ON BOCHNER FLAT ALMOST KÄHLER MANIFOLDS

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Abstract. The main purpose of this article is to prove that there exist no proper $AK_3$-manifold of dimension $2n$, $n \geq 3$, with vanishing Tricerri-Vanhecke Bochner curvature tensor and constant scalar curvature.

1. Introduction

The Riemannian manifolds, that are conformal to a flat Riemannian manifold, form a very important class. As it is well known, a characteristic property of these manifolds (in dimension $\geq 3$) is that they have vanishing Weil curvature tensor.

Studying the Betti numbers of Kähler manifolds, Bochner defined for them a tensor, as an algebraic analogue of the Weil tensor. Although we don’t yet know the exact geometric meaning of the Bochner curvature tensor, it is an object of special interest, because in many cases in the Kähler geometry (not only in the study of Betti numbers) this tensor plays a role similar to that of the Weil tensor for Riemannian manifolds. For example a Riemannian manifold is of constant sectional curvature if and only if it is Einsteinian and has vanishing Weil curvature tensor. Analogously a Kähler manifold is of constant holomorphic sectional curvature if and only if it is Einsteinian and has vanishing Bochner curvature tensor.

In 1981 Tricerri and Vanhecke defined a Bochner-type curvature tensor for an arbitrary almost Hermitian manifold. In particular, in the Kähler case their Bochner tensor coincides with the classical Bochner one. Since then, many studies have been also made about this tensor for different classes of almost Hermitian manifolds.

Kähler manifolds with vanishing Bochner curvature tensor and constant scalar curvature are classified in [9]. The same problem for nearly Kähler manifolds is studied in [8], see also [3]. In the present paper we consider the case of almost Kähler manifolds satisfying the third curvature condition. Namely, in section 4 we prove that such a manifolds of dimension $2n$, $n \geq 3$, must be Kähler (Theorem 3).

In section 3 we prove for semi-Kähler manifolds a result, similar to the well known theorem of Tricerri and Vanhecke for the so-called generalized complex space forms [11].

2. Preliminaries

In this section $x, y, z, u, v$ will be arbitrary vectors in a point $p$ of a $2n$-dimensional almost Hermitian (AH) manifold $M$ with metric tensor $g$ and almost complex structure $J$. The curvature tensor (of type $(1,3)$ or $(0,4)$), the Ricci tensor (of type $(1,1)$ or $(0,2)$) and the scalar curvature are denoted by $R$, $S$ and $\tau$, respectively.

We will use the second Bianchi identity

\[(\nabla_x R)(y, z, u, v) + (\nabla_y R)(z, x, u, v) + (\nabla_z R)(x, y, u, v) = 0\]

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as well as the Ricci identities

\[
(\nabla_x (\nabla_y J))(z) - (\nabla_y (\nabla_x J))(z) = R(x, y) Jz - JR(x, y) z
\]

\[
(\nabla_x (\nabla_y P))(z, u) - (\nabla_y (\nabla_x P))(z, u) = -P(R(x, y) z, u) - P(z, R(x, y) u)
\]

where \( \nabla \) is the covariant differentiation with respect to the Riemannian connection of \( M \) and \( P \) is a tensor field of type (0, 2).

In the almost Hermitian geometry it is convenient to use the following operators:

\[
\varphi(P)(x, y, z, u) = g(x, u)P(y, z) - g(x, z)P(y, u)
+ g(y, z)P(x, u) - g(y, u)P(x, z),
\]

\[
\psi(P)(x, y, z, u) = g(x, Ju) P(y, Jz) - g(x, Jz) P(y, Ju) - 2g(x, Jy) P(z, Ju)
+ g(y, Jz) P(x, Ju) - g(y, Ju) P(x, Jz) - 2g(z, Jy) P(x, Jy)
\]

for a tensor field \( P \) of type (0, 2). Put also \( \pi_1 = 1/2 \varphi(g) \), \( \pi_2 = 1/2 \psi(g) \).

