CONFORMAL AND CP TYPES OF SURFACES OF CLASS $S$

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Abstract. In this paper we describe how to define the circle packing (cp) type (either cp parabolic or cp hyperbolic) of a Riemann surface of class $S$, and study the relation between this type and the conformal type of the surface.

1. Introduction and the definition of cp type

Let $X$ be a simply-connected Riemann surface, and suppose $h : X \to \mathbb{C}$ is a meromorphic function. The pair $(X, h)$ is called a Riemann surface of class $S$ if there exist $q$ points $a_1, a_2, \ldots, a_q \in \mathbb{C}$ such that the restriction map
\[
h : X \setminus \{h^{-1}(a_j) : j = 1, 2, \ldots, q\} \to \mathbb{C}\setminus\{a_1, \ldots, a_q\}
\]
is a topological covering map. In this case $(X, h) \in F_q(a_1, \ldots, a_q)$ is a common notation, but the simplified notation $X \in F_q$ is preferred when there is no confusion about the meromorphic function $h$ and the base points $a_1, \ldots, a_q \in \mathbb{C}$. Function theoretically $X \in F_q$ means that all the critical and asymptotic points of $h$ lie over only the finitely many points, say $a_1, \ldots, a_q$.

Consider a closed Jordan curve $\Gamma'_0 \subset \mathbb{C}$, called the base curve, passing through $a_1, a_2, \ldots, a_q$ in this order. Then we can think of $\Gamma'_0$ as a finite connected planar graph with vertices $a_1, \ldots, a_q$ and edges $[a_i, a_{i+1}]$, $i = 1, \ldots, q \mod q$. Therefore the dual graph of $\Gamma'_0$ is well-defined. Now we denote the dual graph by $\Gamma_0$, and observe that the pull-back of $\Gamma_0$ via $h$, $\Gamma := h^{-1}(\Gamma'_0)$, is a connected planar graph that is bipartite and homogeneous of degree $q$. This graph $\Gamma$ is called the Speiser graph or the line complex of $X \in F_q$. Conversely, for a given connected planar graph $\Gamma$ that is bipartite and homogeneous of degree $q$, it is possible to define a Riemann surface of class $S$ whose Speiser graph is $\Gamma$. Furthermore, it is known that for fixed base points $a_1, \ldots, a_q$ and the base curve $\Gamma'_0$, there is a one-to-one correspondence between (labeled) Speiser graphs and Riemann surfaces of class $S$. For more details, see for example [8, Chap. XI] or [3].

By the famous uniformization theorem [1, Chap. X], every simply-connected Riemann surface $X$ is conformally equivalent to one, and only one, of the following: the unit disk $\mathbb{D}$, or the whole complex plane $\mathbb{C}$, or the Riemann sphere $\overline{\mathbb{C}}$. Accordingly we say that the conformal type of $X$ is hyperbolic, parabolic, or elliptic, respectively. Thus when we study a Riemann surface of class $S$, $(X, h) \in F_q$, the conformal type of $X$ should have been already determined even before the meromorphic function $h$ was considered. However, this does not mean that $h$ has nothing to do with the type of $X$. In many cases it is even possible to determine the type of $X$ from the information about $h$. For example, we know from the Picard’s theorem.

Date: May 11, 2014.
2000 Mathematics Subject Classification. 30F20, 52C26.
Key words and phrases. type problem, Speiser graph, circle packing.
that $X$ must be hyperbolic if $h$ omits three points in $\mathbb{C}$. This kind of process, or problem, that is, determining the conformal type of $X$ using the information about $h : X \to \mathbb{C}$, is called the type problem.

The purpose of this paper is to study the type problem for Riemann surfaces of class $\mathcal{S}$, and compare the conformal type of $X \in F_q$ with its circle packing type (cp type) which we will define later.

In the type problem the elliptic case is often excluded from the beginning and $X$ is assumed to be open. This is because when the Riemann surface is conformally equivalent to $\mathbb{C}$, then it is compact and consequently distinguished from the other two cases very easily. Thus we always assume that $X$ is conformally equivalent to either the whole plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. Moreover, to define the cp type appropriately, we assume that the meromorphic function $h$ has no asymptotic values. Then one can show that the restriction map in (1.1) is a covering map of the carrier of $\mathcal{P}$ asymptotic value, then the pull-back $h^{-1}(\Gamma'_0)$ is not even a graph. Note that a vertex in $\Gamma'_0$ can be lifted to a point at infinity.)

Let $V$ be an index set, and recall that an indexed circle packing $\mathcal{P} = (P_v : v \in V)$ in the plane $\mathbb{C}$ is a collection of closed geometric disks with disjoint interiors. The contacts graph, or nerve, of a circle packing $\mathcal{P}$ is a graph whose vertex set is $V$ and such that an edge $[v, w]$ appears in the graph if and only if $P_v$ and $P_w$ intersects. An interstice of $\mathcal{P}$ is a connected component of the complement of $\bigcup_{v \in V} P_v$, and the carrier is the union of the packed disks and the finite interstices.

