Weak Gibbs and Equilibrium Measures for Shift Spaces

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Abstract: For a large class of irreducible shift spaces $X \subset \mathbb{A}^{\mathbb{Z}^d}$, with $\mathbb{A}$ a finite alphabet, and for absolutely summable potentials $\Phi$, we prove that equilibrium measures for $\Phi$ are weak Gibbs measures. In particular, for $d = 1$, the result holds for irreducible sofic shifts.

1 Introduction

Equilibrium measures and Gibbs measures are basic concepts which occur naturally in statistical mechanics and dynamical systems. For shift spaces $X$ (or lattice models) with a finite alphabet $\mathbb{A}$ and for absolutely convergent potentials $\Phi$ Ruelle proved that Gibbs measures and equilibrium measures are equivalent provided that $X \subset \mathbb{A}^{\mathbb{Z}^d}$ is a subshift of finite type and verifies a supplementary condition (condition $(D)$, [Ru] chapter 4) see comment after definition 2.3. In this note we consider shift spaces verifying only a condition weaker than condition $(D)$ of Ruelle, called below decoupling condition, and prove that equilibrium measures are equivalent to weak Gibbs measures (in the sense of dynamical systems theory). When $d = 1$ we can weaken the decoupling condition and obtain the same result for a larger class of shift spaces. Our results apply for example to all irreducible sofic shifts. The equivalence of equilibrium measures with weak Gibbs measures implies that the empirical measures on the probability space $(X, \mu)$, with $\mu$ an equilibrium measure, have good large deviations estimates [PS]. As a consequence of the contraction principle one has good large deviations estimates for macroscopic observables, which are defined as averages of local functions with respect to the shift action.

In the next section the precise setting and the results are formulated. The main theorem is proved in the last section.

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## 2 Setting and main result

Let $L = \mathbb{Z}^d$ and $\Lambda$ be a finite set (with discrete topology). The full shift $\mathbb{A}^L$ is a compact (metric) space for the product topology. The natural action by translation of $L$ on $\mathbb{A}^L$ is denoted by $T : L \times \mathbb{A}^L \rightarrow \mathbb{A}^L$,

$$(j, x) \mapsto T^j x \quad \text{where} \quad (T^j x)(k) := x(k + j),$$

$x(k)$ being the $k^{th}$ coordinate of $x$. A shift space $X$ is a closed $T$-invariant subset of $\mathbb{A}^L$. $C(X)$ is the set of continuous functions on $X$ with the sup-norm $\| \cdot \|_\infty$, $M_1(X)$ is the set of Borel probability measures on $X$ (with the topology of weak convergence) and $M_1(X, T)$ the subset of $T$-invariant probability measures.

For $\Lambda \subset L$ we set $\Lambda^c := L \setminus \Lambda$ and $|\Lambda|$ denotes the cardinality of $\Lambda$ when $\Lambda$ is finite. For each integer $m$

$$\Lambda_m := \{i \in L : \max\{|i_k| : k = 1, \ldots, d\} \leq m\}.$$  

For $d = 1$, we use also the notations

$$[m_1, m_2] = \{i \in \mathbb{Z} : m_1 \leq i \leq m_2\}, \; (-\infty, m) = \{i \in \mathbb{Z} : i < m\}, \; (m, \infty) = \{i \in \mathbb{Z} : i > m\}.$$  

We use $J_\Lambda$ for the projection map

$$J_\Lambda : \mathbb{A}^{\Lambda'} \rightarrow \mathbb{A}^\Lambda, \; J_\Lambda(x) := (x(k) : k \in \Lambda) \quad \text{for} \; \Lambda \subset \Lambda' \subset L.$$  

Let $\Lambda \subset L$, $|\Lambda| < \infty$. The empirical measure $\mathcal{E}_\Lambda(x)$ is the discrete measure

$$\mathcal{E}_\Lambda(x) := \frac{1}{|\Lambda|} \sum_{j \in \Lambda} \delta_{T^j x} \quad \text{and} \quad \langle \mathcal{E}_\Lambda(x), f \rangle := \int f \; d\mathcal{E}_\Lambda(x).$$

We also write $J_m$ for $J_{\Lambda_m}$, $X_n$ for $J_n(X)$ and $\mathcal{E}(x)$ for $\mathcal{E}_{\Lambda_n}(x)$.

An absolutely summable potential $\Phi = \{\Phi_A\}$ is a family of continuous functions $\Phi_A : X \rightarrow \mathbb{R}$ indexed by the finite subsets of $L$, such that $\Phi_A$ is a local function, that is $\Phi_A(x) = \Phi_A(y)$ whenever $J_A(x) = J_A(y)$. $\Phi_A \cdot T^a = \Phi_{A+a}$ and $\|\Phi\| := \sum_{A \neq \emptyset} \|\Phi_A\|_\infty < \infty$.

