THE MOVABLE CONE OF CALABI–YAU THREEFOLDS IN RULED FANO MANIFOLDS

ATSUSHI ITO, CHING-JUI LAI, SZ-SHENG WANG

ABSTRACT. We describe explicitly the chamber structure of the movable cone for a general complete intersection Calabi–Yau threefold in a non-split \((n+4)\)-dimensional \(\mathbb{P}^n\)-ruled Fano manifold of index \(n+1\) and Picard number two. Moreover, all birational minimal models of such Calabi–Yau threefolds are found whose number is finite.

1. INTRODUCTION

In the classical Mirror Symmetry, the mirror map conjecturally identifies a neighborhood of a large volume limit point in the Kähler moduli of a smooth Calabi–Yau threefold \(X\) with that of a special boundary point in the complex moduli of a mirror, which is characterized by unipotent monodromy. If the nef cone \(\text{Nef}(X)\) is a finite rational polyhedral cone, there is a partial compactification with a finite number of large volume limit points of the complexified Kähler cone of \(X\). In general, a cone conjecture proposed by Morrison and Kawamata \([\text{Mor93, Mor96, Kaw97}]\) predicts that the nef cone has only finitely many \(\text{Aut}(X)\)-orbits of edges.

In the present article, by a smooth Calabi–Yau threefold, we mean a smooth projective threefold \(X\) with \(K_X \sim 0\) and \(H^1(O_X) = 0\). We focus on the birational version of the cone conjecture for such \(X\). Recall that a divisor \(D\) is movable if for some positive number \(m\) the linear system \(|mD|\) has no fixed components and the movable cone \(\text{Mov}(X)\) is the closure of the convex hull of movable divisor classes. The movable cone conjecture is the following:

**Conjecture 1.1** ([\text{Mor96, Kaw97}]). There is a finite rational polyhedral cone which is a fundamental domain for the action of the birational automorphism group \(\text{Bir}(X)\) on the movable effective cone \(\text{Mov}(X) \cap \text{Eff}(X)\).

For a survey of this widely open conjecture, we refer the reader to \([\text{Tot10, LOP18}]\).

A normalized \(\mathbb{P}^n\)-ruled Fano manifold \(P\) over \(M\) is the projective bundle associated to an ample bundle \(\mathcal{F}\) of rank \(n+1\) with \(c_1(\mathcal{F}) = c_1(T_M)\). Such a pair \((M, \mathcal{F})\) is called a Mukai pair in \([\text{Kan19a}]\). The classification of Mukai pairs with rank(\(\mathcal{F}\)) \(\geq \dim M - 2\) has been completed recently in \([\text{Kan19b}]\), and in this series of work we mainly consider the case when \(\dim M = 4\) and Picard number \(\rho(M) = 1\). When \(\mathcal{F}\) splits as a direct sum of line bundles, we proved Conjecture 1.1 in \([\text{LW22}]\) for a general complete intersection Calabi–Yau threefold in such \(P\) except when \(M\) is del Pezzo of degree one. In this paper, we treat the non-splitting \(\mathcal{F}\) as follows, see Theorem 3.1, 4.1 and 5.1 for the details.

**Theorem 1.2.** Let \(P = \mathbb{P}(\mathcal{F})\) be a normalized \(\mathbb{P}^n\)-ruled Fano manifold over a smooth projective fourfold \(M\) with \(\rho(M) = 1\). Assume that \(\mathcal{F}\) is not isomorphic to a direct sum of line bundles nor \(T_{\mathbb{P}^4}\). Then a complete intersection \(X_\mathcal{F}\) of \(n+1\) general hypersurfaces

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in $|\mathcal{O}_F(1)|$ is a smooth Calabi–Yau threefold of Picard number two. Moreover, all the minimal models of $X_F$ are constructed and the movable cone $\overline{\text{Mov}}(X_F)$ is a rational polyhedral cone.

Note that the case $F = T_{p4}$ is non-splitting but it can be reduced to the case $\mathcal{O}_{\mathbb{P}^4}(1)^{\oplus 5}$ (see [LW22 §6.1]). Therefore, Theorem 1.2 does imply that Conjecture 1.1 holds for a general complete intersection Calabi–Yau threefold in a normalized $\mathbb{P}^n$-ruled Fano manifold of Picard number two associated to a non-splitting bundle over a smooth Fano fourfold.

The proof of Theorem 1.2 is based on the construction of the determinantal contraction in [Wan20] and then modeled on the proof for splitting cases in [LW22], but requires many new ideas. We briefly discuss the technical issues below.

For $F$ being splitting, we can find a special surface in $X_F$ by applying the geometric construction of Eagon–Northcott complexes (see [LW22 Proposition 4.4]). When $\text{Bir}(X_F)$ is finite, studying the geometry of such surfaces is sufficient for finding all the other minimal models of $X_F$. However, this construction does not work when $F$ is non-splitting.

When $F$ is not isomorphic to $T_{p4}$ and non-splitting, there are three cases (see Theorem 2.1), which are bundles on the Grassmannian $\text{Gr}(2, 4)$, and del Pezzo fourfolds $V_4$ and $V_5$ of degrees 4 and 5 respectively. Except for the case $V_4$, the other two cases happen to have very different technical difficulties from the splitting cases:

In each case, we have the associated small contraction from $X_F$ to a determinantal variety in $M$ and its flop $X_F \dashrightarrow X_F$ (see (1)). Since there is a natural fibration $X_F \to \mathbb{P}^n$ with $n = 1$ or 2, we obtain one edge of $\overline{\text{Mov}}(X_F)$. Thus what we need to do first is to find another contraction $X_F \to Y_F$ and its flop.

(i) The constructions of $X_F \to Y_F$ and its flop in the case $M = V_4$ are similar to those in the case $(\text{Gr}(2, 4), \mathcal{O}(1)^{\oplus 4})$ (see [LW22 §6.2]). By construction, we have a generically $2 : 1$ morphism $X_F \to \mathbb{P}^3$. We can check that its Stein factorization $X_F \to Y_F$ is a small contraction. Since there is an involution $X_F \dashrightarrow X_F$ over $\mathbb{P}^3$, which is the flop of $X_F \to Y_F$, we obtain another edge of $\overline{\text{Mov}}(X_F)$. Thus we can determine $\overline{\text{Mov}}(X_F)$ and its decomposition by nef cones of all minimal birational models (Theorem 3.1).

(ii) For the case $M = V_5$, we give an alternative description of $X_F$ as a suitable zero locus in $\mathbb{P}_{\mathbb{P}^4}(\wedge^3 T_{p4})$. Moreover, by finding out an analog of $(\mathcal{E}, F)$, we can apply the determinantal construction of [Wan20] to this description to obtain a small contraction $X_F \to Y_F$ and its flop $X_F \dashrightarrow X_F^\perp$. Finally, using the method in [Ito14] of constructing Mori dream spaces with Picard number two, we can construct a small contraction of $X_F^\perp$, its flop $X_F^\perp \dashrightarrow X_F^{++}$ and a fibration $X_F^{++} \to \mathbb{P}^1$. Thus we can determine $\overline{\text{Mov}}(X_F)$ and its decomposition (Theorem 4.1).

(iii) For the case $M = \text{Gr}(2, 4)$, we give an alternative description of $X_F$ as in (1), which induces a small contraction $X_F \to Y_F \subseteq \mathbb{P}^4$. However, this description involves a morphism $\tau$ from a vector bundle to a reflexive sheaf, which is not locally free at a unique point $p \in \mathbb{P}^4$ (see (23)). Hence we “resolve the singularity” of $\tau$, that is, consider a relevant morphism $\rho$ between vector bundles on the blow-up $\widetilde{\mathbb{P}} \to \mathbb{P}^4$ at $p$ (see (27)). By the determinantal construction, the morphism $\rho$ induces a small contraction $\widetilde{X}_F \to \widetilde{Y}_F \subseteq \widetilde{\mathbb{P}}$, which descends to the flop $X_F^\perp \to Y_F$ of $X_F \to Y_F$. Since there is a fibration $X_F^\perp \to \mathbb{P}^2$, we obtain another edge of $\overline{\text{Mov}}(X_F)$ (Theorem 5.1).

\footnote{It is called a Bănică sheaf, see [Kan19b Definition 1.5] and references therein.
We remark that Conjecture 1.1 has been verified for several special cases, see [Bor91, Kaw97, Fry01, LP13, Ogu14, CO15, BN16, Yn22, Wan22, LW22] and references therein. For the case of Calabi–Yau manifold $X$ with Picard number two, Lazić and Peternell [LP13, Theorem 1.4] proved that there is a polyhedral cone $\Pi$ which is a fundamental domain for the action of Bir($X$) on the movable effective cone of $X$, and that the cone $\Pi$ is rational under the additional assumption that Bir($X$) is infinite, i.e., Conjecture 1.1 holds for such $X$. However, Conjecture 1.1 is still open when Bir($X$) is finite. The fundamental difficulty is whether $\text{Mov}(X)$ is rational. Our main result provides new positive evidence in this case.

The paper is organized as follows: In §2 we recall some foundational material concerning the classification of Fano bundles and the determinantal construction. We also give Table 1 and 2 of intersection numbers and Hodge numbers on our Calabi–Yau threefolds respectively. We devote the remaining sections §3–§5 to the proof of our main result, Theorem 1.2.

Notation 1.3. Throughout this paper, we work over the complex field $\mathbb{C}$. For a coherent sheaf $F$ on a variety $M$ and an integer $r \geq 1$, we write $\text{Gr}_M(F, r)$ for the Grassmannian of $r$-dimensional quotients of $F$. For the case $r = 1$, we denote by $\mathbb{P}_M(F)$ the projective bundle with the tautological line bundle $\mathcal{O}_{\mathbb{P}(F)}(1)$. When $F$ is a vector bundle of rank $f$, we set $\text{Gr}_M(r, F) := \text{Gr}_M(F, f-r)$. Moreover, we will use the notation $\text{Gr}_M(r, f)$ if $F$ is trivial and omit the subscript “$M$” when no confusion can arise.

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2. Preliminaries

2.1. Non-split Fano bundles. Let us briefly recall the classification of Fano bundles, i.e., vector bundles whose projectivization are Fano manifolds. For simplicity, we only list such bundles which will be needed in this paper.

Theorem 2.1 ([PSW92, Occ01, NO07, Kan19b]). Let $F$ be an ample bundle on a smooth projective fourfold $M$ with $c_1(F) = c_1(T_M)$ and $\rho(M) = 1$. If $F$ is not isomorphic to a direct sum of line bundles nor to $T_{\mathbb{P}^4}$, then the pair $(M, F)$ is isomorphic to one of the following:

$$(V_4, p^*S^V(1)), \quad (V_5, S^V_5(1)) \quad \text{or} \quad (\text{Gr}(2, 4), S(2) \oplus \mathcal{O}(1)).$$

Here we use the following symbols:

- $S$ is the universal subbundle on $\text{Gr}(2, 4)$.
- $V_4$ is a quartic del Pezzo fourfold obtained as a double cover $V_4 \xrightarrow{p} \text{Gr}(2, 4) \cong Q_4$ of the hyperquadric of dimension 4 branched along a smooth divisor $B \in |\mathcal{O}_{Q_4}(2)|$.
- $V_5$ is a general 2-codimensional linear section of the Grassmannian $\text{Gr}(2, 5)$ embedded into the projective space $\mathbb{P}^9$ via the Plücker embedding.
• \( S_{V_5} \) is the restriction of the universal subbundle on \( \text{Gr}(2, 5) \) to \( V_5 \).

The Fano bundle appearing above defines a normalized Fano manifold ruled by projective spaces. Let us recall the definition of such manifolds from [NO07, Definition 3.2].

**Definition 2.2.** Let \( \mathcal{F} \) be a vector bundle of rank \( r \geq 2 \) on a projective manifold \( M \). We will call \( \mathbb{P}(\mathcal{F}) \) a normalized \( \mathbb{P}^{r-1} \)-ruled Fano manifold if \( \mathcal{F} \) is an ample bundle with \( c_1(\mathcal{F}) = c_1(T_M) \).

Note that such Fano manifold \( \mathbb{P}(\mathcal{F}) \) has index \( r \), i.e., \( O(K_{\mathbb{P}(\mathcal{F})}) \cong O_{\mathcal{F}}(-r) \). Conversely, each \( \mathbb{P}^{r-1} \)-ruled Fano manifold \( \mathbb{P}(\mathcal{F}) \) of index \( r \) is isomorphic to a normalized one (see [NO07, Proposition 3.3]).

**2.2. Determinantal varieties.** This subsection reviews briefly some results of degeneracy loci, which will be needed in the later sections. For more details we refer the reader to [LW22, §3 and §4].

Fix a morphism \( \sigma: \mathcal{E}^\vee \to \mathcal{F} \) of vector bundles on a variety \( M \) of rank \( e \) and \( f \), respectively. For each \( k \leq \min\{e, f\} \), the \( k \)th degeneracy locus of \( \sigma \) is

\[
D_k(\sigma) = \{ x \in M \mid \text{rank}(\sigma(x)) \leq k \},
\]

with the convention \( D_{-1}(\sigma) = \emptyset \). It can be described as the zero locus of the global section \( \wedge^k \sigma \) of the bundle \( \wedge^{k+1} \mathcal{E} \otimes \wedge^{k+1} \mathcal{F} \). Note that the expected codimension of \( D_k(\sigma) \) in \( M \) is \( (e-k)(f-k) \), though \( D_k(\sigma) \) can be empty or have strictly smaller codimension. However, it is known that \( D_k(\sigma) = \emptyset \) if and only if \( D_k(\sigma) \) coincides with the expected one if \( \mathcal{E} \otimes \mathcal{F} \) is globally generated and \( \sigma \) is general (see [Ban91] for example).

The following lemma will be needed in Section 5.

**Lemma 2.3.** For an integer \( r \geq 1 \), consider the Grassmannian \( \pi: \text{Gr}(\mathcal{F}, r) \to M \) as a Quot scheme. Let \( \pi^* \mathcal{F} \to Q \) be the universal quotient bundle of rank \( r \) on \( \text{Gr}(\mathcal{F}, r) \) and \( \mathcal{C} = \text{coker} \sigma \). Then the closed subscheme

\[
\text{Gr}(\mathcal{C}, r) \subseteq \text{Gr}(\mathcal{F}, r)
\]

is the zero locus of the composition \( \pi^* \mathcal{E}^\vee \xrightarrow{\pi^* \sigma} \pi^* \mathcal{F} \to Q \) in \( \text{Gr}(\mathcal{F}, r) \).

**Proof.** Recall that a morphism \( g: T \to \text{Gr}(\mathcal{F}, r) \) corresponds to a quotient bundle \( h^* \mathcal{F} \to Q \) of rank \( r \) over \( h: T \to M \). In this correspondence, \( h^* \mathcal{F} \to Q \) coincides with the pullback of \( \pi^* \mathcal{F} \to Q \) by \( g \). Then \( g \) factors through \( \text{Gr}(\mathcal{C}, r) \hookrightarrow \text{Gr}(\mathcal{F}, r) \) if and only if \( h^* \mathcal{F} \to Q \) factors through \( h^* \mathcal{F} \to h^* \mathcal{C} \) if and only if the composite \( h^* \mathcal{E}^\vee \to h^* \mathcal{F} \to Q \) is zero. Since this composite coincides with \( g^*((\pi^* \mathcal{E}^\vee \to \pi^* \mathcal{F} \to Q)) \), we see that \( g \) factors through \( \text{Gr}(\mathcal{C}, r) \) if and only if \( g \) factors through the zero locus of \( \pi^* \mathcal{E}^\vee \to \pi^* \mathcal{F} \to Q \).