Now we are ready to describe our problem. For a given Riemann surface $(X, h) \in F_q$ of class $\mathcal{S}$, let $T_0$ be a triangulation of the Riemann sphere $\overline{\mathbb{C}}$ such that all the base points $a_1, \ldots, a_q$ are contained in the vertex set of $T_0$. We further assume that the pull-back graph $T := h^{-1}(T_0)$ is a disk triangulation. Then there exists a circle packing $\mathcal{P}$ in $\mathbb{C}$ whose nerve is combinatorially equivalent to (the 1-skeleton of) $T$ and whose carrier is either the whole complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$ [5 Corollary 0.5]. The graph $T$ is called circle packing parabolic type (cp parabolic) if the carrier of $\mathcal{P}$ is $\mathbb{C}$, and otherwise it is called cp hyperbolic. Our question is, does this cp type depend on the triangulation $T_0$ of $\overline{\mathbb{C}}$?

**Proposition 1.** Suppose $(X, h)$ is a Riemann surface of class $\mathcal{S}$ and $T_0$ and $T_0'$ are triangulations of $\overline{\mathbb{C}}$ whose vertex sets contain all the base points and such that their pull-back graphs $T := h^{-1}(T_0)$ and $T' := h^{-1}(T_0')$ are disk triangulations. Then the cp types of $T$ and $T'$ coincide.

The condition that $T$ (or $T'$) is a disk triangulation is a minor one which can be easily satisfied. In fact, if $T_0$ has no edge of the form $[a_i, a_j]$ for $i, j \in \{1, \ldots, q\}$, then $T$ must be a disk triangulation. A bad case happens only when we have multiple edges in $T$ between two vertices lying over some base points $a_i$’s (Figure 1).

Proposition 1 basically deals with the case when $\Gamma'$ is not of bounded valence, since it is trivially verified when the local degree of $h$ is uniformly bounded; i.e., when the dual Speiser graph $\Gamma'$ of bounded valence. This is because the pull-back triangulation $T$ is roughly isometric (see [11, p. 160, for the definition of rough isometries, which are also called quasi-isometries depending on the literature]) to the dual Speiser graph $\Gamma'$, hence $T$ is recurrent if and only if $\Gamma'$ is recurrent [11 Theorem 7.18] (we say that a graph is recurrent or transient if the simple random walk on the graph is recurrent or transient, respectively). Note that in this case $T$
is also of bounded valence. Now from [6, Theorem 1.1], where He and Schramm showed that a bounded valence disk triangulation graph $T$ is recurrent if and only if it is cp parabolic, we conclude that $T$ is cp parabolic if and only if $\Gamma'$ is recurrent. Similarly $\mathcal{T}$ is cp parabolic if and only if $\Gamma'$ is recurrent, hence the cp types of $T$ and $\mathcal{T}$ must coincide in this case.

In the course of a proof for Proposition 1 we encountered the following statement, which might be interesting by itself.

**Proposition 2.** Suppose $G$ is a disk triangulation graph and $G'$ is a semi-bounded refinement graph of $G$. Then $G$ is VEL-parabolic if and only if $G'$ is VEL-parabolic.

The definitions for VEL-parabolicity and refinement graphs are given in §2 and §3, respectively. After completion of this work we learned that the same result as Proposition 2 was obtained independently by Wood [13] using a different method.

**Definition 3.** A Riemann surface $(X, h)$ of class $\mathcal{S}$ is called cp parabolic if $T = h^{-1}(T_0)$ is cp parabolic, where $T_0$ is as in Proposition 1. Otherwise it is called cp hyperbolic.

**Remark 4.** We excluded the elliptic case from our consideration, since in this case everything becomes very trivial. Note that if $X$ is the Riemann sphere, then $h$ must be a rational map, hence the graph $T = h^{-1}(T_0)$ is a finite graph. Definitely this case is distinguished from the other two cases. If needed, however, one can define that $(X, h)$ is cp elliptic if $T$ is a finite graph.

As mentioned earlier, the conformal type of $X$ is uniquely determined by the uniformization theorem. Then a natural question is, does the conformal type of $X$ and cp type of $(X, h)$ agree? The answer for this question should be positive under some “nice” conditions, since certain finite circle packings approximate the Riemann map as conjectured by Thurston [12] and proved by Rodin and Sullivan [9]. For example, we have the affirmative answer if the local degree of $h$ is uniformly bounded. In this case one can show that $T$ is roughly isometric to the Speiser graph $\Gamma$ as well as the dual Speiser graph $\Gamma'$, hence $T$ must be cp parabolic if and only if $\Gamma$ is recurrent as discussed preceding Definition 3. (Note that every Speiser graph is of bounded valence, since it is a homogeneous graph.) Thus by applying Doyle’s criterion [2] (cf. [7]) to this case, we see that the Speiser graph $\Gamma$ is recurrent if and only if $X$ is conformally equivalent to $\mathbb{C}$. In general, however, the conformal type of $X$ and cp type of $(X, h)$ do not have to match.
Theorem 5. There exists a parabolic Riemann surface of class \( S \) that is \( cp \) hyperbolic.

If we extend the definition of \( cp \)-type to the surfaces with asymptotic values (Remark 14, p. 10), then the surface in [4] serves as a counterexample to the other part of the implication; that is, the surface constructed in Section 2 of [4], although it has some asymptotic values, is the example of \( X \in F_q \) such that \( X \) is of \( cp \) parabolic but conformally equivalent to the disk.

