The stability analysis of an SVEIR model with continuous age-structure in the exposed and infectious classes

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\textit{(Received 14 October 2014; accepted 8 January 2015)}

In this paper, an Susceptible-Vaccines-Exposed-Infectious-Recovered model with continuous age-structure in the exposed and infectious classes is investigated. These two ages are assumed to have arbitrary distributions that are represented by age-specific rates leaving the exposed and the infectious classes. We investigate the global dynamics of this model in the sense of basic reproduction number via constructing Lyapunov functions. The asymptotic smoothness of solutions and uniform persistence of the system is shown from reformulating the system as a system of Volterra integral equations.

\textbf{Keywords:} age-structured model; asymptotic smoothness; uniform persistence; global stability; Lyapunov function

1. Introduction

Since the foundation works of Sharpe and Lotka \cite{19} and McKendrick \cite{17}, significant progress has been made for a partial differential equations approach to modelling continuous age structure in an evolving population. Due to the fact that nonlinear age structured models can be viewed as a dynamical system in a state space, nonlinear Volterra integral equations approach and the method of semigroups of linear and nonlinear operators have been frequently used to analyse the nonlinear models from both theoretical developments and biological applications viewpoints. We refer the reader to the nice books and survey for this topic \cite{1,2,5–7,11,13,22,23,26}.

It is of interest from both mathematical and biological viewpoints to investigate whether the changing of dynamical behaviour in population models are the result of nonlinear incidence rate, intracellular delays, or age structure. Recently, some advances have been made in investigating dynamics on Susceptible-Exposed-Infectious-Recovered (SEIR) epidemic model, where an exposed compartment has been incorporated into the classic Susceptible-Infectious-Recovered (SIR) model representing the individuals infected but not infectious. We would like to mention a recent work of Röst and Wu \cite{18} in the year of 2008. They formulated an SEIR model including infected individuals with infection-age structure to allow for varying infectivity. By

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reformulating the model as an infinite delay differential equations, the authors established that asymptotic smoothness, persistence, and local asymptotic stability of the disease-free equilibrium and endemic equilibrium in terms of basic reproduction number, but no analytic proof is given to obtain the global stability of endemic equilibrium. In 2009, McCluskey \[15\] have provided a confirmed answer to open problem left in \[18\], that is, the endemic equilibrium is globally asymptotically stable whenever it exists.

In a subsequent work, Wang et al. \[25\] develop model studied in \[15,18\] to an Susceptible-Vaccines-Exposed-Infectious-Recovered (SVEIR) epidemiological model with infinite delay to account for varying infectivity and vaccination, by introducing vaccination compartment. It is assumed in \[8–10,25\] that before obtaining immunity, the vaccinees still have the possibility of infection while contacting with infected individuals, and the vaccinating individuals may have some partial immunity during the vaccination process or they may recognize the transmission characters of the disease and hence decrease the effective contacts with infected individuals. As similar arguments in \[18\], due to the evolution of infected individuals takes the form

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i(t,a) = -\left( d + \delta + \gamma \right) i(t,a),
\]

which subject the boundary condition \(i(t,0) = \mu E(t)\), the density of individuals with infection-age \(a\) at time \(t\) is given by

\[
i(t,a) = \mu e^{-\left( d + \delta + \gamma \right) a} E(t-a).
\]

It is this assumption that simplifies significantly the analysis of the models of \[18,25\] by turning age structure to be infinite delay differential equations. By employing direct Lyapunov method and LaSalle’s invariance principle, the threshold scenario for the global asymptotic stability of equilibria are completely determined by the basic reproduction number \(R^V_0\). Mathematical results suggest that vaccination is helpful for disease control by decreasing the basic reproduction number. However, there is a necessary condition for successful elimination of disease.

Mathematically, the models studied in \[18,25\] was the case that the exit rate of infectious class is a constant, but not a function of the time spent in that class. It is more realistic situation that the distribution of waiting times in the exposed class and in the infectious class should be formulated by allowing the activation and exit rate of exposed and infectious class to depend on time spent in that class, respectively. This leads to a partial differential equation (PDE) formulation \[26\]. It is very important to highlight the fact that the infectivity of infectious individuals is also different at the differential age of infection. It is also known that variability with time-since-infection in host infectivity can cause qualitative changes in the dynamics of infectious diseases \[24\].

In 2012, McCluskey \[16\] investigated a model of disease transmission with continuous age-structure SEIR model for latently infected individuals and for infectious individuals. The asymptotic smoothness of the orbit generated by system and uniform persistence are proven by reformulating the system as a system of Volterra integral equations. The global stability scenario depending on basic reproduction number are obtained by constructing suitable Lyapunov functions.

In this paper we develop model studied in \[25\] to a more general case, where continuous age-structure for both the exposed and the infectious classes is included, that is, a two-dimensional age-structured variable is used. One can describe risk of activation as a function of \(a\), allowing more generality in the distribution of waiting times or latency periods. The other can describe the exit rate to be a function of the time spent in that class. The model to be studied in this paper is the following initial-boundary-value problem for a nonlinear system of ordinary and PDEs:

\[
\frac{dS(t)}{dt} = \Lambda - \mu S(t) - \alpha S(t) - \beta_1 S(t) \int_0^\infty k(a)i(t,a) \, da,
\]
\[
\frac{dV(t)}{dt} = \alpha S(t) - (\mu + \gamma_1)V(t) - \beta_2 V(t) \int_0^\infty p(a)i(t,a) \, da,
\]

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) e(t,a) = -\theta(a)e(t,a),
\]

\[
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i(t,a) = -\delta(a)i(t,a),
\]

\[
\frac{dR(t)}{dt} = \gamma_1 V(t) + \int_0^\infty \delta(a)i(t,a) \, da - \mu R(t),
\]

with boundary and initial conditions

\[
e(t,0) = \beta_1 S(t) \int_0^\infty k(a)i(t,a) \, da + \beta_2 V(t) \int_0^\infty p(a)i(t,a) \, da,
\]

\[
i(t,0) = \int_0^\infty \xi(a)e(t,a) \, da,
\]

\[
S(0) = S_0 \geq 0, \quad V(0) = V_0 \geq 0, \quad R(0) = \varphi_R \geq 0,
\]

\[
e(0,a) = \varphi_e(a) \in L_+^1(0,\infty), \quad i(0,a) = \varphi_i(a) \in L_+^1(0,\infty),
\]

where the variables \(S(t)\), \(V(t)\), \(R(t)\) represent the numbers of susceptible, vaccinees, and recovered at time \(t\), respectively. The parameters of model (3) are biologically explained as in Table 1.

\[
e(t,a) \text{ represents the density of exposed individuals with age of latency } a \text{ at time } t. \text{ Thus the number of exposed individuals within the exposed subpopulation at time } t \text{ is } E(t) = \int_0^\infty e(t,a) \, da.
\]

\[
i(t,a) \text{ represents the density of infectious individuals with age of infection } a \text{ at time } t. \text{ Thus the number of infectious individuals within this subpopulation at time } t \text{ is } I(t) = \int_0^\infty i(t,a) \, da.
\]

Individuals who have been in the exposed class for duration \(a\) are removed at rate \(\theta(a)\). Individuals who have been in the infectious class for duration \(a\) are removed at rate \(\delta(a)\) and infect susceptibles with the scaled probability of infection \(k(a)\) and vaccinees with the scaled probability of infection \(p(a)\). \(\xi(a)\) is the rate of individuals who have been in the exposed class for duration \(a\), progress to class \(i\).

Since the variable \(R(t)\) does not appear in the first four equations of (3), we can consider the following reduced system

\[
\frac{dS(t)}{dt} = \Lambda - \mu S(t) - \alpha S(t) - \beta_1 S(t) \int_0^\infty k(a)i(t,a) \, da,
\]

\[
\frac{dV(t)}{dt} = \alpha S(t) - (\mu + \gamma_1)V(t) - \beta_2 V(t) \int_0^\infty p(a)i(t,a) \, da,
\]

Table 1. Parameters and their biological meaning in model (3).

| Parameter | Interpretation |
|-----------|----------------|
| \(\Lambda\) | Constant recruitment rate; |
| \(\beta_1, \beta_2\) | The baseline transmission rates for susceptibles or vaccinees, respectively. |
| \(\mu\) | Natural death rate; |
| \(\alpha\) | The rate at which susceptible individuals are moved into the vaccination process; |
| \(\gamma_1\) | The average rate for vaccinees to obtain immunity and move into recovered population |

Note: All these constants are assumed to be positive.
subject to the boundary and initial conditions

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) e(t, a) &= -\theta(a)e(t, a), \\
\left( \frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) i(t, a) &= -\delta(a)i(t, a),
\end{align*}
\]

(5)

The main objective of the paper is to prove that the global results are also true for the continuous age case in the sense of basic reproduction number. Because the continuous age model is described by first order PDEs, it is not easy to analyse the dynamics of the PDE models, particularly the dynamical behaviour of global stability of equilibria. In this paper, by using a class of global Lyapunov functions we prove that two-dimensional age-structured SVEI model exhibits the traditional threshold behaviour as in [25]. Thus the global stability results obtained for distributed delay system are extended to the general continuous age-structure model. For the proof, a main technical tool to be used in our study is the Lyapunov functional on the global attractor for semi-flow restricted to the interior region developed by the functionals used in [4,12,14,16]. We first prove that the semi-flow generated by system is positive invariant and asymptotically smooth. The uniform persistence property is proved by applying Theorem 4.2 in Hale and Waltman [3]. We can obtain that there exists a global attractor. We also refer the reader to a recent advances for more results about this topic.

The remaining part of this paper is organized as follows. In Section 2, we reformulate the system as a Volterra integral equations and introduce some basic results, including state space, assumptions, equilibria, basic reproduction number and boundedness of the solutions. Section 3 is devoted to prove that system (5) is asymptotically smooth. In Section 4, we show that the disease is uniformly persistent if the basic reproduction number is greater than one. In Sections 5 and 6, we give the proof of global stability of equilibria in the sense of basic reproduction number and endemic equilibrium is globally asymptotically stable when it exists.

2. Preliminaries

We first list the assumption that the basic functions \(k(a), p(a), \theta(a), \delta(a), \xi(a)\) are supposed to fulfill in order to be biologically significant and to allow the mathematical treatment of (5) with (6).

**Assumption 2.1** We assume that

(i) \(k(a), p(a), \theta(a), \delta(a), \xi(a)\) are non-negative and belong to \(L_+^\infty(0, \infty)\) with respective essential upper bounds \(\bar{k}, \bar{p}, \bar{\theta}, \bar{\delta}, \bar{\xi}\) \(\in (0, \infty)\);

(ii) \(k(a), p(a), \xi(a)\) are Lipschitz continuous on \(\mathbb{R}_+\) with coefficients \(M_k, M_p, M_\xi\), respectively;

(iii) There exist positive constants \(a_k, a_p, a_\xi \in (0, \infty)\) such that \(k(a), p(a), \xi(a)\) are positive in some neighbourhoods of \(a_k, a_p, a_\xi\), respectively;
(iv) There exists a positive constant $\mu_0 \in (0, \mu]$ such that $\theta(a) - \xi(a) \geq \mu_0$ and $\delta(a) \geq \mu_0$ for all $a > 0$;
(v) For some $a > 0$, $\int_0^a k(a - \eta)\xi(\eta) \, d\eta > 0$.

