1. Introduction

Fundamental and deep connections have been developed in recent years between the geometry of Hilbert schemes $X[n]$ of points on a (quasi-)projective surface $X$ and combinatorics of symmetric functions. Among distinguished classes of symmetric functions, let us mention the monomial symmetric functions, Schur polynomials, Jack polynomials (which depend on a Jack parameter), and Macdonald polynomials, etc (cf. [Mac]). The monomial symmetric functions can be realized as certain ordinary cohomology classes of the Hilbert schemes associated to an embedded curve in a surface (cf. [Na1]). Nakajima [Na2] further showed that the Jack polynomials whose Jack parameter is a positive integer $\gamma$ can be realized as certain $T$-equivariant cohomology classes of the Hilbert schemes of points on the surface $X(\gamma)$ which is the total space of the line bundle $O_{\mathbb{P}^1}(-\gamma)$ over the complex projective line $\mathbb{P}^1$. Here and below $T$ stands for the one-dimensional complex torus. In other words, the Jack parameter is interpreted as minus the self-intersection number of the zero-section in $X(\gamma)$. With very different motivations, Haiman (cf. [Hai] and the references therein) developed connections between the Macdonald polynomials and the geometry of Hilbert schemes, and in particular realized the Macdonald polynomials as certain $T$-equivariant $K$-homology classes of the Hilbert schemes of points on the affine plane $\mathbb{C}^2$ (A similar result has been conjectured in [Na2]).

In this note, we shall establish a link somewhat different from [Na2] between equivariant cohomology of Hilbert schemes and Jack polynomials, and then use this to describe the equivariant and ordinary cohomology rings of the Hilbert schemes of points on the surface $X(\gamma)$. We first show that the Jack polynomials can be realized in terms of certain $T$-equivariant cohomology classes of the Hilbert schemes of points on the affine plane, and the Jack parameter comes from the ratio of the $T$-weights on the two affine lines preserved by the $T$-action. In our view, the present construction is conceptually simpler than the original one in [Na2]. This result is probably not very surprising however and could be well anticipated by experts (as it is done by elaborating the ideas of [Na2] with new inputs from [Vas]). But we
feel it is nice to formulate it precisely and to make it accessible to the public. In the case when the ratio mentioned above is one, the Jack polynomials are the Schur polynomials, and our current construction specializes to \cite{Vas} (also cf. \cite{LQW}).

In addition, we study the $T$-equivariant cohomology ring of $X(\gamma)[n]$ with respect to a certain $T$-action. The $T$-action used in this note has the property that the $T$-fixed points are isolated, and is quite different from the one used in \cite{Na2}. Generalizing an idea in \cite{Vas}, we can define a ring structure on $H^*_T X$ modified from and in turn encodes the $T$-equivariant cohomology groups of middle degree. This construction generalizes the one in \cite{Vas}, which is in turn a modification of the construction in \cite{Na1} (also cf. \cite{Gro}).

Finally, we note that the ordinary cohomology ring of the Hilbert scheme $X(\gamma)[n]$ can be shown to be isomorphic to the graded ring associated to a natural filtration on the ring $H^*_T X(\gamma)[n]$. In this way, we obtain an algorithm for computing the ordinary cup product of cohomology classes in $X(\gamma)[n]$. 

2. Equivariant cohomology rings of Hilbert schemes

2.1. Surfaces with torus actions.

Let $T = \mathbb{C}^*$, and $\theta$ be the 1-dimensional standard $T$-module. For an algebraic variety $M$ with a $T$-action, let $H^*_T(M)$ be the equivariant cohomology ring of $M$ with $\mathbb{C}$ coefficients. Then $H^*_T(M)$ is a $\mathbb{C}[t]$-module if we identify $H^*_T(pt)$ and $\mathbb{C}[t]$ ($t$ is an element of degree-2). For a $T$-equivariant and proper morphism $f: N \to M$ of algebraic varieties, there is a Gysin homomorphism $f_!: H^*_T(N) \to H^*_T(M)$ of $T$-equivariant cohomology groups. If $N$ is a $T$-equivariant codimension-$k$ closed subvariety of $M$ and $i: N \to M$ is the inclusion map, define $[N] = i_!(1_N) \in H^*_T(M)$ where $1_N \in H^0_T(N)$ is the unit of the algebra $H^*_T(N)$.

In this note, we shall consider two types of surfaces with $T$-actions, and the Hilbert schemes of points on these surfaces. The first one is the complex plane $\mathbb{C}^2$, while the second is the total space of a line bundle over $\mathbb{P}^1$. The $T$-actions on these surfaces are specified in the following two examples.