For the case \( r = 1 \), we get the projective bundle \( \mathbb{P}(\mathcal{F}) \) and write \( p_{\mathcal{F}} \) for the projection of \( \mathbb{P}(\mathcal{F}) \) to \( M \). We can also view the composition of \( p_{\mathcal{F}}^* \sigma: p_{\mathcal{F}}^* \mathcal{E}^\vee \to p_{\mathcal{F}}^* \mathcal{F} \) with the canonical map \( p_{\mathcal{F}}^* \mathcal{F} \to O_{\mathcal{F}}(1) \) as a global section \( s_{\sigma} \) of the bundle

\[
\mathcal{H}om(p_{\mathcal{F}}^* \mathcal{E}^\vee, O_{\mathcal{F}}(1)) \cong \mathcal{E} \boxtimes O_{\mathcal{F}}(1).
\]

Here and subsequently, we use \( \mathcal{V} \boxtimes O_{\mathcal{F}}(\ell) \) to denote \( p_{\mathcal{F}}^* \mathcal{V} \otimes O_{\mathcal{F}}(\ell) \) for any bundle \( \mathcal{V} \) over \( M \) and \( \ell \in \mathbb{Z} \) by abuse of notation.

The following lemma is proved in [LW22, Lemma 3.5 and 3.6].

**Lemma 2.4 ([LW22]).** For \( e \geq f \), we have:
(1) The projective bundle $\mathbb{P}(\text{coker} \sigma)$ coincides with the zero scheme $Z(s_\sigma)$ in $\mathbb{P}(\mathcal{F})$.

(2) The restriction of $p_\sigma$ to $Z(s_\sigma)$ maps onto $D_{f-1}(\sigma)$ and it is an isomorphism if $D_{f-2}(\sigma) = \emptyset$.

Note that the expected codimension of $Z(s_\sigma)$ in $\mathbb{P}(\mathcal{F})$ is $e$, and given $x \in D_{f-1}(\sigma)$ the fiber of $Z(s_\sigma)$ over $x$ is $\mathbb{P}(\text{coker} \sigma(x))$. Also, Lemma 2.4 (1) is the case $r = 1$ in Lemma 2.3.

2.3. Calabi–Yau threefolds. We recall a notion of the generality of morphisms of bundles used in [Wan20]. Let $M$ be a smooth projective variety and let $\sigma: \mathcal{E}^\vee \to \mathcal{F}$ be a morphism of vector bundles on $M$ of rank $e$ and $f$ respectively.

**Definition 2.5.** For a given integer $r \geq 0$, the $\sigma$ is said to be $r$-general if $D_i(\sigma) \setminus D_{i-1}(\sigma)$ is smooth of (expected) codimension $(e-i)(f-i)$ in $M$ for all $i = 0, \ldots, r$.

In the remainder of this subsection we assume that $\mathcal{E}$ and $\mathcal{F}$ have the same rank $n+1$ with the Calabi–Yau condition

$$\det(\mathcal{E}) \otimes \det(\mathcal{F}) \cong \mathcal{O}(-K_M)$$

and $\dim M = 4$. To construct Calabi–Yau threefolds, we need the following two propositions, see [Wan20] Proposition 3.6, Theorem 4.4 and [LW22] Remark 3.4, Proposition 4.4 (i) and B.1.

**Proposition 2.6 ([Wan20]).** With the above notation, we fix an $n$-general morphism $\sigma$. Let us denote by $X_\mathcal{F}$ the zero scheme $Z(s_\sigma) \subseteq \mathbb{P}(\mathcal{F})$ and by $\pi_\mathcal{F}$ the restriction of $p_\mathcal{F}$ to $X_\mathcal{F}$. Then $X_\mathcal{F}$ is smooth and $\mathbb{P}(\text{coker} \sigma(x)) = \mathbb{P}^1$ for $x \in D_{n-1}(\sigma)$.

Hence the dimension of the exceptional set $\pi_\mathcal{F}$ is one, that is, the resolution is small.

**Proposition 2.7 ([LW22]).** With the above notation, we assume that the bundle $\mathcal{E} \otimes \mathcal{F}$ is ample and globally generated. Then there exists a Zariski open set $U$ in $\text{Hom}(\mathcal{E}^\vee, \mathcal{F})$ such that for each $\sigma \in U$, the morphism $\sigma$ is $n$-general and the $X_\mathcal{F} = Z(s_\sigma)$ is a Calabi–Yau threefold with Picard number $\rho(\mathbb{P}(\mathcal{F}))$. Moreover, the $D_n(\sigma) \subseteq M$ is a nodal hypersurface with the singular locus $D_{n-1}(\sigma)$ and

$$
\chi_{\text{top}}(X_\mathcal{F}) = \chi_{\text{top}}(\tilde{Y}) + 2|D_{n-1}(\sigma)|,
$$

$$h^{1,1}(X_\mathcal{F}) = h^{1,1}(\tilde{Y}) - |D_{n-1}(\sigma)| + 1,
$$

where $\tilde{Y} \in |-K_M|$ is a smooth member and $\chi_{\text{top}}(-)$ is the topological Euler number.

In Section 3.4 and 5, we will apply Proposition 2.7 with $\mathcal{E} = \mathcal{O}^{\oplus(n+1)}$ and Fano bundles $\mathcal{F}$ listed in Theorem 2.4. We remark that, by the classification, such a Fano bundle $\mathcal{F}$ is globally generated (cf. the proof of [LW22] Proposition 2.7).

The $\pi_\mathcal{F}: X_\mathcal{F} \to D_n(\sigma)$ is called the determinantal contraction of $X_\mathcal{F}$. The singular locus $D_{n-1}(\sigma)$ consists of ordinary double points (ODPs for short).

The following is from [Wan20] Remark 3.3, Proposition 4.5. It computes some invariants of $X_\mathcal{F}$ in terms of the virtual bundle $\mathcal{E} - \mathcal{F}^\vee$ and its dual. For the formulas for Chern classes of such bundles, we refer the reader to [Ful98] Example 3.2.7 or [LW22] Appendix A.

**Proposition 2.8 ([Wan20]).** With the above notation, we fix an $n$-general morphism $\sigma$. Let $H_M$ be any Cartier divisor on $M$, $H_\mathcal{F} = \pi_\mathcal{F}^*H_M$ and $L_\mathcal{F} = \mathcal{O}_\mathcal{F}(1)|_{X_\mathcal{F}}$. Then we
have
\[ \int_{X_F} H_F^k \cdot L_F^{n-k} = \int_M H_M^k \cdot c_{4-k}(E - F') \text{ for } 0 \leq k \leq 3, \]
\[ \int_{X_F} c_2(T_{X_F}) \cdot H_F = \int_M c_2(T_M) \cdot c_1(E - F') \cdot H_M, \]
\[ \int_{X_F} c_2(T_{X_F}) \cdot L_F = \int_M c_2(T_M) \cdot c_2(E - F') - |\text{Sing}(D_n(\sigma))|, \]
and the number of ODPs of $D_n(\sigma)$ is
\[ \int_M c_2(F - E')^2 - c_1(F - E') \cdot c_3(F - E'). \]

There is the other determinantal contraction $\pi_E: X_E \rightarrow D_n(\sigma')$ via the dual morphism $\sigma' : F' \rightarrow E$. By the observation that $D_n(\sigma) = D_n(\sigma')$, we get a birational map $\chi := \pi_{E}^{-1} \circ \pi_{F}$. The following lemma gives some information of the proper transform $\chi_* L_F$, see [LVW22] Lemma 4.8 for a proof.

Lemma 2.9. Under the assumptions as in Proposition 2.8, we assume that $\chi_* L_F = \alpha D_1 + \beta D_2$ in $\text{Pic}(X_E) \text{Q}$ and $D_1, D_2$ are base point free. If $\chi$ is not an isomorphism, then $\alpha \beta < 0$,
\[ L_F \cdot H_F^2 = \chi_* L_F \cdot (\chi_* H_F)^2 \text{ and } L_F^2 \cdot H_F = (\chi_* L_F)^2 \cdot \chi_* H_F. \]

2.4. Flops via dual morphisms. Let $(M, F)$ be a pair as in Theorem 2.1 where rank $F = n + 1$, and $n = 1$ or 2. Let $E = O_M^{\oplus n+1}$. We shall apply Proposition 2.7 to construct our Calabi–Yau threefolds, which will be used in the following sections. It gives rise to smooth Calabi–Yau threefolds $X_F$ and $X_E$ with Picard number two for a general morphism $\sigma : E' \rightarrow F$. Since $E$ is trivial, the Calabi–Yau $X_F \subseteq \mathbb{P}(F)$ is a complete intersection of $n + 1$ general hypersurfaces in $|O_F(1)|$.

The determinantal contractions and the restriction of the projection $\mathbb{P}(E) = M \times \mathbb{P}^n \rightarrow \mathbb{P}^n$ to $X_E$ give the following diagram
\[ X_F \xrightarrow{\chi} X_E \]
\[ \xrightarrow{\pi_F} D_n(\sigma) \xrightarrow{\pi_E} \mathbb{P}^n. \]

We assume that there exists another small contraction $X_F \rightarrow Y_F$.

We let $H_M$ be the fundamental divisor of the Fano fourfold $M$, and write $L_F$ and $H_F$ for the restrictions of $O_F(1)$ and $p_* F_H$ to the Calabi–Yau $X_F$, and similarly for $L_E$ and $H_E$.

Recall that $n = 1$ if $M = V_4$ or $V_5$, and $n = 2$ if $M = \text{Gr}(2, 4)$. According to $\text{dim } X_E = 3 > n$, it follows that the fibration $X_E \rightarrow \mathbb{P}^n$ gives one edge of the nef cone $\text{Nef}(X_E)$ of $X_E$. Since $X_E$ has Picard number 2, the diagram (1) implies that $\text{Nef}(X_E)$ is spanned by $L_E$ and $H_E$. To describe the cone in $N^1(X_E)$, we need to compute some intersection numbers. For abbreviation, we write $L$ and $H$ instead of $L_F$ and $H_F$ and derive Table 1 by using Proposition 2.8.
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Proposition 2.10. Under the above assumptions, the matrix representation of the map \( \chi_*: N^1(X_F) \to N^1(X_E) \) with respect to \( \{L,H\} \) and \( \{L_E,H_E\} \) is given by

\[
[\chi_*] = \begin{bmatrix} -1 & 0 \\ r_M & 1 \end{bmatrix} = [(\chi^{-1})_*].
\]

where \( r_M \) is the index of the smooth Fano fourfold \( M \). Moreover, in \( N^1(X_F) \) we have

\[
\text{Nef}(X_E) = \mathbb{R}_{>0} [r_M H - L] + \mathbb{R}_{>0} [H].
\]

Proof. We first claim that \( \chi \) is not an isomorphism. If the claim is not true, then \( X_F \cong X_E \) admits a fibration over \( \mathbb{P}^n \), where \( n = 1 \) or \( 2 \). This is contrary to the assumption that \( X_F \) admits another small contraction \( X_F \to Y_F \).

Write \( \chi_*L = \alpha L_E + \beta H_E \). Since \( \chi \) is an isomorphism in codimension one, the proper transform \( \chi_*H \) is \( H_E \). By Proposition 2.8 we get that \( L_E \cdot H^2_E = 10, 12, 11 \) and \( L^2_E \cdot H_E = 0, 0, 5 \) for \( M = V_4, V_5, \text{Gr}(2,4) \) respectively. From Table 1 and Lemma 2.9 we get a system of equations for \( \alpha \) and \( \beta \). Solving such system yields \( \alpha = \pm 1 \). According to \( \alpha \beta < 0 \), it follows that \( (\alpha, \beta) = (-1, r_M) \). Note that \( r_{V_4} = r_{V_5} = 3 \) and \( r_{\text{Gr}(2,4)} = 4 \).

Since the projection \( X_F \to \mathbb{P}^n \) and the determinantal contraction \( \pi_E \) are induced by \( L_E \) and \( H_E \) respectively, we find that \( \mathbb{R}_{>0} [r_M H - L] \) and \( \mathbb{R}_{>0} [H] \) are the two boundary rays of \( \text{Nef}(X_E) \) in \( N^1(X_F) \), which completes the proof.

Finally, we deduce the Hodge numbers of \( X_F \) in Table 2 by applying Proposition 2.7, Table 1 and [LW22, Table 6].

3. The Del Pezzo fourfold of degree 4

We consider the case \( (M, F, E) = (V_4, p^* S^\vee(1), \mathcal{O}^{\oplus 2}) \) and will use the same notation as in Section 2.4. For a general morphism \( \sigma: E^\vee \to F \), there are two smooth Calabi-Yau threefolds \( X_F, X_E \) with the determinantal contractions \( \pi_F, \pi_E \) and the diagram (4).

Recall that \( p: V_4 \to \text{Gr}(2,4) \) is a double cover branching along a smooth divisor \( B \in |\mathcal{O}_{\text{Gr}(2,4)}(2)| \). Furthermore, the \( S \) is the universal subbundle on \( \text{Gr}(2,4) \). Hence we...
have the following diagram:

Here \( \tilde{p} \) is the double cover branching along \( p_2^{-1}(B) \) and \( p_1 \) is the morphism defined by \(|\mathcal{O}_{\mathcal{S}^\vee}(1)|\). We note that \( \mathbb{P}_{\text{Gr}(2,4)}(\mathcal{S}^\vee) \) is the flag variety \( F(1,2;4) \) and \( p_1 \) is nothing but the natural morphism \( F(1,2;4) \to \text{Gr}(1,4) = \mathbb{P}^3 \). The morphism \( q \) is just the composite \( p_1 \circ \tilde{p} \).

Since \( \mathcal{F} = p^*\mathcal{S}^\vee(1) \), we have \( q^*\mathcal{O}_{\mathbb{P}^3}(1) \cong \mathcal{O}_{\mathcal{F}}(1) \boxtimes \mathcal{O}_{V_4}(-1) \). Consider the restriction of \( q \) to \( X_\mathcal{F} \) and let \( X_\mathcal{F} \to Y_\mathcal{F} \to \mathbb{P}^3 \) be the Stein factorization. In this section, we prove the following theorem:

**Theorem 3.1.** Let \((M, \mathcal{F}, \mathcal{E}) = (V_4, p^*\mathcal{S}^\vee(1), \mathcal{O}_{\mathbb{P}^3}^\oplus)\). Then for a general morphism \( \sigma: \mathcal{E}^\vee \to \mathcal{F} \), the scheme \( X_\mathcal{F} \) is a smooth Calabi–Yau threefold of Picard number two with

\[
\text{Nef}(X_\mathcal{F}) = \mathbb{R}_{\geq 0}[L - H] + \mathbb{R}_{\geq 0}[H],
\]

such that

(i) the determinantal contraction \( \pi_\mathcal{F} \) is induced by \( \mathcal{H} \);

(ii) the morphism \( X_\mathcal{F} \to \mathbb{P}^3 \) is generically \( 2:1 \), which induces an involution \( \iota: X_\mathcal{F} \to X_\mathcal{F} \) over \( \mathbb{P}^3 \);

(iii) the Stein factorization \( X_\mathcal{F} \to Y_\mathcal{F} \to \mathbb{P}^3 \) of \( X_\mathcal{F} \to \mathbb{P}^3 \) is a small contraction induced by \( L - H \);

(iv) the \( X_\mathcal{E} \) admits a \( K3 \) fibration induced by \( 3\mathcal{H} - \mathcal{L} \).

Moreover, the movable cone \( \overline{\text{Mov}}(X_\mathcal{F}) \) is the convex cone generated by the divisors \( 15\mathcal{L} - 17\mathcal{H} \) and \( 3\mathcal{H} - \mathcal{L} \) which is covered by the nef cones of \( X_\mathcal{F} \) and \( X_\mathcal{E} \), and there are no more minimal models of \( X_\mathcal{F} \), which we summarize in the following diagram:

![Diagram](image)

Figure 1 is the slice of the movable cone of \( X_\mathcal{F} \). It is a subdivision of a closed interval, which comes from the chamber structure of the cone. We depict \( X_\mathcal{F} \) and \( X_\mathcal{E} \) inside their nef cones.

![Table](image)

**Figure 1.** The slice of the movable cone \( \overline{\text{Mov}}(X_\mathcal{F}) \) for \( M = V_4 \).

In the rest of this section, we prove Theorem 3.1.
Lemma 3.2. If $\sigma : E^\vee \to F$ is chosen in general, then the natural projection $X_F \to \mathbb{P}^3$ is generically $2 : 1$ and the Stein factorization $X_F \to Y_F$ of $X_F \to \mathbb{P}^3$ is a small contraction induced by $L - H$.

