $\psi \sim \varphi$ and $\psi$ spans $W$. Consequently, every wedge space can be associated with a polarization class with respect to $\approx_0$. Details are given in Section 2.1.

On the other hand, assuming $\varphi$ spans $V$, not all Dirac seas $\Lambda \psi$ such that $\psi$ spans $W \approx_0 V$ are in $\mathcal{F}_{S(\varphi)}$ because one can obviously find an orthonormal basis $\psi$ of $W$ for which $\{(\psi_n, \varphi_m)\}_{h, m \in \mathbb{N}}$ differs from the identity by more than a trace class operator. Because of this we consider below operations (the operations from the right) that mediate between all wedge spaces belonging to the same polarization class with respect to $\approx_0$. These operations are needed to later define the physically relevant transition probabilities.

On any element of $S$ the action of any unitary map $U$ on $\mathcal{H}$ is then naturally defined by having it act on each factor of the wedge product. Consequently we have a (left) operation on any $\mathcal{F}_S$, namely $L_U : \mathcal{F}_S \rightarrow \mathcal{F}_{US}$, such that

$$L_U (\psi_1 \wedge \psi_2 \wedge \psi_3 \wedge \ldots) = U\psi_1 \wedge U\psi_2 \wedge U\psi_3 \wedge \ldots, \quad \psi \in S,$$

which then incorporates a “lift” of $U$ as a unitary map from one wedge space to another. Now, this can of course also be done for the one-particle time evolution $U = U^A(t, t_0)$ for fixed times $t_0$ and $t$. However, we need to find a way to relate the Dirac seas in $\mathcal{F}_S$ to the ones in $\mathcal{F}_{US}$ by considering the “net balance” between them. As we discussed already what we mathematically have at hand are the polarization classes $C(t_0)$ and $C(t)$ at times $t_0$ and $t$. We choose any two polarizations $V \in C(t_0)$ and $W \in C(t)$ and orthonormal bases $\varphi(t_0)$ of $V$ and likewise $\varphi(t)$ of $W$ and denote their equivalence classes with respect
to \( \sim \) by \( S(\varphi(t_0)) \) and \( S(\varphi(t)) \). This way physical “in” and “out” states can be described by elements in \( \psi^{in} \in F_{S(\varphi(t_0))} \) and \( \psi^{out} \in F_{S(\varphi(t))} \), respectively. But in general \( US(\varphi(t_0)) \) will not be equal \( S(\varphi(t)) \) so that \( L_U \psi^{in} \) and \( \psi^{out} \) are likely to lie in different wedge spaces. However, we show that the polarization classes of \( F_{US(\varphi(t_0))} \) and \( F_{S(\varphi(t))} \) are the same. Therefore, the only difference between those two spaces may come from our specific choice of bases \( \varphi(t_0) \) and \( \varphi(t) \). In order to make them compatible we introduce another operation (from the right): For all unitary \( N \times N \)-matrices \( R = (R_{nm})_{n,m \in \mathbb{N}} \), we define the operation from the right as follows. For \( \psi \in S \), let \( \psi R := (\sum_{n \in \mathbb{N}} \psi_n R_{nm})_{m \in \mathbb{N}} \). In this way, every unitary \( R \) gives rise to a unitary map \( R \) : \( F_{S} \to F_{SR} \), such that

\[
\mathcal{R}_R \psi = \Lambda(\psi R), \quad \psi \in S. \tag{1.6}
\]

By construction the operations from the left and from the right commute. We show that two such unitary matrices \( R, R' \) yield \( F_{SR} = F_{SR'} \) if and only if \( R^{-1} R' \) has a determinant. Furthermore, in Subsection 3.1 we prove that there always exists a unitary matrix \( R \) for which \( US(t_0)R = S(t) \). Now we have all we need to compute the transition probabilities:

\[
|\langle \psi^{out}, \mathcal{R}_R \circ L_U \psi^{in}\rangle|^2. \tag{1.7}
\]

The matrix \( R \) is not unique because for any \( R' \) with \( \det R' = 1 \) one has \( US(t_0)R = US(t_0)RR' = S(t) \) so that the arbitrariness in \( R' \) gives rise to a phase. However, this has clearly no effect on the transition probabilities.

As an example let us consider \( \psi^{in} \) and \( \psi^{out} \) being Hartree-Fock states: Let \( V \in C(t_0) \) and \( W \in C(t_1) \) be appropriate polarizations describing the one-particle wave functions present in a given experimental setup. Furthermore, let \( \varphi \) and \( \psi \) to be orthogonal bases in \( V \) and \( W \), respectively, and \( \psi^{in} = \Lambda \varphi \) as well as \( \psi^{out} = \Lambda \phi \). Using (1.7), (1.4) and the notation \( |A|^2 = A^* A \) one can express the transition probability as

\[
|\langle \Lambda \psi, \mathcal{R}_R \circ L_U \Lambda \varphi \rangle|^2 = |\det \langle \psi_n, (U \varphi R)_m \rangle_{n,m}|^2 \tag{1.8}
\]

\[
= |\det |P_W U P_V|^2|_{V \to V} = |1 - |P_W U P_V|^2|_{V \to V}. \tag{1.9}
\]

where we have used that \( R \) and \( U \) are unitary. This determinant is well-defined whenever its argument differs from the identity only by a trace class operator. Hence, the above expression is only well-defined for \( P_W U P_V \) being a Hilbert-Schmidt operator. The leading order of this determinant is given by \( 1 - \|P_W U P_V\|_2 \), which agrees with the intuition given along (1.2). The operations from the left and from the right are introduced in Subsection 2.2 while in Subsection 2.3 we identify the conditions under which \( R \) exists.

The next step would be to derive the charge current of the created pairs. It must be also defined in a neat way so that it accounts only for the “net description” and not for a whole Dirac sea. For doing that one would e.g. need to analyze the behavior of the phase, cf. [Sch95], which will be the content of our subsequent paper.
2 Infinite Wedge Spaces

2.1 Construction

In this section we give a rigorous construction of infinite wedge products which we described in the introduction. Throughout this work the notion Hilbert space stands for separable, infinite dimensional, complex Hilbert space. Let $\mathcal{H}$ and $\ell$ be Hilbert spaces with corresponding scalar products $\langle \cdot, \cdot \rangle$. For a typical example think of $\mathcal{H} = L^2(\mathbb{R}^3, \mathbb{C}^4)$ and $\ell = \ell^2(\mathbb{N})$, the space of square summable sequences in $\mathbb{C}$. The space $\ell$ will only play the role of an index space. We refer to $\mathcal{H}$ as the one-particle Hilbert space. Furthermore, we denote the space of so-called trace class operators on $\ell$, i.e. bounded operators $T$ on $\ell$ for which $\|T\|_{I_1} := \text{tr} \sqrt{T^*T}$ is finite, by $I_1(\ell)$, the superscript $*$ denoting the Hilbert space adjoint. We say a bounded linear operator $T$ on a Hilbert space $\ell$ has a determinant if it differs from the identity operator $\text{id}_\ell$ on $\ell$ only by a trace class operator, i.e. $T - \text{id}_\ell \in I_1(\ell)$; see [GGK90]. We also need the space of Hilbert-Schmidt operators, i.e. the space of bounded operators $T : \ell \to \mathcal{H}$ such that the Hilbert-Schmidt norm $\|T\|_{I_2} := \text{tr} T^*T$ is finite. The space of Hilbert-Schmidt operators is denoted by $I_2(\ell, \mathcal{H})$, and we write $I_2(\mathcal{H}) = I_2(\mathcal{H}, \mathcal{H})$.

At first let us define the notions: polarizations, polarization classes and the set of Dirac seas from the introduction.

**Definition 2.1 (Polarizations and Polarization Classes).** (a) Let $\text{Pol}(\mathcal{H})$ denote the set of all closed, linear subspaces $V \subset \mathcal{H}$ such that $V$ and $V^\perp$ are both infinite dimensional. Any $V \in \text{Pol}(\mathcal{H})$ is called a polarization of $\mathcal{H}$. For $V \in \text{Pol}(\mathcal{H})$, let $P_V : \mathcal{H} \to V$ denote the orthogonal projection of $\mathcal{H}$ onto $V$.

(b) For $V, W \in \text{Pol}(\mathcal{H})$, $V \approx W$ means $P_W - P_V \in I_2(\mathcal{H})$.

The space $\text{Pol}(\mathcal{H})$ is a kind of Grassmann space of all infinite dimensional closed subspaces with infinite dimensional complement. Obviously, the relation $\approx$ is an equivalence relation on $\text{Pol}(\mathcal{H})$. Its equivalence classes $C \in \text{Pol}(\mathcal{H})/\approx$ are called polarization classes. Its basic properties are collected in the following lemma. We denote by $|X \to Y|$ the restriction to a map from $X$ to $Y$.

**Lemma 2.2 (Properties of $\approx$).** For $V, W \in \text{Pol}(\mathcal{H})$, the following are equivalent:

(a) $V \approx W$

(b) $P_W P_V \in I_2(\mathcal{H})$ and $P_W P_V^\perp \in I_2(\mathcal{H})$

(c) The operators $P_V P_W P_V^\perp |_{V \to V}$ and $P_W P_V P_W^\perp |_{W \to W}$ both have determinants.

(d) The operators $P_W P_V P_V^\perp |_{V \to V}$ and $P_V P_W P_V^\perp |_{V \to V}$ both have determinants.

(e) $P_W |_{V \to W}$ is a Fredholm operator and $P_W^\perp |_{V \to W} \in I_2(V)$.
Proof.

(a)⇒(b): Let \( V, W \in \text{Po1}(\mathcal{H}) \) fulfill \( P_V - P_W \in I_2(\mathcal{H}) \). We conclude that

\[
P_{V \perp} P_V = (\text{id}_{H} - P_W)P_V = (P_V - P_W)P_V \in I_2(\mathcal{H}) \quad \text{and} \quad (2.1)
\]

\[
P_W P_{V \perp} = P_W (\text{id}_{H} - P_V) = -P_W (P_V - P_W) \in I_2(\mathcal{H}). \quad (2.2)
\]

(b)⇒(c): Assuming (b), we conclude

\[
P_V - P_V P_W P_V = P_V P_{V \perp} P_V = (P_{V \perp} P_V)^*(P_{V \perp} P_V) \in I_1(\mathcal{H}) \quad \text{and} \quad (2.3)
\]

\[
P_W - P_W P_V P_W = P_W P_{V \perp} P_W = (P_{V \perp} P_V)^*(P_{V \perp} P_V) \in I_1(\mathcal{H}). \quad (2.4)
\]

This implies \( (P_V - P_V P_W P_V)|_{V \rightarrow V} \in I_1(V) \) and \( (P_W - P_W P_V P_W)|_{W \rightarrow W} \in I_1(W) \) and thus the claim (c).

(c)⇒(d): Assuming (c), we need to show that \( P_{V \perp} P_{W \perp} P_{V \perp}|_{V \rightarrow V \perp} \) has a determinant. Indeed: As \( P_W P_V P_W|_{W \rightarrow W} \) has a determinant, we know that

\[
(P_{V \perp} P_V)^*(P_{V \perp} P_V) = P_W P_{V \perp} P_W = P_W - P_W P_V P_W \in I_1(\mathcal{H}) \quad (2.5)
\]

and thus \( P_{V \perp} P_W \in I_2(\mathcal{H}) \). This implies

\[
P_{V \perp} - P_{V \perp} P_W P_{V \perp} = P_{V \perp} P_W P_{V \perp} = (P_{V \perp} P_V)(P_{V \perp} P_V)^* \in I_1(\mathcal{H}). \quad (2.6)
\]

The claim \( (P_{V \perp} P_{W \perp} P_{V \perp}|_{V \rightarrow V \perp}) \in \text{id}_{V \perp} + I_1(V \perp) \) follows.

(d)⇒(b): Assuming (d), we know

\[
(P_W P_{V \perp})^*(P_W P_{V \perp}) = P_{V \perp} P_W P_{V \perp} = P_{V \perp} - P_{V \perp} P_W P_{V \perp} \in I_1(\mathcal{H}) \quad \text{and} \quad (2.7)
\]

\[
(P_{V \perp} P_V)^*(P_{V \perp} P_V) = P_V P_{V \perp} P_V = P_V - P_V P_W P_V \in I_1(\mathcal{H}). \quad (2.8)
\]

This implies the claim (b).

(b)⇒(a): Assuming \( P_{V \perp} P_V \in I_2(\mathcal{H}) \) and \( P_W P_{V \perp} \in I_2(\mathcal{H}) \), we conclude that

\[
P_V - P_W = (P_V - P_W P_V) - (P_W - P_W P_V) = P_{V \perp} P_V - P_W P_{V \perp} \in I_2(\mathcal{H}). \quad (2.9)
\]

(b)⇒(e): We write the identity on \( \mathcal{H} \) in matrix form

\[
\text{id}_{\mathcal{H}} : V \oplus V^\perp \rightarrow W \oplus W^\perp, \quad (x, y) \mapsto \begin{pmatrix} P_W|_{V \rightarrow W} & P_W|_{V \rightarrow W}^\perp \\ P_{V \perp}|_{V \rightarrow W} & P_{V \perp}|_{V \rightarrow W}^\perp \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \quad (2.10)
\]
Assuming (b) we know that the non-diagonal operators $P_{W^⊥|V→W^⊥}$ and $P_{W|V^⊥→W}$ are Hilbert-Schmidt operators. Subtracting the non-diagonal from the identity we get a new map

$$Q : V ⊕ V^⊥ → W ⊕ W^⊥, \quad (x, y) → \begin{pmatrix} P_{W|V→W} & 0 \\ 0 & P_{W^⊥|V^⊥→W^⊥} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$  (2.11)

which is by construction a perturbation of the identity by a compact operator and, thus, a Fredholm operator. However, this holds if and only if both $P_{W|V→W}$ and $P_{W^⊥|V^⊥→W^⊥}$ are Fredholm operators which implies (e).

(e)⇒(b): Assuming (e) we compute

$$0 = P_{V^⊥|V} = P_{V^⊥}(P_W + P_{W^⊥})|V$$

$$= P_{V^⊥|W}P_W|V→W + P_{V^⊥|W^⊥}P_{W^⊥}|V→W^⊥$$  (2.12)

from which follows that $P_{V^⊥|W}P_W|V$ is a Hilbert-Schmidt operator since $P_{V^⊥|W^⊥}$ is a bounded operator and by assumption $P_{W^⊥|V} ∈ I_2(V)$. Furthermore, by assumption $P_{W|V→W}$ is a Fredholm operator so that $P_{V^⊥|W}P_{W|V→W} ∈ I_2(V)$ yields $P_{V^⊥|W} ∈ I_2(W)$. Finally, we have $P_{W^⊥|V} = P_{W^⊥}P_{V^⊥}|V$ and $P_{V^⊥|W} = P_{V^⊥}P_{W^⊥}|W$ so that $P_{W^⊥}P_V, P_{V^⊥}P_W ∈ I_2(ℋ)$ which implies the claim (b).

Note that in general $P_{W^⊥}P_V ∈ I_2(ℋ)$ is not equivalent to $P_WP_{V^⊥} ∈ I_2(ℋ)$. As an example, take $V$ and $W$ such that $V ⊂ W$ and $V$ has infinite codimension in $W$; compare with condition (c) of the above lemma. Condition (c) appears in Chapter 7 of [PS86] where an equivalence class $C ∈ Pol(ℋ)/\approx$ is endowed with the structure of a complex manifold modeled on infinite dimensional separable Hilbert spaces. Consequently, the space $Pol(ℋ)$ is a complex manifold – the Grassmann manifold of $ℋ$ – which decomposes into the equivalence classes $C ∈ Pol(ℋ)/\approx$ as open and closed submanifolds.

Where exactly the one-particle Hilbert space $ℋ$ is cut into parts by a choice of a polarization in a polarization class will determine the relative charge between two Dirac seas. Within one polarization class the charge may only differ by an integer from one chosen polarization to another. Given $V, W ∈ Pol(ℋ)$ with $V ≈ W$, we know from Lemma 2.2 (c) that $P_{W|V→W}$ and $P_{V|W→V}$ are Fredholm operators. So we are led to the definition of the relative charge:

**Definition 2.3** (Relative Charge). For $V, W ∈ Pol(ℋ)$ with $V ≈ W$, we define the relative charge of $V, W$ to be the Fredholm index of $P_{W|V→W}$:

$$\text{charge}(V, W) := \text{ind}(P_{W|V→W}) = \dim \ker(P_{W|V→W}) - \dim \ker(P_{W|V→W})^*$$

$$= \dim \ker(P_{W|V→W}) - \dim \coker(P_{W|V→W})$$  (2.13)
Let $U(\mathcal{H},\mathcal{H}')$ be the set of unitary operators $U : \mathcal{H} \rightarrow \mathcal{H}'$. We collect some basic properties of the relative charge:

**Lemma 2.4 (Relative Charge Properties).** Let $C \in \text{Pol}(\mathcal{H})/\approx$ be a polarization class and $V, W, X \in C$. Then the following hold:

(a) \[\text{charge}(V, W) = -\text{charge}(W, V)\]

(b) \[\text{charge}(V, W) + \text{charge}(W, X) = \text{charge}(V, X)\]

(c) Let $\mathcal{H}'$ be another Hilbert space and $U \in U(\mathcal{H}, \mathcal{H}')$. Then $\text{charge}(V, W) = \text{charge}(UV, UW)$.

(d) Let $U \in U(\mathcal{H}, \mathcal{H})$ such that $UC = C$. Then $\text{charge}(V, UV) = \text{charge}(W, UW)$.

**Proof.**

(a) $P_W|_{V \rightarrow W}$ and $P_V|_{W \rightarrow V}$ are Fredholm operators with

\[
\text{charge}(V, W) + \text{charge}(V, W) = \text{ind}(P_V|_{W \rightarrow V}) + \text{ind}(P_W|_{V \rightarrow W})
= \text{ind}(P_V P_W|_{V \rightarrow V}) = \text{ind}(P_V P_W P_V|_{V \rightarrow V}) = 0, \tag{2.14}
\]

as $P_V P_W P_V|_{V \rightarrow V}$ is a perturbation of the identity map on $V$ by a compact operator.

(b) As $P_W$ and $P_X$ differ only by a compact operator, we get

\[
\text{charge}(V, W) + \text{charge}(W, X) = \text{ind}(P_W|_{V \rightarrow W}) + \text{ind}(P_X|_{W \rightarrow X})
= \text{ind}(P_X P_W|_{V \rightarrow X}) = \text{ind}(P_X P_X|_{V \rightarrow X}) = \text{charge}(V, X). \tag{2.15}
\]

(c) This follows immediately, since unitary transformations do not change the Fredholm index.

(d) We know $UV \approx V \approx W \approx UW$ by assumption. Using parts (a), (b) and (c) of the lemma, this implies

\[
\text{charge}(V, UV) = \text{charge}(V, W) + \text{charge}(W, UW) + \text{charge}(UW, UV)
= \text{charge}(V, W) + \text{charge}(W, UW) + \text{charge}(W, V) = \text{charge}(W, UW). \tag{2.16}
\]

With the notion of relative charge we refine the polarization classes further into classes of polarizations of equal relative charge:

**Definition 2.5 (Equal Charge Classes).** For $V, W \in \text{Pol}(\mathcal{H})$, $V \approx_0 W$ means $V \approx W$ and $\text{charge}(V, W) = 0$.

By Lemma 2.4 (Relative Charge Properties) the relation $\approx_0$ is an equivalence relation on $\text{Pol}(\mathcal{H})$. This finer relation is better adapted for the lift of unitary one-particle operators like the Dirac time-evolution which conserve the charge.

Next, we introduce the mathematical representation of the Dirac seas:
Definition 2.6 (Dirac Seas). (a) Let $\text{Seas}(\mathcal{H}) = \text{Seas}_\ell(\mathcal{H})$ be the set of all linear, bounded operators $\Phi : \ell \to \mathcal{H}$ such that range $\Phi \in \text{Pol}(\mathcal{H})$ and $\Phi^*\Phi : \ell \to \ell$ has a determinant, i.e. $\Phi^*\Phi \in \text{id}_\ell + I_1(\ell)$.

(b) Let $\text{Seas}^\perp(\mathcal{H}) = \text{Seas}_\ell^\perp(\mathcal{H})$ denote the set of all linear isometries $\Phi : \ell \to \mathcal{H}$ in $\text{Seas}_\ell(\mathcal{H})$.

(c) For any $C \in \text{Pol}(\mathcal{H})/\approx_0$ let $\text{Ocean}(C) = \text{Ocean}_\ell(C)$ be the set of all $\Phi \in \text{Seas}_\ell^\perp(\mathcal{H})$ such that range $\Phi \in C$.

Figuratively speaking, an ocean consists of a collection of related seas. To connect to the introduction in Subsection 1.2, consider the following example: In the case of $\ell = \ell_2(\mathbb{N})$ we encode this map in an orthonormal basis $(\varphi_n)_{n \in \mathbb{N}}$ of $V$ such that for the canonical basis $(e_n)_{n \in \mathbb{N}}$ in $\ell^2$ one has $\varphi_n = e_n$ for all $n \in \mathbb{N}$.

The set $\text{Seas}(\mathcal{H})$ can naturally be structured by the relation introduced now:

Definition 2.7 (Relation between Dirac Seas). For $\Phi, \Psi \in \text{Seas}(\mathcal{H})$, $\Phi \sim \Psi$ means $\Phi^*\Psi \in \text{id}_\ell + I_1(\ell)$, i.e. $\Phi^*\Psi$ has a determinant.

In the forthcoming Corollary 2.9 we show that $\sim$ is an equivalence relation. For its proof we need the following lemma, which will also be frequently used later because it allows us to work for most purposes with Dirac seas in $\text{Seas}^\perp(\mathcal{H})$ instead of $\text{Seas}(\mathcal{H})$:

Lemma 2.8 (Isometries are good enough). For every $\Psi \in \text{Seas}(\mathcal{H})$ there exist $\Upsilon \in \text{Seas}^\perp(\mathcal{H})$ and $R \in \text{id}_\ell + I_1(\ell)$ which fulfill $\Psi = \Upsilon R$, $\Upsilon^*\Psi = R \geq 0$, $\Upsilon \sim \Psi$, and $R^2 = \Psi^*\Psi$.

Proof. Let $\Psi \in \text{Seas}(\mathcal{H})$. The operator $\Psi^*\Psi : \ell \to \ell$ has a determinant and is hence a Fredholm operator. In particular, ker$(\Psi^*\Psi) = \text{ker} \Psi$ is finite dimensional. Let $\Psi = VR$ be the polar decomposition of $\Psi$, with $R = \sqrt{\Psi^*\Psi}$ and $V : \ell \to \mathcal{H}$ being a partial isometry with ker$V = (\text{range } R)^\perp = \text{ker } \Psi$. Then $V$ and $\Psi$ have the same range, and this range has infinite codimension in $\mathcal{H}$. Since ker$V$ has finite dimension, we can extend the restriction of $V$ to $(\text{ker } V)^\perp$ to an isometry $\Upsilon : \ell \to \mathcal{H}$. We get: $\Upsilon^*\Psi = V^*\Psi = V^*VR = R \geq 0$ and $\Upsilon R = VR = \Psi$. Now, as $R^2 = \Psi^*\Psi$ has a determinant, its square root $R$ has also a determinant. This implies $\Upsilon \sim \Psi$. 

