Spectral properties from Matsubara Green’s function approach — application to molecules

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We present results for many-body perturbation theory for the one-body Green’s function at finite temperatures using the Matsubara formalism. Our method relies on the accurate representation of the single-particle states in standard Gaussian basis sets, allowing to efficiently compute, among other observables, quasiparticle energies and Dyson orbitals of atoms and molecules. In particular, we challenge the second-order treatment of the Coulomb interaction by benchmarking its accuracy for a well-established test set of small molecules, which includes also systems where the usual Hartree-Fock treatment encounters difficulties. We discuss different schemes how to extract quasiparticle properties and assess their range of applicability. With an accurate solution and compact representation, our method is an ideal starting point to study electron dynamics in time-resolved experiments by the propagation of the Kadanoff-Baym equations.

I. INTRODUCTION

Many-body perturbation theory (MBPT) is one of the most important tools for the prediction of electronic structures from first principles. The controllability of approximations derived from diagrammatic techniques, the wealth of information about the spectroscopic observables contained in the single-particle Green’s function, and the compatibility of the method with the time-propagation and the description of transport properties are believed to be the strong points of the Green’s function approach. However, technical realization of these advantages is proved to be difficult.

In this work we focus on the extraction of spectral information encoded in the Matsubara Green’s function and on the benchmarking of a popular second Born approximation (2BA) for the self-energy. This study is motivated by the fact that the solution of the Dyson equation on the imaginary time-track is the first step of a typical nonequilibrium Green’s function approach in the two-times plane. The power of the NEGF approach has been demonstrated by the description of ultrafast carrier dynamics and time-resolved photoemission and photoionization of atoms and molecules. Some of these results were obtained by starting from the non-interacting reference state and switching on the interaction adiabatically. The numerical scheme simplifies significantly in this case. However, it requires the propagation up to longer times, increasing the computer memory requirements and computational time considerably. Therefore, an efficient KBE solver necessarily incorporates the vertical track of the Keldysh contour in the propagation scheme (Fig. 1).

The choice of an approximation for the self-energy — the second Born approximation — has been shown to be relevant for molecular systems. In particular, exchange effects (the 2BA includes exchange up to second order) are known to have a significant contribution to the total energy and a fact that is elevated for molecular systems. In contrast to the broadly used for equilibrium calculations) and the $T$-matrix approximations, it possesses an additional benefit of the time-locality. By this we mean that the self-energy is a functional of the Green’s function with the same time-arguments. This simplifies the time-propagation considerably.

Many-body approximations have been tested extensively in the energy domain and used as the initial step for the time-propagation. Direct construction of the initial propagator on the Keldysh contour is less common. Extensive study of the spectral properties of the 2BA theory have been performed on finite lattice systems based on the Pariser–Parr–Pople model. 2BA shows a clear improvement over the HF electronic structure: predicting in accordance with the exact diagonalization results the correlation-induced satellites, and yielding correct shifts of spectral features as a function of the interaction strength. However, oscillatory noise-like features, as well as broadening of peaks in frequency domain appear due to the finite propagation length in time-domain. How spectral features are reproduced in real-like systems, and how 2BA compares with standard quantum chemistry methods are still two open prominent questions. Some steps in answering these questions have been undertaken recently focusing on the integral properties. The importance of proper frequency grid for the representation of Matsubara GF has been demonstrated by Kananenka et al. in a study of simple molecular systems.
on spline interpolation and established a criterion determining the accuracy of the results. However, here we focus on the intrinsic limitations of the 2BA rather than on the errors induced by the numerical procedure.

In addition to already mentioned restrictions induced by the finite grid representation of the Matsubara GF, the extraction of the spectral properties from the imaginary time propagation is a nontrivial mathematical problem. The maximum entropy and the generalized Padé approximation have been compared with the direct Laplace transform by Dirks et al. While on the conceptual level the former method is superior, it was suggested that the actual performance for realistic systems is rather subtle depending on the choice of the self-energy approximation. This is our second motivation for the benchmarking of various approaches for small molecules from the well established G2 test set and comparing the performances of the 2BA and the coupled-cluster approach. The G2 test set also contains a number of molecules (such as the dimers Li₂, F₂, Na₂ and P₂) for which the electronic structure within the Hartree-Fock (HF) treatment differs considerably from the accurate coupled-cluster results. Thus, the limits of the 2BA as such and the extraction of quasiparticle properties can be assessed in an unbiased way.

