Zero sets of Lie algebras of analytic vector fields on real and complex 2-manifolds

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Abstract

Contents

1 Introduction 1
   Terminology . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 3
   Application to group actions . . . . . . . . . . . . . . . . . . . 4
   Background: Fixed points of surface actions . . . . . . . . . . . 4
   Notations and conventions . . . . . . . . . . . . . . . . . . . . . . 6

2 Consequences of tracking 6

3 The index function 10

4 Proof of Theorems 1.3 and 1.7 17

1 Introduction

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*I thank Professors Paul Chernoff, Robert Gompf, Henry King, Robion Kirby, Helena Reis and Joel Robbin for helpful discussions.
Let $M$ denote a manifold with empty boundary $\partial M$, and $X$ a (continuous) vector field on $M$ with zero set $Z(X)$. A block of zeros for $X$ is a compact set $K \subset Z(X)$ having an open neighborhood $U \subset M$ that is isolating: $\overline{U}$ is compact and $\overline{U} \cap Z(X) = K$. We say that $U$ isolating for $X$, and for $(X, K)$. Such a neighborhood is also isolating for all vector fields sufficiently near $X$ in the compact-open topology.

The collection of $X$-blocks is locally finite, and closed under formation of intersections and finite unions.

The index of $X$ at an $X$-block $K$ is the integer $i_K(X) : = i(X, U)$ defined the Poincaré-Hopf index of any sufficiently close approximation to $X$ having only finitely many zeros in $U$ (see Definitions 3.4, 3.7). This number is independent of the choice of $U$, and is stable under perturbations of $X$ (Theorem 3.9).\(^1\) The index is finitely additive over $X$-blocks: If $K, L$ are disjoint $X$-blocks, then $i_{K \cup L}(X) = i_K(X) + i_L(X)$.

$K$ is essential (for $X$) if $i_K(X) \neq 0$. When this holds $Z(X) \cap K \neq \emptyset$, because every isolating neighborhood of $K$ meets $Z(X)$.

An essential $X$-block $K$ is stable under perturbations of $X$, in the following sense: For any neighborhood $W$ of $K$, every vector field sufficiently close to $X$ in $W$ has a zero in $W$ (Theorem 3.9(a)). Conversely, a connected stable $X$-block is essential.

This paper was inspired by a remarkable result of Christian Bonatti establishing common zeros for certain pairs of commuting vector fields:\(^2\)

**Theorem 1.1 (Bonatti [3])** Assume $M$ is a real manifold of dimension 3 or 4, and $X, Y$ are analytic vector fields on $M$ such that $[X, Y] = 0$. Then $Z(Y)$ meets every essential $X$-block.\(^3\)

Another way of stating the conclusion is:

If $\mathfrak{g}$ is 2-dimensional abelian Lie algebra of analytic vector fields on $M$ and $K$ is an essential block for $X \in \mathfrak{g}$, there exists $p \in K$ such that $Y_p = 0$ for all $Y \in \mathfrak{g}$.

Our chief result, Theorem 1.3, extends this to a broader class of Lie algebras of vector fields.

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\(^1\)Professor Bonatti [3] defines the index as the intersection number of $X|U$ with the zero section of the tangent bundle. This requires an independent definition of intersection number, which is rather complicated if $M$ is noncompact or nonorientable. It can be shown that homological definitions (e.g., Golod [4, 13.26]) agree with Definition 3.7.

\(^2\)“The demonstration of this result involves a beautiful and quite difficult local study of the set of zeros of $X$, as an analytic $Y$-invariant set.” —P. Molino [16]

\(^3\)The same conclusion for $\dim(M) = 2$ is proved by applying the theorem to the vector fields $X \times t^2 \frac{\partial}{\partial t}, Y \times t^3 \frac{\partial}{\partial t}$ on $M \times \mathbb{R}$.
Terminology

$M$ always denotes a real or complex manifold, with empty boundary unless the contrary is stated. $\mathbb{F}$ denotes the corresponding ground field, $\mathbb{R}$ or $\mathbb{C}$. The dimension of $M$ over $\mathbb{F}$ is denoted by $\dim(M)$, or $\dim_\mathbb{F}(M)$ to specify the ground field. When $M$ is compact its Euler characteristic is $\chi(M)$.

$v(M)$ denotes the space of vector fields on $M$ with the compact-open topology, a topological vector space over $\mathbb{F}$. When $M$ is a submanifold of $\mathbb{F}^n$ we identify $v(M)$ with the space of continuous maps $M \to \mathbb{F}^n$, The subspace of $C^r$ vector fields is $v^r(M)$, where $r$ is a positive integer, $\infty$, or $\omega$ (meaning analytic over $\mathbb{F}$). When $M$ is complex $v^\omega(M) = v^\omega(M)$. A linear subspace of $v^r(M)$ is referred to as a Lie algebra if it is closed under Lie brackets.

The set of common zeros of a set $s \subset v(M)$ is $Z(s) := \bigcap_{X \in s} Z(X)$.

If $X, Y$ are vector fields on $M$, the tensor field $X \wedge Y \in \Lambda^2(M)$ is denoted by $X \wedge^\mathbb{F} Y$ to emphasize the ground field.

$N$ denotes the set of nonnegative integers. $|S|$ denotes the cardinality of $S$. Further notation is given at the end of this section.

**Hypothesis 1.2** $M$ is a real or complex manifold with empty boundary having dimension 2 over the ground field.

This is our main result:

**Theorem 1.3** Assume Hypothesis 1.2 and let $\mathfrak{g} \subset v^\omega(M)$ be a Lie algebra. If $X \in \mathfrak{g}$ spans a 1-dimensional ideal, $Z(\mathfrak{g})$ meets every essential $X$-block.

The proof, in Section 4, uses criteria developed in Section 3 that guarantee $i(X, U) = i(Y, U)$ when $U$ is isolating for both $X$ and $Y$.

We broaden Bonatti’s assumption that $[X, Y] = 0$ as follows:

**Definition 1.4** $Y$ tracks $X$ provided $Y, X \in v^1(M)$ and $[Y, X] \wedge X = 0$. Equivalently: If $X, Y$ are $C^r$, $r \geq 1$, there is a unique function $f : M \setminus Z(X) \to \mathbb{F}$, necessarily $C^r$, such that $[Y, X]_p = f(p)X_p$, $\quad (X_p \neq 0)$.

If $(x_1, \ldots, x_n)$ are flowbox coordinates for $X$ in a domain $N \subset M \setminus Z(X)$, representing $X|N$ as $\frac{\partial}{\partial x_1}$ and $Y|N$ as $= \sum_j Y_j(x) \frac{\partial}{\partial x_j}$, then

$$Y|N \text{ tracks } X|N \iff \frac{\partial Y_k}{\partial x_1} = 0, \quad (k = 2, \ldots, n).$$

A set $s \subset v(M)$ is said to track $X$ when each of its elements tracks $X$. When $s$ tracks every element of $t \subset v(M)$, we say that $s$ tracks $t$.  

