Optimal Dynamic Measurement Method Using Digital Moving Average Filter

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Abstract. The theory of optimal dynamic measurements is based on the use of methods of optimal control theory to find a dynamically distorted input signal from the observed output signal. The model of the measuring system contains a Leontief type system, the initial Showalter–Sidorov condition. Inertia of the measuring system and interference lead to distortion of the input signal. We present a numerical method to use a digital moving average filter for the observed signal with subsequent application of the method of optimal dynamic measurement, as well as the results of computational experiments.

1. Introduction

The problem of recovering a dynamically distorted signal from the observed one is the second inverse problem of dynamic measurements [1]. In order to solve the problem, until recently, the methods of the theory of inverse problems were used as the main mathematical tools. In the last decade, the theory of optimal dynamic measurements was created [2] and actively developed [3-5]. The theory is based on the synthesis of optimal control methods, Sobolev and Leontief type equations, dynamic measurements and automatic control. The work [6] proposes to apply the methods of the theory of automatic control to problems of dynamic measurements. The mathematical model of the measuring system is based on a Leontief type system that is a finite-dimensional analogue of a Sobolev type equation [7].

At present, methods of the theory of optimal dynamic measurements are developed. The methods are based on the development of the theory of stochastic Sobolev type equations [8] and numerical methods [9].

The report is devoted to a new numerical method, which is based on the application of a digital moving average filter to the observation data, followed by the application of the spline method of optimal dynamic measurements. The results of computational experiments and recommendations for using a digital moving average filter are presented.

2. Use of a digital moving average filter to solve problem of optimal dynamic measurement

2.1. Statement of the main problem of optimal dynamic measurement

Consider the space of states
\[ \mathcal{N} = \{ x \in L_2 \left( (0, \tau), \mathbb{R}^n \right) : \dot{x} \in L_2 \left( (0, \tau), \mathbb{R}^n \right) \}, \]
the space of observations

\[ \mathcal{O} = \{ y \in L_2 \left( (0, \tau), \mathbb{R}^m \right) \}, \]
and the space of measurements

\[ \mathfrak{A} = \left\{ u \in L_2\left((0, \tau), \mathbb{R}^n\right) : u^{(p+1)} \in L_2\left((0, \tau), \mathbb{R}^n\right) \right\}. \]

The measuring device is simulated by the system [6]

\[
\begin{align*}
L \dot{x} &= Ax + Bu, \\
y &= Cx + D \eta,
\end{align*}
\]

and the initial Showalter-Sidorov condition [2]

\[
\left( (\alpha L - A)^{-1} L \right)^{p+1} (x(0) - x_0) = 0,
\]

where \( L \) and \( A \) are square matrices of the order \( n \) (note that there are measuring systems such that \( \det L = 0 \) [7]), the matrix \( A \) is \((L; p)\)-regular [3], \( x(t), y(t), u(t) \) are vector functions of the states of the measuring system, the observed signal and the input signal, respectively, \( x_0 \in \mathbb{R}^n, \alpha \in \rho^1(M) \), the matrix \( B \) characterizes the relationship between the input of the system and its state, the matrices \( C \) and \( D \) characterize the relationship between the state of the system and observations, \( \eta(t) \) is the vector function of noises at the output of the measuring system.

In \( \mathfrak{A} \), consider a closed convex set of admissible measurements \( \mathfrak{A}_\varepsilon \subset \mathfrak{A} \)

\[
\mathfrak{A}_\varepsilon = \left\{ u \in \mathfrak{A} : \sum_{q=0}^r \left\| y^{(q)}_0(t) \right\|^2 dt \leq \varepsilon \right\}.
\]

It is required to find an optimal dynamic measurement \( v \in \mathfrak{A}_\varepsilon \) at which the minimum value

\[
J(v) = \min_{u \in \mathfrak{A}_\varepsilon} J(u)
\]

of the functional

\[
J(u) = \sum_{q=1}^r \left\| y^{(q)}(u, t) - y^{(q)}_0(t) \right\|^2 dt
\]

is achieved, where \( y_0(t), t \in [0, \tau] \), is a continuously differentiable function (we consider the function as a «real observation») constructed on the basis of the values \( Y_{0i} \) observed at the output of the measuring system. The quality functional reflects an estimate of the proximity of the real observation \( y_0(t) \) and the observation \( y(t) \) obtained on the basis of the mathematical model of the measuring device. The presented problem (1) – (5) is called the main problem of optimal dynamic measurements.

2.2. Numerical method algorithm

Suppose that the following is given: matrices included in system (1); the initial value \( x_0 \in \mathbb{R}^n \); the array of observed values \( Y_{0i} \) of the output signal at times \( t_i, i = 0, 1, \ldots, n \), and \( t_{i+1} - t_i = \delta, t_0 = 0, t_n = \tau \); the maximum frequency \( f_c \), which limits the spectrum of the desired input signal.

1. Determine the type of the digital moving average filter and apply the filter to the array of values \( Y_{0i} \) to get smoothed values \( y_{0i}, i = 0, 1, \ldots, n \). At that, as it will be shown in computational experiments, it is important to determine the number of points \( N \) used to calculate the moving average. Assume that the effect of noise at the output is leveled out by smoothing with a filter and proceed to problem (1) – (5) at \( D=0 \).

