On the Entropy of a Quantum Field in the Rotating Black Holes

Min-Ho Lee and Jae Kwan Kim
Department of Physics, Korea Advanced Institute of Science and Technology
373-1 Kusung-dong, Yusung-ku, Taejon 305-701, Korea.

Abstract

By using the brick wall method we calculate the free energy and the entropy of the scalar field in the rotating black holes. As one approaches the stationary limit surface rather than the event horizon in comoving frame, those become divergent. Only when the field is comoving with the black hole (i.e. \( \Omega_0 = \Omega_H \)) those become divergent at the event horizon. In the Hartle-Hawking state the leading terms of the entropy are \( A \frac{1}{h} + B \ln(h) + \text{finite} \), where \( h \) is the cut-off in the radial coordinate near the horizon. In term of the proper distance cut-off \( \epsilon \) it is written as \( S = NA_H/\epsilon^2 \). The origin of the divergence is that the density of state on the stationary surface and beyond it diverges.

\[ \text{e-mail: mhlee@chep6.kaist.ac.kr} \]
1 Introduction

By comparing the black hole physics with the thermodynamics and discovering of the black hole evaporation by Hawking, it was shown that the black hole entropy is proportional to the horizon area\cite{1, 2}.

\[ S_{BH} = \frac{A_H}{4} \]  

in unit \( h = c = G = 1 \). In Euclidean path integration approach it was shown the tree level contribution of the gravitation action gives the black hole entropy\cite{3}. However the exact statistical origin of the Bekenstein-Hawking black hole entropy is unclear.

Recently many efforts have been concentrated to understand the statistical origin of black hole thermodynamics, specially the black hole entropy by various methods (for review see\cite{4}): 't Hooft was calculated the entropy of a quantum field propagating the outside of the black hole. After the regularization he obtained \( S = \frac{1}{4}A_H \) (the brick wall method)\cite{5, 9, 10, 11}. Another approach is to identify the black hole entropy with the entanglement entropy \( S_{\text{ent}} \). Entanglement entropy arises from ignoring the degree of freedom of a proper region of space: \( S = -Tr\rho \ln \rho \). It is found that the entropy is proportional to the area of the boundary\cite{6}. In fact the entanglement entropy and the brick wall method are equivalent. Frolov and Novikov argued that the black hole entropy can be obtained by identifying the dynamical degrees of freedom with the states of all fields which are located inside the black hole\cite{7}. The leading term of the entropy obtained by those methods is proportional to the surface area of the horizon. However the proportional coefficient diverges as the cut off goes to zero. The conical approach also gives similar result with others\cite{8}. The divergence is because of an infinite number of states near the horizon, which can be explained by the equivalence principle\cite{12}. An alternative approach by Frolov is to identify the black hole entropy with the thermodynamic one. In this approach the entropy is finite\cite{13}. However they all treat the only spherical symmetrical black hole.

If the black hole has a rotation, what is changed? It is well known that in a rotating black hole spacetime a particle with a zero angular momentum dropped from infinity is dragged just by the influence of gravity so that it acquires an angular velocity in the same direction in which the black hole rotates. The dragging becomes more and more extreme the nearer one approaches the horizon of the black hole. This effect is called the dragging of inertial frames\cite{25}.

Thus the field at equilibrium with the rotating black hole must also be rotating. The rotation is not rigid but locally is different. So the velocity of the radiation does not exceed the velocity of light. However we do not know how to treat the equilibrium state with a locally different angular velocity. More precisely there is no global static coordinates. So we assume that the radiation has a rigid rotation \( \Omega_0 \) small than or equal to the extremum value of the local rotation. In a rotating black hole the extremum value of it is \( \Omega_H \), which is the angular velocity of the event horizon.

Recently we considered the black hole entropy by the brick wall method in the charged Kerr black hole in\cite{14} and showed the entropy is proportional to the event horizon in Hartle-Hawking states.
this paper to more deeply understand the black hole entropy we shall investigate the black hole entropy by the brick wall method in various stationary black holes: the Kaluza-Klein black hole \[15\] which is the solution of the 4-dimensional effective theory reduced from the 5-dimensional Kaluza-Klein theory, and the Sen black hole\[16\] which is the solution of the Einstein-Maxwell-dilaton-antisymmetric tensor gauge field theory came from the heterotic string theory, and the Kerr-Newman black hole\[17\] which is the solution of the Einstein-Maxwell theory.

In order to understand the equilibrium state of the radiation (the field) in the rotating black hole spacetime in Sec.2 we will first consider the rotating heat bath in the flat spacetime. In Sec.3 we will consider the radiation in equilibrium state in Rindler spacetime with rotation, which is the most simple spacetime having the event horizon and a rotation. In Sec.4 we will investigate the entropy of the quantum field in the stationary black hole background. We find the condition to give the finite value to the free energy and the entropy. In Sec.5 we calculate the entropy in Hartle-Hawking state for the rotating black holes. Final section is devoted to the summary.

2 A Rotating Heat Bath

Let us consider a massless scalar field with a constant angular velocity \(\Omega_0\) about \(z\)-axis at thermal equilibrium with a temperature \(T = 1/\beta\) in Minkowski spacetime, of which line element in cylindrical coordinate is given by

\[
ds^2 = -dt^2 + r^2d\phi^2 + dr^2 + dz^2.
\]

(2)

In this spacetime the positive frequency field mode can be written as \(\Phi_{q,m}(x) = f_{q,m}(r,z)e^{-i\omega t + im\phi}\), where \(q\) denotes a quantum number and \(m\) is the azimuthal quantum number.

For such an equilibrium ensemble of the states of the scalar field the partition function is given by

\[
Z = \sum_{n_q,m} e^{-n_q(\omega_q - m\Omega_0)\beta}
\]

(3)

and the free energy is given by

\[
\beta F = \sum_m \int_0^\infty d\omega g(\omega, m) \ln \left(1 - e^{-\beta(\omega - m\Omega_0)}\right),
\]

(4)

where \(g(\omega, m)\) is the density of state for a fixed \(\omega\) and \(m\).

Following 't Hooft we assume that all possible modes of a scalar field vanish at \(r = r_1\) (\(r_1\) is very small.) and at \(r = L\). In the WKB approximation with \(\Phi = e^{iS(r) - i\omega t + im\phi + ikz}\) the radial wave number \(K(x, \omega, m) = \partial_r S\) is given by

\[
K^2(x, \omega, m) = \omega^2 - \frac{m^2}{r^2} - k^2.
\]

(5)

This expression denotes the ellipsoid in momentum phase space at a fixed frequency \(\omega\). The total number of modes with energy less than \(\omega\) and a fixed \(m\) is obtained by integrating over the volume of
phase space, which is determined by (6)

\[ \Gamma(\omega, m) = \frac{1}{\pi} \sum_m \int dφ dz \int_{r_1}^{L} L r_1 \int dk K(x, \omega, m) \]

\[ = \frac{1}{\pi} \sum_m \int dφ dz \int_{r_1}^{L} \int dk \left( \omega^2 - \frac{m^2}{r^2} - k^2 \right)^{1/2}. \] (6)

