The Generalized Randers Change of the More Generalized
\(m\)-th Root Metrics

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Abstract

A change of Finsler metric \(F(x,y) \rightarrow \tilde{F}(x,y)\) is called a generalized Randers change of \(F\), if \(\tilde{F}(x,y) = F(x,y) + b_i(x,y)y^i\), where \(b_i(x,y)\) is \(h\)-vector in \((M,F)\). The purpose of the present paper is devoted to studying the conditions for more generalized \(m\)-th root metrics \(\tilde{F}_1 = \sqrt[2m_1]{A_1} + B_1 + C_1\) and \(\tilde{F}_2 = \sqrt[2m_2]{A_2} + B_2 + C_2\), when \(m_1, m_2\) are even numbers. Then, we prove that under these conditions generalized Randers metric reduces to Randers metric. Finally, in the special case, we will give conditions for more generalized \(\tilde{F}_1, \tilde{F}_2\), when \(m_1, m_2\) are even numbers.

Keywords: \(m\)-th root metric; more generalized \(m\)-th root metric; generalized Randers change.

1. Introduction

Let \((M,F)\) be an \(n\)-dimensional Finsler manifold. Various Finsler changes have been studied by many distinguished mathematicians. For a differential one-form \(\beta(x,y) = b_i(x,y)y^i\) on \(M\), G. Randers [1], in 1941, introduced a special Finsler space defined by the change

\[
\tilde{F}(x,y) = F(x,y) + \beta(x,y),
\]

where \(F\) is Riemannian. Randers metrics are among the simplest non-Riemannian Finsler metrics. M. Matsumoto [2], in 1974, studied Randers space and generalized Randers space in which \(F\) is
Finslerian. In 1980, H. Izumi [3] introduced the concept of an $h$-vector $b_i$, while studying the conformal transformation of Finsler spaces. Then, instead of the function $b_i$ of coordinates $x^i$ only, we will use the $h$-vector $b_i(x,y)$ and define the generalized Randers change

$$ F(x, y) = F(x, y) + b_i(x, y)y^i. \tag{1.2} $$

We can find some results regarding the generalized Randers change in B. N. Prasad [4] and M. Gupta and P. Pandey [5]. In 1979, Shimada [6] introduced the $m$-th root metric on the differentiable manifold $M$ defined as:

$$ F = \sqrt[2]{\sum a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m}} \tag{1.3} $$

where the coefficients $a_{i_1i_2...i_m}$ are the components of symmetric covariant tensor field of order $(0,m)$ being the functions of positional co-ordinates only. Since then various geometers such as [7], [8] etc. have explored the theory of $m$-th root metric and studied its transformations.

There exist the following important two classes of Finsler metrics,

$$ F = \sqrt[2]{A^m + B}, $$

$$ \tilde{F} = \sqrt[2]{A^m + B + C}, \tag{1.4} $$

where $A = a_{i_1i_2...i_m}(x)y^{i_1}y^{i_2}...y^{i_m}$, $B = b_j(x)y^j y^j$ and $C = c_k(x)y^k$, that is one 1-form. These forms are called a generalized $m$-th root metric and more general generalized $m$-th root metric, respectively. Obviously, $\tilde{F}$ is not reversible Finsler metric and is Randers change of generalized $m$-th root metric $F$. In [9], the authors have studied the geometric properties of locally projectively flat $m$-th root in the form $F = \sqrt[m]{A}$ and generalized $m$-th root in the form $\tilde{F} = \sqrt[2]{A^m + B}$. In [10], Tayebi-
Najafi characterizes locally dually flat and Antonelli $m$-th root metrics. They prove that every $m$-th root metric of isotropic mean Berwald curvature (resp., isotopic Landsberg curvature) reduces to a weakly Berwald metric (resp., Landsberg metric). They show that $m$-th root metric with almost vanishing H-curvature has vanishing H-curvature \[11\]. In \[12\], the authors expresses a necessary and sufficient condition for the metric $F = \sqrt{A^m + B}$ to be locally dually flat. In \[13\], the authors have studied Berwald $m$-th root metrics. Y. Yu and Y. You show that an $m$-th root Einstein Finsler metric is Ricci-flat \[14\].

