Cover Systems for the Modalities of Linear Logic

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Abstract

Ono’s modal FL-algebras are models of an extension of Full Lambek logic that has the modalities \(!\) and \(?\) of linear logic. Here we define a notion of modal FL-cover system that combines aspects of Beth-Kripke-Joyal semantics with Girard’s interpretation of the \(!\) modality, and has structured subsets that interpret propositions. We show that any modal FL-algebra can be represented as an algebra of propositions of some modal FL-cover system.

1 Introduction

Hiroakira Ono pioneered the development of Kripke-style semantic interpretations of substructural logics, beginning with work on logics that lack the contraction rule [Ono and Komori, 1985; Ono, 1985]. His fundamental article [Ono, 1993] then gave a detailed analysis, involving both algebraic and Kripke-type models, for extensions of Full Lambek logic (FL), described roughly as the Gentzen sequent calculus obtained from that for intuitionistic logic by deleting all the structural rules.

Included in this analysis were connectives \(!\) and \(?\) corresponding to the storage and consumption modalities of the linear logic of Girard (1987). The Kripke models for these in [Ono, 1993] were certain relational structures based on semilattice-ordered monoids that carried binary relations to interpret \(!\) and 

In the present paper we give an alternative modelling of this modal FL logic using cover systems that are motivated by the topological ideas underlying the Kripke-Joyal semantics for intuitionistic logic in topoi. A cover system assigns to each point certain sets of points called “covers” in a way that is formally similar to the neighbourhood semantics of modal logics. Covers are used to give non-classical interpretations of disjunction and existential quantification, and in that sense are also reminiscent of Beth’s intuitionistic semantics. The present author has previously developed cover system semantics for the (non-distributive) non-modal FL-logic, as well as for relevant logics and intuitionistic modal logics.

Our treatment of the storage modality \(!\) abstracts from that of the phase space semantics of Girard (1995), which is based on commutative monoids with a certain closure operator on its subsets. Propositions are interpreted there as closed subsets, called facts, and \(!\) \(X\) is defined to be the least fact including \(X \cap I\), where \(I\) is the set of all monoid idempotents that belong to 1, the least fact containing the monoid identity \(\varepsilon\). Here, as well as abandoning the commutativity in order to model FL-logic in general, we allow \(I\) to be a submonoid of this set of idempotents that forms a cover of \(\varepsilon\). We also require \(I\) to be central, i.e. its elements commute with all elements. Our models, which are called modal FL-cover systems, also have a quasiordering that is used to interpret the consumption modality \(?\) by the Kripkean existential clause for a classical \(\diamond\)-style modality. But it should be appreciated that in this non-commutative and non-distributive setting,

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1Bibliographical references for these are given at the end of Section 7
a modality with this existential interpretation need not distribute over disjunction in the way that a classical \(\Diamond\) does.

Propositions for us are “localised up-sets” that are defined by the cover system structure (see Section 3). We show that the set of propositions of a modal FL-cover system satisfies Ono’s axioms for a modal FL-algebra. Any order-complete modal FL-algebra is shown to be isomorphic to the algebra of propositions of some modal FL-cover system, while an arbitrary modal FL-algebra can be embedded into the algebra of propositions of a modal FL-cover system by an embedding that preserves any existing joins and meets. We also show that in any FL-algebra with a storage modality \(!\), the term function \(- ! - a\) defines a modality satisfying the axioms for \(\otimes\) so gives rise to a modal FL-algebra. Here the two occurrences of \(-\) can stand separately for either of the two negation operations that exist in any FL-algebra.

2 Modalities on Residuated Lattices

A residuated partially ordered monoid (or residuated pomonoid), can be defined as an algebra of the form

\[ L = (L, \sqsubseteq, \otimes, 1, \Rightarrow_l, \Rightarrow_r), \]

such that:

- \(\sqsubseteq\) is a partial ordering on the set \(L\).
- \((L, \otimes, 1)\) is a monoid, i.e. \(\otimes\) is an associative binary operation (called fusion) on \(L\), with identity element \(1\), that is \(\sqsubseteq\)-monotone in each argument: \(b \sqsubseteq c\) implies \(a \otimes b \sqsubseteq a \otimes c\) and \(b \otimes a \sqsubseteq c \otimes a\).
- \(\Rightarrow_l\) and \(\Rightarrow_r\) are binary operations on \(L\) called the left and right residuals of \(\otimes\), satisfying the residuation law

\[ a \sqsubseteq b \Rightarrow_l c \iff a \otimes b \sqsubseteq c \iff b \sqsubseteq a \Rightarrow_r c. \]

A residuated lattice is a residuated pomonoid that is a lattice under \(\sqsubseteq\), with binary join operation \(\sqcup\) and meet operation \(\sqcap\). We also write \(\sqcup\) and \(\sqcap\) for the join and meet operations on subsets of \(L\), when these operations are defined.

Galatos et al. (2007) give an extensive treatment of the theory of residuated lattices and its application to substructural logic. They define an FL-algebra (Full Lambek algebra) to be a residuated lattice with an additional distinguished element \(0\). We will mainly deal with lattices that are bounded, i.e. have a greatest element \(T\) and least element \(F\). For this it suffices that there be a least element \(F\), for then there is a greatest element \(F \Rightarrow_l F = F \Rightarrow_r F\).

**Definition 2.1.** A storage modality on a residuated lattice \(L\) is a unary operation \(!\) on \(L\) such that

\(\begin{align*}
(s1) & \quad !a \sqsubseteq a, \\
(s2) & \quad !a \sqsubseteq !a, \\
(s3) & \quad !1 = 1, \\
(s4) & \quad !(a \sqcap b) = !a \otimes !b, \\
(s5) & \quad a \otimes b = b \otimes !a.
\end{align*}\)

**Lemma 2.2.** Any storage modality satisfies the following.

\(\begin{align*}
(1) & \quad !a \sqsubseteq 1, \\
(2) & \quad ! is monotone, i.e. \(a \sqsubseteq b\) implies \(!a \sqsubseteq !b\), \\
(3) & \quad !a = !a \otimes !a.
\end{align*}\)

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Notation: in the literature on residuation, \(a \Rightarrow_l b\) is often written as \(b/a\), and \(a \Rightarrow_r b\) as \(a\setminus b\).
Proof. (1) Using (s3), (s4) and then (s1), with lattice properties, we observe that together are equivalent to (s1)–(s4) in any IL-algebra, which automatically satisfies (s5) because of Definition 2.3.

(2) If \( a \subseteq b \), then \( !a = !(a \cap b) = !a \otimes !b \) by (s4). But by (1) and monotonicity of \( \otimes \), we get \( !a \otimes !b \subseteq 1 \otimes !b = !b \).

(3) Put \( a = b \) in (s4).

(4) By (s1) and (s2), \( !(a \cap b) = !(a \cap b) \). This becomes \( !a \otimes !b = !(a \otimes !b) \) by (s4). But \( !a \otimes !b \subseteq a \otimes b \) by (s1) and monotonicity of \( \otimes \). So \( !(a \otimes !b) \subseteq !(a \otimes b) \) by (2).

(5) \( (a \Rightarrow_b b) \otimes a \subseteq b \), so \( !(a \Rightarrow_b b) \otimes a \subseteq !b \) by (2). Then using (4), \( !(a \Rightarrow_b b) \otimes !a \subseteq !(a \Rightarrow_b b) \subseteq !a \Rightarrow_b !b \).

The case of \( \Rightarrow_r \) is similar, using \( a \otimes (a \Rightarrow_r b) \subseteq b \).

(6) \( \mathcal{T} \subseteq 1 \) by (1). But \( 1 = !1 \subseteq \mathcal{T} \) by (s3) and (2).

