The Attenuated Space Poset $A_q(N, M)$

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Abstract

In this paper, we study the incidence algebra $T$ of the attenuated space poset $A_q(N, M)$. We consider the following topics. We consider some generators of $T$: the raising matrix $R$, the lowering matrix $L$, and a certain diagonal matrix $K$. We describe some relations among $R, L, K$. We put these relations in an attractive form using a certain matrix $S$ in $T$. We characterize the center $Z(T)$. Using $Z(T)$, we relate $T$ to the quantum group $U_{\tau}(\mathfrak{sl}_2)$ with $\tau^2 = q$. We consider two elements $A, A^*$ in $T$ of a certain form. We find necessary and sufficient conditions for $A, A^*$ to satisfy the tridiagonal relations. Let $W$ denote an irreducible $T$-module. We find necessary and sufficient conditions for the above $A, A^*$ to act on $W$ as a Leonard pair.

Keywords. Attenuated space; tridiagonal relations; distance-regular graph; Leonard pair

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1 Introduction

In [22], Terwilliger introduced the incidence algebra of a uniform poset. This algebra is motivated by the Terwilliger algebra of a $Q$-polynomial distance-regular graph. For many $Q$-polynomial distance-regular graphs, the Terwilliger algebra is related to a quantum group [4, 5, 10, 34, 36]. So for a uniform poset, it is natural to ask whether its incidence algebra is related to a quantum group. In this paper we will show that this is the case for certain uniform posets.

We recall some examples of uniform posets. In [22], Terwilliger used the classical geometries to obtain eleven families of uniform posets. The polar spaces give one of the families [22, Example 3.1]. In [36], Worawannotai found another family of uniform posets using the polar spaces. For each bipartite $Q$-polynomial distance-regular graph, Miklavić and Terwilliger [18] considered a poset on its vertex set. They found necessary and sufficient conditions for this poset to be uniform. In [11], Kang and Chen obtained a family of uniform posets using the nonisotropic subspaces of a unitary polar space.

The incidence algebra $T$ of a uniform poset is finite-dimensional and semisimple [22]. In [22], Terwilliger gave a method for computing the irreducible $T$-modules. To describe these modules, it is convenient to use the notion of a Leonard pair. This notion was introduced by Terwilliger [19, 21] as an abstraction of some work of Leonard [16] concerning the orthogonal polynomials in the terminating branch of the Askey scheme [13]. Leonard pairs are closely
related to quantum groups \cite{124,22,27,30,32}. Leonard pairs are also related to $Q$-polynomial distance-regular graphs \cite{15,31,36}. We mentioned a similarity between the Terwilliger algebra for a $Q$-polynomial distance-regular graph and the incidence algebra for a uniform poset. Given a uniform poset, it is natural to look for a Leonard pair structure on each irreducible $T$-module. These Leonard pairs were found for certain examples \cite{11,23,25}.

There is a family of classical geometries called the attenuated spaces. An attenuated space admits the structure of a uniform poset \cite{22}. Bonoli and Melone \cite{2} gave a geometrical characterization of an attenuated space. Wang, Guo, Li \cite{7,35} constructed an association scheme based on an attenuated space. They computed all its intersection numbers and studied its incidence matrices. Kurihara \cite{14} computed the character table of this association scheme. Gao and Wang \cite{6} constructed some error-correcting codes based on an attenuated space. Liu and Wang \cite{17} characterized the full automorphism group of some graphs based on an attenuated space.

An attenuated space gives a uniform poset called $\mathcal{A}_q(N, M)$. In this paper we study the incidence algebra $T$ of $\mathcal{A}_q(N, M)$. We consider the following topics. We consider some generators of $T$: the raising matrix $R$, the lowering matrix $L$, and a certain diagonal matrix $K$. We describe some relations among $R, L, K$. We put these relations in an attractive form using a certain matrix $S$ in $T$. We characterize the center $Z(T)$. Using $Z(T)$, we relate $T$ to the quantum group $U_{\tau}(\mathfrak{sl}_2)$ with $\tau^2 = q$. We consider two elements $A, A^*$ in $T$ of a certain form. We find necessary and sufficient conditions for $A, A^*$ to satisfy the tridiagonal relations \cite{8}. Let $W$ denote an irreducible $T$-module. We find necessary and sufficient conditions for the above $A, A^*$ to act on $W$ as a Leonard pair.

The paper is organized as follows: In Section 2, we recall some basic definitions and facts about the attenuated space poset $\mathcal{A}_q(N, M)$ and its incidence algebra $T$. In Section 3 we display some relations among $R, L, K$. In Section 4 we introduce the matrix $S$ and use it to simplify these relations. In Section 5, we recall some facts from \cite{22} about the irreducible $T$-modules and the center $Z(T)$. In Sections 6, 7 we display two families of central elements of $T$. We show that each family generates $Z(T)$. In Section 8 we relate $T$ to the quantum group $U_{\tau}(\mathfrak{sl}_2)$ with $\tau^2 = q$. In Section 9, we obtain some results about $R, L$ that will be used in Section 11. In Section 10, we recall some definitions and facts about Leonard pairs. In Section 11 we consider two elements $A, A^*$ in $T$ of a certain form. We find necessary and sufficient conditions for $A, A^*$ to satisfy the tridiagonal relations. In Section 12 we consider the actions of $A, A^*$ on the irreducible $T$-modules.

## 2 The attenuated space poset and its incidence algebra

In this section, we first recall some basic definitions and facts about the attenuated space poset $\mathcal{A}_q(N, M)$. We then recall the incidence algebra $T$ of $\mathcal{A}_q(N, M)$. This material is mainly taken from \cite{23}.

Throughout this section fix positive integers $N, M$. Fix a finite field $\mathbb{F}_q$ of order $q$. We will be discussing the square root of $q$. Throughout the paper, fix a square root $q^{1/2}$. Let $H$ be a vector space over $\mathbb{F}_q$ that has dimension $N + M$. Fix an $M$-dimensional subspace $h$ of $H$. Let $P$ denote the set of subspaces of $H$ whose intersection with $h$ is zero. For $x, y \in P$ write $x \leq y$ whenever $x \subseteq y$. This relation $\leq$ is a partial order on $P$. The poset $P$ is called
the attenuated space poset and often denoted by $\mathcal{A}_q(N, M)$.

For $x, y \in P$ write $x < y$ whenever $x \leq y$ and $x \neq y$. We say that $y$ covers $x$ whenever $x < y$ and there is no $z \in P$ such that $x < z < y$. For $0 \leq i \leq N$, let $P_i$ denote the set of elements in $P$ that have dimension $i$. The sequence $\{P_i\}_{i=0}^N$ is a grading of $P$ in the sense of [22]. Let $\mathbb{C}P$ denote the vector space over $\mathbb{C}$ consisting of all formal $\mathbb{C}$-linear combinations of elements in $P$. The set $P$ is a basis for $\mathbb{C}P$, so the dimension of $\mathbb{C}P$ is equal to the cardinality of $P$.

The lowering matrix $L \in \text{Mat}_P(\mathbb{C})$ and the raising matrix $R \in \text{Mat}_P(\mathbb{C})$ have entries

\[
L_{xy} = \begin{cases} 1, & \text{if } y \text{ covers } x; \\ 0, & \text{if } y \text{ does not cover } x \end{cases} \quad R_{xy} = \begin{cases} 1, & \text{if } x \text{ covers } y; \\ 0, & \text{if } x \text{ does not cover } y \end{cases}
\]

for $x, y \in P$. Note that the transpose $R^t = L$. For $0 \leq i \leq N$ let $F_i$ denote the diagonal matrix in $\text{Mat}_P(\mathbb{C})$ with diagonal entries

\[
(F_i)_{yy} = \begin{cases} 1, & \text{if } y \in P_i; \\ 0, & \text{if } y \notin P_i \end{cases} \quad y \in P.
\]

We have

\[
F_i F_j = \delta_{ij} F_i \quad (0 \leq i, j \leq N)
\]

and

\[
I = \sum_{i=0}^N F_i.
\]

We refer to $F_i$ as the $i$th projection matrix for $P$. The $\{F_i\}_{i=0}^N$ are related to the $R, L$ by

\[
RF_i = F_{i+1}R \quad (0 \leq i \leq N-1), \quad RF_N = 0, \quad F_0R = 0,
\]

\[
LF_i = F_{i-1}L \quad (1 \leq i \leq N), \quad LF_0 = 0, \quad F_NL = 0.
\]

Define

\[
K = \sum_{i=0}^N q^{N+M-i} F_i.
\]

Note that $K$ is diagonal. Moreover, $K$ is invertible and

\[
K^{-1} = \sum_{i=0}^N q^{i-N-M} F_i.
\]

The incidence algebra $T$ of $P$ is the subalgebra of $\text{Mat}_P(\mathbb{C})$ generated by $R, L, K^{\pm 1}$. The vector space $\mathbb{C}P$ is a $T$-module. By [22, Theorem 2.5] the algebra $T$ is semisimple. Therefore the $T$-module $\mathbb{C}P$ is a direct sum of irreducible $T$-submodules.

We mention two basic facts for later use.

**Lemma 2.1.** For $0 \leq i, j, k \leq N$,

\[
F_i R^k F_j \neq 0 \quad \text{iff} \quad k = i - j,
\]

\[
F_i L^k F_j \neq 0 \quad \text{iff} \quad k = j - i.
\]
Lemma 2.2. For $D \in \text{Mat}_P(\mathbb{C})$ the following are equivalent:

(i) $D = 0$;

(ii) $F_iDF_j = 0$ for $0 \leq i, j \leq N$.

Proof. (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (i) By (4),

$$D = IDI = \sum_{i=0}^{N} \sum_{j=0}^{N} F_iDF_j.$$ 

The result follows.

3 How $R, L, K$ are related

In this section, we describe some relations among $R, L, K$.

Lemma 3.1. The matrices $R, L, K$ satisfy the following relations:

$$RK = qKR, \quad LK = q^{-1}KL,$$

$$q(q+1)^{-1}RL^2 - LRL + (q+1)^{-1}L^2R + LK = 0,$$  \hspace{1cm} (10)

$$q(q+1)^{-1}R^2L - RLR + (q+1)^{-1}LR^2 + KR = 0.$$  \hspace{1cm} (11)

Proof. To obtain (9), in each equation compare the $(x,y)$-entries of each side for $x, y \in P$. The equation (10) is obtained by [22, Theorem 3.2]. The equation (11) is obtained from (10) by applying the transpose to each side.

