Some notes on string theory

Karl-Georg Schlesinger
Erwin Schrödinger Institute for Mathematical Physics
Boltzmanngasse 9
A-1090 Vienna, Austria
e-mail: kgschles@esi.ac.at

Abstract
We start from a quantum computational principle suggested for string theory in a previous paper and discuss how it might lead to a dynamical principle implying the correct classical gravitational limit. Besides this, we briefly look at some structural properties of moduli space.

1 Introduction

In [Sch] we suggested a quantum computational principle for string theory which was formulated as follows:

Principle:

1. All physical systems should be amenable to a real time simulation on a quantum computer and quantum computers should be described as physical systems by deformation quantization of classical Turing machines.

2. The observable quantities of the world should be those which can be determined by observation of quantum computers on quantum computers.
Remark 1 We should say more precisely at this point what we mean by the assumption that the quantum Turing machine should be describable by deformation quantization. For our purpose, here, it is sufficient to assume that the quantum Turing machine can be described by a quantum field theory, the arithmetic content of the Feynman diagrams of which is equivalent to the data of the weights appearing in deformation quantization. The work of Freedman et al. (see [FLW]) shows that, indeed, one can describe quantum computation in terms of low dimensional quantum field theory (this description being equivalent to the one in terms of spin systems plus an automatic inclusion of error correction). The quantum field theory used in [FLW] is determined by a quantum group and $q$-deformation is known to appear as a special case of deformation quantization (see [MSSW]).

It was discussed there that the first part of this principle leads to the algebra $P_{Z,Tate}$ as a fundamental object of the theory and therefore - under the assumption of the partly conjectural scenario of [Kon 1999] - to the Grothendieck-Teichmüller group $GT$ as a fundamental symmetry. The second part of the principle implies a ladder of quantization and, especially, a deformation quantization of $GT$ (or $P_{Z,Tate}$) to be relevant for the - yet to be discovered - full fledged quantized string theory. We will first discuss the semi-classical setting where the (undeformed) $GT$ is relevant and come to the question of a deformation quantization of the mathematical structures involved only later. In order for the above principle to make sense, it is decisive that the Grothendieck-Teichmüller group as a symmetry plus possibly some natural requirements should basically determine the structural elements and the dynamical principle of the theory. We will therefore investigate in this paper some general features a theory incorporating $GT$ as a symmetry should have.

Throughout this paper, we will assume the scenario of [Kon 1999] to be valid.

2 Dynamics

The algebra $P_{Z,Tate}$ is the algebra of periods of mixed Tate motives unramified over $\mathbb{Z}$. Let $\mathcal{M}$ denote the moduli space of mixed Tate motives unramified over $\mathbb{Z}$. There is a quite concrete definition of motives as so called framed motives, the category of framed motives being equivalent to a more abstractly
defined category of mixed motives (see [Kon 1999]). A framed motive of rank 
$n$ is a matrix

$$A \in GL(n, P)$$

(where $P$ is the algebra of periods) such that

$$\Delta (A_{ij}) = \sum_{k,l} A_{ik} \otimes A^{-1}_{kl} \otimes A_{lj}$$

with $\Delta$ the triple coproduct induced from the torsor which corresponds to 
the isomorphism of Betti and de Rham cohomology. So, we can understand 
$M$ as the moduli space of framed motives with

$$A_{ij} \in P_{Z,Tate}$$

In [Kon 1999] the moduli space $DM$ of deformation quantizations of a finite 
dimensional manifold $M$ is suggested to be a proalgebraic variety which is a 
principal homogenous space of $GT$. This space should have several structural 
features in common with $M$ (as a moduli space of motives, $M$ should also 
be a proalgebraic variety and there should also be a natural action of $GT$). 
Since $M$ is universal while $DM$ depends on the given manifold $M$, one 
suspects each $DM$ to be a kind of “representation” of $M$. We will make the 
assumption in the sequel that in this sense structural features (like e.g. a 
metric) on $M$ induce corresponding structures on the “representation” $DM$. 
One would suspect that even beyond this there should be a natural duality 
between the class of all “representations” $DM$ and $M$, as e.g. known from 
the Doplicher-Roberts theorem for the case of compact groups and from many 
other examples.

