MULTILINEAR HAUSDORFF OPERATORS ON SOME FUNCTION SPACES WITH VARIABLE EXPONENT

NGUYEN MINH CHUONG, DAO VAN DUONG, AND KIEU HUU DUNG

Abstract. The aim of the present paper is to give necessary and sufficient conditions for the boundedness of a general class of multilinear Hausdorff operators that acts on the product of some weighted function spaces with variable exponent such as the weighted Lebesgue, Herz, central Morrey and Morrey-Herz type spaces with variable exponent. Our results improve and generalize some previous known results.

1. Introduction

The one dimensional Hausdorff operator is defined by

$$H_\Phi(f)(x) = \int_0^\infty \frac{\Phi(y)}{y} f\left(\frac{x}{y}\right) dy,$$

where $\Phi$ is an integrable function on the positive half-line. The Hausdorff operator may be originated by Hurwitz and Silverman [25] in order to study summability of number series (see also [24]). It is well known that the Hausdorff operator is one of important operators in harmonic analysis, and it is used to solve certain classical problems in analysis. It is worth pointing out that if the kernel function $\Phi$ is taken appropriately, then the Hausdorff operator reduces to many classical operators in analysis such as the Hardy operator, the Cesàro operator, the Riemann-Liouville fractional integral operator and the Hardy-Littlewood average operator (see, e.g., [2], [16], [21], [34] and references therein).

In 2002, Brown and Móricz [5] extended the study of Hausdorff operator to the high dimensional space. Given $\Phi$ be a locally integrable function on $\mathbb{R}^n$, the $n$-dimensional Hausdorff operator $H_{\Phi,A}$ associated to the kernel function $\Phi$ is then defined in terms of the integral form as follows

$$H_{\Phi,A}(f)(x) = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} f(A(t)x) dt, \ x \in \mathbb{R}^n,$$

2010 Mathematics Subject Classification. Primary 42B30; Secondary 42B20, 47B38.

Key words and phrases. Multilinear operator, Hausdorff operator, weighted Hardy-Littlewood operator, Lebesgue space, Morrey-Herz space, variable exponent.
where \( A(t) \) is an \( n \times n \) invertible matrix for almost everywhere \( t \) in the support of \( \Phi \). It should be pointed out that if we take \( \Phi(t) = |t|^{n}\psi(t_{1})\chi_{[0,1]^{n}}(t) \) and \( A(t) = t_{1}.I_{n} \) (\( I_{n} \) is an identity matrix), for \( t = (t_{1}, t_{2}, ..., t_{n}) \), where \( \psi : [0, 1] \rightarrow [0, \infty) \) is a measurable function, \( H_{\Phi, A} \) then reduces to the weighted Hardy-Littlewood average operator due to Carton-Lebrun and Fosset [8] defined as the following

\[
U_{\psi}(f)(x) = \int_{0}^{1} f(tx)\psi(t)dt, \ x \in \mathbb{R}^{n}.
\] (1.2)

Similarly, by taking \( \Phi(t) = |t|^{n}\psi(t_{1})\chi_{[0,1]^{n}}(t) \) and \( A(t) = s(t_{1}).I_{n} \), with \( s : [0, 1] \rightarrow \mathbb{R} \) being a measurable function, it is easy to see that \( H_{\Phi, A} \) reduces to the weighted Hardy-Cesàro operator \( U_{\psi, s} \) defined by Chuong and Hung [12] as follows

\[
U_{\psi, s}(f)(x) = \int_{0}^{1} f(s(t)x)\psi(t)dt, \ x \in \mathbb{R}^{n}.
\] (1.3)

In recent years, the theory of weighted Hardy-Littlewood average operators, Hardy-Cesàro operators and Hausdorff operators have been significantly developed into different contexts (for more details see [5], [12], [13], [14], [35], [41] and references therein). Also, remark that Coifman and Meyer [9], [10] discovered a multilinear point of view in their study of certain singular integral operators. Thus, the research of the theory of multilinear operators is not only attracted by a pure question to generalize the theory of linear ones but also by their deep applications in harmonic analysis. In 2015, Hung and Ky [26] introduced the weighted multilinear Hardy-Cesàro type operators, which are generalized of weighted multilinear Hardy operators in [21], defined as follows:

\[
U_{\psi, \vec{s}}^{m,n}(f_{1},...,f_{m})(x) = \int_{[0,1]^{n}} \left( \prod_{i=1}^{m} f_{i}(s_{i}(t)x) \right)\psi(t)dt, \ x \in \mathbb{R}^{n},
\] (1.4)

where \( \psi : [0, 1]^{n} \rightarrow [0, \infty) \), and \( s_{1}, ..., s_{m} : [0, 1]^{n} \rightarrow \mathbb{R} \) are measurable functions. By the relation between Hausdorff operator and Hardy-Cesàro operator as mentioned above, we shall introduce in this paper a more general multilinear operator of Hausdorff type defined as follows.

**Definition 1.1.** Let \( \Phi : \mathbb{R}^{n} \rightarrow [0, \infty) \) and \( A_{i}(t), \) for \( i = 1, ..., m, \) be \( n \times n \) invertible matrices for almost everywhere \( t \) in the support of \( \Phi \). Given \( f_{1}, f_{2}, ..., f_{m} : \mathbb{R}^{n} \rightarrow \mathbb{C} \) be measurable functions, the multilinear Hausdorff operator \( H_{\Phi, \vec{A}} \) is defined by

\[
H_{\Phi, \vec{A}}(\vec{f})(x) = \int_{\mathbb{R}^{n}} \Phi(t)\frac{1}{|t|^{n}} \prod_{i=1}^{m} f_{i}(A_{i}(t)x)dt, \ x \in \mathbb{R}^{n},
\] (1.5)
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for \( f = (f_1, ..., f_m) \) and \( A = (A_1, ..., A_m) \).

It is obvious that when \( \Phi(t) = |t|^n \psi(t) \chi_{[0,1]^n}(t) \) and \( A_i(t) = s_i(t).I_n \), where \( \psi : [0,1]^n \to [0, \infty) \), \( s_1, ..., s_m : [0,1]^n \to \mathbb{R} \) are measurable functions, then the multilinear Hausdorff operator \( H_{\Phi, A} \) reduces to the weighted multilinear Hardy-Cesàro operator \( T^{m,n}_{\psi, s} \) above.

It is also interesting that the theory of function spaces with variable exponents has attracted much more attention because of the necessary in the field of electronic fluid mechanics and its important applications to the elasticity, fluid dynamics, recovery of graphics, and differential equations (see [3], [11], [15], [19], [27], [36], [18]). The foundational results and powerful applications of some function spaces with variable exponents in harmonic analysis and partial differential equations are given in the books [18], [20] and the references therein. It is well-known that the Calderón-Zygmund singular operators, the Hardy-type operators and their commutators have been extensively investigated on the Lebesgue, Herz, Morrey, and Morrey-Herz spaces with variable exponent (see, e.g., [6], [7], [17], [22], [32], [33], [31], [37], [38], and others).

Motivated by above mentioned results, the goal of this paper is to establish the necessary and sufficient conditions for the boundedness of multilinear Hausdorff operators on the product of weighted Lebesgue, central Morrey, Herz, and Morrey-Herz spaces with variable exponent. In each case, the estimates for operator norms are worked out. It should be pointed out that all results in this paper are new even in the case of linear Hausdorff operators.

Our paper is organized as follows. In Section 2, we give necessary preliminaries on weighted Lebesgue spaces, central Morrey spaces, Herz spaces and Morrey-Herz spaces with variable exponent. Our main theorems are given and proved in Section 3.

2. Preliminaries

Before stating our results in the next section, let us give some basic facts and notations which will be used throughout this paper. By \( \|T\|_{X \to Y} \), we denote the norm of \( T \) between two normed vector spaces \( X \) and \( Y \). The letter \( C \) denotes a positive constant which is independent of the main parameters, but may be different from line to line. Given a measurable set \( \Omega \), let us denote by \( \chi_{\Omega} \) its characteristic function, by \( |\Omega| \) its Lebesgue measure, and by \( \omega(\Omega) \) the integral \( \int_{\Omega} \omega(x)dx \). For any \( a \in \mathbb{R}^n \) and \( r > 0 \), we denote by \( B(a, r) \) the ball centered at \( a \) with radius \( r \).

Next, we write \( a \preceq b \) to mean that there is a positive constant \( C \), independent of the main parameters, such that \( a \leq Cb \). The symbol \( f \simeq g \) means that \( f \) is equivalent to \( g \) (i.e. \( C^{-1}f \leq g \leq Cf \)).
In what follows, we denote \( \chi_k = \chi_{C_k}, C_k = B_k \setminus B_{k-1} \) and \( B_k = \{ x \in \mathbb{R}^n : |x| \leq 2^k \} \), for all \( k \in \mathbb{Z} \). Now, we are in a position to give some notations and definitions of Lebesgue, Herz, Morrey and Morrey-Herz spaces with constant parameters. For further information on these spaces as well as their deep applications in analysis, the interested readers may refer to the work \([1]\) and to the monograph \([30]\).

In this paper, as usual, we will denote by \( \omega(\cdot) \) a non-negative weighted function on \( \mathbb{R}^n \).

**Definition 2.1.** Let \( 1 \leq p < \infty \), we define the weighted Lebesgue space \( L^p(\omega) \) of a measurable function \( f \) by

\[
\|f\|_{L^p(\omega)} = \left( \int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} < \infty,
\]

and for \( p = \infty \) by

\[
\|f\|_{L^\infty(\omega)} = \inf \{ M > 0 : \omega(\{ x \in \mathbb{R}^n : |f(x)| > M \}) = 0 \} < \infty.
\]

**Definition 2.2.** Let \( \alpha \in \mathbb{R}, 0 < q < \infty \), and \( 0 < p < \infty \). The weighted homogeneous Herz-type space \( K_{q}^{\alpha,p}(\omega) \) is defined by

\[
K_{q}^{\alpha,p}(\omega) = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, \omega) : \|f\|_{K_{q}^{\alpha,p}(\omega)} < \infty \right\},
\]

where \( \|f\|_{K_{q}^{\alpha,p}(\omega)} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \|f \chi_k\|_{L^q(\omega)}^p \right)^{\frac{1}{p}} \).

**Definition 2.3.** Let \( \lambda \in \mathbb{R} \) and \( 1 \leq p < \infty \). The weighted central Morrey space \( B^{p,\lambda}(\omega) \) is defined by the set of all locally \( p \)-integrable functions \( f \) satisfying

\[
\|f\|_{B^{p,\lambda}(\omega)} = \sup_{R > 0} \left( \frac{1}{\omega(B(0,R))^{1+\lambda p}} \int_{B(0,R)} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} < \infty.
\]

**Definition 2.4.** Let \( \alpha \in \mathbb{R}, 0 < p < \infty, 0 < q < \infty, \lambda \geq 0 \). The homogeneous weighted Morrey-Herz type space \( M K_{p,q}^{\alpha,\lambda}(\omega) \) is defined by

\[
M K_{p,q}^{\alpha,\lambda}(\omega) = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}, \omega) : \|f\|_{M K_{p,q}^{\alpha,\lambda}(\omega)} < \infty \right\},
\]

where \( \|f\|_{M K_{p,q}^{\alpha,\lambda}(\omega)} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0 \lambda} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f \chi_k\|_{L^q(\omega)}^p \right)^{\frac{1}{p}} \).
Remark 1. It is useful to note that \( K_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) \) for \( 0 < p < \infty \); \( K_p^{\alpha,p}(\mathbb{R}^n) = L^p(|x|^\alpha dx) \) for all \( 0 < p < \infty \) and \( \alpha \in \mathbb{R} \). Since \( M K_p^{0,q}(\mathbb{R}^n) = K_p^{q,p}(\mathbb{R}^n) \), it follows that the Herz space is a special case of Morrey-Herz space. Therefore, the Herz spaces are natural generalisations of the Lebesgue spaces with power weights.

Now, we present the definition of the Lebesgue space with variable exponent. For further readings on its deep applications in harmonic analysis, the interested reader may find in the works [18], [19] and [20].

Definition 2.5. Let \( \mathcal{P}(\mathbb{R}^n) \) be the set of all measurable functions \( p(\cdot) : \mathbb{R} \rightarrow [1, \infty] \). For \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), the variable exponent Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^n) \) is the set of all complex-valued measurable functions \( f \) defined on \( \mathbb{R}^n \) such that there exists constant \( \eta > 0 \) satisfying

\[
F_p(f/\eta) := \int_{\mathbb{R}^n \setminus \Omega_\infty} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx + \| f/\eta \|_{L^\infty(\Omega_\infty)} < \infty,
\]

where \( \Omega_\infty = \{ x \in \mathbb{R}^n : p(x) = \infty \} \). When \( |\Omega_\infty| = 0 \), it is straightforward

\[
F_p(f/\eta) = \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\eta} \right)^{p(x)} \, dx < \infty.
\]

The variable exponent Lebesgue space \( L^{p(\cdot)}(\mathbb{R}^n) \) then becomes a norm space equipped with a norm given by

\[
\| f \|_{L^{p(\cdot)}} = \inf \left\{ \eta > 0 : F_p \left( \frac{f}{\eta} \right) \leq 1 \right\}.
\]

Let us denote by \( \mathcal{P}_b(\mathbb{R}^n) \) the class of exponents \( q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) such that

\[
1 < q_- \leq q(x) \leq q_+ < \infty, \quad \text{for all } x \in \mathbb{R}^n,
\]

where \( q_- = \text{ess inf}_{x \in \mathbb{R}^n} q(x) \) and \( q_+ = \text{ess sup}_{x \in \mathbb{R}^n} q(x) \). For \( p \in \mathcal{P}_b(\mathbb{R}^n) \), it is useful to remark that we have the following inequalities which are usually used in the sequel.

