Relativistic quantum mechanics on the light-front consistent with quantum field theory

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Abstract.

A relativistic quantum mechanics (RQM) on the light-front (LF) is derived from quantum field theory (QFT) through the use of a unitary transformation. Cluster separability of the derived RQM is proved by solving the so-called decoupling equation for the disconnected parts of the unitary transformation relating the Poincaré generators of RQM to the ones of QFT. For processes involving baryon-like number conservation ($NN \rightarrow NN$, $Nd \rightarrow Nd$, etc.), we show that the two theories, RQM and QFT, are equivalent.

1. Introduction

Dirac’s work on the three forms of relativistic dynamics [1] has led to the present well established approaches of relativistic quantum mechanics (RQM) whose formulation is based on the realization of Poincaré algebra in the space of a fixed number of particles [2, 3, 4, 5, 6, 7]; in particular, they are derived by constructing 10 Poincaré generators in the space of few-body states. Specific questions addressed in each of the papers [2, 3, 4, 5, 6, 7] are briefly discussed in [8], while an overview of the whole subject is given in [9].

Although this RQM has been applied to a number of relativistic few-body problems, the question remains as to how RQM is related to quantum field theory (QFT) where, as is well known, Lorentz invariance requires an interaction Hamiltonian, $H_I$, that allows for particle creation and annihilation. Indeed, the basic question is whether RQM is consistent with QFT. An important step in answering this question has been taken in [10, 11] where a unitary transformation (UT) was introduced to block diagonalizes the QFT Hamiltonian, $H$, in such a way that the subspace, $\eta$, of a fixed number of “nucleons” is decoupled from the rest of the space, $\lambda$. We will consider the somewhat simpler case where the subspace $\eta$ consists of particles (we shall refer to them “nucleons”) obeying baryon-like number conservation, so that the QFT in question forbids transitions between states of different numbers of nucleons (if there are no anti-nucleons in these states).

Here we consider the RQM based on the front form of dynamics which stands out by having the maximal number of kinematical generators (seven versus six in other forms of dynamics). These generators make up the stability subgroup corresponding to the transformations which do not change the hyperplane of constant LF time, $x^+ = t + z = 0$, and consist of the three components of vector $\vec{P} = (P^+, P_\perp) = (P^0 + P^3, P^1, P^2)$ where $P^\mu$ is the four-momentum, plus four generators constructed from the components of the angular momentum $\vec{J}$ and boost $\vec{K}$.
operators, as follows: \( J^3, K^3, E^1 = -K^1 + J^2 \) and \( E^2 = -K^2 - J^1 \). RQM in the front form of dynamics has been formulated and applied to the study of mesons and baryons in [12, 13, 14, 15]. Note that in the instant form of dynamics (i.e. with the usual time) there are only six kinematical generators, \( \vec{P} \) and \( \vec{J} \).

In the present paper we show that the theory corresponding to the diagonalized Hamiltonian, \( H' = UHU^\dagger \), is equivalent to the initial one corresponding to \( H \), and moreover, that the Hamiltonian of the new theory \( (H') \) in the \( \eta \) subspace, has all the essential properties of a Hamiltonian of a correct RQM. By the latter, we mean a number of fundamental properties of Quantum Mechanics (discussed below), including so-called "cluster separability" which guarantees, for example, that a two-body potential in the three-body system is the same as the two-body potential in an isolated two-body system. The achievement of cluster separability is particularly significant as it has been a long-standing problem in RQM theories [2, 3, 4, 5, 6, 7]. Our proof of cluster separability is based on an exact solution of the so-called decoupling equation for the disconnected parts of the UT.

Although the issues addressed in this paper are of a fundamental theoretical nature, it is worth noting that they may be of practical interest as well; in particular, the results of the paper should be useful for the modern theory of nuclear forces (see [16] and the references therein) whose construction in most studies is based on the UT method [10, 11].

2. Unitary transformation method

In 1954, Okubo introduced a UT \([10, 11]\) to block diagonalize the Hamiltonian of usual equal-time QFT so that the subspace of a fixed number of nucleons is decoupled from the rest of the space. Since then, it has been shown [8] that the UT actually diagonalizes all ten Poincaré generators of the QFT. As a result, it has become apparent that the unitary transformed generators realize Poincaré algebra in the subspace of the fixed number of nucleons.

In the present paper, we apply the UT method to the case of LF QFT, and in this way, derive LF RQM by block diagonalizing the 10 Poincaré generators given by LF QFT (which of course do not conserve the number of nucleons). However, further steps need to be performed in order to establish that these diagonalized generators indeed constitute what can rightfully be called relativistic Quantum Mechanics on the LF; in particular, the diagonalized generators should: (i) be able to be split into free and interaction parts\(^2\) in such a way that the interaction parts are diagonal in the number of nucleons, and the resultant theory generates the same scattering amplitude as the one determined by the QFT we started with, and (ii) produce matrix elements of the transformed Hamiltonian whose disconnected parts have a "quantum mechanical structure". The latter requirement is clarified in the discussions below, but means, for example, that the fully disconnected part would be the sum of the LF energies of the nucleons, the pair interaction potential in the three-body system would be the same as in the isolated two-body system, etc. To emphasise the significance of requirement (ii), note that it is badly violated in the Tamm-Dankoff diagonalization method [17, 18]. It is important to emphasize that this "quantum mechanical structure" does not follow from Lorentz invariance alone; indeed, in order to establish this structure, we needed to calculate the disconnected parts of the Hamiltonian \( H' \) non-perturbatively, using a previously unknown factorisation property of the UT derived from the decoupling equation, Eq. (4).

