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A Model for Spatial Spreading and Dynamics of Fox Rabies on a Growing Domain*

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Abstract: In order to explore the impact of the growth rate of the habitat on the transmission of rabies, we consider a SEI model for fox rabies on a growing spatial domain. The basic reproduction number is introduced using the next infection operator, spectral analysis and the corresponding eigenvalue problem. The stability of equilibria is also established using the upper and lower solutions method in terms of this number. Our results show that a large growth rate of the domain has a negative impact on the prevention and control of rabies. Numerical simulations are presented to verify our theoretical results.

Keywords: SEI model; Fox rabies; Growing domain; Basic reproduction number; Stability

MSC : 35K57, 37L15, 92D25

1 Introduction

Rabies, an acute infectious disease caused by virus infecting the central nervous system, is mainly transmitted by direct contact such as biting [3]. Most mammals are susceptible to the disease, and although only very few human

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fatalities occur every year, rabies is still a considerable threat to human beings on account of inefficient treatment and a nearly 100% mortality rate once it reaches the clinical stage [9]. In order to develop public policies for prevention and control of rabies, various mathematical models have been established to study the transmission mechanism of rabies.

The red fox is the main carrier of rabies in Europe [2]. The following SEI model for fox rabies was proposed and studied by Murray et al. in [15]:

\[
\begin{align*}
E_t &= \beta IS - \sigma E - [b + (a - b)\frac{N}{K}]E, \\
I_t &= D\Delta I + \sigma E - \alpha I - [b + (a - b)\frac{N}{K}]I, \\
S_t &= (a - b)S(1 - \frac{N}{K}) - \beta IS,
\end{align*}
\]

where \(S(x,t), E(x,t)\) and \(I(x,t)\) are the densities of susceptible foxes, infected but non-infectious foxes and rabid foxes at location \(x\) and time \(t\), respectively. \(N = E + I + S\) is the total fox population. On account of the random wandering of the rabid foxes, the diffusion coefficient \(D\) is introduced in the equation for \(I\). \(\alpha\) represents the mortality rate of the rabid foxes and \(\beta\) is the disease transmission coefficient. We assume that infected foxes become infectious at the per capita rate \(\sigma\). \(a\) is the birth rate, \(b\) is the intrinsic death rate and \(K\) is the environmental carrying capacity. The term \((a - b)\frac{N}{K}\) denotes the depletion of the food supply by all foxes, where \(a > b\) ensures a sustainable population size. All coefficients in the model (1.1) are nonnegative constants.

Letting \(W = K - S\), model (1.1) becomes

\[
\begin{align*}
E_t &= \beta I(K - W) - \sigma E - [b + (a - b)\frac{N}{K}]E, \\
I_t &= D\Delta I + \sigma E - \alpha I - [b + (a - b)\frac{N}{K}]I, \\
W_t &= -(a - b)(K - W)(1 - \frac{N}{K}) + \beta I(K - W),
\end{align*}
\]

where \(N = E + I + K - W\) is the total fox population.

Problems describing ecological models on fixed spatial domains have been extensively investigated in the litterature. However, the habitats of species in nature are not invariable. Some habitats are affected by climate, temperature and rainfall, and the shifting boundaries are known, for example the area of Dongting Lake in China changes by season, that is, Dongting lake covers an average area of 1814 square kilometres in summer while it covers only 568 square kilometres in winter in the period 1996 to 2016, see [10], [13], [14], [16], [20] and [24] and references therein. Some habitats are influenced by the species itself and the boundaries are moving and unknown. Such boundaries have recently been described by free boundaries, which have been studied in [8], [11],[21]
and [22] for invasive species and in [12] for the transmission of disease. Domain growth, as one possibility for domain evolution, plays an important role in the formation of living patterns.

Inspired by the aforementioned works, we consider a SEI model (1.2) on a growing domain as in [7]. Let \( \Omega_t \subset \mathbb{R}^2 \) be a bounded growing domain at time \( t \), and its growing boundary is denoted \( \partial \Omega_t \). Also we assume that \( E(x(t), t), I(x(t), t) \) and \( W(x(t), t) \) are the densities of the three kinds of fox population at location \( x(t) \in \Omega_t \) and time \( t \). Additionally, the growth of the domain \( \Omega_t \) generates a flow velocity \( \mathbf{a} = \dot{x}(t) \), that is, the flow velocity is identical to the domain velocity. According to the principle of mass conservation and the Reynolds transport theorem [1], we can formulate the problem on a growing domain related to (1.2) as

