Ricci–Flat Branes

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Abstract

Up to overall harmonic factors, the D8-brane solution of the massive type IIA supergravity theory is the product of nine–dimensional Minkowski space (the worldvolume) with the real line (the transverse space). We show that the equations of motion allow for the worldvolume metric to be generalised to an arbitrary Ricci–flat one. If this nine–dimensional Ricci–flat manifold admits Killing spinors, then the resulting solutions are supersymmetric and satisfy the usual Bogomol’nyi bound, although they preserve fewer than the usual one half of the supersymmetries. We describe the possible choices of such manifolds, elaborating on the connection between the existence of Killing spinors and the self–duality condition on the curvature two–form. Since the D8-brane is a domain wall in ten dimensions, we are led to consider the general case: domain walls in any supergravity theory. Similar considerations hold here also. Moreover, it is shown that the worldvolume of any magnetic brane — of which the domain walls are a specific example — can be generalised in precisely the same way. The general class of supersymmetric solutions have gravitational instantons as their spatial sections. Some mention is made of the worldvolume solitons of such branes.
1 Introduction

The discovery of branes in string theory has brought about some revolutionary advances in recent years. From what was initially a purely perturbative theory, it has been possible to extract non-perturbative results which have led to new and deep insights into the nature of string theory. Perhaps the most far-reaching of these has been the discovery of string theory’s eleven-dimensional origin, ‘M-Theory’, with its low-energy limit of eleven-dimensional supergravity.

Ignoring for the time being the D8-brane, the ten-dimensional type IIA and IIB supergravity theories have solutions which describe the various $p$-branes of type IIA and IIB string theory. The field content of the supergravity theories is precisely what is needed to imply the existence of such branes: the relevant $(p+1)$-form potentials are all present $[1]$. The fundamental string and its magnetic dual, the 5-brane, couple to the Neveu–Schwarz–Neveu–Schwarz (NS-NS) potentials; these being present in both the type IIA and IIB theories. In addition to these are the $Dp$-branes, which couple to the Ramond–Ramond (R-R) $(p+1)$-form potentials: the type IIA theory has $p$ even, and type IIB has $p$ odd. For $p < 3$, the branes couple to an electric field strength — these are the fundamental, or electric, branes. In $D$ dimensions, Hodge duality allows for a $(p+2)$-form field strength to be interchanged with a $(D-(p+2))$-form field strength, so the branes with $p > 3$ couple to the dual, magnetic field strength. These are the solitonic, or magnetic, branes. (The D3-brane is a special case, being self-dual in ten dimensions.) All such branes preserve one-half of the space-time supersymmetries. Moreover, they all have a clear and well-defined eleven-dimensional origin.

The only aspect of this unified picture which is somewhat unclear is that of the D8-brane of type IIA string theory. This is a domain wall in ten dimensions, which should couple to a ten-form field strength or, by Hodge duality, a scalar. As first pointed out by Polchinski $[2]$, the once relatively obscure generalisation of the type IIA supergravity theory found by Romans $[3]$ — the massive IIA theory — has the necessary field content $[4]$. Since type IIA string theory necessarily includes a D8-brane, it would seem that it is the Romans theory which is the natural low-energy limit of type IIA string theory.

The Romans theory has some unusual properties: ten-dimensional Minkowski space is not a solution. Indeed, none of the Kaluza–Klein compactifications of the theory originally considered by Romans $[3]$ are supersymmetric. This is in contrast to the D8-brane solution $[3]$. 
which preserves the usual one half of the ten–dimensional supersymmetries. Moreover, the eleven–dimensional origin of the Romans theory is quite mysterious (although see [4]). If the theory is to find a place within M-Theory, then presumably the D8-brane would be the dimensional reduction of an M9-brane [4]. It is unclear, however, how this works in detail.

Although we do not consider it here, mention should be made of an alternative ten–dimensional massive supergravity theory, that of Howe, Lambert and West [8]. Unlike the Romans theory, this does have a well–defined eleven–dimensional origin [8, 9], but its connection with string theory is unclear: the NS-NS two–form which couples to the fundamental strings can be gauged away, so it would seem that this theory does not contain such strings at all.

All the p-brane solutions of the various supergravity theories, including the D8-brane, have a common form: in the Einstein frame the line element is

$$ds^2 = H^{-\frac{\Delta d}{D-2}} dx^i \cdot dx^i + H^{\frac{\Delta d}{D-2}} dy^\alpha \cdot dy^\alpha, \quad (1)$$

where \{x^i\} and \{y^\alpha\} are the worldvolume and transverse coordinates respectively. The function \(H(r)\) is harmonic on the transverse space, \(r\) being the radial coordinate in these directions. \(d = p + 1, \tilde{d} = D - d - 2\) denotes the number of dimensions of the dual brane and, for D-branes and M-branes, we have \(\Delta = 4\).

Now we can ask the question whether it is possible to generalise the metrics on the worldvolume and transverse space in these solutions. That is, whether the field equations, and the supersymmetry conditions, admit general metrics instead of the flat ones appearing above. Indeed, there has been some work to show that the transverse space of the electric branes can be generalised, as long as the metric is Ricci–flat [10, 11, 12]. The resulting solution will be supersymmetric if and only if the transverse space admits Killing spinors.

In the following section of this paper, we show that the worldvolume, as opposed to the transverse space, of the D8-brane solution can be generalised in just such a manner. That is, the solution (4) with \(p = 8\) and \(D = 10\) can be generalised so as to include a non–trivial Ricci–flat worldvolume metric. As long as this metric admits Killing spinors, the solution is supersymmetric, and satisfies the usual Bogomol’nyi bound, as we will show in subsection 2.2. We hope that such considerations will shed some light on the eleven–dimensional origin of the D8-brane. There are numerous examples in the literature of possible supersymmetric manifolds which could be taken as the worldvolume of the D8-brane; and we discuss some of these in subsections 2.3 and 2.4. This involves a consideration of holonomy groups.
The spatial sections of the D8-brane can take the form of eight–dimensional gravitational instantons — manifolds with self–dual curvature — of holonomy Spin(7). Section three is concerned with the dimensional reduction of the D8-brane, a 7-brane domain wall in nine dimensions. Here we show that the manifolds of $G_2$ holonomy described in the literature are also self–dual; and that these can be taken as possible spatial sections of this 7-brane.

Since the D8-brane is a domain wall, we are led to consider whether the same sort of generalisations can be made for domain walls in any supergravity theory. We show, in section four, that this is indeed the case. In particular, it is possible for the 3-brane domain walls of Hořava–Witten theory to have general Ricci–flat worldvolumes. In section five we extend the analysis to arbitrary magnetic branes, of which the domain wall is but a special case. The general statement is, then, that all magnetic branes can have Ricci–flat worldvolumes. We concentrate on the M5-brane here. Finally, some mention is made of the worldvolume solitons of such magnetic branes; those of the D4-brane in particular.

The conventions we use are as follows. The signature of the metric is $(-, +, \ldots, +)$, the sign of the Riemann curvature tensor is defined by $R_{abcd}X^d = [D_a, D_b]X_c$, and the gamma matrices satisfy $\{\Gamma^a, \Gamma^b\} = 2g^{ab}$. As to indices, we take $a, b = 0, \ldots, D−1$ to denote spacetime directions, $i, j = 0, \ldots, p$ to denote worldvolume directions, and $\alpha, \beta = p + 1, \ldots, D − 1$ to denote transverse directions. We are mainly concerned with the case $D = 10$ and $p = 8$. Where necessary, we use underlined indices to denote an (pseudo–)orthonormal basis and we work exclusively in the Einstein frame. In the following section, we use the notation of forms as in [4]. That is, a $q$–form $Q$ has components $Q_{a_1 \ldots a_q}$ given by

$$Q = Q_{a_1 \ldots a_q}dx^{a_1} \wedge \ldots \wedge dx^{a_q},$$

and $|Q|^2 = Q_{a_1 \ldots a_q}Q^{a_1 \ldots a_q}$.

