A Sparse Beta Regression Model for Network Analysis

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March 29, 2022

Abstract

Network data are ubiquitous in modern science and humanity. This paper concerns a new generative model, suitable for sparse networks commonly observed in practice, to capture degree heterogeneity and homophily, two stylized features of a typical network. The former is achieved by differentially assigning parameters to individual nodes, while the latter is materialized by incorporating covariates. Similar models in the literature for heterogeneity often include as many nodal parameters as the number of nodes, leading to over-parametrization. For parameter estimation, we use the penalized likelihood method with an $\ell_1$ penalty on the nodal parameters, immediately connecting our estimation procedure to the LASSO literature. We highlight the differences of our approach to the LASSO method for logistic regression, emphasizing the feasibility of our model to conduct inference for sparse networks, study the finite-sample error bounds on the excess risk and the $\ell_1$-error of the resulting estimator, and develop a central limit theorem for the parameter associated with the covariates. Simulation and data analysis corroborate the developed theory. As a by-product of our main theory, we study what we call the Erdős-Rényi model with covariates and develop the associated statistical inference for sparse networks, which can be of independent interest.

Key words: $\beta$-model, asymptotic normality, consistency, degree heterogeneity, excess risk, homophily, LASSO, sparse networks.

1 Introduction

Network data are ubiquitous in today’s society thanks to unprecedented advances of data collection techniques. This paper concerns a new generative model to simultaneously capture degree heterogeneity and homophily, two stylized features of a typical network (Kolaczyk 2009, Newman 2018). The former refers to the fact that the nodes of a network differ, sometimes drastically, in their propensity in making connections, while the latter states that similar nodes are more likely to attach to each other than dissimilar ones, based on node attributes or covariates. Above all, our model is designed for sparse networks, which we define as networks in which the total number of connections can scale sub-quadratically in the order of $o(n^2)$ with respect to $n$, the number of nodes. A network is called dense otherwise. As is standard in the literature, the properties of our network model are studied under the setting where $n$ goes to infinity.

To fix ideas, assume that we have observed data organized as $\{A_{ij}, Z_{ij}\}_{i,j=1,i\neq j}^n$, where $A = (A_{ij})_{i,j=1}^n$ is the adjacency matrix with $A_{ij} = 1$ if node $i$ and $j$ are connected and $A_{ij} = 0$ otherwise, and $Z_{ij} \in \mathbb{R}^p$ are $p$-dimensional covariates associated with these two nodes. Our model assumes that links are independently made with the probability of a connection between node $i$ and $j$ being

$$P(A_{ij} = 1 | Z_{ij}) = p_{ij} = \frac{\exp(\beta_i + \beta_j + \mu + Z_{ij}^T \gamma)}{1 + \exp(\beta_i + \beta_j + \mu + Z_{ij}^T \gamma)},$$

(1)
where $\beta = (\beta_1, ..., \beta_n)^T \in \mathbb{R}^n$ with $\beta_i$ associated with the $i$th node is the heterogeneity parameter, $\gamma \in \mathbb{R}^p$ is the parameter for the covariates, and $\mu \in \mathbb{R}$ is a parameter common to all the nodes. For identifiability, we assume $\min_i \beta_i = 0$, so that $\beta \in \mathbb{R}_+^n$ with $\mathbb{R}_+ = [0, +\infty)$. Central to our model is the idea that the vector $\beta$ is sparse, although we do not assume that its support is known. If $\beta_i = 0$ for all $i$, model (1) becomes the Erdős-Rényi model with covariates as discussed in Section 2.3 for which we can handle very sparse networks. If a node contains ‘some signal’, a positive $\beta$ parameter implies that it will be better connected than the baseline nodes whose $\beta$ parameters equal zero. Our model is a generalization of the sparse $\beta$-model ($S\beta M$) proposed in Chen et al. (2021) that does not consider covariates. As such, we name our model sparse $\beta$-regression model or $S\beta RM$ for short. Below we highlight informally the main features of this new model:

1. The parameter $\mu$ can be interpreted as the global density parameter of the network and is allowed to diverge to $-\infty$. As a result, $p_{ij}$ can go to 0. In this sense, $S\beta RM$ is well suited for modeling sparse networks commonly seen in practice.

2. The parameter $\beta$ can be understood as the local density parameter, where $\beta_i$ is a heterogeneity parameter distinguishing how node $i$ participates in network formation. Assuming a sparse $\beta$ allows us to differentially assign non-zero local parameters only to those nodes active in making connections, thus avoiding over-parametrization due to the number of the parameters in $\beta$. In the most extreme case when $\beta$ is a zero vector, we will have a logistic regression model. The logistic regression in our context is different from the usual logistic regression model because we allow $\mu$ to go to $-\infty$. Details of this special case are presented in Section 2.3.

3. The parameter $\gamma$ can be seen as the covariate parameter to capture the effect of the covariate $Z_{ij}$ for initiating connections. Here $Z_{ij}$ can be either node-similarities or edge-covariates. For the former, one popular approach is to construct $Z_{ij}$ using $X_i$ and $X_j$, the nodal covariates at node $i$ and $j$ that can be multivariate. For example, we can define $Z_{ij} = g(X_i, X_j)$, where $g(\cdot, \cdot)$ is a function of its arguments. One possible choice of $g$ is $g(X_i, X_j) = -\|X_i - X_j\|_1$, or some other metric, for measuring the similarity between node $i$ and $j$. For this case, $\gamma$ is a scalar parameter for characterizing the tendency of two similar nodes making a connection and a positive $\gamma$ indicates homophily. Another choice is $g(X_i, X_j) = -|X_i - X_j|$, the vector of absolute differences of $X_i$ and $X_j$, where $\gamma$ takes the role of weighting the importance of each covariate.

Our model is high-dimensional with $n + p + 1$ unknown parameters, where the heterogeneity parameter $\beta$ admits a sparse representation with an unknown support. This fact immediately motivates the use of a penalized likelihood approach for estimation. A first idea is to employ an $\ell_0$ penalty on $\beta$ as in Chen et al. (2021) for $S\beta M$ when no covariates are considered. However, their argument for developing a computationally efficient algorithm is no longer applicable. This leads us to the use of an $\ell_1$ penalty on $\beta$, where our estimator is obtained by solving

$$
\min_{\beta \in \mathbb{R}_+^n, \mu \in \mathbb{R}, \gamma \in \mathbb{R}^p} \frac{1}{2} \mathcal{L}(\beta, \mu, \gamma) + \lambda \|\beta\|_1,
$$

(2)

where $\mathcal{L}(\beta, \mu, \gamma)$ is the negative log-likelihood defined in (3), $\lambda$ is a tuning parameter, and $\|\beta\|_1$ is the $\ell_1$-norm of $\beta$. This formulation immediately connects our approach to the LASSO methodology (Tibshirani 1996) developed for variable selection, enabling us to draw upon the vast literature on high-dimensional data analysis, especially for logistic regression.

We highlight that the model in (1) is sparse in terms of its parametrization and the density of the resulting network. The word sparse in $S\beta RM$ refers to the former. The latter is a natural consequence of the sparse representation of the $\beta$ parameter. For example, when $\beta$ is sparse with a finite support, by allowing $\mu$ to grow to $-\infty$ at appropriate rates, the networks generated from this model will be sparse. When $\beta = 0_0$, where $0_0$ is a zero vector of dimension $n$, we show in Section 2.3 that this special version of $S\beta RM$ can model any network whose expected number of edges scales as $O(n^{2-\xi})$ with $\xi \in [0, 2)$. That is, the network modeled by this special case of $S\beta RM$ can be almost arbitrarily sparse.
1.1 Main contributions

Our methodological contribution comes from proposing the SβRM as the first model capable of capturing node heterogeneity differentially while accounting for covariates in the presence of network sparsity. In the literature, closely related models allowing node-specific parameters either ignore covariates and thus homophily (Chen et al. 2021), or overly parametrize by assigning parameters indistinguishably to all the nodes (Chatterjee et al. 2011, Graham 2017, Yan et al. 2019, e.g.), leading to theoretical and practical difficulties in applying these models; See Chen et al. (2021) for some discussion.

Our first theoretical contribution is to analyze the performance of our estimator by establishing its consistency in terms of excess risk and $\ell_1$-norm. Despite the somewhat superficial similarity of our estimator to the penalized logistic regression with an $\ell_1$ penalty, great care needs to be taken when applying results from LASSO theory to our estimator. Firstly, the design matrix of our model associated with $\beta$ is deterministic while that with $\gamma$ is random, making the common assumptions made on the eigenvalues of the design matrix typically seen in LASSO not applicable. Furthermore, our approach differs from classical LASSO theory for logistic regression insofar that we do not assume that the linking probabilities $p_{ij}$ between two nodes stay uniformly bounded away from zero, because otherwise the network will be dense. This assumption is often made in LASSO theory; see, for example, van de Geer & Bühlmann (2011), Theorem 6.4; Buena (2008), Theorem 2.4; or van de Geer (2008), Theorem 2.1. To our best knowledge, we are not aware of similar conditions explicitly stated in the literature, at least not to a model similar to ours. Importantly, our approach differs from classical LASSO theory in that the various parameters in the SβRM have differing effective sample sizes, resulting in different rates of convergence. Loosely speaking, the effective sample size for each $\beta_i$ depends on the number of possible connections that the $i$th node has, while $\mu$ and $\gamma$ are both global parameters depending on the total number of edges. Remarkably, we recover almost the classical LASSO rate of convergence for excess risk and $\ell_1$-error, up to an additional factor having an explicit relation to the expected edge density of a network, which is the price to pay for allowing vanishing link probabilities.

For statistical inference, the homophily parameter $\gamma$ is often of major interest, as the heterogeneity parameter $\beta$ can be seen as nuisance. Our second theoretical contribution is to provide a central limit theorem for $\gamma$ in the face of vanishing link probabilities. Remarkably, we show that this theorem holds without the need to apply the kind of debiasing usually required for LASSO estimators due to shrinkage (Zhang & Zhang 2014). Crucially, inference for LASSO type estimators relies on finding a good approximation to the precision matrix, the inverse of the population Gram matrix, which can be challenging. In particular, it is routinely assumed in the proof of LASSO inference results that the minimum eigenvalue of the Gram matrix is bounded away from zero, uniformly in $n$ (van de Geer et al. 2014, e.g.). In our case, however, this matrix depends on the link probabilities $p_{ij}$ and since we allow $p_{ij} \to 0$ for many $i$ and $j$, such a uniform lower bound assumption becomes invalid. Remarkably, we can overcome this difficulty as long as rates are chosen carefully. The ability to conduct inference with an asymptotically non-invertible Gram matrix and vanishing link probabilities is a significant improvement over many existing methods and a prerequisite for dealing with sparse networks.

As byproducts of our theory, we provide the theory for two special cases of SβRM. First, we provide results analogous to Chen et al. (2021) in Section 2.2 when covariates are not considered, by replacing the $\ell_0$-penalty on $\beta$ in SβM by an $\ell_1$-penalty. We also consider a simplified model of SβRM in Section 2.3 when the heterogeneity parameter is not present, i.e. when $\beta = 0_n$, and present asymptotic normality results for its maximum likelihood estimator (MLE). We remark that the setup in the latter case is different to the usual logistic regression as we allow the global density parameter $\mu$ to diverge to $-\infty$ and thus allow for sparse networks. The model studied in Section 2.3 can be seen as an extension of the Erdős-Rényi model by incorporating covariates, with an emphasis to model those networks that are sparse. This model does not appear analyzed previously and can be of independent interest.

From a computational viewpoint, the similarity between our formulation and LASSO logistic regression enables us to invoke standard algorithms developed for the latter, and thus the estimation of our model parameters can be done extremely fast. Our final contribution is to demonstrate the usefulness of our model via extensive numerical simulation and two real data applications.
1.2 Prior work

Real-world networks are often found to be sparse, with nodes exhibiting different numbers of links and nodes similar in attributes more likely to connect, among many other features. To understand the stochastic nature of these data, statistical analysis of networks has seen an increasing research interest in both theory and applications (Kolaczyk 2009, Goldenberg et al. 2009, Fienberg 2012, Kolaczyk 2017).

The $S\beta$RM becomes the Erdős-Rényi model if the heterogeneity and covariate parameters are absent (Erdős & Rényi 1959, 1960, Gilbert 1959). To generalize the Erdős-Rényi model to include degree heterogeneity, one intuitive idea is to assign node-specific parameters, one for each node. This gives rise to the $\beta$-model which can be dated back to Holland & Leinhardt (1981) and has been thoroughly studied by Chatterjee et al. (2011) who proved the consistency of its MLE. Yan & Xu (2013) further proved the asymptotic normality of that MLE. See also Rinaldo et al. (2013), Karwa & Slavković (2016) and Yan, Qin & Wang (2016) for further results, and Yan, Leng & Zhu (2016) for a directed version of the $\beta$-model. Since the $\beta$-model associates each node with its own parameter, they are over-parametrized. To overcome this, Chen et al. (2021) proposed a sparse $\beta$-model ($S\beta\text{M}$) by assuming that a subset of the node-specific parameters are zero, while Zhang et al. (2021) applied a ridge penalty on the parameters. Another popular idea for incorporating degree heterogeneity is to assume that nodes in a network can be clustered into communities that share same connection patterns. This gives rise to the so-called stochastic block model (SBM); see Holland et al. (1983) for its formalization and Abbe (2018) for a review and recent developments.

Modeling the tendency for similar nodes to link up, also known as homophily, is best achieved by including covariates. For the $\beta$-model, Graham (2017) first included nodal-covariates, giving rise to a models similar to (1) but with a dense $\beta$. See Jochmans (2018) for further results and Yan et al. (2019) for a generalization to directed networks. For the SBM with covariates, Huang & Feng (2018) considered spectral clustering with adjustment, Binkiewicz et al. (2017) applied a modification of spectral clustering, Zhang et al. (2016) proposed to use a joint community detection criterion, while Yan & Sarkar (2021) resorted to convex relaxation that can be used for sparse networks.

Our estimation is closely connected to the LASSO methodology (Tibshirani 1996), especially that developed for generalized linear models (van de Geer 2008, Buena 2008). For inference for LASSO type of estimators, it is found that debiasing is necessary to overcome the bias caused by shrinkage (Zhang & Zhang 2014, van de Geer et al. 2014). One contribution of this work is that debiasing is shown to be not necessary for inference of the covariate parameter under suitable conditions.

1.3 Notations and the plan of the paper

We introduce the notations used in this paper. A network on $n$ nodes is represented as an undirected graph $G_n = (V, E)$, consisting of a node set $V = \{1, \ldots, n\}$ with cardinality $n$ and an edge set $E$, which is a subset of all the two-element subsets of $V$. We assume that the graphs studied are simple. Thus given a graph $G_n$, we can identify it with a binary adjacency matrix $A \in \mathbb{R}^{n \times n}$, where $A_{i,j} = A_{j,i} = 1$, if $\{i, j\} \in E$ and $A_{i,j} = A_{j,i} = 0$ otherwise. We write $d_i = \sum_{j=1}^{n} A_{i,j}$ as the degree of node $i$, $d = (d_1, \ldots, d_n)^T$ as the degree sequence, and $d_+ = \sum_{i=1}^{n} d_i/2 = \sum_{i<j} A_{i,j}$ as the total number of edges. By $a_n \sim b_n$ we mean $0 < \liminf_{n \to \infty} a_n/b_n \leq \limsup_{n \to \infty} a_n/b_n < \infty$ for two sequences of positive numbers $a_n$ and $b_n$. We call a network sparse if $\|d_+\|_\infty \approx n^\kappa$ for some $\kappa \in (0, 2)$, where $\| \cdot \|$ is the expectation with regard to the data generating process. A network is dense if $\|d_+\|_2 \approx n^2$.

For a vector $v \in \mathbb{R}^n$, we use $S(v) = \{i : v_i \neq 0\}$ to denote its support and $\|v\|_0 = |S(v)|$ as the cardinality of $S(v)$. Let $\| \cdot \|_1, \| \cdot \|_2, \| \cdot \|_\infty$ denote the vector $\ell_1$, $\ell_2$- and $\ell_\infty$-norm respectively. For any subset $S \subset \{1, \ldots, n\}$, we denote by $v_S$ the vector $v$ with components not belonging to $S$ set to zero.

For convenience of notation, when dealing with a vector $v \in \mathbb{R}^{n(\ell)}$, we will number its elements as $v = (v_{ij})_{i<j}$. Also, for any square matrix $A$, we denote by $\lambda_{\text{max}}(A)$ its maximum eigenvalue and by $\lambda_{\text{min}}(A)$ its minimum eigenvalue. We use $C$ for some generic, strictly positive constant that may change between displays. For brevity, we denote the set of parameters collectively as $\theta = (\beta^T, \mu, \gamma^T)^T$ and its true value as $\theta_0 = (\beta_0^T, \mu_0, \gamma_0^T)^T$. We write $S_0 = S(\beta_0)$ as the support of $\beta_0$. For ease of presentation, we
introduce the shorthand notation \( s_0 = |S_0| \) and \( S_{0,+} := S_0 \cup \{n + 1, n + 2, \ldots n + p\} \) with cardinality \( s_{0,+} = |S_{0,+}| = s_0 + p + 1 \) to refer to all active indices including \( \mu \) and \( \gamma \).

The rest of the paper is structured as follows. In Section 2, we introduce the S\( \beta \)RM and derive the consistency of the estimator in (2) in terms of excess risk and \( \ell_1 \)-norm. We then zoom in on two special cases of S\( \beta \)RM. In Section 2.2, we show how the results from Section 2.1 can be applied to the S\( \beta \)M without covariates. This is the model studied in Chen et al. (2021) where the parameters are estimated by penalizing the \( \ell_0 \)-norm of \( \beta \). Section 2.3 presents the theory for S\( \beta \)RM when the heterogeneity parameter is not present. In Section 3, we derive a central limit theorem for our estimator of the homophily parameter \( \gamma \) without debiasing. We present extensive simulation results in Section 4 and apply our model to a friendship network of a corporate law firm and the world trade network in Section 5. Conclusion remarks are presented in Section 6. All the proofs are relegated to the Appendix.

2 Sparse \( \beta \)-Regression Model

Recall the S\( \beta \)RM as defined in (1). Given an observed adjacency matrix \( A \) and the associated covariates \( \{Z_{ij}\} \), the negative log-likelihood of the model is seen as

\[
\mathcal{L}(\theta) = \mathcal{L}(\beta, \mu, \gamma) = -\sum_{i=1}^{n} \beta_i d_i - d_+ \mu - \sum_{i<j} (Z_{ij}^T \gamma) A_{ij} + \sum_{i<j} \log(1 + \exp(\beta_i + \beta_j + \mu + Z_{ij}^T \gamma)). \tag{3}
\]

Since \( \beta \) is assumed sparse, it may be tempting to estimate the parameters of the model via the following regularized likelihood by penalizing the \( \ell_0 \) norm of \( \beta \)

\[
\min_{\beta \in \mathbb{R}^n, \mu \in \mathbb{R}, \gamma \in \mathbb{R}^p} \frac{1}{2} \mathcal{L}(\beta, \mu, \gamma) + \lambda \|\beta\|_0.
\]

For the sparse \( \beta \)-model without covariates, Chen et al. (2021) found that this non-convex optimization problem is computationally tractable, thanks to a key monotonicity lemma. The arguments leading to the conclusion of this lemma, however, do not extend to the current setting where covariates are included. This effectively means that the \( \ell_0 \)-norm penalized likelihood becomes a combinatorial problem for the S\( \beta \)RM and an exhaustive search, which is computationally intractable, in the model space is inevitable. The above discussion is reminiscent of the familiar all best-subset selection procedure in regression.

One approach popular in high-dimensional data analysis is to replace the \( \ell_0 \) penalty on \( \beta \) by its \( \ell_1 \) penalty, leading to our proposed estimation procedure in (2). An attractive feature of this formulation is that the resulting objective function is convex. On the computational side, the formulation in (2) is the same as penalized logistic regression with the LASSO penalty. Thus, to solve it in practice, we can use existing algorithms developed for LASSO and in particular, we can use the functions in the \texttt{glmnet} R package (Friedman et al. 2010) by properly setting up the design matrix and the constraints on \( \beta \). Our experience shows that this algorithm can effectively compute the estimator for a network with the number of nodes up to a few thousand. We remark that more scalable algorithms can be explored by exploiting the special structure of the design matrix for \( \beta \) in (3).

In this paper, we focus on the finite-dimensional covariate case by assuming that \( p \), the dimension of the covariates \( Z_{ij} \), is fixed. We assume that \( Z_{ij} \) are independent realizations from centered, uniformly bounded random variables. We do not require \( Z_{ij} \) to be i.i.d. and \( Z_{ij} \) may have correlated entries. These assumptions imply in particular, that there exist constants \( \kappa, c > 0 \) such that \( |Z_{ij}^T \gamma_0| \leq \kappa \) for all \( 1 \leq i < j \leq n \) and \( |Z_{ij,k}| \leq c \) for all \( 1 \leq i < j \leq n, k = 1, \ldots, p \). We assume further that \( \gamma_0 \), the homophily parameter associated with \( Z_{ij} \), lies in a compact, convex set \( \Gamma \subset \mathbb{R}^p \), which means we may choose a universal \( \kappa \) independent of \( \gamma_0 \). Recalling the notation \( \theta = (\beta^T, \mu, \gamma^T)^T \), we let \( \Theta := \mathbb{R}^+_n \times \mathbb{R} \times \Gamma \) denote the parameter space.
2.1 Theory
Since we aim to develop a theory for sparse networks, we allow $\mu_0$ to go to $-\infty$ as $n$ tends to infinity. As a result, as $\min_i \beta_{0,i} = 0$, some link probabilities may go to zero as $n$ tends to infinity. In order to perform consistent estimation, it is clear that we need to restrict the rate at which probabilities may go to zero. Therefore, we assume that there is a non-random sequence $1/2 \geq \rho_{n,0} > 0$, $\rho_{n,0} \rightarrow 0$ as $n \rightarrow \infty$, such that almost surely for all $i,j$:

$$1 - \rho_{n,0} \geq p_{ij} \geq \rho_{n,0}.$$ 

Since a smaller $\rho_{n,0}$ allows sparser networks, we refer to $\rho_{n,0}$ as the network sparsity parameter. It effectively characterizes the maximum permissible sparsity of our network. Applying logit$(x) = \log(x/(1-x))$ to the inequality above we get for all $i,j$:

$$-\logit(\rho_{n,0}) = \logit(1 - \rho_{n,0}) \geq \beta_{0,i} + \beta_{0,j} + \mu_0 + \gamma_0^T Z_{ij} \geq \logit(\rho_{n,0}),$$

which is equivalent to

$$|\beta_{0,i} + \beta_{0,j} + \mu_0 + \gamma_0^T Z_{ij}| \leq -\logit(\rho_{n,0}) =: r_{n,0}, \quad \forall i, j.$$

Note that since $\rho_n \leq 1/2$, we have $r_{n,0} \geq 0$. The previous inequality can also be expressed in terms of the design matrix $D$ associated with the corresponding logistic regression problem, for which we give an explicit formula in (6) below, and is equivalent to $\|D\theta_0\|_{\infty} \leq r_{n,0}$. This motivates the following estimation procedure: Given a sufficiently large constant $r_n$, we define the local parameter space

$$\Theta_{loc} = \Theta_{loc}(r_n) := \{\theta \in \Theta : \|D\theta\|_{\infty} \leq r_n\}$$

and propose to perform estimation via

$$\hat{\theta} = (\hat{\beta}^T, \hat{\mu}, \hat{\gamma}^T)^T = \arg \min_{\theta = (\beta^T, \mu, \gamma)^T \in \Theta_{loc}} \frac{1}{m} \mathcal{L}(\beta, \mu, \gamma) + \lambda \|\beta\|_1,$$  \hspace{1cm} (5)

where $\lambda$ is a tuning parameter. As we have seen in the equations above, any $r_n > 0$ used in the definition of $\Theta_{loc}$ corresponds to some $\rho_n$, which uniformly lower bounds the connection probability and thus can be seen as a proxy for the sparsity of our network. This type of restriction of the parameter space is similar to what was done in Chen et al. (2021), although they restricted the parameter values of $\beta$ and $\mu$ directly. The condition in (4) is slightly more general and somewhat more natural. Noting that $\Theta_{loc}$ is convex, we have a convex optimization problem in (5).

Note that, in (5), we replaced the condition $\min_i \beta_i = 0$ by the less strict condition $\beta \in \mathbb{R}_+^n$. The following Lemma, which is proved in Appendix A, shows that this is viable and that as long as the observed graph is neither empty nor complete and $\lambda > 0$, a solution $\hat{\beta}$ to (5) always exists and automatically fulfills $\min_{1 \leq i \leq n} \hat{\beta}_i = 0$.

**Lemma 1.** Assume that $0 < d_+ < (n \choose 2)$. Then, for any $0 < \lambda < \infty$ there exists a minimizer for the optimization problem (5) and any solution $\hat{\beta} = (\hat{\beta}^T, \hat{\mu}, \hat{\gamma}^T)^T$ of (5) must satisfy $\min_{1 \leq i \leq n} \hat{\beta}_i = 0$.

Following the empirical risk literature (cf. Greenshtein & Ritov (2004), Koltchinskii (2011)) we will analyze the performance of our estimator in terms of excess risk. Define the (global) excess risk as

$$\mathcal{E}(\theta) := \frac{1}{m} \mathbb{E}[\mathcal{L}(\theta) - \mathcal{L}(\theta_0)].$$

Since we define the local parameter space $\Theta_{loc}$ with respect to some rate $r_n$, in our derivations we must account for the fact that this $r_n$ may be smaller than the true $r_{n,0}$. In that case there is no way for us to find the true parameter $\theta_0$ and the best we can hope to achieve is to find the best local approximation
\( \theta^* \) of the truth \( \theta_0 \), which we define as

\[
\theta^* = \arg\min_{\theta \in \Theta_0} \frac{1}{2} \mathbb{E}[\mathcal{L}(\theta)].
\]

Note that the truth \( \theta_0 \) fulfills

\[
\theta_0 = \arg\min_{\theta \in \Theta} \frac{1}{2} \mathbb{E}[\mathcal{L}(\theta)] = \arg\min_{\theta \in \Theta_{(r_n,0)}} \frac{1}{2} \mathbb{E}[\mathcal{L}(\theta)].
\]

Hence, if \( r_{n,0} \leq r_n \), \( \theta^* = \theta_0 \). In general, however, estimating \( \theta^* \) is the best we can achieve when solving (5). Thus, we introduce the notion of local excess risk as in Chen et al. (2021), which measures how close a parameter \( \theta \) is to the best local approximation \( \theta^* \) in terms of excess risk:

\[
\mathcal{E}_{\text{loc}}(\theta) := \mathcal{E}(\theta) - \mathcal{E}(\theta^*).
\]

Clearly, \( \theta^* \) also fulfills \( \theta^* = \arg\min_{\theta \in \Theta_{loc}} \mathcal{E}(\theta) \) and we may consider the excess risk of the best local approximation, \( \mathcal{E}(\theta^*) \), as the approximation error of our model. It accounts for the fact that our model might be misspecified, in the sense that the parameter \( r_n \) is not large enough. As is usual in LASSO theory (cf. van de Geer & Bühlmann (2011), Chapter 6), it is tacitly assumed that this approximation error is small, i.e., we assume that \( r_n \) is sufficiently large. Note that the global excess risk of our estimator \( \hat{\theta} \) decomposes as

\[
\mathcal{E}(\hat{\theta}) = \mathcal{E}(\theta^*) + \mathcal{E}_{\text{loc}}(\hat{\theta}),
\]

where we can consider the approximation error \( \mathcal{E}(\theta^*) \) as a deterministic bias. This is similar to the derivations in Chen et al. (2021).

As is commonly assumed in LASSO theory (cf. van de Geer & Bühlmann (2011), Chapter 6), we assume that the unpenalized parameters of \( \theta^* \) are active. That is, \( \mu^* \neq 0, \gamma_i^* \neq 0, i = 1, \ldots, p \). Denote the set of true active indices by \( S^* = S(\beta^*) = \{i : \beta_i^* > 0\} \) with cardinality \( s^* = |S^*| \). For ease of notation, we introduce the set \( S_+^* = S^* \cup \{n+1, n+2, \ldots, n+p\} \) with cardinality \( s_+^* = |S_+^*| = s^* + p + 1 \) to refer to all active indices including those of \( \mu^* \) and \( \gamma^* \).

We set up our problem in the language of LASSO theory for logistic regression. For each pair \( i < j \), denote by \( X_{ij} \in \mathbb{R}^n \) the vector containing one at the \( i \)th and \( j \)th position and zeros everywhere else. Define the matrices

\[
X = \begin{bmatrix}
X_{12}^T \\
\vdots \\
X_{ij}^T \\
\vdots \\
X_{(n-1),n}^T
\end{bmatrix} \in \mathbb{R}^{(n-1)\times m}, 
Z = \begin{bmatrix}
Z_{12}^T \\
\vdots \\
Z_{ij}^T \\
\vdots \\
Z_{n-1,n}^T
\end{bmatrix} \in \mathbb{R}^{(n-1)\times p}.
\]

Let \( 1 \in \mathbb{R}^{(n-1)} \) be the vector containing only ones. Then the design matrix of (1) can be written as

\[
D = \begin{bmatrix} X & 1 & Z \end{bmatrix} \in \mathbb{R}^{(n-1)\times (n+p+1)},
\]

where \( D \), consisting of the matrices \( X, 1 \) and \( Z \) written next to each other, is the analogue to the design matrix in logistic regression. We number the rows of \( D \) as \( D_{ij}^T, i < j \). Here we see a crucial feature of our design matrix \( D \): While the parameters \( \mu \) and \( \gamma \) appear in the link probability of all \( \binom{n}{2} \) node pairs, each \( \beta_i \) only appears in \( (n-1) \) such probabilities. That means, while the effective sample size for \( \mu \) and \( \gamma \) is \( \binom{n}{2} \), it is only \( n-1 \) for each entry of \( \beta \), i.e. it is of order \( n \) smaller. This is also reflected in the different rates of convergence we obtain in Theorem 1 below.

A compatibility condition: A crucial assumption in LASSO theory is the so called compatibility condition (van de Geer & Bühlmann 2011, van de Geer et al. 2014). It relates the quantities \( \| (\hat{\theta} - \theta^*)_{S^*_+} \|_1 \) and

\[
\frac{1}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[(\beta_i - \beta_i^* + \beta_j - \beta_j^* + \mu - \mu^* + (\gamma - \gamma^*)^T Z_{ij})^2]
\]
Proposition 1. Under Assumption 1, for $c$ fulfil

de the minimum eigenvalue.

To show that the compatibility condition also holds for it holds that

\[ \theta \in \mathbb{R}^n \]

assumption restriction, which effectively quantifies how strongly the columns of $\Sigma$ are close to each other in an appropriate sense. To this end, it is sufficient to impose the following

\[ E \lbrack D^T D \rbrack \]

sizes. We need to account for this fact and therefore have to use a sample-size adjusted Gram matrix.

To prove that $\Sigma$ has this property, we will use techniques similar to the ones used in Kock & Tang (2019). Their matrix structure is somewhat simpler than ours as they obtain an identity matrix where we obtain a special Toeplitz matrix. More precisely, we will first show that the compatibility condition holds for the matrix

\[ \Sigma := S^{-1} \mathbb{E}[D^T D] S^{-1} = \frac{1}{n^2} \begin{bmatrix} \sqrt{n} X^T X & \sqrt{n} X^T 1 & 0 \\ \sqrt{n} 1^T X & 1^T 1 & 0 \\ 0 & 0 & \mathbb{E}[Z^T Z] \end{bmatrix}. \]

We consider the limit of the matrix $\Sigma$ entrywise.

We say the compatibility condition holds if the sample-size adjusted Gram matrix $\Sigma$ has the following property: There is a constant $b$ independent of $n$ such that for every $\theta \in \mathbb{R}^{n+1+p}$ with $\|\theta_{S^+}\|_1 \leq 3\|\theta_{S^+}\|_1$ it holds that

\[ \|\theta_{S^+}\|_1^2 \leq \frac{s^*_1}{b} \theta^T \Sigma \theta. \]

To prove that $\Sigma$ has this property, we will use techniques similar to the ones used in Kock & Tang (2019). Their matrix structure is somewhat simpler than ours as they obtain an identity matrix where we obtain a special Toeplitz matrix. More precisely, we will first show that the compatibility condition holds for the matrix

\[ \Sigma_A := \begin{bmatrix} \frac{1}{n-1} X^T X & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathbb{E}[Z^T Z/(n^2)] \end{bmatrix} \in \mathbb{R}^{(n+1+p) \times (n+1+p)}. \]

To show that the compatibility condition also holds for $\Sigma$, it will then suffice to show that $\Sigma$ and $\Sigma_A$ are close to each other in an appropriate sense. To this end, it is sufficient to impose the following eigenvalue restriction, which effectively quantifies how strongly the columns of $Z$ may be correlated.

Assumption 1. There are universal constants $C > c_{\min} > 0$, independent of $n$, such that for all $n \in \mathbb{N}$, the minimum eigenvalue $\lambda_{\min} = \lambda_{\min}(n)$ and the maximum eigenvalue $\lambda_{\max} = \lambda_{\max}(n)$ of $\frac{1}{n^2} \mathbb{E}[Z^T Z]$ fulfil $c_{\min} \leq \lambda_{\min} \leq \lambda_{\max} \leq C < \infty$. Without loss of generality we assume $c_{\min} < 1/2$.

We summarize these results in the following proposition which is proved in Appendix B.1.

Proposition 1. Under Assumption 1, for $s^* = o(\sqrt{n})$ and $n$ large enough, it holds that for every $\theta \in \mathbb{R}^{n+1+p}$ with $\|\theta_{S^+}\|_1 \leq 3\|\theta_{S^+}\|_1$,

\[ \|\theta_{S^+}\|_1^2 \leq \frac{2s^*_1}{c_{\min}} \theta^T \Sigma \theta. \]

Proposition 1 requires $s^* = o(\sqrt{n})$. The $n$ large enough condition is made precise in the proof and requires that $n$ be such that $1/\sqrt{n} < 1/s^*_1$, which is implied by $s^*_1 = o(\sqrt{n})$ and sufficiently large $n$. Let us put this in the context of general LASSO theory. In general LASSO theory, to show that the $\ell_1$-error
goes to zero in probability for increasing $n$, it is imposed that the sparsity $s$ of the true parameter fulfills
\[ s \cdot \sqrt{\frac{\log\left(\text{number of columns of design matrix}\right)}{\text{effective sample size}}} \xrightarrow{n \to \infty} 0, \]
see for example van de Geer & Bühlmann (2011), Chapter 6. In our case the sparsity refers to $\beta$ and we thus should expect that the restrictions we have to impose on $s^*$ are based on the sample size associated with $\beta$. To make our conditions on $s^*$ precise, define $\eta := 2r_n + 2\|\beta^* - \beta_0\|_\infty + |\mu^* - \mu_0| + 2\kappa$ and let
\[ K_n = K_n(\eta) = \frac{2(1 + \exp(r_n,0 + \eta))^2}{\exp(r_n,0 + \eta)}. \]

Notice that $\eta$ essentially quantifies the approximation error we commit. We make the following assumption on $s^*$.

