Applications in physics of the multiplicative anomaly formula involving some basic differential operators

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April 1998

Abstract: In the framework leading to the multiplicative anomaly formula—which is here proven to be valid even in cases of known spectrum but non-compact manifold (very important in Physics)—zeta-function regularisation techniques are shown to be extremely efficient. Dirac like operators and harmonic oscillators are investigated in detail, in any number of space dimensions. They yield a non-zero anomaly which, on the other hand, can always be expressed by means of a simple analytical formula. These results are used in several physical examples, where the determinant of a product of differential operators is not equal to the product of the corresponding functional determinants. The simplicity of the Hamiltonian operators chosen is aimed at showing that such situation may be quite widespread in mathematical physics. However, the consequences of the existence of the determinant anomaly have often been overlooked.

PACS numbers: 02.30.Tb, 02.70.Hm,04.62.+v

Keywords: Zeta function-regularisation, multiplicative anomaly, Wodzicki residue.
1 Introduction

The importance of zeta-function regularisation for the definition of functional determinants [1] is, without discussion, a powerful tool to deal with the ambiguities (ultraviolet divergences) present within the one-loop or external field approximation, in relativistic quantum field theory (see the seminal papers [2, 3] and for recent reviews [4, 5]). Let us remember that the Euclidean partition related to a quantum scalar field can be formally written as

$$Z = (\det L_D)^{-1/2},$$  \hspace{1cm} (1.1)

with $L_D$ an elliptic differential operator. The latter quantity is ill defined and, with regard to this, we briefly recall how zeta-function regularisation works: it gives a precise meaning, in the sense of analytic continuation, to the determinant of a differential operator which, as the product of its eigenvalues, is formally divergent. When the ($D$-dimensional) manifold is smooth and compact, the spectrum is discrete and one has, for $\text{Re } s > D/l$, $l$ being the order of the differential operator $L_D$, the definition of the zeta-function

$$\zeta(s|L_D) = \sum_i \lambda_i^{-s},$$

where $\lambda_i$ are the eigenvalues of the elliptic operator $L_D$. Making use of the relationship between the zeta-function and the heat-kernel trace via the the Mellin transform, when $\text{Re } s > D/l$, one can write

$$\zeta(s|L_D) = \text{Tr } L_D^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K(t|L_D) \, dt,$$  \hspace{1cm} (1.2)

where $K(t|L_D) = \text{Tr } \exp(-tL_D)$ is the trace of the heat operator. The previous relations are valid also in the presence of zero modes, with the replacement $K(t|L_D) \rightarrow K(t|L_D) - P_0$, $P_0$ being the projector onto the set of zero modes.

A well known heat-kernel expansion argument leads to the meromorphic structure of $\zeta(s|L_D)$. It is found that the analytically continued zeta-function is regular at $s = 0$ and thus its derivative in that point is well defined. As a consequence, the one-loop Euclidean partition function, regularised by means of the zeta-function, reads [3]

$$\ln Z = -\frac{1}{2} \ln \det(\ell^2 L_D) = \frac{1}{2} \zeta'(0|\ell^2 L_D) = \frac{1}{2} \zeta'(0|L_D) - \frac{1}{2} \zeta(0|L_D) \ln \ell^2,$$

where $\zeta'$ is the derivative with respect to $s$ and $\ell$ is a renormalization scale.

However, in general things are not so simple. Sometimes it happens that one has to deal with the product of two or more differential operators and one is directly confronted with the validity (or not) of the multiplicative property:

$$\ln \det(AB) = \ln \det A + \ln \det B,$$  \hspace{1cm} (1.3)

which is of course known to hold for non singular matrices in finite dimensional vector spaces.

A first elementary example is the following. Consider a free vector-valued scalar field $\phi_i$ in $R^4$, with a broken $O(N)$ symmetry —owing to the mass terms $m_i^2$. The Euclidean action is

$$S = \int dx^4 \phi_i \left[ \left(-\Delta + m_i^2\right) \phi_i \right],$$  \hspace{1cm} (1.4)
and the related Euclidean operator reads

\[ L_{ij} = \left( -\Delta + m_i^2 \right) \delta_{ij} , \] (1.5)

in which \( \Delta \) is the Laplace operator. Thus, one is actually dealing with a matrix-valued elliptic differential operator. In this case, the Euclidean partition function is given by

\[ \ln Z = -\frac{1}{2} \ln \det \left| \ell^2 L_{ij} \right| = -\frac{1}{2} \ln \left[ \ell^2 (-\Delta + m_1^2) \cdots \ell^2 (-\Delta + m_N^2) \right] . \] (1.6)

Another, less trivial, example concerns the finite temperature effects for a gas of free relativistic charged bosons. With regard to this case, it is possible to show that the logarithm of the grand canonical partition function, choosing as parametrization of the charged boson field the two real scalar fields \( \phi_i \), can be expressed by

\[ \ln Z_{\beta,\mu} = -\frac{1}{2} \ln \det \left( \ell^4 L_\pm \right) , \] (1.7)

with

\[ L_\pm = -\partial^2 \tau - \Delta + m^2 + \mu^2 \pm 2\mu \sqrt{-\Delta + m^2} , \] (1.8)

\( \tau \) being the imaginary time compactified with period \( \beta \), the inverse of equilibrium temperature, and \( \mu \) the chemical potential. Note however that, if one chooses as parametrization of the charged boson field the complex scalar fields \( \phi \) and \( \phi^* \), one has

\[ \ln Z_{\beta,\mu} = -\frac{1}{2} \ln \det \left( \ell^4 K_\pm K_\mp \right) , \] (1.9)

with

\[ K_\pm = -\partial^2 \tau - \Delta + m^2 + \mu^2 \pm 2\mu \partial \tau . \] (1.10)

In a recent work, Stuart Dowker [7] has shed doubt on the recipe used in computing the partition functions, Eqs. (1.6) and (1.7). In our opinion, the above prescription is correct and we refer to Ref. [8], where this issue has been discussed in detail.

As further examples, we simply recall that in the evaluation of the one-loop Vilkovisky-DeWitt effective action (see, for example, Refs. [9]–[12] for details) and in some GUT-like models (see, for example, [13]), one has to deal with matrices of higher order differential operators, which give rise to products of the same, when one has to compute functional determinants.

