NON–MINIMAL $q$–DEFORMATIONS AND ORTHOGONAL SYMMETRIES: $\mathcal{U}_q(\text{SO}(5))$ EXAMPLE

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Non–minimal $q$–deformations are defined. Their role in the explicit construction of the matrix elements of the generators of $\mathcal{U}_q(\text{SO}(5))$ on suitably parametrized bases are exhibited. The implications are discussed.

Symmetry is one theme of this workshop. Suppose one $q$–deforms some classical symmetry (unitary, orthogonal, · · ·) intending to explore the possibilities of applications of the symmetry thus generalized. Given such a goal, one should go further than the formally $q$–deformed Hopf algebra. One should construct explicitly the representations irreducible ones to start with but also, so far as possible, non-decomposable ones for $q$ a root of unity. By explicit construction I mean a complete set of suitably parametrized basis states spanning the space of the representation in question and the matrix elements of the generators acting on these state vectors. The invariant parameters and the variable indices labelling the states will each (some very directly while others less so) have their specific significance in the description of the phenomenon studied. the matrix elements will measure the response of the states to the constraints of the symmetry (the action of the generators). They will also yield the values of the crucial invariants. Unless all these elements are obtained a central problem remains unsolved. One is not fully equipped to explore possible applications.

As one proceeds with this program one encounters, among others, the following fact. The $q$–deformed unitary algebras are relatively docile while the corresponding orthogonal ones are suprisingly refractory. To give this statement more precise content let me introduce at this point some definitions and terminology.

Let us start with a classical quantity $x$, typically a factor in some classical matrix element.

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a Talk presented at the Nankai workshop, Tianjin, 1995 by A. Chakrabarti.
bURA 14-36 du CNRS, associée à l’E.N.S. de Lyon, et au L.A.P.P. d’Annecy-le-Vieux.
Minimal $q$-deformation:

\[
q = 1 \quad q \neq 1
\]
\[
x \rightarrow [x]_p \equiv \frac{x^p - q^{-p}}{q^p - q^{-p}}
\] (1)

One may further refine this by defining the deformation to be strictly minimal only for $p = 1$ (when the subscript $p = 1$ will be omitted) and to be pseudo-minimal for $p \neq 1$.

Non-minimal $q$-deformation:

An unlimited number of more complicated deformations (retaining the symmetry $q \leftrightarrow q^{-1}$ and the same classical limit $x$) is evidently possible.

Example. 1.

\[
x \rightarrow [x_1]_{p_1} - [x_2]_{p_2}, \quad (x_1 - x_2 = x)
\] (2)

Example. 2.

\[
x \rightarrow [x]_p \frac{[y]_{p_1}}{[y]_{p_2}}
\] (3)

Apart from simple $q$-factors ($x \rightarrow [x]_p q^{f(x)}$, $f(x)$ being some non-singular function of $x$ and possibly other parameters) we have, as yet, encountered only the types (2) and (3) (but possibly with more than one $y$-factors in (3)). Further study may permit the classification of all the relevant ones.

Let us now go back to our initial statement. The well-known Gelfand–Zetlin matrix elements [1] for irreps. of $SU(N)$ are square roots of ratios of products of integer factors. A minimal $q$-deformation of each factor

\[
\left( \frac{x_1 x_2 x_3 \ldots}{y_1 y_2 y_3 \ldots} \right)^{1/2} \rightarrow \left( \frac{[x_1] [x_2] [x_3] \ldots}{[y_1] [y_2] [y_3] \ldots} \right)^{1/2}
\] (4)

gives the corresponding element for $U_q(SU(N))$ for generic $q$. For $q$ a root of unity periodic representations are obtained by introducing suitable fractional parts for each $x$ and $y$ [2]. Relatively simple modifications yield other classes of representations [3, 4]. One can of course introduce a unitary transformation after deforming as in (4) to obtain complicated matrix elements. But all representations, at least for real positive $q$, can be obtained as in (4). Transformations can only introduce spurious non-minimalities masking the basic simplicity.

In $q$-deforming orthogonal algebras the well-known prescriptions for $U_q(SU(2))$ suffice for $SO(3)(\approx SU(2))$ and $SO(4) \approx (SU(2) \otimes SU(2))$. But even for $SO(4)$ problems (and non-minimalities) arise if one tries to $q$-deform
directly the canonical Gelfand–Zetlin matrix elements [5]. The first intrinsically non-trivial case is \( \mathcal{U}_q(\text{SO}(5)) \) which we discuss here showing exactly where and how non-minimalities enter and analysing their implications.