**Assumption 2.** $s^* = o\left(\frac{\sqrt{n}}{\sqrt{\log(n) \cdot K_n}}\right)$.

That means, up to an additional factor $K_n$ – which is the price we have to pay for allowing our link probabilities to go to zero – the permissible sparsity for $\beta^*$ is the permissible sparsity in classical LASSO theory for an effective sample size of order $n$. Clearly, assumption 2 is stronger than the condition $s = o(\sqrt{n})$ in Proposition 1, which thus is not a major restriction.

In our proof, we reformulate our likelihood problem in the language of the sample size adjusted design matrix. This new formulation is entirely equivalent to the previous one in (5), but gives a different interpretation to the sample size adjusted Gram matrix. In particular, we introduce vectors $\tilde{X}_{ij} = \frac{\sqrt{n}}{\sqrt{2}} X_{ij}$ and define
\[ \tilde{X} = \frac{\sqrt{n}}{\sqrt{2}} X = \left[ \tilde{X}_{12}^T \ldots \tilde{X}_{ij}^T \ldots \tilde{X}_{(n-1),n}^T \right] \in \mathbb{R}^{\binom{n}{2} \times (n+1)}, \quad \tilde{D} = [\tilde{X}|1|Z]. \]

We may consider $\tilde{D}$ as a sample-size adjusted design matrix, in the sense that
\[ \Sigma = \frac{1}{\binom{n}{2}} \mathbb{E}[\tilde{D}^T \tilde{D}]. \]

Likewise, we re-define sample-size adjusted parameters. Here, we are effectively blowing up those columns of the design matrix corresponding to $\beta$ to compensate for the fact that $\beta$ has effective sample size of order $n$ smaller than $\mu$ and $\gamma$. The details can be found in Appendix B.2. Naturally, these changes will also result in a sample-size adjusted penalty parameter $\tilde{\lambda}$. For now, we simply remark that $\tilde{\lambda} = \frac{\sqrt{n}}{\sqrt{2}} \lambda$ and refer the readers to Appendix B.2 for the details.

We now state our first main theorem. Its proof is developed in Appendices B.1–B.7.

**Theorem 1.** Assume Assumptions 1 and 2. Fix a confidence level $t$ and let
\[ a_n := \sqrt{\frac{2\log(2(n + p + 1))}{\binom{n}{2}}} (1 \lor c). \]
Choose $\lambda_0 = \lambda_0(t, n)$ as
\[ \lambda_0 = 8a_n + 2\sqrt{\frac{t}{\binom{n}{2}}} (11(1 \lor (e^2 p)) + 8\sqrt{2}(1 \lor c) \sqrt{n} a_n) + \frac{2\sqrt{2}t(1 \lor c)\sqrt{n}}{3\binom{n}{2}}. \]
Let $\bar{\lambda} = \frac{\sqrt{n}}{\sqrt{2}} \lambda \geq 8\lambda_0$ and let $K_n$ be defined as in (7). Then, with probability at least $1 - \exp(-t)$ we have

$$\mathcal{E}(\hat{\theta}) + \bar{\lambda} \left( \frac{\sqrt{n}}{\sqrt{2}} \| \hat{\beta} - \beta^* \|_1 + |\hat{\mu} - \mu^*| + \| \hat{\gamma} - \gamma^* \|_1 \right) \leq 6\mathcal{E}(\theta^*) + 32 \frac{s^*_+ K_n \bar{\lambda}^2}{c_{\min}}.$$ 

Theorem 1 has especially interesting implications if no approximation error is committed, that is in the case that $\theta^* = \theta_0$.

**Corollary 1.** Under the assumptions and with the definitions in Theorem 1, assume that no approximation error is made, i.e. $\theta^* = \theta_0$. Then, with probability at least $1 - \exp(-t)$ we have

$$\mathcal{E}(\hat{\theta}) + \bar{\lambda} \left( \frac{\sqrt{n}}{\sqrt{2}} \| \hat{\beta} - \beta^* \|_1 + |\hat{\mu} - \mu^*| + \| \hat{\gamma} - \gamma^* \|_1 \right) \leq C \frac{s^*_+ \bar{\lambda}^2}{\rho_{n,0}}$$

with constant $C = 128/c_{\min}$.

Corollary 1 gives us an explicit formula for how the sparsity of our network will affect our rate of convergence, which is particularly nice, since in many related works the conditions on network density enter the rate of convergence only indirectly as assumptions on the norm of the true parameter vector, see for example Chatterjee et al. (2011), Yan & Xu (2013). Also, notice that this is essentially the rate of convergence we would expect in the classical LASSO setting for logistic regression up to an additional factor $\rho_{n,0}^{-1}$. Let us consider the implications of Theorem 1 in more detail.

Note that $\lambda_0 \asymp \sqrt{\log(n)/(\binom{n}{r})}$. Hence, we may choose $\bar{\lambda}$ also of order $\sqrt{\log(n)/(\binom{n}{r})}$. Recall that in the classical LASSO setting for logistic regression (cf. van de Geer & Bühlmann (2011)), when no approximation error is committed, when probabilities stay bounded away from zero and when we have the same effective sample size for each parameter, we obtain the rates

$$O_P \left( \text{sparsity} \cdot \frac{\log(\text{number of columns of design matrix})}{\text{effective sample size}} \right)$$

for the excess risk and

$$O_P \left( \text{sparsity} \cdot \sqrt{\frac{\log(\text{number of columns of design matrix})}{\text{effective sample size}}} \right)$$

for the $\ell_1$-error. In the setting of Corollary 1, we obtain

$$\mathcal{E}(\hat{\theta}) = O_P \left( s^*_+ \cdot \frac{1}{\rho_{n,0}} \cdot \frac{\log(n)}{\binom{n}{r}} \right),$$

$$\frac{\sqrt{n}}{\sqrt{2}} \| \hat{\beta} - \beta_0 \|_1 + |\hat{\mu} - \mu_0| + \| \hat{\gamma} - \gamma_0 \|_1 = O_P \left( s^*_+ \cdot \frac{1}{\rho_{n,0}} \cdot \sqrt{\frac{\log(n)}{\binom{n}{r}}} \right),$$

$$\| \hat{\beta} - \beta_0 \|_1 = O_P \left( s^*_+ \cdot \frac{1}{\rho_{n,0}} \cdot \sqrt{\frac{\log(n)}{\sqrt{n - 1}}} \right).$$

That is, up to an additional factor $1/\rho_{n,0}$, we obtain the LASSO rate of convergence for sample size $\binom{n}{r}$ for the global excess risk. By the second line of the display above, we have immediately $\hat{\mu} \stackrel{P}{\to} \mu_0$ and $\hat{\gamma} \stackrel{P}{\to} \gamma_0$ at the rate expected from a LASSO type estimator with effective sample size $\binom{n}{r}$ (up to an additional factor). Furthermore, the third line implies that, again, up to an additional factor, for the error of $\hat{\beta}$, we obtain the rate of convergence we would expect for a LASSO type estimator with sample size $n - 1$. In particular, the assumptions we have to impose to obtain $\ell_1$-consistency include the case $\| \hat{\beta}_0 \|_{\infty} = o(\log(\log(n)))$, which is the condition that had to be imposed in the original $\beta$-model for their strong consistency result (cf. Yan & Xu (2013), Theorem 1).
2.2 Sparse $\beta$-model without covariates

By letting $p = 0$, $\gamma = 0$ and consequently $\kappa = 0$, the results for S$\beta$RM derived in the previous sections have implications for the S$\beta$M without covariates in Chen et al. (2021). In the case without covariates, the negative log-likelihood is given by

$$\mathcal{L}(\beta, \mu) = -\sum_i \beta_i d_i - d_+ \mu + \sum_{i<j} \log(1 + e^{\beta_i + \beta_j + \mu})$$

and our design matrix is simply $D = [X|1] \in \mathbb{R}^{(n \times (n+1)}$. The definitions of $\rho_{n,0}$ and $r_{n,0}$ do not change, as we can simply set $\gamma = 0$ in their original definitions. In this section we will abuse notation slightly by reusing the names from S$\beta$RM, but redefining them to have the components corresponding to $\gamma$ removed. For example, we will use $\theta = (\beta^T, \mu)^T$ for a generic parameter, $\theta_0 = (\beta_0^T, \mu_0)^T$ to denote the truth, $S_+^* = S^* \cup \{n+1\}$ to denote the sparsity including the $\mu$ component etc. We think this is justified as it makes the connection to the respective objects in the model with covariates clearer. Our estimator reduces to

$$\hat{\theta} = (\hat{\beta}^T, \hat{\mu})^T = \arg \min_{(\beta^T, \mu)^T \in \Theta_{\text{loc}}} \frac{1}{2} \mathcal{L}(\beta, \mu) + \lambda \|\beta\|_1,$$

where by slight abuse of notation, for this section only, we define $\Theta_{\text{loc}} = \Theta_{\text{loc}}(r_n) := \{\theta = (\beta^T, \mu)^T \in \mathbb{R}^+_n \times \mathbb{R} : ||D\theta||_\infty \leq r_n\}$, for the reduced design matrix $D$ defined above and a rate $r_n$.

We make definitions completely analogous to the case in which we observe covariates. We adapt the definitions of the excess risk $\mathcal{E}(\theta)$ in the canonical way by letting the components corresponding to $\gamma$ and $Z_{ij}$ equal zero. We define the best local approximation $\theta^*$ as

$$\theta^* = \arg \min_{\theta \in \Theta_{\text{loc}}} \mathcal{E}(\theta)$$

and as before, we assume that all unpenalized parameters, i.e. $\mu^*$ in this case, are active. Since the sparsity assumptions of our parameter only concern $\beta$, it is natural that we should need the same assumptions on $s^*_+$ as before, most notably Assumption 2. We have the analogue to Theorem 1:

**Theorem 2.** Assume Assumption 2. Fix a confidence level $t$ and let

$$a_n = \sqrt{\log(2(n+1)) \binom{n}{2}}$$

and

$$\lambda_0 = 8a_n + 2 \sqrt{\frac{t}{\binom{n}{2}}} (9 + 8\sqrt{2}na_n) + \frac{2\sqrt{2}t\sqrt{n}}{3\binom{n}{2}}.$$

Let $\tilde{\lambda} = \sqrt{\frac{\eta}{\sqrt{2}}} \lambda \geq 8\lambda_0$ and define $\eta$ and $K_n$ as in (7) with $\kappa$ set to zero. Then, with probability at least $1 - \exp(-t)$ we have

$$\mathcal{E}(\hat{\theta}) + \tilde{\lambda} \left( \frac{\sqrt{2}}{\sqrt{n}} \|\hat{\beta} - \beta^*\|_1 + |\hat{\mu} - \mu^*| \right) \leq 6\mathcal{E}(\theta^*) + 4s^*_+ K_n \tilde{\lambda}^2. \quad (8)$$

A proof, which follows almost immediately from the case in which we do observe covariates, is given in Appendix B.7. It is interesting to put this result into context by comparing it with Theorem 2 in Chen et al. (2021). The parameter space over which Chen et al. (2021) are optimizing is not convex and the analogous notion of best local approximation we are using need not be well-defined in their setting. Thus, it is not possible to derive $\ell_1$-error bounds for their estimator, as we do in Theorem 2. Nonetheless and quite remarkably, they are able to prove an existence criterion for their $\ell_0$-constrained estimator and a high-probability, finite sample bound on its excess risk. To compare their results to ours, we consider a special case that they discuss at length. In particular, they consider the situation in which $\mu_0 = -\xi \cdot \log(n) + O(1)$ for some $\xi \in [0, 2)$ and $\beta_{0,i} = \alpha \cdot \log(n) + O(1)$ for some $\alpha \in [0, 1)$ and
all \( i \in S_0 \), where \( \alpha \) and \( \xi \) are such that \( 0 \leq \xi - \alpha < 1 \). It is easy to see that under these assumptions we have \( \rho_{n, 0} \sim n^{-\xi} \). Consider the regime in which no approximation error is committed. Then, using an analogous argument as in the proof of Corollary 1, \( K_n \) is of order \( \rho_{n, 0}^{-1} \). Recalling Assumption 2, we see that to obtain \( \ell_1 \)-consistency of our estimator, we need \( \xi < 1/2 \), which restricts the degree of network sparsity that our estimator can handle. Chen et al. (2021) need no such condition and only need to balance the global sparsity parameter \( \xi \) with the local density parameter \( \alpha \) to have convergence of their excess risk to zero. This illustrates that to obtain our more refined consistency result in terms of \( \ell_1 \)-error, we understandably need to impose stricter assumptions on the permissible sparsity. We now compare the bounds on the excess risk. Note that Chen et al. (2021) scale their excess risk by \( \mathbb{E}[d_+]^{-1} \sim n^{-2+\xi} \), rather than \( \binom{n}{2}^{-1} \sim n^{-2} \) as we do. To put the excess risk on the same scale, we denote by \( \mathcal{E}(\hat{\theta}) = n^2 \mathcal{E}(\hat{\theta}) \) the excess risk rescaled to their setting. With this notation, we see that by Theorem 2 the error rate for the rescaled excess risk of our \( \ell_1 \) constrained estimator becomes
\[
\mathcal{E}(\hat{\theta}) = O_P(s^*_+ \cdot \log(n) \cdot n^{-2+2\xi}),
\]
which by Assumption 2 is \( o_P(\sqrt{\log(n)} \cdot n^{-3/2+\xi}) \). From Chen et al. (2021), Theorem 2, it is seen that the rate for the excess risk of their \( \ell_0 \) constrained estimator is
\[
O_P(\log(n) \cdot n^{-1+\xi/2}).
\]
This shows that in the regime \( \xi \in [0, 1/2) \) necessary for \( \ell_1 \)-consistent parameter estimation, our estimator will always achieve a rate faster than the one in Chen et al. (2021). When we leave this regime, however, consistent estimation with respect to the \( \ell_1 \)-norm may no longer be possible and the estimator in Chen et al. (2021) can outperform our estimator.

### 2.3 S\( \beta \)RM without \( \beta \)

When \( \beta = 0_n \), the linking probability in S\( \beta \)RM becomes
\[
P(A_{ij} = 1|Z_{ij}) = p_{ij} = \frac{\exp(\mu + Z_{ij}^T \gamma)}{1 + \exp(\mu + Z_{ij}^T \gamma)},
\]
which can be seen as a generalized Erdős-Rényi model with covariates incorporated. For this reason, we will abbreviate this model as ER-C.

We study the properties of the MLE of \( \mu \) and \( \gamma \) under the sparse network regime. Towards this, following Chen et al. (2021), we encode the sparsity of the model (9) explicitly by assuming that a reparametrization of the global sparsity parameter \( \mu \) takes the form
\[
\mu = -\xi \log(n) + \mu^1,
\]
where \( \xi \in [0, 2) \) effectively takes the role of \( \rho_{n, 0} \) from the previous sections and \( \mu^1 \in [-M, M] \) for a fixed \( M < \infty \) independent of \( n \). To appreciate this reformulation, we see that the expected total number of edges of ER-C is of the order \( O(n^{2-\xi}) \). When \( \xi = 0 \), ER-C becomes a standard logistic regression model with fixed parameters. It can generate arbitrarily sparse networks when \( \xi > 0 \). To the best of our knowledge, a model of this type that also accounts for covariates has not been studied in the literature before and thus the results below can be of independent interest.

We denote the true parameters \( \mu_0^1 \) and \( \gamma_0 \) respectively. As in the previous section, we abuse notation slightly and denote a generic parameter as \( \theta = (\mu^1, \gamma) \), the true parameter as \( \theta_0 = (\mu_0^1, \gamma_0) \) and our estimator (defined below) as \( \hat{\theta} = (\hat{\mu}^1, \hat{\gamma}) \). We think this abuse of notation is justified as it allows a consistent notation with the other sections. We make the following assumptions.

**Assumption 3.** The true parameter \( \theta_0 = (\mu_0^1, \gamma_0^T)^T \) lies in the interior of \([-M, M] \times \Gamma\).

**Assumption 4.** The \( Z_{ij} \) are i.i.d. realizations of the same random variable. The covariance matrix of \( Z_{12} \), that is the matrix \( \mathbb{E}[Z_{12}Z_{12}^T] \), is strictly positive definite with minimum eigenvalue \( \lambda_{\min} > 0 \).
Assumption 4 is analogue to Assumption 1 in the case with non-zero $\beta$. We remark that the i.i.d. condition is used to simplify parts of the proofs and can be relaxed at the expense of lengthier proofs.

We consider the following function which is proportional to the negative log-likelihood of the ER-C up to a summand independent of the parameter

$$L^\dagger(\mu^\dagger, \gamma) = -d + \mu^\dagger - \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} + \sum_{i<j} \log \left(1 + n^{-\xi} \exp(\mu^\dagger + \gamma^T Z_{ij})\right).$$

(10)

In the ER-C, the dimension of the parameter is fixed. Therefore, it is not necessary to employ a penalized likelihood approach as in the S$\beta$RM and we estimate $\theta$ via maximum likelihood

$$\hat{\theta} = (\hat{\mu}^\dagger, \hat{\gamma})^T = \arg\min_{\theta=(\mu^\dagger, \gamma^T)^T} L^\dagger(\mu^\dagger, \gamma),$$

(11)

where the argmin is taken over $[-M, M] \times \Gamma$. The design matrix $D$ now takes the simplified form

$$D = [1 \mid Z] \in \mathbb{R}^{(\binom{n}{2}) \times (p+1)}.$$

As before, we enumerate the rows of $D$ as $D_{ij}^T$, $i < j$, where each $D_{ij}$ is treated as a column vector, i.e. $D = [D_{ij}]_{1<j}$. Define the matrix $\Sigma \in \mathbb{R}^{(p+1) \times (p+1)}$ as

$$\Sigma := \mathbb{E} \left[(D_{12}D_{12}^T) \exp(\mu_0^\dagger) \exp(\gamma_0^T Z_{12})\right],$$

which is invertible by Assumption 4. We have the following central limit theorem for $\hat{\theta}$, the proof of which can be found in Appendix D. Denote by $\mathcal{N}(0, \Sigma^{-1})$ the law of the multivariate normal distribution with zero mean vector and covariance matrix $\Sigma^{-1}$.

**Theorem 3.** Under Assumptions 3 and 4, it holds, as $n \to \infty$,

$$\sqrt{\binom{n}{2}/n^\xi} (\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma^{-1}).$$

Since the expected number of observed edges in the ER-C is of order $n^{2-\xi}$, the factor $\sqrt{\binom{n}{2}/n^\xi}$ in Theorem 3 corresponds to the square root of the effective sample size. This means, having the link probabilities go to zero reduces the information we gain about $\theta_0$ and this information loss is made explicit in a rate of convergence slower than what we would obtain in a classical parametric setting. This finding is in line with the results in Chen et al. (2021), Proposition 1 and Theorem 1, in which a similar phenomenon was observed.

While we consider Theorem 3 to be interesting from a theoretical point of view, in practice, the sparsity rate parameter $\xi$ will not be known, which makes solving (11) and finding the MLE $(\hat{\mu}^\dagger, \hat{\gamma})$ impossible. Remarkably, it is possible, though, to circumvent this problem with the following argument.

Notice that from Theorem 3 we obtain for any $k = 1, \ldots, (p + 1)$,

$$\sqrt{\binom{n}{2}/n^\xi} \cdot \frac{\hat{\theta}_k - \theta_{0,k}}{\sqrt{\Sigma^{-1}_{kk}}} \xrightarrow{d} \mathcal{N}(0, 1),$$

where $\mathcal{N}(0, 1)$ denotes the law of the univariate standard-normal distribution. We also may make use of the identity

$$\hat{\mu} = -\xi \log(n) + \hat{\mu},$$

(12)

where $\hat{\mu}$ is the MLE of the global sparsity parameter before reparametrization. In particular, $\hat{\mu}$ can be
found without knowledge of $\xi$. Define the matrix

$$
\hat{\Sigma} = \frac{1}{(2)^{\nu}} D^T \text{diag} \left( \frac{\exp(\mu + \gamma^T Z_{ij})}{(1 + \exp(\mu + \gamma^T Z_{ij}))^2}, i < j \right) D.
$$

In Appendix D we show that $n^\xi \hat{\Sigma} = \Sigma + o_P(1)$, which in turn allows us to show $(n^\xi \hat{\Sigma})_{k,k}^{-1} = \Sigma_{k,k}^{-1} + o_P(1)$ for all $k = 1, \ldots, (p + 1)$. Then, by Slutsky’s theorem,

$$
\sqrt{\frac{n}{2}} \frac{\hat{\mu} - \mu_{0,k}}{\Sigma_{k,k}^{-1}} = \sqrt{\frac{n}{2}} \frac{\hat{\mu} - \mu_{0,k}}{n^{-\xi} \Sigma_{k,k}^{-1}} = \sqrt{\frac{n}{2}} \frac{\hat{\mu} - \mu_{0,k}}{\Sigma_{k,k}^{-1} + o_P(1)} \overset{d}{\rightarrow} \mathcal{N}(0,1).
$$

In other words, the matrix $\hat{\Sigma}$ will be singular in the limit as the link probabilities $p_{ij}$ go to zero. The rate $n^\xi$ is precisely the rate with which we need to multiply $\hat{\Sigma}$ to stabilize it and make it converge to the non-singular matrix $\Sigma$, whose inverse is the asymptotic covariance matrix in Theorem 3. Thus, the stabilizing rate $n^\xi$ for the asymptotic covariance and the sample size reducing rate $n^{-\xi}$ will cancel out, allowing us to derive the component-wise limiting distribution of each $\hat{\theta}_k$ without the knowledge of $\xi$. In particular, looking at the case $k = 2, \ldots, (p + 1)$, we are able to calculate confidence intervals for the components of the covariate parameter $\gamma$ without having to know $\xi$. In summary, Theorem 3 allows the following corollary which is proved in Appendix D.

**Corollary 2.** Under Assumptions 3 and 4, the following componentwise asymptotic normality results hold as $n \to \infty$ for $k = 1, \ldots, p$,

$$
\sqrt{\frac{n}{2}} \frac{\hat{\gamma}_k - \gamma_{0,k}}{\Sigma_{k+1,k+1}^{-1}} \overset{d}{\rightarrow} \mathcal{N}(0,1).
$$

Simulation results corroborating the claims in Corollary 2 are shown in Section 4.2.

### 3 Inference for the Homophily Parameter

In this section we derive the limiting distribution for our estimator of the covariates weights $\hat{\gamma}$ when $\rho_{n,0} \geq \rho_n$ and thus $\theta^* = \theta_0$. We will see that the same arguments used for deriving the limiting distribution for $\hat{\gamma}$ also work for $\hat{\mu}$ and as a by-product of our proofs we also obtain an analogous limiting result for $\hat{\mu}$.

Our strategy will be inverting the KKT conditions, similar to van de Geer et al. (2014). The estimation in (5) is a convex optimization problem. Hence, by subdifferential calculus, we know 0 has to be contained in the subdifferential of $\frac{1}{(2)^{\nu}} \mathcal{L}(\theta) + \lambda \| \beta \|_1$ at $\hat{\theta}$. That is, there exists some $v \in \mathbb{R}^{n+1+p}$ such that

$$
0 = \frac{1}{(2)^{\nu}} \nabla \mathcal{L}(\theta)|_{\theta = \hat{\theta}} + \lambda v, \quad \text{(13)}
$$

where $\nabla \mathcal{L}(\theta)|_{\theta = \hat{\theta}}$ is the gradient of $\mathcal{L}(\theta)$ evaluated at $\hat{\theta}$ and for $i = 1, \ldots, n, v_i = 1$ if $\hat{\beta}_i > 0$ and $v_i \in [-1,1]$ if $\hat{\beta}_i = 0$, and $v_i = 0$ for $i = n + 1, \ldots, n + 1 + p$.

To ease notation a little we will use $\hat{\nu} = (\mu, \gamma^T)^T$ to refer to the unpenalized parameter subvector of $\theta$. Thus, denoting $\nabla_{\hat{\nu}} \mathcal{L}(\theta)|_{\theta = \hat{\theta}} \in \mathbb{R}^{p+1}$ the gradient of $\mathcal{L}$ with respect to the unpenalized parameters $(\mu, \gamma^T)^T$ only, evaluated at $\hat{\theta}$, we have

$$
0 = \nabla_{\hat{\nu}} \mathcal{L}(\theta)|_{\theta = \hat{\theta}}. \quad \text{(14)}
$$

Denote by $H(\hat{\theta}) := H_{\hat{\nu} \times \hat{\nu}}(\hat{\theta})|_{\theta = \hat{\theta}}$ the Hessian of $\frac{1}{(2)^{\nu}} \mathcal{L}(\theta)$ with respect to $\theta$ only, evaluated at $\hat{\theta}$. Denote
\[ p_{ij}(\theta) = \frac{\exp(\beta_i + \beta_j + \mu + \gamma T Z_{ij})}{1 + \exp(\beta_i + \beta_j + \mu + \gamma T Z_{ij})}. \]

Now, consider the entries of \( H(\hat{\theta}) \). For all \( k, l = 1, \ldots, (p + 1) \),
\[
H(\hat{\theta})_{k,l} = \frac{1}{2} \partial_{\theta_k \theta_l} L(\hat{\theta}) = \frac{1}{2} \sum_{i<j} D_{ij,n+k} D_{ij,n+l} p_{ij}(1 - p_{ij}(\hat{\theta})),
\]
where \( D_{ij} \) is the \((i,j)\)-th row of the design matrix \( D \), i.e. in particular \( D_{ij,n+k} = 1 \) if \( k = 1 \) and \( D_{ij,n+k} = Z_{ij,k-1} \) for \( k = 2, \ldots, (p + 1) \). In particular, we have the following matrix representation of \( H(\hat{\theta}) \). Let \( D_\theta = [1:Z] \) be the part of the design matrix corresponding to \( \theta \) with rows \( D_{\theta,ij} = (1, Z_{ij}^T), i < j \). Also let \( \hat{W} = \text{diag}(\sqrt{p_{ij}(1 - p_{ij}(\hat{\theta}))), i < j) \). Then we have
\[
H(\hat{\theta}) = \frac{1}{n} D_\theta^T \hat{W}^2 D_\theta.
\]

Let \( W_0 = \text{diag}(\sqrt{p_{ij}(\theta_0)(1 - p_{ij}(\theta_0))), i < j) \) and consider the corresponding population version:
\[
\mathbb{E}[H(\theta_0)] = \frac{1}{n} \mathbb{E}[D_\theta^T W_0^2 D_\theta].
\]

To be consistent with commonly used notation, call \( \hat{\Sigma}_\theta = H(\hat{\theta}) = \frac{1}{n} D_\theta^T \hat{W}^2 D_\theta \) and \( \Sigma_\theta = \mathbb{E}[H(\theta_0)] = \frac{1}{n} \mathbb{E}[D_\theta^T W_0^2 D_\theta] \) and \( \hat{\Theta}_0 := \hat{\Sigma}_0^{-1}, \Theta_0 := \Sigma_0^{-1} \).

We will need to invert \( \hat{\Sigma}_\theta \) and \( \Sigma_\theta \) and show that these inverses are close to each other in an appropriate sense. It is commonly assumed in LASSO theory (cf. van de Geer et al. (2014)) that the minimum eigenvalues of these matrices stay bounded away from zero. In our case, however, such an assumption is invalid.

Indeed, since \( \rho_n \leq 1/2 \), we find that for all \( i < j, p_{ij}(\theta_0)(1 - p_{ij}(\theta_0)) \geq 1/2 \cdot \rho_n \). Also, recall that by Assumption 1, the minimum eigenvalue \( \lambda_{\min} \) of \( \mathbb{E}[Z^T Z/(2_n)] \) stays uniformly bounded away from zero for all \( n \). Then, for any \( n \) and \( v \in \mathbb{R}^{p+1} \setminus \{0\} \) with components \( v = (v_1, v_R)^T, v_R \in \mathbb{R}^p \), we have
\[
v^T \Sigma_\theta v \geq \frac{1}{2} \rho_n v^T \frac{1}{n} \mathbb{E}[D_\theta^T D_\theta] v = \frac{1}{2} \rho_n v^T \left( \frac{1}{n} \mathbb{E}[Z^T Z] \right) v
\]
\[
= \frac{1}{2} \rho_n \left( v_1^2 + v_R^T \frac{1}{n} \mathbb{E}[Z^T Z] v_R \right)
\]
\[
\geq \frac{1}{2} \rho_n (v_1^2 + \lambda_{\min} \|v_R\|_2^2) \geq \frac{1}{2} \rho_n (1 \wedge \lambda_{\min}) \|v\|_2^2 > 0.
\]

Hence, for finite \( n \) all eigenvalues of \( \Sigma_\theta \) are strictly positive and consequently this matrix is invertible. Using similar techniques as in the proof of Proposition 1 in the Appendix we can now show that with high probability the minimum eigenvalue of \( D_\theta^T \Sigma_\theta^{-1} D_\theta / (n_2) \) is also strictly larger than zero and thus for any \( v \in \mathbb{R}^{p+1} \setminus \{0\} \) and any finite \( n \) (the exact derivations are given in Appendix C.1),
\[
\frac{1}{n} v^T D_\theta^T \hat{W}^2 D_\theta v \geq C \rho_n \lambda_{\min} \left( \frac{1}{n} Z^T Z \right) \|v\|_2^2 > 0.
\]

Thus, for every finite \( n, \hat{\Sigma}_\theta \) is invertible with high probability. Since these lower bounds tend to zero with increasing \( n \), a careful argument is needed and we have to impose stricter assumptions than for our consistency result alone.

**Assumption 5.** \( s^* \sqrt{\frac{\log(n)}{\sqrt{n} \rho_n}} \to 0, n \to \infty \).

Assumption 5 is a slightly stricter version of the previously imposed Assumption 2. Previously we only needed a factor of \( 1/\rho_n \) to ensure that the \( \ell_1 \)-error for \( \hat{\beta} \) in Theorem 5 goes to zero. Notice, though,
that these assumptions still allow sparsity rates for $\rho_{n,0}$ of small polynomial order. More precisely, up to a log-factor and depending on the speed of $s_k^\ast$, $\rho_{n,0}$ may still go to zero at a speed of order up to $n^{-1/4}$.

**Theorem 4.** Under Assumptions 1 and 5, when $\theta^* = \theta_0$, we have for any $k = 1, \ldots, p$, as $n \to \infty$,

$$\sqrt{\frac{n}{2}} \frac{\hat{\gamma}_k - \gamma_0,k}{\hat{\Theta}_{\theta,k+1,k+1}} \overset{d}{\to} N(0,1).$$

We also have for our estimator of the global sparsity parameter, $\hat{\mu}$, as $n \to \infty$,

$$\sqrt{\frac{n}{2}} \frac{\hat{\mu} - \mu_0}{\hat{\Theta}_{\theta,1,1}} \overset{d}{\to} N(0,1).$$

Remarkably, Theorem 4 does not require debiasing the estimates $\hat{\mu}$ and $\hat{\gamma}$ for their inference, in comparison to the need for such for the usual LASSO estimates for other models due to the bias incurred by shrinkage (van de Geer et al. 2014). This bias is made explicit in equation (13): The penalized parameter values do not fulfill the first-order estimating equations exactly, but rather a bias of the form $\lambda v$ is incurred as prescribed by subdifferential calculus. While the unpenalized parameter estimates $(\hat{\mu}, \hat{\gamma}^T)^T$ do fulfill the first-order estimating equations exactly, in standard settings this alone would still not be enough to ensure the asymptotic normality of $\hat{\nu}$. However, in our special case, due to the differing sample sizes between $\beta$ and $\theta$, this is enough to allow us to derive a limiting distribution without a debiasing step. More precisely, deriving the limiting distribution of $\hat{\theta}$ relies on a Taylor expansion of the negative log-likelihood $L$. To derive Theorem 4, it is necessary, that in said Taylor expansion the bias incurred from the part of the likelihood relating to $\beta$ vanishes in probability. This essentially is a condition on the asymptotic correlation between the columns of the design matrix $D$ corresponding to $\beta$ and those corresponding to $\mu$ and $\gamma$. Due to each column in $X$, the deterministic part of the design matrix relating to $\beta$, being very sparse and having only $n - 1$ non-zero entries, this bias vanishes fast enough to allow Theorem 4. See Appendix C.5 for details.

4 Simulation

4.1 S$\beta$RM: Sparse $\beta$-regression model

In this section we illustrate the finite sample performance of our penalized likelihood estimator with an extensive set of Monte Carlo simulations. We only show results for S$\beta$RM and the estimator (5), as where applicable the results in the case without covariates are very similar. We check both the $\ell_1$-convergence of our parameter estimates to the true parameter, as well as the asymptotic normality of $\hat{\gamma}$ by documenting the empirical coverage of 95% confidence intervals.

Since our estimation involves the choice of a tuning parameter, we explored the use of the Bayesian Information Criterion (BIC) for model selection as well as a heuristic based on the theory developed in the previous sections to specify its value. We check both the $\ell_1$-convergence of our parameter estimates to the true parameter, as well as the asymptotic normality of $\hat{\gamma}$ by documenting the empirical coverage of 95% confidence intervals.

Since our estimation involves the choice of a tuning parameter, we explored the use of the Bayesian Information Criterion (BIC) for model selection as well as a heuristic based on the theory developed in the previous sections to specify its value. While the former criterion is purely data-driven, the use of the latter is to ensure that our theoretical results are about right in terms of the rates. To make the dependence of our estimator (5) on the penalty parameter explicit, we denote the solution of (5) when using penalty $\lambda$ by $\hat{\theta}(\lambda) = (\hat{\beta}(\lambda)^T, \hat{\mu}(\lambda), \hat{\gamma}(\lambda)^T)^T$ and write $s(\lambda) = |\{i : \hat{\beta}_i(\lambda) > 0\}$ for its sparsity. The value of the BIC at $\lambda$ is given by

$$\text{BIC} = 2\mathcal{L}(\hat{\theta}(\lambda)) + s(\lambda) \log(n(n - 1)/2)$$

and the penalty $\lambda$ was chosen to minimize BIC.

