On the Schur Lie-multiplier and Lie-covers of Leibniz $n$-algebras

Hesam Safa and Guy R. Biyogmam

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Abstract. In this article, we study the notion of central extension of Leibniz $n$-algebras relative to $n$-Lie algebras to study properties of Schur Lie-multiplier and Lie-covers on Leibniz $n$-algebras. We provide a characterization of Lie-perfect Leibniz $n$-algebras by means of universal Lie-central extensions. It is also provided some inequalities on the dimension of the Schur Lie-multiplier of Leibniz $n$-algebras. Analogue to Wiegold [38] and Green [17] results on groups or Moneyhun [26] result on Lie algebras, we provide upper bounds for the dimension of the Lie-commutator of a Leibniz $n$-algebra with finite dimensional Lie-central factor, and also for the dimension of the Schur Lie-multiplier of a finite dimensional Leibniz $n$-algebra.

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1 Introduction

The concept of Schur multiplier was introduced by Schur [37] in 1904 in his study of projective representations of a group. Since then, it has appeared in many studies related to mathematical concepts such as efficient presentations, homology and projective representations of various algebraic structures. The Schur multiplier is very useful as a tool to classify $p$-groups and nilpotent Lie algebras, thanks to the results obtained by Green [17] and Moneyhun [26], respectively. In fact, Green showed that for every finite $p$-group $G$ of order $p^k$, there is a non-negative integer $t(G)$ such that $|M(G)| = p^{\frac{1}{2}k(k-1)-t(G)}$, and Moneyhun showed that if $L$ is a Lie algebra of dimension $k$, then $\dim M(L) \leq \frac{1}{2}k(k-1)$.

In this paper, we aim to provide a similar upper bound for Leibniz $n$-algebras, which is a generalization of both Leibniz algebras [25] and $n$-Lie algebras (also known as Filippov algebras) [15]. This upper bound can be useful in characterizing Lie-nilpotent Leibniz algebras, and more generally, Lie-nilpotent Leibniz $n$-algebras.

Some research interest has focussed on investigating properties on Leibniz $n$-algebras analogue to results obtained on Leibniz algebras and $n$-Lie algebras. It turns out that
several of these properties fail to extend to Leibniz $n$-algebras. The present paper provides a few counter-examples.

The concept of isoclinism was initiated in 1939 by P. Hall in his work on the classification of $p$-groups using an equivalence relation weaker than the notion of isomorphism [18]. Studies of this concept on various algebraic structures can be found in [4, 14, 21, 29, 36].

Following a philosophy that comes from the categorical theory of central extensions relative to a chosen Birkhoff subcategory of a semi-abelian category [11], and applying on the semi-abelian category of Leibniz algebras with respect to its Birkhoff subcategory of Lie algebras, the concepts of Schur multiplier, isoclinism, and cover have been recently considered on Leibniz algebra in the relative context, i.e. with respect to the Liezation functor $(-)_{\text{Lie}}: \text{Leib} \to \text{Lie}$ which assigns the Lie algebra $q_{\text{Lie}} = \langle [x,x] : x \in q \rangle$ to a given Leibniz algebra $q$. This yielded the notions of Schur Lie-multiplier, Lie-isoclinism, Lie-cover [5, 6, 7, 9] as well as other notions such as Lie-stem Leibniz algebras, Lie-perfect Leibniz algebras and Lie-abelian Leibniz algebras. Note that these new notions provide a solid framework in which one can identify properties on $n$-Lie algebras and Leibniz algebras that cannot be extended to Leibniz $n$-algebras. For instance, the concepts of isomorphism and isoclinism coincide for finite dimensional stem $n$-Lie algebras (see [14, 26, 35]). However, this does not hold for finite dimensional Lie-stem Leibniz $n$-algebras ($n \geq 2$), as discussed in [31]. Also, the Schur Lie-multiplier, and more generally, the $c$-nilpotent Schur Lie-multiplier of a Leibniz algebra are both Lie algebras. But, this is not true for Leibniz $n$-algebras with $n \geq 3$, as discussed in Remark 2.1. Moreover, it is shown in [12] that a $k$-dimensional $n$-Lie algebra $L$ is abelian if and only if the dimension of its Schur multiplier $\dim M(L) = \binom{k}{n}$. This does not hold for Leibniz $n$-algebras with $n \geq 3$, as discussed in Example 4.9.

In this article, we continue the study of Lie-central extensions on Leibniz $n$-algebras, initiated in [33, 34]. Concretely, we study the notions of Schur Lie-multiplier and Lie-cover on Leibniz $n$-algebras, and their interplay with Lie-isoclinism. In doing so, we organize the paper as follows: in section 2, we recall the Lie-notions from [33], define the concept of Schur Lie-multiplier on Leibniz $n$-algebras and present some preliminaries including useful results from [34]. In section 3, we study several properties of Lie-covers of Leibniz $n$-algebras. In particular, we show that every Lie-perfect Leibniz $n$-algebra admits at least one Lie-cover. Also, using a homological method analogue to Loday’s discussion in [23] on a necessary and sufficient condition for a pair of groups to be perfect, we provide a characterization of Lie-perfect Leibniz $n$-algebras by means of universal Lie-central extensions. The last section is devoted to some inequalities on the dimension of the Schur Lie-multiplier of Leibniz $n$-algebras. In particular, we establish analogues of Wiegold [38] (resp. Moneyhun [26]) result on an upper bound for the order (resp. dimension) of the commutator subgroup (resp. derived algebra) of a group with finite central factor (resp. Lie algebra with finite dimensional central factor), and also Moneyhun [26] result on an upper bound for the dimension of the Schur multiplier of a finite dimensional Lie algebra. More precisely, we show that for a Leibniz $n$-algebra $q$ with $k$-dimensional Lie-central
factor $q/Z_{\text{Lie}}(q)$, Lie-commutator $q^n_{\text{Lie}}$,
\[
\dim q^n_{\text{Lie}} \leq \sum_{i=1}^{n} \binom{n-1}{i-1} \binom{k}{i},
\]
and also for a $k$-dimensional Leibniz $n$-algebra $q$, Schur Lie-multiplier $M_{\text{Lie}}(q)$,
\[
\dim M_{\text{Lie}}(q) \leq \sum_{i=1}^{n} \binom{n-1}{i-1} \binom{k}{i}.
\]

2 Preliminaries

Throughout this paper, $n \geq 2$ is a fixed integer and all Leibniz $n$-algebras are considered over a fixed field $K$ of characteristic zero.

Recall that a Leibniz algebra \([24, 25]\) is a vector space $q$ equipped with a bilinear map $[-,-] : q \times q \to q$, usually called the Leibniz bracket of $q$, satisfying the Leibniz identity:
\[
[x, [y, z]] = [[x, y], z] - [[x, z], y], \quad x, y, z \in q.
\]

A Leibniz $n$-algebra \([10]\) is a vector space $q$ together with the following $n$-linear map $[-,\ldots,-] : q \times \cdots \times q \to q$ such that
\[
[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, x_{i-1}, [x_i, y_2, \ldots, y_n], x_{i+1}, \ldots, x_n],
\]
for all $x_i, y_j \in q$, $1 \leq i \leq n$ and $2 \leq j \leq n$. A subspace $h$ of a Leibniz $n$-algebra $q$ is called a subalgebra, if $[x_1, \ldots, x_n] \in h$, for any $x_i \in h$. Also, a subalgebra $I$ of $q$ is said to be an ideal, if the brackets $[I, q, q, \ldots, q], [q, I, q, q, \ldots, q], \ldots, [q, \ldots, q, I]$ are all contained in $I$.

