Classical $q$-deformed dynamics

A. Lavagno$^{1,2}$, A.M. Scarfone$^{1,3}$ and P. Narayana Swamy$^4$

$^1$Dipartimento di Fisica, Politecnico di Torino, Italy

$^2$Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Torino, Italy

$^3$Istituto Nazionale di Fisica della Materia (CNR–INFM), Sezione di Torino, Italy and

$^4$Southern Illinois University, Edwardsville, IL 62026, USA

Abstract

On the basis of the quantum $q$-oscillator algebra in the framework of quantum groups and non-commutative $q$-differential calculus, we investigate a possible $q$-deformation of the classical Poisson bracket in order to extend a generalized $q$-deformed dynamics in the classical regime. In this framework, classical $q$-deformed kinetic equations, Kramers and Fokker-Planck equations, are also studied.

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I. INTRODUCTION

The study of quantum algebras and quantum groups has attracted a lot of interest in the last few years, and stimulated intensive research in several physical fields in view of a broad spectrum of applications, ranging from cosmic strings and black holes to the fractional quantum Hall effect and high-$T_c$ superconductors \cite{1}.

From the seminal work of Biedenharn \cite{2} and Macfarlane \cite{3} it is clear that the $q$-calculus, originally introduced at the beginning of last century by Jackson \cite{4} in the study of the basic hypergeometric function, plays a central role in the representation of the quantum groups \cite{5}. In fact it has been shown that it is possible to obtain a “coordinate” realization of the Fock space of the $q$-oscillators by using the deformed Jackson derivative (JD) or the so-called $q$-derivative operator \cite{6,7,8}.

In this paper we want to introduce a $q$-deformation of the PB ($q$-PB) in order to define a generalized $q$-deformed dynamics in a $q$-commutative phase-space. For this purpose we begin with the observation that the creation and annihilation operators in the quantum $q$-deformed SU$_q$(2) algebra corresponds classically to $q$-commuting coordinates in a $q$-phase space and that the commutation relation between the standard quantum operators corresponds classically to the Poisson bracket (PB).

The motivation for our goal lies in the fact that a full understanding of the physical origin of $q$-deformation in classical physics is still lacking because it is not clear if there exists a classical counterpart to the $q$-deformed quantum mechanics inspired by the study of quantum groups. The problem of a possible $q$-deformation of classical mechanics was dealt with in Ref. \cite{9} where a $q$-PB was obtained starting from a point of view different from the one adopted in this paper. In order to introduce the classical correspondence of the quantum $q$-oscillator, we shall follow the main approach based on the following idea. The (undeformed) quantum commutation relations are invariant under the action of SU(2) and, as a consequence, the $q$-deformed commutation relations are invariant under the action of SU$_q$(2). Analogously, since the (undeformed) PB is invariant under the action of the symplectic group Sp(1), we have to require that $q$-PB must satisfy invariance under the action of the $q$-deformed symplectic group Sp$_q$(1).
II. NON-COMMUTATIVE DIFFERENTIAL CALCULUS

Since the creation and annihilation operators in the quantum $q$-deformed SU$_q$(2) algebra correspond classically to non-commuting coordinates in a $q$-phase-space, in this section we introduce the $q$-deformed plane which is generated by the non-commutative elements $\hat{x}$ and $\hat{p}$ fulfilling the relation

$$\hat{p} \hat{x} = q \hat{x} \hat{p}, \quad (1)$$

which is invariant under GL$_q$(2) transformations. Henceforward, for simplicity, we shall limit ourselves to consider the two-dimensional case.

From Eq.(1) the $q$-calculus on the $q$-plane can be obtained formally through the introduction of the $q$-derivatives $\hat{\partial}_x$ and $\hat{\partial}_p$

$$\hat{\partial}_p \hat{p} = \hat{\partial}_x \hat{x} = 1, \quad (2)$$
$$\hat{\partial}_p \hat{x} = \hat{\partial}_x \hat{p} = 0. \quad (3)$$

They fulfill the $q$-Leibniz rule

$$\hat{\partial}_p \hat{p} = 1 + q^2 \hat{p} \hat{\partial}_p + (q^2 - 1) \hat{x} \hat{\partial}_x, \quad (4)$$
$$\hat{\partial}_p \hat{x} = q \hat{x} \hat{\partial}_p, \quad (5)$$
$$\hat{\partial}_x \hat{p} = q \hat{p} \hat{\partial}_x, \quad (6)$$
$$\hat{\partial}_x \hat{x} = 1 + q^2 \hat{x} \hat{\partial}_x, \quad (7)$$

together with the $q$-commutative derivative

$$\hat{\partial}_p \hat{\partial}_x = q^{-1} \hat{\partial}_x \hat{\partial}_p. \quad (8)$$

It is easy to see that in the $q \to 1$ limit one recovers the ordinary commutative calculus.

