Mathisson’s helical motions demystified

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Abstract

The motion of spinning test particles in general relativity is described by Mathisson-Papapetrou-Dixon equations, which are undetermined up to a spin supplementary condition, the latter being today still an open question. The Mathisson-Pirani (MP) condition is known to lead to rather mysterious helical motions which have been deemed unphysical, and for this reason discarded. We show that these assessments are unfounded and originate from a subtle (but crucial) misconception. We discuss the kinematical explanation of the helical motions, and dynamically interpret them through the concept of hidden momentum, which has an electromagnetic analogue. We also show that, contrary to previous claims, the frequency of the helical motions coincides exactly with the zitterbewegung frequency of the Dirac equation for the electron.

Keywords: Center of mass, Frenkel-Mathisson-Pirani spin condition, helical motions, hidden momentum, zitterbewegung.

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1 Introduction. Mathisson’s helical motions.

(Note: for an explanation of the notation herein, see [1].) In a multipole expansion, a body is represented by a set of moments of its energy-momentum tensor \( T^{\alpha\beta} \), taken about a reference worldline \( z^\alpha(\tau) \). Spinning pole-dipole particles correspond to truncating the expansion at dipole order; the equations of motion resulting from \( T^{\alpha\beta} = 0 \) involve only two moments of \( T^{\alpha\beta} \): the momentum \( P^\alpha \), and the angular momentum \( S^{\alpha\beta} \) (see definitions in [1]); and read for a free particle in flat spacetime:

\[
\frac{DP^\alpha}{d\tau} = 0 \quad (a); \quad \frac{DS^{\alpha\beta}}{d\tau} = 2P^{[\alpha}U^{\beta]} \quad (b); \quad P^\alpha = mU^\alpha - \frac{DS^{\alpha\beta}}{d\tau}U^\beta \quad (c) \tag{1}
\]

(1c) following from (1b); \( U^\alpha = dz^\alpha/d\tau \) and \( m \equiv -P^\alpha U_\alpha \) is the proper mass. There are three more unknowns than equations; to form a determined system, these equations require thus a supplementary condition, which amounts to specifying \( z^\alpha(\tau) \). Mathisson’s helical solutions [2] arise when one uses the condition \( S^{\alpha\beta}U_\alpha = 0 \). In this case \( S^{\alpha\beta} = \epsilon^{\alpha\beta\mu\nu}S_\mu U_\nu \); (1b) becomes \( P^\alpha = mU^\alpha + S^{\alpha\beta}a_\beta \), where
$a^\alpha = DU^\alpha/d\tau$; and $DS^\alpha/d\tau = 0$. The general solution of (1) under this condition describes the famous helical motions, which, in the $P^i = 0$ frame, correspond to clockwise (i.e. opposite to the spin direction) circular motions with radius $R = v_\gamma^2 S/m$ and speed $v$ on the $xy$ plane; taking their center as the spatial origin of the frame, they read:

$$z^\alpha(\tau) = \left(\gamma\tau, -R \cos\left(\frac{v_\gamma}{R}\tau\right), R \sin\left(\frac{v_\gamma}{R}\tau\right), 0\right)$$  \hspace{1cm} (2)

These motions were interpreted [2] (for the case of the electron) as the classical counterpart of Dirac’s equation ‘zitterbewegung’. However, the fact that $\gamma$ can be arbitrarily large has led some authors (see e.g. [3, 4]), to believe that according to (2) a given free body might move along circular trajectories with any radius; for this reason these solutions have been deemed unphysical. The same arguments were used to imply that the the frequency $\omega = m/\gamma^2 S$, for an electron, only coincides with Dirac’s zitterbewegung frequency $\omega = 2M_e/h$ in the limit $\gamma \to 1$. Both these assessments are misconceptions as we will see next.

