CLASSIFICATION OF BOTT MANIFOLDS UP TO DIMENSION EIGHT

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Abstract. We show that three- and four-stage Bott manifolds are classified up to diffeomorphism by their integral cohomology rings. In addition, any cohomology ring isomorphism between two three-stage Bott manifolds can be realized by a diffeomorphism between the Bott manifolds.

1. Introduction

A Bott tower of height $n$ is a sequence of projective bundles

$$B_n: B_n \xrightarrow{\pi_n} B_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} B_1 \xrightarrow{\pi_1} B_0 = \{\text{a point}\},$$

where, for $i = 1, \ldots, n$, $\xi_i$ is a complex line bundle and $\mathbb{C}$ is a complex line bundle over $B_{i-1}$, and $\pi_i: B_i = P(\mathbb{C} \oplus \xi_i) \rightarrow B_{i-1}$ is a projective bundle over $B_{i-1}$. We call $B_n$ an $n$-stage Bott manifold, and $B_n$ a Bott tower structure of $B_n$. Note that an $n$-stage Bott manifold is of real dimension $2n$. A one-stage Bott manifold is the complex projective space $\mathbb{C}P^1$ of complex dimension one. A two-stage Bott manifold is known as a Hirzebruch surface. Hirzebruch [5] has shown that the topological type of a Hirzebruch surface $\Sigma_a = P(\mathbb{C} \oplus \gamma \otimes a)$ is completely determined by the parity of $a$, where $\gamma$ is the tautological line bundle over $\mathbb{C}P^1$; i.e., $\Sigma_a$ is homeomorphic to $\Sigma_b$ if and only if $a \equiv b (\mod 2)$. In addition, one can easily see that $H^\ast(\Sigma_0)$ and $H^\ast(\Sigma_1)$ are not isomorphic as graded rings. Later, it is shown that this classification also holds in the smooth category (see [7]), and stimulates the following conjecture (see [3]).

**Conjecture 1.1** (Cohomological rigidity conjecture for Bott manifolds). Let $B_n$ and $B'_n$ be $n$-stage Bott manifolds. Then, $B_n$ is diffeomorphic to $B'_n$ if and only if $H^\ast(B_n)$ is isomorphic to $H^\ast(B'_n)$ as graded rings.

More strongly, we conjecture the following:

**Conjecture 1.2** (Strong cohomological rigidity conjecture for Bott manifolds). For any cohomology ring isomorphism $\varphi$ between two Bott manifolds, there is a diffeomorphism which induces $\varphi$.

Conjecture [1.1] is known to be true for $n \leq 3$ (see [4]), and Conjecture [1.2] is known to be true for $n \leq 2$ (see [2] or Theorem 2.2). However, they have been open for the higher cases. In this paper, we shall show that Conjecture [1.2] is true for three-stage Bott manifolds, and that Conjecture [1.1] is true for four-stage Bott manifolds; namely, we have the following theorems.

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Theorem A (Theorem 3.1). For any cohomology ring isomorphism \( \varphi \) between two three-stage Bott manifolds, there is a diffeomorphism between them, which induces \( \varphi \).

Theorem B (Theorem 3.3). Let \( B_4 \) and \( B'_4 \) be four-stage Bott manifolds. Then, \( B_4 \) is diffeomorphic to \( B'_4 \) if and only if \( H^*(B_4) \) is isomorphic to \( H^*(B'_4) \) as graded rings.

2. Cohomology rings and square vanishing elements

We recall a Bott tower in (I), and one can express
\[
B_j = P(\mathbb{C} \oplus \gamma^{\alpha_j}) \text{ with } \alpha_j \in H^2(B_{j-1}),
\]
where \( \mathbb{C} \) denotes the trivial complex line bundle and \( \gamma^{\alpha_j} \) denotes the complex line bundle over \( B_{j-1} \) with \( \alpha_j \) as the first Chern class for \( j = 1, \ldots, n \). Using the Borel-Hirzebruch formula [I] for the cohomology ring of the projective bundle, we have that \( H^*(B_j) \) is a free module over \( H^*(B_{j-1}) \) via the map \( \pi_j^* \) on the two generator 1 and \( x_j \) of degree 0 and 2, respectively. The ring structure is determined by the single relation
\[
x_j^2 = \pi_j^*(\alpha_j)x_j,
\]
where \( x_j \) is the first Chern class of the tautological line bundle over \( B_j \).

