A unified thermodynamic formalism describing the efficiency of learning is proposed. First, we derive an inequality, which is more strength than Clausius’s inequality, revealing the lower bound of the entropy-production rate of a subsystem. Second, the inequality is transformed to determine the general upper limit for the efficiency of learning. In particular, we exemplify the bound of the efficiency in nonequilibrium quantum-dot systems and networks of living cells. The framework provides a fundamental trade-off relationship between energy and information inheriting in stochastic thermodynamic processes.

Thermodynamic inequalities have laid the foundations of achieving useful limits on entropy production and energy. Tightening the bounds on these quantities will not only help us to develop a deeper understanding of thermodynamics but also provide insights into micro systems. It has been found that thermodynamic inequalities may strengthen Clausius’s inequality of the second law of thermodynamics [1–13].

In recent years, many studies have begun to focus on the role of information in thermodynamics. The most celebrated relationship between information and thermodynamics is Landauer’s principle [14], which explicitly shows the amount of heat dissipation required to erase information in the slow quasistatic limit [15–19]. Experimental verifications of this principle have been conducted in classical and quantum regimes [20, 21]. As modern computing alliances demand high-speed information processing, an important advance is to investigate the energy cost associated with the finite-time information era-
Theoretical bound of the efficiency of learning

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(Dated: September 20, 2022)

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The coarse-grained dynamics of subsystem $X$ is formally written as [24, 25]
the transition rate \( V_{xx'} = \sum_y W_{xx'y}p(y|x') \) with \( p(y|x') = p(x', y) / p(x') \) being the conditional probability, and \( J^v_{xx'y} = W_{xx'y}p(x', y) - W_{xx'y}p(x, y) \) is the current from state \( x' \) to state \( x \) provided that subsystem \( Y \) is at state \( y \).

Starting from the time derivative of Shannon’s entropy \( S^X = -\sum_x p(x) \ln p(x) \) and inserting Eq. (3), one obtains the entropy-production rate \( \dot{\mathcal{S}}^X \) of subsystem \( X \) (see Supplemental Material, Sec. I and Sec. II)

\[
\dot{\mathcal{S}}^X = \dot{S}^X + \dot{S}_r^X - \dot{J}^X \geq 0,
\]

(4)

where

\[
\dot{\mathcal{S}}^X = \frac{1}{2} \sum_{x,x',y,v} J^v_{xx'y} \ln \frac{p(x', y) W_{x'x'y}}{p(x) W_{xx'y}} \geq 0,
\]

(5)

\[
\dot{S}_r^X = \frac{1}{2} \sum_{x,x',y,v} J^v_{xx'y} \ln \frac{W_{x'x'y}}{p(x) W_{xx'y}} \geq 0,
\]

(6)

\[
\dot{J}^X = \frac{1}{2} \sum_{x,x',y,v} J^v_{xx'y} \ln \frac{p(y|x')}{p(y|x)} \geq 0.
\]

(7)

Note that the overdot notation, as in \( \dot{\mathcal{S}}^X, \dot{S}_r^X, \) and \( \dot{J}^X \), is used to emphasize that such quantities are rates. The rate \( \dot{\mathcal{S}}^X \) is always positive due to the fact that \( p(x,y) \frac{W_{x'x'y}}{W_{xx'y}} \geq 0 \) \( \forall x \neq x' \) and \( J^v_{xx'y} \ln \frac{p(x',y) W_{x'x'y}}{p(x) W_{xx'y}} \geq 0 \) [24], satisfying the second law of thermodynamics. \( \dot{\mathcal{S}}^X \) is the entropy flow to the environment, which is associated with the energy flux from \( X \) into the environment. \( \dot{J}^X \) quantifies the rate of information learned. When \( \dot{J}^X > 0 \), \( X \) acts as a sensor, creating information by learning \( Y \). On the other hand, \( \dot{J}^X < 0 \) means that \( Y \) is learning \( X \) and the information learned is being erased or consumed by \( X \) for extracting energy or doing work [27]. Equivalent expressions from Eq. (4) to Eq. (7) can be developed for subsystem \( Y \). In Ref. [29], Hartich interpreted \( \dot{J}^X \) as the rate of the entropy reduction of subsystem \( X \) due to its coupling with \( Y \).

