The extremal process of a cascading family of branching Brownian motion

Mohamed Ali Belloum

Abstract

We study the asymptotic behaviour of the extremal process of a cascading family of branching Brownian motions. This is a particle system on the real line such that each particle has a type in addition to his position. Particles of type 1 move on the real line according to Brownian motions and branch at rate 1 into two children of type 1. Furthermore, at rate $\alpha$, they give birth to children too of type 2. Particles of type 2 move according to standard Brownian motion and branch at rate 1, but cannot give birth to descendants of type 1. We obtain the asymptotic behaviour of the extremal process of particles of type 2.

Keywords: Extremal process, Branching Brownian motion, multitype branching process.

MSC 2020: Primary: 60G55; 60j80. Secondary: 60G70; 92D25.

1 Introduction

The branching Brownian motion is a particle system on $\mathbb{R}$ that can be described as follows. Start with on particle at the origin at time $t = 0$. After an exponential random time of mean one, this particle splits in two children. The new particles then start independent copies of the branching Brownian motion from their positions. Denote by $\mathcal{N}_t$ the set of particles alive at time $t$. For $u \in \mathcal{N}_t$, we denote by $X_u(t)$ the position at time $t$ of that particle. We define

$$M_t = \max_{u \in \mathcal{N}_t} X_u(t)$$

the position of the right most particle in the Branching Brownian motion.

There is a fundamental link between the BBM and the well known Fisher-Kolmogorov-Petrovsky-Piskunov (F-KPP) reaction-diffusion equation

$$\partial_t u = \frac{1}{2} \Delta u - u(1 - u). \quad (1.1)$$

If we denote by $u(t, x) = \mathbb{P}(M_t \leq x)$, the function $u$ is the solution of the F-KPP equation $(1.1)$ with Heaviside initial condition. It is also known [14] that there exists a function $m_t$ such that

$$u(t, m_t + x) \to w(x)$$

uniformly in $x$, where $w$ is a travelling wave solution of the F-KPP equation, that satisfies

$$\frac{1}{2} w_{xx} + \sqrt{2} w_x + w(w - 1) = 0. \quad (1.2)$$

Using these observations, Kolmogorov, Petrovskii and Piskunov [14] proved that $\lim_{t \to \infty} \frac{m_t}{t} = \sqrt{2}$. Bramson [9] showed, using the connection with the BBM, that

$$m_t = \sqrt{2} t - \frac{3}{2\sqrt{2}} \log(t) + O(1).$$

The convergence in distribution of the centred maximal displacement was obtained by Lalley and Sellke [15]. They proved that the centered maximum $M_t - m_t$ converges in law to a randomly shifted Gumbel distribution. More precisely, if we denote by

$$Z_t = \sum_{u \in \mathcal{N}_t} (\sqrt{2} t - X_u(t)) e^{-\sqrt{2}(\sqrt{2} t - X_u(t))}$$

...
the so-called derivative martingale of the BBM, they proved that there exists a constant $C^* > 0$ such that for all $x \geq 0$

$$
\lim_{t \to \infty} u(t, m_t + x) = \mathbb{E} \left( \exp(-C^* Z_{\infty} e^{-\sqrt{2t}}) \right)
$$

where

$$
Z_{\infty} := \lim_{t \to \infty} Z_t \text{ a.s.}
$$

One of the most interesting questions in the context of branching Brownian motion, is the study of the asymptotic behaviour of the extremal process defined by

$$
\mathcal{E}_t = \sum_{u \in \mathcal{N}_t} \delta_{X_u(t) - m_t}.
$$

It was shown by Aidékon, Berestycki, Brunet and Shi [1] as well as Arguin, Bovier and Kistler [3] that this point measure converges in law to a decorated Poisson point process with intensity $\sqrt{2C^* } Z_{\infty} e^{-\sqrt{2t}} dx$. A description of the law of decoration was given by Arguin, Bovier and Kistler [3], they proved that there exists a point measure $\mathcal{D}$ such that

$$
\lim_{t \to \infty} \mathbb{E} \left( \exp \left( - \sum_{u \in \mathcal{N}_t} \varphi(X_u(t) - M_t) \right) \bigg| M_t \geq \sqrt{2t} \right) = \mathbb{E} \left( \exp \left( - \left\langle \mathcal{D}, \varphi \right\rangle \right) \right),
$$

for all continuous function $\varphi$ with a bounded support on the left. Moreover, the point measure $\mathcal{D}$ is supported on $\mathbb{R}_-$, with an atom on 0. The limiting law of the extremal process of the branching Brownian motion, that we denote $\mathcal{E}_{\infty}$, can be described as follows. Let $\xi_i \in \mathbb{N}$ be the atoms of a Poisson point process with intensity $C^* \sqrt{2} e^{-\sqrt{2t}} dx$. For each atom $\xi_j$, we attached a point measure $\mathcal{D}_j$ where $(\mathcal{D}_j, j \in \mathbb{N})$ are i.i.d copies of the point measure $\mathcal{D}$ which are independents of $(\xi_i)_{i \in \mathbb{N}}$, then we set

$$
\mathcal{E}_{\infty} = \sum_{j \in \mathbb{N}} \sum_{d \in \mathcal{D}_j} \delta_{\xi_j + d + \log Z_{\infty}},
$$

where $\sum_{d \in \mathcal{D}_j}$ is the sum on the set of atoms of the point measure $\mathcal{D}_j$.

In this paper we study the asymptotic behaviour of the extremal process of $a$ of a cascading family of branching Brownian motion. This is a particle system on the real line such that each particle has a type in addition to his position. Particles of type 1 move on the real line according to Brownian motion with variance 1 and branch at rate 1 into two children of type 1. Additionally, at rate $\alpha$, they give birth to children of type 2. Particles of type 2 move according to standard Brownian motion and branch at rate 1, but cannot give birth to descendants of type 1.

In a recent paper [4], we studied the asymptotic behaviour of the extremal process of a two-type reducible branching Brownian motion where particles move and reproduce at a different rate. Note that the model we considered here can be seen as a critical case of the one studied in [1].

For all $t \geq 0$ we write $\mathcal{N}_t$ for the set of particle alive at time $t$, separated into $\mathcal{N}_t^1$ (respectively $\mathcal{N}_t^2$) the set of particles of type 1 (respectively type 2). If $u \in \mathcal{N}_t^2$, we denote by $T(u)$ the time at which the oldest ancestor of type 2 of $u$ was born from a particle of type 1. We also write $X_u(t)$ the position at time $t$ of $u \in \mathcal{N}_t$ and $\tilde{M}_t = \max_{u \in \mathcal{N}_t^2} X_u(t)$. We studied the asymptotic behaviour of the two-type reducible branching Brownian motion according to a phase diagram containing three regions. However, we didn’t consider the boundary case.

Multitype branching processes are widely used in biology and ecology. For example, when modeling certain diseases, such processes can be used to describe the evolution of cells that have carried out different numbers of mutations [10]. In epidemiology, multi-type continuous time Markov branching process may be used to describe the dynamics of the spread of parasites of two types that can mutate into each other in a common population [8]. Many applications of multi-type branching processes in biology can be found in [11, 13].