2. Vertex Extremal Length and Circle Packings

Let \( G = (V, E) \) be a graph, where \( V \) is the vertex set and \( E \) is the edge set of \( G \). Every edge \( e \in E \) is associated with two vertices \( v, w \in V \), saying that \( e \) is incident to \( v \) and \( w \), or \( e \) connects \( v \) and \( w \). In this case we write \( e = [v, w] \), and \( v \) and \( w \) are called the endpoints of \( e \). This notation might be confusing, however, since we allow multiple edges between two vertices. (Most Speiser graphs have multiple edges; for example see Figure 5.) On the other hand, we always assume that there is no self-loop.

A closed subset of the plane is called a face of \( G \) if it is the closure of a component of \( \mathbb{C} \setminus G \). Two vertices \( v \) and \( w \) are neighbors if \([v, w] \in E\), and we denote by \( N(v) \) the set of neighbors of \( v \in V \). The degree or valence of \( v \in V \) is defined by the number of edges from \( v \) to its neighbors, and denoted by \( \deg(v) \). We say that the graph \( G \) is of bounded valence if \( \sup \{ \deg(v) : v \in V \} < \infty \). A path \( \gamma \) in \( G \) is a finite or infinite sequence \([v_0, v_1, \ldots]\) of vertices such that \([v_i, v_{i+1}] \in E\) for every \( i = 0, 1, \ldots \). We denote by \( V(\gamma) \) the set of vertices which the path \( \gamma \) visits, and an infinite path is called transient if it visits infinitely many distinct vertices. A graph is called connected if every two vertices in the graph can be connected by a finite path, and infinite if it contains infinitely many vertices. Throughout this paper, we always assume that every graph is connected and infinite, unless otherwise stated.

(For given \((X, h) \in F_q\), the base curve \( \Gamma_0' \) and its dual \( \Gamma_0 \) are finite, though.)

A graph \( G \) is called planar if there is an embedding \( \iota \) from (the 1-complex of) \( G \) into \( \mathbb{C} \), and its image \( \iota(G) \) is called the embedded graph. In general there are some differences between a planar graph and its embedded graphs, since there could be two or more topologically different embeddings of the same planar graph, but we do not distinguish a planar graph from its embedded graph. A planar graph is called locally finite if the cardinality of \( V \cap K = \iota(V) \cap K \) is finite for every compact set \( K \subset \mathbb{C} \).

Definition 6. A graph \( G = (V, E) \) is called Vertex Extremal Length parabolic, or VEL-parabolic, if there exists a function \( m : V \to \mathbb{R}^+ \cup \{0\} \) such that

\[
\begin{align*}
(a) & \sum_{v \in V} m(v)^2 < \infty, \\
(b) & \sum_{v \in V(\gamma)} m(v) = \infty \quad \text{for every transient path } \gamma.
\end{align*}
\]

A function \( m \) satisfying these properties will be called parabolic \( v \)-metric. If there is no such \( v \)-metric, \( G \) is called VEL-hyperbolic.

He and Schramm showed in [6] that a disk triangulation graph \( G \) is VEL-parabolic if and only if \( G \) is \( cp \) parabolic. If \( G \) is of bounded valence, the VEL-parabolicity and recurrence of \( G \) are equivalent ([6], Theorem 8.1 and Theorem 2.6).
The following definition is due to Schramm [10].

**Definition 7.** Let $\tau > 0$. A Lebesgue measurable set $A \subset \mathbb{C}$ is called $\tau$-fat if for every $x \in A$ and $r > 0$ such that $D(x, r) := \{z : |z - x| < r\}$ does not contain $A$, the following inequality holds:

$$\text{area}(A \cap D(x, r)) \geq \tau \cdot \text{area}(D(x, r)).$$

Here $\text{area}(\cdot)$ denotes the 2-dimensional Lebesgue measure. A typical example of a fat set is an Euclidean disk. For instance, if $D = D(0, 1)$, $0 \leq x < 1$, and $0 < r < 1 + x$, then $D(x - r/2, r/2) \subset D \cap D(x, r)$. Therefore,

$$\text{area}(D \cap D(x, r)) \geq \text{area}(D(x - r/2, r/2)) = (1/4) \cdot \text{area}(D(x, r)),$$

showing that $D$ is $1/4$-fat.

**Lemma 8.** Suppose $A$ and $B$ are $\tau$-fat sets for some $\tau > 0$ and $A \cap B \neq \emptyset$. Then $A \cup B$ is $\tau/4$-fat.

**Proof.** Suppose $x \in A \cup B$. Without loss of generality, we assume that $x \in A$. If $D(x, r/2)$ does not contain $A$, then

$$\text{area}((A \cup B) \cap D(x, r)) \geq \text{area}(A \cap D(x, r)) \geq \text{area}(A \cap D(x, r/2))$$

$$\geq \tau \cdot \text{area}(D(x, r/2)) = (\tau/4) \cdot \text{area}(D(x, r))$$

as desired.

If $D(x, r/2)$ contains $A$ but $D(x, r)$ does not contain $A \cup B$, there exists $y \in A \cap B \subset B$ such that $|x - y| < r/2$. Since $D(y, r/2) \subset D(x, r)$, $A \subset D(x, r/2)$, and $A \cup B \not\subseteq D(x, r)$, one can easily see that $D(y, r/2)$ does not contain $B$. Therefore,

$$\text{area}((A \cup B) \cap D(x, r)) \geq \text{area}(B \cap D(x, r)) \geq \text{area}(B \cap D(y, r/2))$$

$$\geq \tau \cdot \text{area}(D(y, r/2)) = (\tau/4) \cdot \text{area}(D(x, r)),$$

which completes the proof. \qed

We conclude this section with the following lemma.