\( \mathcal{U}_q(\text{SO}(5)) \): Corresponding to the two unequal roots one has two \( q \)-deformed Chevalley triplets. The standard Drinfeld-Jimbo Hopf algebra (omitting the coproducts, counits and antipodes) becomes in our conventions [6, 7]

\[
q^{\pm h_1} e_1 = q^{\pm 1} e_1 q^{\pm h_2}, 
q^{\pm h_1} e_2 = q^{\mp 1} e_2 q^{\pm h_2},
q^{\pm h_2} e_1 = q^{\mp 1} e_1 q^{\pm h_2}, 
q^{\pm h_2} e_2 = q^{\mp 1} e_2 q^{\pm h_2},
q^{\pm h_1} f_1 = q^{\pm 1} f_1 q^{\pm h_1}, 
q^{\pm h_2} f_1 = q^{\pm 1} f_1 q^{\pm h_2},
q^{\pm h_1} f_2 = q^{\pm 1} f_2 q^{\pm h_1}, 
q^{\pm h_2} f_2 = q^{\pm 1} f_2 q^{\pm h_2},
\]

(5)

\[
[e_1, f_2] = 0, 
[q e_1, f_1] = 2h_1 
\]

and

\[
e_2 e_3^{(\pm)} = q^{\mp 2} e_3^{(\pm)} e_2, 
f_3^{(\pm)} = q^{\mp 1} f_2 f_3^{(\pm)} 
\]

(6)

where

\[
e_3^{(\pm)} = q^{\mp 1} e_1 e_2 - q^{\mp 1} e_2 e_1, 
f_3^{(\pm)} = q^{\mp 1} f_2 f_3^{(\pm)} - q^{\mp 1} f_1 f_2,
q e_1 e_2^{(+)} - q e_2 e_1^{(-)} = q e_1 e_2^{(+)} - q e_1 e_2^{(-)},
f_3^{(+)} f_1 - q f_1 f_3^{(+)} = q f_3^{(-)} f_1 - q f_3^{(-)} f_1 
\]

(7)

We define \((M, K, M_2, M_4)\) through

\[
q^{\pm M} = q^{\pm h_1}, 
q^{\pm (K-M)} = q^{\pm h_2},
q^{\pm M_2} = q^{\pm h_2}, 
q^{\pm M_4} = q^{\frac{1}{2}(h_1+h_2)}
\]

(8)

The fundamental Casimir (classically quadratic in the Cartan-Weyl generators) can now be written [6, 7] for arbitrary \( q \) as

\[
A = \frac{1}{2} \left\{ (f_1 e_1 + [M][M]+1) \frac{[2M+3]}{[2M+3]} + [K][K+3] \right\}
+ (f_2 e_2 + \frac{M}{2} f_4 e_4) + \frac{M}{2} (f_3^{(+)})^{2M+1} + f_3^{(-)} e_3^{(-)} q^{-2M-1}.
\]

(9)

For brevity we consider in this talk only generic \( q \), real positive. The case \( q \) a root of unity has been discussed in [6]. For generic \( q \) the irreps. are labelled by two (half) integers

\[
n_1 \geq n_2
\]

(10)
One has the following general result [7]. The state annihilated by $e_1, e_2$ corresponds to eigenvalue $n_2$ of $M$ and $n_1$ of $K$. Hence on this space

$$A = \frac{1}{n_1^2} \left\{ [n_1][n_1 + 3] + [n_2][n_2 + 1][2n_1 + 3n_2] \right\} 1$$

(11)

where $1$ is the unit matrix corresponding to the dimension

$$\frac{1}{2}(2n_2 + 1)(2n_2 + 3)(n_1 + n_2 + 2)(n_1 - n_2 + 1)$$

(12)

For $n_2 = 0, \frac{1}{2}$ and $n_1$ (11) reduces to the respective results in [6]. The factor $[2n_1 + 3]/[2n_2 + 3]$ in (11) is a particularly interesting example of the non-minimal case (3). Its implications will be analysed at the end.

After all the $SU(N)$ and $SO(3), SO(4)$ one encounters unequal roots for the first time for $SO(5)$. In constructing matrix elements one can associate the well-known $SU(2)$ structure either with the Chevalley triplet $(e_1, f_1, h_1)$ or with the triplet $(e_2, f_2, h_2)$. The consequences are very different and even more so concerning $q$-deformations. Non-minimality and non-simple lacing (unequal roots) will appear, at least in this example, associated together.