To motivate the heuristic approach to tuning parameter selection, recall that Theorem 1 suggests
that based on a confidence level $t$ picked by us, we should first define

$$a_n = \sqrt{\frac{2 \log(2(n + p + 1))}{\binom{n}{2}}} (1 \lor c),$$

and then based on that, choose $\lambda_0 = \lambda_0(t, n)$ as

$$\lambda_0 = 8a_n + 4 \frac{t (3(1 \lor (c^2p)) + 2\sqrt{2}(1 \lor c)\sqrt{n}a_n) + 2\sqrt{2t}(1 \lor c)\sqrt{n}}{3\binom{n}{2}}.$$  

Finally, the consistency results derived hold for any $\bar{\lambda} \geq 8\lambda_0$, where $\bar{\lambda}$ is the penalty parameter in the rescaled penalized likelihood problem, which relates to the penalty parameter $\lambda$ in the original penalized problem (5) as $\bar{\lambda} = \sqrt{n}/\sqrt{2} \cdot \lambda$. Looking back at the proof of Theorem 1, we see that the factor eight in the relation between $\lambda_0$ and $\bar{\lambda}$ is a technical artifact we had to introduce to prove that the sample-size adjusted estimator $\hat{\theta}$ as defined in Appendix B.2 was close enough to the sample-size adjusted best local approximation $\theta^*$. If we assume that our estimator is close enough to the truth, we may ignore that factor and set $\lambda = \frac{n}{n^2}\lambda_0$. We pick $t = 2$ and set $c$ to the maximum observed covariate value. It is known that in high-dimensional settings the penalty values prescribed by mathematical theory in practice tend to over-penalize the parameter values, see, for example, Yu et al. (2021). Decreasing the penalty by removing the factor eight is thus in line with these empirical findings.

For our simulation, we fixed $p = 2$ by setting the covariate weights as $\gamma_0 = (1, 0.8)^T$ and generated the covariates from a centered Beta $(2, 2)$ distribution as $Z_{ij,k} \sim \text{Beta}(2, 2) - 1/2$. We consider networks of sizes $n = 300, 500, 800$ and 1000 in which the sparsity of $\beta_0$ is set as 7, 9, 10, and 12 respectively. We tested our estimator on three different model configurations with different combinations of $\beta_0$ and $\mu_0$, resulting in networks with varying degrees of sparsity. For each simulation configuration, 1000 data sets are simulated. Specifically,

**Model 1:** We pick $\beta_0 = (1.2, 0.8, 1, \ldots, 1, 0, \ldots, 0)^T$, where the number of ones increases with the network size to match the aforementioned sparsity level, and set $\mu_0 = -0.5 \log(\log(n))$;

**Model 2:** We pick $\beta_0 = \log(\log(n)) \cdot (1.2, 0.8, 1, \ldots, 1, 0, \ldots, 0)^T$ and set $\mu_0 = -1.2 \cdot \log(\log(n))$;

**Model 3:** We pick $\beta_0 = \log(\log(n)) \cdot (2.0, 8, 1, \ldots, 1, 0, \ldots, 0)^T$ and set $\mu_0 = -0.5 \cdot \log(n)$.

In these three models, we allow $\mu_0$ to get progressively more negative to generate networks that are increasingly sparse, and allow the sparsity of $\beta_0$ to increase with network size $n$. All three models get progressively sparser with increasing $n$. Model 3 gives the sparsest networks when $n = 1000$, with only around 3.6% of all possible edges being present on average.

**Consistency:** We calculated the mean absolute error (MAE) for estimating $\beta_0$, the absolute error for estimating $\mu_0$ and the $\ell_1$-error for estimating $\gamma_0$. For Model 1 the results are shown in Figures 1a–1c. While BIC performs slightly better for estimating $\beta_0$ and $\mu_0$ for smaller network sizes, our heuristic performs better for larger network sizes. The $\ell_1$-error for estimating $\gamma$ is almost the same between both model selection schemes across all network sizes. For both methods we can see that the various errors decrease with increasing network size. Model 2 gives similar results with slightly smaller errors produced by BIC for $\beta_0$ and $\mu_0$ for smaller networks and similar or slightly better errors produced by the heuristic for large networks (Figures 2a, 2b). The error for $\gamma_0$ is similar between both methods (Figure 2c). For Model 3, the various errors for parameter estimation are shown in Figures 3a, 3b and 3c. The error values are generally higher than in the other network models, which is to be expected due to the much higher sparsity of the network. Also, for this very sparse case, BIC is performing better than the heuristic. The heuristic consistently selects higher penalty values than BIC and we can see how this results in worse estimates for very sparse networks. Also, for the heuristic we choose one predefined penalty value for any network of a given size $n$, while BIC can adapt to the observed sparsity. This illustrates the point made by Yu et al. (2021), that the penalty prescribed by mathematical theory tends to over-penalize the model. It is to be noted, though, that even in this very sparse regime both model selection techniques produce reasonable estimates that are close to the truth.
Figure 1: Errors for estimating the true parameter $\theta_0$ in Model 1 across various network sizes and 1000 repetitions. Comparison between model selection via BIC and a heuristic approach. The results when model selection is done with BIC are displayed in red (left boxes), those for the pre-determined $\lambda$ in green (right boxes).

Figure 2: Errors for estimating the true parameter $\theta_0$ in Model 2.

Figure 3: Errors for estimating the true parameter $\theta_0$ in Model 3.
Asymptotic normality: Next, we consider the normal approximation for our estimator \( \hat{\gamma} \). We calculate the standardized \( \gamma \)-values

\[
\sqrt{\frac{n}{2}} \frac{\hat{\gamma}_k - \gamma_{0,k}}{\sqrt{\Theta_{\hat{\gamma},k+1,k+1}}}, \quad k = 1, 2,
\]

which by Theorem 4 asymptotically follow a \( \mathcal{N}(0,1) \) distribution. This allows us to construct approximate 95%-confidence intervals for \( \gamma_{0,k} \)

\[
CI_k = \left[ \hat{\gamma}_k - z_{1-\alpha/2} \cdot \sqrt{\frac{n}{2}}, \hat{\gamma}_k + z_{1-\alpha/2} \cdot \sqrt{\frac{n}{2}} \right], \quad k = 1, 2,
\]

where \( z_{1-\alpha/2} \) is the \( 1 - \alpha/2 \) quantile of the standard-normal distribution and we use \( \alpha = 0.05 \). We present the empirical coverage of these intervals and their median length for the different network sizes.

Table 1 shows the results for \( \gamma_{0,1} \) across the different models and sample sizes. The results for \( \gamma_{0,2} \) are similar and are omitted to save space. The coverage is very close to the 95%-level across all network sizes and all models and independent of which model selection criterion we use. This empirically illustrates the validity of the asymptotic results derived in Theorem 4. The median length of the confidence interval decreases with increasing network size and is similar between BIC and the heuristic. This is what we would expect since the estimates for \( \gamma_0 \) are very similar between both methods as shown in Figures 1c, 2c, and 3c. Comparing the length of the confidence intervals between Models 1, 2 and 3, we see that as the models become sparser, the median length increases, which is also to be expected.

| \( n \) | Coverage | CI | Coverage | CI |
|-------|----------|----|----------|----|
|       | Pre-determined \( \lambda \) | BIC |
| Model 1 | 300 | 0.949 | 0.182 | 0.950 | 0.182 |
|       | 500 | 0.944 | 0.110 | 0.944 | 0.110 |
|       | 800 | 0.953 | 0.069 | 0.954 | 0.069 |
|       | 1000 | 0.945 | 0.056 | 0.942 | 0.056 |
| Model 2 | 300 | 0.927 | 0.251 | 0.937 | 0.252 |
|       | 500 | 0.958 | 0.158 | 0.961 | 0.158 |
|       | 800 | 0.940 | 0.103 | 0.940 | 0.103 |
|       | 1000 | 0.945 | 0.083 | 0.947 | 0.083 |
| Model 3 | 300 | 0.931 | 0.333 | 0.939 | 0.335 |
|       | 500 | 0.937 | 0.225 | 0.942 | 0.226 |
|       | 800 | 0.939 | 0.159 | 0.942 | 0.159 |
|       | 1000 | 0.941 | 0.133 | 0.942 | 0.134 |

Table 1: Empirical coverage under nominal 95% coverage and median lengths of confidence intervals.

### 4.2 ER-C: The Erdős-Rényi model with covariates

In this section we illustrate the finite sample performance of the MLE in (11) in the ER-C (9). We focus on inference for the covariate weights, \( \gamma \), in the more realistic case of unknown \( \xi \), that is, we use the identity (12) to estimate \( \mu_0 \) rather than \( \mu_0^\dagger \). Our emphasis is on illustrating that the MLE can be used to perform inference in extremely sparse network settings. To that end we fixed the covariate dimension \( p \) and a true parameter vector \( (\mu_0^\dagger, \gamma_0^T)^T \) and varied the sparsity parameter \( \xi \). The exact model setup was as follows. We set \( p = 20 \) and sampled the covariate values \( Z_{ij,k}, k = 1, \ldots, p, i < j \) from a centered Beta \( (2, 2) \) distribution. We used \( \mu_0^\dagger = 1 \) and \( \gamma_0 = (1.5, 1.2, 0.8, 1, \ldots, 1)^T \). For the sparsity parameter \( \xi \) we used the values \( \xi = 0.3, 1.0, \) or \( 1.5 \). As before, we sampled networks of sizes \( n = 300, 500, 800, 1000, \) and for each configuration we drew 1000 realizations of the ER-C and analyzed the performance of the MLE (11). The sparsest case \( \xi = 1.5 \) is close to the maximum theoretically permissible sparsity and
results in extremely sparse networks. For \( n = 1000 \), on average, only 73 out of the almost half million possible edges are observed in this setting.

The asymptotic normality for each component of \( \hat{\gamma} \) allows us to construct confidence intervals at the 95%-level as prescribed by Corollary 2 and we assess the performance of our MLE by calculating the empirical coverage for each component. There is no significant difference in the empirical coverage or the average length of the confidence intervals between the various components of \( \gamma \), which is why we only present them for \( \gamma_1 \) in Table 2. As we can see, coverage is very close to the nominal confidence level of 95% and the length of the confidence intervals decrease with increasing network size. As expected, confidence intervals are larger for sparse networks. For \( \xi = 1.5 \) we observe very wide confidence intervals, which is due to the very low effective sample size.

| \( n \) | \( \xi = 0.3 \) | \( \xi = 1.0 \) | \( \xi = 1.5 \) |
|---------|---------------|---------------|---------------|
| 300     | 0.941         | 0.956         | 0.944         |
| 500     | 0.955         | 0.938         | 0.967         |
| 800     | 0.943         | 0.950         | 0.951         |
| 1000    | 0.949         | 0.935         | 0.948         |

Table 2: Empirical coverage under nominal 95% coverage and median lengths of confidence intervals for \( \gamma_1 \). The results are similar for the other components of \( \gamma \).

5 Data Analysis

We illustrate our results further by applying our estimator to two real world data sets:

**Lazega’s lawyer friendship data.** In this data set, the 71 lawyers of a New England Law Firm were asked to indicate with whom in the firm they regularly socialized outside of work (Lazega 2001). This is a frequently used network data set that was also analyzed, for example, in Yan et al. (2019), Jochmans (2018) and Snijders et al. (2006). For our analysis we focus on mutual friendships between lawyers as in Snijders et al. (2006), that is, we consider the network in which an undirected edge is placed between two lawyers when they both indicated to socialize with one another. The degrees of the resulting network range from 0 to 16, with eight isolated nodes. The average degree is 4.96 and the edge density is 7%. It is to note that we did not remove the isolated nodes before doing inference. Alongside the network, the following variables were collected: The status of the lawyer (partner or associate), their gender (man or woman), which of three offices they worked in, the years they had spent with the firm, their age, their practice (litigation or corporate) and the law school they had visited (Harvard and Yale, UConn or other).

We fitted the S\( \beta \)RM to this data set, by using as covariates between two nodes the positive absolute difference between these seven variables, where for categorical variables the difference is defined as the indicator whether the values are equal. Since our simulation studies suggest that BIC performs better for smaller networks, we use it for model selection. Model selection with the heuristic results in a slightly larger penalty and slightly different estimates, but overall very similar results. We constructed confidence intervals for the estimated covariate values at the 95%-level. The resulting weights and confidence intervals for the covariates are shown in Table 3.

These findings are in line, both in terms of magnitude of estimated weight as well as, more importantly, the sign of each weight, with what we would expect and with the results in the aforementioned papers. In order of importance, working in the same office, having the same status, being of the same practice and having the same gender have a positive effect on friendship formations, whereas a big difference in age or tenure has a negative effect on friendship formation. While our point estimate for having gone to the same law school is positive, its confidence interval extends to the negative real line and we thus cannot make a definite statement about its effect on friendship formation. This effect is also present when doing model selection with our heuristic. To appreciate how the covariates influence the connection pattern, we visualize the network in Figure 4 by examining the effect of office in Figure 20.
| Covariate                  | Point estimate | Confidence Interval |
|---------------------------|----------------|---------------------|
| Same status               | 0.91           | (0.54, 1.28)        |
| Same gender               | 0.46           | (0.12, 0.81)        |
| Same office               | 2.21           | (1.81, 2.60)        |
| Years with firm difference| -0.073         | (-0.11, -0.040)     |
| Age difference            | -0.031         | (-0.060, -0.0023)   |
| Same practice             | 0.57           | (0.25, 0.89)        |
| Same law school           | 0.30           | (-0.090, 0.62)      |

Table 3: Covariate weights for Lazega’s Lawyer friendship network and 95% confidence intervals.

We can see indeed that these two covariates have played important roles in shaping how connections were made.

![Network diagrams](images)

**Figure 4:** Visualization of Lazega’s friendship network among 71 lawyers. The size of the nodes is proportional to their degree. For better visibility we set the size of all nodes with a degree of five or lower to the size corresponding to a degree of five. In 4a the different colors indicate different offices (blue: Boston, yellow: Hartford, black: Providence; notice that only four lawyers are based in the small Providence office) and in 4b different statuses (red: partner, green: associate). The positions of the vertices are the same in both plots.

**Trade partnerships network.** For our second data set, we analyzed mutually important trade partnerships between 136 countries/regions in 1990. This data was originally analyzed by Silva & Tenreyro (2006) and further analyzed in Jochmans (2018). Even back in 1990 almost every country would trade with every other country, resulting in a very dense network. To be able to make the underlying network formation mechanisms visible, we decided to only focus on important trade partnerships in which the trade volume exceeds a certain limit. More precisely, we place an undirected edge between two countries if the trade volume makes up at least 3% of the importing countries total imports or if it makes up at least 3% of the exporting countries total exports. This leaves us with an undirected network with 136 nodes and 1279 edges, meaning that we have an edge density of 13.9%. The minimum degree of the resulting network was 3 (Dominican Republic), the maximum degree was 126 (USA), and the median degree was 13.

We analyze the same covariates as Jochmans (2018). That is, we have indicator variables common language and common border that take the value one if countries $i$ and $j$ share a common language or border and zero otherwise, log distance which is the log of the geographic distance between the countries, colonial ties which is one if at some point $i$ colonized $j$ or vice versa and zero otherwise, and preferential trade agreement which is an indicator whether or not a preferential trade agreement exists between the countries. Again, we chose BIC for model selection for the reasons outlined above. The results are
summarized in Table 4. These results are in line with what one would expect. Having a preferential trade agreement has the strongest positive effect on mutual trade between countries. Speaking the same language, sharing a border or having colonial ties also has a positive effect, while a large geographical distance has a strong negative effect.

| Covariate                        | Estimated weight | Confidence Interval     |
|----------------------------------|------------------|-------------------------|
| Log distance                     | −1.03            | (−1.04, −1.02)          |
| Common border                    | 0.45             | (0.10, 0.79)            |
| Common language                  | 0.31             | (0.086, 0.54)           |
| Colonial ties                    | 0.42             | (0.17, 0.66)            |
| Preferential trade agreement     | 0.81             | (0.36, 1.27)            |

Table 4: Covariate estimation for world trade data and 95% confidence intervals.

Notice that the confidence intervals for the categorical variables are all much larger than the one for the continuous variable log distance between countries. This is due to the fact that all the columns corresponding to categorical covariates are quite sparse, while the column corresponding to log distance contains only non-zero entries. Only 142 dyads are part of a preferential trade agreement and only 180 share a common border. Consequently the confidence intervals corresponding to these covariates are largest. Note that 1565 node pairs have colonial ties with one another and 1925 speak a common language. While the columns corresponding to these covariates are thus much more populated, they are still relatively sparse when compared to the total number of dyads.

BIC selected 32 active $\beta$-entries, which are visualized on a map in Figure 5. We presented the top half of these countries/regions with their degree and GDP in Table 5. The ranking of the $\beta$ values correlates with our intuition of the economic power of the countries. However, we also pick up underlying network formation mechanisms that go beyond sheer economic power and that are neither explainable by only looking at network summary statistics (such as degree of a node) nor by only looking at economic metrics such as a country’s GDP. More precisely, we note that the top six positions are occupied by six of the seven G7 countries, which serves to show that the $S\beta$RM works well for identifying the most important nodes in a network. Note however, that Japan has the largest $\beta$, albeit having a smaller degree (122) and a significantly smaller GDP than the USA (degree = 126), which comes in second place. In general, the order of degrees no longer aligns exactly with the order of the $\beta$-values as would have been predicted by the $S\beta$M without covariates in Chen et al. (2021). Norway, for example, has a $\beta$-value of zero, even though its degree of 17 and GDP of US$1.22 \times 10^{11}$ exceeds the degree and the GDP of several nodes with an active $\beta$-value. An examination of Norway’s neighboring nodes reveals that it was trading mostly with countries that either are close geographically or have a large $\beta$-value themselves (such as USA and Japan), meaning that the observed covariates are sufficient to explain the linking behavior of Norway. This illustrates that the $S\beta$M with covariates is able to pick up subtleties in network formation that one might miss if one relied solely on network summary statistics such as the degree of a node or solely on non-relational summary statistics such as a country’s GDP.

| $\hat{\beta}$ | Degree | GDP (US$)      | $\hat{\beta}$ | Degree | GDP (US$)      |
|----------------|--------|----------------|----------------|--------|----------------|
| Japan          | 5.85   | 122            | 4.95e+12       | Korea  | 2.11           | 34             | 3.42e+11       |
| USA            | 5.82   | 126            | 6.51e+12       | Singapore | 2.06    | 37             | 5.39e+10       |
| Germany        | 5.17   | 120            | 2.27e+12       | Hong Kong | 2.05    | 40             | 1.07e+11       |
| France         | 4.16   | 103            | 1.47e+12       | Spain   | 1.80           | 41             | 5.46e+11       |
| UK             | 4.15   | 104            | 1.04e+12       | Thailand | 1.78    | 33             | 1.11e+11       |
| Italy          | 3.92   | 95             | 1.03e+12       | China   | 1.58           | 30             | 3.98e+11       |
| Netherlands    | 3.15   | 73             | 3.75e+11       | Russia  | 1.53           | 28             | 5.43e+11       |
| Belgium-Lux    | 2.60   | 59             | 2.36e+11       | India   | 1.33           | 32             | 2.75e+11       |

Table 5: The top 16 active beta-values for the world trade network.
Figure 5: Visualization of the estimated $\beta$ values in the world trade network in 1990 between 136 countries/regions. The color of the country/region corresponds to the magnitude of the estimated $\beta$. Countries in grey either have an estimated $\beta$ value of zero or were not present in the data set.

6 Conclusion

We have presented a new model named $S_{\beta}$RM that simultaneously captures homophily and degree heterogeneity in a network. We have shown that $S_{\beta}$RM is well suited to model sparse networks, thanks to the sparsity assumption on the nodal parameter that can effectively reduce the dimensionality of the model. We have presented theory for the penalized likelihood estimator based on an $\ell_1$ penalty on the nodal parameter, including consistency of the excess risk and the central limit theorem for the homophily parameter. Built on the LASSO theory, our theoretical contributions go beyond existing theory for LASSO as we have argued. The computation of our estimator leverages the recent vast algorithmic development on solving LASSO type problems. Thus, $S_{\beta}$RM represents an attractive model for networks with statistical guarantees and computational feasibility.

There are many important issues for future research. First, we assume fixed-dimensional covariates. Recent data deluge brings more and more data sets that have more variables than observations. How to generalize $S_{\beta}$RM to include growing dimensional covariates is worth further investigation. Second, it will be interesting to incorporate a low rank component in $S_{\beta}$RM in order to capture transitivity, the phenomenon that nodes with common neighbors are more likely to connect, as is done in Ma et al. (2020). Third, it will be interesting to see how $S_{\beta}$RM can be used to model networked data under privacy constraint, along the line of research initiated by Karwa & Slavković (2016) for the $\beta$-model. Lastly, a growing list of networked data are observed along a temporal dimension (Jiang et al. 2020) and it will be interesting to extend our model to a time series context. These issues are beyond the scope of the current paper and will be explored elsewhere.

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Supplementary Materials

The supplementary material contains the main proofs.

A Proof of Lemma 1 in Section 2

Proof of Lemma 1. We first show that a solution exists. Using duality theory from convex optimization (cf. Bertsekas (1995), Chapter 5), we know that for any $\lambda > 0$ there exists a finite $s > 0$ such that the penalized likelihood problem is equivalent to the primal optimization problem

\[
\min_{\beta, \mu, \gamma} \frac{1}{n} L(\beta, \mu, \gamma),
\]

subject to: $(\beta^T, \mu, \gamma)^T \in \Theta, \sum_{i=1}^{n} |\beta_i| \leq s.$ \hspace{1cm} (15)

Let $\beta = (\beta_1, \ldots, \beta_n)^T, \gamma = (\gamma_1, \ldots, \gamma_p)^T$ be fixed. To obtain an estimate for $\mu$, we minimize the function

\[
g_{\beta, \gamma}(\mu) = \frac{1}{n} L(\beta, \mu, \gamma)
\]

\[
= \frac{1}{n} \left( -\sum_{i=1}^{n} \beta_i d_i - d_+ \mu - \sum_{i<j} (Z_{ij}^T \gamma) A_{ij} + \sum_{i<j} \log(1 + \exp(\beta_i + \beta_j + \mu + Z_{ij}^T \gamma)) \right).
\]

It has derivative

\[
g'_{\beta, \gamma}(\mu) = \frac{1}{n} \left( -d_+ + \sum_{i<j} \frac{e^{\beta_i + \beta_j + \mu + Z_{ij}^T \gamma}}{1 + e^{\beta_i + \beta_j + \mu + Z_{ij}^T \gamma}} \right).
\]

We observe that

\[
\lim_{\mu \to \infty} g'_{\beta, \gamma}(\mu) = \frac{1}{n} \left( -d_+ + \left( \frac{n}{2} \right) \right) > 0
\]

and

\[
\lim_{\mu \to -\infty} g'_{\beta, \gamma} = -d_+ \frac{n}{2} < 0.
\]

Furthermore, $g'_{\beta, \gamma}$ is continuous and strictly increasing in $\mu$. Hence, there exists a unique value $\mu^* = \mu^*(\beta, \gamma)$, such that $g'_{\beta, \gamma}(\mu^*) = 0$. Since

\[
g''_{\beta, \gamma}(\mu) = \sum_{i<j} \frac{e^{\beta_i + \beta_j + \mu + Z_{ij}^T \gamma}}{(1 + e^{\beta_i + \beta_j + \mu + Z_{ij}^T \gamma})^2} > 0
\]

for all $\mu, \mu^*$ is a minimizer of $g_{\beta, \gamma}$. Since $g''_{\beta, \gamma}$ is invertible, we can apply the implicit function theorem with function $F(\beta, \gamma, \mu) = g'_{\beta, \gamma}(\mu)$, which gives us that the corresponding function $\mu^* = \mu^*(\beta, \gamma)$ is continuously differentiable. Plugging in $\mu^*(\beta, \gamma)$ for $\mu$ in (15), we are left with the minimization problem

\[
\min_{\beta, \gamma} L(\beta, \mu^*(\beta, \gamma), \gamma),
\]

s.t.: $(\beta^T, \mu^*(\beta, \mu), \gamma^T)^T \in \Theta, \sum_{i=1}^{n} |\beta_i| \leq s.$ \hspace{1cm} (16)

Since $\Gamma$ is compact, we are minimizing a continuous function over a compact set in (16). Hence it attains a minimum $L^*$. By the definition of $\mu^*$, $L^*$ must also be a solution of (15).

For the second claim of the Lemma, suppose there is an $1 \leq i_0 \leq n$ such that $\hat{\beta}_{i_0} = \min_{1 \leq i \leq n} \hat{\beta}_i > 0$. Consider the following vector $\tilde{\theta} = (\beta^T, \tilde{\mu}, \tilde{\gamma}^T)^T$: for all $k$ let $\tilde{\beta}_k = \beta_k - \hat{\beta}_{i_0}$ and $\tilde{\mu} = \hat{\mu} + 2\hat{\beta}_{i_0}$, while
keeping $\tilde{\gamma} = \gamma$. Then, $\tilde{\beta}_k \geq 0$ for all $k$, i.e. $\tilde{\theta}$ is a feasible point for the penalized likelihood problem (5). Furthermore $\min_k \tilde{\beta}_k = 0$ and $L(\tilde{\theta}) = L(\hat{\theta})$. However,

$$\|\tilde{\beta}\|_1 = \sum_{i=1}^{n} |\tilde{\beta}_i - \hat{\beta}_{i_0}| < \|\hat{\beta}\|_1,$$

where the inequality follows from the minimality of $\hat{\beta}_{i_0}$. This gives

$$L(\hat{\theta}) + \|\hat{\beta}\|_1 < L(\tilde{\theta}) + \|\hat{\beta}\|_1.$$

A contradiction to the optimality of $\hat{\theta}$. $\Box$

B Consistency with covariates

B.1 Proof of Proposition 1

Notice that the compatibility condition is clearly equivalent to the condition that

$$\kappa^2(\Sigma, s^*) := \min_{\theta \in \mathbb{R}^{n \times p} \setminus \{0\}} \frac{\theta^T \Sigma \theta}{\|\theta\|_1^2} \geq C > 0$$

stays uniformly bounded away from zero. Recall the definition of $\Sigma_A$. The key to proving Proposition 1 is to show that $\Sigma$ is close to $\Sigma_A$ in an appropriate sense and that $\Sigma_A$ fulfills $\kappa^2(\Sigma_A, s^*) \geq C > 0$ for all $n$ and some universal $C > 0$. We then show that $\kappa^2(\Sigma, s^*)$ is bounded away from zero. Let us analyze the top left block matrix of $\Sigma_A$, i.e. $1/(n-1) \cdot X^T X$, first:

$$\frac{1}{n-1} X^T X = \begin{bmatrix}
1 & \frac{1}{n-1} & \frac{1}{n-1} & \ldots & \frac{1}{n-1} \\
\frac{1}{n-1} & 1 & \frac{1}{n-1} & \ldots & \frac{1}{n-1} \\
\frac{1}{n-1} & \frac{1}{n-1} & 1 & \ldots & \frac{1}{n-1} \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\frac{1}{n-1} & \frac{1}{n-1} & \frac{1}{n-1} & \ldots & 1
\end{bmatrix},$$

that is, $1/(n-1) \cdot X^T X$ has all ones on the diagonal and $1/(n-1)$ everywhere else. This is a special kind of Toeplitz matrix; a circulant matrix to be precise. It is known (see for example Kra & Simanca (2012)), that every circulant matrix $M$ has an associated polynomial $p$ and that the eigenvalues of $M$ are given by $p(\xi_j), j = 0, \ldots, n-1$, where $\xi_j = \exp(i 2\pi j/n)$, where $i$ is the imaginary unit and $\xi_0 = 1$. The associated polynomial of the matrix $1/(n-1) X^T X$ is

$$p(x) = 1 + \frac{1}{n-1} (x + x^2 + \cdots + x^{n-1})$$

and thus the eigenvalues of $1/(n-1) X^T X$ are

$$p(1) = 2, \quad p(\xi_j) = 1 + \frac{1}{n-1} (-1) = \frac{n-2}{n-1}, \quad j = 1, \ldots, n-1,$$

where the eigenvalue $(n-2)/(n-1)$ has multiplicity $n-1$. Hence, we observe the following: For any vector $\theta = (\beta^T, \mu, \gamma^T)^T$,

$$\theta^T \Sigma_A \theta = \beta^T \left( \frac{1}{n-1} X^T X \right) \beta + \mu^2 + \frac{1}{(n/2)^2} \gamma^T E[Z^T Z] \gamma \geq \frac{n-2}{n-1} \beta^T \beta + \mu^2 + \frac{1}{(n/2)^2} \gamma^T E[Z^T Z] \gamma,$$

where for the inequality we have used that for any semi-positive definite, symmetric matrix $M$ with smallest eigenvalue $\lambda$ and any vector $x \neq 0$ of appropriate dimension, we have $x^T M x \geq \lambda x^T x$. Thus,
for any $\theta = (\beta^T, \mu, \gamma^T)^T$,

$$
\frac{\theta^T \Sigma_A \theta}{\frac{1}{s_+^*} \|\theta S_+^*\|_1^2} \geq \frac{n - \frac{2}{n} \|\beta\|_2^2 + \mu^2 + \frac{1}{n} \gamma^T \mathbb{E}[Z^T Z] \gamma}{\frac{1}{s_+^*} \|\theta S_+^*\|_1^2} \geq \frac{n - \frac{2}{n} \|\beta\|_2^2 + \mu^2 + \frac{1}{n} \gamma^T \mathbb{E}[Z^T Z] \gamma}{\|\beta\|_2^2 + \mu^2 + \|\gamma\|_2^2},
$$

by Cauchy-Schwarz

$$
= \frac{n - 2}{n - 1} \frac{\|\beta\|_2^2 + \mu^2 + \frac{n - 1}{n} \frac{1}{2} \gamma^T \mathbb{E}[Z^T Z] \gamma}{\|\beta\|_2^2 + \mu^2 + \|\gamma\|_2^2}.
$$

Now, notice that for any $a, b, c \in \mathbb{R}$, we have $\frac{a + b}{a + c} \geq \min\{1, b/c\}$. This is easily seen by considering the cases $\min\{1, b/c\} = 1$ and $\min\{1, b/c\} = b/c$ separately and rearranging. Thus,

$$
\frac{\theta^T \Sigma_A \theta}{\frac{1}{s_+^*} \|\theta S_+^*\|_1^2} \geq \frac{n - 2}{n - 1} \min \left\{ 1, \frac{n - 1}{n - 2} \frac{1}{2} \gamma^T \mathbb{E}[Z^T Z] \gamma \right\} = \min \left\{ \frac{n - 2}{n - 1}, \frac{\gamma^T \left( \frac{1}{2} \mathbb{E}[Z^T Z] \right) \gamma}{\|\gamma\|_2^2} \right\}
$$

$$
\geq \min \left\{ \frac{n - 2}{n - 1}, \lambda_{\min} \right\},
$$

where $\lambda_{\min}$ is the minimum eigenvalue of $\frac{1}{2} \mathbb{E}[Z^T Z]$. By Assumption 1, we now have for $n \geq 3$,

$$
\kappa^2(\Sigma_A, s^*) = \min_{\theta \in \mathbb{R}^{n+1+p}, \{0\}} \frac{\theta^T \Sigma_A \theta}{\frac{1}{s_+^*} \|\theta S_+^*\|_1^2} \geq c_{\min} > 0.
$$

(17)

Now, we need to show that with high probability $\kappa(\Sigma, s^*) \geq \kappa(\Sigma_A, s^*)$, which would imply that the compatibility condition holds with high probability for the sample size adjusted Gram matrix $\Sigma$ and the associated sample size adjusted design matrix. To that end, we have the following auxiliary Lemma found in Kock & Tang (2019). For completeness, we give the short proof of it. The notation is adapted to our setting.

**Lemma 2** (Lemma 6 in Kock & Tang (2019)). Let $A$ and $B$ be two positive semi-definite $(n + 1 + p) \times (n + 1 + p)$ matrices and $\delta = \max_{i,j} |A_{ij} - B_{ij}|$. For any set $s^* \subset \{1, \ldots, n\}$ with cardinality $s^*$, one has

$$
\kappa^2(B, s^*) \geq \kappa^2(A, s^*) - 16\delta(s^* + p + 1).
$$

**Proof.** Denote by $S_+^* = S^* \cup \{n + 1, \ldots, n + 1 + p\}$ and $s_+^* = s^* + (1 + p)$. Let $\theta = (\beta^T, \mu, \gamma^T)^T \in \mathbb{R}^{n+1+p} \setminus \{0\}$, with $\|\theta_{S_+^*}\|_1 \leq 3\|\theta_{S_+^*}\|_1$. Then,

$$
|\theta^T A \theta - \theta^T B \theta| = |\theta^T (A - B) \theta| \leq \|\theta\|_1 \|(A - B) \theta\|_\infty \leq \delta \|\theta\|_1^2
$$

$$
= \delta (\|\theta_{S_+^*}\|_1 + \|\theta_{S_+^*}\|_1^2) \leq \delta (\|\theta_{S_+^*}\|_1 + 3\|\theta_{S_+^*}\|_1)^2
\leq 16\delta \|\theta_{S_+^*}\|_1^2.
$$

Hence, $\theta^T B \theta \geq \theta^T A \theta - 16\delta \|\theta_{S_+^*}\|_1^2$ and thus

$$
\frac{\theta^T B \theta}{\frac{1}{s_+^*} \|\theta S_+^*\|_1^2} \geq \frac{\theta^T A \theta}{\frac{1}{s_+^*} \|\theta S_+^*\|_1^2} - 16\delta s_+^* \geq \kappa^2(A, s^*) - 16\delta s^*.
$$

Minimizing the left-hand side over all $\theta \neq 0$ with $\|\theta_{S_+^*}\|_1 \leq 3\|\theta_{S_+^*}\|_1$ proves the claim. \qed

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This shows that to control $\kappa^2(\Sigma, s^*)$, we need to control the maximum element-wise distance between $\Sigma$ and $\Sigma_A$: \[ \max_{ij} |\Sigma_{ij} - \Sigma_{A,ij}| \leq \frac{c_{\min}}{32s^+_n}, \]
and thus, by Lemma 2, we have $\kappa^2(\Sigma, s^*) \geq \kappa^2(\Sigma_A, s^*) - \frac{c_{\min}}{2} \geq \frac{c_{\min}}{2} > 0$ and i.e. the compatibility condition holds for $\Sigma$.

