Asymptotic series and inequalities associated to some expressions involving the volume of the unit ball

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Abstract

The aim of this work is to expose some asymptotic series associated to some expressions involving the volume of the \( n \)-dimensional unit ball. All proofs and the methods used for improving the classical inequalities announced in the final part of the first section are presented in an extended form in a paper submitted by the author to a journal for publication.

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1 Introduction and Motivation

In the recent past, inequalities about the volume of the unit ball in \( \mathbb{R}^n \):

\begin{equation}
\Omega_n = \frac{\pi^{n/2}}{\Gamma \left( \frac{n}{2} + 1 \right)} \quad (n \in \mathbb{N})
\end{equation}

have attracted the attention of many authors. See, e.g., [2]-[17]. Here \( \Gamma \) denotes the Euler’s gamma function defined for every real number \( x > 0 \), by the formula:

\[ \Gamma (x + 1) = \int_0^\infty t^x e^{-t} dt, \]

while \( \mathbb{N} \) denotes the set of all positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \).

Our products improve the following classical results:

- Chen and Lin [9] (\( a = \frac{2}{3} - 1 \), \( b = \frac{1}{3} \)):

\[ \frac{1}{\sqrt{\pi (n + a)}} \left( \frac{2\pi e}{n} \right)^{\frac{2}{3}} \leq \Omega_n < \frac{1}{\sqrt{\pi (n + b)}} \left( \frac{2\pi e}{n} \right)^{\frac{2}{3}} \quad (n \in \mathbb{N}); \]
• Borgwardt \[7\] \((a = 0, b = 1)\), Alzer \[3\] and Qiu and Vuorinen \[17\] \((a = \frac{1}{2}, b = \frac{\pi}{2} - 1)\):

\[
\sqrt{\frac{n+a}{2\pi}} \leq \frac{\Omega_{n-1}}{\Omega_n} \leq \sqrt{\frac{n+b}{2\pi}} \quad (n \in \mathbb{N}) ;
\]

• Alzer \[3\] \((\alpha^* = \frac{3\sqrt{\pi}}{4\pi+8}, \beta^* = \sqrt{2\pi})\):

\[
\frac{\alpha^*}{\sqrt{n}} \leq \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \frac{\beta^*}{\sqrt{n}} \quad (n \in \mathbb{N}) ;
\]

• Chen and Lin \[9\] \((a = \frac{\pi(1+\pi)^2}{2} - 1, b = \frac{1}{2} + 4\pi)\):

\[
\sqrt{\frac{2\pi}{n+a}} \leq \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \sqrt{\frac{2\pi}{n+b}} \quad (n \in \mathbb{N}) ;
\]

• Chen and Lin in \[9\] \((\lambda = 1, \mu = \frac{2\ln2-\ln\pi}{2\ln3-3\ln2})\):

\[
\left(1 + \frac{1}{2n} - \frac{3}{8n^2}\right)^\lambda \leq \frac{\Omega_n^2}{\Omega_{n-1} \Omega_{n+1}} \leq \left(1 + \frac{1}{2n} - \frac{3}{8n^2}\right)^\mu \quad (n \in \mathbb{N}) ;
\]

• Anderson et al. \[5\] and Klain and Rota \[12\]:

\[
1 < \frac{\Omega_n^2}{\Omega_{n-1} \Omega_{n+1}} < 1 + \frac{1}{n} \quad (n \in \mathbb{N}) ;
\]

• Alzer \[3\] \((a = 2 - \log_2\pi, \beta = \frac{1}{2})\):

\[
\left(1 + \frac{1}{n}\right)^\alpha \leq \frac{\Omega_n^2}{\Omega_{n-1} \Omega_{n+1}} < \left(1 + \frac{1}{n}\right)^\beta \quad (n \in \mathbb{N}) ;
\]

• Merkle \[13\]:

\[
\left(1 + \frac{1}{n+1}\right)^\frac{1}{2} \leq \frac{\Omega_n^2}{\Omega_{n-1} \Omega_{n+1}} \quad (n \in \mathbb{N}) ;
\]