\[
\begin{align*}
E_t + \mathbf{a} \cdot \nabla E + E(\nabla \cdot \mathbf{a}) &= \beta I(K - W) - \sigma E - [b + (a - b)\frac{N}{K}]E & \text{in } \Omega_t, \\
I_t - D \Delta I + \mathbf{a} \cdot \nabla I + I(\nabla \cdot \mathbf{a}) &= \sigma E - \alpha I - [b + (a - b)\frac{N}{K}]I & \text{in } \Omega_t, \\
W_t + \mathbf{a} \cdot \nabla W + W(\nabla \cdot \mathbf{a}) &= -(a - b)(K - W)(1 - \frac{N}{K}) + \beta I(K - W) & \text{in } \Omega_t, \\
E(x(0), t) &= I(x(0), t) = W(x(0), t) = 0 & \text{on } \partial \Omega_t, \\
E(x(0), 0) &= E_0(x), I(x(0), 0) = I_0(x), W(x(0), 0) = W_0(x) & \text{in } \overline{\Omega}_0.
\end{align*}
\]

Here \( \mathbf{a} \cdot \nabla E, \mathbf{a} \cdot \nabla I \) and \( \mathbf{a} \cdot \nabla W \) are called advection terms related to the transport of material across \( \partial \Omega_t \) with the flow \( \mathbf{a} \), and other extra terms introduced by the growth of the domain \( \Omega_t \) are the dilution terms \( E(\nabla \cdot \mathbf{a}), I(\nabla \cdot \mathbf{a}) \) and \( W(\nabla \cdot \mathbf{a}) \) due to the local volume expansion [5].

In order to simplify the problem (1.3), we assume that the growth of the domain \( \Omega_t \) is uniform and isotropic, that is, \( \Omega_t \) grows at the same proportion in all directions as time \( t \) increases. Mathematically we can formulate this as

\[
x(t) = \rho(t)y \text{ for all } x(t) \in \Omega_t \text{ and } (y, t) \in \Omega_0 \times [0, +\infty),
\]

where \( \rho(t) \in C^1[0, +\infty) \) is called the growth function and satisfies

\[
\rho(0) = 1, \quad \dot{\rho}(t) > 0, \quad \lim_{t \to \infty} \rho(t) = \rho_\infty > 1 \text{ and } \lim_{t \to \infty} \dot{\rho}(t) = 0.
\]

By Lagrangian transformations (see e.g. [4]), we define \( E(x(t), t) = u_1(y, t), I(x(t), t) = u_2(y, t) \) and \( W(x(t), t) = u_3(y, t) \). Then,

\[
\begin{align*}
u_{1t} &= E_t + \mathbf{a} \cdot \nabla E, \quad u_{2t} = I_t + \mathbf{a} \cdot \nabla I, \quad u_{3t} = W_t + \mathbf{a} \cdot \nabla W; \\
\mathbf{a} &= \dot{x}(t) = \frac{\dot{\rho}(t)}{\rho(t)}x(t), \\
\nabla \cdot \mathbf{a} &= \frac{n\dot{\rho}(t)}{\rho(t)}, \quad \Delta I = \frac{1}{\rho^2(t)}\Delta u_2.
\end{align*}
\]
Using this, the problem (1.3) can be transformed into the following reaction-diffusion model on the fixed domain $\Omega_0$

$$
\begin{align*}
&\begin{cases}
  u_{1t} = \beta u_2(K - u_3) - \sigma u_1 - [b + (a - b)K]u_1 - \frac{n\hat{\rho}(t)}{\rho(t)}u_1, & y \in \Omega_0, \ t > 0, \\
  u_{2t} - \frac{D}{\rho^2(t)}\Delta u_2 = \sigma u_1 - \alpha u_2 - [b + (a - b)K]u_2 - \frac{n\hat{\rho}(t)}{\rho(t)}u_2, & y \in \Omega_0, \ t > 0, \\
  u_{3t} = -(a - b)(K - u_3)(1 - \frac{N}{K}) + \beta u_2(K - u_3) - \frac{n\hat{\rho}(t)}{\rho(t)}u_3, & y \in \Omega_0, \ t > 0, \\
  u_1(y, t) = u_2(y, t) = u_3(y, t) = 0, & y \in \partial \Omega_0, \ t > 0,
\end{cases}
\end{align*}
$$

where $N = u_1 + u_2 + K - u_3$ is the total fox population.

The rest of the paper is organized as follows: Section 2 is devoted to the basic reproduction number of the problem (1.4) as well as its analytic properties. In Section 3, we investigate the stability of the disease-free steady state. Numerical simulations and the discussion are finally presented in Sections 4 and 5, respectively.

## 2 The basic reproduction number

In this section, we will determine the basic reproduction number $R_0$ and analyze its properties. Epidemiologically, the basic reproduction number is a critical threshold that reflects whether the disease will be spread or disappear.

