Random Function Priors for Correlation Modeling

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\textbf{Abstract}

The likelihood model of high dimensional data \(X_n\) can often be expressed as \(p(X_n|Z_n, \theta)\), where \(\theta := (\theta_k)_{k \in [K]}\) is a collection of hidden features shared across objects, indexed by \(n\), and \(Z_n\) is a non-negative factor loading vector with \(K\) entries where \(Z_{nk}\) indicates the strength of \(\theta_k\) used to express \(X_n\). In this paper, we introduce random function priors for \(Z_n\) for modeling correlations among its \(K\) dimensions \(Z_{n1}\) through \(Z_{nK}\), which we call \textit{population random measure embedding} (PRME). Our model can be viewed as a generalized paintbox model (Broderick et al., 2013) using random functions, and can be learned efficiently with neural networks via amortized variational inference. We derive our Bayesian nonparametric method by applying a representation theorem on separately exchangeable discrete random measures.

1. Introduction

Let \(X = [X_1, \ldots, X_N]\) be a group of exchangeable high dimensional observations, where \(X_n \in \mathbb{R}^d\). In this paper, we assume \(X\) is generated by the model

\[
p(X) = \int p(X, Z, \theta) dZ d\theta
= \int p(Z) p(\theta) \prod_{n \in [N]} p(X_n|Z_n, \theta) dZ d\theta,
\]

where \(p(X_n|Z_n, \theta)\) is a likelihood model conditioned on latent features \(\theta := (\theta_k)_{k \in [K]}\) that are shared across the population. \(Z_n := \{Z_{n1}, \ldots, Z_{nK}\}\) is a non-negative vector for the \(n\)th observation, where \(Z_{nk}\) determines the extent to which \(\theta_k\) is used to express \(X_n\). For example, in topic models (Blei et al., 2003), \(Z_n\) is a discrete distribution over topics, where \(Z_{nk}\) represents the proportion of words in document \(n\) sampled from topic \(k\). In sparse factor models (Griffiths & Ghahramani, 2011), \(Z_n\) is a binary such that latent feature \(\theta_k\) contributes to the likelihood if and only if \(Z_{nk} = 1\). We generically refer to \(Z_n\) as a “non-negative feature loading vector.” For exchangeable \(X\), it is often assumed the \(Z_n\) are exchangeable as well. If we take \(Z\) as a feature loading matrix with \(Z_n\) as its rows, then \(Z\) is \textit{row exchangeable}. By de Finetti’s theorem, we can represent

\[
p(Z) = \int \prod_{n \in \mathbb{N}} p(Z_n|\zeta) p(\zeta) d\zeta,
\]

for some random object \(\zeta\). (We let \(N = \infty\) in order to apply de Finetti’s theorem.) The goal of this paper is to model complex correlations among entries of \(Z_n\). Following a common practice, we put an independent prior on \(\theta\), \(p(\theta) = \prod_k p(\theta_k)\) and focus on modeling \(p(Z)\).

A straightforward way to model correlation structure is to let \(p(Z_n|\zeta)\) be a \textit{parametric} exponential family model. By defining the mean/natural parameters for the model, one can handle correlations to various degrees. For example, \(Z_n\) may follow a log-normal distribution (Lafferty & Blei, 2006), where correlations are modeled through a covariance matrix. However, exponential family models (Wainwright et al., 2008) can be rigid, since the number of free parameters is fixed for a certain \(K\). To get a more flexible model, it is tempting to consider higher-order moments \(\mathbb{E}[Z_{n1}^{\otimes M}]\) for a large \(M\) up to \(K\), where \(u^{\otimes M}\) denotes an \(M\)-th order outer product of a vector \(u\). But in this case the number of free parameters increases exponentially, leading to intractable inference.

In this paper, we use an alternative \textit{Bayesian nonparametric} method to model \(Z_n\) as an outcome of random functions, which can handle complex correlations even when \(K\) and \(M\) go to infinity. Moreover, those random functions can be learned efficiently through inference/decoder networks via amortized variational inference (Kingma & Welling, 2013). In principle, arbitrarily complex neural networks can be applied to model correlations in our setting.

To give intuition why random function priors are powerful, we first show in Figure 1 an existing feature paintbox model for binary \(Z_n\) that illustrates how to model arbitrarily...
complex correlations using binary random functions (Broderick et al., 2013). For simplicity, let $K = 3$. First, select a compact set $S$ in Euclidean space, on which we can define a uniform distribution. For example let $S = [0, 1]^2$. Then randomly partition $S$ into eight regions. Each partition represents a possible value for $Z_n = [Z_{n1}, Z_{n2}, Z_{n3}]$, as shown in Figure 1. Given the partition, we uniformly sample a point $u_n \sim U(S)$ and assign $Z_n$ be the value defined by the region in which $u_n$ falls. Thus, we translate the problem of modeling distributions on $Z_n$ to modeling the random partition of $S$. Following the classic analogy, we call this a partition paintbox model (Kingman, 1978; Pitman, 2006). One can further factorize the partition paintbox into “feature paintboxes” (Broderick et al., 2013). According to Figure 1, each feature paintbox for the $k$-th feature is randomly partitioned into two regions denoted as $S_k$ (black) and $S_k^c$ (white). Let $Z_{nk} = 1(u_n \in S_k)$. One can check that the feature paintbox model is the equivalent to the partition paintbox model for arbitrary finite $K$. (Note that here $Z_n$ is a random indicator function.)

The feature paintbox model is redundant but flexible. The arbitrary order moment $E[\prod_{k \in J} Z_{nk}] = E[\text{vol}(\cap_{k \in J} S_k)]$ for any $J \subset [K]$ can be modeled once we have enough freedom for $S_k$. We summarize the generative process for the feature paintbox model in Algorithm 1 for arbitrary $K$, including $K = \infty$.

We propose a model that can be treated as a generalization of the feature paintbox model from binary to non-negative $Z$ according to a function $Z_{nk} = f_n(\vartheta_k)$. There are two key differences between our model and the feature paintbox model. First, we use data-specific random functions $f_n$, instead of points $u_n$, to represent each observation. Second, we use points $\vartheta_k$ from a Poisson process, instead of $S_k$, to index each latent feature. A nice property of our model compared to the paintbox model is that we can use deep learning to model $f_n$ through inference and decoder networks (Kingma & Welling, 2013), allowing for efficient amortized variational inference.