The axioms (s1)–(s5) form part of Ono’s definition of a modal FL-algebra, which we come to shortly. Note that (s5) was not used at all in Lemma 2.4. Troelstra (1992) deals with IL-algebras (intuitionistic linear algebras), which are essentially bounded residuated lattices in which \( \otimes \) is commutative, and hence \( \Rightarrow_l \) and \( \Rightarrow_r \) are identical. He defines an ILS-algebra (intuitionistic linear algebra with storage), to be an IL-algebra with a unary operator \( ! \) having \( a \subseteq a \otimes !a = a \otimes b \) only if \( a \subseteq !b \); \( !T = 1 \); and \( !(a \cap b) = !a \otimes !b \). These conditions together are equivalent to (s1)–(s4) in any IL-algebra, which automatically satisfies (s5) because it has commutative \( \otimes \).

Another equivalent definition of ILS-algebras is given by Bucalo (1994, Def. 3.5). It has \( !T \subseteq 1 \) and \( 1 \subseteq !1 \) in place of (s3), and \( !a \otimes !a \subseteq a \) and \( !a \otimes !b \subseteq !(a \otimes !b) \) in place of (s4).

The following notion was introduced in Ono (1993, Definition 6.1).

Definition 2.3. A modal FL-algebra is a bounded FL-algebra with a storage modality \( ! \) and an additional unary operation \( ? \) satisfying

- \((c1)\) \( !(a \Rightarrow_i b) \subseteq ?a \Rightarrow_i ?b \) \( \text{ for } i = l, r \).
- \((c2)\) \( a \subseteq ?a \).
- \((c3)\) \( ?a \subseteq ?a \).
- \((c4)\) \( 0 \subseteq 0 \).
- \((c5)\) \( 0 \subseteq ?a \).

Lemma 2.4. In any modal FL-algebra, \( ? \) is a monotone operation satisfying

\( (c6) \) \( !a \otimes ?b \subseteq ?(a \otimes b) \) and \( (c7) \) \( ?a \otimes !b \subseteq ?(a \otimes b) \).

Proof. We have \((a \Rightarrow_l b) \otimes a \subseteq b \). Now let \( a \subseteq b \). Then \( 1 \otimes a \subseteq b \), so \( 1 \subseteq a \Rightarrow_l b \) and hence by (s3) and monotonicity of \( ! \),

\[
1 = !1 \subseteq !(a \Rightarrow_l b) \subseteq ?a \Rightarrow_l ?b,
\]

with the last inequality given by (c1) with \( i = l \). Thus \( ?a = 1 \otimes ?a \subseteq ?b \), establishing that \( ? \) is monotone.

For (c6), since \( a \subseteq b \Rightarrow_l a \otimes b \), we get \( !a \subseteq !(b \Rightarrow_l a \otimes b) \subseteq ?b \Rightarrow_l ?(a \otimes b) \) using (c1) for \( i = l \). Hence \( !a \otimes ?b \subseteq ?(a \otimes b) \). The proof of (c7) is similar, using \( b \subseteq a \Rightarrow_r a \otimes b \) and (c1) for \( i = r \).
The conditions on $?$ in this Lemma are equivalent to (c1) in any FL-algebra. In fact more strongly:

**Lemma 2.5.** Let $L$ be a residuated pomonoid with a monotone operation $!$ having $!1 = 1$. Then a unary operation $\text{?}$ on $L$ satisfies (c1) if, and only if it is monotone and satisfies (c6) and (c7).

**Proof.** The only-if part is shown by the proof of the last Lemma. Conversely, assume $?$ is monotone and satisfies (c6) and (c7). By (c6) and then $?$-monotonicity,

$$!(a \Rightarrow b) \subseteq ?^2 a \subseteq !((a \Rightarrow b) \otimes a) \subseteq ?b$$

from which residuation gives (c1) for $i = l$. The case of $i = r$ is similar, using (c7). \qed

Any modal FL-algebra $L$ can be embedded into an order-complete modal FL-algebra by an embedding that preserves any joins and meets that exist in $L$. This was shown in Section 4 of [Ond\v{r} 1993], by an extension of the MacNeille completion construction. We make use of the result below in representing modal FL-algebras over cover systems.

## 3 Cover Systems

FL-algebras will be represented as algebras of subsets of structures of the form $S = (S, \preceq, \lhd, \ldots)$, in which $\preceq$ is a preorder (i.e. reflexive transitive relation) on $S$, and $\lhd$ is a binary relation from $S$ to its powerset $\mathcal{P}S$. We sometimes write $y \rhd x$ when $x \preceq y$, and say that $y$ refines $x$. When $x \lhd C$, where $x \in S$ and $C \subseteq S$, we say that $x$ is covered by $C$, and write this also as $C \triangleright x$, saying that $C$ covers $x$ or that $C$ is an $x$-cover.

An up-set is a subset $X$ of $S$ that is closed upwardly under refinement: $y \rhd x \in X$ implies $y \in X$. For an arbitrary $X \subseteq S$,

$$\uparrow X = \{y \in S : (\exists x \in X) x \preceq y\}$$

is the smallest up-set including $X$. For $x \in S$, we write $\uparrow x$ for $\uparrow\{x\} = \{y : x \preceq y\}$, the smallest up-set containing $x$.

The collection $\text{Up}(S)$ of all up-sets of $S$ is a complete poset under the partial order $\subseteq$ of set inclusion, with the join $\bigcup X$ and meet $\bigcap X$ of any collection $X$ of up-sets being its set union $\bigcup X$ and intersection $\bigcap X$ respectively, while $\emptyset = \emptyset$ and $\top = S$.

A subset $Y$ of $S$ refines a subset $X$ if $Y \subseteq \uparrow X$, i.e. if every member of $Y$ refines some member of $X$. We call $S$ a cover system if it satisfies the axioms, for all $x \in S$:

- **Existence:** there exists an $x$-cover $C \subseteq \uparrow x$;
- **Transitivity:** if $x \lhd C$ and for all $y \in C$, $y \lhd C_y$, then $x \lhd \bigcup_{y \in C} C_y$;
- **Refinement:** if $x \preceq y$, then every $x$-cover can be refined to a $y$-cover, i.e. if $C \triangleright x$, then there exists a $C' \triangleright y$ with $C' \subseteq \uparrow C$.

For each subset $X$ of $S$, define

$$jX = \{x \in S : \exists C (x \lhd C \subseteq X)\}. \quad (3.1)$$

A property is thought of as being *locally true* of $x$ if $x$ is covered by a set of members that have this property, i.e. if there is some $C$ such that $x \lhd C$ and each member of $C$ has the property. In this sense, $x$ belongs to $jX$ just when the property of being a member of $X$ is locally true of $x$. So $jX$ can be thought of as the collection of “local members” of $X$. $X$ is called *localised* if $jX \subseteq X$, i.e. if every local member of $X$ is an actual member of $X$.

It was shown in [Goldblatt 2006](#) Theorem 5) and [Goldblatt 2011a](#) Lemma 3.3) that in any cover system, the function $j$ defined by (3.1) is a closure operator on the complete poset $(\text{Up}(S), \subseteq)$ of up-sets, i.e. $j$ is monotonic and has $X \subseteq jX = j(jX)$.
A proposition in a cover system is an up-set $X$ that is localised, i.e. $jX \subseteq X$, hence $jX = X$. In general, a set $X$ is a proposition iff $X = \uparrow X = jX$. $j\uparrow x$ is the smallest proposition that includes an arbitrary $X$, and $j\uparrow x$ is the smallest proposition containing the element $x$. The smallest proposition including an up-set $X$ is just $jX$, so in fact $j$ maps $Up(S)$ onto the set $Prop(S)$ of all localised up-sets of a cover system $S$. Indeed, $Prop(S)$ precisely the set of fixed points of this map.

Requiring propositions to be localised amounts to making truth a property of \emph{local character}, i.e. it holds whenever it does so locally. For further discussion of this see Goldblatt (2011a), or (Goldblatt 2011c, Section 6.3) where an information-theoretic interpretation of the cover relation is also given.

Our cover systems have some formal similarities with the notion of a pretopology of Sambin (1989), but there are some basic differences, including the presence here of the preorder $\prec$, and the absence of Sambin’s reflexivity condition that $x \prec C$ whenever $x \in C$. Our systems are motivated by the topological ideas underlying the Beth-Kripke-Joyal semantics for logic in sheaf categories (Mac Lane and Moerdijk 1992), and relate more to the \emph{cover schemes} of preordered sets (Bell 2005).