Problem (1.4) admits a disease-free steady state $(0, 0, 0)$. Linearizing system (1.4) at $(0, 0, 0)$ and recalling that $\hat{\rho}(t) \to 0$ as $t \to \infty$, we are led to consider the system

$$
\begin{align*}
&\begin{cases}
  u_t = \beta Kv - (\sigma + a)u, & y \in \Omega_0, \ t > 0, \\
  v_t - \frac{D}{\rho^2(t)}\Delta v = \sigma u - (\alpha + a)v, & y \in \Omega_0, \ t > 0, \\
  w_t = (a - b)(u + v - w) + \beta Kv, & y \in \Omega_0, \ t > 0.
\end{cases}
\end{align*}
$$

Since the first two equations of (1.4) are decoupled from the last equation, we consider the following eigenvalue problem

$$
\begin{align*}
&\begin{cases}
  0 = \frac{\beta K\psi}{R_0^*} - (\sigma + a)\phi, & y \in \Omega_0, \\
  -\frac{D\Delta \psi}{\rho^2}\frac{\sigma\phi}{R_0^*} - (\alpha + a)\psi, & y \in \Omega_0, \\
  \phi(y) = \psi(y) = 0, & y \in \partial \Omega_0.
\end{cases}
\end{align*}
$$

By the variational method, the principal eigenvalue of problem (1.4) can be
calculated to be
\[
R^*_0 = \sup_{\psi \in H^1_0(\Omega_0), \psi \neq 0} \sqrt{\frac{\sigma \beta K \int_{\Omega_0} \psi^2 \, dx}{(\sigma + a) \int_{\Omega_0} \frac{D}{\rho^2} |\nabla \psi|^2 + (\alpha + a) \psi^2 \, dx}},
\]
where \((\lambda_1, \zeta(y))\) is the principal eigen-pair of the eigenvalue problem
\[
\begin{align*}
-\Delta \zeta &= \lambda_1 \zeta, & y &\in \Omega_0, \\
\zeta(y) &= 0, & y &\in \partial \Omega_0.
\end{align*}
\] (2.4)

Similarly as in [23] and [25], in order to define the basic reproduction number \(R^*_0\), we write the first two equations of (2.1) as the following equivalent single equation:
\[
\begin{align*}
U_t &= d \Delta U + FU - VU, & y &\in \Omega_0, \; t > 0, \\
u &= v = 0, & y &\in \partial \Omega_0, \; t > 0,
\end{align*}
\]
where \(U = (u, v)^T\), \(d = (0, D)^T\),
\[
F = \begin{pmatrix} 0 & \beta K \\ 0 & 0 \end{pmatrix},
\]
\[
V = \begin{pmatrix} \sigma + a & 0 \\ -\sigma & \alpha + a \end{pmatrix}.
\]

Let \(X_1 = C(\overline{\Omega}_0, \mathbb{R}^2)\) and \(X^+_1 := C(\overline{\Omega}_0, \mathbb{R}^2_+)\), and let \(T(t)\) be the solution semigroup of the following system on \(X_1\)
\[
\begin{align*}
U_t &= d \Delta U - VU, & y &\in \Omega_0, \; t > 0, \\
u &= v = 0, & y &\in \partial \Omega_0, \; t > 0,
\end{align*}
\]
and let \(\phi(y)\) be the density of the initial infectious fox population. Define the next infection operator \(L\) by
\[
L(\phi)(y) := \int_0^\infty F(y)[T(t)\phi](y) \, dt = F(y) \int_0^\infty [T(t)\phi](y) \, dt.
\]
Then \(R_0 = r(L)\), where \(r(L)\) is the spectral radius of \(L\). We have the following result, we refer to Theorem 11.3.3 in [25] for more details:
Lemma 2.1. \( R_0 = R^*_0 \) and \( \text{sign}(1 - R_0) = \text{sign}\lambda^* \), where \( \lambda^* \) is the principal eigenvalue of the following eigenvalue problem

\[
\begin{aligned}
0 &= \beta K \psi - (\sigma + a)\phi + \lambda \phi, & y \in \Omega_0, \\
-\frac{D \Delta \psi}{\rho^*_\infty} &= \sigma \phi - (\alpha + a)\psi + \lambda \psi, & y \in \Omega_0, \\
\phi(y) &= \psi(y) = 0, & y \in \partial \Omega_0.
\end{aligned}
\]

(2.5)

According to the explicit expression of \( R_0 \), we can list some properties of \( R_0 \).

Theorem 2.1. The following assertions hold.

(i) \( R_0(\rho_\infty, \Omega) \) is a positive and strictly increasing function with respect to \( \Omega \), that is, \( R_0(\rho_\infty, \Omega_1) \leq R_0(\rho_\infty, \Omega_2) \) provided that \( \Omega_1 \subseteq \Omega_2 \), with strict inequality if \( \Omega_2 \setminus \Omega_1 \) is a non-empty open set;

(ii) \( R_0(\rho_\infty, \Omega) \) is a monotonically increasing function with respect to \( \rho_\infty \), in the sense that \( R_0(\rho_\infty, \Omega) < R_0(\rho^*_\infty, \Omega) \) provided that \( \rho_\infty < \rho^*_\infty \).