Proof. Since $\mathcal{O}_F(1) \otimes \mathcal{O}_{V_4}(-1) \cong q^* \mathcal{O}_{\mathbb{P}^3}(1)$, it holds that $L - H \cong q|_{X_F}^* \mathcal{O}_{\mathbb{P}^3}(1)$. Hence the morphism $X_F \to Y_F$ is induced by $L - H$.

Let $\mathcal{V} = \Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}$. By [NO07, Theorem 1.3 (4)] (also see p.236, Proof of Theorem 1.3, Case 2) and [Kan19b, Remark 4.2], there is a divisor $Q \in |\mathcal{O}_V(2)|$ such that

$$\mathbb{P}_{V_4}(\mathcal{F}) \cong Q \subseteq \mathbb{P}_{\mathbb{P}^3}(\mathcal{V}).$$

According to $c_1(\Omega_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}) = c_1(\mathcal{O}_{\mathbb{P}^3}(2))$, it follows that

$$\mathcal{O}(K_{\mathcal{V}(\mathcal{F})}) \cong \mathcal{O}_V(-4) \otimes \mathcal{O}_{\mathbb{P}^3}(-2) \quad \text{and} \quad \mathcal{O}(K_Q) \cong \mathcal{O}_V(-2) \otimes \mathcal{O}_{\mathbb{P}^3}(-2)|_Q.$$

Since $\mathcal{O}_F(-2) = \mathcal{O}(K_{\mathcal{V}(\mathcal{F})}) = \mathcal{O}(K_Q)$ under the identification $\mathbb{P}_{V_4}(\mathcal{F}) \cong Q$, we have $\mathcal{O}_F(1) \cong \mathcal{O}_V(1) \otimes \mathcal{O}_{\mathbb{P}^3}(1)|_Q$ as the Picard group of a Fano manifold is torsion free. Note that $\mathcal{V}$ is globally generated because we have

$$0 \to \Omega^2_{\mathbb{P}^3}(2) \to \mathcal{O}^{\oplus 6}_{\mathbb{P}^3} \to \Omega_{\mathbb{P}^3}(2) \to 0$$

by dualizing the Euler sequence and taking the 2nd exterior power. Hence $|\mathcal{O}_V(1) \otimes \mathcal{O}_{\mathbb{P}^3}(1)|$ is ample and base point free (see [LW22, Lemma 2.6]). Furthermore, the natural map $H^0(\mathcal{O}_V(1) \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \to H^0(\mathcal{O}_V(1) \otimes \mathcal{O}_{\mathbb{P}^3}(1)|_Q)$ is surjective by

$$H^1(\mathcal{O}_V(-1) \otimes \mathcal{O}_{\mathbb{P}^3}(1)) = 0,$$

which follows from Kodaira vanishing theorem since

$$\mathcal{O}_V(-1) \otimes \mathcal{O}_{\mathbb{P}^3}(1) = \mathcal{O}(K_{\mathcal{V}(\mathcal{F})}) \otimes (\mathcal{O}_V(3) \otimes \mathcal{O}_{\mathbb{P}^3}(3)).$$

Hence the Calabi–Yau threefold $X_F \subseteq \mathbb{P}_{V_4}(\mathcal{F}) \cong Q$ is isomorphic to a complete intersection $D_1 \cap D_2 \cap Q$ where $D_1$ and $D_2$ are general members of $|\mathcal{O}_V(1) \otimes \mathcal{O}_{\mathbb{P}^3}(1)|$.

Let $V \subseteq \mathbb{P}_{\mathbb{P}^3}(\mathcal{V})$ be the complete intersection fourfold defined by $D_1$ and $D_2$, i.e., it is the zero locus of $U \otimes \mathcal{O}_{\mathbb{P}_{\mathbb{P}^3}(\mathcal{V})} \to \mathcal{O}_V(1) \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ induced by a general two-dimensional subspace $U$ of $H^0(\mathcal{O}_V(1) \otimes \mathcal{O}_{\mathbb{P}^3}(1))$. Let $\tilde{q} : \mathbb{P}_{\mathbb{P}^3}(\mathcal{V}) \to \mathbb{P}^3$ be the natural projection. Then $U$ also defines a morphism $\tau : U \otimes \mathcal{O}_{\mathbb{P}^3} \to \mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^3}(1)$ by the natural isomorphism $\tilde{q}_* \mathcal{O}_U(1) \cong \mathcal{V}$.

Consider the restriction $q_V : V \to \mathbb{P}^3$ of $\tilde{q}$ on $V$. For each $x \in D_k(\tau) \setminus D_{k-1}(\tau)$, the fiber $q_V^{-1}(x)$ is $\mathbb{P}(\text{coker } \tau(x)) \cong \mathbb{P}^{1-k-1} = \mathbb{P}^{1-k}$ for $0 \leq k \leq 2$ by Lemma 2.4. Note that $D_1(\tau)$ and $D_0(\tau)$ have the expected codimension $(2 - 1)(4 - 1) = 3$ and $2 \cdot 4 = 8$ respectively. If $U$ is chosen in general, then $D_0(\tau) = \emptyset$ and $D_1(\tau)$ consists of finitely many (smooth) points whose number can be determined by Giambelli–Thom–Porteous formula [Ful98, Theorem 14.4],

$$|D_1(\tau)| = c_3(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) \cap [\mathbb{P}^3] \in A_3(\mathbb{P}^3).$$

According to $c_3(\mathcal{V} \otimes \mathcal{O}_{\mathbb{P}^3}(1)) = 14$, the projection $q_V : V \to \mathbb{P}^3$ has exactly 14 projective planes $\mathbb{P}^2$ as fibers, and the other fibers of $q_V$ are $\mathbb{P}^1$. Then $\tilde{q}_V^{-1}(x) \cap Q$ is a quadric in $\mathbb{P}^2$ or a (possibly) double point in $\mathbb{P}^1$. In particular, $X_F = V \cap Q \to \mathbb{P}^3$ is generically 2 : 1. We note that this also follows from $(L - H)^3 = 2$ computed by Table 1.

Claim 3.3. The exceptional set of $X_F \to Y_F$ has dimension at most one.

Proof of Claim 3.3. We denote $V, \tau$ by $V_U, \tau_U$ when we need to clarify the choice of $U$. By [Kan19b, Theorem 4.7], the morphism $q : Q \to \mathbb{P}^3$ is a quadric bundle, that is, the fiber $q^{-1}(x) = Q \cap \tilde{q}^{-1}(x)$ is a quadric in $\tilde{q}^{-1}(x) \cong \mathbb{P}^3$ for any $x \in \mathbb{P}^3$. Let $Gr$ be the
Grassmannian of two-dimensional subspaces in $H^0(\mathcal{O}_V(1) \boxtimes \mathcal{O}_\mathbb{P}^3(1))$ and $\Sigma \subseteq \mathbb{P}^3$ the discriminant locus of $q: Q \to \mathbb{P}^3$. We set

$$\text{Gr}^o := \{ [U] \in \text{Gr} \mid \dim D_1(\tau_U) = 0, D_1(\tau_U) \cap \Sigma = D_0(\tau_U) = \emptyset \}.$$ 

This is a non-empty open subset of $\text{Gr}$. Indeed, we already saw that $\dim D_1(\tau_U) = 0$ and $D_0(\tau_U) = \emptyset$ are satisfied if $U$ is general. Since $\dim \Sigma = 2$ and the expected codimension of $D_1(\tau_U)$ in $\Sigma$ is three, we have $D_1(\tau_U) = D_1(\tau) \cap \Sigma = \emptyset$ if $U$ is general.

For each $x \in \mathbb{P}^3$, the natural map

$$(5) \quad H^0(\mathcal{O}_V(1) \boxtimes \mathcal{O}_\mathbb{P}^3(1)) \to H^0(\mathcal{O}_V(1) \boxtimes \mathcal{O}_\mathbb{P}^3(1)|_{\tilde{q}^{-1}(x)}) \cong H^0(\mathcal{O}_\mathbb{P}^3(1))$$

is surjective. This is because this map can be identified with $H^0(\mathcal{V} \boxtimes \mathcal{O}_\mathbb{P}^3(1)) \to \mathcal{V} \boxtimes \mathcal{O}_\mathbb{P}^3(1) \otimes k(x)$, which is surjective since $\mathcal{V}$ is globally generated. Furthermore, the restriction map

$$H^0(\mathcal{O}_V(1) \boxtimes \mathcal{O}_\mathbb{P}^3(1)|_{\tilde{q}^{-1}(x)}) \to H^0(\mathcal{O}_V(1) \boxtimes \mathcal{O}_\mathbb{P}^3(1)|_{q^{-1}(x)})$$

is an isomorphism for any $x \in \mathbb{P}^3$ since $q^{-1}(x)$ is a quadric in $\tilde{q}^{-1}(x)$ $\cong \mathbb{P}^3$.

To prove Claim 3.3, it suffices to show that $X_{\mathcal{F}} = Q \cap V_U \to \mathbb{P}^3$ contracts at most finitely many curves for general $U$. Observe that the fiber of $Q \cap V_U \to \mathbb{P}^3$ over $x \in \mathbb{P}^3$ is

$$q^{-1}(x) \cap q_{V_U}^{-1}(x) \subseteq \tilde{q}^{-1}(x) \cong \mathbb{P}^3.$$ 

If $[U] \in \text{Gr}^o$, we have $\dim q_{V_U}^{-1}(x) \cap q^{-1}(x) \leq 1$ for any $x \in \mathbb{P}^3$. In fact, this inequality is trivial if $x \not\in D_1(\tau_U)$ since $q_{V_U}^{-1}(x) \cong \mathbb{P}^1$ in this case. If $x \in D_1(\tau_U)$, the $q_{V_U}^{-1}(x) \cong \mathbb{P}^2$ is a hyperplane of $\tilde{q}^{-1}(x)$. By $D_1(\tau_U) \cap \Sigma = \emptyset$ and $x \in D_1(\tau_U)$, the $q^{-1}(x)$ is a smooth quadric and hence the hyperplane section $q^{-1}(x) \cap q_{V_U}^{-1}(x)$ is one-dimensional for $x \in D_1(\tau_U)$.

Thus the rest is to show that $q^{-1}(x) \cap q_{V_U}^{-1}(x)$ is one-dimensional for at most finitely many $x \in \mathbb{P}^3$ if $U$ is general. To see this, we consider

$$Z := \{ ([U], x) \in \text{Gr}^o \times \mathbb{P}^3 \mid \dim q^{-1}(x) \cap q_{V_U}^{-1}(x) = 1 \}.$$ 

For $[U] \in \text{Gr}^o$, the morphism $X_{\mathcal{F}} = Q \cap V_U \to \mathbb{P}^3$ contracts a curve over $x \in \mathbb{P}^3$ if and only if $([U], x)$ is contained in $Z$. Thus if $\dim Z \leq \dim \text{Gr}^o$, the fiber of $Z \to \text{Gr}^o$ over a general $[U]$ is at most zero-dimensional and we obtain the finiteness of $x \in \mathbb{P}^3$ with $\dim q^{-1}(x) \cap q_{V_U}^{-1}(x) = 1$.

We are left with the task to check $\dim Z \leq \dim \text{Gr}^o$. For $x \in \mathbb{P}^3$, let

$$K_x \subseteq H^0(\mathcal{O}_V(1) \boxtimes \mathcal{O}_\mathbb{P}^3(1))$$

be the kernel of (5), whose codimension is $\dim H^0(\mathcal{O}_V(1) \boxtimes \mathcal{O}_\mathbb{P}^3(1)|_{\tilde{q}^{-1}(x)}) = 4$. Hence the codimension of the Schubert variety

$$B_x := \{ [U] \in \text{Gr} \mid U \cap K_x \neq \{0\} \} = \{ [U] \in \text{Gr} \mid x \in D_1(\tau_U) \}$$

in $\text{Gr}$ is three. For $[U] \in \text{Gr}^o \setminus B_x$, the $q_{V_U}^{-1}(x)$ is a line since $x \not\in D_1(\tau_U)$. Hence we can define the morphism

$$(6) \quad \text{Gr}^o \setminus B_x \to \text{Gr}(2, H^0(\mathcal{O}_V(1) \boxtimes \mathcal{O}_\mathbb{P}^3(1)|_{\tilde{q}^{-1}(x)})) = \{ \text{lines in } \tilde{q}^{-1}(x) \}$$

by sending $[U] \in \text{Gr}^o \setminus B_x$ to the line $q_{V_U}^{-1}(x)$ in $\tilde{q}^{-1}(x) \cong \mathbb{P}^3$. If $x \not\in \Sigma$, the family of lines in the smooth quadric $q^{-1}(x) \cong \mathbb{P}^3 \times \mathbb{P}^1$ is one-dimensional and hence the codimension of $\{ \text{lines in } q^{-1}(x) \}$ in $\{ \text{lines in } \tilde{q}^{-1}(x) \}$ is three. Similarly, the codimension of $\{ \text{lines in } q^{-1}(x) \}$ in $\{ \text{lines in } \tilde{q}^{-1}(x) \}$ is two or three if $x \in \Sigma$. Thus $\{ [U] \in \text{Gr}^o \setminus B_x \mid q_{V_U}^{-1}(x) \subseteq q^{-1}(x) \}$, if it is non-empty, is of codimension three in $\text{Gr}^o \setminus B_x$ when $x \not\in \Sigma$, and of codimension at least two when $x \in \Sigma$. Note that
the morphism (3) is dominant and all the fibers are isomorphic to open subsets of \( \text{Gr}(2,h^0(O_V(1)\boxtimes O_{p3}(1))-2) \). Hence the codimension does not change after the pullback by (3).

By definition, the fiber of \( Z \to \mathbb{P}^3 \) over \( x \in \mathbb{P}^3 \) is contained in
\[
\{ [U] \in \text{Gr}^\circ \setminus B_x \mid q_{V_4}^{-1}(x) \subseteq q^{-1}(x) \} \cup (B_x \cap \text{Gr}^\circ),
\]
whose codimension in \( \text{Gr}^\circ \) is three (resp. at least two) if \( x \notin \Sigma \) (resp. \( x \in \Sigma \)), if it is non-empty. Since \( x \) moves on \( \mathbb{P}^3 \) and \( \dim \Sigma = 2 \), the dimension of \( Z \) is at most that of \( \text{Gr}^\circ \), and Claim 3.3 follows. \( \Box \)

We note that \( X_F \to Y_F \) is not an isomorphism. Indeed, we have already seen that \( X_F \) is smooth, irreducible and has Picard number two. Notice that \( F \to \mathbb{P}^3 \) is smooth, irreducible and has Picard number two. Hence \( X_F \to Y_F \) is a small contraction by Claim 3.3. \( \Box \)

*Remark* 3.4. Recall that \( B \in |O_{\text{Gr}(2,4)}(2)| \) is the branched divisor of the double cover \( p: \mathbb{P}^4 \to \text{Gr}(2,4) \). If we assume that \( B \) is general, so is the \( Q \in |O_V(2)| \). In this case, we can check Claim 3.3 more easily, and show that \( Y \) has only 102 ODPs.

There is an involution on \( X_F \) over \( \mathbb{P}^3 \) induced from the generically 2 : 1 morphism \( X_F \to \mathbb{P}^3 \). We denote the involution by \( \iota: X_F \dashrightarrow X_F \).

**Lemma 3.5.** For the involution \( \iota: X_F \dashrightarrow X_F \), the matrix representation of \( \iota^*: N^1(X_F) \to N^1(X_F) \) with respect to \( \{ L, H \} \) are given by
\[
[i^*] = \begin{bmatrix} 9 & 8 \\ -10 & -9 \end{bmatrix} = [(\iota^{-1})^*]
\]

*Proof.* Note that \( \iota^*(L - H) = L - H \) since \( L - H \) is the pullback of \( O_{p3}(1) \) by \( X_F \to \mathbb{P}^3 \).

We let \( \bar{q} = q_{X_F}: X_F \to \mathbb{P}^3 \) and \( \bar{q}_*H = aH_{p3} \), where \( a \in \mathbb{Z} \) and \( H_{p3} \) is the hyperplane class of \( \mathbb{P}^3 \). Then
\[
a = (\bar{q}_*H \cdot H_{p3}^2)_{p3} = (H \cdot (L - H)^2)_{X_F} = 8
\]

by the projection formula and Table 11. Hence
\[
H + \iota^*H = \bar{q}^*\bar{q}_*H = \bar{q}^*(8H_{p3}) = 8(L - H).
\]

Then \( \iota^*H = 8L - 9H \) and the rest is clear. \( \Box \)

We are now in a position to prove Theorem 3.1.