3
Example 1.5
If \( g \subset \mathfrak{v}^1(M) \) is a Lie algebra of vector fields then \( g \) tracks \( X \in g \) iff \( X \) spans an ideal. In particular, \( g \) tracks its center.

The the set of analytic vector fields that track \( X \) is a Lie algebra (Proposition 2.2). Therefore Theorem 1.3 has the following corollary:

th:liealgcor

**Corollary 1.6** Assume Hypothesis 1.2. If \( K \) is an essential block of zeros for \( X \in \mathfrak{v}^\omega(M) \), there exists \( p \in K \) such that \( Y_p = 0 \) for all \( Y \in \mathfrak{v}^\omega(M) \) that track \( X \).

Application to group actions

Let \( \mathcal{G} \) be the class comprising those Lie groups \( G \), real or complex, whose Lie algebra \( \mathfrak{L}(G) \) contain an ideal that is 1-dimensional over the ground field.

This class is rather large. It contains every \( G \) such that \( \mathfrak{L}(G) \) has nontrivial center. Therefore it contains every complex solvable \( G \), and every real \( G \) that is supersolvable, i.e., every element in the adjoint representation of \( \mathfrak{L}(G) \) has real spectrum. (See Lie’s Theorem, Jacobson [12, Ch. 2, Th. 12]). \( \mathcal{G} \) is closed under direct products, and contains \( GL(n, \mathbb{R}) \) and \( GL(n, \mathbb{C}) \).

Let \( \text{Fix}(\alpha) \) denote the set of fixed points of a group action \( \alpha \). The following corollary of Theorem 1.3— essentially an application of Zorn’s Lemma— is derived from Theorem 1.3 at the end of Section 4:

th:liegroup

**Theorem 1.7** Assume Hypothesis 1.2 with \( M \) compact. Let \( G \in \mathcal{G} \) be connected and \( \alpha \) an effective analytic action of \( G \) on \( M \). If \( \chi(M) \neq 0 \) then \( \text{Fix}(\alpha) \neq \emptyset \).

This result is reminiscent of Borel’s fixed-point theorem for solvable algebraic actions on complex projective varieties [2, 11], and its extension to solvable holomorphic actions on Kaehler manifolds by Sommese [22]. While these results have strong algebraic hypotheses, they impose no restrictions on dimensions or Euler characteristics. For the special case of supersolvable actions on real surfaces, Theorem 1.7 is due to Hirsch & Weinstein [9].

Theorem 1.8, below, implies:

- **Theorems 1.3 and 1.7 do not extend to \( C^\infty \) vector fields and group actions on real 2-manifolds.**

Background: Fixed points of surface actions

Here we collect some earlier results on fixed points of actions by connected Lie groups \( G \) on manifolds whose boundaries may be nonempty. A manifold is closed if it is compact with empty boundary.

Let \( ST_s(k, \mathbb{R}) \) denote the identity component of the group of \( k \times k \) upper triangular matrices of determinant 1. Elon Lima, in his pioneering paper [14] of fifty years ago, constructed fixed-point free continuous actions of \( ST_s(2, \mathbb{R}) \) on
the compact disk and the sphere $S^2$. He also showed that every action of $G$ on $M$ has a fixed point if $G$ is abelian and $M$ is compact with $\chi(M) \neq 0$. These results were generalized by Joseph Plante:

**Proposition 1.8** (Plante [19])

- $ST_\phi(2, \mathbb{R})$ has an effective, fixed-point free $C^\infty$ actions on all surfaces.
- Assume $G$ is nilpotent and $M$ is a compact surface, $\chi(M) \neq 0$. Then every continuous action of $G$ on $M$ has a fixed point.

Other results include the following:

**Proposition 1.9**  
Let $\alpha$ denote a fixed-point free continuous action of $G$ on a closed surface $M$.

- If $\alpha$ is analytic, $\chi(M) \geq 0$. (Turiel [24])
- If $\chi(M) < 0$, $ST_\phi(2, \mathbb{R})$ is a quotient group of $G$. (Belliart [1])
- If $G$ is supersoluble and $\alpha$ is analytic and effective, $\chi(M) = 0$. (Hirsch & Weinstein [9])

Turiel also proved:

- $ST_\phi(3, \mathbb{R})$ has effective analytic actions on all closed surfaces.

He constructed these actions by starting from standard actions on the projective plane and $S^2$ and successively blowing up fixed points. Other actions are obtained from these by passing to covering spaces. Keeping track of fixed points and their indices leads to the following result:

**Proposition 1.10**  
Hirsch [8, Thm. 22] Let $M$ be a closed surface of genus $g$. For every $k \in \mathbb{N}$ there is an effective analytic action $\beta$ of $ST_\phi(3, \mathbb{R})$ on $M$ such that

$$|\text{Fix}(\beta)| = \begin{cases} 2(g + k + 1) & \text{if } M \text{ is orientable}, \\ g + k & \text{if } M \text{ is nonorientable and } g \geq 1. \end{cases} \quad (1)$$

I don’t know whether the right hand side of Equation (1) can be lowered.

Several results for higher dimensional manifolds are proved in [8], including:

**Proposition 1.11**  
[8, Ex. 20] Assume $N$ is a real $n$-manifold, $n \geq 1$, $\partial N = \emptyset$. Let $\alpha$ be an effective analytic action of $ST(2k, \mathbb{R})$ on $N$, $k \in \mathbb{N}_+$. 

- If $n \leq 2k$, then $\text{Fix}(\alpha) \neq \emptyset$.
- If $n = 2k$ and $N$ is compact, then $\chi(N) = |\text{Fix}(\alpha)| > 0$. 

5
Notations and conventions

Maps are continuous unless otherwise characterized. \( f \sim g \) means \( f \) and \( g \) are homotopic maps. A map is null homotopic if it is homotopic to a constant map.

If \( S \) is a topological space, the closure of \( \Lambda \subset S \) is denoted by \( \overline{\Lambda} \), the frontier by \( \text{fr}(\Lambda) := \Lambda \cap S \setminus \Lambda \), and the interior by \( \text{Int}(\Lambda) \).

\( \|\xi\| \) denotes the Euclidean norm of \( \xi \in \mathbb{R}^n \). \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \).

