BIRATIONALLY SUPERRIGID FANO 3-FOLDS
OF CODIMENSION 4

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Abstract. We determine birational superrigidity for a quasi-smooth prime Fano 3-fold of codimension 4 with no projection centers. In particular we prove birational superrigidity for Fano 3-folds of codimension 4 with no projection centers which are recently constructed by Coughlan and Ducat. We also pose some questions and a conjecture regarding the classification of birationally superrigid Fano 3-folds.

1. Introduction

A prime Fano 3-fold is a normal projective $\mathbb{Q}$-factorial 3-fold $X$ with only terminal singularities such that $-K_X$ is ample and the class group $\text{Cl}(X) \cong \mathbb{Z}$ is generated by $-K_X$. For such $X$, there corresponds the anticanonical graded ring

$$R(X, -K_X) = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} H^0(X, -mK_X),$$

and by choosing generators we can embed $X$ into a weighted projective space. By the codimension of $X$ we mean the codimension of $X$ in the weighted projective space. Based on the analysis by Altınok, Brown, Iano-Fletcher, Kasprzyk, Prokhorov, Reid, etc. (see for example [4]), there is a database [5] of numerical data (such as Hilbert series) coming from graded rings that can be the anticanonical graded ring of a prime Fano 3-fold. Currently it is not a classification, but it serves as an overlist, meaning that the anticanonical graded ring of a prime Fano 3-fold appears in the database.

The database contains a huge number of candidates, which suggests difficulty of biregular classification of Fano 3-folds. The aim of this paper is to shed light on the classification of birationally superrigid Fano 3-folds. Here, a Fano 3-fold of Picard number 1 is said to be birationally superrigid if any birational map to a Mori fiber space is biregular. We remark that, in [1], a possible approach to achieving birational classification of Fano 3-folds is suggested by introducing notion of solid Fano 3-folds, which are Fano 3-folds not birational to neither a conic bundles nor a del Pezzo fibration.

Up to codimension 3, we have satisfactory results on the classification of quasi-smooth prime Fano 3-folds: the classification is completed in codimensions 1 and 2 ([12], [8], [3]) and in codimension 3 the existence is known for all 70 numerical data in the database. Moreover birational superrigidity of quasi-smooth prime Fano 3-folds of codimension at most 3 has been well studied as well (see [14], [9], [7], [19], [2], [1], and see also [20], [21] for solid cases in codimension 2).

For quasi-smooth prime Fano 3-folds of codimension 4, there are 145 candidates of numerical data in [5]. In [6], existence for 116 data is proved, where the construction
is given by birationally modifying a known variety. This process is called unprojection and, as a consequence, a constructed Fano 3-fold corresponding to each of the 116 data admits a Sarkisov link to a Mori fiber space, hence it is not birationally superrigid. The 116 families of Fano 3-folds are characterized as those that possesses a singular point which is so called a type I projection center (see [6] for details). There are other types of projection centers (such as types II, ..., IV according to the database [5]). Through the known results in codimensions 1, 2 and 3, we can expect that the existence of a projection center violates birational superrigidity. Therefore it is natural to consider prime Fano 3-folds without projection centers for the classification of birational superrigid Fano 3-folds (see also the discussion in Section 5).

According to the database [5], there are 5 candidates of quasi-smooth prime Fano 3-folds of codimension 4 with no projection centers. Those are identified by database numbers #25, #166, #282, #308 and #29374. Among them, #29374 corresponds to smooth prime Fano 3-folds of degree 10 embedded in \( \mathbb{P}^7 \), and it is proved in [11] that they are not birationally superrigid (not even birationally rigid, a weaker notion than superrigidity). Recently Coughlan and Ducat [10] constructed many prime Fano 3-folds including those corresponding to #25 and #282 and we sometimes refer to these varieties as cluster Fano 3-folds. There are two constructions, \( G_2^{(4)} \) and \( C_2 \) formats (see [10, Section 5.6] for details and see Section 4.1 for concrete descriptions) for #282 and they are likely to sit in different components of the Hilbert scheme.

**Theorem 1.1.** Let \( X \) be a quasi-smooth prime Fano 3-fold of codimension 4 and of numerical type #282 which is constructed in either \( G_2^{(4)} \) format or \( C_2 \) format. If \( X \) is constructed in \( C_2 \) format, then we assume that \( X \) is general. Then \( X \) is birationally superrigid.

For the remaining three candidates #25, #166 and #282, we can prove birational superrigidity in a stronger manner; we are able to prove birational superrigidity for these 3 candidates by utilizing only numerical data. Here, by numerical data for a candidate Fano 3-fold \( X \), we mean the weights of the weighted projective space, degrees of the defining equations, the anticanonical degree \( (−K_X)^3 \) and the basket of singularities of \( X \) (see Section 3). Note that we do not know the existence of Fano 3-folds for #166 and #308.

**Theorem 1.2.** Let \( X \) be a quasi-smooth prime Fano 3-fold of codimension 4 and of numerical type #25, #166 or #308. Then \( X \) is birationally superrigid.

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## 2. Birational superrigidity

### 2.1. Basic properties.

Throughout this subsection, we assume that \( X \) is a Fano 3-fold of Picard number 1, that is, \( X \) is a normal projective Q-factorial 3-fold such that \( X \) has only terminal singularities, \( −K_X \) is ample and rank \( \text{Pic}(X) = 1 \).

**Definition 2.1.** We say that \( X \) is birationally superrigid if any birational map \( \sigma: X \dashrightarrow Y \) to a Mori fiber space \( Y \rightarrow T \) is biregular.
By an extreme divisorial extraction \( \varphi: (E \subset Y) \to (\Gamma \subset X) \), we mean an extreme divisorial contraction \( \varphi: Y \to X \) from a normal projective \( \mathbb{Q} \)-factorial variety \( Y \) with only terminal singularities such that \( E \) is the \( \varphi \)-exceptional divisor and \( \Gamma = \varphi(E) \).

**Definition 2.2.** Let \( \mathcal{H} \sim -nK_X \) be a movable linear system, where \( n \) is a positive integer. A maximal singularity of \( \mathcal{H} \) is an extreme extraction \( \varphi: (E \subset Y) \to (\Gamma \subset X) \) such that
\[
c(X, \mathcal{H}) = \frac{a_E(K_X)}{m_E(\mathcal{H})} < \frac{1}{n},
\]
where
- \( c(X, \mathcal{H}) := \max \{ \lambda \mid K_X + \lambda \mathcal{H} \text{ is canonical} \} \) is the canonical threshold of \((X, \mathcal{H})\),
- \( a_E(K_X) \) is the discrepancy of \( K_X \) along \( E \), and
- \( m_E(\mathcal{H}) \) is the multiplicity along \( E \) of the proper transform \( \varphi^{-1}_* \mathcal{H} \) on \( Y \).

We say that an extreme divisorial extraction is a maximal singularity if there exists a movable linear system \( H \) such that the extraction is a maximal singularity of \( H \).

A subvariety \( \Gamma \subset X \) is called a maximal center if there is a maximal singularity \( Y \to X \) whose center is \( \Gamma \).

**Theorem 2.3.** If \( X \) admits no maximal center, then \( X \) is birationally superrigid.

For a proof of birational superrigidity of a given Fano 3-fold \( X \) of Picard number 1, we need to exclude each subvariety of \( X \) as a maximal center. In the next subsection we explain several methods of exclusion under a relatively concrete setting. Here we discuss methods of excluding terminal quotient singular points in a general setting.

