DOUBLE CHARACTER SUMS WITH INTERVALS 
AND ARBITRARY SETS IN FINITE FIELDS

ILYA D. SHKREDOV AND IGOR E. SHPARLINSKI

Abstract. We obtain a new bound on certain double sums of 
multiplicative characters improving the range of several previous 
results. This improvement comes from new bounds on the number 
of collinear triples in finite fields, which is a classical object of study 
of additive combinatorics.

1. Introduction

1.1. Motivation and background. For a prime $p$, let $\mathbb{F}_p$ be the finite 
field of $p$ elements.

Given a multiplicative character $\chi$ of the multiplicative group $\mathbb{F}_p^*$ 
(see [18, Chapter 3] for a background on characters), we define the bilinear multiplicative character sums 

$$W_{\chi}(\mathcal{I}, S; \alpha, \beta) = \sum_{s \in S} \sum_{x \in \mathcal{I}} \alpha_s \beta_x \chi(s + x),$$

where $\mathcal{I} = [1, X]$ is an interval, $S \subseteq \mathbb{F}_p$ is an arbitrary set of cardinality 
$\#S = S$, and $\alpha = \{\alpha_s\}_{s \in S}$ and $\beta = \{\beta_x\}_{x \in \mathcal{I}}$ are two sequence of complex weights with values inside of the unit disk:

$$|\alpha_s| \leq 1, \quad s \in S, \quad \text{and} \quad |\beta_x| \leq 1, \quad x \in \mathcal{I}.$$ (1.1)

In particular, the sums

$$\sum_{s \in S} \left| \sum_{x \in \mathcal{I}} \beta_x \chi(s + x) \right| \quad \text{and} \quad \sum_{x \in \mathcal{I}} \left| \sum_{s \in S} \alpha_s \chi(s + x) \right|$$

are both of the same shape, and the other way around: to estimate $W_{\chi}(\mathcal{I}, S; \alpha, \beta)$ it is enough to estimate either of these sums.

First we remark that if $X \geq p^{1/4+\varepsilon}$ with some fixed $\varepsilon > 0$, in the case when of the trivial weights $\beta_x = 1$, $x \in \mathcal{I}$, the Burgess bound implies

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that
\[ \sum_{s \in S} \left| \sum_{x \in I} \chi(s + x) \right| = O(SXp^{-\delta}) \]
for some \( \delta > 0 \) that depends only on \( \varepsilon > 0 \).

Furthermore, the result of Karatsuba [19] (see also [20, Chapter VIII, Problem 9]) which applies to general bilinear sums gives a nontrivial estimate on \( W_\chi(I, S; \alpha, \beta) \) when
\[ \min\{S, X\} \geq p^\varepsilon \text{ and } \max\{S, X\} \geq p^{1/2+\varepsilon}. \]

Finally, by a result of Chang [7, Theorem 9], there is a nontrivial bound
\[ \sum_{x \in I} \left| \sum_{s \in S} \chi(s + x) \right| = O(SXp^{-\delta}) \]
provided that for some fixed real \( \xi > 0 \) and \( \zeta > 0 \) we have \( S = p^{\xi+o(1)} \), \( X = p^{\zeta+o(1)} \) and for
\[ k = \left\lfloor \zeta^{-1} \right\rfloor \]
we have
\[ \xi > \frac{3k - 2 - 4k\zeta}{6k - 8} \tag{1.2} \]
(where \( \delta > 0 \) depends only on \( \xi \) and \( \zeta \)).

On the other hand, the method of Karatsuba [19] gives a similar bound under the condition
\[ \xi > \frac{1 - \zeta}{2} \tag{1.3} \]
which is worse than (1.2) if \( \xi \) and \( \zeta \) are close to each other. For example, for \( \zeta = \xi \) the conditions (1.2) and (1.3) become
\[ \zeta = \xi > 7/22 \quad \text{and} \quad \zeta = \xi > 1/3, \tag{1.4} \]
respectively.

On the other hand, the condition (1.3) is better than (1.2) provided that
\[ 1/4 < \zeta < 2/7. \tag{1.5} \]
This indicates that the approach of Karatsuba [19] is still competitive and deserves further investigation. Because of this, and because it does not seem to be ever presented in full detail, we do this here. Furthermore, we complement this approach with some new ingredients coming from recent advances in additive combinatorics which lead to a stronger bound.
There are also several bounds \([2, 3, 5, 7, 15]\) but they apply only for some special sets \(S\), such sets with well-spaced elements or sets that are contained in short intervals.

Various bounds of character sums sums with more than two arbitrary sets can be found here \([16, 30, 31]\).

1.2. Main results. We formulate our main result in terms of the quantity \(E_+^3(U, V, W)\) which for sets \(U, V, W \subseteq \mathbb{F}_p\) is defined as the number of the solutions to the equation

\[
\begin{align*}
    u_1 - u_2 = v_1 - v_2 = w_1 - w_2, \\
    u_1, u_2 \in U, \ v_1, v_2 \in V, \ w_1, w_2 \in W.
\end{align*}
\]

Assuming that \(#U \geq #V \geq #W\), the trivial upper bound for \(E_+^3(U, V, W)\) is

\[
E_+^3(U, V, W) \leq #U#V#W \min\{#U, #V, #W\}.
\]

We recall that the notations \(U = O(V)\), \(U \ll V\) and \(V \gg U\) are all equivalent to the statement that \(|U| \leq cV\) holds with some constant \(c > 0\), which throughout this work may depend on the integer parameter \(r \geq 1\).

**Theorem 1.1.** For any interval \(I = [1, X]\) of length \(X\) and any set \(S \subseteq \mathbb{F}_p^*\) of size \(#S = S\) such that

\[
S^2 X \leq p^3 \quad \text{and} \quad X < p^{1/2}
\]

and complex weights \(\alpha = \{\alpha_s\}_{s \in S}\) and \(\beta = \{\beta_x\}_{x \in I}\) satisfying (1.1), for any fixed integer \(r \geq 1\) such that \(X \geq p^{1/r}\), we have

\[
W_\chi(I, S; \alpha, \beta) \ll SX \left( \frac{E_+^3(S, S, \overline{I})}{S^4 X^3} + \frac{p^{(r+1)/r}}{SX^{5/2}} + \frac{p^{(r+2)/r}}{S^2 X^2} \right)^{1/4r} p^{o(1)} + S^{1/2}X,
\]

where \(\overline{I} = [-X, X]\).

Combining Theorem 1.1 with the trivial bound (1.7) we obtain:

**Corollary 1.2.** Under the condition of Theorem 1.1, we have

\[
W_\chi(I, S; \alpha, \beta) \ll SX \left( \frac{M p^{(r+1)/r}}{S^2 X^2} + \frac{p^{(r+2)/r}}{SX^{5/2}} + \frac{p^{(r+2)/r}}{S^2 X^2} \right)^{1/4r} p^{o(1)} + S^{1/2}X,
\]

where \(M = \min\{S, X\}\).
The result of Chang [7, Theorem 9] is not fully explicit so it may not be straightforward to compare it with Corollary 1.2, however, see (1.5) for the range of parameters, where Corollary 1.2 is certainly stronger. We also note that compared to the original method of Karatsuba [19], our main technical innovation is Lemma 2.5 which can be of independent interest (and thus we present in a more general form than we need in this work), see also Remark 2.7.

Furthermore, given an interval $I = [1, X]$, we have the trivial inequality

$$E^+_{3}(S, S, I) \ll XE^+(S)$$

where $E^+(S)$ is the additive energy of the set $S$, that is,

$$E^+(S) = \# \{ u_1 + u_2 = v_1 + v_2 : u_1, u_2, v_1, v_2 \in S \}.$$

Thus we have the following bound which underlines one of the bounds used in the proof of Theorem 1.4 (see the proof of Lemma 2.9 below).

**Corollary 1.3.** Under the condition of Theorem 1.1, we have

$$W_{\chi}(I, S; \alpha, \beta) \ll SX \left( \frac{E^+(S)}{S^4X^2} + \frac{p^{(r+2)/r}}{S^{6/2}} + \frac{p^{(r+2)/r}}{S^2X^2} \right)^{1/4r} p^{o(1)} + S^{1/2}X,$$

If $E^+_{3}(S, S, \mathcal{T}) \gg S^3X$, then by (1.9) the additive energy $E^+(S)$ of $S$ is large and thus $S$ is very structured. On the other hand, one can easily give examples of sets with small quantity $E^+_{3}(S, S, \mathcal{T})$, for example, for random sets $S$ or for sets with some prescribed algebraic structure.

