SMOOTH $K$-THEORY OF LOCALLY CONVEX ALGEBRAS

H. INASSARIDZE AND T. KANDELAKI

Abstract. Smooth $K$-functors are introduced and the smooth $K$-theory of locally convex algebras is developed. It is proved that the algebraic and smooth $K$-functors are isomorphic on the category of quasi $\hat{\otimes}$-stable real (or complex) Fréchet algebras.

Introduction

We develop a new topological $K$-theory for arbitrary locally convex $k$-algebras called smooth $K$-theory and constructed by using smooth maps.

The category of locally convex $k$-algebras is a wide class of topological $k$-algebras, containing Fréchet $k$-algebras, Michael’s $k$-algebras which are isomorphic to projective limits of Banach $k$-algebras [13], many important examples of differential operator algebras and differential forms, closely related to noncommutative geometry.

The definition of smooth $K$-functors is motivated by our purpose to extend Karoubi’s Conjecture on the isomorphism of algebraic and topological $K$-functors [11] to a wide class of topological $k$-algebras containing those for which Karoubi’s Conjecture was already confirmed, namely the category of $C^*$-algebras [17], the category of generalized operator algebras and their polynomial extensions [10], by using the important notion of stability of these algebras. For more details see the summarizing article [16] about the relationship between algebraic and topological $K$-theory for Banach algebras and $C^*$-algebras. The real case will be also treated. In connection with this problem the smooth $K$-functors appear in a natural way and play a fundamental role in establishing our main result (Theorem 4.4) confirming what we call the Smooth Karoubi’s Conjecture:

The algebraic and smooth $K$-functors are isomorphic on the category of quasi $\hat{\otimes}$-stable real (or complex) Fréchet algebras.

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Consequently the use of smooth maps leads to an unexpected relationship between algebraic $K$-theory and topologically defined smooth $K$-theory via Grothendieck projective tensor product of locally convex algebras.

The proof of this theorem is spread over Sections 1, 2, 3 and 4.

It should be noted that Phillips [15] has defined a $K$-theory for complex Frechet algebras that are locally multiplicatively convex, extending the topological $K$-theory of complex Banach algebras and showing that many important $K$-theoretical properties are preserved such as homotopy invariance, exactness, periodicity and stability properties. A bivariant $K$-functor on the category of complex locally multiplicatively convex algebras has been constructed by Cuntz [5] defining a bivariant multiplicative character to bivariant periodic cyclic cohomology. When the first variable is trivial, Cuntz’s construction provides a $K$-theory isomorphic to Phillips’ $K$-theory. Both $K$-theories of complex locally multiplicatively convex Fréchet algebras are constructed in different context.

To prove the Smooth Karoubi’s Conjecture we have restricted ourselves by considering Fréchet $k$-algebras which topologies are determined by countably many seminorms. We don’t assume that these seminorms possess the multiplicative property. The proof is essentially based on the countability property of determining seminorms. It seems that our main theorem is not valid in the category of arbitrary (non-unital) locally convex $k$-algebras. This is due to the fact that algebraic $K$-functors in general case are not smooth homotopy functors for quasi $\hat{\otimes}$-stable locally convex $k$-algebras having a bounded approximate unit, since they don’t possess probably the Suslin-Wodzicki’s TF-property of rings and the excision property.

In the case of Fréchet $k$-algebras, in particular for Banach $k$-algebras, the $k$-algebra $A^I$ of all continuous maps $I = [0, 1] \rightarrow A$ into a Fréchet $k$-algebra $A$ is not compatible with the Grothendieck projective tensor product and we don’t have the following needed isomorphisms:

\[(A \hat{\otimes} k)^I \approx A \hat{\otimes} k^I \approx A^I \hat{\otimes} k.\]

That is the reason why the usual construction of topological $K$-functors seems to be difficult to be used to confirm Karoubi’s Conjecture if the stability is expressed in terms of the Grothendieck projective tensor product. On the other hand, the $k$-algebra $A^{\infty(I)}$ of all smooth maps $I \rightarrow A$ possesses the property (0.1) and many other functorial properties helping us to prove our main theorem.

In Section 1 smooth $K$-functors are introduced and investigated in the category of locally convex $k$-algebras. Their main homological
properties are established such as exactness, smooth homotopy property, compatibility with finite products, relationship with algebraic and topological \( K \)-theories in terms of sufficient conditions implying isomorphisms of smooth \( K \)-theory with these \( K \)-theories. It is shown that the smooth \( K \)-theory agrees to Phillips’ \( K \)-theory.

Section 2 deals with the Cohen-Hewitt factorization theorem. It is well-known that the excision property in algebraic \( K \)-theory is closely related with the triple factorization (\( TF \)-) property of rings [17]. The \( TF \)-property for Banach algebras is established in [17] by using a theorem of Hewitt [6] motivated by Cohen’s result [2]. According to Cohen-Hewitt theorem any element of a Banach module can be factorized into a product of two elements, when the Banach algebra possesses a bounded approximate unit which is an approximate unit of the module. In [20] Wodzicki showed that every multiplicatively convex Fréchet \( k \)-algebra with uniformly bounded approximate unit is \( H \)-unital as a consequence of the Cohen-Hewitt factorization theorem extended to the case of multiplicatively convex Fréchet algebras. In this section the Cohen-Hewitt factorization theorem will be proved for Fréchet modules over a Banach \( k \)-algebra which have a bounded approximate unit.

In Section 3 we give an account on Higson’s homotopy invariance theorem. This important theorem plays an essential role for setting the homotopy invariance of functors (for example, using this theorem Karoubi’s Conjecture on the isomorphism of algebraic and topological \( K \)-theories was confirmed on the category of stable \( C^\ast \)-algebras [17]). The natural question arises whether this theorem is true or not for real \( C^\ast \)-algebras, since we could not extend the proof of Lemma 3.1.2 and Theorem 3.1.1 of [7] to the case of real \( C^\ast \)-algebras. In this section we confirm that Higson’s homotopy invariance theorem holds also for functors defined on the category of real \( C^\ast \)-algebras.

Section 4 is devoted to the Smooth Karoubi’s Conjecture. We prove that Fréchet \( k \)-algebras with bounded approximate unit have the triple factorization property and therefore they possess the excision property in algebraic \( K \)-theory. Using Higson’s homotopy invariance theorem we conclude that the functors \( K_n(-\hat{\otimes}K) \), \( n \geq 1 \), are smooth homotopy functors implying the confirmation of the Smooth Karoubi’s Conjecture for real and complex cases.

We close this introduction with a list of terminological and notational conventions used in the present article:

1) \( k \) denotes the field of real or complex numbers;

2) algebras are always associative and are not assumed in general to possess unit;
3) $A^+ = A + k$ denotes the $k$-algebra obtained by adjoining unit to a $k$-algebra $A$;
4) $\mathcal{K}$ denotes the $C^*$-algebra of compact operators on the standard infinite-dimensional separable Hilbert space;
5) $\hat{\otimes}$ denotes the Grothendieck projective tensor product of locally convex $k$-algebras;
6) $\mathcal{Gr}$ and $\mathcal{Ab}$ denote respectively the category of groups and the category of abelian groups;
7) $\text{Ob} \mathcal{A}$ denotes the class of objects of the category $\mathcal{A}$;
8) the determining seminorms of Fréchet $k$-algebras are not assumed to possess the multiplicative property;
9) without loss of generality a locally convex $k$-algebra with bounded approximate unit always means with left bounded approximate unit and having the TF-property always means having the left triple factorization property.

1. Smooth $K$-theory

The smooth $K$-functors $K_{sm}^n$, $n \geq 0$, will be defined on the category $\mathcal{A}$ of locally convex $k$-algebras and their continuous $k$-homomorphisms.

By a locally convex $k$-algebra we mean a $k$-algebra $A$ equipped with a complete locally convex topology such that the multiplication map $A \times A \to A$ is jointly continuous. The category $\mathcal{A}$ is close under the Grothendieck projective tensor product $\hat{\otimes}$. We recall its definition.

Let $A$ and $B$ be two locally convex $k$-algebras. Their projective tensor product $A \hat{\otimes} B$ is given by a family $I$ of seminorms $\mu \otimes \nu$ on $A \otimes B$,

\[(\mu \otimes \nu)(x) = \inf \left( \sum_{k=1}^{n} \mu(a_k)\nu(b_k) \right),\]

where infimum is taken over all representations of $x$ of the form

\[x = \sum_{k=1}^{n} a_k \otimes b_k, \quad a_k \in A, \quad b_k \in B,\]

the locally convex $k$-algebra $A \hat{\otimes} B$ being the completion of $A \otimes B$ with respect to the family $I = \{\mu \otimes \nu\}$ of seminorms.

The projective tensor product possesses exactness property with respect to proper short exact sequences in the category of locally convex $k$-algebras that we shall need in Section 4.

Let

\[0 \to I \xrightarrow{f} B \xrightarrow{g} A \to 0\]
be a sequence of morphisms in the category $\mathcal{LC}$ of locally convex linear topological spaces and continuous linear maps. It will be said that this sequence is a proper exact sequence if $f$ is a homeomorphism of $I$ on $\text{Im} f$ and $g$ is an open surjective map. It will be said proper split exact if $g$ has a right inverse in $\mathcal{LC}$. A short exact sequence in the category $\mathcal{A}$ is said proper exact sequence if it is proper split exact in the category $\mathcal{LC}$.

**Lemma 1.1.** Let

$$0 \to I \overset{f}{\to} B \overset{g}{\to} A \to 0$$

be a proper exact sequence of locally convex $k$-algebras and $H$ be a locally convex $k$-algebra. Then the sequence

$$0 \to H \hat{\otimes} I \overset{H\hat{\otimes}f}{\to} H \hat{\otimes} B \overset{H\hat{\otimes}g}{\to} H \hat{\otimes} A \to 0$$

is a proper exact sequence.

**Proof.** Let $l : A \to B$ be a continuous linear map such that $gl = 1_A$. It is clear that $lg : B \to B$ is a continuous projection and $\text{Im}(lg)$ is a closed linear subspace of $B$ which is isomorphic to $A$. It is easily checked that the natural continuous linear map $I \times A \to B$ given by $(x, y) \mapsto f(x) + l(y)$ is an isomorphism of linear topological spaces. The converse map is defined by $z \mapsto (f^{-1}(z - lg(z)), g(z))$. One has the following commutative diagram

$$
\begin{array}{ccc}
0 & \to & I \hat{\otimes} H \\
\| & & \| \\
0 & \to & I \hat{\otimes} H
\end{array}
\begin{array}{ccc}
\overset{(1,0)}{\longrightarrow} & \to & (I \hat{\otimes} H) \times (A \hat{\otimes} H) \\
\| & \downarrow & \| \\
B \hat{\otimes} H & \overset{g \hat{\otimes} 1_H}{\longrightarrow} & A \hat{\otimes} H
\end{array}
\begin{array}{ccc}
\longrightarrow & \to & A \hat{\otimes} H \\
\| & & \| \\
\longrightarrow & \to & 0
\end{array}
$$

where the top row is the trivial proper split exact sequence of locally convex $k$-algebras and the middle vertical arrow is the composite of the natural isomorphisms $(I \hat{\otimes} H) \times (A \hat{\otimes} H) \overset{\text{pr}_2}{\longrightarrow} (I \times A) \hat{\otimes} H \xrightarrow{\sim} B \hat{\otimes} H$. Therefore the bottom row is a proper split short exact sequence of locally convex $k$-algebras. \hfill $\square$

It is said that a locally convex $k$-algebra $A$ possesses a left bounded approximate unit if there exists a bounded direct set $\{e_\lambda\}$ of elements in $A$ such that

$$\lim_i \| e_\lambda a - a \| = 0$$

for all $a \in A$ and $i \in I$. The right bounded approximate unit in $A$ is defined similarly. It is easily checked that the category of locally convex $k$-algebras with left bounded approximate unit is closed under the projective tensor product.
A Fréchet $k$-algebra is a locally convex complete $k$-algebra such that its topology is given by a countable family of seminorms.

The category $\mathcal{B}$ of Fréchet $k$-algebras is a full subcategory of the category $\mathcal{A}$ and is closed under the Grothendieck tensor product.