The stability subgroup Poincaré generators of LF QFT, \( P^+, P^\perp, J^3, K^3, E^1 \) and \( E^2 \), commute

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\(^1\) Recall that in LF dynamics the Hamiltonian is the difference of the zero'th and the third components of the four-momentum generator of the Poincaré group: \( H = P^- = P^0 - P^3 \).

\(^2\) Eigenstates of the free part of the four-momentum generator determine the asymptotic states of the nucleons which are needed apart from the Hamiltonian to construct the scattering amplitude.
with the UT, so they do not even change under this transformation:\(^3\)

\[
UPU^\dagger = P, \quad UJ^3U^\dagger = J^3, \quad UK^3U^\dagger = K^3, \quad UE^1U^\dagger = E^1, \quad UE^2U^\dagger = E^2 \tag{1}
\]

All the three remaining dynamical generators, the Hamiltonian \(P^-, \quad F^1 = -K^1 - J^2, \) and \(F^2 = -K^2 + J^1, \) can be block diagonalized by the same UT. Thus any of these generators, and in particular the transformed Hamiltonian, has the following block diagonal form

\[
H' = UHU^\dagger = \begin{pmatrix}
\eta H'\eta & 0 \\
0 & \lambda H'\lambda
\end{pmatrix}
\tag{2}
\]

We follow the commonly used parametrization of the UT as originally introduced in [10]:

\[
U = \begin{pmatrix}
\eta U\eta & \eta U\lambda \\
\lambda U\eta & \lambda U\lambda
\end{pmatrix} = \begin{pmatrix}
(1 + A^1\lambda)^{-1/2}\eta & (1 + A^1\lambda)^{-1/2}A^\dagger \\
-(1 + A^1\lambda)^{-1/2}A & (1 + A^1\lambda)^{-1/2}\lambda
\end{pmatrix}
\tag{3}
\]

where \(A = \lambda A\eta \) (i.e. \(\eta A\lambda = \eta A\eta = \lambda A\lambda = 0\)) is determined from the decoupling equation,

\[
(\lambda - A)H(\eta + A) = 0. \tag{4}
\]

The first problem is to show that the theories corresponding to the Hamiltonians \(H'\) and \(H\) are equivalent to each other. This is a trivial problem in the standard setting where \(H_0'\), the free part of the Hamiltonian \(H' = UHU^\dagger\), is obtained from the initial free Hamiltonian \(H_0\), using the same transformation, \(H_0' = UH_0U^\dagger\); in that case, the whole transformation is equivalent to a transition to new fundamental fields, \(\Psi'(x) = U\Psi(x)U^\dagger\), which satisfy the same field-theoretical equations and the same commutation relations as the fields \(\Psi(x)\) of the initial theory. Another way of seeing equivalence in this setting (with \(H_0' = UH_0U^\dagger\)) is to notice that the inhomogeneous term in the equation for an in-state (or out-state), is the UT-ed eigenstate of \(H_0\), which in turn, leads to the equation \(\langle \Psi^{(in)}| = U|\Psi^{(in)}\rangle\). This relation between scattering states of theories corresponding to the Hamiltonians \(H'\) and \(H\), leaves the S-matrix, \(S = \langle \Psi^{(out)}|\Psi^{(in)}\rangle\), unchanged.

However, in order to stay in the subspace \(\eta\) of a fixed number of particles, the nonstandard setting is needed where the free part of the transformed Hamiltonian \(H'\) is identified with the initial free Hamiltonian, \(H_0' = H_0 (\neq UH_0U^\dagger)\). The way the Hamiltonian is split into free and interaction parts makes a difference for the scattering problem, but not for the bound state problem because the latter has an unambiguous solution (up to a normalization) for the bound state vector, \(|B'| = U|B\rangle\). The bound-state energy spectrum thus does not change under the UT.

One needs to show that the on-shell scattering amplitude corresponding to the interaction \(V' = H' - H_0\), coincides with the one produced by the initial Hamiltonian \(H\). This can be done by comparing the scattering states of the two theories. For the \(NN\) scattering in-states one has the straightforward relation

\[
|\Psi_q^{(+)}\rangle = \lim_{\epsilon \to 0} \frac{ie}{P_q - H' + i\epsilon}|q_1, q_2\rangle = U \lim_{\epsilon \to 0} \frac{ie}{P_q - H + i\epsilon}|U^\dagger q_1, q_2\rangle \tag{5}
\]

where

\[
P_q^- = \frac{q_1^2 + m^2}{q_1^2} + \frac{q_2^2 + m^2}{q_2^2}, \tag{6}
\]

