Three-Dimensional Approach for Construction of Low-Momentum Effective Interaction from Realistic Potentials

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Abstract

The low-momentum effective interaction $V_{\text{low } k}$ has been formulated in the three-dimensional momentum-helicity representation as a function of the magnitude of momentum vectors and the angle between them. As an application, AV18 potential has been used in the model space of Lee-Suzuki method and it has been shown that the low-momentum effective interaction, $V_{\text{low } k}$ reproduces the same two-body observables obtained by the bare potential $V_{NN}$.

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I. INTRODUCTION

Several methods have been developed to derive the energy independent low-momentum effective interaction such as the renormalization group (RG) and the model space techniques. These approaches are mainly based on a partial wave (PW) decomposition and the details have been given in references [1]-[11]. Recently Bogner et al. have developed a low-momentum effective interaction which describes the two-nucleon system at low energy successfully. This effective interaction is independent of the potential models as the cutoff is lowered to $\Lambda = 2.1 \text{ fm}^{-1}$ [10, 11].

During the past years, the three-dimensional (3D) approach has been developed for few-body bound and scattering problems [12]-[26]. In this approach momentum helicity basis states have been used for the representation of the nuclear forces. The motivation for developing this approach is to introduce a direct solution of the integral equations avoiding the very involved angular momentum algebra occurring for the permutations, transformations and especially for the three-body forces. Conceptually the 3D formalism considers all partial wave channels automatically.

Recently we have developed a 3D formalism for construction of low-momentum effective interaction neglecting the spin and isospin degrees of freedom [27]. Considering the spin and isospin is a major additional task, which we intend to present in this article. Our aim is to extend the low-momentum effective interaction directly in a spin-isospin dependent 3D approach and to formulate the low-momentum effective interaction in the momentum-helicity representation.

This article is structured in the following way: In section II the model space of Lee-Suzuki method has been used to derive the energy independent model space effective interaction in the 3D momentum-helicity representation. In section III the reduced forms of the equations have been displayed by choosing suitable coordinate systems. Section IV describes the numerical calculations of the low-momentum effective interaction $V_{\text{low } k}$ in the model space Lee-Suzuki method by using the AV18 potential. Finally a summary and an outlook have been provided in section V.
II. LEE-SUZUKI METHOD IN THE 3D MOMENTUM-HELICITY REPRESENTATION

The Lee-Suzuki method has been applied to the free space nucleon-nucleon problem in the 3D momentum-helicity representation and the low-momentum effective interaction $V_{\text{low } k}$ has been obtained as a function of the momentum vectors. In the model space methods the projection operators onto the physically important low-energy model space, the $P$ space, and the high-energy complement, the $Q$ space, have been introduced on momentum-helicity basis state $|k; \hat{k}S\lambda; t\rangle^{\pi a}$ as:

$$P = \sum_{S\lambda t} \int dk |k; \hat{k}S\lambda; t\rangle^{\pi a} \frac{1}{4} \pi a \langle k; \hat{k}S\lambda; t|, \quad |k| \leq \Lambda,$$

$$Q = \sum_{S\lambda t} \int dk |k; \hat{k}S\lambda; t\rangle^{\pi a} \frac{1}{4} \pi a \langle k; \hat{k}S\lambda; t|, \quad |k| > \Lambda,$$

where $\Lambda$ is a momentum cutoff which divides the Hilbert space into the low and high momentum states. The antisymmetrized momentum-helicity basis state which is parity eigenstate is given by [25]:

$$|k; \hat{k}S\lambda; t\rangle^{\pi a} = \frac{1}{\sqrt{2}} (1 - P_{12}) |k; \hat{k}S\lambda\rangle_{\pi} |t\rangle$$

$$= \frac{1}{\sqrt{2}} (1 - \eta_{\pi} (-)^{S+t}) |k; \hat{k}S\lambda\rangle_{\pi} |t\rangle,$$

(2)

Here $S$ is the total spin, $\lambda$ is the spin projection along relative momentum of two nucleons, $t$ is the total isospin and $|t\rangle \equiv |tm_t\rangle$ is the total isospin state of the two nucleons. $m_t$ is the isospin projection along its quantization axis which reveals the total electric charge of system. For simplicity $m_t$ is suppressed since electric charge is conserved. In Eq. (2) $P_{12}$ is the permutation operator which exchanges the two nucleons labels in all spaces i.e. momentum, spin and isospin spaces, and $|k; \hat{k}S\lambda\rangle_{\pi}$ is parity eigenstate which is given by:

$$|k; \hat{k}S\lambda\rangle_{\pi} = \frac{1}{\sqrt{2}} (1 + \eta_{\pi} P_{\pi}) |k; \hat{k}S\lambda\rangle,$$

