Warped Product Submanifolds in Locally Golden Riemannian Manifolds with a Slant Factor

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Abstract: In the present paper, we study some properties of warped product pointwise semi-slant and hemi-slant submanifolds in Golden Riemannian manifolds, and we construct examples in Euclidean spaces. Additionally, we study some properties of proper warped product pointwise semi-slant (and, respectively, hemi-slant) submanifolds in a locally Golden Riemannian manifold.

Keywords: Golden Riemannian structure; warped product submanifold; pointwise slant; semi-slant; hemi-slant; bi-slant submanifold

MSC: 53B20; 53B25; 53C42; 53C15
hemi-slant submanifolds in locally metallic Riemannian manifolds. We also provide examples of pointwise slant and pointwise bi-slant submanifolds, of warped product semi-slant, hemi-slant and pointwise bi-slant submanifolds in Golden Riemannian manifolds.

2. Preliminaries

The Golden number \( \phi = \frac{1+\sqrt{5}}{2} \) is the positive solution of the equation

\[
x^2 - x - 1 = 0.
\]

(1)

It is a member of the metallic numbers family introduced by Spinadel [18], given by the positive solution \( \sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2} \) of the equation \( x^2 - px - q = 0 \), where \( p \) and \( q \) are positive integer values.

The **Golden structure** \( J \) is a particular case of polynomial structure on a manifold [19,20], which satisfies

\[
J^2 = J + I,
\]

(2)

where \( I \) is the identity operator on \( \Gamma(TM) \).

If \((\overline{M}, \overline{g})\) is a Riemannian manifold endowed with a Golden structure \( J \) such that the Riemannian metric \( \overline{g} \) is \( J \)-compatible, i.e.,

\[
\overline{g}(JX, Y) = \overline{g}(X, JY),
\]

(3)

for any \( X, Y \in \Gamma(TM) \), then \((\overline{M}, \overline{g}, J)\) is called a **Golden Riemannian manifold** [7].

In this case, \( \overline{g} \) verifies

\[
\overline{g}(JX, JY) = \overline{g}(J^2X, Y) = \overline{g}(JX, Y) + \overline{g}(X, Y),
\]

(4)

for any \( X, Y \in \Gamma(TM) \).

Let \( M \) be an isometrically immersed submanifold in a Golden Riemannian manifold \((\overline{M}, \overline{g}, J)\). The tangent space \( T_xM \) of \( M \) in a point \( x \in M \) can be decomposed into the direct sum \( T_xM = T_x\overline{M} \oplus T^\perp_xM \), for any \( x \in M \), where \( T^\perp_xM \) is the normal space of \( M \) in \( x \). Let \( i_x \) be the differential of the immersion \( i : M \rightarrow \overline{M} \). Then, the induced Riemannian metric \( g \) on \( M \) is given by \( g(X, Y) = \overline{g}(i_xX, i_xY) \), for any \( X, Y \in \Gamma(TM) \). In the rest of the paper, we shall denote by \( X \) the vector field \( i_xX \) for any \( X \in \Gamma(TM) \).

For any \( X \in \Gamma(TM) \), let \( TX := (JX)^T \) and \( NX := (JX)^\perp \) be the tangential and normal components, respectively, of \( JX \), and for any \( V \in \Gamma(T^\perp M) \), let \( tv := (JV)^T \) and \( nV := (JV)^\perp \) be the tangential and normal components, respectively, of \( JV \). Then, we have

\[
JX = TX + NX, \quad JV = tv + nV,
\]

(5)

for any \( X \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \).

The maps \( T \) and \( n \) are \( \overline{g} \)-symmetric [10]:

\[
\overline{g}(TX, Y) = \overline{g}(X, TY), \quad \overline{g}(nU, V) = \overline{g}(U, nV),
\]

(6)

\[
\overline{g}(NX, V) = \overline{g}(X, tv),
\]

(7)

for any \( X, Y \in \Gamma(TM) \) and \( U, V \in \Gamma(T^\perp M) \). Moreover, from [12] for \( p = q = 1 \) in the metallic structure, we obtain

\[
T^2X = TX + X - tNX, \quad NX = NTX + nNX,
\]

(8)

\[
n^2V = nV + V - NtV, \quad tv = TtV + tnV,
\]

(9)

for any \( X \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \).
Let $\nabla$ and $\nabla$ be the Levi-Civita connections on $(\overline{M}, \overline{g})$ and on its submanifold $(M, g)$, respectively. The Gauss and Weingarten formulas are given by
\[
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X V = -A_V X + \nabla_X^\perp V,
\]
for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where $h$ is the second fundamental form and $A_V$ is the shape operator, which satisfy
\[
\overline{g}(h(X, Y), V) = \overline{g}(A_V X, Y).
\]
For any $X, Y \in \Gamma(TM)$, the covariant derivatives of $T$ and $N$ are given by
\[
(\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y), \quad (\nabla_X N)Y = \nabla_X^\perp NY - N(\nabla_X Y).
\]
For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, the covariant derivatives of $t$ and $n$ are given by
\[
(\nabla_X t)V = \nabla_X tV - t(\nabla_X^\perp V), \quad (\nabla_X n)V = \nabla_X nV - n(\nabla_X^\perp V).
\]
From (2), we obtain
\[
\overline{g}((\nabla_X J)Y, Z) = \overline{g}(Y, (\nabla_X J)Z),
\]
for any $X, Y, Z \in \Gamma(T\overline{M})$, which implies [21]
\[
\overline{g}((\nabla_X T)Y, Z) = \overline{g}(Y, (\nabla_X T)Z), \quad \overline{g}((\nabla_X N)Y, V) = \overline{g}((\nabla_X t)V, Y),
\]
for any $X, Y, Z \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$.

The analogue concept of locally product manifold is considered in the context of Golden geometry, having the name of locally Golden manifold [14]. Thus, we say that the Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ is locally Golden if $J$ is parallel with respect to the Levi-Civita connection $\nabla$ on $\overline{M}$, i.e., $\nabla J = 0$.

**Remark 1.** Any almost product structure $F$ on $\overline{M}$ induces two Golden structures on $\overline{M}$ [9]:
\[
J = \pm \frac{2\phi - 1}{2} J + \frac{1}{2} I,
\]
where $\phi$ is the Golden number.

In addition, for an almost product structure $F$, the decompositions into the tangential and normal components of $FX$ and $FV$ are given by
\[
FX = fX + \omega X, \quad FV = BV + CV,
\]
for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^\perp M)$, where $fX := (FX)^T$, $\omega X := (FX)^\perp$, $BV := (FV)^T$ and $CV := (FV)^\perp$.

Moreover, the maps $f$ and $C$ are $\overline{g}$-symmetric [22]:
\[
\overline{g}(fX, Y) = \overline{g}(X, fY), \quad \overline{g}(CU, V) = \overline{g}(U, CV),
\]
for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(T^\perp M)$.

