Precise nondivergent analytic formulas for the radiative corrections to the $\beta$ energy spectrum in hyperon semileptonic decays over the entire Dalitz plot

S. R. Juárez W. and F. Guzmán A.
Escuela Superior de Física y Matemáticas. Instituto Politécnico Nacional. Edificio 9, Unidad Prof. Adolfo López Mateos, Col. Lindavista, C.P. 07738 México D.F., Mexico

Very accurate analytical expressions for the radiative corrections of unpolarized hyperons semileptonic decays of charged and neutral baryons have been obtained in the recent past. Some of these formulas contain logarithmic singularities at the edges of the Dalitz plot for the three and four-body decays. These singularities are analyzed and integrated analytically to obtain new divergentless formulas for the energy spectrum of the produced $\beta$ particle. The new equations contain terms of the order $\alpha$ times the momentum transfer, are applicable to any beta decay process and are suitable for a model-independent experimental analysis.

13.40.Ks, 13.30.Ce

I. INTRODUCTION

The subject of hyperon decays is of unquestionable importance in high energy physics. The precise description of the semileptonic weak decays and in particular the neutron decay, is relevant in cosmology, astrophysics, solar physics, the solar neutrino problem, and in other areas of particle physics. A precise formula for the energy spectrum of the decay products (fermions) in the hyperon semileptonic decay (HSD) requires the knowledge of the radiative corrections (RC) in the whole region of the Dalitz plot (DP). The total decay rate can be computed directly through analytical formulas for the RC, if these formulas do not contain any divergences. The subject of this paper is to show how the logarithmic kinematical divergences, that appeared in the previously calculated model-independent bidimensional distributions in Refs. [1]-[3], obtained in an analytical way for the RC to baryon $\beta$ decays in the three body region (TBR) and the four body region (FBR) of the DP, are canceled after performing the integration over the energy of the final hyperon that emerges in the process. The new results are valid in any charged or neutral hyperon beta decay. The important feature of the obtained analytical forms of the RC’s is that they are written in a simple form as products of two factors. One of these factors is a model-independent function and the other one does depend on the form factors whose determination (through the experimental data) is useful to obtain information about the underlying interactions in the decay processes, basic symmetries and the internal structure of hadrons. The new formulas are suitable for a direct evaluation of the RC’s for any event at any point of the allowed physical region.

The structure of this paper is as follows. In Sec. II we exhibit the kinematical region in which the three and the four body decays take place. All the amplitudes, the one without any RC, the bremsstrahlung and the virtual ones, for the charged and neutral HSD, with all the $\alpha q/\pi M_1$ terms included, are displayed in this section. The bidimensional observable energy distributions in the FBR and TBR are described for the charged and neutral HSD processes in Sec. III. We devote Sec. IV to display the formulas for the energy spectrum of the emitted charged lepton in a non-divergent form for the pure bremsstrahlung correction in the FBR, and the formulas for the energy spectrum in the TBR with both kinds of RC’s (the combination of the bremsstrahlung and the virtual contributions). In Sec. V we present our final formulas in each region for the charged and the neutral decay processes. Conclusions are given in Sec. VI, with the formulas that contain all the up-to-date improvements and new features for the energy spectrum, together with a comparison with other previously published numerical values assuming their values for the form factors. Several graphs for the partial radiative corrections (the Coulomb effect is included for the neutral HSD process at the TBR) and for the complete electron spectrum are presented in order to illustrate our results. Finally, for self-containment, we include several appendices with the definitions of the coefficients and model-independent functions that appear in the final analytical results, among other relevant relations. The procedure of integration with the explicit cancellations of the kinematical divergences is also included.

II. KINEMATICS AND AMPLITUDES

We start with the presentation of the main features and the notation to describe the decay processes we are interested in:

$$A^s(p_1) \rightarrow B(p_2) + e^-(\ell) + \bar{\nu}_e(p_{\nu})$$

1
where the emission of a virtual or a real photon $\gamma(k)$ takes place. The emission of the photon is described as a radiative correction to the semileptonic decay of the hyperon. $A^*$ corresponds to the neutral ($s = n$) or charged ($s = c$) decaying baryon, $B$ is the produced baryon, $e^-$ and $\tilde{\nu}_e$ denote the lepton and its antineutrino counterpart, respectively. The four-momenta and masses of the particles involved in the baryon semileptonic decay are denoted by

$$p_1 = (E_1, \vec{p}_1), \quad p_2 = (E_2, \vec{p}_2), \quad \ell = (E, \vec{\ell}), \quad p_\nu = (E_\nu, \vec{p}_\nu), \quad \text{and} \quad k = (k_0, \vec{k}),$$

and by $M_1, M_2, m, m_\nu,$ and $m_k$ respectively. We assume throughout this paper that $m_\nu = 0,$ and $m_k = 0$ for the real photons.

The conservation of energy and momentum determines the physical region where the HSD takes place. For the three body decay as in the HSD (without any radiative correction) the physical region is bounded by the hyperon minimal and a maximal energy $(E_2^{\min}, E_2^{\max})$ for each value of $E$. When a real photon is considered as an additional product of the decay the physical four body region for this process contains the former region (TBR) and an additional portion (FBR) with energies below the $E_2^{\min}$ of the physical four body region for this process contains the former region (TBR) and an additional portion (FBR) with energies below the $E_2^{\max}$.

$$M_2 \leq E_2 \leq E_2^{\min}, \quad m \leq E \leq E_c, \quad E_c = \frac{(M_1 - M_2)^2 + m^2}{2 (M_1 - M_2)},$$

and at the TBR by

$$E_2^{\min} \leq E_2 \leq E_2^{\max}, \quad m \leq E \leq E_m, \quad E_m = \frac{M_1^2 - M_2^2 + m^2}{2 M_1},$$

where

$$E_2^{\max} = \frac{1}{2} (M_1 - E \pm \beta E) + \frac{M_2}{2 (M_1 - E \pm \beta E)}, \quad \text{and} \quad \ell = \beta E.$$

The $z$-axis is chosen along the electron three-momentum and the $x$-axis oriented so that the final baryon three-momentum is in the first or fourth quadrants of the $x-z$ plane. To be more explicit,

$$p_1 = (M_1, 0), \quad \vec{\ell} = \beta E \, (0, 0, 1), \quad \vec{p}_2 = |\vec{p}_2| \left( \sqrt{1 - y^2}, \, 0, \, y \right).$$

In the TBR, the radiative correction is due to a virtual or a real photon emission, meanwhile in the FBR the emitted photon is a real photon exclusively, i.e., the radiative correction is due only to the bremsstrahlung effect in the semileptonic decay.

The uncorrected matrix element $M_0$ (without any emission of a photon at the TBR) for the decay in Eq. (1) is given by the product of the matrix elements of the baryonic weak current and of the leptonic current:

$$M_0 = \frac{G_v}{\sqrt{2}} \bar{u}_B W_\mu \bar{u}_A \bar{\nu}_e O_\mu v_\nu,$$

where $G_v = G_\mu V_{ij}$ and $G_\mu$ is the muon decay coupling constant, $V_{ij}$ is the corresponding Cabibbo-Kobayashi-Maskawa matrix element, and

$$W_\mu = f_1(q^2) \gamma_\mu + \frac{f_2(q^2)}{M_1} \sigma_{\mu\nu} q_\nu + \frac{f_3(q^2)}{M_1} q_\mu + \frac{g_1(q^2)}{M_1} \sigma_{\mu\nu} q_\nu + \frac{g_2(q^2)}{M_1} q_\mu,$$

and $O_\mu = \gamma_\mu (1 + \gamma_5).$ (7)

The $q = p_1 - p_2$ denotes the four-momentum transfer.

To obtain the precise decay rates, one has to consider all the involved amplitudes of the processes with and without RC. It has been shown in Ref. [4] that the order $\alpha$ amplitude can be obtained in a model independent fashion by using the Low-theorem, Refs. [5], [6].

For completeness, we reproduce here the model independent bremsstrahlung amplitudes, in terms of the Dirac form factors, given for the charged HSD in Eqs. (18)-(20) of Ref. [7], and for the neutral HSD in Eq. (23) of Ref. [8].

2
The total \textit{bremsstrahlung} transition amplitudes for the charged HSD \((s = c)\) and the neutral HSD \((s = n)\) decays in the entire Dalitz plot region are

\[
M_B^s = M_1^s + M_2^s + M_3^s,
\]
with

\[
M_1^c = e M_0 \left( \frac{\epsilon \cdot l}{l \cdot k} - \frac{\epsilon \cdot p_1}{p_1 \cdot k} \right), \quad M_2^c = e G \sqrt{2} \epsilon \mu_B W A \bar{u}_B \gamma \frac{l}{k} O_{\lambda} v_{\nu},
\]

\[
M_3^c = \frac{G}{\sqrt{2}} \bar{u}_B O_{\lambda} v_{\nu} \epsilon \mu_B \left[ \frac{e W \gamma k_{\mu}}{2 p_1 \cdot k} - \kappa_1 W \gamma \frac{\gamma + M_1}{2 p_1 \cdot k} \sigma_{\mu \nu} k_{\nu} + \kappa_2 \sigma_{\mu \nu} k_{\nu} \frac{2 p_2}{2 p_2 \cdot k} \right] \omega_{\lambda} + e \left( \frac{p_1 k_{\mu}}{p_1 \cdot k} - g_{\mu \nu} \right) \left( \frac{2 + g_{\gamma \gamma}}{M_1} \right) u_A,
\]

and

\[
M_1^n = e M_0 \left( \frac{\epsilon \cdot l}{l \cdot k} - \frac{\epsilon \cdot p_2}{p_2 \cdot k} \right), \quad M_2^n = M_2^c,
\]

\[
M_3^n = \frac{G}{\sqrt{2}} \bar{u}_B O_{\lambda} v_{\nu} \epsilon \mu_B \left[ \frac{e W \gamma k_{\mu}}{2 p_2 \cdot k} - \kappa_1 W \gamma \frac{\gamma + M_1}{2 p_1 \cdot k} \sigma_{\mu \nu} k_{\nu} + \kappa_2 \sigma_{\mu \nu} k_{\nu} \frac{2 p_2}{2 p_2 \cdot k} \right] \omega_{\lambda} + e \left( \frac{p_2 k_{\mu}}{p_2 \cdot k} - g_{\mu \nu} \right) \left( \frac{2 + g_{\gamma \gamma}}{M_1} \right) u_A.
\]

\(\kappa_1\) and \(\kappa_2\) are the anomalous magnetic moments of \(A^s\) and \(B\), respectively. \(\epsilon_\mu\) is the photon polarization four-vector.

The total \textit{virtual} transition amplitudes (present only at the TBR) with terms up to order of \(\alpha q/\pi M_1\) can be written as

\[
M_v^s = M_0^s + M_v^s,
\]
where \(M_0^s\) is the uncorrected matrix element in terms of effective form factors Ref. \[6\].

