Seperability Properties for a Class of Block Matrices.

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It is shown that, for the block matrices belonging to $M(nd, \mathbb{C})$ with commuting and normal block entries of dimension $d$, the separability of such a block matrix is equivalent to its semi-positive definity. The separability decomposition of length equal to the dimension of the block matrix (which is smaller then Carathéodory theorem implies) is given. The separability decomposition depends only on eigenvalues of block entries in the first part and on eigenvectors of the block entries in the second part of the tensor product. It is shown that semi-positive definity of considered block matrices is equivalent to semi-positive definity $d$ smaller matrices of dimension $n$.

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I. INTRODUCTION

The phenomenon of quantum correlations, such as quantum entanglement, quantum discord or quantum steering show in the one of the most amazing way difference between classcal world and the quantum one. Even now, after eighty years from famous paper\textsuperscript{[1]} new ideas and concepts arise giving new opportunities for researchers, we mention here only the milestones such as quantum teleportation\textsuperscript{[2]}, quantum dense coding\textsuperscript{[3]} or pioneering work in quantum cryptography\textsuperscript{[4]} (for more applications see\textsuperscript{[5]}). Because of the multiplicity of possible applications of various types of quantum correlations, is required to know whether given quantum state exhibits desired type of correlations, or complementarily for example whether is separable, when we have to deal with entanglement. Thanks to this checking separability of quantum states plays the key role in quantum information theory. Despite of this all arguments of importance till now we do not have any general method(s), which allows us to checking whether given state is separable or constructing separable decomposition. Another notable fact is that even when we are given with some separable state it is really hard to present their explicit separable decomposition. There are only few non-trivial examples where such decomposition can be done in some general regime (see for example\textsuperscript{[13]}), so every new result for wide class of states is in our opinion welcome.

Here we focus on special class of quantum states represented by block matrices. Namely in the paper\textsuperscript{[6]} it has been shown that if a block Toeplitz matrix if it is positive semidfinite then it is separable, so in fact for these block matrices separability is equivalent to semi-positivity. In this paper we prove that for any block matrix belonging to $M(nd, \mathbb{C})$ (so of dimension $nd$) with commuting and normal block entries, the separability of such a matrix is equivalent to its semi-positivity. The structure of considered block matrix implies that its semi-positivity of considered block matrices is equivalent to semi-positivity $d$ smaller matrices belonging to $M(n, \mathbb{C})$, where $d$ is the block dimension and $n$ is the number of blocks in a row of block matrix.

Before we go further, let us say here a few words more about notation used in this manuscript to which we will refer in next sections. In this section and in our further considerations by $B(\mathbb{C}^d)$ or by $B(\mathcal{H})$ we denote the algebra of all bounded linear operators on $\mathbb{C}^d$ or on $\mathcal{H}$. Using this notation let us define the following set:

$$S(\mathcal{H}) = \{ \rho \in B(\mathcal{H}) \mid \rho \geq 0 \}, \quad (1)$$

which is set of all unnormalised states\textsuperscript{1} on space $\mathcal{H}$. Let us now suppose that we are given with two finite dimensional Hilbert spaces $\mathcal{K}, \mathcal{H}$. Matrix in the bipartite composition system $\rho \in S(\mathcal{H} \otimes \mathcal{K})$ is said to be separable whenever it can be written as $\rho = \sum_i \rho_i \otimes \sigma_i$, where $\rho_i, \sigma_i$ are unnormalised states on $\mathcal{H}$ and $\mathcal{K}$ respectively. If above conditions are not fulfilled we say that matrix $\rho$ is entangled.

The paper is organised as follows: In the Section\textsuperscript{[9]} we define our problem in the details and present the main results for this manuscript. Namely after introductory part we present Proposition\textsuperscript{[4]} and Lemma\textsuperscript{[4]} which are crucial to formulate the Theorem\textsuperscript{[4]} in the mentioned theorem we present necessary and sufficient conditions for separability in the language of eigenvalues and eigenvectors of proper components. At the end of the Section\textsuperscript{[4]} we give three

\textsuperscript{1}If we consider separability it is enough to deal with unnormalized states.
examples in which we present how our main theorem works in practice. It is worth to mention here about Example [3] which has explicit connection with know class of matrices, knowing as circulant ones [10]. Manuscript contains also Appendix [IV] where all the most important proofs from Section II are presented.