**Remark 2** ([11]). If $M$ is a submanifold in the almost product Riemannian manifold $(\overline{M}, \overline{g}, F)$ and $J$ is the Golden structure induced by $F$ on $\overline{M}$, then for any $X \in \Gamma(TM)$, we have
\[
TX = \frac{1}{2} X \pm \frac{2\phi - 1}{2} fX, \quad NX = \pm \frac{2\phi - 1}{2} \omega X.
\]
3. Pointwise Slant Submanifolds in Golden Riemannian Manifolds

We shall state the notion of pointwise slant submanifold in a Golden Riemannian manifold, following Chen’s definition [23,24] of pointwise slant submanifold of an almost Hermitian manifold.

**Definition 1.** A submanifold $M$ of a Golden Riemannian manifold $(\mathcal{M}, \bar{\mathcal{g}}, f)$ is called pointwise slant if, at each point $x \in M$, the angle $\theta_x$ between $JX$ and $T_xM$ (called the Wirtinger angle) is independent of the choice of the nonzero tangent vector $X \in T_xM \setminus \{0\}$, but it depends on $x \in M$. The Wirtinger angle is a real-valued function $\theta_x$ (called the Wirtinger function), verifying

$$\cos \theta_x = \frac{\bar{\mathcal{g}}(JX, TX)}{\|JX\| \cdot \|TX\|} = \frac{\|TX\|}{\|JX\|},$$

for any $x \in M$ and $X \in T_xM \setminus \{0\}$.

A pointwise slant submanifold of a Golden Riemannian manifold is called slant submanifold if its Wirtinger function $\theta$ is globally constant.

In a similar manner as in [23], we obtain

**Proposition 1.** If $M$ is an isometrically immersed submanifold in the Golden Riemannian manifold $(\mathcal{M}, \bar{\mathcal{g}}, f)$, then $M$ is a pointwise slant submanifold if and only if

$$T^2 = (\cos^2 \theta_x)(T + I),$$

for some real-valued function $x \mapsto \theta_x$, for $x \in M$.

From (8) and (21), we have

**Proposition 2.** Let $M$ be an isometrically immersed submanifold in the Golden Riemannian manifold $(\mathcal{M}, \bar{\mathcal{g}}, f)$. If $M$ is a pointwise slant submanifold with the Wirtinger angle $\theta_x$, then

$$\bar{\mathcal{g}}(NX, NY) = (\sin^2 \theta_x)\bar{\mathcal{g}}(TX, Y) + \bar{\mathcal{g}}(X, Y),$$

$$tNX = (\sin^2 \theta_x)(TX + X),$$

for any $X, Y \in T_xM \setminus \{0\}$ and any $x \in M$.

From (21), by a direct computation, we obtain

**Proposition 3.** Let $M$ be an isometrically immersed submanifold in the Golden Riemannian manifold $(\mathcal{M}, \bar{\mathcal{g}}, f)$. If $M$ is a pointwise slant submanifold with the Wirtinger angle $\theta_x$, then

$$(\nabla_X T^2)Y = (\cos^2 \theta_x)(\nabla_X T)Y - \sin(2\theta_x)X(\theta_x)(TY + Y),$$

for any $X, Y \in T_xM \setminus \{0\}$ and any $x \in M$.

4. Pointwise Bi-Slant Submanifolds in Golden Riemannian Manifolds

In this section, we introduce the notion of pointwise bi-slant submanifold in the Golden Riemannian context.

**Definition 2.** Let $M$ be an immersed submanifold in the Golden Riemannian manifold $(\mathcal{M}, \bar{\mathcal{g}}, f)$. We say that $M$ is a pointwise bi-slant submanifold of $\mathcal{M}$ if there exists a pair of orthogonal distributions $D_1$ and $D_2$ on $M$ such that

(i) $TM = D_1 \oplus D_2$;

(ii) $J(D_1) \perp D_2$ and $J(D_2) \perp D_1$;

(iii) $\pi_1 \perp \pi_2$;

(iv) $\pi_1 \perp J\pi_1$ and $\pi_2 \perp J\pi_2$.

Here, $\pi_1$ and $\pi_2$ denote the orthogonal projections of $TM$ onto $D_1$ and $D_2$, respectively.
Let \( M \) be a pointwise bi-slant submanifold in a Golden Riemannian manifold.

Now, we provide an example of a pointwise bi-slant submanifold in a Golden Riemannian manifold.

**Example 1.** Let \( \mathbb{R}^6 \) be the Euclidean space endowed with the usual Euclidean metric \( \langle \cdot, \cdot \rangle \). Let \( i : M \to \mathbb{R}^6 \) be the immersion given by
\[
i(u, v) := \{ \cos u \cos v, \cos u \sin v, \sin u \cos v, \sin u \sin v, \sin v, \cos v \},
\]
where \( M := \{ (u, v) \mid u, v \in (0, \frac{\pi}{2}) \} \).

We define the Golden structure \( J : \mathbb{R}^6 \to \mathbb{R}^6 \) by
\[
J(X_1, X_2, X_3, X_4, X_5, X_6) := (\phi X_1, \overline{\phi} X_2, \phi X_3, \overline{\phi} X_4, \phi X_5, \overline{\phi} X_6),
\]
where \( \phi := \frac{1 + \sqrt{5}}{2} \) is the Golden number and \( \overline{\phi} = 1 - \phi \).

We remark that \( J \) verifies \( J^2 = I + J \) and \( \langle JX, Y \rangle = \langle X, JY \rangle \), for any \( X, Y \in \mathbb{R}^6 \). Additionally, we have
\[
JZ_1 = -\phi \sin u \cos v \frac{\partial}{\partial x_1} - \overline{\phi} \sin u \sin v \frac{\partial}{\partial x_2} + \phi \cos u \cos v \frac{\partial}{\partial x_3} + \overline{\phi} \cos u \sin v \frac{\partial}{\partial x_4},
\]
\[
JZ_2 = -\phi \sin u \sin v \frac{\partial}{\partial x_1} + \overline{\phi} \cos u \cos v \frac{\partial}{\partial x_2} - \phi \cos u \sin v \frac{\partial}{\partial x_3} + \overline{\phi} \sin u \cos v \frac{\partial}{\partial x_4} + \overline{\phi} \sin u \sin v \frac{\partial}{\partial x_5} - \overline{\phi} \sin v \frac{\partial}{\partial x_6}.
\]

We remark that \( \langle JZ_1, Z_1 \rangle = \langle JZ_2, Z_2 \rangle = 0 \), \( \langle JZ_1, Z_1 \rangle = \phi \cos^2 v + \overline{\phi} \sin^2 v \) and \( \langle JZ_2, Z_2 \rangle = 1 \).

On the other hand, we get
\[
\|Z_1\|^2 = 1, \quad \|Z_2\|^2 = 2.
\]
\[
\|JZ_1\|^2 = \phi^2 \cos^2 v + \overline{\phi}^2 \sin^2 v = \phi \cos^2 v + \overline{\phi} \sin^2 v + 1, \quad \|JZ_2\|^2 = \phi^2 + \overline{\phi}^2 = 3.
\]

We denote by \( D_1 := \text{span}\{Z_1\} \) the pointwise slant distribution with the slant angle \( \theta_1 \), where \( \cos \theta_1 = \frac{f(u, v)}{\sqrt{f(u, v) + 1}} \) for \( f(u, v) := \phi \cos^2 v + \overline{\phi} \sin^2 v \) a real function on \( M \). In addition, we denote by \( D_2 := \text{span}\{Z_2\} \) the slant distribution with the slant angle \( \theta_2 \), where \( \cos \theta_2 = \frac{1}{\sqrt{6}} \).
The distributions $D_1$ and $D_2$ satisfy the conditions from Definition 2.

