Resonant solitons in a polydisperse bubble medium

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Abstract. The effect of polydispersity of bubbles on formation of the spatial structure of the wave field, consisting of a sound forerunner, resonant solitons and trace, is investigated. It is shown that in a polydisperse medium the general structure of the wave field is the same as in the medium with identical bubbles, but the energy and spectral characteristics of all wave field components vary substantially.

1. Introduction
Solitons represent a widespread phenomenon. Numerous examples of soliton appearance in different physical systems are found. Examples are the waves on the surface of liquid, in plasma, in crystals and energy transmission lines, as well as in many other media. Formation of solitons is a nonlinear process. Solitons arise in the systems, where vibrations are possible, and this is a very wide class of physical processes [1]. It is often impossible to measure simultaneously a change in the wave field, where solitons arise, and trace the dynamic behavior of structural elements in these media under the influence of a soliton. Theoretical studies of the properties of solitons are carried out on the basis of equations with weak nonlinearity and dispersion. The most famous equations used to study solitons are the equations of Korteweg-de Vries, sine-Gordon, and the nonlinear Schrodinger equation.

Liquid with gas bubbles is one of the most interesting physical object for the study of solitons. Formation of solitons is caused by mutually consistent dynamics of the wave field and reaction of bubbles. Modern experimental methods allow simultaneous measurements of the wave field parameters and dynamics of bubbles. This possibility is due to the fact that the sensors for measuring the pressure field are much smaller than the spatial scale of the waves, and dynamics of the bubbles is well fixed by high-speed photography.

The theoretical models consider usually the media with bubbles of one size. Averaging methods are used at derivation of all model equations for liquid with bubbles. It follows that in the averaged equations it is basically impossible to take into account the influence of bubble size distribution on the structure and characteristics of the wave field [2]. Application of the averaging method for deriving the equations narrows the possibilities of these equations for the analysis of a large class of problems arising in various fields of physics. In experiments, it is impossible to create the medium with bubbles of one size. In spite of this, the soliton-like structures were observed repeatedly by different authors in experiments. In addition, very strong nonlinear effects are observed in the experiments, and analysis of the proceeding processes cannot be described by the Korteweg de Vries equation.

Nonlinear processes and influence of the bubble size distribution on the characteristics of the wave field can be investigated using a nonlinear wave system of equations obtained on the basis of a microscopic model of dynamics of heterogeneous media [3]. In this nonlinear self-consistent system of
equations, each bubble is precisely located at a certain point in space, all bubbles can have different sizes and each bubble affects wave field formation.

2. Nonlinear wave system of equations

The aim of this paper is to investigate the effect of bubble polydispersity on wave field formation in the bubble layer of a finite width, when one of the wave field elements is a resonant soliton. The nonlinear wave system of equations is used for calculation. The wave system of equations is derived from the general equations of the microscopic model of heterogeneous medium dynamics [3]. In the general system of equations, the medium is represented by liquid or gaseous carrier phase with arbitrary particle distribution over space and size. The particles consist of phases different from the carrier medium. Liquid with distributed bubbles is a particular case of a heterogeneous medium.

The system of equations includes the equations of conservation of mass, momentum, angular momentum, energy and the equation of motion for each particle. Particles of the discrete phase act on the carrier medium as the sources of mass, momentum, angular momentum, and energy. The particles move inside the carrier medium due to the forces of interfacial interaction with the carrier phase. The motion of particles can be influenced by other internal and external forces. For example, they are the forces associated with generation of energy within particles or electromagnetic forces. As a particular case, the nonlinear wave system of equations is derived from the general microscopic model for investigation of the wave fields in liquid with gas bubbles. Verification of the wave system of equations was performed by comparing numerical solutions with experimental data [3]. The model quantitatively describes the effects observed in the experiments. This justifies application of the proposed system of equations for studying the propagation of waves in liquid with bubbles in the regions for which there is no enough complete experimental data.

