The obstacle problem for stochastic porous media equations

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November 23, 2021

Abstract

We prove the existence and uniqueness of non-negative entropy solutions of the obstacle problem for stochastic porous media equations. The core of the method is to combine the entropy formulation with the penalization method.

1 Introduction

Consider the following obstacle problem with an upper obstacle $S$:

\[
\begin{aligned}
\frac{du}{dt} &= \left[ \Delta \Phi(u) + F(t, x, u) \right]dt - \nu(dt, x) \\
&\quad + \sum_{k=1}^{\infty} \sigma_k(u) dW^k_t, \quad (t, x) \in (0, T) \times \mathbb{T}^d; \\
u(t, x) &\leq S(t), \quad dP \otimes dt \otimes dx - \text{a.e.}; \\
&\quad u(0, x) = \xi(x), \quad x \in \mathbb{T}^d; \\
\int_{Q_T} (u - S) \nu(dtdx) &= 0, \quad \text{a.s. } \omega \in \Omega,
\end{aligned}
\]

where $\mathbb{T}^d$ is $d$-dimensional torus, and $\{W^k\}_{k \in \mathbb{N}^+}$ is a sequence of independent Brownian motions. $\Phi$ is a monotone function, and a typical type is $\Phi(u) = |u|^{m-1}u$ with $m > 1$. The solution of (1) is a pair $(u, \nu)$.

The initial physical model of this work is fluid flow in a container with a limitation on the density of the fluid. That is, the least amount of the fluid will be pumped out of the container, which makes sure that the density of the fluid is lower than the limitation $S$.

Porous media equations arise in the flow of an ideal gas through a homogeneous porous medium, and the solution $u$ is the scaled density of the gas [37]. These equations have applications in various fields, such as population dynamics [24] and the theory of ionized gases at high temperature [50]. Since there are quite a lot of studies on these equations, we only introduce relevant works, and other results can be found in [12, 46, 4] and references therein.

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By transforming into a porous media equation with random coefficients, [32, 5] proved the existence and uniqueness result for the equation with linear multiplicative noise. In [44, 2, 3, 6], with the monotone operator method [39, 31, 42, 46], they obtained the well-posedness under the condition that the diffusion \(\sigma^k\) is Lipschitz continuous in \(H^{-1}\). Under the condition \(m > 2\) and the Lipschitz continuity of \(\sigma^k\), [8] used an entropy formulation to prove the well-posedness, and [23] gave the existence and uniqueness of the kinetic solution of the stochastic porous media equations. When \(m > 1\) and \(\sigma^k\) has linear growth and locally \(1/2\)-Hölder continuity, [13] obtained the existence and uniqueness of the entropy solution on torus with a probabilistic approach. Using a weighted \(L^1\)-norm, [15] extended the results to the bounded domain.

Obstacle problems for deterministic partial differential equations have been studied extensively in the early stage using variational inequality (see [36] and references therein). [28, 7, 9, 10, 11, 45] generalized to the porous media equations. Avelin [1] proposed the potential theory for porous media equation, and proved that the smallest supersolution is also a variational weak solution. [29] proved the existence of supersolutions under weakened conditions on the obstacle.

Haussmann and Pardoux [26] firstly studied the obstacle problem for stochastic heat equation on the interval \([0, 1]\) by stochastic variational inequalities. Nualart and Pardoux [38] gave the existence of solution of the heat equation driven by the space-time white noise using the penalization method, while [18, 47] proved for general diffusion term. However, these works only considered the special obstacle \(S \equiv 0\). Yang and Tang [49] used the penalization method on the backward equation with two obstacles. In order to deal with general obstacle, [16, 48, 19] studied the quasilinear equations using the parabolic potential theory [40, 41]. Qiu [43] expanded to backward stochastic partial differential equations. [35, 34, 33, 27, 20, 17] applied the method of probabilistic interpretation of the solution using backward doubly stochastic differential equation. It is worth noting that the probabilistic interpretation method is still feasible for nonlinear stochastic partial differential equations.

Our objective is to study the well-posedness of non-negative solution of the obstacle problem for stochastic porous media equations. A major technical difficulty encountered is that we cannot directly apply Itô’s formula on the entropy solution \(u_\epsilon\) of the penalized equation (see (7)), which is necessary to a priori estimates of both \(u_\epsilon\) and the penalty term. To overcome this difficulty, we merge the penalization method with the \(L^1\) technique of stochastic porous media equations. We follow [13] to approximate \(\Phi\) with \(\Phi_n\) in the penalized equation, which is nondegenerate and thus has a unique \(L_2\)-solution \(u_{n, \epsilon}\) (see Theorem 3.7). Furthermore, the \(L_2\) norm of the penalty term can be estimated if the the difference \(u_{n, \epsilon} - S\) has bounded variation when \(u_{n, \epsilon} = S\). This estimate ensures the existence of the weak limit \(\nu\) in the entropy solution (see Definition 2.10). Moreover, using \(L_1\) technique, the existence of \(u\) comes from the limit of \(u_{n, \epsilon}\). The uniqueness of the entropy solution \((u, \nu)\) is derived from a direct \(L_1\) estimate as in [13]. To our best knowledge, this is the first study to the obstacle problem under the entropy formulation of degenerate stochastic partial differential equations.

This paper is organized as follows. Section 2 states the main theorem after introducing notations and assumptions to formulate the entropy solution. We also prove the non-negativity of the entropy solution. In Section 3, we approximate the equation by non-degenerate ones, and obtain the well-posedness of \(L_2\)-solution \(u_{n, \epsilon}\) to the penalized equations. A priori estimates for both \(u_{n, \epsilon}\) and penalty term are derived. In Section 4, we introduce some Lemma and prove the \((\ast)\)-property of \(u_{n, \epsilon}\). Then the \(L_1^+\) estimates are given for two different entropy solutions in Section 5. In Section 6, we pass to the limit \(n \to \infty\) and then \(\epsilon \to 0^+\) to acquire the existence of the entropy solution \((u, \nu)\). To prove the uniqueness, we give another \(L_1\) estimate which can reduce the limitation on \((\ast)\)-property.
2 Entropy formulation

We firstly introduce some notations and settings. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})\) be a complete filtered probability space and \(\mathcal{P}\) be the predictable \(\sigma\)-algebra generated by \(\{\mathcal{F}_t\}\). The noise \(W = \{W^k_t : t \in [0, \infty), \ k \in \mathbb{N}^+\}\) is a sequence of independent \(\mathcal{F}_t\)-adapted Wiener processes on \(\Omega\). For fixed \(T > 0\), denote \(Q_T := [0, T] \times \mathbb{T}^d\). \(L_p\) and \(H^s\) are the usual Lebesgue and Sobolev space with \(p \geq 2\) and \(s > 0\). When \(p = 2\), we simplify \(H^s\) as \(H^s\) (cf. [21]). Given the obstacle \(S\), the obstacle problem denoted by \(\Pi_S(\Phi, F, \xi)\) is to seek a pair \((u, \nu)\) such that

\[
\begin{cases}
  du = [\Delta \Phi(u) + F(t, x, u)]dt - \nu(dt, x) \\
  \hspace{0.5cm} + \sum_{k=1}^{\infty} \sigma^k(u) dW^k_t, & (t, x) \in (0, T) \times \mathbb{T}^d; \\
  u(t, x) \leq S(t), & \text{d}\mathbb{P} \otimes dt \otimes dx \text{ a.e.}; \\
  u(0, x) = \xi(x), & x \in \mathbb{T}^d; \\
  \int_{Q_T} (u - S) \nu(dt, dx) = 0, & \text{a.s. } \omega \in \Omega.
\end{cases}
\]

The nonlinear function \(\Phi\) is of porous media type. The measure \(\nu\) is introduced to ensure that \(u(t, x) \leq S(t)\), and the last condition is the so-called Skohorod condition which requires that the force \(\nu\) we apply to the equation is “minimal”.

We denote by \(\Pi(\Phi, F, \xi)\) the following stochastic porous media equation:

\[
\begin{cases}
  du = [\Delta \Phi(u) + F(t, x, u)]dt + \sum_{k=1}^{\infty} \sigma^k(u) dW^k_t, & (t, x) \in (0, T) \times \mathbb{T}^d; \\
  u(0, x) = \xi(x), & x \in \mathbb{T}^d.
\end{cases}
\]

The well-posedness of the entropy solution of \(\Pi(\Phi, 0, \xi)\) is available in [13].

Define \(Q_T := [0, T] \times \mathbb{T}^d\). Given a smooth function \(\rho : \mathbb{R} \to [0, 2]\), which is supported in \((0, 1)\) and integrates to 1. For \(\theta > 0\), we set \(\rho_\theta(r) := \theta^{-1} \rho(\theta^{-1} r)\) as a sequence of mollifiers. For any function \(g : \mathbb{R} \to \mathbb{R}\), we use the notation \([g]\) \(r\) := \(\int_0^r g(s) ds, \ r \in \mathbb{R}\).

Define the set of functions

\[\mathcal{E} := \{\eta \in C^2(\mathbb{R}) : \eta \text{ is convex with } \eta'' \text{ compactly supported}\}.
\]

Fix two constants \(\kappa \in (0, 1/2]\) and \(\bar{k} \in (0, 1)\). For fixed \(m > 1\), there exists constants \(K \geq 1\) and \(N_0 \geq 0\) such that the following assumptions hold:

**Assumption 2.1.** The function \(\Phi : \mathbb{R} \to \mathbb{R}\) is differentiable, strictly increasing and odd. The function \(\zeta(r) := \sqrt{\Phi'(r)}\) is differentiable away from the origin such that

\[|\zeta(0)| \leq K, \quad |\zeta'(r)| \leq K |r|^{\frac{m-1}{2}}, \quad \forall r \in (0, \infty)\]

and

\[K \zeta(r) \geq 1_{\{r \geq 1\}}, \quad K |[\zeta] (r) - [\zeta] (s)| \geq \begin{cases}
  |r - s|, & \text{if } |r| \vee |s| \geq 1; \\
  |r - s|^{\frac{m+1}{2}}, & \text{if } |r| \vee |s| < 1.
\end{cases}\]
Assumption 2.2. The initial condition $\xi \geq 0$ is an $F_0$-measurable $L_{m+1}(\mathbb{T}^d)$-valued random variable such that $\mathbb{E} \|\xi\|_{L_{m+1}(\mathbb{T}^d)}^{m+1} < \infty$.

Assumption 2.3. The function $\sigma : \mathbb{R} \to l_2$ satisfies $\sigma(0) = 0$ and

$$|\sigma(r) - \sigma(s)|_{l_2} \leq K(|r - s|^{1/2+\kappa} + |r - s|), \quad \forall r, s \in \mathbb{R}.$$  

Assumption 2.4. The function $F : Q_T \times \mathbb{R} \to \mathbb{R}$ satisfies $F(t, x, 0) = 0$ for any $(t, x) \in Q_T$, and

$$|F(t, x_1, r_1) - F(t, x_2, r_2)| \leq K|x_1 - x_2|^\kappa + N_0|r_1 - r_2|$$

Assumption 2.5. The obstacle $S$ satisfies the following equation

$$dS = h_S dt + \sum_{k=1}^{\infty} \sigma^k(S) dW^k_t, \quad t \in [0, T];$$

where $h_S \in L_2(\Omega_T)$ and $S_0 \in L_2(\Omega)$. Further, $S(t) \geq 0$, $\forall t \in [0, T]$ and $S_0 \geq \xi(x)$, $\forall (\omega, x) \in \Omega \times T^d$.

Remark 2.6. It is natural that the functions $\sigma^k(\cdot)$ and $F(t, x, \cdot)$ vanish at zero in Assumptions 2.3 and 2.4 since the equation $\Pi(\Phi, F, \xi)$ describes the density of the gas flow through a porous media. In particular, $u \equiv 0$ is a solution of $\Pi(\Phi, F, 0)$. Moreover, Assumption 2.3 also yields the linear growth:

$$|\sigma(r)|_{l_2} \leq K (1 + |r|), \quad \forall r \in \mathbb{R}.$$  

Remark 2.7. In Assumption 2.5 if $h_S \geq 0$ and $S_0 \geq 0$, the barrier $S$ which satisfies (3) is non-negative. Moreover, a constant barrier $S$ satisfies Assumption 2.5 if $\sigma(S) = 0$.

Remark 2.8. Assumption 2.5 is strong enough such that the measure $\nu$ is absolutely continuous with respect to Lebesgue measure, and for convenience, we still denote by $\nu$ the density function.

Define $\Omega_T := \Omega \times [0, T]$. We now introduce the definition of the entropy solution.

Definition 2.9. An entropy solution of the stochastic porous media equation $\Pi(\Phi, F, \xi)$ is a predictable stochastic process $u : \Omega_T \to L_{m+1}(\mathbb{T}^d)$ such that

(i) $u \in L_{m+1}(\Omega_T; L_{m+1}(\mathbb{T}^d))$;

(ii) For all $f \in C_b(\mathbb{R})$, we have $[\zeta f](u) \in L_2(\Omega_T; H^1(\mathbb{T}^d))$ and

$$\partial_x [\zeta f](u) = f(u) \partial_x [\zeta](u);$$

(iii) For all $(\eta, \varphi, \varrho) \in \mathcal{E} \times C_c^\infty([0, T)) \times C_c^\infty(\mathbb{T}^d)$ and $\phi := \varphi \varrho \geq 0$, we have almost surely

$$-\int_0^T \int_{\mathbb{T}^d} \eta(u) \partial_t \phi dx dt$$

$$\leq \int_{\mathbb{T}^d} \eta(0) \phi dx + \int_0^T \int_{\mathbb{T}^d} \|\zeta^2 \eta^\prime\| (u) \Delta \phi dx dt + \int_0^T \int_{\mathbb{T}^d} \eta^\prime (u) F(t, x, u) \phi dx dt$$

$$+ \int_0^T \int_{\mathbb{T}^d} \left[ \frac{1}{2} \eta^\prime (u) \sum_{k=1}^{\infty} |\sigma^k(u)|^2 \phi - \eta^\prime (u) \nabla [\zeta](u)^2 \phi \right] dx dt$$

$$+ \sum_{k=1}^{\infty} \int_0^T \int_{\mathbb{T}^d} \eta^\prime (u) \phi \sigma^k(u) dx dW^k_t.$$
Definition 2.10. An entropy solution of the obstacle problem \( \Pi_S(\Phi, F, \xi) \) is a pair \((u, \nu)\) such that

(i) The functions \( u \) and \( \nu \) are two predictable stochastic processes and satisfy \((u, \nu) \in \mathcal{M}_{m+1}(\Omega_T; L^2(\mathbb{R}^d) \times L^2(\Omega_T; L^2(\mathbb{R}^d))) \) and \( \nu \geq 0; \)

(ii) For all \( f \in C_b(\mathbb{R}) \), we have \( \|f\|_2(u) \in L^2(\Omega_T; H^1(\mathbb{R}^d)) \) and

\[
\partial_x, \|f\|_2(u) = f(u)\partial_x, \|\xi\|_2(u); \]

(iii) For all \( (\eta, \varphi, \varrho) \in \mathcal{E} \times C^\infty_c([0, T]) \times C^\infty(\mathbb{R}^d) \) and \( \phi := \varphi \varrho \geq 0 \), we have almost surely

\[
-\int_0^T \int_{\mathbb{R}^d} \eta(t, x) \partial_t \phi(x) \, dx \, dt \leq \int_0^T \int_{\mathbb{R}^d} \eta(t, x) \phi(t, x) \, dx \, dt + \int_0^T \int_{\mathbb{R}^d} \eta'(t, x) (F(t, x, u(t)) - \nu(t)) \phi(x) \, dx \, dt
\]

\[
+ \int_0^T \int_{\mathbb{R}^d} \left( \frac{1}{2} \eta''(t, x) \sum_{k=1}^\infty |\sigma^k(t, x)|^2 \phi - \eta''(t, x) |\nabla \|\xi\|_2(u)|^2 \phi \right) \, dx \, dt
\]

\[
+ \sum_{k=1}^\infty \int_0^T \int_{\mathbb{R}^d} \eta'(t, x) \phi(t, x) \sigma^k(t, x) \, dx \, dW^k_t;
\]

(iv) We have \( u \leq S \) almost everywhere in \( Q_T \), almost surely, and the following Skohorod condition holds

\[
\int_{Q_T} (u - S) \nu \, dt \, dx = 0, \quad \text{a.s.} \ \omega \in \Omega.
\]

Our main result is stated as follows.

Theorem 2.11. Let Assumptions 2.1, 2.2 hold. Then, there exists a unique entropy solution \((u, \nu)\) to \( \Pi_S(\Phi, F, \xi) \). Moreover, if \((\bar{u}, \bar{\nu})\) is the entropy solution of \( \Pi_S(\Phi, F, \xi) \), we have

\[
\text{ess sup}_{t \in [0, T]} \mathbb{E} \|u(t) - \bar{u}(t)\|_{L^1(\mathbb{R}^d)} \leq C \mathbb{E} \left\| \xi - \bar{\xi} \right\|_{L^1(\mathbb{R}^d)}
\]

for a constant \( C \) depending only on \( K, N_0, d \) and \( T \).

Remark 2.12. The same assertion holds true for the lower barrier case under the additional conditions

\[
S_0 \in L^m_{m+1}(\Omega) \cap L^4(\Omega), \quad h_S \in L^m_{m+1}(\Omega_T) \cap L^4(\Omega; L^2(0, T)).
\]

In fact, applying Itô's formula to calculate the terms

\[
\|u - S\|_{L^2(\mathbb{R}^d)}^2, \quad \|u - S\|_{L^m_{m+1}(\mathbb{R}^d)}^2, \quad \|u - S\|_{L^2(\mathbb{R}^d)}^2, \quad |S|^{m-1},
\]

\[
\|u - S - 1\|_{L^2(\mathbb{R}^d)}^2, \quad \int_{\mathbb{R}^d} \Phi(r) \, dr, \quad \text{and} \quad \|(u - S)^-\|_{L^2(\mathbb{R}^d)}^2,
\]

we obtain a priori estimates in Section 3 with \( p \leq 4 \), which are sufficient conditions for Theorem 2.11.
Proposition 2.13. Under Assumptions 2.1-2.5, if \((u, \nu)\) is the entropy solution of \(\Pi_S(\Phi, F, \xi)\), we have \(u \geq 0\) almost everywhere in \(Q_T\), almost surely.