Explicitly, for the charged HSD case

\[
M_v^c = \frac{\alpha}{2 \pi} \left[ M_0 \Phi_c + M_{p_1} \Phi_c' \right],
\]
where

\[
\Phi_c = 2 \left[ \frac{1}{\beta} \arctan \left( \frac{\beta}{1 - \beta} \right) - 1 \right] \ln \left[ \frac{\lambda}{m} \right] - \frac{1}{\beta} \left( \arctan \left( \frac{\beta}{1 - \beta} \right) \right)^2 + \frac{1}{\beta} L \left[ \frac{2 \beta}{1 + \beta} \right] - \frac{1}{\beta} L \left[ \frac{2 \beta}{M_1 / E - 1 + \beta} \right] + \frac{3}{2} \ln \left[ \frac{M_1}{m} \right]
\]

and

\[
\Phi_c' = \frac{1 - \beta^2}{\beta} \left[ -\arctan \left( \beta \right) \left[ 1 + \frac{2 \beta}{M_1 / E - 1 + \beta} \right] + \frac{\beta E}{M_1 / E - 2 E} \ln \left[ \frac{M_1}{m} \right] \right],
\]

with

\[
L(x) = \int_0^x \frac{\ln \left( 1 - t \right)}{t} dt \quad \text{is the Spence function,} \quad \text{and} \quad M_{p_1} = \frac{G}{\sqrt{2} m M_1} W^{\mu} u_A \bar{u}_1 O_{\mu} v_{\nu}.
\]

The formulas for the neutral HSD are:

\[
M_v^n = \frac{\alpha}{2 \pi} \left[ M_0 \Phi_n + M_{p_2} \Phi_n' \right], \quad \text{where} \quad M_{p_2} = \frac{G}{\sqrt{2} m (l + p_2)^2} \bar{u}_B W^{\mu} u_A \bar{u}_1 O_{\mu} v_{\nu}.
\]

The \(\Phi_n\) and \(\Phi_n'\) are given in Ref. \[8\], Eqs. (8)-(10), and can be written as
\[ \Phi_n = \Phi_n^{TR}(E, E_2, \lambda) + \Phi_n^{ND}(E, E_2) + \Phi_{\text{Coulomb}}(E, E_2) \]

where

\[ \Phi_n^{TR}(E, E_2, \lambda) = 2 \frac{1}{\beta_N} \arctanh(\beta_N) - 1 \right) \ln \left| \frac{\lambda}{m} \right|, \quad \Phi_{\text{Coulomb}}(E, E_2) = \frac{\pi^2}{\beta_N} \]

\[ \Phi_n^{ND}(E, E_2) = -\frac{1}{\beta_N} \left[ \arctanh(\beta_N) \right]^2 + \frac{1}{\beta_N} \left[ \frac{\Delta V}{x_2^+} \right] - \frac{1}{\beta_N} \left[ \frac{-\Delta V}{1 - x_2^+} \right] + \frac{1}{\beta_N} \arctanh(\beta_N) \left[ \frac{M_2^2 + (1 + \beta_N^2) a}{2 (l + p_2)^2} \right] \]

\[ + \frac{3}{2} \ln \left[ \frac{M_2}{m} \right] - \frac{11}{8} \frac{m^2}{(l + p_2)^2} \ln \left[ \frac{M_2}{m} \right] - \frac{1}{\beta_N} \ln \left[ 1 + \frac{\Delta V}{1 - x_2^+} \right] \left[ \ln \left[ \frac{M_2}{m} \right] - \arctanh(\beta_N) \right], \]

\[ \Phi_n = \frac{1}{\beta_N} \left( 1 - \beta_N^2 \right) \left[ - \left( 1 + \frac{a}{2M_2^2} \right) \arctanh(\beta_N) - \frac{a}{2M_2^2} \beta_N \ln \left[ \frac{M_2}{m} \right] \right], \]

where

\[ \beta_N = \left( 1 - \frac{4m^2M_2^2}{a^2} \right)^{1/2}, \quad a = 2E (E_2 - \beta |\vec{p}_2| y_0) = 2l.p_2, \]

and

\[ \Delta V = x_2^+ - x_2^-, \quad x_2^\pm = \frac{a\beta_N}{(l + p_2)^2}, \quad x_2^\pm = \frac{m^2 + \frac{1}{2} (1 \pm \beta_N) a}{(l + p_2)^2}. \]

The contribution due to the Coulomb interaction in the neutral HSD is included in Eq. (17). The infrared divergence which takes place in Eqs. (19) and (20) when \( \lambda \rightarrow 0 \) in the virtual radiative correction is canceled by another one of the same kind which appears in the bremsstrahlung correction, see Ref. [8]. \( \lambda \) is the small mass which is assigned to the photon to cut off the contribution of very soft photons.

In general, the \( \Phi_s(E) \) and \( \Phi'_s(E) \) are model independent functions (do not depend on the form factors) that contain terms of order \( aq/\pi M_1 \).

### III. BIDIMENSIONAL DISTRIBUTIONS

The evaluation of the differential decay rate that gives the DP with RC of process in Eq. (1) is performed by standard trace calculations, leaving as the relevant independent variables the energies \( E_2 \) and \( E \) of the emitted baryon and the electron, respectively. The phase space integration is accomplished according to the kinematical limits given in Eqs. (3) and (4), for the TBR and FBR respectively.

#### A. Four body region

In this region, the result up to order \( aq/\pi M_1 \) for the HSD is compactly given by products of coefficients \( H_i^{\prime \prime} \) that depend on the form factors and model independent functions \( \theta_i^{TT} \) as

\[ d\Gamma_B^{\prime \prime}(A' \rightarrow Be\nu\gamma) dE = \frac{\alpha}{\pi} d\Omega \sum_{i=0}^{17} H_i^{\prime \prime} \theta_i^{TT}, \quad d\Omega = \frac{G_F^2}{2} \frac{1}{2\pi^3} M_1 dE_2 dE, \]

where the coefficients \( H_i^{\prime \prime} \) (see Appendix A) are given by

\[ H_1 = H_0 = H_1, \quad H_i = H_1, \quad H_i = H_i^{\prime} + N_i^{\prime}, \quad \text{for} \quad i = 0, 2, ..., 17, \quad H_1^{\prime} = 0. \]

The \( H_i^{\prime} \)'s do depend on the form factors through functions \( Q_i, \ i = 1, .., 4 \). The explicit form of the \( Q_i \)'s (Ref. [10]) is also given in Appendix A. The model independent functions are:
\[ \theta^T_i = \theta^{0T}_i = \theta^T_i, \quad \text{for} \quad i = 0, 2, ..., 17. \]  

The \( \theta^T_i, \theta^{0T}_i \), and \( \theta^T_i \)'s with \( i = 0, ..., 17 \) are shown in the Appendix B.

The specific form of Eq. (23) for the cases that correspond to the semileptonic decay of a charged charged HSD \((s = c)\) or a neutral neutral HSD \((s = n)\) decaying hyperon, respectively, becomes:

For the charged HSD process, such as \( \Sigma^-(p_1) \to n(p_2) + e^-(\ell) + \bar{\nu}_e(p_{\nu}) + \gamma(k) \),

\[ d\Gamma_B^- \left( A^- \to B^0 e^{-} \bar{\nu}_e \gamma \right) dE = \frac{\alpha}{\pi} d\Omega \sum_{i=0}^{16} H'_i \theta^T_i, \]  

where the model-independent function \( \theta^T_i \) is

\[ \theta^T_i = (I_1 - 2) \ln \left| \frac{y_0 + 1}{y_0 - 1} \right|, \quad I_1 = \frac{2}{\beta} \arctanh(\beta), \quad y_0 = \frac{(E_0^v)^2 - E^2 \beta^2 - |p_2|^2}{2 |p_2| E \beta}, \quad \text{and} \quad E_0^v = M_1 - E_2 - E. \]  

For the neutral HSD case, such as the \( \Lambda(p_1) \to p^+(p_2) + e^- (\ell) + \bar{\nu}_e(p_{\nu}) + \gamma(k) \),

\[ d\Gamma_B^0 \left( A^0 \to B^+ e^- \bar{\nu}_e \gamma \right) dE = \frac{\alpha}{\pi} d\Omega \left[ (H_0^s + N_0^s) \theta^T_0 + H'_1 \theta^T_1 + \sum_{i=2}^{16} (H'_i + N'_i) \theta^T_i + N'_1 \theta^T_1 \right], \]  

where \( \theta^T_1 \) is separated, for further convenience, into two parts as follows:

\[ \theta^T_1 = \theta_1^{0T} + \theta_1^{TND}, \quad \text{where} \quad \theta_1^{0T} \left( E, E_2 \right) = -2 \ln \left| \frac{y_0 + 1}{y_0 - 1} \right|. \]  

The equations for the \( \theta_1^{0T}, \theta_1^{TND} \) and \( \theta^T_1 \) are also included in the Appendix B.

The result for both types of unpolarized decays given in Eq. (23) is of high precision, model independent and useful for processes where the momentum transfer is not small and therefore cannot be neglected. Terms of order \( \alpha q^2/\pi M_1^2 \) and higher are neglected.

**B. Three body region**

The result up to order \( \alpha q/\pi M_1 \) for the hyperon semileptonic decay (HSD) in this region is given by

\[ d\Gamma_{TBR}^s \left( A^s \to Be\nu \right) dE = \left[ A_0^s + \frac{\alpha}{\pi} \sum_{i=0}^{16} H_i^s \theta^s_i + A^s_\nu \Phi^s_\nu \right] d\Omega, \quad d\Omega = \frac{G^2_F}{2 \pi^3} M_1 M_2 dE dz, \quad z = \frac{E_2}{M_2}. \]  

The coefficients \( H_i^s \)'s are the same as the ones at the FBR (Appendix A), but the model independent functions are different, here

\[ A_0^s = B_1^0, \quad A^s_\nu = A^s_{\nu N}, \quad \theta^s_1 = \Phi_s + \theta^s_1, \quad \theta^s_1^c = \theta_1^{n, n} = \theta_1, \quad \text{for} \quad i = 0, 2, ..., 17. \]  

The \( A_0^s \) and \( A^s_\nu \)'s (given in Appendix A) also depend on the form factors through functions \( Q_i, i = 1, ..., 4 \). The model independent functions \( \theta^s_i \), \( \Phi^s_\nu \) and the \( \theta^s_i \)'s \((i = 0, 2, ..., 17)\) are exhibited in Appendix C.

In order to illustrate the result in Eq. (33), let us consider two cases that correspond to the semileptonic decay of a charged and a neutral decaying hyperon:

For the charged HSD

\[ d\Gamma^c \left( A^- \to B^0 e^- \bar{\nu}_e \right) dE = \left[ A_0^c + \frac{\alpha}{\pi} \left( H_0^c \theta_0 + H'_1 \Theta_c + \sum_{i=2}^{16} H'_i \theta_i \right) \right] d\Omega, \]  

(31)

For the neutral HSD,

\[ d\Gamma^n \left( A^0 \to B^+ e^- \bar{\nu}_e \right) dE = \left[ A_0^n + \frac{\alpha}{\pi} \left( H_0^n + N_0^n \right) \theta_0 + H'_1 \Theta_n + \sum_{i=2}^{17} (H'_i + N'_i) \theta_i \right] d\Omega. \]  

(32)

The \( \theta^s_i \) are analyzed in the Appendix C.
IV. ENERGY SPECTRUM

To obtain the energy spectrum of the beta particle, the integration over $E_2$ has to be performed in the corresponding kinematical regions, $(E_{2b} < E_2 < E_{2\pi})$, according to Eq. (2) and Eq. (3)

$$\Gamma^s(E) = \int_{E_{2b}}^{E_{2\pi}} d\Gamma(A^s \rightarrow Be\nu)/dE.$$  \hspace{1cm} (33)

The result for the case without any radiative corrections becomes

$$\Gamma_o^s(A^s \rightarrow Be\nu) = \frac{G^2_v}{2\pi^3} \frac{M_1}{2} \int_{E_{2b}^\text{max}}^{E_{2b}^\text{min}} A'_0 \, dE_2, \quad \text{where} \quad \sum_{\text{spin}} |M_0|^2 = \frac{G^2_v}{2} \frac{4M_1}{M_2 m_{\nu}} A'_0.$$ \hspace{1cm} (34)

A. Pure Bremsstrahlung correction in the FBR

The $d\Gamma_B^s(A^s \rightarrow Be\nu\gamma)$ in Eq. (23) depends on $y_0$ through $\theta_i^{\text{cT}}$ [see Eq. (27) and Eq. (29)] in the following way:

$$\theta_i^{\text{cT}} \propto \ln \left| \frac{y_0 + 1}{y_0 - 1} \right|.$$ \hspace{1cm} (35)

For collinear events, which take place when

$$E_2 \rightarrow E_{2\min}^c, \quad m \leq E \leq E_c$$ \hspace{1cm} (36)

the $y_0 \rightarrow 1$. This gives rise to a logarithmic divergence. This divergence is an obstacle to perform a direct numerical integration. However, this difficulty is controlled by performing an analytical integration. To proceed, it is convenient to take the following definitions:

$$\frac{(y_0 + 1)}{(y_0 - 1)} = \frac{a_0 + b_0 z + s}{a_0 + b_0 z - s}, \quad a_0 = \frac{M_1 (E_{\text{in}} - E) + M_2^2}{M_2 E_\beta}, \quad b_0 = \frac{E - M_1}{E_\beta} \quad \text{and} \quad s = \sqrt{z^2 - 1}$$ \hspace{1cm} (37)

and to consider separately the terms where the difficulties arise.