For a terminal quotient singular point \( p \in X \) of type \( \frac{1}{r}(1, a, r-a) \), where \( r \) is coprime to \( a \) and \( 0 < a < r \), there is a unique extreme divisorial extraction \( \varphi: (E \subset Y) \to (p \in X) \), which is the weighted blowup with weight \( \frac{1}{r}(1, a, r-a) \), and we call it the Kawamata blowup (see [17] for details). The integer \( r > 1 \) is called the index of \( p \in X \). For the Kawamata blowup \( \varphi: (E \subset Y) \to (p \in X) \), we have \( K_Y = \varphi^* K_X + \frac{1}{r} E \) and
\[
(E^3) = \frac{r^2}{a(r-a)}.
\]

For a divisor \( D \) on \( X \), the order of \( D \) along \( E \), denoted by \( \text{ord}_E(D) \), is defined to be the coefficient of \( E \) in \( \varphi^* D \).

We first explain the most basic method.

**Lemma 2.4 ([9, Lemma 5.2.1]).** Let \( p \in X \) a terminal quotient singular point and \( \varphi: (E \subset Y) \to (p \in X) \) the Kawamata blowup. If \( (-K_Y)^2 \notin \text{Int NE}(Y) \), then \( p \) is not a maximal center.

For the application of the above lemma, we need to find a nef divisor on \( Y \). The following result, which is a slight generalization of [20, Lemma 6.6], is useful.

**Lemma 2.5.** Let \( p \in X \) be a terminal quotient singular point and \( \varphi: (E \subset Y) \to (p \in X) \) the Kawamata blowup. Assume that there are effective Weil divisors \( D_1, \ldots, D_k \) such that the intersection \( D_1 \cap \cdots \cap D_k \) does not contain a curve through \( p \). We set
\[
e := \min \{ \text{ord}_E(D_i)/n_i \mid 1 \leq i \leq k \},
\]
where \( n_i \) is the positive rational number such that \( D_i \sim_q -n_iK_X \). Then \(-\varphi^*K_X - \lambda E\) is a nef divisor for \( 0 \leq \lambda \leq e \).

**Proof.** We may assume \( e > 0 \), that is, \( D_i \) passes through \( p \) for any \( i \). For an effective divisor \( D \sim_q -nK_X \), we call \( \text{ord}_E(D)/n \) the vanishing ratio of \( D \) along \( E \). For \( 1 \leq i \leq k \), we choose a component of \( D_i \), denoted \( D_i' \), which has maximal vanishing ratio along \( E \) among the components of \( D_i \). Clearly we have \( D_1' \cap \cdots \cap D_k' \) does not contain a curve through \( p \) and we have

\[
e' := \min \{ \text{ord}_E(D_i')/n'_i \mid 1 \leq i \leq k \} \geq e,
\]

where \( n'_i \in \mathbb{Q} \) is such that \( D_i' \sim_q -n'_iK_X \). Since \( D_1', \ldots, D_k' \) are prime divisors, we can apply [20, Lemma 6.6] and conclude that \(-\varphi^*K_X - e'E\) is nef. Then so is 

\(-\varphi^*K_X - \lambda E\) for any \( 0 \leq \lambda \leq e' \) since \(-\varphi^*K_X\) is nef, and the proof is completed. \( \square \)

We have another method of exclusion which can be sometimes effective when Lemma 2.4 is not applicable.

**Lemma 2.6.** Let \( p \in X \) be a terminal quotient singular point and \( \varphi: (E \subset Y) \rightarrow (p \in X) \) the Kawamata blowup. Suppose that there exists an effective divisor \( S \) on \( X \) passing through \( p \) and a linear system \( \mathcal{L} \) of divisors on \( X \) passing through \( p \) with the following properties.

1. \( S \cap \text{Bs} \mathcal{L} \) does not contain a curve passing through \( p \), and
2. For a general member \( L \in \mathcal{L} \), we have

\[
(-K_Y \cdot \tilde{S} \cdot \tilde{L}) \leq 0,
\]

where \( \tilde{S}, \tilde{L} \) are the proper transforms of \( S, L \) on \( Y \), respectively.

Then \( p \) is not a maximal center.

**Proof.** We write \( D \sim -nK_X \). Write \( S = \sum m_iS_i + T \), where \( m_i > 0 \), \( S_i \) is a prime divisor and \( T \) is an effective divisor which does not pass through \( p \). We have \( T \sim -lK_X \) for some \( l \geq 0 \) and

\[
(-K_Y \cdot \tilde{T} \cdot \tilde{L}) = nl(-K_X)^3 \geq 0.
\]

Since

\[
0 \geq (-K_Y \cdot \tilde{S} \cdot \tilde{L}) = \sum m_i(-K_Y \cdot \tilde{S}_i \cdot \tilde{L}) + (-K_Y \cdot \tilde{T} \cdot \tilde{L}),
\]

there is a component \( S_i \) for which \(-K_Y \cdot \tilde{S}_i \cdot \tilde{L}) \leq 0. Since \( p \in S_i \cap \text{Bs} \mathcal{L} \subset S \cap \text{Bs} \mathcal{L} \), we may assume that \( S \) is a prime divisor by replacing \( S \) by \( S_i \).

Write \( \mathcal{L} = \{ L_\lambda \mid \lambda \in \mathbb{P}^1 \} \). For \( \lambda \in \mathbb{P}^1 \), we write \( S \cdot L_\lambda = \sum c_iC_{\lambda,i} \), where \( c_i \geq 0 \) and \( C_{\lambda,i} \) is an irreducible and reduced curve on \( X \). For a curve or a divisor \( \Delta \) on \( X \), we denote by \( \tilde{\Delta} \) its proper transform on \( Y \). Then,

\[
\tilde{S} \cdot \tilde{L}_\lambda = \sum i c_i\tilde{C}_{\lambda,i} + \Xi,
\]

where \( \Xi \) is an effective 1-cycle supported on \( E \). Since any component of \( \Xi \) is contracted by \( \varphi \) and \(-K_Y \) is \( \varphi \)-ample, we have \((-K_Y \cdot \Xi) \geq 0 \). Thus, for a general \( \lambda \in \mathbb{P}^1 \), we have

\[
0 \geq (-K_Y \cdot \tilde{S} \cdot \tilde{L}_\lambda) \geq \sum c_i(-K_Y \cdot \tilde{C}_{\lambda,i}).
\]
It follows that \((-K_Y \cdot \tilde{C}_{\lambda,i}) \leq 0\) for some \(i\). We choose such a \(\tilde{C}_{\lambda,i}\) and denote it as \(\hat{C}_{\lambda}\). By the assumption (1), the set 
\[
\{ \hat{C}_{\lambda} \mid \lambda \in \mathbb{P}^1 \text{ is general} \}
\]
consists of infinitely many distinct curves. We have \((-K_Y \cdot \hat{C}_{\lambda,i}) \leq 0\) for some \(i\). We choose such a \(\hat{C}_{\lambda,i}\) and denote it as \(\hat{C}_{\lambda}\). By the assumption (1), the set 
\[
\{ \hat{C}_{\lambda} \mid \lambda \in \mathbb{P}^1 \text{ is general} \}
\]
consists of infinitely many distinct curves. We have \((-K_Y \cdot \hat{C}_{\lambda,i}) \leq 0\) by the construction. We see that \((E \cdot \hat{C}_{\lambda,i}) > 0\) since \(\hat{C}_{\lambda}\) is the proper transform of a curve passing through \(p\). Therefore \(p\) is not a maximal center by [20, Lemma 2.20].

2.2. Fano varieties in a weighted projective space. Let \(\mathbb{P} = \mathbb{P}(a_0, \ldots, a_n)\) be a weighted projective space with homogeneous coordinates \(x_0, \ldots, x_n\) of deg \(x_i = a_i\). We assume that \(\mathbb{P}\) is well formed, that is, 
\[
\gcd\{ a_i \mid 0 \leq i \leq n, i \neq j \} = 1
\]
for \(j = 0, 1, \ldots, n\). Throughout the present subsection, let \(X \subset \mathbb{P}\) be a normal projective 3-fold defined by the equations
\[
F_1 = F_2 = \cdots = F_N = 0,
\]
where \(F_i \in \mathbb{C}[x_0, \ldots, x_{n+1}]\) is a homogeneous polynomial of degree \(d_i\) with respect to the grading deg \(x_i = a_i\).