Example 2.1. Fix two nonzero integers $\alpha$ and $\beta$ with the same signs. Let $u, v$ be the standard coordinate functions on $\mathbb{C}^2$. We define the action of $T$ on $\mathbb{C}^2$ by

$$s \cdot (u, v) = (s^\alpha u, s^{-\beta} v), \quad s \in T. \quad (2.1)$$

The origin of $\mathbb{C}^2$ is the only fixed point, which will be denoted by $x$. Let $\Sigma$ and $\Sigma'$ be the $u$-axis and $v$-axis respectively in $\mathbb{C}^2$. As $T$-modules, we have $T_x \Sigma = \theta^{-\alpha}$
and $T_{2}Σ′ = θ^{3}$. By the localization theorem (see [C-K]), we get

$$[Σ] = -α^{-1}t^{-1}[x], \quad [Σ′] = β^{-1}t^{-1}[x].$$

**Example 2.2.** Fix an integer $γ > 1$. Let $X(γ)$ be the total space of the line bundle $𝒪_{ℙ^{1}}(−γ)$ over $ℙ^{1}$. The quasi-projective surface $X(γ)$ can be regarded as the quotient space of $ℂ \times (ℂ^{2} − \{0\})$ by the $ℂ^{*}$-action defined by

$$s \cdot (b, b_{1}, b_{2}) = (s^{−γ}b, sb_{1}, sb_{2}), \quad s \in ℂ^{*}.$$

We use $[(b, b_{1}, b_{2})]$ to denote the equivalence class. Define a $T$-action on $X(γ)$ by

$$s \cdot [(b, b_{1}, b_{2})] = [(sb, s^{−1}b_{1}, b_{2})], \quad s \in T.$$

(2.4)

For $i = 1, 2$, let $X_{i}$ be the open subset of $X(γ)$ given by

$$X_{1} = \{(b, b_{1}, b_{2}) | b_{2} = 1\}, \quad X_{2} = \{(b, b_{1}, b_{2}) | b_{1} = 1\}.$$  

(2.5)

Then $X_{1}$ and $X_{2}$ form an affine open cover of $X(γ)$. Moreover, each $X_{i}$ is $T$-invariant. For simplicity, denote the point $[(b, b_{1}, 1)] \in X_{1}$ by $(b, b_{1})$. Similarly, denote $[(b, 1, b_{2})] \in X_{2}$ by $(b, b_{2})$. Then $T$ acts on the points of $X_{1}$ by $s \cdot (b, b_{1}) = (sb, s^{−1}b_{1})$, i.e., $T$ acts on the coordinate functions $u_{1}$ and $v_{1}$ of $X_{1}$ by

$$s \cdot (u_{1}, v_{1}) = (s^{−1}u_{1}, sv_{1}), \quad s \in T.$$

(2.6)

Similarly, $T$ acts on the coordinate functions $u_{2}$ and $v_{2}$ of $X_{2}$ by

$$s \cdot (u_{2}, v_{2}) = (s^{−1}u_{2}, s^{−1}v_{2}), \quad s \in T.$$  

(2.7)

Let $x_{i}$ be the origin of $X_{i}$. Then $X(γ)^{T} = \{x_{1}, x_{2}\}$. Let $ρ : X(γ) → ℙ^{1}$ be the projection sending $[(b, b_{1}, b_{2})]$ to $[b_{1}, b_{2}]$. Let $Σ_{0} \cong ℙ^{1}$ be the zero section of $ρ$, and

$$Σ_{1} = ρ^{-1}(0, 1), \quad Σ_{2} = ρ^{-1}(1, 0).$$

(2.8)

Then as $T$-modules, we have $T_{x_{1}}Σ_{1} = θ$, $T_{x_{1}}Σ_{0} = θ^{−1}$, $T_{x_{2}}Σ_{0} = θ$, and $T_{x_{2}}Σ_{2} = θ^{1−γ}$. By the localization theorem, we get

$$[Σ_{1}] = t^{-1}[x_{1}], \quad [Σ_{0}] = −t^{-1}[x_{1}] + t^{-1}[x_{2}], \quad [Σ_{2}] = (1 − γ)^{-1}t^{-1}[x_{2}].$$

(2.9)

**Remark 2.3.** (i) Let $X(1)$ be the total space of $𝒪_{ℙ^{1}}(−1)$. Using $[(b, b_{1}, b_{2})] ∈ X(1)$ to denote the equivalence class defined by (2.3), we define a $T$-action on $X(1)$ by

$$s \cdot [(b, b_{1}, b_{2})] = [(sb, s^{−2}b_{1}, b_{2})], \quad s \in T.$$

Then our methods and results below apply to $X(1)$ as well.

(ii) Other $T$-actions on the surfaces $X(γ)$, $γ ≥ 1$, with isolated fixed points can be treated similarly.

### 2.2. Distinguished equivariant cohomology classes.

In the rest of this note, let $X$ be a surface in Example 2.1 or Example 2.2. Our goal in this subsection is to define some distinguished equivariant cohomology classes for the Hilbert schemes. Let $X^{[n]}$ be the Hilbert scheme parametrizing all the 0-dimensional closed subschemes $ξ$ of $X$ with $\dim_{ℂ} H^{0}(𝒪_{ξ}) = n$. The $T$-action on $X$ induces a $T$-action on $X^{[n]}$. The support of a $T$-fixed point in $X^{[n]}$ is contained in $X^{T}$. By the results in [ES], the $T$-fixed points of $X^{[n]}$ are isolated and parametrized in terms of (multi-)partitions.
Next, let $X = \mathbb{C}^2$ as in Example 2.1. The $T$-fixed points of $X^{[n]}$ are supported in $X^T = \{x\}$ and indexed by partitions $\lambda$ of $n$. We use $\xi_\lambda$ to denote the fixed point in $(X^{[n]})^T$ corresponding to a partition $\lambda$ of $n$. Then for $\lambda \vdash n$, the tangent space of $X^{[n]}$ at the fixed point $\xi_\lambda$ is $T$-equivariantly isomorphic to (see [E-S, Na1, Na2]):

$$T_{\xi_\lambda}X^{[n]} = \bigoplus_{\square \in D_\lambda} (\theta^{a(\square) + \beta(\square) - \alpha(\square) - (\alpha(\square) + 1)})$$