Here we give one of such examples, namely when $S$ is a multiplicative subgroup $G \subseteq F_p$ for which the sums $W_{\chi}(I, S; \alpha, \beta)$ have been considered in [10, 28] (in the case of constant weights $\alpha$ and $\beta$). To simplify the exposition, and enable us to apply a result of Cilleruelo and Garaev [11, Theorem 1] we always assume that $\#G \leq p^{2/5}$. Note that for large subgroups one can use the results and methods of [10, 28].

**Theorem 1.4.** For any fixed positive $\zeta < 1/2$ and $\xi < 2/5$ satisfying

$$\xi > \begin{cases} 
1 - 5\zeta/2, & \text{if } 6/25 < \zeta < 10/31, \\
(6 - 9\zeta)/16, & \text{if } 10/31 \leq \zeta < 134/361, \\
(20 - 40\zeta)/31, & \text{if } 134/361 \leq \zeta < 1/2.
\end{cases}$$

there exists some $\delta > 0$ such that for any interval $I = [1, X]$ of length $X = p^{\xi + o(1)}$ and a multiplicative subgroup $G \subseteq F_p$ of order $T = p^{\xi + o(1)}$,
and complex weights $\alpha = \{\alpha_s\}_{s \in G}$ and $\beta = \{\beta_x\}_{x \in I}$ satisfying (1.1), we have

$$W_\chi(I, G; \alpha, \beta) \ll TXp^{-\delta}.$$  

We note that the bound of [28] is nontrivial only under the condition (1.3). Furthermore, if $\zeta = \xi$, the bound of Theorem 1.4 is nontrivial for

$$\zeta = \xi > 2/7$$

improving on what one can derive from (1.4).

**Remark 1.5.** Clearly the quantity $E_3^+(S, S, \mathcal{T})$ is invariant under translations $S \to S + a$ of $S$ by $a \in \mathbb{F}_p$. Thus the bound of Theorem 1.4 also holds for the sums $W_\chi(I, G; \alpha, \beta)$ where $I = [a + 1, a + X]$ is an arbitrary interval.

**Remark 1.6.** The additive energy of sets plays a central role in additive combinatorics, see [33], and has been studied in a vast number of works. For example, using the bound of Corvaja and Zannier [13, Theorem 2] one derives

$$E_3^+(f(G), f(G), \mathcal{T}) \ll T^{8/3}X$$

for a polynomial image $f(G)$ of a multiplicative subgroup $G \subseteq \mathbb{F}_p$ of order $T \leq p^{3/4}$ and a polynomial $f(Z) \in \mathbb{F}_p[Z]$ (under some mild conditions on $f$). Together with Theorem 1.1 this leads to new bounds on the sums $W_\chi(I, f(G); \alpha, \beta)$ complementing the bound of [10, Theorem 1.2].

Similarly to polynomial images of subgroups in Remark 1.6, one can use bounds on the additive energy of polynomial images of intervals in a combination with Theorem 1.1. We given only two very concrete applications of this type to character sums over primes.

**Theorem 1.7.** Let $f$ be a polynomial over $\mathbb{F}_p$ of degree $d \geq 2$. For any $Q = p^{\zeta + o(1)}$ and $R = p^{\xi + o(1)}$ an fixed positive $\zeta$ and $\xi \leq \min\{1/2, 2 - 2\zeta\}$ satisfying

$$5\zeta/4 + 2\xi > 1 \quad \text{and} \quad \zeta + 5\xi/2 > 1,$$

for $d = 2$ and

$$(1 + 2^{-d+1}) \zeta + 2\xi > 1 \quad \text{and} \quad \zeta + 5\xi/2 > 1,$$

for $d \geq 3$, there exists some $\delta > 0$ such that

$$\left| \sum_{q \leq Q} \left| \sum_{r \leq R \text{ prime}} \chi(f(q) + r) \right| \right|, \quad \left| \sum_{r \leq R \text{ prime}} \sum_{q \leq Q} \chi(f(q) + r) \right| \ll QRp^{-\delta}.$$
Remark 1.8. The bound of Theorem 1.7 makes use of a bound on the additive energy of polynomial images given in Corollary 2.12, which in turn relies on a new result from additive combinatorics given in Lemma 2.10. Furthermore, Lemma 2.10 implies an improvement of a result of Bukh and Tsimerman [6, Theorem 1], see Remark 2.13. On the other hand, it seems that for polynomials of high degree the approaches of [8,9,12] are likely to become more efficient.

2. Background from Additive Combinatorics

2.1. Points–planes incidences. Now we recall some notions about points–planes incidences, in which we follow [29].

First of all, we need a general design bound for the number of incidences. Let \( Q \subseteq \mathbb{F}_p^3 \) be a set of points and \( \Pi \) be a collection of planes in \( \mathbb{F}_p^3 \). Having \( q \in Q \) and \( \pi \in \Pi \) we write

\[
I(q, \pi) = \begin{cases} 
1 & \text{if } q \in \pi, \\
0 & \text{otherwise.}
\end{cases}
\]

So, \( I \) is a \((\#Q \times \#\Pi)\)-matrix.

If \( Q = \mathbb{F}_p^3 \) and \( \Pi \) is the family of all planes in \( \mathbb{F}_p^3 \), then we obtain the matrix \( G \) and thus \( I \) is a submatrix of \( G \). One can easily calculate \( G^tG \) and \( GG^t \) (where \( G^t \) is the transposition of \( G \)) embedding \( \mathbb{F}_p^3 \) into the projective space \( \mathbb{P}\mathbb{F}_p^3 \) and check that both of these matrices are of the form \( a\text{Id} + b1 \), where \( a, b \) are some scalar coefficients, \( \text{Id} \) and \( 1 \) and identity matrix and all-ones matrices of corresponding dimensions, see, for example, [34,35]. Moreover, one can check that in our case of points and planes the following holds

\[
a = p^2 \quad \text{and} \quad b = p + 1.
\]

In other words, \( GG^t = p^2\text{Id} + (p + 1)1 \). In view of these facts and using the singular decomposition (see, for example, [17]), denoting

\[
Q = \#Q,
\]

we see that

\[
G(q, \pi) = \sum_{j=1}^{Q} \mu_j u_j(q)v_j(\pi),
\]

where \( \mu_j \geq 0 \) are square–roots of the eigenvalues of \( GG^t \) (which coincide with square–roots of nonzero eigenvalues of \( G^tG \)) and \( u_j \) and \( v_j \), are the eigenfunctions of \( GG^t \) and \( G^tG \), respectively, \( j = 1, \ldots, Q \). From \( GG^t = p^2\text{Id} + (p + 1)1 \), we obtain

\[
\mu_1^2 = p^2 + (p + 1)Q \quad \text{and} \quad \mu_2 = \ldots = \mu_Q = p
\]

and

\[
u_1(q) = (1, \ldots, 1) \in \mathbb{R}^Q \quad \text{and} \quad v_1(\pi) = (1, \ldots, 1) \in \mathbb{R}^P,
\]
where $P = \#\Pi$. Hence we derive that for any functions $f : Q \to \mathbb{C}$ and $g : \Pi \to \mathbb{C}$, supported only $Q$ and $\Pi$, respectively, one has

$$\left| \sum_{q \in Q} \sum_{\pi \in \Pi} I(q, \pi) f(q) g(\pi) \right| = \left| \sum_{q \in Q} \sum_{\pi \in \Pi} G(q, \pi) f(q) g(\pi) \right|$$

$$= \left| \sum_{j=2}^Q \mu_j \langle f, u_j \rangle \langle g, v_j \rangle \right|$$

$$\leq p \sum_{j=2}^Q |\langle f, u_j \rangle \langle g, v_j \rangle|,$$

provided that

(2.1) \quad \sum_{q \in Q} f(q) = 0 \text{ or } \sum_{\pi \in \Pi} g(\pi) = 0.

