Exactly solvable models play an important role as limiting cases of more complex system or for testing numerical algorithms. Moreover their physical properties can generally be calculated exactly and traced back to simple mechanism that can be used in more complicated scenarios. In this article we consider two notable solvable models: the anisotropic XY model—originally introduced in [1] with the aim of gaining insights on the long range properties of the Heisenberg model—and the dimerized XX model—used sometimes as a prototype model to describe spin-Peierls distortion—. We prove the equivalence of these two models, despite in the literature they are generally considered as separate. The equivalence is shown directly by means of a unitary transformation for their fermionic counterparts and traced back to the spin models, carefully taking care of the boundary conditions.

Introduction

For a chain of length $L$ the dimerized XX and anisotropic XY models are given by the following Hamiltonians

$$H^d = \frac{1}{2} \sum_{i=1}^{L} \left[ \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y \right]$$

(1)

$$H_{XY}^\eta = \frac{1}{2} \sum_{i=1}^{L} \left[ (1+\gamma) \sigma_i^x \sigma_{i+1}^x + (1-\gamma) \sigma_i^y \sigma_{i+1}^y \right].$$

(2)

The superscript $\eta$ denotes different kind of boundary conditions (BCs), $\eta^{x,y} = \eta^{x,y}, \eta = 1,-1,0$, corresponding to periodic (PBC), antiperiodic (ABC), and open (OBC) respectively. Since both Hamiltonians Eqs. (1) and (2) commute with the parity operator $P = \prod_i \sigma_i^z$, we define the parity sectors $\sigma = \pm 1$ and the corresponding projection operators $\Pi_\sigma = (\mathbb{I} + \sigma P)/2$. The central result of this article establishes that $H^d$ and $H_{XY}$ are unitarily equivalent ($\equiv$) up to at most a border term as precisely stated by the following

Theorem 1. For $L$ odd and OBC the models (1) and (2) are unitarily equivalent. For $L$ even and PBC or ABC the equivalence holds in given parity blocks depending on the boundary conditions, according to the relation:

$$\Pi_\sigma H^\eta_d \Pi_\sigma \equiv \Pi_{\eta(-1)^L/2} H_{XY}^{\alpha(-1)^L/2} \Pi_{\eta(-1)^L/2},$$

In other words, for $L$ even, the boundary index in one model sets the parity sector in the other (times a modulation factor $(-1)^{L/2}$), i.e., $\sigma_{XY} = (-1)^{L/2} \eta_d$ and $\sigma_d = (-1)^{L/2} \eta_{XY}$. An immediate consequence of this result is that the two models share the same thermodynamics, since for $L \to \infty$ the effect of boundary terms disappear.

The reason for considering OBC is partly due to the possibility of using models Eqs. (1) and (2) to implement quantum information devices. It has been shown in [2] that in the ground state of the dimer model (though with OBC and $L$ even), the end spins tend to entangle considerably already for small values of the dimerization $\gamma$. Moreover the entanglement survives in the infinite length limit (long distance entanglement). In a similar fashion, it was already observed in [1] that the end-spins of the anisotropic model order and such order survives in the thermodynamic limit (TDL). However this kind of order is of classical nature and no entanglement is present between the end-spins of the open anisotropic chain [6].

Before proceeding to examine the proof of the Theorem, let us spend few words on some benefits of such result. First, let us note that both models commute with $\pi$-rotations around axis $x$ and $y$.

$$\mathcal{R}_\alpha = \prod_i e^{i \pi \sigma_i^\alpha / 2}, \quad \alpha = x, y.$$  

However, the dimer model $H^d$ manifests a much larger symmetry, the total magnetization $M^z = \sum_i \sigma_i^z$. This means that $H^d$ is block diagonal in sectors with given magnetization $M^z$, a feature which is especially useful in case of non-integrable extensions of $H^d$ (which maintain this symmetry) where one has to resort to numerical diagonalization. Thanks to Theorem 1 such a symmetry (or an approximate one) must exist also for the anisotropic model $H_{XY}$. As we will see the magnetization in the dimer model is mapped onto a non-local operator which we are able to compute. Clearly this operator has the same spectrum of $M^z$ and commutes with $H_{XY}$.