(2.10)

where $D_\lambda$ is the Young diagram associated to the partition $\lambda$, $\square$ is a cell in $D_\lambda$, $\ell(\square)$ is the leg of $\square$, and $a(\square)$ is the arm of $\square$ (see [Mac] for the notations). So

$$e_T(T_{\xi_\lambda}X^{[n]}) = (-1)^n c_\lambda(\alpha, \beta)c'_\lambda(\alpha, \beta)t^{2n}.$$  

(2.11)

where $e_T(\cdot)$ stands for the equivariant Euler class and

$$c_\lambda(\alpha, \beta) = \prod_{\square \in D_\lambda} (\alpha(\ell(\square) + 1) + \beta(a(\square))),$$

(2.12)

$$c'_\lambda(\alpha, \beta) = \prod_{\square \in D_\lambda} (\alpha(\ell(\square) + \beta(a(\square) + 1))).$$

(2.13)

Note that $[\xi_\lambda] \in H^T_X(X^{[n]})$. We define the following distinguished class:

$$[\lambda] = \frac{(-1)^n}{c_\lambda(\alpha, \beta)} t^{-n}[\xi_\lambda].$$  

(2.14)

Now let $X = X(\gamma)$ as in Example 2.2. In view of (2.1) and (2.10), there is a $T$-equivariant identification between $X_1$ and the complex plane in Example 2.1 with $\alpha = \beta = -1$; similarly for $X_2$ and the complex plane in Example 2.1 with $\alpha = \gamma - 1$ and $\beta = 1$. Then the $T$-fixed points of $X^{[n]}$ are of the form $\xi_{\lambda^1} + \xi_{\lambda^2}$ where $\lambda^1$ and $\lambda^2$ are partitions with $|\lambda^1| + |\lambda^2| = n$, and $\xi_{\lambda^1}$ and $\xi_{\lambda^2}$ are defined in the previous paragraph as we identify $X_1$ and $X_2$ with $\mathbb{C}^2$. For simplicity, put

$$\xi_{\lambda^1, \lambda^2} = \xi_{\lambda^1} + \xi_{\lambda^2}.$$  

We have a $T$-equivariant splitting $T_{\xi_{\lambda^1, \lambda^2}}X^{[n]} \cong T_{\xi_{\lambda^1}}X^{[\lambda^1]} \oplus T_{\xi_{\lambda^2}}X^{[\lambda^2]}$. By (2.11),

$$e_T(T_{\xi_{\lambda^1, \lambda^2}}X^{[n]}) = (-1)^n c_{\lambda^1}(-1, -1)c'_{\lambda^1}(-1, -1)c_{\lambda^2}(\gamma - 1, 1)c'_{\lambda^2}(\gamma - 1, 1)t^{2n}.$$  

Also, as in (2.14), we introduce the distinguished class

$$[\lambda^1, \lambda^2] = \frac{(-1)^n}{c_{\lambda^1}(-1, -1)c_{\lambda^2}(\gamma - 1, 1)} t^{-n}[\xi_{\lambda^1, \lambda^2}].$$  

(2.15)

2.3. Bilinear forms on the equivariant cohomology.

It is known from [E-S, Got] that the odd Betti numbers of $X^{[n]}$ are equal to zero and $H^k(X^{[n]}) = 0$ for $k > 2n$. Hence the spectral sequence associated with the fibration $X^{[n]} \times_T ET \to BT$ degenerates at the $E_2$-term. We have

$$H^k_T(X^{[n]}) = t^{k-n} \cup H^2_T(X^{[n]})$$

for $k \geq n$. Therefore, the classes defined in (2.14) and (2.15) are contained in

$$H_T^n \overset{\text{def}}{=} H^2_T(X^{[n]}).$$
Moreover, we can define a product structure \( \star \) on \( \mathbb{H}_n \) as follows (also cf. [Vas]):

\[
t^n \cup (A \star B) = A \cup B \in H^2_{\mathbb{T}}(X^n[n])
\]  

(2.16)

for \( A, B \in H^2_{\mathbb{T}}(X^n[n]) \). We see that \( (H^2_{\mathbb{T}}(X^n[n]), \star) \) is a ring.

Let \( H^*_{\mathbb{T}}(\cdot)' = H^*_{\mathbb{T}}(\cdot) \otimes_{\mathbb{C}[t]} \mathbb{C}(t) \) be the localization, and let

\[
\iota : (X^n[n])^T \to X^n[n]
\]

be the inclusion map. By abusing notations, we also use \( \iota_t \) to denote the induced Gysin map on the localized equivariant cohomology groups:

\[
\iota_t : H^*_T((X^n[n])^T) \to H^*_T(X^n[n])',
\]

which is an isomorphism by the localization theorem.

