A Linear Algebra Formulation for Boolean Satisfiability Testing

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Abstract

Boolean satisfiability (SAT) is a fundamental problem in computer science, which is one of the first proven NP-complete problems. Although there is no known theoretically polynomial time algorithm for SAT, many heuristic SAT methods have been developed for practical problems. For the sake of efficiency, various techniques were explored, from discrete to continuous methods, from sequential to parallel programings, from constrained to unconstrained optimizations, from deterministic to stochastic studies. Anyway, showing the unsatisfiability is a main difficulty in certain sense of finding an efficient algorithm for SAT. To address the difficulty, this article presents a linear algebra formulation for unsatisfiability testing, which is a procedure dramatically different from DPLL. Somehow it gives an affirmative answer to an open question by Kautz and Selman in their article “The State of SAT”. The new approach could provide a chance to disprove satisfiability efficiently by resorting to a linear system having no solution, if the investigated formula is unsatisfiable. Theoretically, the method can be applied to test arbitrary formula in polynomial time. We are not unclear whether it can also show satisfiability efficiently in the same way. If so, \( \text{NP} = \text{P} \). Whatever, our novel method is able to deliver a definite result to uniquely positive 3-SAT in polynomial time. To the best of our knowledge, this constitutes the first polynomial
time algorithm for such problem. Anyway, the new formulation could provide a complementary choice to ad hoc methods for SAT problem.

Index Terms

SAT, NP-complete, Pvs.NP, Linear Algebra, Boolean Function, Boolean Solution

I. INTRODUCTION

Boolean satisfiability (SAT) is a fundamental problem in mathematical logic and computing theory, which is one of the first proven NP-complete problems \cite{1}, \cite{2}. The study of SAT is of significance on both of theory and practice. From the theoretical aspect, the famous open problem NP vs. P is equivalent to whether SAT has a polynomial time decision procedure. Therefore, if SAT is showed in class P then $\text{NP} = \text{P}$. In practice, SAT is widely used in artificial intelligence, automatic theorem proving, electronic design automation, formal verification, model checking, computer architecture design, scheduling problems, and so on \cite{3}. Thus, smart methods to solve SAT play an important role in the development of efficient computing systems. Despite the worst-case exponential running time of all known algorithms for SAT, a lot of impressive progresses have been made in solving satisfiability testing for practical SAT problems with up to a million variables \cite{4}, \cite{5}.

In the last decades, based on the DPLL method \cite{6}, \cite{7}, there were developed a large number of high-performance algorithms: local search algorithms \cite{8}–\cite{11}, stochastic algorithms \cite{12}–\cite{14}, conflict-driven clause learning algorithms \cite{15}–\cite{17}, and so on. These algorithms are somehow logic search methods. Another interesting class of methods are based on various optimization strategies, for example, Lagrangian techniques \cite{18}, \cite{19}, Newton’s method and descent method for Universal SAT \cite{20}. There are too many elegant work to mention, we apologize for missing references, and for more detail please refer to some survey papers such as \cite{20}, \cite{21} and the references therein.

Note that, the inherent intractability of SAT is not caused by the variation of the numbers of it’s solution \cite{22}. In fact, the fastest known worst-case algorithm for unique $k$-SAT is the PPSZ \cite{12}, but it’s expected running time is still exponential in the number of variables. So we might explore the solution of uniquely satisfiable formula for understanding SAT. On the other side, unsatisfiability might be another obstacle for SAT especially when a minimal unsatisfiable
CNF formula contain too many clauses so that it’s searching space is intractably huge. Thus, it was eagerly concerned whether a procedure dramatically different from DPLL can be found for handing unsatisfiable instances. In practice, it was ever for a long time that there was no local search algorithm for unsatisfiable formula before GUNSAT was proposed. For simplifying SAT, researchers are used to restrict SAT to a special categories such as $k$-SAT, XOR-SAT, Horn-SAT, one-in-three SAT and so on. By the Schaefer’s dichotomy theorem, each restriction is either in $\text{P}$ or $\text{NP}$-complete. The $\text{NP}$-complete problem one-in-three SAT receives our attention due to it naturally providing us a conversion from CNF formulas to linear Boolean equations.

Therefore, we start with investigating the uniquely one-in-three SAT. Herein, one-in-three problem is to determine whether there is a truth assignment to Boolean variables of a CNF 3-SAT formula such that each clause has exactly one true literal. Accordingly, the uniquely one-in-three SAT is to determine whether there exists a unique such assignment. Let’s look the following toy examples for some inspiration.

**Example 1.1:** Consider the formula

\[(X_1 \lor X_2 \lor X_3) \land (X_2 \lor X_4 \lor X_5) \land (X_2 \lor X_6) \land (X_3 \lor X_4 \lor X_6)\]  

(1)

which can be naturally transformed into a linear system, with an abuse of notations of Boolean variable and equation variable by the same symbol,

\[X_1 + X_2 + X_3 = 1\]  

(2a)

\[X_2 + X_4 + X_5 = 1\]  

(2b)

\[X_2 + X_6 = 1\]  

(2c)

\[X_3 + X_4 + X_6 = 1\]  

(2d)

so that (1) is one-in-three satisfiable iff (2) has Boolean solution (BoS) which consists of Boolean values \{0, 1\}. Furthermore, (1) is uniquely one-in-three satisfiable iff (2) has a unique BoS. It is easy to verify that (1) is uniquely one-in-three satisfiable by the truth assignment $(T, F, F, F, T, T)$ where $T$ and $F$ denote truth values “True” and “False”, respectively; and (2) has a unique BoS $(1, 0, 0, 0, 1, 1)$. This solution can be completely fixed by non-square quadratic
monomials of all products of variables $X_1, \cdots, X_6$ as follows

\begin{align*}
X_1 \cdot X_5 &= 1 \\ (3a) \\
X_1 \cdot X_6 &= 1 \\ (3b) \\
X_5 \cdot X_6 &= 1 \\ (3c) \\
X_1 \cdot X_2 &= 0 \\ (3d) \\
X_1 \cdot X_3 &= 0 \\ (3e) \\
X_1 \cdot X_4 &= 0 \\ (3f) \\
X_2 \cdot X_3 &= 0 \\ (3g) \\
& \vdots \\ (3h) \\
X_4 \cdot X_6 &= 0 \\ (3i)
\end{align*}

where (3) are the assignments to values 1, and (4) are the assignments to value 0.