II. SEPARABILITY PROPERTIES OF BLOCK MATRICES.

In this section we focus on separable decomposition properties of block matrices. The main result is contained in Theorem [I] when conditions if and only if for separability are formulated for certain block matrices. In this same theorem authors also present explicit separable decomposition for mentioned class. At the end of this section we give also three exemplary examples, which show how our result works in practice. We consider the following block matrix

\[
T = \begin{pmatrix}
B_{11} & B_{12} & B_{13} & \cdots & B_{1n} \\
B_{21} & B_{22} & B_{23} & \cdots & B_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B_{n-11} & B_{n-12} & B_{n-13} & \cdots & B_{nn-1} \\
B_{n1} & B_{n2} & B_{n3} & \cdots & B_{nn}
\end{pmatrix},
\]

(2)

where \(B_{ij} \in M(d, \mathbb{C})\) for \(i, j = 1, \ldots, n\). We assume that the matrices \(B_{ij}\) in \(T\) are normal and commute. Such a matrices have the following decomposition property:

Proposition 1. Consider a block matrix as in equation (2) which is not necessarily positive semidefinite and where \(B_{ij} \in M(d, \mathbb{C})\) for \(i, j = 1, \ldots, n\) are normal and all of them form a commutative set of matrices. From these assumptions it follows that there exists for \(B_{ij} \in M(d, \mathbb{C})\), for \(i, j = 1, \ldots, n\) a common set of orthonormal eigenvectors \(\{u_k\}_{k=1}^d\) such that

\[
B_{ij} u_k = \beta_{ij}^{k} u_k, \quad i, j = 1, \ldots, n, \quad k = 1, \ldots, d,
\]

(3)

where \(\beta_{ij}^{k}\) are eigenvalues of the matrix \(B_{ij}\). Then we have the following decomposition of the matrix \(T\)

\[
T = \sum_{k=1}^d M(\beta_{ij}^{k}) \otimes u_k u_k^\dagger,
\]

(4)

where

\[
M(\beta_{ij}^{k}) = \begin{pmatrix}
\beta_{11}^{k} & \beta_{12}^{k} & \beta_{13}^{k} & \cdots & \beta_{1n}^{k} \\
\beta_{21}^{k} & \beta_{22}^{k} & \beta_{23}^{k} & \cdots & \beta_{2n}^{k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\beta_{n-11}^{k} & \beta_{n-12}^{k} & \beta_{n-13}^{k} & \cdots & \beta_{nn-1}^{k} \\
\beta_{n1}^{k} & \beta_{n2}^{k} & \beta_{n3}^{k} & \cdots & \beta_{nn}^{k}
\end{pmatrix} \in M(n, \mathbb{C})
\]

(5)

is a matrix whose entries are eigenvalues of matrices \(B_{ij}\) for \(i, j = 1, \ldots, n\) corresponding the eigenvector \(u_k\) for \(k = 1, \ldots, d\) and \(u_k u_k^\dagger \in M(d, \mathbb{C})\).

Remark 1. The matrix decomposition as in equation (4) has the property that it can be written as a tensor product of eigen-vectors of matrices \(B_{ij}\) and the eigen-values of \(B_{ij}\), which are in different positions in the tensor product.

The matrix decomposition of the form (4) in the Proposition has the following interesting property:

Lemma 1. Let \(\{u_i\}_{i=1}^d\) be an orthonormal basis in the space \(\mathbb{C}^d\) and \(M_i \in M(n, \mathbb{C})\) for \(i = 1, \ldots, d\) then the matrix

\[
T = \sum_{k=1}^d M_k \otimes u_k u_k^\dagger \in M(nd, \mathbb{C})
\]

(6)

is semi-positive definite if and only if all matrices \(M_k\) are semi-positive definite i.e.

\[
\forall k = 1, \ldots, d \quad M_k \geq 0.
\]

(7)
Directly from Lemma 1 follows

**Corollary 1.** Let \( \{u_i\}_{i=1}^d \) be an orthonormal basis in the space \( \mathbb{C}^d \) and \( M_i \in M(n, \mathbb{C}) \) for \( i = 1, \ldots, d \) then if the matrix

\[
T = \sum_{k=1}^d M_k \otimes u_k u_k^\dagger \in M(nd, \mathbb{C})
\]  

is semi-positive definite, then it is separable and has the following separability decomposition

\[
T = \sum_{k=1}^d \sum_{j=0}^{n-1} \lambda_j^k v_j^k v_j^k \otimes u_k u_k^\dagger,
\]

where \( \{\lambda_j^k\}_{j=0}^{n-1} \) and \( \{v_j^k\}_{j=0}^{n-1} \) are eigenvalues and eigenvectors of the semi-positive definite (so hermitian) matrix \( M_k \in M(n, \mathbb{C}) \). So in fact, for matrices with decomposition of the form (11) separability equivalent to semi-positivity.