2 Center of mass. Significance of the spin condition. Kinematical origin of the helical motions.

In order for (1) to be equations of motion for the body, $z^\alpha(\tau)$ must be taken as its representative point — its center of mass (CM); however, in relativity, the CM of a spinning body is an observer dependent point. This is illustrated in Fig. 1 of [1]. A spin condition of the type $S^\alpha u_\beta = 0$ (for some unit time-like vector $u^\alpha$) amounts to choosing $z^\alpha(\tau)$ as the center of mass $x^\alpha_{CM}(u)$ measured by the observer $O(u)$ of 4-velocity $u^\alpha$, see [1] for details. The Mathisson-Pirani condition $S^\alpha u_\beta = 0$ amounts to choosing for $z^\alpha$ the center of mass $x^\alpha_{CM}(U)$ as measured in its own rest frame, i.e., the frame $U^i = 0$. Such CM is dubbed a “proper center of mass”. It turns out that, contrary to what one might expect, such point is not unique. Let $x^\alpha_{CM}(P)$ be the CM measured in the $P^i = 0$ frame; for a free particle in flat spacetime it is one of the proper CM’s, corresponding to $R = 0$ in Eq. (2). This solution corresponds to uniform straightline motion. The center of mass $x^\alpha_{CM}(\bar{u})$ measured by an observer $O(\bar{u})$ moving with 3-velocity $\bar{v}$ in the $P^i = 0$ frame is shifted by a vector $\Delta x^i$, Eq. (3b), relative to $x^\alpha_{CM}(P)$. Hence the set of all possible CM’s measured by all observers $O(\bar{u})$ fills a disk of radius $R_{max} = S_\alpha/M$ centered at $x^\alpha_{CM}(P)$.

$$\Delta x^i = \left(\frac{S_\alpha \times \bar{v}^i}{M}\right) \hspace{1cm} (a); \hspace{1cm} \frac{D\Delta x^\alpha}{dt_P} = -\frac{S^\alpha_\beta}{M} \frac{Dv_\beta}{dt_p} \hspace{1cm} (b) \hspace{1cm} \Rightarrow \hspace{0.5cm} \frac{d\Delta x}{dt} = \frac{1}{M} S_\alpha \times \frac{d\bar{v}}{dt} \hspace{1cm} (c).$$ \hspace{1cm} (3)

Note: $M \equiv \sqrt{-P^\alpha P_\alpha}$; $S^\alpha_\beta$ is the angular momentum taken about $x^\alpha_{CM}(P)$. If $O(\bar{u})$ is inertial, $x^\alpha_{CM}(\bar{u})$ is a point at rest relative to $x^\alpha_{CM}(P)$, c.f. Eqs. (3b)-(3c); thus not at rest relative to $O(\bar{u})$, i.e., it is not a proper CM. But if $\bar{v}$ is not constant, then $x^\alpha_{CM}(\bar{u})$ acquires a non-trivial velocity $\bar{v}_{CM} = d\Delta x/dt$ (as measured in the $P^i = 0$ frame). If $O(\bar{u})$ itself moves with $\bar{v} = \bar{v}_{CM}$, i.e. if $\bar{v}$ is a solution of Eq. (3b), then it is a proper CM (i.e., it is a CM at rest relative to the frame where it is computed). The solutions (in the $P^i = 0$ frame) are circular motions in the plane orthogonal to $\bar{S}_\alpha$, with radius $R = \Delta x = |\bar{v} \times \bar{S}_\alpha|/M$, and constant (independent of $R$) angular velocity $\omega = -M/S_\alpha$ in opposite sense to the rotation of the body. These are precisely the solutions (2), and this is origin of the helical motions [5]. Hence their radius is not arbitrarily large; they are contained within the disk of CM’s, of radius $R_{max} = S_\alpha/M$; which is actually the minimum size a particle can have without violating
the dominant energy condition (i.e., without possessing matter/energy flowing faster than light). The latter implies \( \rho > |\vec{J}| \), where \( \rho \equiv T^{00} \) and \( J^i \equiv T^{0i} \); let \( b \) be the largest dimension of the body. Using the definition of \( S_{\alpha \beta} \) in [1], we may write, in the \( P^i = 0 \) frame:

\[
S_\ast = \left| \int \vec{v} \times \vec{J} d^3x \right| \leq \int r|\vec{J}|d^3x < \int \rho d^3x \leq Mb \iff b > \frac{S_\ast}{M} = R_{\text{max}}
\]

Thus the disk of CM’s, within which all the helical motions are contained, is always smaller than the body.