For the charged HSD the separation of terms in Eq. (26) is such that the ones that contain a logarithmic divergence (at the integration limits) are accumulated in $I_{C}^{T,D}(E)$ and all the other non-divergent ones are accommodated in $I_{C}^{T,ND}(E)$, then

$$\Gamma_B^s(E) = \frac{G^2_v}{2} \frac{1}{2\pi^3} M_1 \left[ I_{C}^{T,D}(E) + I_{C}^{T,ND}(E) \right],$$ \hspace{1cm} (38)

where

$$I_{C}^{T,D}(E) = \frac{\alpha}{\pi} \int_{M_2}^{E_{2\min}} H'_1(E, E_2) \, dE_2 = \frac{\alpha}{\pi} M_2 \sum_{k=0}^{2} \varepsilon_k M_2^k R_F^{CB}.$$ \hspace{1cm} (39)

The explicit $z$ dependence of $H'_1(E, E_2)$ is given by

$$H'_1 = \sum_{k=0}^{2} \varepsilon_k M_2^k z^k, \quad \varepsilon_k = (I_1 - 2) \varepsilon_k, \quad \text{and} \quad R_F^{CB} = \int_{z=1}^{z_{\pi}} z^k \ln \left| \frac{a_0 + b_0 z + \sqrt{z^2 - 1}}{a_0 + b_0 z - \sqrt{z^2 - 1}} \right| dz.$$ \hspace{1cm} (40)

The $\varepsilon_k$'s are shown in Appendix D. The second term in Eq. (38) is

$$I_{C}^{T,ND}(E) = \frac{\alpha}{\pi} \int_{M_2}^{E_{2\min}} \left( H'_0 \theta'_0 + \sum_{i=2}^{16} H'_i \theta'_i \right) \, dE_2.$$ \hspace{1cm} (38)

For the neutral HSD, we follow the same strategy and we obtain
\[ \Gamma_B^{n}(E) = \frac{G^2_e}{2} \frac{1}{2\pi^3} M_1 \left[ I_N^{TD}(E) + I_N^{TND}(E) \right] \]  

where

\[ I_N^{TD}(E) = \frac{\alpha}{\pi} \int_{M_2}^{E_{2}^{\text{min}}} H_{1}^{\prime} \theta_{1}^{TD} \, dE_2 = \frac{\alpha}{\pi} M_2 \sum_{k=0}^{2} \varepsilon_k^n M_2^k R_k^{TB}, \quad \varepsilon_k^n = -2\varepsilon_k \]  

and

\[ I_N^{TND}(E) = \frac{\alpha}{\pi} \int_{M_2}^{E_{2}^{\text{min}}} \left( (H_0^\prime + N_0^\prime) \theta_0^T + H_1^\prime \theta_{1}^{TND} + \sum_{i=2}^{16} (H_i^\prime + N_i^\prime) \theta_i^T + N_{17}^\prime \theta_{17}^T \right) \, dE_2. \]  

Notice that, we obtain the same integral \( R_k^{TB} \) in both cases. After performing a very subtle analysis (illustrated in Refs. [11] and [12]) we get the following non-divergent analytical result

\[ R_k^{TB} = \frac{1}{k+1} \left[ z_t^{k+1} - (z_t^B)^{k+1} \right] T_0^{B} - \frac{1}{k+1} \left[ \sum_{r=0}^{k} (s_t^r - z_t^r (z_t^B)^r) \int_{z_t}^{z_t^B} z^{-r} \, \frac{dz}{s} \right], \]  

where

\[ T_0^{B} = \ln \left| \frac{M_1 (E_m - E)}{E \beta M_2} \right|, \quad s = \sqrt{z^2 - 1}, \]  

and \( E_m \) is given in Eq. (3). The values of \( z \) and \( s \) (see Eq. (3)) at the boundaries are

\[ z_t = \frac{1}{2} \left[ \frac{M_1 - E (1 - \beta)}{M_2} + \frac{M_2}{M_1 - E (1 - \beta)} \right], \quad z_t^B = z_b = \frac{1}{2} \left[ \frac{M_1 - E (1 + \beta)}{M_2} + \frac{M_2}{M_1 - E (1 + \beta)} \right], \]

\[ s_t = \frac{1}{2} \left( \frac{M_1 - E (1 - \beta)}{M_2} - \frac{M_2}{M_1 - E (1 - \beta)} \right), \quad s_t^B = s_b = \frac{1}{2} \left[ \frac{M_1 - E (1 + \beta)}{M_2} - \frac{M_2}{M_1 - E (1 + \beta)} \right]. \]  

An analogous integral is discussed in Appendix E.

**B. Virtual plus Bremsstrahlung corrections at the three body region**

Following a parallel development for the charged HSD we obtain

\[ \Gamma_c^{c}(E) = \frac{G^2_e}{2} \frac{1}{2\pi^3} M_1 \left[ I_D^{c}(E) + I_N^{c}(E) \right], \]  

where

\[ I_D^{c}(E) = \frac{\alpha}{\pi} M_2 \left[ \sum_{k=0}^{2} \varepsilon_k^c M_2^k R_k \right], \quad \varepsilon_k^c = (I_1 - 2) \varepsilon_k \]  

and

\[ I_N^{c}(E) = \int_{E_{2}^{\text{min}}}^{E_{2}^{\text{max}}} \left[ A_0^c + \frac{\alpha}{\pi} \left( H_{1}^{\prime} \theta_{1}^{c\text{nd}} + B_{1}^{c} \Phi_{c}^{\prime} + H_0^\prime \theta_0 + \sum_{i=2}^{16} H_i^\prime \theta_i \right) \right] \, dE_2, \]  

\[ R_k(z) = \int_{z_h}^{z} z^k \ln \left| a_0 + b_0 z + \sqrt{z^2 - 1} \right| \, dz. \]
The $\varepsilon_k$'s are given in Appendix D, and the calculation of the integrated form of $R_k(z)$ is illustrated in Appendix E. A similar procedure is performed for the neutral HSD, and the following results are obtained:

$$\Gamma^a(E) = \frac{G^2_c}{2} \frac{1}{2\pi^3} M_1 \left[ I^a_D(E) + I^a_{ND}(E) \right], \tag{51}$$

$$I^a_D(E) = \frac{\alpha}{\pi} M_2 \left[ \varepsilon_0 + \sum_{n=1}^{2} (\varepsilon_n' + \varepsilon_n'') R_n + \varepsilon_3' R_3 \right], \tag{52}$$

$$I^a_{ND}(E) = \int_{E_{min}^2}^{E_{max}^2} A_0' + \frac{\alpha}{\pi} \left( \varepsilon_0' \right) \left( \frac{H_1^0 + N_0' \theta_0 + \sum_{i=2}^{16} \varepsilon_i' \theta_i + \sum_{i=2}^{17} N_i' \theta_i} {E^2/2} \right) dE_2. \tag{53}$$

The $\theta_1^{\alpha nd}$ and the other model independent functions are given in the Appendix C. Here we have considered the approximation

$$\arctan(\beta_N) \approx \frac{I_1}{2} - \frac{1}{y_0 |p_2|} \frac{1}{M_1^2 \beta_1} \left( 1 - \frac{m^2 I_1}{E^2/2} \right) = \frac{I_1}{2} - \frac{M_2 a_0}{M_1^2 \beta_1} \left( 1 - \frac{m^2 I_1}{E^2/2} \right) \approx \frac{b_0 M_2}{M_1^2 \beta_1} \left( 1 - \frac{m^2 I_1}{E^2/2} \right) \approx \frac{M^2}{M_1^2 \beta_1}. \tag{54}$$

According to Eq. (C10), the model dependent coefficients in Eq. (52), defined as $\varepsilon_k^{1,n}$, become

$$\varepsilon_0^{1,n} = \varepsilon_0' = \varepsilon_0 \left[ \left( I_1 - 2 \right) - \left( \frac{M_2 a_0}{M_1^2 \beta_1} \right) \left( 1 - \frac{m^2 I_1}{E^2/2} \right) \right],$$

$$\varepsilon_1^{1,n} = \varepsilon_1' + \varepsilon_1'' = \varepsilon_1 \left( I_1 - 2 \right) - \left( \frac{b_0 a_0 + M_2 a_0 \varepsilon_1}{M_1^2 \beta_1} \right) \left( 1 - \frac{m^2 I_1}{E^2/2} \right),$$

$$\varepsilon_2^{1,n} = \varepsilon_2' + \varepsilon_2'' = \varepsilon_2 \left( I_1 - 2 \right) - \left( b_0 \varepsilon_1 + M_2 a_0 \varepsilon_2 \right) \left( 1 - \frac{m^2 I_1}{E^2/2} \right),$$

$$\varepsilon_3^{1,n} = \varepsilon_3' = \varepsilon_2 \left( \frac{b_0}{M_1^2 \beta_1} \right) \left( 1 - \frac{m^2 I_1}{E^2/2} \right). \tag{55}$$

The analytical result for the $R_k$'s in Eqs. (48) and (52) is

$$R_k = \left( \frac{z^{k+1} - z_0^{k+1}}{2(k+1)} \right) T_R - \frac{1}{2(k+1)} \sum_{r=0}^{k} \left[ z_r^{k+1} \left( z_{k-r+1}^{k+1} - z_{k-r+1}^{k+1} \right) \left( k-r+1 \right) + (s_z z_r^{k+1} - s_b z_r^{k+1}) \int_{z_b}^{z_r^{k+1}} \frac{z^{k-r}}{s} dz \right] \tag{56}$$

where

$$T_R = \left[ \ln \left( \frac{4(M_1 - M_2)^2}{M_1^2 \left( 1 - \frac{2k}{M_1^2} + \frac{m^2}{M_2^2} \right) \right) \right] + 2 \Theta(E - E_c) \ln \left[ \frac{M_1 (E_m - E)}{M_2 E} \right], \tag{57}$$

the $\Theta(E - E_c)$ is the Heaviside function, and the $z_t$, $z_t^B$, $s_t$, and $s_t^B$ are given in Eq. (46)

$$z_b = z_t^B, \quad s_b = -s_b^B = s_t^B. \tag{58}$$

V. FINAL RESULTS

A. Four body region

Gathering and refining previous results, we obtain the whole spectrum of events in the FBR. The FBR formula for the lepton energy spectrum becomes in general,
\[
d\Gamma_B^s(E) = \frac{\alpha G^2}{\pi} \frac{1}{2\pi s} M_1 M_2 \left[ \sum_{k=0}^{2} \varepsilon_k^s M_k^s R_k^s + \int_{z=1}^{z_i^B} \left( H_0^s \theta_0^T + \sum_{i=2}^{17} H_i^s \theta_i^T + H_i^s \theta_i^T N D \delta_i^s \right) dz \right], \quad (59)
\]

the \( \delta_i^s \) indicates that the last term appears only in the neutral HSD case \((s = n)\). The \( \varepsilon_k^s \) is given in Eq. (44) for \( s = c \), and in Eq. (42) for the \( s = n \) case.

For \( k = 0, 1, 2 \) the explicit values for \( R_k^T \), in Eq. (44) are

\[
R_0^T = \Delta_z T_0^B - \Delta_s \ln |z_s^B + s_t^B|, \quad R_1^T = \frac{1}{2} \left[ z_t^B - (z_i^B)^2 \right] T_0^B - \frac{1}{2} \Delta_s |s_t^B - s_i^B| \ln |z_i^B + s_t^B|, \quad R_2^T = \frac{1}{2} \left[ z_t^B - (z_i^B)^2 \right] T_0^B - \frac{1}{3} \left[ \left( s_t z_t - s_i^B z_t^B + \Delta_s z_t^B \right) s_t^B - \frac{1}{3} \left( s_t z_t^2 - s_i^B (z_i^B)^2 + \frac{\Delta_s}{2} \right) \right] \ln |z_i^B + s_t^B|, \quad (60)
\]

where \( T_0^B \) is given in Eqs. (58), \( s_t, z_t, s_t^B, z_t^B \), are given in Eqs. (74), and

\[
\Delta_z = z_t^B - z_i^B = \frac{2 M_1 (E_m - E) E}{M_2 M_2^2 \left( 1 - \frac{2 E M_1 + m_2^2}{M_2^2} \right)}, \quad \Delta_s = s_t^B - s_i^B = \frac{E \beta}{M_2} \left( 1 + \frac{M_2^2}{M_2^2} \left( 1 - \frac{2 E M_1 + m_2^2}{M_2^2} \right) \right). \quad (61)
\]

### B. Three body region

The TBR formula for the electron energy spectrum becomes

\[
d\Gamma_B^s(E) = \frac{G^2}{\pi} \frac{1}{2\pi s} M_1 M_2 \left\{ \int_{z=1}^{z_i^B} \frac{A_0^s dz}{2\pi s} + \frac{\alpha}{\pi} \left[ \sum_{k=0}^{r_s} \varepsilon_k^s M_k^s R_k + \int_{z=1}^{z_i^B} \left( H_0^s \theta_0^T + A_s^s \Phi_s^T + \sum_{i=2}^{17} H_i^s \theta_i^T + H_i^s \theta_i^T N D \delta_i^s \right) dz \right] \right\}, \quad (62)
\]

where \( r_s = 2 \), \( r_n = 3 \), and the \( \varepsilon_k^s \)'s are displayed in Eq. (18) and in Eq. (53).