A continuous map $f : I \to A$ with values in a locally convex $k$-algebra $A$ is called smooth, if it has all derivatives $f^{(1)}, f^{(2)}, ..., f^{(n)}, ....$

The space $A^\infty(I)$ of all smooth maps from $I$ to $A$ is also a locally convex $k$-algebra. The seminorms $\mu_{s,r}$ on $A^\infty(I)$ are given by

\begin{align}
\mu_{s,r}(f) = \sup_t \left\{ \sum_{k=1}^r ||| f^{(k)}(t) |||_s, \, t \in I \right\},
\end{align}

where $\{||| \cdot |||_s\}_{s \in S}$ is the set of the determining seminorms on $A$. If $A$ is a Michael $k$-algebra or a Fréchet $k$-algebra, then $A^\infty(I)$ is also a Michael $k$-algebra or a Fréchet $k$-algebra respectively.

Any continuous $k$-homomorphism $\phi : A \to A'$ of locally convex $k$-algebras induces in a natural way a continuous $k$-homomorphism $\phi_{\infty} : A^\infty(I) \to A'^{\infty(I)}$.

Now on the category $\mathcal{A}$ we will define the smooth path cotriple $\mathcal{I}$.

The evolution maps at $t = 0$ and $t = 1$, $\varepsilon_i : A^\infty(I) \to A$, $i = 0, 1$, $\varepsilon_0(f) = f(0)$, $\varepsilon_1(f) = f(1)$, will play an important role. Denote by $\mathcal{I}(A)$ the kernel of $\varepsilon_0$ and by $\tau_A : \mathcal{I}(A) \to A$ the restriction of $\varepsilon_1$ on $\mathcal{I}(A)$. There is a natural continuous $k$-homomorphism $\delta_A : \mathcal{I}(A) \to \mathcal{I}^2(A) = \mathcal{I}(\mathcal{I}(A))$ sending $f \in \mathcal{I}(A)$ to $\delta_A(f)(s, t) = f(st)$. It is easily checked that $\delta_A(f)$ is a smooth map.

By taking $\mathcal{I} = (\mathcal{I}, \tau, \delta)$ one gets a cotriple on the category $\mathcal{A}$ which will be called the smooth path cotriple. The cotriple $\mathcal{I}$ induces the augmented simplicial locally convex $k$-algebra

\begin{align}
\mathcal{I}_s^+(A) = \mathcal{I}_s(A) \to A,
\end{align}

where $\mathcal{I}_0(A) = \mathcal{I}(A)$, $\mathcal{I}_n(A) = \mathcal{I}(\mathcal{I}_{n-1}(A))$, $n \geq 1$, $\delta^n_i = \mathcal{I}^i \tau \mathcal{I}^{n-i}$, $s_i = \mathcal{I}^i \delta \mathcal{I}^{n-i}$, $n \geq 1$, $0 \leq i \leq n$.

Applying the general linear group functor one obtains an augmented simplicial group

$GL(\mathcal{I}_s^+(A)) = GL(\mathcal{I}_s(A)) \to GL(A)$.

**Definition 1.2.** We define the smooth $K$-functors $K_{n}^{\text{sm}}$, $n \geq 0$, by setting

\begin{align}
K_{n}^{\text{sm}}(A) = \begin{cases} 
\pi_{n-2} GL(\mathcal{I}_s(A)) & \text{for } n \geq 3, \\
K_0(A), & \text{for } n = 0,
\end{cases}
\end{align}
and $K_1^{sm}(A)$, $K_2^{sm}(A)$ make exact the following sequence
$$0 \to K_2^{sm}(A) \to \pi_0 GL(I_\ast(A)) \xrightarrow{\gamma_0} GL(A) \to K_1^{sm}(A) \to 0,$$
where $\gamma_0$ is induced by $GL(\tau_A)$, or equivalently
$$K_n^{sm}(A) = \pi_{n-2} GL(I_\ast^+(A)), \quad n \geq 1.$$

From the definition of smooth $K$-functors follows immediately
$$K_n^{sm}(I(A)) = 0$$
for all $n \geq 1$ and any locally convex $k$-algebras, since the augmented simplicial group $GL(I^+(I(A)))$ is right contractible and therefore aspherical according to Lemma 1.1 and Lemma 1.2 [18].

It will be shown that all smooth $K$-groups are abelian groups. This is obvious for $n = 0$ and $n \geq 3$. First we will proof that $K_1^{sm}(A)$ is a group or equivalently that $\text{Im} GL(\tau_A)$ is a normal subgroup of $GL(A)$. The exact sequence
$$0 \to I(A) \xrightarrow{\sigma} A^{\infty(I)} \xrightarrow{\varepsilon_0} A \to 0$$
yields the short exact sequence
$$0 \to GL(I(A)) \xrightarrow{GL(\varepsilon_0)} GL(A^{\infty(I)}) \xrightarrow{GL(\sigma)} GL(A) \to 0,$$
since $GL(\varepsilon_0)$ is splitting. Thus $GL(I(A))$ is a normal subgroup of $GL(A^{\infty(I)})$. The assertion follows from the fact that $GL(\varepsilon_1)$ is surjective.

Consider the free cotriple $F = (F, \tau, \delta)$ in the category of rings. Then for any locally convex $k$-algebra $A$ one has a homomorphism $\alpha_0 : F(A) \to I(A)$ given by $|a| \mapsto (t \mapsto at), \quad a \in A, \quad t \in I$. It is clear that the map $t \mapsto at, \quad t \in I$, is a smooth map. We have also the inclusion $\beta_0 : I(A) \to J(A)$, where $J = (J, \tau, \delta)$ is the continuous path cotriple in the category $A$. As a consequence one has morphisms of cotriples
$$F \xrightarrow{\alpha} I \xrightarrow{\beta} J$$
such that the composition $\beta \alpha : F \to J$ is the well-known morphism from the free cotriple to the continuous path cotriple.

For any locally convex $k$-algebra $A$ the groups
$$K_n^{top}(A) = \pi_{n-2} GL(L_\ast^+(A)), \quad n \geq 1,$$
and $K_0^{top}(A) = K_0(A)$ are called the topological $K$-groups of $A$. In fact these topological $K$-groups have been already defined by Swan for any real topological algebra [18]. When $A$ is a Banach $k$-algebra we recover the well-known topological $K$-groups of $A$ [11].
The morphisms $\alpha$ and $\beta$ induce respectively functorial homomorphisms
\[ K^S_n(A) \xrightarrow{\alpha^*_n} K^{sm}_n(A) \xrightarrow{\beta^*_n} K^{top}_n(A) \]
for $n \geq 0$, where $K^S_*$ are Swan’s $K$-functors which are isomorphic to Quillen’s $K$-functors $K_*$. It is clear that the homomorphisms
\[ \alpha^*_1 : K^S_1(A) \to K^{sm}_1(A), \quad \beta^*_1 : K^{sm}_1(A) \to K^{top}_1(A) \]
are surjective and therefore $K^{sm}_1(A)$ is an abelian group.

**Definition 1.3.** A continuous $k$-homomorphism $f : A \to A''$ of locally convex $k$-algebras is called $GL$-fibration with respect to the smooth path cotriple $I$, if the induced homomorphisms
\[ GL(I(A)) \to GL(I(A'')) \]
are surjective for all $i \geq 0$.

It is obvious that any splitting continuous $k$-homomorphism $f : A \to A''$ is a $GL$-fibration with respect to $I$.

Thus the sequence
\[ 0 \to A \xrightarrow{i} A^+ \xrightarrow{p} k \to 0 \]
is a $GL$-fibration with respect to the smooth path cotriple $I$, where $i(a) = (a,0)$ and $p(a,t) = t$, $a \in A$, $t \in k$ and this short exact sequence yields the splitting exact sequence
\[ 0 \to K^{sm}_n(A) \to K^{sm}_n(A^+) \to K^{sm}_n(k) \to 0 \]
for $n \geq 0$.

**Proposition 1.4.** Any short exact sequence of locally convex $k$-algebras
\[ 0 \to A' \xrightarrow{\sigma} A \xrightarrow{\eta} A'' \to 0, \]
where $\eta$ is a $GL$-fibration with respect to the smooth path cotriple $I$, induces a long exact sequence of smooth $K$-functors
\[ ... \to K^{sm}_{n+1}(A'') \to K^{sm}_n(A') \to K^{sm}_n(A) \to K^{sm}_n(A'') \to K^{sm}_{n-1}(A') \to ... \]
\[ ... \to K^{sm}_2(A'') \to K^{sm}_1(A') \to K^{sm}_1(A) \to K^{sm}_1(A'') \to K^0(A') \to K^0(A) \to K^0(A''). \]

**Proof.** The given short exact sequence of locally convex $k$-algebras yields the short exact sequence of augmented simplicial groups
\[ 0 \to GL(I^+_s(A')) \to GL(I^+_s(A)) \to GL(I^+_s(A'')) \to 0 \]
which implies the required long exact sequence of smooth $K$-functors ending with $K_1^{sm}(A)$. It remains to show the exactness of

$$K_1^{sm}(A) \to K_1^{sm}(A') \xrightarrow{\partial_1} K_0(A') \to K_0(A) \to K_0(A').$$

The following commutative diagram

$$
\begin{array}{ccccccc}
GL(I(A)) & \to & GL(I(A')) & \to & 0 \\
\downarrow & & \downarrow & & \\
\vdots & \to & K_1(A) & \to & K_1(A') & \to & K_0(A) \\
\downarrow & & \downarrow & & \\
K_1^{sm}(A') & \to & K_1^{sm}(A) & \to & K_1^{sm}(A') & \to & 0 \\
\downarrow & & \downarrow & & \\
0 & & 0 & & \\
\end{array}
$$

with exact rows and columns defines $\partial_1$ in a natural way and implies the exactness of the remaining sequence. □

An assertion similar to Proposition 1.4 holds also for the topological $K$-functors $K_n^{top}$, $n \geq 0$, with respect to the continuous path cotriple $J$.

It is easy to show that the sequence (1.6)

$$0 \to \Omega_{sm}(A) \to I(A) \xrightarrow{\tau_A} A \to 0$$

is a $GL$-fibration with respect to $I$, where $\Omega_{sm}(A) = \text{Ker } \tau_A$, the homomorphisms $GL_*(A) \to GL_{i+1}(A)$ given by $I_i(\tau_A), \ I_0(\delta_A) = \delta_A, \ I_i(\delta_A) = I(I_{i-1}(\delta_A))$, for $i > 0$, being the splitting homomorphisms for $GL_*(\tau_A)$, $i \geq 0$.

Applying Proposition 1.4 to the sequence 1.6 one obtains the isomorphism $K_2^{sm}(A) \approx K_1^{sm}(\Omega_{sm}(A))$ showing that $K_2^{sm}(A)$ is an abelian group.

**Definition 1.5.** Two continuous $k$-homomorphisms $f, g : A \to B$ are called smooth homotopic if there exists a continuous $k$-homomorphism $h : A \to B^{\infty(I)}$ such that $\varepsilon_0 h = \varepsilon_1 h$, which is called the smooth homotopy between $f$ and $g$.

**Definition 1.6.** A functor $T : \mathcal{A} \to \mathcal{G}r$ is called a smooth homotopy functor if $T(f) = T(g)$ for smooth homotopic continuous $k$-homomorphisms $f$ and $g$.

It is obvious that $T$ is a smooth homotopy functor if $T(\varepsilon_0) = T(\varepsilon_1)$.

The topological $K$-functors $K_n^{top}$ satisfy the condition $K_n^{top}(\varepsilon_0) = K_n^{top}(\varepsilon_1), n \geq 1$, and therefore are smooth $K$-functors for all $n \geq 1$. It is well-known that on the subcategory of Banach $k$-algebras the Grothendieck $K$-functor $K_0$ is a homotopy functor implying by the
same reason that $K_0$ is a smooth homotopy $K$-functor on the category of Banach $k$-algebras.

**Proposition 1.7.** A functor $T : \mathcal{A} \to \mathcal{Gr}$ is a smooth functor if and only if the inclusion $i : A \to A^{\infty(I)}$ induces an isomorphism $T(i) : T(A) \xrightarrow{\cong} T(A^{\infty(I)})$ for all $A \in \text{Ob}\mathcal{A}$.

