Exact degenerate ground states for the F-AF spin chain with bond alternation

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We investigate the $J_1$-$J_2$ spin chain consisting of spins with magnitude $\frac{1}{2}$. The nearest-neighbor and the next-nearest-neighbor exchange interactions are ferromagnetic and antiferromagnetic, respectively, and induce strong frustration. Both these interactions involve the bond alternation. We find exact solutions for all the degenerate ground states on the phase boundary of the ferromagnetic phase. The degeneracy remains irrespective of two parameters representing the bond alternation. The exact solutions are of closed forms for no bond alternation and of recursion formulae in general. The exact solutions are applicable to the $\Delta$ chain as a special case.

KEYWORDS: quantum spin chain, frustration, ferromagnetic interaction, exact ground state

The low-dimensional quantum spin system has been an interesting subject of numerous studies from long years ago. In particular, the interplay of quantum fluctuation and geometric frustration is of current interest. In one-dimensional spin systems, various exotic phases and phenomena are reported such as quantum chiral phases,$^1$–$^9$ $\frac{1}{3}$-plateau$^{10}$–$^{13}$ and singlet cluster solid.$^{14}$

The $J_1$-$J_2$ spin chain is a well-known one-dimensional spin model, which has the nearest-neighbor (NN) and the next-nearest-neighbor (NNN) interactions with exchange parameters $J_1$ and $J_2$, respectively. We restrict ourselves to the case that the spin magnitude $s$ is $\frac{1}{2}$. For $J_1 > 0$ and $J_2 > 0$, because of the frustration, the spin chain is known to exhibit a quantum phase transition from a gapless Tomonaga-Luttinger-liquid phase to a gapped dimerized phase at $J_1/J_2 \simeq 4.15$. At the Majumdar-Ghosh point of $J_1/J_2 = 2$, the ground states are of exact tensor product forms of singlet NN dimers.$^{17}$

Another type of frustration is induced in the $J_1$-$J_2$ spin chain, if the NN interaction is ferromagnetic ($J_1 < 0$) and the NNN interaction is antiferromagnetic ($J_2 > 0$). We call this $J_1$-$J_2$ spin chain the F-AF chain. The F-AF chain is realized in, e.g., Rb$_2$Cu$_2$Mo$_3$O$_{12}$ with $J_1/J_2 \simeq -3^{18}$ and LiCuVO$_4$ with $J_1/J_2 \simeq -0.3$. Relatively less attention has been paid to the F-AF chain until such materials are discovered. We are interested in the difference between frustration effects of the F-AF chain and of the $J_1$-$J_2$ chain with both $J_1 > 0$ and $J_2 > 0$. Further, the uniform F-AF chain is extended to a model including bond alternation, if the NN and/or the NNN interactions have alternative strengths. The competition between frustration and bond alternation is of another physical interest.

In the uniform F-AF chain, the ground state is fully ferromagnetic for $J_1 < -4J_2$. For $-4J_2 < J_1 < 0$, numerical studies suggest a gapless singlet ground state,$^{16,19}$ while a detailed analysis based on the field theory predicts a tiny but non-zero spin-gap for small $|J_1|$. At the phase boundary of $J_1 = -4J_2$, Hamada et al.$^{25}$ found the exact singlet ground state under the periodic boundary condition (PBC). The exact solution is of a resonating-valence-bond (RVB) form.

At the phase boundary for the F-AF chain with the NN bond alternation, Dmitriev et al. obtained an exact singlet ground state for the PBC.$^{26,27}$ In the derivation process, they also found a special ground state for the open boundary condition (OBC), although it is not an eigenstate of the total spin. They further claimed that all the ground states are degenerate with respect to the magnitudes and the $z$-components of the total spin, and that the ground state for each total spin and each $z$-component of the total spin is unique. However, the explicit forms of all the degenerate ground states have not been shown.

In this letter, we report all the exact degenerate ground states for the uniform F-AF chain under the OBC; they are written down in explicit forms. Moreover, when both the NN and the NNN bond alternations exist, we obtained all the exact degenerate ground states in simple recursion relations with respect to the system size $N$. The nondegeneracy of the ground state in each sector with the fixed total spin and its $z$-component is also shown.

