Topological Mechanism of Superconductivity

P.B Wiegmann

James Frank Institute and Enrico Fermi Institute of the University of Chicago,
5640 S.Ellis Ave., Chicago IL, 60637,
and Landau Institute for Theoretical Physics
e-mail:wiegmann@control.uchicago.edu

September 19, 1994

Lectures given at the XI South African Summer School in Theoretical Physics, January 1994

Abstract

We outline the basic ideas of the topological mechanisms of superconductivity. A gauged model of correlated electronic system where a topological fluid is formed as a result of a strong interaction is discussed.
1 Criteron for Superconductivity

1.1 London Equations as Hydrodynamics of an Ideal Liquid

Our understanding of superconductivity starts from the London equations. They describe the linear response of superconductive matter to an external electromagnetic field:

\[ \vec{\nabla} \times \vec{j} = -\frac{1}{\lambda^2} \vec{B}^{\text{ext}}, \]  
\[ \vec{\nabla} \rho + v_0^{-2} \partial_t \vec{j} = \kappa \vec{E}^{\text{ext}}. \]  

(1)

(2)

The first equation implies the Meissner effect (\( \lambda \) is the London penetration depth). The second means that the matter is compressible \( (\kappa \) is a compressibility) and implies an ideal conductivity \( (\sigma(\omega) \sim i\omega^{-1}) \). Of course, due to gauge invariance (or the Maxwell condition \( \vec{\nabla} \times \vec{E} = \partial_t \vec{B} \)), the penetration depth and compressibility are related by \( \kappa = (\lambda v_0)^{-2} \).

These two equations may be written as one. In the Lorentz gauge \( \vec{\nabla} A + v_0^{-2} \partial_t A_0 = 0 \), for example, one has

\[ \vec{j} = -\lambda^{-2} \vec{A}^{\text{ext}}. \]  

(3)

These equations describe the hydrodynamics of the superconductor at low temperatures.

If we replace \( \vec{E}^{\text{ext}} \) in (2) by the gradient of pressure \( \vec{\nabla} p \), and \( \vec{B}^{\text{ext}} \) in (1) by a torque, we obtain the linearized Euler equation for an ideal compressible liquid. Therefore the Meissner effect is a direct consequence of the fact that superconductive matter is a liquid.

Let us elaborate this simple but important observation a bit further. The London equations can also be written in terms of current-current correlators at small \( \omega \) and \( k \). Varying these equations over \( \vec{A}^{\text{ext}} \) and taking into account the continuity equation \( \partial_t \rho + \vec{\nabla} j = 0 \), we find

\[ \langle \rho(k, \omega) \rho(-k, \omega) \rangle = -\kappa \frac{v_0^2 k^2}{\omega^2 - v_0^2 k^2}. \]  

(4)

\footnote{As a result of Coulomb interactions the liquid is in fact incompressible. Below we neglect the Coulomb interaction between charge carriers and consider the electromagnetic field as external.}
and
\[ \langle j_{\perp}(k, \omega)j_{\perp}(k, \omega) \rangle = \frac{1}{\lambda^2}, \quad \vec{\nabla} j_{\perp} = 0 , \quad (5) \]
where \( j_{\perp} \) is the transverse component of the current. Now it is obvious that our matter is a liquid. The second equation (which equivalently reflects the Meissner effect) implies that there is no gapless transverse mode (transverse sound), i.e. that there is no shear modulus – this is the definition of a liquid. A transverse sound is a feature of a solid for which, instead of (5), one has
\[ \langle j_{\perp}(k, \omega)j_{\perp}(k, \omega) \rangle = -\frac{1}{\chi^2} \omega^2 - \nu^2 k^2 k^2. \quad (6) \]
The density-density correlator (4) shows a gapless mode (longitudinal sound) which means that the liquid is ideal and compressible, i.e. may flow without dissipation.

All these equations can be linked to a Hamiltonian:
\[ H = \frac{m}{2\tilde{\rho}} \left[ j^2 + \nu_0^2 (\rho - \bar{\rho})^2 \right] + \vec{A}_{\text{ext}} \cdot \vec{j} + A_{0\text{ext}} (\rho - \bar{\rho}) \quad (7) \]
where \( \bar{\rho} \) is an average density. This is the Hamiltonian of linear hydrodynamics.

### 1.2 Landau Criterion

To be a superfluid a system must have no dissipation. It seems difficult to combine a soft mode in density modulation and the absence of dissipation. A Fermi liquid, for example, shows Landau damping,
\[ \langle j_{\perp}(k, \omega)j_{\perp}(k, \omega) \rangle = -\frac{1}{\chi k^2 + i\gamma \omega / k} , \quad (8) \]
due to gapless particle-hole excitations (\( \chi \) and \( \gamma \) are constants). To have no dissipation a liquid must have no gapless density modes rather than longitudinal sound. In particular the single particle spectrum of the homogeneous liquid must have a gap. In known examples it is sufficient that all other excitations, which change a local density also have a gap. This is the subject of the Landau criterion [1] of superfluidity. In its grotesque form it states that
the spectrum must contain longitudinal sound and the single particle spectrum must have a gap.

It is believed that this condition on the spectrum of quantum liquid is sufficient to derive the Meissner effect.

All of this is true for a homogeneous liquid. The true check whether a liquid is superconductive may be made only if some non-zero concentration of impurities does not result in resistivity. A further belief is that weak impurities do not lead to resistivity whenever the Meissner effect holds in the absence of impurities.

The Landau criterion seems to be sufficient in spatial dimensions greater than one. In one dimension it fails for a very simple reason, namely, in one dimension we cannot distinguish between a liquid and a solid since there are no properties such as transversal modes and shear modulus. There is no Meissner effect either. As a result a single impurity pins down the flow in the same way as a single impurity pins down a solid.

In the early days of superconductivity, Frohlich noticed that in a one-dimensional metal an incommensurate charge density wave (CDW) will slide through the lattice unattenuated. Since it carries an electric charge and since a gap has developed in the electronic spectrum, Frohlich concluded that the ground state of his system is superconductive. Although, a sliding charge density wave indeed contributes to conductivity, the Frohlich superconductivity in one dimension was considered nothing more than a theoretical curiosity, because of a variety of pinning mechanisms.

The failure of the charge density wave mechanism in one dimension does not devaluate Frohlich’s ideas, which may be valid in higher dimensions. The distinctive property of Frohlich superconductivity is a topological order – a topological character of the ground state. We therefore call it the topological mechanism of superconductivity. One of the best known examples of this phenomenon in two dimensions is the superconductivity of particles with fractional statistics or anyon superconductivity (see e.g.).

There are two reasons why the topological mechanism of superconductivity is of interest. First of all it is a fundamentally new mechanism of superconductivity and superfluidity. Secondly it has been found in models of strongly correlated electronic systems, namely in the doped Mott insulator (some people consider it to be a mechanism of high temperature superconductivity).
Below we discuss some models of topological superconductivity and discuss why they are relevant to the physics of correlated electronic systems.

1.3 Topological order

Jumping ahead, let us discuss some features of the ground state with a topological order. Let the wave function of the system be $\psi(\{x_i\})$ where $\{x_i\}$ are coordinates of particles. Its defines a fiber bundle. We shall refer the Chern number of this fiber bundle as a topological order.

Consider for example a two dimensional system. A topological order implies that the ground state wave function has zeros with a non-vanishing signature. In this case the Chern’s number is

$$Q = \sum_{a=1}^{M} \int \varepsilon_{ij} \frac{\partial}{\partial x^i} \psi^* \frac{\partial}{\partial x^j} \psi \frac{d^2S_a}{8\pi}$$

where the integral goes over the space. This equation can be also applied for 3-dimensional systems. In this case the integral goes over the boundary of the system. In one dimension, let us define the Chern number as follows. Consider a closed one-dimensional space as a boundary of a domain of the complex plane and the wave function as the value of some analytical function in the domain. Then (9) gives the Chern number.

Perhaps the simplest example were a topological order occurs is particles in a magnetic field. Then the Chern number is merely Hall conductance. However a topological order may occur in correlated electronic systems even without magnetic field, but as a result of a strong interaction. These systems are not exotic. They were known from the early days of quantum physics in one dimension. They have also recently been discovered in quantum magnetism [7, 8, 9, 10, 11] and intensively discussed with respect to fractional statistics in two dimensions. A topological order also occurs in some models of electronic systems with strong interaction in two and three dimensions. Some people even think that a topological order is a universal feature of correlated quantum systems.

An illustrative example is given by the simplest solvable model of particles in one dimension with a $1/r^2$ interaction:

$$H = -\sum_{i=1}^{N} \frac{\partial^2}{\partial x^2_i} + \sum_{i,j=1}^{N} \frac{\lambda(\lambda - 1)}{(x_i - x_j)^2}.$$
The exact wave function of this model is

\[ \psi(x_1 \ldots x_N) = \prod_{i<j} (x_i - x_j)^{\lambda} . \] (11)

This wave function tells us that each collision contributes \( \pi \) to the phase shift. The Chern class, therefore, is the sum of jumps of the scattering phase,

\[ Q = N(N - 1)^{\lambda/2} . \] (12)

The same property takes place in almost all one-dimensional systems with any interaction. The reason is that in one-dimension any small bare interaction is effectively strong. Below we discuss the possibility that the topological order itself may be developed as a result of an effectively strong interaction in any spatial dimension.

Several peculiar properties follow immediately from the assumption of a topological order. For example, one-particle excitations in the one- and three-dimensional quantum antiferromagnet have fermionic statistics, while in two dimensions the statistics is fractional (half fermionic).

We start from reviewing the one dimensional case were topological character of the ground state is the most transparent.

## 2 Frohlich Superconductivity – One Dimension

Let us now review Frohlich’s ideas (see e.g., Ref. [12, 13]). We start from the Frohlich model of an incommensurate electron-phonon system with

\[ H = \sum \varepsilon_k a_k^+ a_k + \sum_q \omega_q b_q^+ b_q + g \sum_{x,q} a_{k+q} a_q (b_{-q}^+ + b_q) \] (13)

where \( a_k \) and \( b_q \) are electron and phonon operators, respectively.

In one-dimensional electron-phonon systems the Peierls instability causes a lattice displacement

\[ \langle u(x) \rangle = \text{Re} \ b_{2k_F} e^{2k_Fx} = \frac{\Delta}{g} \cos (2k_Fx + \varphi) \] (14)
and a modulation of the electronic density $\rho \sim \cos(2k_Fx + \varphi)$, where $\Delta \sim \exp(-\text{const}/g^2) << E_F$ is the value of the phonon condensate at $2k_F$ and $\varphi$ is its phase.

