Systematic derivation of boundary Lax pairs

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Abstract

We systematically derive the Lax pair formulation for both discrete and continuum integrable classical theories with consistent boundary conditions.

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1 The discrete case

Quadratic Poisson structures first appeared as the well-known Sklyanin bracket [1]. A more general form, characterized by a pair of respectively skew symmetric and symmetric matrices \((r, s)\) appeared in [2] in the formulation of consistent Poisson structures for non-ultralocal classical integrable field theories. Finally it was shown [3] that this was the natural quadratic form a la Sklyanin for a non-skew-symmetric \(r\)-matrix, reading:

\[
\{L_1, L_2\} = [r - r^\pi, L_1L_2] + L_1(r + r^\pi)L_2 - L_2(r + r^\pi)L_1. \quad (1.1)
\]

A typical situation when one considers naturally a quadratic Poisson structure for the Lax matrix occurs when considering discrete or continuous integrable systems where the Lax matrix depends on either a discrete or a continuous variable; the Lax pair is thus associated to a point on the space-like lattice or continuous line [4, 5]. Let us first examine the discrete case where one considers a finite set of Lax matrices \(L_n\) labelled by \(n \in \mathbb{N}\).

Lax representation of classical dynamical evolution equations [6] is one key ingredient in the modern theory of classical integrable systems [7]–[11] together with the associated notion of classical \(r\)-matrix [12, 13]. Introduce the Lax pair \((L, A)\) for discrete integrable models [9] (see also [14] for statistical systems), and the associated auxiliary problem (see e.g. [4])

\[
\psi_{n+1} = L_n \psi_n \\
\dot{\psi}_n = A_n \psi_n. \quad (1.2)
\]

From the latter equations one may immediately obtain the discrete zero curvature condition:

\[
\dot{L}_n = A_{n+1} L_n - L_n A_n. \quad (1.3)
\]

The monodromy matrix arises from the first equation (1.2) (see e.g. [4])

\[
T_a(\lambda) = L_{aN}(\lambda) \ldots L_{a1}(\lambda) \quad (1.4)
\]

where index \(a\) denotes the auxiliary space, and the indices \(1, \ldots, N\) denote the sites of the one dimensional classical discrete model.

Consider now a skew symmetric classical \(r\)-matrix which is a solution of the classical Yang-Baxter equation [12, 13]

\[
\begin{bmatrix} r_{12}(\lambda_1 - \lambda_2), & r_{13}(\lambda_1) + r_{23}(\lambda_2) \end{bmatrix} + \begin{bmatrix} r_{13}(\lambda_1), & r_{23}(\lambda_2) \end{bmatrix} = 0, \quad (1.5)
\]

and let \(L\) satisfy the associated Sklyanin bracket

\[
\left\{ L_a(\lambda), \ L_b(\mu) \right\} = \begin{bmatrix} r_{ab}(\lambda - \mu), & L_a(\lambda)L_b(\mu) \end{bmatrix}. \quad (1.6)
\]
It is then immediate that (1.4) also satisfies (1.6). Use of the latter equation shows that the quantities $trT(\lambda)^n$ provide charges in involution, that is

$$\left\{ tr^m(\lambda), \ tr^m(\mu) \right\} = 0$$

(1.7)

which again is trivial by virtue of (1.6). In the simple $sl_2$ case the only non trivial quantity is $trT(\lambda) = t(\lambda)$, that is the usual “bulk” transfer matrix. In the bulk case in particular the zero curvature condition (1.3) is realized by $L_n$ and $[4]$:

$$A_n(\lambda, \mu) = t^{-1}(\lambda) \ tr_a\{T_a(N, n; \lambda) \ r_{ab}(\lambda - \mu) \ T_a(n - 1, 1; \lambda)\}$$

(1.8)

where we define

$$T_a(n, m; \lambda) = L_{an}(\lambda) L_{an-1}(\lambda) \ldots L_{am}(\lambda)$$

(1.9)