**Proof.** If $T(i)$ is an isomorphism, then the equality $T(\varepsilon_0)T(i) = T(\varepsilon_1)T(i)$ implies $T(\varepsilon_0) = T(\varepsilon_1)$ and therefore $T$ is a smooth homotopy functor. Conversely, let $T$ be a smooth homotopy functor. Then the continuous $k$-homomorphism $i\varepsilon_0 : A^{\infty(I)} \to A^{\infty(I)}$ is smooth homotopic to the identity map, where the map $A^{\infty(I)} \to (A^{\infty(I)})^{\infty(I)}$, sending the smooth map $f : I \to A$ to the smooth map $\varphi : I \to A^{\infty(I)}$, $\varphi(t)(x) = f(tx)$, $i, x \in I$, provides a smooth homotopy between $i\varepsilon_0$ and $1_{A^{\infty(I)}}$. Thus $T(i)T(\varepsilon_0) = id = T(\varepsilon_1)T(i)$. The proof is complete. \hfill \Box

**Proposition 1.8.** The smooth $K$-functors $K^{sm}_n$, $n \geq 1$, are smooth homotopy functors.

**Proof.** The exact sequence (1.5) applied to the smooth $GL$-fibration

$$0 \to \mathcal{I}(A) \to A^{\infty(I)} \xrightarrow{\varepsilon_0} A \to 0$$

yields the short exact sequence

$$0 \to K^{sm}_n(\mathcal{I}(A)) \to K^{sm}_n(A^{\infty(I)}) \xrightarrow{\varepsilon_0} K^{sm}_n(A) \to 0$$

for $n \geq 1$. It remains to recall that $K^{sm}_n(\mathcal{I}(A)) = 0$, $n \geq 1$, and one obtains the isomorphism $K^{sm}_n(A^{\infty(I)}) \xrightarrow{\cong} K^{sm}_n(A^{\infty(I)})$, $n \geq 1$, showing by Proposition 1.7 that the $K$-functors $K^{sm}_n$, $n \geq 1$, are smooth homotopy functors. \hfill \Box

The smooth homotopization of a functor $T : \mathcal{A} \to \mathcal{Gr}$ is a functor $h^{sm}T : \mathcal{A} \to \mathcal{Gr}$ given by

$$(1.7) \quad h^{sm}T(A) = \text{Coker}(T(A^{\infty(I)}) \Rightarrow T(A))$$

for any $A \in \text{Ob}\mathcal{A}$. The canonical morphism $\eta : T \to h^{sm}T$ is universal for morphisms of $T$ into smooth homotopy functors. It is evident that $h^{sm}T = T$ if and only if $T$ is a smooth homotopy functor. The homotopization $hT$ of a functor $T$ is defined similarly $A^{\infty(I)}$ replaced by $A^I$.

Since $K^{sm}_n$ are smooth homotopy functors, the homomorphism $\alpha_n^*(A) : K_n(A) \to K^{sm}_n(A)$ yields a homomorphism

$$(1.8) \quad h^{sm} \alpha_n^*(A) : h^{sm}K_n(A) \to K^{sm}_n(A), \quad n \geq 1,$$

$A \in \text{Ob}\mathcal{A}$. 
Theorem 1.9. Let $A$ be a locally convex $k$-algebra. Then

(i) there is an isomorphism

$$h^{sm}\alpha_1^*(A) : h^{sm}K_1(A) \xrightarrow{\cong} K_1^{sm}(A),$$

(ii) $\alpha_1^*(A) : K_1(A) \to K_1^{sm}(A)$ is an isomorphism if $K_1(A) \to K_1(A^{\infty(I)})$ is an isomorphism.

Proof. (i) The commutative diagram

$$
\begin{array}{ccc}
GL(\mathcal{I}(A)) & \to & GL(A) \\
\downarrow & & \downarrow \\
h^{sm}K_1(\mathcal{I}(A)) & \to & h^{sm}K_1(A) \\
\end{array}
$$

with exact top row and vertical surjective homomorphisms implies the exactness of the bottom row. The local convex $k$-algebra $\mathcal{I}(A)$ is contractible, the trivial map $0_{\mathcal{I}(A)}$ and the identity map $1_{\mathcal{I}(A)}$ being smooth homotopic with smooth homotopy $\delta_A; \mathcal{I}(A) \to \mathcal{I}^2(A)$ between them. Since $h^{sm}K_1$ is a smooth homotopy functor, one gets $h^{sm}K_1(\mathcal{I}(A)) = 0$. Hence $h^{sm}\alpha_1^*(A)$ is an isomorphism.

(ii) Follows from (i), since in this case $h^{sm}K_1(A) = K_1(A)$. □

It is easily checked that the smooth $K$-functors $K_n^{sm}, \ n \geq 1,$ are compatible with finite products of locally convex $k$-algebras, namely there is a natural isomorphism

$$K_n^{sm} \left( \prod_{i=1}^m A_i \right) \xrightarrow{\cong} \prod_{i=1}^m K_n^{sm}(A_i), \ n \geq 1.$$

Proposition 1.10. (i) Let $T : A \to Gr$ be a functor such that $\text{Im}(T(\varepsilon_0) \times T(\varepsilon_1))$ is a normal subgroup of $T(A) \times T(A), \ A \in \text{Ob}A$. Then there is an isomorphism

$$\psi : h^{sm}T(A) \xrightarrow{\cong} \text{Coker}(T(\varepsilon_0) \times T(\varepsilon_1)),$$

(ii) if an addition $T(0) = \{1\}$, then there is a surjection

$$\text{Coker}(T(\varepsilon_0 \times \varepsilon_1)) \to h^{sm}T(A), \ A \in \text{Ob}A.$$

Proof. Completely similar to the proof of Proposition 6 and Corollary 7 [9] and will be omitted. □

Theorem 1.11. Let $A$ be a locally convex $k$-algebra. Then for fixed $i \geq 0$ there is an isomorphism

$$\alpha_{i+1} : K_{i+1}(A) \to K_{i+1}^{sm}(A),$$

if $K_{i+1}(A) \xrightarrow{\cong} K_{i+1}(A^{\infty(I)})$ and $K_j(B^{\infty(I-j)}) \xrightarrow{\cong} K_j(B^{\infty(I-j+l)})$ with $B = A^2$ for all $0 \leq j \leq i$, $0 \leq l \leq j - 1$. 

Proof. We will give the proof for Swan’s $K$-functors $K_i^*$. The case $i = 0$ is already proved (see Theorem 1.9 (ii)).

Consider the short exact sequence of locally convex $K$-algebras

$$0 \to \Omega_{sm}(A) \to A^{\infty(I)} \xrightarrow{\varepsilon} A \times A \to 0,$$

where the map $GL(K)\ast K$ for $k \geq 1$ we set

$$GL(I_i(A)) \to GL(I_i(A^{\infty(I)})) \to GL(I_i(A)) \times GL(I_i(A)),$$

which is a $GL$-fibration with respect to the smooth path cotriple $I$. In effect, it is easy to see that the image of the composites

$$GL(I_i(A)) \to GL(I_i(A^{\infty(I)})) \to GL(I_i(A)) \times GL(I_i(A)),$$

is, respectively, the diagonal subgroup and the subgroup $1 \times GL(I_i(A))$, where the map $GL(I_i(A)) \to GL(I_i(A^{\infty(I)}))$ is induced by the inclusion $i : A \to A^{\infty(I)}$ and the map $GL(I_i(A^{\infty(I)})) \to GL(I_i(A)) \times GL(I_i(A))$ is induced by $\varepsilon_0 \times \varepsilon_1$. Since the diagonal subgroup and the subgroup $1 \times GL(I_i(A))$ generate the group $GL(I_i(A)) \times GL(I_i(A))$, it follows that $GL(I_i(\varepsilon_0 \times \varepsilon_1))$ is surjective.

Therefore the sequence 1.9 induces the long exact sequence of smooth $K$-functors and yields the following exact sequence

$$0 \to K_i^{\ast sm}(A^{\infty(I)}) \to K_i^{\ast sm}(A \times A) \to K_i^{\ast sm}(\Omega_{sm}(A)) \to 0$$

for $i \geq 1$.

Now we will introduce auxiliary $K$-functors $M_kK_i^*$, $k \geq 0$, $i \geq 1$, needed in the sequel.

Let us define the augmented simplicial rings $M_kF_i^+(A)$, $A \in \text{Ob}\mathcal{A}$, as follows:

$$M_0F_i^+(A) = F_i^+(A), \quad M_1F_i^+(A) = \text{Ker}F_i^+(\varepsilon_0 \times \varepsilon_1).$$

Any continuous $k$-homomorphism $f : A \to C$ of locally convex $k$-algebras induces in a natural way a morphism $M_1F_i^+(f) : M_1F_i^+(A) \to M_1F_i^+(C)$ and one gets a functor $M_1F_i^+$ from the category $\mathcal{A}$ to the category of augmented simplicial rings. For $k > 1$ we set $M_kF_i^+(A) = M_1(M_{k-1}F_i^+(A))$, $A \in \text{Ob}\mathcal{A}$.

Define $M_kK_i^*(A) = \pi_{i-2}GL(M_kF_i^+(A))$ for $i \geq 1$, $k \geq 0$.

It is clear that any continuous $k$-homomorphism $f : A \to C$ of locally convex $k$-algebras induces in a natural way a morphism $M_kF_i^+(f) : M_kF_i^+(A) \to M_kF_i^+(C)$ for all $k \geq 0$ $j \geq 0$, where $\{M_kF_i^+(f)\}_j = M_kF_i^+(f)$. It is also obvious that every augmented simplicial ring $M_kF_i^+(A)$, $k \geq 0$, is a simplicial resolution of $\Omega_{sm}^k(A) = \Omega_{sm}(\Omega_{sm}^{k-1}(A))$.

Let $f : A \to C$ be a continuous $k$-homomorphism of locally convex $k$-algebras such that there exists a $k$-linear continuous linear map $f' : C \to A$ with $ff' = 1$. It is easily checked that $f'$ induces $k$-linear
continuous maps $\Omega_{sm}(f') : \Omega_{sm}(C) \to \Omega_{sm}(A)$ and $f^{\infty(I)} : C^{\infty(I)} \to A^{\infty(I)}$ such that $\Omega_{sm}(f)\Omega_{sm}(f') = 1$ and $f^{\infty(I)}f^{\infty(I)} = 1$. It is also clear that the $k$-linear continuous map $f'$ gives rise to a homomorphism of rings $M_kF_j(f') : M_kF_j(C) \to M_kF_j(A)$ such that $M_kF_j(f)M_kF_j(f') = \text{identity}$ for all $k, j \geq 0$. Therefore the induced homomorphism

$$GL(M_kF_j(f)) : GL(M_kF_j(A)) \to GL(M_kF_j(C))$$

is surjective for all $k, j \geq 0$ and one obtains a long exact sequence

$$\ldots \to M_kK_{i+1}^s(C) \to M_kK_i^s(A, \text{Ker}(f)) \to M_kK_i^s(A) \to M_kK_i^s(C) \to M_kK_{i-1}^s(A, \text{Ker}(f)) \to \ldots$$

for all $k \geq 0$, where

$$M_kK_i^s(A, \text{Ker}(f)) = \pi_{i-2}GL(\text{Ker}(M_kF_i^s(f))), \quad i \geq 1.$$

We can apply this exact sequence to the homomorphism $\varepsilon_0 \times \varepsilon_1 : A^{\infty(I)} \to A \times A$ of the sequence (1.9). In effect, there is a $k$-linear continuous map $\varepsilon' : A \times A \to A^{\infty(I)}$ given by $(a, a') \mapsto (1 - t)a + ta'$ such that $(\varepsilon_0 \times \varepsilon_1)\varepsilon' = 1_{A \times A}$. Thus the sequence (1.9) yields the following long exact sequence

$$(1.10) \quad \ldots \to M_{j+1}K_{i+1}^s(A) \to M_jK_{i+1}^s(A \times A) \to M_{j+1}K_i^s(A) \to M_jK_i^s(A \times A) \to \ldots$$

for all $j \geq 0$.

