SELFADHESIVITY IN GAUSSIAN
CONDITIONAL INDEPENDENCE STRUCTURES

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Abstract. Selfadhesivity is a property of entropic polymatroids which guarantees that the polymatroid can be glued to an identical copy of itself along arbitrary restrictions such that the two pieces are independent given the common restriction. We show that positive definite matrices satisfy this condition as well and examine consequences for Gaussian conditional independence structures. New axioms of Gaussian CI are obtained by applying selfadhesivity to the previously known axioms of structural semigraphoids and orientable gaussoids.

1. Introduction

In matroid theory, the term amalgam refers to a matroid in which two smaller matroids are glued together along a common restriction, similar to how four triangles can be glued together along edges to form the boundary of a tetrahedron. This concept is meaningful for conditional independence (CI) structures as well. The bridge from the geometric (matroid-theoretical) concept to probability theory (conditional independence) was built by Matúš [Mat07a] who defined a special kind of amalgam, the adhesive extension, for polymatroids and proved that such extensions always exist for entropic polymatroids with a common restriction.

The purpose of this article is two-fold: First, it is to extend this methodology beyond polymatroids and to introduce a derived collection of amalgamation properties known as selfadhesivity for general conditional independence structures. Second, this general treatment of selfadhesivity is driven by its applications to Gaussian instead of discrete CI inference. The main result, Theorem 3.1, shows that, also in the Gaussian setting, adhesive extensions (of covariance matrices) exist and are even unique. We use the non-trivial structural constraints implied by this result to derive new axioms for Gaussian conditional independence structures. These results are heavily based on computations. All source code and further details on computations are provided on the mathematical research data repository MathRepo hosted by the Max-Planck Institute for Mathematics in the Sciences; the output data, due to its size, is available only on the archiving service KEEPER of the Max-Planck Society:

MathRepo: https://mathrepo.mis.mpg.de/SelfadhesiveGaussianCI/
KEEPER: https://keeper.mpdl.mpg.de/d/fbfe453162e94a14ac28/

2. Preliminaries

Gaussian conditional independence. Let $N$ be a finite ground set indexing jointly distributed random variables $\xi = (\xi_i : i \in N)$. By convention, elements of $N$ are denoted by $i, j, k, \ldots$ and subsets by $I, J, K, \ldots$. Elements are identified with singleton subsets of $N$ and juxtaposition of subsets abbreviates set union. Thus, an expression such as $iK$ is shorthand for $\{i\} \cup K$ as a subset of $N$. The complement of $K \subseteq N$ is $K^c$. The set of all $k$-element subsets of $N$ is $\binom{N}{k}$ and the powerset of $N$ is $2^N$.

We are mostly interested in Gaussian (i.e., multivariate normal) distributions. These distributions are specified by a small number of parameters, namely by the mean vector $\mu \in \mathbb{R}^N$ and the

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covariance matrix $\Sigma \in \text{PD}_N$, where $\text{PD}_N$ is the set of positive definite matrices. Throughout this article, “Gaussian” means “regular Gaussian”, i.e., the covariance matrix is strictly positive definite. For positive semidefinite covariance matrices, which lie on the boundary of $\text{PD}_N$, the CI theory is algebraically more complicated and valid inference properties for regular Gaussians can fail to be valid for singular ones; see [Stu05, Section 2.3.6].

The following result summarizes basic facts from algebraic statistics relating subvectors of $\xi$ and their (positive definite) covariance matrices. It can be found, for instance, in §2.4 of [Sul18]. For $\Sigma \in \text{PD}_N$ and $I, J, K \subseteq N$, let $\Sigma_{I,J}$ denote the submatrix with rows indexed by $I$ and columns by $J$. Submatrices of the form $\Sigma_K := \Sigma_{K,K}$ are principal. Dual to a principal submatrix is its Schur complement $\Sigma^K := \Sigma_K - \Sigma_{K,K}^{-1}\Sigma_{K,J}\Sigma_{J,K}$ in $\Sigma$. Its rows and columns are indexed by $K\subseteq I,J$ and its entries are functions of all the entries of $\Sigma$. Principal submatrices of and Schur complements in positive definite matrices are also positive definite. The Schur complement construction is valid in greater generality which we will need below as well. Let $A$ be any (not necessarily positive definite, or even square) matrix whose rows are indexed by $IK$ and columns by $JK$, where $I, J, K$ are pairwise disjoint, and suppose that the principal submatrix $A_K$ is invertible. Then the Schur complement $A^K = A_{I,J} - A_{I,K}A_K^{-1}A_{K,J}$ is well-defined and its rows are indexed by $I$ and its columns by $J$. See [Zha05] for an introduction to theory of Schur complements in matrix analysis.

**Theorem.** Let $\xi$ be distributed according to the (regular) Gaussian distribution with mean $\mu \in \mathbb{R}^N$ and covariance $\Sigma \in \text{PD}_N$. Let $K \subseteq N$.

- The marginal vector $\xi_K = (\xi_k : k \in K)$ is a regular Gaussian in $\mathbb{R}^K$ with mean vector $\mu_K$ and covariance $\Sigma_K$.

- Given $y \in \mathbb{R}^K$, the conditional $\xi_{K^c} \mid \xi_K = y$ is a regular Gaussian in $\mathbb{R}^{K^c}$ with mean vector $\mu_{K^c} + \Sigma_{K^c,K}^{-1}(y - \mu_K)$ and covariance $\Sigma_{K^c}$.

- Let a Gaussian distribution over $N = IJ$ be given with covariance $\Sigma \in \text{PD}_{IJ}$.

Then the marginal independence $[\xi_I \perp \xi_J \mid \xi_K]$ holds if and only if $\Sigma_{I,J,K} = 0$.

The general CI statement $[\xi_I \perp \xi_J \mid \xi_K]$, with $I, J, K$ pairwise disjoint, is the result of marginalizing $\xi$ to $IJK$, conditioning on $K$ and then checking for independence of $I$ and $J$. The previous theorem implies the following algebraic CI criteria for regular Gaussians:

$$[\xi_I \perp \xi_J \mid \xi_K] \iff (\Sigma_{I,J} - \Sigma_{I,K}\Sigma_{K,J}^{-1}\Sigma_{K,I})_{I,J} = 0$$

\((\perp_1)\)

$$\iff \Sigma_{I,J} - \Sigma_{I,K}\Sigma_{K,J}^{-1}\Sigma_{K,J} = 0$$

\((\perp_2)\)

$$\iff \text{rk} \Sigma_{K,I,K,J} = |K|.$$

Here, rk denotes the rank of a matrix and the last equivalence follows from rank additivity of the Schur complement (see [Zha05]). Indeed, the matrix in \((\perp_1)\) is the Schur complement of $K$ in $\Sigma_{I,K,J}$ and must have rank zero since the principal submatrix $\Sigma_K$ has full rank $|K|$ already because it is positive definite. In particular, the truth of a conditional independence statement does not depend on the conditioning event and it does not depend on the mean $\mu$. Hence, for CI purposes in this article, we identify regular Gaussians with their covariance matrices $\Sigma \in \text{PD}_N$.