Historically, questions about the extreme values of spatial multitype branching processes were not a main subject of interest. Biggins in [6] gave an explicit formula for the speed of the reducible multitype process of the branching random walk. The irreducible case (i.e. when for all pair of types $i$ and $j$, particles of type $i$ have positive probability of having at least one descendant of type $j$ after an
exponential time) has been studied by Ren and Yang [19]. They showed that the asymptotic behaviour of the maximal displacement is similar to that of classical branching Brownian motion. However, the extremal process has not yet been studied. We expect that in this case, results should be similar to what is observed in the standard BBM.

Blath, Jacobie and Nie [7] studied the asymptotic speed in a modified version of the standard BBM, called the On/Off BBM. It is a branching Brownian motion on \( \mathbb{R} \) such that each particle has an active or dormant state. They studied the asymptotic behaviour of the maximal displacement of the On/Off BBM using the McKean representation [18] of the position of the rightmost particle as a solution of F-KPP equation. We observe here that this model can be seen as a two-type active/dormant BBM.

We now introduce our main result.

**Theorem 1.1.** Setting \( m_t = \sqrt{2t - \frac{1}{2\sqrt{2}}} \log(t) \), and \( \hat{E}_t = \sum_{u \in \mathcal{N}_t^1} \delta_{X_u(t) - m_t} \), we have

\[
\lim_{t \to \infty} \hat{E}_t = \hat{E}_\infty \quad \text{for the topology of the vague convergence,}
\]

where \( \hat{E}_\infty \) is a decorated Poisson point process with intensity \( C^* \alpha \sqrt{2} Z_{\infty} e^{-\sqrt{2} x} dx \), the constant \( C^* \) is the one introduced in (1.3) and

\[
Z_{\infty} := \lim_{t \to \infty} \sum_{u \in \mathcal{N}_t^1} (\sqrt{2t} - X_u(t)) e^{\sqrt{2} X_u(t) - 2t} \quad \text{a.s.}
\]

Moreover, we have \( \lim_{t \to \infty} P(M_t \leq m_t + x) = E \left( e^{-\alpha C^* Z_{\infty} e^{-\sqrt{2} x}} \right) \) for all \( x \in \mathbb{R} \).

The random variable \( Z_{\infty} \) is the same limit as in (1.4) since \( (X_u(t), u \in \mathcal{N}_t^1) \) is a standard branching Brownian motion.

An extension of this model is to consider the cascading BBM. It is a particle system that can be described as follows. Particles of type \( i \), \( i \geq 1 \) move according standard Brownian motion and branch at rate \( 1 \) into two children of type \( i \). Additionally, at rate \( \alpha \), they give birth to one particle of type \( i \) and one particle of type \( i + 1 \).

For all \( t \geq 0 \), we write \( \mathcal{N}_t^{(i)} \), \( i \geq 1 \) the set of particles of type \( i \). Fix \( k \geq 2 \), we conjecture the asymptotic behaviour of the extremal process of particles of type \( k \).

**Conjecture 1.2.** Setting \( m_t^{(k)} = \sqrt{2t - \frac{3}{2\sqrt{2}}} \log(t) + \frac{k-1}{\sqrt{2}} \log(t) \) and \( \hat{E}_t^{(k)} = \sum_{u \in \mathcal{N}_t^{(k)}} \delta_{X_u(t) - m_t^{(k)}} \), we have

\[
\lim_{t \to \infty} \hat{E}_t^{(k)} = \hat{E}_\infty^{(k)} \quad \text{for the topology of the vague convergence,}
\]

where \( \hat{E}_\infty^{(k)} \) is a decorated Poisson point process with intensity \( C^* \alpha^{k-1} \sqrt{2} Z_{\infty} e^{-\sqrt{2} x} dx \), with same notation as in Theorem 1.1.

**Remark 1.3.** We observe a change in the logarithmic correction of the median \( m_t \) comparing it to that of a classical branching Brownian motion. More precisely, we pass from a multiplicative factor \( \frac{k-1}{2\sqrt{2}} \) in the case of a standard BBM to a factor \( \frac{k-1}{2\sqrt{2}} + \frac{k}{\sqrt{2}} \) for particles of type \( k \).

**Notation.** Throughout the paper, we use \( C \) and \( c \) to denote a generic positive constant, that may change from line to line. We say that \( f_n \sim g_n \) if \( \lim_{n \to \infty} \frac{f_n}{g_n} = 1 \). For \( x \in \mathbb{R} \), we write \( x_+ = \max(x, 0) \).

**Organisation of the paper.** The rest of the paper is organised as follows. In the next section, we recall a version of a multitype many-to-one lemma that was introduced in [4]. In Section 3, we introduce some useful lemmas in the context of standard BBM. Finally, we conclude the paper with a proof of the main result.

## 2 Multitype many-to-one formula and Brownian motion estimates

The classical many-to-one lemma was first introduced by Kahane and Peyrière [12]. This lemma links an additive functional of branching Brownian motion with a simple function of Brownian motion. Let us recall the standard version of many-to-one in the context of classical BBM.
Lemma 2.1 (Many-to-on-lemma). For any \( t \geq 0 \), and measurable positive function \( f \), we have
\[
E \left( \sum_{u \in \mathcal{N}_t} f(X_u(s), s \leq t) \right) = e^t E(f(B_s, s \leq t)),
\]
(2.1)
where \( B \) is a standard Brownian motion.

Before we introduce a multitype version, we will set some notation. We write
\[
B = \{ u \in \cup_{t \geq 0} \mathcal{N}_t^2, T(u) = b_u \}
\]
for the set of particles of type 2 that are born from a particle of type 1. Recall that \( T(u) \) is the time at which the oldest ancestor of type 2 of \( u \) was born. The following proposition was introduced in [4, Corollary 4.3].

Proposition 2.2. For any measurable non-negative function \( f \), we have
\[
E \left( \sum_{u \in B} f(X_u(s), s \leq T(u)) \right) = \alpha \int_0^\infty e^{\beta t} E(f(B_s, s \leq t)) dt,
\]
\[
E \left( \exp \left( - \sum_{u \in B} f(X_u(s), s \leq T(u)) \right) \right) = E \left( \exp \left( - \alpha \int_0^\infty \sum_{u \in \mathcal{N}_t^1} 1 - e^{-f(X_u(s), s \leq t)} dt \right) \right).
\]
(2.2)

2.1 Brownian motion estimates

We now introduce some Brownian motion estimates that will be needed in the proof of the main result. Let \((B_s)_{s \geq 0}\) be a standard Brownian motion. The quantity \( \sup_{0 \leq s \leq t} B_s \) has the same law as \(|B_t|\). As a consequence, there exists \( C > 0 \) such that for all \( t \geq 1 \), \( y \geq 1 \) we have
\[
P(B_s \geq y, s \leq t) = P(|B_t| \leq y/\sqrt{t}) \leq Cy \wedge \sqrt{t}.
\]
(2.3)

We need also an estimate for the Brownian motion to stay below a line and end up in a finite interval. For all \( K \geq 1 \) and \( y \geq 1 \) we have
\[
P(B_s \leq K, s \leq t, B_t > K - y) \leq C \frac{(1 + K)(1 + y)}{(t + 1)^{3/2}}
\]
(2.4)
This estimate can be obtained using similar computations to those used in [17, Lemma 3.8] for random walks.