**Conjecture:**
There should be a natural duality between $M$ and some suitable notion 
of a moduli space of all deformation quantizations (a kind of union of the 
$DM$).

We have not just taken the disjoint union of the $DM$ because presumably 
compactification of the individual proalgebraic varieties $DM$ plays a role, 
here. In an appendix we will give a slightly more involved argument in 
 favour of the above conjecture.

The work of [Kon 1997] and [CF] shows that $DM$ can always be understood 
as a moduli space of two dimensional conformal field theories. Since
the Deligne conjecture (a proof of which is announced in [Kon 1999]) plays a decisive role in the construction of $\mathcal{D}M$, the fact that the tangential structure of the extended moduli space of string theory introduced by Witten is given by a total Hochschild complex (see [Wil], [Kon 1994]) suggests that, conversely, moduli spaces of two dimensional conformal field theories should be structurally similar to the spaces $\mathcal{D}M$. Also, the so called Cohomology Comparison Theorem (CCT) of [GeSch] leads to an argument to the effect that moduli spaces of two dimensional conformal field theories should be interpretable as moduli spaces of deformation quantizations. Together with the above conjecture, we take the view in this paper that $\mathcal{M}$ should be naturally a dual description of the moduli space $S$ of classical backgrounds of string theory. So, we suppose that one has the freedom to switch between the two descriptions, e.g. when considering a dynamical principle.

Accepting the suggested quantum computational principle, $\mathcal{M}$ is the straightforward candidate for the state space of the theory. So, assuming the duality between $\mathcal{M}$ and $S$ to hold, the principle would at least imply the correct kinematical arena for the theory. It is then a decisive question if we get a natural suggestion for the dynamics.

**Remark 2** The conformal field theories corresponding to deformation quantizations in [CF] are in no way restricted to the critical dimension of string theory. So, it seems that one needs some additional requirements, like critical dimension and the type of supersymmetry, in order to restrict $S$ to really give superstring theory. This is what we meant above when we spoke of GT symmetry “plus possibly some natural requirements”.

**Remark 3** We will exclusively discuss dynamics in a Euclidean sense in this paper. We will not deal with the question of how to introduce a physically sensible time parameter, here.

As a first step, we will see that the GT symmetry also fixes a Riemannian metric on $\mathcal{M}$. Remember that $\mathcal{M}$ is a projective limit of algebraic varieties of framed motives of fixed rank. Denote by $\mathcal{M}_n$ the moduli space of framed mixed Tate motives unramified over $\mathbb{Z}$ of rank $n \in \mathbb{N}$. $\mathcal{M}_n$ is, of course, a subvariety of $(P_{\mathbb{Z},Tate})^{n^2}$. On $(P_{\mathbb{Z},Tate})^{n^2}$ we can introduce the usual Euclidean distance. Let $\gamma_n$ be the induced Riemannian metric on $\mathcal{M}_n$ (i.e. the metric defined as infimum of the length of curves in $\mathcal{M}_n$). The $\gamma_n$ induce a Riemannian metric $\gamma$ on $\mathcal{M}$. 

4
Lemma 1 The metric $\gamma$ is invariant under the action of $GT$ on $\mathcal{M}$ and is up to a normalization factor uniquely determined by $GT$ invariance.

Proof. Considering the action of $GT$ on $\mathcal{M}$ in terms of actions on the components $\mathcal{M}_n$, we get a representation in terms of tensors $T_{i_1j_1i_2j_2}$ with $n^4$ coefficients in $P_{\mathbb{Z},Tate}$ acting on framed motives $A_{ij}$ of rank $n$ by

$$(TA)_{i_1j_1} = T_{i_1j_1i_2j_2}A_{i_2j_2}$$

(here, and in the sequel, we apply the Einstein summation convention).