Using this formula inductively on \( j \) and regarding \( H^*(B_j) \) as a graded subring of \( H^*(B_n) \) through the projections in (I), namely, setting \( x_i := \pi_i^* \circ \cdots \circ \pi_{i+1}^*(x_i) \), we see that
\[
H^*(B_n) = \mathbb{Z}[x_1, \ldots, x_n]/(x_j^2 = \alpha_j x_j \mid j = 1, \ldots, n),
\]
where \( \alpha_1 = 0 \), and \( \alpha_j = \sum_{i=1}^{j-1} A_j^i x_i \) with \( A_j^i \in \mathbb{Z} \) for \( j = 2, \ldots, n \). Since complex line bundles are classified by their first Chern classes, as is well-known, a Bott tower \( B_\bullet \) in (I) is completely determined by the list of integers \( A_j^i \) (\( 1 \leq i < j \leq n \)). In addition, we note that there is the natural filtration of \( H^*(B_n) \):

\[
H^*(B_1) \xrightarrow{\pi_2^*} H^*(B_2) \xrightarrow{\pi_3^*} \cdots \xrightarrow{\pi_n^*} H^*(B_n).
\]

Now, let us consider an element in \( H^2(B_n) \) whose square vanishes. Assume that a primitive element \( z = ax_j + u \) in \( H^2(B_n) \) satisfies \( z^2 = 0 \), where \( a \) is a non-zero integer and \( u \) is a linear combination of \( x_i \)'s for \( i < j \). Then, \( z^2 = a^2 x_j^2 + 2ax_ju + u^2 = 0 \in H^*(B_n) \); i.e., \( 2au = -a^2 \alpha_j \) and \( u^2 = 0 \). This implies that a square vanishing element should be of the form \( z = ax_j - \frac{2}{a^2} \alpha_j \) with \( \alpha_j^2 = 0 \). Therefore, a primitive element in \( H^2(B_n) \) whose square vanishes is either \( x_j - \frac{1}{2} \alpha_j \) or \( 2x_j - \alpha_j \) up to sign for some \( j \), where \( \alpha_j^2 = 0 \) in both cases. Let \( X(B_n) \) be the set of all primitive square vanishing elements of \( H^*(B_n) \) up to sign. Then, \( |X(B_n)| \) is equal to the number of \( j \)'s satisfying \( \alpha_j^2 = 0 \), and, hence, is less than \( n \). We say that \( B_n \) is \( \mathbb{Q} \)-trivial if its cohomology ring is isomorphic to that of \( (\mathbb{C}P^1)^n \) with \( \mathbb{Q} \)-coefficients as graded rings.

Proposition 2.1. \( B_n \) is \( \mathbb{Q} \)-trivial if and only if \( \alpha_j^2 = 0 \) in \( H^*(B_n) \) for all \( j = 1, \ldots, n \).
Proof. If $\alpha_j^2 = 0$, then $(x_j - \alpha_j)^2 = 0$ in $H^*(B_n; \mathbb{Q})$ because $x_j^2 = \alpha_j x_j$. Since $x_j - \frac{\alpha_j}{2}$ for $j = 1, \ldots, n$ generate $H^*(B_n; \mathbb{Q})$ as a graded ring, this shows that $B_n$ is trivial. Conversely, if $B_n$ is trivial, there are $n$ primitive elements in $H^2(B_n)$ up to sign whose square vanish, which implies the converse by the above discussion. \qed

It is known that the strong cohomological rigidity holds for the class of $\mathbb{Q}$-trivial Bott manifolds; namely, we have the following theorem.

**Theorem 2.2** (Choi-Masuda [2]). Any cohomology ring isomorphism between two $\mathbb{Q}$-trivial Bott manifolds is realizable by a diffeomorphism.

Put $t = |X(B_n)|$. A Bott tower $B_\bullet$ is said to be well-ordered if $\alpha_j^2 = 0$ for $j = 1, \ldots, t$, and $\alpha_j^2 \neq 0$ for $j = t + 1, \ldots, n$.