The set of absolutely summable potentials $\Phi$ with the norm $\|\Phi\|$ is a Banach space $\mathbb{B}$. To each $\Phi$ we associate a continuous function

$$\varphi_\Phi := \sum_{A \neq \emptyset} \frac{\Phi_A}{|A|}.$$  

Let $\Lambda$ and $M$ be disjoint subsets of $L$, $\Lambda$ finite. We set

$$U_\Lambda := \sum_{A \subset \Lambda} \Phi_A \quad \text{and} \quad W_{\Lambda, M} := \sum_{A \subset \Lambda \cup M : A \not\subset \Lambda \cup M \not\subset \emptyset} \Phi_A.$$  

The pressure for a continuous function $\psi$ is denoted by $P(\psi)$ ([Ru], [Wa]). If $\psi = \varphi_\Phi$, with $\Phi \in \mathbb{B}$, then $P(\varphi_\Phi) = \lim_{n \rightarrow \infty} P_n(\Phi)$, where

$$P_n(\Phi) := \frac{1}{|\Lambda_n|} \ln \sum_{x \in X_n} \exp U_{\Lambda_n}(x).$$

Let

$$B_m(x) := \{y \in X : J_{\Lambda_m}(y) = J_{\Lambda_m}(x)\}.$$
**Definition 2.1.** A probability measure \( \nu \in M_1(X) \) is a weak Gibbs measure for the continuous function \( \psi \) if for any \( \delta > 0 \) there exists \( N_\delta \) such that

\[
\sup_{x \in X} \left| \frac{1}{|A_m|} \ln \nu(B_m(x)) - \langle E_m(x), \psi \rangle \right| \leq \delta \quad \forall m \geq N_\delta . \tag{2.1}
\]

The set of weak Gibbs measures for \( \psi \) is convex. Indeed, if \( \nu_1 \) and \( \nu_2 \) verify (2.1), then this is also true for \( \min_{i=1,2} \nu_i(B_m(x)) \) and \( \max_{i=1,2} \nu_i(B_m(x)) \) in place of \( \nu(B_m(x)) \). Hence (2.1) holds for \( a \nu_1 + (1-a) \nu_2, 0 < a < 1 \). If \( \nu \in M_1(X,T) \) is a weak Gibbs measure for \( \psi \), then \( P(\psi) = 0 \) and \( \nu \) is an equilibrium measure for \( \psi \) (see [PS]).

**Remark.** Definition 2.1 is equivalent to that of [PS] when \( X \) is a shift space. The terminology is that in usage in dynamical systems theory. In statistical physics there is another notion of “weak Gibbs measure”. See e.g. [Le].

**Definition 2.2.** A probability measure \( \nu \in M_1(X,T) \) is a tangent functional to the pressure \( P \) at the continuous function \( \varphi \) if it is an element of

\[
\hat{\partial}P(\varphi) := \left\{ \nu \in M_1(X,T) : P(\varphi + f) \geq P(\varphi) + \int f \, d\nu , \forall f \in C(X) \right\} .
\]

In the present setting equilibrium measures and tangent functionals to the pressure are equivalent (see [Wa] theorems 9.15 and 8.2).

**Definition 2.3.** A shift space \( X \subset \mathbb{A}^L \) satisfies the decoupling condition if

1) there exists a function \( q : \mathbb{N} \to \mathbb{N} \), written \( q_m \), such that \( \lim_{m \to \infty} q_m/m = 0 \);
2) for \( m \in \mathbb{N} \) and \( x, y \in X \) there exist \( z \in X \) and \( \ell \in \Lambda_{q_m} \) such that

\[
J_{\Lambda_m}(T^{-\ell}z) = J_{\Lambda_m}(y) \quad \text{and} \quad J_{\Lambda_m(q_m)}(z) = J_{\Lambda_m(q_m)}(x) .
\]

Condition 2.3 with no translation, that is \( \ell \equiv 0 \), is the condition \((D)\) of Ruelle. When the dimension \( d = 1 \), the decoupling condition can be stated in a different and more general way.

**Definition 2.4.** A shift space \( X \subset \mathbb{A}^\mathbb{Z} \) satisfies the decoupling condition if

1) there exists a function \( \bar{q} : \mathbb{N} \to \mathbb{N} \), written \( \bar{q}_m \), such that \( \lim_{m \to \infty} \bar{q}_m/m = 0 \);
2) for \( m \in \mathbb{N} \) and \( x, y \in X \) there exist \( \ell^+ \) and \( \ell^- \), \( |\ell^+|, |\ell^-| \leq \bar{q}_m \), and \( z \in X \) such that

\[
J_{[-m,m]}(z) = J_{[-m,m]}(y) \quad \text{and} \quad J_{[-\infty,-m-q_m]}(z) = J_{[-\infty,-m-q_m]}(T^{-\ell^-}x) \quad \text{and} \quad J_{(m+q_m,\infty)}(z) = J_{(m+q_m,\infty)}(T^{-\ell^+}x) .
\]

If \( d = 1 \), then condition 2.3 implies condition 2.4 with \( \ell^- = \ell^+ = \ell \) and \( \bar{q}_m = 2q_m \). Without restricting the generality, we assume from now on that \( q_m \) and \( \bar{q}_m \) are monotone non-decreasing.