*Proof of Theorem 3.1.* The (ii) follows from the construction of \( \pi_F \), and (iii), (iv) from Lemma 3.2. By Proposition 2.10, the morphism \( X_{\mathcal{E}} \to \mathbb{P}^1 \) is induced by \( 3L - H \). Furthermore, \( X_{\mathcal{E}} \to \mathbb{P}^1 \) is a K3 fibration. Indeed, by Proposition 2.8 we have
\[
(7) \quad \int_{X_{\mathcal{E}}} c_2(T_{X_{\mathcal{E}}}) \cdot L_{\mathcal{E}} = \int_{V_4} c_2(T_{V_4}) \cdot c_2(\mathcal{F}) - c_2(\mathcal{F})^2 = 24
\]

and then the intersection number of \( c_2(T_{X_{\mathcal{E}}}) \) with the general fiber \( F \in |L_{\mathcal{E}}| \) of \( X_{\mathcal{E}} \to \mathbb{P}^1 \) is 24. Hence \( F \) is a K3 surface by [Ogu93, Lemma 3.3], which proves (iv).

Finally, we compute the boundaries of \( \text{Mov}(X_F) \). By Lemma 3.3 and Proposition 2.10, we find that \( \iota_*H = 8L - 9H \) and \( \iota_*\chi_*L_{\mathcal{E}} = 15L - 17H \), which completes the proof. \( \Box \)
4. The Del Pezzo fourfold of degree 5

We consider the case \((M, \mathcal{F}, \mathcal{E}) = (V_5, S_{V_5}^1(1), \mathcal{O}^{\oplus 2})\). Recall that \(V_5\) is a general linear section of the Grassmannian \(\text{Gr}(2, 5) = \text{Gr}(2, W) \subseteq \mathbb{P}(\wedge^3 W)\) of codimension two, and \(S_{V_5}\) is the restriction of the universal subbundle on \(\text{Gr}(2, W)\) to \(V_5\), where \(W\) is a five-dimensional vector space over \(\mathbb{C}\). In Section 2.4, we have already seen that for a general morphism \(\sigma : \mathcal{E}^\vee \to \mathcal{F}\), there are two smooth Calabi-Yau threefolds \(X_F, X_E\) and the diagram \(\mathbb{I}\). In this section, we prove the following theorem.

**Theorem 4.1.** Let \((M, \mathcal{F}, \mathcal{E}) = (V_5, S_{V_5}^1(1), \mathcal{O}^{\oplus 2})\). Then for a general morphism \(\sigma : \mathcal{E}^\vee \to \mathcal{F}\), the scheme \(X_F\) is a smooth Calabi-Yau threefold of Picard number two with

\[
\text{Nef}(X_F) = \mathbb{R}_{\geq 0}[L - H] + \mathbb{R}_{\geq 0}[H],
\]

such that

(i) the determinantal contraction \(\pi_F\) is induced by \(H\);

(ii) the \(L - H\) defines a small contraction \(X_F \to Y_F\) and \(Y_F \subseteq \mathbb{P}(W^\vee)\) is a quintic hypersurface with 54 ODPs;

(iii) the flop \(X_F^+\) of \(X_F\) admits a small contraction \(\psi : X_F^+ \to Z_F\) induced by \(9L - 11H\) such that the exceptional locus \(\text{Exc}(\psi)\) is a smooth rational curve \(\Sigma^+\) contracted to an ODP;

(iv) the flop \(X_F^{++}\) of \(X_F^+\) admits a K3 fibration induced by \(4L - 5H\);

(v) \(X_E\) admits a K3 fibration induced by \(3H - L\).

Moreover, the movable cone \(\overline{\text{Mov}}(X_F)\) is the convex cone generated by the divisors \(4L - 5H\) and \(3H - L\) which is covered by the nef cones of \(X_E, X_F, X_F^+\) and \(X_F^{++}\), and there are no more minimal models of \(X_F\), which we summarize in the following diagram:

\[
\begin{array}{c}
X_F^{++} \xrightarrow{\theta} X_F^+ \xrightarrow{\chi} X_F \xrightarrow{\pi_E} \mathbb{P}^1 \\
\downarrow \quad \psi \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qa
\end{array}
\]

The slice of the chamber structure of \(\overline{\text{Mov}}(X_F)\) is given in Figure 2. We depict \(X_E, X_F, X_F^+\) and \(X_F^{++}\) inside their nef cones.

![Figure 2](image-url)

**Figure 2.** The slice of the movable cone \(\overline{\text{Mov}}(X_F)\) for \(M = V_5\).

**Remark 4.2.** It is immediate that \(X_E\) and \(X_F^{++}\) are not isomorphic. Indeed, if they were isomorphic, then \(Z_F\) and \(D_1(\sigma)\) would be isomorphic. But this is impossible because the number of singularities of \(Z_F\) and \(D_1(\sigma)\) are 1 and 29 by Theorem 4.1 (iii) and Table \(\mathbb{I}\) respectively.

4.1. Another description of \(\mathbb{P}(\mathcal{F})\). Let \(W\) be a five-dimensional vector space over \(\mathbb{C}\). Fix a two-dimensional subspace \(U \subseteq \wedge^3 W = H^0(\text{Gr}(2, W), \mathcal{O}(1))\) such that

\[
V_5 = \text{Gr}(2, W) \cap \mathbb{P}(\wedge^3 W/U) \hookrightarrow \mathbb{P}(\wedge^3 W)
\]

is a smooth del Pezzo fourfold of degree 5.
Set $\mathcal{T} = T_{\mathbb{P}(W^\vee)}(-1)$. We define a coherent sheaf $\mathcal{C}$ on $\mathbb{P}(W^\vee)$ as the cokernel of
\[
\tau: U \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \hookrightarrow \wedge^3 W \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \wedge^3 \mathcal{T}.
\]
Since $\mathbb{P}_{\text{Gr}(2,W)}(\mathcal{S}_{\text{Gr}(2,W)}^\vee)$ is the flag variety $F(1,2;W)$, we have the following diagram:
\[
\begin{array}{ccc}
\mathbb{P}_{\text{Gr}(2,W)}(\mathcal{S}_{\text{Gr}(2,W)}^\vee) & \xrightarrow{p_1} & \mathbb{P}(W^\vee) \\
\mathcal{G}(1,W) = \mathbb{P}(W^\vee) & \xrightarrow{p_2} & \text{Gr}(2,W).
\end{array}
\]
Here $p_1$ and $p_2$ are the two projection maps. We note that
\[
\mathbb{P}_{\text{Gr}(2,W)}(\mathcal{S}_{\text{Gr}(2,W)}^\vee) = \mathbb{P}_{\mathbb{P}(W^\vee)}(\mathcal{T}^\vee) \cong \mathbb{P}_{\mathbb{P}(W^\vee)}(\wedge^3 \mathcal{T}),
\]
\[
p^*_2 O_{\mathbb{P}(W^\vee)}(1) = O_{\mathcal{S}_{\text{Gr}(2,W)}^\vee}(1).
\]
Since $V_5 \subseteq \text{Gr}(2,W)$ is the zero locus of $U \otimes \mathcal{O}_{\text{Gr}(2,W)} \to \mathcal{O}_{\text{Gr}(2,W)}(1)$, the projective bundle $\mathbb{P}_{V_5}(\mathcal{S}_{V_5}^\vee) \cong \mathbb{P}(\mathcal{F})$ is the zero locus of
\[
U \otimes \mathcal{O} \to p^*_2 O_{\text{Gr}(2,W)}(1)
\]
in $\mathbb{P}_{\text{Gr}(2,W)}(\mathcal{S}_{\text{Gr}(2,W)}^\vee)$.

**Lemma 4.3.** The projective bundle $\mathbb{P}_{V_5}(\mathcal{S}_{V_5}^\vee)$ coincides with $\mathbb{P}_{\mathbb{P}(W^\vee)}(\mathcal{C}) \subseteq \mathbb{P}_{\mathbb{P}(W^\vee)}(\wedge^3 \mathcal{T})$ under the identification (9).

**Proof.** We apply Lemma 2.4 (11) to $\tau: U \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \wedge^3 \mathcal{T}$ on $\mathbb{P}(W^\vee)$. Under the identification (9), we have $p^*_2 O_{\text{Gr}(2,W)}(1) = \mathcal{O}_{\mathcal{S}_{\text{Gr}(2,W)}^\vee}(1)$. Then (11) is nothing but the section $s_\tau$ induced by $\tau$, and hence this lemma follows. □

**Remark 4.4.** By Lemma 4.3, the fiber of $\mathbb{P}_{V_5}(\mathcal{S}_{V_5}^\vee) = \mathbb{P}_{\mathbb{P}(W^\vee)}(\mathcal{C}) \to \mathbb{P}(W^\vee)$ over $x \in D_k(\tau) \setminus D_{k-1}(\tau)$ is $\mathbb{P}^{4-k-1} = \mathbb{P}^{3-k}$ for $k = 0, 1, 2$. Then we see that $D_0(\tau) = \emptyset$ and $\dim D_1(\tau) = 1$ by [Kan96] Proposition 1.8, Theorem 4.7.

4.2. Another description of $X_\mathcal{F}$ and Construction of the flop $X_\mathcal{F}$. We are going to give another description of $X_\mathcal{F}$ using Lemma 4.3. By this description, we will construct a small contraction $X_\mathcal{F} \to Y_\mathcal{F}$ and its flop $X^+_\mathcal{F} \to Y_\mathcal{F}$.

Recall that, by Proposition 2.7, the $X_\mathcal{F} \subseteq \mathbb{P}_{V_5}(\mathcal{F}) \cong \mathbb{P}_{V_5}(\mathcal{S}_{V_5})$ is induced by a general morphism $\sigma: \mathcal{E}^\vee \to \mathcal{F}$, i.e., $X_\mathcal{F} = Z(s_\sigma)$. Observe that $\mathcal{E} = \mathcal{O}^\oplus 2$ and thus a general morphism $\sigma$ corresponds to a general two-dimensional subspace of $H^0(\mathcal{F}) = H^0(\mathcal{S}_{V_5}^\vee(1))$, say $U_\sigma$. Note that $H^i(\mathcal{S}_{\text{Gr}(2,W)}^\vee(1-i)) = 0$ for $i = 1, 2$ and thus the natural map
\[
W^\vee \otimes \wedge^3 W = H^0(\mathcal{S}_{\text{Gr}(2,W)}^\vee(1)) \to H^0(\mathcal{S}_{V_5}^\vee(1))
\]
is surjective. Hence we can choose a two-dimensional subspace $U' \subseteq W^\vee \otimes \wedge^3 W$ which maps onto $U_\sigma$. In other words, we can take general $U' \subseteq W^\vee \otimes \wedge^3 W$ and identify $\sigma: \mathcal{E}^\vee \to \mathcal{F}$ with the natural morphism
\[
U' \otimes \mathcal{O}_{V_5} \to W^\vee \otimes \mathcal{O}_{V_5}(1) \to \mathcal{S}_{V_5}^\vee(1) = \mathcal{F}.
\]

Let $U' \subseteq W^\vee \otimes \wedge^3 W$ be a general two-dimensional subspace as above. It induces $U' \otimes \mathcal{O} \to \mathcal{O}_{\text{Gr}(2,W)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(W^\vee)}(1)$ on $\mathbb{P}_{\text{Gr}(2,W)}(\mathcal{S}_{\text{Gr}(2,W)}^\vee)$ and $U' \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \wedge^3 \mathcal{T}(1)$ on $\mathbb{P}(W^\vee)$. We denote by $\tau': U' \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \to \wedge^3 \mathcal{T}$.
the twisted morphism. Set
\[ \mathcal{H} := (U^\vee \otimes \mathcal{O}_{F(W^\vee)}) \oplus (U'^\vee \otimes \mathcal{O}_{F(W^\vee)}(1)), \]
where \( U \subseteq \wedge^3 W \) is the same two-dimensional subspace as in the beginning of Section 4.1.

**Proposition 4.5.** Let \( \varrho: \mathcal{H}^\vee \to \wedge^3 \mathcal{T} \) be the sum of the morphisms \( \tau \) and \( \tau' \).

(i) The Calabi–Yau threefold \( X_F \subseteq \mathbb{P}_{V_5}(S^V_{V_5}) \) coincides with
\[ \mathbb{P}_{F(W^\vee)}(\text{coker } \varrho) \subseteq \mathbb{P}_{F(W^\vee)}(C) \]
under the identification \( \mathcal{O} \) if \( U' \) is chosen in general.

(ii) The image \( p_1(X_F) \) coincides with \( Y_F := D_3(\varrho) \) and \( \pi_{\wedge^3 F}: X_F \to Y_F \subseteq \mathbb{P}(W^\vee) \) is a small resolution of a quintic hypersurface with 54 ODPs.

**Proof.** (i) Applying Lemma 2.4 (1) to \( \varrho \) on \( \mathbb{P}(W^\vee) \), we have \( \mathbb{P}_{F(W^\vee)}(\text{coker } \varrho) = Z(s_\varrho) \).
Hence it suffices to show \( Z(s_\varrho) = X_F \).

To shorten notation, we write \( U'_Y \) instead of \( U^\vee \otimes \mathcal{O}_Z \) for a variety \( Z \), and similarly for \( U'_Z \).
Then the section \( s_\varrho \in H^0(\mathcal{H} \otimes \mathcal{O}_{\wedge^3 \mathcal{T}}(1)) \) is the sum of
\[ s_{\tau} \in H^0(U'^{\vee}_{F(W^\vee)} \otimes \mathcal{O}_{\wedge^3 \mathcal{T}}(1)) \quad \text{and} \quad s_{\tau'} \in H^0(U'^{\vee}_{F(W^\vee)}(1) \otimes \mathcal{O}_{\wedge^3 \mathcal{T}}(1)). \]
Under the identification \( \mathbb{P}_{\text{Gr}(2,W)}(S'_{\text{Gr}(2,W)}) = \mathbb{P}_{F(W^\vee)}(\wedge^3 \mathcal{T}) \) in (9), it holds that
\[ \mathbb{P}_{\text{Gr}(2,W)}(1) \otimes \mathcal{O}_{\wedge^3 \mathcal{T}}(1) = p_1^{\vee} \mathbb{P}_{F(W^\vee)}(1) \otimes \mathcal{O}_{\wedge^3 \mathcal{T}}(1) = \mathcal{O}_{S'_{\text{Gr}(2,W)}}(1) \oplus p_2^{\vee} \mathbb{P}_{\text{Gr}(2,W)}(1). \]
Hence \( s_{\tau'} \) is a section in \( H^0(U'^{\vee}_{F(W^\vee)}(1) \otimes \mathcal{O}_{S'_{\text{Gr}(2,W)}}(1)) \) on \( \mathbb{P}_{\text{Gr}(2,W)}(S'_{\text{Gr}(2,W)}) \).

By Lemmas 2.3 and 4.3 we have \( Z(s_{\tau}) = \mathbb{P}(F(W^\vee) \otimes \mathcal{O}) = \mathbb{P}_{V_5}(S^V_{V_5}) \).
By the choice of \( U' \), the restriction of \( s_{\tau'} \) on \( \mathbb{P}_{V_5}(S^V_{V_5}) \), which is a section in
\[ H^0(U'^{\vee}_{F(W^\vee)}(1) \otimes \mathcal{O}_{S^V_{V_5}}(1)) = H^0(U'^{\vee}_{V_5} \otimes \mathcal{O}_{F}(1)), \]
can be identified with \( s_{\sigma} \in H^0(\mathcal{E} \otimes \mathcal{O}_{F}(1)) \) on \( \mathbb{P}_{V_5}(\mathcal{E}) \cong \mathbb{P}_{V_5}(S^V_{V_5}) \). Thus we have
\[ Z(s_\varrho) = Z(s_{\tau}) \cap Z(s_{\tau'}) = \mathbb{P}_{V_5}(S^V_{V_5}) \cap Z(s_{\tau'}) = Z(s_{\tau'}|_{\mathbb{P}_{V_5}(S^V_{V_5})}) = Z(s_{\sigma}) = X_F \]
and (i) follows.