The well-known Gelfand-Zetlin basis and matrix elements [1] are quite unsuitable (for the orthogonal case) as starting point for $q$-deformation. The situation is entirely different from that of the unitary case. The reasons were discussed in [6]. After this remark I now introduce the two bases starting with the Chevalley triplets 1 and 2 respectively.

**Basis I.** Standard $U_q(SU(2))$ structure for $(e_1, f_1, q^{\pm h_1})$, invariants $(n_1, n_2)$, variable indices $(j, m, k, l)$. The domain of the indices are [7]:

(i) For $(n_1, n_2)$ integers

$$j = 0, 1, \cdots, n_1 - 1, n_1, \quad m = -j, -j + 1, \cdots, j - 1, j$$

$$k = -l, -l + 2, \cdots, l - 2, l, \quad l = 0, 1, 2 \cdots$$

$$j + l = n_1 - n_2, n_1 - n_2 + 1, \cdots, n_1 + n_2$$

$$j - l = \frac{1}{2}(1 - (-1)^{n_1 + n_2 - j}) = -n_1 + n_2, -n_1 + n_2 + 2, \cdots, n_1 - n_2.$$  

(13)

(ii) For $(n_1, n_2)$ half-integers

$$j = \frac{1}{2}, \frac{3}{2}, \cdots, n_1 - 1, n_1, \quad m = -j, -j + 1, \cdots, j - 1, j$$

$$k = -l, -l + 1, \cdots, l - 1, l, \quad l = \frac{1}{2}, \frac{3}{2}, \cdots$$

$$j + l = n_1 - n_2 + 1, n_1 - n_2 + 3, \cdots, n_1 + n_2$$

$$j - l = -n_1 + n_2, -n_1 + n_2 + 2, \cdots, n_1 - n_2.$$  

(14)

The dimension is given by (12) for both cases.
The matrix elements are (suppressing $n_1$, $n_2$ in the state labels)

$$q^{\pm M}|j m k l\rangle = q^{\pm m}|j m k l\rangle$$

$$q^{\pm K}|j m k l\rangle = q^{\pm k}|j m k l\rangle$$

$$e_1|j m k l\rangle = ([j - m][j + m + 1])^{1/2}|j m + 1 k l\rangle$$

$$e_2|j m k l\rangle = ([j - m + 1][j + m + 2])^{1/2} \sum_{l'} a(j, k, l, l')|j + 1 m - 1 k + 1 l'\rangle$$

$$+ ([j + m][j + m - 1])^{1/2} \sum_{l'} b(j, k, l, l')|j - 1 m - 1 k + 1 l'\rangle$$

$$+ ([j + m][j + m + 1])^{1/2} \sum_{l'} c(j, k, l, l')|j m - 1 k + 1 l'\rangle$$

(15)

We consider only real solutions of the matrix elements when for any two states $|x\rangle$, $|y\rangle$

$$\langle y|f_i|x\rangle = \langle x|e_i|y\rangle, \quad (i = 1, 2)$$

(16)

As yet solutions have been obtained [6] for arbitrary (half) integer $n_1$ only for the extreme values of $n_2$,

$$n_2 = 0 \quad \text{or} \quad \frac{1}{2}$$

(17)

and

$$n_2 = n_1$$

(18)

For these cases $l$-dependence is trivial. One labels the states as $|j m k\rangle$. Even classical ($q = 1$) solutions are not available for the general case. Referring to [6] for complete solutions of the cases (17) I now present the solution of (18) for comparing it to the corresponding one in Basis II to follow.

For $n_2 = n_1 = n$ (integer or half-integer), suppressing trivial $l$-dependence

$$a(j, k) = b(j + 1, -k - 1) = (q + q^{-1})^{-1} ([n - 1][n + 1][j - k] [j + k + 1][j + k + 2])^{1/2}$$

$$c(j, k) = (q + q^{-1})^{-1} [n + 1][j - k][j + k + 1][j + k + 2][j + 1][j + 2][j + 3][j + 4][j + 5]$$

(19)

The dimension is

$$\frac{1}{3}(n + 1)(2n + 1)(2n + 3)$$

(20)

Comparing with the limit $q = 1$, it is evident that each factor undergoes a (pseudo) minimal deformation of type (1). The situation will change in the basis to follow.