In this paper, we have considered a transformation of the more generalized $m$-th root metric such that it transforms to a similar metric as the generalized Randers one defined in (1.2) in a way that the Finslerian metric $F$ is replaced with more generalized $m$-th root metric $\tilde{F}$ defined in (1.4). Then, we obtain the conditions among two more generalized $m$-th root metrics $\tilde{F}_1 = \sqrt{A_1^m + B_1 + C_1}$ and $\tilde{F}_2 = \sqrt{A_2^m + B_2 + C_2}$ due to generalized Randers change, when $m_1, m_2$ are even numbers. Next, we prove that under these conditions generalized Randers metric reduces to Randers metric.

**Theorem 1.** Let $\tilde{F}_1 = \sqrt{A_1^m + B_1 + C_1}$ and $\tilde{F}_2 = \sqrt{A_2^m + B_2 + C_2}$ are two more generalized $m$-th root metrics on an open subset $U \subset \mathbb{R}^n$. Suppose that $m_1, m_2$ are even numbers and $m_1, m_2 > 2$. If $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$, then $\tilde{F}_1$ reduces to a Randers $\beta$-change of $\tilde{F}_2$.

In overall this paper,

\[
\begin{align*}
A_1 &= \alpha_{i_1i_2...i_{m_1}}(x) y^{i_1} y^{i_2} ... y^{i_{m_1}}, \\
A_2 &= \bar{\alpha}_{i_1i_2...i_{m_2}}(x) y^{i_1} y^{i_2} ... y^{i_{m_2}}, \\
B_1 &= b_{ij}(x) y^{i} y^{j}, B_2 = \bar{b}_{ij}(x) y^{i} y^{j}, \\
C_1 &= c_k(x) y^k, C_2 = \bar{c}_k(x) y^k,
\end{align*}
\]

and $m_1, m_2$ are belongs to natural numbers and $b_i(x, -y) = b_i(x, y)$. 

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2. Preliminaries

Let $M$ be an $n$-dimensional $C^\infty$ manifold. Denote by $T_x M$ the tangent space at $x \in M$, and by $TM = \bigcup_{x \in M} T_x M$, the tangent bundle of $M$. A Finsler metric on $M$ is a function $F: TM \to [0, \infty)$ which has the following properties: (i) $F$ is $C^\infty$ on the slit tangent bundle $TM_0 = TM - \{0\}$; (ii) $F$ is positively 1-homogeneous on the fibers of the tangent bundle $TM$; (iii) for each $y \in T_x M$, the following quadratic form $g_y$ on $T_x M$ is positive definite,

$$g_y(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]|_{s,t=0}, \ u, v \in T_x M.$$  \hspace{1cm} (2.1)

Let $x \in M$ and $F_x := F|_{T_x M}$. For $y \in T_x M_0$, define $C_y: T_x M \otimes T_x M \otimes T_x M \to R$ by

$$C_y(u, v, w) := \frac{1}{2} \frac{d}{dt} [g_{y + tw}(u, v)]|_{t=0}, \ u, v, w \in T_x M.$$  \hspace{1cm} (2.2)

