A CALDERO-CHAPOTON FORMULA FOR GENERALIZED CLUSTER CATEGORIES

SALOMÓN DOMÍNGUEZ AND CHRISTOF GEISS

1. Introduction

The aim of this note is to extend the range of a formula due to Caldero and Chapoton [1, Prp. 3.10] from module categories over hereditary algebras of finite type to triangulated 2-Calabi-Yau categories with a cluster tilting object. In view of the work of Palu [6] it appeared natural to expect such a formula in this context. However, the proof of the original statement does not carry over directly. We had to reorganize the proof of the key statement in our Lemma 2.1 and to review the relevant results from [6].

Theorem. Let $\mathcal{C}$ be a triangulated 2-Calabi-Yau $\mathbb{C}$-category with suspension functor $\Sigma$ and basic cluster tilting object $T = T_1 \oplus \ldots \oplus T_n$. For an Auslander-Reiten triangle $\Sigma Z \to Y \to Z \to \Sigma^2 Z$ in $\mathcal{C}$ we have

$$C^T_{\Sigma Z} \cdot C^T_Z = C^T_Y + 1,$$

where $C^T_Y : \text{Obj}(\mathcal{C}) \to \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$ denotes Palu’s cluster character [6, Sec. 1].

Notice that in case $\dim \mathcal{C}(Z, \Sigma^2 Z) = 1 = \dim \mathcal{C}(Z, Z)$ our result follows from Palu’s multiplication formula [7], otherwise our result looks rather surprising in view of the complexity of this multiplication formula.

Our formula reduces in many cases the complexity of explicit cluster character computations: let $Z$ be an (indecomposable) object in an Auslander-Reiten component which is of type $\mathbb{Z}A_\infty$ or a tube in a cluster category. Based on the formula of Caldero and Chapoton, Dupont pointed out how to express $X^T_Z$ in terms of certain Chebyshev polynomials which take as arguments the values of $X^T_Y$ on objects in the “mouth” of the corresponding component in case $\Lambda := \text{End}_\mathcal{C}(T)^{\text{op}}$, see for example [2, Thm. 5.1]. Our result shows, that formulas of this kind hold without any restriction on $\mathcal{C}$ or $T$.

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1.1. Preliminaries. Let $k$ be an algebraically closed field, and $\mathcal{C}$ be a triangulated 2-Calabi-Yau $k$-category with suspension functor $\Sigma$. This means that $\mathcal{C}$ admits a Serre functor which we may identify for our purpose with $\Sigma^2$. Thus, at each indecomposable object $Z \in \mathcal{C}$ there ends and Auslander-Reiten triangle, and the Auslander-Reiten translate $\tau$ may be identified with $\Sigma$, see for example [8, I.2].

Suppose that $\mathcal{C}$ has a basic cluster tilting object $T = T_1 \oplus \cdots \oplus T_n$ and set $\Lambda := \text{End}_C(T)^{op}$. We consider the functor $E := \mathcal{C}(T, \Sigma^{-}): \mathcal{C} \to \Lambda\text{-mod}$. Recall that $E$ induces an equivalence of $k$-categories $\mathcal{C}/(\text{add}(T)) \sim \Lambda\text{-mod}$, see [5] Prp. 2.1. Moreover, each exact sequence in $\Lambda\text{-mod}$ can be “lifted” to an distinguished triangle in $\mathcal{C}$ by [G] Lemma 3.1.

For each $X \in \mathcal{C}$ there exists an distinguished triangle

$$
\oplus_{i=1}^n T_{i}^{m(i,X)} \to \oplus_{i=1}^n T_{i}^p(i,X) \to X \to \Sigma \left( \oplus_{i=1}^n T_{i}^{m(i,X)} \right)
$$

and $\text{ind}_T(X) := (p(i,X) - m(i,X))_{i=1,\ldots,n} \in \mathbb{Z}^n$ is well-defined, see [G, Sec. 2.1].