The explicit values for \( R_k \), in Eq. (56) for \( k = 0, 1, 2, 3 \) are

\[
R_0 = \frac{1}{2} \left[ \Delta_z (T_R - 2) - \Delta_s \ln \left| \frac{z_t + s_t}{z_b + |s_b|} \right| \right], \quad (63)
\]

\[
R_1 = \frac{1}{4} \left\{ \left( z_t^2 - z_i^B \right) (T_R - 2) - \Delta_s \Delta_{|s|} - (s_t z_t - s_i^B z_b) \ln \left| \frac{z_t + s_t}{z_b + |s_b|} \right| \right\}, \quad \Delta_{|s|} = s_t - |s_b| \quad (64)
\]

\[
R_2 = \frac{1}{6} \left\{ \left( z_t^3 - z_i^B \right) (T_R - 2) - \Delta_s \Delta_{|s|} + \left( s_t z_t^2 - s_i^B z_b^2 \right) \Delta_s \ln \left| \frac{z_t + s_t}{z_b + |s_b|} \right| \right\}, \quad (65)
\]

\[
R_3 = \frac{1}{8} \left\{ \left( z_t^4 - z_i^B \right) (T_R - 2) - \Delta_s \Delta_{|s|} + \left( s_t z_t^3 - s_i^B z_b^3 \right) \Delta_s \ln \left| \frac{z_t + s_t}{z_b + |s_b|} \right| \right\}, \quad (66)
\]

where \( T_R \) is given in Eq. (57), and \( s_t, z_t, s_b, z_b, \Delta_z, \) and \( \Delta_s \) are given in Eqs. (40), and Eqs. (41), respectively.
VI. CONCLUSIONS

The singularities contained in the precise analytical formulas at the level of the Dalitz plot are isolated and well understood. The logarithmic divergences disappear and the lepton energy spectrum becomes finite when the formulas are integrated in an analytical way. The final and general results in Eq. (59) and Eq. (62) are precise formulas suitable to be evaluated numerically, without ambiguity, at any energy in which the charged lepton is emitted. They contain the bremsstrahlung in the four body decay region where the three body decay does not take place and also include events in which the electron is collinear to the produced hadron (at the edge of the DP).

Other authors Ref. [13] have published numerical data for the percent contributions due to the radiative corrections to the HSD decay. Only as a test, we have compared both results. We consider the numerical values obtained with the formulas in Eq. (59) and Eq. (62) in six Tables. Tables I-III are devoted to the charged HSD process, considering the case: $\Sigma^− \rightarrow n + e^− + \bar{\nu}_e$. In the Tables I, III, IV and VI we exhibit the data given in Ref. [13] for the relative RC in $\%$, caused by bremsstrahlung events, which fall inside and outside the TBR Dalitz plot, and the numerical values obtained by means of Eq. (59) and Eq. (62). The results are in good agreement taking into account the fact that the earlier numerical results contain unspecified approximations especially at the Dalitz plot boundary. In addition we illustrate our final results with four graphs. For the $(\Sigma^−, n)$ case, in Fig. 1, the RC at the TBR (which contains the virtual and bremsstrahlung corrections) and the RC in FBR (which contains only the bremsstrahlung corrections) contributions are displayed separately. In Fig. 2, the complete energy spectrum is plotted with and without the total radiative corrections. For the $(\Lambda, p)$ case, Fig. 3 contains the radiative contributions including the Coulomb effect and in Fig. 4 we consider the energy spectrum as in the previous case. The Coulomb effect is included in the analysis. It is incorporated into the virtual radiative corrections to the energy spectrum for the neutral HSD in the TBR.

In summary, the analytical results are useful to obtain information derived from experimental data, about the underlying interactions in the decay processes, the basic symmetries, and the internal structure of hadrons, through the derivation of precise values of the form factors involved in the effective interaction.

The knowledge of the energy spectrum of the electron is fundamental for the determination of the decay rate in these processes. For the complete determination of the decay rate it is necessary to add the contributions of the events in the three body and in the four body regions.

Let us mention that the RC were also computed by other means for photon bremsstrahlung calculations in semileptonic decays Ref. [14]. At last, though the evaluation of the radiative corrections for the HSD is a complex and an old problem (see list of references in Ref. [14]), the results in Eqs. (59) and (62) are new, and are the culmination of a systematical approach to the decays in the whole region of the Dalitz plot.

ACKNOWLEDGMENTS

We thank Professor P. Kielanowski for reviewing the manuscript and providing valuable suggestions. S.R.J.W. also acknowledges partial support by Comisión de Operación y Fomento de Actividades Académicas (Instituto Politécnico Nacional), and by CoNaCyT-México.

APPENDIX A

The coefficients $Q_i$’s $(i = 1, \ldots, 5)$ do depend on the form factors.

$$Q_1 = F_1^2 \left( \frac{2E_2 - M_2}{M_1} \right) + \frac{1}{2} F_2^2 \left( \frac{M_2 + E_2}{M_1} \right) + F_1 F_2 \left( \frac{M_2 + E_2 - Q}{M_1} \right) + F_1 F_3 \left( \frac{M_2 + E_2}{M_1} \right) + F_1 F_3 \left( 1 - \frac{E_2 - Q}{M_1} \right)$$

$$+ F_2 F_3 \left( \frac{M_2 + E_2}{M_1} \right) \left( 1 - \frac{E_2 - Q}{M_1} \right) + G_1^2 \left[ \frac{2E_2 + M_2}{M_1} \right] - \frac{1}{2} G_2^2 \left( \frac{M_2 - E_2}{M_1} \right) + G_1 G_2 \left( \frac{M_2 - E_2}{M_1} \right)$$

$$+ G_1 G_3 \left( \frac{M_2}{M_1} - 1 \right) \left( 1 - \frac{E_2}{M_1} \right) - G_2 G_3 \left( \frac{M_2 - E_2}{M_1} \right) \left( 1 - \frac{E_2}{M_1} \right) + M_1^2 Q_5 \left[ \left( \frac{M_2 - E_2}{M_1} \right) - \frac{1}{2} \frac{q^2}{M_1^2} \right]$$

$$Q_2 = \frac{-F_2^2}{M_1} + \frac{F_1 F_2}{M_1} \left( 1 + \frac{M_2}{M_1} \right) + \frac{F_2 F_3}{M_1} \left( \frac{M_2 + E_2}{M_1} \right) - \frac{G_2^2}{M_1} + \frac{G_1 G_2}{M_1} + \frac{G_1 G_3}{M_1} \left( \frac{M_2}{M_1} - 1 \right)$$

$$- \frac{G_2 G_3}{M_1} \left( \frac{M_2 - E_2}{M_1} \right) + 2 F_1 G_1 \left( \frac{M_1 - E_2}{M_1} \right) + M_1 Q_5 \left( \frac{M_1 - E_2}{M_1} \right)$$

10
\[ Q_3 = Q_1 - 2F_1^2 \left( \frac{E_2 - M_2}{M_1} \right) - 2G_1^2 \left( \frac{E_2 + M_2}{M_1} \right) - M_1^2 Q_5 \left[ \left( 1 - \frac{E_2}{M_1} \right)^2 - \frac{g^2}{M_1^2} \right], \]

\[ Q_4 = Q_2 - 4 \frac{F_1 G_1}{M_1}, \quad Q_5 = \frac{F_2^2}{M_1^2} \left( \frac{M_2 + E_2}{M_1} \right) - \frac{G_2^2}{M_1^2} \left( \frac{M_2 - E_2}{M_1} \right) - 2 \frac{F_1 F_3}{M_1^2} + \frac{G_1 G_3}{M_1^2}, \]  

(A1)

with

\[ F_1 = f_1' + \left( 1 + \frac{M_2}{M_1} \right) f_2, \quad F_2 = -2 f_2, \quad F_3 = f_2 + f_3, \]

\[ G_1 = g_1' - \left( 1 - \frac{M_2}{M_1} \right) g_2, \quad G_2 = -2 g_2, \quad G_3 = g_2 + g_3, \]  

(A2)

\( f_1', g_1' \) are effective form factors, see Ref. [10].

For completeness, we explicitly show the coefficients \( H_i' \) s:

\[ A_0' = Q_1 A_0 E E_0^0 - Q_2 E \mid \vec{p}_2 \rangle \left( \mid \vec{p}_2 \rangle + \mid l \rangle \mid y_0 \rangle \right) - Q_5' \mid \vec{p}_2 \rangle \left( \mid \vec{p}_2 \rangle \mid y_0 \rangle + \mid \vec{p}_2 \rangle \mid y_0 \rangle \right) + Q_4' E E_0^0 \mid \vec{p}_2 \rangle \mid y_0 \rangle - Q_5' \mid \vec{p}_2 \rangle \mid y_0 \rangle, \]

\[ H_0' = E \beta \mid \vec{p}_2 \rangle \left[ \frac{1}{2} \left( Q_3 - Q_4 E_0^0 \right) - \frac{E}{2 M_1} \right], \]

\[ H_1' = A_1 N = E \left\{ E_0^0 Q_1 - Q_2 \mid \vec{p}_2 \rangle^2 - Q_3 \beta^2 E + \beta \mid \vec{p}_2 \rangle y_0 \left( E_0^0 Q_4 - Q_3 - E Q_2 \right) \right\}, \]

\[ H_2' = \frac{m^2}{2 E} \beta \mid \vec{p}_2 \rangle \left[ - \left( Q_1 + Q_3 \right) E_0^0 + \left( Q_2 + Q_4 \right) E \mid \vec{p}_2 \rangle y_0 + Q_4 \left( E + E_0^0 \right)^2 + Q_2 \mid \vec{p}_2 \rangle^2 - 2 \mid \vec{p}_2 \rangle^2 \right], \]

\[ H_3' = \beta \mid \vec{p}_2 \rangle \left\{ \frac{E}{4} \left[ \left( 2 E_0^0 - E \left( 1 + \beta^2 \right) \right) \left( Q_1 + Q_3 \right) + 2 \beta \mid \vec{p}_2 \rangle y_0 \left( E_0^0 Q_4 - Q_3 - E Q_2 \right) - 2 \mid \vec{p}_2 \rangle^2 \right] \]  

\[ + \left( E_0^0 + E \right) \left[ E \left( 1 + \beta^2 \right) \left( Q_2 + 3 Q_4 \right) - 2 \left( Q_3 + 2 E Q_4 \right) \right] \right\}, \]

\[ H_4' = \frac{1}{2} E \beta \mid \vec{p}_2 \rangle \left\{ \frac{1}{2} E \left[ Q_1 - E_0^0 Q_2 - E \left( Q_2 + Q_4 \right) + 3 \left( Q_3 - E_0^0 Q_4 \right) \right] + \frac{1}{2} \left[ -E \beta^2 Q_2 + \mid \vec{p}_2 \rangle^2 Q_4 + E_0^0 \left( 2 Q_3 - 3 E_0^0 Q_4 \right) \right] \right\}, \]

\[ - E E_0^0 \left( \frac{(f_1 - g_1)^2 + 2 f_1 f_3}{M_1} + \frac{1}{M_1} \left[ f_1^2 + g_1^2 + 2 \left( g_1 g_2 + f_1 f_3 \right) \right] \right) \mid \vec{p}_2 \rangle \beta E y_0 + \frac{2 E}{M_1} g_1 g_2 \left[ 2 E_0^0 + E \left( 4 - 3 \beta^2 \right) \right] \]

\[ + \frac{2 E}{M_1} g_1 \left[ 2 E \left( f_1 + f_2 - g_2 \right) + f_2 \left( E_0^0 + \beta^2 E \right) \right] + \frac{2 h^-}{e} \beta^2 E^2 - 2 E \left[ E_0^0 + 2 E \left( 1 - \beta^2 \right) \right] \frac{h^+}{e} \right\}, \]

\[ H_5' = \frac{1}{4} E^2 \beta^2 \mid \vec{p}_2 \rangle \left\{ Q_1 - \left( E + E_0^0 \right) Q_2 + 3 Q_3 - \left( 3 E + 7 E_0^0 \right) Q_4 - 4 \left( 2 E + E_0^0 \right) \frac{h^+}{e} - 8 E_0^0 \frac{h^-}{e} + \frac{4}{M_1} \left[ E_0^0 \left( f_1^2 + 2 f_1 f_3 \right) \right], \]