Dráglain (1988, p. 72) gave a method of constructing closure operators over preordered sets that is motivated by the features of Beth’s models. He defined an operation $D$ on down-sets of a preorder by taking a function $Q$ assigning to each $x \in S$ a collection $Q(x)$ of subsets of $S$, and putting $DY = \{x \in S : \forall C \in Q(x), C \cap Y \neq \emptyset\}$. He gave conditions on $Q$ ensuring that $D$ is a closure operator, and interpreted $C \in Q(x)$ by saying that ‘$C$ is a path starting from the moment $x$’. Now if we define $x \prec C$ to mean $C \in Q(x)$, then for any up-set $X$ it follows that $S \setminus X$ is a down-set and $j_aX = S \setminus (D(S \setminus X))$, so in this sense Dráglain’s approach is dual to that of cover systems. Bezhanishvili and Holliday (2010) give a detailed discussion of the relationship between these approaches.

Every topological space has the cover system in which $S$ is the set of open subsets of the space, with $x \in y$ iff $x \supseteq y$ and $x \prec C$ iff $x = \bigcup C$. This system has the property that every $x$-cover is included in $\uparrow x$, as do the cover schemes of (Bell 2005). But this property makes $Prop(S)$ into a distributive lattice. Indeed even the weaker constraint that every $x$-cover can be \emph{refined} to an $x$-cover included in $\uparrow x$ is enough to force $Prop(S)$ to be a complete Heyting algebra (Goldblatt 2011a, Theorem 3.5), and hence a model of intuitionistic logic. Since we are interested in non-distributive residuated lattices, any such constraint must be abandoned.

4 Residuated Cover Systems

To make $Prop(S)$ into a residuated pomonoid we add a a binary operation $\cdot$ on $S$, which will also be called \emph{fusion} (hopefully without causing confusion). This is lifted to a $\subseteq$-monotone binary operation on subsets of $S$ by putting, for $X,Y \subseteq S$,

$$X \cdot Y = \{x \cdot y : x \in X \text{ and } y \in Y\}.$$  

We write $x \cdot Y$ for the set $\{x\} \cdot Y$, and $X \cdot y$ for $X \cdot \{y\}$.

Define operations $\Rightarrow_\gamma$ and $\Rightarrow_\cdot$ on subsets of $S$ by

$$X \Rightarrow_\gamma Y = \{z \in S : z \cdot X \subseteq Y\}, \quad X \Rightarrow_\cdot Y = \{z \in S : X \cdot z \subseteq Y\}. \quad (4.1)$$

These provide left and right residuals to the fusion operation on the complete poset $(PS, \subseteq)$, i.e. for all $X,Y,Z \subseteq S$ we have

$$X \subseteq Y \Rightarrow_\gamma Z \quad \text{iff} \quad X \cdot Y \subseteq Z \quad \text{iff} \quad Y \subseteq X \Rightarrow_\cdot Z. \quad (4.2)$$

Next define $X \circ Y$ to be the up-set $\uparrow(X \cdot Y)$ generated by $X \cdot Y$. Then if $Z$ is an up-set, we have $X \cdot Y \subseteq Z$ iff $X \circ Y \subseteq Z$, and hence (4.2) implies

$$X \subseteq Y \Rightarrow_\gamma Z \quad \text{iff} \quad X \circ Y \subseteq Z \quad \text{iff} \quad Y \subseteq X \Rightarrow_\cdot Z. \quad (4.3)$$
for any \(X\) and \(Y\). If the fusion operation \(\cdot\) is \(\preceq\)-monotone in each argument, then \(Y \Rightarrow_l Z\) and \(X \Rightarrow_r Z\) are up-sets when \(Z\) is an up-set. In particular, this implies that \(Up(S)\) is closed under \(\Rightarrow_l\) and \(\Rightarrow_r\), and so by (4.3), these operations are left and right residuals to \(\circ\) on \(Up(S)\).

A residuated cover system was defined in (Goldblatt, 2011b) to be a structure of the form

\[
S = (S, \preceq, \lhd, \cdot, \varepsilon),
\]

such that:

- \((S, \preceq, \lhd)\) is a cover system.
- \((S, \cdot, \varepsilon)\) is a pomonoid, i.e. \(\cdot\) is an associative operation on \(S\) that is \(\preceq\)-monotone in each argument, and has \(\varepsilon \in S\) as identity.
- Fusion preserves covering: \(x \lhd C\) implies \(x \cdot y \lhd C\cdot y\) and \(y \cdot x \lhd y \cdot C\).
- Refinement of \(\varepsilon\) is local: \(x \lhd C \subseteq \uparrow \varepsilon\) implies \(\varepsilon \preceq x\).

The last condition states that if \(x\) locally refines \(\varepsilon\), in the sense that it has a cover consisting of points refining \(\varepsilon\), then \(x\) itself refines \(\varepsilon\). This means that the up-set \(\uparrow \varepsilon\) of points refining \(\varepsilon\) is localised, i.e. \(\uparrow \varepsilon \subseteq \uparrow \varepsilon\), and therefore is a proposition. The condition that fusion preserves covering implies, more strongly, that

\[
\text{if } x \lhd C \text{ and } y \lhd D, \text{ then } x \cdot y \lhd C \cdot D. \tag{4.4}
\]

This was shown in (Goldblatt, 2011b), where the following was also established:

**Theorem 4.1.** The set \(\text{Prop}(S)\) of propositions of a residuated cover system \(S\) forms a complete residuated lattice under a monoidal operation \(\otimes\) with identity 1, where

\[
X \otimes Y = j(X \circ Y) = j(\uparrow (X \cdot Y))
\]

\[
1 = \uparrow \varepsilon
\]

\[
\bigcap X = \bigcap jX
\]

\[
\bigcup X = j(\bigcup X)
\]

\[
X \Rightarrow_l Y = \{z \in S : z \cdot X \subseteq Y\}
\]

\[
X \Rightarrow_r Y = \{z \in S : X \cdot z \subseteq Y\}
\]

\[
T = S
\]

\[
F = j\emptyset = \{x : x \lhd \emptyset\}.
\]
proposition including a set \( X \) is \( j \uparrow X \), and we will define \( !X \) to be \( j \uparrow (X \cap I) \), where \( I \) is a designated subset of \( \uparrow \varepsilon (=1) \) that is a cover of \( \varepsilon \) and a submonoid of \((S, \cdot, \varepsilon)\), as well as consisting of idempotents that commute with all members of \( S \). But whereas \( ! \) and \( ? \) are interdefinable in linear logic, here we treat them as independent, and model \( ? \) by the same interpretation that Kripke gave to the classical modality \( \Diamond \). We take a binary relation \( R \) on \( S \) and let \( ? X \) be the set

\[
\{x \in S : \exists y(xRy \in X)\}. \tag{5.1}
\]

In (Goldblatt, 2006, 2011a) we showed that any monotone operation on \( \text{Prop}(S) \) can be given such a modelling, provided the cover system interacts in specified ways with the relation \( R \).

**Definition 5.1.** A modal FL-cover system is a structure

\[
\mathcal{S} = (\mathcal{S}, \preceq, \preceq, \cdot, \varepsilon, 0, I, R),
\]

with \( 0 \in \text{Prop}(S) \), \( I \subseteq \varepsilon \) and \( R \subseteq S \times S \), such that:

- \( (S, \preceq, \preceq, \cdot, \varepsilon) \) is a residuated cover system.
- \( I \) is a submonoid of \((S, \cdot, \varepsilon)\), i.e. \( I \) is closed under \( \cdot \) and contains \( \varepsilon \).
- \( I \) is an \( \varepsilon \)-cover: \( \varepsilon \prec I \).
- \( I \) is idempotent and central: if \( x \in I \), then \( x = x \cdot x \) and \( x \cdot y = y \cdot x \) for all \( y \in S \).
- \( \preceq \) and \( R \) are confluent: if \( x \preceq y \) and \( xRz \), then there exists \( w \) with \( z \preceq w \) and \( yRw \);
- Modal Localisation: if there exists an \( x \)-cover included in \( \langle R \rangle X \), then there exists a \( y \) with \( xRy \) and a \( y \)-cover included in \( X \).
- \( R \)-Monotonicity: If \( x \in I \) and \( yRx \), then \( x \cdot yRx \cdot z \) and \( y \cdot xRx \cdot x \).
- \( R \) is reflexive and transitive.
- \( xRy \in 0 \) implies \( x \in 0 \).
- \( x \in 0 \) implies that for some \( y \), \( xRy \prec 0 \).