The family $C := \{C_y\}_{y \in T_x M_0}$ is called the Cartan torsion. It is well known that $C = 0$ if and only if $F$ is Riemannian. The $h$-vector $b_i$ is $v$-covariant constant with respect to the Cartan connection and satisfies

$$FC^h_{i,j} b_h = \rho h_{ij}, \rho \neq 0,$$  \hspace{1cm} (2.3)

where, $C^h_{ij} := g^{-1} C_{hjk}$ (where, $g^{-1}$ and $C_{hjk}$ are components inverse of $g_{rh}$ and Cartan torsion, respectively.) is the Cartan's $C$-tensor, $h_{ij}$ is the angular metric tensor and $\rho$ is given by

$$\rho = \frac{FC^1_{i} b_i}{(n-1)},$$  \hspace{1cm} (2.4)

where, $C^i$ is the torsion vector $C^i_{jk} g^{jk}$. Then, we have
Where, \( \hat{\beta}_j = \frac{\partial}{\partial y_j} \) and \( \rho \) is independent of directional arguments. We will use from equation (2.5), for the proof of theorem 1.

### 3. Proof of theorem 1

In this section, we prove Theorem 1. To prove it, we need the following:

**Theorem 2.** Let \( \sqrt[2]{A_1 y + B_1} + C_1 \) and \( \sqrt[2]{A_2 y + B_2} + C_2 \) are two more generalized \( m \)-th root metrics on an open subset \( U \subset \mathbb{R}^n \). Suppose that \( m_1, m_2 \) are even numbers with \( m_1 = m_2 \) and \( m_1 (or m_2) > 2 \). If \( \sqrt[2]{A_1 y + B_1} \) is generalized Randers change of \( \sqrt[2]{A_2 y + B_2} \), then \( A_1 = \pm A_2, B_1 = B_2 \) and \( C_1 = C_2 + b_1(x, y) y^i \).

**Proof.** For simplicity, we put \( m_1 = m_2 = m \). Under the assumption, we have

\[
\sqrt[2]{A_1 y + B_1} + C_1 = \sqrt[2]{A_2 y + B_2} + C_2 + b_1(x, y) y^i.
\]  

(3.1)

By putting \((-y)\) instead of \((y)\) in (3.1), we have

\[
\sqrt[2]{A_1 y + B_1} - C_1 = \sqrt[2]{A_2 y + B_2} - C_2 - b_1(x, -y) y^i.
\]  

(3.2)

Summing sides the two equations (3.1) and (3.2), we have

\[
A_1 y + B_1 = A_2 y + B_2.
\]  

(3.3)

Consequently, because of \( m > 2 \), we get the proof.
Theorem 3. Let $\tilde{F}_1 = \sqrt{\frac{2}{m_1}} A_1^{m_1} + B_1 + C_1$ and $\tilde{F}_2 = \sqrt{\frac{2}{m_2}} A_2^{m_2} + B_2 + C_2$ are two more generalized $m$-th root metrics on an open subset $U \subset R^n$. Suppose that $m_1$, $m_2$ are even numbers with $m_1 = m_2$ and $m_1, m_2 > 2$. If $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$, then $\tilde{F}_1$ reduces to a Randers $\beta$-change of $\tilde{F}_2$.

Proof. Suppose that $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$. Then

$$C_1 = C_2 + b_i(x,y) y^i. \quad (3.4)$$

Differentiating (3.4) with respect to $y^k$, we have

$$c_k(x) = \tilde{c}_k(x) + b_k(x,y). \quad (3.5)$$

Then

$$\dot{\beta}_i b_k(x,y) = 0. \quad (3.6)$$

Therefore, $b_i$ are functions of coordinates $x^i$ alone and from (2.4), $b_i$ is not a $h$-vector.

Theorem 4. Let $\sqrt{A_1^{m_1} + B_1} + C_1$ and $\sqrt{A_2^{m_2} + B_2} + C_2$ are two more generalized $m$-th root metrics on an open subset $U \subset R^n$. Suppose that $m_1$, $m_2$ are even numbers with $m_1 \neq m_2$ and $m_1 > m_2 > 2$. If $\tilde{F}_1$ is generalized Randers change of $\tilde{F}_2$, then $A_1 = \pm \sqrt[2]{A_2^{m_1}}$, $B_1 = B_2$ and $C_1 = C_2 + b_i(x,y) y^i$.