Using the Cauchy inequality we now see that under the condition (2.1) we have

(2.2) \quad \left| \sum_{q \in Q} \sum_{\pi \in \Pi} I(q, \pi) f(q) g(\pi) \right| \leq p \|f\|_2 \|g\|_2,$$

where, as usual $\|f\|_2$ and $\|g\|_2$ are the $L^2$-norms of functions $f$ and $g$, respectively.

Furthermore, a deep result on incidences of Rudnev [26] (or see [24, Theorem 8] and the proof of [23, Corollary 2]) combined with the incidence bound from [22, Section 3] leads to the following asymptotic formula:

**Lemma 2.1.** Let $Q \subseteq \mathbb{F}_p^3$ be a set of points and let $\Pi$ be a collection of planes in $\mathbb{F}_p^3$. Suppose that $\#Q \leq \#\Pi$ and that $k$ is the maximum number of collinear points in $Q$. Then

$$\sum_{q \in Q} \sum_{\pi \in \Pi} I(q, \pi) - \frac{\#Q \#\Pi}{p} \ll (\#Q)^{1/2} \#\Pi + k \#Q.$$

**2.2. On the number of collinear triples.** Given three sets $A, B, C \subseteq \mathbb{F}_p$ we denote by $T(A, B, C)$ the number of the solutions to the equation

$$\frac{a_1 - c_1}{b_1 - c_1} = \frac{a_2 - c_2}{b_2 - c_2}, \quad a_1, a_2 \in A, b_1, b_2 \in B, c_1, c_2 \in C.$$

Geometrically, $T(A, B, C)$ is the number of collinear triples of points $((a_1, a_2), (b_1, b_2), (c_1, c_2)) \in A^2 \times B^2 \times C^2$. 
Let \( \mathcal{L} \) be the set of all lines in \( \mathbb{F}_p^2 \). We observe that there are exactly \( p + 1 \) lines passing via any point in \( \mathbb{F}_p^2 \).

Given \( \ell \in \mathcal{L} \), we denote 
\[
\iota_{\mathcal{A}}(\ell) = \# (\ell \cap \mathcal{A}^2).
\]

We denote by \( \mathcal{L}^* (\mathcal{A}, \mathcal{B}, \mathcal{C}) \) the set of lines \( \ell \) having at least two points from \((\mathcal{A} \times \mathcal{A}) \cup (\mathcal{B} \times \mathcal{B}) \cup (\mathcal{C} \times \mathcal{C})\). Then we see that
\[
T(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \sum_{\ell \in \mathcal{L}^* (\mathcal{A}, \mathcal{B}, \mathcal{C})} \iota_{\mathcal{A}}(\ell) \iota_{\mathcal{B}}(\ell) \iota_{\mathcal{C}}(\ell).
\]

Finally, for any real \( M > 0 \) put
\[
L_{\mathcal{A}}(M) = \{ \ell : M < \iota_{\mathcal{A}}(\ell) \leq 2M \}.
\]

We need \cite[Lemma 14]{24}.

**Lemma 2.2.** Let \( \mathcal{A} \subseteq \mathbb{F}_p \) be a set and let \( M \) be a real number with \( \#\mathcal{A} \geq M \geq 2 (\#\mathcal{A})^2 / p \), then
\[
L_{\mathcal{A}}(M) \ll \min \left\{ \frac{p (\#\mathcal{A})^2}{M^2}, \frac{(\#\mathcal{A})^5}{M^4} \right\}.
\]

First we record the trivial identity
\[
(2.3) \quad \sum_{\ell \in \mathcal{L}} \iota_{\mathcal{A}}(\ell) = (p + 1) (\#\mathcal{A})^2,
\]
which holds for any set \( \mathcal{A} \subseteq \mathbb{F}_p \), (as there are exactly \( p + 1 \) lines passing through any point \((a_1, a_2) \in \mathbb{F}_p^2\)).

The next identity is well–known however, we give a short proof for the sake of the completeness.

**Lemma 2.3.** For \( \mathcal{A}, \mathcal{B} \subseteq \mathbb{F}_p \), we have
\[
\sum_{\ell \in \mathcal{L}} \iota_{\mathcal{A}}(\ell) \iota_{\mathcal{B}}(\ell) = (\#\mathcal{A} \#\mathcal{B})^2 + p\# (\mathcal{A}^2 \cap \mathcal{B}^2).
\]

**Proof.** We have
\[
\sum_{\ell \in \mathcal{L}} \iota_{\mathcal{A}}(\ell) \iota_{\mathcal{B}}(\ell) = \sum_{\ell \in \mathcal{L}} \sum_{q \in \mathcal{A}^2} \sum_{r \in \mathcal{B}^2} 1 \quad \sum_{q,r \in \ell, q \neq r} 1 + \sum_{q \in \mathcal{A}^2 \cap \mathcal{B}^2} \sum_{\ell \in \mathcal{L}} 1.
\]
Clearly, since two distinct points define a unique line, we have
\[
\sum_{(q,r) \in A^2 \times B^2 \atop q \neq r} \sum_{\ell \in \mathcal{L} \atop q,r \in \ell} 1 = (\#A \#B)^2 - \# (A^2 \cap B^2).
\]
Furthermore, using again that there are exactly \( p + 1 \) lines passing via any point in \( \mathbb{F}_p^2 \) we also have
\[
\sum_{q \in A^2 \cap B^2} \sum_{\ell \in \mathcal{L} \atop q \in \ell} 1 = (p + 1) \# (A^2 \cap B^2).
\]
The result now follows.

For a set \( A \subseteq \mathbb{F}_p \) we now define the function
\[
(2.4) \quad f_A(\ell) = \iota_A(\ell) - \frac{(\#A)^2}{p}.
\]
In particular, we see from Lemma 2.3 that
\[
(2.5) \quad \sum_{\ell \in \mathcal{L}} |f_A(\ell)|^2 \leq p (\#A)^2.
\]
Finally, for any real \( M > 0 \) put
\[
K_A(M) = \{ \ell \mid |f_A(\ell)| > M \}.
\]

Lemma 2.4. Let \( A \subseteq \mathbb{F}_p \) be a set and let \( M \) be a real number, then
\[
K_A(M) \ll \min \left\{ \frac{p (\#A)^2}{M^2}, \frac{(\#A)^5}{M^4} \right\}.
\]

Proof. For \( M \geq 2 (\#A)^2 / p \) the result follows from Lemma 2.2 as in this case \( |f_A(\ell)| > M \) implies
\[
2|f_A(\ell)| \geq \iota_A(\ell) \geq |f_A(\ell)| / 2.
\]
For \( M < 2 (\#A)^2 / p \) we derive from (2.5)
\[
K_A(M) \ll \frac{p (\#A)^2}{M^2}.
\]
Since for \( M < 2 (\#A)^2 / p \leq 2 (\#A)^{3/2} / p^{1/2} \) we have
\[
\frac{p (\#A)^2}{M^2} \ll \frac{(\#A)^5}{M^4}
\]
and the result follows.

Finally, we are ready to establish one of our main technical result which we believe is of independent interest.
Lemma 2.5. Let \( \mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathbb{F}_p \) be sets. Put \( Z = \max\{\#\mathcal{A}, \#\mathcal{B}, \#\mathcal{C}\} \). Then

\[
T(\mathcal{A}, \mathcal{B}, \mathcal{C}) - \frac{\left(\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}\right)^2}{p} \ll \begin{cases}
p\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}, \\
\left(\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}\right)^{3/2} + \#\mathcal{A}\#\mathcal{B}\#\mathcal{C}Z \\
\sqrt{p}\left(\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}\right)^{7/6} + Z^4.
\end{cases}
\]

Proof. We basically repeat the arguments from [29].

Put

\[
g_A(x) = \chi_A(x) - \frac{\#A}{p},
\]

where \( \chi_A(x) \) is the characteristic function of the set \( \mathcal{A} \). Thus

\[
\sum_{x \in \mathbb{F}_p} g_A(x) = 0. \tag{2.6}
\]

It is easy to see [32] that the quantity \( T(\mathcal{A}, \mathcal{B}, \mathcal{C}) \) equals the number of incidences between the planes

\[
\frac{1}{b_1 - c_1}X - a_2Y + Z = -\frac{c_1}{b_1 - c_1}, \tag{2.7}
\]

and the points

\[
\left(\frac{a_1}{b_2 - c_2}, \frac{c_2}{b_2 - c_2}\right)
\]

with \( a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}, c_1, c_2 \in \mathcal{C} \).