The integration over \( k \) must be carried out over the phase space that satisfies \( K^2 \geq 0 \). \( \Gamma(\omega, m) \) can be obtained by investigating the shape of the expression (5) in momentum phase space. Thus the free energy, after the integration by parts, becomes

\[ \beta F = -\beta \sum_m \int_0^\infty dω \Gamma(\omega, m) \frac{1}{e^{\beta(\omega - m\Omega_0)} - 1} \]

\[ = -\beta \sum_m \int_0^\infty dω \int_{-r\omega}^{r\omega} dm (\omega^2 - \frac{m^2}{r^2}) \frac{1}{e^{\beta(\omega - m\Omega_0)} - 1}, \] (7)

where we assume that the azimuthal quantum number \( m \) is a continuous parameter. By making the change of variable \( m = r\omega u \) we obtain the free energy

\[ \beta F = -\frac{N}{\beta^3} \int dφ dz \int_{r_1}^{L} \frac{r}{(1 - v^2)^2} dr, \] (8)

where \( N \) is a constant and \( v = r\Omega_0 \). Note that as \( L \) goes to \( 1/\Omega_0 \) this partition function diverges as \( \gamma^4 \), where \( \gamma = (1 - v^2)^{-1/2} \).

From the expression (8) it is easy to obtain expressions for the energy \( E \), angular momentum \( J \), and entropy \( S \) of radiation

\[ J = \langle m \rangle_{av} = \frac{1}{\beta \partial \Omega_0} (\beta F) = 4N \frac{1}{\beta^4} \Omega_0 \int r^2 \gamma^6 rdrdφdz, \] (9)

\[ E = \langle \omega \rangle_{av} = \Omega_0 \cdot J - \frac{\partial}{\partial \beta} (\beta F) = N \frac{1}{\beta^4} \int (3 + v^2) \gamma^4 rdrdφdz, \] (10)

\[ S = \beta^2 \frac{\partial}{\partial \beta} F = 4N \frac{1}{\beta^3} \int \gamma^4 rdrdφdz. \] (11)

These coincides with those in ref. [23]. Similarly to the free energy \( F \) these expressions \( J, E, \) and \( S \) diverge as \( L \to 1/\Omega_0 \). The divergence is related to the rigid rotation. In rigid rotating system the velocity of the comoving observer grows as one move from the origin to infinity. So beyond some point the velocity exceeds the velocity of the light. This is unphysical. Thus a rotating system cannot have the size greater than \( 1/\Omega_0 \). Therefore to obtain a finite value for \( J, E, \) and \( S, \) we must take \( L < 1/\Omega_0 \). In such a finite system \( \omega > m\Omega_0 \).

Now let us consider above problem in co-moving coordinate that are rotating with angular velocity \( \Omega_0 \). The line element in comoving frame is given by

\[ ds^2 = -(1 - \Omega_0^2 r^2) dt^2 + 2\Omega_0 r dφ' dt + dr^2 + dz^2, \] (12)
where we have used $\phi' = \phi - \Omega_0 t$. In this coordinate the positive frequency field mode is written as $\Phi_{qm}(x) = \tilde{f}_{qm}(r, z)e^{-i\omega't+i\Phi'}$.

Because in comoving frame the field has no rotation the free energy is given by

$$\beta F = \int_0^\infty d\omega' g'(\omega') \ln \left(1 - e^{-\beta \omega'}\right),$$

(13)

where $g'(\omega')$ is the density of state for a fixed $\omega'$. In WKB approximation the Klein-Gordon equation $\Box \Phi = 0$ yields the constraint

$$g^{ab}k_ak_b = 0$$

(14)

or

$$-(\omega' - \Omega_0 m)^2 + \left(\frac{1}{r^2} m^2 + k^2 + p^2\right) = 0,$$

(15)

where $p = \frac{\partial S}{\partial r}$. In region where $\Omega_0 r < 1$, for a fixed $\omega'$, this expression represents the ellipsoid in momentum space. Therefore the total number of modes with energy less than $\omega'$ is given by

$$\Gamma'(\omega') = \frac{1}{\pi} \sum_m d\phi dz \int dr \int dk \left((\omega' - m\Omega_0)^2 - \frac{m^2}{r^2} - k^2\right)^{1/2}$$

(16)

which is the volume of the ellipsoid. The expression (16) is just the same form as Eq.(11) when $\omega \to \omega - m\Omega_0$. The phase volume (17) diverges as $L \to 1/\Omega_0$. Inserting the expression (17) into (13) and integrating we get

$$\beta F = -\frac{N}{\beta^3} \int d\phi dz \int_{r_1}^L dr \frac{r}{(1 - \Omega_0^2 r^2)^2}.$$

(18)

This expression is the same with Eq.(11). From this we get the energy $E'$ and the entropy $S$:

$$E' = \langle \omega' \rangle_{av} = -\frac{\partial}{\partial \beta}(\beta F) = \frac{3N}{\beta^4} A \int_{r_1}^L dr \frac{r}{(1 - \Omega_0^2 r^2)^2},$$

(19)

$$S = \frac{\beta^2}{\beta^3} \frac{\partial}{\partial \beta}(\beta F) = \frac{4N}{\beta^5} A \int_{r_1}^L dr \frac{r}{(1 - \Omega_0^2 r^2)^2},$$

(20)

where $A = \int d\phi dz$. It is noted that the entropy $S$ is the same with Eq.(11) and the energy $E'$ is satisfied with $E' = E - \Omega_0 J$. This fact show that the coordinate transformation to comoving frame only change the energy and not change the entropy in WKB approximation. Thus in the case of calculating the entropy or the free energy it is convenient to choose the comoving frame. It is noted that in co-moving frame the divergence is related to the time component $g_{tt}$ of the metric (12).

3 A Thermal Bath in Rindler Spacetime with a Rotation

In this section we will consider the thermal equilibrium state of the scalar field with the mass $\mu$ and an uniform rotation about $z-$axis in the Rindler spacetime. The line element of the Rindler spacetime
in cylindrical coordinates is given by
\[ ds^2 = -\xi^2 d\eta^2 + d\xi^2 + r^2 d\phi^2 + dr^2. \tag{21} \]

In this spacetime the event horizon is at \( \xi = 0 \), and \( \xi = constant \) represent the trajectory of the uniform acceleration\[18\]. The importance of the Rindler space-time is that in the large black hole mass limit the metric of the black space-time reduces to that of the Rindler space-time\[10\].

As in Sec.2, the WKB approximation with \( \Phi(x) = e^{-i\omega t + im\phi + iS(\xi,r)} \) yields
\[ K^2(\xi,r,\omega,m) = \frac{\omega^2}{\xi^2} - \frac{1}{r^2} m^2 - p_r^2 - \mu^2, \tag{22} \]
where \( K = \partial_\xi S \) and \( p_r = \partial_r S \). In this section we will calculate the free energy by using the slightly different method with that in section 2.