2.1. State space

System (5) should be studied in a suitable phase space. Following the line of [26], we set the phase space for system (5)–(6) as

$$\mathcal{Y} = \mathbb{R}_+ \times \mathbb{R}_+ \times L^1_+(0, \infty) \times L^1_+(0, \infty)$$

with norm

$$\|(x, y, \varphi, \phi)\|_{\mathcal{Y}} = |x| + |y| + \int_0^\infty |\varphi(a)| \, da + \int_0^\infty |\phi(a)| \, da.$$

Then, the initial values of system (5) are taken to be included in $\mathcal{Y}$:

$$(S(0), V(0), e(0, a), i(0, a)) = (S_0, V_0, \varphi_e(a), \varphi_i(a)) \in \mathcal{Y}.$$  

In what follows, we consider a continuous semi-flow $\Phi_t : \mathcal{Y} \to \mathcal{Y}$, $t \geq 0$ defined by the solution of system (5)–(6), that is,

$$\Phi_t(X_0) := X(t) = (S(t), V(t), e(t, \cdot), i(t, \cdot)) \in \mathcal{Y}, \quad t \geq 0,$$

where $X_0 := (S_0, V_0, \varphi_e(a), \varphi_i(a)) \in \mathcal{Y}$ is the initial condition. We define the state space for system (5) as

$$\Upsilon := \left\{ (S(t), V(t), e(t, \cdot), i(t, \cdot)) \in \mathcal{Y} : 0 \leq S(t) + V(t) + \int_0^\infty e(t, a) \, da + \int_0^\infty i(t, a) \, da \leq \frac{1}{\mu_0} \right\},$$

which is the positive invariant set for system (5).

2.2. Notations

For ease of notation and mathematical tractable, denote

$$\Theta(a) = e^{-\int_0^a \theta(t) \, dt} \quad \text{and} \quad \Gamma(a) = e^{-\int_0^a \delta(t) \, dt}. \quad (8)$$

From the expressions of $\Theta(a)$ and $\Gamma(a)$, it is easy to see from the (i) and (iv) of Assumption 2.1 that for all $a \geq 0$,

$$0 \leq \Theta(a), \quad \Gamma(a) \leq e^{-\mu_0 a},$$

$$\Theta'(a) = -\theta(a) \Theta(a), \quad \Gamma'(a) = -\delta(a) \Gamma(a). \quad (9)$$

Furthermore, in what follows, we still denote

$$A = \int_0^\infty k(a) \Gamma(a) \, da, \quad B = \int_0^\infty p(a) \Gamma(a) \, da, \quad C = \int_0^\infty \xi(a) \Theta(a) \, da. \quad (10)$$

It can be verified that $A, B$ and $C$ are positive and finite. It is useful to denote

$$P(t) = \int_0^\infty k(a) i(t, a) \, da, \quad Q(t) = \int_0^\infty p(a) i(t, a) \, da, \quad M(t) = \int_0^\infty \xi(a) e(t, a) \, da,$$

for $t \geq 0$, for proving the asymptotic smoothness of the semi-flow $\Phi$ in the later section.
2.3. Volterra formulation

The third and fourth equation of system (5) can be reformulated as a Volterra equation by use of Volterra formulation (see Webb [26] and Iannelli [5]). By integrating the terms \( e(t, a) \) and \( i(t, a) \) along the characteristic line \( t - a = \text{const.} \), respectively, they read as the following expressions:

\[
e(t, a) = \begin{cases} 
(\beta_1 S(t - a) P(t - a) + \beta_2 V(t - a) Q(t - a)) \Omega(a) & \text{for } 0 \leq a \leq t; \\
\varphi_e(a - t) \frac{\Omega(a)}{\Omega(a - t)} & \text{for } 0 \leq t \leq a,
\end{cases}
\]

and

\[
i(t, a) = \begin{cases} 
M(t - a) \Gamma(a) & \text{for } 0 \leq a \leq t; \\
\varphi_i(a - t) \frac{\Gamma(a)}{\Gamma(a - t)} & \text{for } 0 \leq t \leq a.
\end{cases}
\]

Thus system (5) can be written as the following Volterra type equations

\[
\frac{dS(t)}{dt} = \Lambda - \mu S(t) - \alpha S(t) - \beta_1 S(t) \int_0^\infty k(a) i(t, a) \, da,
\]

\[
\frac{dV(t)}{dt} = \alpha S(t) - (\mu + \gamma_1) V(t) - \beta_2 V(t) \int_0^\infty p(a) i(t, a) \, da,
\]

\[
e(t, a) = \begin{cases} 
(\beta_1 S(t - a) P(t - a) + \beta_2 V(t - a) Q(t - a)) \Omega(a) & \text{for } 0 \leq a \leq t; \\
\varphi_e(a - t) \frac{\Omega(a)}{\Omega(a - t)} & \text{for } 0 \leq t \leq a,
\end{cases}
\]

\[
i(t, a) = \begin{cases} 
M(t - a) \Gamma(a) & \text{for } 0 \leq a \leq t; \\
\varphi_i(a - t) \frac{\Gamma(a)}{\Gamma(a - t)} & \text{for } 0 \leq t \leq a.
\end{cases}
\]

For the later use, we note that

\[
e(t, a) = e(t - a, 0) \Omega(a) \quad \text{and} \quad i(t, a) = i(t - a, 0) \Gamma(a) \quad \text{for } 0 \leq a \leq t.
\]

2.4. Equilibria and basic reproduction number

System (5) always has the disease free equilibrium \( E_0 = (S_0, V_0, 0, 0) \), where

\[
S_0 = \frac{\Lambda}{\mu + \alpha}, \quad V_0 = \frac{\alpha \Lambda}{(\mu + \alpha)(\mu + \gamma_1)}.
\]

The basic reproduction number for system (5)–(6) can be defined as the expected number of secondary cases produced by a typical infective individual during its entire period of infectiousness, which is given by

\[
R_0 := (\beta_1 S_0 \int_0^\infty k(a) \Gamma(a) \, da + \beta_2 V_0 \int_0^\infty p(a) \Gamma(a) \, da) \int_0^\infty \xi(a) \Omega(a) \, da \\
= (\beta_1 S_0 A + \beta_2 V_0 B) C.
\]

Now let us investigate the existence of endemic equilibrium of system (5). For any endemic equilibrium \( E^* = (S^*, V^*, e^*(a), i^*(a)) \in \mathcal{Y} \) of system (5)–(6), it should satisfy the following
Solving the third and fourth equations of (16) yields that
\[ e^*(a) = e^*(0) \Omega(a), \quad \beta^*(a) = i^*(0) \Gamma(a). \]
Putting it into the last two equation gives
\[ e^*(0) = (\beta_1 S^*A + \beta_2 V^*B)i^*(0), \quad i^*(0) = e^*(0)C, \]
where \( A, B, \) and \( C \) are defined in Equation (10). Multiplying these two equations of (17) yields
\[ (\beta_1 S^*A + \beta_2 V^*B)C = 1. \]
It follows from the first and second equations of (16) that
\[ S^* = \frac{\Lambda}{\mu + \alpha + \beta_1 Ai^*(0)}, \quad V^* = \frac{\alpha \Lambda}{(\mu + \alpha + \beta_1 Ai^*(0))\alpha + \beta_2 Ai^*(0)}; \]
Denote \( I^* = i^*(0) \). Plugging it into Equation (18) yields
\[ H(I^*) = a_0(I^*)^2 + a_1I^* + a_2 = 0, \quad \text{(19)} \]
where \( a_0 = \beta \beta_1 AB, \quad a_1 = \beta_1 A(\mu + \gamma_1) + \beta_2 B(\mu + \alpha) - \beta_1 \beta_2 \Lambda ABC, \quad a_2 = (\mu + \alpha)(\mu + \gamma_1) \).
Since \( a_0 > 0 \), it has \( G(\pm \infty) = +\infty \). When \( \eta_0 \leq 1 \), we know that \( H(0) \geq 0 \) and
\[ H'(I^*) = 2a_0I^* + a_1 = 2a_0I^* + \beta \beta_1 AB(\mu + \alpha) \left\{ \frac{\mu + \gamma_1}{(\mu + \alpha)B\beta_2} + \frac{1}{\beta_1 A} - \frac{\Lambda C}{\mu + \alpha} \right\}. \]
And \( \eta_0 \leq 1 \) is equivalent to
\[ \frac{\Lambda C}{\mu + \alpha} \leq \frac{\mu + \gamma_1}{(\mu + \alpha)B\beta_2 + \varepsilon}, \quad \text{where} \ \varepsilon = (\beta_1 A - \beta_2 B)\mu + \beta_1 A\gamma_1, \]
which implies that \( \Lambda C/(\mu + \alpha) < (\mu + \gamma_1)/(\mu + \alpha)B\beta_2 \). Therefore, \( H'(I^*) > 0 \) for any \( I^* \geq 0 \) when \( \eta_0 \leq 1 \). In this case, it is obvious that Equation (19) has not positive root.
On the other hand, when \( \eta_0 > 1 \), it has that \( H(0) = a_2 < 0 \). From the second order function properties of \( H(I^*) \), Equation (19) has a unique positive real root \( I^* \) (i.e., \( i^*(0) \)). So there exists only endemic equilibrium of system, \( E^* = (S^*, V^*, e^*(a), i^*(a)) \) if \( \eta_0 > 1 \).
2.5. **Boundedness**

For \( \Phi_t \) defined by Equation (7), we have the following proposition, which implies the boundedness of the system (5)–(6).

**Proposition 2.1** Let \( \Phi_t \) be defined by Equation (7), the following statements hold true:

(i) \( \frac{d}{dt} \| \Phi_t(X_0) \|_Y \leq \Lambda - \mu_0 \| \Phi_t(X_0) \|_Y \) for all \( t \geq 0 \);

(ii) \( \| \Phi_t(X_0) \|_Y \leq \max \{ \Lambda/\mu_0, \Lambda/\mu_0 + e^{-\mu_0 t} \|X_0\|_Y \} \leq \max \{ \Lambda/\mu_0, \|X_0\|_Y \} \) for all \( t \geq 0 \);

(iii) \( \limsup_{t \to \infty} \| \Phi_t(X_0) \|_Y \leq \Lambda/\mu_0 \);

(iv) \( \Phi_t \) is point dissipative: there is a bounded set that attracts all points in \( Y \).