Let $q$ be a Leibniz $n$-algebra. Following \([33]\), we define the bracket
\[
[-,\ldots,-]_{\text{Lie}} : q \times \cdots \times q \to q
\]
\[
[x_1, \ldots, x_n]_{\text{Lie}} = \sum_{1 \leq i_j \leq n} [x_{i_1}, \ldots, x_{i_n}],
\]
which is including $n!$ brackets. More precisely,
\[
[x_1, \ldots, x_n]_{\text{Lie}} = [(x_1 + \cdots + x_n), \ldots, (x_1 + \cdots + x_n)]
- \sum_{1 \leq i_j \leq n} [x_{i_1}, \ldots, x_{i_j}, \ldots, x_{i_n}], \quad (2.1)
\]

Also, the Lie-center and the Lie-commutator of a Leibniz $n$-algebra $q$ are defined as $Z_{\text{Lie}}(q) = \{ x \in q \mid [x, q_{n-1}]_{\text{Lie}} = 0 \}$ and $q^n_{\text{Lie}} = \{ [x_1, \ldots, x_n]_{\text{Lie}} \mid x_i \in q \}$, where $[x_{n-1}]_{\text{Lie}} = [x, q, \ldots, q]_{\text{Lie}}$. Clearly, these are ideals of $q$ (see \([33]\), Remark 2.1)).
Let \( 0 \to \mathfrak{r} \to \mathfrak{f} \to \mathfrak{q} \to 0 \) be a free presentation of a Leibniz \( n \)-algebra \( \mathfrak{q} \). The Schur Lie-multiplier of \( \mathfrak{q} \) is defined as

\[
\mathcal{M}_{\text{Lie}}(\mathfrak{q}) = \frac{\mathfrak{r} \cap \mathfrak{f}^{n}_{\text{Lie}}}{[\mathfrak{r}, n-1]^{\text{Lie}}}. 
\]

Clearly, the Schur Lie-multiplier is independent of the chosen free presentation of \( \mathfrak{q} \). Also if \( n = 2 \), then this definition coincides with the notion of the Schur Lie-multiplier of a Leibniz algebra given in [9]. Note that two other notions of the Schur multiplier, namely the \( c \)-nilpotent Schur Lie-multiplier of a Leibniz algebra and the Schur multiplier of a pair of Leibniz algebras are already discussed in [7, 8], respectively.

**Remark 2.1** In the case where the \( n \)-linear map \([-\ldots,-]\) is anti-symmetric in each pair of variables, i.e. \([x_{1}, \ldots, x_{i}, \ldots, x_{j}, \ldots, x_{n}] = -[x_{1}, \ldots, x_{j}, \ldots, x_{i}, \ldots, x_{n}]\), or equivalently \([x_{1}, \ldots, x_{i}, \ldots, x_{j}, \ldots, x_{n}] = 0\) when \( x_{i} = x_{j} \), the Leibniz \( n \)-algebra becomes an \( n \)-Lie algebra. In this case, the equality (2.3) implies that \( q^{n}_{\text{Lie}} = 0 \) and \( Z_{\text{Lie}}(\mathfrak{q}) = \mathfrak{q} \). A Leibniz \( n \)-algebra \( \mathfrak{q} \) is called Lie-abelian, if \( q^{n}_{\text{Lie}} = 0 \) or equivalently \( Z_{\text{Lie}}(\mathfrak{q}) = \mathfrak{q} \). Thus every \( n \)-Lie algebra is a Lie-abelian Leibniz \( n \)-algebra.

Conversely, if \( \mathfrak{q} \) is a Lie-abelian Leibniz \( 2 \)-algebra over a field \( \mathbb{K} \supseteq \frac{1}{2} \), then \([x, x]_{\text{Lie}} = 0\) and so \([x, x] = 0\). Hence \( \mathfrak{q} \) is actually a Lie algebra. Therefore, the Schur Lie-multiplier [9] and also the \( c \)-nilpotent Schur Lie-multiplier [7] of a Leibniz algebra are both Lie algebras, since they are Lie-abelian. But the Schur Lie-multiplier of a Leibniz \( n \)-algebra \((n \geq 3)\) is a Lie-abelian Leibniz \( n \)-algebra which is not an \( n \)-Lie algebra, in general. In fact, for a Leibniz \( 2 \)-algebra \( \mathfrak{q} \), we have \( q^{2}_{\text{Lie}} = \langle [x, x] \mid x \in \mathfrak{q} \rangle \), while if \( \mathfrak{q} \) is a Leibniz \( n \)-algebra with \( n \geq 3 \), then \( q^{2}_{\text{Lie}} \subseteq \mathfrak{s}_{\text{Leib}} \), where

\[
\mathfrak{s}_{\text{Leib}}(\mathfrak{q}) = \langle [x_{1}, \ldots, x_{i}, \ldots, x_{j}, \ldots, x_{n}] \mid \exists i, j : x_{i} = x_{j} \text{ with } x_{1}, \ldots, x_{n} \in \mathfrak{q} \rangle.
\]

In the following example, we give a Lie-abelian Leibniz \( 3 \)-algebra which is not an \( n \)-Lie algebra.

**Example 2.2** Let \( \mathfrak{q} = \text{span}\{x, y\} \) be the 2-dimensional complex Leibniz \( 3 \)-algebra with non-zero multiplications \([x, y, y] = -2x\) and \([y, y, x] = [y, x, y] = x\). It is easy to see that \( Z_{\text{Lie}}(\mathfrak{q}) = \mathfrak{q} \) and \( q^{3}_{\text{Lie}} = 0 \).

Recall from [34] that an extension of Leibniz \( n \)-algebras \( 0 \to \mathfrak{m} \subseteq \mathfrak{g} \xrightarrow{\sigma} \mathfrak{q} \to 0 \) is said to be Lie-central, if \( \mathfrak{m} \subseteq Z_{\text{Lie}}(\mathfrak{g}) \) or equivalently \([m, n-1]_{\text{Lie}} = 0\). Moreover, a Lie-central extension is called a Lie-stem extension, whenever \( \mathfrak{m} \subseteq Z_{\text{Lie}}(\mathfrak{g}) \cap q^{n}_{\text{Lie}} \). In addition, a Lie-stem extension is said to be a Lie-stem cover of \( \mathfrak{q} \), if \( \mathfrak{m} \cong \mathcal{M}_{\text{Lie}}(\mathfrak{q}) \). In this case, \( \mathfrak{g} \) is called a Lie-cover of \( \mathfrak{q} \).

