Equiangular tight Frames in digital processing of sparse signals

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Abstract. Compressed sensing is a new way of compressing information. When it is realized, an important role is played by the special properties of a rectangular matrix. On the other hand, rectangular matrices are matrices of frame synthesis operators. The extremal properties of the equiangular tight frames are shown in this circle of questions. We observed connection between full spark systems and equiangular tight frames.

1. Introduction

Compressed sensing is a new way of reconstructing sparse signal vectors using a small number of linear measurements. It is the requirement of signal sparsity that makes the compressed sensing effective. The situation where the signal of interest lies in the union of subspaces of small dimensions makes it possible to recover from an under-defined system. Of course, a real signal rarely has many zero components. More often it is possible to obtain its sparse representation or approximation. Traditional methods of processing information require the stages of collection, compression and recovery. The presence of the first stage requires the creation of large storage facilities. The method of the compressed sensing assumes the localization of the most significant measurements, thereby combining the first two traditional stages into one.

Wavelet and Fourier transforms, redundant systems and other dictionaries are used to transform the signal into a sparse form. If such a dictionary is not known in advance, dictionary learning method is applied.

2. Sparse signals

Definition 1 A signal \( x \in \mathbb{R}^N \) is called \( k \)-sparse, if it has at most \( k \) nonzero components:

\[ \|x\|_0 := |\{ j : x(j) \neq 0 \}| \leq k. \]

The set of all \( k \)-sparse vectors is denoted by \( \Sigma_k \), i.e.,

\[ \Sigma_k := \{ x \in \mathbb{R}^N : \|x\|_0 \leq k \}. \]

Although \( \|x\|_0 \) is not a norm by definition, this notion is useful for the formulation of the sparse recovery problems. First, it is important to note that often signals are not exactly \( k \)-sparse, in the sense that many coefficients are not exactly zero, but they are small or negligible.
In this case, an important measure is the best $k$-term approximation, defined for $x \in \mathbb{R}^N$ as

$$\sigma_k(x)_1 := \min_{\tilde{x} \in \mathbb{R}^N : \|x - \tilde{x}\|_1 \leq k} \|x - \tilde{x}\|_1.$$  \hfill (1)

If a signal $x$ is $k$-sparse, then $\sigma_k(x)_1 = 0$, and the signal $x$ is called $k$-compressible, if $\sigma_k(x)_1$ is sufficiently small, or more precisely, if there exist $C, r > 0$ such that $\sigma_k(x)_1 \leq Ck^{-r}$.

The mathematical model of a compressed sensing can be described as follows: given a matrix $A \in \mathbb{R}^{M \times N}$ and linear measurements of the sparse vector $y = Ax \in \mathbb{R}^M$, recover $x$. The sparsity $k$ or the positions of the nonzero coefficients are not known.

A first and most intuitive approach to find a sparse vector $x$ would be to solve

$$\min \|x\|_0 \quad \text{subject to} \quad Ax = y,$$  \hfill (P_0)

but this problem is NP-hard.

In (1), how close is $x$ to being sparse is measured in the $\ell_1$ norm, defined for $x \in \mathbb{R}^N$ as

$$\|x\|_1 = \sum_{n=1}^N |x_n|.$$  

This led to the idea of Chen, Donoho and Saunders [1] to substitute $(P_0)$ with an alternative, which can be rewritten as easy as linear programming:

$$\min \|x\|_1 \quad \text{subject to} \quad Ax = y.$$  \hfill (P_1)

This formulation is known as the $\ell_1$ minimization problem, or Basis Pursuit (BP). Such an approach to recovering sparse vectors from linear measurements is probably the most common and most investigated one.

In order to successfully recover $x$ from $y = Ax$, it is not sufficient that $x$ is sparse, special properties of the matrix $A$ need to be satisfied.

Although the $\ell_0$ minimization $(P_0)$ is not suitable for recovering $x$ practically, it is important to understand under which conditions the representation $y = Ax_0$ is unique for $k$-sparse $x_0$, or equivalently, when a $k$-sparse $x_0$ is the unique solution to $(P_0)$. An answer to this question is given by using the spark; a notion introduced in [2] and defined as follows.

**Definition 2** Let $A \in \mathbb{R}^{M \times N}$. The spark of $A$ is defined as the smallest number of linearly dependent columns of $A$.