(ii) By (i) and Lemma 2.4 (2), the restriction of \( p_1 \) to \( X_F \) maps onto \( Y_F = D_3(\varrho) \)
and the fiber of \( \pi_{\wedge^3 F}: X_F \to Y_F \) over \( x \in D_1(\varrho) \setminus D_{k-1}(\varrho) \) is \( \mathbb{P}^{4-k-1} = \mathbb{P}^{3-k} \). Thus \( Y_F \setminus D_2(\varrho) \cong X_F \setminus \pi_{\wedge^3 F}(D_2(\varrho)) \) is smooth of dimension three and the smallness of \( X_F \to Y_F \) holds if \( D_1(\varrho) = \emptyset \) and dim \( D_2(\varrho) = 0 \). Indeed, we shall prove that \( \varrho \) is 3-general.

Recall that \( D_k(\varrho) \) is the locus where the rank of coker \( \varrho \) is at least \( 4-k \). Furthermore, coker \( \varrho \) coincides with the cokernel of the composite
\[ \tau'': U' \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \rightarrow \wedge^3 \mathcal{T} \rightarrow C. \]

Though \( C = \text{coker } \tau \) is not locally free, the restriction of \( C \) on \( D_l(\tau) \setminus D_{l-1}(\tau) \) is locally free of rank \( 4-l \). Since the composite \( (W^\vee \otimes \wedge^3 W) \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \wedge^3 \mathcal{T}(1) \to C \) is surjective, so is the restriction of it on \( D_l(\tau) \setminus D_{l-1}(\tau) \). Hence we can apply Bertini-type theorem (cf. [LW22 Theorem 3.1]) to compute the codimension of \( D_k(\tau''|_{D_l(\tau) \setminus D_{l-1}(\tau)}) \) in \( D_l(\tau) \setminus D_{l-1}(\tau) \).

Note that \( D_2(\tau) = \mathbb{P}(W^\vee) \) and dim \( D_1(\tau) = 1, D_0(\tau) = \emptyset \) by Remark 4.4. On \( D_2(\tau) \setminus D_1(\tau) = \mathbb{P}(W^\vee) \setminus D_1(\tau) \), the \( C \) is locally free of rank two and hence \( D_k(\varrho) \setminus D_1(\tau) \) coincides with \( D_{k-2}(\tau''|_{\mathbb{P}(W^\vee) \setminus D_1(\tau)}) \). Thus \( D_1(\varrho) \setminus D_1(\tau) = \emptyset \) and \( D_2(\varrho) \setminus D_1(\tau) \) is smooth with at most zero-dimension by Bertini-type theorem. On the other hand, \( C \)
is locally free of rank three on $D_1(\tau) \setminus D_0(\tau) = D_1(\tau)$. Hence $D_2(\rho) \cap D_1(\tau)$ coincides with $D_1(\tau''|D_1(\tau))$. Then the expected codimension of $D_1(\tau''|D_1(\tau))$ is two and hence $D_2(\rho) \cap D_1(\tau) = D_1(\tau''|D_1(\tau)) = \emptyset$ by dim $D_1(\tau) = 1$.

Thus we have $D_1(\rho) = \emptyset$ and $D_2(\rho) = D_2(\rho) \cap D_1(\tau)$ is smooth with dim $D_2(\rho) \leq 0$. Note that $Y_F \subseteq \mathbb{P}(W^\vee)$ is a quintic hypersurface since $Y_F = D_3(\rho)$ is the zero locus of $\det(\rho)$, which is a global section of $\det(H) \otimes \det(\wedge^3T) \cong \mathcal{O}_F(5)$. Then $\rho(Y_F) = 1$ and $\pi_{\lambda^3T}$ is not an isomorphism by $\rho(X_F) = 2$, i.e., $D_2(\rho) \neq \emptyset$. Hence $\rho$ is 3-general and $\pi_{\lambda^3T}$ is a small birational morphism. Applying Proposition 2.8 we find that the number of ODPs of $Y_F$ is

$$\left|D_2(\rho)\right| = \int_{\mathbb{P}(W^\vee)} c_2(\wedge^3T - \mathcal{H}^\vee)^2 - c_1(\wedge^3T - \mathcal{H}^\vee) \cdot c_3(\wedge^3T - \mathcal{H}^\vee) = 54.$$  

This completes the proof. \hfill \Box

To find the flop of $X_F$, we can use the same trick as in Section 2.4. Let

$$X_F^+ := \mathbb{P}(W^\vee)(\text{coker } \rho^\vee) \subseteq \mathbb{P}(W^\vee)(H).$$

Note that $D_k(\rho) = D_k(\rho^\vee)$ for all $k$. Therefore $\rho^\vee$ is also 3-general. By Proposition 2.6 we find that the determinantal contraction $\pi_H: X_F^+ \to Y_F$ is small and the $X_F^+$ is smooth. Let $H'$ be the pullback of $\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$ via $X_F^+ \to \mathbb{P}(W^\vee)$ and $L' = \mathcal{O}_H(1)|_{X_F^+}$. Applying Proposition 2.8 we get Table 3.

### Table 3. The intersection numbers of $X_F^+$.

| $L'^3$ | $L'^2H'$ | $L'H'^2$ | $H'^3$ | $L'.c_2(T_{X_F^+})$ | $H'.c_2(T_{X_F^+})$ | # of ODPs |
|--------|----------|----------|--------|-----------------|-----------------|---------|
| 34     | 23       | 13       | 5      | 76              | 50              | 54      |

4.3. **Construction of another birational model $X_F^{++}$.** In the previous subsection, we construct a small contraction $\pi_H: X_F^+ \to Y_F \subseteq \mathbb{P}(W^\vee)$. We now turn to find another small contraction $X_F^+ \to \mathbb{Z}_F$ and its flop $X_F^{++}$.

Recall that $U \subseteq \wedge^3W$ is a two-dimensional subspace such that $V_5 = \text{Gr}(2, W) \cap \mathbb{P}(\wedge^3W/U)$ is the smooth del Pezzo fourfold $V_5$ of degree 5. By $\wedge^3W \cong \wedge^2W^\vee$, we might think that $U \subseteq \wedge^2W^\vee$.

Each $\omega \in \wedge^2W^\vee$ defines an alternating form $f_\omega: W \times W \to \mathbb{C}$ with the kernel

$$\ker f_\omega = \{w \in W \mid f_\omega(w, w') = 0 \text{ for any } w' \in W\}.$$

Since rank $f_\omega$ is even and dim $W = 5$, the dimension of $\ker f_\omega$ is 1 or 3 if $\omega \neq 0$.

**Lemma 4.6.** For any non-zero $u \in U$, we have dim $\ker f_u = 1$.

**Proof.** For $\omega \in \wedge^2W^\vee$, let $H_\omega \subseteq \mathbb{P}(\wedge^2W^\vee) = \mathbb{P}(\wedge^3W)$ be the hyperplane defined by $\omega$. Let $x = [V] \in \text{Gr}(2, W)$ be a point corresponding to a two-dimensional subspace $V \subseteq W$. Then it is known that $H_\omega$ is tangent to $\text{Gr}(2, W) \subseteq \mathbb{P}(\wedge^2W^\vee)$ at $x$ if and only $V \subseteq \ker f_\omega$ (see [BC09] Proposition 1.5 for example). Hence the singular locus of $\text{Gr}(2, W) \cap H_\omega$ coincides with

$$\{[V] \in \text{Gr}(2, W) \mid V \subseteq \ker f_\omega\},$$

which is empty or isomorphic to $\mathbb{P}^2$ according to dim $\ker f_\omega = 1$ or 3.

If dim $\ker f_u = 3$ for some non-zero $u \in U$, the $\text{Gr}(2, W) \cap H_u$ is singular along $\mathbb{P}^2$. Since $V_5$ is a linear section of $\text{Gr}(2, W) \cap H_u$ of codimension one, the $V_5$ is singular along...
\( \mathbb{P}^1 \), which contradicts to the smoothness of \( V_5 \). Hence we have \( \dim \ker f_u = 1 \) for any non-zero \( u \in U \).

By \( \wedge^3 W \cong \wedge^2 W^\vee \) and \( \wedge^3 \mathcal{T} \cong \mathcal{T}^\vee(1) \), we might think that \( \mathcal{C} \) is the cokernel of \n\begin{equation}
\tau : U \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \hookrightarrow \wedge^2 W^\vee \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \mathcal{T}^\vee(1).
\end{equation}

Set \( \Sigma := D_1(\tau) = D_1(\tau^\vee) \subseteq \mathbb{P}(W^\vee) \). By (2), we get the zero scheme
\[ \Sigma^+ = \mathbb{P}_{\mathbb{P}(W^\vee)}(\text{coker } \tau^\vee) \subseteq \mathbb{P}_{\mathbb{P}(W^\vee)}(U^\vee \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}) = \mathbb{P}(U^\vee) \times \mathbb{P}(W^\vee) \]
induced by \( \tau^\vee \) and a natural morphism \( \Sigma^+ \to \Sigma \). We can also regard \( \Sigma^+ \subseteq \mathbb{P}_{\mathbb{P}(W^\vee)}(\mathcal{H}) \) by the natural inclusion \( \mathbb{P}_{\mathbb{P}(W^\vee)}(U^\vee \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}) \subseteq \mathbb{P}_{\mathbb{P}(W^\vee)}(\mathcal{H}) \). By Remark 4.4 we know that \( \dim \Sigma = 1 \). In fact, we have an explicit description of \( \Sigma \) as follows:

**Lemma 4.7.** With notation as above:

(i) The natural projection \( \Sigma^+ \to \mathbb{P}(U^\vee) \cong \mathbb{P}^1 \) is an isomorphism.

(ii) The natural projection \( \Sigma^+ \to \Sigma \) is an isomorphism and it holds that
\n\begin{equation}
\Sigma = \{(\ker f_u)^\vee \mid u \in U \setminus \{0\}\}.
\end{equation}

Furthermore, the degree of \( \Sigma \) in \( \mathbb{P}(W^\vee) \) is two.

(iii) The \( \Sigma^+ = X_F^\vee \cap \mathbb{P}_{\mathbb{P}(W^\vee)}(U^\vee \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}) \) holds scheme-theoretically, where we take the intersection in \( \mathbb{P}_{\mathbb{P}(W^\vee)}(\mathcal{H}) \).

**Proof.** [1] For simplicity, we sometimes denote \( W \otimes \mathcal{O}_Y \) by \( \mathcal{W}_Y \) for a variety \( Y \). The subspace \( U \subseteq \wedge^3 W^\vee \) induces a two-form \( \wedge^3 \mathcal{W}_{\mathbb{P}(U^\vee)} \to \mathcal{O}_{\mathbb{P}(U^\vee)}(1) \) on \( \mathbb{P}(U^\vee) \). Let \( \mathcal{K} \subseteq \mathcal{W}_{\mathbb{P}(U^\vee)} \) be the kernel of this two-from, that is, the kernel of the induced morphism \( \mathcal{W}_{\mathbb{P}(U^\vee)} \to \mathcal{W}_{\mathbb{P}(U^\vee)} \otimes \mathcal{O}_{\mathbb{P}(U^\vee)}(1) \). By Lemma 4.6 the \( \mathcal{K} \subseteq \mathcal{W}_{\mathbb{P}(U^\vee)} \) is a subbundle of rank one and \( \mathcal{K}[u^\vee] = (\ker f_u \subseteq W \text{ holds for any non-zero } u \in U) \). Here \( [u^\vee] \in \mathbb{P}(U^\vee) \) is the point corresponding to the quotient \( U^\vee \twoheadrightarrow (Cu)^\vee \).

Let \( p_{\mathbb{P}(U^\vee)} \) and \( p_{\mathbb{P}(W^\vee)} \) be the natural projections from \( \mathbb{P}(U^\vee) \times \mathbb{P}(W^\vee) \) to the first and second factors respectively. By Lemma 2.4 (1), the \( \Sigma^+ \subseteq \mathbb{P}(U^\vee) \times \mathbb{P}(W^\vee) \) is the zero locus of
\n\begin{equation}
\text{\( s_{\mathcal{T}^\vee} : p^*_{\mathbb{P}(W^\vee)}(\mathcal{T}^\vee(-1)) \to \wedge^2 \mathcal{W}_{\mathbb{P}(U^\vee)} \times \mathbb{P}(W^\vee) \to p^*_{\mathbb{P}(U^\vee)} \mathcal{O}_{\mathbb{P}(U^\vee)}(1) \).}
\end{equation}

By the natural exact sequence
\n\begin{align}
0 \to \mathcal{T}^\vee(-1) \to \wedge^2 W \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \wedge^2 \mathcal{T} \to 0
\end{align}

obtained from \( 0 \to \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \mathcal{T} \to 0 \), the zero locus of \( s_{\mathcal{T}^\vee} \) is the maximum closed subscheme on which the two-form \( \wedge^2 \mathcal{W}_{\mathbb{P}(U^\vee)} \times \mathbb{P}(W^\vee) \to p^*_{\mathbb{P}(U^\vee)} \mathcal{O}_{\mathbb{P}(U^\vee)}(1) \) factors through \( \wedge^2 \mathcal{W}_{\mathbb{P}(U^\vee)} \times \mathbb{P}(W^\vee) \to \wedge^2 (p^*_{\mathbb{P}(W^\vee)} \mathcal{T}) \). By the definition of the kernel \( \mathcal{K} \) and \( \mathcal{T} = \mathcal{W}_{\mathbb{P}(W^\vee)} / \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \), the zero locus \( \Sigma^+ = Z(s_{\mathcal{T}^\vee}) \) is nothing but the maximum locus on which \( p^*_{\mathbb{P}(W^\vee)} \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \subseteq p^*_{\mathbb{P}(U^\vee)} \mathcal{K} \), that is, the zero locus of
\n\begin{align}
p^*_{\mathbb{P}(W^\vee)} \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \hookrightarrow \mathcal{W}_{\mathbb{P}(U^\vee)} \times \mathbb{P}(W^\vee) \to p^*_{\mathbb{P}(U^\vee)}(\mathcal{W}_{\mathbb{P}(U^\vee)} / \mathcal{K}) \end{align}

By Lemma 2.4 (1), this zero locus coincides with \( \mathbb{P}_{\mathbb{P}(U^\vee)}(\mathcal{K}^\vee) \). Since \( \mathcal{K} \) is locally free of rank one, the projection \( \Sigma^+ = \mathbb{P}_{\mathbb{P}(U^\vee)}(\mathcal{K}^\vee) \to \mathbb{P}(U^\vee) \) is an isomorphism.

[1] By the proof of (1), we know that
\n\begin{equation}
p^*_{\mathbb{P}(W^\vee)} \mathcal{O}_{\mathbb{P}(W^\vee)}(-1)|_{\Sigma^+} \subseteq p^*_{\mathbb{P}(U^\vee)} \mathcal{K}|_{\Sigma^+}
\end{equation}

holds on \( \Sigma^+ \). Since both sides of (16) are subbundles of \( \mathcal{W}_{\mathbb{P}(U^\vee)} \times \mathbb{P}(W^\vee) \) of the same rank, this shows that (16) is actually an equality. Thus \( \mathbb{P}(U^\vee) \cong \Sigma^+ \to \Sigma \subseteq \mathbb{P}(W^\vee) \) is induced by the quotient \( \mathcal{W}_{\mathbb{P}(U^\vee)} = W^\vee \otimes \mathcal{O}_{\mathbb{P}(U^\vee)} \to \mathcal{K}^\vee \) and hence \( y = [u^\vee] \in \mathbb{P}(U^\vee) \) is
whose zero locus is nothing but $X + \Sigma$, which completes the proof. □

By Lemma 2.4 (2), the map $\Sigma^+ \to \Sigma$ is an isomorphism if $D_0(\tau') = \emptyset$. If $D_0(\tau')$ is non-empty, then the fiber of $\Sigma^+ = \mathbb{P}(\mathcal{V}_\tau)_{\text{coker } \tau} \to \Sigma$ over a point $x \in D_0(\tau')$ is $\mathbb{P}_1$, that is, the fiber coincides with the whole $\Sigma^+ \cong \mathbb{P}(U') = \mathbb{P}_1$. Therefore the surjective morphism $\Sigma^+ \to \Sigma$ is an isomorphism if it is not a constant map. We already saw that $\mathbb{P}(U') \cong \Sigma^+ \to \Sigma \subseteq \mathbb{P}(W')$ is induced by $W' \otimes \mathcal{O}_{\mathbb{P}(U')} \rightarrow K'$. Hence $\Sigma^+ \to \Sigma$ is not a constant map if and only if $K' \not\cong \mathcal{O}_{\mathbb{P}(U')}$. In fact, we can show $K' \cong \mathcal{O}_{\mathbb{P}(U')}(2)$ as follows: By the definition of $K$, the two-form $\wedge^2 \mathcal{W}_{\mathbb{P}(U')/K} \to \mathcal{O}_{\mathbb{P}(U')}(1)$ induces

\begin{equation}
\wedge^2(\mathcal{W}_{\mathbb{P}(U')/K}) \to \mathcal{O}_{\mathbb{P}(U')}(1),
\end{equation}

which is non-degenerate at any point in $\mathbb{P}(U')$. Hence it induces an isomorphism $\mathcal{W}_{\mathbb{P}(U')/K} \to (\mathcal{W}_{\mathbb{P}(U')/K})' \otimes \mathcal{O}_{\mathbb{P}(U')}(1)$. By taking the determinant, we see that $K' \cong \mathcal{O}_{\mathbb{P}(U')}(2)$.