The Hamiltonian for the F-AF chain with bond alternation is written as

$$\mathcal{H} = \sum_n \left( J_1 s_{2n-1} \cdot s_{2n} + J'_1 s_{2n} \cdot s_{2n+1} + J_2 s_{2n-1} \cdot s_{2n+1} + J'_2 s_{2n} \cdot s_{2n+2} \right),$$

(1)

where $s_n$ is the spin-$\frac{1}{2}$ operator at the $n$-th site. $J_1$ and $J'_1$ are ferromagnetic exchange parameters for NN interactions, and $J_2$ and $J'_2$ are antiferromagnetic exchange parameters for NNN interactions. Different values of $J_1$ and $J'_1$ ($J_2$ and $J'_2$) represent the bond alternation in the NN (NNN) interactions. The lattice described by the Hamiltonian is illustrated in Fig. 1. We express the NN bond alternation by $\gamma$ and the NNN bond alternation by $\delta$ as

$$J_1 = \frac{J_1}{1 + \gamma}, \quad J'_1 = \frac{J_1}{1 - \gamma},$$
$$J_2 = J_2(1 - \delta), \quad J'_2 = J_2(1 + \delta),$$

(2)
where $-1 < \gamma < 1$ and $-1 \leq \delta \leq 1$. Then we have the relations as $J_1^{-1} = \frac{1}{2} (J_1^{-1} + J_2^{-1})$ and $J_2 = \frac{1}{2} (J_1 + J_2)$. Owing to the parameterization (2), the phase boundary of the ferromagnetic phase is independent of $\gamma$ and $\delta$ as will be seen. We also define the total spin as $S_{\text{tot}} = \sum_{n=1}^{N} s_n$, and denote the quantum numbers of the magnitude and the $z$-component by $S_{\text{tot}}$ and $S_{\text{tot}}^z$, respectively.

Hamiltonian (1) is decomposed into triangular spin units each of which consists of three spins. To completely decompose it, we adopt the special OBC where the exchange parameter in the left end is $J_1(1 - \delta)/2$ and that in the right end is $J_1(1 + \delta)/2$ as shown in Fig. 1. The Hamiltonian (1) is then rewritten as $\mathcal{H} = \sum_{n=1}^{N-2} \mathcal{H}_n$, where

$$\mathcal{H}_n = \frac{1 - \delta_n}{2} \left[ J_1 \left( \frac{1}{1 - \gamma_n} s_n \cdot s_{n+1} \right) + J_2 \left( \frac{1}{1 + \gamma_n} s_{n+1} \cdot s_{n+2} + 2J_2 s_n \cdot s_{n+2} \right) \right] + \frac{1}{2} \left[ J_1^2 + (1 - \gamma^2) J_2^2 \right] + \frac{1}{2} \left[ J_1^2 + (1 + \gamma^2) J_2^2 \right] \pm \sqrt{D} \frac{1}{4} \left( 1 + \gamma^2 \right)$$

with $\delta_n \equiv (-1)^s \delta$ and $\gamma_n \equiv (-1)^s \gamma$.

Solving the Hamiltonian $\mathcal{H}_n$ for a single triangular unit, all the different eigenvalues divided by $1 - \delta_n$ are

$$J_1 + (1 - \gamma^2) J_2 \quad \frac{J_1}{4(1 - \gamma^2)}, \quad J_1 + (1 + \gamma^2) J_2 \pm \sqrt{D}$$

with $D = 3\gamma^2 J_2^2 + (J_1 - 2 (1 - \gamma^2) J_2)^2$. The composite spin of the three spins is $\frac{3}{2}$ for the first and $\frac{1}{2}$ for the second in eq. (4). For $J_1 < -4J_2$, the first one in eq. (4) is the lowest. We can incorporate the spin-$\frac{3}{2}$ states of all the triangular units into a ferromagnetic state of the total Hamiltonian $\mathcal{H}$. This means that the ferromagnetic state is the ground state of $\mathcal{H}$. Similar argument has been done in the case of no bond alternation. For $J_1 > -4J_2$, the lowest eigenvalue for a triangular unit is the second one with the upper sign in eq. (4). We cannot incorporate the spin-$\frac{1}{2}$ states of all the triangular units into an eigenstate of $\mathcal{H}$. In particular, the ground state is not ferromagnetic for $J_1 > -4J_2$. Thus the condition $J_1 = -4J_2$ represents the phase boundary of the ferromagnetic phase irrespective of the values of $\gamma$ and $\delta$.