Define a bilinear form \( \langle - , - \rangle : H^*_T(X^n[n])' \times H^*_T(X^n[n])' \to \mathbb{C}(t) \):

\[
\langle A, B \rangle = (-1)^n p_t \iota_t^{-1}(A \cup B)
\]

(2.18)

where \( p \) is the projection \( (X^n[n])^T \to \text{pt} \). This induces a bilinear form \( \langle - , - \rangle \) on

\[
\mathbb{H}_X' \overset{\text{def}}{=} \bigoplus_{n=0}^{\infty} H^*_T(X^n[n])'.
\]

Next, we study the restriction of the bilinear form \( \langle - , - \rangle \) to \( \mathbb{H}_n = H^2_{\mathbb{T}}(X^n[n]) \).

When \( X = \mathbb{C}^2 \), we see from the projection formula and (2.11) that

\[
\begin{align*}
[\xi_\lambda] \cup [\xi_\mu] &= i_{\lambda!}(1_{\xi_\lambda}) \cup i_{\mu!}(1_{\xi_\mu}) = i_{\lambda!}(1_{\xi_\lambda} \cup i_\gamma i_{\mu!}(1_{\xi_\mu})) \\
&= \delta_{\lambda,\mu} c_T(T_{\xi_\lambda}X^n[n])[\xi_\lambda] = \delta_{\lambda,\mu} (-1)^n c_\lambda(\alpha, \beta)c_\lambda'(\alpha, \beta)t^2n[\xi_\lambda].
\end{align*}
\]

(2.20)

It follows from (2.14) and (2.18) that for \( \lambda, \mu \vdash n \), we have

\[
\langle [\lambda], [\mu] \rangle = \delta_{\lambda,\mu} \frac{c_\lambda'(\alpha, \beta)}{c_\lambda(\alpha, \beta)}.
\]

(2.21)

By the localization theorem, we see that the classes \([\lambda], \lambda \vdash n\) form a linear basis of the \( \mathbb{C} \)-vector space \( \mathbb{H}_n \). Similarly, when \( X = X(\gamma) \) is from Example 2.2

\[
\langle [\lambda^1, \lambda^2], [\mu^1, \mu^2] \rangle = \delta_{\lambda^1,\mu^1} \delta_{\lambda^2,\mu^2} \frac{c_{\lambda^1}(1 - \gamma, -1)}{c_{\lambda^2}(1 - \gamma, -1)} = \langle [\lambda^1], [\mu^1] \rangle \cdot \langle [\lambda^2], [\mu^2] \rangle,
\]

(2.22)

and the classes \([\lambda^1, \lambda^2] \), where \( |\lambda^1| + |\lambda^2| = n \), form a linear basis of \( \mathbb{H}_n \).

It follows that the restriction to \( \mathbb{H}_n \) of the bilinear form \( \langle - , - \rangle \) on \( H^*_T(X^n[n])' \) is a nondegenerate bilinear form \( \langle - , - \rangle : \mathbb{H}_n \times \mathbb{H}_n \to \mathbb{C} \). This induces a nondegenerate bilinear form \( \langle - , - \rangle : \mathbb{H}_X \times \mathbb{H}_X \to \mathbb{C} \) where the space \( \mathbb{H}_X \) is defined by

\[
\mathbb{H}_X = \bigoplus_{n=0}^{\infty} \mathbb{H}_n.
\]

(2.23)
3. Heisenberg algebras, equivariant cohomology and Jack polynomials

3.1. Heisenberg algebra actions.

Let $X$ be a surface in Example 2.1 or Example 2.2. Fix a positive integer $i$. For a $\mathbb{T}$-invariant closed curve $Y \subset X$, we define

$$Y_{n,i} = \{(\zeta, \eta) \in X^{[n+i]} \times X^{[n]} | \eta \subset \zeta, \text{ Supp}(I_\eta/I_\xi) = \{y\}, y \in Y\}$$

where $I_\eta$ and $I_\xi$ are the sheaves of ideals corresponding to $\eta$ and $\xi$ respectively. Let $p_1$ and $p_2$ be the projections of $X^{[n+i]} \times X^{[n]}$ to the two factors. As in [Vas], we define a linear operator $p_{-i}([Y]) \in \text{End}(\mathbb{H}_X^{[n]})$ by

$$p_{-i}([Y])(A) = p_{11}(p_{-i}^2 A \cup [Y_{n,i}])$$

(3.1)

for $A \in H^*_\mathbb{T}(X^{[n]})'$. Note that the restriction of $p_1$ to $Y_{n,i}$ is proper. We define $p_i([Y]) \in \text{End}(\mathbb{H}_X^{[n]})$ to be the adjoint operator of $p_{-i}([Y])$ with respect to the bilinear form $\langle -, - \rangle$ on $\mathbb{H}_X$. For $A \in H^*_\mathbb{T}(X^{[n]})'$, we have

$$p_i([Y])(A) = (-1)^i p_{-i}^2 (\iota \times \text{Id})_1^{-1} (p_{2}^* A \cup [Y_{n,i}])$$

(3.2)

where $p_2$ is the projection of $(X^{[n]} \times X^{[n-i]})$ to $X^{[n-i]}$. Finally we put $p_0([Y]) = 0$.