Let \( \dim_{\mathbb{R}}(M) = n \). The tangent bundle \( \tau(M) \) is a fibre bundle over \( M \) with total space \( T(M) \), projection map \( \pi_M : T(M) \to M \), standard fibre \( \mathbb{F}^n \), and structure group \( GL(n, \mathbb{F}) \) (see Steenrod [23]). The tangent space to \( M \) at \( p \) is the fibre \( T_p(M) := \pi_M^{-1}(p) \). When \( M \) is an open set in \( \mathbb{F}^n \) we identify \( T_p(M) \) with \( \mathbb{F}^n \), \( \tau(M) \) with the trivial vector bundle \( M \times \mathbb{F}^n \to M \), and \( X \in v(M) \) with the continuous map \( M \to \mathbb{F}^n \), \( p \mapsto X_p \).

When \( M \) is a complex \( n \)-manifold, its real form \( M^\mathbb{R} \) is the underlying real \( 2n \)-manifold. The differential structure on \( M^\mathbb{R} \) is defined by viewing the \( C^n \)-valued charts as \( \mathbb{R}^{2n} \)-valued charts. The topological spaces \( T(M) \) and \( T(M^\mathbb{R}) \) are identical, as are \( v(M) \) and \( v(M^\mathbb{R}) \).

If \( X \) is \( C^1 \), the \( X \)-trajectory of \( p \in M \) is the maximally defined solution \( t \mapsto x(t) \) to the initial value problem

\[
\frac{dx}{dt} = X(x), \quad x(0) = p,
\]

The image of an \( X \)-trajectory is a \( X \)-orbit. An \( X \)-arc is an arc contained in an \( X \)-orbit.

Assume \( X \in v(M) \), \( r \geq 1 \). The local flow on \( M \) whose trajectory through \( p \) is the \( X \)-trajectory of \( p \) is denoted by \( \Phi^X := \{ \Phi^X_t \}_{t \in \mathbb{R}} \). When \( \partial M = \emptyset \) the maps \( \Phi^X_t \) are \( C^r \) diffeomorphisms between open subsets of \( M \). The local flow on \( T(M) \) induced by \( \Phi^X_t \) is \( T\Phi^X := \{ T\Phi^X_t \}_{t \in \mathbb{R}} \).

A set \( S \subset M \) is \( X \)-invariant if it contains the \( X \)-orbits of its points. If this holds for all \( X \) in a set \( s \subset v(M) \) then \( S \) is \( s \)-invariant.

## 2 Consequences of tracking

Throughout this section we assume:

**Hypothesis 2.1**

- \( M \) is a real or complex manifold, \( \dim_{\mathbb{R}}(M) = n \geq 1 \), \( \partial M = \emptyset \).

- \( X, Y \in v(M) \).

Recall that \( Y \) tracks \( X \) iff \([Y, X] \wedge X = 0\).
Proposition 2.2 If $Y$ and $Z$ track $X$ and $[Y, Z]$ is $C^1$, then $[Y, Z]$ tracks $X$.

Proof Follows from the Jacobi identity.

We say that $\Phi^Y$ respects $X$ if the maps $\Phi^Y_t$ send $X$-curves to $X$-curves, and $T\Phi^Y$ respects $X$ if

$$T\Phi^Y_t(X_p) \wedge X_{\Phi^Y_t(p)} = 0.$$ (2)

Proposition 2.3 Assume $Y$ tracks $X$. Then:

(a) $Z(X)$ is $Y$-invariant,

(b) $\Phi^Y$ respects $X$.

Proof As the theorem local, we assume $M$ is an open set in $\mathbb{F}^n$.

(a) Clearly $Z(X) \cap Z(Y)$ is $Y$-invariant. To show that $Z(X) \setminus Z(Y)$ is $Y$-invariant, fix $p \in Z(X) \setminus Z(Y)$. Let $(y_1, \ldots, y_n)$ be flowbox coordinates in a neighborhood $V_p$ of $p$, representing $Y|_{V_p}$ as $\frac{\partial}{\partial y_1}$ in a convex open subset of $\mathbb{R}^n$, and the $Y$-trajectory of $p$ as

$$t \mapsto y(t) := p + te_1.$$ where $e_1, \ldots, e_n \in \mathbb{F}^n$ are the standard basis vectors.

Let $J_p \subset \mathbb{R}$ be an open interval around 0 such that

$$y(t) \in V_p, \quad (t \in J_p).$$

Then

$$\frac{d}{dt}(T\Phi^Y_t(X_p)) = [Y, X]_{y(t)}, \quad (t \in J_p).$$ (3)

Tracking implies there is a complex-valued function $g(t)$ such that in the flowbox coordinates for $Y$, the vector function $X_{y(t)}$ satisfies

$$\frac{d}{dt} X_{y(t)} = [Y, X]_{y(t)}$$

$$= g(t)X_{y(t)}.$$ As this vector-valued linear differential equation has initial value $X_p = 0$, therefore $X_{y(t)}$ vanishes identically in $t$.

(b) By continuity of $T\Phi^Y$, it suffices to validate (2) for all $p$ in the open, dense $Y$-invariant set $\text{Int}(Z(Y)) \cup M \setminus Z(Y)$. Equation (2) is obvious for $p \in \text{Int}(Z(Y))$. Assume $Y_p \neq 0$ and fix flowbox coordinates for $Y$ at $p$. With notation as in paragraph (a), it suffices to verify (2) for all $t \in J_p$. Equation (3) implies

$$\frac{d}{dt}(T\Phi^Y_t(X_p) \wedge X_{y(t)}) = ([Y, X] \wedge X)_{y(t)} = 0$$ identically in $t$. This shows that 2) holds for all $t \in J_p$ because it holds at $t = 0$. ■
Definition 2.4 The dependency set of $X$ and $Y$ is

$$\text{Dep}(X, Y) := \{ p \in M : X_p \wedge Y_p = 0 \}.$$ 

Proposition 2.5 If $Y$ tracks $X$, $\text{Dep}(X, Y)$ is invariant under $\Phi^X$ and $\Phi^Y$.

Proof For every $A \in \mathfrak{v}^1(M)$ let $\mathcal{L}_A$ denote the Lie derivative of $A$, a derivation sending $C^{r+1}$ tensor fields to $C^r$ tensor fields. For all $A, B, C \in \mathfrak{v}^1(M)$:

$$\mathcal{L}_A(A) = 0, \quad \mathcal{L}_A(B) = [A, B] \quad (4)$$

and

$$\mathcal{L}_A(B \wedge C) = \mathcal{L}_A(B) \wedge C + \mathcal{L}_A(C) \wedge B$$

$$= [A, B] \wedge C + [A, C] \wedge B. \quad (5)$$

To prove $X$-invariance of $\text{Dep}(X, Y)$ it suffices to prove the stronger statement:

$$\mathcal{L}_X(X \wedge Y) = 0. \quad (6)$$

Setting $(A, B, C) := (X, X, Y)$ and using Equations (4) and (5), we see that (6) is equivalent to

$$[X, Y] \wedge X = 0,$$

which holds because $Y$ tracks $X$.