**Theorem 2.1.** Let \( X \subset \mathbb{A}^\mathbb{N} \) be a shift-space satisfying the decoupling condition 2.3 or 2.4 when \( d = 1 \). Let \( \varphi \in \mathbb{B} \). If \( \nu \in M_1(X,T) \) is a tangent functional to the pressure at \( \varphi_\Phi \), then \( \nu \) is a weak Gibbs measure for \( \psi = \varphi_\Phi - P(\varphi_\Phi) \).
Example. Let $X$ be an irreducible sofic shift (see e.g. [LM]). There exists a directed irreducible finite graph $G = (V,E)$ and a labeling of the edges $L: E \to A$ so that $x \in X$ if and only if there is a bi-infinite path on $G$, $(\ldots,e(1),e(0),e(1),\ldots)$ with $x(k) = L(e(k))$ for all $k \in \mathbb{Z}$. Since the finite graph $G$ is irreducible, there exists $q \in \mathbb{N}$ such that there is a path of length smaller than $q$ from any vertex $P \in V$ to any vertex $Q \in V$. Let $x, y \in X$ and $[-m, m]$ be given. There is a bi-infinite path $(\ldots,e(1),e(0),e(1),\ldots)$ which presents $x$ and a bi-infinite path $(\ldots,e'(1),e'(0),e'(1),\ldots)$ which presents $y$. We have

$$J_{[-m,m]}(y) = (y(-m), \ldots, y(m)) = (L(e'(-m)), \ldots, L(e'(m))).$$

The path $(e'(-m), \ldots, e'(m))$ goes from some vertex $Q$ to some vertex $R$. The infinite sequence $(\ldots,x(-(m+q) - 2),x(-(m+q) - 1))$ is presented by the infinite path $(\ldots,e(-(m+q) - 2),e(-(m+q) - 1))$ which ends at some vertex $P$. Similarly the infinite sequence $(x(m+q + 1),x(m+q + 2),\ldots)$ is presented by the infinite path $(e(m+q + 1),e(m+q + 2),\ldots)$ which starts at some vertex $S$. There is a path of length $q^- \leq q$ from $P$ to $Q$ and a path of length $q^+ \leq q$ from $R$ to $S$. The concatenation of these paths define a bi-infinite path presenting some $z' \in X$. Then $z := T^{-q^-}z'$ verifies the properties of definition 2.4 with $q_m = q, \ell^- = q - q^-$ and $\ell^+ = q^+ - q$.

If the shift is aperiodic, then there exists $q$ such that there exists a path of length $q$ from $P$ to $Q$, for any $P, Q \in V$. Hence the shift verifies condition 2.3 with $\ell = 0$. □

Proposition 2.1. Suppose that $X$ is a subshift that verifies the decoupling condition 2.3 or 2.4 and that $Y$ is a factor of $X$. Then $Y$ verifies the decoupling condition 2.3 or 2.4.

Proof. There is a continuous surjective map $\phi: X \to Y$ such that $\phi \circ \sigma_X = \sigma_Y \circ \phi$, where $\sigma_X$ and $\sigma_Y$ are the shift-maps on $X$ and $Y$. Hence $\phi$ is a sliding block-code ([LM] theorem 6.2.9), i.e. there exists $\Lambda_m$ such that the $\ell$th coordinate of $x' = \phi(x), x'(\ell), depends only on the restriction of $x$ to $\Lambda_k + \ell$. Assume that $X$ verifies condition 2.3 with $q_m$ and let $\Lambda_m, x'$ and $y'$ be given. Choose $x, y \in X$, such that $\phi(x) = x'$ and $\phi(y) = y'$. For $\Lambda_{m+k}, x$ and $y$, condition 2.3 implies the existence of $z \in X$ and $\ell \in \Lambda_{q_m+k}$ such that

$$J_{\Lambda_m+k}(T^{-\ell}z) = J_{\Lambda_m+k}(y) \quad \text{and} \quad J_{\Lambda_{\ell+q_m+k}}(z) = J_{\Lambda_{\ell+q_m+k}}(x).$$

Let $z' := \phi(z)$. Therefore

$$(T^{-\ell}z')(j) = (\phi(T^{-\ell}z))(j) = (\phi(y))(j) = y'(j) \quad \text{for all } j \in \Lambda_m$$

and

$$z'(j) = (\phi(z))(j) = (\phi(x))(j) = x'(j) \quad \text{for all } j \in \Lambda_{\ell+q_m+k}.$$ 

Hence $Y$ verifies condition 2.3 with $q'_m = 2k + q_m$ and $\ell' = \ell$. □

Assume that $X$ verifies condition 2.4 with $q_m$ and let $\Lambda_m, x'$ and $y'$ be given. Choose $x, y \in X$, such that $\phi(x) = x'$ and $\phi(y) = y'$. Applying condition 2.4 to $[-(m+k),(m+k)], x$ and $y$, one verifies as above that $Y$ satisfies condition 2.4 with $q'_m = \bar{q}_{m+k} + 2k$ and $(\ell')^\pm = \ell^\pm$. □
3 Proof theorem 2.1

We first state two general lemmas.