Following Palu [G], we have in case $k = \mathbb{C}$ a cluster character

$$C_T^\tau: \text{Obj}(\mathcal{C}) \to \mathbb{Q}[x_1^\pm, \ldots, x_n^\pm], \ X \mapsto x_1^{\text{ind}_T(X)} \sum_{e} \chi(\text{Gr}_e(EX)) x_1^{B_T \cdot e},$$

which naturally extends the expression introduced by Caldero and Chapoton in [11, Sec. 3.1]. Here, $\text{Gr}_e(EX)$ denotes the quiver grassmanian of $\Lambda$-submodules of $EX$ with dimension vector $e$ and $\chi$ is the topological Euler characteristic. Finally, $B_T \in \mathbb{Z}^{n \times n}$ is the matrix with entries

$$(B_T)_{i,j} := \dim \text{Ext}_\Lambda^1(S_i, S_j) - \dim \text{Ext}_\Lambda^1(S_j, S_i),$$

where $S_i := E\Sigma^{-1}T_i / \text{rad } E\Sigma^{-1}T_i$ for $i = 1, \ldots, n$ represents the simple $\Lambda$-modules. Observe, that this is indeed Palu’s cluster character pre-composed with the suspension $\Sigma$.

1.2. Example. Let $\Lambda$ be a finite dimensional basic $\mathbb{C}$-algebra as above. Set

$$g_M := (\dim \text{Ext}_\Lambda^1(S_i, M) - \dim \text{Hom}_\Lambda(S_i, M))_{i=1,\ldots,n}$$

and

$$C'_M := \sum_{a} [F_M(\hat{a}) \cdot \mathbb{Z}^m] = \sum_{a} [F_M(\hat{a}) \cdot \mathbb{Z}^m].$$

see Definition 2.2 and Section 2.6 below for the missing definitions. For $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$ set $T^\mathbf{m} := T_{1}^{m_1} \oplus \cdots \oplus T_{n}^{m_n}$. If $Z = Z' \oplus T^\mathbf{m}$ with $Z'$ having no direct summand in $\text{add}(T)$ we see that $C_Z^\tau = C'_Z \cdot \mathbb{Z}^m$.

Let $0 \to \tau M \to N \to M \to 0$ be an Auslander-Reiten sequence in $\Lambda\text{-mod}$, and $Z \in \mathcal{C}$ indecomposable with $EZ = M$. Then, for the Auslander-Reiten triangle $\Sigma Z \to Y \to Z \to \Sigma^2 Z$ in $\mathcal{C}$ we have $EY = N$ and $E\Sigma Z = \tau M$. Thus, if $Y$ has a non-trivial direct summand from $\text{add}(T)$, we do not have $X'_M X_M = X'_M + 1$. An example of this behaviour can be found for $T$ being a non-acyclic cluster tilting object in a cluster category of type $A_3$ and $M$ a simple $\Lambda = \text{End}_C(T)^{op}$-module.
2. Proof of the Theorem

The following key result and the strategy for its proof are essentially due to Caldero and Chapoton [1, Lemma 3.11]. We review their proof, since we need the result for arbitrary finite dimensional $k$-algebras, rather than only for hereditary algebras of finite type. The only digression from their argument is at the beginning of step (3) of the proof below, since we cannot assume in general that $\dim \text{Ext}^1_{\Lambda}(M, \tau M) = 1$ for an indecomposable module $M$.

2.1. Lemma. Let $\Lambda$ be a finite dimensional basic $k$-algebra and $0 \to L \to M \xrightarrow{\pi} N \to 0$ an Auslander-Reiten sequence in $\Lambda$-mod. Consider the morphism of projective varieties
\[
\xi_g : \text{Gr}_g(M) \to \bigsqcup_{e+f=g} \text{Gr}_e(L) \times \text{Gr}_f(N), U \mapsto (i^{-1}(U), \pi(U)).
\]
Then the fiber $\xi^{-1}_g(A, C)$ is an affine space isomorphic to $\text{Hom}_{\Lambda}(C, L/A)$ if $(A, C) \neq (0, N)$, and $\xi^{-1}_g(0, N) = \emptyset$.