Proof. Under the assumption, we have

$$\sqrt{A_1^{m_1} + B_1} + C_1 = \sqrt{A_2^{m_2} + B_2} + C_2 + b_i(x,y) y^i. \quad (3.7)$$

By putting $(-y)$ instead of $y$ in (3.7), we have

$$\sqrt{A_1^{m_1} + B_1} - C_1 = \sqrt{A_2^{m_2} + B_2} - C_2 - b_i(x,-y) y^i. \quad (3.8)$$
Summing sides the two equations (3.7) and (3.8), we have

\[ \sqrt[\frac{2}{m_1}]{A_1} + B_1 = \sqrt[\frac{2}{m_2}]{A_2} + B_2. \]  

Consequently, because of \( m_1 > m_2 > 2 \), we get the proof.

Similar to theorem 3, we have the following result:

**Corollary 5.** Let \( \tilde{F}_1 = \sqrt[\frac{2}{m_1}]{A_1} + B_1 + C_1 \) and \( \tilde{F}_2 = \sqrt[\frac{2}{m_2}]{A_2} + B_2 + C_2 \) are two more generalized \( m \)-th root metrics on an open subset \( U \subset \mathbb{R}^n \). Suppose that \( m_1, m_2 \) are even numbers with \( m_1 \neq m_2 \) and \( m_1 > m_2 > 2 \). If \( \tilde{F}_1 \) is generalized Randers change of \( \tilde{F}_2 \), then \( \tilde{F}_1 \) reduces to a Randers \( \beta \)-change of \( \tilde{F}_2 \).

**Proof of theorem 1.** From Theorems 3 and Corollary 5, we get the proof.

### 4. Randers and conformal \( \beta \)-change

In 1976, M. Hashiguchi [15] studied the conformal change of Finsler metrics, namely, \( \tilde{F} = e^{\sigma(x)}F \). In particular, he also dealt with the special conformal transformation named \( C \)-conformal. This change has been studied by many authors ([16], [17]). In 2008, S. Abed ([18],[19]) introduced the transformation

\[ \tilde{F}(x, y) = e^{\sigma(x)}F(x, y) + \beta \]  

Moreover, he established the relationships between some important tensors associated with \( (M, F) \) and the corresponding tensors associated with \( (M, \tilde{F}) \). He also studied some invariant and \( \sigma \)-invariant properties and obtained a relationship between the Cartan connection associated with \( (M, F) \) and the transformed Cartan connection associated with \( (M, \tilde{F}) \).
In this section, we have considered two transformations of the more generalized $m$-th root metric such that it transforms to a similar metric as the Randers and conformal $\beta$-change one defined in (1.1) and (4.1), respectively, in a way that the Riemannian metric $F$ is replaced with relation (1.4) and $m_1, m_2$ are even numbers.

**case 1**: $m_1, m_2$ are even numbers and $m_1 = m_2$.

**Theorem 6.** Let $\tilde{F}_1 = \sqrt[2]{A_1^{m_1} + B_1 + C_1}$ and $\tilde{F}_2 = \sqrt[2]{A_2^{m_2} + B_2 + C_2}$ are two more generalized $m$-th root metrics on an open subset $U \subset \mathbb{R}^n$. Suppose that $m_1, m_2$ are even numbers with $m_1 = m_2$ and $m_1 (or \ m_2) > 2$. Then

(i) If $\tilde{F}_1$ is Randers $\beta$-change of $\tilde{F}_2$, then $A_1 = \pm A_2$, $B_1 = B_2$ and $C_1 = C_2 + \beta$.

(ii) If $\tilde{F}_1$ is conformal $\beta$-change of $\tilde{F}_2$, then $A_1 = \pm e^{m_1\sigma(x)}A_2$ (or $A_1 = \pm e^{m_2\sigma(x)}A_2$), $B_1 = e^{2\sigma(x)}B_2$ and $C_1 = e^{\sigma(x)}C_2 + \beta$.