**Proof of Proposition 1.** To make referencing of sections of $\Sigma$ easier, we number its blocks as follows

$$
\Sigma = \frac{1}{(n^2)} \begin{bmatrix}
\frac{n}{2} X^T X & \frac{\sqrt{n}}{2} X^T 1 & 0 & 0 \\
\frac{\sqrt{n}}{2} 1^T X & \frac{\sqrt{n}}{2} 1^T 1 & 0 & 0 \\
0 & 0 & \mathbb{E}[Z^T Z] & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

For $i, j = 1, \ldots, n$, we have $\Sigma_{ij} = \Sigma_{A,ij}$ (block 1). The entry at position $(n + 1), (n + 1)$ (block 5) is also equal and so are blocks (3), (6), (7), (8) and (9). For the entries at positions $i, j$ with $i = n + 1$ and $j = 1, \ldots, n$ as well as positions with $i = 1, \ldots, n$ and $j = n + 1$ (blocks (2) and (4)), we have:

$$
\Sigma_{ij} - \Sigma_{A,ij} = \Sigma_{ij} = \frac{(n-1)^{\frac{1}{2}}}{(n-1)^{\frac{1}{2}}} = \frac{\sqrt{2}}{\sqrt{n}} \leq \frac{c_{\min}}{32s^+_n}
$$

for $n \gg 0$, since we assume that $s^* = o(\sqrt{n})$. The claim now follows from Lemma 2.

In the SβM without covariates we define the matrices $\Sigma$ and $\Sigma_A$ completely analogously to the SβRM by setting the blocks corresponding to the covariates $Z$ to zero.

$$
\Sigma = \frac{1}{(n^2)} \begin{bmatrix}
\frac{n}{2} X^T X & \frac{\sqrt{n}}{2} X^T 1 & 0 & 0 \\
\frac{\sqrt{n}}{2} 1^T X & \frac{\sqrt{n}}{2} 1^T 1 & 0 & 0 \\
0 & 0 & \mathbb{E}[Z^T Z] & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \Sigma_A := \begin{bmatrix}
\frac{1}{n-1} X^T X & 0 \\
0 & 1
\end{bmatrix}.
$$

To be consistent with our numbering scheme from before, we number these four blocks from top left to bottom right as (1), (2), (4) and (5), skipping (3). The proof of Proposition 1 simplifies considerably when no covariates are present and we have the following lemma.

**Lemma 3.** Under Assumption 2 and for $n$ large enough, the compatibility condition holds for the sample-size adjusted Gram matrix $\Sigma$ in the sparse $\beta$ model without covariates. That is, for every $\theta \in \mathbb{R}^{n+1}$ with $\|\theta_{S^+_n}\|_1 \leq 3\|\theta_{S^+_n}\|_1$,

$$
\|\theta_{S^+_n}\|_1^2 \leq \frac{1}{4}s^+_n \theta^T \Sigma \theta.
$$

**Proof.** Following the exact same steps as above for SβRM, we find for any $\theta = (\beta^T, \mu)^T$

$$
\kappa^2(\Sigma_A, s^+_n) = \frac{\theta^T \Sigma_A \theta}{\frac{1}{s^+_n} \|\theta_{S^+_n}\|_1^2} = \frac{n-2}{n-1} > \frac{1}{2} > 0,
$$

for any $n \geq 3$. Using line by line the same arguments as in the proof of Proposition 1 we prove that the compatibility condition holds for the sparse $\beta$ model without covariates. More precisely, by Lemma 2, we know that $\kappa^2(\Sigma, s^+_n) \geq \kappa^2(\Sigma_A, s^+_n) - 16s^+_n$, where $\delta = \max_{ij} |\Sigma_{ij} - \Sigma_{A,ij}|$. Looking back at the proof of Proposition 1, we only need the part of it that deals with blocks (2) and (4), since $\Sigma$ and $\Sigma_A$...
coincide in the blocks (1) and (3). Doing the same calculation as in said proof, we see that for any entry $(i, j)$ in blocks (2) or (5),

$$|\Sigma_{ij} - \Sigma_{A,ij}| \leq \frac{\sqrt{2}}{\sqrt{n}}.$$ 

Under Assumption 2, that is $s^* = o\left(\frac{\sqrt{n}}{\log(n)}K_n\right)$ where we define $K_n$ with the components corresponding to $\gamma$ set to zero, this expression will be smaller than $\frac{1}{64\epsilon_n}$ for $n$ large enough. Thus, for $n$ large enough, by Lemma 2 and inequality (18), $\kappa^2(\Sigma, s^*_n) \geq 1/2 - 1/4 = 1/4.$ \hfill $\Box$

Notice that the bound 1/4 in the lemma above are somewhat arbitrary and an artifact of how we pick our constants in the proof of that Lemma.

**B.2 A rescaled penalized likelihood problem**

We already mentioned in Section 2 that it is possible to present an equivalent formulation of the problem (5) in terms of a rescaled likelihood problem using the sample-size adjusted design matrix $\bar{D}$. We will rely heavily on this formulation which we now make precise.

Recall that in the definition of $\bar{D}$ we effectively blew up the entries belonging to $\beta$. The blow-up factor was chosen precisely such that we can now reformulate our problem as a problem in which each parameter effectively has sample size $(n^2)^\frac{1}{4}$. That is, our original penalized likelihood problem can be rewritten as

$$\hat{\theta} = (\hat{\beta}, \hat{\mu}, \hat{\gamma}) = \text{arg min}_{\beta, \mu, \gamma} \left( \frac{1}{n^2} \left( -\sum_{i=1}^n \frac{\sqrt{n}}{\sqrt{2}} \beta_i d_i - d_+ \mu - \sum_{i<j} (Z_{ij}^T \gamma) A_{ij} \right. \right.$$ 

$$\left. + \sum_{i<j} \log \left( 1 + \exp \left( \frac{\sqrt{n}}{\sqrt{2}} \beta_i + \frac{\sqrt{n}}{\sqrt{2}} \beta_j + \mu + Z_{ij}^T \gamma \right) \right) \right) + \tilde{\lambda} \|\tilde{\beta}\|_1,$$

where $\tilde{\lambda} = \frac{n^2}{\sqrt{2}} \lambda$ and the argmin is taken over $\Theta_{loc} = \{ \tilde{\theta} \in \Theta : \|\bar{D}\tilde{\theta}\|_{\infty} \leq r_n \}$. Note that by the same arguments as before, $\Theta_{loc}$ is convex. Then, given a solution $(\hat{\beta}, \hat{\mu}, \hat{\gamma})$ for a given penalty parameter $\tilde{\lambda}$ to this modified problem (19), we can obtain a solution to our original problem (5) with penalty parameter $\lambda = \tilde{\lambda} \sqrt{2}/\sqrt{n}$, by setting

$$(\hat{\beta}, \hat{\mu}, \hat{\gamma}) = \left( \frac{n^2}{\sqrt{2}} \hat{\beta}, \hat{\mu}, \hat{\gamma} \right).$$

For a compacter way of writing, introduce the following notation: For any parameter $\theta = (\beta^T, \mu, \gamma^T)^T \in \Theta$, we introduce the notation

$$\bar{\theta} = \left( \frac{\sqrt{2}}{\sqrt{n}} \beta, \mu, \gamma \right)$$

and also write $\bar{\beta} = \frac{\sqrt{2}}{\sqrt{n}} \beta$. In particular we use the notation $\bar{\theta}_0 = (\bar{\beta}_0^T, \mu_0, \gamma_0^T)^T, \bar{\beta}_0 = \frac{\sqrt{2}}{\sqrt{n}} \beta_0$, to denote the re-parametrized truth and $\bar{\theta}^* = (\bar{\beta}^T, \mu^*, \gamma^T)^T, \bar{\beta}^* = \frac{\sqrt{2}}{\sqrt{n}} \beta^*$ to denote the re-parametrized best local approximation. Note that for any $\theta \in \Theta$, $D\theta = \bar{D}\bar{\theta}$ and hence the bound $r_n$ is the same in the definitions of $\Theta_{loc}$ and $\Theta_{loc}$. Also, since rescaling the set $\mathbb{R}^n_+$ still results in $\mathbb{R}^n_+$, there is no need to introduce a set $\emptyset$. Note that $\theta \in \Theta_{loc}$ if and only if $\bar{\theta} \in \bar{\Theta}_{loc}$.

For any $\bar{\theta} = (\bar{\beta}^T, \mu, \gamma)^T$, denote the negative log-likelihood function corresponding to the rescaled
problem (19) as
\[
\tilde{\mathcal{L}}(\bar{\theta}) = -\sum_{i=1}^{n} \frac{\sqrt{n}}{\sqrt{2}} \beta_i d_i - d_+ \mu - \sum_{i<j} (Z^T_{ij} \gamma) A_{ij} \\
+ \sum_{i<j} \log \left( 1 + \exp \left( \frac{\sqrt{n}}{\sqrt{2}} \beta_i + \frac{\sqrt{n}}{\sqrt{2}} \beta_j + \mu + Z^T_{ij} \gamma \right) \right).
\]
Then, clearly \( \tilde{\mathcal{L}}(\bar{\theta}) = \mathcal{L}(\theta) \) and
\[
\tilde{\mathcal{E}}(\bar{\theta}) := \frac{1}{(\frac{n}{2})} (\mathbb{E}[\tilde{\mathcal{L}}(\bar{\theta})] - \mathbb{E}[\tilde{\mathcal{L}}(\bar{\theta}^*)]) = \mathcal{E}(\theta).
\]
Thus, \( \bar{\theta}^* \) fulfills
\[
\bar{\theta}^* = \arg \min_{\theta \in \Theta_{\text{loc}}} \tilde{\mathcal{E}}(\bar{\theta}),
\]
i.e. \( \bar{\theta}^* \) is the best local re-parametrized solution.

To give us a more compact way of writing, for any \( \theta \in \Theta \) we introduce functions \( f_\theta : \mathbb{R}^{n+1+p} \to \mathbb{R}, f_\theta(v) = v^T \theta \) and denote the function space of all such \( f_\theta \) by \( \mathbb{F} := \{ f_\theta : \theta \in \Theta \} \). We endow \( \mathbb{F} \) with two norms as follows. Denote the law of the rows of \( D \) on \( \mathbb{R}^{n+1+p} \), i.e. the probability measure induced by \( (X^T_{ij}, 1, Z^T_{ij})^T, i < j, \) by \( \bar{Q} \). That is, for a measurable set \( A = A_1 \times A_2 \subset \mathbb{R}^{n+1} \times \mathbb{R}^p \),
\[
\bar{Q}(A) = \frac{1}{(\frac{n}{2})} \sum_{i<j} P(D_{ij} \in A) = \frac{1}{(\frac{n}{2})} \sum_{i<j} \delta_{ij}(A_1) \cdot P(Z_{ij} \in A_2),
\]
where \( \delta_{ij}(A_1) = 1 \) if \( (X^T_{ij}, 1)^T \in A_1 \) and zero otherwise, is the Dirac-measure. We are interested in the \( L_2 \) and \( L_\infty \) norm on \( \mathbb{F} \) with respect to the measure \( \bar{Q} \) on \( \mathbb{R}^{n+1} \times \mathbb{R}^p \). Denote the \( L_2(\bar{Q}) \)-norm of \( f \in \mathbb{F} \) simply by \( \| f \|_{\bar{Q}} \) and let \( \mathbb{E} Z \) be the expectation with respect to \( Z \):
\[
\| f \|^2_{\bar{Q}} := \| f \|^2_{L_2(\bar{Q})} = \int_{\mathbb{R}^{n+1} \times \mathbb{R}^p} f(v)^2 \bar{Q}(dv) = \frac{1}{(\frac{n}{2})} \sum_{i<j} \mathbb{E} Z [f((X^T_{ij}, 1, Z^T_{ij})^T)^2]
\]
and define the \( L_\infty(\bar{Q}) \)-norm as usual as the \( \bar{Q} \)-a.s. smallest upper bound of \( f \):
\[
\| f \|_{\bar{Q}, \infty} = \inf \{ C \geq 0 : |f(v)| \leq C \text{ for } \bar{Q} \text{-almost every } v \in \mathbb{R}^{n+1+p} \}.
\]
Notice in particular, that for any \( f_\theta \in \mathbb{F}, \theta = (\beta^T, \mu, \gamma^T)^T \in \Theta_{\text{loc}} \), \( \| f_\theta \|_{\infty} \leq \sup_{Z_{ij}} \| D \theta \|_{\infty} \leq r_n \).

We make the analogous definitions for the unscaled design matrix. Define the probability measure induced by the rows of \( D \) on \( \mathbb{R}^{n+1+p} \) as \( \bar{Q} \). It is easy to see that we can switch between these norms as follows. Given a parameter \( \theta \) and its rescaled version \( \bar{\theta} \), then clearly
\[
\| f_\theta \|_{\bar{Q}} = \| f_\theta \|_{\bar{Q}}, \quad \| f_\theta \|_{\bar{Q}, \infty} = \| f_\theta \|_{\bar{Q}, \infty}.
\]
Also note that for any \( \bar{\theta} \)
\[
\| f_{\bar{\theta}} \|^2_{\bar{Q}} = \mathbb{E} Z \left[ \frac{1}{(\frac{n}{2})} \sum_{i<j} (\bar{D}^T_{ij} \bar{\theta})^2 \right] = \bar{\theta}^T \Sigma \bar{\theta}.
\]
(20)
Recall that we want to apply the compatibility condition to vectors of the form \( \bar{\theta} = \bar{\theta}_1 - \bar{\theta}_2, \bar{\theta}_1, \bar{\theta}_2 \in \Theta_{\text{loc}} \). We have the following corollary which follows immediately from Proposition 1.

**Corollary 3.** Under Assumption 1, for \( s^* = o(\sqrt{n}) \) and \( n \) large enough, it holds that for every \( \bar{\theta} = \bar{\theta}_1 - \bar{\theta}_2, \bar{\theta}_1, \bar{\theta}_2 \in \Theta_{\text{loc}} \).
\[ \tilde{\theta}_1 - \tilde{\theta}_2, \tilde{\theta}_1, \tilde{\theta}_2 \in \hat{\Theta}_{\text{loc}} \text{ with } \| \tilde{\theta}_{S^*} \|_1 \leq 3\| \tilde{\theta}_{S^*} \|_1, \]

\[ \| \tilde{\theta}_{S^*} \|_1^2 \leq \frac{2s^*_1}{c_{\min}} \| f_{\tilde{\theta}_1} - f_{\tilde{\theta}_2} \|_{Q^*}^2. \]

**Proof.** Follows from Proposition 1 and identity (20). \qed

### B.3 Two basic inequalities

A key result in the consistency proofs in classical LASSO settings is the so called **basic inequality** (cf. van de Geer & Bühlmann (2011), Chapter 6). We give two formulations of it, one for the original penalized likelihood problem (5) and one, completely analogous result, for the rescaled problem (19).

To that end, let \( P_n \) denote the empirical measure with respect to our observations \((A_{ij}, Z_{ij})\), that is, for any suitable function \( g \), \( P_n g := \sum_{i<j} g(A_{ij}, Z_{ij})/\binom{n}{2} \). In particular, if we let for each \( \theta \in \Theta \),

\[ l_\theta(A_{ij}, Z_{ij}) = -A_{ij}(\beta_i + \beta_j + \mu + \gamma^T Z_{ij}) + \log(1 + \exp(\beta_i + \beta_j + \mu + \gamma^T Z_{ij})), \]

then \( P_n l_\theta = \mathcal{L}(\theta)/\binom{n}{2} \). Similarly, we define \( P = \mathbb{E}P_n \). In particular, \( P l_\theta = \mathbb{E}P_n l_\theta = \mathbb{E}[\mathcal{L}(\theta)/\binom{n}{2}] \), where we suppress the dependence on \( n \) in our notation. We define the **empirical process** as

\[ \{ v_n(\theta) = (P_n - P)l_\theta : \theta \in \Theta \}. \]

Which can also be written in re-parametrized form as

\[ \tilde{v}_n(\hat{\theta}) := \frac{1}{\binom{n}{2}} (\mathcal{L}(\hat{\theta}) - \mathbb{E}[\mathcal{L}(\hat{\theta})]) = v_n(\theta). \]

**Lemma 4.** For any \( \theta = (\beta^T, \mu, \gamma^T)^T \in \hat{\Theta}_{\text{loc}} \) it holds

\[ \mathcal{E}(\hat{\theta}) + \lambda\| \hat{\beta} \|_1 \leq -[v_n(\hat{\theta}) - v_n(\theta)] + \mathcal{E}(\theta) + \lambda\| \beta \|_1. \]

**Proof.** Plugging in the definitions, the above equation is equivalent to

\[ \frac{1}{\binom{n}{2}} (\mathbb{E}[\mathcal{L}(\hat{\theta})] - \mathbb{E}[\mathcal{L}(\theta^*)]) + \lambda\| \hat{\beta} \|_1 \leq -\frac{1}{\binom{n}{2}} \mathcal{L}(\hat{\theta}) + \frac{1}{\binom{n}{2}} \mathbb{E}[\mathcal{L}(\theta)] + \frac{1}{\binom{n}{2}} \mathcal{L}(\theta) - \frac{1}{\binom{n}{2}} \mathbb{E}[\mathcal{L}(\theta)] + \lambda\| \beta \|_1 \]

\[ + \frac{1}{\binom{n}{2}} (\mathbb{E}[\mathcal{L}(\theta)] - \mathbb{E}[\mathcal{L}(\theta^*)]). \]

Rearranging shows that this is true if and only if

\[ \frac{1}{\binom{n}{2}} \mathcal{L}(\hat{\theta}) + \lambda\| \hat{\beta} \|_1 \leq \frac{1}{\binom{n}{2}} \mathcal{L}(\theta) + \lambda\| \beta \|_1, \]

which is true by definition of \( \hat{\theta} \). \qed

**Remark.** For any \( 0 < t < 1 \) and \( \theta \in \hat{\Theta}_{\text{loc}} \), let \( \tilde{\theta} = t\tilde{\theta} + (1 - t)\theta \). Since \( \Gamma \) is convex, \( \tilde{\theta} \in \hat{\Theta}_{\text{loc}} \) and since \( \theta \to l_\theta \) and \( \| \cdot \|_1 \) are convex functions, we can replace \( \hat{\theta} \) by \( \tilde{\theta} \) in the basic inequality and still obtain the same result. Plugging in the definitions, we see that the basic inequality is equivalent to the following:

\[ \mathcal{E}(\tilde{\theta}) + \lambda\| \tilde{\beta} \|_1 \leq -[v_n(\tilde{\theta}) - v_n(\theta)] + \mathcal{E}(\theta) \]

\[ \iff \frac{1}{\binom{n}{2}} \mathcal{L}(\tilde{\theta}) + \lambda\| \tilde{\beta} \|_1 \leq \frac{1}{\binom{n}{2}} \mathcal{L}(\theta) + \lambda\| \beta \|_1 \]

and by convexity

\[ \frac{1}{\binom{n}{2}} \mathcal{L}(\tilde{\theta}) + \lambda\| \tilde{\beta} \|_1 \leq \frac{1}{\binom{n}{2}} t\mathcal{L}(\hat{\theta}) + \frac{1}{\binom{n}{2}} (1 - t)\mathcal{L}(\theta) + t\lambda\| \tilde{\beta} \|_1 + (1 - t)\lambda\| \beta \|_1 \leq \frac{1}{\binom{n}{2}} \mathcal{L}(\theta) + \lambda\| \beta \|_1 \]

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where the last inequality follows by definition of $\hat{\theta}$. In particular, for any $M > 0$, choosing

$$t = \frac{M}{M + \|\hat{\theta} - \theta\|_1},$$

gives $\|\hat{\theta} - \theta\|_1 \leq M$.

**Lemma 5.** For any $\bar{\theta} \in \Theta_{\text{loc}}$ it holds

$$\mathcal{E}(\hat{\theta}) + \lambda \|\hat{\beta}\|_1 \leq -[\bar{v}_n(\hat{\theta}) - \bar{v}_n(\bar{\theta})] + \mathcal{E}(\bar{\theta}) + \lambda \|\bar{\beta}\|_1.$$

Since the proof of Lemma 4 only relies on the argmin property of $\hat{\theta}$, the proof of Lemma 5 is line by line the same as for Lemma 4. We also get the same property for convex combinations of $\hat{\theta}$ and $\bar{\theta}$: For any $t \in (0, 1)$ the rescaled basic inequality Lemma 5 holds for $\hat{\theta}$ replaced by $\tilde{\theta} = t \hat{\theta} + (1 - t) \bar{\theta}$. Note in particular, that $\tilde{\theta} \in \Theta_{\text{loc}}$. Take note that in the basic inequalities we are controlling the global excess risk of any local parameters $\theta \in \Theta_{\text{loc}}$.

**B.4 Lower quadratic margin for $\mathcal{E}$**

In this section we will derive a lower quadratic bound on the excess risk $\mathcal{E}(\theta)$ if the parameter $\theta$ is close to the truth $\theta_0$. This is a necessary property for the proof to come and is referred to as the margin condition in classical LASSO theory (cf. van de Geer & Bühlmann (2011)). We will conduct our derivations for the original parameter space $\Theta_{\text{loc}}$. Since $L(\theta) = \bar{L}(\bar{\theta})$ and $\mathcal{E}(\theta) = \bar{\mathcal{E}}(\bar{\theta})$, we will find that the same results hold in the rescaled model.

The proof mainly relies on a second order Taylor expansion of the function $l_\theta$ of introduced in Section 2.1. Given a fixed $\theta$, we treat $l_\theta$ as a function in $\theta^T x$ and define new functions $l_{ij} : \mathbb{R} \rightarrow \mathbb{R}$, $i < j$,

$$l_{ij}(a) = \mathbb{E}[l_\theta(A_{ij}, a)|Z_{ij}] = -p_{ij}a + \log(1 + \exp(a)),$$

where $p_{ij} = P(A_{ij} = 1|Z_{ij})$ and by slight abuse of notation we use $l_\theta(A_{ij}, a) := -A_{ij}a + \log(1 + \exp(a))$.

Taking derivations, it is easy to see that

$$f_{\theta_0}((X_{ij}^T, 1, Z_{ij}^T)^T) \in \arg\min_a l_{ij}(a).$$

Note that we are using the actual truth $\theta_0$ in the above equation, not the best local approximation $\theta^*$. Write $f_0 = f_{\theta_0}$.

All $l_{ij}$ are clearly twice continuously differentiable with derivative

$$\frac{\partial^2}{\partial a^2} l_{ij}(a) = \frac{\exp(a)}{(1 + \exp(a))^2} > 0, \forall a \in \mathbb{R}.$$

Using a second order Taylor expansion around $a_0 = f_0((X_{ij}^T, 1, Z_{ij}^T)^T)$ we get

$$l_{ij}(a) = l_{ij}(a_0) + l'(a_0)(a - a_0) + \frac{l''(\bar{a})}{2}(a - a_0)^2 = l_{ij}(a_0) + \frac{l''(\bar{a})}{2}(a - a_0)^2,$$

with an $\bar{a}$ between $a$ and $a_0$. Note that $\frac{\exp(a)}{(1 + \exp(a))^2}$ is symmetric and monotone decreasing for $a \geq 0$. 

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Then, for any $\eta > 0$ and any $a$ with $|a - a_0| \leq \eta$, we get
\[
 l_{ij}(a) - l_{ij}(a_0) = \frac{\exp(a)}{(1 + \exp(a))^2} \frac{(a - a_0)^2}{2} \quad (\text{symmetry})
\]
\[
\geq \frac{\exp(|a|)}{(1 + \exp(|a|))^2} \frac{(a - a_0)^2}{2}
\]
\[
\geq \frac{\exp(|f_0((X_{ij}^T,1,Z_{ij}^T)^T)| + \eta)}{(1 + \exp(|f_0((X_{ij}^T,1,Z_{ij}^T)^T)| + \eta))^2} \frac{(f_0((X_{ij}^T,1,Z_{ij}^T)^T) - f_0((X_{ij}^T,1,Z_{ij}^T)^T))^2}{2}
\]
\[
\geq \frac{\exp(r_{n,0} + \eta)}{(1 + \exp(r_{n,0} + \eta))^2} \frac{(f_0((X_{ij}^T,1,Z_{ij}^T)^T) - f_0((X_{ij}^T,1,Z_{ij}^T)^T))^2}{2}.
\]

In particular, for any $\theta$ with $|f_\theta((X_{ij}^T,1,Z_{ij}^T)^T) - f_0((X_{ij}^T,1,Z_{ij}^T)^T)| \leq \eta$, we have
\[
l_{ij}(f_\theta((X_{ij}^T,1,Z_{ij}^T)^T)) - l_{ij}(f_0((X_{ij}^T,1,Z_{ij}^T)^T))
\]
\[
\geq \frac{\exp(|f_0((X_{ij}^T,1,Z_{ij}^T)^T)| + \eta)}{(1 + \exp(|f_0((X_{ij}^T,1,Z_{ij}^T)^T)| + \eta))^2} \frac{(f_\theta((X_{ij}^T,1,Z_{ij}^T)^T) - f_0((X_{ij}^T,1,Z_{ij}^T)^T))^2}{2}
\]
\[
\geq \frac{\exp(r_{n,0} + \eta)}{(1 + \exp(r_{n,0} + \eta))^2} \frac{(f_\theta((X_{ij}^T,1,Z_{ij}^T)^T) - f_0((X_{ij}^T,1,Z_{ij}^T)^T))^2}{2}.
\]

Define the function
\[
\tau = \frac{\exp(r_{n,0} + \eta)}{2(1 + \exp(r_{n,0} + \eta))^2}
\]
and notice that for $K_n$ defined in (7) we now have
\[
K_n = K_n(\eta) = \frac{1}{\tau}.
\]

Define a subset $\mathbb{F}_{local} \subset \mathbb{F}$ as $\mathbb{F}_{local} = \{f_\theta \in \mathbb{F} : \|f_\theta - f_0\|_\infty \leq \eta\}$. Now, for all $f_\theta \in \mathbb{F}_{local}$:
\[
\mathcal{E}(\theta) = \frac{1}{(n)} \sum_{i < j} \mathbb{E}[(l_{ij}(A_{ij}, (X_{ij}^T, Z_{ij}^T)^T) - l_{ij}(A_{ij}, (X_{ij}^T, Z_{ij}^T)^T))]
\]
\[
= \frac{1}{(n)} \sum_{i < j} \mathbb{E}[(l_{ij}(f_\theta((X_{ij}^T, Z_{ij}^T)^T)) - l_{ij}(f_\theta((X_{ij}^T, Z_{ij}^T)^T)))]
\]
\[
\geq \frac{1}{(n)} \sum_{i < j} \tau \mathbb{E}[(f_\theta((X_{ij}^T, Z_{ij}^T)^T) - f_0((X_{ij}^T, Z_{ij}^T)^T))^2]
\]
\[
\geq \frac{1}{K_n}\|f_\theta - f_0\|_{Q}^2.
\]

Thus, we have obtained a lower bound for the excess risk given by the quadratic function $G_n(\|f_\theta - f_0\|)$ where $G_n(u) = 1/K_n \cdot u^2$. Since $\mathcal{E}(\theta) = \mathcal{E}(\tilde{\theta})$ and $\|f_\theta\|_Q^2 = \|f_\theta\|_{Q,\infty}^2$ and $\|f_\theta\|_{Q,\infty} = \|f_\theta\|_{Q,\infty}$, we obtain the same result for the rescaled problem (19): For any $\tilde{\theta} \in \tilde{\Theta}_{loc}$ with $\|f_{\tilde{\theta}} - f_{\tilde{\theta}_0}\|_{Q,\infty} \leq \eta$, we have
\[
\mathcal{E}(\tilde{\theta}) \geq \frac{1}{K_n}\|f_{\tilde{\theta}} - f_{\tilde{\theta}_0}\|_{Q}^2.
\]

Recall that the convex conjugate of a strictly convex function $G$ on $[0, \infty)$ with $G(0) = 0$ is defined as the function
\[
H(v) = \sup_u \{uv - G(u)\}, \quad v > 0
\]
and in particular, if $G(u) = c u^2$ for a positive constant $c$, we have $H(v) = v^2/(4c)$. Hence, the convex
That is, the quadratic margin condition holds for any \( \bar{\theta} \in \Theta_{\text{loc}} \) in spirit van de Geer & Bühlmann (2011), Theorem 6.4. We will first define a set \( I \) and show that consistency holds on \( I \). It will then suffice to show that the probability of \( I \) tends to one as well. The proof follows in spirit van de Geer & Bühlmann (2011), Theorem 6.4.

The proof will use the language of re-parametrized likelihood problem (19). We define some objects that will need for the proof of consistency. We want to use the quadratic margin condition derived in appendix B.4. Recall that for any rescaled \( \bar{\theta} \) defined in (7),

\[
\|f_\theta - f_{\theta_0}\|_{Q, \infty} = \|f_\theta - f_{\theta_0}\|_{Q,\infty} + \|f_{\theta*} - f_{\theta_0}\|_{Q,\infty}
\]

\[
\leq \|f_\theta\|_{Q,\infty} + \|f_{\theta*}\|_{Q,\infty} + \inf\{C : \max_{i<j} |\beta_i^* + \beta_j^* + \mu^* - \beta_{0,i} - \beta_{0,j} - \mu_0 + (\gamma^* - \gamma_0)^T Z_{ij}| \leq C, \ a.s.\}
\]

\[
\leq 2r_n + 2\|\beta^* - \beta_0\|_{\infty} + |\mu^* - \mu_0| + 2\kappa = \eta.
\]

That is, the quadratic margin condition holds for any \( \bar{\theta} \in \Theta_{\text{loc}} \). With that definition of \( \eta \), we have for \( K_n \) defined in (7),

\[
K_n \leq \frac{2(1 + \exp(r_{n,0} + 2r_n + 2\|\beta^* - \beta_0\|_{\infty} + |\mu^* - \mu_0| + 2\kappa))^2}{\exp(r_{n,0} + 2r_n + 2\|\beta^* - \beta_0\|_{\infty} + |\mu^* - \mu_0| + 2\kappa)}.
\]

Define

\[
\epsilon^* = \frac{3}{2} \bar{\mathcal{E}}(\theta^*) + H_n \left( \frac{4\sqrt{2s + \lambda}}{\sqrt{c_{\min}}} \right).
\]

Remember that \( \bar{\mathcal{E}}(\theta^*) = \mathcal{E}(\theta^*) \) corresponds to the approximation error of our model. Let for any \( M > 0 \)

\[
Z_M := \sup_{\bar{\theta} \in \Theta_{\text{loc}}} \| \bar{\nu}_n(\bar{\theta}) - \bar{\nu}_n(\theta^*) \|
\]

where \( \bar{\nu}_n \) denotes the re-parametrized empirical process. Recall that for any rescaled \( \bar{\theta} \) we have \( \bar{\nu}_n(\bar{\theta}) = v_n(\theta) \). Also, by construction \( \bar{\theta} \in \Theta_{\text{loc}} \) if and only if \( \theta \in \Theta_{\text{loc}} \). Hence, the set over which we are maximizing in the definition of \( Z_M \) can be expressed in terms of parameters \( \theta \) on the original scale as

\[
\left\{ \theta = (\beta^T, \mu, \gamma^T)^T \in \Theta_{\text{loc}} : \frac{\sqrt{2}}{\sqrt{n}}\|\beta - \beta^*\|_1 + |\mu - \mu^*| + \|\gamma - \gamma^*\|_1 \leq M \right\}.
\]

Set

\[
M^* := \epsilon^*/\lambda_0,
\]

where \( \lambda_0 \) is a lower bound on \( \lambda \) that will be made precise in the proof showing that \( I \) has large probability. Define

\[
I := \{Z_{M^*} \leq \lambda_0 M^*\} = \{Z_{M^*} \leq \epsilon^*\}.
\]
Theorem 5. Assume that assumptions 1 and 2 hold and that $\tilde{\lambda} \geq 8\lambda_0$. Then, on the set $\mathcal{I}$, we have

$$\mathcal{E}(\hat{\theta}) + \tilde{\lambda} \left( \frac{\sqrt{2}}{\sqrt{n}} \|\hat{\beta} - \beta^*\|_1 + |\mu - \mu^*| + \|\gamma - \gamma^*\|_1 \right) \leq 4\epsilon^* = 6\mathcal{E}(\theta^*) + 4H_n \left( \frac{4\sqrt{2s^*_1\lambda}}{\sqrt{c_{\min}}} \right).$$

Proof of theorem 5. We assume that we are on the set $\mathcal{I}$ throughout. Set

$$t = \frac{M^*}{M^* + \|\hat{\theta} - \theta^*\|_1},$$

and $\tilde{\theta} = (\beta^T, \mu^*, \gamma^T) = t\hat{\theta} + (1 - t)\theta^*$. Then,

$$\|\hat{\theta} - \theta^*\|_1 = t\|\hat{\theta} - \theta^*\| \leq M^*.$$

Since $\hat{\theta}, \theta^* \in \Theta_{loc}$ and by the convexity of $\Theta_{loc}$, $\hat{\theta} \in \Theta_{loc}$, and by the remark after Lemma 5, the basic inequality holds for $\tilde{\theta}$:

$$\tilde{\mathcal{E}}(\tilde{\theta}) + \tilde{\lambda}\|\tilde{\beta}\|_1 \leq -(\bar{v}_n(\tilde{\theta} - \bar{v}_n(\theta^*))) + \tilde{\mathcal{E}}(\tilde{\theta}) + \tilde{\lambda}\|\tilde{\beta}\|_1$$

$$\leq Z_{M^*} + \tilde{\lambda}\|\tilde{\beta}\|_1 + \tilde{\mathcal{E}}(\theta^*)$$

$$\leq \epsilon^* + \tilde{\lambda}\|\tilde{\beta}\|_1 + \tilde{\mathcal{E}}(\theta^*).$$