The proof of Theorem 1 relies on a similar theorem holding for the fermionic version of the models (denoted here with a tilde),

$$\tilde{H}^d = \sum_{i=1}^{L} \left[ 1 + \gamma (-1)^i \right] d_i^\dagger d_{i+1} + d_{i+1}^\dagger d_i$$

(3)

$$\tilde{H}_{XY} = \sum_{i=1}^{L} \left[ a_i^\dagger a_{i+1} + \gamma a_i^\dagger a_{i+1}^\dagger \right] + \text{h.c.}.$$  

(4)
Here $\epsilon = 1, -1, 0$ distinguishes among PBC, ABC, and OBC for the fermions, i.e. $d_{l+1} = \epsilon d_l$ and $a_{L+1} = \epsilon a_1$. As we will see later, the spin systems (1) and (2) are connected to the quadratic fermionic models $\hat{H}_d$ and $\hat{H}_{XY}$ via a Jordan-Wigner (JW) transformation, after careful reshuffling of the boundary conditions. The result for the fermionic models is

**Theorem 2.** In the following cases: L even and PBC or ABC, L odd and OBC, the models (3) and (4) are unitarily equivalent, i.e. there exists a unitary operator $U$ (a “mapping”) such that $U \hat{H}_d U^\dagger = \hat{H}_{XY}$.

A simple way to remind the different cases in which the theorem applies is given by the following argument. Sending $a_j \rightarrow ia_j$ in $\hat{H}_{XY}$, one realizes that the spectrum of $\hat{H}_{XY}$ is invariant under the transformation $\gamma \rightarrow -\gamma$. By relabeling the sites of the dimer model, one sees that $\hat{H}_d$ possess the same invariance only when it contains an even number of bonds. This occurs for L even in case of PBC or ABC, while for L odd only in case of OBC.

**Proof of Theorem 2** Since the fermionic Hamiltonians are quadratic, one way of proving the equivalence between them is to show that they have the same one-body spectrum. To diagonalize the anisotropic model we rewrite the Hamiltonians following the conventions of [1]:

$$\hat{H}_{XY} = \sum_{i,j} a_i^\dagger A_{ij} a_j + (1/2) \left[ \sum_{i,j} a_i^\dagger B_{ij} a_j^\dagger \right] + h.c.$$ and

$$\hat{H}_d = \sum_{i,j} d_i^\dagger M_{ij} d_j$$

with matrices given by

$$A = \begin{pmatrix} 0 & 1 & \epsilon \\ 1 & 0 & \cdots \\ \epsilon & \cdots & 1 \end{pmatrix}, \quad B = \gamma \begin{pmatrix} 0 & 1 & \epsilon \\ -1 & 0 & \cdots \\ -\epsilon & \cdots & -1 \end{pmatrix},$$

while $M$ is

$$\begin{pmatrix} 0 & 1 - \gamma & \cdots & \epsilon [1 + (-1)^L \gamma] \\ 1 - \gamma & 0 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \epsilon [1 + (-1)^L \gamma] & \cdots & 1 - (-1)^L \gamma & 0 \end{pmatrix}.$$ The one particle energies of $\hat{H}_{XY}$ are given by the (positive) square root of the eigenvalues of $(A - B)(A + B)$. Calling $\Lambda_k$ such roots, since $A$ is traceless, one arrives at [1]

$$\hat{H}_{XY} = \sum_k \Lambda_k n_k^\dagger n_k - \frac{1}{2} \sum_k \Lambda_k.$$

The equivalence of the two models now stems from the fact that, for L even and PBC or ABC, and for L odd and OBC $M^2 = (A - B)(A + B)$. Moreover, under the same hypothesis, the eigenvalues of $M$ are symmetric around zero (for L odd and OBC there is one zero eigenvalue). To write $\hat{H}_d$ in the same form as $\hat{H}_{XY}$, perform a particle-hole transformation on the negative eigenvalues of $M$. We arrive then at

$$\hat{H}_d = \sum_k \Lambda_k \beta_k \beta_k^\dagger - \sum_{\text{neg}} \Lambda_k,$$ where $\sum_{\text{neg}} \Lambda_k$ is the sum over the negative eigenvalues of $M$. To complete the proof note that, in the specified cases, $\sum_{\text{neg}} \Lambda_k = (1/2) \sum_k \Lambda_k$. □

**The mapping** The above proof does not give the explicit form of the mapping. We will now provide a physically more compelling proof which has the additional advantage of revealing an exact form of the mapping. For simplicity we will stick to L even and PBC/ABC for the fermionic models. The first step is to write both models in Fourier space

$$\tilde{\hat{H}}_d = \sum_k \left[ 2 \cos(k) d_k^\dagger d_k + 2i \gamma \sin(k) d_k^\dagger d_{k+\pi} \right]$$

$$\tilde{\hat{H}}_{XY} = \sum_k \left\{ 2 \cos(k) a_k^\dagger \right. \left[ i \sin(k) a_k^\dagger a_{-k} - i \sin(k) a_{-k} a_{k} \right\}$$