Using the function \( g_A(x) \), we have

\[
T(\mathcal{A}, \mathcal{B}, \mathcal{C}) = \frac{\left(\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}\right)^2}{p} + \sigma, \tag{2.8}
\]

where the sum \( \sigma \) counts the number of incidences (2.7) with the weight \( g_A(a_1)g_A(a_2) \). Hence by (2.2), which implies as we see from (2.6) that the condition (2.1) is satisfied, we get

\[
|\sigma| \leq p\#\mathcal{A}\#\mathcal{B}\#\mathcal{C} \tag{2.9}
\]

and by Lemma 2.1, we have

\[
\sigma \ll \frac{\left(\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}\right)^2}{p} + \left(\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}\right)^{3/2} + \#\mathcal{A}\#\mathcal{B}\#\mathcal{C}Z. \tag{2.10}
\]

We now observe that if \( \#\mathcal{A}\#\mathcal{B}\#\mathcal{C} \geq p^2 \), then

\[
p\#\mathcal{A}\#\mathcal{B}\#\mathcal{C} \leq \left(\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}\right)^{3/2}.
\]

and thus the bound (2.9) is stronger than (2.10). On the other hand, for \( \#\mathcal{A}\#\mathcal{B}\#\mathcal{C} < p^2 \), the first term in the bound (2.10) never dominates and it simplifies as

\[
\sigma \ll \left(\#\mathcal{A}\#\mathcal{B}\#\mathcal{C}\right)^{3/2} + \#\mathcal{A}\#\mathcal{B}\#\mathcal{C}Z. \tag{2.11}
\]
Now we always use $\vartheta_j$ to denote some real numbers with $|\vartheta_j| \leq 1$, $j = 1, 2, \ldots$.

Let $f_A(\ell)$ be defined by (2.4), we also define $f_B(\ell)$ and $f_C(\ell)$ similarly. We have

$$
T(A, B, C) = \sum_{\ell \in \mathcal{L}} \iota_A(\ell) \iota_B(\ell) \iota_C(\ell)
$$

$$
= \sum_{\ell \in \mathcal{L}} f_A(\ell) \iota_B(\ell) \iota_C(\ell) + \frac{(\#A)^2}{p} \sum_{\ell \in \mathcal{L}} \iota_B(\ell) \iota_C(\ell).
$$

Hence, by Lemma 2.3, estimating $\# (B^2 \cap C^2)$ trivially as $Z^2$, we obtain

$$
T(A, B, C) = \sum_{\ell \in \mathcal{L}} f_A(\ell) \iota_B(\ell) \iota_C(\ell) + \frac{(\#A \#B \#C)^2}{p} + \vartheta_1 (\#A)^2 Z^2.
$$

Thus, also using $\#A \leq Z$ for $\sigma$ defined by (2.8), we obtain

$$
\sigma = \sum_{\ell \in \mathcal{L}} f_A(\ell) \iota_B(\ell) \iota_C(\ell) + \vartheta_2 Z^4.
$$

Furthermore, from the definition of $f_B(\ell)$,

$$
\sum_{\ell \in \mathcal{L}} f_A(\ell) \iota_B(\ell) \iota_C(\ell)
$$

$$
= \sum_{\ell \in \mathcal{L}} f_A(\ell) f_B(\ell) \iota_C(\ell) + \frac{(\#B)^2}{p} \sum_{\ell \in \mathcal{L}} f_A(\ell) \iota_C(\ell).
$$

Using (2.3) and then Lemma 2.3 again, together with trivial bound $\# (B^2 \cap C^2) \leq Z^2$, we see that

$$
\sum_{\ell \in \mathcal{L}} f_A(\ell) \iota_C(\ell) = (\#A \#C)^2 + \vartheta_2 p Z^2 - \frac{(\#A \#B \#C)^2 (p + 1)}{p}
$$

$$
= - \frac{(\#A \#C)^2}{p} + \vartheta_2 p Z^2,
$$

where in fact $\vartheta_2 \in [0, 1]$. Clearly,

$$
\frac{(\#A \#C)^2}{p} \leq p Z^2.
$$

Hence

$$
\frac{(\#B)^2}{p} \sum_{\ell \in \mathcal{L}} f_A(\ell) \iota_C(\ell) = \vartheta_3 Z^4,
$$
which after substitution in (2.13) and then in (2.12) yields
\begin{equation}
\sigma = \sum_{\ell \in \mathcal{L}} f_A(\ell) f_B(\ell) \iota_C(\ell) + 2\vartheta_4 Z^4.
\end{equation}

Finally, the same arguments as in the above and an analogue of (2.14), lead us to the bound
\begin{equation}
\sum_{\ell \in \mathcal{L}} f_A(\ell) f_B(\ell) \iota_C(\ell) = \sum_{\ell \in \mathcal{L}} f_A(\ell) f_B(\ell) f_C(\ell) + \frac{(#C)^2}{p} \sum_{\ell \in \mathcal{L}} f_A(\ell) f_B(\ell)
= \sum_{\ell \in \mathcal{L}} f_A(\ell) f_B(\ell) f_C(\ell) + \vartheta_5 Z^4.
\end{equation}

Now, recalling (2.15), we obtain
\begin{equation}
\sigma = \sigma_0 + 3\vartheta_6 Z^4,
\end{equation}
where
\begin{equation}
\sigma_0 = \sum_{\ell \in \mathcal{L}} f_A(\ell) f_B(\ell) f_C(\ell).
\end{equation}

It remains to estimate \( \sigma_0 \).

We fix some real numbers \( \Delta_A, \Delta_B, \Delta_C > 0 \) and write
\begin{equation}
\sigma_0 \ll p (\Delta_A\#B\#C + \Delta_B\#A\#C + \Delta_C\#A\#B)
+ \sum_{\|f_\ell(t)\| \geq \Delta_\ell, \ u = A, B, C} |f_A(\ell) f_B(\ell) f_C(\ell)|.
\end{equation}

By the Hölder inequality
\begin{equation}
\sum_{\ell \in \mathcal{L}, \ |f_\ell(t)| \geq \Delta_\ell, \ u = A, B, C} |f_A(\ell) f_B(\ell) f_C(\ell)| \leq \prod_{u = A, B, C} \left( \sum_{\ell \in \mathcal{L}, \ |f_\ell(t)| \geq \Delta_\ell} |f_A(\ell)|^3 \right)^{1/3}.
\end{equation}

Given \( F > 0 \) we note that by Lemma 2.4 we have
\begin{equation}
\sum_{F \leq |f_\ell(t)| \leq 2F} |f_\ell(t)|^3 \ll \frac{F^3 (#A)^5}{F^4} = \frac{(#A)^5}{F}.
\end{equation}

It follows that
\begin{equation}
\sum_{\|f_\ell(t)\| \geq \Delta_\ell} |f_A(\ell)|^3 \ll \frac{(#A)^5}{\Delta_\ell}.
\end{equation}

Hence
\begin{equation}
\sigma_0 \ll p (\Delta_A\#B\#C + \Delta_B\#A\#C + \Delta_C\#A\#B) + \frac{#A^{5/3} #B^{5/3} #C^{5/3}}{(\Delta_A \Delta_B \Delta_C)^{1/3}}.
\end{equation}
We now choose
\[ \Delta_u = \#u \left( \#A \#B \#C \right)^{1/6}, \quad u = A, B, C, \]
and obtain
\[ \sigma_0 \ll \sqrt{p} \left( \#A \#B \#C \right)^{7/6} \]
which together with (2.16) implies
\[ (2.17) \quad \sigma \ll \sqrt{p} \left( \#A \#B \#C \right)^{7/6} + Z^4. \]

Combining the bounds (2.9), (2.11) and (2.17) with (2.8), we conclude the proof.

2.3. Bounds on the number of solutions to some equations with general sets. For sets \( S, \mathcal{X}, \mathcal{Y} \subseteq \mathbb{F}_p^* \), we define \( N(S, \mathcal{X}, \mathcal{Y}) \) to be the number of solutions to the system of equations:
\[
\begin{align*}
\frac{x_1 + s_1}{y_1} &= \frac{x_2 + s_2}{y_2} & \text{and} & & \frac{x_1 + t_1}{y_1} \neq \frac{x_2 + t_2}{y_2}, \\
s_1, s_2, t_1, t_2 &\in S, & s_1 \neq t_1, & & S \subseteq \mathcal{X}, & x_1, x_2 \in \mathcal{X}, & y_1, y_2 \in \mathcal{Y}.
\end{align*}
\]

We also recall the definition (1.6).