If $M_1$ and $M_2$ are the integral manifolds of the distributions $D_1$ and $D_2$, respectively, then $M := M_1 \times \sqrt{2} M_2$ with the Riemannian metric tensor

$$g := du^2 + 2dv^2$$

is a pointwise bi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)$.

**Example 2.** If, in Example 1, we consider that $f$ is a Golden function (i.e., $f^2 = f + 1$), then $\cos \theta_1 = 1$, and we remark that $M$ is a semi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)$, with the slant angle $\theta = \arccos \frac{1}{\sqrt{6}}$.

**Example 3.** On the other hand, if, in Example 2, we consider $f = 0$ (i.e., $\tan \nu = \pm \phi$), then $\cos \theta_1 = 0$, and we remark that $M$ is a semi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^6, \langle \cdot, \cdot \rangle, J)$, with the slant angle $\theta = \arccos \frac{1}{\sqrt{6}}$.

If we denote by $P_i$ the projections from $TM$ onto $D_i$ for $i \in \{1, 2\}$, then $X = P_1 X + P_2 X$ for any $X \in \Gamma(TM)$. In particular, if $X \in D_i$, then $X = P_i X$, for $i \in \{1, 2\}$.

If we denote by $T_i = P_i \circ T$ for $i \in \{1, 2\}$, then, from (5), we obtain

$$JX = T_1 X + T_2 X + NX. \quad (25)$$

In a similar manner as in [24], we obtain

**Lemma 1.** Let $M$ be a pointwise bi-slant submanifold of a locally Golden Riemannian manifold $(\mathcal{M}, g, J)$ with pointwise slant distributions $D_1$ and $D_2$ and slant functions $\theta_1$ and $\theta_2$, respectively. Then

(i) for any $X, Y \in D_1$ and $Z \in D_2$, we have

$$(\sin^2 \theta_1 - \sin^2 \theta_2) \overline{g}(\nabla_X Y, T_2 Z + Z) \quad (26)$$

$$= \overline{g}(\nabla_X Y, T_2 Z) + \overline{g}(\nabla_X Z, T_1 Y) + \frac{\sin \theta_1}{\sin \theta_2} \overline{g}(A_{NT_1 Y} Z + A_{NT_2 Y} Z, X) \quad (27)$$

$$- \overline{g}(A_{NT_1 Y} Z + A_{NT_2 Y} Z, X) - \overline{g}(A_{NZ} T_1 Y + A_{NZ} T_2 Z, X);$$

(ii) for any $X \in D_1$ and $Z, W \in D_2$, we have

$$(\sin^2 \theta_2 - \sin^2 \theta_1) \overline{g}(\nabla_Z W, T_1 X + X)$$

$$= \overline{g}(\nabla_Z W, T_1 X) + \overline{g}(\nabla_Z W, T_2 W) + \frac{\sin \theta_2}{\sin \theta_1} \overline{g}(A_{NW} W + A_{NW} X, Z)$$

$$- \overline{g}(A_{NW} W + A_{NW} X, Z) - \overline{g}(A_{ NZ} T_1 Y + A_{ NZ} T_2 Z, W).$$

**Proof.** From (2), we have

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla_X Y, Z) = \overline{g}(f^2 \nabla_X Y, Z) - \overline{g}(J \nabla_X Y, Z), \quad (28)$$

for any $X, Y \in D_1$ and $Z \in D_2$.

By using (3) and $(\nabla_X J) Y = 0$, we obtain

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla_X J Y, Z) - \overline{g}(\nabla_X J Y, Z). \quad (29)$$

From (25), we obtain $JX = T_1 X + NX, JY = T_1 Y + NY$ and $JZ = T_2 Z + NZ$ for any $X, Y \in D_1$ and $Z \in D_2$ and, from here, we obtain

$$\overline{g}(\nabla_X Y, Z) = \overline{g}(\nabla_X J T_1 Y, Z) + \overline{g}(\nabla_X J NY, Z) - \overline{g}(\nabla_X (T_1 Y + NY), Z)$$

$$- \overline{g}(\nabla_X (T_1 Y + NY), Z). \quad (30)$$
Thus, we obtain

\[
\bar{g}(\nabla X, \nabla Y, Z) = \cos^2 \theta_1 \bar{g}(\nabla X(T_1 Y + Y), Z) - \sin(2\theta_1)X(\theta_1)\bar{g}(T_1 Y + Y, Z) - \bar{g}(A_{NT_i} Y, T_2 Z) - \bar{g}(\nabla X N, Y) + \bar{g}(\nabla X N, T_1 Y) + \bar{g}(T_1 Y, \nabla X Z) + \bar{g}(A_{NY} X, Z).
\]

By using \(\bar{g}(T_1 Y + Y, Z) = 0\), we obtain

\[
\sin^2 \theta_1 \bar{g}(\nabla X, Y, Z) = \cos^2 \theta_1 \bar{g}(\nabla X, T_1 Y, Z) - \bar{g}(A_{NT_i} Y + A_{NY} T_2 Z, X) + \bar{g}(J N, \nabla X Y) - \bar{g}(A_{NZ} X, T_1 Y) + \bar{g}(T_1 Y, \nabla X Z) + \bar{g}(A_{NY} X, Z).
\]

By using (8) and (23), we find

\[
\bar{g}(J N, \nabla X Y) = \bar{g}(J N + n N, \nabla X Y)
\]

and from

\[
\bar{g}(\nabla X T_1 Y, Z) = -\bar{g}(J Y - N Y, \nabla X Z) = \bar{g}(\nabla X, J Z) - \bar{g}(\nabla X N, Y, Z)
\]

we have

\[
(\sin^2 \theta_1 - \sin^2 \theta_2)\bar{g}(\nabla X Y, Z) = (1 - \sin^2 \theta_1)\bar{g}(\nabla X Y, T_2 Z) + \cos^2 \theta_1 \bar{g}(A_{NZ} Y + A_{NY} Z, X) + \sin^2 \theta_2 \bar{g}(\nabla X Y, T_2 Z) - \bar{g}(A_{NT_i} Y + A_{NY} T_2 Z + A_{NZ} T_1 Y + A_{NT_i} Z Y, X) - \bar{g}(Y, \nabla X N Z) + \bar{g}(T_1 Y, \nabla X Z) + \bar{g}(A_{NY} X, Z)
\]

and from here, we obtain (26).

In the same manner, we find (27).

Lemma 2. Let \(M\) be a pointwise semi-slant submanifold in a locally Golden Riemannian manifold \((\bar{M}, \bar{g}, \bar{f})\), with pointwise slant distributions \(D_1\) and \(D_2\) having slant functions \(\theta_1\) and \(\theta_2\).

