Explicit Formula for Conditional Expectations of Product of Polynomial and Exponential Function of Affine Transform of Extended Cox-Ingersoll-Ross Process

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Abstract. In this study, an explicit formula for conditional expectations of the product of polynomial and exponential function of an affine transform is derived under the extended Cox-Ingersoll-Ross (ECIR) process. Moreover, we simplify the result to derive an explicit formula for the CIR process.

1. Introduction

For bond pricing, to improve accuracy in calculation of the present value of all cash flows, equilibrium models are offered to be more suitable than models based on arbitrage arguments. Since equilibrium models come from price behavior, one of the interesting things of construction of the models is to predict a long term equilibrium. Vasicek \cite{4} introduced equilibrium model for interest rate based on the idea of the Black-Scholes model by considering arbitrage argument for option pricing, which is

\begin{equation}
\frac{dr_t}{r_t} = \kappa (\theta - r_t)dt + \sigma dW_t,
\end{equation}

where $\kappa$ is the means reversion rate, $\theta$ is the equilibrium interest rate, $\sigma$ is the volatility and $W_t$ refers to the standard Brownian motion. The Vasicek model is a straightforward model to illustrate the equilibrium as the mean and variance of $r_t$ that converge to finite values when $t$ approaches infinity. One of the earliest way to control positivity of $r_t$ and to fix undesirable feature of (1) was introduced by Cox, Ingersoll and Ross \cite{2}, known as CIR model, which is suitable for describing the evolution and structure of interest rate, namely:

\begin{equation}
\frac{dr_t}{r_t} = \kappa (\theta - r_t)dt + \sqrt{r_t} \sigma dW_t.
\end{equation}

To address the inconveniences of the CIR model, the extended Cox-Ingersoll-Ross (ECIR) model \cite{3} is considered for time dependent parameters. In addition, the ECIR model is
frequently used to explain behavior of structure term of interest rates, especially prices of zero-coupon bonds, where the price varies over time. The general form of ECIR is:

\[ dr_t = \kappa(t)(\theta(t) - r_t)dt + \sqrt{\theta(t)}\sigma(t)dW_t, \]

where \( \kappa, \theta \) and \( \sigma \) are functions depending on \( t \).

The expectation rate at time \( t \) for maturity date \( T \geq t \), \( E^P[r_T | F_t] \), is useful in areas of finance. In particular, in estimating fair price, fitting interest rates structure, forecasting, as well as pricing coupon bonds. In practice, Monte Carlo simulation is used to obtain the expectation which requires a lot of computation time, see more in [1]. For the ECIR model (3) with positive real-valued, bounded, continuous functions, \( \kappa(t), \theta(t), \sigma(t) \), Maghsoodi [6] derived the closed form formula of the expectation of \( r_t \) based on the assumption to guarantee positive \( r_t \):

\[ 2\kappa(t)\theta(t) \geq \sigma^2(t). \]

Furthermore, the derived closed form was applied to price the \( T \)-maturity discount bond:

\[ P(t,T) = A(t,T)e^{-B(t,T)r_t}. \]

Rujivan [5] applied (4) with Feynman-Kac theorem to obtain a closed-form formula for the conditional moments for the ECIR model:

\[ E^P[r_T^\gamma | F_t] = E^P[r_T^\gamma | r_t = r], \]

for \( 0 \leq t \leq T \), and any real \( \gamma \) and positive \( r \).

In this study, we extend the result of [5] with a suitable construct of transformation to obtain explicit formulas for conditional expectations of the product of polynomial and exponential function of an affine transform,

\[ E^P \left[ r_T^\gamma e^{\alpha r_T + \beta} | r_t = r \right]. \]

The rest of this paper is structured as follows. The explicit formulas for conditional expectations of the product of polynomial and exponential function of an affine transform is introduced in Section 2. Moreover, its consequences are explored, especially in some cases, the explicit formula can be reduced to a closed-form formula. The aim of the paper are recapitulated in Section 3.

