Polling systems and multitype branching processes in random environment counted by random characteristics

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Abstract

By the methods of multitype branching processes in random environment counted by random characteristics we study the tail distribution of busy periods and some other characteristics of the branching type polling systems in which the service disciplines, input parameters and service time distributions are changing in a random manner.

Key words and phrases: polling systems, multitype branching processes in random environment, final product, busy period, random matrices

1 Polling systems with service policies of branching type

We consider a polling system consisting of a single server and $m$ stations with infinite-buffer queues indexed by $i \in \{1, \ldots, m\}$. Initially there are no customers in the system. When customers arrive to the system the server starts immediately the service by visiting the stations in cyclic order ($1 \rightarrow 2 \rightarrow \cdots \rightarrow m \rightarrow 1 \rightarrow \cdots$) starting at station 1 according to a selected service policy (to be described later on) and with zero switchover times between queues. Later on the initial stage of services ($1 \rightarrow 2 \rightarrow \cdots \rightarrow m$) will be called the zero cycle. The subsequent routes of the server will be called the first cycle, the second cycle and so on. When the system is empty the server waits their arrival at a parking place $R$. Customers arrive to the queues in accordance with a point process

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whose parameters are changing in a random manner each time when the server switches from station to station.

To give a rigorous description of the arrival and service processes for the system in question we need some notions. Let \( s := (s_1, \ldots, s_m) \in [0,1]^m \) be a \( m \)-dimensional variable,

\[
s^k := s_1^{k_1} \cdots s_m^{k_m}, \quad k_i \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},
\]
every \( s \) be a sequence selected in iid manner from \( F \) and let \( \sigma \)-algebra of \( \mathcal{F} \) be a sequence selected in iid manner from \( F \) and let \( \mathcal{F} = \{ \phi(s; \lambda) \} \) be the set of all such m.p.g.f. Assume that a probability measure \( \mathbb{P} \) is specified on the natural \( \sigma \)-algebra of \( \mathcal{F} \) and let

\[
\phi(i)(s; \lambda) := E \left[ s_1^{\theta_{11}} s_2^{\theta_{12}} \cdots s_m^{\theta_{im}} e^{-\lambda \phi_i} \right], \quad i = 1, \ldots, m
\]

be the mixed probability generating function (m.p.g.f.'s) of \( (\theta_{11}, \ldots, \theta_{im}; \phi_i) \), where \( \theta_{ij} \) are nonnegative integer-valued random variables and \( \phi_i \) is a nonnegative random variable. Denote

\[
\phi(s; \lambda) := \left( \phi^{(1)}(s; \lambda), \ldots, \phi^{(m)}(s; \lambda) \right)
\]

the respective vector-valued m.p.g.f. and let \( \mathcal{F} = \{ \phi(s; \lambda) \} \) be the set of all such m.p.g.f. Assume that a probability measure \( \mathbb{P} \) is specified on the natural \( \sigma \)-algebra of \( \mathcal{F} \) and let

\[
\phi_0(s; \lambda), \phi_1(s; \lambda), \phi_2(s; \lambda), \ldots,
\]

be a sequence selected in iid manner from \( \mathcal{F} \) in accordance with \( \mathbb{P} \), where

\[
\phi_n(s; \lambda) := \left( \phi^{(1)}(s; \lambda), \ldots, \phi^{(m)}(s; \lambda) \right)
\]

with

\[
\phi_n(i)(s; \lambda) := E \left[ s_1^{\theta_{11}(n)} s_2^{\theta_{12}(n)} \cdots s_m^{\theta_{im}(n)} e^{-\lambda \phi_i(n)} \right].
\]

In the present paper we investigate such polling systems whose arrival and service procedures of customers meet the following property.

**Branching property.** When the server arrives to station \( i \) for the \( n \)-th time and find there, say, \( k_i \) customers labelled \( 1, 2, \ldots, k_i \), then, during the course of the server’s visit, the arrival of customers is arranged in such a way that after the end of each stage of service of customer \( j \) (the number of such stages may be more than one if the service discipline admits feedback) the queues at the system will be increased by a random population of customers \( (\theta_{11}(n,j), \ldots, \theta_{im}(n,j)) \), where \( \theta_{ji}(n,j) \) is the number of customers added to the \( l \)-th station, and, in addition, a final product of size \( \phi_i(n,j) \geq 0 \) will be added to the system. It is assumed that the vectors \( (\theta_{11}(n,j), \ldots, \theta_{im}(n,j); \phi_i(n,j)) \), \( j = 1, 2, \ldots, k_i \) are iid and such that

\[
E \left[ s_1^{\theta_{11}(n,j)} s_2^{\theta_{12}(n,j)} \cdots s_m^{\theta_{im}(n,j)} e^{-\lambda \phi_i(n,j)} \right] = \phi^{(1)}_n(s; \lambda).
\]
Note that $\phi_n(s, \lambda)$ is a random m.p.g.f. and, therefore, the parameters of the polling system are changed from cycle to cycle in a random manner. Since in the paper we are interested in the characteristics related with busy periods of the system whose work starts by the arrival of a single customer to a station $J \in \{1, \ldots, m\}$ at moment 0, the law of arrival of customers to an idle system plays no role for subsequent arguments.

The aim of the article is to study the distribution of the total size of the final product accumulated in the system during a busy period of the server. In particular, letting the final product $\phi_i(n, j)$ be the service time of the $j$-th customer served during the $n$-th visit of the server to station $i$, we provide conditions under which the tail distribution of the length of the busy period decays, as $y \to \infty$, like $\text{const} \times y^{-\kappa}$ for some $\kappa > 0$.

Before we proceed to the rigorous statements of our results, consider two examples of branching type polling systems.

Let $T_+ = \{T\}$ be the set of all probability distributions of nonnegative random variables, $T_m^+ := \{(T_1, \ldots, T_m) : T_i \in T_+\}$ be the set of all $m$-dimensional tuples of such distributions, $\mathcal{M}_\varepsilon = \{\varepsilon\}$ be the set of all $m \times m$ matrices $\varepsilon = (\varepsilon_{ij})_{i,j=1}^m$ with nonnegative elements, and $\mathcal{M}_\gamma = \{\Gamma\}$ be the set of all $m \times (m+1)$ matrices $\Gamma = (\gamma_{ij})_{i=1,j=0}^m$ with nonnegative elements such that

$$\sum_{j=0}^m \gamma_{ij} = 1, \quad i = 1, \ldots, m.$$ 

Let $P$ be a measure on the Borel $\sigma$-algebra of the space $\mathcal{M}_\varepsilon \times \mathcal{M}_\gamma \times T_m^+$.

**Example 1** (motivated by [34]). Consider a polling system with $m$ stations and a single server performing cyclic service of the customers at the stations. Assume that initially there are no customers in the system and the server is located at parking place $R$. Assume that given the idle system the flow of customers arriving to station $i$ is Poisson with, say, a deterministic rate $\varepsilon_i$. When the first customer appears in the system the server selects a random element $(\varepsilon_0, \Gamma_0, T_0) \in \mathcal{M}_\varepsilon \times \mathcal{M}_\gamma \times T_m^+$ with

$$\varepsilon_0 = (\varepsilon_{ij}(0))_{i,j=1}^m, \quad \Gamma_0 = (\gamma_{ij}(0))_{i=1,j=0}^m, \quad T_0 = (T_{i0}, \ldots, T_{m0})$$

and immediately starts its zero service cycle ($1 \to 2 \to \cdots \to m$) adopting the gated server policy with zero switchover times. Namely, the server serves all the customers that were queueing at a station when the server arrived and then instantly jumps to the next (in cyclic order) station. For the period while the server performs the batch of services at station $i$, new customers arrive to the system according to independent Poisson flows with intensities given by the vector $(\varepsilon_{i1}(0), \varepsilon_{i2}(0), \ldots, \varepsilon_{im}(0))$ (some the components may be equal to zero) and the service times of customers are iid and distributed according to $T_{i0}(x) := P(\tau_i(0) \leq x)$. Each served customer either goes to station $j \in \{1, \ldots, m\}$ with probability $\gamma_{ij}(0)$ or leaves the system with probability $\gamma_{i0}(0)$ independently of other events. Besides, after the end of each service period of a customer the
customer contributes to the system its service time as the final product. It is assumed that given \((E_0, \Gamma_0, T_0)\) the service times and the arrival process of new customers are independent.

The subsequent routes \(n = 1, 2, \ldots\) have the same probabilistic structure specified by the tuples

\[
E_n = (\varepsilon_{ij}(n))_{i,j=1}^m, \quad \Gamma_n = (\gamma_{ij}(n))_{i=1,j=0}^m, \quad T_n = (T_{1n}, \ldots, T_{mn})
\]

with only difference that at the beginning of cycle \(n \geq 1\) there is a possibility to have more than one customer in the system.

Let us show that this system possesses a branching property. To this aim denote

\[
t_{in}(\lambda) := \int_0^\infty e^{-\lambda x} dT_{in}(x)
\]

the Laplace transform of the distribution \(T_{in}(x)\) of the random variable \(\tau_i(n)\) representing the service time of a customer at station \(i\) during the \(n\)-th visit of the server.

It is not difficult to check that the m.p.g.f. \(\phi_n(s;\lambda)\) has the components (in our setting and the service time of customers as the final product)

\[
\phi_n^{(i)}(s;\lambda) := \mathbb{E}\left[ s_1^{\varepsilon_{i1}(n)} s_2^{\varepsilon_{i2}(n)} \cdots s_m^{\varepsilon_{im}(n)} e^{-\lambda \tau_i(n)} \right]
\]

\[
= \int_0^\infty \mathbb{E}\left[ s_1^{\varepsilon_{i1}(n)} s_2^{\varepsilon_{i2}(n)} \cdots s_m^{\varepsilon_{im}(n)} \mid \tau_i(n) = x \right] e^{-\lambda x} dT_{in}(x)
\]

\[
= \left( \gamma_{i0}(n) + \sum_{j=1}^m \gamma_{ij}(n) s_j \right) \int_0^\infty \prod_{j=1}^m e^{\varepsilon_{ij}(n)(s_j-1)x} e^{-\lambda x} dT_{in}(x)
\]

\[
= \left( \gamma_{i0}(n) + \sum_{j=1}^m \gamma_{ij}(n) s_j \right) t_{in} \left( \lambda + \sum_{j=1}^m \varepsilon_{ij}(n)(1-s_j) \right)
\]

(1)

and

\[
h_n^{(i)}(s) := \phi_n^{(i)}(s;0) = \left( \gamma_{i0}(n) + \sum_{j=1}^m \gamma_{ij}(n) s_j \right) t_{in} \left( \sum_{j=1}^m \varepsilon_{ij}(n)(1-s_j) \right).
\]

(2)

Example 2 (compare with [34]). Consider the same polling system as earlier but assume now that at each station the server adopts the exhaustive server policy: it serves all the customers that were queueing at the station when the server arrived together with all subsequent arrivals up until the queue becomes empty and then instantly jumps to the next station.

Let us show that this system possesses a branching property as well.