**Lemma 9** (Lemma 3.4 of [6]). Suppose $G = (V, E)$ is a locally finite planar graph with infinitely many vertices, and $\mathcal{P} = (P_v : v \in V)$ is a collection of $\tau$-fat sets satisfying the following properties:

1. For every $v \in V$, $P_v$ is a compact connected set in $\mathbb{C}$;
2. $\mathcal{P}$ is locally finite in $\mathbb{C}$; that is, for every compact set $K \subset \mathbb{C}$, there are only finitely many $v \in V$ such that $P_v \cap K \neq \emptyset$;
3. every $x \in \mathbb{C}$ is contained in $P_v$ for at most $M < \infty$ vertices $v \in V$, where $M$ does not depend on $x$;
4. if $[v, w] \in E$, then $P_v \cap P_w \neq \emptyset$.

Then $G$ is a VEL-parabolic graph.

In fact, He and Schramm showed the above lemma when $\mathcal{P}$ is a $\tau$-fat packing, but one can easily check that their proof also works in this case.
3. **Refinement Graphs and VEL-Parabolicity**

Let $G$ and $G'$ be two locally finite planar graphs. We say that $G'$ is a refinement graph of $G$ if every face and edge of $G$ are unions of a finite number of faces and edges, respectively, of $G'$. A refinement graph $G'$ of $G$ is called bounded (or semi-bounded) if there exists an absolute constant $M > 0$ such that every face (or every edge, respectively) of $G$ contains at most $M$ vertices of $G'$.

**Lemma 10.** Suppose $G' = (V', E')$ is a semi-bounded refinement graph of $G = (V, E)$. If $G'$ is VEL-parabolic, so is $G$. (Equivalently, if $G$ is VEL-hyperbolic, so is $G'$.)

**Proof.** For every given $v \in V$, we define the starlike set centered at $v$ by

$$E_v := \left( \bigcup_{w \in N(v)} [v, w] \right) \setminus N(v).$$

Here $N(v)$ is the set of neighbors in $G$, not in $G'$, and note that $E_v$ is a subset contained in (the 1-skeleton of) $G$. For future reference, we also let $\overline{E}_v = E_v \cup N(v)$.

Suppose that $m'$ is a parabolic $v$-metric that is obtained from the VEL-parabolicity of $G'$. Since $G'$ is a semi-bounded refinement graph of $G$, there are at most $M > 0$ vertices of $G'$ that are contained in one edge of $G$. Let

$$m(v) := 2M \cdot \max\{m'(w) : w \in E_v \cap V'\} \quad \text{for all } v \in V,$$

and we claim that $m$ is a parabolic $v$-metric of $G$.

First we show that $m$ is square-summable. But this is easy because every vertex $w \in V'$ is contained in $E_v$ for at most two $v \in V$, thus we have

$$\sum_{v \in V} m(v)^2 = \sum_{v \in V} 4M^2 \cdot \max\{m'(w)^2 : w \in E_v \cap V'\}$$

$$\leq 4M^2 \sum_{v \in V} \left( \sum_{w \in E_v \cap V'} m'(w)^2 \right) \leq 8M^2 \sum_{w \in V'} m'(w)^2 < \infty,$$

as desired.

Now suppose $e = [u, v]$ is an edge of $G$. Then because $e \subset E_u \cup E_v$,

$$\sum_{w \in [e \cap V']} m'(w) \leq M \cdot \max\{m'(w) : w \in (E_u \cup E_v) \cap V'\}$$

$$\leq M \cdot \max\{m'(w) : w \in E_u \cap V'\} + M \cdot \max\{m'(w) : w \in E_v \cap V'\}$$

$$= \frac{1}{2} (m(u) + m(v)).$$

Therefore, because any transient path $\gamma = [v_0, v_1, v_2, \ldots]$ in $G$ also can be realized as a transient path $\gamma' = [v_0, v_0, \ldots, v_0, v_1, \ldots, v_2, \ldots]$ in $G'$, we have

$$\infty = \sum_{w \in V(\gamma')} m'(w) \leq \sum_{i=0}^{\infty} \left( \sum_{w \in [v_i, v_{i+1} \cap V']} m'(w) \right)$$

$$\leq \sum_{i=0}^{\infty} \frac{1}{2} \{m(v_i) + m(v_{i+1})\} \leq \sum_{v \in V(\gamma)} m(v),$$

which completes the proof. \qed
Suppose $G = (V, E)$ is a planar graph and let $K$ be a positive integer. We say that $G$ satisfies the property $p(K)$ if $\min\{\deg(v), \deg(w)\} \leq K$ for all $[v, w] \in E$.

**Lemma 11.** Suppose $G = (V, E)$ is a disk triangulation graph satisfying the property $p(K)$ for some $K > 0$ and let $G' = (V', E')$ be a semi-bounded refinement graph of $G$. If $G$ is VEL-parabolic, then $G'$ is also VEL-parabolic. (Equivalently, VEL-hyperbolicity of $G'$ implies VEL-hyperbolicity of $G$.)