**Lemma 3.1.** Let $\Phi \in \mathcal{B}$. Then

$$\lim_{n \to \infty} \frac{1}{|\Lambda_n|} \|W_{\Lambda_n, \Lambda_n^c}\|_{\infty} = 0.$$ 

**Proof.** Let $\partial \Lambda := \{j \in \Lambda : A + j \notin \Lambda\}$. Since $\|\Phi_A\|_{\infty} = \|\Phi_{A+j}\|_{\infty}$,

$$\|W_{\Lambda_n, \Lambda_n^c}\|_{\infty} \leq \sum_{A \cap \Lambda_n^c = \emptyset} \|\Phi_A\|_{\infty} \leq \sum_{j \notin \Lambda_n} \sum_{A \ni j : A \ni \Lambda_n^c} \|\Phi_A\|_{\infty} = \sum_{A \ni \Lambda_n^c} |\partial \Lambda_n| \cdot \|\Phi_A\|_{\infty}.$$ 

By the dominated convergence theorem

$$\lim_{n \to \infty} \sum_{A \ni \Lambda_n^c} \frac{|\partial \Lambda_n|}{|\Lambda_n|} \|\Phi_A\|_{\infty} = \sum_{A \ni \Lambda_n^c} \left( \lim_{n \to \infty} \frac{|\partial \Lambda_n|}{|\Lambda_n|} \right) \|\Phi_A\|_{\infty} = 0.$$

□

**Lemma 3.2.** Let $n > m + q_m$, $q_m > 0$, and $\ell, \ell'$ such that $\Lambda_m + \ell \subset \Lambda_{m+q_m}$ and $\Lambda_m + \ell' \subset \Lambda_{m+q_m}$. Given $\varepsilon > 0$, there exists $m_\varepsilon$ such that for $m \geq m_\varepsilon$ and $x, y \in X_n$ such that $J_{\Lambda_n \setminus (\Lambda_{m+q_m})}(x) = J_{\Lambda_n \setminus (\Lambda_{m+q_m})}(y)$,

$$\exp \left( \frac{U_{\Lambda_m + \ell}(x) - U_{\Lambda_m + \ell'}(y)}{C_m(\varepsilon)} \right) \leq \exp \left( \frac{U_{\Lambda_n}(x) - U_{\Lambda_n}(y)}{C_m(\varepsilon)} \right) \leq C_m(\varepsilon) \exp \left( \frac{U_{\Lambda_m + \ell}(x) - U_{\Lambda_m + \ell'}(y)}{C_m(\varepsilon)} \right)$$

where

$$C_m(\varepsilon) = \exp |\Lambda_m| \left( \frac{2|\Lambda_{m+q_m} \setminus \Lambda_m|}{|\Lambda_m|} \|\Phi\| + 2\varepsilon \right).$$

The same results are true for $\Lambda_n \supset (\Lambda_{m+q_m} + j)$, $(\Lambda_m + j + \ell) \subset (\Lambda_{m+q_m} + j)$ and $(\Lambda_m + j + \ell') \subset (\Lambda_{m+q_m} + j)$.

**Proof.** By lemma 3.1 there exists $m_\varepsilon$ such that for all $m \geq m_\varepsilon$,

$$\|W_{\Lambda_m + j, (\Lambda_m + j)^c}\|_{\infty} \leq \varepsilon |\Lambda_m|.$$  

$$U_{\Lambda_n}(x) = U_{\Lambda_n \setminus (\Lambda_{m+q_m})}(x) + U_{\Lambda_m + \ell}(x) + \sum_{A \subset \Lambda_n \setminus (\Lambda_{m+\ell})} \Phi_A(x) + W_{\Lambda_m + \ell, \Lambda_n \setminus (\Lambda_{m+\ell})}(x).$$

By hypothesis $U_{\Lambda_n \setminus (\Lambda_{m+q_m})}(x) = U_{\Lambda_n \setminus (\Lambda_{m+q_m})}(y)$. Therefore

$$\left| \left( U_{\Lambda_n}(x) - U_{\Lambda_n}(y) \right) - \left( U_{\Lambda_m + \ell}(x) - U_{\Lambda_m + \ell'}(y) \right) \right| \leq 2|\Lambda_{m+q_m} \setminus \Lambda_m| \|\Phi\| + 2\varepsilon |\Lambda_m|.$$  

□
3.1 Proof of theorem [2.1] under condition [2.3]

We prove the theorem [2.1] when \( \partial P(\varphi) = \{ \nu \} \) and condition [2.3] holds. We define a potential \( \Psi \). Let \( \bar{u} \in J_m(X) \);

\[
\Psi_A(x) := \begin{cases} 
1 & \text{if } A = \Lambda_m + j \text{ and } J_{\Lambda_m+j}(x) = \bar{u} \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( \nu \in M_1(X,T) \) is a tangent functional to the convex function \( P \) at \( \varphi_F \),

\[
\frac{P(\varphi_F) - P(\varphi_F - t\varphi)}{t} \leq \langle \nu, t \Psi_{\Lambda_m} \rangle \leq \frac{P(\varphi_F + t\varphi) - P(\varphi_F)}{t}.
\]