2 The Romans Theory

The ten–dimensional type IIA supergravity theory contains the metric, the dilaton $\phi$ and a two–form $B$ in the NS-NS sector, and a one–form potential $A$ and a three–form potential $C$ in the R-R sector. The massive type IIA theory found by Romans [3] is constructed by allowing the two-form to ‘eat’ the one–form, thereby generating a massive two-form via a generalised Higgs mechanism. With the field strengths given by

$$F = 4dC + 6m(B)^2,$$
the Lagrangian for the bosonic sector as given in [4] is

\[
\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2} |\partial \phi|^2 - \frac{1}{3} e^{-\phi} |H|^2 - \frac{1}{12} e^{\frac{5}{2}\phi} |F|^2 - m^2 e^{\frac{5}{2}\phi} |B|^2 - \frac{1}{2} m^2 e^{\frac{5}{2}\phi} \right) \\
+ \frac{1}{9} \varepsilon \left( dC dCB + mdC(B) + \frac{9}{20} m^2 (B)^5 \right),
\]  

(2)

where \( m \) is the mass of the two–form \( B \). The potential, or ‘cosmological term’, is of Liouville type, given by

\[
V(\phi) = \frac{1}{2} m^2 e^{\frac{5}{2}\phi}. 
\]

(3)

In ten dimensions, we can consider a ten–form field strength which, by the equations of motion, must have a constant zero–form Hodge dual. This constant is just the mass parameter, \(*F_{[10]} \sim m\), and the Romans theory can be rewritten to include explicitly the required nine–form potential [4]. This is an R-R potential, as required to couple to the D8-brane \( \mathbb{R} \), which can be seen by transforming to the string frame, \( g^{(s)}_{ab} = e^{\frac{1}{2}\phi} g^{(E)}_{ab} \). The potential is then just a constant

\[
V = \frac{1}{2} m^2, 
\]

(4)

the absence of a dilaton factor indicating the R-R nature of the field.

The D8-brane provides a natural, and non–trivial, background in Type IIA string theory. Indeed, the Romans theory has ‘massive’ fundamental string solutions, coupling to the massive \( B \) field, in just such a background [3]: a solution which describes the intersection of a fundamental string with a D8-brane over a D0-brane. This solution reduces to the usual string solution of the type IIA theory if the mass parameter \( m = 0 \).

### 2.1 Ricci–Flat D8-branes

Since we are considering D8-branes alone, we turn off all gauge fields. The equations of motion following from the Lagrangian (2) are then

\[
R_{ab} = \frac{1}{2} \partial_a \phi \partial_b \phi + \frac{1}{16} m^2 e^{\frac{5}{2}\phi} g_{ab},
\]

(5a)

\[
D^2 \phi = \frac{5}{4} m^2 e^{\frac{5}{2}\phi},
\]

(5b)

where \( D_a \) is the covariant derivative with respect to the metric \( g_{ab} \). Note that due to the Liouville form of the potential, the dilaton equation (3b) implies that \( \phi \) cannot be a constant.
A manifestly conformally flat D8-brane solution was first discussed in [5]. A more convenient, but equivalent, one was given in [4]; in the Einstein frame, it is
\[
ds^2 = H^\frac{1}{2} dx \cdot dx + H^\frac{2}{3} dy^2,
\]
\[
e^\phi = H^{-\frac{3}{4}},
\]
where \(\{x^i\}\) are the worldvolume coordinates. \(H(y)\) is harmonic on the single transverse direction \(y\). The precise form of this function has implications for the singularity structure of the metric as discussed in [14]. With
\[
H(y) = 1 + m |y - y_0|,
\]
the spacetime is free of curvature singularities which would otherwise be present if we dropped the constant. There is still a delta–function singularity at \(y = y_0\) however, although this can be removed, as usual, by adding a source term. The solution then describes a D8-brane situated at \(y = y_0\). Note that the mass parameter \(m\) can be positive or negative here. Performing the coordinate transformation
\[
d \tilde{y}^2 = H^\frac{9}{2} dy^2,
\]
we have
\[
ds^2 = (1 + \frac{25}{16} m |y|) \frac{2}{3} dx \cdot dx + dy^2,
\]
\[
e^\phi = (1 + \frac{25}{16} m |y|)^{-\frac{3}{4}}.
\]
Consider a generalisation of this solution, taking
\[
ds^2 = F^2 \gamma_{ij}(x) dx^i dx^j + dy^2,
\]
as our ansatz for the metric. We also take \(F = F(y)\) and \(\phi = \phi(y)\). The zehnbeins are \(e^{\bar{2}} = dy\) and \(\hat{e}^i = F\hat{e}^i(x)\), where \(\hat{e}^i\) is the neunbein for the metric \(\gamma_{ij}\). The components of the Ricci tensor are
\[
R_{yy} = -9 \frac{F''}{F},
\]
\[
R_{ij} = \hat{R}_{ij} - \gamma_{ij} (FF'' + 8F'^2),
\]
where \(\hat{R}_{ij}\) is the Ricci tensor constructed from the metric \(\gamma_{ij}\) and a prime denotes a derivative with respect to \(y\). The non–trivial components of (10a,b) are then
\[
-18 \frac{F''}{F} = \phi'^2 + \frac{1}{8} m^2 e^{\frac{5}{2}\phi},
\]
\[
R_{ij} = \gamma_{ij} \left( FF'' + 8F'^2 + \frac{1}{16} F^2 m^2 e^{\frac{5}{2}\phi} \right),
\]
\[
\phi'' + 9 \frac{F'}{F}\phi' = \frac{5}{4} m^2 e^{\frac{5}{2}\phi}.
\]
The equations (10a,c) are solved by
\[
\begin{align*}
F(y) &= (1 + \frac{25}{16} m |y - y_0|)^{\frac{1}{25}}, \\
e^\phi &= F(y)^{-20},
\end{align*}
\]
(11)
in which case, the final equation (10b) becomes
\[
\hat{R}_{ij} = 0.
\]
(12)
Thus, for any Ricci–flat metric $\gamma_{ij}(x)$,
\[
\begin{align*}
ds^2 &= F^2 \gamma_{ij}(x) dx^i dx^j + dy^2, \\
e^\phi &= F^{-20}, \\
F(y) &= (1 + \frac{25}{16} m |y - y_0|)^{\frac{1}{25}},
\end{align*}
\]
(13)
is a solution of the Romans theory. The interpretation is obvious: it describes a D8-brane with Ricci–flat worldvolume. The flat case (8) is then a special case of this more general family of solutions. It should be noted, however, that the isometry group may no longer be the nine–dimensional Poincaré group. Since the overall harmonic function is unchanged, this generalisation has not altered the singularity structure of the D8-brane, as long as the metric $\gamma_{ij}$ does not have any singularities itself. As an example of a solution which does alter the singularity structure, the worldvolume of the D8-brane could take the form of a nine–dimensional Schwarzschild black hole, although this solution will not be supersymmetric.

### 2.2 Supersymmetric D8-Branes and the Bogomol’nyi Bound

Ricci–flatness of $\gamma_{ij}$ is a necessary condition for the solutions (13) to be supersymmetric. To see this, consider the supersymmetry transformations of the spacetime fermionic fields [3]
\[
\begin{align*}
\delta \psi_a &= D_a \epsilon = \left(D_a - \frac{1}{32} me^4 \frac{\Gamma_a}{32} \right) \epsilon, \\
\delta \lambda &= -\frac{1}{2\sqrt{2}} \left( \Gamma^a \partial_a \phi + \frac{5}{4} me^4 \phi \right) \epsilon,
\end{align*}
\]
(14a,b)
where $\epsilon$ is an arbitrary 32–component Majorana spinor. Setting the dilatino variation (14b) to zero gives
\[
\left(1 \mp \Gamma_0 \right) \epsilon = 0,
\]
(15)
so $\epsilon$ must have a definite chirality, in the sense that $\Gamma_0 \epsilon = \pm \epsilon$. This removes one half of the components of $\epsilon$. The sign here is set by the sign of $(y - y_0)$, so the chirality of $\epsilon$ changes as
we pass through the location of the brane [1]. The vanishing of the gravitino variation (14a) implies

\[ \epsilon' = \frac{1}{32} m F^{25} \Gamma_y \epsilon, \]  
\[ \hat{D}_i \epsilon = 0, \]  
(16a)

(16b)

where \( \hat{D}_i \) is the covariant derivative with respect to the metric \( \gamma_{ij} \). A solution of (16a) is

\[ \epsilon = F(y)^{1/2} \hat{\epsilon}(x), \]  
(17)

where \( \hat{\epsilon}(x) \) is an \( SO(8,1) \) ‘worldvolume spinor’ which does not depend on \( y \).