Given $n$ many Boolean variables $X = \{X_1, \cdots, X_n\}$, there are $\frac{n(n-1)}{2}$ many non-square quadratic monomials $\{X_iX_j\}_{i \neq j \leq n}$. In the sequel, we assume $n \geq 3$, that is, each formula we are interested contains at least three Boolean variables. In general, each satisfiable assignment for a given one-in-three satisfiable CNF 3-SAT formula determine uniquely a complete Boolean assignment to all non-square quadratic monomials generated by productions among the variables. However, it’s inverse is not true.

**Example 1.2:** In the following formula

$$X_1 \lor X_2 \lor X_3$$ (4)

three satisfiable assignments $(T, F, F), (F, T, F)$ and $(F, F, T)$ generate same Boolean assignment $(0, 0, 0)$ to non-square quadratic monomials $X_1X_2, X_1X_3, X_2X_3$.

As this example contains only one clause, it is natural to exclude such easy case in our investigation. So we assume the clause number, $m$ says, of studied CNF formulas is at least 2 in the follows.

Notice that, the formula (4) is not uniquely a one-in-three satisfiable at all. Anyway, if a CNF 3-SAT formula $\mathcal{F}(X)$ is uniquely a one-in-three satisfiable then the Boolean assignment to all non-square quadratic monomials corresponding with it’s solution can determine the unique
solution. A complete Boolean assignment (CBA) to \( \{X_iX_j\}_{i \neq j \leq n} \) is a Boolean value function \( \Theta : \{X_iX_j\}_{i \neq j \leq n} \mapsto \mathbb{B} \) such that each \( \Theta(X_iX_j) \) is well defined. According to a CBA \( \Theta(X_iX_j) \), we set a Boolean assignment on \( X \) as follows: \( X_i = 1 \) if \( \Theta(X_iX_j) = 1 \) for some \( X_j \), otherwise \( X_i = 1 \), which is called a drawback Boolean assignment (DBA) from non-square quadratic assignment, and denoted by \( \Delta(X) \). A DBA \( \Delta(X) \) from a CBA \( \Theta(X_iX_j) \) is called a Boolean root (BR) of \( \Theta(X_iX_j) \) if \( \Theta(X_iX_j) = \Delta(X)\Delta(X_j) \) for all \( i \neq j \). In term of such terminologies, we may say

**Observation 1.3:** If a CNF 3-SAT formula \( \mathcal{F}(X) \) can deduce a unique CBA \( \Theta(X_iX_j) \) to all non-square quadratic monomials \( \{X_iX_j\}_{i \neq j \leq n} \), then there is probably a unique one-in-three satisfiable solution \( \Delta(X) \) to the given CNF 3-SAT formula only when \( \Delta(X) \) is a BR of \( \Theta(X_iX_j) \).

**Remark 1.4:** We would like to point out that, it is not suggested to apply CBA to generic one-in-three satisfiable problem. This is because that the DBA of CBA \( \Theta(X_iX_j) \) may not provide a solution while there is a solution whose CBA is \( \Theta(X_iX_j) \).

**Example 1.5:** Look at the following formula

\[
(X_1 \lor X_2 \lor X_3) \land (X_1 \lor X_4 \lor X_5)
\]

has a one-in-three satisfiable truth assignment \((T, F, F, F, F)\). The corresponding CBA on \( X_1X_3, X_1X_4, X_1X_5, X_2X_3, X_2X_4, X_2X_5, X_3X_4, X_3X_5, X_4X_5 \) is \((0, 0, 0, 0, 0, 0, 0, 0, 0)\). It’s DBA is \((0, 0, 0, 0, 0)\) corresponds to truth assignment \((F, F, F, F)\) which is not a one-in-three satisfiable truth assignment to [5].

By Observation [1.3], deciding whether a CNF 3-SAT formula is uniquely one-in-three satisfiable amounts to showing the existence of one and only one complete assignments to all non-square quadratic monomials. How can we achieve such goal? Our basic idea is to exploit the algebraic constraint \( X = X^2 \) on Boolean variable \( X \) and linear constraints on variables like [2] to discover the constraints on non-square quadratic monomials. In the following, we illustrate this through another toy example which studies a non-one-in-three satisfiable CNF 3-SAT formula also using such quadratic constraints.

**Example 1.6:** Let’s consider the following CNF 3-SAT formula

\[
(X_1 \lor X_2 \lor X_3) \land (X_2 \lor X_3 \lor X_4) \land (X_1 \lor X_4)
\]
It's linear transformation is

\begin{align}
X_1 + X_2 + X_3 &= 1 \quad (7a) \\
X_2 + X_3 + X_4 &= 1 \quad (7b) \\
X_1 + X_4 &= 1 \quad (7c)
\end{align}

The quadratic constraints on variables \(X_1, \ldots, X_4\) are

\begin{align}
X_1 &= X_1^2 \quad (8a) \\
X_2 &= X_2^2 \quad (8b) \\
X_3 &= X_3^2 \quad (8c) \\
X_4 &= X_4^2 \quad (8d)
\end{align}