Since \( \nu \) is the unique tangent functional to \( P \) at \( \varphi \), \( t \mapsto P(\varphi + t\varphi) \) is differentiable at \( t = 0 \) and one may interchange \( \frac{d}{dt} \) and \( \lim_n \) (see theorem 25.7 [Ro]),

\[\langle \nu, \Psi_{\Lambda_m} \rangle = \left. \frac{d}{dt} P(\varphi + t\varphi) \right|_{t=0} = \lim_{n \to \infty} \left. \frac{d}{dt} P_n(\Phi + t\Psi) \right|_{t=0}\]

and

\[
\left. \frac{d}{dt} P_n(\Phi + t\Psi) \right|_{t=0} = \frac{1}{|\Lambda_n|} \sum_{A \in \Lambda_n} \sum_{x \in X_n} \Psi_A(x) \exp U_{\Lambda_n}(x) \sum_{x \in X_n} \exp U_{\Lambda_n}(x) = \frac{1}{|\Lambda_n|} \sum_{j \in \Lambda_{n-m}} \sum_{x \in X_n} \Psi_{\Lambda_m+j}(x) \exp U_{\Lambda_n}(x) \sum_{x \in X_n} \exp U_{\Lambda_n}(x).
\]

The key step of the proof is to obtain upper and lower bounds independent of \( n \) and \( j \in \Lambda_{n-(m+q_m)} \) for

\[
\frac{\sum_{x \in X_n} \Psi_{\Lambda_m+j}(x) \exp U_{\Lambda_n}(x)}{\sum_{x \in X_n} \exp U_{\Lambda_n}(x)}.
\]

(The terms with \( j \in \Lambda_{n-m} \setminus \Lambda_{n-(m+q_m)} \) do not affect the limit \( n \to \infty \) in (3.1).) For \( j \in \Lambda_{n-(m+q_m)} \) and \( \ell \in \Lambda_{q_m} \), let

\[
E^{j}_\ell(v) := \{ x \in X_n : J_{(\Lambda_m+j)+\ell}(x) = v \} \quad \text{and} \quad Z^{j,j}_n(v) := \sum_{x \in E^{j}_\ell(v)} \exp U_{\Lambda_n}(x).
\]

Hence

\[
\frac{\sum_{x \in X_n} \Psi_{\Lambda_m+j}(x) \exp U_{\Lambda_n}(x)}{\sum_{x \in X_n} \exp U_{\Lambda_n}(x)} = \frac{Z^{j,j}_n(\bar{u})}{\sum_{x \in X_n} \exp U_{\Lambda_n}(x)}.
\]

For any \( \ell \in \Lambda_{q_m} \) and \( j \in \Lambda_{n-(m+q_m)} \),

\[
\sum_{x \in X_n} \exp U_{\Lambda_n}(x) = \sum_{v \in J_{(\Lambda_m+j)+\ell}(X)} Z^{j,j}_n(v).
\]

**Lemma 3.3.** Let \( \varepsilon > 0 \), \( n > m + q_m \), \( j \in \Lambda_{n-(m+q_m)} \) and \( \bar{u} \in X_{\Lambda_m+j} \). Let

\[
K_m(\varepsilon) = \left| \Lambda_{q_m} \right| \left| A \right| \left| \Lambda_{m+q_m} \setminus \Lambda_m \right| C_m(\varepsilon).
\]
Then there exists \( m_e \) such that for all \( m \geq m_e \),
\[
\frac{Z_{n,0}^j(\bar{u})}{\sum_{x \in X_n} \exp U_{\Lambda_n}(x)} \leq \frac{|\Lambda_{q_m}| K_m(\varepsilon) \exp U_{\Lambda_m}(\bar{u})}{\exp |\Lambda_m| P_m(\Phi)}
\]
and
\[
\frac{\sum_{\ell \in \Lambda_{q_m}} Z_{n,\ell}^j(\bar{u})}{\sum_{x \in X_n} \exp U_{\Lambda_n}(x)} \geq \frac{\exp U_{\Lambda_m}(\bar{u})}{K_m(\varepsilon) \exp |\Lambda_m| P_m(\Phi)}.
\]