• Chen and Lin \[9\] \((\alpha = \frac{1}{2}, \beta = \frac{2\ln2-\ln\pi}{2\ln3-3\ln2})\):

\[
\left(1 + \frac{1}{n+1}\right)^\alpha \leq \frac{\Omega_n^2}{\Omega_{n-1} \Omega_{n+1}} \leq \left(1 + \frac{1}{n+1}\right)^\beta \quad (n \in \mathbb{N}) .
\]
2 Classical results and new achievements

2.1 Asymptotic series and estimates for $\Omega_n$ and $\Omega_n^{1/n}$

Mortici [15, Rel. 17] established the following asymptotic series as $n \to \infty$:

$$
\frac{1}{n} \ln \Omega_n \sim \frac{n + 1}{2n} \ln \frac{n}{2} + \frac{1}{2} \ln (\pi e) - \frac{\ln 2\pi}{2n} - \left( \frac{1}{6n^2} - \frac{1}{45n^4} + \frac{8}{315n^6} - \frac{8}{105n^8} + \frac{1}{128n^{10}} + \ldots \right).
$$

The entire series is given below:

**Theorem 1** The following asymptotic series holds true, as $n \to \infty$:

$$
(2) \quad \frac{1}{n} \ln \Omega_n \sim -\frac{n}{2n} \ln \frac{n}{2} + \frac{1}{2} \ln (\pi e) - \frac{\ln 2\pi}{2n} - \sum_{j=1}^{\infty} \frac{2^{2j-2}B_{2j}}{j (2j-1)n^{2j}}.
$$

($B_j$ are the Bernoulli numbers).

The following double inequality [15, Theorem 2] was presented:

$$
(3) \quad \alpha(n) \leq \frac{1}{n} \ln \Omega_n \leq \beta(n) \quad (n \in \mathbb{N}),
$$

where

$$
\alpha(n) = -\frac{n + 1}{2n} \ln \frac{n}{2} + \frac{1}{2} \ln (\pi e) - \frac{\ln 2\pi}{2n} - \lambda(n),
$$

$$
\beta(n) = -\frac{n + 1}{2n} \ln \frac{n}{2} + \frac{1}{2} \ln (\pi e) - \frac{\ln 2\pi}{2n} - \mu(n),
$$

with

$$
\mu(n) = \frac{1}{6n^2} - \frac{1}{45n^4} + \frac{8}{315n^6} - \frac{8}{105n^8} + \frac{1}{128n^{10}},
$$

$$
\lambda(n) = \mu(n) + \frac{1}{128n^{10}}.
$$

**Theorem 2** The following inequality holds true:

$$
\alpha(n) - \beta(n+1) > 0 \quad (n \in \mathbb{N}).
$$

In consequence, the sequence $\left\{\Omega_n^{1/n}\right\}_{n \geq 1}$ decreases monotonically (to 0).

**Theorem 3** The following double inequality holds true, for every integer $n \geq 3$ in the left-hand side and $n \geq 1$ in the right-hand side:

$$
(4) \quad \frac{1}{\sqrt{\pi (n + \theta(n))}} \left( \frac{2\pi e}{n} \right)^{\frac{\theta}{2}} < \Omega_n < \frac{1}{\sqrt{\pi (n + \nu(n))}} \left( \frac{2\pi e}{n} \right)^{\frac{\nu}{2}},
$$

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where
\[ \theta (n) = \frac{1}{3} + \frac{1}{18n} - \frac{31}{810n^2} \]
\[ \nu (n) = \theta (n) - \frac{139}{9720n^3}. \]

Next we construct an asymptotic series for the ratio $\Omega_{n}^{1/n}/\Omega_{n+1}^{1/(n+1)}$, then we give some lower and upper bounds.

**Theorem 4** The following asymptotic series holds true, as $n \to \infty$:

\[ \frac{1}{n} \ln \Omega_{n} - \frac{1}{n + 1} \ln \Omega_{n+1} \sim \Psi (n) - \sum_{j=1}^{\infty} \frac{\psi_j}{n^j} \]

where
\[ \Psi (n) = -\frac{n + 1}{2n} \ln \frac{n}{2} + \frac{n + 2}{2n + 2} \ln \frac{n + 1}{2} - \frac{\ln 2\pi}{2n(n + 1)} \]

and the coefficients $\psi_j$ are given by:
\[ \psi_{2t+1} = \sum_{k+2s=2t+1} (-1)^{k+1} \binom{k + 2s - 1}{k} \quad (t, k, s \in \mathbb{N}) \]
\[ \psi_{2t} = \frac{2^{2t-2}B_{2t}}{t(2t-1)} + \sum_{k+2s=2t} (-1)^{k+1} \binom{k + 2s - 1}{k} \quad (t, k, s \in \mathbb{N}), \]

with
\[ \binom{v}{k} = \frac{v(v-1) \cdots (v-k+1)}{k!} \quad (v \in \mathbb{R}, k \in \mathbb{N}_0). \]