As the last ingredient of this study we consider the extraction of the spectral information from the Matsubara GF using the extended Koopman’s theorem (EKT). While extensive tests of this possibility have been performed in the quantum chemistry framework, there has also been proposals to use EKT within the Green’s function approach. However, the EKT cannot be considered an equivalent substitute of the aforementioned spectral methods. While they use in principle all dynamical information encapsulated in the MGF, the latter approach solely relies on the one and two-particle density matrices and is sensitive to the asymptotic behavior of bound state wave-functions. Because of this restriction, only the first ionization potential and electron affinity in each symmetry class can be obtained. However, the advantage is that EKT can be applied to any correlated ground state, e.g., from coupled cluster approaches.

The work is organized as follows. In Sec. II summarize well known self-energy expressions specializing on the Matsubara formalism, recall basic facts about the extended Koopman’s theorem and describe the numerical implementation of these methods for molecular systems including the analytic continuation and the Padé approximation. In Sec. III the results of benchmark calculations are presented and compared with reference experimental and coupled cluster numerical data. In Sec. IV we discuss in details our main finding that the 2BA can compete with accurate quantum chemistry methods and thus endorse the method as accurate and extendible approach to equilibrium and excited-state properties of molecules.

II. METHODS

Our calculations are performed in the molecular orbital basis. Its size will be denoted as \( N_{\text{bas}} \). Correspondingly the Matsubara Green’s function (MGF) \( G(\tau) \) and self-energy \( \Sigma(\tau) \) are matrices related by the Dyson equation

\[
G(\tau) = g(\tau) + \int_{-\beta}^{0} d\tau_1 \int_{-\beta}^{0} d\tau_2 g(\tau - \tau_1) \Sigma(\tau_1 - \tau_2) G(\tau_2). \quad (1)
\]

Here \( g(\tau) \) is the reference Green’s function

\[
g_{ij}(\tau) = \delta_{ij} \left[ (n_i - 1) \theta(\tau) + n_i \theta(-\tau) \right] e^{-\left(\epsilon_i - \mu\right)\tau}, \quad (2)
\]

with \( n_i = n_F(\epsilon_i - \mu) \) \((n_F(\omega) \equiv (1 + e^{\beta\omega})^{-1}) \) denotes the Fermi distribution function, and \( \epsilon_i \) standing for the HF eigenvalues. Introducing the Coulomb matrix elements

\[
(i|k) = \int d\mathbf{r} d\mathbf{r'} \phi_i^{*}(\mathbf{r}) \phi_k^{*}(\mathbf{r}) \phi_j(\mathbf{r}) \phi_k(\mathbf{r'}),
\]

the constituent second order self-energy can be efficiently computed using the matrix multiplication:

\[
\Sigma_{ij}(\tau) = \sum_{klmpq} \left\{ (ik|mq)(lj|pn) \left[ 2G_{kl}(\tau) G_{mn}(\tau) G_{pq}(-\tau) + G_{kt}(\tau) G_{mn}(\tau) G_{pl}(-\tau) \right] \right. \quad (3)
\]

While this does not change the complexity proportional to the sixth power of the number of basis functions \( O(N_{\text{bas}}^6) \), the use of specialized libraries and parallelization allows to achieve a substantial speed up (the brute force approach leads to \( O(N_{\text{bas}}^8) \) scaling). Other possibility to increase the performance would be to use the finite-element discrete variable representation, which was shown to lead to \( O(N_{\text{bas}}^4) \) scaling.