[i] If \( F_p(f) \leq C \), then \( \| f \|_{L^{p(\cdot)}} \leq \max \{ C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}} \} \), for all \( f \in L^{p(\cdot)}(\mathbb{R}^n) \),

[ii] If \( F_p(f) \geq C \), then \( \| f \|_{L^{p(\cdot)}} \geq \min \{ C^{\frac{1}{q_-}}, C^{\frac{1}{q_+}} \} \), for all \( f \in L^{p(\cdot)}(\mathbb{R}^n) \). (2.1)

The space \( \mathcal{P}_\infty(\mathbb{R}^n) \) is defined by the set of all measurable functions \( q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \) and there exists a constant \( q_\infty \) such that

\[
q_\infty = \lim_{|x| \to \infty} q(x).
\]

For \( p(\cdot) \in \mathcal{P}(\mathbb{R}^n) \), the weighted variable exponent Lebesgue space \( L_w^{p(\cdot)}(\mathbb{R}^n) \) is
the set of all complex-valued measurable functions \( f \) such that \( f \omega \) belongs to the \( L^p(\mathbb{R}^n) \) space, and the norm of \( f \) in \( L^p_{\omega}(\mathbb{R}^n) \) is given by

\[
\|f\|_{L^p_{\omega}} = \|f\omega\|_{L^p}.
\]

Let \( C^0_{\omega}(\mathbb{R}^n) \) denote the set of all log-Hölder continuous functions \( \alpha(\cdot) \) satisfying at the origin

\[
|\alpha(x) - \alpha(0)| \leq \frac{C_0^\alpha}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.
\]

Denote by \( C^n_{\omega}(\mathbb{R}^n) \) the set of all log-Hölder continuous functions \( \alpha(\cdot) \) satisfying at infinity

\[
|\alpha(x) - \alpha_{\infty}| \leq \frac{C_\infty^\alpha}{\log(e + |x|)}, \text{ for all } x \in \mathbb{R}^n.
\]

Next, we would like to give the definition of variable exponent weighted Herz spaces \( K_{q(\cdot),\omega}^{\alpha(\cdot),p} \) and the definition of variable exponent weighted Morrey-Herz spaces \( M K_{p,q(\cdot),\omega}^{\alpha(\cdot),\lambda} \) (see [31], [38] for more details).

**Definition 2.6.** Let \( 0 < p < \infty, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n) \) and \( \alpha(\cdot) : \mathbb{R}^n \to \mathbb{R} \) with \( \alpha(\cdot) \in L_{\infty}(\mathbb{R}^n) \). The variable exponent weighted Herz space \( K_{q(\cdot),\omega}^{\alpha(\cdot),p} \) is defined by

\[
K_{q(\cdot),\omega}^{\alpha(\cdot),p} = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{K_{q(\cdot),\omega}^{\alpha(\cdot),p}} < \infty \right\},
\]

where \( \|f\|_{K_{q(\cdot),\omega}^{\alpha(\cdot),p}} = \left( \sum_{k=-\infty}^{\infty} \|2^{k\alpha(\cdot)}f\chi_k\|_{L^q(\cdot)}^p \right)^{\frac{1}{p}} \).

**Definition 2.7.** Assume that \( 0 \leq \lambda < \infty, 0 < p < \infty, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n) \) and \( \alpha(\cdot) : \mathbb{R}^n \to \mathbb{R} \) with \( \alpha(\cdot) \in L_{\infty}(\mathbb{R}^n) \). The variable exponent weighted Morrey-Herz space \( M K_{p,q(\cdot),\omega}^{\alpha(\cdot),\lambda} \) is defined by

\[
M K_{p,q(\cdot),\omega}^{\alpha(\cdot),\lambda} = \left\{ f \in L^q_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{M K_{p,q(\cdot),\omega}^{\alpha(\cdot),\lambda}} < \infty \right\},
\]

where \( \|f\|_{M K_{p,q(\cdot),\omega}^{\alpha(\cdot),\lambda}} = \sup_{k_0 \in \mathbb{Z}} 2^{-k_0\lambda} \left( \sum_{k=-\infty}^{k_0} \|2^{k\alpha(\cdot)}f\chi_k\|_{L^q(\cdot)}^p \right)^{\frac{1}{p}} \).

Note that, when \( p(\cdot), q(\cdot) \) and \( \alpha(\cdot) \) are constant, it is obvious to see that

\[
L^p_{\omega,1/p} = L^p(\omega), \quad K_{q,\omega,1/p}^{\alpha,p} = K_{\omega}^{\alpha,p}(\omega) \text{ and } M K_{p,q,\omega,1/p}^{\alpha,\lambda} = M K_{p,q}^{\alpha,\lambda}(\omega).
\]

Because of defining of weighted Morrey-Herz spaces with variable exponent and Proposition 2.5 in [31], we have the following result. The proof is trivial and may be found in [38].
Lemma 2.8. Let $\alpha(\cdot) \in L^\infty(\mathbb{R}^n)$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, $p \in (0, \infty)$ and $\lambda \in [0, \infty)$. If $\alpha(\cdot)$ is log-Hölder continuous both at the origin and at infinity, then

$$\|f\chi_j\|_{L^p_j(\cdot)} \leq C.2^{j(\lambda-\alpha(0))}\|f\|_{MK^{\alpha(\cdot),\lambda}_{p,q}(\cdot)}, \text{ for all } j \in \mathbb{Z}^-,$$

and

$$\|f\chi_j\|_{L^p_j(\cdot)} \leq C.2^{j(\lambda-\alpha(\cdot))}\|f\|_{MK^{\alpha(\cdot),\lambda}_{p,q}(\cdot)}, \text{ for all } j \in \mathbb{N}.$$

We also extend to define two-weight $\lambda$-central Morrey spaces with variable-exponent as follows.

Definition 2.9. For $\lambda \in \mathbb{R}$ and $p \in \mathcal{P}_\infty(\mathbb{R}^n)$, we denote $B_{\omega_1,\omega_2}^{p(\cdot),\lambda}$ the class of locally integrable functions $f$ on $\mathbb{R}^n$ satisfying

$$\|f\|_{B_{\omega_1,\omega_2}^{p(\cdot),\lambda}} = \sup_{R > 0} \frac{1}{\omega_1(B(0,R))^{\lambda/p_\infty}} \|f\|_{L^p_{\omega_1}(B(0,R))} < \infty,$$

where $\|f\|_{L^p_{\omega_1}(B(0,R))} = \|f\chi_{B(0,R)}\|_{L^p_{\omega_1}(\cdot)}$ and $\omega_1$, $\omega_2$ are non-negative and local integrable functions. Moreover, as $p(\cdot)$ is constant and $\omega_1 = \omega$ and $\omega_2 = \omega^{1/p}$, it is natural to see that $B_{\omega,\omega^{1/p}}^{p(\cdot),\lambda} = B_{\omega}^{p(\cdot),\lambda}(\cdot)$.

Later, the next theorem is stated as an embedding result for the Lebesgue spaces with variable exponent (see, for example, Theorem 2 in [6], Theorem 2.45 in [18], Lemma 3.3.1 in [20]).

Theorem 2.10. Let $p(\cdot), q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $q(x) \leq p(x)$ almost everywhere $x \in \mathbb{R}^n$, and

$$\frac{1}{r(\cdot)} := \frac{1}{q(\cdot)} - \frac{1}{p(\cdot)} \text{ and } \|1\|_{L^{r(\cdot)}} < \infty.$$ 

Then there exists a constant $K$ such that

$$\|f\|_{L^{p(\cdot)}_{\omega}(\cdot)} \leq K\|1\|_{L^{r(\cdot)}}\|f\|_{L^{q(\cdot)}_{\omega}(\cdot)}.$$ 

3. Main results and their proofs

Before stating the next main results, we introduce some notations which will be used throughout this section. Let $\gamma_1, ..., \gamma_m \in \mathbb{R}$, $\lambda_1, ..., \lambda_m \geq 0$, $p, p_i \in (0, \infty)$, $q_i \in \mathcal{P}_b(\mathbb{R}^n)$ for $i = 1, ..., m$ and $\alpha_1, ..., \alpha_m \in L^\infty(\mathbb{R}^n) \cap C_0^{\log}(\mathbb{R}^n) \cap C_\infty^{\log}(\mathbb{R}^n)$. The functions $\alpha(\cdot), q(\cdot)$ and numbers $\gamma, \lambda$ are defined by

$$\alpha_1(\cdot) + \cdots + \alpha_m(\cdot) = \alpha(\cdot),$$

$$\frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)} = \frac{1}{q(\cdot)},$$

$$\gamma_1 + \cdots + \gamma_m = \gamma,$$

$$\lambda_1 + \lambda_2 + \cdots + \lambda_m = \lambda.$$
Thus, it is clear to see that the function $\alpha$ also belongs to the $L^\infty(\mathbb{R}^n) \cap C_0^{\log}(\mathbb{R}^n) \cap C_0^{\log}(\mathbb{R}^n)$ space.

For a matrix $A = (a_{ij})_{n \times n}$, we define the norm of $A$ as follow

$$\|A\| = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$  \hspace{1cm} (3.1)

As above we conclude $|Ax| \leq \|A\| |x|$ for any vector $x \in \mathbb{R}^n$. In particular, if $A$ is invertible, then we find

$$\|A\|^{-n} \leq |\det(A^{-1})| \leq \|A^{-1}\|^n.$$  \hspace{1cm} (3.2)

In this section, we will investigate the boundedness of multilinear Hausdorff operators on variable exponent Herz, central Morrey and Morrey-Herz spaces with power weights associated to the case of matrices having the important property as follows: there exists $\rho_A \geq 1$ such that

$$\|A_i(t)\| \cdot \|A_i^{-1}(t)\| \leq \rho_A^i, \text{ for all } i = 1, ..., m,$$  \hspace{1cm} (3.3)

for almost everywhere $t \in \mathbb{R}^n$. Thus, by the property of invertible matrix, it is easy to show that

$$\|A_i(t)\|^\sigma \lesssim \|A_i^{-1}(t)\|^{-\sigma}, \text{ for all } \sigma \in \mathbb{R},$$  \hspace{1cm} (3.4)

and

$$|A_i(t)x|^\sigma \gtrsim \|A_i^{-1}(t)\|^{-\sigma}|x|^\sigma, \text{ for all } \sigma \in \mathbb{R}, x \in \mathbb{R}^n.$$  \hspace{1cm} (3.5)

**Remark 2.** If $A(t) = (a_{ij}(t))_{n \times n}$ is a real orthogonal matrix for almost everywhere $t$ in $\mathbb{R}^n$, then $A(t)$ satisfies the property (3.3). Indeed, we know that the definition of the matrix norm (3.1) is the special case of Frobenius matrix norm. We recall the property of the above norm as follows

$$\sqrt{\rho(B^*B)} \leq \|B\| \leq \sqrt{n} \cdot \sqrt{\rho(B^*B)}, \text{ for all } B \in M_n(\mathbb{C}),$$

where $B^*$ is the conjugate matrix of $B$ and $\rho(B)$ is the largest modulus of the eigenvalues of $B$. Thus, since $A^{-1}(t)$ is also a real orthogonal matrix, we get

$$\|A(t)\| \leq \sqrt{n} \text{ and } \|A^{-1}(t)\| \leq \sqrt{n},$$

which immediately obtain the desired result. Now, we are ready to state our first main result in this paper.

**Theorem 3.1.** Let $\omega_1(x) = |x|^{\gamma_1}, ..., \omega_m(x) = |x|^{\gamma_m}, \omega(x) = |x|^\gamma$, $q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n)$, $\zeta > 0$ and the following conditions are true:

$$q_i(A_i^{-1}(t)\cdot) \leq \zeta q_i(\cdot) \text{ and } \|1\|_{L^q(\varpi \cdot)} < \infty, \text{ a.e. } t \in \text{supp}(\Phi),$$  \hspace{1cm} (3.6)

$$C_1 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} c_{\alpha_i, \eta_i, \gamma_i}(t) \|1\|_{L^q(\varpi \cdot)} dt < \infty,$$  \hspace{1cm} (3.7)
Consequently, we obtain
\[
\frac{1}{r_i(t, \cdot)} = \frac{1}{q_i(A_i^{-1}(t) \cdot)} - \frac{1}{\zeta q_i(\cdot)},
\]
for all \( i = 1, \ldots, m \).

Then, \( H_{\Phi, A} \) is a bounded operator from \( L^{q_1(\cdot)} \times \cdots \times L^{q_m(\cdot)} \) to \( L^{q(\cdot)} \).

**Proof.** By using the versions of the Minkowski inequality for variable Lebesgue spaces from Corollary 2.38 in [18], we have
\[
\| H_{\Phi, A}(\tilde{f}) \|_{L^{q(\cdot)}} \lesssim \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \| f_i(A_i(t) \cdot) \|_{L^{q_i(\cdot)}} \, dt. \tag{3.8}
\]

By assuming \( \sum_{i=1}^{m} \frac{1}{q_i(\cdot)} = \frac{1}{q(\cdot)} \) and applying the Hölder inequality for variable Lebesgue spaces (see also Corollary 2.28 in [18]), we imply that
\[
\| \prod_{i=1}^{m} f_i(A_i(t) \cdot) \|_{L^{q(\cdot)}} \lesssim \prod_{i=1}^{m} \| f_i(A_i(t) \cdot) \|_{L^{q_i(\cdot)}}.
\]

Consequently, we obtain
\[
\| H_{\Phi, A}(\tilde{f}) \|_{L^{q(\cdot)}} \lesssim \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \| f_i(A_i(t) \cdot) \|_{L^{q_i(\cdot)}} \, dt. \tag{3.9}
\]

For \( \eta > 0 \), we see that
\[
\int_{\mathbb{R}^n} \left( \frac{|f_i(A_i(t) \cdot)| \omega_i(x)}{\eta} \right)^{q_i(x)} \, dx
\]
\[
= \int_{\mathbb{R}^n} \left( \frac{|f_i(z)| |A_i^{-1}(t)z|^{\gamma_i}}{\eta} \Phi(A_i^{-1}(t)z) \right)^{q_i(A_i^{-1}(t)z)} |\det A_i^{-1}(t)| \, dz
\]
\[
\leq |\det A_i^{-1}(t)| \int_{\mathbb{R}^n} \left( \max \left\{ \| A_i^{-1}(t) \|^{\gamma_i}, \| A_i(t) \|^{-\gamma_i} \right\} |f_i(z)| \omega_i(z) \right)^{q_i(A_i^{-1}(t)z)} \, dz
\]
\[
\leq \int_{\mathbb{R}^n} \left( \frac{c_{A_i,q_i,\gamma_i}(t)}{\eta} |f_i(z)| \omega_i(z) \right)^{q_i(A_i^{-1}(t)z)} \, dz.
\]

Hence, it follows from the definition of the Lebesgue space with variable exponent that
\[
\| f_i(A_i(t) \cdot) \|_{L^{q_i(\cdot)}} \leq c_{A_i,q_i,\gamma_i}(t). \| f_i \|_{L^{q_i}(A_i^{-1}(t) \cdot)}, \tag{3.10}
\]
In view of (3.6) and Theorem 2.10 we deduce
\[
\|f\|_{L^{q_1,\cdot}(A_i^{-1}(t))} \lesssim \|1\|_{L^{r_1,\cdot}(c)} \|f\|_{L^{q_1,\cdot}(c)}.
\] (3.11)
Therefore, by (3.9)-(3.11), we obtain
\[
\|H_{\Phi,A}(\tilde{f})\|_{L^{q_1,\cdot}(c)} \lesssim C_1 \prod_{i=1}^{m} \|f_i\|_{L^{q_i,\cdot}(c)},
\]
which finishes the proof of this theorem. \(\square\)

In particular, when \(\zeta \leq 1\), we have the following important result to the case of matrices having property (3.3) above.