In the uniform case of $\gamma = \delta = 0$, we found all the exact degenerate ground states in explicit forms at the phase boundary of $J_1 = -4J_2$. The exact ground state with $S_{\text{tot}} = S_{\text{tot}}^z = N/2 - p$ is written down as

$$|\Phi_p\rangle = c_p \sum_{\{i_k,j_k\}} \left| \phi_p(i_1,j_1; i_2,j_2; \ldots; i_p,j_p) \right\rangle,$$

where $c_p$ is a normalization constant, and $|\phi_p(i_1,j_1; i_2,j_2; \ldots; i_p,j_p)\rangle = (|i_1\downarrow|j_1\uparrow - |i_1\uparrow|j_1\downarrow)/\sqrt{2}$ is the singlet pair of spins at $i_1$ and $j_1$. Here, $|i\downarrow\rangle$ and $|i\uparrow\rangle$ for the single spin labeled by $i$ are eigenstates of $s_i^z$ with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. The summation in eq. (5) has been taken over all possible combinations $\{i_1,j_1; i_2,j_2; \ldots; i_p,j_p\}$, or shortly $\{i_k,j_k\}$, under the condition that $i_k < j_k$ ($k = 1, 2, \ldots, p$). Clearly the state (5) with $p = 0$ is fully ferromagnetic. In the special case of $p = N/2$ with even $N$, the state (5) is the same as the exact singlet ground state derived by Hamada et al.\(^{25}\) By operating $s_{n_1} \cdot s_{n_2}$ on $|\phi_p(i_1,j_1; \cdots)\rangle$ in eq. (5), we find

$$s_{n_1} \cdot s_{n_2} |\phi_p(i_1,j_1; \cdots)\rangle = \frac{1}{4} |\phi_p(i_1,j_1; \cdots)\rangle,$$

where $c_p$ is a normalization constant, and $|\phi_p(i_1,j_1; i_2,j_2; \ldots; i_p,j_p)\rangle = (|i_1\downarrow|j_1\uparrow - |i_1\uparrow|j_1\downarrow)/\sqrt{2}$ is the singlet pair of spins at $i_1$ and $j_1$. Here, $|i\downarrow\rangle$ and $|i\uparrow\rangle$ for the single spin labeled by $i$ are eigenstates of $s_i^z$ with eigenvalues $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. The summation in eq. (5) has been taken over all possible combinations $\{i_1,j_1; i_2,j_2; \ldots; i_p,j_p\}$, or shortly $\{i_k,j_k\}$, under the condition that $i_k < j_k$ ($k = 1, 2, \ldots, p$). Clearly the state (5) with $p = 0$ is fully ferromagnetic. In the special case of $p = N/2$ with even $N$, the state (5) is the same as the exact singlet ground state derived by Hamada et al.\(^{25}\) By operating $s_{n_1} \cdot s_{n_2}$ on $|\phi_p(i_1,j_1; \cdots)\rangle$ in eq. (5), we find

$$s_{n_1} \cdot s_{n_2} |\phi_p(i_1,j_1; \cdots)\rangle = \frac{1}{4} |\phi_p(i_1,j_1; \cdots)\rangle,$$

where $i_k$ and $j_k$ ($k = 1, 2, \ldots, p$) are different from $n_1$ and $n_2$. By the aid of these equations, we obtain

$$\mathcal{H}_n |\Phi_p\rangle = -\frac{3J_2}{4} |\Phi_p\rangle$$

for all $n$. Summing up this equation from $n = 1$ to $N - 2$, we have the following eigenvalue equation:

$$\mathcal{H} |\Phi_p\rangle = -\frac{3J_2}{4} (N - 2) |\Phi_p\rangle.$$

This eigenvalue is the ground state energy, because it is just the lower bound which is the sum of the lowest energies of the $N - 2$ triangular units as known from eq. (4) with $\gamma = \delta = 0$ and $J_1 = -4J_2$. Therefore $|\Phi_p\rangle$ for any $p$ is an exact ground state.