Thus $\Sigma^+ \to \Sigma$ is an isomorphism induced by $K' \cong \mathcal{O}_{\mathbb{P}(U')}(2)$. In particular, the degree of $\Sigma$ is two.

Recall $\varphi = (\tau, \tau')$. By definition and Lemma 2.4 (1), we have $\Sigma^+ = \mathbb{P}(W')_{\text{coker } \tau'} = Z(s_{\tau'})$ and $X^+ = \mathbb{P}(W')_{\text{coker } \varphi} = Z(s_{\varphi})$. Since the restriction of $s_{\varphi} = (s_{\tau'}, s_{\tau'})$ on $\mathbb{P}(U') \cong \mathbb{P}(W')_{\text{coker } \varphi}$ is $(s_{\tau'}, 0)$, we have

$X^+ \cap \mathbb{P}(W')\vert_{(U') \otimes \mathcal{O}_{\mathbb{P}(W')})} = Z(s_{\varphi}) \cap \mathbb{P}(W')\vert_{(U') \otimes \mathcal{O}_{\mathbb{P}(W')}} = Z(s_{\tau'}) = \Sigma^+$,

which completes the proof. □

We will construct a small contraction $X^+ \to Z$ whose exceptional locus is the curve $\Sigma$. The task is now to find a divisor $D$ of $X^+ \to Z$ not meeting $\Sigma^-$. Using the description (13) of $\tau$ and similarly for $\tau'$, we find that $\varphi$ is decomposed as

$\varphi: \mathcal{H} \to \wedge^2 W' \otimes \mathcal{O}_{\mathbb{P}(W')} \to T'(1)$.

On $p_\mathcal{H}: \mathbb{P}(W')(\mathcal{H}) \to \mathbb{P}(W')$, we have

$\mathcal{O}_{\mathcal{H}}(-1) \hookrightarrow p_\mathcal{H}_* \mathcal{H} \to \wedge^2 W' \otimes \mathcal{O} \twoheadrightarrow p_\mathcal{H}_* T'(1),$

whose zero locus is nothing but $X^+ = \mathbb{P}(W')_{\text{coker } \varphi} \subseteq \mathbb{P}(W')(\mathcal{H})$. Hence the restriction of $\mathcal{O}_{\mathcal{H}}(-1) \hookrightarrow p_\mathcal{H}_* \mathcal{H} \to \wedge^2 W' \otimes \mathcal{O}$ on $X^+$ factors as

$s: \mathcal{O}_{\mathcal{H}}(-1) \vert_{X^+} \hookrightarrow \pi_\mathcal{H}_* (\mathcal{H}) \twoheadrightarrow \pi_\mathcal{H}_* (\wedge^2 \mathcal{T}) \subseteq \wedge^2 W' \otimes \mathcal{O}

by the dual of (13), where we recall that $\pi_\mathcal{H}: X^+ \to Y \subseteq \mathbb{P}(W')$ is the restriction of $p_\mathcal{H}$. By taking wedge product of $s$, we have

$s \wedge s: (\mathcal{O}_{\mathcal{H}}(-1) \otimes \mathcal{O}_{\mathcal{H}}(-1)) \vert_{X^+} \twoheadrightarrow \pi_\mathcal{H}_* (\wedge^2 \mathcal{T}) \otimes \wedge^2 \mathcal{T} \twoheadrightarrow \pi_\mathcal{H}_* (\wedge^4 \mathcal{T}).$

From the isomorphism $\wedge^4 \mathcal{T} \cong \mathcal{O}_{\mathbb{P}(W')}(-1)$, we get a section

$s \wedge s \in H^0(X^+ \vert_{X^+} \otimes \pi_\mathcal{H}_* \mathcal{O}_{\mathbb{P}(W')}(-1)).

Let $D \subseteq X^+$ denote the zero locus of the section $s \wedge s$. By the following lemma, the $s \wedge s$ is a non-trivial section and hence $D$ is a true (effective) divisor in the linear system $\mathcal{O}_{\mathcal{H}}(2)\vert_{X^+} \otimes \pi_\mathcal{H}_* \mathcal{O}_{\mathbb{P}(W')}(-1)$.

Lemma 4.8. Let $\Sigma^+$ be as in Lemma 4.7. Then $D \cap \Sigma^+ = \emptyset$.\hspace{1cm}
Proof. Notice that the zero locus of \( s \land s \) is the locus where the corresponding two-form \( s : \mathcal{O}_S(1)|_{X^+_F} \to \pi_*^s(\wedge^2 T^\vee) \) is degenerate.

We use the notation in the proof of Lemma 17. Recall that an isomorphism \( \mathbb{P}(U^\vee) \to \Sigma \) is defined by \([u^\vee] \mapsto [(\ker f_u)^\vee] \) and \( \Sigma^+ \subseteq \mathbb{P}(U^\vee) \times \mathbb{P}(W^\vee) \) coincides with the graph of the isomorphism. By the isomorphism \( \mathbb{P}(U^\vee) \to \Sigma \), the restriction \( T|_\Sigma \) is identified with \( W_{\mathbb{P}(U^\vee)}/\mathcal{K} = (W \otimes \mathcal{O}(U^\vee))/\mathcal{K} \).

Under the identification \( \Sigma^+ \cong \mathbb{P}(U^\vee) \), the restriction of \( s \) on \( \Sigma^+ \) is identified with
\[
\mathcal{O}_{\mathbb{P}(U^\vee)}(-1) \to \wedge^2 (W_{\mathbb{P}(U^\vee)}/\mathcal{K}),
\]
which is nothing but the dual of (17). It is non-degenerate at any point in \( \mathbb{P}(U^\vee) \) by the definition of the kernel \( \mathcal{K} \). Thus \( s \) is non-degenerate at any point in \( \Sigma^+ \) and hence \( \Sigma^+ \) does not intersect with the zero locus of \( s \land s \).

Recall that \( L' = \mathcal{O}_S(1)|_{X^+_F} \) and \( H' \) is the pullback of \( \mathcal{O}_{\mathbb{P}(W^\vee)}(1) \) via \( X^+_F \to \mathbb{P}(W^\vee) \).

Using the effective divisor \( D \), we can construct a small contraction \( X^+_F \to Z_F \) and its flop \( X^{++}_F \) as follows:

**Proposition 4.9.** There exists a birational model \( X^{++}_F \) of \( X^+_F \) such that
\[
\text{Nef}(X^+_F) = \mathbb{R}_{\geq 0}[H'] + \mathbb{R}_{\geq 0}[2L' - H'],
\]
\[
\text{Nef}(X^{++}_F) = \mathbb{R}_{\geq 0}[2L' - H'] + \mathbb{R}_{\geq 0}[L' - H'].
\]

Furthermore, \( L' - H' \) and \( 2L' - H' \) define a fibration \( X^{++}_F \to \mathbb{P}^1 \) and a small contraction \( \psi : X^+_F \to Z_F \) with \( \text{Exc}(\psi) = \Sigma^+ \) being contracted to an ODP respectively.

**Proof.** Note that
\[
H^0 \left( \mathcal{O}_S(1) \otimes s^* \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \right) = H^0 \left( \mathcal{H}' \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \right) = U^\vee.
\]

Take a basis \( t_1, t_2 \in U^\vee \) and let \( D_1', D_2' \subseteq \mathbb{P}(W^\vee)(\mathcal{H}) \) be the corresponding divisors. Then \( D_1' \cap D_2' = \mathbb{P}(W^\vee)(U^\vee \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}(1)) \subseteq \mathbb{P}(W^\vee)(\mathcal{H}) \). Hence \( X^+_F \cap D_1' \cap D_2' = \Sigma^+ \) by Lemma 17. By Lemma 18, we have \( X^+_F \cap D_1' \cap D_2' \cap D = \Sigma^+ \cap D = \emptyset \). Set \( D_i := D_i'|_{X^+_F} \) for \( i = 1, 2 \). Hence, on \( X^+_F \), we have effective divisors
\[
D_1, D_2 \in [L' - H'] \quad \text{and} \quad D \in [2L' - H'].
\]

Since \( D_1 \cap D_2 \cap D = \emptyset \), we can apply [Ito14, Lemma 2.4, Proposition 2.5] and obtain a model \( X^{++}_F \). The description of the nef cones and the morphism \( X^{++}_F \to \mathbb{P}^1 \) is obtained by [Ito14, Proposition 2.5].

We review the construction of the flop for the benefit of the reader. First, since \( H' \) is base point free, we have
\[
\text{Bs}(2L' - H') \subseteq D_1 \cap D_2 \cap D = \emptyset.
\]
This means that \( 2L' - H' \) is also base point free. Let \( \psi : X^+_F \to Z_F \) be the morphism defined by \( [m(2L' - H')] \) for \( m \gg 0 \). We claim that the exceptional set of \( \psi \) is equal to \( \Sigma^+ \), i.e. \( \psi \) is small.

Indeed, the curve \( \Sigma^+ \) is contained in \( \text{Exc}(\psi) \) since \( \Sigma^+ \cdot (2L' - H') = \Sigma^+ \cdot D = 0 \) by Lemma 18. On the other hand, if there exists a curve \( C \subseteq \text{Exc}(\psi) \) such that \( C \neq \Sigma^+ \), then \( C \cdot (L' - H') \) and \( C \cdot H' \) are nonnegative because \( H' \) is base point free and \( D_1 \cap D_2 = \Sigma^+ \neq C \). Therefore \( C \cdot (2L' - H') = 0 \) implies \( C \cdot L' = C \cdot H' = 0 \). This contradicts the fact that \( X^+_F \) is projective and has Picard number two.

Next we construct the flop \( X^{++}_F \) of \( \psi : X^+_F \to Z_F \). Let \( \mu : \text{Bl}_{\Sigma^+} X^+_F \to X^+_F \) be the blow-up along \( \Sigma^+ \) and \( E \) the exceptional divisor of \( \mu \). Since \( D_1 \cap D_2 = \Sigma^+ \subseteq X^+_F \), the
rational map $X^+ \rightarrow \mathbb{P}(U/V)$ defined by the pencil $\langle D_1, D_2 \rangle \subseteq |L' - H'|$ can be resolved by $\mu$ to get the morphism $\pi: \text{Bl}_{\Sigma} X^+ \rightarrow \mathbb{P}(U/V) \cong \mathbb{P}^1$. Note that

$$E = \mathbb{P}_{\Sigma}(\mathcal{O}(-D_1)|_{\Sigma^+} \oplus \mathcal{O}(-D_2)|_{\Sigma^+}) = \mathbb{P}^1 \times \Sigma^+ \cong \mathbb{P}^1 \times \mathbb{P}^1.$$ 

Furthermore, we have $\mathcal{O}(-E)|_{E} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$ since $\Sigma^+ \cdot D_i = -1$ by $\Sigma^+ \cdot (2L' - H') = 0$ and $\Sigma^+ \cdot H' = \Sigma \cdot \mathcal{O}_{\mathbb{P}(W^\vee)}(1) = 2$. Hence $\mathcal{O}(\mu^* D_i - E)|_{E} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1)$ and $f|_E: E = \mathbb{P}^1 \times \Sigma^+ \rightarrow \mathbb{P}^1$ is the projection to the first factor.

We write $X^+ \rightarrow \mathbb{P}^1 \times Z_F$, and the natural projection gives rise to the fibration $X^+ \rightarrow Z_F$. Since $\psi \circ \mu(E) = \psi(\Sigma^+) \in Z_F$ is a point, $(f, \psi \circ \mu)|_E$ contracts the divisor $E = \mathbb{P}^1 \times \Sigma^+$ to a curve $\mathbb{P}^1 \times \psi(\Sigma^+)$. On the other hand, $(f, \psi \circ \mu)$ is an isomorphism onto the image outside $E$ since $\text{Bl}_{\Sigma} X^+ \setminus E \cong X^+ \setminus \Sigma^+ \cong Z_F \setminus \psi(\Sigma^+)$. Therefore the natural projection $X^+ \rightarrow Z_F$ is the flop of $X^+ \rightarrow Z_F$. In fact, it is an Atiyah flop because of $\mathcal{O}(E)|_E \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1).

By construction, $H'$ and $2L' - H'$ define small contractions $X^+ \rightarrow Y_F$ and $X^+ \rightarrow Z_F$ respectively. Hence $H'$ and $2L' - H'$ span $\text{Nef}(X^+)$.

**Proof of Theorem 4.1** According to [10] and $S_{\mathbb{P}^2}^\vee = \mathcal{F}(-1)$, it follows that the determinantal contraction $\pi_{\mathbb{P}^2}^\vee: X_F \rightarrow Y_F$ is induced by $L - H$. Then (ii) follows from this and (i) and (iii).

The (ii) and (iii) follow from the construction of $\pi_F$ and Proposition 2.10. Remark that, as in (iv), we find that $c_2(T_{\mathbb{P}^2}) \cdot c_2(\mathcal{F}) = 53$ and $c_2(T_{X_F}) \cdot L_E = 24$ by the Schubert calculus for $\text{Gr}(2, W)$.

Set $\theta := (\pi_{\mathbb{P}^2}^\vee)^{-1} \circ \pi_H$ and $\theta_s L' = aL + bH$. Notice that $\theta_s H' = L - H$, and $\theta$ is not an isomorphism because the numbers of singular points of $D_1(\sigma)$, $Y_F$ and $Z_F$ are distinct. By Lemma 2.9 Table II and III, we have

$$23 = 47a^2 + 60ab + 18b^2,$$

$$13 = 17a + 12b.$$ 

The only solution is $(a, b) = (5, -6)$ because $ab < 0$. Then (iii) and (iv) follow from Proposition 4.9 except that a general fiber $\Sigma$ of $X^+ \rightarrow \mathbb{P}^1$ is K3. To prove this, we shall compute $c_2(T_{X^+}) \cdot S$ as before. By Proposition 4.9, [Fri91, Lemma 7.5] and Table III the intersection number equals

$$c_2(T_{X^+}) \cdot (L' - H') + 2(L' - H') \cdot \Sigma^+ = 26 - H' \cdot \Sigma^+.$$ 

Notice that $H' \cdot \Sigma^+ = \mathcal{O}_{\mathbb{P}(W^\vee)}(1) \cdot \Sigma = 2$ since $\Sigma \subseteq \mathbb{P}(W^\vee)$ is a conic by Lemma 4.7. Hence the intersection number $c_2(T_{X^+}) \cdot S = 24$ and the proof is completed.