Since during the \(n\)-th visit of the server to station \(i\) each customer at this station is served by the server a random number of times having shifted by 1
geometric distribution with parameter $\gamma_{ii}(n)$, the Laplace transform $w_{in}(\lambda)$ of the distribution of the random variable $\eta_i(n)$, the total service time of a customer at station $i$ during the $n$-th cycle, has the form

$$w_{in}(\lambda) = \frac{1 - \gamma_{ii}(n)}{1 - \gamma_{ii}(n)\lambda}.$$  

(3)

Let now $(\sigma_{i1}(n), \ldots, \sigma_{im}(n))$ be a random vector distributed as the vector of the number of customers arriving to the stations of the polling system during the total service time of a customer at queue $i$ which, after the end of its service at station $i$ moves either to a station $j \neq i$ or leaves the system. Denote

$$y_{in}(s) := \frac{\gamma_{ii0}(n) + \sum_{j \neq i} \gamma_{ij}(n)s_j}{1 - \gamma_{ii}(n)}.$$

Similarly to (1) one can show that

$$E \left[ s_{\sigma_{i1}(n)} \ldots s_{\sigma_{im}(n)} e^{-\lambda \tau_i(n)} \right] = y_{in}(s)w_{in} \left( \lambda + \sum_{j=1}^{m} \varepsilon_{ij}(n)(1-s_j) \right).$$  

(4)

Let now $l_{in}(\lambda)$ be the Laplace transform of the distribution of a busy period $\eta_{i,tot}(n)$ of the system generated by a single customer in an $M/G/1$ queue with arrival rate $\varepsilon_i(n)$ and the service time distributed as $\eta_i(n)$, and let $(\theta_{i1}(n), \ldots, \theta_{im}(n))$ be the total number of new customers arriving to the $i$-th stations of our polling system within the time-interval distributed as $\eta_{i,tot}(n)$. In this case $l_{in}(\lambda)$ is a unique solution of the equation

$$l_{in}(\lambda) = w_{in}(\lambda + \varepsilon_{ii}(n)(1-l_{in}(\lambda)))$$

and (compare with [36]) the functions

$$\phi_{n}^{(i)}(s;\lambda) := E \left[ s_{\theta_{i1}(n)}^{\sigma_{i1}(n)} \ldots s_{\theta_{im}(n)}^{\sigma_{im}(n)} e^{-\lambda \eta_{i,tot}(n)} \right], \quad i = 1, 2, \ldots, m$$

are (in our setting) unique solutions of the equations

$$\phi_{n}^{(i)}(s;\lambda) = y_{in}(s)w_{in} \left( \lambda + \sum_{j \neq i} \varepsilon_{ij}(n)(1-s_j) + \varepsilon_{ii}(n)(1-\phi_{n}^{(i)}(s;\lambda)) \right).$$

In particular,

$$h_{n}^{(i)}(s) = y_{in}(s)w_{in} \left( \sum_{j \neq i} \varepsilon_{ij}(n)(1-s_j) + \varepsilon_{ii}(n)(1-h_{n}^{(i)}(s)) \right).$$  

(5)

Note that in the framework of the suggested approach one can consider also models with batch arrivals of customers as well. In this case one should replace, for instance, everywhere in $t_{in}(\cdot)$ variables $s_1, \ldots, s_m$ by the respective
probability generating functions of the sizes of batches of customers arriving to stations $j = 1, \ldots, m$.

Since we imposed the branching property on the service disciplines of the systems it is not a surprise for the reader that we conduct our investigation by the methods of the theory of branching processes. This approach is not new. Polling systems possessing the branching property in which the probability generating functions $h_n^{(i)}(s)$, $n = 1, 2, \ldots$ are nonrandom and the same for all cycles were considered in particular, in [24], [36], [42] and quite recently in [9], [12] and [13]. These models cover many classical service policies, including the exhaustive, gated, binomial-gated and their feedback modifications (see surveys [4] and [35] for definitions and more details).

Polling systems with input parameters and service disciplines changing in a random manner and(or) depending on the states of systems are not studies yet in full generality. The analysis of such systems uses rather often the fluid method ([21]–[23]) or a method based on the construction of appropriate Lyapunov functions ([32] - [34]) (in the last case without reference to branching processes). The authors of [34] write that it is possible to generalize their models. However "...This leads to a lot of complexities in the proofs." In the present paper we show that the reduction of the problems related to the branching type polling systems with final product and evolving in random environment (BTPSFPRE) to the respective problems for the multitype branching processes counted by random characteristics and evolving in random environment (MBPRCRE) gives the desired answers by a unique method and in a general situation. Since m.p.g.f.'s are selected for each cycle independently of the past, different service disciplines are allowed at different stations in our model. Moreover, the service disciplines at the stations may be changed at random from visit to visit. In particular, we allow for the mixture of exhaustive and gated service disciplines. Here we consider the models with zero switchover times. The study of BTPSFPRE with positive (and random) switchover times may be conducted by a similar method. This, however, requires more efforts and will be done elsewhere.

The scheme of the remaining part of the paper looks as follows. Section 2 is devoted to a detailed description of multitype branching processes counted by random characteristics and evolving in random environment. In Section 3 we recall some known results for ordinary multitype branching processes in random environment (MBPRE) and formulate the main results of our paper describing the asymptotic behavior of the total size of the final product for subcritical MBPRCRE. Section 4 recalls an important statement related with the asymptotic properties of the tail distribution of infinite sums of products of random matrices. In Section 5 we deduce some estimates for the moments of the population sizes of subcritical MBPRE's. The proofs of the main results of our paper are collected in Section 7. And, finally, we demonstrate in Section 8 how one can use the obtained results for MBPRCRE to make conclusions about the probabilistic properties of various characteristics of the branching type polling systems.
2 Branching processes in random environment counted by random characteristics

As we claimed in the previous section, the main goal of the present paper is to analyze properties of busy periods and some other characteristics of the branching type polling systems by means of MBPRCRE. However, to start such analysis we need to pass through a relatively long way of notation and statements which is practically always the case when one consider multitype branching processes.

Let \( (\xi_1, \ldots, \xi_m; \varphi) \) be a \((m+1)\)-dimensional vector where the components \( \xi_1, \ldots, \xi_m \) are integer-valued nonnegative random variables and \( \varphi \) is a nonnegative random variable and let

\[
F(s; \lambda) = E \left[ s_1^{\xi_1} s_2^{\xi_2} \cdots s_m^{\xi_m} e^{-\lambda \varphi} \right], \quad s = (s_1, \ldots, s_m) \in [0, 1]^m, \lambda \geq 0,
\]

be the respective m.p.g.f. Denote \( F_\lambda := \{ F(s; \lambda), s \in [0, 1]^m \} \) the set of all such m.p.g.f.'s and let

\[
F^m_\lambda := F_\lambda \times F_\lambda \times \cdots \times F_\lambda = \left\{ F(s; \lambda) = (F^{(1)}(s; \lambda), \ldots, F^{(m)}(s; \lambda)) \right\}
\]

be the \( m \)-times direct product of \( F_\lambda \). Let, further,

\[
F_0 := \{ f(s) = F(s; 0), s \in [0, 1]^m \}
\]

be the set of all ordinary probability generating functions (p.g.f.'s)

\[
f(s) = E \left[ s_1^{\xi_1} s_2^{\xi_2} \cdots s_m^{\xi_m} \right]
\]

and

\[
F^m_0 := F_0 \times \cdots \times F_0 = \left\{ f(s) = (f^{(1)}(s), \ldots, f^{(m)}(s)) \right\}
\]

be the set of all \( m \)-dimensional (vector-valued) p.g.f.'s. Assume that a probability measure \( P \) is specified on the natural \( \sigma \)-algebra \( \mathcal{A} \) generated by the subsets of \( F^m_\lambda \). Let

\[
F_0(s; \lambda), F_1(s; \lambda), \ldots, F_k(s; \lambda), \ldots \text{ with } F_n(s; \lambda) := \left( F^{(1)}_n(s; \lambda), \ldots, F^{(m)}_n(s; \lambda) \right)
\]

be a sequence of vector-valued m.p.g.f.'s selected from \( F^m_\lambda \) in an iid manner in accordance with measure \( P \). The sequence \( \{F_n(s; \lambda), n \geq 0\} \) is called a random environment. With m.p.g.f. \( F^{(i)}_n(s; \lambda) \) we associate a random vector of offsprings \( \xi_i(n) = (\xi_{i1}(n), \xi_{i2}(n), \ldots, \xi_{im}(n)) \) and a random variable \( \varphi_i(n) \) such that

\[
F^{(i)}_n(s; \lambda) = E \left[ s_1^{\xi_{i1}(n)} s_2^{\xi_{i2}(n)} \cdots s_m^{\xi_{im}(n)} e^{-\lambda \varphi_i(n)} \right].
\]

Now we may give an informal description of the MBPRCRE

\[
R(n) = (Z(n); \Phi(n)), n = 0, 1, \ldots,
\]
which may be treated as the process describing the evolution of a $m$-type population of particles with accumulation of a final product.

The starting conditions of the process are: a vector (may be random) of particles $Z(0) = (Z_1(0), \ldots, Z_m(0))$ where $Z_i(0)$ denotes the number of particles of type $i \in \{1, \ldots, m\}$ in the process at moment 0, and an amount $\Phi(0)$ (may be random) of a final product. All the particles have the unit life length and just before the death produce children and final products independently of each other. For instance, a particle, say, of type $i$ produces particles of different types and adds some amount of the final product to the existing amount of the final product in accordance with m.p.g.f. $F_{i}^{(i)}(s;\lambda)$. The newborn particles constitute the first generation of the MBPRCRE, have the unit life-length and dying produce, independently of each other, offsprings and final products in accordance with their types and subject to the m.p.g.f. $F_{i}^{(i)}(s;\lambda), i = 1, 2, \ldots, m$ and so on.

A rigorous definition of the process we are interesting in looks as follows.

**Definition 3** A $(m+1)$-dimensional Galton-Watson branching process

$$ R_{\varphi}(n) = R(n) := (Z(n); \Phi(n)) = (Z_1(n), \ldots, Z_m(n); \Phi(n)), n \in \mathbb{N}_0, $$

counted by a random characteristics $\varphi$ in a fixed (but picked at random) environment $\{F_n(s;\lambda), n \geq 0\}$ is a time-inhomogeneous Markov process with the state space

$$ \mathbb{N}_0^m \times \mathbb{R}_0 := \{z = (z_1, \ldots, z_m; w), z_i \in \mathbb{N}_0; w \in [0, \infty)\} $$

defined as

$$ E_{F_0, \ldots, F_n}[s^{Z(n+1)}e^{-\lambda\Phi(n+1)} R(0), \ldots, R(n)] = e^{-\lambda\Phi(n)} (F_n(s;\lambda))^{Z(n)} . \tag{6} $$

Note that the initial value $R(0)$ may be random, and, for the reason of applications to queueing systems we do not exclude the case $z = 0$.

In what follows, to simplify notation, we write

$$ E_{F}[s^{Z(n+1)}e^{-\lambda\Phi(n+1)}] := E[s^{Z(n+1)}e^{-\lambda\Phi(n+1)} R(0), \ldots, R(n)] = e^{-\lambda\Phi(n)} (F_n(s;\lambda))^{Z(n)} , $$

$$ \text{or} $$

$$ R(n + 1) = (0; \Phi(n)) + \sum_{i=1}^{m} \sum_{k=1}^{Z_i(n)} (\xi_i(n;k); \varphi_i(n;k)) , $$

$$ \Phi(n + 1) = \Phi(0) + \sum_{i=0}^{n} \sum_{i=1}^{m} Z_i(l) \varphi_i(l;k) \tag{7} $$

8
where the random vector \((\xi_i(n; k); \varphi_i(n; k))\) represents the offspring vector and the size of the final product of the \(k\)-th particle of type \(i\) of the \(n\)-th generation of the process. Given the environment and \(n = 0, 1, \ldots\), and \(i \in \{1, \ldots, m\}\) the vectors
\[
(\xi_i(n; k); \varphi_i(n; k)), \quad k = 1, 2, \ldots, Z_i(n)
\]
are independent and identically distributed: \((\xi_i(n; k); \varphi_i(n; k)) \overset{d}{=} (\xi_i(n); \varphi_i(n))\).

Observe that if \(\Phi(0) = 0\) and \(\varphi_i(n; k) \equiv 1\) then \(\Phi(n)\) is the total number of particles born in the process within generations \(0, 1, \ldots, n - 1\); if \(\varphi_i(n; k) = I\{\sum_{j=1}^{m} \xi_{ij}(n; k) \geq t\}\) for some positive integer \(t\) (here and in what follows \(I\{A\}\) means the indicator of the event \(A\)) and \(\Phi(0) = 0\), then \(\Phi(n)\) is the total number of particles of all types in generations \(0, 1, \ldots, n - 1\) each of which had at least \(t\) children, and so on.

Letting \(\lambda = 0\) we arrive to the definition of the ordinary multitype branching process in random environment (MBPRE) which we call the underlying MBPRE for the initial MBPRCRE.