\(^3\) Note that the UT introduced by Okubo [10] was for the instant form of dynamics and therefore commutes with only six of the Poincaré generators, the 3-momentum, \(\vec{P}\) and the angular momentum \(\vec{J}\), so that, \(UPU^\dagger = \vec{P}\) and \(UJU^\dagger = \vec{J}\).
\( m \) is a dressed nucleon mass. It is obvious from Eq. (5) that the operator \( U^\dagger \) on the right hand side may make a non-trivial contribution to the connection between \( |\Psi_q^{(+)}\rangle \) and the in-state of the initial theory

\[
|\Psi_q^{(+)}\rangle = \lim_{\epsilon \to 0} \frac{i\epsilon}{P_q - H + i\epsilon} |q_1, q_2\rangle.
\] (7)

As described below, we have derived the contribution of \( U^\dagger \) by solving the decoupling equation, Eq. (4), and found that its contribution is indeed non-trivial, so that \( |\Psi_q^{(+)}\rangle \neq U|\Psi_q^{(+)}\rangle \). This is in sharp contrast to all previous studies where \( U^\dagger \) was assumed to be replaceable by the identity operator. We will see that \( U^\dagger \) contributes via the disconnected part of the matrix element:

\[
\langle q'_1, q'_2 | \frac{i\epsilon}{P_q - H + i\epsilon} U^\dagger |q_1, q_2\rangle_{\text{disc}} \neq \langle q'_1, q'_2 | \frac{i\epsilon}{P_q - H + i\epsilon} |q_1, q_2\rangle_{\text{disc}}
\] (8)

where the disconnected part of any matrix element is defined as the sum of all terms corresponding to the transition between two given subgroups of particles, \((q_1,...,q_s)(q_{s+1},...,q_m)\) and \((p_1,...,p_l)\), in the initial and final states respectively, such that there are no particle exchanges between these two independent transitions. Correspondingly the total momentum is conserved in each of these two transitions:

\[
\langle p_1,...,p_l; p_{l+1},...,p_n | O |q_1,...,q_s; q_{s+1},...,q_m\rangle_{\text{disc}}
\]

\[
= \delta^3(p_1 + ... + p_l - q_1 - ... - q_s)\delta^3(p_{l+1} + ... + p_n - q_{s+1} - ... - q_m)O(p, q)
\] (9)

where the semicolons separate the quantum numbers of particles from the independent subgroups and \( \delta^3(p) = \delta(p^+\lambda)\delta^3(p^\perp) \).

The paper is based on the conjecture about factorization of the disconnected parts of the matrix elements of the \( \eta \) projected part, \( \eta U \), of the unitary transformation \( U \),

\[
\langle p_1,...,p_l; p_{l+1},...,p_n | \eta U |q_1,...,q_s; q_{s+1},...,q_m\rangle_{\text{disc}}
\]

\[
= \langle p_{l+1},...,p_n | \eta U |q_{s+1},...,q_m\rangle \langle p_1,...,p_l | \eta U |q_1,...,q_s\rangle.
\] (10)

It results from the rigorous analysis of the decoupling Eq. (4) (see appendix A) and holds for only matrix elements of \( U \) corresponding to the transitions from arbitrary initial state to only \( \eta \) final subspace. As we will see below matrix elements of \( \lambda U\eta \) factorize in a different way,

\[
\langle \lambda, \eta |\eta U|\eta, \eta_2\rangle_{\text{disc}} = \langle \lambda |\eta U|\eta_1\rangle \langle \eta |\eta U|\eta_2\rangle
\]

\[
\langle \lambda_1, \lambda_2 |\eta U|\eta_1, \eta_2\rangle_{\text{disc}} = -\langle \lambda_1 |\eta U|\eta_1\rangle \langle \lambda_2 |\eta U|\eta_2\rangle
\] (11)

where we considered two different type of \( \lambda \) states, \( \langle \lambda, \eta \rangle \) and \( \langle \lambda_1, \lambda_2 \rangle \). The difference from Eq. (10) is in the minus sign in the second of the equations (11).

The factorization property (10) leads to the following connection between the two-body scattering states, \( |\Psi_q^{(+)}\rangle \) and \( |\Psi_q^{(-)}\rangle \) (see appendix B):

\[
\frac{1}{Z} \lim_{\epsilon \to 0} \frac{i\epsilon}{P_q - H' + i\epsilon} |q_1, q_2\rangle = \frac{1}{Z} \lim_{\epsilon \to 0} \frac{i\epsilon}{P_q - H + i\epsilon} |q_1, q_2\rangle,
\] (12)

where the renormalization constants \( Z \) and \( Z' \) are defined as the residues of the single particle propagators (at the pole energy \( q^- = \frac{q_1^2 + m^2}{q^*} \)) corresponding to the single particle matrix elements of the resolvent operators

\[
\langle p | \frac{1}{q^- - H' + i\epsilon} |q\rangle \sim \delta^3(p - q) \frac{Z'}{q^- - \frac{q_1^2 + m^2}{q^*} + i\epsilon}
\] (13)
\[
\langle p | \frac{1}{q^- - H + i\epsilon} | q \rangle \sim \delta^3(p - q) \frac{Z}{q^- - \frac{q_1^2}{q^+} + i\epsilon}. \tag{14}
\]

Note that since \( H' \) is diagonal (with respect to the number of particles) in the \( \eta \) subspace, the corresponding self-energy, \( \Sigma'(q^-) \), does not depend on the LF "energy", \( q^- \), and therefore the single particle wave function does not get renormalized, \( Z'_q = 1 \) (see Eq. (B.8) in the Appendix).

