Cosmological evolution in Weyl conformal geometry

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Abstract

We discuss the cosmological evolution of the Weyl conformal geometry and its associated Weyl quadratic gravity. The Einstein gravity (with a positive cosmological constant) is recovered in the spontaneously broken phase of Weyl gravity; this happens after the Weyl gauge field ($\omega_\mu$) of scale symmetry, that is part of the Weyl geometry, becomes massive by Stueckelberg mechanism and decouples. This breaking is a natural result of the cosmological evolution of Weyl geometry, in the absence of matter. Of particular interest in the analysis is the special limiting case of Weyl integrable geometry. Both this case as well as the general one provide an accelerated expansion of the Universe, controlled by the scalar mode of the $\tilde{R}^2$ term in the action and by $\omega_0$. Their comparison to the ΛCDM model shows a very good agreement to this model for the (dimensionless) Hubble function $h(z)$ and the deceleration $q(z)$ for redshift $z \leq 3$. Therefore, the Weyl conformal geometry and its associated Weyl quadratic gravity provide an interesting alternative to the ΛCDM model and to the Einstein gravity.

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1 Introduction

The original Weyl quadratic gravity based on the Weyl conformal geometry [1–3] provided an alternative to Einstein’s general relativity. The theory has a gauged scale symmetry known as Weyl gauge symmetry that follows from the underlying Weyl geometry. Hence, both the action and the geometry (connection) have this symmetry. The Weyl quadratic gravity was soon disregarded by Einstein’s critique of its non-metricity [1] related to the (apparently massless) dynamical Weyl gauge field \( \omega_\mu \). Dirac revived the interest in this theory but considered Weyl’s quadratic action “too complicated to be satisfactory” and introduced instead a simplified, linear version of this theory [8]. Subsequent studies followed this approach [9–24] (for a review see [27]). These models are limited to Lagrangians linear in the scalar curvature \( \tilde{R} \) of Weyl geometry and thus need additional states (scalar fields beyond the Higgs) to maintain the Weyl gauge symmetry and to generate the mass scales of the theory (Planck, etc) by vacuum expectation values (vev’s) of these scalar fields.

The original Weyl quadratic gravity was re-considered in [28,29] where it was shown that the Weyl field \( \omega_\mu \) is actually massive, possibly near the Planck scale \( M_p \), after a geometric Stueckelberg mechanism; in this, \( \omega_\mu \) “absorbs” the scalar field \( \phi_0 \) extracted from the \( \tilde{R}^2 \) term in the action. Hence, non-metricity is not a problem since it is suppressed by the (large) mass of the Weyl gauge boson and the theory is then viable. Actually, it is the non-metricity of the underlying geometry that ensures a spontaneous breaking of the Weyl gauge symmetry and the mass generation, like Planck scale or the Weyl field mass \( M_\omega \). This breaking takes place in the absence of matter fields. Below \( m_\omega \), the field \( \omega_\mu \) decouples, to restore metricity and leave in the broken phase the Einstein gravity and a positive cosmological constant. For ultraweak coupling, \( m_\omega \) may be much lighter, of few TeV, which is the current lower bound on the non-metricity scale [30]. Similar results exist in Palatini quadratic gravity [6,7] and may apply to metric affine gravity [31–33].

The Weyl gauge symmetry is preferable since it has a non-trivial current [28,29] which is unlike the Weyl local symmetry (without \( \omega_\mu \)) [34,35], and it has a geometric interpretation (in Weyl geometry) which does not seem possible for Weyl symmetry (without \( \omega_\mu \)) [15,36,37]. It is also preferable to the global scale symmetry which is broken by black-hole physics [38]. The (geometric) field \( \omega_\mu \) may also bring a geometric solution to the dark matter problem.

Interestingly, Weyl geometry provides a natural embedding of the Standard Model (with a vanishing Higgs mass) without new degrees of freedom required beyond the SM spectrum and Weyl geometry [29]. Mass generation (Higgs vev, \( M_p \), fermions masses, etc) then follow from the Stueckelberg breaking of the Weyl gauge symmetry. Models in Weyl geometry also have successful inflation [7,39,40] with predictions similar to those in the Starobinsky model [41]. Briefly, Weyl geometry is a viable frame for model building beyond the SM that automatically includes the Einstein gravity and a positive cosmological constant.

Motivated by these results, here we consider the cosmological evolution in Weyl geometry and its associated Weyl quadratic gravity. The study continues that in [28,29] at the level of the equations of motion and sheds new light on the spontaneous breaking of the Weyl gauge symmetry: we show (Section 2) that the breaking of the symmetry and the

\[1\] We ignore here Weyl’s unfortunate wrong interpretation of \( \omega_\mu \) as the real photon.

\[2\] Actually, quadratic gravity in so-called “Palatini approach” due to Einstein [4,5] is also non-metric [37].
“gauge fixing” condition ($\nabla_\mu \omega^\mu = 0$) specific to a massive gauge field, are a natural result of the cosmological evolution in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe. An interesting aspect is that all mass scales of the theory (Planck scale $M_p$, cosmological constant $\Lambda$, $m_\omega$) have a geometric origin, due to $\phi_0$ propagated by the $\tilde{R}^2$ term. We then show (Section 3) that, in the absence of matter, the Weyl geometry and its associated Weyl quadratic gravity provide an accelerated expansion of the Universe in agreement with recent results [42–48]. The scalar mode (Stueckelberg field $\phi_0$) in the $\tilde{R}^2$ term of the action contributes (positively) to this acceleration together with the time-like component of $\omega_\mu$, which also gives a dark matter-like contribution (of opposite sign).

A particularly interesting case is the limit of a Weyl integrable geometry when $\omega_\mu$ is “pure gauge”, giving an isotropic solution. This case is discussed and compared numerically to the general case. A good agreement is found of both these Weyl cases with the $\Lambda$CDM model (based on the Einstein gravity with a cosmological constant) for the Hubble function $h(z)$ and the deceleration $q(z)$ (Section 3). These results indicate that the Weyl conformal geometry and its associated Weyl quadratic gravity can provide an interesting alternative to the $\Lambda$CDM model and to the Einstein gravity. This suggests that, ultimately, the underlying geometry of our Universe may actually be the Weyl conformal geometry. Our conclusions are found in Section 4.

2 Weyl action and spontaneous symmetry breaking

2.1 Brief review of Weyl action

Weyl geometry is defined by classes of equivalence $(g_{\alpha\beta}, \omega_\mu)$ of the metric $(g_{\alpha\beta})$ and the Weyl gauge field $(\omega_\mu)$, related by the Weyl gauge transformation, see (a) below. If matter is present, (a) must be extended by transformation (b) of the scalars $(\phi)$ and fermions $(\psi)$

\[ (a) \hspace{1cm} \hat{g}_{\mu\nu} = \Sigma^d g_{\mu\nu}, \hspace{1cm} \hat{\omega}_\mu = \omega_\mu - \frac{1}{\alpha} \partial_\mu \ln \Sigma, \hspace{1cm} \sqrt{\hat{g}} = \Sigma^{2d/3} \sqrt{g}, \]
\[ (b) \hspace{1cm} \hat{\phi} = \Sigma^{-d/2} \phi, \hspace{1cm} \hat{\psi} = \Sigma^{-2d/3} \psi, \hspace{1cm} (d = 1). \]  

Here $d$ is the Weyl charge of $g_{\mu\nu}$, $\alpha$ is the Weyl gauge coupling, $g = |\det g_{\mu\nu}|; \Sigma > 0$; without loss of generality, we set $d = 1$. The Weyl connection $\tilde{\Gamma}$ is a solution to $\nabla_\lambda g_{\mu\nu} = -\alpha \omega_\lambda g_{\mu\nu}$, where $\nabla_\mu$ is defined by $\tilde{\Gamma}_\mu^\lambda$ with:

\[ \tilde{\Gamma}_\mu^\lambda = \Gamma_\mu^\lambda + (1/2) \alpha \left[ \delta_\mu^\lambda \omega_\nu + \delta_\nu^\lambda \omega_\mu - g_{\mu\nu} \omega^\lambda \right], \hspace{1cm} \Rightarrow \omega_\mu \propto \tilde{\Gamma}_\mu - \Gamma_\mu. \]  

with the notation $\Gamma_\mu^\nu = \Gamma_\mu^\nu$, $\tilde{\Gamma}_\mu^\nu = \Gamma_\mu^\nu$. Hence, the Weyl gauge field $\omega_\mu$ measures the departure of the (trace of) the Weyl connection $\tilde{\Gamma}$ from the Levi-Civita connection $\Gamma$. If $\omega_\mu$ is massive and decouples ($\omega_\mu \to 0$), $\tilde{\Gamma} \to \Gamma$ and Weyl geometry becomes Riemannian.