Corollary 2.9. The relation $\sim$ is an equivalence relation on $\text{Seas}(\mathcal{H})$.

Proof. By definition of $\text{Seas}(\mathcal{H})$, the relation $\sim$ is reflexive. To show symmetry, take $\Phi, \Psi \in \text{Seas}(\mathcal{H})$ with $\Phi \sim \Psi$. We conclude $\Psi^*\Phi - \text{id}_\ell = (\Phi^*\Psi - \text{id}_\ell)^* \in I_1(\ell)$ and thus $\Psi \sim \Phi$. To show transitivity, let $\Phi, \Psi, \Gamma \in \text{Seas}(\mathcal{H})$ with $\Phi \sim \Psi$ and $\Psi \sim \Gamma$. By Lemma 2.8 (Isometries are good enough), take $\Upsilon \in \text{Seas}^\perp(\mathcal{H})$ and $R \geq 0$ corresponding to $\Psi$. Let $P : \mathcal{H} \to \mathcal{H}$ denote the orthogonal projection having the same range as $\Upsilon$, and let
Let \( P^c = \text{id}_H - P \) denote the complementary projection. In particular, one has \( P = \Upsilon^* \Upsilon^* \). Then
\[
\Phi^* \Gamma = \Phi^* P \Gamma + \Phi P^c \Gamma = (\Phi^* \Upsilon)(\Gamma^* \Upsilon)^* + \Phi^* P^c \Gamma. \tag{2.17}
\]
Now, since \( \Phi \sim \Psi \) we know that \( \Phi^* \Psi = \Phi^* \Upsilon \) has a determinant. Since \( R \) has also a determinant, we conclude that \( \Phi^* \Gamma \) has a determinant, too. Using \( \Psi \sim \Gamma \), the same argument shows that \( \Gamma^* \Upsilon \) has a determinant, and thus \( (\Phi^* \Upsilon)(\Gamma^* \Upsilon)^* \) has a determinant. Next we show that \( P^c \Gamma \) is a Hilbert-Schmidt operator. Indeed, \( (P^c \Gamma)^* (P^c \Gamma) = \Gamma^* \Gamma - \Gamma^* P \Gamma = \Gamma^* \Gamma - (\Gamma^* \Upsilon)(\Gamma^* \Upsilon)^* \) is a difference of two operators having a determinant, since \( \Gamma^* \Gamma \) and \( \Gamma^* \Upsilon \) both have determinants. Hence, \( (P^c \Gamma)^* (P^c \Gamma) \in I_1(\ell) \), which implies \( P^c \Gamma \in I_2(\ell) \). We conclude \( \Phi^* P^c \Gamma = (P^c \Phi)^* (P^c \Gamma) \in I_1(\ell) \). Using (2.17) this yields that \( \Phi^* \Gamma \) has a determinant, as the sum of an operator having a determinant and a trace class operator. This proves that \( \Phi \sim \Gamma \).

For \( \Phi \in \text{Seas}(H) \), the equivalence class of \( \Phi \) with respect to \( \sim \) turns out to form an affine space. The following definition and lemma characterize these equivalence classes. These properties will later be used to show that the wedge spaces to be constructed (in forthcoming Definition 2.17 (Infinite Wedge Spaces)) are separable spaces.

**Definition 2.10 (Dirac Sea Classes).** Let \( \Phi \in \text{Seas}(H) \).

(a) Let \( S(\Phi) \subset \text{Seas}(H) \) denote the equivalence class of \( \Phi \) with respect to \( \sim \).

(b) For bounded linear, operators \( L: \ell \to H \), we define
\[
\| L \|_{\phi} := \| \Phi^* L \|_{I_1} + \| L \|_{I_2}
\]
and the vector space
\[
\mathcal{V}(\Phi) := \{ L: \ell \to H \mid L \text{ is linear and bounded with } \| L \|_{\phi} < \infty \}.
\]

**Lemma 2.11 (Dirac Sea Class Properties).** Let \( \Phi \in \text{Seas}(H) \).

(a) It holds that \( S(\Phi) = \Phi + \mathcal{V}(\Phi) \).

(b) For \( \Psi \in \text{Seas}(H) \) with \( \Phi \sim \Psi \), one has \( \mathcal{V}(\Phi) = \mathcal{V}(\Psi) \), and the norms \( \| \cdot \|_{\phi} \) and \( \| \cdot \|_{\psi} \) are equivalent.

**Proof.** (a) Take \( \Psi \sim \Phi \). By definition, \( \Phi^* \Psi \in \text{id}_\ell + I_1(\ell) \) and \( \Phi^* \Phi \in \text{id}_\ell + I_1(\ell) \). The difference yields \( \Phi^* (\Psi - \Phi) \in I_1(\ell) \). Similarly, \( \Psi^* \Psi \in \text{id}_\ell + I_1(\ell) \) and \( \Psi^* \Phi \in \text{id}_\ell + I_1(\ell) \). Combining all this, we get \( (\Psi - \Phi)^* (\Psi - \Phi) \in I_1(\ell) \), and hence \( \Psi - \Phi \in I_2(\ell, H) \). This shows \( \Psi - \Phi \in \mathcal{V}(\Phi) \).

Conversely, take \( B \in \mathcal{V}(\Phi) \). We set \( \Psi = \Phi + B \). First we show that range \( \Psi \in \text{Pol}(H) \), i.e. that it is closed and has infinite dimension and codimension. To do this we use the following general fact: A Fredholm operator between two Hilbert spaces maps closed,
infinite dimensional, and infinite codimensional subspaces, respectively. Consider
\[
\tilde{\Phi} : \ell \oplus \text{range } \Phi^\perp \rightarrow \mathcal{H}, \quad (x, y) \mapsto \Phi x + y \tag{2.18}
\]
\[
\tilde{\Psi} : \ell \oplus \text{range } \Phi^\perp \rightarrow \mathcal{H}, \quad (x, y) \mapsto \Phi x + Bx + y \tag{2.19}
\]
with the direct sum is understood as orthogonal direct sum. Since range \( \Phi \) is in Pol(\( \mathcal{H} \)) and therefore closed, the map \( \tilde{\Phi} \) is onto. Furthermore, \( \Phi^*\Phi \in \text{id}_{\mathcal{H}} + I_1(\mathcal{H}) \) is a perturbation of the identity by a compact operator and therefore a Fredholm operator. In particular, this implies \( \dim \ker \Phi = \dim \ker \Phi^*\Phi < \infty \). Thus, \( \tilde{\Phi} \) is also a Fredholm operator. Now, \( \tilde{\Psi} \) is a perturbation of \( \tilde{\Phi} \) by the compact operator \( (x, y) \mapsto Bx \) and therefore is Fredholm operator, too. Since \( \ell \oplus 0 \) is closed, infinite dimensional, infinite and codimensional, so is range \( \tilde{\Psi} = \tilde{\Psi}(\ell \oplus 0) \).

Using \( \Phi \in \text{Seas}(\mathcal{H}) \) and the definition of \( \mathcal{V}(\Phi) \), we get \( \Psi^*\Psi = \Phi^*\Phi + \Phi^*B + (\Phi^*B)^* + B^*B \in (\text{id}_\ell + I_1) + I_1 + I_2, \) hence \( \Phi^*\Phi \in \text{Seas}(\mathcal{H}) \). Furthermore, \( \Phi^*\Psi = \Phi^*\Phi + \Phi^*B \in (\text{id}_\ell + I_1) + I_1 = \text{id}_\ell + I_1(\ell) \) holds. This yields \( \Psi \sim \Phi \).

(b) Since \( \Phi \sim \Psi \) there is a \( L \in \mathcal{V}(\Phi) \) such that \( \Phi = \Psi + L \). Let \( M \in \mathcal{V}(\Psi) \). Using the triangle inequality in \( I_1(\ell) \) and \( \|L^*M\|_{I_1} \leq \|L\|_{I_2}\|M\|_{I_2} \), we get
\[
\|M\|_\Phi = \|\Phi^*M\|_{I_1} + \|M\|_{I_2} \leq \|\Phi^*M\|_{I_1} + \|L^*M\|_{I_1} + \|M\|_{I_2} \leq (1 + \|L\|_{I_2})(\|\Phi^*M\|_{I_1} + \|M\|_{I_2}) = (1 + \|L\|_{I_2})\|M\|_\Psi. \tag{2.20}
\]
In the same way, we get \( \|M\|_\Psi \leq (1 + \|L\|_{I_2})\|M\|_\Phi \).

The equivalence classes of \( \text{Seas}(\mathcal{H})/\sim \) and \( \text{Pol}(\mathcal{H})/\sim_0 \) go hand in hand quite naturally as the following lemma shows.

**Lemma 2.12** (Connection between \( \sim \) and \( \approx_0 \)). Given \( C \in \text{Pol}(\mathcal{H})/\approx_0 \) and \( \Phi \in \text{OceN}(C) \) we have
\[
C = \{\text{range } \Psi \mid \Psi \in \text{Seas}^\perp(\mathcal{H}) \text{ such that } \Psi \sim \Phi \}.
\]

**Proof.** Let \( C' := \{\text{range } \Psi \mid \Psi \in \text{Seas}^\perp(\mathcal{H}) \text{ such that } \Psi \sim \Phi \} \) and \( V := \text{range } \Phi \).

\( C' \subseteq C \): Let \( W \in C' \), then there is a \( \Psi \sim \Phi \) such that \( \text{range } \Psi = W \). One has
\[
P_V P_W P_V|_{V \rightarrow V} = \Phi(\Phi^*\Psi^*\Phi)\Phi^*|_{V \rightarrow V}. \tag{2.21}
\]
But \( \Phi^*\Psi^*\Phi \in \text{id}_\ell + I_1(\ell) \), hence, \( \Phi^*\Psi^*\Phi \in \text{id}_\ell + I_1(\ell) \) and \( \Phi^*|_V \) is unitary, so we conclude that \( P_V P_W P_V|_{V \rightarrow V} \) has a determinant. Analogously, we get that \( P_W P_V P_W|_{W \rightarrow W} = \Psi(\Psi^*\Phi^*\Psi)\Psi^* \) has a determinant because, again, \( \Psi^*|_W \) is unitary. Lemma 2.2.
(Properties of \(\approx\)) then states \(V \approx W\). We still need to show \(V \approx_0 W\). Therefore, consider charge\((W,V) = \text{ind}(P_V|_{W\to V})\) and \(P_V|_{W\to V} = \Phi\Phi^*\Psi\Psi^*|_{W\to V}\). Since \(\Phi\Psi \in \text{id}_\ell + I_1(\ell)\) and \(\Psi^*|_W\) is unitary, \(\Psi^*P_V|_{W\to V}\Psi = \Psi^*\Phi\Phi^*\Psi \in \text{id}_\ell + I_1(\ell)\), which is a perturbation of the identity by a compact operator. Therefore \(\text{ind}(P_V|_{W\to V}) = 0\).

Hence, we have shown that \(V \approx_0 W\) and therefore \(W \in C\).

\(C' \supseteq C\): Let \(W \in C\), then \(W \approx V\) and charge\((V,W) = 0\). We need to find an isometry \(\Psi \sim \Phi\) such that range \(\Psi = W\). We make a polar decomposition of \(P_W\Phi\).

By Lemma 2.2.14 we know that range \(P_W\Phi\) is closed. There is a partial isometry \(U : \ell \to \text{range } P_W\Phi = \text{range } P_W^|_V = \text{range } P_W^\perp \subset \mathcal{H}\) with \(\ker U = \ker P_W\Phi\) such that \(P_W\Phi = U|P_W\Phi|\) where \(|P_W\Phi|\) is given by the square root of the positive semi-definite operator \((P_W\Phi)^*(P_W\Phi)\). Furthermore, \((P_W\Phi)^*(P_W\Phi) = \Phi^*P_VP_WP_V\Phi \in \text{id}_\ell + I_1(\ell)\) holds by Lemma 2.2.14. Hence, this operator is a Fredholm operator. That means also that ker \(((P_W\Phi)^*(P_W\Phi)) = \ker P_W\Phi\) is finite dimensional. Moreover, 0 = charge\((V,W) = \text{ind}(P_W|_{W\to V})\) implies that \(\dim \ker P_W\Phi = \dim W/(\text{range } P_W^|_V) = \dim W \cap (\text{range } P_W^\perp)\). Thus, there is another partial isometry \(\tilde{U} : \ell \to \mathcal{H}\) of finite rank such that \(\tilde{U}|_{\ker P_W\Phi}\) maps \(P_W\Phi\) unitarily onto \(W \cap (\text{range } P_W^\perp)\) and \(\tilde{U}|_{(\ker P_W\Phi)^\perp} = 0\). We set \(\Psi := U + \tilde{U} : \ell \to \mathcal{H}\), and get \(\Psi^*\Psi = U^*U + \tilde{U}^*\tilde{U} = 1\) and therefore \(\Psi \in \text{Seas}^\perp(\mathcal{H})\). By construction, range \(\Psi = W\) holds. Furthermore, we have the identities

\[ U^*\Phi = U^*P_W\Phi = U^*U|P_W\Phi| = |P_W\Phi|. \quad (2.22) \]

This operator has a determinant since \(|P_W\Phi| \geq 0\) and

\[ |P_W\Phi|^2 = \Phi^*P_VP_WP_V\Phi \in \text{id}_\ell + I_1(\ell) \quad (2.23) \]

hold. The last identity follows since \(P_VP_WP_V|_{W\to V}\) has a determinant by Lemma 2.2.14. On the other hand, \(U^*\Phi\) has finite rank since \(\tilde{U}\) does. Hence, \(\Psi^*\Phi = U^*\Phi + \tilde{U}^*\Phi \in \text{id}_\ell + I_1(\ell)\), i.e. \(\Psi \sim \Phi\), which means that \(W \in C'\).

\[ \square \]

Now we begin with the construction of the infinite wedge spaces for each equivalence class of Dirac seas \(S \in \text{Seas}(\mathcal{H})/\sim\). We follow the standard linear algebra method: First, we construct with the elements of \(S\) a space of formal linear combinations \(\mathbb{C}^S\) which we equip with a semi-definite sesquilinear form that in turn induces a semi-norm. Completion with respect to this semi-norm yields the infinite wedge space of \(S\).

**Construction 2.13** (Formal Linear Combinations). (a) For any set \(S\), let \(\mathbb{C}^S\) denote the set of all maps \(\alpha : S \to \mathbb{C}\) for which \(\{\Phi \in S \mid \alpha(\Phi) \neq 0\}\) is finite. For \(\Phi \in S\), we define \(\Phi \in \mathbb{C}^S\) to be the map fulfilling \(\Phi|\Phi| = 1\) and \(\Phi|\Psi| = 0\) for \(\Phi \neq \Psi \in S\). Thus, \(\mathbb{C}^S\) consists of all finite formal linear combinations \(\alpha = \sum_{\Psi \in S} \alpha(\Psi)|\Psi|\) of elements of \(S\) with coefficients in \(\mathbb{C}\).
(b) Now, let \( S \in \text{Seas}(\mathcal{H})/\sim \) as in Definition 2.10 (Dirac Sea Classes). We define the map \( \langle \cdot, \cdot \rangle : S \times S \to \mathbb{C}, (\Phi, \Psi) \mapsto \langle \Phi, \Psi \rangle := \det(\Phi^* \Psi) \). Note that this is well defined since for \( \Phi, \Psi \in S \) the fact \( \Phi \sim \Psi \) implies that \( \Phi^* \Psi \) has a determinant.

(c) Taking \( S \) as before, let \( \langle \cdot, \cdot \rangle : C(S) \times C(S) \to \mathbb{C} \) denote the sesquilinear extension of \( \langle \cdot, \cdot \rangle : S \times S \to \mathbb{C} \), defined as follows: For \( \alpha, \beta \in C(S) \),

\[
\langle \alpha, \beta \rangle = \sum_{\Phi \in S} \sum_{\Psi \in S} \overline{\alpha(\Phi)} \beta(\Psi) \det(\Phi^* \Psi). \tag{2.24}
\]

Here, the bar denotes the complex conjugate. Note that the sums consist of at most finitely many nonzero summands. In particular we have \( \langle [\Phi], [\Psi] \rangle = \langle \Phi, \Psi \rangle \) for \( \Phi, \Psi \in S \).

Lemma 2.14. The sesquilinear form \( \langle \cdot, \cdot \rangle : C(S) \times C(S) \to \mathbb{C} \) is hermitean and positive semi-definite, i.e. \( \langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle} \) and \( \langle \alpha, \alpha \rangle \geq 0 \) hold for all \( \alpha, \beta \in C(S) \).

Proof. For \( \Phi, \Psi \in S \), we have

\[
\langle \Phi, \Psi \rangle = \det(\Phi^* \Psi) = \overline{\det(\Psi^* \Phi)} = \overline{\langle \Psi, \Phi \rangle} \tag{2.25}
\]

This implies that \( \langle \cdot, \cdot \rangle : C(S) \times C(S) \to \mathbb{C} \) is hermitean. Let \( \alpha \in S \). We get

\[
\langle \alpha, \alpha \rangle = \sum_{\Phi \in S} \sum_{\Psi \in S} \overline{\alpha(\Phi)} \alpha(\Psi) \det(\Phi^* \Psi). \tag{2.26}
\]

Let \( (e_i)_{i \in \mathbb{N}} \) be an orthonormal basis in \( \ell \). In the following, we abbreviate \( N_m = \{1, \ldots, m\} \). Fredholm determinants are approximated by finite-dimensional determinants (see Section VII.3, Theorem 3.2 in [GGK90]), therefore

\[
\det(\Phi^* \Psi) = \lim_{m \to \infty} \det(\langle e_i, \Phi^* \Psi e_j \rangle)_{i,j \in N_m}. \tag{2.27}
\]

Let \( (f_k)_{k \in \mathbb{N}} \) be an orthonormal basis of \( \mathcal{H} \). For every \( i, j \in \mathbb{N} \), we get

\[
\langle e_i, \Phi^* \Psi e_j \rangle = \lim_{n \to \infty} \sum_{k=1}^{n} \langle \Phi e_i, f_k \rangle \langle f_k, \Psi e_j \rangle, \tag{2.28}
\]

and, hence, for every \( m \in \mathbb{N} \)

\[
\det(\langle e_i, \Phi^* \Psi e_j \rangle)_{i,j \in N_m} = \lim_{n \to \infty} \det \left( \sum_{k=1}^{n} \langle \Phi e_i, f_k \rangle \langle f_k, \Psi e_j \rangle \right)_{i,j \in N_m} = \lim_{n \to \infty} \sum_{I \subseteq \mathbb{N}_n, |I| = m} \det(\langle \Phi e_i, f_k \rangle)_{k \in I, i \in N_m} \det(\langle f_k, \Psi e_j \rangle)_{k \in I, j \in N_m}. \tag{2.29}
\]
Substituting this in (2.26) and (2.27), we conclude
\[
\langle \alpha, \alpha \rangle = \lim_{m \to \infty} \lim_{n \to \infty} \sum_{\Phi \in \mathcal{S}} \sum_{\Psi \in \mathcal{S}} \sum_{|I| = m} \alpha(\Phi) \alpha(\Psi) \det((f_k, \Phi e_i))_{k \in I, i \in \mathbb{N}_n} \det((f_k, \Psi e_j))_{j \in \mathbb{N}_m} \\
= \lim_{m \to \infty} \lim_{n \to \infty} \sum_{I \subseteq \mathbb{N}_n, |I| = m} \sum_{\Phi \in \mathcal{S}} \alpha(\Phi) \det((f_k, \Phi e_i))_{k \in I, i \in \mathbb{N}_m} \geq 0. \tag{2.30}
\]

**Definition 2.15.** Let \( \| \cdot \| : \mathbb{C}^{(\mathcal{S})} \to \mathbb{R}, \alpha \mapsto \| \alpha \| = \sqrt{\langle \alpha, \alpha \rangle} \) denote the semi-norm associated to \( \langle \cdot, \cdot \rangle \), and \( N_\mathcal{S} = \{ \alpha \in \mathbb{C}^{(\mathcal{S})} \mid \langle \alpha, \alpha \rangle = 0 \} \) denote the null space of \( \mathbb{C}^{(\mathcal{S})} \) with respect to \( \| \cdot \| \).

This null space \( N_\mathcal{S} \) is quite large. The following lemma identifies a few elements of this null space and is also the key ingredient to Corollary 2.18 (Null space) and therewith to Lemma 2.23 (Uniqueness up to a Phase).

**Lemma 2.16.** For \( \Phi \in \mathcal{S} \) and \( R \in \text{id}_\ell + I_1(\ell) \), one has \( \Phi R \in \mathcal{S} \) and \( [\Phi R] - (\det R)[\Phi] \in N_\mathcal{S} \).

**Proof.** First, we observe that \( (\text{range}(\Phi R))^\perp \supseteq (\text{range} \Phi)^\perp \) is infinite-dimensional. Since \( \Phi^* \Phi \in \text{id}_\ell + I_1(\ell) \) and \( R \in \text{id}_\ell + I_1(\ell) \), we have \( \Phi^*(\Phi R) \in \text{id}_\ell + I_1(\ell) \) and \( (\Phi R)^*(\Phi R) = R^*(\Phi^* \Phi) R \in \text{id}_\ell + I_1(\ell) \). This shows \( \Phi R \in \text{Seas}(\mathcal{H}) \) and \( \Phi \sim \Phi R \), and thus \( \Phi R \in \mathcal{S} \). We calculate:
\[
\| [\Phi R] - (\det R)[\Phi] \|^2 = \det((\Phi R)^*(\Phi R)) - (\det R) \det((\Phi R)^* \Phi) \\
- \det R \det(\Phi^* \Phi R) + |\det R|^2 \det(\Phi^* \Phi) \\
= 2|\det R|^2 \det(\Phi^* \Phi) - 2|\det R|^2 \det(\Phi^* \Phi) = 0. \tag{2.31}
\]

Now we have everything needed to define some key objects in this work: The infinite wedge spaces. These spaces shall make up the playground for the second quantized Dirac time-evolution:

**Definition 2.17** (Infinite Wedge Spaces). Let \( \mathcal{F}_\mathcal{S} \) be the completion of \( \mathbb{C}^{(\mathcal{S})} \) with respect to the semi-norm \( \| \cdot \| \). We refer to \( \mathcal{F}_\mathcal{S} \) as infinite wedge space over \( \mathcal{S} \). Let \( \iota : \mathbb{C}^{(\mathcal{S})} \to \mathcal{F}_\mathcal{S} \) denote the canonical map. The sesquilinear form \( \langle \cdot, \cdot \rangle : \mathbb{C}^{(\mathcal{S})} \times \mathbb{C}^{(\mathcal{S})} \to \mathbb{C} \) induces a scalar product \( \langle \cdot, \cdot \rangle : \mathcal{F}_\mathcal{S} \times \mathcal{F}_\mathcal{S} \to \mathbb{C} \). Let \( \Lambda : \mathcal{S} \to \mathcal{F}_\mathcal{S} \) denote the canonical map \( \Lambda \Phi = \iota([\Phi]) \), \( \Phi \in \mathcal{S} \).