**Theorem 3.2.** Let us have the given supposition of Theorem 3.1 and assume that
\[
q_i(A_i^{-1}(t) \cdot) \leq q_i(\cdot), \|1\|_{L^{r_1,\cdot}(c)} < \infty,
\] (3.12)
where \(\frac{1}{r_{1i}(t, \cdot)} = \frac{1}{q_i(A_i^{-1}(t) \cdot)} - \frac{1}{q_i(\cdot)}\), a.e. \(t \in \text{supp}(\Phi)\), for all \(i = 1, \ldots, m\).

(a) If \(C_2 = \int \Phi(t) \cdot \prod_{i=1}^{m} \max \left\{ \|A_i^{-1}(t)\|_{\frac{n_i}{n_i^+}+\gamma_i}, \|A_i^{-1}(t)\|_{\frac{m_i}{m_i-}+\gamma_i} \right\} \|1\|_{L^{r_1,\cdot}(c)} dt < \infty\), then
\[
\|H_{\Phi,A}(\tilde{f})\|_{L^{q_1,\cdot}(c)} \lesssim C_2 \prod_{i=1}^{m} \|f_i\|_{L^{q_i,\cdot}(c)}.
\]

(b) Let
\[
C_2^* = \int_{\mathbb{R}^n} \Phi(t) \cdot \prod_{i=1}^{m} \min \left\{ \|A_i^{-1}(t)\|_{\frac{n_i}{n_i^+}+\gamma_i}, \|A_i^{-1}(t)\|_{\frac{m_i}{m_i-}+\gamma_i} \right\} dt.
\]
Assume that \(H_{\Phi,A}\) is a bounded operator from \(L^{q_1,\cdot}_{\omega_1} \times \cdots \times L^{q_m,\cdot}_{\omega_m}\) to \(L^{q,\cdot}_\omega\) and the following condition is satisfied:
\[
\frac{1}{q_1^-} + \frac{1}{q_2^-} + \cdots + \frac{1}{q_m^-} = \frac{1}{q^+}.
\] (3.13)
Then, \(C_2^*\) is finite. Moreover,
\[
\|H_{\Phi,A}\|_{L^{q_1,\cdot}_{\omega_1} \times \cdots \times L^{q_m,\cdot}_{\omega_m} \rightarrow L^{q,\cdot}_\omega} \gtrsim C_2^*.
\]

**Proof.** We begin with the proof for the case (a). From (3.12), by using the Theorem 2.10 again, we have
\[
\|f\|_{L^{q_1,\cdot}(A_i^{-1}(t))} \lesssim \|1\|_{L^{r_1,\cdot}(c)} \|f\|_{L^{q_1,\cdot}(c)}, \text{for all } f \in L^{q_i,\cdot}(A_i^{-1}(t))\). (3.14)
On the other hand, by \((3.2)\) and \((3.4)\), for \(i = 1, 2, \ldots, m\), we find
\[
c_{A_i, q_i, \gamma_i}(t) \lesssim \max \left\{ \left\| A_i^{-1}(t) \right\|^\frac{1}{n} + \gamma_i, \left\| A_i^{-1}(t) \right\|^\frac{n}{n} + \gamma_i \right\}.
\] (3.15)

By the similar arguments as Theorem \(3.1\) by \((3.14)\) and \((3.15)\), we also obtain
\[
\left\| H_{q_i, A_i}(\vec{f}) \right\|_{L^{q_i}_{\gamma_i}} \lesssim C_2 \prod_{i=1}^m \left\| f_i \right\|_{L^{q_i}_{\gamma_i}}.
\]

Now, for the case (b), we make the functions \(f_i\) for all \(i = 1, \ldots, m\) as follows:
\[
f_i(x) = \begin{cases} 
0, & \text{if } |x| < \rho_A^{-1}, \\
|x| - \frac{n}{n-1} \gamma_i - \varepsilon, & \text{otherwise}.
\end{cases}
\]

This immediately deduces that \(\left\| f_i \right\|_{L^{q_i}_{\gamma_i}} > 0\). Besides that, we also compute
\[
F_{q_i}(f_i \omega_i) = \int_{|x| \geq \rho_A^{-1}} |x|^{-n-q_i} d\sigma(x) dx = \int_{\rho_A^{-1} S^{n-1}} r^{-q_i(rx')^{-1}} d\sigma(x') dr
\]
\[
= \int_{\rho_A^{-1} S^{n-1}} r^{-q_i(rx')^{-1}} d\sigma(x') dr + \int_{1 S^{n-1}} r^{-q_i(rx')^{-1}} d\sigma(x') dr.
\]

Thus, \(F_{q_i}(f_i \omega_i)\) is dominated by
\[
\int_{\rho_A^{-1} S^{n-1}} r^{-1-q_i} d\sigma(x') dr + \int_{1 S^{n-1}} r^{-1-q_i} d\sigma(x') dr \lesssim \left( (\rho_A^{-q_i} - 1)q_{i-} + q_{i+} \right) \varepsilon^{-1}.
\]

From the above estimation, by \((2.1)\), we get
\[
\left\| f_i \right\|_{L^{q_i}_{\gamma_i}} \lesssim \left( (\rho_A^{-q_i} - 1)q_{i-} + q_{i+} \right)^\frac{1}{n} \frac{1}{\varepsilon^{q_i}}
\] (3.16)

Next, let us denote two useful sets as follows
\[
S_x = \bigcap_{i=1}^m \{ t \in \mathbb{R}^n : |A_i(t) x| \geq \rho_A^{-1} \},
\]
and
\[
U = \{ t \in \mathbb{R}^n : \left\| A_i(t) \right\| \geq \varepsilon, \text{ for all } i = 1, \ldots, m \}.
\]

Then, we claim that
\[
U \subset S_x, \text{ for all } x \in \mathbb{R}^n \setminus B(0, \varepsilon^{-1}).
\] (3.17)

Indeed, let \(t \in U\). This leads that \(\left\| A_i(t) \right\| |x| \geq 1\), for all \(x \in \mathbb{R}^n \setminus B(0, \varepsilon^{-1})\). Therefore, it follows from applying the condition \((3.3)\) that
\[
|A_i(t) x| \geq \left\| A_i^{-1}(t) \right\|^{-1} |x| \geq \rho_A^{-1},
\]
which finishes the proof of the relation (3.17).

Now, by letting \( x \in \mathbb{R}^n \setminus B(0, \varepsilon^{-1}) \) and using (3.17), we see that

\[
H_{\Phi, i}(\vec{f})(x) \geq \int_{S_x} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m |A_i(t)x|^{-\frac{n}{\nu} \gamma_i - \gamma_i - \varepsilon} dt \geq \int_U \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m |A_i(t)x|^{-\frac{n}{\nu} \gamma_i - \gamma_i - \varepsilon} dt.
\]

Thus, by (3.13), we find

\[
H_{\Phi, i}(\vec{f})(x) \geq \left( \int_U \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \|A_i^{-1}(t)\|^{\frac{n}{\nu} \gamma_i + \gamma_i + \varepsilon} dt \right) |x|^{-\frac{n}{\nu} \gamma_i - \gamma_i} \chi_{\mathbb{R}^n \setminus B(0, \varepsilon^{-1})}(x).
\]

For convenience, we denote

\[
\Gamma_{\varepsilon} = \int_U \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \min \left\{ \|A_i^{-1}(t)\|^{\frac{n}{\nu} \gamma_i}, \|A_i^{-1}(t)\|^{\frac{n}{\nu} \gamma_i + \gamma_i} \right\} \prod_{i=1}^m \|A_i^{-1}(t)\|^{\varepsilon \chi_{\mathbb{R}^n \setminus B(0, \varepsilon^{-1})}} dt.
\]

Hence, by (3.19), we arrive at

\[
\left\| H_{\Phi, i}(\vec{f}) \right\|_{L^q_i} \geq \varepsilon^{-\varepsilon \chi_{\mathbb{R}^n \setminus B(0, \varepsilon^{-1})}} \left\| h \right\|_{L^q_i} \Gamma_{\varepsilon},
\]

where \( h(x) = |x|^{-\frac{n}{\nu} \gamma_i - \gamma_i} \chi_{\mathbb{R}^n \setminus B(0, \varepsilon^{-1})} \). Next, we will prove the following result

\[
\left\| h \right\|_{L^q_i} \geq \varepsilon^{\min \left\{ \frac{q_+}{q_-}, \frac{1}{q_+} \right\}}.
\]

Indeed, for \( \varepsilon \) sufficiently small such that \( \varepsilon^{-1} > 1 \), we compute

\[
F_q(h \omega) = \int_{|x| \geq \varepsilon^{-1}} |x|^{-n - meq} dx = \int_{\varepsilon^{-1}}^{+\infty} \int_{S^{n-1}} r^{-1 - meq} \sigma (x') dr d\sigma(x')
\]

\[
\geq \int_{\varepsilon^{-1}}^{+\infty} \int_{S^{n-1}} r^{-1 - meq} d\sigma(x') dr \geq \varepsilon^{meq_+} \varepsilon^{-1}.
\]

From this, by the inequality (2.1), we immediately obtain the inequality (3.20).

By writing \( \vartheta(\varepsilon) \) as

\[
\vartheta(\varepsilon) = \prod_{i=1}^m \left( \frac{\varepsilon^{meq_+}}{\varepsilon^{q_-}} \frac{1}{\varepsilon^{q_+}} \right) \left( \frac{\rho_{\varepsilon q_i} - 1}{q_i} \right) \frac{1}{\varepsilon^{q_-}} \frac{1}{\varepsilon^{q_+}},
\]

then, by (3.19) and (3.20), we estimate

\[
\left\| H_{\Phi, i}(\vec{f}) \right\|_{L^q_i} \geq \varepsilon^{-\varepsilon \chi_{\mathbb{R}^n \setminus B(0, \varepsilon^{-1})}} \vartheta(\varepsilon) \prod_{i=1}^m \|f_i\|_{L^{q_i}_i}. \tag{3.22}
\]
Note that, by letting \( \varepsilon \) sufficiently small and \( t \in U \), we find
\[
\prod_{i=1}^{m} \| A_i^{-1}(t) \|^{\varepsilon \varepsilon_{me}} \leq \rho_A^{\varepsilon} \lesssim 1. \tag{3.23}
\]

By the relation (3.13), we get the limit of function \( \varepsilon^{-me} \) is a positive number when \( \varepsilon \) tends to zero. Thus, by (3.22), (3.23) and the dominated convergence theorem of Lebesgue, we also have
\[
\int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \min \left\{ \| A_i^{-1}(t) \|^{\frac{1}{\max_{i}^{\gamma_i + \gamma_i}}, \| A_i^{-1}(t) \|^{\frac{1}{\min_{i}^{\gamma_i + \gamma_i}}} \right\} dt < \infty.
\]
which completes the proof of the theorem.

Next, we discuss the boundedness of the multilinear Hausdorff operators on the product of weighted Morrey-Herz spaces with variable exponent.

**Theorem 3.3.** Let \( \omega_1(x) = |x|^{\gamma_1}, \ldots, \omega_m(x) = |x|^{\gamma_m}, \omega(x) = |x|^{\gamma}, q(\cdot) \in \mathcal{P}_b(\mathbb{R}^n), \lambda_1, \ldots, \lambda_m, \zeta > 0, \) and the hypothesis (2.6) in Theorem 2.1 hold. Suppose that for all \( i = 1, \ldots, m \), we have
\[
\alpha_i(0) - \alpha_{i\infty} \geq 0. \tag{3.24}
\]
At the same time, let
\[
\mathcal{C}_3 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} c_{A_{i,\alpha_i,\gamma_i}}(t) \bigg| 1 \bigg|_{L^{r(\cdot),1}} \max \left\{ \| A_i(t) \|^{\lambda_i - \alpha_i(0)}, \| A_i(t) \|^{\lambda_i - \alpha_{i\infty}} \right\} \times
\]
\[
\times \max \left\{ \sum_{r=\Theta^*_n-1}^{0} 2^{r(\lambda_i - \alpha_i(0))}, \sum_{r=\Theta^*_n-1}^{0} 2^{r(\lambda_i - \alpha_{i\infty})} \right\} dt < \infty,
\tag{3.25}
\]
where \( \Theta^*_n = \Theta^*_n(t) \) is the greatest integer number satisfying
\[
\max_{i=1,\ldots,m} \left\{ \| A_i(t) \|, \| A_i^{-1}(t) \| \right\} < 2^{-\Theta^*_n}, \text{ for a.e. } t \in \mathbb{R}^n.
\]

Then, \( H_{\Phi, A} \) is a bounded operator from \( M_{K_{p_1,\zeta q_1(\cdot),\omega_1}, \ldots, M_{K_{p_m,\zeta q_m(\cdot),\omega_m}}} \) to \( M_{K_{p,q(\cdot),\omega}} \).