Rank additivity of the Schur complement also shows that the “$\geq$” part of the rank condition in \((\perp_2)\) always holds. Hence, the minimal rank $|K|$ is attained if and only if all minors of $\Sigma_{I,K,J}$ of size $|K| + 1$ vanish. But only a subset of these minors is necessary: by \((\perp_1)\) the rank of $\Sigma_{I,K,J}$ is $|K|$ if and only if $\Sigma_{I,J} = \Sigma_{I,K}\Sigma_{K,J}^{-1}\Sigma_{K,J}$ holds. This is one polynomial condition for each $i \in I$ and $j \in J$, namely $\text{det} \Sigma_{i,K,j,K} = 0$ — again by Schur complement expansion of the determinant. These minors correspond to CI statements of the form $[\xi_i \perp \xi_j \mid \xi_K]$. This proves the following “localization rule” for Gaussian conditional independence:

$$[\xi_I \perp \xi_J \mid \xi_K] \iff \bigwedge_{i \in I, j \in J} [\xi_i \perp \xi_j \mid \xi_K].$$
We adopt the form $[I \perp J \mid K]$ for CI statements $[\xi_I \perp \xi_J \mid \xi_K]$ without the mention of a random vector. These symbols are treated as combinatorial objects and $\mathcal{A}_N := \{ [i \perp j \mid K] : ij \in (N)_2, K \subseteq N \setminus ij \}$ is the set of all elementary CI statements. The CI structure of $\Sigma$ is the set

$$ [\Sigma] := \{ [i \perp j \mid K] \in \mathcal{A}_N : \det \Sigma_{iK,jK} = 0 \}.$$  

The localization rule shows that $[\Sigma]$ encodes the entire set of true CI statements for a Gaussian with covariance matrix $\Sigma$ and with slight abuse of notation we employ statements such as $[I \perp J \mid K] \in [\Sigma]$. It is important to note in this context that we treat only pure CI statements, i.e., $[I \perp J \mid K]$ where $I, J, K$ are pairwise disjoint. Any general CI statement with overlaps between the three sets decomposes, analogously to the localization rule, into a conjunction of pure CI statements and functional dependence statements. For a regular Gaussian, functional dependences are always false, so this is no restriction in generality. In particular, the general statement $[N \perp M \mid L]$, which frequently appears later, is equivalent to $((N \setminus L) \perp (M \setminus L) \mid L)$ which is pure provided that $L \supseteq N \cap M$.

Polymatroids and selfadhesivity. A polymatroid over the finite ground set $N$ is a function $h : 2^N \to \mathbb{R}$ assigning to every subset $K \subseteq N$ a real number, such that $h$ is

- normalized: $h(\emptyset) = 0$,
- isotone: $h(I) \leq h(J)$ for $I \subseteq J$,
- submodular: $h(I) + h(j) \geq h(I \cup j) + h(I \cap j)$.

With the linear functional $\triangle(I, J|K) : h := h(IK) + h(JK) - h(IJK) - h(K)$, submodularity can be restated as $\triangle(I, J|K) \cdot h \geq 0$ for all pairwise disjoint $I, J, K$. If $h_\xi$ is the entropy vector of a discrete random vector $\xi$, i.e., $h_\xi(K)$ is the Shannon entropy of the marginal vector $\xi_K$, then it is a polymatroid and the quantity $\triangle(I, J|K) \cdot h_\xi$ is known as the conditional mutual information $I(\xi_I; \xi_J|\xi_K)$. Its vanishing is equivalent to the conditional independence $[\xi_I \perp \xi_J \mid \xi_K]$. Hence we may define the CI structure of a polymatroid as $[h] := \{ [i \perp j \mid K] \in \mathcal{A}_N : \triangle(ij|K) \cdot h = 0 \}$. These structures are called (elementary) semimatroids in [Mat94] and (equivalently, but based on properties of multinformation instead of entropy vectors) structural semigraphoids in [Stu94]. Again, per [Mat94] a localization rule holds for them which we use to interpret the containment of non-elementary CI statements:

$$(L') \quad [I \perp J \mid K] \in [h] \iff \bigwedge_{i \in I, j \in J, K \subseteq L \subseteq IJK \setminus ij} [i \perp j \mid L] \in [h].$$

This rule can be proved from the semigraphoid axioms and hence it holds true also for Gaussians. In this case, it is equivalent to the shorter rule (L) using that Gaussians are compositional graphoids.

Matuš in [Mat07a] introduced the notions of adhesive extensions and selfadhesive polymatroids to mimic a curious amalgamation property of entropy vectors. The underlying construction is the Copy lemma of [ZY98], also known as the conditional product; see [Stu05, Section 2.3.3]. For any polymatroid $g : 2^N \to \mathbb{R}$ and subset $L \subseteq N$ the restriction $g|_L : 2^L \to \mathbb{R}$ given by $g|_L(K) := g(K)$. Let $g$ and $h$ be two polymatroids on ground sets $N$ and $M$, respectively, and suppose that their restrictions $g|_L$ and $h|_L$ to $L = N \cap M$ coincide. A polymatroid $f$ on $NM$ is an adhesive extension of $g$ and $h$ if:

- $f|_N = g$ and $f|_M = h$,
- $[N \perp M \mid L] \in [f]$. 