**Proof** (i). For simplicity, we put $m_1 = m_2 = m$. Under the assumption, we have

$$\sqrt[2]{A_1^m + B_1 + C_1} = \sqrt[2]{A_2^m + B_2 + C_2 + \beta}.$$  \hspace{1cm} (4.2)

By putting (-y) instead of (y) in (4.2), we have

$$\sqrt[2]{A_1^m + B_1 - C_1} = \sqrt[2]{A_2^m + B_2 - C_2 - \beta}.$$ \hspace{1cm} (4.3)

Summing sides the two equations (4.2) and (4.3), we have
Consequently, because of \( m > 2 \), we get the proof.

**Proof (ii).** For simplicity, we put \( m_1 = m_2 = m \). Under the assumption, we have

\[
\sqrt[2]{A_1^m} + B_1 = \sqrt[2]{A_2^m} + B_2.
\]

By putting \((-y)\) instead of \((y)\) in (4.5), we have

\[
\sqrt[2]{A_1^m} - B_1 = \sqrt[2]{A_2^m} - B_2.
\]

Summing sides the two equations (4.5) and (4.6), we have

\[
A_1^m + B_1 = e^{2\sigma(x)} A_2^m + e^{2\sigma(x)} B_2.
\]

Consequently, because of \( m > 2 \), we get the proof.

**Corollary 7.** Let \( \tilde{F}_1 = \sqrt[2]{A_1^m} + B_1 + C_1 \) and \( \tilde{F}_2 = \sqrt[2]{A_2^m} + B_2 + C_2 \) are two more generalized \( m \)-th root metrics on an open subset \( U \subset R^n \). Suppose that \( m_1, m_2 \) are even numbers with \( m_1 = m_2 \). Then

(i) If \( B_1 = B_2 \) and \( m_1 \sqrt{A_1^m} \) are Riemannian metrics, then \( \tilde{F}_1 = \tilde{F}_2 \) if and only if \( C_1 = C_2 \).

(ii) If \( B_1 = e^{2\sigma(x)} B_2 \) and \( m_2 \sqrt{A_2^m} \) are Riemannian metrics, then \( \tilde{F}_1 = \tilde{F}_2 \) if and only if \( C_1 = e^{\sigma(x)} C_2 \).

**Case 2:** \( m_1, m_2 \) are even numbers and \( m_1 \neq m_2 \).
Theorem 8. Let $\tilde{F}_1 = \sqrt{A_1^{m_1} + B_1 + C_1}$ and $\tilde{F}_2 = \sqrt{A_2^{m_2} + B_2 + C_2}$ are two more generalized m-th root metrics on an open subset $U \subset \mathbb{R}^n$. Suppose that $m_1, m_2$ are even numbers with $m_1 \neq m_2$ and $m_1 > m_2 > 2$. Then

(i) If $\tilde{F}_1$ is Randers $\beta$-change of $\tilde{F}_2$, then $A_1 = \pm \sqrt{A_2^{m_1}}$, $B_1 = B_2$ and $C_1 = C_2 + \beta$.

(ii) If $\tilde{F}_1$ is conformal $\beta$-change of $\tilde{F}_2$, then $A_1 = (e^{m_2 \sigma(x) A_2})^{m_1}$, $B_1 = e^{2 \sigma(x) B_2}$ and $C_1 = e^{\sigma(x) C_2} + \beta$.

Proof (i). Under the assumption, we have

$$\sqrt{A_1^{m_1} + B_1 + C_1} = \sqrt{A_2^{m_2} + B_2 + C_2 + \beta}. \quad (4.8)$$

By putting $(-y)$ instead of $(y)$ in (4.8), we have

$$\sqrt{A_1^{m_1} + B_1 + C_1} = \sqrt{A_2^{m_2} + B_2 + C_2 - \beta}. \quad (4.9)$$

Summing sides the two equations (4.8) and (4.9), we have

$$A_1^{m_1} + B_1 = A_2^{m_2} + B_2. \quad (4.10)$$

Consequently, because of $m_1 > m_2 > 2$, we get the proof.