Within the physical literature, in all the examples we have recalled, and in many other that we do not mention here, the way one usually proceeds is by formally assuming the validity of the multiplicative relation, indiscriminately. Needless to say, this may be dangerous. One has to use always some regularisation procedure and it turns out, in fact, that the regularised determinants do not satisfy in general the multiplicative property, Eq. (1.3). Even when one is dealing with commuting operators, there exists the so-called multiplicative anomaly [14, 15]. In terms of \( F(A,B) \equiv \det(AB)/(\det A \det B) \) [13], it is simply defined as

\[ a(A,B) = \ln F(A,B) = \ln \det(AB) - \ln \det(A) - \ln \det(B) . \] (1.11)

It should be noted by passing that the non vanishing of the multiplicative anomaly implies that the relation

\[ \ln \det A = \operatorname{Tr} \ln A \] (1.12)
does not hold, in general, for elliptic operators. The formal use of the above operator identity
is not justified, since the multiplicative anomaly may be present. In fact, if one assumes that
$\text{Tr} \ln A$ is a linear functional, in the simplest case of $[A, B] = 0$, one has

$$\text{Tr} \ln AB = \text{Tr} (\ln A + \ln B) = \text{Tr} \ln A + \text{Tr} \ln B.$$  \hspace{1cm} (1.13)

Thus, if Eq. (1.12) holds, one arrives at a contradiction with Eq. (1.11), as soon as the multi-
plicative anomaly is not vanishing. With regard to this issue, taking for granted Eq. (1.12), one
might start with the definition

$$Z = \exp \left( -\frac{1}{2} \text{Tr} \ln L_D \right),$$  \hspace{1cm} (1.14)

instead of Eq. (1.1) and then make use of some regularisation while preserving the linear property
of the trace. In this case, in Ref. [16] the absence of the multiplicative anomaly has been claimed
and doubt has been shed on the use of zeta-function regularisation. In our opinion, this is not
justified (see Ref. [17]).

The presence of the multiplicative anomaly in the commuting case might turn out as a
surprise. However, in Ref. [18] it has been shown that its existence is actually unavoidable in
order to have independence of the partition function on the choice of bosonic degrees of freedom
(see, also [8]). In some cases, the multiplicative anomaly gives a contribution to the effective
action which can be readily absorbed into the renormalization procedure, but in some systems it
may certainly produce physical consequences, which have to be carefully analysed. A non trivial
example has been discussed in Ref. [18], where after a renormalization of the charge operator, it
has been shown that, starting from the natural definition Eq. (1.1), the multiplicative anomaly
gives rise to a new contribution —overlooked in previous treatments— to the high temperature
expansion of the free energy of the relativistic boson gas in the symmetric unbroken phase.

We will show, and illustrate with the help of several examples, that this multiplicative
anomaly appears already in very simple situations (one-dimensional, first order differential op-
erators differing in a constant term). And that it can be most conveniently expressed by means
of the non-commutative residue associated with a classical pseudo-differential operator, known
as the Wodzicki residue [19]. In fact, the purpose of the present paper is to continue the analysis
of the emergence of the multiplicative anomaly, that was started in Refs. [18, 20]—where the
commutative case was discussed— and to extend it to the more general non-commutative case,
working out explicitly the multiplicative anomaly formula for a large class of elliptic operators
—and to investigate some very basic examples with care. In particular, in some of these ex-
amples we will deal with non compact manifolds, where, strictly speaking, the Wodzicki theory
has not yet been developed. Nevertheless, we will see in such examples that the formula for the
multiplicative anomaly turns out to be valid too.

The content of the paper is as follows. In Sect. 2 a perturbative derivation of the multiplica-
tive anomaly for a particular class of differential operators in arbitrary dimensions is presented,
while the general formula in the case of lower dimensions $D \leq 4$ is given in 2.1. In Sect. 3
we shall recall the definition of the Wodzicki residue, together with some related results, which
will be used in Sect. 4 in order to render explicit its relation with the multiplicative anomaly
formula. Then, we show in 4.1 that such a general formula can be notably simplified in lower
dimensions, confirming completely the result obtained by using perturbation theory. In Sect. 5,
we show how the multiplicative anomaly might be used in order to compute the heat-kernel coeffi-
cients. Finally, in Sect. 6 we discuss some basic physical examples, formulated in non-compact
manifolds, but for which the spectrum of the Hamiltonian operator is exactly known, and we compute the anomaly explicitly. The paper ends with some conclusions in Sect. 7.

2 Perturbative derivation of the the multiplicative anomaly

In this section we will consider a particular case involving two self-adjoint elliptic invertible operators $H$, $H_V = H + V$, and the related product

$$A = H(H + V) .$$

By definition, the corresponding multiplicative anomaly reads

$$a(H, H_V) = \ln \det A - \ln \det H - \ln \det H_V$$

where the functional determinants are evaluated by using $\zeta$-function regularisation. It is convenient to introduce the quantity

$$\mathcal{A}(s) = \zeta(s|H) + \zeta(s|H_V) - \zeta(s|A) ,$$

where $\zeta(z|L)$ is the zeta-function associated with the elliptic operator $L$. Thus

$$a(H, H_V) = \mathcal{A}'(0) = \lim_{s \to 0} ds \left[ \zeta(s|H) + \zeta(s|H_V) - \zeta(s|A) \right] .$$

Now we also suppose $V$ to be a small perturbation potential. We indicate by $G = H^{-1}$ the inverse operator of $H$ and by $G_V$ the inverse of $H_V$. One has the well known operatorial equation $G_V = G - GVG_V$, whose solution is formally given by

$$G_V = G \sum_{n=0}^{\infty} (-VG)^n .$$

To begin with, let us consider the special, but physically important case of a constant $V$. Then $H$ and $H_V$ are commuting operators, and we may compute the complex power by means of the binomial expansion,

$$G^s_V = G^s \left[ 1 + \sum_{n=1}^{\infty} f_n(s) V^n G^n \right] , \quad f_n(s) = \frac{(-1)^n \Gamma(s + n)}{n! \Gamma(s)} .$$

Using the definition of the zeta-function and the properties of the trace, we obtain

$$\zeta(s|H_V) \equiv \text{Tr } G^s_V = \zeta(s|H) + \sum_{n=1}^{\infty} f_n(s) \text{Tr } (V^n H^{-s-n})$$

and, in a similar way,

$$\zeta(s|A) = \zeta(2s|H) + \sum_{n=1}^{\infty} f_n(s) \text{Tr } (V^n H^{-2s-n}) .$$

Finally

$$\mathcal{A}(s) = \sum_{n=1}^{\infty} f_n(s) [\zeta(s + n|H) - \zeta(2s + n|H)] \text{ tr } V^n ,$$
where \( \text{tr} \) is the trace on internal indices. Under the assumptions above, all operators here involved are invertible. If \( H \) has a zero eigenvalue this must be excluded from the definition of the zeta functions for the operators \( H \) and \( A \), but it gives a contribution to \( \zeta(s|HV) \). Such a contribution modifies the anomaly by an additive logarithmic term since, in this case,

\[
\mathcal{A}(s) = g_0 V^{-s} + \sum_{n=1}^{\infty} f_n(s) [\zeta(s + n|H) - \zeta(2s + n|H)] \text{ tr } V^n ,
\]

\( g_0 \) being the number of zero modes.