We can rewrite this definition using the $\ell_0$ notation as

$$\text{spark}(A) = \min\{\|x\|_0 : x \in \mathbb{R}^N\{0 \}, \text{ such that } Ax = 0\}.$$  

**Theorem 1** [2]. Let $A \in \mathbb{R}^{M \times N}$, and let $k \in \mathbb{N}$. Then the following conditions are equivalent.

1. If a solution $x$ of $(P_0)$ satisfies $\|x\|_0 \leq k$, then this is the unique solution.
2. $k < \text{spark}(A)/2$.

The idea of this theorem is as follows. If the spark of the matrix is large enough, then, choosing any columns in an amount smaller than the spark, we obtain linearly independent columns. Therefore, the restriction on these columns will be an injection. However, computing the spark is an NP-hard problem, therefore other easily computable properties are used in practice instead.

One example is the mutual coherence.
Definition 3 Let $A \in \mathbb{R}^{M \times N}$ be a matrix with columns $a_n \in \mathbb{R}^M, n = 1, \ldots, N$. Then its mutual coherence is defined as

$$\mu(A) := \max_{n \neq n'} \frac{|\langle a_n, a_{n'} \rangle|}{\|a_n\| \cdot \|a_{n'}\|}. \quad (2)$$

Theorem 2 [2, 3]. Let $A \in \mathbb{R}^{M \times N}$ and let $x_0 \in \mathbb{R}^N \setminus \{0\}$ be a common solution of $(P_0)$ and $(P_1)$. If $\|x_0\|_0 < \frac{1}{2} \left(1 + \frac{1}{k} \right)$, then $x_0$ is the unique solution of both $(P_0)$ and $(P_1)$.

Theorem 3 [2, 3]. Let $A \in \mathbb{R}^{M \times N}$ have a unit norm columns and a mutual coherence which satisfies $(2k - 1)\mu(A) < 1$. Then $(P_1)$ recovers every $k$-sparse vector $x$ from $y = Ax$.

Another relatively easily computable property is introduced by the following

Definition 4 Let $A \in \mathbb{R}^{M \times N}$. Then $A$ has the null space property (NSP) of order $k$, if for all $h \in \ker(A) \setminus \{0\}$ and all index sets $T \subset \{1, \ldots, n\}$ with $|T| \leq k$, $\|h_T\|_1 < \frac{1}{2}\|h\|_1$, where $h_T$ is the vector $h$ restricted to the indices in $T$.

Theorem 4 [4]. Let $A \in \mathbb{R}^{M \times N}$. Then the following conditions are equivalent.

1. If a solution $x$ of $(P_1)$ satisfies $\|x\|_0 \leq k$, then it is the unique solution.
2. $A$ satisfies the NSP of order $k$.

Maybe the best-known property of the matrix $\Phi$ in this circle of questions is the so-called RIP-property (restricted isometry property).

Definition 5 [5]. Let $A \in \mathbb{R}^{M \times N}$. Then $A$ has the restricted isometry property (RIP) of order $k$, if there exists $\delta_k \in (0,1)$ such that

$$(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2$$

for every $k$-sparse vector $x$. The smallest $\delta_k$ is called the restricted isometry constant (RIC).

From the definition of RIP it follows that each subset of $k$ columns is almost isometric, and hence these columns are linearly independent, that is, the spark of such a matrix must be sufficiently large.

RIC may be calculated by the equality

$$\delta_k(A) = \max_{\mathcal{R} \subseteq \{1, \ldots, N\}, |\mathcal{R}| \leq k} \|A_{\mathcal{R}}^* A_{\mathcal{R}} - I_{\mathcal{R}}\|_{2 \to 2},$$

where $A_{\mathcal{R}}$ is a submatrix with columns from $A$ with indexes from the set $\mathcal{R}$.

The checking RIP requires the search of all subsets $\{1, \ldots, N\}$ of size at most $k$, so that it is almost impossible to implement it.

The following theorem is a classical result on the $\ell_1$-reconstruction of sparse vectors using RIP.

