Quantization via hopping amplitudes: Schrödinger equation and free QED

L. Polley

FB Physik
Oldenburg University
26111 Oldenburg, Germany

E-mail: polley@uni-oldenburg.de

April 1, 2022

Abstract

Schrödinger’s equation with scalar and vector potentials is shown to describe “nothing but” hopping of a quantum particle on a lattice; any spatial variation of the hopping amplitudes acts like an external electric and/or magnetic field. The main point of the argument is the superposition principle for state vectors; Lagrangians, path integrals, or classical Hamiltonians are not (!) required. Analogously, the Hamiltonian of the free electromagnetic field is obtained as a twofold continuum limit of unitary hopping in $\mathbb{Z}(N)$ link configuration space, if gauge invariance and $\mathcal{C}$ and $\mathcal{P}$ symmetries are imposed.

PACS: 03.65.Bz,03.70.+k,11.15.Ha
1 Introduction

For at least two standard quantum systems, canonical quantization (or other classical-to-quantum substitution rules) can be avoided; it can be replaced by an intrinsic quantum mechanical consideration of “hopping” in a discrete configuration space. This only requires to interpret a familiar tool from model building—hopping amplitudes—as a first-principle concept.

Hopping amplitudes have a long tradition particularly in solid-state theory [1,2]. On a fundamental level they have been used in lattice gauge theory [3,4] for discretizing (not avoiding) path-integral actions. More recently, in the field of quantum computation, hopping parameters are being used as collision constants in unitary cellular automata [5,6] designed for efficient simulation of the Schrödinger equation [6] or 1-photon and Weyl equation [3]. These latter applications differ in a crucial way from the viewpoint taken here, by assuming locality in conjunction with a finite, irreducible time step. It has proven to be a major challenge to design algorithms satisfying that computational requirement. Apart from technical complications, however, unitary cellular automata in some cases require configuration spaces larger than the physical ones. For example, local hopping rules in $d$ spatial dimensions are found to require $2^d$-component wave functions [3]. Consequently, a real spinless particle (as opposed to its computer simulation) can have a unitary and local equation of motion only with respect to continuous time.

Hopping amplitudes can do more than approximate or discretize processes originally defined otherwise. They necessarily emerge as coefficients of a superposition when a particle is prepared in a position eigenstate. The crucial axiom here is that the state of a quantum particle is completely specified by a position at one instant of time. To illustrate the idea, consider a particle confined to a 1-dimensional array of discrete positions at a spacing $a$. Let us work in the Heisenberg picture and denote by $|n,t\rangle$ the eigenstate of position $x = na$ at time $t$.

To prepare a position $n$ at time $t$ means to prepare a state with an uncertain position at time $t + dt$, because any motional information is lacking from $|n,t\rangle$. For $dt$ small enough, the uncertainty only relates to positions $n$, $n + 1$, and $n - 1$. Furthermore, $n + 1$ and $n - 1$ will occur symmetrically if we assume the symmetries of a free particle. Thus

$$|n,t\rangle = \alpha|n,t + dt\rangle + \beta|n + 1,t + dt\rangle + \beta|n - 1,t + dt\rangle$$ (1)

where $\alpha$ and $\beta$ are some numbers dependent on the size of the time step. For
\[ dt \rightarrow 0 \] we must have \( \alpha \rightarrow 1 \) and \( \beta \rightarrow 0 \), hence

\[ \begin{align*}
\alpha &= 1 + \alpha_1 dt + O(dt^2) \\
\beta &= \beta_1 dt + O(dt^2)
\end{align*} \]

Thus the basic hopping equation (1) converges to the differential equation

\[ -\frac{d}{dt}|n,t\rangle = \alpha_1 |n,t\rangle + \beta_1 |n+1,t\rangle + \beta_1 |n-1,t\rangle \quad (2) \]

We now use the statistical interpretation of the scalar product. From

\[ \langle n,t|n',t\rangle = \delta_{n,n'} \]

we find by differentiating with respect to \( t \) and using (2) that the coefficients \( \alpha_1 \) and \( \beta_1 \) must be purely imaginary. Finally, we consider a general state vector in the Heisenberg picture,

\[ |\psi\rangle = \sum_n \psi(n,t) |n,t\rangle \quad (3) \]