Since the image of an element of $\mathcal{M}_n$ under $T$ has to be an element of $\mathcal{M}_n$, we have

$$\Delta (TA)_{i_1j_1} = (TA)_{i_1k_1} \otimes (TA)^{-1}_{k_1l_1} \otimes (TA)_{l_1j_1}$$

Since $\Delta$ is an algebra morphism on $P_{\mathbb{Z},Tate}$, we have

$$\Delta (TA)_{i_1j_1} = \Delta (T_{i_1j_1i_2j_2}A_{i_2j_2})$$

Since $\Delta$ is a quotient of the motivic Galois group, $T$ commutes with $\Delta$, i.e.

$$\Delta (TA)_{i_1j_1} = \Delta (T_{i_1j_1i_2j_2}) \Delta (A_{i_2j_2})$$

On the other hand,

$$(TA)_{i_1k_1} \otimes (TA)^{-1}_{k_1l_1} \otimes (TA)_{l_1j_1}$$

$$= T_{i_1k_1i_2k_2}A_{i_2k_2} \otimes A_{k_2l_2}^{-1} \otimes T_{k_2l_2k_1l_1} \otimes T_{l_1j_1i_2j_2}A_{i_2j_2}$$

$$= \left(T_{i_1k_1i_2k_2} \otimes (T^{-1})^T_{k_1l_1k_2l_2} \otimes T_{l_1j_1i_2j_2}\right) \left(A_{i_2k_2} \otimes A_{k_2l_2}^{-1} \otimes A_{i_2j_2}\right)$$

where $T$ denotes the transposed matrix on interpreting $T$ as a matrix - with a first and a second double index - acting on $(P_{\mathbb{Z},Tate})^{n^2}$. So,

$$\Delta (T_{i_1j_1i_2j_2}) = T_{i_1k_1i_2k_2} \otimes \left(T^{-1}ight)^T_{k_1l_1k_2l_2} \otimes T_{l_1j_1i_2j_2}$$

holds on the span of the tangent spaces of $\mathcal{M}_n$.

Now, by the definition of the torsor underlying $\Delta$ and since $GT$ is a quotient of the motivic Galois group, $T$ commutes with $\Delta$, i.e.

$$\Delta (T_{i_1j_1i_2j_2}A_{i_2j_2}) = (T_{i_1k_1i_2k_2} \otimes T_{k_1l_1k_2l_2} \otimes T_{l_1j_1i_2j_2}) \Delta (A_{i_2j_2})$$
and we get
\[ \Delta (T_{i_1j_1i_2j_2}) = T_{i_1k_1i_2k_2} \otimes T_{k_1l_1k_2l_2} \otimes T_{i_1j_1i_2j_2} \]
on the span of the tangent spaces of \( \mathcal{M}_n \). But since in the orthogonal complement of the span of the tangent spaces of \( \mathcal{M}_n \) we are free to make a choice for \( T \) (under the restriction that
\[ T \in GL(n^2, P_{\text{Tate}}) \]
holds), we can assume that both equations for \( \Delta (T_{i_1j_1i_2j_2}) \) hold without restriction. But this means \( T \) is an orthogonal \( n^2 \times n^2 \) matrix. In consequence, the metric on \( \mathcal{M}_n \) which is invariant under the transformations \( T \) is up to a normalization factor uniquely determined and is given by \( \gamma_n \).

Assuming the duality of \( \mathcal{M} \) and \( S \), there should be a unique \( GT \) invariant Riemannian metric on \( S \), then. As is well known, there is a natural metric on the moduli space of two dimensional conformal field theories if one remembers that infinitesimal deformations of conformal field theories are parametrized by local operators. The metric is then simply defined by the two point function. So, in view of the above uniqueness result, it would suffice to show \( GT \) invariance of the two point function metric \( d \), in order to identify \( \gamma \) and \( d \) as dual to each other.

Remember that we assumed above that \( S \) carries a transitive action of \( GT \), too. But, again, using the fact that infinitesimal deformations of conformal field theories are parametrized by local operators, we conclude that the Lie algebra of \( GT \) (see [Dri]) should naturally act as a symmetry on two dimensional conformal field theories. But if this is true, i.e. if the action of \( GT \) on \( S \) induces a universal symmetry property of two dimensional conformal field theories, the two point function metric - as based on observables - would have to be invariant.

