ALL-ORDERS WORMHOLE VERTEX OPERATORS
FROM THE WHEELER-DEWITT EQUATION

ALEX LYONS*

Theoretical Physics Institute, Department of Physics
University of Alberta
Edmonton, Alberta, T6G 2J1, CANADA

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Abstract

We discuss the calculation of semi-classical wormhole vertex operators from wave functions which satisfy the Wheeler-deWitt equation and momentum constraints, together with certain ‘wormhole boundary conditions’. We consider a massless minimally coupled scalar field, initially in the spherically symmetric ‘mini-superspace’ approximation, and then in the ‘midi-superspace’ approximation, where non-spherically symmetric perturbations are linearized about a spherically symmetric mini-superspace background. Our approach suggests that there are higher derivative corrections to the vertex operator from the non-spherically symmetric perturbations. This is compared directly with the approach based on complete wormhole solutions to the equations of motion where it has been claimed that the semi-classical vertex operator is exactly given by the lowest order term, to all orders in the size of the wormhole throat. Our results are also compared with the conformally coupled case.

* Electronic mail: ALYONS@FERMI.PHYS.UALBERTA.CA
1 Introduction

Wormholes can be thought of as euclidean solutions to the field equations for gravity, possibly coupled to some matter system, which connect two asymptotically flat regions.* These solutions can be thought of as contributing to the semi-classical (zero-loop) approximation to the partition function for gravity, and thereby to the Green functions for fields in the asymptotic regions. There are, however, problems with just considering real euclidean solutions. Firstly, it is known that a complex contour is needed in order to make the path integral for gravity well defined, and it is not clear how that contour should be defined, particularly when matter fields are present. Thus it would seem that arbitrary complex saddle-points, in which both geometry and matter are complex, might be just as relevant to the semi-classical evaluation of the partition function as real euclidean solutions. Secondly, it is known that in order for real euclidean wormhole solutions to exist, one needs very particular types of matter fields, such as a conformally coupled, or imaginary minimally coupled, scalar field. However, for wormholes to provide a viable solution to the cosmological constant problem [2], or to be relevant to the late stages of black hole evaporation [3], one would like their existence to be independent of the particular matter fields in the system.

It is for these reasons that Hawking and Page [4] introduced the idea that wormholes should be considered instead as solutions to the Wheeler-deWitt equation. The wave functions of wormholes would satisfy particular boundary conditions corresponding to the fact that they describe states which, semi-classically, correspond to four-geometries connecting two asymptotically flat regions. Hawking and Page show that one can find a complete spectrum of such wave functions which generalizes to a rather large class of matter field models. However, they did not extract a low-energy effective action from their wave functions. It is generally expected that at low energies it is possible to replace the wormhole ends by vertex operators in the effective field theory, which would then give rise to an effective bilocal action. The aim of this paper is to show that it is possible to extract the vertex operators directly from the wave functions. The approach based on wormhole wave functions turns out to be more general than using euclidean solutions to the field equations, in that it potentially allows a wider class of operators in the effective action. In general we would expect the effective action to include all possible gauge-invariant and Lorentz-covariant operators. The euclidean solution approach fails to find all these operators, due to the restricted nature of possible solutions. For example, in the imaginary scalar field model,

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* By ‘asymptotically flat’ we mean ‘asymptotically euclidean in the sense of Gibbons and Pope [1]’.
the euclidean wormhole solutions have to be $O(4)$-symmetric. Grinstein, Maharana and Sudarsky have calculated the semi-classical vertex operator corresponding to these wormholes, to all orders in the wormhole throat radius [5], and they show that there are no derivative terms in the effective action. However, other work based on the wormhole wave functions suggests that one obtains these derivative terms [3, 6–8].

We would like to point out that considering just solutions to the Wheeler-deWitt equation is, itself, probably an approximation. It is now realized (using a simple two-dimensional model) [9] that the wormhole wave functions need not actually be ‘on-shell’, i.e. satisfying the Wheeler-deWitt equation. In the two-dimensional model considered in [9], the ‘on-shell’ states only provide the dominant contribution in the situation where the wormholes are long and thin. The ‘on-shell’ states appear as poles in the vertex operator, when the integration over the size of the wormhole throat is performed. In four-dimensional gravity we do not know the measure in the integration over the size of the wormhole. However, we shall assume that it is such that ‘on-shell’ states provide the dominant contribution to the low-energy effective action, as in the two-dimensional model.

The plan of this paper is as follows: In Sec. 2, we consider the spatially homogeneous, $O(4)$-symmetric, massless scalar field model. We show how the $\varepsilon^{ik\phi}$: effective interactions, usually associated with the euclidean axionic (or imaginary scalar field) wormhole solutions, can be thought of as arising from a basis of solutions to the Wheeler-deWitt equation. Thus the wave function approach (in the $O(4)$-symmetric mini-superspace truncation) reproduces the results usually obtained by considering euclidean wormhole solutions. In Sec. 3, we discuss the non-spherically symmetric perturbations of the axion wormhole in the wave function approach. We treat the problem in the ‘midi-superspace approximation’ as used in [10]. Excitations of the lowest spatially inhomogeneous mode ($n = 2$) would be expected to produce a tower of effective interactions of the form : $(\partial \phi)^m e^{ik\phi}$., for even $m$. This is based on the fall-off behaviour of the modes in the asymptotic region. However, a more careful analysis of the linearized lapse and shift constraints shows that only the ground state wave function is allowed. This leads to just the vertex operator with $m = 0$. In contrast, for the higher spatially inhomogeneous scalar modes one obtains higher derivative corrections to the vertex operator. Our findings for the lowest mode are in agreement with the ‘non-renormalization theorem’ of Grinstein et al [5]. However, it is claimed in [5] that the vertex operator $e^{ik\phi}$: is correct to all orders in the size of the wormhole. In our formalism, one would expect higher derivative corrections to this operator. We discuss the differences between our conclusions and those of [5] and also compare our results with those of Dowker [11] in the conformally coupled case. We summarize our main results in
Finally, in the Appendix, we discuss in more detail the construction of the solutions to the euclidean Schrödinger equation which are used in Sec. 3.