The integrability condition of the remaining equation (16b) is, as usual,

\[ [\hat{D}_i, \hat{D}_j] \hat{\epsilon} = \hat{R}_{ijkl} \hat{\Gamma}^{kl} \hat{\epsilon} = 0, \]  
(18)

where \( \hat{\Gamma}^i = F \Gamma^i \) satisfy \( \{ \hat{\Gamma}^i, \hat{\Gamma}^j \} = 2 \gamma^{ij} \). By contracting (18) with \( \hat{\Gamma}^j \), we find

\[ \hat{R}_{ij} = 0, \]  
(19)

a necessary condition for the solutions (13) to be supersymmetric.

This can also be seen by a consideration of the integrability condition of the equation

\[ \delta \psi_a = \mathcal{D}_a \epsilon = 0. \]  
This is

\[ [\mathcal{D}_a, \mathcal{D}_b] \epsilon = \left( R_{abcd} + \frac{1}{128} m^2 e^2 \phi g_{ac} g_{bd} \right) \Gamma^{cd} \epsilon + \frac{5}{32} m e^2 \phi (\Gamma_a \partial_b \phi - \Gamma_b \partial_a \phi) \epsilon = 0. \]  
(20)

The \( \{y,y\} \) component of (20) is trivial. The \( \{y,i\} \) component is just the chirality condition (15); and the \( \{i,j\} \) component gives

\[ \hat{R}_{ijkl} \hat{\Gamma}^{kl} \hat{\epsilon} = 0, \]  
as in (18).

Either way, Ricci–flatness is a necessary condition for the D8-brane (13) to be supersymmetric. It is not, however, a sufficient condition since we must ensure (18), as opposed to (19), is satisfied. In other words, we must be able to construct a solution of

\[ \hat{D}_i \hat{\epsilon} = 0, \]  
(21)

a Killing spinor with respect to the metric \( \gamma_{ij} \). The trivial solution is, of course, to set \( \gamma_{ij} = \eta_{ij} \), in which case \( \hat{\epsilon}(x) = \hat{\epsilon}_0 \) is just a constant sixteen–component Majorana spinor. This
flat case is the maximally supersymmetric solution, breaking one half of the ten-dimensional supersymmetries due to the chirality requirement (15). Other, more general choices of $\gamma_{ij}$ will, however, break a greater fraction. If (21) admits $N$ solutions, then $N/32$ of the ten-dimensional supersymmetries are preserved.

It is of interest to note that all such supersymmetric solutions saturate the usual Bogomol’nyi bound on the mass and charge densities of the brane. To see this, consider the supercharges per unit eight–volume

$$Q_\epsilon = \int_{\partial \Sigma} \bar{\epsilon} \Gamma^a \Gamma_b \psi_c d\Sigma_{ab},$$

where $\Sigma$ is a nine–dimensional space–like surface, the integral over which reduces to one over the one–dimensional space transverse to the brane. The variation of $Q_\epsilon$ is

$$\delta_{\epsilon_1} Q_{\epsilon_2} = [Q_{\epsilon_1}, Q_{\epsilon_2}] = \int_{\partial \Sigma} N^{ab} d\Sigma_{ab},$$

where $N^{ab} = \bar{\epsilon}_1 \Gamma^{ab} \epsilon_2 \psi_\Sigma$ is the Nester form [17]. With (14a), we have

$$N^{ab} = \bar{\epsilon}_1 \Gamma^{ab} \epsilon_2 D_\Sigma \epsilon_2 - \frac{1}{4} m \epsilon_1 \Gamma^{ab} \epsilon_2.$$  

The surface integral (23) is evaluated on both sides of the domain wall, i.e. as $y \to y_0^\pm$. In this limit, we have $F(y) \to 1$, $e^{\frac{5}{4} \phi} \to 1$, and $\epsilon \to \hat{\epsilon}(x)$. Then

$$\delta_{\epsilon_1} Q_{\epsilon_2} = N^{ab} \bigg|_{y=y_0^+}$$

and with $\hat{D}_i \hat{\epsilon}_2 = 0$, this becomes

$$\delta_{\epsilon_1} Q_{\epsilon_2} = \frac{1}{4} m \epsilon_1 \left( 1 \mp \Gamma_y \right) \hat{\epsilon}_2 \bigg|_{y=y_0^+},$$

where the relative signs are set by the sign of $(y - y_0)$. As long as we can construct Killing spinors on the worldvolume, we thus have D8–branes with both the mass per unit volume, $M$, and the charge per unit volume, $Z$, proportional to $m$. Indeed, if we take $\hat{\epsilon}_1 = \hat{\epsilon}_2 = \hat{\epsilon}$, then (26) becomes $\hat{\epsilon}^\dagger \left( 1 \mp \Gamma_y \right) \hat{\epsilon}$ which vanishes due to the chirality requirement (13). For an arbitrary configuration, it can be shown that there is a Bogomol’nyi–type bound $M \geq Z$ [16]; and our Ricci–flat solutions saturate this bound. Since all of these latter are thus BPS states, it is not the case that the flat solution is energetically favoured over the general Ricci–flat one.
2.3 Supersymmetric Manifolds, Holonomy Groups and All That

If by the D8-brane worldvolume, we mean the nine–manifold with line element

$$ds^2 = \gamma_{ij}(x)dx^i dx^j,$$  \hspace{1cm} (27)

then we are interested in Ricci–flat worldvolumes which admit Killing spinors. We therefore consider manifolds of the form $\mathbb{R}^{n+1} \times \mathcal{M}^{8-n}$, with line element

$$ds^2 = -dt^2 + dx_1^2 + \ldots + dx_n^2 + d\tilde{s}^2,$$  \hspace{1cm} (28)

where $d\tilde{s}^2$ is the line element on $\mathcal{M}^{8-n}$. The spatial sections of the D8-brane worldvolume thus have the form $\mathbb{R}^n \times \mathcal{M}^{8-n}$. We can either take this as a potential D8-brane worldvolume or, by throwing away the $\mathbb{R}^n$ factor (and ignoring the overall harmonic function in the bulk metric), we effectively generate solutions which describe Ricci–flat domain walls in $D = (10-n)$: $(8-n)$-branes, with worldvolumes of the form $\mathbb{R} \times \mathcal{M}^{8-n}$. These could be dimensional reductions of the D8-brane, along the lines of [4], or domain walls in other types of supergravity theory. As we shall see below, all such domain walls can be generalised to this Ricci–flat case, so it is indeed of relevance to study the general D8-brane worldvolume (28).

As is well known, the classification of possible manifolds which admit Killing spinors is given in terms of their holonomy groups. Since the D8-brane admits Killing spinors as long $\mathcal{M}^{8-n}$ does, we are interested in manifolds $\mathcal{M}^d$ with dimensions $d = 8, 7, 6, 5, 4$. The Killing spinors will then have the form

$$\hat{\epsilon}(x) = \epsilon_0 \otimes \eta$$  \hspace{1cm} (29)

where $\epsilon_0$ is a constant $SO(n,1)$ spinor and $\eta$ transforms under the holonomy group of $\mathcal{M}^d$.