Proof. The proof of the monotonicity in (i) is similar to Corollary 2.3 in [6]. The proof of (ii) follows directly from (2.3).

\[\Box\]

3 The stability of the disease-free equilibrium

In this section we will investigate the stability of the disease-free equilibrium \( (0, 0, 0) \) in terms of the threshold \( R_0 \). First we introduce the definition of the pair of coupled upper and lower solutions.

Definition 3.1. Let \( (\bar{u}_1(y,t), \bar{u}_2(y,t), \bar{u}_3(y,t)), (\hat{u}_1(y,t), \hat{u}_2(y,t), \hat{u}_3(y,t)) \) be a pair of (triplets of) functions in \( C^2(\Omega_0 \times (0, +\infty)) \) satisfying \( (0, 0, 0) \leq (\bar{u}_1, \bar{u}_2, \bar{u}_3) \leq (\hat{u}_1, \hat{u}_2, \hat{u}_3) \leq (K, K, K) \). The pair (of triplets) is called coupled upper and lower solutions of (1.4), if the following relations
are satisfied:

\[
\begin{aligned}
\dot{u}_1 &\leq \beta \hat{u}_2(K - \hat{u}_3) - \sigma \hat{u}_1 - [b + (a - b)\frac{\hat{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K}]\hat{u}_1 - \frac{n\rho(t)}{\rho(t)} \hat{u}_1, \\
\dot{u}_2 - \frac{D(t)}{\rho(t)} \Delta \hat{u}_2 &\leq \sigma \hat{u}_1 - \alpha \hat{u}_2 - [b + (a - b)\frac{\hat{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K}]\hat{u}_2 - \frac{n\rho(t)}{\rho(t)} \hat{u}_2, \\
\dot{u}_3 &\leq -(a - b)(K - \hat{u}_3)(1 - \frac{\hat{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K}) + \beta \hat{u}_2(K - \hat{u}_3) - \frac{n\rho(t)}{\rho(t)} \hat{u}_3, \\
\dot{\hat{u}}_1 &\geq \beta \hat{u}_2(K - \hat{u}_3) - \sigma \hat{u}_1 - [b + (a - b)\frac{\hat{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K}]\hat{u}_1 - \frac{n\rho(t)}{\rho(t)} \hat{u}_1, \\
\dot{\hat{u}}_2 &\geq \sigma \hat{u}_1 - \alpha \hat{u}_2 - [b + (a - b)\frac{\hat{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K}]\hat{u}_2 - \frac{n\rho(t)}{\rho(t)} \hat{u}_2, \\
\dot{\hat{u}}_3 &\geq -(a - b)(K - \hat{u}_3)(1 - \frac{\hat{u}_1 + \hat{u}_2 + K - \hat{u}_3}{K}) + \beta \hat{u}_2(K - \hat{u}_3) - \frac{n\rho(t)}{\rho(t)} \hat{u}_3,
\end{aligned}
\]

(3.1)

\[
\begin{aligned}
\hat{u}_1(y,t) = 0 &\leq \hat{u}_1(y,t), \hat{u}_2(y,t) = 0 &\leq \hat{u}_2(y,t), \hat{u}_3(y,t) = 0 &\leq \hat{u}_3(y,t), \\
&\leq \hat{u}_1(y,0) \leq \eta_1(y), \hat{u}_2(y,0) \leq \eta_2(y), \hat{u}_3(y,0) \leq \eta_3(y), \\
&\geq \hat{u}_1(y,0) \geq \eta_1(y), \hat{u}_2(y,0) \geq \eta_2(y), \hat{u}_3(y,0) \geq \eta_3(y), \\
&\leq \hat{u}_3(y,t) \leq \bar{u}_3(y,t), \hat{u}_3(y,t) \leq \bar{u}_3(y,t), \hat{u}_3(y,t) \leq \bar{u}_3(y,t).
\end{aligned}
\]

(3.2)

Let

\[
(\hat{u}_1, \hat{u}_2, \hat{u}_3)(y,t) = (0, 0, 0), \quad (\bar{u}_1, \bar{u}_2, \bar{u}_3)(y,t) = (\varepsilon \phi(y), \varepsilon \psi(y), \varepsilon \xi(y)),
\]

(3.3)

where \( \varepsilon \) is sufficiently small, \( \phi(y) \) and \( \psi(y) \) are the normalized positive eigenfunctions in problem (2.2), and \( \xi(y) \) satisfies

\[
0 = \frac{(a - b)\phi + (a - b + \beta K)\psi}{R_0} - (a - b)\xi.
\]