As is well known \[21\], for elliptic operators the zeta function has only simple poles. Then the multiplicative anomaly assumes the form

\[
a(H, HV) = A'(0) = -g_0 \ln V + \sum_{n \geq 2} (-1)^n \frac{\gamma + \psi(n)}{2n} \text{ Res } \zeta(s|H)|_{s=n} \text{ tr } V^n ,
\]

where

\[
\gamma + \psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} ,
\]

the sum in Eq. (2.11) being extended to all positive integers where the zeta-function has a simple pole. Here \( \gamma \) is the Euler-Mascheroni constant, while \( \psi(x) \) is the digamma function. This expression has limited validity, mainly because it assumes that the operators involved in the product commute, but nevertheless, one should note that it holds as soon as one can define the zeta-functions involved. That includes, for example, the case when the manifold is non-compact or when it is compact with boundary.

Furthermore, when one deals with the last case, i.e. a compact \( D \)-dimensional manifold with boundary, one can certainly proceed, since the meromorphic structure of the zeta function corresponding to an elliptic operator \( H \) of order \( h \) is well known. In fact, the short-\( t \) heat-kernel asymptotic expansion of \( \text{Tr } e^{-tH} \) is known (see, for example, \[22\]). Making then use of the Mellin transform

\[
\zeta(s|H) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt t^{s-1} \text{ Tr } e^{-tH} ,
\]

a standard procedure yields the analytical continuation

\[
\zeta(s|H) = \frac{1}{\Gamma(s)} \sum_n \frac{K_n}{s + n - D/h} + \text{ analytic part } ,
\]

where

\[
K_n = \frac{1}{(4\pi)^{D/2}} \int k_n(x|H) d^D x
\]

are the heat-kernel coefficients related to \( H \), which enter in the asymptotic expansion of \( \text{Tr } e^{-tH} \). They are, in principle, computable (see, for example, for the case of a Laplace-type operator, the recent paper \[23\] and references therein). From this we immediately obtain

\[
\text{Res } \zeta(s|H)|_{s=n} = \frac{K_{D-hn}(H)}{(n-1)!} \]

\( 6 \)
and thus,

\[ a(H, H_V) = -g_0 \ln V + \sum_{n \geq 2, h n \leq D} (-1)^n \frac{\gamma + \psi(n)}{2n!} K_{D-h n}(H) \; \text{tr} \; V^n. \]  

(2.17)

We close this section with some remarks. The multiplicative anomaly formula above is valid only for constant \( V \). For a manifold without boundary one has \( K_r = 0 \), when \( r \) is odd. Thus, for a second order differential operator (\( h = 2 \)) in any odd-dimensional compact manifold without boundary, the multiplicative anomaly vanishes. It is also vanishing in two dimensions, but it is actually present for \( D = 4 \), being proportional to the Seeley-DeWitt coefficient \( K_0 \). In the case of first order differential operators (\( h = 1 \)), the anomaly is non-vanishing for \( D = 2 \), and for the physically more interesting case \( D = 4 \) one has (assuming zero modes to be absent)

\[ a(H, H_V) = \frac{1}{4} K_2(H) \; \text{tr} \; V^2 + \frac{11}{288} K_0(H) \; \text{tr} \; V^4. \]

(2.18)

We note the presence of the first non trivial Seeley-DeWitt coefficient \( K_2(H) \), which for a Dirac-like operator depends on the scalar curvature of the manifold one is dealing with and \( V \) may be a mass difference. This could be interpreted as a (potentially) interesting, finite effect of induced gravity –according to Sakharov– caused by the quantum fluctuation of the matter spinor field.

It is also interesting to note that the linear term in \( V \) does never contribute to the multiplicative anomaly. If one invokes dimensional arguments, this fact is not really surprising in the case of constant \( V \). However, as we shall see in the following, this result is also true for an arbitrary potential. To show this important fact we shall make use of standard perturbation theory.

2.1 Explicit expression of the anomaly from perturbation theory

As above, we assume \( H \) to be an elliptic differential operator of known, non-degenerate spectral decomposition \( \{ \lambda_i, \varphi_i \}_{i \in I} \) and \( V \) a small (non constant in general) perturbation potential. Let us denote by \( \mu_i \) and \( a_i \) the eigenvalues of the operator \( H_V \) and \( A \) respectively. By using perturbation theory up to second order in \( V \), we have

\[
\begin{align*}
\mu_i &= \lambda_i + V_{ii} + \sum_{j \neq i} \frac{V_{ij} V_{ji}}{\lambda_i - \lambda_j} + O(V^3), \\
a_i &= \lambda_i^2 + \lambda_i V_{ii} + \sum_{j \neq i} \frac{\lambda_i \lambda_j V_{ij} V_{ji}}{\lambda_i^2 - \lambda_j^2} + O(V^3), \\
V_{ij} &= (\varphi_i, V \varphi_j). 
\end{align*}
\]

(2.19) (2.20) (2.21)

Let us assume that \( \text{Re} \; s \) is sufficiently large. Then, by definition \( \zeta(s|H) = \sum_i \lambda_i^{-s} \) while, using again the binomial expansion

\[
\begin{align*}
\zeta(s|H_V) &= \sum_i \mu_i^{-s} = g_0 V^{-s} + \zeta(s|H) - s \sum_i V_{ii} \lambda_i^{-s-1} \\
&\quad + \frac{s(s+1)}{2} \sum_i V_{ii}^2 \lambda_i^{-s-2} - s \sum_{i,j,i \neq j} \frac{V_{ij} V_{ji}}{\lambda_i - \lambda_j} \lambda_i^{-s-1} + O(V^2), 
\end{align*}
\]

(2.22) (2.23)
\[ \zeta(s|A) = \sum_i a_i^{-s} = \zeta(2s|H) - s \sum_i V_{ii} \lambda^{-2s-1}_i \]
\[ + \frac{s(s+1)}{2} \sum_i V_{ii}^2 \lambda^{-2s-2}_i - s \sum_{i,j,i \neq j} \frac{\lambda_j V_{ij} V_{ji}}{\lambda^2_i - \lambda^2_j} \lambda^{-2s-1}_i + O(V^2). \]

As a result, the quantity \( A(s) \) can be written as
\[ A(s) = g_0 V^{-s} + 2\zeta(s|H) - \zeta(2s|H) + s[\Phi_1(2s) - \Phi_1(s)] \]
\[ - \frac{s(s+1)}{2} [\Phi_2(2s) - \Phi_2(s)] + s[\Psi_1(2s) - \Psi_1(s) + \Psi_2(2s)], \]
where we have introduced the functions
\[ \Phi_k(s) = \sum_i V_{ii}^k \lambda^{-s-k}_i, \]
\[ \Psi_1(s) = \sum_{i,j,i \neq j} V_{ij} V_{ji} \lambda^{-s-1}_i, \]
\[ \Psi_2(s) = \sum_{i,j,i \neq j} V_{ij} V_{ji} \lambda^{-s}_i. \]

Since we are interested in the derivative at \( s = 0 \), let us assume that the behaviour of the analytic continuation of these functions, in a neighborhood of the origin, has the form
\[ \Phi_k(s) = \frac{B_k}{s} + A_k + O(s), \]
\[ \Psi_k(s) = \frac{C_k}{s} + D_k + O(s), \quad k = 1, 2. \]

in agreement with the fact that the zeta function has only simple poles. From Eqs. (2.26)-(2.31), we easily obtain
\[ a(H, H V) = \frac{B_2}{4} + D_2, \]
again up to second order of perturbation theory. As anticipated before, at first order in \( V \) there is no contribution to the multiplicative anomaly (observe that the potential here is not necessarily constant).