Then we square and multiply the equations in (7) each other side by side

\begin{align}
(X_1 + X_2 + X_3)^2 &= 1 \quad (9a) \\
(X_2 + X_3 + X_4)^2 &= 1 \quad (9b) \\
(X_1 + X_4)^2 &= 1 \quad (9c) \\
(X_1 + X_2 + X_3) \cdot (X_2 + X_3 + X_4) &= 1 \quad (9d) \\
(X_1 + X_2 + X_3) \cdot (X_1 + X_4) &= 1 \quad (9e) \\
(X_2 + X_3 + X_4) \cdot (X_1 + X_4) &= 1 \quad (9f)
\end{align}
and multiply the equations between (7) and (8) side by side as follows

\begin{align*}
X_1 \cdot (X_2 + X_3 + X_4) &= X_1^2 & (10a) \\
X_2 \cdot (X_1 + X_3) &= X_2^2 & (10b) \\
X_3 \cdot (X_1 + X_4) &= X_3^2 & (10c) \\
X_4 \cdot (X_1 + X_2 + X_3) &= X_4^2 & (10d) \\
X_1 \cdot (X_1 + X_2 + X_3) &= X_1^2 & (10e) \\
X_2 \cdot (X_1 + X_2 + X_3) &= X_2^2 & (10f) \\
X_3 \cdot (X_1 + X_2 + X_3) &= X_3^2 & (10g) \\
X_2 \cdot (X_2 + X_3 + X_4) &= X_2^2 & (10h) \\
X_3 \cdot (X_2 + X_3 + X_4) &= X_3^2 & (10i) \\
X_4 \cdot (X_2 + X_3 + X_4) &= X_4^2 & (10j) \\
X_1 \cdot (X_1 + X_4) &= X_1^2 & (10k) \\
X_4 \cdot (X_1 + X_4) &= X_4^2 & (10l)
\end{align*}

All equations from (9) to (11) constitute the quadratic constraints for (7). It is not hard to see that (7) has BoS iff the system of quadratic constraints has BoS. From equations (11), we obtain

\begin{align*}
X_1X_2 &= 0 & (11a) \\
X_1X_3 &= 0 & (11b) \\
X_2X_3 &= 0 & (11c) \\
X_2X_4 &= 0 & (11d) \\
X_3X_4 &= 0 & (11e) \\
X_1X_4 &= 0 & (11f)
\end{align*}

Note that, the equations (3) and (4) in Example 1.1 were derived by a similar procedure. Furthermore, it is a definite result. And the solution can be obtained by assigning \( X_1 = 1, X_5 = 1, X_6 = 1 \), since \( X_1X_5 = 1, X_1X_6 = 1 \) and \( X_5X_6 = 1 \) and other \( X_i \)s being 0. In fact, this solution is already showed by another form that \( X_1^2 = 1, X_2^2 = 0, \cdots, X_6^2 = 1 \).
Now, using equations (10) and (11) will get

\[ X_1^2 = 0 \quad (12a) \]
\[ X_2 = 0 \quad (12b) \]
\[ X_3^2 = 0 \quad (12c) \]
\[ X_4^2 = 0 \quad (12d) \]

It is evidently contradicting with the following equations (13) derived from (9) and (11)

\[ X_1^2 + X_2^2 + X_3^2 = 1 \quad (13a) \]
\[ X_2^2 + X_3^2 + X_4^2 = 1 \quad (13b) \]
\[ X_1^2 + X_4^2 = 1 \quad (13c) \]

Then we can immediately infer that (6) is not one-in-three satisfiable.

**Remark 1.7:** Note that, if we use \( Z_k \) taking place of \( X_iX_j \) for \( 1 \leq i, j \leq 4 \), then the resulted linear system \( L(Z) \) about variables \( Z_k \) is inconsistent. The inconsistency of \( L(Z) \) implies that the quadratic system consisting of equations (9)-(11) has no real solution, and so has no BoS. As a result, this way can also show that (6) is not one-in-three satisfiable.

In the following, we just formalize and generalize the basic idea and techniques in above two toy examples. For the sake of clarity, through the article we take the notations roughly as following: capital letters with subindex are used for Boolean and equational variables; script letters \( \ell, C \) and \( \mathcal{F} \), etc, stand for literals, clauses and formulas; low case letters \( f, g, h \) and capital Greek letters \( \Delta, \Theta \), etc, are used for Boolean or real functions; \( \mathbb{B} = \{0, 1\} \) be the set of Boolean values, accordingly \( \mathbb{B}^n \) is the \( n \)-dimensional Boolean space; \( \mathbb{R} \) is the set of real numbers.

The rest of article is organized as follows. Section II provides the basic notions and notations. In section III our novel algorithms are presented and analyzed briefly. Some simple optimizations on the algorithms will be discussed in the section IV. At last, we conclude the article in section V.

II. PRELIMINARIES

This section presents the basic notions and notations for our novel method to unsatisfiability test. By means of them, we give the basic idea of the new method. And also here are some
simple examples to explain the notions and illustrate the basic idea. Roughly, the basic idea is to convert a one-in-three SAT problem into the existence of real solution of some linear system. It consists of several crucial processes as follows

1) First one is *linearization transformation* (LT) that converts a Boolean formula into a linear system such that there is an equivalent relation between the satisfiability of Boolean formula and the existence of Boolean solution of the converted linear system;

2) Second is *quadratic propagation* (QP) that extends a linear systems into some quadratic system such that they have the same Boolean solutions;

3) Third is *linearization abstraction* (LA) that abstracts the quadratic system as a linear system such that they have the same Boolean solutions.

In the procedure, it contains linearization twice which plays crucial role in the conversion from logical to algebraic problem and respectively efficiently solving problem, and one time quadratic algebraic operation that plays a central role in the efficiently BoS searching. So we call this method *linear algebra formulation* summarily. In the article, we follow the standard concepts of propositional logic formula in terms of *literals* and *clauses*, *conjunctive normal form* (CNF), *Boolean satisfiability* (SAT) and *Boolean function* (BF). Given $n$-many Boolean variables $X = \{X_1, \cdots, X_n\}$, a $m$-clauses CNF formula $F$ is denoted by

$$F = \bigwedge_{i \leq m} C_i$$

$$= \bigwedge_{i \leq m} \bigvee_{j \leq j_i} \ell_{ij}(X)$$

where $C_i = \bigvee_{j \leq j_i} \ell_{ij}(X)$, and each literal $\ell_{ij}(X)$ is of form $X_k$ or $\neg X_k$ for some $1 \leq k \leq n$. As convention, $F$ has *pure polarity* if at most one of $X_k$ and $\neg X_k$ can occur in $F$ for any $1 \leq k \leq n$, $F$ is a *positive formula* if merely $X_k$ can occurs in $F$ and is a 3-SAT formula if $j_i \leq 3$ for all $i \leq m$.