Frohlich noticed that the periodic density fluctuations of electrons are fixed only relative to the lattice and can easily travel with some velocity such that $\rho \sim \cos(2k_F(x - vt) + \varphi)$. This could be compensated by changing the phase according to $\dot{\varphi} = -2k_Fv$. Therefore the current is $j = \rho v = v(Nk_F/\pi)$, i.e.

$$j_x = -\frac{N}{2\pi} \dot{\varphi},$$

and from continuity

$$\rho = \rho_0 + \frac{N}{2\pi} \varphi',$$

where $N$ is the degeneracy of an electronic state (spin, for example). These are Frohlich’s equations.

Assuming that the charge density wave (CDW) is not pinned, i.e., $(\frac{1}{v_0^2} - v_0 \partial_x^2)\varphi = -E_x$, we obtain the eq. (4) with $\lambda^{-2} = \frac{N}{2\pi} v_0$, where $v_0 = \frac{\pi \rho}{mN}$ is the Fermi velocity.

Since there is a gap in the electronic spectrum, and the only gapless mode reflects the ability of the CDW to move, Frohlich concluded that his system is superconductive. In fact the CDW is an ideal conductor rather than a superconductor due to pinning mechanisms. We already mentioned that the Landau criterion does not work in one dimension - there is no space to flow around an obstacle.

Let us now derive the Frohlich equations formally. The first step is to pick out the fast variables and keep only the slow variables. In this case the slow variables are associated with electrons in the vicinity of two Fermi points $\pm k_F$,

$$a(x) \sim e^{ikFx} \psi_L + e^{-ikFx} \psi_R,$$

and phonons with momentum close to

$$\sum_q b_q e^{iqx} \sim \frac{\Delta}{g} e^{i(2kFx + \varphi)}.$$

In the continuum limit we then obtain the so-called linear $\sigma$-model

$$L = \frac{|\Delta|^2}{g^2 \omega^2} - \frac{|\Delta|^2}{g^2} + \bar{\psi}(i\hat{D} - |\Delta| e^{i\gamma_5 \varphi})\psi,$$
where $\hat{D} = \hat{\gamma}_\mu (\partial_\mu + A_\mu^{\text{ext}})$ and $\hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_5$ are two-dimensional Dirac matrices, and $\bar{\sigma}$ is a characteristic frequency of phonons. The modulus of the phonon field does not fluctuate much and is determined by its mean field value $\Delta \sim \exp(-\text{const}/g^2) \ll E_F$. This is the Peierls instability – as a result of the interaction a gap $\Delta$ has opened at the Fermi level. Let us stress that position of the gap is always at the Fermi level, so that the spectrum strongly depends on the number of particles (filling factor).

Consider now a linear response to an electric field (there is no magnetic field in one dimension), which brings us to the subject of the axial current anomaly. The point is that the continuum model (19) possesses a local axial gauge symmetry, $\psi(x) \rightarrow \psi(x) \exp i\gamma_5 \alpha(x)$, in addition to the ordinary gauge symmetry, $\psi(x) \rightarrow \psi(x) \exp i\alpha'(x)$. This would mean conservation of the axial current, $\partial_\mu j_5^\mu = \nabla \rho + v_0^{-2} \partial_t j = 0$ as well as the ordinary current, $\partial_\mu j_\mu = \partial_t \rho + \nabla j = 0$. As a result of this “symmetry” an external electric field may be gauged away which in turn turn means that the system does not respond to an electric field. This of course not true, due the axial anomaly – in the quantum theory the axial current is not conserved.

The axial current anomaly tells us that an external electric field gives rise to the non-conservation of the axial current $j_5^\mu = \bar{\psi}\gamma_\mu \gamma_5 \psi$,

$$\partial_\mu j_5^\mu = \frac{N}{2\pi} E_{\text{ext}}$$

(20)

where we have set $v_0 = e = c = 1$ and $\bar{\sigma} \rightarrow \infty$. Using that in 1D, $j_5^\mu = \epsilon_{\mu\nu} j_\nu$, we get a one-dimensional version of the (19).

$$\epsilon_{\mu\nu} \partial_\mu j_\nu = \frac{N}{2\pi} E_{\text{ext}}.$$  

(21)

In particular this equation means that the electronic system is compressible, even though the single particle spectrum has a gap. Compressibility is an inherit feature of the Peierls instability in incommensurate systems – we already stressed that the gap always opens at the Fermi level. Therefore, if one adds a particle to the system, it will not go to the upper band to occupy the lowest empty state. Instead it rearranges the period of the CDW, so as to create one more level in the lower band. The energy of the system does not change much by adding a particle, so the system is compressible.

This phenomenon implements the so-called level crossing. The fact is that in the presence of a kink in the phase, $\phi(\infty) - \phi(-\infty) = 2\pi$, the electronic
spectrum remains unchanged, except that one level appears at the top of the lower band with an energy $E = -\Delta$. When we add a particle to the system, it will therefore create a kink in the spatial configuration of $\phi$ and an extra level to be absorbed in the lower band. The anomaly equation (21) reflects this phenomenon. We derive this well-known result further on.

Strictly speaking, we cannot distinguish between solid and liquid in one dimension – there is no transversal current. Nevertheless, there is a global version of the Meissner effect – a current in a closed wire must satisfy

$$\int j \, dx = -\frac{1}{\lambda^2} \Phi^{\text{ext}}$$  \hspace{1cm} (22)

where $\Phi^{\text{ext}}$ is the magnetic flux within the wire.

Combining (20, 21) with the Frohlich equations (15, 16) and using the relation $j_\mu = -\partial L/\partial A^{\text{ext}}_\mu$ we obtain a bosonized version of the incommensurate CDW:

$$L_\phi = \frac{N}{4\pi} \left( \frac{1}{2} (\partial_m \phi)^2 - \frac{2}{\pi} E^{\text{ext}} \phi \right).$$  \hspace{1cm} (23)

Let us now turn to the commensurate CDW and consider an electronic system close to a half-filled band. At exact half filling, the CDW is two-fold commensurate and so the vacuum is two-fold degenerate. A canonical tight-binding model is

$$H = \sum_n \Delta_{n,n+1} (a_n^+ a_{n+1} + h.c) + H_\Delta$$  \hspace{1cm} (24)

where $\Delta_{n,n+1}$ is a fluctuating hopping amplitude and $H_\Delta$ is a phonon energy. In the continuum limit the half-filled case is described by the same $\sigma$-model (17) but with a real $\Delta$:

$$L = \bar{\psi} (i \dot{\psi} - i \gamma_5 \Delta (x)) \psi - \frac{\Delta^2}{g^2}$$  \hspace{1cm} (25)

In this case there is no soft translational mode since the CDW is commensurate and is pinned by the lattice. The excitations are kinks of $\Delta (x)$ which connect two-fold degenerate mean field vacua: $\Delta \to \pm \Delta_0$ when $x \to \pm \infty$. In the presence of the kink the electronic spectrum remains approximately unchanged, except for appearance of the so-called zero mode, a state with a zero energy, located exactly in the middle of the gap. This mode is $1/2$
degenerate (the index theorem), i.e. it can accommodate one-half of a particle. Fractional degeneracy means that if the particle has an isospin, only one particle with spin, say up, can occupy the zero mode. In coordinate space the wave function is located in the core of the kink. The axial anomaly in this case tells us that the density of extra states is equal to the density of zeros of $\Delta(x)$ (see, e.g.,[14])

$$\rho(x) = \frac{1}{2} \delta(\Delta(x)) \frac{\partial \Delta}{\partial x}.$$  

(26)

Suppose now we dope the system by adding additional particles $\delta$. The system will lower its energy by creating the number of solitons (zeros in $\Delta(x)$) which is necessary to absorb all dopants.

Due to the interaction between solitons they form a periodic lattice. As a result of that, zero modes develop a narrow band in the middle of the gap with a width of the order $\Lambda \sim \Delta_0 \exp(-\text{const} \delta \rho/\bar{\rho})$ where $\delta \rho$ is the doping density and $\bar{\rho}$ the density of the undoped system. This band absorbs all the dopants and is always completely full.

As in Frohlich’s case, these solitons have a translation mode due to their topological origin: a soliton lattice can slide along the atomic lattice without dissipation. Let $\bar{x}_i$ be the zeros of $\Delta(x)$. Then the density of extra particles (dopants) is $\delta \rho(x) = \frac{1}{2} \sum_i \delta(x - \bar{x}_i)$. Displacement of the positions of zeros around their mean field values, $x_i = \bar{x}_i + \varphi(x_i, t)/2\pi$, give rise to fluctuations of the density $\delta \rho(x) = \rho(x) - \bar{\rho}$. According to (26) they obey the same Frohlich equations (15,16),

$$j_\mu = -\frac{\bar{\rho}}{2\pi} \epsilon_{\mu \nu} \partial_\nu \varphi.$$  

(27)

The anomaly equations (21), with an extra $\bar{\varphi}$, apply again$^\text{2}$

$$\epsilon_{\mu \nu} \partial_\nu j_\mu = \frac{\bar{\rho}}{2} E_{\text{ext}}.$$

(28)

Each twist of $\varphi$ adds one additional state in the middle of the gap. Therefore, adding $n_e$ additional particles gives rise to the topological charge of $n_e = Q$ where $Q = \int \frac{d\varphi}{2\pi}$. All of this is true in incommensurate cases when the

---

2 In the lower density limit the factor $\bar{\varphi}$ is hidden in the velocity $v_0$. Here and later on we neglect its space dependence.
system, after doping, has infinitely degenerate classical vacua. If the doping is a rational number, say, \( p/q \), then the number of degenerate vacua is finite, namely \( q \). The CDW is pinned, generally by an exponentially small potential 

\[ \Delta_0^2 (\Delta_0 / \varepsilon_f)^{q-2} \cos q \phi. \]

Let us list some important messages that follow from this picture:

(i) An energy gap in the electronic spectrum eliminates elastic scattering.

(ii) The system has infinitely many degenerate vacua. They are characterized by the number of solitons, i.e., by a topological charge \( |Q \rangle \).

(iii) A topological configuration \( \varphi(x, t) \) such as \( \varphi(x, t = 0) = 0 \) and \( \int \partial_x \varphi dx = 2\pi \) at \( t \to \infty \), which transforms one vacuum \( |Q \rangle \) into another \( |Q + 1 \rangle \), is a low energy excitation with a dispersion \( \omega(k) = v_o k \). Then “the whole system, electrons and solitons, can move through the system without being disturbed” \[2\].

(iv) This, however, is not sufficient in dimensions greater than one. To achieve superconductivity in higher dimensions, it must have all these properties and in addition be a liquid.