2. Main results

This section derives explicit formula for conditional expectations of product of polynomial and exponential function of affine transform of ECIR process for parameters \( \gamma, \alpha, \beta \in \mathbb{R} \). Furthermore, the result is simplified to standard CIR model with constants \( \kappa, \theta \) and \( \sigma \). In order to guarantee \( r_t \geq 0 \) for all \( t \in [0, \infty] \), the condition of Maghsoodi [6] is needed.

**Assumption.** The parameter functions \( \theta(t), \kappa(t) \) and \( \sigma(t) \) are positive and continuous on \([0,T]\) such that the dimension parameters \( \delta(t) := \frac{4\theta(t)\kappa(t)}{\sigma^2(t)} \) of the ECIR process (3) is bounded and \( \delta(t) \geq 2 \) for all \( t \in [0,T] \).

**Theorem 1.** Suppose that \( V_t \) follows the ECIR process (3) and let \( \gamma, \alpha, \beta \in \mathbb{R} \). By assuming the above assumption holds, we obtain

\[ U^{(\gamma)}_E(v,\tau) := E^P \left[ V_T^\gamma e^{\alpha V_T + \beta} | V_t = v \right] \]
In addition:

\[ B(\tau) = \frac{\alpha \exp \left[ - \int_0^\tau \kappa(T-u)du \right]}{1 - \alpha \int_0^\tau \frac{1}{2} \sigma^2(T-s) \exp \left[ - \int_0^s \kappa(T-u)du \right] ds}. \]  

**Proof**  
Given the expectation:

\[ U_E^{(\gamma)}(v, \tau) := E \left[ V_T^{\gamma} e^{\alpha V_T + \beta} \mid V_t = v \right], \]

for \( v > 0 \) and \( \tau = T - t \geq 0 \). Applying the Feynman-Kac Theorem, \( U := U_E^{(\gamma)} \) satisfies the PDE:

\[
0 = \frac{\partial U}{\partial t} + \frac{1}{2} \sigma^2(t,v) \frac{\partial^2 U}{\partial v^2} + \mu(t,v) \frac{\partial U}{\partial v} \\
= -e^{B(\tau)v + \beta} \sum_{k=0}^{\infty} \left[ \frac{d}{d\tau} A_{\gamma-k}(\tau)v^{\gamma-k} + \frac{d}{d\tau} B(\tau) A_{\gamma-k}(\tau)v^{\gamma-k+1} \right] \\
+ \frac{1}{2} \sigma^2(T-\tau) ve^{B(\tau)v + \beta} \sum_{k=0}^{\infty} \left[ A_{\gamma-k}(\tau)(\gamma-k)(\gamma-k-1)v^{\gamma-k-2} \right. \\
+ 2B(\tau)A_{\gamma-k}(\tau)(\gamma-k)v^{\gamma-k-1} + B^2(\tau)A_{\gamma-k}(\tau)v^{\gamma-k} \\
+ \left. \kappa(T-\tau) |\theta(T-\tau) - v| e^{B(\tau)v + \beta} \sum_{k=0}^{\infty} [A_{\gamma-k}(\tau)(\gamma-k)v^{\gamma-k-1} + B(\tau)A_{\gamma-k}(\tau)v^{\gamma-k}] \right].
\]
Since $U_E^{(\gamma)}(v, 0) = v^\gamma e^{\alpha v + \beta}$, we consider the form of $U_E^{(\gamma)}(v, \tau)$ as:

\[
U_E^{(\gamma)}(v, \tau) = \sum_{k=0}^{\infty} A_{\gamma-k}(\tau) v^{\gamma-k} e^{B(\gamma)v + \beta}.
\] (17)

with condition:

\[
B(0) = \alpha, A_\gamma(0) = 1, A_{\gamma-k}(0) = 0,
\] (18)

for all $k \in \mathbb{N}$.