**Proof.** For $x \in \mathbb{C}$ contained in the 1-skeleton of $G$, we will use the expression $x \in G$. Also for $w \in (G \cap V')$, we define $V_w = \{v \in V : w \in E_v\}$. Note that if $w \in (G \cap V') \setminus V$, then $V_w$ consists of two vertices of $V$, the endpoints of the edge containing $w$, while $V_w = N(w) \cup \{w\}$ if $w \in V$. Also let $Z := \{v \in V : \deg(v) > K\}$. Here $\deg(v)$ is the number of edges incident to $v$ in the graph $G$, not in $G'$, but we will consider $Z$ a subset of $V'$ as well as $V$, since $V \subset V'$.

Let $m$ be the $v$-metric defined on $V$, which is assumed to exist by the VEL-parabolicity of $G$, and we define for all $w \in V'$

$$m'(w) := \begin{cases} m(w), & \text{if } w \in Z; \\ 3 \cdot \max\{m(v) : v \in V_w \setminus Z\}, & \text{if } w \in G \setminus Z; \\ 0, & \text{otherwise}. \end{cases}$$

Note that $V_w \setminus Z$ is always nonempty since $G$ satisfies the property $p(K)$. Consequently, $m'$ is well-defined.

There are at most $M > 0$ vertices of $G'$ contained in one edge of $G$, since $G'$ is a semi-bounded refinement graph of $G$. Thus if $v \in (V \setminus Z)$, $\overline{E_v}$ contains at most $KM$ vertices of $G'$. In other words, every $v \in (V \setminus Z)$ is contained in $V_w$ for at most $KM$ different vertices $w \in G \cap V'$. Therefore,

$$\sum_{w \in V'} m'(w)^2 = \sum_{w \in Z} m'(w)^2 + \sum_{w \in (G \cap V') \setminus Z} m'(w)^2 + \sum_{w \in V'} m'(w)^2 \leq \sum_{v \in Z} m(v)^2 + 9 \sum_{w \in (G \cap V') \setminus Z} \left( \sum_{v \in (V_w \setminus Z)} m(v)^2 \right) + 0 \leq \sum_{v \in Z} m(v)^2 + 9KM \sum_{v \in (V \setminus Z)} m(v)^2 \leq 9KM \sum_{v \in V} m(v)^2 < \infty,$$

which shows that $m'$ is square summable.

Let $F$ be the face set of $G$, and for each $v \in Z$ let $F_v$ be the union of the faces of $G$ containing $v$ on their boundaries. We define

$$\mathcal{F}_Z := \{F_v : v \in Z\} \cup \{f : f \in F \text{ and } f \not\subseteq F_v \text{ for any } v \in Z\}.$$

In other words, an element of $\mathcal{F}_Z$ is a face of $G$ if none of the vertices on its boundary belongs to $Z$, and other elements of $\mathcal{F}_Z$ are $F_v$’s for $v \in Z$. Definitely the interiors of the elements in $\mathcal{F}_Z$ are mutually disjoint and we have $\bigcup_{f \in \mathcal{F}_Z} f = C$. Now suppose $\gamma' = [w_0, w_1, \ldots]$ is a transient path in $G'$. Let $\{w_0 = w_{i_0}, w_{i_1}, w_{i_2}, \ldots\}$ be a subset of $G \cap V(\gamma') \subset G \cap V'$ whose elements are indexed so that the finite sub-path $\gamma'_k = [w_{i_k}, w_{i_k+1}, w_{i_k+2}, \ldots, w_{i_k+1}, -1, w_{i_k+1}]$ is contained in $f_k$ for some $f_k \in \mathcal{F}_Z$. Also note that $w_{i_k}, w_{i_k+1} \in \partial f_k$, where $\partial f_k$ denotes the boundary of $f_k$. 


Figure 2. the cases when $\gamma'_k$ passes through $v \in Z$ (left) and passes near but not through $v \in Z$ (right)

For each $k$, we will find a compact set $\Lambda_k$ which is contained in the 1-skeleton of $G$ and whose $m'$-length is comparable to the $m'$-length of $\gamma'_k$. Let $\Lambda_k = \partial f_k$ if $f_k$ is a triangle; i.e., if $\partial f_k \cap Z = \emptyset$. If $f_k = F_v$ for some $v \in Z$, we first consider the case that the subarc $\gamma'_k$ passes through the vertex $v$. In this case, we define $\Lambda_k$ as the union of all edges of $G$ with one end at $v$ and the other end at a vertex in $\partial F_v \cap (\{w_{ik} \cup w_{ik+1}\} \cup V_w)$ (Figure 2). Note that there are at most six such edges, and $\Lambda_k$ is a union of exactly six edges only when both $w_{ik}$ and $w_{ik+1}$ are in $V_w$.

Finally if $f_k = F_v$ for some $v \in Z$ but the subarc $\gamma'_k$ does not pass through $v$, there exist $f_1, f_2, \ldots, f_l \in F$ which are contained in $F_v = f_k$ such that $f_j \cap \gamma'_k \neq \emptyset$, $j = 1, 2, \ldots, l$. In this case we define $\Lambda_k = \partial F_v \cap (\partial f_1 \cup \partial f_2 \cup \cdots \cup \partial f_l)$.