In [34], it is discussed the following results on the Schur Lie-multiplier of Leibniz \( n \)-algebras, using their Lie-central extensions.
Lemma 2.3 Let $0 \to r \to f \xrightarrow{\pi} q \to 0$ be a free presentation of a Leibniz $n$-algebra $q$, and $0 \to m \to g \xrightarrow{\theta} p \to 0$ be a Lie-central extension of another Leibniz $n$-algebra $p$. Then for each homomorphism $\alpha : q \to p$, there exists a homomorphism $\beta : \frac{f}{[r, n-1]_{\text{Lie}}} \to g$ such that the following diagram is commutative:

$$
\begin{array}{cccc}
0 & \to & r & \xrightarrow{\beta} & q & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & m & \xrightarrow{\theta} & g & \to & p & \to & 0,
\end{array}
$$

where $\pi$ is the natural epimorphism induced by $\pi$.

Theorem 2.4 Let $q$ be a Leibniz $n$-algebra whose Schur Lie-multiplier is finite dimensional and $q \cong f/r$, where $f$ is a free Leibniz $n$-algebra. Then the extension $0 \to m \to g \to q \to 0$ is a Lie-stem cover if and only if there exists an ideal $s$ of $f$ such that

(i) $g \cong f/s$ and $m \cong r/s$.

(ii) $\frac{r}{[r, n-1]_{\text{Lie}}} \cong M_{\text{Lie}}(q) \oplus \frac{s}{[r, n-1]_{\text{Lie}}}$.

Corollary 2.5 Any finite dimensional Leibniz $n$-algebra has at least one Lie-cover.

Corollary 2.6 All Lie-stem covers of a Leibniz $n$-algebra with finite dimensional Schur Lie-multiplier are Lie-isoclinic.

Corollary 2.7 Let $q$ be a Leibniz $n$-algebra with finite dimensional Schur Lie-multiplier, and $g$ be a Lie-cover of $q$. Then $g \sim \frac{f}{[r, n-1]_{\text{Lie}}}$, where $f$ is a free Leibniz $n$-algebra such that $q \cong f/r$.

3 On Lie-covers of Leibniz $n$-algebras

This section is devoted to obtain some properties of Lie-covers of Leibniz $n$-algebras.

Lemma 3.1 Let $q$ be a Leibniz $n$-algebra and

$$
\begin{array}{cccc}
0 & \to & m' & \xrightarrow{\alpha} & g' & \xrightarrow{\beta} & q & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & m & \xrightarrow{\theta} & g & \xrightarrow{\gamma} & q & \to & 0,
\end{array}
$$

be a commutative diagram of Leibniz $n$-algebras such that the first row is exact and the second one is a Lie-stem extension of $q$. If the homomorphism $\gamma$ is onto, then so is $\beta$. 


Proof. Clearly, \( g = m + \text{Im}\beta \) and hence \( g^n_{\text{Lie}} = [m_{n-1}g]_{\text{Lie}} + [\text{Im}\beta]_{n-1}g_{\text{Lie}} \). Now, since \( m \subseteq Z_{\text{Lie}}(g) \), we have \( g^n_{\text{Lie}} = [\text{Im}\beta]_{n-1}g_{\text{Lie}} \), and since \( m \subseteq g^n_{\text{Lie}} \), we get \( m \subseteq [\text{Im}\beta]_{n-1}g_{\text{Lie}} \subseteq \text{Im}\beta \). Therefore, \( g = \text{Im}\beta \).

Proposition 3.2 Let \( 0 \to m \to g \xrightarrow{\pi} q \to 0 \) be a Lie-stem extension of a finite dimensional Leibniz \( n \)-algebra \( q \). Then \( g \) is a homomorphic image of a Lie-cover of \( q \).

Proof. Let \( 0 \to r \to f \xrightarrow{\pi} q \to 0 \) be a free presentation of \( q \). By Lemma 2.3, there exists a homomorphism \( \beta: \frac{f}{[r_{n-1}f]_{\text{Lie}}} \to g \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \to & r \\
\downarrow & & \downarrow \\
\frac{r}{[r_{n-1}f]_{\text{Lie}}} & \xrightarrow{\beta} & \frac{f}{[r_{n-1}f]_{\text{Lie}}} \\
\downarrow & & \downarrow \\
m & \to & g \\
\downarrow & & \downarrow \\
q & \xrightarrow{\pi} & 0 \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}
\]

Lemma 3.1 implies that \( \beta \) is an epimorphism and hence \( \beta(\frac{r}{[r_{n-1}f]_{\text{Lie}}}) = m \). Now, put \( \ker\beta = \ker\beta = u/[r_{n-1}f]_{\text{Lie}} \), for some ideal \( u \) in \( r \). Thus \( g \cong f/u \) and \( m \cong r/u \). Clearly,

\[
\beta((r \cap f^u_{\text{Lie}})/[r_{n-1}f]_{\text{Lie}}) \subseteq \beta(r/[r_{n-1}f]_{\text{Lie}}) \cap \beta(f^u_{\text{Lie}}/[r_{n-1}f]_{\text{Lie}}) = m \cap g^n_{\text{Lie}} = m.
\]

Conversely, let \( y \in \beta(r/[r_{n-1}f]_{\text{Lie}}) \cap \beta(f^u_{\text{Lie}}/[r_{n-1}f]_{\text{Lie}}) \). Then \( y = \beta(r + [r_{n-1}f]_{\text{Lie}}) = \beta(x + [r_{n-1}f]_{\text{Lie}}) \), for some \( r \in r \) and \( x \in f^u_{\text{Lie}} \). Therefore, \( x-r + [r_{n-1}f]_{\text{Lie}} \in \ker\beta = u/[r_{n-1}f]_{\text{Lie}} \subseteq r/[r_{n-1}f]_{\text{Lie}} \). Hence, \( x \in r \) and \( y \in \beta((r \cap f^u_{\text{Lie}})/[r_{n-1}f]_{\text{Lie}}) \). This shows that \( \beta \) may be restricted to an epimorphism from \( (r \cap f^u_{\text{Lie}})/[r_{n-1}f]_{\text{Lie}} \) onto \( m \).

Thus

\[
\frac{r}{u} \simeq \frac{r}{[r_{n-1}f]_{\text{Lie}}} \simeq m \cong \frac{(r \cap f^u_{\text{Lie}})/[r_{n-1}f]_{\text{Lie}}}{(u \cap f^u_{\text{Lie}})/[r_{n-1}f]_{\text{Lie}}} \simeq \frac{(r \cap f^u_{\text{Lie}}) + u}{u},
\]

and by finite dimensionality, we have \( r = (r \cap f^u_{\text{Lie}}) + u \). Now, let \( s/[r_{n-1}f]_{\text{Lie}} \) be a complement of \( (u \cap f^u_{\text{Lie}})/[r_{n-1}f]_{\text{Lie}} \) in \( u/[r_{n-1}f]_{\text{Lie}} \). Then \( s \cap (r \cap f^u_{\text{Lie}}) = [r_{n-1}f]_{\text{Lie}} \) and \( s + (r \cap f^u_{\text{Lie}}) = r \). Therefore,

\[
\frac{r}{[r_{n-1}f]_{\text{Lie}}} \cong M_{\text{Lie}}(q) \oplus \frac{s}{[r_{n-1}f]_{\text{Lie}}}.
\]

Now, Theorem 2.4 implies that \( f/s \cong \frac{f}{u}/s/u = \frac{g}{s/u} \) is a Lie-cover of \( q \).