At the steady state, the system reaches the stationary probability distribution \( \{\hat{p}(x) = 0\} \), and thus \( \dot{S}^X = 0 \). Therefore, Eq. (5) is simplified into

\[
\dot{\mathcal{S}}^X = \dot{S}_r^X - \dot{J}^X \geq 0.
\]

(8)

In the case of \( \dot{J}^X > 0 \), Eq. (8) dictates that the rate of information learned is bounded by the entropy flow to the environment \( \dot{S}_r^X \), which characterizes the dissipation necessary for \( X \) learning about \( Y \). Information will provide the resource for extracting energy or doing work through the feedback on \( Y \). From inequality (8), the efficiency of learning is defined as

\[
\eta^X = \frac{\dot{J}^X}{\dot{S}_r^X} \leq 1.
\]

(9)

It determines the effectiveness of learning and has been widely adopted [29-31].

**Theoretical bound of the efficiency of learning**—Our aim is to evaluate the upper bound of the efficiency of learning. To this end, we first present an alternative lower limit on \( \dot{\mathcal{S}}^X \), i.e.,

\[
\sqrt{\dot{\mathcal{S}}^X} \geq \frac{|\dot{S}_r^X|}{\sqrt{\dot{\mathcal{S}}^X}},
\]

(10)

The Cauchy-Schwartz inequality [5, 6] and the inequality \( \frac{(x-y)^2}{x+y} \leq \frac{x^2 y^2}{x+y} \) for non-negative \( x \) and \( y \) have been applied in the proof (see Supplemental Material, Sec. III). Eq. (10) provides a restriction stronger than the value claimed by Eq. (8). By supposing that \( \dot{S}_r^X \geq 0 \) and \( \dot{J}^X \geq 0 \), a universal lower bound for the efficiency of learning is given by (see Supplemental Material, Sec. III)

\[
\eta^X \leq 1 - \frac{\dot{S}_r^X}{\dot{\mathcal{S}}^X}.
\]

(12)

This bound, which is the main result of this Letter, is applicable to every physical model as long as the principle of detailed balance is satisfied. The physical meaning of the inequalities (10) and (12) is obvious. If bath \( v \) induces a transition from state \( (x', y) \) to state \( (x, y) \), the energy absorbed by the system is simply written as \( E(x, y) - E(x', y) = -k_B T_v \ln \frac{W_{x'x'y}}{W_{xx'y}} \), where \( E(x, y) \) is the energy of state \( (x, y) \), \( k_B \) is the Boltzmann constant, and \( T_v \) denotes the temperature of bath \( v \). \( W_{x'x'y}p(x', y) \) quantifies the frequency of the jump between given states. Therefore, the coefficient \( \Theta^X \) can be regarded as a dynamic activity factor. On the other hand, the rate \( \dot{S}_r^X \) originates from the irreversible energy loss to the environment. The inequalities (10) and (12) manifest a trade-off between the speed of the energy exchange and the dissipation.

**Numerical demonstration**.—In the following, a quantum system and a cellular network, where information and thermodynamics can be intuitively related via the Clausius inequality, will be used to exemplify our findings. We demonstrate that the efficiency of learning is well characterized by the inequality in (12).

**The double-quantum-dot system.**—The double-quantum-dot (DQD) system consists of two quantum dots interacting via a Coulomb repulsion \( U \). The DQD [see Fig. 1(a)] is modeled by the Hamiltonian

\[
H_S = \varepsilon_X d_X^d X + \varepsilon_Y d_Y^d Y + U d_X^d X d_Y^d Y,
\]

(13)

where \( d_X^d (d_X) \) creates (annihilates) one electron on QD \( X \) with energy \( \varepsilon_X \), and \( d_Y^d (d_Y) \) creates (annihilates) one electron on QD \( Y \) with energy \( \varepsilon_Y \). QD \( X (Y) \) is weakly coupled to

![Figure 1. (a) The schematic diagram of a double-quantum-dot system. (b) The schematic representation of a cellular network.](image-url)
two Fermi reservoirs $XL$ and $XR$ ($YL$ and $YR$). Thus, electrons are transported through parallel interacting channels. The Hamiltonian of the reservoirs is given by

$$H_R = \sum_k \sum_{\alpha \in \{X,Y\}} \sum_{\beta \in \{L,R\}} \varepsilon_{\alpha \beta} c_{\alpha \beta}^{\dagger} c_{\alpha \beta},$$

(14)

where $c_{\alpha \beta}^{\dagger}$ ($c_{\alpha \beta}$) is the creation (annihilation) operator at energy level $\varepsilon_{\alpha \beta}$ in bath $\alpha \beta$. The interaction between the DQD and the environment reads

$$H_I = \sum_k \sum_{\alpha \in \{X,Y\}} \sum_{\beta \in \{L,R\}} \left( t_{\alpha \beta} d_{\alpha} c_{\alpha \beta}^{\dagger} + t_{\beta \alpha} c_{\beta \alpha} d_{\alpha}^{\dagger} \right),$$