When \( V \) is constant, one has
\[ \Phi_k(s) = V^k \zeta(s + k|H), \quad \Psi_1(s) = \Psi_2(s) = 0 \]
and then
\[ a(H, H V) = \frac{V^2}{4(4\pi)^{D/2}} K_{D-2h}(H), \]
which is the leading term in \( V \) in Eq. (2.11). To summarize, up to second order of perturbation theory and for the special class of product of two operators, the multiplicative anomaly depends only on \( V^2 \) and not on the derivatives of \( V \). Moreover, for dimensional reasons, in a 4-dimensional manifold and for a second order differential operator, the second order approximation gives the exact result and thus, in such case one may argue that the form of the anomaly reads
\[ a(H, H V) = \frac{1}{4(4\pi)^2} \int k_0(x|H) \text{ tr } (V^2) \, d^4 x, \]
whatever be the potential (e.g. not necessarily constant). We will confirm this result making use of a more powerful technique we are going to introduce in the next section.
3 The Wodzicki residue

For the reader’s convenience, we will update in this section some basic information concerning the Wodzicki residue [19] (see also [14] and the references to Wodzicki quoted therein) that will be used in the rest of the paper.

Let us consider a D-dimensional, smooth (compact) manifold without boundary, \( M_D \), and a (classical) ΨDO, \( Q \), acting on sections of vector bundles on \( M_D \). To any classical ΨDO, it corresponds a complete symbol \( \sigma(Q) = Q(x,k) = e^{ikx}Qe^{-ikx} \), such that, modulo infinitely smoothing operators, one has

\[
(Qf)(x) \sim \int_{R^D} \frac{dk}{(2\pi)^D} \int_{R^D} dy \, e^{i(x-y)k} Q(x,k)f(y).
\]

The complete symbol of \( Q \) admits an asymptotic expansion for \( |k| \to \infty \), given by

\[
Q(x,k) \sim \sum_{j=0}^{\infty} Q_{q-j}(x,k),
\]

where the coefficients fulfill the homogeneity property \( Q_{q-j}(x,tk) = t^{q-j}Q_{q-j}(x,k) \), for \( t > 0 \), being \( Q_{q-j}(x,k) \neq 0 \). The number \( q \) is called the order of \( Q \). We shall deal mainly with self-adjoint operators and thus we will assume that their complex powers, beside the semigroup property, also satisfy \( (A^c)^{-s} = A^{-cs} \), the \( c \) and \( s \) being arbitrary complex numbers. As an example that we will encounter, let us consider \( \ln A \), where \( A \) is an elliptic operator of order \( a \). Then

\[
\sigma(\ln A) \sim a \ln |k| + \sum_{j=0}^{\infty} A_{-j}(x,k).
\]

Observe that \( \ln A \) is not a classical ΨDO operator, since in general its symbol differs from that of a zero order operator in the presence of the \( \ln |k| \) term.

In order to introduce the definition of the non-commutative residue of \( Q \), let us consider an elliptic operator \( A \), with \( a > q \), and form the family of ΨDOs \( A_Q(u) = A + uQ \), \( u \) being a real parameter. The associated zeta-function reads

\[
\zeta(s|A_Q(u)) = \text{Tr} \left( A + uQ \right)^{-s}.
\]

The meromorphic structure of the above zeta-function can be obtained from the short-\( t \) asymptotics of \( \text{Tr} \, e^{-A_Q(u)} \) [24], namely

\[
\text{Tr} \, e^{-tA_Q(u)} \simeq \sum_{j=1}^{\infty} \alpha_j(u)t^{(j-D)/a} + \sum_{k=1}^{\infty} \beta_k(u)t^k \ln t.
\]

Note the presence of logarithmic terms that lead, using the Mellin transform, to double poles in the meromorphic expansion of \( \zeta(s|A_Q(u)) \), i.e.

\[
\zeta(s|A_Q(u)) = \frac{1}{\Gamma(s)} \left( \int_0^1 + \int_1^{\infty} \right) dt \, t^{s-1} \text{Tr} \, e^{-A_Q(u)}
\]

\[
= \frac{1}{\Gamma(s)} \left[ \sum_{j=1}^{\infty} \frac{\alpha_j(u)}{s + \frac{D-j}{a}} - \sum_{k=1}^{\infty} \frac{\beta_k(u)}{(s+k)^2} + J(s,u) \right] ,
\]

\[9\]
where $J(s,u)$ is the analytic part. Taking the derivative with respect to $u$ and then the limit $u \to 0$, one gets

$$
\lim_{u \to 0} \frac{d}{du} \text{Tr} \left( A + uQ \right)^{-s} = -s \text{Tr} \left( QA^{-s-1} \right)
$$

$$
= \frac{1}{\Gamma(s)} \left[ \sum_{j=1}^{\infty} \frac{\alpha'_j(0)}{s + jD/a} - \sum_{k=1}^{\infty} \frac{\beta'_k(0)}{(s+k)^2} + J'(s,0) \right].
$$

(3.7)

By definition, the non-commutative residue of $Q$ is

$$
\text{res}(Q) = \text{Res} \left[ a \lim_{u \to 0} \frac{d}{du} \text{Tr} \left( A + uQ \right)^{-s} \right]_{s=-1} = a\beta'_1(0),
$$

(3.8)

where $\text{Res}$ is the usual Cauchy residue. It is possible to show that $\text{res}(Q)$ is independent on the elliptic operator $A$ and that it is a trace in the algebra of classical $\Psi$DOs (actually, the only trace up to multiplicative constants). From the above definition and taking the derivative with respect to $u$ at $u = 0$ of Eq. (3.5), one obtains a possible way to compute the non-commutative residue. In fact

$$
\text{Tr} \left( Qe^{-tA} \right) \simeq - \sum_{j=1}^{\infty} \frac{\alpha'_j(0)}{a^j} t^{(j-D)/a-1} - \frac{\text{res}(Q)}{a} \ln t + O(t \ln t)
$$

(3.9)

and so the non-commutative residue of $Q$ can be read off from the short $t$ asymptotics of the quantity $\text{Tr} \left( Qe^{-tA} \right)$, just picking the coefficient associated with $\ln t$. When the manifold is non-compact, this is one of the methods that we have at hand for evaluating the Wodzicki residue, as long as all the traces involved exist.