With $\tilde{\Gamma}$ of (2) one defines the tensor and scalar curvature of Weyl geometry, via the usual formulae of the Riemannian case, see e.g. Appendix A in [29]. With this, one finds that the

\[ 1 \text{Our convention is } g_{\mu\nu} = (+, -, -, -) \text{ while the curvature tensors are defined as in [19].} \]
scalar curvatures $\tilde{R}$ of Weyl geometry and $R$ of the Riemannian geometry are related by

$$\tilde{R} = R - 3 \alpha \nabla_\mu \omega^\mu - \frac{3}{2} \alpha^2 \omega_\mu \omega^\mu. \quad (3)$$

The rhs of (3) is in a Riemannian notation, so $\nabla_\mu \omega^\lambda = \partial_\mu \omega^\lambda + \Gamma^\lambda_{\mu\rho} \omega^\rho$. The advantage of Weyl geometry is that $\tilde{R}$ transforms covariantly, just like the square of a scalar field.

The gravity action in Weyl geometry was introduced in [13] and here we follow [28]

$$\mathcal{L}_0 = \sqrt{g} \left[ \frac{1}{12} \frac{1}{\xi^2} \tilde{R}^2 - \frac{1}{4} F_{\mu\nu}^2 \right], \quad (5)$$

with perturbative coupling $\xi < 1$. Here $F_{\mu\nu} = \tilde{\nabla}_\mu \omega_\nu - \tilde{\nabla}_\nu \omega_\mu$ is the field strength of $\omega_\mu$, with $\tilde{\nabla}_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\chi_{\mu\nu} \omega_\chi$. Since $\Gamma^\alpha_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$ is symmetric, then $F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu$.

To simplify the calculations, in $\mathcal{L}_0$ one can replace $\tilde{R}^2 \rightarrow -2\phi_0^2 \tilde{R} - \phi_0^4$ where $\phi_0$ is a scalar field. This gives a classically equivalent Lagrangian since by using the solution $\phi_0^2 = -\tilde{R}$ of the equation of motion of $\phi_0$ back in the modified $\mathcal{L}_0$, one recovers onshell eq. (5). Hence

$$\mathcal{L}_0 = \sqrt{g} \left[ -\frac{1}{12} \frac{1}{\xi^2} \phi_0^2 \tilde{R} - \frac{1}{4!} \frac{1}{\xi^2} \phi_0^2 - \frac{1}{4} F_{\mu\nu}^2 \right]. \quad (6)$$

This is the simplest action with Weyl gauge symmetry that we shall use, equivalent to (5).

The advantage of (6) over (5) is that the equations of motion simplify considerably since (6) is now linear in the curvature while the field $\phi_0$ becomes dynamical, see later.

### 2.2 From Weyl to Einstein

As shown in [28, 29], $\mathcal{L}_0$ has spontaneous breaking to an Einstein-Proca Lagrangian of the Weyl gauge field. Here we briefly review this result. In $\mathcal{L}_0$ replace $\tilde{R}$ by eq. (3).

$$\mathcal{L}_0 = \sqrt{g} \left\{ -\frac{1}{12} \frac{1}{\xi^2} \phi_0^2 \tilde{R} - \frac{1}{4} \phi_0^4 \left( R - 3\alpha \nabla_\mu \omega^\mu - \frac{3}{2} \alpha^2 \omega_\mu \omega^\mu \right) - \frac{1}{4!} \frac{1}{\xi^2} \phi_0^2 \right\}. \quad (7)$$

This can be re-written as

$$\mathcal{L}_0 = \sqrt{g} \left\{ -\frac{1}{2} \phi_0^2 \frac{R}{\xi^2} + (\partial_\mu \phi_0)^2 - \frac{\alpha}{2} \nabla_\mu (\omega^\mu \phi_0^2) - \frac{\phi_0^4}{4!} \frac{1}{\xi^2} + \frac{\alpha^2}{8} \phi_0^2 \left[ \omega_\mu - \frac{1}{\alpha} \partial_\mu \ln \phi_0^2 \right]^2 - \frac{1}{4} F_{\mu\nu}^2 \right\} \quad (8)$$

Each term multiplied by $1/\xi^2$ and $\mathcal{L}_0$ are invariant under (1). One would like to “fix the gauge” of this Weyl gauge symmetry. To do so, apply to $\mathcal{L}_0$ transformation (1) with a

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4 $\mathcal{L}_0$ may also contain an additional term due to the Weyl tensor of Weyl geometry ($\tilde{C}_{\mu\nu\rho\sigma}$)

$$\mathcal{L}'_0 = -(\sqrt{g}/\eta^2) \tilde{C}^2_{\mu\nu\rho\sigma}, \quad \eta < 1, \quad \text{where} \quad \tilde{C}^2_{\mu\nu\rho\sigma} = C^2_{\mu\nu\rho\sigma} + (3/2) \alpha^2 F_{\mu\nu}^2. \quad (4)$$

where $\tilde{C}_{\mu\nu\rho\sigma}$ is related to its Riemannian counterpart $C_{\mu\nu\rho\sigma}$ as shown above. This term may be needed at a quantum level and brings a ghost degree of freedom and a renormalization of the coupling of the $F^2$ term. We do not include this term in our present study. For an analysis of the $C_{\mu\nu\rho\sigma}$ term see [30, 37].

5 Similar to the Riemannian case, $\tilde{R}^2$ propagates a spin-zero mode ($\phi_0$) beyond the graviton, because it contains the higher derivative $\tilde{R}^2$, see [28, 38]; that $\phi_0$ is dynamical is also seen from its eq of motion, eq. (16).
scale-dependent $\Sigma = \phi_0^2 / \langle \phi_0^2 \rangle$; this is fixing $\phi_0$ to its vev; naively, one simply sets $\phi_0 \rightarrow \langle \phi_0 \rangle$ in eq. (5). We discuss shortly (Section 2.3) how $\phi_0$ acquires a vev and how this gauge is fixed by the cosmological evolution. In terms of the new, transformed fields (with a “hat”), $L_0$ becomes

$$L_0 = \sqrt{g} \left[ -\frac{1}{2} M_p^2 \hat{R} + \frac{3}{4} M_p^2 \alpha^2 \hat{\omega}_\mu \hat{\omega}^\mu - \Lambda M_p^2 - \frac{1}{4} \hat{F}_{\mu\nu}^2 \right], \quad M_p^2 \equiv \frac{\langle \phi_0^2 \rangle}{6 \xi^2}; \quad \Lambda \equiv \frac{1}{4} \langle \phi_0^2 \rangle. \quad (9)$$

where we ignored a total derivative in the action. This is the Einstein-Proca Lagrangian for the Weyl vector [28, 29], in the Einstein gauge (“frame”). The Weyl gauge field has absorbed the derivative of the field $\ln \phi_0$ in a Stueckelberg mechanism: the massless $\omega_\mu$ and real, massless $\phi_0$ are replaced by a massive Weyl gauge field, with a mass $m_\omega^2 = (3/2) M_p^2 \alpha^2$. This mass is close to the Planck scale, unless one is tuning $\alpha \ll 1$; hence, any non-metricity effects are strongly suppressed by $m_\omega \sim M_p$. Current lower bounds on this mass (which sets the non-metricity scale) are actually very mild, close to the TeV scale [30]. Since $\omega_\mu$ is massive it can now decouple in eq. (9) to leave in the broken phase (below $m_\omega$) the Einstein action with a positive cosmological constant. Hence, the Einstein action is a “low energy” broken phase limit of the original Weyl quadratic gravity. At the same time the connection $\tilde{\Gamma}$ of (2) becomes Levi-Civita ($\Gamma$) and the geometry becomes Riemannian. All mass scales of the theory ($M_p, m_\omega, \Lambda$) have geometric origin, being proportional to the vev of $\phi_0$ propagated by the $\hat{R}^2$ term in the action. For details see [28, 29].