**Remark 2.** Let us notice that for every matrix \( X \in M(d, \mathbb{C}) \) positivity of \( X \) implies its hermiticity.

The above results allow to formulate the main statement of our paper.

**Theorem 1.** Suppose that a block matrix

\[
T = \begin{pmatrix}
B_{11} & B_{12} & B_{13} & \cdots & B_{1n} \\
B_{21} & B_{22} & B_{23} & \cdots & B_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
B_{n-11} & B_{n-12} & B_{n-13} & \cdots & B_{n-1n} \\
B_{n1} & B_{n2} & B_{n3} & \cdots & B_{nn}
\end{pmatrix} \in M(nd, \mathbb{C})
\]  

is such that all matrices \( B_{ij} \in M(d, \mathbb{C}) \) are normal and commute, then:

1. the matrix \( T \) is separable, if and only if it is semi-positive definite,

2. the matrix \( T \) is semi-positive definite if and only if all \( d \) matrices \( M(\beta_{ij}^k) \in M(n, \mathbb{C}) \) in its decomposition

\[
T = \sum_{k=1}^d M(\beta_{ij}^k) \otimes u_k u_k^\dagger
\]

are semi-positive definite,

3. if the matrix \( T \) is separable, then it has the following separability decomposition

\[
T = \sum_{k=1}^d \sum_{j=0}^{n-1} \lambda_j^k v_j^k v_j^k \otimes u_k u_k^\dagger,
\]

where \( \{\lambda_j^k\}_{j=0}^{n-1} \) and \( \{v_j^k\}_{j=0}^{n-1} \) are eigenvalues and eigenvectors of matrices \( M(\beta_{ij}^k) \) and

\[
B_{ij} u_k = \beta_{ij}^k u_k, \quad i,j = 1, \ldots, n, \quad k = 1, \ldots, d,
\]

i.e. \( \beta_{ij}^k \) and \( u_k \) are eigenvalues and eigenvectors of \( B_{ij} \).

So for this class of block matrices the separability decomposition may by constructed directly from the eigenvectors of matrices \( B_{ij} \in M(d, \mathbb{C}) \) and from eigenvalues and eigenvectors of matrices \( M(\beta_{ij}^k) \in M(n, \mathbb{C}) \), which are positive semi-definite if \( T \) is separable.

**Remark 3.** Theorem 1 implies that maximal length of separable decomposition given by formula (11) is bounded by \( dn \). Reader notices that mentioned maximal length is smaller than bound given by Carathéodory theorem, which is in this case equal to \( d^2 n^2 \).

**Remark 4.** If the block matrix \( T \) has some special structure, for example it is a Toeplitz matrix, then the same structure appears in the eigen-value matrices \( M(\beta_{ij}^k) \), which will be seen in examples below.
Directly from elementary properties of semi-positive matrices it follows

**Proposition 2.** A necessary condition on the eigenvalues of the matrices $B_{ij}$ for $i, j = 0, 1, \ldots, n - 1$ for positivity of matrices $M(β_k^i)$ is the following

$$
∀k = 1, \ldots, d \quad ∀i, j = 1, \ldots, n - 1 \quad |β_k^i| \leq β_k^i.
$$  \hspace{1cm} (14)

Let us consider some examples of block matrices in the considered class.