Note that \( \iota[N_\mathcal{S}] = \{0\} \). Hence, the null space is automatically factored out during the completion procedure. In fact, the null space of the canonical map \( \iota : \mathbb{C}^{(\mathcal{S})} \to \mathcal{F}_\mathcal{S} \) equals \( \ker \iota = N_\mathcal{S} \). Thus we can rewrite Lemma 2.16 in the following way:
Corollary 2.18 (Null space). For $\Phi \in S$ and $R \in \text{id}_\ell + I_1(\ell)$, one has $\Lambda(\Phi R) = (\det R)\Lambda\Phi$.

Combining Corollary 2.18 above with Lemma 2.8 (Isometries are good enough), we get the following: For every $\Phi \in S$ there are $\Upsilon \in S \cap \text{Seas}^+(\mathcal{H})$ and $R \in \text{id}_\ell + I_1(\ell)$ with $r = \det R \in \mathbb{R}_0^+$ such that $\Lambda\Phi = r\Lambda\Upsilon$. As a consequence, $\{\Lambda\Psi \mid \Psi \in S \cap \text{Seas}^+(\mathcal{H})\}$ spans a dense subspace of $\mathcal{F}_S$. The scalar product $\langle \cdot , \cdot \rangle$ gives $\mathcal{F}_S$ the structure of a separable Hilbert space:

Lemma 2.19 (Separability). The inner product space $(F_S, \langle \cdot , \cdot \rangle)$ is separable.

Proof. It suffices to show that there exists a countable dense subset of $\Lambda S$ with respect to the norm $\|\cdot\|_{F_S}$ in $F_S$. Choose $\Phi \in S$. Then by Lemma 2.11 (Dirac Sea Class Properties) we know that $S = \Phi + \mathcal{V}(\Phi)$. Now, the set of operators of finite rank is dense and separable in $(\mathcal{V}(\Phi), \|\cdot\|_{\Phi})$. Hence, we can choose a countable, dense subset $D$ in $(\mathcal{V}(\Phi), \|\cdot\|_{\Phi})$. We show now that $\Lambda(\Phi + D)$ is dense in $\Lambda S$ with respect to the norm $\|\cdot\|_{F_S}$. Let $\Psi = \Phi + L \in S$ with $L \in \mathcal{V}(\Phi)$. We find a sequence $(L_n)_{n \in \mathbb{N}}$ in $D$ with $\|L_n - L\|_{\Phi} \to 0$ for $n \to \infty$ and define $\Psi_n := \Phi + L_n$. One then obtains the following estimate for all large $n$:

$$\|\Lambda\Psi - \Lambda\Psi_n\|_{F_S}^2 = \langle \Lambda\Psi - \Lambda\Psi_n, \Lambda\Psi - \Lambda\Psi_n \rangle_{F_S}$$

$$= \det(\Psi^*\Psi) - \det(\Psi^*\Psi_n) - \det(\Psi_n^*\Psi) + \det(\Psi_n^*\Psi_n)$$

$$\leq C_1(\Psi)(\|\Psi^*(\Psi - \Psi_n)\|_{I_1} + \|\Psi_n^*(\Psi - \Psi_n)\|_{I_1})$$

(2.32)

by local Lipschitz continuity of the Fredholm determinant with respect to the norm in $I_1(\ell)$; see [Sim05, Theorem 3.4 p. 34]. The constant $C_1(\Psi) < \infty$ depends only on $\Psi$. Next, the triangle inequality applied to the second term gives

$$\ldots \leq C_1(\Psi)\left(2\|\Psi^*(\Psi - \Psi_n)\|_{I_1} + \|(\Psi - \Psi_n)^*(\Psi - \Psi_n)\|_{I_1}\right)$$

$$\leq 2C_1(\Psi)\|\Psi - \Psi_n\|_\Psi = 2C_1(\Psi)\|L - L_n\|_\Psi$$

$$\leq C_2(\Psi, \Phi)\|L - L_n\|_\Phi \xrightarrow{n \to \infty} 0$$

(2.33)

for some constant $C_2(\Psi, \Phi) < \infty$ depending only on $\Psi$ and $\Phi$ since the norms $\|\cdot\|_\Psi$ and $\|\cdot\|_\Phi$ are equivalent by Lemma 2.11 (Dirac Sea Class Properties). This shows that $\Lambda(\Phi + D)$ is a countable, dense subset of $\Lambda(S)$. \qed
The following diagram summarizes the setup:

Note that, by Lemma 2.12 (Connection between $\sim$ and $\approx_0$), $\mathcal{F}_S$ carries the whole information of the polarization class $C \in \text{Pol}(\mathcal{H})/\approx_0$; however, it depends on a choice of basis. In this sense we say that the wedge space $\mathcal{F}_S$ belongs to the polarization class $C$.

### 2.2 Operations from the Left and from the Right

Having constructed the infinite wedge spaces $\mathcal{F}_S$ for each $S \in \text{Seas}(\mathcal{H})/\sim$ we now introduce two types of operations on them which are the tools needed in the next subsection. In the following let $\mathcal{H}'$, $\ell'$ be also two Hilbert spaces.

**Construction 2.20** (The Left Operation).

(a) The following operation from the left is well-defined:

$$ U(\mathcal{H}, \mathcal{H}') \times \text{Seas}_\ell(\mathcal{H}) \to \text{Seas}_\ell(\mathcal{H}'), \quad (U, \Phi) \mapsto U\Phi. $$

(b) This operation from the left is compatible with the equivalence relation $\sim$ in the following sense: For $U \in U(\mathcal{H}, \mathcal{H}')$ and $\Phi, \Psi \in \text{Seas}_\ell(\mathcal{H})$, one has $\Phi \sim \Psi$ if and only if $U\Phi \sim U\Psi$ in $\text{Seas}_\ell(\mathcal{H}')$. Thus, the action of $U$ on $\text{Seas}_\ell(\mathcal{H})$ from the left induces also an operation from the left on equivalence classes modulo $\sim$ as follows. For $S \in \text{Seas}_\ell(\mathcal{H})/\sim$ and $U \in U(\mathcal{H}, \mathcal{H}')$,

$$ US = \{U\Phi \mid \Phi \in S\} \in \text{Seas}_\ell(\mathcal{H}')/\sim. $$

(c) For $U \in U(\mathcal{H}, \mathcal{H}')$ and $S \in \text{Seas}_\ell(\mathcal{H})/\sim$, the induced operation $\mathcal{L}_U : C(S) \to C(US)$, given by

$$ \mathcal{L}_U \left( \sum_{\Phi \in S} \alpha(\Phi)[\Phi] \right) = \sum_{\Phi \in S} \alpha(\Phi)[U\Phi], $$

is an isometry with respect to the hermitean forms $\langle \cdot, \cdot \rangle$ on $C(S)$ and on $C(US)$. In particular one has $\mathcal{L}_U[N_S] \subseteq N_{US}$. 

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(d) For every \( U \in \mathcal{U}(\mathcal{H}, \mathcal{H}') \), the operation from the left \( \mathcal{L}_U : \mathbb{C}^{(S)} \to \mathbb{C}^{(UV)} \) induces a unitary map \( \mathcal{L}_U : \mathcal{F}_S \to \mathcal{F}_{UV} \), characterized by \( \mathcal{L}_U(\Lambda \Phi) = \Lambda(U \Phi) \) for \( \Phi \in \mathcal{S} \). This operation is functorial in the following sense. Let \( \mathcal{H}' \) be another Hilbert space. For \( U \in \mathcal{U}(\mathcal{H}, \mathcal{H}') \), \( V \in \mathcal{U}(\mathcal{H}', \mathcal{H}'') \) and \( S \in \mathcal{Seas}_l(\mathcal{H})/\sim \), one has \( \mathcal{L}_U \mathcal{L}_V = \mathcal{L}_{UV} : \mathcal{F}_S \to \mathcal{F}_{UV} \) and \( \mathcal{L}_{id_H} = id_{\mathcal{F}_S} \).

In complete analogy to the operation from the left, we introduce next an operation from the right. Let \( \ell' \) be another Hilbert space, and let \( GL_-(\ell', \ell) \) denote the set of all bounded invertible linear operators \( R : \ell' \to \ell \) with the property \( R^* R \in id_{\ell'} + I_{1}(\ell') \). Note that \( GL_-(\ell) := GL_-(\ell, \ell) \) is a group with respect to composition.

**Construction 2.21** (Operation from the Right).

(a) The following operation from the right is well-defined:

\[
\mathcal{Seas}_r(\mathcal{H}) \times GL_-(\ell', \ell) \to \mathcal{Seas}_r(\mathcal{H}), \quad (\Phi, R) \mapsto \Phi R.
\]

(b) This operation from the right is compatible with the equivalence relations \( \sim \): For \( \Phi, \Psi \in \mathcal{Seas}_r(\mathcal{H}) \) and \( R \in GL_-(\ell', \ell) \), one has \( \Phi \sim \Psi \) if and only if \( \Phi R \sim \Psi R \) in \( \mathcal{Seas}_r(\mathcal{H}) \). Thus, the operation of \( R \) from the right induces also an operation from the right on equivalence classes modulo \( \sim \) as follows: For \( S \in \mathcal{Seas}_r(\mathcal{H})/\sim \) and \( R \in GL_-(\ell', \ell) \),

\[
SR = \{ \Phi R | \Phi \in S \} \in \mathcal{Seas}_r(\mathcal{H})/\sim.
\]

(c) For \( S \in \mathcal{Seas}_r(\mathcal{H})/\sim \) and \( R \in GL_-(\ell', \ell) \) the induced operation from the right \( \mathcal{R}_R : \mathbb{C}^{(S)} \to \mathbb{C}^{(SR)} \), given by

\[
\mathcal{R}_R \left( \sum_{\Phi \in S} \alpha(\Phi)[\Phi] \right) = \sum_{\Phi \in S} \alpha(\Phi)[\Phi R],
\]

is an isometry up to scaling with respect to the hermitean forms \( \langle \cdot, \cdot \rangle \) on \( \mathbb{C}^{(S)} \) and on \( \mathbb{C}^{(SR)} \). More precisely, one has for all \( \alpha, \beta \in \mathbb{C}^{(S)} \):

\[
\langle \mathcal{R}_R \alpha, \mathcal{R}_R \beta \rangle = \det(R^* R) \langle \alpha, \beta \rangle.
\]

In particular one has \( \mathcal{R}_R[N_S] \subseteq N_{SR} \).

(d) For every \( R \in GL_-(\ell', \ell) \), the operation \( \mathcal{R}_R : \mathbb{C}^{(S)} \to \mathbb{C}^{(SR)} \) induces a bounded linear map, again called \( \mathcal{R}_R : \mathcal{F}_S \to \mathcal{F}_{SR} \), characterized by \( \mathcal{R}_R(\Lambda \Phi) = \Lambda(\Phi R) \) for \( \Phi \in \mathcal{S} \). Up to scaling, this map is unitary. More precisely, for \( \Phi, \Psi \in \mathcal{F}_S \), one has

\[
\langle \mathcal{R}_R \Phi, \mathcal{R}_R \Psi \rangle = \det(R^* R) \langle \Phi, \Psi \rangle.
\]

The operation \( \mathcal{R}_R \) is contra-variantly functorial in the following sense. Let \( \ell'' \) be another Hilbert space. For \( Q \in GL_-(\ell'', \ell') \), \( R \in GL_-(\ell', \ell) \) and \( S \in \mathcal{Seas}(\mathcal{H}, \ell)/\sim \), one has \( \mathcal{R}_Q \mathcal{R}_R = \mathcal{R}_{QR} : \mathcal{F}_S \to \mathcal{F}_{SR} \) and \( \mathcal{R}_{id_{\ell}} = id_{\mathcal{F}_S} \).
The associativity of composition \((U\Phi)R = U(\Phi R)\) immediately yields:

**Lemma 2.22** (Left and Right Operations Commute). The operations from the left and from the right commute: For \(U \in U(\mathcal{H}, \mathcal{H}')\), \(R \in \text{GL}_{-}(\ell', \ell)\), and \(S \in \text{Seas}_{\ell}(\mathcal{H})/\sim\), one has \(L_U R = R L_U : \mathcal{F}_S \to \mathcal{F}_{USR}\).

We conclude this subsection with a last lemma that states an important property of the infinite wedge spaces. Essentially, it says that for any \(R \in \text{GL}_{-}(\ell)\) such that \(R\) has a determinant we have \(F_S \rightarrow \mathcal{F}_S R\). We introduce \(\text{SL}(\ell)\) to denote the set of all operators \(R \in \text{id}_{\ell} + I_1(\ell)\) with the property \(\det R = 1\).

**Lemma 2.23** (Uniqueness up to a Phase). (a) For all \(R \in \text{GL}_{-}(\ell)\) and \(S \in \text{Seas}_{\ell}(\mathcal{H})/\sim\), one has \(S = SR\) if and only if \(\det R\) holds for all \(\Psi \in \mathcal{F}_S\). As a special case, if \(R \in \text{SL}(\ell)\), then \(R: \mathcal{F}_S \rightarrow \mathcal{F}_S\) is the identity map.

(b) For all \(Q, R \in \text{GL}_{-}(\ell', \ell)\) and \(S \in \text{Seas}_{\ell}(\mathcal{H})/\sim\), we have \(SR = SQ\) if and only if \(Q^{-1}R \in \text{GL}_{-}(\ell')\) has a determinant. In this case, one has for all \(\Psi \in \mathcal{F}_S\):

\[
R_R \Psi = \det(Q^{-1}R)R_Q \Psi
\]

**Proof.** (a) Given \(R \in \text{GL}_{-}(\ell)\) and \(S \in \text{Seas}_{\ell}(\mathcal{H})/\sim\), take any \(\Phi \in S\). Then, as \(\Phi^* \Phi\) has a determinant, \(\Phi^* \Phi R\) has a determinant if and only if \(R\) has a determinant. This is equivalent to \(\Phi \sim \Phi R\) and to \(S = SR\). In this case, Lemma 2.18 (Null space) implies \(R_R \Psi = (\det R) \Psi\) for all \(\Psi \in \mathcal{F}_S\).

(b) Let \(\Phi \in S \cap \text{Seas}_{\ell}(\mathcal{H})\). Then \(SR = SQ\) holds if and only if \(\Phi R \sim \Phi Q\), i.e. if and only if \(Q^* R = (\Phi Q)^* \Phi R\) has a determinant. Since \(Q^* Q\) has a determinant and is invertible, this is equivalent to \(Q^{-1} R \in \text{id}_{\ell} + I_1(\ell')\). Using part (a), for any \(\Psi \in \mathcal{F}_S\), we have in this case: \(R_R \Psi = R_{Q^{-1} R} R_Q \Psi = \det(Q^{-1} R) R_Q \Psi\).

### 2.3 Lift Condition

Given two Hilbert spaces \(\mathcal{H}\) and \(\mathcal{H}'\) and two polarization classes \(C \in \text{Pol}(\mathcal{H})/\approx_0\) and \(C' \in \text{Pol}(\mathcal{H}')/\approx_0\) we now identify conditions under which a unitary operator \(U : \mathcal{H} \to \mathcal{H}'\) can be lifted to a unitary map between two wedge spaces.

By Lemma 2.2 (Properties of \(\approx\)) it is clear how any unitary \(U : \mathcal{H} \to \mathcal{H}'\) acts on polarization classes, and we do not prove the following simple lemma:

**Lemma 2.24** (Action of \(U\) on Polarization Classes). The natural operation

\[
U(\mathcal{H}, \mathcal{H}') \times \text{Pol}(\mathcal{H}) \to \text{Pol}(\mathcal{H}'), \quad (U, V) \mapsto UV = \{Uv \mid v \in V\}
\]
is compatible with the equivalence relations \(\approx\) in the following sense: For \(U \in \mathcal{U}(\mathcal{H}, \mathcal{H}')\) and \(V, W \in \text{Pol}(\mathcal{H})\), one has \(V \approx W\) if and only if \(UV \approx UW\). As a consequence, this operation from the left induces a natural operation on polarization classes \(U(\mathcal{H}, \mathcal{H}') \times (\text{Pol}(\mathcal{H})/\approx) \rightarrow \text{Pol}(\mathcal{H}')/\approx\), \((U, [V]_\approx) \mapsto [UV]_\approx\).

In order to describe charge-preserving time-evolutions \(U \in \mathcal{U}(\mathcal{H}, \mathcal{H}')\), the following subclass of \(\mathcal{U}(\mathcal{H}, \mathcal{H}')\), the restricted set of unitary operators, will be convenient:

**Definition 2.25** (Restricted Set of Unitary Operators). Given the polarization classes \(C \in \text{Pol}(\mathcal{H})/\approx_0\) and \(C' \in \text{Pol}(\mathcal{H}')/\approx_0\) we define

\[
U^0_{\text{res}}(\mathcal{H}, C; \mathcal{H}', C') := \{U \in \mathcal{U}(\mathcal{H}, \mathcal{H}') \mid \text{ for all } V \in C \text{ holds } UV \in C'\}
\]

\[
= \{U \in \mathcal{U}(\mathcal{H}, \mathcal{H}') \mid \text{ there exists } V \in C \text{ such that } UV \in C'\}.
\]

As a special case, we yield a group \(U^0_{\text{res}}(\mathcal{H}, C) := U^0_{\text{res}}(\mathcal{H}, C; \mathcal{H}, C)\).

Note that for a third Hilbert space \(\mathcal{H}''\) with a polarization class \(C'' \in \text{Pol}(\mathcal{H}'')/\approx_0\), one has

\[
U^0_{\text{res}}(\mathcal{H}', C'; \mathcal{H}'', C'')U^0_{\text{res}}(\mathcal{H}, C; \mathcal{H}', C') = U^0_{\text{res}}(\mathcal{H}, C; \mathcal{H}'', C'').
\]

For unitary operations that change the relative charge of two polarizations by \(c \in \mathbb{Z}\), there is a natural generalization of \(U^0_{\text{res}}\) to \(U^c_{\text{res}}\); this is discussed in a different context in Section 2.4.

Now we have all what is needed to prove the main result of this section: The following theorem is our version of the classical Shale-Stinespring theorem [SS65], and hence not completely new. The connection is explained in Section 2.4.

**Theorem 2.26** (Lift Condition). For given polarization classes \(C \in \text{Pol}(\mathcal{H})/\approx_0\) and \(C' \in \text{Pol}(\mathcal{H}')/\approx_0\), let \(S \in \text{Ocean}_e(C)/\sim\) and \(S' \in \text{Ocean}_e(C')/\sim\). Then, for any unitary map \(U : \mathcal{H} \rightarrow \mathcal{H}'\), the following are equivalent:

(a) There is \(R \in U(\ell)\) such that \(USR = S'\), and hence \(R L_U \) maps \(\mathcal{F}_S\) to \(\mathcal{F}_{S'}\).

(a') There is \(R \in GL_-(\ell)\) such that \(USR \sim S'\).

(b) \(U \in U^0_{\text{res}}(\mathcal{H}, C; \mathcal{H}', C')\).

**Proof.** We take \(\Phi \in S, \Phi' \in S'\) and set \(V = \text{range } \Phi, V' = \text{range } \Phi'\).

(a) \(\Rightarrow\) (b) : Take \(R \in U(\ell)\) such that \(US(\Phi)R = S(\Phi')\). In particular, \(U \Phi R \sim \Phi'\), and hence \(\Phi^* U \Phi R \in \text{id}_\ell + I_1(\ell)\). This implies

\[
(\Phi^* U \Phi R)^* \Phi^* U \Phi R \in \text{id}_\ell + I_1(\ell).
\] (2.34)

Because \(U \Phi R : \ell \rightarrow UV\) is unitary and \(\Phi' \Phi^* = P_{V'}\), we conclude that \(P_{UV} P_{V'} P_{UV} = P_{UV} \Phi' \Phi^* P_{UV} P_{UV} [UV \rightarrow UV]\) has a determinant. Similarly,

\[
\Phi^* P_{UV} \Phi^* = \Phi^* U \Phi R (\Phi^* U \Phi R)^* \in \text{id}_\ell + I_1(\ell)
\] (2.35)
implies that \( P_{V'} P_{UV} P_{V'} \mid_{V' \to V'} \) has also a determinant. Together this yields \( UV \approx V' \) by Lemma 2.24 (Properties of \( \approx \)).

Furthermore, because of \( U \Phi R \sim \Phi' \), we know that \( \Phi'^* U \Phi R \) is a Fredholm operator with index 0. Since \( \Phi R : \ell \to V \) and \( \Phi' : \ell \to V' \) are unitary, \( P_{V'} \mid_{UV \to V'} \) is also a Fredholm operator with index 0, i.e. \( \text{charge}(UV, V') = 0 \). This shows \( UV \approx_0 V' \), and the claim \( U \in U_0^{\text{res}}(H, C; H', C') \) follows.

\[(b) \Rightarrow (a') : \text{We abbreviate } A = P_{V'} \mid_{UV \to V'} . \] The assumption \( U \in U_0^{\text{res}}(H, C; H', C') \) implies \( A^* A \in \text{id}_{UV} + I_1(UV) \), and \( A \) is a Fredholm operator with index \( \text{ind} A = 0 \).