A nonlinear wave equation describing propagation of sound in liquid with arbitrarily distributed bubbles and random bubble sizes is obtained by isolating the linear wave operator for the carrier phase from the general equations. The linear wave operator is the left-hand side of the inhomogeneous wave equation. All terms that describe the presence of bubbles in the carrier medium are the right-hand side of the inhomogeneous wave equation. For study the propagation of waves in a bubbly medium are sufficient to take into account only the terms on the right-hand side of this equation describing the volume pulsations of the bubbles. The inhomogeneous wave equation is supplemented by the Rayleigh equations numerically equal to the number of bubbles in the considered region. The pulsations of each bubble in a bubble medium are described by its own Rayleigh equation. This condition is basic - each bubble can have its own size, different from the rest. Equations of state are written separately for the carrier medium and gas inside the bubbles. The wave system of equations in the one-dimensional case has the form [3]:

\[
\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = - \frac{1}{c^2} \frac{\partial}{\partial t} \left( \rho \frac{\partial}{\partial t} \ln (1 - \alpha) \right), \tag{1}
\]

\[
R_k \frac{d^2 R_k}{dt^2} + \frac{3}{2} \left( \frac{dR_k}{dt} \right)^2 + \frac{4\mu}{\rho_0 R_k} \frac{dR_k}{dt} + \frac{2\sigma}{\rho_0 R_k} = \frac{1}{\rho_0} \left[ P_0 + 2\sigma \left( \frac{R_0}{R_k} \right)^3 \right] - \frac{P_0}{\rho_0} - \frac{\rho(\vec{r},t)}{\rho_0}, \tag{2}
\]

\[
\alpha(\vec{r},t) = \sum_k v_k(t) \cdot \delta(\vec{r} - \vec{r}_k(t)), \tag{3}
\]

\[
v_k(t) = \frac{4}{3} \pi R_k^3(t), \tag{4}
\]

where \( k = 1, ..., N \), determines the total number of bubbles in the area under investigation, \( \delta(\vec{x}-\vec{x}_k) \) characterizes the position of the \( k \)-th bubble from the region occupied by the bubbles, \( P(x,t) \) is pressure in the wave, \( P_0 \) is initial pressure in the medium, \( c \) is the speed of sound in pure liquid, \( \alpha \) is volumetric gas content, \( R_k \) is radius of the \( k \)-th bubble, \( v_k \) is volume of the \( k \)-th bubble, \( \rho \) is density of liquid, \( \sigma \) is
surface tension of liquid, \( \mu \) is water viscosity, \( t \) is time, \( x \) is space coordinate. Water under normal conditions, most often used in experiments, was chosen as liquid for calculations.

The system of equations (1) - (4) is self-consistent. The differential derivative on the right-hand side of equation (1) depends on the volume of bubbles. This derivative describes a change in the volume of each bubble under the influence of wave at the bubble localization point in the medium. The change in bubble volumes, in turn, changes the right-hand side of the wave equation (2), which causes the change in the wave itself.

The numerical solution of the system of equations (1) - (4) is carried out by finite difference methods. For numerical solutions of this system of equations, the algorithms have been developed and original programs were written. To solve the wave equation (1), an explicit of the cross-type scheme and implicit Krank-Nicholson type scheme were used. Both schemes yield convergent results with accuracy of order 2. The solution of the Rayleigh equation for each bubble was carried out by the Runge-Kutta method of the fourth order of accuracy.

3. Discussion of results

This section presents the results of numerical study of the influence of the bubble size distribution on characteristics of the wave field produced by a wave passing into the bubble layer from pure water. The solutions of the nonlinear wave system of equations (1) - (4) are represented as a spatial distribution of the pressure wave field and corresponding bubble radius values at the time, when the front of the sound precursor reaches the far boundary of the bubble layer. The speed of the front of the sonic precursor is equal to the speed of sound \( c \) in pure liquid. Changes in the wave field and bubble radii in a layer with different bubbles are determined with respect to the wave field formed in a medium with identical bubbles [4] at the same value of the volume gas content.