For $Y$-invariance, suppose $X_p, Y_p$ are linearly dependent and $\Phi^Y_Y(p) = q$. Since $Y_q = T\Phi^Y_Y(Y_p)$, linearity of $T\Phi^Y_Y$ implies $Y_q$ and $T\Phi^Y_Y(X_p)$ are linearly dependent. Consequently $Y_q$ and $X_q$ are linearly dependent by Proposition 2.3(b).

Theorem 2.6 If $M$ is real the following conditions are equivalent: ***[OK?]***

(i) $Y$ tracks $X$,

(ii) $\Phi^Y$ respects $X$,

(iii) $T\Phi^Y$ respects $X$.

Proof Proposition 2.3 shows that (i) $\implies$ (ii), (iii).

To prove (iii) $\implies$ (ii), fix an $X$-curve $\Gamma$. If $\Gamma$ is a singleton, so is $\Phi^Y_Y(q)\Gamma$ by 2.3(a). Assume $\Gamma$ is not a singleton and $p \in \Gamma$, so that $X_p$ is a nonzero tangent vector to $\Gamma$. Set

$$\Gamma' := \Phi^Y_Y(\Gamma), \quad p' := \Phi^Y_Y(p).$$

Equation (2) shows that $T\Phi^Y_Y(X_p)$, which is a nonzero tangent vector to $\Gamma'$, is a scalar multiple of $X_{p'}$. Thus $X$ is tangent to $\Gamma'$, hence $\Gamma'$ is an $X$-curve by the uniqueness theorem for trajectories of $C^1$ vector fields.
A similar argument shows that (ii) \(\implies\) (iii).

We complete the proof by proving (ii) \(\implies\) (i). We can work locally, taking \(M\) to be an open set in \(\mathbb{R}^n\). It suffices to verify the tracking condition

\[
[Y, X]_x \wedge X_x = 0 
\]  

(7)

for all \(x\) in some convex open neighborhood \(V_p\) of each \(p \in M \setminus ZX\). Therefore there are flowbox coordinates on \(V_p M\) in which \(X\) takes the constant nonzero value \(e \in \mathbb{R}^n\), and

\[
\Phi_t^X(y) = y + te. 
\]

(8)

The \(X\)-curves are open intervals in lines parallel to \(e\).

Consider the maps defined for each \(t\) by

\[
H_t := \Phi_t^Y \circ \Phi_t^X - \Phi_t^X \circ \Phi_t^Y 
\]

(9)

parametrized by \(t \in \mathbb{R}\). We will use the classical formula\(^4\)

\[
[X, Y]_y \wedge X_y = \lim_{t \to 0} t^{-2} H_t(y). 
\]

(10)

By (9),

\[
H_t(y) = \Phi_t^Y(y + te) - (\Phi_t^Y(y) + te). 
\]

(11)

Both terms on the right hand side lie in the line through \(\Phi_t^Y(y)\) parallel to \(e\), because \(\Phi_t^Y\) permutes \(X\)-curves. Therefore by (11) there exist \(\lambda(t, y) \in \mathbb{R}\) such that \(H_t(y) = \lambda(t, y)e\). Wedging both sides of Equation (11) with \(e\) yields and using (10) yields

\[
[X, Y]_y \wedge X_y = 0 = \lim_{t \to 0} (t^{-2} \lambda(t, y) e \wedge e) = 0. 
\]

Thus (ii) \(\implies\) (i).

The next result is not used in the proofs of the main results. Consider the set of vector fields

\[
\mathcal{T}^1(X) := \{ Y \in \mathfrak{v}^1(M) : Y \text{ tracks } X \}. 
\]

\[
\text{th:trackclosed}
\]

**Theorem 2.7** If \(M\) is real, \(\mathcal{T}^1(X)\) is closed in the compact-open topology.

**Proof** We show that \(\mathfrak{v}^1(M) \setminus \mathcal{T}^1(X)\) is open. Suppose \(Y \in \mathfrak{v}^1(M) \setminus \mathcal{T}^1(X)\). Fix \(p \in M \setminus Z(X)\), a compact arc \(A\) contained in the \(X\)-orbit of \(p\), and \(s > 0\) such that:

\[
\Phi_s^Y(A) \text{ is defined and does not lie in any } X\text{-orbit}. 
\]

(12)

\(^4\)In many differential geometry texts this is proved only for \(X, Y \in \mathfrak{v}^r(M)\) for some \(r > 1\), typically \(r = \infty\). The general case follows because \(C^1\) vector fields can be approximated by \(C^r\) vector fields in the \(C^r\) topology.
Let $D \subset M$ be a smooth open $(n - 1)$-cell transverse to $X$. There is an open neighborhood $N \subset M \setminus Z(X)$ of $A$ that is a union of $X$-arcs, and admits a retraction $R: N \to D \cap N$ that is constant on the $X$-arcs in $N$. By Equation (12) there exist $u, v \in A$ such that
\[ R \circ \Phi^Y_s(u) \neq R \circ \Phi^Y_s(v). \]

The set
\[ \mathcal{U} := \{ Y' \in v^1(M) : \Phi^Y_s'(A) \subset N, \ R \circ \Phi^Y_s'(u) \neq R \circ \Phi^Y_s'(v) \} \]
is an open neighborhood of $Y$ in $v^1(M)$ for the compact-open topology. If $Y' \in \mathcal{U}$, then $\Phi^Y_s'(A)$ is an arc in $N$ that is not contained in any $X$-arc, and therefore not contained in any $X$-orbit. This shows that $\mathcal{U}$ is a neighborhood of $Y$ in $v^1(M) \setminus T^1(X)$.

\section{The index function}

In this section $M$ is real and satisfies the following conditions:

\begin{hyp}
\textbf{Hypothesis 3.1}
\begin{itemize}
  \item $\partial M = \emptyset$, $\dim(M) = n \geq 1$,
  \item $X, Y \in v(M)$,
  \item $K$ is a block of zeros of $X$,
  \item $U \subset M$ is isolating for $(X, K)$.
\end{itemize}
\end{hyp}

\begin{defn}
\textbf{Definition 3.2} A \textit{deformation} from $X$ to $X'$ is path $\{X_t\}_{t \in [0,1]}$ in $v(M)$ of the form
\[ t \mapsto X', \quad X^0 := X, \quad X^1 = Y. \]

The deformation is \textit{nonsingular} in a set $S \subset M$ provided $Z(X') \cap S = \emptyset$.
\end{defn}

\begin{prop}
\textbf{Proposition 3.3} $X$ has arbitrarily small convex open neighborhoods $N \subset v(M)$ such that for all $Y, Z \in N$:
\begin{itemize}
  \item[(i)] $U$ is isolating for $Y$,
  \item[(ii)] the deformation $Y' := (1 - t)Y + tZ$, $(0 \leq t \leq 1)$ is nonsingular in $\text{fr}(U)$.
\end{itemize}

Moreover:
\begin{itemize}
  \item[(iii)] The set of $Y \in N$ such that $Z(Y) \cap U$ is finite contains a dense open subset of $N$.
\end{itemize}
\end{prop}
Proof (i) and (ii) follow from the definition of the compact-open topology on \( v(M) \). Standard approximation theory gives (iii).

\[ \text{Definition 3.4} \]
When \( K \) is finite, the \textit{Poincaré-Hopf index} of \( X \) at \( K \), and in \( U \), is the integer \( i^\text{PH}_K(X) = i^\text{PH}(X, U) \) defined as follows. Let \( \dim \mathbb{R}(M) = n \). For each \( p \in K \) choose an open set \( W \subset U \) meeting \( K \) only at \( p \), such that \( W \) is the domain of a \( C^1 \) chart \( \phi: W \approx W' \subset \mathbb{R}^n \), \( \phi(p) = p' \).