A Leibniz \( n \)-algebra \( q \) is said to be Hopfian, if every epimorphism \( \varphi: q \to q \) is an isomorphism (see also [35]). Here, we give certain conditions under which any Lie-cover of \( q \) is a Hopfian Leibniz \( n \)-algebra.
Proposition 3.3 Let \( q \) be a Leibniz \( n \)-algebra with finite dimensional Schur Lie-multiplier, and \( 0 \to m_i \to g_i \to q \to 0 \) \( (i = 1, 2) \) be two Lie-stem covers of \( q \). If \( \gamma : g_1 \to g_2 \) is an epimorphism such that \( \gamma(m_1) = m_2 \), then \( \gamma \) is an isomorphism.

Proof. Let \( 0 \to r \to f \xrightarrow{\pi} q \to 0 \) be a free presentation of \( q \). By Theorem \([2.4]\) there exist ideals \( s_i \) \( (i = 1, 2) \) of \( f \) such that \( g_i \cong f/s_i, m_i \cong r/s_i \), and \( r/[r_{m-1}]_{\text{Lie}} \cong \mathcal{M}_{\text{Lie}}(q) \oplus s_i/[r_{m-1}]_{\text{Lie}} \). Hence, one may consider the epimorphism \( \gamma : f/s_1 \to f/s_2 \) with \( \gamma(r/s_1) = r/s_2 \). Now, by the proof of Theorem \([2.4]\) there exists an epimorphism \( \beta : f/[r_{m-1}]_{\text{Lie}} \to f/s_2 \) such that \( \ker \beta = s_2/[r_{m-1}]_{\text{Lie}} \). Since \( f \) is free, there exists a homomorphism \( \bar{\alpha} : f \to f/s_1 \) such that \( \gamma \circ \bar{\alpha} = \beta \circ \pi' \), where \( \pi' : f \to f/[r_{m-1}]_{\text{Lie}} \) is the natural epimorphism. Clearly, \( \bar{\alpha} \) induces a homomorphism \( \alpha : f/[r_{m-1}]_{\text{Lie}} \to f/s_1 \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
0 & \to & r \\
\downarrow & & \downarrow \\
0 & \to & f \\
\downarrow & \downarrow & \downarrow \\
0 & \to & q \\
\downarrow & \downarrow & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

where \( \alpha_1, \beta_1 \) and \( \gamma_1 \) are the restriction homomorphisms of \( \alpha, \beta \) and \( \gamma \), respectively, and also \( \beta_2 \) and \( \gamma_2 \) are the induced isomorphisms of \( \beta \) and \( \gamma \), respectively. Now, Lemma \([3.1]\) implies that \( \alpha \) is onto. Put \( \ker \alpha = u/[r_{m-1}]_{\text{Lie}} \), for some ideal \( u \) of \( f \). Then \( f/u \cong f/s_1 \). On the other hand, since \( \ker \alpha \subseteq \ker \beta \), we have \( u \subseteq s_2 \), and since \( u + (r \cap f_{m-1}) = r \), we get \( u = s_2 \), which shows that \( \gamma \) is an isomorphism.

Corollary 3.4 If \( q \) is a Lie-abelian Leibniz \( n \)-algebra with finite dimensional Schur Lie-multiplier, then every Lie-cover of \( q \) is Hopfian.

Proof. Suppose that \( 0 \to m \to g \to q \to 0 \) is a Lie-stem cover of \( q \). Since \( g^n_{m_{\text{Lie}}} = (g^n_m)_{\text{Lie}} = 0 \), we have \( g^n_{m_{\text{Lie}}} = m \). Now, if \( \gamma : g \to q \) is an epimorphism, then \( \gamma(m) = \gamma(g^n_{m_{\text{Lie}}}) = g^n_{m_{\text{Lie}}} = m \) and hence by the above theorem, \( \gamma \) is an isomorphism.

Proposition 3.5 Let \( q \) be a Leibniz \( n \)-algebra with finite dimensional Schur Lie-multiplier, and \( 0 \to m_i \to g_i \to q \to 0 \) \( (i = 1, 2) \) be two Lie-stem covers of \( q \). Then \( Z_{\text{Lie}}(g_1)/m_1 \cong Z_{\text{Lie}}(g_2)/m_2 \).

Proof. Let \( 0 \to r \to f \to q \to 0 \) and \( 0 \to m \to g \to q \to 0 \) be a free presentation and a Lie-stem cover of \( q \), respectively. Theorem \([2.4]\) implies that there exists an ideal
\( s \) of \( \mathfrak{f} \) such that \( \mathfrak{g} \cong \mathfrak{f}/s \), \( \mathfrak{m} \cong \mathfrak{r}/s \) and \( \mathfrak{r}/[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \cong \mathcal{M}_{\text{Lie}}(\mathfrak{q}) \oplus s/[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \). Now, put \( Z_{\text{Lie}}(\mathfrak{f}/[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}}) = \mathfrak{u}/[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \), for some ideal \( \mathfrak{u} \) of \( \mathfrak{f} \). Clearly, \([\mathfrak{u},_{n-1}\mathfrak{f}]_{\text{Lie}} \subseteq [\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \) and hence \( \mathfrak{u}/s \subseteq Z_{\text{Lie}}(\mathfrak{f}/s) \). Conversely, let \( x + s \in Z_{\text{Lie}}(\mathfrak{f}/s) \). Then \([x,_{n-1}\mathfrak{f}]_{\text{Lie}} \subseteq s \cap f^m_{\text{Lie}} = [\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \), which implies that \( x + [\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \in Z_{\text{Lie}}(\mathfrak{f}/[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}}) \). Therefore, \( x + s \in \mathfrak{u}/s \) and hence \( Z_{\text{Lie}}(\mathfrak{f}/s) = \mathfrak{u}/s \). Thus \( Z_{\text{Lie}}(\mathfrak{g})/\mathfrak{m} \cong \mathfrak{u}/\mathfrak{r} \), which completes the proof.

In what follows, we obtain some results on Lie-covers of a Lie-perfect Leibniz \( n \)-algebra. A Leibniz \( n \)-algebra \( \mathfrak{q} \) is said to be Lie-perfect, whenever \( \mathfrak{q} = q^n_{\text{Lie}} \).

Let \( G_i : 0 \rightarrow \mathfrak{m}_i \rightarrow \mathfrak{g}_i \xrightarrow{\sigma} \mathfrak{q} \rightarrow 0 \) \( (i = 1, 2) \) be two Lie-central extensions a Leibniz \( n \)-algebra \( \mathfrak{q} \). Recall from [31] that \( (\beta|, \beta, 1_q) : G_1 \rightarrow G_2 \) is a homomorphism of Lie-central extensions, if the following diagram of homomorphisms is commutative:

\[
\begin{array}{ccc}
G_1 : 0 & \xrightarrow{\beta} & \mathfrak{m}_1 \\
& \sigma_1 \downarrow \beta \downarrow & \mathfrak{g}_1 \\
& \downarrow & \mathfrak{q} \\
G_2 : 0 & \xrightarrow{\beta} & \mathfrak{m}_2 \\
& \sigma_2 \downarrow 1_q \downarrow & \mathfrak{g}_2 \\
& & \mathfrak{q} \\
\end{array}
\]

Moreover, the Lie-central extension \( G_1 \) is said to be universal, if for every Lie-central extension \( G_2 \), there exists a unique homomorphism \( (\beta|, \beta, 1_q) : G_1 \rightarrow G_2 \).