Lemma 3.2. We have

$$R^2LR = q^{-2}(q+1)R^2K + \frac{q(q^2-1)R^3L + (q-1)L^3R}{q^3 - 1},$$  \hspace{1cm} (12)

$$RLR^2 = q^{-2}(q+1)R^2K + \frac{q^2(q-1)R^3L + (q^2-1)L^3R}{q^3 - 1}.$$  \hspace{1cm} (13)

Moreover,

$$L^2RL = (q+1)L^2K + \frac{q^2(q-1)RL^3 + (q^2-1)L^3R}{q^3 - 1},$$  \hspace{1cm} (14)

$$LRL^2 = (q+1)L^2K + \frac{q(q^2-1)RL^3 + (q-1)L^3R}{q^3 - 1}.$$  \hspace{1cm} (15)
Proof. In equation (11) multiply each term on the left by \( R \) and simplify the result using (9) to obtain
\[
q(q + 1)^{-1}R^3L - R^2LR + (q + 1)^{-1}RLR^2 + qKR^2 = 0. \tag{16}
\]
In equation (11) multiply each term on the right by \( R \) to obtain
\[
q(q + 1)^{-1}R^2LR - RLR^2 + (q + 1)^{-1}LR^3 + KR^2 = 0. \tag{17}
\]
Combining (16), (17) we obtain (12), (13). Apply the transpose map to (12), (13) to get (14), (15).

For a nonzero \( \tau \in \mathbb{C} \) such that \( \tau^2 \neq 1 \), define
\[
[n]_\tau = \frac{\tau^n - \tau^{-n}}{\tau - \tau^{-1}}, \quad n = 0, 1, 2, \ldots
\]

**Lemma 3.3.** The matrices \( R, L \) satisfy the cubic \( q^{1/2} \)-Serre relations:
\[
R^3L - [3]_{q^{1/2}}R^2LR + [3]_{q^{1/2}}RLR^2 - LR^3 = 0,
\]
\[
L^3R - [3]_{q^{1/2}}L^2RL + [3]_{q^{1/2}}LRL^2 - RL^3 = 0.
\]

*Proof.* By Lemma 3.2. \( \square \)

**Lemma 3.4.** The matrices \( RL, LR, K, K^{-1} \) mutually commute.

*Proof.* By (9), each of \( RL, LR \) commutes with \( K \). To see that \( RL, LR \) commute, use (9)–(11). \( \square \)

## 4 The matrix \( S \)

In this section, we define a matrix \( S \in \text{Mat}_P(\mathbb{C}) \) and we discuss how \( S \) is related to \( R, L, K \).

**Definition 4.1.** For \( 1 \leq i \leq N + 1 \), denote by \( \bar{P}_i \) the set of all \( i \)-dimensional subspaces of \( H \) whose intersection with \( h \) has dimension one.

Define \( S \in \text{Mat}_P(\mathbb{C}) \) as follows.

**Definition 4.2.** For \( x, y \in P \) we give the \((x, y)\)-entry of \( S \). Write \( x \in P_i \) and \( y \in P_j \) with \( 0 \leq i, j \leq N \). Then
\[
S_{xy} = \begin{cases} 
1, & \text{if } i = j \text{ and } x + y \in \bar{P}_{i+1}; \\
0, & \text{otherwise}.
\end{cases}
\]

It is clear that \( S^t = S \).

**Lemma 4.3.** The matrix \( S \) is related to \( R, L, K \) as follows:
\[
S + LR - RL = \frac{K - q^{N+M}K^{-1} + (1 - q^M)I}{q - 1}, \tag{18}
\]
\[
LS - qSL = (q^M - 1)L, \tag{19}
\]
\[
SR - qRS = (q^M - 1)R. \tag{20}
\]
Proof. We first obtain (18). To do this, for \( x, y \in P \) compute the \((x, y)\)-entry of each term. We have

\[
(x, y)\text{-entry of LR} = (q - 1)^{-1}(q^{N+M-1} - q^N)
\]

\[
(x, y)\text{-entry of RL} = (q - 1)^{-1}(q' - 1)
\]

| \((x, y)\)-entry of LR | \((x, y)\)-entry of RL | condition |
|------------------------|------------------------|-----------|
| \((q - 1)^{-1}(q^{N+M-1} - q^N)\) | \((q - 1)^{-1}(q' - 1)\) | \(x = y \in P_i\) |
| 1 | 1 | \(x, y \in P_i, \ x + y \in P_{i+1}\) |
| 0 | 1 | \(x, y \in P_i, \ x + y \in \tilde{P}_{i+1}\) |
| 0 | 0 | otherwise |

By the above arguments,

\[
(x, y)\text{-entry of LR} - (x, y)\text{-entry of RL} = (q - 1)^{-1}(q^{N+M-1} - q^N - q' + 1)
\]

| \((x, y)\)-entry of LR - RL | condition |
|---------------------------|-----------|
| \((q - 1)^{-1}(q^{N+M-1} - q^N - q' + 1)\) | \(x = y \in P_i\) |
| \(-1\) | \(x, y \in P_i, \ x + y \in \tilde{P}_{i+1}\) |
| 0 | otherwise |

Equation (18) follows from this along with (7), (8) and Definition 4.2. Combining (9), (10), (18) we obtain (19). In (19), apply the transpose to each side and use \(S^t = S, L^t = R\) to get (20).

Lemma 4.4. The matrix \(S\) commutes with each of \(RL, LR, K, K^{-1}\).

Proof. Combining Lemma 3.4 and (18), we find that \(S\) commutes with \(K, K^{-1}\). Combining (19), (20) we find that \(S\) commutes with \(RL, LR\).

5 The irreducible \(T\)-modules and the center \(Z(T)\)

In this section, we recall some facts from \[22, 23\], about the irreducible \(T\)-modules and the center \(Z(T)\).

By a \(T\)-module we mean a \(T\)-submodule of \(CP\). Let \(W\) be an irreducible \(T\)-module. By the endpoint of \(W\) we mean \(\min\left\{i \mid 0 \leq i \leq N, F_iW \neq 0\right\}\). By the diameter of \(W\) we mean \(|\{i \mid 0 \leq i \leq N, F_iW \neq 0\}\} - 1.

Lemma 5.1. \[22\] Theorems 2.5, 3.3) For \(0 \leq r, d \leq N\), there exists an irreducible \(T\)-module with endpoint \(r\) and diameter \(d\), if and only if

\[N - 2r \leq d \leq N - r, \quad d \leq N + M - 2r.\]

Lemma 5.2. \[22\] Theorem 2.5] Let \(W\) be an irreducible \(T\)-module with endpoint \(r\) and diameter \(d\). Then the isomorphism class of \(W\) is determined by the sequence \((r, d)\).

Let \(Z(T)\) denote the center of \(T\). We now give a basis for the vector space \(Z(T)\).

Let \(\Psi\) denote the set of isomorphism classes of irreducible \(T\)-modules. The elements of \(\Psi\) are called types. By Lemmas 5.1, 5.2, we view

\[\Psi = \{(r, d) \mid 0 \leq r, d \leq N, \ N - 2r \leq d \leq N - r, \ d \leq N + M - 2r\}\]

For \(\lambda = (r, d) \in \Psi\), let \(V_{\lambda}\) denote the subspace of \(CP\) spanned by the irreducible \(T\)-modules of type \(\lambda\). Then \(V_{\lambda}\) is a \(T\)-module, and the sum \(CP = \sum_{\lambda \in \Psi} V_{\lambda}\) is direct. For \(\lambda \in \Psi\), define a linear map \(e_{\lambda} : CP \to CP\) such that \((e_{\lambda} - I)V_{\lambda} = 0\) and \(e_{\lambda}V_{\mu} = 0\) for all \(\mu \in \Psi\) with \(\mu \neq \lambda\).

According to the Wedderburn theory \[3\], the elements \(\{e_{\lambda}\}_{\lambda \in \Psi}\) form a basis for \(Z(T)\).
Definition 5.3. [23 line(16)] For \((r, d) \in \Psi\) define
\[ x_{r+i}(r, d) = \frac{q^{N+M-r-d}(q^i - 1)(q^{d+1-i} - 1)}{(q - 1)^2} \quad (1 \leq i \leq d). \]

Note 5.4. Referring to Definition 5.3, \(x_{r+i}(r, d) \neq 0\) for \(1 \leq i \leq d\).

Lemma 5.5. [23 p.78] Let \(W\) denote an irreducible \(T\)-module with endpoint \(r\) and diameter \(d\). Then there exists a basis \(\{w_i\}_{i=0}^d\) for \(W\) such that
(i) \(w_i \in F_{r+i}W\) \(\quad (0 \leq i \leq d),\)
(ii) \(Rw_i = w_{i+1}\) \(\quad (0 \leq i \leq d - 1), \quad Rw_d = 0,\)
(iii) \(Lw_i = x_{r+i}(r, d)w_{i-1}\) \(\quad (1 \leq i \leq d), \quad Lw_0 = 0.\)

6 The central elements \(\Phi, \Omega\)

In this section, we display two elements \(\Phi, \Omega\) in \(Z(T)\). Define
\[ \Phi = \sum_{\lambda=(r,d) \in \Psi} q^r e_\lambda, \quad \Omega = \sum_{\lambda=(r,d) \in \Psi} q^{d/2} e_\lambda. \]

By construction \(\Phi, \Omega\) are in \(Z(T)\). Note that \(\Phi, \Omega\) are invertible, with
\[ \Phi^{-1} = \sum_{\lambda=(r,d) \in \Psi} q^{-r} e_\lambda, \quad \Omega^{-1} = \sum_{\lambda=(r,d) \in \Psi} q^{-d/2} e_\lambda. \]

Lemma 6.1. Let \(W\) denote an irreducible \(T\)-module with endpoint \(r\) and diameter \(d\). Then on \(W\),
\[ \Phi = q^r I, \quad \Omega = q^{d/2} I. \]

Proof. By construction. \(\square\)

Proposition 6.2. The center \(Z(T)\) is generated by \(\Phi, \Omega\).

Proof. Let \(Z\) denote the subalgebra of \(Z(T)\) generated by \(\Phi, \Omega\). We show \(Z = Z(T)\). To do this, it suffices to show that \(e_\lambda \in Z\) for all \(\lambda \in \Psi\). Suppose we are given two distinct elements in \(\Psi\), denoted by \((r, d)\) and \((r', d')\). Observe that \(q^r \neq q^{r'}\) or \(q^{d/2} \neq q^{d'/2}\). By this and Lemma 6.1, we see that \(e_\lambda \in Z\) for all \(\lambda \in \Psi\). \(\square\)

7 The central elements \(C_1, C_2\)

In this section, we display two elements \(C_1, C_2\) in \(Z(T)\) and discuss their combinatorial meaning.
Definition 7.1. Define

\[ C_1 = q^{-1}(q - 1)^{-1}(q + 1)K + RL - q^{-1}LR, \]
\[ C_2 = K^2 + (q - 1)RLK - (q - 1)LRK. \]

Lemma 7.2. We have \( C_1^t = C_1 \) and \( C_2^t = C_2 \).