In the process of LT, the notion of exactly one satisfiability plays a key role. A formula $F$ of form (14) is called *exactly one satisfiable* (EOS) if there is a truth assignment $X^*$ to the variables $X$ such that each clause $C_i$ has exactly one true literal. EOS is a generalization of one-in-three 3-SAT [24]. Two formulas $F$ and $H$ are said *equi-exactly-one-satisfiable* (equi-EOS) if $F$ is exactly one satisfiable whenever $H$ is and vice versa. If the $F$ defined by (14) is of pure polarity, then we could construct an equi-EOS positive formula $F^*$ by simply substituting
all negation literal \( \neg X_k \) with \( X_k \). As \( X_k \) does not occur in \( F \), \( F \) and \( H \) must be equi-EOS. Therefore, we can conclude that

**Proposition 2.1:** Each pure polarity formula is equi-EOS to a positive formula.

To describe the intrinsic structure of resulted linear system from LT, we introduce the following notions and notations. A **Boolean function** (BF) is defined as usual meaning by a map \( f : \mathbb{B}^n \mapsto \mathbb{B} \), and said **linear Boolean summation**, **polynomial Boolean summation** and **homogeneous Boolean summation**, respectively, if \( f(X) \) is a positive summation of one degree, arbitrary degree monomials and homogeneous monomials with respect to variables \( X = \{X_1, \ldots, X_n\} \) such that all coefficients and constants are Boolean values.

**Example 2.2:** For instance, \( X_1 + X_2 + X_3 + X_4 \) is a linear Boolean summation, but is neither \( X_1 + X_2 - X_3 + X_4 \) nor \( X_1 + X_2 + X_3 + 2 \); \( X_1X_2 + X_3 + 1 \) is a polynomial Boolean summation, but is neither \( X_1X_2 - X_3 + 1 \) nor \( X_1X_2 + X_3 - 1 \); \( X_1X_2 + X_3X_4 \) is a homogeneous Boolean summation, but is not \( X_1X_2 + X_3X_4 + 1 \).

A **Boolean equation** (BE) is of form \( f(X) - b = 0 \) such that \( f \) is a BF and \( b \in \mathbb{B} \). Accordingly, a BE \( f(X) - b = 0 \) is called a **linear Boolean summation equation**, **polynomial Boolean summation equation** and **homogeneous Boolean summation equation** if \( f(X) \) is linear Boolean summation, polynomial Boolean summation and homogeneous Boolean summation, respectively. And a BE \( f(X) - g(X) = 0 \) is called a **variational linear Boolean summation equation** (or polynomial Boolean summation equation, or homogeneous Boolean summation equation) if \( f(X) \) is linear Boolean summation (or polynomial Boolean summation, or homogeneous Boolean summation) and \( g(X) \) is a single Boolean variable (or monomial, or monomial of degree as same as \( f(X) \)). Furthermore, a system of BEs \( \{f_i(X) - b_i = 0\}_{i \leq m} \) is called **linear Boolean summation system** (LBS), **polynomial Boolean summation systems** (PBS) and **homogeneous Boolean summation systems** (HBS), if the corresponding BEs is linear Boolean summation equations, polynomial Boolean summation equations, homogeneous Boolean summation equations, respectively. Similarly, we define \( \{f_i(X) - g_i(X) = 0\}_{i \leq m} \) as **variational linear Boolean summation system** (VLBS), **variational polynomial Boolean summation systems** (VPBS) and **variational homogeneous Boolean summation systems** (VHBS). As usual, a **Boolean solution** (BoS) to a given LBS/VLBS, PBS/VPBS or HBS/VHBS, is a Boolean vector which satisfies all BEs in the corresponding systems.
For a given positive formula $F$ of form (14), we would like to make a reduction of its EOS to existence of BoS of an LBS (15) as follows.

$$\sum_{j \leq j_i} \ell_{i,j}(X) = 1 \quad 1 \leq i \leq m,$$

(15)

where all $\ell_{i,j}(X) = X_k$ become equation variables from positive literals in $F$ for some $1 \leq k \leq n$. If we take a one-to-one relation between truth values \{True, False\} and Boolean values \{1, 0\} as True $\leftrightarrow$ 1, False $\leftrightarrow$ 0. Then it is easy to see

**Proposition 2.3:** A positive formula $F$ of form (14) is EOS iff the corresponding LBS (15) has a BoS.

As a result, we can solve the EOS of a positive formula through investigating the BoS of the corresponding LBS. By Proposition 2.1, the EOS of each pure polarity formula can also be studied by a similar way. In general, a CNF formula $F$ may have no pure polarity property. In this case, we introduce auxiliary variables $Y_k$ for all $\neg X_k$. Then we could construct an equi-EOS positive formula $F^*$ as follows. Let $F(Y/\neg X)$ be the resulted formula by substitute all negative literals $\neg X_k$ with $Y_k$ in $F$. Now, $F^*$ is defined by

$$F^* = \wedge_k (X_k \vee Y_k) \wedge F(Y/\neg X)$$

(16)

where $k$ ranges in an index set $I$ such that $\neg X_k$ occurs in $F$ then $k \in I$. Such $F^*$ is called a positivization of $F$, also denoted by $P(F)$. Therefore, it is actually true that