Rules of this form go back to [Mat92]. A weaker localization rule $(L')$ (discussed below) holds for all semigraphoids, whereas the one presented above can be proved for compositional graphoids; see [LS18] in the context of graphical models. In both cases, a general CI statement is reduced to a conjunction of elementary CI statements $[\xi_i \perp \xi_j \mid \xi_K]$ about the independence of two singletons. We adopt the form $[I \perp J \mid K]$ for CI statements $[\xi_I \perp \xi_J \mid \xi_K]$ without the mention of a random vector. Again, per [Mat94] a localization rule holds for them which we use to interpret the containment of non-elementary CI statements. The CI structure of $\Sigma$ is the set 

$$ [\Sigma] := \{ [i \perp j \mid K] \in \mathcal{A}_N : \det \Sigma_{iK,jK} = 0 \}.$$
Since \( L \subseteq N \) and \( L \subseteq M \), the statement \([N \perp M \mid L]\) is naturally equivalent to the pure CI statement \([N' \perp M' \mid L]\) with \( N' = N \setminus L \) and \( M' = M \setminus L \). In polymatroidal terms, \( N \) and \( M \) are said to form a modular pair in \( f \) if this CI statement holds.

Next, suppose that we have only one polymatroid \( h \) on ground set \( N \) and fix \( L \subseteq N \). An \( L \)-copy of \( N \) is a finite set \( M \) with \(|M| = |N|\) and \( M \cap N = L \). We fix a bijection \( \pi : N \to M \) which preserves \( L \) pointwise. The polymatroid \( h \) is a selfadhesive polymatroid at \( L \) if there exists a polymatroid \( \overline{h} \) which is an adhesive extension of \( h \) and its induced copy \( \pi(h) \) over their common restriction to \( L \). The polymatroid is selfadhesive if it is selfadhesive at every \( L \subseteq N \). The fundamental result of [Mat07a] is:

**Theorem.** Any two of the restrictions of an entropic polymatroid have an entropic adhesive extension. In particular, entropy vectors are selfadhesive.

**Remark 2.1.** The set of polymatroids on \( N \) which are selfadhesive forms a rational, polyhedral cone in \( \mathbb{R}^{2^N} \). To see this, let \( N \), a subset \( L \subseteq N \) and an \( L \)-copy \( M \) of \( N \) with bijection \( \pi \) be fixed. The conditions for a pair \((h, \overline{h})\), where \( h : 2^N \to \mathbb{R} \) and \( \overline{h} : 2^{NM} \to \mathbb{R} \), to be polymatroids and \( \overline{h} \) to be an adhesive extension of \( h \) and \( \pi(h) \) are homogeneous linear equalities and inequalities with integer coefficients in the entries of \( h \) and \( \overline{h} \). Hence, the set of such pairs is a rational, polyhedral cone in \( \mathbb{R}^{2^N} \times \mathbb{R}^{2^{NM}} \). By the Fourier–Motzkin elimination theorem [Zie95, Theorem 1.4], these properties are inherited by the projection down to \( \mathbb{R}^{2^N} \) which consists of all polymatroids \( h \) which are selfadhesive at \( L \). Intersecting these cones for all \( L \) gives the desired set of selfadhesive polymatroids and shows that this set is a rational, polyhedral cone.

**Remark 2.2.** Linear inequalities which are valid for entropic polymatroids are called information inequalities. The above observation implies that selfadhesivity, as a necessary condition for entropicness, captures only finitely many information inequalities for each fixed \( N \). By contrast, Matúš [Mat07b] showed that even for \(|N| = 4\) there are infinitely many irredundant information inequalities.

In the \(|N| = 4\) case, the cone of selfadhesive polymatroids is characterized (in addition to the polymatroid properties) by the validity of the Zhang–Yeung inequalities (see Remark 3.3). In this sense, selfadhesivity is a reformulation of the Zhang–Yeung inequalities using only the notions of restriction and conditional independence. The generalization of the concept of adhesive extension to more than one \( L \)-copy of a polymatroid leads to the book inequalities of [Csi14].

### 3. Adhesive Extensions of Gaussians

The analogous result for Gaussian covariance matrices is our main theorem:

**Theorem 3.1.** Let \( \Sigma \in \text{PD}_N \) and \( \Sigma' \in \text{PD}_M \) be two covariance matrices with common restriction \( \Sigma_L = \Sigma'_L \), where \( L = N \cap M \). There exists a unique \( \Phi \in \text{PD}_{NM} \) such that:

- \( \Phi_N = \Sigma \) and \( \Phi_M = \Sigma' \),
- \([N \perp M \mid L] \in \|\Phi\|\).

**Proof.** Let \( N' = N \setminus L \), \( M' = M \setminus L \). We use the following names for blocks of \( \Sigma \) and \( \Sigma' \):

\[
\Sigma = \begin{pmatrix}
L & N' \\
N' & A & Y
\end{pmatrix}, \quad \Sigma' = \begin{pmatrix}
L & M' \\
M' & B & Z
\end{pmatrix}.
\]

Consider the matrix

\[
\Phi = \begin{pmatrix}
L & N' & M' \\
N' & A & B \\
M' & Y & Z
\end{pmatrix}.
\]
where $\Lambda$ will be determined shortly. Its restrictions to $N$ and $M$ are clearly equal to $\Sigma$ and $\Sigma'$, respectively. The CI statement $[N \perp \perp M \mid L]$ is equivalent to the rank requirement $\text{rk} \Phi_{N,M} = |N \cap M| = |L|$, but then rank additivity of the Schur complement shows

$$|L| = \text{rk} \Phi_{N,M} = \text{rk} \left( \begin{array}{cc} X & B \\ A^T & \Lambda \end{array} \right) = \text{rk} X + \text{rk}(\Lambda - A^T X^{-1} B).$$

This implies $\Lambda = A^T X^{-1} B$ and thus $\Phi$ is uniquely determined by $\Sigma$ and $\Sigma'$ via the two conditions in the theorem. To show positive definiteness, consider the transformation

$$P = \begin{pmatrix} L & N' & M' \\ 1 & -X^{-1} A & -X^{-1} B \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

of the bilinear form $\Phi$:

$$P^T \Phi P = \begin{pmatrix} X & 0 & 0 \\ 0 & Y - A^T X^{-1} A & 0 \\ 0 & 0 & Z - B^T X^{-1} B \end{pmatrix} = \begin{pmatrix} \Sigma_L & 0 & 0 \\ 0 & \Sigma_L & 0 \\ 0 & 0 & \Sigma' \end{pmatrix}.$$ 

The result is clearly positive definite and since $P$ is invertible, this shows $\Phi \in \text{PD}_{N,M}$. □

**Remark 3.2.** An alternative proof of this theorem was kindly pointed out by one of the referees. It relies on viewing the existence of $\Phi$ as a positive definite matrix completion problem where the entries of $\Phi_N$ and $\Phi_M$ are prescribed and the submatrix $\Phi_{N',M'}$ is left unspecified. The machinery developed in [GJSW84] shows that a positive definite completion exists and that there is a unique completion $\Psi$ with maximal determinant. This matrix satisfies $(\Psi^{-1})_{N',M'} = 0$ which is equivalent to $[N \perp \perp M \mid L]$ by the duality concept in Gaussian CI theory; cf. [Boe22, Proposition 3.10].