The Dyson equation (1) is typically solved by Fourier-transforming to imaginary frequencies \( G(\tau) \to \tilde{G}(i\omega_n) \),

\[
\tilde{G}(i\omega_m) = \int_0^\beta d\tau G(\tau) e^{i\omega_n\tau}, \quad \omega_m = \frac{(2m + 1)\pi}{\beta}. \quad (4)
\]
yielding an algebraic Dyson equation. The self-energy is, however, most efficiently evaluated in \( \tau \)-space. Hence, switching back and forth between time and frequency representation is the standard implementation in the self-consistency cycle. Due to the non-continuous behavior of the MGF at \( \tau = 0 \), the Fourier coefficients \( G(i\omega_m) \) behave as \( (i\omega_m)^{-1} \) for large \( |m| \). This slow convergence introduces significant numerical errors which are countered by tail corrections. However, the standard first-order correction scheme still requires a typical number of thousands of frequencies \( i\omega_m \) to achieve accurately converged results. Higher-order tail corrections is a promising perspective to improve the efficiency of this solution scheme.

An alternative approach is to solve the Dyson equation directly as integral equation. By replacing the integration over the imaginary time arguments by a suitable quadrature with points \( \tau_p \) and weights \( w_p \), the integral equation (1) is recasted into a system of linear equations:

\[
\sum_{q=1}^{N_{\text{quad}}} \left[ I_{pq} - w_q \mathbf{Z}(\tau_p, \tau_q) \right] G(\tau_p) = \mathbf{g}(\tau_p) \tag{5}
\]

with integral kernel

\[
\mathbf{Z}(\tau, \tau') = \int_0^\infty d\tau'' g(\tau - \tau'') \Sigma(\tau'' - \tau') \tag{6}
\]

Combining \( N_{\text{bas}} \) basis and \( N_{\text{quad}} \) grid indices into a multi-index, Eq. (5) is transformed into a \( N_{\text{bas}} \times N_{\text{quad}} \)-dimensional system of linear equations with \( N_{\text{bas}} \) right-hand sides. Solving the Dyson equation directly as integral equation yields a solution free of high-frequency artefacts. By evaluating eq. (5) and (6) by higher-order quadrature schemes, we obtain a highly accurate solution even for moderate number of grid points \( \tau_p \).

The numerical bottleneck of the method is the additional computational cost of constructing the kernel (6). However, its calculation can be efficiently incorporated into a distributed memory scheme for solving the linear equation (5), giving rise to excellent scaling with the number of processing cores. We remark that the MGF obtained by the solving the frequency space can also be used provide a good initial guess for \( G(r) \) to be inserted in the right-hand side of eq. (1). Constructing an improved MGF from the left-hand side and substituting back into the convolution on the right-hand side constitutes an iterative solution of the Dyson equation.

In order to deal with cusps at the boundaries \( \tau = -\beta \) and \( \tau = 0 \) (Fig. 2) while retaining a compact representation of the MGF, we employ a grid \( \tau_p \) with exponentially increased density at the boundaries (see Fig. 2). The exponential scaling is optimized to best represent the (noninteracting) MGF corresponding to highest molecular orbital (HOMO). For achieving converged results, we typically need \( N_{\text{quad}} = 200 \) to \( N_{\text{quad}} = 300 \) grid points.

We have implemented three different levels of self-consistency at which the Matsubara GF is determined:

(i) the non-self-consistent (non-sc) treatment, where the Dyson equation is solved only once using the self-energy constructed from the reference Hartree-Fock (HF) Green’s function:

(ii) the partially self-consistent scheme where only the mean-field part of the Hamiltonian is updated (HF-sc) until the convergence on the density matrix;

(iii) the fully self-consistent scheme (full-sc), where the Dyson equation is solved and the self-energy is constructed repeatedly until the convergence on the Green’s function is achieved.

Direct extraction of the spectral function \( A_{ik}(\omega) \) from the MGF amounts to solving the integral equation

\[
G(\tau) = \int \frac{d\omega}{2\pi} A(\omega) \frac{e^{-\tau\omega}}{1 - e^{-\beta\omega}}, \tag{7}
\]

which is a nontrivial task. That is why we explore the analytic continuation and the extended Koopman theorem routes.

a. Analytic continuation transforms the Green’s function of the imaginary time argument \( \tau \) into the function of complex frequency \( \zeta \) in a sequence of two steps \( G(\tau) \rightarrow \tilde{G}(i\omega_m) \rightarrow \tilde{G}(\zeta) \). For \( \zeta \) in the vicinity of real axis, the latter quantity relates to retarded/advanced GFs \( \tilde{G}(\omega \pm i\eta) = G^{R/A}(\omega + \mu) \) and yields the spectral function according to

\[
A(\omega) = i \left[ \tilde{G}(\omega - \mu + i\eta) - \tilde{G}(\omega - \mu - i\eta) \right]_{\eta \rightarrow 0^+}.
\]