2 Wormholes, scalar fields, and the Wheeler-deWitt equation

The aim of this section is to relate the wave function description of wormholes to the more usual description in terms of real euclidean wormhole solutions. We start by considering the simplest mini-superspace model which exhibits wormhole solutions. This is the model consisting of gravity minimally coupled to a massless scalar field. The scalar field can be thought of as the Goldstone boson corresponding to a spontaneously broken global $U(1)$ symmetry [12]. Alternatively there is a dual description of this model (related by a simultaneous canonical and Hodge duality transformation) in terms of an anti-symmetric three-index field strength. It was in this form that wormhole solutions were first found [13]. However, we shall here (for simplicity) work with the scalar field formulation. In this model the spatial sections are taken to be three-spheres and the metric ansatz takes the form:

$$ds^2 = \sigma^2 \left( N^2(t)dt^2 + a^2(t)d\Omega_3^2 \right), \quad (2.1)$$

with the matter field taken to be homogeneous on the spatial sections. The constant $\sigma^2 = 2/3\pi m_p^2$ is included for later convenience, and $d\Omega_3^2$ is the usual round metric on the unit three-sphere.

With this ansatz the euclidean action for this mini-superspace model is

$$I = -\frac{1}{2} \int dt \, N a^3 \left( \frac{1}{N^2 a^2} \left( \frac{da}{dt} \right)^2 + \frac{1}{a^2} - \frac{1}{N^2} \left( \frac{d\phi}{dt} \right)^2 \right) + \frac{1}{2} \left( a(t) \right)^2. \quad (2.2)$$

This is the action appropriate for the boundary value problem where $a$ and $\phi$ are fixed on the two boundaries, labelled by $t_\pm$. The boundary term in the action is present in the case where the boundary labelled by $t_-$ is taken to become asymptotically flat space. This is the case which we shall be interested in throughout this paper. The limit $t_- \to -\infty$ is the asymptotically flat region of the space-time. Varying the action with respect to $N$ and $\phi$ leads to the constraint equation (the hamiltonian constraint for gravity) and the equation of motion for $\phi$ which are:

$$\frac{1}{N^2} \left( \frac{da}{dt} \right)^2 = 1 + \frac{a^2}{N^2} \left( \frac{d\phi}{dt} \right)^2$$

$$\frac{d}{dt} \left( \frac{a^3}{N} \frac{d\phi}{dt} \right) = 0. \quad (2.3)$$
These form a complete set of equations for the system, since if these hold at all times then the $a$ equation of motion is automatically satisfied.

The simplest general form of the solution to these equations can be obtained in the gauge where $N = 1/a$. In this case (2.3) can be integrated trivially to yield the solution
\[
\begin{align*}
    ds^2 &= \sigma^2 \left( a^{-2}(t)dt^2 + a^2(t)d\Omega_3^2 \right) \\
    a^2 &= \sqrt{4t^2 - Q^2} \\
    \phi - \phi_\pm &= \frac{1}{4} \ln \left( \frac{2t - Q}{2t + Q} \right),
\end{align*}
\]
where the constant of the motion (the euclidean momentum conjugate to $\phi$) is
\[
    Q = a^4 \frac{d\phi}{dt} = a^2 \sinh 2(\phi - \phi_\pm).
\]

Here $\phi_\pm$ is the asymptotic value of $\phi$ at infinity, and a constant of integration can also be added to $t$.

Flat space is recovered as the special case $Q = 0$, while the wormhole solution is found by taking $Q$ to be purely imaginary. The metric then has two asymptotically flat regions, as $t \to \pm \infty$, with a throat at $t = 0$. This solution will also have a purely imaginary $\dot{\phi}$. One might argue that this solution is a bit of a cheat, since the matter field has negative definite energy density and seems rather unphysical. However, there are two good reasons for arguing that such a configuration may have relevance. Firstly, from the point of view of providing a semi-classical contribution to the partition function, it is known that the contour of integration must be distorted into the complex plane for convergence of the path integral. In that case one might argue that any saddle-point may be important, even one in which the metric and matter field become complex [14]. The second argument comes from viewing the semi-wormhole solution ($-\infty < t \leq 0$) as providing a semi-classical contribution to the amplitude for a tunnelling process in quantum gravity. The idea is that if one wants the amplitude to tunnel between states of definite real lorentzian momentum for the scalar field, then the boundary data ensures that $Q$ must be purely imaginary, as in this solution [15].

In the alternative treatment of wormholes which uses the Wheeler-deWitt equation, we can bypass the argument over whether or not one should consider complex solutions. One might expect that a WKB approximation to a solution of the Wheeler-deWitt equation could be written in the form $e^{-I_{sp}(a,\phi)}$, where $I_{sp}(a,\phi)$ is the euclidean action of the classical solution (2.4) between asymptotically flat space as $t \to -\infty$, and an inner
boundary given by \( t_+ \), on which \( a \) and \( \phi \) are fixed and real. The action of this classical solution, given by substituting (2.4) into (2.2), is

\[
I_{sp}(a, \phi) = -t_+ = \frac{1}{2} a^2 \cosh 2(\phi - \phi_+) , \tag{2.6}
\]

which reproduces the flat space action in the case \( \phi = \phi_- \). Notice that the solution which we use has real \( \phi \). We do not need to go to complex \( \phi \), because we do not need to find a classical solution which has two asymptotically flat regions. Instead, we are interested in the wave function for real values of its arguments. The Wheeler-deWitt equation for this model is

\[
\frac{1}{a} \frac{\partial}{\partial a} \left( a \frac{\partial \psi}{\partial a} \right) - \frac{1}{a^2} \frac{\partial^2 \psi}{\partial \phi^2} - a^2 \psi = 0 , \tag{2.7}
\]

where the factor ordering has been chosen to be the one which is covariant under general co-ordinate transformations in the \( (a, \phi) \) mini-superspace. The family of wave functions

\[
\psi_{\phi_-}(a, \phi) = e^{-\frac{1}{2} a^2 \cosh 2(\phi - \phi_-)} \tag{2.8}
\]

in fact solve this equation exactly, although the derivation from the euclidean action of a classical solution suggests that one would have only expected to satisfy this equation to leading order in the WKB approximation.