From the definition of $m'$ and $\Lambda_k$, it is easy to see that

$$\sum_{v \in (\Lambda_k \cap V)} m(v) \leq \sum_{w \in (\gamma'_k \cap V')} m'(w) \leq \sum_{j=ik}^{ik+1} m'(w_j)$$

for all $k = 0, 1, 2, \ldots$. Furthermore, $\Lambda := \bigcup_{k=1}^{\infty} \Lambda_k$ is a connected unbounded set, hence there exists a transient path $\gamma \subset \Lambda \cap G$. Therefore by (3.1),

$$\sum_{v \in V(\gamma)} m(v) \leq \sum_{k=0}^{\infty} \left( \sum_{v \in (\Lambda_k \cap V)} m(v) \right) \leq 2 \sum_{w \in V(\gamma')} m'(w),$$

which completes the proof.

4. Proof of Propositions 1 and 2

Suppose $G = (V, E)$ is a disk triangulation graph. We divide each face of $G$ into four triangles by connecting the midpoints of the edges, and get a new disk triangulation graph $G'$. Formally, the vertex set of $G'$ is $V \sqcup E$, the disjoint union of $V$ and $E$, and an edge $[v, w]$ appears in $G'$ if and only if (i) $v, w$ are two different edges of $G$ belonging to the boundary of the same face of $G$; or (ii) $v \in V, w \in E$, and $v$ is an end point of $w$; or (iii) $v \in E, w \in V$, and $w$ is an end point of $v$. 
Trivially $G^f$ is a disk triangulation and refinement graph of $G$. Also note that $G^f$ satisfies the property $p(K)$ with $K = 6$.

**Lemma 12.** Suppose $G = (V, E)$ is a disk triangulation graph and $G^f = (V^f, E^f)$ is as above. Then $G$ is VEL-parabolic if and only if $G^f$ is VEL-parabolic.

**Proof.** Because $G^f$ is a (semi-)bounded refinement graph of $G$, VEL-parabolicity of $G^f$ implies VEL-parabolicity of $G$ by Lemma 10.

Conversely, suppose $G$ is VEL-parabolic. Then by Corollary 0.5 of [5] and Theorem 1.2 of [6], there exists a circle packing $P = (P_v : v \in V)$ whose nerve is combinatorially equivalent to $G$ and whose carrier is $\mathbb{C}$. Therefore $G$ can be embedded in $\mathbb{C}$ so that each $v \in V$ is the center of $P_v$ and each edge $[v, w] \in E$ is a straight line segment connecting the centers of $P_v$ and $P_w$.

In this embedding, each face of $G$ is a Euclidean triangle, and if $f$ is a face of $G$ with vertices $u, v, w$, then the inscribed circle of $f$ passes through the points $P_u \cap P_v, P_v \cap P_w,$ and $P_w \cap P_u$ (Figure 3). Therefore, if $e = [v, w]$ is an edge of $G$ and $f_1$ and $f_2$ are the faces of $G$ sharing $e$, the union of closed inscribed disks of $f_1$ and $f_2$ is a connected compact set in $\mathbb{C}$, because these two disks meet at $P_v \cap P_w$. We denote by $P_e$ the union of these two disks.

![Figure 3. inscribed circles and packed disks](image)

Now let $P' = (P_w : w \in V^f = V \cup E)$; i.e., for $w \in V$ we assign the disk $P_w$ of the circle packing $P$, and for $w \in V^f \setminus V = E$ we assign the union of the inscribed disks tangent to $w$. Then because each disk is $(1/4)$-fat, every $P_w$ is $(1/16)$-fat by Lemma 8. Moreover, for every $x \in \mathbb{C}$ one can easily check that there are at most seven vertices $w \in V^f$ such that $x \in P_w$. (Seven overlapping can actually occur when $\{x\} = P_u \cap P_v$ for some $[u, v] \in E$.) It is also trivial that $P'$ is locally finite and that if $[w, w'] \in E^f$ then $P_w \cap P_{w'} \neq \emptyset$. Thus by Lemma 9 we conclude that $G^f$ is VEL-parabolic. □

**Proof of Propositions 1 and 2.** We first prove Proposition 2. Suppose $G$ is a disk triangulation graph and $G^f$ is a semi-bounded refinement graph of $G$. If $G^f$ is VEL-parabolic, so is $G$ by Lemma 10. Conversely, suppose $G$ is VEL-parabolic. Let $G^f$ be the graph as in Lemma 12, and $G''$ a common semi-bounded refinement graph of both $G^f$ and $G'$. Then Lemma 12 implies that $G^f$ is VEL-parabolic, hence we know from Lemma 11 that $G''$ is also VEL-parabolic, since $G^f$ satisfies the property $p(6)$. Now the VEL-parabolicity of $G'$ follows from the VEL-parabolicity of $G''$ and Lemma 10. This completes the proof of Proposition 2.
Proposition 1 is actually an easy corollary of Proposition 2. Suppose $T_0$ and $T'_0$ are finite triangulations of $\mathbb{C}$ containing all the base points of $(X, h)$ in their vertex sets and such that their pull-back graphs $T = h^{-1}(T_0)$ and $T' = h^{-1}(T'_0)$ are disk triangulation graphs. Let $\Lambda_0$ be a finite common refinement graph of $T_0$ and $T'_0$, and note that $\Lambda_0$ must be a (semi-)bounded refinement graph of both $T_0$ and $T'_0$ since it is finite. Then the pull-back graph $\Lambda = h^{-1}(\Lambda_0)$ is a semi-bounded refinement graph of $T$ and $T'$. Then $T$ is VEL-parabolic if and only if $\Lambda$ is VEL-parabolic by Proposition 2 and similarly $T'$ is VEL-parabolic if and only if $\Lambda$ is VEL-parabolic. Since VEL-parabolicity is equivalent to cp parabolicity for disk triangulation graphs, this proves Proposition 1.