### 3 Fermionic Number and Solitons

#### 3.1 Level Crossing

Let us consider fermions in a static (vector or scalar) potential. When the potential changes adiabatically, the fermionic energy levels also shift. If the chemical potential \( \mu \) lies in the gap, then for an adiabatic change of the potential the levels generally cannot cross the energy level \( E = \mu \) and therefore all occupied levels remain below \( \mu \). However, there are certain potentials which create some unoccupied levels below \( \mu \) or force some occupied levels to cross the level of the chemical potential. This phenomenon is known as level crossing. Unoccupied levels appear below (or occupied levels appear above) the chemical potential and separated from the continuous part of the spectrum are what is called zero modes.

There is an important theorem (the index theorem) which states that potentials which give rise to crossing levels, must have a topological charge.
We refer to them below as solitons. In Sect. 2 we discussed solitons (kinks) in one-dimensional models. Below we review them and discuss solitons in some models in 2 and 3 dimensions which will illustrate the essential physics of the topological mechanism.

3.2 Models and Solitons

3.2.1 One Dimension – Kinks.

The most general Hamiltonian of the Peierls model is

\[ H = \alpha_x i \partial_x + \beta \pi_1 + i \gamma_5 \pi_2 \] (29)

where the Dirac matrices \( \alpha_x, \beta, \gamma_5 = \alpha \beta \) may chosen as the Pauli matrices: \( \alpha_x = \sigma_3, \beta = \sigma_1, \gamma_5 = -i \sigma_2 \). We also assume that the modulus of the vector \((\pi_1, \pi_2)\) takes a fixed value at infinity.

The soliton here is a field \( \vec{\pi}(x) \) which forms a homotopy class \( \pi_1(S^1) \). Its topological charge is

\[ Q = -\int \frac{\vec{\pi} \times \partial \vec{\pi}}{\vec{\pi} \cdot \vec{\pi}} \frac{dx}{2\pi} . \] (30)

With a given value of the \( |\vec{\pi}| \) at infinity \( Q \) is an integer number.

3.2.2 Two Dimensions – Vortices.

In dimensions greater than one it is necessary to include a vector gauge field in addition to a scalar field.

We consider two models:

A. QED\(_3\)

\[ H = \sum_{\sigma=1,2} \psi_\sigma^\dagger [\alpha_\mu (i \partial_\mu + a_\mu) + \beta m] \psi_\sigma . \] (31)

A soliton for this model is a vertex or magnetic flux with the charge

\[ Q = -\int \frac{\vec{\nabla} \times \vec{a}}{2\pi} dx . \] (32)

B. Another model is non-abelian (SU(2)) QCD\(_3\) coupled to a meson field

\[ H = \alpha_\mu (i \partial_\mu + \vec{A}_\mu \vec{\tau}) + \beta \vec{\tau} \vec{\pi} . \] (33)
Solitons in this model are a combination of the hedgehog of the meson field (an element of the class $\pi_2(S^2)$) and a vortex of the gauge field. Its topological charge is

$$Q = \int \left[ -\frac{\vec{F}}{|\vec{F}|} + \frac{1}{4} \frac{\vec{D}\vec{\pi} \times \vec{D}\vec{\pi}}{|\vec{D}\vec{\pi}|^3} \right] d^2x$$

(34)

where $\vec{F} = \partial_1 \vec{A}_2 - \partial_2 \vec{A}_1 + \vec{A}_1 \times \vec{A}_2$ and $\vec{D}\vec{\pi} = \partial \vec{\pi} + 2\vec{A} \times \vec{\pi}$ is the covariant derivative. A soliton with a topological charge of 1 has the asymptotic form

$$\pi_3 = 0 \quad (\pi_1, \pi_2) \to \frac{\vec{r}}{r} \quad A_i^3 \to \frac{1}{2} \varepsilon_{ij} \frac{r_j}{r^2} .$$

(35)

at infinity. A general form of a vortex with an arbitrary topological charge is given by the homotopy class $\pi_2(S^2)$.

Let $\hat{\pi} = \vec{\pi}/|\vec{\pi}|$ be an element of this class. We shall set

$$A_i = \frac{1}{2} \hat{\pi} \partial_i \hat{\pi} = \vec{A}_i \pi ,$$

(36)

a vortex with a charge

$$Q = \frac{1}{8\pi} \int \text{tr}(\hat{\pi}d\hat{\pi}d\hat{\pi} .$$

(37)

A unit of the topological charge is carried by a zero of $\vec{\pi}$. The density of the topological charge is given by the “magnetic field” projected onto $\vec{\pi}$

$$B^{\text{vortex}} \equiv \hat{\pi} \vec{F}^{\text{vortex}} = \frac{1}{2} \varepsilon^{abc} \hat{\pi} a \partial_1 \hat{\pi} b \partial_2 \hat{\pi} c$$

(38)

so that $Q = \frac{1}{2\pi} \int B^{\text{vortex}} d\vec{r}$ is a total flux.

3.2.3 Three Dimensions – Monopoles.

As a model in three spatial dimension we consider the SU(2) gauge theory with chiral bosons:

$$H = \alpha_\mu (i\partial_\mu + \vec{A}_\mu \vec{\pi}) + i\gamma_5 \vec{\pi} \vec{\pi} + m .$$

(39)

The spectrum of this model is symmetric since the matrix $\gamma_5$ anticommutes with the Hamiltonian.
Solitons for this model are the Polyakov-'t Hooft monopoles. At \( m = 0 \), and at \( r \to \infty \) the monopole configuration is

\[
\vec{\pi}(r) \to \frac{\vec{r}}{r} \quad A_i^a \to \frac{1}{2} \varepsilon_{iaj} \frac{r_j}{r^2} .
\] (40)

Similar to the two-dimensional case the soliton configuration is given by (36) and is an element of \((\pi_2(S^2))\). Then the monopole charge is carried by zeros of \( \vec{\pi} \). It has the form of (37) where the integral now goes over a boundary surrounded zeros of \(|\vec{\pi}|\). The density of the monopole’s charge is

\[
q = \frac{1}{8\pi} \varepsilon_{ijk} \mathrm{tr}(\hat{\pi}_i \hat{\pi}_j \hat{\pi}_k) \nu_k
\] (41)

where

\[
\nu_k = \partial_k |\vec{\pi}| \delta(|\vec{\pi}|)
\] (42)
is the density of zeros of \( \vec{\pi} \).

The “magnetic field” projected onto \( \vec{\pi} \) is a gauge invariant,

\[
B_i^{\text{mon}} \equiv \frac{1}{2} \varepsilon_{ijk} \hat{\pi}_j \hat{\pi}_k = \varepsilon_{abc} \hat{\pi}^a \partial_j \hat{\pi}^b \partial_k \hat{\pi}^c .
\] (43)

So \( q = \frac{1}{4\pi} \nu_i B_i^{\text{mon}} \), which is equivalent to the Dirac equation for the monopole charge \( q = \frac{1}{4\pi} \partial_i B_i^{\text{mon}} \).

Rotation of the vector \( \hat{\pi} \) implies a gauge transformation \([15]\). It is convenient to describe a multimonopole system in the unitary gauge. If \( g \) is a transformation which rotates \( \hat{\pi} \) to the third axis, \( \hat{\pi} = g^{-1} \tau_3 g \), then the gauge transformation

\[
\hat{\pi} \to g \hat{\pi} g^{-1} = \tau_3
\] (44)

\[
A_i \to g A_i g^{-1} - \partial_i g g^{-1} = A_i^D \tau_3
\] (45)

converts the non-abelian monopole to the Dirac monopole:

\[
A_i^D = -\mathrm{tr}(\tau_3 \partial_i g g^{-1}) .
\] (46)

This gauge is singular. It transfers a topological charge from the \( \vec{\pi} \)-field to the gauge field. A general form of the topological charge can be written in a compact form if one includes the mass \( m \) into the 4-vector \( \pi_a = (m, \vec{\pi}) \):

\[
Q = \frac{1}{2\pi^2} \int \varepsilon_{ijk} \varepsilon_{abcd} \frac{\tau_a D_i \pi_b D_j \pi_c D_k \pi_d}{|\vec{\pi}|^4} - \frac{3\pi_a D_i \pi_b F_{jk}^{cd}}{4|\vec{\pi}|^2} d^3 x .
\] (47)
3.3 Anomalous Calculus

In this section we calculate the so-called “fermionic number” or the Atiah-Patodi-Singer invariant $\eta$ for the models discussed in the previous section. It is simply the number of new states appearing above (or below) a certain energy when one switches on the potential. For Dirac particles it is the number of states appearing inside the gap. These states are referred to as zero modes or midgap states. Since without a potential the spectrum of the Dirac operator is symmetric, this number also measures the difference between the number of states with positive and negative energy $\eta = N_+ - N_-$ (spectral asymmetry). If $N = N_+ + N_-$ is the total number of levels, then the number of negative levels would be $N_- = \frac{N}{2} - \frac{1}{2} \eta$. The “regularized” number of negative levels i.e. the change of the negative levels due to the potential, $\Delta N = N_- \frac{\eta}{2} = -\frac{1}{2} \eta$, is the fermionic number.

The index theorem (see e.g. [14]) states that the fermionic number of the Dirac operator, or the number of extra states induced by a soliton, equals the topological charge of the soliton, $\Delta N = Q$. Moreover the density of the extra states is equal to the topological density $q(x)$:

$$\rho(x) \equiv \langle |\psi(x)|^2 \rangle = q(x) \ .$$

We illustrate this general fact for the models listed in this section.

The fermionic number for the Dirac operator may be defined as

$$\Delta N = -\frac{1}{2} \text{Tr}[\text{sgn } H] \ .$$

This divergent quantity has to be regularized. One of the standard ways of regularization is to replace it by

$$\Delta N = -\frac{1}{2\pi} \int \text{Tr}\left[\frac{H}{H^2 + z^2}\right]$$

The integrand now converges.