Proof. For sufficiently small \(\delta > 0\), we introduce a function \(\eta_\delta \in C^2(\mathbb{R})\) defined by

\[\eta_\delta(0) = \eta'_\delta(0) = 0, \quad \eta''_\delta(r) = \rho_\delta(r).\]

Applying entropy formulation (4) with \(\eta(\cdot) = \eta_\delta(\cdot - \cdot)\) and \(\phi\) independent of \(x\), we get

\[\begin{align*}
- \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \eta_\delta(-u) \partial_t \phi \, dx \, dt \\
\leq \mathbb{E} \int_0^T \int_{\mathbb{T}^d} - \eta'_\delta(-u) F(t, x, u) \phi \, dx \, dt + \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \eta'_\delta(-u) \nu \phi \, dx \, dt \\
+ \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \frac{1}{2} \eta''_\delta(-u) \sum_{k=1}^\infty |\sigma_k(u)|^2 \phi - \eta''_\delta(-u) |\nabla \zeta(u)|^2 \phi \, dx \, dt.
\end{align*}\]

In view of the Skohorod condition and the non-negativity of \(\nu\) and \(S\), we have

\[\begin{align*}
\mathbb{E} \int_0^T \int_{\mathbb{T}^d} \eta'_\delta(-u) \nu \phi \, dx \, dt \\
= \mathbb{E} \int_0^T \int_{\mathbb{T}^d} 1_{\{\nu=0\}} \eta'_\delta(-u) \nu \phi \, dx \, dt + \mathbb{E} \int_0^T \int_{\mathbb{T}^d} 1_{\{\nu>0\}} \eta'_\delta(-S) \nu \phi \, dx \, dt = 0.
\end{align*}\]

Combining inequality (6) with Assumptions 2.3 and 2.4 and \(|\eta'_\delta(r) \cdot r - r^+| \leq \delta\), we have

\[\begin{align*}
- \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \eta_\delta(-u) \partial_t \phi \, dx \, dt \\
\leq N_0 \mathbb{E} \int_0^T \int_{\mathbb{T}^d} (-u)^+ \phi \, dx \, dt + C\delta^{2\kappa}.
\end{align*}\]

Since

\[\left| \mathbb{E} \int_0^T \int_{\mathbb{T}^d} \eta_\delta(-u) \partial_t \phi \, dx \, dt - \mathbb{E} \int_0^T \int_{\mathbb{T}^d} (-u)^+ \partial_t \phi \, dx \, dt \right| \leq C\delta,
\]

we get

\[\begin{align*}
- \mathbb{E} \int_0^T \int_{\mathbb{T}^d} (-u)^+ \partial_t \phi \, dx \, dt \\
\leq C \mathbb{E} \int_0^T \int_{\mathbb{T}^d} (-u)^+ \phi \, dx \, dt + C\delta^{2\kappa}
\end{align*}\]

for sufficiently small \(\delta > 0\) and a constant \(C\) which is independent of \(\delta\). Setting \(\delta \to 0^+\), as the proof of (31), using Grönwall’s inequality, we have

\[\mathbb{E} \int_{\mathbb{T}^d} (-u(t,x))^+ \, dx \leq 0, \quad \text{a.e. } t \in [0, T].\]

Therefore, we have \(u \geq 0\) almost everywhere in \(Q_T\), almost surely. \(\square\)
3 Approximation

A natural method to deal with the obstacle problem is to consider the penalized equation

\[
\begin{aligned}
du_{\epsilon} &= \left[\Delta \Phi(u_{\epsilon}) + F(t,x,u_{\epsilon}) - \frac{1}{\epsilon}(u_{\epsilon} - S)^+ \right]dt \\
&+ \sum_{k=1}^{\infty} \sigma^k(u_{\epsilon})dW^k_t, \quad (t,x) \in (0,T) \times \mathbb{T}^d; \\
u_{\epsilon}(0,x) &= \xi(x), \quad x \in \mathbb{T}^d.
\end{aligned}
\] (7)

We expect that both \(u_{\epsilon}\) and \((u_{\epsilon} - S)^+ / \epsilon\) have limits \(u\) and \(\nu\) when \(\epsilon \to 0^+\), and the pair \((u, \nu)\) is a solution of \(\Pi_S(\Phi, F, \xi)\). However, for the entropy solutions of the stochastic porous media equations, the lack of uniform estimates to the penalty term \((u_{\epsilon} - S)^+ / \epsilon\) will make it difficult to get the existence of the limit \(\nu\). To solve this problem, we use a sequence of smooth functions \(\{\Phi_n\}_{n \in \mathbb{N}}\) to approximate \(\Phi\) as in [13].

With the well-posedness and properties of solutions of penalized equations, we prove the existence and uniform estimates to the penalty term \((u_{\epsilon} - S)^+ / \epsilon\).

**Proposition 3.1.** [13] Proposition 5.1] Let \(\Phi\) satisfy Assumption 2.4 with a constant \(K > 1\). Then, for all \(n \in \mathbb{N}\), there exists an increasing function \(\Phi_n \in C^\infty(\mathbb{R})\) with bounded derivatives, satisfying Assumption 2.4 with constant \(3K\), such that \(\zeta_n(r) \geq 2/n\), and

\[
\sup_{|r| \leq n} |\zeta(r) - \zeta_n(r)| \leq 4/n.
\]

Define

\[
\xi_n := \rho^{\otimes d}_{1/n} * ((-n) \vee (\xi \wedge n)).
\] (8)

Then \(\xi_n\) also satisfies Assumption 2.2 and 2.5. For any \(\epsilon > 0\), we define the penalty term

\[
G_{\epsilon}(r, \tilde{r}) := \frac{(r - \tilde{r})^+}{\epsilon},
\]

which is Lipschitz continuous with Lipschitz constant \(1/\epsilon\) in both \(r\) and \(\tilde{r}\). Moreover, the non-negativity of the barrier \(S\) indicates that \(G_{\epsilon}(0, S(\omega, t)) \equiv 0\) almost surely in \(\Omega_T\). In this section, we study the penalized equation \(\Pi(\Phi_n, F - G_{\epsilon}(\cdot, S), \xi_n)\) which reads,

\[
\begin{aligned}
du_{n,\epsilon} &= \left[\Delta \Phi_n(u_{n,\epsilon}) + F(t,x,u_{n,\epsilon}) - G_{\epsilon}(u_{n,\epsilon}, S) \right]dt \\
&+ \sum_{k=1}^{\infty} \sigma^k(u_{n,\epsilon})dW^k_t, \quad (t,x) \in Q_T; \\
u_{n,\epsilon}(0,x) &= \xi_n(x), \quad x \in \mathbb{T}^d.
\end{aligned}
\]

**Definition 3.2.** An \(L_2\)-solution of equation \(\Pi(\Phi_n, F - G_{\epsilon}(\cdot, S), \xi_n)\) is a continuous \(L_2(\mathbb{T}^d)\)-valued process \(u_{n,\epsilon}\), such that \(u_{n,\epsilon} \in L_2(\Omega_T; H^1(\mathbb{T}^d))\), \(\nabla \Phi_n(u_{n,\epsilon}) \in L_2(\Omega_T; L_2(\mathbb{T}^d))\), and for all \(\phi \in C^\infty(\mathbb{T}^d)\), we have

\[
\int_{\mathbb{T}^d} u_{n,\epsilon}(t,x)\phi dx = \int_{\mathbb{T}^d} \xi_n \phi dx - \int_0^t \left[ \int_{\mathbb{T}^d} \nabla \Phi_n(u_{n,\epsilon}) \nabla \phi dx \\
+ \int_{\mathbb{T}^d} [F(s,x,u_{n,\epsilon}) - G_{\epsilon}(u_{n,\epsilon}, S)] \phi dx \right] ds
\]
Proof. Applying Itô’s formula (cf. [12, Lemma 2]), we have

\[ -\sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{T}^d} \sigma^k(u_{n,\epsilon}) \phi dxdW^k_s, \quad \text{a.e. } t \in [0,T]. \]

We first prove a priori estimates of \( u_{n,\epsilon} \).

**Theorem 3.3.** Let Assumptions 2.4[2.5] hold. Then, for all \( n \in \mathbb{N}, \epsilon > 0 \) and \( p \in [2, \infty) \), there exists a constant \( C \) independent of \( n \) and \( \epsilon \) such that

\[
\begin{align*}
\mathbb{E} \sup_{t \leq T} \|u_{n,\epsilon}(t)\|_{L^p_{2}(\mathbb{T}^d)}^p &+ \mathbb{E} \|\nabla \|\zeta_n\| (u_{n,\epsilon})\|_{L^p_{2}(\mathbb{T}^d)}^p + \frac{1}{\epsilon} \mathbb{E} \|u_{n,\epsilon} - S\|^p_{L^p_{2}(\mathbb{T}^d)} \leq C \left(1 + \mathbb{E} \|\xi_n\|_{L^p_{2}(\mathbb{T}^d)}^p\right), \quad \text{and} & \quad (9) \\
\mathbb{E} \sup_{t \leq T} \|u_{n,\epsilon}(t)\|_{L^{m+1}_{m+1}(\mathbb{T}^d)}^{m+1} &+ \frac{1}{\epsilon} \mathbb{E} \int_0^T \int_{\mathbb{T}^d} |(u_{n,\epsilon} - S)^+|^2 |u_{n,\epsilon}|^{m-1} dxd \leq C \left(1 + \mathbb{E} \|\xi_n\|_{L^p_{m+1}(\mathbb{T}^d)}^{m+1}\right). & \quad (10)
\end{align*}
\]

**Proof.** Applying Itô’s formula (cf. [12, Lemma 2]), we have

\[ \|u_{n,\epsilon}(t)\|_{L^2_{2}(\mathbb{T}^d)}^2 = \|\xi_n\|_{L^2_{2}(\mathbb{T}^d)}^2 - 2 \int_0^t \langle \partial_x, \Phi_n(u_{n,\epsilon}), \partial_x, u_{n,\epsilon}\rangle_{L^2_{2}(\mathbb{T}^d)} ds \]

\[ + 2 \int_0^t \langle F(s, x, u_{n,\epsilon}) - \frac{1}{\epsilon} (u_{n,\epsilon} - S)^+, u_{n,\epsilon}\rangle_{L^2_{2}(\mathbb{T}^d)} ds \]

\[ + 2 \sum_{k=1}^{\infty} \int_0^t \langle \sigma^k(u_{n,\epsilon}), u_{n,\epsilon}\rangle_{L^2_{2}(\mathbb{T}^d)} dW^k_s \]

\[ + \int_0^t \sum_{k=1}^{\infty} \|\sigma^k(u_{n,\epsilon})\|^2_{L^2_{2}(\mathbb{T}^d)} ds, \quad \text{a.e. } t \in [0,T]. \]

In view of the definition of \( \zeta_n \) and Assumptions 2.3 and 2.4 we have

\[ \|u_{n,\epsilon}(t)\|_{L^2_{2}(\mathbb{T}^d)}^2 \leq C + \|\xi_n\|_{L^2_{2}(\mathbb{T}^d)}^2 + C \int_0^t \|u_{n,\epsilon}\|_{L^2_{2}(\mathbb{T}^d)}^2 ds \]

\[ - 2 \int_0^t \|\nabla \|\zeta_n\| (u_{n,\epsilon})\|_{L^2_{2}(\mathbb{T}^d)}^2 + \frac{1}{\epsilon} \langle (u_{n,\epsilon} - S)^+, u_{n,\epsilon}\rangle_{L^2_{2}(\mathbb{T}^d)} ds \]

\[ + 2 \sum_{k=1}^{\infty} \int_0^t \langle \sigma^k(u_{n,\epsilon}), u_{n,\epsilon}\rangle_{L^2_{2}(\mathbb{T}^d)} dW^k_s. \]

Since the barrier \( S \) is non-negative, we have

\[ -\frac{2}{\epsilon} \int_0^t \int_{\mathbb{T}^d} u_{n,\epsilon}(u_{n,\epsilon} - S)^+ dxd \leq -\frac{2}{\epsilon} \int_0^t \|u_{n,\epsilon} - S\|^p_{L^2_{p}(\mathbb{T}^d)} ds. \]

Raising to the power \( p/2 \), taking suprema up to time \( t' \) and expectations, gives

\[ \mathbb{E} \sup_{t \leq t'} \|u_{n,\epsilon}(t)\|^p_{L^2_{p}(\mathbb{T}^d)} + \mathbb{E} \left( \int_0^{t'} \|\nabla \|\zeta_n\| (u_{n,\epsilon})\|^2_{L^2_{2}(\mathbb{T}^d)} ds \right)^{\frac{p}{2}} \]

8
Since the obstacle $S$ for sufficiently small $\bar{\epsilon}$, we have

\[
\frac{1}{\epsilon} \int_0^{t'} \left( \left\| (u_{n, \epsilon} - S)^+ \right\|^2_{L^2(T^d)} \right) ds \leq C \left[ 1 + \mathbb{E} \left\| \xi_n \right\|_{L^2(T^d)}^p + \int_0^{t'} \mathbb{E} \sup_{t \leq s} \left\| u_{n, \epsilon}(t) \right\|_{L^2(T^d)}^p ds 
+ \mathbb{E} \sup_{t \leq t'} \left( \int_0^t \langle \sigma^k(u_{n, \epsilon}), u_{n, \epsilon} \rangle_{L^2(T^d)} dW_s^k \right)^2 \right].
\]

Since

\[
\mathbb{E} \sup_{t \leq t'} \left( \int_0^t \langle \sigma^k(u_{n, \epsilon}), u_{n, \epsilon} \rangle_{L^2(T^d)} dW_s^k \right)^2 \leq CE \left[ \int_0^{t'} \left( \int_{T^d} \sum_{k=1}^{\infty} |\sigma^k(u_{n, \epsilon})|^2 dx \right) \left( \int_{T^d} |u_{n, \epsilon}|^2 dx \right) ds \right]^{\frac{p}{2}} \leq CE \left[ 1 + \int_0^{t'} \left( \int_{T^d} |u_{n, \epsilon}|^2 dx \right)^2 ds \right]^{\frac{p}{4}} 
\leq C + \varepsilon C \mathbb{E} \sup_{t \leq t'} \left\| u_{n, \epsilon}(t) \right\|_{L^2(T^d)}^p + \frac{C}{\varepsilon} \int_0^{t'} \mathbb{E} \sup_{t \leq s} \left\| u_{n, \epsilon}(t) \right\|_{L^2(T^d)}^p ds
\]

for sufficiently small $\varepsilon > 0$, applying Grönwall’s inequality, we have the first desired estimate (9).

We now prove inequality (10). Using Itô’s formula (cf. [12, Lemma 2]), we have

\[
\left\| u_{n, \epsilon}(t) \right\|_{L^{m+1}(T^d)}^{m+1} = \left\| \xi_n \right\|_{L^{m+1}(T^d)}^{m+1} + (m + 1) \int_0^t \int_{T^d} \partial_x \Phi_n(u_{n, \epsilon}) |u_{n, \epsilon}|^{m-1} \partial_x u_{n, \epsilon} dx ds 
+ (m + 1) \int_0^t \int_{T^d} \left[ F(s, x, u_{n, \epsilon}) - \frac{1}{\epsilon} (u_{n, \epsilon} - S)^+ \right] \cdot u_{n, \epsilon} |u_{n, \epsilon}|^{m-1} dx ds 
+ (m + 1) \sum_{k=1}^{\infty} \int_0^t \int_{T^d} \sigma^k(u_{n, \epsilon}) u_{n, \epsilon} \cdot |u_{n, \epsilon}|^{m-1} dxdW_s^k 
+ \frac{(m^2 + m)}{2} \sum_{k=1}^{\infty} \int_0^t \int_{T^d} |\sigma^k(u_{n, \epsilon})|^2 |u_{n, \epsilon}|^{m-1} dx ds.
\]

Since the obstacle $S$ is non-negative and $\Phi_n$ is monotone, in view of Assumptions 2.3 and 2.4, we have

\[
\left\| u_{n, \epsilon}(t) \right\|_{L^{m+1}(T^d)}^{m+1} \leq C + \left\| \xi_n \right\|_{L^{m+1}(T^d)}^{m+1} + C \int_0^t \int_{T^d} |u_{n, \epsilon}|^{m+1} ds 
+ (m + 1) \sum_{k=1}^{\infty} \int_0^t \int_{T^d} \sigma^k(u_{n, \epsilon}) u_{n, \epsilon} \cdot |u_{n, \epsilon}|^{m-1} dxdW_s^k.
\]
Since
\[(m^2 + m)\mathbb{E}\sup_{t \leq t'} \left| \sum_{k=1}^{\infty} \int_{t_0}^{t} \sigma^k(u_{n,\epsilon})u_{n,\epsilon} \cdot |u_{n,\epsilon}|^{m-1} dW_s^k \right| \]
\[\leq CE \left[ \left( \int_{t_0}^{t'} \sum_{k=1}^{\infty} \left( \int_{\mathbb{T}^d} \sigma^k(u_{n,\epsilon}) \cdot |u_{n,\epsilon}|^{m-1} dx \right)^2 ds \right]^{\frac{1}{2}} \right] \]
\[\leq C + CE \left[ \left( \int_{t_0}^{t'} \|u_{n,\epsilon}(s)\|_{L^{m+1}(\mathbb{T}^d)}^{2(m+1)} ds \right)^{\frac{1}{2}} \right] \]
\[\leq C + \frac{1}{2} \mathbb{E}\sup_{t \leq t'} \|u_{n,\epsilon}(t)\|_{L^{m+1}(\mathbb{T}^d)}^{m+1} + C\mathbb{E} \left[ \int_{0}^{t'} \|u_{n,\epsilon}(s)\|_{L^{m+1}(\mathbb{T}^d)}^{m+1} ds \right] , \]
using Grönwall’s inequality, we obtain the second desired inequality (10). \(\square\)

**Remark 3.4.** In view of Definition 3.2, Theorem 3.3 and the smoothness of \(\zeta_n\), with Itô’s formula, the \(L_2\)-solution \(u_{n,\epsilon}\) is also an entropy solution of \(\Pi(\Phi_n, F - G_\epsilon(\cdot, S), \xi_n)\) in the sense of Definition 2.9.

