ON PRODUCTS OF SYMMETRIES IN VON NEUMANN ALGEBRAS

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Abstract. Let $R$ be a type $II_1$ von Neumann algebra. We show that every unitary in $R$ may be decomposed as the product of six symmetries (that is, self-adjoint unitaries) in $R$, and every unitary in $R$ with finite spectrum may be decomposed as the product of four symmetries in $R$. Consequently, the set of products of four symmetries in $R$ is norm-dense in the unitary group of $R$. Furthermore, we show that the set of products of three symmetries in a von Neumann algebra $M$ is not norm-dense in the unitary group of $M$. This strengthens a result of Halmos-Kakutani which asserts that the set of products of three symmetries in $B(ℋ)$, the ring of bounded operators on a Hilbert space $ℋ$, is not the full unitary group of $B(ℋ)$.

1. Introduction

In [5], Kadison initiated the study of infinite unitary groups (that is, the unitary group of factors not of type $I_n$ for $n \in \mathbb{N}$), acknowledging the difficulties arising from the fact they are not locally compact in the norm topology. Having shown various results of a topological flavor, he wonders about the algebraic nature of these groups with particular interest in the case of $II_1$ factors. In response[1], Halmos and Kakutani showed in [4] that in a type $I_\infty$ factor, every unitary may be decomposed into the product of four symmetries; We remind the reader that a symmetry or reflection is a synonym for a self-adjoint unitary. Moreover, they showed that for purely algebraic reasons the scalar unitary $\exp(\frac{2\pi i}{3})I$, being a central element of order 3, cannot be decomposed as the product of three symmetries.

It is natural to wonder whether similar conclusions hold for other von Neumann algebras. For $n \in \mathbb{N}$, since every symmetry in the type $I_n$ factor $M_n(\mathbb{C})$ has determinant $\pm 1$, every product of symmetries in $M_n(\mathbb{C})$ must be an $n \times n$ unitary matrix with determinant $\pm 1$. Conversely, in [9, Theorem 3], Radjavi showed that any $n \times n$ unitary matrix with determinant $\pm 1$ can be decomposed as the product of four symmetries in $M_n(\mathbb{C})$. In [3], Fillmore showed that any unitary in a properly infinite von Neumann algebra can be decomposed as the product of four symmetries; in particular, this takes care of the case of type $I_\infty$, type $II_\infty$, and type $III$ von Neumann algebras. In their approach, Halmos and Kakutani, and Fillmore essentially use the well-known fact that every permutation of a non-empty set can be written as the product of two involutions (permutations of order two); In the context of permutations of an orthonormal basis $B$ of a Hilbert space $ℋ$, it tells us that every unitary on $ℋ$ that permutes some orthonormal basis of $ℋ$ (such as “shift” operators) can be written as the product of two symmetries.

From the work of Broise (see [1]), it is known that every unitary in a type $II_1$ factor is the product of finitely many symmetries. From Dowker and Thom’s refinement of Broise’s

1This may be inferred from the comments in [5] pg. 399 and [9].
result (see [2, Theorem 3.1]), we may infer that every unitary in a type $II_1$ factor may be decomposed as the product of 16 symmetries. In this article, our main goal is to investigate the minimal such number in this context. Furthermore, we directly work with von Neumann algebras of type $II_1$ rather than factors of type $II_1$.

The key results of this article are summarized below.

(i) [Theorem 2.4] A unitary in a type $I_n$ von Neumann algebra $R$ can be decomposed as the product of four symmetries in $R$ if and only if its center-valued determinant (see Definition 2.1) is a central symmetry.

(ii) [Theorem 3.9] In a type $II_1$ von Neumann algebra $R$, every unitary is the product of six symmetries in $R$.

(iii) [Theorem 3.12] In a type $II_1$ von Neumann algebra $R$, every unitary with finite spectrum is the product of four symmetries in $R$.

(iv) [Theorem 4.7] In a von Neumann algebra $M$, the set of unitaries which can be decomposed as the product of three symmetries in $M$ is not norm-dense in the unitary group of $M$.

Before proceeding further, we set up some notation used throughout this article.

Notation: The set of complex numbers of unit modulus is denoted by $S^1$. For $n \in \mathbb{N}$, we denote the set of $n \times n$ complex matrices by $M_n(\mathbb{C})$ and the group of unitary matrices in $M_n(\mathbb{C})$ by $U(n)$. The spectrum of an operator $T$ is denoted by $\text{sp}(T)$. For a subset $A$ of a von Neumann algebra $\mathcal{M}$, we denote the norm-closure of $A$ by $\overline{A}$. The symbol $\sim$ describes the Murray-von Neumann equivalence relation for projections in a given von Neumann algebra. Let $R$ be a finite von Neumann algebra with center $C$ and $\tau : R \to C$ denote the canonical center-valued trace. The restriction of $\tau$ to the set of projections in $R$ is called the (center-valued) dimension function (see [8]) which we denote by $\dim \tau$. For a von Neumann algebra $\mathcal{M}$, we denote the group of unitary operators in $\mathcal{M}$ by $U(\mathcal{M})$, and the set of symmetries in $U(\mathcal{M})$ by $S(\mathcal{M})$.

For a positive integer $n$, let $S(\mathcal{M})^n$ denote the set of unitaries that can be decomposed as the product of $n$ symmetries in $\mathcal{M}$. Since the identity operator is a symmetry, it is straightforward to see that

$$S(\mathcal{M}) \subseteq S(\mathcal{M})^2 \subseteq \cdots \subseteq S(\mathcal{M})^n \subseteq \bigcup_{n \in \mathbb{N}} S(\mathcal{M})^n \subseteq U(\mathcal{M}).$$

Recasting our previous discussion in this notation, we observe that when $\mathcal{M}$ is a von Neumann algebra of type $I_\infty$, type $II_\infty$ or type $III$ we have $S(\mathcal{M})^4 = U(\mathcal{M})$. In Theorem 2.4 of this article, by appropriately adapting Radjavi’s strategy (cf. [9]) for a type $I_n$ von Neumann algebra $\mathcal{M}$, we provide a simple characterization of $S(\mathcal{M})^4$ and note that $S(\mathcal{M})^4 = \bigcup_{n \in \mathbb{N}} S(\mathcal{M})^n$.

Let $R$ be a type $II_1$ von Neumann algebra. In view of the previously discussed results for the other “types” of von Neumann algebras, it is tempting to conjecture that $U(R) = S(R)^4$. In Theorem 3.9, we show that every unitary in $R$ is the product of six symmetries in $R$, that is, $U(R) = S(R)^6$. Furthermore, in Theorem 3.12 we observe that every unitary in $R$ with finite spectrum belongs to $S(R)^4$. Since every unitary in $R$ may be norm-approximated
arbitrarily closely by a unitary in $\mathcal{B}$ with finite spectrum, we conclude that $\mathcal{S}(\mathcal{B})^4$ is norm-dense in $\mathcal{U}(\mathcal{B})$. The sequence $\{\mathcal{S}(\mathcal{B})^n\}_{n\in\mathbb{N}}$ of subsets of $\mathcal{U}(\mathcal{B})$ transitions from a norm-dense subset of $\mathcal{U}(\mathcal{B})$ at $n=4$ to all of $\mathcal{U}(\mathcal{B})$ at $n=6$. This motivates us to investigate the evolution of the sets $\mathcal{S}(\mathcal{B})^n$ starting from $n=1$.