**Proof.** By estimating as (3.9) above, we have
\[
\| H_{\Phi, A}(\vec{f}) \|_{L^{\gamma}(\cdot)} \lesssim \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \| f_i(A_i(t)) \|_{L^{\gamma}(\cdot)} dt. \tag{3.26}
\]
Let us now fix \( i \in \{1,2,\ldots,m\} \). Since \( \| A_i(t) \| \neq 0 \), there exists an integer number \( \ell_i = \ell_i(t) \) such that \( 2^{\ell_i-1} < \| A_i(t) \| \leq 2^{\ell_i} \). For simplicity of notation,
we write $\rho^*_A(t) = \max_{i=1}^{n} \{ \| A_i(t) \| \cdot \| A_i^{-1}(t) \| \}$. Then, by letting $y = A_i(t).z$ with $z \in C_k$, it follows that
\[
|y| \geq \| A_i^{-1}(t) \|^{-1} |z| \geq \frac{2^{k+\ell_i-2}}{\rho^*_A} > 2^{k+\ell_i-2+\Theta^*_n},
\]
and
\[
|y| \leq \| A_i(t) \| \cdot |z| \leq 2^{k+\ell_i}.
\]
These estimations can be used to get
\[
A_i(t).C_k \subset \{ z \in \mathbb{R}^n : 2^{k+\ell_i-2+\Theta^*_n} < |z| \leq 2^{k+\ell_i} \}. \tag{3.27}
\]
Now, we need to prove that
\[
\| f_i(A_i(t).)\chi_k \|_{L^q_{\omega_i}^{(t)}} \lesssim c_{A_i,\bar{q},\gamma_i}(t) \| 1 \|_{L^{\bar{q}}_{\omega_i}(t)} \cdot \sum_{r=\Theta^*_n}^{0} \| f_i\chi_k+\ell_i+r \|_{L^q_{\omega_i}^{(t)}}. \tag{3.28}
\]
Indeed, for $\eta > 0$, by (3.27), we find
\[
\begin{aligned}
\int_{\mathbb{R}^n} \left( \frac{|f_i(A_i(t)x)|}{\eta} \chi_k(x) \omega_i(x) \right)^{\eta(x)} dx \\
= \int_{A_i(t)C_k} \left( \frac{|f_i(z)||A_i^{-1}(t)z|}{\eta} \right)^{\eta(A_i^{-1}(t).z)} |\det A_i^{-1}(t)| dz \\
\leq |\det A_i^{-1}(t)| \int_{A_i(t)C_k} \left( \max \{ \| A_i^{-1}(t) \|^\gamma_i, \| A_i(t) \|^{-\gamma_i} \} |f_i(z)| \omega_i(z) \right)^{\eta(A_i^{-1}(t),z)} dz.
\end{aligned}
\]
So, we have that
\[
\begin{aligned}
\int_{\mathbb{R}^n} \left( \frac{|f_i(A_i(t)x)|}{\eta} \chi_k(x) \omega_i(x) \right)^{\eta(x)} dx \\
\leq \int_{\mathbb{R}^n} \left( \frac{c_{A_i,\bar{q},\gamma_i}(t)}{\eta} \sum_{r=\Theta^*_n}^{0} f_i(z) \chi_k+\ell_i+r(z) \omega_i(z) \right)^{\eta(A_i^{-1}(t),z)} dz.
\end{aligned}
\]
Therefore, by the definition of Lebesgue space with variable exponent, it is easy to get that
\[
\| f_i(A_i(t).)\chi_k \|_{L^q_{\omega_i}^{(t)}} \leq c_{A_i,\bar{q},\gamma_i}(t) \cdot \sum_{r=\Theta^*_n}^{0} \| f_i\chi_k+\ell_i+r \|_{L^q_{\omega_i}^{(t),A_i^{-1}(t)}}.
\]
which completes the proof of the inequalities (3.28), by (3.11). Now, it immediately follows from (3.26) and (3.28) that

\[
\| H_{\Phi,\tilde{A}}(\tilde{f}) \chi_k \|_{L^q_t(\cdot)} \lesssim \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t) \|1\|_{L^{r_i}(\cdot)} \times \\
\times \prod_{i=1}^{m} \left( \sum_{r=\Theta_n^* - 1}^{0} \| f \chi_{k+\ell_i+r} \|_{L^{q_i}(\cdot)} \right) dt. \tag{3.29}
\]

Consequently, by applying Lemma 2.3 in Section 2, we get

\[
\| H_{\Phi,\tilde{A}}(\tilde{f}) \chi_k \|_{L^q_t(\cdot)} \lesssim \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \mathcal{L}(t) \prod_{i=1}^{m} c_{A_i,q_i,\gamma_i}(t) \|1\|_{L^{r_i}(\cdot)} \times \\
\times \prod_{i=1}^{m} \| f_i \|_{M_{K_{p_i,q_i,\omega_i}(t)}^{\alpha_i(t),\lambda_i}} dt, \tag{3.30}
\]

where

\[
\mathcal{L}(t) = \prod_{i=1}^{m} \left( 2^{(k+\ell_i)(\lambda_i - \alpha_i(0))} \sum_{r=\Theta_n^* - 1}^{0} 2^r(\lambda_i - \alpha_i(0)) + 2^{(k+\ell_i)(\lambda_i - \alpha_{i-\infty})} \sum_{r=\Theta_n^* - 1}^{0} 2^r(\lambda_i - \alpha_{i-\infty}) \right).
\]

By having \(2^{\ell_i - 1} \leq \| A_i(t) \| \leq 2^{\ell_i}\), for all \(i = 1, \ldots, m\), it implies that

\[
2^{\ell_i(\lambda_i - \alpha_i(0))} + 2^{\ell_i(\lambda_i - \alpha_{i-\infty})} \lesssim \max \left\{ \| A_i(t) \|_{\lambda_i - \alpha_i(0)}, \| A_i(t) \|_{\lambda_i - \alpha_{i-\infty}} \right\}.
\]

Thus, we can estimate \(\mathcal{L}\) as follows

\[
\mathcal{L}(t) \lesssim \prod_{i=1}^{m} \max \left\{ \| A_i(t) \|_{\lambda_i - \alpha_i(0)}, \| A_i(t) \|_{\lambda_i - \alpha_{i-\infty}} \right\} \times \\
\times \left\{ 2^{k(\lambda_i - \alpha_i(0))} \sum_{r=\Theta_n^* - 1}^{0} 2^r(\lambda_i - \alpha_i(0)) + 2^{k(\lambda_i - \alpha_{i-\infty})} \sum_{r=\Theta_n^* - 1}^{0} 2^r(\lambda_i - \alpha_{i-\infty}) \right\} \tag{3.31}
\]

From this, by (3.30), it is not difficult to show that

\[
\| H_{\Phi,\tilde{A}}(\tilde{f}) \chi_k \|_{L^q_t(\cdot)} \leq C_3 \prod_{i=1}^{m} \left( 2^{k(\lambda_i - \alpha_i(0))} + 2^{k(\lambda_i - \alpha_{i-\infty})} \right) \prod_{i=1}^{m} \| f_i \|_{M_{K_{p_i,q_i,\omega_i}(t)}^{\alpha_i(t),\lambda_i}}.
\]
On the other hand, using Proposition 2.5 in [31], we get

\[ \| H_{f,A}(\tilde{f}) \|_{M_{K^{(1),\lambda}}^p} \lesssim \max \left\{ \sup_{k_0 < 0, k_0 \in \mathbb{Z}} E_1, \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} (E_2 + E_3) \right\}, \quad (3.32) \]

where

\[ E_1 = 2^{-k_0} \left( \sum_{k = -\infty}^{k_0} 2^{k\alpha(0)p} \| H_{f,A}(\tilde{f}) \chi_k \|_{L^p(\mathbb{R})}^p \right)^{1/p}, \]

\[ E_2 = 2^{-k_0} \left( \sum_{k = -\infty}^{-1} 2^{k\alpha(0)p} \| H_{f,A}(\tilde{f}) \chi_k \|_{L^p(\mathbb{R})}^p \right)^{1/p}, \]

\[ E_3 = 2^{-k_0} \left( \sum_{k = 0}^{k_0} 2^{k\alpha_{\infty}p} \| H_{f,A}(\tilde{f}) \chi_k \|_{L^p(\mathbb{R})}^p \right)^{1/p}. \]

In order to complete the proof, it remains to estimate the upper bounds for \( E_1, E_2 \) and \( E_3 \). Note that, using (3.31), \( E_1 \) is dominated by

\[ C_3 2^{-k_0} \lambda \left( \sum_{k = -\infty}^{k_0} 2^{k\alpha(0)p} \left( \prod_{i=1}^{m} (2^{k(\lambda_i - \alpha_i(0))) + 2^{k(\lambda_i - \alpha_i(\infty)))} \right) \right)^{1/p}. \]

This implies that

\[ E_1 \lesssim C_3 T_0 \prod_{i=1}^{m} \| f_i \|_{M_{K^{(1),\lambda_i}}^p}, \quad (3.33) \]

where \( T_0 = 2^{-k_0} \lambda \left( \sum_{k = -\infty}^{k_0} 2^{k\alpha(0)p} \prod_{i=1}^{m} (2^{k(\lambda_i - \alpha_i(0))) + 2^{k(\lambda_i - \alpha_i(\infty)))} \right)^{1/p}. \) By some simple computations, we obtain

\[ T_0 = 2^{-k_0} \lambda \left( \sum_{k = -\infty}^{k_0} \prod_{i=1}^{m} (2^{k\lambda_i p} + 2^{k(\lambda_i - \alpha_i(\infty)))} \right)^{1/p} \lesssim \left( \prod_{i=1}^{m} 2^{-k_0} \alpha_i p \left( \frac{k_0}{2^{k\lambda_i p}} + \frac{2^{k(\lambda_i - \alpha_i(\infty)))}}{1 - 2^{-k(\lambda_i - \alpha_i(\infty)))}} \right)^{1/p} \right)^{1/p}. \]

Hence, by assuming that \( \lambda_i > 0 \), for all \( i = 1, \ldots, m \) and (3.24), we see at once that

\[ T_0 \lesssim \left( \prod_{i=1}^{m} \frac{2^{k_0 \lambda_i p}}{1 - 2^{-\lambda_i p}} + \frac{2^{k_0 (\lambda_i - \alpha_i(\infty)))}}{1 - 2^{-k(\lambda_i - \alpha_i(\infty)))}} \right)^{1/p} \lesssim \prod_{i=1}^{m} \left( 1 + 2^{k_0 (\alpha_i(0) - \alpha_i(\infty)))} \right). \]
Then, from (3.33), we have
\[ E_1 \lesssim C_3 \prod_{i=1}^{m} \left( 1 + 2^{k_{0}}(\alpha_{i}(0) - \alpha_{i0}) \right) \prod_{i=1}^{m} \| f_i \|_{MK_{\gamma_{i},\xi_{i} \gamma_{i}}^{\alpha_{i}(-\lambda_{i})}}. \tag{3.34} \]
By estimating in the same way as \( E_1 \), we also get
\[ E_2 \lesssim C_3 \cdot 2^{-k_{0} \lambda_{i}} \prod_{i=1}^{m} \| f_i \|_{MK_{\gamma_{i},\xi_{i} \gamma_{i}}^{\alpha_{i}(-\lambda_{i})}}. \tag{3.35} \]
For \( i = 1, ..., m \), we denote
\[ K_i = \begin{cases} 2^{k_{0}(\alpha_{i0} - \alpha_{i}(0))} + \left| 2^{\lambda_{i}p} - 1 \right|^{\frac{1}{p}} + 2^{-k_{0} \lambda_{i}}, & \text{if } \lambda_{i} + \alpha_{i0} - \alpha_{i}(0) \neq 0, \\ 2^{-k_{0} \lambda_{i}}(k_{0} + 1)^{\frac{1}{p}} + \left| 2^{\lambda_{i}p} - 1 \right|^{\frac{1}{p}}, & \text{otherwise}. \end{cases} \]
Then, we may show that
\[ E_3 \lesssim C_3 \left( \prod_{i=1}^{m} K_i \right) \prod_{i=1}^{m} \| f_i \|_{MK_{\gamma_{i},\xi_{i} \gamma_{i}}^{\alpha_{i}(-\lambda_{i})}}. \tag{3.36} \]
The proof of inequality (3.36) is not difficult, but for convenience to the reader, we briefly give here. By employing (3.31) again, we make
\[ E_3 \lesssim C_3 \cdot \mathcal{T}_{\infty} \prod_{i=1}^{m} \| f_i \|_{MK_{\gamma_{i},\xi_{i} \gamma_{i}}^{\alpha_{i}(-\lambda_{i})}}, \tag{3.37} \]
where \( \mathcal{T}_{\infty} = 2^{-k_{0} \lambda_{i}} \left( \sum_{k=0}^{k_{0}} 2^{k \lambda_{i}p} \prod_{i=1}^{m} \left( 2^{k(\lambda_{i} - \alpha_{i}(0))p} + 2^{k(\lambda_{i} - \alpha_{i0})p} \right) \right)^{\frac{1}{p}}. \]
By a similar argument as \( \mathcal{T}_{0} \), we also get
\[ \mathcal{T}_{\infty} \lesssim \prod_{i=1}^{m} 2^{-k_{0} \lambda_{i}} \left( \sum_{k=0}^{k_{0}} 2^{k \lambda_{i}p} + \sum_{k=0}^{k_{0}} 2^{k(\lambda_{i} + \alpha_{i0} - \alpha_{i}(0))p} \right)^{\frac{1}{p}} \equiv \prod_{i=1}^{m} \mathcal{T}_{i,\infty}. \]
In the case \( \lambda_{i} + \alpha_{i0} - \alpha_{i}(0) \neq 0 \), we deduce that \( \mathcal{T}_{i,\infty} \) is dominated by
\[ \mathcal{T}_{i,\infty} \leq 2^{-k_{0} \lambda_{i}} \left( \frac{2^{k_{0} \lambda_{i}p} - 1}{2^{\lambda_{i}p} - 1} + \frac{2^{k_{0}(\lambda_{i} + \alpha_{i0} - \alpha_{i}(0))p} - 1}{2^{(\lambda_{i} + \alpha_{i0} - \alpha_{i}(0))p} - 1} \right)^{\frac{1}{p}} \]
\[ \lesssim 2^{k_{0}(\alpha_{i0} - \alpha_{i}(0))} + \left| 2^{\lambda_{i}p} - 1 \right|^{-1/p} + 2^{-k_{0} \lambda_{i}}. \]
Otherwise, we have
\[ \mathcal{T}_{i,\infty} \leq \left( \frac{2^{k_{0} \lambda_{i}p} - 1}{2^{\lambda_{i}p} - 1} + (k_{0} + 1) \right)^{\frac{1}{p}} \lesssim 2^{-k_{0} \lambda_{i}}(k_{0} + 1)^{\frac{1}{p}} + \left| 2^{\lambda_{i}p} - 1 \right|^{-1/p}. \]
This leads that \( \mathcal{T}_{\infty} \lesssim \prod_{i=1}^{m} K_i \). Hence, by (3.37), we obtain the inequality (3.36). From (3.32) and (3.34) - (3.36), we conclude that the proof of Theorem 3.3 is finished. \( \square \)
Next, we will discuss the interesting case when \( \lambda_1 = \cdots = \lambda_m = 0 \). Remark that these special cases of variable exponent Morrey-Herz spaces are variable exponent Herz spaces. Hence, we also have the boundedness for the multilinear Hausdorff operators on the product of weighted Herz spaces with variable exponent as follows.

**Theorem 3.4.** Suppose that we have the given supposition of Theorem 3.3 and \( \alpha_i(0) = \alpha_i, \) for all \( i = 1, \ldots, m \). Let \( 1 \leq p, p_i < \infty \) such that
\[
\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p}.
\] (3.38)

At the same time, let
\[
C_4 = \int_{\mathbb{R}^n} (2 - \Theta_n^*)^m \frac{\Phi(t)}{t^n} \prod_{i=1}^m c_{A_i, q_i, \gamma_i}(t) \| f \|_{L^{p_i}((\cdot))} \times
\]
\[
\times \| A_i(t) \|^{-\alpha_i(0)} \left( \sum_{r=\Theta_n^*-1}^0 2^{-r\alpha_i(0)} \right) dt < \infty.
\] (3.39)

Then, \( H_{\Phi, \vec{A}} \) is a bounded operator from \( K_{\zeta_{q_1}^{\alpha_1}, \omega_1}^{\alpha_1, \cdot, p_1} \times \cdots \times K_{\zeta_{q_m}^{\alpha_m}, \omega_m}^{\alpha_m, \cdot, p_m} \) to \( K_{\zeta_{q_1}^{\alpha_1}, \omega_1}^{\alpha_1, \cdot, p} \).