(i) If the slant functions are \(\theta_1 = 0\) and \(\theta_2 = \theta\), we obtain

\[
\sin^2 \theta_1 \bar{g}(\nabla X Y, T_2 Z + Z) = -\bar{g}(\nabla X Y, T_2 Z) - \bar{g}(\nabla X Z, T_1 Y) \quad (30)
\]

for any \(X, Y \in D^T\) and \(Z \in D^\theta\), and

\[
\sin^2 \theta_1 \bar{g}(\nabla Z W, T_1 X + X) = \bar{g}(\nabla Z W, T_1 X) + \bar{g}(\nabla Z X, T_2 W) \quad (31)
\]

\[
+ (\cos^2 \theta + 1)\bar{g}(A_{NW} X, Z) - \bar{g}(A_{NT_2} W X + A_{NW} T_1 X, Z),
\]

\[
+ \bar{g}(A_{NT_1} Y + A_{NY} T_2 Z + A_{NZ} T_1 Y + A_{NT_2} Z Y, X) - \bar{g}(Y, \nabla X N Z) + \bar{g}(T_1 Y, \nabla X Z) + \bar{g}(A_{NY} X, Z)
\]
for any $X \in D^\vartheta$ and $Z, W \in D^\theta$.

(ii) If the slant functions are $\theta_1 = \vartheta$ and $\theta_2 = 0$, we obtain

$$\sin^2 \theta \tilde{\varphi}(\nabla_X Y, T_2 Z + Z) = \tilde{\varphi}(\nabla_X Y, T_2 Z)$$

$$+ (\cos^2 \theta + 1)\tilde{\varphi}(A_{NY} Z, Z) - \tilde{\varphi}(A_{NT_1 Y} Z + A_{NT_2 Z} Y, X)$$

for any $X, Y \in D^\vartheta$ and $Z \in D^T,\vartheta$.

Let $M$ be a pointwise hemi-slant submanifold in a locally Golden Riemannian manifold $(\tilde{\mathcal{M}}, \tilde{\varphi}, J)$, with pointwise slant distributions $D_1$ and $D_2$ having slant functions $\theta_1$ and $\theta_2$.

(i) If the slant functions are $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \theta$, we obtain

$$\cos^2 \theta \tilde{\varphi}(\nabla_X Y, T_2 Z + Z) = \tilde{\varphi}(\nabla_X Y, T_2 Z)$$

$$+ \tilde{\varphi}(A_{NZ} Y + A_{NY} Z, X) - \tilde{\varphi}(A_{NT_2 Z} Y + A_{NY} T_2 Z, X),$$

for any $X, Y \in D^\vartheta$ and $Z \in D^\theta$, and

$$\cos^2 \theta \tilde{\varphi}(\nabla_Z W, X) = -\tilde{\varphi}(\nabla_Z W, T_2 W)$$

$$- (\cos^2 \theta + 1)\tilde{\varphi}(A_{NX} W + A_{NW} X, Z) + \tilde{\varphi}(A_{NT_2 W} X + A_{NX} T_2 W, Z),$$

for any $X \in D^\vartheta$ and $Z, W \in D^\theta$.

(ii) If the slant functions are $\theta_1 = \theta$ and $\theta_2 = \frac{\pi}{2}$, we obtain

$$\cos^2 \theta \tilde{\varphi}(\nabla_X Y, Z) = -\tilde{\varphi}(\nabla_X Z, T_1 Y)$$

$$- (\cos^2 \theta + 1)\tilde{\varphi}(A_{NZ} Y + A_{NY} Z, X) + \tilde{\varphi}(A_{NT_1 Z} Y + A_{NZ} T_1 Y, X),$$

for any $X, Y \in D^\vartheta$ and $Z \in D^\vartheta$, and

$$\cos^2 \theta \tilde{\varphi}(\nabla_Z W, T_1 X + X) = \tilde{\varphi}(\nabla_Z W, T_1 X)$$

$$+ \tilde{\varphi}(A_{NX} W + A_{NW} X, Z) - \tilde{\varphi}(A_{NT_1 W} X + A_{NW} T_1 X, Z),$$

for any $X \in D^\vartheta$ and $Z, W \in D^\vartheta$.

5. Warped Product Pointwise Bi-Slant Submanifolds in Golden Riemannian Manifolds

In [13], the authors of this paper introduced the Golden warped product Riemannian manifold and provided a necessary and sufficient condition for the warped product of two locally Golden Riemannian manifolds to be locally Golden. Moreover, the subject was continued in the papers [14,25], where the authors characterized the metallic structure on the product of two metallic manifolds in terms of metallic maps and provided a necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic.

Let $(M_1, g_1)$ and $(M_2, g_2)$ be two Riemannian manifolds (of dimensions $n_1 > 0$ and $n_2 > 0$, respectively) and let $\pi_1, \pi_2$ be the projection maps from the product manifold $M_1 \times M_2$ onto $M_1$ and $M_2$ respectively. We denote by $\tilde{\varphi} := \varphi \circ \pi_1$ the lift to $M_1 \times M_2$ of a smooth function $\varphi$ on $M_1$. Then, $M_1$ is called the base, and $M_2$ is called the fiber of $M_1 \times M_2$. The unique element $\tilde{X}$ of $\Gamma(T(M_1 \times M_2))$ that is $\pi_1$-related to $X \in \Gamma(TM_1)$ and to the zero vector field on $M_2$ will be called the horizontal lift of $X$, and the unique element $\tilde{V}$ of $\Gamma(T(M_1 \times M_2))$ that is $\pi_2$-related to $V \in \Gamma(TM_2)$ and to the zero vector field on $M_1$ will
be called the \textit{vertical lift of }V. We denote by $\mathcal{L}(M_1)$ the set of all horizontal lifts of vector fields on $M_1$ and by $\mathcal{L}(M_2)$ the set of all vertical lifts of vector fields on $M_2$.

For $f : M_1 \to (0, \infty)$, a smooth function on $M_1$, we consider the Riemannian metric $g$ on $M := M_1 \times M_2$:

\[
g := \pi_1^*g_1 + (f \circ \pi_1)^2 \pi_2^*g_2. \tag{38}
\]

**Definition 3.** The product manifold $M$ of $M_1$ and $M_2$ together with the Riemannian metric $g$ is called the warped product of $M_1$ and $M_2$ by the warping function $f$ [26].

A warped product manifold $M := M_1 \times_f M_2$ is called trivial if the warping function $f$ is constant. In this case, $M$ is the Riemannian product $M_1 \times M_2$, where the manifold $M_2$ is equipped with the metric $f^2g_2$ (which is homothetic to $g_2$) [4].

In the next considerations, we shall denote by $(f \circ \pi_1)^2 =: f^2$, $\pi_1^*g_1 =: g_1$ and $\pi_2^*g_2 =: g_2$, respectively.

**Lemma 4** ([4]). For $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, we have on $M := M_1 \times_f M_2$ that

\[
\nabla_X Z = \nabla_Z X = X(\ln f) Z, \tag{39}
\]

where $\nabla$ denotes the Levi-Civita connection on $M$.

The warped product $M_1 \times_f M_2$ of two pointwise slant submanifolds $M_1$ and $M_2$ in a Golden Riemannian manifold $(\mathcal{M}, \mathcal{g}, J)$ is called a \textit{warped product pointwise bi-slant submanifold}. Moreover, the pointwise bi-slant submanifold $M_1 \times_f M_2$ is called \textit{proper} if both of the submanifolds $M_1$ and $M_2$ are proper pointwise slant in $(\mathcal{M}, \mathcal{g}, J)$.