**Definition 2.7.** We say that \(X\) is quasi-smooth if the affine cone 
\[
(F_1 = F_2 = \cdots = F_N = 0) \subset \mathbb{A}^{n+1} = \text{Spec} \mathbb{C}[x_0, \ldots, x_n]
\]
is smooth outside the origin.

In the following we assume that \(X\) is a quasi-smooth prime Fano 3-fold. For \(0 \leq i \leq n\), we define \(p_{x_i} = (0 : \cdots : 1 : \cdots : 0) \in \mathbb{P}\), where the unique 1 is in the \((i + 1)\)st position, and we define \(D_i = (x_i = 0) \cap X\) which is a Weil divisor such that 
\[
D_i \sim -a_i K_X.
\]

**Lemma 2.8.** If \((-K_X)^3 \leq 1\), then no curve on \(X\) is a maximal center.

**Proof.** The same proof of [1, Lemma 2.1] applies in this setting without any change.

**Lemma 2.9.** Assume that \(a_0 \leq a_1 \leq \cdots \leq a_n\). If \(a_{n-1} a_n (-K_X)^3 \leq 4\), then no nonsingular point of \(X\) is a maximal center.

**Proof.** The proof is almost identical to that of [1, Lemma 2.6].

**Definition 2.10.** Let \(C \subset \{x_0, \ldots, x_n\}\) be a set of homogeneous coordinates. We define
\[
\Pi(C) := \bigcap_{z \in C} (z = 0) \subset \mathbb{P},
\]
\[
\Pi_X(C) := \Pi(C) \cap X \subset X.
\]
We also denote
\[
\Pi(C) = \Pi(x_{i_1}, \ldots, x_{i_m}), \quad \Pi_X(C) = \Pi_X(x_{i_1}, \ldots, x_{i_m}),
\]
when \(C = \{x_{i_1}, \ldots, x_{i_m}\}\).
Lemma 2.11. Let $p \in X$ be a singular point of type $\frac{1}{2}(1, 1, 1)$ and let
\[
b := \max\{ a_i \mid 0 \leq i \leq n, a_i \text{ is odd} \}.
\]
If $2b(-K_X)^3 \leq 1$, then $p$ is not a maximal center.

Proof. Let $C = \{x_1, \ldots, x_m\}$ be the set of homogeneous coordinates of odd degree. The set $\Pi_X(C) = D_{i_1} \cap \cdots \cap D_{i_m}$ consists of singular points since $X$ is quasi-smooth and has only terminal quotient singularities (which are isolated). In particular $\Pi_X(C)$ is a finite set of points. Let $\varphi: (E \subset Y) \to (p \in X)$ be the Kawamata blowup. Then $\mathrm{ord}_E(D_{i_j}) \geq 1/2$ since $2D_{i_j}$ is a Cartier divisor passing through $p$ and thus $-b\varphi^*K_X - \frac{1}{2}E$ is nef by Lemma 2.5. We have
\[
(-b\varphi^*K_X - \frac{1}{2}E)(-K_Y) = b(-K_X)^3 - \frac{1}{2} \leq 0.
\]
This shows that $(-K_Y)^2 \notin \overline{NE}(Y)$ and $p$ is not a maximal center by Lemma 2.4. □

Definition 2.12. Let $p = p_{x_k} \in X$ be a terminal quotient singular point of type $\frac{1}{a_k}(1, c, a_k - c)$ for some $c$ with $1 \leq c \leq a_k/2$. We define
\[
i vr_p(C) := \min_{1 \leq j \leq m} \left\{ \frac{a_j}{a_j a_k} \right\},
\]
where $C = \{x_i, \ldots, x_m\}$ and $a_j$ is the integer such that $1 \leq a_j \leq a_k$ and $a_j$ is congruent to $a_j$ modulo $a_k$, and call it the initial vanishing ratio of $C$ at $p$.

Definition 2.13. For a terminal quotient singularity $p$ of type $\frac{1}{r}(1, a, r-a)$, we define
\[
wp(p) := a(r-a),
\]
and call it the weight product of $p$.

Lemma 2.14. Let $p = p_{x_k} \in X$ be a terminal quotient singular point. Suppose that there exists a subset $C \subset \{x_0, \ldots, x_n\}$ satisfying the following properties.

1. $p \in \Pi_X(C)$, or equivalently $x_k \notin C$.
2. $\Pi_X(C \cup \{x_k\}) = \emptyset$.
3. $\mathrm{ivr}_p(C) \geq \wp(p)(-K_X)^3$.

Then $p$ is not a maximal center.

Proof. We write $C = \{x_i, \ldots, x_m\}$. We claim that $\Pi_X(C) = D_{i_1} \cap \cdots \cap D_{i_m}$ is a finite set of points. Indeed, if it contains a curve, then $\Pi_X(C \cup \{x_k\}) = \Pi_X(C) \cap D_k \neq \emptyset$ since $D_k$ is an ample divisor on $X$. This is impossible by the assumption (2). Note that we have $\mathrm{ord}_E(D_{i_j}) \geq a_j/a_k$ (cf. [1, Section 3]) so that
\[
e := \min\{ \mathrm{ord}_E(D_{i_j})/a_j \mid 1 \leq j \leq m \} \geq \mathrm{ivr}_p(C).
\]
By Lemma 2.5, $-\varphi^*K_X - \mathrm{ivr}_p(C)E$ is nef and we have
\[
(-\varphi^*K_X - \mathrm{ivr}_p(C)E)(-K_Y)^2 = (-K_X)^3 - \frac{\mathrm{ivr}_p(C)}{\wp(p)} \leq 0
\]
by the assumption (3). Therefore $(-K_Y)^2 \notin \overline{NE}(Y)$ and $p$ is not a maximal center. □
Let \( p \in X \) be a singular point such that it can be transformed to \( p_{x_0} \) by a change of coordinates. For simplicity of the description we assume \( p = p_{x_0} \) and we set \( r = a_0 > 1 \). Let \( \varphi: (E \subset Y) \to (p \in X) \) be the Kawamata blowup. We explain a systematic way to estimate \( \text{ord}_E(x_i) \) for coordinates \( x_i \) and also an explicit description of \( \varphi \). It is a consequence of the quasi-smoothness of \( X \) that after re-numbering the defining equation we can write

\[
F_l = a_l x_k^m x_i + \text{(other terms)}, \quad \text{for } 1 \leq l \leq n - 3,
\]

where \( a_l \in \mathbb{C} \setminus \{0\} \), \( m_l \) is a positive integer and \( x_0, x_1, \ldots, x_{n-3} \) are mutually distinct so that by denoting the other 3 coordinates as \( x_{j_1}, x_{j_2}, x_{j_3} \) we have

\[
\{ x_0, x_{i_1}, \ldots, x_{i_{n-3}}, x_{j_1}, x_{j_2}, x_{j_3} \} = \{ x_0, \ldots, x_n \}.
\]

In this case we can choose \( x_{j_1}, x_{j_2}, x_{j_3} \) as local orbi-coordinates of \( X \) at \( p \) and the singular point \( p \) is of type

\[
\frac{1}{r}(a_{j_1}, a_{j_2}, a_{j_3}).
\]

**Definition 2.15** ([1, Definitions 3.6, 3.7]). For an integer \( a \), we denote by \( \bar{a} \) the positive integer such that \( \bar{a} \equiv a \pmod{r} \) and \( 0 < \bar{a} \leq r \). We say that

\[
w(x_1, \ldots, x_n) = \frac{1}{r}(b_1, \ldots, b_n)
\]

is an admissible weight at \( p \) if \( b_i \equiv a_i \pmod{r} \) for any \( i \).