For equidistant grids in imaginary time, fast Fourier transformation to the imaginary frequency domain is the standard procedure. However, for our efficient solution scheme of the Dyson equation — which relies on an optimized non-equidistant grid of \( \tau \)-points — it is more efficient to employ an orthogonal polynomial representation. The Fourier coefficients of the Matsubara GF (4) are computed by representing the function in terms of Legendre polynomials \( P_n(x) \):

\[
G(r) = \frac{1}{\beta} \sum_{n=0}^{\infty} \sqrt{2n+1} P_n \left( \frac{2r}{\beta} - 1 \right) C_n, \tag{8}
\]

yielding

\[
\tilde{G}(i\omega_m) = (-1)^m \sum_{n=0}^{\infty} i^{n+1} \sqrt{2n+1} j_n \left( \frac{1}{\beta} \omega_m \right) C_n. \tag{9}
\]

Here \( j_n(x) \) denotes the spherical Bessel function of first kind.

On the second step, the complex function \( \tilde{G}(\zeta) \) is represented by its Padé approximant, constructed from the points \( i\omega_m \). In practice, the order of the Legendre polynomials is truncated at \( \approx 48 \), yielding excellent accuracy. The Padé plays only an minor role.

b. Extended Koopman’s theorem One quantity immediately available from MGF is the density matrix \( \gamma^{\pm} = \lim_{\omega \rightarrow 0^{\pm}} G(r) \) (upper/lower sign for particle/hole density, respectively). Computation of the quasiparticle excitations additionally requires the two-body correlation function as encoded in the first derivative \( \Delta^\pm = -\lim_{\omega \rightarrow 0^{\pm}} \partial_r G(r) \). With these two ingredients a generalized eigenvalue problem

\[
\Delta^\pm u_a = \epsilon_a^\pm \gamma^\pm u_a. \tag{10}
\]
values are in eV. (f) shows the mean average error (MAE) for each method.

yields the quasiparticle energies $e_{s}^{\pm}$ and the corresponding Dyson orbitals $u_{s}^{\pm}$ representing the spectral function as

$$A(\omega) = 2\pi \sum_{i=\pm} \sum_{a} |u_{i}^{a}|^{2} \delta(\omega - e_{i}^{a}). \quad (11)$$

Let us remark on the relation of analytic continuation and the EKT. Provided one has found an exact MGF (by exact diagonalization, for instance), the EKT reproduces (in the limit $\beta \to \infty$) the exact many-body energies. The same is true for analytic continuation. At the level of finite-order MBPT, the relation is less clear. Low-order diagrammatic methods such as the 2BA or the $GW$ approximation result in additional features like satellites and broadening, which cannot be captured by the EKT. A typical example of the different behavior of the simple exponential behavior of the MGF implied by the EKT and the self-consistent solution is shown in Fig. 2. One can expect that in case the effects of the 2BA is primarily given inconsistent results if the above mentioned features of MBPT come into play. For this reason, we employ both methods for obtaining quasiparticle properties and compare them.

III. ELECTRONIC PROPERTIES OF G2 MOLECULES

For a comprehensive benchmark, we study all 36 neutral closed-shell molecules from the G2-1 test set with geometries optimized at the B3LYP/6-31+G** level. The restricted Hartree-Fock (HF) calculation is performed using the aug-cc-pVDZ basis as the starting point, all the matrix elements are transformed from the atomic to molecular orbital basis using our in-house code. In this basis the Dyson equation (1) is subsequently solved for the low-temperature case $\beta = 80$ and spectral properties are determined. We tested the convergence of the results with respect to the basis size by introducing a cut-off energy $E_{\text{cut}}$ such that $\epsilon_{i} < E_{\text{cut}}$ for all states $i$. After some tests we fixed $E_{\text{cut}} = 1.5$ a.u., which yields excellent convergence. The typical size of the linear system of equations (5) is 10000-25000.

