Poisson structures for lifts and periodic reductions of integrable lattice equations

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Abstract
We introduce and study suitable Poisson structures for four-dimensional maps derived as lifts and specific periodic reductions of integrable lattice equations. These maps are Poisson with respect to these structures and the corresponding integrals are in involution.

Keywords: Poisson structures, discrete integrable systems, Yang–Baxter maps, Lax matrices

1. Introduction

Multidimensional consistency (or compatibility) plays an essential role in the study of partial difference equations on quadrilateral lattices and can be considered as a criterion related to integrability. In two dimensions, three-dimensional (3D) consistency provides zero curvature representations as well as an effective way of classifying certain classes of equations [1]. Further developments in this direction include Bäcklund transformations of continuous systems, discrete Lagrangian formalism, connection with Yang–Baxter (YB) maps etc.

3D consistent equations give rise to mappings on the lattices by considering well-posed periodic initial value problems (see e.g. [11]). In this paper we will focus only on the (2, 2) staircase periodic reductions of quad-graph equations [18, 20] with one field on each vertex and on the (1, 1) periodic reductions of a specific kind of systems with two fields on the vertices of any elementary quadrilateral. In this way we always derive four-dimensional maps. From another point of view, 3D consistent equations can be lifted to four-dimensional YB maps as described in [19]. In both cases (periodic reductions and lifts) the corresponding maps preserve the spectrum of their monodromy matrix. So, from this spectrum first integrals are obtained. In order to study the integrability (in the Arnold–Liouville sense) of these maps, one has to consider a suitable Poisson structure such that the maps are Poisson and the corresponding integrals in involution. The main purpose of this paper is to introduce such structures that fulfill these properties.
We present two different ways of obtaining Poisson structures for the lifts and the periodic reductions in question. The first one is from the Sklyanin bracket [22], and it is based on the Lax representation of the initial equation, while the second one is related to the existence of a three-leg form [8]. It seems that we cannot use both ways in all cases, but in the cases that we can do this the two Poisson structures coincide.

The special form of the Lax representations of the lifts as YB maps leads us to an inverse procedure. Instead of starting with a quad-graph equation and lifting it to a YB map, we can start with a YB map of a specific form, that admits a Lax pair, and squeeze it down to an equation (or a system of equations). The latter equation will have the same Lax pair. We apply this method to some known YB maps [16] and in this way we derive multiparametric versions of two well-known integrable lattice equations, the cross-ratio and the lattice non-linear Schrödinger (NLS) system.

The paper is organized as follows. We begin in section 2 by giving the necessary definitions of 3D consistent equations, YB maps, lifts and periodic reductions. We also present the multiparametric cross-ratio equation and its Lax pair. In sections 3 and 4 we study Poisson structures derived from the Sklyanin bracket and from three-leg forms respectively. Under some conditions, suitable Poisson structures for the lifts of quad-graph equations give rise to suitable Poisson structures for the corresponding (2, 2) periodic reductions and vice versa. In section 5, we consider more general cases related to a specific kind of systems on quadrilaterals. We apply our results to the multiparametric NLS system and we derive the Poisson structure for the lift and (1, 1) periodic reduction. In section 6, we talk about the integrability of the presented maps and we conclude in section 7 by giving some comments and perspectives for future work.

2. Integrable lattice equations and YB maps

We review some general facts about 3D consistent lattice equations, YB maps and their Lax representations. 3D consistent quadrilateral equations can be lifted to YB maps and YB maps of specific form can be squeezed down to quad-graph equations with the same Lax pair.
2.1. 3D consistent quad-graph equations and periodic reductions

We consider equations on quadrilaterals of the type

\[ Q(w, w_1, w_2, w_{12}; \alpha, \beta) = 0, \quad (1) \]

that can be uniquely solved for any one of their arguments \( w, w_1, w_2, w_{12} \in \mathbb{C} \).

By considering the initial values (black points) at the vertices of a cube as in the figure 1, we can determine the value \( w_{123} \) in three different ways. If all the three values coincide then we call the equation (1) 3D consistent.

Adler, Bobenko and Suris (ABS) have classified all the 3D consistent equations on \( \mathbb{C} \) with some extra properties into a list [1], commonly referred as the ABS list.

The 3D consistency of a quadrilateral equation gives rise to Lax representations [8, 17], i.e. an equation

\[ L(w_2, w_{12}, \alpha)L(w, w_2, \beta) = L(w_1, w_{12}, \beta)L(w, w_1, \alpha) \quad (2) \]

for some matrix \( L \), equivalent to (1). The matrix \( L \) is called a Lax matrix of equation (1). In a more general setting, quad-graph equations are related to a Lax pair \( L, M \) that gives rise to a Lax representation of the form \( L(w_2, w_{12}, \alpha)M(w, w_2, \beta) = M(w_1, w_{12}, \beta)L(w, w_1, \alpha) \). In all the cases that we deal with in this paper \( L = M \).

Multidimensional maps on quadrilateral lattices are derived from 3D consistent equations by considering well-defined initial value problems. In this paper we will restrict to low dimensional maps derived from the so called staircase periodic initial value problem [12, 18, 20]. We consider initial values at lattice points \( x_{i,j} \) and \( x_{i+1,j} \) which satisfy the periodicity \( x_{i,m} = x_{i+n,m+n} \). By solving the corresponding quad-graph equation (1) at each elementary square of the lattice with respect to \( x_{i,j+1} \), we derive an \( n \)-dimensional map that maps the points \( x_{i,j+1} \) to the points \( x_{i,j+1} \). We will refer to these maps as \((n,n)\) staircase periodic reductions of the initial quadrilateral equation\(^1\).

For each quad-graph equation (1), we can define a function \( F : \mathbb{C}^3 \to \mathbb{C} \), such that (1) is equivalent with

\[ w_2 = F(w_1, w, w_{12}, \alpha, \beta). \quad (3) \]

\(^1\) In general, given \((r,s) \in \mathbb{Z}^2, r \neq 0, s \neq 0\), one can impose the periodicity condition \( x_{i+r,n+s} = x_{i,n} \) to bring an equation on a square to \(|r| + |s|\)-dimensional maps referred as \((r,s)\) periodic reductions of the quad-graph equation [12, 18, 20].
In this way, the map obtained by the (2, 2) periodic reduction (figure 2) can be expressed as

\[ S_{a,\beta} : (x_1, x_2, x_3, x_4) \mapsto (x_1', x_2', x_3', x_4'), \]  

where

\[ x_2' = F(x_2, x_3, x_1, \alpha, \beta), \quad x_4' = F(x_4, x_3, x_1, \alpha, \beta), \]  
\[ x_3' = F(x_3, x_4', x_2', \alpha, \beta), \quad x_1' = F(x_1, x_2', x_4', \alpha, \beta). \]

2.2. YB maps

A YB map is a map \( R : X \times X \to X \times X, R : (x, y) \mapsto (u(x, y), v(x, y)), \) that satisfies the YB equation [4, 6, 26, 28]

\[ R_{23} \circ R_{13} \circ R_{12} = R_{12} \circ R_{13} \circ R_{23}. \]

Here by \( R_{ij} \) for \( i, j = 1, 2, 3, \) we denote the action of the map \( R \) on the \( i \) and \( j \) factor of \( X \times X \times X, \) i.e. \( R_{12}(x, y, z) = (u(x, y), v(x, y), z), \) \( R_{13}(x, y, z) = (u(x, z), v(x, z)) \) and \( R_{23}(x, y, z) = (x, u(y, z), v(y, z)). \) A YB map \( R : (X \times I) \times (X \times I) \to (X \times I) \times (X \times I), \)

\[ R : ((x, \alpha), (y, \beta)) \mapsto ((u, \alpha), (v, \beta)) = ((u(x, \alpha, y, \beta), \alpha), (v(x, \alpha, y, \beta), \beta)), \]  

is called a parametric YB map [26, 27]. We usually keep the parameters separately and denote (7) as \( R_{a,\beta}(x, y) : X \times X \to X \times X. \) Generally, \( X \) can be any set. From our point of view, the sets \( X \) and \( I \) have the structure of an algebraic variety and the maps that we consider are birational.