From now on write $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}(\tilde{\theta})$ and $\mathcal{E}^* = \tilde{\mathcal{E}}(\theta^*)$. Note that $\|\beta^*\|_1 = \|\tilde{\beta}_{S^*}\|_1 + \|\tilde{\beta}_{S^*}\|_1$ and thus, by the triangle inequality,

$$\tilde{\mathcal{E}} + \tilde{\lambda}\|\tilde{\beta}_{S^*}\|_1 \leq \epsilon^* + \tilde{\lambda}(\|\beta^*\|_1 + \|\tilde{\beta}_{S^*}\|_1) + \mathcal{E}^*$$

$$\leq \epsilon^* + \tilde{\lambda}(\|\beta^* - \tilde{\beta}_{S^*}\|_1) + \mathcal{E}^*$$

$$\leq \epsilon^* + \tilde{\lambda}(\|\beta^* - \tilde{\beta}_{S^*}\|_1 + \|\mu^*, \gamma^T\|_1) + \|\mu^*, \gamma^T\|_1 + \mathcal{E}^*$$

$$= \epsilon^* + \tilde{\lambda}\|\theta - \theta^*\|_1 + \mathcal{E}^*$$

$$\leq 2\epsilon^* + \tilde{\lambda}\|\theta - \theta^*\|_1.$$

Where for the equality we have used that by assumption $\mu^*$ and $\gamma^*$ are active and hence $\|(\hat{\theta} - \theta^*)_{S^*_1}\|_1 = \|\beta^* - \tilde{\beta}_{S^*}\|_1 + \|\mu^*\|_1 + \|\gamma^T\|_1 + \|\mu^*, \gamma^T\|_1 + \|\mu^*, \gamma^T\|_1 + \mathcal{E}^*$. (23)

Case i) If $\tilde{\lambda}\|\theta - \theta^*\|_1 \geq \epsilon^*$, then

$$\tilde{\mathcal{E}} + \tilde{\lambda}\|\theta - \theta^*\|_1 \leq 3\tilde{\lambda}\|(\hat{\theta} - \theta^*)_{S^*_1}\|_1.$$ (24)

Since $\|(\hat{\theta} - \theta^*)_{S^*_1}\|_1 = \|\tilde{\beta}_{S^*}\|_1$, we may thus apply the compatibility condition corollary 3 (note that $\tilde{\beta}^* = \tilde{\beta}_{S^*}^*$) to obtain

$$\|\hat{\theta} - \theta^*\|_1 \leq \frac{\sqrt{2s^*_1}}{\sqrt{c_{\min}}}\|f_{\theta} - f_{\theta^*}\|_Q.$$

where we have used that $\theta \mapsto f_{\theta}$ is linear and hence $f_{\theta - \theta^*} = f_{\theta} - f_{\theta^*}$. Observe that

$$\|\hat{\theta} - \theta^*\|_1 = \|\tilde{\beta}_{S^*}\|_1 + \|\tilde{\beta}_{S^*}\|_1.$$ (25)

Hence,

$$\tilde{\mathcal{E}} + \tilde{\lambda}\|\theta - \theta^*\|_1 = \tilde{\mathcal{E}} + \tilde{\lambda}(\|\beta_{S^*}\|_1 + \|\tilde{\beta}_{S^*}\|_1)$$

$$\leq \epsilon^* + \tilde{\lambda}\|\theta - \theta^*\|_1 + \mathcal{E}^*$$

$$\leq \epsilon^* + \mathcal{E}^* + 2\tilde{\lambda}\frac{\sqrt{2s^*_1}}{\sqrt{c_{\min}}}\|f_{\theta} - f_{\theta^*}\|_Q.$$
Recall that for a convex function $G$ and its convex conjugate $H$ we have $uv \leq G(u) + H(v)$. Since $\theta, \theta^* \in \Theta_{\text{loc}}$ it holds $\|f_\theta - f_{\bar{\theta}_0}\|_\infty \leq \eta, \|f_{\bar{\theta}^*} - f_{\bar{\theta}_0}\|_\infty \leq \eta$. Thus, we obtain

$$2\lambda \sqrt{\frac{s^+}{c_{\min}}} \|f_\theta - f_{\bar{\theta}^*}\|_Q = 4\lambda \sqrt{\frac{s^+}{c_{\min}}} \frac{\|f_\theta - f_{\bar{\theta}^*}\|_Q}{2}$$

$$\leq 4\lambda \sqrt{\frac{s^+}{c_{\min}}} \frac{\|f_\theta - f_{\bar{\theta}_0}\|_Q + \|f_{\bar{\theta}^*} - f_{\bar{\theta}_0}\|_Q}{2}$$

$$\leq H_n \left(4\lambda \sqrt{\frac{s^+}{c_{\min}}} \right) + G_n \left( \frac{\|f_\theta - f_{\bar{\theta}_0}\|_Q + \|f_{\bar{\theta}^*} - f_{\bar{\theta}_0}\|_Q}{2} \right)$$

$$\leq H_n \left(4\lambda \sqrt{\frac{s^+}{c_{\min}}} \right) + \frac{G_n^{\text{convex}}(\|f_\theta - f_{\bar{\theta}_0}\|_Q) + G_n(\|f_{\bar{\theta}^*} - f_{\bar{\theta}_0}\|_Q)}{2}$$

It follows

$$\tilde{E} + \lambda \|\tilde{\theta} - \bar{\theta}^*\|_1 \leq \epsilon^* + \frac{3}{2} \|\bar{\theta}^*\|_1 = \frac{2\lambda \|\bar{\theta}^*\|_1}{2} = \frac{2\lambda \|\bar{\theta}^*\|_1}{2} \leq \frac{M^*}{2}.$$}

and therefore

$$\tilde{E} + \lambda \|\tilde{\theta} - \bar{\theta}^*\|_1 \leq 2\epsilon^*.$$}

Finally, this gives

$$\|\tilde{\theta} - \bar{\theta}^*\|_1 \leq \frac{2\epsilon^*}{\lambda} = \frac{2\lambda_0 M^*}{\lambda} \leq \frac{M^*}{2},$$

From this, by using the definition of $\tilde{\theta}$, we obtain

$$\|\tilde{\theta} - \bar{\theta}^*\|_1 = \epsilon \|\tilde{\theta} - \bar{\theta}^*\|_1 = \frac{M^*}{\epsilon \|\bar{\theta}^*\|_1} \|\tilde{\theta} - \bar{\theta}^*\|_1 \leq \frac{M^*}{2}.$$}

Rearranging gives

$$\|\tilde{\theta} - \bar{\theta}^*\|_1 \leq M^*.$$}

**Case ii)** If $\bar{\lambda} \|\bar{\theta}^* - \bar{\theta}\|_{S^+_{\lambda}} \leq \epsilon^*$, then from (23)

$$\tilde{E} + \bar{\lambda} \|\tilde{\bar{\theta}}^*\|_1 \leq 3\epsilon^*.$$}

Using once more (25), we get

$$\tilde{E} + \bar{\lambda} \|\tilde{\theta} - \bar{\theta}^*\|_1 = \tilde{E} + \bar{\lambda} \|\tilde{\bar{\theta}}^*\|_1 + \bar{\lambda} \|\bar{\theta}^* - \bar{\theta}\|_{S^+_{\lambda}} \leq 4\epsilon^*.$$}

Thus,

$$\|\tilde{\theta} - \bar{\theta}^*\|_1 \leq \frac{4\epsilon^*}{\lambda} = \frac{4\lambda_0 M^*}{\lambda} \leq \frac{M^*}{2}$$

by choice of $\lambda \geq 8\lambda_0$. Again, plugging in the definition of $\tilde{\theta}$, we obtain

$$\|\tilde{\theta} - \bar{\theta}^*\|_1 \leq M^*.$$}

Hence, in either case we have $\|\tilde{\theta} - \bar{\theta}^*\|_1 \leq M^*$. That means, we can repeat the above steps with $\tilde{\theta}$ instead of $\tilde{\theta}$: Writing $\tilde{E} := E(\tilde{\theta})$, following the same reasoning as above we arrive once more at (23):

$$\tilde{E} + \bar{\lambda} \|\tilde{\bar{\theta}}^*\|_1 \leq 2\epsilon^* + \bar{\lambda} \|\bar{\theta}^* - \bar{\theta}\|_{S^+_{\lambda}} \leq 2\epsilon^* + \bar{\lambda} \|\bar{\theta}^* - \bar{\theta}\|_{S^+_{\lambda}}.$$}
From this, in case i) we obtain (24) which allows us to use the compatibility assumption to arrive at (26):
\[
\frac{\hat{\xi}}{2} + \lambda \|\hat{\theta} - \theta^*\|_1 \leq 2\epsilon^*,
\]
resulting in
\[
\hat{\xi} + \lambda \|\hat{\theta} - \theta^*\|_1 \leq 4\epsilon^*.
\]
In case ii) on the other hand, we arrive directly at (27), and hence
\[
\hat{\xi} + \lambda \|\hat{\theta} - \theta^*\|_1 \leq 3\epsilon^*.
\]
Plugging in the definitions of \(\hat{\theta}\) and \(\theta^*\) and using the fact that \(\hat{\xi} = E(\hat{\theta}) = E(\bar{\theta})\) proves the claim. \(\square\)

In the SβM without covariates we have an analogous result. Extend the definitions of \(I, Z_M\) to the SβM in the straightforward way by letting \(p = 0, \gamma = 0, \kappa = 0, Z_{ij} = 0, i < j\). We already know that by Lemma 3 the compatibility condition holds for the SβM without covariates. The proof of the following corollary then follows almost by line by line as the proof of theorem 5.

**Corollary 4.** Assume that in the SβM without covariates Assumption 2 holds and that \(\lambda \geq 8\lambda_0\), with \(\lambda_0\) as in theorem 5. Then, on the set \(I\) defined in (22), we have
\[
E(\hat{\theta}) + \lambda \left( \frac{\sqrt{2}}{\sqrt{n}} \|\hat{\beta} - \beta^*\|_1 + |\bar{\mu} - \mu^*|\right) \leq 6E(\theta^*) + 4H_n \left( \frac{4\sqrt{2s^*}}{\sqrt{c_{min}}} \right).
\]

**Proof.** Analogous to the proof of theorem 5. \(\square\)

### B.6 Controlling the special set \(I\)

We now show that \(I\) has probability tending to one. Recall some results on concentration inequalities.

#### B.6.1 Concentration inequalities

We first recall some probability inequalities that we will need. This is based on Chapter 14 in van de Geer & Bühlmann (2011). Throughout let \(Z_1, \ldots, Z_n\) be a sequence of independent random variables in some space \(Z\) and \(G\) be a class of real valued functions on \(Z\).

**Definition 2.** A Rademacher sequence is a sequence \(\epsilon_1, \ldots, \epsilon_n\) of i.i.d. random variables with \(P(\epsilon_i = 1) = P(\epsilon_i = -1) = 1/2\) for all \(i\).

**Theorem 6** (Symmetrization Theorem as in van der Vaart & Wellner (1996), abridged). Let \(\epsilon_1, \ldots, \epsilon_n\) be a Rademacher sequence independent of \(Z_1, \ldots, Z_n\). Then
\[
E \left( \sup_{g \in G} \left| \sum_{i=1}^{n} \{g(Z_i) - E[g(Z_i)]\} \right| \right) \leq 2E \left( \sup_{g \in G} \left| \sum_{i=1}^{n} \epsilon_i g(Z_i) \right| \right).
\]

**Theorem 7** (Contraction theorem as in Ledoux & Talagrand (1991)). Let \(z_1, \ldots, z_n\) be non-random elements of \(Z\) and let \(F\) be a class of real-valued functions on \(Z\). Consider Lipschitz functions \(g_i : \mathbb{R} \to \mathbb{R}\) with Lipschitz constant \(L = 1\), i.e. for all \(i\)
\[
|g_i(s) - g_i(s')| \leq |s - s'|, \forall s, s' \in \mathbb{R}.
\]
Let \(\epsilon_1, \ldots, \epsilon_n\) be a Rademacher sequence. Then for any function \(f^* : Z \to \mathbb{R}\) we have
\[
E \left( \sup_{f \in F} \left| \sum_{i=1}^{n} \epsilon_i \{g_i(f(z_i)) - g_i(f^*(z_i))\} \right| \right) \leq 2E \left( \sup_{f \in F} \left| \sum_{i=1}^{n} \epsilon_i \{f(z_i) - f^*(z_i)\} \right| \right).
\]
The last theorem we need is a concentration inequality due to Bousquet (2002). We give a version as presented in van de Geer (2008).

**Theorem 8** (Bousquet’s concentration theorem). Suppose $Z_1, \ldots, Z_n$ and all $g \in \mathcal{G}$ satisfy the following conditions for some real valued constants $\eta_n$ and $\tau_n$

$$
\|g\|_\infty \leq \eta_n, \forall g \in \mathcal{G}
$$

and

$$
\frac{1}{n} \sum_{i=1}^n \text{Var}(g(Z_i)) \leq \tau_n^2, \forall g \in \mathcal{G}.
$$

Define

$$
Z := \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n g(Z_i) - \mathbb{E}[g(Z_i)] \right|.
$$

Then for any $z > 0$

$$
P \left( Z \geq \mathbb{E}[Z] + z \sqrt{2(\tau_n^2 + 2\eta_n\mathbb{E}[Z])} \right) \leq \exp(-nz^2).
$$

**Remark.** Looking at the original paper of Bousquet (2002), their result looks quite different at first. To see that the above falls into their framework, set the variables in Bousquet (2002) as follows

$$
f(Z_i) = (g(Z_i) - \mathbb{E}[g(Z_i)])/(2\eta_n),
$$

$$
f_k = \arg \sup_f \left| \sum_{i \neq k} f(Z_i) \right|,
$$

$$
\tilde{Z}_k = \left| \sum_{i=1}^n f_k(Z_i) \right| - \tilde{Z}_k,
$$

$$
\bar{Z} = \frac{2\eta_n}{n} Z.
$$

Now apply theorem 2.1 in Bousquet (2002), choosing for their $(Z, Z_1, \ldots, Z_n)$ the above defined $(\bar{Z}, \tilde{Z}_1, \ldots, \tilde{Z}_n)$, for their $(Z'_1, \ldots, Z'_n)$ the above defined $(\tilde{Z}'_1, \ldots, \tilde{Z}'_n)$ and setting $u = 1$ and $\sigma^2 = \frac{\tau_n^2}{4\eta_n^2}$ in their theorem: The result is exactly theorem 8 above.

Finally we have a Lemma derived from Hoeffding’s inequality. The proof can be found in van de Geer & Bühlmann (2011), Lemma 14.14 (here we use the special case of their Lemma for $m = 1$).

**Lemma 6.** Let $\mathcal{G} = \{g_1, \ldots, g_p\}$ be a set of real valued functions on $\mathcal{Z}$ satisfying for all $i = 1, \ldots, n$ and all $j = 1, \ldots, p$

$$
\mathbb{E}[g_j(Z_i)] = 0, \ |g_j(Z_i)| \leq c_{ij},
$$

for some positive constants $c_{ij}$. Then

$$
\mathbb{E} \left[ \max_{1 \leq j \leq p} \left| \sum_{i=1}^n g_j(Z_i) \right| \right] \leq [2 \log(2p)]^{1/2} \max_{1 \leq j \leq p} \left[ \sum_{i=1}^n c_{ij}^2 \right]^{1/2}.
$$

**B.6.2 The expectation of $Z_M$**

Recall the definition of $Z_M$:

$$
Z_M := \sup_{\bar{\theta} \in \Theta_{	ext{base}}, \|\bar{\theta} - \bar{\theta}^*\|_1 \leq M} |\tilde{v}_n(\bar{\theta}) - \tilde{v}_n(\bar{\theta}^*)|,
$$

where $\tilde{v}_n$ denotes the re-parametrized empirical process. Recall, that there is a constant $c \in \mathbb{R}$ such that uniformly $|Z_{ij,k}| \leq c, 1 \leq i < j \leq n, k = 1, \ldots, p$. 

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Lemma 7. For any $M > 0$ we have in $S\beta RM$

$$
\mathbb{E}[Z_M] \leq 8M(1 \vee c)\sqrt{\frac{2\log(2(n+p+1))}{\binom{n}{2}}}
$$

and in the $S\beta M$ without covariates

$$
\mathbb{E}[Z_M] \leq 8M\sqrt{\frac{\log(2(n+1))}{\binom{n}{2}}}.
$$

Proof. We only give the proof for the $S\beta RM$. The proof for the case without covariates is exactly the same with the corresponding parts set to zero. Let $\epsilon_{ij}, i < j$, be a Rademacher sequence independent of $A_{ij}, Z_{ij}, i < j$. We first want to use the symmetrization Theorem 6: For the random variables $Z_1, ..., \text{we choose } T_{ij} = (A_{ij}, X_{ij}^T, 1, Z_{ij}^T) \in \{0, 1\} \times \mathbb{R}^{n+1+p}$. For any $\bar{\theta} \in \Theta_{loc}$ we consider the functions

$$
g_{\bar{\theta}}(T_{ij}) = \frac{1}{2}\left(-A_{ij}\bar{D}_{ij}^T(\bar{\theta} - \bar{\theta}^\ast) + \log(1 + \exp(\bar{D}_{ij}^T\bar{\theta})) - \log(1 + \exp(\bar{D}_{ij}^T\bar{\theta}^\ast))\right)
$$

and the function set $\mathcal{G} = \mathcal{G}(M) := \{g_{\bar{\theta}} : \bar{\theta} \in \Theta_{loc}, \|\bar{\theta} - \bar{\theta}^\ast\|_1 \leq M\}$. Note, that

$$
\bar{v}_n(\bar{\theta}) - \bar{v}_n(\bar{\theta}^\ast) = \sum_{i<j} (g_{\bar{\theta}}(T_{ij}) - \mathbb{E}[g_{\bar{\theta}}(T_{ij})]).
$$

Then, the symmetrization theorem gives us

$$
\mathbb{E}[Z_M] = \mathbb{E}\left[\sup_{g_{\bar{\theta}} \in \mathcal{G}} \left| \sum_{i<j} g_{\bar{\theta}}(T_{ij}) - \mathbb{E}[g_{\bar{\theta}}(T_{ij})]\right|\right]
$$

$$
\leq 2\mathbb{E}\left[\sup_{g_{\bar{\theta}} \in \mathcal{G}} \left| \sum_{i<j} \epsilon_{ij}g_{\bar{\theta}}(T_{ij})\right|\right].
$$

Next, we want to apply the contraction theorem 7. Denote $T = (T_{ij})_{i<j}$ and let $\mathbb{E}_T$ be the conditional expectation given $T$. We need the conditional expectation at this point, because theorem 7 requires non-random arguments in the functions. This does not hinder us, as later we will simply take iterated expectations, canceling out the conditional expectation, see below. For the functions $g_i$ in theorem 7 we choose

$$
g_{ij}(x) = \frac{1}{2}\left(-A_{ij}x + \log(1 + \exp(x))\right)
$$

Note, that $\log(1 + \exp(x))$ has derivative bounded by one and thus is Lipschitz continuous with constant one by the Mean Value Theorem. Thus, all $g_{ij}$ are also Lipschitz continuous with constant 1:

$$
|g_{ij}(x) - g_{ij}(x')| \leq \frac{1}{2}\{|A_{ij}(x - x')| + |\log(1 + \exp(x)) - \log(1 + \exp(x'))|\} \leq |x - x'|.
$$

For the function class $\mathcal{F}$ in theorem 7 we choose $\mathcal{F} = \mathcal{F}_M := \{f_{\bar{\theta}} : \bar{\theta} \in \Theta_{loc}, \|\bar{\theta} - \bar{\theta}^\ast\|_1 \leq M\}$ and pick $f^* = f_{\bar{\theta}^\ast}$. Then, by theorem 7

$$
\mathbb{E}_T\left[\sup_{\bar{\theta} \in \Theta_{loc}, \|\bar{\theta} - \bar{\theta}^\ast\|_1 \leq M} \left| \sum_{i<j} \frac{1}{\binom{n}{2}} \epsilon_{ij}(g_{\bar{\theta}}((X_{ij}^T, 1, Z_{ij}^T)^T)) - g_{ij}(f_{\bar{\theta}^\ast}((X_{ij}^T, 1, Z_{ij}^T)^T))\right|\right]
$$

$$
\leq 2\mathbb{E}_T\left[\sup_{\bar{\theta} \in \Theta_{loc}, \|\bar{\theta} - \bar{\theta}^\ast\|_1 \leq M} \left| \sum_{i<j} \frac{1}{\binom{n}{2}} \epsilon_{ij}(f_{\bar{\theta}}((X_{ij}^T, 1, Z_{ij}^T)^T) - f_{\bar{\theta}^\ast}((X_{ij}^T, 1, Z_{ij}^T)^T))\right|\right].
$$
Recall that we can express the functions $f_{\bar{\theta}} = f_{\beta, \mu, \gamma}$ as

$$f_{\beta, \mu, \gamma}(\cdot) = \mu e_{n+1}(\cdot) + \sum_{i=1}^{n} \beta_i e_i(\cdot) + \sum_{i=1}^{p} \gamma_i e_{n+1+i}(\cdot),$$

where $e_i(\cdot)$ is the projection on the $i$th coordinate. Consider any $\bar{\theta} = (\bar{\beta}^T, \mu, \gamma^T)^T \in \overline{\Theta}_{\text{hoc}}$ with $\|\bar{\theta} - \hat{\theta}^*\|_1 \leq M$. For the sake of a compact representation we use our shorthand notation $\bar{\theta} = (\bar{\theta}_i)_{i=1}^{n+1+p}$ where the components $\theta_i$ are defined in the canonical way and we also simply write $e_k(X_{ij}, 1, Z_{ij})$ for the projection of the the vector $(X_{ij}^T, 1, Z_{ij})^T \in \mathbb{R}^{n+p+1}$ to its $k$th component, i.e. instead of $e_k((X_{ij}^T, 1, Z_{ij})^T)$. Then,

$$\left| \frac{1}{\binom{n}{2}} \sum_{i<j} \epsilon_{ij} \left( f_{\bar{\theta}}((\bar{X}_{ij}^T, 1, Z_{ij})^T) - f_{\hat{\theta}^*}((\bar{X}_{ij}^T, 1, Z_{ij})^T) \right) \right|$$

$$= \left| \frac{1}{\binom{n}{2}} \sum_{i<j} \epsilon_{ij} \left( n+p+1 \sum_{k=1}^{n} (\bar{\theta}_k - \hat{\theta}_k^*) e_k(\bar{X}_{ij}, 1, Z_{ij}) \right) \right|$$

$$\leq \frac{1}{\binom{n}{2}} \sum_{k=1}^{n+p+1} \max_{1 \leq i \leq n+1} \left| \sum_{i<j} \epsilon_{ij} e_k(\bar{X}_{ij}, 1, Z_{ij}) \right|$$

$$\leq M \max_{1 \leq i \leq n+p+1} \left| \frac{1}{\binom{n}{2}} \sum_{i<j} \epsilon_{ij} e_k(\bar{X}_{ij}, 1, Z_{ij}) \right|. $$

Note, that the last expression no longer depends on $\bar{\theta}$. To bind the right hand side in the last expression we use Lemma 6: In the language of the Lemma, choose $Z_1, \ldots, Z_n$ as $T_{ij} = (e_{ij}, \bar{X}_{ij}^T, 1, Z_{ij})^T$. We choose for the $p$ in the formulation of the Lemma $n+p+1$ and pick for our functions

$$g_k(T_{ij}) = \frac{1}{\binom{n}{2}} \epsilon_{ij} e_k(\bar{X}_{ij}, 1, Z_{ij}), k = 1, \ldots, n+p+1.$$

Note, that then $\mathbb{E}|g_k(T_{ij})| = 0$. We want to employ Lemma 6 which requires us to bound $|g_k(T_{ij})| \leq c_{ij, k}$ for all $i < j$ and $k = 1, \ldots, n+1+p$.

For any fixed $1 \leq k \leq n$ we have

$$|g_k(T_{ij})| \leq \begin{cases} \sqrt{\frac{\pi}{2}} \frac{1}{(n-1)\sqrt{n}}, & i \text{ or } j = k \\ 0, & \text{otherwise} \end{cases}.$$  

Note that the first case occurs exactly $(n-1)$ times for each $k$. Thus, for any $k \leq n$,

$$\sum_{i<j} c_{ij, k} = \left( \frac{\sqrt{2}}{(n-1)\sqrt{n}} \right)^2 (n-1) = \frac{1}{\binom{n}{2}}.$$

If $k = n+1$, $|g_k(T_{ij})| = 1/(\binom{n}{2})$ and hence

$$\sum_{i<j} c_{ij, n+1} = \frac{1}{\binom{n}{2}}.$$

Finally, if $k > n+1$, $|g_k(T_{ij})| \leq c/\binom{n}{2}$ and therefore,

$$\sum_{i<j} c_{ij, k} \leq \frac{c^2}{\binom{n}{2}}.$$
In total, this means
\[
\max_{1 \leq k \leq n + 1 + p} \sum_{i < j} c_{ij,k}^2 \leq \frac{1 \lor c^2}{(n/2)}.
\]

Therefore, an application of Lemma 6 results in
\[
E \left[ \max_{1 \leq l \leq n + p + 1} \left| \frac{1}{n/2} \sum_{i < j} \epsilon_{ij} \ell(X_{ij}, Z_{ij}) \right| \right] \leq \sqrt{2 \log(2(n + 1 + p))} \max_{1 \leq k \leq n + 1 + p} \left( \sum_{i < j} c_{ij,k}^2 \right)^{1/2}
\]
\[
\leq \sqrt{2 \log(2(n + 1 + p))} \frac{1 \lor c^2}{(n/2)}
\]
\[
= \sqrt{2 \log(2(n + 1 + p))} \frac{1 \lor c}{(n/2)}.
\]

Putting everything together, we obtain
\[
E[Z_M] \leq 2E \left[ \sup_{\bar{\theta} \in \Theta_{loc}} \left| \frac{1}{n/2} \sum_{i < j} \epsilon_{ij} (-A_{ij}(f_{\bar{\theta}}(X_{ij}, Z_{ij}) - f_{\bar{\theta}^*}(X_{ij}, Z_{ij}))) \right| \right]
\]
\[
= 2E \left[ E_T \left[ \sup_{\bar{\theta} \in \Theta_{loc}} \left| \frac{1}{n/2} \sum_{i < j} \epsilon_{ij} (f_{\bar{\theta}}(X_{ij}, Z_{ij}) - f_{\bar{\theta}^*}(X_{ij}, Z_{ij})) \right| \right] \right]
\]
\[
\leq 8E \left[ E_T \left[ \sup_{\bar{\theta} \in \Theta_{loc}} \left| \frac{1}{n/2} \sum_{i < j} \epsilon_{ij} (f_{\bar{\theta}}(X_{ij}, Z_{ij}) - f_{\bar{\theta}^*}(X_{ij}, Z_{ij})) \right| \right] \right]
\]
\[
\leq 8ME \left[ \max_{1 \leq i \leq n + p + 1} \left| \frac{1}{n/2} \sum_{i < j} \epsilon_{ij} \ell(X_{ij}, Z_{ij}) \right| \right]
\]
\[
\leq 8M \sqrt{2 \log(2(n + 1 + p))} \frac{1 \lor c}{(n/2)}.
\]
This concludes the proof.

We now want to show that $Z_M$ does not deviate too far from its expectation. The proof relies on the concentration theorem due to Bousquet, theorem 8.

**Corollary 5.** Pick any confidence level $t > 0$. Let
\[
a_n := \sqrt{\frac{2 \log(2(n + p + 1))}{n/2}} (1 \lor c),
\]
and choose $\lambda_0 = \lambda_0(t, n)$ as
\[
\lambda_0 = 8a_n + 2 \sqrt{\frac{t}{n/2}} (11(1 \lor (c^2p)) + 8\sqrt{2}(1 \lor c)\sqrt{n}a_n) + \frac{2\sqrt{2}t (1 \lor c)\sqrt{n}}{3\sqrt{n/2}}.
\]

Then, we have the inequality
\[
P(Z_M \geq M\lambda_0) \leq \exp(-t).
\]
In the $S3M$ without covariates set

$$a_n = \sqrt{\frac{\log(2(n+1))}{\binom{n}{2}}}, \quad \lambda_0 = 8a_n + 2\sqrt{\frac{t}{(n/2)}}(9 + 8\sqrt{2}na_n) + \frac{2\sqrt{2t\sqrt{n}}}{3\binom{n}{2}}.$$  

In either case we have

$$P(Z_M > \lambda_0 M) \leq \exp(-t).$$

**Proof.** Again, we only give the proof for the case with covariates. The case without covariates is completely analogous by setting the corresponding parts to zero. We want to apply Bousquet’s concentration theorem 8. For the random variables $Z_i$ in the formulation of the theorem we choose once more $T_{ij} = (A_{ij}, X_{ij}, 1, Z_{ij}), i < j$, and as functions we consider

$$g_\theta(T_{ij}) = -A_{ij}D_{ij}^T(\bar{\theta} - \theta^*) + \log(1 + \exp(D_{ij}^T\bar{\theta}^*)) - \log(1 + \exp(D_{ij}^T\theta^*)),

\mathcal{G} = \mathcal{G}_M := \{g_\theta : \bar{\theta} \in \Theta_{loc}, ||\bar{\theta} - \theta^*||_1 \leq M\}.$$  

Then, by definition we have

$$Z_M = \sup_{g_\theta \in \mathcal{G}} \frac{1}{\sqrt{n}} \left| \sum_{i<j} \{g_\theta(T_{ij}) - \mathbb{E}[g_\theta(T_{ij})]\} \right|.$$  

To apply theorem 8, we need to bound the infinity norm of $g_\theta$. Recall that we denote the distribution of $[\bar{X}|1|Z]$ by $Q$ and the infinity norm is defined as the $Q$-almost sure smallest upper bound on the value of $g_\theta$. We have for any $g_\theta \in \mathcal{G}$, using the Lipschitz continuity of $\log(1 + \exp(x))$:

$$|g_\theta(T_{ij})| \leq |D_{ij}^T(\bar{\theta} - \theta^*)| + |\log(1 + \exp(D_{ij}^T\bar{\theta}^*))| - \log(1 + \exp(D_{ij}^T\theta^*))|

\leq 2|D_{ij}^T(\bar{\theta} - \theta^*)|

\leq 2||\beta - \beta^*||_1 + ||\mu - \mu^*|| + c||\gamma - \gamma^*||_1.$$  

Thus,

$$||g_\theta||_\infty \leq 2||\beta - \beta^*||_1 + ||\mu - \mu^*|| + c||\gamma - \gamma^*||_1

\leq 2(1 \lor c)||\theta - \theta^*||_1

\leq \sqrt{2}(1 \lor c)\sqrt{n}M =: \eta_n.$$  

For the last inequality we used that for any $\theta$ with $||\bar{\theta} - \theta^*||_1 \leq M$ it follows that $||\theta - \theta^*||_1 \leq \sqrt{n}/\sqrt{2}M$, which is possibly a very generous upper bound. This does not matter, however, as the term associated with the above bound will be negligible, as we shall see.