This definition looks formidable but has a certain inevitability. Its conditions are those that are needed to show that the lattice of propositions of \( S \) is a complete modal FL-algebra. The details of how this works are given in the proof of the next theorem, but first we give a summary. We have already observed that \( \text{Prop}(S) \) is a complete residuated lattice when \( \mathcal{S} \) is based on a residuated cover system. The specified proposition \( 0 \) then serves as the distinguished element making \( \text{Prop}(S) \) into an FL-algebra. The listed conditions on \( I \) are used\(^1\) to show that the operation \( !X = j \uparrow (X \cap I) \) is a storage modality on \( \text{Prop}(S) \), i.e. satisfies (s1)–(s5) of Definition\(^2\,2.1\) In particular, while (s1) and (s2) hold for any \( I \), the proof of (s3) uses both \( I \subseteq \varepsilon \) and \( \varepsilon \prec I \); that of (s4) uses that \( I \) is closed under \( \cdot \) and idempotent; and (s5) uses that \( I \) is central. The listed conditions on \( R \) are used to show that the operation \( ?X \) belongs to \( \text{Prop}(S) \) whenever \( X \) does; the proof of (c1) uses \( \cdot \)-monotonicity; those of (c2) and (c3) use reflexivity and transitivity of \( R \), respectively; and the last two conditions on \( R \) are used for (c4) and (c5) respectively.

We turn now to the details.

**Theorem 5.2.** If \( \mathcal{S} = (\mathcal{S}, \preceq, \preceq, \cdot, \varepsilon, 0, I, R) \) is a modal FL-cover system, then

\[
\mathbf{L}_{\mathcal{S}} = (\text{Prop}(\mathcal{S}), \preceq, \otimes, 1, \rightarrow_{t}, \rightarrow_{r}, !, ?),
\]

is a complete modal FL-algebra, where the operations \( \otimes, 1, \rightarrow_{t}, \rightarrow_{r} \) are as given in Theorem\(^3\,4.1\); the storage modality is defined by \( !X = j \uparrow (X \cap I) \); and the consumption modality is given by \( ?X = \langle R \rangle X \) as defined in (5.1).

\(^1\)Except for the condition \( \varepsilon \in I \); see the note at the end of this section.
Proof. By Theorem 1, \((\text{Prop}(S), \subseteq, \circ, 1, \Rightarrow, \Rightarrow, 0)\) is a complete FL-algebra. That it is closed under \(\Rightarrow\) follows because \(jY\) is a proposition for any \(Y \subseteq S\), hence in particular \(X = j\uparrow (X \cap I) \in \text{Prop}(S)\). Confluence of \(\Rightarrow\) and \(R\) ensures that \((R)X\) is an up-set if \(X\) is, while Modal Localisation ensures that \((R)X\) is localised if \(X\) is \((\text{Goldblatt} 2011, p1047)\). Thus \(\text{Prop}(S)\) is closed under the modality ?.

We verify the axioms of Definitions 2.1 and 2.4 for any propositions \(X, Y \in \text{Prop}(S)\).

(s1): If \(x \in !X\), then for some \(C, x \prec C \subseteq \uparrow(X \cap I) \subseteq \uparrow X = X\) (as \(X\) is an up-set). So \(x \prec C \subseteq X\), showing \(x \in jX \subseteq X\) (as \(X\) is localised). This proves \(!X \subseteq X\).

(s2): Since \(X \cap I \subseteq j\uparrow(X \cap I) = !X\), we get \(j\uparrow(X \cap I) \subseteq j\uparrow(!X \cap I)\), i.e. \(!X \subseteq !X\).

(s3): We want \(j! \equiv 1\), i.e. \(j\uparrow(\varepsilon \cap I) = \varepsilon\). Since \(I \subseteq \varepsilon\), this simplifies to \(jI = \varepsilon\). Now from \(I \subseteq \varepsilon\) we get \(jI \subseteq j\uparrow \varepsilon = j\varepsilon \subseteq \varepsilon\) (since \(\varepsilon\) is a proposition). For the converse inclusion, as \(I\) is an \(\varepsilon\)-cover, \(\varepsilon \cap I \subseteq \varepsilon\), hence \(e \in jI\) and so \(\varepsilon \subseteq jI\) as \(jI\) is an up-set.

(s4): We want \(!X \otimes Y = !X \otimes !Y\), i.e. \(j\uparrow(X \otimes Y \cap I) = j\uparrow(X \otimes !Y)\).

Now if \(x \in !X \otimes !Y\), then \(x \prec C \subseteq \uparrow(X \otimes Y \cap I)\) for some \(C\). If \(C\) is any member of \(C\), then there is some \(d\) with \(c \succ d \subseteq X \otimes Y \cap I \subseteq !X \otimes !Y\). Then \(d = d \cap d\) by idempotence, this leads to \(c \in \uparrow(\varepsilon \otimes X \otimes !Y)\). Altogether this shows that \(x \prec C \subseteq \uparrow(\varepsilon \otimes X \otimes !Y)\), hence \(x \in j\uparrow(\varepsilon \otimes X \otimes !Y) = !X \otimes !Y\) as required.

For the converse inclusion it suffices to show that \(!X \otimes Y \subseteq !X \otimes Y\), since this forces \(!X \otimes Y = j\uparrow(\varepsilon \otimes X \otimes !Y)\) !\(X \otimes Y\), because \(!X \otimes Y\) is a proposition and \(j\uparrow(\varepsilon \otimes X \otimes !Y)\) is the least proposition including \(!X \otimes Y\). So suppose \(x \cdot y \in !X \otimes Y\), where \(x \in X\) and \(y \in Y\). Then \(x \in C_x \subseteq \uparrow(X \cap I)\) and \(y \in C_y \subseteq \uparrow(Y \cap I)\) for some \(C_x\) and \(C_y\). Hence \(y \in C_y \subseteq \uparrow(Y \cap I)\) and \(x \in C_x \subseteq \uparrow(X \cap I)\), as fusion preserves covering. Now take any element \(c\cdot \cdot d\) of \(C_x \cdot C_y\). Then there exist \(d\) with \(e \succeq d\) \(\subseteq X \otimes Y\) and \(e \succeq d\) \(\subseteq Y \otimes I\). Hence \(x \cdot y \in C_x \cdot C_y\) by the strong form \(\lceil\cdot\rceil\) of preservation of covering by fusion. Now take any \(x \cdot y \in \uparrow(X \otimes Y \cap I)\). Then \(x \cdot y \in X \otimes Y \subseteq \uparrow(X \otimes Y \cap I)\) as \(I\) is closed under fusion. It follows that \(x \cdot y \in \uparrow(X \otimes Y \cap I)\). Altogether we now have \(x \cdot y \in X \otimes Y \subseteq \uparrow(X \otimes Y \cap I)\), so \(x \cdot y \in j\uparrow(X \otimes Y \cap I) = !X \otimes !Y\) as required.

(s5): We want \(!X \otimes Y = Y \otimes !X\). First, to show \(!X \otimes Y \subseteq Y \otimes !X\) it suffices to show that \(!X \otimes Y \subseteq Y \otimes !X\) since \(!X \otimes Y\) is the least proposition including \(!X \otimes Y\). So take \(x \in X\) and \(y \in Y\). Then \(x \prec C \subseteq \uparrow(X \cap I)\) for some \(C\). Now for any \(c \in C\), there is some \(c\) \(\subseteq X \cap I\) with \(c \succ c\). Then \(c \cdot y \succ c \cdot y = y \cdot c\), with the last equality holding because \(c\) \(\subseteq I\) and \(I\) is central. But \(y \cdot c \in \uparrow(Y \cap \circ I) \subseteq Y \cdot X\), so \(y \cdot c \in \uparrow(Y \otimes !X)\). This proves that \(C \cdot y \subseteq \uparrow(Y \otimes !X)\). But \(x \cdot y \in C \cdot y\), as fusion preserves covering, so then \(x \cdot y \in j\uparrow(Y \otimes !X) = Y \otimes !X\).