By the definition of $p_{-i}([Y])$ for $i > 0$, its restriction to $\mathbb{H}_X$ gives a linear operator in $\text{End}(\mathbb{H}_X)$, denoted by $p_{-i}([Y])$ as well. Next, we recall from Subsection 2.3 that there is a nondegenerate bilinear form $\langle -, - \rangle : \mathbb{H}_X \otimes \mathbb{H}_X \to \mathbb{C}$, which is the restriction of the bilinear form $\langle -, - \rangle$ on $\mathbb{H}_X$. Thus, the restriction of $p_i([Y])$ to $\mathbb{H}_X$ is the adjoint operator of $p_{-i}([Y])$ with respect to the bilinear form $\langle -, - \rangle$ on $\mathbb{H}_X$, and hence is an operator in $\text{End}(\mathbb{H}_X)$ which will again be denoted by $p_i([Y])$.

By [222] and [229], $H^*_\mathbb{T}(X)$ is linearly spanned by the classes $[Y]$ where $Y$ denotes $\mathbb{T}$-invariant closed curves in $X$. So we can extend the notion $p_k([Y])$ linearly to obtain the operator $p_k(\omega) \in \text{End}(\mathbb{H}_X)$ for an arbitrary class $\omega \in H^*_\mathbb{T}(X)$. Note from Example 2.1 (respectively, Example 2.2) that $\Sigma$ and $\Sigma'$ (respectively, $\Sigma_0$ and $\Sigma_i$ where $i = 1$ or 2) intersect transversely at one point. This (simple but crucial) observation together with an argument parallel to [Na1] [Vas] leads to the following.

**Proposition 3.1.** The operators $p_k(\omega), k \in \mathbb{Z}$ and $\omega \in \mathbb{H}_1 = H^*_T(X)$, acting on $\mathbb{H}_X$ satisfy the following Heisenberg commutation relation:

$$[p_k(\omega_1), p_\ell(\omega_2)] = k \delta_{k,-\ell} \langle \omega_1, \omega_2 \rangle \text{ Id.}$$

(3.3)

Furthermore, $\mathbb{H}_X$ becomes the Fock space over the Heisenberg algebra modeled on $H^*_\mathbb{T}(X)$ with the unit $|0\rangle \in H^*_\mathbb{T}(X^{[0]})$ of $H^*_\mathbb{T}(X^{[0]})$ being a highest weight vector. $\square$

3.2. The case of the complex plane $\mathbb{C}^2$.

In this subsection, we consider $X = \mathbb{C}^2$ from Example 2.1. Let $p_i = p_i([\Sigma])$ for $i \in \mathbb{Z}$. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_\ell) = (1^{m_1} 2^{m_2} \ldots)$, define

$$\delta_\lambda = \prod_{i \geq 1} r_i^{m_i} m_i!,$$

$$p_{-\lambda} = \frac{1}{\delta_\lambda} \prod_{i \geq 1} p_i^{m_i}.$$  

(3.4)
By (2.2), (2.14) and (2.21), we have
\[ \langle \Sigma, \Sigma \rangle = \frac{\beta}{\alpha} \cdot \frac{1}{\delta_{\lambda, \mu}} (\beta/\alpha)^{\ell(\lambda)}. \] (3.5)

Let \( S^nX \) (respectively, \( S^n\Sigma \)) be the \( n \)-th symmetric product of \( X \) (respectively, of \( \Sigma \)). Let \( \pi: X[n] \to S^nX \) be the Hilbert-Chow morphism. For \( \lambda \vdash n \), define
\[ S^n_{\lambda} \Sigma = \left\{ \sum_{i=1}^{\ell} \lambda_i y_i \in S^n \Sigma \mid y_i \in \Sigma, \text{ and } y_i \neq y_j \text{ if } i \neq j \right\}. \]

Let \( \Sigma^{(\lambda)} \) be the closure of \( \pi^{-1}(S^n_{\lambda} \Sigma) \) in \( X[n] \). An argument parallel to the proof of the Corollary 6.10 in [Na2] (also cf. [Vas]) shows that
\[ [\Sigma^{(\lambda)}] = [\lambda] + \sum_{\mu < \lambda} c_{\lambda, \mu} [\mu] \] (3.6)

where \( c_{\lambda, \mu} \in \mathbb{C} \) and "<" denotes the dominance partial ordering of partitions.

Let \( \Lambda \) be the ring of symmetric polynomials in infinitely many variables, \( \Lambda^n \) be the space of degree-\( n \) symmetric polynomials, \( m_\lambda \) be the monomial symmetric function associated to a partition \( \lambda \), and \( p_k \) be the \( k \)-th power sum symmetric function. Given a partition \( \lambda = (1^{m_1} 2^{m_2} \ldots) \), we define
\[ p_\lambda = \frac{1}{\delta_{\lambda}} \prod_{i \geq 1} p_{m_i}^i. \] (3.7)

It is known that the symmetric functions \( p_\lambda \) form a linear basis of the \( \mathbb{C} \)-vector space \( \Lambda \). So we can define a bilinear form \( \langle -, - \rangle \) on \( \Lambda \) by
\[ \langle p_\lambda, p_\mu \rangle = \delta_{\lambda, \mu} \frac{1}{\delta_{\lambda}} (\beta/\alpha)^{\ell(\lambda)}. \] (3.8)