**Remark 3.3.** Zhang and Yeung [ZY98] proved the first information inequality for entropy vectors which is not a consequence of the Shannon inequalities (equivalently, the polymatroid properties). It can be expressed as the non-negativity of the functional

$$\nabla(i, j|kl) := \Delta(kl|i) + \Delta(kl|j) + \Delta(ij) - \Delta(kl) - \Delta(ikl) - \Delta(il|k) + \Delta(kl|i).$$

Matuš [Mat07a] characterized the selfadhesive polymatroids over a 4-element ground set as those polymatroids satisfying $\nabla(i, j|kl) \geq 0$ for all choices of $i, j, k, l$. As a corollary to Theorem 3.1 we obtain that the multiinformation vectors and hence the differential entropy vectors of Gaussian distributions satisfy the Zhang–Yeung inequalities. This is one half of the result proved by Lučička [Lne03]. However, that result also follows from the metatheorem of Chan [Cha03] since $\nabla(i, j|kl)$ is balanced.

In the theory of regular Gaussian conditional independence structures, it is natural to relax the positive definiteness assumption on $\Sigma$ to that of principal regularity, i.e., all principal minors, instead of being positive, are required not to vanish. Principal regularity is the minimal technical condition which allows the formation of all Schur complements and the property is inherited by principal submatrices and Schur complements, hence enabling analogues of marginalization and conditioning over general fields instead of the field $\mathbb{R}$; see [Boe21] for applications. However, the last step in the above proof of Theorem 3.1 requires positive definiteness and does not work for principally regular matrices.
Example 3.4. Consider the following principally regular matrix over \( N = i j k l \):

\[
\Gamma = \begin{pmatrix}
  i & j & k & l \\
  1 & 0 & 0 & \frac{1}{\sqrt{2}} \\
  0 & 1 & \frac{1}{2\sqrt{2}} & 0 \\
  0 & \frac{1}{\sqrt{2}} & 1 & \frac{\sqrt{3}}{2} \\
  \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{2} & 1
\end{pmatrix}
\]

and fix \( L = ij \). By the proof of Theorem 3.1, the submatrix and rank conditions uniquely determine an adhesive extension of \( \Gamma \) with an \( L \)-copy of itself over the ground set \((ijklk'l')\). This unique candidate matrix is

\[
\begin{pmatrix}
  i & j & k & l & k' & l' \\
  1 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
  0 & 1 & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & 0 \\
  0 & \frac{1}{\sqrt{2}} & 1 & \frac{\sqrt{3}}{2} & 1 & 0 \\
  \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{2} & 1 & 0 & \frac{1}{2} \\
  \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{3}}{2} & 1 & 0 & \frac{1}{2}
\end{pmatrix}
\]

But this matrix is not principally regular, as the \((lk'l')\)-principal minor is zero. However, the CI structure \( G = \mathbb{L}[\Gamma] \) is the dual of the graphical model for the undirected path \( i - l - k - j \); cf. [LM07, Section 3]. This implies that \( G \) is representable by a positive definite matrix with rational entries and even though the particular matrix representation \( \Gamma \) does not have a selfadhesive extension (in the sense of Theorem 3.1), another representation of \( G \) exists which is positive definite and hence selfadhesive.

4. Structural selfadhesivity

The existence of adhesive extensions and in particular selfadhesivity of positive definite matrices induces similar properties on their CI structures, since the conditions in Theorem 3.1 can be formulated using only the concepts of restriction and conditional independence. On the CI level, we sometimes use the term structural selfadhesivity to emphasize that it is generally a weaker notion than what is proved for covariance matrices above. Selfadhesivity can be used to strengthen known properties of CI structures: if it is known that all positive definite matrices have a certain distinguished property \( p \), then the fact that \( \Sigma \) and any \( L \)-copy of it fit into an adhesive, positive definite extension obeying \( p \) says more about the structure of \( \Sigma \) than \( p \) alone. We begin by making precise the notion of a property:

Definition 4.1. Let \( \mathfrak{A}_N = 2^{\mathcal{A}_N} \) be the set of all CI structures over \( N \). For \( N = [n] = \{ 1, \ldots, n \} \) we use abbreviations \( \mathcal{A}_n \) and \( \mathfrak{A}_n \). A property of CI structures is an element \( p \) of the property lattice

\[
\mathcal{P} := \bigotimes_{n=1}^{\infty} 2^{\mathfrak{A}_n}.
\]

A property \( p \) consists of one set \( p(n) \subseteq \mathfrak{A}_n \) per finite cardinality \( n \). This is the set of CI structures over \([n]\) which “have property \( p \”. CI structures \( \mathcal{L} \) and \( \mathcal{M} \) over \( N \) and \( M \), respectively, are isomorphic if there is a bijection \( \pi : N \to M \) such that under the induced map \( \mathcal{M} = \pi(\mathcal{L}) \). We are only interested in properties which are invariant under isomorphy. Hence, the choice of ground sets \([n]\) presents no restriction. Moreover, we freely identify isomorphic CI structures in the following. In particular, each \( k \)-element subset \( K \subseteq [n] \) will be tacitly identified with \([k]\) and we use notation such as \( p(K) \).
Example 4.2. By the localization rule \((L')\), the well-known semigraphoid axioms of \cite{PP85} reduce to the single inference rule
\[ (S) \quad [i 
abla j \mid L] \land [i 
abla k \mid jL] \Rightarrow [i 
abla j \mid kL] \land [i 
abla k \mid L]. \]
Being a semigraphoid is a property defined by
\[ sg(n) := \left\{ \mathcal{L} \subseteq \mathcal{A}_n : (S) \text{ holds for } \mathcal{L} \text{ for all } ijk \in \binom{[n]}{3} \text{ and } L \subseteq [n] \setminus ijk \right\}. \]
Being realizable by a Gaussian distribution is another property
\[ g^+(n) := \{ \llbracket \Sigma \rrbracket \in \mathcal{A}_n : \Sigma \in \text{PD}_n \}. \]
Both are closed under restriction, which can be expressed as follows: for every \( \mathcal{L} \in \mathcal{p}(N) \) and every \( K \subseteq N \) we have \( \mathcal{L}|_K := \mathcal{L} \cap \mathcal{A}_K \in \mathcal{p}(K) \).