To find the ground state for arbitrary $S_{\text{tot}}$ and $S_{\text{tot}}^z$, we tilt $S_{\text{tot}}$ by symmetrically flipping some of half spins in the ferromagnetic part of $|\phi_p\rangle$. Then we have

$$|\Phi_{pp'}\rangle = c_{pp'} \sum_{\{i_k,j_k\}} \left[ \prod_{q=1}^{p} \left( |\phi_q(i_q,j_q)\rangle - |\phi_q(i_q,j_q)\rangle \right) \prod_{l \neq \{i_k,j_k\}} |l\rangle \right],$$

where $c_{pp'}$ is the normalization constant and the primed summand is taken over all $N - 2p - p'$ down spins and $N - 2p - p'$ up spins for each $\{i_k,j_k\}$. This ground state $|\Phi_{pp'}\rangle$ has $S_{\text{tot}} = N/2 - p$ and $S_{\text{tot}}^z = N/2 - p - p'$. Equation (12) and the spin rotational symmetry of $\mathcal{H}$ guarantee that $|\Phi_{pp'}\rangle$ is a ground state. The number of states in the form of eq. (13) is $\frac{1}{4} (N + 2)^2$ for even $N$ and $\frac{1}{4} [(N + 2)^2 - 1]$ for odd $N$.

For the general F-AF chain with bond alternation, it is difficult to write down all the degenerate ground states in explicit forms. We however found that the ground states...
are exactly expressed in simple recursion relations with respect to the system size \( N \) on the phase boundary of \( J_1 = -4J_2 \). The recursion relations are derived from the fact that a ground state of the total chain is simultaneously the ground states of \( H_a \) for all the triangular units. The derivation is explained in what follows.

Let \([j, m]_N\) be the ground state of the \( N \)-site chain for \( S_{tot} = j \) and \( S_{tot}^z = m \) at \( J_1 = -4J_2 \). The \((N + 2)\)-sites ground state \([j, m]_{N+2}\) is expressed in terms of the \( N \)-sites ground states for \( S_{tot} = j \pm 1 \) and \( j \) with states of two extra \( \frac{1}{2} \) spins at the \((N + 1)\)-th and the \((N + 2)\)-th sites. By denoting the Clebsch-Gordan coefficient as \( C(\nu, \mu) = (\nu; \mu; \nu + \nu, \mu \pm \mu) \), it is written as follows:

\[
[j, m]_{N+2} = a_N(j) \sum_{\mu} C(1, \mu)[j+1, m-\mu]_N \otimes |t_\mu) \\
+ b_N(j) \sum_{\mu} C(0, \mu)[j, m-\mu]_N \otimes |t_\mu) \\
+ c_N(j) \sum_{\mu} C(-1, \mu)[j-1, m-\mu]_N \otimes |t_\mu) \\
+ d_N(j)[j, m-\mu]_N \otimes |s),
\]

where \(|s)\) is the singlet state for the extra spins, and \(|t_\mu)\) is the triplet state for them with quantum number \( \mu \) of the \( z \)-component of the composite spin; then each summation takes over \( \mu = -1, 0, 1 \). The coefficients \( \{a_N, b_N, c_N, d_N\} \) are independent of \( m \) because of the rotational symmetry of \( \mathcal{H} \). Since \( [\frac{N}{2} + 1, m]_{N+2} \) is fully ferromagnetic, we immediately find \( a_N(\frac{N}{2} + 1) = 0 \), \( b_N(\frac{N}{2} + 1) = 0 \), \( c_N(\frac{N}{2} + 1) = 1 \) and \( d_N(\frac{N}{2} + 1) = 0 \). Since there do not exist \( \frac{N}{2} + 1, m \) of \( j \leq \frac{N}{2} \), we have \( a_N(\frac{N}{2}) = 0 \) and \( c_N(0) = c_N(\frac{N}{2}) = 0 \). Further, since \( |0, 0\rangle_{N+2} \) cannot be produced from \(|0, 0\rangle_N \) and \(|t_0)\), we have \( b_N(0) = 0 \).