Lemma 2.6. Let \( S, \mathcal{X} \subseteq \mathbb{F}_p^* \) be arbitrary sets of cardinalities \( S \) and \( X \leq p^{1/2} \) such that \( S^2 X \leq p^2 \), and let \( \mathcal{Y} \) be the set of primes of the interval \([1, Y]\) for some \( Y \leq p^{1/2} \). Then we have
\[
N(S, \mathcal{X}, \mathcal{Y}) \ll Y \mathbb{E}_3^+ (S, \mathcal{X}) + S^3 X^{3/2} + S^2 X^2.
\]

Proof. We derive from (2.18) that
\[ \frac{x_1 + s_1}{x_2 + s_2} = \frac{x_1 + t_1}{x_2 + t_2} = \frac{y_1}{y_2} \neq 0. \]

First we consider the case when the common value \( \lambda \) of every ratio in the equation (2.19) is \( \lambda = 1 \). In this case we derive \( s_1 - s_2 = t_1 - t_2 = x_2 - x_1 \). Thus the vector \((s_1, s_2, t_1, t_2, x_1, x_2) \in S \times \mathcal{X}^2 \) can be chosen in \( \mathbb{E}_3^+ (S, \mathcal{X}) \) ways. So we see from (2.19) (and discarding the conditions that \( y_1 \) and \( y_2 \) are primes) that such vectors contribute in total
\[ N_1 = Y \mathbb{E}_3^+ (S, \mathcal{X}) \]
to \( N(S, \mathcal{X}, \mathcal{Y}) \).

By the second inequality of Lemma 2.5, using that \( S^2 X \leq p^2 \), we see that the equation
\[ \frac{x_1 + s_1}{x_2 + s_2} = \frac{x_1 + t_1}{x_2 + t_2}, \quad s_1, s_2, t_1, t_2 \in S, \quad x_1, x_2 \in \mathcal{X}, \]
has $O(S^3X^{3/2} + S^2XZ)$ solutions, where $Z = \max\{S, X\}$. Using $Z \leq S + X$ we derive

$$S^3X^{3/2} + S^2XZ \leq S^3X^{3/2} + S^2X(S + X)$$

$$= S^3X^{3/2} + S^3X + S^2X^2 \ll S^3X^{3/2} + S^2X^2.$$ 

However now we consider only solutions when the common value $\lambda$ of both sides of the equation (2.20) satisfies $\lambda \neq 1$. For each $\lambda \neq 1$ we get an equation of the type $y_1 = \lambda y_2$ which has at most 1 solutions in primes $y_1, y_2 \leq p^{1/2}$. Hence such vectors contribute in total

$$N_{\neq 1} \ll S^3X^{3/2} + S^2X^2$$

to $N(S, X, Y)$.

Collecting contributions $N_1$ and $N_{\neq 1}$ of both types, that is, writing $N(S, X, Y) = N_1 + N_{\neq 1}$, we obtain the result. □

**Remark 2.7.** We note that there are no details how this quantity $N(S, X, Y)$ is estimated in [19]. However, judging by the range where the main result of [19] is nontrivial it seems that instead of Lemma 2.5 the number of solutions to the equation (2.20) is estimated in [19] trivially as $O(S^3X^2)$.

**Remark 2.8.** The condition $S^2X \leq p^2$ of Lemma 2.6 is imposed so that the second inequality of Lemma 2.5 simplifies. One can certainly drop this restriction and apply Lemma 2.5 in full generality. In turn this may lead to some alternative forms of Theorem 1.1.

### 2.4. Bounds on the number of solutions to some equations with multiplicative subgroups

Here we give some bound on $E^+_3(G, G, I)$ with a multiplicative subgroup $G \subseteq \mathbb{F}_p^*$. We always assume that $G$ is of order $T \leq p^{2/5}$ as otherwise other, more standard methods work better.

**Lemma 2.9.** Let $G \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup of order $T \leq p^{2/5}$ and $I = [1, X]$. Then we have

$$E^+_3(G, G, I) \leq p^{o(1)} \begin{cases} T^{49/20}X, \\ T^2X + T^{4/3}X^{3/2} + T^{11/6}X^2p^{-1/2} + T^{41/24}X^{3/2}p^{-1/8}. \end{cases}$$

**Proof.** By [25] we have

$$E^+(S) = \#\{u_1 + u_2 = v_1 + v_2 \in G^4 : u_1 + u_2 = v_1 + v_2\} \leq T^{49/20}p^{o(1)},$$

which together with (1.9) immediately implies the first bound (see also Remark 1.6).
We now derive the second bound. Given a set \( A \subseteq \mathbb{F}_p \), we write \( r_-(A; x) \) and \( r_+(A; x) \) for the number of ways \( x \in \mathbb{F}_p \) can be expressed as a sum \( a - b \) and \( a/b \) with \( a, b \in A \), respectively.

Put

\[
\mathcal{T} = \mathcal{I} - \mathcal{I} = \{ x - y : x, y \in \mathcal{I} \} = [-X, X].
\]

Then

\[ (2.21) \quad E_3^+(\mathcal{G}, \mathcal{G}, \mathcal{I}) \leq T^2 X + 4RX, \]

where

\[
R = \sum_{x \in \mathcal{I}^*, x \neq 0} r_-(\mathcal{G}; x).
\]

We note that for \( x \in \mathcal{T} \) the value of \( r_-(\mathcal{G}; x) \) depends only on the coset \( \lambda \mathcal{G} \) with \( x \in \lambda \mathcal{G} \).

Let \( h = (p - 1)/T \).

Consider cosets \( C_j = \lambda_j \mathcal{G}, j = 1, \ldots, h \), such that for \( x_j \in C_j \) one has

\[
r_-(\mathcal{G}; x_1) \geq \ldots \geq r_-(\mathcal{G}; x_h).
\]

Let

\[
c_j = \# (C_j \cap \mathcal{T}) \quad \text{and} \quad t_j = r_-(\mathcal{G}; x_j),
\]

where \( C_j \cap \mathcal{T} \) is considered in \( \mathbb{F}_p \) (that is, after reducing elements modulo \( p \)), \( j = 1, \ldots, h \). First we note that

\[ (2.22) \quad \sum_{j=1}^h c_j^2 \leq \sum_{x \in \mathcal{G}} r_{\mathcal{I}^*/\mathcal{I}^*}(x) = N(\mathcal{I}^*, \mathcal{G}), \]

where

\[
N(\mathcal{I}^*, \mathcal{G}) = \# \{(x, y) \in \mathcal{I}^* : x/y \in \mathcal{G}\}
\]

The quantity \( N(\mathcal{I}^*, \mathcal{S}) \) has been introduced and studied in [4] however the argument in [4] is optimised for the case of rather large subgroups of order \( T \) around \( p^{1/2} \). Thus here we use a bound of Cilleruelo and Garaev [11, Theorem 1] which implies that for \( T \leq p^{2/5} \) we have

\[ (2.23) \quad N(\mathcal{I}^*, \mathcal{S}) \leq (X + TX^2/p + T^{3/4}Xp^{-1/4}) p^{o(1)} \]

and we have followed the scheme of the proof from paper [4].

By [21, Equation (3.13)] we have

\[
t_j \ll T^{2/3}j^{-1/3}, \quad j = 1, \ldots, h.
\]

Hence

\[ (2.24) \quad \sum_{j=1}^h t_j^4 \ll T^{8/3}. \]
Finally, using the Cauchy inequality, we derive from (2.22), (2.23) and (2.24) that

\[
R^2 \ll \sum_{j=1}^{h} t_j^4 \sum_{j=1}^{h} c_j^2 \ll T^{8/3} \left( X + TX^2/p + T^{3/4}X/p^{1/4} \right) p^\rho(1),
\]

which after substitution in (2.21) implies the desired result. \( \square \)

2.5. On the additive energy of polynomial images. Given an integer \( k \geq 2 \) and sets \( S_1, \ldots, S_k \subseteq \mathbb{F}_p \) one can generalize the additive energy as

\[
T_k^+(S_1, \ldots, S_k) = \# \{ u_1 + \cdots + u_k = v_1 + \cdots + v_k : u_i, v_i \in S_i, \ i = 1, \ldots, k \}.
\]

If \( S_i = S, \ i = 1, \ldots, k \) then we write \( T_k^+(S) \) for \( T_k^+(S, \ldots, S) \). Clearly, the energy \( T_k^+(S) \) is translation/dilation invariant and \( T_2^+(S) = E^+(S) \).