**Proof.** It follows from Proposition 3.8 in [4] that
\[
\| H_{\Phi, \vec{A}}(\vec{f}) \|_{K_{\zeta_{q_1}^{\alpha_1}, \omega_1}^{\alpha_1, \cdot, p}} \lesssim \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \| H_{\Phi, \vec{A}}(\vec{f}) \chi_k \|_{L^{p_1}((\cdot))} \right)^{\frac{1}{p}}
\]
\[
+ \left( \sum_{k=0}^{\infty} 2^{k\alpha\infty p} \| H_{\Phi, \vec{A}}(\vec{f}) \chi_k \|_{L^{p_1}((\cdot))} \right)^{\frac{1}{p}}.
\] (3.40)

From this, by \( \alpha(0) = \alpha_\infty \), we conclude that
\[
\| H_{\Phi, \vec{A}}(\vec{f}) \|_{K_{\zeta_{q_1}^{\alpha_1}, \omega_1}^{\alpha_1, \cdot, p}} \lesssim \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \| H_{\Phi, \vec{A}}(\vec{f}) \chi_k \|_{L^{p_1}((\cdot))} \right)^{\frac{1}{p}}.
\] (3.40)

For convenience, let us denote by
\[
\mathcal{H} = \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \| H_{\Phi, \vec{A}}(\vec{f}) \chi_k \|_{L^{p_1}((\cdot))} \right)^{\frac{1}{p}}.
\]

Next, we need to estimate the upper bound of \( \mathcal{H} \). By (3.29) and using the Minkowski inequality, we get
\[
\mathcal{H} \leq \int_{\mathbb{R}^n} \frac{\Phi(t)}{t^n} \prod_{i=1}^m c_{A_i, q_i, \gamma_i}(t) \| f \|_{L^{p_i}((\cdot))} \times
\]
\[
\times \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \prod_{i=1}^m \left( \sum_{r=\Theta_n^*-1}^{0} \| f_i \chi_k + \epsilon_i + r \|_{L^{q_i}_{\omega_i}((\cdot))} \right)^{\frac{1}{p}} \right\} dt.
\] (3.41)
By (3.38) and the H"{o}lder inequality, it follows that
\[
\left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha(0)p} \prod_{i=1}^{m} \left( \sum_{r=\Theta_{n}^{*}-1}^{0} \| f_{i}X_{k+\ell_{i}+r} \|_{L_{\Sigma_{i}}^{\eta_{i}}} \right)^{p_{i}} \right\}^{\frac{1}{p}} \leq \prod_{i=1}^{m} \left\{ \sum_{k=-\infty}^{\infty} 2^{k\alpha_{i}(0)p_{i}} \left( \sum_{r=\Theta_{n}^{*}-1}^{0} \| f_{i}X_{k+\ell_{i}+r} \|_{L_{\Sigma_{i}}^{\eta_{i}}} \right)^{p_{i}} \right\}^{\frac{1}{p_{i}}}.
\] (3.42)

On the other hand, by \( p_{i} \geq 1 \), we have
\[
\left( \sum_{r=\Theta_{n}^{*}-1}^{0} \| f_{i}X_{k+\ell_{i}+r} \|_{L_{\Sigma_{i}}^{\eta_{i}}} \right)^{p_{i}} \leq (2 - \Theta_{n}^{*})^{p_{i}-1} \sum_{r=\Theta_{n}^{*}-1}^{0} \| f_{i}X_{k+\ell_{i}+r} \|_{L_{\Sigma_{i}}^{\eta_{i}}}^{p_{i}}.
\]
Hence, combining (3.41) and (3.42), we obtain
\[
\mathcal{H} \leq \int_{\mathbb{R}^{n}} (2 - \Theta_{n}^{*})^{m-1} \frac{\Phi(t)}{|t|^{n}} \prod_{i=1}^{m} \frac{C_{A_{i},q_{i},\gamma_{i}}(t)}{\| f_{i}X_{k+\ell_{i}+r} \|_{L_{\Sigma_{i}}^{\eta_{i}}}} \prod_{i=1}^{m} \mathcal{H}_{i}dt,
\] (3.43)
where \( \mathcal{H}_{i} = \sum_{r=\Theta_{n}^{*}-1}^{0} \left( \sum_{k=-\infty}^{\infty} 2^{k\alpha_{i}(0)p_{i}} \| f_{i}X_{k+\ell_{i}+r} \|_{L_{\Sigma_{i}}^{\eta_{i}}}^{p_{i}} \right)^{\frac{1}{p_{i}}}, \) for all \( i = 1, 2, \ldots, m. \)

Then, we find
\[
\mathcal{H}_{i} = \sum_{r=\Theta_{n}^{*}-1}^{0} \left( \sum_{k=-\infty}^{\infty} 2^{(l_{i}-r)\alpha_{i}(0)p_{i}} \| f_{i}X_{k+\ell_{i}+r} \|_{L_{\Sigma_{i}}^{\eta_{i}}}^{p_{i}} \right)^{\frac{1}{p_{i}}}
= \sum_{r=\Theta_{n}^{*}-1}^{0} 2^{-(l_{i}+r)\alpha_{i}(0)} \left( \sum_{t=-\infty}^{\infty} 2^{t\alpha_{i}(0)p_{i}} \| f_{i}X_{k\ell_{i}} \|_{L_{\Sigma_{i}}^{\eta_{i}}}^{p_{i}} \right)
\leq \sum_{r=\Theta_{n}^{*}-1}^{0} 2^{-r\alpha_{i}(0)} 2^{-\ell_{i}\alpha_{i}(0)} \| f_{i} \|_{K_{\Sigma_{i},\omega_{i}}^{\alpha_{i}(0)}}.
\] (3.44)

Since \( 2^{\ell_{i}-1} \leq \| A_{i}(t) \| \leq 2^{\ell_{i}} \), we imply that \( 2^{-\ell_{i}\alpha_{i}(0)} \leq \| A_{i}(t) \|^{-\alpha_{i}(0)} \). Thus, by (3.43) and (3.44), we get
\[
\| H_{\Phi,\tilde{A}}(\tilde{f}) \|_{K_{\Phi}^{\omega(p)}(\alpha_{i})} \lesssim C_{4} \prod_{i=1}^{m} \| f_{i} \|_{K_{\Sigma_{i},\omega_{i}}^{\alpha_{i}(0),p_{i}}}^{\alpha_{i}(0)},
\]
which finishes our desired conclusion. \( \square \)

**Remark 3.** We would like to give several comments on Theorem 3.3 and Theorem 3.4. If we suppose that
\[
\operatorname{ess sup}_{t \in \operatorname{supp}(\Phi)} \| A_{i}(t) \| < \infty, \text{ for all } i = 1, \ldots, m,
\]
then we do not need to assume the conditions \( \alpha_i(0) - \alpha_i \infty \geq 0 \) in Theorem 3.3 and \( \alpha_i(0) = \alpha_i \infty \) in Theorem 3.4. Indeed, by putting

\[
\beta = \operatorname{ess sup}_{t \in \text{supp}(\Phi)} \ell_i(t),
\]

and applying Lemma 2.8 in Section 2, we refine the estimation as follows:

In the case \( k < \beta \), we get

\[
\left\| f_i x_{k+\ell_i+r} \right\|_{L_q^{\alpha_i}(\cdot)} \lesssim 2^{(k+\ell_i+r)(\lambda_i - \alpha_i(0))} \left\| f_i \right\|_{M_{K_p,\alpha_i,\lambda_i}^{\alpha_i,\lambda_i}(\cdot,\omega_i)}.
\]

In the case \( k \geq \beta - \Theta_n^* + 1 \), we have

\[
\left\| f_i x_{k+\ell_i+r} \right\|_{L_q^{\alpha_i}(\cdot)} \lesssim 2^{(k+\ell_i+r)(\lambda_i - \alpha_i(0))} \left\| f_i \right\|_{M_{K_p,\alpha_i,\lambda_i}^{\alpha_i,\lambda_i}(\cdot,\omega_i)}.
\]

Otherwise, we obtain

\[
\left\| f_i x_{k+\ell_i+r} \right\|_{L_q^{\alpha_i}(\cdot)} \lesssim \left( 2^{(k+\ell_i+r)(\lambda_i - \alpha_i(0))} + 2^{(k+\ell_i+r)(\lambda_i - \alpha_i \infty)} \right) \left\| f_i \right\|_{M_{K_p,\alpha_i,\lambda_i}^{\alpha_i,\lambda_i}(\cdot,\omega_i)}.
\]

Also, the other estimations can be done by similar arguments as two theorems above. From this we omit details, and their proof are left to reader.

**Theorem 3.5.** Suppose that the given supposition of Theorem 3.3 and the hypothesis (3.2) in Theorem 3.2 are true.

(a) If

\[
\mathcal{C}_5 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \max \left\{ \left\| A_i^{-1}(t) \right\|^{\frac{m_i}{n_i} + \gamma_i}, \left\| A_i^{-1}(t) \right\|^{\frac{m_i}{n_i} - \gamma_i} \right\} \left| A_i^{-1}(t) \right|^{-\lambda_i} \times
\]

\[\times \max \left\{ \left\| A_i^{-1}(t) \right\|^{\alpha_i(0)}_1, \left\| A_i^{-1}(t) \right\|^{\alpha_i \infty}_1 \right\} \left\| f_i \right\|_{L^{n_i}(\cdot,\omega_i)} dt < \infty,
\]

then

\[
\left\| H_{\Phi,A}(\vec{f}) \right\|_{M_{K_p,\alpha_i,\lambda_i}^{\alpha_i,\lambda_i}(\cdot,\omega_i)} \lesssim \mathcal{C}_5 \prod_{i=1}^m \left\| f_i \right\|_{M_{K_p,\alpha_i,\lambda_i}^{\alpha_i,\lambda_i}(\cdot,\omega_i)}.
\]

(b) Denote by

\[
\mathcal{C}^*_5 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \min \left\{ \left\| A_i^{-1}(t) \right\|^{\frac{m_i}{n_i} + \gamma_i}, \left\| A_i^{-1}(t) \right\|^{\frac{m_i}{n_i} - \gamma_i} \right\} \left| A_i^{-1}(t) \right|^{-\lambda_i} \times
\]

\[\times \min \left\{ \left\| A_i^{-1}(t) \right\|^{\alpha_i(0)} + C_0^a, \left\| A_i^{-1}(t) \right\|^{\alpha_i(0) - C_0^a} \right\} dt.
\]

Suppose that \( H_{\Phi,A} \) is a bounded operator from \( M_{K_p,\alpha_i,\lambda_i}^{\alpha_i,\lambda_i}(\cdot,\omega_i) \times \cdots \times M_{K_p,\alpha_i,\lambda_i}^{\alpha_i,\lambda_i}(\cdot,\omega_i) \) to \( M_{K_p,\alpha_i,\lambda_i}^{\alpha_i,\lambda_i}(\cdot,\omega_i) \) and one of the following conditions holds:

(b1) \( q_i = q_i -, C_0^a \) and \( C_0^a \leq \alpha_i(0) - \alpha_i \infty \), for all \( i = 1, \ldots, m \);

(b2) \( q_i \neq q_i -, C_0^a = C_0^a = 0, \lambda_i = \alpha_i(0) = \alpha_i \infty \), for all \( i = 1, \ldots, m \);

(b3) \( q_i \neq q_i -, \) both \( C_0^a \) and \( C_0^a \) are less than \( \alpha_i(0) - \alpha_i \infty \), \( C_0^a + C_0^a \leq C_0^a \), \( \lambda_i \in [\eta_i^0, \eta_i^1] \cap [\zeta_i^0, \zeta_i^1] \), for all \( i = 1, \ldots, m \).
Here $C^{\alpha_i} = \frac{q_i - (\alpha_i(0) - \alpha_i(\infty))(1 + \frac{\eta_i}{q_i})}{q_i + 1}$ and $\eta_i^0, \eta_i^1, \zeta_i^0, \zeta_i^1$ are defined by

$$
\eta_i^0 = \frac{C_0^{\alpha_i}}{q_i - 1} - \alpha_i(0) \frac{q_i}{q_i + 1} + \alpha_i(\infty), \quad \eta_i^1 = \frac{C_0^{\alpha_i}}{q_i - 1} - \alpha_i(0) \frac{q_i}{q_i + 1},
$$

$$
\zeta_i^0 = \frac{C_0^{\alpha_i}}{q_i - 1} - \alpha_i(0) + \alpha_i(\infty) \frac{q_i}{q_i - 1}, \quad \zeta_i^1 = \frac{C_0^{\alpha_i}}{q_i - 1} - \alpha_i(0) + \alpha_i(\infty) \frac{q_i}{q_i - 1}.
$$

Then, we have that $C_5^*$ is finite. Furthermore,

$$
\|H_{\Phi, \tilde{A}}\|_{MK_{p_1, q_1}(\omega_1) \times \cdots \times MK_{p_m, q_m}(\omega_m) \to MK_{p(\cdot), \omega}} \geq C_5^*.
$$

**Proof.** Firstly, we prove for the case (a). From (3.3), we call $\Theta_n$ the greatest integer number such that $p_\delta < 2^{-\Theta_n}$. Now, we replace $\Theta_n$ by $\Theta_n$ in the proof of Theorem 3.3 and the other results are estimated in the same way. Then, by (3.14), we get

$$
\|H_{\Phi, \tilde{A}}(\tilde{f})\|_{MK_{p(\cdot), \omega}} \leq \left( \int \frac{\Phi(t)}{t^n} \prod_{i=1}^m c_{A_i, q_i, \gamma_i} (t) \right)^{\frac{1}{2}} L^{L_2(t, \cdot)} \times \max \left\{ \|A_i(t)\|^{\lambda_i - \alpha_i(0)}, \|A_i(t)\|^{\lambda_i - \alpha_i(\infty)} \right\} \prod_{i=1}^m \|f_i\|_{MK_{p_i, q_i}(\omega_i)}.
$$

By the inequality (3.4), we have

$$
\max \left\{ \|A_i(t)\|^{\lambda_i - \alpha_i(0)}, \|A_i(t)\|^{\lambda_i - \alpha_i(\infty)} \right\} \lesssim \|A_i^{-1}(t)\|^{-\lambda_i} \max \left\{ \|A_i^{-1}(t)\|^{\alpha_i(0)}, \|A_i^{-1}(t)\|^{\alpha_i(\infty)} \right\}.
$$

Thus, by (3.15), (3.45) and $C_3 < \infty$, we finish the proof for this case.