Using that \( \Phi : \ell \to V \) and \( \Phi' : \ell \to V' \) are unitary maps, we rewrite this in the form
\[(\Phi'^* U \Phi)^* \Phi'^* U \Phi \in \text{id}_{\ell} + I_1(\ell), \text{ and } \Phi'^* U \Phi \text{ is a Fredholm operator with } \text{ind}(\Phi'^* U \Phi) = 0. \]

We now use a polar decomposition of \( \Phi'^* U \Phi \) in the form \( \Phi'^* U \Phi = BQ \), where \( B : \ell \to \ell \) is positive semi-definite and \( Q : \ell \to \ell \) is unitary. Note that we can take \( Q \) to be unitary, not only a partial isometry, as \( \Phi'^* U \Phi \) has the Fredholm index 0. Taking \( R = Q^{-1} \), we get \( \Phi'^* U \Phi R = B \). Now \( B^2 = B^* B \) is a determinant because \( Q^* B^* B Q = (\Phi'^* U \Phi)^* \Phi'^* U \Phi \) has a determinant. Since \( B \geq 0 \), this implies that \( B \) has also a determinant. We conclude \( U \Phi R \sim \Phi' \).

\[(a') \Rightarrow (a) : \text{We take } R \in \text{GL}_-(\ell) \text{ with } U \Phi R \sim \Phi'. \] By polar decomposition, we write \( R \) in the form \( R = R' Q \), where \( R' : \ell \to \ell \) is unitary and \( Q : \ell \to \ell \) is invertible, positive definite, and has a determinant. As \( \Phi'^* U \Phi R = \Phi'^* U \Phi R' Q \) and \( Q \) both have determinants, \( \Phi'^* U \Phi R' \) has also a determinant. This shows \( U \Phi R' \sim \Phi' \) and hence \( S(U \Phi R') = U S(\Phi) R' = S(\Phi') \). In particular, \( \mathcal{R}_R \mathcal{L}_U \) maps \( \mathcal{F}_S(\Phi) \) to \( \mathcal{F}_{US(\Phi) R V} = \mathcal{F}_{S(\Phi')} \).

\[\square\]

For \( U = \text{id}_H \) we immediately get:

**Corollary 2.27 (Orbits in \( \text{Ocean} \)).** Given \( C \in \text{Pol}(H)/\approx_0 \) and \( S \in \text{Ocean}(C)/\sim \) we have
\[\text{Ocean}(C)/\sim = \{ S R \mid R \in U(\ell) \}. \]

This is the counterpart to Lemma 2.22 (Connection between \( \sim \) and \( \approx_0 \)) which stated that for every polarization class \( C \), the equivalence class with respect to \( \sim \) of any single element \( \Phi \in \text{Ocean}(C) \) suffices to recover \( C \). For every \( S \in \text{Ocean}(C)/\sim \) we constructed a wedge space \( \mathcal{F}_S \). Now, the above corollary states that all these wedge spaces \( \{ \mathcal{F}_S \mid S \in \text{Ocean}(C)/\sim \} \) are related to each other by unitary operations from the right.

Also together with Lemma 2.23 (Uniqueness up to a Phase) one gets:

**Corollary 2.28 (Uniqueness of the Lift up to a Phase).** Given \( C, C', S, S' \) as in Theorem 2.26 (Lift Condition), let \( U \in U_0^{\text{res}}(H, C; H', C') \). Take an \( R \in U(\ell) \) as in Theorem 2.26 (Lift Condition). Then the elements of the set
\[\{ \mathcal{R}_Q \mathcal{R}_R \mathcal{L}_U \mid Q \in U(\ell) \cap (\text{id}_\ell + I_1(\ell)) \} = \{ e^{i \varphi} \mathcal{R}_R \mathcal{L}_U \mid \varphi \in \mathbb{R} \}\]
are the only unitary maps from $\mathcal{F}_S$ to $\mathcal{F}_{S'}$ in the set $\{R_T L_U \mid T \in U(\ell)\}$.

In this sense we refer to the lift $L_U R_R$ as being unique up to a phase. A typical situation is this: Consider for example the one-particle Dirac time-evolution $U : \mathcal{H} \to \mathcal{H}$ and assume that $U \in U_{\text{res}}^0(\mathcal{H}, C; \mathcal{H}, C')$ for two given polarization classes $C, C' \in \text{Pol}(\mathcal{H})/\approx_0$; we justify this assumption in Section 3 p. 31 below. We choose $\Phi, \Phi' \in \text{Seas}^+(\mathcal{H})$ such that $\text{range } \Phi \in C$ and $\text{range } \Phi' \in C'$. By Lemma 2.12 (Connection between $\sim$ and $\approx_0$) it follows that $S = S(\Phi) \in \text{ocean}(C)/\sim$ and $S' = S(\Phi) \in \text{ocean}(C')/\sim$ hold. The elements of the associated wedge spaces $\mathcal{F}_S$ and $\mathcal{F}_{S'}$ represent the “in” and “out” states, respectively. Theorem 2.26 (Lift Condition) and Corollary 2.28 (Uniqueness of the Lift up to a Phase) assure for the $S, S'$ there is an $R \in U(\ell)$ such that

$$\mathcal{F}_S \xrightarrow{L_U} \mathcal{F}_{US} \xrightarrow{R_R} \mathcal{F}_{USR} = \mathcal{F}_{S'} \cup e^{i\varphi} \quad \varphi \in \mathbb{R}.$$  

We have illustrated this situation in Figure 1.

Figure 1: A sketch of how the equivalence classes are related.

### 2.4 Comparison with Standard Fock Spaces

In this subsection we sketch the relation of infinite wedge spaces to standard Fock spaces omitting complete proofs.

#### 2.4.1 Connection with Constant Charge Sectors of Fock Spaces

As before, let $\mathcal{H}$ be a complex separable Hilbert space. We pick an orthonormal basis $\varphi = (\varphi_j)_{j \in \mathbb{Z}}$ of $\mathcal{H}$ which will be fixed throughout this Subsection 2.4. An infinite form is
a formal expression of the form

$$\psi = \varphi_{j_1} \wedge \varphi_{j_2} \wedge \ldots = \bigwedge_{n \in \mathbb{N}} \varphi_{j_n},$$

where \((j_n)_{n \in \mathbb{N}}\) is a strictly increasing sequence with the property that \(j_{n+1} = j_n + 1\) for all \(n\) larger than a suitable \(n_1 \in \mathbb{N}\). Let \(B = B(\varphi)\) be the set of all such forms. These forms \(\psi \in B(\varphi)\), \(\psi = \varphi_{j_1} \wedge \varphi_{j_2} \wedge \ldots\), are infinite formal exterior products of basis elements \(\varphi_j\), where only finitely many \(\varphi_j\) with \(j < 0\) occur and all except for finitely many of the \(\varphi_k\) with \(k > 0\) occur. This description is Dirac's original picture of his sea [Dir34b].

**Definition 2.29.** The charge \(C(\psi)\) of a form \(\psi \in B\) is the value \(c \in \mathbb{Z}\) with \(j_n = n - c\) for suitably large \(n\). Denote \(B_c = B_c(\varphi) := \{\psi \mid C(\psi) = c\}\) for \(c \in \mathbb{Z}\).

Now one can define a natural Fock space (attached to \(\varphi\)) as the uniquely determined Hilbert space generated by \(B\) and having \(B\) as an orthonormal basis. However, we want to get directly into contact with the infinite wedge spaces of Subsections 2.1 and 1.2. To do this we observe that a form \(\psi \in B(\varphi)\) can also be regarded as to be the sequence \(\psi = (\psi_n)_{n \in \mathbb{N}}\) with \(\psi_n = \varphi_{j_n}\). The condition \(C(\psi) = 0\) means \(\psi_n = \varphi_n\) for \(n \geq n_1\) and therefore implies that the \(\mathbb{N} \times \mathbb{N}\) matrices \((\langle \psi_n, \varphi_m \rangle)_{m,n \in \mathbb{N}}\) and \((\langle \psi_n, \psi_m \rangle)_{m,n \in \mathbb{N}}\) differ from the identity only by a trace class operator and thus have a determinant. Hence, \(B_0(\varphi)\) regarded as a set of sequences in \(\mathcal{H}\) is a subset of the equivalence class \(\mathcal{S}(\varphi)\), which we defined in Subsection 1.2 of the introduction. We abbreviate \(\mathcal{F}_\mathcal{S}(\varphi)\) by \(\mathcal{F}_\varphi\) here, and in the following. By a slight abuse of notation the form \(\psi\) can be regarded as to be the element \(\Lambda \psi\) of \(\mathcal{F}_\varphi\). With this identification \(B_0 \subset \mathcal{F}_\varphi\) is orthonormal since \(\langle \psi, \psi' \rangle = \det(\langle \varphi_{j_n}, \varphi_{j'_n} \rangle)_{n \leq n_2}\) whenever \(\psi' = \varphi_{j_1} \wedge \varphi_{j_2} \wedge \ldots \in B_0\) with \(j'_n = n\) for \(n \geq n'_1 \geq n_1\). Using similar techniques as in Lemma 2.19 (Separability) one can show the following statement which we do not prove here:

**Lemma 2.30.** \(B_0(\varphi) \subset \mathcal{F}_\varphi\) spans a dense subspace of \(\mathcal{F}_\varphi\) and therefore is an orthonormal basis.

This result extends to arbitrary charges \(c \in \mathbb{Z}\). For each \(c \in \mathbb{Z}\) the sector \(\mathcal{F}_{\varphi,c}\) is defined in analogy to \(\mathcal{F}_\varphi\): In the notation of Sect. 2.1 let \(\Phi_c \in \text{Seas}_c(\mathcal{H})\), where \(c = f_2(\mathbb{N})\), be defined by \(\Phi_c(e_i) := \varphi_{i-c}\) for \(i \in \mathbb{N}\). We obtain the Hilbert space \(\mathcal{F}_{\varphi,c} := \mathcal{F}_\mathcal{S}(\Phi_c)\) (see Definition 2.17 (Infinite Wedge Spaces)) as the Fock space sector of charge \(c\). Note that \(\mathcal{F}_{\varphi,0} = \mathcal{F}_\varphi\). In the same way as before \(B_c(\varphi)\) is an orthonormal basis of \(\mathcal{F}_{\varphi,c}\). We call the Hilbert space sum

$$\mathcal{F}_\varphi^\infty := \bigoplus_{c \in \mathbb{Z}} \mathcal{F}_{\varphi,c}$$

(2.37)

to be the (full) Fock space associated to \(\varphi\) with the sectors \(\mathcal{F}_{\varphi,c}\) of charge \(c\). Observe, that \(B = B(\varphi)\) is an orthonormal basis of \(\mathcal{F}_\varphi^\infty\). The given basis \((\varphi_j)_{j \in \mathbb{Z}}\) induces a polarization
\( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) of \( \mathcal{H} \), where \( \mathcal{H}_+ \) respectively \( \mathcal{H}_- \) is the closed subspace generated by \( \{ \varphi_j \mid j \in \mathbb{Z}, j \leq 0 \} \) respectively \( \{ \varphi_j \mid j \in \mathbb{Z}, j > 0 \} \).

Let \( \bigwedge^\bullet \mathcal{H}_+ \) and \( \bigwedge^\bullet \mathcal{H}_- \) be their exterior algebras and let \( \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) := \bigwedge^\bullet \mathcal{H}_+ \otimes \bigwedge^\bullet \mathcal{H}_- \) be the \textit{standard Fock space}, in both cases in the Hilbert space sense. Then \( B(\varphi) \) can be interpreted as a special listing of an orthonormal basis of the standard Fock space \( \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) \).

Namely, to an element \( \psi = \bigwedge_{n=1}^\infty \varphi_{j_n} \) (in the sense of equation (2.36)) corresponds the vector

\[
\psi := \varphi_{j_1} \wedge \ldots \wedge \varphi_{j_M} \otimes \varphi_{i_1} \wedge \ldots \wedge \varphi_{i_N} \in \mathcal{H}_+^\Lambda^M \otimes \mathcal{H}_-^\Lambda^N, \tag{2.38}
\]

where \( j_1 < j_2 < \ldots < j_M \leq 0 \leq j_{M+1} \) and \( \{i_1, \ldots, i_N\} = \mathbb{N} \setminus \{j_n \mid n > M\} \), \( i_1 < \ldots < i_N \).

Note that \( C(\psi) = M - N \) in this case.

We obtain:

**Proposition 2.31.** The subset \( i_\psi(B) \in \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) \) is an orthonormal basis of \( \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) \).

Consequently, \( \psi \mapsto i_\psi(\psi) \), induces an isometric isomorphism

\[
i_\psi : \mathcal{F}_\psi^\infty \to \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) \tag{2.39}
\]

of Hilbert spaces. The c-sector \( \mathcal{F}_\psi^\infty \) is mapped by \( i_\psi \) onto the c-sector

\[
\mathcal{F}_c(\mathcal{H}_+, \mathcal{H}_-) := \bigoplus_{M,N\in\mathbb{N},M=c+N} \mathcal{H}_+^\Lambda^M \otimes \mathcal{H}_-^\Lambda^N \tag{2.40}
\]

of \( \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) \), and the vacuum \( \Omega_\psi = \varphi_1 \wedge \varphi_2 \wedge \ldots \in \mathcal{F}_\psi \) is mapped to the vacuum \( \Omega = 1 \otimes 1 \in \mathcal{F}_\psi,0(\mathcal{H}_+, \mathcal{H}_-) \).

### 2.4.2 Creation Operators

For every \( \chi \in \mathcal{H} \) the \textit{creation operator} \( a_\chi^* \) on the c-sector \( \mathcal{F}_\psi,c \) is the map \( a_\chi^* : \mathcal{F}_\psi,c \to \mathcal{F}_\psi,c+1 \) induced by

\[
a_\chi^* : \varphi_{j_1} \wedge \varphi_{j_2} \wedge \ldots \mapsto \chi \wedge \varphi_{j_1} \wedge \varphi_{j_2} \wedge \ldots \tag{2.41}
\]

and extended appropriately.

Then \( a_\chi^* \) is a well-defined linear operator \( a_\chi^* : \mathcal{F}_\psi,c \to \mathcal{F}_\psi,c+1 \) of norm \( \|\chi\| \) depending complex linearly on \( \chi \) which induces the norm \( \|\chi\| \) operator \( a_\chi^* : \mathcal{F}_\psi^\infty \to \mathcal{F}_\psi^\infty \) on the full Fock space. If we use for the standard creation operator on the standard Fock space \( \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) \) the same symbol \( a_\chi^* : \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) \to \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) \) we obtain the commutativity \( i_\psi \circ a_\chi^* = a_\chi^* \circ i_\psi \):

\[
\begin{array}{ccc}
    \mathcal{F}_\psi^\infty & \xrightarrow{a_\chi^*} & \mathcal{F}_\psi^\infty \\
    i_\psi \downarrow & & \downarrow i_\psi \\
    \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-) & \xrightarrow{a_\chi^*} & \mathcal{F}(\mathcal{H}_+, \mathcal{H}_-)
\end{array} \tag{2.42}
\]
The commutativity in Diagram (2.42) holds similarly for the annihilation operator \( a_{\chi} : F^\infty_\varphi \rightarrow F^\infty_\varphi \) which is the adjoint of \( a^*_{\chi'} \). Note that for all \( \chi, \chi' \in \mathcal{H} \), \( a_{\chi}, a^*_{\chi'} \) satisfy the canonical anti-commutation relations (CAR)

\[
\{ a_{\chi}, a^*_{\chi'} \} = a_{\chi} a^*_{\chi'} + a^*_{\chi'} a_{\chi} = \langle \chi, \chi' \rangle .
\]

(2.43)

The actions of the creation and annihilation operators on \( F^\infty_\varphi \) yield another description of the space \( F^\varphi_\varphi \). Each \( \psi = \psi_1 \wedge \psi_2 \wedge \ldots \in \Lambda \mathcal{S}(\varphi) \subset F^\varphi_\varphi \) can be expressed as the following limit:

\[
\psi = \lim_{n \to \infty} a^*_{\psi_1} \ldots a^*_{\psi_n} a_{\varphi_n} \ldots a_{\varphi_1} \Omega^\varphi
\]

(2.44)

with the vacuum \( \Omega^\varphi = \varphi_1 \wedge \varphi_2 \wedge \ldots \) of \( F^\varphi_\varphi \). Using the commutativity of Diagram (2.42) and the corresponding one for \( a_{\chi} \) this property establishes:

\[
i_{\varphi}(\psi) = \lim_{n \to \infty} a^*_{\psi_1} \ldots a^*_{\psi_n} a_{\varphi_n} \ldots a_{\varphi_1} \Omega^\varphi.
\]

(2.45)

2.4.3 Connection to the Shale-Stinespring Criterion

In general, Fock spaces arise as representation spaces of the CAR algebra \( \mathfrak{A} = \mathfrak{A}(\mathcal{H}) \) of a given Hilbert space \( \mathcal{H} \). \( \mathfrak{A} \) is the natural \( C^* \)-algebra with 1 generated by the elements of \( \mathcal{H} \) and respecting the relations (2.43).

Let \( \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_- \) a polarization and \( \varphi = (\varphi_n)_{n \in \mathbb{Z}} \) an adapted orthonormal basis, i.e. \( (\varphi_n)_{n \in \mathbb{N}} \) is an orthonormal basis of \( \mathcal{H}_- \) and \( (\varphi_n)_{n \leq 0} \) is an orthonormal basis of \( \mathcal{H}_+ \). Then the annihilation and creation operators induce a representation

\[
\pi_{\varphi} : \mathfrak{A}(\mathcal{H}) \rightarrow B(F^\infty_\varphi)
\]

(2.46)

by \( \pi_{\varphi}(\chi) := a_{\chi} + a^*_{\chi} : F^\infty_\varphi \rightarrow F^\varphi_\varphi \), \( \chi \in \mathcal{H} \). These representations are irreducible.

The question answered by the theorem of Shale-Stinespring [SS65] is the following: Given two orthonormal basis \( \varphi, \varphi' \) (or two polarizations) of \( \mathcal{H} \), when are the representations \( \pi_{\varphi}, \pi_{\varphi'} \) are equivalent? That is, when does there exist a unitary (intertwining) operator \( T : F^\infty_\varphi \rightarrow F^\varphi_\varphi \) such that \( T \circ \pi_{\varphi}(\chi) = \pi_{\varphi'}(\chi) \circ T \) for all \( \chi \in \mathcal{H} \)?

\[
\begin{array}{c}
F^\infty_\varphi \downarrow \pi_{\varphi}(\chi) \uparrow T \downarrow \pi_{\varphi'}(\chi) \uparrow F^\varphi_\varphi
\end{array}
\]

(2.47)

This question is closely related to the implementation problem: Given a unitary operator \( U \in U(\mathcal{H}) \), when does there exist a unitary operator \( U^\sim \in U(F^\varphi_\varphi) \) such that \( U^\sim \circ \pi_{\varphi}(\chi) = \)
\[ \pi_{\varphi}(U\chi) \circ U^\sim \text{ for all } \chi \in H? \]

\[
\begin{array}{cccc}
F^\infty_{\varphi} & U^\sim & F^\infty_{\varphi} \\
\pi_{\varphi}(\chi) & \downarrow & \pi_{\varphi}(U\chi) \\
F^\infty_{\varphi} & U^\sim & F^\infty_{\varphi}
\end{array}
\]

If the first question can be solved for \( \varphi' = U\varphi \), i.e. for \( \varphi'_n = U(\varphi_n) \), then \( U^\sim := L_{U^{-1}} \circ T \), see Construction 2.20 is an answer to the second problem, where \( L_U : F^\infty_{\varphi} \to F^\infty_{\varphi'} \) is the natural left action induced by \( U \). This holds since \( L_U \circ \pi_{\varphi}(U\chi) = \pi_{U\varphi}(\chi) \circ L_U = \pi_{\varphi'}(\chi) \circ L_U \), hence

\[ U^\sim \circ \pi_{\varphi}(\chi) = L_{U^{-1}} \circ T \circ \pi_{\varphi}(\chi) = L_{U^{-1}} \circ \pi_{U\varphi}(\chi) \circ T = \pi_{\varphi}(U\chi) \circ L_{U^{-1}} \circ T = \pi_{\varphi}(U\chi) \circ U^\sim. \]

\[
\begin{array}{cccc}
F^\infty_{\varphi} & T & F^\infty_{\varphi} & \frac{L_{U^{-1}}}{F^\infty_{\varphi}} \\
\pi_{\varphi}(\chi) & \downarrow & \pi_{\varphi}'(\chi) & \pi_{\varphi}(U\chi) \\
F^\infty_{\varphi} & T & F^\infty_{\varphi} & \frac{L_{U^{-1}}}{F^\infty_{\varphi}}
\end{array}
\]

Conversely, considering the second question for the unitary map \( U \) defined by \( U\varphi_i = \varphi'_i, i \in \mathbb{Z} \), the implementation \( U^\sim \) composed with the left action \( L_U : F_{\varphi} \to F_{\varphi'} \) yields an intertwining operator thus answering the first question.

The answer to each of the two questions is that \( U \) has to be in the restricted unitary group; see [SS65].

We obtain this result in a slightly different situation with our methods in the following simple way. We ask for an implementation of the unitary map \( U \in U(H) \) with the additional requirement that the implementation \( U^\sim \) should not change the charge, i.e. \( U^\sim(F_{\varphi,c}) \subset F_{\varphi',c} \), in particular \( U^\sim \in U(F_{\varphi}) \).

Let us, first of all, consider the left operation \( L_U : F^\infty_{\varphi} \to F^\infty_{U\varphi} \) for a given unitary \( U : H \to H \). Altering \( U\varphi \) to \( \varphi' := U\varphi R \) by the operation from the right induced by an appropriate \( R \), see Construction 2.21 we obtain a unitary map \( R_U \circ L_U : F^\infty_{\varphi} \to F^\infty_{\varphi'} \). This approach gives an implementation \( U^\sim = R_U \circ L_U \) if and only if \( R \in U(\ell) \) can be chosen such that \( F^\infty_{\varphi} = F^\infty_{\varphi'} \) or, equivalently such that \( US(\varphi)R = S(\varphi) \). According to Theorem 2.26 (Lift Condition) this holds together with the charge conservation exactly when \( U \) is in the group \( U^{0\text{res}}(H, [H_-]_{\approx 0}) \), see Definition 2.25. As a result we have:

**Theorem 2.32.** The following conditions are equivalent for \( U \in U(H) \):

- \( U \) has an implementation \( U^\sim : F_{\varphi} \to F_{\varphi} \) (and in the same way \( U^\sim \in U(F_0(H_+, H_-)) \)).

- \( U \) has an implementation \( U^\sim : F^\infty_{\varphi} \to F^\infty_{\varphi} \) with \( U^\sim(F_{\varphi,c}) \subset F_{\varphi',c} \) for all \( c \in \mathbb{Z} \) (and in the same way \( U^\sim \in U(F(H_+, H_-)) \)).
• $U \in U^0_{\text{res}}(H_+, [H_-]_{\approx 0})$.

Disregarding the charge condition one can show the original Shale-Stinespring theorem [SS65] in a similar straight-forward way.

3 Application to the External Field Problem in QED

We now come to the one-particle Dirac time-evolution in an external four-vector field $A \in C^\infty_c(\mathbb{R}^4, \mathbb{R}^4)$, i.e. the set of infinitely often differentiable $\mathbb{R}^4$ valued functions on $\mathbb{R}^4$ with compact support. Recall the discussion at the end of Subsection 2.3. We order to apply Theorem 2.26 (Lift Condition) to the one-particle Dirac time-evolution $U^A(t_1, t_0)$ for fixed $t_0, t_1 \in \mathbb{R}$ and in this way to obtain a lift to unitary maps from one wedge space to another (the second quantized time-evolution) we need to show that $U^A(t_1, t_0) \in U^0_{\text{res}}(\mathcal{H}, C(t_0); \mathcal{H}, C(t_1))$ for appropriate $C(t_0), C(t_1) \in \text{Pol}(\mathcal{H})/\approx_0$. To ensure this condition holds is the main content of this last section.