We now generalize the procedure described in [4] for periodic boundary conditions to the case of generic integrable “boundary conditions”. We propose a construction of two types of monodromy and transfer matrices, and associated Lax-type evolution equations, albeit incorporating a supplementary set of non-dynamical parameters encapsulated into a “reflection” matrix $K(\lambda)$. Any physical interpretation of the $K$-matrix as a description of the “boundary properties” may not be appropriate in all cases. We should stress that this is the first time to our knowledge (see also [15]) that such an investigation is systematically undertaken. There are several related studies regarding particular examples of open spin chains [16, 17], however the derivation of the corresponding Lax pair is restricted to the Hamiltonian only and not to all associated integrals of motion. In this study we present a generic description independent of the choice of model, and we derive the Lax pair for each one of the entailed boundary integrals of motion.

Subsequently we shall deal with two types of classical algebras, which are derived from two known types of consistent quantum boundary conditions. These boundary conditions are known as soliton preserving (SP), (see e.g. [18–22]), and soliton non-preserving (SNP) [23, 24, 25]. SNP boundary conditions have been also introduced and studied for integrable quantum lattice systems [26–30]. From the algebraic perspective the two types of boundary conditions are associated with two distinct algebras, i.e. the reflection algebra [18] and the twisted Yangian respectively [31, 32] (see also relevant studies [25, 29, 30, 33, 34]). It will be convenient for our purposes here to introduce some useful notation:

$$\hat{r}_{ab}(\lambda) = r_{ba}(\lambda) \quad \text{for SP}, \quad \hat{r}_{ab}(\lambda) = r_{ba}^{t_{ab}}(\lambda) \quad \text{for SNP}$$

$$r_{ab}^*(\lambda) = r_{ab}(\lambda) \quad \text{for SP}, \quad r_{ab}^*(\lambda) = r_{ab}^{t_{ab}}(-\lambda) \quad \text{for SNP}$$

$$\hat{r}_{ab}^*(\lambda) = r_{ba}(\lambda) \quad \text{for SP}, \quad \hat{r}_{ab}^*(\lambda) = r_{ab}^{t_{ab}}(-\lambda) \quad \text{for SNP}$$

(1.10)
The two types of monodromy matrices will respectively represent the classical version of the reflection algebra $\mathbb{R}$, and the twisted Yangian $\mathbb{T}$ written in the compact form: (see e.g. \cite{18, 21}):

$$\begin{align*}
\left\{ T_1(\lambda_1), T_2(\lambda_2) \right\} &= r_{12}(\lambda_1 - \lambda_2)T_1(\lambda_1)T_2(\lambda_2) - T_1(\lambda_1)T_2(\lambda_2)\hat{r}_{12}(\lambda_1 - \lambda_2) \\
+ T_1(\lambda_1)\hat{r}^*_{12}(\lambda_1 + \lambda_2)T_2(\lambda_2) - T_2(\lambda_2)r^*_{12}(\lambda_1 + \lambda_2)T_1(\lambda_1)
\end{align*}$$

(1.11)

where $\hat{r}$, $r^*$, $\hat{r}^*$ are defined in (1.10). In most cases, such as the $A^{(1)}_{N-1}$ $r$-matrices $r^t_{12} = r_{21}$ implying that in the SNP case $r^*_{ab} = \hat{r}^*_{ab}$. In the case of the Yangian $r$-matrix $r_{12} = r_{21}$, hence all the expressions above may be written in a more symmetric form.

In order to construct representations of (1.11) yielding a generating function of integrals of motion one now introduces $c$-number (non-dynamical) representations satisfying the purely algebraic condition (1.11) since:

$$\left\{ K^\pm_1(\lambda_1), K^\pm_2(\lambda_2) \right\} = 0.$$  

(1.12)

Taking now as $T(\lambda)$ any bulk monodromy matrix (1.4) built from local $L$ matrices obeying (1.6) and defining in addition

$$\hat{T}(\lambda) = T^{-1}(-\lambda) \quad \text{for SP,} \quad T(\lambda) = T^t(-\lambda) \quad \text{for SNP.}$$