For the case of a compact manifold, Wodzicki has obtained a useful local form of the non-commutative residue, that is, a density which can be integrated to yield the non-commutative residue, namely

$$
\text{res}(Q) = \int_{M_D} \frac{dx}{(2\pi)^D} \int_{|k|=1} Q_{-D}(x,k)dk .
$$

(3.10)

Here the component of order $-D$ (remember that $D$ is the dimension of the manifold) of the complete symbol appears.

Let us now consider an elliptic (self-adjoint) operator $B$ of order $b > q$. From Eq. (3.7) and the requirement $\text{Tr} \left[ Q(B^c)^{-z} \right] = \text{Tr} \left( QB^{-cz} \right)$. One has the following

**Lemma 1.** In a neighborhood of $z = 0$,

$$
\text{Tr} \left( QB^{-z} \right) = \frac{\text{res}(Q)}{zb} + \frac{\gamma \text{res}(Q)}{b} - R_Q(B) + O(z),
$$

(3.11)

where $\gamma$ is the Euler-Mascheroni constant and the quantity $R_Q(B)$ satisfies the relation

$$
R_Q(B^c) - R_Q(B) = \frac{\gamma \text{res}(Q)}{b} \frac{1-c}{c}, \quad c > 0 .
$$

(3.12)

The latter equation gives another way to compute the non-commutative residue, namely

$$
\text{res}(Q) = b \left[ \text{Res} \text{Tr} \left( QB^{-z} \right) \right]_{z=0} .
$$

(3.13)
Making use of Lemma 1, one also obtains (with \( c_1 > 0 \) and \( c_2 > 0 \)):

\[
\text{Tr} \left[ Q(A^{-c_1 z} - B^{-c_2 z}) \right] = \frac{\text{res}(Q)}{z} \left( \frac{1}{ac_1} - \frac{1}{bc_2} \right) + \gamma \frac{\text{res}(Q)}{z} \left( \frac{1}{a} - \frac{1}{b} \right) - R_Q(A) + R_Q(B) + O(z).
\] (3.14)

In particular, if \( ac_1 = bc_2 = 1 \) there is no pole at \( z = 0 \), and then

\[
\text{Tr} \left[ Q(A^{-\frac{a}{\eta}} - B^{-\frac{b}{\eta}}) \right] = \gamma \frac{\text{res}(Q)}{z} \left( \frac{1}{a} - \frac{1}{b} \right) - R_Q(A) + R_Q(B) + O(z).
\] (3.15)

On the other hand, we also have [15]

**Lemma 2**

\[
\text{Tr} \left[ Q(A^{-\frac{a}{\eta}} - B^{-\frac{b}{\eta}}) \right] = -\text{res} \left[ Q \left( \frac{\ln A}{a} - \frac{\ln B}{b} \right) \right] + O(z).
\] (3.16)

This Lemma is a consequence of Lemma 1. In fact, if \( C \) is a suitable elliptic operator, making use of the relation \( Cz = I + z \ln C + O(z^2) \), one obtains

\[
\text{Tr} \left[ Q(A^{-z} - B^{-z}) \right] = \text{Tr} \left[ Q(A^{-\frac{a}{\eta}} - B^{-\frac{b}{\eta}}) Cz C^{-z} \right] = -z \text{Tr} \left[ Q \left( \frac{\ln A}{a} - \frac{\ln B}{b} \right) C^{-z} \right] + O(z^2).
\] (3.17)

Now \( Q(\frac{\ln A}{a} - \frac{\ln B}{b}) \) is a classical \( \Psi \)DO and we can make use of Lemma 1 to obtain the desired result. It is easy to show that Lemmas 1 and 2 allow us to rewrite Eq. (3.15) as

\[
\text{Tr} \left[ Q(A^{-z} - B^{-z}) \right] = \frac{\text{res}(Q)}{z} \left( \frac{1}{a} - \frac{1}{b} \right) - \text{res} \left[ Q \left( \frac{\ln A}{a} - \frac{\ln B}{b} \right) \right] + O(z).
\] (3.18)

As a result, we have

**Lemma 3**

\[
\lim_{z \to 0} \frac{d}{dz} \left\{ z \text{Tr} \left[ Q(A^{-z} - B^{-z}) \right] \right\} = -\text{res} \left[ Q \left( \frac{\ln A}{a} - \frac{\ln B}{b} \right) \right].
\] (3.19)

### 4 The multiplicative anomaly formula

In this section we will present a quick proof of the multiplicative anomaly formula, following the derivation in [15]. We consider two invertible, elliptic, self-adjoint operators \( A \) and \( B \) on \( M_D \) of orders \( a \) and \( b \) respectively, and the quantity

\[
F(A, B) = \frac{\det(AB)}{(\det A)(\det B)} = e^{a(A, B)}.
\] (4.1)

Moreover, we introduce the family of \( \Psi \)DOs

\[
A(t) = \eta^t B^\frac{a}{\eta}, \quad A(0) = B^\frac{a}{\eta}, \quad A(1) = A, \\
\eta = A B^{-\frac{a}{\eta}}, \quad \text{ord} \, \eta = 0, \quad \text{ord} \, A(t) = a, \quad \text{ord} \, (A(t)B) = a + b.
\] (4.2)
and the function

\[ F(A(t), B) = \frac{\det(A(t)B)}{(\det A(t))(\det B)}. \]  

(4.3)

One trivially gets

\[ F(A(0), B) = 1, \quad F(A(1)), B) = \frac{\det(AB)}{(\det A)(\det B)} = F(A, B). \]  

(4.4)

As a consequence, one is led to deal with the following expression for the anomaly

\[ a(A(t), B) = \ln F(A(t), B) = -\lim_{s \to 0} \frac{\partial}{s} \left[ s \ln \left[ (A(t)B)^{-s} - \ln \left[ (A(t)B)^{-s} - \ln B^{-s} \right] \right] \right]. \]  

(4.5)

This quantity has the properties: \( a(A(0), B) = 0 \) and \( a(A(1), B) = a(A, B) \).