This section is structured as follows: In the first Subsection, we show that for any $t_0, t_1 \in \mathbb{R}$ there exist $C(t_0), C(t_1) \in \text{Pol}(\mathcal{H})/\approx_0$, depending only on $A(t_0)$ and $A(t_1)$, respectively, such that $U^A(t_1, t_0) \in U^0_{\text{res}}(\mathcal{H}, C(t_0); \mathcal{H}, C(t_1))$. In the second Subsection we identify the polarization classes $C(t)$ uniquely by the magnetic components of $A(t)$ for all $t \in \mathbb{R}$. The third Subsection combines these results with Section 2 and shows the existence of the second quantized Dirac time-evolution for the external field problem in QED. Finally, the fourth Subsection concludes with the analysis of second quantized gauge transformations as unitary maps between varying Fock spaces.

3.1 One-Particle Time-Evolution

Throughout this section we work with $\mathcal{H} = L_2(\mathbb{R}^3, \mathbb{C}^4)$, with $\mathbb{R}^3$ being interpreted as momentum space. The free Dirac equation in momentum representation is given by

$$i\frac{d}{dt}\psi^0(t) = H^0\psi^0(t) \tag{3.1}$$

for $\psi^0(t) \in \text{domain}(H^0) \subset \mathcal{H}$, where $H^0$ is the multiplication operator with

$$H^0(p) = \alpha \cdot p + \beta m = \sum_{\mu=1}^3 \alpha^\mu p_\mu + \beta m, \quad p \in \mathbb{R}^3 \tag{3.2}$$

and the $\mathbb{C}^{4\times 4}$ Dirac matrices $\beta$ and $\alpha^\mu$, $\mu = 1, 2, 3$, fulfill

$$\beta^2 = 1 \quad \{\alpha^\mu, \beta\} := \alpha^\mu \beta + \beta \alpha^\mu = 0 \quad \{\alpha^\mu, \alpha^\nu\} := \alpha^\mu \alpha^\nu + \alpha^\nu \alpha^\mu = 2\delta^{\mu\nu}. \tag{3.3}$$
Which specific representation of this matrix algebra with hermitean matrices is used does not affect any of the following arguments. For convenience we introduce also

\[ a^0 = 1 \in \mathbb{C}^{4 \times 4}. \]

\( H^0 \) is a self-adjoint multiplication operator which generates a one-parameter group of unitary operators

\[ U^0(t_1, t_0) = U^0(t_1 - t_0) := \exp(-i(t_1 - t_0)H^0) \]  \hspace{1cm} (3.4)

on \( \mathcal{H} \) for all \( t_0, t_1 \in \mathbb{R} \). The matrix \( H^0(p) \) has double eigenvalues \( \pm E(p) \), where \( E(p) = \sqrt{|p|^2 + m^2} > 0, p \in \mathbb{R}^3 \). Therefore, the spectrum of the free Dirac operator is \( \sigma(H^0) = (-\infty, -m] \cup [+m, +\infty) \) and the corresponding free spectral projectors \( P_{\pm} \) are multiplication operators with the matrices

\[ P_{\pm}(p) = \frac{1}{2} \left( 1 \pm \frac{H^0(p)}{E(p)} \right). \]  \hspace{1cm} (3.5)

We define \( \mathcal{H}_\pm := P_{\pm}\mathcal{H} \) for which \( \mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+ \). For any linear operator \( L \) on \( \mathcal{H} \) and signs \( \sigma, \tau \in \{-, +\} \) we write \( L_{\sigma\tau} = P_\sigma LP_\tau \). Furthermore, \( L_{\text{ev}} = L_{++} + L_{--} \) denotes the even (diagonal) part, and \( L_{\text{odd}} = L_{+-} + L_{-+} \) for the odd (non-diagonal) part of \( L \). If \( L \) has an integral kernel \( (q, p) \mapsto L(p, q) \), the kernel of \( L_{\sigma\tau} \) is given by \( (p, q) \mapsto L_{\sigma\tau}(p, q) = P_\sigma(p)L(p, q)P_\tau(q). \)

Now, let \( A = (A_\mu)_{\mu=0,1,2,3} \in C_c^\infty(\mathbb{R}^4, \mathbb{R}^4) \) be a smooth, compactly supported, external four-vector field. We denote its time slice at time \( t \in \mathbb{R} \) by \( A(t) = (\mathbb{R}^3 \ni x \mapsto (A_\mu(t, x))_{\mu=0,1,2,3}) \). The Dirac equation with the external field \( A \) in momentum representation is then given by

\[ i\frac{d}{dt}\psi(t) = H^{A(t)}\psi(t) = \left( H^0 + iZ^{A(t)} \right)\psi(t) \]  \hspace{1cm} (3.6)

where for \( A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec{A}) \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4) \), the operator \( Z^A \) on \( \mathcal{H} \) is defined as follows:

\[ iZ^A = e \sum_{\mu=0}^3 \alpha^\mu \hat{A}_\mu, \]  \hspace{1cm} (3.7)

denoting the elementary charge by \( e \). Here we understand \( \hat{A}_\mu, \mu = 0, 1, 2, 3 \), as convolution operators

\[ (\hat{A}_\mu\psi)(p) = \int_{\mathbb{R}^3} \hat{A}_\mu(p - q)\psi(q) \, dq, \quad p \in \mathbb{R}^3, \]  \hspace{1cm} (3.8)

for \( \psi \in \mathcal{H} \) and \( \hat{A}_\mu \) being the Fourier transform of \( A_\mu \) given by

\[ \hat{A}_\mu(p) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-ipx}A_\mu(x) \, dx. \]  \hspace{1cm} (3.9)
Therefore, in momentum representation, $Z^A$ is an integral operator with integral kernel

$$(p,q) \mapsto Z^A(p-q) = -ie^{\sum_{\mu=0}^3 a^\mu \tilde{A}_\mu(p-q)}, \quad p,q \in \mathbb{R}^3.$$ (3.10)

The Dirac equation with external field $A$ gives also rise to a family of unitary operators $(U^A(t_1,t_0))_{t_0,t_1 \in \mathbb{R}}$ on $H$ which fulfill

$$\frac{\partial}{\partial t_1} U^A(t_1,t_0) = -iH^A(t_1)U^A(t_1,t_0),$$ (3.11)

$$\frac{\partial}{\partial t_0} U^A(t_1,t_0) = iU^A(t_1,t_0)H^A(t_0)$$ (3.12)
on the appropriate domains, such that for every solution $\psi(t)$ of equation (3.6 p.32) one has $\psi(t_1) = U^A(t_1,t_0)\psi(t_0)$; see [Tha92].

We now introduce key objects of this work:

**Definition 3.1** (Induced Polarization Classes). For $A \in C^\infty_c(\mathbb{R}^3,\mathbb{R}^4)$, we define the integral operator $Q^A : H \rightarrow H$ by its integral kernel, also denoted by $Q^A$:

$$\mathbb{R}^3 \times \mathbb{R}^3 \ni (p,q) \mapsto Q^A(p,q) := \frac{Z^A_{+}(p,q) - Z^A_{-}(p,q)}{i(E(p) + E(q))}$$ (3.13)

with $Z^A_{\pm}(p,q) := P_{\pm}(p)Z^A(p-q)P_{\pm}(q)$.

Furthermore, we define the polarization class $C(0) := [H_-]|_{_{\mathbb{R}^3}}$ belonging to the negative spectral space $H_-$ of the free Dirac operator $H^0$, and therewith the polarization classes

$$C(A) := e^{Q^A}C(0) = \{e^{Q^A}V \mid V \in C(0)\}.$$ (3.14)

The operators $Q^A$ are bounded and skew-adjoint, and thus, the operators $e^{Q^A}$ are unitary. They will appear naturally in the iterative scheme that we use to control the time-evolution, and their origin will become clear as we go along (Lemma 3.6 p.40).

We now state the main result of this section, using the notation of Section 2.1.

**Theorem 3.2** (Dirac Time-Evolution with External Field). For all four-vector potentials $A \in C^\infty_c(\mathbb{R}^4,\mathbb{R}^4)$ and times $t_1, t_0 \in \mathbb{R}$ it is true that

$$U^A(t_1,t_0) \in U^0_{res}(H,C(A(t_0)); H,C(A(t_1))).$$

We do not focus on finding the weakest regularity conditions on the external four-vector potential $A$ under which this theorem holds, although much weaker conditions will suffice. Actually, the theorem and also its proof remain valid for four-vector potentials $A$ in the following class $A \supset C^\infty_c(\mathbb{R}^4,\mathbb{R}^4)$:
**Definition 3.3** (Class of External Four-Vector Potentials). Let \( \mathcal{A} \) be the class of four-vector potentials \( A = (A_\mu)_{\mu=0,1,2,3} : \mathbb{R}^4 \to \mathbb{R}^4 \) such that for all \( \mu = 0,1,2,3, m = 0,1,2 \) and \( p = 1,2 \) the integral

\[
\int_{\mathbb{R}} \left\| \frac{d^m}{dt^m} \hat{A}_\mu(t) \right\|_p dt
\]  

exists and is finite. Here \( \hat{A}_\mu(t) \) denotes the Fourier transform of a time slice \( A_\mu(t) \) with respect to the spatial coordinates.

This class of four-vector potentials has also been considered by Scharf in his analysis of the second-quantized scattering operator in an external potential (Theorem 5.1 in [Sch95]). We remark that the class \( \mathcal{A} \) does not contain the Coulomb potential, not even when one truncates it at large times.

Since quite some computation is involved in the proof of the above theorem, we split it up into a series of small lemmas, to separate technicalities from ideas. Here is the skeleton of the proof:

![Diagram showing the relationship between the theorems and lemmas](image)

The key ideas are worked out in Lemma 3.5 (Grönwall Argument). The other lemmas have a more technical character.

In the following, when dealing with a given external vector potential \( A \in \mathcal{A} \), we abbreviate \( U(t_1,t_0) = U^A(t_1,t_0), H(t) = H^A(t), Z(t) = Z^A(t), \) and \( Q(t) = Q^A(t) \). We start with putting things together:

**Proof of Theorem 3.2** (Dirac Time-Evolution with External Field). By Lemma 2.2(b), we need only to show that for some \( V \in C(A(t_1)) \) and some \( W \in C(A(t_0)) \) it is true that

\[
P_V U(t_1,t_0)P_W, P_V U(t_1,t_0)P_W \in I_2(\mathcal{H}).
\]  

(3.16)

Let us choose \( V = e^{Q(t_1)}H_- \in C(A(t_1)) \) and \( W = e^{Q(t_0)}H_- \in C(A(t_0)) \). Then, claim (3.16) is equivalent to

\[
e^{Q(t_1)}P_\pm e^{-Q(t_1)}U(t_1,t_0)e^{Q(t_0)}P_\pm e^{-Q(t_0)} \in I_2(\mathcal{H}).
\]  

(3.17)
Since \( e^{Q(t_1)} \) and \( e^{-Q(t_0)} \) are both unitary operators, this claim is equivalent to

\[
P_\pm e^{-Q(t_1)}U(t_1,t_0)e^{Q(t_0)}P_\mp \in I_2(\mathcal{H}). \tag{3.18}
\]

Now \( Q(t) \) is a bounded operator, and Lemma \( 3.7 \) (I\(_2\) Estimates) states that \( Q^2(t) \in I_2(\mathcal{H}) \) for any time \( t \in \mathbb{R} \). Therefore, by expanding \( e^{\pm Q(t)} \) in its series, we find that

\[
e^{\pm Q(t)} -(\text{id}_\mathcal{H} \pm Q(t)) \in I_2(\mathcal{H}). \tag{3.19}
\]

Hence it suffices to prove

\[
P_\pm (\text{id}_\mathcal{H} - Q(t_1))U(t_1,t_0)(\text{id}_\mathcal{H} + Q(t_0))P_\mp \in I_2(\mathcal{H}). \tag{3.20}
\]

This is just the claim \( 3.26 \) of Lemma \( 3.5 \) (Grönwall Argument) and concludes the proof.

The following stenographic notation will be very convenient: For families of operators \( A = (A(t_1,t_0))_{t_1 \geq t_0} \) and \( B = (B(t_1,t_0))_{t_1 \geq t_0} \), indexed by time intervals \( [t_0,t_1] \subset \mathbb{R} \), we set

\[
AB = \left( \int_{t_0}^{t_1} A(t_1,t)B(t,t_0) \, dt \right)_{t_1 \geq t_0},
\]

whenever this is well-defined. Furthermore, if \( C = (C(t))_{t \in \mathbb{R}} \) and \( D = (D(t))_{t \in \mathbb{R}} \) denote families of operators indexed by time points, we abbreviate \( AC = (A(t_1,t_0)C(t_0))_{t_1 \geq t_0} \), \( CA = (C(t_1)A(t_1,t_0))_{t_1 \geq t_0} \), and \( CD = (C(t)D(t))_{t \in \mathbb{R}} \). The operator norm on bounded operators on \( \mathcal{H} \) is denoted by \( \| \cdot \| \). We set

\[
\|A\|_\infty := \sup_{s,t \in \mathbb{R} : s \geq t} \|A(s,t)\|, \quad \|A\|_{I_2,\infty} := \sup_{s,t \in \mathbb{R} : s \geq t} \|A(s,t)\|_{I_2},
\]

\[
\|C\|_1 := \int_{\mathbb{R}} \|C(t)\| \, dt, \quad \|C\|_{I_2,\infty} := \sup_{t \in \mathbb{R}} \|C(t)\|_{I_2},
\]

whenever these quantities exist. Recall Definition \( 3.1 \) (Induced Polarization Classes) of the operators \( Q(t) = Q^{A(t)} \), and let \( (Q'(t) : \mathcal{H} \to \mathcal{H})_{t \in \mathbb{R}} \) denote their time derivative, defined by using the time derivative of the corresponding kernels

\[
(p,q) \mapsto Q'(t,p,q) := \frac{\partial}{\partial t} Q^{A(t)}(p,q), \quad p,q \in \mathbb{R}^3, t \in \mathbb{R}. \tag{3.22}
\]

We now provide the lemmas in the above diagram: For our purposes, the following fixed point form \( 3.23 \) of the Dirac equation is technically more convenient to handle than the Dirac equation in its differential form \( 3.6 \), as the fixed point equation gives rise to iterative approximation methods and deals only with bounded operators. We could have used it as our starting point.
Lemma 3.4 (Fixed Point Form of the Dirac Equation). The one-particle Dirac time-
evolution $U$ fulfills the fixed point equation

$$U = U^0 + U^0 Z U.$$  \hspace{1cm} (3.23)

we only sketch its proof:

Proof. Using the Dirac equation in the form \(3.11\) - \(3.12\), we get for \(t_0, t_1 \in \mathbb{R}\) on an
appropriate domain:

$$\frac{\partial}{\partial t_1} [U^0(t_1,t)U(t,t_0)] = -iU^0(t_1,t)[H_A(t) - H^0] U(t,t_0) = U^0(t_1,t)Z(t)U(t,t_0).$$  \hspace{1cm} (3.24)

Note that although $H_A(t)$ and $H^0$ are unbounded operators, their difference $iZ(t)$ is a
bounded operator. Integrating \(3.24\), and using $U(t,t) = id = U^0(t,t)$, we get

$$U(t_1,t_0) = U^0(t_1,t_0) + \int_{t_0}^{t_1} U^0(t_1,t)Z(t)U(t,t_0) \, dt.$$  \hspace{1cm} (3.25)

This equation recast in our stenographic notation is the fixed point equation \(3.23\) for $U$.

Iterating this fixed point equation leads to the well-known Born series.

Lemma 3.5 (Grönwall Argument). For all $t_0, t_1 \in \mathbb{R}$, the following holds:

$$P_\pm (id_H - Q(t_1))U(t_1,t_0)(id_H + Q(t_0))P_\mp \in \mathcal{I}_2(H)$$  \hspace{1cm} (3.26)

Proof. Without loss of generality and to simplify the notation, we treat only the case
$t_1 \geq t_0$. Let

$$R := (id_H - Q)U(id_H + Q).$$  \hspace{1cm} (3.27)

The strategy is to expand $R$ in a series and to check the Hilbert-Schmidt properties of the
non-diagonal part term by term. Lemma \(3.4\) (Fixed Point Form of the Dirac Equation)
states that the Dirac time-evolution $U$ fulfills the fixed-point equation $U = U^0 + U^0 Z U$
(equation \(3.23\) below); recall that $U^0$ is the free Dirac time-evolution introduced in
\(3.4\). Iterating this fixed point equation once yields

$$U = U^0 + U^0 Z U^0 + U^0 Z U^0 Z U.$$
therefore take a closer look at this term in Lemma 3.6 (Partial Integration). Equation (3.51) in this lemma states
\[ U^0 Z U^0 = QU^0 - U^0 Q - U^0 Q' U^0 + U^0 Z_{ev} U^0. \]

One finds that the non-diagonal part \((U^0 Z U^0)_{odd}\) does in general not consist of Hilbert-Schmidt operators because of the first two terms \(QU^0 - U^0 Q\) on the right hand side, which are the boundary terms of the partial integration. However, we show now that the transformation induced by \(Q\) remedies these terms such that the non-diagonal part of \(R\) consists of Hilbert-Schmidt operators.

Substituting the formula (3.51), cited above, into the fixed point equation \(U = U^0 + U^0 Z U\), we get
\[ U = U^0 + QU^0 - U^0 Q - U^0 Q' U^0 + U^0 Z_{ev} U^0 \]
\[ + QU^0 Z U - U^0 Q Z U - U^0 Q' U Z U + U^0 Z_{ev} U^0 Z U \]
\[ = U^0 + QU - U^0 Q - U^0 Q' U + U^0 Z_{ev} U - U^0 Q Z U. \] (3.28)

We rewrite this as
\[ (\text{id}_H - Q)U = U^0 (\text{id}_H - Q) + U^0 (-Q' + Z_{ev} - QZ) U. \] (3.29)

Multiplying (3.29) with \(\text{id}_H + Q\) from the right and using the equation
\[ U (\text{id}_H + Q) = (\text{id}_H + Q) R + Q^2 U (\text{id}_H + Q), \]
which follows from the Definition (3.27) of \(R\), we get
\[ R = U^0 (\text{id}_H - Q^2) + U^0 (-Q' + Z_{ev} - QZ) U (\text{id}_H + Q) \]
\[ = U^0 (-Q' + Z_{ev} - QZ) (\text{id}_H + Q) R \]
\[ + U^0 (\text{id}_H - Q^2) + U^0 (-Q' + Z_{ev} - QZ) Q^2 U (\text{id}_H + Q). \] (3.30)

We view (3.30) also as a fixed point equation for \(R\). In order to control the Hilbert-Schmidt norm of the non-diagonals of \(R\), we solve this fixed point equation for \(R\) by iteration. Using the abbreviation
\[ F := (-Q' + Z_{ev} - QZ) (\text{id}_H + Q), \] (3.31)
\[ G := -U^0 Q^2 + U^0 (-Q' + Z_{ev} - QZ) Q^2 U (\text{id}_H + Q), \] (3.32)
we rewrite (3.30) as \(R = U^0 FR + U^0 + G\) and define recursively for \(n \in \mathbb{N}_0\):
\[ R^{(0)} := 0, \quad R^{(n+1)} := U^0 FR^{(n)} + U^0 + G. \] (3.33)
Although our main interest is to control the Hilbert-Schmidt norm \( \| R(t_1, t_0)_{\text{odd}} \|_{L_2} \), we need also some control of the \( R^{(n)} \) in the operator norm. We show first that \( \| R^{(n)} - R \|_\infty \to 0 \) as \( n \to \infty \). We have for all \( n \in \mathbb{N}_0 \)

\[
R^{(n+1)} - R = U^0 F(R^{(n)} - R),
\]

which implies

\[
R^{(n)} - R = (U^0 F)^n (R^{(0)} - R) = -(U^0 F)^n R.
\]

Now for \( s \geq t \), we know \( \| U^0(s, t) F(t) \| = \| F(t) \| \), because \( U^0(s, t) \) is unitary. Let \( t_1 \geq t_0 \).