Let us consider first PBC. The momenta in the Brillouine zone (BZ) are given by $k = 2\pi \eta/L, \eta = -L/2 + 1, \ldots, L/2$. Note that only for PBC and ABC if $k \in \text{BZ}$ then $-k \in \text{BZ}$. Moreover only for L even $k \in \text{BZ} \Rightarrow k + \pi \in \text{BZ}$. In particular Eqs. (5) and (6) are not correct if L is odd. The unitary transformation that maps the dimer model onto the XY is

$$d_k^\dagger = \begin{cases} a_{-k-\pi} & -\pi < k < 0 \\ a_k & 0 \leq k \leq \pi, \end{cases}$$

Notice that the particle hole transformation does not involve neither $k = 0$ nor $k = \pi$. In fact, for these two momenta, the dimer model is given by $2(\delta_{00} - d_0^\dagger d_0)$ and the anisotropic one by $2(\delta_{00} a_0^\dagger a_0 \mp \pi)$. The same mapping Eq. (7) transforms $\tilde{\hat{H}}_d$ into $\tilde{\hat{H}}_{XY}$ also in the case of ABC where the momenta satisfy $k = \pi/L (2n-1), n = -L/2 + 1, \ldots, L/2$.

The mapping Eq. (7) can be written in a compact form as $d_k^\dagger = f_+ (k) a_k^\dagger + f_- (k) a_{-k-\pi}$ with the help of two auxiliary functions $f_\pm (k) : = \theta (\pm \sin(k)) \pm \delta_{\sin(k), 0}/2$, where $\theta$ is the Heaviside function with the convention $\theta (0) = 1/2$.

Thanks to Eq. (7) the equivalence between (fermionic) dimer and anisotropic models can be generalized. In fact, the mapping transforms an $r$-nearest neighbor hopping term into itself, provided $r$ is odd. Instead, an alternating hopping of the form $\sum_i (-1)^i d_i^\dagger d_{i+r} + h.c.$ becomes $\sum_i (a_i^\dagger a_{r+i} + a_{r+i} a_{r})$, again for $r$ odd. When $r$ is even the mapping introduces non-analitics in Fourier space and correspondingly the transformed model becomes long-ranged in real space. These findings can also be obtained directly in real space Fourier transforming back Eq. (7):

$$d_m^\dagger = \sum_k \left[ f_+ (m-x) a_k^\dagger + (-1)^r f_- (m-x) a_k \right],$$

with the definition $f_\pm (x) = L^{-1} \sum_k e^{-ikx} f_\pm (k)$. Writing simply $f_\pm$ in place of the matrix $(f_\pm)_{i,j}$ : $= f_\pm (i-j)$ the following relations hold: $f_\pm f_\mp = f_\pm, f_\pm^\dagger f_- = f_-^\dagger f_\pm = 0$.

**Proof of Theorem 1** The first step is to map the spin models Eqs. (1) and (2) to fermionic models via the JW transformation. In terms of ladder operator $\sigma_i^\pm = (\sigma_i^x \pm i \sigma_i^y)/2$, the JW is given by $\sigma_i^\pm = c_i^\dagger e^{i\phi} \sum_{j=1}^{N_c} c_j$ (this in turn implies $\sigma_i^x = c_i^\dagger e^{-i\pi} \sum_{j=1}^{N_c} c_j$, $\sigma_i^y = 2c_i^\dagger c_i - \mathbb{1}$). The dimer and...
anisotropic boundary terms become respectively

\begin{align*}
H_d^\eta &\rightarrow - \eta_d \left(1 + \gamma (-1)^L\right) \left[d_x^d d_x^d + d_y^d d_y^d\right] e^{i\pi N_d} \\
H_{XY}^\eta &\rightarrow - \eta_a \left[\gamma a_L^d a_L^d + 1\right] e^{i\pi N_a} + \text{h.c.},
\end{align*}