**Lemma 3.5.** Let Assumptions 2.1-2.5 hold. Then, for all \(n \in \mathbb{N}\) and \(\epsilon > 0\), if \(u_{n,\epsilon}\) is an \(L_2\)-solution of \(\Pi(\Phi_n, F - G_\epsilon(\cdot, S), \xi_n)\), we have \(u_{n,\epsilon} \geq 0\) almost everywhere in \(Q_T\), almost surely.

**Proof.** For sufficiently small \(\delta > 0\), we introduce a function \(\eta_\delta \in C^2(\mathbb{R})\) defined by
\[\eta_\delta(0) = \eta_\delta'(0) = 0, \quad \eta_\delta''(r) = \rho_\delta(r).\]
Based on Remark 3.4, applying entropy formulation (4) with \(\eta(\cdot) = \eta_\delta(-\cdot)\) and \(\phi\) which is independent of \(x\), we have
\[- \mathbb{E} \int_{0}^{T} \int_{\mathbb{T}^d} \eta_\delta(-u_{n,\epsilon}) \partial_t \phi dxdt \]
\[\leq \mathbb{E} \int_{0}^{T} \int_{\mathbb{T}^d} \eta''_\delta(-u_{n,\epsilon}) \left[ \frac{1}{\epsilon} (u_{n,\epsilon} - S)^+ - F(t, x, u_{n,\epsilon}) \right] \phi dxdt \]
\[+ \mathbb{E} \int_{0}^{T} \int_{\mathbb{T}^d} \frac{1}{2} \eta''_\delta(-u_{n,\epsilon}) \sum_{k=1}^{\infty} |\sigma^k(u_{n,\epsilon})|^2 \phi - \eta''_\delta(-u_{n,\epsilon}) |\nabla \|\zeta_n\| (u_{n,\epsilon})|^2 \phi dxdt.\]
Since \(\text{supp} \ \eta_\delta(-\cdot) \subset (\infty, 0]\) and the barrier \(S\) is non-negative, we have
\[\eta_\delta(-u_{n,\epsilon}(t, x)) \cdot (u_{n,\epsilon}(t, x) - S(t))^+ = 0, \quad \text{a.s.} \ \ (\omega, t, x) \in \Omega \times Q_T.\]
Therefore, proceeding as in the proof of Proposition 2.13, we have
\[\mathbb{E} \int_{\mathbb{T}^d} (-u_{n,\epsilon}(t, x))^+ dx \leq 0 \quad \text{a.e.} \ t \in [0, T].\]
Then, the proof is complete. \(\square\)
Theorem 3.6. Let Assumptions 2.1-2.5 hold. Then, for all $n \in \mathbb{N}$ and $\epsilon > 0$, we have
\[
\mathbb{E} \|\nabla \Phi_n(u_{n,\epsilon})\|_{L^2(Q_T)}^2 \leq C \left( 1 + \mathbb{E} \|\xi_n\|_{L^{m+1}(T^d)}^{m+1} \right)
\] (11)
for a constant $C$ independent of $n$ and $\epsilon$.

Proof. Applying Itô’s formula (cf. [30]), we have
\[
\int_{T^d} \int_0^{u_{n,\epsilon}(t)} \Phi_n(r)drdx = \int_{T^d} \int_0^t \Phi_n(r)drdx - \int_0^t \int_{T^d} \partial_x \Phi_n(u_{n,\epsilon}) \partial_x \Phi_n(u_{n,\epsilon}) dxds
\]
\[
+ \int_0^t \int_{T^d} \left( F(s, x, u_{n,\epsilon}) - \frac{1}{\epsilon}(u_{n,\epsilon} - S)^+ \right) \Phi_n(u_{n,\epsilon}) dxds
\]
\[
+ \frac{1}{2} \int_0^t \sum_{k=1}^\infty \int_{T^d} |\sigma^k(u_{n,\epsilon})|^2 \Phi_n(u_{n,\epsilon}) dxds
\]
\[
+ \sum_{k=1}^\infty \int_0^t \int_{T^d} \sigma^k(u_{n,\epsilon}) \Phi_n(u_{n,\epsilon}) dx dW^k_s.
\] (12)

From Assumption 2.1, we have
\[
\Phi_n'(r) = \zeta^2(r) \leq \left( |\zeta(0)| + \int_0^r |\zeta'(t)| dt \right)^2 \leq C(1 + |r|^{m-1}).
\]

Then, from Assumptions 2.3 and 2.4, we have
\[
\int_{T^d} \int_0^{\xi_n} \Phi_n(r)drdx + \int_0^T \int_{T^d} F(s, x, u_{n,\epsilon}) \Phi_n(u_{n,\epsilon}) dxds
\]
\[
+ \int_0^t \sum_{k=1}^\infty \int_{T^d} |\sigma^k(u_{n,\epsilon})|^2 \Phi_n(u_{n,\epsilon}) dxds \leq C \left( 1 + \|\xi_n\|_{L^{m+1}(T^d)} + \|u_{n,\epsilon}\|_{L^{m+1}(Q_T)} \right)
\] (13)

For the last term in the right hand side of (12), applying Burkholder-Davis-Gundy inequality and Hölder’s inequality, we have
\[
\mathbb{E} \sup_{t \leq T} \left| \sum_{k=1}^\infty \int_0^t \int_{T^d} \sigma^k(u_{n,\epsilon}) \Phi_n(u_{n,\epsilon}) dx dW^k_s \right|
\]
\[
\leq C \mathbb{E} \left| \left( \int_0^T \sum_{k=1}^\infty \left( \int_{T^d} \sigma^k(u_{n,\epsilon}) dx \right)^2 ds \right)^{\frac{1}{2}} \right|
\]
\[
\leq C \mathbb{E} \left| \left( \int_0^T \sum_{k=1}^\infty \left( \int_{T^d} \sigma^k(u_{n,\epsilon})^2 dx \right) ds \right)^{\frac{1}{2}} \right|
\]
\[
\cdot \left( \int_{T^d} \Phi_n(u_{n,\epsilon})^{m-1} dx \right) ds \right)^{\frac{1}{2}}
\] (14)
projection operator, which means

\[ \langle \cdot, \cdot \rangle \]

The Galerkin approximation of \( \Pi(\bar{\Phi}_C) \) for any independent of \( \ell \in \mathbb{N} \), admits a unique solution \( \Pi(\Phi) \) on \( \{0 \leq t \leq T\} \), almost surely, and

\[ \ell \sup_{t \leq T} \|u_{n, \epsilon}\|_{L^{m+1}(\mathbb{T}^d)} + C \left( 1 + \mathbb{E} \|u_{n, \epsilon}\|_{L^{m+1}(Q_T)} \right). \]

Furthermore, since \( u_{n, \epsilon} \) is non-negative and \( \Phi_n \) is strictly increasing and odd, we have \( \Phi_n(u_{n, \epsilon}) \geq 0 \) almost everywhere in \( Q_T \), almost surely, and

\[ -\frac{1}{\epsilon} \int_0^T \int_{\mathbb{T}^d} (u_{n, \epsilon} - S)^+ \Phi_n(u_{n, \epsilon}) dx ds \leq 0. \quad (15) \]

Combining with (10) and (12)-(15), we obtain the desired estimate. \( \square \)

Using Galerkin approximation method as in [14], we give the existence and uniqueness theorem, which extends [14, Proposition 5.4] to incorporate the barrier \( S \).

**Theorem 3.7.** Let Assumptions 2.7, 2.5 hold. Then, for all \( n \in \mathbb{N} \) and \( \epsilon > 0 \), \( \Pi(\Phi_n, F - G_r(\cdot, S), \xi_n) \) admits a unique \( L_2 \)-solution \( u_{n, \epsilon} \).

**Proof.** Since we fix \( n \in \mathbb{N} \) and \( \epsilon > 0 \), we relabel \( \bar{\Phi} := \Phi_n, \bar{\xi} := \xi_n \) and \( \bar{G} := G_r \) in order to ease the notation. Let \( \{e_i\}_{i \in \mathbb{N}^+} \subset C^\infty(\mathbb{T}^d) \) be an orthonormal basis of \( L_2(\mathbb{T}^d) \) consisting of eigenvectors of \( (I - \Delta) \). Define \( \mathcal{H}^{-1}(\mathbb{T}^d) \) as the dual of \( H^1_0(\mathbb{T}^d) \), equipped with the inner product of \( \langle \cdot, \cdot \rangle_{\mathcal{H}^{-1}(\mathbb{T}^d)} := \langle (I - \Delta)^{-i/2}, (I - \Delta)^{-i/2} \rangle_{L_2(\mathbb{T}^d)}. \) For any \( l \in \mathbb{N}^+ \), let \( \Pi_l : \mathcal{H}^{-1}(\mathbb{T}^d) \rightarrow V_l := \text{span}\{e_1, \ldots, e_l\} \) be the projection operator, which means

\[ \Pi_l v := \sum_{i=1}^l \langle v, e_i \rangle_{\mathcal{H}^{-1}(\mathbb{T}^d)} e_i, \quad \forall v \in \mathcal{H}^{-1}(\mathbb{T}^d). \]

The Galerkin approximation of \( \Pi(\bar{\Phi}, F - \bar{G}(\cdot, S), \bar{\xi}) \)

\[
\begin{align*}
\left\{
\begin{array}{l}
\frac{d}{dt} u_l = \Pi_l \left( \Delta \bar{\Phi}(u_l) + F(t, x, u_l) - \bar{G}(u_l, S) \right) dt \\
\quad + \sum_{k=1}^\infty \Pi_l \sigma^k(u_l) dW^k_t, \quad (t, x) \in Q_T; \\
\end{array}
\right.
\end{align*}
\]

is an equation on \( V_l \), whose coefficients are locally Lipschitz continuous and have a linear growth. Therefore, it admits a unique solution \( u_l \) satisfying

\[ u_l \in L_2(\Omega; H^1(\mathbb{T}^d)) \cap L_2(\Omega; C([0, T]; L_2(\mathbb{T}^d))). \]

Following the proof of Theorem 3.3, there exists a constant \( C \) independent of \( l \in \mathbb{N}^+ \) such that

\[ \mathbb{E} \int_0^T \|u_l\|_{H^1(\mathbb{T}^d)}^2 dt \leq C \left( 1 + \mathbb{E} \|\bar{\xi}\|_{L_2(\mathbb{T}^d)}^2 \right), \quad \text{and} \quad (17) \]

\[ \mathbb{E} \sup_{t \leq T} \|u_l(t)\|_{L_2(\mathbb{T}^d)}^p \leq C \left( 1 + \mathbb{E} \|\bar{\xi}\|_{L_2(\mathbb{T}^d)}^p \right), \quad \forall p \in [2, \infty). \quad (18) \]
Moreover, we have almost surely, for all $t \in [0, T]$
\[ u_t(t) = J^1_t + J^2_t(t) + J^3_t(t) \quad \text{in } H^{-1}(\mathbb{T}^d), \]
with
\[ J^1_t := \Pi I \xi, \]
\[ J^2_t(t) := \int_0^t \Pi I (\Delta \tilde{\Phi}(u_t) + F(s, \cdot, u_t) - \tilde{G}(u_t, S)) \, ds, \]
\[ J^3_t(t) := \sum_{k=1}^{\infty} \int_0^t \Pi I \sigma_k(u_t) d\mathcal{W}_k. \]

Using Sobolev's embedding theorem, inequality (17) and the Lipschitz continuity of $F$ and $\tilde{G}$, we have
\[ \sup_t \mathbb{E} \left\| J^2_t \right\|_{H^2_x([0, T]; H^{-1}(\mathbb{T}^d))}^2 \leq \sup_t \mathbb{E} \left\| J^2_t \right\|_{H^1([0, T]; H^{-1}(\mathbb{T}^d))}^2 < \infty. \]

By [22] Lemma 2.1, the linear growth of $\sigma$ and (18), we have
\[ \sup_t \mathbb{E} \left\| J^3_t \right\|_{H^2_p([0, T]; H^{-1}(\mathbb{T}^d))}^p < \infty, \quad \forall \alpha \in (0, \frac{1}{2}), \quad p \in [2, \infty). \]

Using these two estimates and (17), we get
\[ \sup_t \mathbb{E} \left\| u_t \right\|_{H^2_x([0, T]; H^{-1}(\mathbb{T}^d)) \cap L_2([0, T]; H^1(\mathbb{T}^d))} < \infty. \]

Then, [22] Theorem 2.1, Theorem 2.2] yield the embedding
\[ H^2_x([0, T]; H^{-1}(\mathbb{T}^d)) \cap L_2([0, T]; H^1(\mathbb{T}^d)) \rightarrow \mathcal{X} := L_2([0, T]; L_2(\mathbb{T}^d)) \cap C([0, T]; H^{-2}(\mathbb{T}^d)) \]
is compact. Then, for any sequences $\{l_q\}_{q \in \mathbb{N}}, \{l_q\}_{q \in \mathbb{N}} \subset \mathbb{N}^+$, the laws of $(u_{l_q}, u_{l_q})$ are tight on $\mathcal{X} \times \mathcal{X}$. Define
\[ W(t) := \sum_{k=1}^{\infty} \frac{1}{\sqrt{2^k}} W^k_t \psi_k, \]
where $\{\psi_k\}_{k \in \mathbb{N}^+}$ is the standard orthonormal basis of $l_2$. Moreover, from Assumption 2.5 it is easy to find $S \in L_2(\Omega; C[0, T])$. By Prokhorov's theorem, there exists a (non-relabeled) subsequence $(u_{l_q}, u_{l_q})$ such that the laws of $\{(u_{l_q}, u_{l_q}, W(\cdot), S(\cdot))\}_{q \in \mathbb{N}}$ on $\mathcal{Z} := \mathcal{X} \times \mathcal{X} \times C([0, T]; l_2) \times C([0, T])$ are weakly convergent. By Skorokhod's representation theorem, there exist $\mathcal{Z}$-valued random variables $(\tilde{\tilde{u}}, \tilde{\tilde{u}}, W(\cdot), \tilde{S}(\cdot))$\rangle_{q \in \mathbb{N}}$ on a probability space $(\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathcal{F}}}, \tilde{\tilde{P}})$ such that in $\mathcal{Z}$, we have $\tilde{\tilde{P}}$-almost surely
\[ (\tilde{u}_{l_q}, \tilde{u}_{l_q}, W_q(\cdot), S_q(\cdot)) \xrightarrow{q \to \infty} (\bar{u}, \bar{u}, \tilde{W}(\cdot), \tilde{S}(\cdot)) \]
and
\[ (\tilde{u}_{l_q}, \tilde{u}_{l_q}, W_q(\cdot), S_q(\cdot)) \overset{d}{=} (u_{l_q}, u_{l_q}, W(\cdot), S(\cdot)), \quad \forall q \in \mathbb{N}. \]

Therefore for all $q \in \mathbb{N}$, we have
\[ \tilde{S}_q(t), \tilde{S}(t) \geq 0, \quad \forall t \in [0, T], \quad \text{a.s. } \tilde{\omega} \in \tilde{\Omega}. \]
Moreover, after passing to a non-relabeled subsequence \( \{l_q\}_{q \in \mathbb{N}} \) and \( \{\tilde{l}_q\}_{q \in \mathbb{N}} \), we may assume that
\[
(u_{l_q}, u_{\tilde{l}_q}) \xrightarrow{q \to \infty} (\bar{u}, \tilde{u}), \quad \text{a.s.} \quad (\bar{\omega}, t, x) \in \bar{\Omega} \times \mathbb{T}^d.
\]

Let \( \{\tilde{F}_t\} \) be the augmented filtration of \( \mathcal{G}_t := \sigma(\bar{u}(s), \tilde{u}(s), \bar{W}(s), \tilde{S}(s) | s \leq t) \), and define \( \tilde{W}^k_{q,t} := \sqrt{2}E(\tilde{W}_q(t, \epsilon_k)_t) \) and \( \tilde{\tilde{W}}^k_t := \sqrt{2}E(\tilde{W}(t, \epsilon_k)_t) \). As in the proof of \cite{14} Proposition 5.4|, it is easy to see that \( \{\tilde{W}^k_{q,t}\} \in \mathbb{N}^+ \) are independent, standard and real-valued \( \{\tilde{F}_t\} \)-adapted Wiener processes. Moreover, Note that \( \tilde{G}(0, \tilde{S}) = 0 \), \( F(t, x, 0) = 0 \) and \( \{\tilde{S}_q\}_{q \in \mathbb{N}} \) is uniformly integrable. Combining the Lipschitz continuity of \( F \) and \( \tilde{G} \) with the proof of \cite{14} Proposition 5.4, we can prove that both \( \bar{u} \) and \( \tilde{u} \) are \( L_2 \)-solutions of \( \Pi(\bar{\Phi}, F - \tilde{G}(\cdot, \tilde{S}), \tilde{\xi}) \), where \( \tilde{\xi} := \tilde{u}(0) \). By Remark 3.4 functions \( \bar{u} \) and \( \tilde{u} \) are also entropy solutions under the Definition 2.3.

Then, applying the Lipschitz continuity of \( \tilde{G} \) and \( \tilde{G}(0, \tilde{S}) = 0 \) instead of the \( L_2 \) estimate of \( \tilde{G} \) in \cite{23}, from Theorem 4.6 we know that both \( \bar{u} \) and \( \tilde{u} \) have the \((\ast)\)-property. By Theorem 5.3 (choose \( G(t, x, r) = \tilde{G}(t, x, r) = \tilde{G}(r, \tilde{S}(t)) \)) and Grönwall’s inequality, we have \( \bar{u} = \tilde{u} \) as in the proof of Lemma 6.3. Based on \cite{25} Lemma 1, we have that the initial sequence \( \{u_i\}_{i=1} \) converges in probability to some \( u \in \mathcal{X} \); From this convergence and the uniform estimates on \( u_i \), one can pass to the limit in (16) and obtain that \( u \) is an \( L_2 \)-solution of \( \Pi(\bar{\Phi}, F - \tilde{G}(\cdot, \tilde{S}), \tilde{\xi}) \).