The misconception in the literature. — Different representations of the same extended body must yield the same moments \( (P^\alpha \) and \( S_{\alpha \beta} \) with respect to the same observer and the same reference worldline. As shown in [1], it is the quantities \( S_\ast = \gamma S \) and \( M = m/\gamma \), not \( m \) and \( S \) (which depend, via \( U^\alpha \) and \( z^\alpha \), respectively, on the particular helix chosen), that we must fix in order to ensure that we are dealing with the same particle. Thus, \( R = v^2 S/m = v S_\ast/M \leq R_{\text{max}} \), for all the helical representations corresponding to a given particle. Moreover, the frequency \( \omega = m/\gamma^2 S = M/S_\ast \) is the same for all helices corresponding to the same particle, and coincides exactly (even in the relativistic limit) with Dirac’s zitterbewegung frequency, identifying \( S_\ast = \hbar/2 \) and \( M = M_e \).

3 Dynamical Interpretation of the Helical Motions

We see from Eq. (3b) that the CM \( x_{\text{CM}}^\alpha(u) \) is not at rest in the \( \vec{F} = 0 \) frame when the 4-velocity \( u^\alpha \) of the observer measuring it changes; conversely, \( \vec{F} \) will not be zero in the CM frame (where, by definition, the particle is at rest); thus \( P^\alpha \) is not parallel to \( U^\alpha \), and the particle is said to possess hidden momentum [6]. This is a key concept for the understanding of the dynamics of the helical solutions; namely how the CM of a spinning particle can accelerate in the absence of any force without violating the conservation laws. Consider a generic spin condition \( S_{\alpha \beta} u_\beta = 0 \); contracting (1b) with \( u_\beta \), leads to

\[
S_{\alpha \beta} \frac{Du_\beta}{d\tau} = \gamma(u, U) P^\alpha - m(u) U^\alpha; \tag{4}
\]