Plugging (3.2) back into (3.1), it is easy to verify that the first three inequalities in (3.1) hold. The fourth inequality becomes

\[
0 \geq \beta K \psi - \sigma \phi - [b + (a - b)\frac{\varepsilon \phi + K - \varepsilon \xi}{K}]\phi - \frac{n\rho(t)}{\rho(t)} \phi.
\]

According to the first equation in (2.2), we only need to prove that

\[
b + (a - b)\frac{\varepsilon \phi + K - \varepsilon \xi}{K} + \sigma + \frac{n\rho(t)}{\rho(t)} \geq R_0(\sigma + a).
\]

(3.4)
Since $R_0 < 1$ and $\varepsilon$ is sufficiently small, (3.4) holds and the fourth inequality in (3.1) holds. The fifth inequality becomes

$$-\frac{D\Delta \psi}{\rho(t)^2} \geq \sigma \phi - \alpha \psi - [b + (a - b)\frac{\varepsilon \psi + K - \varepsilon \xi}{K}]\psi - \frac{n\dot{\rho}(t)}{\rho(t)}\psi. \quad (3.5)$$

It is easy to check that $\psi(y) = \zeta(y)$, where $\zeta(y)$ satisfies (2.4). We have $-\frac{D\Delta \psi}{\rho(t)^2} \geq -\frac{D\Delta \phi}{\rho_{\infty}^2}$ due to $\Delta \psi = \Delta \zeta = -\lambda_1 \zeta \leq 0$. We have that (3.5) is satisfied if

$$-\frac{D\Delta \psi}{\rho_{\infty}^2} \geq \sigma \phi - \alpha \psi - [b + (a - b)\frac{\varepsilon \psi + K - \varepsilon \xi}{K}]\psi - \frac{n\dot{\rho}(t)}{\rho(t)}\psi \quad (3.6)$$

holds. From the second equation in (2.2), (3.5) becomes

$$\left(\frac{1}{R_0} - 1\right)\sigma \phi \geq \{a - [b + (a - b)\frac{\varepsilon \psi + K - \varepsilon \xi}{K}]\} \psi. \quad (3.7)$$

Since $R_0 < 1$ and that the right of (3.7) tends to 0 as $\varepsilon \to 0$, the fifth inequality in (3.1) holds for sufficiently small $\varepsilon$. The sixth inequality in (3.1) is equivalent to

$$0 \geq (a - b)(K - \varepsilon \xi)\frac{\phi + \psi - \xi}{K} + \beta \phi(K - \varepsilon \xi) - \frac{n\dot{\rho}(t)}{\rho(t)}. \quad (3.8)$$

Due to (3.3), (3.8) becomes

$$(a - b)(1 - R_0) + \frac{n\dot{\rho}(t)}{\rho(t)} \geq -\varepsilon [(a - b)\frac{\phi + \psi - \xi}{K}] + \beta \psi. \quad (3.9)$$

Since $R_0 < 1$ and $\dot{\rho}(t) > 0$, (3.8) is also true for sufficiently small $\varepsilon$.

Therefore, the function-pairs

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3)(y, t) = (0, 0, 0), \quad (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)(y, t) = (\varepsilon \phi(y), \varepsilon \psi(y), \varepsilon \xi(y))$$

are the upper and lower solutions of problem (1.4). This implies that the solutions of problem (1.4) lies between the lower solutions and the upper solutions as long as the initial values belong to the prescribed intervals. Therefore, given the condition $R_0 < 1$, we can conclude local stability of the disease-free equilibrium $(0, 0, 0)$. \[\square\]

The next result shows that the disease-free equilibrium $(0, 0, 0)$ is unstable if $R_0 > 1$. 


Theorem 3.2. If $R_0 > 1$, then there exists a $\delta_0 > 0$ such that any positive solution of problem (1.4) satisfies \( \limsup_{t \to \infty} \| (u_1(\cdot, t), u_2(\cdot, t), u_3(\cdot, t)) - (0, 0, 0) \| \geq \delta_0 \).

Proof. We argue by contradiction and assume that for any $\delta \in (0, K)$, there exists a $T_\delta > 0$ such that

\[
0 < u_1(y, t), u_2(y, t), u_3(y, t) < \delta \quad \text{for all } y \in \Omega_0, t \geq T_\delta. \tag{3.10}
\]

We consider the following eigenvalue problem:

\[
\begin{cases}
0 = \beta u_2(K - \delta) - \sigma u_1 - [b + (a - b)\frac{K + 3\delta}{K}]u_1 - \delta u_1 + \lambda u_1, & y \in \Omega_0, \\
-\frac{D\Delta u_1}{(\rho_\infty - \delta)^2} = \sigma u_1 - \alpha u_2 - [b + (a - b)\frac{K + 3\delta}{K}]u_2 - \delta u_2 + \lambda u_2, & y \in \Omega_0, \\
u_1 = u_2 = 0, & y \in \partial \Omega_0.
\end{cases} \tag{3.11}
\]