There is a hint that the conclusion we have just drawn might indeed be correct. The structure of the Connes-Kreimer Hopf algebra of renormalization is in accordance with the arithmetic properties of deformation quantization and of the Drinfeld associator, i.e. with the data of the Grothendieck-Teichmüller group (see [CK], [Kon 1999]). So, it seems that the symmetry properties behind the renormalization scheme might be intimately linked to \( GT \) invariance of moduli space.
A given metric canonically determines a dynamical principle by geodesic motion. So, the suggested quantum computational principle has a natural dynamical law corresponding to it as geodesic motion with respect to $\gamma$ on $\mathcal{M}$, respectively, $\delta$ on $\mathcal{S}$. For a quantum theory a Klein-Gordon type equation is the counterpart of geodesic motion, i.e. for quantized string theory one would anticipate a Klein-Gordon equation on $\mathcal{M}$ (or $\mathcal{S}$). The reader acquainted with work in quantum gravity will notice that this is similar to the situation, there: The Wheeler-De Witt equation is also formally a Klein-Gordon equation. Indeed, this is not by accident, as we can see from the following lemma which shows that the dynamics suggested here by abstract arguments has the correct classical gravity limit.

**Lemma 2** Geodesic motion with respect to the metric $\delta$ on $\mathcal{S}$ induces Einstein dynamics in the classical limit of string theory if one restricts to backgrounds which allow for a natural slicing by a global time parameter.

**Proof.** In the classical limit, the two point function metric has to induce a metric $\tilde{\delta}$ on the space of Riemannian $(n-1)$-metrics (where $n = 10$ for string theory) which are Ricci flat backgrounds with a slicing by a global time parameter. But by the properties of the two point function, $\tilde{\delta}$ has to be a four index tensor field satisfying the locality requirement of [DeW]. But then there is a one parameter family of metrics satisfying these criteria, only, and there is a natural value for the parameter (see [DeW]). So, basically there is a unique candidate for the classical limit of the two point function metric, only. But the Klein-Gordon equation with respect to this metric $\tilde{\delta}$ is just the Wheeler-De Witt equation of Einstein gravity.

In conclusion, there are indications that the suggested quantum computational principle does not only lead to a natural choice for the kinematical structure and for a dynamical principle for the theory but also that the resulting dynamical principle has the correct classical limit in the form of Einstein dynamics.

**Remark 4** Though the above proof uses the fact that in the classical limit backgrounds given by two dimensional conformal field theories go to Ricci flat metrics, i.e. metrics satisfying the Einstein vacuum equations, the result
is not just a reformulation of this fact. The Ricci flat limit of two dimensional conformal field theories refers to the consideration of single fixed backgrounds. Our result shows that there is a dynamics on moduli space (i.e. on the space of backgrounds) which is also compatible with Einstein dynamics. A dynamics on moduli space is what one ultimately expects for the yet to be discovered full formulation of string theory which gives the motivation for studying qualitative properties and possible choices for dynamical principles on moduli space. Especially, the above result shows that the quantization of the suggested dynamics on moduli space would have (in a formal sense) the Wheeler-De Witt equation as a special limit. While the Ricci flat limit of conformal field theories shows that string theory unifies Einstein gravity and quantum mechanics consistently on a perturbative level, we get an indication, here, that string theory might also have a limit in which it reproduces the nonperturbative approach to canonical quantum gravity (see e.g. AL 1994, AL 1996 and the literature cited therein). A further elaboration on the role the Grothendieck-Teichmüller group plays in the structure of string theory moduli space seems to be a promising candidate for the investigation of such a limit (we are planning to undertake further work in this direction).