(1.13)

one shows that representations of the corresponding algebras $\mathbb{R}$, $\mathbb{T}$, are given by the following expression see e.g. \cite{18, 35}:

$$T(\lambda) = T(\lambda) \ K^-(\lambda) \ \hat{T}(\lambda).$$

(1.14)

For a detailed proof see e.g. \cite{35},

Define now as generating function of the involutive quantities

$$t(\lambda) = tr\{K^+(\lambda) \ T(\lambda)\}.$$  

(1.15)

Due to (1.11) it is shown that \cite{18, 35}

$$\left\{ t(\lambda_1), t(\lambda_2) \right\} = 0, \quad \lambda_1, \ \lambda_2 \in \mathbb{C}.$$  

(1.16)

Usually one considers the quantity $\ln t(\lambda)$ to get local integrals of motion, however for the examples we are going to examine here the expansion of $t(\lambda)$ is enough to provide the associated local quantities as will be transparent in the subsequent section. Finally one shows that time evolution of the local Lax matrix $L_n$ under generating Hamiltonian action of $t(\lambda)$ is given by:

$$\dot{L}_n(\mu) = A_{n+1}(\lambda, \mu) \ L_n(\mu) - A_n(\lambda, \mu) \ L_n(\mu),$$  

(1.17)
where $A_n$ is the modified (boundary) quantity,

$$A_n(\lambda, \mu) = \text{tr}_a \left( K_a^+(\lambda) T_a(N, n; \lambda) r_{ab}(\lambda - \mu) T_a(n - 1, 1; \lambda) K_a^-(\lambda) \hat{T}_a(\lambda) \right)$$

$$+ K_a^+(\lambda) T_a(n; \lambda) K_a^-(\lambda) \hat{T}_a(1, n - 1; \lambda) \hat{r}_{ab}^*(\lambda + \mu) \hat{T}_a(n, N; \lambda) \right)$$

(1.18)

where $T(n, m; \lambda)$ is defined in (1.9) and

$$\hat{T}(m, n; \lambda) = \hat{L}_{an}(\lambda) \ldots \hat{L}_{am}(\lambda) \quad n > m.$$

(1.19)

To prove (1.18) we need in addition to (1.6) one more fundamental relation i.e.

$$\left\{ \hat{L}_a(\lambda), L_b(\mu) \right\} = \hat{L}_a(\lambda) \hat{r}_{ab}^*(\lambda) L_b(\mu) - L_b(\mu) \hat{r}_{ab}^*(\lambda + \mu) \hat{L}_a(\lambda).$$

(1.20)

Taking into account (1.6) and the latter expressions we derive:

$$\left\{ t(\lambda), L_{bn}(\mu) \right\} = \text{tr}_a \left( K_a^+(\lambda) T_a(N, n + 1; \lambda) r_{ab}(\lambda - \mu) T_a(n, 1; \lambda) K_a^-(\lambda) \hat{T}(\lambda) \right)$$

$$+ K_a^+(\lambda) T_a(n; \lambda) K_a^-(\lambda) \hat{T}_a(1, n; \lambda) \hat{r}_{ab}^*(\lambda + \mu) \hat{T}_a(n + 1, N; \lambda) \right) L_{bn}(\mu)$$

$$- L_{bn}(\mu) \text{tr}_a \left( K_a^+(\lambda) T_a(N, n; \lambda) r_{ab}(\lambda - \mu) T_a(n - 1, 1; \lambda) K_a^-(\lambda) \hat{T}(\lambda) \right)$$

$$+ K_a^+(\lambda) T_a(\lambda) K_a^-(\lambda) \hat{T}_a(1, n - 1; \lambda) \hat{r}_{ab}^*(\lambda + \mu) \hat{T}_a(n, N; \lambda) \right).$$

(1.21)

Expression (1.18) is readily extracted from (1.21).