Perusing Berger’s list [17], and ignoring the trivial case, the relevant holonomy groups are as follows. For $d = 8$, we have $H = SU(4), Sp(2)$ or $Spin(7)$, the first two corresponding to Kähler and Hyper–Kähler manifolds respectively. Such choices will preserve $\frac{1}{16}, \frac{3}{32}, \frac{1}{32}$ of the ten–dimensional supersymmetries respectively. For $d = 7, 6, 4$, we have $H = G_2, SU(3), SU(2)$ respectively. (We ignore the $d = 5$ case here, since the holonomy group is then just $H = SU(2) \times 1$, so this effectively reduces to the $d = 4$ case.) Manifolds with these holonomy groups will preserve $\frac{1}{16}, \frac{1}{8}, \frac{1}{4}$ of the ten–dimensional supersymmetries respectively.
In the Euclidean regime, there is a connection between supersymmetric manifolds — those which admit Killing spinors — and self–dual manifolds — those which have self–dual curvature two–forms [18]. The four–dimensional self–duality condition on the curvature two–form can be generalised to

$$\Theta_{ab} = \frac{1}{2} \phi_{abcd} \Theta^{cd},$$

(30)

where $\phi_{abcd}$ is a duality operator, identified with the components of some fundamental, nowhere–vanishing four–form on $\mathcal{M}^d$. Just as in four dimensions, this second order equation on the metric is equivalent to a first order one on the vielbein [18]:

$$\omega_{ab} = \frac{1}{2} \phi_{abcd} \omega^{cd},$$

(31)

at least for some specific choice of gauge. Manifolds which satisfy this can be thought of as $d$–dimensional gravitational instantons. This is a direct generalisation of the similar considerations for self–dual Yang–Mills fields in $d$ dimensions: for $d > 4$, the operator $\phi$ belongs to an irreducible representation of $SO(d)$; and if a subgroup $H$ of $SO(d)$ can be found such that the decomposition of this representation under $H$ contains a singlet, then the corresponding tensor invariant $\phi$ can be constructed [19]. Manifolds with self–dual connections, in the sense of (31), will then have holonomy group $H$.

In eight dimensions, the duality operator $\phi$ can be chosen to be invariant under one of two maximal subgroups of $SO(8)$: $(SU(4) \times U(1))/Z_4$ or Spin(7) [19]. The former choice would generate the eight–dimensional Kähler manifolds, although it would seem that we cannot generate the Hyper–Kähler case using this method. The Spin(7) case would seem to be the more interesting, however: starting with $\phi$ as the unique Spin(7)–invariant Hodge self–dual four–form, it has been shown that any manifold which satisfies (30) in $d = 8, 7, 6, 5, 4$ has holonomy group $H = \text{Spin}(7), G_2, SU(3), SU(2) \times 1, SU(2)$ respectively [18]. This is just the list of holonomy groups given above. Moreover, the $d = 8$ equation is a ‘master’ equation from which the equations in $d < 8$ can be derived simply by assuming $\mathcal{M}^8$ to be the product of $\mathcal{M}^{8-n}$ with $T^n$ or $\mathbb{R}^n$, since the holonomy groups of the latter are trivial. This exactly parallels the remarks made above: by throwing away the $\mathbb{R}^n$ piece of the metric (28), we generate domain walls in lower dimensions. The ‘dimensional reduction’ is the same in both cases. So the spatial sections of the D8-brane worldvolume can be identified with an eight–dimensional gravitational instanton, or can be a product of flat space with a lower dimensional instanton.
Compact manifolds $\mathcal{M}^d$ with the required properties are well-known. The $d$–dimensional torus $T^d$ is the trivial example, since it has a trivial holonomy group. In $d = 8, 7$, Joyce has constructed compact manifolds with holonomy groups $H = \text{Spin}(7), \text{G}_2$ respectively [20, 21, 22]. In six dimensions, the Calabi–Yau manifolds have $H = SU(3)$, and in four dimensions the Hyper–Kähler manifold $K3$ has $H = SU(2)$. All such spaces are Ricci–flat as required. Making use of such manifolds as the spatial sections of the D8-brane would give a solution, the interpretation of which would be of a D8-brane wrapped on $\mathcal{M}^d$.

2.4 Explicit Non–Compact Examples

We are more interested, however, in the unwrapped brane, so we turn our attention to non–compact manifolds $\mathcal{M}^d$ for $d = 8, 7, 6, 5, 4$. Starting with $d = 8$, $\mathcal{M}^8$ can be a non–compact Kähler or Hyper–Kähler manifold with $H = SU(4), Sp(2)$ respectively. The latter includes the product of two Euclidean Taub–NUT spaces, an asymptotically locally Euclidean manifold with holonomy $H = Sp(1) \times Sp(1)$.

More generally, we could make use of the asymptotically locally Euclidean ‘toric’ Hyper–Kähler manifolds, with a tri–holomorphic $T^2$ isometry. The eight–dimensional line element has the local form [11]

$$
\hat{s}^2 = U_{AB}d\vec{x}^A d\vec{x}^B + U^{AB}(d\phi_A + A_A)(d\phi_B + A_B),
$$

(32)

where $\vec{x}^A = \{x^A_s, \ A = 1, 2, \ s = 1, 2, 3\}$ are coordinates on two copies of $\mathbb{E}^3$ and $U_{AB}$ are the entries of a positive definite symmetric $2 \times 2$ matrix function $U$ of these coordinates. $U^{AB}$ are the entries of $U^{-1}$, and the $\phi_A$ are periodically identified with period $2\pi$. The two one–forms $A_A = d\vec{x}^B \cdot \omega_{BA}$, where $\omega_{AB}$ are a triplet of $2 \times 2$ matrix functions of the $\vec{x}^A$. The one-forms satisfy the constraint

$$
F_{ABC}^{rs} = \varepsilon^{rst} \partial_r^A U_{BC},
$$

(33)

where

$$
F_{ABC}^{rs} = \partial_{\vec{x}^A} \omega_{BC}^s - \partial_{\vec{x}^B} \omega_{AC}^s,
$$

(34)

are the components of the two–form field strength $F_A = dA_A$. Here

$$
\partial^r_{\vec{x}^A} = \frac{\partial}{\partial \vec{x}^A}.
$$

A solution is [11]

$$
U_{AB} = U_{AB}^{\infty} + \frac{\rho_{APB}}{2|\sum_C \rho_{PC} \vec{x}_C - a|},
$$

(35)
where, if the metric is to be non–singular, \( \{p_A\} \) is a set of two coprime integers. \( \underline{a} \) is an arbitrary three–vector, specifying the ‘location’ of a 3-plane in \( \mathbb{E}^6 \), and \( U^\infty_{AB} \) is a constant. The metric is entirely non–singular, as should be the case for the D8–brane worldvolume. More general solutions, consisting of superpositions of (35) can be constructed \[11\]. With \( U^\infty_{AB} = \delta_{AB} \), the solution is asymptotic to \( \mathbb{E}^6 \times T^2 \). With \( p_{APB} = \delta_{AB} \), and \( \underline{a} = 0 \), the solution reduces to a product of two Euclidean Taub–NUT manifolds, with holonomy group \( H = Sp(1) \times Sp(1) \). The general solution, with \( H = Sp(2) \), admits three \( SO(8) \) Killing spinors, so will preserve \( \frac{3}{32} \) of the ten–dimensional supersymmetries. The isometry group in this case is just \( U(1)^2 \), generated by the Killing vectors \( \frac{\partial}{\partial \phi^A} \).