Using the abbreviation

\[
I(t_1, t_0) := \{ (s_1, \ldots, s_n) \in \mathbb{R}^n \mid t_1 > s_n > \ldots > s_1 > t_0 \},
\]

we get

\[
\| R^{(n)}(t_1, t_0) \| = \| (U^0 F)^n R \|_{L_2(t_1, t_0)} \leq \int_{I(t_0, t_1)} \| F(s_n) \| \| F(s_{n-1}) \| \ldots \| F(s_1) \| \| R(s_1, t_0) \| ds_1 \ldots ds_n \leq \frac{\| F \|_1^n}{n!} \| R \|_\infty \to_n 0; \quad (3.37)
\]

we use here the bounds \( \| F \|_1 < \infty \) and \( \| R \|_\infty < \infty \) from (3.57, 3.35) in Lemma 3.7 below. Note that the convergence in (3.37) is uniform in the time variables \( t_0 \) and \( t_1 \). This proves the claim

\[
\| R^{(n)} - R \|_\infty \to_n 0. \quad (3.38)
\]

As a consequence, we find

\[
\sup_{n \in \mathbb{N}_0} \| R^{(n)} \|_\infty \leq \sup_{n \in \mathbb{N}_0} \| R^{(n)} - R \|_\infty + \| R \|_\infty < \infty. \quad (3.39)
\]

Now we split \( F \) into its diagonal and non-diagonal parts: \( F = F_{\text{ev}} + F_{\text{odd}}, \) where

\[
F_{\text{ev}} = Z_{\text{ev}} - QZ_{\text{odd}} - QQ' - QZ_{\text{ev}} Q,
\]

\[
F_{\text{odd}} = Z_{\text{ev}} Q - QZ_{\text{ev}} - Q' - QZ_{\text{odd}} Q;
\]

recall that \( Q \) is odd: \( Q = Q_{\text{odd}} \). We calculate for \( n \geq 1 \):

\[
R^{(n+1)} = U^0 F R^{(n)} + U^0 + G
= U^0 F_{\text{ev}} R^{(n)} + U^0 F_{\text{odd}} R^{(n)} + U^0 + G
= U^0 F_{\text{ev}} R^{(n)} + U^0 F_{\text{odd}} U^0 F R^{(n-1)} + U^0 F_{\text{odd}} G + U^0 F_{\text{odd}} U^0 + U^0 + G.
\]

(3.42)
Estimating the Hilbert-Schmidt norm for the non-diagonals in each summand on the right hand side in (3.42) now gives:

\[
\| R^{(n+1)}(t_1, t_0) \|_{I_2} \leq \int_{t_0}^{t_1} \| [U^0 F_{ev}] (t_1, t) \| \| R^{(n)} (t, t_0) \|_{I_2} dt \\
+ \int_{t_0}^{t_1} \| [U^0 F_{odd} U^0] (t_1, t) \| \| F R^{(n-1)} (t, t_0) \| dt \\
+ \int_{t_0}^{t_1} \| [U^0 F_{odd}] (t_1, t) \| \| G (t, t_0) \|_{I_2} dt \\
+ \| [U^0 F_{odd} U^0] (t_1, t_0) \|_{I_2} + \| G (t_1, t_0) \|_{I_2}
\]

where we have abbreviated

\[
C_3 := \| U^0 F_{odd} U^0 \|_{I_2, \infty} \| F \|_1 \sup_{n \in \mathbb{N}} \| R^{(n-1)} \|_\infty + \| F_{odd} \|_1 \| G \|_{I_2, \infty}
\]

\[
+ \| U^0 F_{odd} U^0 \|_{I_2, \infty} + \| G \|_{I_2, \infty}.
\]

Lemma 3.1.20 (I2 Estimates) states that \( U^0 F_{odd} U^0 \) and \( G \) consist of Hilbert-Schmidt operators, with \( \| U^0 F_{odd} U^0 \|_{I_2, \infty} < \infty \) and \( \| G \|_{I_2, \infty} < \infty \). Furthermore, it also states that \( \| F \|_1 < \infty \), which implies also \( \| F_{ev} \|_1 < \infty \) and \( \| F_{odd} \|_1 < \infty \). Combining these facts with the bound (3.43), it follows that

\[
C_3 < \infty.
\]

We claim that the following bound holds for all \( n \geq 1 \):

\[
\| R^{(n)} (t_1, t_0) \|_{I_2} \leq C_3 \exp \left( \int_{t_0}^{t_1} \| F_{ev} (t) \| dt \right).
\]

We prove it by induction. For \( n = 1 \), we have \( R^{(1)} = U^0 + G \). Using \( U^0_{odd} = 0 \) and \( t_1 \geq t_0 \), we conclude

\[
\| R^{(1)} (t_1, t_0) \|_{I_2} \leq \| G \|_{I_2, \infty} \leq C_3 \leq C_3 \exp \left( \int_{t_0}^{t_1} \| F_{ev} (t) \| dt \right).
\]

For the induction step \( n \sim n + 1 \), we calculate, using the estimate (3.43) in the first step and the induction hypothesis in the second step:

\[
\| R^{(n+1)} (t_1, t_0) \|_{I_2} \leq \int_{t_0}^{t_1} \| F_{ev} (t) \| \| R^{(n)} (t, t_0) \|_{I_2} dt + C_3
\]

\[
\leq C_3 \int_{t_0}^{t_1} \| F_{ev} (t) \| \exp \left( \int_{t_0}^{t} \| F_{ev} (s) \| ds \right) dt + C_3
\]

\[
= C_3 \exp \left( \int_{t_0}^{t_1} \| F_{ev} (t) \| dt \right).
\]
Finally, we get \( \| R^{(n)}_{\text{odd}} \|_{L^2,\infty} \leq C_3 \epsilon \| F_{ev} \|_1 < \infty \), which is a uniform bound in \( n \). We now use following general fact, which follows from Fatou’s lemma: If \((L_n)_{n \in \mathbb{N}}\) is a sequence of Hilbert-Schmidt operators converging to a bounded operator \( L \) with respect to the operator norm, then the following bound holds:

\[
\| L \|_{L^2} \leq \lim \inf_{n \to \infty} \| L_n \|_{L^2}.
\] (3.49)

An application of this fact to the sequence \( (R^{(n)}_{\text{odd}}(t_1, t_0))_{n \in \mathbb{N}} \), using the uniform convergence stated in (3.38), yields the result:

\[
\sup_{t_1 \geq t_0} \| \{(id-H-Q(t_1))U(t_1, t_0)(id-H+Q(t_0))\}_{\text{odd}} \|_{L^2} = \| R_{\text{odd}} \|_{L^2,\infty} \leq C_3 \epsilon \| F_{ev} \|_1 < \infty. \] (3.50)

This proves the claim (3.26).

\[ \square \]

**Lemma 3.6** (Partial Integration). *The following integration-by-parts formula holds true:*

\[
U^0ZU^0 = QU^0 - U^0Q - U^0Q'U^0 + U^0Z_{ev}U^0. \] (3.51)

**Proof.** We split \( Z = Z_{ev} + Z_{\text{odd}} \) into even and odd pieces:

\[
U^0ZU^0 = U^0Z_{\text{odd}}U^0 + U^0Z_{ev}U^0 \] (3.52)

Now, \( U^0Z_{\text{odd}}U^0 \) consists of integral operators with the following integral kernels: The component \( U^0Z_{+}U^0 \) has the integral kernel

\[
(p, q) \mapsto \int_0^{t_1} e^{-i(t_1-t)H(p)}P_+(p)Z^{A(t)}(p-q)P_-(q)e^{-i(t-t_0)H(p)} dt
\]
\[
= \int_0^{t_1} e^{-i(t_1-t)E(p)}P_+(p)Z^{A(t)}(p-q)P_-(q)e^{i(t-t_0)E(q)} dt
\]
\[
= e^{-it_1E(p)}P_+(p) \int_0^{t_1} e^{i(E(p)+E(q))}Z^{A(t)}(p-q) dt P_-(q)e^{-it_0E(q)}. \] (3.53)

Recall that the function \( E : \mathbb{R}^3 \to \mathbb{R} \) is defined by \( E(p) = +\sqrt{m^2 + p^2} \). The crucial point is that the frequencies \( E(p) \) and \( E(q) \) have equal signs; they do not partially cancel each other, giving rise to a highly oscillatory integral at high momenta. Note that this works only for the odd part of \( Z \). Integrating by parts, the right hand side in (3.53) equals

\[
\cdots = \frac{P_+(p)Z^{A(t)}(p-q)P_-(q)e^{i(t_1-t_0)E(q)} - e^{-i(t_1-t_0)E(p)}P_+(p)Z^{A(t)}(p-q)P_-(q)}{i(E(p) + E(q))}
\]
\[
- e^{-it_1E(p)}P_+(p) \int_0^{t_1} e^{i(E(p)+E(q))} \frac{\partial}{\partial t} Z^{A(t)}(p-q) dt P_-(q)e^{-it_0E(q)}. \] (3.54)
Similarly, the integral kernel of the \(-+\) component \(U^0Z_{-}U^0\) can be rewritten by an integration by parts as

\[
\frac{P_-(p)Z^{A(t)}(p-q)P_+(q)}{-i(E(p) + E(q))}e^{-i(t_1 - t_0)E(q)} - e^{i(t_1 - t_0)E(q)}\frac{P_-(p)Z^{A(t)}(p-q)P_+(q)}{-i(E(p) + E(q))} - e^{it_1E(p)}P_-(p)\int_{t_0}^{t_1}e^{-i(E(p) + E(q))\frac{\partial}{\partial t}}Z^{A(t)}(p-q)\,dt\,P_+(q)e^{it_0E(q)},
\]

(3.55)

The sum of \([3.54, 3.55]\) is just the integral kernel of \(QU^0 + U^0Q - U^0Q'U^0\). Substituting this into \([3.52, 3.54]\) proves the claim \([3.51, 3.55]\).

Recall that the class \(\mathcal{A} \supset C^\infty_c(\mathbb{R}^4, \mathbb{R}^4)\) of vector potentials was introduced in Definition \(3.3\) (Class of External Four-Vector Potentials).

**Lemma 3.7 (I2 Estimates).** Assume that the external vector potential \(A\) belongs to the class \(\mathcal{A}\). Then the operators \(U^0Z_{ev}QU^0, U^0QZ_{ev}U^0, U^0Q'U^0, Q^2, Q'Q, \) and \(QZQ\), constructed with this potential \(A\), are Hilbert-Schmidt operators. Furthermore, their Hilbert-Schmidt norm is uniformly bounded in the time variables. Finally, the family of operators \(F = (-Q' + Z_{ev} - QZ)(id_H + Q), G = -U^0Q^2 + U^0(-Q' + Z_{ev} - QZ)Q^2U(id_H + Q)\) and \(R = (id_H - Q)U(id_H + Q)\), introduced in \(3.31, 3.32, 3.27\), respectively, fulfill the following bounds in the Hilbert-Schmidt norm:

\[
\|U^0F_{odd}U^0\|_{1,\infty} < \infty \quad \text{and} \quad \|G\|_{1,\infty} < \infty,
\]

(3.56)

and the following bounds in the operator norm:

\[
\|F\|_1 < \infty \quad \text{and} \quad \|R\|_\infty < \infty.
\]

(3.57)

**Proof.** Preliminarily, we estimate for any \(A \in \mathcal{A}\), \(\mu = 0, 1, 2, 3, m = 0, 1, \) and \(n = 1, 2\), using the fundamental theorem of calculus and averaging the starting point \(s\) uniformly over the unit interval:

\[
\sup_{t \in \mathbb{R}} \left\| \frac{d^m}{dt^m} \hat{A}_\mu(t) \right\|_n = \sup_{t \in \mathbb{R}} \left\| \int_0^t \left[ \frac{d^m}{ds^m} \hat{A}_\mu(s) + \int_s^t \frac{d^{m+1}}{du^{m+1}} \hat{A}_\mu(u) \,du \right] \,ds \right\|_n \leq \int_\mathbb{R} \left\| \frac{d^m}{dt^m} \hat{A}_\mu(t) \right\|_n \,dt + \int_\mathbb{R} \left\| \frac{d^{m+1}}{dt^{m+1}} \hat{A}_\mu(t) \right\|_n \,dt < \infty.
\]

(3.58)

At first let us examine the operators \(U^0Z_{ev}QU^0, U^0QZ_{ev}U^0, U^0Q'U^0\). All of these operators have in common that the operator \(Q\) or its derivative are sandwiched between two free time-evolution operators \(U^0\). The kernel of \(Q\), equation \(3.13\), appeared the first time after a partial integration in the time variable, Lemma \(3.51\) (Partial Integration), which gave rise to the factor \([i(E(p) + E(q))]^{-1}\). The idea is that with another partial
integration in the time variable, we will gain another such factor, giving enough decay to see the Hilbert-Schmidt property of the kernel.

In order to treat a part of the cases simultaneously, let $V$ denote $Z_{ev}Q$, $QZ_{ev}$, or $Q'$. Note that in each of these cases, for $t \in \mathbb{R}$, $V(t) : \mathcal{H} \rightarrow \mathcal{H}$ is an odd integral operator. We denote its integral kernel by $(p,q) \mapsto V(t,p,q)$. For any $t_0, t_1 \in \mathbb{R}$, we have

$$
\|(U^0VU^0)(t_1,t_0)\|_2 \leq \|(U^0V_{-}U^0)(t_1,t_0)\|_2 + \|(U^0V_{+}U^0)(t_1,t_0)\|_2,
$$
(3.59)

$$
\|(U^0V_{\pm}U^0)(t_1,t_0)\|_2 = \left\| \int_{t_0}^{t_1} dt \ e^{\mp iE(p)(t_1-t)}V_{\pm}(t,p,q)e^{\pm iE(q)(t-t_0)} \right\|_{2,(p,q)}. \tag{3.60}
$$

Using a partial integration, the last expression (3.60) is estimated as follows.

$$
\ldots = \left\| \int_{t_0}^{t_1} dt \ \left[ \frac{d}{dt} \ e^{\mp iE(p)+E(q)t} \right] V_{\pm}(t,p,q)e^{\mp iE(p)t}e^{\pm iE(q)t_0} \right\|_{2,(p,q)}
$$

$$
\leq 2 \sup_{t \in \mathbb{R}} \left\| \frac{V_{\pm}(t,p,q)}{E(p)+E(q)} \right\|_{2,(p,q)} + \int_{\mathbb{R}} dt \ \left\| \frac{V_{\pm}'(t,p,q)}{E(p)+E(q)} \right\|_{2,(p,q)}
$$

$$
=: f[V_{\pm}] + g[V'_{\pm}]. \tag{3.61}
$$

The first summand comes from the two boundary terms for $t = t_0$ and $t = t_1$. In the following, we show that $f[V_{\pm}]$ and $g[V'_{\pm}]$ are finite. Then, $U^0Z_{ev}QU^0$, $U^0QZ_{ev}U^0$, $U^0Q'U^0$ are in $I_2$ with a Hilbert-Schmidt norm uniformly bounded in the time variable.

**Case $V = Z_{ev}Q$:** The 2-norm of the kernel of $V_{\pm}(t)$ is estimated as follows:

$$
|V_{\pm}(t,p,q)| = \left| \int_{\mathbb{R}^3} dk \ \frac{Z_{\pm}(t,p,k)Z_{\pm}(t,k,q)}{E(k)+E(q)} \right|
$$

$$
= \left| \int_{\mathbb{R}^3} dk \ \sum_{\mu,\nu=0}^{3} \frac{P_{\pm}(p)\alpha^\mu P_{\pm}(k)\alpha^\nu P_{\pm}(q)\hat{A}_\mu(t,p-k)\hat{A}_\nu(t,k-q)}{E(k)+E(q)} \right|
$$

$$
\leq \sum_{\mu,\nu=0}^{3} \int_{\mathbb{R}^3} dk \ |P_{\pm}(p)\alpha^\mu P_{\pm}(k)\alpha^\nu P_{\pm}(q)| \frac{|\hat{A}_\mu(t,p-k)\hat{A}_\nu(t,k-q)|}{E(k)+E(q)}
$$

$$
\leq C_4 \sum_{\mu,\nu=0}^{3} \int_{\mathbb{R}^3} dk \ \frac{|\hat{A}_\mu(t,p-k)\hat{A}_\nu(t,k-q)|}{E(k)+E(q)} \tag{3.62}
$$

with the constant

$$
C_4 := \sum_{\mu,\nu=0}^{3} \sup_{p,k,q \in \mathbb{R}^3} |P_{\pm}(p)\alpha^\mu P_{\pm}(k)\alpha^\nu P_{\pm}(q)| < \infty; \tag{3.63}
$$

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note that \( \sup_{p \in \mathbb{R}^3} |P_\pm(p)| < \infty \) holds, because \( P_\pm \) are orthogonal projections. An analogous argument for \( V' \) yields

\[
|V'_{\pm\pm}(t, p, q)| = \left| \int_{\mathbb{R}^3} dk \frac{Z'_{\pm\pm}(t, p, k) Z_{\pm\pm}(t, k, q) + Z_{\pm\pm}(t, p, k) Z'_{\pm\pm}(t, k, q)}{E(k) + E(q)} \right| \\
\leq C_4 \sum_{\mu, \nu=0}^3 \sup_{t \in \mathbb{R}} \left| \int_{\mathbb{R}^3} \frac{|\hat{A}_\mu(t, p - k) \hat{A}_\nu(t, k - q)| + |\hat{A}_\mu(t, p - k) \hat{A}'_\nu(t, k - q)|}{E(k) + E(q)} \right|. \tag{3.64}
\]

With the bound (3.62), we compute

\[
f[V_{\pm\pm}] = 2 \sup_{t \in \mathbb{R}} \| V_{\pm\pm}(t, p, q) \|_{2,(p,q)} \\
\leq 2C_4 \sum_{\mu, \nu=0}^3 \sup_{t \in \mathbb{R}} \left| \int_{\mathbb{R}^3} \frac{|\hat{A}_\mu(t, p - k) \hat{A}_\nu(t, k - q)|}{E(p) + E(q)} \right| \| E(k) + E(q) \|_{2,(p,q)} \tag{3.65}\]

Lemma (Integral Estimates), applied to the present situation, states that the norm in the last expression is bounded by \( C_8 \| \hat{A}_\mu(t, \cdot) \|_1 \| \hat{A}_\nu(t, \cdot) \|_2 \) with a finite constant \( C_8 \). Applying this yields

\[
f[V_{\pm\pm}] \leq 2C_4C_8 \sum_{\mu, \nu=0}^3 \sup_{t \in \mathbb{R}} \| \hat{A}_\mu(t, \cdot) \|_1 \| \hat{A}_\nu(t, \cdot) \|_2. \tag{3.66}\]

The fact \( A \in \mathcal{A} \) and inequality (3.58) ensure that this expression is finite.

The second summand on the right hand side of (3.61) is estimated with the help of the bound (3.64) as follows:

\[
g[V'_{\pm\pm}] = \int_{\mathbb{R}} dt \left| \frac{V'_{\pm\pm}(t, p, q)}{E(p) + E(q)} \right|_{2,(p,q)} \\
\leq C_4 \sum_{\mu, \nu=0}^3 \int_{\mathbb{R}} \left| \int_{\mathbb{R}^3} \frac{|\hat{A}_\mu(t, p - k) \hat{A}_\nu(t, k - q)|}{E(p) + E(q)} \right| \| E(k) + E(q) \|_{2,(p,q)} \\
+ C_4 \sum_{\mu, \nu=0}^3 \int_{\mathbb{R}} \left| \int_{\mathbb{R}^3} \frac{|\hat{A}_\mu(t, p - k) \hat{A}'_\nu(t, k - q)|}{E(p) + E(q)} \right| \| E(k) + E(q) \|_{2,(p,q)}. \tag{3.67}\]

Again by Lemma (Integral Estimate) we then find

\[
g[V'_{\pm\pm}(t)] \leq C_4C_8 \sum_{\mu, \nu=0}^3 \int_{\mathbb{R}} \left( \| \hat{A}_\mu'(t) \|_1 \| \hat{A}_\nu(t) \|_2 + \| \hat{A}_\mu(t) \|_1 \| \hat{A}_\nu'(t) \|_2 \right), \tag{3.68}\]

while the fact \( A \in \mathcal{A} \) together with its consequence (3.58) ensure the finiteness of this expression. Summarizing, we have shown that \( \| U^0 Z_{ev} Q U^0 \|_{1,\infty} < \infty \).
For any linear operator $A$, we estimate $A$. The fact that $V$ is self-adjoint and $Q$ is skew-adjoint, we compute

$$
\|U^0 Q Z_{ev} U^0 \|_{1_2, \infty} = \| (U^0 Q Z_{ev} Q U^0)^* \|_{1_2, \infty} = \| U^0 Q Z_{ev} Q U^0 \|_{1_2, \infty},
$$

which we have already shown to be finite.

**Case $V = Q'$:** In this case we get

$$|V_{\pm}(t, p, q)| = \frac{|Z_{\pm}(t, p, q)|}{E(p) + E(q)} \leq \sum_{\mu=0}^{3} |P_{\pm}(p)\alpha^\mu P_{\pm}(q)| \frac{\hat{A}_{\mu}'(t, p - q)}{E(p) + E(q)} \leq C_5 \sum_{\mu=0}^{3} \frac{|\hat{A}_{\mu}'(t, p - q)|}{E(p) + E(q)} \quad (3.70)
$$

with the finite constant

$$C_5 := \sum_{\mu=0}^{3} \sup_{p, q \in \mathbb{R}^3} |P_{\pm}(p)\alpha^\mu P_{\pm}(q)|, \quad (3.71)
$$

A similar bound holds for the derivative

$$|V_{\pm}'(t, p, q)| \leq C_5 \sum_{\mu=0}^{3} \frac{\hat{A}_{\mu}'(t, p - q)}{E(p) + E(q)}. \quad (3.72)
$$

**Lemma 3.8:** Integral Estimates, applied to the present situation, states the following bound:

$$
\left\| \frac{\hat{A}_{\mu}'(t, p - q)}{E(p) + E(q)} \right\|_{2,(p, q)} \leq C_8 \| \hat{A}_{\mu}'(t) \|_2. \quad (3.73)
$$

Using this yields the following estimate:

$$
f[V_{\pm}(t)] = 2 \sup_{t \in \mathbb{R}} \left\| \frac{V_{\pm}(t, p, q)}{E(p) + E(q)} \right\|_{2,(p, q)} \leq 2C_5 \sum_{\mu=0}^{3} \sup_{t \in \mathbb{R}} \left\| \frac{\hat{A}_{\mu}'(t, p - q)}{E(p) + E(q)} \right\|_{2,(p, q)} \leq 2C_5C_8 \sum_{\mu=0}^{3} \sup_{t \in \mathbb{R}} \| \hat{A}_{\mu}'(t) \|_2. \quad (3.74)
$$

The fact that $A \in \mathcal{A}$ and inequality [3.58] ensures the finiteness of this expression. Furthermore, we estimate

$$
g[V_{\pm}'(t)] = \int_{\mathbb{R}} dt \left\| \frac{V_{\pm}'(t, p, q)}{E(p) + E(q)} \right\|_{2,(p, q)} \leq C_5 \sum_{\mu=0}^{3} \int_{\mathbb{R}} dt \left\| \frac{\hat{A}_{\mu}'(t, p - q)}{E(p) + E(q)} \right\|_{2,(p, q)}. \quad (3.75)
$$
Again Lemma 3.8 (Integral Estimates) gives that the last expression is bounded as follows:

\[ \ldots \leq C_5 C_8 \sum_{\mu=0}^{3} \int_{\mathbb{R}} dt \| \hat{A}^{(n)}_{\mu}(t) \|_2 \]  

(3.76)

which is also finite since \( A \in A \). Summarizing, we have shown \( \| U^0 Q' U^0 \|_{2,\infty} < \infty \).

Next we examine the operators \( Q^2 \), \( Q'Q \) and \( QZQ \). All of them have in common that \( Q \) or its derivatives appear twice, and therefore we have two of such factors \([E(p) + E(q)]^{-1}\) in the kernel of these operators. We shall see that these facts give enough decay to ensure the finiteness of the Hilbert-Schmidt norms of these operators.