We have:
\[ \frac{1}{n} \ln \Omega_{n} - \frac{1}{n + 1} \ln \Omega_{n+1} \sim \Psi (n) - \frac{\ln 3n^3}{3n^3} + \frac{\ln 2n^2}{2n^2} - \frac{26}{45n^5} + \frac{11}{18n^6} + \cdots. \]

**Theorem 5** The following double inequality holds true:
\[ \Psi (n) - \frac{1}{3n^3} < \frac{1}{n} \ln \Omega_{n} - \frac{1}{n + 1} \ln \Omega_{n+1} < \Psi (n) - \frac{1}{3n^3} + \frac{1}{2n^4} \quad (n \in \mathbb{N}). \]

### 2.2 Asymptotic series and estimates for $\frac{\Omega_{n-1}}{\Omega_{n}}$

**Theorem 6** The following asymptotic series holds true, as $n \to \infty$:

\[ \ln \frac{\Omega_{n-1}}{\Omega_{n}} = \frac{1}{2} \ln \frac{n}{2\pi} + \sum_{j=1}^{\infty} \frac{\mu_j}{n^j} \]

where
\[ \mu_j = (-1)^j \left[ B_{j+1} \left( \frac{1}{2} \right) - B_{j+1} (1) \right] \cdot \frac{2^j}{j(j+1)} \quad (j \in \mathbb{N}) \]

($B_j$ are the Bernoulli polynomials).
In a concrete form, (6) can be written as:

\[
\ln \frac{\Omega_{n-1}}{\Omega_n} = \frac{1}{2} \ln \frac{n}{2\pi} + \left( \frac{1}{4n} - \frac{1}{24n^3} + \frac{1}{20n^5} - \frac{17}{112n^7} + \frac{31}{36n^9} + \cdots \right). 
\]

Remark that only odd powers of \( n^{-1} \) appear in this series. This can be justified using the following representation formulas of the Bernoulli polynomials in terms of the Bernoulli numbers:

\[
B_t(1) = (-1)^t B_t, \quad B_t \left( \frac{1}{2} \right) = (2^{1-t} - 1) B_t \quad (t \in \mathbb{N}).
\]

See, e.g., [18]. As the Bernoulli numbers of odd order vanish, it results that

\[
B_{j+1} \left( \frac{1}{2} \right) = B_{j+1}(1) = 0 \quad (j \in 2\mathbb{N})
\]

and consequently, \( \mu_j = 0 \), whenever \( j \) is a positive even integer.

We present the following estimates:

**Theorem 7** The following double inequality holds true:

\[
(7) \quad a(n) < \ln \frac{\Omega_{n-1}}{\Omega_n} < b(n) \quad (n \in \mathbb{N}),
\]

where

\[
a(n) = \frac{1}{2} \ln \frac{n}{2\pi} + \frac{1}{4n} - \frac{1}{24n^3} + \frac{1}{20n^5} - \frac{17}{112n^7},
\]

\[
b(n) = a(n) + \frac{31}{36n^9}.
\]

Further we deduce a new asymptotic series and one of the resulting double inequality:

**Theorem 8** The following asymptotic series holds true, as \( n \to \infty \):

\[
(8) \quad \frac{\Omega_{n-1}}{\Omega_n} = \sqrt{\frac{n}{2\pi}} \left\{ \sum_{j=0}^{\infty} \frac{c_j}{n^j} \right\},
\]

where \( c_0 = 1 \) and

\[
c_j = \frac{1}{j} \sum_{k=1}^{j} (-1)^k \left[ B_{k+1} \left( \frac{1}{2} \right) - B_{k+1}(1) \right] \frac{2^k}{k+1} c_{j-k} \quad (j \in \mathbb{N}).
\]