We take \( d/dt \), use (2), put \( x = na \), and reexpress \( \alpha_1 \) and \( \beta_1 \) by

\[ U = (\alpha_1 + 2\beta_1) i\hbar \quad \frac{1}{2m} = \frac{a^2 \beta_1}{i\hbar} \]

Thus we find

\[ i\hbar \frac{d}{dt}\psi(x,t) = U\psi(x,t) - \frac{\hbar^2}{2m} \left( \psi(x+a,t) + \psi(x-a,t) - 2\psi(x,t) \right) \]

This equation converges to the free Schrödinger equation in the continuum limit \( a \rightarrow 0 \).

In Section 2, the hopping-parameter description of a Schrödinger particle is discussed in full generality. Hopping amplitudes will not be restricted to next neighbours, and it will only be assumed that the hopping amplitudes realise the full translational and cubical symmetries of the lattice in \( O(1/a^2) \) while any inhomogeneities in the hopping process are at most of \( O(1/a) \). Then a (trivial) renormalization scheme exists for the continuum limit \( a \rightarrow 0 \) which leads to the standard nonrelativistic Schrödinger equation, with a vector potential and a scalar potential.

In Section 3, the hopping-parameter approach is applied to quantum electrodynamics without charges and currents. This requires the discretization
of both, the values of a field \( u(x) \) and its spatial variable \( x \). The reader of section 3 is assumed to be somewhat familiar with lattice gauge theory \([4]\). In fact, the model considered in this section is a Hamiltonian version of the intensively studied \( Z(N) \) lattice gauge theory \([7]\). The Hamilton operator of the electromagnetic field is recovered in the twofold limit of \( N \to \infty \) and zero lattice spacing. Section 4 contains some concluding remarks.

2 Schrödinger particle in 3 dimensions

Consider a simple cubic lattice where \( \vec{x} = a\vec{n} \) is the position vector of a site, \( a \) is the lattice spacing, and \( \vec{n} \) an integer vector. The most general hopping equation for a single-component wave function as defined in (3) is

\[
\frac{i\hbar}{\hbar} \frac{d}{dt} \psi(\vec{x}, t) = \sum_{\vec{n}} \kappa(\vec{x}, \vec{n}, t) \psi(\vec{x} + a\vec{n}, t)
\]

The factor of \( i\hbar \) is only cosmetic, since the hopping parameters \( \kappa(\vec{x}, \vec{n}) \) can be any complex numbers, so far. Conservation of probability requires

\[
\kappa(\vec{x} - a\vec{n}, \vec{n}, t) = \kappa(\vec{x}, -\vec{n}, t)
\]

An important case of reference is that of a free particle, characterized by hopping parameters with the full symmetry of the lattice. Then \( \kappa(\vec{x}, \vec{n}, t) = \kappa_0(\vec{n}) \) because of translational invariances. Cubic symmetry implies

\[
\kappa_0(\vec{n}) = \kappa_0(-\vec{n})
\]

so that all \( \kappa_0(\vec{n}) \) are real because of (3). Most importantly, the symmetry also implies \( \sum_{\vec{n}} \kappa_0(\vec{n}) n_i n_j \propto \delta_{ij} \). A convenient parametrization is

\[
\sum_{\vec{n}} \kappa_0(\vec{n}) n_i n_j = -\frac{\hbar^2}{ma^2} \delta_{ij}
\]

The reduced parameter \( m \) will be identified as the particle mass later on; the sign of \( m \) is discussed in the Conclusions. In general, the sum in equation (7) need not converge. Assuming convergence here is the basis for the non-relativistic physics as it emerges in the form of the Schrödinger equation in the continuum limit.
To recover the Schrödinger equation, we Taylor-expand the displaced wave functions on the RHS of (4),

$$\psi(\vec{x} + a\vec{n}, t) = \psi(\vec{x}, t) + a n_i \nabla_i \psi(\vec{x}, t) + \frac{1}{2} a^2 n_i n_j \nabla_i \nabla_j \psi(\vec{x}, t) + \mathcal{O}(a^3)$$  (8)