We turn, now, to the case in which the holonomy group of \( \mathcal{M}^8 \) is \( \text{Spin}(7) \). Complete, non–compact manifolds with \( H = \text{Spin}(7) \) have been constructed by Gibbons et al \[23\]; these take the form of \( \mathbb{R}^4 \) bundles over \( S^4 \), and have the line element

\[
ds^2 = \left( 1 - \left( \frac{M}{r} \right)^{\frac{4}{3}} \right)^{-1} dr^2 + \frac{9}{20} r^2 \left( d\mu^2 + \frac{1}{4} \sin^2 \mu \Sigma^2_s \right) + \frac{9}{100} r^2 \left( 1 - \left( \frac{M}{r} \right)^{\frac{4}{3}} \right) (\sigma_s - A^*)^2,
\]

where \( s = 1, 2, 3 \) and \( M \) is an integration constant. The \( \{\sigma_s\} \) and \( \{\Sigma_s\} \) are left–invariant one–forms on the principal \( SU(2) \) bundle over \( S^4 \), with the single–instanton connection

\[
A^* = \cos^2 \frac{\mu}{2} \Sigma_s.
\]

This metric is Ricci–flat as required and has the isometry group \( SO(5) \times SU(2) \). The singularity at \( r = M \) is a removable ‘bolt’ singularity which is topologically \( S^4 \). In the limit \( r \to M \), the metric is that on \( \mathbb{R}^4 \), up to an overall numerical constant, and the boundary at infinity is the squashed seven–sphere \[24\]. The D8–brane with such spatial sections preserves \( \frac{1}{32} \) of the ten–dimensional supersymmetries, since the \( \text{Spin}(7) \) manifold admits a single Killing spinor.

The fact that (36) is a self–dual manifold has been underlined in \[12\]: the same metric is a solution of the self–duality condition (31), with \( \omega_8^{abc} \) the components of the unique Hodge self–dual \( \text{Spin}(7) \)–invariant four–form. In this case, the self–duality conditions become \[19,12\]

\[
\omega_8^{abc} = \frac{1}{2} c_{\underline{a} \underline{b} \underline{c}} \omega^{\underline{d} \underline{e}},
\]

where now \( a, b = 1, \ldots 7 \) and \( c_{\underline{a} \underline{b} \underline{c}} \) are the octonionic structure constants. The D8–brane with such a worldvolume thus has spatial sections which are eight–dimensional gravitational
instantons. Since $S^7$ can be thought of as an $S^3$ bundle over $S^4$, and $S^3 = SU(2)$, we see why the $SU(2)$ connection appears in the metric. In some sense, this manifold is the eight-dimensional generalisation of the Eguchi–Hanson space, the latter making use of the fact that $S^3$ is an $U(1)$ bundle over $S^2$, and having a squashed three-sphere as its boundary at infinity.

3 7-Brane Domain Walls with Holonomy $G_2$

As explained in [4], the Romans theory can be dimensionally reduced to generate a massive nine-dimensional supergravity theory, despite the fact that the product of nine-dimensional Minkowski space with a circle is not a solution of the Romans theory. All one needs is a solution with a $U(1)$ isometry, and this is provided by the D8-brane solution. The new nine-dimensional theory has a ‘cosmological constant’ in exactly the same way as the Romans theory does; and this allows for the existence of a 7-brane domain wall solution — the double dimensional reduction of the D8-brane. Moreover, it was shown in [4] that this 7-brane solution is T–dual to the direct dimensional reduction of the D7-brane of type IIB supergravity, although the dimensional reduction in this case must be of a Scherk–Schwarz type [25].

To discuss the 7-brane domain wall of this massive nine-dimensional supergravity theory consider, first, a D8-brane solution of the form

$$ds^2 = F^2 (\gamma_{ij}(x)dx^i dx^j + dy_1^2) + dy_2^2,$$

where now $\gamma_{ij}$ is the metric on an eight-dimensional manifold. By taking $y_1$ as the coordinate on a circle, and with $F(y_2)$ and the dilaton as in (13), this is the generalisation of the dimensionally reduced D8-brane of [4]. By throwing away the circular dependance, we have 7-branes with general worldvolumes. These are solutions of the massive nine-dimensional theory of [4] if and only if the metric $\gamma_{ij}$ is Ricci–flat, exactly as in the case above. If the 7-brane worldvolume of the form

$$d\tilde{s}^2 = \gamma_{ij}(x)dx^i dx^j = -dt^2 + d\tilde{s}^2,$$

is to be supersymmetric, the holonomy group of $\mathcal{M}^7$, with line element $d\tilde{s}^2$, must be $H = G_2$. Such manifolds have again been constructed by Gibbons et al [23], and have a form similar to the Spin(7) manifold [2] discussed above. That is, $\mathcal{M}^7$ can be an $\mathbb{R}^3$ bundle over $S^4$, or an $\mathbb{R}^4$ bundle over $S^3$.
With an appropriate duality operator, and with the caveat to be discussed below, these manifolds solve the seven–dimensional self–duality condition, just as the Spin(7) manifold (30) does in eight dimensions. To see this, consider the self–duality condition (31) with \( \phi^a_{bc} \), the components of the \( G_2 \)–invariant, seven–dimensional Hodge dual of the octonionic structure constants. Explicitly [26],

\[
\phi^a_{bc} = \frac{1}{3!} \varepsilon^a_{bcdefgh} c^{efgh},
\]

where

\[
c_{abc} = +1 \quad \text{for} \quad abc = 123, 516, 624, 435, 471, 673, 572.
\]

We thus have

\[
\phi^a_{bc} = +1 \quad \text{for} \quad abc = 1245, 2671, 3526, 4273, 5764, 6431, 7531.
\]

The self–duality conditions (31) are then

\[
\omega_71 = \omega_{26} + \omega_{53}, \quad \omega_72 = \omega_{61} + \omega_{34}, \quad \omega_73 = \omega_{42} + \omega_{15},
\]

\[
\omega_74 = \omega_{23} + \omega_{65}, \quad \omega_75 = \omega_{46} + \omega_{31}, \quad \omega_76 = \omega_{12} + \omega_{54},
\]

\[
\omega_{63} = \omega_{25} + \omega_{14}.
\]

With the ansatz

\[
d\tilde{s}^2 = f^2(r) dr^2 + g^2(r) \Sigma_s^2 + h^2(r) (\sigma_s - A^s)^2,
\]

the metric on \( \mathcal{M}^7 \) has the form of an \( \mathbb{R}^4 \) principal bundle over \( S^3 \). Since \( S^3 \) is parallelizable, this bundle is trivial. \( \{ \Sigma_s \} \) are the left–invariant one–forms on the base space and \( \{ \sigma_s \} \) the left–invariant one–forms on the fibres. They satisfy

\[
\Sigma_s = -\frac{1}{2} \varepsilon_{stu} \Sigma_t \wedge \Sigma_u, \quad \sigma_s = -\frac{1}{2} \varepsilon_{stu} \sigma_t \wedge \sigma_u.
\]

\( A^s \) is the connection on the bundle, given by

\[
A^s = \frac{1}{2} \Sigma_s.
\]

With the orthonormal one–forms \( e^7 = fdr, \ e^s = g\Sigma_s, \) and \( e^\hat{s} = h (\sigma_s - A^s) \), where \( \hat{s} = 4, 5, 6 = 1, 2, 3 \), the connection one–forms are given by

\[
\omega_{\hat{7}} = \frac{g'}{fg} e^\hat{7}, \quad \omega_{\hat{7}} = \frac{h'}{fh} e^\hat{7}, \quad \omega_{\hat{s}} = \frac{1}{8g^2} \varepsilon_{stu} e^u.
\]
\[ \omega^a_L = -\frac{11}{2} g_{stu} \left( e^a + \frac{1}{4} h e^a \right), \quad \omega^a_L = -\frac{11}{2} g_{stu} \left( e^a + \frac{h}{g} e^a \right). \] (46)

Substituting into the self–duality conditions (43), we find the following first order differential equations

\[
\begin{align*}
\frac{1}{4} g^2 + \frac{g'}{fg} &= 0, \\
\frac{11}{2} h - \frac{1}{8} g^2 + \frac{h'}{f h} &= 0,
\end{align*}
\] (47a, b)

Using the reparametrisation invariance of the metric (44) under \( r \to r' = r'(r) \), we can set

\[ g^2(r) = \frac{1}{12} r^2, \] (48)

in which case, the solution of (47a, b) is

\[ f^2(r) = \left( 1 - \left( \frac{M}{r} \right)^3 \right)^{-1}, \quad h^2(r) = \frac{1}{9} r^2 \left( 1 - \left( \frac{M}{r} \right)^3 \right), \] (49)

where \( M \) is an arbitrary integration constant. This is precisely the metric found by solving the Einstein equations in [23]; here, we have derived it from first order equations alone. In direct analogy with the Spin(7) case (36) above, the ‘bolt’ singularity here is topologically \( S^3 \), and as \( r \to M \), the metric (44) reduces to the metric on \( \mathbb{R}^4 \). The boundary at infinity is \( S^3 \times S^3 \) and the isometry group is \( SO(4) \times SU(2) \).