According to [25], a Lax matrix of the parametric YB map (7) is a matrix \( L \) that depends on a point \( x \in X, \) a parameter \( \alpha \in I \) and a spectral parameter \( \zeta \in \mathbb{C}, \) such that

\[ L(u, \alpha, \zeta)L(v, \beta, \zeta) = L(y, \beta, \zeta)L(x, \alpha, \zeta). \]  

Furthermore, if equation (8) is equivalent to \( (u, v) = R_{a,\beta}(x, y) \) then \( L \) is called strong Lax matrix. We often omit the spectral parameter \( \zeta \) and denote the Lax matrix \( L(x, \alpha, \zeta) \) just by \( L(x, \alpha). \) The next proposition, presented in [14] and essentially also in [26], provides a sufficient condition for solutions of equation (8) to satisfy the YB equation.

**Proposition 2.1.** If \( u = u_{a,\beta}(x, y), v = v_{a,\beta}(x, y) \) satisfy (8), for a matrix \( L \) and the equation

\[ L(\hat{x}, \alpha)L(\hat{y}, \beta)L(\hat{z}, \gamma) = L(x, \alpha)L(y, \beta)L(z, \gamma) \]

implies that \( \hat{x} = x, \hat{y} = y \) and \( \hat{z} = z, \) for every \( x, y, z \in X, \) then \( R_{a,\beta}(x, y) \mapsto (u, v) \) is a parametric YB map with Lax matrix \( L. \)

The dynamical aspects of YB maps have been studied by Veselov [26, 27]. For any YB map there is a hierarchy of commuting transfer maps which preserve the spectrum of the corresponding monodromy matrix. Furthermore, YB maps and their Lax matrices are related to Bäcklund transformations of continuous and discrete integrable systems (see e.g. [13, 24]).

2.3. Lift of 3D consistent quad-graph equations to YB maps

3D consistent equations on quad-graphs can be lifted to YB maps. This lifting procedure has been described in [19].
Let $Q(w, w_1, w_2, w_{12}, \alpha, \beta) = 0$ be a quad-graph equation affine linear in each argument $w, w_1, w_2, w_{12}$. As before, we define the function $F$ by solving this equation with respect to $w_2$, by $w_2 = F(w, w_1, w_{12}, \alpha, \beta)$. The lift of the equation $Q$ is defined as the map $R_{\alpha, \beta}(x_1, x_2, y_1, y_2) \mapsto (u_1, u_2, v_1, v_2)$, with

$$u_1 = F(y_1, x_1, y_2, \alpha, \beta), \quad u_2 = y_2, \quad v_1 = x_1, \quad v_2 = F(x_2, x_1, y_2, \alpha, \beta). \quad (9)$$

As was shown in [19], under some additional conditions, $R_{\alpha, \beta}$ is a YB map. All the lifts of the equations of the ABS list are YB maps. The proof of the YB property of these maps follows from the 3D consistency of the initial equations.

The name lift of the quad-graph equation is justified from the following observation. If we set $x_2 = y_1$, then $u_1 = v_2$, and by labeling the variables as $x_2 = y_1 = w_1, v_1 = x_1 = w, u_2 = y_2 = w_{12}$ and $u_1 = v_2 = w_2$, both first and last equations of (9) reduce to the initial quad-graph equation $Q(w, w_1, w_2, w_{12}, \alpha, \beta) = 0$. Having in mind this observation, we can easily prove the next proposition.

**Proposition 2.2.** If $L$ is a Lax matrix of the lift of a quad-graph equation then $L$ is also a Lax matrix of the quad-graph equation.

**Proof.** Let $L$ be a Lax matrix of the lift of a quad-graph equation $w_2 = F(w_1, w, w_{12}, \alpha, \beta)$. Then

$$L(u_1, u_2, \alpha)L(v_1, v_2, \beta) = L(y_1, y_2, \beta)L(x_1, x_2, \alpha), \quad (10)$$

for $u_1, u_2, v_1$ and $v_2$ given by (9). By setting $x_2 = y_1 = w_1, v_1 = x_1 = w, u_2 = y_2 = w_{12}$, we have $u_1 = v_2 = w_2$. If we substitute these to (10), we derive the Lax representation (2). □

In many cases, including the examples that will be presented here, the converse of this proposition also holds. That means that the Lax matrix of the initial equation is a Lax matrix of its lift (not always strong). In this cases, the YB property of the map (9) can be proved from proposition 2.1.

Using proposition 2.2 we can reverse the procedure and derive 3D consistent systems from YB maps of the form (9) as the next example shows.

**Multiparametric cross-ratio equation.** The map $R_{\alpha, \beta}(x_1, x_2, y_1, y_2) = (F(y_1, x_1, y_2, \alpha, \beta), y_2, x_1, F(x_2, x_1, y_2, \alpha, \beta))$, with

$$F(x, x_1, x_2, \alpha, \beta) = \frac{\beta x_2 x_2 (\alpha x_1 - \alpha x_2) x_2 + \alpha x_1 (\beta x_1 - \beta x_2)}{\beta x_1 (\alpha x_1 - \alpha x_2) + \alpha x_2 (\beta x_1 - \beta x_2)}$$
and vector parameters \( \tilde{a} = (\alpha, \alpha_1, \alpha_2) \) and \( \tilde{\beta} = (\beta, \beta_1, \beta_2) \) is a YB map\(^2\) with Lax matrix

\[
L(x_1, x_2, \tilde{a}) = \begin{pmatrix}
\alpha_1 \zeta + \frac{\alpha x_2}{\alpha_1 x_1 - \alpha_2 x_2} & \frac{\alpha x_1 x_2}{\alpha_1 x_1 - \alpha_2 x_2} \\
\frac{\alpha x_1 x_2}{\alpha_1 x_1 - \alpha_2 x_2} & \alpha_2 \zeta - \frac{\alpha x_1}{\alpha_1 x_1 - \alpha_2 x_2}
\end{pmatrix}.
\]  

(11)

According to proposition 2.2, the quad-graph equation \( w_2 \equiv F(w_1, w, w_{12}, \tilde{a}, \tilde{\beta}) \) satisfies the Lax equation (2). This equation can be seen as a multiparametric version of the cross-ratio equation (special case of equation (33) in [9]) and is given as follows

\[
\alpha (\beta_1 w - \beta_2 w_2) (\beta_1 w_1 - \beta_2 w_{12}) - \beta (\alpha_1 w - \alpha_2 w_1) (\alpha_1 w_2 - \alpha_2 w_{12}) = 0.
\]  

(12)

This equation satisfies the 3D consistency and the tetrahedron properties [1]. Therefore, one can derive its Lax pair which contains three spectral parameters. For \( \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1 \), equation (12) is reduced to \( Q_1 \) with \( \delta = 0 \) of ABS classification (cross-ratio equation). However, equation (12) does not possess the symmetries of the square \( (D_4\text{-symmetry}) \) in the normal way, but it satisfies the following symmetry property

\[
Q(w_1, w_2, w_{12}, \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2) = -Q(w_{12}, w_1, w_2, w, \beta, \beta_2, \beta_1, \alpha, \alpha_1, \alpha_2).
\]

(13)

Also, by using the non-autonomous transformation \( w_{l,m} \mapsto A^{-l} B^{m-m} w_{l,m} \), where \( A = \alpha_1/\alpha_2 \) and \( B = \beta_1/\beta_2 \), equation (12) is brought to the cross-ratio equation. The map \( R_{\alpha,\beta} \) is a Poisson map with respect to the Sklyanin bracket. We will give the corresponding Poisson structure in the next section (example 3.3).