Let $\mathcal{M}$ be a von Neumann algebra. In Proposition 4.1 we characterize $\mathcal{S}(\mathcal{M})^2$ as the set of unitaries in $\mathcal{M}$ that are unitarily equivalent in $\mathcal{M}$ to their adjoint; in particular, the spectrum of a unitary in $\mathcal{S}(\mathcal{M})^2$ is symmetric about the real axis. In Theorem 4.7, we use the above characterization of $\mathcal{S}(\mathcal{M})^2$ to show that a unitary with spectrum contained in exactly one of the four connected components of $S^1\setminus\{1, i, -1, -i\}$ does not belong to $\mathcal{S}(\mathcal{M})^3$. Since such unitaries form a norm-open subset of $\mathcal{U}(\mathcal{M})$, we conclude that $(\mathcal{S}(\mathcal{M})^3)^\sim \neq \mathcal{U}(\mathcal{M})$, that is, $\mathcal{S}(\mathcal{M})^3$ is not norm-dense in $\mathcal{U}(\mathcal{M})$. This gives us a stronger version of the conclusion by Halmos and Kakutani in [4] that $\mathcal{S}(\mathcal{B}(\mathcal{H}))^3 \neq \mathcal{U}(\mathcal{B}(\mathcal{H}))$, where $\mathcal{B}(\mathcal{H})$ is the ring of bounded operators acting on a Hilbert space $\mathcal{H}$.

At the very end, we discuss some open questions and conjectures for further exploration, the main question of interest being whether $\mathcal{S}(\mathcal{B})^4 = \mathcal{U}(\mathcal{B})$ for every type $II_1$ von Neumann algebra $\mathcal{B}$.

2. Products of symmetries in a type $I_n$ von Neumann algebra

For a positive integer $n$, let $\mathcal{B}$ be a type $I_n$ von Neumann algebra with center $\mathcal{C}$. It is well-known that $\mathcal{B} \cong M_n(\mathcal{C})$ (see [8, Theorem 6.6.5]). Since $\mathcal{C}$ is an abelian von Neumann algebra, there is a hyperstonean space $X$ such that $\mathcal{C} \cong C(X)$, the space of complex-valued continuous functions on $X$; hence, $\mathcal{B} \cong M_n(C(X)) \cong C(X; M_n(\mathcal{C}))$, the space of $M_n(\mathcal{C})$-valued continuous functions on $X$.

**Definition 2.1.** Let $\det : M_n(\mathcal{C}) \to \mathbb{C}$ denote the usual determinant function. For $f \in C(X; M_n(\mathcal{C}))$, the function $\det \circ f$ is in $C(X)$. We call the map $f(\in \mathcal{B}) \mapsto \det \circ f(\in \mathcal{C})$ as the center-valued determinant on $\mathcal{B}$ and denote it by $\det_c : \mathcal{B} \to \mathcal{C}$.

**Remark 2.2.** It is straightforward to verify the following properties of $\det_c : \mathcal{B} \to \mathcal{C}$ from analogous properties of $\det : M_n(\mathcal{B}) \to \mathbb{C}$:

(i) $\det_c(AB) = \det_c(A) \det_c(B)$ for $A, B \in \mathcal{B}$.
(ii) $\det_c(A^*) = \det_c(A)^*$ for $A \in \mathcal{B}$.

**Remark 2.3.** For $f = (f_{jk})_{1 \leq j, k \leq n} \in M_n(C(X))$ with $f_{jk} \in C(X)$, note that

$$\det_c(f)(x) = \sum_{\sigma \in S_n} \left(\text{sgn}(\sigma) \prod_{j=1}^{n} f_{j\sigma(j)}(x)\right), \text{ for } x \in X.$$ 

From the above formula, it is straightforward to see that $\det_c : \mathcal{B} \to \mathcal{C}$ is norm-continuous.

Using Radjavi’s strategy in the case of $M_n(\mathcal{C})$ (cf. [9, Theorem 3] and Kadison’s diagonalization theorem for normal matrices over von Neumann algebras, in the theorem below we completely characterize all unitaries in a type $I_n$ von Neumann algebra which can be written as the product of finitely many symmetries.

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**Remark 2.4.** For a positive integer $n$, let $\mathcal{B}$ be a type $I_n$ von Neumann algebra with center $\mathcal{C}$. It is well-known that $\mathcal{B} \cong M_n(\mathcal{C})$ (see [8, Theorem 6.6.5]). Since $\mathcal{C}$ is an abelian von Neumann algebra, there is a hyperstonean space $X$ such that $\mathcal{C} \cong C(X)$, the space of complex-valued continuous functions on $X$; hence, $\mathcal{B} \cong M_n(C(X)) \cong C(X; M_n(\mathcal{C}))$, the space of $M_n(\mathcal{C})$-valued continuous functions on $X$.

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$$\det_c(f)(x) = \sum_{\sigma \in S_n} \left(\text{sgn}(\sigma) \prod_{j=1}^{n} f_{j\sigma(j)}(x)\right), \text{ for } x \in X.$$ 

From the above formula, it is straightforward to see that $\det_c : \mathcal{B} \to \mathcal{C}$ is norm-continuous.
Theorem 2.4. Let $\mathcal{R}$ be a type $I_n$ von Neumann algebra with center $\mathcal{C}$. Then a unitary $U$ in $\mathcal{R}$ can be decomposed as the product of four symmetries if and only if $\text{det}_c(U)$ is a symmetry in $\mathcal{C}$.

Proof. As discussed above, let $X$ be the hyperstonean space such that $\mathcal{C} \cong C(X)$ and $\mathcal{R} \cong C(X; M_n(\mathbb{C}))$. Note that every unitary in $\mathcal{R}$ corresponds to a $U(n)$-valued continuous function on $X$. Let $R : X \to U(n)$ be a symmetry in $\mathcal{R}$. Since $R^2 = I$ and $R = R^*$, from Remark 2.2, $\text{det}_c(R)^2 = I$ and $\text{det}_c(R)$ is self-adjoint. Thus $\text{det} R(x) \in \{\pm 1\}$ for all $x \in X$, or equivalently, the range of $\text{det}_c(R) : X \to S^1$ is in $\{\pm 1\}$. Again from Remark 2.2, it is straightforward to see that if the unitary $U : X \to U(n)$ in $\mathcal{R}$ is the product of finitely many symmetries in $\mathcal{R}$, then the range of $\text{det}_c(U) : X \to S^1$ is $\{\pm 1\}$ (or equivalently, $\text{det}_c(U)$ is a symmetry in $\mathcal{C}$). This proves one direction of the assertion.