Again, let us consider a free particle first. Inserting $\kappa(\vec{x}, \vec{n}, t) = \kappa_0(\vec{n})$ in (4) and using (8), (6), and (7) we find

$$i\hbar \frac{d}{dt} \psi(\vec{x}, t) = E_0 \psi(\vec{x}, t) - \frac{\hbar^2}{2m} \nabla \cdot \nabla \psi(\vec{x}, t) + \mathcal{O}(a)$$  (9)

where $E_0 = \sum_{\vec{n}} \kappa_0(\vec{n})$ is certainly infinite but does not affect the shape of the wavefunctions. In contrast, the parameter $m$ determines the particle mass and must be finite, as anticipated in definition (7).

Now we “turn on” deviations of the hopping parameters from $\kappa_0(\vec{n})$. Let us put

$$\kappa(\vec{x}, \vec{n}, t) = \kappa_0(\vec{n}) + \kappa_1(\vec{x}, \vec{n}, t)$$  (10)

Again, we insert (8) in (4). The multiplicative terms on the RHS of (4) now are $E_0 \psi(\vec{x}, t) + \sum_{\vec{n}} \kappa_1(\vec{x}, \vec{n}, t) \psi(\vec{x}, t)$. The inhomogeneous term can be rewritten as

$$\frac{1}{2} \sum_{\vec{n}} (\kappa_1(\vec{x}, \vec{n}, t) + \kappa_1(\vec{x}, -\vec{n}, t)) \psi(\vec{x}, t)$$

Using (3) and expanding the ensuing displaced argument, we obtain the following form of the multiplication operator:

$$\frac{1}{2} \sum_{\vec{n}} (\kappa_1(\vec{x}, \vec{n}, t) - \overline{\kappa_1(\vec{x}, \vec{n}, t)}) + \frac{1}{2} \sum_{\vec{n}} a \vec{n} \cdot \nabla \kappa_1(\vec{x}, \vec{n}, t) + \mathcal{O}(a^2 \kappa_1)$$

This shows that for a finite $\vec{x}$-dependent contribution, the real part of $\kappa_1$ must be of $\mathcal{O}(1)$ while the imaginary part can be of $\mathcal{O}(1/a)$. Hence, if we define a vector potential

$$\vec{A}(\vec{x}, t) = \frac{ma}{e\hbar} \sum_{\vec{n}} \vec{n} \Im \kappa_1(\vec{x}, \vec{n}, t)$$  (11)

then the multiplicative terms of (4) take the form

$$\left( E_0 + \sum_{\vec{n}} \Re \kappa_1(\vec{x}, \vec{n}, t) \right) \psi(\vec{x}, t) + i \frac{e\hbar}{2m} \left( \nabla \cdot \vec{A}(\vec{x}, t) \right) \psi(\vec{x}, t)$$  (12)
The gradient terms on the RHS of (4) can be written as

\[ \frac{a}{2} \tilde{\nabla} \psi(\vec{x}, t) \cdot \sum_{\vec{n}} \vec{n} \left( \kappa(\vec{x}, \vec{n}, t) - \kappa(\vec{x}, -\vec{n}, t) \right) \]

By (10) and (5) this is equal to

\[ \frac{a}{2} \tilde{\nabla} \psi(\vec{x}, t) \cdot \sum_{\vec{n}} \vec{n} \left( \kappa_{1}(\vec{x}, \vec{n}, t) - \kappa_{1}(\vec{x} - a\vec{n}, \vec{n}, t) \right) \]

The displacement of \(\vec{x}\) in \( \kappa_{1}(\vec{x} - a\vec{n}, \vec{n}, t) \) produces a term of higher order in \(a\) which can be neglected in the limit \(a \to 0\). Thus the only relevant contribution to the gradient terms comes from the imaginary part of \(\kappa_{1}(\vec{x}, \vec{n}, t)\) and is of the form

\[ i \frac{\hbar}{m} \left( \tilde{\nabla} \psi(\vec{x}, t) \right) \cdot \vec{A}(\vec{x}, t) \] (13)

where \(\vec{A}(\vec{x}, t)\) is the same as in (11).