For comparison, the second-order Møller-Plesset perturbation theory (MP2) and the coupled-cluster method including single and double excitations (CCSD) is used. Using these two reference methods, the total energy of the neutral and the positively/negatively charged ions were computed, yielding accurate estimates to the vertical ionization potential (IP) and the electron affinity (EA) according to the energy difference method. It should be noted, however, that for some molecules the underlying HF calculation suffers from multi-configuration instabilities. In such cases, the HF ground state of the neutral or ionized system differs significantly from the true electronic state. We will come back to this point later.

A. Ionization potentials and electron affinities

In order to assess the performance of the 2BA, as compared to the reference methods, we computed the IPs and compared them to the experimental values in Fig. 3. The EKT (10) was used to extract the IPs from the MGF. Generally, the 2BA provides a quite accurate picture. Typical deviations from experimental values, which occur within MP2 and CCSD, as well, are not cured by the 2BA. This can be related to the
above mentioned multi-configuration problems. In principle, these deficiencies can be rectified by starting from a multi-configurational HF to construct the reference GF $g(\tau)$. As can already observed from the distribution of the IPs in Fig. 3(a)–(c), the non-sc scheme severely overestimates the IPs. Comparing with the initial restricted Hartree-Fock (RHF) values (which are mostly located under the diagonal), the non-sc treatment moves most of the points up and thus "overshoots" the QP shifts. The HF-sc level, on the other hand, yields much better results, which can be seen from the small distance of the points from the diagonal. Visually, the predictions of the IPs by the HF-sc scheme is very similar to the MP2 or CCSD reference. Switching to full self-consistency, Fig. 3(c), the values are slightly deteriorating with respect to the HF-sc level. Such oscillatory behavior of the MBPT and the levels of self-consistency is very typical. Similar behavior is also known for the $GW$ approximation, where partly self-consistent schemes such as $GW_0$ or quasiparticle self-consistency are typically superior to full-sc treatment.

For a quantitative analysis, we computed the mean average error (MAE) for each of the methods (Fig. 3(f)). As inferred from Fig. 3(f), the 2BA on non-sc level is not even better than the RHF (IPs from Koopman’s theorem), because of the overestimated QP shifts, while the accuracy of the HF-sc scheme is comparable to the MP2 method. The quality of the full-sc treatment is on the intermediate level between the RHF and the MP2. In principle, the 2BA is expected to perform similarly to MP2, as both methods are of the second order in the Coulomb interaction. Due to the oscillatory nature of MBPT theory\textsuperscript{30}, however, the partially self-consistent (HF-sc) level performs the best.

So far the IPs were computed using the EKT. As the next step we compared them to the IPs extracted from the analytic continuation accomplished by the Padé approximation (Fig. 4). While both methods yield identical values for non-sc case, there some small deviations for the HF-sc and the full-sc level.

The situation changes substantially for the electron affinities (Fig. 5). For a large class of molecules, the EKT provides a good estimate of the EA (we take CCSD as the reference). However, for some molecules (CH\textsubscript{2}, SiH\textsubscript{2}, Li\textsubscript{2}, F\textsubscript{2}, CO\textsubscript{2}, Na\textsubscript{2}, P\textsubscript{2}, Cl\textsubscript{2}, SiO, CIF, SO\textsubscript{2}) the EA obtained by the EKT applied to the 2BA (HF-sc level) is very different from the reference. These discrepancies are reminiscent of the errors of Koopman’s theorem (KT) for the EAs within RHF. In fact, the EKT gives only small QP shifts from the initial RHF energy levels entering the reference MGF $g(\tau)$. Hence, the EAs differ only little from $-e_{LUMO}$. The above molecules are typical cases where $-e_{LUMO}$ is a poor estimate for the EA (even within the RHF). Fig. 5 demonstrates that this behavior transfers to the EKT: the molecules where the KT prediction differs substantially from the more accurate estimation based on the total energy differences (the so-called $\Delta HF$ method) are iden-
tical to those where the EA obtained by the EKT is quite off the CCSD reference value. However, employing the Padé approximation yields a substantial improvement, as the EAs obtained within the 2BA are much closer to the CCSD values. In particular, except for the $F_2$ and $P_2$ dimers, the Padé approximation always reproduces the correct sign of the respective EAs. In cases where the modulus of the EAs is underestimated as compared to CCSD, taking the second EAs (i.e. the second QP peak) leads to almost perfect agreement. This is a clear indication of the multi-configurational instability of the ground state of either the neutral or the negatively charged molecule. We remark such deficiencies of the HF starting MGF can, in principle, be overcome by the full-sc treatment (as the dependence on the starting points disappears). However, converging the Dyson equation towards self-consistency can be hindered by the multi-valuedness of the solution.\textsuperscript{51,52}