An interesting limit of Weyl geometry is the case $\omega_\mu = (1/\alpha) \partial_\mu \ln \phi_0^2$ i.e. $\omega_\mu$ is actually “pure gauge”. This is the so-called Weyl integrable limit of the Weyl geometry action considered.

In this case the kinetic term of $\omega_\mu$ is vanishing, so the action is then given by the first term in eq. (5). Then from eq. (6) without the last term, one can analytically integrate out $\omega_\mu$ and finds

$$L_0 = \sqrt{g} \left[ -\frac{1}{2} \frac{1}{6} \phi_0^2 R + g^{\mu\nu} \partial_\mu \phi_0 \partial_\nu \phi_0 \right] - \frac{1}{4!} \phi_0^4 \right]. \quad (10)$$

The action has now a Weyl local symmetry only (no $\omega_\mu$) (see also (1) with $\hat{\omega}_\mu = 0$ for suitable $\Sigma$). Notice that, just like in the general case, after gauge fixing in which $\phi_0 \rightarrow \langle \phi_0 \rangle$, one obtains the usual Einstein term and also the cosmological constant term from last term in (10), with a positive sign. The cosmological constant comes from the scalar mode “extracted” from $\hat{R}^2$, just like in the general case. So both the Einstein action and a positive cosmological constant are obtained in the broken phase and originate from the initial $\tilde{R}^2$. Hence this limiting case is not really conformal to the Einstein action (where $\Lambda$ can be added with arbitrary sign and size).

An action and results similar to (10) are also obtained from the general case when the mass of $\omega_\mu$ is large enough, $\sim$ Planck scale; then $\omega_\mu$ can be integrated out and one obtains again action (10) after gauge fixing, up to corrections suppressed by $M_p$.

6The cosmological implications of the Weyl integrable geometry were considered in [58], while for the analysis of other physical and geometrical aspects of the theory see [59, 60].

7We discuss in the next section how $\phi_0$ acquires a non-zero vev.

8A similar conclusion applies to $R^2$ gravity in the Palatini formalism, see [6] (Section 2) and [61].
2.3 Equations of motion

The above breaking of Weyl gauge symmetry discussed at the level of the Lagrangian can also be understood from the equations of motion, as a natural result of the cosmological evolution in an FLRW universe. To this purpose, let us write the equations of motion for our action, eq. (7). To simplify the notation below, let us denote:

\[ K = \frac{\phi_0^2}{\xi^2}, \quad V = \frac{1}{4!} \frac{\phi_0^4}{\xi^4}. \] (11)

Variation of (7) with respect to the metric gives

\[ \frac{1}{\sqrt{g}} \delta \mathcal{L}_0 \delta g^{\mu \nu} = \frac{1}{12} K \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \right) + \frac{1}{12} \left( g_{\mu \nu} \Box - \nabla_\mu \nabla_\nu \right) K - \frac{\alpha^2}{16} K \left( g_{\mu \nu} \omega^\rho \omega_\rho - 2 \omega_\mu \omega_\nu \right) + \frac{\alpha}{8} K \left( \nabla_\mu \omega_\nu + \nabla_\nu \omega_\mu - g_{\mu \nu} \nabla_\rho \omega^\rho \right) + \frac{1}{2} g_{\mu \nu} V + \frac{1}{2} \left( \frac{1}{4} g_\mu \omega_\beta F^{\alpha \beta} - g^{\alpha \beta} F_\alpha \omega_\beta \right). \] (12)

Taking the trace gives

\[ \frac{1}{12} KR + \frac{\alpha^2}{8} K \omega_\rho \omega^\rho - \frac{\alpha}{4} K \nabla_\rho \omega^\rho + 2V = 0. \] (13)

The equation of motion of \( \phi_0 \) is

\[ \frac{1}{12} KR - \frac{\alpha^2}{8} K \omega_\rho \omega^\rho - \frac{\alpha}{4} K \nabla_\rho \omega^\rho + \frac{1}{2} \phi_0 \frac{\partial V}{\partial \phi_0} = 0. \] (14)

On the ground state this gives \( \langle \phi_0^2 \rangle = -\tilde{R} = -[R - (3/2)\alpha^2 \omega_\rho \omega^\rho] \), which we already know.

The equation of motion of \( \omega_\mu \) is

\[ \frac{\alpha^2}{4} K \omega^\rho - \frac{\alpha}{4} g^{\rho \sigma} \nabla_\sigma K + \nabla_\rho F^{\rho \sigma} = 0. \] (15)

From eqs. (13), (14) then

\[ \Box K = 0, \quad \Rightarrow \quad \partial^\mu \left( \sqrt{g} \partial_\mu \phi_0^2 \right) = 0. \] (16)

where \( \Box = \nabla^\mu \nabla_\mu \). There is thus an onshell conserved current \( K_\mu \equiv \sqrt{g} \partial_\mu \phi_0^2 \). The equation of motion of \( \phi_0 \), eq. (16), is non-trivial showing that this field, after “linearising” \( \tilde{R}^2 \), became dynamical and corresponds to the spin-zero mode that \( \tilde{R}^2 \) propagates beyond the graviton (similar to the Riemannian \( R^2 \) that it actually contains).

From the equation of motion of \( \omega_\mu \) by applying \( \nabla_\sigma \) we find a conserved current:

\[ \nabla_\mu J^\mu = 0, \quad J^\mu = -\frac{\alpha}{4} g^{\mu \nu} \left( \partial_\nu - \alpha \omega_\nu \right) K = -\frac{\alpha}{4 \xi^2} g^{\mu \nu} \left( \partial_\nu - \alpha \omega_\nu \right) \phi_0^2, \] (17)

\[ ^9 \text{In the global scale invariant case there is a non-trivial current \cite{62,63}, as above but with } \omega_\mu = 0. \]
where we used the antisymmetry of $F_{\mu\nu}$. But using that $\Box K = 0$ we also find

$$\nabla_\mu J^\mu = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} J^\mu) = \frac{\alpha^2}{4\xi^2} \sqrt{g} \nabla_\mu (\phi_0^2 \omega^\mu). \quad (18)$$

Hence $\nabla_\mu (\phi_0^2 \omega^\mu) = 0$ which will be used in Section 2.4. Notice that in the Weyl integrable limit $\omega_\mu = (1/\alpha) \partial_\mu \ln \phi_0^2$ the current is vanishing, while $\nabla_\mu (\phi_0^2 \omega^\mu) = 0$ becomes $\Box \phi_0^2 = 0$ which is already seen in (16).

2.4 Gauge fixing and symmetry breaking by cosmological evolution

Consider hereafter the FLRW metric $g_{\mu\nu} = (1, -a(t)^2, -a(t)^2, -a(t)^2)$ and with $\phi_0 = \phi_0(t)$ only, then eq.(16) can be written as below, with $H = \dot{a}/a$:

$$\ddot{K} + 3H \dot{K} = 0. \quad (19)$$

This gives

$$\dot{\phi}_0 = \frac{c_0}{a(t)^3 \phi_0}, \quad \text{and} \quad \phi_0^2(t) = c_0 \int_0^t \frac{d\tau}{a(\tau)^3} + c_1, \quad (20)$$

where $c_{0,1}$ are some constants. For $t \to \infty$, similar to a global Weyl symmetry [62–65], $\phi_0$ evolves to a constant. What happens then to this degree of freedom? In this case we find from eq.(18) a gauge fixing condition specific to massive gauge fields

$$\nabla_\mu \omega^\mu = 0. \quad (21)$$

This means that the field $\omega_\mu$ has become a massive Proca field, by “absorbing” the $\phi_0$ degree of freedom which thus disappears from the spectrum. To see this in more detail, consider the quasi-homogeneous case of $\partial_i \omega_\mu = 0$. Eq.(15) for $\omega_\mu$ gives for temporal $\mu = 0$ and spatial $\mu = i$ components

$$\frac{\alpha}{2} \omega_0 = \partial_0 \ln \phi_0, \quad \ddot{\omega}_i + H \dot{\omega}_i + \frac{\alpha^2}{4\xi^2} \omega_i \phi_0^2 = 0. \quad (22)$$