The property lattice is equipped with a natural order relation of component-wise set inclusion from the boolean lattices \( 2^{\mathcal{A}_n} \). This order relation \( \leq \) compares properties by generality: if \( p \leq q \), then for all \( n \geq 1 \) we have \( p(n) \subseteq q(n) \), and \( p \) is sufficient for \( q \) and, equivalently, \( q \) is necessary for \( p \). A function \( \varphi \) on the property lattice is recessive if for every \( p \in \mathcal{p} \) we have \( \varphi(p) \leq p \). It is monotone if \( p \leq q \) entails \( \varphi(p) \leq \varphi(q) \).

Definition 4.3. Let \( p \) be a property of CI structures. The selfadhesion \( p^{sa}(N) \) of \( p \) is the set of CI structures \( \mathcal{L} \) such that for every \( L \subseteq N \) together with an \( L \)-copy \( M \) of \( N \) and bijection \( \pi : N \to M \) there exists \( \mathcal{Z} \in \mathcal{p}(NM) \) satisfying the conditions:
\[ \begin{align*}
- & \mathcal{Z}_{\mid N} = \mathcal{L}, \mathcal{Z}_{\mid M} = \pi(\mathcal{L}), \text{ and} \\
- & [N \nabla M \mid L] \in \mathcal{Z}.
\end{align*} \]
A property is selfadhesive if \( p = p^{sa} \).

The following is a direct consequence of Theorem 3.1:

Corollary 4.4. The property \( g^+ \) of being regular Gaussian is selfadhesive.

Proof. Let \( \mathcal{L} \in g^+(N) \) be Gaussian and \( \Sigma \in \text{PD}_N \) a realizing matrix. For any \( L \subseteq N \), Theorem 3.1 applies with \( \Sigma' = \Sigma \) and gives a matrix \( \Phi \) whose CI structure is a witness for the structural selfadhesivity of \( \mathcal{L} \) at \( L \). \( \square \)

Lemma 4.5. The operator \( ^{sa} \) is recessive and monotone on the property lattice.

Proof. Let \( p \) be a property and \( \mathcal{L} \in p^{sa}(N) \). In particular, \( \mathcal{L} \) is selfadhesive with respect to \( p \) at \( L = N \). The \( L \)-copy \( M \) of \( N \) in the definition must be \( M = N \) and it follows that \( \mathcal{L} \in \mathcal{p}(NM) = \mathcal{p}(N) \). This proves recessiveness \( p^{sa} \leq p \). For monotonicity, let \( p \leq q \) and \( \mathcal{L} \in \mathcal{p}^{sa}(N) \). Then for every \( L \) with \( L \)-copy \( M \) of \( N \) there exists a certificate for the existence of \( \mathcal{L} \) in \( p^{sa} \). This certificate lives in \( \mathcal{p}(NM) \subseteq q(NM) \) which proves \( \mathcal{L} \in q^{sa}(N) \). \( \square \)

Thus, from monotonicity and the fact that \( g^+ \) is a fixed point of selfadhesion, we can conclude that a property which is necessary for Gaussianity remains necessary after selfadhesion. Since selfadhesivity makes properties more specific, this allows us to take known necessary properties of Gaussian CI and to derive new, stronger properties from them.

Corollary 4.6. If \( g^+ \leq p \), then \( g^+ \leq p^{sa} \). \( \square \)
Iterated application of selfadhesion gives rise to a chain of ever more specific properties $g^+ \leq \cdots \leq p^{k \cdot \text{sa}} \leq \cdots \leq p^{2 \cdot \text{sa}} := (p^{\text{sa}})_{\text{sa}} \leq p$. For each fixed component $n$ of the property, this results in a descending chain in the finite boolean lattice $2^{\mathcal{A}}$ which must stabilize eventually. However, the whole property $p$ has a countably infinite number of components and it is not clear if iterated selfadhesions converge after finitely many steps to the limit $p^{(\omega \cdot \text{sa})} := \bigwedge_{k=1}^{\infty} p^{k \cdot \text{sa}}$ in the property lattice.

**Question 4.7.** Does $p^{\cdot \text{sa}}$ stabilize after the first application to “well-behaved” properties like $sg$, i.e., is $sg^{\text{sa}} = sg^{\omega \cdot \text{sa}}$? Under which assumptions on a property does $p^{\cdot \text{sa}}$ stabilize after a finite number of applications?

We now turn to the question which closure properties of $p$ are recovered for $p^{\text{sa}}$. For example, if for every $\mathcal{L}, \mathcal{L}' \in p(N)$ we have $\mathcal{L} \cap \mathcal{L}' \in p(N)$, then $p$ is closed under intersection. Semigraphoids enjoy this closure property because they are axiomatized by the Horn clauses $(S)$. The following lemma shows that all iterated selfadhesions inherit closure under intersection.

**Lemma 4.8.** If $p$ is closed under intersection, then so is $p^{\cdot \text{sa}}$.

**Proof.** Let $\mathcal{L}, \mathcal{L}' \in p^{\text{sa}}(N)$ and fix a set $L \subseteq N$ and an $L$-copy $M$ of $N$ with bijection $\pi$. There are $\mathcal{L}$ and $\mathcal{L}'$ in $p(\mathcal{N}M)$ witnessing the selfadhesivity of $\mathcal{L}$ and $\mathcal{L}'$, respectively, at $L$. Their intersection $\mathcal{L} \cap \mathcal{L}'$ is in $p(\mathcal{N}M)$ by assumption and we have

1. $(\mathcal{L} \cap \mathcal{L}')|_N = \mathcal{L}|_N \cap \mathcal{L}'|_N = \mathcal{L} \cap \mathcal{L'},$
2. $(\mathcal{L} \cap \mathcal{L}')|_M = \mathcal{L}|_M \cap \mathcal{L}'|_M = \pi(\mathcal{L}) \cap \pi(\mathcal{L}') = \pi(\mathcal{L} \cap \mathcal{L'}),$
3. $[N \perp M | L] \in \mathcal{L} \cap \mathcal{L'}.$

Thus it proves selfadhesivity of $\mathcal{L} \cap \mathcal{L'}$ with respect to $p$ at $L$. \hfill $\square$

Similarly to matroid theory, minors are the natural subconfigurations of CI structures. They are the CI-theoretic abstraction of marginalization and conditioning on random vectors.