We next introduce the 3-dimensional Bessel process that we denote \((R_s)_{s \geq 0}\). We have the following link between the process \((R_s)_{s \geq 0}\) and the Brownian motion: For all \( t \geq 0 \), \( x > 0 \) and any measurable positive function \( g \), we have
\[
E_x \left( g(B_s, s \in [0, t]) 1_{\{B_t > 0, s \leq t\}} \right) = E_x \left( \frac{x}{R_s} g(R_s, s \in [0, t]) \right).
\]
(2.5)
In other words, \( R \) corresponds to the law of the Brownian motion conditioned on not hitting 0 in the sense of Doob’s \( h \)-transform. Let \( p_s(x, z) \) be the transition density of \( R_s \) started from \( x \) at time \( s \). We have
\[
p_s(x, z) = \sqrt{\frac{2}{\pi}} e^{-(z-x)^2/2s} \mathbb{1}_{\{z > 0\}} \times \begin{cases} \frac{z}{2x^2} (1 - e^{-2xz/s} & \text{if } x > 0 \, \text{ and } \, z > 0, \\ \frac{z}{2x^2} & \text{if } x = 0. \end{cases}
\]

2.2 Branching Brownian motion estimates

In this section, we denote by \((X_t(u), u \in \mathcal{N}_t)\) a standard branching Brownian motion. We recall here some useful estimates on this process, that will be used to prove Theorem 1.1.

We know that with high probability all particles in the one-type BBM are smaller than \( \sqrt{2t} + y \), for all \( y \geq 0 \). More precisely, we have the following upper bound.
Proposition 2.3. There exists a constant $C > 0$ such that for any $t \geq 1$ and $K \geq 1$
\[ P \left( \exists s \geq 0, u \in \mathcal{N}_u^t : X_u(s) \geq \sqrt{2}s + K \right) \leq C(K + 1)e^{-\sqrt{2}K}. \]

Proof. Let $l \geq 1$ be an integer. Define $\tau = \inf \{ t \leq l, \exists u \in \mathcal{N}_u^t : X_u(t) \geq \sqrt{2}s + K \}$ and $Z_l$ to be the number of particles in $\mathcal{N}_u^t$ that stay below the barrier $s \mapsto \sqrt{2}s + K$ for all $s \leq l - 1$ and such that $X_u(t) > \sqrt{2}s + K$ for some $t \in [l - 1, l]$. Then, by the Markov inequality, we have
\[ P \left( \exists s \geq 0, u \in \mathcal{N}_u^t : X_u(s) \geq \sqrt{2}s + K \right) \leq \sum_{l=1}^{\infty} P(\tau \in [l - 1, l]) = \sum_{l=1}^{\infty} E(Z_l). \]

Using Lemma 2.1, we obtain
\[ E(Z_l) \leq P \left( B_s \leq \sqrt{2}s + K, s \leq l - 1, B_r > \sqrt{2}r + K \text{ for some } r \in [l - 1, l] \right). \] (2.6)

Applying the Markov property at time $l - 1$, we get
\[ P \left( B_s \leq \sqrt{2}s + K, s \leq l - 1, B_r > \sqrt{2}r + K \text{ for some } r \in [l - 1, l] \right) \leq E(g(\sup_{0 \leq s \leq 1} B_s)) \]
where $g(x) = P \left( B_s \leq \sqrt{2}s + K, s \leq l - 1, B_{l-1} > \sqrt{2}(l - 1) + K - x \right)$. Moreover, using Girsanov theorem we have
\[ P \left( B_s \leq \sqrt{2}s + K, s \leq l - 1, B_{l-1} > \sqrt{2}(l - 1) + K - x \right) \leq e^{-t}e^{\sqrt{2}(x-K)}P \left( B_s \leq K, s \leq l - 1, B_{l-1} > K - x \right) \]
\[ \leq C e^{-t}e^{\sqrt{2}(x-K)} \frac{(1 + K)(1 + x)}{(l + 1)^{3/2}} \]
where in the last inequality we used (2.4). Plugging all this in (2.6) and using that $\sup_{0 \leq s \leq 1} B_s$ has the same law as $|B_1|$ (see Section 2.1), then easy computations lead to
\[ P \left( \exists s \geq 0, u \in \mathcal{N}_u^t : X_u(s) \geq \sqrt{2}s + K \right) \]
\[ \leq C(K + 1)e^{-\sqrt{2}K} \sum_{l=1}^{\infty} \frac{1}{(l + 1)^{3/2}} \leq C(K + 1)e^{-\sqrt{2}K}, \]
which completes the proof. \hfill \Box

We also have an upper bound on the tail of the maximal displacement that was introduced by Bramson [9] and refined by Arguin, Bovier and Kestler [2]. We write $\tilde{m}_t = \sqrt{2}t - \frac{1}{\sqrt{2}\log(t)}$.

Proposition 2.4. [2, Corollary 10] There exists $t_0 > 0$ such that $\forall t \geq t_0$ and $y \in \mathbb{R}_+$
\[ P(M_t > \tilde{m}_t + y) \leq C(1 + y)e^{-\sqrt{2}y^{3/2}} \]
for some constant $C > 0$.

We next recall a link between the FKPP equation and the branching Brownian motion.

Lemma 2.5. Let $f : \mathbb{R} \mapsto [0, 1]$ a measurable function and
\[ u_f(t, x) = 1 - E[\prod_{u \in \mathcal{N}_u} (1 - f(x - X_u(t)))]. \] (2.7)
Then $u_f$ solves the FKPP equation with the initial condition $u_f(0, x) = f(x)$. 

5
In our work we need an uniform estimate of general solutions of the F-KPP equation that is useful for the computation of the asymptotics of the Laplace transform of the extremal process of the BBM. Before that, let us recall a result of Bramson \cite{Bramson} on the convergence of the solutions of F-KPP equation to travelling wave (see also Theorem 4.2 in [3]).

**Theorem 2.6.** \cite[Theorems A, B]{Bramson} Let $u_f$ be a solution of the F-KPP equation in the form of \eqref{2.7} with the initial condition $u(0, x) = f(x)$, where the function $f$ satisfying

\begin{align*}
(ii) & \quad 0 \leq f(x) \leq 1 \\
(iii) & \quad \text{for some } y > 0, N > 0, M > 0, \int_{-N}^{x+N} u(0, z) dz > y \text{ for all } x \leq -M, \quad \text{\eqref{2.8}} \\
(iv) & \quad \sup \{x \in \mathbb{R}, f(x) > 0\} < \infty,
\end{align*}

then

\[ u_f(t, \bar{m}_t + x) \to w(x), \text{ uniformly in } x \text{ as } t \to \infty, \]

where $w$ is the unique solution (up to translation) of the equation \eqref{1.2}.

The next proposition follows from Proposition 4.3 and Lemma 4.5 in \cite{Bramson}.