For an admissible weight \( w \) at \( p \) and a polynomial \( f = f(x_0, \ldots, x_n) \), we denote by \( f^w \) the lowest weight part of \( f \), where we assume that \( w(x_0) = 0 \).

We say that an admissible weight \( w \) at \( p \) satisfies the KBL condition if \( x_0^e x_i \in F_l^w \) for \( 1 \leq l \leq n - 3 \) and

\[
(b_{j_1}, b_{j_2}, b_{j_3}) = (\bar{a}_{j_1}, \bar{a}_{j_2}, \bar{a}_{j_3}).
\]

Let \( w(x_1, \ldots, x_n) = \frac{1}{r}(b_1, \ldots, b_n) \) be an admissible weight at \( p \) satisfying the KBL condition. We denote by \( \Phi_w: Q_w \to \mathbb{P} \) at \( p \) with weight \( w \), and by \( Y_w \) the proper transform of \( X \) via \( \Phi_w \). Then the induced morphism \( \varphi_w = \Phi_w|_{Y_w}: Y_w \to X \) coincides with the Kawamata blowup at \( p \). From this we see that the exceptional divisor \( E \) is isomorphic to

\[
E_w := (f_1 = \cdots = f_{n-3} = 0) \subset \mathbb{P}(b_1, \ldots, b_n),
\]

where \( f_l = F_l^w(1, x_1, \ldots, x_n) \). We refer readers to [1, Section 3] for details.

**Lemma 2.16** ([1, Lemma 3.9]). Let \( w(x_1, \ldots, x_n) = \frac{1}{r}(b_1, \ldots, b_n) \) be an admissible weight at \( p \in X \) satisfying the KBL condition. Then the following assertions hold.

1. We have \( \text{ord}_E(D_i) \geq b_i/r \) for any \( i \).
2. If \( F_l^w = \alpha_l x_0^{m_l} x_i \), where \( \alpha_l \in \mathbb{C} \setminus \{0\} \), for some \( 1 \leq l \leq n - 3 \), then the weight

\[
w'(x_1, \ldots, x_n) = \frac{1}{r}(b'_1, \ldots, b'_n),
\]

where \( b'_j = b_j \) for \( j \neq l \) and \( b'_l = b_l + r \), satisfies the KBL condition. In particular, \( \text{ord}_E(D_l) \geq (b_l + r)/r \).

We will use the following notation for a polynomial \( f = f(x_0, \ldots, x_n) \).

- For a monomial \( p = x_0^{e_0} \cdots x_n^{e_n} \), we write \( p \in f \) if \( p \) appears in \( f \) with non-zero (constant) coefficient.
For a subset \( C \subset \{ x_0, \ldots, x_n \} \) and \( \Pi = \Pi(C) \), we denote by \( f|_\Pi \) the polynomial in variables \( \{ x_0, \ldots, x_n \} \setminus C \) obtained by putting \( x_i = 0 \) for \( x_i \in C \) in \( f \).

**Remark 2.17.** We explain some consequences of quasi-smoothness, which will be frequently used in Section 3. We keep the above notation and assumption. In particular \( X \subset \mathbb{P}(a_0, \ldots, a_n) \) is assumed to be quasi-smooth.

1. Let \( C = \{ x_{i_1}, \ldots, x_{i_m} \} \) be the coordinates such that \( r := \gcd\{ a_{i_1}, \ldots, a_{i_m} \} > 1 \) and \( a_j \) is coprime to \( r \) for any \( j \neq i_1, \ldots, i_m \). Then
   \[
   \Sigma := \Pi_X(\{ x_0, \ldots, x_n \} \setminus C)
   \]
   is contained in the singular locus of \( X \) and \( X \) has a quotient singular point of index \( r \) at each point of \( \Sigma \). In particular, if \( X \) has only isolated singularities (e.g. \( \dim X = 3 \) and \( X \) has only terminal singularities), then either \( \Sigma = \emptyset \) or \( \Sigma \) consists of finite set of singular points of index \( r \).

2. Let \( x_k \) be the coordinate such that \( a_j \neq a_k \) for any \( j \neq k \). If \( X \) does not contain a singular point of index \( r \), then \( p_k \notin X \), that is, a power of \( x_k \) appears in one of the defining polynomials with non-zero coefficient.

### 3. Proof of birational superrigidity by numerical data

We prove birational superrigidity of codimension 4 quasi-smooth prime Fano 3-folds with no projections by utilizing only numerical data. The numerical data for each Fano 3-fold will be described in the beginning of the corresponding subsection. The Fano 3-folds are embedded in a weighted projective 7-space, denoted by \( \mathbb{P} \), and we use the symbol \( p, q, r, s, t, u, v, w \) for the homogeneous coordinates of \( \mathbb{P} \). We use the following terminologies: Let \( X \subset \mathbb{P} \) be a codimension 4 quasi-smooth prime Fano 3-fold. For a homogeneous coordinate \( z \in \{ p, q, \ldots, w \} \),

- \( D_z := (z = 0) \cap X \) is the Weil divisor on \( X \) cut out by \( z \), and
- \( p_z \in \mathbb{P} \) is the point at which only the coordinate \( z \) does not vanish.

Note that Theorem 1.2 will follow from Theorems 3.1, 3.2, 3.4.

#### 3.1. Fano 3-folds of numerical type \#25

Let \( X \) be a quasi-smooth prime Fano 3-fold of numerical type \#25, whose data consist of the following.

- \( X \subset \mathbb{P}(2p, 5q, 6r, 7s, 8t, 9u, 10v, 11w) \).
- \( (-K_X)^3 = 1/70 \).
- \( \deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (16, 17, 18, 18, 19, 20, 20, 21, 22) \).
- \( \mathcal{B}_X = \{ 7 \times \frac{1}{2}(1, 1, 1), \frac{5}{3}(1, 1, 4), \frac{1}{6}(1, 2, 5) \} \).

Here the subscripts \( p, q, \ldots, w \) of the weights means that they are the homogeneous coordinates of the indicated degrees, and \( \mathcal{B}_X \) indicates the numbers and the types of singular points of \( X \).

**Theorem 3.1.** Let \( X \) be a quasi-smooth prime codimension 4 Fano 3-fold of numerical type \#25. Then \( X \) is birationally superrigid.

**Proof.** By Lemmas 2.8 and 2.9, no curve and no nonsingular point on \( X \) is a maximal center. By Lemma 2.11, singular points of type \( \frac{1}{2}(1, 1, 1) \) are not maximal centers.
Let \( p \) be the singular point of type \( \frac{1}{9}(1, 1, 4) \). Replacing the coordinate \( v \) if necessary, we may assume \( p = p_q \). We set \( C = \{ p, s, u, v \} \). We have
\[
\text{ivr}_p(C) = \frac{2}{35} = \wp(p)(-K_X)^3.
\]
By Lemma 2.14, it remains to show that \( \Pi_X := \Pi_X(C \cup \{ q \}) = \emptyset \). We set \( \Pi := \Pi(C \cup \{ q \}) \subset \mathbb{P} \) so that \( \Pi_X = \Pi \cap X \). Since \( p_t \notin X \), one of the defining polynomials contain a power of \( t \). By looking at the degrees of \( F_1, \ldots, F_9 \), we have \( t^2 \in F_1 \).
Similarly, we have \( r^3 \in F_3 \) and \( w^2 \in F_9 \) after possibly interchanging \( F_3 \) and \( F_4 \). The monomial \( t^2 \) (resp. \( r^3 \)) is the only monomial of degree 16 (resp. 18) consisting of the variables \( r, t, w \). The monomials \( w^2 \) and \( t^2 r \) are the only monomials of degree 22 consisting the variables \( r, t, w \). Hence, re-scaling \( r, t, w \), we can write
\[
F_1|\Pi = t^2, \quad F_3|\Pi = r^3, \quad F_9|\Pi = w^2 + \alpha t^2 r,
\]
for some \( \alpha \in \mathbb{C} \). The set \( \Pi_X \) is contained in the common zero loci of the above 3 polynomials inside \( \Pi \). The equations have only trivial solution and this shows that \( \Pi_X = \emptyset \). Thus \( p \) is not a maximal center.