With inhomogeneities of \(O(1)\) in the real part, and of \(O(1/a)\) in the imaginary part, it is clear that the double-gradient terms of equation (4) are the same as in the free-particle case (1). Collecting all the terms discussed above, we recover from (4) the general, nonrelativistic Schrödinger equation

\[ i \frac{\hbar}{m} \frac{\partial}{\partial t} \psi(\vec{x}, t) = \frac{1}{2m} \left( \frac{\hbar}{i} \tilde{\nabla} - e \vec{A}(\vec{x}, t) \right) \psi(\vec{x}, t) + U(\vec{x}, t) \psi(\vec{x}, t) \] (14)

with the vector potential of equation (11) and the scalar potential

\[ U(\vec{x}, t) = E_{0} + \sum_{\vec{n}} \Re \kappa_{1}(\vec{x}, \vec{n}, t) - \frac{e^{2}}{2m} \vec{A}(\vec{x}, t)^{2} \] (15)

In canonical quantization, the prescription is to identify \(U(\vec{x}, t)\) and \(\vec{A}(\vec{x}, t)\) with the corresponding functions of the classical Hamiltonian. This amounts to an extrapolation into the microscopic domain. The corresponding procedure in the present context is as follows. By Ehrenfest’s theorem, eq. (14) will reproduce the classical equations of motion for the centre of a wave packet in the limit \(\hbar \to 0\). The classical \(U(\vec{x}, t)\) and \(\vec{A}(\vec{x}, t)\) then coincide with those in the Schrödinger equation. Thus, if desired, \(U(\vec{x}, t)\) and \(\vec{A}(\vec{x}, t)\) can be extrapolated as with canonical quantization.

In concluding the section, it should be noted that the order-of-magnitude assumptions for the hopping parameters depend on the further assumption
that no dramatic cancellations occur between $\kappa(\vec{x}, \vec{n}, t)$ for different $\vec{n}$. Of course, those cancellations would require some extra reason for a fine-tuning. In the absence of a reason, the assumptions describe the most general and, hence, the most likely set of parameters consistent with the constraints.

3 Free electromagnetic field

This section is to demonstrate that “unitary hopping” can be a useful concept also for quantum field theories. We here consider source-free $U(1)$ gauge theory. Its Hamilton operator in the temporal gauge is an $\infty$-dimensional version of (14). A “hopping” scenario requires the configuration space to be discrete. Thus local gauge invariance will have to be discretized, too. In case of $U(1)$ this can be done in a way that preserves an exact local gauge group, namely $Z(N)$, whose limit $N \to \infty$ reproduces $U(1)$.

In lattice gauge theory, a gauge field lives on the links between next-neighbour lattice sites. A link can be specified by the site $\vec{s} = (n_x, n_y, n_z)$ from which it emanates in a positive direction, and by the corresponding $k = 1, 2, 3$. In $Z(N)$ gauge theory the link variables are phase factors of the form

$$e^{2\pi il/N} \quad l = 0, 1, \ldots, N - 1 \quad (16)$$

They are related to the electromagnetic vector potential $A(\vec{s}, k)$, integrated along the link, by

$$\exp (2\pi il/N) = \exp (iaeA/\hbar) \quad (17)$$

Thus a $Z(N)$ gauge field configuration is determined by the numbers

$$l(\vec{s}, k) \equiv l(n_x, n_y, n_z, k) \quad n_i = 0, \pm 1, \pm 2, \ldots \quad k = 1, 2, 3 \quad (18)$$

We shall indicate by omitting the arguments $\vec{s}$ and $k$ that we mean the configuration as a whole.

