AIRCRAFT ROUTING AND CREW PAIRING: UPDATED ALGORITHMS AT AIR FRANCE

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Abstract. Aircraft routing and crew pairing problems aims at building the sequences of flight legs operated respectively by airplanes and by crews of an airline. Given their impact on airlines operating costs, both have been extensively studied for decades. Our goal is to provide reliable and easy to maintain frameworks for both problems at Air France. We propose simple approaches to deal with Air France current setting. For routing, we introduce a compact MIP formulation that can be solved by current MIP solvers in at most a few minutes even on Air France largest instances. Regarding crew pairing, we provide a standard methodology to model the column generation pricing subproblem within a new resource constrained shortest path framework recently introduced by the first author. This new framework, which can be used as a black-box, leverages on bounds on paths' resource to discard partial solutions and speed-up the resolution. The resulting approach enables to solve to optimality Air France largest instances. Recent literature has focused on integrating aircraft routing and crew pairing problems. As a side result, we have been able to solve to near optimality large industrial instances of the integrated problem by combining the aforementioned algorithms within a simple cut generating method.

1. Introduction

1.1. Context. Interactions between Operations Research and Air Transport Industry have been successful for at least five decades [6]. These interactions have taken various forms: yield management, airplane timetabling, ground operations scheduling, air traffic management, etc. Key applications are notably the construction of sequences of flight legs operated by airplanes and crews. As airplane sequences of legs are routes and crew sequences pairings, this construction is therefore called aircraft routing for airplanes, and crew pairing for crews.

The present paper focuses on these two applications and is the fruit of a research partnership with Air France, the main French airline. We aim at providing a reliable and easy to maintain framework that can cope with the specific and challenging industrial context of the company. Air France working rules are much richer than the IATA standards: collective agreements reach hundreds of pages, and two of the most cited references on crew pairing [12, 22] develop ad-hoc approaches to build pairings satisfying the company’s rules. While the aircraft routing remains easy, this turns the exact resolution of Air France crew pairing into a challenge.

As crews need time to cross airports if they change of airplane, the two problems are linked and the sequential resolution currently in use in the industry is suboptimal. Solving the integrated problem has been identified by academics as a difficult problem. A request
also formulated by Air France was to propose a solution scheme for that integrated problem
easy to adapt and maintain in their industrial setting.

1.2. Literature review. Aircraft routing is the easiest of the two problems. Usually, costs
are not taken into account when routes are built, making aircraft routing a feasibility prob-
lem. The key aspect is that routes must regularly visit some specific airports to enable
maintenance operations. Aircraft routing is sometimes turned into an optimization problem,
either to maximize profit when the fleet is heterogeneous [5, 13], or to minimize delay propa-
gation along sequences of flights [21]. As Air France fleets are homogeneous and delay is not
taken into account, we stick to the tradition and model the aircraft routing as a feasibility
problem in this paper. State-of-the-art solution approaches rely on column generation [5, 13],
where columns are sequences of flight legs between airports where maintenance checks can
be performed. They can solve to optimality large instances of the optimization version in a
few hours. Alternative approaches include heuristics [15] and Lagrangian relaxation [9].

Crew variables wages and hotel rooms are among airlines first sources of variable costs.
As both depend on the sequences of flight legs crews operate, crew pairing is an intensively
studied optimization problem, see Gopalakrishnan and Johnson [16] for a review. Two
elements make crew pairing difficult. First, major airlines must solve huge instances with
up to several thousand flight legs. Second, regulatory agencies and collective bargaining
agreements list numerous working rules that make the crew pairing problem highly non-
linear. For instance, the cost structure of pairings is typically represented as a non-linear
function of the flying duration, the total working time, the resting time, and the total time
the crews spend away from home. Since the seminal work of Minoux [27], state-of-the-art
approaches solve the crew pairing by column generation [1, 4, 8, 12, 17, 19, 20, 22, 32, 35].
They consider set partitioning formulations where columns are possible pairings. These
methods hide the non-linearity in the pricing subproblem, which can be efficiently solved
using resource constrained shortest path approaches [18]. As a large part of the working
rules apply to duties, i.e., subsequences of a pairing formed by the flight legs operated on
a same day, two strategies are used to solve the pricing subproblem. The first strategy
starts by building duties and then combine them to obtain a pairing, while the second
directly builds a pairing. The first strategy enables to split the complexity of the working
rules, and therefore makes the modeling easier. It is therefore the most common and has
been adopted by Desaulniers et al. [12] to model Air France working rules. However, as
the number of non-dominated pairings is huge, solving the pricing subproblem becomes
costly on large instances. Formulations where columns are the duties can be used when
working rules are not too complicated [34]. When they are really simple, compact integer
programming approaches [7] where variables indicate if a given connection is used can even
be considered. However, this is generally not the case, and such models are generally turned
into initialization heuristics [4] where only a subset of working rules are considered. Other
alternatives to column generation include heuristics [3] and metaheuristics [23].

During the last decade, integrated Crew Pairing and Aircraft Routing problem has raised
an increasing attention. Moving from a sequential to an integrated approach enables to
reduce the cost by 5% on average according to Cordeau et al. [11], and 1.6% according to
Papadakos [28]. Solution methods are column generation based matheuristics [10, 25, 26, 28].
Ruther et al. [30] considered the integration of these problems with tail assignment,
which decides which physical aircraft operates which sequence of legs, and Salazar-González
[31] with crew rostering, which decides which sequences of legs each physical crew member operates.

1.3. Contribution and methods. As emphasized in the literature review above, the traditional approach to aircraft routing is column generation, and so was also the approach at Air France. Using a natural integer program to model the problem, we show that state-of-the-art MIP solvers are able to solve any industrial instance met by the company in less than 4 minutes on a standard computer. It is of interest for the company since it greatly simplifies the algorithmic machinery for their aircraft routing problems. It was quite surprising for us that this approach does not seem to have been used elsewhere. An explanation could be that the recent dramatic improvements of the MIP solvers allow now to deal with integer program sizes that were completely out-of-reach ten years ago.

A second contribution is the design of an efficient column generation method for the crew pairing. It leverages in the shortest path framework proposed by the first author [29] to avoid the construction of the duties network [12] and speed-up the resolution of the pricing subproblem. Contrary to the integer program we propose for the aircraft routing problem, the techniques we use are quite involved, but taking the shortest path framework as a “black-box” (which is intended to), the overall method is easy to implement and can be used in practice by the company. Perhaps more interesting, we are able to solve any industrial instance to optimality within a few minutes.

We finally combine the two approaches to get a simple method for solving the so-called integrated aircraft routing crew pairing problem. Again, our objective is to propose a method as simple as possible, to make it usable and maintainable by the company. Experiments show that the method is able to find solutions within an optimality gap of 10 euros on instances with up to 1626 flight legs.