Let \( p = p_s \) be the singular point of type \( \frac{1}{7}(1, 2, 5) \) and set \( C = \{ p, q, r \} \). We have
\[
\text{ivr}_p(C) = \frac{1}{7} = \wp(p)(-K_X)^3.
\]
By Lemma 2.14, it remains to show that \( \Pi_X := \Pi_X(C \cup \{ s \}) = \emptyset \). We set \( \Pi := \Pi(C \cup \{ s \}) \subset \mathbb{P} \) so that \( \Pi_X = \Pi \cap X \). Since \( p_t, p_u, p_v, p_w \notin X \), we may assume \( t^2 \in F_1, u^2 \in F_3, v^2 \in F_6 \) and \( w^2 \in F_9 \) after possibly interchanging defining polynomials of the same degree. Then we can write
\[
F_1|\Pi = t^2, \quad F_3|\Pi = u^2 + \alpha vt, \quad F_6|\Pi = v^2 + \beta wu, \quad F_9|\Pi = w^2 + \gamma t^2 r,
\]
for some \( \alpha, \beta, \gamma \in \mathbb{C} \). This shows that \( \Pi_X = \emptyset \) and thus \( p \) is not a maximal center. This completes the proof. \( \square \)

### 3.2. Fano 3-folds of numerical type #166

Let \( X \) be a quasi-smooth prime Fano 3-fold of numerical type #166, whose data consist of the following.

- \( X \subset \mathbb{P}(2p, 2q, 3r, 3s, 4t, 4u, 5v, 5w) \).
- \(( -K_X )^3 = 1/6 \).
- \( \deg(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (8, 8, 8, 9, 9, 9, 10, 10, 10) \).
- \( B_X = \{ 11 \times \frac{1}{2}(1, 1, 1), \frac{1}{2}(1, 1, 2) \} \).

**Theorem 3.2.** Let \( X \) be a quasi-smooth prime codimension 4 Fano 3-fold of numerical type #166. Then \( X \) is birationally superrigid.

**Proof.** By Lemmas 2.8 and 2.9, no curve and no nonsingular point is a maximal center.

Let \( p \) be a singular point of type \( \frac{1}{2}(1, 1, 1) \). After replacing coordinates, we may assume \( p = p_p \). We set \( C = \{ q, r, s, t, u \} \). We have
\[
\text{ivr}_p(C) = \frac{1}{6} = \wp(p)(-K_X)^3.
\]
Moreover we have \( \Pi_X(C \cup \{ p \}) = \emptyset \) because \( X \) is quasi-smooth and it does not have a singular point of index 5. Thus, by Lemma 2.14, \( p \) is not a maximal center.
Let \( p \) be the singular point of type \( \frac{1}{3}(1, 1, 2) \). After replacing \( r \) and \( s \), we may assume \( p = p_*. \) We set \( C = \{p, q, r\} \). Then we have
\[
\text{ivp}(C) = \frac{1}{3} = \text{wp}(p)(-K_X)^3.
\]
By Lemma 2.14, it remains to show that \( \Pi_X = \Pi_X(\mathcal{C} \cup \{s\}) = \emptyset \). We set \( \Pi := \Pi(\mathcal{C} \cup \{s\}) \subset \mathbb{P} \) so that \( \Pi_X = \Pi \cap X \). We have
\[
\Pi_X = (F_1|_\Pi = F_2|_\Pi = F_3|_\Pi = F_7|_\Pi = F_9|_\Pi = 0) \cap \Pi.
\]
We see that \( F_1|_\Pi, F_2|_\Pi, F_3|_\Pi \) consist only of monomials in variables \( t, u \), and \( X \) does not have a singular point of index 4. Hence the equation
\[
F_1|_\Pi = F_2|_\Pi = F_3|_\Pi = 0
\]
implies \( t = u = 0 \). Similarly, \( F_7|_\Pi, F_8|_\Pi, F_9|_\Pi \) consist only of the monomials in variables \( v, w \), and \( X \) does not contain a singular point of index 5. Hence the equation
\[
F_7|_\Pi = F_8|_\Pi = F_9|_\Pi = 0
\]
implies \( v = w = 0 \). It follows that \( \Pi_X = \emptyset \) and \( p \) is not a maximal center. Therefore \( X \) is birationally superrigid. \( \square \)

### 3.3. Fano 3-folds of numerical type \#282

Let \( X \) be a quasi-smooth prime Fano 3-fold of numerical type \#282, whose data consist of the following.

- \( X \subset \mathbb{P}(1_p, 6_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w) \).
- \( (K_X)^3 = 1/42 \).
- \( \text{deg}(F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (16, 17, 18, 19, 20, 21, 22) \).
- \( \mathcal{B} = \{2 \times \frac{1}{2}(1, 1, 1), 2 \times \frac{1}{3}(1, 1, 2), \frac{1}{6}(1, 1, 5), \frac{1}{7}(1, 1, 6)\} \).

**Proposition 3.3.** Let \( X \) be a quasi-smooth prime codimension 4 Fano 3-fold of numerical type \#282. Then no curve and no point is a maximal center except possibly for the singular point of type \( \frac{1}{6}(1, 1, 5) \).

**Proof.** By Lemmas 2.8, 2.9 and 2.11, it remains to exclude singular points of type \( \frac{1}{3}(1, 1, 2) \) and \( \frac{1}{7}(1, 1, 6) \) as maximal centers.

Let \( p \) be a singular point of type \( \frac{1}{6}(1, 1, 2) \) and let \( \varphi: (E \subset Y) \rightarrow (p \in X) \) be the Kawamata blowup. We claim that \( \Pi_X(p, s, t, w) = D_p \cap D_s \cap D_t \cap D_w \) is a finite set of points (containing \( p \)). Since \( X \) does not contain a singular point of index 10, we may assume that \( u^2 \in F_6 \). Then, by re-scaling \( v \), we have
\[
F_6(0, q, r, 0, 0, u, v, 0) = v^2
\]
and this shows that \( \Pi_X(p, s, t, w) = \Pi_X(p, s, t, v, w) \). The latter set consists of singular points \( \{2 \times \frac{1}{2}(1, 1, 2), \frac{1}{6}(1, 1, 5)\} \) and thus \( \Pi_X(p, s, t, w) \) is a finite set of points. We have
\[
\text{ord}_E(D_p), \text{ord}_E(D_s), \text{ord}_E(D_t), \text{ord}_E(D_w) \geq \frac{1}{3}, \quad \text{ord}_E(D_t), \text{ord}_E(D_w) \geq \frac{2}{3}.
\]
By Lemma 2.5, \( N := -\varphi^*K_X - \frac{21}{21}E \) is a nef divisor on \( Y \) and we have \( (N \cdot (-K_Y)^2) = 0 \). Thus \( p \) is not a maximal center.

Let \( p = p_* \) be the singular point of type \( \frac{1}{7}(1, 1, 6) \) and set \( C = \{p, q, r\} \). We have
\[
\text{ivp}(C) = \frac{1}{7} = \text{wp}(p)(-K_X)^3.
\]
Lemma 2.14. Where \( \alpha, \beta \) is a finite set of points. We have

\[
F_1|_\Pi = t^2, \quad F_3|_\Pi = \alpha vt + u^2, \quad F_6|_\Pi = \beta uv + v^2, \quad F_9|_\Pi = w^2,
\]

where \( \alpha, \beta \in \mathbb{C} \). This shows that \( \Pi_X(C \cup \{s\}) = \Pi \cap X = \emptyset \). Thus \( p \) is not a maximal center by Lemma 2.14 and the proof is completed. \( \square \)

3.4. Fano 3-folds of numerical type \#308. Let \( X \) be a quasi-smooth prime Fano 3-fold of numerical type \#308, whose data consist of the following.