The second requirement of theorem 8 is that the average variance of $g_\theta(T_{ij})$ has to be uniformly bounded. To that end we calculate

$$\frac{1}{\binom{n}{2}} \sum_{i<j} \text{Var}(g_\theta(T_{ij})) = \frac{1}{\binom{n}{2}} \sum_{i<j} \text{Var}(-A_{ij}D_{ij}^T(\theta - \theta^*))

+ \frac{1}{\binom{n}{2}} \sum_{i<j} \text{Var}(\log(1 + \exp(D_{ij}^T\bar{\theta}^*)) - \log(1 + \exp(D_{ij}^T\theta^*)))

+ \frac{2}{\binom{n}{2}} \sum_{i<j} \text{Cov}(-A_{ij}D_{ij}^T(\theta - \theta^*), \log(1 + \exp(D_{ij}^T\bar{\theta}^*)) - \log(1 + \exp(D_{ij}^T\theta^*))).$$  

Let us look at these terms in term. For the first term, we obtain

$$\frac{1}{\binom{n}{2}} \sum_{i<j} \text{Var}(-A_{ij}D_{ij}^T(\theta - \theta^*)) \leq \frac{1}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[-A_{ij}D_{ij}^T(\theta - \theta^*)^2] \leq \mathbb{E} \left[ \frac{1}{\binom{n}{2}} \sum_{i<j} (D_{ij}^T(\theta - \theta^*))^2 \right].$$  

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For the second term we get
\[
\frac{1}{\binom{n}{2}} \sum_{i<j} \text{Var}(\log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*)))
\]
\[
\leq \frac{1}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[(\log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*)))^2]
\]
\[
\leq \mathbb{E} \left[ \frac{1}{2} \sum_{i<j} (D_{ij}^T (\theta - \theta^*))^2 \right].
\]
The last term decomposes as
\[
\frac{2}{\binom{n}{2}} \sum_{i<j} \text{Cov}(-A_{ij} D_{ij}^T (\theta - \theta^*), \log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*)))
\]
\[
= \frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[-A_{ij} D_{ij}^T (\theta - \theta^*) \cdot (\log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*)))]
\]
\[
- \frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[-A_{ij} D_{ij}^T (\theta - \theta^*)] \cdot \mathbb{E}[\log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*)]
\]
For the first term in that decomposition we have
\[
\frac{2}{\binom{n}{2}} \sum_{i<j} |\mathbb{E}[-A_{ij} D_{ij}^T (\theta - \theta^*) \cdot (\log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*)))]| \]
\[
\leq \frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[|D_{ij}^T (\theta - \theta^*)|] \cdot |\log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*))|
\]
\[
\leq \frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[|D_{ij}^T (\theta - \theta^*)|^2]
\]
and for the second term, using the same arguments, we get
\[
\frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[|A_{ij} D_{ij}^T (\theta - \theta^*)|] \cdot \mathbb{E}[\log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*))]
\]
\[
\leq \frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[|D_{ij}^T (\theta - \theta^*)|^2],
\]
meaning that in total
\[
\frac{2}{\binom{n}{2}} \sum_{i<j} |\text{Cov}(-A_{ij} D_{ij}^T (\theta - \theta^*), \log(1 + \exp(D_{ij}^T \bar{\theta})) - \log(1 + \exp(D_{ij}^T \theta^*)))|
\]
\[
\leq \frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[|D_{ij}^T (\theta - \theta^*)|^2] + \frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[|D_{ij}^T (\theta - \theta^*)|]^2.
\]
In total, we thus get
\[
\frac{1}{\binom{n}{2}} \sum_{i<j} \text{Var}(g_{ij}(T_{ij})) \leq 4 \cdot \mathbb{E} \left[ \frac{1}{\binom{n}{2}} \sum_{i<j} (D_{ij}^T (\theta - \theta^*))^2 \right] + \frac{2}{\binom{n}{2}} \sum_{i<j} \mathbb{E}[|D_{ij}^T (\theta - \theta^*)|^2]. \tag{28}
\]
Furthermore,
\[
\frac{1}{n} \sum_{i < j} (D_{ij}^T(\theta - \theta^*))^2 = \frac{1}{n} \sum_{i < j} (\beta_i + \beta_j + \mu - \beta_i^* - \beta_j^* - \mu^* + (\gamma - \gamma^*)^T Z_{ij})^2
\]
\[
\leq \frac{4}{n} \sum_{i < j} \left\{ (\beta_i - \beta_i^*)^2 + (\beta_j - \beta_j^*)^2 + (\mu - \mu^*)^2 + ((\gamma - \gamma^*)^T Z_{ij})^2 \right\},
\]
where the inequality follows from the Cauchy-Schwarz inequality. Recall that for any \( x \in \mathbb{R}^p, \| x \|_2 \leq \| x \|_1 \leq \sqrt{n} \| x \|_2 \) and note that
\[
| (\gamma - \gamma^*)^T Z_{ij} | \leq c \| \gamma - \gamma^* \|_1 \leq c \sqrt{n} \| \gamma - \gamma^* \|_2.
\]

Then, from the above
\[
\frac{1}{n} \sum_{i < j} (D_{ij}^T(\theta - \theta^*))^2 \leq \frac{4}{n} \sum_{i < j} \left\{ (\beta_i - \beta_i^*)^2 + (\beta_j - \beta_j^*)^2 + (\mu - \mu^*)^2 + c^2 \| \gamma - \gamma^* \|_2^2 \right\}
\]
\[
= 4 \left( (\mu - \mu^*)^2 + c^2 \| \gamma - \gamma^* \|_2^2 + \frac{1}{n} \sum_{i < j} \left\{ (\beta_i - \beta_i^*)^2 + (\beta_j - \beta_j^*)^2 \right\} \right)
\]
\[
= 4 \left( (\mu - \mu^*)^2 + c^2 \| \gamma - \gamma^* \|_2^2 + \frac{1}{n} (n - 1) \| \beta - \beta^* \|_2^2 \right)
\]
\[
= 4 \left( (\mu - \mu^*)^2 + c^2 \| \gamma - \gamma^* \|_2^2 + \| \frac{\sqrt{\beta}}{\sqrt{n}} \|_2 \right)
\]
\[
= 4 \left( (\mu - \mu^*)^2 + c^2 \| \gamma - \gamma^* \|_2^2 + \| \beta - \beta^* \|_2^2 \right)
\]
\[
\leq 4(1 \lor (c^2p)) \| \theta - \theta^* \|_2^2
\]
\[
\leq 4(1 \lor (c^2p)) \| \theta - \theta^* \|_2^2
\]
\[
\leq 4(1 \lor (c^2p)) \| \theta - \theta^* \|_2^2
\]
\[
\leq 4(1 \lor (c^2p)) M^2.
\]

Notice that for the second term in (28) we have
\[
\frac{2}{n} \sum_{i < j} E[|D_{ij}^T(\theta - \theta^*)|^2] = \frac{2}{n} \sum_{i < j} (\beta_i - \beta_i^* + \beta_j - \beta_j^* + \mu - \mu^* + (\gamma - \gamma^*)^T E[Z_{ij}])^2
\]
\[
= \frac{2}{n} \sum_{i < j} (\beta_i - \beta_i^* + \beta_j - \beta_j^* + \mu - \mu^*)^2
\]
\[
\leq \frac{6}{n} \sum_{i < j} \left\{ (\beta_i - \beta_i^*)^2 + (\beta_j - \beta_j^*)^2 + (\mu - \mu^*)^2 \right\},
\]
so that we may use the same steps as in (29) to conclude that
\[
\frac{2}{n} \sum_{i < j} E[|D_{ij}^T(\theta - \theta^*)|^2] \leq 6M^2 \leq 6(1 \lor (c^2p)) M^2.
\]

Such that in total,
\[
\frac{1}{n} \sum_{i < j} \text{Var}(g_{j}(T_{ij})) \leq 22(1 \lor (c^2p)) M^2 =: \gamma_n^2.
\]
Applying Bousquet’s concentration theorem 8 with \( \eta_n, \tau_n \) defined above, we obtain for all \( z > 0 \)
\[
\exp\left(-\left(\frac{n}{2}\right)z^2\right) \geq P\left(Z_M \geq \mathbb{E}[Z_M] + z\sqrt{2\tau_n^2 + 2\eta_n\mathbb{E}[Z_M]} + \frac{2z^2\eta_n}{3}\right)
\]
\[
= P\left(Z_M \geq \mathbb{E}[Z_M] + z\sqrt{22(1 \lor (c^2p))M^2 + 2\sqrt{2}(1 \lor c)\sqrt{nM}\mathbb{E}[Z_M]} + \frac{2\sqrt{2}z^2(1 \lor c)\sqrt{nM}}{3}\right)
\]
(30)

From Lemma 7, we know
\[
\mathbb{E}[Z_M] \leq 8M\sqrt{\frac{2\log(2(n+p+1))}{(n/2)}(1 \lor c)} = 8Ma_n.
\]

Using this, we obtain from (30)
\[
\exp\left(-\left(\frac{n}{2}\right)z^2\right) \geq P\left(Z_M \geq 8Ma_n + z\sqrt{22(1 \lor (c^2p))M^2 + 16\sqrt{2}(1 \lor c)\sqrt{nM^2a_n}}
\]
\[
+ \frac{2\sqrt{2}z^2(1 \lor c)\sqrt{nM}}{3}\right)
\]
\[
= P\left(Z_M \geq M\left(8a_n + 2z\sqrt{11(1 \lor (c^2p)) + 8\sqrt{2}(1 \lor c)\sqrt{n}a_n} + \frac{2\sqrt{2}z^2(1 \lor c)\sqrt{n}}{3}\right)\right).
\]

Now, pick \( z = \sqrt{\frac{t}{(n/2)}} \) to get
\[
\exp(-t) \geq P\left(Z_M \geq M\left(8a_n + 2z\sqrt{11(1 \lor (c^2p)) + 8\sqrt{2}(1 \lor c)\sqrt{n}a_n} + \frac{2\sqrt{2}t(1 \lor c)\sqrt{n}}{3}\right)\right),
\]
which is the claim. \( \square \)

### B.7 Proof of theorems 1 and 2 and corollary 1

**Proof of theorem 1.** Follows immediately from theorem 5 and corollary 5. \( \square \)

**Proof of corollary 1.** We are in the case in which no approximation error is committed, that is in the case \( r_{n,0} \leq r_n \). Let \( \rho_n \) be the lower bound on the link probabilities corresponding to \( r_n \). In that case \( \theta^* = \theta_0 \) and hence \( \mathcal{E}(\theta^*) = 0 \). By increasing \( r_{n,0} \) if needed, we may assume without loss of generality that \( r_{n,0} = r_n \) and \( \rho_{n,0} = \rho_n \). Also, the definition of \( \eta \) may be simplified. Looking back at the derivation of \( K_n \) in (21) we see that for \( a_0 = f_0((X_{ij}^T, 1, Z_{ij}^T)^T) \) and \( a = f_\beta((X_{ij}^T, 1, Z_{ij}^T)^T) \), we have \( |a_0|, |a| \leq r_{n,0} \).

Thus, for the intermediate point \( \tilde{a} \) between \( a_0 \) and \( a \) we must also have \( |\tilde{a}| \leq r_{n,0} \) and we may use that upper bound in (21) instead of \( |\tilde{a}| \leq |a_0| + \eta \). \( K_n \) then simplifies to
\[
K_n = 2\left(1 + \exp(r_{n,0})\right)^2 = 2\left(1 + \exp(-\text{logit}(\rho_{n,0}))\right)^2 \leq \frac{2}{\rho_{n,0}}.
\]

Thus, under the conditions of theorem 1, we have with high probability
\[
\mathcal{E}(\hat{\theta}) + \hat{\lambda}\left(\frac{\sqrt{2}}{\sqrt{n}}\|\hat{\beta} - \beta^*\|_1 + |\hat{\mu} - \mu^*| + \|\hat{\gamma} - \gamma^*\|_1\right) \leq C\frac{s_*\lambda^2}{\rho_{n,0}}.
\]
with constant $C = 64/c_{\min}$.

Proof of theorem 2. Follows immediately from corollary 4 and corollary 5.

C Proof of theorem 4

C.1 Inverting population and sample Gram matrices

Recall that by Assumption 1, the minimum eigenvalue $\lambda_{\min}$ of $\frac{1}{n} \mathbb{E}[Z^T Z]$ stays uniformly bounded away from zero for all $n$. Consequently the minimum eigenvalue of $\frac{1}{n} \mathbb{E}[D_\theta^T D_\theta]$ is lower bounded by $1 \wedge \lambda_{\min}$ which is bounded away from zero uniformly for all $n$. We now show that under Assumption 1, with high probability the minimum eigenvalue of $\frac{1}{(2)} D_\theta^T D_\theta$ is bounded away from zero. More precisely, recall the definition of $\kappa(A, m)$ for square matrices $A$ and dimensions $m$. We want to consider the expression

$$\kappa^2 \left( \frac{1}{(2)} \mathbb{E}[D_\theta^T D_\theta], p + 1 \right)$$

which simplifies to

$$\kappa^2 \left( \frac{1}{(2)} \mathbb{E}[D_\theta^T D_\theta], p + 1 \right) := \min_{v \in \mathbb{R}^{p+1}(0)} \frac{v^T \frac{1}{(2)} \mathbb{E}[D_\theta^T D_\theta] v}{\|v\|_2^2} \geq (1 \wedge \lambda_{\min}).$$

and compare it to $\kappa^2 \left( \frac{1}{(2)} D_\theta^T D_\theta, p + 1 \right)$. By Assumption 1 and the argument above, we have

$$\kappa^2 \left( \frac{1}{(2)} \mathbb{E}[D_\theta^T D_\theta], p + 1 \right) \geq C > 0$$

for a constant $C$ independent of $n$. With $\delta = \max_{k,l} \left| \left( \frac{1}{(2)} D_\theta^T D_\theta \right)_{kl} - \left( \frac{1}{(2)} \mathbb{E}[D_\theta^T D_\theta] \right)_{kl} \right|$, by Lemma 2, we have

$$\kappa^2 \left( \frac{1}{(2)} D_\theta^T D_\theta, p + 1 \right) \geq \kappa^2 \left( \frac{1}{(2)} \mathbb{E}[D_\theta^T D_\theta], p + 1 \right) - 16\delta(p + 1).$$

By looking at the proof of Lemma 2, we see that in this particular case we do not even need the factor $16(p + 1)$ on the right hand side above, but this does not matter anyways, so we keep it. We notice that

Lemma 8.

$$\delta = \max_{k,l} \left| \left( \frac{1}{(2)} D_\theta^T D_\theta \right)_{kl} - \left( \frac{1}{(2)} \mathbb{E}[D_\theta^T D_\theta] \right)_{kl} \right| = O_P \left( \left( \frac{n}{2} \right)^{-1/2} \right).$$

Proof. To make referencing submatrices of $1/(n) D_\theta^T D_\theta$ and its expectation easier, write

$$B := \frac{1}{(2)} D_\theta^T D_\theta = \frac{1}{(2)} \begin{bmatrix} 1^T 1 \\ Z^T Z \end{bmatrix}, \quad A := \frac{1}{(2)} \mathbb{E}[D_\theta^T D_\theta] = \frac{1}{(2)} \begin{bmatrix} 1^T 1 \\ 0 \\ \mathbb{E}[Z^T Z] \end{bmatrix}$$

where we have chosen our numbering to be consistent with the notation used in the proof of Proposition 1. The matrices $A$ and $B$ are equal in block $\circled{5}$. For $i,j$ corresponding to the blocks $\circled{6}$ and $\circled{8}$, $B_{ij} - A_{ij} = B_{ij}$ is the sum of all the entries of some column $Z_k$ of the matrix $Z$ for an appropriate $k$. That is, there is a $1 \leq k \leq p$ such that

$$B_{ij} - A_{ij} = \frac{1}{(2)} Z_k^T 1 = \frac{1}{(2)} \sum_{k \leq t} Z_{k,t}.$$
Note, that thus by model assumption \( \mathbb{E}[B_{ij} - A_{ij}] = 0 \). We know that for each \( k, s, t : Z_{k,st} \in [-c, c] \). Hence, by Hoeffding’s inequality, for all \( \eta > 0 \),
\[
P (|B_{ij} - A_{ij}| \geq \eta) = P \left( \sum_{s < t} Z_{k,st} \geq \left( \frac{n}{2} \right) \eta \right) \leq 2 \exp \left( -\frac{2 \eta^2}{\sum_{i < j} (2c)^2} \right) \leq 2 \exp \left( -\frac{n \eta^2}{2c^2} \right).
\]
For \( i, j \) from block \( \mathcal{B} \), a typical element has the form
\[
B_{ij} - A_{ij} = \frac{1}{(\frac{n}{2})^2} \sum_{s < t} (Z_{k,st}Z_{l,st} - \mathbb{E}[Z_{k,st}Z_{l,st}]),
\]
for appropriate \( k, l \). In other words, \( B_{ij} - A_{ij} \) is the inner product of two columns of \( Z \), minus their expectation, scaled by \( 1/(\frac{n}{2})^2 \). Since \( Z_{k,st}Z_{l,st} \in [-c^2, c^2] \) for all \( k, l, s, t \), we have that for all \( k, l, s, t \) :
\[
Z_{k,st}Z_{l,st} - \mathbb{E}[Z_{k,st}Z_{l,st}] \in [-2c^2, 2c^2].
\]
Thus, by Hoeffding’s inequality, for all \( \eta > 0 \),
\[
P (|B_{ij} - A_{ij}| \geq \eta) = P \left( \sum_{s < t} (Z_{k,st}Z_{l,st} - \mathbb{E}[Z_{k,st}Z_{l,st}]) \geq \left( \frac{n}{2} \right) \eta \right) \leq 2 \exp \left( -\frac{n \eta^2}{8c^4} \right).
\]
Thus, with \( \bar{c} = c^2 \vee (2c^4) \), we have for any entry in blocks \( \mathcal{B}, \mathcal{S}, \mathcal{B} \), that for any \( \eta > 0 \),
\[
P (|B_{ij} - A_{ij}| \geq \eta) \leq 2 \exp \left( -\frac{n \eta^2}{2\bar{c}} \right).
\]
The claim will follow from a union bound: Because block \( \mathcal{B} \) is the transpose of block \( \mathcal{S} \), it is sufficient to control one of them. By symmetry of block \( \mathcal{B} \) it suffices to control the upper triangular half, including the diagonal, of block \( \mathcal{B} \). Thus, we only need to control the entries \( B_{ij} - A_{ij} \) for \( i, j \) in the following index set
\[
A = \{(i, j) : i, j \text{ belong to block } \mathcal{S} \text{ or the upper triangular half or the diagonal of block } \mathcal{B} \}.
\]
Keep in mind that block \( \mathcal{S} \) has \( p \) elements, while the upper triangular part of block \( \mathcal{B} \) plus its diagonal has \( (\frac{p}{2}) + p = \left( \frac{p^2}{2} + 1 \right) \) elements. Thus, for any \( \eta > 0 \),
\[
P \left( \max_{i,j} |B_{ij} - A_{ij}| \geq \eta \right) \leq \sum_{(i,j) \in A} P (|B_{ij} - A_{ij}| \geq \eta) \\
\leq 2p \exp \left( -\frac{n \eta^2}{2c^4} \right) + 2 \left( \frac{p+1}{2} \right) \exp \left( -\frac{n \eta^2}{8c^4} \right) \\
\leq 2 \left( p + \left( \frac{p+1}{2} \right) \right) \exp \left( -\frac{n \eta^2}{2\bar{c}} \right) \\
= p(p+3) \exp \left( -\frac{n \eta^2}{2\bar{c}} \right).
\]
This proves the claim. \( \square \)

Thus, for \( n \) large enough, we have with high probability \( \delta \leq \frac{(1 \wedge \lambda_{\text{min}})}{32(p+1)} \). Then, by Lemma 2, with high probability and uniformly in \( n \),
\[
\kappa^2 \left( \frac{1}{(\frac{n}{2})^2} D^T \vartheta D \vartheta, p + 1 \right) \geq \kappa^2 \left( \frac{1}{(\frac{n}{2})^2} \mathbb{E}[D^T \vartheta D \vartheta], p + 1 \right) - 16 \delta (p+1) \geq \frac{(1 \wedge \lambda_{\text{min}})}{2} \geq C > 0.
\]
Yet, if \( \kappa^2 \left( \frac{1}{(\frac{n}{2})^2} D^T \vartheta D \vartheta, p + 1 \right) \geq C > 0 \) uniformly in \( n \), then for any \( v \neq 0, v^T \frac{1}{(\frac{n}{2})^2} D^T \vartheta D \vartheta v \geq C \|v\|^2 \). But we also know that the minimum eigenvalue of \( \frac{1}{(\frac{n}{2})^2} D^T \vartheta D \vartheta \) is the largest possible \( C \) such that this bound holds (it is actually tight with equality for the eigenvectors corresponding to the minimum eigenvalue).
Therefore, with high probability, the minimum eigenvalue of $\frac{1}{(\alpha^2)} D_\theta^T D_\theta$ stays uniformly bounded away from zero. Thus, for any $v \in \mathbb{R}^{p+1}\backslash\{0\}$ and any finite $n$:

$$
\frac{1}{(\alpha^2)} v^T D_\theta^T \hat{W}^2 D_\theta v \geq \min_{i \neq j} \{p_{ij}(\hat{\theta})(1 - p_{ij}(\hat{\theta}))\} \left( v^T \frac{1}{(\alpha^2)} D_\theta^T D_\theta v \right) \geq C \rho_n \|v\|_2^2 > 0.
$$

Thus, $\lambda_{\text{min}} \left( \frac{1}{(\alpha^2)} D_\theta^T \hat{W}^2 D_\theta \right) \geq C \rho_n \lambda_{\text{min}} \left( \frac{1}{(\alpha^2)} D_\theta^T D_\theta \right) > 0$. That means, for every finite $n$, $\frac{1}{(\alpha^2)} D_\theta^T \hat{W}^2 D_\theta$ is invertible with high probability.

### C.2 Goal and approach

**Goal:** We want to show that for $k = 1, \ldots, p+1$,

$$
\sqrt{\frac{n}{2}} \frac{\hat{\theta}_k - \theta_0,k}{\sqrt{\hat{\Theta}_{\theta,k,k}} \rightarrow \mathcal{N}(0,1)}.
$$

**Approach:** Recall the definition of the "one-sample-version" of $\mathcal{L}$, i.e. $l_\theta : \{0,1\} \times \mathbb{R}^{n+1+p} \rightarrow \mathbb{R}$, for $\theta = (\beta^T, \mu, \gamma^T)^T \in \Theta$,

$$
l_\theta(y, x) := -y \theta^T x + \log(1 + \exp(\theta^T x)).
$$

Then, the negative log-likelihood is given by

$$
\mathcal{L}(\theta) = \sum_{i < j} l_\theta(A_{ij}, (X_{ij}^T, 1, Z_{ij}^T)^T)
$$

and

$$
\nabla \mathcal{L}(\theta) = \sum_{i < j} \nabla l_\theta(A_{ij}, (X_{ij}^T, 1, Z_{ij}^T)^T), \quad H \mathcal{L}(\theta) = \sum_{i < j} H l_\theta(A_{ij}, (X_{ij}^T, 1, Z_{ij}^T)^T),
$$

where $H$ denotes the Hessian with respect to $\theta$. Consider $l_\theta$ as a function in $\theta^T x$ and introduce:

$$
l(y, a) := -ya + \log(1 + \exp(a)), \quad (31)
$$

with second derivative: $\hat{l}(y, a) = \partial_a l(y, a) = \frac{\exp(a)}{(1 + \exp(a))^2}$. Note, that $\partial_a l(y, a)$ is Lipschitz continuous (it has bounded derivative $|\partial_a l(y, a)| \leq 1/(6\sqrt{3})$; Lipschitz continuity then follows by the Mean Value Theorem). Doing a first order Taylor expansion in $a$ of $\hat{l}(y, a) = \partial_a l(y, a)$ in the point $(A_{ij}, D_{ij}^T \theta_0)$ evaluated at $(A_{ij}, D_{ij}^T \hat{\theta})$, we get

$$
\partial_a l(A_{ij}, D_{ij}^T \hat{\theta}) = \partial_a l(A_{ij}, D_{ij}^T \theta_0) + \partial_a l(A_{ij}, \alpha) D_{ij}^T (\hat{\theta} - \theta_0), \quad (32)
$$

for an $\alpha$ between $D_{ij}^T \hat{\theta}$ and $D_{ij}^T \theta_0$. By Lipschitz continuity of $\partial_a l$, we also find

$$
|\partial_a l(A_{ij}, \alpha) D_{ij}^T (\hat{\theta} - \theta_0) - \partial_a l(A_{ij}, D_{ij}^T \hat{\theta}) D_{ij}^T (\hat{\theta} - \theta_0)| \leq |\alpha - D_{ij}^T \hat{\theta}| D_{ij}^T (\hat{\theta} - \theta_0)| \leq |D_{ij}^T (\hat{\theta} - \theta_0)|^2, \quad (33)
$$

where the last inequality follows, because $\alpha$ is between $D_{ij}^T \hat{\theta}$ and $D_{ij}^T \theta_0$.
Consider the vector $P_n \nabla l_{\theta}$: By equation (32), with $\alpha_{ij}$ between $D^T_{ij} \hat{\theta}$ and $D^T_{ij} \theta_0$,

$$P_n \nabla l_{\theta} = \frac{1}{(\eta/2)} \sum_{i<j} \left( \partial_{\theta_k} l(A_{ij}, D^T_{ij} \hat{\theta}) \right)_{k=1,\ldots,n+1+p},$$

as a $(n + 1 + p) \times 1$-vector

$$= \frac{1}{(\eta/2)} \sum_{i<j} \hat{\theta}(A_{ij}, D^T_{ij} \hat{\theta}) D_{ij}$$

$$= \frac{1}{(\eta/2)} \sum_{i<j} (\hat{\theta}(A_{ij}, D^T_{ij} \theta_0) + \hat{\theta}(A_{ij}, \alpha_{ij}) D^T_{ij}(\hat{\theta} - \theta_0)) D_{ij}$$

which by (33) gives

$$= P_n \nabla l_{\theta_0} + \frac{1}{(\eta/2)} \sum_{i<j} D_{ij} \left\{ \hat{\theta}(A_{ij}, D^T_{ij} \hat{\theta}) D^T_{ij}(\hat{\theta} - \theta_0) + O(|D^T_{ij}(\hat{\theta} - \theta_0)|^2) \right\}.$$

Noticing that $\hat{\theta}(A_{ij}, D^T_{ij} \hat{\theta}) = p_{ij}(\hat{\theta})(1 - p_{ij}(\hat{\theta}))$ and we thus have $\sum_{i<j} \hat{\theta}(A_{ij}, D^T_{ij} \hat{\theta}) D_{ij} D^T_{ij}(\hat{\theta} - \theta_0) = D^T \hat{W}^2 D(\hat{\theta} - \theta_0)$:

$$= P_n \nabla l_{\theta_0} + P_n H l_{\theta}(\hat{\theta} - \theta_0) + O \left( \frac{1}{(\eta/2)} \sum_{i<j} D_{ij} |D^T_{ij}(\hat{\theta} - \theta_0)|^2 \right)$$

$$= P_n \nabla l_{\theta_0} + \frac{1}{(\eta/2)} D^T \hat{W}^2 D(\hat{\theta} - \theta_0) + O \left( \frac{1}{(\eta/2)} \sum_{i<j} D_{ij} |D^T_{ij}(\hat{\theta} - \theta_0)|^2 \right),$$

where the $O$ notation is to be understood componentwise. Above, we have equality of two $((n+1+p) \times 1)$-vectors. We are only interested in the portion relating to $\theta = (\mu, \gamma)^T$, that is, in the last $p + 1$ entries. Introduce the $((n + 1 + p) \times (n + 1 + p))$-matrix

$$M = \begin{pmatrix} 0 & 0 \\ 0 & \hat{\Theta}_\theta \end{pmatrix},$$

where 0 are zero-matrices of appropriate dimensions. Multiplying the above with $M$ on both sides gives:

$$M P_n \nabla l_{\theta} = M P_n \nabla l_{\theta_0} + M \frac{1}{(\eta/2)} D^T \hat{W}^2 D(\hat{\theta} - \theta_0) + MO \left( \frac{1}{(\eta/2)} \sum_{i<j} D_{ij} |D^T_{ij}(\hat{\theta} - \theta_0)|^2 \right).$$

Let us consider these terms in turn: Multiplication by $M$ means that the first $n$ entries of any of the vectors above are zero. Hence we only need to consider the last $p + 1$ entries. The left-hand side of (34) is equal to zero by (14). The last $p + 1$ entries of the first term on the right-hand side are $\hat{\Theta}_\theta P_n \nabla l_{\theta_0}$. For the second term on the right-hand side, notice that

$$\frac{1}{(\eta/2)} D^T \hat{W}^2 D = \frac{1}{(\eta/2)} \begin{bmatrix} X^T \hat{W}^2 X & X^T \hat{W}^2 1 \\ 1^T \hat{W}^2 X & 1^T \hat{W}^2 1 \\ Z^T \hat{W}^2 X & Z^T \hat{W}^2 1 \end{bmatrix}.$$

$\hat{\Theta}_\theta$ is the exact inverse of $\hat{\Sigma}_\theta$ which is the lower-right $(p + 1) \times (p + 1)$ block of above matrix. Thus,

$$M \frac{1}{(\eta/2)} D^T \hat{W}^2 D = \begin{bmatrix} \hat{\Theta}_\theta & 0 \\ 0 & I_{(p+1)\times(p+1)} \end{bmatrix}.$$

Then, for the last $p + 1$ entries of $M \frac{1}{(\eta/2)} D^T \hat{W}^2 D(\hat{\theta} - \theta_0)$,

$$\left( M \frac{1}{(\eta/2)} D^T \hat{W}^2 D(\hat{\theta} - \theta_0) \right)_{\text{last } p + 1 \text{ entries}} = \hat{\Theta}_\theta \frac{1}{(\eta/2)} D^T \hat{W}^2 X(\hat{\beta} - \beta_0) + \left( \frac{\hat{\beta} - \beta_0}{\gamma - \gamma_0} \right).$$
Thus, (34) implies
\[ 0 = \hat{\Theta}_\theta P_n \nabla_{\gamma} \theta_0 + \hat{\Theta}_\theta \left( \frac{1}{n^2} \right) \hat{D}_\theta^T \hat{W}^2 X (\hat{\beta} - \beta_0) + \left( \frac{\hat{\mu} - \mu_0}{\gamma - \gamma_0} \right) + O \left( \frac{1}{n^2} \sum_{i<j} \left( \frac{1}{Z_{ij}} \right) |D_{ij}^T (\hat{\theta} - \theta_0)|^2 \right), \]
which is equivalent to
\[ \left( \frac{\hat{\mu} - \mu_0}{\gamma - \gamma_0} \right) = -\hat{\Theta}_\theta P_n \nabla_{\theta} \theta_0 - \hat{\Theta}_\theta \left( \frac{1}{n^2} \right) \hat{D}_\theta^T \hat{W}^2 X (\hat{\beta} - \beta_0) + O \left( \frac{1}{n^2} \sum_{i<j} \left( \frac{1}{Z_{ij}} \right) |D_{ij}^T (\hat{\theta} - \theta_0)|^2 \right). \]

Our goal is now to show that for each component $k = 1, \ldots, p + 1$,
\[ \sqrt{\left( \frac{n}{2} \right)} \frac{\hat{\theta}_k - \theta_{0,k}}{\sqrt{\hat{\Theta}_{\theta,k,k}}} \xrightarrow{d} N(0, 1), \]
as described in the Goal section. To that end, by equation (35), we now need to solve the following three problems: Writing $\hat{\Theta}_{\theta,k}$ for the $k$th row of $\hat{\Theta}_\theta$,

1. \[ \sqrt{\left( \frac{n}{2} \right)} \frac{\hat{\Theta}_{\theta,k} P_n \nabla_{\theta} \theta_0}{\sqrt{\hat{\Theta}_{\theta,k,k}}} \xrightarrow{d} N(0, 1), \]
2. \[ \frac{1}{\sqrt{\hat{\Theta}_{\theta,k,k}}} \hat{\Theta}_{\theta,k} \left( \frac{1}{n^2} \right) \hat{D}_\theta^T \hat{W}^2 X (\hat{\beta} - \beta_0) = o_P \left( \left( \frac{n}{2} \right)^{-1/2} \right). \]
3. \[ O \left( \frac{1}{\sqrt{\hat{\Theta}_{\theta,k,k}}} \hat{\Theta}_{\theta,k} \left( \frac{1}{n^2} \right) \sum_{i<j} \left( \frac{1}{Z_{ij}} \right) |D_{ij}^T (\hat{\theta} - \theta_0)|^2 \right) = o_P \left( \left( \frac{n}{2} \right)^{-1/2} \right). \]

C.3 Bounding inverses

The problems (1) - (3) above suggest that it will be essential to bound the norm and the distance of $\hat{\Theta}_\theta$ and $\Theta_\theta$ in an appropriate manner. Notice that for any invertible matrices $A, B \in \mathbb{R}^{m \times m}$ we have
\[ A^{-1} - B^{-1} = (A^{-1} - I) (B - A), \]
Thus, for any sub-multiplicative matrix norm $\| \cdot \|$, we get
\[ \| A^{-1} - B^{-1} \| \leq \| A^{-1} \| \| B^{-1} \| \| B - A \|. \]

We are particularly interested in the matrix $\infty$-norm, defined as
\[ \| A \|_{\infty} := \sup \left\{ \frac{\| Ax \|_\infty}{\| x \|_\infty, \| x \| \neq 0} \right\} = \sup \left\{ \| Ax \|_\infty, \| x \|_\infty = 1 \right\} = \max_{1 \leq i \leq m} \sum_{j=1}^{m} |A_{i,j}|, \]
i.e. $\| A \|_{\infty}$ is the maximal row $\ell_1$-norm of $A$. It is well-known, that any such matrix norm induced by a vector norm is sub-multiplicative ($\| AB \|_{\infty} \leq \| A \|_{\infty} \| B \|_{\infty}$) and consistent with the inducing vector norm ($\| Ax \|_{\infty} \leq \| A \|_{\infty} \| x \|_{\infty}$ for any vector $x$ of appropriate dimension). We first want to bound the matrix $\infty$-norm in terms of the largest eigenvalue.

**Lemma 9.** For any symmetric, positive semi-definite $(m \times m)$-matrix $A$ with maximal eigenvalue $\lambda > 0$, we have $\| A \|_{\infty} \leq \sqrt{m} \lambda$. 
Proof.

\[ \| A \|_\infty = \sup \{ \| Ax \|_\infty, \| x \|_\infty = 1 \} \]
\[ \leq \sup \{ \| Ax \|_2, \| x \|_\infty = 1 \}, \quad \| Ax \|_\infty \leq \| Ax \|_2 \]
\[ = \sup \left\{ \frac{\| Ax \|_2}{\| x \|_2}, \| x \|_\infty = 1 \right\} \]
\[ \leq \sqrt{m} \sup \left\{ \frac{\| Ax \|_2}{\| x \|_2}, \| x \|_\infty = 1 \right\}, \quad \text{if} \quad \| x \|_\infty = 1, \then \| x \|_2 \leq \sqrt{m}, \]
\[ \leq \sqrt{m} \sup \left\{ \frac{\| Ax \|_2}{\| x \|_2}, x \neq 0 \right\} \]
\[ = \sqrt{m} \| A \|_2 = \sqrt{m} \lambda, \]

where \( \| A \|_2 \) is the spectral norm of the matrix \( A \) and we have used that for symmetric matrices, the spectral norm is equal to the modulus of the largest eigenvalue of \( A \).