**Lemma 2.3.** Every Bott manifold $B_n$ admits a well-ordered Bott tower structure.

Proof. Consider any Bott tower structure of $B_n$ which is not well-ordered. In other words, there exists at least one $j$ such that $\alpha_j^2 \neq 0$ but $\alpha_{j+1}^2 = 0$. Remember that $\alpha_{j+1} = \sum_{i=1}^{j-1} A_{i+1} x_i + A_{j+1} x_j$. If $A_{j+1} \neq 0$ and, as assumed, $\alpha_{j+1}^2 = 0$, then we get $\alpha_{j+1} = A_{j+1} x_j - \frac{\alpha_j}{2} \alpha_j$ with $\alpha_{j+1}^2 = 0$, which is a contradiction. Hence, $A_{j+1} = 0$. We can interchange the label $j$ and $j + 1$, which proves the lemma by following procedure; since $A_{j+1}^2 = 0$, $\gamma_{j+1}^j$ can be regarded as a complex bundle over $B_{j-1}$. Let $\pi : P(\mathbb{C} \oplus \gamma_{j+1}^j) \to B_{j-1}$ be the corresponding projection. Then,

$$
\begin{array}{ccc}
\pi^* B_j & \cong & P(\mathbb{C} \oplus \gamma_{j+1}^j) = B_{j+1} \\
\downarrow \pi & & \downarrow \pi_j \\
P(\mathbb{C} \oplus \gamma_{j+1}^j) & \to & P(\mathbb{C} \oplus \gamma_{j}^j) = B_j \\
\downarrow \pi & & \downarrow \pi_j \\
B_{j-1}.
\end{array}
$$

where $\pi^* B_j$ is the pullback of $B_j \to B_{j-1}$ by $\pi$. Then, one can see that $\pi^* B_j$ is diffeomorphic to $B_{j+1}$, and it gives another Bott tower structure of $B_n$, which is obtained from $B_\bullet$ by interchanging the $j$ and $j + 1$ stages. \qed

From now on, we only consider Bott manifolds with well-ordered Bott tower structure; namely, we assume that any Bott tower which appears in this paper is well-ordered.

Let $B'_n$ be another Bott manifold. Suppose that $H^*(B_n)$ and $H^*(B'_n)$ are isomorphic as graded rings. A graded ring isomorphism $\varphi : H^*(B_n) \to H^*(B'_n)$ is said to be $k$-stable if there is a graded ring isomorphism $h_k : H^*(B_k) \to H^*(B'_k)$ which makes the diagram

$$
\begin{array}{ccc}
H^*(B_k) & \xrightarrow{\pi^* \circ \cdots \circ \pi^*} & H^*(B_n) \\
\downarrow h_k & & \downarrow \varphi \\
H^*(B'_k) & \xrightarrow{\pi^* \circ \cdots \circ \pi^*} & H^*(B'_n)
\end{array}
$$
commute. We note that \( \varphi \) should send elements in \( X(B_n) \) to elements in \( X(B_n') \) up to sign, and \( X(B_n) \) forms an basis of \( \pi_n^* \circ \cdots \circ \pi_{i+1}^*(H^2(B_i)) \). It implies that \( |X(B_n)| = |X(B_n')| \) (say, \( t \)), and \( \varphi \) is \( t \)-stable.

**Theorem 2.4** (Ishida [6]). Let \( B_n \) and \( B'_n \) be two Bott manifolds. If there is an isomorphism \( \varphi: H^*(B_n) \to H^*(B'_n) \) which is \( (n-1) \)-stable, and if \( h_{n-1} \) is a realizable by a diffeomorphism between \( B_{n-1} \) and \( B'_{n-1} \), then so is \( \varphi \) by a diffeomorphism between \( B_n \) and \( B'_n \).

### 3. Classification of low-stage Bott manifolds

Note that there is only one one-stage Bott manifold \( \mathbb{C}P^1 \), and every two-stage Bott manifold is \( \mathbb{Q} \)-trivial. Hence, by Theorem 2.2, the strong cohomological rigidity holds for one- and two-stage Bott manifolds.

**Theorem 3.1.** For any cohomology ring isomorphism \( \varphi \) between two three-stage Bott manifolds, there is a diffeomorphism between them, which induces \( \varphi \).

**Proof.** If three-stage Bott manifolds are \( \mathbb{Q} \)-trivial, then, by Theorem 2.2, \( \varphi \) can be realized by diffeomorphism. Otherwise, namely, they are not \( \mathbb{Q} \)-trivial, then \( \varphi \) should be 2-stable. Since the strong cohomological rigidity holds for two-stage Bott manifolds, by Theorem 2.4, \( \varphi \) is realizable.

Now, we prepare one lemma for proving the cohomological rigidity for four-stage Bott manifolds.