Using the orthogonality of exponential functions, we write

\[
T_k^+(S_1, \ldots, S_k) = \frac{1}{p} \sum_{\lambda=0}^{p-1} \prod_{j=1}^{k} \left| \sum_{v_j \in S_j} e_p(\lambda v_j) \right|^2
\]

where \( e_p(v) = \exp(2\pi i v/p) \) and applying the Hölder inequality, one sees that

\[
(2.25) \quad T_k^+(S_1, S_2, \ldots, S_k) \leq \prod_{j=2}^{k} \left( T_k^+(S_1, S_j \ldots, S_j) \right)^{1/(k-1)}.
\]

Furthermore, for \( k = 2, 3, \ldots \) we write

\[
E^+(S) = \frac{1}{p} \sum_{\lambda=0}^{p-1} \sum_{v \in S} e_p(\lambda v) \left| e_p(\lambda v) \right|^4 \leq \frac{1}{p} \sum_{\lambda=0}^{p-1} \sum_{v \in S} e_p(\lambda v) \left| \sum_{v \in S} e_p(\lambda v) \right|^{2k/(k-1)} \left| \sum_{v \in S} e_p(\lambda v) \right|^{2(k-2)/(k-1)},
\]

and using that

\[
\frac{1}{k-1} + \frac{k-2}{k-1} = 1
\]

by the Hölder inequality we derive

\[
(2.26) \quad (E^+(S))^{k-1} \leq T_k^+(S) T_1^+(S)^{k-2} = T_k^+(S)(\#S)^{k-2}.
\]
Now we obtain a nontrivial upper bound for the energy $T_k^+(f(A))$ and hence for $E^+(f(A))$ for a non-linear polynomial $f$ over $\mathbb{F}_p$ in the case when our set $A \subseteq \mathbb{F}_p$ has small sum and difference set
\[ A \pm A = \{a \pm b : a, b \in A\}. \]
A similar question has been studied, in particular, in [6] and [1]. In the proof we follow the method from [1].

**Lemma 2.10.** Let $f$ be a polynomial over $\mathbb{F}_p$ of degree $d \geq 2$. Then for $d = 2$ and sets $A_1, A_2, A_3 \subseteq \mathbb{F}_p$ with $\#A_3 \leq \#A_1 \leq \#A_2 \#(A_2 + A_3)$ we have
\[
T_3^+(f(A_1), f(A_2), f(A_3)) \ll \frac{(\#A_1 \#A_2 \#(A_2 + A_3))^2}{p} + (\#A_1 \#A_2 \#(A_2 + A_3))^{3/2},
\]
and for $d \geq 3$ and a set $A \subseteq \mathbb{F}_p$ and we have
\[
T_{2d-2+1}^+(f(A)) \ll \frac{(\#(A - A))^{2d-1-2}(\#A \#(A + A))^2}{p} + (\#(A - A))^{2d-1-5/2}(\#A \#(A + A))^{3/2},
\]
where the implied constant may depend on $d$.

**Proof.** For $d = 2$, making a linear changing of the variables one can assume that $f(Z) = \alpha Z^2 + \beta \in \mathbb{F}_p[Z]$ with $\alpha \neq 0$. Then, clearly, the quantity $T_3^+(f(A))$ is equal to the number of solutions to
\[
f(a_1) + 2ab_1 (b_1 + b_2) - \alpha (b_1 + b_2)^2 = f(a_2) + 2\alpha c_1 (c_1 + c_2) - \alpha (c_1 + c_2)^2,
\]
where $a_1, a_2 \in A_1$, $b_1, c_1 \in A_2$, $b_2, c_2 \in A_3$. Changing variables $b_1 + b_2 = u$, $c_1 + c_2 = v$, leads to the equation
\[
f(a_1) + 2ab_1 u - \alpha u^2 = f(a_2) + 2\alpha c_1 v - \alpha v^2,
\]
where $a_1, a_2 \in A_1$, $b_1, c_1 \in A_2$, $u, v \in A_2 + A_3$.

We now consider the set of points
\[ Q = \{(f(a_1) - \alpha u^2, 2\alpha u, c_1) : a_1 \in A_1, c_1 \in A_2, u \in A_2 + A_3\} \]
and the set of planes
\[ \Pi = \{Z_1 + b_1 Z_2 - 2\alpha v Z_3 = f(a_2) - \alpha v^2 : a_2 \in A_1, b_1 \in A_2, v \in A_2 + A_3\} \]
as in Lemma 2.1. Clearly, $\#Q = \#\Pi \ll (\#A_2 + A_3) \#A_1 \#A_2$. Examining the second and then the first and the third components of points in $Q$ we see the maximum number of collinear points in $Q$ is
\[ k \leq \max\{\#A_1, (\#A_2 + A_3)\}. \]
Using \( \#(A_2 + A_3) \leq \#A_2 \#A_1 \leq \#A_1 \#A_3 \), and \( \#A_1 \leq \#A_2 \#(A_2 + A_3) \) one can check that the second term in the bound of Lemma 2.1 never dominates and we obtain the desired result for \( d = 2 \).

Thus, we start with the case \( d = 3 \). In fact for the purpose of the induction, we need to establish a more general result. In particular, we assume that we are given another set \( H \subseteq \mathbb{F}_p \) with

\[
\#A \ll \#H \ll \#(A + A)\#(A - A),
\]

say, and we estimate \( T_3^+ (H, f(A), f(A)) \). We proceed as in the proof of [1, Proposition 2.12, Claim (b)]. As above making a change of the variables one can also assume that \( f(x) = \alpha x^3 + \beta x + \gamma \in \mathbb{F}_p[Z] \) with \( \alpha \neq 0 \).

Then, clearly, the quantity \( T_3^+ (H, f(A), f(A)) \) is equal to the number of solutions to

\[
h_1 + 3\alpha(b_1 - b_2) \left( \frac{(b_1 + b_2)^2}{4} + \frac{(b_1 - b_2)^2}{12} \right) + \beta(b_1 - b_2)
= h_2 + 3\alpha(c_1 - c_2) \left( \frac{(c_1 + c_2)^2}{4} + \frac{(c_1 - c_2)^2}{12} \right) + \beta(c_1 - c_2),
\]

\( h_1, h_2 \in H, b_1, b_2, c_1, c_2 \in A \).

Changing variables \( b_1 + b_2 = u_1, b_1 - b_2 = u_2, c_1 + c_2 = v_1, c_1 - c_2 = v_2 \) we obtain the equation

\[
h_1 + 3\alpha u_2 \left( \frac{u_1^2}{4} + \frac{u_2^2}{12} \right) + \beta u_2 = h_2 + 3\alpha v_2 \left( \frac{v_1^2}{4} + \frac{v_2^2}{12} \right) + \beta v_2,
\]

\( h_1, h_2 \in A, u_1, v_1 \in A + A, u_2, v_2 \in A - A \).

Now using Lemma 2.1, with the set of points

\[
Q = \{(h_1 + \beta u_2 + \alpha u_2^2/4, 3\alpha u_2/4, v_1^2) : h_1 \in H, u_2 \in A - A, v_1 \in A + A\}
\]

and the set of planes

\[
\Pi = \{ \frac{Z_1 + u_1^2 Z_2 - 3\alpha v_2 Z_3}{4} = h_2 + \beta v_2 + \alpha v_2^2/4 : h_2 \in H, v_2 \in A - A, u_1 \in A + A \}.
\]

Clearly,

\[
\#Q = \#\Pi \ll \#H \#(A - A) \#(A + A).
\]

We now apply Lemma 2.1, where we have

\[
k \leq \max\{\#(A - A), \#(A + A), \#H\}.
\]
Thus under the condition (2.27) Lemma 2.1 implies that

\[ T_3^+ (\mathcal{H}, f(\mathcal{A}), f(\mathcal{A})) \ll \frac{(\#\mathcal{H}\#(\mathcal{A} - \mathcal{A})\#(\mathcal{A} + \mathcal{A}))^2}{p} + (\#\mathcal{H}\#(\mathcal{A} - \mathcal{A})\#(\mathcal{A} + \mathcal{A}))^{3/2}, \]

which in particular gives the desired bound on \( T_3^+ (f(\mathcal{A})) \) for \( d = 3 \).