The Hamiltonian will be postulated below to be invariant under charge conjugation $C$, and under space inversion $P$ about any point $\vec{s}_0$. As it follows from the relation (17) to the vector potentials (see also (3)), $C$ and $P_{\vec{s}_0}$ are characterized by their action on the link variables,

$$C l(\vec{s}, k) = -l(\vec{s}, k) \quad (19)$$

$$P_{\vec{s}_0} l(\vec{s}, k) = -l(2\vec{s}_0 - \vec{s} - \hat{k}, k) \quad (20)$$
We also postulate invariance under local $Z(N)$ gauge transformations. These are characterized by a number $g(\vec{s}) = 0, 1, \ldots, N - 1$ on each lattice site. The link field configuration transforms according to

$$l'(\vec{s}, k) = l(\vec{s}, k) + g(\vec{s} + \hat{k}) - g(\vec{s})$$

The elementary gauge-invariant construct on a time slice is the plaquette variable

$$p(\vec{s}, i, k) = l(\vec{s}, i) + l(\vec{s} + \hat{i}, k) - l(\vec{s} + \hat{k}, i) - l(\vec{s}, k) \quad (21)$$

Gauge-invariant, too, is any shift of a link variable; in particular,

$$l(\vec{s}, k) \to l(\vec{s}, k) \pm 1 \quad \text{if and only if} \quad l'(\vec{s}, k) \to l'(\vec{s}, k) \pm 1$$

The gauge field is quantized by assigning a probability amplitude $\psi(l, t)$ to each link-field configuration $l$. For this “wavefunction” the general form of a unitary-hopping equation is

$$i\hbar \frac{d}{dt} \psi(l, t) = \sum_{\Delta l} \kappa(l, \Delta l) \psi(l + \Delta l, t) \quad (22)$$

**Gauge invariance** of the process requires, in the notation of (21),

$$\kappa(l, \Delta l) = \kappa(p, \Delta l)$$

**Locality** of link interactions is not as uniquely defined—a fact being utilized with the “improved actions” of numerical lattice gauge theories [10]. We shall only consider the simplest realization of locality, assuming

- Link-changing processes are independent on different links.
- A plaquette can influence a change on its own links, at most.

These assumptions correspond to a pre-relativistic, purely spatial notion of locality—no reference whatsoever is made to the phenomenon of light. By the assumption of independence, a change on $k$ links within the same time interval $dt$ will come with a factor of $(dt)^k$ and will contribute to the time derivative in equation (22) only for $k = 1$. Thus the sum over all link-changes $\Delta l$ reduces to a sum over one-link changes. For further simplification, we only consider a change by one unit, corresponding to nearest-neighbour hopping in configuration space. Thus (22) takes the form

$$i\hbar \frac{d}{dt} \psi(l, t) = \sum_{\text{links} \hat{s}, \hat{i}} \sum_{\pm} \kappa_{\pm}(p; \vec{s}, i) \psi(l \pm u_{\vec{s}, i}, t) \overset{\text{def}}{=} H \psi(l, t) \quad (23)$$
where
\[ u_{\vec{s},i} = \begin{cases} 
1 & \text{on link } \vec{s},i \\
0 & \text{elsewhere} 
\end{cases} \]

We intend to Taylor-expand the wavefunction. Instead of the derivative \( \partial/\partial l \) on each link we prefer to use the lattice version of the functional derivative \( \delta/\delta A \) with respect to the vector potential. \( l \) and \( A \) are related through equation (17). Hence, \( \partial/\partial l \) equals the partial derivative \((2\pi\hbar/eNa)\delta/\delta A\).

Now \( \partial/\partial A \) can be expressed by the functional derivative \( \delta/\delta A \) essentially by introducing factors so that in the characteristic relation \( \partial A(\vec{s},i)/\partial A(\vec{s},i') = \delta_{\vec{x}\vec{y}}\delta_{ii'} \) the \( \delta_{\vec{x}\vec{y}} \) is changed into the lattice delta function \( a^{-3}\delta_{\vec{x}\vec{y}} \). Thus,

\[ \frac{\partial}{\partial l(\vec{s},i)} = \frac{2\pi\hbar a^2}{eN} \frac{\delta}{\delta A(\vec{s},i)} \]