(3)

where $P_{\pi}$ is parity operator, $\eta_{\pi} = \pm 1$ are the parity eigenvalues and $|k; \hat{k}S\lambda\rangle$ is momentum-helicity state. The normalization of the momentum-helicity basis state is given by [25]:

$$\pi' a \langle k'; \hat{k}' S'\lambda'; t'|k; \hat{k}S\lambda; t\rangle^{\pi a} = (1 - \eta_{\pi} (-)^{S+t}) \delta_{t't} \delta_{\eta_{\pi}'\eta_{\pi}} \delta_{\lambda'\lambda} \times \{ \delta(k' - k) \delta_{\lambda'\lambda} + \eta_{\pi} (-)^{S} \delta(k' + k) \delta_{\lambda'\lambda} \},$$

(4)
and the completeness relation of this state is defined by:

\[
\sum_{S_{\lambda t}} \int d|k; \hat{k}S_{\lambda t}; t\rangle^\pi_4 \frac{1}{4} \pi_4 \langle k; \hat{k}S_{\lambda t}; t | = 1. \quad (5)
\]

It is clear that the projection operators \( P \) and \( Q \) satisfy the following relations:

\[
P + Q = 1, \quad P Q = Q P = 0, \quad P^2 = P, \quad Q^2 = Q, \quad (6)
\]

and they act on the full-space two-body problem states as:

\[
P |\psi_k^\lambda \rangle = |\Psi^\lambda \rangle, \quad Q |\psi_k^\lambda \rangle = \omega |\Psi^\lambda \rangle,
\]

where \(|\psi_k^\lambda \rangle \) and \(|\Psi^\lambda \rangle \) denote the states of the full and model spaces respectively and \( \omega \) is an operator which transforms the states of the \( P \)-space to the \( Q \)-space. The non-hermitian low-momentum effective potential in the model space that reproduces the model space component of the wave function from the full-space wave function is given by \( V_{NN} \):

\[
V_{low k} = PV_{NN}(P + Q\omega P), \quad (8)
\]

where \( V_{NN} \) denotes the bare two-body interaction. By using the integral form of the projection operators \( P \) and \( Q \), the low-momentum effective interaction \( V_{low k} \) can be written in the 3D representation on momentum-helicity basis state as:

\[
\pi(a'|\hat{p}' S\lambda'; t|V_{low k}|\hat{p} S\lambda; t) = \pi(a'|\hat{p}' S\lambda'; t|V_{NN}|\hat{p} S\lambda; t)^{\pi a} + \frac{1}{4} \sum_{\lambda''} \int d q'' \int d \hat{q} \pi(a'|\hat{p}' S\lambda'; t|V_{NN}|q S\lambda''; t)^{\pi a} \times \pi(a|q S\lambda''; t|\omega|\hat{p} S\lambda; t)^{\pi a}, \quad (9)
\]

where we use the property that \( V_{NN} \) conserves parity, spin and isospin. By defining matrix elements of each arbitrary operator \( A \) as:

\[
A^{\piSt\lambda}(p', p) \equiv \pi(a'|\hat{p}' S\lambda'; t|A|\hat{p} S\lambda; t)^{\pi a}. \quad (10)
\]
Eq. (9) can be rewritten as:

$$V_{\text{low } k}^{\pi St, \lambda'}(p', p) = V_{\text{low } k}^{\pi St, \lambda'}(p', p) + \frac{1}{4} \lambda'' \int_\Lambda dq q^2 \int dq V_{\text{low } k}^{\pi St, \lambda''}(p', q) \omega_{\pi St, \lambda''}(q, p),$$

(11)

where $p$ and $q$ are the momentum vectors in the $P$ and $Q$ spaces respectively. As we know, the total spin states of two nucleons are singlet ($S = 0$) and triplet ($S = 1$) states. For the singlet case, Eq. (11), there is only one equation which reads as:

$$V_{\text{low } k}^{\pi 0t,00}(p', p) = V_{\text{low } k}^{\pi 0t,00}(p', p) + \frac{1}{4} \lambda'' \int_\Lambda dq q^2 \int dq V_{\text{low } k}^{\pi 0t,00}(p', q) \omega^{\pi 0t,00}(q, p),$$

(12)

however, for the triplet case there are three equations for each initial helicity $\lambda = -1, 0, 1$ as:

$$V_{\text{low } k}^{\pi 1t, \lambda'}(p', p) = V_{\text{low } k}^{\pi 1t, \lambda'}(p', p) + \frac{1}{4} \lambda'' \int_\Lambda dq q^2 \int dq \{V_{\text{low } k}^{\pi 1t, \lambda'}(p', q) \omega^{\pi 1t, \lambda'}(q, p)
+ V_{\text{low } k}^{\pi 1t, \lambda'}(p', q) \omega^{\pi 1t, \lambda'}(q, p)
+ V_{\text{low } k}^{\pi 1t, \lambda'}(p', q) \omega^{\pi 1t, \lambda'}(q, p)\}. \quad (13)$$