Special care should be taken at the boundary points $n = 1$ and $n = N + 1$. Indeed we set: $T(N, N + 1, \lambda) = T(0, 1, \lambda) = \hat{T}(1, 0, \lambda) = \hat{T}(N + 1, N, \lambda) = \mathbb{I}$. We should stress that the derivation of the boundary Lax pair is universal, namely the expressions (1.18) are generic and independent of the choice of $L$, $r$. Note that a different construction of representations of (1.11) was already given in a very general setting in [36]. It is related to the formulation of non-ultralocal integrable field theories on a lattice and extends the analysis of [2].

### 1.1 Example

We shall now examine a simple example, i.e. the open generalized DST model, which may be seen as a lattice version of the generalized (vector) NLS model, (see also [37, 38, 39, 40, 35] for further details). The open Toda chain will also be discussed as a limit of the DST model. We shall explicitly evaluate the “boundary” Lax pairs for the first integrals of motion. We focus here on the special case of the simplest rational non-dynamical $r$-matrices [41]

$$r(\lambda) = \frac{P}{\lambda} \quad \text{where} \quad P = \sum_{i,j=1}^N E_{ij} \otimes E_{ji}$$

(1.22)
\( P \) is the permutation operator, and \((E_{ij})_{kl} = \delta_{ik} \delta_{jl}\).

The Lax operator of the \( gl(N) \) DST model has the following form:

\[
L(\lambda) = (\lambda - \sum_{j=1}^{N-1} x^{(j)} X^{(j)}) E_{11} + b \sum_{j=2}^{N} E_{jj} + b \sum_{j=2}^{N} x^{(j-1)} E_{1j} - \sum_{j=2}^{N} X^{(j-1)} E_{j1} \tag{1.23}
\]

with \( x^{(j)}_n, X^{(j)}_n \) being canonical variables. In [35] the first non-trivial integral of motion for the SNP case, choosing the simplest consistent value \( K^\pm = \mathbb{I} \) was explicitly computed:

\[
\mathcal{H} = -\frac{1}{2} \sum_{n=1}^{N} N_n^2 - b \sum_{n=1}^{N} \sum_{j=1}^{N-1} X^{(j)}_n x^{(j)}_{n+1} - \frac{1}{2} \sum_{j=1}^{N-1} (X^{(j)}_N X^{(j)}_N + b^2 x^{(j)}_1)
\]

where \( N_n = \sum_{j=1}^{N-1} x^{(j)}_n X^{(j)}_n \).

Our aim is now to determine the modified Lax pair induced by the non-trivial integrable boundary conditions. We shall focus here on the case of SNP boundary conditions, basically because in the particular example we consider here such boundary conditions are technically easier to study. Taking into account (1.18) we explicitly derive the modified Lax pair for the generalized DST model with SNP boundary conditions. Indeed, after expanding (1.18) in powers of \( \lambda^{-1} \), and recalling (1.22) we obtain the quantity associated to the Hamiltonian (1.24)

\[
A^{(2)}_n = \lambda E_{11} - \sum_{j \neq 1} X^{(j-1)}_{n-1} E_{j1} + b \sum_{j \neq 1} x^{(j-1)}_n E_{1j}, \quad n \in \{2, \ldots, N\}
\]

\[
A^{(2)}_1 = \lambda E_{11} - b \sum_{j \neq 1} x^{(j-1)}_1 E_{j1} + b \sum_{j \neq 1} x^{(j-1)}_1 E_{1j},
\]

\[
A^{(2)}_{N+1} = \lambda E_{11} - \sum_{j \neq 1} X^{(j-1)}_N E_{j1} + \sum_{j \neq 1} X^{(j-1)}_N E_{1j} \tag{1.25}
\]

It is worth stressing that in the \( sl_2 \) case the SP and SNP boundary conditions coincide given that

\[
L^{-1}(-\lambda) = V L^t(-\lambda) V, \quad V = \text{antid}(1, \ldots, 1). \tag{1.26}
\]

The equations of motion associated to the Hamiltonian (1.24) may be readily extracted by virtue of

\[
\dot{L} = \{\mathcal{H}^{(2)}, L\}. \tag{1.27}
\]