Recall that the manifold is conformal flat if and only if its Weil tensor \( C(R) \) vanishes, where

\[
C(R) = R - \frac{1}{2n - 2} \varphi(S) + \frac{\tau}{(2n - 1)(2n - 2)} \pi_1.
\]

The classes of Kähler (\( K \)), nearly Kähler (\( NK \)), almost Kähler (\( AK \)), quasi Kähler (\( QK \)), semi-Kähler (\( SK \)) manifolds are defined respectively by \( \nabla J = 0 \), \( (\nabla_x J)x = 0 \),

\[
dF = 0 \quad \text{or} \quad g((\nabla_x J)y, z) + g((\nabla_y J)z, x) + g((\nabla_z J)x, y) = 0,
\]

\[
(\nabla_x J)y + (\nabla_{Jx} J)y = 0, \quad \delta F = 0, \quad \text{where} \ F(x, y) = g(x, Jy) \ \text{is the fundamental form and} \ \delta \ \text{denotes the coderivative. The following inclusions are strict}
\]

\[
K \subset NK \subset QK \subset SK \quad K \subset AK \subset QK \subset SK,
\]

see e.g. [1]. Moreover \( K = NK \cap AK \).

For a class \( L \) of almost Hermitian manifolds its subclass \( L_i \) is defined by the \( i \)-th of the following identities for its curvature tensor

1) \( R(x, y, z, u) = R(x, y, Jz, Ju); \)

2) \( R(x, y, z, u) = R(x, y, Jz, Ju) + R(x, Jy, z, Ju) + R(Jx, y, z, Ju); \)

3) \( R(x, y, z, u) = R(Jx, Jy, Jz, Ju). \)

Then \( AH_1 \subset AH_2 \subset AH_3, \ K = K_1, NK = NK_2. \)

For almost Hermitian manifolds a second Ricci tensor and a second scalar curvature are introduced. Namely, the \( * \)-Ricci tensor \( S^* \) and the \( * \)-scalar curvature \( \tau^* \) are given by

\[
S^*(x, y) = \sum_{i=1}^{2n} R(x, e_i, Je_i, Jy) \quad \tau^*(x, y) = \sum_{i=1}^{2n} S^*(e_i, e_i)
\]

where \( \{e_i, i = 1, \ldots, 2n\} \) is an orthonormal basis of \( T_pM \). Then for \( AH_3 \)-manifolds the Tricerri-Vanhecke Bochner curvature tensor \( B(R) \) of \( M \) is defined by

\[
B(R) = R - (\varphi + \psi)(T) - \varphi(Q) + \mu(\pi_1 + \pi_2) + \nu\pi_1,
\]

where

\[
T = \frac{1}{8(n + 2)}(S + 3S^*) - \frac{1}{8(n - 2)}(S - S^*) \quad Q = \frac{1}{2(n - 2)}(S - S^*)
\]

\[
\mu = \frac{\tau + 3\tau^*}{16(n + 1)(n + 2)} - \frac{\tau - \tau^*}{16(n - 1)(n - 2)} \quad \nu = \frac{\tau - \tau^*}{4(n - 1)(n - 2)}.
\]
see [11]. In particular, if the manifold is Kähler, this tensor coincides with the classical Bochner curvature tensor. Note that for any $AH_3$-manifold the tensors $S$, $S^*$ (and so also $T$ and $Q$) are symmetric and $J$-invariant (hybrid). Note also that for a manifold $M \in AH_3$ with $B(R) = 0$ it follows $M \in AH_2$.