Using eq. (14) two times successively, we obtain \([j, m]_{N+2}\) in the following form:

\[
[j, m]_{N+2} = \sum_{\{s^\pm = \pm 1/2\}} |A_N(s^\pm_{N-1}, s^\pm_N, s^+_N, s^+_{N+2})\rangle \\
\otimes |s^\mp_{N-1} s^\mp_N s^+_N s^+_{N+2} \rangle.
\]

(15)

Here, \( |A_N(s^\pm_{N-1}, s^\pm_N, s^+_N, s^+_{N+2})\rangle \) is expressed by a summation of the ground states of \((N-2)\)-site chain and contains \( \{a_N, b_N, c_N, d_N\} \) and \( \{a_{N-2}, b_{N-2}, c_{N-2}, d_{N-2}\} \). The recursion relations for \( \{a_N, b_N, c_N, d_N\} \) are derived by imposing the condition that eq. (15) is the ground state of the local Hamiltonian \( \mathcal{H}_{N-1} \) and \( \mathcal{H}_N \). Then we have

\[
(\mathcal{H}_{N-1} + \mathcal{H}_N)[j, m]_{N+2} = -\frac{(3 + \gamma^2)J_2}{2(1 - \gamma^2)} [j, m]_{N+2}.
\]

(16)

This eigenvalue equation with eq. (15) stands, only if the following recursion relations are satisfied:

\[
a_N(j) = a_{N-2}(j),
\]

(17)

\[
b_N(j) = b_{N-2}(j),
\]

(18)

\[
c_N(j) = c_{N-2}(j),
\]

\[
d_N(j) = d_{N-2}(j).
\]

Table 1. Initial coefficients \( \{a_4, b_4, c_4, d_4\} \) for the recursion relations (17) to (19). Here, \( a_0 = (51 + 10\gamma^2 + 3\gamma^4)^{-1/2} \), \( a_1 = (3(385 + 200\gamma + 86\gamma^2 + 24\gamma^3 + 9\gamma^4))^{1/2} \), \( a_2 = (3(385 + 200\gamma + 86\gamma^2 + 24\gamma^3 + 9\gamma^4))^{1/2} \), \( b_0 = (5 + 2\gamma + \gamma^2)^{1/2} \), \( b_1 = (1 + \gamma)(2(3 + \gamma^2))^{1/2} \).

| j | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| a_4(j) | (3 - \gamma)\alpha_0\beta_0 | (5 - \gamma)(3 + \gamma)\sqrt{2}\alpha_1 | 0 | 0 |
| b_4(j) | 0 | 2\sqrt{30}\alpha_1\beta_0 | 2\sqrt{6}\alpha_2 | 0 |
| c_4(j) | 0 | \sqrt{5}\alpha_1\beta_1 | \sqrt{15}\alpha_1\beta_0 | 1 |
| d_4(j) | \alpha_0\beta_0 | (1 + \gamma)\sqrt{15}\alpha_1\beta_0 | (1 + \gamma)\alpha_2 | 0 |

Further, the following relations for the same \( N \) should be satisfied:

\[
b_N(j)/d_N(j) = \sqrt{j - 1b_N(j - 1) + \frac{4}{1 + \gamma}} \sqrt{j + 1},
\]

(20)

\[
c_N(j)/d_N(j) = \left[ \sqrt{j - 1b_N(j - 1) + \sqrt{jd_N(j - 1)}} \right]
\]

\[
\times \left[ \sqrt{j - 1b_N(j - 1) + \left( \frac{3 - \gamma}{1 + \gamma} \right)} \sqrt{jd_N(j - 1)} \right]^{-1}.
\]

(21)

With the normalization condition

\[
a_N^2 + b_N^2 + c_N^2 + d_N^2 = 1,
\]

the recursion relations (17) to (19) determine \( \{a_N, b_N, c_N, d_N\} \) by starting from initial values for, e.g., \( N = 4 \) except for coefficients with special values of \( j \). Since the denominator of the right hand side of eq. (17) for \( j = \frac{N}{2} - 1 \) and of eq. (18) for \( j = \frac{N}{2} \) becomes zero, we separately evaluate \( a_N(\frac{N}{2} - 1) \) and \( b_N(\frac{N}{2}) \) by using eqs. (20) and (21). Parameter \( \gamma \) representing the NN bond alternation only affects eqs. (20) and (21). All the above equations are independent of \( \delta \).