The following questions now arise: Firstly, in what way are the wave functions \( \psi_{\phi_-} \) related to the euclidean wormhole solution to the field equations with an imaginary scalar field? If there is a straightforward connection between them, one can then ask whether the low-energy vertex operators for these wormholes can be obtained directly from the wave functions. In the past vertex operators have been obtained by considering Green functions on the euclidean background solution [16]. One motivation for using the wave functions instead comes from the possibility of generalizing this model to include spatially inhomogeneous perturbations. One expects that these extra degrees of freedom would enlarge the space of wormhole quantum states. The vertex operators corresponding to these extra states would perhaps be missed in a treatment which just considers the field theory on a background \( O(4) \) symmetric solution. These will be the questions we shall consider in this paper.

To address the first question, we first consider the Fourier transform (with respect to \( \phi \)) of our wave function. In other words we consider

\[
\hat{\psi}_{\phi_-}(a, k) = \int_{-\infty}^{\infty} d\phi e^{ik\phi} \psi_{\phi_-}(a, \phi) . \tag{2.9}
\]
This can be thought of as changing from the \((a, \phi)\) to the \((a, k)\) representation of the state \(|\phi_-\rangle\). \(\hat{\psi}\) is given explicitly by a modified Bessel function of imaginary order, \(K_{\frac{1}{2}ik}(a^2/2)\), multiplied by \(e^{ik\phi_-}\). It has the approximately exponential form \([17]\)

\[
\hat{\psi} \sim (a^4 - k^2)^{-1/4} \exp \left[ -\frac{1}{2} \left( \sqrt{a^4 - k^2} + k \arcsin(k/a^2) \right) + ik\phi_- \right], \tag{2.10}
\]

in the region \(a^4 \gg k^2\) and \(k^2 \gg 1\).

This form is suggestive of a euclidean saddle-point approximation to a path integral representation for \(\hat{\psi}\). Indeed, consider the complex canonical transformation from \((a(t), \phi(t))\) to \((a(t), k(t))\), where \(k = -ia^3\dot{\phi}/N\). For \(a^4 > k^2\), the classical variational problem with \((a, k)\) fixed on the inner boundary and \(\phi \to \phi_-\) at infinity has a euclidean stationary point whose action is precisely the negative of the value in the exponent above. The action (2.2) has to be supplemented by a matter field dependent boundary term on the inner boundary given by \(-ik(t_+)\phi(t_+)\), since the classical variational problem has changed from one in which \(\phi\) is fixed on the boundary to one in which \(k\) is fixed. This boundary term is fixed by the mathematical requirement that the variational principle subject to the new boundary conditions should yield the equations of motion. It is this boundary term which yields the \(\frac{1}{2}k\arcsin(k/a^2) - ik\phi_-\) term in the evaluated euclidean action. The \(\frac{1}{2}\sqrt{a^4 - k^2}\) term comes from (2.2) evaluated at the stationary point.

The saddle-point four-geometry in the \((a, k)\) boundary value problem is precisely a part of the imaginary scalar field wormhole outside a three-sphere of given radius \(\sigma a\), carrying ‘charge’ \(k\). (This would be the lorentzian \(U(1)\) charge carried by the wormhole in the Goldstone boson interpretation of the theory.) The boundary value problem has two solutions however, unlike in the case where the field \(\phi\) is specified on the boundary. These correspond to taking more than or less than half of the wormhole. The evaluated action for each solution differs only in the sign taken for the square root in \(\frac{1}{2}\sqrt{a^4 - k^2}\). The solution which is related to the Fourier transform of the wave function \(e^{-\frac{1}{2}a^2 \cosh 2(\phi - \phi_-)}\) corresponds to taking less than half the wormhole, and the positive square root. This means that the wave function goes like \(e^{-a^2/2}\) as \(a \to \infty\). The \((a, \phi)\) representation has the advantage that the path integral appears to pick out the semi-classical wave function uniquely whereas the \((a, k)\) representation does not. This is similar to the situation encountered by Hartle and Hawking in defining the semi-classical approximation to the wave function of the universe in pure Einstein gravity with a cosmological constant \([18]\). In that
case, there are two euclidean four-geometries satisfying the field equations with a threesphere boundary of given radius $\sigma a$ and no other boundary, but if instead the conjugate momentum to $a$ is specified on the boundary, then there is only one.

The analysis above shows a connection between the Fourier transformed wave function $\hat{\psi}$ and the imaginary scalar field wormholes. We see from it that it is natural to take the boundary condition that the wave functions fall off like $e^{-a^2/2}$ at large $a$. These wave functions correspond to saddle-point evaluations of path integrals where the fourgeometries are taken to be asymptotically flat. We obtained a representation of the states $|\phi_-\rangle$ in terms of either $(a, \phi)$ or $(a, k)$. The $(a, k)$ representation of $|\phi_-\rangle$ can be related to imaginary scalar field wormhole solutions of the euclidean field equations. The $(a, \phi)$ representation suggested the correct boundary conditions for wormhole wave functions at large $a$. The next step is to ask whether it is possible to use the wave functions we have found to evaluate the vertex operators, and thus the effective action, corresponding to the euclidean wormhole solutions. We proceed by analogy with the conformally coupled scalar field [3] and the case of massless fermions, photons or gravitons [6–8]. Namely, we assume that there is a Hilbert space spanned by wormhole quantum states, $|\phi_-\rangle$. The states $|k\rangle = \int d\phi_- e^{-ik\phi_-} |\phi_-\rangle$ also span this space. Each state has a vertex operator associated to it, which reproduces the effect of the state on low-energy $n$-point functions in the asymptotic region. One can try to calculate the vertex operators corresponding to the basis $|\phi_-\rangle$, where one fixes $\phi \to \phi_-$ at infinity. But the vertex operators turn out to be especially simple if the basis $|k\rangle$ is used. In the $(a, \phi)$ representation the wave functions for these states are 'plane wave' products, of the form

$$\psi_k(a, \phi) = K_{-\frac{1}{2}ik}(a^2/2) e^{-ik\phi}.$$ (2.11)

These wave functions satisfy the Wheeler-deWitt equation and are exponentially damped at large $a$. This boundary condition is suggested by the previous discussion of the saddlepoint approximation to a path integral representation of the relevant wave functions. We are, however, being more general than [4] in our choice of boundary conditions in that we do not require any particular regularity condition at small $a$. Our view is that the Wheeler-deWitt equation is only an effective theory of quantum gravity, which ceases to be strictly valid at small $a$. (The semi-classical approximation breaks down there.) Thus any boundary conditions at small $a$ should come from a more fundamental theory of quantum gravity, and not be imposed without justification.