2. Compact integer program for aircraft routing

2.1. Problem formulation. Building the sequences of flight legs required for aircraft routing formally corresponds to solving the following problem.

The input is formed by a set of airports, a collection \( \mathcal{L} \) of flight legs, and a number \( n_a \) of airplanes. Some airports are bases in which maintenance checks can be performed. A flight leg is characterized by departure and arrival airports, as well as departure and arrival times (it is of course assumed that departure time is smaller than arrival time for any flight leg). A route is a sequence of flight legs such that any two consecutive flight legs \( \ell, \ell' \) satisfy:

- the arrival airport of \( \ell \) is the departure airport of \( \ell' \)
- the departure time of \( \ell' \) minus the arrival time of \( \ell \) is bounded from below by a fixed quantity (which can depend on the airport, the time, and the fleet).

We consider the flight legs on a weekly horizon: the departure and arrival times are given for a typical week. The task consists in determining a collection of disjoint routes covering \( \mathcal{L} \) such that the number of airplanes needed to operate these routes is non-greater than \( n_a \), and such that airplanes following these routes in a cyclic way (repeating the solution week after week) spend a night in a base every \( \delta_{\text{maint}} \) days, where \( \delta_{\text{maint}} \) is a fixed parameter.

Deciding whether there is a feasible solution is NP-complete [33].
2.2. **Modeling as an integer program.** We first explain how the problem can be modeled as a disjoint cycle problem in a directed graph. This will make easier the description of the integer program.

Define the directed graph $D = (V, A)$ as follows. Its vertex set is $L \times [\delta_{\text{maint}}]$. Each vertex corresponds to a flight leg with the number of days since the last visit to a base. An ordered pair $((\ell, k), (\ell', k'))$ is in $A$ if the arrival airport of $\ell$ is the departure airport of $\ell'$ and we are in one of the three following situations:

- $\ell$ and $\ell'$ are performed during the same day and $k = k'$,
- $\ell$ and $\ell'$ are not performed on the same day, the airport is a base, and $k' = 1$,
- $\ell$ and $\ell'$ are not performed on the same day, the airport is not a base, and $k' - k$ is the number of days between the arrival of $\ell$ and the departure of $\ell'$.

An arc corresponds to two flight legs that can be consecutive in a route, with the suitable restrictions on the number of days since the last visit to a base. A route corresponds to a cycle in this graph. We choose arbitrarily one instant in the week and we denote by $A_0$ the set of arcs $((\ell, k), (\ell', k'))$ “crossing this instant”, i.e., such that at this instant $\ell$ or the waiting time between the arrival of $\ell$ and the departure of $\ell'$ contains the instant. With the notation $V_\ell = \{((\ell, k) \in V : k \in [\delta_{\text{maint}}]\}$, a feasible solution of the aircraft routing problem induces vertex disjoint cycles in $D$, visiting each $V_\ell$ exactly once and having at most $n^a$ arcs in $A_0$. (Recall that $n^a$ is the number of airplanes.)

Conversely, any collection of vertex disjoint cycles satisfying these properties forms a feasible solution: any cycle provides a route, possibly of several week. If it intersects $A_0$ a number $k$ times, then it can be operated by $k$ airplanes. Indeed, the number of times it intersects $A_0$ is equal to the number of weeks this cycle lasts, as illustrated on Figure 1. Thus the numbers of arcs selected in $A_0$ is an upper bound on the number of airplanes required to operate such a solution.

Therefore, the aircraft routing problem is equivalent to decide whether the following integer program has a feasible solution

\[
\begin{align*}
\sum_{a \in \delta^-(v)} x_a &= \sum_{a \in \delta^+(v)} x_a \quad \forall v \in V \\
\sum_{a \in \delta^-(\ell)} x_a &= 1 \quad \forall \ell \in \mathcal{L} \\
\sum_{a \in A_0} x_a &\leq n^a \\
x_a &\in \{0, 1\} \quad \forall a \in A.
\end{align*}
\]

**Figure 1.** A two-weeks route crosses $A_0$ twice
3. Column generation approach to crew pairing

3.1. Problem formulation. A pairing $p$ is a sequence of flight legs that can be operated by a crew. We denote by $P$ the set of pairings that satisfy the working rules. The definition of $P$ is technical, and we do not need it in this section. Examples of working rules that must be satisfied by pairings are given in Section 4. The period between the departure of the first leg and the arrival of the last leg of a pairing can span at most four days. A pairing is long if it spans more than three days. A duty is the subsequence of a pairing formed by the flight legs operated on the same day. A duty is long if it contains more than three legs, and short otherwise. Operating a pairing $p$ in $P$ has a cost $c_p$ that corresponds to crew wages and hotel nights.

Given a set $L$ of flight legs, solving the Crew Pairing problem consists in selecting pairings so that each leg $\ell$ in $L$ belongs to exactly one of them. Air France working rules impose that the proportion of long pairings in the solution is non-greater than a quantity $\alpha$, and that the proportion of long duties is non-greater than a quantity $\beta$.

3.2. Modeling as an integer program. Contrary to the aircraft routing, we do not use a specific frontal MIP to solve the crew pairing, but a column generation approach. The binary variable $y_p$ indicates if a pairing $p$ in $P$ belongs to the solution.

\[
\begin{align*}
\text{(CP)} & \quad \min & \sum_{p \in P} c_p y_p \\
\text{s.t.} & \quad \sum_{p \ni \ell} y_p = 1 \quad \forall \ell \in L \\
& \quad \sum_{p \in P} y_p \leq \alpha \sum_{p \in P} y_p \\
& \quad \sum_{p \in P} \left( (1 - \beta) \Delta^l(p) - \beta \Delta^s(p) \right) y_p \leq 0 \\
& \quad y_p \in \{0,1\} \quad \forall p \in P,
\end{align*}
\]

where $p \ni \ell$ means that the flight leg $\ell$ is present in $p$, where $P^l$ is the set of long pairings, and where $\Delta^s(p)$ (resp. $\Delta^l(p)$) is the number of short (resp. long) duties in a pairing $p$. The first constraint ensures that each leg is covered, the second that the proportion of long pairings is non-greater than $\alpha$, and the third that the proportion of long duties is non-greater than $\beta$.

3.3. Column generation approach. We propose an exact method for solving the program (CP). It is based on column generation, topic in which we assume some basic knowledge from the reader. More details on that topic can be found for instance in a survey by Lübbecke [24].

The algorithm maintains a subset of pairings $P' \subseteq P$.

(i) Initialize $P'$ in such a way that (CP) restricted to $P'$ is feasible (e.g., taking all possible pairings of two flight legs makes the job).