In Appendix B it is proved that the RQM corresponding to the Hamiltonian \( H' \) produces the same on-shell \( NN \) scattering amplitude as the one produced by the initial QFT.

The factorization property (10) is crucial for the analysis of the structure of the few-body interaction, \( H' \). To this end note the following property of the disconnected parts of the matrix elements of \( H \),

\[
\begin{align*}
\langle p_1, \ldots, p_l; p_{l+1}, \ldots, p_n | H | q_1, \ldots, q_s; q_{s+1}, \ldots, q_m \rangle^{\text{disc}} \\
= \langle p_{l+1}, \ldots, p_n | H | q_{s+1}, \ldots, q_m \rangle \langle p_1, \ldots, p_l | q_1, \ldots, q_s \rangle \\
+ \langle p_{s+1}, \ldots, p_n | q_{s+1}, \ldots, q_m \rangle \langle p_1, \ldots, p_l | H | q_1, \ldots, q_s \rangle,
\end{align*}
\tag{15}
\]

which follows from the general representation for the Hamiltonian in the QFT on the LF

\[
H = \int \mathcal{H}(\psi(x)) d^3x,
\tag{16}
\]

where \( d^3x = dx^- d^2x^\perp \). Lorentz invariance does not allow an interaction Hamiltonian in the form of the product of "usual" interaction Hamiltonians

\[
\left[ \int \mathcal{H}_1(\psi(x)) d^3x \right] \left[ \int \mathcal{H}_2(\psi(x)) d^3x \right]
\tag{17}
\]

as in the latter the two operators are at the same LF time \( x^+ \), yet at different 3-coordinate \( x \).

The property (15) is at the heart of the convolution approach \([17, 18]\) where it enables one to sum all possible terms contributing to the disconnected parts of the Green functions (including ones corresponding to so-called same-time dressing). Just the full sum solves the longstanding renormalization problem in time ordered perturbation theory (TOPT), see \([17, 18]\) and references therein. Using the property (15) in combination with Eq. (10) one gets the same property for \( H' \). In the subspace \( \eta \) we have

\[
\begin{align*}
\langle p_1, \ldots, p_s; p_{s+1}, \ldots, p_n | \eta H' | q_1, \ldots, q_s; q_{s+1}, \ldots, q_m \rangle^{\text{disc}} \\
= \langle p_{s+1}, \ldots, p_n | H' | q_{s+1}, \ldots, q_m \rangle \langle p_1, \ldots, p_s | q_1, \ldots, q_s \rangle \\
+ \langle p_{s+1}, \ldots, p_n | q_{s+1}, \ldots, q_m \rangle \langle p_1, \ldots, p_s | H' | q_1, \ldots, q_s \rangle.
\end{align*}
\tag{18}
\]

Eq. (18) is all one needs to analyse the structure of the equations in the \( \eta \)-space. Such analysis shows that the "free" part of the equations is determined by the sum of the LF energies of the particles with physical (not bare) masses. It is determined by the fully disconnected part of the corresponding matrix element of the Hamiltonian \( H' \) and in the two nucleon case it is

\[
\langle p_1; p_2 | \eta H' | q_1; q_2 \rangle^{\text{disc}} = \langle p_1 | H' | q_1 \rangle \delta^3(p_2 - q_2) + \langle p_2 | H' | q_2 \rangle \delta^3(p_1 - q_1)
\]

\[
= \delta^3(p_1 - q_1) \delta^3(p_2 - q_2) \left( \frac{q_1^2}{q_1^+ + m^2} + \frac{q_2^2}{q_2^+ + m^2} \right)
\tag{19}
\]
The pair interaction (say of particles 2 and 3) in the three-body system is determined by the sum of all disconnected diagrams corresponding to the first particle in flight

\[
\langle p_1; p_2, p_3 | H' | q_1; q_2, q_3 \rangle_{\text{disc}} = \langle p_1 | H' | q_1 \rangle \delta^3(p_2 - q_2) \delta^3(p_3 - q_3) + \langle p_2, p_3 | H' | q_2, q_3 \rangle \delta^3(p_1 - q_1)
\]

\[
= \delta^3(p_1 - q_1) \delta^3(p_2 - q_2) \delta^3(p_3 - q_3) \left( \frac{q_{1+}^2 + m^2}{q_1^2} \right) + \left( \langle p_2, p_3 | H' | q_2, q_3 \rangle_{\text{disc}} + V_{23}^{(2)} \right) \delta^3(p_1 - q_1)
\]

where here and below \( V^{(n)} \) is the \( n \)-body force potential defined as the connected part of the \( n \)-body matrix element of \( H' \). It is remarkable that the pair interaction in the three-body system is just the same as in the case of the isolated system of two particles. Recall that in the corresponding equations of TOPT, the pair interaction kernel (the sum of irreducible diagrams where the third particle is in flight) cannot even be expressed in terms of the isolated two-body kernel.