**Proof** Note that

\[
\frac{d}{dt} \| \Phi_t(X_0) \|_Y = \frac{dS(t)}{dt} + \frac{dV(t)}{dt} + \frac{d}{dt} \int_0^\infty e(t,a) \, da + \frac{d}{dt} \int_0^\infty i(t,a) \, da.
\]

By Equation (11), we get

\[
\int_0^\infty e(t,a) \, da = \int_0^t (\beta_1 S(t-a)P(t-a) + \beta_2 V(t-a)Q(t-a)) \Omega(a) \, da \\
+ \int_t^\infty \varphi_c(a-t) \frac{\Omega(a)}{\Omega(a-t)} \, da.
\]

Taking the substitution \( \sigma = t-a \) and \( \tau = a-t \) in the first and second integral, respectively, and differentiating by \( t \), we get

\[
\frac{d}{dt} \int_0^\infty e(t,a) \, da = \frac{d}{dt} \int_0^t (\beta_1 S(\sigma)P(\sigma) + \beta_2 V(\sigma)Q(\sigma)) \Omega(t-\sigma) \, d\sigma + \frac{d}{dt} \int_0^\infty \varphi_c(\tau) \frac{\Omega(t+\tau)}{\Omega(\tau)} \, d\tau \\
= (\beta_1 S(t)P(t) + \beta_2 V(t)Q(t)) \Omega(0) + \int_0^\infty \varphi_c(\tau) \frac{\Omega(t+\tau)}{\Omega(\tau)} \, d\tau \\
+ \int_0^t (\beta_1 S(\sigma)P(\sigma) + \beta_2 V(\sigma)Q(\sigma)) \Omega'(t-\sigma) \, d\sigma.
\]

Notes that

\( \Omega(0) = 1 \) and \( \Omega'(a) = -\theta(a) \Omega(a) \)

and hence, we have

\[
\frac{d}{dt} \int_0^\infty e(t,a) \, da = \beta_1 S(t)P(t) + \beta_2 V(t)Q(t) - \int_0^\infty \theta(a)e(t,a) \, da.
\]

Similarly, we have

\[
\frac{d}{dt} \int_0^\infty i(t,a) \, da = \int_0^\infty \xi(a)e(t,a) \, da - \int_0^\infty \delta(a)i(t,a) \, da.
\]
Thus, from (iv) of Assumption 2.1, we can get

\[
\Phi_1 = \Lambda - \mu S(t) - \alpha S(t) - \beta_1 S(t) \int_0^\infty k(a) i(t, a) \, da + \alpha S(t) - (\mu + \gamma_1) V(t)
\]

\[
- \beta_2 V(t) \int_0^\infty p(a) i(t, a) \, da
\]

\[
+ \beta_1 S(t) P(t) + \beta_2 V(t) Q(t) - \int_0^\infty \theta(a) e(t, a) \, da + \int_0^\infty \xi(a) e(t, a) \, da
\]

\[
- \int_0^\infty \delta(a) i(t, a) \, da
\]

\[
= \Lambda - \mu S(t) - (\mu + \gamma_1) V(t) - \int_0^\infty (\theta(a) - \xi(a)) e(t, a) \, da - \int_0^\infty \delta(a) i(t, a) \, da.
\]

Thus, from (iv) of Assumption 2.1, we can get

\[
\frac{d}{dt} \| \Phi_1(X_0) \|_{\mathcal{Y}} \leq \Lambda - \mu_0 (S(t) + V(t)) - \mu_0 \int_0^\infty e(t, a) \, da - \mu_0 \int_0^\infty i(t, a) \, da
\]

\[
= \Lambda - \mu_0 \| \Phi_1(X_0) \|_{\mathcal{Y}}.
\]

Hence, it follows from the variation of constants formula that for \( t \geq 0 \),

\[
\| \Phi_1(X_0) \|_{\mathcal{Y}} \leq \frac{\Lambda}{\mu_0} - e^{-\mu_0 t} \left( \frac{\Lambda}{\mu_0} - \| \Phi_1(X_0) \|_{\mathcal{Y}} \right).
\]

The third and fourth statement directly follows. Therefore, \( \Phi_1 \) is point dissipative and \( \mathcal{Y} \) attracts all points in \( \mathcal{Y} \), which completes the proof.

The following propositions are direct consequences of Proposition 2.1, which is similar to [16, Proposition 2, 3].

**Proposition 2.2** If \( X_0 \in \mathcal{Y} \) and \( \| X_0 \|_{\mathcal{Y}} \leq K \) for some \( K \geq \Lambda / \mu_0 \), then the following hold for all \( t \geq 0 \):

1. \( S(t), V(t), \int_0^\infty e(t, a) \, da, \int_0^\infty i(t, a) \, da \leq K \);
2. \( P(t) \leq \tilde{k} K, Q(t) \leq \tilde{p} K \) and \( M(t) \leq \tilde{\xi} K \);
3. \( e(t, 0) \leq \tilde{\beta} \tilde{k}^2 \) and \( i(t, 0) \leq \tilde{\xi} K \) where \( \tilde{\beta} = \beta_1 \tilde{k} + \beta_2 \tilde{p} \).

**Proposition 2.3** Let \( C \in \mathcal{Y} \) be bounded, then

1. \( \Phi_1(C) \) is bounded;
2. \( \Phi_1 \) is eventually bounded on \( C \);

### 2.6. Main results

Let \( \mathfrak{n}_0 \) is defined by Equation (15). Now, we are ready to state the main results of this paper as follows.

**Theorem 2.1** If \( \mathfrak{n}_0 < 1 \), then the disease-free equilibrium \( E_0 \) is the unique equilibrium of system (5), and it is globally stable.
Theorem 2.2 If $R_0 > 1$, then the unique endemic equilibrium $E^*$ of system (5) is globally asymptotically stable.

The purpose of this paper is to study the global stability results of general continuous age-structure model. To achieve it, we need several necessary arguments in next two sections, including asymptotic smoothness, uniform persistence and compact attractor of the orbits of the solutions generated by system. The proof of Theorems 2.1 and 2.2 can be found in Sections 5 and 6.

3. Asymptotic smoothness

In what follows, we regard $\Phi$ as an operator on $\mathbb{R}_+ \times \mathcal{Y}$ satisfying

$$
\Phi(t, X) = \Phi_t(X), \quad t \in \mathbb{R}_+, \ X \in \mathcal{Y}.
$$

First, we introduce some lemmas which will be used in the proofs of the main results.

**Lemma 3.1** [16] Let $D \subseteq \mathbb{R}$. For $j = 1, 2$, suppose $f_j : D \to \mathbb{R}$ is a bounded Lipschitz continuous function with bound $K_j$ and Lipschitz coefficient $M_j$. Then the product $f_1 f_2$ is also Lipschitz continuous with coefficient $K_1 M_2 + K_2 M_1$.

By using Lemma 3.1, we can prove the following Lemma.

**Lemma 3.2** The functions $P(t), Q(t)$ and $M(t)$ are Lipschitz continuous on $\mathbb{R}_+$.

**Proof** Let $K$ be a positive constant such that $K \geq \max \{ \Lambda / \mu_0, \|X_0\|_{\mathcal{Y}} \}$. Then, by (ii) of Proposition 2.1, we have $\|X(t)\|_{\mathcal{Y}} \leq K$ for all $t \geq 0$.

For $t \geq 0$ and $h > 0$, we have

$$
P(t + h) - P(t) = \int_0^\infty k(a) i(t + h, a) \, da - \int_0^\infty k(a) i(t, a) \, da
$$

$$
= \int_0^h k(a) i(t + h, a) \, da + \int_h^\infty k(a) i(t + h, a) \, da - \int_0^\infty k(a) i(t, a) \, da
$$

$$
= \int_0^h k(a) i(t + h - a, 0) \Gamma(a) \, da + \int_h^\infty k(a) i(t + h, a) \, da - \int_0^\infty k(a) i(t, a) \, da.
$$

By applying $k(\tau) \leq \tilde{k}$, $i(t + h - a, 0) \leq \tilde{x} K$ and $\Gamma(a) \leq 1$ to the first integral and changing the variable in the second integral as $\sigma = a - h$, we have

$$
P(t + h) - P(t) \leq \tilde{k} \tilde{x} K h + \int_0^\infty k(\sigma + h) i(t + h, \sigma + h) \, d\sigma - \int_0^\infty k(a) i(t, a) \, da.
$$

It follows from Equation (14) that

$$
i(t + h, \sigma + h) = i(t, \sigma) \frac{\Gamma(\sigma + h)}{\Gamma(\sigma)}.
$$
Thus, we have
\[
P(t+h) - P(t) \leq \tilde{k}\tilde{\xi}Kh + \int_0^\infty \left( k(a+h) \frac{\Gamma(a+h) - k(a)}{\Gamma(a)} i(t,a) \right) da
\]
\[
= \tilde{k}\tilde{\xi}Kh + \int_0^\infty \left( k(a+h) e^{-\int_a^{a+h} \delta(r) dr} - k(a) \right) i(t,a) da
\]
\[
= \tilde{k}\tilde{\xi}Kh + \int_0^\infty k(a+h) \left( e^{-\int_a^{a+h} \delta(r) dr} - 1 \right) i(t,a) da
\]
\[+ \int_0^\infty (k(a+h) - k(a)) i(t,a) da.
\]

By (i) of Assumption 2.1, we have \(0 \geq -\int_a^{a+h} \delta(r) dr \geq -\tilde{\delta}h\). Thus,
\[
1 \geq e^{-\int_a^{a+h} \delta(r) dr} \geq e^{-\tilde{\delta}h} \geq 1 - \tilde{\delta}h.
\]
Furthermore,
\[
0 \leq k(a+h)|e^{-\int_a^{a+h} \delta(r) dr} - 1| \leq \tilde{k}\tilde{\delta}h
\]
and
\[
\int_0^\infty i(t,a) da \leq \|X(t)\|_Y \leq K.
\]

By using (ii) of Assumption 2.1, we have \(\int_0^\infty (k(a+h) - k(a))i(t,a) da \leq M_khK\) and hence,
\[
P(t+h) - P(t) \leq \tilde{k}\tilde{\xi}Kh + \tilde{k}\tilde{\delta}Kh + M_khK. \quad (21)
\]
This implies that \(P(t)\) is Lipschitz continuous with coefficient \(M_p = (\tilde{k}\tilde{\xi} + \tilde{k}\tilde{\delta} + M_k)K\). In a similar manner, we can prove the Lipschitz continuity of \(Q(t)\) and \(M(t)\).

Now we are in a position to state the main results of this section. To this end, we introduce the following theorem, which comes from Proposition 3.13 of [26].

**Theorem 3.1** The semi-flow \(\Phi : \mathbb{R}_+ \times Y \rightarrow Y\) is asymptotically smooth if there are maps \(\Theta, \psi : \mathbb{R}_+ \times Y \rightarrow Y\) s.t. \(\Phi(t,X) = \Theta(t,X) + \psi(t,X)\), and the following hold for any bounded closed set \(C\) that is forward invariant under \(\Phi\):

(i) \(\lim_{t \to \infty} \text{diam} \Theta(t,C) = 0\);
(ii) There exists \(t_C \geq 0\) such that \(\psi(t,C)\) has compact closure for each \(t \geq t_C\).

To verify the two conditions are fulfilled for system (5), we first to divide the semi-flow \(\Phi\) into two parts: For \(t \geq 0\), let \(\psi(t,X_0) = (\tilde{S}(t),\tilde{V}(t),\tilde{\epsilon}(t,\cdot),\tilde{i}(t,\cdot))\) and \(\Theta(t,X_0) = \)
(0, 0, \tilde{\phi}_e(t, \cdot), \tilde{\phi}_i(t, \cdot)), where

\[
\tilde{e}(t, a) = \begin{cases}
(\beta_1 S(t - a) P(t - a) + \beta_2 V(t - a) Q(t - a)) \Omega(a), & 0 \leq a \leq t; \\
0, & t < a;
\end{cases}
\]

\[
\tilde{i}(t, a) = \begin{cases}
M(t - a) \Gamma(a), & 0 \leq a \leq t; \\
0, & t < a;
\end{cases}
\]

\[
\tilde{\phi}_e(t, a) = \begin{cases}
0, & 0 \leq a \leq t; \\
\varphi_e(a - t) \frac{\Omega(a)}{\Omega(a - t)}, & t < a;
\end{cases}
\]

\[
\tilde{\phi}_i(t, a) = \begin{cases}
0, & 0 \leq a \leq t; \\
\varphi_i(a - t) \frac{\Gamma(a)}{\Gamma(a - t)}, & t < a.
\end{cases}
\]

Then, for \( t \geq 0 \), we have \( \Phi = \Theta + \Psi \). So we can estimate \( \|\tilde{\phi}_e(t, \cdot)\|_1, \|\tilde{\phi}_i(t, \cdot)\|_1 \) such that (i) of Theorem 3.1 holds true. To verify the condition (ii) of Theorem 3.1, we need a criterion for compactness in the state space of system (5).