For an $AK_2$-manifold the following identities hold:

\begin{equation}
2(\nabla_x Q)(y, z) = Q((\nabla_x J)y, Jz) + Q(Jy, (\nabla_x J)z),
\end{equation}

\begin{equation}
R(x, y, z, u) - R(x, y, Jz, Ju) = \frac{1}{2}g(K(x, y), K(z, u)),
\end{equation}

where $K(x, y) = (\nabla_x J)y - (\nabla_y J)x$, see [1], [4]. Note that (2.3) and $M \in QK$ imply

\begin{equation}
(\nabla_x Q)(y, z) + (\nabla_x Q)(Jy, Jz) = 0,
\end{equation}

\begin{equation}
(\nabla_x Q)(y, z) + (\nabla_{Jx} Q)(y, Jz) = 0.
\end{equation}

Moreover, if the scalar curvature $\tau$ of $M$ is a constant, then by (2.5) $\tau^*$ is also constant. So $\mu$ and $\nu$ are constants, too.

### 3. A Theorem for semi-Kähler manifolds.

**Theorem 1.** Let $M$ be a $2n$-dimensional semi-Kähler manifold, $n > 2$, whose curvature tensor has the form

\begin{equation}
R = \varphi(P) + f\pi_1 + h\pi_2,
\end{equation}

where $f$ and $h$ are functions and $P$ is a symmetric tensor field of type $(0,2)$. Then $h$ is a constant. If $h = 0$, then $M$ is conformal flat. If $h \neq 0$, then $M$ is a Kähler manifold of constant holomorphic sectional curvature.

**Proof.** First of all we note that (3.1) implies (with a contraction) that the tensor field $P$ is symmetric. Suppose that $x, y, z$ are unit vectors in $T_pM$ such that $x, y, z, Jx, Jy, Jz$ are mutually orthogonal.

From the second Bianchi identity

$$(\nabla_{Jx} R)(y, Jy, Jz, z) + (\nabla_y R)(Jy, x, Jz, z) + (\nabla_{Jy} R)(Jx, y, Jz, z) = 0$$

it follows

\begin{equation}
Jx(h) - h\left(g((\nabla_y J)y, x) + g((\nabla_J y)Jy, x)\right) = 0.
\end{equation}

Since $M$ is semi-Kähler, this implies easily that $h$ is a constant. If $h = 0$ by a standard way we obtain from (3.1)

$$R = \frac{1}{n - 2}\varphi(S) - \frac{\tau}{2(n - 1)(2n - 2)}\pi_1,$$

so $M$ is conformal flat.

Let $h \neq 0$. Then (3.2) becomes

$$g((\nabla_y J)y, x) + g((\nabla_{Jy} J)y, x) = 0.$$

Now we change in (2.1) $(x, y, z, u, v)$ with $(Jx, y, z, Jz, Jy)$ and because of $h = const. \neq 0$ we find

$$g((\nabla_J y)y, x) + g((\nabla_J z)z, x) = 0.$$

From the last two equalities we find $(\nabla_y J)y = 0$, so $M$ is a nearly Kähler manifold.
We replace in (2.1) \((z,u,v)\) by \((Jz,Jz,z)\). The result is
\[
(\nabla_x P)(y,z) - (\nabla_y P)(x,z) + h\left(3g((\nabla_x J)y,Jz) + 3g((\nabla_y J)Jz,x) + 2g((\nabla_z J)x,y)\right) = 0 .
\]
Making a cyclic sum in the above equality and using that \(M\) is nearly Kähler we derive
\[
g((\nabla_x J)y,z) + g((\nabla_y J)Jz,x) + g((\nabla_z J)x,y) = 0
\]
for all \(x,y,z\) in \(T_pM\) such that \(x,y,z,Jx,Jy,Jz\) are mutually orthogonal. Note that the last equality is true also when \(z=y\) or \(z=Jy\). Hence it follows that it is true for arbitrary \(x,y,z\) in \(T_pM\), so \(M\) is also almost Kähler. Consequently \(M\) is Kählerian, thus proving the assertion. □

**Remark.** An almost Hermitian manifold is said to be a *generalized complex space form* if its curvature tensor has the form
\[
R = f\pi_1 + h\pi_2 ,
\]
where \(f\) and \(h\) are functions. For such a manifold Tricerri and Vanhecke [11] proved that it is of constant sectional curvature or a Kähler manifold of constant holomorphic sectional curvature.