Next, we will prove for case (b). By choosing $f_i(x) = |x|^{-\alpha_i(x)\gamma_i} \sim \frac{1}{q_i(x) - \gamma_i + \lambda_i}$, it is evident that $\|f_i\|_{MK_{p_i, q_i}(\omega_i)} > 0$, for all $i = 1, \ldots, m$. Now, we need to show that

$$
\|f_i\|_{MK_{p_i, q_i}(\omega_i)} < \infty, \quad \text{for all } i = 1, \ldots, m.
$$

Indeed, we find

$$
F_{q_i}(f_i \omega_i, \chi_k) = \int_{C_k} |x|^{(\lambda_i - \alpha_i(x)\gamma_i - n} dx = \int_{2^{k-1}g_{n-1}} \int y^{(\lambda_i - \alpha_i(x)\gamma_i - n} \sigma (x') dx' dr.
$$

**Case 1:** $k \leq 0$. Since $\alpha_i \in C_{\infty}^{\log} (\mathbb{R}^n)$, it follows that

$$
-C_{\alpha_i} + \alpha_i(x) \leq \alpha_i(x) \leq \alpha_i(x) + C_{\alpha_i}.
$$
As a consequence, we get
\[
F_q(f_1 \omega_1 \chi_k) \leq \int_{2^{k-1} \mathbb{S}^{n-1}} f(r, x')^{-1} d\sigma(x') dr
\]
\[
\leq \max \{ 2^k (\lambda_i - \alpha_{i,\infty} - C^\alpha_{i,\infty}) q_i, 2^k (\lambda_i - \alpha_{i,\infty} - C^\alpha_{i,\infty}) q_i^{-1} \}.
\]
Thus, by (2.1), we obtain
\[
\| f_1 \chi_k \|_{L^{q_i}(\cdot)} \leq \max \{ 2^k (\lambda_i - \alpha_{i,\infty} - C^\alpha_{i,\infty}) q_i, 2^k (\lambda_i - \alpha_{i,\infty} - C^\alpha_{i,\infty}) q_i^{-1} \}
\]
where
\[
\beta_{i,\infty} = \begin{cases} 
q_i^+, & \text{if } \lambda_i - \alpha_{i,\infty} - C^\alpha_{i,\infty} < 0, \\
q_i^-, & \text{otherwise.}
\end{cases}
\]
Case 2: $k > 0$. Since $\alpha_i \in C_0^{\log}(\mathbb{R}^n)$, we have
\[
- C^\alpha_{i,\infty} + \alpha_i(0) \leq \alpha_i(x) \leq \alpha_i(0) + C^\alpha_{i,\infty}.
\]
Denote
\[
\beta_{i,0} = \begin{cases} 
q_i^+, & \text{if } \lambda_i - \alpha_i(0) + C^\alpha_0 \geq 0, \\
q_i^-, & \text{otherwise.}
\end{cases}
\]
By having (3.48) and estimating in the same way as the case 1, we deduce
\[
\| f_1 \chi_k \|_{L^{q_i}(\cdot)} \leq 2^k (\lambda_i - \alpha_i(0) + C^\alpha_0) \beta_{i,0}.
\]
Next, it follows from Proposition 2.5 in [31] that
\[
\| f_i \|_{M^{p_i,q_i}(\cdot)} \leq \max \left\{ \sup_{k_0 < 0, k_0 \in \mathbb{Z}} E_{i,1}, \sup_{k_0 \geq 0, k_0 \in \mathbb{Z}} (E_{i,2} + E_{i,3}) \right\},
\]
where
\[
E_{i,1} = 2^{-k_0 \lambda_i} \left( \sum_{k=-\infty}^{k_0} 2^{k \alpha_i(0) p_i} \| f_i \|_{L^{q_i}(\cdot)}^{p_i} \right)^{\frac{1}{p_i}},
\]
\[
E_{i,2} = 2^{-k_0 \lambda_i} \left( \sum_{k=-\infty}^{-1} 2^{k \alpha_i(0) p_i} \| f_i \|_{L^{q_i}(\cdot)}^{p_i} \right)^{\frac{1}{p_i}},
\]
\[
E_{i,3} = 2^{-k_0 \lambda_i} \left( \sum_{k=0}^{k_0} 2^{k \alpha_i p_i} \| f_i \|_{L^{q_i}(\cdot)}^{p_i} \right)^{\frac{1}{p_i}}.
\]
Notice that the relation $\alpha_t(0) + (\lambda_i - \alpha_{i\infty} - C_{t\infty}^\alpha)\beta_{i\infty}$ is required positively which is proved later, because $E_{i,1}$ is infinite otherwise. Thus, because of \((3.47)\), we have $E_{i,1}$ and $E_{i,2}$ are dominated by

\[
E_{i,1} \lesssim 2^{-k_0\lambda_i} \left( \sum_{k=-\infty}^{k_0} 2^{k\alpha_i(0)} p_k 2^{kp_i(\lambda_i - \alpha_{i\infty} - C_{t\infty}^\alpha)\beta_{i\infty}} \right)^{\frac{1}{p_i}} \\
E_{i,2} \lesssim 2^{-k_0\lambda_i} 2^{-\alpha_t(0) - (\lambda_i - \alpha_{i\infty} - C_{t\infty}^\alpha)\beta_{i\infty}} \lesssim 2^{-k_0\lambda_i}.
\]

By \((3.49)\), we have $E_{i,3}$ is controlled by

\[
E_{i,3} \lesssim 2^{-k_0\lambda_i} + 2^{-k_0\lambda_i} \left( \sum_{k=1}^{k_0} 2^{k\alpha_{i\infty}} p_k \| f_i \|_{L^p_{\theta_i}(\cdot)}^{\frac{1}{p_i}} \right) \\
\lesssim 2^{-k_0\lambda_i} + 2^{-k_0\lambda_i} \left( \sum_{k=1}^{k_0} 2^{kp_i(\alpha_{i\infty} + (\lambda_i - \alpha_t(0) + C_{t0}^\alpha)\beta_{t0})} \right)^{\frac{1}{p_i}} \\
\lesssim \begin{cases} 
2^{-k_0\lambda_i} (k_0^{\frac{1}{p_i}} + 1), & \text{if } \alpha_{i\infty} + (\lambda_i - \alpha_t(0) + C_{t0}^\alpha)\beta_{t0} = 0, \\
2^{-k_0\lambda_i} + 2^{-k_0(\lambda_i - \alpha_{i\infty} - (\lambda_i - \alpha_t(0) + C_{t0}^\alpha)\beta_{t0})}, & \text{otherwise.}
\end{cases}
\]

This implies that

\[
E_{i,3} \lesssim 2^{-k_0\lambda_i} (k_0^{\frac{1}{p_i}} + 1) + 2^{-k_0(\lambda_i - \alpha_{i\infty} - (\lambda_i - \alpha_t(0) + C_{t0}^\alpha)\beta_{t0})},
\]

For convenience, we set

\[
\begin{align*}
\theta_{t0} &= \lambda_i - \alpha_{i\infty} - (\lambda_i - \alpha_t(0) + C_{t0}^\alpha)\beta_{t0}, \\
\theta_{i\infty} &= \alpha_t(0) + (\lambda_i - \alpha_{i\infty} - C_{t\infty}^\alpha)\beta_{i\infty} - \lambda_i.
\end{align*}
\]

Combining \((3.50)-(3.53)\), we get that

\[
\| f_i \|_{MK^{\alpha_i(\cdot),\lambda}_{p_i,q_i(\cdot)\omega_i}(\cdot)} \lesssim \max \left\{ \sup_{k_0<0,k_0\in\mathbb{Z}} 2^{k_0\theta_{t0}}, \sup_{k_0\geq 0,k_0\in\mathbb{Z}} (2^{-k_0\lambda_i} (k_0^{\frac{1}{p_i}} + 1) + 2^{-k_0\theta_{t0}}) \right\}.
\]

From the above estimation, we will finish the proof of \((3.46)\) if the following result can be proved

\[
\theta_{t0} \geq 0 \text{ and } \theta_{i\infty} \geq 0.
\]

In order to do this, let us consider three cases as follows.

**Case b1.** By $q_{+} = q_{-}$, we have $\beta_{t0} = \beta_{i\infty} = 1$. So, by the information of $C_{t0}^\alpha$ and $C_{t\infty}^\alpha$, it is easy to have the desired result \((3.51)\).

**Case b2.** In this case, we find $\theta_{t0} = \theta_{i\infty} = 0$. This follows immediately that the result \((3.54)\) is true.
Case b3. Because both $C_{\alpha_i}^0$ and $C_{\infty}^0$ are less than $\alpha_i(0) - \alpha_{i,\infty}$, we have $[\eta_0^i, \eta_1^i]$ and $[\zeta_0^i, \zeta_1^i]$ are not empty sets. Also, we obtain

\[ \alpha_i(0) - C_{\alpha_i}^0 \in [\eta_0^i, \eta_1^i] \text{ and } \alpha_{i,\infty} + C_{\alpha_i}^0 \in [\zeta_0^i, \zeta_1^i]. \] (3.55)

From $C_{\alpha_i}^0 + C_{\alpha_i}^0 \leq C_{\alpha_i}$, it implies that $\eta_1^i \geq \zeta_0^i$ and $\zeta_1^i \geq \eta_0^i$. Hence, we also have $[\eta_0^i, \eta_1^i] \cap [\zeta_0^i, \zeta_1^i]$ is not an empty set. Thus, by (3.55), we observe that

\[ \int \frac{\Phi(t)}{|t|^n} \prod_{1}^{m} |A_i(t)x|^{-\alpha_i(x) - \frac{m}{q_i(x)} - \gamma_i \lambda_i} dt \]

From this, because of (3.46) and assuming that $H_{\Phi, \vec{A}}$ is a bounded operator, we conclude

\[ \|H_{\Phi, \vec{A}}\|_{MK_{p_1(\gamma_1, \omega_1)} \times \cdots \times MK_{p_m(\gamma_m, \omega_m)} \rightarrow MK_{p(\gamma, \omega)}} \geq C_{5} \frac{\|\cdot - \alpha_i(x) - \frac{m}{q_i(x)} - \gamma_i \lambda_i\|_{MK_{p(\gamma, \omega)}}}{\prod_{i=1}^{m} \|A_i^{-1}(t)\|_{MK_{K_{q_i(\gamma_i, \omega_i)}}}}. \]

This implies the desired assertion. 

**Theorem 3.6.** Suppose that the assumptions of Theorem 3.4 and the hypotheses (5.12) in Theorem 5.3 are true.

(a) If

\[ C_{5} = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \max \left\{ \|A_i^{-1}(t)\|_{MK_{q_{i}(\omega_i)}}^{\alpha_i(p_i)} \right\} dt < \infty, \]

then $H_{\Phi, \vec{A}}$ is a bounded operator from $K_{q_{1}(\gamma_1, \omega_1)} \times \cdots \times K_{q_{m}(\gamma_m, \omega_m)}$ to $K_{q(\gamma, \omega)}$. 

(b) Denote by
\[
C_6^* = \begin{cases}
\frac{\int \Phi(t) \prod_{i=1}^{m} \| A_i^{-1}(t) \|^{\alpha_i(0) + n_i + \gamma_i} dt}{|t|^n}, \text{ if } q_{i^+} = q_{i^-} \text{ for all } i = 1, \ldots, m,
\end{cases}
\]

Let \( H_{\Phi,A} \) be a bounded operator from \( K_{\alpha_1(\cdot),\omega_1} \times \cdots \times K_{\alpha_m(\cdot),\omega_m} \) to \( K_{\alpha(\cdot),\omega} \) and one of the following conditions is satisfied:

1. \( q_{i^-} = q_{i^+} \), for all \( i = 1, \ldots, m \);
2. The case (b1) is not true and \( \alpha_i(0) < \| \alpha_i \|_{L^\infty} q_{i^-}/q_{i^+} \), for all \( i = 1, \ldots, m \).

Then, we have that \( C_6^* \) is finite. Furthermore, there exists \( C > 0 \) such that the operator norm of \( H_{\Phi,A} \) is not greater than \( C \cdot C_6^* \).

**Proof.** In the case (a), by combining Theorem 3.4 and the part (a) of Theorem 3.5 we immediately imply the desired result.

In the case (b1), we have that \( q_1(\cdot), \ldots, q_m(\cdot) \), and \( q(\cdot) \) are constant. Thus, for all \( i = 1, \ldots, m \), we will choose the function \( f_i \) as follows:

\[
f_i(x) = \begin{cases}
0, & \text{if } |x| < p_A^{-1},
|x|^{-\alpha_i(0) - \frac{n_i}{q_i} - \gamma_i - \varepsilon}, & \text{otherwise}.
\end{cases}
\]

It is obvious to see that when \( k \) is an integer number satisfying \( k \leq \frac{-\log(\rho_{\varepsilon})}{\log(2)} \) then \( \| f_i \chi_k \|_{L^{q_{i^+}}_{\omega_{q_i}}} = 0 \). Otherwise, we have

\[
\| f_i \chi_k \|_{L^{q_{i^+}}_{\omega_{q_i}}} \lesssim 2^{-kq_i(\alpha_i(0) + \varepsilon)} \left( \frac{2^{q_i(\alpha_i(0) + \varepsilon)} - 1}{q_i(\alpha_i(0) + \varepsilon)} \right).
\]

Hence, by applying Proposition 3.8 in [4] again and \( \alpha_i(0) = \alpha_i \omega_i \), we find

\[
\| f_i \|_{K_{\alpha_1(\cdot),\omega_1}} \lesssim \left( \sum_{k=\rho}^{\infty} 2^{kq_i(\alpha_i(0) + \varepsilon)} \| f_i \chi_k \|_{L^{q_{i^+}}_{\omega_{q_i}}} \right)^{\frac{1}{p_i}}
\]

\[
\lesssim \left( \frac{2^{q_i(\alpha_i(0) + \varepsilon)} - 1}{q_i(\alpha_i(0) + \varepsilon)} \right)^{\frac{1}{n_i}} \left( \frac{2^{q_i - \rho p_i} - 1}{2^{q_i - 1}} \right)^{\frac{1}{n_i}} < \infty,
\]

where \( \rho \) is the smallest integer number such that \( \rho > \frac{-\log(\rho_{\varepsilon})}{\log(2)} \). Estimating as (3.18), we have

\[
H_{\Phi,A}(f)(x) \gtrsim \left( \int_{U} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \| A_i^{-1}(t) \|^{\alpha_i(0) + n_i + \gamma_i} dt \right) |x|^{-\alpha_0 - \frac{n}{q} - \gamma - \varepsilon} \chi_{R^n \setminus B(0, \varepsilon^{-1})}(x).
\]