**Theorem 3.2** A set \( C \in L^1_+(0, \infty) \) has compact closure if and only if the following conditions hold:

(i) \( \sup_{f \in C} \int_0^\infty f(a) \, da < \infty; \)

(ii) \( \lim_{r \to \infty} \int_r^\infty f(a) \, da \to 0 \) uniformly with respect to \( f \in C; \)

(iii) \( \lim_{h \to 0^+} \int_0^\infty |f(a + h) - f(a)| \, da \to 0 \) uniformly with respect to \( f \in C; \)

(iv) \( \lim_{h \to 0^+} \int_0^h f(a) \, da \to 0 \) uniformly with respect to \( f \in C. \)

In light of Lemmas 3.1, 3.2 and Theorems 3.1, 3.2 and these preparations, we will derive the following main theorem of this section.

**Theorem 3.3** The semi-flow \( \Phi \) generated by system (5) is asymptotically smooth.

**Proof** Let \( C \subset Y \) be a bounded subset of \( Y \) and \( K > \Lambda/\mu_0 \) be the bound for \( C \). We consider the solution \( \Phi(t, X_0) = (S(t), V(t), e(t, \cdot), i(t, \cdot)) \) with \( X_0 \in C \).

Let us denote the standard \( L^1 \)-norm by \( \| \cdot \|_1 \). Then,

\[
\|\tilde{\phi}_e(t, \cdot)\|_1 = \int_0^\infty |\tilde{\phi}_e(t, a)| \, da
\]

\[
= \int_t^\infty \varphi_e(a - t) \frac{\Omega(a)}{\Omega(a - t)} \, da
\]

\[
= \int_0^\infty \varphi_e(\sigma) \frac{\Omega(\sigma + t)}{\Omega(\sigma)} \, d\sigma
\]

\[
= \int_0^\infty \varphi_e(\sigma) e^{-\int_\sigma^{\sigma+t} \theta(\tau) \, d\tau} \, d\sigma
\]

\[
\leq e^{-\mu_0 t} \int_0^\infty \varphi_e(\sigma) \, d\sigma
\]

\[
\leq Ke^{-\mu_0 t}.
\]
This shows that \( \|\tilde{\phi}_e(t, \cdot)\|_1 \to 0 \) as \( t \to \infty \). Similarly, we can show \( \|\tilde{\phi}_i(t, \cdot)\|_1 \leq Ke^{-\mu_0 t} \) and hence, \( \|\tilde{\phi}_i(t, \cdot)\|_1 \to 0 \) as \( t \to \infty \). Consequently, \( \Theta(t, X_0) \) approaches 0 with exponential decay and therefore, \( \lim_{t \to \infty} \text{diam}(\Theta(t, C)) = 0 \) as required in Theorem 3.1.

Next we verify the (ii) of Theorem 3.1, and this can be done by verifying conditions (1)–(4) of Theorem 3.2.

It follows from (3) of Proposition 2.3, we know that \( S(t) \) and \( V(t) \) remain in the compact set \([0, K]\). Next, we show that \( \tilde{e}(t, a) \) and \( \tilde{i}(t, a) \) remain in a pre-compact subset of \( L^1_+ \) independent of \( X_0 \).

By Proposition 2.2 and (9), it is easy to see that

\[
\tilde{e}(t, a) \leq (\beta_1 \tilde{k} + \beta_2 \bar{p})K^2e^{-\mu_0 a}
\]

from which conditions (1), (2) and (4) of Theorem 3.2 follow directly. Now, it remains only to verify that condition (3) of Theorem 3.2 holds. Since we are interested in the limit as \( h \) tends to \( 0^+ \), we restrict our attention to \( h \in (0, t) \). Then,

\[
\int_0^\infty |\tilde{e}(t, a + h) - \tilde{e}(t, a)| \, da \\
= \int_0^{t-h} |(\beta_1 S(t - a - h)P(t - a - h) + \beta_2 V(t - a - h)Q(t - a - h))\Omega(a + h) \\
- (\beta_1 S(t - a)P(t - a) + \beta_2 V(t - a)Q(t - a))\Omega(a)| \, da \\
+ \int_{t-h}^{t} |0 - (\beta_1 S(t - a)P(t - a) + \beta_2 V(t - a)Q(t - a))\Omega(a)| \, da \\
\leq \int_0^{t-h} |(\beta_1 S(t - a - h)P(t - a - h) + \beta_2 V(t - a - h)Q(t - a - h))\Omega(a + h) \\
- (\beta_1 S(t - a)P(t - a) + \beta_2 V(t - a)Q(t - a))\Omega(a)| \, da + \bar{\beta}K^2h \\
\leq \bar{\beta}K^2h + \Delta + \Xi,
\]

where

\[
\Delta = \int_0^{t-h} (\beta_1 S(t - a - h)P(t - a - h) + \beta_2 V(t - a - h)Q(t - a - h))|\Omega(a + h) - \Omega(a)| \, da,
\]

and

\[
\Xi = \int_0^{t-h} |(\beta_1 S(t - a - h)P(t - a - h) + \beta_2 V(t - a - h)Q(t - a - h)) \\
- (\beta_1 S(t - a)P(t - a) + \beta_2 V(t - a)Q(t - a))|\Omega(a) \, da \\
\leq \int_0^{t-h} |(\beta_1 S(t - a - h)P(t - a - h) - (\beta_1 S(t - a)P(t - a))|\Omega(a) \, da \\
+ \int_0^{t-h} |(\beta_2 V(t - a - h)Q(t - a - h) - \beta_2 V(t - a)Q(t - a))|\Omega(a) \, da.
\]
Note that
\[
\int_0^{t-h} |\Omega(a + h) - \Omega(a)| \, da = \int_0^{t-h} (\Omega(a) - \Omega(a + h)) \, da
\]
\[
= \int_0^{t-h} \Omega(a) \, da - \int_0^{t-h} \Omega(a + h) \, da
\]
\[
= \int_0^{t-h} \Omega(a) \, da - \int_h^t \Omega(a) \, da
\]
\[
= \int_0^t \Omega(a) \, da - \int_h^t \Omega(a) \, da
\]
\[
\leq h.
\]
Hence, it follows that
\[
\Delta \leq \tilde{\beta}K^2 h.
\]
For \( \Xi \), combining Proposition 2.2 with the expression for \( dS(t)/dt \), we find that \( |dS(t)/dt| \) is bounded by \( M_S = \Lambda + (\mu + \alpha)K + \beta_1\tilde{p}K^2 \), and therefore \( S(t) \) is Lipschitz on \([0, \infty)\) with coefficient \( M_S \). Similarly, \( V(t) \) is Lipschitz on \([0, \infty)\) with coefficient \( M_V = (\mu + \alpha)K + \beta_2\tilde{p}K^2 \).
By Lemmas 3.1 and 3.2, there exist two Lipschitz coefficients \( M_P, M_Q \) for \( P, Q \) respectively. Thus, \( S(t)P(t) \) and \( V(t)Q(t) \) are Lipschitz continuous on \([0, \infty)\) with coefficients \( M_{SP} = KM_P + \beta_1KM_S \) and \( M_{VQ} = KM_Q + \beta_2KM_V \). Denote \( M = \beta_1M_{SP} + \beta_2M_{VQ} \). Then,
\[
\Xi \leq Mh \int_0^{t-h} e^{-\mu_0 a} \, da \leq \frac{Mh}{\mu_0},
\]
and hence, we have
\[
\int_0^{\infty} |\tilde{e}(t, a + h) - \tilde{e}(t, a)| \, da \leq \left( 2\tilde{\beta}K^2 + \frac{Mh}{\mu_0} \right) h.
\]
This implies that \( \tilde{e}(t, a) \) remains in a pre-compact subset of \( L^1_+ \) independent of \( X_0 \).
Similarly, \( \tilde{i}(t, a) \) remains in a pre-compact subset of \( L^1_+ \) independent of \( X_0 \).
Thus, (3) of Theorem 3.2 holds and the proof is complete.

4. Uniform persistence

In this section we show the uniform persistence of system (5). Let \( \hat{e}(t) := e(t, 0) \) and \( \hat{i}(t) := i(t, 0) \). Then Equations (11) and (12) can be rewritten as
\[
e(t, a) = \begin{cases} \hat{e}(t - a)\Omega(a), & t \geq a \geq 0; \\ \varphi_e(a - t)\Omega(a) / \Omega(a - t), & a \geq t \geq 0, \end{cases}
\]
and
\[
i(t, a) = \begin{cases} \hat{i}(t - a)\Gamma(a), & t \geq a \geq 0; \\ \varphi_i(a - t)\Gamma(a) / \Gamma(a - t), & a \geq t \geq 0, \end{cases}
\]
where \( \Omega(a) \) and \( \Gamma(a) \) are defined by Equation (8).
Substituting Equations (22) and (23) into the boundary condition (6), we obtain the following system of integral equations of \( \hat{e}(t) \) and \( \hat{i}(t) \):

\[
\hat{e}(t) = \beta_1 S(t) \left\{ \int_0^t k(a) \Gamma(a) \hat{i}(t-a) \, da + \int_t^\infty k(a) \frac{\Gamma(a)}{\Gamma(a-t)} \varphi_i(a-t) \, da \right\} + \beta_2 V(t) \left\{ \int_0^t p(a) \Gamma(a) \hat{i}(t-a) \, da + \int_t^\infty \frac{\Gamma(a)}{\Gamma(a-t)} \varphi_i(a-t) \, da \right\},
\]

(24)

\[
\hat{i}(t) = \int_0^t \xi(a) \Omega(a) \hat{e}(t-a) \, da + \int_t^\infty \xi(a) \frac{\Omega(a)}{\Omega(a-t)} \varphi_e(a-t) \, da.
\]

(25)

In addition, note that the first and second equation of (5) can be rewritten as

\[
\frac{dS(t) + dV(t)}{dt} = \Lambda - \mu S(t) + (\mu + \gamma_1) V(t) - \hat{e}(t).
\]

(26)

We first prove the following lemma.