### 4. The main result.

In the following five lemmas we assume that \(M\) is an \(AK_3\)-manifold with vanishing Tricerri-Vanhecke Bochner curvature tensor and constant scalar curvature. As noted in section 2 in this case the manifold is \(AK_2\) and of constant \(\ast\)-scalar curvature, so \(\mu\) and \(\nu\) are also constants.

We begin by proving that under the above assumptions the tensor \(T\) has the property of the type (2.3).

**Lemma 1.** The tensor \(T\) satisfies
\[
2(\nabla_x T)(y,z) = T((\nabla_x J)y,Jz) + T(Jy,(\nabla_x J)z)
\]
for all \(x,y,z\) in \(T_pM\).

**Proof.** Let \(x,y\) be unit vectors in \(T_pM\) with \(x \perp y, Jy\). Putting in the second Bianchi identity (2.1) \(z = u = Jy, v = y\) and using (2.5) we obtain
\[
(\nabla_x T)(y,y) + (\nabla_x T)(Jy,Jy) = (\nabla_y T)(x,y) + (\nabla_Jy T)(x,Jy) .
\]
Analogously if we put in (2.1) \(z = Jy, u = Jx, v = x\), we find
\[
(\nabla_x T)(x,x) + (\nabla_x T)(Jx,Jx) + (\nabla_x T)(y,y) + (\nabla_x T)(Jy,Jy) = 4(\nabla_y T)(x,y) + 4(\nabla_Jy T)(x,Jy) .
\]
From the last two equalities it follows
\[
(\nabla_x T)(x,x) + (\nabla_x T)(Jx,Jx) = 3(\nabla_x T)(y,y) + 3(\nabla_x T)(Jy,Jy) .
\]
Let \(\{e_i, Je_i, i = 1, \ldots n\}\) be an orthonormal basis of \(T_pM\) such that \(x = e_1\). We put in (1.3) \(y = e_i\) and we add for \(i = 2, \ldots n\). Then using \(x(\tau) = x(\tau^*) = 0\) we obtain
\[
(\nabla_x T)(x,x) + (\nabla_x T)(Jx,Jx) = 0 .
\]
Now (1.3) becomes
\[
(\nabla_x T)(y,y) + (\nabla_x T)(Jy,Jy) = 0
\]
for \( x \perp y, Jy \). By (4.4) and (4.5) we may conclude that (4.5) holds for arbitrary vectors \( x, y \in T_pM \). Since \( T \) is a symmetric tensor, this implies
\[
(\nabla_x T)(y, z) + (\nabla_x T)(Jy, Jz) = 0
\]
for all \( x, y, z \in T_pM \). Hence using
\[
(\nabla_x T)(Jy, Jz) = (\nabla_x T)(y, z) - T((\nabla_x J)y, Jz) - T(Jy, (\nabla_x J)z)
\]
we obtain the assertion. \( \square \)

**Remark.** It follows from Lemma 1 that \( T \) satisfies also the analogues of (2.5) and (2.6).

**Lemma 2.** The tensor \( T \) satisfies
\[
T(R(x, y)z, u) + T(z, R(x, y)u) + T(R(x, y)Jz, Ju) + T(Jz, R(x, y)Ju)
\]
\[
= \frac{1}{2} \left( T((\nabla_x J)(\nabla_y J)z, u) + T((z, (\nabla_x J)(\nabla_y J)u) - T((\nabla_x J)(\nabla_y J)z, u) - T(z, (\nabla_x J)(\nabla_y J)u) \right).
\]