3.3.1 One Dimension – Kinks.

Let us start from the Peierls model (29),

$$H = \alpha_x i \partial_x + \beta \pi_1 + i \gamma_5 \pi_2 \quad \alpha_x = \sigma_3, \ \beta = \sigma_1, \ \gamma_5 = -i \sigma_2 \ .$$

(51)
Then the square of the Hamiltonian is

\[ H^2 = (i\partial_x)^2 + \vec{\pi}^2 + \varepsilon_{ab}\sigma_a\partial_x\pi_b, \quad a, b = 1, 2. \]  (52)

Let us now expand the integrand of (50) in \( \partial\vec{\pi} \). In the first non-vanishing order we have

\[ \Delta N = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} \text{Tr} \left( \frac{\sigma^3 i\partial + \vec{\pi}}{(i\partial)^2 + \vec{\pi}^2 + z^2} \right) \]  (53)

This is the only order which contributes to the fermion number. The trace in spinor space gives

\[ \Delta N = -\frac{1}{\pi} \int_{-\infty}^{+\infty} dz \text{Tr} \left( \frac{\pi^1 \partial\pi^2 - \pi^2 \partial\pi^1}{(i\partial)^2 + \vec{\pi}^2 + z^2} \right). \]  (54)

The trace in momentum space is conveniently calculated in the plane wave basis,

\[ \Delta N = -\frac{1}{\pi} \int_{-\infty}^{+\infty} dz \int dx \int \frac{dp}{2\pi} \epsilon_{\mu\nu}\pi^\mu \partial \pi^\nu \frac{1}{[p^2 + \vec{\pi}^2 + z^2]^2}. \]  (55)

Finally the integrals over \( p \) and \( z \) give

\[ \Delta N = -\frac{1}{2\pi} \int dx \epsilon_{\mu\nu} \frac{\pi^\mu \partial \pi^\nu}{\pi^2}, \]  (56)

the topological charge (30) of kinks. The number of extra states induced by a soliton therefore equals the topological charge of the soliton.

The model with \( \pi_1 = 0 \) deserves special interest. This is commensurate Peierls model \((\pi_2 = \Delta)\). It is defined by

\[ H = \alpha_x i\partial_x + i\gamma_5 \Delta. \]  (57)

The spectrum of the Hamiltonian is symmetric. This means that if there is a state \( \psi_E \) with an energy \( E \) then there is always a state \( \beta\psi \) with the energy \( -E \), except for \( E \neq 0 \). It follows from the fact that the Hamiltonian anticommute with the matrix \( \beta \).

Now the soliton is a kink with \( \Delta(-\infty) = -\Delta(+\infty) = \Delta_0 \). Setting \( \pi_1 = 0 \) in (50), we obtain (42) in the form

\[ Q = -\frac{1}{2} \int \delta(\Delta) \partial\Delta dx. \]  (58)
In this case an extra state appears in the middle of the gap (zero mode). The kink does not respect periodic boundary conditions. As a result of this a zero mode state may accommodate only $1/2$ of the particle. In a system with a periodic boundary conditions kinks may appear only together with antikinks, so the total number of states remains integer.

### 3.3.2 Two Dimensions – Vortices.

**A. QED**

We now consider the model (31)

$$H = \sum_{\sigma=1,2} \psi_{\sigma}^\dagger [\alpha_\mu (i\partial_\mu + a_\mu) + \beta m] \psi_{\sigma}.$$  \hspace{1cm} (59)

The square of its Hamiltonian is

$$H^2 = (i\partial_\mu + a_\mu)^2 + m^2 - \beta(\partial_1 a_2 - \partial_2 a_1)$$  \hspace{1cm} (60)

Let us expand the integrand in (51) in terms of $\beta(\partial_1 a_2 - \partial_2 a_1)$. The first and only non-vanishing order gives:

$$\Delta N = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \int d^2 x \int \frac{d^2 p}{(2\pi)^2} \frac{m(\partial_1 a_2 - \partial_2 a_1)}{[(p_\mu + a_\mu)^2 + m^2 + z^2]^2}.$$  \hspace{1cm} (61)

The trace leaves the integral over $p$ and $z$:

$$\Delta N = -\frac{2}{\pi} \int_{-\infty}^{+\infty} dz \int d^2 x \int \frac{d^2 p}{(2\pi)^2} \frac{m(\partial_1 a_2 - \partial_2 a_1)}{[(p_\mu + a_\mu)^2 + m^2 + z^2]^2}.$$  \hspace{1cm} (62)

As a result we obtain (compare (32)) the relation between the fermion number and the topological charge of the vortices

$$\Delta N = -\frac{1}{2\pi} \text{sgn} m \int d^2 x (\partial_1 a_2 - \partial_2 a_1).$$  \hspace{1cm} (63)

**B. QCD**

**B. QCD\textsubscript{3} coupled with a meson field.** This model is defined by

$$\mathcal{H} = \alpha_\mu (i\partial_\mu + \vec{A}_\mu \vec{\tau}) + \beta \vec{\pi} \vec{\tau}.$$  \hspace{1cm} (64)
and the square of the Hamiltonian is
\[ \mathcal{H}^2 = (i\partial_\mu + \vec{A}_\mu \vec{\pi})^2 - \beta \vec{F} \vec{\tau} + (\vec{\pi})^2 + \alpha_\mu \beta i\partial_\mu \vec{\pi} \vec{\tau} + \alpha_\mu \beta i[\vec{A}_\mu \times \vec{\pi}] 2\vec{\tau}. \] (65)

To simplify calculations we first drop the gauge field. Then, expanding the integrand in (50) in terms of \( \alpha_\mu \beta i\partial_\mu \vec{\pi} \vec{\tau} \) up to the second term, we have
\[
\Delta N = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \text{ Tr} \left[ \frac{\alpha_\mu i\partial_\mu + \beta \vec{\pi} \vec{\tau}}{(i\partial_\mu)^2 + (\vec{\pi})^2 + z^2} \times \right.
\left. \left( 1 - \left( \alpha_\mu \beta i\partial_\mu \vec{\pi} \vec{\tau} \right) \frac{1}{(i\partial_\mu)^2 + (\vec{\pi})^2 + z^2} + \left( \alpha_\mu \beta i\partial_\mu \vec{\pi} \vec{\tau} \right) \frac{1}{(i\partial_\mu)^2 + (\vec{\pi})^2 + z^2} \right)^2 \right] \] (66)

Changing \( i\partial_\mu \) to \( p_\mu \) and taking the trace in the spinor and isospinor spaces we get
\[
\Delta N = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dz \int d^2x \int d^2p \frac{\alpha_\mu p_\mu + \beta \vec{\pi} \vec{\tau}}{(2\pi)^2 [(p_\mu)^2 + (\vec{\pi})^2 + z^2]^3} i\beta \left[ \partial_1 \vec{\pi} \vec{\tau}, \partial_2 \vec{\pi} \vec{\tau} \right]. \] (67)

Finally, integrating over \( p \) and \( z \) we obtain the fermion number
\[
\Delta N = \int \rho(x) d^2x \] (68)
with the density
\[
\rho = \frac{1}{8\pi} \epsilon^{ij} \epsilon^{abc} \vec{\pi}^a \partial_i \vec{\pi}^b \partial_j \vec{\pi}^c \] (69)

Once again we obtain that the density of the extra states is equal to the topological charge of the soliton.

To include the gauge field one must first replace derivatives in (68) by covariant derivatives. Then we notice that for \( \pi_1 = \pi_2 = 0 \) and \( A^1_\mu = A^2_\mu = 0 \) the fermion number must reproduce the QED\(_3\) result(63). Taken together these considerations give
\[
\rho = -\frac{1}{2\pi} \frac{\vec{F}}{|\vec{\pi}|} + \frac{1}{8\pi} \epsilon^{ij} \epsilon^{abc} \vec{\pi}^a (\partial_i \vec{\pi}^b + 2[\vec{A}_i \times \vec{\pi}]^b)(\partial_j \vec{\pi}^c + 2[\vec{A}_j \times \vec{\pi}]^c) \] (70)

where \( \vec{F} = \partial_1 \vec{A}_2 - \partial_2 \vec{A}_1 + [\vec{A}_1 \times \vec{A}_2] \).
A special case of interest is $\pi_3 = 0, A^1_\mu = A^2_\mu = 0$. In this case the spectrum is even as the matrix $\beta\tau_3$ anticommutes with the Hamiltonian. Equation (70) gives

$$\rho = - (\varepsilon_{ij}\partial_i \pi_1 \partial_j \pi_2 + \frac{1}{4\pi} |\vec{\pi}|^2 \vec{\nabla} \times \vec{A}^3) \delta(|\vec{\pi}|) .$$

(71)

In this case all extra states are located exactly in the middle of the gap (zero modes). Each zero mode may accommodate 1/2 of a particle.

### 3.3.3 Three Dimensions – Monopoles.

The fermion number for the model (39) is again equal to the topological charge of the monopole (47) [16]. The simplest way to calculate it is to drop the gauge field and proceed similarly as in the previous section. Now we must keep the third order term in the expansion (66). As a result we obtain

$$\Delta N = \frac{1}{2\pi^2} \int d^3x \varepsilon_{ijk} \varepsilon_{abcd} \frac{\pi_a d_i \pi_b d_j \pi_c d_k \pi_d}{|\vec{\pi}|^4} .$$

(72)

Next we want to make this expression gauge invariant. After a replacement of derivatives by covariant derivatives, it still remains gauge non-invariant. To rectify this one must add the second term in (47).

Similar to the previous examples a very special case arises when $m = \pi = 0$. In this case the matrix $\beta\gamma_5$ anticommutes with the Hamiltonian, and the spectrum is symmetrical. Therefore the extra states appear in the middle of the gap at $E = 0$ (zero modes). As usual each zero mode is 1/2 degenerate. Setting $\pi_0 = 0$ in (47) we obtain

$$\rho(x) = \frac{1}{8\pi} \varepsilon_{ijk} \text{tr} (\hat{\pi} [\partial_i \hat{\pi} \partial_j \hat{\pi} + \frac{1}{4\pi} F_{ij}]) \nu_k$$

(73)

where $\hat{\pi} = \vec{\pi}/|\vec{\pi}|$ and $\nu_k = \partial_k |\vec{\pi}| \delta(|\vec{\pi}|)$ is the density of zeros of $\vec{\pi}$.

### 4 Topological Fluid as a Superfluid

#### 4.1 The Basic Model – Holons and Spinons – Zero Current Theory

The main inspiration for considering topological superconductivity comes from strongly correlated electronic systems. They suggest a general model
to study this phenomenon. Running ahead we discuss this model now and later derive it from a canonical model of the doped Mott insulator.

Consider two sorts of particles “holons” $h$ and “spinons” $\psi$ (the names come from the Mott insulator literature) which obey the so called “zero current” theory. This means that densities and currents obey the local constraint:

(i) their densities are always complementary, $h^\dagger(x)h(x) = \psi(x)\psi^\dagger(x)$, namely the number of holes in spinons is the number of holons, where the total number of holons is given by $\int h^\dagger(x)h(x)dx = \delta$ (doping) and

(ii) their currents are always opposite (the so-called “zero current theory”) $\vec{j}_h(x) + \vec{j}_s(x) = 0$.