**Lemma 4.1** If \( \gamma_0 > 1 \), then there exists a positive constant \( \epsilon > 0 \) independent of the initial condition (6) such that

\[
\limsup_{t \to \infty} \hat{e}(t) > \epsilon.
\]

(27)

**Proof** When \( \gamma_0 > 1 \), without loss of generality, we can assume that \( \epsilon > 0 \) is sufficiently small so that

\[
\left( \frac{\beta_1}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) \, da + \frac{\beta_2}{\mu + \gamma_1} \int_0^\infty p(a) \Gamma(a) \, da \right) \int_0^\infty \xi(a) \Omega(a) \, da > 1.
\]

(28)

In what follows, by way of contradiction, we suppose that there exists a sufficiently large \( T > 0 \) such that

\[
\hat{e}(t) \leq \epsilon \quad \text{for all} \ t \geq T.
\]

(29)

From Equations (24)–(25) and the positivity of coefficients, we obtain inequalities

\[
\hat{e}(t) \geq \beta_1 S(t) \int_0^t k(a) \Gamma(a) \hat{i}(t-a) \, da + \beta_2 V(t) \int_0^t p(a) \Gamma(a) \hat{i}(t-a) \, da
\]

(30)

and

\[
\hat{i}(t) \geq \int_0^t \xi(a) \Omega(a) \hat{e}(t-a) \, da.
\]

(31)

Combining above two equations yields

\[
\hat{e}(t) \geq \beta_1 S(t) \int_0^t k(a) \Gamma(a) \int_0^{t-a} \xi(\sigma) \Omega(\sigma) \hat{e}(t-a-\sigma) \, d\sigma \, da
\]

\[
+ \beta_2 V(t) \int_0^t p(a) \Gamma(a) \int_0^{t-a} \xi(\sigma) \Omega(\sigma) \hat{e}(t-a-\sigma) \, d\sigma \, da.
\]

(32)

On the other hand, from Equation (29), the first equation of (5) leads to the following inequality.

\[
\frac{dS(t)}{dt} \geq \Lambda - (\mu + \alpha) S(t) - \epsilon \quad \text{for all} \ t \geq T.
\]
Performing the variation of constants formula, we have

\[ S(t) \geq \frac{\Lambda - \epsilon}{\mu + \alpha} \quad \text{for all } t \geq T. \]

Similarly, we have

\[ V(t) \geq \frac{\alpha}{\mu + \gamma_1} \frac{\Lambda - \epsilon}{\mu + \alpha} \quad \text{for all } t \geq T. \]

Applying these inequalities to Equation (32), we have that for all \( t \geq T \),

\[
\epsilon \geq \hat{e}(t) \geq \beta_1 \frac{\Lambda - \epsilon}{\mu + \alpha} \int_0^t k(a) \Gamma(a) \int_0^{t-a} \xi(\sigma) \Omega(\sigma) \hat{e}(t-a-\sigma) \, d\sigma \, da \\
+ \beta_2 \frac{\alpha}{\mu + \gamma_1} \frac{\Lambda - \epsilon}{\mu + \alpha} \int_0^t p(a) \Gamma(a) \int_0^{t-a} \xi(\sigma) \Omega(\sigma) \hat{e}(t-a-\sigma) \, d\sigma \, da. \tag{33}
\]

Now, without loss of generality, we can perform the time-shift of system (5) with respect to \( T \) replacing the initial value of system (5) as \( X_1 := \Phi(T, X_0) \). Then, we can consider that the inequality (33) holds for all \( t \geq 0 \).

Let us denote by \( \mathcal{L}[f](\lambda) \) the Laplace transform of function \( f \):

\[
\mathcal{L}[f](\lambda) := \int_0^\infty e^{-\lambda t} f(t) \, dt.
\]

Note that for any \( \lambda > 0 \), the Laplace transform of \( \hat{e} \) is bounded:

\[
\mathcal{L}[\hat{e}](\lambda) = \int_0^\infty e^{-\lambda t} \hat{e}(t) \, dt \leq \epsilon \int_0^\infty e^{-\lambda t} \, dt = \frac{\epsilon}{\lambda} < +\infty.
\]

Then, taking the Laplace transform of both sides of Equation (33) and changing the order and variable of integration yields

\[
\mathcal{L}[\hat{e}](\lambda) \geq \beta_1 \frac{\Lambda - \epsilon}{\mu + \alpha} \int_0^\infty e^{-\lambda t} \int_0^t k(a) \Gamma(a) \int_0^{t-a} \xi(\sigma) \Omega(\sigma) \hat{e}(t-a-\sigma) \, d\sigma \, da \, dt \\
+ \beta_2 \frac{\alpha}{\mu + \gamma_1} \frac{\Lambda - \epsilon}{\mu + \alpha} \int_0^\infty e^{-\lambda t} \int_0^t p(a) \Gamma(a) \int_0^{t-a} \xi(\sigma) \Omega(\sigma) \hat{e}(t-a-\sigma) \, d\sigma \, da \, dt
\]

\[
= \beta_1 \frac{\Lambda - \epsilon}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) \int_0^\infty e^{-\lambda t} \int_0^{t-a} \xi(\sigma) \Omega(\sigma) \hat{e}(t-a-\sigma) \, d\sigma \, dt \, da \\
+ \beta_2 \frac{\alpha}{\mu + \gamma_1} \frac{\Lambda - \epsilon}{\mu + \alpha} \int_0^\infty p(a) \Gamma(a) \int_0^\infty e^{-\lambda t} \int_0^{t-a} \xi(\sigma) \Omega(\sigma) \hat{e}(t-a-\sigma) \, d\sigma \, dt \, da
\]

\[
= \beta_1 \frac{\Lambda - \epsilon}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) \int_0^\infty e^{-\lambda(t+a)} \int_0^\tau \xi(\sigma) \Omega(\sigma) \hat{e}(\tau-\sigma) \, d\sigma \, d\tau \, da \\
+ \beta_2 \frac{\alpha}{\mu + \gamma_1} \frac{\Lambda - \epsilon}{\mu + \alpha} \int_0^\infty p(a) \Gamma(a) \int_0^\infty e^{-\lambda(t+a)} \int_0^\tau \xi(\sigma) \Omega(\sigma) \hat{e}(\tau-\sigma) \, d\sigma \, d\tau \, da
\]
\[ \begin{align*}
&= \beta_1 \frac{\lambda - \alpha}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) e^{-\lambda a} \int_0^\infty e^{-\lambda t} \int_0^T \hat{\xi}(\sigma) \Omega(\sigma) \hat{\varepsilon}(\tau - \sigma) \, d\sigma \, d\tau \, da \\
&+ \beta_2 \frac{\alpha}{\mu + \gamma_1 \mu + \alpha} \int_0^\infty p(a) \Gamma(a) e^{-\lambda a} \int_0^\infty e^{-\lambda t} \int_0^T \hat{\xi}(\sigma) \Omega(\sigma) \hat{\varepsilon}(\tau - \sigma) \, d\sigma \, d\tau \, da \\
&= \beta_1 \frac{\lambda - \alpha}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) e^{-\lambda a} \int_0^\infty \hat{\xi}(\sigma) \Omega(\sigma) \int_0^\infty e^{-\lambda t} \hat{\varepsilon}(\tau - \sigma) \, d\sigma \, da \\
&+ \beta_2 \frac{\alpha}{\mu + \gamma_1 \mu + \alpha} \int_0^\infty p(a) \Gamma(a) e^{-\lambda a} \int_0^\infty \hat{\xi}(\sigma) \Omega(\sigma) \int_0^\infty e^{-\lambda t} \hat{\varepsilon}(\tau - \sigma) \, d\sigma \, da \\
&= \beta_1 \frac{\lambda - \alpha}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) e^{-\lambda a} \int_0^\infty \hat{\xi}(\sigma) \Omega(\sigma) e^{-\lambda \sigma} \, d\sigma \\
&+ \beta_2 \frac{\alpha}{\mu + \gamma_1 \mu + \alpha} \int_0^\infty p(a) \Gamma(a) e^{-\lambda a} \int_0^\infty \hat{\xi}(\sigma) \Omega(\sigma) e^{-\lambda \sigma} \, d\sigma \\
&= \left( \beta_1 \frac{\lambda - \alpha}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) e^{-\lambda a} \, da + \beta_2 \frac{\alpha}{\mu + \gamma_1 \mu + \alpha} \int_0^\infty p(a) \Gamma(a) e^{-\lambda a} \, da \right) \int_0^\infty \hat{\xi}(a) \Omega(a) \, da.
\end{align*} \]

Dividing both sides by \( L[\hat{\varepsilon}](\lambda) \), we have

\[
1 \geq \left( \beta_1 \frac{\lambda - \alpha}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) e^{-\lambda a} \, da + \beta_2 \frac{\alpha}{\mu + \gamma_1 \mu + \alpha} \int_0^\infty p(a) \Gamma(a) e^{-\lambda a} \, da \right) \int_0^\infty \hat{\xi}(a) \Omega(a) \, da.
\]

Since this inequality holds for any \( \lambda > 0 \), we can conclude that

\[
1 \geq \left( \beta_1 \frac{\lambda - \alpha}{\mu + \alpha} \int_0^\infty k(a) \Gamma(a) \, da + \beta_2 \frac{\alpha}{\mu + \gamma_1 \mu + \alpha} \int_0^\infty p(a) \Gamma(a) \, da \right) \int_0^\infty \hat{\xi}(a) \Omega(a) \, da.
\]

This is a contradiction to Equation (28).

From Lemma 4.1, the following corollary immediately follows.

**Corollary 4.1** If \( R_0 > 1 \), then there exists a positive constant \( \epsilon > 0 \) independent of the initial condition (6) such that

\[
\limsup_{t \to \infty} S(t) > \epsilon, \quad \limsup_{t \to \infty} V(t) > \epsilon, \quad \limsup_{t \to \infty} \hat{i}(t) > \epsilon.
\]

**Proof** The first two inequalities can be easily proved by integrating the first and second equations of (5) and making use of the boundedness of \( \int_0^\infty i(t, a) \, da \). The last inequality can be proved by applying the inequality in Lemmas 4.1 to Equation (25).

Lemma 4.1 and Corollary 4.1 implies the uniform weak persistence of the system (5). To derive the uniform (strong) persistence from it, we make use of a technique introduced by Smith and Thieme [21, Chapter 9] (see also McCluskey [16, Section 8]).
Let $\phi : \mathbb{R} \rightarrow \mathcal{Y}$ be a total $\Phi$-trajectory such that $\phi(r) := (S(r), V(r), e(r, \cdot), i(r, \cdot))$, $r \in \mathbb{R}$. Then, it follows that $\phi(r + t) = \Phi(t, \phi(r))$, $t \geq 0$, $r \in \mathbb{R}$ and

$$
e(r, a) = e(r - a, 0)\Omega(a) = \hat{e}(r - a)\Omega(a),$$

$$i(r, a) = i(r - a, 0)\Gamma(a) = \hat{i}(r - a)\Gamma(a), \quad r \in \mathbb{R}, \quad a \geq 0.$$  

Hence, from Equations (5) and (24)-(25), we have

$$\frac{dS(r)}{dr} = \Lambda - \mu S(r) - aS(r) - \beta_1 S(r) \int_0^\infty \Gamma(a)\hat{i}(r - a) \, da,$$

$$\frac{dV(r)}{dr} = aS(r) - (\mu + \gamma_1)V(r) - \beta_2 V(r) \int_0^\infty p(a)\Gamma(a)i(r - a) \, da,$$

$$\hat{e}(r) = \beta_1 S(r) \int_0^\infty k(a)\Gamma(a)\hat{i}(r - a) \, da + \beta_2 V(r) \int_0^\infty p(a)\Gamma(a)\hat{i}(r - a) \, da,$$

$$\hat{i}(r) = \int_0^\infty \xi(a)\Omega(a)\hat{e}(r - a) \, da, \quad r \in \mathbb{R}. \quad (35)$$

Substituting the last equation into the third equation, we obtain the following integral equation of $\hat{e}$.

$$\hat{e}(r) = \beta_1 S(r) \int_0^\infty k(a)\Gamma(a) \int_0^\infty \xi(b)\Omega(b)\hat{e}(r - a - b) \, db \, da$$

$$+ \beta_2 V(r) \int_0^\infty p(a)\Gamma(a) \int_0^\infty \xi(b)\Omega(b)\hat{e}(r - a - b) \, db \, da, \quad r \in \mathbb{R}. \quad (36)$$

We prove the following lemma.

**Lemma 4.2** For total $\Phi$-trajectory $\phi$ in $\mathcal{Y}$, $S(r)$ and $V(r)$ are strictly positive on $\mathbb{R}$. Furthermore, $\hat{e}(r) = 0$ for all $r \geq 0$ if $\hat{e}(r) = 0$ for all $r \leq 0$.

**Proof** Suppose that $S(r^*) = 0$ for a number $r^* \in \mathbb{R}$ and show a contradiction. In this case, it follows from the first equation of (35) that $dS(r^*)/dr = \Lambda > 0$. This implies that $S(r^* - \eta) < 0$ for sufficiently small $\eta > 0$ and it contradicts to the fact that the total $\Phi$-trajectory $\phi$ remains in $\mathcal{Y}$. Consequently, $S(r)$ is strictly positive on $\mathbb{R}$.