- \( X \subset \mathbb{P}(1_p, 5_q, 6_r, 6_s, 7_t, 8_u, 9_v, 10_w) \).
- \( (-K_X)^3 = 1/30 \).
- deg\((F_1, F_2, F_3, F_4, F_5, F_6, F_7, F_8, F_9) = (14, 15, 16, 17, 18, 19, 20) \).
- \( B_X = \left\{ \frac{1}{2}(1, 1, 1), \frac{1}{3}(1, 1, 2), \frac{1}{5}(1, 2, 3), 2 \times \frac{1}{6}(1, 1, 5) \right\} \).

**Theorem 3.4.** Let \( X \) be a quasi-smooth prime Fano 3-fold of numerical type \#308. Then \( X \) is birationally superrigid.

**Proof.** By Lemmas 2.8, 2.9 and 2.11, no curve and no nonsingular point is a maximal center and the singular point of type \( \frac{1}{2}(1, 1, 1) \) is not a maximal center.

Let \( p \) be the singular point of type \( \frac{1}{2}(1, 1, 2) \), which is necessary contained in (\( p = q = t = u = w = 0 \)), and let \( \varphi: (E \subset Y) \to (p \in X) \) be the Kawamata blowup. We set \( C = \{p, q, u\} \) and \( \Pi = \Pi(C) \subset \mathbb{P} \). Since \( p_t, p_w \notin X \), we have \( t^2 \in F_1, w^2 \in F_9 \) and we can write

\[
F_1|_\Pi = t^2, \quad F_9|_\Pi = w^2 + \alpha t^2 r + \beta t^2 s,
\]

where \( \alpha, \beta \in \mathbb{C} \). Thus,

\[
\Pi_X(C) = \Pi \cap X = \Pi_X(p, q, t, u, w),
\]

and this consists of two \( \frac{1}{6}(1, 1, 5) \) points and \( p \). In particular \( D_p \cap D_q \cap D_u = \Pi_X(C) \) is a finite set of points. We have

\[
\text{ord}_E(D_p) \geq \frac{1}{3}, \quad \text{ord}_E(D_q) \geq \frac{2}{3}, \quad \text{ord}_E(D_u) \geq \frac{2}{3},
\]

hence \( N := -8\varphi^* K_X - \frac{2}{3} E \) is a nef divisor on \( Y \) by Lemma 2.5. We have

\[
(N \cdot (-K_Y)^2) = 8(-K_X)^3 - \frac{2}{3} \cdot \frac{3^2}{2} = -\frac{1}{15} < 0.
\]

By Lemma 2.4, \( p \) is not a maximal center.

Let \( p \) be a singular point of type \( \frac{1}{6}(1, 1, 5) \). After replacing \( r \) and \( s \), we may assume \( p = p_s \). We set \( C = \{p, q, r\} \). We have

\[
\text{ivp}(C) = \frac{1}{6} = \wp(p)(-K_X)^3.
\]

Since \( p_t, p_u, p_v, p_w \notin X \), we may assume \( t^2 \in F_1, u^2 \in F_3, v^2 \in F_6, w^2 \in F_9 \) after possibly interchanging \( F_3 \) with \( F_4 \) and \( F_6 \) with \( F_7 \). Then, by setting \( \Pi = \Pi(C \cup \{s\}) \) and by re-scaling \( t, u, v, w \), we have

\[
F_1|_\Pi = t^2, \quad F_3|_\Pi = u^2 + \alpha vt, \quad F_6|_\Pi = v^2 + \beta uv, \quad F_9|_\Pi = w^2,
\]

where \( \alpha, \beta \in \mathbb{C} \). This shows that \( \Pi_X(C \cup \{s\}) = \emptyset \) and \( p \) is not a maximal center by Lemma 2.14.
Finally, let $p$ be a singular point of type $\frac{1}{3}(1, 2, 3)$ and let $\varphi: (E \subset Y) \to (p \in X)$ be the Kawamata blowup. Replacing the coordinate $w$, we may assume $p = p_q$. We write

$$F_3 = \lambda q^3 x + \mu q^2 r + \nu q^2 s + q f_{11} + f_{16},$$
$$F_4 = \lambda' q^3 x + \mu' q^2 r + \nu' q^2 s + q g_{11} + g_{16},$$

where $\lambda, \mu, \nu, \lambda', \mu', \nu' \in \mathbb{C}$ and $f_{11}, f_{16}, g_{11}, g_{16} \in \mathbb{C}[p, r, s, t, u, v, w]$ are homogeneous polynomials of the indicated degrees.

We first consider the case where $\mu \nu' - \nu \mu' \neq 0$. By replacing $r$ and $s$, we may assume that $\mu = \nu' = 1$ and $\lambda = \nu = \lambda' = 0$. We consider the initial weight at $p$

$$w_{\text{in}}(p, r, s, t, u, v, w) = \frac{1}{5}(1, 1, 1, 2, 3, 4, 5).$$

Then $F_3^{\text{in}} = q^2 r$ and $F_4^{\text{in}} = q^2 s$, and this implies $\text{ord}_E(D_r), \text{ord}_E(D_s) \geq 6/5$. Note that $\text{ord}_E(D_p) \geq w_{\text{in}}(x) = 1/5$. We set $C = \{p, r, s\}$ and $\Pi = \Pi(C \cup \{q\})$. By re-scaling $t, u, v, w$, we can write

$$F_1|_{\Pi} = t^2, \quad F_3|_{\Pi} = u^2 + \alpha t, \quad F_6|_{\Pi} = v^2 + \beta wu, \quad F_9|_{\Pi} = w^2,$$

where $\alpha, \beta \in \mathbb{C}$. Hence $\Pi_X(C \cup \{q\}) = \emptyset$. Since $D_q$ is an ampleness divisor, this implies that $D_p \cap D_r \cap D_s$ is a finite set of points (including $p$). By Lemma 2.5, $N := -\varphi^*K_X - \frac{1}{5}E$ is a nef divisor on $Y$. We have

$$(N \cdot (-K_Y)^2) = (-K_X)^3 - \frac{1}{5^3}(E^3) = \frac{1}{30} - \frac{1}{30} = 0,$$

and this shows that $p$ is not a maximal center.

Next we consider the case where $\mu \nu' - \nu \mu' = 0$. By replacing $r$ and $s$ suitably and by possibly interchanging $F_3$ and $F_4$, we may assume that

$$F_3 = q^3 p + q f_{11} + f_{16},$$
$$F_4 = q^2 s + q g_{11} + g_{16}.$$

It is straightforward to see that $q^3 p$ is the unique monomial in $F_3$ with initial weight $1/5$, so that $\text{ord}_E(D_p) \geq 6/5$. Let $\mathcal{L} \subset | -6K_X|$ be the pencil generated by the sections $r$ and $s$. Since $\text{ord}_E(D_r) = 1/5$ and $\text{ord}_E(D_s) \geq 1/5$, a general member $L \in \mathcal{L}$ vanishes along $E$ to order $1/5$ so that $\tilde{L} \sim -6\varphi^*K_X - \frac{1}{5}E$. We have

$$(-K_Y \cdot \tilde{D}_p \cdot \tilde{L}) = 6(-K_X)^3 - \frac{\text{ord}_E(D_p)}{5^2} \cdot (E^3) = \frac{1}{5} - \frac{\text{ord}_E(D_p)}{6} \leq 0$$

since $\text{ord}_E(p) \geq 6/5$. By Lemma 2.6, $p$ is not a maximal center and the proof is completed. \qed

4. Birational superrigidity of cluster Fano 3-folds

In this section, we prove Theorem 1.1, which follow from Theorems 4.1 and 4.2 below.
4.1. #282 by $G_2^{(4)}$ format. Let $X$ be a codimension 4 prime Fano 3-fold of numerical type #282 constructed in $G_2^{(4)}$ format. Then, by [10, Example 5.5], $X$ is defined by the following polynomials in $\mathbb{P}(p, q, r, s, t, u, v, w)$.

$$
F_1 = t^2 - qv + sQ_9,
F_2 = ut -qw + s(v + p^2 t),
F_3 = t(v + p^2 t) - uQ_9 + q(r + p^4 t),
F_4 = (w + p^4 s)s - P_{12}q + u(u + p^2 s),
F_5 = tw - uv + s(qr + p^4 t),
F_6 = (qr + p^4 t)t - Q_9 w + v(v + p^2 t),
F_7 = rs^2 - wu + tP_{12},
F_8 = P_{12}Q_9 - (vw + p^4 qw + p^2 uv + uqr + str - stp^2),
F_9 = rs(u + p^2 s) - vP_{12} + w(w + p^4 s).
$$

Here $P_{12}, Q_9 \in \mathbb{C}[p, q, r, s, t, u, v, w]$ are homogeneous polynomials of the indicated degree. Recall that $(-K_X)^3 = 1/42$.