(ii) Solve the linear relaxation of (CP) restricted to $P'$ with any standard solver. Denote by $c^{\text{LB}}$ its optimal value.

(iii) Find a pairing $p$ of minimum reduced cost $\bar{c}_p$. If $\bar{c}_p < 0$, then add $p$ to $P'$ and go back to (ii).
(iv) Solve \((CP)\) restricted to \(\mathcal{P}'\) with any standard solver. Denote by \(c^{\text{UB}}\) its optimal value.

(v) Find all pairings \(p\) with reduced cost non-larger than \(c^{\text{UB}} - c^{\text{LB}}\) and add them to \(\mathcal{P}'\).

(vi) Solve \((CP)\) restricted to \(\mathcal{P}'\) with any standard solver and return its optimal solution \(y^*\).

The problem in Step (iii) of finding a pairing \(p\) of minimum reduced cost \(\tilde{c}_p\) is the pricing subproblem. Finding all the pairings with reduced cost less than or equal to \(c^{\text{UB}} - c^{\text{LB}}\) in Step (v) is a variant of it. Section 4 describes the proposed method to solve the pricing subproblem.

Since the total number of possible pairings is finite, Step (ii) – which consists in solving the “master problem” – is repeated only a finite number of times, and thus the overall method terminates in finite time. After having performed Step (ii) for the last time, \(c^{\text{LB}}\) is a lower bound on the optimal value of \((CP)\). Step (iv) provides a first feasible solution of \((CP)\). The value \(c^{\text{UB}}\) is thus an upper bound on the optimal value of \((CP)\). Steps (v) and (vi) are less standard with respect to the classical column generation theory. At the end of Step (vi), the solution \(y^*\) is an optimal solution of \((CP)\). This fact relies on the following lemma, due to Baldacci et al. [2, Corollary 1].

**Lemma 1.** Consider an integer program in standard form with variables \((z_i)\), and suppose given an upper bound \(UB\) as well as a lower bound \(LB\) on its optimal value. Let \((\tilde{c}_i)\) be the reduced costs of the variables when the linear relaxation of the program has been solved to optimality. Then for every \(i\) such that \(\tilde{c}_i > UB - LB\), the variable \(z_i\) is equal to 0 in all optimal solutions.

### 4. Pricing subproblem

We now introduce a solution scheme for the pricing subproblem

\[
\min_{p \in \mathcal{P}} \tilde{c}_p.
\]

As pairings can be considered as paths satisfying some constraints in a certain graph, the pricing subproblem is generally solved as a resource constrained shortest path problem. We model it within the **Monoid Resource Constrained Shortest Path Problem** framework [29], which we now briefly describe. This framework is rather abstract, but practically, it only requires to implement a few operators on the resource set.

#### 4.1. Framework and algorithm

A binary operation \(\oplus\) on a set \(M\) is associative if \(q \oplus (q' \oplus q'') = (q \oplus q') \oplus q''\) for \(q, q', q'' \in M\). An element 0 is neutral if \(0 \oplus q = q \oplus 0 = q\) for any \(q \in M\). A set \((M, \oplus)\) is a monoid if \(\oplus\) is associative and admits a neutral element. A partial order \(\preceq\) is compatible with \(\oplus\) if the mappings \(q \mapsto q \oplus q'\) and \(q \mapsto q' \oplus q\) are non-decreasing for all \(q'\) in \(M\). A partially ordered set \((M, \preceq)\) is a lattice if any pair \((q, q')\) of elements of \(M\) admits a greatest lower bound or meet denoted \(q \wedge q'\), and a least upper bound or join denoted \(q \vee q'\). A set \((M, \oplus, \preceq)\) is a lattice ordered monoid if \((M, \oplus)\) is a monoid, \((M, \preceq)\) is a lattice, and \(\preceq\) is compatible with \(\oplus\).

Given a lattice ordered monoid \((M, \oplus, \preceq)\), a digraph \(D = (V, A)\), an origin vertex \(o\), a destination vertex \(d\), and two non-decreasing mappings \(c : M \to \mathbb{R}\) and \(\rho : M \to \{0, 1\}\), the
Monoid Resource Constrained Shortest Path Problem seeks

an $o$-$d$ path $P$ of minimum $c \left( \bigoplus_{a \in P} q_a \right)$ among those satisfying $\rho \left( \bigoplus_{a \in P} q_a \right) = 0$.

The sum $\bigoplus_{a \in P} q_a$ is the resource of a path $P$, and we denote it $q_P$. The real number $c(q_P)$ is its cost, and the path $P$ is feasible if $\rho \left( \bigoplus_{a \in P} q_a \right)$ is equal to 0. We therefore call $c$ and $\rho$ the cost and the infeasibility functions. When the last vertex of $P$ is the first vertex of $Q$, we denote by $P + Q$ the path $P$ followed by the path $Q$.

We now describe an enumeration algorithm for the Monoid Resource Constrained Shortest Path Problem. It follows the standard labeling scheme [18] for resource constrained shortest paths. The lattice ordered monoid structure enables to extend these algorithms to new problems and to speed them up thanks to new tests and keys. A list $L$ of partial paths $P$ and an upper bound $c^{UB}_o$ on the cost of an optimal solution are maintained. Initially, $L$ contains the empty path at the origin $o$, and $c^{UB}_o = +\infty$. While $L$ is not empty, the following operations are repeated.

(i) Extract a path $P$ of minimum “key” from $L$. Let $v$ be the last vertex of $P$.

(ii) If $v = d$, then: if $\rho(q_P) = 0$ and $c(q_P) < c^{UB}_i$, update $c^{UB}_i$ to $c(q_P)$.

(iii) Else if “test(s)” return(s) “yes”, extend $P$: for each arc $a$ outgoing from $v$, add $P + a$ to $L$.

The key in Step (i) and the test(s) in Step (iii) must be specified to obtain a practical algorithm.

From now on, we suppose to have for each $v$ in $V$ a set $B_v$ of bounds such that, for each $v$-$d$ path $Q$, there is a $b \in B_v$ with $b \preceq q_Q$. The key and one test rely on these sets $B_v$, see Section 4.3 for more details on how these sets $B_v$ are built. We use $\min \{ c(q_P \oplus b) : b \in B_v, \rho(q_P \oplus b) = 0 \}$ as key of Step (i), where $v$ is the last vertex of $P$. Experiments have shown that with suitably defined $B_v$ this key is a good lower approximation of the cost of a feasible $o$-$d$ path starting by $P$. A first test used in Step (iii) is based on the notion of dominance. A path $P$ dominates a path $Q$ if $q_P \preceq q_Q$. The dominance test uses a list $L^{nd}_v$ of non-dominated $o$-$v$ paths for each vertex $v$. It can be expressed as follows.