The effect of these wormhole states on the field theory $n$-point functions is given by the matrix element

$$\langle k|\phi(x_1)\cdots\phi(x_n)|0\rangle,$$ (2.12)
as in [3]. This is essentially the dilute wormhole approximation, in that we are considering each wormhole independently and assuming that Green functions with points in the two asymptotically flat regions factorize. (We suspect that wave functions would have to be replaced by density matrices in a description which goes beyond the dilute wormhole approximation [19].) The vacuum state $|0\rangle$ is given by $|\phi_- = 0\rangle$. The matrix element has a path integral representation as

$$
\int da_0 \mu(a_0) \int d\phi_0 \bar{\psi}_k(a_0, \phi_0) \int [dg][d\phi] e^{-I[g,\phi]} \phi(x_1) \cdots \phi(x_n) , \quad (2.13)
$$

where the inner path integral is taken over asymptotically flat four-geometries with an inner boundary on which $a = a_0$ and $\phi = \phi_0$, and with $\phi \to 0$ at infinity. We integrate in the range $0 < a_0 < \infty$, $-\infty < \phi_0 < \infty$ with some unknown measure $\mu(a_0)$. The saddle-point approximation, applied to the inner integral, produces the result

$$
\int d^4 x_0 \int da_0 \Delta(a_0) \int d\phi_0 \bar{\psi}_k(a_0, \phi_0) e^{-\frac{i}{2} a_0^2 \cosh 2\phi_0} \left(\frac{1}{2} a_0^2 \sinh 2\phi_0\right)^n \prod_{j=1}^n (x_j - x_0)^{-2} , \quad (2.14)
$$

for the matrix element, where the new co-ordinates $x$ are related to the old $t$ co-ordinate by

$$
(x - x_0)^2 = -2t , \quad (2.15)
$$

and we use (2.4)–(2.6). The co-ordinates $x^\mu$ become the usual Cartesian co-ordinates in the asymptotically flat region, and $x_0$ can be interpreted as the position of the wormhole end in asymptotically flat space. To obtain (2.14), we have also expanded the logarithm which occurs in the saddle-point solution (2.4) for $\phi$, in the asymptotic region $t_j \to -\infty$. The prefactor $\Delta(a_0)$ comes from the determinant of the fluctuations about the saddle-point, and from $\mu(a_0)$. Its exact form need not concern us, and is irrelevant in the leading semi-classical approximation. The scalar field action being purely quadratic, we know that $\Delta$ is just a function of $a_0$. The variables $x_0$ are the zero-mode co-ordinates. They must be integrated over at the end of the calculation. In the simple model in which $\phi$ is homogeneous on the three-sphere spatial sections there is no additional integration over angular zero modes although in a more general model these integrals would also be present.

Now the integral over $\phi_0$ may also be approximated using the saddle-point method. (This is an ordinary integral over $\phi_0$ in the spatially homogeneous model.) The saddle-point values for $\phi_0$ are the roots of

$$
i k = a^2 \sinh 2\phi_{0_{sp}} , \quad (2.16)$$
so that in this approximation each of the \( n \) factors of \( \frac{1}{2}a_0^2 \sinh 2\phi_0 \) are replaced by their saddle-point values, \( \frac{1}{2}ik \). (There is also another prefactor, which depends on \( k \) and \( a_0 \), but not on \( n \).) The \( a_0 \) integral gives a factor which will depend on \( k \) but not on \( n \). We shall therefore ignore it, as it can be absorbed in the normalization of the vertex operator. The matrix element in the semi-classical limit (and for \( |x_j - x_0| \to \infty \)) therefore assumes the form

\[
\int d^4x_0 \alpha(k) \left( \frac{1}{2}ik \right)^n \prod_{j=1}^{n} (x_j - x_0)^{-2} .
\]

This can be identified with a flat space correlation function \( \langle V_n^k \phi(x_1) \cdots \phi(x_n) \rangle \), provided the operator \( V_n^k \) is taken to be

\[
V_n^k = \int d^4x_0 \frac{\alpha(k)}{n!} : (ik\phi(x_0))^n : .
\]

Thus we obtain that the semi-classical effect of the wave function \( \psi_k(a, \phi) \) on \( n \)-point functions in the asymptotic region is the same as the insertion of the operator \( V_n^k \) in the flat space \( n \)-point functions, for some function \( \alpha(k) \). Crucial to this identification is the fact that the flat space \( \phi \) propagator, \( \langle \phi(x)\phi(0) \rangle = \frac{1}{2}x^{-2} \), falls off with the same power of proper distance in the asymptotic region as the saddle-point solution for \( \phi(x) \), given by (2.4) with \( \phi_\pm = 0 \). The factor of \( n! \) in (2.18) is present because each of the \( n \) factors in the vertex operator can be contracted with any of the \( n \) fields in the asymptotic region, and there are \( n! \) ways of doing this. The vertex operator we have found depends on \( n \), but we require a vertex operator which reproduces the matrix element for \( any \) \( n \). It must take the form of a sum

\[
V^k = \sum_{0}^{\infty} V_n^k = \int d^4x_0 \alpha(k) : e^{ik\phi(x_0)} : .
\]

The effect of this operator on \( n \)-point functions is the same as (2.18), to lowest order in \( \hbar \), which can be reinstated by multiplying each contraction of \( \phi \)'s by \( \hbar \), and dividing \( ik\phi \) in the vertex operator by \( \hbar \). The correct result to lowest order in \( \hbar \) is all we can expect to obtain, since we have used the saddle-point approximation throughout. At this point we would like to remark that this analysis has just taken into account a single wormhole end. The inclusion of an arbitrary number of wormholes will lead to an effective integral over the coupling constants \( \alpha(k) \), for each value of \( k \). This is similar to a functional integral over the field \( \alpha(k) \) on superspace. \( \alpha \) becomes a kind of quantized field on superspace
Whether there is some mechanism such as ‘the Big Fix’ [2] to fix the values of $\alpha(k)$ uniquely, remains unclear.