The morphism $\alpha : F \to I$ of cotriples induces a morphism $F_* : \mathcal{L}_e(A) \to \mathcal{L}_e(A)$ which yields a morphism $M_jF_* : \mathcal{L}_e(\Omega_{sm}^j(A))$ for all $j \geq 0$ implying isomorphisms

$$h^{sm}(M_jK_{i+1}^s(A)) \approx K_{i+1}^s(\Omega_{sm}^j(A)) \approx K_{i+1}^s(A)$$

for any locally convex $k$-algebra $A$ and $j \geq 0$.

By Proposition (1.10), one has a natural surjection

$$\text{Coker}(M_jK_i^s(\varepsilon_0 \times \varepsilon_1)) \to h^{sm}(M_jK_i^s(A))$$

for all $j \geq 0, i \geq 1$.

Thus for any $j$ satisfying $0 \leq j \leq i - 1$ the exact sequence (1.10) provides the following commutative diagram:

$$\begin{array}{ccc}
M_{j+1}K_{i-j}^s(A) & \to & h^{sm}(M_jK_{i-j+1}(A)) \\
\downarrow & & \downarrow \\
K_{i-j}^s(\Omega_{sm}^{j+1}(A)) & \cong & K_{i+1}^s(\Omega_{sm}^j(A))
\end{array}$$

if $M_jK_{i-j}^s(A) \to M_jK_{i-j}^s(A^{\infty(I)})$ is an isomorphism, where the top homomorphism is surjective.
It follows that if \( M_jK^s_{i-j}(A) \rightarrow M_jK^s_{i-j}(A^\infty(I^2)) \) is an isomorphism for \( 0 \leq j \leq i - 1 \), then the diagram (1.11) holds also for \( A^\infty(I) \) and \( 0 \leq j \leq i - 1 \). It is easy to verify that in this case the diagram (1.11) yields the commutative diagram

\[
\begin{array}{ccc}
h^{sm}(M_{j+1}K^s_{i-j})(A) & \rightarrow & h^{sm}(M_jK_{i-j+1})(A) \\
\downarrow & & \downarrow \\
K^s_{i-j}(\Omega^{j+1}_{sm}(A)) & \cong & K^{sm}_{i+1}(A)
\end{array}
\]

(1.12)

for all \( j \) satisfying \( 0 \leq j \leq i - 1 \), where the top homomorphism is surjective.

Since \( h^{sm}(M_iK^s_1)(A) \rightarrow K^{sm}_{i+1}(A) \) is an isomorphism, the diagram

\[
\begin{array}{ccc}
h^{sm}(M_iK^s_1)(A) & \rightarrow & h^{sm}(M_{i-1}K^s_2)(A) \\
\downarrow & & \downarrow \\
K^s_{i}(\Omega^{i-1}_{sm}(A)) & \cong & K^{sm}_{i+1}(A)
\end{array}
\]

(1.13)

implies the isomorphism \( h^{sm}(M_{i-1}K^s_2)(A) \rightarrow K^{sm}_{i+1}(A) \).

Thus the diagram

\[
\begin{array}{ccc}
h^{sm}(M_{i-1}K^s_2)(A) & \rightarrow & h^{sm}(M_{i-2}K^s_3)(A) \\
\downarrow & & \downarrow \\
K^s_{2}(\Omega^{i-1}_{sm}(A)) & \cong & K^{sm}_{i+1}(A)
\end{array}
\]

(1.14)

implies the isomorphism \( h^{sm}(M_{i-2}K^s_3)(A) \rightarrow K^{sm}_{i+1}(A) \).

Continuing in this manner step by step we arrive to the diagram

\[
\begin{array}{ccc}
h^{sm}(M_1K^s_1)(A) & \rightarrow & h^{sm}(K^s_{i+1})(A) \\
\downarrow & & \downarrow \\
K^s_{1}(\Omega_{sm}(A)) & \cong & K^{sm}_{i+1}(A),
\end{array}
\]

(1.15)

where the left vertical homomorphism is an isomorphism and the top homomorphism is surjective. Hence \( h^{sm}K^s_{i+1}(A) \rightarrow K^{sm}_{i+1}(A) \) is an isomorphism, if \( M_jK^s_{i-j}(A) \rightarrow M_jK^s_{i-j}(A^\infty(I^2)) \) is an isomorphism for \( 0 \leq j \leq i - 1 \).

Now conditions on the algebraic \( K \)-functors \( K^s_j \) will be given implying these conditions and therefore the isomorphism \( h^{sm}K^s_{i+1}(A) \rightarrow K^{sm}_{i+1}(A) \).

To this end let us consider the following commutative diagram with exact rows

\[
\begin{array}{cccc}
M_jK^s_{i+1}(A^2) & \rightarrow & M_{j+1}K^s_i(A) & \rightarrow & M_jK^s_i(A^\infty(I)) & \rightarrow & M_jK^s_i(A^2) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_jK^s_{i+1}(A^2)^{\infty(I^2)} & \rightarrow & M_{j+1}K^s_i(A^\infty(I^2)) & \rightarrow & M_jK^s_i(A^\infty(I^2)) & \rightarrow & M_jK^s_i((A^2)^{\infty(I^2)})
\end{array}
\]

(1.16)

induced by the exact sequence (1.10).
The diagram \[1.16\] shows that for \( j = 0 \) one has \( M_i K^s_i(A) \xrightarrow{\approx} M_i K^s_i(A^{\infty(I^2)}) \) if

1. \( K^s_{i+1}(A^2) \xrightarrow{\approx} K^s_{i+1}((A^2)^{\infty(I^2)}), \)
2. \( K^s_i(A^{\infty(I)}) \xrightarrow{\approx} K^s_i(A^{\infty(I^2)}). \)

Similarly the diagram \[1.16\] shows that for \( j = 1 \) one has \( M_i K^s_i(A) \xrightarrow{\approx} M_i K^s_i(A^{\infty(I^2)}) \) if \( M_i K^s_{i+1}(A^2) \xrightarrow{\approx} M_i K^s_{i+1}((A^2)^{\infty(I^2)}) \) and \( M_i K^s_i(A^{\infty(I)}) \xrightarrow{\approx} M_i K^s_i(A^{\infty(I^2)}) \). Thus one has an isomorphism \( M_i K^s_i(A) \xrightarrow{\approx} M_i K^s_i(A^{\infty(I^2)}) \) if

1. \( K^s_i(A^{\infty(I^2)}) \xrightarrow{\approx} K^s_i(A^{\infty(I^4)}), \)
2. \( K^s_{i+1}((A^2)^{\infty(I)}) \xrightarrow{\approx} K^s_{i+1}((A^2)^{\infty(I^3)}), \)
3. \( K^s_i(A^4) \xrightarrow{\approx} K^s_i((A^4)^{\infty(I^2)}). \)

By induction on \( j \) it is easily checked that one has an isomorphism

\[ M_j K^s_i(A) \xrightarrow{\approx} M_j K^s_i(A^{\infty(I^2)}), \quad j \geq 0, \]

if the following isomorphisms hold:

\[ K^s_i(A^{\infty(I)}) \xrightarrow{\approx} K^s_i(A^{\infty(I^2)}), \]
\[ K^s_{i+1}((A^2)^{\infty(I^1)}) \xrightarrow{\approx} K^s_{i+1}((A^2)^{\infty(I^1)}), \]
\[ K^s_i((A^4)^{\infty(I)} \xrightarrow{\approx} K^s_i((A^4)^{\infty(I^2)}), \]

\[ K^s_i(A^2^l) \xrightarrow{\approx} K^s_i((A^2^l)^{\infty(I^2)}), \]

or simply if

\[ K^s_i((A^2^l)^{\infty(I^1)}) \xrightarrow{\approx} K^s_i((A^2^l)^{\infty(I^1+2)}), \]

for all \( l \) satisfying \( 0 \leq l \leq j \).

We deduce that there is an isomorphism

\[ h^{sm} K^s_{i+1}(A) \rightarrow K^s_{i+1}(A) \]

if

\[ K^s_i((A^2^l)^{\infty(I^1)}) \xrightarrow{\approx} K^s_i((A^2^l)^{\infty(I^1+2)}), \]

for \( 1 \leq j \leq i, \ 0 \leq l \leq j - 1. \)

To these conditions it suffices to add the isomorphism \( K^s_{i+1}(A) \rightarrow K^s_{i+1}(A^{\infty(I)}) \) to obtain the required isomorphism

\[ K^s_{i+1}(A) \rightarrow K^s_{i+1}(A). \]

This completes the proof. \( \square \)
Corollary 1.12. Let $A$ be a locally convex $k$-algebra. Assume that Quillen’s $K$-functors $K_i$, $i \geq 1$, are compatible with finite products of $A$ and $A^{\infty I}$ on themselves. Then there is an isomorphism

$$\alpha_* : K_*(A) \to K_*^{sm}(A),$$

if $K_*(A) \approx K_*(A^{\infty I^n})$ for all $n > 0$.

Theorem 1.13. Let $A$ be a locally convex $k$-algebra. Then for fixed $i \geq 0$ there is an isomorphism

$$\beta_{i+1}(A) : K_{i+1}^{sm}(A) \approx K_{i+1}^{top}(A),$$

if

$$K_{i+1}^{sm}(A) \approx K_{i+1}^{sm}(A^I),$$

$$K_{j}^{sm}(A^{I_{i-j}}) \approx K_{j}^{sm}(A^{I_{i-j+2}})$$

for all $0 \leq j \leq i$.

Proof. Similar to the proof of Theorem 1.11. In effect, the sequence

(1.17) $0 \to \Omega A \to A^I \xrightarrow{\varepsilon_0 \times \varepsilon_1} A \times A \to 0$

is a GL-fibration with respect to the smooth path cotriple that follows from the natural commutative diagram

$$\begin{array}{ccc}
A^{\infty I} & \xrightarrow{\varepsilon_0 \times \varepsilon_1} & A \times A \\
\sigma \downarrow & & \downarrow \\
A^I & \xrightarrow{\varepsilon_0 \times \varepsilon_1} & A \times A,
\end{array}$$

where $\Omega(A) = Ker \tau_A$, $\tau_A : J(A) \to A$ and $\sigma$ is the natural inclusion, the continuous $k$-homomorphism $\varepsilon_0 \times \varepsilon_1$ being a GL-fibration with respect to the smooth path cotriple $I$. Replace the auxiliary $K$-functors $M_jK^*_i(-)$ by the $K$-functors $K^{sm}_i(\Omega^{j}_{sm}(-))$, use the natural isomorphisms

$$K^{sm}_*((A \times A)^{I^n}) \approx K^{sm}_*(A^{I^n}) \times K^{sm}_*(A^{I^n}), n \geq 0,$$

the canonical isomorphism $hK^{sm}_1 \approx K^{top}_1$, where $hK^{sm}_1$ is the continuous homotopyzation of the functor $K^{sm}_1$, and note that in this case $M_j\mathcal{I}^+_s(A) = Ker M_{j-1}\mathcal{I}^+_s(\bar{\varepsilon}_0 \times \bar{\varepsilon}_1)$ is the $\mathcal{I}$-projective cotriple resolution of $\Omega^{j}_{sm}(A)$ for all $j \geq 1$. \hfill $\square$

A Fréchet $k$-algebra whose seminorms possess the multiplicative property will be called Fréchet-Michael $k$-algebra. It is easy to see that the smooth and continuous cotriples are both defined on the category of Fréchet-Michael $k$-algebras and their continuous $k$-homomorphisms.
Theorem 1.14. Let $A$ be a complex Fréchet-Michael algebra. Then there is an isomorphism

$$\beta_n(A) : K_n^{sm}(A) \cong K_n^{top}(A)$$

for all $n \geq 0$.

Proof. The case $n = 0$ is trivial and it suffices to prove the theorem for the unital case.

First it will be shown that $\beta_1(A) : K_1^{sm}(A) \cong K_1^{top}(A)$.