**Proposition 2.4:** Each CNF formula $F$ like (14) is equi-EOS to a positive formula $P(F)$ like (16).

Accordingly, we can convert the EOS problem of $F$ into the existence problem of BoS of the following LBS

$$\sum_{j \leq j_i} \ell_{i,j}^*(X) = 1, \quad 1 \leq i \leq m$$

(17a)

$$X_k + Y_k = 1 \quad k \in I$$

(17b)

where $\ell_{i,j}^*(X) = X_k$ if $\ell_{i,j}(X) = X_k$ and $\ell_{i,j}^*(X) = Y_k$ if $\ell_{i,j}(X) = \neg X_k$. As a consequence, we can convert the EOS problem of a generic CNF formula $F$ like (14) into the existence of BoS of (17). The transformation from a CNF formula $F$ into a LBS like (15) or (17) is the so-called linearization transformation. Without loss of generality, we study the BoS of LBS (15) for positive formula in stead of generic ones in the sequel.
Generally, it is NP-hard to decide if an LBS (17) has a BoS. Anyway, we could exploit the idea behind the toy examples in the last section to approximate BoS. The basic idea is to resort to finding some easily decidable BoS-equisolvable LBS, where two LBSs \( L_1 \) and \( L_2 \) are BoS-equisolvable if they satisfy: \( L_1 \) has BoS iff \( L_2 \) has BoS. In light of this, we extend a given LBS \( L \) to some BoS-equisolvable super LBS \( L^* \) containing \( L \) as a subsystem. To this end, the equations in \( L^* \) are obtained by two types of algebraic operations on \( L \), which are called quadratic propagation (QP) and linearization abstraction (LA). Herein, the QP of (15) consists of two sorts of operations. One is done by mutually multiplying equations inside \( L \) side by side, which is called inner quadratic propagation (IQP), another is accomplished through side by side multiplications over equations of \( L \) and quadratic constraints on Boolean variables which restrict the Boolean variables by

\[
X_k = X_k^2, \quad 1 \leq k \leq n
\]  

which is called constraint quadratic propagation (CQP). Formally, IQP and CQP are carried out respectively by

\[
\left( \sum_{j \leq j_i} \ell_{i,j}(X) \right) \cdot \left( \sum_{j \leq j_t} \ell_{t,j}(X) \right) = 1, \quad 1 \leq i, t \leq m
\]  

\[
X_k \cdot \sum_{j \leq j_i} \ell_{i,j}(X) = X_k^2, \quad 1 \leq k \leq n, 1 \leq i \leq m
\]

The idea of LA is to substitute all quadratic monomials \( X_i X_j \) in the VHBS (19) with new variables, \( Z_u \) says, in order to transform such VHBS into a VLBS. Herein, \( u \leq \frac{n(n+1)}{2} \).

**Example 2.5:** Let’s consider the following CNF formula

\[
F(X) := (X_1 \lor X_2 \lor X_3) \land (\neg X_1 \lor X_2 \lor X_3)
\]  

It is easy to verify that \( F(X) \) is not EOS. We could convert it into an LBS through LT as follows

\[
X_1 + X_2 + X_3 = 1
\]  

\[
Y_1 + X_2 + X_3 = 1
\]  

\[
X_1 + Y_1 = 1
\]
It is easy to see that (21) has real number solutions whose space dimension is one but no BoS. Multiplying the equations in ((21) each other obtains \( \frac{3(3+1)}{2} = 6 \) quadratic equations

\[
X_1^2 + X_2^2 + X_3^2 + 2X_1X_2 + 2X_1X_3 + 2X_2X_3 = 1
\]

\[
Y_1^2 + X_2^2 + X_3^2 + 2Y_1X_2 + 2Y_1X_3 + 2X_2X_3 = 1
\]

\[
X_1^2 + Y_1^2 + 2X_1Y_1 = 1
\]

\[
X_1Y_1 + X_1X_2 + X_1X_3 + X_2Y_1 + X_2^2 + 2X_2X_3 + X_3Y_1 + X_3^2 = 1
\]

\[
Y_1^2 + X_1Y_1 + Y_1X_2 + X_1X_2 + X_3Y_1 + X_1X_3 = 1
\]

\[
Y_1^2 + X_1Y_1 + Y_1X_2 + X_1X_2 + X_3Y_1 + X_1X_3 = 1
\]

and we employ the following constraints on Boolean variables \( X_1, X_2, X_3, Y_1 \)

\[
X_1 = X_1^2, \quad X_2 = X_2^2, \quad X_3 = X_3^2, \quad Y_1 = Y_1^2
\]

Then we multiply equations of (21) and (23) side by side, and get \( 3 \times 4 = 12 \) quadratic equations such as

\[
X_1^2 + X_1X_2 + X_1X_3 = X_1^2
\]

\[
X_1Y_1 + X_1X_2 + X_1X_3 = X_1^2
\]

\[
X_1^2 + X_1Y_1 = X_1^2
\]

\[
X_1X_2 + X_2^2 + X_2X_3 = X_2^2
\]

\[
\vdots
\]

Based on equations (22) and (24), the LA carry out by replacing totally \( \frac{4(4+1)}{2} = 10 \) quadratic monomials through

\[
Z_1 = X_1^2, \quad Z_2 = X_2^2, \quad Z_3 = X_3^2, \quad Z_4 = Y_1^2
\]

\[
Z_5 = X_1X_2, \quad Z_6 = X_1X_3, \quad Z_7 = X_1Y_1
\]

\[
Z_8 = X_2X_3, \quad Z_9 = X_2Y_1, \quad Z_{10} = Y_1X_3
\]
herein, $X_iX_j$ and $X_jX_i$ are replaced by the same variable since their multiplications are same. As a result, it gets a linear system consisting of 18 linear equations about $Z_k$ like

\[ Z_1 + Z_2 + Z_3 + 2Z_5 + 2Z_6 + 2Z_8 = 1 \]  \hspace{1cm} (26a)

\[ Z_4 + Z_1 + Z_3 + 2Z_9 + 2Z_{10} + 2Z_8 = 1 \]  \hspace{1cm} (26b)

\[ Z_1 + Z_4 + 2Z_7 = 1 \]  \hspace{1cm} (26c)

\[ Z_7 + Z_5 + Z_6 + Z_9 + Z_2 + 2Z_8 + Z_{10} + Z_3 = 1 \]  \hspace{1cm} (26d)

\[ \vdots \]

This linear system has no real number solution at all. As a consequence, it has no BoS. Therefore, (21) has no BoS. That is, the EOS of $\mathcal{F}(X)$ might be reduced to the existence of real solution of such linear system of variables $Z_k$.