Proof. By \( R^t = L \) along with Lemma 3.4 and Definition 7.1.

Lemma 7.3. The matrices \( C_1 \) and \( C_2 \) are in \( Z(T) \).

Proof. We first show that \( C_1 \in Z(T) \). To do this, we show that \( C_1 \) commutes with each of \( R, L, K \). To see that \( C_1 \) commutes with \( R \), use \( RK = qKR \) and (11). To see that \( C_1 \) commutes with \( L \), use \( R^t = L \) and Lemma 7.2. The matrix \( C_1 \) commutes with \( K \) by Lemma 3.4. We have shown that \( C_1 \in Z(T) \). In a similar way, one can show that \( C_2 \in Z(T) \).

Lemma 7.4. Let \( W \) denote an irreducible \( T \)-module with endpoint \( r \) and diameter \( d \). Then on \( W \),

\[ C_1 = (q - 1)^{-1}q^{N+M-r}(1 + q^{-d-1})I, \quad C_2 = q^{2N+2M-2r-d}I. \]

Proof. We consider the actions of \( C_1 \) and \( C_2 \) on the basis \( \{ w_i \}_{i=0}^{d} \) given in Lemma 5.5. By the definition of \( K \) along with Definition 5.3, Lemma 5.5, Definition 7.1, we find that for \( 0 \leq i \leq d \),

\[ C_1w_i = (q - 1)^{-1}q^{N+M-r}(1 + q^{-d-1})w_i, \]  
\[ C_2w_i = q^{2N+2M-2r-d}w_i. \]

The result follows.

Proposition 7.5. The center \( Z(T) \) is generated by \( C_1, C_2 \).

Proof. The proof is similar to that of Lemma 6.2. Let \( Z \) denote the subalgebra of \( Z(T) \) generated by \( C_1, C_2 \). We show that \( Z = Z(T) \). To do this, it suffices to show that \( e_\lambda \in Z \) for all \( \lambda \in \Psi \). Suppose we are given two distinct elements in \( \Psi \), denoted by \( (r, d) \) and \( (r', d') \). We claim that

\[ q^{-r} + q^{-r-d-1} \neq q^{-r'} + q^{-r'-d'-1} \]

or

\[ q^{-2r-d} \neq q^{-2r'-d'}. \]

Suppose the claim is false. We have

\[ q^{-r} + q^{-r-d-1} = q^{-r'} + q^{-r'-d'-1}, \]  
\[ q^{-2r-d} = q^{-2r'-d'}. \]
Suppose for the moment that \( r = r' \). Then by (24), \( q^{d-d'} = 1 \). But \( q \) is not a root of unity, so \( d = d' \) for a contradiction. Therefore \( r \neq r' \). Without loss of generality we may assume \( r < r' \). Eliminating \( q^{-d} \) in (23) using (24) we find
\[
(1 - q^{r-r'})(q^{-r} - q^{-r'-d-1}) = 0.
\]
Note that \( 1 - q^{r-r'} \) is nonzero, so \( q^{-r} = q^{-r'-d-1} \). Therefore \( r = r' + d' + 1 \), so
\[
d' = r - r' - 1.
\] (25)

Now \( d' < 0 \) for a contradiction. The claim is proved. By the claim and Lemma 7.4 \( e_\lambda \in Z \) for all \( \lambda \in \Psi \). The result follows.

**Lemma 7.6.** The following (i), (ii) hold.

(i) \( RL = -q(q - 1)^{-2}K + \frac{q(q - 1)^{-1}C_1}{(q - 1)^{-2}C_2K^{-1}} \),

(ii) \( LR = -(q - 1)^{-2}K + \frac{q(q - 1)^{-1}C_1 - q(q - 1)^{-2}C_2K^{-1}}{q^{-1}} \).

**Proof.** Solve the two equations in Definition 7.1 for \( RL, LR \).

We now discuss how \( C_1, C_2 \) are related to \( \Phi, \Omega \).

**Lemma 7.7.** We have
\[
C_1 = q^{N+M-1}\Phi^{-1}\Omega^{-1}\frac{q^{1/2}\Omega + q^{-1/2}\Omega^{-1}}{q^{1/2} - q^{-1/2}},
\] (26)
\[
C_2 = q^{2N+2M}\Phi^{-2}\Omega^{-2}.
\] (27)

**Proof.** Combine Lemma 6.1 and Lemma 7.4.

We now interpret our results so far in terms of augmented down-up algebras [33]. Fix a nonzero \( \tau \in \mathbb{C} \) which is not a root of unity.

**Definition 7.8.** [33, Definition 2.3] Let \( s, t \) denote distinct integers. Let \( \phi \) denote a Laurent polynomial over \( \mathbb{C} \) in a variable \( \vartheta \). The *augmented down-up algebra* \( \mathcal{A}_\tau(s, t, \phi) \) is the \( \mathbb{C} \)-algebra with generators \( K, E, F, C_s, C_t \) and relations
\[
KK^{-1} = K^{-1}K = 1,
\]
\( C_s, C_t \) are central,
\[
KE = \tau^2EK, \quad KF = \tau^{-2}FK,
\]
\[
FE = C_s\tau^sK^s + C_t\tau^tK^t + \phi(\tau K),
\]
\[
EF = C_s\tau^{-s}K^s + C_t\tau^{-t}K^t + \phi(\tau^{-1}K).
\]

**Lemma 7.9.** Let \( \tau = q^{1/2} \). The vector space \( \mathbb{C}P \) has an \( \mathcal{A}_\tau(s, t, \phi) \)-module structure on which \( K, E, F, C_s, C_t \) act as follows:

| generator | \( K \) | \( E \) | \( F \) | \( C_s \) | \( C_t \) |
|-----------|-----------|-----------|-----------|-----------|-----------|
| action    | \( K \) | \( L \) | \( R \) | \( \frac{q^s}{(q^2-1)^2}C_2 \) | \( \frac{q^t}{q^2-1}C_1 \) |

Here \( s = -1, t = 0, \phi = -\tau(\tau^2 - 1)^{-2}\vartheta \).

**Proof.** Combine (9), Lemmas 7.3 7.6 and Definition 7.8.

**Remark 7.10.** The \( \mathcal{A}_\tau(s, t, \phi) \)-module structure in Lemma 7.9 is discussed further in [33].
8 Some $U_\tau(sl_2)$-module structures on $\mathbb{C}P$

Throughout this section, fix a nonzero $\tau \in \mathbb{C}$ that is not a root of unity. In this section we recall the quantum enveloping algebra $U_\tau(sl_2)$. We then display some $U_\tau(sl_2)$-module structures on $\mathbb{C}P$.

**Definition 8.1.** Let $U_\tau(sl_2)$ denote the $\mathbb{C}$-algebra with generators $e, f, k^{\pm 1}$ and the following relations:

\begin{align*}
kk^{-1} &= k^{-1}k = 1, \\
ke &= \tau^2 ek, \\
 kf &= \tau^{-2}fk, \\
ef - fe &= \frac{k - k^{-1}}{\tau - \tau^{-1}}.
\end{align*}

We call $e, f, k^{\pm 1}$ the Chevalley generators for $U_\tau(sl_2)$.

**Theorem 8.2.** Assume that $\tau = q^{1/2}$ and let $\Theta$ denote an invertible element in $\mathbb{Z}(T)$. Then $\mathbb{C}P$ becomes a $U_\tau(sl_2)$-module on which $e, f, k$ act as follows:

| generator | $e$ | $f$ | $k$ |
|-----------|-----|-----|-----|
| action    | $\Theta L$ | $q^{-N-M+1}\Theta^{-1}\Phi\Omega R$ | $q^{-N-M}\Phi\Omega K$ |

**Proof.** Since $K$ is invertible, (28) holds. By (9) we obtain (29). To obtain (30), combine Lemmas 7.6, 7.7.

9 Some results about $R, L$

For the rest of this paper, assume that $N \geq 6$. In this section, we obtain some results about $R, L$ that we will use later in the paper.

**Lemma 9.1.** The following hold in the algebra $T$:

\begin{align*}
LR^3F_0 &= (q - 1)^{-2}q^M(q^3 - 1)(q^{N-2} - 1)R^2F_0, \\
L^3RF_2 &= (q - 1)^{-2}q^M(q^3 - 1)(q^{N-2} - 1)L^2F_2.
\end{align*}

**Proof.** We first obtain (31). Let $y$ be the unique element in $P_0$. For $x \in P_2$, we calculate the $(x, y)$-entry of each matrix:

\begin{align*}
(x, y)\text{-entry of } R^2F_0 &= q + 1 \\
(x, y)\text{-entry of } LR^3F_0 &= (q - 1)^{-2}q^M(q + 1)(q^3 - 1)(q^{N-2} - 1)
\end{align*}

Line (31) follows. Apply the transpose map to each side of (31) to get (32).

**Lemma 9.2.** In the poset $\mathcal{A}_q(N, M)$,

(i) there exists $x \in P_N$ and $y \in P_{N-2}$ such that $x + y \in \tilde{P}_{N+1}$,

(ii) there exists $x \in P_N$ and $y \in P_{N-2}$ such that $y < x$. 

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Proof. By the arguments about \( A_q(N, M) \) given in Section 2.

Lemma 9.3. For \( 1 \leq i \leq N - 3 \),

(i) there exists \( x \in P_{i+2} \) and \( y \in P_i \) such that \( x + y \in P_{i+3} \),

(ii) there exists \( x \in P_{i+2} \) and \( y \in P_i \) such that \( x + y \in \tilde{P}_{i+3} \),

(iii) there exists \( x \in P_{i+2} \) and \( y \in P_i \) such that \( y < x \).

Proof. By the arguments about \( A_q(N, M) \) given in Section 2.

Lemma 9.4. The following hold in the algebra \( T \):

(i) \( R^2 F_{N-2}, R^3 LF_{N-2} \) are linearly independent,

(ii) \( L^2 F_N, RL^3 F_N \) are linearly independent.