Let $f : A \to B$ be a surjective homomorphism of arbitrary complex Fréchet-Michael algebras. According to Lemma 1.14 [15] the induced map $GL^0(f) : GL^0(A) \to GL^0(B)$ is surjective, where $GL^0(-)$ denotes the component of the unit of $GL(-)$. Therefore the canonical map $\tau_A : \mathcal{I}(A) \to A$ induces a surjective map $GL^0(\tau_A) : GL^0(\mathcal{I}(A)) \to GL^0(A)$. On the other hand $\mathcal{I}(A)$ is continuously contractible. In effect, take the map $\delta_A : \mathcal{I}(A) \to (\mathcal{I}(A))^{\infty(I)}$ which is the composite of

$$\mathcal{I}(A) \xrightarrow{\delta_A} \mathcal{I}^2(A) \to (\mathcal{I}(A))^{\infty(I)}.$$ 

One has the following commutative diagram

$$\begin{array}{ccc}
\mathcal{I}(A) & \xrightarrow{\delta_A} & (\mathcal{I}(A))^{\infty(I)} \\
\theta \downarrow & & \downarrow \xi_i \\
\mathcal{I}(A) & \xrightarrow{\theta \delta_A} & (\mathcal{I}(A))^I \\
\end{array}$$

(1.18)

$i = 0, 1$, where $\xi_i = f(i)$ and $\theta$ is the natural inclusion. Diagram (1.18) implies the equalities $\xi_0 \theta \delta_A = 0$ and $\xi_1 \theta \delta_A = 1_{\mathcal{I}(A)}$, showing that $\mathcal{I}(A)$ is a contractible Fréchet-Michael algebra. Thus, $GL(\mathcal{I}(A))$ is a connected space and therefore one has a surjection $GL(\tau_A) : GL(\mathcal{I}(A)) \to GL^0(A)$. We deduce

$$K_1^{sm}(A) = GL(A)/\text{Im}GL(\tau_A) = GL(A)/GL^0(A) = K_1^{top}(A)$$

for any complex Fréchet-Michael algebra $A$.

To obtain the required isomorphism in higher dimensions it should be noted that any surjective homomorphism $f : A \to B$ of complex Fréchet-Michael algebras is a $GL$-fibration with respect to the continuous path cotriple. In fact by Lemma 1.9 [15] one has surjections $J_n(A) \to J_n(B)$, $n \geq 0$, induced by $f$, and therefore the induced map $GL^0(J_n(f)) : GL^0(J_n(A)) \to GL^0(J_n(B))$ is surjective for all $n \geq 0$. Since $J_n(A)$ and $J_n(B)$ are contractible complex Fréchet-Michael algebras, $GL(J_n(A))$ and $GL(J_n(B))$ are connected spaces. We conclude that the homomorphism $GL(J_n(f))$ is surjective for all $n \geq 0$ and by definition $f$ is a $GL$-fibration with respect to the path cotriple $J$. 


The smooth and continuous contractibility of $\mathcal{I}(A)$ implies $K_n^{sm}(\mathcal{I}(A)) = K_n^{top}(\mathcal{I}(A)) = 0$ for $n > 0$ and any complex Fréchet-Michael algebra $A$. It remains to apply to the short exact sequence (1.6) of complex Fréchet-Michael algebras the long exact sequence of smooth and topological $K$-functors respectively and one obtains finally the following isomorphisms

$$K_n^{sm}(A) \approx K_1^{sm}(\Omega^{n-1}_{sm}(A)) \approx K_1^{top}(\Omega^{n-1}_{sm}(A)) \approx K_n^{top}(A)$$

for $n > 1$ and any complex Fréchet-Michael algebra $A$. This completes the proof. □

**Corollary 1.15.** Let $A$ be a complex Fréchet-Michael algebra such that $GL_1(A^+)$ is open in $A^+$. Then the $K$-groups $K_n^{sm}(A)$, $K_n^{top}(A)$ and $RK_n(A)$ (defined in [15]) are isomorphic for all $n \geq 0$. In particular, on the category of complex Banach algebras the smooth $K$-theory is isomorphic to the well-known topological $K$-theory of Banach algebras.

We can interpret smooth $K$-functors $K_*^{sm}$ as "smooth homotopy groups" of the general linear group.

The group $GL_m(A) = \text{Ker}(GL_m(A^+) \to GL_m(k))$ with topology induced from $M_m(A^+)$ in general is not a topological group, since the map $(-)^{-1}: GL_m(A) \to GL_m(A)$ is not always continuous, depending on the locally convex $k$-algebra $A$.

Denote by $R^n GL_m(A)$ the group of all continuous maps $f : I^n \to GL_m(A)$ satisfying the following conditions:

(i) $f(t_1, t_2, ..., t_n) = 1$ if any $t_i = 0$;

(ii) the composite maps $\varphi_{ij}.f : I^n \to A$ are smooth maps for all $1 \leq i, j \leq m$, where the map $\varphi_{ij} : GL_m \to A$ is given by $\varphi_{ij}(M) = a_{ij}$ with $a_{ij}$ the element of $A$ staying at the intersection of the $i$-th row and the $j$-th column of the matrix $M$;

(iii) the map $f^{-1} : I^n \to GL_m(A)$ given by $f^{-1}(t_1, t_2, ..., t_n) = (f(t_1, t_2, ..., t_n))^{-1}$ is continuous and satisfies condition (ii).

Using the homotopy groups of the simplicial group $R_*GL_m(A)$, whose boundary and degeneracy maps coincide with those of $GL_m(I_*(A))$, we define by setting

$$\pi_n^{sm}(GL_m(A)) = \pi_{n-1}R_*GL_m(A), \ n \geq 2,$$

and by the exact sequence

$$0 \to \pi_1^{sm}GL_m(A) \to \pi_0R_*GL_m(A) \to GL_m(A) \to \pi_0^{sm}GL_m(A) \to 0.$$

Finally, define

$$\pi_n^{sm}GL(A) = \lim_{\rightarrow} \pi_n^{sm}GL_m(A)$$
for \( n \geq 0 \). It is obvious that one has
\[
K_{n}^{sm}(A) = \pi_{n-1}^{sm}GL(A), \quad n \geq 1.
\]

2. The Cohen-Hewitt factorization for Fréchet\emph{k}-algebras

It is intended to extend the Cohen-Hewitt theorem factorization theorem to the category of Fréchet \( k \)-algebras which will be used in the sequel.

We will assume that \( A \) is a \( k \)-algebra, \( L \) is a left \( A \)-module equipped with a \( k \)-linear space structure and
\[
(t\mu)x = t(\mu x) = \mu(tx)
\]
for \( t \in k, \mu \in A \) and \( x \in L \).

Particularly we will consider \( A \) as a Banach \( k \)-algebra and \( L \) as a Fréchet space. Since \( L \) is a Fréchet space, its topology is given by an increasing sequence of seminorms \( || \cdot ||_{n} \) satisfying the following conditions:

- For any seminorm \( || \cdot ||_{n} \) there exist a number \( C \geq 1 \) independent of \( n \) and a seminorm \( || \cdot ||_{m(n)} \) such that
\[
||\mu x||_{n} \leq C||\mu|| \cdot ||x||_{m(n)}
\]
for all \( \mu \in A, x \in L \) and \( n \). It is obvious that the module structure \( A \times L \to L \) is jointly continuous. Therefore \( L \) is a topological \( A \)-module which will be called a Fréchet module over a Banach \( k \)-algebra.

Thus a metric \( \rho \) compatible with the Fréchet topology can be defined on \( L \) given by
\[
\rho(x, y) = \sup_{n=1}^{\infty} \frac{1}{2^n} \frac{||x - y||_n}{1 + ||x - y||_n}.
\]

**Definition 2.1.** Let \( L \) be a Fréchet module over a Banach \( k \)-algebra \( A \). It will be said that \( L \) possesses a left bounded approximate unit (bounded by a positive constant \( d \)) in \( A \), if for any finite subset \( \{\mu_1, \ldots, \mu_m\} \subset A \), any element \( x \in L \) and \( \varepsilon > 0 \) there exists \( \nu \in A \) such that
\[
||\nu|| \leq d, \quad ||\nu \mu_i - \mu_i|| < \varepsilon, \quad \rho(\nu x; x) < \varepsilon.
\]

This is equivalent to the existence of a bounded direct set \( \{\nu_{\alpha}\} \) in \( A \) such that
\[
\lim_{\alpha} \nu_{\alpha} \mu = \mu \quad \text{and} \quad \lim_{\alpha} (\nu_{\alpha} x) = x,
\]
\( \mu \in A, x \in L \).
Denote by $A^+$ the Banach algebra $k$-algebra $A + k = \{(a, t) \mid a \in A, t \in k\}$ with usual sum, with product given by $(a, t)(a', t') = (aa' + t'a + ta', tt')$ and with norm $\|(a, t)\| = |a| + |t|$.

For $\mu \in A$ with $\|\mu\| \leq d$ and $d \geq 1$, define an element $\varphi(\mu)$ of $A^+$ by

$$\varphi(\mu) = \frac{2d + 1}{2d} \left(1 + \sum_{k=1}^{\infty} (-1)^k (2d)^{-k} \mu^k\right).$$

Then one has (see [6])

$$\varphi(\mu) = \left[\frac{2d}{2d + 1} + \frac{1}{2d + 1} \mu\right]^{-1} \text{ and } \frac{2}{3} + d^{-1} \leq \|\varphi(\mu)\| \leq 2 + d^{-1}.$$ 

The following lemma is a generalization of Lemma 2.1 [6].

**Lemma 2.2.** Let $L$ be a Fréchet module over a Banach $k$-algebra. Then for any element $\mu \in A$ with $\|\mu\| \leq d \geq 1$ one has

$$\rho(\varphi(\mu)x; x) \leq C(2 + d^{-1})\rho(\mu x; x).$$

**Proof.** The following inequality is a consequence of the inequality (2.1) and the properties (2.5):

$$\|\varphi(\mu)x - x\|_n = \|\varphi(\mu)x - \varphi(\mu) \cdot \left(\frac{2d}{2d + 1} + \frac{1}{2d + 1} \mu\right)x\|_n \leq \|\varphi(\mu)\| \cdot \|\mu x - x\|_m(n) \leq C(2 + d^{-1})\|\mu x - x\|_m(n).$$

Since $(2 + d^{-1})C \geq 1$, one has

$$\frac{\|\varphi(\mu)x - x\|_n}{1 + \|\varphi(\mu)x - x\|_n} \leq C(2 + d^{-1}) \cdot \frac{\|\mu x - x\|_m(n)}{1 + \|\mu x - x\|_m(n)}.$$ 

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\|\varphi(\mu)x - x\|_n}{1 + \|\varphi(\mu)x - x\|_n} \leq C(2 + d^{-1}) \cdot \sum_{i=1}^{\infty} \frac{1}{2^i} \cdot \frac{\|\mu x - x\|_i}{1 + \|\mu x - x\|_i}.$$ 

Recall the definition of the metric $\rho$ to obtain the required inequality. \hfill $\square$

Let $d > 0$ and $\{\mu_n\}$ be a sequence of elements of $A$ such that $\|\mu_n\| \leq d, \ n = 1, 2, \ldots$. Define an element $\sigma_n \in A^+$ by

$$\sigma_n = \sum_{k=1}^{n} (2d)^{k-1} (2d + 1)^{-k} \mu_k + (2d)^n (2d + 1)^n$$
for \( n = 1, 2, ..., \) and \( \sigma_0 = 1 \).

It is known that \( \sigma_n \) is invertible in \( A^+ \).

**Proposition 2.3.** Let \( L \) be a Fréchet module over a Banach \( k \)-algebra \( A \) having a bounded approximate unit in \( A \), and let \( z \in L \). Let \( d \) be an upper bound on the norms of approximate units in \( A \). Then for \( \varepsilon > 0 \) there exists a sequence \( \mu_n \) of elements of \( A \) with \( ||\mu_n|| \leq d \), such that the following inequality holds

\[
\rho(\sigma_n^{-1} \cdot z; \sigma_{n-1}^{-1} \cdot z) \leq \frac{\varepsilon}{2^n}.
\]

**Proof.** Since \( A \) has an approximate unit bounded by \( d > 0 \), for \( \varepsilon > 0 \) there exists \( \mu_1 \in A \) with \( ||\mu_1|| \) such that

\[
\rho(\mu_1 z; z) \leq (2C(2 + d^{-1}))^{-1}\varepsilon.
\]

Then

\[
\sigma_1 = \frac{1}{2d + 1} \mu_1 + \frac{2d}{2d + 1}
\]

and \( \sigma_1^{-1} = \varphi(\mu_1) \).