The solution for $\omega_0(t)$ is

$$\omega_0(t) = 2c_0 \left[ \alpha a(t)^3 \phi_0(t)^2 \right]^{-1} \quad (23)$$

When $\phi_0$ becomes a constant (vev), $\omega_0 \to 0$ while $\omega_i$ satisfies eq.(22) but with $\phi_0 \to \langle \phi_0 \rangle$:

$$\ddot{\omega}_i + H \dot{\omega}_i + \frac{\alpha^2}{4\xi^2} \omega_i \langle \phi_0^2 \rangle = 0. \quad (24)$$

\textsuperscript{10}The other possibility consistent with a FLRW metric is $\omega_i(t) = 0$, with $i = 1, 2, 3$, see next section.
Hence $\omega_i$ satisfies the equations of an oscillator with a mass\footnote{Since the equation of motion is linear in $\omega_\mu$, the perturbations about $\omega_\mu(t)$ respect the same relation.}

\[ m^2_\omega = \frac{\alpha^2}{4 \xi^2} \langle \phi^2_0 \rangle. \]  

(25)

This result is also obvious from eq. (15) with $\phi_0 \rightarrow \langle \phi_0 \rangle$, which shows a Proca field equation. From (7) the Planck scale

\[ M^2_p = \frac{\langle \phi_0 \rangle^2}{6 \xi^2}, \quad \Rightarrow \quad m^2_\omega = \frac{3 \alpha^2}{2} M^2_p, \]  

(26)

in agreement with the mass of $\omega_\mu$ shown in the Lagrangian of eq. (4).

Therefore, the breaking of the symmetry, Proca mass generation for $\omega_\mu$ and the gauge fixing of the Weyl gauge symmetry are natural results of the cosmological evolution in the FLRW universe. After $\omega_\mu$ decouples the connection (2) evolves into Levi-Civita and then the geometry becomes Riemannian. These results, obtained from the equations of motion, complement the Lagrangian picture reviewed in Section 2.2. This breaking mechanism is entirely geometrical: there is no scalar field added to this purpose: $\phi_0$ has geometrical origin, from the $\tilde{R}^2$ term, while $\omega_\mu$ is an intrinsic part of the underlying Weyl geometry.

Further, the solution to (24), assuming $H \approx \text{constant}$, is of the form

\[ \omega_i(t) = \frac{1}{\sqrt{\alpha}} [A \cos \theta(t) + B \sin \theta(t)], \quad \theta(t) = \gamma t, \quad \gamma^2 = m^2_\omega - \frac{1}{4} H^2. \]  

(27)

with $A, B$ constants; for $m^2_\omega \gg H^2$ this solution oscillates rapidly. In the general case ($H$ not constant) $A, B$ become functions of time. This will be used in cosmological applications.

Finally, note that eq. (12) also gives for $i \neq j$ that

\[ \frac{\alpha^2}{4} K \omega_i \omega_j = \dot{\omega}_i \dot{\omega}_j, \quad i \neq j, \quad i, j = 1, 2, 3. \]  

(28)

One immediate solution to (28) consistent with the isotropy of the FLRW metric is $\omega_\mu(t) = (\omega_0(t), 0, 0, 0)$. There is a second, “anisotropic” solution, with $\omega_{1,2} = 0, \omega_3 \neq 0$ so $\omega_\mu(t) = (\omega_0(t), 0, 0, \omega_3(t))$; then eq. (27) actually applies to $\omega_3$. This gives a diagonal stress-energy tensor for the contribution of $\omega_\mu(t)$ but with a different value along OZ (as expected). Since the contribution of $\omega_i, i = 1, 2, 3$ to the stress energy tensor in eq. (12) is suppressed by the scale factor (see (27)) and $\omega_i$ oscillates rapidly, the time average of this contribution may be small and the overall anisotropy may be mild enough, while the contribution of $\phi_0$ to the stress energy tensor may dominate.

Note that, when taking account of the first equation in (22), then the first solution above (“isotropic” case) corresponds to the limiting case of a Weyl integrable geometry mentioned earlier, when $\omega_\mu$ is “pure gauge”. The second (“anisotropic”) solution is the most general in Weyl geometry. The cosmological implications of both solutions are discussed shortly.
3 Cosmological applications of Weyl geometry

The present-day Universe is in a state of accelerating expansion [42-48]. The analysis of temperature fluctuations of the cosmic microwave background radiation (CMB) by the Planck mission [66, 67] has revealed that the matter content of the Universe consists of 5% baryonic matter and 95% accounted for by two mysterious components: the dark energy (with negative pressure) and dark matter [68-71], respectively.

To explain these cosmological observations the ΛCDM model was proposed, based on the introduction in the Einstein gravitation field equation of the cosmological constant Λ, first used by Einstein [72] to obtain a static (unstable) model of the Universe. The ΛCDM model gives a good fit of the data, but its foundations are questionable due to the lack of solid theoretical basis; this is due to the uncertainties in the physical and geometrical interpretation of Λ (for a discussion see [73, 74]).

The Weyl conformal geometry may provide a solution to this problem. Firstly, we saw in the previous sections that it can naturally recover Einstein gravity and predicts a positive cosmological constant in the broken phase. In this section we examine the implications for cosmology of Weyl quadratic gravity in its symmetric phase, together with its underlying Weyl geometry, and compare the results to those in the ΛCDM model.

Our study below considers first the Weyl model with the solution that is compatible with the isotropy of the FLRW metric, i.e.

\[ \dot{\omega}_0(t) = (\omega_0(t), 0, 0, 0). \]

In the Appendix we re-do the analysis below for the Weyl model using the second solution \( \omega_\mu(t) = (\omega_0(t), 0, 0, \omega_3(t)) \), and provide the technical details; the formalism is similar and in this case we gain a good insight into the impact of the effect of space-like components of \( \omega_\mu \) relative to the isotropic solution, in a first approximation. The numerical results of the two cases will then be compared to those of the ΛCDM, see later (Figures 1 and 2).

3.1 Accelerated expansion

From eq.(12) we find from the “00” and “ij” components, respectively

\[ \frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} - \frac{\phi_0}{\phi_0} + 3H \frac{\phi}{\phi_0} - \frac{\phi_0^2}{12} = 0, \]  

(29)

\[ \frac{\ddot{a}^2}{a^2} + 2 \frac{\ddot{a}}{a} + \frac{\kappa}{a^2} + 3 \frac{\dot{\phi}_0}{\phi_0} + 9H \frac{\dot{\phi}_0}{\phi_0} - \frac{\phi_0^2}{4} = 0, \]  

(30)

The last term in eqs.(29), (30) is due to the potential of \( \phi_0 \). From eq.(19)

\[ -\frac{\ddot{\phi}_0}{\phi_0} = \frac{\phi_0^2}{\phi_0^2} + 3H \frac{\dot{\phi}_0}{\phi_0}, \]  

(31)

\footnote{Strictly speaking, this case would also demand a suitably modified (“anisotropic”) FLRW metric along OZ, but that would introduce an additional parameter (scale factor) in the theory and that would make the analysis less predictable. Including this case in the analysis here was motivated by recent results in [75] that may question the usual FLRW metric assumption and, secondly, by the fact that the formalism is similar to that of the main, isotropic case.}
Then eqs. (29) and (30) become
\[
\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} + \frac{\dot{\phi}^2_0}{\phi^2_0} + 6H\frac{\dot{\phi}_0}{\phi_0} - \frac{\phi^2_0}{12} = 0 \tag{32}
\]
\[
\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + \frac{\kappa}{a^2} - 3\frac{\dot{\phi}^2_0}{\phi^2_0} - \frac{\phi^2_0}{4} = 0, \tag{33}
\]
Subtracting these
\[
\frac{\ddot{a}}{a} - \frac{2\dot{\phi}^2_0}{\phi^2_0} - 3H\frac{\dot{\phi}_0}{\phi_0} - \frac{\phi^2_0}{12} = 0. \tag{34}
\]
This shows there is a time-dependent \(\ddot{a}(t)\) of the Universe expansion. There are three terms contributing to \(\ddot{a}\): the terms depending on \(\dot{\phi}_0\) are due to \(\omega_0\); of these, the term \(\propto \dot{\phi}^2_0\) gives a positive contribution to \(\ddot{a}\), while the term \(\propto \phi_0\) may give a positive (negative) contribution, depending on the positive (negative) sign of \(c_0\) in eq. (20), respectively. This means it depends on the initial condition imposed on \(\dot{\phi}_0(0)\). Further, the term involving \(\phi^2_0\) is due to the potential of \(\phi_0\) and gives a positive contribution to \(\ddot{a}\) - it is related to the cosmological constant \(\Lambda = 1/4\langle \phi^2_0 \rangle\) after symmetry breaking.