The next step is to compute the first derivative of \( a(A(t), B) \) with respect to \( t \), the result being

\[ \partial_t a(A(t), B) = \lim_{s \to 0} \frac{\partial}{s} \left\{ s \ln \left[ (A(t)B)^{-s} - \ln \left[ (A(t)B)^{-s} - \ln B^{-s} \right] \right] \right\}. \]  

(4.6)

Furthermore, making use of the Lemma 3, i.e. Eq. (3.19), with \( Q = \ln \eta \), one obtains

\[ \partial_t a(A(t), B) = \text{res} \left[ \ln \left( \frac{\ln A(t)}{a} - \frac{\ln [A(t)B]}{a + b} \right) \right]. \]  

(4.7)

Finally, performing the integration with respect to \( t \) from 0 to 1, one gets the Kontsevich-Vishik multiplicative anomaly formula [15], namely

\[ a(A, B) = \int_0^1 dt \text{ res} \left[ \ln \left( \frac{\ln A(t)}{a} - \frac{\ln [A(t)B]}{a + b} \right) \right]. \]  

(4.8)

The multiplicative anomaly formula, Eq. (4.8), notably simplifies in the special case of commuting operators. In fact one has

\[ \frac{\ln [A(t)B]}{a + b} = t \ln \eta + \frac{\ln B}{b}, \quad \frac{\ln A(t)}{a} = t \ln \eta + \frac{\ln B}{b} \]  

(4.9)

and

\[ \frac{\ln [A(t)B]}{a + b} - \frac{\ln A(t)}{a} = t \frac{bt \ln \eta}{a(a + b)}. \]  

(4.10)

As a result,

\[ a(A, B) = \frac{b}{2a(a + b)} \text{res} \left[ (\ln (AB^{-a}))^2 \right], \]  

(4.11)

which can be rewritten as the Wodzicki multiplicative formula [13]

\[ a(A, B) = \frac{\text{res} \left[ (\ln (A^b B^{-a}))^2 \right]}{2ab(a + b)} = a(B, A), \]  

(4.12)

where the symmetry property in \( A \) and \( B \) is manifest.

We would like to end this section with the following simple (but important) remark. The notion of non-commutative residue can be introduced only for classical \( \Psi DOs \), namely the ones
whose symbol admits the asymptotics Eq. (3.2). For example, in general $\ln A$ with $A$ elliptic and $a > 0$, is not a classical $\Psi$DO, but $\ln \frac{A}{a} - \frac{\ln A}{b}$ is, as well as $\ln \eta$, since $\operatorname{ord}(\eta) = 0$. Thus, all the formulae in which the non-commutative residue appears are well defined.

For more than two operators, the recurrence equation

$$a(A, B, C) = a(AB, C) + a(A, B)$$

holds and thus, the knowledge of the multiplicative formula for the product of two operators is sufficient for computing the multiplicative anomaly associated with the product of an arbitrary number of operators.

### 4.1 The multiplicative anomaly formula in lower dimensions

In the commutative case, examples of the evaluation of the multiplicative anomaly have been already presented (see [18, 20]). For the non-commutative one, we have obtained some insights by using perturbative arguments. Here we would like to compute the multiplicative anomaly in the lower dimensional cases $D \leq 4$, by making use of the Kontsevich-Vishik formula Eq. (4.8).

We shall consider a compact manifold, in order to make use of the Wodzicki local formula for the evaluation of the non-commutative residue. With this aim, we start from the following property for the product of the two non-commuting self-adjoint elliptic $\Psi$DOs that we shall consider, $Q$ and $P$ (Baker-Campbell-Haussdorf formula)

$$\ln (PQ) = \ln P + \ln Q + [\ln P, G] + \frac{1}{12} [\ln Q, [\ln Q, \ln P]] + O(\ln^3 Q \ln P) ,$$

where $G$ is a suitable nested commutator. Making use of this operator identity, we have that the zero order $\Psi$DO $Q_t$, defined by

$$Q_t = \frac{\ln(\eta^t B)}{a} - \frac{\ln(\eta^t B^{a+b})}{a+b},$$

may be rewritten as

$$Q_t = -\frac{bt \ln \eta}{a(a+b)} + [\ln \eta, G] - \frac{t}{12b} [\ln B, [\ln B, \ln \eta]] + O(\ln^3 B \ln \eta) .$$

Since $\operatorname{res}(\ln \eta Q_t)$ is the integrand in the multiplicative anomaly formula, Eq. (4.8), and the non-commutative residue is a trace in the algebra of classical $\Psi$DOs, one has

$$\operatorname{res}(\ln \eta [\ln \eta, G]) = 0 .$$

As a consequence, the Kontsevich-Vishik formula Eq. (1.8) yields

$$a(A, B) = \frac{\operatorname{res} (\ln^2 \eta)}{2ab(a+b)} - \frac{1}{12b} \operatorname{res} (\ln \eta [\ln B, [\ln B, \ln \eta]])$$

$$+ O \left( \operatorname{res} (\ln^2 \eta \ln^3 B) \right) ,$$

in which the term which is non vanishing in the commutative case has been conveniently isolated.
Now we recall that for a zero order $\Psi$DO, $\eta$, one has $\sigma(F(\eta)) = F(\sigma(\eta))$ (for $F(.)$ analytic) and, moreover, that the symbol $\sigma$ of the commutator of two $\Psi$DOs reads

$$\sigma([Q,P]) = 2 \left[ \frac{\partial}{\partial x^\mu} \sigma(Q) \frac{i\partial}{\partial k_\mu} \sigma(P) - \frac{i\partial}{\partial k_\mu} \sigma(Q) \frac{\partial}{\partial x^\mu} \sigma(P) \right] + \text{higher order derivatives}. \quad (4.19)$$

If we assume, as in Sect. 2, that $A$ and $B$ have the same principal and sub-leading symbols except for the homogeneous one of degree zero, namely $a = b$ and $\sigma(A) - \sigma(B) = V(x)$, then

$$\sigma(\ln \eta) = O(1/|k|^a), \quad \frac{\partial}{\partial x^\mu} \sigma(\ln \eta) = O(1/|k|^a), \quad (4.20)$$

$$\frac{\partial}{\partial x^\mu} \sigma(\ln B) = O(|k|^0), \quad \frac{\partial}{\partial k_\mu} \sigma(\ln B) = O(1/|k|), \quad (4.21)$$

and it follows that

$$\sigma(\ln \eta [\ln B, [\ln B, \ln \eta]]) = O(1/|k|^{2a+2}). \quad (4.22)$$

This means that if the dimension of the compact manifold $D$ satisfies $D < 2a + 2$, only the first term on the right-hand side of Eq. (4.18) gives a non vanishing contribution to the anomaly. Such a condition is satisfied in the particular but important case of second order differential operators in four dimensions. On the other hand, for first order differential operators $a = b = 1$ and $D = 4$, also the second term of Eq. (4.18) gives a contribution. This exact analysis confirms the perturbative results of Sect. 2.

5 Heat kernel coefficients from the multiplicative anomaly

As we have shown in Section 2, Eq. (2.17), in the case of constant $V$ the multiplicative anomaly $a(H, H_V)$ can be directly related to the heat kernel coefficients $K_n(H)$, but it can also be computed by means of Wodzicki formula, Eq. (4.12). This means that one could use the Wodzicki residue in order to compute heat-kernel coefficients (see Ref. [14, 25]). Now we illustrate the method for an invertible differential operator $H$, of order $h$, on a manifold without boundary.