By Lemma 2.2,

\[
\rho(\sigma_1^{-1} z; \sigma_0^{-1}) \leq C(2 + d^{-1})\rho(\mu_1 z; z) \leq \frac{\varepsilon}{2}.
\]

That means Proposition 2.3 is true for \( n = 1 \).

Suppose Proposition 2.3 holds for \( n = m \) and it will be shown that it remains true for \( n = m + 1 \).

Let us consider an element \( \mu' \in A \) such that \( ||\mu'|| \leq d \) and define for \( \varepsilon > 0 \)

\[
\sigma'_{m+1} = \sum_{k=1}^{m} (2d)^{k-1}(2d + 1)^{-k} \mu_k + (2d)^m(2d + 1)^{-m} \mu' + (2d)^{m+1}(2d + 1)^{-m-1}.
\]

It is clear that (2.9) yields \( \sigma_{m+1} \) when \( \mu' \) is replaced by \( \mu_{m+1} \). Rewrite (2.9) in the following form

\[
\sigma'_{m+1} = \left( \frac{1}{2d + 1} \mu' + \frac{2d}{2d + 1} \right) \tau_m,
\]

where

\[
\tau_m = \sum_{k=1}^{m} (2d)^{k-1}(2d + 1)^{-k} \varphi(\mu') \mu_k + (2d)^m(2d + 1)^{-m}.
\]

The element \( \mu' \) can be chosen such that \( \tau_m \) becomes invertible in \( A^+ \). In effect, since the group of invertible elements in \( A^+ \) is an open subset and the map \( x \mapsto x^{-1} \) is an homeomorphism, for \( ||\mu' \mu_k - \mu_k|| \) sufficiently small (this can be realized \( A \) having a bounded approximate unit)
\[ ||\tau_m - \sigma_m|| \text{ is sufficiently small and therefore } \tau_m \text{ is invertible. It follows that } ||\tau_m^{-1} - \sigma_m^{-1}|| \text{ is also arbitrary small. Further, } \mu' \text{ can be chosen such that } \rho(\mu' z; z) \text{ will be sufficiently small. For such } \mu' \text{ one has} \]

\[ (2.11) \quad \rho((\sigma_{m+1}^{-1})z; \sigma_m^{-1}z) = \rho(\tau_m^{-1}\varphi(\mu')z; \sigma_m^{-1}z) \leq \rho(\tau_m^{-1}\varphi(\mu')z; \tau_m^{-1}z) + \rho(\tau_m^{-1}z; \sigma_m^{-1}z). \]

By Lemma 2.2 combined with above, \( \rho(\varphi(\mu')z; z) \) is arbitrary small. Hence, since \( \tau_m^{-1} : L \to L \) is a homeomorphism given by \( x \mapsto \tau_m^{-1}x \), this implies that

\[ \rho((\tau_m^{-1}\varphi(\mu')z; \tau_m^{-1}z) \]

is also arbitrary small.

We have also to show that the second summand \( \rho((\tau_m^{-1}z; \sigma_m^{-1}z) \) of (2.11) is arbitrary small. In fact, any \( z \in L \) induces a continuous map \( A \to L \) of metric spaces given by \( a \mapsto az \). Hence, if \( ||\tau_m^{-1}z - \sigma_m^{-1}z|| \) is sufficiently small, then \( \rho((\tau_m^{-1}z; \sigma_m^{-1}z) \) is also sufficiently small. Therefore the element \( \mu_{m+1} = \mu' \) can be chosen such that

\[ (2.12) \quad \rho(\sigma_{m+1}^{-1}z; \sigma_m^{-1}z) \leq \frac{\varepsilon}{2^{m+1}}. \]

This completes the proof. \( \square \)

Now we are ready to prove the theorem generalizing Cohen-Hewitt’s result (Theorem 2.5 [4])

**Theorem 2.4.** Let \( L \) be a Fréchet module over a Banach \( k \)-algebra \( A \) having a bounded approximate unit bounded by a number \( d > 0 \). Then for \( z \in L \) and \( \varepsilon > 0 \) there exist elements \( \sigma \in A \) and \( y \in L \) with the following properties

(i) \( z = \sigma y \);
(ii) \( y \in A \cdot z \) (closure in \( L \));
(iii) \( \rho(y; z) \leq \varepsilon \);
(iv) \( ||\sigma|| \leq d \).

**Proof.** Take \( \sigma_n \) defined as above (see equality 2.4) which is invertible in \( A^+ \). Consider the elements \( y_n = \sigma_n z, \ n = 1, 2, \ldots \) By summing the inequality

\[ (2.13) \quad \rho(\sigma_n^{-1}z; \sigma_{n-1}^{-1}z) \leq \frac{\varepsilon}{2^n} \]

from \( m \) to \( m + k \), one obtains the inequality

\[ \rho(y_m; y_{m+k}) \leq \frac{\varepsilon}{2^m}, \]

showing that the sequence \( \{y_n\} \) is a Cauchy sequence and denote \( y = \lim y_n \).
Since \( L \) possesses an approximate unit, one has \( z \in \overline{Az} \) implying that the elements \( y_n \) and \( y \) belong to \( \overline{Az} \). By summing the inequality \[2.13\] from 1 to \( m \), and taking into account that \( \sigma_0 = 1 \), one gets the inequality \( \rho(y_m; z) < \varepsilon \) and therefore
\[
\rho(y; z) \leq \varepsilon.
\]

By the definition of \( \sigma_n \) it follows that \( \lim_{n \to \infty} \sigma_n \) exists in \( A^+ \) and in fact it is an element \( \sigma \) of \( A \),
\[
\sigma = \sum_{k=1}^{\infty} (2d)^{k-1}(2d+1)^k \mu_k.
\]
Clearly \( ||\sigma|| \leq d \). Since the module map is jointly continuous, this implies
\[
z = \lim_{n \to \infty} (\sigma_n \cdot y_n) = \lim_{n \to \infty} \sigma_n \cdot \lim_{n \to \infty} y_n = \sigma \cdot y.
\]
This completes the proof. \( \square \)

3. Homotopy invariance in \( K \)-homology and Higson’s Theorem

The purpose of this section is, according to the homotopy invariance of \( K \)-homology, to present Higson’s homotopy invariance theorem for both real and complex cases. Higson’s theorem asserts

**Theorem 3.1.** ([7] (Theorem 3.2.2)) Let \( E \) be a stable and split additive functor from an admissible sub-category \( S \) of the category of complex \( C^* \)-algebras into the category of abelian groups. Then it is homotopy invariant.

Higson’s proof is a consequence of the following proposition (cf. Theorem 3.1.4 in [7].)

**Proposition 3.2.** Let \( E \) be a functor from the category \( S \) into the category \( Ab \) admitting a pairing with the set of Fredholm \( B \)-pairs, \( B \in \text{Ob}S \). Then \( E \) is a homotopy functor.

We recall the definition of a pairing of a functor \( E : S \to Ab \) with the set of Fredholm pairs, defined in [7], where \( S \) is an admissible subcategory of the category \( C^* \)-algebras. A subcategory \( S \) of the category of \( C^* \)-algebras and \( * \)-homomorphisms is said to be admissible if

1. \( k \) belongs to \( S \);
2. if \( A \) belongs to \( S \), then so is \( A \otimes k^! \);
3. if \( A \) belongs to \( S \), then so is \( A \otimes K \);
4. if \( 0 \to A \to B \to C \to 0 \) is a split exact sequence with \( A \) and \( C \) in \( S \), then so is \( B \).
Here ”⊗” is the minimal $C^*$-tensor product. A Fredholm $B$-pair is a pair $(\varphi, \psi)$ of $\ast$-homomorphisms from $B$ into $\mathcal{L}_k(\mathcal{H})$ such that $\varphi(b) - \psi(b) \in \mathcal{K}(\mathcal{H})$ for any $b \in B$, where $\mathcal{H}$ is a countably generated Hilbert space over $k$. Here $\mathcal{K}(\mathcal{H})$ is the $C^*$-algebra of compact operators. A pairing of $E$ with the set of Fredholm $B$-pairs assigns to each Fredholm $B$-pair $(\varphi, \psi)$ a homomorphism $\times(\varphi, \psi) : E(A \otimes B) \to E(A \otimes k)$ for any $A, B \in \text{Ob}\mathcal{S}$ with the following properties:

1. **Functoriality.** If $(\varphi, \psi)$ is a Fredholm $B'$-pair and if $f : B \to B'$ is a $\ast$-homomorphism of $\mathcal{S}$, then the diagram

\[
\begin{array}{ccc}
E(A \otimes B) & \xrightarrow{\times(\varphi f, \psi f)} & E(A \otimes k) \\
E(id_A \otimes f) \downarrow & & \downarrow 1 \\
E(A \otimes B') & \xrightarrow{\times(\varphi, \psi)} & E(A \otimes k)
\end{array}
\]

commutes.

2. **Additivity.** If $(\varphi, \chi)$ and $(\chi, \psi)$ are Fredholm $B$-pairs, then

\[\times(\varphi, \chi) + \times(\chi, \psi) = \times(\varphi, \psi).\]

3. **Stability.** If $(\varphi, \psi)$ is a Fredholm $B$-pair and $\eta : B \to \mathcal{L}_k(\mathcal{H})$ is any $\ast$-homomorphism, then

\[\times(\varphi, \psi) = \times\left(\begin{pmatrix} \varphi & 0 \\ 0 & \eta \end{pmatrix}, \begin{pmatrix} \psi & 0 \\ 0 & \eta \end{pmatrix}\right).\]

4. **Non-degeneracy.** If $(e, \theta)$ is a Fredholm $B$-pair, $e : k \to \mathcal{K}(\mathcal{H})$ maps $1 \in k$ to $p$, where $p$ is a rank one projection in $\mathcal{K}(\mathcal{H})$ and $\theta$ is the zero homomorphism, then

\[\times(e, \theta) : E(A \otimes k) \to E(A \otimes k)\]

is the identity morphism.

5. **Unitary equivalence.** If $U \in \mathcal{L}(\mathcal{H})$ is a unitary operator, then

\[\times(\varphi, \psi) = \times(U\varphi U^*, U\psi U^*).\]

6. **Compact perturbations.** If $U \in \mathcal{L}(\mathcal{H})$ is a unitary operator equal to the identity modulo compacts, then

\[\times(\varphi, U\varphi U^*) = 0.\]

Let $\text{hom}(E(A \otimes -), E(A \otimes k))$ be a represented contravariant functor from an admissible category to the category of abelian groups.

The following proposition is crucial for the proof of proposition 3.2:

**Proposition 3.3.** Let $E$ be a functor from an admissible category of $C^*$-algebras to the category of abelian groups and assume there is a
pairing of $E$ with the set of Fredholm $B$-pairs. Then there is a natural transformation of functors

$$\phi : KK(\cdot, k) \to \text{hom}(E(A \otimes \cdot), E(A \otimes k)).$$

sending the identity of $KK(k, k)$ to the identity of $\text{hom}(E(A \otimes k), E(A \otimes k))$.