Also, recall that the inverse of a symmetric matrix \( A \) is itself symmetric:

\[ I = AA^{-1} = A^T A^{-1 \text{ transpose}} \rightarrow I = (A^{-1})^T A^T \quad \text{symmetry} \quad (A^{-1})^T A = A^{-1}. \]

Hence, \( \hat{\Theta}_\theta \) and \( \Theta_\theta \) are symmetric and we may apply Lemma 9. Using that \( \lambda_{\text{max}}(\Sigma_\theta^{-1}) = \frac{1}{\lambda_{\text{min}}(\Sigma_\theta)} \), we get

\[ \| \Theta_\theta \|_\infty \leq \sqrt{p} \cdot \lambda_{\text{max}}(\Sigma_\theta^{-1}) \leq C \frac{1}{\rho_n}, \]

and with high probability

\[ \| \hat{\Theta}_\theta \|_\infty \leq \sqrt{p} \cdot \lambda_{\text{max}}(\hat{\Sigma}_\theta^{-1}) \leq C \frac{1}{\rho_n}, \]

with some absolute constant \( C \). Finally, by (36),

\[ \| \hat{\Theta}_\theta - \Theta_\theta \|_\infty \leq \| \hat{\Theta}_\theta \|_\infty \| \Theta_\theta \|_\infty \| \Sigma_\theta - \Sigma_\theta \|_\infty \leq \frac{C}{\rho_n^2} \| \Sigma_\theta - \Sigma_\theta \|_\infty. \]

It remains to control \( \| \hat{\Sigma}_\theta - \Sigma_\theta \|_\infty \). We have

\[ \hat{\Sigma}_\theta - \Sigma_\theta = \frac{1}{(\frac{n}{2})} \left( D_\theta^T \hat{W}^2 D_\theta - \mathbb{E}[D_\theta^T W_\theta^2 D_\theta] \right) \]
\[ = \frac{1}{(\frac{n}{2})} \left( D_\theta^T \hat{W}^2 - W_\theta^2 \right) D_\theta + \frac{1}{(\frac{n}{2})} \left( D_\theta^T W_\theta^2 D_\theta - \mathbb{E}[D_\theta^T W_\theta^2 D_\theta] \right). \]

Recall that \( \hat{w}_{ij}^2 = p_{ij}(\hat{\theta})(1 - p_{ij}(\hat{\theta})) = \frac{\exp(D_{ij}^T \hat{\theta})}{(1 + \exp(D_{ij}^T \hat{\theta}))^2} = \partial_{\theta} \cdot (A_{ij}, D_{ij}^T \hat{\theta}), \) with the function \( l \) defined in (31). Also recall that \( \partial_{\theta} \cdot l \) is Lipschitz with constant one, by the Mean Value Theorem and the fact that
it has derivative \( \partial_{a\ell} l \) bounded by one. Thus, considering the \((k, l)\)-th element of \((I)\) above, we get:

\[
\left| \frac{1}{n} \left( D_{ij}^2 W^2 - W_{ij}^2 \right) D_{ij} \right| = \left| \frac{1}{n} \sum_{i<j} D_{ij,n+k} D_{ij,n+l} (\hat{w}_{ij}^2 - w_{0,ij}^2) \right| \\
\leq C \frac{1}{n} \sum_{i<j} |\hat{w}_{ij}^2 - w_{0,ij}^2|, \text{ by uniform boundedness of } Z_{ij} \\
\leq C \frac{1}{n} \sum_{i<j} |D_{ij}^2 (\theta - \theta_0)|, \text{ by Lipschitz continuity} \\
\leq C \frac{n}{n} \sum_{i<j} \left\{ |\beta_i - \beta_0,i| + |\beta_j - \beta_0,j| + |\hat{\mu} - \mu_0| + |Z_{ij}^T (\gamma - \gamma_0)| \right\} \\
\leq C \left\{ \sum_{i<j} |\beta_i - \beta_0,i| + |\beta_j - \beta_0,j| \right\} + C |\hat{\mu} - \mu_0| + C \|\gamma - \gamma_0\|_1 \\
= (n-1)\|\hat{\theta} - \theta_0\|_1 \\
\leq C \left\{ \frac{1}{n} \|\hat{\theta} - \theta_0\|_1 + |\hat{\mu} - \mu_0| + \|\gamma - \gamma_0\|_1 \right\} \\
= O_P \left( s^* \sqrt{\frac{\log(n)}{(n/2)^2} \rho_n^{-1}} \right), \text{ under the conditions of theorem 1.}
\]

Since the dimension of \((I)\) is \((p + 1) \times (p + 1)\) and thus remains fixed, any row of \((I)\) has \(\ell_1\) norm of order \(O_P \left( s^* \sqrt{\frac{\log(n)}{(n/2)^2} \rho_n^{-1}} \right)\) and thus

\[
\| (I) \|_\infty = O_P \left( s^* \sqrt{\frac{\log(n)}{(n/2)^2} \rho_n^{-1}} \right).
\]

Taking a look at the \((k, l)\)-th element in \((II)\):

\[
\left| \frac{1}{n} \left( D_{ij}^2 W^2 - \mathbb{E}[D_{ij}^2 W_{ij}^2 D_{ij}] \right) D_{ij} \right| = \left| \frac{1}{n} \sum_{i<j} \left\{ D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2 - \mathbb{E}[D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2] \right\} \right|.
\]

Note that the random variables \(D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2\) are bounded uniformly in \(i, j, k, l\). Thus, by Hoeffding’s inequality, for any \(t \geq 0\),

\[
P \left( \left| \frac{1}{n} \sum_{i<j} \left\{ D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2 - \mathbb{E}[D_{ij,n+k} D_{ij,n+l} w_{0,ij}^2] \right\} \right| \geq t \right) \leq 2 \exp \left( -C \left( \frac{n}{2} \right)^2 \right).
\]

This means, \(\frac{1}{n} \left( D_{ij}^2 W_{ij}^2 D_{ij} - \mathbb{E}[D_{ij}^2 W_{ij}^2 D_{ij}] \right) \) = \(O_P \left( \frac{n}{2} \right)^{-1/2}\). Again, since the dimension \(p + 1\) is fixed, we get by a simple union bound

\[
\| (II) \|_\infty = O_P \left( \frac{n}{2} \right)^{-1/2}.
\]

In total, we thus get

\[
\| \hat{\Sigma} - \Sigma_\theta \|_\infty = O_P \left( s^* \sqrt{\frac{\log(n)}{(n/2)^2} \rho_n^{-1}} + \frac{1}{\sqrt{(n/2)}} \right) = O_P \left( s^* \sqrt{\frac{\log(n)}{(n/2)^2} \rho_n^{-1}} \right).
\]

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We can now obtain a rate for $\|\hat{\Theta}_\theta - \Theta_\theta\|_\infty$.

$$\|\hat{\Theta}_\theta - \Theta_\theta\|_\infty \leq \frac{C}{\rho_n^2} \|\Sigma_\theta - \Sigma_\theta\|_\infty = O_P \left( s_+ \sqrt{\frac{\log(n)}{n}} \rho_n^{-3} \right).$$

By Assumption 5, we have $s_+ \sqrt{\frac{\log(n)}{n}} \rho_n \to 0$, $n \to \infty$, which in particular also implies that the above is $o_P(1)$. Notice in particular, that we have now managed to get for $k = 1, \ldots, p + 1,$

- $\|\hat{\Theta}_{\theta,k} - \Theta_{\theta,k}\|_1 = o_P(1),$
- $\hat{\Theta}_{\theta,k,k} = \Theta_{\theta,k,k} + o_P(1)$.

### C.4 Problem 1

We can now take a look at the problems (1) - (3) outlined above. For problem (1), we want to show:

$$\sqrt{\left( \frac{n}{2} \right)} \frac{\hat{\Theta}_{\theta,k} P_n \nabla \theta_{\theta_0}}{\sqrt{\hat{\Theta}_{\theta,k,k}}} \to N(0, 1).$$

**Step 1:** Show that

$$\hat{\Theta}_{\theta,k} P_n \nabla \theta_{\theta_0} = \Theta_{\theta,k} P_n \nabla \theta_{\theta_0} + o_P \left( \frac{n}{2} \right)^{-1/2}. \quad (37)$$

We have

$$|\hat{\Theta}_{\theta,k} - \Theta_{\theta,k}| P_n \nabla \theta_{\theta_0} | \leq \|\hat{\Theta}_{\theta,k} - \Theta_{\theta,k}\|_1 \left\| \frac{1}{2} \sum_{i<j} \left( \frac{1}{2} Z_{ij} \right) (p_{ij}(\theta_0) - A_{ij}) \right\|_\infty \leq \left\| \hat{\Theta}_\theta - \Theta_\theta \right\|_\infty \left\| \frac{1}{2} \sum_{i<j} D_{\theta,ij} (p_{ij}(\theta_0) - A_{ij}) \right\|_\infty.$$ 

Consider the vector $\sum_{i<j} D_{\theta,ij} (p_{ij}(\theta_0) - A_{ij}) \in \mathbb{R}^{p+1}$. The $k$th component of it has the form $\sum_{i<j} (p_{ij}(\theta_0) - A_{ij})$ for $k = 1$ and $\sum_{i<j} Z_{ij,k=1} (p_{ij}(\theta_0) - A_{ij})$, $k = 2, \ldots, p + 1$. Notice that these components are all centered:

$$\mathbb{E}[D_{\theta,ij,k}(p_{ij}(\theta_0) - A_{ij})] = \mathbb{E}[D_{\theta,ij,k} \mathbb{E}(p_{ij}(\theta_0) - A_{ij}) | Z_{ij}] = \mathbb{E}[D_{\theta,ij,k} \cdot 0] = 0,$$

as well as $|D_{\theta,ij,k}(p_{ij}(\theta_0) - A_{ij})| \leq c$, where $c > 1$ is a universal constant bounding $|Z_{ij,k}|$ for all $i, j, k$. Thus, by Hoeffding’s inequality, for any $t > 0$,

$$P \left( \left| \frac{1}{2} \sum_{i<j} D_{\theta,ij,k} (p_{ij}(\theta_0) - A_{ij}) \right| \geq t \right) \leq 2 \exp \left( - \frac{2 \left( \frac{n}{2} \right) t^2}{c^2} \right)$$

and thus,

$$\frac{1}{2} \sum_{i<j} D_{\theta,ij} (p_{ij}(\theta_0) - A_{ij}) = O_P \left( \frac{n}{2} \right)^{-1/2}.$$ 

Since we have $\|\hat{\Theta}_\theta - \Theta_\theta\|_\infty = o_P(1)$, by Section C.3, step 1 is now concluded.

**Step 2:** Show that

$$\hat{\Theta}_{\theta,k,k} = \Theta_{\theta,k,k} + o_P(1).$$
Since $\|\hat{\Theta}_\vartheta - \Theta_\vartheta\|_\infty = o_P(1)$, by Section C.3, for all $k$

$$|\hat{\Theta}_{\vartheta,k,k} - \Theta_{\vartheta,k,k}| \leq \|\hat{\Theta}_\vartheta - \Theta_\vartheta\|_\infty = o_P(1)$$

and step 2 is concluded.

**Step 3:** Show that

$$\left| \frac{1}{\Theta_{\vartheta,k,k}} \right| \leq C < \infty,$$

for some universal constant $C > 0$. Then, we may conclude from step 1 and step 2 that

$$\sqrt{\frac{n}{2}} \frac{\hat{\Theta}_{\vartheta,k}P_n \nabla_\vartheta \theta_0}{\sqrt{\Theta_{\vartheta,k,k}}} = \sqrt{\frac{n}{2}} \frac{\Theta_{\vartheta,k}P_n \nabla_\vartheta \theta_0}{\sqrt{\Theta_{\vartheta,k,k}}} + o_P(1).$$

To prove step 3, notice that $\Theta_\vartheta$ is symmetric and hence has only real eigenvalues. Therefore it is unitarily diagonalizable and for any $x \in \mathbb{R}^{p+1}$, we have $x^T \Theta_\vartheta x \geq \lambda_{\min}(\Theta_\vartheta) \|x\|_2^2$. We also know that

$$\lambda_{\min}(\Theta_\vartheta) = \frac{1}{\lambda_{\max}(\Sigma_\vartheta)}.$$

Under Assumption 1 we can now deduce an upper bound on the maximum eigenvalue of $\Sigma_\vartheta$: For any $x \in \mathbb{R}^p$,

$$x^T \Sigma_\vartheta x = x^T \frac{1}{(n/2)} E[D_\vartheta^T W_\vartheta^2 D_\vartheta] x \leq x^T \left(\frac{1}{(n/2)} E[D_\vartheta^T D_\vartheta]\right) x \leq (1 \vee \lambda_{\max}) \|x\|_2^2,$$

where used that any entry in $W_\vartheta^2$ is bounded above by one. Since $x^T \Sigma_\vartheta x \leq \lambda_{\max}(\Sigma_\vartheta) \|x\|_2^2$ and since this bound is tight, we can conclude by Assumption 1 that $\lambda_{\max}(\Sigma_\vartheta) \leq (1 \vee \lambda_{\max}) \leq C < \infty$ for some universal constant $C > 0$.

In particular, since $\Theta_{\vartheta,k,k} = e_k^T \Theta_\vartheta e_k$, we get

$$\Theta_{\vartheta,k,k} \geq \lambda_{\min}(\Theta_\vartheta) \|e_k\|_2^2 = \frac{1}{\lambda_{\max}(\Sigma_\vartheta)} \geq C > 0,$$

uniformly for all $n$. Consequently,

$$0 \leq \frac{1}{\Theta_{\vartheta,k,k}} \leq C < \infty.$$

Step 3 is thus concluded.

**Step 4:** Finally, show that

$$\sqrt{\frac{n}{2}} \frac{\hat{\Theta}_{\vartheta,k}P_n \nabla_\vartheta \theta_0}{\sqrt{\Theta_{\vartheta,k,k}}} \to N(0,1),$$

Such that by all the above

$$\sqrt{\frac{n}{2}} \frac{\hat{\Theta}_{\vartheta,k}P_n \nabla_\vartheta \theta_0}{\sqrt{\Theta_{\vartheta,k,k}}} \to N(0,1).$$

For brevity, we write $p_{ij}$ for the true link probabilities $p_{ij}(\theta_0)$. Also keep in mind that $\Theta_{\vartheta,k}$ denotes the $k$th row of $\Theta_\vartheta$, while $D_{\vartheta,ij}$ denote $(p + 1) \times 1$-column vectors. We want to apply the Lindeberg-Feller Central Limit Theorem. The random variables we study are the summands in

$$\sqrt{\frac{n}{2}} \Theta_{\vartheta,k}P_n \nabla_\vartheta \theta_0 = \sum_{i<j} \left\{ \frac{1}{\sqrt{\frac{n}{2}}} \Theta_{\vartheta,k} D_{\vartheta,ij}(p_{ij} - A_{ij}) \right\}.$$
First, notice that these random variables are centered:

$$E \left[ \frac{1}{\sqrt{\binom{n}{2}}} \Theta_{\theta,k} D_{\theta,ij}(p_{ij} - A_{ij}) \right] = E \left[ \frac{1}{\sqrt{\binom{n}{2}}} \Theta_{\theta,k} D_{\theta,ij} E[p_{ij} - A_{ij} | Z_{ij}] \right]$$

$$= E \left[ \frac{1}{\sqrt{\binom{n}{2}}} \Theta_{\theta,k} D_{\theta,ij} \cdot 0 \right] = 0.$$  

For the Lindeberg-Feller CLT we need to sum up the variances of these random variables. We claim that

$$\sum_{i<j} \text{Var} \left( \frac{1}{\sqrt{\binom{n}{2}}} \Theta_{\theta,k} D_{\theta,ij}(p_{ij} - A_{ij}) \right) = \Theta_{\theta,k,k}.$$  

Indeed, consider the vector-valued random variable \( \sum_{i<j} \left\{ \frac{1}{\sqrt{\binom{n}{2}}} D_{\theta,ij}(p_{ij} - A_{ij}) \right\} \in \mathbb{R}^{p+1} \). It has covariance matrix

$$E \left[ \sum_{i<j} \left\{ \frac{1}{\sqrt{\binom{n}{2}}} D_{\theta,ij}(p_{ij} - A_{ij}) \right\} \sum_{i<j} \left\{ \frac{1}{\sqrt{\binom{n}{2}}} D_{\theta,ij}(p_{ij} - A_{ij}) \right\}^T \right]$$

$$= E \left[ \sum_{i<j} \frac{1}{\sqrt{\binom{n}{2}}} D_{\theta,ij}(p_{ij} - A_{ij}) \frac{1}{\sqrt{\binom{n}{2}}} D_{\theta,ij}^T(p_{ij} - A_{ij}) \right], \quad \text{by independence across } i, j$$

$$= \frac{1}{\binom{n}{2}} \sum_{i<j} \left[ E[D_{\theta,i,j,k} D_{\theta,i,j,l}(p_{ij} - A_{ij})^2] \right]_{k,l=1,...,p+1}, \quad \text{as a } ((p+1) \times (p+1))-\text{matrix}$$

$$= \frac{1}{\binom{n}{2}} E[D_{\theta}^T W_{\theta}^2 D_{\theta}]$$

$$= \Sigma_{\theta}.$$  

Thus, by independence across \( i, j \),

$$\sum_{i<j} \text{Var} \left( \frac{1}{\sqrt{\binom{n}{2}}} \Theta_{\theta,k} D_{\theta,ij}(p_{ij} - A_{ij}) \right) = \text{Var} \left( \Theta_{\theta,k} \sum_{i<j} \frac{1}{\sqrt{\binom{n}{2}}} D_{\theta,ij}(p_{ij} - A_{ij}) \right)$$

$$= \Theta_{\theta,k} \Sigma_{\theta} \Theta_{\theta,k}^T = \Theta_{\theta,k,k},$$  

where for the last equality we have used that \( \Theta_{\theta} \) is the inverse of \( \Sigma_{\theta} \) and thus, \( \Sigma_{\theta} \Theta_{\theta,k}^T = e_k \). Now, we need to show that the Lindeberg condition holds. That is, we want that for any \( \epsilon > 0 \),

$$\lim_{n \to \infty} \frac{1}{\Theta_{\theta,k,k}} \sum_{i<j} \left[ \frac{1}{\sqrt{\binom{n}{2}}} \Theta_{\theta,k} D_{\theta,ij}(p_{ij} - A_{ij}) \right] \leq 0.$$  

We have

$$|\Theta_{\theta,k} D_{\theta,ij}(p_{ij} - A_{ij})| \leq p \cdot c \cdot \|\Theta_{\theta,k}\|_1 \leq C \|\Theta_{\theta}\|_\infty \leq C \rho_n^{-1}.$$  

At the same time, we know from step 3 that \( \Theta_{Z,k,k} \geq C > 0 \) for some universal \( C \). Then, as long as \( \rho_n^{-1} \) goes to infinity at a rate slower than \( n \), which is enforced by Assumption 5, we must have for \( n \) large enough

$$|\Theta_{\theta,k} D_{\theta,ij}(p_{ij} - A_{ij})| < \epsilon \sqrt{\binom{n}{2}} \Theta_{\theta,k,k}.$$
uniformly in \(i,j\). Thus, the indicator function and therefore each summand in (38) is equal to zero for \(n\) large enough. Hence, (38) holds. Then, by the Lindeberg-Feller CLT,
\[
\sqrt{n} \frac{\Theta_{\vartheta,k} P_n \nabla_{\varphi \theta_0}}{\sqrt{\Theta_{\vartheta,k,k}}} \xrightarrow{d} N(0,1).
\]
Now, by the steps 1-4 and Slutzky’s Theorem
\[
\sqrt{n} \frac{\tilde{\Theta}_{\vartheta,k} P_n \nabla_{\varphi \theta_0}}{\sqrt{\tilde{\Theta}_{\vartheta,k,k}}} = \sqrt{n} \frac{\Theta_{\vartheta,k} + o_P(1) P_n \nabla_{\varphi \theta_0}}{\sqrt{\Theta_{\vartheta,k,k} + o_P(1)}} = \sqrt{n} \frac{\Theta_{\vartheta,k} + o_P(1) P_n \nabla_{\varphi \theta_0}}{\sqrt{\Theta_{\vartheta,k,k} + o_P(1)}} \xrightarrow{d} N(0,1).
\]
This concludes solving problem 1.

### C.5 Problem 2

For problem 2 we must show
\[
1 \sqrt{\Theta_{\vartheta,k,k}} \tilde{\Theta}_{\vartheta,k,k} \left( \frac{1}{2} D_{\vartheta}^T \tilde{W}^2 X (\tilde{\beta} - \beta_0) \right) = o_P \left( \frac{n^{-1/2}}{2} \right).
\]
Since we have \(\|\Theta_{\vartheta} - \Theta_{\vartheta}\|_\infty = o_P(1)\), we do not need to worry about \(1 \sqrt{\Theta_{\vartheta,k,k}}\), because \(\frac{1}{\sqrt{\Theta_{\vartheta,k,k}}} = \Theta_{\vartheta,k,k} + o_P(1)\) and \(\frac{1}{\sqrt{\Theta_{\vartheta,k,k}}} \leq C < \infty\), i.e. \(\frac{1}{\sqrt{\Theta_{\vartheta,k,k}}} = O_P(1)\). By theorem 1 we also have a high-probability error bound on \(\|\hat{\beta} - \beta_0\|_1\). The problem will be bounding the corresponding matrix norms.

\[
\left| \tilde{\Theta}_{\vartheta,k,k} \left( \frac{1}{2} D_{\vartheta}^T \tilde{W}^2 X (\tilde{\beta} - \beta_0) \right) \right| \leq \left\| \left( \frac{1}{2} \right) X^T \tilde{W}^2 D_{\vartheta} \tilde{\Theta}_{\vartheta,k,k} \right\|_\infty \|\hat{\beta} - \beta_0\|_1.
\]

Notice that in the display above we have the vector \(\ell_\infty\)-norm. Also,
\[
\left\| \left( \frac{1}{2} \right) X^T \tilde{W}^2 D_{\vartheta} \tilde{\Theta}_{\vartheta,k,k} \right\|_\infty \leq \left\| \tilde{\Theta}_{\vartheta,k,k} \right\|_\infty \left\| \left( \frac{1}{2} \right) X^T \tilde{W}^2 D_{\vartheta} \right\|_\infty.
\]

Here we used the compatibility of the matrix \(\ell_\infty\)-norm with the vector \(\ell_\infty\)-norm. The first term is the vector norm, the second the matrix norm. We know,
\[
\|\tilde{\Theta}_{\vartheta,k,k}\|_\infty \leq \|\tilde{\Theta}_{\vartheta}\|_\infty \leq C \rho^{-1},
\]
where on the left hand side we have the vector norm and in the middle display the matrix norm. Finally, \(\left( \frac{1}{2} \right) X^T \tilde{W}^2 D_{\vartheta}\) is a \((n \times (p + 1))\)-matrix. The \((k,l)\)-th element looks like
\[
\left| \left( \frac{1}{2} \right) \sum_{i=1,i \neq l}^n D_{\vartheta,i,k} \tilde{W}_{i,l}^2 \right| \leq \left( \frac{1}{n} \right) \cdot (n - 1) \cdot c = \frac{C}{n}.
\]
Thus, the $\ell_1$-norm of any row of $\frac{1}{(2)} X^T \hat W^2 D_\theta$ is bounded by $C/n$ and thus

$$\left\| \frac{1}{(2)} X^T \hat W^2 D_\theta \right\|_\infty \leq \frac{C}{n}.$$  

Recall that $\|\hat \beta - \beta_0\|_1 = O_P\left(s^*_+ \sqrt{\log(n)} / \sqrt{n} \cdot \hat \rho_n^{-1}\right)$ by theorem 1. Then,

$$\left| \hat \Theta_{\hat \theta, k} \frac{1}{(2)} X^T \hat W^2 D_\theta (\hat \beta - \beta_0) \right| \leq \|\hat \Theta_{\hat \theta, k}\|_\infty \left| \frac{1}{(2)} D^T_\theta \hat W^2 X \right|_\infty \|\hat \beta - \beta_0\|_1 = O_P \left(\frac{s^*_+}{\hat \rho_n^2} \cdot \frac{\sqrt{\log(n)}}{\sqrt{n}}\right).$$

Multiplying by $\sqrt{\frac{(n)}{2}} = O(n)$, gives

$$\sqrt{\frac{(n)}{2}} \left| \hat \Theta_{\hat \theta, k} \frac{1}{(2)} D^T_\theta \hat W^2 X (\hat \beta - \beta_0) \right| = O_P \left(\frac{s^*_+}{\hat \rho_n^2} \cdot \frac{\sqrt{\log(n)}}{\sqrt{n}}\right),$$

which is $o_P(1)$ under Assumption 5.

**C.6 Problem 3**

Finally, we must show

$$O \left(\sqrt{\frac{(n)}{2}} \hat \Theta_{\hat \theta, k} \frac{1}{(2)} \sum_{i < j} \left| D_{ij} (\hat \beta - \theta_0) \right|^2 \right) = O_P \left(\left(\frac{n}{2}\right)^{-1/2}\right).$$

Again, since $\hat \Theta_{\hat \theta, k} = \hat \Theta_{\hat \theta, k} + o_P(1)$ and $\hat \Theta_{\hat \theta, k} \geq C > 0$ uniformly in $n$, we do not need to worry about the factor $\frac{1}{\sqrt{\hat \Theta_{\hat \theta, k}}}$ and it remains to show

$$O \left(\hat \Theta_{\hat \theta, k} \frac{1}{(2)} \sum_{i < j} D_{\theta, i,j} |D_{ij} (\hat \beta - \theta_0)|^2 \right) = O_P \left(\left(\frac{n}{2}\right)^{-1/2}\right).$$

We have

$$\left| \hat \Theta_{\hat \theta, k} \frac{1}{(2)} \sum_{i < j} D_{\theta, i,j} |D_{ij} (\hat \beta - \theta_0)|^2 \right| \leq \frac{1}{(2)} \left\| \hat \Theta_{\hat \theta, k} D_{\theta, i,j} \right\| \left| D^T_{ij} (\hat \beta - \theta_0) \right|^2 \leq c \|\hat \Theta_{\hat \theta, k}\|_1 \frac{1}{(2)} \left\| D^T_{ij} (\hat \beta - \theta_0) \right\|^2 \leq C \rho_n \frac{1}{(2)} \left\| D^T_{ij} (\hat \beta - \theta_0) \right\|^2,$$

where for the last inequality we have used that $\|\hat \Theta_{\hat \theta, k}\|_1 \leq \|\hat \Theta\|_\infty \leq C \frac{1}{\rho_n}$. Now remember from (29) that

$$\frac{1}{(2)} \sum_{i < j} \left| D^T_{ij} (\hat \beta - \theta_0) \right|^2 = C \|\hat \beta - \theta_0\|_1^2.$$
where we make use of the fact that \(\theta^* = \theta_0\) if there is no approximation error (as assumed by theorem 4) and that \(\bar{D}\hat{\theta} = D\theta\). From theorem 1 we know that under the assumptions of theorem 4, \(\|\hat{\theta} - \theta_0\|_1 = O_P\left(s^*_+\sqrt{\frac{\log(n)}{2}} \rho_n^{-1}\right)\). Thus,

\[
\sqrt{\frac{n}{2}} \left| \hat{\theta}_{\sigma, k} - \frac{1}{2} \sum_{i<j} D_{\sigma, ij} |D_{ij}^T(\hat{\theta} - \theta_0)^2 | = O_P\left(s^*_+n\sqrt{\frac{\log(n)}{2}} \rho_n^{-3}\right) .
\]

We see that this is \(o_P(1)\) by applying Assumption 5 twice. Problem 3 is solved.

**Proof of theorem 4.** Theorem 4 now follows from the solved problems (1) - (3). \(\square\)

### D Proofs of section 2.3

We first prove the consistency of the MLE \(\hat{\theta} = (\hat{\mu}^T, \hat{\gamma}^T)^T\) and then its asymptotic normality.

#### D.1 Consistency of \((\hat{\mu}^T, \hat{\gamma})\)

We want to find a limit for an appropriately scaled version of \(\mathcal{L}^\dagger\). To that end, we first prove a concentration result of \(d_+\) around its expectation. Consider

\[
E[d_+] = E[E[d_+ | Z]] = \sum_{i<j} E \left[ \frac{n^{-\xi} \exp(\mu_0^i) \exp(\gamma_0^T Z_{ij})}{1 + n^{-\xi} \exp(\mu_0^i) \exp(\gamma_0^T Z_{ij})} \right] = n^{-\xi} \exp(\mu_0^i) \sum_{i<j} E \left[ \frac{\exp(\gamma_0^T Z_{ij})}{1 + n^{-\xi} \exp(\mu_0^i) \exp(\gamma_0^T Z_{ij})} \right] = n^{-\xi} \exp(\mu_0^i) \frac{n}{2} E \left[ \frac{\exp(\gamma_0^T Z_{12})}{1 + n^{-\xi} \exp(\mu_0^i) \exp(\gamma_0^T Z_{12})} \right], \text{ since } Z_{ij}\text{ are i.i.d.}
\]

By the law of total variance, we may write the variance of \(d_+\) as

\[
\text{Var}(d_+) = E[\text{Var}(d_+ | Z)] + \text{Var}(E[d_+ | Z]).
\]

We have,

\[
\text{Var}(E[d_+ | Z]) = \text{Var} \left( \sum_{i<j} p_{ij} \right) = \sum_{i<j} n^{-2\xi} \text{Var} \left( \frac{\exp(\mu_0^i + \gamma_0^T Z_{ij})}{1 + n^{-\xi} \exp(\mu_0^i) \exp(\gamma_0^T Z_{ij})} \right) = O(n^{2-2\xi}).
\]

Also, by independence of the \(A_{ij}\) given \(Z\),

\[
\text{Var}(d_+ | Z) = \sum_{i<j} \text{Var}(A_{ij} | Z) = \sum_{i<j} p_{ij} (1 - p_{ij}) = O(n^{2-\xi}).
\]

Therefore,

\[
\text{Var}(d_+) = O\left(n^{2-2\xi}\right) + O(n^{2-\xi}) = O(n^{2-\xi}).
\]
By Chebychev’s inequality, for any $t > 0$,
\[ P(|d_+ - E[d_+]| \geq t) \leq \frac{\text{Var}(d_+)}{t^2}. \]

Letting $\epsilon > 0$ and picking $t = n^{2-\xi} \epsilon$, we obtain
\[ P(n^{-2+\xi}|d_+ - E[d_+]| \geq \epsilon) \leq \frac{O(n^{2-\xi})}{n^{4-2\xi}} = O(1) \to 0, \quad n \to \infty, \]
since $\xi \in [0,2)$. This implies
\[ d_+ = E[d_+] + o_P(n^{2-\xi}) = \frac{n^{2-\xi}}{2} \exp(\mu_0^T) \mathbb{E} \left[ \frac{\exp(\gamma^T Z_{12})}{1 + n^{-\xi} \exp(\mu_0^T) \exp(\gamma^T Z_{12})} \right] + o_P(n^{2-\xi}). \]

In particular, this implies
\[ 2n^{-2+\xi} d_+ \xrightarrow{P} \exp(\mu_0^T) \mathbb{E} \left[ \exp(\gamma^T Z_{12}) \right], \quad n \to \infty. \tag{39} \]

Next, we deal with the second term in $L^\dagger$:
\[
\mathbb{E} \left[ \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} \right] = \sum_{i<j} \mathbb{E} \left[ (\gamma^T Z_{ij}) \mathbb{E}[A_{ij}|Z_{ij}] \right] = \sum_{i<j} \mathbb{E} \left[ (\gamma^T Z_{ij}) p_{ij} \right]
\]
\[
= \sum_{i<j} n^{-\xi} \exp(\mu_0^T) \mathbb{E} \left[ (\gamma^T Z_{ij}) \frac{\exp(\gamma_0^T Z_{12})}{1 + n^{-\xi} \exp(\mu_0^T) \exp(\gamma_0^T Z_{12})} \right] \]
\[
= n^{-\xi} \exp(\mu_0^T) \left( \frac{n}{2} \right) \mathbb{E} \left[ (\gamma^T Z_{12}) \frac{\exp(\gamma_0^T Z_{12})}{1 + n^{-\xi} \exp(\mu_0^T) \exp(\gamma_0^T Z_{12})} \right], \quad Z_{ij} \text{ i.i.d.}
\]
where we suppress the dependence of $\tilde{\alpha}_n$ on $\gamma$ in our notation. Pay special attention to the distinction between the generic $\gamma$ and the true parameter $\gamma_0$ here. The last equality in the previous display can be written as
\[
\mathbb{E} \left[ \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} \right] = \frac{n^{2-\xi}}{2} \exp(\mu_0^T) \tilde{\alpha}_n + o(n^{2-\xi}).
\]

We use the law of total variance once more to bound $\text{Var}(\sum_{i<j} (\gamma^T Z_{ij}) A_{ij})$. For any $i,j$,
\[
\text{Var}(\gamma^T Z_{ij}) A_{ij}) = \mathbb{E}[\text{Var}(\gamma^T Z_{ij}) A_{ij}|Z] + \mathbb{V}[\mathbb{E}(\gamma^T Z_{ij}) A_{ij}|Z]).
\]
We have,
\[
\text{Var}(\mathbb{E}(\gamma^T Z_{ij}) A_{ij}|Z]) = \text{Var}(\gamma^T Z_{ij}) p_{ij}) \leq \mathbb{V}(\gamma^T Z_{ij})^2 p_{ij} \leq C n^{-2\xi}
\]
and
\[
\text{Var}(\gamma^T Z_{ij}) A_{ij}|Z]) = (\gamma^T Z_{ij})^2 p_{ij}(1-p_{ij}) \leq C n^{-\xi},
\]
where in both instances we may choose some constant $C > 0$ independent of $i,j$ and $n$. Thus,
\[
\text{Var} \left( \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} \right) \leq \sum_{i<j} C(n^{-2\xi} + n^{-\xi}) = O(n^{2-\xi}).
\]
Using Chebyshev’s inequality, we obtain for any $t > 0$,

$$P\left( \left| \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} - \mathbb{E}\left[ \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} \right] \right| \geq t \right) \leq \frac{\operatorname{Var}\left( \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} \right)}{t^2}.$$  

Letting $\epsilon > 0$ and picking $t = n^{2-\xi} \epsilon$, we obtain

$$P\left( n^{-2+\xi} \left| \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} - \mathbb{E}\left[ \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} \right] \right| \geq \epsilon \right) \leq \frac{O(n^{2-\xi})}{n^{2-\xi} \cdot n^{2-\xi}} \to 0.$$  

This implies

$$\sum_{i<j} (\gamma^T Z_{ij}) A_{ij} = \mathbb{E}\left[ \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} \right] + o_P(n^{2-\xi}) = \frac{n^{2-\xi}}{2} \exp(\mu^t) \overline{\alpha}_n + o(n^{2-\xi}).$$

Since $\overline{\alpha}_n \to \mathbb{E}[\exp(\gamma^T Z_{12})] \exp(\gamma^T Z_{12})]$ almost surely, we end up with

$$2n^{-2+\xi} \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} \overset{P}{\to} \exp(\mu^t) \mathbb{E}[\exp(\gamma^T Z_{12}) \exp(\gamma^T Z_{12})], \quad n \to \infty. \quad (40)$$

It remains to analyze the last term in $\mathcal{L}^t$, i.e. term $\sum_{i<j} \log(1 + n^{-\xi} \exp(\mu^t + \gamma^T Z_{ij}))$. Since $\log(1 + x) \leq x$ for $x > -1$:

$$\sum_{i<j} \log\left( 1 + n^{-\xi} \exp(\mu^t + \gamma^T Z_{ij}) \right) \leq n^{-\xi} \exp(\mu^t) \sum_{i<j} \exp(Z_{ij}^t) \gamma$$

$$= n^{-\xi} \exp(\mu^t) \left( \frac{n}{2} \sum_{i<j} \exp(Z_{ij}^t) \right)$$

$$= \frac{n^{2-\xi}}{2} \exp(\mu^t) \alpha_n + o(n^{2-\xi}).$$

On the other hand, we also have $x/(1 + x) \leq \log(1 + x)$ for all $x > -1$. Also recall that $|\gamma^T Z_{ij}| \leq \kappa$ almost surely. Thus,

$$\sum_{i<j} \log\left( 1 + n^{-\xi} \exp(\mu^t + \gamma^T Z_{ij}) \right) \geq n^{-\xi} \exp(\mu^t) \sum_{i<j} \frac{\exp(\gamma^T Z_{ij})}{1 + n^{-\xi} \exp(\mu^t) \exp(\gamma^T Z_{ij})}$$

$$\geq n^{-\xi} \exp(\mu^t) \frac{1}{1 + n^{-\xi} \exp(\mu^t + \kappa)} \sum_{i<j} \exp(\gamma^T Z_{ij})$$

$$= n^{-\xi} \exp(\mu^t) \frac{1}{1 + n^{-\xi} \exp(\mu^t + \kappa)} \left( \frac{n}{2} \right) \alpha_n$$

$$= \frac{n^{2-\xi}}{2} \exp(\mu^t) \frac{1}{1 + n^{-\xi} \exp(\mu^t + \kappa)} \alpha_n + o(n^{2-\xi}).$$

Notice that since the $Z_{ij}$ are i.i.d. and since $\gamma^T Z_{ij}$ is uniformly bounded,

$$\alpha_n \overset{a.s.}{\to} \mathbb{E}[\exp(\gamma^T Z_{12})].$$

We now have found an upper and a lower bound on $\sum_{i<j} \log(1 + n^{-\xi} \exp(\mu^t + \gamma^T Z_{ij}))$. Multiplying both sides with $2n^{2-\xi}$ and taking the limit $n \to \infty$, we see that both the lower as well as the upper bound converge to $\exp(\mu^t) \mathbb{E}[\exp(\gamma^T Z_{12})]$. But then this already must be the limit for
Now, for any $\epsilon, \eta > 0$, Putting equations (39), (40) and (41) together, we obtain that for any $n \rightarrow \infty$ in probability of

\[ \sup_{\theta} |2n^{-2+\xi} \mathcal{L}^\dagger(\theta) - M(\theta)| = o_P(1), \tag{43} \]

with the supremum taken over all $\theta \in [-M, M] \times \Gamma$.