Note that \( L_2 \)-solution of \( \Pi(\bar{\Phi}, F - \tilde{G}(\cdot, S), \tilde{\xi}) \) is also an entropy solution and has \((\ast)\)-property, then the uniqueness of \( u \) is acquired by applying Theorem 5.3 and Grönwall’s inequality as in the proof of Lemma 6.3.

We have the following estimate for the penalty term \( G_\epsilon \).

**Theorem 3.8.** Let Assumptions 2.2, 2.3 hold. Then, for all \( n \in \mathbb{N} \) and \( \epsilon > 0 \), there exists a constant \( C \) independent of \( n \) and \( \epsilon \) such that
\[
\mathbb{E} \int_0^t \int_{\mathbb{T}^d} G_\epsilon(u_{n, \epsilon}, S) dx ds \leq C \left( 1 + \mathbb{E} \|\xi_n\|_{L_2(\mathbb{T}^d)}^2 \right).
\]

**Proof.** Applying Itô’s formula, we have
\[
\int_{\mathbb{T}^d} (u_{n, \epsilon}(t, x) + 1)^2 dx = \int_{\mathbb{T}^d} \left( \xi_n(x) + 1 \right)^2 dx - 2 \int_0^t \langle \partial_x \Phi_n(u_{n, \epsilon}), \partial_x u_{n, \epsilon} \rangle_{L_2(\mathbb{T}^d)} ds + 2 \int_0^t \int_{\mathbb{T}^d} \left( F(s, x, u_{n, \epsilon}) - \frac{1}{\epsilon} (u_{n, \epsilon} - S)^+ \right) (u_{n, \epsilon} + 1) dx ds + 2 \int_0^t \int_{\mathbb{T}^d} \left( \sigma^k(u_{n, \epsilon}), u_{n, \epsilon} + 1 \right)_{L_2(\mathbb{T}^d)} dW^k_s + \int_0^t \sum_{k=1}^{\infty} \|\sigma^k(u_{n, \epsilon})\|_{L_2(\mathbb{T}^d)}^2 ds.
\]

As in the proof of Theorem 3.3, with Assumptions 2.1, 2.4, we have
\[
2 \int_0^t \int_{\mathbb{T}^d} \frac{1}{\epsilon} ((u_{n, \epsilon} - S)^+) dx ds + 2 \int_0^t \int_{\mathbb{T}^d} \frac{1}{\epsilon} (u_{n, \epsilon} - S)^+ dx ds \leq C + \|\xi_n\|_{L_2(\mathbb{T}^d)}^2 + C \|u_{n, \epsilon}\|_{L_2(\mathbb{T}^d)}^2
\]
(21)
\[ + 2 \sum_{k=1}^{\infty} \int_0^t \langle \sigma^k(u_{n,\epsilon}), u_{n,\epsilon} + 1 \rangle_{L^2(\mathbb{T}^d)} dW_s^k. \]

Since

\[ E \left| \sum_{k=1}^{\infty} \int_0^t \langle \sigma^k(u_{n,\epsilon}), u_{n,\epsilon} + 1 \rangle_{L^2(\mathbb{T}^d)} dW_s^k \right| \]

\[ \leq E \left[ \left( \sum_{k=1}^{\infty} \int_0^t \left| \sigma^k(u_{n,\epsilon}) \right|^2 dx \right)^{\frac{1}{2}} \right] \]

\[ \leq E \left[ \left( \int_0^t \left( \int_{\mathbb{T}^d} |u_{n,\epsilon}|^2 dx \right) \left( \int_{\mathbb{T}^d} |u_{n,\epsilon} + 1|^2 dx \right) ds \right)^{\frac{1}{2}} \right] \]

\[ \leq C + C E \left[ \left( \int_0^t \left\| u_{n,\epsilon} \right\|_{L^2(\mathbb{T}^d)} ds \right)^{\frac{1}{2}} \right] \]

\[ \leq C + C E \sup_{t \in [0,T]} \left\| u_{n,\epsilon}(t) \right\|_{L^2(\mathbb{T}^d)}^2 + C E \left\| u_{n,\epsilon} \right\|_{L^2(\mathbb{T}^d)}^2, \]

using Lemma 3.5 and inequalities (9) and (21), we obtain the desired inequality. \( \square \)

To obtain the \( L_2 \) estimate of \( G_\epsilon(u_{n,\epsilon}, S) \), the specific form of stochastic differential equation in Assumption 2.5 is crucial, which gives that the difference \( u_{n,\epsilon} - S \) has bounded variation when \( u_{n,\epsilon} = S \).

Actually, the local martingale part will make it fail to obtain better a priori estimate for the penalty term. However, this term has no affect to the obstacle problem for backward equations (cf. [43, 49]).

**Theorem 3.9.** Let Assumptions 2.1-2.5 hold. Then, for all \( n \in \mathbb{N} \) and \( \epsilon > 0 \), there exists a constant \( C \) independent of \( n \) and \( \epsilon \) such that

\[ \frac{1}{\epsilon} E \left[ \sup_{t \in [0,T]} \int_{\mathbb{T}^d} |(u_{n,\epsilon} - S)^+(t)|^2 dx \right] + \frac{1}{\epsilon^2} E \int_0^T \left\| (u_{n,\epsilon} - S)^+ \right\|_{L^2(\mathbb{T}^d)}^2 dt \]

\[ \leq C \left( 1 + E \left\| \xi_n \right\|_{L^2(\mathbb{T}^d)}^2 \right). \]

**Proof.** We consider the equation

\[ d(u_{n,\epsilon} - S) = \left[ \Delta \Phi_n(u_{n,\epsilon}) + F(t, x, u_{n,\epsilon}) - \frac{1}{\epsilon} (u_{n,\epsilon} - S)^+ - h_S \right] dt \]

\[ + \sum_{k=1}^{\infty} \left[ \sigma^k(u_{n,\epsilon}) - \sigma^k(S) \right] dW_t^k, \quad (t, x) \in Q_T; \]

\[ u_{n,\epsilon}(0, x) - S(0) = \xi_n(x) - S_0, \quad x \in \mathbb{T}^d. \]

Using Itô’s formula (cf. the proof of [49 Lemma 5.1]), we have

\[ \frac{1}{\epsilon} \int_{\mathbb{T}^d} \left\| (u_{n,\epsilon} - S)^+(t) \right\|^2 dx = \sum_{l=1}^{5} I_l, \]

15
where

\[ I_1 := \frac{1}{\epsilon} \int_{\mathbb{T}^d} |(\xi_n - S_0)|^2 dx, \]

\[ I_2 := \frac{2}{\epsilon} \int_0^t \langle \Delta \Phi_n(u_{n,\epsilon}), (u_{n,\epsilon} - S)^+ \rangle_{L_2(\mathbb{T}^d)} ds, \]

\[ I_3 := \frac{2}{\epsilon} \int_0^t \langle F(s, \cdot, u_{n,\epsilon}) - \frac{1}{\epsilon}(u_{n,\epsilon} - S)^+ - h_S, (u_{n,\epsilon} - S)^+ \rangle_{L_2(\mathbb{T}^d)} ds, \]

\[ I_4 := \frac{2}{\epsilon} \sum_{k=1}^{\infty} \int_0^t \langle \sigma^k(u_{n,\epsilon}) - \sigma^k(S), (u_{n,\epsilon} - S)^+ \rangle_{L_2(\mathbb{T}^d)} dW^k_s, \]

\[ I_5 := \frac{1}{\epsilon} \sum_{k=1}^{\infty} \int_0^t \| (\sigma^k(u_{n,\epsilon}) - \sigma^k(S)) \mathbf{1}_{\{u_{n,\epsilon} \geq S\}} \|_{L_2(\mathbb{T}^d)}^2 ds. \]

They are estimated below. Since \( \xi_n \leq S_0 \), we have \( I_1 \equiv 0 \). In view of \( \partial_x S \equiv 0 \) and Proposition 3.1, we have

\[ I_2 = \frac{2}{\epsilon} \int_0^t \langle \Phi'_n(u_{n,\epsilon}) \partial_x (u_{n,\epsilon} - S), \partial_x (u_{n,\epsilon} - S) \mathbf{1}_{\{u_{n,\epsilon} \geq S\}} \rangle_{L_2(\mathbb{T}^d)} ds \]

\[ \leq - \frac{8}{n^2 \epsilon} \int_0^t \| \mathbf{1}_{\{u_{n,\epsilon} \geq S\}} \partial_x (u_{n,\epsilon} - S) \|^2_{L_2(\mathbb{T}^d)} ds. \]

We have the following estimate for \( I_3 \)

\[ I_3 + \frac{2}{\epsilon^2} \int_0^t \| (u_{n,\epsilon} - S)^+ \|_{L_2(\mathbb{T}^d)}^2 ds \]

\[ = \frac{2}{\epsilon} \int_0^t \langle F(s, \cdot, u_{n,\epsilon}) - h_S, (u_{n,\epsilon} - S)^+ \rangle_{L_2(\mathbb{T}^d)} ds \]

\[ \leq \int_0^t \| F(s, \cdot, u_{n,\epsilon}) - h_S \|^2_{L_2(\mathbb{T}^d)} ds + \frac{1}{\epsilon^2} \int_0^t \| (u_{n,\epsilon} - S)^+ \|^2_{L_2(\mathbb{T}^d)} ds \]

\[ \leq C + C \int_0^t \| u_{n,\epsilon} \|^2_{L_2(\mathbb{T}^d)} ds + \frac{1}{\epsilon^2} \int_0^t \| (u_{n,\epsilon} - S)^+ \|^2_{L_2(\mathbb{T}^d)} ds. \]

Now, we estimate \( I_4 \). Using Burkholder-Davis-Gundy inequality and Assumption 2.3, we have

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} I_4 \right]
\leq \frac{C}{\epsilon} \mathbb{E} \left[ \left( \int_0^t \sum_{k=1}^{\infty} \langle \sigma^k(u_{n,\epsilon}) - \sigma^k(S), (u_{n,\epsilon} - S)^+ \rangle_{L_2(\mathbb{T}^d)} ds \right)^{\frac{1}{2}} \right]
\leq \frac{C}{\epsilon} \mathbb{E} \left[ \left( \int_0^t \sum_{k=1}^{\infty} \left( \int_{\mathbb{T}^d} |\sigma^k(u_{n,\epsilon}) - \sigma^k(S)|^2 \mathbf{1}_{\{u_{n,\epsilon} \geq S\}} dx \right) \right)^{\frac{1}{2}} \right]
\times \left( \int_{\mathbb{T}^d} \| (u_{n,\epsilon} - S)^+ \|^2 dx \right) \frac{1}{2}.\]
Using Hölder’s inequality, we have

\[
\mathbb{E}\left[ \sup_{T \in [0, t]} I_4 \right] \leq \frac{C}{\epsilon} \mathbb{E}\left[ \left( \frac{1}{t} \int_0^t \left| (u_{n, \epsilon} - S)^+ (\tau) \right|^2 dx \right) \cdot \left( \int_0^t (x^2 + |x|_{x=(u_{n, \epsilon} - S)^+}) dx ds \right) \right]^{\frac{1}{2}}
\]

\[
\leq \frac{1}{4\epsilon} \mathbb{E}\left[ \sup_{T \in [0, t]} \int_{T \in [0, t]} \left| (u_{n, \epsilon} - S)^+ (\tau) \right|^2 dx \right] + C \frac{1}{\epsilon} \sup_{T \in [0, t]} \left\| (u_{n, \epsilon} - S)^+ \right\|_{L^2(\Omega_T \times \mathbb{T}^d)}^2
\]

In the same way, using Assumption 2.3, we have

\[
I_5 \leq \frac{C}{\epsilon} \left( \int_0^t \left\| (u_{n, \epsilon} - S)^+ \right\|_{L^2(\mathbb{T}^d)}^2 \ ds + \int_0^t \int_{T \in [0, t]} (u_{n, \epsilon} - S)^+ dx ds \right)
\]

Combining the preceding five estimates, we have

\[
\frac{3}{4\epsilon} \mathbb{E}\left[ \sup_{T \in [0, T]} \int_{T \in [0, t]} \left| (u_{n, \epsilon} - S)^+ (t) \right|^2 dx \right] + \frac{1}{\epsilon^2} \sup_{T \in [0, t]} \left\| (u_{n, \epsilon} - S)^+ \right\|_{L^2(\Omega_T \times \mathbb{T}^d)}^2
\]

\[
\leq C + C \left\| u_{n, \epsilon} \right\|_{L^2(\Omega_T \times \mathbb{T}^d)}^2 + C \frac{1}{\epsilon} \left\| (u_{n, \epsilon} - S)^+ \right\|_{L^2(\Omega_T \times \mathbb{T}^d)}^2
\]

\[
+ C \left\| \frac{1}{\epsilon} (u_{n, \epsilon} - S)^+ \right\|_{L^1(\Omega_T \times \mathbb{T}^d)}^2
\]

for a constant $C$ independent of $n$ and $\epsilon$. Using Theorems 3.3 and 3.8, we obtain the desired inequality.

\[\square\]

### 4 \((\star)\)-property

We introduce the \((\star)\)-property to give an estimate to stochastic integral, which is a key step in the proof of $L^1_\tau$ estimate between two entropy solutions. This method is also used in [13] [14].

For simplicity, we denote the integral to time as $\int_0^T \cdot dt$ and $\int_0^t \cdot dx$, and denote the integral to space as $\int_0^T \cdot dx$ and $\int_0^T \cdot dy$. Let $g \in C^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ and $\varphi \in C^\infty_c((0, T))$. For all $\theta > 0$, we introduce

\[
\phi_\theta(t, x, s, y) := g(x, y) \rho_\theta(t - s) \varphi\left(\frac{t + s}{2}\right), \quad (t, x, s, y) \in Q_T \times Q_T.
\]

For all $\tilde{u} \in L_{m+1}(\Omega_T; L_{m+1}(\mathbb{T}^d))$, $h \in C^\infty(\mathbb{R})$ and $h' \in C^\infty_c(\mathbb{R})$, we further define

\[
H_\theta(t, x, a) := \sum_{k=1}^\infty \int_0^T \int_y h(\tilde{u}(s, y) - a) a^k(y, \tilde{u}(s, y)) \phi_\theta(t, x, s, y) dW_s^k, \quad \forall a \in \mathbb{R}
\]
Lemma 4.4. \( u \) function formally in \( \varepsilon \).

Lemma 4.5. In particular, we have for a constant \( C \) and \( C \)

\[ \text{Definition 4.1. A function } u \in L_{m+1}(\Omega_T \times \mathbb{T}^d) \text{ is said to have the } (\ast)\text{-property if for all } (g, \varphi, h, h') \in C^\infty(\mathbb{T}^d \times \mathbb{T}^d) \times C^\infty_{c}((0, T)) \times L_{m+1}(\Omega_T; L_{m+1}(\mathbb{T}^d)) \times C^\infty(\mathbb{R}) \times C^\infty_{c}(\mathbb{R}) \text{, and for all sufficiently small } \theta > 0, \text{ we have } H_\theta(\cdot, \cdot, u) \in L_1(\Omega_T \times \mathbb{T}^d) \text{ and } \]

\[ \mathbb{E} \int_{t,x} H_\theta(t, x, u(t, x)) \leq C\theta^{1-\mu} + B(u, \hat{u}, \theta) \]

for a constant \( C \) independent of \( \theta \).

Remark 4.2. Notice that \( \varphi \) is supported in \((0, T)\) and \( \rho_0(t - \cdot) \) is supported in \([t - \theta, t]\). For sufficiently small \( \theta > 0 \), we have

\[ H_\theta(t, x, a) = \sum_{k=1}^{\infty} 1_{t > \theta} \int_{t-\theta}^{t} h(\hat{u}(s, y) - a)\sigma^k(y, \hat{u}(s, y))\phi_\theta(t, x, y) dW_s^k. \]

The following three lemmas are introduced from [13, Section 3]. They are essential to the proofs to the \((\ast)\)-property of solution \( u \) and the \( L_1^T \) estimate between two entropy solutions.

Lemma 4.3. For all \( \lambda \in (\frac{m+3}{2(m+1)}, 1), k \in \mathbb{N} \) and sufficiently small \( \theta \in (0, 1) \), we have

\[ \mathbb{E} \| \partial_\theta H_\theta \|_{L_\infty([0, T]; W_{m+1}^k(\mathbb{T}^d \times \mathbb{R}))} \leq C\theta^{-\lambda(m+1)}N_m(\hat{u}), \]

where

\[ N_m(\hat{u}) := \mathbb{E} \int_0^T \left( 1 + \| \hat{u}(t) \|_{L_{m+1}^\infty(\mathbb{T}^d)} + \| \hat{u}(t) \|_{L_{m+1}^2(\mathbb{T}^d)} \right) dt, \]

and \( C \) is a constant depending only on \( N_0, N_1, k, d, T, \lambda, m, \) and the functions \( h, g, \varphi \), but not on \( \theta \). In particular, we have

\[ \mathbb{E} \| \partial_\theta H_\theta \|_{L_\infty([0, T]; W_{m+1}^k(\mathbb{T}^d \times \mathbb{R}))} \leq C\theta^{-\lambda(m+1)} \left( 1 + \| \hat{u} \|_{L_{m+1}^\infty(Q_T)} \right). \]

Lemma 4.4. (i) Let \( \{ u_n \}_{n \in \mathbb{N}} \) be a sequence bounded in \( L_{m+1}(\Omega \times Q_T) \), satisfying the \((\ast)\)-property uniformly in \( n \), which means the constant \( C \) in Definition 4.1 is independent of \( n \). If \( u_n \) converges to a function \( u \) almost surely on \( \Omega \times Q_T \), then \( u \) has the \((\ast)\)-property.