### III. Algorithms and Brief Analysis

In the previous sections, we showed the basic idea of our linear algebra formulation. Therein, the linear algebra method was successful to carry out exactly one unsatisfiability testing on some examples. In fact, it might be applicable to any such test. Furthermore, it can be applied to general Boolean satisfiability problem by additional effort. Anyway, the linear algebra test for exactly one satisfiability plays a central role in this approach. Hence, we present this method in a more formal way by Algorithm 1. Herein, outcome “EOS” means that $\mathcal{F}$ is EOS; outcome “NEOS” means that $\mathcal{F}$ is not exactly one satisfiable; outcome “Unknown” stands for that the true answer is unknown; and a non-square variable means a $Z_k$ standing for a non-square quadratic monomial $X_iX_j$ for $i \neq j$ such as $Z_5$ to $Z_{10}$ in (25).

By the construction of LBS (15) from a positive formula $\mathcal{F}$, there is a bijection between the EOSs of $\mathcal{F}$ and BoSs of (15). Therefore, if a positive formula $\mathcal{F}$ is uniquely EOS then there is a unique BoS of of (15). In this case, it is not hard to verify all non-square variables of $L(Z)$ in Algorithm 1 must be fixed. Therefore, we can conclude the following assertion.

**Proposition 3.1:** If a positive formula $\mathcal{F}$ is uniquely EOS, then Algorithm 1 will output “EOS” definitely.

For the soundness of Algorithm 1, it can be guaranteed by Proposition 2.3 and the following results.
Theorem 3.2: The following three assertions are equivalent:
1) The LBS (15) has BoS;
2) The quadratic VHBS (19) has BoS;
3) The $L(Z)$ in Algorithm 1 has BoS.

Proof: It is easy to show implications from 1) to 2), and from 2) to 3). In fact, $L(Z)$ can be simplified so that it contains (15) as a subsystem. Therefore, 3) naturally implies 1).

Based on this and Proposition 2.3, we obtain a sufficient condition for that a positive formula $F$ has no BoS, as follows.

Proposition 3.3: If $L(Z)$ in Algorithm 1 has no solution in $\mathbb{R}^k$, then it definitely has no BoS. Therefore, $F$ is not EOS.

As a result, the satisfiability problem of a generic CNF formula $F$ can be converted into a real solution existence problem of some LBS/VLBS, if we have a reduction from SAT to EOS. Fortunately, it was well done by 3-SAT reduction that establishes a reduction from generic CNF formula to an equisatisfiable 3-SAT formula, and one-in-three 3-SAT reduction by Schaefer [24] who provided a reduction from 3-SAT formulas to equisatisfiable one-in-three 3-SAT formulas. Moreover, these reductions can be done in polynomial time.

Now it is ready to give the scheme on deciding the generic satisfiability in stead of EOS using linear algebra. The routine is performing a series of equisatisfiable transformations as follows. Given a generic Boolean formula $G(X)$ with variables $X = \{X_1, \ldots, X_n\}$, we carry out the following process:

1) First, we transform $G(X)$ into an equisatisfiable CNF formula, says $G^*(X, Y)$.
2) For $G^*(X, Y)$, we compute an equisatisfiable 3-SAT formula $T(X, Y, Z)$.
3) Based on $T(X, Y, Z)$, we compose a positive formula $F(X, Y, Z, U)$ such that $T(X, Y, Z)$ is satisfiable if and only if $F(X, Y, Z, U)$ is EOS.
4) Now, we apply Algorithm 1 to $F(X, Y, Z, U)$. If the answer is “EOS” for $F(X, Y, Z, U)$ then $G(X)$ is satisfiable, similarly, if “NEOS” outputs then $G(X)$ is unsatisfiable. As for answer “Unknown”, we cannot conclude the satisfiability of $G(X)$.

The whole procedure could be more formally summarized by Algorithm 2.

Naturally, we might concern which kinds of or how many unsatisfiable formulas can be tested by Algorithm 2. First of all, uniquely one-in-three positive 3-SAT can be tested efficiently based
upon Algorithm 1, Schaefer has given a polynomial time reduction \cite{24} from 3-SAT $\mathcal{F}$ to one-in-three 3-SAT $\mathcal{F}^*$ by substituting each clause $X \lor Y \lor Z$ in $\mathcal{F}$ with

$$(X \lor V_1 \lor V_2) \land (Y \lor V_2 \lor V_3) \land (Z \lor V_4) \land$$

$$(V_1 \lor V_3 \lor V_5) \land (V_2 \lor V_4 \lor V_6)$$

(27)

Herein, $\mathcal{F}^*$ is still a positive formula. Using this reduction, it is easy to verify

**Proposition 3.4:** There is a one-to-one relation between the satisfiability of $\mathcal{F}$ and one-in-three satisfiability of $\mathcal{F}^*$. In particular, $\mathcal{F}$ has a unique satisfiable solution iff $\mathcal{F}^*$ has a unique one-in-three satisfiable solution.

By Observation 1.3 and the process of Algorithm 1, the uniquely one-in-three satisfiability can be tested by using Algorithm 1 with a slight modification at step 26 by setting “Answer=UEOS”, e.g. uniquely exact-one-satisfiable. It is not hard to see Algorithm 1 in polynomial time class. Therefore, we actually have showed

**Proposition 3.5:** Unique 3-SAT can be decided in polynomial time.

On the other hand, unique 3-SAT is probably as hard as SAT \cite{22}, \cite{25}. If it is proven true, then $\text{NP} = \text{P}$ will be true by this proposition.

As the final $L(Z)$ in Algorithm 1 show an inclusion relation formulas $\mathcal{F}$ and $\mathcal{F} \land \mathcal{E}$, by the relation between the solutions of such two systems, it can easily infer the following proposition.

**Proposition 3.6:** If Algorithm 2 outputs “UNSAT” for a fixed formula $\mathcal{F}$, then given arbitrary formula $\mathcal{E}$, it will output “UNSAT” for $\mathcal{F} \land \mathcal{E}$.