3 The implications of the second part

So far, we have dealt only with the implications of the first part of the principle. As we mentioned already, the second part implies that actually a quantum deformation of GT should be the relevant symmetry object. Conformal field theories play the role of classical backgrounds around which one can perturbatively expand in string theory (comparable to Ricci flat metrics in Einstein gravity). One can give arguments that the inclusion of nonclassical backgrounds (similar to general Riemannian 3-metrics for the classical gravity case) should lead to quantum deformations of the mathematical structures appearing (see GS). Especially, deformations of conformal field theories to models with noncommutative world sheets should appear, here, as is indirectly also suggested by the work of Gre. A detailed discussion of a deformation of quantum group symmetries (since these are linked to conformal field theories, i.e. to the undeformed case in the sense of our present discussion) has been given elsewhere (see GS 2000).
4 $\zeta$-functions

Consider the action functional $S$ generating the Klein-Gordon equation on $\mathcal{M}$ as the equation of motion. For physical reasons, we should consider only action functionals which are at most second order in the fields and their derivatives and do not contain higher derivatives of the fields. But then the uniqueness result for the metric $\gamma$ implies that $S$ is - up to a normalization factor - the unique $GT$ invariant action satisfying these criteria (a term linear in the fields would destroy $GT$ invariance, i.e. both terms - the one in the fields and the one in their derivatives - have to be quadratic, similarly, there can be no mixed term in fields and their derivatives). It is one of the beliefs of experts in the theory of motives that $\zeta$-functions should be interpretable as a kind of regularized volumes on configuration spaces of quantum fields (see [JKS]). The partition function is, of course, a natural candidate, then (an additional argument in favour of an interpretation of $\zeta$-functions as partition functions of quantum systems is provided by the work of [Con]). Motivated by the uniqueness property of $S$, we suspect that the partition function defined from $S$ is interpretable as $\zeta$-function of the proalgebraic variety $\mathcal{M}$.

Remark 5 We should say that all differential geometric notions used in this section are to be understood in a purely formal sense. To make the ideas sketched precise, one would presumably have to use an approach replacing these by techniques from algebraic geometry.

5 Conclusion

We have investigated the implications of a quantum computational principle for the kinematical and dynamical features of string theory. We have found, especially, that such a principle might be able to determine to a considerable extent the correct moduli space and is consistent with the requirement that Einstein dynamics should appear as a suitable classical limit.
The question of duality of $\mathcal{M}$ and $\mathcal{S}$

Suppose the connection between $GT$ and the Connes-Kreimer Hopf algebra can be rigorously established. Then duality between $\mathcal{M}$ and $\mathcal{S}$ would come down to the following: First, every two dimensional quantum field theory would have to be a representation of the renormalization scheme. This is certainly true. Second, in the other direction we would need a result showing that we can reconstruct the renormalization scheme from the knowledge of all two dimensional quantum field theories. The historical development gives a very strong argument in favour of such a reconstruction possibility. After all, the renormalization scheme was not derived from abstract logical arguments but was distilled from quantum field theories, i.e. the historical development is itself a “reconstruction” (one should better say “construction”) process. This reconstruction possibility should persist even if we restrict to two dimensional theories since one does not have the feeling that we have to be free to consider arbitrary dimensions in order to be able to discover the renormalization scheme. So, we suspect that any fixed dimension should do.

Though the above argument relies on strong historical evidence, it is not a proof, of course (historical evidence may deceive us in spite of being strong). Can we do better? Since we are interested in a low dimensional situation, only, we can try to use the framework of axiomatic quantum field theory. Based on some of the concepts of this framework, we will try to give a sketch for a more adequate mathematical argument, now. The framework of algebraic quantum field theory nowadays makes use of the algebraic language of categories. It is a well known experience in category theory that a category of objects of some type often has a similar algebraic structure than the objects themselves. It is precisely this feature of category theory which we will see at work, here.

Let $\mathcal{Q}$ be the 2-category of $C^*$-quantum categories as introduced in [FK] where a quantum category is an abelian, semisimple, finite, rigid, braided, monoidal category and a $C^*$-quantum category is a quantum category with a compatible $\ast$ structure on it.