(Dom) Is $P$ dominated by no path in $L^{nd}_v$?

If the answer is “yes”, then before extending $P$, we remove all the paths dominated by $P$ from $L^{nd}_v$ and add $P$ to $L^{nd}_v$. This test relies on the fact that there is an optimal solution whose subpaths are all non-dominated. The second test we use is the following lower bound test.

(Low) Is there a bound $b$ in $B_v$ such that $\rho(q_P \oplus b) = 0$ and $c(q_P \oplus b) \leq c^{UB}_o$?

If the answer to this test is “no”, then any $o$-$d$ path starting by $P$ is either infeasible or of cost greater than $c^{UB}_o$. Following the same technique as the one in the preamble of Section 8 of [29], we have the following result.

Proposition 2. Suppose that $D$ is acyclic. Then for any combination of the tests (Low) and (Dom), the algorithm converges after a finite number of iterations, and, at the end of the algorithm, $c^{UB}_o$ is equal to the cost of an optimal solution of the Monoid Resource Constrained Shortest Path Problem if such a solution exists, and to $+\infty$ otherwise.

It is well known that the use of lower bounds is a key element in the performance of the enumeration algorithms [14]. However, in the standard resource constrained shortest path
framework [18], a unique bound is used, and there is no standard way to build the lower bounds for non-linear problems. The Monoid Resource Constrained Shortest Path Problem framework enables to use the lower bound test (Low) in non-linear contexts. Indeed, the monoid structure enables to compute paths resources in both direction, the ordered monoid structure to use bounds, and the lattice structure to compute these bounds. Besides, the originality of the lower bound test (Low) is that it uses sets of bounds $B_v$ instead of single bounds and can thus discard more paths. These features of the lower bound test (Low) happen to be a key element in the performance of our approach to the pricing subproblem.

Step (v) of the algorithm of Section 3.3 requires to solve the following variant of the Monoid Resource Constrained Shortest Path Problem:

$$\text{generate all the } o-d \text{ paths } P \text{ satisfying } \rho(q_P) = 0 \text{ and } c(q_P) \leq c^{UB},$$

where $c^{UB}$ is a given constant. To solve this variant, we only need to make small changes in the algorithm. First, the dominance test should not be used anymore. Second, a list $P$ of paths is maintained. This list is initially empty. Third, $c^{UB}_{od}$ is replaced by $c^{UB}$ everywhere in the algorithm and in the tests. Finally, Step (ii) is replaced by

(ii$'$) If $v = d$, then: if $\rho(q_P) = 0$ and $c(q_P) < c^{UB}$, add $P$ to $P$.

The algorithm still converges after a finite number of iterations when $D$ is acyclic, and at the end of the algorithm, $P$ contains all the $o-d$ paths with cost non-greater than $c^{UB}$.

4.2. **Modeling the pricing subproblem.** We now explain how to model our pricing subproblem in the Monoid Resource Constrained Shortest Path Problem framework. We therefore elaborate on the structure of the pairing set $P$. As a pairing lasts at most four days, we solve a shortest path problem for each period of four consecutive days in the week. This will enable us to work on acyclic digraphs. To comply with the IR-OPS regulation of the European Aviation Safety Agency and with the Air France own working rules, a pairing must satisfy more than 70 rules. Fortunately, these rules share similar structures and can therefore be modeled using a limited number of ideas. Our purpose here is to introduce these ideas and not to get into details of the intricacies of the IR-OPS and Air France regulations. We therefore illustrate these ideas on a toy pricing subproblem with only a few but difficult rules, which we now introduce.

A pair of legs $(\ell_1, \ell_2)$ is a **connection** if a crew can operate $\ell_2$ after $\ell_1$. Practically, this is the case when $\ell_2$ departs after the arrival of $\ell_1$ from the airport where $\ell_1$ has arrived. If the arrival of $\ell_1$ and the departure of $\ell_2$ are on the same day, then $(\ell_1, \ell_2)$ is a **day connection**. Otherwise, it is a **night connection**. During a night connection, a crew goes to a hotel to rest. If the duration of this night connection is smaller than a threshold, then it is a **reduced rest**. A sequence $p$ of flight legs $(\ell_1, \ldots, \ell_k)$ such that $(\ell_i, \ell_{i+1})$ is a connection for every $i$ is in $P$ if it satisfies the following rules.

(a) $\ell_1$ starts and $\ell_k$ ends in one of Paris airports.
(b) The number of legs in any duty in $p$ is non-greater than 4. If the night connection before the first leg of a duty is a reduced rest, then the number of legs in this duty must be non-greater than 3.
(c) The total flying duration of a crew in a duty is non-greater than $F(t)$, where $F$ is a given function and $t$ is the time at which the first leg of the duty departs.
The objective of the toy pricing subproblem is to find a pairing \( p \) of minimum reduced cost \( \tilde{c}_p \).

For simplicity, we omit in the master problem the long pairings and long duties constraints. The reduced cost is then of the form \( \tilde{c}_p = \sum_{\ell \in p} z_\ell \), where \( z_\ell \) is the dual variable associated to the partitioning constraint.

We now model this toy subproblem as a **Monoid Resource Constrained Shortest Path Problem**. Let \( D = (V, A) \) be the digraph defined as follows. The vertex set is \( V = \mathcal{L} \cup \{o, d\} \), where \( \mathcal{L} \) is the set of legs of four consecutive days, \( o \) is an origin vertex, and \( d \) is a destination vertex. The arc set \( A \) contains an arc \((o, \ell)\) for all legs \( \ell \) starting in Paris, an arc \((\ell, d)\) for all legs \( \ell \) ending in Paris, and an arc \((\ell, \ell')\) for each connection \((\ell, \ell')\). Let \( F_m = \max F(t) \).

The monoid \( M \) we use is of the form \( M^\rho \times \mathbb{R} \), where \( M^\rho = (\mathbb{Z} \times \mathbb{R}_+) \cup (\mathbb{Z}_+ \times \mathbb{R}_+) \cup \{\infty\} \). A pairing \( p \) which spans a single day, has \( n \) legs, and flying duration \( f \), has a resource \( r_p \in M^\rho \) of the form \((n, f) \in \mathbb{Z}_+ \times \mathbb{R}_+ \). A pairing which spans several days, has \( n^b \) (resp. \( n^e \)) legs and flying duration \( f^b \) (resp \( f^e \)) on its first (resp. last) day has a resource of the form \( r_p = (n^b, f^b, n^e, f^e) \).