The transform of \( X \) by \( \phi \) is the vector field \( X' \) on \( W' \) defined as \( X' := T \phi \circ X \circ \phi^{-1} \).

There is a map of pairs \( F_p: (W', W' \setminus \{p'\}) \to (\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) \) that expresses \( X' \) by the formula \( X'_p = (x, F_p(x)) \in \{x\} \times \mathbb{R}^n \), \( (x \in W') \).

Let \( i^\text{PH}_p(X) \in \mathbb{Z} \) denote the degree of the map defined for all sufficiently small \( \epsilon > 0 \) as \( S^{n-1} \to S^{n-1} \), \( u \mapsto \frac{F_p(\epsilon u)}{||F_p(\epsilon u)||} \).

This is independent of \( \epsilon \) and \( \phi \) by standard properties of degrees.

\[ i^\text{PH}_K(X) = i^\text{PH}(X, U) := \begin{cases} \sum_{p \in K} i^\text{PH}_K(X) & \text{if } K \neq \emptyset, \\ 0 & \text{if } K = \emptyset. \end{cases} \]

This integer depends only on \( X \) and \( K \).

\[ \text{Proposition 3.5} \]
Let \( \{X^t\} \) be a deformation that is nonsingular in \( \text{fr}(U) \). If both \( Z(X^0) \cap U \) and \( Z(X^1) \cap U \) are finite, then \( i^\text{PH}(X, U) = i^\text{PH}(X^1, U) \).

Proof The proof is similar to that of a standard result on homotopy invariance of intersection numbers in oriented manifolds (compare Hirsch [7, Theorem 5.2.1]).

\[ \text{Definition 3.6} \]
The \textit{support} of a deformation \( \{X^t\} \) is \( \text{supp}\{X^t\} := \{p \in M: X^t_p = X^0_p, \ 0 \leq t \leq 1\} \).

A deformation \( \{X^t\} \) is \textit{compactly supported} in \( S \subset M \) if its support is a compact subset of \( S \).
Definition 3.7 The index of $X$ in $U$ is

$$i(X, U) := i^{PH}(X', U)$$

(13)

where $X'$ is any vector field on $M$ such that $Z(X') \cap U$ is finite, and there is a deformation from $X$ to $X'$ that is compactly supported in $\text{Int}(U)$. This integer is well defined because the right hand side of Equation (13) depends only on $X$ and $U$, by Proposition 3.5. The notation $i(X, U)$ tacitly assumes $U$ is isolating for $X$.

Lemma 3.8 If $U$ and $U_1$ are isolating for $(X, K)$ then $i(X, U) = i(X, U_1)$.

Proof Let $W$ be isolating for $(X, K)$, with $\overline{W} \subset U_1 \cap U$. It suffices to show that $i(X, U) = i(X, W)$, for this also implies $i(X, U_1) = i(X, W)$. By definition, $i(X, W) = i^{PH}(X', W)$ provided $X$ and $X'$ are homotopic by deformation with compact support in $W$ and $Z(Y_1) \cap W$ is finite. Let $\{Y^t\}$ be the deformation defined by

$$Y^t_p = \begin{cases} 
X^t_p & \text{if } p \notin W, \\
Y^t_p & \text{if } p \in \overline{W}.
\end{cases}$$

Therefore $i(X, U) = i(X, W)$, because this deformation is compactly supported in $U$ and $Z(Y^1) \cap U$ is finite.

It follows that $i(X, U)$ depends only on $X$ and $K$. We define the index of $X$ at $K$ to be

$$i_K(X) := i(X, U).$$

The key properties of the index function is:

The Poincaré-Hopf Theorem ([10, 20]) can be stated this way:

Theorem 3.9 (Stability) Let $U \subset M$ be isolating for $X$.

(a) If $i(X, U) \neq 0$ then $Z(X) \cap U \neq \emptyset$.

(b) If $Y$ is sufficiently close to $X$ then $i(Y, U) = i(X, U)$.

(c) Let $\{X^t\}$ be a deformation of $X$ that is nonsingular in $\text{fr}(U)$. Then

$$i(X^t, U) = i(X, U), \quad (0 \leq t \leq 1).$$

Proof If $i(X, U) \neq 0$ By Definition 3.7 shows that $X$ is the limit of a convergent sequence $\{X^n\}$ in $\nu(M)$ such that $Z(X^n) \cap U \neq \emptyset$. Passing to a subsequence and using compactness of $\overline{U}$ shows that $Z(X) \cap \overline{U} \neq \emptyset$, and (a) follows because $Z(X) \cap \text{fr}(U) = \emptyset$. Parts (b) and (c) are implied by Propositions 3.3 and 3.5.

The Poincaré-Hopf Theorem ([10, 20]) can be stated this way:
**Theorem 3.10** Let $N$ be a compact manifold. Assume $Z \in \nu(N)$ has no zeros on $\partial N$, and in local coordinates $Z_p$ points into $N$ if $p \in \partial N$. Then

$$i(Z, N \setminus \partial N) = (-1)^{\dim(N)-1} \chi(N).$$

For calculations of the index in more general settings see Morse [17], Pugh [21], Gottlieb [5], Jübin [13].

**Proposition 3.11** Let $U$ be isolating for $X$ and $Y$. For each component $U'$ of $U$ that meets $Z(X) \cup Z(Y)$, assume one of the following holds:

(a) $X_p \not= \lambda Y_p, \quad (p \in \text{fr}(U'), \lambda < 0)$,

or

(b) $X_p \not= \lambda Y_p, \quad (p \in \text{fr}(U'), \lambda > 0)$.

Then $i(X, U) = i(Y, U)$.