**Algorithm 1 Feature paintboxes model**

1: for $k \in [K]$ do
2: Generate a random subset $S_k \subset S$.
3: end for
4: Guarantee that $\sum_{k \in [K]} \text{vol}(S_k) < \infty$ almost surely.
5: for $n = 1, 2, \ldots$ do
6: Independently generate $u_n \sim U(S)$.
7: Let $Z_n = [Z_{n1}, \ldots, Z_{nK}]$. Set $Z_{nk} = 1(u_n \in S_k)$.
8: end for

In what follows, Section 2 sets up the problem of modeling $Z$ from a random matrix point of view. In Section 3, we embed $Z$ as a random measure and derive the functional form of $Z_{nk} = f_n(\vartheta_k)$ through a representation theorem. In Section 4, we present a concrete example for Bayesian nonparametric topic modeling together with its amortized variational inference algorithm, and show empirical results in Section 5. Finally, we discuss related work in Section 6 and conclude in Section 7.

2. Z as a random matrix?

We will rely on representation theorems to derive the functional form of our models. This usually works out by finding an infinite dimensional random object paired with an exchangeability assumption on that random object. The choice of random objects is the key step, and we will see below that it can be hard to derive an interesting model when choosing a bad random object.

Consider modeling $Z$ as a random matrix. Equation (2) above is one example that derives a mixture representation by assuming row exchangeability of $Z$. However, Equation (2) is uninformative in that, first, it does not tell us what random object $Z_n$ is, and second, it does not determine the connection between $Z_n$ and $\zeta$ through $p(Z_n | \zeta)$. Our discussion in Section 1 will show that this provides too much freedom to choose $\zeta$ and $p(Z_n | \zeta)$.

We further restrict $Z$ by assuming it is column exchangeable as well. This requires allowing both $N$ and $K$ to equal infinity. We call $Z$ separately exchangeable if it is both row and column exchangeable. Once $K = \infty$, we need to guarantee series convergence for rows. That is, $\sum_{k \in \mathbb{N}} Z_{nk} = \infty$ with probability 1, for any $n \in \mathbb{N}$. Row sum convergence is always considered necessary. (For example, in a topic model we want to normalize $Z_n$.) However, the following proposition says that when $Z$ is separately exchangeable, we will get an empty model even for a binary $Z$.

**Proposition 1.** An infinite binary matrix $Z$ (i) is separately exchangeable, and (ii) has finite row sums almost surely, if and only if $Z = 0$ almost surely.

**Proof (Sketch).** One can prove that $Z$ is a graphon model if
it is separately exchangeable (Hoover, 1979; Aldous, 1985; Orbanz & Roy, 2015). A graphon model satisfies finite row sums if and only if $Z = 0$.

When choosing a bad random object, one can either get a vacuous or an empty model through representation theorems. In the next section, we fix this problem by introducing a nice random object $\xi$ generated by embedding $Z$ as a random measure. Then we apply representation theorems on $\xi$.

### 3. Z as a random measure

#### 3.1. Population random measure embedding

In this section, we embed the random matrix $Z$ as a discrete random measure $\xi = \sum_{n,k} Z_{nk} \delta_{\tau_n, \sigma_k}$ on an infinite strip $[0, 1] \times \mathbb{R}_+$, where $(\tau_n)_{n \in \mathbb{N}} \subset [0, 1]$ distinguishes objects and $(\sigma_k)_{k \in \mathbb{N}} \subset \mathbb{R}_+$ distinguishes latent features. Both $(\tau_n)_{n \in \mathbb{N}}$ and $(\sigma_k)_{k \in \mathbb{N}}$ are random as well, and are not necessarily ordered. Note that $\xi$ preserves the matrix structure as demonstrated in Figure 2; the intersection points of horizontal/vertical dashed lines indexed by $(\tau_n)_{n \in \mathbb{N}}$ and $(\sigma_k)_{k \in \mathbb{N}}$ form an “equivalent class” of matrix $Z$ up to a re-ordering of rows and columns. The infinite strip is an abstract space introduced solely for applying representation theorems.

Next, we assume $\xi$ is separately exchangeable. That is, $\xi(T_1(A) \times T_2(B)) =_d \xi(A \times B)$ for any measure-preserving transformations $T_1, T_2$ on $[0, 1]$ and $\mathbb{R}_+$ separately for arbitrary Borel sets $A, B$. Even though the notion of separate exchangeability is different for $\xi$ than for random matrix $Z$, they are conceptually similar, since interchanging row/column indices will not affect the joint distribution. It turns out that we can represent $\xi$ precisely as follows:

**Proposition 2.** A discrete random measure $\xi$ on $[0, 1] \times \mathbb{R}_+$ is separately exchangeable if and only if

$$\xi = \sum_{n,k} f_n(\vartheta_k) \delta_{\tau_n, \sigma_k} + \sum_{m,k} g_m(\vartheta_k) \delta_{\rho_{mk}, \sigma_k},$$

almost surely for some random measurable functions $f_n, g_m \geq 0$ on $\mathbb{R}_+^2$, a unit rate Poisson process $\{(\vartheta_k, \sigma_k)\}$ on $\mathbb{R}_+^2$, and independent $U(0, 1)$ arrays $(\tau_n)$ and $(\rho_{mk})$.

**Proof.** This follows from the general representation theorem for separately exchangeable random measures on $[0, 1] \times \mathbb{R}_+$ (Kallenberg, 2006) by removing the non-atomic parts. Details are given in the appendix.

We briefly look at the two parts of this representation:

1. $\sum_{n,k} f_n(\vartheta_k) \delta_{\tau_n, \sigma_k}$: This is the part we are interested in. Correlations are learned through coupling of random functions $f_n$ with a Poisson process.

2. $\sum_{m,k} g_m(\vartheta_k) \delta_{\rho_{mk}, \sigma_k}$: This part is less important since the double index in $\rho_{mk}$ means each row (object) slice $\xi(\rho_{mk}, \cdot)$ contains at most one atom. We drop this part in our model.