The vertex operator (2.19) is exactly what was found in [16], which uses the imaginary scalar field wormhole solution. However, it is important that nowhere here is it necessary to assume the existence of a complete wormhole solution interpolating between two asymptotically flat regions. We believe that, being not so reliant on exact wormhole solutions to the classical field equations, this derivation of the effective interaction will generalize more readily to more realistic models, such as those in which the restriction of homogeneity is removed. We shall now go on to discuss these more general models.

3 Spatially inhomogeneous perturbations

It has been suggested by Grinstein et al [5] that the vertex operator for massless scalar field wormholes given by (2.19) is correct, to all orders in $l_{pl}$, the Planck length, and to lowest order in $\hbar$. The calculations of [5] are based on the Green functions of test fields in an $O(4)$-symmetric euclidean wormhole background. We shall see in this section that to obtain derivative effective interactions (which will be higher order in $l_{pl}$), it is necessary to consider inhomogeneities on the three-sphere sections. There will not be a complete wormhole solution connecting two asymptotically flat regions, but in the wave function approach this does not matter. What is important is that there is a family of solutions to the Wheeler-deWitt equation with appropriate boundary conditions. We shall start by investigating the wave functions in the midi-superspace approximation. Then the general form of the wave functions and the asymptotic behaviour of the modes will suggest the form that the effective interactions take.

A The Wave Functions

We shall assume a metric of the form

$$ds^2 = \sigma^2 \left[(N^2 + N_i N^i)d\tau^2 + 2N_i dx^i d\tau + h_{ij} dx^i dx^j\right], \quad (3.1)$$

where the three-metric $h_{ij}$ is

$$h_{ij} = e^{2\alpha(\tau)}(\Omega_{ij} + \epsilon_{ij}) \quad (3.2)$$

Here $\Omega_{ij}$ is the round metric on the unit three-sphere, and we expand $\epsilon_{ij}$ in harmonics,

$$\epsilon_{ij} = \sum_{n \ell m} \left[ \sqrt{6} a_{n \ell m} \frac{1}{3} Q^n_{\ell m} \Omega_{ij} + \sqrt{6} b_{n \ell m} (P_{ij})^n_{\ell m} + \sqrt{2} c_{n \ell m} (S_{ij})^n_{\ell m} + 2d_{n \ell m} (G_{ij})^n_{\ell m} \right], \quad (3.3)$$
with the sum starting at \( n = 2 \), and the notation of [10]. The \( n = 2 \) modes are rather special: the \( n = 2 \) traceless tensor harmonics all vanish identically and so the \( n = 2 \) term in the sum is just the \( \frac{\sqrt{6}}{3} a_2 Q^2 \Omega_{ij} \) part. (We shall henceforth drop the labels \( \ell m \).) The lapse, shift and scalar field are similarly expanded:

\[
N = N_0 \left[ 1 + 6^{-1/2} \sum_n g_n Q^n \right]
\]

\[
N_i = e^\alpha \sum_n \left[ 6^{-1/2} k_n (P_i)^n + \sqrt{2} j_n (S_i)^n \right]
\]

\[
\Phi = \sigma^{-1} \left[ \frac{1}{\sqrt{2\pi}} \phi(t) + \sum_n f_n Q^n \right]
\]

The action is expanded to all orders in \( \alpha, \phi, N_0 \) and to second order in the perturbation quantities \( q_n \in (a_n, b_n, c_n, d_n, f_n) \) and \( r_n \in (g_n, k_n, j_n) \). We refer the reader to [10] for the details of the action, constraints and three-sphere harmonics. Conjugate momenta can be defined in the usual manner, and the hamiltonian expressed as a function of coordinates and momenta. (We denote the euclidean momenta conjugate to \( q \) by \( \pi_q \).) The hamiltonian takes the form

\[
H = N_0 \left[ H_0 + \sum_n H^n_2 + \sum_n g_n H^n_1 \right] + \sum_n (k_n^S H^n_{-1} + j_n^V H^n_{-1})
\]

with the subscripts denoting the order of each part of the hamiltonian in the perturbation quantities, and whether each part arises from varying the lapse or shift.

We look for wave functions which can be expressed in the form

\[
\Psi = \Psi_0(\alpha, \phi) \prod_n \psi_n(\alpha, \phi, q_n)
\]

where \( \Psi_0(\alpha, \phi) \) is one of the euclidean WKB solutions of the mini-superspace background Wheeler-deWitt equation discussed in Sec. 2 (for instance one of the \( \psi_k(e^\alpha, \phi) \) defined in (2.11)). The full Wheeler-deWitt equation \((H_0 + \sum_n H^n_2)\Psi = 0\) reduces to a set of euclidean Schrodinger equations for \( \psi_n \), along the euclidean trajectories defined by the background wave function. The partial wave functions \( \psi_n \) must also satisfy the linearized hamiltonian and momentum constraints. The euclidean trajectories which are defined by the background wave functions \( \psi_k \) are the solutions to the background classical equations of motion (2.4), with \( Q = ik \). We shall consider this choice of background wave function from now on. The vertex operator corresponding to this choice will contain a factor of
$e^{ik\phi}$, from $\psi_k$, and a factor from each of the $\psi_n$, which will depend on which solution to the euclidean Schrodinger equation is chosen for each $n$ mode. This framework, with a WKB background plus quantized perturbations satisfying a Schrodinger-type equation, is the ‘$M$-expansion’, which is discussed further in [20]. We shall later discuss qualitatively the type of wave functions one might expect from the modes with $n > 2$, but first we consider the $n = 2$ modes. (We drop the mode label 2 on the perturbation quantities for ease of notation.)