Let $U$ be a unitary in $\mathcal{R} \cong M_n(\mathcal{C})$ such that $\text{det}_c(U)$ is a symmetry in $\mathcal{C}$. By Kadison’s diagonalization theorem (see [6, Lemma 3.7, Theorem 3.19]), $U$ is unitarily equivalent to a diagonal matrix in $M_n(\mathcal{C})$. Hence we may assume that $U = \text{diag}(\lambda_1, \ldots, \lambda_n)$ where $\lambda_j : X \to S^1$ is in $\mathcal{C} \cong C(X)$ for $1 \leq j \leq n$. Define unitary matrices $V$ and $W$ in $M_n(\mathcal{C})$ by

$$V := \text{diag}(\lambda_1, \lambda_1^*, \ldots, \prod_{j=1}^{2k+1} \lambda_j, \left(\prod_{j=1}^{2k+1} \lambda_j\right)^*, \ldots),$$

$$W := \text{diag}(1, (\lambda_1\lambda_2), (\lambda_1\lambda_2)^*, \ldots, \prod_{j=1}^{2k} \lambda_j, \left(\prod_{j=1}^{2k} \lambda_j\right)^*, \ldots).$$

Clearly $V, W \in \mathcal{R}$ and it is straightforward to verify by entry-wise multiplication of the diagonals that $U = VW$. If $n$ is even (with $n = 2\ell$), then all the entries in $V$ appear in conjugate pairs and the corresponding principal diagonal blocks may be decomposed as the product of two symmetries in $M_{2\ell}(\mathcal{C})$ (see Remark 3.4). The first diagonal entry of $W$ is the constant function 1, which is a central symmetry, and the last diagonal entry is $\lambda_1\lambda_2\cdots\lambda_{2\ell} = \text{det}_c(U)$ which is a central symmetry by hypothesis. The remaining entries of $W$ appear in conjugate pairs and as observed before, the corresponding principal diagonal blocks may be decomposed as the product of two symmetries in $M_{2\ell}(\mathcal{C})$.

If $n$ is odd, then the last diagonal entry of $V$ is $\text{det}_c(U)$, which is a central symmetry, and the remaining entries appear in conjugate pairs. For $W$, the first diagonal entry is 1, which is a central symmetry, and the remaining entries appear in conjugate pairs. A similar argument as in the previous paragraph proves the desired result.

In summary, both $V$ and $W$ may be decomposed as the product of two symmetries in $M_n(\mathcal{C})$ and thus $U = VW$ may be decomposed as the product of four symmetries in $\mathcal{R}$. □

Corollary 2.5. For a type $I_n$ von Neumann algebra $\mathcal{R}$, $S(\mathcal{R})^4$ is not norm-dense in $U(\mathcal{R})$.

Proof. Let $m$ be an integer coprime to $n$. Note that $\text{det}_c \left( \exp\left(\pi i \frac{1}{m}\right)I \right) = \exp\left(\pi i \frac{n}{m}\right)I$, which is not a central symmetry. From Theorem 2.4, $\exp(\pi i \frac{n}{m})I \in U(\mathcal{R}) \setminus S(\mathcal{R})^4$.

Since $\text{det}_c : \mathcal{R} \to \mathcal{C}$ is a norm-continuous map (see Remark 2.3) and the central symmetries form a norm-closed subset of $\mathcal{C}$, from Theorem 2.4 we conclude that $S(\mathcal{R})^4$ is norm-closed. Thus the assertion follows. □
3. Products of symmetries in a type $II_1$ von Neumann algebra

3.1. Decomposition of a unitary as a product of six symmetries. En route to Theorem 3.9, we begin with some preparatory results.

**Lemma 3.1.** Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert $\mathcal{H}$. Let $(E_n)_{n \in \mathbb{N}}$ be a sequence of mutually orthogonal projections in $\mathcal{M}$, and $(A_n)_{n \in \mathbb{N}}$ be a sequence of operators in $\mathcal{M}$ whose operator-norms are uniformly bounded above, that is, there exists an $\alpha > 0$ such that $\|A_n\| \leq \alpha$ for all $n \in \mathbb{N}$. Then $\lim_{k \to \infty} \sum_{j=1}^{k} E_j A_j E_j(=: \sum_{j=1}^{\infty} E_j A_j E_j)$ exists in the strong-operator topology (and hence, in the weak-operator topology) and is an operator in $\mathcal{M}$.

**Proof.** Define $A^{(k)} := \sum_{j=1}^{k} E_j A_j E_j$. Let $x \in \mathcal{H}$. For positive integers $k_1 < k_2$, note that

$$\| (A^{(k_2)} - A^{(k_1)}) x \|^2 = \| \sum_{j=k_1+1}^{k_2} E_j A_j E_j x \|^2 = \sum_{j=k_1+1}^{k_2} \| E_j A_j E_j x \|^2$$

$$\leq (\alpha^2) \sum_{j=k_1+1}^{k_2} \| E_j x \|^2$$

Furthermore, we have $\sum_{j=1}^{k} \| E_j x \|^2$ is a Cauchy sequence converging to $\| (\sum_{j=1}^{\infty} E_j) x \|^2$. Thus $A^{(k)} x$ is Cauchy in norm and converges to a vector which we label $Ax$. In this notation, we observe that $A = \sum_{j=1}^{\infty} E_j A_j E_j$, is the strong-operator limit of the sequence of operators, $(A^{(k)})_{k \in \mathbb{N}}$, in $\mathcal{M}$. As $\mathcal{M}$ is strong-operator closed, $A$ is an operator in $\mathcal{M}$. \qed

**Proposition 3.2.** Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert $\mathcal{H}$ and $(E_n)_{n \in \mathbb{N}}$ be a sequence of mutually orthogonal projections in $\mathcal{M}$. Let $(A_n)_{n \in \mathbb{N}}, (B_n)_{n \in \mathbb{N}}$ be two sequences of operators in $\mathcal{M}$ whose operator-norms are uniformly bounded above, that is, there exists an $\alpha > 0$ such that $\|A_n\| \leq \alpha, \|B_n\| \leq \alpha$ for all $n \in \mathbb{N}$. Let $A := \sum_{j=1}^{\infty} E_j A_j E_j$, $B := \sum_{j=1}^{\infty} E_j B_j E_j$. Then

$$AB = \sum_{j=1}^{\infty} E_j (A_j E_j B_j) E_j.$$

**Proof.** Clearly $\|A_kE_kB_k\| \leq \alpha^2$ for all $k \in \mathbb{N}$ so that by Lemma 3.1, $\sum_{j=1}^{\infty} E_j (A_j E_j B_j) E_j$ may be interpreted as the strong-operator limit of the partial sums of the series.

Note that multiplication $(S,T) \in \mathcal{M} \times \mathcal{M} \mapsto ST \in \mathcal{M})$ is separately continuous in the strong-operator topology. Thus for every $k \in \mathbb{N}$, we have

$$E_k A_k E_k B = \sum_{j=1}^{\infty} (E_k A_k E_k)(E_j B_j E_j) = E_k A_k E_k B_k E_k,$$

and

$$AB = \sum_{j=1}^{\infty} E_j A_j E_j B = \sum_{j=1}^{\infty} E_j (A_j E_j B_j) E_j.$$

\qed
Corollary 3.3. Let $\mathcal{M}$ be a von Neumann algebra acting on the Hilbert $\mathcal{H}$ and $(E_n)_{n \in \mathbb{N}}$ be a sequence of mutually orthogonal projections in $\mathcal{M}$ such that $\sum_{j=1}^{\infty} E_j = I$.

(i) Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of operators in $\mathcal{M}$ such that $U_n E_n = E_n U_n = U_n$, and $U_n^* U_n = U_n U_n^* = E_n$. In other words, $U_n$ is a unitary operator in the von Neumann algebra $\mathcal{M}$. Then $\sum_{j=1}^{\infty} E_j U_j^* E_j$ is a unitary operator in $\mathcal{M}$.

(ii) Let $(H_n)_{n \in \mathbb{N}}$ be a sequence of self-adjoint operators in $\mathcal{M}$ whose operator-norms are uniformly bounded above, that is, there exists an $\alpha > 0$ such that $\|H_n\| \leq \alpha$ for all $n \in \mathbb{N}$. Then $\sum_{j=1}^{\infty} E_j H_j^* E_j$ is a self-adjoint operator in $\mathcal{M}$.