Proof. (i) For \( x \in P_N \) and \( y \in P_{N-2} \), we calculate the \((x, y)\)-entry of each matrix:

\[
\begin{array}{ccc}
(x, y)\text{-entry of } R^2 F_{N-2} & (x, y)\text{-entry of } R^3 LF_{N-2} & \text{condition} \\
q + 1 & (q - 1)^{-2}(q + 1)(q^3 - 1)(q^{N-2} - 1) & y < x \\
0 & (q - 1)^{-1}(q + 1)(q^3 - 1) & x + y \in \tilde{P}_{N+1}
\end{array}
\]

In the above table the entries form a 2 \( \times \) 2 matrix. This matrix is upper triangular with nonzero diagonal entries, so it is invertible. The result follows in view of Lemma 9.2.

(ii) Apply the transpose map to the matrices in (i) and use \( R^t = L \) along with (5), (6).

Lemma 9.5. For \( 1 \leq i \leq N - 3 \), the following hold in the algebra \( T \):

(i) \( R^2 F_i, R^3 LF_i, LR^3 F_i \) are linearly independent;

(ii) \( L^2 F_{i+2}, L^3 RF_{i+2}, RL^3 F_{i+2} \) are linearly independent.

Proof. (i) For \( x \in P_{i+2} \) and \( y \in P_i \), we calculate the \((x, y)\)-entry of each matrix:

\[
\begin{array}{ccc}
(x, y)\text{-entry of } R^2 F_i & (x, y)\text{-entry of } LR^3 F_i & (x, y)\text{-entry of } R^3 LF_i & \text{condition} \\
q + 1 & (q - 1)^{-1}q^{M(q^{i-1} - 1)} & (q - 1)^{-1}(q^{i+1} - 1) & y < x \\
0 & \gamma & \gamma & x + y \in P_{i+3} \\
0 & 0 & \gamma & x + y \in \tilde{P}_{i+3}
\end{array}
\]

where

\[
\gamma = (q - 1)^{-1}(q + 1)(q^3 - 1).
\]

In the above table the entries form a 3 \( \times \) 3 matrix. This matrix is upper triangular with nonzero diagonal entries, so it is invertible. The result follows in view of Lemma 9.3.

(ii) Apply the transpose map to the matrices in (i) and use \( R^t = L \) along with (5), (6).
10 Leonard pairs

In this section, we recall the definition of a Leonard pair and discuss some basic facts about these objects.

Through this section, fix an integer $d \geq 0$. Let $V$ denote a vector space over $\mathbb{C}$ with dimension $d + 1$. Denote by $\text{End}(V)$ the $\mathbb{C}$-algebra of all $\mathbb{C}$-linear maps $V \to V$. Let $\{v_i\}_{i=0}^d$ denote a basis of $V$. For $A \in \text{End}(V)$ and $X \in \text{Mat}_{d+1}(\mathbb{C})$, we say that $X$ represents $A$ with respect to $\{v_i\}_{i=0}^d$ whenever $Av_j = \sum_{i=0}^d X_{ij}v_i$ for $0 \leq j \leq d$. For $X \in \text{Mat}_{d+1}(\mathbb{C})$, $X$ is called upper bidigonal whenever every nonzero entry appears on the diagonal or the superdiagonal. The matrix $X$ is called lower bidigonal whenever every nonzero entry appears on the diagonal or the subdiagonal. Assume that $X$ is tridiagonal. Then $X$ is called irreducible whenever the entries on the superdiagonal and subdiagonal are all nonzero.

**Definition 10.1.** [24, Definition 1.1] By a Leonard pair on $V$, we mean an ordered pair $A, A^*$ of elements in $\text{End}(V)$ that satisfy the following conditions:

(i) there exists a basis for $V$ with respect to which the matrix representing $A$ is irreducible tridiagonal and the matrix representing $A^*$ is diagonal;

(ii) there exists a basis for $V$ with respect to which the matrix representing $A^*$ is irreducible tridiagonal and the matrix representing $A$ is diagonal.

We call $V$ the underlying vector space, and call $d$ the diameter.

**Definition 10.2.** Referring to Definition 10.1 consider a basis for $V$ from part (ii). With respect to this basis, the matrix representing $A$ is diagonal, denoted by $\text{diag}(\theta_0, \theta_1, \ldots, \theta_d)$. We call $\{\theta_i\}_{i=0}^d$ an eigenvalue sequence for $A, A^*$. By a dual eigenvalue sequence for $A, A^*$, we mean an eigenvalue sequence for the Leonard pair $A^*, A$.

**Note 10.3.** Let $A, A^*$ denote a Leonard pair on $V$. Let $\{\theta_i\}_{i=0}^d$ denote an eigenvalue sequence for $A, A^*$. Then the sequence $\{\theta_{d-i}\}_{i=0}^d$ is an eigenvalue sequence for $A, A^*$ and $A, A^*$ has no other eigenvalue sequence. A similar comment applies to the dual eigenvalue sequence.

**Definition 10.4.** [26, Definition 22.1] By a parameter array over $\mathbb{C}$ of diameter $d$ we mean a sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$ of elements in $\mathbb{C}$ that satisfy the following conditions:

(i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ if $i \neq j$ $(0 \leq i, j \leq d)$,

(ii) $\varphi_i \neq 0$, $\phi_i \neq 0$ $(1 \leq i \leq d)$,

(iii) $\varphi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0)(\theta_{i-1} - \theta_d)$ $(1 \leq i \leq d)$,

(iv) $\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} + (\theta_i^* - \theta_0)(\theta_{d-i+1} - \theta_0)$ $(1 \leq i \leq d)$,
Lemma 10.7. \[27, \text{Theorem 17.1}\] Let \[A, A^*\] denote matrices in \[\text{Mat}_{d+1}(\mathbb{C})\]. Assume that \(A\) is lower bidiagonal and \(A^*\) is upper bidiagonal. Then the following (i), (ii) are equivalent:

(i) the pair \(A, A^*\) is a Leonard pair;

(ii) there exists a parameter array \((\{\theta_i\}_{i=0}^d, \{\varphi_i\}_{i=0}^d, \{\varphi_i\}_{j=1}^d, \{\phi_j\}_{j=1}^d)\) over \(\mathbb{C}\) such that

\[
A_{ii} = \theta_i, \quad A_{ii}^* = \theta_i^* \quad (0 \leq i \leq d),
\]

\[
A_{i,i-1} A_{i-1,i}^* = \varphi_i \quad (1 \leq i \leq d).
\]

Lemma 10.6. \[8, \text{Theorem 10.1}\] Let \(A, A^*\) denote a Leonard pair over \(\mathbb{C}\). Then there exists a sequence of scalars \(\beta, \gamma, \gamma^*, \varphi, \varphi^*\) taken from \(\mathbb{C}\) such that both

\[
[A, A^2 A^* - \beta A A^* A + A^* A^2 - \gamma(A A^* + A^* A) - \varphi A^*] = 0, \quad (33)
\]

\[
[A^*, A^2 A - \beta A^* A A^* + A A^2 - \gamma^*(A^* A + A A^*) - \varphi^* A] = 0, \quad (34)
\]

where \([r,s]\) means \(rs - sr\). The sequence is uniquely determined by the Leonard pair \(A, A^*\) provided that the diameter is at least 3.

We call (33), (34) the tridiagonal relations.

Lemma 10.7. \[32, \text{Corollary 4.4, Theorem 4.5}\] Let \(A, A^*\) denote a Leonard pair over \(\mathbb{C}\). Let \(\{\theta_i\}_{i=0}^d\) (resp. \(\{\theta_i^*\}_{i=0}^d\)) denote an eigenvalue sequence (resp. dual eigenvalue sequence) of \(A, A^*\). Let \(\beta, \gamma, \gamma^*, \varphi, \varphi^*\) denote scalars in \(\mathbb{C}\). Then these scalars satisfy (33), (34) if and only if the following (i)–(v) hold:

(i) \(\beta + 1 = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} = \frac{\theta_{i-2} - \theta_{i+1}^*}{\theta_{i-1}^* - \theta_i^*} \quad (2 \leq i \leq d - 1),\)

(ii) \(\gamma = \theta_{i+1} - \beta \theta_i + \theta_{i-1} \quad (1 \leq i \leq d - 1),\)

(iii) \(\gamma^* = \theta_{i+1}^* - \beta \theta_i^* + \theta_{i-1}^* \quad (1 \leq i \leq d - 1),\)

(iv) \(\varphi = \theta_{i-1}^2 - \beta \theta_{i-1} \theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) \quad (1 \leq i \leq d),\)

(v) \(\varphi^* = \theta_{i-1}^{*2} - \beta \theta_{i-1}^* \theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i-1}^* + \theta_i^*) \quad (1 \leq i \leq d).\)

Let \(n\) denote a positive integer. For the rest of this section, let \(\{\theta_i\}_{i=0}^n\) denote a sequence of scalars in \(\mathbb{C}\). For \(\beta \in \mathbb{C}\), \(\{\theta_i\}_{i=0}^n\) is called \(\beta\)-recurrent whenever

\[
\theta_{i-2} - (\beta + 1) \theta_{i-1} + (\beta + 1) \theta_i - \theta_{i+1} = 0 \quad (2 \leq i \leq n - 1).
\]

Lemma 10.8. \[24, \text{Lemma 9.2}\] Given a sequence \(\{\theta_i\}_{i=0}^n\) of scalars in \(\mathbb{C}\) and given \(\beta \in \mathbb{C}\).
(i) Assume $\beta = 2$. Then $\{\theta_i\}_{i=0}^n$ is $\beta$-recurrent if and only if there exist scalars $a, b, c \in \mathbb{C}$ such that
\[
\theta_i = a + bi + ci^2 \quad (0 \leq i \leq n).
\]
(ii) Assume $\beta = -2$. Then $\{\theta_i\}_{i=0}^n$ is $\beta$-recurrent if and only if there exist scalars $a, b, c \in \mathbb{C}$ such that
\[
\theta_i = a + b(-1)^i + ci(-1)^i \quad (0 \leq i \leq n).
\]
(iii) Assume $\beta \neq 2$, $\beta \neq -2$. Then $\{\theta_i\}_{i=0}^n$ is $\beta$-recurrent if and only if there exist scalars $a, b, c \in \mathbb{C}$ such that
\[
\theta_i = a + bQ^i + cQ^{-i} \quad (0 \leq i \leq n),
\]
where $\beta = Q + Q^{-1}$.