**Definition 4** A \(m\)-type Galton-Watson process
\[
Z(n) = (Z_1(n), \ldots, Z_m(n)), \quad n \in \mathbb{N}_0
\]
in a fixed (but selected at random) environment \(\{f_n(s), n \geq 0\}\) is a time-inhomogeneous Markov chain with the state space
\[
\mathbb{N}_0^m := \{z = (z_1, \ldots, z_m), z_i \in \mathbb{N}_0\}
\]
defined as
\[
Z(0) = z, \quad \mathbb{E}[s^{Z(n+1)}|f_0, \ldots, f_n, Z(0), \ldots, Z(n)] = (f_n(s))^{Z(n)}. \quad (8)
\]

To simplify notation we write
\[
\mathbb{E}_f[s^{Z(n+1)}] := \mathbb{E}[s^{Z(n+1)}|f_0, \ldots, f_n].
\]
It follows from (8) that \(Z(0) = z\) and
\[
Z(n + 1) := \sum_{i=1}^{m} \sum_{k=1}^{Z_i(n)} \xi_i(n; k)
\]
where \(\xi_i(n; k) \overset{d}{=} \xi_i(n), k = 1, 2, \ldots, Z_i(n)\), and, given the environment and \(n = 0, 1, \ldots\), and \(i \in \{1, \ldots, m\}\) the mentioned random vectors are independent.

Let
\[
A_n = (a_{ij}(n))_{i,j=1}^{m} := \left(\frac{\partial F_n^{(i)}(s, \lambda)}{\partial s_j} \bigg|_{s=1, \lambda=0}\right)_{i,j=1}^{m} = \left(\frac{\partial f_n^{(i)}(s)}{\partial s_j} \bigg|_{s=1}\right)_{i,j=1}^{m} \quad (9)
\]
be the mean matrix of the vector-valued p.g.f. \( f_n \) and
\[
C_n := (E_F \varphi_1(n), ..., E_F \varphi_1(n))' = \left( \frac{\partial F_n^{(1)}(s, \lambda)}{\partial \lambda} \bigg|_{s=1, \lambda=0}, ..., \frac{\partial F_n^{(m)}(s, \lambda)}{\partial \lambda} \bigg|_{s=1, \lambda=0} \right)'.
\]
(10)
By our assumptions the pairs \((A_n, C_n)\), \( n = 0, 1, \ldots \) are iid: \((A_n, C_n) \overset{d}{=} (A, C)\).
Suppose that
\[
E \log^+ \|A\| < \infty.
\]
(11)
It is known (see, for instance, [31]) that given condition (11) the limit
\[
\lim_{n \to \infty} \frac{1}{n} \log \|A_{n-1}A_{n-2} \cdots A_0\| =: \alpha
\]
exists with probability 1 and, moreover,
\[
\lim_{n \to \infty} \frac{1}{n} E \log \|A_{n-1}A_{n-2} \cdots A_0\| = \alpha.
\]
(13)
In what follows we call a MBPRE subcritical if \( \alpha < 0 \) and supercritical if \( \alpha > 0 \).
Recall that the single-type BPRE with iid offspring p.g.f.'s were introduced by Smith and Wilkinson in [37] and, in a more general setting, in [14], [15], [16] and have been investigated by many authors (see survey [39] for a list of references up to 1985 and [1], [2], [3], [7], [10], [11], [20], [25], [40] and [41] for some more recent results). MBPRE were analyzed, in particular, in [14], [30] and [38].
Ordinary single-type Galton-Watson branching processes counted by random characteristics where investigated by Sevastyanov [8] (for integer-valued \( \varphi(n; k) \)) and by Grishechkin [6] (for the general \( \varphi(n; k) \)). Grishechkin used the Galton-Watson and continuous time Markov branching processes counted by random characteristics (with or without immigration) to study queueing systems with processor sharing discipline [5].

3 Limit theorems for MBPRCRE

Introduce the notation
\[
\Pi_{l,n} := \prod_{i=l}^{n-1} A_i, \quad 1 \leq l \leq n,
\]
with the agreement that \( \Pi_{n,n} := E \) is the unit \( m \times m \) matrix. For vectors \( u = (u_1, \ldots, u_m), v = (v_1, \ldots, v_m)' \in \mathbb{R}^m \) denote
\[
\langle u, v \rangle := \sum_{k=1}^{m} u_k v_k
\]
their inner product.

For a \( m \times m \) matrix \( A = (a_{ij})_{i,j=1}^m \) and a \( m \)-dimensional vector \( u = (u_1, \ldots, u_m) \) introduce the norms

\[
\|A\| := \sum_{i,j=1}^m |a_{ij}|, \quad \|u\| := \sum_{i=1}^m |u_i|
\]

and

\[
\|A\|_2 := \sqrt{\sum_{i,j=1}^m |a_{ij}|^2}, \quad \|u\|_2 := \sqrt{\sum_{i=1}^m |u_i|^2}.
\]

Now we formulate an important statement concerning properties of MBPRE. Let

\[
q_i(f) := \lim_{n \to \infty} P_f(\|Z(n)\| = 0 | Z(0) = e_i)
\]

be the extinction probability of a MBPRE initiated at time 0 by a single individual of type \( i \) and

\[
q(f) := (q_1(f), \ldots, q_m(f)).
\]

**Theorem 5** ([38]) If the mean matrices of a MBPRE meet condition (11) and there exists a positive integer \( L \) such that

\[
P\left( \min_{1 \leq i,j \leq m} (A_{L-1}A_{L-2} \cdots A_0)_{ij} > 0 \right) = 1
\]

and \( 1 \leq l \leq m \) such that

\[
E |\log (1 - P_f(Z_l(L) = 0 | Z(0) = e_i))| < \infty,
\]

then, for \( \alpha \) specified by (13)

1) \( \alpha < 0 \) implies \( P_f(q(f) = 1) = 1 \);

2) \( \alpha > 0 \) implies \( P_f(q(f) < 1) = 1 \) and

\[
P_f\left( \lim_{n \to \infty} n^{-1} \log \|Z(n)\| = \alpha | Z(0) = e_i \right) = 1 - q_i(f)
\] (14)

with probability 1 for \( 1 \leq i \leq m \).

Let \( \tau \) be the extinction moment of a MBPRE which starts by (may be random) vector of the number of particles \( Z(0) \) with \( E \|Z(0)\| < \infty \). Clearly,

\[
P(\tau > n) = P(\|Z(n)\| \geq 1) \leq E \|Z(n)\| = E \|Z(0)A_0A_1 \cdots A_{n-1}\|
\]
\[
\leq E \|Z(0)\| E \|A_0A_1 \cdots A_{n-1}\|.
\]

Hence we see that if \( \alpha < 0 \) then for any \( \alpha^* \in (0, -\alpha) \) there exists a constant \( K_* = K_*(\alpha^*) \in (0, \infty) \) such that for each \( n = 0, 1, 2, \ldots \)

\[
P(\tau > n) \leq K_* e^{-\alpha^* n}.
\] (15)
Let
\[ \Phi := \lim_{n \to \infty} \Phi(n) = \Phi(0) + \sum_{n=0}^{\infty} \sum_{i=1}^{m} \sum_{k=1}^{Z_i(n)} \varphi_i(n; k) \]
be the total size of the final product produced by the particles of the MBPRCRE up to the extinction moment (if any). It is easy to see that if the underlying MBPRE is supercritical, satisfies conditions of Theorem 5 and, in addition, the final product of the MBPRCRE meets the condition
\[ P \left( \min_{1 \leq i \leq m} E_i \varphi_i(n) > 0 \right) > 0 \] (16)
then, by (14) and the law of large numbers, for each \( i = 1, \ldots, m \)
\[ P (\Phi = \infty | Z(0) = e_i) \geq P \left( \liminf_{n \to \infty} \left( \sum_{j=1}^{m} Z_j(n) \sum_{k=1}^{\varphi_j(n)} \right) > 0 \right) > 0. \] (17)

Note, finally, that if \( \Phi(0) = 0 \) and \( \varphi_j(n; k) \equiv 1 \) then
\[ \Phi := \lim_{n \to \infty} \Phi(n) = \sum_{n=0}^{\infty} \| Z(n) \| \]
is the total number of individuals ever existed in the MBPRCRE or, what is the same, in the underlying MBPRE.

For a given \( x \geq 0 \) set
\[ s(x) := \lim_{n \to \infty} (E \| A_{n-1} \cdots A_0 \|)^{1/n} = \lim_{n \to \infty} (E \| \Pi_{0,n} \|)^{1/n} \] (18)
and let
\[ s'(0) = \frac{1}{n} E \log \| A_{n-1} \cdots A_0 \| = \lim_{n \to \infty} \frac{1}{n} E \log \| \Pi_{0,n} \| \] (19)
be the top Lyapunov exponent for this sequence of matrices.

Denote \( D := \{ x > 0 : E \| A_0 \|^x < \infty \} \). It is known that the limits in (18) and (19) exist and, moreover, \( s(x) \) is a log-convex continuous function in \( D \) (see, for instance, [31]). Put
\[ \kappa := \inf \{ x > 0 : s(x) > 1 \} \] (20)
and \( \kappa = \infty \) if \( s(x) \leq 1 \) for all \( x > 0 \). Observe that \( s(0) = 1 \) and, therefore, \( \kappa = 0 \) if \( s'(0) > 0 \) and \( \kappa \in (0, \infty) \) if \( s'(0) < 0 \).

In the last case (which will be our main concern) the series \( \sum_{n=0}^{\infty} E \| \Pi_{0,n} \|^x \) converges if \( 0 < x < \kappa \) and diverges if \( x > \kappa \).

Introduce the set
\[ U_+ = \{ u = (u_1, \ldots, u_m) \in \mathbb{R}^m : u_i \geq 0, 1 \leq i \leq m, \| u \|_2 = 1 \} \]
and associate with the tuple \((A_n, C_n), n = 0, 1, 2, \ldots\) of iid pairs the series

\[
\Xi_l := \sum_{k=l}^{\infty} A_l A_{l+1} \ldots A_{k-1} C_k = \sum_{k=l}^{\infty} \Pi_{l,k} C_k, \ l = 0, 1, \ldots; \quad \Xi := \Xi_0.
\]

Our main results are established under the following hypothesis.

**Condition T.** There exist positive constants \(\kappa\) and \(K_0\) and a continuous strictly positive function \(l(u)\) on \(U_+\) such that for all \(u \in U_+\)

\[
\lim_{y \to \infty} y^\kappa P(\langle u, \Xi \rangle > y) = K_0 l(u).
\]

In Section 4 we list sufficient conditions on the distributions of the pairs \((A_n, C_n)\) which provide the validity of Condition T. These conditions are extracted from paper [28] where the behavior of the tail distribution of sums and products of random matrices were investigated.

The following theorem is the main result of the article.

**Theorem 6** Let a MBPRCRE satisfy the following hypotheses:

1) the underlying MBPRE is subcritical and meet conditions of Theorem 5;
2) for \(\kappa\) specified by (20) the following assumptions fulfill:
   - if \(\kappa > 1\) then
     \[
     \max_{1 \leq i \leq m} \mathbb{E} \left( \sum_{j=1}^{m} (\xi_{ij} - a_{ij}) \right)^\kappa < \infty \text{ and } \mathbb{E} \left( \sum_{i=1}^{m} (\varphi_i(n) - \mathbb{E}_F \varphi_i(n)) \right)^\kappa < \infty, \tag{21}
     \]
   - if \(\kappa \leq 1\) then
     \[
     \max_{1 \leq i \leq m} \mathbb{E} \left( \sum_{j=1}^{m} \text{Var}_F \xi_{ij} \right)^\kappa < \infty \text{ and } \mathbb{E} \left( \sum_{i=1}^{m} \text{Var}_F \varphi_i(n) \right)^\kappa < \infty; \tag{22}
     \]
3) there exists \(\delta > 0\) such that \(0 < \mathbb{E} \varphi_i^{\kappa + \delta}(n) < \infty, \quad i = 1, \ldots, m.\)

If, in addition, the mean matrix (9) and the vector (10) are such that Condition T is valid, then, as \(y \to \infty\)

\[
\mathbb{P}(\Phi > y) \sim Cy^{-\kappa}, \quad C \in (0, \infty).
\]

An evident corollary of Theorem 6 is the following statement.