**5. The Grassmannian $\text{Gr}(2, 4)$**

We consider the case $(M, F, E) = (\text{Gr}(2, 4), S(2) \oplus \mathcal{O}(1), \mathcal{O}^{\oplus 3})$. As before, for a general morphism $\sigma: \mathcal{E}^\vee \rightarrow \mathcal{F}$, there are two smooth Calabi-Yau threefolds $X_F$, $X_E$ and the diagram IV. In this section, we prove the following theorem.
Theorem 5.1. Let \((M, F, E) = (\text{Gr}(2,4), S(2) \oplus \mathcal{O}(1), \mathcal{O}^{\oplus 3})\). Then for a general morphism \(\sigma : E' \to F\), the scheme \(X_F\) is a smooth Calabi–Yau threefold of Picard number two with
\[
\text{Nef}(X_F) = \mathbb{R}_{\geq 0}[L - H] + \mathbb{R}_{\geq 0}[H],
\]
such that

(i) the determinantal contraction \(\pi_F\) is induced by \(H\);
(ii) the \(L - H\) induces a small contraction \(\varphi : X_F \to Y_F\) and \(Y_F \subseteq \mathbb{P}^4\) is a quintic hypersurface;
(iii) the flop \(X_F^+\) of \(\varphi\) admits an elliptic fibration over \(\mathbb{P}^2\) induced by \(4L - 5H\);
(iv) the \(X_E\) admits an elliptic fibration over \(\mathbb{P}^2\) induced by \(4H - L\).

Moreover, the movable cone \(\text{Mov}(X_F)\) is the convex cone generated by the divisors \(4L - 5H\) and \(4H - L\) which is covered by the nef cones of \(X_F\), \(X_F^+\) and \(X_E\), and there are no more minimal models of \(X_F\), which we summarize in the following diagram:

\[
\begin{array}{cccc}
X_F^+ & \overset{\theta}{\longrightarrow} & X_F & \overset{\chi}{\longrightarrow} & X_E \\
\downarrow{\varphi^+} & & \downarrow{\varphi} & & \downarrow{\pi_F} \quad \downarrow{\pi_E} \\
\mathbb{P}^2 & & Y_F & & D_2(\sigma) & & \mathbb{P}^2
\end{array}
\]

The slice of the chamber structure of \(\text{Mov}(X_F)\) is given in Figure 3. We depict \(X_E\), \(X_F\), and \(X_F^+\) inside their nef cones.

\[
\begin{array}{cccc}
4L - 5H & \quad L - H & \quad H & \quad 4H - L \\
\text{ } & \text{ } & \text{ } & \text{ }
\end{array}
\]

\[
\begin{array}{c}
X_F^+ \\
X_F \\
X_E
\end{array}
\]

\text{Figure 3.} The slice of the movable cone \(\text{Mov}(X_F)\) for \(M = \text{Gr}(2,4)\).

Remark 5.2. It is easily seen that \(X_E\) and \(X_F^+\) are not isomorphic. If the assertion were false, then we could find that \(Y_F\) and \(D_2(\sigma)\) are isomorphic. The \(Y_F\) is a hypersurface in \(\mathbb{P}^4\) of degree 5 by Theorem 5.1 (iii). On the other hand, we may regard \(D_2(\sigma)\) as a complete intersection of two hypersurfaces in \(\mathbb{P}^5\) of degree 2 and 4. Therefore the Picard group of \(Y_F\) (resp. \(D_2(\sigma)\)) is isomorphic to that of \(\mathbb{P}^4\) (resp. \(\mathbb{P}^5\)) by Lefschetz theorem for Picard groups, see, e.g., [Laz04a, Example 3.1.25]. Let the generator of \(\text{Pic}(Y_F)\) and \(\text{Pic}(D_2(\sigma))\) be \(A_1\) and \(A_2\) respectively. We deduce the contradiction \(5 = A_1^3 = A_2^3 = 8\).

5.1. Another description of \(\mathbb{P}(F)\). As in Section 4, let \(W\) be a five-dimensional vector space over \(\mathbb{C}\). Fix a nonzero vector \(w_0 \in W\) and let \(V = W/\mathbb{C}w_0\). Then there is a natural morphism
\[
0 \to \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}
\]
via the Euler sequence
\[
0 \to \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \to W \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to T_{\mathbb{P}(W^\vee)}(-1) \to 0
\]
and the quotient map \(W \to V\). We define a coherent sheaf \(\mathcal{C}'\) on \(\mathbb{P}(W^\vee)\) to be the cokernel of (18). Then \(\mathcal{C}'\) is locally free of rank three on \(\mathbb{P}(W^\vee)\) \(\setminus \{[w_0^\vee]\}\).

Recall that \(0 \to \mathcal{S} \to V \otimes \mathcal{O}_{\text{Gr}(2,V)} \to \mathcal{Q} \to 0\) is the universal exact sequence on \(\text{Gr}(2,V)\) and define \(\tilde{S}\) to be the kernel of the surjection
\[
0 \to \mathcal{S} \to V \otimes \mathcal{O}_{\text{Gr}(2,V)} \to \mathcal{Q} \to 0
\]
and the quotient map \(W \to V\). We define a coherent sheaf \(\mathcal{C}'\) on \(\mathbb{P}(W^\vee)\) to be the cokernel of (18). Then \(\mathcal{C}'\) is locally free of rank three on \(\mathbb{P}(W^\vee)\) \(\setminus \{[w_0^\vee]\}\).
Hence we have an exact sequence

(20) \[ 0 \to \tilde{S} \to W \otimes \mathcal{O}_{\text{Gr}(2,V)} \to \mathcal{Q} \to 0. \]

Note that we have \( \tilde{S} \cong S \oplus \mathcal{O}_{\text{Gr}(2,V)} \) by choosing a splitting \( W \cong V \oplus \mathcal{C} \). Because \( S \) has rank two and \( \det(\tilde{S}) = \mathcal{O}_{\text{Gr}(2,V)}(-1) \), we find that \( \tilde{S} \cong \mathcal{S}' \otimes \det(\tilde{S}) = \mathcal{S}'(-1) \) and thus \( \tilde{S}'(1) \cong \mathcal{F} \). In particular, we get \( \mathbb{P}(\mathcal{S}') \cong \mathbb{P}(\mathcal{F}) \) over \( \text{Gr}(2,V) \).

By the surjection \( W' \otimes \mathcal{O}_{\text{Gr}(2,V)} \to \mathcal{S}' \), we have a closed immersion

\[ \mathbb{P}_{\text{Gr}(2,V)}(\mathcal{S}') \hookrightarrow \mathbb{P}_{\text{Gr}(2,V)}(W' \otimes \mathcal{O}_{\text{Gr}(2,V)}) = \mathbb{P}(W') \times \text{Gr}(2,V). \]

Let \( p_{\mathbb{P}(W')} \) and \( p_{\text{Gr}(2,V)} \) be the natural projections from \( \mathbb{P}(W') \times \text{Gr}(2,V) \) to the first and the second factors respectively. Set \( p_1 \) be the restriction of \( p_{\mathbb{P}(W')} \) to \( \mathbb{P}_{\text{Gr}(2,V)}(\mathcal{S}') \), and similarly for \( p_2 \).

Note that

(21) \[ \mathcal{O}_{\mathcal{S}'}(1) = p_1^* \mathcal{O}_{\mathbb{P}(W')}(1). \]

The following gives another description of the projective bundle associated to \( \mathcal{S}' \).

**Lemma 5.3.** The Grassmannian \( \mathbb{P}_{\mathbb{P}(W')}(\mathcal{C}', 2) \) coincides with the projective bundle \( \mathbb{P}_{\text{Gr}(2,V)}(\mathcal{S}') \) in \( \mathbb{P}(W') \times \text{Gr}(2,V) \):

\[ \begin{array}{ccc}
\mathbb{P}(W') & \xrightarrow{p_1} & \mathbb{P}_{\text{Gr}(2,V)}(\mathcal{S}') \\
\mathbb{P}(W') & \xrightarrow{p_2} & \text{Gr}(2,V).
\end{array} \]

In particular, \( p_1 \) is a \( \mathbb{P}^2 \)-bundle \( \mathbb{P}(\mathcal{C}') \) over \( \mathbb{P}(W') \setminus \{[w'_0]\} \), and the fiber \( p_1^{-1}([w'_0]) \) is \( \text{Gr}(V, 2) = \text{Gr}(2,V) \).

**Proof.** By [19] and Lemma 2.3, the projective bundle \( \mathbb{P}_{\text{Gr}(2,V)}(\mathcal{S}') \) is the zero locus of

(22) \[ p_{\text{Gr}(2,V)}^* \mathcal{Q}' \to W' \otimes \mathcal{O}_{\mathbb{P}(W') \times \text{Gr}(2,V)} \to p_{\mathbb{P}(W')}^* \mathcal{O}_{\mathbb{P}(W')}(1) \]

in \( \mathbb{P}(W') \times \text{Gr}(2,V) \). Since the zero locus of (22) coincides with that of

\[ p_{\mathbb{P}(W')}^* \mathcal{O}_{\mathbb{P}(W')}(1) \to V \otimes \mathcal{O}_{\mathbb{P}(W') \times \text{Gr}(2,V)} \to p_{\text{Gr}(2,V)}^* \mathcal{Q}, \]

the bundle \( \mathbb{P}_{\text{Gr}(2,V)}(\mathcal{S}') \) coincides with

\[ \text{Gr}_{\mathbb{P}(W')}(\mathcal{C}', 2) \hookrightarrow \text{Gr}_{\mathbb{P}(W')}(V \otimes \mathcal{O}_{\mathbb{P}(W')}, 2) = \mathbb{P}(W') \times \text{Gr}(2,V) \]

again by [18] and Lemma 2.3.

To see the last statement, we note that \( \mathcal{C}' \) is locally free of rank 3 on \( \mathbb{P}(W') \setminus \{[w'_0]\} \).

Hence the Plücker embedding

\[ \text{Gr}_{\mathbb{P}(W')}(\mathcal{C}', 2) \hookrightarrow \mathbb{P}(\mathcal{C}') \]

is an isomorphism over \( \mathbb{P}(W') \setminus \{[w'_0]\} \). On the other hand, we have \( p_1^{-1}([w'_0]) = \text{Gr}(\mathcal{C}'([w'_0]), 2) = \text{Gr}(V, 2). \) \( \square \)
5.2. **Construction of the flop.** Recall that the defining section $s_\sigma$ of the Calabi–Yau $X_F \subseteq \mathbb{P}(\mathcal{F}) \cong \mathbb{P}(\tilde{S}^\vee)$ is induced by a general morphism $\sigma: \mathcal{E}^\vee = \mathcal{O}^{\oplus 3} \to \mathcal{F}$. From (20), we have the surjection

$$W^\vee \otimes \wedge^2 V = H^0(W^\vee \otimes \mathcal{O}_{\mathbb{P}(\tilde{S}^\vee)}(1)) \to H^0(\tilde{S}^\vee(1)) = H^0(\mathcal{O}_{\tilde{S}^\vee}(1) \boxtimes \mathcal{O}_{\mathbb{P}(S^\vee)}(1)).$$

As in the proof of Proposition 5.3 there exists a general three-dimensional subspace

$$U_\sigma \subseteq W^\vee \otimes \wedge^2 V$$

such that the image of $U_\sigma$ under (23) corresponds to the subspace induced by the section $s_\sigma$. Hence we can identify

$$s_\sigma: p_2^*\mathcal{E}^\vee = \mathcal{O}^{\oplus 3}_{\mathbb{P}(\tilde{S}^\vee)} \to \mathcal{O}_F(1) = \mathcal{O}_{\tilde{S}^\vee}(1) \otimes p_2^*\mathcal{O}_{\mathbb{P}(\tilde{S}^\vee)}(1)$$

with the composite

$$U_\sigma \otimes \mathcal{O}_{\tilde{S}^\vee}(-1) \hookrightarrow U_\sigma \otimes W \otimes \mathcal{O}_{\mathbb{P}(\tilde{S}^\vee)} \to \wedge^2 V \otimes \mathcal{O}_{\mathbb{P}(\tilde{S}^\vee)} \to p_2^*\mathcal{O}_{\mathbb{P}(\tilde{S}^\vee)}(1)$$

of the natural morphisms up to the twist by $\mathcal{O}_{\tilde{S}^\vee}(1)$.

On the other hand, the subspace $U_\sigma$ also induces a morphism

$$U_\sigma \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \hookrightarrow U_\sigma \otimes W \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \wedge^2 V \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}$$

on $\mathbb{P}(W^\vee)$. By composing this with $\wedge^2 V \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \wedge^2 \mathcal{C}'$, which is induced by the natural surjection $V \otimes \mathcal{O}_{\mathbb{P}(W^\vee)} \to \mathcal{C}'$, we obtain a morphism

$$\tau: U_\sigma \otimes \mathcal{O}_{\mathbb{P}(W^\vee)}(-1) \to \wedge^2 \mathcal{C}'.$$

We set $\mathbb{P}^o = \mathbb{P}(W^\vee) \setminus \{[w^\vee]\}$. By Lemma 5.3 we have an identification

$$p_1^{-1}(\mathbb{P}^o) = \text{Gr}_{\mathbb{P}^o}(\mathcal{C}'|_{\mathbb{P}^o}, 2) = \mathbb{P}_{\mathbb{P}^o}(\wedge^2 \mathcal{C}'|_{\mathbb{P}^o}).$$

Now we can give another description of $X_F \cap p_1^{-1}(\mathbb{P}^o)$:

**Proposition 5.4.** The Calabi–Yau threefold $X_F \subseteq \mathbb{P}_{\text{Gr}(2,V)}(\tilde{S}^\vee) = \text{Gr}_{\mathbb{P}(W^\vee)}(\mathcal{C}', 2)$ coincides with

$$\mathbb{P}_{\mathbb{P}(W^\vee)}(\text{coker } \tau) \subseteq \mathbb{P}_{\mathbb{P}(W^\vee)}(\wedge^2 \mathcal{C}')$$

on the open subset $p_1^{-1}(\mathbb{P}^o) \subseteq \text{Gr}_{\mathbb{P}(W^\vee)}(\mathcal{C}', 2)$ under the identification (26).

**Proof.** Applying Lemma 2.4 (1) to $\tau|_{\mathbb{P}^o}$, we see that $\mathbb{P}_{\mathbb{P}(W^\vee)}(\text{coker } \tau)$ coincides with the zero locus of

$$s_{\tau|_{\mathbb{P}^o}}: U_\sigma \otimes p_1^*\mathcal{O}_{\mathbb{P}^o}(-1) \to p_1^*(\wedge^2 \mathcal{C}'|_{\mathbb{P}^o}) \to \mathcal{O}_{\wedge^2 \mathcal{C}'|_{\mathbb{P}^o}}(1)$$

on $p_1^{-1}(\mathbb{P}^o) = \text{Gr}_{\mathbb{P}^o}(\mathcal{C}'|_{\mathbb{P}^o}, 2) = \mathbb{P}_{\mathbb{P}^o}(\wedge^2 \mathcal{C}'|_{\mathbb{P}^o})$. By the definition of the Plücker embedding, the $\mathcal{O}_{\wedge^2 \mathcal{C}'|_{\mathbb{P}^o}}(1)$ coincides with $p_2^*\mathcal{O}_{\text{Gr}(2,V)}(1)|_{p_1^{-1}(\mathbb{P}^o)}$. Furthermore, we have $p_1^*\mathcal{O}_{\mathbb{P}(W^\vee)}(1) = \mathcal{O}_{\tilde{S}^\vee}(1)$ by (21). Then $s_{\tau|_{\mathbb{P}^o}}$ is nothing but the restriction of (24) on $p_1^{-1}(\mathbb{P}^o)$. Since (24) can be identified with $s_\sigma$, the zero locus of $s_{\tau|_{\mathbb{P}^o}}$ coincides with $X_F = Z(s_\sigma)$ in $p_1^{-1}(\mathbb{P}^o)$ and the proposition follows. \qed

Using the above description of $X_F$, we can construct the other small contraction of it as follows.