Observing the above proof, we can get the following additional result. For given $(X, h) \in F_q$, let $\Gamma'_0$ be the base curve as defined in the introduction, and $\Gamma_0$ its dual. Then the graph $\Gamma_0$ has exactly two vertices, say $\circ$ and $\times$, and the vertices of $\Gamma'_0$ is nothing but the base points $a_1, \ldots, a_q$ of $(X, h)$. For each $j = 1, \ldots, q$, we draw a Jordan arc $\ell_j$ from $\circ$ to $\times$ so that it intersects $\Gamma_0 \cup \Gamma'_0$ only at $a_j$ and the endpoints $\circ, \times$. Now we define $\Lambda_0$ by

$$\Lambda_0 := \Gamma_0 \cup \Gamma'_0 \cup \ell_1 \cup \cdots \cup \ell_q.$$  

Then $\Lambda_0$ is a triangulation of $\mathbb{C}$ which contains all the base points in its vertex set, and the pull-back graph $\Lambda := h^{-1}(\Lambda_0)$ is a disk triangulation graph.

Note that the graph $\Lambda$ can be obtained from the Speiser graph $\Gamma = h^{-1}(\Gamma_0)$ or its dual graph $\Gamma' = h^{-1}(\Gamma'_0)$ by dividing each face with $k$ edges into $2k$ triangles; i.e., if $f$ is a face of $\Gamma$ or $\Gamma'$, then we pick point in the interior of $f$, and connect it to the vertices on $\partial f$ and the midpoints of the edges surrounding $f$, so that $f$ is divided into $2k$ triangles. Therefore one can see that $\Lambda$ is a semi-bounded refinement graph of $\Gamma$ and $\Gamma'$. (In fact, $\Lambda$ is a bounded refinement graph of $\Gamma'$ since every face of $\Gamma'$ is a $q$-gon. Also note that $\Lambda$ satisfies the property $p(K)$ with $K = \max\{4, 2q\} = 2q$.) Now by Lemma 10 VEL-hyperbolicity of $\Gamma$ or $\Gamma'$ implies that of $\Lambda$, and we have:

**Corollary 13.** Suppose $(X, h)$ is a Riemann surface of class $S$. If either the Speiser graph corresponding to $(X, h)$ or its dual is VEL-hyperbolic, $(X, h)$ is of cp hyperbolic type.

**Remark 14.** It would be more natural to define cp type for every surfaces of class $S$, even in the presence of asymptotic values. Thus suppose $(X, h) \in F_q$ and let $\Gamma$ be its corresponding Speiser graph. The graph $\Gamma$ would have some infinite faces when $h$ has asymptotic values, but we can still divide each finite face of $\Gamma$ into triangles, and obtain a graph similar to $\Lambda$ described before Corollary 13.

The obtained graph, which we still denote by $\Lambda$, is no longer a disk triangulation graph, but it is possible to check if $\Lambda$ is VEL-parabolic or hyperbolic. Thus we can naturally extend the definition of cp types, even in the presence of asymptotic values, according to the cp type of $\Lambda$: i.e., we can define that $(X, h) \in F_q$ is cp parabolic if and only if $\Lambda$ is VEL-parabolic.

5. **Proof of Theorem 5**

Suppose a Speiser graph $\Gamma$ is given. If $\Gamma$ is VEL-hyperbolic, then Corollary 13 implies that the corresponding surface $X \in F_q$ is cp hyperbolic. On the other hand, since every Speiser graph is of bounded valence, VEL-hyperbolicity of $\Gamma$ is equivalent to its transiency, which then implies hyperbolicity of $X$ (2 or 7); see
the Doyle’s criterion below). Therefore the conformal and cp types agree in this case.

If \( \Gamma \) is VEL-parabolic (hence recurrent), however, the cp type of \( X \) could be either one. Note that the disk triangulation graph \( \Lambda \) described before Corollary 13 is a semi-bounded refinement graph of \( \Gamma \), but the proof for Lemma 11 does not work in this case since \( \Gamma \) is not a triangulation. Definitely neither does the VEL-parabolicity of \( \Gamma \) guarantee the parabolicity of \( X \) in the conformal type, so one has to investigate recurrent Speiser graphs for a Riemann surface of class \( S \) with different cp and conformal types. Our strategy is to construct a Speiser graph whose growth rate is very slow so that the corresponding surface is conformally parabolic, while its dual graph is VEL-hyperbolic. Then the surface will be cp hyperbolic by Corollary 13.