**Proof.** Using Lemma 1 we calculate
\[
(\nabla_x (\nabla_y T))(z, u) = \frac{1}{2} \left( T((\nabla_x (\nabla_y J))z, Ju) + T(Jz, (\nabla_x (\nabla_y J))u) \right)
\]
\[
+ \frac{1}{4} \left( T((\nabla_x J)u, (\nabla_y J)z) + T((\nabla_x J)z, (\nabla_y J)u) - T((\nabla_x J)(\nabla_y J)z, u) - T(z, (\nabla_x J)(\nabla_y J)u) \right).
\]
Hence
\[
(\nabla_x (\nabla_y T))(z, u) - (\nabla_y (\nabla_x T))(z, u)
\]
\[
= \frac{1}{2} \left( T((\nabla_x (\nabla_y J)z, Ju) + T(Jz, (\nabla_x (\nabla_y J))u) - T((\nabla_y (\nabla_x J))z, Ju) - T(Jz, (\nabla_y (\nabla_x J))u) \right)
\]
\[
+ \frac{1}{4} \left( T((\nabla_y J)(\nabla_x J)z, u) + T(z, (\nabla_y J)(\nabla_x J)u) - T((\nabla_x J)(\nabla_y J)z, u) - T(z, (\nabla_x J)(\nabla_y J)u) \right).
\]
Applying the Ricci identity for the tensors \( T \) and \( J \) in the above equality we obtain the assertion. \( \square \)

In the rest of this section \( x, y, z \) will be mutually orthogonal eigenvectors of \( T \), which span a 3-dimensional antiholomorphic plane in \( T_pM \). For any eigenvector \( x \) of \( T \) denote by \( \lambda_x \) the corresponding eigenvalue.

**Lemma 3.** If for a triple \( \{x, y, z\} \)
\[
g((\nabla_x J)y, z) \neq 0,
\]
then the tensor \( T \) is proportional to the metric tensor in the point \( p \).

**Proof.** The substitution of \((u, v)\) by \((z, Jz)\) in (2.1) with the use of Lemma 1 and \( M \in AK \) gives
\[
(\nabla_x (T + Q))(y, Jz) - (\nabla_y (T + Q))(x, Jz)
\]
\[
+ (\lambda_x - \frac{1}{2} \lambda_y - \frac{5}{2} \lambda_z + \mu)g((\nabla_x J)y, z) + (\frac{1}{2} \lambda_x - \lambda_y + \frac{5}{2} \lambda_z - \mu)g((\nabla_y J)x, z) = 0.
\]
Analogously from
\[
(\nabla_{Jx} R)(y, x, z) + (\nabla_y R)(x, Jx, z) + (\nabla_x R)(Jx, y, x, z) = 0
\]
we find
\[
(\nabla_{x}(T+Q))(y,z) - (\nabla_{y}(T+Q))(Jx,z)
\]
\[
+ (\lambda_{x} + \frac{1}{2}\lambda_{y} + \frac{1}{2}\lambda_{z} - \mu)g((\nabla_{x}J)y,z) + ( - \frac{9}{2}\lambda_{x} - \frac{3}{2}\lambda_{z} + 3\mu)g((\nabla_{y}J)x,z) = 0 .
\]
From (4.6) and (4.7), using (2.6) and its analogue for $D$ we obtain
\[
(\lambda_{y} + 3\lambda_{z} - 2\mu)g((\nabla_{x}J)y,z) + ( - 5\lambda_{x} + \lambda_{y} - 4\lambda_{z} + 4\mu)g((\nabla_{y}J)x,z) = 0 .
\]
Changing the places of $x$ and $y$ we have also
\[
(\lambda_{x} - 5\lambda_{y} - 4\lambda_{z} + 4\mu)g((\nabla_{x}J)y,z) + (\lambda_{x} + 3\lambda_{z} - 2\mu)g((\nabla_{y}J)x,z) = 0 .
\]
Since $g((\nabla_{x}J)y,z) \neq 0$, the last two equalities imply
\[
D(x,y,z) = 5\lambda_{x}^{2} + 5\lambda_{y}^{2} - 7\lambda_{z}^{2} - 25\lambda_{x}\lambda_{y} - 13\lambda_{x}\lambda_{z} - 13\lambda_{y}\lambda_{z} + (14\lambda_{x} + 14\lambda_{y} + 20\lambda_{z})\mu - 12\mu^{2} = 0 .
\]
Since $M$ is almost Kähler, at least one of $g((\nabla_{y}J)z,x)$ and $g((\nabla_{x}J)x,y)$ must also be different from zero. Let e.g. $g((\nabla_{y}J)z,x) \neq 0$. Then it follows $D(y,z,x) = 0$.