11 The tridiagonal relations and $A_q(N, M)$

We continue to discuss the poset $A_q(N, M)$ from Section 2. For the rest of the paper, we fix matrices $A, A^*$ of the following form:

\[
A = \sum_{i=0}^{N-1} \alpha_i RF_i + \sum_{i=0}^{N} \theta_i F_i, \quad (35)
\]
\[
A^* = \sum_{i=1}^{N} \alpha_i^* LF_i + \sum_{i=0}^{N} \theta_i^* F_i. \quad (36)
\]

The $\alpha_i, \alpha_i^*, \theta_i, \theta_i^*$ are scalars in $\mathbb{C}$ such that

\[
\begin{align*}
\alpha_i &\neq 0 \quad (0 \leq i \leq N - 1), \\
\alpha_i^* &\neq 0 \quad (1 \leq i \leq N), \\
\theta_i &\neq \theta_j \quad \theta_i^* \neq \theta_j^* \quad \text{if } i \neq j \quad (0 \leq i, j \leq N). \quad (37, 38, 39)
\end{align*}
\]

The matrices $R, L$ are from (1) and $\{F_i\}_{i=0}^{N}$ are from (2).

For convenience, let
\[
\alpha_{-1} = 0, \quad \alpha_N = 0, \quad \alpha_0^* = 0, \quad \alpha_N^* = 0.
\]

Let $\theta_{-1}, \theta_{N+1}, \theta_{-1}^*, \theta_{N+1}^*$ denote indeterminates. Define
\[
\xi_i = \alpha_i \alpha_{i+1}^* \quad (-1 \leq i \leq N). \quad (40)
\]

Observe that $\xi_{-1} = 0$, $\xi_N = 0$ and $\xi_i \neq 0$ for $0 \leq i \leq N - 1$.

Fix some scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ in $\mathbb{C}$ and define
\[
B = [A, A^2A^* - \beta AA^*A + A^* A^2 - \gamma (AA^* + A^* A) - \varrho A^*], \quad (41)
\]
\[
B^* = [A^*, A^* A - \beta A^* AA^* + AA^*^2 - \gamma^* (A^* A + AA^*) - \varrho^* A^*]. \quad (42)
\]

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We now find necessary and sufficient conditions for $B$ and $B^*$ to be zero. Expand $B$ and $B^*$ as follows:

$$B = A^3A^*(\beta + 1)A^2A^* + (\beta + 1)AA^*A^2 - A^*A^3$$
$$B^* = A^3A^* - (\beta + 1)A^2A^* + (\beta + 1)AA^*A^2 - AA^*$$

\[ \text{(43)} \]

\[ \text{(44)} \]

**Lemma 11.1.** For $0 \leq i, j \leq N$, we have

(i) $F_iBF_j = 0$ if $i - j < -1$ or $i - j > 3$,

(ii) $F_iB^*F_j = 0$ if $j - i < -1$ or $j - i > 3$.

**Proof.** Evaluating (43), (44) using (35), (36).

**Lemma 11.2.** In the algebra $T$,

(i) for $0 \leq i \leq N - 3$, $F_{i+3}BF_i$ is equal to $\alpha_i\alpha_{i+1}\alpha_{i+2}(\theta_i^* - \theta_{i+3}^* - (\beta + 1)(\theta_{i+1}^* - \theta_{i+2}^*))$ 

\[ \text{times } R^3F_i, \]

(ii) for $1 \leq i \leq N$, $F_{i-1}BF_i$ is equal to $\alpha_i^*(\theta_{i-1}^* - \theta_i^*)(\theta_{i-2} - \beta\theta_{i-1}\theta_i + \theta_{i}^2 - \gamma(\theta_{i-1} + \theta_i) - \rho)$ 

\[ \text{times } LF_i, \]

(iii) for $0 \leq i \leq N - 1$, $F_{i+1}BF_i$ is a weighted sum involving the following terms and coefficients:

| Term       | Coefficient |
|------------|-------------|
| $RF_i$     | $\alpha_i(\theta_i^* - \theta_{i+1}^*)(\theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \rho)$ |
| $R^2LF_i$ | $\alpha_i\alpha_{i-1}\alpha_i^*(\theta_{i-1} - \beta\theta_i\theta_{i+1} + \theta_i^2)$ |
| $LR^2F_i$ | $\alpha_i\alpha_{i+1}\alpha_{i+2}(\theta_i^2 + \beta\theta_{i+1} + \theta_{i+2}) + \gamma(\theta_i + \theta_{i+1}) + \rho)$ |

(iv) for $0 \leq i \leq N$, $F_iBF_i$ is a weighted sum involving the following terms and coefficients:

| Term       | Coefficient |
|------------|-------------|
| $RLF_i$    | $\alpha_i^*(\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1} + \theta_i) - \rho)$ |
| $LRF_i$    | $\alpha_i^*(\theta_i^2 - \beta\theta_i\theta_{i+1} + \theta_{i+1}^2 - \gamma(\theta_i + \theta_{i+1}) - \rho)$ |

(v) for $0 \leq i \leq N - 2$, $F_{i+2}BF_i$ is a weighted sum involving the following terms and coefficients:

| Term       | Coefficient |
|------------|-------------|
| $R^2F_i$   | $\alpha_i\alpha_{i+1}(\theta_i^2 - \theta_{i+2}^* - (\beta + 1)(\theta_{i+1}^* - \theta_{i+2}^* - (\beta + 1)(\theta_i^* - \theta_{i+2}^*)\theta_{i+2} + \theta_{i+2}^*\theta_{i+1})$ |
| $R^3LF_i$  | $\alpha_i\alpha_{i+1}\alpha_{i+2}\alpha_i^*$ |
| $R^2LRF_i$ | $-(\beta + 1)\alpha_i^2\alpha_{i+1}\alpha_{i+2}$ |
| $RLR^2F_i$ | $(\beta + 1)\alpha_i\alpha_{i+1}\alpha_{i+2}^*$ |
| $LR^3F_i$  | $-\alpha_i\alpha_{i+1}\alpha_{i+2}\alpha_{i+3}^*$ |

**Proof.** Combine (3), (5), (6), (35), (36), (43).

\[ \text{\Box} \]
Lemma 11.3. In the algebra $T$, 

(i) for $3 \leq i \leq N$, $F_{i-3}B^*F_i$ is equal to $\alpha^+_i \alpha^*_{i-1} \alpha^*_{i-2} (\theta_i - \theta_{i-3} - (\beta + 1)(\theta_{i-1} - \theta_{i-2}))$ times $L^3 F_i$, 

(ii) for $0 \leq i \leq N - 1$, $F_{i+1}B^*F_i$ is equal to $\alpha_i (\theta^*_{i+1} - \theta^*_i)(\theta^*_{i+1} - \beta \theta^*_i \theta^*_{i+1} + \theta^*_{i+2} - \gamma^*(\theta^*_{i+1} + \theta^*_i) - \varrho^*)$ times $RF_i$, 

(iii) for $1 \leq i \leq N$, $F_{i-1}B^*F_i$ is a weighted sum involving the following terms and coefficients:

| term   | coefficient |
|--------|-------------|
| $LF_i$ | $\alpha^*_i (\theta_i - \theta_{i-1})(\theta^*_{i-1} - \gamma^*(\theta^*_i + \theta^*_{i-1}) - \varrho^*)$ |
| $L^2 RF_i$ | $\alpha_i \alpha^*_i \alpha^*_{i-1} (\theta^*_{i-1} - \beta \theta^*_i \theta^*_{i+1} + \theta^*_{i+2} - \gamma^*(\theta^*_{i+1} + \theta^*_i) - \varrho^*)$ |
| $RL^2 F_i$ | $\alpha^*_{i-1} \alpha_i (\theta^*_{i-1} - \beta \theta^*_i \theta^*_{i+1} + \theta^*_{i+2} - \gamma^*(\theta^*_i + \theta^*_{i+1}) - \varrho^*)$ |

(iv) for $0 \leq i \leq N$, $F_i B^* F_i$ is a weighted sum involving the following terms and coefficients:

| term   | coefficient |
|--------|-------------|
| $RLF_i$ | $-\alpha^*_i \alpha_i (\theta^*_{i-1} - \beta \theta^*_i \theta^*_{i+1} + \theta^*_{i+2} - \gamma^*(\theta^*_i + \theta^*_{i+1}) - \varrho^*)$ |
| $LRF_i$ | $\alpha^*_{i-1} \alpha_i (\theta^*_{i-1} - \beta \theta^*_i \theta^*_{i+1} + \theta^*_{i+2} - \gamma^*(\theta^*_i + \theta^*_{i+1}) - \varrho^*)$ |

(v) for $2 \leq i \leq N$, $F_{i-2}B^*F_i$ is a weighted sum involving the following terms and coefficients:

| term   | coefficient |
|--------|-------------|
| $L^2 F_i$ | $\alpha^*_i \alpha^*_{i-1} (\theta_i - \theta_{i-2})(\theta^*_{i-2} + \theta^*_{i-3} - \beta \theta^*_i \theta^*_{i-1} + \theta^*_{i+2} - \gamma^*(\theta^*_i + \theta^*_{i-1}) - (\beta + 1)(\theta_{i-1} - \theta_{i-2})\theta^*_{i-2} + (\theta_i - \theta_{i-1})\theta^*_i)$ |
| $L^3 RF_i$ | $\alpha^*_i \alpha^*_{i-1} \alpha^*_{i-2} \alpha_i$ |
| $L^2 RLF_i$ | $-(\beta + 1) \alpha^*_i \alpha^*_{i-1} \alpha_{i-2}$ |
| $LRL^2 F_i$ | $(\beta + 1) \alpha_i \alpha^*_i \alpha^*_{i-1} \alpha^*_{i-2}$ |
| $RL^3 F_i$ | $-\alpha^*_i \alpha^*_{i-1} \alpha^*_{i-2} \alpha_{i-3}$ |

**Proof.** Combine (3), (5), (6), (35), (36), (11). \qed

At the beginning of Section 11, we defined some scalars $\{\theta_i\}_{i=0}^N$, $\{\theta^*_i\}_{i=0}^N$ and $\beta, \gamma, \gamma^*, \varrho, \varrho^*$. As we proceed, we often make an assumption about these scalars called the standard assumption.