It is important to note that in WKB approximation the density of state \( g(\omega,m) \) is determined by the constraint (22), and that the free energy is singular at \( \omega = m\Omega_0 \). In particular if \( \omega - m\Omega_0 < 0 \) the free energy becomes an imaginary number. However in the WKB approximation we can easily see \( \bar{\omega} = \omega - m\Omega_0 > 0 \) in the region such that \( \xi - \Omega_0 r > 0 \). But in the region such that \( \xi - \Omega_0 r < 0 \) it is possible that \( \omega - m\Omega_0 < 0 \). (More details are in Sec.4.) Therefore to obtain the finite value for the free energy we must require the system to be in the region such that \( \xi - \Omega_0 r > 0 \). Then the free energy is written as
\[ \beta F = \sum_m \int_{m\Omega_0}^{\infty} d\omega g(\omega,m) \ln \left( 1 - e^{-\beta(\omega - m\Omega_0)} \right) \]
\[ = \int_0^{\infty} d\omega \sum_m g(\omega + m\Omega_0,m) \ln \left( 1 - e^{-\beta\omega} \right) \]
\[ = -\beta \int_0^{\infty} d\omega \frac{1}{e^{\beta\omega} - 1} \int dm \Gamma(\omega + m\Omega_0,m), \tag{23} \]
where we have integrated by parts and assumed that the quantum number \( m \) is a continuous variable.

The total number of modes with energy less than \( \omega \) is obtained by integrating over the volume of phase space
\[ \Gamma(\omega) = \int dm \Gamma(\omega + m\Omega_0,m) \]
\[ = \int dm \int d\phi dr \int_{r_1}^{L} d\xi \frac{1}{\pi} \int dp_r K(\xi,r,\omega + m\Omega_0,m) \]
\[ = \frac{1}{\pi} \int dm \int d\phi dr \int_{r_1}^{L} d\xi \int dp_r \left( \frac{\omega^2}{\xi^2} + \frac{2}{\xi^2} m\Omega_0 \omega + \frac{m^2 \Omega_0^2}{\xi^2} - \frac{1}{r^2} m^2 - p_r^2 - \mu^2 \right)^{1/2}. \tag{24} \]

The integrations over \( m \) and \( p_r \) must be carried out over the phase space that satisfies \( K^2(\omega + m\Omega_0,m) \geq 0 \). After the integration we obtain the number of states with energy less than \( \omega \), which is given by
\[ \Gamma(\omega) = \frac{4}{3} \int d^3x \frac{\xi r}{\sqrt{(\xi^2 - \Omega_0^2 r^2)}} \left( \frac{\omega^2}{\xi^2 - \Omega_0^2 r^2} - \mu^2 \right)^{3/2}. \tag{25} \]
Thus the free energy becomes

\[ \beta F = -\frac{4}{3} \beta \int d^3x \int_{\mu}^{\infty} \frac{d\omega}{\omega} \frac{1}{e^{\beta \omega} - 1} \frac{\xi r}{\sqrt{\xi^2 - \Omega_0^2 r^2}} \left( \frac{\omega^2}{\xi^2} - \mu^2 \right)^{3/2}. \]  

(26)

For a massless scalar field (\( \mu = 0 \)) the free energy becomes

\[ \beta F = -\frac{N}{\beta^3} \int d\phi dr \int_{\xi_1}^{L} d\xi \frac{\xi r}{(\xi^2 - \Omega_0^2 r^2)^2}. \]  

(27)

From this we get the energy \( E \), the angular momentum \( J \), and the entropy \( S \) of the field

\[ J = \langle m \rangle_{av} = \frac{4N}{\beta^4} \Omega_0 \int \frac{r^2}{(\xi^2 - \Omega_0^2 r^2)^3} \xi r d\xi dr dz, \]  

(28)

\[ E = \langle E \rangle_{av} = \frac{N}{\beta^4} \int \frac{3\xi^2 + \Omega_0^2 r^2}{(\xi^2 - \Omega_0^2 r^2)^3} \xi r d\xi dr dz, \]  

(29)

\[ S = \frac{4N}{\beta^3} \int \frac{1}{(\xi^2 - \Omega_0^2 r^2)^2} \xi r d\xi dr dz. \]  

(30)

It is noted that the thermodynamic quantities \( F, E, \) and \( S \) are divergent as \( \xi \to \Omega_0 r \) rather than the event horizon. Only in \( \Omega_0 = 0 \) case the divergence occurs at the horizon \( \xi = 0 \). Such a fact can be easily understand in the co-moving frame, of which line element is given by

\[ ds^2 = -\xi^2 d\eta^2 + r^2 (d\phi' + \Omega_0 d\eta)^2 + d\xi^2 + dr^2 \]  

(31)

\[ = -(\xi^2 - \Omega_0^2 r^2) d\eta^2 + 2\Omega_0 r^2 d\eta d\phi' + r^2 d\phi'^2 + d\xi^2 + dr^2, \]

where we used \( \phi' = \phi - \Omega_0 \eta \). In this spacetime the event horizon is at \( \xi = 0 \). In addition to the event horizon there is a stationary limit surface at \( \xi = \Omega_0 r \), where the Killing vector \( \partial_\eta \) becomes null. That surface is the elliptic hyper-surface [19]. In the interval \( 0 < \xi < \Omega_0 r \), the Killing vector is spacelike. We can also show that the entropy in the co-moving frame is the same form with Eq. (30). These facts imply that the divergence of the thermodynamic quantities is deeply related to the stationary limit surface in the co-moving frame rather than the event horizon.

## 4 A Entropy of a Scalar Field in a Rotating Black Hole

### 4.1 General Formalism

Let us consider a scalar field with mass \( \mu \) in thermal equilibrium at temperature \( 1/\beta \) in the rotating black hole background, of which line element is generally given by

\[ ds^2 = g_{tt}(r, \theta) dt^2 + 2 g_{t\phi}(r, \theta) dt d\phi + g_{\phi\phi}(r, \theta) d\phi^2 + g_{rr}(r, \theta) dr^2 + g_{\theta\theta}(r, \theta) d\theta^2. \]  

(32)

This metric has two Killing vector fields: the timelike Killing vector \( \xi^\mu = (\partial_t)^\mu \) and the axial Killing vector \( \psi^\mu = (\partial_\phi)^\mu \). The metrics, we concern, of the Kaluza-Klein, the Sen, and the Kerr-Newman black holes are in the appendix. The properties of those metrics are

\[ g_{tt} g_{\phi\phi} - g_{t\phi}^2 = -\Delta(r) \sin^2 \theta \to 0 \]  

(33)
and
\[
\left(g_{tt}g_{\phi\phi} - g_{t\phi}^2\right) g_{rr} \to \text{finite}
\] (34)
as one approaches the horizon. Another property is that there are two important surfaces (the event horizon and the stationary limit surface), and the two surfaces do not coincide. On the stationary limit surface the Killing vector \(\xi^\mu\) vanishes, and the Killing vector \(\xi^\mu + \Omega_H \psi^\mu\) is null on the horizon, where \(\Omega_H\) is the angular velocity of the horizon.

The equation of motion of the field with mass \(\mu\) and arbitrary coupled to the scalar curvature \(R(x)\) is
\[
\left[\nabla_\mu \nabla^\mu - \xi R - \mu^2\right] \Psi = 0,
\] (35)
where \(\xi\) is an arbitrary constant. \(\xi = 1/6\) and \(\mu = 0\) case corresponds to the conformally coupled one. We assume that the scalar field is rotating with a constant azimuthal angular velocity \(\Omega_0\). The associated conserved quantities are angular momentum \(J\). The free energy of the system is then given by
\[
F = \frac{1}{\beta} \sum_m \int_0^\infty dE g(E, m) \ln \left(1 - e^{-\beta(E - m\Omega_0)}\right),
\] (36)
where \(g(E, m)\) is the density of state for a given \(E\) and \(m\).