(ii) Let \( u \in L_2(\Omega \times Q_T) \). Then, for sufficiently small \( \theta \in (0, 1) \), we have

\[ \mathbb{E} \int_{t,x} H_\theta(t, x, u(t, x)) = \lim_{\lambda \to 0} \mathbb{E} \int_{t,x,a} H_\theta(t, x, a)\rho_\lambda(u(t, x) - a). \]

Lemma 4.5. Let Assumption 2.1 holds and \( u \in L_1(\Omega \times Q_T) \). For some \( \varepsilon \in (0, 1) \), let \( \varrho : \mathbb{R}^d \to \mathbb{R} \) be a non-negative function integrating to one and supported on a ball of radius \( \varepsilon \). Then, we have

\[ \mathbb{E} \int_{t,x,y} |u(t, x) - u(t, y)|\varrho(x - y) \leq C\varepsilon^{-\frac{2d}{d+1}}(1 + \mathbb{E} \| \nabla \varrho \|_{L_1(Q_T)}) \]

for a constant \( C \) independent of \( d, K \) and \( T \).
Define \( g_\zeta := \rho_\zeta^d \) for all \( \zeta > 0 \). Now we prove that the solution \( u_{n,\epsilon} \) has the uniform \((*)\)-property.

**Theorem 4.6.** Let Assumptions 2.1-2.5 hold. For any \( n \in \mathbb{N} \) and \( \epsilon > 0 \), let \( u_{n,\epsilon} \) be the \( L_2 \)-solution of \( (\Phi_n, F - G_{\epsilon}(\cdot, S), \xi_n) \). Then, \( u_{n,\epsilon} \) has the \((*)\)-property. If in addition \( \|\xi\|_{L_2(\mathbb{R}^d)} \) has moments of order \( 4 \), then the constant \( C \) in Definition 4.4 is independent of \( n \) and \( \epsilon \).

**Proof.** Fixed \( \theta > 0 \) small enough so that Remark 4.2 holds. We now apply the approximation method in the proof of [13, Lemma 5.2]. For a function \( f \in L_2(\mathbb{T}^d) \) and \( \gamma > 0 \), let \( f^{(\gamma)} := g_\gamma * f \) be the mollification. Then, the function \( u_n^{(\gamma)} \) satisfies (pointwise) the equation

\[
d u_n^{(\gamma)} = \left[ (\Delta \Phi_n(u_{n,\epsilon}) + F(t, u_{n,\epsilon}) - \frac{1}{\epsilon}(u_{n,\epsilon} - S)^+) \right] dt + \sum_{k=1}^{\infty} (\sigma^k(u_{n,\epsilon}))^{(\gamma)} dW^k_t.
\]

Applying Itô’s formula, we have

\[
\int_{t,x,a} H_\theta(t, x, a) \left( \rho_\lambda(u_n^{(\gamma)}(t, x) - a) - \rho_\lambda(u_n^{(\gamma)}(t, x) - a) \right) = \sum_{\lambda} N_{\lambda,\gamma}^{(1)}
\]

where

\[
N_{\lambda,\gamma}^{(1)} := \int_{t,x,a} H_\theta(t, x, a) \int_{t-\theta}^{t} \rho_\lambda(u_n^{(\gamma)}(s, x) - a) \Delta(\Phi_n(u_{n,\epsilon}))^{(\gamma)} ds,
\]

\[
N_{\lambda,\gamma}^{(2)} := \int_{t,x,a} H_\theta(t, x, a) \sum_{k=1}^{\infty} \int_{t-\theta}^{t} \rho_\lambda(u_n^{(\gamma)}(s, x) - a) (\sigma^k(u_{n,\epsilon}))^{(\gamma)} dW^k_s,
\]

\[
N_{\lambda,\gamma}^{(3)} := \int_{t,x,a} H_\theta(t, x, a) \frac{1}{2} \int_{t-\theta}^{t} \rho_\lambda''(u_n^{(\gamma)}(s, x) - a) \sum_{k=1}^{\infty} (\sigma^k(u_{n,\epsilon}))^{(\gamma)} ds,
\]

\[
N_{\lambda,\gamma}^{(4)} := \int_{t,x,a} H_\theta(t, x, a) \int_{t-\theta}^{t} \rho_\lambda'(u_n^{(\gamma)}(s, x) - a) (F(s, \cdot, u_{n,\epsilon}) - \frac{1}{\epsilon}(u_{n,\epsilon} - S)^+)^{(\gamma)} ds.
\]

For \( N_{\lambda,\gamma}^{(4)} \), using integration by parts formula (in a), we have

\[
E \left| N_{\lambda,\gamma}^{(4)} \right|
\]

\[
\leq E \left| \int_{t,x,a} \partial_a H_\theta(t, x, a) \int_{t-\theta}^{t} \rho_\lambda(u_n^{(\gamma)}(s, x) - a) (F(s, \cdot, u_{n,\epsilon}) - \frac{1}{\epsilon}(u_{n,\epsilon} - S)^+)^{(\gamma)} ds \right|
\]

\[
\leq N_1 + N_2,
\]

where

\[
N_1 := E \left| \int_{t,x,a} \partial_a H_\theta(t, x, a) \int_{t-\theta}^{t} \rho_\lambda(u_n^{(\gamma)}(s, x) - a) (F(s, \cdot, u_{n,\epsilon}))^{(\gamma)} ds \right|
\]

and

\[
N_2 := E \left| \int_{t,x,a} \partial_a H_\theta(t, x, a) \int_{t-\theta}^{t} \rho_\lambda(u_n^{(\gamma)}(s, x) - a) \left( \frac{1}{\epsilon}(u_{n,\epsilon} - S)^+ \right)^{(\gamma)} ds \right|.
\]

Since

\[
\int f^{(\gamma)}(x) = \int_y f(y) \int_x \varrho_\gamma(x - y) = \int_y f(y) \int_x \varrho_\gamma(x - y) = 2 \int_y f(y),
\]

19
applying Assumptions 2.2 and 2.4, Lemma 4.3 and Theorem 3.3 with [14, Remark 3.2], we have

\[ N_1 \leq \mathbb{E} \left\| \partial_a H_\theta \right\|_{L^\infty(Q_T \times \mathbb{R})} \int_{t,x} \int_{t-\theta}^t \left( F(s,\cdot, u_{n,\epsilon}) \right)^{(\gamma)} ds \int_a \left( \rho_\lambda (u_{n,\epsilon}^{(\gamma)}(s,x) - a) \right) \]

\[ \leq 2\mathbb{E} \left\| \partial_a H_\theta \right\|_{L^\infty(Q_T \times \mathbb{R})} \theta \int_{t,x} \left( F(t,\cdot, u_{n,\epsilon}) \right)^{(\gamma)} \]

\[ \leq C\theta \left( \mathbb{E} \left\| \partial_a H_\theta \right\|_{L^\infty(Q_T \times \mathbb{R})} \right)^{1/2} \left( \mathbb{E} \left\| F(\cdot,\cdot, u_{n,\epsilon}) \right\|_{L^2(Q_T)} \right)^{1/2} \]

\[ \leq C\theta \left( \mathbb{E} \left\| \partial_a H_\theta \right\|_{L^\infty(Q_T \times \mathbb{R})} \right)^{1/2} \left( 1 + \mathbb{E} \left\| u_{n,\epsilon} \right\|_{L^2(Q_T)} \right)^{1/2} \leq C(n,\epsilon) \theta^{1-\mu}. \]

Similarly, using Theorem 3.9, we have

\[ N_2 \leq \mathbb{E} \left\| \partial_a H_\theta \right\|_{L^\infty(Q_T \times \mathbb{R})} \int_{t,x} \int_{t-\theta}^t \left( \frac{1}{\epsilon} (u_{n,\epsilon} - S)^+ \right)^{(\gamma)} ds \int_a \left( \rho_\lambda (u_{n,\epsilon}^{(\gamma)}(s,x) - a) \right) \]

\[ \leq \mathbb{E} \left\| \partial_a H_\theta \right\|_{L^\infty(Q_T \times \mathbb{R})} \theta \int_{t,x} \left( \frac{1}{\epsilon} (u_{n,\epsilon} - S)^+ \right)^{(\gamma)} \]

\[ \leq C\theta \left( \mathbb{E} \left\| \partial_a H_\theta \right\|_{L^\infty(Q_T \times \mathbb{R})} \right)^{1/2} \left( \mathbb{E} \left\| \frac{1}{\epsilon} (u_{n,\epsilon} - S)^+ \right\|_{L^2(Q_T)}^2 \right)^{1/2} \leq C(n,\epsilon) \theta^{1-\mu}. \]

The estimates for \( N_{\lambda,\gamma}^{(1)}, N_{\lambda,\gamma}^{(2)}, \) and \( N_{\lambda,\gamma}^{(3)} \) can be obtained as in the proof of [13, Lemma 5.2]. Combining these estimates and following the proof of [14, Lemma 5.2], we have

\[ \mathbb{E} \int_{t,x,a} H_\theta(t, x, u_{n,\epsilon}(t, x)) \]

\[ \leq \limsup_{\lambda \to 0} \limsup_{\gamma \to 0} \mathbb{E} \left( N_{\lambda,\gamma}^{(1)} + N_{\lambda,\gamma}^{(3)} \right) + \limsup_{\lambda \to 0} \limsup_{\gamma \to 0} \mathbb{E} \left| N_{\lambda,\gamma}^{(2)} \right| \]

\[ + \limsup_{\lambda \to 0} \limsup_{\gamma \to 0} \mathbb{E} N_{\lambda,\gamma}^{(2)} \leq C(n,\epsilon) \theta^{1-\mu} + B(u_{n,\epsilon}, \ddot{u}, \theta). \]

Moreover, if \( \mathbb{E} \left\| \phi \right\|_{L^2(T^d)}^4 < \infty \), with Theorems 3.3 and 3.9, the constant \( C(n,\epsilon) \) in the above can be selected to be independent of \( n \) and \( \epsilon \).

**5 \( L_1^+ \) estimate**

Note that \( \rho_\gamma = \rho_\gamma^0 \).

**Lemma 5.1.** Let \( G(t, x, r) \) and \( \bar{G}(t, x, r) \) be two functions, which are Lipschitz continuous in \( r \), and satisfy \( G(\cdot, \cdot, 0), \bar{G}(\cdot, \cdot, 0) \in L_2(\Omega_T; L_2(T^d)) \). Suppose that \( u \) and \( \bar{u} \) are entropy solutions of \( \Pi(\Phi, F - G, \xi) \) and \( \Pi(\bar{\Phi}, F - G, \bar{\xi}) \), respectively. Let Assumptions 2.4 hold for both \( (\Phi, F, \sigma, \xi) \) and \( (\bar{\Phi}, F, \sigma, \bar{\xi}) \). If \( u \) has the \((\star)\)-property, then for every non-negative \( \varphi \in C_c^\infty((0, T)) \) such that

\[ \left\| \varphi \right\|_{L^\infty(0, T)} \vee \left\| \partial_t \varphi \right\|_{L^1(0, T)} \leq 1, \]

\[ 20 \]
and $\zeta, \delta \in (0, 1], \lambda \in [0, 1]$ and $\alpha \in (0, 1 \wedge (m/2))$, we have

$$-\mathbb{E} \int_{t,x,y} (u(t,x) - \bar{u}(t,y))^+ \varphi_\zeta(x-y) \partial_\zeta \varphi(t)$$

$$\leq C \zeta^2 \left( \mathbb{E} \left\| \mathbf{1}_{\{|u| \geq R_\delta\}} (1 + |u|) \right\|_{L^m(Q_T)}^m + \mathbb{E} \left\| \mathbf{1}_{\{|\bar{u}| \geq R_\Delta\}} (1 + |\bar{u}|) \right\|_{L^m(Q_T)}^m ight)$$

$$+ C \left( \delta^{2\alpha} + \zeta^2 + \zeta^{-2}\lambda^2 + \zeta^{-2}\delta^{2\alpha} \right) \cdot \mathbb{E} \left( 1 + \|u\|_{L^{m+1}(Q_T)}^m + \|\bar{u}\|_{L^{m+1}(Q_T)}^m \right)$$

$$+ C \epsilon \int_{t,x,y} (u(t,x) - \bar{u}(t,y))^+ \varphi_\zeta(x-y) \varphi(t)$$

$$+ C \epsilon \int_{t,x,y} \mathbf{1}_{\{|\bar{u}(t,y)| \leq u(t,x)\}} \left( \tilde{G}(t,y, \bar{u}(t,y)) - G(t,x,u(t,x)) \right)^+ \varphi_\zeta(x-y)$$

for a constant $C$ depending only on $N_0, K, d$ and $T$. The parameter $R_\lambda$ is defined by

$$R_\lambda := \sup \left\{ R \in [0, \infty) : \left| \zeta(r) - \tilde{\zeta}(r) \right| \leq \lambda, \forall |r| < R \right\}.$$ 

**Proof.** For sufficiently small $\theta > 0$, we introduce

$$\phi_{\theta, \zeta}(t, x, y, z) := \rho_\theta (t-s) \varphi_\zeta(x-y) \varphi\left( \frac{t+s}{2} \right), \quad \phi_\zeta(t, x, y) = \varphi_\zeta(x-y) \varphi(t).$$

Furthermore, for each $\delta > 0$, we define the function $\eta_\delta \in C^2(\mathbb{R})$ by

$$\eta_\delta(0) = \eta_\delta'(0) = 0, \quad \eta_\delta''(r) = \rho_\delta(r).$$

Thus, we have

$$|\eta_\delta(r) - r^-| \leq \delta, \quad \sup \eta_\delta'' \subset [0, \delta], \quad \int_\mathbb{R} \eta_\delta''(r) dr \leq 2, \quad |\eta_\delta''| \leq 2\delta^{-1}.$$

Fix $(a, s, y) \in \mathbb{R} \times Q_T$. Since $u$ is the entropy solution of $\Pi(\Phi, F - G, \xi)$, using the entropy inequality of $u$ in Definition 2.9 with $\eta_\delta(r-a)$ and $\phi_{\theta, \zeta}(\cdot, \cdot, s, y)$ instead of $\eta(r)$ and $\phi$, we have

$$- \int_{t,x} \eta_\delta(u-a) \partial_\zeta \phi_{\theta, \zeta}$$

$$\leq \int_{t,x} \left[ \Delta_x \eta_\delta(u-a) \right] (u) \Delta_x \phi_{\theta, \zeta} + \int_{t,x} \eta_\delta(u-a) (F(t,x,u) - G(t,x,u)) \phi_{\theta, \zeta}$$

$$+ \int_{t,x} \left( \frac{1}{2} \eta_\delta''(u-a) \sum_{k=1}^\infty |\sigma_k(u)|^2 \phi_{\theta, \zeta} - \eta_\delta''(u-a) |\nabla_x [\zeta(u)]|^2 \phi_{\theta, \zeta} \right)$$

$$+ \sum_{k=1}^\infty \int_0^T \int_{x} \eta_\delta(u-a) \phi_{\theta, \zeta} \sigma_k(u) dW_t^k,$$

where $u = u(t,x)$. Notice that all the expressions are continuous in $(a, s, y)$. We take $a = \bar{u}(s, y)$ by convolution and integrate over $(s, y) \in Q_T$. By taking expectations, we have

$$- \mathbb{E} \int_{t,x,s,y} \eta_\delta(u-a) \partial_\zeta \phi_{\theta, \zeta}$$

21
\[ \begin{align*}
&\leq E \int_{t,x,y} \left[ \zeta^2 \eta_0'(\cdot - \tilde{u}) \right](u) \Delta_x \phi_{\theta,\zeta} + E \int_{t,x,y} \eta_0'(u - \tilde{u}) \left( F(t, x, u) - G(t, x, u) \right) \phi_{\theta,\zeta} \\
&\quad + E \int_{t,x,y} \left( \frac{1}{2} \eta_0''(u - \tilde{u}) \sum_{k=1}^{\infty} |\sigma^k(u)|^2 \phi_{\theta,\zeta} - \eta_0''(u - \tilde{u}) |\nabla_x \left[ \zeta \right](u)|^2 \phi_{\theta,\zeta} \right) \\
&\quad - E \int_{t,x,y} \left[ \sum_{k=1}^{\infty} \int_0^T \int_x \eta_0'(u - a) \phi_{\theta,\zeta} \sigma^k(u) dW_t^k \right]_{a=\tilde{u}},
\end{align*} \]

where \( u = u(t, x) \) and \( \tilde{u} = \tilde{u}(s, y) \). Similarly, for each \( (a, t, x) \in \mathbb{R} \times Q_T \) and entropy solution \( \tilde{u} \), we apply the entropy inequality of \( \tilde{u} \) with \( \eta(r) := \eta_0(a - r) \) and \( \phi(s, y) := \phi_{\theta,\zeta}(t, x, s, y) \). After substituting \( a = u(t, x) \) by convolution, integrating over \( (t, x) \in Q_T \) and taking expectations, we have

\[ -E \int_{t,x,y} \eta_0(u - \tilde{u}) \partial_t \phi_{\theta,\zeta} \]

\[ -E \int_{t,x,y} \left[ \zeta^2 \eta_0'(u - \cdot) \right](u) \Delta_x \phi_{\theta,\zeta} + E \int_{t,x,y} \eta_0'(u - \tilde{u}) \left( F(s, y, \tilde{u}) - \tilde{G}(s, y, \tilde{u}) \right) \phi_{\theta,\zeta} \\
+ E \int_{t,x,y} \left( \frac{1}{2} \eta_0''(u - \tilde{u}) \sum_{k=1}^{\infty} |\sigma^k(u)|^2 \phi_{\theta,\zeta} - \eta_0''(u - \tilde{u}) |\nabla_x \left[ \zeta \right](u)|^2 \phi_{\theta,\zeta} \right) \\
- E \int_{t,x} \left[ \sum_{k=1}^{\infty} \int_0^T \int_y \eta_0'(a - u) \phi_{\theta,\zeta} \sigma^k(u) dW_t^k \right]_{a=\tilde{u}}. \]