Thus, we can represent $\xi = \sum_{n,k} f_n(\vartheta_k) \delta_{\tau_n, \sigma_k}$ as a coupling of a 2d Poisson process $(\vartheta_k, \sigma_k)$ and random functions $f_n$. As mentioned in Section 1, we derive $Z_{nk} = f_n(\vartheta_k)$. Since we model the entire population through $Z$ by a random measure embedding, we call our model *population random measure embedding (PRME)*.

#### 3.2. Construction via completely random measures

Once we have a representation for $Z_{nk}$, we still need to guarantee series convergence $\sum_k Z_{nk} = \sum_k f_n(\vartheta_k) < \infty$. This is not obvious, since $\vartheta_k$ spans uniformly on $\mathbb{R}_+$. One remedy is to introduce a transformation $\tilde{\vartheta}_k = T(\vartheta_k)$ that maps almost every $\tilde{\vartheta}_k$ close to zero, leaving only finite number of $\tilde{\vartheta}_k$ above any positive threshold. The method to introduce such a transformation $T$ is via completely random measures (CRM) (Kingman, 1967). In the appendix, we show the construction of $T$ via CRMs. In addition, we show that the well-known Indian buffet process (Ghahramani & Griffiths, 2006; Griffiths & Ghahramani, 2011), its extensions (Teh & Gurur, 2009), hierarchical Dirichlet processes (HDP) (Teh et al., 2005) and the discrete infinite logistic normal distribution (DILN) (Paisley et al., 2012b) are instances of population random measure embeddings. However, these models have restrictions in their model capacity. For example, (Paisley et al., 2012b) relies on a linear kernel to model

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**Figure 2.** Decoupling a separately exchangeable discrete random measure $\xi$ into two parts.
correlations and there is no obvious extension to complex kernels. As we will show, a PRME can be more flexible by using nonlinear object-specific functions \( f_n \) such as deep neural networks.

4. An illustration on topic modeling

4.1. The model

In a topic model, we use \( Z_n \) to represent an un-normalized discrete distribution over topics, where \( Z_{nk} \) is the strength of topic \( k \) for document \( n \). We use a PRME to model \( Z_{nk} \), with the following construction,

\[
Z_{nk} \sim \text{Gamma}(\beta p_k, \exp(f(h_n, \ell_k))),
\]

\[
p_k = V_k \prod_{k' \neq 1} (1 - V_{k'}), \quad V_k \sim \text{Beta}(1, \alpha),
\]

\[
h_n \sim \mathcal{N}(0, aI), \quad \ell_k \sim \mathcal{N}(0, bI),
\]

\[
f(h_n, \ell_k) \sim \mathcal{N}(\mu_f(h_n, \ell_k), \sigma_f^2(h_n, \ell_k)).
\]

(4)

We now explain how Equation (4) relates to the original PRME equation \( Z_{nk} = f_n(\vartheta_k) \), via four steps.

1. \[ f_n(\vartheta_k) \rightarrow f(h_n, \vartheta_k) \]

   We use a parametric function \( f(h_n, \cdot) \) to represent \( f_n(\cdot) \), where \( f \) is a random function, and \( h_n \) is an observation-specific random vector. This decomposition is necessary, since we model \( f \) as a normal distribution parameterized by \( \text{decoder networks} \mu_f, \sigma_f^2 \), and \( h_n \) as the output of an inference network.

2. \[ f(h_n, \vartheta_k) \rightarrow f(h_n, \tilde{\vartheta}_k) \]

   We transform \( \vartheta_k = T(\vartheta_k) \) by transforming the original Poisson process \( (\vartheta_k, \sigma_k) \) to a hierarchical Gamma process (Teh et al., 2005; Wang et al., 2011). Then we use a stick-breaking construction over \( \tilde{\vartheta}_k \) (Sethuraman, 1994), where \( \vartheta_k \sim \text{Gamma}(\beta p_k, 1) \). \( \beta \) is a hyperparameter and \( p_k \) is generated by the second line of Equation (4).

3. \[ f(h_n, \tilde{\vartheta}_k) \rightarrow f(h_n, \tilde{\vartheta}_k, \ell_k) \]

   We augment \( \tilde{\vartheta}_k \) to \( (\tilde{\vartheta}_k, \ell_k) \) to introduce extra randomness via \( \ell_k \). This operation is equivalent to augmenting the original 2d Poisson process \( (\vartheta_k, \sigma_k) \) to a higher dimensional Poisson process \( (\vartheta_k, \sigma_k, \ell_k) \).

4. \[ f(h_n, \tilde{\vartheta}_k, \ell_k) \rightarrow \tilde{\vartheta}_k \cdot \exp(f(h_n, \ell_k)) \]

   We represent \( f(h_n, \vartheta_k, \ell_k) \) as \( \vartheta_k \cdot \exp(f(h_n, \ell_k)) \) and assign priors for \( h_n \) and \( \ell_k \) (line 3 in Equation (4)). We get Equation (4) by absorbing \( \exp(f(h_n, \ell_k)) \) into the Gamma scale parameter.

In our construction, series convergence \( \sum_{k=1}^{\infty} Z_{nk} < \infty \) can be achieved by bounding \( \mu_f \) and \( \sigma_f^2 \) through a truncation layer in the decoder network. Given \( Z_n \), we sample words in a document, \( X_{nm} \) for \( m \in [M_n] \), by first sampling its topic assignment \( C_{nm} \sim \text{Dis}(\frac{Z_{nm}}{\sum_{n=1}^{N} Z_{nm}}) \), and then sampling the word from that topic, \( X_{nm} \sim \text{Dis}(\theta_{C_{nm}}) \), with topic prior \( \theta_k \sim \text{Dir}(\gamma_k) \). We recall that in topic models, \( \theta_k \) (topic \( k \)) is a discrete distribution over the vocabulary.