A similar situation is present in the more complicated four-body sector; using Eq. (18) one can show that there is no two-pair simultaneous interaction in the four-body system - one has only the sum of pair interactions:

\[
\langle p_1, p_2; p_3, p_4 | H' | q_1, q_2, q_3, q_4 \rangle_{\text{disc}} = \langle p_1, p_2 | H' | q_1, q_2 \rangle \delta^3(p_3 - q_3) \delta^3(p_4 - q_4) + \langle p_3, p_4 | H' | q_3, q_4 \rangle \delta^3(p_1 - q_1) \delta^3(p_2 - q_2)
\]

\[
= \delta^3(p_1 - q_1) \delta^3(p_2 - q_2) \delta^3(p_3 - q_3) \left( \frac{\sum q_{i+}^2 + m^2}{q_i^2} \right) + V_{12}^{(2)} \delta^3(p_3 - q_3) \delta^3(p_4 - q_4) + V_{34}^{(2)} \delta^3(p_1 - q_1) \delta^3(p_2 - q_2).
\]

The properties observed in Eqs. (19), (20), and (21) hold for the general case of arbitrary number of particles and lead to the equations of quantum mechanical form. They can be summarized in the compact form of a second quantized QM:

\[
H' = \int d^3p \ a_p^+ \frac{p^2 + m^2}{p^3} a_p + \int d^3p_1 d^3p_2 d^3q_1 d^3q_2 a_{p_1}^+ a_{p_2}^+ V^{(2)}(p_1, p_2, q_1, q_2) a_{q_1} a_{q_2}
\]

\[
+ \int d^3p_1 d^3p_2 d^3q_1 d^3q_2 a_{p_1}^+ a_{p_2}^+ V^{(3)}(p_1, p_2, p_3, q_1, q_2, q_3) a_{q_1} a_{q_2} a_{q_3}
\]

\[
+ \int d^3p_1 d^3p_2 d^3q_1 d^3q_2 a_{p_1}^+ a_{p_2}^+ a_{p_3}^+ V^{(4)}(p_1, p_2, p_3, q_1, q_2, q_3, q_4) a_{q_1} a_{q_2} a_{q_3} a_{q_4} + \ldots
\]

where \( \delta^3p = dp^+ d^2p^\perp \) (which should not be confused with a similar notation for the coordinate phase space, \( d^3x = dx^- d^2x^\perp \)), \( a_p^+ \), \( a_p \) are the nucleon creation and annihilation operators, \( V^{(n)} \) are \( n \)-body force potentials. Note that all these potentials depend only on \( p_i^+, q_j \), the three-dimensional part of the momenta, which are integration variables in Eq. (22); in particular, they do not depend on the total off-shell LF energy, in contrast to the Tamm-Dankoff or quasipotential approaches. This is also the property of usual QM.

3. Comments

As mentioned in Sect. 2, all ten Poincaré generators of QFT are block-diagonalized by a unique UT. Seven of the unitary transformed generators, \( P^+, P^\perp, J^3, K^3, E^1 \) and \( E^2 \), are kinematic
(as are the initial ones, \(P^+, P^-\), \(J^3, K^3, E^1\) and \(E^2\)), because they are diagonal in the number of particles and commute with the UT, see Eq. (1). The three remaining generators, the Hamiltonian, \(P^\text{off}\) and the generators \(F^{\text{off}}_1, F^{\text{off}}_2\), are dynamical. Note that the factorisation property of the UT allows one to show that the few-body structure of \(F^{\text{off}}_1\) and \(F^{\text{off}}_2\) is similar to the structure of \(P^\text{off}\) studied in the paper and represented in a compact form in Eq. (22).

Note that the property of cluster separability is violated in [2] because, by construction, pair interaction parts of a three-body potential in the momentum representation are not diagonal with respect to the three-momentum of a particle in flight. They are accompanied with \(\delta\)-functions which makes them diagonal with respect to the space components of the boosted momenta of particles in flight. These boosts are different in the initial and final states because they are determined by total momentum of the three on-mass-shell particles in the initial and final states respectively, which is why the momentum of a particle in flight is not conserved. This violation is important in problems where the three-body system cannot be fixed at rest, like, for example, in the case of electron scattering off a three-body bound state. It would be interesting to discuss our results in the context of ”packing operators” [6, 7] which were devised to handle the clustering problem present in [2].

**Appendix A. Disconnected parts of the matrix elements of \(A\)**

To prove Eq. (10), we first need to use the decoupling equation, Eq. (4), to derive expressions for the disconnected parts of the matrix elements of \(A\). We begin by proving that the disconnected parts of the matrix elements of \(A^\dagger\) are given by

\[
\begin{align*}
\langle \eta_1, \eta_2 | A^\dagger | \lambda_1, \lambda_2 \rangle^\text{disc} &= \langle \eta_1 | A^\dagger | \lambda_1 \rangle \langle \eta_2 | \lambda_2 \rangle \\
\langle \eta_1, \eta_2 | A^\dagger | \lambda_1, \lambda_2 \rangle^\text{disc} &= \langle \eta_1 | A^\dagger | \lambda_1 \rangle \langle \eta_2 | A^\dagger | \lambda_2 \rangle.
\end{align*}
\]

(A.1)