We know that for any \( u \in \mathbb{F}_p \) the following holds

\[ f(Z + u) - f(Z) = d u g_u(Z), \]

with some polynomial \( g_u \in \mathbb{F}_p[Z] \) depending on \( f \) and \( u \), of degree
\( \deg g_u = \deg f - 1 = d - 1 \). Thus \( T_{2d+1}^+ (f(\mathcal{A})) \) equals the number of solutions to the equation

\[ f(a_1) + du_1 g_u(b_1) + \ldots + du_{d+1} g_{u_{d+1}}(b_{d+1}) = f(a_2) + dv_1 g_v(c_1) + \ldots + dv_{d+1} g_{v_{d+1}}(c_{d+1}), \]

where \( a_1, a_2, b_j, c_j \in \mathcal{A} \) and \( u_j, v_j \in \mathcal{A} - \mathcal{A} \). Hence by (2.25), we see that

\[ T_{2d+1}^+ (f(\mathcal{A})) \leq (\#(\mathcal{A} - \mathcal{A}))^{2d+1} \max_{u \in \mathcal{A} - \mathcal{A}} \max_{a \in \mathcal{A}} T_{2d+1}^+ (f(\mathcal{A}), g_u(\mathcal{A}), \ldots, g_u(\mathcal{A})) \]

\[ = (\#(\mathcal{A} - \mathcal{A}))^{2d+1} T_{2d+1}^+ (f(\mathcal{A}), f_{d-1}(\mathcal{A}), \ldots, f_{d-1}(\mathcal{A})) \]

for some polynomial \( f_{d-1} \in \mathbb{F}_p[Z] \) of degree \( d - 1 \). Clearly, using the same argument (which does not make any use of the first set) one can obtain in a similar way that for any set \( \mathcal{H} \subseteq \mathbb{F}_p \), we have

\[ T_{2d+1}^+ (\mathcal{H}, f_{d-1}(\mathcal{A}), \ldots, f_{d-1}(\mathcal{A})) \]

\[ \leq (\#(\mathcal{A} - \mathcal{A}))^{2d} T_{2d-1+1}^+ (\mathcal{H}, f_{d-2}(\mathcal{A}), \ldots, f_{d-2}(\mathcal{A})) \]

for some polynomial \( f_{d-2} \in \mathbb{F}_p[Z] \) of degree \( d - 2 \). Iteratively, using (2.29) with \( \mathcal{H} = f(\mathcal{A}) \), we derive

\[ T_{2d-2+1}^+ (f(\mathcal{A})) \leq (\#(\mathcal{A} - \mathcal{A}))^{2d-2+\ldots+4} T_3(f(\mathcal{A}), f_3(\mathcal{A}), f_3(\mathcal{A})) \]

\[ = (\#(\mathcal{A} - \mathcal{A}))^{2d-1-4} T_3(f(\mathcal{A}), f_3(\mathcal{A}), f_3(\mathcal{A})), \]

for some cubic polynomial \( f_3 \in \mathbb{F}_p[Z] \). Finally, applying bound (2.28) we obtain

\[ T_{2d-2+1}^+ (f(\mathcal{A})) \ll (\#(\mathcal{A} - \mathcal{A}))^{2d-1-2} \frac{(\#\mathcal{A}\#(\mathcal{A} + \mathcal{A}))^2}{p} + (\#(\mathcal{A} - \mathcal{A}))^{2d-1-5/2} (\#\mathcal{A}\#(\mathcal{A} + \mathcal{A}))^{3/2} \]

as required. \( \square \)
For intervals $\mathcal{A} = \mathcal{I}$, the statement of Lemma 2.10 simplifies as follows. Clearly, it is enough to present these bounds only for initial intervals $\mathcal{I} = [1, X]$.

**Corollary 2.11.** Let $f$ be a polynomial over $\mathbb{F}_p$ of degree $d \geq 2$ and let $\mathcal{I} = [1, X]$ be an interval of length $X \leq p^{2/3}$. Then for $d = 2$ we have

$$T^+_3 (f(\mathcal{I})) \ll X^{9/2},$$

and for $d \geq 3$ we have

$$T^+_{2d-2+1} (f(\mathcal{I})) \ll X^{2d-1+1/2},$$

where the implied constant may depend on $d$.

We now record the bounds on the additive energy of polynomial images which are implied by Corollary 2.11 combined with (2.26).

**Corollary 2.12.** Let $f$ be a polynomial over $\mathbb{F}_p$ of degree $d \geq 2$ and let $\mathcal{I} = [1, X]$ be an interval of length $X \leq p^{2/3}$. Then for $d = 2$ we have

$$E^+ (f(\mathcal{I})) \ll X^{11/4}$$

and for $d \geq 3$ we have

$$E^+ (f(\mathcal{I})) \ll X^{3-1/2d-1}$$

where the implied constant may depend on $d$.

**Remark 2.13.** We recall that

$$\#(\mathcal{A} - \mathcal{A}) \leq \frac{\#(\mathcal{A} + \mathcal{A})^2}{\#\mathcal{A}}, \quad (2.30)$$

which follows from the Ruzsa triangle inequality [27, Chapter 1, Theorem 8.1], see also [6, Lemma 9]). From Lemma 2.10 together with (2.30) one can derive that if $\#\mathcal{A} \#(\mathcal{A} + \mathcal{A}) \#(\mathcal{A} - \mathcal{A}) \leq p^2$, then for $d \geq 3$

$$\#(\mathcal{A} + \mathcal{A})^{4-14/2d} \#(f(\mathcal{A}) + f(\mathcal{A})) \gg |\mathcal{A}|^{5-6/2d}$$

and thus

$$\#(\mathcal{A} + \mathcal{A}) + \#(f(\mathcal{A}) + f(\mathcal{A})) \gg |\mathcal{A}|^{1+1/(5\cdot2d-1-7)}.$$ 

This improves [6, Theorem 1] which gives the exponent $1 + 1/(16 \cdot 6^4)$ and under a more stringent condition $\#\mathcal{A} \leq p^{1/2}$. We also have a similar result for $d = 2$. 
3. Proofs of main results

3.1. Proof of Theorem 1.1. We have

\[ |W_{\chi}(I, S; \alpha, \beta)| \leq \sum_{x \in I} \left| \sum_{s \in S} \alpha_s \chi(s + x) \right|. \]

Thus, using \( \overline{\chi} \) to denote the complex conjugate character to \( \chi \), by the Cauchy inequality we derive

\[
|W_{\chi}(I, S; \alpha, \beta)|^2 \leq X \sum_{x \in I} \left| \sum_{s \in S} \alpha_s \chi(s + x) \overline{\chi}(t + x) \right|^2
= X \sum_{x \in I} \sum_{s, t \in S} \alpha_s \overline{\alpha_t} \chi(s + x) \overline{\chi}(t + x)
= XV + O(SX^2),
\]

where

\[ V = \sum_{s, t \in S \atop s \neq t} \alpha_s \overline{\alpha_t} \sum_{x \in I} \chi(s + x) \overline{\chi}(t + x). \]

We fix some integers \( Y, Z \geq 1 \) with \( 4YZ \leq X \) and denote by \( \mathcal{Y} \) the set of primes of the interval \([Y, 2Y]\).