Similarly, if there exists a number $r^* \in \mathbb{R}$ such that $V(r^*) = 0$, then $dV(r^*)/dr = \alpha S(r^*) > 0$ follows from the second equation of (35) and the strict positivity of $S(r)$. However, it implies that $V(r^* - \eta) < 0$ for sufficiently small $\eta > 0$ and this is a contradiction.

We prove the last part of this lemma. By changing the variables, we can rewrite Equation (36) as follows.

$$\hat{e}(r) = \beta_1 S(r) \int_0^\infty k(a)\Gamma(a) \int_{-\infty}^{r-a} \xi(r - a - c)\Omega(r - a - c)\hat{e}(c) \, dc \, da$$

$$+ \beta_2 V(r) \int_0^\infty p(a)\Gamma(a) \int_{-\infty}^{r-a} \xi(r - a - c)\Omega(r - a - c)\hat{e}(c) \, dc \, da,$$

$$= \beta_1 S(r) \int_{-\infty}^{r} k(r - \sigma)\Gamma(r - \sigma) \int_{-\infty}^{\sigma} \xi(\sigma - c)\Omega(\sigma - c)\hat{e}(c) \, dc \, d\sigma$$

$$+ \beta_2 V(r) \int_{-\infty}^{r} p(r - \sigma)\Gamma(r - \sigma) \int_{-\infty}^{\sigma} \xi(\sigma - c)\Omega(\sigma - c)\hat{e}(c) \, dc \, d\sigma.$$
Hence, if \( \hat{e}(r) = 0 \) for all \( r \leq 0 \), then we have
\[
\hat{e}(r) = \beta_1 S(r) \int_0^r k(r - \sigma) \Gamma(r - \sigma) \int_0^\sigma \xi(\sigma - c) \Omega(\sigma - c) \hat{e}(c) \, dc \, d\sigma \\
+ \beta_2 V(r) \int_0^r p(r - \sigma) \Gamma(r - \sigma) \int_0^\sigma \xi(\sigma - c) \Omega(\sigma - c) \hat{e}(c) \, dc \, d\sigma \\
\leq (\beta_1 \tilde{S} \tilde{k} \hat{\xi} + \beta_2 \tilde{V} \tilde{p} \hat{\xi}) \int_0^r \int_0^\sigma \hat{e}(c) \, dc \, d\sigma + \int_0^r \hat{e}(c) \, dc \\
\leq \max(\beta_1 \tilde{S} \tilde{k} \hat{\xi} + \beta_2 \tilde{V} \tilde{p} \hat{\xi}, 1) \left( \int_0^r \hat{e}(c) \, dc + \int_0^r \int_0^\sigma \hat{e}(c) \, dc \, d\sigma \right) \\
= \max(\beta_1 \tilde{S} \tilde{k} \hat{\xi} + \beta_2 \tilde{V} \tilde{p} \hat{\xi}, 1) \hat{E}(r) .
\] (37)

where \( \tilde{S} \in (0, \infty) \) and \( \tilde{V} \in (0, \infty) \) are some upper bounds for \( S(r) \) and \( V(r) \), respectively, which existence is guaranteed by Proposition 2.3. Let
\[
\hat{E}(r) := \int_0^r \hat{e}(c) \, dc + \int_0^r \int_0^\sigma \hat{e}(c) \, dc \, d\sigma, \quad r \geq 0 .
\]

Then, from inequality (37), we have
\[
\frac{d\hat{E}(r)}{dr} = \hat{e}(r) + \int_0^r \hat{e}(c) \, dc \\
\leq (\beta_1 \tilde{S} \tilde{k} \hat{\xi} + \beta_2 \tilde{V} \tilde{p} \hat{\xi}) \int_0^r \int_0^\sigma \hat{e}(c) \, dc \, d\sigma + \int_0^r \hat{e}(c) \, dc \\
\leq \max(\beta_1 \tilde{S} \tilde{k} \hat{\xi} + \beta_2 \tilde{V} \tilde{p} \hat{\xi}, 1) \left( \int_0^r \hat{e}(c) \, dc + \int_0^r \int_0^\sigma \hat{e}(c) \, dc \, d\sigma \right) \\
= \max(\beta_1 \tilde{S} \tilde{k} \hat{\xi} + \beta_2 \tilde{V} \tilde{p} \hat{\xi}, 1) \hat{E}(r) .
\]

Hence, integration above yields
\[
\hat{E}(r) \leq \hat{E}(0) e^{\max(\beta_1 \tilde{S} \tilde{k} \hat{\xi} + \beta_2 \tilde{V} \tilde{p} \hat{\xi}, 1) r} = 0, \quad r \geq 0 ,
\]
since \( \hat{E}(0) = 0 \). This implies \( \hat{e}(r) = 0 \) for all \( r \geq 0 \) and the proof is complete. \( \blacksquare \)

We next prove the following lemma.

**Lemma 4.3** For total \( \Phi \)-trajectory \( \phi \) in \( \mathcal{V} \), \( \hat{e}(r) \) is strictly positive or identically zero on \( \mathbb{R} \).

*Proof* From the second statement of Lemma 4.2, by performing appropriate shifts, we see that \( \hat{e}(r) = 0 \) for all \( r \geq r^* \) if \( \hat{e}(r) = 0 \) for all \( r \leq r^* \), where \( r^* \in \mathbb{R} \) is arbitrary. This implies that either \( \hat{e}(r) \) is identically zero on \( \mathbb{R} \) or there exists a decreasing sequence \( \{r_j\}_{j=1}^\infty \) such that \( r_j \to -\infty \) as \( j \to \infty \) and \( \hat{e}(r_j) > 0 \). In the latter case, letting \( \hat{e}_j(t) := \hat{e}(t + r_j) \), \( t \geq 0 \), we have from Equation (36) that
\[
\hat{e}_j(t) = \beta_1 S(t + r_j) \int_0^t k(a) \Gamma(a) \int_0^\infty \xi(b) \Omega(b) \hat{e}_j(t - a - b) \, db \, da \\
+ \beta_2 V(t + r_j) \int_0^t p(a) \Gamma(a) \int_0^\infty \xi(b) \Omega(b) \hat{e}_j(t - a - b) \, db \, da + \hat{j}_j(t) ,
\] (38)

where
\[
\hat{j}_j(t) = \beta_1 S(t + r_j) \int_t^\infty k(a) \Gamma(a) \int_0^\infty \xi(b) \Omega(b) \hat{e}_j(t - a - b) \, db \, da \\
+ \beta_2 V(t + r_j) \int_t^\infty p(a) \Gamma(a) \int_0^\infty \xi(b) \Omega(b) \hat{e}_j(t - a - b) \, db \, da .
\] (39)
Let $\underline{S} := \inf_{r \in \mathbb{R}} S(r) > 0$. Then, we have from Equation (38) that
\[
\hat{e}_j(t) \geq \beta_1 \underline{S} \int_0^t k(a) \Gamma(a) \int_0^{t-a} \xi(b) \Omega(b) \hat{e}_j(t-a-b) \, db \, da + \hat{j}_j(t)
\]
\[
= \beta_1 \underline{S} \int_0^t k(t-\sigma) \Gamma(t-\sigma) \int_0^\sigma \xi(b) \Omega(b) \hat{e}_j(\sigma-b) \, db \, d\sigma + \hat{j}_j(t)
\]
\[
= \beta_1 \underline{S} \int_0^t k(t-\sigma) \Gamma(t-\sigma) \int_0^\sigma (\sigma-\rho) \Omega(\sigma-\rho) \hat{e}_j(\rho \, d\rho + \hat{j}_j(t)
\]
\[
= \beta_1 \underline{S} \int_0^t (\int_0^\sigma k(t-\sigma) \Gamma(t-\sigma) \xi(\sigma-\rho) \Omega(\sigma-\rho) \, d\sigma) \hat{e}_j(\rho) \, d\rho + \hat{j}_j(t)
\]
Hence, defining
\[
\tilde{k}(s) := \beta_1 \underline{S} \int_0^s k(s-\eta) \Gamma(s-\eta) \xi(\eta) \Omega(\eta) \, d\eta, \quad s \geq 0,
\] (40)
we have from the above inequality that
\[
\hat{e}_j(t) \geq \int_0^t \tilde{k}(t-\rho) \hat{e}_j(\rho) \, d\rho + \hat{j}_j(t) = \int_0^t \tilde{k}(\rho) \hat{e}_j(t-\rho) \, d\rho + \hat{j}_j(t).
\]
Note that from Equation (39), $\hat{j}_j(0) = \hat{e}(r_j) > 0$ and $\hat{j}_j(r)$ is continuous at 0 and from (v) of Assumption 2.1 and (40), $\tilde{k}(s)$ is not zero a.e. Therefore, we can apply Corollary B.6 of Smith and Thieme [21] to conclude that there exists a number $t^* > 0$, which only depends on $\tilde{k}(s)$, such that $\hat{e}_j(t) > 0$ for all $t > t^*$. From the definition of $\hat{e}_j$, this implies that $\hat{e}(t) > 0$ for all $t > t^* + r_j$. Since $r_j \to -\infty$ as $j \to \infty$, this implies that $\hat{e}(r) > 0$ for all $r \in \mathbb{R}$. Consequently, $\hat{e}(r)$ is strictly positive on $\mathbb{R}$. 

Incidentally, we are concerning the existence of a compact attractor. The boundedness and asymptotic smoothness of semi-flow $\Phi$ allows us to draw the following conclusion.

**Proposition 4.1** There exists a compact attractor $A$ of bounded sets in $\mathcal{Y}$.

**Proof** The point dissipativity and eventual boundedness of semi-flow $\Phi$ are guaranteed by Propositions 2.1 and 2.3. The asymptotic smoothness of $\Phi$ was shown by Theorem 3.3. Thus, Theorem 2.33 of Simith and Thieme [21] can be applied to complete the proof. 

Now, let us define a function $\rho : \mathcal{Y} \to \mathbb{R}_+$ on $\mathcal{Y}$ by
\[
\rho(x, y, \varphi, \psi) := \beta_1 x \int_0^\infty k(a) \psi(a) \, da + \beta_2 y \int_0^\infty p(a) \psi(a) \, da, \quad (x, y, \varphi, \psi) \in \mathcal{Y}.
\]
Then, it follows from the previous argument that
\[
\rho(\Phi_t(Y_0)) = \hat{e}(t).
\]
Then, Lemma 4.1 implies the uniform weak $\rho$-persistence of semi-flow $\Phi$ for $\forall t > 1$. Moreover, from Lemmas 4.2–4.3 and Proposition 4.1 and the Lipschitz continuity of $\hat{e}$ (which immediately
follows from Proposition 2.3), we can apply Theorem 5.2 of Smith and Thieme [21] to conclude that the uniform weak $\rho$-persistence of semi-flow $\Phi$ implies the uniform (strong) $\rho$-persistence. In conclusion, we obtain the following theorem.