We define the operator \( \oplus \) on \( M^\rho \) as follows.

\[
\begin{align*}
  r \oplus \infty &= \infty \oplus r = \infty \quad \text{for all } r \\
  (n, f) \oplus (\tilde{n}, \tilde{f}) &= (n + \tilde{n}, f + \tilde{f}) \\
  (n, f) \oplus (\tilde{n}^b, \tilde{f}^b, \tilde{n}^e, \tilde{f}^e) &= (n + \tilde{n}^b, f + \tilde{f}^b, \tilde{n}^e, \tilde{f}^e) \\
  (n^b, f^b, n^e, f^e) \oplus (\tilde{n}, \tilde{f}) &= (n^b, f^b, n^e + \tilde{n}, f^e + \tilde{f}) \\
  (n^b, f^b, n^e, f^e) \oplus (\tilde{n}^b, \tilde{f}^b, \tilde{n}^e, \tilde{f}^e) &= \begin{cases} 
    \infty & \text{if } n^e + \tilde{n}^b > 4 \text{ or } f^e + \tilde{f}^b > F_m, \\
    (n^b, f^b, \tilde{n}^e, \tilde{f}^e) & \text{otherwise.}
  \end{cases}
\end{align*}
\]

We define \( \preceq \) on \( M^\rho \) by

\[
\begin{align*}
  (0, 0) &\preceq q \quad \text{and} \quad q \preceq \infty \quad \text{for all } r \in M^\rho \\
  (n, f) &\preceq (\tilde{n}, \tilde{f}) \quad \text{if} \quad n \leq \tilde{n} \quad \text{and} \quad f \leq \tilde{f} \\
  (n^b, f^b, n^e, f^e) &\preceq (\tilde{n}^b, \tilde{f}^b, \tilde{n}^e, \tilde{f}^e) \quad \text{if} \quad n^b \leq \tilde{n}^b, \quad f^b \leq \tilde{f}^b, \quad n^e \leq \tilde{n}^e, \quad \text{and} \quad f^e \leq \tilde{f}^e,
\end{align*}
\]

and a pair \((n, f) \neq (0, 0)\) is not comparable with \((n^b, f^b, n^e, f^e)\).

With these definitions, it can be checked that \((M^\rho, \oplus, \preceq)\) is a lattice ordered monoid. As \((\mathbb{R}, +, \leq)\) is a lattice ordered monoid, the product \( M^\rho \times \mathbb{R} \) is a lattice ordered monoid when endowed with the product sum and order. Finally, given \( q = (r, z) \in M \), we define

\[
\rho((r, z)) = \rho_{M^\rho}(r) \quad \text{and} \quad c((r, z)) = z
\]

where \( \rho_{M^\rho} \) is defined on \( M^\rho \) by

\[
\begin{align*}
  \rho_{M^\rho}((n, f)) &= \max (1_{(4, \infty)}(n), 1_{(F_m, \infty)}(f)), \\
  \rho_{M^\rho}((n^b, f^b, n^e, f^e)) &= \max (1_{(4, \infty)}(n^b), 1_{(F_m, \infty)}(f^b), 1_{(4, \infty)}(n^e), 1_{(F_m, \infty)}(f^e)), \\
  \rho_{M^\rho}(\infty) &= 1.
\end{align*}
\]

where \( 1_I \) denotes the indicator function of set \( I \).

Let \((\ell_1, \ell_2)\) be a connection. If it is a day connection, then the resource of its arcs is \(((1, f(\ell_2)), z_{\ell_2})\), where \( f(\ell_2) \) is the flying duration of leg \( \ell_2 \), and \( z_{\ell_2} \) is the dual variable of
the cover constraint associated to $\ell_2$ in (CP). If it is a night connection $(\ell_1, \ell_2)$, then its resource is $((0,0, n^e, f^r, z^\ell_2))$, where $n^e = 2$ if $(\ell_1, \ell_2)$ is a reduced rest, and 1 otherwise, and $f^e = f(\ell_2) + F_m - F(t)$, where $t$ is the departure time of $\ell_2$.

It can be checked that the definition of $\oplus$ ensures that, given a pairing $p$, the path $P$ in $D$ corresponding to its sequence of flight legs, and its resources $r_p$ in $M^p$ as defined, we have $\bigoplus_{a \in p} r_a = r_p$. The definitions of $\oplus$, $\rho$, and $c$ then naturally lead to the following proposition.

**Proposition 3.** The sequence of flight legs $p$ corresponding to an o-d path $P$ is a pairing if and only if $\rho(q_P) = 0$. In that case, $c(q_P) = \overline{c}_p$.

### 4.3. Bounds on resources

As we already mentioned, the use of sets $B_v$ of bounds instead of single bounds $\{b_v\}$ is a specificity of our approach. In this section, we explain why sets of bounds enable to discard more paths than single bounds, and we describe a practically efficient method to compute these sets of bounds in the context of column generation.

**Why sets of bounds.** Suppose that $B_v$ is a singleton $\{b_v\}$. An o-v path $P$ will be discarded by (Low) if $q_P + b_v$ is large enough to satisfy $\rho(q_P + b_v) = 1$ of $c(q_P + b_v) > c_{o,d}^{UB}$. Thus, larger bounds $b_v$ enable to discard more paths. However, a single bound $b_v$ is always non-greater than the greatest lower bound $\bigwedge q_P$ on the resources of all the v-d paths. On Figure 2, we illustrate paths resources $q_P \in \mathbb{R}^2$ with crosses $\times$, and $\bigwedge q_P$ by the red diamond $\diamond$. On the contrary, if we use a set of bounds $B_v$ instead of a single bound $b_v$, we can partition the set of v-d paths into clusters of similar paths and obtain larger lower bounds on the resources of paths in each cluster, which are illustrated by the blue circles $\circ$ on Figure 2.

**Building the sets $B_v$ of bounds.** Suppose that we have a digraph $\mathcal{D}$, and a digraph homomorphism $\theta : \mathcal{D} \rightarrow D$ satisfying the following properties: $\theta^{-1}(d)$ is a singleton $\{d'\}$, for all $(u, v)$ in $A$, $\theta^{-1}(v)$ is not empty, and for all $v'$ in $\theta^{-1}(v)$, there is a single $u'$ in $\theta^{-1}(u)$ such that $(u', v')$ is an arc of $\mathcal{D}$. Define the resource of an arc $a'$ in $\mathcal{D}$ to be the resource $q_{\theta(a')}$ of its image by $\theta$.

**Lemma 4.** Let $b_{v'}$ be a lower bound on the resource of any $v'$-d' path in $\mathcal{P}'$ and define $B_v = \{b_{v'} : \theta(v') = v\}$. Then, for each v-d path $Q$, there is a $b \in B_v$ such that $b \leq q_Q$.