### 3. Poisson structure derived from the Sklyanin bracket

In this section, we derive Poisson structures for the lifts and \((2,2)\) periodic reductions from the Sklyanin bracket [22] that is related to their Lax representations.

#### 3.1. The Sklyanin bracket on the lifts of quad-graph equations

In some cases, Lax pairs of YB maps are derived by reduction of polynomial matrices to the symplectic leaves of the Sklyanin bracket and the YB maps turn out to be symplectic with respect to this bracket [14, 16]. Furthermore, the Sklyanin bracket ensures that the integrals that we derive from the trace of the corresponding monodromy matrix of periodic initial value problems will be in involution [3, 26, 27].

We denote by \( L^m \) the set of \( m \times m \), \( n \)-degree polynomial matrices of the form

\[
L(x, \zeta) = X_0 + \zeta X_1 + \ldots + \zeta^n X_n.
\]

(14)

here \( x = (X_0, \ldots, X_n) \), \( X_i \in \text{Mat}_{m \times m} \) and \( \zeta \in \mathbb{C} \). The functions that depend on the coefficients \( X_i \) form a Poisson algebra with respect to the \( r \)-matrix quadratic Poisson bracket or Sklyanin bracket, which in tensor notation is given by the formula

\[
\{L(x, \zeta), L(x, h)\} = \{r(\zeta, h), L(x, \zeta) \otimes L(x, h)\}.
\]

(15)

Generally, \( r(\zeta, h) \) is a classical \( r \)-matrix, i.e. a solution of the classical YB equation [3, 5, 21–23]. We will consider the simple case \( r(\zeta, h) = \frac{r}{\zeta - h} \), where \( r \) denotes the permutation matrix:

\( r(x \otimes y) = y \otimes x \).

\(^2\) Case I YB map of the classification in [16], after a change of variables.
The $m^2$ elements of the highest degree term $X_n$ and the $mn$ coefficients of the determinant of $L(x, \zeta)$ are Casimir functions. By restricting to the common level set of the Casimir functions, we derive $m^2(n + 1)$ dimensional matrices that satisfy (13).

The Sklyanin bracket can be extended to the Cartesian product $\mathbb{L}_m \times \mathbb{L}_m$ in the natural way by setting

$$\{ L(x, \zeta) \otimes L(y, h) \} = [r(\zeta, h), L(x, \zeta) \otimes L(y, h)],$$

which satisfies (14).

**Proposition 3.1.** Let $R_{a,b}: (x, y) \mapsto (u, v)$ be a YB map with Lax matrix $L$ that satisfies (14). Then

$$\{ L(u, \alpha, \beta) L(v, \beta, \zeta) \otimes L(v, \alpha, h) L(v, \beta, h) \} = [r(\zeta, h), L(u, \alpha, \beta) L(v, \beta, \zeta) \otimes L(u, \alpha, h) L(v, \beta, h)].$$

**Proof.** From equation (14) we derive

$$\{ L(y, \beta, \zeta) L(x, \alpha, \zeta) \otimes L(y, \beta, h) L(x, \alpha, h) \} = [r(\zeta, h), L(y, \beta, \zeta) L(x, \alpha, \zeta) \otimes L(y, \beta, h) L(x, \alpha, h)].$$

Using (8), we have

$$\{ L(u, \alpha, \beta) L(v, \beta, \zeta) \otimes L(u, \alpha, h) L(v, \beta, h) \} = [r(\zeta, h), L(y, \beta, \zeta) L(x, \alpha, \zeta) \otimes L(y, \beta, h) L(x, \alpha, h)].$$

Equation (15) is a necessary condition for a YB map to be Poisson with respect to the bracket (14). In many cases, the Poisson property of these maps follows from the uniqueness of the refactorization of the Lax matrices ([14, 16]).

**Example 3.2.** The map

$$R_{a,b}(x_1, x_2, y_1, y_2) = \left( y_1 + \frac{a - \beta}{x_1 - y_2}, x_1 + \frac{a - \beta}{x_1 - y_2}, x_2 + \frac{a - \beta}{x_1 - y_2} \right)$$

is a well known property of the Sklyanin bracket, often referred to as the comultiplication property.

3 This is a well known property of the Sklyanin bracket, often referred to as the comultiplication property.
is a parametric YB map with (not strong) Lax matrix
\[ L(x_1, x_2, \alpha) = \begin{pmatrix} x_1 & \alpha + x_1 x_2 - \zeta \\ -1 & -x_2 \end{pmatrix}. \]

This map was derived in [14] and can be considered as a lift of KdV quad-graph equation \((H_1 \text{ equation of the ABS classification list [1]})\)
\[ (w_{12} - w)(w_1 - w_2) = \alpha - \beta. \]

We can verify that equations (14) are equivalent to
\[ \{ x_1, x_2 \} = 1, \{ y_1, y_2 \} = 1, \{ x_i, y_j \} = 0, \text{ for } i, j = 1, 2, \]
and that the YB map (16) is Poisson with respect to this bracket.

**Example 3.3.** The Lax matrix of the multiparametric cross-ratio equation (11), presented in the previous section, satisfies the Sklyanin bracket. In this case the extended Sklyanin bracket (14) is equivalent to
\[ \{ x_1, x_2 \} = \frac{(\alpha_1 x_1 - \alpha_2 x_2)^2}{\alpha}, \{ y_1, y_2 \} = \frac{(\beta_1 y_1 - \beta_2 y_2)^2}{\beta}, \{ x_i, y_j \} = 0. \]

The corresponding lift of this equation \(R_{\alpha, \beta}\) is Poisson with respect to this bracket.

### 3.2. The Sklyanin bracket on \((2, 2)\) staircase periodic reductions

Next, we establish a connection between the lifts and the \((2, 2)\) periodic reductions of quad-graph equations. In this way, and under some additional conditions, the Poisson structure of the lift of a quad-graph equation gives rise to a suitable Poisson structure for the periodic reduction. We begin with the following lemma.

We consider a function \(F : \mathbb{C}^5 \to \mathbb{C}\), such that
\[ F(x, x_1, x_2, \alpha, \beta) = F(x, x_2, x_1, \beta, \alpha), \]
as well as the three parametric maps
\[ R_{\alpha, \beta}(x_1, x_2, y_1, y_2) = \left( F(y_1, x_1, y_2, \alpha, \beta), y_2, x_1, \left( x_2, x_1, y_2, \alpha, \beta \right) \right) := (u_1, u_2, v_1, v_2). \]
\[ T_{\alpha, \beta}^1(x_1, x_2, y_1, y_2) = \left( F(y_1, x_1, y_2, \alpha, \beta), y_2, x_1, \left( x_2, y_2, x_1, \alpha, \beta \right) \right) := (u'_1, u'_2, v'_1, v'_2). \]
\[ T_{\alpha, \beta}^2(x_1, x_2, y_1, y_2) = \left( F(y_1, y_2, x_1, \alpha, \beta), y_2, x_1, \left( x_2, x_1, y_2, \alpha, \beta \right) \right) := (\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2). \]

**Lemma 3.4.** Let \(R_{\alpha, \beta}\) be a Poisson map with respect to a Poisson structure of the form
\[ \pi = J_{\alpha, \beta}(x_1, x_2) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + J_{\beta, \alpha}(y_1, y_2) \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}, \]

where
\[ J_{\alpha, \beta}(x_1, x_2) = x_1 \alpha + x_2 \beta - \zeta. \]
then the map \( S_{\alpha, \beta} = T^2_{\alpha, \beta} \circ T^1_{\alpha, \beta} \) is Poisson with respect to
\[
\pi_2 = J_{\beta, a}(x_1, x_2) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - J_{\beta, a}(y_1, y_2) \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2},
\]
if and only if
\[
J_{\beta, a}(x_1, x_2) \frac{\partial u'_1}{\partial x_1} \frac{\partial u'_2}{\partial x_2} = J_{\beta, a}(y_1, y_2) \frac{\partial v'_1}{\partial y_1} \frac{\partial v'_2}{\partial y_2}.
\]

In the case that the function \( F \) is defined by a quad-graph equation \( Q(w, w_1, w_2, w_{12}) \) from (3), then the map \( R_{\alpha, \beta} \) is the lift of this equation and the map
\[
S_{\alpha, \beta}(x_1, x_2, x_3, x_4) = S_{\alpha, \beta}(x_1, x_2, x_4, x_3)
\]
is its \((2, 2)\) staircase periodic reduction (4). In the coordinates \((x_1, x_2, x_3, x_4)\), the corresponding Poisson structure \( \pi_2 \) of the last lemma becomes
\[
\pi_2 = J_{\beta, a}(x_1, x_2) \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + J_{\beta, a}(x_3, x_4) \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}
\]
and condition (24) can be written as \( \pi_2(dx'_2, dx'_4) = 0 \), where \( x'_2, x'_4 \) are given in (5).