On this account we might concentrate on so-called minimal unsatisfiable formulas. Given a CNF unsatisfiable $\mathcal{F} = \land_{i \leq m} \mathcal{C}_i$, a minimal unsatisfiable core \cite{26}, \cite{27} of $\mathcal{F}$ is a subformula $\mathcal{M}$ of it such that $\mathcal{M}$ is unsatisfiable and any proper subformula of $\mathcal{M}$ is satisfiable. Similarly, for a non-exactly-one-satisfiable CNF $\mathcal{F} = \land_{i \leq m} \mathcal{C}_i$, a minimal non-exactly-one-satisfiable core of $\mathcal{F}$ is a subformula $\mathcal{M}$ of it such that $\mathcal{M}$ is non-exactly-one-satisfiable and any proper subformula of $\mathcal{M}$ is EOS. A formula $\mathcal{F}$ is said an atomic non-exactly-one-satisfiable formula if $\mathcal{F}$ is a minimal non-exactly-one-satisfiable core of itself, moreover, if such $\mathcal{F}$ is also positive then we say $\mathcal{F}$ a positively atomic non-exactly-one-satisfiable formula. Let $L(\mathcal{F})$ be the linear system like (15) converted from $\mathcal{F}$ through LT, then we have
Proposition 3.7: If a CNF $F$ is positively atomic non-exactly-one-satisfiable, then the system $L(F)$ has no BoS and it’s coefficient matrix and augmented matrix both are linearly independent respectively.

As a consequence, such positively atomic non-exactly-one-satisfiable CNF formula $F$ always has small number of clauses, which is no more than the number of variables of $F$. This takes some insights of the non-exactly-one-satisfiable property from algebraic aspects. However, the numbers of equations of $L(F)$ could not be the decisive factor as for whether the final linear system $L(Z)$ in Algorithm 1 has BoS.

Example 3.8: For any $k \geq 8$, take a positive CNF formula
\[(X_1 \lor X_2) \land (X_1 \lor X_3) \land (X_2 \lor X_3 \lor X_4) \land (X_1 \lor X_4 \lor \cdots \lor X_k)\] (28)

After LT, it’s LBS consists of the following equations
\[X_1 + X_2 = 1\] (29a)
\[X_1 + X_3 = 1\] (29b)
\[X_2 + X_3 + X_4 = 1\] (29c)
\[X_1 + X_4 + \cdots + X_k = 1\] (29d)

It is easy to compute the number $K = 10 + 4k$ of final linear system $L(Z)$ after LA, but the number of variables $Z$ is $\frac{k(k+1)}{2} > K$ whenever $k \geq 10$. However, $L(Z)$ is always inconsistent no matter how big $k$ is. That is, Algorithm 1 can always output a definite answer.

Importantly, if Algorithm 1 is able to output “NEOS” for each positively atomic non-exactly-one-satisfiable CNF formula, then Algorithm 1 is able to output “NEOS” for all non-exactly-one-satisfiable CNF formula by Proposition 3.6. Accordingly, Algorithm 2 is able to output a definite answer “SAT” or “UNSAT”. Various examples indicate that the intrinsic nature of Algorithm 1 may result in a definite answer. Nevertheless, all examples we tried support

Conjecture 3.9: For each positively atomic non-exactly-one-satisfiable CNF $F$, Algorithm 1 can always output “NEOS”.

In other words, we conjecture that $L(Z)$ in Algorithm 1 being inconsistent is a necessary and sufficient condition for a positive CNF $F$ formula that is atomic non-exactly-one-satisfiable. If
Conjecture 3.9 is true, we could modify Algorithm 1 through replacing “Unknown” by “EOS”, then it can always get a definite answer “EOS” or “NEOS”. Accordingly, we could modify Algorithm 2 and then obtain a definite answer “SAT” or “UNSAT”. As all computations in Algorithms 1 and 2 can be done in polynomial time, if Conjecture 3.9 is true, then SAT can be solved in polynomial time by the modified algorithms and so \( \text{NP} = \text{P} \), although it is not consistent with popular hypothesis in cryptography and most complexity theorists’ opinion [28].

**Remark 3.10:** Note that, it could choose different sufficient conditions as criteria for “NEOS” in Algorithm 1 and so for “UNSAT” in Algorithm 2. For instance, it suffices to show \( L(Z) \) has no solution in any finite field space \( \mathbb{F}_p^k \), particularly in \( \mathbb{F}_2^k \) and \( \mathbb{F}_3^k \), where \( k \) is the dimension of space. Accordingly, we might investigate different versions of Conjecture 3.9. Hopefully, it could give us some insight over filed \( \mathbb{F}_2^k \) or \( \mathbb{F}_3^k \).

**IV. OPTIMIZATION DISCUSSION**

In the previous sections, we just present the basic method and algorithms. For high quality performance, these algorithms can be optimized by various ways. This section will proposes two such optimizations. One of them exploits a close relation among linear equations, the linear dependence, to reduce the size of LBS used for QP. For the same purpose, another optimization technique make use of a loose relation among linear equations, system blocking.

The first optimization is to reduce the LBS (15) obtained by LT step, if it is possible. Let \( L(Z) = \{ L_i(Z) = 1 \} \) be the linear system consists \( M = mn + \frac{m(m+1)}{2} \) many linear equations \( L_i(Z) = 1 \), which are obtained by LA on quadratic HBS/VHBS (19). Then, it is not hard to verify

\[
\text{Proposition 4.1: If (15) is linearly dependent then } L(Z) \text{ must be so.}
\]

Furthermore, assume \( \sum_{j \leq j_m} \ell_{m,j}(X) = 1 \) is a linear combination of \( \sum_{j \leq j_m} \ell_{i,j}(X) = 1 \) for \( i \neq m \), that is, there are \( c_i \in \mathbb{R} \) for \( 1 \leq i < m \) such that

\[
\sum_{i<m} c_i = 1 \quad (30a)
\]

\[
\sum_{j \leq j_m} \ell_{m,j}(X) = \sum_{i<m} c_i \sum_{j \leq j_i} \ell_{i,j}(X) \quad (30b)
\]

Let’s list the linear equations \( L_k(Z) \) obtained by LA on equations including \( \sum_{j \leq j_m} \ell_{m,j}(X) \) as a factor to the left sides of (19a) and (19b) in the end of the linear system \( L(Z) \), that is, \( u \leq k \leq M \) for some \( u \). Then we can show that
Proposition 4.2: Under the linear dependence assumption (30), there are $m + n$ such linear equations $L_k(Z)$, i.e., $M = u + m + n - 1$, and each $L_k(Z)$ for $u \leq k \leq M$ is a linear combination of $L_j(Z)$ for $1 \leq j < u$. Therefore, two linear systems $\{L_k(Z)\}_{1 \leq i < u}$ and $\{L_i(Z)\}_{1 \leq i \leq M}$ have the same solution.