**Lemma 3.2.** Let \( B_n = P(\mathbb{C} \oplus \gamma^\alpha) \) and \( B'_n = P(\mathbb{C} \oplus \gamma^\beta) \) be two projective bundles over an \( (n-1) \)-stage Bott manifold \( B_{n-1} \). If there exists \( u \in H^2(B_{n-1}) \) such that \( \alpha - 2u \) and \( u(u - \beta) = 0 \), then \( B_n \) is isomorphic to \( B'_n \) as bundles.

**Proof.** Note that \( P(\mathbb{C} \oplus \gamma^\beta) \) is isomorphic to \( P(\gamma^u \oplus \gamma^{\beta-u}) \). The total Chern class of \( \gamma^{-u} \oplus \gamma^{\beta-u} \) is \((1-u)(1+\beta-u) = 1 + \beta - 2u + u(u - \beta) = 1 + \alpha \). Hence, \( \gamma^{-u} \oplus \gamma^{\beta-u} \) and \( \mathbb{C} \oplus \gamma^\alpha \) are isomorphic by [6, Theorem 3.1]. So are \( P(\mathbb{C} \oplus \gamma^\beta) \) and \( P(\mathbb{C} \oplus \gamma^\alpha) \).

**Theorem 3.3.** Let \( B_4 \) and \( B'_4 \) be four-stage Bott manifolds. Then, \( B_4 \) is diffeomorphic to \( B'_4 \) if and only if \( H^*(B_4) \) is isomorphic to \( H^*(B'_4) \) as graded rings.

**Proof.** Let \( \varphi: H^*(B_4) \to H^*(B'_4) \) be a graded ring isomorphism. If both \( B_4 \) and \( B'_4 \) are \( \mathbb{Q} \)-trivial, then, by Theorem 2.2, \( \varphi \) can be realized by diffeomorphism. If \( |X(B_4)| = 3 \), then, combining Theorem 3.1 and Theorem 2.4, \( \varphi \) also can be realized. Hence, for the above two cases, \( B_4 \) and \( B'_4 \) are diffeomorphic.

Assume that \( |X(B_4)| = 2 \). We denoted by \( y_j, \beta_j \) and \( B'_j \) those elements in \( H^*(B'_4) \) which correspond to \( x_j, \alpha_j \) and \( A'_j \) in \( H^*(B_4) \) for \( j = 1, \ldots, 4 \). Since \( \varphi \) is 2-stable, \( \varphi \) induces a ring isomorphism

\[
\begin{array}{ccc}
H^*(B_4)/\pi_1^* \circ \pi_3^*(H^*(B_2)) & \to & H^*(B'_4)/\pi_1^* \circ \pi_3^*(H^*(B'_2)) \\
\mathbb{Z}[x_3, x_4]/(x_3^2 = 0, x_4^2 = A_2^2 x_3 x_4) & \to & \mathbb{Z}[y_3, y_4]/(y_3^2 = 0, y_4^2 = B_2^2 y_3 y_4)
\end{array}
\]
Hence, since it preserves the set of primitive square vanishing elements, we conclude
$A_4^3$ and $B_4^3$ have the same parity, and $\varphi(x_3)$ is either $\epsilon y_3 + w$, $\epsilon(y_4 - \frac{b_3^1}{2}y_3) + w$ (if $B_4^3$ is even) or $\epsilon(2y_4 - B_4^3y_3) + w$ (if $B_4^3$ is odd), where $\epsilon = \pm 1$ and $w$ is a linear combination of $y_1$ and $y_2$.

**CASE 1**: $\varphi(x_3) = \epsilon y_3 + w$. Note that $\varphi$ is 3-stable. Hence, $\varphi$ can be realized by diffeomorphism.