Basis II. Standard $U_q^2(SU(2))$ structure for $(e_2, f_2, q^{\pm h_2})$, invariants $(n_1, n_2)$, variable indices $(j_2, m_2, j_4, m_4)$. The domain of the indices are [7] (for
The dimension is given by (12).

The matrix elements are

\[ q^{\pm M_2}[j_2 \, m_2 \, j_4 \, m_4] = q^{\pm m_2}[j_2 \, m_2 \, j_4 \, m_4] \]
\[ q^{\pm M_4}[j_2 \, m_2 \, j_4 \, m_4] = q^{\pm m_4}[j_2 \, m_2 \, j_4 \, m_4] \]
\[ e_2[j_2 \, m_2 \, j_4 \, m_4] = (j_2 - m_2)[j_2 + m_2 + 1/2]^{1/2}[j_2 + m_2 + 1] \]
\[ e_1[j_2 \, m_2 \, j_4 \, m_4] = \sum_{\epsilon, \epsilon'} ((j_2 - \epsilon m_2 + \frac{1+\epsilon'}{2})^{1/2} \]
\[ \times c_{(\epsilon, \epsilon')} (j_2, j_4, m_4) j_2 + \frac{\epsilon'}{2} m_2 - \frac{1}{2}, j_4 + \frac{\epsilon'}{2} m_4 + \frac{1}{2}) \]

with, as in (16),
\[ \langle y | f_i | x \rangle = \langle x | e_i | y \rangle, \quad (i = 1, 2) \]

Now a classical solution is available [8]. In our notation this is
\[ c_{(\epsilon, \epsilon')} (j_2, j_4, m_4) = (j_4 + \epsilon' m_4 + \frac{1+\epsilon'}{2})^{1/2} c_{(\epsilon, \epsilon')} (j_2, j_4) \quad (\epsilon, \epsilon' = \pm 1) \]
\[ c_{(\epsilon, \epsilon')} (j_2, j_4) = \epsilon \epsilon' c_{(-\epsilon, -\epsilon')} (j_2 + \frac{\epsilon}{2}, j_4 + \frac{\epsilon'}{2}) \]

where
\[ c_{(++)}(j_2, j_4) = \left( \frac{(n_1 + j_2 + j_4 + 3)(n_1 - j_2 - j_4)(j_2 + j_4 + n_2 + 2)(j_2 + j_4 - n_2 + 1)}{(2j_2 + 1)(2j_2 + 2)(2j_4 + 1)(2j_4 + 2)} \right)^{1/2} \]
\[ c_{(+-)}(j_2, j_4) = \left( \frac{(n_1 + j_2 - j_4 + 2)(n_1 - j_2 + j_4)(j_2 + j_4 + n_2 + 2)(j_2 + j_4 - n_2 + 1)}{(2j_2 + 1)(2j_2 + 2)(2j_4 + 1)(2j_4 + 2)} \right)^{1/2}. \]

(see the comments in [7] concerning the relation to [8]).

But now q-deformation is the problem. So far solutions have been obtained for
\[ n_2 = 0 \]
\[ n_2 = n_1 = n \]

Referring to [7] for (26) I now reproduce only the solution for (27).

For \( n_1 = n_2 = n \)
\[ j_2 + j_4 = n, \quad j_2 = 0, \frac{1}{2}, \cdots, n \]
\[ c_{(ee)} (j_2, j_4, m_4) = 0. \]
\[
c_{(\epsilon,-\epsilon)}(j_2, j_4, m_4) = \left( [n + 1]_2 - [j_2 + \epsilon m_4 + \frac{1}{2}(1 + \epsilon)]_2 \right)^{1/2} c_{(\epsilon,-\epsilon)}(j_2)
\]

with
\[
c_{(+ -)}(j_2) = -c_{(- +)}(j_2 + \frac{1}{2}) = \left( \frac{[2j_2 + 1]_2 [2j_2 + 2]}{[2j_2 + 1]_2 [2j_2 + 2]} \right)^{1/2}.
\]

The \(m_4\)-dependence in (28) is of type (2) with
\[
x_1 = n + 1, \quad x_2 = j_2 + \epsilon m_4 + \frac{1}{2}(1 + \epsilon)
\]
so that
\[
x = x_1 - x_2 = j_4 + \epsilon' m_4 + \frac{1}{2}(1 + \epsilon') \quad (29)
\]
consistently with (24). The \(c(j_2)\)'s in (28) are of type (3) with \(x = 1\) but double \(y\)-factors.