Symmetry relations for the potential matrix elements in the momentum-helicity representation are denoted by [25]:

$$V_{\text{NN}}^{\pi St, -\lambda'}(p', p) = \eta_{\text{NN}}(-)^S V_{\text{NN}}^{\pi St, \lambda'}(-p', p),$$
$$V_{\text{NN}}^{\pi St, \lambda'}(p', p) = \eta_{\text{NN}}(-)^S V_{\text{NN}}^{\pi St, \lambda'}(p', -p),$$
$$V_{\text{NN}}^{\pi St, -\lambda'}(p', p) = V_{\text{NN}}^{\pi St, \lambda'}(-p', -p). \quad (14)$$

A corresponding symmetry relations are also valid for $\omega$-matrix elements. By applying this symmetry relations for matrix elements of $V_{\text{NN}}$ and $\omega$, Eq. (13) can be reduced to:

$$V_{\text{low } k}^{\pi 1t, \lambda'}(p', p) = V_{\text{low } k}^{\pi 1t, \lambda'}(p', p) + \frac{1}{4} \int_\Lambda dq q^2 \int dq \{2 V_{\text{NN}}^{\pi 1t, \lambda'}(p', q) \omega^{\pi 1t, \lambda'}(q, p)
+ V_{\text{NN}}^{\pi 1t, \lambda'}(p', q) \omega^{\pi 1t, \lambda'}(q, p)\}. \quad (15)$$

Hence for the triplet case one needs only two equations for $\lambda' = 1, 0$ for each $\lambda$. To calculate the low-momentum effective potential $V_{\text{low } k}$ we need to determine $\omega$-matrix elements. The key aspect of the Lee-Suzuki method is the determination of the $\omega$ operator defined by the following equation [28]:

$$Q |\psi_{k'kS\lambda,t}\rangle^{\pi a} = Q \omega P |\psi_{k'kS\lambda,t}\rangle^{\pi a}. \quad (16)$$
Applying $\pi_a(q; \hat{q}S\lambda'; t)$ to the left hand side and using the integral form of projection operators $P$ and $Q$, this equation can be rewritten as:

$$
\pi_a(q; \hat{q}S\lambda'; t|\Psi_{k,\hat{k}S\lambda'; t})^{\pi_a} = \frac{1}{4} \sum_{\lambda'} \int_0^\Lambda dp \int d\hat{p} \pi_a(q; \hat{q}S\lambda'; t|\omega|p; \hat{p}S\lambda'; t)^{\pi_a} \times \pi_a(p; \hat{p}S\lambda'; t|\Psi_{k,\hat{k}S\lambda'; t})^{\pi_a}.
$$

(17)

We introduce the completeness relation for the scattering wave function in the model space on momentum-helicity basis state:

$$
\frac{1}{4} \sum_{\lambda'} \int_0^\Lambda dk d\hat{k} \Psi_{k}^{\pi St,\lambda\lambda'}(p') \tilde{\Psi}_{k}^{\pi St,\lambda'\lambda}(p) = (1 - \eta_r(-)^{S+t})\{\delta(p' - p)\delta_{\lambda\lambda'} + \eta_r(-)^S \delta(p' + p)\delta_{\lambda'\lambda'}\},
$$

(18)

where we use the following notations for the matrix elements of the wave functions:

$$
\Psi_{k}^{\pi St,\lambda\lambda'}(p) \equiv \pi_a(p; \hat{p}S\lambda'; t|\Psi_{k,\hat{k}S\lambda'; t})^{\pi_a},
\tilde{\Psi}_{k}^{\pi St,\lambda'\lambda}(p) \equiv \pi_a(\tilde{\Psi}_{k,\hat{k}S\lambda'; t}|p; \hat{p}S\lambda'; t)^{\pi_a}.
$$

(19)

After implementing Eq. (18), Eq. (17) can be written as:

$$
\omega^{\pi St,\lambda\lambda'}(q,p) = \frac{1}{4} \sum_{\lambda''} \int_0^\Lambda dk d\hat{k} \Psi_{k}^{\pi St,\lambda'\lambda''}(q) \tilde{\Psi}_{k}^{\pi St,\lambda''\lambda}(p),
$$

(20)

where $\Psi_{k}^{\pi St,\lambda\lambda'}(p)$ and $\Psi_{k}^{\pi St,\lambda\lambda'}(q)$ are the wave function components of the $P$ and $Q$ spaces of the full-space, respectively. These can be written in the form of the half-on-shell (HOS) two-body $T$-matrix as:

$$
\Psi_{k}^{\pi St,\lambda\lambda'}(p) = (1 - \eta_r(-)^{S+t})\{\delta(p - k)\delta_{\lambda\lambda'} + \eta_r(-)^S \delta(p + k)\delta_{\lambda'\lambda'} + \frac{1}{2} \frac{T^{\pi St,\lambda\lambda'}(p, k, k^2)}{k^2 - p^2 + i\epsilon}\},
$$