The other seven–dimensional \( G_2 \) metric considered by Gibbons et al [23] is that of an \( \mathbb{R}^3 \) bundle over \( S^4 \). This is somewhat more complicated than the above example since the bundle is no longer trivial. It can be shown that \( G_2 \) holonomy implies self–dual curvature in the above (\( G_2 \)) sense and, for this reason, it should be expected that the \( \mathbb{R}^3 \) bundle over \( S^4 \) is self–dual in the same way as the \( \mathbb{R}^4 \) bundle over \( S^3 \). This is not the case, however, at least naively. That is, although the \( \mathbb{R}^3 \) bundle over \( S^4 \) is Ricci–flat and has \( G_2 \) holonomy, its connection one–form does not satisfy the relations (43). Moreover, neither does its curvature two–form satisfy the same relations with \( \omega_{ab} \) replaced by \( \Theta_{ab} \). A possible explanation of this is as follows. Transforming the orthonormal basis by an arbitrary \( SO(7) \) rotation will leave the line element unchanged. Such a rotation will have an effect on the curvature two form however: schematically, if \( e \to Ge \), where \( G \in SO(7) \), we have \( \Theta \to G^T \Theta G \). Then the question is whether this transformation commutes with the duality operator \( \frac{1}{2} \phi \) in (34). Indeed, since the duality operator is \( G_2 \)–invariant, and not \( SO(7) \)–invariant, it is unlikely to commute with such a rotation. It would seem, then, that a judiciously chosen \( SO(7) \) rotation
of the basis will ensure self-duality of the curvature two-form. This is similar to the four-dimensional case in which, via an $SO(4)$ rotation, a manifold with self-dual curvature can always be brought into a form such that its connection is also self-dual; the difference being, of course, that here we do not have a self-dual curvature in the first place.

At any rate, since both the $\mathbb{R}^4$ bundle over $S^3$ and the $\mathbb{R}^3$ bundle over $S^4$ are Ricci-flat manifolds of holonomy $H = G_2$, either manifold can be used as the spatial sections of a supersymmetric 7-brane in nine dimensions.

4 The General Domain Wall

Considerations similar to the above hold for domain walls in any supergravity theory. For example, we can dimensionally reduce the D8-brane along the lines of \cite{4}, generating domain walls in $D < 10$ massive supergravity theories. Or we can consider domain walls in massive gauged supergravity theories \cite{27}.

In $D$ dimensions, the domain wall is a $(D - 2)$-brane, which couples to a potential, or ‘cosmological constant’

$$V(\phi) = 2\Lambda e^{\alpha \phi}.$$  \hspace{1cm} (50)

$\alpha$ is a constant which can be parametrised as

$$\alpha^2 = \Delta + 2 \frac{(D - 1)}{(D - 2)}.$$ \hspace{1cm} (51)

The value of $\Delta$ varies from case to case \cite{14}. In certain vacua of gauged supergravities, we have $\Delta = -2(D - 1)/(D - 2)$, in which case $\alpha = 0$, and the potential is a true cosmological constant. On the other hand, some gauged seven- and four-dimensional supergravities have $\Delta = -2$. We will take $\Delta = 4$, however, since this is the value of $\Delta$ for the Romans theory and all its dimensional reductions. (Incidentally, the Howe-Lambert-West massive supergravity theory \cite{8} has $\Delta = 0$. It would seem, then, that this theory does not have the usual brane solutions, as in \cite{11}.)

The relevant Lagrangian is

$$\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2} |\partial \phi|^2 - V(\phi) \right),$$ \hspace{1cm} (52)

the equations of motion of which are

$$R_{ab} = \frac{1}{2} \partial_a \phi \partial_b \phi + \frac{V(\phi)}{D - 2} g_{ab},$$ \hspace{1cm} (53a)

$$D^2 \phi = \frac{dV}{d\phi}.$$ \hspace{1cm} (53b)
These have the domain wall solution [14]

\[
\begin{align*}
ds^2 &= H^{\frac{1}{b-2}}d\varphi \cdot d\varphi + H^{\frac{b-1}{b-2}} dy^2, \\
\exp\phi &= H^{-\frac{1}{2}},
\end{align*}
\]

where \( H(y) \) is harmonic on the single transverse direction. As explained in [14], we can take

\[
H(y) = 1 + m|y - y_0|,
\]

to avoid potential curvature singularities, as for the D8-brane above.

Keeping the dilaton field and the specific form of the harmonic function \( H(y) \) as above, we generalise the metric to

\[
ds^2 = H^{\frac{1}{b-2}} \gamma_{ij}(x) dx^i dx^j + H^{\frac{b-1}{b-2}} dy^2.
\]

This is a solution of the equations of motion (53a,b), as long as the metric \( \gamma_{ij} \) is Ricci–flat. That is, any domain wall — not just the D8-brane of above — can in general have a Ricci–flat worldvolume.

The remarks concerning supersymmetry that we made above can be applied here. That is, the general domain wall will preserve supersymmetry if and only if the worldvolume manifold with line element

\[
ds^2 = \gamma_{ij}(x) dx^i dx^j,
\]

admits Killing spinors. As already mentioned, the dimensional reduction of the Romans theory leads to a massive supergravity theory in nine dimensions, with a 7-brane domain wall. The worldvolume metric of this can then have spatial sections with holonomy group \( H = G_2 \). In \( D = 8 \), we can have a 6-brane supersymmetric domain wall, the spatial sections of which take the form of non–compact Calabi–Yau manifolds with holonomy group \( H = SU(3) \) [28].

Moving down in dimension, we can have 4-brane domain walls in \( D = 6 \) supergravity theories. These solutions will be supersymmetric if the spatial sections of their worldvolumes have holonomy group \( H = SU(2) \). In the compact case, we could consider

\[
ds^2 = -dt^2 + \tilde ds^2_{K3},
\]

where \( \tilde ds^2_{K3} \) is the metric on \( K3 \). This has the interpretation of a 4-brane wrapped on \( K3 \). Or, in the more interesting non–compact case,

\[
ds^2 = -dt^2 + \tilde ds^2_{TN},
\]
where $d\tilde{s}^2_{TN}$ is the asymptotically locally Euclidean Taub–NUT metric. The 4-brane worldvolume is then just the five–dimensional Kaluza–Klein monopole [29, 30].

An interesting example is that of the domain walls in Hořava–Witten theory. As is well known, the strongly coupled $E_8 \times E_8$ heterotic string theory is just M-Theory compactified on an $S^1/\mathbb{Z}_2$ orbifold with a set of $E_8$ gauge fields on each of the orbifold fixed planes [31, 32]. The compactification of the eleven–dimensional theory on a Calabi–Yau manifold leads to a gauged five–dimensional supergravity theory with two four–dimensional boundaries. This has a solution describing two parallel supersymmetric 3-brane domain walls located at the orbifold fixed planes [33]. On a further dimensional reduction, four–dimensional spacetime is identified with the 3-brane worldvolume — the universe as a domain wall scenario [33]. This being a specific example of the general case considered here, it should be obvious that the usual flat 3-brane domain walls can be generalised to any Ricci–flat worldvolume.