Then, given $e^{B(\gamma)v + \beta} > 0$ and conditional on $A_\gamma \neq 0$ for all $\gamma \in \mathbb{R}$, the coefficient of $v^{\gamma+1}$ can be written as a deterministic PDE:

\[
\frac{d}{d\tau} B(\tau) = \frac{1}{2} \sigma^2 (T - \tau) B^2(\gamma) - \kappa (T - \tau) B(\gamma),
\] (19)

where the solution for initial condition (18) is:

\[
B(\tau) = \frac{\alpha \exp \left[ - \int_0^\tau \kappa(T - u) du \right]}{1 - \alpha \int_0^\tau \frac{1}{2} \sigma^2(T - s) \exp \left[ - \int_0^s \kappa(T - u) du \right] ds}.
\] (20)

Given (16), we obtain functional relationship between $A_\gamma(\tau), A_{\gamma-1}(\tau)$ and $B(\gamma)$, which is the term structure coefficient of $v^{\gamma}$ as follows:

\[
\frac{d}{d\tau} A_\gamma(\tau) = -\frac{d}{d\tau} B(\gamma) A_{\gamma-1}(\tau)
+ \frac{1}{2} \sigma^2 (T - \tau) B(\gamma) A_\gamma(\gamma) + \frac{1}{2} \sigma^2 (T - \tau) B(\gamma) A_\gamma(\gamma)
+ \frac{1}{2} \sigma^2 (T - \tau) B^2(\gamma) A_{\gamma-1}(\gamma) + \kappa (T - \tau) \theta (T - \tau) B(\gamma) A_\gamma(\gamma)
- \kappa (T - \tau) A_{\gamma-1}(\gamma) - \kappa (T - \tau) B(\gamma) A_{\gamma-1}(\gamma).
\] (21)

Using (20) and initial condition (18) yields:

\[
A_\gamma(\tau) = \exp \left[ \int_0^\tau \gamma \sigma^2(T - u) B(u) + \kappa (T - u) \theta (T - u) B(u) - \gamma \kappa (T - u) du \right].
\] (22)

Moreover by (16) and initial conditional (18), the coefficients of $v^{\gamma-k+1}$ for all $k = 2, 3, 4, \ldots$ are found as follows:

\[
\frac{d}{d\tau} A_{\gamma-k+1}(\tau) = Q_{\gamma-k+1}(T - \tau) A_{\gamma-k+1}(\tau) + P_{\gamma-k+2}(T - \tau) A_{\gamma-k+2}(\tau),
\] (23)

where:

\[
P_{\gamma-k+2}(\tau) = (\gamma - k + 2) \left[ \frac{1}{2} (\gamma - k + 1) \sigma^2(\tau) + \kappa(\tau) \theta(\tau) \right],
\] (24)

\[
Q_{\gamma-k+1}(\tau) = (\gamma - k + 1) \sigma^2(\tau) B(T - \tau) + \kappa(\tau) \theta(\tau) B(T - \tau),
\] (25)

\[-(\gamma - k + 1) \kappa(\tau).
\] (26)

This results to:

\[
A_{\gamma-k+1}(\tau) = \exp \left[ \int_0^\tau Q_{\gamma-k+1}(T - u) du \right] \int_0^\tau \exp \left[ - \int_0^s Q_{\gamma-k+1}(T - u) du \right] P_{\gamma-k+2}(T - s) A_{\gamma-k+2}(s) ds,
\] (27)
which completes the proof.

The result of Theorem 1 can be simplified into a finite sum in the case when \( \gamma \) is a non-negative integer, as stated in the following result.