The following result shows that every Lie-perfect Leibniz \( n \)-algebra admits at least one Lie-cover.

**Theorem 3.6** Let \( \mathfrak{q} \) be a Lie-perfect Leibniz \( n \)-algebra with a free presentation \( \mathfrak{q} \cong \mathfrak{f}/\mathfrak{r} \). Then the Lie-central extension

\[
0 \rightarrow \mathcal{M}_{\text{Lie}}(\mathfrak{q}) \rightarrow \mathfrak{f}^n_{\text{Lie}}/[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \xrightarrow{\rho} \mathfrak{q} \rightarrow 0
\]

(3.2)

where \( \rho(x + [\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}}) = x + \mathfrak{r} \), is a Lie-stem cover of \( \mathfrak{q} \). Moreover, it is a universal Lie-central extension.

**Proof.** Since \( \mathfrak{q} \) is Lie-perfect, we have \( \mathfrak{f} = f^n_{\text{Lie}} + \mathfrak{r} \) and hence \( f^n_{\text{Lie}} = (f^n_{\text{Lie}} + \mathfrak{r})_{\text{Lie}} = (f^n_{\text{Lie}})_{\text{Lie}} + [\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \). Thus

\[
\left( \frac{f^n_{\text{Lie}}}{[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}}} \right)^n = \frac{f^n_{\text{Lie}}}{[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}}},
\]

which actually shows that \( f^n_{\text{Lie}}/[\mathfrak{r},_{n-1}\mathfrak{f}]_{\text{Lie}} \) is a Lie-perfect Leibniz \( n \)-algebra. Therefore, the Lie-central extension (3.2) is a Lie-stem cover of \( \mathfrak{q} \).

Now, let \( G : 0 \rightarrow \mathfrak{m} \rightarrow \mathfrak{g} \xrightarrow{\sigma} \mathfrak{q} \rightarrow 0 \) be an arbitrary Lie-central extension of \( \mathfrak{q} \). By
Lemma 2.3, we have the following commutative diagram:

\[
\begin{array}{ccccccc}
F : 0 & \rightarrow & \mathcal{M}_{\text{Lie}}(q) & \rightarrow & f^n_{\text{Lie}} & \rightarrow & q & \rightarrow & 0 \\
0 & \rightarrow & r & \rightarrow & f & \rightarrow & q & \rightarrow & 0 \\
G : 0 & \rightarrow & m & \rightarrow & g & \rightarrow & q & \rightarrow & 0 \\
\end{array}
\]

which shows that there exists a homomorphism from $F$ to $G$. We show that this homomorphism is unique. So, let $\varphi_i : f^n_{\text{Lie}}/[r_{n-1}f]_{\text{Lie}} \rightarrow g$ ($i = 1, 2$) be two homomorphisms such that the following diagram is commutative:

\[
\begin{array}{ccccccc}
F : 0 & \rightarrow & \mathcal{M}_{\text{Lie}}(q) & \rightarrow & f^n_{\text{Lie}} & \rightarrow & q & \rightarrow & 0 \\
0 & \rightarrow & r & \rightarrow & f & \rightarrow & q & \rightarrow & 0 \\
G : 0 & \rightarrow & m & \rightarrow & g & \rightarrow & q & \rightarrow & 0 \\
\end{array}
\]

Therefore, $\sigma \circ \varphi_1 = \rho = \sigma \circ \varphi_2$ and hence $\varphi_1(x) - \varphi_2(x) \in \ker \sigma \subseteq Z_{\text{Lie}}(g)$, for all $x \in f^n_{\text{Lie}}/[r_{n-1}f]_{\text{Lie}}$. Now, we prove that $\varphi_1([x_1, \ldots, x_n]_{\text{Lie}}) = \varphi_2([x_1, \ldots, x_n]_{\text{Lie}})$, for all $x_i \in f^n_{\text{Lie}}/[r_{n-1}f]_{\text{Lie}}$ and since $f^n_{\text{Lie}}/[r_{n-1}f]_{\text{Lie}}$ is Lie-perfect, we get $\varphi_1 = \varphi_2$, which completes the proof. Clearly,

\[
[\varphi_1(x_1) - \varphi_2(x_1), \varphi_1(x_2), \ldots, \varphi_1(x_n)]_{\text{Lie}} = 0,
\]

and so

\[
\varphi_1([x_1, \ldots, x_n]_{\text{Lie}}) = [\varphi_2(x_1), \varphi_1(x_2), \ldots, \varphi_1(x_n)]_{\text{Lie}} = *.
\]

On the other hand, since

\[
[\varphi_2(x_1), \varphi_1(x_2) - \varphi_2(x_2), \varphi_1(x_3), \ldots, \varphi_1(x_n)]_{\text{Lie}} = 0,
\]

we have

\[
* = [\varphi_2(x_1), \varphi_2(x_2), \varphi_1(x_3), \ldots, \varphi_1(x_n)]_{\text{Lie}}.
\]

By repeating the above process, we finally obtain

\[
* = [\varphi_2(x_1), \varphi_2(x_2), \ldots, \varphi_2(x_n)]_{\text{Lie}} = \varphi_2([x_1, \ldots, x_n]_{\text{Lie}}).
\]

In [23], Loday discussed a necessary and sufficient condition for a pair of groups to be perfect, using a homological method. In the next theorem, we show this result for a Leibniz $n$-algebra.
**Theorem 3.7** A Leibniz n-algebra q is Lie-perfect if and only if it has a universal Lie-central extension.

**Proof.** If q is Lie-perfect, then the Lie-central extension \( \Theta \) in Theorem 3.6 is universal. Conversely, let \( G : 0 \to m \to q \to 0 \) be a universal Lie-central extension of q. Clearly, 
\[
0 \to m \times \frac{g}{\mathfrak{g}_{\text{Lie}}} \to \frac{g}{\mathfrak{g}_{\text{Lie}}} \to q \to 0,
\]
where \( \delta(x, y + \mathfrak{g}_{\text{Lie}}) = \sigma(x) \), is a Lie-central extension of q. Define homomorphisms \( \psi_i : \mathfrak{g} \to \mathfrak{g}/\mathfrak{g}_{\text{Lie}} \) by \( \psi_1(x) = (x, 0) \) and \( \psi_2(x) = (x, \sigma(x) + \mathfrak{g}_{\text{Lie}}) \). It is easy to see that the following diagram is commutative \((i = 1, 2)\):
\[
\begin{array}{ccc}
G : 0 & \to & m \\
\psi_1 & \downarrow & \sigma \\
0 & \to & m \times \frac{g}{\mathfrak{g}_{\text{Lie}}} \to \mathfrak{g} \to q \to 0.
\end{array}
\]

Then the universal property of \( G \) implies that \( \psi_1 = \psi_2 \). Therefore, \( g \) is Lie-perfect and since \( \sigma \) is onto, q is a Lie-perfect Leibniz n-algebra, as well.