There is a direct way to carefully check whether the anzatz (A.1) satisfies Eq. (4), but this is a tedious and error-prone task. A more efficient way is to identify our disconnectedness problem with the one generated by the sum of two independent hamiltonians. This trick proved very efficient in the construction of disconnected amplitudes in TOPT [17, 18]. In the current context, a simple example of this approach is to express Eq. (15) for the disconnected matrix elements of the Hamiltonian \(H\) as

\[
\langle s'_1, s'_2 | H | s_1, s_2 \rangle^\text{disc} = \langle s'_1, s'_2 | (H_1 + H_2) | s_1, s_2 \rangle
\]

(A.2)

where \(H_1\) and \(H_2\) are independent hamiltonians that depend on their own nucleon and meson fields, \(\psi_1 \equiv \{ \Psi, \Phi_1 \}\) and \(\psi_2 \equiv \{ \Psi, \Phi_2 \}\), respectively, in exactly the same way that \(H\) depends on its fields \(\psi \equiv \{ \Psi, \Phi \}\). Because particles of ”type 1” do not interact with particles of ”type 2”, the corresponding Hamiltonian is \(H_1 + H_2\). The state, \(| s_1, s_2 \rangle\) on the right hand side of Eq. (A.2) is given by the direct product \(| s_1, s_2 \rangle = | s_1 \rangle \times | s_2 \rangle\) where \(| s_1 \rangle\) and \(| s_2 \rangle\) are particle Fock states associated with ”type 1” and ”type 2” particles, respectively. Similarly \(| s'_1, s'_2 \rangle = | s'_1 \rangle \times | s'_2 \rangle\). It follows that

\[
\begin{align*}
\langle s'_1, s'_2 | (H_1 + H_2) | s_1, s_2 \rangle &= \langle s'_1 | H_1 | s_1 \rangle \langle s'_2 | s_2 \rangle + \langle s'_2 | H_2 | s_2 \rangle \langle s'_1 | s_1 \rangle \\
&= \langle s'_1 | H | s_1 \rangle \langle s'_2 | s_2 \rangle + \langle s'_2 | H | s_2 \rangle \langle s'_1 | s_1 \rangle
\end{align*}
\]

(A.3)

which is just Eq. (15) in compact notation. In a similar way we express Eq. (A.1) for the disconnected matrix elements of \(A^\dagger\) as

\[
\begin{align*}
\langle \eta_1, \eta_2 | A^\dagger | s_1, s_2 \rangle^\text{disc} &= \langle \eta_1 | A^\dagger | s_1 \rangle \langle A^\dagger_1 + A^\dagger_2 + A^\dagger_3 | s_2 \rangle
\end{align*}
\]

(A.4)

where like \(H_i\) (\(i = 1\) or \(2\)), \(A_i\) is determined in the theory of ”type \(i\)” nucleons and mesons in the same way as \(A\) is determined by Eq. (4) in the initial theory with the fields \(\Psi\) and \(\Phi\).
There is an important subtlety regarding Eq. (A.4) - it is valid only if the final state is from the \( \eta \)-subspace; e.g., it is not valid for the final state \( \langle \lambda_1, \eta_2 \rangle \). Indeed,
\[
\langle \lambda_1, \eta_2 | A^\dagger | s_1, s_2 \rangle^{\text{disc}} = 0
\neq \langle \lambda_1, \eta_2 | (A_1^\dagger + A_2^\dagger + A_1^\dagger A_2^\dagger) | s_1, s_2 \rangle = \langle \lambda_1, \eta_2 | A_1^\dagger | s_1, s_2 \rangle
= \langle \eta_2 | A_2^\dagger | s_2 \rangle \langle \lambda_1 | s_1 \rangle.
\] (A.5)

Eq. (A.4) can be written for an arbitrary final state in the following form
\[
\langle s_1', s_2' | A | s_1, s_2 \rangle^{\text{disc}} = \langle s_1', s_2' | (\eta A_1^\dagger + \eta A_2^\dagger + A_1^\dagger A_2^\dagger) | s_1, s_2 \rangle
\] (A.6)
where we have taken ito account that \( \langle s_1', s_2' | \eta A_1^\dagger A_2^\dagger = \langle s_1', s_2' | A_1^\dagger A_2^\dagger \). It is straightforward to rewrite Eq. (A.6) for the operator \( A \)
\[
\langle s_1', s_2' | A | s_1, s_2 \rangle^{\text{disc}} = \langle s_1', s_2' | (A_1 \eta + A_2 \eta + A_1 A_2) | s_1, s_2 \rangle.
\] (A.7)

To prove Eq. (A.7), we show that it leads to the decoupling equation for disconnected matrix elements; indeed, recalling the left hand side of Eq. (4), we have that
\[
[(\lambda | (\lambda - A) H (\eta + A) | \eta)])^{\text{disc}} = [(\lambda | (\lambda - A \eta) H (\eta + A) | \eta)]^{\text{disc}} =
\langle \lambda | (1 - A \eta - A_2 \eta - A_1 A_2 \eta)(H_1 + H_2)(1 + A + A_2 + A_1 A_2) | \eta \rangle.
\] (A.8)