Problem (3.11) has a principal eigenvalue $\lambda_0^*$ and a pair of positive corresponding eigenfunctions $(\phi_0^*(y), \psi_0^*(y))$. It is easy to check that $\psi_0^*(y) = \zeta(y)$, where $\zeta(y)$ satisfies (2.4). By Lemma 2.1, $R_0 > 1$ implies that $\lambda^* < 0$. Therefore, $\lim_{\delta \to 0} \lambda_0^* = \lambda^* < 0$. We can fix a small $\delta_0 \in (0, K)$ such that $\lambda_0^* < 0$. Then there exists a $T_1 > 0$ such that

\[
0 < u_1(y, t), u_2(y, t), u_3(y, t) < \delta_0 \quad \text{for all } y \in \Omega_0, t \geq T_1.
\]

Since $\lim_{t \to \infty} \rho(t) = \rho_\infty$, there exists a $T_2 > 0$ such that

\[
\rho_\infty - \delta_0 < \rho(t) \leq \rho_\infty \quad \text{for } t \geq T_2.
\]

Similarly, the limit $\lim_{t \to \infty} \frac{n\rho(t)}{\rho(t)} = 0$ implies that there exists a $T_3 > 0$ such that

\[
\frac{n\rho(t)}{\rho(t)} < \delta_0 \quad \text{for } t \geq T_3.
\]

Now choose a large $T^* = \max\{T_1, T_2, T_3\}$. Note that any positive solution $(u_1, u_2, u_3)$ of the problem (1.4) satisfies

\[
\begin{cases}
u_1 \geq \beta u_2(K - \delta_0) - \sigma u_1 - [b + (a - b)\frac{K + 3\delta_0}{K}]u_1 - \delta_0 u_1, & y \in \Omega_0, t \geq T^*, \\
u_2 - \frac{D\Delta u_1}{\rho(t)^2} \geq \sigma u_1 - \alpha u_2 - [b + (a - b)\frac{K + 3\delta_0}{K}]u_2 - \delta_0 u_2, & y \in \Omega_0, t \geq T^*, \\
u_1 = u_2 = 0, & y \in \partial \Omega_0, t \geq T^*. \tag{3.12}
\end{cases}
\]

Define $(u_1(y, t), u_2(y, t))$ to be a positive solution of the problem

\[
\begin{cases}
u_1 = \beta u_2(K - \delta_0) - \sigma u_1 - [b + (a - b)\frac{K + 3\delta_0}{K}]u_1 - \delta_0 u_1, & y \in \Omega_0, t \geq T^*, \\
u_2 - \frac{D\Delta u_1}{\rho(t)^2} = \sigma u_1 - \alpha u_2 - [b + (a - b)\frac{K + 3\delta_0}{K}]u_2 - \delta_0 u_2, & y \in \Omega_0, t \geq T^*, \\
u_1 = u_2 = 0, & y \in \partial \Omega_0, t \geq T^*, \\
u_1(y, T^*) = u_1(y, T^*), u_2(y, T^*) = u_2(y, T^*), & y \in \Omega_0.
\end{cases}
\]
It then follows from the comparison principle that

$$(u_1(y, t), u_2(y, t)) \geq (u_1(y, t), u_2(y, t)) > (0, 0) \text{ for all } y \in \Omega_0, t \geq T^*.$$ (3.13)

Now we conclude that $(u_1(y, T^*), u_2(y, T^*)) \geq (\mu \phi_{\delta_0}(y), \mu \psi_{\delta_0}(y))$ in $\Omega_0$ for sufficiently small $\varepsilon$. In fact, since $u_1(y, T^*), u_2(y, T^*), \phi_{\delta_0}(y)$ and $\psi_{\delta_0}(y)$ are all $> 0$ for $y \in \Omega_0$, we have

$$\frac{\partial u_1(y, T^*)}{\partial \eta} < \frac{1}{2} \frac{\partial u_1(y, T^*)}{\partial \eta} |_{\partial \Omega_0} < 0, \quad \frac{\partial u_2(y, T^*)}{\partial \eta} < \frac{1}{2} \frac{\partial u_2(y, T^*)}{\partial \eta} |_{\partial \Omega_0} < 0,$$

for $y \in B(y_0, \varepsilon(y_0)) \cap \Omega_0$. Set $\mu_1 = \min \left \{ \frac{\partial u_1(y, T^*)}{\partial \eta}, \frac{\partial u_2(y, T^*)}{\partial \eta} \right \}$, then

$$\frac{\partial u_1(y, T^*)}{\partial \eta} \geq \mu_1 \frac{\partial \phi_{\delta_0}(y)}{\partial \eta}, \quad \frac{\partial u_2(y, T^*)}{\partial \eta} \geq \mu_1 \frac{\partial \psi_{\delta_0}(y)}{\partial \eta} \text{ for } y \in B(y_0, \varepsilon(y_0)) \cap \Omega_0.$$