Case 1. $g((\nabla_{x}J)x,y)$ also does not vanish. The $D(z,x,y) = 0$. The system
\[
D(x,y,z) = D(y,z,x) = D(z,x,y) = 0
\]
has a solution $x = y = z = \mu/2$.

Case 2. $g((\nabla_{x}J)x,y) = 0$. Then (4.8) and $M \in AK$ imply
\[
5\lambda_{x} - 2\lambda_{y} + \lambda_{z} - 2\mu = 0 .
\]
The system
\[
D(x,y,z) = D(y,z,x) = 5\lambda_{x} - 2\lambda_{y} + \lambda_{z} - 2\mu = 0
\]
also has a solution $x = y = z = \mu/2$. If $n \geq 3$, then $T = \mu/2g$ and the Lemma is proved.

Let $n \geq 4$ and $u$ be an eigenvector of $T$, orthogonal to $\text{span}\{x,y,z,Jx,Jy,Jz\}$. Replacing in (2.1) $(x,y,z,u,v)$ by $(y,z,u,Ju,x)$ we find
\[
(\lambda_{x} + \lambda_{z} + 2\lambda_{u} - 2\mu)g((\nabla_{y}J)z,x) - (\lambda_{x} + \lambda_{y} + 2\lambda_{u} - 2\mu)g((\nabla_{x}J)y,x) = 0
\]
and using $x = y = z = \mu/2$:
\[
(2\lambda_{u} - \mu)g((\nabla_{y}J)z,x) - (2\lambda_{u} - \mu)g((\nabla_{x}J)y,x) = 0
\]
Now because of (2.2) it follows
\[
(2\lambda_{u} - \mu)g((\nabla_{x}J)y,z) = 0
\]
so $\lambda_{u} = \mu/2$, thus proving the Lemma.

\begin{lemma}
If
\[
g((\nabla_{y}J)y,x) \neq 0 ,
\]
then $\lambda_{x} = \lambda_{z}$.
\end{lemma}

\begin{proof}
We put in (2.1) $u = z, v = y$. Then we find
\[
(\nabla_{x}(T+Q))(y,y) + (\nabla_{z}(T+Q))(z,z) = (\nabla_{y}(T+Q))(x,y) + (\nabla_{z}(T+Q))(x,z) .
\]
We replace here $z$ by $Jz$ and add the result with the above, using (2.5) and its analogue for the tensor $T$. The result is
\[
(\nabla_{x}(T+Q))(y,y) = (\nabla_{y}(T+Q))(x,y) .
\]
\end{proof}
Now from the second Bianchi identity
\[(\nabla_{Jx}R)(x, y, y, x) + (\nabla_{y}R)(y, Jx, y, x) + (\nabla_{y}R)(Jx, x, y, x) = 0\]
using (4.9), (2.6) and Lemma 1 we obtain
\[3\lambda_{x} + \lambda_{y} - 2\mu = 0.\]
Analogously we put in (2.1) \(u = Jz\), \(v = y\) and we derive
\[\lambda_{x} + \lambda_{y} + 2\lambda_{z} - 2\mu = 0.\]
The last two equalities imply the assertion. □

**Lemma 5.** Let \(M\) be non Kähler in a point \(p\), i.e. \((\nabla J)_{p} \neq 0\). Then there exists a number \(\lambda\), such that \(T = \lambda g\) in \(p\).