Let \( P_{\lambda}^{(\beta/\alpha)} \) denote the Jack polynomials (see page 379 in [Mac] for their characterization). We introduce a ring structure \( \circ \) on \( \Lambda^n \) defined by
\[ \frac{P_{\lambda}^{(\beta/\alpha)}}{c_{\lambda}(\alpha, \beta)} \circ \frac{P_{\mu}^{(\beta/\alpha)}}{c_{\mu}(\alpha, \beta)} = \delta_{\lambda, \mu} \frac{P_{\lambda}^{(\beta/\alpha)}}{c_{\lambda}(\alpha, \beta)} \] (3.9)

**Theorem 3.2.** There exists a linear isomorphism \( \phi: \mathbb{H}_X \to \Lambda \) preserving bilinear forms such that \( \phi(p_{-\lambda}[0]) = p_\lambda, \phi([\Sigma^{(\lambda)}]) = m_\lambda, \) and \( \phi([\lambda]) = P_{\lambda}^{(\beta/\alpha)} \). Furthermore, the restriction \( \phi_n \) of \( \phi \) to \( \mathbb{H}_n \) is an isomorphism of rings, i.e., \( \phi_n: (\mathbb{H}_n, \circ) \cong (\Lambda^n, \circ) \).

**Proof.** The linear isomorphism \( \phi: \mathbb{H}_X \to \Lambda \) is defined by mapping \( p_{-\lambda}[0] \) to \( p_\lambda \). So
\[ \phi(p_{-\lambda} A) = p_\lambda A, \quad i > 0, \quad A \in \mathbb{H}_X. \] (3.10)

Next, an argument similar to the proof of the Theorem 4.6 in [Na2] (also cf. [Vas]) verifies that for all \( i > 0 \), we have
\[ p_{-i} \cdot [\Sigma^{(\lambda)}] = \sum_{\mu} a_{\lambda, \mu} [\Sigma^{(\mu)}] \] (3.11)
where the coefficients $a_{\lambda,\mu}$ are the same as in $p_i m_\lambda = \sum_\mu a_{\lambda,\mu} m_\mu$. Therefore we conclude from an induction, (3.11) and (3.10) that $\phi([\Sigma(\lambda)]) = m_\lambda$.

By (3.4) and (3.8), $\phi$ preserves the bilinear forms. So we see from (2.21) that

$$\langle \phi([\lambda]), \phi([\mu]) \rangle = \delta_{\lambda,\mu} \frac{c'_\lambda(\alpha, \beta)}{c_\lambda(\alpha, \beta)} = \delta_{\lambda,\mu} \frac{c'_\lambda(1, \beta/\alpha)}{c_\lambda(1, \beta/\alpha)}.$$  

By (3.6), we have $[\lambda] = [\Sigma(\lambda)] + \sum_{\mu < \lambda} d_{\lambda,\mu} [\Sigma(\mu)]$. It follows that

$$\phi([\lambda]) = m_\lambda + \sum_{\mu < \lambda} d_{\lambda,\mu} m_\mu.$$  

By the characterization of the Jack polynomials in [Mac], $\phi([\lambda]) = P^{(\beta/\alpha)}_{\lambda}$. Finally, by (2.20) and the definition of $\star$, we get

$$\frac{[\lambda]}{c_\lambda(\alpha, \beta)} \star \frac{[\mu]}{c'_\lambda(\alpha, \beta)} = \delta_{\lambda,\mu} \frac{[\lambda]}{c'_\lambda(\alpha, \beta)}, \quad \lambda, \mu \vdash n.$$  

In view of (3.9), $\phi_n : (\mathbb{H}_n, \star) \to (\Lambda^n, \circ)$ is an isomorphism of rings. \hfill \qed

3.3. The case of the surface $X(\gamma)$.

Let $X = X(\gamma)$ be the surface from Example 2.2. Recall that there is a $\mathbb{T}$-equivariant identification between the affine open subset $X_1$ of $X$ and the complex plane in Example 2.1 with $\alpha = \beta = -1$; similarly for $X_2$ and the complex plane in Example 2.1 with $\alpha = \gamma - 1$ and $\beta = 1$. By the discussions in Subsection 3.1, $\mathbb{H}_X$ is the Fock space of the Heisenberg algebra generated by $p_i([\Sigma_1])$ and $p_i([\Sigma_2])$ with $i \in \mathbb{Z}$, where $\Sigma_1$ and $\Sigma_2$ are the two $\mathbb{T}$-equivariant fibers in $X$ defined by (2.8). Note also that $\mathbb{H}_{X_1}$ is the Fock space of the Heisenberg algebra generated by $p_i([\Sigma_1])$ for $i \in \mathbb{Z}$, where $\Sigma_1$ is considered as a $\mathbb{T}$-equivariant closed curve in the affine open subset $X_1 \subset X$. Similarly, $\mathbb{H}_{X_2}$ is the Fock space of the Heisenberg algebra generated by $p_i([\Sigma_2])$ for $i \in \mathbb{Z}$. To avoid confusions, we use $p_i^{X_1}$ to denote the operators $p_i([\Sigma_1])$ acting on $\mathbb{H}_{X_1}$, $p_i^{X_2}$ to denote the operators $p_i([\Sigma_2])$ acting on $\mathbb{H}_{X_2}$, and $p_i([\Sigma_1])$ and $p_i([\Sigma_2])$ for the Heisenberg operators acting on $\mathbb{H}_X$. As in (3.4), for a partition $\lambda = (1^{m_1} 2^{m_2} \ldots)$ and for $j = 1$ or $2$, we define

$$p_{-\lambda}([\Sigma_j]) = \frac{1}{\delta_{\lambda}} \prod_{i \geq 1} p_{-i}([\Sigma_j])^{m_i}.$$  