(21)

$$
\Psi_{k}^{\pi St,\lambda\lambda'}(q) = \frac{T^{\pi St,\lambda\lambda'}(q, k, k^2)}{k^2 - q^2}.
$$

(22)

The HOS two-body $T$-matrix can be obtained from the Lippmann-Schwinger (LS) equation in the 3D momentum-helicity representation, which is given by ($\hbar = m_N = 1$) [25]:

$$
T^{\pi St,\lambda\lambda'}(k', k, k^2) = V_{NN}^{\pi St,\lambda\lambda'}(k', k) + \frac{1}{4} \sum_{\lambda''} \int dk'' V_{NN}^{\pi St,\lambda''\lambda'}(k', k'') \frac{T^{\pi St,\lambda''\lambda'}(k'', k, k^2)}{k^2 - k''^2 + i\epsilon},
$$

(23)
The equation for the singlet case is:

\[
T^{\pi 0t,00}(k', k, k^2) = V_{NN}^{\pi 0t,00}(k', k) + \frac{1}{4} \int dk'' \frac{V_{NN}^{\pi 0t,00}(k', k'') T^{\pi 0t,00}(k'', k, k^2)}{k^2 - k''^2 + i\varepsilon},
\]

and for the triplet case we have two coupled equations for \(\lambda' = 1, 0\) for each \(\lambda\) as:

\[
T^{\pi 1t,\lambda\lambda}(k', k, k^2) = V_{NN}^{\pi 1t,\lambda\lambda}(k', k) + \frac{1}{2} \int dk'' \frac{V_{NN}^{\pi 1t,\lambda\lambda}(k', k'') T^{\pi 1t,\lambda\lambda}(k'', k, k^2)}{k^2 - k''^2 + i\varepsilon}
+ \frac{1}{4} \int dk'' \frac{V_{NN}^{\pi 1t,0,0}(k', k'') T^{\pi 1t,0,0}(k'', k, k^2)}{k^2 - k''^2 + i\varepsilon}.
\]

Eqs. (18) and (20) for the singlet case can be written as:

\[
\frac{1}{4} \int_0^\Lambda dk' k^2 \int d\hat{k} \Psi_k^{\pi 0t,00}(p') \tilde{\Psi}_k^{\pi 0t,00}(p) = (1 - \eta_\pi (-)^t) \{\delta(p' - p) + \eta_\pi \delta(p' + p)\},
\]

\[
\omega^{\pi 0t,00}(q, p) = \frac{1}{4} \int_0^\Lambda dk' k^2 \int d\hat{k} \Psi_k^{\pi 0t,00}(q) \tilde{\Psi}_k^{\pi 0t,00}(p).
\]

It is clear that the same symmetry relations as we mentioned for potential matrix elements are valid for matrix elements of \(\Psi\) and \(\tilde{\Psi}\). Thus by applying the symmetry relations for triplet case, we write two equations for \(\lambda' = 1, 0\) for each \(\lambda\) respectively:

\[
\frac{1}{4} \int_0^\Lambda dk' k^2 \int d\hat{k} \{2 \Psi_k^{\pi 1t,\lambda'\lambda}(p') \tilde{\Psi}_k^{\pi 1t,\lambda\lambda}(p) + \Psi_k^{\pi 1t,0,0}(p') \tilde{\Psi}_k^{\pi 1t,0,0}(p)\}
= (1 + \eta_\pi (-)^t) \{\delta(p' - p) \delta_{\lambda\lambda'} - \eta_\pi \delta(p' + p) \delta_{\lambda'-\lambda}\},
\]

\[
\omega^{\pi 1t,\lambda\lambda}(q, p) = \frac{1}{4} \int_0^\Lambda dk' k^2 \int d\hat{k} \{2 \Psi_k^{\pi 1t,\lambda'\lambda}(q) \tilde{\Psi}_k^{\pi 1t,\lambda\lambda}(p) + \Psi_k^{\pi 1t,0,0}(q) \tilde{\Psi}_k^{\pi 1t,0,0}(p)\}.
\]

III. CHOOSING SUITABLE COORDINATE SYSTEMS FOR NUMERICAL CALCULATIONS

In this section suitable coordinate systems are chosen in order to write the Eqs. (11), (18) and (20) in reduced forms for numerical calculations. The azimuthal behavior of potential and two-body \(T\)-matrix elements for special case where the vector \(k\) is along \(z\) axis are given by [25]:

\[
T^{\pi St,\lambda\lambda}(k', kz) = e^{i\lambda\varphi'} T^{\pi St,\lambda\lambda}(k', k, x'),
\]

\[
V_{NN}^{\pi St,\lambda\lambda}(k', kz) = e^{i\lambda\varphi'} V_{NN}^{\pi St,\lambda\lambda}(k', k, x'),
\]