\[ H_6' = \frac{\mid \vec{p}_2 \rangle \left( 1 - \beta^2 \right) E \beta}{4} \left[ Q_1 + Q_3 - \left( Q_2 + Q_4 \right) \left( E_0^0 + E \right) \right], \]
\[ H''_7 = -\frac{E\beta|\vec{p}_2|}{4} \left\{ \frac{(2E - E_\nu^0)}{E}(Q_1 + Q_3) + \frac{|\vec{p}_2|^2}{E}(Q_2 + Q_4) + 2(\beta^2 - 1)EQ_4 + |\vec{p}_2|\beta y_0 \left( Q_2 + 3Q_4 - 2\frac{h^+}{e} \right) \right\} \]
\[ -2\left( E_\nu^0 + E\beta^2 \right) \frac{h^+}{e} + [(\beta^2 - 3)E - 2E_\nu^0]Q_2 + \frac{E}{M_1}(1 - \beta^2) \left[ f_1^2 + g_1^2 - g_1(2f_1 + 3f_2 + g_2) \right] \]
\[ + g_1(f_2 - g_2) \left( M_2 \right) \left[ \frac{2E_2 - M_1}{M_2} - \frac{M_2}{M_1} \right] + \frac{2m^2}{EM_1}(f_1f_3 - g_1g_2) \right\}, \]

\[ H''_8 = \frac{E\beta|\vec{p}_2|}{4} \left\{ Q_1 + Q_3 - (2E + E_\nu^0)Q_2 + (E_\nu^0 - E)Q_4 + 2E_\nu^0 \left( \frac{2h^- - h^+}{e} \right) + [(f_1 - g_1)^2 + 2f_1f_3] \left( \frac{E - 2E_\nu^0}{M_1} \right) \right\} \]
\[ + g_1\left( \frac{(2f_2 + g_2)(E_\nu^0 - 2E) + g_2E_\nu^0}{M_1} \right) \bigg], \quad H''_9 = \frac{|\vec{p}_2|\beta}{8} \left[ -(Q_1 + Q_3) + (Q_2 + Q_4)(E + E_\nu^0) \right], \]

\[ H''_{10} = \frac{1}{4}(E\beta)^3|\vec{p}_2| \left\{ -(Q_2 + 5Q_4) - 4 \left[ \frac{3h^- + 2h^+}{e} \right] + \frac{2}{M_1} \left[ 3f_1(f_1 + 2f_3) + g_1(g_1 - 4f_1 - 6f_2 + 8g_2) \right] \right\}, \]

\[ H''_{11} = 0, \quad H''_{12} = (E\beta)^2|\vec{p}_2|^2 \left\{ \frac{h^+}{e} - \frac{Q_2}{2} + \frac{1}{2M_1} \left[ (f_1 + g_1)^2 + 2g_1(3f_2 - 2g_2) + 2f_1f_3 \right] \right\}, \]

\[ H''_{13} = \frac{(E\beta)^2|\vec{p}_2|^2}{2} \left\{ Q_4 - \frac{2h^+}{e} + \frac{1}{M_1} \left[ 2f_2(f_1 - g_1) + g_1^2 - f_1^2 - 2f_2^2 - 2f_1f_3 \right] \right\}, \]

\[ H''_{14} = \frac{(E\beta)^2|\vec{p}_2|^2}{4M_1} \left[ (f_1 - g_1)^2 - 4f_2g_1 + 2(f_1f_3 - g_1g_2) - M_1Q_2 \right], \]

\[ H''_{15} = \frac{E\beta|\vec{p}_2|}{8} \left\{ -(Q_2 + Q_4) - 4\frac{h^-}{e} + \frac{2}{M_1} \left[ (f_1 - g_1)^2 - 2f_2g_1 + 2f_1f_3 \right] \right\}, \quad H''_{16} = \frac{|\vec{p}_2|\beta}{4M_1} \left[ -M_1\frac{h^+}{e} + g_1(g_2 - f_2) \right]. \]

We have used the definition
\[ h^\pm = g_1^2(\kappa_1 + \kappa_2) \pm f_1g_1(\kappa_2 - \kappa_1). \]

The coefficients \( N_i \)'s are given by
\[ N'_0 = -|\vec{p}_2| \frac{E\beta}{2M_1} \left[ 2(E - E_\nu^0)R^+ + (E + 2E_\nu^0)R^- + (1 - y_0)|\vec{p}_2|\beta R^- \right], \quad N'_2 = N'_6 = N'_9 = N'_11 = N'_15 = 0, \]

\[ N'_4 = \frac{E\beta m^2}{M_1}|\vec{p}_2| R^+, \quad N'_4 = -\frac{E\beta m^2}{M_1}|\vec{p}_2| \left[ 2E^2R^+ + E(\beta + 4|\vec{p}_2|y_0)R^- \right], \quad N'_5 = -\frac{(E\beta)^2}{M_1}|\vec{p}_2| \left[ ER^+ + 2E_\nu^0R^- \right], \]

\[ N'_7 = |\vec{p}_2|\frac{\beta}{2M_1} \left[ 2m^2 + EE_\nu^0(1 - \beta x_0) \right] R^+, \quad N'_8 = -|\vec{p}_2| \frac{E\beta}{2M_1} \left( 2E + E_\nu^0 \right) R^+, \]

\[ N'_{10} = -\frac{3|\vec{p}_2|(E\beta)^3}{2M_1}R^-, \quad N'_{12} = |\vec{p}_2|^2 \frac{E\beta m^2}{M_1} \left( 2E - E_\nu^0 \right) R^+, \quad N'_{13} = -|\vec{p}_2|^2 \frac{(E\beta)^2}{2M_1} \left( 2R^+ - R^- \right), \]

12
\[ N'_{14} = -|\vec{p}_2| \frac{(E \beta)^2}{M_i} R^+, \quad N'_{16} = -|\vec{p}_2| \frac{\beta}{4M_i} R^+, \quad N'_{17} = |\vec{p}_2| \frac{E \beta}{4M_2} [2E_0 + (1 - y_0) |\vec{p}_2| \beta] R^- \] 

(A4)

and

\[ R^+ = |f_1|^2 \pm |g_1|^2, \quad B''_4 = Q_1 E E_0^0 - Q_2 E |\vec{p}_2| (|\vec{p}_2| + \beta E y_0), \] 

(A5)

\[ A''_{1N} = \frac{a}{2 M_i^2} \frac{1}{(1 - 2 E_0^0/M_1)} \left[ E_0^0 (Q_1 E_2 + Q_4 \vec{p}_2^2) - (\vec{p}_2^2 + |\vec{p}_2|^2 \beta E y_0) (Q_2 E_2 + Q_3) \right]. \] 

(A6)

APPENDIX B

The FBR model independent functions \( \theta^c_i \) and \( \theta^{T}_i \) are given in Eqs. (27) and (28), respectively,

\[ \theta^T_0 = 2 (I_1 - 2), \quad \text{and} \quad \theta^{cT}_i = \theta^{PT}_i = \theta^T_i, \quad \text{for} \quad i = 0, 2, \ldots 17. \] 

(B1)

\[ \theta^{cT}_1 = \theta^T_1 = (I_1 - 2) \ln \frac{y_0 + 1}{y_0 - 1}, \quad \theta^{cT}_2 = \theta^T_2 = \theta^{cT}_1 + \theta^{TND}_1, \quad \theta^{TND}_1 = -2 \ln \frac{y_0 + 1}{y_0 - 1}. \] 

(B2)

The FBR \( \theta^{TND}_1 \) in Eqs. (29) is

\[ \theta^{TND}_1 = \frac{1}{2} \left( \ln^2 v^+_{\text{max}} - \ln^2 v^+_{\text{min}} \right) - \frac{1}{2} \left( \ln^2 v^-_{\text{min}} - \ln^2 v^-_{\text{max}} \right), \]

\[ + \frac{1}{\beta N} \left[ \ln v^+ \ln \left| \frac{v^+ - a (1 + \beta N)}{v^+ - a (1 - \beta N)} \right| + \ln v^- \ln \left| \frac{v^+ - a (1 + \beta N)}{v^- - a (1 - \beta N)} \right| \right] - \frac{1}{\beta N} \left[ I^0_1 - I^2_2 - I^2_3 + I^4_2 \right]. \] 

(B3)

For \( i = 2, \ldots 17 \)

\[ \theta^T_2 = \frac{1}{|\vec{p}_2|} \left[ \frac{I^2_2}{1 + \beta a^+} - \frac{I^2_2}{1 + \beta a^-} + \frac{E^2}{m^2} \left( I^0_2 - I^2_2 + \beta \ln \left| I^0_3 / I^2_3 \right| \right) \right] + \frac{2 I_1}{E (1 + \beta a^-) (1 + \beta a^+)}, \]

\[ \theta^T_3 = \frac{I_1}{|\vec{p}_2|} \ln \left[ \frac{1 + \beta a^+}{1 + \beta a^-} \right] + \frac{1}{|\vec{p}_2|} \left( L \left[ \frac{1 - \beta}{1 + \beta a^-} \right] - L \left[ \frac{1 - \beta}{1 + \beta a^+} \right] + L \left[ \frac{1 + \beta}{1 + \beta a^-} \right] - L \left[ \frac{1 + \beta}{1 + \beta a^+} \right] \right), \]

\[ \theta^T_4 = \frac{1}{|\vec{p}_2|} \left[ \frac{a^+ I^2_2 - a^- I^2_2 + \ln \left| \frac{I^0_2}{I^2_3} \right|}{I^0_2} \right], \quad \theta^T_5 = \frac{1}{2 |\vec{p}_2|} \left[ (1 - a^2) I^2_2 - (1 - a^2) I^2_2 + I^0_2 \right], \]

\[ \theta^T_6 = \frac{2 (y_0 - a^-)}{(1 + \beta a^-)^2} (I^2_2 + \beta I_1) - \frac{2 (y_0 + a^+)}{(1 + \beta a^+)^2} (I^2_2 + \beta I_1) + 2 \left[ 2 + \beta \left( \frac{y_0 - a^-}{1 + \beta a^-} - \frac{y_0 + a^+}{1 + \beta a^+} \right) I_4 \right], \]

\[ \theta^T_7 = 2 \left[ 2 I_1 + \frac{y_0 - a^-}{1 + \beta a^-} (\beta I_1 + I^2_2) - \frac{y_0 + a^+}{1 + \beta a^+} (\beta I_1 + I^2_2) \right], \quad \theta^T_8 = 2 \left[ 4 + (y_0 - a^-) I^2_2 - (y_0 + a^+) I^2_2 \right], \]

\[ \theta^T_9 = 24 E + 2 \left[ 6 (E_0^0 - E) + \beta (G^T - G^T^+) I_1 + 2 (G^T - I^2_2 + G^T + I^2_2) + 2 |\vec{p}_2| \right] \left[ \frac{(y_0 - a^-)^2}{1 + \beta a^-} - \frac{(y_0 + a^+)^2}{1 + \beta a^+} \right], \]

\[ \theta^{T}_0 = \frac{1}{3 |\vec{p}_2|} \left[ 2 (a^2 - a^2) - a^- I^2_2 + a^+ I^2_2 + \ln \left| \frac{I^0_2}{I^2_3} \right| \right], \quad \theta^{T}_{11} = 2 (I_4 - I_1) \frac{1}{|\vec{p}_2|}, \quad \theta^{T}_{12} = \frac{1}{|\vec{p}_2|} \theta^{T}_0, \quad \theta^{T}_{13} = 0, \]
\[
\theta_{14}' = 2 \left[ (2 - a^- I_2^-) (y_0 - a^-) - (2 - a^+ I_2^+) (y_0 + a^+) \right], \\
\theta_{15}' = 24E'_0 + 4\beta E \left[ a^- (y_0 - a^-) I_2^- - a^+ (y_0 + a^+) I_2^+ \right] + 2|\vec{p}_2| \left[ (y_0 - a^-)^2 I_3^- - (y_0 + a^+)^2 I_3^+ \right], \\
\theta_{16}' = 24E^2 (I_1 - 2) + 8 \left( E'_0^2 - 2E^2\beta^2 \right) I_3 + 4E\beta |\vec{p}_2| \left[ \frac{(y_0 - a^-)^2}{1 + \beta a^-} (\beta I_1 + I_2^-) - \frac{(y_0 + a^+)^2}{1 + \beta a^+} (\beta I_1 + I_2^+) \right], \\
\theta_{17}' = 2I_1, \quad \text{and} \quad a^\pm = \frac{E'_0 \pm |\vec{p}_2|}{E\beta}.
\]