**Cases \( Q^2 \) and \( Q'Q \):** We denote the \( n \)th derivative with respect to time \( t \) by a superscript \( (n) \). For \( n = 0, 1 \) we estimate

\[
\sup_{t \in \mathbb{R}} \| Q^{(n)}(t) Q(t) \|_2 \\
\leq \sup_{t \in \mathbb{R}} \sum_{\mu,\nu=0}^{3} \sum_{\pm} \left\| \int_{\mathbb{R}^3} dk |P_{\pm}(p)\alpha^\mu P_\mp(k)\alpha^\nu P_{\pm}(q)| \frac{\| \hat{A}^{(n)}_{\mu}(t, p - k) \hat{A}_{\nu}(t, k - q) \|_2}{[E(p) + E(k)][E(k) + E(q)]} \right\|_{2,(p,q)} \\
\leq C_6 \sum_{\mu,\nu=0}^{3} \sup_{t \in \mathbb{R}} \left\| \int_{\mathbb{R}^3} dk \frac{|\hat{A}^{(n)}_{\mu}(t, p - k) \hat{A}_{\nu}(t, k - q)|}{[E(p) + E(k)][E(k) + E(q)]} \right\|_{2,(p,q)} 
\]  

(3.77)

with the finite constant

\[ C_6 := \sum_{\mu,\nu=0}^{3} \sup_{p,k,q \in \mathbb{R}^3} \sum_{\pm} |P_{\pm}(p)\alpha^\mu P_\mp(k)\alpha^\nu P_{\pm}(q)|. \]  

(3.78)

Lemma 3.8 (Integral Estimates) provides the upper bound \( C_8 \| \hat{A}^{(n)}_{\mu}(t) \|_1 \| \hat{A}_{\nu}(t) \|_2 \) for the norm of the integral on the right hand side of (3.77). Thus, the right hand side of (3.77) is bounded by:

\[ \ldots \leq C_8 C_6 \sum_{\mu,\nu=0}^{3} \sup_{t \in \mathbb{R}} \| \hat{A}^{(n)}_{\mu}(t) \|_1 \| \hat{A}_{\nu}(t) \|_2, \]  

(3.79)

which is finite because of \( A \in A \) and inequality (3.58). Hence, we have shown \( \| Q^2 \|_{2,\infty} < \infty \) and \( \| Q'Q \|_{2,\infty} < \infty \).
Case $QZQ$: In this case we find

$$\|Q(t)ZQ(t)\|_2$$

$$= \left\| \sum_{\sigma,\tau \in \{-,\}} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \ Q_{-\sigma,\sigma}(t,p,k)Z_{\sigma,\tau}(t,k,j)Q_{\tau,-\tau}(t,j,q) \right\|_{2,(p,q)}$$

$$\leq 4 \sup_{\sigma,\tau \in \{-,\}} \sum_{\lambda,\mu,\nu = 0}^3 \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \ |P_{-\sigma}(p)\alpha^\lambda P_\sigma(k)\alpha^\mu P_\tau(j)\alpha^\nu P_{-\tau}(q)| \times$$

$$\times \frac{|\hat{A}_\lambda(t,p-k)\hat{A}_\mu(t,k-j)\hat{A}_\nu(t,j-q)|}{[E(p)+E(k)][E(j)+E(q)]} \right\|_{2,(p,q)}$$

$$\leq C_7 \left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \ \frac{|\hat{A}_\lambda(t,p-k)\hat{A}_\mu(t,k-j)\hat{A}_\nu(t,j-q)|}{[E(p)+E(k)][E(j)+E(q)]} \right\|_{2,(p,q)}$$

(3.80)

with the finite constant

$$C_7 := 4 \sup_{\sigma,\tau \in \{-,\}} \sum_{\lambda,\mu,\nu = 0}^3 \sup_{p,k,j,q \in \mathbb{R}^3} \ |P_{-\sigma}(p)\alpha^\lambda P_\sigma(k)\alpha^\mu P_\tau(j)\alpha^\nu P_{-\tau}(q)|.$$  (3.81)

By Lemma (3.8) (Integral Estimates) we find the following bound for the right hand side in (3.80):

$$\ldots \leq C_7 C_8 \sum_{\lambda,\mu,\nu = 0}^3 \sup_{I \in \mathbb{R}} \|A_\lambda(t)\|_1 \|A_\mu(t)\|_2 \|A_\nu(t)\|_2$$

(3.82)

which is finite because $A \in A$ and inequality (3.58). This proves the claim $\|QZQ\|_{1,\infty} < \infty$.

Finally, we prove the claims (3.56) and (3.57). As a consequence of $A \in A$ and the bound (3.58), using the definition of the operators $Z(t)$, $Q(t)$, and $Q'(t)$ by their integral kernels given in the equations (3.10), (3.13), and (3.22), we observe the following operator norm bounds:

$$\|L\|_1 < \infty \quad \text{and} \quad \|L\|_\infty < \infty \quad \text{for} \quad L \in \{Z, Z_{ev}, Q, Q'\};$$

(3.83)

recall the definition (3.21) of the norms used here. Furthermore, we know $\|U\|_\infty = 1$, since the one-particle Dirac time evolution $U$ consists of unitary operators. Combining these facts proves the claim (3.57). To prove the first claim in (3.56), we calculate:

$$U^0 F_{odd} U^0 = U^0 Z_{ev} Q U^0 - U^0 Q Z_{ev} U^0 - U^0 Q' U^0 - U^0 Q Z_{odd} Q U^0;$$

(3.84)

see also equation (3.41). Using the bounds in the Hilbert-Schmidt norm proven before, this implies the claim $\|U^0 F_{odd} U^0\|_{1,\infty} < \infty$. Finally, using $\|U^0\|_1 = 1$, $\|Q^2\|_{1,\infty} < \infty$,
||U||_\infty = 1$, and the bounds (3.83), the second claim \( ||G||_{12,\infty} < \infty \) in (3.56) follows also. This finishes the proof of the lemma.

We now state and prove the integral estimates that were used in the previous proof. Recall that the function \( E : \mathbb{R}^3 \rightarrow \mathbb{R} \) is defined by \( E(p) = \sqrt{|p|^2 + m^2} \).

**Lemma 3.8 (Integral Estimates).** For \( C_8 := \|E^{-2}\|_2 < \infty \), the following bounds hold for all \( A_1, A_3 \in L_1(\mathbb{R}^3, \mathbb{C}) \) and \( A_2 \in L_2(\mathbb{R}^3, \mathbb{C}) \):

\[
\left\| \frac{A_2(p - q)}{|E(p) + E(q)|^2} \right\|_{2,(p,q)} \leq C_8 \|A_2\|_2 \quad (i)
\]
\[
\left\| \int_{\mathbb{R}^3} dk \frac{A_1(p - k)A_2(k - q)}{|E(p) + E(q)||E(k) + E(q)|} \right\|_{2,(p,q)} \leq C_8 \|A_1\|_1 \|A_2\|_2 \quad (ii)
\]
\[
\left\| \int_{\mathbb{R}^3} dk \frac{A_1(p - k)A_2(k - q)}{|E(p) + E(k)||E(k) + E(q)|} \right\|_{2,(p,q)} \leq C_8 \|A_1\|_1 \|A_2\|_2 \quad (iii)
\]
\[
\left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} d_r \frac{A_1(p - j)A_2(j - k)A_3(k - q)}{|E(p) + E(j)||E(k) + E(q)|} \right\|_{2,(p,q)} \leq C_8 \|A_1\|_1 \|A_2\|_2 \|A_3\|_1 \quad (iv)
\]

**Proof.** Inequality (i): Substituting \( r := p - q \) and and using \( E(p) + E(q) \geq E(p) \), one finds

\[
\left\| \frac{A_2(p - q)}{|E(p) + E(q)|^2} \right\|_{2,(p,q)} \leq \left\| \frac{A_2(r)}{E(p)^2} \right\|_{2,(p,r)} = \|E^{-2}\|_2 \|A_2\|_2. \quad (3.85)
\]

Inequality (ii): Let \( B = \{ \chi \in L_2(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{C}) \mid \|\chi\|_2 \leq 1 \} \) denote the unit ball in \( L_2(\mathbb{R}^3 \times \mathbb{R}^3, \mathbb{C}) \). Using a dual representation of the norm \( \| \cdot \|_2 \), we get

\[
\left\| \int_{\mathbb{R}^3} dk \frac{A_1(p - k)A_2(k - q)}{|E(p) + E(q)||E(k) + E(q)|} \right\|_{2,(p,q)} \leq \left\| \int_{\mathbb{R}^3} dk \frac{A_1(p - k)A_2(k - q)}{E(q)^2} \right\|_{2,(p,q)} \leq \sup_{\chi \in B} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \frac{A_1(p - k)A_2(k - q)}{E(q)^2} \chi(p,q). \quad (3.86)
\]

Substituting \( j := p - k \), we bound the right hand side in (3.86) as follows:

\[
\ldots = \sup_{\chi \in B} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dj \left| \frac{A_1(j)A_2(p - j) - q)}{E(q)^2} \chi(p,q) \right|
\leq \|A_1\|_1 \sup_{\chi \in B} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \left| \frac{A_2(p - j) - q)}{E(q)^2} \chi(p,q) \right|. \quad (3.87)
\]

Substituting \( r := p - j - q \) and changing the order of integration turns this into

\[
\ldots = \|A_1\|_1 \sup_{\chi \in B} \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dr \left| \frac{A_2(r)}{E(q)^2} \chi(r + q + j, q) \right|. \quad (3.88)
\]

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By the Cauchy-Schwarz inequality, we bound the last expression as follows:
\[
\ldots \leq \|A_1\|_1 \sup_{\chi \in B, j \in \mathbb{R}^3} \left\| \frac{A_2(r)}{E(q)^2} \right\|_{2,(q,r)} \|\chi(r + q, j, q)\|_{2,(q,r)}
= \|E^{-2}\|_2 \|A_1\|_1 \|A_2\|_2. \tag{3.89}
\]

**Inequality (iii):** Similarly, we estimate
\[
\left\| \int_{\mathbb{R}^3} dk \frac{A_1(p - k)A_2(k - q)}{[E(p) + E(k)][E(k) + E(q)]} \right\|_{2,(p,q)} \leq \left\| \int_{\mathbb{R}^3} dk \frac{|A_1(p - k)A_2(k - q)|}{E(k)^2} \chi(p, q) \right\|_{2,(p,q)} \tag{3.90}
\]
Although these terms looks similar to (3.86), there seems to be no substitution which enables us to use the result (3.86) directly.
Interchanging the order of integration and substituting first $j := p - k$ and then $r = k - q$, the right hand side in (3.90) equals
\[
\ldots = \sup_{\chi \in B} \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dj \left| \frac{A_1(j)A_2(k - q)}{E(k)^2} \chi(j + k, q) \right|
\leq \|A_1\|_1 \sup_{\chi \in B, j \in \mathbb{R}^3} \left( \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \left| \frac{A_2(k - q)}{E(k)^2} \chi(j + k, q) \right| \right)
= \|A_1\|_1 \sup_{\chi \in B, j \in \mathbb{R}^3} \left( \int_{\mathbb{R}^3} dr \left| \frac{A_2(r)}{E(k)^2} \chi(j + k, k - r) \right| \right)
\leq \|A_1\|_1 \sup_{\chi \in B} \left( \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dr \int_{\mathbb{R}^3} dk \left| \frac{A_2(r)}{E(k)^2} \chi(j + k, k - r) \right| \right)
= \|E^{-2}\|_2 \|A_1\|_1 \|A_2\|_2. \tag{3.91}
\]

**Inequality (iv):** Again, we get
\[
\left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dr \left( A_1(p - j)A_2(j - k)A_3(k - q) \right) E(p) + E(j)[E(k) + E(q)] \right\|_{2,(p,q)}
\leq \left\| \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dr \left( \frac{A_1(p - j)A_2(j - k)A_3(k - q)}{E(j)E(k)} \right) \right\|_{2,(p,q)}
= \sup_{\chi \in B} \int_{\mathbb{R}^3} dp \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dr \int_{\mathbb{R}^3} ds \left| \frac{A_1(r)A_2(j - k)A_3(s)}{E(j)E(k)} \chi(r + j, k - s) \right|. \tag{3.92}
\]
Interchanging the integration and substituting $r := p - j$ and $s := k - q$, this equals
\[
\ldots = \sup_{\chi \in B} \int_{\mathbb{R}^3} dk \int_{\mathbb{R}^3} dq \int_{\mathbb{R}^3} dr \int_{\mathbb{R}^3} ds \left| \frac{A_1(r)A_2(j - k)A_3(s)}{E(j)E(k)} \chi(r + j, k - s) \right|. \tag{3.93}
\]
We apply Hölder’s inequality twice to bound (3.93) as follows:

\[ \ldots \leq \|A_1\|_1 \|A_3\|_1 \sup_{\chi \in B, r,s \in \mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{A_2(j-k)}{E(j)E(k)} \chi(r+j,k-s) \]  

(3.94)

Using the Cauchy-Schwarz inequality and then the substitution \( u := j - k \), this term is bounded from above by

\[ \ldots \leq \|A_1\|_1 \|A_3\|_1 \left\| \frac{A_2(u)}{E(u+k)E(k)} \right\|_{2,(u,k)} \]

\[ \leq \|A_1\|_1 \|A_3\|_1 \left\| \frac{1}{E(u+k)E(k)} \right\|_{2,k} \]

\[ \leq \|A_1\|_1 \|A_2\|_2 \|A_3\|_1 \left\| E^{-2} \right\|_2. \]  

(3.95)

In the last step, we have once more used the Cauchy-Schwarz inequality.

\[ \square \]

3.2 Identification of Polarization Classes

In this Subsection we show that there is a one-to-one correspondence of the magnetic components \( \vec{A} \) of the four-vector fields \( A = (A_\mu)_{\mu=0,1,2,3} = (A^0, -\vec{A}) \) to the physically relevant polarization classes \( C(A) = \{ e^{Q^A} V \mid V \in C(0) = [\mathcal{H}_-]|_{\approx_0} \} \), introduced in Definition 3.1 (Induced Polarization Classes).

Theorem 3.9 (Identification of the Polarization Classes). For \( A, A' \in C_c^\infty(\mathbb{R}^3, \mathbb{R}^4) \), the following are equivalent:

(a) \( C(A) = C(A') \)

(b) \( \vec{A} = \vec{A'} \)

On this ground the following notation makes sense:

Definition 3.10 (Physical Polarization Classes). For \( A = (A_\mu)_{\mu=0,1,2,3} = (A^0, -\vec{A}) \) in \( C_c^\infty(\mathbb{R}^3, \mathbb{R}^4) \), we define

\[ C(\vec{A}) := C(A). \]

For this Subsection it is convenient to use the four-vector notation of special relativity. To avoid confusion, in this section, three-vectors are labeled with an arrow. Define the Lorentz metric \( (g_{\mu\nu})_{\mu,\nu=0,1,2,3} = \text{diag}(1, -1, -1, -1) \). Raising and lowering of
such that the corresponding projection operators introduced in (3.5) are Hilbert-Schmidt operators. As a consequence, \( P \) need to show that \( P \) holds if and only if \( \tilde{A} = \tilde{A}' \).

Now, \( P' = e^{Q^A} P_- e^{-Q^A} \) holds. Just as in (3.19), we know that \( e^{\pm Q^A} - (id_{\mathcal{H}} \pm Q^A) \) are Hilbert-Schmidt operators. As a consequence, \( P' \) differs from \( (id_{\mathcal{H}} + Q^A) P_- (id_{\mathcal{H}} - Q^A) \) only by a Hilbert-Schmidt operator. Using that \( Q^A \) is odd, we know \( Q^A P^- Q^A = [(Q^A)^2]_{++} \). Because \( (Q^A)^2 \) is a Hilbert-Schmidt operator by Lemma 3.7 (I_2 Estimates), it follows that \( Q^A P^- Q^A \in I_2(\mathcal{H}) \). We conclude that \( P' - id_{\mathcal{H}} - Q^A P_- + P^- Q^A \in I_2(\mathcal{H}) \). The same
argument, applied to $A'$, shows that
\[
P_W - \text{id}_H - Q^A P_- + P_- Q^{A'} \in I_2(H).
\]
Taking the difference, this implies
\[
P_V - P_W = (Q^A - Q^{A'}) P_- - P_- (Q^A - Q^{A'}) + I_2(H)
\]
\[
= Q^{A-A'} P_- - P_- Q^{A-A'} + I_2(H) = Q^{A-A'}_{+-} + I_2(H); \tag{3.103}
\]
recall that $Q^A$ is linear in the argument $A$. Using once more that $Q^{A-A'}$ is odd, this yields the following equivalences:
\[
V \approx W \iff P_V - P_W \in I_2(H) \iff Q^{A-A'}_{+-} - Q^{A-A'}_{+-} \in I_2(H) \iff Q^{A-A'}_{+-} \in I_2(H) \tag{3.104}
\]
Now Lemma 3.11 (Hilbert-Schmidt Condition for $Q$) below, applied to $A - A'$, states that $Q^{A-A'}_{+-} \in I_2(H)$ is equivalent to $\vec{A} = \vec{A}'$. Summarizing, we have shown that $V \approx W$ holds if and only if $\vec{A} = \vec{A}'$.

In order to show that in this case $V \approx 0$ holds also, it remains to show $\text{charge}(V, W) = 0$. Now, because $e^{Q^A}|_{H_- \to V}$ and $e^{-Q^{A'}}|_{W \to H_-}$ are unitary maps, we get
\[
\text{charge}(V, W) = \text{ind}(P_W|_{V \to W}) = \text{ind}
\left(\begin{array}{c}
\begin{array}{r}
e^{-Q^A'} P_W e^{Q^A}\
|H_- \to H_-
\end{array}
\end{array}\right)
\]
\[
= \text{ind}
\left(\begin{array}{c}
\begin{array}{r}
P_- e^{-Q^A'} e^{Q^A}\
|H_- \to H_-
\end{array}
\end{array}\right) = \text{ind}
\left(\begin{array}{c}
\begin{array}{r}(e^{-Q^A'} e^{Q^A})_{-}\
|H_- \to H_-
\end{array}
\end{array}\right). \tag{3.105}
\]
Because $Q^A$ is skew-adjoint and its square $(Q^A)^2$ is a Hilbert-Schmidt operator, $e^{Q^A}$ is a compact perturbation of the identity $\text{id}_H$. The same argument shows that $e^{-Q^{A'}}$ is also a compact perturbation of the identity. Hence, $(e^{-Q^A'} e^{Q^A})_{-} |_{H_- \to H_-}$ is a compact perturbation of $\text{id}_{H_-}$ and thus has Fredholm index 0. This shows that $\text{charge}(V, W) = 0$ and finishes the proof.

The lemma used in the proof of Theorem 3.9 (Identification of the Polarization Classes) is:

**Lemma 3.11** (Hilbert-Schmidt Condition for $Q$). For $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec{A})$ in $\mathcal{C}_2^0(\mathbb{R}^3, \mathbb{R}^4)$, the following are equivalent:

(a) $Q^A \in I_2(H)$,

(b) $\vec{A} = 0$.

**Proof.** We calculate the squared Hilbert-Schmidt norm $\|Q^A\|_{I_2}^2$ of $Q^A$. Using the abbreviations $Q^A_{+-}(\vec{p}, \vec{q}) = P_+(\vec{p}) Q^A(\vec{p}, \vec{q}) P_-(\vec{q})$ and $Q^A_{-+}(\vec{p}, \vec{q}) = P_-(\vec{p}) Q^A(\vec{p}, \vec{q}) P_+(\vec{q})$, we get
\[
\|Q^A\|_{I_2}^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \text{tr}[Q^A(\vec{p}, \vec{q}) Q^A(\vec{p}, \vec{q})^*] \, dp \, dq
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left[\text{tr}[Q^A_{+-}(\vec{p}, \vec{q}) Q^A_{-+}(\vec{p}, \vec{q})^*] + \text{tr}[Q^A_{-+}(\vec{p}, \vec{q}) Q^A_{+-}(\vec{p}, \vec{q})^*]\right] \, dp \, dq. \tag{3.106}
\]
Inserting the Definition (3.13) of $Q^A$, using that $[\gamma^0 \hat{A}(\vec{p} - \vec{q})]^* = \gamma^0 \hat{A}(\vec{q} - \vec{p})$ and that $P_+(p)$ and $P_-(q)$ are orthogonal projections having the representation (3.102), we express the first summand as follows:

$$
\text{tr}[Q_{A+}^*(\vec{p}, \vec{q})Q_{A+}^*(\vec{p}, \vec{q})^*] = \frac{e^2}{4p_+0q_0(p_+0 - q_0)^2} \text{tr} \left( [(\vec{p}_+ + m)\gamma^0] [\gamma^0 \hat{A}(\vec{p} - \vec{q})][((\vec{q}_- + m)\gamma^0] \cdot [(\vec{q}_- + m)\gamma^0]^* [\gamma^0 \hat{A}(\vec{p} - \vec{q})]^* [(\vec{p}_+ + m)\gamma^0] \right)
$$

Now we use the following formulas for traces of products of $\gamma$-matrices:

$$
\text{tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu},
\text{tr}(\gamma^\mu \gamma^\nu \gamma^\kappa) = 0,
\text{tr}(\gamma^\mu \gamma^\nu \gamma^\kappa \gamma^\lambda) = 4(g^{\mu\nu}g^{\kappa\lambda} + g^{\mu\kappa}g^{\nu\lambda} - g^{\mu\lambda}g^{\nu\kappa}).
$$

We obtain

$$
0 \leq \text{tr}[Q_{A+}^*(\vec{p}, \vec{q})Q_{A+}^*(\vec{p}, \vec{q})^*]
$$

$$
= \frac{e^2}{4p_+0q_0(p_+0 - q_0)^2} \text{tr} \left( (\vec{p}_+ + m)\hat{A}(\vec{p} - \vec{q})(\vec{q}_- + m)\hat{A}(\vec{q} - \vec{p}) \right)
$$

$$
= \frac{e^2}{p_+0q_0(p_+0 - q_0)^2} \left( (m^2 - p_+ \cdot q_-)\hat{A}(\vec{p} - \vec{q}) \cdot \hat{A}(\vec{q} - \vec{p}) + (p_+ \cdot \hat{A}(\vec{p} - \vec{q}))(q_+ \cdot \hat{A}(\vec{q} - \vec{p})) + (p_- \cdot \hat{A}(\vec{q} - \vec{p}))(q_- \cdot \hat{A}(\vec{q} - \vec{p})) \right). \tag{3.111}
$$

The second summand on the right hand side in (3.105) can be calculated in a similar way by exchanging the indices “+” and “-”:

$$
0 \leq \text{tr}[Q_{A-}^*(\vec{p}, \vec{q})Q_{A-}^*(\vec{p}, \vec{q})^*]
$$

$$
= \frac{e^2}{p_-0q_0(p_-0 - q_0)^2} \left( (m^2 - p_- \cdot q_+)\hat{A}(\vec{p} - \vec{q}) \cdot \hat{A}(\vec{q} - \vec{p}) + (p_- \cdot \hat{A}(\vec{q} - \vec{p}))(q_+ \cdot \hat{A}(\vec{q} - \vec{p})) + (p_+ \cdot \hat{A}(\vec{p} - \vec{q}))(q_- \cdot \hat{A}(\vec{q} - \vec{p})) \right)
$$

$$
= \text{tr}[Q_{A-}^*(\vec{q}, \vec{p})Q_{A+}^*(\vec{q}, \vec{p})^*]. \tag{3.112}
$$

Thus, the two summands in (3.106) are the same up to exchanging $\vec{p}$ and $\vec{q}$. In particular,
this yields
\[
\|Q^A\|_{I_2}^2 = 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \text{tr}[Q^A_{+ -}(\vec{p}, \vec{q})Q^A_{+ -}(\vec{p}, \vec{q})^*] \, dp \, dq
\]
\[
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{2e^2}{p_{++} q_{- -} (p_{++} - q_{- -})^2} \left( (m^2 - p_+ \cdot q_-) \hat{A}(\vec{p} - \vec{q}) \cdot \hat{A}(\vec{q} - \vec{p}) + (p_+ \cdot \hat{A}(\vec{p} - \vec{q})) (q_- \cdot \hat{A}(\vec{q} - \vec{p})) \right) \, dp \, dq. \tag{3.113}
\]