(30)
where \( x' = \mathbf{k}' \cdot \mathbf{\hat{k}} \). Inserting these relations into the LS equation one can obtain \[25\]:

\[
T^{\pi S t, \lambda \lambda}(k', k, x', k^2) = V^{\pi S t, \lambda \lambda}_{NN}(k', k, x') + \frac{1}{4} \sum_{\lambda''} \int_{0}^{\infty} dk''k'' \int_{1}^{1} dx'' \frac{V^{\pi S t, \lambda \lambda''}_{\lambda''}(k', k'', x', x'')} {k^2 - k''^2 + i\varepsilon} T^{\pi S t, \lambda'' \lambda}(k'', k, x'', k^2),
\]

(31)

where \( x'' = \mathbf{k}'' \cdot \mathbf{\hat{k}} \) and \( \varphi'' \)-integration is carried out independently by defining:

\[
\nu^{\pi S t, \lambda \lambda''}_{\lambda''}(k', k'', x', x'') = \int_{0}^{2\pi} d\varphi'' e^{-i\lambda' (\varphi'' - \varphi')} \ V^{\pi S t, \lambda \lambda''}_{NN}(k', k'').
\]

(32)

The integrand is periodic with respect to \( \varphi'' \). Thus we can set \( \varphi' = 0 \) and Eq. \[31\] for the singlet and the triplet cases can be written as:

\[
T^{\pi 0t, 00}(k', k, x', k^2) = V^{\pi 0t, 00}_{NN}(k', k, x') + \frac{1}{4} \int_{0}^{\infty} dk''k'' \int_{1}^{1} dx'' \frac{V^{\pi 0t, 00}_{\lambda''}(k', k'', x', x'')} {k^2 - k''^2 + i\varepsilon} T^{\pi 0t, 00}(k'', k, x'', k^2),
\]

(33)

\[
T^{\pi 1t, \lambda \lambda}(k', k, x', k^2) = V^{\pi 1t, \lambda \lambda}_{NN}(k', k, x') + \frac{1}{2} \int_{0}^{\infty} dk''k'' \int_{1}^{1} dx'' \frac{V^{\pi 1t, \lambda \lambda}(k', k'', x', x'')} {k^2 - k''^2 + i\varepsilon} T^{\pi 1t, \lambda \lambda}(k'', k, x'', k^2)
\]

\[
+ \frac{1}{4} \int_{0}^{\infty} dk''k'' \int_{1}^{1} dx'' \frac{V^{\pi 1t, \lambda 0}_{\lambda''}(k', k'', x', x'')} {k^2 - k''^2 + i\varepsilon} T^{\pi 1t, \lambda 0}(k'', k, x'', k^2),
\]

(34)

As we know the T-matrix elements \( T^{\pi S t, \lambda \lambda}(k', k, k^2) \) are not the solution of the LS equation. The solution of the LS equation would be \( T^{\pi S t, \lambda \lambda}(k', k, x', k^2) \), which are T-matrix elements in momentum-helicity basis with initial momentum in the z-direction and without its azimuthal dependence. Therefore \( T^{\pi S t, \lambda \lambda}(k', k, k^2) \) can be connected to the solution of the LS equation as follows \[26\]:

\[
T^{\pi S t, \lambda \lambda}(k', k, k^2) = \sum_{N=-S}^{S} e^{iN(\varphi'-\varphi)} d_{NN\lambda}^{S}(x') d_{NN\lambda}(x) T^{\pi S t, \lambda \lambda}(k', k, y, k^2),
\]

(35)

where:

\[
y = k' \cdot \mathbf{\hat{k}} = x' x + \sqrt{1 - x'^2} \sqrt{1 - x^2} \cos(\varphi' - \varphi),
\]

(36)

and \( d_{NN\lambda}(x) \) are rotation matrices \[29\]. Now we consider the azimuthal behavior of \( \Psi \) and \( \omega \)-matrix elements:

\[
\omega^{\pi S t, \lambda \lambda}(q \mathbf{\hat{z}}, p) = e^{-i\lambda' \varphi} \omega^{\pi S t, \lambda \lambda}(q, p, x),
\]

\[
\Psi_{p}^{\pi S t, \lambda \lambda}(q \mathbf{\hat{z}}) = e^{-i\lambda' \varphi} \Psi_{p}^{\pi S t, \lambda \lambda}(q, x),
\]

(37)
where the vector $\mathbf{q}$ is along the $z$ axis and $x = \mathbf{p} \cdot \mathbf{q}$. The Eq. (20) can be reduced to:

$$\omega^{\pi St,\lambda\lambda}(q, p, x) = \frac{1}{4} \sum_{\lambda'} \int_0^\Lambda dk k^2 \int_{-1}^1 dx'' \Psi_k^{\pi St,\lambda'\lambda'}(q, x'') \tilde{\Psi}_k^{\pi St,\lambda'\lambda'}(p, x'', x),$$

(38)

where $x'' = \mathbf{k} \cdot \mathbf{q}$ and:

$$\Psi_k^{\pi St,\lambda'\lambda'}(q, x'') = \frac{T^{\pi St,\lambda'\lambda'}(q, k, x'', k^2)}{k^2 - q^2}.$$  