First, we start with a Speiser graph \( \Psi = (V_\Phi, E_\Phi) \) such that both the graph \( \Psi \) and its dual graph \( \Psi' \) are VEL-hyperbolic. More specifically, for \( \Psi \) we pick the regular graph of degree 3 such that every face of it is an octagon, so that the dual graph \( \Psi' \) is a regular graph of degree 8 which is a disk triangulation. Fix a vertex \( v_0 \) of \( \Psi \), and let \( S_\Phi(n) \) be the combinatorial sphere of radius \( n \) centered at \( v_0 \); that is, \( S_\Phi(n) \) is the set of vertices in \( V_\Phi \) whose combinatorial distance from \( v_0 \) is equal to \( n \). Let \( B_\Phi(n) = \bigcup_{k=0}^n S_\Phi(k) \) be the combinatorial ball of radius \( n \) centered at \( v_0 \), and let \( E_\Phi(n) \) be the set of edges in \( E_\Phi \) with one end on \( S_\Phi(n) \) and the other end on \( S_\Phi(n+1) \).

We next choose a sequence of natural numbers (odd numbers) \( \{l_n\}_{n=0}^{\infty} \) which will be determined later. Then for each edge in \( E_\Phi(0) \), we replace it by an unbranched tree of length \( l_0 \) so that the graph remains to be a Speiser graph. Note that every Speiser graph is bipartite, so the vertex set \( V_\Phi \) is divided into two classes, say vertices tagged with \( \circ \) and those with \( \times \), and every edge must connect two vertices with different tags. Moreover, \( \Psi \) is the regular graph of degree 3 with octagonal faces, thus in order to make the modified graph remain to be a Speiser graph, the number \( l_0 \) must be odd and every second edge in this unbranched tree of length \( l_0 \) must be a double edge (Figure 4).

![Figure 4](image-url)  
**Figure 4.** Replacing an edge with an unbranched tree of length 5

Similarly we replace every edge in \( E_\Phi(1) \) by an unbranched tree of length \( l_1 \) so that the graph remains to be a Speiser graph, and repeat this process for all edges in \( E_\Phi(n) \), \( n = 2, 3, \ldots \).

The resulting graph \( \Gamma = (V_\Gamma, E_\Gamma) \) we have just obtained from this process is a Speiser graph whose dual graph \( \Gamma' \) is VEL-hyperbolic. This is because \( \Gamma' \) contains \( \Psi' \) as a subgraph, and we chose \( \Psi' \) to be VEL-hyperbolic. We conclude that the surface \( X \in F_q \) associated with \( \Gamma \) is cp parabolic by Corollary 13.

It remains to show that the surface \( X \in F_q \) associated with \( \Gamma \) is conformally parabolic, for which we use the criterion of Doyle ([2]; cf. [7]): ‘the Riemann surface of class \( S \) associated with a given Speiser graph is parabolic if and only if the corresponding extended Speiser graph is recurrent’. Note that the extended Speiser graph of a given Speiser graph is a planar graph with infinitely many ends, obtained by adding infinitely many square grids to all the faces of the Speiser graph (Figure 5).
Let $\Upsilon = (V_\Upsilon, E_\Upsilon)$ be the extended Speiser graph obtained from $\Gamma$, and note that the vertex $v_0$ of $\Psi$, the center of the spheres and balls in the construction of $\Gamma$, may be regarded as a vertex in $\Gamma$, or even as a vertex in $\Upsilon$. Now we let $S_\Gamma(n)$, $B_\Gamma(n)$, $B_\Upsilon(n)$ be combinatorial spheres and balls in $\Gamma$ and $\Upsilon$, respectively, with centered at $v_0$ and radius $n$.

Depending on the growth rate of $|B_\Psi(n)|$, the number of vertices in $B_\Psi(n)$, we can choose the sequence $l_n$ increasing so fast that $|B_\Gamma(k)| \leq k \log k$ for sufficiently large $k$. More specifically, we have $|S_\Psi(n)| \leq 3^n$ since $\Psi$ is a regular graph of degree 3. This means that $|S_\Gamma(k)| \leq 3^n$ for $k \leq \sum_{j=0}^n l_j$. Thus if we define $l_n := \exp(3^{n+1})$, then for $1 + \sum_{j=0}^{n-1} l_j \leq k \leq \sum_{j=0}^n l_j$ we have

$$|B_\Gamma(k)| \leq k3^n = k \log l_{n-1} \leq k \log k.$$

On the other hand, $\Upsilon$ is a graph whose vertices have degree at most 6 such that for each $m \in \mathbb{N}$ and $w \in V_\Upsilon$ there are exactly 3 vertices in $V_\Upsilon \setminus V_\Gamma$ which are directly $m$ steps above $w$. In other words, if $w$ has distance $m - k$ from $v_0$, then only 3 vertices in $V_\Upsilon \setminus V_\Gamma$ are directly above $w$ and have distance $m$ from $w$. Therefore we must have

$$|S_\Upsilon(k)| \leq |B_\Upsilon(k)| + |S_\Upsilon(k) \setminus B_\Upsilon(k)| \leq k \log k + 3k \log k = 4k \log k$$

for sufficiently large $k$. Therefore there exists a constant $C$ such that $|B_\Upsilon(k)| \leq Ck^2 \log k$ for sufficiently large $k$. Since $\Upsilon$ is of bounded valence, the Nash-Williams criterion (cf. [11 p. 56]) shows that $\Upsilon$ is recurrent. We conclude that the surface $X \in F_q$ associated with $\Gamma$ is parabolic by the Doyle’s criterion mentioned above, and this completes the proof of Theorem 5.

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