By listing the first terms, we get:

\[
\frac{\Omega_{n-1}}{\Omega_n} = \sqrt{\frac{n}{2\pi}} \left( 1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} + \frac{869}{65536n^6} + \cdots \right).
\]

Related to this asymptotic expansion, we prove the following estimates:
Theorem 9  The following double inequality holds true, for every integer \( n \geq 12 \) in the left-hand side and \( n \geq 1 \) in the right-hand side:

\[
\sqrt{\frac{n}{2\pi}} c(n) < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n}{2\pi}} d(n),
\]

where

\[
c(n) = 1 + \frac{1}{4n} + \frac{1}{32n^2} - \frac{5}{128n^3} - \frac{21}{2048n^4} + \frac{399}{8192n^5} + \frac{869}{65536n^6},
\]

\[
d(n) = c(n) + \frac{869}{65536n^6}.
\]

Theorem 10  The following asymptotic formula holds true as \( n \to \infty \):

\[
\frac{\Omega_{n-1}}{\Omega_n} = \sqrt{\frac{n}{2\pi} + \frac{1}{2\pi} + \frac{1}{16\pi n} - \frac{1}{32\pi n^2} - \frac{5}{256\pi n^3} + \frac{23}{512\pi n^4} + \frac{53}{2048\pi n^5} - \frac{593}{4096\pi n^6} - \frac{5165}{65536\pi n^7} + \cdots}
\]

The first terms are indicated below:

\[
\frac{\Omega_{n-1}}{\Omega_n} = \left( \frac{n}{2\pi} + \frac{1}{16\pi n} - \frac{1}{32\pi n^2} - \frac{5}{256\pi n^3} + \frac{23}{512\pi n^4} + \frac{53}{2048\pi n^5} - \frac{593}{4096\pi n^6} - \frac{5165}{65536\pi n^7} + \cdots \right)^{1/2}.
\]

By truncation of this series, increasingly accurate under- and upper- approximations for the ratio \( \frac{\Omega_{n-1}}{\Omega_n} \) are obtained. As an example, we show the following:

Theorem 11  The following double inequality holds true, for every integer \( n \geq 1 \) in the left-hand side and \( n \geq 2 \) in the right-hand side:

\[
\sqrt{\frac{n}{2\pi} + \frac{1}{16\pi n} - \frac{1}{32\pi n^2} - \frac{5}{256\pi n^3}} < \frac{\Omega_{n-1}}{\Omega_n} < \sqrt{\frac{n}{2\pi} + \frac{1}{16\pi n} - \frac{1}{32\pi n^2}}.
\]

Theorem 12  The following double inequality holds true:

\[
\sqrt{\frac{2\pi}{n + 4\pi + \frac{1}{2}}} + \varepsilon_1(n) < \frac{\Omega_n}{\Omega_{n-1} + \Omega_{n+1}} < \sqrt{\frac{2\pi}{n + 4\pi + \frac{1}{2}}} + \varepsilon_2(n) \quad (n \in \mathbb{N}),
\]

where

\[
\varepsilon_1(n) = -\frac{1}{2} \pi - 4\pi^2 + 8\pi^3
\]

\[
\varepsilon_2(n) = \varepsilon_1(n) + \frac{3}{2} \pi - 7\pi^2 - 12\pi^3 + 64\pi^4.
\]
2.3 Asymptotic series and estimates for $\frac{\Omega^2}{\Omega_{n-1}\Omega_{n+1}}$

We start this section by establishing new asymptotic expansions for the ratio $\frac{\Omega^2}{\Omega_{n-1}\Omega_{n+1}}$ and some associated inequalities.