**Theorem 2.** Suppose that \( V_t \) follows the ECIR process (3) with \( \gamma, \alpha, \beta \in \mathbb{R} \). Let \( n \) be a non-negative integer. Then:

\[
U^{(n)}_E(v, \tau) := E^{P^\tau} \left[ V^n_T e^{\alpha V_T + \beta} \mid V_t = v \right] = e^{B(\tau)v + \beta} \sum_{j=0}^{n} A_j(\tau)v^j, \tag{28}
\]

for \( (v, \tau) \in D(y) \subset (0, \infty) \times [0, \infty) \), where

\[
A_n(\tau) = \exp \left[ \int_0^\tau n\sigma^2(T-u)B(u) + \kappa(T-u)\theta(T-u)B(u) - n\kappa(T-u)du \right], \tag{29}
\]

\[
A_j(\tau) = \exp \left[ \int_0^\tau Q_j(T-u)du \right] \int_0^\tau \exp \left[ - \int_0^s Q_j(T-u)du \right] P_{j+1}(T-s) A_{j+1}(s)ds, \tag{30}
\]

\[
P_{j+1}(\tau) = (j+1) \left[ \frac{1}{2} j \sigma^2(\tau) + \kappa(\tau)\theta(\tau) \right], \tag{31}
\]

\[
Q_j(\tau) = j \sigma^2(\tau)B(T - \tau) + \kappa(\tau)\theta(\tau)B(T - \tau) - j\kappa(\tau), \tag{32}
\]

for \( j = n - 1, \ldots, 0 \), and \( B(\tau) \) satisfies equality (14). In addition, \( U^{(n)}_E(v, \tau) \) is strictly increasing with respect to \( v \) for any positive \( \tau \) and \( j = n - 1, n - 2, \ldots, 0 \).

**Proof** Let \( \gamma = n \) be a non-negative integer and \( k = n + 1 \). Given \( P_0(\tau) = 0 \), from (12), we have \( A_{-1}(\tau) = 0 \), and \( A_{n-k}(\tau) = 0 \) for all \( k = n + 1, n + 2, \ldots \). Then, (9) can be written as:

\[
U^{(n)}_E(v, \tau) = e^{B(\tau)v + \beta} \sum_{k=0}^{n} A_{n-k}(\tau)v^{n-k}. \tag{33}
\]

Setting \( k = n - j \), the exact term can be rewritten as:

\[
U^{(n)}_E(v, \tau) = e^{B(\tau)v + \beta} \sum_{j=0}^{n} A_j(\tau)v^j. \tag{34}
\]

Furthermore, since from (31) \( P_{j+1}(\tau) > 0 \) for all \( \tau > 0 \), and (29) and (30) guarantee that \( A_j(\tau) > 0 \) for \( j = 0, 1, \ldots, n \), therefore, \( U^{(n,\alpha,\beta)}_E(v, \tau) \) is strictly increasing with respect to \( v \) for \( \tau > 0 \) (for \( v > 0 \)).

Derivation of (7) when \( \kappa(t), \theta(t) \) and \( \sigma(t) \) are constant for all \( 0 \leq t \leq T \), reduces the ECIR model (3) to the CIR model (2) as stated in Theorems 3 and 4. The proofs of Theorems 3 and 4 are similar to Theorems 1 and 2 thus omitted.