The partial wave function $\psi_2$ depends just on $\alpha, k, a, f$. The $n = 2$ linear constraint equations to leading order in the $M$-expansion are

$$S\hat{H}_1\psi_2 = \frac{1}{3}e^{-3\alpha}(-\hat{\pi}_a + a\pi_\alpha + 3f\pi_\phi)\psi_2 = 0$$

$$\hat{H}_{\mid 1}\psi_2 = \frac{1}{2}e^{-3\alpha}(a(\pi_\alpha^2 + 3\pi_\phi^2 - 3e^{4\alpha}) - 2(\pi_\phi\hat{\pi}_f - \pi_\alpha\hat{\pi}_a))\psi_2 = 0 .$$

(3.7)

The carets on $\pi_a, \pi_f$ denote the operator form of the euclidean momenta conjugate to $a, f$ (represented as $-\partial/\partial a, -\partial/\partial f$), while the quantities $\pi_\alpha$ and $\pi_\phi$ are the euclidean momenta of $\alpha, \phi$ in the background solution. These are given by $\pi_\alpha = \sqrt{e^{4\alpha} - k^2}$ and $\pi_\phi = ik$. We can use the first of equations (3.7) to substitute for $\hat{\pi}_a\psi_2$ into the second equation, which we can then solve on the surface $a = 0$. This can then be used as an initial condition for the first equation, to obtain $\psi_2(\alpha, k, a, f)$ at non-zero values of $a$. We see that $\psi_2$ has the form of a Gaussian at $a = 0$,

$$\psi_2(\alpha, k, 0, f) = c_2(\alpha, k)e^{-\frac{\hat{\pi}_a(\alpha, k)}{2}f^2} ,$$

(3.8)

although it is not clear what $\psi_2$ represents since both $a$ and $f$ are gauge quantities. The main point is that the linearized constraint equations have picked out one solution to the euclidean Schrodinger equation for these modes, which appears to be in a kind of ‘ground state’.

For $n > 2$ the situation is completely different from the above. In this case there are three linearized constraint equations for five perturbation quantities, so there are two physical degrees of freedom remaining; one scalar (say $s_n$, originating from $a_n, b_n, f_n$) and one tensor ($d_n$). We can perform a reduction to the physical modes before quantization (see [21,22]). Then each of the $n > 2$ partial wave functions $\psi_n$ depends on $\alpha, k, s_n, d_n$ with the $s_n$ and $d_n$ parts being separable. Each part obeys a euclidean Schrodinger equation of the form

$$N_0^{ph}\hat{H}_{\mid 2}\psi_n = -\frac{\partial}{\partial \tau}\psi_n ,$$

(3.9)
where the operator $\hat{H}_n^{ph}$ is a diagonalizable homogeneous quadratic in $q_n$ and $\hat{\pi}_n$, and $q_n \in (s_n, d_n)$. The problem thus reduces to the euclidean Schrödinger equation for a harmonic oscillator with time-dependent frequency. (For the tensor modes $d_n$ in pure gravity this equation reduces to the simple harmonic oscillator, as shown in [8].) The method of Salusti and Zirilli [23] (also discussed in [24]) can be used to find a complete set of orthonormal solutions. We describe how this can be done, and the inner product defined in the euclidean sector, in the Appendix. The states can be obtained from a Gaussian ground state by the application of suitable ‘raising operators’, in a similar manner to the simple harmonic oscillator. The action of $m$ raising operators on the ground state yields a state which is interpreted as describing a closed universe containing $m$ particles in that mode. It takes the form

$$\psi_{nm}(\tau, q_n) = \gamma_{nm}(\tau) H_m(\beta_n(\tau)q_n)e^{-\alpha_n(\tau)q_n^2},$$  

where $H_m$ is an $m$th order Hermite polynomial. The solutions to the euclidean Schrödinger equation along the classical trajectories of fixed $k$ can then be ‘lifted’ to full solutions of the Wheeler-deWitt equation, using the method described in [20].

B The Classical Solutions and the Vertex Operators

There are two ingredients in the path integral formula for the matrix element (2.12). One is the wave function, which describes the quantum state of the wormhole. We have described above the general form that such wave functions take. The other is the path integral over asymptotically flat four geometries with an inner boundary on which the three-geometry and matter field are specified. For the semi-classical evaluation of this path integral we need the classical euclidean solutions for the perturbation modes. We shall now discuss these.

The classical euclidean solutions including the $n = 2$ modes can be obtained from the background solution (2.4) by slicing the background in a different way. The background solution can be expressed in the form ($\infty > r > r_0 > \sqrt{|Q|}/2$)

$$ds^2 = \sigma^2 \Omega(r^2)(dr^2 + r^2 d\Omega_3^2)$$

$$\phi = \phi(r^2),$$

where

$$\Omega(X) = 1 - \frac{1}{4}Q^2 X^{-2}$$

$$\phi(X) = \phi_- + \frac{1}{2} \ln \left[ \frac{X + \frac{1}{2}Q}{X - \frac{1}{2}Q} \right].$$

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We now change co-ordinates from $r, \chi$ to $r', \chi'$, with $r \cos \chi = r' \cos \chi' - \epsilon$ and $r \sin \chi = r' \sin \chi'$ (so that $r^2 = r'^2 - 2 \epsilon r' \cos \chi' + \epsilon^2$), for a constant parameter $\epsilon$. This change is an isometry of the flat metric $dr^2 + r^2 d\Omega^2_3$. However $\Omega$ and $\phi$ develop non-spherically symmetric parts in the new slicing of $\mathbb{R} \times S^3$ in three-spheres of constant $r'$. (We also change the boundary to one at constant $r'$, instead of at constant $r$.) We see that

$$\Omega(r^2) = \Omega(r'^2) - \epsilon Q_2 r'^{-5} \cos \chi' + O(\epsilon^2)$$
$$\phi(r^2) = \phi(r'^2) + \epsilon Q r'^{-3} \Omega^{-1}(r'^2) \cos \chi' + O(\epsilon^2)$$

(3.13)

as $\epsilon \to 0$. We thereby obtain an exact solution to the linearized equations for the $n = 2$ modes, since $\sqrt{2/\pi} \cos \chi$ is an $n = 2$ scalar harmonic. We shall now remove the primes on $r', \chi'$. (Notice that in this form the solution is obtained in the particular gauge $k = 0, g = a, N_0 = e^a r^{-1}$.)