**Proof** $U \cap (Z(X) \cup Z(Y))$ is the compact set $\overline{U} \cap (Z(X) \cup Z(Y))$. This implies only finitely many components of $U$ meet $Z(X) \cup Z(Y)$. The union $U_1$ of these components is isolating for $X$ and $Y$. The index function is additive over disjoint unions, and both $X$ and $Y$ have index zero in the open set $U \setminus U_1$, which is disjoint from $Z(X) \cup Z(Y)$. Therefore

$$i(X, U) = i(X, U_1),$$

$$i(Y, U) = i(Y, U_1).$$

Replacing $U$ by $U_1$, we assume $U$ has only finitely many components. As it suffices to prove $X$ and $Y$ have the same index in each component of $U$, we also assume $U$ is connected.

Let $p \in \text{fr}(U)$ be arbitrary. If (a) holds, consider the deformation

$$X' := (1 - t)X + tY.$$

If $t = 0$ or $1$ then $X'_p \neq 0$ because $U$ is isolating for $X$ and $Y$, while if $0 < t < 1$ then $X'_p \neq 0$ by (a). Therefore the conclusion follows from the Stability Theorem 3.9. If (b) holds the same argument works for the deformation $(1 - t)X - tY$. 

**Proposition 3.12** Assume $U$ is isolating for both $X$ and $Y$. Let $V \subset M$ be an open neighborhood of $\text{fr}(U)$ such that

$$X_p \wedge Y_p = 0, \quad (p \in V).$$

(14)

If $M$ is even dimensional then $i(X, U) = i(Y, U)$. 


Proof We can assume $U$ is connected, arguing as in the proof of Proposition 3.11. The compact set $\overline{U} \cap (Z(X) \cup Z(Y))$ lies in $U$. Choose an isolating neighborhood $W$ for $(X, K)$ such that

$$N := \overline{W} \text{ is a compact, connected } n\text{-manifold in } U, \quad \partial N = \text{fr}(W). \tag{15}$$

As neither $X$ nor $Y$ has a zero in $\partial N$, by (14) there is a map

$$f: \partial N \to \mathbb{R}\setminus\{0\}, \quad X_p = f(p)Y_p,$$

such that $f$ has constant nonzero sign in each of the finitely many components of $\partial N$.

Consider the decomposition

$$\partial N = (\partial_+, N) \cup (\partial_-, N), \quad \text{where } f|\partial_+ N > 0, \quad f|\partial_- N < 0.$$

To fix ideas we assume $\partial_- N \neq \emptyset$, the case $\partial_+ N \neq \emptyset$ being similar.

Because each tangent space $T_p(M)$ is even dimensional, the antipodal map of $T_p(M)\setminus\{0\}$ is isotopic to the identity map. It follows that there is a homotopy of nonsingular sections of $T_{\partial, N}M$ connecting $X|\partial_- N$ to $-X|\partial_- N$. Since $\partial_- N$ is a subcomplex of a smooth triangulation of $M$ (Whitehead [25], Munkres [18]), this homotopy extends to a deformation $\{X^t\}$ of $X$ from $X = X^0$ to a vector field $X^1$ that agrees with $X$ outside a compact neighborhood of $\partial_- N$ in $U \setminus \partial_- N$. Thus

$$X^1_p = -X_p, \quad (p \in \partial_- N), \quad X^1_p = X_p, \quad (p \in \partial_+ N).$$

For each $p \in \partial N$ there exists $\lambda(p) \in \mathbb{R}$ such that

$$X^1_p = \lambda(p)Y_p, \quad \lambda(p) > 0,$$

hence

$$i(X^1, W) = i(Y, W)$$

by Proposition 3.11(a). The conclusion follows because

$$i(X, W) = i(X^1, W)$$

by the Stability Theorem 3.9.

Fix $N$ as in (15), so that $i(X, U) = i(X, N \setminus \partial N)$. An orientation of $N$ corresponds to a generator

$$\nu_N \in H_n(N, \partial N) \cong \mathbb{Z}.$$

Let $\nu^N \in H^n(N, \partial N)$ denote the dual generator. Denote the canonical dual pairing (the Kronecker Index) by

$$H^n(N, \partial N) \times H_n(N, \partial N) \to \mathbb{Z}, \quad \kappa(c, \mu) = c \cdot \mu.$$

Let $c_{X,N} \in H^n(N, \partial N)$ denote the obstruction to extending $X|\partial N$ to a nonsingular vector field on $N$.

With $N$ as in Equation (15), unwinding definitions yields:
Proposition 3.13 If $N$ is oriented, $i(X, U) = c_{X,N} \cdot \nu_N$. 

A similar result holds for nonorientable manifolds, using homology with coefficients twisted by the orientation sheaf.

Theorem 3.14 $i(X, U) = 0$ iff $X$ can be approximated by vector fields $X'$ with no zeros in $\overline{U}$.

Proof If the approximation is possible then the index vanishes, thanks to the Stability Theorem 3.9. To prove the converse, fix a Riemann metric on $M$ and $\epsilon > 0$. There exists an isolating neighborhood $U'$ of $K$ whose closure is a compact submanifold $N \subset U$ and

$$\|X_p\| < \epsilon, \quad (p \in N).$$

Define

$$E_\epsilon := \{x \in (N) : 0 < \|x\| < \epsilon, \quad (p \in N)\}.$$ 

This is the total space of a fibre bundle $\eta$ over $N$ that is fibre homotopically equivalent to the sphere bundle associated to the tangent bundle of $N$. Thanks to Proposition 3.13, $X|\partial N$ extends to a section $X'' : N \to E_\epsilon$ of $\eta$. Let $X' \in \mathfrak{v}(M)$ be the extension of $X''$ that agrees with $X$ outside $N$. Then $X'$ is an $\epsilon$-approximation to $X$ with no zeros in $\overline{U}$.

Examination of the proof, together with standard approximation theory, yields the following addenda to 3.14:

Corollary 3.15 Assume $i(X, U) = 0$.

(i) If $X$ is analytic, the approximations in Theorem 3.14 can be chosen to be analytic.

(ii) If $X$ is $C^r$ with $0 \leq r \leq \infty$, the approximations can be chosen to be $C^r$, and to agree with $X$ in $M \setminus U$.

Definition 3.16 Let $\eta$ denote a real or complex vector bundle, with total space $E$ and $n$-dimensional fibres. A trivialization of $\eta$ is a map $\psi : E \to \mathbb{F}^n$ which restricts to a linear isomorphism on each fibre.

Proposition 3.17 Assume $N \subset U$ is a compact, connected real $n$-manifold whose interior is isolating for $(X, K)$. Let $\psi$ be a trivialization of $\tau_{\partial N}(M)$. Then $i(X, U)$ equals the degree $\deg(F_X)$ of the map

$$F_X : \partial N \to S^{n-1}, \quad p \mapsto \frac{\psi(X_p)}{\|\psi(X_p)\|}.$$
Proof: Follows from Proposition 3.13, because $\deg(F_X) = c_{X,N} \cdot \nu_N$ by obstruction theory.