\(I_1\) is given in Eq. (27), \(I_2^\pm, I_3^\pm, I_4,\) and \(G^T\) are and \(\beta_N\) is defined in Eq. (21).

\[
I_2^\pm = \ln \left[ \frac{1 + a^\pm}{a^\pm - 1} \right], \quad I_3^\pm = \frac{2}{(a^\pm)^2 - 1}, \quad I_4 = \frac{2}{1 - \beta^2}, \quad G^T = \mp \beta \left[ \frac{2Ea^\pm (y_0 \pm a^\pm)}{(1 + \beta a^\pm)} + \frac{|\vec{p}_2| (y_0 \pm a^\pm)^2}{(1 + \beta a^\pm)^2} \right],
\]

\[
v_{\min}^\pm = 2 (E_2 \pm |\vec{p}_2|) \left( E \pm |\vec{E}| \right), \quad v_{\max}^\pm = 2 \left\{ EE_2 - |\vec{p}_2| |\vec{E}| \pm E_2 |\vec{E}| - |\vec{p}_2| E \right\},
\]

\[
a = M_1^2 - H^2 - q^2, \quad H^2 = (p_1 - \ell)^2, \quad q^2 = (p_1 - p_2)^2.
\]

The \(I_i^\prime\)'s in Eq. (B3) are defined in terms of the Heaviside function \(\theta(x)\). Then for \(i = 1, ..., 4\)

\[
I_i^\prime = I_{iA} \theta (r_{A_i}) + I_{iB} \theta (r_{B_i}) + I_{iC} \theta (r_{C_i}),
\]

where

\[
I_{1A,2A} = L \left( \frac{a (1 + \beta_N)}{v_{\min}^+} \right) - L \left( \frac{a (1 + \beta_N)}{v_{\max}^+} \right) + \frac{1}{2} \left( \ln^2 v_{\max}^+ - \ln^2 v_{\min}^+ \right),
\]

\[
I_{1B,2B} = -\frac{\pi^2}{3} L \left( \frac{v_{\min}^+}{a (1 + \beta_N)} \right) - L \left( \frac{a (1 + \beta_N)}{v_{\max}^+} \right) + \frac{1}{2} \ln^2 a (1 + \beta_N) \frac{1}{v_{\min}^+} + \frac{1}{2} \left( \ln^2 v_{\max}^+ - \ln^2 v_{\min}^+ \right),
\]

\[
I_{1C,2C} = L \left( \frac{v_{\max}^+}{a (1 + \beta_N)} \right) - L \left( \frac{v_{\min}^+}{a (1 + \beta_N)} \right) + \ln |a (1 + \beta_N)| \ln \left| v_{\max}^+/v_{\min}^+ \right|,
\]

\[
I_{3A,4A} = L \left( \frac{a (1 + \beta_N)}{v_{\min}^-} \right) - L \left( \frac{a (1 + \beta_N)}{v_{\max}^-} \right) - \frac{1}{2} \left( \ln^2 v_{\min}^- - \ln^2 v_{\max}^- \right),
\]

\[
I_{3B,4B} = \frac{\pi^2}{3} + L \left( \frac{v_{\max}^-}{a (1 + \beta_N)} \right) + L \left( \frac{a (1 + \beta_N)}{v_{\min}^-} \right) - \frac{1}{2} \ln^2 a (1 + \beta_N) \frac{1}{v_{\max}^-} + \frac{1}{2} \left( \ln^2 v_{\max}^- - \ln^2 v_{\min}^- \right),
\]

\[
I_{3C,4C} = L \left( \frac{v_{\max}^-}{a (1 + \beta_N)} \right) - L \left( \frac{v_{\min}^-}{a (1 + \beta_N)} \right) + \ln |a (1 + \beta_N)| \ln \left| v_{\max}^-/v_{\min}^- \right|,
\]

and the arguments of the Heaviside function are given by the expressions

\[
r_{A1,A2} = -r_{B1,B2} = v_{\min}^- - a (1 + \beta_N), \quad r_{B1,B2} = -r_{C1,C2} = v_{\max}^- - a (1 + \beta_N), \quad r_{A3,A4} = -r_{B3,B4} = v_{\max}^- - a (1 + \beta_N), \quad r_{B3,B4} = -r_{C3,C4} = v_{\min}^- - a (1 + \beta_N).
\]

The left (right) subindex in the left hand side (LHS) in the former equations corresponds to the upper (lower) sign in the right hand side (RHS) in the same equations. \(L(x)\) is the Spence function.
APPENDIX C

The TBR $\theta'^s$ and $\theta'^n$ in Eqs. (3) are

$$\theta'^s = \Phi_s + \theta'^1_s.$$  \hfill (C1)

For the charged HSD, after we split the $\theta'^d$ and $\theta'^{nd}$ contributions, the model-independent function becomes

$$\theta'^c = (I_1 - 2) \left[ \ln |a_0 + b_0z + \sqrt{z^2 - 1}| \right] + \theta'^{nd},$$  \hfill (C2)

$$\theta'^{nd} = -\frac{1}{\beta} L \left( \frac{2\beta E}{M_1 - E + \beta E} \right) + \frac{I_1}{2} \left( 1 + \frac{m^2}{E(M_1 - 2E)} \right) - \frac{1}{\beta} \ln |1 - \frac{2\beta E}{M_1 - E + \beta E}| \left( \ln \frac{M_1}{m} | - \beta \frac{I_1}{2} \right)$$

$$+ \left( 1 - \frac{I_1}{2} \right) \ln \left| \frac{M_1mE_0^3(M_1 - E_2 - E_m)}{E_0^2\beta^2M_2^2} \right| - \frac{1}{2\beta} \ln \left| 1 - \beta x_0 \right| \ln \left| 1 - \beta x_0 \right| + \frac{3}{2} \ln \frac{M_1}{m} - \frac{11}{8}$$

$$+ \frac{1}{\beta} \left[ L \left( \frac{\beta(1 - x_0)}{1 - \beta x_0} \right) + L \left( \frac{\beta(1 + x_0)}{1 + \beta} \right) \right] - \frac{1}{4} \beta I_1^2,$$  \hfill (C3)

where

$$x_0 = -\frac{(|\nu_1| + |\nu_2| + \beta E)}{E_0^3}, \quad 1 - \beta x_0 = \frac{M_1(M_1 - E_2 - E_m)}{EE_0^3}.$$  \hfill (C4)

To obtain the results in Eq. (22) and in Eq. (23), for the neutral HSD, the $\theta'^n$ is separated for convenience into a $\theta'^d$ and $\theta'^{nd}$. These model-independent functions are

$$\theta'^n = \theta'^d + \theta'^{nd}, \quad \theta'^d = \theta'^I_0, \quad \theta'^{nd} = \Phi_{\text{Coulomb}} + C'_{\text{N}}.$$  \hfill (C5)

After considering the approximation in Eq. (54)

$$\theta'^I_0 = \psi_{D1} + E_2 \psi_{D2},$$  \hfill (C6)

and

$$H'_1 \theta'^I_0 = \sum_{n=0}^{\infty} \varepsilon_n M_2^n z^n \left[ \psi_{D1} + E_2 \psi_{D2} \right] = \left[ \sum_{n=0}^{\infty} \varepsilon'_n M_2^n z^n + \sum_{m=1}^{\infty} \varepsilon''_m M_2^m z^m \right] \ln |a_0 + b_0z + \sqrt{z^2 - 1}|,$$  \hfill (C7)

where

$$\psi_{D1} = \left[ I_1 - 2 - \frac{M_2a_0}{M_1\beta} \left( 1 - \frac{m^2 I_1}{E^2} \right) \right] \ln |a_0 + b_0z + \sqrt{z^2 - 1}|.$$  \hfill (C8)

$$\psi_{D2} = -\frac{b_0}{M_1\beta} \left( 1 - \frac{m^2 I_1}{E^2} \right) \ln |a_0 + b_0z + \sqrt{z^2 - 1}|,$$  \hfill (C9)

$$\varepsilon'_n = \varepsilon_n \left[ I_1 - 2 - \frac{M_2a_0}{M_1\beta} \left( 1 - \frac{m^2 I_1}{E^2} \right) \right], \quad \varepsilon''_m = -\varepsilon_{m-1} \frac{b_0}{M_1\beta} \left( 1 - \frac{m^2 I_1}{E^2} \right),$$  \hfill (C10)

The $\Phi_{\text{Coulomb}}$ is given in Eq. (18) and

$$C'_N = C_N + 2 \left( \frac{\arctanh(\beta N)}{\beta N} - 1 \right) \ln |2M_2E\beta| + \frac{2 \arctanh(\beta N)}{\beta N} \ln \left( \frac{1 + \beta}{\beta} \right).$$  \hfill (C11)
\[ C_N = \frac{1}{\beta_N} \left\{ - (\text{arctanh } \beta_N)^2 + L \left( \frac{2a \beta_N}{a(1 + \beta_N) + 2m^2} \right) + L \left( \frac{2a \beta_N}{a(1 + \beta_N) + 2M_2^2} \right) - 2L \left( \frac{(1 - \beta_N)}{2 \beta_N} \right) \right\} + 2L \left( \frac{-2E|\vec{p}_2|(1 + \beta)(1 + y_0)}{a(1 + \beta_N)} \right) + 2L \left( \frac{2E(1 + \beta)(E_2 + |\vec{p}_2|) + (\beta_N - 1)a}{a(1 - \beta_N)} \right) \]

\[ - \frac{1}{2}L \left( \frac{a(1 + \beta_N)}{\omega_0} \right) + \frac{1}{2}L \left( \frac{a(1 - \beta_N)}{\omega_0} \right) + 2mM_2 \frac{2M_2}{\omega_{\text{min}}^2 \omega_0} \text{arctanh } (\beta_N) \]

\[ - \ln \frac{2E(1 + \beta)(E_2 + |\vec{p}_2|) + (\beta_N - 1)a}{2a \beta_N} \ln \left[ \frac{2E(1 + \beta)(E_2 + |\vec{p}_2|) + (\beta_N - 1)a}{a \beta_N} \right] \]

\[ - \ln \left( \frac{a(1 + \beta_N) + 2M_2^2}{a(1 - \beta_N) + 2M_2^2} \right) \left[ \ln \frac{M_2}{m} - \text{arctanh } (\beta_N) + \frac{1}{2} \ln \left( \frac{a(1 + \beta_N) + 2M_2^2}{a(1 - \beta_N) + 2M_2^2} \right) \right] \]

\[ + \left( \frac{1}{\beta_N} \text{arctanh } (\beta_N) \right) \left[ - \ln \left( \frac{m^2}{\beta_N} \right) + \frac{2M_2^2 + (1 + \beta_N)^2 a}{2(p_2 + \ell)^2} \right] + \left( \frac{1}{2} - \frac{m^2}{(p_2 + \ell)^2} \right) \ln \left( \frac{M_2}{m} \right) \]

\[ - \frac{11}{8} + \ln \left( H^2 - M_2^2 \right) \left( q^2 - m^2 \right) + \left( \ln \frac{E(1 + \beta)(E_2 + |\vec{p}_2|)}{mM_2} \right)^2 - (\text{arctanh } (\beta_N))^2 \],

(C12)

where

\[ \omega_0 = \omega_1 + \sqrt{\omega_1^2 - 4m^2M_2^2}, \quad \omega_1 = \frac{ab_r - 4m^2M_2^2}{a - b_r}, \quad b_r = \frac{M_2^2(q^2 - m^2)}{H^2 - M_2^2} + \frac{m^2}{q^2 - m^2}, \]

\[ q^2 = M_1^2 + M_2^2 - 2E_2M_1, \quad H^2 = M_1^2 + m^2 - 2EM_1, \quad \omega_{\text{min}} = (1 + \beta_N) \sqrt{(a - b_r)(H^2 - M_2^2)(q^2 - m^2)} \frac{a\beta^3_N}{a\beta^3_N} . \]