The next result easily follows from Theorem 3.6 and Corollary 2.7.

**Corollary 3.8** Let q be a Lie-perfect Leibniz n-algebra with finite dimensional Schur Lie-multiplier, and \( g \) be a Lie-cover of q. Then \( g \simeq \frac{\mathfrak{f}_{\text{Lie}}}{[r, m-1, f]_{\text{Lie}}} \sim \frac{f}{[r, m-1]_{\text{Lie}}} \), where \( \mathfrak{f} \) is a free Leibniz n-algebra such that \( q \simeq f/r \).

It is well-known that isomorphism and isoclinism are equivalent for finite dimensional stem Lie algebras and stem n-Lie algebras (see [26, 14]). But this is not true for finite dimensional Lie-stem Leibniz n-algebras \((n \geq 2)\) (see [31]). Moreover in [31, Corollary 3], it is discussed certain conditions under which these concepts are equivalent.

**Lemma 3.9** Let q and p be two finite dimensional Lie-isoclinic Lie-stem complex Leibniz 2-algebras and for all elements \( x_1, x_2 \in q \), there exists \( \varepsilon_{12} \in \mathbb{C} \) such that \( [x_1, x_2] = \varepsilon_{12}[x_2, x_1] \). Then q and p are isomorphic.

Note that in the proof of above result, authors use the fact that \( Z_{\text{Lie}}(q) = Z(q) \), for any Lie-stem Leibniz 2-algebra q, where \( Z(q) = \{ z \in q \mid [x, z] = [z, x] = 0, \forall x \in q \} \) is the center of q (see [31, Lemma 4]). But this is not true for Lie-stem Leibniz n-algebras for \( n \geq 3 \). In [33, Example 4.5], it is given a Lie-stem complex Leibniz 3-algebra whose Lie-center is not central. Hence, we discuss the following result for Leibniz 2-algebras.

**Corollary 3.10** Let q be a Lie-perfect complex Leibniz 2-algebra with finite dimensional Schur Lie-multiplier with a free presentation \( q \simeq f/r \), in which for all elements \( x_1, x_2 \in [f, f]_{\text{Lie}}/[r, f]_{\text{Lie}} \) there exists \( \varepsilon_{12} \in \mathbb{C} \) such that \( [x_1, x_2] = \varepsilon_{12}[x_2, x_1] \). If \( g \) is a Lie-cover of q, then \( g \simeq [f, f]_{\text{Lie}}/[r, f]_{\text{Lie}} \).

**Proof.** Clearly, every Lie-cover of q is Lie-perfect. Now, Corollary 3.8 and Lemma 3.9 complete the proof.
On dimension of the Schur Lie-multiplier of a Leibniz $n$-algebra

In this section, we give some equalities and inequalities for the dimension of the Schur Lie-multiplier of a Leibniz $n$-algebra.

**Proposition 4.1** [24, Proposition 4.1] Let $0 \rightarrow r \rightarrow f \xrightarrow{\pi} q \rightarrow 0$ be a free presentation of a Leibniz $n$-algebra $q$. Also, let $m$ be an ideal of $q$ and $s$ an ideal of $f$ such that $m \cong s/r$. Then the following sequences are exact:

(i) $0 \rightarrow \frac{r \cap [s, n-1 f]_{\text{Lie}}}{[r, n-1 f]_{\text{Lie}}} \rightarrow M_{\text{Lie}}(q) \rightarrow M_{\text{Lie}}(\frac{q}{m}) \rightarrow \frac{m \cap q^n_{\text{Lie}}}{[m, n-1 q]_{\text{Lie}}} \rightarrow 0$,

(ii) $M_{\text{Lie}}(q) \rightarrow M_{\text{Lie}}(\frac{q}{m}) \rightarrow \frac{m \cap q^n_{\text{Lie}}}{[m, n-1 q]_{\text{Lie}}} \rightarrow \frac{m}{[m, n-1 q]_{\text{Lie}}} \rightarrow q \rightarrow q^m \rightarrow 0$.

**Corollary 4.2** Under the notation of the above result, if $m \subseteq Z_{\text{Lie}}(q)$ then the following sequence is exact:

$$m \otimes^{n-1} \frac{q}{n_{\text{Leib}}(q)} \rightarrow M_{\text{Lie}}(q) \rightarrow M_{\text{Lie}}(\frac{q}{m}) \rightarrow m \cap q^n_{\text{Lie}} \rightarrow 0,$$

where $n_{\text{Leib}}(q) = \langle [x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n] | \exists i, j : x_i = x_j \text{ with } x_1, \ldots, x_n \in q \rangle$.

**Proof.** Since $m \subseteq Z_{\text{Lie}}(q)$, we have $[s, n-1 f]_{\text{Lie}} \subseteq r$. Then by Proposition 4.1 (i), the following sequence is exact:

$$0 \rightarrow \frac{[s, n-1 f]_{\text{Lie}}}{[r, n-1 f]_{\text{Lie}}} \rightarrow M_{\text{Lie}}(q) \rightarrow M_{\text{Lie}}(\frac{q}{m}) \rightarrow m \cap q^n_{\text{Lie}} \rightarrow 0.$$

Now, define

$$\varphi : m \otimes^{n-1} \frac{q}{n_{\text{Leib}}(q)} \rightarrow \frac{r \cap f^n_{\text{Lie}}}{[r, n-1 f]_{\text{Lie}}} = M_{\text{Lie}}(q)$$

$$m \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_{n-1} \mapsto [s, f_1, \ldots, f_{n-1}]_{\text{Lie}} + [r, n-1 f]_{\text{Lie}},$$

where $\pi(s) = m$ and $\pi(f_i) = x_i$, for all $1 \leq i \leq n-1$. Since $m$ is Lie-central, $\varphi$ is a well-defined homomorphism. Clearly, $\text{Im} \varphi = [s, n-1 f]_{\text{Lie}}/[r, n-1 f]_{\text{Lie}}$, which completes the proof.

Note that the $n$-Lie algebra $q/_{n_{\text{Leib}}(q)}$ is actually a Lie-abelian Leibniz $n$-algebra (see Remark 2.1). The next result easily follows from Proposition 4.1 and the following com-
Corollary 4.3 Under the notation of Proposition 4.1, if $q$ is a finite dimensional Leibniz $n$-algebra, then

(i) $\mathcal{M}_{\text{Lie}}(q)$ is finite dimensional,

(ii) $\dim \mathcal{M}_{\text{Lie}}(\frac{q}{m}) \leq \dim \mathcal{M}_{\text{Lie}}(q) + \dim \frac{m \cap q^n}{\lbrack m, n-1 \rbrack_{\text{Lie}}}$,

(iii) $\dim \mathcal{M}_{\text{Lie}}(q) + \dim (m \cap q^n_{\text{Lie}}) = \dim \mathcal{M}_{\text{Lie}}(\frac{q}{m}) + \dim [m, n-1]_{\text{Lie}} + \dim \frac{\mathcal{M}_{\text{Lie}}(\frac{q}{m})}{\lbrack m, n-1 \rbrack_{\text{Lie}}}$,