(39)

Integration over $\varphi''$ can be performed independently by defining:

$$\tilde{\Psi}_k^{\pi St,\lambda'\lambda'}(p, x''', x) \equiv \int_0^{2\pi} d\varphi'' e^{-i\lambda'(\varphi''-\varphi)} \Psi_k^{\pi St,\lambda'\lambda'}(p).$$

(40)

Eq. (38) for the singlet and the triplet cases can be written as:

$$\omega^{\pi 0t,00}(q, p, x) = \frac{1}{4} \sum_{\lambda'} \int_0^\Lambda dk k^2 \int_{-1}^1 dx'' \Psi_k^{\pi 0t,00}(q, x'') \tilde{\Psi}_k^{\pi 0t,00}(p, x'', x),$$

(41)

$$\omega^{\pi 1t,\lambda\lambda}(q, p, x) = \frac{1}{4} \sum_{\lambda'} \int_0^\Lambda dk k^2 \int_{-1}^1 dx'' \left\{ 2 \Psi_k^{\pi 1t,\lambda\lambda}(q, x'') \tilde{\Psi}_k^{\pi 1t,\lambda\lambda}(p, x'', x) + \Psi_k^{\pi 1t,\lambda\lambda}(q, x'') \tilde{\Psi}_k^{\pi 1t,\lambda\lambda}(p, x'', x) \right\}.$$ 

(42)

Multiplying both sides of Eq. (13) by $e^{-i\lambda''(\varphi'-\varphi)}$ and integrating over $\varphi'$ yields:

$$\frac{1}{4} \sum_{\lambda''} \int_0^\Lambda dk k^2 \int_{-1}^1 dx'' \tilde{\Psi}_k^{\pi St,\lambda''\lambda''}(p', x', x'') \tilde{\Psi}_k^{\pi St,\lambda''\lambda''}(p, x'', x) = \left\{ 1 - \eta_\pi (-)^{S+1} \right\} \delta(p' - p) \{ \delta(x' - x) \delta_{\lambda\lambda} + \eta_\pi (-)^{S+\lambda''} \delta(x' + x) \delta_{\lambda-\lambda} \},$$

(43)

where:

$$\tilde{\Psi}_k^{\pi St,\lambda''\lambda''}(p', x', x'') \equiv \int_0^{2\pi} d\varphi' e^{-i\lambda''(\varphi'-\varphi')} \Psi_k^{\pi St,\lambda''\lambda''}(p'),$$

(44)

and:

$$\Psi_k^{\pi St,\lambda''\lambda''}(p') = \left\{ 1 - \eta_\pi (-)^{S+1} \right\} \{ \delta(p' - k) [ \delta(x' - x'') \delta(\varphi' - \varphi'') \delta_{\lambda\lambda''} + \eta_\pi (-)^{S} \delta(x' + x'') \delta(\varphi' - \varphi'' - \pi) \delta_{\lambda-\lambda''} ] + \sum_{N=-S}^{S} e^{iN(\varphi'-\varphi'')} d_{N\lambda\lambda'}^{S}(x') d_{N\lambda''\lambda''}(x'') \frac{2 \delta_{S\lambda''}(y)}{k^2 - p'^2 + i\varepsilon} \right\}. $$

(45)

Eq. (43) for the singlet and the triplet cases can be written as:

$$\frac{1}{4} \int_0^\Lambda dk k^2 \int_{-1}^1 dx'' \tilde{\Psi}_k^{\pi 0t,00}(p', x', x'') \tilde{\Psi}_k^{\pi 0t,00}(p, x'', x) = \left\{ 1 - \eta_\pi (-)^{S+1} \right\} \delta(p' - p) \{ \delta(x' - x) + \eta_\pi \delta(x' + x) \},$$

(46)
Finally by considering the vector $\mathbf{p}$ along $z$ axis and using azimuthal behavior of potential and $\omega$-matrix elements, Eq. (11) can be rewritten as:

$$V_{\text{low} k}^{\pi St, \lambda}(p', p, x') = V_{NN}^{\pi St, \lambda}(p', p, x') + \frac{1}{4} \sum_{\lambda''} \int_{-1}^{1} dq \int_{-1}^{1} dx'' \nu_{\lambda}^{-\pi St, \lambda''}(p', q, x', x'') \omega_{\pi St, \lambda''}(q, p, x''),$$

where $x' = \mathbf{p}' \cdot \mathbf{p}$ and $x'' = \mathbf{q} \cdot \mathbf{p}$. This equation for the singlet and the triplet cases can be written as:

$$V_{\text{low} k}^{\pi 0t, 00}(p', p, x') = V_{NN}^{\pi 0t, 00}(p', p, x') + \frac{1}{4} \int_{-1}^{1} dq \int_{-1}^{1} dx'' \nu_{0}^{-\pi 0t, 00}(p', q, x', x'') \omega_{\pi 0t, 00}(q, p, x''),$$

$$V_{\text{low} k}^{\pi 1t, \lambda}(p', p, x') = V_{NN}^{\pi 1t, \lambda}(p', p, x') + \frac{1}{4} \int_{-1}^{1} dq \int_{-1}^{1} dx'' \{2 \nu_{\lambda}^{-\pi 1t, \lambda'}(p', q, x', x'') \omega_{\pi 1t, \lambda'}(q, p, x'') + \nu_{\lambda}^{-\pi 1t, 00}(p', q, x', x'') \omega_{\pi 1t, 00}(q, p, x'').$$