The holons carry an electric charge and spinons are neutral, so that the holon current is in fact the electromagnetic current. The Hamiltonian of the basic model is

$$\mathcal{H} = \frac{1}{m}h^\dagger H^2 h(x) + \psi(x)H\psi(x) + \vec{A}^\text{ext}\vec{j}_h,$$  \hspace{1cm} (74)

where $H$ may be any Hamiltonian which exhibits a level crossing. For example, one may consider any one of the models considered in the previous section. The local constraints are automatically enforced if $H$ contains an abelian gauge field, as in (31) of QED$_3$, otherwise a gauge field (Lagrangian multiplier) must be added to (74): $\vec{A}(\vec{j}_h + \vec{j}_s) + A_0(h^\dagger(x)h(x) - \psi(x)\psi^\dagger(x))$. Later we derive the Hamiltonian $\hat{H}$ for the Mott insulator. This Hamiltonian is in fact different from any of the standard models listed in the Sect. 3, but has essential common features with models of the sec.3.

### 4.2 Topological Instability

Let us consider the basic model of topological fluids (74). We argue that the perturbative vacuum of the model is unstable with respect to the creation of topological charge and solitons, namely we show that the vacuum has a topological charge such that the fermionic number equals the number of dopants (holons),

$$-\frac{1}{2}\eta = \Delta N = \delta.$$  \hspace{1cm} (75)

This the topological charge of the ground state.
To see this, let us apply the adiabatic strategy: first consider all potentials \( \vec{A} \) and \( \pi \) static and find the best spatial configuration to minimize the energy for a given doping, and then consider fluctuations around the minimal static configuration. First assume that there are no solitons and start adding dopants. Consider first the spinon part of the model. Dopants start to fill the upper band of the Dirac spectrum of spinons. This costs the energy of the gap, plus the Fermi energy of spinons, plus some small radiative corrections of order of \( \delta \). If, however, there are enough solitons to absorb by their zero modes all dopants, we pay less than the gap and gain all that energy. In addition we gain the Fermi energy, since the zero modes form a narrow band inside the gap. The same arguments hold for the holon part of the Hamiltonian as well. The solitons form a narrow band for the holons at zero energy, so they also favour the topological vacuum. As a result the number of solitons in the vacuum must provide that number of zero modes which will absorb all dopants, \( h\dagger h = \delta \).

To formalize these arguments, one can show that the perturbative vacuum is indeed unstable. The topological condensate appears in the one-loop correction. In leading order of the chemical potential, the free energy is

\[
E - \frac{1}{2}|\mu\eta| - \mu\delta
\]

where \( E \) is a perturbative energy. Equations (75) minimize this free energy.

We conclude that a small doping requires a configuration of \( \pi \) and \( \vec{A} \) with the relevant topological charge determined by (75). In turn the wave function of the state with a topological charge establishes the Chern class (7) equal to the charge. We refer to this phenomenon as topological instability.

### 4.3 Topological Field Theory coupled with the Matter Field and Superfluidity

A simple version of the basic theory occurs when the gap in the Dirac spectrum is the largest parameter in the system. Integrating over spinons we can then neglect all terms except the so called the Wess-Zumino-Witten term, denoted WZW hereafter. It is the only term which does not vanish when the gap approaches infinity, and is given by

\[
WZW = \lim_{\text{gap} \to \infty} \ln \text{Det}(i\partial_t + H)
\]
The WZW term captures in the Lagrangian language the phenomenon of anomalies, namely, anomalous currents may be obtained as the Lagrangian equation $j_\mu = -\delta WZW/\delta A_\mu$. For the 1D model (29), the topological current (see (30))

$$j_\mu = -\frac{1}{2\pi} \varepsilon_{\mu\nu} \varepsilon_{ab} \pi^a \partial_\nu \pi^b |\vec{\pi}|^2$$

(78)

Therefore,

$$WZW = -\int \frac{dx}{2\pi} \varepsilon_{\mu\nu} \varepsilon_{ab} A_\mu \pi^a \partial_\nu \pi^b |\vec{\pi}|^2$$

(79)

The WZW term for QED$_3$ it is known as the Chern-Simons term:

$$WZW = -\int \frac{dx}{4\pi} \text{sgn} \varepsilon_{\mu\nu\lambda} A_\mu d_\nu A_\lambda$$

(80)

For QCD$_3$ and for the 3D model $WZW = -\int A_\mu q_\mu dx$, where $q_\mu$ is the topological current given by the integrand of (32) and (42) [17].

The WZW term itself gives rise to and is descriptive of topological field theories. A topological field theory describes only the zero mode sector of a field theory and the zero mode sector is invariant under all diffeomorphisms of the volume – positions of the solitons are not fixed unless they interact with matter. This means that any external perturbation, such as inhomogeneous pressure or twist, does not change the energy of the system. As a result the zero mode sector is topological in the sense that it has no physical states in the bulk. Physical states appear only as a result of a boundary and they are confined to a boundary (the edge states).

Our basic model is a topological field theory coupled with matter (holons),

$$L = \int dx \ h^\dagger (i\partial_t + A_0 + H^2) h + WZW .$$

(81)

In this case the zero modes are occupied by particles (holons) and possess some dynamics. Nevertheless, this dynamics is limited as a result of the overall gauge symmetry of the unperturbed topological field theory. Matter of course destroys all diffeomorphisms which change the density of the matter, so that only diffeomorphisms which preserve the density will survive – but matter which is invariant under density preserving diffeomorphisms is an ideal liquid i.e. a superfluid.
4.4 Superconductivity

As in the one-dimensional case, it is almost obvious that a dilute incommensurate system of solitons forms a compressible liquid. Indeed, a soliton position is not fixed relatively to the rest of the system. It can be translated to any place in the system without changing the energy – zero modes are translationally degenerate. Therefore, if a soliton moves slowly, say with a momentum \( \vec{k} \), the energy of the system changes by \( v \vec{k} \). Since the interaction between solitons is short range, the same remains true for dilute system of solitons \( \square \). Thus we assume the solitons form a liquid. Each soliton carries a density of charge carriers and an electric charge according to its topological charge \( \square \). Translation degeneracy of the zero modes therefore implies longitudinal sound in modulation of electronic density and this conforms to the first part of the Landau criterion.

Furthermore, the narrow band formed by electrons trapped in solitons is always filled and separated from the rest of the spectrum. Therefore all excitations, except coherent modulation of charge density and topological density, cost non-zero energy. Indeed, one particle excitations consist of a particle being taken away from the soliton core. Since all particles form bound states with solitons, this costs energy i.e. the excitation has a gap. This conforms to the second part of the Landau criterion. To summarize, we conclude that the Landau criterion is satisfied and the topological liquid is a superfluid.

To formalize these arguments let us consider for example the basic model \( \square \) with

\[
H = \sum_{\sigma=1,2} \psi_\sigma^\dagger \left[ \alpha_\mu (i\partial_\mu + a_\mu) + \beta m \right] \psi_\sigma \quad (82)
\]

in two dimensions. At low energy the system develops a density of magnetic flux determined by the density of extra particles:

\[
\rho(x) = -\frac{1}{\pi} \vec{\nabla} \times \vec{a} = -\frac{1}{\pi} F(x) \quad .
\]

\( ^{3} \) At larger density solitons may form a Wigner crystal, i.e. a solid, or the interaction may destroy a bound state between a soliton and electrons (whatever comes first). It may happen that the soliton density is commensurate with the lattice. In this case solitons form a solid and are pinned by the lattice. If commensurability is weak, the solid may be melted by quantum fluctuations. We do not discuss these questions here.
When the solitons move, they are followed by particles, implying
\[ j^* = \frac{1}{\pi} (\partial_t a - \nabla a_0) = - \frac{1}{\pi} E , \]  
(84)

where \( j^*_i = \varepsilon_{ik} j_k \), and an additional factor 2 comes from the spin. A homogeneous density \( \bar{\rho} \) gives rise to a homogeneous magnetic field \( \langle F \rangle \) and no electric field. This is the first level of adiabatic approximation. To study modulation of the density and to see that there is no dissipation, we want to see how the gauge field fluctuates. Its fluctuations are determined by the motion of electrons in the presence of magnetic flux \( F/\pi = \bar{\rho} \). The relation between the density and topology implies that the first two Landau levels are completely filled and in addition they separated by the gap \( m \) from the upper Dirac band. In other words, all single particle excitations have a gap, which means that excitations over the gap produce an energy \( \frac{1}{2} \varepsilon \bar{E}^2 + \frac{1}{2 \chi} (F - \langle F \rangle)^2 \). Here \( \varepsilon \) and \( \chi \) are the dielectric constant and diamagnetic susceptibility of electronic zero modes, respectively. As a result static fluctuations of the topological charge \( F(x) - \langle F \rangle \) and topological current \( \bar{E}(x) \) are short range:

\[ \langle (F(x) - \langle F \rangle), (F(x') - \langle F \rangle) \rangle = \chi \delta(x - x') , \]  
(85)
\[ \langle \bar{E}(x), \bar{E}(x') \rangle = \varepsilon^{-1} \delta(x - x') . \]  
(86)

We can now obtain density and current correlation functions since they are directly related to fluctuations of the topological charge. Combining them with the continuity condition, we obtain the linear response function for a superfluid with \( \frac{\nu^2}{\lambda^2} = \chi \) and \( \nu^2 = \varepsilon \chi \) in (1) and (2).

A similar strategy applies for the 2D model (33). Analysis of the three dimensional model is more complicated and involves some new features [4]. The physical picture, however, remains the same, as briefly indicated below.

Since all single particle excitations have a gap, we expect that all low energy excitations are fluctuations of the topological charge \( q_0 \) and topological current \( q_i \). For the same reason, we know that static fluctuations are short range. We also know that there is a longitudinal sound since the topological configuration possesses a soft translational mode. Assuming that solitons do not form a crystal, we must conclude that the low energy topological excitations obey hydrodynamics:

\[ H = \frac{\nu^2}{2\kappa} [q_0^2 + v_0^{-2} q_i^2] . \]  
(87)
where \((q_0, \vec{q})\) is a density of the topological charge and topological current of solitons.

Finally, as a result of the topological instability, the density of electric charge (current) is directly related to the topological charge (current)

\[
\dot{j}_\mu = \nu q_\mu ,
\]

which, together with (87), gives the hydrodynamics of the ideal charged liquid,

\[
H = \frac{1}{2\kappa}((\rho - \rho_0)^2 + v_0^{-2}j^2)
\]

i.e. hydrodynamics of the superconductor (7).