**Definition 4.9.** Let $\mathcal{L} \subseteq \mathcal{A}_N$ and $x \in N$. The marginal and the conditional of $\mathcal{L}$ on $N \setminus x$ are, respectively,

$$\mathcal{L} \setminus x := \{ [i \perp j | K] \in \mathcal{A}_{N \setminus x} : [i \perp j | K] \in \mathcal{L} \} = \mathcal{L} \cap \mathcal{A}_{N \setminus x},$$

$$\mathcal{L} / x := \{ [i \perp j | K] \in \mathcal{A}_{N \setminus x} : [i \perp j | xK] \in \mathcal{L} \}.$$ 

A minor of $\mathcal{L}$ is any CI structure which is obtained by a sequence of marginalizations and conditionings.

If for every $\mathcal{L} \in p(N)$ and every minor $\mathcal{K}$ of $\mathcal{L}$ on ground set $M \subseteq N$ we have $\mathcal{K} \in p(M)$, then $p$ is minor-closed. Minor-closedness is necessary for the existence of a finite axiomatization of a property $p$. More concretely, [Mat97] studied descriptions of properties by finitely many “forbidden minors”, which is under natural regularity assumptions equivalent to having a finite axiomatic description by boolean CI inference formulas; cf. [Boe22, Section 4.4] for details.

**Lemma 4.10.** If $p \leq sg$ is minor-closed, then so is $p^{\cdot \text{sa}}$.

**Proof.** By induction it suffices to prove closedness under marginals and conditionals. Let $\mathcal{L} \in p^{\text{sa}}(N)$ and $x \in N$. First, we prove that $\mathcal{L} \setminus x \in p^{\text{sa}}(N \setminus x)$. Fix $L \subseteq N \setminus x$ and an $L$-copy $M$ of $N$ with bijection $\pi$ and let $\mathcal{L}$ be the witness for selfadhesivity of $\mathcal{L}$ at $L$. The minor $\mathcal{L} \setminus \{ x, \pi(x) \}$ is in $p(\mathcal{N}M \setminus \{ x, \pi(x) \})$ by assumption of minor-closedness; and note that $M \setminus \pi(x)$ is an $L$-copy of $N \setminus x$. Moreover, $\mathcal{L} \setminus \{ x, \pi(x) \}|_{N \setminus x} = \mathcal{L} \setminus x$ which is isomorphic to $\pi(\mathcal{L} \setminus x) = \pi(\mathcal{L}) \setminus \pi(x) = (\mathcal{L} \setminus \{ x, \pi(x) \})|_{M \setminus \pi(x)}$. For the last argument we need the semigraphoid
property to hold for p. This ensures by [Stu05, Lemma 2.2] that the localization rule (L') applies.

This rule shows that \([N \perp M \mid L] \in \overline{\mathcal{L}}\) is equivalent to

\[
\bigwedge_{i \in N', j \in M', L \subseteq P \subseteq N \mid x \setminus j} [i \perp j \mid P] \in \overline{\mathcal{L}}.
\]

Applying the rule (L') again in reverse to a subset of these elementary CI statements shows that \(((N \setminus x) \perp (M \setminus \pi(x)) \mid L) \in \overline{\mathcal{L}}\) holds, which finishes the proof that \(\overline{\mathcal{L}} \setminus \{x, \pi(x)\}\) is a witness for the selfadhesiveness of \(\mathcal{L} \setminus x\) at \(L\).

To prove that \(\mathcal{L} / x \in p^{sa}(N \setminus x)\), pick any \(L \subseteq N \setminus x\) and let \(M\) be an \(Lx\)-copy of \(N\) with bijection \(\pi\). Note that \(M \setminus x\) is an \(L\)-copy of \(N \setminus x\) with bijection \(\pi|_{N \setminus x}\). Let \(\overline{\mathcal{L}} \in p(NM)\) be a witness for the selfadhesiveness of \(\mathcal{L}\) at \(Lx\) and consider the conditional \(\overline{\mathcal{L}} / x\):

\[
(\overline{\mathcal{L}} / x)|_{N \setminus x} = \{(ij|K) \in A_{N \setminus x} : (ij|Kx) \in \overline{\mathcal{L}}\}
= (\overline{\mathcal{L}}|_N) / x = \mathcal{L} / x.
\]

An analogous computation shows \((\overline{\mathcal{L}} / x)|_{M \setminus x} = \pi(\mathcal{L} / x)\) using that \(x\) is fixed by \(\pi\). Moreover, we have \([N \perp M \mid Lx] \in \overline{\mathcal{L}}\) which is equivalent to \(((N \setminus x) \perp (M \setminus x) \mid Lx) \in \overline{\mathcal{L}}\) since \(x \in N \cap M\). But this entails \(((N \setminus x) \perp (M \setminus x) \mid L) \in \overline{\mathcal{L}} / x\) and hence \(\mathcal{L} / x\) is selfadhesive at \(L\) with witness \(\overline{\mathcal{L}} / x\).

\[\square\]

**Question 4.11.** Does \(sg^{sa}\) have a finite axiomatization? Is finite axiomatizability or finite non-axiomatizability in general preserved by selfadhesiveness?

**4.1. Selfadhesivity testing.** Whether or not a CI structure \(\mathcal{L} \subseteq A_N\) is in \(p^{sa}(N)\) can be checked algorithmically if an oracle \(p(\mathcal{L})\) for the property \(p\) is available. This oracle is a subroutine which receives a partially defined CI structure \(\mathcal{L}\) over \(N\), i.e., a set of CI statements or negated CI statements specifying constraints on some statements from \(A_N\). Then \(p\) decides if \(\mathcal{L}\) can be extended to a member of \(p(N)\).