Before showing this proposition, which will be based on an investigation in $KK$-theory due to J. Cuntz and G. Scandalis [4], we need some remarks about functional calculus for both real and complex $C^*$-algebra cases. Since we have not seen functional calculus for real $C^*$-algebras in the literature, we will explain what we mean. The functional calculus for a self-adjoint element $x$ in a complex $C^*$-algebra $A$ is the $*$-monomorphism $\Phi : C(\text{sp}x) \to A$, defined by $id_{\text{sp}x} \mapsto x$. Let $A$ be a real $C^*$-algebra and consider the complex involutive algebra $A \otimes_R C$ with involution $(a \otimes c)^* = a^* \otimes \bar{c}$. Then there exist a $C^*$-norm on $A \otimes_R C$ and a canonical $*$-embedding $A \hookrightarrow A \otimes_R C$ defined by $a \mapsto a \otimes 1$ (cf. Theorem 2 and Corollary 2 in [14]. Let $r \in A$ be a self-adjoint element in the real $C^*$-algebra $A$ and $R(sp(r \otimes 1))$ be the real algebra of continuous real functions on $sp(r \otimes 1)$, then the map $id_{sp(r \otimes 1)} \mapsto r$ defines a homomorphism $\psi : R(sp(r \otimes 1)) \to A$ such that diagram

$$
\begin{array}{ccc}
R(sp(r \otimes 1)) & \xrightarrow{\psi} & A \\
\cap & & \cap \\
C(sp(r \otimes 1)) & \xrightarrow{\Psi} & A \otimes_R C
\end{array}
$$

commutes. This implies that $\psi : R(sp(r \otimes 1)) \to A$ is a monomorphism which is called real functional calculus of the self-adjoint element $r$ in a real algebra $A$.

Now we return to the proof of Proposition 3.3.

Proof. Since the functional calculus exists for real $C^*$-algebras, the technique of 17.4 and 17.6 in [1] can be applied not only for complex $C^*$-algebras, but also for real $C^*$-algebras. Further, $KK(\cdot, k)$ can be replaced naturally by $KK_c(\cdot, k)$ ([1], Theorem 17.10.7). Now we construct a natural transformation

$$\vartheta : KK_c(\cdot, k) \to \text{hom}(E(A \otimes \cdot), E(A \otimes k)).$$

as follows. Recall that if $(E, \varphi, F)$ is a Kasparov $(B, k)$-module, according to the results of subsections 17.4 and 17.6 in [1], one can construct a Fredholm pair $(\varphi_0, \varphi_1)$ having the following properties (see Chapter 17, 6.2-6.3 in [1]):

1. if $(E, \varphi, F)$ is degenerated, then $\varphi_0 = \varphi_1$;
(2) Unitary equivalence corresponds to conjugation between $\varphi_0$ and $\varphi_1$ by the same unitary operator.

(3) ”compact perturbation” corresponds to conjugation of $\varphi_1$ by a unitary operator which is a compact perturbation of the identity.

Comparing properties (3), (5), (6) of the pairing with the properties (1), (2), (3), one immediately concludes that there is a map $\theta_B : KK_c(B, k) \to \text{hom}(E(A \otimes B), E(A \otimes k))$ defined by

$$(E, \varphi, F) \mapsto (\varphi_0, \varphi_1).$$

Properties (1) and (2) of the pairing guarantee that $\{\theta_B\}$ is a natural transformation of functors with values in the category of abelian groups. The last requirement of the proposition follows from the property (4) of the pairing.

Now we are ready to prove Proposition 3.3.

**Proof.** Since $K$-homology has the homotopy invariance property with respect to the first variable, the diagram

$$(3.3) \quad KK(k, k) \quad \begin{array}{c} \epsilon_0 = \epsilon_1 \downarrow \\ \theta_k \quad \end{array} \quad KK(k[0, 1], k)$$

$$\text{hom}(E(A \otimes k), E(A \otimes k)) \quad \begin{array}{c} \epsilon_0 = \epsilon_1 \\ \theta_{k[0, 1]} \quad \end{array} \quad \text{hom}(E(A \otimes k[0, 1]), E(A \otimes k))$$

commutes, where $\epsilon_0$, $\epsilon_1$, $\tilde{\epsilon}_0$, $\tilde{\epsilon}_1$ are induced by the evolution maps $ev_0$, $ev_1 : k[0, 1] \to k$ at 0 and 1 respectively. Let $\iota$ be a class in $KK(k, k)$ of the Fredholm pair $(e, \theta)$ having property (4) of the Fredholm pairing. Then $\theta_k(\iota) = id_{E(A)}$. Since $\theta_{k[0, 1]}\epsilon_0 = \theta_{k[0, 1]}\epsilon_1$, this implies

$$\epsilon_0 = \tilde{\epsilon}_0(id_{E(A)}) = \tilde{\epsilon}_0(\theta_k(\iota)) = \tilde{\epsilon}_1(\theta_k(\iota)) = \tilde{\epsilon}_1(id_{E(A)}) = \epsilon_1,$$

where $\epsilon_0 : E(A \otimes k[0, 1]) \to E(A \otimes k)$ and $\epsilon_1 : E(A \otimes k[0, 1]) \to E(A \otimes k)$ are homomorphisms induced by the evolutions. $\square$

**Corollary 3.4.** Let $E$ be a stable and split additive functor from an admissible subcategory of the category of real or complex $C^*$-algebras to the category of abelian groups. Then it is homotopy invariant.

This result can be similarly deduced from Proposition 3.2 as it is done in subsection 3.2 of [7].

4. **Smooth Karoubi’s Conjecture for Fréchet $k$-algebras**

The problem of the isomorphism of algebraic and smooth $K$-theories on the category $\mathcal{A}$ of Fréchet $k$-algebras which we call Smooth Karoubi’s Conjecture can be formulate as follows.
Smooth Karoubi’s Conjecture:

For any Fréchet $k$-algebra $A$ with a bounded approximate unit there is an isomorphism

$$\alpha_n^* : K_n(A \hat{\otimes} K) \xrightarrow{\sim} K_n^{sm}(A \hat{\otimes} K)$$

for all $n \geq 0$.

Using results of Sections 1-3 the aim of this section is to confirm this conjecture.

To this end we will investigate the functors $K_n(A \hat{\otimes} (- \otimes K))$, $n \in \mathbb{Z}$, on the category of $C^*$-algebras, where $\otimes$ is the well-known tensor product defined on this category. It will be shown that for any $A$ belonging to a wide class of locally convex $k$-algebras these functors have important homological properties such as exactness, stability, homotopy property and Bott periodicity.

First of all the exactness property will be considered. According to Corollary 3.12 [17] any locally convex $k$-algebra has the excision property in algebraic $K$-theory if it has the TF-property. The Cohen-Hewitt generalized theorem proved in Section 2 allows us to establish the TF-property for a wide class of locally convex $k$-algebras, namely for Fréchet $k$-algebras with bounded approximate unit.

Let $A$ be locally convex $k$-algebra. An element $a \in A$ is said to be bounded with respect to a family $\{| | \cdot \rangle_\alpha\}$ of seminorms if there exists a positive constant $C$ such that

$$| |a| |_\alpha < C.$$  

Denote by $A_b$ the space of bounded elements in $A$ with respect to a determining family $\mathcal{F}$ of seminorms. One introduce on $A_b$ a norm given by

$$| |a| | = \sup_\alpha | |a| |_\alpha,$$  

$a \in A_b, \alpha \in \mathcal{F}$.

**Theorem 4.1.** Let $A$ be a Fréchet $k$-algebra. Then

(i) $A_b$ is a Banach $k$-algebra with respect to the above defined norm and $A$ is a Fréchet $A_b$-module in the sense of Section 2.

(ii) If $A$ is a Fréchet $k$-algebra with bounded approximate unit (in the sense of Definition 2.1 as a Fréchet $A_b$-module), then it possesses the TF-property and therefore the excision property in algebraic $K$-theory.

**Proof.** (i) Since $A$ is a Fréchet algebra, there exists a determining countable subset $\{| | \cdot \rangle_\alpha\}$ of seminorms. For any elements $a, b \in A_b$ and a seminorm $| | \cdot | |_n$ there exists a seminorm $| | \cdot | |_m$ such that

$$| |ab| |_n \leq C | |a| |_m \cdot | |b| |_m.$$
Then one has
\[ ||ab|| \leq C||a|| \cdot ||b||. \]
Therefore \( ||ab|| \leq C||a|| \cdot ||b|| \). Thus \( A_b \) is a normed \( k \)-algebra.

Now we have to show that \( A_b \) is complete. Let \( (a_i) \) be a Cauchy sequence in \( A_b \). If \( || \cdot ||_n \) is a seminorm and \( \varepsilon > 0 \), there exists \( i_0 \in \mathbb{N} \) such that \( ||a_i - a_j||_n < \varepsilon \) for \( i \geq i_0 \). Furthermore we can find \( i_1 \) such that \( ||a_i - a_j|| < \varepsilon \) for \( i, j > i_1 \). If we choose \( l \geq i_0, i_1 \), one has
\[ ||a_i - a_l||_n \leq ||a_i - a_l|| + ||a_l - a_i|| + ||a_i||_n \leq 2\varepsilon \]
and
\[ ||a||_n \leq ||a - a_l||_n + ||a_l - a_i|| + ||a_i||_n \leq 2\varepsilon + ||a_i||_n \]
for any \( || \cdot ||_n \), implying \( a \in A_b \).

It is clear that the action of \( A_b \) on \( A \) satisfies condition 2.4 and that is an easy consequence of
\[ ||ax||_n \leq C||a||_m \cdot ||x||_m \]
for some seminorm \( || \cdot ||_m \), where \( \mu \in A_b \) and \( x \in A \).

(ii) In Theorem 2.4 replace \( A \) by \( A_b \) and \( L \) by \( A^m \). Then for an element \( x = (x_1, ..., x_m) \in A^m \) there exist elements \( \sigma \in A_b \) and \( y = (y_1, ..., y_m) \in A_b \cdot x \subset A \cdot x \) such that \( x = \sigma y \). Applying again the generalized Cohen-Hewitt factorization Theorem 2.4 to the \( A_b \)-module \( A \), we obtain the factorization \( \sigma = \gamma \delta \) for some \( \gamma, \delta \in A \) such that \( \delta \in A \gamma \delta \). Therefore the right annihilator \( r(\delta) \) contains the right annihilator \( r(\gamma \delta) \). The inclusion \( r(\delta) \subset r(\gamma \delta) \) is always true and trivial. This proves that \( A \) possesses the TF-property. \( \square \)

Let \( D \) be a real or complex \( C^* \)-algebra. Denote by \( SB \) the \( C^* \)-tensor product \( k^{(0,1)} \otimes D \) and by \( CB \) the \( C^* \)-tensor product \( k^{(0,1)} \otimes D \). If \( D \) is a real \( C^* \)-algebra, denote by \( \mathcal{B}D \) the \( C^* \)-tensor product \( C_0^R(i\mathbb{R}) \otimes D \), where \( C_0^R(i\mathbb{R}) \) is the real \( C^* \)-algebra defined in [3]. Now we give some slight generalization of Cuntz-Bott Periodicity Theorem 4.4 [3], which will be useful below.

**Theorem 4.2.** Let \( E : C^* \rightarrow Ab \) be a functor defined on the category \( C^* \) of \( C^* \)-algebras and *-homomorphisms satisfying the following properties:

1. \( E \) is homotopy invariant;
2. \( E \) is stable invariant;
3. \( E \) is half-exact that means for any proper exact sequence of \( C^* \)-algebras
\[ 0 \rightarrow I \rightarrow B \rightarrow C \rightarrow 0 \]
the sequence of abelian groups

\begin{equation}
E(I) \to E(B) \to E(C)
\end{equation}

is exact.

Then there are natural isomorphisms

- \( E(S^2 D) \approx E(D) \) for any complex \( C^* \)-algebra \( D \);
- \( E(\mathbb{H}SD) \approx E(D) \) for any real \( C^* \)-algebra \( D \).

**Proof.** Note that the exact sequences

\begin{equation}
0 \to K_C \to T_C \to C_0(\mathbb{R}) \to 0 \quad \text{(complex case)}
\end{equation}

\begin{equation}
0 \to K_R \to T_R \to C_0^E(i\mathbb{R}) \to 0 \quad \text{(real case)}
\end{equation}

defined in [3] have bounded linear sections, since \( C_0(\mathbb{R}) \) and \( C_0^E(i\mathbb{R}) \) are \( C^* \)-nuclear \( C^* \)-algebras. Taking into account this observation the proof completely coincides with the proof of the similar result in [3]. □

We arrive to the following result.