As we saw in the previous section, in many cases the lift of a quad-graph equation is Poisson with respect to the extended Sklyanin bracket. Obviously the bracket (14) is of the form of (22) for some function \( J_{\alpha, \beta} \) (that depends only in one parameter). In these cases, if the conditions of lemma 3.4 are fulfilled, a modification of the Sklyanin bracket from \( \pi_1 \) to \( \pi_2 \) gives rise to a suitable Poisson structure for the \((2, 2)\) periodic reduction of the initial quad-graph equation.

Finally, we have to remark that the property (18), follows from the following symmetry of the square
\[
Q(w, w_1, w_2, w_{12}, \alpha, \beta) = \pm Q(w_{12}, w_1, w_2, w, \beta, \alpha).
\]

We can summarise all these facts in the following proposition.

**Proposition 3.5.** We consider a quad-graph equation \( Q(w, w_1, w_2, w_{12}, \alpha, \beta) \), that satisfies the symmetry condition (27), with corresponding lift \( R_{\alpha, \beta} \) a Poisson YB map with respect to the Sklyanin bracket (14) and Lax matrix \( L \). The \((2, 2)\) periodic reduction \( S_{\alpha, \beta} : (x_1, x_2, x_3, x_4) \mapsto (x'_1, x'_2, x'_3, x'_4) \) is Poisson with respect to the bracket defined by
\[
\{ L(x_1, x_2, \beta, \zeta) \otimes L(x_1, x_2, \beta, h) \}_{2}
\]
\[
= \left[ r(\zeta, h), L(x_1, x_2, \beta, \zeta) \right] \otimes L(x_1, x_2, \beta, h),
\]
\[
\{ L(x_3, x_4, \beta, \zeta) \otimes L(x_3, x_4, \beta, h) \}_{2}
\]
\[
= \left[ r(\zeta, h), L(x_3, x_4, \beta, \zeta) \right] \otimes L(x_3, x_4, \beta, h),
\]
and \( \{ x_1, x_3 \} = \{ x_1, x_4 \} = \{ x_2, x_3 \} = \{ x_2, x_4 \} = 0 \), if and only if \( \{ x'_2, x'_4 \} = 0 \).
Example 3.6. We consider the lift of KdV quad-graph equation of example 3.2

\[ R_{\alpha,\beta}(x_1, x_2, y_1, y_2) = \left( F \left( y_1, x_1, y_2, \alpha, \beta \right), y_2, x_1, F \left( x_2, x_1, y_2, \alpha, \beta \right) \right), \]

where \( F(x, x_1, x_2, \alpha, \beta) = x + \frac{\alpha - \beta}{x_1 - x_2} \). As we mentioned before, this is a Poisson map with respect to the Poisson structure (22), for \( J_{\alpha,\beta}(x_1, x_2) = 1 \) and satisfies the condition (24). So, the corresponding (2, 2) staircase periodic reduction \( S_{\alpha,\beta} \) (25) is Poisson with respect to the bracket \( \pi_2 = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \).

Example 3.7. We consider the 3D consistent quad-graph equation

\[ \alpha (w - w_2)(w_1 - w_1) - \beta (w - w_1)(w_2 - w_2) = 0. \] (28)

This is the \( Q_1 \) quad-graph equation of the ABS list for \( \delta = 0 \). The lift of the equation gives rise to the YB map

\[ R_{\alpha,\beta}(x_1, x_2, y_1, y_2) = \left( F \left( y_1, x_1, y_2, \alpha, \beta \right), y_2, x_1, F \left( x_2, x_1, y_2, \alpha, \beta \right) \right), \]

with

\[ F(x, x_1, x_2, \alpha, \beta) = \frac{\alpha x_1(x - x_2) + \beta x_2(x - x_1)}{\alpha(x - x_2) + \beta(x_1 - x)}. \]

and (strong) Lax matrix

\[ L(x_1, x_2, \alpha) = \begin{pmatrix} \zeta + \frac{\alpha x_2}{x_1 - x_2} & -\frac{\alpha x_1 x_2}{x_1 - x_2} \\ \frac{\alpha}{x_1 - x_2} & \frac{x_1}{x_1 - x_2} & \zeta - \frac{\alpha x_1}{x_1 - x_2} \end{pmatrix}. \]

In this case, the corresponding Sklyanin bracket (14) is equivalent to

\[ \{ x_1, x_2 \} = -\frac{(x_1 - x_2)^2}{\alpha}, \quad \{ y_1, y_2 \} = -\frac{(y_1 - y_2)^2}{\beta}, \]

\[ \{ x_i, y_j \} = 0, \quad \text{for } i, j = 1, 2, \]

i.e. the Poisson structure (22) for \( J_{\alpha,\beta}(x_1, x_2) = -\frac{(x_1 - x_2)^2}{\alpha} \). The YB map \( R_{\alpha,\beta} \) is Poisson with respect to this structure and the condition (24) is satisfied. So, the corresponding periodic reduction (25) is Poisson with respect to

\[ \pi_2 = -\frac{(x_1 - x_2)^2}{\beta} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} - \frac{(x_3 - x_4)^2}{\beta} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}. \]

We have to remark that there are cases where the lift of a quad-graph equation is Poisson with respect to the Sklyanin bracket but the (2, 2) periodic reduction is not Poisson with respect to the corresponding bracket of proposition 3.5, because some of the conditions are not satisfied. For example, the lift of the multiparametric cross-ratio equation (12) is a Poisson map with respect to the Sklyanin bracket (17), but this equation does not satisfy the symmetry condition (27) and the corresponding (2, 2) periodic reduction is not Poisson with respect to the bracket of proposition 3.5.
Furthermore, we cannot define a Poisson bracket from (14) for any Lax representation of a lift of a quad-graph equation. In the cases that the quad-graph equations have an equivalent three-leg form (e.g. the equations of the ABS list), we can derive directly a Poisson structure for the (2, 2) periodic reductions and subsequently (except for the $Q_4$ case) a Poisson structure for the corresponding lifts. We study these cases in the next section.

4. Poisson structure derived from three-leg forms

In this section, we derive Poisson brackets for the (2, 2) periodic reduction (figure 2) of the ABS equations from the so-called three-leg forms.

Recall that a three-leg form of equation (1) centred at $w$ is defined as follows

$$\psi(w, w_1, \alpha) - \psi(w, w_2, \beta) = \phi(w, w_{12}, \alpha, \beta),$$

for some functions $\psi$ and $\phi$ such that equation (29) is equivalent to equation (1). In this equation, $\psi$ is called a short leg and $\phi$ a long leg. It is noted that all equations in the ABS list [1] (after some transformations) have a three leg-form. Due to the $D_4$ symmetry properties of these equations, their three-leg forms can be centred at any vertex of the quadrilateral.