Now, we provide an example of a warped product pointwise bi-slant submanifold in a Golden Riemannian manifold.

**Example 4.** Let $\mathbb{R}^6$ be the Euclidean space endowed with the usual Euclidean metric $\langle \cdot, \cdot \rangle$. Let $i : M \to \mathbb{R}^6$ be the immersion given by

\[
i(f, u) := (f \sin u, f \cos u, f, f \cos u, f \sin u, u),
\]

where $M := \{(f, u) \mid f > 0, u \in (0, \frac{\pi}{2}) \}$. We can find a local orthogonal frame on $TM$ given by

\[
Z_1 = \sin u \frac{\partial}{\partial x_1} + \cos u \frac{\partial}{\partial x_2} + \cos u \frac{\partial}{\partial x_3} + \sin u \frac{\partial}{\partial x_5},
\]

\[
Z_2 = f \cos u \frac{\partial}{\partial x_1} - f \sin u \frac{\partial}{\partial x_2} + f \sin u \frac{\partial}{\partial x_4} + f \cos u \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}.
\]

We define the Golden structure $J : \mathbb{R}^6 \to \mathbb{R}^6$ by

\[
J(X_1, X_2, X_3, X_4, X_5, X_6) := (\phi X_1, \phi X_2, \phi X_3, \bar{\phi} X_4, \bar{\phi} X_5, \bar{\phi} X_6),
\]

where $\phi$ is the Golden number and $\bar{\phi} = 1 - \phi$.

We remark that $J$ verifies $J^2 = I + J$ and $\langle JX, Y \rangle = \langle X, JY \rangle$, for any $X, Y \in \mathbb{R}^6$. Additionally, we have

\[
JZ_1 = \phi \sin u \frac{\partial}{\partial x_1} + \phi \cos u \frac{\partial}{\partial x_2} + \phi \cos u \frac{\partial}{\partial x_3} + \bar{\phi} \cos u \frac{\partial}{\partial x_4} + \bar{\phi} \sin u \frac{\partial}{\partial x_5},
\]

\[
JZ_2 = \phi f \cos u \frac{\partial}{\partial x_1} - \phi f \sin u \frac{\partial}{\partial x_2} + \bar{\phi} f \sin u \frac{\partial}{\partial x_4} + \bar{\phi} f \cos u \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}.
\]

We remark that $\langle JZ_1, Z_2 \rangle = \langle JZ_2, Z_1 \rangle = 0$, $\langle JZ_1, Z_1 \rangle = 2\phi + \bar{\phi} = \phi + 1$ and $\langle JZ_2, Z_2 \rangle = f^2(\phi + \bar{\phi}) + \bar{\phi} = f^2 + 1 - \phi$. 


On the other hand, we obtain
\[ \|Z_1\|^2 = 3, \|Z_2\|^2 = 2f^2 + 1, \]
\[ \|JZ_1\|^2 = 2\phi^2 + \bar{\phi}^2 = \phi + 4, \|JZ_2\|^2 = f^2(\phi^2 + \bar{\phi}^2) + \bar{\phi}^2 = 3f^2 + 2 - \phi. \]

We denote by \( D_1 := \text{span}\{Z_1\} \) the slant distribution with the slant angle \( \theta_1 \) and \( D_2 := \text{span}\{Z_2\} \) the pointwise slant distribution with the slant angle \( \theta_2 \), where
\[ \cos \theta_1 = \frac{\phi + 1}{\sqrt{3(\phi + 4)}} \]
and
\[ \cos \theta_2 = \frac{f^2 + 1 - \phi}{\sqrt{(2f^2 + 1)(3f^2 + 2 - \phi)}}. \]

The distributions \( D_1 \) and \( D_2 \) are integrable and, if \( M_1 \) and \( M_2 \) are the integral manifolds of the distributions \( D_1 \) and \( D_2 \), respectively, then \( M := M_1 \times \sqrt{2f^2 + 1} M_2 \) with the Riemannian metric tensor
\[ g := 3df^2 + (2f^2 + 1)du^2 \]
satisfy the conditions of the warped product of \( M_1 \) and \( M_2 \) by the warping function \( \sqrt{2f^2 + 1} \). Thus, we obtain a warped product pointwise bi-slant submanifold in the Golden Riemannian manifold \((\mathbb{R}^6, \langle \cdot, \cdot \rangle, f)\).

In a similar manner as in [25], we obtain

**Proposition 4.** Let \( M := M_1 \times f M_2 \) be a warped product submanifold in a locally Golden Riemannian manifold \((\mathbb{M}, g, J)\) with warping function \( f \). Then, for any \( X, Y \in \Gamma(TM_1) \) and \( Z, W \in \Gamma(TM_2) \), we have
\[ \tilde{g}(h(X, Y), NZ) = -\tilde{g}(h(X, Z), NY), \tag{40} \]
\[ \tilde{g}(h(X, Z), NW) = 0, \tag{41} \]
and
\[ \tilde{g}(h(Z, W), NX) = T_1X(\ln f)\tilde{g}(Z, W) - X(\ln f)\tilde{g}(Z, T_2W). \tag{42} \]

**Proof.** For any \( X, Y \in \Gamma(TM_1) \) and \( Z \in \Gamma(TM_2), \) by using (3), (5), (10), (39) and \( \nabla J = 0 \), we obtain
\[ \tilde{g}(h(X, Y), NZ) = \tilde{g}(\nabla_X Y, JZ) - \tilde{g}(\nabla_X Y, T_2 Z) \]
\[ = \tilde{g}(\nabla_X T_1 Y, Z) + \tilde{g}(\nabla_X NY, Z) + \tilde{g}(\nabla_X T_2 Z, Y) \]
\[ = -\tilde{g}(\nabla_X Z, T_1 Y) - \tilde{g}(\nabla_X NY, Z) + \tilde{g}(\nabla_X T_2 Z, Y) \]
\[ = -X(\ln f)\tilde{g}(T_1 Y, Z) - \tilde{g}(h(X, Z), NY) + X(\ln f)\tilde{g}(Y, T_2 Z). \]

On the other hand, \( \tilde{g}(T_1 Y, Z) = \tilde{g}(JY, Z) = \tilde{g}(Y, JZ) = \tilde{g}(Y, T_2 Z). \) Thus, we obtain (40). For any \( X \in \Gamma(TM_1) \) and \( Z, W \in \Gamma(TM_2), \) by using (3), (5), (10), (39) and \( \nabla J = 0 \), we obtain
\[ \tilde{g}(h(X, Z), NW) = \tilde{g}(\nabla_X Z, JW) - \tilde{g}(\nabla_X Z, T_2 W) \]
\[ = \tilde{g}(\nabla_X T_2 Z, W) - \tilde{g}(A_{NY} X, W) - \tilde{g}(\nabla_X T_2 Z, W) \]
\[ = X(\ln f)\tilde{g}(T_2 Z, W) - \tilde{g}(Z, T_2 W) - \tilde{g}(h(X, W), NZ) \]
and using
\[ \tilde{g}(T_2 Z, W) - \tilde{g}(Z, T_2 W) = \tilde{g}(JZ, W) - \tilde{g}(Z, JW) = 0, \]
we obtain
\[ \tilde{g}(h(X, Z), NW) = -\tilde{g}(h(X, W), NZ). \tag{43} \]
On the other hand, after interchanging $Z$ by $X$, we have
\begin{align*}
\mathcal{G}(h(Z, X), NW) &= \mathcal{G}(\nabla_Z T_1 X, W) - \mathcal{G}(A_{NX} Z, W) - \mathcal{G}(\nabla_Z X, T_2 W) \\
&= T_1 X(\ln f) \mathcal{G}(Z, W) - X(\ln f) \mathcal{G}(Z, T_2 W) - \mathcal{G}(h(Z, W), NX) = \mathcal{G}(h(X, W), NZ)
\end{align*}
and using (43), we obtain (41).