The relevant five–dimensional Lagrangian is [33]

$$S = S_{\text{bulk}} + S_{\text{boundary}},$$

$$S_{\text{bulk}} = \frac{1}{2} \int_{M^5} \sqrt{-g} \left( R - \frac{1}{2} \frac{1}{V^2} |\partial V|^2 - \frac{1}{3} \frac{1}{V^2} \alpha^2 \right),$$

$$S_{\text{boundary}} = \sqrt{2} \left( \int_{M^4(1)} \sqrt{-g} \frac{\alpha}{V} - \int_{M^4(2)} \sqrt{-g} \frac{\alpha}{V} \right),$$

where $V$ is a modulus which encodes the variation of the Calabi–Yau volume and $\alpha$ is a constant ‘mass term’, for the definition of which we refer the reader to [33]. The equations of motion will admit 3-brane solutions of the form

$$\begin{align*}
    ds^2 &= H_{ij}(x) dx^i dx^j + H^4 dy^2, \\
    V(y) &= H^3, \\
    H(y) &= 1 + \frac{2}{3} \alpha |y|,
\end{align*}$$

if and only if $\gamma_{ij}$ is Ricci–flat. (In the above we have set three arbitrary constants to unity.) We have checked this solution explicitly.

These 3-branes can have worldvolumes that are four–dimensional Schwarzschild black holes. Although not supersymmetric, since the Schwarzschild solution does not admit Killing spinors, this is of interest nonetheless. It explains how our four–dimensional universe, upon a further dimensional reduction, could have the form of a Schwarzschild black hole. It would seem that this embedding of the Schwarzschild solution in five dimensions violates the no–go theorem of [34] but in fact it does not. The theorem is that it is impossible to embed
the Schwarzschild solution in a flat five-dimensional spacetime; and our five-dimensional manifold is certainly not flat, so there is no contradiction.

5 Magnetic Branes in General

There has been some discussion in the literature of solutions of supergravity theories which describe generalised electric branes, in which the transverse space is no longer flat. Such solutions preserve fewer than the usual one half of the spacetime supersymmetries. The eleven-dimensional membrane solution can be generalised so as to interpolate between eleven-dimensional Minkowski space and $AdS_4 \times M^7$, where $M^7$ is any Einstein space [10]; the squashed seven-sphere for example, which admits Killing spinors and so allows for generalised supermembranes. Indeed, the eight-dimensional Spin(7) and Hyper-Kähler manifolds (36) and (32) above have also been considered as possible transverse spaces of the eleven-dimensional supermembrane [12, 11]; or as the transverse space of the fundamental string in ten dimensions [12]. Here we show that the worldvolume, as opposed to the transverse space, of the magnetic branes can also be generalised in just such a manner.

A general magnetic $p$-brane, with $p = D - n - 2$ couples to a rank $n$ field strength $F_{[n]}$. The domain wall is then a specific example of the more general magnetic brane, one with $n = 0$ (or, by Hodge duality, $n = D$) [14]. This leads us to consider whether we can generalise the worldvolume metric of any magnetic brane in the above manner; and the answer is that we can. Heuristically, since the field strength which couples to a magnetic brane is non-zero in the transverse directions only and, since we change the worldvolume metric only, the generalisation will go through in precisely the same way as above. Indeed, it was noted in [34] that the M5-brane worldvolume can be generalised to

$$ds^2 = -dt^2 + dx^2 + ds_{K3}^2.$$  \hspace{1cm} (59)

This solution has the interpretation of an M5-brane wrapped on $K3$, as considered from the point of view of the worldvolume action in [36]. It is a specific example of the more general claim that we are making here.

The Lagrangian relevant to the study of a general magnetic brane is

$$\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2} \partial \phi \cdot \partial \phi - \frac{1}{2(n!)} e^{\alpha \phi} F_{[n]}^2 \right),$$  \hspace{1cm} (60)

where the dimensionality of the brane is given by $d = p + 1 = D - n - 1$, and where $\alpha$ is as in (51) (we set $\Delta = 4$ here). $F_{[n]}$ is the field strength which couples to the $p$-brane. The
equations of motion

\[ R_{ab} = \frac{1}{2} \partial_a \phi \partial_b \phi + \frac{1}{2(2n-1)!} e^{\alpha \phi} \left( F_{a\alpha_1...\alpha_n} F^{\alpha_1...\alpha_n} - \frac{(n-1)}{n(D-2)} F^{2} g_{ab} \right), \]  

(61a)

\[ D^2 \phi = \frac{\alpha}{2n!} e^{\alpha \phi} F^2, \]  

(61b)

\[ D_a \left( e^{\alpha \phi} F^{a\alpha_1...\alpha_n} \right) = 0, \]  

(61c)

have the solution

\[
\begin{align*}
    ds^2 &= H^{-1/2} \gamma_{ij}(x) dx^i dx^j + H^{1/2} dy \cdot dy, \\
    e^\phi &= H^{-\alpha/2}, \\
    F_{\alpha_1...\alpha_n} &= \lambda \varepsilon_{\alpha_1...\alpha_n \beta} y^{\beta}, \\
\end{align*}
\]

(62)

as long as the worldvolume metric \( \gamma_{ij} \) is Ricci-flat. Here, \( \{ y^\alpha \} \) are the coordinates on the transverse space, \( r \) being the radial coordinate in these directions and \( H(r) \) is, as usual, harmonic on this space. \( \tilde{d} = D - d - 2 = n - 1 \) is the dimensionality of the electric brane dual to the magnetic one and the alternating tensor in the expression for the field strength has components \( \pm 1 \).

Consider the prototypical example: the M5-brane. The general solution of the eleven–dimensional equations of motion is

\[
\begin{align*}
    ds^2 &= H^{-\kappa} \gamma_{ij}(x) dx^i dx^j + H^{1/2} dy \cdot dy, \\
    F_{\alpha\beta\gamma\delta} &= \pm 3 k \varepsilon_{\alpha\beta\gamma\delta} \frac{y^\rho}{r^2}, \\
    H(r) &= 1 + \frac{k}{r^3}, \\
\end{align*}
\]

(63)

where \( k \) can be related to the tension of the brane by the inclusion of a source term in the action. These solutions are supersymmetric if and only if the worldvolume admits Killing spinors, just as for the cases considered above. The supersymmetry transformation of the gravitino is

\[
\delta \psi_a = \left( D_a - \frac{1}{288} (\Gamma_a \delta_{\beta\gamma\delta} - 8 \delta^\beta_a \Gamma_{\beta\gamma\delta}) F_{\beta\gamma\delta} \right) \epsilon, 
\]

(64)

where \( \epsilon \) is an arbitrary 32–component Majorana spinor. We make the usual 6 + 5 split

\[
\Gamma_a = (\gamma_\perp \otimes \Sigma, 1 \otimes \Sigma_a), 
\]

(65)

where \( \gamma_\perp \) and \( 1 \) are \( SO(5,1) \) matrices, \( \Sigma \) and \( \Sigma_a \) are \( SO(5) \) matrices, \( \Sigma = \Sigma_6 \ldots \Sigma_{10} \), so that \( \Sigma^2 = 1 \), and we take

\[
\epsilon(x, y) = \hat{\epsilon}(x) \otimes \eta(r). 
\]

(66)
Substituting for the solution (63), and setting the variation \( \delta \psi_\alpha = 0 \) gives

\[
\eta(r) = H^{-\frac{1}{12}}(r) \eta_0, \tag{67}
\]

for a constant spinor \( \eta_0 \), in addition to the usual chirality condition

\[
(1 \mp \Sigma) \eta = 0, \tag{68}
\]

which removes one half of the supersymmetries. The remaining condition, \( \delta \psi_i = 0 \), is satisfied if and only if

\[
\hat{D}_i \epsilon = 0, \tag{69}
\]

as promised. All solutions which satisfy this condition will saturate the usual Bogomol’nyi bound, as in the D8-brane case above.