where \(N_{d(a)}\) is the total number operator for the \((a)\) fermions and \(\eta_{d(XY)}\) specifies the spin BC for the dimer and XY model. For OBC, \(\eta_d = \eta_{XY} = 0\), we can directly apply the result of Theorem 2 and deduce that also the spin models are unitarily equivalent for \(L\) odd. To study the remaining cases we first need to compute \(\exp (i\pi N_d)\) under the action of the mapping Eq. (7). Writing the number operator in Fourier space we get \(N_d = \sum_{0 \leq k \leq \pi} a_k^d a_k^d + \sum_{-\pi < k < 0} a_k^d a_k^d\). The sum over negative momenta contains a different number of terms depending on the boundary conditions. For PBC the sum contains \(L/2 - 1\) terms while for ABC it contains \(L/2\) terms. Calling \(N_{d+} = \sum_{0 \leq k \leq \pi} a_k^d a_k^d\) and \(N_{d-} = \sum_{-\pi < k < 0} a_k^d a_k^d\) we can write compactly \(N_d = N_{d+} - N_{d-} + L/2 - (1 + \epsilon)/2\), where \(\epsilon = \pm 1\) defines the boundary conditions of the fermions. Since \(N_{d+}\) are integers, under the action of the mapping we obtain \(\exp (i\pi N_d) = -e^{-\epsilon L/2} \exp (i\pi N_a)\). Let us now consider the spin models \(H_d^\eta\) (\(H_{XY}^\eta\)) in the parity sector \(\sigma_d (\sigma_{XY})\). Thanks to the JW transformation, for \(L\) even, the parity operator is \(P = e^{i\pi N_{d(a)}}\) and in each sector \(e^{i\pi N_{d(a)}} = \sigma_{d(XY)}\). This means, that in the parity sector \(\sigma_d\) the spin model with BC \(\eta_d\) has boundary conditions \(-\eta_d\sigma_d\) in the fermions. The same clearly holds for the anisotropic XY model. The equivalence of the fermionic models (Theorem 2) holds when they have the same BCs, that we denote with \(\epsilon\). So, we arrive at the relation \(\eta_d \sigma_d = \eta_d \sigma_a = -\epsilon\). Now we use the mapping of \(\exp (i\pi N_d)\) = \(\sigma_d\), obtaining \(\sigma_d = -e^{-\epsilon L/2} \sigma_{XY}\). Solving these last two equations, we finally obtain the parity sectors and boundary conditions, under which the equivalence of the spin models apply: \(\sigma_{XY} = \eta_d (-1)^{L/2}\) and \(\eta_{XY} = \sigma_d (-1)^{L/2}\).

In the above proof we have partly seen what happens to the conserved quantity \(N_d\) after the action of the mapping. The precise form also depends on the boundary conditions \(\epsilon\). In Fourier space we can write \(N_d = \sum_k [f_+(k) - f_-(k)] a_k^d a_k^d + L/2 - (1 + \epsilon)/2\). Since the functions \(f_+ (k)\) and \(f_- (k)\) are analytic, and using \(f_+ (k) + f_- (k) = 1, \forall k\), the number operator becomes non local in real space. For example its explicit form for PBC (\(\epsilon = 1\)) is

\[N_d = \frac{2}{L} N_a + \frac{L}{2} - 1 + \sum_{x \neq y} (i)^x-y \frac{2 \sin \left[\left(x-y \left(\frac{\pi}{L} + \frac{\pi}{2}\right)\right]}{\sin \left[\left(x-y \frac{\pi}{L}\right)\right]} a_x^d a_y^d.\]

A corollary of our proof is that such operator commutes with the anisotropic Hamiltonian Eq. (6) (\(\epsilon = 1\)) and its spectrum is made of integers from zero to \(L\).

**Majorana fermions** The possibility of mapping the dimer model into an anisotropic one is not restricted to the mapping Eq. (7). Another mapping is obtained directly in real space by introducing Majorana fermions \(\zeta_\alpha (j)\), \(\alpha = 1, 2\),

\[
\left(\begin{array}{c}
\zeta_1 (j) \\
\zeta_2 (j)
\end{array}\right) = \frac{1}{\sqrt{2}} \left(\begin{array}{c}
1 \\
-i
\end{array}\right) \left(\begin{array}{c}
a_j^+ \\
a_j
\end{array}\right),
\]

that satisfy the commutation relations \(\{\zeta_\alpha (j), \zeta_\beta (j')\} = \delta_{\alpha \beta} \delta_{jj'}\). This way, it is possible to show that each model gets transformed onto two separate Ising chains in transverse field each consisting of \(L/2\) sites. Then, assuming \(L\) even and PBC or ABC, the two pairs of Ising chains are made identical by translating by one site one of the two chains obtained from \(H_{XY}\). The composition of all these steps yields to the following mapping

\[d_j^\eta = \frac{1}{2} \left[i a_{j+1}^d + a_j^d - (-1)^j (a_{j+1}^d + a_j^d)\right].\]