**Theorem 4.3.** Let \( A \) be a Fréchet \( k \)-algebra with a bounded approximate unit. Then the functors

\[ K_n^{A,K} = K_n(A \hat{\otimes} (- \otimes K)) : C^* \to Ab, \]

\( n \in \mathbb{Z} \),

(1) have the excision property in the following sense: if

\begin{equation}
0 \to I \to B \to C \to 0
\end{equation}

is a proper exact sequence of \( C^* \)-algebras, then there is a long exact sequence of abelian groups

\begin{equation}
\ldots \to K_n^{A,K}(C) \to K_n^{A,K}(I) \to K_n^{A,K}(B) \to K_n^{A,K}(C) \to \ldots
\end{equation}

(2) are stable invariant;
(3) are homotopy invariant;
(4) satisfy the following relations:

- \( K_{n+1}^{A,K}(D) \approx K_n^{A,K}(SD) \) for any \( C^* \)-algebra \( D \);
- \( K_{n}^{A,K}(D) \approx K_{n+1}^{A,K}(SD) \) for any complex \( C^* \)-algebra \( D \);
- \( K_{n}^{A,K}(D) \approx K_{n+1}^{A,K}(\mathbb{H}D) \) for any real \( C^* \)-algebra \( D \).

**Proof.** (1). Since \( K \) is a \( C^* \)-nuclear algebra, one has the proper exact sequence of \( C^* \)-algebras

\[ 0 \to I \otimes K \to B \otimes K \to C \otimes K \to 0. \]
According to Lemma 1.1 this implies the exactness of the following sequence of Fréchet $k$-algebras

\[(4.4) \quad 0 \rightarrow A \hat{\otimes}(I \otimes \mathcal{K}) \rightarrow A \hat{\otimes}(B \otimes \mathcal{K}) \rightarrow A \hat{\otimes}(C \otimes \mathcal{K}) \rightarrow 0.\]

Since every $C^*$-algebra has a bounded approximate unit and by assumption $A$ has a bounded approximate unit too, one concludes that $A \hat{\otimes}(I \otimes \mathcal{K})$ has also a bounded approximate unit. Thus by Theorem 4.1 the Fréchet $k$-algebra $A \hat{\otimes}(I \otimes \mathcal{K})$ has the TF-property and therefore the excision property in algebraic $K$-theory. This implies the long exact sequence of algebraic $K$-groups associated with the short exact sequence\[(4.4)\] of Fréchet $k$-algebras.

(2). Straightforward;

(3). (1) and (2) allow us to apply Higson’s homotopy invariance theorem (see [7] for complex case and Corollary 3.4 for real $C^*$-algebras) to show that the functors $K_n^{A,K}$, $n \in \mathbb{Z}$, are homotopy invariant.

(4) Since the sequence

\[0 \rightarrow Sk \rightarrow Ck \rightarrow k \rightarrow 0\]

is a proper short exact sequence, the first isomorphism is a consequence of (1) applied to the proper short exact sequence

\[0 \rightarrow A \hat{\otimes}(SB \otimes \mathcal{K}) \rightarrow A \hat{\otimes}(CB \otimes \mathcal{K}) \rightarrow A \hat{\otimes}(B \otimes \mathcal{K}) \rightarrow 0.\]

The last two isomorphisms are immediate consequences of (1)-(3) and Theorem 4.2. □

A Fréchet $k$-algebra $B$ will be called quasi $\hat{\otimes}$-stable if it has the form $A \hat{\otimes}\mathcal{K}$ for some Fréchet $k$-algebra $A$ with bounded approximate unit.

We are ready to prove the Smooth Karoubi’s Conjecture.

**Theorem 4.4.** The functors $K_n(-\hat{\otimes}\mathcal{K})$ and $K_n^{sm}(-\hat{\otimes}\mathcal{K})$, $n \geq 0$, are isomorphic on the category of quasi $\hat{\otimes}$-stable Fréchet $k$-algebras.

**Proof.** According to Corollary 1.12 it suffices to prove that the functors $K_n(-\hat{\otimes}\mathcal{K})$ are smooth homotopy functors for all $n \geq 1$. Consider the commutative diagram

\[
\begin{array}{ccc}
K_n((A^{\infty}(I^n) \hat{\otimes} k) \hat{\otimes} \mathcal{K}) & \Rightarrow & K_n((A \hat{\otimes} k) \hat{\otimes} \mathcal{K}) \\
\downarrow \approx & & \downarrow \approx \\
K_n((A \hat{\otimes}(k^{\infty}(l^n)) \hat{\otimes} \mathcal{K})) & \Rightarrow & K_n(A \hat{\otimes}(k \hat{\otimes} \mathcal{K})) \\
\downarrow & & \downarrow \| \\
K_n((A \hat{\otimes}(k^n \otimes \mathcal{K}))) & \Rightarrow & K_n(A \hat{\otimes}(k \otimes \mathcal{K})),
\end{array}
\]

where the horizontal homomorphisms are induced by evolution maps. Since the functors $K_n(A \hat{\otimes}(- \hat{\otimes} \mathcal{K}))$, $n \in \mathbb{Z}$, are homotopy invariant, the bottom two horizontal homomorphisms are equal, implying the
equality of the top two horizontal homomorphisms. This shows that the functors $K_n(-\hat{\otimes}K)$ are smooth homotopy invariant. □

**Theorem 4.5.** On the category $\mathcal{A}$ of Fréchet $k$-algebras with bounded approximate unit and continuous $k$-homomorphisms the functors

$$K^k_n = K_n(-\hat{\otimes}K): \mathcal{A} \to Ab,$$

$n \in \mathbb{Z}$,

(1) are smooth homotopy invariant;

(2) have the excision property in the following sense: if

(4.6) $0 \to I \to B \to C \to 0$

is a proper exact sequence in $\mathcal{A}$, then there is a long exact sequence of abelian groups

(4.7) $\ldots \to K^k_n(C) \to K^k_n(I) \to K^k_n(B) \to K^k_n(C) \to \ldots$

(3) $K^k_{n+1}(A) \approx K^k_n(\Omega_{sm}A)$ for any $k$-algebra $A \in \text{Ob}\mathcal{A}$;

(4) satisfy the following relations:

- $K^k_n(A) \approx K^k_{n+1}(Sk)$ for any complex algebra $A \in \text{Ob}\mathcal{A}$;
- $K^k_n(A) \approx K^k_{n+1}(\mathfrak{U}k)$ for any real algebra $A \in \text{Ob}\mathcal{A}$.

**Proof.** (1) is already proved for $n \geq 1$ (see proof of Theorem 4.4) and for $n < 1$ the proof is similar.

(2) If (4.6) is a proper short exact sequence in $\mathcal{A}$, then so is the sequence

(4.8) $0 \to I\hat{\otimes}K \to B\hat{\otimes}K \to C\hat{\otimes}K \to 0$

of Fréchet $k$-algebras implying the long exact sequence of algebraic $K$-groups.

(3) Consider the short exact sequence 1.6

$$0 \to \Omega_{sm}A \to \mathcal{I}(A) \xrightarrow{\tau_A} A \to 0,$$

where the epimorphism $\tau_A$ has a natural bounded section given by $a \mapsto a \cdot t$, $t \in [0,1]$. Then the exact sequence

(4.9) $0 \to \Omega_{sm}A\hat{\otimes}K \to \mathcal{I}(A)\hat{\otimes}K \xrightarrow{\tau_A} A\hat{\otimes}K \to 0$

is a proper short exact sequence. The Fréchet $k$-algebra $\Omega_{sm}A\hat{\otimes}K$ has a bounded approximate unit and therefore (4.9) induces the long exact sequence of algebraic $K$-groups. Since $K_n(\mathcal{I}(A)\hat{\otimes}K) = 0$, one gets the required isomorphism $K^k_{n+1}(A) \approx K^k_n(\Omega_{sm}A)$.

(4) is an immediate consequence of (4) of Theorem 4.3. □
Remark 4.6. There is another way to prove Theorem 4.4. Since $K_n^s(A \hat{\otimes} \mathcal{K})$, $n \geq 1$, are smooth homotopy functors, one has isomorphisms

$$K_n^s(A \hat{\otimes} \mathcal{K}) \approx K_1^s(\Omega_{sm}^{n-1} A \hat{\otimes} \mathcal{K}), \quad n \geq 1,$$

and $K_1^s(\Omega_{sm}^{n-1} A \hat{\otimes} \mathcal{K}) \approx K_1(\Omega_{sm}^{n-1} A \hat{\otimes} \mathcal{K})$ by Theorem 1.9. On the other hand, $K_1(\Omega_{sm}^{n-1} A \hat{\otimes} \mathcal{K}) \approx K_n(A \hat{\otimes} \mathcal{K})$ by Theorem 4.5(3). Therefore

$$K_n^s(A \hat{\otimes} \mathcal{K}) \approx K_1(\Omega_{sm}^{n-1} A \hat{\otimes} \mathcal{K}) \approx K_n(A \hat{\otimes} \mathcal{K}), \quad n \geq 1.$$

Remark 4.7. A locally convex $k$-algebra $A$ has the smooth homotopy (the homotopy) property in algebraic $K$-theory if there is an isomorphism $K_*(A) \cong K_*(A^{\infty(I^n)})$ for all $n > 0$. It seems that ”the homotopy property in algebraic $K$-theory” is more appropriate than the corresponding ”$K$-stability” used in [9], Definition 12. By Theorem 4.5(1) any quasi $\hat{\otimes}$-stable Fréchet $k$-algebra has the smooth homotopy property in algebraic $K$-theory and it is clear that any stable $C^*$-algebra has the homotopy property in algebraic $K$-theory.

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References

[1] Blackadar B., $K$-theory for Operator Algebras, M.S.R.I. Publ. 5. Springer-Verlag, 1986.
[2] Cohen P.J., Factorization in Group Algebras, Duke Math. J. 26. (1959), 199-205.
[3] Cuntz J., $K$-theory and $C^*$-algebras, Lect. Notes in Math. 1046 (1983), 55-79.
[4] Cuntz J., Skandalis G., Mapping cones and exact sequences in $KK$-theory, J. operator theory, 15 (1986), 163-180.
[5] Cuntz J., Bivariante K-Theorie fur lokalconexe Algebren und der Chern-Connes Charakter, Doc. Math. J. DMV 2 (1997), 139-182.
[6] Hewitt E., The Ranges of Certain Convolution Operators, Math. Scand. 15 (1964), 147-155.
[7] Higson N., Algebraic $K$-theory of stable $C^*$-algebras, Adv. Math. 67 (1988), 1-140.
[8] Inassaridze H., Algebraic $K$-theory, Kluwer Acad. Publ. Dordrecht, Boston, London, 1995.
[9] Inassaridze H., Algebraic $K$-theory of Normed Algebras, $K$-Theory 21(1)(2000), 25-56.
[10] Inassaridze H. and Kandelaki T., K-theory of Stable Generalized Operator Algebras, $K$-Theory 27 (2002), 103-110.
[11] Karoubi M., K-theorie algebrique de certaines algebres d’operateurs, Lect. Notes in Math. 725 (1979), 254-290.
[12] Kasparov G., The operator K-functor and extensions of C*-algebras, Izv. A.N. SSSR. 44, (1980), 571-636.
[13] Michael E.A., Locally multiplicatively convex topological algebras, Memoirs of the AMS, No 11, 1952.
[14] Palmer T.W., Real C*-algebras, Pacific J. Math. 35, (1970), 195-204.
[15] Phillips C., K-theory of Frechet algebras, International J. of Math. 2(1) (1991), 77-129.
[16] Rosenberg J., Comparison between algebraic and topological K-theory for Banach algebras and C*-algebras, Handbook of K-theory edited by E.M.Friedlander and D.Grayson, 2005 (to appear), (see also http://www.math.umd.edu/~jmr/).
[17] Suslin A. and Wodzicki M., Excision in Algebraic K-theory, Ann. of Math. 136 (1) (1992), 51-122.
[18] Swan R.G., Some Relations between Higher K-Functors, J.Algebra 21 (1) (1972), 113-136.
[19] Treves F., Topological vector spaces, distributions and kernels, Academic Press, New York, London, 1967.
[20] Wodzicki M., Excision in cyclic homology and in rational algebraic K-theory 129 (1989), 591-639.

M.ALEXIDZE Str.1, Tbilisi 0193, GEORGIA
E-mail address: hvedri@rmi.acnet.ge and kandel@rmi.acnet.ge