4.5 Topological Superconductivity and Bosonization in Higher Dimensions

Bosonization is a fascinating phenomenon of one-dimensional physics – collective modes of interactive fermionic systems may be described by bosonic particles. Moreover, fermions themselves can be treated as solitons of bosonic non-linear waves. The physical reasons behind bosonization are the same as for the Frohlich superconductivity. In fact, a generic one-dimensional incommensurate system develops some density wave (CDW, SDW, etc.) as a low energy excitation. These are bosonized fermionic collective modes with

\[
H = \frac{\hbar}{2}[\pi^2 + v_0^2(\partial_x \varphi)^2] .
\]

(Here \(\pi\) is the canonical momentum of the bosonic field \(\varphi\).) In other words, bosonization in one dimension implies hydrodynamics. If the one-dimensional system is commensurate, the Hamiltonian of the density wave acquires some non-linear terms, as in the Sine-Gordon model, for example.

This interpretation allows the extension of bosonization into higher dimensions. We shall say that a fermionic system can be bosonized if its low energy excitations can be described entirely in terms of currents, such as when an effective Hamiltonian of low energy dynamics has the Sugawara form, i.e. may be written as a bilinear form of currents. Recall that this form implies the hydrodynamics of the superfluid (6). The Sugawara form is
a linear form of quantum hydrodynamics [4]:

\[ H = \frac{1}{2} \rho(x) \bar{v}(y)^2, \quad [\rho(x), \bar{v}(y)] = i \vec{\nabla} \delta(x - y), \quad \vec{j} = \rho \bar{v}. \quad (90) \]

The bosonized version of linear hydrodynamics appears in terms of a displacement \( \vec{u} \). Let us solve the continuity condition by setting \( \rho - \bar{\rho} = \bar{\rho} \vec{\nabla} \vec{u} \) and \( \vec{j} = \bar{\rho} \partial_t \vec{u} \). Then (6) becomes

\[ L = \frac{\bar{\rho}}{2} [((\partial_t \vec{u})^2 - v_0^2(\nabla \vec{u})^2]. \quad (91) \]

In contrast to one dimension, not every two- or three-dimensional fermionic system can be bosonized, but only those which are superfluids. For instance a fundamental property of the Fermi liquid is that it cannot be bosonized. Dissipation and gapless excitons are its own characteristic features.

We have indicated that a fermionic theory with a topological charge in the ground state can be bosonized, regardless of its dimension. From this point of view bosonization can be treated as a projection onto a zero mode narrow band.

5 A Gauge Model for Correlated Electronic Systems

Strongly correlated electronic systems represent physical systems where topological fluids may appear. In this section we sketch the derivation of a basic model of a topological fluid from the Hubbard model or the “t - J” model – a canonical model of strongly correlated electronic systems. We also emphasize a set of assumptions and approximations which are necessary to separate the physics of topological fluids from a variety of other physical phenomena which occur in a correlated electronic system.

5.1 A Model for the Doped Mott Insulator – Resonant Valence Bonds and Chirality

The “t - J” model is defined by

\[ H = \sum_{ij} t_{ij} c_{i\sigma}^{\dagger} c_{j\sigma} + J_{ij} \vec{S}_i \cdot \vec{S}_j \quad (all \ J_{i,j} > 0), \quad (92) \]
where $\vec{S}_i = c_{i\sigma} \vec{\sigma}_{\sigma\sigma'} c_{i\sigma'}$ is a spin operator of an electron at the lattice site $i$. A hopping amplitude $t_{ij}$ and an exchange amplitude $J_{ij}$ connect the nearest sites and the total number of electrons is close to the number of lattice sites: $N_e = N_0(1 - \bar{\rho})$, while a strong Coulomb interaction does not allow doubly occupied states:

$$n_i = \sum_{\sigma=\uparrow,\downarrow} c_{i\sigma}^\dagger c_{i\sigma} = 0 \text{ or } 1 .$$

At zero doping the model describes the Heisenberg antiferromagnet

$$H = \sum_{ab} J_{ab} \vec{S}_a \vec{S}_b ,$$

with spins $\vec{S}_a$ and $\vec{S}_b$ on sublattices A and B, respectively.

In addition to an average value of spin (classical description) there are two operators which characterize the ground state of an antiferromagnet, namely density of energy

$$\varepsilon_{ij} = (1/4 + \vec{S}_i \vec{S}_j)$$

and chirality or measure of topological order

$$W(C) = \text{tr} \prod_{i \in C} (1/2 + \vec{\sigma} \vec{S}_i) ,$$

where $\vec{\sigma}$ are Pauli matrices, and $C$ is a lattice contour.

The latter operator is of particular importance for the doped case since it determines the correlation of electronic phases at different spatial points.

These two operators are related to the amplitude and phase $\Delta_{ij}$ of Anderson’s Resonance Valence Bond (RVB) through

$$\varepsilon_{ij} = |\Delta_{ij}|^2$$

and

$$W(C) = \prod_C \Delta_{ij} .$$

It follows from this definition that the RVB is a gauge field. It can be locally transformed by a $U(1)$ transformation

$$\Delta_{ij} \rightarrow \Delta_{ij} e^{i(\alpha_i - \alpha_j)} .$$
and by a symmetry group of the crystal class

$$\Delta \rightarrow C^{-1} \Delta C,$$  \hspace{1cm} (100)

without changing the physical properties. In terms of this field, the topological order parameter $W(C)$ acquires the form of a lattice Wilson loop:

$$W(C) = e^{i \phi(C)}.$$  \hspace{1cm} (101)

Roughly speaking, the flux of the RVB field

$$e^{i \phi(C)} = \prod_C e^{i A_{ij}},$$  \hspace{1cm} (102)

where $A_{ij}$ is a phase of $\Delta_{ij}$, represents a “magnetic” flux, and penetrates through a surface enclosed by the contour $C$. The RVB field can be formally introduced by fermionic representation of a spin operator:

$$\vec{S} = c_{ij}^+ \bar{\sigma}_{\sigma \sigma'} c_{i \sigma'}$$  \hspace{1cm} (103)

with the condition

$$\sum_{\sigma} c_{\sigma \sigma}^+ c_{i \sigma} = 1$$  \hspace{1cm} (104)

so that

$$\Delta_{ij} = \sum_{\sigma} c_{\sigma \sigma}^+ c_{j \sigma}.$$  \hspace{1cm} (105)

Using this representation the Heisenberg Hamiltonian can be explicitly written in terms of the RVB field \[18, 19\] as

$$H_A = \sum_{\sigma, \langle ab \rangle} c_{\sigma i}^+ U_{ij} c_{\sigma j} + A_0(i)(c_{\sigma i}^+ c_{\sigma i} - 1) + U_{ij} J_{jk}^{-1} U_{ki}.$$  \hspace{1cm} (106)

The field $A_0$ is a Lagrange multiplier which ensures the constraint (93), while the phase of the conjugated RVB field $U_{ij}$ forms a time and space component of a gauge field. The constraint (104) is relaxed after doping, when a small density of empty sites appear on the lattice,

$$\sum_{\sigma} c_{\sigma i}^+ c_{\sigma i} = 1 - h_i^+ h_i,$$  \hspace{1cm} (107)
where $h_i$ is a dopant (hole) operator. An electron operator is then represented by the product of a spinon and a holon, $c_{\sigma i}h_i^\dagger$. The hopping Hamiltonian is therefore

$$H = t \sum_{\langle \vec{a}, \vec{b} \rangle} h_i^\dagger(\vec{a})h_i(\vec{b})\Delta_{\vec{a},\vec{b}} + \text{c.c.}$$

(108)

where $\vec{a}$ and $\vec{b}$ are sites of sublattices $A$ and $B$ and $\Delta_{\vec{a},\vec{b}} = \langle \sum_{\sigma} c_{\sigma i}^\dagger(\vec{a})c_{\sigma i}(\vec{b}) \rangle$. So far all manipulations were exact.

Depending on parameters this model may describe physically different situations. We are interested in a situation when fluctuations of the bond energy (modulus of $\Delta_{i,j}$) are much less and much slower that fluctuations of the bond phase (chirality). If their scales are separated we refer to this state as quantum antiferromagnetism. Let us stress that in the Ne\'el state the modulus and phase of bonds fluctuate similarly and cannot be separated. In this case holes form the Fermi-liquid and interact weakly with spin fluctuations.

5.1.1 Adiabatic Approximation.

Some progress can be made if the magnetic background can be treated adiabatically. This means that a hole moves in a slowly varying spin configuration, similar to the Peierls-Frohlich problem where electrons move with a slowly varying distortion. In this case we may use a semi-classical strategy: first find a static $\bar{\Delta}$ and $\bar{\Delta}^0$ which minimize the energy of the system with a given doping,

$$\Delta_{ij} = \frac{\delta \langle H \rangle}{\delta U_{ij}},$$

(109)

and then take into account quantum fluctuations around the static mean field.

To apply the semi-classical strategy one more step is required. The point is that even if the hopping amplitude is bigger than the exchange energy, the dynamics of spins cannot be considered adiabatically – in the antiferromagnet each jump of a hole sharply changes the spin configuration by flipping a spin on a sublattice. However, two consecutive jumps bring a hole to the same sublattice, so the spin configuration remains approximately unchanged. Thus, to apply adiabatic arguments we must first integrate over fast and sharp processes. Perhaps, the easiest way to implement the adiabatic approximation is to introduce a difference between energies of a hole on different sublattices.
by adding the term $\mu (\sum \vec{a} c^\dagger \vec{a} - \sum \vec{b} c^\dagger \vec{b})$ to the Hamiltonian. If the hopping energy $t$ is less than $\mu$ (adiabatic parameter), a holon and a doublon appear on different sublattices only virtually. Therefore in leading order in $t/\mu$ one may consider only processes of two consecutive hoppings described by the effective holon Hamiltonian

$$H = t' \left( \sum_{\vec{a},\vec{a}'} h^\dagger(\vec{a}) \Delta_{\vec{a},\vec{a}'}^2 h(\vec{a}') + \sum_{\vec{b},\vec{b}'} h^\dagger(\vec{b}) \Delta_{\vec{b},\vec{b}'}^2 h(\vec{b}') + \text{c.c.} \right) \tag{110}$$

where

$$\Delta_{\vec{a},\vec{a}'}^2 = \sum_{\vec{b}'} \Delta_{\vec{a},\vec{b}'} \Delta_{\vec{b}',\vec{a}'}^* \quad \Delta_{\vec{b},\vec{a}'}^2 = \sum_{\vec{a}'} \Delta_{\vec{b},\vec{a}'} \Delta_{\vec{a}'',\vec{b}}^* \tag{111}$$

are hopping operators within the same sublattice. After all, we suggest that this Hamiltonian be considered as phenomenological.