Alternatively the equations of motion may be derived from the zero curvature condition, which the modified Lax pair satisfies. It is clear that to each one of the higher local charges
a different quantity $A_n^{(i)}$ is associated. Both equations (1.27), (1.3) lead naturally to the same equations of motion, which for example in the $sl_2$ case read as:

\[
\begin{align*}
\dot{x}_n &= x_n^2 X_n + bx_{n+1}, \\
\dot{X}_n &= -x_n X_n^2 - bX_{n-1}, \quad n \in \{2, \ldots N - 1\} \\
\dot{x}_1 &= x_1^2 X_1 + bx_2, \\
\dot{X}_1 &= -x_1 X_1^2 - bx_1 \\
\dot{x}_N &= x_N^2 X_N + X_N, \\
\dot{X}_N &= -x_N X_N^2 - bX_{N-1}.
\end{align*}
\] (1.28)

Note that the Toda model [42] may be seen as an appropriate limit of the $sl_2$ DST model (see also [43]), and the corresponding boundary Hamiltonian, Lax pair and equations of motion are easily obtained from the expressions above (for more details see [35]).

2 The continuous case

Let us now recall the basic notions regarding the Lax pair and the zero curvature condition for a continuous integrable model following essentially [4]. Define $\Psi$ as being a solution of the following set of equations (see e.g. [4])

\[
\begin{align*}
\frac{\partial \Psi}{\partial x} &= U(x, t, \lambda)\Psi \quad (2.1) \\
\frac{\partial \Psi}{\partial t} &= V(x, t, \lambda)\Psi \quad (2.2)
\end{align*}
\]

$U$, $V$ being in general $n \times n$ matrices with entries defined as functions of complex valued dynamical fields, their derivatives, and the spectral parameter $\lambda$. The monodromy matrix from (2.1) may be written as:

\[
T(x, y, \lambda) = \mathcal{P} \exp\left\{ \int_y^x U(x', t, \lambda)dx' \right\}. \quad (2.3)
\]

The fact that $T$ also satisfies equation (2.1) will be extensively used to get the relevant integrals of motion. Compatibility conditions of the two differential equations (2.1), (2.2) lead to the zero curvature condition [8]–[10]

\[
\dot{U} - V' + [U, V] = 0, \quad (2.4)
\]

giving rise to the corresponding classical equations of motion of the system under consideration.

Hamiltonian formulation of the equations of motion is available again under the $r$-matrix approach. In this picture the underlying classical algebra is manifestly analogous to the quantum case. The existence of the Poisson structure for $\bar{U}$ realized by the classical $r$-matrix, satisfying the classical Yang-Baxter equation (1.5), guarantees the integrability of
the classical system. Indeed assuming that the operator $U$ satisfies the following ultralocal form of Poisson brackets
\[ \{ U_a(x, \lambda), U_b(y, \mu) \} = \left[ r_{ab}(\lambda - \mu), U_a(x, \lambda) + U_b(y, \mu) \right] \delta(x - y), \] (2.5)
then $T(x, y, \lambda)$ satisfies (1.6), and consequently one may readily show for a system on the full line:
\[ \left\{ \ln tr\{T(x, y, \lambda_1)\}, \ln tr\{T(x, y, \lambda_2)\} \right\} = 0 \] (2.6)
i.e. the system is integrable, and the charges in involution –local integrals of motion– are obtained by expansion of the generating function $\ln tr\{T(x, y, \lambda)\}$, based essentially on the fact that $T$ satisfies (2.1).