**Proposition 2.7.** Let $u_f$ be a solution of the F-KPP equation in the form of \eqref{2.7} with the initial condition $u(0, x) = f(x)$ and satisfying the assumptions of Theorem 2.6. Then, for any fixed $\varepsilon > 0$, uniformly in $x \in [-\frac{1}{\sqrt{t}}, -\varepsilon \sqrt{t}]$, we have the convergence

\[ \lim_{t \to \infty} \frac{e^{-\sqrt{2t}}}{(-x)} t^{3/2} e^{x^2/2t} u_f(t, \sqrt{2t} - x) = \gamma(f), \quad \text{\eqref{2.9}} \]

where $\gamma(f) = \lim_{r \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_f(r, z + \sqrt{2t}) e^{z^2/2t} dz$.

**Proof.** Fix $\varepsilon > 0$, using Proposition 4.3 in \cite{Bramson} for $r$ large enough, $t \geq 8r$ and $-x \geq 8r - \frac{3}{2\sqrt{2}} \log(t)$, we have

\[ \rho^{-1}(r) \psi(r, t, -x + y + \sqrt{2}) \leq u_f(t, -x + y + \sqrt{2}) \leq \rho(r) \psi(r, t, -x + y + \sqrt{2}) \]

where $\rho(r) \to 1$ as $r \to \infty$ and

\[ \psi(r, t, -x + y + \sqrt{2}) = \frac{e^{-\sqrt{2t}}}{\sqrt{2\pi}(t-r)} \int_{0}^{\infty} u_f(r, z + \sqrt{t-r}) e^{z^2/2(t-r)} \left( 1 - e^{-2z \sqrt{t-r}} \right)^3 dz. \]

Using Lemma 4.5 in \cite{Bramson}, and since $\rho(r) \to 1$ we have

\[ \lim_{t \to \infty} \sup_{x \in [-\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}]} \frac{e^{\sqrt{2t}(y-x)}}{-x} t^{3/2} e^{-x^2/2t} u_f(t, -x + y + \sqrt{2t}) \]

\[ \leq \lim_{t \to \infty} \inf_{x \in [-\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}]} \sup_{x \in [-\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}]} \frac{e^{\sqrt{2t}(y-x)}}{-x} t^{3/2} e^{-x^2/2t} \psi(r, t, -x + y + \sqrt{2t}) \leq \gamma(f) \]

and similarly

\[ \lim_{t \to \infty} \inf_{x \in [-\frac{1}{\sqrt{t}}, \frac{1}{\sqrt{t}}]} \frac{e^{\sqrt{2t}(y-x)}}{-x} t^{3/2} e^{-x^2/2t} u_f(t, -x + y + \sqrt{2t}) \geq \gamma(f) \]

for some constant $\gamma(\varphi)$ given in Lemma 4.5 in \cite{Bramson}, which completes the proof.

In particular, by setting $f(x) = \mathbb{1}_{\{x \leq 0\}}$, we have $u(t, \sqrt{2t} - x + y) = P(M_t > \sqrt{2t} - x + y)$, and the following uniform estimate of the tail of $M_t$.

**Corollary 2.8.** For all $\varepsilon > 0$ and $y \in \mathbb{R}_+$, we have

\[ P(M_t > \sqrt{2t} - x + y) \sim_{t \to \infty} \frac{C^*}{t^{3/2}} e^{-\sqrt{2}(y-x)} e^{-x^2/2t} \quad \text{\eqref{2.10}} \]

uniformly in $x \in [-\frac{1}{\sqrt{t}}, -\varepsilon \sqrt{t}]$, where the constant $C^*$ is the one introduced in \eqref{1.3}.
We end this section by an uniform estimate of the Laplace transform of the extremal process of the BBM that generalizes (2.10). Denote by $\mathcal{T}$ the set of non-negative, continuous, bounded functions $\varphi : \mathbb{R} \to \mathbb{R}_+$ with support bounded on the left.

**Corollary 2.9.** Fix $\varepsilon > 0$. Setting

$$E_t(x) = \sum_{u \in N_t} \delta_{X_u(t) - \sqrt{2t} + x},$$

we have for all $\varphi \in \mathcal{T}$

$$E \left( 1 - e^{-\sum_{u \in N_t} \varphi(x + X_u(t) - \sqrt{2t})} \right) = C^* e^{\frac{\sqrt{2}x - \frac{2^2}{3}2^2}{\sqrt{t}}} \int e^{-\sqrt{2\|D\|} \left( 1 - E(e^{-\langle D, \varphi \rangle}) \right)} \, dz(1 + o(1)),$$

uniformly in $x \in [-\frac{1}{2}\sqrt{t}, -\varepsilon \sqrt{t}]$, as $t \to \infty$.

**Proof.** The proof follows from Proposition 2.7. By setting $f(x) = 1 - e^{-\varphi(-x)}$, we have

$$u_f(t, \sqrt{2t} - x - X_u(t)) = E \left( 1 - e^{-\sum_{u \in N_t} \varphi(x + X_u(t) - \sqrt{2t})} \right).$$

Now observe that for all $\varphi \in \mathcal{T}$ the function $x \mapsto f(x) = 1 - e^{-\varphi(x)}$ satisfies assumptions of Theorem 2.6, then in view of Proposition 2.7, we obtain

$$E \left( 1 - e^{-\sum_{u \in N_t} \varphi(x + X_u(t) - \sqrt{2t})} \right) = \frac{e^{\sqrt{2x} - \frac{2^2}{3}2^2}}{\sqrt{t}} \gamma(\varphi)(1 + o(1)).$$

On the other hand, it is known, using Corollary 4.12 in [3], that the constant $\gamma(\varphi)$ can expressed through the decoration $D$ defined in (1.5), as follows

$$\gamma(\varphi) = C^* \sqrt{2} \int e^{-\sqrt{2\|D\|} \left( 1 - E(e^{-\langle D, \varphi \rangle}) \right)} \, dz,$$

where the constant $C^*$ is introduced in (1.3), which completes the proof. \(\square\)

## 3 Proof of the main result

Using [5, Lemma 4.1], it is enough to show that for all $\varphi \in \mathcal{T}$

$$\lim_{t \to \infty} E \left( e^{-\langle \tilde{Z}_t, \varphi \rangle} \right) = E \left( \exp(-\alpha C^* \sqrt{2Z_\infty} \int e^{-\sqrt{2\|D\|} \left( 1 - E(e^{-\langle D, \varphi \rangle}) \right)} \, dz) \right),$$

where $D$ is the law of the point measure defined in (1.5).

The first step of the proof of Theorem 1.1 is to show that for all $A \geq 0$ and $\varepsilon > 0$, every particle $u$ of type 2 to the right of $m_t - A$ at time $t$ satisfy $T(u) \in [\varepsilon t, (1 - \varepsilon)t]$ with high probability.