The $n = 2$ scalar field perturbation falls off like $r^{-3} \cos \chi$ in the asymptotic region ($r \to \infty$), while the background part of the scalar field falls off like $r^{-2}$. It is essentially the fact that $\phi$ falls off with proper distance like the two-point function in flat space which led to the vertex operator being a function of $\phi$ (no derivatives of $\phi$) for the spherically symmetric model of Sec. 2. We see from (3.13) that the $n = 2$ mode falls off like a single derivative of the two-point function $\epsilon^\mu \partial_\mu r^{-2} = -2 r^{-3} \cos \chi$, where $\epsilon^\mu$ is a constant vector in flat space. Thus the vertex operator coming from these modes would be expected to be a function of $\partial_\mu \phi$. By considering the asymptotic form of the linearized equations for the general $n$ mode, we see that the scalar part falls off like $r^{-n-1} Q_n$, for a scalar harmonic $Q_n$. Thus the contribution from the $n$th mode will produce a vertex operator which is made up of $n - 1$ derivatives of $\phi$. There is an integration over the $O(4)$ rotations at the end of the calculation (a zero mode integration, similar to the integral over $x_0$ in Sec. 2). This means that the total vertex operator will be Lorentz covariant. In general, a wave function in the scalar sector of the form (3.10) would be expected to lead to a vertex operator $(\langle V_n \rangle)^m$ which is the $m$th power of a basic operator $V_n$. The operator $V_n$ will be made up of $n - 1$ derivatives of $\phi$. However, we have found for the $n = 2$ mode, that the wave function satisfying the linearized hamiltonian and momentum constraints is forced to have Gaussian form. For such wave functions the vertex operator will be restricted to have $m = 0$. Thus we have appeared to rule out the operators $(\partial \phi)^m$, for $m > 0$, but not corresponding operators involving higher derivatives of $\phi$. This appears to contradict work in [11], which considers the conformally coupled massless scalar field $n = 2$ mode. In [11] it appears that the vertex operator $(\partial \phi)^2$ is present, corresponding to the lowest allowed excited solutions to the Wheeler-deWitt equation. We expect however, that the
conformally and minimally coupled massless scalar fields should have the same number of degrees of freedom for each level $n$, and that our result on the absence of first derivative terms in the vertex operator would apply also in the conformally coupled case.

Our argument fixing the form of the $n = 2$ wave function to be in its ground state relied on the fact the $n = 2$ modes are pure gauge. It does not apply to the modes $n > 2$. For the $n > 2$ modes excited states will exist which satisfy the Wheeler-deWitt equation and linearized constraints. It is expected that the part of the vertex operator (in the scalar part of the theory) coming from the $m$th excited state will consist of $m$ powers of operators involving $n − 1$ derivatives of $\phi$. This will yield corrections to the vertex operator $e^{ik\phi}$ in the massless scalar field model which cannot be obtained using the Green functions of fields in the spherically symmetric wormhole background [5]. Thus the issue of whether the semi-classical vertex operator has really been found to be $e^{ik\phi}$ to all orders in the wormhole scale, is still a matter of debate.

4 Conclusions

We have discussed the form of the vertex operator for wormholes in the massless minimally coupled scalar field model. We used the wave function approach of Hawking [3], and described the evaluation of vertex operators corresponding to a basis of solutions to the hamiltonian and momentum constraints. The $O(4)$-symmetric mini-superspace truncation yielded a correspondence between a particular basis of wave functions which separated in the $(a, \phi)$ mini-superspace variables, and effective semi-classical vertex operators of the form $\int d^4x_0 \alpha(k):e^{ik\phi(x_0)}:. These vertex operators had previously been obtained by considering the field theory in the background of an imaginary scalar field euclidean wormhole solution [16]. Using that method it appears that there are no further corrections to the vertex operator of higher order in $\Lambda_{\text{pl}}$ (or the wormhole characteristic scale). However, using instead the wave function approach it appears that such corrections exist, and would come from non-spherically symmetric perturbation modes on the spherically symmetric background. The $n = 2$ modes, which would yield a factor in the vertex operator like $(\partial \phi)^m$, appear to be pure gauge, and so the linearized hamiltonian and momentum constraints restrict this part of the wave function to be in its ground state. Thus these terms appear not to be present. This appears to contradict earlier work on the conformally coupled scalar field model [11]. However the $n > 2$ modes can be excited, and would appear to lead to higher derivatives in the effective interaction. This contrasts with the results obtained in [5] using the field theory in the imaginary scalar field wormhole background, and
suggests that the two different approaches to wormholes correspond to physically different situations.

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Appendix

We show here how to construct the basis of solutions to the euclidean Schrodinger equation which are used in Sec. 3. We also construct an inner product which is particularly natural for wormholes, and with respect to which the wave functions are orthonormal.

Consider the case where the euclidean WKB background wave functions are the \( \psi_k(e^\alpha, \phi) \) defined in (2.11). Each wave function \( \psi_k \) defines a euclidean Hamilton-Jacobi function

\[
I_k(\alpha, \phi) = \frac{1}{2} \left( \sqrt{e^{4\alpha} - k^2} + k \arcsin(k e^{-2\alpha}) \right) + ik \phi .
\]  

(A.1)

We shall denote the partial derivatives by \( I_\alpha = \sqrt{e^{4\alpha} - k^2} \) and \( I_\phi = ik \). There are associated euclidean trajectories, which are the solutions to the equation of motion (2.4), with \( Q = ik \). The classical euclidean momenta are given by \( \pi_\alpha = I_\alpha, \pi_\phi = I_\phi \). Since the background wave functions have WKB form, in the region \( e^{4\alpha} \gg k^2 \gg 1 \), we can reduce the quantization of the \( n > 2 \) perturbations to the study of the Schrodinger equation for one tensor and one scalar mode, in the background solution [22]. These are the physical degrees of freedom, once all linearized constraints have been solved. In our case, the background classical geometries are euclidean, and one obtains a euclidean Schrodinger, or heat, equation. The equation is generally of the form

\[
H_n \psi_k(\tau, q_n) = -\frac{1}{N_0} \frac{\partial}{\partial \tau} \psi_k(\tau, q_n) ,
\]  

(A.2)

where the ‘time parameters’ \( \tau \) along the trajectories have been introduced via \( \dot{\phi} e^{3\alpha} = N_0 I_\phi \) and \(-\dot{\alpha} e^{3\alpha} = N_0 I_\alpha \). This can be conveniently solved in the gauge \( N_0 = e^{-\alpha} \) by \( \tau = -\frac{1}{2} I_\alpha \).