Our aim here is to consider integrable models on the interval with consistent “boundary conditions”, and derive rigorously the Lax pairs associated to the entailed boundary local integrals of motion as a continuous extension of the discrete case described previously. We briefly describe this process below for any classical integrable system on the interval. In this case one constructs a modified transition matrix $\hat{T}$, based on Sklyanin’s formulation and satisfying again the Poisson bracket algebras $\mathbb{R}$ or $\mathbb{T}$. To construct the generating function of the integrals of motion one also needs $c$-number representations of the algebra $\mathbb{R}$ or $\mathbb{T}$ satisfying (1.11), (1.12). The modified transition matrices, realizing the corresponding algebras $\mathbb{R}$, $\mathbb{T}$ are given by (1.14), where now $T$ defined in (2.3) and $\hat{T}$ in (1.13). The generating function of the involutive quantities is defined in (1.15) and one shows in this case as well:
\[ \{ t(x, y, t, \lambda_1), t(x, y, t, \lambda_2) \} = 0, \quad \lambda_1, \lambda_2 \in \mathbb{C}. \] (2.7)

In the case of open boundary conditions, exactly as in the discrete integrable models, we prove (for more details on the proof see [15])
\[ \left\{ T_a(0, -L, \lambda), U_b(x, \mu) \right\} = M'_a(x, \lambda, \mu) + \left[ M_a(x, \lambda, \mu), U_b(x, \mu) \right] \] (2.8)
where we define
\[ M_a(x, \lambda, \mu) = T(0, x, \lambda)r_{ab}(\lambda - \mu)T(x, -L, \lambda)K^{-}(\lambda)\hat{T}(0, -L, \lambda) + T(0, -L, \lambda)K^{-}(\lambda)\hat{T}(x, -L, \lambda)\hat{r}_{ab}(\lambda + \mu)\hat{T}(0, x, \lambda). \] (2.9)
Finally bearing in mind the definition of $t(\lambda)$ (1.15), and (2.8) we conclude with:
\[ \left\{ \ln t(\lambda), U(x, \mu) \right\} = \frac{\partial V(x, \lambda, \mu)}{\partial x} + \left[ V(x, \lambda, \mu), U(x, \mu) \right] \] (2.10)
where

\[ V(x, \lambda, \mu) = t^{-1}(\lambda) \operatorname{tr}_a \left( K^+(\lambda) M_a(x, \lambda, \mu) \right). \tag{2.11} \]

As in the discrete case particular attention should be paid to the boundary points \( x = 0, -L \). Indeed, for these two points one has to simply take into account that \( T(x, x, \lambda) = \hat{T}(x, x, \lambda) = I \). Moreover, the expressions derived in \[2.9], \[2.11\] are universal, that is independent of the choice of model.

\subsection{2.1 Example}

We shall now examine a particular example associated to the rational \( r \)-matrix \[1.22\], that is the \( gl_N \) NLS model. Although in \[35\] an extensive analysis for both types of boundary conditions is presented, here we shall focus on the simplest diagonal \((K^\pm = I)\) boundary conditions. The Lax pair is given by the following expressions \[4, 44\]:

\[ U = U_0 + \lambda U_1, \quad V = V_0 + \lambda V_1 + \lambda^2 V_2 \tag{2.12} \]

where

\[ U_1 = \frac{1}{2i} \left( \sum_{i=1}^{N-1} E_{ii} - E_{N,N} \right), \quad U_0 = \sum_{i=1}^{N-1} (\bar{\psi}_i E_{N,i} + \psi_i E_{iN}) \]

\[ V_0 = i \sum_{i,j=1}^{N-1} (\bar{\psi}_i \psi_j E_{ij} - |\psi_i|^2 E_{N,N}) - i \sum_{i=1}^{N-1} (\bar{\psi}_i E_{iN} - \psi_i E_{N,i}), \]

\[ V_1 = -U_0, \quad V_2 = -U_1 \tag{2.13} \]

and \( \psi_i, \bar{\psi}_j \) satisfy \[4\].

\[ \{ \psi_i(x), \psi_j(y) \} = \{ \bar{\psi}_i(x), \bar{\psi}_j(y) \} = 0, \quad \{ \psi_i(x), \bar{\psi}_j(y) \} = \delta_{ij} \delta(x - y). \tag{2.15} \]