That completes the proof that \(!X \cdot Y \subseteq Y \otimes !X\), and hence that \(!X \otimes Y \subseteq Y \otimes !X\). The proof of the converse inclusion \(!X \otimes Y \subseteq Y \otimes !X\) is similar.

(e1): We want \(!X \Rightarrow \Rightarrow Y \subseteq X \Rightarrow \Rightarrow Y\) for \(i = l, r\). Taking the case of \(i = l\), let \(x \in !X \Rightarrow \Rightarrow Y\), so \(x \prec C \subseteq \uparrow(X \Rightarrow \Rightarrow Y) \cap I\) for some \(C\). Then \(x \cdot y \in C \cdot y\) as fusion preserves covering. To show that \(x \in X \Rightarrow \Rightarrow Y\), take any \(y \in X\). We must then prove that \(x \cdot y \in \Rightarrow Y\). We have \(yRz \in \Rightarrow Y\) for some \(z\). For each \(c \in C\) there exists \(d \in (\Rightarrow Y) \cap I\) with \(c \succeq d\), hence \(c \cdot y \succeq d \cdot y\). Since \(d \in I\), by \(R\)-monotonicity this implies \(d \cdot y \in (\Rightarrow Y) \cdot X \subseteq \Rightarrow Y\), so \(y \cdot c \in \Rightarrow Y\). Altogether this shows that \(x \cdot y \in \Rightarrow Y\), so \(x \cdot y \in \Rightarrow Y\). As \(\Rightarrow Y\) is a proposition.

The case of \(i = r\) is similar, using the facts that that \(x \in C\) implies \(y \cdot x \in y \cdot C\), and \(yRz\) implies \(y \cdot dRz \cdot d\) when \(d \in I\).

(e2): That \(X \subseteq ?X\) follows because \(R\) is reflexive.

(e3): That \(?X \subseteq X\) follows because \(R\) is transitive.

(e4): That \(0 \subseteq 0\) corresponds exactly to the \(S\)-condition that \(xRy \in 0\) implies \(x \in 0\).
To show \( 0 \subseteq X \), let \( x \in 0 \). Then for some \( y, x R y \not< \emptyset \subseteq X \). Now \( y \in jX = X \) as \( X \) is a proposition, hence \( x \in (R)X = ? X \). □

It is notable that the only part of idempotence of \( I \) that was used was the condition \( d \geq d \cdot d \) for \( d \in I \) in the proof of (s4). The reverse inequality \( d \leq d \cdot d \) holds independently of idempotence, for if \( d \in I \) then \( \epsilon \leq d \), so \( d = d \cdot \epsilon \leq d \cdot d \). Thus if \( \leq \) is a partial order (i.e. also anti-symmetric) it is enough to require that \( d \leq d \cdot d \) in order to have \( d = d \cdot d \) for \( d \in I \).

Note also that we made no of use the requirement that \( I \) contains \( \epsilon \) in the above result. But we will see that any modal FL-algebra is representable as an algebra of propositions based on a cover system that does have \( \epsilon \in I \) and \( \leq \) anti-symmetric.

## 6 Representation of Modal FL-Algebras

Let \( L = (L, \sqsubseteq, \otimes, 1, \Rightarrow_1, \Rightarrow_r) \) be an order-complete residuated lattice. Define a structure \( S^L = (S, \preceq, \preceq_1, \epsilon, r) \) by putting \( S = L; x \preceq y \iff y \sqsubseteq x; x \preceq_1 \epsilon \iff x \sqsubseteq \bigcup C; x \cdot y = x \otimes y; \) and \( \epsilon = 1 \).

Then \( S^L \) is a residuated cover system, and moreover is one in which the monoid operation \( \otimes \) on \( \text{Prop}(S^L) \) has \( X \sqsubseteq Y = X \circ Y = \uparrow(X \cdot Y) \). A cover system will be called strong if it satisfies this last condition \( X \otimes Y = X \circ Y \).

Note that in \( S^L \), the up-set \( \uparrow x = \{ y : x \leq y \} \) is equal to \( \{ y : y \sqsubseteq x \} \), which is the down-set of \( x \) in \( (L, \sqsubseteq) \). In fact the propositions of \( S^L \) are precisely these up-sets: if \( x \in \text{Prop}(S^L) \), then \( X = \uparrow x \) where \( x = \bigcup X \). The map \( x \mapsto \uparrow x \) is order-invariant, in the sense that \( x \subseteq y \) iff \( \uparrow x \subseteq \uparrow y \), and is an isomorphism between the complete posets \( (L, \sqsubseteq) \) and \( (\text{Prop}(S^L), \subseteq) \), preserving all joins and meets. It also has \( \uparrow(x \otimes y) = \uparrow x \circ \uparrow y \) and \( \uparrow(1) = \uparrow \epsilon \), and so is an isomorphism between \( L \) and the complete residuated lattice \( \text{Prop}(S^L) \) as described in Theorem 4.1.

In this way we see that every order-complete residuated lattice is isomorphic to the full algebra of all propositions of some residuation cover system. The proofs of these claims about the relationship between \( L \) and \( S^L \) are set out in detail in (Goldblatt, 2011b, Section 3).

Now suppose \( L \) is a modal FL-algebra \( (L, \sqsubseteq, \otimes, 1, \Rightarrow_1, \Rightarrow_r, 0, !, ?) \). Expand the above residuated cover system \( S^L \) to a structure \( (S, \preceq, \preceq_1, \epsilon, o^L, I, R) \) by adding the definitions

\[
0^L = \uparrow 0 = \{ x : x \sqsubseteq 0 \},
I = \{ ! x : x \in L \},
R = \{ (x, y) : x \sqsubseteq ? y \}.
\]

**Lemma 6.1.** \( S^L \) is a modal FL-cover system in which, for all \( x \in L \),

\[
\text{Lemma 6.1.} \quad \uparrow(\neg x) = j(\uparrow x) \cap I, \quad \text{and} \quad \uparrow(\neg x) = (R)(\uparrow x).
\]

**Proof.** We will apply the axioms for \( L \) given in Definitions 2.1 and 2.3 and the properties derived in Lemmas 2.2 and 2.4. Observe that \( 0^L \) belongs to \( \text{Prop}(S^L) \) because it is of the form \( \uparrow x \); and that \( I \subseteq \uparrow \epsilon \) because in general \( ! x \subseteq 1 = \epsilon \), so \( \epsilon \preceq ! x \). We show that \( \epsilon \preceq x \).

- \( I \) is a submonoid of \( (S, \cdot, \epsilon) \): the equation \( x \otimes ! y = ! (x \otimes ! y) \) ensures that \( I \) is closed under \( \cdot \), while \( \epsilon = 1 = ! 1 \) ensures that \( \epsilon \in I \).
- \( I \) is an \( \epsilon \)-cover: \( \epsilon \in I \) implies that \( \epsilon \subseteq \bigcup I \), showing \( \epsilon \preceq I \).
- \( I \) is idempotent because \( ! x \otimes ! x = ! x \); and central because \( ! x \otimes y = y \otimes ! x \) (s5).
- Confluence of \( \preceq \) and \( R \): if \( x \preceq y \) and \( x R z \), then \( y \subseteq x \subseteq ? z \), so \( y R z \). Putting \( w = z \) gives \( z \preceq w \) and \( y R w \).
• Modal Localisation: suppose there is a $C$ with $x \triangleleft C \subseteq (R)X$. Then $x \subseteq \uparrow C$. Let $C' = \{z \in X : \exists y \in C (cRz)\}$ and put $y = \bigcup C'$. Then $C'$ is a $y$-cover included in $X$, and so it remains only to show that $xRy$. But if $c \in C$, then by supposition there exists $z$ with $cRz \in X$. Then $c \subseteq z$ and $z \subseteq y$, hence $z \subseteq z$ as $z$ is monotone, and thus $c \subseteq y$. Therefore $x \subseteq \bigcup C \subseteq y$, giving $xRy$ as required.