**Theorem 4.1** If $\Re_0 > 1$, then semi-flow $\Phi$ is uniformly (strongly) $\rho$-persistent.

The uniform persistence of system (5) for $\Re_0 > 1$ immediately follows from Theorem 4.1. In fact, it follows from Equation (22) that

$$\|e(t, \cdot)\|_{L^1} \geq \int_0^t \hat{e}(t-a) \Omega(a) \, da$$

and hence, from a variation of the Lebesgue–Fatou lemma [20, Section B.2], we have

$$\liminf_{t \to \infty} \|e(t, \cdot)\|_{L^1} \geq \hat{e}^\infty \int_0^\infty \Omega(a) \, da,$$

where $\hat{e}^\infty := \liminf_{t \to \infty} \hat{e}(t)$. Under Theorem 4.1, there exists a positive constant $\epsilon > 0$ such that $\hat{e}^\infty > \epsilon$ if $\Re_0 > 1$ and hence, the persistence of $e(t, a)$ with respect to $\| \cdot \|_{L^1}$ follows. By a similar argument, we can prove that $i(t, a)$ is also persistent with respect to $\| \cdot \|_{L^1}$. Consequently, we have the following theorem.

**Theorem 4.2** If $\Re_0 > 1$, the semi-flow $\{\Phi(t)\}_{t \geq 0}$ generated by Equation (5) is uniformly persistent in $\mathcal{Y}$, that is, there exists a constant $\epsilon > 0$ such that for each $Y_0 \in \mathcal{Y}$,

$$\liminf_{t \to +\infty} S(t) \geq \epsilon, \quad \liminf_{t \to +\infty} V(t) \geq \epsilon, \quad \liminf_{t \to +\infty} \|e(t, \cdot)\|_{L^1} \geq \epsilon, \quad \liminf_{t \to +\infty} \|i(t, \cdot)\|_{L^1} \geq \epsilon.$$ 

In the next two sections, we prove the stability of equilibria of model (5). The basic reproduction number introduced in Equation (15) will be used for formulate the stability conditions.

5. **Proof of the Theorem 2.1**

Denote $g(x) = x - 1 - \ln x \geq g(1) = 0$, which is used for ease of notation. First, system (5) always admits infection-free equilibria $E^0 = (S_0, V_0, 0, 0)$, where

$$S_0 = \frac{\Lambda}{\mu + \alpha} \quad \text{and} \quad V_0 = \frac{\alpha \Lambda}{(\mu + \gamma_1)(\mu + \alpha)}.$$

Now we use the approach in [4,12,14,16], define the following Lyapunov function:

$$L_{DFE} = L_1 + L_2 + L_3,$$

where

- $L_1 = S_0 g(S(t)/S_0) + V_0 g(V(t)/V_0)$;
- $L_2 = \int_0^\infty \Phi(a) e(t, a) \, da$;
- $L_3 = \int_0^\infty \Psi(a) i(t, a) \, da$. 


where $\Phi(a)$ and $\Psi(a)$ are defined later. The function $L_{\text{DFE}}$ is non-negative defined with respect to the disease-free steady state $E_0$, which is a global minimum. Now, calculating the time derivative of $L_1$ along Equation (5), we have

$$
\frac{dL_1}{dr} = \left(1 - \frac{S_0}{S(t)}\right) \frac{dS(t)}{dr} + \left(1 - \frac{V_0}{V(t)}\right) \frac{dV(t)}{dr}
$$

$$
= \mu S_0 \left(2 - \frac{S_0}{S(t)} - \frac{S(t)}{S_0}\right) + \alpha S_0 \left(3 - \frac{V(t)}{V_0} - \frac{S_0}{S(t)} - \frac{S(t) V_0}{S_0 V(t)}\right)
$$

$$
- \beta_1 S(t) \int_0^\infty k(a) i(t, a) \, da + \beta_1 S_0 \int_0^\infty k(a) i(t, a) \, da
$$

$$
- \beta_2 V(t) \int_0^\infty p(a) i(t, a) \, da + \beta_2 V_0 \int_0^\infty p(a) i(t, a) \, da.
$$

Here we use the fact that $\alpha S_0 = (\mu + \gamma_1)V_0$. The derivative of $L_2$ along the solutions of Equation (5) can be calculated as

$$
\frac{dL_2}{dr} = \frac{d}{dr} \left[\int_0^t \Phi(a) e(t - a, 0)e^{-\int_0^a \theta(r) \, dr} \, da + \int_t^\infty \Phi(a) \varphi e(a - t)e^{-\int_0^a \theta(r) \, dr} \, da\right]
$$

$$
= \frac{d}{dr} \left[\int_0^t \Phi(t - r) e(r, 0)e^{-\int_0^r \theta(r) \, dr} \, dr + \int_r^\infty \Phi(t + r) \varphi e(r)e^{-\int_0^r \theta(r) \, dr} \, dr\right]
$$

$$
= \Phi(0) e(t, 0) + \int_0^\infty (\Phi'(a) - \theta(a) \Phi(a)) e(t, a) \, da.
$$

Similarly, we have

$$
\frac{dL_3}{dr} = \Psi(0) \int_0^\infty \xi(a) e(t, a) \, da + \int_0^\infty (\Psi'(a) - \delta(a) \Psi(a)) i(t, a) \, da.
$$

Then

$$
\frac{dL_{\text{DFE}}}{dr} = \mu S_0 \left(2 - \frac{S_0}{S(t)} - \frac{S(t)}{S_0}\right) + \alpha S_0 \left(3 - \frac{V(t)}{V_0} - \frac{S_0}{S(t)} - \frac{S(t) V_0}{S_0 V(t)}\right)
$$

$$
- \beta_1 S(t) \int_0^\infty k(a) i(t, a) \, da + \beta_1 S_0 \int_0^\infty k(a) i(t, a) \, da
$$

$$
- \beta_2 V(t) \int_0^\infty p(a) i(t, a) \, da + \beta_2 V_0 \int_0^\infty p(a) i(t, a) \, da
$$

$$
+ \Phi(0) e(t, 0) + \int_0^\infty (\Phi'(a) - \theta(a) \Phi(a)) e(t, a) \, da
$$

$$
+ \Psi(0) \int_0^\infty \xi(a) e(t, a) \, da + \int_0^\infty (\Psi'(a) - \delta(a) \Psi(a)) i(t, a) \, da
$$

$$
= \mu S_0 \left(2 - \frac{S_0}{S(t)} - \frac{S(t)}{S_0}\right) + \alpha S_0 \left(3 - \frac{V(t)}{V_0} - \frac{S_0}{S(t)} - \frac{S(t) V_0}{S_0 V(t)}\right) - e(t, 0) + \Phi(0) e(t, 0)
$$

$$
+ \int_0^\infty (S_0 \beta_1 k(a) + V_0 \beta_2 p(a) + \Psi'(a) - \delta(a) \Psi(a)) i(t, a) \, da
$$

$$
+ \int_0^\infty (\Phi'(a) - \theta(a) \Phi(a) + \Psi(0) \xi(a)) e(t, a) \, da.
$$
Furthermore, we have
\[
\Phi_1(\Psi(0)\xi(\tau)) \leq 0
\]
when we choose the above \( \Phi_1 \) and \( \Psi_1 \), then we have
\[
\frac{dL_{\text{DFE}}}{dt} = \mu S_0 \left( 2 - \frac{S_0}{S(t)} - \frac{S(t)}{S_0} \right) + a S_0 \left( 3 - \frac{V(t)}{V_0} - \frac{S(t)}{S(t)} - \frac{S(t)V_0}{S_0V(t)} \right) + (3\Re_0 - 1)e(t, 0).
\]

Therefore, \( \Re_0 \leq 1 \) ensures that \( \frac{dL_{\text{DFE}}}{dt} \leq 0 \) holds. Furthermore, the strict equality holds only if \( S = S_0 \), \( V = V_0 \), \( e(t, a) = 0 \) and \( i(t, a) = 0 \), simultaneously. Thus, \( M_0 = \{ E_0 \} \subset \Upsilon \) is the largest invariant subset of \( \{ \frac{dL_{\text{DFE}}}{dt} = 0 \} \), and by the Lyapunov–LaSalle invariance principle, the infection-free equilibrium \( E_0 \) is globally asymptotically stable when \( \Re_0 \leq 1 \).

6. Proof of the Theorem 2.2

Before addressing stability of endemic equilibrium, let us summarize some straightforward lemmas, which will be used in the proof of Theorem 2.2. For ease of notation, we will denote
\[
\beta_1(a) = S^* \beta_1 k(a), \quad \beta_2(a) = V^* \beta_2 p(a), \quad A' = A \beta_1 S^*, \quad B' = B \beta_2 V^*,
\]
throughout the proof of the Theorem 2.2.

**Lemma 6.1** Each solution of Equation (5) satisfies:
\[
\int_0^\infty \beta_1(a)i^*(a) \frac{S(t)}{S^*} \frac{i(t, a)}{i^*(a)} \, da + \int_0^\infty \beta_2(a)i^*(a) \frac{V(t)}{V^*} \frac{i(t, a)}{i^*(a)} \, da = (A' + B') \int_0^\infty \xi(a)e^*(a) \frac{e(t, 0)}{e^*(0)} \, da.
\]

Furthermore, we have
\[
(A' + B') \int_0^\infty \xi(a)e^*(a) \, da = \int_0^\infty (\beta_1(a) + \beta_2(a))i^*(a) \, da.
\]

**Proof** The left-hand side of equality can be calculated as
\[
\int_0^\infty \beta_1(a)i^*(a) \frac{S(t)}{S^*} \frac{i(t, a)}{i^*(a)} \, da + \int_0^\infty \beta_2(a)i^*(a) \frac{V(t)}{V^*} \frac{i(t, a)}{i^*(a)} \, da
\]
\[
= \int_0^\infty \beta_1 k(a)S^* i^*(a) \frac{i(t, a)}{i^*(a)} \, da + \int_0^\infty \beta_2 V^* p(a)i^*(a) \frac{i(t, a)}{i^*(a)} \, da
\]
\[
= e(t, 0).
\]
With the aid of boundary condition (6) and equilibrium equations (16) and (17), we get
\[
(A' + B') \int_0^\infty \xi(a)e^*(a) \frac{e(t,0)}{e^*(0)} da = (A' + B')i^*(0) \frac{e(t,0)}{e^*(0)} \\
= e(t,0).
\]
Hence, the first part of (6.1) follows. We next verify the second part of Lemma 6.1. From Equation (17) and \((A' + B')C = 1\), one can see that
\[
\int_0^\infty (\beta_1(a) + \beta_2(a))i^*(a) da = e^*(0) \\
= (A' + B') \frac{i^*(0)}{C} \\
= (A' + B') \int_0^\infty \xi(a)e^*(a)H da.
\]
This completes the proof.