To evaluate the free energy we will follow the brick wall method of \(\text{t Hooft}\) [5]. Following the brick wall method we impose a small radial cut-off \(h\) such that
\[
\Psi(x) = 0 \quad \text{for} \quad r \leq r_H + h,
\] (37)
where \(r_H\) denotes the coordinate of the event horizon. To remove the infra-red divergence we also introduce another cut-off \(L \gg r_H\) such that
\[
\Psi(x) = 0 \quad \text{for} \quad r \geq L.
\] (38)
It is noted that the brick wall is spherically symmetric. In the WKB approximation with \(\Psi = e^{-iE t + im\phi + iS(r, \theta)}\) the equation (35) yields the constraint
\[
p_r^2 = \frac{1}{g^{rr}} \left[-g^{tt} E^2 + 2g^{t\phi} E m - g^{\phi\phi} m^2 - g^{\theta\theta} p_\theta^2 - V(x)\right],
\] (39)
where \(p_r = \partial_r S,\) \(p_\theta = \partial_\theta S,\) and \(V(x) = \xi R(x) + \mu^2.\) In WKB approximation it is important to note that the number of state for a given \(E\) is determined by \(p_\theta, p_r\) and \(m.\) The number of mode with energy less than \(E\) and with a fixed \(m\) is obtained by integrating over \(p_\theta\) in phase space.
\[
\Gamma(E, m) = \frac{1}{\pi} \int d\phi d\theta \int dr \int dp_\theta p_r(E, m, x)
\]
\[
\quad = \frac{1}{\pi} \int d\phi d\theta \int dr \int dp_\theta \left[\frac{1}{g^{rr}} \left(-g^{tt} E^2 + 2g^{t\phi} E m - g^{\phi\phi} m^2 - g^{\theta\theta} p_\theta^2 - V(x)\right)\right]^{\frac{1}{2}}.
\] (40)
The integration over \(p_\theta\) must be carried over the phase space such that \(p_r \geq 0.\)
In this point we need some remarks. In a rotating system, in general, there is a superradiance effect, which occurs when \( 0 < \mathcal{E} < m \Omega_0 \). For this range of the frequency the free energy \( F \) becomes a complex number. In case \( \mathcal{E} = m \Omega_0 \) the free energy is divergent. Therefore to obtain a real finite value for the free energy \( F \), we must require that \( \mathcal{E} > m \Omega_0 \). ( For \( 0 < \mathcal{E} < m \Omega_0 \) the free energy diverges. See below.) This requirement say that we must restrict the system to be in the region such that \( g'_{tt} \equiv g_{tt} + 2 \Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} < 0 \). In this region \( \mathcal{E} - m \Omega_0 > 0 \), so the free energy is a finite real value. It is easily showed as follows. Let us define \( E = \mathcal{E} - m \Omega_0 \). Then it is written as

\[
E = \left( \frac{g^{t\phi}}{g_{tt} - \Omega_0} \right) m + \frac{1}{g^{tt}} \left[ \left( g^{t\phi} m \right)^2 + \left( -g^{tt} \right) \left( V + g^{\phi\phi} m^2 + g^{rr} p_r^2 + g^{\theta\theta} p_\theta^2 \right) \right]^{1/2},
\]

where we used \( g^{tt} = \frac{g_{\phi\phi}}{D}, \ g^{t\phi} = \frac{-g_{t\phi}}{D}, \ g^{\phi\phi} = \frac{g_{tt}}{D}. \) and \( \Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} \). Here \( -D = g^{2}_{t\phi} - g_{tt} g_{\phi\phi} \). From Eq. (41), for all \( m, p_r \) and \( p_\theta \), one can see the condition such that \( E > 0 \) is

\[
\sqrt{-D} g_{\phi\phi} \pm (\Omega - \Omega_0) > 0
\]

or

\[
g'_{tt} \equiv g_{tt} + 2 \Omega_0 g_{t\phi} + \Omega_0^2 g_{\phi\phi} < 0.
\]

Therefore in the region such that \( -g'_{tt} > 0 \) (called region I) the free energy is a real, but in the region such that \( -g'_{tt} < 0 \) (called region II) the free energy is complex. However in the region I the integration over the momentum phase space is convergent. But in the region II the integration over the momentum phase is divergent. These facts become more apparent if we investigate the momentum phase space. In the region I the possible points of \( p_i \) satisfying \( \mathcal{E} = \Omega_0 p_\phi = E \) for a given \( E \) are located on the following surface

\[
\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} + \frac{g'_{tt}}{-D} \left( p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{g'_{tt}} E \right)^2 = \left( \frac{E^2}{-g'_{tt}} - V \right),
\]

which is the ellipsoid, a compact surface. Here \( p_\phi = m \). So the density of state \( g(E) \) for a given \( E \) is finite and the integrations over \( p_i \) give a finite value. But in the region II the possible points of \( p_i \) are located on the following surface

\[
\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} - \frac{g'_{tt}}{-D} \left( p_\phi + \frac{g_{t\phi} + \Omega_0 g_{\phi\phi}}{g'_{tt}} E \right)^2 = -\left( \frac{E^2}{g'_{tt}} + V \right),
\]

which is the hyperboloid, a non-compact surface. So \( g(E) \) diverges and the integration over \( p_i \) diverges. In case of \( g'_{tt} = 0 \), the possible points are given by the surface

\[
\frac{p_r^2}{g_{rr}} + \frac{p_\theta^2}{g_{\theta\theta}} = \frac{p_\phi - \left( \frac{g_{\phi\phi} E^2}{D} + V \right) / \left( \frac{2 g_{\phi\phi}}{D} E \right)}{\frac{g_{tt} E}{2 g_{\phi\phi}}}.
\]
which is elliptic paraboloid and also non-compact. Therefore the value of the \( p_i \) integration are divergent. Actually the surface such that \( g'_{tt} = 0 \) is the velocity of the light surface (VLS). Beyond VLS (in region II) the co-moving observer must move more rapidly than the velocity of light. Thus we will assume that the system is in the region I. (For the possible region I see Sec. 4.2.) For example, in the case of \( \Omega_0 = 0 \) the points satisfying \( g'_{tt} = 0 \) are on the stationary limit surface. The region of the outside (inside) of the stationary limit surface corresponds to the region I (II). In the rotating system in Sec. 2 the region I is \( r < 1/\Omega_0 \) and \( r > 1/\Omega_0 \) corresponds to the region II. In the Rindler spacetime with a rotation \( \xi > \Omega_0 r \) corresponds to the region I, and \( \xi < \Omega_0 r \) to the region II.