In conclusion, the acceleration is controlled by \(\omega_0(t) \sim \partial_0 \ln \omega_0\) and the scalar mode \(\phi_0\) of \(R^2\) term, and is thus of geometric origin. It is intriguing to see the multiple role of \(\phi_0\): it induces the Stueckelberg mechanism of symmetry breaking and subsequently becomes part of the Weyl-Proca massive field; its vev generates \(M_p = \langle \phi^2_0 \rangle / (6\xi^2)\), the cosmological constant \(\Lambda = \langle \phi^2_0 \rangle / 4\) and an acceleration of the expansion, giving a dark energy - like contribution.

Let us also consider the limit \(t \to \infty\), then \(\phi_0 \to \langle \phi_0 \rangle\) (broken phase), then from eq. (32)
\[
\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} - \frac{1}{3} \Lambda \approx 0, \quad \Lambda \equiv \frac{1}{4}\langle \phi^2_0 \rangle. \tag{35}
\]
In the same limit, from (34)
\[
\frac{\ddot{a}}{a} - \frac{1}{3} \Lambda = 0. \tag{36}
\]
Hence, in this limit the acceleration is given by the cosmological constant itself. We simply recovered from the Weyl model the usual de Sitter exponentially expanding Universe.

For completeness, let us also present the form of eqs. (29), (30) in the presence of matter
\[
\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} + \frac{\dot{\phi}^2_0}{\phi^2_0} + 6H\frac{\dot{\phi}_0}{\phi_0} - \frac{\phi^2_0}{12} = \frac{1}{3}\xi^2 \mathcal{T}_{00} \tag{37}
\]
\[
\frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} + \frac{\kappa}{a^2} - \frac{3}{2}\frac{\dot{\phi}^2_0}{\phi^2_0} + 2\frac{\ddot{\phi}_0}{\phi_0} + 6H\frac{\dot{\phi}_0}{\phi_0} - \frac{\phi^2_0}{4} = \frac{1}{3}\xi^2 \mathcal{T}^i_i. \tag{38}
\]
Using (31). With \(\mathcal{T}_{00}, \mathcal{T}^i_i\) denoting the stress energy tensor matter matter contributions. Eqs. (37), (38) have similarities to eqs. (27), (28) of [76] which where written without actually providing
a Lagrangian, by additional assumptions, in an attempt to uplift the Einstein’s equations
to a scale invariant form (the coefficient 6 in (37) and eqs. (38) is replaced in (76) by 2 and
4 respectively and $\phi_0 \to \lambda$).

3.2 Friedmann equations and Weyl cosmology

The general Friedmann equations are shown in eqs. (32), (33). Using eq. (20) we replace the
derivatives of $\phi_0$ in terms of $\phi_0(t)$ itself. Then

$$3H^2 = -\frac{3c_0^2}{a^6\phi_0^4} - 18H \frac{c_0}{a^3\phi_0^2} + \frac{\phi_0^2}{4},$$

and

$$2\dot{H} + 3H^2 = \frac{3c_0^2}{a^6\phi_0^4} + \frac{\phi_0^2}{4}.\tag{40}$$

To study the cosmology of the model defined by eqs. (39) and (40) we first re-express the
time coordinate, the Hubble function and $\phi_0$ in terms of dimensionless variables ($\tau, h, \phi$)

$$\tau = H_0 t, \quad H = H_0 h, \quad \phi_0 = H_0 \phi, \tag{41}$$

where $h, \phi$ are functions of $\tau$; $H_0$ is the present value of the Hubble function. Therefore

$$\dot{\phi}_0(t) = \phi'(\tau) H_0^2, \quad \dot{\phi}_0(t)|_{t=0} = \phi'(\tau)|_{\tau=0} H_0^2, \quad \phi_0(t)|_{t=0} = \phi(\tau)|_{\tau=0} H_0. \tag{42}$$

From eq. (20)

$$\frac{d\phi}{dt} - \frac{c_h}{a^3\phi} = 0, \quad \text{where} \quad c_h \equiv \frac{c_0}{H_0^3} = \left[\phi(\tau)\phi'(\tau)\right]|_{\tau=0}. \tag{43}$$

The Friedmann equations become

$$3h^2(\tau) = -\frac{3c_h^2}{a^6\phi^4} - 18c_h \frac{h}{a^3\phi^2} + \frac{\phi^2}{4}, \quad (= \rho_{\text{eff}}). \tag{44}$$

$$2\frac{dh(\tau)}{d\tau} + 3h^2(\tau) = \frac{3c_h^2}{a^6\phi^4} + \frac{\phi^2}{4}, \quad (= -p_{\text{eff}}). \tag{45}$$

To compare the cosmological predictions of the Weyl geometry-based model to the
ΛCDM and to the observations, we introduce the redshift $z$ via $1 + z = 1/a$. With
this notation and eqs. (43) to (45) we have a system of first order differential equations of the
cosmological evolution in the redshift space

$$\frac{d\phi(z)}{dz} + (1 + z)^2 \frac{c_h}{\phi(z) h(z)} = 0, \tag{46}$$

As a result, we have $d/dt = (dz/dt)(d/dz) = -(1 + z) H(z) (d/dz)$.
with \( c_h = -\phi(z=0)\phi'(z=0) \) and
\[
(1 + z)^3 \frac{d}{dz} \left\{ \frac{h^2(z)}{(1 + z)^3} \right\} + \frac{3c_h^2(1 + z)^5}{\phi^4(z)} + \frac{\phi^2(z)}{4(1 + z)} = 0,
\]
and the constraint (closure relation)
\[
h^2(z) + \frac{6c_h(1 + z)^3}{\phi^2(z)} h(z) + \frac{c_h^2(1 + z)^6}{\phi^4(z)} - \frac{\phi^2(z)}{12} = 0.
\]
Eqs. (46) to (48) define our Weyl cosmological model. This is solved numerically for various initial conditions \( (\phi(z=0), \phi'(z=0), h(z=0)) \). From eq. (48), one can also express analytically \( h(z) \) in terms of \( \phi(z) \) (or vice-versa) and replace it in (47). At large field values \( \phi(z)^6 \gg 96c_h^3(1 + z)^6 \), the middle terms in (48) are suppressed and then \( h(z) \approx \phi(z)/(2\sqrt{3}) \).

Finally, introduce the deceleration function \( q(z) \)
\[
q = \frac{d}{d\tau} \frac{1}{h(\tau)} - 1 = (1 + z) \frac{1}{h(z)} \frac{dh(z)}{dz} - 1,
\]
with \( h(z) \) a solution to the above system. Eq. (49) will be used for the numerical analysis.

With the notation in eqs. (44), (45), we find
\[
6h(\tau)\frac{d\tau}{d\tau} = \frac{d\rho_{\text{eff}}(\tau)}{d\tau}.
\]
and
\[
\frac{d\rho_{\text{eff}}(\tau)}{d\tau} + 3h(\tau) \left[ p_{\text{eff}}(\tau) + \rho_{\text{eff}}(\tau) \right] = 0.
\]
This gives the energy conservation equation for the Weyl cosmological model.

### 3.3 Weyl cosmology versus ΛCDM

In this section we compare the ΛCDM model to the Weyl cosmological model defined above. In the ΛCDM model the simplifying hypothesis that the matter content of the late Universe contains only dust matter is generally adopted. Therefore the matter in the present day Universe has negligible thermodynamic pressure. Hence, the energy conservation equation, \( \dot{\rho} + 3H (\rho + p) = 0 \), of standard cosmology gives for the time variation of the energy density of the dust matter with \( p = 0 \) the simple expression \( \rho = \rho_0/a^3 = \rho_0(1 + z)^3 \), where \( \rho_0 \) is the present day matter density.