For this case, Eq. (4.12) reduces to

$$a_D(H_V, H) = \frac{1}{4h} \text{res} \left[ \ln(H_V H^{-1}) \right]^2 = \frac{1}{4h} \text{res} \left[ \ln(I + VH^{-1}) \right]^2, \quad (5.1)$$

If $V$ is a constant, from Eq. (2.17) one also obtains

$$a_D(H_V, H) = \frac{1}{4} K_{D-2h}(H) V^2 + O(V^3). \quad (5.2)$$

Now, taking the limit $V \to 0$, one may show that

$$K_{D-2h}(H) = \lim_{V \to 0} \frac{a_D(H_V, H)}{4V^2} = \frac{1}{4h} \text{res} H^{-2}, \quad (5.3)$$

in agreement with Ref. [25].

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6 Some physical examples

Here we are going to consider some basic examples in which the multiplicative anomaly may actually have physical consequences. Our aim has been, in fact, to look for the most simple situations (from the point of view of the number of space dimensions and of the order and nature of the differential operators involved) that might already exhibit a non-zero anomaly. It should not come as a surprise that, among them, the harmonic oscillator plays an important role. It yields non-trivial results easy to calculate analytically in any number of dimensions.

6.1 Presence of the anomaly for Dirac-like operators in one space dimension

Consider the square root of the harmonic oscillator obtained by Delbourgo in Ref. [26]. This example has potentially some interesting physical applications, for it is well known that a fermion in an external constant electromagnetic field has a similar spectrum (Landau spectrum). Exactly in the same way as when going from the Klein-Gordon to the Dirac equation and at the same price of doubling the number of components (e.g., introducing spin), Delbourgo has constructed a model for which there exists a square root of its Hamiltonian, which is very close to the one for the harmonic oscillator. It is in fact different from the Dirac oscillator introduced by several other authors, corresponding to the minimal substitution $p \to p - i\alpha r$. The main difference lies in the introduction now of the parity operator, $Q$. Whereas creation and destruction operators for the harmonic oscillator, $a^\pm = P \pm iX$, are non-hermitian, the combinations $D^\pm = P \pm iQX$ are hermitian and

$$H^\pm \equiv (D^\pm)^2 = P^2 + X^2 \mp Q = 2H_{\text{osc}} \mp Q.$$  \hspace{1cm} (6.1)

Notice that the parity term commutes with $H_{\text{osc}}$. Doubling the components ($\sigma_i$ are the Pauli matrices)

$$P \to -i\sigma_1 \frac{\partial}{\partial x}, \quad X \to \sigma_1 x, \quad Q \to \sigma_2,$$  \hspace{1cm} (6.2)

the operators $D^\pm$ are represented by

$$D^\pm \to -i\sigma_1 \frac{\partial}{\partial x} \pm \sigma_3 x.$$  \hspace{1cm} (6.3)

In the sequel, we will only consider the operator $D \equiv D^+$. It has for eigenfunctions and eigenvalues, respectively,

$$\psi_n^\pm(x) = \frac{-i e^{-x^2/2}}{\sqrt{2^n (n-1)! \sqrt{\pi}}} \left( \begin{array}{c} -i \left[ H_{n-1}(x) \pm H_n(x)/\sqrt{2n} \right] \\ H_{n-1}(x) \mp H_n(x)/\sqrt{2n} \end{array} \right), \quad \lambda_n = \pm \sqrt{2n}, \quad n \geq 1,$$

$$\psi_0(x) = \frac{e^{-x^2/2}}{\sqrt{2\sqrt{\pi}}} \left( \begin{array}{c} 1 \\ i \end{array} \right), \quad \lambda_0 = 0,$$  \hspace{1cm} (6.4)

where the $H_n(x)$ are Hermite polynomials.

The two operators we shall consider for the calculation of the anomaly are $D$ and $D_V = D + V$, $V$ being a real, constant potential with $|V| < \sqrt{2}$, that goes multiplied with the identity matrix in
the two (spinorial) dimensions (omitted here). Notice that \( \mathcal{D} \) and \( \mathcal{D}_V \) are hermitian, commuting operators. The zeta function for the operator \( \mathcal{D} \) reads

\[
\zeta(s|\mathcal{D}) = \sum_i \lambda_i^{-s} = \sum_{n=1}^{\infty} [1 + (-1)^{-s}] \left(\sqrt{2n}\right)^{-s} = [1 + (-1)^{-s}]2^{-s/2}\zeta_R(s/2),
\]

(6.5)

\( \zeta_R(s) \) being the usual Riemann zeta function, which has a simple pole at \( s = 1 \). Furthermore, the manifold is not compact, thus, by direct use of Eq. (2.11) we readily get

\[
a(\mathcal{D}, \mathcal{D}_V) = \frac{V^2}{2} - \ln V.
\]

(6.6)

The logarithmic term is due to the presence of a zero mode.

How does this match with the Wodzicki formula? First of all, we point out that one has to deal with zero modes. Thus some care must be taken to properly treat them. Secondly, since we are working in a non-compact manifold, we cannot used the local formula to evaluated the non-commutative residue. A direct check, modulo the zero mode problem, that makes use of Eq. (3.9), yields the multiplicative anomaly above. It thus seems that Wodzicki’s expression requires only small modification in order to deal with spinorial operators as the ones we have here.

We conclude by pointing out that we have here, before us, the first and most simple example of the presence of a non-trivial anomaly for operators of degree one in a space of dimension one (spinorial, however).

### 6.2 Generalization of the Dirac-like operators to \( D \) dimensions

Referring again to the work by Delbourgo in Ref. [26], the above operators \( \mathcal{D}^+ \) and \( \mathcal{D}^- \) admit possible extensions to \( D \) dimensions, which have for eigenvalues, respectively,

\[
\lambda_+ = \pm \sqrt{2(2n_r + l)}, \quad \lambda_- = \pm \sqrt{2(2n_r + l + D)}, \quad n_r, l = 0, 1, 2, \ldots
\]

(6.7)

Notice that \( \mathcal{D}^+ \) exhibits a zero mode (for \( n_r = l = 0 \)), what is not the case with \( \mathcal{D}^- \). Each of these two operators provides a different example for the calculation of the anomaly, and we will denote the corresponding partners by \( \mathcal{D}_V^\pm \) (see the preceding subsection).