Leaving out the suffices \( n \), and choosing a convenient operator ordering, the ‘hamiltonian’ \( H \) is

\[
H = \frac{1}{2} e^{-3\alpha} \left( A \partial_q^2 + \frac{1}{2} B(q \partial_q + \partial_q q) + C q^2 \right) ,
\]  

(A.3)
where, for the tensor modes \((q = d)\), \(A = -1\), \(B = 0\), \(C = e^{4\alpha}(n^2 - 1)\), and for the scalar modes \((q = s = f + \frac{I}{I_\alpha}(a + b))\),

\[
A = - \left( 1 + \frac{3}{n^2 - 4} \left( \frac{I_\phi}{I_\alpha} \right)^2 \right)
\]

\[
B = 6 \left( \frac{n^2 - 1}{n^2 - 4} \right) \frac{I_\phi^2}{I_\alpha}
\]

\[
C = (n^2 - 1) \left( \frac{I_\alpha^2 - n^2}{n^2 - 4} \right).
\]

The same procedure as used in [22] has been used to obtain these equations, except for the difference that this analysis is in euclidean time.

By writing

\[
\psi_k(\tau, q) = e^{-\frac{3}{n^2} q^2} \chi_k(\tau, q),
\]

the equation for \(\chi_k\) takes the form of (A.2) with \(A \to A\), \(B \to 0\) and

\[
C \to C' = C - \frac{B^2}{4A} - \frac{e^{-3\alpha}}{2N_0} \frac{d}{d\tau} \left( \frac{B}{A} \right).
\]

Solutions exist for \(\chi_k\) which are of Gaussian form

\[
\chi_k(\tau, q) = u^{-1/2} \exp \left( \frac{\dot{u} e^{3\alpha}}{2uN_0 A} q^2 \right),
\]

for any \(u\) satisfying the linear equation

\[
\frac{1}{N_0} \frac{d}{d\tau} \left( \frac{\dot{u} e^{3\alpha}}{N_0 A} \right) + C' e^{-3\alpha} u = 0.
\]

The equation (A.8) has a number of important properties. Firstly, it is real and so it possesses two linearly independent real solutions. This is not the case for its lorentzian counterpart. Secondly, in the asymptotic region, \(\tau \to -\infty\), \(A \to -1\) and \(C' \sim e^{4\alpha}(n^2 - 1)\), so (A.8) reduces to the equation for the tensor modes. One of the solutions to (A.8) tends to zero while the other tends to infinity, in this region. We can use this to define a preferred vacuum wave function, \(\chi_{k0}(\tau, q)\) which is obtained by choosing the solution to (A.8), \(u(\tau) = u_0(\tau)\), which tends to zero in the asymptotic region. (In the gauge \(N_0 = 1\) we have \(u_0(\tau) \sim \tau^{-n-1}\).) Thirdly, we note that (A.8) is unchanged with respect
to $\tau \rightarrow -\tau$. (If we choose, for instance, the gauge $N_0 = e^{-\alpha}$, with the choice of time parameter $\tau = -\frac{1}{2}I_0$, this is made somewhat more explicit.) The origin of this symmetry is the discrete isometry of the wormhole geometry, which interchanges the two asymptotic regions. This symmetry means that if $u(\tau)$ is a solution to (A.8), then so is $\tilde{u}(\tau) = u(-\tau)$.

The ‘vacuum’ wave function, $\chi_{k0}$, is annihilated by the operator

$$a(u) = u\partial_q - \frac{\dot{u}e^{3\alpha}}{N_0A}q$$ \hspace{1cm} (A.9)

with $u = u_0$. The operator $a(\tilde{u}_0)$, when applied to $\chi_{k0}$, yields another solution to the euclidean Schrodinger equation, (non-vanishing, because $u_0(\tau)$ and $\tilde{u}_0(\tau)$ are linearly independent). We call this solution $\chi_{k1}$. This process can be continued indefinitely to produce $\chi_{km}$. The operator $a(\tilde{u}_0)$ is the ‘generalized raising operator’. We can then construct $\psi_{km}$ from $\chi_{km}$ using (A.5).

To construct the appropriate inner product, in which these wave functions are all orthogonal, we first use $\tilde{u}_0(\tau)$ to define a ‘dual vacuum’

$$\tilde{\chi}_{k0}(\tau, q) = \tilde{u}_0^{-1/2}\exp\left(-\frac{\dot{\tilde{u}}_0e^{3\alpha}}{2\tilde{u}_0N_0A}q^2\right)$$ \hspace{1cm} (A.10)

$$\tilde{\psi}_{k0}(\tau, q) = e^{\frac{B}{4A}q^2}\tilde{\chi}_{k0}(\tau, q)$$

which satisfies the ‘time reversed’ euclidean Schrodinger equation

$$\frac{1}{2}e^{-3\alpha}\left(A\dot{q}_q^2 - \frac{1}{2}B(q\partial_q + \partial_qq) + Cq^2\right)\tilde{\psi}_k(\tau, q) = +\frac{1}{N_0}\frac{\partial}{\partial\tau}\tilde{\psi}_k(\tau, q)$$ \hspace{1cm} (A.11)

(Under time-reversal we note that $B \rightarrow -B$.) The operator

$$\tilde{a}(u) = -u\partial_q - \frac{\dot{u}e^{3\alpha}}{N_0A}q$$ \hspace{1cm} (A.12)

annihilates $\tilde{\chi}_{k0}$, if $u = \tilde{u}_0$, and if $u = u_0$ it can be applied $m$ times to $\tilde{\chi}_{k0}$, to produce $\tilde{\chi}_{km}$ and hence the ‘dual $m$th excited state’, $\tilde{\psi}_{km}$.

With these definitions, the inner product

$$(\phi, \psi) = \int_{-\infty}^{+\infty} dq\bar{\phi}(\tau, q)\psi(\tau, q)$$ \hspace{1cm} (A.13)

is independent of $\tau$, and with suitable normalization constants the wave functions $\psi_{km}$ are orthonormal with respect to this inner product.
The wave functions $\psi_{km}(\tau, q_n)$ have the form quoted in the text,

$$\psi_{km}(\tau, q_n) = \gamma_{knm}(\tau) H_m(\beta_{kn}(\tau) q_n) e^{-\alpha_{kn}(\tau) q_n^2}, \quad (A.14)$$

where $H_m$ is an $m$th order Hermite polynomial. They are interpreted in terms of quantum states corresponding to a closed universe containing $m$ particles in the $n$th harmonic mode.

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