Proof (ii). Under the assumption, we have

$$\sqrt{A_1^{m_1} + B_1 + C_1} = e^{\sigma(x)}(\sqrt{A_2^{m_2} + B_2 + C_2} + \beta). \quad (4.11)$$

By putting $(-y)$ instead of $(y)$ in (4.11), we have
\[
\sqrt{A_1^{2/m_1} + B_1 \cdot C_1} = e^{\phi(x)} \left( \sqrt{A_2^{2/m_2} + B_2 \cdot C_2} \right) - \beta. \tag{4.12}
\]

Summing sides the two equations (4.11) and (4.12), we have

\[
A_1^{2/m_1} + B_1 = e^{2\phi(x)} A_2^{2/m_2} + e^{2\phi(x)} B_2. \tag{4.13}
\]

Consequently, because of \(m_1 > m_2 > 2\), we get \(A_1 = \pm (e^{m_2 \phi(x)} A_2)^{m_1/m_2}, B_1 = e^{2\phi(x)} B_2\) and then \(C_1 = e^{\phi(x)} C_2 + \beta\).

In above theorem, for sections of (i),(ii) we have the followings:

For (i). if \(m_1 - m_2 = k\), where \(k\) is even number, then by (4.10), we get

\[(a_1)\] if \(\frac{k}{m_2} > 1\), then

Case 1. \(\frac{k}{m_2} = 2t\). Therefore, from theorem 8, \(A_1 = \pm A_2^{1+2t}\).

Case 2. \(\frac{k}{m_2} = 2t + 1\). Therefore, from theorem 8, \(A_1 = \pm A_2^{2(1+t)}\).

Case 3. \(m_2 \nmid k\). Because of \(k = m_2q + r\), from theorem 8, \(A_1 = \pm A_2^{1+q+\frac{r}{m_2}}\).

\[(a_2)\] if \(\frac{k}{m_2} < 1\), then

Case 1. \(\frac{m_2}{k} = 2t\). Therefore, from theorem 8, \(A_1 = \pm A_2^{\frac{1+2t}{2t}}\).

Case 2. \(\frac{m_2}{k} = 2t + 1\). Therefore, from theorem 8, \(A_1 = \pm A_2^{\frac{2+2t}{2t+1}}\).

Case 3. \(k \nmid m_2\). Because of \(m_2 = kq' + r'\), from theorem 8, \(A_1 = \pm A_2^{1+q'+\frac{k}{r'}}\).

\[(a_3)\] if \(\frac{k}{m_2} = 1\), then , from theorem 8, \(A_1 = \pm A_2\).

For (ii). if \(m_1 - m_2 = k\), where \(k\) is even number, then by (4.13), we get
(a₁) if \( \frac{k}{m_2} > 1 \), then

Case 1. \( \frac{k}{m_2} = 2t \). Therefore, from theorem 8, \( A_1 = \pm (e^{\frac{k}{2+2t}} A_2)^{1+2t} \).

Case 2. \( \frac{k}{m_2} = 2t + 1 \). Therefore, from theorem 8, \( A_1 = \pm (e^{\frac{k}{2+t}} A_2)^{2(1+2t)} \).

Case 3. \( m_2 \nmid k \). Because of \( k = m_2q + r \), from theorem 8, \( A_1 = \pm (e^{\frac{k-r}{q}} A_2)^{1+q+r/m_2} \).

(a₂) if \( \frac{k}{m_2} < 1 \), then

Case 1. \( \frac{m_2}{k} = 2t \). Therefore, from theorem 8, \( A_1 = \pm (e^{2kt} \sigma(x) A_2)^{1+2t} \).