**Theorem 13** The following asymptotic series holds true, as $n \to \infty$:

$$\ln \frac{\Omega^2}{\Omega_{n-1}\Omega_{n+1}} = \sum_{j=1}^{\infty} \frac{\lambda_j}{n^j},$$

where

$$\lambda_j = (-1)^j \left\{ 2B_{j+1} - B_{j+1} \left( \frac{1}{2} \right) - B_{j+1} \left( \frac{3}{2} \right) \right\} \frac{2^j}{j(j+1)} \quad (j \in \mathbb{N}).$$

As the first terms in this series are $\ln \frac{\Omega^2}{\Omega_{n-1}\Omega_{n+1}} = 1 + \frac{1}{2n} - \frac{3}{8n^2} + \frac{3}{16n^3} + \frac{3}{128n^4} + \cdots$, we are entitled to present the following estimates:

**Theorem 14** The following double inequality holds true:

$$p(n) < \ln \frac{\Omega^2}{\Omega_{n-1}\Omega_{n+1}} < q(n) \quad (n \in \mathbb{N}),$$

where

$$p(n) = \frac{1}{2n} - \frac{1}{2n^2} + \frac{5}{12n^3} - \frac{1}{4n^4} + \frac{1}{16n^5} - \frac{1}{6n^6},$$

$$q(n) = p(n) + \frac{1}{6n^6}.$$

**Theorem 15** The following asymptotic series holds true, as $n \to \infty$:

$$\frac{\Omega^2}{\Omega_{n-1}\Omega_{n+1}} = \sum_{j=0}^{\infty} \frac{d_j}{n^j},$$

where $d_0 = 1$ and $d_j$s, $j \in \mathbb{N}$, are defined by the recursive relation:

$$d_j = \frac{1}{j} \sum_{k=1}^{j} (-1)^k \left[ 2B_{k+1} - B_{k+1} \left( \frac{1}{2} \right) - B_{k+1} \left( \frac{3}{2} \right) \right] \frac{2^k}{k+1} d_{j-k}.$$

More exactly, we have:

$$\frac{\Omega^2}{\Omega_{n-1}\Omega_{n+1}} = 1 + \frac{1}{2n} - \frac{3}{8n^2} + \frac{3}{16n^3} + \frac{3}{128n^4} + \cdots$$

We propose the following estimates associated to the series $d_j$:
Theorem 16 The following double inequality holds true, for every integer \( n \geq 6 \) in the left-hand side and \( n \geq 1 \) in the right-hand side:

\[
(15) \quad r(n) < \frac{\Omega^2_n}{\Omega_{n-1}\Omega_{n+1}} < s(n),
\]

where

\[
\begin{align*}
 r(n) &= 1 + \frac{1}{2n} - \frac{3}{8n^2} + \frac{3}{16n^3} \\
 s(n) &= r(n) + \frac{3}{128n^4}.
\end{align*}
\]

Theorem 17 The following asymptotic formula holds true as \( n \to \infty \):

\[
(16) \quad \frac{\Omega^2_n}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1 + \frac{1}{n}\right)^{\sum_{j=0}^{\infty} \frac{t_j}{j!}},
\]

where \( t_0 = \frac{1}{2} \) and \( t_j^s, j \in \mathbb{N} \), are the solution of the infinite system:

\[
\sum_{j=1}^{m} (-1)^{j+1} \frac{t_{m-j}}{j} = \lambda_m \quad (m \in \mathbb{N}).
\]

Theorem 18 The following double inequality holds true, for every integer \( n \geq 5 \) in the left-hand side and \( n \geq 1 \) in the right-hand side:

\[
(17) \quad \left(1 + \frac{1}{n}\right)^{\frac{1}{2} + \frac{1}{8n} + \frac{1}{8n^2}} < \frac{\Omega^2_n}{\Omega_{n-1}\Omega_{n+1}} < \left(1 + \frac{1}{n}\right)^{\frac{1}{2} - \frac{3}{8n^2} + \frac{23}{48n^3} - \frac{15}{32n^4} + \cdots}.
\]

Here we only list the following results:

\[
\frac{\Omega^2_n}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1 + \frac{1}{n+1}\right)^{\frac{1}{2} + \frac{1}{8n+1} - \frac{3}{8n^2} + \frac{23}{48n^3} - \frac{15}{32n^4} + \cdots}
\]

and

\[
\frac{\Omega^2_n}{\Omega_{n-1}\Omega_{n+1}} \sim \left(1 + \frac{1}{n+1}\right)^{\frac{1}{2} + \frac{1}{8n+1} - \frac{3}{8(n+1)^2} + \frac{23}{48(n+1)^3} - \frac{15}{32(n+1)^4} + \cdots}.
\]

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2. All proofs and the methods used for improving the classical inequalities announced in the final part of the first section are presented in an extended form in a paper submitted by the author to a journal for publication.
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