A bubble layer with the length of 0.8 m is considered. A sound pulse with an amplitude of 1 MPa falls on the interface between pure liquid and bubble layer. Part of the wave pulse energy is reflected from the boundary of the layer, and part of the pulse passing into the layer forms a wave pressure field, which is the subject of study. The initial data for calculations correspond to the experimental conditions, where the structures under study were observed [5]. The bubble size distribution is described by the Gaussian function. In calculations, the rms deviation was set in the range from 0 to 50% relative to the mean radius of the bubble. The bubble radius at each point of the bubble layer is determined by the random number generator. The average radius of bubbles \( R_0 = 0.25 \times 10^{-3} \) m, and the volumetric gas content, typical of many experiments, is \( 10^{-3} \). The pulse duration is equal to \( 30 \times 10^{-6} \) s.

Figure 1 shows the spatial profile of the wave field in a medium with identical bubbles. This structure of the wave field is the standard used to determine the changes in the wave field due to

![Figure 1. Structure of the wave field in a layer with uniformly distributed identical bubbles. \( P_0=1MPa \) – amplitude of the incident wave of a pure liquid, \( \varphi=10^{-3} \) – volume gas content, \( R_0=0.25 \times 10^{-3} \) – the initial radius of bubbles, 1 – sound precursor, 2 – resonant soliton, 3 – trace, 4 – radiation into a clean liquid. Red line – the value of pressure in the wave field \( P \), blue line – the value of the deviation of bubble radii from the equilibrium value \( R_0 \).]
polydispersity. The graph shows two profiles. The first profile (red line) describes the spatial distribution of the pressure field. The second profile (blue line) describes the value of the bubble radius at the point of their localization.

In the structure of the wave field, three zones are distinguished with clearly different characteristics. The first zone is a sound precursor. Due to dispersion, high frequencies are located closer to the front of the precursor. The front of the precursor moves with the speed of sound in pure liquid. Bubbles in a sound precursor pulsate in phase with a change in the pressure field.

The second zone is a resonant soliton and compression of bubbles occurs in antiphase with respect to the pressure in the wave. The resonance soliton is formed at a distance of its own spatial scale near the interface of pure liquid and bubble layer. At the maximum compression in the pressure field of the resonant soliton, the bubble radius decreased by a factor of two, and volumetric gas content decreased by eight times. The average velocity of the resonant soliton on the passed interval, when the front of the sound precursor reached the far boundary of the bubble layer is 654 m/s.

Bubbles in the zone of the trace are also pulsating in antiphase. This zone expands with time, but with a much lower speed. In this zone, radiation is generated, and it decreases with time [6]. The propagation velocity of the emitted wave is larger than the resonant soliton velocity. The wave passes through the resonant soliton and forms a sound precursor.

Figure 2 shows the spatial profile of the pressure field and the bubble radius values at each point of the polydisperse bubbly medium. The root-mean-square deviation – 50%. $P_0 = 1 \text{ MPa}$ – amplitude of the incident wave of a pure liquid, $\varphi = 10^{-3}$ - volume gas content, $R_0 = 0,25 \ 10^{-3}$ – the average radius of bubbles. 1 – sound precursor, 2 – resonant soliton, 3 – trace, 4 – radiation into a clean liquid. Red line – the value of pressure in the wave field $P$, blue line – the value of the deviation of bubble radii from the equilibrium value $R_{0i}$.

![Figure 2. Structure of the wave field in a layer with different bubbles. The root-mean-square deviation – 50%. $P_0 = 1 \text{ MPa}$ – amplitude of the incident wave of a pure liquid, $\varphi = 10^{-3}$ - volume gas content, $R_0 = 0,25 \ 10^{-3}$ – the average radius of bubbles. 1 – sound precursor, 2 – resonant soliton, 3 – trace, 4 – radiation into a clean liquid. Red line – the value of pressure in the wave field $P$, blue line – the value of the deviation of bubble radii from the equilibrium value $R_{0i}$.

In a polydisperse medium, the structure of the pressure wave field also consists of three characteristic regions: sound precursor, resonant soliton, and trace. The structure of the pressure field in each of the zones differs significantly from the corresponding structure of the pressure field for a medium with identical bubbles.