Adding them together, we have

\[ -E \int_{t,x,y} \eta_0(u - \tilde{u}) \left( \partial_t \phi_{\theta,\zeta} + \partial_x \phi_{\theta,\zeta} \right) \leq \sum_{i=1}^{6} B_i, \]

with

\[ B_1 := E \int_{t,x,y} \left[ \zeta^2 \eta_0'(\cdot - \tilde{u}) \right](u) \Delta_x \phi_{\theta,\zeta} + E \int_{t,x,y} \left[ \zeta^2 \eta_0'(u - \cdot) \right](u) \Delta_y \phi_{\theta,\zeta}, \]

\[ B_2 := E \int_{t,x,y} \eta_0'(u - \tilde{u}) \left( F(t, x, u) - G(t, x, u) \right) \phi_{\theta,\zeta}, \]

\[ B_3 := E \int_{t,x,y} \left( \frac{1}{2} \eta_0''(u - \tilde{u}) \sum_{k=1}^{\infty} |\sigma^k(u)|^2 \phi_{\theta,\zeta} - \eta_0''(u - \tilde{u}) |\nabla_x \left[ \zeta \right](u)|^2 \phi_{\theta,\zeta} \right), \]

\[ B_4 := E \int_{t,x,y} \left( \frac{1}{2} \eta_0''(u - \tilde{u}) \sum_{k=1}^{\infty} |\sigma^k(u)|^2 \phi_{\theta,\zeta} - \eta_0''(u - \tilde{u}) |\nabla_x \left[ \zeta \right](u)|^2 \phi_{\theta,\zeta} \right), \]

\[ B_5 := E \int_{t,x,y} \left[ \sum_{k=1}^{\infty} \int_0^T \int_x \eta_0'(u - a) \phi_{\theta,\zeta} \sigma^k(u) dW_t^k \right]_{a=\tilde{u}}, \]

\[ B_6 := E \int_{t,x} \left[ \sum_{k=1}^{\infty} \int_0^T \int_y \eta_0'(a - u) \phi_{\theta,\zeta} \sigma^k(u) dW_t^k \right]_{a=\tilde{u}}. \]
Since the integrand of the stochastic integral in $B_{\delta}$ vanish on $[0, s]$, we have $B_{\delta} \equiv 0.$

For $B_{\delta}$, applying the (•)-property of $u$ with $h(r) := -\eta'_{\delta}(-r)$ and $g(x, y) := \phi_{\xi}(x - y)$, we have

$$B_{\delta} \leq C\theta^{1-\mu} - \sum_{k=1}^{\infty} \mathbb{E} \int_{t,x,s,y} \phi_{\theta,\varsigma} \sigma^{k}(u)\sigma^{k}(\bar{u})\eta^{\prime\prime}_{\delta}(u - \bar{u}),$$

for a constant $C$ independent of $\theta$, and we have $\mu = \frac{3m+5}{4(m+1)} < 1$. Therefore, taking $\theta \to 0^+$ as in the proof of [13 Theorem 4.1], we have

$$-\mathbb{E} \int_{t,x,y} \eta_{\delta}(u - \bar{u})\partial_{t}\phi_{\varsigma}$$

$$\leq \mathbb{E} \int_{t,x,y} \|\xi^{2}\eta_{\delta}'(\cdot - \bar{u})\|_{(u)}\Delta_{x}\phi_{\varsigma} + \mathbb{E} \int_{t,x,y} \|\xi^{2}\eta_{\delta}'(u - \cdot)\|_{(\bar{u})}\Delta_{y}\phi_{\varsigma}$$

$$+ \mathbb{E} \int_{t,x,y} \left(\frac{1}{2}\eta_{\delta}''(u - \bar{u}) \sum_{k=1}^{\infty} |\sigma^{k}(u)|^{2}\phi_{\varsigma} - \eta_{\delta}''(u - \bar{u})|\nabla_{x} \|\xi\|_{(u)}|^{2}\phi_{\varsigma}\right)$$

$$+ \mathbb{E} \int_{t,x,y} \left(\frac{1}{2}\eta_{\delta}''(u - \bar{u}) \sum_{k=1}^{\infty} |\sigma^{k}(\bar{u})|^{2}\phi_{\varsigma} - \eta_{\delta}''(u - \bar{u})|\nabla_{y} \|\xi\|_{(\bar{u})}|^{2}\phi_{\varsigma}\right)$$

$$- \sum_{k=1}^{\infty} \mathbb{E} \int_{t,x,y} \phi_{\varsigma}\sigma^{k}(u)\sigma^{k}(\bar{u})\eta^{\prime\prime}_{\delta}(u - \bar{u}),$$

where $u = u(t, x)$ and $\bar{u} = \bar{u}(t, y)$. For the terms including $\sigma^{k}$, using the property of $\eta_{\delta}$ and Assumption 2.3 we have

$$\frac{1}{2} \mathbb{E} \int_{t,x,y} \eta_{\delta}''(u - \bar{u}) \sum_{k=1}^{\infty} (|\sigma^{k}(u)|^{2} - 2\sigma^{k}(u)\sigma^{k}(\bar{u}) + |\sigma^{k}(\bar{u})|^{2}) \phi_{\varsigma}$$

$$\leq C \mathbb{E} \int_{t,x,y} \eta_{\delta}''(u - \bar{u}) \left(\sum_{k=1}^{\infty} |\sigma^{k}(u) - \sigma^{k}(\bar{u})|^{2}\right) \phi_{\varsigma}$$

$$\leq C \mathbb{E} \int_{t,x,y} \eta_{\delta}''(u - \bar{u}) \|u - \bar{u}\|^{1+2\kappa}\phi_{\varsigma} \leq C\delta^{2\kappa}.$$

Furthermore, since $\partial_{x,\phi_{\varsigma}} = -\partial_{y,\phi_{\varsigma}}$ and $\partial_{x,\int_{0}^{\bar{u}} \eta'_{\delta}(r - \bar{u})\zeta^{2}(r)dr = 0$, we have

$$N_{1} := \mathbb{E} \int_{t,x,y} \|\xi^{2}\eta_{\delta}'(\cdot - \bar{u})\|_{(u)}\Delta_{x}\phi_{\varsigma}$$

$$= -\mathbb{E} \int_{t,x,y} 1_{\{\bar{u} \leq u\}} \partial_{x,y,\phi_{\varsigma}} \int_{\bar{u}}^{u} \eta'_{\delta}(r - \bar{u})\zeta^{2}(r)dr$$

$$= -\mathbb{E} \int_{t,x,y} 1_{\{\bar{u} \leq u\}} \partial_{x,y,\phi_{\varsigma}} \int_{\bar{u}}^{u} \int_{\bar{u}}^{r} \eta''_{\delta}(r - \bar{u})\zeta^{2}(r)drd\bar{u}$$

$$= -\mathbb{E} \int_{t,x,y} 1_{\{\bar{u} \leq u\}} \partial_{x,y,\phi_{\varsigma}} \int_{\bar{u}}^{u} \int_{\bar{u}}^{u} 1_{\{\bar{u} \leq r\}} \eta''_{\delta}(r - \bar{u})\zeta^{2}(r)d\bar{u}d\bar{u}.$$
Similarly, we have

\[ N_2 := \mathbb{E} \int_{t,x,y} \| \tilde{\zeta} \eta''_0(u - \cdot)(\tilde{u}) \| \Delta_y \phi_z \]

\[ = -\mathbb{E} \int_{t,x,y} \mathbf{1}_{\{\tilde{u} \leq u\}} \partial_x \phi_z \int_{\tilde{u}}^u \eta''_0(u - \tilde{r}) \tilde{\zeta}^2(\tilde{r}) d\tilde{r} \]

\[ = -\mathbb{E} \int_{t,x,y} \mathbf{1}_{\{\tilde{u} \leq u\}} \partial_x \phi_z \int_{\tilde{u}}^u \int_{\tilde{r}}^u \eta''_0(r - \tilde{r}) \tilde{\zeta}^2(\tilde{r}) dr d\tilde{r} \]

\[ = -\mathbb{E} \int_{t,x,y} \mathbf{1}_{\{\tilde{u} \leq u\}} \partial_x \phi_z \int_{\tilde{u}}^u \int_{\tilde{r}}^u \mathbf{1}_{\{r \leq \tilde{r}\}} \eta''_0(r - \tilde{r}) \tilde{\zeta}^2(\tilde{r}) d\tilde{r} dr. \]

Notice also that

\[ N_3 := -\mathbb{E} \int_{t,x,y} \eta''_0(u - \tilde{u}) |\nabla_x [\zeta](u)|^2 \phi_z - \mathbb{E} \int_{t,x,y} \eta''_0(u - \tilde{u}) |\nabla_y [\tilde{\zeta}](\tilde{u})|^2 \phi_z \]

\[ \leq -2 \mathbb{E} \int_{t,x,y} \eta''_0(u - \tilde{u}) \nabla_x [\zeta](u) \cdot \nabla_y [\tilde{\zeta}](\tilde{u}) \phi_z \]

\[ = -2 \mathbb{E} \int_{t,x,y} \mathbf{1}_{\{\tilde{u} \leq u\}} \phi_z \partial_x [\zeta](u) \partial_y, \int_{\tilde{u}}^u \eta''_0(u - \tilde{r}) \tilde{\zeta}(\tilde{r}) d\tilde{r} \]

\[ = 2 \mathbb{E} \int_{t,x,y} \mathbf{1}_{\{\tilde{u} \leq u\}} \phi_z \partial_x [\zeta](u) \int_{\tilde{u}}^u \eta''_0(u - \tilde{r}) \tilde{\zeta}(\tilde{r}) d\tilde{r}. \]

Applying \[13\] Remark 3.1 with

\[ f(u) := \int_{\tilde{u}}^u \eta''_0(u - \tilde{r}) \tilde{\zeta}(\tilde{r}) d\tilde{r} \]

and using \( \partial_x [\zeta f](u) = f(u) \partial_x [\zeta](u) \), we have

\[ N_3 \leq -2 \mathbb{E} \int_{t,x,y} \mathbf{1}_{\{\tilde{u} \leq u\}} \partial_x \phi_z \int_{\tilde{u}}^u \eta''_0(r - \tilde{r}) \tilde{\zeta}(\tilde{r}) (r) d\tilde{r} dr \]

\[ = 2 \mathbb{E} \int_{t,x,y} \mathbf{1}_{\{\tilde{u} \leq u\}} \partial_x \phi_z \int_{\tilde{u}}^u \int_{\tilde{r}}^u \mathbf{1}_{\{r \leq \tilde{r}\}} \eta''_0(r - \tilde{r}) \tilde{\zeta}(\tilde{r}) (r) d\tilde{r} dr. \]

Then, we have

\[ \sum_{i=1}^3 N_i \leq \mathbb{E} \int_{t,x,y} \mathbf{1}_{\{\tilde{u} \leq u\}} |\partial_x \phi_z| \int_{\tilde{u}}^u \int_{\tilde{r}}^u \mathbf{1}_{\{r \leq \tilde{r}\}} \eta''_0(r - \tilde{r}) |\zeta(r) - \tilde{\zeta}(\tilde{r})|^2 d\tilde{r} dr. \]

With \[13\] estimates (4.13)-(4.17), we have

\[ \sum_{i=1}^3 N_i \leq C \zeta^{-2}(\delta^{2\alpha} + \lambda^2) \mathbb{E} \left( 1 + \|u\|_{L^{m+1}(Q_T)}^{m+1} + \|\tilde{u}\|_{L^{m+1}(Q_T)}^{m+1} \right) \]

\[ + C \zeta^{-2} \mathbb{E} \| \mathbf{1}_{\{|u| \geq R_T\}} (1 + |u|) \|_{L^m(Q_T)}^m \]

\[ + C \zeta^{-2} \mathbb{E} \| \mathbf{1}_{\{|	ilde{u}| \geq R_T\}} (1 + |	ilde{u}|) \|_{L^m(Q_T)}^m \].

(29)
For the other terms in the right hand side of (27), from Assumption 2.4 we have
\[
\begin{align*}
E \int_{t,x,y} \eta_{\delta}(u - \bar{u}) \left[ F(t, x, u) - G(t, x, u) - F(t, y, \bar{u}) + \bar{G}(t, y, \bar{u}) \right] \phi_{\zeta} \\
\leq E \int_{t,x,y} \eta_{\delta}(u - \bar{u}) \left( F(t, x, u) - F(t, y, u) \right) \phi_{\zeta} \\
+ E \int_{t,x,y} \eta_{\delta}(u - \bar{u}) \left( F(t, y, u) - F(t, y, \bar{u}) \right) \phi_{\zeta} \\
+ E \int_{t,x,y} \eta_{\delta}(u - \bar{u}) \left( \bar{G}(t, y, \bar{u}) - G(t, x, u) \right) \phi_{\zeta} \\
\leq C\tilde{\phi} + CE \int_{t,x,y} (u - \bar{u})^+ \phi_{\zeta} (x - y) \varphi(t) \\
+ CE \int_{t,x,y} 1_{\{u > \bar{u}\}} \left( G(t, y, \bar{u}) - G(t, x, u) \right) \phi_{\zeta} (x - y) \varphi(t).
\end{align*}
\]

Combining inequalities (27)-(30) with
\[
\left| E \int_{t,x,y} \eta_{\delta}(u - \bar{u}) \partial_t \phi_{\zeta} - E \int_{t,x,y} (u - \bar{u})^+ \partial_t \phi_{\zeta} \right| \leq C\delta,
\]
we obtain the desired inequality.

\[\Box\]

**Lemma 5.2.** Let Assumptions 2.1 - 2.4 hold for \((\Phi, F, \sigma, \xi)\). Let \(G(t, x, r)\) be a function satisfying \(G(\cdot, \cdot, 0) \in L_2(\Omega_T; L_2(\mathbb{T}^d))\), and be Lipschitz continuous in \(r\) with Lipschitz constant \(\bar{K}\). If \(u\) is an entropy solution of \(\Pi(\Phi, F - G, \xi)\), we have
\[
\lim_{\tau \to 0^+} \frac{1}{\tau} E \int_0^\tau \int_x |u(t, x) - \xi(x)|^2 dt = 0.
\]

The proof of Lemma 5.2 is similar to that of [13, Lemma 3.2] under the Lipschitz continuity of \(F\) and \(G\). Therefore, we omit the proof here.

**Lemma 5.3.** Let Assumptions 2.1 - 2.4 hold for both \((\Phi, F, \sigma, \xi)\) and \((\tilde{\Phi}, F, \sigma, \tilde{\xi})\). Let \(G(t, x, r)\) and \(\bar{G}(t, x, r)\) be two functions, which are Lipschitz continuous in \(r\) with Lipschitz constant \(\bar{K}\), and satisfy \(G(\cdot, \cdot, 0), \bar{G}(\cdot, \cdot, 0) \in L_2(\Omega_T; L_2(\mathbb{T}^d))\). Suppose that \(u\) and \(\bar{u}\) are entropy solutions of \(\Pi(\Phi, F - G, \xi)\) and \(\Pi(\tilde{\Phi}, F - G, \tilde{\xi})\), respectively. If \(u\) has the \((\ast)\)-property, then the following two assertions are true:
(i) if furthermore \(\Phi = \tilde{\Phi}\), then for all \(\zeta, \delta \in (0, 1)\) and \(\alpha \in (0, 1 \wedge (m/2))\), we have
\[
E \int_{t,x,y} (u(t, x) - \bar{u}(t, y))^+ \phi_{\zeta}(x - y) \\
\leq E \int_{t,x,y} (\xi(x) - \tilde{\xi}(y))^+ \phi_{\zeta}(x - y) \\
+ CE(\zeta, \delta)E \left( 1 + \|u\|_{L^{m+1}_t(Q_T)}^{m+1} + \|\bar{u}\|_{L^{m+1}_t(Q_T)}^{m+1} \right) \\
+ CE \int_0^T \int_{t,x,y} (u(t, x) - \bar{u}(t, y))^+ \phi_{\zeta}(x - y)dt \\
+ CE \int_0^T \int_{t,x,y} 1_{\{u(t,x) > \bar{u}(t,y)\}} (G(t, y, \bar{u}(t, y)) - G(t, x, u(t, x)))^+ \phi_{\zeta}(x - y)dt,
\]
25
where
\[ C(\zeta, \delta) = \delta^{2\kappa} + \zeta^2 + \zeta^{-2} \delta^{2\alpha}. \]

(ii) for all \( \zeta, \delta \in (0, 1] \), \( \lambda \in [0, 1] \) and \( \alpha \in (0, 1 \wedge (m/2)) \), we have
\[
E \int_{t,x} (u(t, x) - \tilde{u}(t, x))^+ \\
\leq CE \int_{x \in (\xi(x) - \tilde{\xi}(x))^+ + C \sup_{|h| \leq \xi} E \| \xi(x) - \tilde{\xi}(x) \| L_1(T^2) \\
+ C\zeta^{-2} \left( E \| 1_{\{ |u| \leq R_\lambda \}} (1 + |u|) \|_{L_m(Q_T)}^m \\
+ E \| 1_{\{ |\tilde{u}| \leq R_\lambda \}} (1 + |\tilde{u}|) \|_{L_m(Q_T)}^m \right) \\
+ C\zeta^{m+1} \left( 1 + E \| |\xi(x)| \|_{L_1(T^2)} \right) \\
+ CE \int_{t,x,y} 1_{\{u(t,x) > \tilde{u}(t,y)\}} (\tilde{G}(t, y, \tilde{u}(t,y)) - G(t, x, u(t,x)))^+ \zeta(x - y),
\]

where
\[ \zeta(\xi, \delta, \lambda) = \delta^{2\kappa} + \zeta^2 + \zeta^{-2} \lambda^2 + \zeta^{-2} \delta^{2\alpha}, \]
\[ R_\lambda = \sup \{ R \in [0, \infty] : |\zeta(r) - \tilde{\zeta}(r)| \leq \lambda, \forall |r| < R \}, \]
and the constant \( C \) depends only on \( N_0, K, d, T \) and \( \phi \).