**Theorem 7** If the conditions of Theorem 6 are valid then

\[
\mathbb{E} \Phi^x < \infty
\]

if and only if \(x < \kappa.\)
4 Auxiliary results

The proof of Theorem 6 is heavily based on Condition T whose validity is not easy to check. We list here a set of assumptions given in [28] which imply Condition T.

Let \( \Lambda(A) \) be the spectral radius of the matrix \( A \). The following statement is a refinement of a Kesten theorem from [28].

**Theorem 8** (see [18]) Let \( \{A_n, n \geq 0\} \) be a sequence of iid matrices generated by a measure \( \mathbb{P}_A \) with support concentrated on nonnegative matrices and \( A = (a_{ij})_{i,j=1}^m \). Assume that the following conditions are valid:

1) there exists \( \varepsilon > 0 \) such that \( E \|A\|^\varepsilon < \infty \);
2) \( A \) has no zero rows a.s.;
3) the group generated by

\[
\{ \log \Lambda(a_n \cdots a_0) : a_n \cdots a_0 > 0 \text{ for some } n \text{ and } a_i \in \text{supp}(\mathbb{P}_A) \}
\]

is dense in \( \mathbb{R} \);
4) there exists \( \kappa_0 > 0 \) for which

\[
E \left[ \min_{1 \leq i \leq m} \left( \sum_{j=1}^{m} a_{ij} \right)^{\kappa_0} \right] \geq m^{\kappa_0/2}
\]

and

\[
E \|A\|^{\kappa_0} \log^+ \|A\| < \infty.
\]

Then there exists a \( \kappa \in (0, \kappa_0] \) such that

\[
s'(\kappa) = \lim_{n \to \infty} \frac{1}{n} \log E \|A_{n-1} \cdots A_0\|^\kappa = 0.
\]

If, in addition, the tuple of \( m \)-dimensional vectors \( \{C_n, n \geq 0\} \) is such that the pairs \( (A_n, C_n), n = 0, 1, \ldots \) are iid: \( (A_n, C_n) \overset{d}{=} (A, C) \) and such that

\[
P(C = 0) < 1, \quad P(C \geq 0) = 1, \quad E \|C\|^\kappa < \infty,
\]

then there exist a constant \( K_0 \in (0, \infty) \) and a continuous strictly positive function \( l(u) \) on \( U_+ \) such that

\[
\lim_{y \to -\infty} y^k P((u, \Xi) > y) = K_0 l(u), \quad u \in U_+.
\]

The next lemma will be of importance for subsequent arguments.

**Lemma 9** ([27], Theorem 1.5.1) If \( X_i, i = 1, 2, \ldots \) is a sequence of iid random variables such that \( E |X_i|^p < \infty \) and \( E X_i = 0 \) if \( p \geq 1 \), and \( N \) is a stopping time for the sequence \( S_n = X_1 + \ldots + X_n \), then there exists a constant \( R_p \in (0, \infty) \) such that

\[
E |S_N|^p \leq R_p E |X_i|^p E N^{p/2}.
\]
5 Properties of the underlying MBPRE

In this section we assume that the conditions of Theorem 6 are valid. This means, in particular, that we deal with subcritical MBPRE. Let us agree to denote by \( K, K_x, x \in (0, \infty) \) positive constants which may be different from formula to formula.

First we evaluate the expectation of the random variable \( \|Z(n)\|^x \), \( 0 < x < \kappa \), from above.

**Lemma 10** If \( Z(0) = z \) and \( \kappa > 1 \) then for each \( x \in [1, \kappa) \) there exist \( \rho_x \in (0, 1) \) and \( K_x < \infty \) such that

\[
E \|Z(n)\|^x \leq K_x \rho_x^n \|z\|^x
\]

for all \( n = 1, 2, \ldots \).

**Proof.** Clearly, for any nonrandom vector \( b \in \mathbb{R}^m \)

\[
E (Z(n), b) = E \left( \sum_{i=1}^{m} \sum_{k=1}^{Z_i(n-1)} \xi_i(n-1; k), b \right)
\]

\[
= E (Z(n-1), A_{n-1}b) = E (Z(n-j), \Pi_{n-j,n}b)
\]

\[
= \ldots = E (z, \Pi_{0,n}b), \quad 0 \leq j \leq n.
\]

Thus, if \( \|b\| \leq K < \infty \) then for any \( \delta > 0 \) with \( s(1) (1 + \delta) < 1 \) there exists a constant \( K_1 \in (0, \infty) \) such that for all \( n = 1, 2, \ldots \) and \( 0 \leq j \leq n \)

\[
E (Z(n-j), \Pi_{n-j,n}b)| \leq \|z\| E \|\Pi_{0,n}\| E \|b\| \leq K_1 (s(1)(1 + \delta))^n \|z\|. \quad (24)
\]

Now we apply arguments similar to those used in [19]. It is easy to check that for any \( y, w \geq 0 \) and any \( \varepsilon \in (0, 1) \)

\[
y^x \leq (1 + \varepsilon)w^x + c_{x,\varepsilon} |y - w|^x,
\]

where \( c_{x,\varepsilon} := \left(1 - (1 + \varepsilon)^{-1/x}\right)^{-x} \). Hence we have

\[
E \|Z(n)\|^x = E \|Z(n-1)A_{n-1} + Z(n) - Z(n-1)A_{n-1}\|^x
\]

\[
\leq (1 + \varepsilon)E \|Z(n-1)A_{n-1}\|^x + c_{x,\varepsilon}E \|Z(n) - Z(n-1)A_{n-1}\|^x.
\]

(25)

Recalling the definition \( a_{ij}(n) = E_F \xi_{ij}(n) \), set

\[
\beta_i(n) := \sum_{j=1}^{m} (\xi_{ij}(n) - a_{ij}(n))
\]

(26)

and let

\[
M_x(n; i) := E_F |\beta_i(n)|^x, \quad M_x := \max_{1 \leq i \leq m} E |\beta_i(n)|^x.
\]
By Lemma 9 with \( p = x > 1 \) we conclude
\[
\begin{align*}
\mathbb{E}_F \|Z(n) - Z(n-1)A_{n-1}\| & \leq \mathbb{E}_F \left| \sum_{i=1}^{m} \sum_{k=1}^{m} \sum_{j=1}^{m} \xi_{ij}(n-1; k) - a_{ij}(n-1) \right|^{x} \\
\leq m^x \sum_{i=1}^{m} \mathbb{E}_F \left| \sum_{k=1}^{(n-1)} \sum_{j=1}^{m} \xi_{ij}(n-1; k) - a_{ij}(n-1) \right|^{x} \\
\leq R_x m^x \sum_{i=1}^{m} M_x(n-1, i) \mathbb{E}_F Z_i^{x/2}Z_i^{1/2}(n-1).
\end{align*}
\]

These estimates and \((25)\) give
\[
\begin{align*}
\mathbb{E} \|Z(n)\|^{x} & \leq (1 + \varepsilon)\mathbb{E} \|Z(n-1)A_{n-1}\|^{x} \\
& \quad + c_{x, \varepsilon} R_x m^x M_x \sum_{i=1}^{m} \mathbb{E} Z_i^{x/2Z_i^{1/2}}(n-1).
\end{align*}
\]

Now we are ready to demonstrate \((23)\). First we establish \((23)\) for all integer \( x \in [1, \kappa) \). For \( x = 1 \) we have proved \((23)\) by \((24)\) with \( b = 1 \). Now we use induction for \( x \geq 2 \). Observing that \( x/2 \leq 1 \leq x-1 \) in this case, we see by \((28)\) and the estimate
\[
\sum_{i=1}^{m} Z_i^{x/2Z_i^{1/2}}(n-1) \leq \mathbb{E} \sum_{i=1}^{m} Z_i^{x-1}(n-1) \leq \mathbb{E} \|Z(n-1)\|^{x-1}
\]
that
\[
\begin{align*}
\mathbb{E} \|Z(n)\|^{x} & \leq (1 + \varepsilon)\mathbb{E} \|Z(n-1)A_{n-1}\|^{x} \\
& \quad + c_{x, \varepsilon} R_x m^x M_x \mathbb{E} \|Z(n-1)\|^{x-1} \leq (1 + \varepsilon)^2 \mathbb{E} \|Z(n-2)A_{n-2}A_{n-1}\|^{x} \\
& \quad + (1 + \varepsilon)c_{x, \varepsilon} R_x m^x M_x \|A_{n-1}\| \mathbb{E} \|Z(n-2)\|^{x-1} \\
& \quad + c_{x, \varepsilon} R_x m^x M_x \mathbb{E} \|Z(n-1)\|^{x-1} \leq \ldots \leq (1 + \varepsilon)^n \mathbb{E} \|Z_{n-1}\|^{x} \\
& \quad + c_{x, \varepsilon} R_x m^x M_x \sum_{j=0}^{n-1} (1 + \varepsilon)^j \mathbb{E} \|\Pi_{n-j, n}\|^{x} \mathbb{E} \|Z(n-j-1)\|^{x-1}.
\end{align*}
\]

Since \( x \in [1, \kappa) \), for any \( \delta > 0 \) there exists a constant \( L_x \) such that
\[
\mathbb{E} \|\Pi_{0, n}\|^{x} \leq L_x (s(x)(1 + \delta))^n
\]
for all \( n = 0, 1, 2, \ldots \). By induction hypothesis there exist constants \( K_{x-1} \) and \( \rho_{x-1} \in (s(x), 1) \) such that
\[
\mathbb{E} \|Z(n-j)\|^{x-1} \leq K_{x-1}\rho_{x-1}^{n-j} \|z\|^{x-1} \leq K_{x-1}\rho_{x-1}^{n-j} \|z\|^{x}.
\]

16
for all $j = 0, 1, \ldots, n$. Thus,
\[
\begin{align*}
E \| \mathbf{Z}(n) \|^x & \leq (1 + \varepsilon)^n L_x (s(x)(1 + \delta))^n \| \mathbf{z} \|^x \\
& + c_{x, \varepsilon} R_x m^x M_x L_{x-1} \| \mathbf{z} \|^x \sum_{j=0}^{n-1} (1 + \varepsilon)^j (s(x)(1 + \delta))^j \rho_{x-1}^{n-j-1} \\
& = L_x s^n(x) (1 + \varepsilon)^n (1 + \delta)^n \| \mathbf{z} \|^x \\
& + K s^n(x) (1 + \varepsilon)^n (1 + \delta)^n \| \mathbf{z} \|^x \sum_{j=1}^{n} \frac{\rho_{x-1}^{j-1}}{s^j(x)(1 + \delta)^j (1 + \delta)^j}.
\end{align*}
\]

(30)

Now selecting $\delta$ and $\varepsilon$ in such a way that $s(x)(1 + \varepsilon)(1 + \delta) \in (\rho_{x-1}, 1)$ we get (23).

To treat the case of noninteger $x \in [1, \kappa)$ observe that $x^* = \lceil x \rceil \in [1, \kappa)$ is an integer for which (23) is valid. Thus, it remains to demonstrate (23) for $x = x^* + \gamma < \kappa$, where $\gamma \in (0, 1)$. Since $x^* \geq x/2 \lor 1$ one can use the same arguments as earlier with $x^*$ for $x - 1$.

The lemma is proved.

Let $r$ be an integer and
\[
\zeta = \zeta(r) := \min \{ n \geq 0 : \| \mathbf{Z}(n) \| > r \}
\]
with the natural agreement that $\zeta = \infty$ if $\max_n \| \mathbf{Z}(n) \| \leq r$.