(15)

where $t_{\alpha \beta}$ denotes the coupling strength of the transition between QD $\alpha$ and reservoir $\alpha \beta$ at energy level $\varepsilon_{\alpha \beta}$. We use $x_0$ and $y_0$ ($x_1$ and $y_1$) to denote that $X$ and $Y$ are in empty (filled) states, respectively. The energy eigenstates of the DQD coincide with the localized Fock states $| (x_0, y_0) \rangle, | (x_1, y_1) \rangle, | (x_0, y_1) \rangle,$ and $| (x_1, y_0) \rangle$, where their respective eigenvalues are $0, \varepsilon_X, \varepsilon_Y,$ and $\varepsilon_X + \varepsilon_Y + U$.

For a non-degenerate system weakly coupled to different environmental modes, the dynamics of the populations satisfies Eq. (1) [35, 36]. The transition rates follow from Fermi’s golden rule and are given by

$$W_{x_1 x_0 | y_1} = \Gamma_{y_1} f_{x_1}^{y_1}, \quad W_{x_0 x_1 | y_1} = \Gamma_{y_1} (1 - f_{x_1}^{y_1}),$$

$$W_{y_0 y_1 | x_1} = \Gamma_{x_1} f_{y_1}^{x_1}, \quad W_{y_1 y_0 | x_1} = \Gamma_{x_1} (1 - f_{y_1}^{x_1}),$$

where $f_{x_1}^{y_1} = \{1 + \exp[\beta_{x} (\varepsilon_{y} + iU - \mu_{y})]\}^{-1}$ and $f_{y_1}^{x_1} = \{1 + \exp[\beta_{y} (\varepsilon_{x} + iU - \mu_{x})]\}^{-1}$ ($i = 0, 1$) are the Fermi distribution functions, $\beta_{x,y} = 1/(k_B T_v)$, and $\Gamma_{x_1}$ ($\Gamma_{y_1}$) is a positive constant that characterizes the height of the potential barrier between $X$ ($Y$) and reservoir $v$. The potential barrier of $X$, characterized by $\Gamma_{x_1}$, depends on the state of $X$, and vice versa.

For the purpose of reducing the number of parameters, the energy dependences of the tunneling rates are parametrized by dimensionless parameters $\Delta$, $\alpha$, and $\lambda$, i.e.,

$$\Gamma_{Y x_0}^{Y} = \Gamma_{x_0} e^{\Delta \cos(\delta)}, \quad \Gamma_{Y x_1}^{Y} = \Gamma_{x_1} e^{-\Delta \cos(\delta)};$$

$$\Gamma_{Y x_0}^{R} = \Gamma_{x_0} e^{-\Delta \cos(\delta)}, \quad \Gamma_{Y x_1}^{R} = \Gamma_{x_1} e^{\Delta \cos(\delta)};$$

$$\Gamma_{X y_0}^{X} = \Gamma_{y_0} e^{\alpha \cos(\delta)}, \quad \Gamma_{X y_1}^{X} = \Gamma_{y_1} e^{-\alpha \cos(\delta)};$$

$$\Gamma_{X y_0}^{R} = \Gamma_{y_0} e^{-\alpha \cos(\delta)}, \quad \Gamma_{X y_1}^{R} = \Gamma_{y_1} e^{\alpha \cos(\delta)}.$$

According to Eqs. (6), (7), and (11), the entropy flow associated with subsystem $X$

$$S^{X}_r = -\beta \Delta \mu_{X} \left( J_{Y x_1 | y_0}^{XR} + J_{X y_1 | x_0}^{YR} \right) - \beta U \left( J_{Y y_0 | x_0}^{YL} + J_{X y_1 | y_0}^{YR} \right),$$

(17)

the rate of information learned

$$\dot{X} = \left( J_{Y l x_1 | y_0}^{YL} + J_{X y_1 | y_0}^{YR} \right) \ln \frac{p(x_0, y_1)}{p(x_1, y_0)},$$