(3.56)
Let \( k_0 \) be the smallest integer number such that \( 2^{k_0-1} \geq \varepsilon^{-1} \). Using Proposition 3.8 in [4] again, \( \alpha_i(0) = \alpha_i \infty \) and (3.56), we obtain
\[
\| H_{\Phi,A}(\tilde{f}) \|_{K_{q_i}^{\alpha_i(\cdot),p}}^{p} \geq \sum_{k=k_0}^{\infty} 2^{k\alpha(0)p} \left( \int_{2^{k-1} < |x| \leq 2^k} |x|^{-\varepsilon m q - \alpha(0) q - n} \, dx \right)^{\frac{p}{n}} \times \left( \int_U \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \| A_i^{-1}(t) \|_{\alpha_i(0) + \frac{\alpha_i}{q_i} + \gamma_i + \varepsilon} \, dt \right)^{p}. \tag{3.57}
\]
An elementary calculation leads that
\[
\sum_{k=k_0}^{\infty} 2^{k\alpha(0)p} \left( \int_{2^{k-1} < |x| \leq 2^k} |x|^{-\varepsilon m q - \alpha(0) q - n} \, dx \right)^{\frac{p}{n}} \geq \left( \frac{2^{k_0 m \varepsilon p}}{1 - 2^{-\varepsilon m p}} \right)^{\frac{1}{n}} \left( \frac{2^{k_0 \varepsilon m + \alpha(0)}}{q(\varepsilon m + \alpha(0))} \right)^{1 - \frac{1}{n}}. \tag{3.58}
\]
For simplicity of notation, we write
\[
\vartheta^\ast(\varepsilon) = \frac{2^{k_0 \varepsilon m + \alpha(0)}}{q(\varepsilon m + \alpha(0))} \left( \frac{2^{k_0 m \varepsilon p}}{1 - 2^{-\varepsilon m p}} \right)^{1 - \frac{1}{n}} \prod_{i=1}^{m} \left( \frac{2^{k_0 \varepsilon m i + \alpha_i(0) + \varepsilon}}{q_i(\alpha_i(0) + \varepsilon)} \right)^{1 - \frac{1}{n_i}}.
\]
Therefore, by (3.57) and (3.58), we estimate
\[
\| H_{\Phi,A}(\tilde{f}) \|_{K_{q_i}^{\alpha_i(\cdot),p}}^{p} \geq \varepsilon^{-m \varepsilon} \vartheta^\ast \prod_{i=1}^{m} \| f_i \|_{K_{q_i}^{\alpha_i(\cdot),p_i}}^{p} \times \left( \int_U \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \| A_i^{-1}(t) \|_{\alpha_i(0) + \frac{\alpha_i}{q_i} + \gamma_i + \varepsilon} \prod_{i=1}^{m} \| A_i^{-1}(t) \|_{\varepsilon} \, dt \right). \tag{3.59}
\]
By (3.58), it is easy to show that
\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-m \varepsilon} \vartheta^\ast(\varepsilon) = a > 0.
\]
Thus, by (3.59), (3.23) and the dominated convergence theorem of Lebesgue, we complete the proof for this case.

Next, let us consider the case (b2). We now choose the functions \( f_i \) for all \( i = 1, \ldots, m \) as follows:
\[
f_i(x) = \begin{cases} 0, & \text{if } |x| < \rho^{-1}_A, \\ |x|^{-\alpha_i} - \frac{n}{q_i(x)} - \gamma_i - \varepsilon, & \text{otherwise}. \end{cases}
\]
Thus, we have
\[
F_{q_i}(f \omega_i, \chi_k) = \int_{C^k} |x|^{-\alpha_i} q_i(x) \, dx = \int_{2^{k-1}}^2 \int_{S^{n-1}} r^{-\alpha_i} q_i(r x') r dx' \, dr.
\]
Hence, by letting $k \leq 0$, $F_{q_i}(f_{i\omega_i},\chi_k)$ is controlled as follows

$$\int_{2^{k-1}S^{n-1}}\int r^{-\langle \alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+1}^{-1}} d\sigma(x')dr \lesssim 2^{-k(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+1} - 1}$$

As a consequence of the above estimate, by (2.1), we get

$$\left\| f_i \chi_k \right\|_{L^q_{q_i}(\varepsilon)} \lesssim \left( \eta_{j+} \right)^{\frac{1}{p_i}} 2^{-k(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+} - 1}, \quad (3.60)$$

where $\eta_{j+} = \frac{2^{\langle \|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+} - 1}}{q_{i+}(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+} - 1)}$. Otherwise, by the similar argument as above, we also obtain

$$\left\| f_i \chi_k \right\|_{L^q_{q_i}(\varepsilon)} \lesssim \left( \eta_{j-} \right)^{\frac{1}{p_i}} 2^{-k(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1}, \quad (3.61)$$

where $\eta_{j-} = \frac{2^{\langle \|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1}}{q_{i-}(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1)}$. From defining $\rho$ and assuming $\alpha_i(0) = \alpha_{i\infty}$, by Proposition 3.8 in [4] again, we get

$$\left\| f_i \right\|_{K^{\alpha_i(0),p_i}_{q_i(\varepsilon)}} \leq \left\{ \sum_{k=0}^{\infty} 2^{k(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+} - 1} \right\}^{\frac{1}{p_i}} \left\{ \sum_{k=1}^{\infty} 2^{k(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1} \right\}^{\frac{1}{p_i}} \quad (3.62)$$

Notice that, from assuming in this case, we deduce

$$\alpha_i(0) - (\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+} - 1) < 0, \quad \text{for all } \varepsilon \in \mathbb{R}^+.$$ 

Thus, by (3.60)-(3.62), $\left\| f_i \right\|_{K^{\alpha_i(0),p_i}_{q_i(\varepsilon)}}$ is dominated by

$$\left( \frac{2^{(-p+1)p_i(\alpha_i(0))+(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+} - 1)}}{2^{p_i(\alpha_i(0))+(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1)} - 1} \right)^{\frac{1}{p_i}} + \eta_{j-}^{\frac{1}{p_i}} \left( \frac{2^{p_i(\alpha_i(0))-(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1)}}{1 - 2^{p_i(\alpha_i(0))-(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1)}} \right)^{\frac{1}{p_i}}.$$ 

This implies that

$$\left\| f_i \right\|_{K^{\alpha_i(0),p_i}_{q_i(\varepsilon)}} \lesssim \frac{I_i(\varepsilon)}{\left( 1 - 2^{p_i(\alpha_i(0))-(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1)} \right)^{\frac{1}{p_i}}} \quad (3.63)$$

where

$$I_i(\varepsilon) = \frac{\frac{1}{\eta_{j+}} \left( 2^{(-p+1)p_i(\alpha_i(0))+(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i+} - 1)} \right)^{\frac{1}{p_i}}}{\left( 1 - 2^{p_i(\alpha_i(0))-(\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1)} \right)^{\frac{1}{p_i}}} + \eta_{j-}^{-\alpha_i(0) - (\|\alpha_i, \|L_{\infty} + \varepsilon\rangle q_{i-} - 1)}.$$
On the other hand, by the similar estimating as (3.18), we also obtain

\[ H_{\Phi, A}(f^\sim)(x) \geq \left( \int_U \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m |A_i(t,x)|^{-\alpha_i L_\infty - \frac{n}{q_i} \gamma_i - n^{-\epsilon}} dt \right) \chi_{\mathbb{R}^n \setminus B(0,\epsilon^{-1})}(x). \]

For convenience, we put

\[ \Gamma^*_\epsilon = \int_U \frac{\Phi(t)}{|t|^n} \prod_{i=1}^m \min \left\{ \| A_i^{-1}(t) \|_{L_\infty}^{-\frac{n}{q_i} \gamma_i}, \| A_i^{-1}(t) \|_{L_\infty}^{-\frac{n}{q_i} \gamma_i - \frac{n}{q_i} \epsilon} \right\} \| A_i^{-1}(t) \|_{L_\infty}^{\alpha_i} dt. \]

From this, by (3.3), it is not hard to see that

\[ H_{\Phi, A}(f^\sim)(x) \geq \Gamma^*_\epsilon |x|^{-\left( \sum_{i=1}^m \| \alpha_i L_\infty \right) - \frac{n}{q_i} \gamma_i - \frac{n}{q_i} \epsilon} \chi_{\mathbb{R}^n \setminus B(0,\epsilon^{-1})} =: \Gamma^*_\epsilon g(x), \]

where we denote \( g(x) = |x|^{-\left( \sum_{i=1}^m \| \alpha_i L_\infty \right) - \frac{n}{q_i} \gamma_i - \frac{n}{q_i} \epsilon} \chi_{\mathbb{R}^n \setminus B(0,\epsilon^{-1})}. \)

Since \( \alpha(0) = \alpha_{\infty} \), we deduce that

\[ \left\| H_{\Phi, A}(f^\sim) \right\|_{K_{\Phi(\cdot), \omega}^{\alpha(\cdot), p}} \geq \Gamma^*_\epsilon \left( \sum_{k=k_0}^{k} 2^{k_0 \alpha(0)p} \| g\chi_k \|_{L_\omega^{p'}} \right)^{\frac{1}{p}}, \] \hspace{2cm} (3.64)

by using Proposition 3.8 in [4] again. Here we recall that \( k_0 \) is the smallest integer number so that \( 2^{k_0-1} \geq \epsilon^{-1} \). Let us now show that

\[ \left( \sum_{k=k_0}^{k} 2^{k_0 \alpha(0)p} \| g\chi_k \|_{L_\omega^{p'}} \right)^{\frac{1}{p}} \geq \eta^*_x \left( \frac{2}{\left(1 - 2^{k_0 \alpha(0) - \left( \sum_{i=1}^m \| \alpha_i L_\infty + \epsilon m \right) q_+ \right)} \right)^{\frac{1}{p}}, \] \hspace{2cm} (3.65)

where \( \eta^*_x = \frac{1}{\left( \sum_{i=1}^m \| \alpha_i L_\infty + \epsilon m \right) q_+}. \) Indeed, by \( k \geq k_0 > 1 \), we get

\[ F_x(g\omega, \chi_k) = \int_{C^k} |x|^{-\left( \sum_{i=1}^m \| \alpha_i L_\infty + \epsilon m \right) q_+ - n} dx \]

\[ \geq \int_{2^{k-1}}^{2^k} \int_{S^{n-1}} r^{-\left( \sum_{i=1}^m \| \alpha_i L_\infty + \epsilon m \right) q_+ - 1} d\sigma(x') dr \geq \eta^*_x \cdot 2^{-k \left( \sum_{i=1}^m \| \alpha_i L_\infty + \epsilon m \right) q_+}. \]

Thus, by (2.1), we have \( \| g\chi_k \|_{L_\omega^{p'}} \geq \eta^*_x \cdot 2^{-k \left( \sum_{i=1}^m \| \alpha_i L_\infty + \epsilon m \right) q_+}. \) This finishes the proof of the estimation (3.65).
Now, we define
\[
\psi^*(\varepsilon) = \frac{1}{n+2} \prod_{i=1}^{n} (1 - 2^{\alpha_i(0)} - \frac{\sum_{i=1}^{n} \alpha_i}{\sum_{i=1}^{n} (\|\alpha_i\|_{L^\infty} + \varepsilon)})^{\frac{1}{p_{i+1}}}. 
\]

By (3.63)-(3.65), we estimate
\[
\|H_{\Phi,A}(\tilde{f})\|_{K^{\alpha(\cdot),p}_{q_i(\cdot)}(\omega)} \geq \varepsilon^{-m_\varepsilon} \psi^*(\varepsilon) \cdot \left( \int_{\mathbb{R}^n} \Phi(t)^{-1} m \prod_{i=1}^{m} \|A_i^{-1}(t)\|^\frac{1}{p_{i+1}} \right) \cdot \prod_{i=1}^{m} \|f_i\|_{K^{\alpha_i(\cdot),p_i}_{q_i(\cdot)}(\omega_i)}.
\]

Because of assuming \(\alpha_i(0) < \|\alpha_i\|_{L^\infty} q_{i+1}\), we have \(\alpha(0) < \sum_{i=1}^{m} \|\alpha_i\|_{L^\infty}\). From this, the limit of function \(\varepsilon^{-m_\varepsilon} \psi^*\) is a positive number when \(\varepsilon\) tends to zero. Therefore, by (3.23), (3.66) and the dominated convergence theorem of Lebesgue, we obtain
\[
\|H_{\Phi,A}(\tilde{f})\|_{K^{\alpha(\cdot),p}_{q_i(\cdot)}(\omega)} \geq C_6 \cdot \prod_{i=1}^{m} \|f_i\|_{K^{\alpha_i(\cdot),p_i}_{q_i(\cdot)}(\omega_i)},
\]
which ends the proof for this case.

When all of \(\alpha_1(\cdot), \ldots, \alpha_m(\cdot)\) and \(q_1(\cdot), \ldots, q_m(\cdot)\) are constant, we obtain the following useful result which is seen as an extension of Theorem 3.1 and Theorem 3.2 in the work [14] to the case of matrices having property (3.3) as mentioned above.

**Theorem 3.7.** Let \(\omega(x) = |x|^\gamma, \gamma_1, \ldots, \gamma_m \in \mathbb{R}, \lambda_1, \ldots, \lambda_m \in \mathbb{R}^+, \alpha_1, \ldots, \alpha_m \in \mathbb{R}, 1 \leq q_i, q < \infty, 0 < p_i, p < \infty\) and \(\omega_i(x) = |x|^{\gamma_i}\) for all \(i = 1, \ldots, m\).

Simultaneously, let
\[
\frac{\gamma}{q} = \frac{\gamma_1}{q_1} + \cdots + \frac{\gamma_m}{q_m}.
\]

Then \(H_{\Phi,A}\) is a bounded operator from \(M K^{\alpha_1,\lambda_1}_{p_1,q_1}(\omega_1) \times \cdots \times M K^{\alpha_m,\lambda_m}_{p_m,q_m}(\omega_m)\) to \(M K^{\alpha,\lambda}_{p,q}(\omega)\) if and only if
\[
C_7 = \int_{\mathbb{R}^n} \Phi(t)^{-1} m \prod_{i=1}^{m} \|A_i^{-1}(t)\|^{-\lambda_i + \alpha_i + \frac{n + \gamma_i}{q_i}} dt < +\infty.
\]

Moreover,
\[
\|H_{\Phi,A}\|_{M K^{\alpha_1,\lambda_1}_{p_1,q_1}(\omega_1) \times \cdots \times M K^{\alpha_m,\lambda_m}_{p_m,q_m}(\omega_m) \rightarrow M K^{\alpha,\lambda}_{p,q}(\omega)} \simeq C_7.
\]
Proof. It is clear to see that the results of Theorem 3.7 can be viewed as consequence of Theorem 3.5. Indeed, we put \( \gamma^* = \frac{\gamma}{m} \) for \( i = 1, \ldots, m \) and \( \omega^* = |x|^{\gamma^*}, \omega_i^* = |x|^{\gamma_i} \) for \( i = 1, \ldots, m \). By having (3.8) and assuming that \( \alpha_1(\cdot), \ldots, \alpha_m(\cdot) \) and \( q_1(\cdot), \ldots, q_m(\cdot) \) are constant, we have

\[
C_5 = C_5^* = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \left\| A_i^{-1}(t) \right\|^{-\lambda_i + \alpha_i + \frac{m}{m} + \gamma^*_i} dt = C_7.
\]

Thus, combining the case (a) and case (b1) of Theorem 3.5, we deduce

\[
\left\| H_{\Phi, \vec{A}} \right\|_{MK_{p_1,q_1}^{\alpha_1,\lambda_1} \times \cdots \times MK_{p_m,q_m}^{\alpha_m,\lambda_m} \rightarrow MK_{p,q}^{\alpha,\lambda}} \simeq C_7.
\]

At this point, by relation (2.2), we immediately get the desired result. \( \square \)

As a consequence of Theorem 3.6, we also obtain the analogous result for the constant parameters case as follows.