Theorem 5 [4, 5] Let $A \in \mathbb{R}^{M \times N}$ satisfy the RIP of order $2k$ with $\delta_{2k} < \sqrt{2} - 1$. Let $x \in \mathbb{R}^N$ and $\hat{x}$ be a solution of the associated $\ell_1$-problem $(P_1)$. Then

$$\|x - \hat{x}\|_2 \leq C \cdot \frac{\sigma_k(x)_1}{\sqrt{k}},$$

for some constant $C$ depending only on $\delta_{2k}$.
3. Frame theory

One of the important dictionaries for solving the problems of compressed sensing is the frame. There are several reasons for this importance.

Frame for Hilbert space $\mathcal{H}$ is a sequence $\{\varphi_i\}_{i \in I}$, which satisfies

$$A \|x\|^2 \leq \sum_{i \in I} |(x, \varphi_i)|^2 \leq B \|x\|^2, \quad x \in \mathcal{H},$$

numbers $0 < A \leq B < \infty$ are called frame bounds.

In a finite-dimensional space the concept of a frame is equivalent to the completeness of the system, i.e. $\text{span}\{\varphi_i\}_{i \in I} = \mathcal{H}$ [6].

Frame is tight, if $A = B$. Tight frames are useful for the reconstruction of a signal $x$ by measurements $(x, \varphi_i), \ i \in I:

$$x = \frac{1}{A} \sum_{i \in I} (x, \varphi_i) \varphi_i.$$

$\Phi$ — Parseval frame $\iff A = B = 1$.

$\Phi$ — equal-norm frame $\iff \|\varphi_{n'}\| = \|\varphi_{n''}\|$ for all $n' \neq n''$.

$\Phi$ — equiangular frame $\iff \Phi$ is an equal-norm frame and there exists a $\alpha \geq 0$ such that $|\langle \varphi_{n'}, \varphi_{n''}\rangle| = \alpha$ for all $n' \neq n''$.

$\Phi$ — equiangular Parseval frame (EPF) $\iff \Phi$ is an equiangular + Parseval frame.

In finite dimensional case $\mathcal{H} = \mathbb{H}^M$ frame is defined by the columns of $M \times N$-matrix with full rank

$$\Phi = [\varphi_1, \ldots, \varphi_N], \quad N \geq M.$$

The analysis operator of the frame is the map $\Phi^*: \mathbb{H}^M \to \mathbb{H}^N$ given by $(V x)_n = \langle x, \varphi_n \rangle$, $n = 1, \ldots, N$. It is adjoint to

the synthesis operator: $\Phi: h \in \mathbb{H}^N \to \sum_{n=1}^N h_n \varphi_n \in \mathbb{H}^M$.

The frame operator is the positive, self adjoint invertible operator $S = \Phi \Phi^*$ on $\mathbb{H}^M$.

The Gramian is the operator $G = \Phi^* \Phi$ on $\mathbb{H}^N$.

Frame bounds are the extreme eigenvalues of the matrix $S = \Phi \Phi^*$. The tight frame has equal frame bounds. For the tight frame $\Phi$ the rows of $\Phi$ have equal norms and they are orthogonal. Tight frames are effective for linear coding and reconstruction of a signal: if $y = \Phi^* x$, then

$$x = \frac{1}{A} \Phi y,$$

where $A$ is the frame bound.

Hilbert-Schmidt norm of the Gram matrix of $\{\varphi_i\}_{i=1}^N$ is the number

$$\|\Phi^* \Phi\|_{HS}^2 := \sum_{n=1}^N \sum_{n'=1}^N |\langle \varphi_n, \varphi_{n'} \rangle|^2.$$

Sometimes this number is called the frame potential. If the columns of $\Phi$ are unit-norm, the matrix $\Phi^* \Phi$ can’t have more than $M$ non-zero eigenvalues, we have
\[ N^2 = (\text{Tr}(\Phi^*\Phi))^2 = \left( \sum_{m=1}^{M} \lambda_m (\Phi^*\Phi) \right)^2 \leq M \left( \sum_{m=1}^{M} \lambda_m (\Phi^*\Phi) \right)^2 = M \|\Phi^*\Phi\|_{HS}^2. \]

The equality here is possible only for tight frames.