Define a linear map $\Psi : \mathbb{H}_{X_1} \otimes \mathbb{H}_{X_2} \rightarrow \mathbb{H}_X$ as follows:

$$\Psi([\lambda^1] \otimes [\lambda^2]) = [\lambda^1, \lambda^2].$$  

(3.12)

The linear map $\Psi$ is an isomorphism since the classes $[\lambda^1], [\lambda^2], [\lambda^1, \lambda^2]$ defined in (2.11) and (2.15) form linear bases of $\mathbb{H}_{X_1}, \mathbb{H}_{X_2}, \mathbb{H}_X$ respectively. In addition, we see from (2.22) that $\Psi$ preserves the bilinear forms.

**Lemma 3.3.** The linear isomorphism $\Psi$ commutes with Heisenberg operators, i.e.,

$$\Psi \circ (p_i^{X_1} \otimes \text{Id}) = p_{-i}([\Sigma_1]) \circ \Psi,$$

$$\Psi \circ (\text{Id} \otimes p_i^{X_2}) = p_{-i}([\Sigma_2]) \circ \Psi.$$
Proof. By the symmetry between \( \Sigma_1 \) and \( \Sigma_2 \), we need only to prove the first identity. Also, since \( \Psi \) preserves the bilinear forms and \( p_i \) is the adjoint operator of \( p_{-i} \), it suffices to prove the first identity for \( i > 0 \).

Since the two fibers \( \Sigma_1 \) and \( \Sigma_2 \) do not intersect, we see that for partitions \( \lambda^1 \) and \( \lambda^2 \) with \( |\lambda^1| + |\lambda^2| = n \), the \( \mathbb{T} \)-equivariant closed subvariety \( \Sigma_1^{(\lambda^1)} \times \Sigma_2^{(\lambda^2)} \subset X^{[n]} \) is the closure of \( \pi^{-1}(S_{\lambda^1}^{(\lambda^1)} \times S_{\lambda^2}^{(\lambda^2)}) \) in \( X^{[n]} \), where \( \pi \colon X^{[n]} \to S^n X \) denotes the Hilbert-Chow morphism. We conclude that

\[
\Psi([\Sigma_1^{(\lambda^1)}] \otimes [\Sigma_2^{(\lambda^2)}]) = [\Sigma_1^{(\lambda^1)} \times \Sigma_2^{(\lambda^2)}]
\]

by writing \( [\Sigma_1^{(\lambda^1)} \times \Sigma_2^{(\lambda^2)}] \) in terms of \( [\lambda^1, \lambda^2] \) and \( [\mu^1, \mu^2] \) where \( \mu^1 < \lambda^1 \) or \( \mu^2 < \lambda^2 \), similarly as in (3.6). It implies that the classes \( [\Sigma_1^{(\lambda^1)} \times \Sigma_2^{(\lambda^2)}] \) form a linear basis of \( \mathbb{H}_\lambda \). By (3.11) and a similar computation for \( p_{-i}([\Sigma_1]) [\Sigma_1^{(\lambda^1)} \times \Sigma_2^{(\lambda^2)}] \), we have

\[
\Psi \left( \left( p_{X,i}^{(\lambda^1)} \otimes \text{Id} \right) ([\Sigma_1^{(\lambda^1)}] \otimes [\Sigma_2^{(\lambda^2)}]) \right) = \Psi \left( \left( p_{-X,i}^{(\lambda^1)} [\Sigma_1^{(\lambda^1)}] \right) \otimes [\Sigma_2^{(\lambda^2)}] \right)
\]

\[
= p_{-i}([\Sigma_1]) [\Sigma_1^{(\lambda^1)} \times \Sigma_2^{(\lambda^2)}]
\]

\[
= p_{-i}([\Sigma_1]) \Psi([\Sigma_1^{(\lambda^1)}] \otimes [\Sigma_2^{(\lambda^2)}]).
\]

It follows that \( \Psi \circ \left( p_{X,i}^{(\lambda^1)} \otimes \text{Id} \right) = p_{-i}([\Sigma_1]) \circ \Psi \) for \( i > 0 \). \( \Box \)

Let \( \Lambda_1 \) be the ring \( \Lambda \) of symmetric functions with the bilinear form

\[
\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \frac{1}{3\lambda},
\]

\( \Lambda_2 \) be the same ring \( \Lambda \) equipped with a different bilinear form

\[
\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} \frac{1}{3\lambda} \left( 1/(\gamma - 1) \right)^{\ell(\lambda)},
\]

and \( \Lambda^{n_i} \) be the space of degree-\( n_i \) symmetric polynomials in \( \Lambda_i \). The tensor product \( \Lambda_1 \otimes \Lambda_2 \) has an induced bilinear form. Let

\[
(\Lambda_1 \otimes \Lambda_2)^n = \bigoplus_{n_1 + n_2 = n} \Lambda_1^{n_1} \otimes \Lambda_2^{n_2}.
\]

We define a ring structure on \( (\Lambda_1 \otimes \Lambda_2)^n \) by declaring the elements

\[
\frac{P_\lambda^{(1)}}{c_\lambda(-1,-1)} \otimes \frac{P_\mu^{(1/\gamma-1)}}{c_\mu(\gamma-1,1)} \in (\Lambda_1 \otimes \Lambda_2)^n,
\]

where \( |\lambda| + |\mu| = n \), to be idempotents.