Proof. (1) Suppose first we had $B \in \xi^{-1}(0, N)$. Then $B \cong N$ and we would obtain a section to $\pi$, a contradiction. Thus $\xi^{-1}_g(0, N) = 0$.

(2) Now, let $C \subset N$ be a proper submodule and denote by $\gamma : C \hookrightarrow N$ the inclusion. We obtain the following commutative diagram with exact rows:

\[
\begin{array}{ccccccccc}
0 & \to & L & \xrightarrow{i} & M' & \xrightarrow{\pi'} & C & \to & 0 \\
0 & \to & L & \xrightarrow{i} & M & \xrightarrow{\pi} & N & \to & 0 \\
\end{array}
\]

The first row splits since $\gamma$ factors over $\pi$. Since $\gamma'$ is injective by the snake lemma, it induces an isomorphism between
\[
\text{Gr}_{A,C}(M') := \{B \in \text{Gr}_g(M') \mid i^{-1}(B) = A \text{ and } \pi'(B) = C\}
\]
and $\xi^{-1}_g(A, C)$. Thus, in this case we are reduced to the corresponding statement in the easy case of a split exact sequence, see for example [1, Lemma 3.8]. Note, that the proof of this Lemma does not depend on any particular property of $\Lambda$.

(3) Finally, let $C = N$ and $A \neq 0$. Denote by $\alpha : A \hookrightarrow L$ the inclusion, and by $\Phi := [0 \to L \to M \to N \to 0] \in \text{Ext}^1_{\Lambda}(N, L) \setminus \{0\}$ the class of our Auslander-Reiten sequence. We claim that there exists $\Psi \in \text{Ext}^1_{\Lambda}(N, A)$ with $\text{Ext}_{\Lambda}^1(N, \alpha)(\Psi) = \Phi$, i.e. we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & A & \xrightarrow{\nu} & M' & \xrightarrow{\nu'} & N & \to & 0 \\
0 & \to & L & \xrightarrow{i} & M & \xrightarrow{\pi} & N & \to & 0 \\
\end{array}
\]
with exact (non-split) rows. The class of the top row is then $\Psi \neq 0$. Indeed, consider the short exact sequence $0 \to A \overset{\alpha}{\to} L \overset{\beta}{\to} L/A \to 0$; since $\Phi$ is an almost split sequence and the projection $\beta: L \to L/A$ is not a section, we have

$$\Phi \in \text{Ker}(\text{Ext}^1_\Lambda(N, \beta)) = \text{Im}(\text{Ext}^1_\Lambda(N, \alpha)).$$

Note that $\alpha'$ is injective by the snake lemma. In particular, $\text{Im}(\alpha') \in \xi_g^{-1}(A, N)$ for $g = \dim A + \dim N$. Now we can argue as [1] in the end of the proof of Lemma 3.11, where the authors show that

$$\text{Hom}_\Lambda(N, L/A) \to \xi_g^{-1}(A, N),$$

$$\varphi \mapsto \{\iota(l) + \alpha'(m) \mid (l, m) \in L \times M', \beta(l) = \varphi(\nu(m))\}$$

is the required bijection. Note, that this elementary argument does not require $\Lambda$ to be hereditary or of finite representation type. \qed

2.2. Definition. Let $\Lambda$ be a finite dimensional basic $\mathbb{C}$-algebra. For a $\Lambda$-module $M$ we define the $F$-polynomial to be the generating function for the Euler characteristic of all possible quiver grassmanians, i.e.

$$F_M := \sum_e \chi(\text{Gr}_e(M))y^e \in \mathbb{Z}[y_1, \ldots, y_n]$$

where the sum runs over all possible dimension vectors of submodules of $M$. Moreover, we assume that $S_1, \ldots, S_n$ is a complete system of representatives of the simple $\Lambda$-modules, and we identify the classes $[S_i] \in K_0(\Lambda)$ with the natural basis of $\mathbb{Z}^n$.