**CASE 2**: $\varphi(x_3) = \epsilon(y_4 - \frac{b_3^1}{2}y_3) + w$. Note that $B_4^3$ (say, $b$) is even. If $b = 0$, then we may interchange the third and fourth stages of its Bott tower structure as in Lemma 3.2. Hence, $\varphi(x_3)$ would be 3-stable, and, hence, it can be realized. Suppose that $b \neq 0$. Since $x_3(x_3 - \alpha_3) = 0$,

$$
0 = \varphi(x_3(x_3 - \alpha_3)) = (\epsilon y_4 - \frac{eb}{2}y_3 + w)(\epsilon y_4 - \frac{eb}{2}y_3 + w - \varphi(\alpha_3))
$$

$$
= y_4(y_4 - by_3 + 2\epsilon w - \varphi(\alpha_3)) + \frac{by_3}{4}(by_3 - 4\epsilon w + 2\epsilon \varphi(\alpha_3)) + w^2 - w\varphi(\alpha_3)
$$

Because $\varphi(\alpha_3)$ is a linear combination of $y_1$ and $y_2$ and $b \neq 0$, we have that

(2) $y_4(y_4 - by_3 + 2\epsilon w - \varphi(\alpha_3)) = 0 \in H^*(B'_3)$, and

(3) $\frac{by_3}{2}(-by_3 - 2\epsilon w + \varphi(\alpha_3)) = 0 \in H^*(B'_3)$.

Hence, by (2), $\beta_4 = by_3 - 2\epsilon w + \varphi(\alpha_3)$. Let $u = \frac{by_3}{2}$. Then, by (3), $u(\beta_4 - u) = 0$. Hence, by Lemma 3.2, we have an isomorphism $f: B'_4 \to P(C \oplus \gamma^3 - 2w)$ as bundles over $B'_3$. This isomorphism gives a new Bott tower structure of $B'_4$ whose 3rd and 4th stages are interchangeable. The interchange map is denoted by $g$. The new Bott tower structure obtained by $g \circ f(B'_4)$ is denoted by $B''_4$. Note that $f$ and $g$ are diffeomorphisms, and $B''_4$ is well-ordered. Hence, one can easily check that $g^* \circ f^* \circ \varphi: H^*(B_4) \to H^*(B'_4)$ is 3-stable. Therefore, $g^* \circ f^* \circ \varphi$ is realizable, and, hence, so is $\varphi$.

**CASE 3**: $\varphi(x_3) = \epsilon(2y_4 - B_4^3y_3) + w$. Note that both $A_4^3$ (say, $a$) and $B_4^3$ (say, $b$) are odd. We may also assume that $\varphi^{-1}(y_3) = \epsilon(2x_4 - ax_3) + z$, where $\epsilon = \pm 1$ and $z$ is a linear combination of $x_1$ and $x_2$. Since $x_3(x_3 - \alpha_3) = 0$,

$$
0 = \varphi(x_3(x_3 - \alpha_3)) = (2\epsilon y_4 - by_3 + w)(2\epsilon y_4 - by_3 + w - \varphi(\alpha_3))
$$

$$
= 4y_4(y_4 - by_3 + \epsilon w - \frac{\varphi(\alpha_3)}{2}) + b^2y_3(y_3 - \frac{2b}{b}w + \frac{1}{b}\epsilon \varphi(\alpha_3)) + w^2 - w\varphi(\alpha_3)
$$

Hence, $\beta_4 = by_3 - \epsilon w + \epsilon \varphi(\alpha_3)/2$, $\beta_3 = \frac{2aw}{b} - \frac{\varphi(\alpha_3)}{b}$ and $w^2 = w\varphi(\alpha_3)$. Note that $\beta_3^2 = \frac{1}{b} \varphi(\alpha_3)^2 \neq 0 \in H^4(B'_4)$. Similarly, we also have $\alpha_3^2 = \frac{1}{a} \varphi^{-1}(\beta_3^2)$. Thus, $\alpha_3^2 = \frac{1}{a \epsilon} \varphi(\alpha_3)^2$. Since $\alpha_3^2$ does not vanish, $a^2b^2 = 1$. Hence, $|a| = |b| = 1$. We may assume that $a = b = 1$. Then, $\beta_3 = 2\epsilon w - \varphi(\alpha_3)$, and $\beta_4 = y_3 - \epsilon w + \frac{\varphi(\alpha_3)}{2} = y_3 - \frac{\beta_4}{2}$. Similarly, we have $\alpha_4 = x_3 - \frac{\alpha_4}{4}$. By Lemma 3.2, we have a bundle isomorphism $f: P(C \oplus \gamma^{\varphi(\alpha_3)}) \to B'_3$ over $B'_2$. Then, we obtain the pullback $f^* B'_4 = P(C \oplus \gamma^{\varphi(\alpha_3)})$ of $B'_4$ by $f$, and we obtain the induced diffeomorphism $\tilde{f}: P(C \oplus \gamma^{\varphi(\alpha_3)}) \to B'_4$. On the other hand, since any cohomology ring isomorphism between two Hirzebruch surfaces is realizable, we consider a diffeomorphism $g: B'_2 \to B_2$ which induces $f$ restricted by
$H^*(B_2)$. Then, we also obtain the pullback $g^{-1}(f^*B'_k) = P(\mathbb{C} \oplus \gamma^{x_3 - \frac{3\alpha}{2}})$ of $f^*B'_k$ by $g^{-1}$, and we also have the induced diffeomorphism $g^{-1}: P(\mathbb{C} \oplus \gamma^{x_3 - \frac{3\alpha}{2}}) \to f^*B'_k$; see the following diagram