Proof. (i) From Lemma 3.1 we observe that $U := \sum_{j=1}^{\infty} E_j U_j E_j$, $V := \sum_{j=1}^{\infty} E_j U_j^* E_j$ are well-defined. From Proposition 3.2 (and the hypothesis), we conclude that $UV = \sum_{j=1}^{\infty} E_j (U_j E_j U_j^*) E_j = \sum_{j=1}^{\infty} E_j (E_j U_j U_j^*) E_j = \sum_{j=1}^{\infty} E_j = I$.

Similarly $VU = I$. Thus $U$ is a unitary operator in $\mathcal{M}$ and $V = U^*$.

(ii) This follows from the fact that the limit of self-adjoint operators, in the strong-operator topology, is self-adjoint. □

Remark 3.4. Let $\mathcal{M}$ be a von Neumann algebra with identity $I$ and let $U_1$ be a unitary operator in $\mathcal{M}$. We observe that the unitary

$$
\begin{bmatrix}
U_1 & 0 \\
0 & U_1^*
\end{bmatrix} = 
\begin{bmatrix}
0 & I \\
I & 0
\end{bmatrix} 
\begin{bmatrix}
0 & U_1^* \\
U_1 & 0
\end{bmatrix}
$$

in $M_2(\mathcal{M})$ is a product of two symmetries in $M_2(\mathcal{M})$.

Lemma 3.5. Let $\mathcal{R}$ be a type $\text{II}_1$ von Neumann algebra, and $U$ be a unitary in $\mathcal{R}$. Then for every $n \in \mathbb{N}$ there are projections $E_1, E_2, \ldots, E_n$ in $\mathcal{R}$ commuting with $U$ such that $E_1 \sim E_2 \sim \cdots \sim E_n$ and $E_1 + E_2 + \cdots + E_n = I$.

Proof. By [6] Corollary 3.15, for every positive integer $n$, each maximal abelian self-adjoint subalgebra of $\mathcal{R}$ contains $n$ orthogonal equivalent projections with sum $I$. Considering a maximal abelian self-adjoint subalgebra $\mathcal{A}$ of $\mathcal{R}$ containing $U$, the assertion follows immediately with projections $E_1, E_2, \ldots, E_n \in \mathcal{A}$. □

Remark 3.6. Let $\mathcal{R}$ be a type $\text{II}_1$ von Neumann algebra acting on the Hilbert space $\mathcal{H}$. By [6] Corollary 3.15, for every $n \in \mathbb{N}$, there are equivalent mutually orthogonal projections $E_1, E_2, \ldots, E_n$ partitioning the identity, that is, $E_1 + E_2 + \cdots + E_n = I$. There is a system of matrix units $\{V_{j,k}\}_{1 \leq j, k \leq n}$ where $V_{j,k}$ is a partial isometry in $\mathcal{M}$ with initial projection $E_k$ and final projection $E_j$ so that:

1. $V_{j,j} = E_j$ for $1 \leq j \leq n$;
2. $V_{j,k}^* = V_{k,j}$ for $1 \leq j, k \leq n$;
3. $V_{j,k} V_{\ell,m} = \delta_{k,\ell} V_{j,m}$ for $1 \leq j, k, \ell, m \leq n$.

Note that the mapping $\psi : \mathcal{R} \to M_n(E_1 \mathcal{R} E_1)$ given by $\psi(A) = (V_{1,j} A V_{k,1})_{1 \leq j, k \leq n}$ is a $*$-isomorphism of von Neumann algebras. With this $*$-isomorphism in hand, we will take the liberty of viewing $\mathcal{R}$ as a matrix algebra over a type $\text{II}_1$ von Neumann algebra $(E_1 \mathcal{R} E_1)$.
acting on the Hilbert space $E_1(\mathcal{H})$, in this case) which helps avail various standard algebraic techniques involving matrices. Furthermore, as noted in Lemma 3.5 for a given unitary $U$ in $\mathcal{R}$, using Kadison’s diagonalization theorem we may assume that $U$ is in diagonal form in this matrix representation.

In the study of the decomposition of a unitary as the product of symmetries, the basic ingredient in previous investigations (see [1], [9], [2], and Theorem 2.4 above) is the identification of an appropriate “basis” which “diagonalizes” the unitary under consideration. This allows for the usage of appropriate matrix-algebraic techniques. Lemma 3.8 below sets the stage for such a (infinitary) “diagonalization” process in the context of type $II_1$ von Neumann algebras to facilitate the use of the matrix decomposition mentioned in Remark 3.6

**Lemma 3.7.** Let $\mathcal{R}$ be a type $II_1$ von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and $\tau$ be the canonical center-valued trace on $\mathcal{R}$. Let $U,V$ be unitaries in $\mathcal{R}$. Then there is a projection $E \in \mathcal{R}$ with $\dim_{\tau}(E) = \frac{1}{2}I$, a unitary $V'$ in the von Neumann algebra $E\mathcal{R}E$ (acting on the Hilbert space $E(\mathcal{H})$), and symmetries $R_1, R_2 \in \mathcal{R}$ such that $UV = R_1R_2U(V'E + I - E)$.

**Proof.** Let $\mathcal{S}$ be a type $II_1$ von Neumann algebra such that $\mathcal{R} \cong M_2(\mathcal{S})$. By Kadison’s diagonalization theorem, there are unitaries $V_1, V_2 \in \mathcal{S}$ and a unitary $W \in M_2(\mathcal{S})$ such that $V = W\text{diag}(V_1, V_2)W^*$. Note that the projection $E := W\text{diag}(0, I)W^* \in M_2(\mathcal{S})$ and the unitary $V' := W\text{diag}(I, V_1V_2)W^*$ in $M_2(\mathcal{S})$ commute. Furthermore,

$$UV = U(W\text{diag}(V_1, V_2)W^*)$$

$$= (UW\text{diag}(V_1, V_1^*)W^*U^*)U(W\text{diag}(I, V_1V_2)W^*).$$

$$= (UW\text{diag}(V_1, V_1^*)W^*U^*)U(V'E + I - E).$$

From Remark 3.4 it is straightforward to see that $UW\text{diag}(V_1, V_1^*)W^*U^*$ is the product of two symmetries in $M_2(\mathcal{S})$. By appropriately interpreting these operators and our computations in the context of $\mathcal{R}$, we see that $\dim_{\tau}(E) = \frac{1}{2}I$ and the required conditions in the assertion are satisfied. \square

**Lemma 3.8.** Let $\mathcal{R}$ be a type $II_1$ von Neumann algebra acting on the Hilbert space $\mathcal{H}$ and $\tau$ be the canonical center-valued trace on $\mathcal{R}$. Let $E_1 \in \mathcal{R}$ be a projection with $\dim_{\tau}(E_1) = \frac{2}{3}I$, and $B_1$ be a unitary operator in the von Neumann algebra $E_1\mathcal{R}E_1$ (acting on the Hilbert space $E_1(\mathcal{H})$). Let $U$ be a unitary operator $\mathcal{R}$ commuting with $E_1$. Then there exist four symmetries $R_1, R_2, R_3, R_4 \in \mathcal{R}$, a projection $E_2 \in \mathcal{R}$ orthogonal to $E_1$ with $\dim_{\tau}(E_2) = \frac{1}{6}I$, and a unitary operator $B_2$ in the von Neumann algebra $E_2\mathcal{R}E_2$ (acting on the Hilbert space $E_2(\mathcal{H})$) such that

$$U = R_1R_2R_3R_4(B_1E_1 + B_2E_2 + (I - E_1 - E_2)).$$

**Proof.** By considering the unitary operator $U(B_1^*E_1 + I - E_1)$ in $\mathcal{R}$ (in lieu of $U$), without loss of generality, we may assume that $B_1 = E_1$.