Instead of the compact manifold in (59), we can make use of the asymptotically locally Euclidean Taub–NUT metric, giving a worldvolume of the form

\[
d\hat{s}^2 = -dt^2 + dx^2 + ds_{TN}^2, \tag{70}
\]

which is supersymmetric since the holonomy group of the Taub–NUT space is \( H = SU(2) \). There do not seem to be many other possibilities. The Ricci–flat M5-branes no longer interpolate between eleven–dimensional Minkowski space and \( AdS_7 \times S^4 \). For a worldvolume of the form (70), the solution at infinity in the transverse space is \( Mink_7 \times \text{Taub–NUT} \), i.e. the Kaluza–Klein monopole oxidised to eleven dimensions. As \( r \to 0 \), the line element is

\[
ds^2 = R^2(-dt^2 + dx^2) + 4k^2dR^2 + R^2ds_{TN}^2 + k^2d\Omega_4^2, \tag{71}
\]

which is the metric on the warped product of Taub–NUT with \( AdS_3 \times S^4 \), where the \( AdS_3 \) piece has cosmological constant \( \Lambda = -1/(4k^2/3) \).

The above considerations can be applied to any magnetic brane, although the specific form of the supersymmetric solutions must be considered case by case.

5.1 Worldvolume Solitons

Asymptotically, the spacetime metrics of the usual brane solutions (1) are flat and it is well–known that both D-branes and M-branes admit worldvolume solitons when embedded in such a flat spacetime (37). That is, the Dirac–Born–Infeld (DBI) worldvolume lagrangian is linearised and the energy of the branes is minimised for solitonic configurations of the
worldvolume fields. Now the more general solutions we have been discussing have a Ricci–
flat spacetime metric at infinity; and it is of interest to note that branes embedded in such
Ricci–flat spacetimes also have worldvolume solitons, as long as an (anti–)self–dual gauge
field can be constructed on the worldvolume manifold.

Consider, then, the simplest example, that of the D4-brane, the double dimensional
reduction of the M5-brane considered above. At infinity, the spacetime line element has the
form

$$ds^2 = -dt^2 + \gamma_{AB} dx^A dx^B + dy \cdot dy = -dt^2 + ds^2_{TN} + dy \cdot dy,$$

(72)

where $A, B = 1, \ldots, 4$ and $ds^2_{TN}$ is the asymptotically locally Euclidean Taub–NUT metric,
given by

$$ds^2_{TN} = M^2 V^{-1}(d\psi \pm \cos \theta d\phi)^2 + V(dR^2 + R^2 d\Omega^2),$$

$$V(R) = 1 + \frac{M}{R}.$$  

(73)

The plus (minus) sign corresponds to a self–dual (anti–self–dual) manifold respectively and
$M$ is an integration constant.

We work in an orthonormal basis, take all worldvolume scalars to be constant and split the
worldvolume coordinates $\xi^i = \{\xi^0, \xi^A\}$. For static configurations, $F_{0A} = 0$, and $-\det(g_{ij} + F_{ij}) = (-g_{00}) \det(g_{AB} + F_{AB})$, where $g_{ij}$ is the pullback of the spacetime metric. The DBI
lagrangian has the form

$$\mathcal{L} = e \left( 1 - \sqrt{1 + \frac{1}{4} F^2 + \frac{1}{4} \tilde{F}^2 + \frac{1}{16} (F \cdot \tilde{F})^2} \right)$$

$$= e \left( 1 - \sqrt{\left(1 \pm \frac{1}{4} F \cdot \tilde{F}\right)^2 - \frac{1}{4} \text{tr} \left| F \mp \tilde{F} \right|^2} \right),$$

(74)

where we have set the brane tension and the inverse string tension $2\pi\alpha'$ to unity. $e$ is the
determinant of $e_A^A$, the vierbein of $g_{AB}$. $F^2 = F_{AB} F^{AB}$, $\tilde{F}$ is the Hodge dual of $F$ with
respect to the worldspace directions and $F \cdot \tilde{F} = F_{AB} \tilde{F}^{AB}$. The lagrangian is linearised for
configurations which satisfy $F_{AB} = \pm \tilde{F}_{AB}$ and, since we are dealing with the purely static
case the energy density, $T^{00} = -\mathcal{L}$, is minimised for such (anti–)self–dual field strengths [37].

D4-branes embedded in a general Ricci–flat spacetime will thus have worldvolume solitons
— in this case, abelian (anti–)instantons — if and only if a field strength which is (anti–)
self–dual with respect to $g_{AB}$ can be constructed.

Since no worldvolume scalars are excited, the spatial component of the pullback of the
spacetime metric in the static gauge is just $g_{AB} = \gamma_{AB}$. The gauge field which gives an (anti–
self–dual field strength with respect to this metric is well–known. With the orthonormal
one–forms \( e^0 = V^{1/2}dR, \ e^1 = V^{1/2}Rd\theta \) and \( e^2 = V^{1/2}R\sin \theta d\phi, \ e^3 = MV^{-1/2}(d\psi \pm \cos \theta d\phi) \), it is

\[
A = \frac{1}{M} V^{-1/2} e^3, \tag{75}
\]

which has the field strength

\[
F = \frac{1}{(R + M)^2}(e^0 \wedge e^3 \mp e^1 \wedge e^2). \tag{76}
\]

This is manifestly (anti–)self–dual. Although the gauge field has the usual string–like singu-
laritiy along the \( z \)–axis, the energy of the instanton is finite.

D4–branes embedded in the spacetime (72) thus have worldvolume solitons in the same
way that the standard flat D4–branes do, since (anti–)self–dual gauge fields can still be
constructed on their spatial sections. This is of interest for two reasons. Firstly, it should
be the case that the M5–brane with worldvolume metric (70) discussed above will have
worldvolume string solitons along similar lines as for the flat case. Secondly, it should
be possible to generalise our reasoning to the non–abelian DBI action describing
multiple D4–branes. In particular, since it is known how to construct (anti–)self–dual \( SU(2) \)
gauge fields on Taub–NUT space, we could consider the action describing two D4–branes
with a two–centered Taub–NUT space as the spatial sections of the worldvolume. The energy
of such configurations will be minimised by these more general non–abelian instantons is
precisely the same way as for the flat case. This might provide some clues as to the
nature of D–branes in curved spacetime, a subject which is far from being fully understood.

6 Conclusions

We have shown that the worldvolumes of all the magnetic branes of string theory can take
the form of any Ricci–flat manifold. A specific case is the domain wall, the ten–dimensional
example being that of the D8–brane of string theory. We have shown in detail that the
D8–brane solution of Romans’ massive type IIA supergravity theory can be generalised to
include a large class of possible solutions. These are supersymmetric if and only if the
worldvolume manifold admits Killing spinors, a necessary, but not sufficient, requirement for
which is Ricci–flatness.

We have described some of the eight–dimensional manifolds in the literature which do
admit Killing spinors, and elaborated on the fact that these are just the self–dual eight–
dimensional gravitational instantons. The dimensional reduction of the D8-brane is a seven–dimensional domain wall in nine dimensions. In this case, we have shown that the known manifolds which admit Killing spinors satisfy the self–duality condition in seven dimensions, although the connection between supersymmetric and self–dual manifolds would seem to be a subtle one in certain cases.

The fact that the magnetic branes can have Ricci–flat worldvolumes is perhaps to be expected, given the similar results concerning the transverse spaces of the electric branes: roughly speaking, the transverse space of an electric brane is interchangeable with the worldvolume of the dual magnetic brane. More speculatively, perhaps the requirement of Ricci–flatness in these cases is a consequence of the beta functions of string theory. After all, to first order in the inverse string tension $\alpha'$, the beta functions of the conformal field theory on the closed string worldsheet imply that the ambient spacetime must be Ricci–flat [4].

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