With the assumption that the system is in the region I we can obtain the free energy as follows

\[
\beta F = \sum_m \int_{m\Omega_0}^{\infty} d\mathcal{E} g(\mathcal{E}, m) \ln \left( 1 - e^{-\beta(\mathcal{E} - m\Omega_0)} \right)
\]

\[
= \int_0^{\infty} d\mathcal{E} \sum_m g(\mathcal{E} + m\Omega_0, m) \ln \left( 1 - e^{-\beta\mathcal{E}} \right)
\]

\[
= -\beta \int_0^{\infty} d\mathcal{E} \frac{1}{e^{\beta\mathcal{E}} - 1} \int dm \Gamma(\mathcal{E} + m\Omega_0, m),
\]

where we have integrated by parts and we assume that the quantum number \( m \) is a continuous variable. The integrations over \( m \) and \( p_0 \) yield

\[
F = -\frac{4}{3} \int d\phi d\theta \int_{r_H + h}^L dr \int_{V(x)}^{\infty} d\mathcal{E} \frac{1}{e^{\beta\mathcal{E}} - 1} \left( \frac{\mathcal{E}^2}{-g'_{tt}} - V(x) \right)^{3/2}.
\]

In particular when \( \Omega_0 = 0 \) and non-rotating case \( g_{\phi\phi} = 0 \), the free energy (II) coincides with the expression in ref. [5, 11] and it is proportional to the volume of the optical space in the limit \( V(x) = 0 \) [20]. It is easy to see that the integrand diverges as \( r_H + h \) or \( L \) approach the surface such that \( g'_{tt} = 0 \). In that case the contribution of the \( V(x) \) can be negligible.

For a massless and minimally coupled scalar field case (\( \mu = \xi = 0 \)) the free energy reduces to

\[
\beta F = -\frac{N}{\beta^3} \int d\theta d\phi \int_{r_H + h}^L dr \frac{\sqrt{g_4}}{(-g_{tt})^2} = -N \int_0^\beta d\tau \int d\theta d\phi \int_{r_H + h}^L dr \frac{\sqrt{g_4}}{\beta_{\text{local}}},
\]

where \( \beta_{\text{local}} = \sqrt{-g_{tt}/\beta} \) is the reciprocal of the local Tolman temperature [20] in the comoving frame. This form is just the free energy of a gas of massless particles at local temperature \( 1/\beta_{\text{local}} \).

From this expression (50) it is easy to obtain expressions for the total energy \( U \), angular momentum \( J \), and entropy \( S \) of a scalar field

\[
J = \langle m \rangle = -\frac{1}{\beta} \frac{\partial}{\partial \Omega_0} \beta F = \frac{4N}{\beta^4} \int d\theta d\phi \int_{r_H + h}^L dr \frac{\sqrt{g_4}}{(-g_{tt})^2} \frac{g_{\phi\phi}}{(-g'_{tt})} (\Omega_0 - \Omega),
\]

\[
U = \langle \mathcal{E} \rangle = \Omega_0 J + \frac{\partial}{\partial \beta} \beta F = \frac{N}{\beta^3} \int d\theta d\phi \int_{r_H + h}^L dr \frac{\sqrt{g_4}}{(-g_{tt})^2} \left[ 3 + 4 \frac{\Omega_0 (\Omega_0 - \Omega) g_{\phi\phi}}{(-g'_{tt})} \right],
\]

\[
S = \beta^2 \frac{\partial}{\partial \beta} F = \beta (U - F - \Omega_0 J) = \frac{4N}{\beta^3} \int d\theta d\phi \int_{r_H + h}^L dr \frac{\sqrt{g_4}}{(-g_{tt})^2},
\]

which are also divergent as one approach the surface such that \( g'_{tt} = 0 \).
4.2 The region such that \(-g'_{tt} > 0\).

In this section we study where is the possible region I for three black hole, the Kaluza-Klein, and the Sen, the Kerr-Newman black holes, for \(\Omega_0 = \Omega_H\), \(\Omega_0 < \Omega_H\) and the extreme case with \(\Omega_0 = \Omega_H\).

4.2.1 The Kaluza-Klein black hole

A) \(\Omega_0 = \Omega_H\) case: In \(\Omega_0 = \Omega_H\) case the position of the light of velocity surface is exactly found. In such a case \(g'_{tt}\) can be written as

\[
g'_{tt} = g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi}
\]

\[
= \frac{\mu^2}{B\Sigma} (x - \bar{r}_H) \left\{ \frac{y^2 \sin^2 \theta}{4r_H^2} (1 - v^2)x^3 + \frac{y^2 \sin^2 \theta}{4r_H^2} \left( 2 - \bar{r}_-(1 - v^2) \right) x^2 
+ \left[ -1 + \frac{y^2 \sin^2 \theta}{4r_H^2} \left( 4 + y^2(1 - v^2) \cos^2 \theta - 2\bar{r}_- \right) \right] x 
+ \left[ \bar{r}_- + \frac{y^2 \sin^2 \theta}{4r_H^2} \left( -4\bar{r}_H - \bar{r}_-y^2(1 - v^2) \cos^2 \theta \right) \right] \right\}
\]

\[
= \frac{\mu^2}{B\Sigma} (x - \bar{r}_H) \frac{y^2 \sin^2 \theta}{4r_H^2} (1 - v^2) \left( x^3 + a_1 x^2 + a_2 x + a_3 \right)
\]

for \(\theta \neq 0\), where \(x = \frac{\mu}{\rho}, y = \frac{\rho}{\mu}, \bar{r}_H = \frac{\rho H}{\mu}\), and \(\bar{r}_- = \frac{\rho H}{\mu}\). From this we can see that there are two VLS. One is the horizon \((r = r_H)\), and another light of velocity surface (call outer VLS) is given by

\[
r_{VLS} = 2\mu \sqrt{-Q} \cos \left( \frac{1}{3} \Theta \right) - \frac{1}{3} a_1 \mu,
\]

where

\[
\Theta = \arccos \left( \frac{P}{\sqrt{-Q^2}} \right)
\]

with

\[
Q = \frac{3a_2 - a_1^2}{9}, \quad P = \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54}.
\]

In case of the slowly rotating black hole \((a \text{ is small})\) the VLS is approximately given by

\[
r_{VLS} \sim 2\mu \frac{r_H}{a\sqrt{1 - v^2} \sin \theta} - \frac{1}{3} \left( \frac{2}{1 - v^2} - \frac{r_-}{\mu} \right) \mu,
\]

which is an open, roughly, cylindrical surface. As \(v \to 1\) or \(a \to 0\) the VLS become more distant, which came from the fact that as \(v \to 1\) or \(a \to 0\) the coordinate angular velocity \(\frac{d\phi}{dt} = -\frac{g_{t\phi}}{g_{\phi\phi}}\) becomes vanish. For \(\theta = 0\) it is always that \(g'_{tt} < 0\) for \(r > r_H\). As \(a \to \mu\) the outer VLS approaches horizon. See Fig.1.

B) \(\Omega_0 < \Omega_H\) case: In this case \(g'_{tt} = 0\) is a fourth order polynomial equation in \(r\) for a given \(\theta\). The region I corresponds to \(r_{in} < r < r_{VLS}\). At \(\theta = \pi/2\) \(r_{in}\) is between the stationary limit surface and the event horizon, and at \(\theta = 0\) \(r_{in}\) contacts with the event horizon. Actually the inner VLS \(r_{in}\)
places between the stationary limit surface and the event horizon for all $\theta$. The particular point is that as $\Omega_0 \rightarrow \Omega_H$, $r_{in}$ approaches the horizon. However it does attach the horizon only when $\Omega_0 = \Omega_H$. While, the outer velocity of light surface locates at the very far distance from the horizon, and it is a roughly cylindrical surface as in case $\Omega_0 = \Omega_H$. For the position of the inner VLS see Fig.2.