The following result will be used in the proof Theorem 4.3.

**Proposition 3.18** Assume:

(a) $W \subset M$ is a connected isolating neighborhood for both $X$ and $Y$,

(b) $\Phi: T(W \setminus K) \to \mathbb{R}^n$ is a trivialization of $\tau(W \setminus K)$,

(c) $E \subset \mathbb{R}^{n \times n}$ is a linear space of matrices, $\dim(E) < n$,

(d) $A: W \setminus K \to E$ is a map such that

\[ \Phi(X_q) = A(q)\Phi(Y_q), \quad (q \in W \setminus K). \quad (16) \]

Then \( i(Y, W) = 0 \implies i(X, W) = 0. \)

Proof: Let $N$ be as in (15). Consider the maps

\[ F_X: \partial N \to S^{n-1}, \quad p \mapsto \frac{\Phi(X_p)}{\|\Phi(X_p)\|}, \]

\[ F_Y: \partial N \to S^{n-1}, \quad p \mapsto \frac{\Phi(Y_p)}{\|\Phi(Y_p)\|}. \]

Corollary 3.17 implies $\deg(F_Y) = 0$, hence $F_Y$ is null homotopic, and it suffices to prove $F_X$ null homotopic.

Degree theory shows that $F_Y$ is homotopic to a constant map

\[ \tilde{F}_Y: \partial N \to S^{n-1}, \quad \tilde{F}_Y(p) = c \in S^{n-1}. \]

By Equation (16) there exists $\lambda: \partial N \to \mathbb{R}$ such that

\[ F_X(p) = \lambda(p)A(p)F_Y(p), \quad \lambda(p) > 0. \]

Consequently $F_X$ is homotopic to

\[ \tilde{F}_X: \partial N \to S^{n-1}, \quad \tilde{F}_X(p) = \lambda(p)A(p)c. \]

The map

\[ H: E \setminus \{0\} \to S^{n-1}, \quad B \mapsto \frac{B(c)}{\|B(c)\|} \]

satisfies:

\[ \tilde{F}_X(\partial N) \subset H(E \setminus \{0\}) \subset S^{n-1}. \quad (17) \]
Since the unit sphere $\Sigma \subset E \setminus \{0\}$ is a deformation retract of $E \setminus \{0\}$, Equation (17) shows that $\tilde{F}_X$ is homotopic to a map

$$ G: \partial N \to H(\Sigma) \subset S^{n-1}. $$

Now $\dim(\Sigma) = \dim(E) - 1 \leq n - 2$. As $H$ is Lipschitz, $\dim(H(\Sigma)) \leq n - 2$. Therefore $H(\Sigma)$ is a proper subset of $S^{n-1}$ containing $G(\partial N)$, implying $G$ is null homotopic. The conclusion follows because the maps

$$ F_X, \tilde{F}_X, G: \partial N \to S^{n-1} $$

are homotopic and therefore have the same degree.

**Example 3.19**

Let $\mathcal{A}$ denote a finite dimensional algebra over $\mathbb{R}$, perhaps nonassociative, with multiplication $(a, b) \mapsto a \cdot b$. Let $X, Y$ be vector fields on a connected open set $U \subset \mathcal{A}$, whose respective zero sets $K, L$ are compact. Assume there is a map $A: U \to \mathcal{A}$ such that

$$ X_p = A(p) \cdot Y_p, \quad (p \in U). $$

Then $K \cap L \neq \emptyset$, by Proposition 3.18.

## 4 Proof of Theorems 1.3 and 1.7

**Definition 4.1** A set $S \subset M$ is an *analytic space*, or *subvariety*, if each point of $S$ has an open neighborhood $V \subset M$ such that $S \cap V$ is the zero set of an analytic map $V \to \mathbb{R}^k$.

The local topology of analytic spaces is rather simple, owing to the theorem of Łojasiewicz [15]:

**Theorem 4.2 (Triangulation)** If $S$ is a locally finite collection of subvarieties of $M$, there is a triangulation of $M$ with the elements of $S$ as subcomplexes.

Recall the hypotheses of Theorem 1.3, assumed henceforth:

- $M$ is a connected real or complex 2-manifold with empty boundary,
- $X$ is an analytic vector field on $M$,
- $\mathfrak{g} \subset \mathfrak{so}(M)$ is a Lie algebra tracking $X$,
- $K$ is an essential $X$-block,

and therefore
• $K$ is a nonempty compact subvariety.

Being compact and essential, $K$ has only finitely many components and at least one of them is essential. Replacing $K$ by such a component, we assume:

• $K$ is connected.

Proposition 2.3 implies:

• $K$ is $g$-invariant.

Most of the proof of Theorem 1.3 goes into the special case that $g$ is generated by $X$ and $Y$:

\[ \text{Theorem 4.3} \quad \text{If } Y, X \in \nu^\omega(M) \text{ and } Y \text{ tracks } X, \text{ then } Z(Y) \cap K \neq \emptyset. \]

\[ \text{Proof} \quad \text{We justify four simplifying assumptions:} \]

(A1) $\dim_{\mathbb{R}}(K) = 1$.

If $\dim_{\mathbb{R}}(K) = 0$ then $K$ is a singleton and the conclusion is obvious. If $\dim_{\mathbb{R}}(K) = 2$ then $K = M$ because $M$ is connected, hence $\chi(M) \neq 0$, and the Poincaré-Hopf Theorem 3.10 implies $Z(Y) \cap K = Z(Y) \neq \emptyset$.

(A2) $K$ is an analytic submanifold of $M$.

Otherwise the singular set of the 1-dimensional analytic space $K$ is nonempty and finite, and being invariant under analytic local diffeomorphisms, it lies in $Z(Y)$.

(A3) $\chi(K) = 0$.

If $\chi(K) \neq 0$ then $Z(Y) \cap K \neq \emptyset$ by $Y$-invariance of $K$ and Poincaré-Hopf.

(A4) $Z(Y) \cap \overline{U} \subset K$, hence $U$ is isolating for $Y$.

For if $U$ cannot be chosen with this property, then $Z(Y)$ meets every neighborhood of $K$, hence it also meets $K$.

In view of (A4) it is enough to prove:

\[ i(Y, V) = i(X, V) \text{ for some open neighborhood } V \text{ of } K \text{ in } U \quad (18) \]

for then $i(Y, V) \neq 0$, whence $Z(Y) \cap K \neq \emptyset$.