Now,

\[ \theta_0 = (1 + y_0)(I_1 - 2), \quad \text{and } \theta_i = \frac{1}{|\vec{p}_2|} (T_i^+ + T_i^-) \quad \text{for } i = 0, 2, \ldots, 16, \quad \theta_{17} = (1 + y_0)I_1 \] (C13)

\[ T_2^\pm = \pm \frac{1 \pm \alpha^\pm}{(1 + \beta)(1 + \alpha^\pm)} \ln \left( \frac{1 \pm \beta}{1 - \beta x_0} \right) \pm \frac{(1 \pm x_0) \ln(1 \pm x_0)}{(1 \pm \beta)(1 - \beta x_0)} + \frac{(1 \pm \alpha^\pm)(1 \pm \beta x_0)}{(1 \pm \beta)(1 + \alpha^\pm)} \ln(1 \pm a^\pm) - \frac{(x_o + a^\pm) \ln(\pm x_o \pm a^\pm)}{(1 + \beta a^\pm)(1 - \beta x_0)} , \]

\[ T_3^\pm = \frac{1}{2\beta} \left[ L \left( \frac{1 - \beta}{1 - \beta x_0} \right) - L \left( \frac{1 + \beta}{1 + \beta} \right) \right] - L \left( \frac{1 - \beta}{1 + \beta} \right) + L \left( \frac{1 + \beta a^\pm}{1 + \beta} \right) \]

\[ + L \left( \frac{1 - \beta x_0}{1 + \beta a^\pm} \right) - L \left( \frac{1 - \beta}{1 - \beta} \right) + \ln \left( \frac{1 + \beta x_0}{1 - \beta} \right) \ln \left( \frac{1 + \beta a^\pm}{1 - \beta} \right) \] ,

\[ T_4^\pm = (x_0 \pm 1) \ln(1 \pm x_0) \pm (1 \pm \alpha^\pm) \ln(1 \pm a^\pm) - (x_0 + a^\pm) \ln(\pm x_0 \pm a^\pm) , \]

\[ T_5^\pm = \frac{1}{2} \left[ (1 - x_0^2) \ln(1 \pm x_0) + (x_0 \mp 1)a^\pm + 1 -(a^\pm)^2 \ln(1 \pm a^\pm) + (x_0^2 - a^\pm)^2 \ln[\pm(x_0 + a^\pm)] \right] \] ,

\[ T_6^\pm = \left( -\beta E + |\vec{p}_2| \pm \frac{\beta E_0^0(x_0 + a^\pm)}{1 + \beta a^\pm} \right) I_4 \pm \frac{\beta E_0^0(x_0 + a^\pm)}{(1 + \beta a^\pm)^2} I_4 + \left( E_0^0 - \frac{\beta E_0^0(x_0 + a^\pm)}{(1 + \beta a^\pm)} \right) J_1 - \frac{\beta E_0^0(x_0 + a^\pm)}{(1 + \beta a^\pm)^2} J_1 \]

\[ \pm \frac{E_0^0(x_0 + a^\pm)}{(1 + \beta a^\pm)^2} J_2 - E_0^0(x_0 + a^\pm) J_2^\pm , \]

\[ T_7^\pm = \left( |\vec{p}_2| - \beta E \mp \frac{\beta E_0^0(x_0 + a^\pm)}{1 + \beta a^\pm} \right) I_1 \mp \frac{E_0^0(x_0 + a^\pm)}{(1 + \beta a^\pm)^2} I_2 + \left( E_0^0 - \frac{\beta E_0^0(x_0 + a^\pm)}{(1 + \beta a^\pm)} \right) J_1 - \frac{E_0^0(x_0 + a^\pm)}{(1 + \beta a^\pm)^2} J_2^\pm , \]

\[ T_8^\pm = -2(\beta E - |\vec{p}_2| + E_0^0(x_0 + a^\pm) \mp E_0^0(x_0 + a^\pm) I_1^\pm - E_0^0(x_0 + a^\pm) J_2^\pm ) , \]
Finally, simplifying Eq. (14) and Eq. (20)

\[ J_1 = T_0 \]

\[ T_0 = \frac{1}{3} (x_0 \mp 1) \ln(1 \mp x_0) + \frac{1}{3} (a^\mp \mp 1) \ln(1 \mp a^\mp) - \frac{1}{3} (x_0 + a^\mp) \ln(\mp(x_0 + a^\mp)) \]

\[ + \frac{1}{6} (1 - x_0^2)(a^\mp + 1) - \frac{1}{3} (x_0 \pm 1)(1 - a^\mp^2), \]

\[ T_{11} = \frac{1}{2} \frac{E_0^0}{|p_2|/\beta} (E_0^0(1 - \beta x_0)) J_4 - J_1 - (\beta E_0^0 + \beta E - |p_2|) I_4 + (\beta E - |p_2| I_1), \]

\[ T_{12} = T_{13} = \frac{1}{2} \frac{E_0^0}{|p_2|/\beta} (E_0^0(1 - \beta x_0)) J_1 + 2 E_0^0 x_0 + 2(\beta E - |p_2|) - (\beta E_0^0 + \beta E - |p_2| I_1), \]

\[ T_{15} = 3 E_0^0(2 |p_2| (1 + y_0) + \beta E(1 - x_0^2)) - E_0^0 x_0 + a^\pm^2 (J_3^\pm + I_3^\pm) - 2 \beta E E_0^0 x_0 + a^\pm a^\pm (J_2^\pm + I_2^\pm), \]

\[ T_{16} = 4 \beta^2 E^2 \left\{ \frac{3}{2 \beta^2} \left( 2(\beta E - |p_2| + E_0^0 x_0) + \beta E_0^0(1 - x_0^2) \right) + \left( 3 \beta E - |p_2| + \beta E_0^0 \right) - |p_2| (1 + y_0) + \frac{|p_2| E_0^0}{2 \beta^2 E^2} \right\} I_1 \]

\[ - \frac{E_0^0}{2 \beta E(1 + a^\pm)} (\beta J_1 + J_2^\pm + \beta I_1 + I_2^\pm) + \left( \frac{3 E_0^0(1 - \beta x_0)}{2 \beta^2} + \frac{E_0^0(\beta E - |p_2|) x_0}{2 \beta^2 E^2} \right) J_1 \} . \]

The \( x_0, a^\pm, I_1, I_2, I_3^\pm, I_4 \), are given in Eq. (C4), Eq. (B4), Eq. (23) and Eqs. (D3). The \( J_1, J_2^\pm, J_3^\pm, J_4 \) and \( G^\pm \) are

\[ J_1 = -\frac{1}{\beta} \ln \left( \frac{1 + \beta}{1 - \beta x_0} \right) + \ln \left( \frac{1 - \beta}{1 - \beta x_0} \right), \quad J_2^\pm = \ln \left( \frac{a^\pm - 1}{a^\pm + x_0} \right) + \ln \left( \frac{a^\pm + 1}{a^\pm + x_0} \right), \quad J_3^\pm = -2 \left( \frac{a^\pm}{a^{\mp 2} - 1} - \frac{1}{a^\pm + x_0} \right), \]

\[ J_4 = \frac{2}{\beta} \left( \frac{1}{1 - a^\pm} - \frac{1}{1 - \beta x_0} \right), \quad G^\pm = \frac{\beta E_0^0 x_0 + a^\pm}{4 \beta^2 E^2(1 + a^\pm)^2} + \frac{a^\pm(a^{\mp 2} - 1)}{4(1 + a^\mp)} \]

Finally, simplifying Eq. (L4) and Eq. (24)

\[ \Phi'_c = \frac{m^2}{E(M_1 - 2E)} \left( 1 - \frac{M_1}{E} \right) I_2 + \ln \left( \frac{M_1}{m} \right) \right), \quad \Phi'_n = \frac{-2m^2}{a} \left( 1 + \frac{2M_2^2}{a} \right) \frac{1}{\beta N} \arctanh(\beta N) + \ln \left( \frac{M_2}{m} \right) \right) \]

**APPENDIX D**

\[ H_0' = \sum_{k=0}^2 \varepsilon_k E_k^2, \]

\[ \frac{\varepsilon_0}{M_1^2} = \left( F_1^2 - G_1^2 + F_1 F_2 + G_1 G_2 + \frac{F_2^2 - G_2^2}{2} \right) \left[ M^0 - E \left( \frac{1}{M_1} \right) \right] \frac{M_2}{2 M_1} - 2 \left( F_1^2 - G_1^2 \right) \left( 1 - \frac{E}{M_1} \right) \frac{M_2}{2 M_1^2} \]

\[ + \left( F_1 + G_1 \right) \left[ \frac{1}{2} M^0 + E \left( 2 \left( 1 - \frac{E}{M_1} \right) + \frac{m^2}{M_1^2} \right) \right] + 2 F_1 G_1 E \left[ 2 \left( \frac{M_2^2}{M_1^2} + \frac{E}{M_1} \right) - M^0 \right] \]
\[
\frac{\varepsilon_1}{M_1} = \left( F_1 F_2 + G_1 G_2 + \frac{F_2^2 - G_2^2}{2} \right) \left( 1 - 2 E \right) M_1 + \left( F_1^2 - G_1^2 \right) M_1 + \left( F_2 F_3 + G_2 G_3 \right) \left( 1 - 2 E \right) M_1^2
\]
\[
+ \left( F_2^2 + G_2^2 \right) \left( 1 - 2 E \right) \frac{E}{M_1} + \left[ \left( F_1 + G_1 \right)^2 + F_1 F_2 - G_1 G_2 \right] \frac{1}{2} \left( M^+ + 2 \left( 1 - 2 \frac{E}{M_1} \right) \right)
\]
\[
+ \left( F_1 F_2 - G_1 G_2 + \frac{F_2^2 + G_2^2}{2} \right) \frac{1}{2} M^- - \left( F_1 F_3 + F_2 F_3 + G_1 G_3 - G_2 G_3 \right) \frac{m^2 M_2}{M_1^3}
\]
\[
+ |F_2 F_3 + G_2 G_3 + 2 (G_1 G_3 - F_1 F_3)| \frac{m^2}{2 M_1^2}.
\]

Terms of \( O(q^2/M_1^2) \) are neglected and

\[
M^- = \frac{m^2 - M_2^2 - M_2}{M_1^2} \quad \text{and} \quad M^+ = \frac{m^2 + M_2^2 + M_2}{M_1^2}.
\]

**APPENDIX E**

In this appendix we describe the integration procedure we follow to obtain the result in Eq. (56), Ref. [17]. The first step is an integration by parts

\[
\int z^n \ln \left| a_0 + b_0 z + \sqrt{z^2 - 1} \right| dz = \frac{z^{n+1}}{n+1} \ln \left| a_0 + b_0 z + \sqrt{z^2 - 1} \right| - \frac{1}{n+1} \int D^{(n)} dz,
\]

where

\[
D^{(n)} = \frac{z^{n+1}}{s} \left( \frac{b_0 s + z}{a_0 + b_0 s + z} \right), \quad s = \sqrt{z^2 - 1}.
\]

After multiplying the numerator and denominator of \( D^{(n)} \) by \( (a_0 + b_0 z - s) \) it becomes

\[
D^{(n)} = z^{n+1} \left( \frac{\frac{a z}{a z^2 + bz + c}}{\frac{\frac{b}{2a}}{\frac{1}{2} \frac{1}{z t - z}} + \frac{z b + \frac{b}{2a}}{z - z b}} + \frac{\frac{a_0 z}{a z^2 + bz + c}}{\frac{\frac{1}{2a}}{\frac{1}{z t - z}} + \frac{z b + \frac{b}{2a}}{z - z b}} \right),
\]

\[
a = b_0^2 - 1, \quad b = 2a_0 b_0, \quad c = a_0^2 + 1. \quad \text{Due to} \quad az^2 + bz + c = a(z - z_t)(z - z_b),
\]

it is reduced to a suitable expression which can be integrated very easily:

\[
D^{(n)} = -\frac{z^{n+1}}{\Delta z} \left( \frac{z + \frac{b}{2a}}{z_t - z} + \frac{z b + \frac{b}{2a}}{z - z b} \right) \left( \frac{1}{s (z t - z)} + \frac{\frac{a_0}{a} z_t + \frac{b_0}{a}}{z_b} \right)
\]

where \( z_t \), \( z_b \) and \( \Delta z \) are defined in Eqs.(40) and (41). \( D^{(n)} \) is simplified to

\[
D^{(n)} = \frac{z^{n+1}}{2} \left[ \frac{1}{z_t - z} - \frac{1}{z - z_b} + \frac{1}{s} \left( \frac{s_t}{(z_t - z)} + \frac{s_b}{(z - z_b)} \right) \right],
\]