**Theorem 4.1.** Let $X$ be a codimension 4 Fano 3-fold of numerical type #282 constructed in $G_2^{(4)}$ format. Then $X$ is birationally superrigid.

**Proof.** By Proposition 3.3, it remains to exclude the singular point $p \in X$ of type $\frac{1}{5}(1, 1, 5)$ as a maximal center. The point $p$ corresponds to the unique solution solution of the equations

$$p = s = t = u = v = w = F_3 = F_4 = 0,$$

and we have $p = p_r$. We set $C = \{p, q\}$, $\Pi = \Pi(C)$ and $\Gamma := \Pi_X(C) = \Pi \cap X$.

We will show that $\Gamma$ is an irreducible and reduced curve. We can write

$$P_{12}|_{\Pi} = \lambda r^2, \quad Q_9|_{\Pi} = \mu u,$$

where $\lambda, \mu \in \mathbb{C}$. By the quasi-smoothness of $X$ at $p$, we see that $\lambda, \mu \neq 0$. Then we have

$$F_1|_{\Pi} = t^2 + mu, \quad F_4|_{\Pi} = ws + u^2, \quad F_7|_{\Pi} = rs^2 - wu + \lambda tr^2,$$

$$F_2|_{\Pi} = ut + sv, \quad F_5|_{\Pi} = tw - uv, \quad F_8|_{\Pi} = \lambda v^2 u - (vw + str),$$

$$F_3|_{\Pi} = tv - \mu u^2, \quad F_6|_{\Pi} = -\mu uw + v^2, \quad F_9|_{\Pi} = rsu - \lambda rv^2 + w^2.$$

We work on the open subset $U$ on which $w \neq 0$. Then $\Gamma \cap U$ is isomorphic to the $\mathbb{Z}/11\mathbb{Z}$-quotient of the affine curve

$$(\lambda r^2 v + \mu^3 rv^6 - 1 = 0) \subset \mathbb{A}^2_{r,v}.$$

It is straightforward to check that the polynomial $\lambda r^2 v + \mu^3 rv^6 - 1$ is irreducible. Thus $\Gamma \cap U$ is an irreducible and reduced affine curve. It is also straightforward to check that

$$\Gamma \cap (w = 0) = (p = q = w = 0) = \{p_r, p_s\}.$$

This shows that $\Gamma$ is an irreducible and reduced curve.
Let \( \varphi : (E \subset Y) \to (p \in X) \) be the Kawamata blowup and let \( \tilde{\Delta} \) be the proper transform via \( \varphi \) of a divisor or a curve on \( X \). We show that \( \tilde{D}_p \cap \tilde{D}_q \cap E \) does not contain a curve. Consider the initial weight
\[
w_{in}(p, q, s, t, u, v, w) = \frac{1}{6}(1, 6, 1, 2, 3, 4, 5).
\]
We set \( f_i = F_i^{w_{in}}(p, q, 1, s, t, u, v, w) \). We have
\[
f_4 = (w + p^2)s - \lambda q + u(u + p^2s),
f_7 = s^2 + \lambda t,
f_8 = \lambda \mu u - st,
f_9 = s(u + p^2s) - \lambda v.
\]
Since \( E \) is isomorphic to the subvariety \( (f_4 = f_7 = f_8 = f_9 = 0) \subset \mathbb{P}(1_p, 6_q, 1_s, 2_t, 3_u, 4_v, 5_w) \), it is straightforward to check that \( \tilde{D}_p \cap \tilde{D}_q \cap E \) consists of finite set of points (in fact, 2 points). Thus we have \( \tilde{D}_p \cdot \tilde{D}_q = \Gamma \) since \( D_p \cdot D_q = \Gamma \).

We have
\[
\tilde{D}_p \sim -\varphi^*K_X - \frac{1}{6}E, \quad \tilde{D}_q \sim -6\varphi^*K_X - \frac{e}{6}E,
\]
for some integer \( e \geq 6 \) and hence
\[
(D_p \cdot \Gamma) = (\tilde{D}_p^2 \cdot D_q) = \frac{1}{7} - \frac{e}{30} < 0.
\]
By [20, Lemma 2.18], \( p \) is not a maximal center.

4.2. #282 by \( \mathbb{C}_2 \) format. Let \( X \) be a codimension 4 prime Fano 3-fold of numerical type \#282 constructed in \( \mathbb{G}_2^{(4)} \) format. Then, by [10, Example 5.5], \( X \) is defined by the following polynomials in \( \mathbb{P}(1_p, 6_q, 6_r, 7_s, 8_t, 9_u, 10_v, 11_w) \).

\[
F_1 = tR_8 - S_6Q_{10} + su,
F_2 = tu - wS_6 + sv,
F_3 = rS_6^2 - vR_8 + u^2,
F_4 = tQ_{10} - S_6P_{12} + sw,
F_5 = rsS_6 - wR_8 + uQ_{10},
F_6 = rs^2 - P_{12}R_8 + Q_{10}^2,
F_7 = rtS_6 - vQ_{10} + uw,
F_8 = rst - wQ_{10} + uP_{12},
F_9 = rt^2 - vP_{12} + w^2.
\]

Here \( P_{12}, Q_{10}, R_8, S_6 \in \mathbb{C}[p, q, r, s, t, u, v, w] \) are homogeneous polynomials of the indicated degree.

**Theorem 4.2.** Let \( X \) be a quasi-smooth prime codimension 4 Fano 3-fold of numerical type \#282 constructed by \( \mathbb{C}_2 \) format. We assume that \( q \in S_6 \). Then \( X \) is birationally superrigid.
Proof. By Proposition 3.3, it remains to exclude the singular point \( p \) of type \( \frac{1}{6}(1,1,5) \) as a maximal center.

Replacing \( q \), we may assume that \( S_6 = q \). The singular point \( p \) corresponds to the solution of the equation

\[
p = s = t = u = v = w = S_6 = 0,
\]

and thus \( p = p_r \). We set \( C = \{p, q\} \) and \( \Pi = \Pi(C) \).