**Example 1.** Let $B ∈ M(d, C)$ be a normal matrix and $P_{ij} ∈ C[x]$ for $i, j = 1, \ldots, n$ be arbitrary polynomials, then the block matrix $T$

$$
T = \begin{pmatrix}
P_{11}(B) & P_{12}(B) & P_{13}(B) & \cdots & P_{1n}(B) \\
P_{21}(B) & P_{22}(B) & P_{23}(B) & \cdots & P_{2n}(B) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
P_{n-11}(B) & P_{n-12}(B) & P_{n-13}(B) & \cdots & P_{n-1n}(B) \\
P_{n1}(B) & P_{n2}(B) & P_{n3}(B) & \cdots & P_{nn}(B)
\end{pmatrix} = (P_{ij}(B))
$$  \hspace{1cm} (15)

satisfies the assumptions of Theorem \[\text{?}\], such a matrix is separable if and only if it is semi-positive. The eigen-value matrices $M(β_k^i)$ have the following form

$$
M(β_k^i) = \begin{pmatrix}
P_{11}(β_k) & P_{12}(β_k) & P_{13}(β_k) & \cdots & P_{1n}(β_k) \\
P_{21}(β_k) & P_{22}(β_k) & P_{23}(β_k) & \cdots & P_{2n}(β_k) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
P_{n-11}(β_k) & P_{n-12}(β_k) & P_{n-13}(β_k) & \cdots & P_{n-1n}(β_k) \\
P_{n1}(β_k) & P_{n2}(β_k) & P_{n3}(β_k) & \cdots & P_{nn}(β_k)
\end{pmatrix},
$$  \hspace{1cm} (16)

where $\{β_k\}_{k=1}^d$ are eigenvalues of the matrix $B$. The matrices $M(β_k^i)$ have the same structure as the structure of blocks in the matrix $T$. The matrix $T$ is semi-positive definite iff all matrices $M(β_k^i)$ are positive semidefinite and then the matrix $T$ is separable with the following separability decomposition

$$
T = \sum_{k=1}^{d} \sum_{j=0}^{n-1} λ_j^k v_j^k u_j^k \otimes u_k u_k^\dagger,
$$  \hspace{1cm} (17)

where

$$
Bu_k = β_k u_k, \quad k = 1, \ldots, d,
$$  \hspace{1cm} (18)

i.e. $u_k$ are eigenvalues and eigenvectors of $B$ and $\{λ_j^k\}_{j=0}^{n-1}$ and $\{v_j^k\}_{j=0}^{n-1}$ are eigenvalues and eigenvectors of the matrices $M(β_k^i)$

**Example 2.** Let us consider the particular case of the previous example.

$$
T = \begin{pmatrix}
1 & B & B^2 & \cdots & B^{n-1} \\
(B)^\dagger & 1 & B & \cdots & B^n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(B^{n-2})^\dagger & (B^{n-3})^\dagger & (B^{n-4})^\dagger & \cdots & B \\
(B^{n-1})^\dagger & (B^{n-2})^\dagger & (B^{n-3})^\dagger & \cdots & 1
\end{pmatrix},
$$  \hspace{1cm} (19)

where the matrix $B$ is normal. It is clear that in this case all block matrix entries commute and we have

$$
M(β_k) = \begin{pmatrix}
1 & β_k & (β_k)^2 & \cdots & (β_k)^{n-1} \\
β_k & 1 & β_k & \cdots & (β_k)^{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
(β_k)^{n-2} & (β_k)^{n-3} & (β_k)^{n-4} & \cdots & β_k \\
(β_k)^{n-1} & (β_k)^{n-2} & (β_k)^{n-3} & \cdots & 1
\end{pmatrix}.
$$  \hspace{1cm} (20)
One can check that
\[ \forall n \in \mathbb{N} \quad \det(M(\beta_k)) = (1 - |\beta_k|^2)^{n-1}. \] (21)

Therefore from the Sylvester’s criterion we it follows that if
\[ \forall k = 1, \ldots, d \quad |\beta_k|^2 \leq 1, \] (22)
then all matrices \( M(\beta_k) \) are positive semidefinite and the matrix \( T \) in this example is separable with the following separability decomposition
\[ T = \sum_{k=1}^{d} M(\beta_k) \otimes u_k u_k^*, \] (23)

where
\[ Bu_k = \beta_k u_k, \quad k = 1, \ldots, d, \] (24)
i.e. \( \beta_k \) and \( u_k \) are eigenvalues and eigenvectors of \( B \). Note that the unitary matrices (i.e. when \( B \in U(d) \)) satisfies the condition \( \forall k = 1, \ldots, d \quad |\beta_k|^2 \leq 1 \) and for unitary matrices all matrices \( M(\beta_k) \) are of rank one.

The next example is more explicite.