Thanks to Lemma 4, we can leverage on efficient algorithms to compute a single lower-bound $b_v$ on the resource of all the v-d paths to build the sets $B_v$. The first author [29] introduced a procedure in two steps. Given an integer $\kappa$, the first step builds a digraph $\mathcal{D}$ and a homomorphism $\theta$ as above satisfying $|\theta^{-1}(v)| \leq \kappa$ for all $v$. This procedure is time-consuming, and its execution time increases with $\kappa$. The second step is a dynamic programming algorithm that builds the bounds $b_{v'}$. Its execution time increases linearly.
with $\kappa$ and is practically negligible compared to the time required by the construction of $D$ or by the enumeration algorithm. Lemma 4 gives sets $B_v$ satisfying $|B_v| \leq \kappa$. The parameter $\kappa$ enables to balance the time spent in the computation of the sets $B_v$ and in the enumeration algorithm: a large $\kappa$ leads to large $B_v$ that takes more time to be computed but enables to discard more paths in the enumeration algorithm. Numerical results in Section 6 show that a well chosen $\kappa$ enables to divide by 3 the computing time. We also exploit the fact that the pricing subproblem digraph remains unchanged along the column generation to speed-up bounds computations: we build once and for all the digraph $D$ at the first step of the column generation and then only update the $b_v$’s with the new reduced costs, which is practically very fast.

Regarding the practical choice of $\kappa$ for the crew pairing pricing subproblem, the following rule of thumbs ensures good results in practice: use $\kappa = 1$ if there are fewer than 100 vertices, $\kappa = 50$ is there are fewer than 300 vertices, $\kappa = 150$ if there are fewer than 1500 vertices, and $\kappa = 250$ if there are more.

5. Integrated problem

If a crew changes of airplane during a connection between two flight legs $\ell$ and $\ell'$, its members need time to cross the airport between the arrival of $\ell$ and the departure of $\ell'$. This is not possible if the time between the arrival of $\ell$ and the departure of $\ell'$ is too short. A short connection is an ordered pair $(\ell, \ell')$ of flight legs that can be operated by a crew only if $\ell$ and $\ell'$ are operated by the same airplane. Due to short connections, aircraft routing and crew pairing problems are linked. More precisely, for every short connection $\alpha = (\ell, \ell')$, the solutions $x$ of (AR) and $y$ of (CP) must satisfy

$$
\sum_{p \in P_\alpha} y_p \leq \sum_{a \in A_\alpha} x_a,
$$

where we denote by $A_\alpha$ (resp. $P_\alpha$) the sets of arcs (resp. pairings) using the short connection $\alpha$. For any feasible solution of the aircraft routing problem, there is a solution of the crew pairing problem compatible with it since there is no constraint on the number of crews, but solving the two problems simultaneously allows to spare these additional crews and to reduce the costs, as explained in Section 1.2. The integrated problem aims at performing this task and is thus modeled by the following integer program

$$
\text{(Int)} \quad \min \sum_{p \in P} c_p y_p \\
\text{s.t.} \quad x \text{ satisfies constraints of (AR)} \\
\quad y \text{ satisfies constraints of (CP)} \\
\quad x \text{ and } y \text{ satisfy constraints (2) for all short connections } \alpha.
$$

Given the performance of our solution schemes for the aircraft routing and crew pairing problems, it is natural to test the ability of a simple combination of these approaches to tackle with the integrated problem. Instead of solving directly Program (Int), we adopt a cut generating approach using the methods proposed in the previous sections in a rather independent way.

Let $S(y)$ denote the sets of short connections used in a solution $y$ of (CP). Given a feasible solution $y$ of (CP), if there is no feasible solution $x$ of (AR) satisfying (2), then any
solution $y'$ such that $S(y) \subseteq S(y')$ leads to a similar infeasibility. To avoid such solutions in (CP), we set $S = S(y)$ and add the constraint

$$
\sum_{p \in P} |p \cap S| y_p \leq |S| - 1,
$$

where $|p \cap S|$ denotes the cardinality of $\{\alpha \in S : p \in P_\alpha\}$. It prevents a solution to use all short connections in $S$ but does not restrict otherwise the set of solutions.

We can now describe the algorithm for the integrated problem. The algorithm maintains a set $S$ of short connection cuts. Initially, $S$ is empty. The following steps are repeated.

(i) Solve (CP) with additional constraints (3) for $S \in S$. Let $y^*$ be the optimal solution.

(ii) Solve (AR) with the additional constraints (2).

- If it is feasible, then stop (we have found the optimal solution of (Int)).
- Otherwise, add $S(y^*)$ to $S$ and go back to (i).

Because of the cuts added along the algorithm, a solution $y^*$ is considered at most once. The number of solutions to the crew pairing problem being finite, the cut generation algorithm terminates after a finite number of iterations. The solutions of the last call to (i) and (ii) form an optimal solution to (Int): at each iteration, the only solutions of (CP) that are forbidden by the additional constraints (3) are not feasible for (Int) and at the last iteration, $y^*$ is the optimal solution of a relaxation of (Int).

In practice, the algorithm does not converge after thousands of iterations on industrial instances. We therefore replace $|S| - 1$ by $\gamma|S|$ with $\gamma < 1$ in the constraints (3), loosing the optimality of the solution returned. Numerical experiments in Section 6 show that $\gamma = 0.9$ is a good compromise: we obtain near optimal solutions after a few dozens of iterations.

6. Experimental results

6.1. Instances. Table 1 describes six industrial instances of Air France. Each instance contains the legs of a fleet on a weekly horizon. The two first columns provide the name of the instance and the number of legs it contains. The next columns give the number of connections that can be done by airplanes, i.e., the number of ordered pair of legs $(\ell, \ell')$ that can be operated consecutively in a route, and the number of airplanes $n^a$ available. The two last columns provide the number of connections that can be taken by crews, and the order of magnitude of the number of pairings in a good solution. These instances are large: for instance, the largest instance considered by Mercier and Soumis [25] has 523 legs. Table 2 provides six smaller artificial instances, which we have built by considering only a subset of the legs of the instance A318. Instances CP50, CP70, and CP90 are interesting pairing instances but make no sense for aircraft routing: they require a huge number of airplanes compared to the number of legs.