IV. DISCUSSION AND NUMERICAL RESULTS

We have chosen AV18 phenomenological potential for our calculations. This potential is fitted to $pp$ as well as $np$ data below 350 MeV laboratory energy. In addition the AV18 potential is fitted also to low-energy $nn$ scattering parameters and deuteron properties. With this interaction in the first step we have calculated two-body $T$-matrix by solving the LS Eqs. (33) and (34) for the singlet and the triplet cases respectively [25]. In the next step we have calculated $\tilde{\Psi}_{k, \lambda''}^{\pi St, \lambda'}(p, x'', x)$ as an inverse of $\Psi_{k, \lambda''}^{\pi St, \lambda'}(p', x', x'')$ for each $\lambda''$ from Eqs. (46) and (47) by using the LU decomposition method. In the numerical calculations we have used the Lapack library [30], for to solve a system of linear equations for the calculation of $\tilde{\Psi}_{k, \lambda''}^{\pi St, \lambda'}(p, x'', x)$ and the two-body $T$-matrix elements. Then by solving Eqs. (11) and (12) we have obtained $\omega_{\pi St, \lambda'}(q, p, x)$ and finally we have inserted the $\omega$-matrix elements into
Eqs. (49) and (50) to obtain the low-momentum effective interaction $V_{\text{low } k}^{\pi \text{St}, \lambda'}(p', p, x')$ for the singlet and the triplet cases respectively.

In numerical calculations we have used the Gaussian quadrature grid points to discrete the momentum and the angle variables. The integration interval for the $P$ and $Q$ spaces are covered by two different hyperbolic and linear mappings of the Gauss-Legendre points from the interval $[-1,+1]$ to the intervals $[0,a]$ and $[a,b]$: 

$$k = \frac{1 + x}{\Lambda - \left(\frac{1}{\Lambda} - \frac{2}{a}\right)x}, \quad k = \frac{b - a}{2}x + \frac{b + a}{2}. \quad (51)$$

The typical values for $a$ and $b$ are $10 \, fm^{-1}$ and $150 \, fm^{-1}$, respectively. As we mentioned in the introduction section we have used the value of $2.1 \, fm^{-1}$ for the cutoff $\Lambda$ in our calculations. The $\varphi''$-integration within an interval $[0, 2\pi]$ has been rewritten within an interval $[0, \frac{\pi}{2}]$ as shown in the following notation:

$$I = \int_{0}^{2\pi} d\varphi'' f(\cos(\varphi' - \varphi'')) e^{im(\varphi' - \varphi'')} = \int_{0}^{2\pi} d\varphi'' f(\cos \varphi'') e^{im\varphi''}$$

$$= \int_{0}^{\pi} d\varphi'' \{ f(\cos \varphi'') e^{im\varphi''} + f(-\cos \varphi'') e^{im(\varphi'' + \pi)} \}$$

$$= \int_{0}^{\pi} d\varphi'' \{ f(\cos \varphi'')(e^{im\varphi''} + e^{im(2\pi - \varphi'')}) + f(-\cos \varphi'')(e^{im(\pi + \varphi'')} + e^{im(\pi - \varphi'')} \}. \quad (52)$$

The second equality has been justified by the periodicity of the integrand within $2\pi$. Thus the number of integration points for polar angle has been reduced.

In our calculations we have chosen sixty grid points for the momentum variables in the interval $[0,a]$, and twenty two grid points for the momentum variables in the interval $[a,b]$. Also forty and ten grid points for the spherical and the polar angle variables have been used respectively. The solutions of the integral Eqs. (33) and (34) require a one-dimensional interpolation. We have used the cubic hermitian splines of Ref. [31] for its accuracy and high computational speed.

In Fig. 1 we have shown the total two-body cross section for $np$ scattering by using $V_{\text{low } k}$ and $V_{NN}$. Also in Fig. 2 we have compared the calculated $^1S_0$ and $^3S_1$-wave phase shifts from $V_{\text{low } k}$ and $V_{NN}$. For calculation of the phase shifts we have used the relation between the PW and 3D representations of the on-shell two-body $T$-matrix [26]:

$$T^{Sj,l}(p, p, p^2) = \frac{\pi}{2} \sqrt{\frac{2l' + 1}{2j + 1}} \sum_{\lambda \lambda'} C(l'Sj; 0\lambda') C(lSj; 0\lambda) \times \int_{-1}^{1} dx' d_{\lambda \lambda'}(x') T^{\pi \text{St}, \lambda \lambda}(p, p, x', p^2). \quad (53)$$
FIG. 1: The total cross section from the low-momentum effective potential $V_{\text{low } k}$ (circles) and the bare potential $V_{NN}$ (solid line) as a function of kinetic energy in the lab frame.