2.3. Proposition. Let $\Lambda$ be a finite dimensional basic $\mathbb{C}$-algebra. Then the following holds:

(a) If $0 \to L \overset{\iota}{\to} M \overset{\pi}{\to} N \to 0$ is an Auslander-Reiten sequence in $\Lambda$-mod, then

$$F_L \cdot F_N = F_M + \xi_g^{\dim N}.$$

(b) For the indecomposable projective $\Lambda$-module $P_i$ with top $S_i$ we have

$$F_{P_i} = F_{\text{rad} P_i} + \xi_g^{\dim P_i}$$

for $i = 1, \ldots, n$.

(c) For the indecomposable injective $\Lambda$-module $I_j$ module with socle $S_j$ we have

$$F_{I_j} = y_j \cdot F_{I_j/s_j} + 1$$

for $j = 1, 2, \ldots, n$.

Proof. (a) In view of the well-known properties of Euler characteristics, see for example [3, 7.4], this is a direct consequence of Lemma 2.1 and the definition of $F$-polynomials.

Statements (b) and (c) are easy. \qed
2.4. Remarks. Let $\mathcal{C}$ be a triangulated 2-Calabi-Yau category with cluster tilting object $T$ and $\Lambda := \text{End}_\Lambda(T)^{\text{op}}$ as in the introduction. For an Auslander-Reiten triangle $\Sigma Z \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma^2 Z$ in $\mathcal{C}$, clearly $Z$ is indecomposable. If we apply the functor $E := \mathcal{C}(T, \Sigma -)$ the following three cases can occur:

(a) If $Z \notin \text{add}(T \oplus \Sigma^{-1}T)$ the sequence
\[ 0 \to E\Sigma Z \xrightarrow{E\alpha} EY \xrightarrow{E\beta} EZ \to 0 \]

is an Auslander-Reiten sequence in $\Lambda$-mod.

(b) If $Z \cong \Sigma^{-1}T_i$ for some $i$, then
\[ EY \cong \text{rad} P_i, \quad EZ \cong P_i \quad \text{(and } E\Sigma Z = 0) \]

(c) If $Z \cong T_j$ for some $j$, then
\[ E\Sigma Z \cong I_j, \quad EY \cong I_j/S_j \quad \text{(and } EZ = 0) \]

In particular, always one of the situations of Proposition 2.3 applies. We leave the easy proof as an exercise.

2.5. Proposition. Let $\mathcal{C}$ be a triangulated 2-Calabi-Yau $k$-category with a basic cluster tilting object $T = T_1 \oplus \cdots \oplus T_n$.

(a) For each object $Z \in \mathcal{C}$ we have
\[ (2.1) \quad -B_T \cdot \dim_\Lambda(EZ) = \text{ind}_T(\Sigma Z) + \text{ind}_T(Z). \]

(b) If $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is a distinguished triangle in $\mathcal{C}$, then
\[ (2.2) \quad \text{ind}_T(Y) = \text{ind}_T(X) + \text{ind}_T(Z) + B_T \cdot \dim_\Lambda(\text{Ker}(E\alpha)). \]

(c) If $\Sigma Z \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma^2 Z$ is an Auslander-Reiten triangle then moreover
\[ (2.3) \quad \text{ind}_T(Y) = \begin{cases} 
\text{ind}_T(\Sigma Z) + \text{ind}_T(Z) & \text{if } Z \notin \text{add}(T), \\
B_T \cdot \dim_\Lambda S_i & \text{if } Z \cong T_i 
\end{cases} \]

holds.