Since $X$ and $Y$ are analytic, their dependency set $D := \text{Dep}_{\mathbb{R}}(X, Y) \subset M$ (see 2.4) is a subvariety containing $K$, and (A1) implies $\dim_{\mathbb{R}}(D) = 1$ or 2. Because $\dim_{\mathbb{R}}(M) = 2$, the Triangulation Theorem 4.2 implies there are only three possibilities:
(B1) $\dim_{\mathbb{F}}(D) = 1$ and $K$ is a component of $D$.

(B2) $\dim_{\mathbb{F}}(D) = 1$ and $K$ is not a component of $D$.

(B3) $\dim_{\mathbb{F}}(D) = 2$.

Assume (B1): There is a neighborhood $V \subset U$ of $K$ such that $\text{fr}(V) \cap D = \emptyset$, implying
$$X_p \wedge_{\mathbb{F}} Y_p \neq 0, \quad (p \in \text{fr}(V)),$$
and (18) follows from Proposition 3.11.

***[FIX BELOW]:

Assume (B2): Then (A1) and (A2) imply $K$ is a 1-dimensional analytic submanifold of $M$, while (B2) implies $D$ contains an irreducible subvariety $L \neq K$ such that $L \cap K$ is $Y$-invariant and 0-dimensional, hence a nonempty subset of $Z(Y) \cap K$.

Assume (B3): In this case $D = M$ because $X$ and $Y$ are analytic and $M$ is connected, therefore $X \wedge_{\mathbb{F}} Y = 0$.

If $M$ is real, $i(X, U) = i(Y, U)$ by Proposition 3.12, implying (18).

Let $M$ be complex. (A2) and (A3) imply

(C1) $K$ is a compact connected Riemann surface of genus 0, holomorphically embedded in $U$,

and therefore

(C2) $\tau(K)$ is a holomorphically trivial line bundle.

By (B3) and (A4) there is a holomorphic function
$$f : W \to \mathbb{C}, \quad X_p = f(p)Y_p, \quad (p \in W), \quad f^{-1}(0) = K. \quad (19)$$

This implies the complex line bundle $\nu(K) := \tau_K(W)/\tau(K)$ is holomorphically trivial (Griffiths & Harris [6, p. 134]).

Since $\tau_K(W) \cong \tau(K) \oplus \nu(K)$ as a complex vector bundle, $\tau_K(W)$ is a trivial complex vector bundle. By the Triangulation Theorem 4.2, $K$ is a deformation retract of $W$. Therefore the Homotopy Extension Theorem (Hirsch [7, Chap. 4, Thm. 1.5]) implies:

(C3) $\tau(W)$ is a trivial complex vector bundle.

Assume per contra: $Z(Y) \cap K = \emptyset$. Then (A4) implies
$$i(Y, U) = 0. \quad (20)$$
Define
\[ \theta: \mathbb{C} \to \mathbb{R}^{2 \times 2}, \quad a + b \sqrt{-1} \mapsto \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \quad (a, b \in \mathbb{R}). \]

Denote by
\[ \Theta: \mathbb{C}^{2 \times 2} \to \mathbb{R}^{4 \times 4} \]
the \( \mathbb{R} \)-linear isomorphism that replaces each matrix entry \( z \) by the block \( \theta(z) \).

Define
\[ H: \mathbb{C} \to \mathbb{R}^{4 \times 4}, \quad a + b \sqrt{-1} \mapsto \begin{bmatrix} a & 0 & 0 & -b \\ b & a & 0 & 0 \\ 0 & 0 & a & -b \\ 0 & 0 & b & a \end{bmatrix}, \quad a, b \in \mathbb{R}. \]

Note that \( E := H(\mathbb{C}) \) is a 2-dimensional linear subspace of \( \mathbb{R}^{4 \times 4} \).

Let
\[ \Psi: T(W) \to \mathbb{C}^2. \]
be a trivialization of the complex vector bundle \( \tau(W) \) (see Definition 3.16). The real vector bundle \( \tau(W^\mathbb{R}) \) has the trivialization
\[ \Phi := \Theta \circ \Psi: T(W) \to \mathbb{R}^4. \]

With \( f: W \to \mathbb{C} \) as in (19), define
\[ A := H \circ f: W \to E. \]

Then
\[ \Phi(X_q) = A(q)\Phi(Y_q), \quad (q \in W). \]

Equation (20) and Proposition 3.18 imply the contradiction \( i_K(X) = 0 \).

We return to the general case of Theorem 1.3. Let \( \mathcal{H}_K \) be the collection of sets \( s \subset g \) satisfying:
\[ Z(s) \cap K \text{ is } g \text{-invariant and nonempty, and } g \text{ tracks } s. \]

\( \mathcal{H}_K \) is nonempty because it contains the singleton \( \{X\} \). By Proposition 2.2,
\[ s \in \mathcal{H}_K \implies Z(s) \text{ is } g \text{-invariant.} \]

There is a maximal element \( \hat{s} \in \mathcal{H}_K \) by Zorn’s Lemma and compactness of \( K \). We need to prove \( \hat{s} = g \).

If the nonempty, compact subvariety \( Z(\hat{s}) \subset K \) is 0-dimensional and hence finite, then \( Z(\hat{s}) \subset Z(g) \cap K \) by Proposition 2.2, entailing \( \hat{s} = g \). Therefore we can assume \( \dim(Z(\hat{s})) = 1 \). This implies \( Z(\hat{s}) = K \) because \( K \) is a connected, 1-dimensional analytic submanifold. By Theorem 4.3,
\[ Y \in g \implies K \cap Z(Y) \neq \emptyset. \]
Therefore

\[
K \cap Z(\hat{s} \cup \{Y\}) = K \cap (Z(\hat{s}) \cap Z(Y))
\]
\[
= (K \cap Z(\hat{s})) \cap Z(Y)
\]
\[
= K \cap Z(Y)
\]
\[
\neq \emptyset,
\]

hence

\[
Y \in \mathfrak{g} \implies \hat{s} \cup \{Y\} \in \mathcal{H}_K.
\]

Therefore maximality of \( \hat{s} \) implies

\[
Y \in \mathfrak{g} \implies Y \in \hat{s},
\]

which finishes the proof of Theorem 1.3.

Proof of Theorem 1.7 The analytic action \( \alpha \) of \( G \) on \( M \) induces an isomorphism from the Lie algebra \( \mathcal{L}(G) \) of \( G \) onto a Lie subalgebra \( \mathfrak{g} \subset \mathcal{v}^{\alpha}(M) \). Some element \( X \in \mathfrak{g} \) spans a 1-dimensional ideal because \( G \in \mathcal{G} \). As \( \chi(M) \neq 0 \), some component of \( Z(X) \) is an essential \( X \)-block by the Poincaré-Hopf Theorem 3.10. Therefore \( Z(\mathfrak{g}) \neq \emptyset \) by Theorem 1.3, proved above. Since \( G \) is connected, \( \text{Fix}(\alpha) = Z(\mathfrak{g}) \).

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