(18)

and the coefficient

$$\Theta^{X} = \frac{1}{2} \left( \varepsilon - \mu_{XL} \right)^2 \left[ W_{10}^{XL} p(0,0) + W_{01}^{XL} p(1,0) \right]$$

$$+ \frac{1}{2} \left( \varepsilon - \mu_{XR} \right)^2 \left[ W_{10}^{XR} p(0,0) + W_{01}^{XR} p(1,0) \right]$$

$$+ \frac{1}{2} \left( \varepsilon + U - \mu_{XL} \right)^2 \left[ W_{10}^{YL} p(0,1) + W_{01}^{YL} p(1,1) \right]$$

$$+ \frac{1}{2} \left( \varepsilon + U - \mu_{XR} \right)^2 \left[ W_{10}^{YR} p(0,1) + W_{01}^{YR} p(1,1) \right].$$

(19)

The cellular network.—In a living cell, a protein inside the cell may be phosphorylation and dephosphorylation through the consumption of energy provided by adenosine triphosphate (ATP). The receptors on the surface of the cell are either bound by ligands or empty. It is assumed that the receptors are occupied by ligands and immediately change into the inactive state $y_0$ when the concentration of the ligand is high. On the other hand, if the concentration is low, the receptors are unbound and become active represented by state $y_1$. The transitions between state $y_0$ and $y_1$ happen at the rate $\gamma$, because the concentration of the ligand in the environment jumps between low and high values. The rates of the reactions of the protein depend on the concentration of the ligand. For simplicity, we consider the following reaction equation [31, 37, 38]

$$x_0 + ATP \xrightarrow{\kappa_-} x_1 + ADP \xrightarrow{\omega_-} x_0 + ADP + P_1,$$

(20)

where $\kappa_+$ and $\omega_+$ are, respectively, the rates of phosphorylation and dephosphorylation, and $\kappa_-$ and $\omega_-$ represent the rates of the corresponding reversed processes. The phosphorylation allows the protein to change from the inactive state $x_0$ to the active state $x_1$ by the attachment of a phosphate $P_1$, while ATP is simultaneously converted to adenosine diphosphate (ADP).

The dephosphorylation occurs when the protein gives a phosphate $P_1$ to ADP and turns from the active state $x_1$ back to the inactive state $x_0$. The reactions are driven by the chemical potential difference $\Delta \mu \equiv \mu_{ATP} - \mu_{ADP} - \mu_P$, which is related to the rates in the reactions through $\Delta \mu = k_B T \ln [\kappa_+ \omega_+ / (\kappa_- \omega_-)]$ with $T$ being the temperature of the cell. The active receptors arouse the process of phosphorylation in the cell. If the concentration is high, only dephosphorylation occurs due to the inactive receptors. As a summary, the low concentration implies that the receptors become active and only phosphorylation takes place. The full network of the four-state model is shown in Fig. 1 (b). The transition rates are represented by

$$W_{x_1 x_0 | y_0} = \omega_-,$$

$$W_{x_0 x_1 | y_0} = \omega_+,$$

$$W_{x_1 x_0 | y_1} = \kappa_+,$$

$$W_{x_0 x_1 | y_1} = \kappa_-,$$

$$W_{y_0 y_1 | x_0} = W_{y_1 y_0 | x_1} = W_{y_0 y_1 | x_1} = \gamma.$$
\[ J = \gamma \left[ p(x_1, y_1) - p(x_1, y_0) \right] \]
\[ = \frac{\gamma}{2} \left[ \kappa_+ \omega_+ - \kappa_- \omega_- \right] \]
\[ = \frac{\gamma}{2} \left[ \kappa_+ + \kappa_- + \omega_+ + \omega_- \right] + \left( \kappa_+ + \kappa_- \right) \left( \omega_+ + \omega_- \right). \] (21)

According to Eqs. (6), (7), and (11), the entropy flow to the reservoir associated with the protein (subsystem X)
\[ \dot{S}_r^X = J \Delta \mu, \] (22)
the rate of information learned
\[ j^X = J \ln \frac{p(x_1, y_1) p(x_0, y_0)}{p(x_0, y_1) p(x_1, y_0)}, \] (23)
and the coefficient
\[ \Theta^X = \frac{1}{2} \left( \ln \frac{\kappa_+}{\kappa_-} \right)^2 \left[ \kappa_- p(x_1, y_1) + \kappa_+ p(x_0, y_1) \right] \]
\[ + \frac{1}{2} \left( \ln \frac{\omega_+}{\omega_-} \right)^2 \left[ \omega_+ p(x_1, y_1) + \omega_- p(x_0, y_0) \right]. \] (24)