The time evolution of the Hubble function in terms of the scale factor and of the redshift \( z \) is given by
\[
H = H_0 \sqrt{(\Omega_b + \Omega_{DM}) a^{-3} + \Omega_{\Lambda}}
\]
\[
= H_0 \sqrt{(\Omega_b + \Omega_{DM}) (1 + z)^3 + \Omega_{\Lambda}},
\]
where $\Omega_b$, $\Omega_{DM}$, and $\Omega_\Lambda$ denote the density of the baryonic matter, of the cold (pressureless) dark matter, and of the dark energy (modeled by a cosmological constant), respectively, while $H_0$ is the present-day value of the Hubble function. These three densities obey the closure relation $\Omega_b + \Omega_{DM} + \Omega_\Lambda = 1$, which shows that the geometry of the Universe is flat, a relation that was confirmed by observations.

The deceleration parameter in the standard general relativistic cosmology is given by

$$q(z) = \frac{3(1+z)^3(\Omega_{DM} + \Omega_b)}{2(\Omega_\Lambda + (1+z)^3(\Omega_{DM} + \Omega_b))} - 1. \quad (53)$$

To compare the predictions of the Weyl cosmological model to the ΛCDM model, we adopt for the density parameters the values $\Omega_{DM} = 0.2589 \pm 0.0057$, $\Omega_b = 0.0486 \pm 0.0010$, and $\Omega_\Lambda = 0.6911 \pm 0.0062$, respectively, which follow from the Planck data. Hence, the total matter density $\Omega_m = \Omega_{DM} + \Omega_b \approx 0.31$. With the help of the density parameters we obtain for the present-day value of the deceleration parameter the value $q(0) = -0.5381$. This indicates that at present the Universe is in an accelerating phase. In our comparison below of the Weyl model versus ΛCDM we also include the observational data for the redshift dependence of the Hubble function, by using the data quoted in Table IV of [78] (see references therein for the observational results and their error bars).

In Figure 1, the Hubble function $h(z)$ for the Weyl cosmological model is compared to its evolution in the ΛCDM standard model and to the data [78]. For a chosen set of initial conditions shown in this figure, we see that the Weyl model gives a very good description of the data and is in good agreement with the ΛCDM model. This is true for the main case considered here of solution $\omega_\mu = (w_0, 0, 0, 0)$, but also for the case with solution $\omega_\mu = (\omega_0, 0, 0, \omega_3)$ discussed in the Appendix, for suitable initial conditions of the fields detailed in this figure.

In Figure 2 a comparison is shown of the deceleration function $q(z)$ in the Weyl cosmological model versus that in the ΛCDM. One can notice that there is a good agreement between the predicted evolution of $q(z)$ by the Weyl model and the ΛCDM for the main case here with solution $\omega_\mu = (w_0, 0, 0, 0)$; for the case with solution $\omega_\mu = (\omega_0, 0, 0, \omega_3)$ differences emerge for larger values of $z$, that depend on the initial conditions for the fields; for example, the deceleration can return to negative values near $z \approx 3$.

The numerical analysis also shows that the scalar field $\phi(z)$ is a monotonically increasing function of the redshift (monotonically decreasing function of time), in a range between 2.5 and 3.5 (for the considered $\phi(0)$ values shown in the figures) and for $0 \leq z \leq 3$. A similar (monotonically increasing) behaviour and values exist for $\tilde{\omega}(z)$ in the case $\omega_\mu = (\omega_0, 0, 0, \omega_3)$, with $0 \leq z \leq 3$ and with the corresponding initial conditions of this case shown in Figures 1, 2.

Unlike for the case of $\omega_\mu = (\omega_0, 0, 0, 0)$, the “anisotropic” case $\omega_\mu = (\omega_0, 0, 0, \omega_3)$ brings in a dependence of the predictions discussed above, on the couplings $\alpha$ and $\xi$. We checked that these couplings remain in a perturbative regime for a suitable choice of $\omega_3(0)$ and $\omega_3'(0)$, see the Appendix.

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14 For the Hubble constant we take the value $H_0 = 67.74 \pm 0.46$ km/s/Mpc [66,67].
Figure 1: **Left plot:** The case $\omega_\mu = (\omega_0, 0, 0, 0)$. The dimensionless Hubble function $h(z)$ in $\Lambda$CDM (red curve) and in Weyl cosmology as a function of the redshift for initial conditions: $h(0) = 1$, $\phi'(z = 0) = 0.06$ and with different $\phi(z = 0) = 2.67$ (dotted curve), $\phi(z = 0) = 2.75$ (short dashed curve), $\phi(z = 0) = 2.81$ (dashed curve) and $\phi(z = 0) = 2.89$ (long-dashed curve).

**Right plot:** The case $\omega_\mu = (\omega_0, 0, 0, \omega_3)$ (detailed in the Appendix). The dimensionless Hubble function $h(z)$ in $\Lambda$CDM (red curve) and in Weyl cosmology as a function of the redshift for initial conditions: $h(0) = 1$, $\phi'(z = 0) = 0.06$, $\phi(z = 0) = 2.67$ (dotted curve), $\phi(z = 0) = 2.75$ (short dashed curve), $\phi(z = 0) = 2.81$ (dashed curve) and $\phi(z = 0) = 2.89$ (long-dashed curve). Here $\tilde{\omega}$ and $\omega_3$ are related by [41]. The experimental data [78] are presented with their error bars.

Figure 2: **Left plot:** The case $\omega_\mu = (\omega_0, 0, 0, 0)$. The deceleration $q(z)$ in $\Lambda$CDM (red curve) and in Weyl cosmology as a function of the redshift $z$ for initial conditions $h(z = 0) = 1$, $\phi'(z = 0) = 0.06$, with $\phi(z = 0) = 2.67$ (dotted curve), $\phi(z = 0) = 2.75$ (short dashed curve), $\phi(z = 0) = 2.81$ (dashed curve) and $\phi(z = 0) = 2.89$ (long-dashed curve).

**Right plot:** The case $\omega_\mu = (\omega_0, 0, 0, \omega_3)$ (detailed in the Appendix): The deceleration $q(z)$ in $\Lambda$CDM (red curve) and in Weyl cosmology as a function of redshift $z$ for initial conditions: $h(0) = 1$, $\phi'(z = 0) = 0.06$, $\phi(z = 0) = 2.81$ and $\tilde{\omega}(z = 0) = 4.3$, hence $\lambda = 0.057/\tilde{\omega}(z = 0)^2 = 0.003$, for different values of $u(z = 0)$: $-0.7$ (dotted curve), $-0.8$ (short dashed curve), $-0.9$ (dashed curve) and $-1$ (long dashed curve); $\tilde{\omega}$ and $\omega_3$ are related by [41]. The deceleration can return to negative values at higher $z$.  

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4 Conclusions

The Weyl conformal geometry is relevant in the early Universe where all states are essentially massless and effective theories at short distances may become conformal or Weyl invariant. Therefore, this geometry provides the natural framework for studying cosmology. Theories built in Weyl geometry have the unique feature that both the action and the underlying geometry (connection) are Weyl (gauge) invariant. Theories built in the Riemannian space-time do not have this feature. This geometry also allows a natural embedding of the Standard Model without any additional degrees of freedom beyond those of the SM and Weyl geometry. Models based on Weyl geometry have successful inflation with predictions similar to those of the Starobinsky model.

With this motivation, we studied the cosmological evolution of the Weyl conformal geometry and its associated Weyl quadratic gravity, in the absence of matter. In previous work we showed at the level of the Lagrangian how the Weyl gauge field becomes massive, by a Stueckelberg mechanism in which the Weyl field is “absorbing” the massless scalar field $\phi_0$ present in the $\tilde{R}^2$ term in the action. This mechanism is of geometric nature i.e. it takes place in the absence of matter, since both the Weyl vector $\omega_\mu$ and the scalar $\phi_0$ have a geometric origin. In this work we re-examined this mechanism at the level of the equations of motion. We showed how the symmetry breaking and the gauge fixing condition specific to massive Proca fields $\nabla_\mu \omega^\mu = 0$ are a natural result of the cosmological evolution. This relation emerges dynamically in a FLRW Universe, from the Weyl current conservation, after $\phi_0$ becomes constant (acquires a vev) at late times. This shows the spontaneous breaking of the Weyl gauge symmetry, in the absence of matter, to an Einstein-Proca action. After the massive Weyl gauge field decouples, the Einstein gravity is recovered with a positive cosmological constant. The mass of $\omega_\mu$ is near the Planck scale, unless one tunes $\alpha \ll 1$.