C) the extreme black hole case with $\Omega_0 = \Omega_H$: The extreme black hole for the Kaluza-Klein black hole occurs when $\mu^2 = a^2$. In this case the inner horizon and outer horizon are at the same place. At $\theta = 1/2\pi$, $g_{tt}'$ is written as

$$g_{tt}' = \frac{\mu^2}{\Sigma} (x - \bar{r}_H)^2 x \left( x + \frac{2}{1 - \nu^2} \right) \frac{1 - \nu^2}{4},$$

which shows that the possible region such that $g_{tt}' < 0$ does not exist at $\theta = 1/2\pi$. Therefore in the extreme black hole case it is impossible to consider the brick wall model of 't Hooft.

4.2.2 The Sen black hole

A) $\Omega_0 = \Omega_H$ case: In $\Omega_0 = \Omega_H$ case $g_{tt}'$ can be written as

$$g_{tt}' = g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi}$$

$$= \frac{\mu^2}{\Sigma} \left( x - \bar{r}_H \right) \left( \frac{y^2 \sin^2 \theta}{4r_H^2 \cosh^4 \gamma} x^3 + \frac{y^2 \sin^2 \theta}{4r_H^2 \cosh^4 \gamma} \left( 2 \cosh 2\gamma - \bar{r}_- \right) x^2 + \left[ -1 + \frac{y^2 \sin^2 \theta}{\bar{r}_H^2} + \frac{y^2 \sin^2 \theta}{4r_H^2 \cosh^4 \gamma} \left( y^2 \cos^2 \theta - 2\bar{r}_- \cosh 2\gamma \right) \right] x + \left[ \bar{r}_- + \frac{y^2 \sin^2 \theta}{r_H^2} - \frac{y^2 \sin^2 \theta}{4r_H^2 \cosh^4 \gamma} \left( \bar{r}_- y^2 \cos^2 \theta \right) \right] \right)$$

$$= \frac{\mu^2}{\Sigma} \left( x - \bar{r}_H \right) \frac{y^2 \sin^2 \theta}{4r_H^2 \cosh^4 \gamma} \left( x^3 + a_1 x^2 + a_2 x + a_3 \right)$$

for $\theta \neq 0$, where $x = \frac{r}{\mu}, y = \frac{a}{\mu}, \bar{r}_H = \frac{r}{\mu}$, and $\bar{r}_- = \frac{r}{\mu}$. Then the exact position of the inner VLS and outer VLS are given by

$$r_{in} = r_H, \quad r_{VLS} = 2\mu \sqrt{-Q} \cos \left( \frac{1}{3} \Theta \right) - \frac{1}{3} a_1 \mu.$$ \hspace{1cm} (63)

The position of the outer VLS for small $a$ is approximately given by

$$r_{VLS} \sim \frac{2\mu r_H \cosh^2 \gamma}{a \sin \theta} - \frac{1}{3} \left( 2 \cosh (2\gamma) - \frac{r_-}{\mu} \right) \mu,$$

which is an open, roughly, cylindrical surface. As $a \rightarrow 0$ the VLS goes to the infinity, and it disappears when $a = 0$. As $\gamma$ or $a$ is increasing the VLS approaches the horizon. At $\theta = \frac{1}{2}\pi$, similarly to the Kaluza-Klein black hole, $g_{tt}' < 0$ for $r > r_H$. See Fig.3.

B) $\Omega_0 < \Omega_H$ case: In this case $g_{tt}' = 0$ is also a fourth order equation in $r$ for a given $\theta$. Similarly to the Kaluza-Klein black hole the region I is $r_{in} < r < r_{VLS}$. At $\theta = 0$ the inner VLS $r_{in}$ is at the
horizon, and at $\theta = \pi/2$ $r_{in}$ locates at the between the stationary limit surface and the event horizon. See Figure 4. As $\Omega_0 \rightarrow \Omega_H$, $r_{in}$ approaches to the horizon. Only when $\Omega_0 = \Omega_H$ it coincides with the event horizon. The outer velocity of light surface, in case of small $a$, locates at the very far distance from the horizon, and it is a roughly cylindrical surface.

C) *the extreme black hole case with* $\Omega_0 = \Omega_H$: The extreme black hole for the Kaluza-Klein black hole occurs when $\mu^2 = a^2$. In this case the inner horizon and outer horizon are at the same place. At $\theta = \frac{1}{2}\pi$ $g'_{tt}$ is written as

$$g'_{tt} = \frac{\mu^2}{\Sigma} (x - \bar{r}_H)^2 x (x + 2 \cosh 2\gamma)$$

which shows that the possible region such that $g'_{tt} < 0$ does not exist at $\theta = 1/2\pi$. Therefore in the extreme black hole case it is impossible to consider the brick wall model of 't Hooft.

### 4.2.3 The Kerr-Newman black hole

A) $\Omega_0 = \Omega_H$ case: In $\Omega_0 = \Omega_H$ case we can exactly find the position of the light of velocity surface. In such a case $g'_{tt}$ can be written as

$$g'_{tt} = g_{tt} + 2\Omega_H g_{t\phi} + \Omega_H^2 g_{\phi\phi}$$

$$= \frac{M^2}{\Sigma} (x - \bar{r}_H) \left\{ \bar{\Omega}_H^2 \sin^2 \theta \cdot x^3 + \bar{r}_H \bar{\Omega}_H^2 \sin^2 \theta \cdot x^2 + \left[ -1 + \bar{\Omega}_H^2 \sin^2 \theta \left( y^2 + y^2 \cos^2 \theta + \bar{r}_H^2 \right) \right] x \right. + \left. 2 \left( 1 - \bar{\Omega}_H \bar{r}_H \sin^2 \theta \right)^2 - \bar{r}_H + \bar{r}_H \bar{\Omega}_H^2 \sin^2 \theta \left( \bar{r}_H^2 + y^2 + y^2 \cos^2 \theta \right) \right\}$$

$$\equiv \frac{M^2}{\Sigma} (x - \bar{r}_H) \bar{\Omega}_H^2 \sin^2 \theta \left( x^3 + a_1 x^2 + a_2 x + a_3 \right)$$

for $\theta \neq 0$, where $x = r/M, y = a/M, z = e/M, \bar{\Omega}_H = M\Omega_H, \bar{r}_H = r_H/M$. Then the exact position of the outer light of velocity surface is given by

$$r_{VLS} = 2M \sqrt{-Q} \cos \left( \frac{1}{3} \Theta \right) - \frac{1}{3} a_1 M.$$  

(69)

For small $a$ Eq. (69) is approximately given by

$$r_{VLS} \sim \frac{1}{\Omega_H \sin \theta} \frac{r_H}{3},$$

(70)

which is an open, roughly, cylindrical surface. For $\theta = 0$ it is always that $g'_{tt} < 0$ for $r > r_H$. As $a \rightarrow 0$, $r_{VLS}$ goes to infinity, and as $a \rightarrow \sqrt{M^2 + e^2}$ it approaches the event horizon. See Fig.5. The inner VLS $r_{in}$ is the event horizon.