**Lemma 11** Under the conditions of Theorem 6 for any fixed $r \geq 1$

\[
E \| \mathbf{Z}(\zeta(r)) \|^x I \{ \zeta(r) < \infty \} < \infty.
\]

**Proof.** Similar to (28) we have for $x > 1$
\[
E_F [\| \mathbf{Z}(n) \|^x | \mathbf{Z}(0), \ldots, \mathbf{Z}(n-1)] \leq (1 + \varepsilon) \| \mathbf{Z}(n-1) \| A_{n-1} \|^x
\]
\[
+ c_{x, \varepsilon} R_x m^x \sum_{i=1}^{m} M_x(n-1, i) Z_i^{x/2v} (n-1) =: \Psi_x(n-1),
\]
while Jensen’s inequality yields for $x \leq 1$
\[
E_F [\| \mathbf{Z}(n) \|^x | \mathbf{Z}(0), \ldots, \mathbf{Z}(n-1)] \leq (E_F [\| \mathbf{Z}(n) \| | \mathbf{Z}(0), \ldots, \mathbf{Z}(n-1)])^x
\]
\[
= \| \mathbf{Z}(n-1) \| A_{n-1} \|^x \leq \Psi_x(n-1).
\]

(32)

Clearly,
\[
\Psi_{\kappa}(n-1) I \{ n < \zeta \} \leq Q_{n-1}(r) := (1 + \varepsilon)^r A_{n-1} \|^x
\]
\[
+ c_{x, \varepsilon} R_x m^r A_{n-1} \sum_{i=1}^{m} M_x(\zeta - 1, i).
\]
Using these estimates we have on the event \( \{ \zeta < \infty \} : \)

\[
\| Z(\zeta) \|^{\kappa} = \Psi_\kappa(\zeta - 1) \frac{\| Z(\zeta) \|^{\kappa}}{\Psi_\kappa(\zeta - 1)} \leq Q_{\kappa - 1}(r) \frac{\| Z(\zeta) \|^{\kappa}}{\Psi_\kappa(n - 1)} \leq \sum_{\zeta \leq n < r} Q_n(r) \frac{\| Z(n) \|^{\kappa}}{\Psi_\kappa(n - 1)}. \tag{33}
\]

By (31) and (32) we now see that

\[
E \left[ \| Z(\zeta) \|^{\kappa} I \{ \zeta < \infty \} \right] \leq \sum_{n \geq 1} E \left[ Q_n(r) \frac{\| Z(n) \|^{\kappa}}{\Psi_\kappa(n - 1)} I \{ \tau \geq n \} \right]
\leq \sum_{n \geq 1} E [Q_n(r) I \{ \tau \geq n \}] = \sum_{n \geq 1} P(\tau \geq n) E [Q_n(r)]
= \left[ (1 + \varepsilon)^{r} E \| A \|^\kappa + c_{x,\varepsilon} r^{\kappa/2 + 1} \right] \sum_{n \geq 1} P(\tau \geq n) < \infty,
\]

since for \( \kappa_1 := \min(\kappa/2, 1) \)

\[
\sum_{n \geq 1} P(\tau \geq n) = \sum_{n \geq 1} P(\| Z(n - 1) \|^\kappa \geq 1) = \sum_{n \geq 1} E[P_F(\| Z(n - 1) \|^\kappa \geq 1)]
\leq \sum_{n \geq 1} E [E_F \| Z(n - 1) \|^\kappa_1] \leq \sum_{n \geq 1} E \| E_F Z(n - 1) \|^{\kappa_1}
\leq \sum_{n \geq 1} \| z \|^\kappa_1 E \| A_0 \cdots A_{n-1} \|^\kappa_1 < \infty.
\]

The lemma is proved.

Let \( B_n \) \((n = 1, 2, \ldots)\) be the \( \sigma \)-algebra generated by the tuple

\[ F_0(s, \lambda), F_1(s, \lambda), \ldots, F_{n-1}(s, \lambda), Z(0), \ldots, Z(n) \]

and let

\[ C_n := (E_F \varphi_1(n), \ldots, E_F \varphi_m(n))^f, n = 0, 1, 2, \ldots \]

Recall that the pairs \((A_n, C_n), n = 0, 1, \ldots,\) are iid according to the definition of our MBPRCRE and, in particular, \((A_n, C_n)\) is independent on \( B_n \). Set

\[ S(\zeta) := \sum_{n=\zeta}^{\infty} \langle Z(n), C_n \rangle. \tag{34} \]

The next lemma shows that for large \( r \) the random variable \( S(\zeta) = S(\zeta(r)) \) is, in a sense, close to the conditional expectation \( E [S(\zeta) | B_\zeta] = \langle Z(\zeta), \Xi_\zeta \rangle. \)

**Lemma 12** Under the conditions of Theorem 6 for any \( \varepsilon > 0 \) there exists \( r = r(\varepsilon) \) such that for all \( y \geq y_0 \)

\[
P \left( |S(\zeta) - \langle Z(\zeta), \Xi_\zeta \rangle| > \varepsilon y; \zeta < \infty \right) \leq \frac{\varepsilon}{y^\kappa} E \left[ \| Z(\zeta) \|^\kappa I \{ \zeta < \infty \} \right]. \tag{35}
\]
Proof. Evidently, for \( n \geq \zeta + 1 \)

\[
\langle Z(n) - Z(\zeta)\Pi_{\zeta,n}, C_n \rangle = \sum_{l=\zeta+1}^{n} \langle Z(l)\Pi_{l,n} - Z(l-1)\Pi_{l-1,n}, C_n \rangle
\]

\[
= \sum_{l=\zeta+1}^{n} \langle Z(l) - Z(l-1)\Lambda_{l-1}, \Pi_{l,n}C_n \rangle
\]

which implies

\[
|S(\zeta) - \langle Z(\zeta), \Xi_\zeta \rangle| = \left| \sum_{n=\zeta+1}^{\infty} \langle Z(n) - Z(\zeta)\Pi_{\zeta,n}, C_n \rangle \right|
\]

\[
\leq \sum_{n=\zeta+1}^{\infty} |\langle Z(n) - Z(\zeta)\Pi_{\zeta,n}, C_n \rangle|
\]

\[
= \sum_{n=\zeta+1}^{\infty} \left| \sum_{l=\zeta+1}^{n} \langle Z(l) - Z(l-1)\Lambda_{l-1}, \Pi_{l,n}C_n \rangle \right|
\]

\[
\leq \sum_{l=\zeta+1}^{\infty} \sum_{n=\zeta+1}^{\infty} |\langle Z(l) - Z(l-1)\Lambda_{l-1}, \Pi_{l,n}C_n \rangle|
\]

\[
\leq \sum_{l=\zeta+1}^{\infty} \|Z(l) - Z(l-1)\Lambda_{l-1}\| \left\| \sum_{n=l}^{\infty} \Pi_{l,n}C_n \right\|
\]

\[
= \sum_{l=\zeta+1}^{\infty} \|Z(l) - Z(l-1)\Lambda_{l-1}\| \|\Xi_l\|. \quad (36)
\]

Hence, on account of \( \sum_{j=1}^{\infty} j^{-2} = \pi^2/6 \leq 2 \), we may apply on the event \( \{\zeta < \infty\} \) the arguments used in [29], Lemma 3, to conclude that

\[
P_F \left( |S(\zeta) - \langle Z(\zeta), \Xi_\zeta \rangle| \geq \varepsilon y \mid B_\zeta \right)
\]

\[
\leq P_F \left( \sum_{l=\zeta+1}^{\infty} \|Z(l) - Z(l-1)\Lambda_{l-1}\| \|\Xi_l\| \geq 6\pi^{-2}\varepsilon y \sum_{l=\zeta+1}^{\infty} \frac{1}{(l-\zeta)^2} \mid B_\zeta \right)
\]

\[
\leq \sum_{l=\zeta+1}^{\infty} P_F \left( \|Z(l) - Z(l-1)\Lambda_{l-1}\| \|\Xi_l\| \geq \frac{\varepsilon y}{2(l-\zeta)^2} \mid B_\zeta \right). \quad (37)
\]

Since \( Z(l) - Z(l-1)\Lambda_{l-1} \) and \( \Xi_l \) are independent random vectors on the event \( \zeta \leq l < \infty \), we get

\[
P_F \left( \|Z(l) - Z(l-1)\Lambda_{l-1}\| \|\Xi_l\| \geq \frac{\varepsilon y}{2(l-\zeta)^2} \mid B_\zeta \right)
\]

\[
= \int_{0}^{\infty} P_F \left( \|Z(l) - Z(l-1)\Lambda_{l-1}\| \in dt \mid B_\zeta \right) P \left( \|\Xi\| \geq \frac{\varepsilon y}{2t(l-\zeta)^2} \right).
\]

19
According to Condition $T$ there exists a constant $K \in (0, \infty)$ such that for all $l > \zeta$

\[
P_F \left( \|Z(l) - Z(l-1)A_{l-1}\| \geq \frac{\varepsilon y}{2(l - \zeta)^2} |B_{\zeta}| \right) \leq \int_0^\infty P_F (\|Z(l) - Z(l-1)A_{l-1}\| \in dB_{\zeta}) \frac{K t^\kappa}{\varepsilon \gamma y^\kappa} (l - \zeta)^{2\kappa} dt \leq \frac{K}{\varepsilon \gamma y^\kappa} (l - \zeta)^{2\kappa} E_F \|Z(l) - Z(l-1)A_{l-1}\|^\kappa |B_{\zeta}|.
\]

(38)

Now we consider the cases $\kappa \leq 1$ and $\kappa > 1$ separately.

For the first case we use the estimate

\[
E_F \|Z(l) - Z(l-1)A_{l-1}\|^\kappa |B_{\zeta}| \leq \left( E_F \left[ \|Z(l) - Z(l-1)A_{l-1}\|^2 |B_{\zeta}| \right] \right)^{\kappa/2}.
\]

(39)

Further we have

\[
E_F \left[ \|Z(l) - Z(l-1)A_{l-1}\|^2 |B_{\zeta}| \right] = E_F \left[ \left( \sum_{i=1}^m \sum_{j=1}^m [\xi_{ij}(l-1; k) - a_{ij}(l-1)] \right)^2 |B_{\zeta}| \right]
\]

\[
= \sum_{i=1}^m E_F \beta_i^2 (l-1) E_F [Z_i(l-1) |B_{\zeta}]
\]

\[
= \sum_{i=1}^m E_F \beta_i^2 (l-1) \mathbb{P}(Z_i(l-1) \leq \|Z(\zeta)\Pi_{\zeta,l-1}\| \sum_{i=1}^m E_F \beta_i^2 (l-1).)
\]

(40)

Thus, for $\kappa \leq 1$

\[
\left( E_F \left[ \|Z(l) - Z(l-1)A_{l-1}\|^2 |B_{\zeta}| \right] \right)^{\kappa/2} \leq \|Z(\zeta)\Pi_{\zeta,l-1}\|^\kappa \left( \sum_{i=1}^m E_F \beta_i^2 (l-1) \right)^{\kappa/2}.
\]

(41)

This, in view of the inequality

\[
s(\kappa/2) = \lim_{n \to \infty} \left( E \|\Pi_{0,n}\|^{\kappa/2} \right)^{1/n} < 1,
\]
the first part of condition (22), and relations (37)-(41) leads to the estimate
\[
\mathbb{P}(\|S(\zeta) - \langle Z(\zeta), \Xi_\zeta \rangle \| > \varepsilon y; \zeta < \infty) \\
\leq \frac{K}{\varepsilon^n y^n} \mathbb{E}\left[ \sum_{l=\zeta+1}^{\infty} (l - \zeta)^{2\kappa} \|Z(\zeta)\|^{\kappa/2} \|\Pi_{l-1}\|^{\kappa/2} \mathbb{I}\{\zeta < \infty\} \right] \\
= \frac{K}{\varepsilon^n y^n} \mathbb{E}\left[ \|Z(\zeta)\|^{\kappa/2} \mathbb{I}\{\zeta < \infty\} \right] \sum_{l=1}^{\infty} (l - \zeta)^{2\kappa} \|\Pi_{l-1}\|^{\kappa/2} \\
\leq \frac{\text{const}}{\varepsilon^n y^n \kappa^{\kappa/2}} \mathbb{E}\|Z(\zeta)\|^{\kappa \{\zeta < \infty\}} \leq \frac{\varepsilon}{y^n} \mathbb{E}\|Z(\zeta)\|^{\kappa \{\zeta < \infty\}} \tag{42}
\]
for all \( r \geq r_0(\varepsilon) \), proving the lemma for \( \kappa \leq 1 \).