Let $\mathcal{S}$ be a type $II_1$ von Neumann algebra such that $\mathcal{R} \cong M_3(\mathcal{S})$. We denote the identity of $\mathcal{S}$ by $I_\mathcal{S}$ and the center-valued trace on $\mathcal{S}$ by $\tau_\mathcal{S}$. Since $U$ and $E_1$ commute, without loss
of generality, we may assume that $U$ and $E_1$ are both in diagonal form (via simultaneous diagonalization using Kadison’s diagonalization theorem) so that $U = \text{diag}(U_1, U_2, U_3)$, for unitaries $U_1, U_2, U_3$ in $\mathcal{J}$, and $E_1 = \text{diag}(I_{\mathcal{J}}, I_{\mathcal{J}}, 0)$.

Using Lemma 3.7 in the context of $\mathcal{J}$, we have a projection $E \in \mathcal{J}$ with $\text{dim}_{\tau, \mathcal{J}}(E) = \frac{1}{2}I_{\mathcal{J}}$, a unitary $V' \in \mathcal{J}$ commuting with $E$, and symmetries $S_1, S_2 \in \mathcal{J}$ such that

$$U_3 = (U_2^*U_1^*)(U_1U_2U_3) = S_1S_2(U_2^*U_1^*)(V'E + I - E).$$

Thus we have

$$\text{diag}(U_1, U_2, U_3) = \text{diag}(U_1, U_1^*, I_{\mathcal{J}}) \cdot \text{diag}(I_{\mathcal{J}}, U_1U_2, U_2^*U_1^*) \cdot \text{diag}(I_{\mathcal{J}}, I_{\mathcal{J}}, U_1U_2U_3) = \text{diag}(U_1, U_1^*, S_1S_2) \cdot \text{diag}(I_{\mathcal{J}}, U_1U_2, U_2^*U_1^*) \cdot \text{diag}(I_{\mathcal{J}}, I_{\mathcal{J}}, V'E + I - E).$$

From Remark 3.4 it is straightforward to see that the matrices,

$$\text{diag}(U_1, U_1^*, S_1S_2), \quad \text{diag}(I_{\mathcal{J}}, U_1U_2, U_2^*U_1^*),$$

can each be decomposed as the product of two symmetries in $M_3(\mathcal{J})$. We choose $E_2 := \text{diag}(0, 0, E)$ and $B_2 := \text{diag}(0, 0, V'E)$. By appropriately interpreting these operators and our computations in the context of $\mathcal{R}$, we see that $\text{dim}_{\tau}(E_2) = \frac{1}{6}I$ and the required conditions in the assertion are satisfied.

**Theorem 3.9.** Every unitary in a type $II_1$ von Neumann algebra $\mathcal{R}$ may be decomposed as the product of six symmetries in $\mathcal{R}$.

**Proof.** Let $U$ be a unitary in $\mathcal{R}$, and $\mathcal{A}$ be a maximal abelian self-adjoint subalgebra of $\mathcal{M}$ containing $U$. Using [6, Proposition 3.13, Corollary 3.14], we may inductively choose a sequence of mutually orthogonal projections $F^{(n)}$ in $\mathcal{A}$ (and thus, commuting with $U$) such that $\text{dim}_{\tau}(F^{(n)}) = \frac{2}{4}I$. Note that $\sum_{n \in \mathbb{N}} F^{(n)} = I$.

For $n \in \mathbb{N}$, we choose $E_1^{(n)}$ to be a subprojection of $F^{(n)}$ with $\text{dim}_{\tau}(E_1^{(n)}) = \frac{2}{3} \text{dim}_{\tau}(F^{(n)}) = \frac{2}{3}I$. With the aim of bringing Lemma 3.8 into action, below we inductively define three relevant sequences (with some desirable properties):

(i) a sequence of projections $\{E_2^{(n)}\}_{n \in \mathbb{N}}$ such that $E_2^{(n)}$ is a subprojection of $F^{(n)}$ orthogonal to $E_1^{(n)}$ with $\text{dim}_{\tau}(E_2^{(n)}) = \frac{1}{6} \text{dim}_{\tau}(F^{(n)}) = \frac{1}{3}I$,

(ii) a sequence $\{B_2^{(n)}\}$ of elements in $E_1^{(n)} \mathcal{R} E_1^{(n)}$,

(iii) a sequence $\{B_2^{(n)}\}$ of elements in $E_2^{(n)} \mathcal{R} E_2^{(n)}$.

Set $B_1^{(1)} := E_1^{(1)}$. For $n \in \mathbb{N}$, based on our knowledge of $B_1^{(n)}$, we define $E_2^{(n)}, B_2^{(n)}$ as follows. For the unitary $UF^{(n)}$ in the type $II_1$ von Neumann algebra $F^{(n)} \mathcal{R} F^{(n)}$, by Lemma 3.8 we have four symmetries $R_i^{(n)} \in F^{(n)} \mathcal{R} F^{(n)}$ for $1 \leq i \leq 4$, a projection $E_2^{(n)} \leq F^{(n)}$ orthogonal to $E_1^{(n)}$ with $\text{dim}_{\tau}(E_2^{(n)}) = \frac{1}{6} \text{dim}_{\tau}(F^{(n)})$ and a unitary operator $B_2^{(n)}$ in $E_2^{(n)} \mathcal{R} E_2^{(n)}$ such that

$$UF^{(n)} = R_1^{(n)}R_2^{(n)}R_3^{(n)}R_4^{(n)}(B_1^{(n)}E_1^{(n)} + B_2^{(n)}E_2^{(n))} + F^{(n)} - E_1^{(n)} - E_2^{(n)}).$$

For $n \in \mathbb{N}$, based on our knowledge of $E_2^{(n)}$ and $B_2^{(n)}$, we define $B_1^{(n+1)}$ as follows. Since $\text{dim}_{\tau}(E_2^{(n)}) = \frac{2}{3}I = \text{dim}_{\tau}(E_1^{(n+1)})$, there is a partial isometry $V_n$ in $\mathcal{R}$ with initial projection $E_2^{(n)}$ and final projection $E_1^{(n+1)}$. Define $B_1^{(n+1)} := V_n(B_2^{(n)})^*V_n^*$ and note that it is a unitary in $E_1^{(n+1)} \mathcal{R} E_1^{(n+1)}$. 


For \( n \in \mathbb{N} \), let \( E_3^{(n)} := F^{(n)} - E_1^{(n)} - E_2^{(n)} \). In this notation, we have
\[
UF^{(n)} = R_1^{(n)} R_2^{(n)} R_3^{(n)} R_4^{(n)} (E_1^{(n)} E_1^{(n)} + B_2^{(n)} E_2^{(n)} + E_3^{(n)}),
\]
where \( F^{(n)} = E_1^{(n)} + E_2^{(n)} + E_3^{(n)} \).