FIG. 2: The $^1S_0$ and $^3S_1$-wave phase shifts from the low-momentum effective potential $V_{\text{low } k}$ (circles) and the bare potential $V_{NN}$ (solid line) as a function of kinetic energy in the lab frame. The results are in good agreement with high accuracy. In Figs. 3-6, the calculated low-momentum effective potential $V_{\text{low } k}^{\pi S_{L},\lambda}(p, p', x')$ and the bare potential $V_{NN}^{\pi S_{L},\lambda}(p, p', x')$ as well as differences between them have been shown as a function of the momentum variable $p$ and the angle variable $x$.

As a test of our calculations we have compared the obtained results for the low-momentum
FIG. 3: (a) The comparison of the low-momentum effective potential $V_{\text{low } k}$ (solid lines) with the bare potential $V_{NN}$ (dashed lines) and (b) differences between them, for $S = 0$ and $t = 0$.

FIG. 4: (a) The comparison of the low-momentum effective potential $V_{\text{low } k}$ (solid lines) with the bare potential $V_{NN}$ (dashed lines) and (b) differences between them, for $S = 0$ and $t = 1$.

effective potential and the $\omega$ operator in the 3D and the PW approaches. As a first step we have calculated the $\omega$ operator and the low-momentum effective potential in the PW approach directly. We have then obtained the PW projection of the $\omega$ operator and the low-momentum effective potential from their corresponding 3D representation by the following relations:
FIG. 5: (a) The comparison of the low-momentum effective potential $V_{\text{low } k}$ (solid lines) with the bare potential $V_{NN}$ (dashed lines) and (b) differences between them, for $S = 1$ and $t = 1$. 
FIG. 6: (a) The comparison of the low-momentum effective potential $V_{\text{low } k}$ (solid lines) with the bare potential $V_{NN}$ (dashed lines) and (b) differences between them, for $S = 1$ and $t = 0$. 
FIG. 7: The comparison of $\omega(q, p)(p^2 - q^2)$ for the $^1S_0$-wave calculated in the PW approach (dotes) and in the 3D approach (solid lines).

\[
V_{\text{low } k}^{S_{jl}, l'_{l'}}(p, p) = \frac{\pi}{2} \sqrt{\frac{2l' + 1}{2j + 1}} \sqrt{\frac{2l + 1}{2j + 1}} \sum_{\lambda' \lambda} C(l'S; 0\lambda') C(lS; 0\lambda) \\
\times \int_{-1}^{1} dx' \frac{d_{\lambda' \lambda}^{j'}(x')}{d_{\lambda \lambda'}^{j}(x)} V_{\text{low } k}^{\pi S_{jl} S_{jl'}}(p, p, x'),
\] (54)

\[
\omega_{\text{low } k}^{S_{jl}, l'_{l'}}(q, p) = \frac{\pi}{2} \sqrt{\frac{2l' + 1}{2j + 1}} \sqrt{\frac{2l + 1}{2j + 1}} \sum_{\lambda' \lambda} C(l'S; 0\lambda') C(lS; 0\lambda) \\
\times \int_{-1}^{1} dx \frac{d_{\lambda' \lambda}^{j'}(x)}{d_{\lambda \lambda'}^{j}(x)} \omega_{\pi S_{jl} S_{jl'}}^{\pi}(q, p, x).
\] (55)

The obtained results for the channel $^1S_0$ in the 3D and PW approaches have been given in Fig. 7 and Fig. 8. The agreement between the two approaches is quite satisfactory.
FIG. 8: The comparison of the low-momentum effective potential $V_{\text{low } k}$ for the $^1S_0$-wave calculated in the PW approach (dotes) and in the 3D approach (solid lines).

V. SUMMARY AND OUTLOOK

In this article the 3D formulation of the model space Lee-Suzuki method based on momentum-helicity representation has been presented on momentum-helicity basis states. The low-momentum effective interaction $V_{\text{low } k}$ has been derived as a function of the magnitude of momenta and the angle between them for the singlet and the triplet cases respectively without using the partial wave decomposition. The calculated two-body observables from the low-momentum effective interaction and the bare interaction have been presented. In addition, a comparison between the calculated $V_{\text{low } k}$ from the PW and the 3D approach has been demonstrated as a test of our calculations.

The advantage of our formulation in the 3D representation in comparison with the PW representation is that we have calculated the low-momentum effective interaction by considering all partial waves automatically. The implementation of the obtained 3D low-momentum effective interaction in the few-body bound and scattering calculations is a major
task that can be considered for the future investigations.

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