**Proposition 3.1.** Fix $A > 0$, $m_t = \sqrt{2t} - \frac{1}{2\sqrt{2}} \log(t)$. We have

$$\lim_{\varepsilon \to 0} \limsup_{t \to \infty} P(\exists u \in N^2_t : T(u) \notin [\varepsilon t, (1 - \varepsilon)t], X_u(t) \geq m_t - A) = 0 \quad (3.1)$$

**Proof.** We first set, for $\varepsilon, A, K \geq 0$ and $t \geq 0$:

$$Z_t(A, \varepsilon, K) = \sum_{u \in B} \| T(u) \leq \varepsilon t \| \{ X_u(r) \leq r \sqrt{2 + K}, r \leq T(u) \} \| \{ M^*_r \geq m_t - A \};$$

and

$$\tilde{Z}_t(A, \varepsilon, K) = \sum_{u \in B} \| T(u) \geq (1 - \varepsilon)t \| \{ X_u(r) \leq r \sqrt{2 + K}, r \leq T(u) \} \| \{ M^*_r \geq m_t - A \};$$
where \( M^n_t \) is the position of the rightmost descendant at time \( t \) of the individual \( u \). Observe that by Markov inequality and Proposition 2.3 we have

\[
P(\exists u \in A^l_2 : T(u) \notin [st, (1 - \epsilon) t]), X_u(t) \geq m_t - A)
\leq P(\exists u \in A^l_2 : X_u(t) \geq \sqrt{2s} + K) + P(Z_t(A, \epsilon, K) \geq 1)
\leq C(K + 1) + E(Z_t(A, \epsilon, K)) + E(\tilde{Z}_t(A, \epsilon, K)).
\]

Hence by fixing \( K \) large enough, it is enough to prove that \( \limsup_{t \to \infty} E(Z_t(A, \epsilon, K)) \) and \( \limsup_{t \to \infty} E(\tilde{Z}_t(A, \epsilon, K)) \) are both \( o(1) \) to complete the proof.

Using the branching property and Corollary 2.2, we have

\[
E(Z_t(A, \epsilon, K)) = E \left( \sum_{u \in B} \mathbb{1}_{T(u) \leq ct} \mathbb{1}_{X_u(t) \leq \sqrt{2s} + K} \mathbb{1}_{r \leq T(u)} F(t - T(u), X_u(T(u))) \right)
\leq \alpha \int_0^{ct} e^{-\sqrt{2s} t} F(t - s, B_s) \mathbb{1}_{B_s \leq \sqrt{2s} + K} ds
\leq \alpha \int_0^{ct} e^{-\sqrt{2s} t} F(t - s, B_s + \sqrt{2s}) \mathbb{1}_{B_s \leq K} ds.
\]

where we have set \( F(r, x) = \mathbb{P}^{(2)}(x + M_r \geq 2 - A) \).

By Proposition 2.3, there exists \( C > 0 \) such that for all \( x \in \mathbb{R} \) and \( t \geq 0 \), we have

\[
\mathbb{P}^{(2)}(M_t \geq m_t + x) \leq C(1 + x_+) e^{-\sqrt{2t} x},
\]

so that for all \( s \leq t \),

\[
F(t - s, x) = \mathbb{P}^{(2)}(M_{t-s} \geq \sqrt{2}(t - s) - \frac{1}{\sqrt{2}} \log(t) - A - (x - \sqrt{2} s))
\leq C \frac{\sqrt{t + 1}}{(t - s + 1)^{\frac{1}{2}}} \left( 1 + \frac{\log(t)}{\sqrt{2}} + (-x)_+ \right) e^{-\sqrt{2}(t-s) - A}.
\]

As a result using that \( s \leq ct \)

\[
E(Z_t(A, \epsilon, K)) \leq \alpha \frac{2C}{t} \int_0^{ct} \mathbb{E}(c + \frac{\log(t)}{\sqrt{2}} + (-B_s)_+ \mathbb{1}_{B_s \leq K}) ds.
\]

Using (2.3) and the definition of Bessel process (2.5), we get

\[
E(Z_t(A, \epsilon, K)) \leq \alpha \frac{2C}{t} \int_0^{ct} \frac{1}{\sqrt{s}} ds + 2K \alpha C \epsilon = \alpha \frac{C}{t} \left( c + \frac{\log(t)}{\sqrt{2}} \right)^{\frac{1}{2}} + 2K \alpha C \epsilon
\]

We now estimate \( E(\tilde{Z}_t(A, \epsilon, K)) \). Using similar calculation we have

\[
E(\tilde{Z}_t(A, \epsilon, K)) \leq \alpha \int_{(1-\epsilon) t}^{t} \mathbb{E}(e^{-\sqrt{2s} t} F(t - s, B_s + \sqrt{2s}) \mathbb{1}_{B_s \leq K, r \leq s}) ds
\]

where again \( F(r, x) = \mathbb{P}^{(2)}(x + M_r \geq 2 - A) \). Using Proposition 2.8 we have the following upper bound

\[
E(\tilde{Z}_t(A, \epsilon, K)) \leq \alpha C \int_{(1-\epsilon) t}^{t} \frac{\sqrt{t + 1}}{(t - s + 1)^{\frac{1}{2}}} \mathbb{E}K \left( \frac{K}{R_s} e^{-\frac{n^2}{2(t-s)^{\frac{1}{2}}}} + \frac{\log(t)}{\sqrt{2}} + R_s \right) ds
\leq 2\alpha C \int_{(1-\epsilon) t}^{t} \frac{1}{\sqrt{s}} \mathbb{E}K \left( \frac{K}{R_s} e^{-\frac{n^2}{2(t-s)^{\frac{1}{2}}}} + \frac{\log(t)}{\sqrt{2}} + R_s \right) ds.
\]

By the definition of a Bessel process we obtain
where we used Bessel scaling in (3.4).

On the one hand, we know that the density of $R_1$ under $P_x$ for $x > 0$ is equal to

$$y \mapsto \frac{y e^{-(y-x)^2/2}}{x \sqrt{2\pi}} (1 - e^{-2xy}) \mathbf{1}_{\{y > 0\}}.$$

Using that for $x, y > 0$, $1 - e^{-2xy} \leq 2xy$ we have

$$B(x, y) := \mathbb{E}_x \left( \frac{e^{-x^2 y^2}}{\pi (x^2 + y^2)} \right) \leq \frac{1}{\pi x^2 y^2} \int_0^\infty \int_0^\infty \frac{e^{-u^2}}{u^2} e^{-(y-x)^2/2} (1 - e^{-xy}) dy dx.$$

Plugging this in equation (3.4) and by the change of variable $u = \frac{t}{y}$, we have

$$\int_{(1-\epsilon)t}^t \frac{\sqrt{t+1}}{\sqrt{(s-t+1)^2}} e^{-\frac{x^2 t^2}{s-t} - \frac{c}{t}} ds \leq C \frac{c + \log(t)}{\sqrt{t}} \int_{1-\epsilon}^1 \frac{e^{-\frac{x^2 t^2}{s-t} - \frac{c}{t}}}{s} y(x(1-u)) e^{-x^2 u/2} dy du \leq \frac{e^{-\frac{x^2(t+1)}{t}}}{(1-u)^2} + x.$$

Plugging this in (3.6), for $t$ large enough we deduce that

$$\int_{(1-\epsilon)t}^t \frac{\sqrt{t+1}}{\sqrt{(s-t+1)^2}} e^{-\frac{x^2 t^2}{s-t} - \frac{c}{t}} ds \leq C \frac{c + \log(t)}{\sqrt{t}} \int_{1-\epsilon}^1 \frac{e^{-\frac{x^2 t^2}{s-t} - \frac{c}{t}}}{s} du + C \epsilon.$$ 