Recall that \( \{F^{(n)}\} \) is a sequence of mutually orthogonal projections partitioning the identity operator. From Corollary 3.3, we conclude that for \( 1 \leq j \leq 4 \) the operator
\[
R_j := \sum_{n \in \mathbb{N}} P_j^{(n)}
\]
is a symmetry in \( \mathcal{R} \), and \( W := \sum_{n \in \mathbb{N}} (B_1^{(n)} E_1^{(n)} + B_2^{(n)} E_2^{(n)} + E_3^{(n)}) \) is a unitary in \( \mathcal{R} \).

**Claim 1:** The unitary \( W \) is a product of two symmetries in \( \mathcal{R} \).

**Proof of Claim 1.** For \( n \in \mathbb{N} \), we define \( E^{(n)} := E_2^{(n)} + E_1^{(n+1)} \). Recall that \( E_2^{(n)} \sim E_1^{(n+1)} \) with the equivalence implemented by the partial isometry \( V_n \). Using the corresponding \( 2 \times 2 \) system of matrix units, it is straightforward to see that \( B_2^{(n)} E_2^{(n)} + B_1^{(n+1)} E_1^{(n+1)} \) is of the form \( \text{diag}(B, B^*) \), which by Remark 3.6, is a product of two symmetries in \( E^{(n)} \mathcal{R} E^{(n)} \). With \( G^{(n)} := E_2^{(n)} + E_1^{(n+1)} + E_3^{(n+1)} \), we observe that
\[
W_n := B_2^{(n)} E_2^{(n)} + B_1^{(n+1)} E_1^{(n+1)} + E_3^{(n+1)},
\]
is the product of two symmetries in \( G^{(n)} \mathcal{R} G^{(n)} \). Furthermore, \( \{G^{(n)}\} \) is a sequence of mutually orthogonal projections such that
\[
\sum_{n \in \mathbb{N}} G^{(n)} = \sum_{n \in \mathbb{N}} (E_2^{(n)} + E_1^{(n+1)} + E_3^{(n+1)}) = E_2^{(1)} + \sum_{n \geq 2} (E_1^{(n)} + E_2^{(n)} + E_3^{(n)}) = I - E_1^{(1)} - E_3^{(1)}.
\]

Note that,
\[
W = \sum_{n \in \mathbb{N}} (B_1^{(n)} E_1^{(n)} + B_2^{(n)} E_2^{(n)} + E_3^{(n)}) = E_1^{(1)} E_1^{(1)} + E_3^{(1)} + \sum_{n \in \mathbb{N}} W_n = E_1^{(1)} E_1^{(1)} + E_3^{(1)} + \sum_{n \in \mathbb{N}} W_n.
\]

From Corollary 3.3, we conclude that \( W \) is the product of two symmetries in \( \mathcal{R} \).

Using Lemma 3.1, we have
\[
R_1 R_2 R_3 R_4 W = \sum_{n \in \mathbb{N}} UF^{(n)} = U.
\]
Thus \( U \) can be decomposed as the product of six symmetries in \( \mathcal{R} \). \( \square \)
3.2. Approximation of a unitary by products of four symmetries.

**Proposition 3.10.** Let $\theta \in [0, 1]$. Let $\mathfrak{A}$ be a unital $C^*$-algebra with identity $I$ with unitaries $U, V$ in $\mathfrak{A}$ such that $UV = \exp(2\pi i \theta)VU$. Then there exist symmetries $R_1, R_2, R_3, R_4$ in $M_2(\mathfrak{A})$ such that

$$
\exp(\pi i \theta) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = R_1 R_2 R_3 R_4.
$$

**Proof.** Note that for the following four elements in $M_2(\mathfrak{A})$,

$$
R_1 := \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix}, \quad R_2 := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad R_3 := \exp(\pi i \theta) \begin{bmatrix} 0 & U^*V \\ UV^* & 0 \end{bmatrix}, \quad R_4 := \begin{bmatrix} 0 & V \\ V^* & 0 \end{bmatrix},
$$

we have

$$
R_1 R_2 R_3 R_4 = \exp(\pi i \theta) \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.
$$

Clearly $R_1, R_2, R_4$ are symmetries and $R_3$ is a unitary. Since $UV = \exp(2\pi i \theta)VU$, we have

$$
R_3^2 = \exp(2\pi i \theta) \begin{bmatrix} U^*VUV^* & 0 \\ 0 & UV^*U^*V \end{bmatrix}
$$

$$
= \exp(2\pi i \theta) \begin{bmatrix} \exp(-2\pi i \theta) & 0 \\ 0 & \exp(-2\pi i \theta) \end{bmatrix}
$$

$$
= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix},
$$

and hence $R_3$ is also a symmetry. □

**Proposition 3.11.** Let $\mathcal{R}$ be a type $II_1$ von Neumann algebra. For every $\theta \in [0, 1]$, there are four symmetries $R_1, R_2, R_3, R_4$ in $\mathcal{R}$ such that $\exp(2\pi i \theta)I = R_1 R_2 R_3 R_4$.

**Proof.** Let $\mathfrak{A}$ be the hyperfinite $II_1$ factor. Recall that $\mathfrak{A}$ unitally embeds into every type $II_1$ von Neumann algebra. Thus we need only prove the assertion for $\mathcal{R} = \mathfrak{A}$.

**Case I:** $\theta$ is rational.

Let $\theta = \frac{m}{n}$. There are Weyl unitaries $U, V$ in $M_n(\mathbb{C})$ such that $UV = \exp(2\pi i \theta)VU$. By Proposition 3.10, $\lambda I_{2n}$ can be decomposed as the product of four symmetries in $M_{2n}(\mathbb{C})$. Note that there is a unital embedding $M_{2n}(\mathbb{C}) \hookrightarrow \mathcal{R}$. Thus $\lambda I$ can be decomposed as the product of four symmetries in $\mathcal{R}$.

**Case II:** $\theta$ is irrational.

Let $A_\theta$ denote the irrational rotation algebra corresponding to $\theta$. There are unitaries $U, V \in A_\theta$ such that $UV = \exp(2\pi i \theta)VU$. By Proposition 3.10, $\lambda I$ can be decomposed as the product of four symmetries in $M_2(A_\theta)$. Note that there is a unital embedding $M_2(A_\theta) \hookrightarrow \mathcal{R}$. Thus $\lambda I$ can be decomposed as the product of four symmetries in $\mathcal{R}$. □

**Theorem 3.12.** Let $\mathcal{R}$ be a type $II_1$ von Neumann algebra and $U$ be a unitary operator in $\mathcal{R}$ with finite spectrum. Then there are four symmetries $R_1, R_2, R_3, R_4$ in $\mathcal{R}$ such that $U = R_1 R_2 R_3 R_4$. 
Proof. Let the spectrum of \( U \) be \( \{\lambda_1, \cdots, \lambda_n\} \). There are mutually orthogonal projections \( E^{(1)}, \cdots, E^{(n)} \) such that \( U = \sum_{k=1}^{n} \lambda_k E^{(k)} \) and \( \sum_{k=1}^{n} E^{(k)} = I \). For \( 1 \leq k \leq n \), the von Neumann algebra \( E^{(k)}(\mathcal{R})E^{(k)} \) acting on \( E^{(k)}(\mathcal{H}) \) is a type \( II_1 \) von Neumann algebra. From Proposition 3.11, there are four symmetries \( R_1^{(k)}, R_2^{(k)}, R_3^{(k)}, R_4^{(k)} \) in \( E^{(k)}(\mathcal{R})E^{(k)} \) such that
\[
\lambda_k E^{(k)} = R_1^{(k)} R_2^{(k)} R_3^{(k)} R_4^{(k)}, \quad 1 \leq k \leq n.
\]
We identify \( \bigoplus_{k=1}^{n} E^{(k)}(\mathcal{R})E^{(k)} \) in the canonical manner as a von Neumann subalgebra of \( \mathcal{R} \). For \( 1 \leq j \leq 4 \), we define \( R_j := \sum_{k=1}^{n} R_j^{(k)} \). Note that \( R_j \) is a symmetry in \( \mathcal{R} \) as it is self-adjoint and \( R_j^2 = \sum_{k=1}^{n} (R_j^{(k)})^2 = \sum_{k=1}^{n} E^{(k)} = I \), and \( U = R_1 R_2 R_3 R_4 \).