For any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$, by using (3), (5), (10), (39) and $\nabla f = 0$, we obtain
\begin{align*}
\mathcal{G}(h(Z, W), NX) &= \mathcal{G}(\nabla_Z W, JX) - \mathcal{G}(\nabla_Z W, T_1 X) \\
&= \mathcal{G}(\nabla_Z T_2 W, X) + \mathcal{G}(\nabla_Z NW, X) - \mathcal{G}(\nabla_Z W, T_1 X) \\
&= -\mathcal{G}(T_2 W, \nabla_Z X) - \mathcal{G}(A_{NW} Z, X) + \mathcal{G}(W, \nabla_Z T_1 X) \\
&= -X(\ln f) \mathcal{G}(Z, T_2 W) + T_1 X(\ln f) \mathcal{G}(Z, W)
\end{align*}
and we obtain (42).

6. Warped Product Pointwise Semi-Slant or Hemi-Slant Submanifolds in Golden Riemannian Manifolds

In this section, we obtain some properties of the distributions in the case of pointwise semi-slant and pointwise hemi-slant submanifolds in locally Golden Riemannian manifolds.

**Definition 4.** Let $M := M_1 \times_f M_2$ be a warped product bi-slant submanifold in a Golden Riemannian manifold $(\mathcal{M}, \mathcal{G}, J)$ such that one of the components $M_i$ ($i \in \{1, 2\}$) is an invariant submanifold (respectively, anti-invariant submanifold) in $\mathcal{M}$ and the other one is a pointwise slant submanifold in $\mathcal{M}$, with the Wirtinger angle $\theta_\mathcal{M} \in [0, \frac{\pi}{2}]$. Then, we call the submanifold $M$ warped product pointwise semi-slant submanifold (respectively, warped product pointwise hemi-slant submanifold) in the Golden Riemannian manifold $(\mathcal{M}, \mathcal{G}, J)$.

Now, we provide an example of a warped product semi-slant submanifold in a Golden Riemannian manifold.

**Example 5.** Let $\mathbb{R}^7$ be the Euclidean space endowed with the usual Euclidean metric $\langle \cdot, \cdot \rangle$ and consider the immersion $i : M \rightarrow \mathbb{R}^7$, given by
\begin{align*}
i(f, u, v) := (f \cos u, f \sin u, f \cos v, f \sin v, f, u, v),
\end{align*}
where $M := \{(f, u, v) \mid f > 0, u, v \in [0, \frac{\pi}{2}]\}$. The local orthogonal frame on $TM$ is given by
\begin{align*}
Z_1 &= \cos u \frac{\partial}{\partial x_1} + \sin u \frac{\partial}{\partial x_2} + \cos v \frac{\partial}{\partial x_3} + \sin v \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_1}, \\
Z_2 &= -f \sin u \frac{\partial}{\partial x_1} + f \cos u \frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_2}, \\
Z_3 &= -f \sin v \frac{\partial}{\partial x_3} + f \cos v \frac{\partial}{\partial x_4} + \frac{\partial}{\partial y_3},
\end{align*}
We define the Golden structure $J : \mathbb{R}^7 \rightarrow \mathbb{R}^7$ by
\begin{align*}
J \left( \frac{\partial}{\partial x_k}, \frac{\partial}{\partial y_l} \right) := (\phi \frac{\partial}{\partial x_1}, \phi \frac{\partial}{\partial x_2}, \phi \frac{\partial}{\partial x_3}, \phi \frac{\partial}{\partial x_4}, \phi \frac{\partial}{\partial y_1}, \phi \frac{\partial}{\partial y_2}, \phi \frac{\partial}{\partial y_3}),
\end{align*}
for $k \in \{1, 2, 3, 4\}$ and $l \in \{1, 2, 3\}$, where $\phi$ is the Golden number and $\overline{\phi} = 1 - \phi$. Since

\[ JZ_1 = \phi \cos u \frac{\partial}{\partial x_1} + \phi \sin u \frac{\partial}{\partial x_2} + \overline{\phi} \cos v \frac{\partial}{\partial x_3} + \overline{\phi} \sin v \frac{\partial}{\partial x_4} + \overline{\phi} \frac{\partial}{\partial y_1}, \]

\[ JZ_2 = \phi Z_2, \quad JZ_3 = \overline{\phi} Z_3, \]

we remark that $\langle JZ_k, Z_l \rangle = 0$, for any $k \neq l$, where $k, l \in \{1, 2, 3\}$, and $\langle JZ_1, Z_1 \rangle = \phi + 2\overline{\phi} = 2 - \phi$.

We find that

\[ \|Z_1\|^2 = 3, \quad \|Z_2\|^2 = \|Z_3\|^2 = f^2 + 1, \]

\[ \|JZ_1\|^2 = \phi^2 + 2\overline{\phi}^2 = 5 - \phi, \quad \|JZ_2\|^2 = \phi^2(f^2 + 1), \quad \|JZ_3\|^2 = \overline{\phi}^2(f^2 + 1). \]

Denote by $D_1 := \text{span}\{Z_1\}$ the slant distribution with the slant angle $\theta$, where $\cos \theta = \frac{2-\phi}{\sqrt{3(5-\phi)}}$ and by $D_2 := \text{span}\{Z_2, Z_3\}$ the invariant distribution (with respect to $J$).

If $M_\theta$ and $M_T$ are the integral manifolds of the distributions $D_1$ and $D_2$, respectively, then $M := M_\theta \times \sqrt{f^2 + 1} M_T$ with the metric

\[ g := 3df^2 + (f^2 + 1)(du^2 + dv^2) = g_{M_\theta} + (f^2 + 1)g_{M_T} \]

is a warped product semi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^7, \langle \cdot, \cdot \rangle, J)$.

Now, we provide an example of a warped product hemi-slant submanifold in a Golden Riemannian manifold.

**Example 6.** Let $\mathbb{R}^5$ be the Euclidean space endowed with the usual Euclidean metric $\langle \cdot, \cdot \rangle$ and consider the immersion $i : M \to \mathbb{R}^5$, given by

\[ i(f, u) := (f \sin u, f \cos u, \phi f \sin u, \phi f \cos u, -f), \]

where $M := \{(f, u) \mid f > 0, u \in (0, \frac{\pi}{2})\}$ and $\phi$ is the Golden number.