Please note that we inserted the \( \eta \)-projector next to \( A \) to validate the use of Eq. (A.4). There are two types of \( \lambda \)-states, \( \langle \lambda_1, \eta_2 \rangle \) and \( \langle \lambda_1, \lambda_2 \rangle \); we should proceed separately for each of them in Eq. (A.8). For the final \( \langle \lambda_1, \lambda_2 \rangle \) state we have
\[
[(\lambda_1, \lambda_2 | (\lambda - A) H (\eta + A) | \eta)])^{\text{disc}}
= \langle \lambda_1, \lambda_2 | (1 - A_1 \eta - A_2 \eta - A_1 A_2 \eta)(H_1 + H_2)(1 + A + A_2 + A_1 A_2) | \eta \rangle
= \langle \lambda_1, \lambda_2 | (\lambda - A_1)(H_1(1 + A_1)A_2 + (1 \leftrightarrow 2)) | \eta \rangle
= \langle \lambda_1 | (1 - A) H (1 + A) | \eta \rangle \langle \lambda_2 | A | \eta_2 \rangle + (1 \leftrightarrow 2) = 0
\] (A.9)
where we have used Eq. (4) for the full amplitude, and the fact that
\[
\langle \lambda_1, \lambda_2 | A_1 \eta \rangle = 0, \quad A_1 A_1 = \lambda A_1 \eta \lambda A_1 \eta = 0,
\langle \lambda_1, \lambda_2 | X_1 N_1 B_1 | \eta \rangle = \langle \lambda_1, \lambda_2 | X_2 N_2 B_2 | \eta \rangle = 0.
\] (A.10)

Note that Eq. (A.9) is just the decoupling equation for the case of disconnected matrix elements. In the same way for the final \( \langle \lambda_1, \eta_2 \rangle \) state we have
\[
[(\lambda_1, \eta_2 | (\lambda - A) H (\eta + A) | \eta)])^{\text{disc}}
= \langle \lambda_1, \eta_2 | (1 - A - A_2 - A_1 A_2)(H_1 + H_2)(1 + A + A_2 + A_1 A_2) | \eta \rangle
= \langle \lambda_1, \eta_2 | (\lambda - A_1 )H_1(1 + A_1) | \eta_1, \eta_2 \rangle
= \langle \lambda_1 | (1 - A) H (1 + A) | \eta_1 \rangle \langle \eta_2 | \rangle = 0
\] (A.11)

Having established Eq. (A.4) for operator \( A \), we now derive a similar factorization for \( A^\dagger A \):
\[
\langle p_1, ..., p_l | p_{l+1}, ..., p_n | A^\dagger A | q_1, ..., q_l, q_{l+1}, ..., q_n \rangle^{\text{disc}}
= \langle p_1, ..., p_l | p_{l+1}, ..., p_n | A_1^\dagger A_1 + A_2^\dagger A_2 + (A_1^\dagger A_1)(A_2^\dagger A_2) | q_1, ..., q_l, q_{l+1}, ..., q_n \rangle
\] (A.12)
where in the second line the nucleons $p_1, \ldots, p_l; q_1, \ldots, q_l$ in the final and initial state are of "type 1", the nucleons $p_{l+1}, \ldots, p_n; q_{l+1}, \ldots, q_n$ in the final and initial state are of "type 2", and the operators $A_i$ are defined via the fields $\psi_i$ in the same way as $A$ is defined via the fields $\psi$. Indeed, writing (A.12) as

$$\langle \eta_1', \eta_2' | A^\dagger A | \eta_1, \eta_2 \rangle^{\text{disc}} = \langle \eta_1', A^\dagger A | \eta_1 \rangle \langle \eta_2' | \eta_2 \rangle + \langle \eta_1', A^\dagger A | \eta_2 \rangle \langle \eta_2' | \eta_1 \rangle$$

we have that

$$\langle \eta_1', \eta_2' | A^\dagger A | \eta_1, \eta_2 \rangle^{\text{disc}} = \sum_\lambda \langle \eta_1', \eta_2' | A^\dagger A | \lambda \rangle^{\text{disc}} \langle \lambda | \eta_1 \rangle \langle \eta_2 | \eta_2 \rangle$$

Next, one can prove the factorization (10) of the matrix elements of $\eta U$. The most nontrivial part is to show the factorisation of $\eta U$, using Eq. (A.12) in

$$\langle \eta_1, \eta_2 | U | \eta_1, \eta_2 \rangle^{\text{disc}} = \langle \eta_1, \eta_2 | (1 + A^\dagger A)^{-1/2} | \eta_1, \eta_2 \rangle^{\text{disc}}.$$  

Let us check Eq. (10) in the $\eta$ subspace, i.e., for the case of $n$ nucleon states

$$\langle p_1, \ldots, p_l; p_{l+1}, \ldots, p_n | U | q_1, \ldots, q_l; q_{l+1}, \ldots, q_n \rangle^{\text{disc}}$$

where we have used the notation $[A^\dagger A]^{\text{disc}}$

$$\langle p_1, \ldots, p_l; p_{l+1}, \ldots, p_n | [A^\dagger A]^{\text{disc}} | q_1, \ldots, q_l; q_{l+1}, \ldots, q_n \rangle^{\text{disc}} = \langle p_1, \ldots, p_l; p_{l+1}, \ldots, p_n | A^\dagger A | q_1, \ldots, q_l; q_{l+1}, \ldots, q_n \rangle^{\text{disc}}.$$  