**Proof.** To simplify the notations we consider the case \( j = 0 \) since the other cases are treated exactly in the same manner. We write \( Z_{n,\ell}^0 = Z_{n,\ell} \) and \( E_{\ell}^0(v) = E_{\ell}(v) \). We introduce an auxiliary partition function \( Z_n \). For \( \ell \in \Lambda_{q_m} \), we decompose \( E_{\ell}(v) \) into
\[
E_{\ell}(v, w) := \{ x \in X_n : J_{\Lambda_m+\ell}(x) = v, J_{\Lambda_{n_m+\Lambda_{q_m}}}(x) = w \}.
\]
Let
\[X_{n,m} := J_{\Lambda_{n_m+\Lambda_{q_m}}}(X)\]
Then
\[Z_{n,\ell}(v) = \sum_{w \in X_{n,m}} \sum_{x \in E_{\ell}(v, w)} \exp U_{\Lambda_n}(x)\]
It may happen that \( E_{\ell}(v, w) = \emptyset \) for some \((v, w)\). But, by the decoupling condition 2.3, for any \((v, w)\) there exists \( \ell \in \Lambda_{q_m} \) such that \( E_{\ell}(v, w) \neq \emptyset \). By a slight abuse of notations, if \( J_{\Lambda_m}(T^{\ell}z) = v \) we also write \( J_{\Lambda_m+\ell}(z) = v \). With this convention we define for \( v \in X_m \),
\[
\bar{E}(v, w) := \bigcup_{\ell \in \Lambda_{q_m}} E_{\ell}(v, w) \quad \text{and} \quad \bar{Z}_n(v) := \sum_{w \in X_{n,m}} \sum_{x \in E_{\ell}(v, w)} \exp U_{\Lambda_n}(x).
\]
For any \( \ell \)
\[
Z_{n,\ell}(v) \leq \bar{Z}_n(v) \leq \sum_{\ell' \in \Lambda_{q_m}} Z_{n,\ell'}(v). \tag{3.3}
\]
By \((3.2)\)
\[
\sum_{x \in X_n} \exp U_{\Lambda_n}(x) \leq \sum_{v \in X_{n,m}} \bar{Z}_n(v) \leq |\Lambda_{q_m}| \sum_{x \in X_n} \exp U_{\Lambda_n}(x). \tag{3.4}
\]
Let \( x \in \bar{E}(v, w) \) and \( y \in \bar{E}(\bar{u}, w) \). There exist \( \ell \) and \( \ell' \) such that \( x \in E_{\ell}(v, w) \) and \( y \in E_{\ell'}(\bar{u}, w) \). By lemma 3.2
\[
\exp U_{\Lambda_n}(x) \leq C_m(\varepsilon) \exp(U_{\Lambda_m+\ell}(v) - U_{\Lambda_m+\ell}(\bar{u})) \exp U_{\Lambda_n}(y) \tag{3.5}
\]
and
\[
C_m(\varepsilon)^{-1} \exp(U_{\Lambda_m+\ell}(v) - U_{\Lambda_m+\ell}(\bar{u})) \exp U_{\Lambda_n}(y) \leq \exp U_{\Lambda_n}(x). \tag{3.6}
\]
By translation invariance of \( \Phi \), if \( J_{\Lambda_m+\ell}(x) = v \) and \( J_{\Lambda_m}(x') = v \), then
\[
U_{\Lambda_m+\ell}(v) = U_{\Lambda_m+\ell}(x) = U_{\Lambda_m}(x') = U_{\Lambda_m}(v).
\]
Similarly, \( U_{\Lambda_m+\ell}(\bar{u}) = U_{\Lambda_m}(\bar{u}) \). For any \( v \), the cardinality of \( E_{\ell}(v, w) \) is smaller than \( |\Lambda||\Lambda_{q_m+\Lambda_{q_m}}| \) and that of \( \bar{E}(v, w) \) smaller than \( |\Lambda_{q_m}| |\Lambda_{q_m+\Lambda_{q_m}}| \). Summing \((3.5)\) over \( y \) and then over \( x \),
\[
\sum_{x \in \bar{E}(v, w)} \exp U_{\Lambda_n}(x) \leq K_m(\varepsilon) \exp(U_{\Lambda_m}(v) - U_{\Lambda_m}(\bar{u})) \sum_{y \in \bar{E}(\bar{u}, w)} \exp U_{\Lambda_n}(y).
\]
Summing (3.6) over $x$ and then over $y$,
\[ \exp(U_{\Lambda_m}(v) - U_{\Lambda_m}({\bar{u}})) \frac{\sum_{y \in E(u,w)} \exp U_{\Lambda_m}(y)}{K_m(\varepsilon)} \leq \sum_{x \in E(v,w)} \exp U_{\Lambda_m}(x). \]

We get, by summing over $w \in X_{n,m}$ and taking into account (3.2),
\[ e^{U_{\Lambda_m}(v)} e^{-U_{\Lambda_m}({\bar{u}})} \frac{\tilde{Z}_n({\bar{u}})}{K_m(\varepsilon)} \leq \tilde{Z}_n(v) \leq e^{U_{\Lambda_m}(v)} e^{-U_{\Lambda_m}({\bar{u}})} K_m(\varepsilon) \tilde{Z}_n({\bar{u}}). \]

Finally we sum over $v$. Taking into account (3.3) we get
\[ \sum_v \frac{Z_{n,0}(v)}{\sum_{\ell \in \Lambda_{n,m}} Z_{n,\ell}({\bar{u}})} = \sum_{x \in X_n} \frac{\exp U_{\Lambda_m}(x)}{\sum_{\ell \in \Lambda_{n,m}} \exp U_{\Lambda_m}(x)} \leq \sum_{v} \frac{\tilde{Z}_n(v)}{\tilde{Z}_n({\bar{u}})} \leq K_m(\varepsilon) \sum_v \frac{\exp U_{\Lambda_m}(v)}{\exp U_{\Lambda_m}({\bar{u}})}, \]
so that
\[ \sum_{\ell \in \Lambda_{n,m}} \frac{Z_{n,\ell}({\bar{u}})}{\sum_{x \in X_n} \exp U_{\Lambda_m}(x)} \geq \exp U_{\Lambda_m}({\bar{u}}) \frac{\exp U_{\Lambda_m}(v)}{K_m(\varepsilon) \exp \Lambda_m P_m(\Phi)}. \]