Proof. Let \( s, \tau \in (0, T) \) with \( s < \tau \), be Lebesgue points of the function
\[ t \mapsto E \int_{x \in (u(t, x) - \tilde{u}(t, y))^+ \zeta(x - y). \]

Fix a constant \( \gamma \in (0, \max\{ \tau - s, T - \tau \}) \). We choose a sequence of functions \( \{ \phi_n \}_{n \in \mathbb{N}} \) satisfying \( \phi_n \in C^\infty_c((0, T)) \) and \( \| \phi_n \|_{L_\infty(0,T)} \vee \| \partial_t \phi_n \|_{L_1(0,T)} \leq 1 \), such that
\[ \lim_{n \to \infty} \| \phi_n - V_\gamma \|_{H_0^1((0,T))} = 0, \]
where \( V_\gamma : [0, T] \to \mathbb{R} \) satisfies \( V_\gamma(0) = 0 \) and \( V_\gamma = \gamma^{-1} 1_{[s, s + \gamma]} - \gamma^{-1} 1_{[\tau, \tau + \gamma]} \). Substituting \( \phi \) with \( \phi_n \) in (24) and taking the limit \( n \to \infty \), we have
\[
\frac{1}{\gamma} \int_{t \in [s,\tau]} \int_{x,y} (u(t, x) - \tilde{u}(t, y))^+ \zeta(x - y)dt \\
\leq C\gamma^{-2} \left( E \| 1_{\{ |u| \leq R_\lambda \}} (1 + |u|) \|_{L_m(Q_T)}^m \\
+ E \| 1_{\{ |\tilde{u}| \leq R_\lambda \}} (1 + |\tilde{u}|) \|_{L_m(Q_T)}^m \right) \\
+ C \left( \gamma^{2\kappa} + \zeta^2 + \zeta^{-2} \lambda^2 + \zeta^{-2} \delta^{2\alpha} \right) \cdot E \left( 1 + \|u\|_{L_{m+1}(Q_T)}^{m+1} + \|\tilde{u}\|_{L_{m+1}(Q_T)}^{m+1} \right) \\
+ CE \int_{x \in (u(t,x) > \tilde{u}(t,y))^+ \zeta(x - y)dydt \\
+ CE \int_{x \in (u(t,x) > \tilde{u}(t,y))^+ \zeta(x - y)dydt.
\]
We have already obtained a priori estimates and properties of $L_2$-solution $u_{n,\varepsilon}$ to $\Pi(\Phi_n, F - G_\varepsilon(\cdot, S), \xi_n)$, and gotten $L_1^\gamma$ estimate of two different entropy solutions based on ($\ast$)-property. Applying these results, we now prove the existence of entropy solution $(u, \nu)$ of the obstacle problem $\Pi_S(\Phi, F, \xi)$ in two steps:

$$\frac{1}{\gamma} \mathbb{E} \int_s^{s+\gamma} \int_{x,y} (u(t, x) - \bar{u}(t, y))^+ \varrho_\xi(x - y)\,dt$$

$$=: M(\gamma) + \frac{1}{\gamma} \mathbb{E} \int_s^{s+\gamma} \int_{x,y} (u(t, x) - \bar{u}(t, y))^+ \varrho_\xi(x - y)\,dt.$$ 

Let $\gamma \to 0^+$, we have

$$\mathbb{E} \int_{x,y} (u(\tau, x) - \bar{u}(\tau, y))^+ \varrho_\xi(x - y) \leq M(0) + \mathbb{E} \int_{x,y} (u(s, x) - \bar{u}(s, y))^+ \varrho_\xi(x - y)$$

holds for almost all $s \in (0, \tau)$. Then, for each $\tilde{\gamma} \in (0, \tau)$, by averaging over $s \in (0, \tilde{\gamma})$, we have

$$\mathbb{E} \int_{x,y} (u(\tau, x) - \bar{u}(\tau, y))^+ \varrho_\xi(x - y)$$

$$\leq M(0) + \frac{1}{\tilde{\gamma}} \mathbb{E} \int_0^{\tilde{\gamma}} \int_{x,y} (u(s, x) - \bar{u}(s, y))^+ \varrho_\xi(x - y)\,ds.$$

Taking the limit $\tilde{\gamma} \to 0^+$ and using Lemma 5.2 we have

$$\mathbb{E} \int_{x,y} (u(\tau, x) - \bar{u}(\tau, y))^+ \varrho_\xi(x - y) \leq M(0) + \mathbb{E} \int_{x,y} (\xi(x) - \tilde{\xi}(y))^+ \varrho_\xi(x - y). \quad (31)$$

Taking $\lambda = 0$ and $R_\lambda = \infty$, we obtain the desired inequality in (i).

For (ii), we fixed $s_1 \in (0, T]$. By integrating inequality (31) over $\tau \in (0, s_1)$, we have

$$\mathbb{E} \int_0^{s_1} \int_{x,y} (u(\tau, x) - \bar{u}(\tau, y))^+ \varrho_\xi(x - y)\,d\tau$$

$$\leq T \mathbb{E} \int_x (\xi(x) - \tilde{\xi}(x))^+ + T \sup_{|h| \leq \varsigma} \mathbb{E} \|\tilde{\xi}(\cdot) - \tilde{\xi}(\cdot + h)\|_{L_1(\mathbb{R}^d)}$$

$$+ C \varsigma^{-2} \left( \mathbb{E} \|1_{|u| \geq R_\lambda}(1 + |u|)\|^m_{L_m(Q_T)} + \mathbb{E} \|1_{|u| \geq R_\lambda}(1 + |\bar{u}|)\|^m_{L_m(Q_T)} \right)$$

$$+ C \left( \delta^{2\varsigma_\rho} + \varsigma^{2\varsigma_\nu} + \varsigma^{-2}\lambda^2 + \varsigma^{-2}\delta^{2\alpha} \right) \cdot \mathbb{E} \left( 1 + \|u\|^m_{L_{m+1}(Q_T)} + \|\bar{u}\|^m_{L_{m+1}(Q_T)} \right)$$

$$+ CE \int_0^{s_1} \int_{x,y} (u(t, x) - \bar{u}(t, y))^+ \varrho_\xi(x - y)\,dtd\tau$$

$$+ CE \int_{t,x,y} 1_{u(t,x) > \bar{u}(t,y)}(\tilde{G}(t, y, \bar{u}(t, y)) - G(t, x, u(t, x)))^+ \varrho_\xi(x - y),$$

Moreover, Using Lemma 4.5 and Grönwall’s inequality, we have Assertion (ii). \hfill \Box

### 6 Existence of solution

We have already obtained a priori estimates and properties of $L_2$-solution $u_{n,\varepsilon}$ to $\Pi(\Phi_n, F - G_\varepsilon(\cdot, S), \xi_n)$, and gotten $L_1^\gamma$ estimate of two different entropy solutions based on ($\ast$)-property. Applying these results, we now prove the existence of entropy solution $(u, \nu)$ of the obstacle problem $\Pi_S(\Phi, F, \xi)$ in two steps:
Firstly, we take the limit $n \to \infty$ to prove the existence and comparison theorem of the entropy solution $u_\epsilon$ of $\Pi(\Phi, F - G_\epsilon(\cdot, S), \xi)$. Then, these results indicate the existence of the entropy solution of the obstacle problem $\Pi(\Phi, F, \xi)$ when $\epsilon \to 0^+$.

Fix $\epsilon > 0$, for any $n, n' \in \mathbb{N}$, suppose that $u_{n, \epsilon}$ and $u_{n', \epsilon}$ are $L_2$-solutions of $\Pi(\Phi_n, F - G_\epsilon(\cdot, S), \xi_n)$ and $\Pi(\Phi_{n'}, F - G_\epsilon(\cdot, S), \xi_{n'})$, respectively. Then, Remark 3.4 shows that they are also entropy solutions of the corresponding equations. Using Theorem 4.6, we know that both $u_{n, \epsilon}$ and $u_{n', \epsilon}$ have the $\ast$-property, which is uniform in $n$ and $\epsilon$ if $\mathbb{E} \|\xi\|_{L_2(\mathbb{T}^d)} < \infty$.

**Theorem 6.1.** Let Assumptions 2.2, 2.3 hold. Then, for fixed $\epsilon > 0$, the equation $\Pi(\Phi, F - G_\epsilon(\cdot, S), \xi)$ has an entropy solution $u_\epsilon$. Moreover, there exists a constant $C$ independent of $\epsilon$ such that

$$
\mathbb{E} \sup_{t \leq T} \|u_\epsilon(t)\|_{L_2(\mathbb{T}^d)}^p + \mathbb{E} \|\nabla [u_\epsilon]\|_{L_2(Q_T)}^p + \left(\frac{1}{\epsilon}\right)^{p/2} \mathbb{E} \|u_\epsilon - S\|_{L_2(Q_T)}^p \leq C \left(1 + \mathbb{E} \|\xi\|_{L_2(\mathbb{T}^d)}^p\right),
$$

for some $C > 0$. Let $\epsilon > 0$ be fixed. Because $n$ and $n'$ have the same status, we obtain a same inequality with swapping $n$ and $n'$. Adding them together, we have

$$
\mathbb{E} \int_{x} |u_{n, \epsilon}(t, x) - u_{n', \epsilon}(t, x)|
$$

$$
\leq C \mathbb{E} \int_{x} |\xi_n(t, x) - \xi_{n'}(t, x)| + C \sup_{|h| \leq \xi} \mathbb{E} \|\xi_n(\cdot + h) - \xi_{n'}(\cdot + h)\|_{L_1(\mathbb{T}^d)}
$$

$$
+ C \sup_{|h| \leq \xi} \mathbb{E} \|\xi_n(\cdot) - \xi_{n'}(\cdot + h)\|_{L_1(\mathbb{T}^d)}
$$

$$
+ C \xi^{-2} \left(\mathbb{E} \|1_{\{\|u_{n, \epsilon}\|_{L_1(\mathbb{T}^d)} \geq R\}} (1 + \|u_{n, \epsilon}\|_{L_1(\mathbb{T}^d)})\right)^m
$$

$$
+ \mathbb{E} \|1_{\{\|u_{n, \epsilon}\|_{L_1(\mathbb{T}^d)} \geq R\}} (1 + \|u_{n', \epsilon}\|_{L_1(\mathbb{T}^d)})\right)^m
$$

$$
+ C \xi^{\frac{m}{p}} \left(1 + \mathbb{E} \|\nabla [\xi_n]\|_{L_2(\mathbb{T}^d)}^p + \mathbb{E} \|\nabla [\xi_{n'}]\|_{L_2(\mathbb{T}^d)}^p\right)
$$

$$
+ C \xi^{\frac{m}{p}} \left(1 + \mathbb{E} \|\nabla [\xi_n]\|_{L_2(\mathbb{T}^d)}^p + \mathbb{E} \|\nabla [\xi_{n'}]\|_{L_2(\mathbb{T}^d)}^p\right)
$$

$$
+ C \mathbb{E} \int_{x,y} \left(G_\epsilon(\tilde{u}(t, y), S(t)) - G_\epsilon(u(t, x), S(t))\right)\mathbb{E}[y, x-y].
$$

Note that $G_\epsilon$ is Lipschitz continuous for fixed $\epsilon > 0$. Using Lemma 4.3 and Grönwall's inequality, we can eliminate the last term and obtain the $L_1$ estimate for $u_{n, \epsilon}$ and $u_{n', \epsilon}$.  

28
Without loss of generality, we can assume \( n \leq n' \). Taking \( \lambda = 8/n \) and using Proposition 3.1, we have \( R_\lambda > n \). We also choose \( \vartheta > (m \land 2)^{-1} \) and \( \alpha \in (1/(2\vartheta), (m \land 2)/2) \). Let \( \delta = \varsigma^{2\vartheta} \). With Theorem 3.3 we have
\[
E \int_{t,x} |u_{n,e}(\tau,x) - u_{n',e}(\tau,x)| \leq M_1(\varsigma) + M_2(\varsigma, n, n')
\]
with
\[
M_1(\varsigma) := C \left( \sup_{|h| \leq \varsigma} E \|\xi(\cdot) - \xi(\cdot + h)\|_{L_1(\mathbb{T}^d)} + \varsigma^{4\delta} + \varsigma^{\delta} + \varsigma^{-2+4\alpha\vartheta} + \varsigma^{2} \right)
\]
and
\[
M_2(\varsigma, n, n') := C \left( E \|\xi - \xi_n\|_{L_1(\mathbb{T}^d)} + E \|\xi - \xi_{n'}\|_{L_1(\mathbb{T}^d)} \right) + C\varsigma^{-2}n^{-2}
\]
\[
\quad \quad \quad \quad \quad \quad \quad \quad \quad + E \left( 1\{||u_{n,e}|| \geq n\}(1 + |u_{n,e}|) \right)_{L_m(\mathbb{T}^d)} + E \left( 1\{||u_{n',e}|| \geq n\}(1 + |u_{n',e}|) \right)_{L_m(\mathbb{T}^d)}.
\]

Since \( M_1(\varsigma) \) converges to 0 when \( \varsigma \to 0^+ \), for any \( \varepsilon_0 > 0 \), it is smaller than \( \varepsilon_0 \) for sufficiently small \( \varsigma \). Then, for fixed \( \varsigma \), we choose \( n_0 \) big enough such that \( M_2(\varsigma, n, n') \) is smaller than \( \varepsilon_0 \) for all \( n_0 \leq n \leq n' \). Therefore, we have
\[
E \|u_{n,e} - u_{n',e}\|_{L_1(\mathbb{T}^d)} \leq 2\varepsilon_0, \quad \forall n_0 \leq n \leq n',
\]
which indicates the sequence \( \{u_{n,e}\}_{n \in \mathbb{N}} \) converges to a limit \( u_\varepsilon \) in \( L_1(\Omega_T; L_1(\mathbb{T}^d)) \). By taking a subsequence, when \( n \to \infty \), we may assume that \( u_{n,e} \to u_\varepsilon \) almost surely in \( \Omega_T \times \mathbb{T}^d \). Moreover, Theorem 3.3 shows that the sequence \( \{\|u_{n,e}\|\}_{n \in \mathbb{N}} \) is uniformly integrable on \( \Omega_T \times \mathbb{T}^d \) for all \( q \in [0, m+1] \).

If the right hand sides of (32)-(34) are bounded, with the definition of \( \xi_n \), the left hand sides of the estimates in Theorems 3.3 and 3.9 are weak convergence in the corresponding Banach space. By applying Banach-Saks Theorem and taking a subsequence, the weak limits are the corresponding terms in the left hand side of (32)-(34). Since
\[
\|f\|_\mathfrak{B} \leq \liminf_{n \to \infty} \|f_n\|_\mathfrak{B}, \quad \forall f_n \in \mathfrak{B}, \quad f_n \rightharpoonup f,
\]
for all Banach space \( \mathfrak{B} \), after taking inferior limit to the estimates in Theorems 3.3 and 3.9, we obtain estimates (32)-(34).

Now we only need to verify that \( u_\varepsilon \) is an entropy solution of \( \Pi(\Phi, F - G_c(\cdot, S), \xi) \) in the sense of Definition 2.9. Firstly, Assertion (i) in Definition 2.9 is a direct consequence of (33).

As for Assertion (ii) in Definition 2.9 for any \( f \in C_b(\mathbb{R}) \), using
\[
\|\zeta_n f\| \leq C \|f\|_{L_\infty} |r|^{(m+1)/2}, \quad \forall r \in \mathbb{R},
\]
and \( \partial_x [\zeta_n f] (u_{n,e}) = f(u_{n,e})\partial_x [\zeta_n] (u_{n,e}) \) and Theorem 3.3 we have
\[
\sup_n E \int t \|\zeta_n f\| (u_{n,e})^2_{H^1(\mathbb{T}^d)} < \infty.
\]

By taking a subsequence, we have that \( \|\zeta_n f\| (u_{n,e}) \) converges weakly to some \( v_f \) in \( L_2(\Omega_T; H^1(\mathbb{T}^d)) \), and \( \|\zeta_n\| (u_{n,e}) \) converges weakly to some \( v \) in \( L_2(\Omega_T; H^1(\mathbb{T}^d)) \). Then, with the pointwise convergence and
Let Assumptions 2.1-2.5 hold. For each Lemma 6.3, acquire the entropy formulation in (iii). The proof is complete.

Similar to the proof of Assertion (ii), we have that 
\[ \partial_x \] 
constructed in Theorem 6.1. Then, we have 
\[ \int_{t,x} (u) \partial_x \| \zeta f \| (u) \phi \]
\[ = \lim_{n \to \infty} \int_{t,x} f(u) \partial_x \| \zeta_n \| (u) \phi \]
\[ = \int_{t,x} f(u) \partial_x \| \zeta \| (u) \phi, \quad \forall \phi \in C^\infty(T^d), B \in F. \]

To prove Assertion (iii), denote \( \eta, \varphi \) and \( \phi \) as test functions in Assertion (iii). Applying Itô’s formula and using Itô’s product rule, we have
\[
- E \left[ B \int_0^T \int_{T^d} \eta(u_n) \partial_t \phi dx dt \right] = E \left[ B \int_0^T \int_{T^d} \eta(u_n) \partial_t \phi dx dt + \int_0^T \int_{T^d} \| \zeta_n \| \Delta \phi dx dt \right]
+ \int_0^T \int_{T^d} \left( \frac{1}{2} \eta''(u_n) \sum_{k=1}^\infty |\sigma^k(u_n)|^2 \phi - \eta''(u_n) |\nabla \| \zeta_n \| (u_n)|^2 \phi \right) dx dt
+ \sum_{k=1}^\infty \int_0^T \int_{T^d} \eta'(u_n) \phi \sigma^k(u_n) dx dW^k_i. \]
(35)

Similar to the proof of Assertion (ii), we have that \( \partial_x \| \zeta_n \sqrt{\eta''} \| (u_n) \) converges weakly to \( \partial_x \| \zeta \sqrt{\eta''} \| (u) \) in \( L^2(\Omega_T; L^2(T^d)) \), which implies
\[
E \left[ B \int_0^T \int_{T^d} \eta''(u_n) |\nabla \| \zeta \| (u_n)|^2 \phi dx dt \right]
\leq \liminf_{n \to \infty} E \left[ B \int_0^T \int_{T^d} \eta''(u_n) |\nabla \| \zeta_n \| (u_n)|^2 \phi dx dt \right].
\]

Therefore, taking inferior limit on (35) and using Assumptions 2.1-2.4 and the convergence of \( u_{n,\varepsilon} \), we acquire the entropy formulation in (iii). The proof is complete.

**Remark 6.2.** If furthermore \( E \| \xi \|_{L^4(T^d)}^4 < \infty \), applying Lemma 4.4 and Theorem 4.6, we have the \((\star)\)-property of \( u_\varepsilon \), and the constant \( C \) in Definition 4.1 is independent of \( \varepsilon \).