Similarly we bound equation (3.5)

$$\int_{(1-\epsilon)t}^t \frac{\sqrt{t+1}}{\sqrt{(t-s+1)^2}} e^{-\frac{x^2 t^2}{t-s} - \frac{c}{t}} ds \leq C \frac{c + \log(t)}{\sqrt{t}} \int_{1-\epsilon}^1 \frac{e^{-\frac{x^2 t^2}{s-t} - \frac{c}{t}}}{s} dy du \leq \int_{1-\epsilon}^1 1 + \sqrt{1 - u} e^{-\frac{x^2(t+1)}{2tu}} du.$$

We finally obtain, for $t$ large enough

$$C \frac{t}{\sqrt{t}} \int_0^t e^{-\frac{x^2 t^2}{s-t} - \frac{c}{t}} du + C \epsilon \leq 2C \alpha \left( \frac{c + \log(t)}{\sqrt{t}} \right) \int_0^t e^{-\frac{x^2 t^2}{s-t} - \frac{c}{t}} du + C \epsilon \leq 2 \alpha C \left( \frac{c + \log(t)}{\sqrt{t}} \right) \sqrt{\epsilon} + 2C \epsilon K \epsilon,$$

letting $t \to \infty$ then $\epsilon \to 0$ in (3.3) and (3.7) we conclude that

$$\lim_{\epsilon \to 0} \lim_{t \to \infty} \mathbb{P} \left( \exists u \in \mathcal{N}_c : X_u(t) \geq m_c - A, T(u) \notin [\epsilon t, (1-\epsilon)t] \right) = 0,$$

which completes the proof.
We now show that, with high probability, every particle of type 2 that contributes to the extremal process of the BBM satisfy \( X_u(T(u)) - \sqrt{2T(u)} \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \).

**Proposition 3.2.** Fix \( A > 0 \). We have

\[
\lim_{\varepsilon \to 0} \limsup_{t \to \infty} \mathbb{P} \left( \exists u \in \mathcal{N}^2_t : X_u(t) \geq m_t - A, X_u(T(u)) - \sqrt{2T(u)} \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right) = 0
\]

**Proof.** We write

\[
\mathbb{P} \left( \exists u \in \mathcal{N}^2_t : X_u(t) \geq m_t - A, X_u(T(u)) - \sqrt{2T(u)} \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right) \\
\leq \mathbb{P} \left( \exists u \in \mathcal{N}^2_t : X_u(t) \geq m_t - A, T(u) \notin [\varepsilon t, (1-\varepsilon)t] \right) \\
+ \mathbb{P} \left( \exists u \in \mathcal{N}^2_t : X_u(t) \geq m_t - A, T(u) \in [\varepsilon t, (1-\varepsilon)t], X_u(T(u)) - \sqrt{2T(u)} \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right).
\]  

(3.8)

Using Proposition 3.1 it is enough to estimate (3.8). We set, for \( \varepsilon, A, K \geq 0 \) and \( t \geq 0 \):

\[
Y_t(A, \varepsilon, K) \equiv \sum_{u \in \mathcal{B}} \mathbb{I} \left( T(u) \in [\varepsilon t, (1-\varepsilon)t], X_u(T(u)) - \sqrt{2T(u)} \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right) \mathbb{I} \left( X_u(t) \geq -\varepsilon A \right) \mathbb{I} \left\{ X_u(r) \leq \sqrt{t} + K, r \leq T(u) \right\} \mathbb{I} \{ m_t^* \geq m_t - A \}.
\]

By the Markov inequality, it is enough to estimate \( \mathbb{E}(Y_t(A, \varepsilon, K)) \). We have

\[
\mathbb{E}(Y_t(A, B, K)) \\
= \mathbb{E} \left( \sum_{u \in \mathcal{B}} \mathbb{I} \left( T(u) \in [\varepsilon t, (1-\varepsilon)t], X_u(T(u)) - \sqrt{2T(u)} \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right) \mathbb{I} \left( X_u(t) \geq -\varepsilon A \right) \right) \mathbb{E} \left( F(t - T(u), X_u(T(u))) \right)
\]

\[
= \alpha \int_{\varepsilon t}^{(1-\varepsilon)t} e^{\varepsilon t} \times \mathbb{E} \left( F \left( t - s, B_x \right) \mathbb{I} \left\{ B_x - \sqrt{2s} \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right], B_x \leq \sqrt{t} + K, r \leq s \right\} \right) ds
\]

\[
= \alpha \int_{\varepsilon t}^{(1-\varepsilon)t} e^{-\varepsilon \sqrt{2s}} F \left( t - s, B_x + \sqrt{2s} \right) \mathbb{I} \left\{ B_x \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right], B_x \leq \sqrt{t} + K, r \leq s \right\} ds,
\]

where we have set \( F(r, x) = \mathbb{P}^{(2)}(x + M_r \geq m_t - A) \). Using Proposition 2.8 we have

\[
\mathbb{E}(Y_t(A, \varepsilon, K)) \leq \alpha C \int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{1+1}}{(t-s+1)^{\frac{3}{2}}} \mathbb{E} \left( e^{-\frac{2}{2(t-s)}} \left( c + \log(\frac{t}{\sqrt{2}}) + (-B_s)_+ \right) \mathbb{I} \left\{ B_s \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right], B_s \leq \sqrt{t} + K, r \leq s \right\} \right) ds.
\]

By (2.5), we get

\[
\int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{1+1}}{(t-s+1)^{\frac{3}{2}}} E_k \left( \frac{K}{R} \mathbb{I} \left\{ R, \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right\} \right) ds
\]

\[
\leq \left( c + \frac{\log(t)}{\sqrt{2}} \right) \int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{1+1}}{(t-s+1)^{\frac{3}{2}}} \mathbb{E}_K \left( \frac{K}{R} \mathbb{I} \left\{ R, \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right\} \right) ds + \int_{t}^{(1-\varepsilon)t} \frac{\sqrt{1+1}}{(t-s+1)^{\frac{3}{2}}} \mathbb{E}_K \left( \mathbb{I} \left\{ R, \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right\} \right) ds.
\]

Using the change of variable \( u = s/t \) and the Bessel scaling we have

\[
\int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{1+1}}{(t-s+1)^{\frac{3}{2}}} \mathbb{E}_K \left( \frac{K}{R} \mathbb{I} \left\{ R, \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right\} \right) ds + \int_{t}^{(1-\varepsilon)t} \frac{\sqrt{1+1}}{(t-s+1)^{\frac{3}{2}}} \mathbb{E}_K \left( \mathbb{I} \left\{ R, \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right\} \right) ds
\]

\[
\leq \frac{1}{\varepsilon t} \int_{\varepsilon}^{1-\varepsilon} \frac{1}{\sqrt{u(1-u)^{3/2}}} \mathbb{E}_K / \sqrt{u} \left( \frac{K}{K} e^{-\frac{2}{2(t-s)}} \mathbb{I} \left\{ R, \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right\} \right) du
\]

\[
+ \frac{2}{(1-u)^{3/2}} \mathbb{E}_K / \sqrt{u} \left( e^{-\frac{2}{2(t-s)}} \mathbb{I} \left\{ R, \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right\} \right) du.
\]

(3.9)