If the columns \( \Phi \) are unit-norm, then mutual coherence
\[ \mu := \max_{n,n' \in \{1, \ldots, N\}, n \neq n'} |\langle \phi_n, \phi'_n \rangle| \]
for the \( M \times N \)-matrix \( \Phi = [\varphi_1, \ldots, \varphi_N] \) satisfies
\[ \frac{N^2}{M} \leq \|\Phi^*\Phi\|_{HS}^2 = \sum_{n=1}^{N} \sum_{n'=1}^{N} |\langle \phi_n, \phi'_n \rangle|^2 \leq N + N (N - 1) \mu^2. \]

The equality in the first inequality is achieved for tight frames. The equality in the second inequality is achieved for equiangular tight frames. These are unit-norm frame with the additional condition \( |\langle \phi_n, \phi'_n \rangle| = \text{const} \) for any \( n \neq n' \).

So we have

**Theorem 6** For any \( M \times N \)-matrix \( \Phi \) with unit-norm columns the inequality
\[ \mu \geq \sqrt{\frac{N - M}{M(N - 1)}} \]
is valid. The equality is possible only for equiangular tight frames.

Equiangular tight frames are used in communication theory [7]. It’s proved, that linear coding by equiangular tight frames has additional stability concerning losses in communication channels.

Now we shall answer the question: does the equiangular tight frame have a full spark? Let’s look on examples from [8], [9].

The first example is in \( \mathbb{C}^2 \) :
\[ \mathcal{F}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 1 & 1 & 1 \\ 0 & \sqrt{2} & \sqrt{2} e^{2\pi i / 3} & \sqrt{2} e^{4\pi i / 3} \end{pmatrix}. \]

We obtain calculating all possible minors of the second order that \( \text{spark}(S) = 3 \), i.e. \( \mathcal{F}_1 \) is a full spark frame.

However, the following examples of equiangular tight frames show that this is not always the case.

The system \( \mathcal{F}_2 \) in \( \mathbb{C}^3 \) is an equiangular tight frame [8]
\[ \mathcal{F}_2 = \begin{pmatrix} \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} & \sqrt{\frac{1}{5}} \\ 0 & \sqrt{\frac{2}{5} e^{2\pi i / 3}} & \sqrt{\frac{2}{5} e^{4\pi i / 3}} \\ \sqrt{\frac{2}{5} e^{-2\pi i / 3}} & 0 & \sqrt{\frac{2}{5} e^{-4\pi i / 3}} \end{pmatrix}. \]
We obtain calculating all possible minors of the third order that $\text{spark}(\mathcal{F}_2) = 3$, i.e. $\mathcal{F}_1$ is not a full spark frame.

One more example was found in [9]

\[
\mathcal{F}_3 = \frac{1}{2} \begin{pmatrix}
2 & -1 & -1 & -1 & -1 & -1 & -1 \\
0 & i\sqrt{3} & -i\sqrt{3} & 1 & 1 & \xi\sqrt{2} & \xi^2\sqrt{2} \\
0 & 0 & 0 & \sqrt{2} & \xi\sqrt{2} & \xi^2\sqrt{2} & \xi^4\sqrt{2}
\end{pmatrix},
\]

where $\xi = e^{2\pi i/3}$.

Acting in a similar way, we obtain that $\text{spark}(\mathcal{F}_3) = 3$.

The question posed is still open in real spaces. All known examples of real equiangular tight frames have full sparks.

### 4. Frames in phase recovery

Phase Recovery (Phase Retrieval, Phaseless Reconstruction, PR) is called the problem of signal reconstruction by absolute values of linear measurements.

Let $\mathbb{H}$ be $\mathbb{R}$ or $\mathbb{C}$, and and $\mathbb{T} = \{ h \in \mathbb{H} : |h| = 1 \}$. We denote by $\mathbb{H}^M/\mathbb{T}$ the set of equivalence classes $x \sim y$ with respect to $h \in \mathbb{T} : x = hy$. A set of vectors $\Phi = \{ \varphi_n \}_{n=1}^N$ admits a Phase Recovery or has the Phase Reconstruction Property (PRP) if the mapping

\[
\beta_\Phi : \mathbb{H}^M/\mathbb{T} \mapsto \mathbb{R}_+^N,
\]

\[
\beta_\Phi = [ |\langle \varphi_1, x \rangle|^2, \ldots, |\langle \varphi_N, x \rangle|^2 ]^T
\]

is injective. Numbers $\{ |\langle \varphi_n, x \rangle|^2 \}_{n=1}^N$ are called signal intensity measurements.