The local orthogonal frame on TM is given by

\[ Z_1 = \sin u \frac{\partial}{\partial x_1} + \cos u \frac{\partial}{\partial x_2} + \phi \sin u \frac{\partial}{\partial x_3} + \phi \cos u \frac{\partial}{\partial x_4} - \frac{\partial}{\partial x_5}, \]

\[ Z_2 = f \cos u \frac{\partial}{\partial x_1} - f \sin u \frac{\partial}{\partial x_2} + \phi f \cos u \frac{\partial}{\partial x_3} - \phi f \sin u \frac{\partial}{\partial x_4} \]

We define the Golden structure $J : \mathbb{R}^5 \to \mathbb{R}^5$ by

\[ J(X_1, X_2, X_3, X_4, X_5) := (\phi X_1, \phi X_2, \overline{\phi} X_3, \overline{\phi} X_4, \phi X_5), \]

where $\overline{\phi} = 1 - \phi$. Since

\[ JZ_1 = \phi \left( \sin u \frac{\partial}{\partial x_1} + \cos u \frac{\partial}{\partial x_2} \right) - \left( \sin u \frac{\partial}{\partial x_3} + \cos u \frac{\partial}{\partial x_4} \right) - \phi \frac{\partial}{\partial x_5}, \]

\[ JZ_2 = \phi f \left( \cos u \frac{\partial}{\partial x_1} - \sin u \frac{\partial}{\partial x_2} \right) - f \left( \cos u \frac{\partial}{\partial x_3} - \sin u \frac{\partial}{\partial x_4} \right), \]

we remark that $\langle JZ_k, Z_l \rangle = 0$, for any $k \in \{1, 2\}$, and $\langle JZ_1, Z_1 \rangle = \phi$.

We find that

\[ \|Z_1\|^2 = \phi^2 + 2 = \phi + 3, \quad \|Z_2\|^2 = f^2(\phi^2 + 1) = f^2(\phi + 2), \]

\[ \|JZ_1\|^2 = 2\phi^2 + 1 = 2\phi + 3, \quad \|JZ_2\|^2 = f^2(\phi^2 + 1) = f^2(\phi + 2). \]
Denote by $D_1 := \text{span}\{Z_1\}$ the slant distribution with the slant angle $\theta$, where $\cos \theta = \frac{\theta}{\sqrt{\langle J\phi + 3 \phi, J\phi + 3 \phi \rangle}}$ and by $D_2 := \text{span}\{Z_2\}$ the anti-invariant distribution (with respect to $J$).

If $M_\theta$ and $M_\perp$ are the integral manifolds of the distributions $D_1$ and $D_2$, respectively, then $M := M_\theta \times f \sqrt{\phi + 2} M_\perp$ with the metric

$$g := (\phi + 3)df^2 + f^2 (\phi + 2)du^2 = g_{M_\theta} + f^2 (\phi + 2)g_{M_\perp}$$

is a warped product hemi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^5, \langle \cdot, \cdot \rangle, J)$.

In a similar manner as in Theorem 2 from [25], we obtain

**Theorem 1.** If $M := M_T \times f M_\theta$ is a warped product pointwise semi-slant submanifold in a locally Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the pointwise slant angle $\theta_x \in (0, \frac{\pi}{2})$, for $x \in M_\theta$, then the warping function $f$ is constant on the connected components of $M_T$.

**Proof.** For any $X \in \Gamma(TM_T)$, $Z \in \Gamma(TM_\theta) \setminus \{0\}$, by using (10) in $\nabla_Z X = J \nabla_Z X$ and (39), we obtain

$$TX(\ln f) Z + h(TX, Z) = T \nabla_Z X + N \nabla_Z X + th(X, Z) + nh(X, Z).$$

From the equality of the normal components of the last equation, it follows

$$h(TX, Z) = X(\ln f)NZ + nh(X, Z)$$

and replacing $X$ with $TX = JX$ (for $X \in \Gamma(TM_T)$) in (44), we obtain

$$h(J^2 X, Z) = TX(\ln f)NZ + nh(TX, Z).$$

Thus, we obtain

$$TX(\ln f)\overline{g}(NZ, NZ) = \overline{g}(h(J^2 X, Z), NZ) - \overline{g}(nh(TX, Z), NZ)$$

$$= \overline{g}(h(TX, Z), NZ) + \overline{g}(h(X, Z), NZ) - \overline{g}(nh(TX, Z), NZ),$$

for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$.

From (41), we have $\overline{g}(h(TX, Z), NZ) = \overline{g}(h(X, Z), NZ) = 0$, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$ and by using (22), we obtain

$$TX(\ln f) \sin^2 \theta [\overline{g}(TZ, Z) + \overline{g}(Z, Z)] = -\overline{g}(nh(TX, Z), NZ).$$

On the other hand, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$, we have $TX \in \Gamma(TM_T)$ and $TZ \in \Gamma(TM_\theta)$, and from (41), we obtain

$$\overline{g}(h(TX, Z), NZ) = \overline{g}(h(TX, Z), NTZ) = 0.$$

Thus, by using (2) and (6), we have

$$\overline{g}(nh(TX, Z), NZ) = \overline{g}(h(TX, Z), nNZ) = \overline{g}(h(TX, Z), J^2 Z - JTZ)$$

$$= \overline{g}(h(TX, Z), NZ) + \overline{g}(h(TX, Z), Z) - \overline{g}(h(TX, Z), NTZ) = 0$$

and using (45), we obtain

$$TX(\ln f) \tan^2 \theta_x \overline{g}(TZ, Z) = 0,$$

for any $Z \in \Gamma(TM_\theta)$ and $x \in M_\theta$.

Since $\theta_x \in (0, \frac{\pi}{2})$ and $TZ \neq 0$, we get $TX(\ln f) = 0$, for any $X \in \Gamma(TM_T)$, which implies that the warping function $f$ is constant on the connected components of $M_T$. 

\end{proof}
Theorem 2. Let \( M := M_\theta \times f M_T \) be a warped product pointwise semi-slant submanifold in a locally Golden Riemannian manifold \( (\overline{M}, \overline{g}, J) \) with the pointwise slant angle \( \theta_x \in (0, \frac{\pi}{2}) \), for \( x \in M_\theta \). Then
\[
(A_{NT_1} Y - A_{NT_1} X) \in \Gamma(TM_\theta),
\]
for any \( X, Y \in \Gamma(TM_\theta) \).

Proof. For any \( X, Y \in \Gamma(TM_\theta) \) and \( Z \in \Gamma(TM_T) \setminus \{0\} \), from (32) and the symmetry of the shape operator, we have
\[\sin^2 \theta \overline{g}(\overline{X}, Y) = \overline{g}(\nabla XY - \nabla YX - \nabla YX + \nabla YX, Z)\]
\[+ (\cos^2 \theta + 1) \overline{g}(h(X, Z), NY) - \overline{g}(h(Y, Z), NX)\]
\[= \overline{g}(\nabla XY, JZ) - \overline{g}(\nabla YX, JZ) + \overline{g}(A NY, Z) - \overline{g}(A NX, Z)\]
\[= \overline{g}([X, Y], JZ) + \overline{g}(h(X, Z), NY) - \overline{g}(h(Y, Z), NX) = \overline{g}([X, Y], T_2 Z)\].