**Example 3.** In the paper [10] a class of circulant matrices has been introduced, which are of the form
\[ T = \sum_{k=0}^{d-1} \sum_{i,j=0}^{d-1} a_{ij} e_{ij} \otimes e_{i+k,j+k} = \sum_{k=0}^{d-1} a_{ij} e_{ij} \otimes S^k e_{ij} S^k, \] (25)
where \( \{e_{ij}\}_{i,j=0}^{d-1} \) is standard matrix basis in \( M(d, \mathbb{C}) \), \( A^k = (a_{ij}^k) \in M(d, \mathbb{C}) \) are arbitrary matrices and \( S \) is a matrix generator of cyclic group of order \( d \), which has the followin properties
\[ S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \] (26)
so is unitary (in fact orthogonal) with eigenvalues \( \{e^k : k = 0, 1, \ldots, d-1, \quad e^d = 1\} \) and eigenvectors \( \{u_k = (e^{kl}) \in \mathbb{C}^d : k, l = 0, 1, \ldots, d-1\} \), so that
\[ S u_k = e^k u_k. \] (27)

The circulant matrices are semi-positive definite iff all matrices \( A^k \) are semi-positive definite. In our example we will consider a special case of circulant matrices when \( A^k = A \) for \( k = 0, 1, \ldots, d-1 \). In this case the circulant matrix takes the form
\[ T = \begin{pmatrix} a_{00} 1 & a_{01} S & a_{02} S^2 & \cdots & a_{0d-1} S^{n-1} \\ a_{10} S^d & a_{11} 1 & a_{12} S & \cdots & a_{1d-1} S^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{d-20} S^{n-2} & a_{d-21} S^{n-3} & a_{d-22} S^{n-4} & \cdots & a_{d-2d-1} S \\ a_{d-10} S^{n-1} & a_{d-11} S^{n-2} & a_{d-12} S^{n-3} & \cdots & a_{d-1d-1} 1 \end{pmatrix} = (a_{ij} S^{i-j}) \] (28)

and it is clear that the block entries are normal and commute, so the matrix \( T \) satisis the assumptions of Theorem 7. In this case the eigen-value matrices \( M(\beta_k^2) \equiv M(\epsilon_k) \) are the following
\[ M(\epsilon_k) = \begin{pmatrix} a_{00} 1 & a_{01} \epsilon^k & a_{02} \epsilon^{2k} & \cdots & a_{0d-1} \epsilon^{(n-1)k} \\ a_{10} \epsilon^k & a_{11} 1 & a_{12} \epsilon^k & \cdots & a_{1d-1} \epsilon^{(n-2)k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{d-20} \epsilon^{(n-2)} & a_{d-21} \epsilon^{(n-3)} & a_{d-22} \epsilon^{(n-4)} & \cdots & a_{d-2d-1} \epsilon^{k} \\ a_{d-10} \epsilon^{(n-1)} & a_{d-11} \epsilon^{(n-2)} & a_{d-12} \epsilon^{(n-3)} & \cdots & a_{d-1d-1} 1 \end{pmatrix} = (a_{ij} \epsilon^{k(i-j)}) \] (29)
and one can check that they satisfy a nice relation

\[ M(\epsilon^k) = A \circ u_k u_k^\dagger, \]  

(30)

where \( \circ \) means the Hadamard product. According to Theorem 1 the circulant matrix \( T \) is separable iff it is semi-positive definite, equivalently when the matrices \( M(\epsilon^k) \) are positive semidefinite, which holds when the matrix \( A \) is semi-positive definite. In this case we have the following separability decomposition of length \( d \)

\[ T = \sum_{k=0}^{d-1} (A \circ u_k u_k^\dagger) \otimes u_k u_k^\dagger. \]  

(31)

**Remark 5.** If in the last example the matrix \( A = (a_{ij}) \) is such that \( a_{ij} = a \) for \( i, j = 0, \ldots, d - 1 \), then the circulant matrix \( T \) is a block Toeplitz matrix and it is known [12] that Toeplitz block matrices are separable only if they are semi-positive definite and for these matrices this holds without any assumptions on the block entries of the matrices.

### III. CONCLUSIONS

Summarizing in this paper we investigate separable decomosition of certain class of block matrices \( T \) given by formula (2), i.e. when all sublocks are normal in \( T \) and commute. Authors for such class of block matrices prove the Theorem containing main result for this manuscript. Mentioned theorem presents if and only if conditions for separability of mentioned class and gives explicit method for construction of separable decomposition. What is the most important here, such decomposition can be written in terms of eigenvalues and eigenvectors in the first part and eigenvectors of the block entries in the second one as in equation (12).