**Lemma 6.2** Each solution of Equation (5) satisfies:
\[
\int_0^\infty \xi(a)e^*(a) \frac{i(t,0)}{i^*(0)} da = \int_0^\infty \xi(a)e^*(a) \frac{e(t,0)}{e^*(0)} da.  \tag{43}
\]

**Proof** It directly follows from equilibrium equations (16), \(i^*(0) = \int_0^\infty \xi(a)e^*(a) da\).

**Lemma 6.3** Each solution of Equation (5) satisfies:
\[
\int_0^\infty \beta_1(a)i^*(a) \frac{S(t)i(t,a)e^*(0)}{S^*i^*(a)e(t,0)} da + \int_0^\infty \beta_2(a)i^*(a) \frac{V(t)i(t,a)e^*(0)}{V^*i^*(a)e(t,0)} da \\
= \int_0^\infty \beta_1(a)i^*(a) da + \int_0^\infty \beta_2(a)i^*(a) da.  \tag{44}
\]

**Proof** It follows from the equilibrium equations (16) that
\[
e^*(0) = \int_0^\infty \beta_1(a)i^*(a) da + \int_0^\infty \beta_2(a)i^*(a) da.
\]
On the other hand, we have
\[
\int_0^\infty \beta_1(a)i^*(a) \frac{S(t)i(t,a)e^*(0)}{S^*i^*(a)e(t,0)} da + \int_0^\infty \beta_2(a)i^*(a) \frac{V(t)i(t,a)e^*(0)}{V^*i^*(a)e(t,0)} da \\
= \int_0^\infty \beta_1S^*k(a)i^*(a) \frac{S(t)i(t,a)e^*(0)}{S^*i^*(a)e(t,0)} da + \int_0^\infty \beta_2V^*p(a)i^*(a) \frac{V(t)i(t,a)e^*(0)}{V^*i^*(a)e(t,0)} da \\
= \int_0^\infty \beta_1k(a) \frac{S(t)i(t,a)e^*(0)}{e(t,0)} da + \int_0^\infty \beta_2p(a) \frac{V(t)i(t,a)e^*(0)}{e(t,0)} da \\
= e^*(0).
\]
This completes the proof.
Proof of the Theorem 2.2

This completes the proof. ■

Proof of the Theorem 2.2  We consider the following Lyapunov function,

\[ L_{EE} = L_4 + L_5 + L_6, \]

where

- \( L_4 = g(S(t)/S^*) + g(V(t)/V^*) \);
- \( L_5 = (A' + B') \int_0^\infty \alpha_\sigma(a) g(e(t,a))/e^*(a) \, da \);
- \( L_6 = \int_0^\infty \alpha_1(a) g(i(t,a))/i^*(a) \, da \),

where

\[ \alpha_\sigma(a) = \int_a^\infty \xi(\sigma) e^*(\sigma) \, d\sigma \quad \text{and} \quad \alpha_1(a) = \int_a^\infty (\beta_1(\sigma) + \beta_2(\sigma)) i^*(\sigma) \, d\sigma. \]

Calculating the time derivative of \( L_4 \) along Equation (5), we have

\[ \frac{dL_4}{dt} = \left(1 - \frac{S^*}{S(t)}\right) \frac{dS(t)}{dt} + \left(1 - \frac{V^*}{V(t)}\right) \frac{dV(t)}{dt} \]

\[ = \mu S^* + \alpha S^* - \mu S(t) - \alpha S(t) + \beta_1 S^* \int_0^\infty k(a)i^*(a) \, da \]

\[ - \beta_1 S(t) \int_0^\infty k(a)i(t,a) \, da - \mu S^* \frac{S^*}{S(t)} - \alpha S(t) + \mu S^* + \alpha S^* \]

\[ - \beta_1 S^* \int_0^\infty k(a)i^*(a) \, da + \beta_1 S^* \int_0^\infty k(a)i(t,a) \, da \]

\[ - \beta_2 V(t) \int_0^\infty p(a)i^*(a) \, da \]

\[ - \alpha S(t) \frac{V^*}{V(t)} + \alpha S^* - \beta_2 V^* \int_0^\infty p(a)i^*(a) \, da + \beta_2 V^* \int_0^\infty p(a)i(t,a) \, da \]

\[ - \alpha S^* \frac{V(t)}{V^*} + \alpha S^* \frac{V(t)}{V^*} \]
\[
\frac{dL_4}{dt} = \mu S^* \left( 2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)} \right) + \alpha S^* \left( 3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)} \right) \\
+ \int_0^\infty \beta_1(a) i^*(a) \, da - \beta_1 S(t) \int_0^\infty k(a) i(t, a) \, da \\
- \frac{S^*}{S(t)} \beta_1 S^* \int_0^\infty k(a) i^*(a) \, da + \beta_1 S^* \int_0^\infty k(a) i(t, a) \, da \\
- \beta_2 V^* \int_0^\infty p(a) i^*(a) \, da + \beta_2 V^* \int_0^\infty p(a) i(t, a) \, da \\
- \beta_2 V(t) \int_0^\infty p(a) i(t, a) \, da - (\mu + \gamma_1) V(t) + \alpha S^* \frac{V(t)}{V^*}.
\]

Note that
\[
\alpha S^* - (\mu + \gamma_1) V^* - \beta_2 V^* \int_0^\infty p(a) i^*(a) \, da = 0. \tag{46}
\]

It follows that
\[
\frac{\alpha S^* V(t)}{V^*} = (\mu + \gamma_1) V(t) + \beta_2 V(t) \int_0^\infty p(a) i^*(a) \, da.
\]

Hence, we have
\[
\frac{dL_5}{dt} = (A' + B') \int_0^\infty \xi(a) e^*(a) \left\{ \frac{e(t, 0)}{e^*(0)} - \frac{e(t, a)}{e^*(a)} + \ln \frac{e(t, a)}{e^*(a)} - \ln \frac{e(t, 0)}{e^*(0)} \right\} \, da, \tag{48}
\]
\[
\frac{dL_6}{dt} = \int_0^\infty (\beta_1(a) + \beta_2(a)) i^*(a) \left\{ \frac{i(t, 0)}{i^*(0)} - \frac{i(t, a)}{i^*(a)} + \ln \frac{i(t, a)}{i^*(a)} - \ln \frac{i(t, 0)}{i^*(0)} \right\} \, da. \tag{49}
\]

Now, collecting the terms in Equations (47), (48) and (49) yields
\[
\frac{dL_{EE}}{dt} = \mu S^* \left( 2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)} \right) + \alpha S^* \left( 3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)} \right) \\
+ \int_0^\infty \beta_1(a) i^*(a) \left\{ 1 - \frac{S(t)i(t, a)}{S^*i^*(a)} - \frac{S^*}{S(t)} + \frac{i(t, 0)}{i^*(0)} + \ln \frac{i(t, a)}{i^*(a)} - \ln \frac{i(t, 0)}{i^*(0)} \right\} \, da
\]
Consequently, by using Lemmas 6.1, 6.3 and 6.4, we get

\[
\frac{dL_{EE}}{dt} = \mu S^* \left( 2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)} \right) + (\mu + \gamma_1)V^* \left( 3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)} \right) \\
+ \int_0^\infty \beta_2(a) \int_0^\infty \xi(a) e^*(a) \left\{ \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{e(t,a)}{e^*(a)} \right\} da \\
+ (A' + B') \int_0^\infty \xi(a) e^*(a) \left\{ \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{e(t,a)}{e^*(a)} \right\} da \\
= \mu S^* \left( 2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)} \right) + \alpha S^* \left( 3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)} \right) \\
+ \int_0^\infty \beta_1(a) i^*(a) \left\{ 1 - \frac{S^*}{S(t)} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)} \right\} da \\
+ \int_0^\infty \beta_2(a) i^*(a) \left\{ -1 + \frac{V^*}{V(t)} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)} \right\} da \\
+ (A' + B') \int_0^\infty \xi(a) e^*(a) \left\{ \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{e(t,a)}{e^*(a)} \right\} da \\
+ \int_0^\infty \beta_1(a) i^*(a) \left\{ 1 - \frac{S^*}{S(t)} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)} \right\} da \\
+ (A' + B') \int_0^\infty \xi(a) e^*(a) \left\{ \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{e(t,a)}{e^*(a)} \right\} da \\
+ \int_0^\infty \beta_2(a) i^*(a) \left\{ -1 + \frac{V^*}{V(t)} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)} \right\} da.
\]

Under the Lemmas 6.1 and 6.2, it is easy to see that the last six terms of the above equation equal to 0. Thus, using Equation (46), we have

\[
\frac{dL_{EE}}{dt} = \mu S^* \left( 2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)} \right) + (\mu + \gamma_1)V^* \left( 3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)} \right) \\
+ \int_0^\infty \beta_2(a) i^*(a) \left\{ 1 - \frac{S^*}{S(t)} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)} \right\} da \\
+ \int_0^\infty \beta_1(a) i^*(a) \left\{ -1 + \frac{V^*}{V(t)} + \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)} \right\} da \\
+ (A' + B') \int_0^\infty \xi(a) e^*(a) \left\{ \ln \frac{i(t,a)}{i^*(a)} - \ln \frac{e(t,a)}{e^*(a)} \right\} da .
\]
\[ + \int_{0}^{\infty} \beta_2(a)i^*(a) \left\{ 1 - \frac{S(t)V^*}{S^*V(t)} + \ln \frac{i(t,a)S(t)}{i^*(a)S^*} - \ln \frac{i(t,0)}{i^*(0)} + 1 - \frac{V(t)i(t,a)e^*(0)}{V^*i^*(a)e(t,0)} \right\} da \\
+ \int_{0}^{\infty} \beta_2(a)i^*(a) \left\{ \ln \frac{S(t)V^*}{S^*V(t)} - \ln \frac{S(t)V^*}{S^*V(t)} + \ln \frac{V(t)i(t,a)e^*(0)}{V^*i^*(a)e(t,0)} - \ln \frac{V(t)i(t,a)e^*(0)}{V^*i^*(a)e(t,0)} \right\} da \\
+ (A' + B') \int_{0}^{\infty} \xi(a)e^*(a) \left\{ \ln \frac{e(t,a)i^*(0)}{e^*(a)i(t,0)} \right\} da \\
+ (A' + B') \int_{0}^{\infty} \xi(a)e^*(a) \left\{ 1 - \frac{e(t,a)i^*(0)}{e^*(a)i(t,0)} \right\} da. \]

Consequently, we have

\[
\frac{dL_{EE}}{dt} = \mu S^* \left( 2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)} \right) + \mu V^* \left( 3 - \frac{V(t)}{V^*} - \frac{V^*}{V(t)} \right) \\
- \int_{0}^{\infty} \beta(a)i^*(a)g \left( \frac{S^*}{S(t)} \right) da - \int_{0}^{\infty} \beta_1(a)i^*(a)g \left( \frac{S(t)i(t,a)e^*(0)}{V^*i^*(a)e(t,0)} \right) da \\
- \int_{0}^{\infty} \beta_2(a)i^*(a)g \left( \frac{S(t)V^*}{V^*i^*(a)e(t,0)} \right) da - \int_{0}^{\infty} \beta_2(a)i^*(a)g \left( \frac{S(t)V^*}{V^*i^*(a)e(t,0)} \right) da \\
- (A' + B') \int_{0}^{\infty} \xi(a)e^*(a)g \left( \frac{e(t,a)i^*(0)}{e^*(a)i(t,0)} \right) da.
\]

Hence, \( \frac{dL_{EE}}{dt} \leq 0 \) holds. Furthermore, the strict equality holds only if \( S = S^* \), \( V = V^* \), \( e(t,a) = e^*(a) \), \( i(t,a) = i^*(a) \). Thus, \( M_* = \{ E^* \} \subset \Omega \) is the largest invariant subset of \( \frac{dL_{EE}}{dt} = 0 \), and by the Lyapunov–LaSalle invariance principle, the endemic equilibrium \( E^* \) is globally asymptotically stable when \( \gamma_0 > 1 \). This completes the proof. 

Acknowledgements

The authors thank the anonymous referee for his (or her) valuable comments and suggestions on the previous version of this paper.

Disclosure statement

No potential conflict of interest was reported by the authors.

Funding

J. Wang is supported by National Natural Science Foundation of China (no. 11401182, 11471089), Natural Science Foundation of Heilongjiang Province (no. A201415), Science and Technology Innovation Team in Higher Education Institutions of Heilongjiang Province (No. 2014TD005), Project funded by China Postdoctoral Science Foundation (No. 2014M552295), Project funded by Chongqing Postdoctoral Foundation (No. Xm2014024) and Youth Foundation of Heilongjiang University. T. Kuniya is supported by Grant-in-Aid for Research Activity Start-up, No.25887011 of Japan Society for the Promotion of Science.

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