\[ \square \]

Corollary 3.13. Let \( \mathcal{R} \) be a type \( II_1 \) von Neumann algebra. Then every unitary in \( \mathcal{R} \) can be approximated in norm by a sequence of unitaries in \( \mathcal{S}(\mathcal{R})^4 \), that is, \( U(\mathcal{R}) = (\mathcal{S}(\mathcal{R})^4)^\ast \).

Proof. Let \( W \) be any unitary operator in \( \mathcal{R} \). By [7] Theorem 5.2.5, there is a self-adjoint operator \( H \) in \( \mathcal{R} \) with spectrum in \([0,1]\) such that \( \exp(2\pi i H) = W \). Using the spectral theorem, \( H \) may be approximated in norm by a sequence of self-adjoint operators \( \{H_n\}_{n \in \mathbb{N}} \) in \( \mathcal{R} \) which have finite spectrum contained in \([0,1]\). Let \( H_n := \sum_{j=1}^{n} \lambda_j E_j \). From Theorem 3.12, for \( n \in \mathbb{N} \), \( W_n := \exp(2\pi i H_n) \) is the product of four symmetries. As the function \( t \to \exp(2\pi it) \) is continuous from \([0,1]\) to \( S^1 \), the sequence of unitaries \( \{W_n\}_{n \in \mathbb{N}} \) converges to \( W \) in norm.

\[ \square \]

4. Products of Symmetries in a von Neumann Algebra

In this section, \( \mathcal{M} \) denotes a von Neumann algebra acting on the Hilbert space \( \mathcal{H} \). We investigate the evolution of the sequence \( \{\mathcal{S}(\mathcal{M})^n\}_{n \in \mathbb{N}} \).

Proposition 4.1 (cf. [10] Theorem 3). Let \( \mathcal{M} \) be a von Neumann algebra. A unitary \( U \) in \( \mathcal{M} \) may be decomposed as the product of two symmetries in \( \mathcal{M} \) if and only if there is a unitary \( W \) in \( \mathcal{M} \) such that \( U^* = W^* U W \), that is, \( U \) and \( U^* \) are unitarily equivalent in \( \mathcal{M} \).

Proof. Let \( U = ST \) where \( S, T \) are symmetries in \( \mathcal{M} \). Since \( U^* = TS = S(ST)S = SUS \), we observe that \( U \) and \( U^* \) are unitarily equivalent with the unitary equivalence implemented by \( S \).

Conversely, let \( W \) be a unitary in \( \mathcal{M} \) such that \( U^* = W^* U W \). By taking adjoint on both sides, we have \( U = W^* U^* W \) which implies that \( U^* = W U W^* \). Thus
\[
W^2 U^* = W^2 (W^* U W) = W U W = (W U W^*) W^2 = U^* W^2.
\]

Thus \( W^2 \) commutes with \( U^* \) (and hence also with its multiplicative inverse \( U \)). Let \( \mathcal{N} \) be the von Neumann subalgebra of \( \mathcal{M} \) generated by \( \{I, W^2\} \). Clearly \( \{U, U^*, W, W^*\} \subseteq \mathcal{N} \cap \mathcal{M} \), the relative commutant of \( \mathcal{N} \) in \( \mathcal{M} \). By [7] Theorem 5.2.5, there is a positive operator \( H \) in \( \mathcal{N} \) such that \( W^2 = \exp(2\pi i H) \). Note that \( V := \exp(\pi i H) \) is a unitary operator in \( \mathcal{N} \) such that \( V^2 = W^2 \). Since \( V \in \mathcal{N} \), we have that \( V \) and \( V^* \) commute with each of \( U, U^*, W, W^* \). Thus \( S = V^* W \) is a symmetry and \( U S = S U^* \). It follows that \( U = S(U S) \) is the product of two symmetries in \( \mathcal{M} \).

\[ \square \]

Remark 4.2. In particular, Proposition 4.1 implies that the spectrum of an element in \( \mathcal{S}(\mathcal{M})^2 \) is symmetric about the real axis.
Lemma 4.3.

(i) Let $\mathcal{M}$ be a von Neumann algebra with no direct summand of type $I_{2k-1}$ ($k \in \mathbb{N}$) in its type decomposition. Then there are two mutually orthogonal equivalent projections $E_1, E_2$ in $\mathcal{M}$ such that $E_1 + E_2 = I$.

(ii) A von Neumann algebra $\mathcal{M}$ has no direct summand of type $I_{2k-1}$ ($k \in \mathbb{N}$) in its type decomposition if and only if $\mathcal{M} \cong M_2(\mathcal{S})$ for some von Neumann algebra $\mathcal{S}$.

Proof. (i) Since $\mathcal{M}$ is a direct sum of a properly infinite von Neumann algebra and von Neumann algebras of types $I_{2k}$ and $II_1$, it suffices to consider the three cases in which either $\mathcal{M}$ is properly infinite, or type $I_{2k}$ ($k \in \mathbb{N}$) or type $II_1$. For a properly infinite von Neumann algebra, the assertion follows from [8, Lemma 6.3.3]. For a type $II_1$ von Neumann algebra or a type $I_{2k}$ von Neumann algebra ($k \in \mathbb{N}$), the assertion follows from [8, Theorems 8.4.3, 8.4.4].

(ii) If $\mathcal{M}$ is of the form $M_2(\mathcal{S})$ for a von Neumann algebra $\mathcal{S}$, then clearly $\mathcal{M}$ has no direct summand of type $I_{2k-1}$ ($k \in \mathbb{N}$) in its type decomposition. The converse follows from part (i) and Remark 3.6.

Corollary 4.4. Let $\mathcal{M}$ be a von Neumann algebra with no direct summand of type $I_{2k-1}$ ($k \in \mathbb{N}$) in its type decomposition, and $\alpha \in S^1 \subset \mathbb{C}$. The scalar unitary $\alpha I$ in $\mathcal{M}$ can be decomposed as the product of three symmetries in $\mathcal{M}$ if and only if $\alpha \in \{1, i, -1, -i\}$.

Proof. Let $R_1, R_2, R_3$ be symmetries in $\mathcal{M}$ such that $\xi I = R_1 R_2 R_3$. We have $\xi R_3 = R_1 R_2 \in $ in case of which we have $\xi = \pm i$.