**Proposition 4.1.** The (2, 2) periodic reduction map $S_{\alpha, \beta} : (x_1, x_2, x_3, x_4) \mapsto (x'_1, x'_2, x'_3, x'_4)$ of the ABS equations, described by (5) and (6), satisfies

$$s(x_1, x_2, \beta)dx_1 \wedge dx_3 + s(x_3, x_4, \beta)dx_3 \wedge dx_4 = s(x'_1, x'_2, \beta)dx'_1 \wedge dx'_3 + s(x'_3, x'_4, \beta)dx'_3 \wedge dx'_4,$$

where the function $s(w, w_1, \alpha)$ is symmetric when we change $w \leftrightarrow w_1$ and is given by

$$s(w, w_1, \alpha) = \frac{\partial \psi(w, w_1, \alpha)}{\partial w_1} = \frac{\partial \phi(w_1, w, \alpha)}{\partial w}.$$

**Proof.** It is known that the function $s$ is symmetric (see lemma 9, [1]). Differentiating three leg-forms centred at $x_1$ and $x_3$ and wedging with $dx_1$ and $dx_3$ respectively, we get

$$\frac{\partial \psi(x_1, x_2, \beta)}{\partial x_2}dx_1 \wedge dx_2 = \frac{\partial \psi(x_1, x'_2, \alpha)}{\partial x'_2}dx'_1 \wedge dx'_2,$$

$$= \frac{\partial \phi(x_1, x_3, \alpha, \beta)}{\partial x_3}dx_1 \wedge dx_3,$$

$$\frac{\partial \psi(x_3, x_4, \beta)}{\partial x_4}dx_3 \wedge dx_4 = \frac{\partial \psi(x_3, x'_4, \alpha)}{\partial x'_4}dx'_3 \wedge dx'_4,$$

$$= \frac{\partial \phi(x_3, x_1, \alpha, \beta)}{\partial x_3}dx_3 \wedge dx_1.$$
Using lemma 9 [1], we get \( \frac{\partial \phi(x_1, x_2, \alpha, \beta)}{\partial x_3} = \frac{\partial \phi(x_3, x_2, \alpha, \beta)}{\partial x_3} \). Therefore, we have

\[
\begin{align*}
&s(x_1, x_2, \beta)dx_1 \wedge dx_2 - s(x_1, x_2', \alpha)dx_1 \wedge dx_2' \\
&= -s(x_3, x_4, \beta)dx_3 \wedge dx_4 + s(x_3, x_4', \alpha)dx_3 \wedge dx_4',
\end{align*}
\]

which implies

\[
\begin{align*}
&s(x_1, x_2, \beta)dx_1 \wedge dx_2 + s(x_3, x_4, \beta)dx_3 \wedge dx_4 \\
&= s(x_1, x_2', \alpha)dx_1 \wedge dx_2' + s(x_3, x_4', \alpha)dx_3 \wedge dx_4'.
\end{align*}
\]

Similarly, we get

\[
\begin{align*}
&s(x_1, x_2', \alpha)dx_1 \wedge dx_2' + s(x_3, x_4', \alpha)dx_3 \wedge dx_4' \\
&= s(x_1', x_2', \beta)dx_1' \wedge dx_2' + s(x_3', x_4', \beta)dx_3' \wedge dx_4'.
\end{align*}
\]

Using (31) and (32), we obtain (30). □

For equations in the ABS list that require point transformations for the field variables and lattice parameters in order to bring their three-leg form to (29), this proposition holds in the new variables. Next we list the function \( s \) for all the cases of the ABS list.

- **H1**: \( s(w, w_1, \alpha) = 1 \)
- **H2**: \( s(w, w_1, \alpha) = \frac{1}{w + w_1 + \alpha} \)
- **H3\( ^{(0)} \)**: \( s(w, w_1, \alpha) = \frac{1}{ww_1} \)
- **H3\( ^{(1)} \)**: \( s(w, w_1, \alpha) = \frac{1}{ww_1 + \alpha} \)
- **Q1\( ^{(0)} \)**: \( s(w, w_1, \alpha) = \frac{1}{(w - w_1)^2} \)
- **Q1\( ^{(1)} \)**: \( s(w, w_1, \alpha) = -\frac{1}{w - w_1 + \alpha} + \frac{1}{w - w_1 - \alpha} \)
- **Q2**: \( s(w, w_1 \alpha) = \frac{(w - w_1)^2 - 2\alpha^2(w + w_1) + \alpha^4}{\alpha^2 - 1} \)
- **Q3\( ^{(0)} \)**: \( s(w, w_1, \alpha) = \frac{1}{(w_1 - aw)(w - aw)} \)
- **Q3\( ^{(1)} \)**: \( s(w, w_1 \alpha) = \frac{1}{aw_1 w_1 + w_1 w - \alpha} \)

\[ N = 2 \left( r(w)w + w^2 - 1 \right) \left( r(w_1)w_1 + w_1^2 - 1 \right) \left( r(w) + r(w_1) \right) \times \left( w_1 + r(w) \right) \left( \alpha^2 - 1 \right) \]

\[ D = \left( ar(w_1)r(w) + ar(w_1) + aw_1 r(w) + aw_1 w - 1 \right) \times \left( ar(w) - r(w_1) + aw - w_1 \right) \]

\[ \left( r(w_1) + r(w) + r(w_1)w + w_1 r(w) + w_1 w - \alpha \right) \times \left( r(w) - ar(w_1) + aw_1 + w \right) \]

and \( r(w) = \sqrt{w^2 - 1} \).
• $Q_4$: we use the Hietarinta form [10] for the $Q_4$ equation and we get
\[
s(w, w_1, \alpha) = \frac{1}{\alpha^2 w^2 w_1^2 + 2\alpha w_1^2 + \alpha^2 - w^2 - w_1^2},
\]
\[a = \sqrt{\alpha^4 + \delta \alpha^2 + 1}.
\] (33)

The 2-form $\omega_2 = s(x_1, x_2, \beta)dx_1 \wedge dx_2 + s(x_3, x_4, \beta)dx_3 \wedge dx_4$ is a symplectic form on the subset where $s(x_1, x_2, \beta)s(x_3, x_4, \beta) \neq 0$. So, proposition 4.1 implies that the maps derived by the (2, 2) periodic reduction of the ABS equations are symplectic with respect to it. Moreover, this proposition still holds when we change the edges $(x_1, x_2)$ and $(x_3, x_4)$ to $(x_4, x_1)$ and $(x_2, x_3)$ respectively and the parameter $\beta$ to $\alpha$. In other words, the map $S_{\alpha, \beta}$ is also symplectic with respect to $\omega'_{\alpha} = s(x_2, x_3, \alpha)dx_2 \wedge dx_3 + s(x_2, x_1, \alpha)dx_4 \wedge dx_1$.

Remark 4.2. The 2-forms that were presented in [1] (proposition 12) are obtained by adding these two symplectic structures, $\omega_2 + \omega'_{\alpha}$. However, it is important to note that these 2-forms are degenerate for the cases of $H_1$ and $H_3^\pm 0$ and not symplectic as $\omega_2$ and $\omega'_{\alpha}$. So they cannot convert to Poisson brackets for the associated maps by inverting their structure matrix. However, proposition 4.1 in this paper gives us a suitable Poisson structure for (2, 2) periodic reductions of these equations, as well as much simpler Poisson structure for the non-degenerate cases.

Corollary 4.3. The map (4) preserves the Poisson bracket which is given by
\[
\frac{1}{s(x_1, x_2, \beta)} \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2} + \frac{1}{s(x_3, x_4, \beta)} \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}.
\] (34)

We also note that one can use the same technique to obtain symplectic forms for two component pKdV [7]. These forms will be reduced to the symplectic forms $\omega_2$ and $\omega'_{\alpha}$ of pKdV if we set the two components to be equal. On the other hand, the multiparametric cross-ratio equation has a three-leg form but it does not give us a similar result as in corollary 4.3, since the corresponding function $s$ is not symmetric, i.e. $s(x_1, x_2, \alpha) \neq s(x_2, x_1, \alpha)$.