**Proposition 5.5.** Let $Y_F \subseteq \mathbb{P}(W^\vee)$ be the image $p_1(X_F)$ and $\varphi := p_1|_{X_F}: X_F \to Y_F$. Then $\varphi$ is a small contraction and $Y_F$ is a quintic hypersurface in $\mathbb{P}(W^\vee)$.
Proof. Recall that the coherent sheaf $\land^2 C'$ has rank three. For $x \in \mathbb{P}^o = \mathbb{P}(W')\setminus\{[w'_0]\}$, the fiber $\varphi^{-1}(x)$ is $\mathbb{P}_{\mathbb{P}(W')}(\text{coker } \tau(x)) \cong \mathbb{P}^{3-k-1} = \mathbb{P}^{2-k}$ for $k = \text{rank } \tau(x) = 1, 2$ by Proposition 5.4. On the other hand, the fiber $\varphi^{-1}([w'_0])$ is a smooth conic, which is a linear section $p_{1}^{-1}([w'_0]) = \text{Gr}(2, V)$ of codimension three, since $U_0$ is general. Note that the codimension of $D_0(\tau|_{\mathbb{P}^o}) \subseteq \mathbb{P}^o$ is 4 and $D_0(\tau|_{\mathbb{P}^o}) = \emptyset$. Therefore, the $\varphi$ is small.

Note that $Y_\tau \setminus \{[w'_0]\}$ is the zero locus of $\text{det}(\land^2 C') \otimes \mathcal{O}_{\mathbb{P}(W')}(3)|_{\mathbb{P}^o} \cong \mathcal{O}_{\mathbb{P}(W')}(5)|_{\mathbb{P}^o}$. Then it is the restriction of a quintic hypersurface on $\mathbb{P}^o$. Since $Y_\tau$ is irreducible, we see that $Y_\tau$ is a quintic hypersurface. In particular, the $Y_\tau$ is Cohen–Macaulay. Since the singular locus of $Y_\tau$ is zero-dimensional, $Y_\tau$ is normal and hence $\varphi$ is a small contraction. □

In the remainder of this subsection, we are going to find the flop of $\varphi$. Let $\mu$ be the blow-up $\mathbb{P} \to \mathbb{P}(W')$ at the point $[w'_0]$ and $E$ the exceptional divisor of $\mu$. This $\mathbb{P}$ is the graph of the rational map $\mathbb{P}(W') \dashrightarrow \mathbb{P}(V')$ given by the projection from the point $[w'_0]$, and we get the morphism $\varphi: \mathbb{P} \to \mathbb{P}(V')$. Set $T' = T_{\mathbb{P}(V')}(-1).

Lemma 5.6. There is an isomorphism $f^*T' \cong C'$ on $\mathbb{P} \setminus E \cong \mathbb{P}^o$

Proof. On $\mathbb{P} \setminus E \cong \mathbb{P}^o$, there is a diagram

$$
\begin{array}{ccc}
\mu^*\mathcal{O}_{\mathbb{P}(V')}(1) & \longrightarrow & \mathcal{O}_{\tilde{\mathbb{P}}} \otimes V \\
\downarrow^\alpha & & \downarrow^\beta \\
f^*\mathcal{O}_{\mathbb{P}(V')}(1) & \longrightarrow & f^*T'
\end{array}
$$

where the top row is from [15], and the bottom row is the pullback of the Euler sequence on $\mathbb{P}(V')$. The map $\alpha$ is induced by the natural projection $W \to V$ and $\beta$ is the induced map from the exact sequences. Since $\alpha$ is an isomorphism on $\mathbb{P} \setminus E$, so is $\beta$ and this lemma follows. □

We write $U_\sigma$ for the trivial bundle $U_0 \otimes \mathcal{O}_{\mathbb{P}(W')}$. From the observation $W' \otimes \land^2 V = H^0(\mu^*\mathcal{O}_{\mathbb{P}(W')}(1)) \otimes H^0(f^*(\land^2 T'))$, the subspace $U_0 \subseteq W' \otimes \land^2 V$ also induces a morphism

$$
(27) \quad \varphi: \mu^*(U_\sigma(-1)) \to f^*(\land^2 T'),
$$

which coincides with $\tau: U_\sigma(-1) \to \land^2 C'$ on $\mathbb{P} \setminus E \cong \mathbb{P}^o$ under the isomorphism $f^*T' \cong C'$ in Lemma 5.6. Set $Y_\tau = D_2(\varphi)$. By Lemma 5.6 the morphism $\mu$ maps $Y_\tau$ onto $Y_\tau$. As in Section 2.4 let

$$
\tilde{X}_\tau \subseteq \mathbb{P}\mathcal{P}(\mu^*(U_\sigma'(1))) \cong \mathbb{P}(U_\sigma') \times \tilde{\mathbb{P}} \cong \mathbb{P}^2 \times \tilde{\mathbb{P}}
$$

be the zero locus of the global section $s_{\varphi^\lor}$ of

$$
(28) \quad \mathcal{O}_{\mathbb{P}^2\mathcal{P}(U_\sigma'(1))} \otimes f^*(\land^2 T') \cong \mathcal{O}_{\mathbb{P}^2}(1) \otimes (f^*(\land^2 T') \otimes \mu^*\mathcal{O}(1))
$$

induced by the dual morphism $\varphi^\lor$. Therefore, we find the determinantal contraction $\tilde{X}_\tau \to Y_\tau \subseteq \tilde{\mathbb{P}}$, which is small, for a general $U_\sigma$ since $\mu^*(U_\sigma(1)) \otimes f^*(\land^2 T')$ is globally generated.
We write $X^\dagger_F \subseteq \mathbb{P}^2 \times \mathbb{P}(W^\vee)$ for the image $(\text{id}_{\mathbb{P}^2} \times \mu)(\tilde{X}_F)$. Via the projections, we have the morphism $\varphi^\dagger: X^\dagger_F \to Y_F$ and following commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}^2 \times \mathbb{P} & \cong & \mathbb{P}(\mu^*(\mathcal{U}_\omega^\vee(1))) \\
\downarrow \text{id}_{\mathbb{P}^2} \times \mu & & \downarrow \mu \\
\mathbb{P}^2 \times \mathbb{P}(W^\vee) & \cong & \mathbb{P}(\mathcal{U}_\omega^\vee(1)) \\
\downarrow \text{id}_{\mathbb{P}^2} \times \mu & & \downarrow \mu \\
\tilde{X}_F & \to & \tilde{Y}_F \\
\end{array}
\]

(29)

**Proposition 5.7.** The morphism $\varphi^\dagger: X^\dagger_F \to Y_F$ is small. In particular, $(\varphi^\dagger)^{-1} \circ \varphi$ is an isomorphism in codimension one.

**Proof.** The morphism $\varphi^\dagger$ is small over $Y_F \setminus \{[w_0^\vee]\}$ since so is $\tilde{X}_F \to \tilde{Y}_F$. Hence it suffices to show that the fiber of $\varphi^\dagger$ over $[w_0^\vee]$ is one-dimensional. By construction, the fiber over the point $[w_0^\vee]$ is $X^\dagger_F \cap (\mathbb{P}^2 \times [w_0^\vee])$, which is the image of $\tilde{X}_F \cap (\mathbb{P}^2 \times E) = Z(s_{\varphi^\dagger}) \cap (\mathbb{P}^2 \times E)$.

Under the isomorphism $f|_E: E \to \mathbb{P}(V^\vee)$ and (28), the restriction of the section $s_{\varphi^\dagger}$ to $\mathbb{P}^2 \times E$ corresponds to a global section $\omega$ of $\mathcal{O}_{\mathbb{P}^2} \boxtimes \wedge^2 T'$ on $\mathbb{P}^2 \times \mathbb{P}(V^\vee)$. Note that we can regard the morphism $\omega$ as an element of

$$H^0(\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \wedge^2 T') = H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbb{P}^2}(1)).$$

Furthermore, $\omega$ is general in $H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbb{P}^2}(1)) = \wedge^2 V \otimes U_\omega^\vee$ since $\omega$ corresponds to the composite $U_\omega \hookrightarrow W^\vee \otimes \wedge^2 V \to (\mathcal{C}w_0)^\vee \otimes \wedge^2 V$ and $U_\omega$ is general.

Let $pr_1$ be the projection of $\mathbb{P}^2 \times \mathbb{P}(V^\vee)$ on the $i$th factor and $Z(\omega) \subseteq \mathbb{P}^2 \times \mathbb{P}(V^\vee)$ the zero locus of $\omega \in H^0(\mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \wedge^2 T')$. Under the identification $E = \mathbb{P}(V^\vee)$ by $f|_E$, we have $\tilde{X}_F \cap (\mathbb{P}^2 \times E) = Z(s_{\varphi^\dagger}) \cap (\mathbb{P}^2 \times E) = Z(\omega)$ and $X^\dagger_F \cap (\mathbb{P}^2 \times [w_0^\vee]) = pr_1(Z(\omega)).$ Hence the rest is to show $\text{dim } pr_1(Z(\omega)) = 1$.

We claim that $pr_1(Z(\omega)) \subseteq \mathbb{P}^2$ is the zero locus of $\wedge^2 \omega \in H^0(\wedge^4 V \otimes \mathcal{O}_{\mathbb{P}^2}(2))$, the Pfaffian of $\omega \in H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbb{P}^2}(1))$. In fact, a point $[u^\vee] \in \mathbb{P}(U_\omega^\vee) \cong \mathbb{P}^2$ is contained in $pr_1(Z(\omega))$ if and only if there exists $[v^\vee] \in \mathbb{P}(V^\vee)$ such that $([u^\vee], [v^\vee]) \in Z(\omega)$, that is,

$$\mathcal{C}u \hookrightarrow U_\omega \xrightarrow{\omega} \wedge^2 V \to \wedge^2 T'([v^\vee]) = \wedge^2 (V/\mathcal{C}v)$$

is the zero map. This condition is equivalent to say that the two-form $\omega([v^\vee]) \in \wedge^2 V$ is degenerate, that is, $\wedge^2 \omega([v^\vee]) = 0$.

Since $\omega \in H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbb{P}^2}(1))$ is general, $\wedge^2 \omega \in H^0(\wedge^2 V \otimes \mathcal{O}_{\mathbb{P}^2}(2))$ is non-zero. Hence $pr_1(Z(\omega)) = Z(\wedge^2 \omega) \subseteq \mathbb{P}^2$ is one-dimensional and this proposition follows. \qed

By Proposition 5.7, we have the following proposition.

**Proposition 5.8.** Let $X^\dagger_F$ be the normalization of $X^\dagger_F$. Then $\varphi^\dagger: X^\dagger_F \to Y_F$ is the flop of $\varphi: X_F \to Y_F$ and the natural projection gives rise to $X^\dagger_F \to \mathbb{P}^2$.

**Proof.** Since $X^\dagger_F \dasharrow X_F$ is small and the Picard number of $X^\dagger_F$ is at least two, $X^\dagger_F$ is $\mathbb{Q}$-factorial. Hence $X^\dagger_F \to Y_F$ is the flop of $\varphi: X_F \to Y_F$. \qed

### 5.3. Birational models

We note that the flop $X^\dagger_F$ has the same Picard number as $X_F$. There are two natural divisors on it. One is the pullback of $\mathcal{O}_{\mathbb{P}(W^\vee)}(1)$ via $X^\dagger_F \to \mathbb{P}(W^\vee)$, denoted by $H'$. The other one, denoted by $L'$, is the pullback of $\mathcal{O}_{\mathcal{U}_\omega^\vee}(1)$ via $X^\dagger_F \to X^\dagger_F \subseteq \mathbb{P}(\mathcal{U}_\omega^\vee(1))$. Set $\theta := (\varphi^\dagger)^{-1} \circ \varphi^\dagger$.

**Lemma 5.9.** For the birational map $\theta: X^\dagger_F \dasharrow X_F$, the matrix representation of $\theta_*: N^1(X^\dagger_F) \to N^1(X_F)$ with respect to $\{L', H'\}$ and $\{L, H\}$ is given by

$$[\theta_*] = \begin{bmatrix} 5 & 1 \\ -6 & -1 \end{bmatrix}.$$
Proof. By (24) and $F = \tilde{S}^\vee(1)$, we see that $\theta_*H' = L - H$. For the other divisor, we set $\theta_*L' = aL + bH$. To apply Lemma 2.9, we need to compute the intersection numbers $L' \cdot H^2$ and $L^2 \cdot H'$. Note that $\theta$ is not an isomorphism because only $X^+_F$ has a fibration over $\mathbb{P}^2$.

From the commutative diagram (22) and the fact that $\tilde{X}_F, X^+_F$ and $X^+_F$ are birational, we can reduce the computation to the birational model $\tilde{X}_F$, which is induced by the dual morphism of (27). We write $\alpha$ and $\xi$ in $A^1(\tilde{P})$ for the pullbacks of the hyperplane classes on $\mathbb{P}(\mathcal{V}^\vee)$ and $\mathbb{P}(W^\vee)$ respectively. Then, applying Proposition 2.8, we find that

$$L^k \cdot H^{3-k} = \int_{\tilde{P}} c_{k+1}(f^*(\wedge^2 \mathcal{T}')) - \mu^*(\mathcal{U}_\sigma(-1))) \cdot \alpha^{3-k}. \tag{30}$$

Note that we have the total Chern class $f^*c(\wedge^2 \mathcal{T}') = 1 + 2\alpha + 2\alpha^2$ and the Chow ring $A(\tilde{P}) = \mathbb{Z}[\alpha, \xi]/(\alpha^4, \xi^2 - \alpha \xi)$ by the Euler sequence of $\mathbb{P}(\mathcal{V}^\vee)$ and $\tilde{P} = \mathbb{P}(\mathcal{V}^\vee)(\mathcal{O}(1) \oplus \mathcal{O})$ respectively. It is easily seen that $L' \cdot H^2 = 14$ and $L^2 \cdot H' = 28$. By Lemma 2.9 and Table 11 we get that

$$\begin{cases} 28 = 40a^2 + 48ab + 13b^2, \\ 14 = 16a + 11b. \end{cases}$$

Then the only solution is $(a, b) = (5, -6)$ because $ab < 0$, which completes the proof. □

We can now prove Theorem 5.1.

Proof of Theorem 5.1. First, we claim that $X_\mathcal{E} \to \mathbb{P}^2$ is an elliptic fibration. To see this, we let $X_\mathcal{E} \to P \to \mathbb{P}^2$ be the Stein factorization. By the adjunction formula and generic smoothness, we know that a general fiber of $X_\mathcal{E} \to \mathbb{P}^2$ is a disjoint union of smooth elliptic curves and is linearly equivalent to the cycle $L^2_\mathcal{E}$. Let $d$ be the degree of $P \to \mathbb{P}^2$ and $C$ a connected component of the general fiber. Since $C \subseteq X_\mathcal{E} \subseteq \text{Gr}(2, 4) \times \mathbb{P}^2$, we get the closed embedding

$$C \hookrightarrow \pi_\mathcal{E}(C) \subseteq D_2(\sigma) \subseteq \text{Gr}(2, 4) \hookrightarrow \mathbb{P}^5.$$  

Notice that $d(C \cdot H_\mathcal{E}) = L^2_\mathcal{E} \cdot H_\mathcal{E} = 5$ as we have seen in Proposition 2.10. On the other hand, $C \cdot H_\mathcal{E} = \pi_\mathcal{E}(C) \cdot \mathcal{O}_\mathbb{P}^5(1) \geq 3$ since $\pi_\mathcal{E}(C) \subseteq \mathbb{P}^5$ is an elliptic curve. Hence $d = 1$ and $P \to \mathbb{P}^2$ is a birational finite morphism, so it is an isomorphism. Thus $X_\mathcal{E} \to \mathbb{P}^2$ has connected fibers. Similarly, $X^+_\mathcal{E} \to \mathbb{P}^2$ is also an elliptic fibration. We note that the natural projection $\pi_\mathcal{E}(\mu^*(\mathcal{U}_\sigma(1))) \to \mathbb{P}^2$ corresponds to the complete linear system $|\mathcal{O}_\mu(\mathcal{U}_\sigma(1))(-1)|$ and $L^3 = 47, H_\mathcal{E}^3 = 5$ by (30). Hence a general fiber of $X^+_\mathcal{E} \to \mathbb{P}^2$ is linearly equivalent to the cycle $(L' - H)^2$ and $(L' - H)^2 \cdot H' = 5$. The rest is the same as the case $X_\mathcal{E} \to \mathbb{P}^2$.

The (iii) and (iv) now follow from Propositions 5.5, 5.8 and Lemma 5.9. The (i) and (iv) follow from the construction of $\pi_\mathcal{F}$ and Proposition 2.10. □

Now the proof of Theorem 1.2 is completed by Theorems 3.1, 4.1 and 5.1.

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