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### IV. APPENDIX: PROOFS OF THE THEOREMS FROM THE MAIN TEXT

In this additional section we present proofs from the main text. First we give proof of the Proposition and then we present argumentation for Lemma 1.

**Proof of the Proposition 1.** The matrix \( T \) has the following tensor structre

\[ T = \sum_{i,j=1}^{n} E_{ij} \otimes B_{ij}, \]  

(32)

where \( \{E_{ij}\}_{i,j=1}^{n} \) is a standard matrix basis and from the assumptions concerning the commutativity of the matrices \( B_{ij} \) for \( i, j = 1, \ldots, n \) we have

\[ \forall i, j = 1, \ldots, n \quad U^\dagger B_{ij} U = \text{Diag}(\beta_{1}^{ij}, \beta_{2}^{ij}, \ldots, \beta_{d}^{ij}) \equiv D_{ij}, \]  

(33)

where

\[ U = [u_1 u_2 \ldots u_d] \]  

(34)

is an unitary matrix whose coloumns are common orthonormal eigenvectors of the matrices \( B_{ij} \). From this we have

\[ \bar{T} \equiv (1 \otimes U^\dagger)T(1 \otimes U) = \begin{pmatrix} D_{11} & D_{12} & D_{13} & \cdots & D_{1n} \\
D_{21} & D_{22} & D_{23} & \cdots & D_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
D_{n-11} & D_{n-12} & D_{n-13} & \cdots & D_{n-1n} \\
D_{n1} & D_{n2} & D_{n3} & \cdots & D_{nn} \end{pmatrix}, \]  

(35)
where all matrices $D_{ij}$ for $i, j = 1, \ldots, n$ are diagonal with eigenvalues of $B_{ij}$ on the diagonal. From the structure of the matrix $\tilde{T}$ it is clear that we have the following decomposition of this matrix

$$\tilde{T} \equiv \sum_{k=1}^{d} M(\beta_{k}^{ij}) \otimes E_{kk}. \quad (36)$$

Now using the identity

$$UE_{kk}U^\dagger = u_k u_k^\dagger \quad (37)$$

in the equation

$$T = (1 \otimes U) \tilde{T} (1 \otimes U^\dagger) = \sum_{k=1}^{d} M(\beta_{k}^{ij}) \otimes UE_{kk}U^\dagger \quad (38)$$

we get the decomposition of the matrix $T$ given in the Proposition.

\[\square\]

**Proof of the Lemma** It is clear, that if the matrices $M_k$ are semi-positive definite then the matrix $T$, as the sum of semi-positive matrices, is semi-positive definite.

Suppose now that the matrix $T$ is semi-positive definite and let $\{v_i\}_{i=0}^{n-1}$ be an orthonormal basis in the space $\mathbb{C}^n$, then

$$\forall X = \sum_{j=1}^{d} \sum_{k=0}^{n-1} x_{kj} v_k \otimes u_j \in \mathbb{C}^{nd} \quad (X, TX) \equiv X^\dagger TX \geq 0, \quad (39)$$

where $x = (x_{ij}) \in M(n \times d, \mathbb{C})$ is an arbitrary matrix. Now we have

$$X^\dagger TX = \left( \sum_{s=0}^{n-1} \sum_{t=1}^{d} x_{st} v_s^\dagger \otimes u_t \right) \left( \sum_{k,j=1}^{d} \sum_{t=0}^{n-1} x_{tj} M_k v_t \otimes u_k u_k^\dagger (u_j) \right) = \sum_{k=1}^{d} \sum_{s,t=0}^{n-1} x_{sk} x_{tk} (v_s, M_k v_t) = \sum_{k=1}^{d} (y_k, M_k y_k), \quad (40)$$

where $y_k = \sum_{p=0}^{n-1} x_{pk} v_p \in \mathbb{C}^n$ and when the matrices $x = (x_{ps})$ run over $M(n \times d, \mathbb{C})$, then they generates all possible sets of $d$ vectors $\{y_k\}_{k=1}^{d}, y_k \in \mathbb{C}^n$. Now we choose a particular vector $X \in \mathbb{C}^{nd}$, defined by a particular matrix

$$x = (x_{sp}) = \begin{cases} x_{sp} = 0 & \text{for } p \neq k, \\ x_{sk} = x_s & k \in \mathbb{C}, \end{cases} \quad (41)$$