Using (3) and (40), we obtain
\[\overline{g}(\nabla X Y, J Z) = \overline{g}(\nabla Y X, J Z) - \overline{g}(A_{NY} X, Z) - \overline{g}(A_{NX} Y, Z)\]
\[= \overline{g}([X, Y], J Z) + \overline{g}(h(X, Z), NY) - \overline{g}(h(Y, Z), NX) = \overline{g}([X, Y], T_2 Z)\].

From (40), we obtain
\[\overline{g}(h(X, Z), NY) = \overline{g}(h(Y, Z), NX) = -\overline{g}(h(X, Y), N Z)\].

Thus, using the symmetry of the shape operator, we have
\[\overline{g}(h(X, T_2 Z), NY) - \overline{g}(h(Y, T_2 Z), NX) = -\overline{g}(h(X, Y) T_2 Z) + \overline{g}(h(Y, X), NT_2 Z) = 0\]
and
\[\overline{g}(h(X, Z), NT_1 Y) - \overline{g}(h(Y, Z), NT_1 X) = \overline{g}(A_{NT_1 Y} X - A_{NT_1 X} Y, Z)\].

Thus, we obtain
\[0 = \sin^2 \theta \overline{g}([X, Y], T_2 Z) = \overline{g}(A_{NT_1 Y} X - A_{NT_1 X} Y, Z)\].

\( \blacksquare \)

A similar result valid for warped product hemi-slant submanifolds in a locally metallic Riemannian manifold [25], which can be proved following the same steps, holds in our setting, too.

Theorem 3. If \( M := M_\perp \times f M_\theta \) or \( M := M_\theta \times f M_\perp \) is a warped product pointwise semi-slant submanifold in a locally Golden Riemannian manifold \( (\overline{M}, \overline{g}, J) \) with the pointwise slant angle \( \theta_x \in (0, \frac{\pi}{2}) \), for \( x \in M_\theta \), then the warping function \( f \) is constant on the connected components of \( M_\perp \) if and only if
\[A_{NZ} X = A_{NX} Z\],
for any \( X \in \Gamma(TM_\perp) \) and \( Z \in \Gamma(TM_\theta) \) (or \( X \in \Gamma(TM_\theta) \) and \( Z \in \Gamma(TM_\perp) \), respectively).

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References
1. Chen, B.-Y. Geometry of warped products as Riemannian submanifolds and related problems. Soochow J. Math. 2020, 28, 125–156.
2. Chen, B.-Y. Geometry of warped product CR-Submanifolds in a Kaehler manifold. Monatsh. Math. 2001, 133, 177–195. [CrossRef]
3. Chen, B.-Y. Geometry of warped product CR-Submanifolds in a Kaehler manifold II. Monatsh. Math. 2001, 134, 103–119. [CrossRef]
4. Chen, B.-Y. Differential Geometry of Warped Product Manifolds and Submanifolds; World Scientific: Singapore, 2017.
5. Hretcanu, C.E.; Blaga, A.M. Submanifolds in metallic Riemannian manifolds. Differ. Geom. Dyn. Syst. 2018, 20, 83–97.
6. Hretcanu, C.E.; Crasmareanu, M. Metallic structures on Riemannian manifolds. Rev. Unión Mat. Argent. 2013, 54, 15–27.
7. Crasmareanu, M.; Hretcanu, C.E. Golden differential geometry. Chaos Solitons Fractals 2008, 38, 1229–1238. [CrossRef]
8. Hretcanu, C.E.; Crasmareanu, M.C. Applications of the Golden ratio on Riemannian manifolds. Turkish J. Math. 2009, 33, 179–191.
9. Hretcanu, C.E.; Crasmareanu, M. On some invariant submanifolds in a Riemannian manifold with Golden structure. An. Stiint. Univ. Cui§a Iasi Mat. (N. S.) 2007, 53 (Suppl. 1), 199–211.
10. Blaga, A.M.; Hretcanu, C.E. Invariant, anti-invariant and slant submanifolds of a metallic Riemannian manifold. Novi Sad J. Math. 2018, 48, 55–80. [CrossRef]
11. Hretcanu, C.E.; Blaga, A.M. Slant and semi-slant submanifolds in metallic Riemannian manifolds. J. Funct. Spaces 2018, 2018, 2864263. [CrossRef]
12. Hretcanu, C.E.; Blaga, A.M. Hemi-slant submanifolds in metallic Riemannian manifolds. Carpathian J. Math. 2019, 35, 59–68. [CrossRef]
13. Blaga, A.M.; Hretcanu, C.E. Golden warped product Riemannian manifolds. Lib. Math. 2017, 37, 39–49.
14. Blaga, A.M.; Hretcanu, C.E. Remarks on metallic warped product manifolds. Facta Univ. 2018, 33, 269–277.
15. Erdogan, F.E. Transversal Lightlike Submanifolds of Metallic Semi-Riemannian Manifolds. Turkish J. Math. 2018, 42, 3133–3148. [CrossRef]
16. Erdogan, F.E.; Perkta§, S.Y.; Acet, B.E.; Blaga, A.M. Screen transversal lightlike submanifolds of metallic semi-Riemannian manifolds. J. Geom. Phys. 2019, 142, 111–120. [CrossRef]
17. Perkta§, S.Y.; Erdogan, F.E.; Acet, B.E. Lightlike submanifolds of metallic semi-Riemannian manifolds. Filomat 2020, 34, 1781–1794.
18. De Spinadel, V.W. The metallic means family and forbidden symmetries. Int. Math. J. 2002, 2, 279–288.
19. Goldberg, S.I.; Petridis, N.C. Differentiable solutions of algebraic equations on manifolds. Kodai Math. Sem. Rep. 1973, 25, 111–128. [CrossRef]
20. Goldberg, S.I.; Yano, K. Polynomial structures on manifolds. Kodai Math. Sem. Rep. 1970, 22, 199–218. [CrossRef]
21. Blaga, A.M.; Hretcanu, C.E. Metallic conjugate connections. Rev. Unión Mat. Argent. 2018, 59, 179–192. [CrossRef]
22. Li, H.; Liu, X. Semi-slant submanifolds of a locally product manifold. Georgian Math. J. 2005, 12, 273–282. [CrossRef]
23. Chen, B.-Y.; Garay, O. Pointwise slant submanifolds in almost Hermitian manifolds. Turkish J. Math. 2012, 36, 630–640.
24. Chen, B.-Y.; Uddin, S. Warped product pointwise bi-slant submanifolds of Kaehler manifolds. Publ. Math. Debr. 2018, 1–17. In-print. [CrossRef]
25. Hretcanu, C.E.; Blaga, A.M. Warped product submanifolds in metallic Riemannian manifolds. Tamkang J. Math. 2020, 51, 161–186. [CrossRef]
26. Bishop, R.L.; O’Neill, B. Manifolds of negative curvature. Trans. Am. Math. Soc. 1969, 145, 1–49. [CrossRef]