Let us outline the asymmetric mixed derivative relations, Eq.(4) and Eq.(7), in $\hat{x}$ and in $\hat{p}$. These properties arise directly from the non-commutative structure of the $q$-plane defined in Eq.(1).

We recall now that the most general function on the $q$-plane can be expressed as a polynomial in the $q$-variables $\hat{x}$ and $\hat{p}$

$$f(\hat{x}, \hat{p}) = \sum_{i,j} c_{ij} \hat{x}^i \hat{p}^j, \quad (9)$$
where we have assumed the \( \hat{x} - \hat{p} \) ordering prescription (it can always be accomplished by means of Eq. (10). Thus, taking into account Eqs. (4)-(7), we obtain the action of the \( q \)-derivatives on the monomials

\[
\hat{\partial}_x(\hat{x}^n \hat{p}^m) = [n]_q \hat{x}^{n-1} \hat{p}^m , \\
\hat{\partial}_p(\hat{x}^n \hat{p}^m) = [m]_q q^n \hat{x}^n \hat{p}^{m-1} ,
\]

where we have introduced the \( q \)-basic number\footnote{\[ [n]_q = \frac{q^{2n} - 1}{q^2 - 1} . \]}

A realization of the above \( q \)-algebra and its \( q \)-calculus can be accomplished by the replacements \footnotetext{\[ \hat{x} \rightarrow x , \]}

\footnotetext{\[ \hat{p} \rightarrow p D_x , \]}

\footnotetext{\[ \hat{\partial}_x \rightarrow D_x , \]}

\footnotetext{\[ \hat{\partial}_p \rightarrow D_p D_x , \]}

where

\[
D_x = q^x \partial_x , \\
D_x f(x, p) = f(qx, p) ,
\]

is the dilatation operator along the \( x \) direction (reducing to the identity operator for \( q \rightarrow 1 \)), whereas

\[
D_x = \frac{q^{2x} \partial_x - 1}{(q^2 - 1) x} , \\
D_p = \frac{q^{2p} \partial_p - 1}{(q^2 - 1) p} ,
\]

are the JD with respect to \( x \) and \( p \). Their action on an arbitrary function \( f(x, p) \) is

\[
D_x f(x, p) = \frac{f(q^2 x, p) - f(x, p)}{(q^2 - 1) x} , \\
D_p f(x, p) = \frac{f(x, q^2 p) - f(x, p)}{(q^2 - 1) p} .
\]

Therefore, as a consequence of the non-commutative structure of the \( q \)-plane, in this realization the \( \hat{x} \) coordinate becomes a \( c \)-number and its derivative is the JD whereas the \( \hat{p} \) coordinate and its derivative are realized in terms of the dilatation operator and JD.
III. \textit{q-POISSON BRACKET AND q-SYMPLECTIC GROUP}

With the formulation of the $q$-differential calculus, we are now able to introduce a $q$-PB. Since the undeformed PB is invariant under the action of the undeformed symplectic group Sp(1), we will assume as previously stated, as a fundamental point, that the $q$-PB must satisfy the invariance property under the action of the $q$-deformed symplectic group Sp$_q$(1) with the same value of the deformed parameter $q$ used in the construction of the quantum plane.

Let us start by recalling the classical definition of a 2-Poisson manifold, which is a two dimensional Euclidean space $\mathbb{R}^2$ generated by the position and momentum variables $x \equiv x^1$ and $p \equiv x^2$ and equipped with a PB. By introducing $f(x, p)$ and $g(x, p)$, two arbitrary smooth functions, the PB is defined as

$$\{f, g\} = \partial_x f \partial_p g - \partial_p f \partial_x g .$$  \hspace{1cm} (23)

Eq. (23) can be expressed in a compact form

$$\{f, g\} = \partial_i f J^{ij} \partial_j g ,$$  \hspace{1cm} (24)

where $J^{ij}$ are the entries of the unitary symplectic matrix $J$ given by

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$  \hspace{1cm} (25)

Remarkably, Eq. (24) does not change under the action of a symplectic transformation Sp(1) on the phase-space. As is well known Eq. (24) can also be expressed as

$$\{f, g\} = \{x^i, x^j\} \partial_i f \partial_j g ,$$  \hspace{1cm} (26)

so that, if we know the PB between the generators $x^i$ we can compute the PB between any pair of functions $f$ and $g$.

By requiring that the $q$-PB must be invariant under the action of the $q$-symplectic group Sp$_q$(1), we are lead to introduce the following $q$-deformed PB between the $q$-generators $\hat{x}^i$

$$\{\hat{x}^i, \hat{x}^j\}_q = \hat{\partial}_x \hat{x}^i \hat{\partial}_p \hat{x}^j - q^2 \hat{\partial}_p \hat{x}^i \hat{\partial}_x \hat{x}^j .$$  \hspace{1cm} (27)
It is easy to verify the following fundamental relations
\[
\begin{align*}
\{\hat{x}, \hat{x}\}_q &= \{\hat{p}, \hat{p}\}_q = 0, \\
\{\hat{x}, \hat{p}\}_q &= 1, \\
\{\hat{p}, \hat{x}\}_q &= -q^2,
\end{align*}
\]
which coincide with the one obtained in Ref. [9]. In particular, from Eqs. (29) and (30) it follows that the \(q\)-PB is not antisymmetric. A similar behavior appears also in the quantum \(q\)-oscillator theory [2, 3].