(iv) $\dim \mathcal{M}_{\text{Lie}}(q) + \dim (m \cap q^n_{\text{Lie}}) = \dim \mathcal{M}_{\text{Lie}}(\frac{q}{m}) + \dim \frac{\mathcal{M}_{\text{Lie}}(\frac{q}{m})}{\lbrack m, n-1 \rbrack_{\text{Lie}}}$,

(v) $\dim \mathcal{M}_{\text{Lie}}(q) + \dim q^n_{\text{Lie}} = \dim \frac{f^n_{\text{Lie}}}{\lbrack f, n-1 \rbrack_{\text{Lie}}}$,

(vi) if $\mathcal{M}_{\text{Lie}}(q) = 0$, then $\mathcal{M}_{\text{Lie}}(\frac{q}{m}) \cong \frac{m \cap q^n}{\lbrack m, n-1 \rbrack_{\text{Lie}}}$,

(vii) if $m$ is a Lie-central ideal of $q$, then

$$\dim \mathcal{M}_{\text{Lie}}(q) + \dim (m \cap q^n_{\text{Lie}}) \leq \dim \mathcal{M}_{\text{Lie}}(\frac{q}{m}) + \dim (m \otimes_{n-1}^{\text{Leib}} q)$$

$$\leq \dim \mathcal{M}_{\text{Lie}}(\frac{q}{m}) + \dim (m \otimes_{n-1}^{\text{Leib}} q)_{\text{Lie}}$$.

Let $q$ be a Leibniz $n$-algebra and $m$ be an ideal of $q$. Then $(m, q)$ is called a pair of Leibniz $n$-algebras (see [31]). Suppose that $q \cong f/\tau$ is a free presentation of $q$ such that $m \cong s/\tau$, for an ideal $s$ of $f$. Then one may define the Schur Lie-multiplier of a pair of Leibniz $n$-algebras as

$$\mathcal{M}_{\text{Lie}}(m, q) = \frac{\mathcal{M}_{\text{Lie}}(q)}{\lbrack f, n-1 \rbrack_{\text{Lie}}}$$. 
which is actually the first non-zero term in the exact sequence given in Proposition 4.1 (i). Clearly if \( m = q \), then this definition coincides with the Schur Lie-multiplier of a Leibniz \( n \)-algebra discussed in this paper, and in addition if \( n = 2 \), then it yields the Schur Lie-multiplier of a Leibniz algebra given in [2].

The next result follows from Corollary 4.3 (iii) and [33, Lemma 2.6 (iii)].

**Corollary 4.4** Let \( q \) be a finite dimensional Leibniz \( n \)-algebra and \( m \) be an ideal of \( q \) such that \( q \) is Lie-isoclinic to \( q/m \). Then

\[
\dim M_{\text{Lie}}(q) = \dim M_{\text{Lie}}(\frac{q}{m}) + \dim M_{\text{Lie}}(m, q).
\]

In what follows, we obtain upper bounds for the dimension of the Lie-commutator of a Leibniz \( n \)-algebra with finite dimensional Lie-central factor as well as for the dimension of the Schur Lie-multiplier of a finite dimensional Leibniz \( n \)-algebra.

In 1904, Schur [37] proved that if the central factor of a group \( G \) is finite, then so is \( G' \), where \( G' \) is the commutator subgroup of \( G \). Also, Wiegold [38] showed that if \( |G/Z(G)| = p^k \), then \( G' \) is a \( p \)-group of order at most \( p^{\frac{k}{2}}(k-1) \).

Nearly a century after Schur, Moneyhun [26] proved that if \( L \) is a Lie algebra with \( \dim L/Z(L) = k \), then \( \dim[L, L] \leq \frac{1}{2}k(k-1) \). In fact, if \( \{\bar{x}_1, \ldots, \bar{x}_k\} \) is a basis for \( L/Z(L) \), then \([L, L]\) can be generated by \( \{[x_i, x_j] : 1 \leq i < j \leq k\} \). Therefore, \( \dim[L, L] \leq \binom{k}{2} \). Furthermore, if \( L \) is an \( n \)-Lie algebra such that \( \dim L/Z(L) = k \), then one can similarly show that the dimension of the commutator of \( L \) is at most \( \binom{k}{n} \) (see [12]).

The structure of a group (resp. Lie algebra) and its central factor, with respect to the order (resp. dimension) of its commutator has been already studied by many authors (see [1, 12, 22, 30], for instance). Now, we obtain an upper bound for the dimension of the Lie-commutator of a Leibniz \( n \)-algebra with finite dimensional Lie-central factor.

**Theorem 4.5** Let \( q \) be a Leibniz \( n \)-algebra such that \( \dim(q/Z_{\text{Lie}}(q)) = k \). Then

\[
\dim q^n_{\text{Lie}} \leq \sum_{i=1}^{n} \binom{n-1}{i-1} \binom{k}{i}.
\]

**Proof.** Let \( \{\bar{x}_1, \ldots, \bar{x}_k\} \) be a basis for \( q/Z_{\text{Lie}}(q) \). We should actually find the cardinal number of the set

\[
B = \{[x_{j_1}, \ldots, x_{j_n}]_{\text{Lie}} : 1 \leq j_1 \leq \cdots \leq j_n \leq k\}.
\]

In fact,

\[
\dim q^n_{\text{Lie}} \leq \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n,
\]

where \( \Gamma_i \) is the number of elements \( [x_{j_1}, \ldots, x_{j_n}]_{\text{Lie}} \) in \( B \) such that the set \( \{x_{j_1}, \ldots, x_{j_n}\} \) contains exactly \( i \) distinct members \( (1 \leq i \leq n) \). Therefore, \( \Gamma_1 = k = \binom{k}{1} \), since we have these \( k \) elements:

\[
[x_1, \ldots, x_1]_{\text{Lie}}, \ldots, [x_k, \ldots, x_k]_{\text{Lie}}.
\]
Also \( \Gamma_2 = \binom{n-1}{1} \binom{k}{2} \), because of elements of the form:

\[
[x_{j_s}, x_{j_t}, \ldots, x_{j_t}]_{\text{Lie}}, \ldots, [x_{j_s}, x_{j_t}, \ldots, x_{j_t}]_{\text{Lie}}, \ldots, [x_{j_s}, x_{j_t}, x_{j_t}]_{\text{Lie}},
\]

in which \( 1 \leq j_s < j_t \leq k \). By a similar combinatorial computation, one can deduce that

\[
\Gamma_i = \binom{n-1}{i-1} \binom{k}{i},
\]

which completes the proof.

**Corollary 4.6** Let \( q \) be a Leibniz 2-algebra such that \( \dim(q/Z_{\text{Lie}}(q)) = k \). Then

\[
\dim[q, q]_{\text{Lie}} \leq \frac{1}{2} k(k+1).
\]

**Remark 4.7** Let \( q \) be a Leibniz \( n \)-algebra,

\[
Z(q) = \{ x \in q \mid [x, q, \ldots, q] = \cdots = [q, \ldots, q, x] = 0 \}
\]

be the center, and \( q^n = [q, \ldots, q] \) be the commutator of \( q \). It is easy to see that if \( \dim(q/Z(q)) = k \), then \( \dim q^n \leq k^n \).