**Definition 11.4.** Under the standard assumption,

(i) $\beta + 1 = \frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}$ \hspace{1cm} ($2 \leq i \leq N - 1$),

(ii) $\beta + 1 = \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$ \hspace{1cm} ($2 \leq i \leq N - 1$),

(iii) $\gamma = \theta_{i-1} - \beta \theta_i + \theta_{i+1}$ \hspace{1cm} ($1 \leq i \leq N - 1$),

(iv) $\gamma^* = \theta^*_{i-1} - \beta \theta^*_i + \theta^*_{i+1}$ \hspace{1cm} ($1 \leq i \leq N - 1$),

(v) $\varrho = \theta^2_{i-1} - \beta \theta_{i-1} \theta_i + \theta^2_i - \gamma(\theta_{i-1} + \theta_i)$ \hspace{1cm} ($1 \leq i \leq N$),

(vi) $\varrho^* = \theta^2_{i-1} - \beta \theta^*_{i-1} \theta^*_i + \theta^2_{i+1} - \gamma^*(\theta^*_{i-1} + \theta^*_i)$ \hspace{1cm} ($1 \leq i \leq N$).
Lemma 11.5. Assume $B = 0$ and $B^* = 0$. Then the standard assumption holds.

Proof. We Refer to Definition 11.4 To get (ii), (v) combine Lemma 2.2, Lemma 11.2(i), (ii) along with (37), (38), (39). Using (v), one gets (iii). To get (i), (vi), combine Lemma 2.2, Lemma 11.3(i), (ii) along with (37), (38), (39). Using (vi), one gets (iv).

For notational convenience define

$$\varpi_i = (\beta + 1)((\theta_i^* - \theta_{i+2}^*)(\theta_{i+1} - \theta_i) + (\theta_{i+1}^* - \theta_{i+2}^*)(\theta_i - \theta_{i+2})) \quad (0 \leq i \leq N - 2).$$

Lemma 11.6. Under the standard assumption, the following hold.

(i) For $0 \leq i \leq N - 2$, the scalar $\varpi_i$ is equal to the expression

$$(\theta_i^* - \theta_{i+2}^*)(\theta_i + \theta_{i+1} + \theta_{i+2} - \gamma) - (\beta + 1)((\theta_{i+1}^* - \theta_{i+2}^*)\theta_{i+2} + (\theta_i^* - \theta_{i+1}^*)\theta_i) \quad (45)$$

that appears in the table from Lemma 11.3(v).

(ii) For $2 \leq i \leq N$, the scalar $-\varpi_{i-2}$ is equal to the expression

$$(\theta_i - \theta_{i-2})(\theta_i^* + \theta_{i-1}^* + \theta_{i-2}^* - \gamma^*) - (\beta + 1)((\theta_{i-1} - \theta_{i-2})\theta_{i-2}^* + (\theta_i - \theta_{i-1})\theta_i^*) \quad (46)$$

that appears in the table from Lemma 11.3(v).

Proof. (i) Evaluate (45) using Lemma 11.5(i), (iii) and simplify the result.

(ii) Evaluate (46) using Lemma 11.5(ii), (iv) and simplify the result.

Lemma 11.7. Under the standard assumption, for $0 \leq i \leq N - 2$, $F_{i+2}BF_i$ is a weighted sum involving the following terms and coefficients:

| term       | coefficient |
|------------|-------------|
| $R^2F_i$  | $-\alpha_i\alpha_{i+1}(q^{N-M-i-2}(q + 1)(\beta + 1)(\alpha_i\alpha_{i+1} - \alpha_{i+2}\alpha_{i+2}) - \varpi_i)$ |
| $R^3LF_i$ | $\alpha_i\alpha_{i+1}(\alpha_{i-1}\alpha_i + q^{-q^2-1}(\beta + 1)\alpha_i\alpha_{i+1} + q^2(q-1)\alpha_{i+1}\alpha_{i+2})$ |
| $LR^3F_i$ | $-\alpha_i\alpha_{i+1}(\frac{3}{q^3-1}(\beta + 1)\alpha_i\alpha_{i+1} - q^2(q-1)(\beta + 1)\alpha_{i+1}\alpha_{i+2} + \alpha_{i+2}\alpha_{i+3})$ |

For $i = 0$, the term $R^3LF_i = 0$ and so the second row of the above table is eliminated. For $i = N - 2$, the term $LR^3F_i = 0$ and so the third row of the above table is eliminated.

Proof. By (12), (13) we obtain

$$R^2LF_i = q^{N+M-i-2}(q + 1)R^2F_i + \frac{q(q^2 - 1)R^3LF_i + (q - 1)L^3F_i}{q^3 - 1}, \quad (47)$$

$$RLR^2F_i = q^{N+M-i-2}(q + 1)R^2F_i + \frac{q^2(q - 1)R^3LF_i + (q^2 - 1)L^3F_i}{q^3 - 1}. \quad (48)$$

Consider the weighted sum description of $F_{i+2}BF_i$ in Lemma 11.2(v). Simplify this description by eliminating $R^2LF_i$ using (47) and $RLR^2F_i$ using (48). Combining this with Lemma 11.6(i), we get the result.

Referring to Lemma 11.7, for $i = 0$ we obtain the following simplification.
Lemma 11.8. Under the standard assumption, the matrix $F_2BF_0$ is equal to $-\alpha_0\alpha_1q^M R^2F_0$ times the scalar

$$
\frac{q^N - 1}{q - 1}(\beta + 1)\alpha_0\alpha_1^* - \frac{q^{N-1} - 1}{q - 1} \frac{q^2 - 1}{q - 1}(\beta + 1)\alpha_1\alpha_2^* + \frac{q^{N-2} - 1}{q - 1} \frac{q^3 - 1}{q - 1}\alpha_2\alpha_3^* - q^{-M}\varnothing_0.
$$

Proof. In Lemma 11.7, set $i = 0$ to obtain $F_2BF_0$ as a weighted sum. In this weighted sum, eliminate $LR^3F_0$ using (31) and simplify the result.

Lemma 11.9. Under the standard assumption, for $2 \leq i \leq N$, $F_{i-2}B^*F_i$ is a weighted sum involving the following terms and coefficients:

| term       | coefficient |
|------------|-------------|
| $L^2F_i$   | $-\alpha_{i-1}^*\alpha_i^*(q^{N+M-i}(\beta + 1)(\alpha_{i-1}^* - \alpha_{i-2}\alpha_{i-1}^*\varnothing) + \varnothing_{i-2})$ |
| $RL^3F_i$  | $-\alpha_{i-1}^*\alpha_i^*(\alpha_{i-3}\alpha_{i-2}^* - \frac{q(q^2-1)}{q-1}(\beta + 1)\alpha_{i-2}\alpha_i^* + \frac{q^3(q-1)}{q-1}(\beta + 1)\alpha_{i-1}\alpha_i^*)$ |
| $L^3RF_i$  | $\alpha_{i-1}^*\alpha_i^*(q^{N-1}(\beta + 1)\alpha_{i-2}\alpha_i^* - \frac{q^2}{q-1}(\beta + 1)\alpha_{i-1}\alpha_i^* + \alpha_i\alpha_{i+1}^*)$ |

For $i = 2$, the term $RL^3F_i = 0$ and so the second row of the above table is eliminated. For $i = N$, the term $L^3RF_i = 0$ and so the third row of the above table is eliminated.

Proof. Similar to the proof of Lemma 11.7.

Referring to Lemma 11.9, for $i = 2$ we obtain the following simplification.

Lemma 11.10. Under the standard assumption, the matrix $F_0B^*F_2$ is equal to $\alpha_1^*\alpha_2^*q^M L^2F_2$ times

$$
\frac{q^N - 1}{q - 1}(\beta + 1)\alpha_0\alpha_1^* - \frac{q^{N-1} - 1}{q - 1} \frac{q^2 - 1}{q - 1}(\beta + 1)\alpha_1\alpha_2^* + \frac{q^{N-2} - 1}{q - 1} \frac{q^3 - 1}{q - 1}\alpha_2\alpha_3^* - q^{-M}\varnothing_0.
$$

Proof. In Lemma 11.9, set $i = 2$ to obtain $F_0B^*F_2$ as a weighted sum. In this weighted sum, eliminate $L^3RF_2$ using (32) and simplify the result.

Lemma 11.11. Assume $B = 0$ and $B^* = 0$. Then the scalar $q^{-M}\varnothing_0$ is equal to

$$
\frac{q^N - 1}{q - 1}(\beta + 1)\alpha_0\alpha_1^* - \frac{q^{N-1} - 1}{q - 1} \frac{q^2 - 1}{q - 1}(\beta + 1)\alpha_1\alpha_2^* + \frac{q^{N-2} - 1}{q - 1} \frac{q^3 - 1}{q - 1}\alpha_2\alpha_3^*.
$$

Proof. Use Lemma 2.2 with $D = B$, along with Lemma 11.8.

Lemma 11.12. Assume $B = 0$ and $B^* = 0$. Then for $1 \leq i \leq N - 2$,

$$
q^{N+i-2}(\beta + 1)(\xi_i - \xi_{i+1}) - \varnothing_i = 0, \quad (49)
$$

$$
\xi_{i-1} - \frac{q(q^2 - 1)}{q^3 - 1}(\beta + 1)\xi_i + \frac{q^3(q - 1)}{q^3 - 1}(\beta + 1)\xi_{i+1} = 0. \quad (50)
$$

Moreover for $1 \leq i \leq N - 3$,

$$
\frac{q - 1}{q^3 - 1}(\beta + 1)\xi_i - \frac{q^2 - 1}{q^3 - 1}(\beta + 1)\xi_{i+1} + \xi_{i+2} = 0. \quad (51)
$$
Proof. First assume that \( 1 \leq i \leq N - 3 \). In the table of Lemma 11.7, each coefficient is zero by Lemma 2.2 and Lemma 9.5(i). This gives (49)–(51). Next assume that \( i = N - 2 \). In the table of Lemma 11.7, the term \( LR^3F_i = 0 \). So the third equation of the table is eliminated. In the remaining two rows, each coefficient is zero by Lemma 2.2 and Lemma 9.4(i). This gives (49), (50).

Next we simplify the equations (49)–(51).

Lemma 11.13. Assume \( B = 0 \) and \( B^* = 0 \). Then for \( 1 \leq i \leq N - 3 \),

(i) \( \xi_{i-1} - (\beta + 1)\xi_i + (\beta + 1)\xi_{i+1} - \xi_{i+2} = 0 \),

(ii) \( q^{-1}\xi_{i-1} - (\beta + 1)\xi_{i+1} + (q + 1)\xi_{i+2} = 0 \),

(iii) \( \xi_{i-1} - \xi_{i+2} - (q + 1)^{-1}q^{2+i-N-M}\triangle_i = 0 \).

Proof. (i) Combine (50), (51).
(ii) Combine (50), (51).
(iii) By (49) and (i).