To shorten notation, introduce $M_n(\theta) := 2n^{-2+\xi} \mathcal{L}^\dagger(\theta)$. Since we already have pointwise convergence in probability of $M_n$ to $M$, it will be suffice to show that for any $\epsilon > 0$

\[ \lim_{\delta \downarrow 0} \limsup_{n \rightarrow \infty} P \left( \sup_{\|\theta_1 - \theta_2\|_2 \leq \delta} \left| M_n(\theta_1) - M_n(\theta_2) \right| \geq \epsilon \right) = 0. \tag{44} \]

Property (43) then follows from the pointwise convergence, the continuity of $M$ and the compactness of the parameter space $[-M, M] \times \Gamma$. To ease notation further, define

\[ \Delta^n_\delta := \sup_{\|\theta_1 - \theta_2\|_2 \leq \delta} |M_n(\theta_1) - M_n(\theta_2)|. \]

Let $\epsilon, \eta > 0$. We have to show that there exists a $\delta > 0$ such that

\[ \limsup_{n \rightarrow \infty} P(\Delta^n_\delta \geq \epsilon) \leq \eta. \tag{45} \]

Consider the following representation of $\mathcal{L}^\dagger(\theta)$:

\[ \mathcal{L}^\dagger(\theta) = -d_+ \mu^\dagger - \sum_{i<j} (\gamma^T Z_{ij}) A_{ij} + \sum_{i<j} \log \left( 1 + n^{-\xi} \exp(\mu^\dagger + \gamma^T Z_{ij}) \right) \]

\[ = \sum_{i<j} (-\mu^\dagger + \gamma^T Z_{ij}) A_{ij} + \log \left( 1 + n^{-\xi} \exp(\mu^\dagger + \gamma^T Z_{ij}) \right) \]

\[ = \sum_{i<j} -D^T_{ij} \theta A_{ij} + \log \left( 1 + n^{-\xi} \exp(D^T_{ij} \theta) \right). \]

Now, for any $\delta > 0$ and any $\theta_1, \theta_2$ with $\|\theta_1 - \theta_2\|_2 < \delta$ and any $i < j$, we obtain:

\[ E|l_{ij}(\theta_1) - l_{ij}(\theta_2)| = E \left| -D^T_{ij}(\theta_1 - \theta_2) A_{ij} + \log \left( 1 + n^{-\xi} \exp(D^T_{ij} \theta_1) \right) - \log \left( 1 + n^{-\xi} \exp(D^T_{ij} \theta_2) \right) \right|. \]
Hence, by the Mean Value Theorem with $\alpha$ between $D_{ij}\theta_1$ and $D_{ij}\theta_2$:  
\[
\mathbb{E}[l_{ij}(\theta_1) - l_{ij}(\theta_2)] \leq \mathbb{E} \left| D_{ij}^T(\theta_1 - \theta_2)|A_{ij} \right| + \frac{n^{-\xi} \exp(\alpha)}{1 + n^{-\xi} \exp(\alpha)} \mathbb{E} \left| D_{ij}^T(\theta_1 - \theta_2) \right|
\]
\[
\leq C\|\theta_1 - \theta_2\|_2 \mathbb{E}[p_{ij}] + Cn^{-\xi}\|\theta_1 - \theta_2\|_2
\]
\[
\leq C\|\theta_1 - \theta_2\|_2 \left( \mathbb{E} \left[ n^{-\xi} \frac{\exp(D_{ij}^T\theta_0)}{1 + n^{-\xi} \exp(D_{ij}^T\theta_0)} \right] + n^{-\xi} \right)
\]
\[
\leq Cn^{-\xi}\|\theta_1 - \theta_2\|_2
\]
\[
\leq Cn^{-\xi} \delta,
\]
where $C > 0$ denotes some generic constant that may change between displays. By the compactness of our parameter space and the resulting uniform boundedness of $|D_{ij}(\theta_1 - \theta_2)|$, we may in particular choose this $C$ independent of $n, i$ and $j$. Then, almost surely,
\[
\mathbb{E}[L^1(\theta_1) - L^1(\theta_2)] \leq C \binom{n}{2} n^{-\xi} \delta
\]
and thus, almost surely,
\[
\mathbb{E}\Delta_n^\delta \leq Cn^{-2+\xi}n^{-\xi} \binom{n}{2} \delta \leq C\delta.
\]
Thus, we can choose a $\delta > 0$ independent of $n$, such that $\mathbb{E}\Delta_n^\delta \leq \epsilon\eta$. But then an application of Markov’s inequality yields for all $n$ large enough
\[
P(\Delta_n^\delta \geq \epsilon) \leq \eta.
\]
It follows (45), which implies (44), which yields (43).

The second condition of theorem 5.7 in van der Vaart (1998) requires that the true parameter be a well-separated extrema of $M$. That is, we must show: For any fixed $\epsilon > 0$,
\[
\sup_{\theta : d(\theta, \theta_0) \geq \epsilon} M(\theta) > M(\theta_0).
\]
(46)

Consider the first partial derivatives of $M$:
\[
\partial_\mu^1 M(\mu^1, \gamma) = -\exp(\mu_0^1)\mathbb{E}[\exp(\gamma^T Z_{12})] + \exp(\mu^1)\mathbb{E}[\exp(\gamma^T Z_{12})],
\]
\[
\partial_{\gamma_k} M(\mu^1, \gamma) = -\exp(\mu_0^1)\mathbb{E}[Z_{12,k} \exp(\gamma_{12}^T Z_{12})] + \exp(\mu^1)\mathbb{E}[Z_{12,k} \exp(\gamma_{12}^T Z_{12})].
\]
Clearly, by Assumption 3 the true parameter is a critical point of $M$, i.e. the first partial derivatives of $M$ evaluated at $\theta_0 = (\mu_{00}^1, \gamma_0^1)^T$ are zero:
\[
\nabla M(\theta_0) = 0.
\]

Consider the Hessian $HM(\mu^1, \gamma)$ of $M$ at the point $(\mu^1, \gamma)$:
\[
\frac{\partial^2}{\partial (\mu^1)^2} M(\mu^1, \gamma) = \exp(\mu^1)\mathbb{E}[\exp(\gamma^T Z_{12})],
\]
\[
\frac{\partial^2}{\partial \mu^1 \partial \gamma_k} M(\mu^1, \gamma) = \exp(\mu^1)\mathbb{E}[Z_{12,k} \exp(\gamma^T Z_{12})],
\]
\[
\frac{\partial^2}{\partial \gamma_k^2} M(\mu^1, \gamma) = \exp(\mu^1)\mathbb{E}[Z_{12,k}^2 \exp(\gamma^T Z_{12})],
\]
\[
\frac{\partial^2}{\partial \gamma_k \partial \gamma_l} M(\mu^1, \gamma) = \exp(\mu^1)\mathbb{E}[Z_{12,k} Z_{12,l} \exp(\gamma^T Z_{12})].
\]
We thus see that $HM(\mu^1, \gamma)$ allows a matrix representation as
\[
HM(\mu^1, \gamma) = \exp(\mu^1)\mathbb{E} \left[ \exp(\gamma^T Z_{12}) \begin{bmatrix} Z_{12} & Z_{12}^T \end{bmatrix} \right] \in \mathbb{R}^{(p+1) \times (p+1)}.
\]
By the compactness of our parameter space and the boundedness of $Z_{12}$, we now obtain for any $v \in \mathbb{R}^{p+1}$:

$$v^T HM(\mu^\dagger, \gamma) v = \exp(\mu^\dagger) \mathbb{E} \left[ \exp(\gamma^T Z_{12}) v^T D_{12} D_{12}^T v \right] \geq C \mathbb{E} \left[ v^T D_{12} D_{12}^T v \right] = C v^T \mathbb{E} \begin{bmatrix} 1 & 0 \\ 0 & Z_{12} Z_{12}^T \end{bmatrix} v \geq C \|v\|^2,$$

where for the last inequality we have used that the matrix is strictly positive definite by Assumption 4. That means, $HM(\mu^\dagger, \gamma)$ is strictly positive definite on the entire parameter space $[-M, M] \times \Gamma$. Hence, $M$ is strictly convex and its minimum $\theta_0$ already must be a global minimum. Now, since our parameter space is compact, $M$ is continuous and $\theta_0$ is a global maximum, it is easy to see that (46) must hold.

Finally, since (43) and (46) hold, we have consistency as $\theta \xrightarrow{P} \theta_0$ (van der Vaart 1998, Theorem 5.7).

### D.2 Asymptotic normality

The proof of asymptotic normality in spirit follows to some extent the proof of theorem 4. By Assumption 3, the MLE $\hat{\theta}$ fulfills the first order estimating equations:

$$0 = \nabla \mathcal{L}^l(\hat{\theta}),$$

which, when looking at the individual components, means that

$$0 = \partial_{\mu^l} \mathcal{L}^l(\hat{\theta}) = -d_+ + n^{-\xi} \exp(\hat{\mu}^l) \sum_{i<j} \frac{\exp(\gamma^T Z_{ij})}{1 + n^{-\xi} \exp(\hat{\mu}^l + \hat{\gamma}^T Z_{ij})},$$

$$0 = \partial_{\gamma^T} \mathcal{L}^l(\hat{\theta}) = \sum_{i<j} Z_{ij,k} A_{ij} + n^{-\xi} \exp(\hat{\mu}^l) \sum_{i<j} \frac{Z_{ij,k} \exp(\gamma^T Z_{ij})}{1 + n^{-\xi} \exp(\hat{\mu}^l + \hat{\gamma}^T Z_{ij})}, \quad k = 1, \ldots, p.$$

We want to make use of a Taylor expansion. Define the functions $l_n(y, a) : \{0, 1\} \times \mathbb{R} \to \mathbb{R}$,

$$l_n(y, a) = -ya + \log(1 + n^{-\xi} \exp(a)).$$

In particular,

$$\mathcal{L}^l(\mu^l, \gamma) = \sum_{i<j} l_n(A_{ij}, (\mu^l, \gamma^T) D_{ij}).$$

The $l_n$ have the following derivatives:

$$\hat{l}_n(y, a) := \partial_a l_n(y, a) = -y + n^{-\xi} \frac{\exp(a)}{1 + n^{-\xi} \exp(a)},$$

$$\hat{l}_n(y, a) := \partial_{\alpha} l_n(y, a) = n^{-\xi} \frac{\exp(a)}{(1 + n^{-\xi} \exp(a))^2},$$

$$\partial_{\alpha} l_n(y, a) = n^{-\xi} \frac{\exp(a)}{(1 + n^{-\xi} \exp(a))^2} \frac{1 - n^{-\xi} \exp(a)}{1 + n^{-\xi} \exp(a)}.$$

Note that $|\partial_{\alpha} l_n(y, a)| \leq C n^{-\xi}$ and hence $\hat{l}_n(y, a)$ is Lipschitz continuous in $a$ with constant $C n^{-\xi}$ by the Mean-Value Theorem. Doing a first order Taylor expansion in $a$ of $\hat{l}_n(y, a) = \partial_a l(y, a)$ in the point $a_0 = (A_{ij}, D_{ij}^T \theta_0)$ evaluated at $a = (A_{ij}, D_{ij}^T \hat{\theta})$, we get

$$\partial_a l(A_{ij}, D_{ij} \hat{\theta}) = \partial_a l(A_{ij}, D_{ij} \theta_0) + \partial_{\alpha} l(A_{ij}, \alpha) D_{ij}^T \hat{\theta} - \partial_{\alpha} l(A_{ij}, \alpha) D_{ij}^T \theta_0,$$

for an $\alpha$ between $D_{ij}^T \hat{\theta}$ and $D_{ij}^T \theta_0$. 

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Consider the vector $1/(n^2) \nabla \mathcal{L}^1(\hat{\theta})$: By equation (47), with $\alpha_{ij}$ between $D_{ij}^T \hat{\theta}$ and $D_{ij}^T \theta_0$,

$$0 = \frac{1}{(2)} \nabla \mathcal{L}^1(\hat{\theta}) = \frac{1}{(2)} \sum_{i<j} \left( \partial_{\theta_{ij}} l(A_{ij}, D_{ij}^T \hat{\theta}) \right)_{k=1,...,p+1},$$

as a $(p + 1) \times 1$-vector

$$= \frac{1}{(2)} \sum_{i<j} \left( \lambda(A_{ij}, D_{ij}^T \hat{\theta}) D_{ij} \right), \quad \text{by the chain rule}$$

$$= \frac{1}{(2)} \sum_{i<j} \left( \lambda(A_{ij}, D_{ij}^T \theta_0) + \lambda(A_{ij}, \alpha_{ij}) D_{ij}^T (\hat{\theta} - \theta_0) \right) D_{ij}, \quad \text{by (47)}$$

$$= \frac{1}{(2)} \nabla \mathcal{L}^1(\theta_0) + \frac{1}{(2)} \sum_{i<j} \lambda(A_{ij}, \alpha_{ij}) D_{ij} D_{ij}^T (\hat{\theta} - \theta_0).$$

Proving theorem 3 now breaks down into three problems.

**D.2.1 Problem 1**

First, we show that under appropriate scaling $1/(n^2) \nabla \mathcal{L}^1(\theta_0)$ is asymptotically normal. We may write the components of $\nabla \mathcal{L}^1(\theta_0)$ more compactly as

$$\nabla \mathcal{L}^1(\theta_0)_k = \sum_{i<j} D_{ij,k}(p_{ij} - A_{ij}),$$

where $D_{ij,k}$ is the $k$th component of the $(i,j)$-th row of $D$, i.e. $D_{ij,k} = 1$, if $k = 1$ and $D_{ij,k} = Z_{ij,k-1}$, if $k = 2, \ldots, p + 1$ and

$$p_{ij} = \mathbb{E}[A_{ij}|Z_{ij}] = n^{-\xi} \cdot \frac{\exp(\mu_0^T + \gamma_0^T Z_{ij})}{1 + n^{-\xi} \exp(\mu_0^T + \gamma_0^T Z_{ij})}.$$

Notice that all components of $\nabla \mathcal{L}^1(\theta_0)$ are centered. Indeed,

$$\mathbb{E}[\nabla \mathcal{L}^1(\theta_0)_k] = \sum_{i<j} \mathbb{E}[D_{ij,k}(p_{ij} - A_{ij})] = \sum_{i<j} \mathbb{E}[D_{ij,k} \mathbb{E}[(p_{ij} - A_{ij})|Z_{ij}]] = \sum_{i<j} \mathbb{E}[D_{ij,k} \cdot 0] = 0.$$

We want to apply the Lindeberg-Feller Central Limit Theorem to the term

$$\sqrt{\frac{n^2}{2}} \cdot \frac{1}{(2)} \nabla \mathcal{L}^1(\theta_0) = \sum_{i<j} D_{ij}(p_{ij} - A_{ij}) \cdot \sqrt{\frac{n^2}{2}}.$$

To that end, define the triangular array $Y_{n,ij} = D_{ij}(p_{ij} - A_{ij}) \cdot \sqrt{\frac{n^2}{2}}, 1 \leq i < j \leq n, n \in \mathbb{N}$. Since the $Y_{n,ij}$ are centered, their covariance matrix is given by

$$\text{Cov}(Y_{n,ij}) = \mathbb{E}[Y_{n,ij} Y_{n,ij}^T] = \mathbb{E} \left[ D_{ij} D_{ij}^T (p_{ij} - A_{ij})^2 \cdot \frac{n^2}{(2)} \right] = \mathbb{E} \left[ D_{ij} D_{ij}^T p_{ij} (1 - p_{ij}) \cdot \frac{n^2}{(2)} \right],$$

where for the last equality we have used that $\mathbb{E}[(p_{ij} - A_{ij})^2|Z_{ij}] = p_{ij}(1 - p_{ij})$. In analogy to the case with non-zero $\beta$, we write $W_0^2 = \text{diag}(p_{ij}(1 - p_{ij}), i < j) \in \mathbb{R}^{(n) \times (\frac{n}{2})}$. Then, we get for the sum of covariance matrices

$$\sum_{i<j} \text{Cov}(Y_{n,ij}) = \sum_{i<j} \mathbb{E} \left[ D_{ij} D_{ij}^T p_{ij} (1 - p_{ij}) \cdot \frac{n^2}{(2)} \right] = \frac{n^2}{(2)} \mathbb{E}[D^T W_0^2 D] =: \Sigma^{(n)}.$$
For any pair $i < j$, we have $p_{ij}(1 - p_{ij}) = n^{-\xi} \exp(\mu_i^\top Z_{ij}) - \frac{\exp(\gamma_i^T Z_{ij})}{(1 + n^{-\xi} \exp(\mu_i^\top + \gamma_i^T Z_{ij}))^2}$. Hence, $n^\xi p_{ij}(1 - p_{ij}) \to \exp(\mu_i^\top + \gamma_i^T Z_{ij})$ as $n \to \infty$. Consider the $(k,l)$-th entry of $\Sigma^{(n)}$:

$$
\Sigma_{k,l}^{(n)} = \frac{1}{(2)} \sum_{i < j} \mathbb{E} \left[ (D_{ij} D_{ij}^T)_{k,l} \exp(\mu_i^\top) \left( \frac{\exp(\gamma_i^T Z_{ij})}{(1 + n^{-\xi} \exp(\mu_i^\top + \gamma_i^T Z_{ij}))^2} \right) \right]
$$

$$
= \mathbb{E} \left[ (D_{12} D_{12}^T)_{k,l} \exp(\mu_i^\top) \left( \frac{\exp(\gamma_i^T Z_{12})}{(1 + n^{-\xi} \exp(\mu_i^\top + \gamma_i^T Z_{12}))^2} \right) \right], \quad Z_{ij} \text{ i.i.d.}
$$

$$
n \to \infty \Rightarrow \mathbb{E} \left[ (D_{12} D_{12}^T)_{k,l} \exp(\mu_i^\top) \exp(\gamma_i^T Z_{12}) \right] = \Sigma_{k,l},
$$

by dominated convergence. Hence, with $\Sigma = (\Sigma_{kl})_{k,l} \in \mathbb{R}^{(p+1) \times (p+1)}$, as $n \to \infty$,

$$
\sum_{i < j} \text{Cov}(Y_{n,ij}) \to \Sigma,
$$

where convergence is to be understood componentwise. We claim that $\Sigma$ is strictly positive definite. Indeed, since $\mu_i^\top + \gamma_i^T Z_{12}$ lies in some compact set there is a constant $C > 0$ such that $\exp(\mu_i^\top) \exp(\gamma_i^T Z_{12}) > C > 0$ almost surely. Then, for any vector $v = (v_1, v_1^T) \in \mathbb{R}^{p+1}$, $v_1 \in \mathbb{R}$,

$$
v^T \Sigma v = \mathbb{E}[(D_{12} v)^2 \exp(\mu_i^\top) \exp(\gamma_i^T Z_{12})] > C v^T \mathbb{E}[D_{12} D_{12}^T] v.
$$

Yet, by Assumption 4,

$$
v^T \mathbb{E}[D_{12} D_{12}^T] v = v^T \mathbb{E} \begin{bmatrix} 1 & Z_{12}^T \\ Z_{12} & Z_{12} Z_{12}^T \end{bmatrix} v
= v^T \begin{bmatrix} 1 & 0 \\ Z_{12} & \mathbb{E}[Z_{12} Z_{12}^T] \end{bmatrix} v
= v_1^2 + v_R^T \mathbb{E}[Z_{12} Z_{12}^T] v_R \geq (1 \wedge \lambda_{\min}) \|v\|_2^2.
$$

Thus, for any $v \neq 0$,

$$
v^T \Sigma v \geq C \|v\|_2^2 > 0
$$

and therefore $\Sigma$ is positive definite.

Furthermore, we clearly have $\mathbb{E}[\|Y_{n,ij}\|^2] < C < \infty$ for any $i, j, n$. Finally, let $\epsilon > 0$. Since $\|D_{ij}(p_{ij} - A_{ij})\|_2$ is uniformly bounded for all $i < j$, we may find an $n_0 \in \mathbb{N}$ such that for all $n > n_0$ we have $\|Y_{n,ij}\| < \epsilon$ for all $i < j$. This gives us that, as $n \to \infty$,

$$
\sum_{i < j} \mathbb{E}[\|Y_{n,ij}\|^2 \mathbbm{1}(\|Y_{n,ij}\| > \epsilon)] \to 0.
$$

Then, by the vector-valued Lindeberg-Feller Central Limit Theorem, we obtain

$$
\sqrt{\left( \frac{n}{2} \right)} n^{\xi/2} \cdot \frac{1}{(\xi)} \nabla L^i(\theta_0) = \sum_{i < j} Y_{n,ij} \xrightarrow{d} \mathcal{N}(0, \Sigma).
$$

(48)

**D.2.2 Problem 2**

Next, we must find a bound on the speed of convergence of $\hat{\theta} - \theta_0$. Recall that we obtained the equality

$$
0 = \frac{1}{(\xi)} \nabla L^i(\theta_0) + \frac{1}{(\xi)} \sum_{i < j} \bar{I}(A_{ij}, \alpha_{ij}) D_{ij} D_{ij}^T(\theta - \theta_0).
$$

(49)

Consider the matrix

$$
\Sigma_{\alpha} := \frac{1}{(\xi)} \sum_{i < j} \bar{I}(A_{ij}, \alpha_{ij}) D_{ij} D_{ij}^T = \left( \frac{1}{(\xi)} \right) D^T \text{diag}(\bar{I}(A_{ij}, \alpha_{ij}), i < j) D.
$$
Since $\alpha_{ij}$ lies between $D_i^T \hat{\theta}$ and $D_j^T \theta_0$ and both of these points lie in some compact set, we have for some universal constant $C > 0$, independent of $i, j$,

$$\hat{l}(A_{ij}, \alpha_{ij}) \geq C n^{-\xi}.$$ 

Thus, for any $v \in \mathbb{R}^{p+1}$,

$$v^T \Sigma_\alpha v \geq C n^{-\xi} v^T \left( \frac{1}{\sqrt{2}} D^T D \right) v.$$ 

Completely analogously to the case with non-zero $\beta$, we can show that $\frac{1}{\sqrt{2}} D^T D$ is positive definite with high probability by using Lemma 6 in Kock & Tang (2019) (cf. section C.1). Therefore, with high probability, $\lambda_{\text{min}}(\Sigma_\alpha) \geq C n^{-\xi} > 0$. Thus,

$$\lambda_{\text{max}}(\Sigma_\alpha^{-1}) = \frac{1}{\lambda_{\text{min}}(\Sigma_\alpha)} \leq C n^{\xi}.$$ 

From (49) we now obtain

$$\Sigma_\alpha (\hat{\theta} - \theta_0) = -\frac{1}{\sqrt{2}} \nabla \mathcal{L}^\dagger(\theta_0)$$

which is equivalent to

$$\hat{\theta} - \theta_0 = -\Sigma^{-1}_\alpha \frac{1}{\sqrt{2}} \nabla \mathcal{L}^\dagger(\theta_0)$$

which after rescaling gives

$$\sqrt{\frac{n}{n^\xi}} (\hat{\theta} - \theta_0) = -\sqrt{\frac{n}{n^\xi}} \Sigma^{-1}_\alpha \frac{1}{\sqrt{2}} \nabla \mathcal{L}^\dagger(\theta_0) = -n^{-\xi} \Sigma^{-1}_\alpha \cdot \sqrt{\frac{n}{n^\xi}} \frac{1}{\sqrt{2}} \nabla \mathcal{L}^\dagger(\theta_0).$$

From the previous section we know $\sqrt{\frac{n}{n^\xi}} \frac{1}{\sqrt{2}} \nabla \mathcal{L}^\dagger(\theta_0)$ $\xrightarrow{d}$ $\mathcal{N}(0, \Sigma)$. Also, the maximum eigenvalue of $n^{-\xi} \Sigma^{-1}_\alpha$ is uniformly bounded by some universal constant $C < \infty$, making the right-hand side above $O_P(1)$. This means

$$\hat{\theta} - \theta_0 = O_P \left( \sqrt{\frac{n^\xi}{n^\xi}} \right).$$

**D.2.3 Problem 3**

Finally, we derive the desired central limit theorem for our estimator. We claim that $n^\xi \Sigma_\alpha = \Sigma + o_P(1)$. To prove this, first consider the functions

$$f_n(x) = \frac{\exp(x)}{\left(1 + n^{-\xi} \exp(x)\right)^2}.$$ 

For every $x$, we have pointwise convergence $f_n(x) \to f(x) := \exp(x)$ as $n \to \infty$. Since $\hat{\theta}$ and $\theta_0$ lie in some compact set and since $Z_{ij}$ is uniformly bounded, the values $\alpha_{ij}$ in (49) and $\mu_0^\dagger + \gamma_0^T Z_{ij}, i < j$ all lie in some compact interval $I \subset \mathbb{R}$ independent of $i, j$ and $n$. Also notice that $f_n(x) \leq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in I$. Recall that by Dini’s theorem a sequence of monotonically increasing, continuous, real-valued functions that converges pointwise to some continuous limit function on a compact topological space, must already converge uniformly. Hence, $f_n$ converges uniformly to $f$ on $I$: $\lim_{n \to \infty} \sup_{x \in I} |f_n(x) - f(x)| = 0$. Furthermore, since $I$ is compact and hence bounded, $f$ has bounded derivative on $I$ and thus is Lipschitz continuous on $I$ with some finite constant $C$ by the Mean-Value Theorem:

$$|f(x) - f(y)| \leq C |x - y|, \quad \text{for all } x, y \in I.$$ 

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Now consider the \((k,l)\)-th entry of \(n^\xi \Sigma_n - \Sigma\):

\[
|n^\xi \Sigma_n - \Sigma|_{kl} = \left| \frac{1}{n} \sum_{i<j} D_{i,k} D_{i,l} \frac{\exp(\alpha_{ij})}{(1 + n^{-\xi} \exp(\alpha_{ij}))^2} - \mathbb{E}[D_{12,k} D_{12,l} \exp(\mu_0^+ + \gamma_0^T Z_{12})]\right|
\]

\[
\leq \frac{1}{n} \sum_{i<j} D_{i,k} D_{i,l} \left\{ \frac{\exp(\alpha_{ij})}{(1 + n^{-\xi} \exp(\alpha_{ij}))^2} - \exp(\mu_0^+ + \gamma_0^T Z_{ij}) \right\}
\]

\[
+ \frac{1}{n} \sum_{i<j} D_{i,k} D_{i,l} \exp(\mu_0^+ + \gamma_0^T Z_{ij}) - \mathbb{E}[D_{12,k} D_{12,l} \exp(\mu_0^+ + \gamma_0^T Z_{12})]
\]

By the strong law of large numbers, \((II)\) goes to zero almost surely. Let us consider \((I)\).

\[
(I) \leq \frac{1}{n} \sum_{i<j} D_{i,k} D_{i,l} \left\{ \frac{\exp(\alpha_{ij})}{(1 + n^{-\xi} \exp(\alpha_{ij}))^2} - \exp(\mu_0^+ + \gamma_0^T Z_{ij}) \right\}
\]

\[
\leq C \cdot \max_{i<j} \left| \frac{\exp(\alpha_{ij})}{(1 + n^{-\xi} \exp(\alpha_{ij}))^2} - \exp(\mu_0^+ + \gamma_0^T Z_{ij}) \right|
\]

\[
= C \cdot \max_{i<j} |f_n(\alpha_{ij}) - f(\mu_0^+ + \gamma_0^T Z_{ij})|
\]

\[
\leq C \cdot \left\{ \max_{i<j} |f_n(\alpha_{ij}) - f(\alpha_{ij})| + \max_{i<j} |f(\alpha_{ij}) - f(\mu_0^+ + \gamma_0^T Z_{ij})| \right\}
\]

\[
\leq C \cdot \left\{ \sup_{x \in I} |f_n(x) - f(x)| + \max_{i<j} |\alpha_{ij} - \mu_0^+ + \gamma_0^T Z_{ij}| \right\},
\]

where we have used the Lipschitz continuity of \(f_n\) to \(f\) on \(I\) for the last inequality. By the uniform convergence of \(f_n\) to \(f\) on \(I\), we know that the first term in the last line goes to zero. For the second term, recall that \(\alpha_{ij}\) is a point between \(D_{i,j}^T \hat{\theta}\) and \(D_{i,j}^T \theta_0 = \mu_0^+ + \gamma_0^T Z_{ij}\). Hence,

\[
\max_{i<j} |\alpha_{ij} - \mu_0^+ + \gamma_0^T Z_{ij}| \leq \max_{i<j} |(\hat{\mu}^T - \mu_0^+) + (\hat{\gamma} - \gamma_0)^T Z_{ij}| \leq C\|\hat{\theta} - \theta_0\|_1 \overset{P}{\to} 0,
\]

by the consistency of \(\hat{\theta}\). Thus, \((I) \overset{P}{\to} 0\) as \(n \to \infty\).

In conclusion, \(|n^\xi \Sigma_n - \Sigma|_{kl} \overset{P}{\to} 0\) and therefore,

\[
n^\xi \Sigma_n = \Sigma + o_P(1),
\]

where \(o_P(1)\) is to be understood as a matrix in which each component is \(o_P(1)\). Now, we get from (49),

\[
0 = \frac{1}{n} \nabla L^1(\theta_0) + \Sigma_n (\hat{\theta} - \theta_0)
\]

which after multiplying with \(n^\xi\) is equivalent to

\[
0 = n^\xi \frac{1}{n} \nabla L^1(\theta_0) + (\Sigma + o_P(1)) (\hat{\theta} - \theta_0).
\]

Rearranging gives

\[
\Sigma(\hat{\theta} - \theta_0) = -n^\xi \frac{1}{n} \nabla L^1(\theta_0) + o_P(1)(\hat{\theta} - \theta_0).
\]

Now, remember that \(\Sigma\) is positive definite and thus invertible, to get

\[
(\hat{\theta} - \theta_0) = -\Sigma^{-1} n^\xi \frac{1}{n} \nabla L^1(\theta_0) + \Sigma^{-1} o_P(1)(\hat{\theta} - \theta_0).
\]
Observe that $\Sigma^{-1}$ has bounded maximum eigenvalue due to Assumption 4 and thus $\Sigma^{-1}o_P(1) = o_P(1)$:

$$(\hat{\theta} - \theta_0) = -\Sigma^{-1}n^{\xi} \frac{1}{(2)} \nabla L(\theta_0) + o_P(1)(\hat{\theta} - \theta_0).$$

Finally, multiply by $\sqrt{\frac{(n)}{n^{\xi}}}$ and remember that $\hat{\theta} - \theta_0 = O_P\left(\sqrt{\frac{n^{\xi}}{n} \xi}\right)$

$$\sqrt{\frac{(n)}{n^{\xi}}}(\hat{\theta} - \theta_0) = -\Sigma^{-1}n^{\xi/2} \frac{1}{(2)} \nabla L(\theta_0) + o_P(1).$$

With this, due to (48), we have proven

$$\sqrt{\frac{(n)}{n^{\xi}}}n^{\xi}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma^{-1}).$$

(50)

**Proof of theorem 3.** By the solved problems 1 - 3 above.

It remains to prove corollary 2.

**Proof of corollary 2.** Notice that from (50) we get: For any $k = 1, \ldots, (p + 1)$,

$$\sqrt{\frac{(n)}{n^{\xi}}} \frac{\hat{\theta}_k - \theta_{0,k}}{\sqrt{\Sigma^{-1}_{k,k}}} \xrightarrow{d} \mathcal{N}(0, 1).$$

(51)

By the exact same arguments that we have used to show that $n^{\xi} \Sigma_\alpha = \Sigma + o_P(1)$, we can also show that

$$n^{\xi} \hat{\Sigma} = \Sigma + o_P(1),$$

where $\hat{\Sigma}$ is the same matrix as $\Sigma_\alpha$ with $\alpha_{ij}$ replaced by $\hat{\mu}^T + \hat{\gamma}^T Z_{ij}$:

$$\hat{\Sigma} = \frac{1}{(2)} D^T \text{diag}\left(\frac{n^{-\xi} \exp(\hat{\mu}^T + \hat{\gamma}^T Z_{ij})}{(1 + n^{-\xi} \exp(\hat{\mu}^T + \hat{\gamma}^T Z_{ij}))^2}, i < j\right) D.$$

By the same arguments as before, we can show that the minimum eigenvalue of $n^{\xi} \hat{\Sigma}$ is bounded away from zero, uniformly in $n$. This implies that the maximum eigenvalue of $(n^{\xi} \hat{\Sigma})^{-1}$ is bounded by some finite constant $C$. We already know that the same property holds for $\Sigma$ and $\Sigma^{-1}$. Therefore, we have for the matrix $\infty$-norm:

$$\|(n^{\xi} \Sigma)^{-1} - \Sigma^{-1}\|_\infty \leq \|(n^{\xi} \hat{\Sigma})^{-1}\|_\infty \|\Sigma^{-1}\|_\infty \|n^{\xi} \hat{\Sigma} - \Sigma\|_\infty \leq C \|n^{\xi} \hat{\Sigma} - \Sigma\|_\infty = o_P(1).$$

This means in particular for the diagonal elements:

$$(n^{\xi} \hat{\Sigma})^{-1}_{k,k} = n^{-\xi} \hat{\Sigma}^{-1}_{k,k} = \Sigma^{-1}_{k,k} + o_P(1).$$

But then, from (51) and by Slutzky’s Theorem,

$$\sqrt{\frac{(n)}{2}} \frac{\hat{\theta}_k - \theta_{0,k}}{\sqrt{\Sigma^{-1}_{k,k}}} = \sqrt{\frac{(n)}{n^{\xi}}} \frac{\hat{\theta}_k - \theta_{0,k}}{\sqrt{n^{-\xi} \hat{\Sigma}^{-1}_{k,k}}} = \sqrt{\frac{(n)}{n^{\xi}}} \frac{\hat{\theta}_k - \theta_{0,k}}{\sqrt{\Sigma^{-1}_{k,k} + o_P(1)}} \xrightarrow{d} \mathcal{N}(0, 1).$$

\qed