The initial coefficients \( \{a_4, b_4, c_4, d_4\} \) for the recursion relations (17) to (19) are given in Table I, and the initial ground states \([j, m]_4\) are uniquely determined as

\[
[2, 2]_4 = (|\uparrow \uparrow \uparrow \uparrow \rangle), \quad [2, 1]_4 = \frac{(1, 1, 1, 1) \cdot u_1}{2},
\]

\[
[1, 1]_4 = \frac{(3 + \gamma, 1 - \gamma, -1 + \gamma, -3 - \gamma) \cdot u_1}{2\sqrt{5 + 2\gamma + \gamma^2}}.
\]

(23)
where \( u_1 \equiv \begin{vmatrix} 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ \end{vmatrix} \), \( u_0 \equiv \begin{vmatrix} 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ \end{vmatrix} \) and \( u_{-1} \equiv \begin{vmatrix} 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ | 1 \ 1 \ 1 \ 1 \ 
abla \ \end{vmatrix} \). Thus the ground state \( |j, m\rangle_N \) with arbitrary even \( N \) is constructed recursively by eqs. (14), (17) to (21). Note that all the coefficients can be determined as positive values by a suitable choice of the sign of each ground state. Thus the coefficients \( \{a_N(j), b_N(j), c_N(j), d_N(j)\} \) are determined uniquely. Further, since the total ground state is also a ground state of each local Hamiltonian \( \mathcal{H}_n \), any total state which is not a local ground state cannot be another total ground state. Therefore, the ground state in the sector of fixed \( S^z_{\text{tot}} = j \) and \( S^z_{\text{tot}} = m \) is non-degenerate. Although we have only shown the initial values for even \( N \), the above recursion relations is valid for odd \( N \) by using the initial coefficients and the ground states for \( N = 3 \). The total degeneracy of the ground states is \( \frac{1}{2}(N + 2)^2 \) for even \( N \) and \( \frac{1}{2}(N + 2)^2 - 1 \) for odd \( N \). Among the ground states for the OBC the state for \( j = m = 0 \) is simultaneously the ground state for the PBC even if \( \gamma \neq 0 \) and \( \delta \neq 0 \).

Using the above recursion relations, we can calculate physical quantities for large \( N \). In Fig. 2, we show the expectation value \( \langle S^z_{n} \rangle \) for the ground state \( |j, j\rangle_{100} \) when \( \gamma = 0 \) (left) and \( \gamma = 0.8 \) (right).

Fig. 2. The expectation value \( \langle S^z_{n} \rangle \) for the ground state \( |j, j\rangle_{100} \) when \( \gamma = 0 \) (left) and \( \gamma = 0.8 \) (right).

\[
\langle S^z_{n} \rangle = \frac{\sqrt{J - 1} b_N((j - 1) + \sqrt{J + 1} d_N(j - 1)}{\sqrt{J + 1} b_N(j - 1) + \sqrt{J + 1} d_N(j - 1)} \cdot \sqrt{J + 1} c_N(j) - \sqrt{J + 1} c_N(j - 1)
\]

These equations are not enough to uniquely determine the coefficients \( \{a_N, b_N, c_N, d_N\} \), since the degeneracy of the ground states increases. We speculate the \((N - 1)/2C_{j - 1/2}\) fold degeneracy for given \( N, j \) and \( m \) by numerical calculations.

To summarize, we found exact solutions of all the degenerated ground states of the F-AF chain on the ferromagnetic phase boundary. For a uniform F-AF chain, the exact ground states for arbitrary \( S^z_{\text{tot}} \) and \( S^z_{\text{tot}} \) is written down explicitly in the closed form (13). In each state, \( p \) singlet pairs are distributed uniformly among the remaining \( N - 2p \) unpaired spins. For a general F-AF chain with bond alternation, the recursion formulæ for the ground states with respect to the system size \( N \) have been derived. This formulæ is independent of the NNN bond alternation. By using the recursion formulæ, we can evaluate various physical quantities of the ground states for large systems. The detailed calculations will be reported elsewhere.

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