We showed that the Weyl conformal geometry alone provides a natural explanation for the accelerated expansion of the Universe. The associated, relevant degrees of freedom are: the scalar mode $\phi_0$ (that “linearises” the $\tilde{R}^2$ term in the action) and the time component of the Weyl gauge field $\omega_\mu$ that give positive contributions to this acceleration; further, $\omega_0$ also has a negative (dark matter-like) contribution, while spatial components $\omega_k, k = 1, 2, 3$, if present, have negative contributions, too. The scalar $\phi_0$ also generates the cosmological constant, that gives a dark energy-like contribution.

We compared the Weyl cosmological model to the data and to the $\Lambda$CDM model based on Einstein gravity with a cosmological constant. The Weyl integrable model, with solution $\omega_\mu = (\omega_0, 0, 0, 0)$ consistent with the FLRW metric, and $\Lambda$CDM model have a similar dependence of the Hubble function $h(z)$ and deceleration function $q(z)$ in terms of the redshift variable ($z \leq 3$). In this case, the agreement with the $\Lambda$CDM and also with the data is independent of the actual values of the couplings $\xi$ and $\alpha$. We also explored the more general Weyl model having an anisotropic solution $\omega_\mu = (\omega_0, 0, 0, \omega_3)$, to gain an insight into this case. We found in general similar results, for perturbative values of the couplings of the theory. In conclusion, Weyl geometry and its associated quadratic gravity can provide an interesting alternative to the $\Lambda$CDM and to the Einstein gravity; this means that, ultimately, the underlying geometry of our Universe may actually be Weyl conformal geometry. These results open a new direction of research in cosmology that deserves further study.
Appendix

A. Cosmological applications: second solution

We present here the implications for cosmology of the second (“anisotropic”) solution discussed in Section 2.4, $\omega_{\mu} = (\omega_0(t), 0, 0, \omega_3(t))$, that leads to the numerical results presented in Figures 1 and 2. The analysis is very similar to that in the text for the isotropic solution.

A.1 Accelerated expansion

From eq.(12) we find the equations for the “00” and “ij” components

\[
\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} \frac{2 \xi^2}{\phi_0^2} T_\omega^+ - \frac{\ddot{\phi}_0}{\phi_0} + 3 H \frac{\dot{\phi}_0}{\phi_0} - \frac{\dot{\phi}_0^2}{\phi_0^2} \frac{12}{12} = 0, \tag{A-1}
\]

\[
\frac{\dot{a}^2}{a^2} + 2 \frac{\ddot{a}}{a} + \frac{\kappa}{a^2} \frac{2 \xi^2}{\phi_0^2} T_\omega^- + 3 \frac{\ddot{\phi}_0}{\phi_0} + 9 H \frac{\dot{\phi}_0}{\phi_0} + \frac{\dot{\phi}_0^2}{\phi_0^2} \frac{4}{4} = 0, \tag{A-2}
\]

with

\[
T_\omega^\pm = \frac{\alpha^2 \phi_0^2}{\omega_k \omega_k} \frac{\dot{\phi}_k \dot{\phi}_k}{a^2} \pm \frac{\dot{\phi}_0^2}{a^2}. \tag{A-3}
\]

$T_\omega^\pm$ is the contribution of space-like components $\omega_k$ ($k = 1, 2, 3$) to the stress-energy tensor, while the similar contribution of $\omega_0$ is given by the two terms involving $\ddot{\phi}_0$, $\dot{\phi}_0$, via eq.(22).

The sum above over $k$ is actually restricted to $k = 3$. Setting $T_\omega^\pm = 0$ one recovers eqs.(29), (30) and subsequent. Using eq.(19), (31) then

\[
\frac{\dot{a}^2}{a^2} + \frac{\kappa}{a^2} - \frac{2 \xi^2}{\phi_0^2} T_\omega^+ \phi_0^2 + 6 H \frac{\dot{\phi}_0}{\phi_0} \phi_0^2 \frac{12}{12} = 0 \tag{A-4}
\]

\[
\frac{\dot{a}^2}{a^2} + 2 \frac{\ddot{a}}{a} + \frac{\kappa}{a^2} \frac{2 \xi^2}{\phi_0^2} T_\omega^- \phi_0^2 + 3 \frac{\ddot{\phi}_0}{\phi_0} \phi_0^2 \frac{4}{4} = 0 \tag{A-5}
\]

Subtracting them

\[
\frac{\ddot{a}}{a} + \frac{\xi^2}{\phi_0^2} \frac{\dot{\phi}_k \dot{\phi}_k}{a^2} - 2 \frac{\ddot{\phi}_0}{\phi_0^2} - 3 H \frac{\dot{\phi}_0}{\phi_0} \phi_0^2 \frac{12}{12} = 0. \tag{A-6}
\]

There is again a time-dependent $\ddot{a}(t)$ of the Universe expansion with an interpretation similar to that in the text after eq.(34); however, now there is an additional contribution from the term $\propto \xi^2$ and due to $\omega_k$ that gives a negative contribution to $\ddot{a}$; this term is suppressed by $a^2$ and by the coupling $\xi \ll 1$ of the $\tilde{R}^2$ term in the action. In conclusion, the acceleration is controlled by $\omega_0(t)$ and the scalar mode $\phi_0$ of $\tilde{R}^2$ term, that later becomes the longitudinal component of massive $\omega_{\mu}$. Finally, in the limit of $t \to \infty$, $\phi_0 \to \langle \phi_0 \rangle$ (constant) and neglecting the scale-suppressed $T_\omega^\pm$ (due to $\omega_3$) then eq.(A-1) and (A-6) recover eqs.(35).
and \([36]\) in the text. In this particular limit the acceleration is given by the cosmological constant and we recover the usual de Sitter exponentially expanding Universe.

### A.2 Friedmann equations and Weyl cosmology

The general Friedmann equations are shown in eqs.\((A-4), (A-5)\). Using again solution \((20)\) we replace the derivatives of \(\phi_0\) in terms of \(\phi_0(t)\) itself. With notation \((A-3)\), the Friedmann equations become

\[
3H^2 = \frac{6\xi^2}{\phi_0^2} T_\omega^+ - \frac{3c_0^2}{a^6\phi_0^4} - 18H \frac{c_0}{a^3\phi_0} + \frac{\phi_0^2}{4}, \tag{A-7}
\]

and

\[
2\dot{H} + 3H^2 = -\frac{2\xi^2}{\phi_0^2} T_\omega^- + \frac{3c_0^2}{a^6\phi_0^4} + \frac{\phi_0^2}{4}. \tag{A-8}
\]

To estimate the behaviour of \(T_\omega^\pm\) in these equations consider for a moment the particular case when \(\phi_0\) is replaced by \(\langle \phi_0 \rangle\), then \(\omega_3\) satisfies \((24)\) with solution \((27)\); then \(T_\omega^\pm\) is replaced by its vev

\[
\langle T_\omega^\pm \rangle = m_\omega^2 \frac{\omega_k \omega_k}{2a^2} \pm \frac{\omega_k \dot{\omega}_k}{2a^2}. \tag{A-9}
\]

With \((27)\), \(\langle T_\omega^\pm \rangle\) are highly oscillatory for \(m_\omega^2 \gg H^2\), hence one could replace \(\langle T_\omega^\pm \rangle\) by its time-averaged value (denoted with a subscript \(t\)), to find

\[
\langle T_\omega^+ \rangle_t = (A^2 + B^2) \frac{m_\omega}{2a^3}, \quad \langle T_\omega^- \rangle_t = 0 \tag{A-10}
\]

This gives an indication of the behaviour of \(T_\omega^\pm\). We see that \(\omega_k\) has vanishing pressure in this approximation and it mimics the dark matter behaviour. In general, there are additional corrections to \(\langle T_\omega^+ \rangle_t \sim 1/a^6\) and \(\langle T_\omega^- \rangle_t \sim 1/a^3\), see e.g. \([25]\). In our numerical analysis \(\phi_0\) is not replaced by its vev and we use and integrate numerically eq.\((22)\) (instead of eq.\((24)\) of solution \((27)\)) and compute exactly the value of \(T_\omega^\pm\).