where $k \in \{1, \ldots, d\}$ and is arbitrary. So in the matrix $x$ only the $k$th column is not equal to zero and it forms an arbitrary vector from the space $\mathbb{C}^n$. It is clear that this particular vector $X$ defines the particular set of vectors $\{y_1 = 0, y_2 = 0, \ldots, y_k, \ldots, y_d = 0\}$, where $y_k \in \mathbb{C}^n$ is arbitrary. From semi-positivity of the matrix $T$ for this particular vector $X$ we have

$$0 \leq X^\dagger TX = \sum_{s=1}^{d} (y_s, M_s y_s) = (y_k, M_k y_k), \quad (42)$$

where $k \in \{1, \ldots, d\}$ and is arbitrary and $y_k \in \mathbb{C}^n$ is also arbitrary so it means that all matrices $M_k$ are semi-positive.

\[\square\]

[1] A. Einstein, B. Podolsky and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. A, vol. 47, 777, 1935
[2] C. H. Bennett, G. Brassard, Crépeau, R. Jozsa, A. Peres, W. K. Wootters, Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels, Phys. Rev. Lett. 70 1895-1899 (1993).
[3] C. H. Bennett, S. Wiesner Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states, Phys. Rev. Lett. 69 (20), 2881, (1992).
[4] A. Ekert, Quantum cryptography based on Bell’s theorem, Phys. Rev. Lett. 67 661-663 (1991).
[5] R. Horodecki, P. Horodecki, M. Horodecki and K. Horodecki, Quantum Entanglement, Rev. Mod. Phys. 81: 865-942, 2009
[6] L. Gurvits and H. Barnum, Largest separable balls around the maximally mixed bipartite quantum state, Phys. Rev. A, vol. 66, p. 062311, 2011
[7] P. Ch. Hansen, Deconvolution and regularization with Toeplitz matrices, Numerical Algorithms, vol. 29, p. 323-378, 2002
[8] N. Johnston, Covariance matrix estimation for stationary time series, The Annals of Statistics, vol. 40, No.1, p. 466-493, 2012
[9] S. S Reddi, Eigenvector properties of Toeplitz matrices and their application to spectral analysis of time series, Signal Processing, vol. 7, p. 45-56, 1984
[10] D. Chruściński and A. Kossakowski, Phys. Rev. A 76, 032308 (2007).
[11] S. J. Cho, S-H. Kye i S. G. Lee. Generalized choi maps in threedimensional matrix algebra, Linear Algebra Appl. 171, 213, (1992).
[12] P. Horodecki, Separability criterion and inseparable mixed states with positive partial transpositionPhys. Rev. Lett. A 232, 333, (1997).
[13] R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, Quantum entanglement, Rev. Mod. Phys. 81, 865, (2009).
[14] R. A. Horn, Ch. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991
[15] K.-Ch. Ha, S.-H. Kye, Separable states with unique decompositions, Communications in Mathematical Physics, Volume 328, Issue 1, pp 131-153 (2014)
[16] D. Chruściński, J. Jurkowski and A. Rutkowski, A class of bound entangled states of two qutrits, Open Sys. Information Dyn. 16, 235-242, (2009).
[17] R. F. Werner, Quantum states with einstein-podolsky-rosen correlations admitting a hidden-variable mode, Phys. Rev. A 40, 4277, (1989).
[18] M. Horodecki and P. Horodecki, Reduction criterion of separability and limits for a class of distillation protocols, Phys. Rev. A 59, 4206, (1999).
[19] L. Chen, E. Chitambar, K. Modi, and G. Vacanti, Multiparticle Classical States and Detecting Quantum Discord, Phys. Rev. A 83, 020101 (2011).
[20] H. Ollivier and W. H. Zurek, Quantum Discord: A Measure of the Quantumness of Correlations, Physics Review Letters vol. 88, 017901 (2001)
[21] B. Baumgartner, B.C. Hismayr and H. Narnhofer, The state space for two qutrits has a phase space structure in its core, Phys. Rev. A 74, 032327, (2006)
[22] B. Baumgartner, B.C. Hismayr and H. Narnhofer, A special simplex in the state space for entangled qudits, J. Phys. A: Math. Theor. 40, 7919, (2007)
[23] B. Baumgartner, B.C. Hismayr and H. Narnhofer, The geometry of bipartite qutrits including bound entanglement, Phys. Lett. A 372, 2190, (2008)