In addition, by considering the Poisson bracket of the (2, 2) periodic reductions we can derive a suitable Poisson structure for the corresponding lifts of the quad-graph equations (as YB maps). First we notice that the Poisson structure (34) is equivalent with the bracket $\pi_2$ in (26) for $J_{\alpha, \beta}(x_1, x_2) = \frac{1}{s(x_1, x_2, \beta)}$. Then we can check if the corresponding YB maps $R_{\alpha, \beta}$ of lemma 3.4 are Poisson with respect to $\pi_1$ in (22). By direct calculation we can prove the next proposition.

Proposition 4.4. The lifts of the quad-graph equations $H_1$, $H_2$, $H_3$, $Q_1$, $Q_2$, $Q_3$, are Poisson YB maps with respect to the Poisson structure $\pi_1$ described in (22), for $J_{\alpha, \beta}(x_1, x_2) = \frac{1}{s(x_1, x_2, \alpha)}$.

Surprisingly, the lift of $Q_4$ with the corresponding structure derived by (33) is not a Poisson map.
5. Poisson structure on specific systems of quad-graph equations

Lifts of 3D consistent quad-graph equations, \( w_2 = F(w_1, w, w_{12}, \alpha, \beta) \), are YB maps of the specific form \( R_{\alpha,\beta}(x_1, x_2, y_1, y_2) = (F(y_1, x_1, y_2, \alpha, \beta), y_2, x_1, F(x_2, x_1, y_2, \alpha, \beta)) \). Next, we consider more general YB maps of the form

\[
R_{\alpha,\beta}(x_1, x_2, y_1, y_2) = \left( F_1(y_1, x_1, y_2, \alpha, \beta), y_2, x_1, F_2(x_2, x_1, y_2, \alpha, \beta) \right)
\]

\[
:= (u_1, u_2, v_1, v_2), \tag{35}
\]

that involve two different functions \( F_1 \) and \( F_2 \). As we are going to show, these maps are related to a specific kind of systems of quadrilateral equations. As before, suitable Poisson structures of the YB maps give rise to suitable Poisson structure of specific periodic reductions of the system. We summarise our results in the next proposition.

**Proposition 5.1.** Let \( R_{\alpha,\beta} \), defined by (35), be a YB map with Lax matrix \( L \). Then

\( w_2 = F_2(w_1, t, w_{12}, \alpha, \beta), \quad t_2 = F_1(t_1, t, w_{12}, \alpha, \beta), \tag{36} \)

satisfies the Lax representation

\[
L(t_2, w_{12}, \alpha)L(t, w_2, \beta) = L(t_1, w_{12}, \beta)L(t, w_1, \alpha). \tag{37}
\]

(\( \beta \)) The \((1, 1)\) periodic reduction \( S_{\alpha,\beta}((x_1, y_1), (x_2, y_2)) = R_{\alpha,\beta} \circ R_{\alpha,\beta}(y_1, x_2, y_2, x_1) \).

(\( \gamma \)) If \( R_{\alpha,\beta} \) is Poisson with respect to the Sklyanin bracket (14), then the \((1, 1)\) periodic reduction \( S_{\alpha,\beta} \) of the system (36) is Poisson with respect to the bracket

\[
\begin{align*}
\{ L(y_1, x_2, \alpha, \zeta) \otimes L(y_1, x_2, \alpha, h) \} &= \left[ r(\zeta, h), L(y_1, x_2, \alpha, \zeta) \otimes L(y_1, x_2, \alpha, h) \right], \\
\{ L(y_2, x_1, \beta, \zeta) \otimes L(y_2, x_1, \beta, h) \} &= \left[ r(\zeta, h), L(y_2, x_1, \beta, \zeta) \otimes L(y_2, x_1, \beta, h) \right]
\end{align*}
\]

and \( \{x_i, x_j\} = \{y_i, y_j\} = \{x_i, y_j\} = \{x_2, y_2\} = 0. \)
Proof. (α) From the Lax representation of the YB map \( R_{\alpha,\beta} \) we have
\[
L(u_1, y_2, \alpha)L(x_1, v_2, \beta) = L(y_1, x_2, \beta)L(x_1, x_2, \alpha).
\]
(38)

Also, from (35), (36), by setting \( x_1 = t, x_2 = w_1, y_1 = t \) and \( y_2 = w_2 \), we derive \( u_1 = t \) and \( v_2 = w_2 \). If we substitute these values to (38), we derive the Lax representation (37).

(β) We consider the \((1, 1)\) periodic reduction of the system (36), with initial values \( (x_1, y_1), (x_2, y_2) \) as in figure 3, \( S_{\alpha,\beta} : (x_1, y_1), (x_2, y_2) \mapsto (x_1', y_1'), (x_2', y_2') \). Then from (36) we have \( (y_1', x_2', y_2', x_1') = (F_1(y_1, y_2, x_2, y_1', x_1, \alpha, \beta), F_2(x_2, y_1, x_1, \alpha, \beta), F_1(y_2, y_1, x_1, \alpha, \beta), F_2(x_2, y_2, x_1, \alpha, \beta)) \) which is equal to \( R_{\alpha,\beta} \circ R_{\alpha,\beta}(y_1, x_2, y_2, x_1) \).

(γ) It follows directly from (α) and (β).

Equivalently, one can start by considering 3D consistent systems of the special form (36) and derive YB maps as (35) and if the \((1, 1)\) periodic reduction of these systems is Poisson, then the composition \( R_{\alpha,\beta} \circ R_{\alpha,\beta} \) will be Poisson too.

Multiparametric lattice NLS system

We will apply the last results to a multiparametric form of the lattice NLS system that is related to a generalization of the Adler–Yamilov map.

We begin with the parametric map \( R_{\alpha,\beta} : (x_1, x_2, y_1, y_2) \mapsto (u_1, u_2, v_1, v_2) \), with
\[
u_1 = x_1, \quad v_2 = \frac{\beta_2}{\alpha_2} x_2 + \frac{\epsilon^2 (\alpha \beta_2^2 - \alpha_2^2 \beta_1^2)}{\alpha_2 (\alpha \beta_2^2 + \epsilon^2 x_1 y_2)^2} y_2.
\]
(39)

For any \( \epsilon \), this map is a YB map with vector parameters \( \alpha = (\alpha_1, \alpha_2), \beta = (\beta_1, \beta_2) \) and strong Lax matrix
\[
L(x_1, x_2; \alpha) = \begin{pmatrix}
\frac{\epsilon}{\alpha_2} (x_1 + x_2) - \frac{x_1}{\epsilon} \\
-\frac{\alpha_2}{\alpha_2} x_2 + \frac{\epsilon}{\alpha_2} (\alpha_2^2 + \epsilon^2 x_1 y_2) y_2
\end{pmatrix}.
\]
(40)

In this case equation (14) are equivalent with the coordinate Poisson brackets
\[
\{x_i, x_j\} = \alpha_2, \quad \{y_i, y_j\} = \beta_2, \quad \{x_i, y_j\} = 0,
\]
(41)

and the map \( R_{\alpha,\beta} \) is Poisson with respect to this bracket. The corresponding functions \( F_1, F_2 \) are defined by the relations \( F_1(y_1, x_1, y_2, \alpha, \beta) = u_1 \) and \( F_2(x_2, y_1, x_1, \alpha, \beta) = v_2 \).