**Algorithm 1** Blackbox selfadhesivity membership test

1: function is-selfadhesive(\(\mathcal{L}, p\))  \(\triangleright\) tests if \(\mathcal{L} \in p^{sa}(N)\)
2: for all \(L \subseteq N\) do
3: \(\langle M, \pi \rangle \leftarrow L\)-copy of \(N\) with bijection \(\pi : N \rightarrow M\)
4: \(\mathcal{L} \leftarrow \emptyset\)
5: for all \(s \in A_N\) do
6: if \(s \in \mathcal{L}\) then \(\mathcal{L} \leftarrow \mathcal{L} \cup \{s, \pi(s)\}\)
7: if \(s \notin \mathcal{L}\) then \(\mathcal{L} \leftarrow \mathcal{L} \cup \{\neg s, \neg \pi(s)\}\)
8: end for
9: \(\mathcal{L} \leftarrow \mathcal{L} \cup \{[N \perp M \mid L]\}\)  \(\triangleright\) or equivalent statements via (L')
10: if \(p(\mathcal{L}) = \text{false}\) then return \text{false}\)
11: end for
12: return \text{true}\)
13: end function

Each component \(p(n)\) of a property \(p\) is a set of subsets of \(A_n\). There are two principal ways of representing this set: explicitly, by listing its elements, or implicitly, by listing a set of abstract axioms in the form of boolean formulas which all its elements and no other CI structures satisfy. A typical application of Algorithm 1 takes in both, an explicit description of \(p(n)\) to iterate over, as well as an implicit description \(p\) of \(p\) to perform selfadhesivity testing for ground sets of sizes between \(n\) and \(2n\). It outputs only an explicit description of \(p^{sa}\) at a given index \(n\). Transforming this explicit description obtained from Algorithm 1 into an implicit description to call the algorithm again is akin to transforming a disjunctive normal form of a boolean formula.
into a conjunctive normal form, which is a hard problem. Moreover, it would be required to compute \( p^{sa}(m) \) explicitly for all \( n \leq m \leq 2n \). This makes it difficult to iterate selfadhesions.

**Remark 4.12.** The proof of Lemma 4.5 shows that a CI structure \( L \) satisfies selfadhesivity with respect to \( p \) at \( L = N \) if and only if \( L \) has property \( p \). In the other extreme case, every structure in \( p \) is selfadhesive at \( L = \emptyset \) if \( p \) is closed under the direct sum operation introduced in [Mat94]. Many useful properties are closed under direct sums because this operation mimics the independent joining of two random vectors; see [Mat04]. If this is known a priori, some selfadhesivity tests can be skipped.

We now proceed to apply Algorithm 1 to two practically tractable necessary conditions for Gaussian realizability. The computational results allow, via Corollary 4.6, the deduction of new CI inference axioms for Gaussians on five random variables.

### 4.2. Structural semigraphoids

It is easy to see that every Gaussian CI structure \( L = [\Sigma] \) can also be obtained from the correlation matrix \( \Sigma' \) of the original distribution \( \Sigma \). Hence, we may assume that \( \Sigma \) is a correlation matrix. In that case, the *multiinformation vector* of \( \Sigma \) is the map \( m_\Sigma : 2^N \to \mathbb{R} \) given by \( m_\Sigma(K) := -\frac{1}{2} \log \det \Sigma_K \). This function satisfies \( m_\Sigma(\emptyset) = m_\Sigma(i) = 0 \) for all \( i \in N \) and it is supermodular by the Koteljanskii inequality; see [JB93]. Similarly to entropy vectors, the equality condition in these inequalities characterizes conditional independence: \( \Delta(i|j|K) \cdot m_\Sigma = 0 \iff [i \perp \!\!\!\perp j \mid K] \in [\Sigma] \).

In the nomenclature of [Stu05, Chapter 5], \( m_\Sigma \) is an \( \ell \)-standardized supermodular function. The functions having these two properties form a rational, polyhedral cone \( S_N \) of codimension \( |N| + 1 \) in \( \mathbb{R}^{2N} \). Each of its facets is given by equality in precisely one of the supermodular inequalities \( \Delta(i|j|K) \leq 0 \) for an elementary CI statement \([i \perp \!\!\!\perp j \mid K] \in A_N \). Since the facets of this cone are in bijection with CI statements, it is natural to identify faces (intersections of facets) dually with CI structures (unions of CI statements). The property of CI structures defined by arising from a face of \( S_N \) is that of *structural semigraphoids*, denoted by \( \text{sg}^* \), and it is necessary for \( \text{sg}^+ \) since every Gaussian CI structure \([\Sigma] \) is associated with the unique face on which \( m_\Sigma \in S_N \) lies in the relative interior.

**Remark 4.13.** Structural semigraphoids can be equivalently defined via the face lattice of the cone of tight polymatroids, i.e., polymatroids \( h \) with \( h(N) = h(N \setminus i) \) for every \( i \in N \). The tightness condition poses no extra restrictions: for every polymatroid, there exists a tight polymatroid inducing the same pure CI statements (only differing in the functional dependences); cf. [MC16, Section III]. A proof of the equivalence is contained in [Boe22, Section 6.3].

Deciding whether a partially defined CI structure \( \tilde{L} \) is consistent with the structural semigraphoid property is a question about the incidence structure of the face lattice of \( S_N \). Such questions reduce to the feasibility of a rational linear program as previously demonstrated by [BHLS10]. Algorithm 2 relies on this insight by setting up the polyhedral description of the structural semigraphoidality test and then delegating the computation to specialized linear programming software.

Equipped with this oracle for \( \text{sg}^* \), Algorithm 1 can be applied to compute membership in \( \text{sg}^{sa}_* \). We run the structural selfadhesivity test for the *gaussoids* of [LM07] because they are easily computable candidates for Gaussian CI structures; see also [BDKS19]. For \( n = 4 \) random variables, the gaussoids which are structural semigraphoids already coincide with the realizable Gaussian structures (as classified in [LM07]) and selfadhesivity offers no improvement. This is no longer the case for five random variables:

**Computation 1.** There are 508817 gaussoids on \( n = 5 \) random variables modulo isomorphy. Of these 336838 are structural semigraphoids and 335047 of them are selfadhesive with respect to \( \text{sg}_* \).
Algorithm 2  Structural semigraphoid consistency test

1: function is-structural$(\mathcal{L})$ \hfill \triangleright tests if $\mathcal{L}$ is consistent with $\mathfrak{s}_N$ \hfill $P \leftarrow \{ m(\emptyset) = m(i) = 0 \text{ for all } i \in N \}$ \hfill $\triangleright H$ description of polyhedron
2: \hfill $\triangleright$ for all $s \in A_N$ do
3: \hfill $\triangleright$ if $s \in \mathcal{L}$ then $P \leftarrow P \cup \{ -\triangle(s) \cdot m = 0 \}$
4: \hfill $\triangleright$ else $P \leftarrow P \cup \{ -\triangle(s) \cdot m \geq 1 \}$
5: \hfill $\triangleright$ $\triangleright$ The condition $-\triangle(s) \cdot m > 0$ is equivalent to $\geq 1$ in a cone
6: \hfill $\triangleright$ end for
7: return is-feasible$(P)$ \hfill $\triangleright$ call an LP solver
8: end function