4.2. Amortized variational inference

Assume we have \( N \) documents and the posterior is truncated to \( K \) topics. The joint likelihood is

\[
p(\ell, V, \theta, h, Z, C, X) = \prod_{k=1}^{K} p(\ell_k)p(V_k)p(\theta_k)\prod_{n=1}^{N} \left[ p(h_n) \prod_{k=1}^{K} p(Z_{nk}|V, h_n, \ell_k)\prod_{m=1}^{M_n} p(C_{nm}|Z_n)p(X_{nm}|C_{nm}, \theta) \right].
\]

(5)

We use variational inference to approximate the model posterior by optimizing the variational objective function

\[
\max_q \mathcal{L} = \max_q \mathbb{E}_q \left[ \ln \frac{p(\ell, V, \theta, h, Z, C, X)}{q(\ell, V, \theta, h, Z, C)} \right],
\]

(6)

where we restrict \( q \) to the factorized family

\[
q(\ell, V, \theta, h, Z, C) = \prod_{k=1}^{K} q(\ell_k)q(V_k)q(\theta_k)\prod_{n=1}^{N} \left[ q(h_n|X_n) \prod_{m=1}^{M_n} q(C_{nm}) \right].
\]

(7)

Further, for global variables we let

\[
q(\ell_k) = \delta_{\ell_k}, \quad q(V_k) = \delta_{V_{\ell_k}} \quad q(\theta_k) = \text{Dir}(\gamma_k).
\]

(8)

For local variables, we introduce an inference network \( g \) and let \( q(h_n|X_n) = \delta_g(X_n) \). For the remaining variables

\[
q(Z_{nk}) = \text{Gam}(\alpha_{nk}, b_{nk}), \quad q(C_{nm}) = \text{Disc}(\phi_{nm}).
\]

(9)

We use coordinate ascent to update \( q \). Each of these updates is guaranteed to improve the objective when the gradient descent step size is small enough (Nesterov, 2013). More details are given in the appendix.

For \( q(Z_{nk}) \), we maximize a lower bound for \( \mathcal{L} \) similar to Paisley et al. (2012b), giving updates

\[
a_{nk} = \beta \tilde{p}_k + \sum_{m=1}^{M_n} \phi_{nm}(k),
\]

\[
b_{nk} = 1 / \left( \mathbb{E} \left[ \exp(-f(h_n, \ell_k)) \right] + \frac{M_n}{\varepsilon_n} \right),
\]

(10)

where \( \varepsilon_n = \sum_{k=1}^{K} \mathbb{E}[Z_{nk}] \).
For $q(C_{nm})$ and $q(\theta)$, we have respective updates
\begin{equation}
\phi_{nm}(k) \propto \exp\left(\mathbb{E}[\ln \theta_{k,X_{nm}}] + \mathbb{E}[\ln Z_{nk}]\right),
\end{equation}
\begin{equation}
\gamma_{kd} = \gamma_0 + \sum_{n=1}^{N} \sum_{m=1}^{M_n} \phi_{nm}(k) \cdot \mathbb{I}(X_{nm} = d).
\end{equation}

For $[\ell, V, g, f]$, we do gradient ascent on $L$. Batch variational inference can be done via coordinate ascent by iteratively updating the above variables. Dependencies among variables are shown in Figure 3(a).

For stochastic inference, in each global iteration we sample a subset $N_i \subset [N]$ and compute the noisy variational objective
\begin{equation}
L_t = \mathbb{E}\left[\ln p(\ell,V,\theta) + \frac{N}{|N_t|} \sum_{n \in N_t} \mathbb{E}\left[\ln p(h_n,Z_n,C_n,X_n)\right]\right] + \mathbb{H}[q(\theta)] + \frac{N}{|N_t|} \sum_{n \in N_t} \mathbb{H}[q(Z_n,C_n)].
\end{equation}

Optimizing local variables $Z, C$ can be done via closed-form updates exactly as in the batch case. For the other parameters we use stochastic gradient methods. Let $\rho(t) \propto (t_0 + t)^{-\kappa}$ be the step size with some constant $t_0$ and $\kappa \in (0, 1]$. We apply the stochastic natural gradient method (Hoffman et al., 2013) for $\theta$
\begin{equation}
\gamma^{(t)}_{kd} = \gamma_0 + \sum_{n \in N_t} M_n \phi_{nm}(k) \cdot \mathbb{I}(X_{nm} = d),
\end{equation}
\begin{equation}
\gamma^{(t)}_{kd} = (1 - \rho(t))\gamma_{kd}^{(t-1)} + \rho(t)\gamma_{kd}^{(t)}.
\end{equation}
and stochastic gradient method for the rest,
\begin{equation}
[\ell, V, g, f]^{(t)}(t) = [\ell, V, g, f]^{(t-1)} + \rho(t)\nabla_{[\ell, V, g, f]} L_t.
\end{equation}
Since in each iteration we only do one gradient step, the cost is low. Note that through the variational autoencoder (VAE) (Kingma & Welling, 2013) we transfer local updates for $h_n$ to global update for $g$, which will significantly speed-up inference. We summarize the stochastic inference algorithm in Algorithm 2.

### 4.3. Network architectures

The flexibility of our model comes from the inference and decoder networks $g$ and $f$. As we show in the experiments, these allow us to learn complex non-linear “paintboxes” in order to capture complex topic correlations. Since optimizing over deep neural networks is still a challenging problem in theory, we design our networks with architectures that work well in practice. Rather than directly applying multilayer perceptrons (Rumelhart et al., 1985), we instead use more complex layer designs such as batch normalization (Ioffe & Szegedy, 2015) and deep residual networks (ResNet) (He et al., 2016) to speed-up training. For inference network $g$, we use the bag-of-words representation of $X_n$ as the input feature. For decoder network $f$, we concatenate $h_n = g(X_n)$ and $\ell_k$ as inputs. Detailed architecture design is shown in Figure 3(b).

### 5. Experiments

#### 5.1. Batch experiments

We show empirical results on three text datasets: a 5K subset of New York Times, 20Newsgroups, and NeurIPS. Their basic statistics are shown in Table 2. For each test document $X_n$, we do a 90%/10% split into training words $X_{n,TR}$ and testing words $X_{n,TS}$. The perplexity is calculated based on
the prediction of $X_{n,TS}$ given the model and $X_{n,TR}$,

$$\text{perplexity} = \exp\left(-\frac{\sum_{m \in X_{n,TS}} \ln p(X_{n,m} | X_{n,TR})}{|X_{n,TS}|}\right).$$

(16)

Lower perplexity means better predictive performance.