Instance A318-9 (resp. A320-fam) contains legs operated by A318 and A319 (resp. A318, A319, A320, and A321) airplanes. As Air France crews can operate legs on planes of different subfleets on the same pairing, there is a common crew pairing problem for all these subfleets. But as the subfleet of an airplane is fixed, when we solve aircraft routing on these instances, we solve it separately for each subfleet.
| Instance | Legs | Airplane Airplanes | Crew | Crew pairings ($\approx$) |
|----------|------|---------------------|------|---------------------------|
| A318     | 669  | 39564               | 18   | 3742                      | 130  |
| A319     | 957  | 45901               | 41   | 3738                      | 240  |
| A320     | 918  | 49647               | 45   | 3813                      | 280  |
| A321     | 778  | 29841               | 25   | 3918                      | 165  |
| A318-9   | 1766 | –                   | (59) | 8070                      | 350  |
| A320-fam | 3398 | –                   | (129)| 21563                     | 690  |

Table 1. Air France industrial instances

| Instance | Legs | Airplane Airplanes | Crew | Crew pairings ($\approx$) |
|----------|------|---------------------|------|---------------------------|
| AR4      | 152  | 2107                | 4    | 389                       | 60   |
| AR8      | 313  | 8723                | 8    | 1112                      | 100  |
| AR12     | 470  | 19536               | 12   | 2055                      | 125  |
| CP50     | 290  | –                   | –    | 1006                      | 50   |
| CP70     | 408  | –                   | –    | 1705                      | 70   |
| CP90     | 516  | –                   | –    | 2490                      | 90   |

Table 2. Artificial aircraft routing and crew pairing instances

6.2. **Experimental setting.** All the numerical experiments are performed on a server with 128 Gb of RAM and 12 cores at 2.4 GHz. **CPLEX 12.1.0** is used to solve the aircraft routing compact integer program. The algorithms are not parallelized.

6.3. **Aircraft Routing.** Table 3 provides the performances of our algorithms on the aircraft routing problem. The first two columns give the name of the instance and the time needed to solve our compact integer program (AR). The solution scheme for the integrated problem in Section 5 solves this problem with additional constraints (2). On all but the last iterations of the integrated problem scheme, aircraft routing is infeasible. The two next columns correspond to the time needed to solve an infeasible and a feasible constrained version. Finally, the last column gives the time needed to find an optimal solution of the optimization problem that consists in finding the minimum number of airplanes needed to operate the instance: we use the left-hand side of the third constraint of (AR) as objective. The typical computing time is a few dozens of seconds on industrial instance. The longest constrained feasible version requires a few minutes.

6.4. **Crew Pairing.** Table 4 provides the results of the algorithm of Section 3.3 on our crew pairing instances. We solve the pricing subproblem using the enumeration algorithm of Section 4.1 with the tests (Low) and (Dom). All instances are solved to optimality. The first column of Table 4 gives the name of the instance. The next column provides the value of $\kappa$ determined using the rule of thumb of Section 4.3 and needed by the algorithm building the sets $B_v$. The two next columns provide the number of iterations in the column generation, and the percentage of time spent in the pricing subproblem. This pricing time includes the time needed by the computation of the sets $B_v$ and by the enumeration algorithm. The
two next columns indicate the percentage of the total CPU time spent solving Step (ii), and solving Steps (iv) and (vi). The next column provides the gap between the linear relaxation at the end of the algorithm and the integer solution returned, and the last column gives the total time needed by the algorithm. We underline that the integrality gap is very small at the end of the column generation, which explains the fast resolution of Step (vi).

Focus on the pricing subproblem: dominance and choice of $\kappa$. Table 5 studies the influence of the settings of the pricing subproblem algorithm. The three first columns give the instance solved, the test(s) used to discard paths in the enumeration algorithm, and the maximum size $\kappa$ of the lower bounds sets introduced in Section 4.3. The next column give the proportion of the column generation CPU time spent in the pricing subproblem, the remaining being spent solving the restricted master problem. The next column gives the average number of paths enumerated by the pricing subproblem algorithm along the column generation. When both (Low) and (Dom) tests are used, the next column provides the proportion of paths that are cut by the dominance test (Dom), the remaining being cut by the lower bound test (Low). The last column provides the total CPU time spent in the column generation. The lower bound test (Low) is the main element in the performance of the algorithm. Indeed, even when the dominance test is used, more than 90% of the paths are cut by the lower bound test. We experimentally remark that the computing time convexly depends on the
### Table 5. Tests and bounds set size influence on pricing subproblem

| Instance | Tests       | \( \kappa \) | Pricing time (hh:mm:ss) | av. nb paths | Cut enum. | Cut Dom. | Total time (hh:mm:ss) |
|----------|-------------|---------------|-------------------------|--------------|----------|----------|-----------------------|
| CP50     | (Low), (Dom) | -             | 00:00:23.0              | 6.016e+03    | 6.89%    |          |                       |
|          | 10          | 59.87%        | 00:00:17.2              | 1.601e+03    | 4.01%    |          |                       |
|          | 50          | 69.45%        | 00:00:24.7              | 7.766e+02    | 4.69%    |          |                       |
|          | 100         | 77.06%        | 00:00:33.3              | 6.467e+02    | 5.03%    |          |                       |
| CP70     | (Low), (Dom) | -             | 00:00:40.7              | 2.752e+04    | 7.69%    |          |                       |
|          | 10          | 58.48%        | 00:01:12.1              | 7.613e+03    | 4.28%    |          |                       |
|          | 50          | 68.89%        | 00:01:23.0              | 3.917e+03    | 5.24%    |          |                       |
|          | 100         | 77.43%        | 00:01:42.6              | 3.085e+03    | 5.77%    |          |                       |
| CP90     | (Low), (Dom) | -             | 00:04:27.1              | 1.488e+05    | 9.81%    |          |                       |
|          | 10          | 81.86%        | 00:12:36.3              | 4.119e+04    | 5.88%    |          |                       |
|          | 50          | 73.42%        | 00:09:39.7              | 2.534e+04    | 4.87%    |          |                       |
|          | 100         | 77.98%        | 00:10:27.5              | 1.879e+04    | 5.60%    |          |                       |
| A318     | (Low), (Dom) | -             | 05:03:41.8              | 2.746e+05    | 8.99%    |          |                       |
|          | 10          | 96.02%        | 05:06:46.6              | 2.420e+05    | 6.62%    |          |                       |
|          | 50          | 88.78%        | 02:06:43.4              | 1.489e+05    | 3.73%    |          |                       |
|          | 100         | 86.97%        | 01:32:49.6              | 1.270e+05    | 3.72%    |          |                       |
|          | 150         | 86.94%        | 01:40:45.4              | 1.138e+05    | 3.75%    |          |                       |
|          | (Low)       | -             | 01:17:08.4              | 1.148e+06    |          |          |                       |
|          | 10          | 97.02%        | 04:54:52.4              | 1.070e+06    |          |          |                       |
|          | 50          | 86.85%        | 01:45:27.4              | 5.735e+05    |          |          |                       |
|          | 100         | 88.52%        | 01:45:18.2              | 4.783e+05    |          |          |                       |

sets of bounds \( B_v \), size \( \kappa \), and that an appropriately chosen \( \kappa \) enables to solve the problem 3 times faster than if a single bound (\( \kappa = 1 \)) is used.