For the case \( \kappa > 1 \) we use Lemma 9 to conclude that for any \( l > \zeta \)
\[
\mathbb{E}_F \left[ \|Z(l) - Z(l - 1)A_{l-1}\|^{\kappa} \right] \\
\leq R_{\kappa} m^\kappa \sum_{i=1}^{m} M_{\kappa}(n; i) \mathbb{E}_F \left[ \|Z_\zeta(l - 1)\|^{\kappa/2} \ | B_\zeta \right] \\
\leq R_{\kappa} m^\kappa \mathbb{E}_F \left[ \|Z(l - 1)\|^{\kappa/2} \ | B_\zeta \right] \sum_{i=1}^{m} M_{\kappa}(n; i).
\]
By Lemma 10 there exist constants \( \rho_{\kappa/2^1} \in (0, 1) \) and \( \Kappa_{\kappa/2^1} < \infty \) such that for all \( l > \zeta \)
\[
\mathbb{E}_F \left[ \|Z(l - 1)\|^{\kappa/2^1} \ | B_\zeta \right] \leq \Kappa_{\kappa/2^1} \rho_{\kappa/2^1}^{l-\zeta - 1} \|Z(\zeta)\|^{\kappa/2^1}.
\]
This yields the estimates
\[
\mathbb{P}(\|S(\zeta) - \langle Z(\zeta), \Xi_\zeta \rangle \| > \varepsilon y; \zeta < \infty) \\
\leq \frac{m^\kappa K}{\varepsilon^n y^n} \mathbb{E}\left[ \sum_{l=\zeta+1}^{\infty} (l - \zeta)^{2\kappa} \mathbb{E}_F \left[ \|Z(l - 1)\|^{\kappa/2} \sum_{i=1}^{m} M_{\kappa}(n; i) \ | B_\zeta \right] \mathbb{I}\{\zeta < \infty\} \right] \\
\leq \frac{m^\kappa + 1}{\varepsilon^n y^n} K \mathbb{E}\left[ \sum_{l=\zeta+1}^{\infty} (l - \zeta)^{2\kappa} \mathbb{E}_F \left[ \|Z(l - 1)\|^{\kappa/2} \ | B_\zeta \right] \mathbb{I}\{\zeta < \infty\} \right] \\
\leq \frac{m^\kappa + 1}{\varepsilon^n y^n} K \mathbb{E}\left[ \sum_{l=\zeta+1}^{\infty} (l - \zeta)^{2\kappa} \|Z(\zeta)\|^{\kappa/2} K_{\kappa/2^1} \rho_{\kappa/2^1}^{l-\zeta - 1} \mathbb{I}\{\zeta < \infty\} \right] \\
= \frac{m^\kappa + 1}{\varepsilon^n y^n} K \mathbb{E}\left[ \|Z(\zeta)\|^{\kappa/2} \mathbb{I}\{\zeta < \infty\} \sum_{l=1}^{\infty} (l - \zeta)^{2\kappa} \rho_{\kappa/2^1}^{l-1} \right] \\
\leq \frac{\text{const}}{\varepsilon^n y^n \kappa^{\kappa-\kappa/2^1}} \mathbb{E}\|Z(\zeta)\|^{\kappa \{\zeta < \infty\}} \leq \frac{\varepsilon}{y^n} \mathbb{E}\|Z(\zeta)\|^{\kappa \{\zeta < \infty\}}
\]
(the last is valid by selecting \( r \) sufficiently large) which justifies the statement of the lemma for \( \kappa > 1 \).

The lemma is proved.
The accumulated amount of the final product

In this section we deduce some estimates related with the total size of the final product accumulated in a subcritical MBPRCRE during its evolution.

Let
\[ \Delta(n) := \sum_{i=1}^{m} \sum_{k=1}^{Z_i(n)} \varphi_i(n; k) \]
be the total size of the final product produced by the individuals of the \( n \)-th generation of a MBPRCRE and
\[ \bar{\Phi}(N) := \sum_{n=N}^{\infty} \Delta(n), \quad \Phi = \bar{\Phi}_0. \]

**Lemma 13** Under the conditions of Theorem 6 for any \( \varepsilon > 0 \) there exists \( r = r(\varepsilon) \) such that for all \( y \geq y_0 \)
\[ \mathbb{P} \left( \left| \bar{\Phi}(\zeta) - S(\zeta) \right| > \varepsilon y; \zeta < \infty \right) \leq \frac{\varepsilon}{y^n} \mathbb{E} \|Z(\zeta)\|^{\kappa} \mathbb{I} \{ \zeta < \infty \}. \] (43)

**Proof.** We have
\[ \Delta(n) - \langle Z(n), C_n \rangle = \sum_{i=1}^{m} \sum_{k=1}^{Z_i(n)} (\varphi_i(n; k) - \mathbb{E}_F \varphi_i(n)). \]

Now we consider separately the cases \( \kappa \leq 1 \) and \( \kappa > 1 \).

For \( \kappa \geq 1 \) we use Lemmas 9 and 10 to get for \( n > \zeta \):
\[ \mathbb{P}_F \left( \frac{\varepsilon}{y^n} \right) \leq \frac{4^\kappa (n - \zeta)^{2\kappa}}{(\varepsilon y)^\kappa} \mathbb{E}_F \left[ \sum_{i=1}^{m} \sum_{k=1}^{Z_i(n)} (\varphi_i(n; k) - \mathbb{E}_F \varphi_i(n)) \right] \]
\[ \leq (4m)^\kappa R_\kappa K_{\kappa/2} (n - \zeta)^{2\kappa} \mathbb{E}_F \left[ \sum_{i=1}^{m} \mathbb{E}_F |\varphi_i(n) - \mathbb{E}_F \varphi_i(n)|^\kappa \right] \]
\[ \leq (4m)^\kappa R_\kappa K_{\kappa/2} (n - \zeta)^{2\kappa} \mathbb{E}_F \left[ \sum_{i=1}^{m} \mathbb{E}_F |\varphi_i(n) - \mathbb{E}_F \varphi_i(n)|^\kappa \mathbb{E}_F \|Z(\zeta)\|^{\kappa/2} \right]. \]
Hence, in view of condition (21)

\[ \mathbb{P} \left( |\Phi(\zeta) - S(\zeta)| > \varepsilon y; \zeta < \infty \right) \]

\[ \leq E \left[ \sum_{n=\zeta}^{\infty} P_F \left( |\Delta(n) - \langle Z(n), C_n \rangle| > \frac{\varepsilon y}{2(n - \zeta)^2} | B_\zeta \right) \right] I (\zeta < \infty) \]

\[ \leq \frac{(4m)^{\kappa} R_k K_{\kappa/2} K_{\kappa}}{(\varepsilon y)^{2m}} E \left[ \left\| Z(\zeta) \right\|^\kappa/(2^1) \sum_{n=\zeta}^{\infty} (n - \zeta)^2 \rho_{\kappa/2}^n I (\zeta < \infty) \right] \]

\[ = \text{const} \left( \frac{\varepsilon y}{\kappa}\right) E \left[ \left\| Z(\zeta) \right\|^\kappa/I (\zeta < \infty) \right] \leq \frac{\varepsilon y}{\kappa} E \left[ \left\| Z(\zeta) \right\|^\kappa/I (\zeta < \infty) \right] \]

for all \( r \geq r(\varepsilon) \).

To analyze the case \( \kappa \leq 1 \) we apply for \( n > \zeta \) the inequality

\[ E_F \left( |\Delta(n) - \langle Z(n), C_n \rangle| \right)^2 | B_\zeta \right) \leq \left( E_F \left( |\Delta(n) - \langle Z(n), C_n \rangle|^2 | B_\zeta \right) \right)^{\kappa/2} \]

Further, we have

\[ E_F \left( |\Delta(n) - \langle Z(n), C_n \rangle|^2 | B_\zeta \right) \leq m^2 \sum_{i=1}^{m} E_F \beta_i^2 \left( \frac{\varepsilon y}{\kappa} \right) E \left[ Z_i(n) | B_\zeta \right] \]

\[ = m^2 \sum_{i=1}^{m} E_F \beta_i^2 \left( \frac{\varepsilon y}{\kappa} \right) \left( Z(\zeta) \Pi_{\zeta,n} \right)_i \leq m^2 \left\| Z(\zeta) \Pi_{\zeta,n} \right\| \sum_{i=1}^{m} E_F \beta_i^2 \left( \frac{\varepsilon y}{\kappa} \right) \]

Thus, for \( \kappa \leq 1 \)

\[ \left( E_F \left( |\Delta(n) - \langle Z(n), C_n \rangle|^2 | B_\zeta \right) \right)^{\kappa/2} \leq m^2 \left\| Z(\zeta) \Pi_{\zeta,n+1} \right\|^{\kappa/2} \left( \sum_{i=1}^{m} E_F \beta_i^2 \left( \frac{\varepsilon y}{\kappa} \right) \right)^{\kappa/2} \]

This combined with the assumption (22) shows that

\[ \mathbb{P} \left( |\Phi(\zeta) - S(\zeta)| > \varepsilon y; \zeta < \infty \right) \]

\[ \leq E \left[ \sum_{n=\zeta}^{\infty} P_F \left( |\Delta(n) - \langle Z(n), C_n \rangle| > \frac{\varepsilon y}{2(n - \zeta)^2} | \zeta, Z(0), \ldots, Z(\zeta) \right) \right] I (\zeta < \infty) \]

\[ \leq \frac{m^\kappa K}{\varepsilon^\kappa y^{\kappa}} E \left[ \sum_{l=\zeta+1}^{\infty} (l - \zeta)^2 \left\| Z(\zeta) \right\|^{\kappa/2} \left\| \Pi_{\zeta,l} \right\|^{\kappa/2} I (\zeta < \infty) \right] \]

\[ = \frac{m^\kappa K}{\varepsilon^\kappa y^{\kappa}} E \left[ \left\| Z(\zeta) \right\|^{\kappa/2} \sum_{l=1}^{\infty} l^{2\kappa} E \left[ \left\| \Pi_{0,l} \right\|^{\kappa/2} I (\zeta < \infty) \right] \right] \]

\[ \leq \text{const} \frac{\varepsilon y}{\kappa} E \left[ \left\| Z(\zeta) \right\|^{\kappa} \left\{ I (\zeta < \infty) \right\} \right] \]

\[ \leq \frac{\varepsilon y}{\kappa} E \left[ \left\| Z(\zeta) \right\|^{\kappa} \left\{ I (\zeta < \infty) \right\} \right] \]
for all $r \geq r(\varepsilon)$.

The lemma is proved.

Up to now we have assumed that the initial number of particles and the initial volume of the final product are nonrandom. Lemmas 14 and 15 are free of this restriction.

**Lemma 14** Let a MBPRCRE be subcritical, $\mathbb{E} \|Z(0)\| < \infty$ and there exist $\delta > 0$ such that

$\mathbb{E} \Phi^{\kappa+\delta}(0) < \infty$, $\max_{1 \leq i \leq m} \mathbb{E} \varphi_i^{\kappa+\delta} < \infty$.