In Table 1, we compare three Bayesian nonparametric models: hierarchical Dirichlet process (HDP) (Teh et al., 2005), discrete infinite logistic normal (DILN) (Paisley et al., 2012b), and our population random measure embedding (PRME) using 4-layer MLP with batch normalization. We tune $\gamma_0$ and fix the truncation level $K = 100$ and set the $a = 1, b = 1, \alpha = 1, \beta = 5$ for fair comparisons. All gradient updates are done via Adam (Kingma & Ba, 2014) with learning rate $10^{-4}$. As Table 1 shows, PRME consistently perform better than HDP and DILN. Where DILN was designed to outperform HDP by learning topic correlation structure, PRME improves upon DILN by learning a more complex kernel structure.

Since PRME encodes complex correlation patterns with a neural network, we further consider the influence of network architecture on perplexity for the New York Times dataset.

We compare four layer designs: multilayer perceptron (MLP), MLP with batch normalization (MLP+BN), ResNet, and ResNet with batch normalization (ResNet+BN); see Figure 3(b) for details. In Table 4 and Table 5, we separately tune the depth of each network and the hidden size of $h/\ell$ while holding other parameters fixed. The details of layer sizes can be found in Table 3. We observe that the perplexity result tend to be better when we scale up the network depth/width. Batch normalization and ResNet both improve performance.

### Online experiments

For the larger one million New York Times dataset, we show “topic paintboxes” learned with stochastic PRME in Figure 4.\footnote{The number of layers includes inference network and decoder network. We ignore the last layer of the decoder network.} In Figure 4, each paintbox corresponds to one topic whose top words are displayed inside the box. The color of a pixel $(x, y)$ in the kth paintbox ranges from blue (small value) to red (large value) and represents mean topic strength $\mathbb{E}[Z_{(x,y),k}] = \beta p_k \mathbb{E}[\exp(f(h_{(x,y)}, \ell_k))]$ as a function of $h_{(x,y)}$ for topic $k$. To define $h_{(x,y)}$ for 2d visualization, we collect the empirical embeddings $H = [h_1, \ldots, h_N] \uparrow = [g(X_1), \ldots, g(X_N)] \uparrow$ on a subset of data, subtract their mean $m_h$, and use the SVD to select the two most informative directions $h_1, h_2$ with singular values $s_1, s_2$. Then we plot each paintbox as the function value $\mathbb{E}[Z_{(x,y),k}] = \beta p_k \mathbb{E}[\exp(f(m_h + x s_1 h_1 + y s_2 h_2, \ell_k))]$ by tuning $(x, y) \in [-0.2, 0.2]^2$.

The correlation between topics can be read out from the paintboxes. Those paintboxes that have overlapping salient regions tend to be more correlated. For example, topic 13 [music, concert, orchestra], topic 20 [film, movie, films],

### Table 3. Network layer configurations for New York Times dataset.

| Depth | Inference Network | Decoder Network |
|-------|-------------------|-----------------|
| 2 layers | $[8000 \times d_h]$ | $[d_h + d_e] \times 80$ | $[80 \times 2]$ |
| 4 layers | $[8000 \times 1000]$ | $[d_h + d_e] \times 80$ | $[80 \times 2]$ |
| 6 layers | $[8000 \times 1000]$ | $[d_h + d_e] \times 80$ | $[80 \times 2]$ |
| 8 layers | $[8000 \times 1000]$ | $[d_h + d_e] \times 80$ | $[80 \times 2]$ |
Figure 4. A paintbox demonstration of salient topics learned from the one million New York Times dataset. In each paintbox on the LHS, pixel \((x, y)\) represents the topic strength \(Z_{(x,y),k}\) as a function of \(h_{(x,y)}\) for a particular topic \(k\). We also show embeddings of three articles in the same space, as well as their projection onto selected paintboxes. Each article is connected to its most-used topics.

1. **Blasts Kill 21 at a Cafe in North Iraq**
   
   **Baghdad** — Two bombers wearing suicide vests blew themselves up in a popular cafe crowded with young people ...

2. **NBA League Pass Fans Think They’re Paying More for Less**
   
   Thanks to the N.B.A. lockout, the regular season is 20 percent shorter this season. As part of the new collective bargaining agreement ...

3. **Any Novel can be Shaped into a Movie**
   
   "The English Patient" proves that any novel can be successfully filmed, though it isn’t easy. Much of the novel focuses on Hana, whose father has been killed in the war ...

and topic 47 [book, books, publishing] share a salient region, which gives a third-order positive correlations over those topics. In principle, the paintbox can explain arbitrary order correlations as the neural network complexity increases. We observe that each paintbox in Figure 4 consists of multiple contiguous salient regions. This is due to the smoothness of neural networks, since \(g(X_{n1}) \approx g(X_{n2})\) when \(X_{n1}\) and \(X_{n2}\) share similar words. Also, the various “modes” in each paintbox demonstrate the greater flexibility of neural networks in explaining different contexts of a topic.

In Figure 4, we also display three documents with their embeddings \(h_n\) projected onto the 2d paintbox space. Each embedding hits salient regions of several paintboxes. Thus, each document can be interpreted as a mixture of these corresponding topics. We again note that we only display the paintbox in 2d via post-processing, but the actual paintbox is in 20 dimension; a higher-dimensional paintbox can be more complex than what is shown.

We can compare the difference between paintboxes for PRME in Figure 4 and paintboxes for binary random measures in Figure 1. First, the paintbox for PRME is real-valued, so it is natural to use smooth functions to model it. In the binary case the paintbox is zero/one valued; in this case one can apply a threshold function over the PRME paintbox to binarize it. Second, in contrast to the binary paintbox, each PRME paintbox is unbounded. We control the area of this salient region through regularization.