So, according to proposition (5.1), the system (36) satisfies the Lax representation \( L(t_2, w_12, \alpha)L(t, w_2, \beta) = L(t, w_12, \beta)L(t, w_1, \alpha) \). We can write this system as
\[
\begin{align*}
\alpha_2^2 \beta_2^2 x_1 t - \beta_2^2 \alpha_2^2 x_2 t &- \frac{\alpha_2}{\alpha_2} \left( (\beta_2^2 x_1 - \alpha_2^2 x_2) t + \alpha_1 \beta_2 - \alpha_2 \beta_1 \right) = 0, \\
\beta_2^2 \alpha_2^2 x_1 &- \alpha_2^2 \beta_2^2 x_2 + \epsilon^2 x_1 w_2 \left( (\beta_2^2 x_1 - \alpha_2^2 x_2) t + \alpha_1 \beta_2 - \alpha_2 \beta_1 \right) = 0.
\end{align*}
\]
(42)

The \((1, 1)\) periodic reduction of this system \( S_{\alpha,\beta}((x_1, y_1), (x_2, y_2)) = R_{\alpha,\beta} \circ R_{\alpha,\beta}(y_1, x_2, y_2, x_1) \) is a Poisson map with respect to
The YB map (39) is a special case\textsuperscript{4} of a generalisation of the Adler–Yamilov map that was presented in [16]. We have checked that this lattice system is not 3D consistent. In fact, given initial values \( t, t_1, w_1, w_2, w_3 \) and parameters \( (\alpha_1, \alpha_2), (\beta_1, \beta_2), (\gamma_1, \gamma_2) \) on the cube, one has three ways of computing \( w_{123} \) and gets different values (two of the three values which are associated with \( w_{11} \) are the same). However, it still gives us the Lax pair given by (40).

For \( \varepsilon = \alpha_2 = \beta_2 = 1 \), the map (39) is reduced to the Adler–Yamilov YB map [2] and the system (42) to the lattice NLS system [7]. In this case the corresponding symplectic structures of the YB map and the \((1, 1)\) reduction becomes \( \wedge + \wedge x y y \) \( \wedge d d d d 1 2 2 1 \) respectively.

6. Integrability

The Lax representation of quadrilateral equations and YB maps provides integrals of periodic reductions. In the case of the lifts of quad-graph equations as YB maps, the corresponding integrals are derived from the trace of the monodromy matrix,

\[
M_1(x_1, x_2, y_1, y_2) = L(y_1, y_2, \alpha)L(x_1, x_2, \beta),
\]

while in the case of \((2, 2)\) periodic reductions from the trace of

\[
M_2(x_1, x_2, x_3, x_4) = L(x_2, x_1, \beta)L(x_3, x_2, \alpha)L(x_4, x_3, \beta)L(x_1, x_4, \alpha). \tag{45}
\]

If the Poisson structure is derived from the Sklyanin bracket, the involutivity of the integrals follows directly [3]. In the rest of the cases we have to check it by computing their brackets.

The trace of the monodromy matrix (45) for \( H_1 \) and \( Q_1^{\text{quad}} \) does not give us enough integrals. Two additional integrals for these equations are \( I_{H_1} := x_1 + x_2 + x_3 + x_4 \) and \( I_{Q_1^{\text{quad}}} := (x_1 - x_2)(x_3 - x_4)/(x_2 - x_3)(x_1 - x_4) \), respectively. In the case of the corresponding lifts, the trace of (44) gives enough integrals for \( Q_1^{\text{quad}} \) but only one for \( H_1 \). One extra integral for the lift of \( H_1 \) is \( J_{H_1} := (x_1 + x_2 - y_1 - y_2)^2 \). Finally, all the necessary integrals of the multilinear cross-ratio equation and NLS system (as YB maps and periodic reductions) are derived from the trace of their monodromy matrices.

The integrals of all the examples that have been presented in this paper are in involution with respect to the corresponding Poisson structures.

Higher dimensional maps can be derived by considering initial value problems with greater periodicity. The YB maps that we studied here as lifts of quadrilateral equations are four-dimensional maps and can be considered as the transfer maps of one periodic initial value problem (see e.g. [15]), with two fields on each edge of every elementary quadrilateral. We can consider \( n \)-periodic initial value problem to derive \( 4n \)-dimensional transfer maps. By extending the Poisson bracket to \( \mathbb{C}^{4n} \), in the natural way, the corresponding transfer maps will be Poisson. In this case the integrals are derived from the trace of the monodromy matrix.

\textsuperscript{4} It is derived by setting \( \alpha_1 = \beta_1 = \varepsilon \) and renaming the parameters \( \alpha_2 \) and \( \alpha_3 \) into \( \alpha_1 \) and \( \alpha_2 \) respectively at the map presented in [16].
\( M_n(x_1, \ldots, x_{2n}, y_1, \ldots, y_{2n}) = L(y_{2n}, y_{2n-1}, \beta)L(x_{2n}, x_{2n-1}, \alpha) \cdots L(x_2, x_1, \alpha)L(y_2, y_1, \beta). \)

As before, if the Poisson structure is coming from the (extended) Sklyanin bracket, they will be in involution.

On the other hand, in the case of \((n, n)\) periodic reductions of the quad-graph equations with fields on the vertices we cannot extend the corresponding Poisson bracket (34) in a similar way and derive Poisson maps. Therefore our analysis in this paper stops at \((2, 2)\) periodic reductions. There might be a way of extending the presented Poisson structure to some kind of higher periodic reductions and this is an issue that we would like to investigate in the near future. However, it is noted that due to the symmetry of Lagrangians of the ABS equations, one can obtain a presymplectic structure for \((n, n)\) periodic reductions. This is a more complicated structure that involves all the short legs and is given as follows (see [1])

\[
\begin{align*}
    s(x_1, x_2, \beta)dx_1 \wedge dx_2 + s(x_2, x_3, \alpha)dx_2 \wedge dx_3 + s(x_3, x_4, \beta)dx_3 \wedge dx_4 + \ldots \\
    + s(x_{2n}, x_1, \alpha)dx_{2n} \wedge dx_1.
\end{align*}
\]

This form is degenerate in the cases of \(H_1\) and \(Q_1^{\delta=0}\) (remark 4.2). A similar form can be derived for the staircase periodic reductions of the multiparametric cross-ratio equation (12).

### 7. Conclusion

We presented two different ways to obtain Poisson structures that are preserved under the four-dimensional maps derived as lifts and periodic reductions of integrable lattice equations. The corresponding integrals are in involution with respect to them.

We applied our results in the cases that can be viewed in the following table. The check mark \(\times\) refer to Poisson structures derived from the Sklyanin bracket, while the \(\checkmark\) to structures derived from three-leg forms. The last two columns refer to multiparametric cross-ratio equation and NLS system respectively.

| Lifts (YB maps) | \(H_1\) | \(H_2\) | \(H_3\) | \(Q_1^{\delta=0}\) | \(Q_1^{\beta=1}\) | \(Q_2\) | \(Q_3\) | \(Q_4\) | m.c-r | m.NLS |
|----------------|--------|--------|--------|----------------|----------------|--------|--------|--------|-------|-------|
| Per. reductions | \(\times\) | \(\checkmark\) | \(\checkmark\) | \(\times\\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(\times\) | \(\times\) |

Regarding the Sklyanin bracket, we have only been able to check a few of the Lax representations of the ABS list. By considering different Lax pairs it might be possible to convert more cases in this framework. In the cases that we can apply both ways like in \(H_1\) and \(Q_1^{\delta=0}\), we derive the same Poisson structure from the Sklyanin bracket and from the corresponding three-leg form. This fact suggests a deeper relation between these two different approaches that we would like to investigate in the future.

Finally, we believe that the study of multiparametric extensions of known integrable lattice equations and YB maps, as well as the significance of the extra parameters on the continuum limits is an interesting issue that deserves further attention.

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Appendix. Proof of lemma 3.4

Let \( S_{\alpha, \beta}(x_1, x_2, y_1, y_2) = (U_1, U_2, V_1, V_2) \). From (19)–(21), we have
\[
\begin{align*}
    u_1 = u_1' = V_1, & \quad u_2 = u_2' = U_2 = y_2, \\
    v_1 = v_1' = \tilde{v}_1 = x_1, & \quad v_2 = \tilde{v}_2 = y_2 \\text{and} \quad U_2 = v_2'.
\end{align*}
\]
If \( S_{\alpha, \beta} \) is Poisson map with respect to \( \pi_2 \), then \( \pi_2(dU_2, dV_1) = \pi_2(dv_2', \tilde{u}_1') = 0 \). From the last equality and from (23) we derive (24).