• $R$ is reflexive as $x \subseteq x$ (c2) gives $xRx$. $R$ is transitive because if $xRyRz$, then $x \subseteq y$ and $y \subseteq z$, hence $y \subseteq z$ (c3), so $x \subseteq z$, i.e. $xRz$.

• $R$-Monotonicity: if $x \in I$ and $yRz$, then $x = !w$ for some $w$ and $y \subseteq ?z$. Using (s2), $\subseteq$-monotonicity of $\otimes$ and then (c6), we reason that

$$x \cdot y = !w \otimes !w \subseteq !w \otimes ?z \subseteq (?w \otimes z) = ?(x \cdot z),$$

so $x \cdot yRx \cdot z$. The proof that $y \cdot xRz \cdot x$ is similar, using (c7).

• If $xRy \in 0^L$, then $x \subseteq y$ and $y \subseteq 0$. Hence $?y \subseteq 0 \subseteq 0$ (c4), implying $x \subseteq 0$ and hence $x \in 0^L$.

• Let $x \in 0^L$. To show $\exists y : xRy \in \emptyset$, put $y = \bigcup \emptyset$. Then indeed $y \in \emptyset$, and since $x \subseteq 0$ and $0 \subseteq ?y$ (c5), we have $xRy$ as required.

That completes the proof that $\mathcal{S}^L$ is a modal FL-cover system. To prove (6.1), we first show

$$!x = \bigcup \uparrow(\uparrow x \cap I).$$

(6.3)

For, since $!x \subseteq x$ (s1) we have $!x \in \uparrow x \cap I$, so $!x \subseteq \bigcup \uparrow(\uparrow x \cap I)$. But conversely, for any $y \in \uparrow(\uparrow x \cap I)$, there is some $z \in \uparrow x \cap I$ such that $y \geq !z$, hence $y \subseteq !z \subseteq x$. Then $y \subseteq x$ and $x \subseteq \uparrow x$. Thus $!x$ is an upper bound of $\uparrow(\uparrow x \cap I)$, implying $\bigcup \uparrow(\uparrow x \cap I) \subseteq !x$ and proving (5.3). Now for any $y \in S$ we reason that $y \in \uparrow(\uparrow x \cap I)$ if there is a $C$ with $y \triangleleft C \subseteq (\uparrow x) \cap I)$. If there is a $C \subseteq (\uparrow x) \cap I$ with $y \subseteq \bigcup C$ if $y \subseteq \bigcup \uparrow(\uparrow x \cap I)$ if $y \subseteq !x$ by (6.3) if $y \subseteq !x$.

That proves (6.1). For (6.2), if $y \in \uparrow(?x)$, then $y \subseteq ?x$ and so $yRx \in \uparrow x$, showing $y \in (R)\uparrow x$. Conversely, if $yRx \subseteq x$ for some $z$, then $y \subseteq ?z \subseteq ?x$, implying $y \in \uparrow(?x)$. ❑

Results (6.1) and (6.2) state that the map $x \mapsto \uparrow x$ preserves the modalities of the modal FL-algebras $L$ and $\text{Prop}(\mathcal{S}^*)$. Thus we have altogether established

Theorem 6.2. Any order-complete modal FL-algebra is isomorphic to the modal FL-algebra of all propositions of some modal FL-cover system that is strong, i.e. satisfies $X \otimes Y = \uparrow(X \cdot Y)$. ❑

Combining this with Ono’s result, mentioned at the end of Section 2 on completions of modal FL-algebras, we have

Theorem 6.3. Any modal FL-algebra is isomorphically embeddable into the modal FL-algebra of propositions of some strong modal FL-cover system, by an embedding that preserves all existing joins and meets. ❑

7 Kripke-Type Semantics

Ono (1993) defined a modal substructural propositional logic that is sound and complete for validity in models on modal FL-algebras, as well as a first-order extension of this logic that is characterised by suitable models on modal FL-algebras. In view of the representation theorems just obtained, these models can be taken to based on algebras of the form $\text{Prop}(S)$, where $S$ is a
strong modal FL-cover system. In any such model $\mathcal{M}$, each sentence $\varphi$ is interpreted as a proposition $|\varphi|^\mathcal{M} \in Prop(\mathcal{S})$, with propositional constants and connectives interpreted by the operations of Theorem 4.1; quantifiers interpreted using the join and meet operations $\sqcap$ and $\sqcup$ of that Theorem; and modalities interpreted by $!X = j \uparrow (X \cap I)$ and $?X = (R)X$. Writing $\mathcal{M}, x \models \varphi$ to mean that $x \in |\varphi|^\mathcal{M}$, and unravelling the definitions of the operations on $Prop(\mathcal{S})$, results in a Kripke-style satisfaction relation between formulas and points in models on cover systems. We now briefly present the inductive clauses specifying such a satisfaction relation in models for a certain type of first-order language.

Let $\mathcal{L}$ be a signature, comprising a collection of individual constants $c$, and predicate symbols $P$ with specified arities $n < \omega$. The $\mathcal{L}$-terms are the individual constants $c \in \mathcal{L}$ and the individual variables $v$ from some fixed denumerable list of such variables. An atomic $\mathcal{L}$-formula is any expression $P\tau_1 \ldots \tau_n$, where $P \in \mathcal{L}$ is $n$-ary, and the $\tau_i$ are $\mathcal{L}$-terms. The set of all $\mathcal{L}$-formulas is generated from the atomic $\mathcal{L}$-formulas and constant formulas $T, F, 1, 0$, using the propositional connectives $\land, \lor, \to$ and $\leftrightarrow$ (interpreted as $\land, \lor, \to_1$ and $\to_r$); the quantifiers $\forall v, \exists v$ for all variables $v$, and the modalities $!$ and $?$. An $\mathcal{L}$-model $\mathcal{M} = (\mathcal{S}, U, | - |^\mathcal{M})$ has $\mathcal{S} = (\mathcal{S}, \leq, \prec, \cdot, \cdot, 0^\mathcal{S}, I, R)$ a modal FL-cover system, $U$ a non-empty set (universe of individuals), and $| - |^\mathcal{M}$ an interpretation function assigning:

1. to each individual constant $c \in \mathcal{L}$ an element $|c|^\mathcal{M} \in U$; and
2. to each n-ary predicate symbol $P \in \mathcal{L}$, a function $|P|^\mathcal{M} : U^n \to Prop(\mathcal{S})$.

Intuitively, $|P|^\mathcal{M}(u_1, \ldots, u_n)$ is the proposition asserting that the predicate $P$ holds of the $n$-tuple of individuals $(u_1, \ldots, u_n)$. Let $\mathcal{L}^U$ be the extension of $\mathcal{L}$ to include the members of $U$ as individual constants. $\mathcal{M}$ automatically extends to an $\mathcal{L}^U$-model by putting $|c|^\mathcal{M} = c$ for all $c \in U$. $\mathcal{M}$ has a truth/satisfaction relation $\mathcal{M}, x \models \psi$ between elements $x \in \mathcal{S}$ and sentences $\psi$ of $\mathcal{L}^U$, with associated truth-sets $|\psi|^\mathcal{M} = \{ x \in \mathcal{S} : \mathcal{M}, x \models \psi \}$. These notions are defined by induction on the length of $\varphi$, as follows.