We split the expectation into two parts

\[
\mathbb{E} \left( \frac{1}{R} e^{-\frac{2}{2(t-s)}} \mathbb{I} \left\{ R, \notin \left[ \frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t} \right] \right\} \right) = \mathbb{E} \left( \frac{1}{R} e^{-\frac{2}{2(t-s)}} \mathbb{I} \left\{ R, \leq \varepsilon \right\} \right) + \mathbb{E} \left( \frac{1}{R} e^{-\frac{2}{2(t-s)}} \mathbb{I} \left\{ R, \geq \varepsilon \right\} \right)
\]
and we will deal with the two quantities in the same way. Using that \(1 - e^{-2xy} \leq 2xy\), for \(x = \frac{K}{\sqrt{nu}}\), \(y > 0\) and \(t \) large enough we have

\[
E_x \left( \frac{1}{R_1} e^{-\frac{u^2}{2\pi \nu t}} I_{(R_1 \leq \varepsilon)} \right) \leq \frac{1}{\sqrt{2\pi}} \int_0^\varepsilon \frac{1}{x} e^{-\frac{x^2}{2\pi \nu t}} e^{-(y-x)^2/2} (1-e^{-2xy}) dy \leq \frac{2}{\sqrt{2\pi}} \int_0^\varepsilon ye^{-\frac{y^2}{2\pi \nu t}} e^{-y^2/2} dy.
\]

Then with the change of variable \(v = y\sqrt{\frac{u}{1-u}}\), we obtain

\[
\int_\varepsilon^1 \frac{1}{\sqrt{u(1-u)^{3/2}}} e^{-\frac{v^2}{2\pi \nu t}} e^{y^2/2} du \leq \int_\varepsilon^y \frac{1}{\sqrt{t}} e^{-\frac{v^2}{2\pi \nu t}} e^{-y^2/2} dv = 2 \int_\varepsilon^y \frac{1}{\sqrt{t}} e^{-\frac{v^2}{2\pi \nu t}} e^{-2v^2/2(\nu^2+y^2)^2} dv \leq 2 \int_\varepsilon^y \frac{1}{\sqrt{t}} e^{-y^2/2} y dv \leq \sqrt{2\pi}/y.
\]

As a result, using Fubini’s theorem in (3.10), we obtain the following upper bound

\[
\int_\varepsilon^1 \frac{1}{\sqrt{u(1-u)^{3/2}}} E_{K/\sqrt{nu}} \left( \frac{K}{R_1} e^{-\frac{u^2}{2\pi \nu t}} I_{(R_1 \leq \varepsilon)} \right) du \leq C(\int_0^\varepsilon e^{-y^2/2} dy + \int_1^{\infty} e^{-y^2/2} dy).
\]

We similarly bound

\[
\int_\varepsilon^1 \frac{1}{(1-u)^{3/2}} E_{K/\sqrt{nu}} \left( \frac{1}{R_1} e^{-\frac{u^2}{2\pi \nu t}} I_{(R_1 \leq \varepsilon)} \right) du \leq C(\int_0^\varepsilon e^{-y^2/2} dy + \int_1^{\infty} e^{-y^2/2} dy).
\]

As a result we obtain

\[
E(Y_t(A, \varepsilon, K)) \leq \alpha C \int_\varepsilon^1 \frac{1}{(t-s+1)} \left( \frac{\sqrt{t + 1}}{\sqrt{2}} \right)^2 E \left( e^{-\frac{v^2}{2\pi \nu t}} \left( c + \log(t) \sqrt{2} + (-B_s) \right) \right) I_{\{B_v \in \{-\sqrt{t}, \varepsilon \sqrt{t}, 2B_v \leq K, v \leq t \}} ds
\]

\leq C(\int_0^\varepsilon e^{-y^2/2} dy + \int_1^{\infty} e^{-y^2/2} dy) \left( \frac{1}{\sqrt{t}} \left( c + \log(t) \sqrt{2} \right) + 1 \right).
\]

Letting \(t \to \infty\) then \(\varepsilon \to 0\), we conclude the proof.

We now turn to the proof of the main theorem.

Proof of Theorem 1.1. Let \(\varepsilon > 0\), we set

\[
E^\varepsilon_t := \sum_{u \in B} I_{\{T(u) \in [t, (1-\varepsilon)t]\}} I_{\{X_u(T(u)) - \sqrt{2}T(u) \in [-\frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t}]\}} \sum_{u' \in N^2_t \atop u' \neq u} \delta_{X_{u'}(t) - m_t}.
\]

Let \(\varphi \in \mathcal{T}\), we assume the support of \(\varphi\) is contained in \([-A, \infty)\) for some \(A > 0\). We set

\[
\mathcal{G}_t(\varepsilon) = \left\{ \exists u \in N^2_t : X_u(t) \geq m_t - A, T(u) \in [t, (1-\varepsilon)t], X_u(T(u)) - \sqrt{2}T(u) \in [-\frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t}] \right\}.
\]

By Propositions 3.1 and 3.2 we have \(\limsup_{t \to \infty} P(\mathcal{G}_t(\varepsilon)^c) = 0\) as \(\varepsilon \to 0\), furthermore we have

\[
\left| E \left( e^{-\langle \mathcal{E}^\varepsilon, \varphi \rangle} \right) - E \left( e^{-\langle \mathcal{E}^\varepsilon_t, \varphi \rangle} \right) \right| \leq P(\mathcal{G}_t(\varepsilon)^c),
\]

therefore it is enough to compute the asymptotic behaviour of \(E(e^{-\langle \mathcal{E}^\varepsilon_t, \varphi \rangle})\).

Using (2.2), for all \(\varphi \in \mathcal{T}\), we have

\[
E \left( e^{-\langle \mathcal{E}^\varepsilon_t, \varphi \rangle} \right) = E \left( \exp \left( -\alpha \int_{t}^{(1-\varepsilon)t} \sum_{u \in N^2_t} I_{\{X_{u}(s) \neq s\in [\frac{1}{\varepsilon} \sqrt{t}, -\varepsilon \sqrt{t}]\}} F_{\varphi}(t-s, X_u(s) - \sqrt{2}s) ds \right) \right).
\]
with \( F_\varphi(r, x) = 1 - \mathbb{E}^{(2)} \left( e^{-\sum_{u \in N^2} \varphi(X_u(r) - m, -x)} \right) \). Additionally, by Corollary 2.9, we have
\[
F_\varphi(r, x) = \frac{\sqrt{t}(x) e^{\sqrt{2t} x}}{r^{\frac{3}{2}}} e^{-x^2 / 2r} (1 + o(1))
\]
as \( r \to \infty \), uniformly in \( t - r \in [\varepsilon t, (1 - \varepsilon) t] \) and \( x \in [-\frac{1}{2} \sqrt{t}, -\varepsilon \sqrt{t}] \).