Lemma 11.14. Assume \( B = 0 \) and \( B^* = 0 \). Then the sequences \( \{\theta_i\}_{i=0}^N \), \( \{\theta_i^*\}_{i=0}^N \), \( \{\xi_i\}_{i=0}^{N-1} \) are all \( \beta \)-recurrent.

Proof. By Lemma 11.5, \( \{\theta_i\}_{i=0}^N \) and \( \{\theta_i^*\}_{i=0}^N \) are \( \beta \)-recurrent. By Lemma 11.13(i), \( \{\xi_i\}_{i=0}^{N-1} \) is \( \beta \)-recurrent.

Lemma 11.15. Assume \( B = 0 \) and \( B^* = 0 \). Then \( \beta \neq 2 \) and \( \beta \neq -2 \).

Proof. First we show \( \beta \neq 2 \). Suppose on the contrary that \( \beta = 2 \). Then by Lemma 11.13(ii) we find that for \( 1 \leq i \leq N - 3 \),

\[
q^{-1}\xi_{i-1} - 3\xi_{i+1} + (q + 1)\xi_{i+2} = 0.
\]

(52)

By Lemma 10.8(i) there exist scalars \( a, b, c \in \mathbb{C} \) such that \( \xi_i = a + bi + ci^2 \) for \( 0 \leq i \leq N - 1 \). Evaluating (52) using this, we find that for \( 1 \leq i \leq N - 3 \), 0 is a weighted sum involving the following terms and coefficients:

| term | coefficient |
|------|-------------|
| \( i^2 \) | \((q - 1)^2c\) |
| \( i \) | \( b - 2c - 3q(b + 2c) + q(q + 1)(b + 4c) \) |
| 1 | \( a - b + c - 3q(a + b + c) + q(q + 1)(a + 2b + 4c) \) |

Since \( N \geq 6 \), in the above table each coefficient is 0. By construction, \( q \) is an integer at least 2. Then \( c = 0 \) by the first row of the table. Combining this with the second row of the table we get \( (q - 1)^2b = 0 \), so \( b = 0 \). Combining \( b = 0 \) and \( c = 0 \) with the third row of the table we get \( (q - 1)^2a = 0 \), so \( a = 0 \). Hence \( \xi_i = 0 \) for \( 0 \leq i \leq N - 1 \). This contradicts the line below (40), so \( \beta \neq 2 \).

Now we show \( \beta \neq -2 \). Suppose on the contrary that \( \beta = -2 \). Then by Lemma 11.13(ii) we find that for \( 1 \leq i \leq N - 3 \),

\[
q^{-1}\xi_{i-1} + \xi_{i+1} + (q + 1)\xi_{i+2} = 0.
\]

(53)
By Lemma 10.8(ii) there exist scalars $a, b, c \in \mathbb{C}$ such that $\xi_i = a + b(-1)^i + ci(-1)^i$ for $0 \leq i \leq N - 1$. Evaluating (53) using this, we find that for $1 \leq i \leq N - 3$, 0 is a weighted sum involving the following terms and coefficients:

| term    | coefficient |
|---------|-------------|
| $i(-1)^i$ | $(q^2 - 1)c$ |
| $(-1)^i$  | $(q^2 - 1)b + (1 + q + 2q^2)c$ |
| 1         | $(1 + q)^2a$ |

Since $N \geq 6$, in the above table each coefficient is 0. By construction, $q$ is an integer at least 2. Then $c = 0$ by the first row of the table and $a = 0$ by the third row of the table. Combining $c = 0$ with the second row of the table we get $(1 - q^2)b = 0$, so $b = 0$. Hence $\xi_i = 0$ for $0 \leq i \leq N - 1$. This contradicts the line below (40), so $\beta \neq -2$.

**Definition 11.16.** Fix a nonzero $Q \in \mathbb{C}$ such that $\beta = Q + Q^{-1}$.

Assume $B = 0$ and $B^* = 0$. Then the scalar $Q$ in Definition 11.16 is not equal to $\pm 1$ by Lemma 11.15.

**Lemma 11.17.** Assume $B = 0$ and $B^* = 0$. Then there exist scalars $a, b, c, a^*, b^*, c^*$ in $\mathbb{C}$ such that for $0 \leq i \leq N$,

$$\theta_i = a + bQ^i + cQ^{-i},$$

$$\theta_i^* = a^* + b^*Q^i + c^*Q^{-i}.\quad (54)$$

Also, there exist scalars $x, y, z$ in $\mathbb{C}$ such that for $0 \leq i \leq N - 1$,

$$\xi_i = x + yQ^i + zQ^{-i}.\quad (56)$$

The scalar $Q$ is from Definition 11.16.

**Proof.** Combine Lemmas 10.8, 11.14, 11.15, Definition 11.16.

**Note 11.18.** Assume $B = 0$ and $B^* = 0$. Then referring to Lemma 11.17, the scalars $b, c$ are not both zero by (39). Similarly, the scalars $b^*, c^*$ are not both zero.

**Note 11.19.** Assume $B = 0$ and $B^* = 0$. Then referring to Lemma 11.17, by (39) we have that $Q^i \neq 1$ for $1 \leq i \leq N$.

**Lemma 11.20.** Assume $B = 0$ and $B^* = 0$. Then the scalars $\beta, \gamma, \gamma^*, \varrho, \varrho^*$ from above (11) are given by $\beta = Q + Q^{-1}$ and

$$\gamma = -Q^{-1}(Q - 1)^2a,$$

$$\gamma^* = -Q^{-1}(Q - 1)^2a^*,$$

$$\varrho = Q^{-1}(Q - 1)^2a^2 - (Q - Q^{-1})^2bc,$$

$$\varrho^* = Q^{-1}(Q - 1)^2a'^2 - (Q - Q^{-1})^2b'^2c'^*.$$

The scalar $Q$ is from Definition 11.16 and the scalars $a, b, c, a^*, b^*, c^*$ are from Lemma 11.17.

**Proof.** Use Lemma 11.5 and (54), (55).

**Lemma 11.21.** Assume $B = 0$ and $B^* = 0$. Then $y = 0$ or $z = 0$.
Proof. Evaluate Lemma 11.13(ii) using (56). Then for $1 \leq i \leq N - 3$, $0$ is equal to a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $1$      | $q^{-1}(1 - qQ)(1 - qQ^{-1})x$ |
| $Q^i$    | $(q^{-1}Q^{-1} - Q)(1 - qQ)y$ |
| $Q^{-i}$ | $(q^{-1}Q - Q^{-1})(1 - qQ^{-1})z$ |

Recall that $Q \neq \pm 1$ and $N \geq 6$. Therefore, in the above table each coefficient is $0$. In other words,

\begin{align*}
(1 - qQ)(1 - qQ^{-1})x &= 0, \tag{57} \\
(q^{-1}Q^{-1} - Q)(1 - qQ)y &= 0, \tag{58} \\
(q^{-1}Q - Q^{-1})(1 - qQ^{-1})z &= 0. \tag{59}
\end{align*}

We now show that $yz = 0$. Suppose not. Then both

\begin{align*}
(q^{-1}Q^{-1} - Q)(1 - qQ) &= 0, \tag{60} \\
(q^{-1}Q - Q^{-1})(1 - qQ^{-1}) &= 0. \tag{61}
\end{align*}

By (60), one gets $Q = q^{-1}$ or $Q^2 = q^{-1}$. By (61), one gets $Q = q$ or $Q^2 = q$. This contradicts Note 11.19. Therefore $yz = 0$ and the result follows. \hfill \square

We now investigate the cases in Lemma 11.21. In what follows, we refer to the scalars in Definition 11.16 and Lemma 11.17.

Lemma 11.22. Assume $B = 0$ and $B^* = 0$.

Case I: Assume $y = 0$ and $z \neq 0$. Then

$$Q = q, \quad bb^* = 0, \quad z = -q^{-1-N-M}(q - 1)^2cc^*, \quad c \neq 0, \quad c^* \neq 0.$$  

Case II: Assume $y \neq 0$ and $z = 0$. Then

$$Q = q^{-1}, \quad cc^* = 0, \quad y = -q^{-1-N-M}(q - 1)^2bb^*, \quad b \neq 0, \quad b^* \neq 0.$$  

Case III: Assume $y = 0$ and $z = 0$. Then

$$Q \in \{q, q^{-1}\}, \quad x \neq 0, \quad bb^* = 0, \quad cc^* = 0.$$  

Proof. Evaluate Lemma 11.13(iii) using (54), (55), (56). Then for $1 \leq i \leq N - 3$, $0$ is equal to a weighted sum involving the following terms and coefficients:

| term     | coefficient |
|----------|-------------|
| $Q^i$    | $Q^{-1}(1 - Q^3)y$ |
| $Q^{-i}$ | $Q(1 - Q^{-3})z$ |
| $(qQ^2)^i$ | $-(q + 1)^{-1}q^{2-N-M}(Q^3 - 1)(Q^2 - 1)(Q - 1)bb^*$ |
| $(qQ^{-2})^i$ | $-(q + 1)^{-1}q^{2-N-M}(Q^{-3} - 1)(Q^{-2} - 1)(Q^{-1} - 1)cc^*$ |
We now consider the cases.

Case I: Using (59) one gets \( Q = q \) or \( Q^2 = q \). We claim that \( Q = q \). Otherwise, we have \( qQ^2 = Q^4 \) and \( qQ^{-2} = 1 \). Then by Note 11.19 the scalars \( Q^{-1}, qQ^2, qQ^{-2} \) are mutually distinct. So in the above table the coefficient of each term is zero. In particular, \( Q(1 - Q^{-3})z = 0 \). By Note 11.19 again, one gets \( z = 0 \), a contradiction. We have shown that \( Q = q \). Using this fact, we simplify the above table, and find that 0 is equal to a weighted sum involving the following terms and coefficients:

| term          | coefficient                                           |
|---------------|-------------------------------------------------------|
| \( q^{-1} \)  | \( q(1 - q^{-3})z - (q + 1)^{-1}q^{1-N-M}(q^{-3} - 1)(q^{-2} - 1)(q^{-1} - 1)cc^* \) |
| \( q^{3i} \)  | \( -(q + 1)^{-1}q^{1-N-M}(q^3 - 1)(q^2 - 1)(q - 1)bb^* \) |

In this table the coefficient of each term is zero. Therefore,

\[
q(1 - q^{-3})z - (q + 1)^{-1}q^{3-N-M}(q^{-3} - 1)(q^{-2} - 1)(q^{-1} - 1)cc^* = 0, \tag{62}
\]

\[
-(q + 1)^{-1}q^{1-N-M}(q^3 - 1)(q^2 - 1)(q - 1)bb^* = 0. \tag{63}
\]

Using (63), we obtain \( bb^* = 0 \). Using (62), we obtain \( z = -q^{-1-N-M}(q - 1)^2cc^* \).