In this case, if we reduce the LBS (15) to its subsystem $\{\sum_{j \leq i \leq u} \ell_{i,j}(X) = 1\}_{i < m}$ then Algorithm 1 would still output the same result, in the mean time, the computational cost will be cut down significantly.

A second optimization is based on system blocking. Let $L_1$ and $L_2$ be two disjoint subsystems of a given LBS $L$ such that their join is $L$, they are said separated subsystems of $L$ if their variable sets, $V_1$ and $V_2$ respectively says, are disjoint. Herein, $L_1$ and $L_2$ are called separated blocks of $L$. Such LBS $L$ is called blockable. Then, as to the BoSs of $L$, $L_1$ and $L_2$, it holds the following relation.

Proposition 4.3: A blockable LBS $L$ has BoS iff both $L_1$ and $L_2$ have BoSs. Furthermore, a Boolean vector $(V_1^*, V_2^*)$ is a BoS of $L$ iff $V_1^*$ is a BoS of $L_1$ and $V_2^*$ is a BoS of $L_2$.

In light of this result, Algorithm could be optimized in what follows. First, we divide LBS (15) into separated blocks $L_1, \ldots, L_k$, if possible. Then on each block we reduce it’s size using the linear dependence relation if possible. Afterwards, we apply Algorithm 1 on each reduced block instead of LBS (15). As each reduced block has less variables and equations, this technique might speed up the process especially on blockable linear Boolean systems.

V. CONCLUSION

In the article, we propose a novel method to unsatisfiability test. This method mainly establishes an equivalent relation between satisfiability of Boolean formulas and Boolean solvability of linear Boolean summation systems, and bring up a new method to solving LBS which exploits the intrinsic nature of LBS and it’s BoS. As it is known now, our new approach is a procedure dramatically different from DPLL. Hence, it gives an affirmative answer to the question in the end of Challenge 1 in [21]. More importantly, we developed two polynomial time algorithms based upon our new method. As a result, if a positive formula is uniquely exact-one-satisfiable, then our Algorithm 1 is able to deliver a definite result in polynomial time rather than exponential time of well known algorithms such as PPSZ algorithm [12]. However, the reasons underlying this remarkable performance are not yet fully understood. Nonetheless, various examples support
that our method would be a hopeful polynomial time algorithm for SAT if \( \text{NP} = \text{P} \). Although the new method could be applied to arbitrary Boolean formula, it can only provide us partial solutions for SAT. Hence it is an incomplete method. Here, we leave a big question whether there is a simply complete solution to SAT via this method, which amounts to whether \( \text{NP} = \text{P} \) is true. Once Conjecture 3.9 is proven true, \( \text{NP} = \text{P} \) would follow immediately.

Due to the open conjecture, the current two algorithms can only provide partial solution for SAT, which is especially expected outperformance on unsatisfiable problems. Therefore, we should explore other skills based on the kernel algorithm to find satisfiable instance. On the other side, only theoretical algorithms were proposed in this article, we are going to test them on benchmarks to investigate their practical performances, in the future.

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Algorithm 1 The Kernel Algorithm: LAT for EOS

**Input:** A CNF positive formula $F(X) = C_i(X) = \bigwedge_i \bigvee_j L_{ij}(X)$ with variables $X = \{X_1, \ldots, X_n\}$;

**Initial:** Answer= “Unknown”;

**Output:** Answer= “EOS”, or “NEOS”, or “Unknown”.

1. Do LT on $F(X)$ and obtain an LBS [15]
2. Decide whether [15] has a unique solution in $\mathbb{R}^n$
3. if [15] has a unique solution, $X^*$ says then
4. Decide whether $X^*$ is a BoS
5. if $X^*$ is a BoS then
6. Set Answer= “EOS”, and go to [56]
7. if $X^*$ is not a BoS then
8. if $X^*$ has no solution in $\mathbb{R}^n$ then
9. Go to [15]
10. if $X^*$ has more than one solution in $\mathbb{R}^n$ then
11. end if
12. end if
13. end if
14. end if
15. end if
16. end if
17. end if
18. Do QP on [15], and obtain a VHBS [19]
19. Do LA by substitution on [19], and obtain a VLBS, says $L(Z)$
20. Decide whether $L(Z)$ has solution in $\mathbb{R}^n$
21. if $L(Z)$ has no solution in $\mathbb{R}^n$ then
22. Set Answer= “NEOS”, and go to [56]
23. if $L(Z)$ has solution in $\mathbb{R}^n$ then
24. Check whether all non-square variables are Boolean constants and it’s DBA is a BR
25. if Yes then
26. Set Answer= “EOS”
27. if All non-square variables are constants but neither Boolean nor is it’s DBA a BR then
28. Set Answer= “NEOS”
29. if Not all non-square variables are constants then
30. Set Answer= “Unknown”, and go to [56]
31. end if
32. end if
33. end if
34. end if
35. end if
36. return Answer
Algorithm 2 Linear Algebraic Approach Test for SAT

Input: A CNF positive formula $G(X)$ with variables $X$;
Initial: Answer = "Unknown";
Output: Answer = "SAT", or "UNSAT", or "Unknown".

1: Transform $G(X)$ into an equisatisfiable CNF formula $G^*$
2: Compute an equisatisfiable 3-SAT formula $T$ for $G^*$
3: Compose a positive formula $F$ for $T$
4: Call Algorithm 1 for $F$
5: if Output is “EOS” then
   6: Set Answer = "SAT", and go to 14
7: if Output is “NEOS” then
   8: Set Answer = "UNSAT", and go to 14
9: if Output is “Unknown” then
   10: Set Answer = "Unknown", and go to 14
11: end if
12: end if
13: end if
14: return Answer