B) $\Omega_0 < \Omega_H$ case: In this case, similarly to other black holes, the inner VLS $r_{in}$ approaches to the horizon as $\Omega_0 \rightarrow \Omega_H$. See Fig.6. The inner VLS is a compact surface, which shrink to horizon as $\Omega_0 \rightarrow \Omega_J$. See Fig.7. The outer VLS is at far place, which disappears when $\Omega_0 = 0$.  

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C) the extreme black hole case with \( \Omega_0 = \Omega_H \): For extreme Kerr-Newman black hole case, which occurs when \( M^2 = a^2 + e^2 \), \( g'_{tt} \) at \( \theta = \frac{7}{2} \pi \) is written as

\[
g'_{tt} = \frac{M^2}{\Sigma} \frac{y}{1 + y^2} (x - 1)^2 \left( x + 1 - \frac{1}{y} \right) \left( x + 1 + \frac{1}{y} \right)
\]  

(71)

From this we obtain the position of VLS at \( \theta = \frac{\pi}{2} \) as

\[
r = M \quad \text{for } \frac{1}{2} M \leq a \leq M \text{ and } a = 0,
\]

(72)

\[
r = \left( -1 + \frac{M}{a} \right) M \quad \text{for } 0 < a < \frac{1}{2} M.
\]

(73)

The second case corresponds to the extreme black hole that is slowly rotating and has many charge. (In this case \( e > \sqrt{3}/2M \approx 0.866M \)). In particular in case of \( e \leq \sqrt{3}/2M \) ( \( a = M \) for \( e = 0 \)) the horizon and the light of velocity surface are at the same position. Therefore in case of the extreme black hole with \( a \geq 1/2M \) it is impossible to consider the brick wall model of ’t Hooft.

5 The Entropy in the Hartle-Hawking Vacuum

The Hartle-Hawking vacuum state is one that the angular velocity \( \Omega_0 \) is equal to that of the event horizon, and the temperature \( \beta \) is equal to the Hawking temperature, where the Hawking temperature and the angular velocity of the horizon are defined as

\[
T_H = \frac{\kappa}{2\pi}, \quad \Omega_H = \lim_{r \to r_H} \left( -\frac{g_{t\phi}}{g_{\phi\phi}} \right).
\]

(74)

Here \( \kappa \) is the surface gravity of the horizon.

First of all let us assume that \( \Omega_0 = \Omega_H \). In this case, as stated in Sec.4, the possible region I is \( r_H < r < L < r_{VLS} \). The outer brick wall must locate inside the outer VLS. This fact was already pointed out by Frolov and Thorne [26] to remove the singular structure of the Hartle-Hawking vacuum and modify it. Now recall that in general \( g'_{tt}|_{r=r_H} = 0 \). This came from that \( g'_{tt} \) is the same form as \( \chi^\mu X_\mu = (\xi^\mu + \Omega_H \psi^\mu)(\xi_\mu + \Omega_H \psi_\mu) \), and \( \chi^\mu \) is null on the horizon. So it follows that \( g'_{tt} = (r - r_H) G(r, \theta) \), where \( G(r, \theta) \) is a non-vanishing function at \( r = r_H \) except the extremal case. (We can not consider the extreme black hole case.)

Therefore for the three black holes the leading behaviors of the free energy \( F \) for very small \( h \) are then given by

\[
\beta F \approx -\frac{N}{\beta^3} \int d\theta d\phi \int_{r_H + h}^{L} dr \frac{\sqrt{g_4}}{(-g_{tt})^2} \int_{r_H + h}^{L} dr \frac{D(r)}{(r - r_H)^2 G^2(r, \theta)}.
\]

(75)

(76)
where $D(r, \theta) = \sqrt{g_{\theta\theta}}$. Since $D(r, \theta)$ and $G(r, \theta)$ are non-vanishing functions at $r = r_H$ we can expand it about $r = r_H$ as follows.

$$D(r, \theta) = D(r_H, \theta) + D'(r_H, \theta)(r - r_H) + O((r - r_H)^2),$$  \hspace{1cm} (77)

$$\frac{1}{G^2(r, \theta)} = \frac{1}{G^2(r_H, \theta)} + \left( \frac{1}{G^2(r_H, \theta)} \right)' + O((r - r_H)^2),$$  \hspace{1cm} (78)

where $'$ denotes the partial derivative for $r$. So the free energy is approximately given by

$$\beta F \approx -N \frac{\beta}{3} \int d\phi d\theta \int dr \left\{ \frac{D(r_H, \theta)}{G^2(r_H, \theta)(r - r_H)^2} + \frac{D(r_H, \theta)}{G^2(r_H, \theta)} \frac{1}{(r - r_H)} + O((r - r_H)^0) \right\}$$

$$= -2\pi N \frac{1}{\beta^3} \left\{ \frac{1}{h} \int d\theta \frac{D(r_H, \theta)}{G^2(r_H, \theta)} - \ln(h) \int d\theta \left( \frac{D(r_H, \theta)}{G^2(r_H, \theta)} \right)' + \ldots \right\},$$  \hspace{1cm} (79)

which show that generally, in addition to the linear divergence term in $h$, there is a logarithmic one in the case of rotating black hole. If we written the free energy in terms of the proper distance cut-off $\epsilon$, it become very simple form.

$$\beta F \approx -N \frac{1}{\beta^3} \int_{r=r_H}^{r=r_H+h} d\phi d\theta \int dr \sqrt{g_{\phi\phi}} \int_{r_H+h}^{L} dr \sqrt{g_{rr}} \left( \frac{g_{\phi\phi}}{g_{\phi\phi} - g_{\theta\theta} - g_{\phi\phi}} \right)^{3/2}$$

$$= N \frac{A_H}{2(\kappa \beta)^3} \frac{A_H}{\epsilon^2},$$  \hspace{1cm} (80)

where $A_H$ is the area of the event horizon, and $\epsilon$ is the proper distance from the horizon to $r_H + h$.

However the proper distance cut-off is dependent on the coordinate $\theta$, which is the general property of the rotating black hole.

From the free energy $F$ we obtain the leading behaviors of the entropy $S$ as

$$S = \beta^2 \frac{\partial}{\partial \beta} F$$

$$\approx N \frac{1}{\beta^3} \left( A \frac{1}{h} + B \ln(h) + finite \right),$$  \hspace{1cm} (82)

where $A$ and $B$ are in c-number in Eq.(79), or

$$S \approx \frac{4N}{2(\kappa \beta)^3} \frac{A_H}{\epsilon^2}.$$  \hspace{1cm} (83)

The entropy $S$ is linearly and logarithmically divergent as $h \to 0$. The divergences arise because the density of state for a given $E$ diverges as $h$ goes to zero.

Now we take $T$ as the Hartle-Hawking temperature $T_H = \frac{\kappa}{2\pi}$. Then the entropy becomes

$$S_H \approx \frac{N8\pi^3}{\kappa^3} \left( A \frac{1}{h} + B \ln(h) + finite \right),$$  \hspace{1cm} (84)
or
\[ S_H \approx \frac{N}{4\pi^3} \frac{A_H}{\epsilon^2}. \] (85)

The entropy of a scalar field in Hartle-Hawking state diverges quadratically in \( \epsilon^{-1} \) as the system approaches the horizon. Or it diverges in \( h^{-1} \) and \( \ln(h) \). In case \( a = 0 \) our result (85) agrees with the result calculated by 't Hooft \[5\] and with one in ref.[21]. These facts imply that the leading behaviors of entropy (85) is general form.