where \( \gamma(u, U) \equiv -U^\beta u_\beta \) and \( m(u) \equiv -P^\beta u_\beta \). We split the momentum \( P^\alpha \) in two parts: “kinetic momentum” \( P_{\text{kin}}^\alpha = m U^\alpha \), which is the projection of \( P^\alpha \) along \( U^\alpha \); and the projection orthogonal to \( U^\alpha \), \( P_{\text{hid}}^\alpha \equiv (h U)^\alpha P^\beta \), which is the hidden momentum. Hence, if \( Du_\beta/d\tau = 0 \), that is, if we take as \( z^\alpha(\tau) \) the CM measured by an observer \( \mathcal{O}(u) \) such that \( u^\alpha \) is parallel transported along it (e.g., an inertial observer in flat spacetime), then \( P_{\text{hid}}^\alpha \parallel U^\alpha \), and \( P_{\text{hid}}^\alpha = 0 \). Otherwise, \( P_{\text{hid}}^\alpha \neq 0 \) in general. This is reciprocal to Eq. (3b); one can obtain one effect from the other, see [1]. Notice the important message encoded herein: in relativity, the motion of a spinning particle is not determined by the force laws given the initial position and velocity; one needs also to determine the field of vectors \( u^\alpha \) relative to which the CM is computed; the variation of \( u^\alpha \) along \( z^\alpha(\tau) \) is enough to possibly cause the CM to accelerate, even in the absence of any force; in this case the variation of \( P_{\text{kin}}^\alpha \) is compensated by an opposite variation of \( P_{\text{hid}}^\alpha \), keeping \( P^\alpha \) constant. If \( u^\alpha \) varies in a way such that the signal in Eq. (3b) oscillates, we may have a bobbing; or if it is such that \( \mathcal{O}(u) \) sees its CM to be at rest (\( u^\alpha = U^\alpha \), i.e, its 3-velocity \( \vec{v} \), in the frame \( P^i = 0 \), is a solution of \( \vec{v} = \vec{v}_{\text{CM}} \), Eq. (3b)), so that the condition \( S_{\alpha \beta} u_\beta = S_{\alpha \beta} U_\beta = 0 \) is obeyed, then we have a helical solution. In this case \( P_{\text{hid}}^\alpha = S_{\alpha \beta} a_\beta = \epsilon_{\beta \gamma \delta} a_\beta S^\gamma U^\delta \), which in vector notation reads \( P_{\text{hid}} = -\vec{S} \times \vec{a} = \vec{S} \times \vec{U} \vec{G} \) where \( \vec{G} \) is the “gravitoelectric field” as measured in the CM frame [7]. This is formally analogous to the hidden momentum \( P_{\text{hid}}^\alpha = \epsilon_{\beta \gamma \delta} E^\gamma \mu^\beta U^\delta \) of electromagnetic
Figure 1: Hidden momentum provides dynamical interpretation for the helical motions: the acceleration results from an interchange between kinetic $P^\alpha_{\text{kin}} = mU^\alpha$ and hidden “inertial” momentum $P^\alpha_{\text{hid}} = S^\alpha{}_{\beta\gamma}a^\beta$, which occurs in a way that their variations cancel out at every instant, keeping $P^\alpha$ constant. This is made manifest in b) panel, representing the $\vec{P} = 0$ frame, wherein $\vec{P}_{\text{hid}} = \vec{a} \times U \vec{S} = -m\vec{U} = -\vec{P}_{\text{kin}}$. Panel c) represents an electromagnetic analogue [6]: a (negatively) charged test particle possessing magnetic dipole moment $\vec{\mu} = (\mu^x, \mu^y, 0)$, orbiting a cylindrical (positively) charged body. The cylinder is along the $z$ axis, and $\vec{E}$ is the electric field it produces (measured in the particle’s CM frame). The $z$ component of the force vanishes for this setup; hence $P^z = 0 = \text{constant}$. But the particle possesses a hidden momentum $\vec{P}_{\text{hid}} = \vec{\mu} \times U \vec{E}$: as it orbits the line charge, $\vec{P}_{\text{hid}}$ oscillates between positive and negative values along the $z$-axis, implying the particle to bob up and down in order to keep the total momentum along $z$ constant: $P^z = P^z_{\text{kin}} + P^z_{\text{hid}} = 0$. (Note however the important distinction: $\vec{a} \times U \vec{S}$, but not $\vec{\mu} \times U \vec{E}$, is pure gauge).

Concluding: there is nothing wrong with Mathisson-Pirani condition, it is as valid as any other of the (infinite) possible choices; and in some applications the most suitable one, see [7]. It is degenerate, and the helical solutions it allows for a free particle, in addition to the expected uniform straightline motion, are alternative and physically consistent descriptions of the motion (only more complicated): in the first case, we have $a^\alpha = P^\alpha_{\text{hid}} = 0$; in the second case we have an helix, but also $P^\alpha_{\text{hid}} \neq 0$ (the latter being pure gauge and the motion effects induced by it confined to the worldtube of CM’s).

References

[1] L. F. Costa, C. Herdeiro, J. Natário, M. Zilhão, Phys. Rev. D 85, 024001 (2012) [arXiv:1109.1019]
[2] M. Mathisson, Acta Phys. Pol. 6, 218 (1937)
[3] J. Weyssenhoff, A. Raabe, Acta Phys. Pol. 9, 7 (1947)
[4] W. G. Dixon, Il Nuovo Cimento, 38, 1616 (1965)
[5] C. Moller, Ann. Inst. Henri Poincaré 11, 251 (1949)

[6] S. E. Gralla, A. I. Harte, R. M. Wald, Phys. Rev. D 81, 104012 (2010) [arXiv:1004.0679]

[7] L. F. O. Costa, J. Natário, M. Zilhão, in preparation