\[
\begin{array}{ccc}
P(\mathbb{C} \oplus \gamma^{x_3 - \frac{3\alpha}{2}}) & \xrightarrow{\pi'_4} & P(\mathbb{C} \oplus \gamma^{y_3 - \frac{2\alpha_3}{2}}) \\
\downarrow g & & \downarrow f \\
P(\mathbb{C} \oplus \gamma^{\alpha_3}) & \xrightarrow{\pi'_3} & P(\mathbb{C} \oplus \gamma^{\alpha_3'}) \\
B_2 & \xrightarrow{\gamma} & B_2'.
\end{array}
\]

Note that $P(\mathbb{C} \oplus \gamma^{x_3 - \frac{3\alpha}{2}}) = P(\mathbb{C} \oplus \gamma^{\alpha_4})$, and, hence, $P(\mathbb{C} \oplus \gamma^{\alpha_4}) \to P(\mathbb{C} \oplus \gamma^{\alpha_3}) \to B_2$ is a Bott tower structure of $B_4$. Hence, $\tilde{f} \circ g^{-1}$ is a diffeomorphism between $B_4$ and $B'_k$.

In the three above cases, we have shown that $B_4$ and $B'_k$ are diffeomorphic, which proves the theorem. \hfill $\Box$

**Example 3.4.** Let $B_4$ be a 4-stage Bott manifold with the Bott tower structure $P(\mathbb{C} \oplus \gamma^{x_3 - \frac{3\alpha}{2}}) \to P(\mathbb{C} \oplus \gamma^{\alpha_3}) \to B_2$. Consider four homomorphisms $\varphi_k: H^*(B_4) \to H^*(B_4)$ ($k = 1, \ldots, 4$) defined by

1. $\varphi_1(x_1) = x_1$, $\varphi_1(x_2) = x_2$, $\varphi_1(x_3) = 2x_4 - x_3 + \alpha_3$, and $\varphi_1(x_4) = x_4$;
2. $\varphi_2(x_1) = x_1$, $\varphi_2(x_2) = x_2$, $\varphi_2(x_3) = 2x_4 - x_3 + \alpha_3$, and $\varphi_2(x_4) = x_4 - x_3 + \frac{\alpha_3}{2}$;
3. $\varphi_3(x_1) = x_1$, $\varphi_3(x_2) = x_2$, $\varphi_3(x_3) = -2x_4 + x_3$, and $\varphi_3(x_4) = -x_4$;
4. $\varphi_4(x_1) = x_1$, $\varphi_4(x_2) = x_2$, $\varphi_4(x_3) = -2x_4 + x_3$, and $\varphi_4(x_4) = -x_4 + x_3 - \frac{\alpha_3}{2}$.

Then, they are all well-defined, and are graded ring isomorphisms. Moreover, they are all under the third case of the proof of Theorem 3.3.

We remark that a cohomology ring isomorphism $\varphi$ is realizable unless it is under the last case of the proof of Theorem 3.3. However, we do not know whether $\varphi$ of the last case is realizable or not. In order to prove the strong cohomological rigidity for 4-stage Bott manifolds, what we need is that any automorphism of the cohomology ring of $B_4$ with the Bott tower structure $P(\mathbb{C} \oplus \gamma^{x_3 - \frac{3\alpha}{2}}) \to P(\mathbb{C} \oplus \gamma^{\alpha_3}) \to B_2$ under the last case is realizable. We note that there are only finitely many such automorphisms. Since we may assume that $\varphi(x_1) = x_1$ and $\varphi(x_2) = x_2$, there are only four essential automorphisms $\varphi_k$ ($k = 1, \ldots, 4$).

**Problem 3.5.** Are $\varphi_k$’s ($k = 1, \ldots, 4$) realizable?

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References

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