Let us now use this to prove that \( \hat{A} = 0 \) implies \( Q^A \in I_2(H) \). In the case \( \hat{A} = 0 \), formula (3.113) boils down to
\[
\|Q^A\|_{I_2}^2 = 2e^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{E(\vec{p})E(\vec{q}) - \vec{p} \cdot \vec{q} - m^2}{E(\vec{p})E(\vec{q})} |\hat{A}_0(\vec{p} - \vec{q})|^2 \, dp \, dq
\]
\[
= 2e^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{E(\vec{p})E(\vec{p} - \vec{k}) - \vec{p} \cdot (\vec{p} - \vec{k}) - m^2}{E(\vec{p})E(\vec{p} - \vec{k})} |\hat{A}_0(\vec{k})|^2 \, dp \, dk
\]
\[
= 2e^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{E(\vec{p})E(\vec{p} - \vec{k}) - \vec{p} \cdot \vec{k}}{E(\vec{p})E(\vec{p} - \vec{k})} |\hat{A}_0(\vec{k})|^2 \, dp \, dk
\]
\[
\leq 2e^2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{E(\vec{p} \cdot \vec{k}) - E(\vec{p}) + \vec{p} \cdot \vec{k}}{E(\vec{p} - \vec{k})E(\vec{p})} |\hat{A}_0(\vec{k})|^2 \, dp \, dk, \tag{3.114}
\]
where we have used \( E(\vec{p})^2 - |\vec{p}|^2 = m^2 \). We expand \( E(\vec{p} - \vec{k}) \) around \( \vec{k} = 0 \): For \( t \in \mathbb{R} \), one has
\[
\frac{\partial}{\partial t} E(\vec{p} - t\vec{k}) = -\frac{\vec{k} \cdot (\vec{p} - t\vec{k})}{E(\vec{p} - t\vec{k})}, \tag{3.115}
\]
\[
\frac{\partial^2}{\partial t^2} E(\vec{p} - t\vec{k}) = \frac{|\vec{k}|^2}{E(\vec{p} - t\vec{k})} - \frac{|\vec{k} \cdot (\vec{p} - t\vec{k})|^2}{E(\vec{p} - t\vec{k})^3}. \tag{3.116}
\]
Using
\[
0 \leq |\vec{k} \cdot (\vec{p} - t\vec{k})|^2 \leq |\vec{k}|^2 |\vec{p} - t\vec{k}|^2 \leq |\vec{k}|^2 E(\vec{p} - t\vec{k})^2 \tag{3.117}
\]
we conclude
\[
0 \leq \frac{\partial^2}{\partial t^2} E(\vec{p} - t\vec{k}) \leq \frac{|\vec{k}|^2}{E(\vec{p} - t\vec{k})}. \tag{3.118}
\]
By Taylor’s formula, we get for some \( t, \vec{p}, \vec{k} \in [0, 1] \):
\[
0 \leq E(\vec{p} - \vec{k}) - E(\vec{p}) + \frac{\vec{p} \cdot \vec{k}}{E(\vec{p})} \leq \frac{|\vec{k}|^2}{2E(\vec{p} - t\vec{p}, \vec{k})}. \tag{3.119}
\]

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Now, using the variable $\vec{q}_t := \vec{p} - t\vec{k}$ with $0 \leq t \leq 1$, we estimate
\[
E(\vec{p})^2 = E(\vec{q}_t + t\vec{k})^2 = |\vec{q}_t + t\vec{k}|^2 + m^2 \leq 2|\vec{q}_t|^2 + 2t^2|\vec{k}|^2 + m^2 \\
\leq \frac{2}{m^2}(|\vec{q}_t|^2 + m^2)(t^2|\vec{k}|^2 + m^2) \leq \frac{2}{m^2}E(\vec{q}_t)^2E(\vec{k})^2.
\]
This yields for $0 \leq t \leq 1$:
\[
\frac{1}{E(\vec{p} - t\vec{k})} \leq \frac{\sqrt{2} E(\vec{k})}{m E(\vec{p})}.
\]
Substituting the bounds (3.119) and (3.121) for $t = 1$ and for $t = t_{\vec{p},\vec{k}}$ in (3.114), we conclude
\[
\|Q^A\|_{L_2}^2 \leq \frac{2e^2}{m^2} \int_{R^3} \frac{dp}{E(\vec{p})} \int_{R^3} |\vec{k}|^2E(\vec{k})^2|\hat{A}_0(\vec{k})|^2 dk < \infty.
\]
Thus $\hat{A} = 0$ implies $\|Q^A\|_{L_2} < \infty$.

We now prove that $\hat{A} \neq 0$ implies $\|Q^A\|_{L_2} = \infty$. We split $A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\hat{A})$ into $A = (A_0, \vec{0}) + (0, -\hat{A})$. Abbreviating $Q^{A_0} = Q^{(A_0, \vec{0})}$ and $Q^{\hat{A}} := Q^{(0, -\hat{A})}$, we conclude
\[
Q^A = Q^{A_0} + Q^{\hat{A}}.
\]
The part (b)⇒(a) implies that the first summand $Q^{A_0}$ is a Hilbert-Schmidt operator. Hence, $Q^A$ is a Hilbert-Schmidt operator if and only if $Q^{\hat{A}}$ is a Hilbert-Schmidt operator. Thus it remains to show that $\hat{A} \neq 0$ implies $\|Q^{\hat{A}}\|_{L_2} = \infty$.

Equation (3.113) in the special case of a vanishing 0-component of the vector potential can be rewritten as follows:
\[
\|Q^{\hat{A}}\|_{L_2}^2 = \int_{R^3} \int_{R^3} \frac{2e^2}{E(\vec{p})E(\vec{q})} \left( (m^2 + E(\vec{p})E(\vec{q}) + \vec{p} \cdot \vec{q})|\hat{A}(\vec{p} - \vec{q})|^2 \\
- (\vec{p} \cdot \hat{A}(\vec{p} - \vec{q})) (\vec{q} \cdot \hat{A}(\vec{q} - \vec{p})) - (\vec{p} \cdot \hat{A}(\vec{q} - \vec{p})) (\vec{q} \cdot \hat{A}(\vec{p} - \vec{q})) \right) dp dq
\]
Using (3.112), we see that the integrand in this integral is non-negative. We substitute $\vec{k} := \vec{p} - \vec{q}$. For any measurable set $S \subseteq R^3 \times R^3$, we get a lower bound by restricting the integration to $S$:
\[
\|Q^{\hat{A}}\|_{L_2}^2 \geq \int_S \frac{2e^2}{E(\vec{p})E(\vec{p} - \vec{k})} \frac{E(\vec{p} - \vec{k})^2}{E(\vec{p} - \vec{k})^2} \left( (m^2 + E(\vec{p})E(\vec{p} - \vec{k})^2 \\
+ \vec{p} \cdot (\vec{p} - \vec{k})) |\hat{A}(\vec{k})|^2 - (\vec{p} \cdot \hat{A}(\vec{k}))(\vec{p} - \vec{k}) \cdot \hat{A}(\vec{k}) \\
- (\vec{p} \cdot \hat{A}(\vec{k}))(\vec{p} - \vec{k}) \cdot \hat{A}(\vec{k}) \right) dp dk.
\]
The following considerations serve to find an appropriate choice of the set $S$. By the assumption $\tilde{A} \neq 0$ we can take $\bar{l} \in \mathbb{R}^3$ such that $\hat{A}(\bar{l}) \neq 0$. For every $\bar{a} \in \mathbb{C}^3 \setminus \{0\}$, there exists a unit vector $\bar{b} \in \mathbb{R}^3$, $|\bar{b}| = 1$, such that $|\bar{b} \cdot \bar{a}| \leq |\bar{a}|/\sqrt{2}$. One can see this as follows. We define $\bar{c} = \bar{a}$ if $|\text{Re} \bar{a}| \geq |\text{Im} \bar{a}|$, and $\bar{c} = i\bar{a}$ otherwise. In particular, $|\bar{c}| = |\bar{a}|$ and

$$2|\text{Im} \bar{a}|^2 \leq |\text{Re} \bar{a}|^2 + |\text{Im} \bar{a}|^2 = |\bar{a}|^2. \quad (3.126)$$

Take any unit vector $\bar{b} \in \mathbb{R}^3$ orthogonal to $\text{Re} \bar{c}$. Using $(3.126)$, we get

$$|\bar{b} \cdot \bar{a}| = |\bar{b} \cdot \bar{c}| = |\bar{b} \cdot \text{Im} \bar{c}| \leq |\bar{b}| |\text{Im} \bar{c}| = |\text{Im} \bar{c}| \leq \frac{|\bar{a}|}{\sqrt{2}}. \quad (3.127)$$

We apply this to $\bar{a} = \hat{A}(\bar{l})$, taking a unit vector $\bar{b} \in \mathbb{R}^3$ with $|\bar{b} \cdot \hat{A}(\bar{l})| \leq |\hat{A}(\bar{l})|/\sqrt{2}$. Take any fixed number $C_9$ such that $1/\sqrt{2} < C_9 < 1$; then $|\bar{b} \cdot \hat{A}(\bar{l})| < C_9|\bar{b}| |\hat{A}(\bar{l})|$ holds because of $|\bar{b}| = 1$ and $|\hat{A}(\bar{l})| > 0$. Now $\hat{A}$ is a continuous function. Therefore, there is a compact ball $B_r(\bar{l}) = \{\bar{k} \in \mathbb{R}^3 \mid |\bar{k} - \bar{l}| \leq r\}$, centered at $\bar{l}$ with some radius $r > 0$, such that

$$C_{10} := \inf_{\bar{k} \in B_r(\bar{l})} |\hat{A}(\bar{k})| > 0 \quad (3.128)$$

is true and $|\bar{b} \cdot \hat{A}(\bar{k})| < C_9|\bar{b}| |\hat{A}(\bar{k})|$ holds for all $\bar{k} \in B_r(\bar{l})$. By compactness of the ball, using continuity of the function $\mathbb{R}^3 \times \mathbb{R}^3 \ni (\bar{p}, \bar{k}) \mapsto |\bar{p} \cdot \hat{A}(\bar{k})| - C_9|\bar{p}| |\hat{A}(\bar{k})|$, the set

$$S_1 := \{\bar{p} \in \mathbb{R}^3 \mid \text{for all } \bar{k} \in B_r(\bar{l}) \text{ holds } |\bar{p} \cdot \hat{A}(\bar{k})| < C_9|\bar{p}| |\hat{A}(\bar{k})|\} \quad (3.129)$$

is an open subset of $\mathbb{R}^3$. The set $S_1$ is nonempty because of $\bar{b} \in S_1$. Furthermore, $S_1$ is a homogeneous set in the following sense: For all $\bar{p} \in \mathbb{R}^3$ and all $\lambda \in \mathbb{R} \setminus \{0\}$, $\lambda \bar{p} \in S_1$ is equivalent to $\lambda \bar{p} \in S_1$. Note that $|\bar{p} \cdot \hat{A}(\bar{k})| = |\bar{p} \cdot \hat{A}(-\bar{k})|$ holds, as $\hat{A}(-\bar{k})$ and $\hat{A}(\bar{k})$ are complex conjugate to each other.

We set $S = S_1 \times B_r(\bar{l})$. For the following considerations, note that $|E(\bar{p} - \bar{k}) - E(\bar{p})| \leq |\bar{k}|$, $|\bar{p}| \leq E(\bar{p})$, and $(\bar{p} \cdot \hat{A}(\bar{k}))(\bar{p} \cdot \hat{A}(-\bar{k})) = |\bar{p} \cdot \hat{A}(\bar{k})|^2$ hold for all $\bar{p}, \bar{k} \in \mathbb{R}^3$, and that $\hat{A}$ is bounded on the ball $B_r(\bar{l})$. Using this, one sees that there is a constant $C_{11} > 0$, depending only on the potential $\hat{A}$ and on the choice of the compact ball $B_r(\bar{l})$, such that for all $\bar{p} \in \mathbb{R}^3$ and all $\bar{k} \in B_r(\bar{l})$, one has

$$\left|\left((m^2 + E(\bar{p}))E(\bar{p} - \bar{k}) + \bar{p} \cdot (\bar{p} - \bar{k}))|\hat{A}(\bar{k})|^2 \right.ight.$$

$$\left. - (\bar{p} \cdot \hat{A}(\bar{k}))((\bar{p} - \bar{k}) \cdot \hat{A}(-\bar{k})) - (\bar{p} \cdot \hat{A}(-\bar{k}))((\bar{p} - \bar{k}) \cdot \hat{A}(\bar{k})) \right)$$

$$- \left[2E(\bar{p})^2|\hat{A}(\bar{k})|^2 - 2|\bar{p} \cdot \hat{A}(\bar{k})|^2\right] \leq C_{11}E(\bar{p}). \quad (3.130)$$

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Furthermore, there is another constant $C_{12} > 0$, depending only on the choice of the compact ball $B_r(\vec{l})$, such that for all $\vec{p} \in \mathbb{R}^3$ and all $\vec{k} \in B_r(\vec{l})$, one has

$$E(\vec{p} - \vec{k})(E(\vec{p}) + E(\vec{p} - \vec{k}))^2 \leq C_{12}E(\vec{p})^3.$$  

(3.131)

Substituting the bounds (3.130), (3.131), the choice (3.129) of $S_1$, and the lower bound (3.128) of $|\vec{A}|$ on $B_r(\vec{l})$ in the lower bound (3.125) of $\|Q\vec{A}\|_{L^2}^2$, we obtain

$$\|Q\vec{A}\|_{L^2}^2 \geq \int_S \frac{2e^2}{C_{12}E(\vec{p})^4} \left(2E(\vec{p})^2|\vec{A}(\vec{k})|^2 - 2|\vec{p}| \cdot \vec{A}(\vec{k})|^2 - C_{11}E(\vec{p}) \right) dp dk$$

$$\geq \int_S \frac{2e^2}{C_{12}E(\vec{p})^4} \left(2(1 - C_9)E(\vec{p})^2|\vec{A}(\vec{k})|^2 - C_{11}E(\vec{p}) \right) dp dk$$

$$\geq |B_r(\vec{l})| \int_{S_1} \frac{2e^2}{C_{12}E(\vec{p})^4} \left(2(1 - C_9)C_{10}E(\vec{p})^2 - C_{11}E(\vec{p}) \right) dp = \infty.$$  

(3.132)

We have used that $1 - C_9 > 0$, and that $S_1$ is a nonempty, open homogeneous subset of $\mathbb{R}^3$. Thus the lemma is proven.

\[ \square \]

### 3.3 Gauge Transformations

As an addendum we briefly discuss gauge transformations. Let $\vec{A} \in C^\infty_c(\mathbb{R}^3, \mathbb{R}^3)$ be a vector potential and $\vec{A}^- = \vec{A} + \vec{\nabla}Y$ be a gauge transform of it with $Y \in C^\infty_c(\mathbb{R}^3, \mathbb{R})$. Let $e^{iY} : \mathcal{H} \rightarrow \mathcal{H}$ be the multiplication operator with $e^{iY}$. We prove:

**Theorem 3.12** (Gauge Transformations). The gauge transformation $e^{iY}$ fulfills:

$$e^{iY} \in U^0_{\text{res}}(\mathcal{H}, C(\vec{A}); \mathcal{H}, C(\vec{A}^-))$$  

(3.133)

Although the statement of this theorem does not involve time, we prove it using the time-evolution from Subsection 3.1.4. A “direct” proof, avoiding time-evolution and using similar techniques as in Subsection 3.1.4, is possible. However, the approach presented here avoids additional analytical considerations.

**Proof.** We switch the gauge transformation on between the times 0 and 1, using a smooth function $f : \mathbb{R} \rightarrow [0, 1]$ with $f(t) = 0$ and $f(t) = 1$ for $t$ in a neighborhood of 0 and 1, respectively. We define $Y : \mathbb{R}^4 \ni (t, \vec{x}) \mapsto f(t)Y(\vec{x}) \in \mathbb{R}$. Take the static vector potential $\vec{A} : \mathbb{R}^4 \ni (t, \vec{x}) \mapsto (0, -\vec{A}(\vec{x})) \in \mathbb{R}^4$ and its gauge-transformed version $\vec{A}^- = (\vec{A}^-)_\mu = 0, 1, 2, 3 = (A_\mu - \partial_\mu Y, -\vec{A} - \vec{\nabla}Y)$. In other words,

$$\vec{A}^-(t, \vec{x}) = (-f'(t)Y(\vec{x}), -\vec{A}(\vec{x}) - f(t)\vec{\nabla}Y(\vec{x})).$$  

(3.134)

(It is no problem that the vector potentials used here do in general not have compact support in time, because we use only times $t \in [0, 1]$.) Note that at time $t = 0$, the gauge
transformation is turned off: \( A(0) = A^\sim(0) = (0, -\vec{A}) \), and at time \( t = 1 \) it is completely turned on: \( A(1) = (0, -\vec{A}) \) and \( A^\sim(1) = (0, -\vec{A}^\sim) \). The one-particle Dirac time-evolutions \( U^A \) and \( U^{A^\sim} \) are also related by a gauge transformation as follows:

\[
e^{iY(t_1)}U^A(t_1, t_0) = U^{A^\sim}(t_1, t_0)e^{iY(t_0)}, \quad t_1, t_0 \in [0, 1]. \tag{3.135}
\]

In particular, this includes \( e^{iY}U^A(1, 0) = U^{A^\sim}(1, 0) \). By Theorem 3.2 (Dirac Time-Evolution with External Field), we have the following:

\[
U^A(0, 1) \in U^0_{\text{res}}(\mathcal{H}, C(A); \mathcal{H}, C(A)) \tag{3.136}
\]

\[
U^{A^\sim}(1, 0) \in U^0_{\text{res}}(\mathcal{H}, C(A); \mathcal{H}, C(A^\sim)) \tag{3.137}
\]

This implies the following:

\[
e^{iY} = U^{A^\sim}(1, 0)U^A(0, 1) \in U^0_{\text{res}}(\mathcal{H}, C(A); \mathcal{H}, C(A^\sim)) \tag{3.138}
\]

Thus the claim is proven.

We infer that in general the gauge transformation \( e^{iY} \) changes the polarization class. Using varying wedge spaces, it can be second quantized as follows. Let \( S \in \text{Ocean}(C(\vec{A})) \) and \( S^\sim \in \text{Ocean}(C(\vec{A}^\sim)) \). By Theorem 2.26 (Lift Condition), there exists \( R \in U(\ell) \) such that we have the following second-quantized gauge transformation from \( F_S \) to \( F_{S^\sim} \):

\[
\begin{array}{c}
F_S \xrightarrow{L_{e^{iY}}} F_{(e^{iY}S)} \xrightarrow{R_R} F_{S^\sim}
\end{array}
\]

### 4 Summary: the Second Quantized Time-Evolution

Combining Theorem 3.9 (Identification of the Polarization Classes), Theorem 3.2 (Dirac Time-Evolution with External Field), Theorem 2.26 (Lift Condition), Corollary 2.28 (Uniqueness of the Lift up to a Phase) and 3.12 (Gauge Transformations) we have proven the following:

**Main Results** (Second Quantized Dirac Time-Evolution). Let \( A = (A_\mu)_{\mu=0,1,2,3} = (A_0, -\vec{A}) \in \mathcal{C}_c^\infty(\mathbb{R}^4, \mathbb{R}^4) \) be an external vector potential. Let \((U^A(t_1, t_0) : \mathcal{H} \to \mathcal{H})_{t_0, t_1 \in \mathbb{R}}\) be the corresponding one-particle Dirac time-evolution.

We have constructed the following:

1. natural polarization classes \( C(\vec{A}(t)) \in \text{Pol}(\mathcal{H})/\approx_0, t \in \mathbb{R}, \), introduced in Definition 3.10, depending only on the three-vector potential \( \vec{A}(t) \) at time \( t \in \mathbb{R} \), but neither on the history \( A(s), s \neq t \), nor on the electric potential \( A^0(t) \), and...
for any choice of a vacuum $\Phi$ a natural family of Fock spaces $F_t$, $t \in \mathbb{R}$, as follows:

Given a (separable, infinite dimensional) Hilbert space $\ell$, for any $\Phi \in \text{Ocean}_\ell(C(0))$ one has the family of equivalence classes

$$S(t) := [e^{Q_A(t)} \Phi]_\sim \in \text{Ocean}_\ell(C(\tilde{A}(t))), \quad t \in \mathbb{R},$$

and, hence, the corresponding family of Fock spaces $F_t = F_{S(t)}$, $t \in \mathbb{R}$.

We have obtained the following:

3. For $t_0, t_1 \in \mathbb{R}$ it holds that

$$U^A(t_1, t_0) \in U^0_{\text{res}}(\mathcal{H}, C(\tilde{A}(t_0)); \mathcal{H}, C(\tilde{A}(t_1))).$$

This yields on the above family of Fock spaces a natural second-quantized Dirac time-evolution

$$\tilde{U}^A(t_1, t_0) : F_{t_0} \to F_{t_1},$$

completely determined up to a phase. It is given as follows: abbreviating $U^A := U^A(t_1, t_0)$, there is $R \in U(\ell)$ such that

$$\tilde{U}^A(t_1, t_0) = R R_1 L_{U^A} : F_{t_0} \to F_{t_1},$$

is a unitary map between the Fock spaces $F_{t_0}$ and $F_{t_1}$. $\tilde{U}^A(t_1, t_0)$ is unique up to a phase in the following sense: for any two such choices $R_1, R_2 \in U(\ell)$ with $R_{R_1} L_{U^A}, R_{R_2} L_{U^A} : F_{t_0} \to F_{t_1}$, the operator $R_1^{-1} R_2$ has a determinant $\det(R_1^{-1} R_2) = e^{i\varphi}$ for some $\varphi \in \mathbb{R}$, and it holds

$$R_{R_2} L_{U^A} = e^{i\varphi} R_{R_1} L_{U^A}.$$  

4. The formalism is gauge invariant although gauge transformations may change the polarization classes, and therefore the induced second-quantized gauge transformations may act between varying Fock spaces.

We emphasize that the family of Fock spaces is defined in terms of the equivalence classes $S(t)$, $t \in \mathbb{R}$. Hence, the dependence on the choice of $\Phi$ is weak. Up to a natural isomorphism, being unique up to a phase, $F_t$ depends only on the polarization class $[\text{range } e^{Q_A(t)} \Phi]_\approx_0$; cf. Lemma 2.12 (Connection between $\sim$ and $\approx_0$). Changing the choice of $\Phi$ within the same equivalence class can be viewed as a Bogolyubov transformation.

An application of this theorem is the computation of transition amplitudes. Consider given $\Lambda \Psi^{\text{in}} \in F_{S(t_0)}$ and $\Lambda \Psi^{\text{out}} \in F_{S(t_1)}$, which represent “in” and “out” states at times $t_0$ and $t_1$, respectively. The transition amplitude is according to the above theorem given by

$$\left| \langle \Lambda \Psi^{\text{out}}, R_{R_1} L_{U^A} \Lambda \Psi^{\text{in}} \rangle \right|^2 = |e^{i\varphi}|^2 \left| \langle \Lambda \Psi^{\text{out}}, R_{R_2} L_{U^A} \Lambda \Psi^{\text{in}} \rangle \right|^2 = \left| \langle \Lambda \Psi^{\text{out}}, R_{R_2} L_{U^A} \Lambda \Psi^{\text{in}} \rangle \right|^2$$

which is therefore independent on our specific choice of the matrix $R_1$ or $R_2$, and also gauge invariant.
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