By means of Eqs. (13)-(16), a realization of our generalized \(q\)-PB can be written as
\[
\{f, g\}_q = \mathcal{D}_x f(x, p D_x) \mathcal{D}_p g(q x, p D_x) - q^2 \mathcal{D}_p f(q x, p D_x) \mathcal{D}_x g(x, p D_x)
\]
where \(f\) and \(g\) are identified with \(x\) or \(p\), respectively.

IV. \(q\)-DEFORMED KINETIC EQUATIONS

On the basis of the above classical \(q\)-deformed theory, we shall now derive the corresponding classical kinetic equations based on the \(q\)-calculus. Starting from the realization of the \(q\)-algebra, defined in Eqs. (13)-(16), we are able to write the Kramers equation corresponding to the equation of motion for the distribution function \(f(x, p; t)\), in position and momentum space, describing the motion of particles of mass \(m\) in an external field \(F(x)\) [15]. In the one-dimensional case it can be generalized as follows
\[
\frac{\partial f(x, p; t)}{\partial t} = \left\{ -\frac{p}{m} \mathcal{D}_x \mathcal{D}_x - \mathcal{D}_p \mathcal{D}_x [J_1^q(p D_x) + F(x)] + J_2^q (\mathcal{D}_p D_x) (\mathcal{D}_p D_x) \right\} f(x, p; t),
\]
where \(J_1^q(p D_x)\) and \(J_2^q\) are the drift and diffusion coefficients, respectively. Specifying the action of the dilatation operator \(D_x\) along the \(x\) direction, the above Kramers equation can be written as
\[
\frac{\partial f(x, p; t)}{\partial t} = -\frac{p}{m} \mathcal{D}_x f(q x, p; t) - \mathcal{D}_p [J_1^q(p D_x) + F(q x)] f(q x, p; t) + J_2^q \mathcal{D}_p^2 f(q^2 x, p; t),
\]
where $D_p^2$ means the double application of the JD in the momentum space. Without any external force, for a homogeneous system undergoing a constant diffusion, the above generalized Kramers equation reduces to the following $q$-deformed Fokker-Planck equation

$$
\frac{\partial f(p; t)}{\partial t} = D_p \left[ - J_1^q(p) + J_2^q D_p \right] f(p; t).
$$

(34)

If we postulate a generalized Brownian motion in a $q$-deformed classical dynamics by the following definition of the drift and diffusion coefficients

$$
J_1^q(p) = -\gamma p \left( \frac{q^2 D_p + 1}{2} \right), \quad J_2^q = \gamma m kT,
$$

(35)

where $\gamma$ is the friction constant, $T$ is the temperature of the system and $D_p$ is the dilatation operator in the momentum space, the stationary solution $f_{st}(p)$ of the above Fokker-Planck equation can be obtained as a solution of the following stationary $q$-differential equation

$$
D_p f(p) = -\frac{p}{2 m kT} \left[ q^2 f(qp) + f(p) \right].
$$

(36)

It is easy to show that the solution of the above equation can be written as

$$
f_{st}(p) = E_q \left[ -\frac{p^2}{2 m kT} \right],
$$

(37)

where $E_q[x]$ is the $q$-deformed exponential function, well-known in $q$-calculus, defined in terms of the series

$$
E_q[x] = \sum_{k=0}^{\infty} \frac{x^k}{[k]_q!},
$$

(38)

where $[k]_q!$ is the $q$-basic factorial defined as $[k]_q! = [k]_q [k-1]_q \cdots [1]_q$.

V. CONCLUSIONS

We have shown that $q$-calculus can play a crucial role in the formulation of a generalized $q$-classical theory, defined by means of the introduction of a $q$-PB. In analogy with quantum group invariance properties of the quantum $q$-oscillator theory, the $q$-PB has been defined by assuming the invariance under the action of $Sp_q(1)$ group with its derivatives acting on the $q$-deformed non-commutative plane invariant under $Gl_q(2)$ transformations. Therefore such a classical $q$-deformation theory can be seen as the analogue of $q$-oscillator deformation in the quantum theory. In this framework, we have studied the classical $q$-deformed
kinetic equations, Kramers and Fokker-Planck equations and we have found, as a stationary solution, the well-known $q$-deformed exponential function defined in terms of a series. This opens the possibility of introducing a classical counterpart of the quantum $q$-deformations and we expect that such a classical $q$-deformed dynamics can be very relevant in several physical applications.

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