Since $I, -I$ are symmetries, we need only prove the assertion for $\alpha = \pm i$. By Lemma 4.3 (ii), we may assume that $\mathcal{M} \cong M_2(\mathcal{S})$ for some von Neumann algebra $\mathcal{S}$ with identity $I_{\mathcal{S}}$. Note that

$$iI = \begin{bmatrix} iI_{\mathcal{S}} & 0 \\ 0 & iI_{\mathcal{S}} \end{bmatrix} = \begin{bmatrix} 0 & I_{\mathcal{S}} \\ I_{\mathcal{S}} & 0 \end{bmatrix} \begin{bmatrix} 0 & -iI_{\mathcal{S}} \\ iI_{\mathcal{S}} & 0 \end{bmatrix} \begin{bmatrix} I_{\mathcal{S}} & 0 \\ 0 & -I_{\mathcal{S}} \end{bmatrix}$$

is a product of three symmetries in $M_2(\mathcal{S}) \cong \mathcal{M}$. We may deduce a similar decomposition of $-I$.

Note that for an odd positive integer $n$, the scalar unitary matrix $iI \in M_n(\mathbb{C})$ has imaginary determinant and thus cannot be decomposed into the product of any finite number of symmetries, let alone the product of three symmetries.

For $1 \leq k \leq 4$, let $C_k := \{ \exp(2\pi i \alpha) : \frac{k-1}{4} < \alpha < \frac{k}{4} \}$ denote the four connected components of $S^1 \setminus \{1, i, -1, -i\}$.

Proposition 4.5. Let $U$ be a unitary operator in $\mathcal{B}(\mathcal{M})$ such that $\text{sp}(U) \subset C_k$ for some $k \in \{1, 2, 3, 4\}$. Then $U$ cannot be decomposed as the product of three symmetries.

Proof. Let $\xi := \exp \left( 2\pi i \left( \frac{2k-1}{8} \right) \right)$. Since $U$ is a normal operator with $\text{sp}(U) \subset C_k$, from the spectral theorem we have that

$$\|U - \xi I\| < \max_{\ell = k, k-1} \left\{ \left| \exp \left( 2\pi i \left( \frac{\ell}{4} \right) \right) - \exp \left( 2\pi i \left( \frac{2k-1}{8} \right) \right) \right| \right\} = 2 \sin \frac{\pi}{8}.$$
Let, if possible, $U = R_1 R_2 R_3$ for symmetries $R_1, R_2, R_3$. We note that \( \| R_1 R_2 - \xi R_3 \| = \| (U - \xi I) R_3 \| < 2 \sin \frac{\pi}{8} \). As $R_3$ is a symmetry, \( \text{sp}(\xi R_3) \subseteq \{ \xi, -\xi \} \).

Recall that if $A$ and $B$ are normal operators and \( \lambda \in \text{sp}(A) \), then the distance from \( \lambda \) to \( \text{sp}(B) \) is at most \( \| A - B \| \). Thus each element of \( \text{sp}(R_1 R_2) \) is at most a distance of \( 2 \sin \frac{\pi}{8} \) away from \( \{ \xi, -\xi \} \). We conclude that either \( \text{sp}(R_1 R_2) \) is contained inside \( C_1 \cup C_3 \) or inside \( C_2 \cup C_4 \) depending on $k$.

For $z \in S^1$, we observe that $z \in C_1 \cup C_3$ if and only if $z \notin C_1 \cup C_3$, and a similar conclusion also holds for $C_2 \cup C_4$. In view of Remark 4.2, this leads to a contradiction. Thus our original assumption that $U$ is the product of three symmetries must be incorrect.

**Remark 4.6.** Let $\mathcal{M}$ be a von Neumann algebra with identity operator $I$. Since the spectrum of \( \exp\left(\frac{2\pi i}{3}\right)I \) is \( \{ \exp\left(\frac{2\pi i}{3}\right) \} \) which is contained in \( S^1 \setminus \{ 1, i, -1 - i \} \), from Proposition 4.5 we conclude that \( \exp\left(\frac{2\pi i}{3}\right)I \) cannot be decomposed as the product of three symmetries in $\mathcal{M}$. In [4 pg. 78], Halmos and Kakutani show this using purely algebraic means whereas we view it through an operator-theoretic lens.

**Theorem 4.7.** Let $\mathcal{M}$ be a von Neumann algebra. The set of unitaries with spectrum contained inside $C_k$ for some $k \in \{ 1, 2, 3, 4 \}$ is an open subset of $\mathcal{U}(\mathcal{M})$ in the uniform topology. Thus \( (\mathcal{S}(\mathcal{M})^3) = \neq \mathcal{U}(\mathcal{M}). \)

**Proof.** Without loss of generality by multiplication, we may consider a unitary $U$ in $\mathcal{M}$ such that \( \text{sp}(U) \subset C_1 \). As \( \text{sp}(U) \) is compact, we have \( \varepsilon := \text{dist}(\text{sp}(U), \{ 1, i \}) > 0 \). If $V$ is a unitary in $\mathcal{M}$ such that \( \| V - U \| < \varepsilon \), then $V$ has spectrum contained in $C_1$. This proves that the set of unitaries with spectrum contained inside $C_1$ is an open set. Further $C_1$ contains \( \exp(i\frac{\pi}{3})I \) and is thus a non-empty open subset of $\mathcal{U}(\mathcal{M}) \setminus \mathcal{S}(\mathcal{M})^3$ by Proposition 4.5. As a result, we have \( (\mathcal{S}(\mathcal{M})^3) = \neq \mathcal{U}(\mathcal{M}). \)
Theorem 4.8. Let $\mathcal{M}$ be a von Neumann algebra with no direct summand of type $I_n$ (for $n \in \mathbb{N}$) in its type decomposition. Then $S(\mathcal{M})^6 = \mathcal{U}(\mathcal{M})$, and $S(\mathcal{M})^4$ is norm-dense in $\mathcal{U}(\mathcal{M})$.

Proof. The assertion follows from Theorem 3.9, Theorem 3.12 and [3, pg. 900]. □

Let $\mathcal{M}$ be a von Neumann algebra with no direct summand of type $I_n$ (for $n \in \mathbb{N}$) in its type decomposition. From the results of this section, we have the following strict inclusions

$$S(\mathcal{M})^n \subset (S(\mathcal{M})^2)^- \subset (S(\mathcal{M})^4)^- = \mathcal{U}(\mathcal{M}).$$

Since $(S(\mathcal{M})^4)^- = (S(\mathcal{M})^n)^-$ for all $n \geq 4$, it looks plausible to conjecture that $S(\mathcal{M})^4 = S(\mathcal{M})^6 = \mathcal{U}(\mathcal{M})$.

We end our discussion with a few open questions as fodder for further investigation. For a type $II_1$ von Neumann algebra $\mathcal{R}$, let $\mathcal{N}_{\text{sym}}(\mathcal{R})$ denote the smallest positive integer $n$ such that $S(\mathcal{R})^n = \mathcal{U}(\mathcal{R})$. By Theorem 3.9 and 4.7, $4 \leq \mathcal{N}_{\text{sym}}(\mathcal{R}) \leq 6$.

**Question 1:** Let $\mathcal{R}$ be the hyperfinite $II_1$ factor. Is $\mathcal{N}_{\text{sym}}(\mathcal{R}) = 4$?

**Question 2:** Can every Haar unitary in a $II_1$ factor $\mathcal{R}$ be decomposed as the product of four symmetries in $\mathcal{R}$?

**Question 3:** Let $\mathcal{R}$ be a $II_1$ von Neumann algebra. Is $\mathcal{N}_{\text{sym}}(\mathcal{R}) = 4$? If $\mathcal{R} \cong M_2(\mathcal{R})$, is $\mathcal{N}_{\text{sym}}(\mathcal{R}) = 4$?

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