$$
\begin{align*}
\mathcal{M}, x \models c_1 \ldots c_n & \iff x \in |P|^\mathcal{M}(c_1, \ldots, c_n) \\
\mathcal{M}, x \models T & \iff x \not\in \emptyset \\
\mathcal{M}, x \models F & \iff x \not\in x \\
\mathcal{M}, x \models 0 & \iff x \in 0^\mathcal{S} \\
\mathcal{M}, x \models \varphi \land \psi & \iff \mathcal{M}, x \models \varphi \text{ and } \mathcal{M}, x \models \psi \\
\mathcal{M}, x \models \varphi \lor \psi & \iff \text{there is an } x\text{-cover } C \subseteq |\varphi|^\mathcal{M} \cup |\psi|^\mathcal{M} \\
\mathcal{M}, x \models \varphi \to_1 \psi & \iff \mathcal{M}, y \models \varphi \text{ implies } \mathcal{M}, x \cdot y \models \psi \\
\mathcal{M}, x \models \varphi \to_r \psi & \iff \mathcal{M}, y \models \varphi \text{ implies } \mathcal{M}, y \cdot x \models \psi \\
\mathcal{M}, x \models \forall \psi & \iff \text{for all } c \in U, \mathcal{M}, x \models \varphi(c/v) \\
\mathcal{M}, x \models \exists \psi & \iff \text{there is an } x\text{-cover } C \subseteq \bigcup_{c \in U} \{ |\varphi(c/v)|^\mathcal{M} \} \\
\mathcal{M}, x \models ! \varphi & \iff \text{there is an } x\text{-cover } C \subseteq \bigcup_{c \not\in U} \{ |\varphi|^\mathcal{M} \cap I \} \\
\mathcal{M}, x \models ? \varphi & \iff \text{for some } y, xRy \text{ and } \mathcal{M}, y \models \varphi.
\end{align*}
$$

A sentence $\varphi$ is true in model $\mathcal{M}$ if it is true at every point, i.e., if $\mathcal{M}, x \models \varphi$ for all $x \in \mathcal{S}$, or equivalently $|\varphi|^\mathcal{M} = S$. A formula $\varphi$ with free variables is true in $\mathcal{M}$ if every $\mathcal{L}^U$-sentence $\varphi(c_1/v_1, \ldots, c_n/v_n)$ that is a substitution instance of $\varphi$ is true in $\mathcal{M}$.

Detailed discussion of this kind of cover system semantics, and associated completeness theorems axiomatising their valid sentences, are presented in (Goldblatt 2006) for the logic of non-modal FL-algebras; in (Goldblatt 2011a) for intuitionistic modal first-order logics; in (Goldblatt 2011c, Chapter 6) for propositional and quantified relevant logics; and in (Goldblatt 2011b) for a ‘classical’ version of bilinear logic that we also discuss below in Section 9.
8 Negation and Orthogonality

Any FL-algebra has two unary ‘negation-like’ operations, \( \neg_l \) and \( \neg_r \), defined by putting \( \neg_l a = a \Rightarrow_l 0 \) and \( \neg_r a = a \Rightarrow_r 0 \). In the algebra \( \text{Prop}(S) \) of a residuated cover system with a distinguished proposition \( 0 \), these operations can be analysed by the ‘orthogonality’ relation \( \bot \) on \( S \) defined by

\[
z \bot y \iff z \cdot y \in 0. \tag{8.1}
\]

Writing \( z \bot X \) when \( z \bot y \) for all \( y \in X \), and \( X \bot z \) when \( y \bot z \) for all \( y \in X \), we get that \( z \bot X \) iff \( z \cdot X \subseteq 0 \) and \( X \bot z \) iff \( X \cdot z \subseteq 0 \), so

\[
\neg_l X = \{ z \in S : z \bot X \} \quad \text{and} \quad \neg_r X = \{ z \in S : X \bot z \}. \tag{8.2}
\]

The operations \( \neg_l \) and \( \neg_r \) interpret left and right negation connectives, defined by taking \( \neg_0 \varphi \) to be \( \varphi \rightarrow_l 0 \) and \( \neg_0 \varphi \) to be \( \varphi \rightarrow_r 0 \). These have the semantics

\[
\mathcal{M}, x \models \neg_0 \varphi \iff x \bot \varphi^\mathcal{M}
\]

\[
\mathcal{M}, x \models \neg_0 \varphi \iff \varphi^\mathcal{M} \bot x. \tag{8.3}
\]

When \( \cdot \) is commutative, the relation \( \bot \) is symmetric and \( \neg_l \) and \( \neg_r \) are identical. But even in the absence of symmetry we do have \( z \bot \varepsilon \) iff \( \varepsilon \bot z \) iff \( z \in 0 \). So \( 0 \) itself is recoverable from \( \bot \) as the set \( \{ z : z \bot \varepsilon \} = \{ z : \varepsilon \bot z \} \).

Now if \( y \in \varepsilon \), then in general \( z \cdot \varepsilon \not\subseteq z \cdot y \), so \( z \bot \varepsilon \) implies \( z \bot y \) as \( 0 \) is an up-set. This shows that \( z \bot \varepsilon \) iff \( z \bot \varepsilon \) iff \( \varepsilon \in \neg_l \varepsilon \). Similarly, \( \varepsilon \bot z \) iff \( \varepsilon \bot z \) iff \( z \in \neg_r \varepsilon \). Thus in \( \text{Prop}(S) \) we have \( 0 = \neg_l 1 = \neg_r 1 \). When \( S \) is a modal FL-cover system, we also have

\[
0 = \neg_l 1 = \neg_r 1. \tag{8.4}
\]

To see why, note that since \( I \subseteq \uparrow \varepsilon \), we have \( 0 = \neg_l \varepsilon \subseteq \neg_l I \). But if \( z \in \neg_l I \), then \( z \bot I \), so \( z \bot \varepsilon \) as \( \varepsilon \in I \), hence \( z \in 0 \). This shows that \( 0 = \neg_l 1 \). The proof that \( 0 = \neg_r 1 \) is similar.

The relation \( \bot \) defined in (8.1) has the following properties:

- \( z \bot y \) iff \( z \cdot y \bot \varepsilon \).
- **Orthogonality to \( \varepsilon \) is monotonic**: \( y \triangleright z \bot \varepsilon \) implies \( y \bot \varepsilon \).
- **Orthogonality to \( \varepsilon \) is local**: \( x \triangleleft C \bot \varepsilon \) implies \( x \bot \varepsilon \).

Vice versa, if we begin with a residuated cover system \( S \) having a binary relation \( \bot \) with these properties, then it follows that \( \{ z : z \bot \varepsilon \} = \{ z : \varepsilon \bot z \} \), and that this set belongs to \( \text{Prop}(S) \). So we can take it as the definition of \( 0 \). Then the sets \( \neg_l X \) and \( \neg_r X \) defined from \( \bot \) as in (8.2) turn out to be \( X \Rightarrow_l 0 \) and \( X \Rightarrow_r 0 \) for this choice of \( 0 \), respectively.

The modelling of negation by an orthogonality relation as in (8.3) first occurred in [Goldblatt 1974], with \( \bot \) symmetric. The idea of defining \( \bot \) from a distinguished subset of a monoid as in (8.1) is due to [Girard 1987].

We will make use of some basic properties of \( \neg_l \) and \( \neg_r \) in any FL-algebra (see e.g., [Galatos et al. 2007, Section 2.2]):

- \( a \subseteq \neg_l a \) and \( a \subseteq \neg_r a \).
- \( a \subseteq b \) implies \( \neg_l b \subseteq \neg_l a \) (antitonicity).
- \( \neg_l 1 = 0 = \neg_r 1 \).
- \( 1 \subseteq \neg_l 0 \cdot \neg_r 0 \).
- \( a \Rightarrow b \subseteq \neg_r b \Rightarrow_l \neg_r a \).
- \( a \Rightarrow b \subseteq \neg_l b \Rightarrow_r \neg_l a \).
For instance, the last ‘contrapositive’ inequality is the case \( c = 0 \) of
\[
\alpha \Rightarrow_I b \sqsubseteq (b \Rightarrow_I c) \Rightarrow_r (a \Rightarrow_I c),
\] (8.5)
which is itself shown by using residuation to reason that
\[
(b \Rightarrow_I c) \otimes (a \Rightarrow_I b) \otimes a \subseteq (b \Rightarrow_I c) \otimes b \subseteq c,
\]
implying that \((b \Rightarrow_I c) \otimes (a \Rightarrow_I b) \subseteq (a \Rightarrow_I c)\), from which (8.5) follows.