The transformation above has the advantage of being local in real space and much simpler than Eq. (8). By using Eq. (10) one can reproduce the results of Theorem 2 for PBC or ABC. However Eq. (10) is more powerful in view of its applications to more general local Fermi models. Using the mapping Eq. (10) the “disordered” tight binding model

\[H_d = \sum_{j=1}^L J_j d_j d_{j+1} + \text{h.c.},\]

with arbitrary hopping rate \(J_j\) can be mapped onto the generalized anisotropic model

\[H_{XY} = \sum_{j=1}^L \left[ J_j^+ a_j^d a_{j+1} + J_j^- a_j^d a_{j+1}^\dagger \right] + \text{h.c.},\]

with \(J_j^+ = (\pm)^j (J_j \pm J_j-1)/2\) and \(J_j^\pm\) must always be considered periodic i.e. \(J_{L+1} = J_1\). This mapping can be further generalized by adding a uniform and staggered chemical potential. After applying the transformation Eq. (10) such terms become

\[
\sum_{j=1}^L \left[ \mu + \mu_{st} (-1)^j \right] d_j^\eta d_j - \mu L/2 =
\]

\[
-\frac{L}{2} \sum_{j=1}^L \left[ \mu + \mu_{st} (-1)^j \right] \left[ a_j^d a_{j+1} + (-1)^j a_j^d a_{j+1}^\dagger \right] + \text{h.c.}
\]

The equivalence between the generalized models Eqs. (11) and (12) has potential applications in the study of disordered systems. To obtain results on the random version of the anisotropic model Eq. (12), it can be favorable to simulate Hamiltonian Eq. (11) which conserves the number of excitations. Moreover, through the JW transformation, apart from a possible border term depending on the BCs, the equivalence between Fermi models can be extended to their spin counterpart. In this way a random XY model can be mapped into a random XX model.
Continuum limit The mappings that we have analyzed so far admit a simple interpretation in the continuum limit. To this end we expand the fermionic fields into chiral components $\psi(x) = e^{i k_F x} R(x) + e^{-i k_F x} L(x)$. For $\gamma = 0$ the two models merge in free massless fermions: $H_0 = \sum_j a_j^\dagger a_{j+1}^\dagger + h.c.$ where the band is half filled, so $k_F = \pi/2$. In the continuum limit, we get \[ H_0 = \int_0^L dx \left[ : R^\dagger(x) \partial_x R(x) - L^\dagger(x) \partial_x L(x) : \right] \] (13)
while the mass-generating terms in $\gamma$, $O_{XY} = a_j^\dagger a_{j+1}^\dagger + h.c.$ and $O_d = (-1)^j a_j^\dagger a_{j+1} + h.c.$ become
\[ O_{XY} = i : L^\dagger(x) R^\dagger(x) - R(x) L(x) : \]
\[ O_d = i : L^\dagger(x) R(x) - R^\dagger(x) L(x) : \]
From these expressions, we see directly that the terms multiplied by $\gamma$ in $H_d$ and $H_{XY}$ are transformed into each other by particle-hole exchange (and a minus sign) on the left movers, $L \rightarrow -L^\dagger$, which is reminiscent of the discrete mapping Eq. (7) where the particle hole transformation was also applied only for negative momenta.

Translating into bosonic language, it is known that the model Eq. (13) is equivalent to the Gaussian model
\[ H_0 = \frac{1}{2} \int dx \left\{ \partial_x \Theta(x)^2 + \partial_x \Phi(x)^2 \right\} \] (14)

The fields $\Phi$ and $\Theta$ are bosonic and reciprocally dual: $\partial_x \Phi = \partial_x \Theta$ and $\partial_x \Phi = \partial_x \Theta$. A nonvanishing value of $\gamma$ has the effect of transforming Eq. (14) in the sine-Gordon model by adding a relevant (in the renormalization group sense) term $O_{XY} = : \sin (\sqrt{4 \pi} \Theta(x)) :$ or $O_d = : \sin (\sqrt{4 \pi} \Phi(x)) :$, respectively in the XY or in the dimer case. Hence, in the bosonic language, the dimer$\leftrightarrow$XY mapping simply acts by swapping $\Phi \leftrightarrow \Theta$. It is interesting to observe that a direct consequence of the mapping is the interchange between density and current density, as it can be readily inferred by their expressions
\[ \rho(x) = : R^\dagger(x) R(x) + L^\dagger(x) L(x) : = -\frac{1}{\sqrt{\pi}} \partial_x \Phi(x) \]
\[ j(x) = : R^\dagger(x) R(x) - L^\dagger(x) L(x) : = \frac{1}{\sqrt{\pi}} \partial_x \Theta(x). \]