A semigraphoid $\mathcal{L}$ is structural if and only if it is induced by a polymatroid, i.e., $\mathcal{L} = [h]$. In this case, two distinct notions of selfadhesivity can be applied to $\mathcal{L}$: the first is Matúš’s definition of selfadhesivity for the inducing polymatroid $h$; and the second is structural selfadhesivity from Definition 4.3 for the CI structure $\mathcal{L}$ with respect to the property $\mathfrak{s}_N$. Analogously to Corollary 4.4, one sees that the second condition is implied by the first. The existence of a selfadhesive inducing polymatroid can be efficiently tested for ground set size four based on the polyhedral description of the cone of selfadhesive 4-polymatroids from [Mat07a, Corollary 6].

Computation 2. Out of the 1 285 isomorphy representatives of $\mathfrak{s}_4$, exactly 1 224 are in $\mathfrak{s}_4^a(4)$. Each of them is induced by a selfadhesive 4-polymatroid.

Question 4.14. Is every element of $\mathfrak{s}_4^a(N)$ induced by a selfadhesive $N$-polymatroid, for every finite set $N$?

4.3. Orientable gaussoids. Recall from [BDKS19] that a gaussoid is orientable if it is the support of an oriented gaussoid. Oriented gaussoids are a variant of CI structures in which every statement $\{ i \perp j \mid K \}$ has a sign $\{ 0, +, - \}$ attached, indicating conditional independence, positive or negative partial correlation, respectively. Oriented gaussoids are axiomatically defined and therefore SAT solvers are ideally suited to decide the consistency of a partially defined CI structure with these axioms. The property of orientability, denoted $\circ$, is obtained from the set of oriented gaussoids by mapping all CI statements oriented as 0 to elements of a CI structure and all statements oriented $+$ or $-$ to non-elements. To facilitate orientability testing, one allocates two boolean variables $V^0_s$ and $V^*_s$ for every CI statement $s$. The former indicates whether $s$ is 0 or not while the latter indicates, provided that $V^0_s$ is false, if $s$ is $+$ or $-$. Further details about oriented gaussoids, their axioms and use of SAT solvers for CI inference are available in [BDKS19]. Algorithm 3 gives a condensed account of the algorithm.

Algorithm 3 Orientable gaussoid consistency test

1: function is-orientable$(\mathcal{L})$ \hfill $\triangleright$ tests if $\mathcal{L}$ is consistent with $\circ(N)$ \hfill $\varphi \leftarrow$ oriented-gaussoid-axioms$(N)$ \hfill $\triangleright$ boolean formula
2: \hfill $\triangleright$ for all $s \in A_N$ do
3: \hfill $\triangleright$ if $s \in \mathcal{L}$ then $\varphi \leftarrow \varphi \land [V^0_s = \text{true}]$
4: \hfill $\triangleright$ else $s \in \mathcal{L}$ then $\varphi \leftarrow \varphi \land [V^*_s = \text{false}]$
5: \hfill $\triangleright$ $\varphi \leftarrow \varphi \land [V^0_s = \text{true} \Rightarrow V^*_s = \text{false}]$
6: \hfill $\triangleright$ there are only three signs $\{ 0, +, - \}$
7: \hfill $\triangleright$ end for
8: return is-satisfiable$(\varphi)$ \hfill $\triangleright$ call a SAT solver
9: end function
**Computation 3.** All orientable gaussoids on \( n = 4 \) are Gaussian. Of the 508,817 isomorphy classes of gaussoids on \( n = 5 \) precisely 175,215 are orientable and 168,010 are selfadhesive with respect to orientability.

4.4. **Structural orientable gaussoids.** The meet \( sg_\ast \land o \) of structural semigraphoids and orientable gaussoids in the property lattice is likewise necessary for Gaussianity and an oracle for it can be combined from the oracles of its two constituents. Its selfadhesion yields no improvement over apparently weaker properties:

**Computation 4.** The properties \( sg_\ast \land o \) and \( sg^a_\ast \land o \) coincide at \( n = 5 \) with 175,139 isomorphy types. On the other hand, \( sg_\ast \land o^a \), \( sg^a_\ast \land o^a \) and \( (sg_\ast \land o)^a \) coincide at \( n = 5 \) with 167,989 types.

Up to a few isolated examples in the literature, this represents the currently best known upper bound in the classification of realizable Gaussian conditional independence structures on five random variables. Examination of the difference \( (sg_\ast \land o)(5) \setminus (sg_\ast \land o)^a(5) \) reveals new axioms for Gaussian CI beyond structural semigraphoids and orientability, e.g.:

\[
\begin{align*}
[i \perp j | km] & \land [i \perp m | l] \land [j \perp k | i] \land [j \perp m] \land [k \perp l] \Rightarrow [i \perp j], \\
[i \perp k | jl] & \land [i \perp l | km] \land [j \perp k | i] \land [j \perp m | k] \land [k \perp l] \Rightarrow [i \perp k], \\
[i \perp k | jl] & \land [i \perp l | jm] \land [j \perp k | il] \land [j \perp m | k] \land [k \perp l] \Rightarrow [i \perp l].
\end{align*}
\]

The MathRepo page corresponding to this paper contains code and more information on how to obtain these inference rules algorithmically. Due to the large amount of data involved and the complexity of minimizing boolean formulas, it is currently not known how many genuinely new and mutually irredundant axioms are encoded in the results.

**Mathematical software and data repository.** SoPlex v4.0.0 was used to solve rational linear programs exactly; see [GSW12, GSW15, GBE+18]. To check orientability, we used the incremental SAT solver CaDiCaL v1.3.1 by [Bie19] and to enumerate satisfying assignments the AllSAT solver nbc_minisat_all v1.0.2 by [TS16]. Example 3.4 was found using Wolfram Mathematica v11.3 [WM]. The source code and results for all computations are available on the supplementary MathRepo website of the MPI-MiS and the KEEPER of the Max-Planck Society:

MathRepo: https://mathrepo.mpdl.mpg.de/SelfadhesiveGaussianCI/
KEEPER: https://keeper.mpdl.mpg.de/d/fbfe463162e94a14ac28/

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