6 Summary and Conclusion

By using the brick wall method we have calculated the entropies of the rotating systems with a rotation \( \Omega_0 \) at thermal equilibrium with temperature \( T \) in the rotating black holes. In WKB approximation to get the real finite free energy and entropy the system must be in the region I. As the system approaches the VLS ( \( r_{in} \) and \( r_{VLS} \) ) the thermodynamic quantities become divergent. From this fact we conclude that the divergence of the thermodynamic quantities including the entropy is related to the stationary limit surface in the co-moving frame. In spherical symmetric black hole the stationary limit surface and the event horizon are coincide. Only when \( \Omega_0 = \Omega_H \) the system can be approach the horizon. The entropy for this case is linearly and logarithmically divergent as the ultraviolet cut-off goes to zero. To remove such a divergence, in addition to the renormalization of the gravitational constant, we need the renormalization of the curvature square term\[21\]. But after the renormalization the entropy does not proportional to the area of the event horizon. If we use the proper distance cut-off the entropy is proportional to the horizon area \( A_H \). But the cut-off depends on the coordinate \( \theta \).

Another particular point is that in the extremal black hole case we can not consider the brick wall method of 't Hooft except for the case \( 0 < a < 1/2M \) in Kerr-Newman black hole.

Appendix

For the three rotating black holes the metrics, the surface gravities, , and the proper distances \( \epsilon \) are given as follows:

1) the Kaluza-Klein black hole \[7\]

\[
ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{B \Sigma} dt^2 - 2a \sin^2 \theta \frac{1}{\sqrt{1 - v^2}} B dtd\phi + \left[ B \left( r^2 + a^2 \right) + a^2 \sin^2 \theta \frac{Z}{B} \right] \sin^2 \theta d\phi^2 + \frac{B \Sigma}{\Delta} dr^2 + B \Sigma d\theta^2,
\] (86)

where
\[
\Delta = r^2 - 2\mu r + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta, \quad Z = \frac{2\mu r}{\Sigma}, \quad B = \left( 1 + \frac{v^2 Z}{1 - v^2} \right)^{\frac{1}{2}}.
\] (87)
The physical mass $M$, the charge $Q$, the angular momentum $J$, and the horizon are expressed by the parameters $v, \mu$, and $a$ as

$$M = \mu \left[ 1 + \frac{v^2}{2(1 - v^2)} \right], \quad Q = \frac{\mu v}{1 - v^2}, \quad J = \frac{\mu a}{\sqrt{1 - v^2}}, \quad r_H = \mu + \sqrt{\mu^2 - a^2}. \quad (88)$$

The surface gravity and proper distance are

$$\kappa_{\text{Kaluza-Klein}} = \frac{\sqrt{(1 - v^2)(\mu^2 - a^2)}}{r_H^2 + a^2}, \quad (89)$$

$$\epsilon_{\text{Kaluza-Klein}} = 2 \left( \frac{B(r_H)\Sigma(r_H)}{2r_H - 2\mu} \right)^{1/2} \sqrt{h}. \quad (90)$$

2) the Sen black hole [16]:

$$ds^2 = -\frac{\Delta - a^2 \sin^2 \theta}{\Sigma} dt^2 - \frac{4 \mu r \cos^2 \gamma \sin^2 \theta}{\Sigma} dtd\phi \quad (91)$$

$$+ \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \frac{\Lambda}{\Sigma} \sin^2 \theta d\phi^2, \quad (92)$$

where

$$\Delta = r^2 - 2\mu r + a^2, \quad \Sigma = r^2 + a^2 \cos^2 \theta + 2 \mu r \sinh^2 \gamma, \quad (93)$$

$$\Lambda = \left( r^2 + a^2 \right) \left( r^2 + a^2 \cos^2 \theta \right) + 2 \mu r \sin^2 \theta \quad (94)$$

$$+ 4 \mu r \left( r^2 + a^2 \right) \sinh^2 \gamma + 4 \mu^2 r^2 \sinh^4 \gamma. \quad (95)$$

The mass $M$, the charge $Q$, the angular momentum $J$, and the horizon are given by parameters $\mu, \beta$, and $a$ as

$$M = \frac{\mu}{2} (1 + \cosh 2\gamma), \quad Q = \frac{\mu}{\sqrt{2}} \sinh 2\gamma, \quad J = \frac{\alpha \mu}{2} (1 + \cosh 2\gamma), \quad r_H = \mu + \sqrt{\mu^2 - a^2}. \quad (96)$$

The surface gravity and proper distance are

$$\kappa_{\text{Sen}} = \frac{\sqrt{(2M^2 - e^2)^2 - 4J^2}}{2M \left[ 2M^2 - e^2 + \sqrt{(2M^2 - e^2)^2 - 4J^2} \right]}, \quad (97)$$

$$\epsilon_{\text{Sen}} = 2 \left( \frac{r_H^2 + a^2 \cos^2 \theta + 2 \mu r \sinh^2 \gamma}{2r_H - 2\mu} \right)^{1/2} \sqrt{h}. \quad (98)$$

3) the charged Kerr black hole [17]

$$ds^2 = -\left( \frac{\Delta - a^2 \sin^2 \theta}{\Sigma} \right) dt^2 - \frac{2a \sin^2 \theta (r^2 + a^2 - \Delta)}{\Sigma} dtd\phi \quad (99)$$

$$+ \left[ (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \right] \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,$$
where
\[ \Sigma = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 + e^2 - 2Mr, \]  
(100)
and \( e, a, \) and \( M \) are charge, angular momentum per unit mass, and mass of the spacetime respectively. The event horizon is
\[ r_H = M + \sqrt{M^2 - a^2 - e^2}. \]  
(101)
The surface gravity and proper distance are
\[ \kappa_{Kerr} = \frac{\sqrt{M^2 - a^2 - e^2}}{2M \left[ M + \sqrt{M^2 - a^2 - e^2} \right] - e^2}, \]  
(102)
\[ \epsilon_{Kerr} = 2 \left( \frac{r_H^2 + a^2 \cos^2 \theta}{2r_H - 2M} \right)^{1/2} \sqrt{\hbar}. \]  
(103)

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Figure 1: The position of the outer velocity of light surface for the Kaluza-Klein black hole.

Figure 2: The position of $r_{in}$ at $\theta = 0.5\pi$ for the Kaluza-Klein black hole. $v = 0.5$. 
Figure 3: The position of the outer velocity of light surface for the Sen black hole. $\gamma = 5.0$.

Figure 4: The position of the inner velocity of light surface for the Sen black hole. $\gamma = 5.0, \theta = 0.5\pi$. 
Figure 5: The position of the outer light of velocity surface for the Kerr-Newman black hole. $e = 0.0$.

Figure 6: The position of the inner light of surface for the Kerr-Newman black hole. $\theta = 0.5\pi$. 

Figure 6: The shape of the inner light of surface for the Kerr-Newman black hole. $a = 0.8M, e = 0$. 

\[ r_{in} \cos \theta / M \]

\[ r_{in} \sin \theta / M \]