By the mean value theorem, we have

$$u_1(y, T^*) \geq \mu_1 \phi_{\delta_0}(y), \quad u_2(y, T^*) \geq \mu_1 \psi_{\delta_0}(y) \text{ for } y \in B(y_0, \varepsilon(y_0)) \cap \Omega_0.$$

Since $\partial \Omega_0$ is bounded, we can find finitely many points $y_0^i \in \partial \Omega_0$, radii $\varepsilon(y_0^i) > 0$ ($i = 1, \ldots, N$) such that $\partial \Omega_0 \subset \bigcup_{i=1}^N B(y_0^i, \varepsilon(y_0^i))$, hence there exists a small $h = \min_i \varepsilon(y_0^i) > 0$ such that

$$u_1(y, T^*) \geq \mu_1 \phi_{\delta_0}(y), \quad u_2(y, T^*) \geq \mu_1 \psi_{\delta_0}(y) \text{ for } y \in \Omega_0 | \text{dist}(y, \partial \Omega_0) \leq h.$$

Meanwhile, for any $y \in \Omega_0 | \text{dist}(y, \partial \Omega_0) > h$, since $u_1(y, T^*), u_2(y, T^*), \phi_{\delta_0}(y)$ and $\psi_{\delta_0}(y)$ are all $> 0$, there exists a small $\mu_2 > 0$ such that $\frac{\mu_2}{\phi_{\delta_0}(y)} \geq \mu_2$ for $y \in \Omega_0 | \text{dist}(y, \partial \Omega_0) > h$. Therefore, a sufficiently small $\mu > 0$ satisfying $\mu \leq \min \{\mu_1, \mu_2\}$ can be chosen to make sure $(u_1(y, T^*), u_2(y, T^*)) \geq (\mu \phi_{\delta_0}(y), \mu \psi_{\delta_0}(y)) \text{ in } \Omega_0$.

Set

$$U_1 = \mu e^{-\lambda_{\delta_0}(t-T^*)} \phi_{\delta_0}(y) \text{ and } U_2 = \mu e^{-\lambda_{\delta_0}(t-T^*)} \psi_{\delta_0}(y).$$
It is easy to verify that \((U_1(y,t), U_2(y,t))\) is a positive solution of the problem

\[
\begin{aligned}
U_{1t} &= \beta U_2(K - \delta_0) - \sigma U_1 - \left[b + (a - b)\frac{K + 3\delta_0}{K}\right]U_1 - \delta_0 U_1, \quad y \in \Omega_0, \; t \geq T^*, \\
U_{2t} &= \frac{D \Delta U_2}{(\rho_0)^2} + \sigma u_1 - \alpha U_2 - \left[b + (a - b)\frac{K + 3\delta_0}{K}\right]U_2 - \delta_0 U_2, \quad y \in \Omega_0, \; t \geq T^*, \\
U_1 &= U_2 = 0, \\
U_1(y,T^*) &= \mu \phi^*(y), U_2(y,T^*) = \mu \psi^*(y), \\
y \in \partial \Omega_0, \; t \geq T^*, \\
y \in \overline{\Omega}_0.
\end{aligned}
\]

Recalling that \(\Delta \psi^*_0(y) = \Delta \zeta(y) = -\lambda_1 \zeta(y) \leq 0\) yields

\[
U_{2t} \leq \frac{D \Delta U_2}{\rho^2(t)} + \sigma u_1 - \alpha U_2 - \left[b + (a - b)\frac{K + 3\delta_0}{K}\right]U_2 - \delta_0 U_2 \text{ for all } y \in \Omega_0, t \geq T^*,
\]

which means that \((U_1(y,t), U_2(y,t))\) is a lower solution of problem (3.12), so by the comparison principle we have that

\[(u_1(y,t), u_2(y,t)) \geq (U_1(y,t), U_2(y,t)) \text{ for all } y \in \Omega_0, t \geq T^*,\]

which together with (3.13) gives

\[(u_1(y,t), u_2(y,t)) \geq (U_1(y,t), U_2(y,t)) = (\mu e^{-\lambda^*_0(t-T^*)}\phi^*_0(y), \mu e^{-\lambda^*_0(t-T^*)}\psi^*_0(y))\]

for all \(y \in \overline{\Omega}_0, t \geq T^*.\) But since \(\lambda^*_0 < 0\), \(u_1(y,t)\) and \(u_2(y,t)\) tends to \(\infty\) as \(t\) goes to \(\infty\), for any fixed \(y \in \overline{\Omega}_0\) which contradicts (3.10). The proof is now completed.

### 4 Numerical simulations

In this section we carry out some numerical simulations in one space dimension to illustrate our theoretical analysis.