Case II: Similar to the proof of Case I.

Case III: By (56) and since \( \xi_i \neq 0 \) for \( 0 \leq i \leq N - 1 \), we find \( x \neq 0 \). Combine this with (57) to get \( Q = q \) or \( Q = q^{-1} \). Now \( Q^2 \neq Q^{-2} \). Moreover, for \( 1 \leq i \leq N - 3 \), 0 is equal to a weighted sum involving the following terms and coefficients:

| term          | coefficient                                           |
|---------------|-------------------------------------------------------|
| \( (qQ^2)^i \) | \( -(q + 1)^{-1}q^{2-N-M}Q(Q^{-3} - 1)(Q^{-2} - 1)(Q^{-1} - 1)cc^* \) |
| \( (qQ^{-2})^i \) | \( -(q + 1)^{-1}q^{2-N-M}Q^{-1}(Q^3 - 1)(Q^2 - 1)(Q - 1)bb^* \) |

In this table the coefficient of each term is zero. Therefore \( bb^* = 0 \) and \( cc^* = 0 \).

**Note 11.23.** Referring to Definition 11.16 the scalar \( Q \) is defined up to inverse. In Case III, after replacing \( Q \) by \( Q^{-1} \) if necessary, we will assume \( Q = q \).

**Definition 11.24.** Consider Cases I, II, III in Lemma 11.22

We partition Case I into subcases:

Case I+:

\[ b \neq 0, \quad b^* = 0; \]

Case I−:

\[ b = 0, \quad b^* \neq 0; \]

Case I0:

\[ b = 0, \quad b^* = 0. \]

We partition Case II into subcases:

Case II+:

\[ c \neq 0, \quad c^* = 0; \]

Case II−:

\[ c = 0, \quad c^* \neq 0; \]

Case II0:

\[ c = 0, \quad c^* = 0. \]

We partition Case III into subcases:

Case III+:

\[ b \neq 0, \quad b^* = 0, \quad c = 0, \quad c^* \neq 0, \quad Q = q; \]
Case III⁻: \( b = 0, \quad b^* \neq 0, \quad c \neq 0, \quad c^* = 0, \quad Q = q. \)

**Theorem 11.25.** Assume \( B = 0 \) and \( B^* = 0 \). Then \( \{\xi_i\}^{N-1}_{i=0}, \{\theta_i\}^N_{i=0}, \{\theta_i^*\}^N_{i=0} \) are given in the table below:

| Case | \( \xi_i \) | \( \theta_i \) | \( \theta_i^* \) |
|------|-------------|-------------|-------------|
| I⁺   | \( x - q^{-1}N^{-1}(q-1)^2cc^* \) | \( a + bq^i + cq^{-i} \) | \( a^* + c^*q^{-i} \) |
| I⁻   | \( x - q^{-1}N^{-1-i}(q-1)^2cc^* \) | \( a + cq^{-i} \) | \( a^* + b^*q^i + c^*q^{-i} \) |
| I⁰   | \( x - q^{-1}N^{-1-M-i}(q-1)^2cc^* \) | \( a + cq^{-i} \) | \( a^* + c^*q^{-i} \) |
| II⁺  | \( x - q^{-1}N^{-1-M-i}(q-1)^2bb^* \) | \( a + bq^{-i} + cq^{i} \) | \( a^* + b^*q^{-1} \) |
| II⁻  | \( x - q^{-1}N^{-1-M-i}(q-1)^2bb^* \) | \( a + bq^{-i} \) | \( a^* + b^*q^{-i} \) |
| II⁰  | \( x - q^{-1}N^{-1-M-i}(q-1)^2bb^* \) | \( a + bq^{-i} \) | \( a^* + b^*q^{-i} \) |
| III⁺ | \( x \) | \( a + bq^i \) | \( a^* + c^*q^{-i} \) |
| III⁻ | \( x \) | \( a + cq^{-i} \) | \( a^* + b^*q^i \) |

**Proof.** By Lemmas [11.17][11.22], Definition [11.24] and [39].

We are done assuming \( B = 0 \) and \( B^* = 0 \). We now go in the other logical direction.

**Theorem 11.26.** Assume \( \{\xi_i\}^{N-1}_{i=0}, \{\theta_i\}^N_{i=0}, \{\theta_i^*\}^N_{i=0} \) satisfy one of the cases I⁺, I⁻, I⁰, II⁺, II⁻, II⁰, III⁺, III⁻ from Theorem 11.25. Define \( \beta, \gamma, \gamma^*, \varrho, \varrho^* \) using Lemma 11.5. Then referring to (11), (42) we have \( B = 0 \) and \( B^* = 0 \).

**Proof.** For \( 0 \leq i, j \leq N \) the matrix \( F_jBF_i = 0 \) for the following reason:

| case | reason |
|------|--------|
| \( i - j > 1 \) | Lemma [11.1] |
| \( i - j = 1 \) | Lemma [11.2(ii)] |
| \( i - j = 0 \) | Lemma [11.2(iv)] |
| \( i - j = -1 \) | Lemma [11.2(iii)] |
| \( i - j = -2 \) and \( i = 0 \) | Lemma [11.8] |
| \( i - j = -2 \) and \( 1 \leq i \leq N - 2 \) | Lemma [11.7] |
| \( i - j = -3 \) | Lemma [11.2(i)] |
| \( i - j < -3 \) | Lemma [11.1] |

Now \( B = 0 \) by Lemma [2.2].

For \( 0 \leq i, j \leq N \) the matrix \( F_jB^*F_i = 0 \) for the following reason:

| case | reason |
|------|--------|
| \( i - j < -1 \) | Lemma [11.1] |
| \( i - j = -1 \) | Lemma [11.3(ii)] |
| \( i - j = 0 \) | Lemma [11.3(iv)] |
| \( i - j = 1 \) | Lemma [11.3(iii)] |
| \( i - j = 2 \) and \( i = 2 \) | Lemma [11.10] |
| \( i - j = 2 \) and \( 3 \leq i \leq N \) | Lemma [11.9] |
| \( i - j = 3 \) | Lemma [11.3(i)] |
| \( i - j > 3 \) | Lemma [11.1] |

Now \( B^* = 0 \) by Lemma [2.2].
12 Leonard pairs based on $A_q(N, M)$

In this section we obtain some Leonard pairs from $A_q(N, M)$. Define $A, A^*$ as in \(35\), \(36\). Assume $\{\xi_i\}_{i=0}^{N-1}$, $\{\theta_i\}_{i=0}^{N-1}$ are from Theorem \(11.25\). Let $W$ denote an irreducible $T$-module with endpoint $r$ and diameter $d$.

**Theorem 12.1.** The elements $A, A^*$ act on $W$ as a Leonard pair if and only if

\[
x \neq q^{i+d-N-M}(q-1)^2(bc^* + cb^*)q^{-i} \hspace{1cm} (1 \leq i \leq d).
\]

**Proof.** Recall the basis $\{w_i\}_{i=0}^{d}$ for $W$ in Lemma \(5.5\) With respect to this basis the matrix representing $A$ is

\[
A : \begin{pmatrix}
\theta_r & \theta_{r+1} \\
\alpha_{r+1} & \theta_{r+2}
\end{pmatrix}
\]

and the matrix representing $A^*$ is

\[
A^* : \begin{pmatrix}
\theta^*_r & \alpha_{r+1}^* x_{r+1}(r, d) \\
0 & \alpha_{r+1}^* x_{r+2}(r, d)
\end{pmatrix}
\]

Referring to Definition \(5.3\), set

\[
\varphi_i = \xi_{r+i-1} x_{r+i}(r, d) \hspace{1cm} (1 \leq i \leq d).
\]

Then $\varphi_i \neq 0$ for $1 \leq i \leq d$ by Note \(5.4\) and \(37\), \(38\). Also, set

\[
\phi_i = \varphi_1 \sum_{h=0}^{i-1} \frac{\theta_{r+h} - \theta_{r+d-h}}{\theta_{r} - \theta_{r+d}} + (\theta^*_{r+i} - \theta^*_{r})(\theta_{r+d-i+1} - \theta_r) \hspace{1cm} (1 \leq i \leq d).
\]

Referring to the cases from Theorem \(11.25\) we have

| Case | $\phi_i$ |
|------|----------|
| I$^+$ | $q^{N-M-r-d}(q-1)x - bc^* q^{-i}(q^i - 1)(q^{d-i+1} - 1)$ |
| I$^-$ | $q^{N-M-r-d}(q-1)x - cb^* q^{-i}(q^i - 1)(q^{d-i+1} - 1)$ |
| I$^0$ | $q^{N-M-r-d}(q-1)^2 x - bc^* q^{-i}(q^i - 1)(q^{d-i+1} - 1)$ |
| II$^+$ | $q^{N-M-r-d}(q-1)x - bc^* q^{-i}(q^i - 1)(q^{d-i+1} - 1)$ |
| II$^-$ | $q^{N-M-r-d}(q-1)x - cb^* q^{-i}(q^i - 1)(q^{d-i+1} - 1)$ |
| II$^0$ | $q^{N-M-r-d}(q-1)^2 x - bc^* q^{-i}(q^i - 1)(q^{d-i+1} - 1)$ |
| III$^+$ | $q^{N-M-r-d}(q-1)x - bc^* q^{-i}(q^i - 1)(q^{d-i+1} - 1)$ |
| III$^-$ | $q^{N-M-r-d}(q-1)x - cb^* q^{-i}(q^i - 1)(q^{d-i+1} - 1)$ |

24
Using (64) we find that in each case, $\phi_i \neq 0$ for $1 \leq i \leq d$.

In each case, it is routine to check using Definition 10.4 that
$(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, \{\varphi_j\}_{j=1}^d, \{\phi_j\}_{j=1}^d)$
is a parameter array. By Lemma 10.5 the pair $A, A^*$ acts on $W$ as a Leonard pair.

\section*{13 Directions for further research}

In this section we mention some open problems.

\textbf{Problem 13.1.} Describe $\Phi, \Omega$ in terms of $C_1, C_2$.

\textbf{Problem 13.2.} Describe $\Phi, \Omega$ in terms of $R, L, K, K^{-1}$.

\textbf{Problem 13.3.} Find a combinatorial interpretation of $\Phi, \Omega$.

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