In [26], Moneyhun proved that if \( L \) is a Lie algebra of dimension \( k \), then \( \dim \mathcal{M}(L) \leq \frac{1}{2} k(k-1) \). Also, for a \( k \)-dimensional \( n \)-Lie algebra \( L \), we have \( \dim \mathcal{M}(L) \leq \binom{k}{n} \) (see [12]). In the next result, we give an upper bound for the dimension of the Schur Lie-multiplier of a finite dimensional Leibniz \( n \)-algebra.

**Theorem 4.8** Let \( q \) be a \( k \)-dimensional Leibniz \( n \)-algebra. Then

\[
\dim \mathcal{M}_{\text{Lie}}(q) \leq \sum_{i=1}^{n} \binom{n-1}{i-1} \binom{k}{i}.
\]

In particular, if the equality occurs, then \( q \) is a Lie-abelian Leibniz \( n \)-algebra.

**Proof.** Let \( 0 \to m \to g \to q \to 0 \) be a Lie-stem cover of \( q \). Clearly, \( \dim(g/Z_{\text{Lie}}(g)) \leq \dim(g/m) = k \), thus \( \dim g^n_{\text{Lie}} \leq \sum_{i=1}^{n} \binom{n-1}{i-1} \binom{k}{i} \), thanks to Theorem 4.5. Hence

\[
\dim \mathcal{M}_{\text{Lie}}(q) = \dim m \leq \dim g^n_{\text{Lie}} \leq \sum_{i=1}^{n} \binom{n-1}{i-1} \binom{k}{i}.
\]

Now, if the equality holds, then the above inequality implies that \( m = g^n_{\text{Lie}} \) and hence \( q = g/m \) is Lie-abelian.
In [12], it is shown that a $k$-dimensional $n$-Lie algebra $L$ is abelian if and only if $\dim \mathcal{M}(L) = \binom{n}{k}$. But the following example shows that the converse of the last statement of Theorem 4.8 is not true when $n \geq 3$.

**Example 4.9** Let $q = \text{span}\{x, y\}$ be the 2-dimensional complex Leibniz 3-algebra with non-zero multiplications $[x, x, y] = -[y, x, x] = y$. Clearly, $q^3_{\text{Lie}} = 0$ and so $q$ is Lie-abelian. Moreover,

$$\dim \frac{q}{3\text{Leib}(q)} = \dim \frac{q}{\text{span}\{y\}} = 1.$$  

Now, in Corollary 4.11 (vii), if we put $m = q$, then

$$\dim \mathcal{M}_{\text{Lie}}(q) \leq \dim (q \otimes^2 \frac{q}{3\text{Leib}(q)}) \leq \dim q = 2,$$

while $\sum_{i=1}^3 \binom{2}{i-1} \binom{2}{i} = 4$. Also, Example 2.2 is another counter-example.

**Corollary 4.10** Let $q$ be a $k$-dimensional Leibniz 2-algebra. Then

$$\dim \mathcal{M}_{\text{Lie}}(q) \leq \frac{1}{2} k(k + 1),$$

and the equality holds if and only if $q$ is Lie-abelian.

**Proof.** If the equality holds then by Theorem 4.8, $q$ is Lie-abelian. Conversely, let $q$ be a Lie-abelian Leibniz 2-algebra. Then for every Lie-stem extension $0 \to m \to g \to q \to 0$ of $q$, we have $m = g^2_{\text{Lie}} \subseteq Z_{\text{Lie}}(g)$. Similar to the Lie case given in [26, Lemma 23], assume that $u$ and $v$ are vector spaces with bases $A = \{x_1, \ldots, x_k\}$ and $B = \{y_{ij} | 1 \leq i \leq j \leq k\}$, respectively, and put $g = u + v$ and consider the multiplications: $[x_i, x_j] = y_{ij}$ for $1 \leq i \leq j \leq k$, $[x_j, x_i] = 0$ for $1 \leq i < j \leq k$ and $[y_{ij}, g] = [g, y_{ij}] = 0$ for every $g \in g$. Therefore, $g$ is a Leibniz algebra (by linear extending the above products to all of $g$). Moreover if $i < j$, then $y_{ij} = [x_i, x_j] = [x_i, x_j]_{\text{Lie}} \in g^2_{\text{Lie}}$ and also $y_{ii} = [x_i, x_i] = \frac{1}{2}[x_i, x_i]_{\text{Lie}} \in g^2_{\text{Lie}}$. Hence $g^2_{\text{Lie}} = v$. Furthermore, $\dim(g/g^2_{\text{Lie}}) = \dim u = k$ and $0 \to g^2_{\text{Lie}} \to g \to q \to 0$ is a Lie-stem extension of $q$ in which $g$ is of maximal dimension. Now, Proposition 3.2 implies that $g$ is a Lie-cover of $q$ and so $\dim \mathcal{M}_{\text{Lie}}(q) = \dim g^2_{\text{Lie}} = |B| = \frac{1}{2} k(k + 1)$.

**Corollary 4.11** If $q$ is a $k$-dimensional Leibniz $n$-algebra, then

$$\dim \mathcal{M}_{\text{Lie}}(q) \leq \sum_{i=1}^n \binom{n-1}{i-1} \binom{k}{i} - \dim q^n_{\text{Lie}}.$$  

**Proof.** Let $0 \to \tau \to \mathfrak{f} \to q \to 0$ be a free presentation of $q$. Since

$$\dim \left(\frac{\mathfrak{f}/[\tau, n-1 \mathfrak{f}]_{\text{Lie}}}{Z_{\text{Lie}}(\mathfrak{f}/[\tau, n-1 \mathfrak{f}]_{\text{Lie}})}\right) \leq \dim(\mathfrak{f}/\tau) = k,$$

by Theorem 4.5 we get $\dim(q^n_{\text{Lie}}/[\tau, n-1 \mathfrak{f}]_{\text{Lie}}) \leq \sum_{i=1}^n \binom{n-1}{i-1} \binom{k}{i}$. Now the result follows from Corollary 4.3 (v).
Remark 4.12 Let $L$ be a Lie algebra of dimension $k$. Then $t(L) = \frac{1}{2}k(k-1) - \dim \mathcal{M}(L)$ is a non-negative integer. One of the most interesting problem in the study of Lie algebras is classifying Lie algebras by $t(L)$ (see [28]). For example in [2], Batten et al. characterized nilpotent Lie algebras with $t(L) = 0, 1, 2$, and later Hardy and Stitzinger [20] when $t(L) = 3, 4, 5, 6$ and then for $t(L) = 7, 8$ in [19]. A similar problem arises in the case of Lie superalgebras [27, 32], and also in finite group theory [3, 13, 16, 39]. Here, the same question appears for Leibniz $n$-algebras. The characterizing of $k$-dimensional Leibniz $n$-algebras by $t(n, q)$, where

$$t(n, q) = \sum_{i=1}^{n} \binom{n-1}{i-1} \binom{k}{i} - \dim \mathcal{M}_{\text{Lie}}(q).$$

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Hesam Safa
Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran.
E-mail address: h.safa@ub.ac.ir

Guy R. Biyogmam
Department of Mathematics, Georgia College & State University, Milledgeville, GA, USA.
E-mail address: guy.biyogmam@gcsu.edu