**Proof.** Let \(\{e_{i}, Je_{i}; \ i = 1, ..., n\}\) be an orthonormal basis of \(T_{p}M\) of eigenvectors of \(T\). According to Lemmas 3 and 4 it suffices to consider the case
\[(\nabla_{e_{1}}J)e_{1} \neq 0 \quad (\nabla_{e_{j}}J)e_{j} = 0\]
for any \(i = 2, ..., n, j = 1, ..., n\). Moreover by Lemma 4 \(\lambda_{2} = \lambda_{3} = ... = \lambda_{n}\). Denote this number \(\lambda\) and suppose \(\lambda \neq \lambda_{1}\). Note that we may assume \(e_{2} \parallel (\nabla_{e_{1}}J)e_{1}\). Then we have also
\[g((\nabla_{e_{1}}J)e_{1}, e_{i}) = g((\nabla_{e_{1}}J)e_{1}, Je_{i}) = 0 \quad \text{for} \ i = 3, ..., n\]
and hence using again Lemma 3
\[(4.10) \quad (\nabla_{e_{1}}J)e_{i} = 0 \quad \text{for} \ i = 3, ..., n.\]
Putting in Lemma 2 \(x = u = e_{1}, y = z = e_{i} \ (i > 1)\) we obtain
\[2\lambda_{1} + 2\lambda + Q(e_{1}, e_{1}) + Q(e_{i}, e_{i}) - 2\mu - \nu = 0.\]
Hence \(Q(e_{i}, e_{i}) = Q(e_{j}, e_{j})\) for any \(i, j = 2, ..., n\). Making the same substitution in (2.4) we find
\[Q(e_{1}, e_{1}) + Q(e_{i}, e_{i}) - \nu = -\frac{1}{2}g((\nabla_{e_{1}}J)e_{i}, (\nabla_{e_{1}}J)e_{i})\]
for any \(i = 2, ..., n\). Now \(Q(e_{2}, e_{2}) = Q(e_{3}, e_{3})\) and (4.10) imply
\[g((\nabla_{e_{1}}J)e_{2}, (\nabla_{e_{1}}J)e_{2}) = 0\]
and hence
\[(\nabla_{e_{1}}J)e_{1} = 0,\]
which is a contradiction. This proves the lemma. □

**Theorem 2.** Let \(M\) be a 2n-dimensional conformal flat \(AK_{3}\)-manifold, \(n \geq 2\). Then \(M\) is a flat Kähler manifold or \(n = 2\) and \(M\) is locally a product of two 2-dimensional Kähler manifolds \(M_{1}\) and \(M_{2}\) of constant sectional curvature \(c\) and \(-c\), \(c > 0\), respectively.

**Proof.** According to \[7\] and \[8\] a conformal flat \(AK_{3}\)-manifold of dimension \(\geq 4\) must be a 4- or a 6-dimensional manifold of constant sectional curvature, or a flat Kähler manifold, or a product of two almost Kähler manifolds \(M_{1}\) and \(M_{2}\) of constant sectional curvature \(c\) and \(-c\), \(c > 0\), respectively. On the other hand by \[10\] an almost Kähler manifold of constant sectional curvature and dimension \(\geq 4\) is a flat Kähler manifold. Hence the assertion follows. □
Remark. There exist conformal flat almost Kähler manifolds which are not Kähler, see e.g. [2]. So the $AH_3$-assumption cannot be removed.

**Theorem 3.** Let $M$ be a $2n$-dimensional $AK_3$-manifold, $n \geq 3$, with vanishing Tricerri-Vanhecke Bochner curvature tensor and constant scalar curvature. Then $M$ is a Kähler manifold.

**Proof.** Assume that $M$ is non Kähler in a point $q$. Then $M$ is non Kähler in a neighborhood $U$ of $q$. By Lemma 5 in $U$ it holds $T = \lambda g$ with a function $\lambda$. Now from $B = 0$ it follows that in $U$ the curvature tensor of $M$ has the form

$$R = \varphi(Q) + \theta(\pi_1 + \pi_2) - \nu\pi_1$$

with a function $\theta$. According to Theorem 1 $U$ must be conformal flat. By Theorem 2 this contradicts the assumption that $U$ is non Kähler. \qed

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