Case 2. \( \frac{m_2}{k} = 2t + 1 \). Therefore, from theorem 8, \( A_1 = \pm (e^{(2t+1)t} \sigma(x) A_2)^{2+2t} \).

Case 3. \( k \nmid m_2 \). Because of \( m_2 = kq + r' \), from theorem 8, \( A_1 = \pm (e^{k(q+r')} \sigma(x) A_2)^{1+q+r'/k} \).

(a₃) if \( \frac{k}{m_2} = 1 \), then, from theorem 8, \( A_1 = \pm (e^{k(\sigma(x))} A_2)^2 \).

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References

[1] G. Randers, {On the asymmetrical metric in the four-space of general relativity}, Phys. Rev. (2) 59 (1941):195-199.
[2] M. Matsumoto, {On Finsler spaces with Randers metric and special forms of important tensors}, J. Math. Kyoto Univ., 14 (1974):477-498.
[3] H. Izumi, {Conformal transformations Finsler spaces II. An h-conformally flat Finsler space}, Tensor (N.S.), 34 (1980):337-359.
[4] B. N. Prasad, {On the torsion tensors $R_{bik}$ and $P_{bik}$ of Finsler spaces with a metric $ds = (g_{ij}(dx^i)dx^j)^{1/2} + b_i(x, y)dx^i$}, *Indian J. Pure Appl. Math.*, 21 (1990):27-39.

[5] M. K. Gupta and P. N. Pandey, {On hypersurface of a Finsler space with a special metric}, *Acta Math. Hungar.*, 120 (2008):165-177.

[6] H. Shimada, {On Finsler spaces with metric $L = \sqrt[n]{a_{m1}a_{m2}...a_{mn}y^{l1}y^{l2}...y^{ln}}$}, *Tensor(N.S)*, 33 (1979):365-372.

[7] B. N. Prasad and A. K. Dwivedi, {On conformal transformation of Finsler spaces with $m$-th root metric}, *Indian J. pure appl. Math.*, 33(6) (2002):789-796.

[8] A. Srivastava and P. Arora, {Kropina change of $m$-th root metric and its conformal transformation} Bull. of *Calcutta Mathematical Society*, 103(3) (2011).

[9] S. Zhang. D. Zu. B. Li, {projected flat $m$-th root Finsler metrics}, *J. Math. Ningbo Univ.*, 24 (3), (2010).

[10] A. Tayebi, B. Najafi, {On $m$-th root Finsler metrics.} *Geometry and Physics*, (2011), 61:1479-1484.

[11] A. Tayebi, B. Najafi, {On $m$-th root metrics with special curvature properties.} *C. R. Acad. Sci, Ser. I*, 349 (2011):691-693.

[12] A. Tayebi, E. Peyghan, M. S. Nia, {On generalized $m$-th root Finsler metrics,} *Linear Algebra Appl.*, 437 (2012):675-683.

[13] D. Zu, S. Zhang and B. Li, {On Berwald $m$-th root Finsler metrics.} *Publ. Math. Debrecen*, 4994 (2012):1-9.

[14] Y. Yu, Y. You, {On Einstein $m$-th root metrics.} *Differential Geometry and its Applications*, 28, (2010):290-294.

[15] M. Hashiguchi, {On conformal transformation of Finsler metrics,} *J. Math. Kyoto Univ.*, 16 (1976):25-50.

[16] H. Izumi, {Conformal transformations of Finsler spaces,} *Tensor, N. S.*, 31 (1977):33-41.

[17] Nabil L. Youssef, S. H. Abed and A. Soleiman, {A global theory of conformal Finsler geometry,} *Tensor, N. S.*, 69 (2008):155-178.

[18] S. H. Abed, {Conformal $\beta$-changes in Finsler spaces.} *Proc. Math. Phys. Soc. Egypt*, 86 (2008):79-89.

[19] S. H. Abed, {Cartan connection associated with a $\beta$-conformal change in Finsler geometry,} *Tensor, N. S.*, 70 (2008):146-158.