Since several factor (besides the multi-configurational stability) contribute to the IPs and EAs measured in experiments, accurate methods like the CCSD can, of course, not yield perfect results. Most importantly, the restricted Gaussian basis set does describe excited orbitals well. In order to compare the methods on equal grounds, we show the IPs, EAs, and the resulting QP gap of the 2BA directly vs. the CCSD in Fig. 6. As for the IPs extracted by the EKT (Fig. 3), one can infer that the HF-sc scheme performs the best throughout; the agreement of the gaps between CCSD and the 2BA is especially good. It does not relies on the errors cancellation for the electron affinities (as illustrated in Fig. 5, see important exceptions) and ionization potentials, but is separately achieved for each quantity. Fig. 6 confirms that the 2BA on HF-sc level is almost comparable to the CCSD method.

IV. CONCLUSIONS

We performed benchmark calculations using the second Born approximation for the electron self-energy of a number of molecular systems and found overall very good performance of the Matsubara Green’s function approach in comparison with correlated quantum chemistry methods. For all methods, the error is substantially smaller if systems with multi-configurational ground state are excluded. Same is true for $GW$ method.\textsuperscript{53}

The partially self-consistent scheme has been demonstrated to perform the best throughout with a predictive power on par with quantum chemistry methods. Nevertheless, the fully self-consistent scheme performs very well for the majority of molecules, too. This is an important requirement for performing time-dependent calculations, allowing to compute, for instance, accurate optical absorption spectra. An alternative possibility would be to completely eliminate the Matsubara step and exploit the adiabatic switching scenario that can be further facilitated by the use of generalized Kadanoff-Baym Ansatz.\textsuperscript{54-57}

For extracting the quasiparticle properties we adopted two methods: the EKT and analytic continuation. They yield almost identical results for electron removal energies, the EKT was found to suffer from similar deficiencies as the KT within HF theory for around one third of the investigated molecules. This drawback can be cured by analytic continuation which yields excellent results for both ionization potentials and electron affinities.

As is mentioned in the introduction, there are several implementations of 2BA for systems ranging from atoms,\textsuperscript{36} to Hubbard,\textsuperscript{25,27} and Anderson,\textsuperscript{58} models, to ultracold gases,\textsuperscript{59} and periodic systems,\textsuperscript{60,61} also addressing the question of the accuracy of such calculations. Our study focuses on one missing aspect of such studies, namely the performance of the method for molecular system, as a quick way to initialize the time-dependent propagation of the Kadanoff-Baym equations. To this end, we specifically focused on the inherent accuracy of 2BA. For improving the general predictive power, two key issues have to be adressed: an accurate solution of the Dyson equation and systematic improvements of the basis. This first requirement is fulfilled by our efficient solution scheme, based on a compact grid representation of the MGF. Further improvements in this regard can be expected by working directly in the basis of orthogonal polynomials or an optimized sparse representation,\textsuperscript{62} which would allow studying considerably larger systems or higher-order MBPT. Promising routes speeding up the calculation for larger systems while keeping a accurate single-particle basis is the already mentioned use of specialized basis to represent the self-energy\textsuperscript{13} or stochastic sampling method.\textsuperscript{63}

Finally we notice that similar to the quantum chemistry and solid-state case,\textsuperscript{17} the explicitly correlated R12/F12 approaches are expected to recover even a larger portion of the correlation energy in the Matsubara Green’s function method. This idea opens new prospects for further investigations.

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The global error of our solution scheme scales as $O(\Delta r^4)$, where $\Delta r$ is the typical grid spacing.

This method is implemented in the computer codes of Martin Eckstein and Philipp Werner.

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