**Theorem 3.** Suppose that \( V_t \) follows the CIR process such that \( \kappa(t) = \kappa \), \( \theta(t) = \theta \) and \( \sigma(t) = \sigma \). Let \( \gamma, \alpha, \beta \in \mathbb{R} \). Then:
\[ U_C^{(\gamma)}(v, \tau) := E^P \left[ V_T^\gamma e^{\alpha V_T + \beta} | V_t = v \right], \]
\[ = \exp \left[ \frac{2\alpha \kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} v + \beta + \gamma \kappa t + \frac{2\theta \kappa^2 \tau}{\sigma^2} \right] \]
\[ \left( \frac{2\kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^\frac{2}{\gamma \kappa t} \sigma^2 v^\gamma \]
\[ + \sum_{k=1}^\infty \left\{ \exp \left[ \frac{2\alpha \kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} v + \beta + (\gamma - k) \kappa t + \frac{2\theta \kappa^2 \tau}{\sigma^2} \right] \right\} \left( \frac{e^{\gamma \kappa t} - 1}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^k v^{\gamma - k} \]
\[ = \exp \left[ \frac{2\alpha \kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} v + \beta + \gamma \kappa t + \frac{2\theta \kappa^2 \tau}{\sigma^2} \right] \]
\[ \left( \frac{2\kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^\frac{2}{\gamma \kappa t} \sigma^2 v^\gamma \]
\[ + \sum_{k=1}^\infty \left\{ \exp \left[ \frac{2\alpha \kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} v + \beta + (\gamma - k) \kappa t + \frac{2\theta \kappa^2 \tau}{\sigma^2} \right] \right\} \left( \frac{e^{\gamma \kappa t} - 1}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^k v^{\gamma - k} \]
\[ \left( \prod_{m=1}^k \bar{P}_{\gamma - m + 1} \right) \left( \frac{2^k}{k!} \right) \left( \frac{2\kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^\frac{2}{\gamma \kappa t} \sigma^2 v^{\gamma - k} \]
\[ \left( \frac{e^{\gamma \kappa t} - 1}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^k v^{\gamma - k} \]
\[
\text{for all } (v, \tau) \in D(\gamma) \text{ where } D(\gamma) \text{ is a subset of } (0, \infty) \times [0, \infty) \text{ and} \]
\[ P_{\gamma - m + 1} = (\gamma - m + 1) \left[ \frac{1}{2} (\gamma - m) \sigma^2 + \kappa \theta \right] \]
\[ \text{for } m = 1, 2, \ldots, k. \]

**Theorem 4.** Suppose that \( V_t \) follows the CIR process (2) such that \( \kappa(t) = \kappa, \theta(t) = \theta \) and \( \sigma(t) = \sigma \). Let \( n \) be a non-negative integer. Then:

\[ U_C^{(n)}(v, \tau) := E^P \left[ V_T^n e^{\alpha V_T + \beta} | V_t = v \right], \]
\[ = \exp \left[ \frac{2\alpha \kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} v + \beta + n \kappa t + \frac{2\theta \kappa^2 \tau}{\sigma^2} \right] \]
\[ \left( \frac{2\kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^\frac{2}{n \kappa t} \sigma^2 v^n \]
\[ + \sum_{j=0}^{n-1} \left\{ \exp \left[ \frac{2\alpha \kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} v + \beta + j \kappa t + \frac{2\theta \kappa^2 \tau}{\sigma^2} \right] \right\} \left( \prod_{m=1}^{n-j} \bar{P}_{n-m+1} \right) \left( \frac{2^{n-j}}{(n-j)!} \right) \left( \frac{2\kappa}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^\frac{2}{j \kappa t} \sigma^2 v^{n-j} \]
\[ \left( \frac{e^{\gamma \kappa t} - 1}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^{n-j} v^j \]
\[ \left( \frac{e^{\gamma \kappa t} - 1}{\alpha \sigma^2 + e^{\gamma \kappa t} (2\kappa - \alpha \sigma^2)} \right)^{n-j} v^j \]
\[
\text{for all } v > 0 \text{ and } \tau = T - t \geq 0 \text{ where } \bar{P}_{n-m+1} = (n - m + 1) \left\{ \frac{1}{2} (n - m) \sigma^2 + \kappa \theta \right\} \text{ for } m = 1, 2, \ldots, n - j. \]

**Remark.** Following Theorem 1, the affine transform \( e^{\alpha \kappa t + \beta} \) is removed by substituting \( \alpha = \beta = 0 \). Theorems (1-4) corresponds to modification of Ruijvan [5].

**3. Conclusion**

For real parameters \( \gamma, \alpha \) and \( \beta \), this work extends the result of Ruijvan [5] to obtain explicit formulas for conditional expectations of the product of polynomial and exponential function of an affine transform (7) for ECIR and CIR processes. The results are simplified into exact formulas for the case that \( \gamma \) is a non-negative integer, and by taking \( \alpha = \beta = 0 \), the results reduced to the results in [5].
Acknowledgments
Financial support provided by the Development and Promotion of Science and Technology Talents Project (DPST), the Institute for the Promotion of Teaching Science and Technology (IPST), Thailand.

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