2. Divide the segment \([0, \tau]\) into \( M \) segments \( [\tau_{m-1}, \tau_m] \), where \( m = 1, 2, \ldots, M \), and \( \tau_0 = 0, \tau_n = \tau_M \). At each segment \([\tau_{m-1}, \tau_m]\), solve the problem on optimal dynamic measurement, while the approximate value of the optimal measurement \( \hat{v}_m(t) \) is found in the form of a polynomial of the degree \( \ell \leq n - 1/M \). The basics of the algorithm are presented in [3]. In this algorithm, the important thing is the condition \( \hat{v}_m^t(\tau_m) = \hat{v}_{m+1}^t(\tau_m) \).
3. Display the obtained array of values \( \hat{\nu}(t_i) \) graphically. In addition, the interpolation function \( \hat{\nu}(t) \) can be constructed by this array.

2.3. Computational experiments

The Metran-43 sensor with an analog electronic converter was tested at the stand. The result of implementing the algorithm without using a digital filter is shown in Fig. 1, where \( u \) is the test signal recorded by the control sensor (black line), \( \nu \) is the optimal measurement (red line). The differences are due to interference at the output of the measuring device and its inertia. In this experiment, \( \delta=0,0002 \) and \( n=701 \).

When conducting computational experiments, we apply the following various types of digital moving average filter to the arrays of observations:

1) simple moving average (type A)

\[
y_{0i} = \frac{1}{N} \sum_{k=0}^{N-1} Y_{i-k};
\]

2) weighted linearly smoothed moving average with an increase in weight coefficients to the current value, which is the largest in the set of \( N \) points (type B)

\[
y_{0i} = \frac{\sum_{k=0}^{N-1}(N - k)Y_{i-k}}{N \sum_{k=0}^{N-1}(k + 1)};
\]

3) weighted linearly smoothed moving average with a decrease in weight coefficients to the current value, which is the average of the set of \( N \) points (type F)

\[
y_{0i} = \frac{\sum_{k=0}^{N-1}(k + 1)Y_{i-k}}{N \sum_{k=0}^{N-1}(k + 1)};
\]

4) weighted linearly smoothed moving average with an increase in weight coefficients to the current value, which is the average of the set of \( N \) points (type C). In addition, various values of \( N \) were considered.

Figs. 2 - 6 show the results of a computational experiment when using the filter of the type A for different values of \( N \).

Fig. 1. The test signal \( u \) and the optimal dynamic measurement \( \nu \) without using a digital filter

Fig. 2. The test signal \( u \) and the optimal dynamic measurement \( \nu \) when using the digital filter of the weighted moving average of the type A at \( N=10 \)

Fig. 3. The test signal \( u \) and the optimal dynamic measurement \( \nu \) when using the digital filter of the weighted moving average of the type A at \( N=20 \)
Note that the number of points $N$ corresponds to the interval over which the average value is calculated and which is an “analog” of the sampling interval. Using the Kotelnikov theorem with a known $f_c = 250$ Hz, the sampling interval $\Delta = 0.002$ is determined. However, taking into account the fact that the average is calculated by the set of points, this interval is doubled. This interval corresponds to a set of 20 points. The results of computational experiments at $N = 20$ (compared with the results at $N = 10$) showed the achievement of a smaller value of the dynamic error when using various types of digital filter. With a further increase in the value of $N$, the error increases due to greater smoothing and the inability to reflect the local peaks of the input signal. As the value of $N$ increases, the lag effect becomes more pronounced; in order to level the effect, the standard technique is to use a shift. For this form of the average, it is empirically established that the best shift in $t$ is the shift in the number of points $m$ such that $m\delta = \Delta$, i.e. equal to the interval, and the high-frequency interference is better filtered in this case. (Figs. 4, 5).

Figs. 6, 7 show the results of applying a weighted linearly smoothed moving average with an increase in weight coefficients to the current value, which is the largest in the set of $N$ points (type $B$). The same patterns are observed when using the filter of the type $A$. However, in this case, the shift to the left is required to be less, i.e. equal to the number of points $m$ such that $m\delta \approx \frac{2\Delta}{3}$.
Figs. 8, 9 show the results of applying a weighted linearly smoothed moving average with a decrease in weight coefficients to the current value, which is the largest in the set of $N$ points (type $F$). The same patterns are observed when using the filter of the type $A$. However, in this case, the shift to the left is required to be more, i.e. equal to the number of points $m$ such that $m\delta \approx \frac{4\Delta}{3}$.

Fig. 8. The test signal $u$ and the optimal dynamic measurement $v$ when using the digital filter of the weighted moving average of the type $F$ at $N=10$ and the shift in $\Delta$

Fig. 9. The test signal $u$ and the optimal dynamic measurement $v$ when using the digital filter of the weighted moving average of the type $F$ at $N=20$ and the shift in $\frac{4\Delta}{3}$

The filter of the type $C$ is the unique non-causal filter among the considered ones. Fig. 10 shows the results of a computational experiment using the filter of the type $C$. In this case, the lag effect is not observed, a smaller dynamic error takes place at $N=20$.

3. Conclusion
As a result of application of a digital moving average filter to the observed signal when recovering a dynamically distorted signal by the optimal dynamic measurement method, we draw the following conclusions. The number of points $N$ used in the moving average method must belong to the segment equal to $2\Delta$, where $0 < \Delta \leq \frac{1}{2f_c}$, $f_c$ is the maximum frequency that limits the spectrum of the desired input signal.

A smaller dynamic error is achieved when using the following two types of digital moving average filters: 1) the type $A$, and we propose to use a shift in values in order to reduce the lag effect, 2) the type $C$.

In the future, we intend to consider other types of digital filters within the similar statement of the problem.

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