Then for any $\varepsilon > 0$ there exists $r = r(\varepsilon)$ such that for all $y \geq y_0$

$$P(\Phi(\zeta - 1) > \varepsilon y, \zeta < \infty) < \frac{1}{y^{\kappa+\delta/2}}. \tag{44}$$

**Proof.** Let, as earlier, $\tau$ be the extinction moment of the underlying MBPRE and let $c$ be a constant such that for all $y \geq y_0(c)$

$$\sum_{t > c \ln y} P(\tau > t) \leq \frac{1}{y^{\kappa+\delta}}.$$

Such a constant, clearly, exists in view of (15). Recalling (7) put

$$\Phi_r(n) = \Phi(0) + \sum_{l=0}^n \sum_{i=1}^m \sum_{k=1}^r \varphi_i(l; k).$$

Now we have for $y \geq y_0(c)$

$$P(\Phi(\zeta - 1) > \varepsilon y, \zeta < \infty) = \sum_{n=1}^{\infty} P(\Phi(n-1) > \varepsilon y, \zeta = n)$$

$$\leq \sum_{n=1}^{\infty} P(\Phi_r(n-1) > \varepsilon y, \zeta = n) \leq \sum_{n=1}^{\infty} P(\Phi_r(n-1) > \varepsilon y, \tau > n)$$

$$\leq P(\Phi(0) > \frac{\varepsilon y}{2}) + \sum_{1 \leq n \leq c \ln y} \sum_{l=0}^n \sum_{i=1}^m \sum_{k=1}^r P(\varphi_i(l; k) > \frac{\varepsilon y}{2mrc \ln^2 y})$$

$$+ \sum_{n > c \ln y} P(\tau > n) \leq \frac{2}{(\varepsilon y)^{\kappa+\delta}} \mathbb{E} \Phi^{\kappa+\delta}(0)$$

$$+ \left(\frac{2mrc \ln^2 y}{(\varepsilon y)^{\kappa+\delta}}\right)^{\kappa+\delta} \sum_{1 \leq n \leq c \ln y} \sum_{l=0}^n \sum_{i=1}^m \sum_{k=1}^r \mathbb{E} \varphi_i^{\kappa+\delta}$$

$$+ \frac{1}{y^{\kappa+\delta}} \leq K \ln^{2(\kappa+\delta+1)} \frac{y}{(\varepsilon y)^{\kappa+\delta}} + \frac{1}{y^{\kappa+\delta}} \leq \frac{1}{y^{\kappa+\delta/2}}.$$

**Lemma 15** Under conditions of Lemma 14 for any $r$ there exists $y_0 = y_0(r)$ such that for all $y > y_0$

$$P(\Phi > y, \zeta = \infty) \leq \frac{1}{y^{\kappa+\delta/2}}. \tag{45}$$

24
Proof. Letting \( c \) be the same as in the previous lemma we have for \( y \geq y_0(c) \)

\[
P(\Phi > y; \zeta = \infty) \leq P(\Phi_r(\tau) > y) \leq P(\Phi_r([c \ln y]) > y) + P(\tau > c \ln y)
\]

\[
\leq P\left( \Phi(0) > \frac{y}{2} \right) + \sum_{i=0}^{\lfloor c \ln y \rfloor} \sum_{k=1}^{r} P\left( \varphi_i(l; k) > \frac{y}{2mr} \right) + P(\tau > c \ln y) \leq K \frac{\ln^{3+\delta+1}}{y^{\delta+\delta}} + \frac{1}{y^{\delta+\delta}}
\]
as desired.

7 Proof of Theorem 6

Now we are ready to prove the main result of the paper, Theorem 6. First observe that by the equivalence of the norms \( \| \cdot \| \) and \( \| \cdot \|_2 \) and estimates (35),(43), (44) and (45), for any \( \varepsilon \in (0, 1/3) \) one can find \( r = r(\varepsilon) \) such that for all \( y \geq y_0(r, \varepsilon) \)

\[
P(\Phi > y) \leq P(\Phi > y; \zeta = \infty) + P\left( (\mathbf{Z}(\zeta), \Xi) > y(1-3\varepsilon); \zeta < \infty \right) + P\left( |\Phi(\zeta) - S(\zeta)| > \varepsilon y; \zeta < \infty \right) + P(\Phi(\zeta - 1) > \varepsilon y; \zeta < \infty)
\]

\[
\leq P\left( (\mathbf{Z}(\zeta), \Xi) > y(1-3\varepsilon); \zeta < \infty \right) + \frac{2\varepsilon}{y^2} E \left[ \| \mathbf{Z}(\zeta) \|_2^2 \mathbf{I} \{ \zeta < \infty \} \right] + \frac{2}{y^{\delta+3/2}}.
\]

\[ \text{(46)} \]

Let \( K(l) := \inf_{u \in U_+} l(u) > 0 \) where \( l(u) \) is the function involved in Condition T. By this condition and the independency of \( \mathbf{Z}(\zeta) \) and \( \Xi \) we conclude

\[
\lim_{y \to \infty} y^k \mathbf{P}\left( (\mathbf{Z}(\zeta), \Xi) > y(1-3\varepsilon); \zeta < \infty \right)
\]

\[
\leq \lim_{y \to \infty} y^k \int_{\|u\|_2 = r}^{\infty} \mathbf{P}\left( \mathbf{Z}(\zeta) \in du; \zeta < \infty \right) \mathbf{P}\left( \left( \frac{u}{\|u\|_2} , \Xi \right) \geq (1-3\varepsilon)y \right)
\]

\[
= K_0(1-3\varepsilon)^{-k} \int_{\|u\|_2 = r}^{\infty} \mathbf{P}\left( \mathbf{Z}(\zeta) \in du; \zeta < \infty \right) \|u\|_2^k l\left( \frac{u}{\|u\|_2} \right)
\]

\[
= K_0(1-3\varepsilon)^{-k} E \left[ \| \mathbf{Z}(\zeta) \|_2^k l\left( \frac{\mathbf{Z}(\zeta)}{\|\mathbf{Z}(\zeta)\|_2} \right) \mathbf{I} \{ \zeta < \infty \} \right] < \infty.
\]

Since \( \gamma^k \mathbf{P}(\Phi > y) \) does not depend on \( r \) and \( \varepsilon \), the previous estimate and (46) yield

\[
\lim_{y \to \infty} y^k \mathbf{P}(\Phi > y) < \infty
\]

and, moreover,

\[
\lim_{y \to \infty} y^k \mathbf{P}(\Phi > y) \leq K_0 \lim_{r \to \infty} E \left[ \| \mathbf{Z}(\zeta) \|_2^k l\left( \frac{\mathbf{Z}(\zeta)}{\|\mathbf{Z}(\zeta)\|_2} \right) \mathbf{I} \{ \zeta < \infty \} \right].
\]

\[ \text{(48)} \]
To get a similar estimate from below we use for $\varepsilon > 0$ the inequality
\[
\mathbb{P}(\Phi > y) \geq \mathbb{P}(\Phi > y; \zeta < \infty) \geq \mathbb{P}(\langle Z(\zeta), \Xi_0 \rangle > y(1 + 3\varepsilon); \zeta < \infty)
\]
\[
- \mathbb{P}(\langle \Phi(\zeta) - S(\zeta) \rangle > \varepsilon y; \zeta < \infty) - \mathbb{P}(|S(\zeta) - \langle Z(\zeta), \Xi_0 \rangle| > \varepsilon y; \zeta < \infty)
\]
\[
- \mathbb{P}(\Phi(\zeta - 1) > \varepsilon y; \zeta < \infty).
\]

Now we select $r$ as large to meet estimates (35), (43) and (44). This gives for sufficiently large $y > r$ the inequality
\[
\mathbb{P}(\Phi > y) \geq \mathbb{P}(\langle Z(\zeta), \Xi_0 \rangle > y(1 + 3\varepsilon); \zeta < \infty) - \frac{2\varepsilon}{y^2} E[\|Z(\zeta)\|^6] I\{\zeta < \infty\} - \frac{1}{y^{n+\delta/2}}.
\]

Letting $y \to \infty$ we obtain
\[
\liminf_{y \to \infty} y^n \mathbb{P}(\langle Z(\zeta), \Xi_0 \rangle > y(1 + 3\varepsilon))
\]
\[
= \liminf_{y \to \infty} y^n \int_{\|u\|_2 = r}^{\infty} \mathbb{P}(Z(\zeta) \in du; \zeta < \infty) \mathbb{P}\left(\left\langle \frac{u}{\|u\|_2}, \Xi_0 \right\rangle \geq \frac{(1 + 3\varepsilon)y}{\|u\|_2}\right)
\]
\[
= K_0(1 + 3\varepsilon)^{-\kappa} E\left[\|Z(\zeta)\|^6 I\left\{\|\frac{Z(\zeta)}{\|Z(\zeta)\|_2}\| \zeta < \infty\right\}\right].
\]

We know that for sufficiently small $\varepsilon > 0$ and an appropriate $r$
\[
K_0(1 + 3\varepsilon)^{-\kappa} E\left[\|Z(\zeta)\|^6 I\left\{\|\frac{Z(\zeta)}{\|Z(\zeta)\|_2}\| \zeta < \infty\right\}\right] - 2\varepsilon E[\|Z(\zeta)\|^6 I\{\zeta < \infty\}]
\]
\[
\geq (K_0 K(l)(1 + 3\varepsilon)^{-\kappa} - 2\varepsilon) E[\|Z(\zeta)\|^6 I\{\zeta < \infty\}] > 0
\]
which implies
\[
\liminf_{y \to \infty} y^n \mathbb{P}(\Phi > y) > 0
\]

leading in turn to
\[
\liminf_{y \to \infty} y^n \mathbb{P}(\Phi > y) \geq K_0 \lim_{r \to \infty} E\left[\|Z(\zeta)\|^6 I\left\{\|\frac{Z(\zeta)}{\|Z(\zeta)\|_2}\| \zeta < \infty\right\}\right].
\]

This combined with (48) gives
\[
\lim_{y \to \infty} y^n \mathbb{P}(\Phi > y) = K_0 \lim_{r \to \infty} E\left[\|Z(\zeta)\|^6 I\left\{\|\frac{Z(\zeta)}{\|Z(\zeta)\|_2}\| \zeta < \infty\right\}\right] \in (0, \infty).
\]

The theorem is proved.

**Theorem 16** Let conditions of Theorem 6 be valid for a subcritical MBPRCRE starting at moment 0 by a random tuple $Z(0) = (Z_1(0), \ldots, Z_m(0))$ of particles with $\mathbb{P}(Z(0) \neq 0) > 0$ and having a random initial size $\Phi(0)$ of the final product. If
\[
E \left[\|Z(0)\|^{\nu_1} + \Phi(0)\right] < \infty
\]
for any $t \in (0, \kappa)$, then
\[
E \Phi^x < \infty
\]
if and only if $x \in (0, \kappa)$.
Proof. Let $\Phi_{ik}$ be the total size of the final product produced by all descendants of the $k$-th particle of type $i$ of the zero generation. Clearly, the accumulated amount $\Phi$ of the final product can be written as

$$\Phi = \Phi(0) + \sum_{i=1}^{m} \sum_{k=1}^{Z_i(0)} \Phi_{ik}.$$ 

Given $Z(0) = z = (z_1, \ldots, z_m)$ we have for $x \leq 1$

$$E[\Phi^x | Z(0) = z] \leq E[\Phi^x(0)] + \sum_{i=1}^{m} \sum_{k=1}^{z_i} E[\Phi^x_{ik}] = E[\Phi^x(0)] + \sum_{i=1}^{m} z_i E[\Phi^x | Z(0) = e_i],$$

while for $x > 1$ there exists a constant $R_x^*$ such that (see, for instance, Theorem 5.2, page 22 in [27])

$$E[\Phi^x | Z(0) = z] \leq (m + 1)^x E[\Phi^x(0)] + R_x^* \max_{1 \leq i \leq m} E[\Phi^x | Z(0) = e_i],$$

By the total probability formula and the estimates above we obtain for $x \in (0, \kappa)$:

$$E[\Phi^x] = \sum_{z \in Z_+ \setminus \{0\}} P(Z(0) = z) E[\Phi^x(0)] + \sum_{z \in Z_+ \setminus \{0\}} P(Z(0) = z) E[\Phi^x | Z(0) = z]$$

$$\leq K_2 E[\Phi^x(0)] + K_3 \max_{1 \leq i \leq m} E[\Phi^x | Z(0) = e_i] E[\|Z(0)\|^{x+1}] < \infty.$$ 

On the other hand, for any $z \in \mathbb{N}_0^m \setminus \{0\}$ such that $P(Z(0) = z) > 0$

$$E[\Phi^x] \geq E[\Phi^x | Z(0) = z] P(Z(0) = z)$$

and the desired result for $x \geq \kappa$ follows from Theorem 6.

8 Polling systems with zero switchover times and MBPRCRE

We start this section by the description of a connection between the BTPSFPRE with zero switchover times and the MBPRCRE.

Consider a polling system with zero switchover times and assume that the operation of the initially idle system starts at the moment when a customer
arrives to a station \( J \in \{1, 2, \ldots, m\}\). We would like to study the distribution of the busy period of the server which performs \( J - 1 \) switches of zero length and then starts the service of the customer arrived to station \( J \).

The length of this busy period is constituted by the time intervals spend by the server to make a random number of complete cycles \((1 \rightarrow 2 \rightarrow \cdots \rightarrow m \rightarrow 1)\) plus the time needed to perform the last (may be, incomplete) route just before the moment when there are no customers in the system for the first time.