**Lemma 6.3.** Let Assumptions 2.1-2.5 hold. For each \( \varepsilon_1 > \varepsilon_2 > 0 \), let \( u_{\varepsilon_1} \) and \( u_{\varepsilon_2} \) be the entropy solutions constructed in Theorem 6.1. Then, we have
\[ u_{\varepsilon_1} \geq u_{\varepsilon_2} \geq 0, \quad \text{a.s.} \ (\omega, t, x) \in \Omega_T \times T^d. \]
Proof. For each $n \in \mathbb{N}$, denote $u_{n,\epsilon_1}$ and $u_{n,\epsilon_2}$ are the $L_2$-solutions of $\Pi(\Phi_n, F - G_{\epsilon_1}(\cdot, S), \xi_n)$ and $\Pi(\Phi_n, F - G_{\epsilon_2}(\cdot, S), \xi_n)$, respectively. Based on the proof of Theorem 6.1 by repeatedly taking subsequences, when $n \to \infty$, we can assume $u_{n,\epsilon_1} \to u_{\epsilon_1}$ and $u_{n,\epsilon_2} \to u_{\epsilon_2}$ almost surely in $\Omega_T \times \mathbb{T}^d$. Therefore, we only need to prove

$$u_{n,\epsilon_1} \geq u_{n,\epsilon_2} \geq 0, \quad \text{a.s. } (\omega, t, x) \in \Omega_T \times \mathbb{T}^d,$$

while the second inequality is shown in Lemma 3.5.

For the first inequality, Using Remark 3.4 we have that $L_2$-solutions $u_{n,\epsilon_1}$ and $u_{n,\epsilon_2}$ are also entropy solutions of the corresponding equations. Therefore, we apply Theorem 5.3 with $u = u_{n,\epsilon_2}$ and $\tilde{u} = u_{n,\epsilon_1}$ and take $\Phi = \Phi_n$, $\xi = \tilde{\xi} := \xi_n$, $G(t, x, r) := G_{\epsilon_2}(r, S(t))$ and $\tilde{G}(t, x, r) := G_{\epsilon_1}(r, S(t))$. Then, we have for all $\tau \in [0, T]$, $\varsigma, \delta \in (0, 1)$, $\lambda \in [0, 1]$ and $\alpha \in (0, 1 \wedge (m/2))$,

$$\mathbb{E} \int_{x,y} (u_{n,\epsilon_2}(\tau, x) - u_{n,\epsilon_1}(\tau, y))^+ \varrho_\varsigma(x - y)$$

$$\leq \mathbb{E} \int_{x,y} (\xi_n(x) - \xi_\delta(y))^+ \varrho_\varsigma(x - y)$$

$$+ C\mathcal{E}(\varsigma, \delta)\mathbb{E} \left(1 + \|u_{n,\epsilon_2}\|_{L^{m+1}(Q_T)}^{m+1} + \|u_{n,\epsilon_1}\|_{L^{m+1}(Q_T)}^{m+1}\right)$$

$$+ CE \int_0^\tau \int_{x,y} (u_{n,\epsilon_2}(t, x) - u_{n,\epsilon_1}(t, y))^+ \varrho_\varsigma(x - y)dt$$

$$+ CE \int_0^\tau \int_{x,y} 1\{u_{n,\epsilon_2}(t, x) \geq u_{n,\epsilon_1}(t, y)\}$$

$$\cdot \left(\frac{1}{\epsilon_1} (u_{n,\epsilon_1}(t, y) - S(t))^+ - \frac{1}{\epsilon_2} (u_{n,\epsilon_2}(t, x) - S(t))^+ \right)$$

$$\varrho_\varsigma(x - y)dt.$$

Since

$$\frac{1}{\epsilon_1} (u_{n,\epsilon_1}(t, y) - S(t))^+ - \frac{1}{\epsilon_2} (u_{n,\epsilon_2}(t, x) - S(t))^+$$

$$\leq \frac{1}{\epsilon_2} (u_{n,\epsilon_1}(t, y) - u_{n,\epsilon_2}(t, x))^+,$$

using the non-negativity of $\varrho_\varsigma$, the last term on the right hand side of (36) is no more than 0. On the other hand, choose $\vartheta > (m \wedge 2)^{-1}$ and then $\alpha < 1 \wedge (m/2)$ such that $-2 + (2\alpha)(2\vartheta) > 0$. Let $\delta = \varsigma^{2\vartheta}$ then yields $\mathcal{E}(\varsigma, \delta) \to 0$ as $\varsigma \to 0^+$. Therefore, with the continuity of translations in $L_1$ and taking the limit $\varsigma \to 0^+$, we have

$$\mathbb{E} \int_{x,y} (u_{n,\epsilon_2}(\tau, x) - u_{n,\epsilon_1}(\tau, x))^+$$

$$\leq CE \int_0^\tau \int_{x,y} (u_{n,\epsilon_2}(t, x) - u_{n,\epsilon_1}(t, x))^+dt.$$

Using Grönwall’s inequality, we have the desired result. \qed

**Theorem 6.4.** Let Assumptions 2.7, 2.3 hold. Then, the obstacle problem $\Pi_S(\Phi, F, \xi)$ has an entropy solution $(u, \nu)$ in the sense of Definition 2.1d.
Proof. Let \( \{ \epsilon_i \}_{i \in \mathbb{N}} \) be a monotone decreasing sequence such that \( \lim_{i \to \infty} \epsilon_i = 0 \). From Theorem 6.1 and Lemma 6.3, equation \( \Pi(\Phi, F - G_{\epsilon_i}(\cdot, S), \xi) \) has an entropy solution \( u_{\epsilon_i} \), and the functions \( u_{\epsilon_i} \) almost surely decrease to a limit \( u \) as \( i \to \infty \). This is also a strong convergence in \( L_{m+1}(\Omega; C([0, T]; L_{m+1}(\mathbb{T}^d))) \) based on the dominated convergence theorem and (33), and we have

\[
E \sup_{t \leq T} \| u(t) \|_{L_{m+1}(\mathbb{T}^d)}^{m+1} \leq C \left( 1 + E \| \xi \|_{L_{m+1}(\mathbb{T}^d)}^{m+1} \right).
\]

On the other hand, using estimate (34) and taking a subsequence, we have that the sequence \( \{ G_{\epsilon_i}(u_{\epsilon_i}, S) \}_{i \in \mathbb{N}} \) converges weakly to some function \( \nu \in L_2(\Omega_T \times \mathbb{T}^d) \) as \( i \to \infty \). The non-negativity of \( \nu \) is easily obtained by taking test function in \( L_2(\Omega_T \times \mathbb{T}^d) \). Applying Banach-Saks Theorem and taking the subsequence again, we have

\[
E \| \nu \|_{L_2(\Omega_T)}^2 \leq C \left( 1 + E \| \xi \|_{L_2(\mathbb{T}^d)}^2 \right).
\]

Now we verify that \( (u, \nu) \) is an entropy solution of obstacle problem \( \Pi_S(\Phi, F, \xi) \). Note that Assertion (i) of Definition 2.10 has been proved, and Assertion (ii) and (iii) can be verified as in the proof of Theorem 6.1 via the strong convergences of \( u_{\epsilon_i} \) and \( \eta'(u_{\epsilon_i}) \) and the weak convergence of \( G_{\epsilon_i}(u_{\epsilon_i}, S) \).

For Assertion (iv), using estimate (34) and the strong convergence of \( u_{\epsilon_i} \), we have

\[
E \left\| (u - S)^+ \right\|_{L_2(\Omega_T)}^2 = \lim_{i \to \infty} E \left\| (u_{\epsilon_i} - S)^+ \right\|_{L_2(\Omega_T)}^2 \leq \lim_{i \to \infty} C_{\epsilon_i} \left( 1 + E \| \xi \|_{L_2(\mathbb{T}^d)}^2 \right) = 0.
\]

Therefore, we have \( u \leq S \) almost everywhere in \( \Omega_T \), almost surely.

Furthermore, using the strong convergence of \( u_{\epsilon_i} - S \) and the weak convergence of \( (u_{\epsilon_i} - S)^+ / \epsilon_i \), we obtain

\[
E \int_{\Omega_T} (u - S) \nu \, dt \, dx = \lim_{i \to \infty} E \int_{\Omega_T} (u_{\epsilon_i} - S) \frac{1}{\epsilon_i} (u_{\epsilon_i} - S)^+ \, dt \, dx \geq 0.
\]

Since \( \nu \geq 0 \) and \( u \leq S \), we have

\[
E \int_{\Omega_T} (u - S) \nu \, dt \, dx \leq 0.
\]

Combining these two inequalities, we have that the entropy solution \( (u, \nu) \) satisfies the Skorhod condition.

\[ \Box \]

Remark 6.5. If furthermore \( E \| \xi \|_{L_2(\mathbb{T}^d)}^4 < \infty \), then Lemma 6.4 and Remark 6.2 show that \( u \) in Theorem 6.4 has (\( \ast \))-property, which will be used in the proof of uniqueness.

7 Uniqueness of solution

Theorem 7.1. Let Assumptions 2.1-2.5 hold. Suppose that \( (u, \nu) \) and \( (\tilde{u}, \tilde{\nu}) \) are two entropy solutions of the obstacle problem \( \Pi_S(\Phi, F, \xi) \) and \( \Pi_S(\Phi, F, \tilde{\xi}) \), respectively. Moreover, \( u \) has (\( \ast \))-property. Then, we have

\[
\text{ess sup}_{t \in [0, T]} \int_{\mathbb{T}} |u(t, x) - \tilde{u}(t, x)| \leq CE \int_{\mathbb{T}} |\xi(x) - \tilde{\xi}(x)|
\]

for a constant \( C \) depending only on \( K, N_0, d \) and \( T \). If furthermore \( \xi = \tilde{\xi} \), we have \( u = \tilde{u} \) almost everywhere in \( \Omega_T \), almost surely.
Proof. To get rid of the ($\ast$)-property of $\hat{u}$, we need to adjust the proof of Lemma 5.1 as in the proof of [13 Theorem 4.1]. We take $\eta_\delta \in C^2(\mathbb{R})$ such that

$$\eta_\delta(0) = \eta'_\delta(0) = 0, \quad \eta''_\delta(r) = \rho_\delta(\lvert r \rvert).$$

Therefore, we have

$$\lvert \eta_\delta(r) - \lvert r \rvert \rvert \leq \delta, \quad \text{supp } \eta''_\delta \subset [\delta, \delta], \quad \int_\mathbb{R} \eta''_\delta(r) \, dr \leq 2, \quad \lvert \eta''_\delta \rvert \leq 2\delta^{-1}.$$  

Based on the symmetry of $\eta_\delta$, we apply entropy formulation [5] on $\eta_\delta(r-a)$ with $(r, a) := (u(t, x), \tilde{u}(s, y))$ or $(r, a) := (\hat{u}(s, y), u(t, x))$ instead of both $\eta_\delta(r - a)$ and $\eta_\delta(a - r)$ in Lemma 5.1. By applying the ($\ast$)-property of $u$ and taking the limit $\theta \to 0^+$, for all $\zeta, \delta \in (0, 1]$, we have

$$- \mathbb{E} \int_{t, x, y} \eta_\delta(u - \hat{u}) \partial_t \phi_\zeta$$

$$\leq \mathbb{E} \int_{t, x, y} \lVert \zeta^2 \eta''_\delta(\cdot - \hat{u}) \rvert(u) \Delta_x \phi_\zeta + \mathbb{E} \int_{t, x, y} \lVert \zeta^2 \eta''_\delta(u - \cdot) \rvert(\tilde{u}) \Delta_y \phi_\zeta$$

$$+ \mathbb{E} \int_{t, x, y} \eta'_\delta(u - \tilde{u}) \left[ (F(t, x, u) - \nu) - (F(t, y, \tilde{u}) - \tilde{\nu}) \right] \phi_\zeta$$

$$+ \mathbb{E} \int_{t, x, y} \left( 1 \frac{\eta''_\delta(u - \tilde{u})}{2} \sum_{k=1}^{\infty} \lVert \sigma^k(u) \rvert^2 \phi_\zeta - \eta''_\delta(u - \hat{u}) \rvert(u) \lvert \nabla_x \lVert \zeta \rvert \rvert^2 \phi_\zeta \right)$$

$$+ \mathbb{E} \int_{t, x, y} \left( 1 \frac{\eta''_\delta(u - \hat{u})}{2} \sum_{k=1}^{\infty} \lVert \sigma^k(\tilde{u}) \rvert^2 \phi_\zeta - \eta''_\delta(u - \tilde{u}) \rvert(\tilde{u}) \lvert \nabla_y \lVert \zeta \rvert \rvert^2 \phi_\zeta \right)$$

$$- \sum_{k=1}^{\infty} \mathbb{E} \int_{t, x, y} \phi_\zeta \sigma^k(u) \sigma^k(\hat{u}) \eta''_\delta(u - \hat{u}).$$

Here $u = u(t, x), \nu = \nu(t, x), \tilde{u} = \hat{u}(t, y)$ and $\tilde{\nu} = \hat{\nu}(t, y)$, and the definition of $\phi_\zeta$ can be found in Lemma 5.1. Since $\eta'_\delta$ is odd and monotone, using the Skohorod condition for $(u, \nu)$ and $(\tilde{u}, \tilde{\nu})$, we have

$$\mathbb{E} \int_{t, x, y} \eta'_\delta(u - \hat{u})(\tilde{\nu} - \nu) \phi_\zeta = \mathbb{E} \int_{t, x, y} \eta'_\delta(S - \hat{u}) 1_{\{u = S, \tilde{u} \neq S\}} (-\nu) \phi_\zeta$$

$$+ \mathbb{E} \int_{t, x, y} \eta'_\delta(u - S) 1_{\{u \neq S, \tilde{u} = S\}} \tilde{\nu} \phi_\zeta \leq 0.$$  

The estimates for other terms in the right hand side of (37) are similar to the proof of [13 Theorem 4.1] or Lemma 5.1. Since

$$\mathbb{E} \int_{t, x, y} \eta_\delta(u - \hat{u}) \partial_t \phi_\zeta - \mathbb{E} \int_{t, x, y} \lvert u - \hat{u} \rvert \partial_t \phi_\zeta \leq C \delta$$

$\kappa \in (0, 1/2]$, we have

$$- \mathbb{E} \int_{t, x, y} \lvert u - \hat{u} \rvert \partial_t \phi_\zeta$$

$$\leq C \mathbb{E} \int_{t, x, y} \lvert u - \hat{u} \rvert \phi_\zeta + C \mathbb{E} (\zeta, \delta) \mathbb{E} \left( 1 + \lVert u \rVert_{L_{m+1}(Q_T)}^{m+1} + \lVert \hat{u} \rVert_{L_{m+1}(Q_T)}^{m+1} \right).$$  

33
For $\vartheta > (m \wedge 2)^{-1}$ and $\alpha \in (1/(2\vartheta), 1 \wedge (m/2))$, we have $-1 + 2\alpha \vartheta > 0$. Choose $\delta = \varsigma^{2\vartheta}$ such that $C(\varsigma, \delta) \to 0$ as $\varsigma \to 0^+$. With the continuity of translations, we have

$$-\mathbb{E} \int_{t,x} |u(t, x) - \bar{u}(t, x)| \partial_t \varphi(t) \leq C \mathbb{E} \int_{t,x} |u(t, x) - \bar{u}(t, x)| \varphi(t)$$

for a constant $C$ depending only on $K$, $N_0$, $d$ and $T$. Similar to the proof of (31), we have

$$-\mathbb{E} \int_{t,x} |u(t, x) - \bar{u}(t, x)| \varphi(t) \leq C \mathbb{E} \int_{t,x} |u(t, x) - \bar{u}(t, x)| \varphi(t)$$

for a constant $C$ depending only on $K$, $N_0$, $d$ and $T$. Using Grönwall’s inequality, we prove the theorem.

Now we prove our main theorem.

**Proof of Theorem 2.11.** The existence is referred to Theorem 6.4. For the uniqueness, we define $\xi_n$ as in (8). Denote $(u_n, \nu_n)$ as the entropy solution of the obstacle problem $\Pi_{S}(\Phi, F, \xi_n)$ constructed in Theorem 6.4. From Remark 6.5 the function $u_n$ has $(\star)$-property. Then, Theorem 7.1 indicates the uniqueness of $u_n$.

On the other hand, for any entropy solution $(u, \nu)$ to the obstacle problem $\Pi_{S}(\Phi, F, \xi)$, applying Theorem 7.1 again, we have

$$\text{ess sup}_{t \in [0, T]} \mathbb{E} \int_{\mathbb{T}^d} |u_n(t, x) - u(t, x)| dx \leq C \mathbb{E} \int_{\mathbb{T}^d} |\xi_n(x) - \xi(x)| dx$$

for a constant $C$ depending only on $K$, $N_0$, $d$ and $T$. Therefore, $u_n$ converges to $u$ in $L^1(\Omega_T \times \mathbb{T}^d)$ when $n \to \infty$. Then, the uniqueness of $u_n$ gives the uniqueness of $u$.

Now, we apply entropy formulation (5) with the functions $\eta(r) := r$ and $\eta(r) := -r$. By taking expectations and combining these two inequalities, we have

$$-\mathbb{E} \int_0^T \int_{\mathbb{T}^d} u \partial_t \phi dx dt = \mathbb{E} \left[ \int_{\mathbb{T}^d} \xi \phi(0) dx + \int_0^T \int_{\mathbb{T}^d} \Phi(u) \Delta \phi dx dt ight. + \int_0^T \int_{\mathbb{T}^d} (F(t, x, u) - \nu) \phi dx dt \right].$$

With the uniqueness of $u$, if there exists another entropy solution $(u, \bar{\nu})$ to the obstacle problem $\Pi_{S}(\Phi, F, \xi)$, we have

$$\int_0^T \int_{\mathbb{T}^d} \nu \phi dx dt = \int_0^T \int_{\mathbb{T}^d} \bar{\nu} \phi dx dt,$$

for all test function $\phi := \varphi \varrho \geq 0$, where $(\varphi, \varrho) \in C_0^\infty([0, T)) \times C^\infty(\mathbb{T}^d)$. Therefore, we have $\nu = \bar{\nu}$ almost everywhere in $Q_T$, almost surely.

34
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37

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