The boundary Hamiltonian for the generalized NLS model may be expressed as (see \[35\])

\[ \mathcal{H} = \int_{-L}^0 dx \sum_{i=1}^{N-1} \left( \kappa |\psi_i(x)|^2 \sum_{j=1}^{N-1} |\psi_j(x)|^2 + \psi_i(x) \bar{\psi}_i(x) \right) \]

\[ - \sum_{i=1}^{N-1} \left( \psi_i(0) \bar{\psi}_i(0) + \psi_i(0) \bar{\psi}_i(0) \right) + \sum_{i=1}^{N-1} \left( \psi_i(-L) \bar{\psi}_i(-L) + \psi_i(-L) \bar{\psi}_i(-L) \right). \tag{2.16} \]

\[ \text{\footnotesize The Poisson structure for the generalized NLS model is defined as:} \]

\[ \{ A, B \} = i \sum_i \int_{-L}^L dx \left( \frac{\delta A}{\delta \psi_i(x)} \frac{\delta B}{\delta \psi_i(x)} - \frac{\delta A}{\delta \bar{\psi}_i(x)} \frac{\delta B}{\delta \bar{\psi}_i(x)} \right) \tag{2.14} \]
One sees here that the $K$-matrix indeed contributes as a genuine boundary effect. The Hamiltonian, obtained as one of the charges in involution (see e.g. [35] for further details) provides the classical equations of motion by virtue of:

$$\frac{\partial \psi_i(x,t)}{\partial t} = \left\{ \mathcal{H}(0,-L), \psi_i(x,t) \right\}, \quad \frac{\partial \bar{\psi}_i(x,t)}{\partial t} = \left\{ \mathcal{H}(0,-L), \bar{\psi}_i(x,t) \right\},$$

$$-L \leq x \leq 0. \tag{2.17}$$

Indeed considering the Hamiltonian $\mathcal{H}$, we end up with the following set of equations with Dirichlet type boundary conditions

$$i \frac{\partial \psi_i(x,t)}{\partial t} = -\frac{\partial^2 \psi_i(x,t)}{\partial^2 x} + 2\kappa \sum_{j=1}^{N-1} |\psi_j(x,t)|^2 \psi_i(x,t)$$

$$\psi_i(0) = \psi_i(-L) = 0 \quad i \in \{1, \ldots, N-1\}. \tag{2.18}$$

For a detailed and quite exhaustive analysis of the various integrable boundary conditions of the NLS model see [35].

As mentioned our ultimate goal here is to derive the boundary Lax pair, in particular the $\mathbb{V}$ operator. Hereafter we shall focus on the SP case with the simplest boundary conditions i.e. $K^\pm = \mathbb{I}$. For any $gl_N$ $r$-matrix we may expand (2.11), taking also into account (1.22), in powers of $\lambda^{-1}$ (we refer the interested reader to [35, 15] for technical details) and we obtain $\mathbb{V}^{(3)}(x,\lambda)$ –the bulk part– coincides with $\mathbb{V}$ defined in (2.12), (2.13), and for the boundary points $x_b \in \{0, -L\}$ in particular:

$$\mathbb{V}^{(3)}(x_b,\lambda) = -\frac{\lambda^2}{2i} \left( \sum_{i=1}^{N-1} E_{ii} - E_{NN} \right) + i \sum_{i,j=1}^{N-1} \bar{\psi}_i(x_b) \psi_j(x_b) E_{ij} - i \sum_{i,j=1}^{N-1} \left( \bar{\psi}_i(x_b) E_{iN} - \psi_i(x_b) E_{Ni} \right). \tag{2.19}$$

We may alternatively rewrite the latter formula as:

$$\mathbb{V}^{(3)}(x_b,\lambda) = \mathbb{V}(x_b,\lambda) + i \sum_{i=1}^{N-1} |\psi_i(x_b)|^2 E_{iN} + \lambda \sum_{i=1}^{N-1} (\bar{\psi}_i(x_b) E_{iN} + \psi_i(x_b) E_{Ni}). \tag{2.20}$$

The last two terms additional to $\mathbb{V}$ (2.12), (2.13) are due to the non-trivial boundary conditions; of course more complicated boundary conditions would lead to more intricate modifications of the Lax operator $\mathbb{V}$.

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