**Theorem 3.8.** Let \( 1 \leq p, p_1, \ldots, p_m < \infty \), the assumptions of Theorem 3.7 and the hypothesis (3.33) in Theorem 3.4 hold. We have that \( H_{\Phi, \vec{A}} \) is a bounded operator from \( K_{q_1}^{\alpha_1,p_1}(\omega_1) \times \cdots \times K_{q_m}^{\alpha_m,p_m}(\omega_m) \) to \( K_q^{\alpha,p}(\omega) \) if and only if

\[
C_8 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \left\| A_i^{-1}(t) \right\|^{\alpha_i + \frac{m}{m} + \gamma^*_i} dt < +\infty.
\]

Furthermore,

\[
\left\| H_{\Phi, \vec{A}} \right\|_{MK_{q_1}^{\alpha_1,p_1}(\omega_1) \times \cdots \times MK_{q_m}^{\alpha_m,p_m}(\omega_m) \rightarrow MK_{q}^{\alpha,p}(\omega)} \simeq C_8.
\]

**Proof.** By putting \( \gamma^*, \gamma_1^*, \ldots, \gamma_m^* \), \( \omega^*, \omega_1^*, \ldots, \omega_m^* \) above, it is not hard to see that \( C_6 = C_6^* = C_8 \). Therefore, by using case a, case b1 of Theorem 3.6 and the relation (2.2), we finish the proof of this theorem. \( \square \)

Now, let us take measurable functions \( s_1(t), \ldots, s_m(t) \neq 0 \) almost everywhere in \( \mathbb{R}^n \). We consider a special case that the matrices \( A_i(t) = \text{diag}[s_{i1}(t), \ldots, s_{in}(t)] \) with \(|s_{i1}| = \cdots = |s_{in}| = |s_i|\), for almost everywhere \( t \in \mathbb{R}^n \), for all \( i = 1, \ldots, m \). It is obvious that the matrices \( A_i(t) \)'s satisfy the condition (3.3). Therefore, since the Lebesgue space with power weights is a special case of the Herz space, we also obtain the following corollary.

**Corollary 3.9.** Let \( 1 \leq p, p_1, \ldots, p_m < \infty \), \( \alpha_1, \ldots, \alpha_m \in \mathbb{R} \), and the hypothesis (3.33) in Theorem 3.4 is true. Then \( H_{\Phi, \vec{A}} \) is a bounded operator from \( L^p(|x|^{\alpha_1} dx) \times \cdots \times L^p(|x|^{\alpha_m} dx) \) to \( L^p(|x|^{\alpha} dx) \) if and only if

\[
C_9 = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} |s_i(t)|^{-\alpha_i - \frac{m}{m}} dt < +\infty.
\]
Furthermore, 
\[ \|H_{\Phi, A}\|_{L^{p_1}(|x|^{\alpha_1}dx) \times \cdots \times L^{p_m}(|x|^{\alpha_m}dx) \rightarrow L^p(|x|^\gamma dx)} = C_9. \]

Proof. By the assumption of the matrices $A_i$’s, it is easy to see that 
\[ |A_i(t)x|^\alpha = |s_i(t)|^\alpha |x|^\alpha, \text{ for all } \alpha \in \mathbb{R}, i = 1, \ldots, m. \]
Hence, we immediately obtain the desired result. \qed

By the relation between the Hausdorff operators and the Hardy-Cesàro operators as mentioned in Section 1, we see that Corollary 3.9 extends and strengthens the results of Theorem 3.1 in [26] with power weights.

Let us now assume that $q(\cdot)$ and $q_i(\cdot) \in \mathcal{P}_\infty(\mathbb{R}^n)$, $\lambda, \gamma, \alpha_i, \lambda_i, \gamma_i$ are real numbers such that $\lambda_i \in \left(\frac{1}{q_i(\infty)}, 0\right)$, $\gamma_i \in (-n, \infty)$, $i = 1, 2, \ldots, m$ and
\[
\frac{1}{q_1(\cdot)} + \frac{1}{q_2(\cdot)} + \cdots + \frac{1}{q_m(\cdot)} = \frac{1}{q(\cdot)},
\]
\[
\frac{\gamma_1}{q_1(\infty)} + \frac{\gamma_2}{q_2(\infty)} + \cdots + \frac{\gamma_m}{q_m(\infty)} = \frac{\gamma}{q(\infty)},
\]
\[
\frac{n + \gamma_1}{n + \gamma} \lambda_1 + \frac{n + \gamma_2}{n + \gamma} \lambda_2 + \cdots + \frac{n + \gamma_m}{n + \gamma} \lambda_m = \lambda,
\]
\[
\alpha_1 + \cdots + \alpha_m = \alpha.
\]

We are also interested in the multilinear Hausdorff operators on the product of weighted $\lambda$-central Morrey spaces with variable exponent. We have the following interesting result.

**Theorem 3.10.** Let $\omega_1(x) = |x|^{\gamma_1}, \ldots, \omega_m(x) = |x|^{\gamma_m}$, $\omega(x) = |x|^\gamma$ and $v_1(x) = |x|^{\alpha_1}, \ldots, v_m(x) = |x|^{\alpha_m}$, $v(x) = |x|^\alpha$. In addition, the hypothesis (3.12) in Theorem 3.2 holds and the following condition is true:
\[
C_{10} = \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^\alpha} \prod_{i=1}^m \|A_i(t)\|^{(n+\gamma_i)(\frac{1}{q_i(\infty)}+\lambda_i)} c_{A_i, q_i, \alpha_i}(t) \|1\|_{L^{p_i}(\cdot)} dt < +\infty, \quad (3.68)
\]
Then, we have $H_{\Phi, \lambda}$ is bounded from $B_{\omega_1, v_1}^{q_1(\cdot), \lambda_1} \times \cdots \times B_{\omega_m, v_m}^{q_m(\cdot), \lambda_m}$ to $B_{\omega, v}^{q(\cdot), \lambda}$.

Proof. For $R > 0$, we denote
\[
\Delta_R = \frac{1}{\omega(B(0, R))^\frac{1}{\gamma} + \lambda} \|H_{\Phi, \lambda}(\tilde{f})\|_{L^q(\omega(B(0, R)))}.
\]
It follows from using the Minkowski inequality for the variable Lebesgue space that
\[
\Delta_R \lesssim \int_{\mathbb{R}^n} \frac{1}{\omega(B(0, R))^\frac{1}{\gamma} + \lambda} \Phi(t) \prod_{i=1}^m f_i(A_i(t)) \|1\|_{L^{p_i}(\omega(B(0, R)))} dt. \quad (3.69)
\]
Corollary 3.11. Let condition holds: In view of $n$ $H$ have that By estimating as (3.10) and (3.14), we have the analogous result for the space to obtain

\[ \left\| \prod_{i=1}^{m} f_i(A_i(t).) \right\|_{L_q^p(B(0,R))} \lesssim \prod_{i=1}^{m} \left\| f_i(A_i(t).) \right\|_{L_q^p(B(0,R))}. \]  

(3.70)

By estimating as (3.10) and (3.14), we have

\[ \left\| f_i(A_i(t).) \right\|_{L_q^p(B(0,R))} \lesssim C_{A_i,q_i,\alpha_i(t).} \left\| f_i \right\|_{L_q^p(B(0,R)||A_i(t)||)}. \]  

(3.71)

In view of $\frac{n+\gamma_1}{n+\gamma} \lambda_1 + \frac{n+\gamma_2}{n+\gamma} \lambda_2 + \cdots + \frac{n+\gamma_m}{n+\gamma} \lambda_m = \lambda$, we estimate

\[ \frac{1}{\omega(B(0,R))^{\frac{1}{n+\gamma}}} \lesssim \frac{\left\| A_i(t) \right\|_{(q_i+\gamma)(\frac{1}{q_i}+\lambda_i)}}{\omega_i(B(0,R)||A_i(t)||)^{\frac{1}{n+\gamma}+\lambda_i}}. \]

Thus, by (3.69) and (3.71), it follows that $\Delta_R \lesssim C_{10} \prod_{i=1}^{m} \left\| f_i \right\|_{B_{\omega_i,v_i}^{q_i,\lambda_i}}$. Consequently, it is straightforward

\[ \left\| H_{\Phi,A}(\vec{f}) \right\|_{B_{\omega,v}^{q,\lambda}} \lesssim C_{10} \prod_{i=1}^{m} \left\| f_i \right\|_{B_{\omega_i,v_i}^{q_i,\lambda_i}}. \]

As a consequence of Theorem 3.10 by the reason (3.2) and (3.4), we also have the analogous result for the $q, q_1, \ldots, q_m$-constant case as follows.

Corollary 3.11. Let $\omega_i, v_i, \omega, v$ be as Theorem 3.10. In addition, the following condition holds:

\[ C_{11} = \int_{\mathbb{R}^n} \frac{\Phi(t)}{t^n} \prod_{i=1}^{m} \left\| A_i^{-1}(t) \right\|_{(q_i+\gamma)(\frac{1}{q_i}+\lambda_i)} \frac{dt}{t} < +\infty. \]  

(3.72)

Then, we have

\[ \left\| H_{\Phi,A}(\vec{f}) \right\|_{B_{\omega,v}^{q,\lambda}} \lesssim C_{11} \prod_{i=1}^{m} \left\| f_i \right\|_{B_{\omega_i,v_i}^{q_i,\lambda_i}}. \]

Proof. By (3.12) and (3.14), it is clear to see that

\[ \left\| A_i(t) \right\|_{(q_i+\gamma)(\frac{1}{q_i}+\lambda_i)} c_{A_i,q_i,\alpha_i(t)} \lesssim \left\| A_i^{-1}(t) \right\|_{(q_i+\gamma)(\frac{1}{q_i}+\lambda_i)} \]

Hence, by Theorem 3.10, the proof is finished.}

Moreover, we also obtain the above operator norm on the product of weighted $\lambda$-central Morrey spaces as follows.

Theorem 3.12. Let $\omega(x) = |x|^{\gamma}$ and $\omega_i(x) = |x|^{\gamma_i}$ for $i = 1, \ldots, m$. Then, we have that $H_{\Phi,A}$ is bounded from $B_{q_1,\lambda_1}(\omega_1) \times \cdots \times B_{q_m,\lambda_m}(\omega_m)$ to $B_{q,\lambda}(\omega)$ if and only if

\[ C_{12} = \int_{\mathbb{R}^n} \frac{\Phi(t)}{t^n} \prod_{i=1}^{m} \left\| A_i^{-1}(t) \right\|_{(q_i+\gamma)(\frac{1}{q_i}+\lambda_i)} dt < +\infty. \]
**Corollary 3.11.** In more details, by letting $\alpha_i = \frac{\gamma_i}{q_i}$ for $i = 1, \ldots, m$, we have $C_1 = C_2 < \infty$. Hence, by Corollary 3.11 we find

$$\|H_{\Phi, \mathbf{A}}(\mathbf{f})\|_{B^{\eta, \lambda}(\omega)} \lesssim C_2 \cdot \prod_{i=1}^{m} \|f_i\|_{B^{\eta_i, \lambda_i}(\omega)}.$$

From this, by $B^{q, \lambda}(\omega)$ and $B^{q_i, \lambda_i}(\omega)$ for $i = 1, \ldots, m$, the proof of sufficient condition of this theorem is ended.

To give the proof for the necessary condition, let us now choose $f_i(x) = |x|^{(n+\gamma_i)\lambda_i}$.

Then, it is not hard to show that

$$\|f_i\|_{B^{q_i, \lambda_i}(\omega)} = \left(\frac{n + \gamma_i}{|S_{n-1}|}\right)^{\lambda_i} \frac{1}{(1 + q_i \lambda_i)^{\frac{1}{q_i}}}.$$

Thus, we have

$$\prod_{i=1}^{m} \|f_i\|_{B^{q_i, \lambda_i}(\omega)} \lesssim \left(\frac{\gamma + n}{|S_{n-1}|}\right)^{\lambda} (1 + q^\frac{1}{q}).$$

(3.73)

By choosing $f_i$'s, we also have

$$\|H_{\Phi, \mathbf{A}}(\mathbf{f})\|_{B^{q, \lambda}(\omega)} = \sup_{R > 0} \left(\frac{1}{\omega(B(0, R))^{1+q\lambda}} \int_{B(0, R)} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} |A_i(t)x|^{(n+\gamma_i)\lambda_i} \, dt \right)^{\frac{q}{q}}.$$

By (3.5), we get $|A_i(t)x|^{(n+\gamma_i)\lambda_i} \gtrsim |A_i^{-1}(t)|^{-(n+\gamma_i)\lambda_i} |x|^{(n+\gamma_i)\lambda_i}$. Therefore, we imply that

$$\|H_{\Phi, \mathbf{A}}(\mathbf{f})\|_{B^{q, \lambda}(\omega)} \gtrsim \left( \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} |A_i^{-1}(t)|^{-(n+\gamma_i)\lambda_i} \, dt \right) \times$$

$$\times \sup_{R > 0} \left(\frac{1}{\omega(B(0, R))^{1+q\lambda}} \int_{B(0, R)} \left( \prod_{i=1}^{m} |x|^{(n+\gamma_i)\lambda_i q} \right) |x|^{\gamma} \, dx \right)^{\frac{q}{q}}$$

$$= \left( \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} |A_i^{-1}(t)|^{-(n+\gamma_i)\lambda_i} \, dt \right) \left(\frac{\gamma + n}{|S_{n-1}|}\right)^{\lambda} (1 + q^\frac{1}{q}).$$
Hence, it follows from (3.73) that
\[
\left\| H_{\Phi, \vec{A}}(\vec{f}) \right\|_{B^{q, \lambda}(\omega)} \gtrsim \left( \int_{\mathbb{R}^n} \frac{\Phi(t)}{|t|^n} \prod_{i=1}^{m} \left\| A_i^{-1}(t) \right\|^{-(n+\gamma_i)\lambda_i} dt \right) \prod_{i=1}^{m} \| f_i \|_{B^{q_i, \lambda_i}(\omega_i)}.
\]
Because of assuming that \( H_{\Phi, \vec{A}} \) is bounded from \( B^{q_1, \lambda_1}(\omega_1) \times \cdots \times B^{q_m, \lambda_m}(\omega_m) \) to \( B^{q, \lambda}(\omega) \), it immediately deduces that \( C_{12} < \infty \), and hence, the proof of the theorem is completed. \( \square \)

**Acknowledgments.** This paper is supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.51.

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