Assume that in the course of the \( n \)–th service cycle within the initial busy period the service discipline at station \( i \) satisfies the branching property with vector-valued m.p.g.f.

\[
\phi_n(s; \lambda) = \left( \phi_n^{(1)}(s; \lambda), \ldots, \phi_n^{(m)}(s; \lambda) \right)
\]

where

\[
\phi_n^{(i)}(s; \lambda) := E \left[ s_1^{\theta_{1i}(n)} s_2^{\theta_{2i}(n)} \cdots s_m^{\theta_{mi}(n)} e^{-\lambda \phi_i(n)} \right], \ i = 1, 2, \ldots, m.
\]

Suppose that the sequence \( \phi_0(s; \lambda), \phi_1(s; \lambda), \ldots \) is selected at random in an iid manner. Let further, for each customer, say \( j \), served at station \( i \) during the \( n \)–th route the final product \( \phi_i(n, j) \) and the vector \((\theta_{i1}(n, j), \ldots, \theta_{im}(n, j))\) of the numbers of new customers arrived to the system during the service time \( \tau_i(n, j) \) of the customer under consideration have the property

\[
(\theta_{i1}(n, j), \ldots, \theta_{im}(n, j); \phi_i(n, j)) \overset{d}{=} (\theta_{i1}(n), \ldots, \theta_{im}(n); \phi_i(n))
\]

Here the final product may be not only \( \tau_i(n, j) \) but any nonnegative random variable being either dependent on the the tuple \((\theta_{i1}(n, j), \ldots, \theta_{im}(n, j); \tau_i(n, j))\) or independent on the performance of the system at all. Set \( h_n^{(i)}(s) := \phi_n^{(i)}(s; 0), i = 1, 2, \ldots, m \) and for \( n = 0, 1, 2, \ldots \) introduce m.p.g.f.’s

\[
F_n^{(i)}(s; \lambda) = E \left[ s_1^{\xi_{1i}(n)} s_2^{\xi_{2i}(n)} \cdots s_m^{\xi_{mi}(n)} e^{-\lambda \phi_i(n)} \right]
\]

and p.g.f.’s

\[
f_n^{(i)}(s) = E \left[ s_1^{\xi_{1i}(n)} s_2^{\xi_{2i}(n)} \cdots s_m^{\xi_{mi}(n)} \right]
\]

by the equalities \( F_n^{(m)}(s; \lambda) = \phi_n^{(m)}(s; \lambda), \)

\[
F_n^{(i)}(s; \lambda) = \phi_n^{(i)} \left( s_1, \ldots, s_i, F_n^{(i+1)}(s; \lambda), \ldots, F_n^{(m)}(s; \lambda); \lambda \right), i < m, \tag{50}
\]

and \( f_n^{(m)}(s) = h_n^{(m)}(s), \)

\[
f_n^{(i)}(s) = h_n^{(i)} \left( s_1, \ldots, s_i, f_n^{(i+1)}(s), \ldots, f_n^{(m)}(s) \right), i < m. \tag{51}
\]

We would like to describe conditions on the branching type polling system under which power moments of the amount of the final product accumulated in the system during its busy period are finite or infinite. Our results are based on
the following a bit long but important statement revealing connections between the behavior of certain characteristics of the busy periods of BTPSFPRE and related characteristics of MBPRCRE whose population and the size of the final product at moment 0 are random.

**Theorem 17** The joint distribution of the number of customers at different stations at the end of the \( n - \)th service cycle and the amount of the final product accumulated in the system to the end of the \( n - \)th service cycle in a BTPSFPRE starting by a single customer at station \( J \) and stopped at the end of the first busy period coincides with the joint distribution of the number of particles in the \( n - \)th generation and the total amount of the final product produced for the \( n - 1 \) generations in a MBPRCRE whose generation 0 is specified by a random number of particles and a random amount of the final product with m.p.g.f. \( F_0^{(J)}(s;\lambda) \) and where the joint distribution of the number of direct descendants and the amount of the final product produced by particles of different types of the \( k - \)th generation is given by the vector-valued m.p.g.f.’s

\[
\mathbf{F}_k(s;\lambda) = \left( F_k^{(1)}(s;\lambda), \ldots, F_k^{(m)}(s;\lambda) \right), \ k = 1, 2, \ldots, n.
\]

To prove this theorem one should repeat almost literally the proof of Theorem 4 in [36] and we omit the respective arguments.

Note that if, instead of a single individual at station \( J \) we would initially have in the system a batch of customers \((k_1, \ldots, k_m)\) with \( k_i \) customers at station \( i \) then the distribution of the initial number of particles and the initial size of the final product in the corresponding MBPRCRE should be specified by the m.p.g.f.

\[
F_0(s;\lambda) = \prod_{J=1}^{m} \left( F_0^{(J)}(s;\lambda) \right)^{k_J}.
\]

We call the MBPRCRE described by Theorem 17 the associated MBPRCRE for the BTPSFPRE.

Now we may reformulate the results of Section 3 in terms of our polling system.

Let \( A_n := (a_{ij}(n))_{i,j=1}^{m} \) be the matrix with elements

\[
a_{ij}(n) := \frac{\partial}{\partial s_j} f_n^{(i)}(s) \mid_{s=1} = E \xi_{ij}(n)
\]

and let \( H_n := (h_{ij}(n))_{i,j=1}^{m} \) be the matrix

\[
h_{ij}(n) := \frac{\partial}{\partial s_j} h_n^{(i)}(s) \mid_{s=1} = E h_{ij}(n).
\]

Then in view of (51) \( a_{mj}(n) = h_{mj}(n), j = 1, 2, \ldots, m \) and for \( i < m \)

\[
a_{ij}(n) = h_{ij}(n) I \{ j \leq i \} + \sum_{k=i+1}^{m} h_{ik}(n) a_{kj}(n).
\]

29
For $i = 1, \ldots, m$ introduce auxiliary matrices

$$H_n^{(i)} := \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & 0 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 0 & \ddots & \ddots & 0 & 0 \\
0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\
\end{pmatrix},$$

where for each $i$ the elements of the matrix $H_n^{(i)}$ are located only in row $i$ of the matrix $H_n$. It is not difficult to check by (53) that

$$A_n = H_n^{(1)} H_n^{(2)} \cdots H_n^{(m)}. \quad (54)$$

Further, let $C_n := (C_1(n), \ldots, C_m(n))'$ be a random vector with components

$$C_i(n) := \frac{d}{d\lambda} F_n^{(i)}(1; \lambda) \big|_{\lambda=0} = E \Phi_i(n), \quad i = 1, \ldots, m$$

and let $c_n := (c_1(n), \ldots, c_m(n))'$ be a random vector with components

$$c_i(n) := \frac{d}{d\lambda} \phi_n^{(i)}(1; \lambda) \big|_{\lambda=0} = E \phi_i(n), \quad i = 1, \ldots, m.$$

Then, by (50) $C_m(n) = c_m(n)$ and, for $i < m$

$$C_i(n) = c_i(n) + \sum_{k=i+1}^m h_{ik}(n)C_k(n).$$

Hence we get $C_n = c_n + H_n^{\Delta} C_n$ where

$$H_n^{\Delta} := (h_{ij}(n)I (i < j))_{i,j=1}^m$$

is the upper triangular matrix generated by $H_n$. Thus, $C_n = (E - H_n^{\Delta})^{-1} c_n$.

The next two statements are easy consequences of Theorem 16 and 17.

The first theorem gives conditions under which a busy period of a BTPSFPRE is infinite with positive probability.

**Theorem 18** Assume that the MBPRCRE associated with a BTPSFPRE is such that its underlying MBPRE satisfies conditions of Theorem 5 with $\alpha > 0$ and, in addition, condition (16) is valid. If $\Phi$ is the total size of the final product accumulated in the BTPSFPRE during a busy period then $P(\Phi = \infty) > 0$. In particular, if the service time of any customer at any station is positive with probability 1 then the busy period of the BTPSFPRE is infinite with positive probability.
The statement of the results for a BTPSFPRE whose associated MBPRCRE is subcritical requires more efforts.

**Theorem 19** Assume that the MBPRCRE associated with a BTPSFPRE is subcritical, satisfies conditions of Theorem 6 and

$$\min_{1 \leq J \leq m} F^{(J)}_0(0;0) > 0.$$ 

If the parameter $\kappa$ specified by (20) is such that

$$\max_{1 \leq J \leq m} \mathbb{E} \left[ (\xi J_1 + \cdots + \xi J_m)^{t+1} + \varphi'_j(n) \right] < \infty$$

for any $t \in (0, \kappa)$, then there exists a constant $C \in (0, \infty)$ such that

$$\mathbb{P} (\Phi > y) \sim Cy^{-\kappa}, \ y \to \infty.$$ 

(55)

In particular, if the final product of any customer is its service time then the tail distribution of the length $\Phi$ of a busy period of the system satisfies (55).

**Corollary 20** Under the conditions of Theorem 19 $\mathbb{E} \Phi^x < \infty$ if and only if $x \in (0, \kappa)$.

Let us come back to Examples 1 and 2 considered at the beginning of the paper.

Differentiating (2) at point $s = 1$ we see that the matrix $H_n$ and the vector $C_n$ in Example 1 have elements

$$h_{ij}(n) = \gamma_{ij}(n) + \varepsilon_{ij}(n) \mathbb{E} \left[ \tau_i(n) | T_{in} \right], \ i, j = 1, \ldots, m$$

and

$$C_i(n) = \mathbb{E} \left[ \tau_i(n) | T_{in} \right], \ i = 1, \ldots, m,$$

while by differentiating (5) at point $s = 1$ and taking into account (3) we conclude after evident transformations that $h_{ii}(n) = 0, i = 1, \ldots, m$ and, for $i \neq j$

$$h_{ij}(n) = \frac{\gamma_{ij}(n) (1 - \gamma_{ii}(n)) + \varepsilon_{ij}(n) \mathbb{E} \left[ \tau_i(n) | T_{in} \right]}{1 - \gamma_{ii}(n) - \varepsilon_{ii}(n) \mathbb{E} \left[ \tau_i(n) | T_{in} \right]},$$

(56)

if

$$\frac{1 - \gamma_{ii}(n)}{\mathbb{E} \left[ \tau_i(n) | T_{in} \right]} > \varepsilon_{ii}(n).$$

(57)

and $h_{ij}(n) = \infty$, otherwise.

Note that if the quantities $\varepsilon_{ij}(n), i, j = 1, \ldots, m$ are nonrandom, $\gamma_{ii}(n) = 0$ with probability 1 for all $i = 1, \ldots, m$, and $T_{in}$ is an exponential distribution with random parameter $\mu_{in}$, relations (56) and (57) look as follows:

$$h_{ij}(n) = \frac{\mu_{in} \gamma_{ij}(n) + \varepsilon_{ij}(n)}{\mu_{in} - \varepsilon_{ii}(n)}$$

31
and

\[ \mu_{in} - \varepsilon_{ii}(n) > 0. \]

They are in complete agreement with the respective formulas and restrictions of Sections 1.1 and 1.2 in [34].

**Concluding remarks.** Our results give a criterion allowing to answer the question: when \( \Phi \), the total amount of the final product accumulated in the polling system with zero switchover times during a busy period has finite or infinite moment of order \( x \)? In this respect Theorem 19 refines and extends in several directions Theorem 1.1 in [34]. For instance, we do not require exponentiality of the service time distributions of customers and prove the mentioned criterion for a wide class of polling systems which are not covered by the results of [34]. Moreover, we even describe the behavior of the tail distribution of \( \Phi \). Unfortunately, such a refinement is achieved for the expense of transparency of the conditions involved. The most essential of our hypotheses is Condition \( T \), whose validity is established up to now only for a restricted class of nonnegative random matrices. The extension of the class of nonnegative random matrices for which Condition \( T \) is valid is an interesting and challenging problem. Some statements related with such circle of problems have been obtained quite recently in a number of papers (see, for instance, [17] and [26]). Unfortunately, they do not fit the case of measures concentrated on nonnegative matrices only.

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