Discussion. —For the DQD system, the efficiency \( \eta^Y \) of learning is always a monotonically increasing of \( \delta \) for fixed \( \beta U \) [Fig. 2(a)]. The transition rates \( \Gamma^{XL}_{y_0} \rightarrow 1, \Gamma^{XR}_{y_0} \rightarrow 0, \Gamma^{XL}_{y_1} \rightarrow 0, \) and \( \Gamma^{XR}_{y_1} \rightarrow 1 \) with a large value of \( \delta \). It parametrizes the case where an electron can only enter and exit subsystem X from reservoir XL at energy \( \varepsilon_X \), whereas the tunneling process to reservoir XR is allowed at energy \( \varepsilon_X + U \). Thus, electron transport is only permitted by exchanging energy with subsystem Y. As a result, the two QDs become highly correlated, since subsystem X is capable of rapidly adapting to the variation of the state in subsystem Y by tracking and giving feedback on Y. When \( \delta \) and \( \beta U \) are both small, \( \eta^X \) is quite small, meaning that the energy provided by reservoirs cannot effectively support the learning process. Fig. 2(a) also displays that the relation \( \eta^X \leq 1 - \dot{S}_r^X / \Theta^X \leq 1 \) always holds, which clearly shows the tightness of the bound in Eq. (12). In the region where \( \delta \) and \( \beta U \) become relatively large, \( \eta^X \) gets closer to 1 and simultaneously the upper limit \( 1 - \dot{S}_r^X / \Theta^X \) may be saturated. The upper bound indeed characterizes the cost necessary for X learning about Y.

For the cellular network, the reaction rates of phosphorylation and dephosphorylation depend on the concentration of the ligand in the environment. Eqs. (22) and (23) demonstrate the tight coupling between \( \dot{S}_r^X \) and \( j^X \), since they are both proportional to the probability current J. Therefore, \( I^X \) is bounded by the rate of ATP consumption, leading to the efficiency \( \eta^X \leq 1 - \dot{S}_r^X / \Theta^X \leq 1 \) always holds, which clearly shows the tightness of the bound in Eq. (12). In the region where \( \delta \) and \( \beta U \) become relatively large, \( \eta^X \) gets closer to 1 and simultaneously the upper limit \( 1 - \dot{S}_r^X / \Theta^X \) may be saturated. The upper bound indeed characterizes the cost necessary for X learning about Y.

The fundamental trade-off inequalities (10) and (12) have been proven to be effective in general Markovian stochastic processes. It is worth investigating the extension of the results to the systems with quantum effects. For example, one could exploit quantum coherence for increasing the efficiency of learning. In addition, by modelling a system as a complex network, a topological analysis could be performed to identify which type of thermodynamic flux and affinity is fundamental to the enhancement of the performance of an informational learning process.

Conclusions. —The fundamental trade-off inequalities (10) and (12) in general Markovian stochastic processes have been established. It is claimed that the entropy-production rate of a subsystem closely depends on the rate of information learned, and has a lower limit associated with the irreversible energy loss and the entropy flow to the environment. The limit imposes a constraint stronger than the conventional second law.

Figure 2. (a) The efficiency \( \eta^X \) and the upper limit \( 1 - \dot{S}_r^X / \Theta^X \) for the double-quantum-dot system varying with the dimensionless Coulomb interaction strength \( \beta U \) and the parameter \( \delta \) associated with the tunneling rate, where \( \beta (\mu_{XR} - \mu_{XL}) = 1, \beta (\mu_{VR} - \mu_{YL}) = 2, \Delta = 1, \Gamma = 1, \varepsilon_X = (\mu_{XR} + \mu_{XL}) - \frac{U}{2}, \) and \( \varepsilon_Y = (\mu_{VR} + \mu_{YL}) - \frac{U}{2}. \) (b) The efficiency \( \eta^X \) and the upper limit \( 1 - \dot{S}_r^X / \Theta^X \) for the cellular network varying with the transition rate \( \gamma \) and the chemical potential difference \( \Delta \mu \), where \( \omega_+ = \kappa_+ + \frac{\exp(\Delta \mu / 2)}{2} \) and \( \omega_- = \kappa_- = 1 \).
which plays a vital role in determining the universal upper bound for the efficiency of learning. A quantum system and a cellular network that both satisfy the stochastic detailed balance condition have been used to exemplify the inequalities. The results reveal a hidden universal law inheriting in general informational learning processes.

Acknowledgments.—This work has been supported by the National Natural Science Foundation (Grants No. 12075197) and the Fundamental Research Fund for the Central Universities (No. 20720210024) of China.

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