Now as long as $t' \sim t^2/\mu >> J$, one can treat magnetic fluctuations adiabatically. A solution of the mean field equation (109) depends on a particular exchange coupling. Here we are interested in a solution where $|\Delta_{ab}|$ slightly varies from bond to bond, but never vanishes: $\Delta_{ab} \neq 0$. Then the mean field values of $\Delta_{ij}$ and $U_{ij}$ are proportional to each other. After all these considerations we end up with the basic Hamiltonian

$$H = \tilde{t} \left( \sum_{\vec{a},\vec{a}'} h^\dagger(\vec{a}) \Delta_{\vec{a},\vec{a}'}^2 h(\vec{a}') + \sum_{\vec{b},\vec{b}'} h^\dagger(\vec{b}) \Delta_{\vec{b},\vec{b}'}^2 h(\vec{b}') + \text{c.c.} \right) + \tilde{J} \sum_{\vec{a},\vec{b}} c^\dagger_{\vec{a}} \sigma \Delta_{\vec{a}}^2 c_{\vec{b}} \sigma + \sum_{\vec{a}} A_0 (c^\dagger_{\vec{a}} c_{\vec{a}} + h^\dagger_{\vec{a}} h_{\vec{a}} - 1), \tag{112}$$

where $\tilde{t} \sim t'J^{-1}$ and $\tilde{J} \sim J$ are phenomenological constants and we set $|\Delta_{ij}| = 1$. To remain within adiabatic approximation we also assume that $\tilde{t} > \tilde{J}$.

We refer this model as the gauge model of correlated electronic systems. In addition to holons and spinons it has a gauge field, the phase of the bond $\Delta_{ij} = \exp i A_{ij}$, and a Lagrange multiplier $A_0$ as a dynamical gauge field. Let us also notice that the gauge field has no energy of its own.

The hopping part of the model has an important symmetry which reflects the symmetry of the square lattice. The spectrum of the Hamiltonian is symmetric, namely, if there is an eigenstate with an energy $E$, then there is an eigenstate with the energy $-E$. To see this one must note that the Hamiltonian (112) changes sign if one replaces $c^\dagger_{\vec{b}}$ and $h_{\vec{b}}$ to $-c^\dagger_{\vec{b}}$ and $-h_{\vec{b}}$ and
keeps \( c_\vec{a} \) and \( h_\vec{a} \) unchanged. It also means that there is a matrix \( \Gamma \) which anticommutes with the Hamiltonian. In the sublattice basis it is a diagonal matrix

\[
\Gamma = \text{diag}(1, -1) .
\] (113)

5.2 Flux Phase

The next step in the adiabatic approximation is to determine a mean value of the gauge field \( A_{ij}, A_0 \). We approach the problem in two stages. At first, we neglect a small doping and find the most favorable configuration of \( U_{ij} \) for the half-filled case. Then we discuss the effect of doping. The first stage is again approached in two parts. Assuming that \( |\Delta_{ab}| \) is a constant, we first find the phase of \( \Delta_{ab} \) or \( U_{ab} \), i.e., the ground state value of the chirality (98) and later consider a small variation of \( |\Delta_{ab}| \).

At the first step, we suppose that a variation of \( |\Delta_{ab}| \) in space is small and may be found on the basis of continuous theory. Let us therefore set \( |U_{ab}| = \Delta_0 J_{ab} = t_s \) in the first approximation. Then the only degree of freedom to be determined is the chirality \( W(C) = \Delta_{0}^{L(C)} \exp(i\phi(C)) \). It has been suggested in Refs. [22, 21, 23] and has been proved recently by E. Lieb [23] that the flux \( \pi \) per elementary plaquette provides the minimum of the energy:

\[
W(P) = \Delta_0^4 (-1) .
\] (114)

This state is known as a flux phase or chiral phase.

5.3 Continuum Limit

As usual, in order to go to the continuum limit we must separate fast and slow variables. The fast variables are represented by the gauge potential of the background flux. In the flux state translations do not commute any longer. They form a group of magnetic translations:

\[
T_{\vec{e}_i} T_{\vec{e}_j} + T_{\vec{e}_j} T_{\vec{e}_i} = 0 ,
\] (115)

where \( \vec{e}_i \) are primitive lattice vectors. Apart from spinor representations of \( \text{SO}_3 \), it is also represented by \( 2^d \) matrices of dimension \( d \times d \) (where \( d \) is a spatial dimension) which obey the Clifford algebra

\[
\{\alpha_i, \alpha_j\} = 2\delta_{ij} .
\] (116)
Setting $\Delta_{ab}$ to its mean field value $\Delta_{ab} = \alpha_{\mu}$, and $A_0(a) = A_0(b) = m$, we get the nearest neighbour part of the Hamiltonian in the form
\[ H = t \sum_{i=1}^{d} T_{\vec{e}_i} e^{2i(\vec{k}\vec{e}_i)} + m\beta = t \sum_{i=1}^{d} \alpha_i \cos 2k_i + m\beta \] (117)
where the wave vector $\vec{k} = (k_x, k_y, ...)$ belongs to the reduced Brillouin zone $-\frac{\pi}{2} < k_i < \frac{\pi}{2}$, while the matrix $\beta = \pm 1$ depending on whether a site belongs to sublattice $A$ or $B$, and $\{\alpha_i, \beta\} = 0$. To find the spectrum it is sufficient to consider the square of $H$. Due to (115) one has
\[ H^2 = \sum_{i=1}^{d} \cos 2k_i^2 + m^2. \] (118)

The spectrum contains two Dirac bands which touch at $k_F = (\frac{\pi}{2}, \frac{\pi}{2}, ...)$. The lower band is completely filled. Expanding in small momentum $\vec{p} = \vec{k} - \vec{k}_F$ around the Dirac point, we get the Dirac Hamiltonian
\[ H = \sum_{\sigma} \psi_\sigma^\dagger \left( \alpha \vec{\sigma} + m \right) \psi_\sigma \] (119)
where the Dirac spinor $\psi_\sigma$ is the slowly varying part of $c_\sigma$. To see this, let us denote a lattice site $\vec{R}$ as
\[ \vec{R} = \sum_{i=1}^{3} (2x_i + (r_i - 1))\vec{e}_i, \] (120)
where $r_i = 1, 2$ and $x_i$ are integers. If $r_i = 1$, the site belongs to the sublattice $A$; if $r_i = 2$ then to a sublattice $B$. Then
\[ \psi_{\vec{R},\sigma}(\vec{x}) \approx \exp(ik_{\sigma}\vec{R}) c_{\vec{R},\sigma}. \] (121)

Let us mention that the dimension of the matrices $\alpha$ is twice that of the irreducible representation of the Clifford algebra, so they belong to a reducible representation. This is the well known lattice doubling – a lattice always produces an even number of fermionic species. To summarize, let us denote $T_{\vec{e}_i} = \alpha_i; \ \gamma_0 = \beta; \ \gamma_i = \beta \alpha_i$. Then $\{\alpha_i, \alpha_j\} = 2\delta_{ij}; \ \beta \alpha_i \beta = \alpha_i; \ \beta^2 = 1$ and the $\gamma_{\mu}$’s form an algebra of Dirac matrices, $\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu}$.

This type of mean field theory is essentially invalid, because it does not manifestly reflect the main physical property of the theory – its gauge invariance. However, the gauge symmetry is restored by fluctuations. Before we discuss gauge fluctuations around the mean field, let us choose a convenient basis of the $\alpha$ matrices on a square lattice.
5.4 Lattice adopted basis

Expansion around the mean field value follows the strategy developed in polyacetylene physics [12, 13] – only processes with a momentum transfer of $2k_F$ are relevant. Others lead to terms with higher order derivatives. For the sake of simplicity we consider here the basis adopted on the 2D square at half filling. Generalization to the three-dimensional cubic lattice is developed in Ref.[4]. Since at half filling the period is doubled, we must consider the four nearest points of a lattice sell as a set of discrete variables. Let us denote them in terms of Cartesian coordinates: $\vec{r} = (r_x, r_y)$: $r_i = 1, 2$. Then $\vec{\alpha}$ can be chosen in the form

$$ (\alpha_x)_{r\bar{r}} = (\sigma_1)_{r_x} (\sigma_3)_{r_y} ,$$  

$$ \alpha_y = \mathbb{1} \otimes \sigma_1 ; \quad \beta = \sigma_1 \otimes \sigma_2 \mathbb{1} ,$$

where the $\sigma$'s are Pauli matrices. In addition there are three more matrices $\tau_x, \tau_y$ and $\tau_z$ which commute with the matrices $\alpha$. We choose them to form a basis of SU(2), $[\tau^i, \tau^j] = i \epsilon_{ijk} \tau^k$:

$$ \tau_1 = \sigma_1 \mathbb{1} ; \quad \tau_2 = \sigma_3 \otimes \sigma_1 ; \quad \tau_3 = \sigma_2 \otimes \sigma_1 .$$

There is of course a direct product basis in which spin and isospin appear as in independent spaces:

$$ \vec{\alpha} \rightarrow M \vec{\alpha} M^{-1} = \mathbb{1} \otimes \alpha_i ; \quad \vec{\beta} \rightarrow \mathbb{1} \otimes \beta ; \quad \vec{\tau} \rightarrow \vec{\tau} \otimes \mathbb{1} .$$

In continuous theory, the amplitude $\Delta_{ab}$ fluctuates around its mean field value such that these fluctuations are smooth, but not necessarily small. Fluctuations of the modulus of $\Delta_{ab}$ are small and we neglect them, but fluctuations of the phase of $\Delta_{ab}$ restore the gauge invariance and may not be small. To describe them we write

$$ \Delta_{ab} = e^{\Lambda_i \alpha_i T_i} ,$$

where $T_i = \exp(-\partial_i)$ is a translation operator and $\Lambda_i$ a diagonal matrix. (Fluctuations may change the value of a bond, but do not introduce a new bond.) Four diagonal matrices which form a basis for fluctuations may be written in terms of the Dirac basis. They are $\mathbb{1} \otimes \mathbb{1}$ and given by

$$ \alpha_1 \tau^1 = \mathbb{1} \otimes \sigma^3 ,$$

$$ \alpha_2 \tau^2 = \sigma^3 \otimes \mathbb{1} ,$$

$$ \beta \tau^3 = \sigma^3 \otimes \sigma^3 .$$
The simplest way to include fluctuations of the phase is to restore the gauge invariance. The original lattice model is invariant under the $U(1)$ gauge transformation $c_i \rightarrow c_i \exp i\phi_i$. In the continuum limit where each of the four sites is treated separately, the gauge group becomes a four parameter group, and accordingly the phase of the gauge transformation is an arbitrary diagonal matrix:

$$
\begin{align*}
\psi & \rightarrow e^{i\phi_1} \psi \\
\psi & \rightarrow e^{i\phi_2} \alpha_1 \tau_1 \psi \\
\psi & \rightarrow e^{i\phi_3} \alpha_2 \tau_2 \psi \\
\psi & \rightarrow e^{i\phi_4} \beta \tau_3 \psi .
\end{align*}
$$