Expanding the wavefunction up to order \( a^4 \) we have

\[ \psi(l \pm u_{\vec{s},i}, t) = \psi(l, t) \pm \frac{2\pi\hbar a^2}{eN} \frac{\delta\psi(l, t)}{\delta A(\vec{s}, i)} + \frac{2\pi^2\hbar^2 a^4}{e^2N^2} \frac{\delta^2\psi(l, t)}{\delta A(\vec{s}, i)^2} \]  

(24)

The first-derivative term is immediately discarded if we postulate that the Hamiltonian be invariant under space inversion \( P \) (cf. (20)). This is because the plaquette variables in the hopping amplitudes \( \kappa^\pm(p; \vec{s},i) \) are invariant under \( P \) whereas \( l \) and hence \( \partial/\partial A \) changes sign.

It remains to discuss the multiplicative terms of (23). To expand the hopping amplitudes in a power series in \( a \), we note [4, 7] that the magnetic flux density \( B_i = \frac{1}{2}\epsilon_{ijk}F_{jk} \) is related to the plaquette variable by

\[ \exp \left( i a^2 eF_{jk}(\vec{s})/\hbar \right) = \exp \left( 2\pi i p(\vec{s},j,k)/N \right) \]

Thus, at a given flux density of \( \mathcal{O}(1) \), the plaquette phase factor deviates from 1 only in \( \mathcal{O}(a^2) \), while the plaquette variable \( p \) is of \( \mathcal{O}(a^2N) \). To be on the safe side, we therefore expand the hopping amplitude as a function of \( a^2F_{ij} \) instead of \( p \). Furthermore, we invoke our locality postulates to restrict plaquettes with an influence on link \((\vec{s},i)\) to the four cases \( p(\vec{s},i,j) \) and \( p(\vec{s} - \hat{j},i,j) \) with \( j \neq i \). Thus, expanding \( \kappa^\pm(p; \vec{s},i) \) to \( \mathcal{O}(a^4) \) we obtain

\[ \kappa^0_{\pm}(\vec{s},i) + \frac{ea^2}{\hbar} \sum_{j \neq i} \left( \kappa^{(1)}_{\pm}(\vec{s},i,j)F_{ij}(\vec{s}) + \kappa^{(1)'}_{\pm}(\vec{s},i,j)F_{ij}(\vec{s} - \hat{j}) \right) \]  

(25)
where in the last term we have discarded any shift of $\vec{s}$ by $\hat{j}$ or $\hat{j}'$ as this would lead to an $O(a^5)$ contribution.

The $a^2$ terms of expression (23) must vanish if the Hamiltonian is to be invariant under charge conjugation. This is because $C$ (cf. (13)) reverses the values of both links and plaquettes, hence reverses the sign of the $a^2$ term in (25), while all remaining terms of (25) and also of (24) are $C$-invariant.

By translation invariance of the hopping process, all $\kappa$’s must be independent of the site vector $\vec{s}$. By invariance under reflections about a coordinate plane, $\kappa^{(2)}_\pm(\vec{s}, i, j, j')$ in the $F^2$ term of (25) must be proportional to $\delta_{jj'}$. Hence, by cubic rotational invariance, it must be independent of $i$. For the same reason, $\kappa^{(0)}_\pm(i)$ as the relevant coefficient of $\delta^2\psi(l, t)/\delta A(\vec{s}, i)^2$ must be independent of $i$.

Inserting in (23) the remaining terms of (24) and (25) we identify the Hamiltonian of free QED as

$$H = v + \frac{e^2 a^4}{\hbar^2} \sum_{j,j' \neq i} \kappa^{(2)}_\pm(\vec{s}, i, j, j') F_{ij}(\vec{s}) F_{ij'}(\vec{s})$$

where $v = \sum_{\vec{s}, i} (\kappa^{(0)}_+ + \kappa^{(0)}_-)$ is the vacuum energy and where

$$\frac{1}{\epsilon_0} = -\frac{4\pi^2 a}{e^2 N^2} (\kappa^{(0)}_+ + \kappa^{(0)}_-) \quad \frac{1}{\mu_0} = \frac{4e^2 a}{\hbar^2} (\kappa^{(2)}_+(1, 1) + \kappa^{(2)}_-(1, 1))$$