Regarding the domain growth, we choose \(\Omega(t) = (0, x(t)) = (0, \rho(t)y), \) where \(\rho(t) = \frac{e^t}{1 + (e - 1)}\) and \(y \in \Omega_0 = (0, 1).\) Then, the domain grows like \(\rho(t)\) from initial rate \(\rho(0) = 1\) to the final rate \(\rho_\infty = m\) with \(m > 1\). To highlight the impacts of the domain growth on the transmission of rabies, we first fix the following parameters

\[D = 1, a = 1, b = 0.2, K = 1000, \alpha = 0.01, \beta = 0.08, \sigma = 0.05\]

and subsequently obtain \(\lambda_1 = \pi^2\). Next, we choose a different growth rate \(\rho(t)\) for the domain and study the asymptotic behavior of the solution to the problem (1.4).
Example 4.1 Set $m = 1.2$ and we have

$$R_0 = \sqrt{\frac{\sigma \beta K}{(\sigma + a)(\frac{D}{\rho_1\lambda_1} + \alpha + a)}} = 0.64 < 1.$$ 

By Theorem 3.1, we know that the disease-free equilibrium of problem (1.4) is stable. One can see from Fig.1 that the solution $(u_1(y,t), u_2(y,t), u_3(y,t))$ decays to zero, which consists with the result of Theorem 3.1.

Example 4.2 Set $m = 4$ and a direct calculation shows that

$$R_0 = \sqrt{\frac{\sigma \beta K}{(\sigma + a)(\frac{D}{\rho_2\lambda_1} + \alpha + a)}} = 1.05 > 1.$$ 

Theorem 3.2 shows that the disease-free equilibrium $(0, 0, 0)$ is now unstable. It is easily seen from Fig.2 that $(u_1, u_2, u_3)$ stabilizes to a positive steady state.

Comparing the above two cases, it can be seen that the infected but non-infectious population $u_1$ and rabid population $u_2$ vanish at small growth rate, but spread at large growth rate.

5 Discussion

Domain growth plays a significant role in the evolution of a biological population, and this has drawn much attention recently. In order to explore the impact of the domain growth on the transmission of fox rabies, we investigate a SEI model for fox rabies with uniform and isotropic domain growth.

We first transform the SEI model on the growing domain into a reaction-diffusion system on a fixed domain, and the basic reproduction number $R_0$ is introduced by spectral analysis and the so-called next infection operator. The relationship between $R_0$ and $\rho_\infty$ directly follows by the explicit expression of $R_0$ which is determined by the variational method. Then, the stability of the disease-free equilibrium in terms of the threshold value $R_0$ is investigated by the upper and lower solutions method. It is proved in Theorem 3.1 that if $R_0 < 1$, the disease-free steady state $(0, 0, 0)$ for the problem (1.4) is locally asymptotically stable), while if $R_0 > 1$, the disease-free equilibrium $(0, 0, 0)$ is unstable according to Theorem 3.2. Finally our analytical results are clearly supported by numerical simulations. When $R_0 < 1$, the solution of (1.4) decays to zero when the domain growth is small (see Fig.1) while when $R_0 > 1$, the disease-free equilibrium is unstable at a large domain growth (see Fig.2). Our results
Figure 1: $\rho_1(t) = \frac{e^t}{1 + \frac{1}{1+2(e^t-1)}}$. For small growth rate $\rho_1(t)$, we have $R_0 < 1$. The first three graphs show that $(u_1, u_2, u_3)$ decays to zero quickly. The last two graphs in line 3 are the cross-sectional view (the left) and contour map (the right) of species $u_1$, respectively. The color bar in the graph of the cross-sectional view shows the density of the species $u_1$. The contour map shows the convergence of the temporal solution $u_1$ to the trivial solution (red dashed line).
Figure 2: $\rho_2(t) = \frac{e^t}{1 + \frac{1}{4}(e^t - 1)}$. In this case, the growth rate $\rho_2(t)$ is now large enough to give that $R_0 > 1$. $(u_1, u_2, u_3)$ tends to a positive steady state from the first three graphs. The last two graphs present the growth of the domain. The color bar in the graph of the cross-sectional view shows the density of the species $u_1$. The contour map shows the convergence of the temporal solution $u_1$ to the positive solution (red dashed line).
show that a large growth of the domain has a negative effect on the stability of disease-free equilibrium, in the sense that it works against the prevention and control of rabies.

However, we can not derive the existence and uniqueness of the positive equilibrium. Moreover, all coefficients except $\rho(t)$ are constants in the problem (1.4), but in fact rabies is mainly affected by spatial heterogeneity and spatial distribution of habitats ([18], [19]), which implies that the diffusion coefficient $D$ and the disease transmission coefficient $\beta$ (and other constants) depend on the location $x$. We plan to investigate these problems in the future.

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