Proof. (a) We note first that we have $\text{ind}_T(\Sigma Z) + \text{ind}_T(Z) = \text{ind}_T(\Sigma Z) - \text{coind}_T(\Sigma Z)$ by [6, Lemma 2.1 (1)]. Next, by [6, Lemma 2.3] and [6, Theorem 3.3] together with our definition of $B_T$ we obtain
\[ \text{ind}_T(\Sigma Z) - \text{coind}_T(\Sigma Z) = B_T \cdot \dim_\Lambda EZ. \]

Note that [6, Lemma 2.3] is stated for indecomposable objects in $\mathcal{C}$. However, for each indecomposable summand of $\Sigma Z$ we obtain the correct difference.

(b) Choose $C \in \mathcal{C}$ with $\mathcal{C}(T, C) \cong \text{Coker} \mathcal{C}(T, \beta) \cong \text{Ker}(E\alpha)$. By comparing with [6, Prop. 2.2] we have only to show that $\text{ind}_T(C) + \text{ind}_T(\Sigma^{-1}C) = -B_T \cdot (\dim_\Lambda \mathcal{C}(T, C))$. This is exactly the statement of (a) with $Z = \Sigma^{-1}C$.

(c) Since we have an Auslander-Reiten triangle, a similar argument as in Remark 2.4 shows that $E\alpha$ is injective unless $Z \cong T_j$ for some $j = 1, \ldots, n$. 
Thus, our claim follows from (b) if we note that $- \text{ind}_T(\Sigma T_j) = \text{ind}_T(T_j) = e_j$. □

2.6. Conclusion of the proof. Let $\hat{y}_j := \prod_{i=1}^n x_i^{B_{i,j}} \in \mathbb{Z}[x_1^\pm, \ldots, x_n^\pm]$, and for $\mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}^n$ write $\hat{y}^\mathbf{e} := \prod_{i=1}^n \hat{y}_{i}^{e_i} = x_{B_T}^\mathbf{e}$. With this notation we have obviously

$$C^T_X = x^{-\text{ind}_T(X)} F_{EX}(\hat{y})$$

for any $X \in \mathcal{C}$.

Let first $Z \not\in \text{add}(T)$, then we have

$$C^T_{\Sigma Z} \cdot C^T_Z = x^{-\text{ind}_T(\Sigma Z) + \text{ind}_T(Z)} F_{E \Sigma Z}(\hat{y}) \cdot F_{E Z}(\hat{y})$$

$$= x^{-\text{ind}_T(\Sigma Z) + \text{ind}_T(Z)} (F_{E Y}(\hat{y}) + y^\text{dim}_\lambda(E Z))$$

$$= x^{\text{ind}_T(Y)} F_{E Y}(\hat{y}) + x^{\text{ind}_T(\Sigma Z) + \text{dim}_T(Z) + B_T \cdot \text{dim}_\lambda(E Z)}$$

$$= C^T_Y + 1.$$  

Here we used consecutively (2.4), Proposition 2.3 (a) resp. (b) and Remark 2.4 (a) resp. (b), equation (2.3), and finally (2.1).

Finally, suppose that $Z \cong T_j$ for some $j$, i.e. we consider the Auslander-Reiten triangle $\Sigma T_j \rightarrow Y \rightarrow T_j \rightarrow \Sigma^2 T_j$. Then we have

$$C^T_{\Sigma T_j} \cdot C^T_{T_j} = x_j^{-1} F_{I_j} (\hat{y}) x_j$$

$$= F_{I_j/S_j} (\hat{y}) \hat{y}_j + 1$$

$$= C^T_Y + 1.$$  

Here, we used consecutively Remark 2.4, Proposition 2.3 (c), and equation (2.3). □

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E-mail address: salomon@matem.unam.mx

E-mail address: christof@matem.unam.mx
URL: http://www.matem.unam.mx/christof

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Ciudad Universitaria, C.P. 04510, México D.F., MEXICO