By the approach followed in [Bae], one can show that there is a tensor product on $\mathcal{Q}$, turning it into a symmetric (weak) monoidal 2-category. In the same way, one proves the existence of a direct sum (note that the finiteness condition of [FK] implies what is called finite dimensionality in [Bae]). Also, by the results of [Bae] one gets a 2-categorical version of rigidity for $\mathcal{Q}$. 

10
The finiteness condition has some additional consequences since it implies that up to isomorphism the morphisms in a $C^*$-quantum category can be seen as finite tuples of matrices. But as a consequence of this, the 1-morphism classes of $\mathcal{Q}$ carry the structure of a $\mathbb{C}$-linear category and this linear structure is compatible with composition of 1-morphisms. Hence, $\mathcal{Q}$ carries the 2-categorical analog of the structure of a $\mathbb{C}$-linear category. Besides this, by redoing the proof that the category of finite dimensional vector spaces is abelian in a categorified setting, one should find a 2-categorical counterpart of the notion of an abelian category and get the result that $\mathcal{Q}$ has this property.

In conclusion, we should find that $\mathcal{Q}$ is the 2-categorical version of a $\mathbb{C}$-linear, abelian, rigid, symmetric monoidal category. In the same way, one expects to find the 2-categorical analog of a $\mathbb{C}$-linear, exact, faithful, monoidal functor from $\mathcal{Q}$ to the category of finite dimensional vector spaces. Now, assuming that there is a 2-categorical analog of the general reconstruction theorems for Hopf algebras (see e.g. [CP]), it would follow that there is a Hopf category $\mathcal{H}$ (in the sense of [CrFr]) generating $\mathcal{Q}$ as its category of either representations or corepresentations on finite dimensional 2-vector spaces. By the symmetry of the monoidal structure, $\mathcal{H}$ would then have to be commutative or cocommutative. But this means - now using the usual reconstruction theorems - $\mathcal{H}$ would basically be a category of representations of some Hopf algebra $\mathbf{H}$. So, there should be a universal Hopf algebra $\mathbf{H}$ having a representation on every quantum field theory. Physically, the symmetries behind the renormalization scheme are the only natural candidate for such a universal Hopf algebra. We summarize this in the following conjecture:

**Conjecture:**

The Hopf algebra $\mathbf{H}$ defined by $\mathcal{Q}$ should be equivalent to the Connes-Kreimer Hopf algebra.

We have followed a purely algebraic approach to the question of duality between the renormalization scheme and the general structure of quantum field theory, here. But it should be possible to derive from such an algebraic approach the more geometric duality between $\mathcal{M}$ and $\mathcal{S}$, too. In addition, this approach relies on a connection between the Connes-Kreimer Hopf algebra and $GT$. But as we mentioned already, there is growing evidence for such a connection to exist (see [CK], [Kon 1999]).
Remark 6 The arguments given, here, are aiming at a direct proof of the assumed duality in the sense of an algebraic reconstruction procedure. Alternatively, one can from existing literature collect together a chain of arguments which might also develop into a proof for the duality of $\mathcal{M}$ and $\mathcal{S}$. Concluding this appendix, we give a sketch of these arguments, now: Surely, $\mathcal{M}$ and the Grothendieck-Teichmüller group, itself, are dual to each other. $\mathcal{M}$ is defined by representations of $GT$ and, conversely, from $\mathcal{M}$ we get the algebra $P_{\text{Tate}}$ as the algebra of periods appearing in $\mathcal{M}$, and $P_{\text{Tate}}$ determines $GT$ (where, as remarked at the beginning, we assume the scenario of [Kon 1999] to be valid). Starting from the space $\mathcal{S}$, we remember that the fusion structure of two dimensional conformal field theories is determined by quasi-triangular quasi-Hopf algebras. So, $\mathcal{S}$ determines a certain moduli space of quasi-triangular quasi-Hopf algebras. But then we can use Drinfeld’s original approach (see [Dri]) to introduce $GT$ from the setting of quasi-Hopf algebras. For the converse direction, we point, again, to the increasing evidence for a deep relationship between $GT$ and the Connes-Kreimer algebra. But, obviously, two dimensional conformal field theories carry representations of the renormalization scheme as abstractly displayed by the Connes-Kreimer algebra. So, we might get back $\mathcal{S}$ as a class of representations.