**Theorem 3.4.** There exists a linear isomorphism \( \Phi \colon \mathbb{H}_X \to \Lambda_1 \otimes \Lambda_2 \) preserving bilinear forms such that \( \Phi(p_{-\lambda^1}([\Sigma_1]) p_{-\lambda^2}([\Sigma_2])|0\rangle) = p_{\lambda^1} \otimes p_{\lambda^2} \),

\[
\Phi([\Sigma_1^{(\lambda^1)} \times \Sigma_2^{(\lambda^2)}]) = m_{\lambda^1} \otimes m_{\lambda^2},
\]

and \( \Phi([\lambda^1,\lambda^2]) = P_{\lambda^1}^{(1)} \otimes P_{\lambda^2}^{(1/\gamma-1)} \). Furthermore, the restriction of the map \( \Phi \) to \( \mathbb{H}_n \) is a ring isomorphism onto \( (\Lambda_1 \otimes \Lambda_2)^n \).

**Proof.** Follows from Lemma 3.3 and arguments similar for Theorem 3.2. \( \Box \)
Next, we discuss the implication of Theorem 3.4 to the ordinary cohomology ring $H^*(X^{[n]})$. With the notations for Heisenberg algebras in the ordinary cohomology setting in [Na2], a linear basis of $H^*(X^{[n]})$ consists of classes of the form

$$\Omega_{\lambda^1,\lambda^2} = \prod_{k \geq 1} P_X[-k]^{m_k(1)} \prod_{k \geq 1} P_{\Sigma_0}[-k]^{m_k(2)}|0\rangle,$$

(3.13)

where $|\lambda^1| + |\lambda^2| = n$, $\lambda^i = (1^{m_1(i)}2^{m_2(i)} \cdots)$, $1_X \in H^0(X)$ is the fundamental cohomology class, and $\Sigma_0$ is the zero-section in $X = X(\gamma)$. Denote

$$\Omega^T_{\lambda^1,\lambda^2} = \prod_{k \geq 1} p_{-k}(|\lambda^1\rangle)^{m_k(1)} \prod_{k \geq 1} p_{-k}(|\Sigma_0\rangle)^{m_k(2)}|0\rangle \in \mathbb{H}_n.$$

(3.14)

The graded element in $H^*(X^{[n]})$ associated to $\Omega^T_{\lambda^1,\lambda^2}$ is $\Omega_{\lambda^1,\lambda^2}$. Since $\{t, [\Sigma_0]\}$ and $\{[\Sigma_1], [\Sigma_2]\}$ are two different linear basis for $H^2_\gamma(X)$, it follows from the definitions that there is a simple explicit transition matrix $M_1$ between the basis of the equivariant cohomology $H^*_\gamma(X^{[n]})$ given by (3.14) and the basis given by

$$\Omega^T_{\lambda^1,\lambda^2} = \prod_{k \geq 1} p_{-k}(|\Sigma_1\rangle)^{m_k(1)} \prod_{k \geq 1} p_{-k}(|\Sigma_2\rangle)^{m_k(2)}|0\rangle.$$

(3.15)

For a fixed $r$, denote by $\left(g^{(r)}_{\lambda,\mu}\right)$ the transition matrix between Jack polynomials $P^{(r)}_{\lambda}$ and the power-sums $p_{\mu}$. By Theorem 3.3, the transition matrix between the basis (3.15) and the basis $\{[\mu^1, \mu^2]\}$ is provided by $M_2 \stackrel{def}{=} \left(g^{(1)}_{\lambda^1,\mu^1}\right) \otimes \left(g^{(1/\gamma-1)}_{\lambda^2,\mu^2}\right)$.

Now the structure of the ordinary cohomology ring $H^*(X^{[n]})$ can be described as follows in terms of the Heisenberg monomials (3.13). Given two Heisenberg monomials $\Omega_{\lambda^1,\lambda^2}$ and $\Omega_{\mu^1,\mu^2}$ in $H^*(X^{[n]})$, we consider the product of $\Omega^T_{\lambda^1,\lambda^2}$ and $\Omega^T_{\mu^1,\mu^2}$ in $\mathbb{H}_n$. The latter can be calculated by transferring over to the basis $\{\Omega^T_{\lambda^1,\lambda^2}\}$ via the transition matrix $M_1$, and then to the basis $\{[\mu^1, \mu^2]\}$ via the transition matrix $M_2$, where the product is explicitly known. Finally, we reverse the above steps and pass to the associated graded ring to express the product of $\Omega_{\lambda^1,\lambda^2}$ and $\Omega_{\mu^1,\mu^2}$ in terms of the Heisenberg monomial basis (3.13) of $H^*(X^{[n]})$.

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