The particular results

\[
D^{(0)} = -\frac{1}{2} \left[ 2 + \frac{z_t}{z_t - z} - \frac{z_b}{z - z_b} + \frac{1}{s} \left( -\Delta s + \frac{s_t z_t}{z_t - z} + \frac{s_b z_b}{z - z_b} \right) \right],
\]

\[
D^{(1)} = -\frac{1}{2} \left[ -2z - (z_t + z_b) + \frac{z_t^2}{z_t - z} - \frac{z_b^2}{z - z_b} + \frac{1}{s} \left( -\Delta s + \left( s_b z_b - s_t z_t \right) + \frac{s_t z_t^2}{z_t - z} + \frac{s_b z_b^2}{z - z_b} \right) \right],
\]
\[ D^{(2)} = -\frac{1}{2} \left\{ -2z^2 - z(\zeta_t + \zeta_b) - (\zeta_t^2 + \zeta_b^2) + \frac{z_t^3}{z_t - \zeta_t} - \frac{z_b^3}{z_b - \zeta_b} \right. \\
\left. + \frac{1}{s} \left[ -\Delta_z z^2 + z(s_{bi}z_b - s_t\zeta_t) + (s_{bi}z_b^2 - s_t\zeta_t^2) + \frac{s_t\zeta_t^3}{z_t - \zeta_t} + \frac{s_{bi}z_b^3}{z_b - \zeta_b} \right] \right\}, \]

inspire the following relation:

\[ D^{(n)} = -\frac{1}{2} \sum_{r=0}^{n} \left\{ -(z_t^r + z_b^r) z^{-n-r} - \frac{1}{s} (s_t z_t^r - s_{bi} z_b^r) z^{-n-r} \right\} \\
+ \frac{z_t^{n+1}}{z_t - \zeta_t} - \frac{z_b^{n+1}}{z_b - \zeta_b} + \frac{1}{s} \left( \frac{s_t z_t^{n+1}}{z_t - \zeta_t} + \frac{s_{bi} z_b^{n+1}}{z_b - \zeta_b} \right). \]

Therefore

\[
\int z^n \ln |a_0 + b_0 z + s| \, dz = T_D - \frac{1}{2} \frac{n}{(n+1)} \sum_{r=0}^{n} \left\{ \left( \frac{z_t^r + z_b^r}{(n-r+1)} \right) + (s_t z_t^r - s_{bi} z_b^r) \int \frac{z^{-r-1}}{s} \, dz \right\},
\]

where

\[
T_D = z_t^{n+1} \ln \left| a_0 + b_0 z + \sqrt{z^2 - 1} \right| - \frac{z_b^{n+1}}{2(n+1)} \left[ \ln |z_t - z| + \ln \left| \frac{z_t z - 1 + s_t s_{bi}}{z_t - \zeta_t} \right| \right] \\
- \frac{z_t^{n+1}}{2(n+1)} \left[ \ln |z - z_b| + \frac{s_{bi}}{\sqrt{z_b^2 - 1}} \ln \left| 1 - \frac{z_t z - 1 + s_t s_{bi}}{z_t - z} \right| \right].
\]

To take into account the integration limits we consider that

\[
\sqrt{\frac{z^2 - 1}{z_b^2 - 1}} = \begin{cases} 
\frac{s_{bi}}{s_{bd}} = -s_{bi}, & m < E < E_c, \\
E_c < E < E_m.
\end{cases}
\]

and obtain

\[
a_0 + b_0 z_b + \sqrt{z_b^2 - 1} = \begin{cases} 
2s_{bi}, & m < E < E_c, \\
0, & E_c < E < E_m.
\end{cases}
\]

\[
a_0 + b_0 z_t + \sqrt{z_t^2 - 1} = \begin{cases} 
0, & m < E < E_c, \\
0, & E_c < E < E_m.
\end{cases}
\]

As a consequence of these results for \( m < E < E_c \) and \( E_c < E < E_m \) we consider two regions.

We define

\[
T_D = \begin{cases} 
T_D^{<} & m < E < E_c, \\
T_D^{>} & E_c < E < E_m.
\end{cases}
\]

At the region \( m < E < E_c \) taking the limit \( z \to z_t \)

\[
T_D^{<}(z_t) = \frac{z_t}{2(n+1)} \lim_{z \to z_t} \left\{ \ln |a_0 + b_0 z + \sqrt{z^2 - 1}/z_t z - 1 - s_{bi} s_t| \right\} \\
= \frac{z_t^{n+1}}{2(n+1)} \left[ \ln |1 - z_b z_t - s_{bi} s_t| \right];
\]

for the limit \( z \to z_b \)

\[
T_D^{<}(z_b) = \frac{z_b^{n+1}}{2(n+1)} \ln |a_0 + b_0 z_b + s_{bi}| \\
- \frac{z_b^{n+1}}{2(n+1)} \ln |z_t z_b - 1 - s_t s_{bi}| \\
= \frac{z_b^{n+1}}{2(n+1)} \ln \left| 2s_{bi}^2 - 2s_{bi} \right| \\
- \frac{z_b^{n+1}}{2(n+1)} \ln |z_t z_b - 1 - s_t s_{bi}|.
\]
The following remains:

$$T_D^<(z_t) - T_D^<(z_b) = \frac{z_t^{n+1}}{2(n+1)} \ln |2 (b_0 s_t + z_t)^2 (z_t z_b - 1 - s_t s_{bd})| - \frac{z_b^{n+1}}{2(n+1)} \ln |2 (1 - z_b z_t - s_{bd} s_t)| \cdot$$

Similarly for the second region in which $E_c < E < E_m$

$$T_D^<(z_t) = \frac{z_t^{n+1}}{2(n+1)} \ln \left\{ \ln \left| \frac{(a_0 + b_0 z + \sqrt{z^2 - 1})}{z_t z - 1 - s_t s} \right|^2 \right\} - \frac{z_b^{n+1}}{2(n+1)} \ln |1 - z_b z_t + s_{bd} s_t|$$

and

$$T_D^<(z_b) = \frac{z_b^{n+1}}{2(n+1)} \ln \left\{ \ln \left| \frac{(a_0 + b_0 z + \sqrt{z^2 - 1})}{z_t z - 1 - z_t z_b + s_{bd} s} \right|^2 \right\} - \frac{z_t^{n+1}}{2(n+1)} \ln |z_t z_b - 1 - s_t s_{bd}|$$

Then

$$T_D^<(z_t) - T_D^<(z_b) = \frac{z_t^{n+1}}{2(n+1)} \ln |2 (b_0 s_t + z_t)^2 (z_t z_b - 1 - s_t s_{bd})| - \frac{z_b^{n+1}}{2(n+1)} \ln |(1 - z_b z_t - s_{bd} s_t) (-2) (b_0 s_{bd} + z_b)|$$

Summarizing

$$T_D^<(z_t) - T_D^<(z_b) = \frac{(z_t^{n+1} - z_b^{n+1})}{2(n+1)} \ln \left| \frac{4 (E_m - E)^2}{M^2 \left( 1 - \frac{2E}{M} + \frac{m^2}{M^2} \right)} \right|$$

$$T_D^<(z_t) - T_D^<(z_b) = \frac{(z_t^{n+1} - z_b^{n+1})}{2(n+1)} \ln \left| \frac{4 (E_m - E)^2}{M^2 \left( 1 - \frac{2E}{M} + \frac{m^2}{M^2} \right)} \right| + \ln \left| \frac{M^2 (E_m - E)^2}{M^2 T^2} \right|$$

In general

$$T_D(z_t) - T_D(z_b) = \frac{(z_t^{n+1} - z_b^{n+1})}{2(n+1)} T_R$$

where

$$T_R = \ln \left| \frac{4 (E_m - E)^2}{M^2 \left( 1 - \frac{2E}{M} + \frac{m^2}{M^2} \right)} \right| + 2 \Theta (E - E_c) \ln \left| \frac{M^2 (E_m - E)}{M^2 T} \right|$$

and $\Theta (E - E_c)$ is the Heaviside function.

Finally,

$$R_n(z) = \frac{(z_t^{n+1} - z_b^{n+1})}{2(n+1)} T_R - \frac{1}{2(n+1)} \sum_{r=0}^n \left[ z_t^r z_b^r \left( \frac{z_t^{n-r+1} - z_b^{n-r+1}}{n-r+1} \right) \right] + (s_t z_t^r - s_{bd} z_b^r) \int_{z_b}^{z_t} \frac{z_t^{n-r}}{s} dz.$$

FIG. 1. Radiative corrections for the CHSD in the three and four body regions of the Dalitz plot.

FIG. 2. Electron energy spectrum for the CHSD with all RC contributions and without any.
FIG. 3. Radiative corrections for the NHSD in the three and four body regions of the Dalitz plot.

FIG. 4. Electron energy spectrum for the NHSD with all RC contributions and without any.

TABLE I. Radiative corrections in the four body region of the Dalitz plot.

| x = E/E_m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| % in Ref. [13] | 7.8 | 1.5 | 0.5 | 0.1 | 0.02 |
| % from Eq. (59) | 7.776 | 1.533 | 0.458 | 0.135 | 0.019 |

TABLE II. Radiative corrections in the three body region of the Dalitz plot.

| x = E/E_m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| % in Ref. [13] | 18.3 | 7.1 | 3.6 | 1.4 | -0.2 | -1.7 | -3.2 | -5.0 | -7.5 |
| % from (Eq. (59) + Eq. (62)) | 18.218 | 7.262 | 3.787 | 1.749 | 0.186 | -1.396 | -2.933 | -4.692 | -7.146 |

TABLE III. Total radiative corrections.

| x = E/E_m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| % in Ref. [13] | 18.2 | 7.1 | 3.5 | 1.4 | -0.3 | -1.8 | -3.4 | -5.2 | -7.7 |
| % from (Eq. (59) + Eq. (62)) | 18.335 | 7.413 | 3.884 | 1.773 | 0.115 | -1.396 | -2.933 | -4.692 | -7.146 |

TABLE IV. Radiative corrections in the four body region of the Dalitz plot.

| x = E/E_m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| % in Ref. [13] | 9.5 | 2.3 | 0.8 | 0.25 | 0.02 |
| % from Eq. (59) | 9.277 | 2.200 | 0.765 | 0.242 | 0.024 |

TABLE V. Radiative corrections in the three body region of the Dalitz plot.

| x = E/E_m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| % in Ref. [13] | 9.058 | 5.213 | 3.119 | 1.531 | 0.091 | -1.396 | -2.933 | -4.692 | -7.146 |
| % from Eq. (59) + Eq. (62) | 18.335 | 7.413 | 3.884 | 1.773 | 0.115 | -1.396 | -2.933 | -4.692 | -7.146 |

TABLE VI. Total radiative corrections.

| x = E/E_m | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|-----------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| % in Ref. [13] | 18.2 | 7.1 | 3.5 | 1.4 | -0.3 | -1.8 | -3.4 | -5.2 | -7.7 |
| % from (Eq. (59) + Eq. (62)) | 18.335 | 7.413 | 3.884 | 1.773 | 0.115 | -1.396 | -2.933 | -4.692 | -7.146 |
[1] S.R. Juárez W., Phys. Rev. D 51, 6267 (1995).
[2] S.R. Juárez W., Phys. Rev. D 53, 3746 (1996).
[3] S.R. Juárez W., Phys. Rev. D 55, 2889 (1997).
[4] S.R. Juárez W. and A. Martínez, Rev. Mex. Fís. 35, 13 (1989).
[5] F.E. Low, Phys. Rev. 110, 974 (1958).
[6] H. Chew, Phys. Rev. 123, 377 (1961).
[7] D.M. Tun, S.R. Juárez W., and A. García, Phys. Rev. D 44, 3589 (1991).
[8] S.R. Juárez W., Phys. Rev. D 48, 5233 (1993); 52, 556 (1995).
[9] A. García and S. R. Juárez W., Phys. Rev. D 22, 1132 (1980); 22, 2923(E) (1980).
[10] D.M. Tun, S.R. Juárez W., and A. García, Phys. Rev. D 40, 2967 (1989).
[11] S.R. Juárez W. and F. Guzmán, Rev. Mex. Fís. 46, 29 (2000).
[12] F. Guzmán A., M.S. thesis, Escuela Superior de Física y Matemáticas, IPN (1998).
[13] F. Glück and K. Tóth, Phys. Rev. D 41, 2160 (1990), K. Tóth et al., ibid. D 33, 3306 (1986).
[14] F. Glück and I. Joó, Comput. Phys. Commun. 107, 92 (1997); 95, 111 (1996).
[15] J. Martínez C., M.S. thesis, Escuela Superior de Física y Matemáticas, IPN (1997); A. León H., M.S. thesis, Escuela Superior de Física y Matemáticas, IPN (1999).
Electron energy in MeV.

SIGMA -> NEUTRON

TBR
FBR
SIGMA -> NEUTRON

Electron energy in MeV.

WITH-R.C. ◊
WITHOUT-R.C. +
Electron energy in MeV.
Electron energy in MeV.

LAMBDA -> PROTON

WITH-R.C.  
WITHOUT-R.C.