In the limit $a \to 0$ we put $\vec{x} = a \vec{s}$ and $d^3x = a^3$ to obtain the familiar form

$$H = v + \frac{\epsilon_0}{2} \int \vec{E}^2(\vec{x}) \, d^3x + \frac{1}{2\mu_0} \int \vec{B}^2(\vec{x}) \, d^3x$$

where

$$E_i(\vec{s}) = \frac{i\hbar}{\epsilon_0} \frac{\delta}{\delta A(\vec{s}, i)}$$

### 4 Conclusions

We have derived the Schrödinger equation for a nonrelativistic scalar particle and for the free electromagnetic field, starting out from the superposition
principle for state vectors, using the statistical interpretation, and exploiting spatial symmetries to a large extent. The ambition was to avoid any use of the distinctly non-quantal concept of trajectories, even in the path-integral sense.

In the case of a free particle, which has all the exploitable symmetries, the approach taken here should be compared with the general, group-theoretical approach to quantum mechanics as exposed, for example, in [11]. The main difference is that we found it unnecessary to consider any classical space-time symmetries (Galilei or Lorentz transformations). Rather, the structure of the dynamics follows from spatial symmetries together with the absence of motional information from states such as $|\vec{x},t\rangle$. That absence induces symmetries of the time evolution which, however, can be realized only by way of a superposition.

As we have seen, Taylor expansions led to 2nd order derivatives and, in the case of QED, to the $B^2$ magnetic energy in the Hamiltonian. The sign of the Taylor coefficients, though, must be determined by extra arguments. For the mass parameter $m$ in equation (7), it is a matter of convention whether kinetic energies are always taken as positive or always negative, so both signs of $m$ would seem to make physical sense. A similar remark applies to the case of free QED, except for the relative sign of the parameters $\epsilon_0$ and $\mu_0$ in (26). Here an additional assumption is required, such as the existence of a ground state, to recover the positive phenomenological sign.

For the definition of the mass in (7), it was essential that a free particle find identical hopping conditions on every site of the lattice. But this is also what characterizes the lattice as a cartesian coordinate system. In case of QED, a cartesian structure is comprised in the local $Z(N)$ gauge invariance. Thus the unitary-hopping scenario may explain why cartesian coordinates play such a preferred role in a wide range of quantum systems [8].

Within the “physical” subspace of locally gauge-invariant states, the Hamiltonian dynamics of the electromagnetic field as described by (27) is automatically Lorentz invariant. This is quite remarkable since we derived the dynamics from quantum-mechanical principles in which the roles of space and time are initially very different. A similar observation was made by Bialynicki-Birula [3] with respect to the Weyl equation.

References
[1] W. Heisenberg, *Zeitschrift für Physik*, 49: 619, 1928.

[2] D. Baeswyl et al. (eds.), *The Hubbard Model*, Proceedings of the Conference on the Mathematics and Physics of the Hubbard Model (San Sebastian), Plenum, New York 1995.

[3] J. Kogut, L. Susskind, *Physical Review D*, 11: 395, 1975.

[4] M. Creutz, *Quarks, Gluons and Lattices*, Cambridge 1983.

[5] I. Bialynicki-Birula, *Physical Review D*, 49: 6920, 1994.

[6] B. Boghosian, W. Taylor, *Physical Review E*, 57: 54, 1998; *Physica D*, 120: 30, 1998; *International Journal of Modern Physics*, 8: 705, 1997.

[7] M. Creutz, L. Jacobs, C. Rebbi, *Physical Review D*, 20: 1915, 1979.

[8] N. H. Christ, T. D. Lee, *Physical Review D*, 22: 939, 1980.

[9] K. Ishikawa, G. Schierholz, M. Teper, *Zeitschrift für Physik C*, 19: 327, 1983.

[10] T. DeGrand et al. (eds.), *Lattice ’98*, conference proceedings, *Nuclear Physics B* (Proc. Suppl.), 73, 1999; Section L.

[11] A. Bohr, O. Ulfbeck, *Reviews of Modern Physics*, 67: 1, 1995.