Acknowledgements:
I would like to thank H. Grosse for numerous discussions and the Erwin Schrödinger Institute for Mathematical Physics, Vienna, for hospitality.

References

[AL 1994] A. Ashtekar, J. Lewandowski, “Differential geometry on the space of connections via graphs and projective limits”, preprint, CGPG-94/12-4.

[AL 1996] A. Ashtekar, J. Lewandowski, “Quantum theory of geometry I: Area operators”, preprint, gr-qc/9602046v2.

[Bae] J. C. Baez, Higher-dimensional algebra II: 2-Hilbert spaces, preprint, q-alg/9609018v2.
[Con] A. Connes, *Trace formula in noncommutative geometry and the zeros of the Riemann zeta function*, math.NT/9811068.

[CF] A. Cattaneo, G. Felder, *A path integral approach to the Kontsevich quantization formula*, math/9902090.

[CK] A. Connes, D. Kreimer, *Lessons from quantum field theory*, hep-th/9904044v2.

[CP] V. Chari, A. Pressley, *Quantum groups*, Cambridge University Press, Cambridge 1994.

[CrFr] L. Crane, I. B. Frenkel, *Four-dimensional topological quantum field theory, Hopf categories and the canonical bases*, J. Math. Phys. 35, 5136-5154 (1994).

[DeW] B. S. De Witt, *Quantum theory of gravity*, I, Phys. Rev. 160, 1113-1148 (1967).

[Dri] V. G. Drinfeld, *On quasi-triangular Quasi-Hopf algebras and a group closely related with Gal(\overline{Q}/Q)*, Leningrad Math. J., 2, 829-860 (1991).

[FK] J. Fröhlich, T. Kerler, *Quantum groups, quantum categories and quantum field theory*, Lecture Notes in Mathematics 1542, Springer, Berlin 1993.

[FLW] M. Freedman, M. Larsen, Z. Wang, *A modular functor which is universal for quantum computation*, quant-ph/0001108v2.

[Gre] M. B. Green, *World sheets for world sheets*, Nucl. Phys. B293 (1987), 593-611.

[GeSch] M. Gerstenhaber, S.D. Schack, *Algebraic cohomology and deformation theory*, in M. Hazewinkel, M. Gerstenhaber, *Deformation theory of algebras and structures and applications*, Nato ASI series 247, Kluwer, Dordrecht 1988.

[GS] H. Grosse, K.-G. Schlesinger, *Deformations of conformal field theories to models with noncommutative world sheets*, preprint.
[GS 2000] H. Grosse, K.-G. Schlesinger, *On second quantization of quantum groups*, to appear in Journal of Mathematical Physics, earlier version also available in the preprint series of the Erwin Schrödinger Institute for Mathematical Physics, Vienna, 841 (available under http://www.esi.ac.at).

[JKS] U. Janssen, S. Kleiman, J.-P. Serre, *Motives*, 2 volumes, Proc. of Symposia in pure mathematics, Vol. 55, American Mathematical Society, Providence 1994.

[Kon 1994] M. Kontsevich, *Homological algebra of mirror symmetry*, alg-geom/9411018.

[Kon 1997] M. Kontsevich, *Deformation quantization of Poisson manifolds I*, math/9709180.

[Kon 1999] M. Kontsevich, *Operads and motives in deformation quantization*, math.QA/9904055.

[MSSW] J. Madore, S. Schraml, P. Schupp, J. Wess, *Gauge theory on noncommutative spaces*, Eur. Phys. J. C16 (2000), 161-167, also available as hep-th/0001203.

[Sch] K.-G. Schlesinger, *On the universality of string theory*, preprint.

[Wit] E. Witten, *Mirror manifolds and topological field theory*, in: S. T. Yau, *Essays on mirror manifolds*, International Press, Hong Kong 1991.