Figure 5(a) demonstrates the perplexity of DILN and PRME with various decay speed \(\kappa\) on a held-out test set of size 3K. PRME converges after seeing one million documents, and it performs better than DILN. Also, online learning is much more efficient than batch learning with various training data size, as shown in Figure 5(b). In Figure 5(c), we compare run times for updating local parameters \([Z,C]\) for PRME) and global parameters \([\theta, \ell, V, g, f]\) for PRME) with batch size 500. Since the cost is very imbalanced between local and global, for demonstration purpose we compare the cost between five local iterations and one global iteration. In our experiments, local updates requires around 20 iterations to converge. Compared with DILN, PRME costs much less in local and costs more in global updates, since it uses the VAE to transfer local updates for \(h_n\) into global updates for \(g\). The extra global cost (~0.35s) is significantly smaller than the reduced local cost (~4s), even when using a deep network architecture. Finally, Figure 5(d) demonstrates the usage proportion for all topics. PRME tends to use a subset of the 100 available topics in the truncated posterior, indicating use by the model of this nonparametric feature.
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6. Discussion

6.1. Connections with other random objects

Another view is to treat \((Z_{nk})_{n \in [N], k \in [K]}\) as a bipartite graph over objects \([N]\) and atoms (features) \([K]\) with edge strength \(Z_{nk}\). An important topic in random graph theory is to study the total strength of edges \(|E| = \sum_{n \in [N], k \in [K]} E[Z_{nk}]\) asymptotically as a function of \(N\). There has been extensive work on random graphs, networks, and relational models (Roy et al., 2008; Miller et al., 2009; Caron, 2012; Lloyd et al., 2012; Veitch & Roy, 2015; Cai et al., 2016; Lee et al., 2016; Crane & Dempsey, 2017; Caron & Rousseau, 2017; Caron & Fox, 2017), but these methods mainly focus on dense graphs where \(|E| \sim O(N^2)\), and sparse graphs where \(|E| \sim O(N^{1+\alpha})\) with \(0 < \alpha < 1\) or \(|E| \sim O(N \log N)\). Our method offers a new solution to extremely sparse hidden graphs where \(|E| \sim O(N)\), by coupling random functions and a Poisson process. Our solution cannot be trivially derived from previous representations in sparse/dense graphs. There is a developed probability theory building connections between exchangeable binary random measures and functions on combinatorial structures among atoms (Pitman, 1995; 2006; Broderick et al., 2013; 2015; Heaukulani et al., 2016; Campbell et al., 2018).

Our topic model construction is motivated by previous research on dependent random measures (Zhou et al., 2011; Paisley et al., 2012b; Chen et al., 2013; Foti et al., 2013; Zhang & Paisley, 2015; 2016). Our focus is to place mild exchangeability assumptions on a population random measure \(\xi\) and derive a very general random function model through representation theorems. Hence our use of neural networks to achieve this task. We mention that our method can also be adapted to non-exchangeable settings.

6.2. Deep hierarchical Bayesian models

One can scale up model capacity by stacking multiple one-layer Bayesian nonparametric models such as Dirichlet processes (Teh et al., 2005), beta processes (Thibaux & Jordan, 2007), and Gamma processes (Zhou et al., 2015; Zhou, 2018). Population random measure embedding uses a different strategy by constructing random measures as a coupling of random functions with a single Poisson process. In this way, we transfer all the model complexity into random functions \(f_n\). Using amortized variational inference, we transfer posterior inference of discrete random measures into optimizing neural networks, which is much more efficient.

6.3. Posterior inference bottleneck

Efficient posterior inference is essential in Bayesian nonparametric methods where conjugacy often does not hold (Broderick et al., 2014; Zhang et al., 2016). In principle, one can apply a simple prior on \(Z\) and still rely on accurate posterior inference to resolve the structure. However, posterior inference for random measures is not simple because complex correlations among atoms leads to slow MCMC mixing. Instead, one can approximate the posterior using variational methods (Blei et al., 2017) and try to learn a \(q\) distribution with good approximation quality (Paisley et al., 2012a; Hoffman & Blei, 2015; Ranganath et al., 2016; Tran et al., 2017). Our method introduced a structured prior to regularize variational inference. Empirical results showed that we get an interpretable posterior.

7. Conclusion and Future Work

We presented random function priors to handle complex correlations among features via a population random measure embedding. We further derived a new Bayesian nonparametric topic model to demonstrate the effectiveness of our method for learning topic correlations through deep neural networks with amortized variational posterior inference. In future work, we will consider the more challenging task of removing the non-differentiable Poisson process and making our model fully differentiable.

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Appendix

Proof of Proposition 1.

Proof. From (Aldous, 1985; Hoover, 1979; Orbanz & Roy, 2015), we can represent every separately exchangeable infinite binary matrix $Z = (Z_{nk})$ if and only if it can be represented as follows: There is a random function $W : [0, 1]^2 \to [0, 1]$ such that

$$(Z_{nk}) \overset{d}{=} (\mathbb{I}(R_{nk} < W(U_n, V_k))).$$

(17)

Thus, one can reconstruct $Z$ by first sample $W, (U_n), (V_k)$ and then sample $Z_{nk} | W, (U_n), (V_k) \sim \text{Bernoulli}(W_{U_n, V_k})$ through independent coin flips. It is straightforward to prove that the finite row sum assumption can only be satisfied when $\int_{[0, 1]^2} W(u, v) du dv = 0$. When that happens, $Z = 0$ almost surely. \(\square\)

Proof of Proposition 2.

Proof. The representation theorem in Proposition 2 is immediate from a more general result of separately exchangeable random measures. We temporarily reload notations $h_k, \beta_n$.