6.5. **Integrated problem.** Table 6 provides numerical results of the integrated aircraft routing and crew pairing solution scheme on our instances. The first column provides the instance solved, and the next column the short connection constraints strength parameter \( \gamma \). The next column provides the number of steps of the integrated problem algorithm
of Section 5 before convergence. The two next columns provide the parameter \( \kappa \) of the label correcting algorithm defined in Section 4.3, and the total number of column generation iterations realized on the successive integrated problem algorithms steps. The column “CP CG time” provides the proportion of the total CPU time spent in the column generation, i.e., solving the pricing subproblem and the linear relaxation of the master problem, and the next column the proportion spent solving the integer version of the crew pairing master problem. The column “AR time” provides the proportion spent solving aircraft routing integer program (AR). The column “Sho. Con.” gives the number of short connections in the final solution. The linear relaxation of the crew pairing master problem (CP) with no short connection constraint is used as the lower bound on the cost of an optimal solution. The gap provided is between the cost of the solution returned and this lower bound. Finally, the last column provides the total CPU time needed by the algorithm.

We emphasize the fact that the solution returned by the approximate algorithm is almost optimal. Practically speaking, the gap obtained is non-greater than 10 euros. The computation time needed to obtain a near optimal solution of the integrated problem is of the same order of magnitude than the time needed to obtain a solution of the crew pairing problem

| Instance | \( \gamma \) | Integ. steps | \( \kappa \) | CG it. | CP CG time | CP MIP time | AR time | Sho. Con. | Gap | Total time (dd:hh:mm:ss) |
|----------|---------------|--------------|-------------|--------|------------|------------|---------|-----------|-----|-------------------------|
| AR4 0.9  | 24            | 20          | 36         | 9.58%  | 54.85%     |            | 35.57%  | 63        | 0.0328% | 00:00:18.4            |
| AR4 0.8  | 6             | 20          | 23         | 24.31% | 43.80%     |            | 31.89%  | 57        | 0.0937% | 00:00:05.4            |
| AR4 0.6  | 3             | 20          | 17         | 26.37% | 44.43%     |            | 29.20%  | 49        | 0.8185% | 00:00:04.5            |
| AR8 0.9  | 61            | 20          | 172        | 20.55% | 50.54%     |            | 28.90%  | 148       | 0.0070% | 00:08:46.7            |
| AR8 0.8  | 4             | 20          | 87         | 48.55% | 29.95%     |            | 21.50%  | 136       | 0.0073% | 00:00:48.9            |
| AR8 0.6  | 5             | 20          | 144        | 55.66% | 29.07%     |            | 15.28%  | 114       | 0.9426% | 00:01:13.3            |
| AR12 0.9 | 55            | 20          | 305        | 51.97% | 31.90%     |            | 16.13%  | 213       | 0.0051% | 00:27:48.3            |
| AR12 0.8 | 35            | 20          | 381        | 19.21% | 76.86%     |            | 3.03%   | 204       | 0.0403% | 01:21:07.1            |
| AR12 0.6 | 30            | 20          | 633        | 0.76%  | 99.07%     |            | 0.17%   | 159       | 1.4285% | 01:11:28:58.2          |
| A318 0.9 | 6             | 150         | 460        | 95.53% | 2.56%      |            | 1.91%   | 323       | 0.0002% | 01:53:47.4            |
| A318 0.8 | 5             | 150         | 461        | 95.62% | 2.71%      |            | 1.67%   | 312       | 0.0029% | 01:48:02.3            |
| A318 0.6 | 14            | 150         | 1030       | 26.27% | 73.20%     |            | 0.53%   | 269       | 0.6857% | 16:02:03.1            |

Table 6. Numeric results on integrated problem

| Instance | \( \gamma \) | Integ. steps | \( \kappa \) | CG it. | CP CG time | CP MIP time | AR time | Sho. Con. | Gap | Total time (dd:hh:mm:ss) |
|----------|---------------|--------------|-------------|--------|------------|------------|---------|-----------|-----|-------------------------|
| AR4 0.9  | 24            | 20          | 36         | 9.58%  | 54.85%     |            | 35.57%  | 63        | 0.0328% | 00:00:18.4            |
| AR4 0.8  | 6             | 20          | 23         | 24.31% | 43.80%     |            | 31.89%  | 57        | 0.0937% | 00:00:05.4            |
| AR4 0.6  | 3             | 20          | 17         | 26.37% | 44.43%     |            | 29.20%  | 49        | 0.8185% | 00:00:04.5            |
| AR8 0.9  | 61            | 20          | 172        | 20.55% | 50.54%     |            | 28.90%  | 148       | 0.0070% | 00:08:46.7            |
| AR8 0.8  | 4             | 20          | 87         | 48.55% | 29.95%     |            | 21.50%  | 136       | 0.0073% | 00:00:48.9            |
| AR8 0.6  | 5             | 20          | 144        | 55.66% | 29.07%     |            | 15.28%  | 114       | 0.9426% | 00:01:13.3            |
| AR12 0.9 | 55            | 20          | 305        | 51.97% | 31.90%     |            | 16.13%  | 213       | 0.0051% | 00:27:48.3            |
| AR12 0.8 | 35            | 20          | 381        | 19.21% | 76.86%     |            | 3.03%   | 204       | 0.0403% | 01:21:07.1            |
| AR12 0.6 | 30            | 20          | 633        | 0.76%  | 99.07%     |            | 0.17%   | 159       | 1.4285% | 01:11:28:58.2          |
| A318 0.9 | 6             | 150         | 460        | 95.53% | 2.56%      |            | 1.91%   | 323       | 0.0002% | 01:53:47.4            |
| A318 0.8 | 5             | 150         | 461        | 95.62% | 2.71%      |            | 1.67%   | 312       | 0.0029% | 01:48:02.3            |
| A318 0.6 | 14            | 150         | 1030       | 26.27% | 73.20%     |            | 0.53%   | 269       | 0.6857% | 16:02:03.1            |

Table 7. Influence of \( \gamma \) on integrated problem
in Table 4. Solving aircraft routing and crew pairing sequentially strongly constrains the solution: indeed, when solved in an integrated fashion, around half of the connections in the solution are short connections.

**Influence of γ.** Table 7 provides numerical results on the same instances for various γ. The second column provides the parameter γ used, the other ones are identical to those of Table 6. The smaller γ, the more constrained is the solution, and thus the higher is its cost. On industrial instances, choosing a constraint strength parameter γ equal to 0.9 enables to obtain solution of excellent qualities in reasonable time. On some instances, using stronger constraints with γ = 0.8 enables to accelerate convergence and thus reduces the computation time. Nonetheless, too strong constraints with γ = 0.6 makes the constrained master problem (CP) harder to solve and increases the computation time.

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