**If one tries to solve using only (pseudo) minimal deformations of type (1) one runs into contradictions.** Thus non-minimality is essential for this basis. This basis, in turn, seems to be essential for providing access to certain interesting sectors. Let me just mention two such points.

(i) Suitably adapting familiar continuation techniques Basis I leads to \(U_q(SO(3,2))\) while Basis II is needed to arrive at \(U_q(SO(4,1))\).

(ii) Under suitable contraction procedures quite different \(q\)-deformed inhomogeneous algebras are obtained from the two bases respectively (see the comments in [6] and [7]).

It is not possible to discuss these aspects here. But they suffice to indicate the potential interest of a general solution of (22) for arbitrary \(q\). (In fact once solutions are found for generic \(q\) our method of fractional parts explained in [3] and [4] and already used for Basis I in [6] will readily yield solutions for \(q\) a root of unity.)

The general solution will also permit a better understanding of the role of non-minimality. This role is not merely formal. If there is indeed some physical application, the physical content of different types of deformations will be different. As \(q\) moves away from unity they will respond differently. Thus, to take an example, for \(q = e^{i\delta}\) and \(x = x_1 - x_2\)
\[
[x_1]_{p_1} - [x_2]_{p_2} = x + \frac{1}{2} \delta^2 (p_1^2 x_1(x_1^2 - 1) - p_2^2 x_2(x_2^2 - 1)) + \cdots \quad (30)
\]
In this context the non-minimality of type (3) noted in (11) also has a striking consequence.
For $q = e^\delta$,

$$\frac{1}{[3]} \left\{ [n_1][n_1 + 3] + [n_2][n_2 + 1] \frac{2n_1 + 3q}{[2n_1 + 3]} \right\} = A_2 + \delta^2 A_4 + \cdots \quad (31)$$

where

$$A_2 = \frac{1}{2} (n_1(n_1 + 3) + n_2(n_2 + 1)) \quad (32)$$
$$A_4 = 4 n_2(n_2 + 1)(n_1 + 1)(n_1 + 2) + \cdots \quad (33)$$

The other terms of $A_4$ are very easily obtained. Let us, however, concentrate on the first term, a direct consequence of the factor $[2n_1 + 3]/[2n_1 + 3]$ in (31).

$A_2$ is just the well-known eigenvalue of the first classical Casimir (quadratic in the Cartan-Weyl generators) for the irrep. $(n_1, n_2)$. The first term of $A_4$ is the classical eigenvalue of the second (quadratic) Casimir operator.

This is an illustration of the general result announced in my first talk [9]. The $q$-deformed quadratic Casimir alone completely characterizes the irreps. $(n_1, n_2)$. We note moreover that this is here achieved through a typical non-minimality.

Consider now the consequence of the same non-minimal structure in the context of contraction. Certain aspects of contraction of $U_q(SO(5))$ are discussed in [6]. Here let us just note that the eigenvalue of the contracted Casimir (for $q > 1$ for example) is obtained by dividing the l.h.s. of (31) by the leading term of $[n_1][n_1 + 3]$ as $n_1 \to \infty$ multiplied by a constant $\lambda^{-2}$, i.e. by

$$\frac{1}{\lambda^2 [2] \frac{2n_1 + 3}{(q^2 - q^{-2})}} \quad (34)$$

and taking the limit. This gives an eigenvalue

$$\lambda^2 \left\{ 1 + \frac{(q^{-1} - 1)^2}{(q + q^{-1})} [n_2][n_2 + 1] \right\}$$

A general solution for $U_q(SO(5))$ on a suitable basis can lead through contraction to a successful construction of representations of $U_q(E(4))$ (the $q$-deformed 4-dimensional Euclidean algebra) for arbitrary $q$. Then one has to see if a suitable analytic continuation to $q$-deformed Poincaré algebra is possible through this approach. This is one of our main goals. This remains to be done. But (35) shows that it will include the following remarkable feature. The $q$-deformed "mass-like" operator (the $q$-deformation of the sum of squares of the translations) will have eigenvalues depending on the "spin-like" parameter $n_2$ as in (35). This is reminiscent of a famous feature of $SU(6)$ type models. This seeping of internal discrete quantum numbers into the "mass-like" spectrum seems to be a typical feature of $q$-deformations [10]. But for the orthogonal symmetry (at least for the present example) this turns out to be an intriguing
consequence of non-minimality. One need not inject ansätze to construct a
mass spectrum depending on internal quantum numbers. One just solves the
mathematical problem of constructing representations explicitly and the result
is there.

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