At each point of a polydisperse medium, the bubbles have a random radius. This leads to significant changes in all zones of the wave field. In the zone of the sound precursor, the effect of dispersion also leads to localization of short wavelengths closer to the front of the precursor, but this effect is less pronounced. The bubbles pulses in phase with the oscillations of the pressure field in the sound precursor. The main difference is that the structures of the sound precursor and trace after the resonance soliton become non-regular. This is especially manifested in the pulsations of bubbles at the trailing edge of the resonant soliton and in the trace. When the resonant soliton passes a specific area, each bubble in this region pulsates independently. This is expressed in figure 2 as a point cloud, where each point represents deviation of the bubble radius at this point from its equilibrium value.
Figure 3. Spectra of amplitudes of sound precursors. a) monodisperse medium. b) polydisperse medium, the root-mean-square deviation is 50%.

Figure 3 shows the spectra of sound precursors in a monodisperse medium (a) and in a polydisperse medium (b). A comparison of the spectra shows that the medium with identical bubbles has a continuous spectrum with one maximum, and for a polydisperse medium the spectrum consists of several overlapping lines.

Figure 4 shows the amplitude spectra of the pressure field in the traces after the resonant solitons. It follows from the graphs that in both cases the spectra are similar. The main difference lies in the intensity of lines in the spectra. On average, in a polydisperse medium, the line intensities are approximately twice as large.

Quantitative changes are associated with a change in the characteristics of the resonant soliton. The soliton velocity in the polydisperse medium is 82 m/s less than the monodisperse medium and it is 572 m/s. This effect is caused by a stronger decrease in the amplitude of the resonant soliton in a polydisperse medium during its motion. This is due to significant energy dissipation by the bubbles.

Tables 1 and 2 show the energy distribution between the components of the wave field and bubbles. A characteristic feature of resonant solitons is an increase in asymmetry, which is characterized by the ratio of the width of the soliton leading front to the width of the trailing front with increasing soliton amplitude. Calculations showed that the size scattering of bubbles increases the effect. For the studied values, asymmetry in the medium with identical bubbles is 0.92, and for a polydisperse medium with bubble size dispersion of 50% this ratio is 0.62. In a polydisperse medium, all other conditions being equal, at the observation point the amplitude of the resonant soliton is less than in a medium with identical bubbles by the factor of 2.06.

Analysis of the data from the tables shows that, in a monodisperse medium, ~ 93% of total energy distributed in the medium is concentrated in the soliton, and ~ 40% of total soliton energy is concentrated in the bubbles. In a polydisperse medium, ~ 50% of total energy is concentrated in the resonant soliton and about the same amount of energy is concentrated in the bubbles in the trace. Sound precursors in both cases contain energy almost two orders of magnitude less.

Figure 4. Spectra of amplitudes of traces. a) monodisperse medium. b) polydisperse medium, the root-mean-square deviation is 50%.
Table 1. Monodisperse medium.

| Energy            | Wave, J | Bubbles, J | Total energy, J |
|-------------------|---------|------------|-----------------|
| Resonant soliton  | 2.48    | 1.70       | 4.18            |
| Sound precursors  | 0.037   | 0.0086     | 0.046           |
| Trace             | 0.08    | 0.19       | 0.27            |
| Total energy      | 2.60    | 1.9        | 4.5             |

Table 2. Polydispersse medium, root-mean-square deviation is 50%.

| Energy            | Wave, J | Bubbles, J | Total energy, J |
|-------------------|---------|------------|-----------------|
| Resonant soliton  | 1.29    | 1.12       | 2.41            |
| Sound precursors  | 1.4 · 10^{-3} | 0.2 · 10^{-3} | 1.2 · 10^{-3} |
| Trace             | 0.13    | 2.22       | 2.35            |
| Total energy      | 1.42    | 3.34       | 4.76            |

4. Conclusions
Polydispersity of bubbles does not fundamentally change the spatial structure of the wave field, consisting of a sound precursor, resonant soliton and trace, but it has a significant effect on the quantitative characteristics and spectrum of all characteristics of the wave-field zones.

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