(128)

To restore the gauge invariance of the mean field Hamiltonian (119) we must add compensating fields:

$$
H = \exp \left[ iE_1 \beta \tau_3 + iG_1 \alpha_1 \tau_1 + iK_1 \alpha_2 \tau_2 \right] \alpha \left( i\partial_1 + a_1 \right) \exp \left[ -iE_1 \beta \tau_3 \right] + \exp \left[ iE_2 \beta \tau_3 + iG_2 \alpha_1 \tau_1 + iK_2 \alpha_2 \tau_2 \right] \alpha \left( i\partial_2 + a_2 \right) \exp \left[ -iE_2 \beta \tau_3 \right] + \beta \tau_3 m .
$$

(129)

These compensating fields are gauge fields. Under the transformation (128) they transforms as follows:

$$
\begin{align*}
A_\mu & \rightarrow A_\mu + \partial_\mu \phi_1 \\
G_\mu & \rightarrow G_\mu + \phi_2 \\
K_\mu & \rightarrow K_\mu + \phi_3 \\
E_\mu & \rightarrow E_\mu + \phi_4
\end{align*}
$$

(130)

The quantities $E = E_1 - E_2, G = G_1 - G_2, K = K_1 - K_2$ and $F = \vec{\nabla} \times \vec{a}$ are therefore gauge invariant objects. They are not small, so the exponents in (129) cannot be expanded. The four types of “magnetic field” describe fluctuations of the flux in four adjacent plaquettes and implement the symmetry of the square lattice.

Fixing a gauge, one may slightly simplify the Hamiltonian (129). One may choose $G_2 = 0, K_1 = 0, E_2 = 0$, in which case the Hamiltonian has the form

$$
H = \exp \left[ iE \beta \tau_3 \right] \alpha_1 \left( i\partial_1 + a_1 \right) \exp \left[ -iE \beta \tau_3 \right] \alpha_2 \left( i\partial_2 + a_2 \right) \exp \left[ -iE_2 \beta \tau_3 \right] + \partial_1 G \tau_1 + \partial_2 K \tau_2 + \beta \tau_3 m
$$

(131)
We note that even though the terms $i\partial_1 G$ and $\partial_2 K$ appear to be temporal components of the SU(2) gauge field, their symmetry is different. They are components of the spinor connection of the original crystal group.

Fluctuations of the modulus can also be obtained from this symmetry. First of all the Hamiltonian must anticommute with the matrix \( \Gamma = \beta \tau^3 \). Secondly, since the Hamiltonian connects only the nearest lattice sites, it must be a tensor product of diagonal and anti-diagonal matrices in the sublattice basis. Except for the matrices $\alpha_i, \alpha_i \tau^3, \tau^1, \tau^2$ which already appeared in the Hamiltonian, three more such matrices, $\beta \alpha_i \tau^3, \beta \tau^1, \beta \tau^2$, describe fluctuations of the modulus of $\Delta_{ab}$. Taking them into account too, finally gives

$$H = \exp[iE\beta \tau^3] \alpha_1 (i\partial_1 + a_1) \exp[-iE\beta \tau^3] \alpha_2 (i\partial_2 + a_2)$$
$$+ \partial_1 \tau^1 + \partial_2 \tau^2 + i\beta \alpha_i \tau^3 \Phi_i + \beta \tau^a \Phi^a + \beta \tau^3 m \quad (132)$$

where $\Phi_i, \Phi^a$ are components of the modulus of $\Delta$ in a proper basis. As a result, the basic model for the doped Mott insulator in the flux phase has a form:

$$\mathcal{H} = \frac{1}{2m} h^\dagger(x) H^2 h(x) + \psi^\dagger(x) H \psi(x) + \frac{1}{2\lambda} (\Phi_i^2 + (\Phi^a)^2) \quad (133)$$

For further explicit reference, let us present the Hamiltonian $H$ for the three dimensional flux phase. In this case translations are $8 \times 8$ matrices. They can be chosen in the form

$$T_1 = \alpha_1 = \sigma_1 \otimes \sigma_3 \otimes 1$$
$$T_2 = \alpha_2 = 1 \otimes \sigma_1 \otimes \sigma_3$$
$$T_3 = \alpha_3 = \sigma_3 \otimes 1 \otimes \sigma_1 \quad (134)$$

The fourth Dirac matrix $\beta$ may be chosen to be diagonal,

$$\beta = \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \quad (135)$$

Four other matrices which commute with the Dirac matrices are translations along face diagonals,

$$T_{12} = \tau^3 = \sigma_2 \otimes \sigma_1 \otimes \sigma_3$$
$$T_{23} = \tau^1 = \sigma_3 \otimes \sigma_2 \otimes \sigma_1$$
$$T_{31} = \tau^2 = \sigma_1 \otimes \sigma_3 \otimes \sigma_2 \quad (136)$$
and translation along the space diagonal,

\[ T_{123} = i\gamma_5\gamma_0 = \gamma_1\gamma_2\gamma_3 = \alpha_1\alpha_2\alpha_3\beta = \sigma_1 \otimes \sigma_1 \otimes \sigma_1 \]  \hspace{1cm} (137)

This last operator is known as the parity operator.

The 8 diagonal matrices which describe fluctuations are

\[
\begin{align*}
O_1 &= 1 \otimes 1 \otimes 1 \\
O_2 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_3 \\
O_3 &= \sigma_3 \otimes 1 \otimes 1 \\
O_4 &= 1 \otimes \sigma_3 \otimes 1 \\
O_5 &= 1 \otimes 1 \otimes \sigma_3 \\
O_6 &= \sigma_3 \otimes \sigma_3 \otimes 1 \\
O_7 &= \sigma_3 \otimes 1 \otimes \sigma_3 \\
O_8 &= 1 \otimes \sigma_3 \otimes \sigma_3
\end{align*}
\]  \hspace{1cm} (138)

Following the procedure we discussed for the 2D case, one obtains

\[
H = \sum_i \exp \left[ iE_i\beta + \sum_{k\neq l\neq m\neq n} (iG_{ij}^k\tau^l\alpha_m\alpha_n\beta + iK_{ij}^k\tau^l\alpha_m\alpha_n) \right] \alpha_i (i\partial_i + a_i) \times \\
\times \exp \left[ iE_i\beta + \sum_{k\neq l\neq m\neq n} (iG_{ij}^k\tau^l\alpha_m\alpha_n\beta + iK_{ij}^k\tau^l\alpha_m\alpha_n) \right] \\
+ i\alpha_i\beta\Phi_i + i\alpha_i\beta\tau_i\Phi_i + \alpha_1\alpha_2\alpha_3\tau_i\Phi_i \hspace{1cm} (139)
\]

Despite some efforts, topology of the numerous fields appeared in the models (133,139) and their anomalous properties have not been elaborated. Nevertheless, there is some confidence that it will exhibit topological instability and topological superconductivity along the scenario outlined in these lectures.

### 6 Conclusion

In summary, we have seen that if the ground state of a quantum system dynamically acquires a non-zero Chern’s number (i.e. a topological order), then it is superconductive. In units where \( e = c = m = 1 \), the London depth is literally the density of the topological charge. This connection is rigid and as is the case with the whole theory, it is determined by the geometry of the phase space. It therefore seems likely that superfluidity can be derived on a more phenomenological (model independent) basis.

It is also clear that topological superconductivity is drastically different from the BCS mechanism. How does this difference manifest itself in observables? For example, what are the tunneling properties of the topological
superconductor, and the symmetry of the order parameter? This is perhaps the most interesting avenue of developing this theory.

Acknowledgement: This work is supported in part by NSF Grant STC-8809854 and by the Japan Society for the Promotion of Science. I acknowledge the help provided by A. Abanov in preparation these lectures and also numerous discussion with him on different aspects of topological mechanism of superconductivity.

References

[1] Landau, L.D.: JETP 11 (1941) 592
[2] Frohlich, G.: Proc. R. Soc. A223 (1954) 296
[3] Lee, P.A., Rice, T.M., Anderson, P.W.: Solid State Commun. 14 (1974) 703
[4] Wiegmann, P.: Prog. Theor. Phys. 107 (1992) 243
[5] Laughlin, R.: Phys. Rev. Lett. 60 (1988) 2677
[6] Wilczek, F. (ed.): Fractional Statistics and Anyon Superconductivity (World Scientific, Singapore, 1990)
[7] Kalmayer, V., Laughlin, R.B.: Phys. Rev. Lett. 59 (1987) 2095
[8] Wen, X.G., Wilczek, F., Zee, A.: Phys. Rev. B 39 (1989) 11413
[9] Khveschenko, D.V., Wiegmann, P.B.: Mod. Phys. Lett. B 3 (1989) 1383; 4 (1990) 17
[10] Laughlin, R.B., Zou, Z.: Phys. Rev. 41 (1989) 664
[11] Wiegmann, P.B.: Physica C153 (1988) 103; Proc. Nobel Symp. 73 Phys. Scripta T27 (1988)
[12] Heeger, A.J., Kivelson, S., Schrieffer, J.R., Su, W.-P.: Rev. Mod. Phys. 60 (1988) 781
[13] Brarovskii, S.P., Kirova, N.: Sov. Sci. Rev. A 5 (1984) 99 (Harwood Academic Publishers)

[14] Niemi, A.J., Semenoff, G.W.: Phys. Rep. 135 (1986) 100

[15] Arafune, J., Freund, P.G.O., Goebel, C.J.: J. Math. Phys. 16 (1975) 433

[16] Goldstone, J., Wilczek, F.: Phys. Rev. Lett. 47 (1981) 986

[17] Witten, E.: Nucl. Phys. B 223 (1983) 422; Nucl. Phys. B 223 (1983) 433

[18] Baskaran, G., Anderson, P.W.: Phys. Rev. B37 (1988) 580

[19] Ioffe, L.B., Larkin, A.I.: Phys. Rev. B 39 (1989) 8988

[20] Affleck, I., Marston, B.: Phys. Rev. B37 (1988) 3774

[21] Kotliar, G.: Phys. Rev. B37 (1988) 3664

[22] Hasegaga, D., Lederer, P., Rice, T.M.: Phys. Rev. Lett. 63 (1989) 907

[23] Lieb, E.: Phys. Rev. Lett. to be published