On the other hand, let us suppose that (24) holds. Since \( R_{\alpha, \beta} \) is a Poisson map with respect to \( \pi_1 \), \( \pi_1(du_1, du_2) = J_{\alpha, \beta}(u_1, u_2) \) and \( \pi_1(dv_1, dv_2) = J_{\beta, \alpha}(v_1, v_2) \). From these relations, using (22), we derive
\[
\frac{\partial u_1'}{\partial y_1} J_{\beta, \alpha}(y_1, y_2) = \frac{\partial u_1}{\partial y_1} J_{\beta, \alpha}(y_1, y_2) = J_{\alpha, \beta}(u_1, u_2) = J_{\alpha, \beta}(u_1', u_2''), \tag{46}
\]
\[
\frac{\partial v_2}{\partial x_2} J_{\beta, \alpha}(x_1, x_2) = \frac{\partial v_2}{\partial x_2} J_{\beta, \alpha}(x_1, x_2) = J_{\beta, \alpha}(v_1, v_2) = J_{\beta, \alpha}(v_1', v_2''). \tag{47}
\]
Equivalently, by interchanging the parameters \( \alpha \) and \( \beta \) in the last two equations, we derive from (18)
\[
\frac{\partial \tilde{u}_1}{\partial y_1} J_{\alpha, \beta}(y_1, y_2) = J_{\beta, \alpha}(\tilde{u}_1, \tilde{v}_2) = J_{\beta, \alpha}(\tilde{u}_1', \tilde{u}_2''), \tag{48}
\]
\[
\frac{\partial \tilde{v}_2}{\partial x_2} J_{\alpha, \beta}(x_1, x_2) = J_{\beta, \alpha}(x_1, \tilde{v}_2') = J_{\beta, \alpha}(v_1', \tilde{v}_2''). \tag{49}
\]
So, from (48) and (49) we have that
\[
J_{\beta, \alpha}(U_1, U_2) = \frac{\partial U_1}{\partial v_1'} J_{\beta, \alpha}(v_1', v_2''), \quad \frac{\partial U_1}{\partial v_1'} J_{\beta, \alpha}(x_1, x_2). \tag{50}
\]
Similarly, from (47), (46)
\[
J_{\beta, \alpha}(V_1, V_2) = \frac{\partial V_2}{\partial u_2'} J_{\beta, \alpha}(u_1', u_2''), \quad \frac{\partial V_2}{\partial u_2'} J_{\beta, \alpha}(y_1, y_2). \tag{51}
\]
Furthermore, condition (24) is equivalent to
\[
J_{\alpha, \beta}(x_1, x_2) \frac{\partial \tilde{u}_1}{\partial y_1} \frac{\partial \tilde{v}_2}{\partial x_2} = J_{\alpha, \beta}(y_1, y_2) \frac{\partial \tilde{u}_1}{\partial y_1} \frac{\partial \tilde{v}_2}{\partial y_2}. \tag{52}
\]
Next, we calculate the Poisson brackets \( \pi_2(dU_1, dU_2), \pi_2(dV_1, dV_2) \) and \( \pi_2(dU_i, dV_j) \), \( i, j = 1, 2 \). From (23) we get
\[
\pi_2 (dU_1, dU_2) = J_{\beta, a}(x_1, x_2) \frac{\partial U_1}{\partial v_1} \frac{\partial v_2}{\partial x_2} + \frac{\partial U_1}{\partial v_1} \left( J_{\beta, a}(x_1, x_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right) \\
\quad - J_{\beta, a}(y_1, y_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_1}{\partial y_2} \\
= J_{\beta, a}(U_1, U_2) + \frac{\partial U_1}{\partial u_1} \left( J_{\beta, a}(x_1, x_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right) \\
\quad - J_{\beta, a}(y_1, y_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_1}{\partial y_2}.
\]

(53)

\[
\pi_2 (dV_1, dV_2) = -J_{\beta, a}(y_1, y_2) \frac{\partial V_2}{\partial y_2} \frac{\partial u_1^\prime}{\partial x_1} \frac{\partial v_1}{\partial x_2} + \frac{\partial V_2}{\partial y_2} \left( J_{\beta, a}(x_1, x_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right) \\
\quad - J_{\beta, a}(V_1, V_2) + \frac{\partial V_2}{\partial u_1} \left( J_{\beta, a}(x_1, x_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right) \\
\quad - J_{\beta, a}(y_1, y_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_1}{\partial y_2}.
\]

(54)

where for the last equalities of (53) and (54) we used (50) and (51) respectively. Also,

\[
\pi_2 (dU_1, dV_1) = -\frac{\partial U_1}{\partial v_1} \left( J_{\beta, a}(x_1, x_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial x_2} - J_{\beta, a}(y_1, y_2) \frac{\partial u_1^\prime}{\partial y_2} \frac{\partial v_2}{\partial y_2} \right).
\]

(55)

\[
\pi_2 (dU_2, dV_2) = -\frac{\partial V_2}{\partial u_1} \left( J_{\beta, a}(x_1, x_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial x_2} - J_{\beta, a}(y_1, y_2) \frac{\partial u_1^\prime}{\partial y_2} \frac{\partial v_2}{\partial y_2} \right).
\]

(56)

\[
\pi_2 (dU_2, dV_1) = \left( J_{\beta, a}(x_1, x_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial x_2} - J_{\beta, a}(y_1, y_2) \frac{\partial u_1^\prime}{\partial y_2} \frac{\partial v_2}{\partial y_2} \right).
\]

(57)

and

\[
\pi_2 (dU_1, dV_2) = \left( \frac{\partial U_1}{\partial u_1^\prime} \frac{\partial V_2}{\partial v_2} - \frac{\partial U_1}{\partial v_1} \frac{\partial V_2}{\partial v_2} \right) \left( J_{\beta, a}(x_1, x_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial x_2} \right) \\
\quad - J_{\beta, a}(y_1, y_2) \frac{\partial u_1^\prime}{\partial x_2} \frac{\partial v_2}{\partial y_2} \\
\quad + \frac{\partial v_2}{\partial x_2} \left( J_{\beta, a}(x_1, x_2) \frac{\partial U_1}{\partial v_1} \frac{\partial V_2}{\partial v_2} - J_{\beta, a}(y_1, y_2) \frac{\partial U_1}{\partial v_1} \frac{\partial V_2}{\partial v_2} \right).
\]
or from (46), (49)
\[
\pi_2(dU_1, dV_2) = \left( \frac{\partial U_1}{\partial v_1} \frac{\partial V_2}{\partial v_2} - \frac{\partial U_1}{\partial u_1} \frac{\partial V_2}{\partial u_2} \right) J_{\beta,a}(x_1, x_2) \frac{\partial u_1}{\partial x_1} \frac{\partial v_2}{\partial x_2}
- J_{\beta,a}(y_1, y_2) \frac{\partial u_1}{\partial y_1} \frac{\partial v_2}{\partial y_2}
+ J_{a,\beta}(v_1', v_2') \frac{\partial U_1}{\partial v_1'} \frac{\partial V_2}{\partial v_2'} = J_{a,\beta}(u_1', u_2') \frac{\partial U_1}{\partial u_1'} \frac{\partial V_2}{\partial u_2'}.
\] (58)

From conditions (24) and (52), (53)–(58) become
\[
\pi_2(dU_i, dU_j) = J_{\beta,a}(U_i, U_j), \quad \pi_2(dV_i, dV_j) = -J_{\beta,a}(V_i, V_j),
\]
\[\pi_2(dU_i, dV_j) = 0, \text{ for } i, j = 1, 2,
\]
i.e. the map \( S_{a,\beta} \) is Poisson.

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