**Theorem 1** (Kallenberg, 2006). A random measure $\xi$ on $[0, 1] \times \mathbb{R}_+$ is separately exchangeable if and only if almost surely

$$\xi = \sum_{n,k} f_n(\vartheta_k) \delta_{\tau_n, \sigma_k} + \sum_{m,k} g_m(\vartheta_k) \delta_{\rho_{mk}, \sigma_k}$$

point masses

$$+ \sum_k h_k(\vartheta_k)(\lambda \otimes \delta_{\sigma_k}) + \sum_n \beta_n(\delta_{\tau_n} \otimes \lambda)$$

line measures

$$+ \gamma \lambda^2,$$

(18)

for some measurable functions $f_n, g_m, h_k, \beta_n \geq 0$ on $\mathbb{R}_+^2$, a unit rate Poisson process $\{(\vartheta_k, \sigma_k)\}$ on $\mathbb{R}_+^2$, some independent $U(0, 1)$ arrays $(\tau_n)$ and $(\rho_{mk})$, an independent set of random variables $\beta_n, \gamma \geq 0$, and the Lebesgue measure $\lambda$. The latter can then be chosen to be non-random if and only if $\xi$ is extreme.

The representation theorem consists of three parts: point masses, line measures, and a diffuse measure. We select the point masses part for discrete separately exchangeable random measures. Decomposition of the entire measure $\xi$ is demonstrated in Fig 6.

Discussion on Section 3.2. Existing models as special cases of PRME model.

Let $\xi = \sum_{n,k} f_n(\vartheta_k) \delta_{\tau_n, \sigma_k}$ be our PRME model. We focus on a specific object $n$, remove the redundant $\tau_n$, and directly work on random measures on $\Theta$. This transformation let us be on the same page of other research on completely random measures.

We have $\xi_n = \sum_k f_n(\vartheta_k) \delta_{\theta_k}$ be a population random measure embedding model, where $(\vartheta_k, \theta_k)$ is a Poisson process on $\mathbb{R}_+ \times \Theta$ with mean measure $\mu(\vartheta) d\vartheta d\theta$. The according CRM is $\lambda = \sum_k \vartheta_k \delta_{\theta_k}$ with Lévy measure $\nu(d\vartheta, d\theta) = \mu(\vartheta) \mu(d\theta) d\vartheta d\theta$. Assume the tail function $T(\vartheta) = \nu((\vartheta, \infty), \Theta)$ is invertible. One can do a transformation between atoms by $(\vartheta_k, \theta_k) \to (T^{-1}(\vartheta_k), \theta_k) = (\tilde{\vartheta}_k, \theta_k)$. The following examples are just special cases of this transformation, as we shall see.

**IBP and extensions**

The Indian buffet process (IBP) take a particular form $\xi_n = \sum_k f \circ T^{-1}(\vartheta_k) \delta_{\theta_k}$, where $f$ are independent Bernoulli random variables with success rate $T^{-1}(\vartheta_k)$. IBP uses a particular transformation $T^{-1}(\vartheta) = e^{-\vartheta}$ (Thibaux & Jordan, 2007). (Teh & Gorur, 2009) gives a power-law extension of IBP with three parameters (3IBP) with an application in language models. However, 3IBP does not enjoy an analytical form for $T^{-1}$. But we can safely work on the CRM directly, given the generality of the existence of $T^{-1}$ (Orbanz & Williamson, 2011). One can observe that the sampling function $f \circ T^{-1}$ does not change with $n$. This is the main limitation for IBP and 3IBP A MCMC sampling solution can be found in (Griffiths & Ghahramani, 2011; Teh & Gorur, 2009).

**Correlated random measures**

The key restrictions of IBP and 3IBP is that $E[\xi_n((\{\theta_k\}) \cdot \xi_n((\{\theta_k\})) \mid \vartheta] = E[\xi_n((\{\theta_k\}) \mid \vartheta] \cdot E[\xi_n((\{\theta_k\}) \mid \vartheta]$. In order to model feature correlations, (Paisley et al., 2012b) model $f_n(\vartheta_k)$ as exchangeable random functions. The extra randomness besides $\vartheta$ can be modelled by augmenting the Poisson process $(\vartheta_k, \theta_k)$ on $\mathbb{R}_+ \times \Theta$ to higher dimension $(\ell_k, \vartheta_k, \theta_k)$ on $\mathbb{R}^d \times \mathbb{R}_+ \times \Theta$ with mean measure $\nu(d\ell, d\vartheta, d\theta) = \nu(d\vartheta, d\theta) \nu(d\ell)$. The discrete infinite logistic normal distribution (DILN) (Paisley et al., 2012b) further proposes an example $\xi_n = \sum_k \vartheta_{nk}(\beta \vartheta_k, \exp(-h_n(\ell_k))) \delta_{\sigma_k, \tau_n}$, where $h_n(\cdot) \sim \text{GP}(m(\cdot), K(\cdot, \cdot))$ and $Z_{nk}$ is a gamma distribution parameterized by its shape and scale parameters. However, DILN is restricted to use linear kernels, which is very restrictive. (Ranganath & Blei, 2018) proposed general correlated random measures with examples for the binary, discrete, and continuous cases.
We expand each term in Eq. (19) as follows.

\[
\mathbb{E}[\ln p(\ell_k)] = \ln p(\ell_k) = -\frac{r_l \ln(2\pi b)}{2} - \ell_k^T \ell_k
\]

\[
\mathbb{E}[\ln p(V_k)] = \ln p(V_k) = \ln \alpha + (\alpha - 1) \ln(1 - \tilde{V}_k).
\]

\[
\mathbb{E}[\ln p(\theta_k)] = \ln \Gamma(D\gamma_0) - D \ln \Gamma(\gamma_0) + \sum_{d=1}^{D} (\gamma_0 - 1) \mathbb{E}[\ln \theta_{kd}],
\]

where \(\mathbb{E}[\ln \theta_{kd}] = \psi(\gamma_{kd}) - \psi(\sum_{d=1}^{D} \gamma_{kd}).\)

\[
\mathbb{E}[\ln p(h_n)] = \ln p(h_n) = -\frac{r_n \ln(2\pi a)}{2} - \frac{\tilde{h}_n^T h_n}{2a}.
\]

\[
\mathbb{E}[\ln p(Z_{nk}|V, h_n, \ell_k)] = -\ln \Gamma(\beta \tilde{p}_k) - \beta \tilde{p}_k \mathbb{E}[f(h_n, \ell_k)] + (\beta \tilde{p}_k - 1) \mathbb{E}[\ln Z_{nk}] - \mathbb{E}[Z_{nk}] \mathbb{E}[\exp(-f(h_n, \ell_k))],
\]

where \(\mathbb{E}[f(h_n, \ell_k)] = \mu_f(h_n, \ell_k),\)