Integrating these densities over the space we obtain two quantum numbers: the total number and the current. In particular, the total number and current are directly related to the two winding numbers $m, n \in \mathbb{Z}$ of respectively $\Theta$ and $\Phi$. Such integers (which determine the scaling dimensions of the primary operators in the Gaussian model) are both good quantum numbers for $\gamma = 0$. For $\gamma \neq 0$, the breaking of translational symmetry in the dimer chain invalidates the conservation of the current, but maintains the particle number conservation. In the XY model the situation is just reversed: the particle number is no more conserved, due to the pair creation-destruction terms, while the current keeps being a good quantum number. Higher dimensions The mapping described in Eq. (7) can be easily generalized to $D$-dimension. In an hypercubic $D$-dimensional lattice the anisotropic model reads
\[ \hat{H}_{XY} = \sum_{i=1}^D \sum_{\mathbf{x}} \left( a_{\mathbf{x}+e_i}^\dagger a_{\mathbf{x}}^\dagger + \gamma a_{\mathbf{x}} a_{\mathbf{x}+e_i}^\dagger \right) + h.c., \] (15)
where $\mathbf{x} = (x_1, \ldots, x_D)$ and $e_i$ is the unit vector along the $i$-th direction. We do not specify BCs here, to fix ideas we can take PBC on a bipartite lattice. After Fourier transforming one realizes that the BZ is contained in $[-\pi, \pi]^D$. Now, let us divide the BZ in two regions according to the sign of the first moment $k_1: A = \{ k \in BZ, : k_1 \in [0, \pi) \}$ and $B = \{ k \in BZ, : k_1 \in (-\pi, 0) \}$. The canonical transformation $a_{\mathbf{k}}^\dagger = d_{\mathbf{k}}^\dagger$ for $k \in A$ and $a_{\mathbf{k}}^\dagger = d_{\mathbf{k}+\pi}^\dagger$ for $k \in B$ with $\pi = (\pi, \pi, \ldots, \pi)$, generalizes the one-dimensional version Eq. (7). This mapping transforms the Hamiltonian Eq. (15) into the $D$-dimensional dimer model:
\[ \hat{H}_{XY} = \sum_{i=1}^D \sum_{\mathbf{k}} \left[ 2 \cos(\mathbf{k} \cdot \mathbf{e}_i) d_{\mathbf{k}}^\dagger d_{\mathbf{k}} \right. \]
\[ \left. + (i \gamma \sin(\mathbf{k} \cdot \mathbf{e}_i) d_{\mathbf{k}}^\dagger d_{\mathbf{k}+\pi} + h.c.) \right] \]
\[ = \sum_{i=1}^D \sum_{\mathbf{x}} \left( 1 + \gamma (-1)^{|\mathbf{x}|} \right) d_{\mathbf{x}}^\dagger d_{\mathbf{x}+e_i} + h.c., \]
where the modulation factor is given by $(-1)^{|\mathbf{x}|} = \exp (i \pi \cdot \mathbf{x})$.

Conclusions In this paper we have analyzed two common spin models (XY and dimerized XX) and showed that they are unitary equivalent apart from at most a border term. By explicitly providing the unitary transformation we have been able to generalize the equivalence in many ways. For example the fully disordered (with site dependent couplings) XY chain can be mapped onto a disordered XX chain. Considering the fermionic counterpart we have also shown that generally a dimerized, $r$-nearest neighbor, hopping term, is mapped onto an $r$-nearest neighbor pair creation term. In one dimension our mappings have a simple interpretation in the continuum limit in terms of bosonic fields. Similar considerations can also be extended to higher dimensions.

Mapping XY models onto XX ones can be useful in view of numerical simulations of disordered models or non-integrable extensions. This is due to the explicit particle number conservation of the XX models which makes them easier to treat numerically. A by-product of our analysis is that particle number symmetry is also present in the XY models although in a hidden fashion.

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[6] For instance, for $\gamma > 0$ the only non-zero correlation surviving in the TDL is $\langle \sigma^z_1 \sigma^z_L \rangle$. Such order is clearly classical.