Similarly, taking into account (3.4) we get
\[ \sum_v \frac{\sum_{\ell \in \Lambda_{n,m}} Z_{n,\ell}(v)}{Z_{n,0}({\bar{u}})} = \frac{|\Lambda_{n,m}|}{\sum_{x \in X_n} \exp U_{\Lambda_m}(x)} \geq \sum_v \frac{\tilde{Z}_n(v)}{\tilde{Z}_n({\bar{u}})} \geq \sum_v \frac{\exp U_{\Lambda_m}(v)}{K_m(\varepsilon) \exp U_{\Lambda_m}({\bar{u}})}, \]
so that
\[ \frac{Z_{n,0}({\bar{u}})}{\sum_{x \in X_n} \exp U_{\Lambda_m}(x)} \leq \frac{|\Lambda_{n,m}| K_m(\varepsilon) \exp U_{\Lambda_m}({\bar{u}})}{\exp \Lambda_m P_m(\Phi)}. \]

\[ \square \]

**Lemma 3.4.** *Uniformly in $z \in X$,*
\[ \lim_{m \to \infty} \left( \langle \mathcal{E}_m(z), \varphi_\Phi \rangle - \frac{U_{\Lambda_m}(z)}{\lambda_m} \right) = 0. \]

**Proof.**
\[ |\lambda_m| \langle \mathcal{E}_m(z), \varphi_\Phi \rangle - U_{\Lambda_m}(z) | = \left| \sum_{k \in \Lambda_m} \varphi_\Phi(T^k z) - \sum_{A \subseteq \Lambda_m} \Phi_A(z) \right| \]
\[ = \left| \sum_{k \in \Lambda_m} \left( \sum_{A \in \Lambda_m} \frac{\Phi_A(z)}{|A|} - \sum_{A \in \Lambda_m} \frac{\Phi_A(z)}{|A|} \right) \right| \]
\[ \leq \left| \sum_{k \in \Lambda_m} \sum_{A \in \Lambda_m} \frac{\Phi_A(z)}{|A|} \right| \]
\[ \leq \left| \sum_{A \in \Lambda_m} \frac{|A \cap \Lambda_m \Phi_A(z)|}{|A|} \right| \leq \|W_{\Lambda_m, \Lambda_m^c}\|_\infty. \]

\[ \square \]
Let \( m \geq m_\varepsilon \) and \( z \in X \), \( J_m(z) = \bar{u} \). Taking the limit \( n \to \infty \) in (3.1), one gets from lemma 3.3 and
\[
\sum_{j \in \Lambda_{n-m}} Z_{n,0}^j(\bar{u}) \leq \sum_{j \in \Lambda_{n-m}} \sum_{\ell \in \Lambda_m} Z_{n,\ell}^j(\bar{u}) \leq |\Lambda_m| \sum_{j \in \Lambda_{n-m}} Z_{n,0}^j(\bar{u}) ,
\]

\[
\frac{\exp U_{\Lambda_m}(z)}{|\Lambda_{qm}|K_m(\varepsilon) \exp |\Lambda_m|P_m(\Phi)} \leq \langle \nu, \Psi_{\Lambda_m} \rangle \leq \frac{|\Lambda_{qm}|K_m(\varepsilon) \exp U_{\Lambda_m}(z)}{\exp |\Lambda_m|P_m(\Phi)}.
\]

Therefore, by lemma 3.4 given \( \delta > 0 \) there exists \( N_\delta \) such that
\[
\sup_{z \in X} \left| \frac{1}{|\Lambda(m)|} \ln \nu(B_m(z)) - \left\langle \mathcal{E}_m(z), (\varphi_\Lambda - P(\varphi_\Lambda)) \right\rangle \right| \leq \delta \quad \forall m \geq N_\delta .
\]

(3.7)

This proves the theorem when \( \partial P(\varphi_\Lambda) = \{\nu\} \).