Now in Boolean modal algebra, a modality \(!\) has the dual modality \(\sim\), where \(\sim\) is the Boolean complement/negation operation. Given the two negations \(\sim_I\) and \(\sim_r\), we would seem to have four possibilities here for defining a term function to which \(\sim\) is dual. But it turns out that they are all the same:

**Lemma 8.1.** Any FL-algebra with a storage modality \(!\) satisfies \(-_I a = -_r a\) and \(-_r a = !\sim_I a\), for all \(a\). Hence
\[
\sim_I a = \sim_r a = -_I a = -_r a = \sim_I a.
\]

**Proof.** By (s5) and then residuation, \((!a \Rightarrow_I 0) \otimes !a = !a \otimes (!a \Rightarrow_I 0) \subseteq 0\), implying that \((!a \Rightarrow_I 0) \subseteq (!a \Rightarrow_I 0)\). Similarly \(!a \otimes (!a \Rightarrow_I 0) = (!a \Rightarrow_I 0) \otimes !a \subseteq 0\), implying \((!a \Rightarrow_I 0) = (!a \Rightarrow_I 0)\). Hence \((!a \Rightarrow_I 0) = (!a \Rightarrow_I 0)\), i.e. \(-_I a = \sim_I a\).

Next, by (s5) and (s1), \(a \otimes (!a \Rightarrow_I 0) = (!a \Rightarrow_I 0) \otimes a \subseteq (a \Rightarrow_I 0) \otimes a \subseteq 0\), so \((a \Rightarrow_I 0) \subseteq a \Rightarrow_I 0\). Hence by \(!\)-monotonicity and (s2), \((!a \Rightarrow_I 0) \subseteq !(a \Rightarrow_I 0)\). The reverse inequality holds similarly, so \((a \Rightarrow_I 0) = (a \Rightarrow_I 0)\), i.e. \(-_r a = \sim_r a\).

The second statement of the Lemma follows from the first.

Thus we can define an operation \(?\) by writing \(?a\) for the element \(-_I \sim_I a\), or any of its three other manifestations as given by this last result.

**Theorem 8.2.** If \(!\) is a storage modality on an FL-algebra \(L\), then the operation \(?\) satisfies the axioms (c1)–(c5) and so, together with \(!\), makes \(L\) into a modal FL-algebra.

**Proof.**

(c1) From the contrapositive inequality \(a \Rightarrow_I b \sqsubseteq -_I b \Rightarrow_r \sim_I a\), by \(!\)-monotonicity and Lemma \(\square\) we get \((!a \Rightarrow_I b) \subseteq -_I b \Rightarrow_r \sim_I a\). But \(-_I b \Rightarrow_r \sim_I a \subseteq -_I \sim_I a \Rightarrow_I -_I \sim_I a \Rightarrow_I \sim_I a\), which says \(\sim_I a \subseteq \sim_I a\).

Similarly we show that \((!a \Rightarrow_I b) \subseteq \sim_I a \Rightarrow \sim_I a \Rightarrow \sim_I a \Rightarrow \sim_I a\).

(c2) Since by (s1) \((a \Rightarrow_I 0) \subseteq a \Rightarrow_r 0\), residuation gives \( a \subseteq !(a \Rightarrow_I 0) \Rightarrow_I 0 = -_I -_r a = -_I -_r a = ?a\).

(c3) We have \(! -_r a \subseteq -_r -_I a\), as an instance of \(b \subseteq -_r -_I b\). Hence by \(!\)-monotonicity and (s2), \(-_r a \subseteq -_I -_r -_I a\). This together with antitonicity gives \(-_I -_r -_r a \subseteq -_r -_r a\), which says \(?a \subseteq ?a\).

(c4) \(1 = !1 \subseteq !-r 0\), hence \(-_I -_r 0 \subseteq -_I 1 = 0\). This says \(?0 \subseteq 0\).

(c5) By Lemma \(\square\), we have \(-_r a \subseteq 1\). Hence \(0 = -_I 1 \subseteq -_I -_r a = ?a\).

\[\square\]

### 9 Classical/Grishin Algebras

Ond \(1993\) defined an FL-algebra to be classical if it satisfies the equations
\[
(a \Rightarrow_r 0) \Rightarrow_I 0 = a = (a \Rightarrow_I 0) \Rightarrow_r 0.
\]
This can be written as $\neg l \neg r a = a = \neg r \neg l a$, and will be called the law of double-negation elimination. Girard’s linear logic is modelled by classical FL-algebras in which the fusion operation $\otimes$ is commutative.

Lambek (1995) defined a Grishin algebra to be a lattice-ordered pomonoid that has two unary operations $\neg l$ and $\neg r$, and a distinguished element 0 that satisfies double-negation elimination and the conditions

$$a \sqsubseteq b \iff a \otimes \neg r b \sqsubseteq 0 \quad \text{iff} \quad \neg l b \otimes a \sqsubseteq 0.$$  

He described such algebras as being “a generalisation of Boolean algebras which do not obey Gentzen’s three structural rules”. His motivation was to study algebraic models for classical bilinear propositional logic, described as “a non-commutative version of linear logic which allows two negations”. Such models were first considered by Grishin (1983).

Lambek showed that a Grishin algebra can be equivalently defined as a residuated lattice with two operations $\neg l$ and $\neg r$ satisfying double-negation elimination and

$$\neg l 1 = \neg r 1, \quad a \Rightarrow b = \neg l (a \otimes \neg r b), \quad a \Rightarrow r b = \neg r (\neg l b \otimes a).$$  

A proof that the notions of classical FL-algebra and Grishin algebra are equivalent is given in Goldblatt (2011b, Theorem 2.2).

A residuated cover system $\mathcal{S}$ will be called classical if it has a distinguished proposition (localised up-set) 0 such that the least proposition containing any given $X$ is equal to both $\neg l \neg r X$ and $\neg r \neg l X$. In other words,

$$j \uplus X = \neg l \neg r X = \neg r \neg l X \tag{9.1}$$  

holds for all $X \subseteq \mathcal{S}$, where $\neg l$ and $\neg r$ are defined from $\Rightarrow l$ and $\Rightarrow r$ using 0. This is equivalent to requiring that $\text{Prop}(\mathcal{S})$ be a Grishin algebra/classical FL-algebra, and is also equivalent to the requirement that (9.1) holds just for all up-sets $X$ (Goldblatt 2011b, Theorem 4.2).

We showed in Goldblatt (2011b) that every Grishin algebra has an isomorphic embedding into the algebra of all propositions of some strong classical residuated cover system, by a map that preserves all existing joins and meets. The method, involving MacNeille completion, can be combined with the constructions of this paper to give a representation of any classical modal FL-algebra as an algebra of propositions of some strong classical modal FL-cover system.

In conclusion we relate our constructions back to the modelling of consumption modalities in Girard (1995), which is based on the notion of a phase space as a commutative monoid with a distinguished subset (but without a preorder). Suppose $\mathcal{S}$ is a classical modal FL-cover system in which is commutative. Then the relation $\perp$ defined in (8.1) is symmetric, and so the sets $\neg l X$ and $\neg r X$ in (8.2) are one and the same. We denote this set by $X^\perp$. The operation $X \mapsto X^\perp$ is a closure operator on the powerset of $\mathcal{S}$ that has $X^{\perp \perp} = X^\perp$. Moreover, $j \uplus X = X^{\perp \perp}$ according to (9.1). Now the modality $? X$ on $\text{Prop}(\mathcal{S})$ is given by

$$? X = (\uplus X)^\perp = (j \uplus (X^\perp \cap I))^\perp = (X^\perp \cap I)^{\perp \perp \perp} = (X^\perp \cap I)^\perp.$$  

$(X^\perp \cap I)^\perp$ is Girard’s definition of $? X$ when $I$ is the set of idempotents belonging to 1 = $\{\varepsilon\}$.

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