\(\mathbb{E}[\exp(-f(h_n, \ell_k))] = \exp(-\mu_f(h_n, \ell_k) + \frac{1}{2} \sigma_f^2(h_n, \ell_k)).\)

\[
\mathbb{E}[\ln Z_{nk}] = \ln(\hat{b}_{nk}) + \psi(a_{nk}), \quad \mathbb{E}[Z_{nk}] = a_{nk} \hat{b}_{nk}.
\]

\[
\mathbb{E}[\ln p(C_{nm}|Z_n)] = \sum_{k=1}^{K} \phi_{nm}(k) \mathbb{E}[-\ln(\sum_{k'=1}^{K} Z_{nk'})].
\]

\[
\mathbb{E}[\ln p(X_{nm}|C_{nm}, \theta)] = \sum_{k=1}^{K} \phi_{nm}(k) \mathbb{E}[\ln \theta_{X_{nm}n}],
\]

where \(\mathbb{E}[\ln \theta_{X_{nm}n}] = \psi(\gamma_{X_{nm}}) - \psi(\sum_{d=1}^{D} \gamma_{kd}).\)

\[
\mathbb{H}[q(\ell_k)] = 0.
\]

\[
\mathbb{H}[q(V_k)] = 0.
\]

\[
\mathbb{H}[q(\theta_k)] = \sum_{d=1}^{D} \ln(\gamma_{kd}) - \ln(\sum_{d=1}^{D} \gamma_{kd}) - \sum_{d=1}^{D} (\gamma_{kd} - 1) \mathbb{E}[\ln \theta_{kd}].
\]

\[
\mathbb{H}[q(h_n)] = 0.
\]

\[
\mathbb{H}[q(Z_{nk})] = a_{nk} + \ln(\hat{b}_{nk}) + \ln(\hat{b}_{nk}) + (1 - a_{nk}) \psi(a_{nk}).
\]

\[
\mathbb{H}[q(C_{nm})] = -\sum_{k=1}^{K} \phi_{nm}(k) \ln \phi_{nm}(k).
\]
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Variational inference for $\ell_k$, $V_k$, and network parameters can be done by directly plug-in and take gradients. Updating $q(\theta_k)$ and $q(C_{nm})$ follows the general variational update rule. Updating $q(Z_{nk})$ requires lower-bounding $\mathcal{L}$.

For $\ell$, we use gradient ascent:

$$\nabla_\ell \mathcal{L} = \sum_{k=1}^{K} \nabla \ell \mathbb{E} \left[ \ln p(\ell_k) \right] + \sum_{n=1}^{N} \sum_{k=1}^{K} \nabla \ell \mathbb{E} \left[ \ln p(Z_{nk} | V, h_n, \ell_k) \right].$$

(33)

For $V$, we use gradient ascent:

$$\nabla_V \mathcal{L} = \sum_{k=1}^{K} \nabla V \mathbb{E} \left[ \ln p(V_k) \right] + \sum_{n=1}^{N} \sum_{k=1}^{K} \nabla V \mathbb{E} \left[ \ln p(Z_{nk} | V, h_n, \ell_k) \right].$$

(34)

For $\theta$, we have a closed-form update:

$$\gamma_{kd} = \gamma_0 + \sum_{n=1}^{N} \sum_{m=1}^{M_n} \phi_{nm}(k) \cdot 1(X_{nm} = d)$$

(35)

For $h_n$, we update the inference network $g$:

$$\nabla_g \mathcal{L} = \sum_{n=1}^{N} \nabla g \mathbb{E} \left[ \ln p(h_n) \right] + \sum_{n=1}^{N} \sum_{k=1}^{K} \nabla g \mathbb{E} \left[ \ln p(Z_{nk} | V, h_n, \ell_k) \right].$$

(36)

For $Z_{nk}$, we maximize a lower bound for $\mathcal{L}$ similar as (Paisley et al., 2012b). Related terms in $\mathcal{L}$ are:

$$\mathcal{L}(q(Z_{nk})) = (\beta \tilde{p}_k - 1 + \sum_{m=1}^{M_n} \phi_{nm}(k)) \mathbb{E}[\ln Z_{nk}]$$

$$- \mathbb{E}[Z_{nk}] \mathbb{E}[\exp(-f(h_n, \ell_k))]$$

$$- M_n \mathbb{E} \left[ \ln \sum_{k'=1}^{K} Z_{nk'} \right] + \mathbb{H}[q(Z_{nk})].$$

(37)

The term that make closed-form update intractable is $\mathbb{E}[\ln \sum_{k'=1}^{K} Z_{nk'}]$. We use the bound:

$$\mathbb{E}[\ln \sum_{k'=1}^{K} Z_{nk'}] \leq \ln \varepsilon_n + \sum_{k'=1}^{K} \mathbb{E}[Z_{nk'}] - \varepsilon_n \varepsilon_n^{-1}.$$  

(38)

This bound is correct for any $\varepsilon_n > 0$, and here we pre-compute $\varepsilon_n = \sum_{k'=1}^{K} \mathbb{E}[Z_{nk'}]$ and treat $\varepsilon_n$ as a constant in the above equation. After Plugging-in the bound and some algebra, we solve $q(Z_{nk})$ as:

$$a_{nk} = \beta \tilde{p}_k + \sum_{m=1}^{M_n} \phi_{nm}(k),$$

$$1/b_{nk} = \mathbb{E} \left[ \exp(-f(h_n, \ell_k)) \right] + M_n \varepsilon_n.$$  

(39)

For decoder network $f$:

$$\nabla_f \mathcal{L} = \sum_{n=1}^{N} \sum_{k=1}^{K} \nabla f \mathbb{E} \left[ \ln p(Z_{nk} | V, h_n, \ell_k) \right].$$

(40)

For $C_{nm}$, we have a closed-form update:

$$\phi_{nm}(k) \propto \exp \left( \mathbb{E}[\ln \theta_k, X_{nm}] + \mathbb{E}[\ln Z_{nk}] \right).$$

(41)