Applying the same transformation as in the work of Fouvry and Michel [14, Equations (4.3) and (4.4)] and write

\[
V \leq \frac{p^{o(1)}}{YZ} \sum_{s, t \in S \atop y \in \mathcal{Y}} \sum_{x \in I} \sum_{z = Z + 1}^{2Z} \eta_z \chi(s + x + yz) \overline{\chi}(t + x + yz)
\]

with some complex numbers \( \eta_z \) satisfying \( |\eta_z| = 1 \) and the new interval

\[ \mathcal{T} = [-X, X]. \]

Now, using the multiplicativity of \( \chi \), we obtain

\[
V \leq \frac{p^{o(1)}}{YZ} \sum_{s, t \in S \atop x \in \mathcal{T}} \sum_{y \in \mathcal{Y}} \sum_{z = Z + 1}^{2Z} \eta_z \chi \left( \frac{s + x}{y} + z \right) \overline{\chi} \left( \frac{t + x}{y} + z \right).
\]

For each pair \((\lambda, \mu) \in \mathbb{F}_p^2\) we denote by \( \nu(\lambda, \mu) \) the number of solutions to the system of equation

\[
\frac{s + x}{y} = \lambda, \quad \frac{t + x}{y} = \mu, \quad (s, t, x, y) \in S^2 \times \mathcal{T} \times \mathcal{Y}, \ s \neq t.
\]
Thus we can re-write (3.2) as

\[
V \leq \frac{p^{\theta(1)}}{YZ} \sum_{(\lambda,\mu) \in F_p^2} \nu(\lambda, \mu) \left| \sum_{z=Z+1}^{2Z} \eta_z \chi(\lambda + z) \overline{\chi}(\mu + z) \right|.
\]

Clearly,

\[
\sum_{(\lambda,\mu) \in F_p^2} \nu(\lambda, \mu) \ll S^2 XY \quad \text{and} \quad \sum_{(\lambda,\mu) \in F_p^2} \nu(\lambda, \mu)^2 = N(\mathcal{S}, \mathcal{I}, \mathcal{Y}).
\]

We now fix some integer \( r \geq 1 \) and write

\[
\nu(\lambda, \mu) = \nu(\lambda, \mu)^{1-1/r} \left( \nu(\lambda, \mu)^2 \right)^{1/2r}.
\]

Applying the H"{o}lder inequality we derive from (3.3) that

\[
V^{2r} \leq \frac{p^{\theta(1)}}{Y^2 Z^{2r}} \left( \sum_{(\lambda,\mu) \in F_p^2} \nu(\lambda, \mu) \right)^{2r-2} \sum_{(\lambda,\mu) \in F_p^2} \nu(\lambda, \mu)^2 \sum_{(\lambda,\mu) \in F_p^2} \left| \sum_{z=Z+1}^{2Z} \eta_z \chi(\lambda + z) \overline{\chi}(\mu + z) \right|^{2r}
\]

\[
\leq \frac{S^{4r-4} X^{2r-2} p^{\theta(1)}}{Y^2 Z^{2r}} N(\mathcal{S}, \mathcal{I}, \mathcal{Y}) \sum_{(\lambda,\mu) \in F_p^2} \left| \sum_{z=Z+1}^{2Z} \eta_z \chi(\lambda + z) \overline{\chi}(\mu + z) \right|^{2r}.
\]

By the condition (1.8) we see that Lemma 2.6 applies and yields

\[
V^{2r} \leq \frac{S^{4r-4} X^{2r-2}}{Y^2 Z^{2r}} \left( YE_3^+ (\mathcal{S}, \mathcal{S}, \mathcal{I}) + S^3 X^{3/2} + S^2 X^2 \right) \sigma p^{\theta(1)},
\]

where

\[
\sigma = \sum_{(\lambda,\mu) \in F_p^2} \left| \sum_{z=Z+1}^{2Z} \eta_z \chi(\lambda + z) \overline{\chi}(\mu + z) \right|^{2r}.
\]
Furthermore, expanding and changing the order of summation, we derive
\[
\sigma = \sum_{(\lambda, \mu) \in \mathbb{F}_p^2} \sum_{z_1, \ldots, z_{2r} = Z + 1} \prod_{i=1}^{r} \eta_{z_i} \chi(\lambda + z_i) \chi(\mu + z_i) \prod_{i=r+1}^{2r} \eta_{z_i} \chi(\lambda + z_i) \chi(\mu + z_i) 
\]
\[
\leq \sum_{z_1, \ldots, z_{2r} = Z + 1} \left| \sum_{\lambda \in \mathbb{F}_p} \prod_{i=1}^{r} \chi(\lambda + z_i) \chi(\lambda + z_{r+i}) \right|^2. 
\]

Using the Weil bound in the form given by [18, Corollary 11.24] if \((z_1, \ldots, z_r)\) is not a permutation of \((z_{r+1}, \ldots, z_{2r})\), and the trivial bound otherwise, we derive
\[
(3.5) \quad \sigma \ll Z^{2r} p + Z^{r} p^2, 
\]
(see also [18, Lemma 12.8] that underlies the Burgess method).

We now choose
\[
Y = \left\lfloor 2XP^{-1/r} \right\rfloor \quad \text{and} \quad Z = \left\lfloor p^{1/r} \right\rfloor, 
\]
(note that due to the condition \(X \geq p^{1/r}\) this is an admissible choice), so that (3.5) becomes
\[
\sigma \ll Z^{2r} p, 
\]
which after the substitution in (3.4) becomes
\[
V^{2r} \leq \frac{S^{r-4} X^{2r-2}}{Y^2} \left( Y \mathcal{E}_3^+(\mathcal{S}, \mathcal{T}) + S^3 X^{3/2} + S^2 X^2 \right) p^{1+o(1)}. 
\]

In turn, substituting this in (3.1) after simple calculations we derive the desired result.

3.2. Proof of Theorem 1.4. Clearly, we need the conditions
\[
(3.6) \quad 5\zeta + 2\xi > 2 \quad \text{and} \quad \zeta + \xi > 1/2 
\]
to make sure that the terms in the bound of Theorem 1.1 that do not depend on \(\mathcal{E}_3^+(\mathcal{S}, \mathcal{T})\) are nontrivial (provided that \(r\) is large enough depending only on \(\zeta\) and \(\xi\)).

Now substituting the first bound of Lemma 2.9 into Theorem 1.1 obtain a nontrivial result provided
\[
(3.7) \quad 40\zeta + 31\xi > 20 
\]
and \(r\) is large enough.
It is also useful to observe that since $TX \leq p^{2/5+1/2+o(1)}$ the second bound of Lemma 2.9 simplifies as

$$E^+_3(\mathcal{G}, \mathcal{G}, \mathcal{I}) \leq (T^2 X + T^{4/3} X^{3/2} + T^{11/24} X^{3/2} p^{-1/8}) p^{o(1)}.$$ 

Hence, substituting this bound into Theorem 1.1 obtain a nontrivial result provided

$$9 \zeta + 16 \xi > 6 \quad \text{and} \quad 36 \zeta + 55 \xi > 21$$

(again for a sufficiently large $r$).

So, to have a nontrivial estimate, the parameters $\zeta$ and $\xi$ must satisfy (3.6) and at least one out of (3.7) and (3.8). Now, after simple, but somewhat tedious, calculations one derives the desired result.

### 3.3. Proof of Theorem 1.7.

For the first sum we write

$$\sum_{q \leq Q} \sum_{r \leq R} \chi(f(q) + r) = \sum_{q \leq Q} e^{i\psi_q} \sum_{r \leq R} \chi(f(q) + r)$$

where $0 \leq \psi_q < 2\pi$ is the argument of the second sum (which depends only on $q$). We introduce the weights $\alpha_s$ and $\beta_x$, where

- $\alpha_s$ is supported on the set $\mathcal{S} = \{f(q) : q \leq Q, q \text{ prime}\}$ and is defined as
  $$\alpha_s = \frac{1}{d} \sum_{q \leq Q, f(q) = s} e^{i\psi_q},$$
  thus $|\alpha_s| \leq 1$;
- $\beta_x$ is the characteristic function of primes $r \leq R$.

With the above notations,

$$\sum_{q \leq Q} \sum_{r \leq R} \chi(f(q) + r) = DW_\chi(\mathcal{I}, \mathcal{S}; \alpha, \beta).$$

We also use a similar representation for the second sum.

The result is instant if one combines Corollary 1.3 with the bound on the additive energy of polynomial images from Corollary 2.12.

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Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, Russia, 119991, and Institute for Information Transmission Problems of Russian Academy of Sciences, Bolshoy Karetny Per. 19, Moscow, Russia, 127994, and MIPT, Institutskii per. 9, Dolgoprudnii, Russia, 141701

E-mail address: ilya.shkredov@gmail.com

Department of Pure Mathematics, University of New South Wales, Sydney, NSW 2052 Australia

E-mail address: igor.shparlinski@unsw.edu.au