We will show that \( \Gamma := \Pi \cap X \) is an irreducible and reduced curve. We have \( \Pi_X(\{p, q, r, s\}) = \emptyset \) (see the proof of Proposition 3.3). Hence \( \Gamma \cap \{s = 0\} = \Pi_X(\{p, q\}) \) does not contain a curve and it remains to show that \( \Gamma \cap U_s \) is irreducible and reduced, where \( U_s := \{s \neq 0\} \subset \mathbb{P} \) is the open subset. We can write

\[
P_{12}|_{\Pi} = \lambda r^2, \quad Q_{10}|_{\Pi} = \mu v, \quad R_8|_{\Pi} = \nu t,
\]

for some \( \lambda, \mu, \nu \in \mathbb{C} \), and we have \( S_6|_{\Pi} = 0 \). We see that \( t^2 \) appears in \( F_1 \) (resp. \( v^2 \) appears in either \( F_6 \) or \( F_7 \)) with non-zero coefficient since \( p_r \notin X \) (resp. \( p_v \notin X \)), which implies that \( \mu \neq 0 \). Note that \( F_i|_{\Pi} = F_i|_{\Pi}(r, s, t, u, v, w) \) is a polynomial in variables \( r, s, t, u, v, w \) and we set \( f_i = F_i|_{\Pi}(r, 1, t, u, v, w) \). Let \( C \subset \mathbb{A}^6_{r, t, u, v, w} \) be the affine scheme defined by the equations

\[
f_1 = f_2 = \cdots = f_9 = 0.
\]

Then \( \Gamma \cap U_s \) is isomorphic to the quotient of \( C \) by the natural \( \mathbb{Z}/7\mathbb{Z} \)-action. We have

\[
\begin{align*}
f_1 &= \nu t^2 + u, \\
f_2 &= tu + v, \\
f_3 &= -\nu tv + u^2, \\
f_4 &= \mu tv + w, \\
f_5 &= -\nu tw + \mu uv, \\
f_6 &= r - \lambda \nu t^2 + \mu^2 v^2, \\
f_7 &= -\mu v^2 + uw, \\
f_8 &= rt - \mu vw + \lambda r^2 u, \\
f_9 &= rt^2 - \lambda r^2 v + w^2.
\end{align*}
\]

By the equations \( f_1 = 0, f_2 = 0 \) and \( f_4 = 0 \), we have

\[
u = -\nu t^2, \quad v = -tu = \nu t^3, \quad w = -\mu tv = -\mu \nu t^4.
\]

By eliminating the variables \( u, v, w \) and cleaning up the equations, \( C \) is isomorphic to the hypersurface in \( \mathbb{A}^5_{r, t, u, v, w} \) defined by

\[
r - \lambda \nu t^2 + t^2 \mu v^2 = 0,
\]

which is an irreducible and reduced curve since \( \mu \nu \neq 0 \), and so is \( \Gamma \cap U_s \). Thus \( \Gamma \) is an irreducible and reduced curve.

Let \( \varphi: (E \subset Y) \to (p \in X) \) be the Kawamata blowup. We have \( e := \text{ord}_E(D_q) \geq 6/6 \) and \( \text{ord}_E(D_p) = 1/6 \) so that we have

\[
\tilde{D}_q \sim -6\varphi^*K_X - e^6 E = -6K_Y + \frac{6-e^6}{6} E, \quad \tilde{D}_p \sim -\varphi^*K_X - \frac{1}{6} E = -K_Y.
\]

We show that \( \tilde{D}_q \cap \tilde{D}_p \cap E \) does not contain a curve. The Kawamata blowup \( \varphi \) is realized as the weighted blowup at \( p \) with the weight

\[
w_{\text{in}}(p, q, s, t, u, v, w) = \frac{1}{6}(1, 6, 1, 2, 3, 4, 5).
\]
We have

\[ F_4^{\text{in}} = -\lambda qr^2 + t(\mu v + h) + sw, \]
\[ F_6^{\text{in}} = -\lambda \mu tr^2 + rs^2, \]
\[ F_8^{\text{in}} = \lambda ur^2 + rst, \]
\[ F_9^{\text{in}} = -\lambda vr^2 + rt^2, \]

where we define \( h := Q_{10}^{\text{in}} - \mu v \). Note that \( h \) is a linear combination of \( up, tp^2, sp^3, rp^4 \) and thus \( h \) is divisible by \( p \). It follows that \( E \) is isomorphic to the subscheme in \( \mathbb{P}(1_p, 6_q, 1_s, 2_t, 3_u, 4_v, 5_w) \) defined by the equations

\[ \lambda q - t(\mu v + h) - sw = \lambda \mu t - s^2 = \lambda u + st = \lambda v + t^2 = 0. \]

It is now straightforward to check that \( \bar{D}_q \cap \bar{D}_p \cap E = (p = q = 0) \cap E \) is a finite set of points (in fact, it consists of 2 points). This shows that \( \bar{D}_q \cdot \bar{D}_p = \bar{\Gamma} \) since \( D_q \cdot D_p = \Gamma \).

We have

\[ (\bar{D}_p \cdot \bar{\Gamma}) = (\bar{D}_q^2 \cdot \bar{D}_q) = 6(-K_X)^3 - \frac{e}{6^3}(E^3) = \frac{1}{7} - \frac{e}{30} < 0 \]

since \( e \geq 6 \). By [20, Lemma 2.18], \( p \) is not a maximal center.

5. On further problems

5.1. Prime Fano 3-folds with no projection centers. We further investigate birational superrigidity of prime Fano 3-folds of codimension \( c \) with no projection centers for \( 5 \leq c \leq 9 \). There are only a few such candidates, which can be summarized as follows.

- In codimension \( c \in \{5, 7, 8\} \), there is a unique candidate and it corresponds to smooth prime Fano 3-folds of degree \( 2c + 2 \). All of these Fano 3-folds are rational (see [15, Corollary 4.3.5 or §12.2]) and are not birationally superrigid.
- In codimension 6, there are 2 candidates; one candidate corresponds to smooth prime Fano 3-folds of degree 14 which are birational to smooth cubic 3-folds (see [25], [13]) and are not birationally superrigid, and the existence is not known for the other candidate which is \#78 in the database.
- In codimension 9, there is a unique candidate of smooth prime Fano 3-folds of degree 20. However, according to the classification of smooth Fano 3-folds there is no such Fano 3-fold (see e.g. [25, Theorem 0.1]).

It follows that, in codimension up to 9, \#78 is the only remaining unknown case for birational superrigidity (of general members).

Question 5.1. Do there exist prime Fano 3-folds which correspond to \#78? If yes, then is a (general) such Fano 3-fold birationally superrigid?

In codimension 10 and higher, there are a lot of candidates of Fano 3-folds with no projection centers. We expect that many of them are non-existence cases and that there are only a few birationally superrigid Fano 3-folds in higher codimensions.

Question 5.2. Is there a numerical type (in other words, Graded Ring Database ID) \#i in codimension greater than 9 such that a (general) quasi-smooth prime Fano 3-fold of numerical type \#i is birationally superrigid?
5.2. Classification of birationally superrigid Fano 3-folds. There are many difficulties in the complete classification of birationally superrigid Fano 3-folds. For example, we need to consider Fano 3-folds which are not necessarily quasi-smooth or not necessarily prime, and also we need to understand subtle behaviors of birational superrigidity in a family, etc.

**Question 5.3.** Is there a birationally superrigid Fano 3-fold which is either of Fano index greater than 1 or has a non-quotient singularity?

**Remark 5.4.** By recent developments [23], [24], [18], it has been known that there exist birationally superrigid Fano varieties which have non-quotient singularities (see [23], [24], [18]) at least in very high dimensions. On the other hand, only a little is known for Fano varieties of index greater than 1 (cf. [22]) and there is no example of birationally superrigid Fano varieties of index greater than 1.

We concentrate on quasi-smooth prime Fano 3-folds. Even in that case, it is necessary to consider those with a projection center, which are not treated in this paper. Let \( X \) be a general quasi-smooth prime Fano 3-fold of codimension \( c \). Then the following are known.

- When \( c = 1 \), \( X \) is birationally superrigid if and only if \( X \) does not admit a type I projection center (see [14], [9], [7]).
- When \( c = 2, 3 \), \( X \) is birationally superrigid if and only if \( X \) is singular and admits no projection center (see [16], [19], [2], [1]).

With these evidences, we expect the following.

**Conjecture 5.5.** Let \( X \) be a general quasi-smooth prime Fano 3-fold of codimension at least 2. Then \( X \) is birationally superrigid if and only if \( X \) is singular and admits no projection centers.

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