The basic zeta functions are now

\[
\zeta_+(s|H) = [1 + (-1)^{-s}] 2^{-s/2} \sum_{n_r,l=0}^{\infty} (2n_r + l)^{-s/2} = [1 + (-1)^{-s}] 2^{-s/2}\zeta_2\left(\frac{s}{2}, 0\right)(2, 1),
\]

(6.8)

\[
\zeta_-(s|H) = [1 + (-1)^{-s}] 2^{-s/2} \sum_{n_r,l=0}^{\infty} (2n_r + l + D)^{-s/2} = [1 + (-1)^{-s}] 2^{-s/2}\zeta_2\left(\frac{s}{2}, D\right)(2, 1).
\]

(6.9)

As usually, the prime means in the first expression that the zero mode has to be excluded from the sum (i.e., the term with \( n_r = l = 0 \)). Here \( \zeta_2(s,b|(a_1,a_2)) \) is the Barnes zeta function in 2 dimensions. It has simple poles at the points \( s = 1, 2 \) (in general at \( s = N, N-1, \ldots \) for \( \zeta_N \))
and the residues are well known, as given by generalized Bernoulli polynomials at those points (see Refs. [27–29]). Again using Eq. (2.11), one gets

\[ a_+ \equiv a(D^+, D^+_V) = \sum_{j=1}^{2} \frac{V^{2j} \psi(2j) + \gamma}{2j} \left( \frac{V^2}{12} - 1 \right) \psi(j, D) + \gamma \left( \frac{V^2}{2j} \right) \psi(2j) + \gamma \left( j, D \right| (2, 1)), \]

where, as before, the logarithmic term is due to the presence of the zero mode, and \( \psi \) is the digamma function. For \( D = 2 \), the result is

\[ a_+ = \frac{V^2}{4} \left( \frac{11V^2}{12} - 1 \right) - \ln V, \]

\[ a_- = \frac{V^2}{4} \left( \frac{11V^2}{12} - 1 \right). \]

Let us now check with the result obtained from Wodzicki’s formula

\[ a_W(A_1, A_2) = \frac{\operatorname{res} \left[ \left( \ln \left( A_1^{a_2} A_2^{-a_1} \right) \right)^2 \right]}{2a_1 a_2 (a_1 + a_2)} = \frac{1}{4} \operatorname{res} \left\{ \left[ \ln \left( I + \frac{V}{D^\pm} \right) \right)^2 \right\}. \]

Here \( a_1 = a_2 = 1 \). Looking for the non-commutative residue, making use again of Eq. (3.9), of course one obtains Eqs. (6.10) and (6.11) (modulo the logarithmic term corresponding to the zero mode, which is immediate to supply). From the physical point of view, it is not completely clear if these operators make sense for any value of \( D \). In fact, in Ref. [26] some doubts were arisen concerning their precise physical meaning in four and higher dimensions.

6.3 Harmonic oscillators in \( D \) dimensions

Let us recall the case of the harmonic oscillators in \( D \) dimensions, with angular frequencies \((\omega_1, \ldots, \omega_D)\). The eigenvalues read

\[ \lambda_n = \bar{n} \cdot \bar{\omega} + b, \quad \bar{n} \equiv (n_1, \ldots, n_D), \quad \bar{\omega} \equiv (\omega_1, \ldots, \omega_D), \quad b = \frac{1}{2} \sum_{k=1}^{D} \omega_k \]

and the related zeta function is the Barnes one \( \zeta_D(s, b|\bar{\omega}) \), whose poles are to be found at the points \( s = k \) \((k = D, D - 1, \ldots, 1)\). Their corresponding residua can be expressed in terms of generalized Bernoulli polynomials \( B_{D-k}^{(D)}(b|\bar{\omega}) \). They are defined by [30]

\[ \frac{t^D e^{-at}}{\prod_{i=1}^{D} (1 - e^{-b_i t})} = \frac{1}{\prod_{i=1}^{D} b_i} \sum_{n=0}^{\infty} B_n^{(D)}(a|b_i) \frac{(-t)^n}{n!}. \]

The residua of the Barnes zeta function are then:

\[ \operatorname{Res} \zeta_D(k, b|\bar{\omega}) = \frac{(-1)^{D+k}}{(k-1)!(D-k)!} \prod_{j=1}^{D} \omega_j B_{D-k}^{(D)}(b|\bar{\omega}), \quad k = D, D - 1, \ldots \]
Now, if $V$ is a constant potential, from Eq. (2.11) we easily obtain
\[
a(H, H_V) = \frac{(-1)^D}{2 \prod_{j=1}^{D/2} \omega_j} \sum_{k=1}^{[D/2]} \frac{\gamma + \psi(D - 2k)}{(2k)! (D - 2k)!} B_{2k}^{(D)}(b|\bar{\omega}) V^{2k}. \tag{6.16}
\]

Notice that in our case the generalized Bernoulli polynomials of odd order vanish: $B_1^{(D)}(b|\bar{\omega}) = B_3^{(D)}(b|\bar{\omega}) = \cdots = 0$, for any $D$. On the other hand, the remaining generalized Bernoulli polynomials are never zero, in fact
\[
B_0^{(D)}(b|\bar{\omega}) = 1, \quad B_2^{(D)}(b|\bar{\omega}) = -\frac{1}{12} \sum_{i=1}^{D} \omega_i^2, \\
B_4^{(D)}(b|\bar{\omega}) = \frac{1}{24} \left[ \frac{7}{10} \sum_{i=1}^{D} \omega_i^4 + \sum_{i<j} \omega_i^2 \omega_j^2 \right], \tag{6.17}
\]
\[
B_6^{(D)}(b|\bar{\omega}) = -\frac{5}{96} \left[ \frac{31}{70} \sum_{i=1}^{D} \omega_i^6 + \frac{7}{10} \sum_{i \neq j} \omega_i^4 \omega_j^2 + \sum_{i<j<k} \omega_i^2 \omega_j^2 \omega_k^2 \right], \ldots
\]
As a consequence, the anomaly does not vanish in any case, not for $D$ odd or $D = 2$, whatever the frequencies $\omega_i$ be. Moreover, only even powers of the potential $V$ appear. Again, since the manifold is not compact, a direct use of Eq. (3.9) confirms the validity of the Wodzicki formula.

7 Conclusion

In this paper, we have considered the issue of the evaluation of functional determinants that typically are present in the one-loop approximation of the Euclidean partition function. We have stressed that the regularised determinant of a product of elliptic operators has a multiplicative anomaly factor. In general, such a factor is not vanishing and it should be taken into account in associated physical applications, since it can play an important role. The non-commutative case of the multiplicative anomaly formula has been discussed in some detail, making use of different techniques. We have started with an elementary application of standard perturbation theory of linear operators to a particular case involving a product of two of them.

Then, more powerful techniques of ΨDOs and the Wodiczki theory of the non-commutative residue have been employed in a useful re-derivation of the general multiplicative anomaly formula.

One of the most interesting results obtained in the paper deals with the possibility to notably simplify the general formula for the non-commuting case in the lower dimensional situations: $D = 2$, $D = 4$, both of which turn out to be very relevant for physical applications.

Several basic examples have been discussed with care, mainly in the non compact case. Here, another result of our paper is the formulation of a conjecture that extends the validity of the Wodzicki formula to the calculation of the multiplicative anomaly in the case of non-compact manifolds.

Acknowledgments

We would like to thank Klaus Kirsten for valuable discussions. This work has been supported by the cooperative agreement INFN (Italy)–DGICYT (Spain). EE has been financed also by DG-
ICYT (Spain), project PB96-0925, and by CIRIT (Generalitat de Catalunya), grant 1995SGR-00602.

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