As in the text eq.\((41)\), to study the cosmology of the model defined by \((A-7), (A-8)\) we introduce the dimensionless variables \((\tau, h, \phi, \tilde{\omega})\)

\[
\tau = H_0 t, \quad H = H_0 h, \quad \phi_0 = H_0 \phi, \quad \omega_3 = \frac{1}{\xi} H_0 \tilde{\omega} \tag{A-11}
\]

where \(h, \phi, \tilde{\omega}\) are functions of \(\tau\); Therefore

\[
\dot{\phi}_0(t) = \dot{\phi}'(\tau) H_0^2, \quad \dot{\phi}_0(t)|_{\tau=0} = \phi'(\tau)|_{\tau=0} H_0^2, \quad \phi_0(t)|_{\tau=0} = \phi(\tau)|_{\tau=0} H_0. \tag{A-12}
\]

Then, from eqs.\((20), (22)\)

\[
\frac{d\phi}{d\tau} - \frac{c_h}{a^3\phi} = 0, \quad \text{where} \quad c_h \equiv \frac{c_0}{H_0^3} = \left[\phi(\tau)\phi'(\tau)\right]|_{\tau=0}, \tag{A-13}
\]
\[
\frac{d^2 \tilde{\omega}}{d\tau^2} + h \frac{d\tilde{\omega}}{d\tau} + \lambda \phi \tilde{\omega} = 0, \quad \text{where} \quad \lambda \equiv \frac{\alpha^2}{4\xi^2}. \tag{A-14}
\]

The Friedmann equations become

\[
3h^2(\tau) = \frac{3}{a^2} \left\{ \lambda \tilde{\omega}^2 + \frac{1}{\phi(\tau)^2} \left[ \frac{d\tilde{\omega}}{d\tau} \right]^2 \right\} - \frac{3c_h^2}{a^6 \phi^4} - \frac{18}{a^3 \phi^2} + \frac{\phi^2}{4}, \quad (= \rho_{\text{eff}}). \tag{A-15}
\]

\[
2 \frac{dh(\tau)}{d\tau} + 3h^2(\tau) = -\frac{1}{a^2} \left\{ \lambda \tilde{\omega}^2 - \frac{1}{\phi^2} \left[ \frac{d\tilde{\omega}}{d\tau} \right]^2 \right\} + \frac{3c_h^2}{a^6 \phi^4} + \frac{\phi^2}{4}, \quad (= -p_{\text{eff}}). \tag{A-16}
\]

Notice now the dependence of these equations on \(\lambda\) and thus on the couplings \(\alpha, \xi\).

To compare the cosmological predictions to \(\Lambda\text{CDM}\) we define

\[
u(\tau) = \frac{d\tilde{\omega}}{d\tau}. \tag{A-17}
\]

and also introduce the redshift \(z\) via \(1 + z = 1/a\). Then eqs. (A-13) to (A-16) show a system of first order differential equations of the cosmological evolution in the redshift space

\[
\frac{d\phi(z)}{dz} + (1 + z)^2 \frac{c_h}{\phi(z) h(z)} = 0, \tag{A-18}
\]

\[
\frac{d\tilde{\omega}(z)}{dz} + \frac{u(z)}{(1 + z) h(z)} = 0, \tag{A-19}
\]

\[
\frac{du(z)}{dz} - \frac{u(z)}{1 + z} - \frac{\lambda \phi^2(z) \tilde{\omega}(z)}{(1 + z) h(z)} = 0, \tag{A-20}
\]

with \(c_h = -\phi(z=0)\phi'(z=0)\) and

\[
(1 + z)^3 \frac{d}{dz} \left\{ \frac{h^2(z)}{(1 + z)^3} \right\} = \frac{\lambda \phi^2(z) \tilde{\omega}^2(z) - u^2(z)}{\phi^2(z)} (1 + z) + \frac{3c_h^2 (1 + z)^5}{\phi^4(z)} + \frac{\phi^2(z)}{4(1 + z)} = 0\tag{A-21}
\]

with the closure relation

\[
h^2(z) - \frac{(1 + z)^2}{\phi^2(z)} \left[ \lambda \phi^2(z) \tilde{\omega}^2(z) + u^2(z) \right] + \frac{c_h^2}{\phi^2(z)} (1 + z)^6 + \frac{6c_h^2 h(z)}{\phi^2(z)} (1 + z)^3 - \frac{\phi^2(z)}{12} = 0. \tag{A-22}
\]

The set of eqs. (A-18) to (A-22) define the Weyl cosmological model for the second solution for \(\omega_\mu\) (compare against eqs. (46) to (48) of “isotropic” solution). This set is solved numerically with initial conditions \((\phi(z = 0), \phi'(z = 0), \tilde{\omega}(z = 0), u(z = 0), h(z = 0))\). The results of this investigation are presented in Figures 1 and 2 in the text (right plots).

Finally, from (A-15)

\[
6h(\tau) \frac{dh(\tau)}{d\tau} = \frac{d\rho_{\text{eff}}(\tau)}{d\tau}. \tag{A-23}
\]
With notation (A-15), (A-16), we find
\[
\frac{d\rho_{\text{eff}}(\tau)}{d\tau} + 3h(\tau)\left[p_{\text{eff}}(\tau) + \rho_{\text{eff}}(\tau)\right] = 0.
\] (A-24)
This gives the energy conservation equation of the model.

**A.3 Constraints on the couplings**

One may ask what constraints the couplings \(\xi\) and \(\alpha\) must respect to have the above agreement(s). To this purpose, one uses the closure relation eq. (A-22) for \(z = 0\), with present-day value \(h(0) = 1\) and finds a constraint
\[
\lambda \tilde{\omega}^2(0) \phi^2(0) + u^2|_{z=0} = \phi^2(0)\left\{1 + \frac{\phi'^2(0)}{\phi^2(0)} - 6 \frac{\phi'(0)}{\phi(0)} - \frac{\phi^2(0)}{12}\right\}.
\] (A-25)
The deceleration function \(q(z)\) gives another constraint for \(z = 0\), from (49)
\[
\lambda \tilde{\omega}^2(0) \phi^2(0) - u^2|_{z=0} = 2 \phi^2(0)\left[q(0) - \frac{1}{2}\right] + 3\phi'^2(0) + \frac{1}{4}\phi^4(0).
\] (A-26)
The last two equations give
\[
\begin{align*}
\lambda \tilde{\omega}^2(0) &= q(0) + 2 \frac{\phi'^2(0)}{\phi^2(0)} - 3 \frac{\phi'^2(0)}{\phi(0)} + \frac{1}{12} \phi^2(0), \\
u^2|_{z=0} &= \phi^2(0)\left[1 - q(0) - \frac{\phi'^2(0)}{\phi^2(0)} - 3 \frac{\phi'(0)}{\phi(0)} - \frac{1}{6} \phi^2(0)\right].
\end{align*}
\] (A-27) (A-28)
The initial (present day) conditions for \(\tilde{\omega}(0)\) and its derivative \(u|_{z=0}\) are determined by the present-day values of the scalar field \(\phi(z = 0)\), of its derivative \(\phi'(z = 0)\) and of the deceleration \(q(z = 0)\). The initial value of the Weyl vector \((\tilde{\omega})\) is also related to the ratio \(\lambda = \alpha^2/(4\xi^2)\); a given value of \(\tilde{\omega}(0)\) is fixing the ratio of the couplings \((\alpha, \xi)\) (and vice-versa).

For example, for generic values considered in Figures 1 and 2 of \(\phi(z = 0) = 2.81, \phi'(z = 0) = 0.06\) (also \(\tilde{\omega}(z = 0) = 4.3\), which reproduce the \(\Lambda CDM\), then \(\lambda \tilde{\omega}^2(0) \approx 0.056\) and hence \(\lambda \approx 0.003\); therefore \(\alpha < \xi\). We also find from (A-28) that \(u^2|_{z=0} = 1.24\). These initial conditions can be re-formulated as constraints for \(\tilde{\omega}(0), \alpha, \xi\), by using definitions (A-11), (A-14) for \(\tilde{\omega}\) and \(\lambda\):
\[
\alpha^2 \omega^2(0) \approx 0.22 H_0^2, \quad \xi^2 \left[ \frac{d \omega}{dz} \right]_{z=0}^2 \approx 1.24 H_0^2,
\] (A-29)
where \(H_0 = 2.1978 \times 10^{-18} s^{-1}\) (corresponding to \(H_0 = 67.8\) Km/s/Mpc). The couplings \(\alpha\) and \(\xi\) can thus be in a perturbative regime for a suitable choice of \(\omega(0)\) and \(\omega(0)\), as mentioned in the text.

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