As a result, recalling that \( \gamma(\varphi) = \alpha C^* \sqrt{2} \int e^{-\sqrt{2t} (1 - \mathbb{E}(e^{-(D, \varphi, +z)}))} dz \) using the notation of Corollary 2.9, we have
\[
\limsup_{t \to \infty} \mathbb{E} \left( e^{-\langle \mathcal{E}_t, \varphi \rangle} \right) \leq \limsup_{t \to \infty} \mathbb{E} \left( \exp \left( -\gamma(\varphi) \int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t}}{(t-s)^{3/2}} \tilde{Z}_s^z ds \right) \right)
\]
and
\[
\liminf_{t \to \infty} \mathbb{E} \left( e^{-\langle \mathcal{E}_t, \varphi \rangle} \right) \geq \liminf_{t \to \infty} \mathbb{E} \left( \exp \left( -\gamma(\varphi) \int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t}}{(t-s)^{3/2}} \tilde{Z}_s^z ds \right) \right)
\]
where
\[
\tilde{Z}_s^z = \sum_{u \in N_s} \left( \sqrt{2} X_u(s) - X_v^z(s) \right) e^{\frac{(v - X_u(s))^2}{2} +} \mathbb{1}_{[|X_u(s) - \sqrt{2} | \leq 1/(\sqrt{t})]} \}
\]
We set \( \lambda t = s \), then we have
\[
\mathbb{E} \left( \exp \left( -\gamma(\varphi) \int_{\varepsilon t}^{(1-\varepsilon)t} \frac{\sqrt{t}}{(t-s)^{3/2}} \tilde{Z}_s^z ds \right) \right) = \mathbb{E} \left( \exp \left( -\gamma(\varphi) \int_{\varepsilon}^{1-\varepsilon} \frac{\tilde{Z}_z^z}{(1-\lambda)^{3/2}} d\lambda \right) \right).
\]
We now observe that by Theorem 1.2 in [16], for all \( \lambda \in [0, 1] \) we have
\[
\lim_{t \to \infty} \tilde{Z}_s^z = \mathbb{E}(h_{\lambda, \varepsilon}(R_1)) Z_\infty
\]
where \( x \mapsto h_{\lambda, \varepsilon}(x) = e^{-\frac{1}{1-\lambda}} x^2 \mathbb{1}_{[x \leq 1/(\sqrt{t})]} \} \), \( (R_\varepsilon)_s \geq 0 \) is a 3-dimensional Bessel process and \( Z_\infty \) is the limit of the critical derivative martingale. As a result, writing
\[
c(\varepsilon) = \int_{\varepsilon}^{1-\varepsilon} \frac{\mathbb{E}(h_{\lambda, \varepsilon}(R_1))}{(1-\lambda)^{3/2}} d\lambda,
\]
by dominated convergence theorem, (3.11) and (3.12) yield
\[
\lim_{t \to \infty} \mathbb{E} \left( e^{-\langle \mathcal{E}_t, \varphi \rangle} \right) = \mathbb{E} \left( \exp(-c(\varepsilon) \gamma(\varphi) Z_\infty) \right),
\]
On the other hand, recall that the density of \( R_1 \) under \( \mathbb{P}_0 \) is
\[
z \mapsto \sqrt{\frac{2}{\pi}} z^2 e^{-z^2 / 2} \mathbb{1}_{\{z > 0\}}.
\]
Hence, using computations with respect to the density of \( R_1 \) and the monotone convergence theorem we obtain \( \lim_{\varepsilon \to 0} \mathbb{E}(h_{\lambda, \varepsilon}(R_1)) = (1 - \lambda)^{3/2} \) leading, using again dominated convergence theorem
\[
\lim_{\varepsilon \to 0} c(\varepsilon) = 1.
\]
Therefore, letting \( t \to \infty \) we deduce
\[
\lim_{t \to \infty} \mathbb{E} \left( e^{-\langle \mathcal{E}_t, \varphi \rangle} \right) = \mathbb{E} \left( \exp(-\gamma(\varphi) Z_\infty) \right) = \mathbb{E} \left( \exp(-\alpha C^* \sqrt{2} Z_\infty \int e^{-\sqrt{2t} (1 - \mathbb{E}(e^{-(D, \varphi, +z)}))} dz \right),
\]
which is the Laplace transform of a decorated PPP with intensity \( \sqrt{2} \alpha C^* Z_\infty e^{-\sqrt{2t}} dz \). As a result using [5, Lemma 4.1], we complete the proof of Theorem 1.1. \( \square \)
References

[1] E. Aïdékon, J. Berestycki, É. Brunet, and Z. Shi. Branching Brownian motion seen from its tip. *Probab. Theory Relat. Fields*, 157(1-2):405–451, 2013.

[2] Louis-Pierre Arguin, Anton Bovier, and Nicola Kistler. Poissonian statistics in the extremal process of branching Brownian motion. *Ann. Appl. Probab.*, 22(4):1693–1711, 2012.

[3] Louis-Pierre Arguin, Anton Bovier, and Nicola Kistler. The extremal process of branching Brownian motion. *Probab. Theory Relat. Fields*, 157(3-4):535–574, 2013.

[4] Mohamed Ali Belloum and Bastien Mallein. Anomalous spreading in reducible multitype branching Brownian motion. *Electronic Journal of Probability*, 26:1–39, 2021.

[5] Julien Berestycki, Éric Brunet, Aser Cortines, and Bastien Mallein. A simple backward construction of branching brownian motion with large displacement and applications. *arXiv preprint arXiv:1810.05809*, 2018.

[6] J. D. Biggins. Spreading speeds in reducible multitype branching random walk. *Ann. Appl. Probab.*, 22(5):1778–1821, 2012.

[7] Jochen Blath, Dave Jacob, and Florian Nie. How the interplay of dormancy and selection affects the wave of advance of an advantageous gene. *arXiv preprint arXiv:2106.08655*, 2021.

[8] Konstantin Borovkov, Robert Day, and Timothy Rice. High host density favors greater virulence: a model of parasite–host dynamics based on multi-type branching processes. *Journal of mathematical biology*, 66(6):1123–1153, 2013.

[9] Maury D. Bramson. Maximal displacement of branching Brownian motion. *Commun. Pure Appl. Math.*, 31:531–581, 1978.

[10] Richard Durrett. Branching process models of cancer. In *Branching process models of cancer*, pages 1–63. Springer, 2015.

[11] Patsy Haccou, Patricia Haccou, Peter Jagers, Vladimir A Vatutin, and Vladimir Vatutin. *Branching processes: variation, growth, and extinction of populations*. Number 5. Cambridge university press, 2005.

[12] J.-P. Kahane and J. Peyriere. Sur certaines martingales de Benoit Mandelbrot. *Adv. Math.*, 22:131–145, 1976.

[13] Marek Kimmel and David E Axelrod. Branching processes with infinitely many types. In *Branching Processes in Biology*, pages 155–205. Springer, 2015.

[14] Andrei N Kolmogorov. Étude de l’´equation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique. *Bull. Univ. Moskow, Ser. Internat., Sec. A*, 1:1–25, 1937.

[15] S. P. Lalley and T. Sellke. A conditional limit theorem for the frontier of a branching Brownian motion. *Ann. Probab.*, 15:1052–1061, 1987.

[16] Thomas Madaule. First order transition for the branching random walk at the critical parameter. *Stochastic Processes Appl.*, 126(2):470–502, 2016.

[17] Bastien Mallein. Maximal displacement in a branching random walk through interfaces. *Electron. J. Probab.*, 20:40, 2015. Id/No 68.

[18] H. P. McKean. Application of Brownian motion to the equation of Kolmogorov-Petrovskii-Piskunov. *Commun. Pure Appl. Math.*, 28:323–331, 1975.

[19] Yan-Xia Ren and Ting Yang. Multitype branching Brownian motion and traveling waves. *Adv. Appl. Probab.*, 46(1):217–240, 2014.