The pressure is convex and continuous on the Banach space \( B \). To prove the theorem in the general case we apply a theorem of Mazur and a theorem of Lanford and Robinson (see [Ru] appendix A.3.7). The set of \( \Phi \) such that \( \partial P(\varphi_\Lambda) = \{\nu\} \) has a unique element \( \nu \) is residual (theorem of Mazur). Let \( \nu \in \partial P(\varphi_\Lambda) \) be such that there exist \( \Phi_k \in B \) with \( \partial P(\varphi_{\Lambda_k}) = \{\nu_k\} \), \( \lim_k \Phi_k = \Phi \) and \( \lim_k \nu_k = \nu \). One can choose \( N_\delta \) so that (3.7) holds for \( \nu_k \) and \( \varphi_{\Lambda_k} \), uniformly in \( k \). Hence (3.7) holds also for such \( \nu \in \partial P(\varphi_\Lambda) \) and \( \varphi_\Lambda \). It also holds for \( \mu \) in the convex hull of such \( \nu \) and for all \( \rho \in \partial P(\varphi) \) which are limits of such \( \mu \). Hence (3.7) is true on the weak-closed convex hull of such \( \nu \), which coincides with \( \partial P(\varphi_\Lambda) \) (theorem of Lanford and Robinson).

\[\square\]

3.2 Proof of theorem 2.1 for \( d = 1 \) under condition 2.4

From now on dimension \( d = 1 \). We introduce some notations specific to the case \( d = 1 \). Let \( n > m + 2\bar{q}_m \) be given and \( j \in [-n + (m + 2\bar{q}_m), n - (m + 2\bar{q}_m)] \). We set
\[
I_{n,m}^{-} := [-n + \bar{q}_m, -(m + \bar{q}_m) + j] \quad I_{n,m}^{+} := ((m + \bar{q}_m) + j, n - \bar{q}_m]
\]
and
\[
X_{n,m}^{-} := J_{I_{n,m}^{-}}(X) \quad X_{n,m}^{+} := J_{I_{n,m}^{+}}(X).
\]

For \( \Lambda \subset \mathbb{Z} \), we also denote by \( T^\ell \) the map from \( J_\Lambda(X) \) to \( J_{\Lambda - \ell}(X) \) defined by
\[
w \mapsto (T^\ell w)(k) := w(k + \ell) , \quad k \in (\Lambda - \ell) .
\]

Let \( v \in J_{[-m,m] + j}(X) \), \( w^- \in X_{n,m}^{-} \), \( w^+ \in X_{n,m}^{+} \) and \( |\ell^-|, |\ell^+| \leq \bar{q}_m \). We set
\[
E_j^\ell(v) := \{ z \in X_n : J_{[-m-j+m+j]}(z) = v \};
\]
\[
E_{\ell^- + \ell^+}(v, w^-, w^+) := \{ z \in E_j^\ell(v) : J_{I_{n,m}^{-} + \ell^-}(z) = T^{-\ell^-} w^-, J_{I_{n,m}^{+} + \ell^+}(z) = T^{-\ell^+} w^+ \}.
\]

The cardinality of \( E_{\ell^- + \ell^+}(v, w^-, w^+) \) is at most \( |\Lambda|^{4\bar{q}_m} \). For any \( \ell^- \), \( \ell^+ \), we have
\[
Z_{n}^j(v) := \sum_{x \in E_j^\ell(v)} \exp U_{\Lambda_n}(x) = \sum_{w^-, w^+ \in E_{\ell^- + \ell^+}(v, w^-, w^+)} \exp U_{\Lambda_n}(x). 
\]
We construct a set $\tilde{E}^j(v, w^-, w^+)$ for $w^- \in X^{j,-}_{n,m}$, $w^+ \in X^{j,+}_{n,m}$ and $v \in J_{[-m,m]+j}(X)$, by the decoupling condition there exist $\ell^-, \ell^+$ such that $E^j_{\ell^-, \ell^+}(v, w^-, w^+) \neq \emptyset$. For each pair $(w^-, w^+)$ we set

$$\tilde{E}^j(v, w^-, w^+) := \bigcup_{\ell^-, \ell^+} E^j_{\ell^-, \ell^+}(v, w^-, w^+)$$

and

$$\tilde{Z}^j_n(v) := \sum_{w^-, w^+ \in \tilde{E}^j(v, w^-, w^+)} \exp U_{n}(x).$$

We have $1 \leq |\tilde{E}^j(v, w^-, w^+)| \leq (2\bar{q}_m + 1)^2|A|^{4\bar{q}_m}$ and

$$Z^j_n(v) \leq \tilde{Z}^j_n(v) \leq (2\bar{q}_m + 1)^2 Z^j_n(v). \quad (3.8)$$

From that point the proof proceeds as that of theorem 2.1 by comparing $\tilde{Z}^j_n(v)$ and $\tilde{Z}^j_n(\bar{u})$, and then, using (3.3), by comparing $Z^j_n(v)$ and $Z^j_n(\bar{u})$. Let

$$\bar{K}_m(\varepsilon) := (2\bar{q}_m + 1)^2|A|^{4\bar{q}_m} C_m(\varepsilon).$$

Then

$$\frac{\exp U_{\Lambda_m+j}(\bar{u})}{(2\bar{q}_m + 1)^2 \bar{K}_m(\varepsilon) \exp |\Lambda_m| P_m(\Phi)} \leq \frac{Z^j_n(\bar{u})}{\sum_v Z^j_n(v)} \leq \frac{(2\bar{q}_m + 1)^2 \bar{K}_m(\varepsilon) \exp U_{\Lambda_m+j}(\bar{u})}{\exp |\Lambda_m| P_m(\Phi)}.$$ 

The rest of the proof is identical to the proof of theorem 2.1. \qed

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