**Definition 6** A set of vectors $\Phi = \{ \varphi_n \}_{n=1}^N$ in $\mathbb{H}^M$ satisfies the complement property (CP) if for every subset $T \subseteq \{1, \ldots, N\}$ either $\{ \varphi_n \}_{n \in T}$ or $\{ \varphi_n \}_{n \in T^c}$ spans $\mathbb{H}^M$.

**Remark 1** The term complement property (CP) becomes common. It seems to us a more expressive combination is alternative completeness.

The equivalence of properties (CP) and (PRP) is known when $\mathbb{H} = \mathbb{R}$.

**Theorem 7** [10]. A set of vectors $\Phi = \{ \varphi_n \}_{n=1}^N$ in $\mathbb{R}^M$ allows (PRP) if and only if $\Phi$ satisfies the complement property.

The complement property is closely connected with full spark systems (every set of $M$ vectors in $\Phi$ is linearly independent). As it is proved in [10] any full spark system with $\geq 2M-1$ elements has (CP). This fact increases interest in studying the relationship between equiangular frames and full spark systems.

In the case $\mathbb{H} = \mathbb{C}$, the (CP) is only a necessary condition to the injectivity $\beta$.

**Theorem 8** [11]. Let $\Phi = \{ \varphi_n \}_{n=1}^N$ be a set of vectors in $\mathbb{C}^M$. If $\beta_\Phi$ is injective, then $\Phi$ satisfies the complement property.
In [11] it was discovered another characterization PRP in complex case. Let \( H^{M \times M} \) be the space of Hermitian \( M \times M \) matrices. For a set of measurement vectors \( \{ \varphi_n \}_{n=1}^N \) in \( \mathbb{C}^M \) the PhaseLift operator is defined as

\[
A : H^{M \times M} \to \mathbb{R}_+^N,
\]

\[
H \mapsto [(H, \varphi_1 \varphi_1^*_{HS}), (H, \varphi_2 \varphi_2^*_{HS}), \ldots, (H, \varphi_n \varphi_n^*_{HS})]^T.
\]

Notice that the mapping \( A \) with \( H = xx^* \) gives exactly the intensity measurements, since

\[
A(xx^*)(n) = \langle xx^*, \varphi_n \varphi_n^* \rangle_{HS} = \text{trace} (xx^* \varphi_n \varphi_n^*) = \varphi_n^* xx^* \varphi_n = |\langle x, \varphi_n \rangle|^2 = \beta_{\Phi}(x)(n).
\]

**Theorem 9** [11]. The mapping \( A \) is not injective if and only if there exists a matrix of rank 1 or 2 in the kernel of \( A \).

An important open problem in Phase Recovery is the search for the smallest possible number of measurements to ensure injectivity of the operator \( \beta \). In the space \( \mathbb{R}^M \) the answer is known, the number is \( 2M - 1 \) [10]. The problem is unsolved in \( \mathbb{C}^M \).

The connection between PR and CS becomes more transparent when a sparse signal \( x \) is considered. In the compressed sensing we are interested in recovering a \( k \)-sparse \( x \) from measurements \( y = Ax \), in PR we have a similar task but we are given only the absolute values of the measurements.

**Definition 7** [12], [13]. A given set \( \Phi = \{ \varphi_n \}_{n=1}^N \) in \( \mathbb{R}^M \) has the \( k \)-complement property, if for all \( T \subseteq \{1, \ldots, N\} \) and all \( K \subseteq \{1, \ldots, M\} \) with \( |K| \leq k \), either \( \{ \varphi_K \}_{n \in T} \) or \( \{ \varphi_K \}_{n \notin T} \) spans \( \mathbb{R}^k \).

Here we denote by \( \varphi^K_n \) the restriction of \( \varphi_n \) to the coefficients in \( K \).

**Theorem 10** [12], [13]. Let \( \Phi = \{ \varphi_n \}_{n=1}^N \) in \( \mathbb{R}^M \) satisfy the \( 2k \)-complement property. If \( x_0 \) is a \( k \)-sparse vector in \( \mathbb{R}^M \) and \( y = \beta_{\Phi}(x_0) \), then \( x_0 \) is the unique real vector satisfying the given intensity measurements with \( k \) or fewer nonzero elements.

5. Conclusion

Thus it is possible to note impressive achievements in both directions, in Compressed Sensing, and in Phase Recovery. Nevertheless, in both sections of applied mathematics there remain many interesting problems which throw us challenges that need to be answered.

6. References

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