Eq. (A.16) follows from the fact that the disconnected part of the matrix element of any iteration of operator $A^\dagger A$ is equal to the iteration of the disconnected parts of the matrix elements of $A^\dagger A$. This statement for the second iteration mathematically looks as follows:

$$\langle f | (A^\dagger A) (A^\dagger A) | i \rangle^{\text{disc}} = \sum_j \langle f | (A^\dagger A) | j \rangle^{\text{disc}} \langle j | (A^\dagger A) | i \rangle^{\text{disc}}.$$  

(A.18)
Using Eq. (A.12) in Eq. (A.16) one gets Eq. (10)

\[ \langle p_1, \ldots; p_{n+1}, \ldots; p_n | U | q_1, \ldots; q_i; q_{i+1}, \ldots; q_n \rangle_{\text{disc}} \]

\[ = \langle p_1, \ldots; p_{n+1}, \ldots; p_n | (1 + A^\dagger A)^{-1/2} | q_1, \ldots; q_i; q_{i+1}, \ldots; q_n \rangle_{\text{disc}} \]

\[ = \langle p_1, \ldots; p_{n+1}, \ldots; p_n | (1 + [A^\dagger A]_{\text{disc}})^{-1/2} | q_1, \ldots; q_i; q_{i+1}, \ldots; q_n \rangle_{\text{disc}} \]

\[ = \langle p_1, \ldots; p_{n+1}, \ldots; p_n | (1 + A^\dagger_1 A_1 + A^\dagger_2 A_2 + A^\dagger_3 A_3 A_3 A_3)^{-1/2} | q_1, \ldots; q_i; q_{i+1}, \ldots; q_n \rangle_{\text{disc}} \]

\[ = \langle p_1, \ldots; p_{n+1}, \ldots; p_n | (1 + A^\dagger_1 A_1)^{-1/2} (1 + A^\dagger_2 A_2)^{-1/2} | q_1, \ldots; q_i; q_{i+1}, \ldots; q_n \rangle \]

\[ = \langle p_1, \ldots; p_{n+1}, \ldots; p_n | (1 + A^\dagger_1 A_1)^{-1/2} q_1, \ldots; q_i \rangle \times \langle p_{i+1}, \ldots; p_n | (1 + A^\dagger_2 A_2)^{-1/2} q_{i+1}, \ldots; q_n \rangle \]

\[ = \langle p_1, \ldots; p_i | U_1 | q_1, \ldots; q_i \rangle \times \langle p_{i+1}, \ldots; p_n | U_2 | q_{i+1}, \ldots; q_n \rangle \]

\[ = \langle p_1, \ldots; p_i | U | q_1, \ldots; q_i \rangle \times \langle p_{i+1}, \ldots; p_n | U | q_{i+1}, \ldots; q_n \rangle \]

(A.19)

There are two types of disconnected parts corresponding to $\eta U \lambda$,

\[ \langle \eta_1, \eta_2 | U | \lambda_1, \lambda_2 \rangle_{\text{disc}} = \langle \eta_2 | U | \eta_2 \rangle \langle \eta_1 | U | \lambda_1 \rangle \]

\[ \langle \eta_1, \eta_2 | U | \lambda_1, \lambda_2 \rangle_{\text{disc}} = \langle \eta_2 | U | \lambda_2 \rangle \langle \eta_1 | U | \lambda_1 \rangle, \]

(A.20)

whose factorisation is not hard to show:

\[ \langle \eta_1, \eta_2 | U | \lambda_1, \eta_2 \rangle_{\text{disc}} = \langle \eta_1, \eta_2 | (1 + A^\dagger A)^{-1/2} A^\dagger | \lambda_1, \eta_2 \rangle_{\text{disc}} \]

\[ = \sum_{\eta'} \langle \eta_1, \eta_2 | (1 + A^\dagger A)^{-1/2} | \eta'_1, \eta'_2 \rangle_{\text{disc}} \langle \eta'_1, \eta'_2 | A^\dagger | \lambda_1, \eta_2 \rangle_{\text{disc}} \]

\[ = \sum_{\eta'} \langle \eta_2 | (1 + A^\dagger A)^{-1/2} | \eta_2 \rangle \langle \eta_1 | (1 + A^\dagger A)^{-1/2} | \eta'_1 \rangle \langle \eta'_1 | A^\dagger | \lambda_1 \rangle \]

\[ = \langle \eta_2 | (1 + A^\dagger A)^{-1/2} | \eta_2 \rangle \langle \eta_1 | (1 + A^\dagger A)^{-1/2} A^\dagger | \lambda_1 \rangle \]

\[ = \langle \eta_2 | U | \eta_2 \rangle \langle \eta_1 | U | \lambda_1 \rangle, \]

\[ \langle \eta_1, \eta_2 | U | \lambda_1, \lambda_2 \rangle_{\text{disc}} = \langle \eta_1, \eta_2 | (1 + A^\dagger A)^{-1/2} A^\dagger | \lambda_1, \lambda_2 \rangle_{\text{disc}} \]

\[ = \sum_{\eta'} \langle \eta_1, \eta_2 | (1 + A^\dagger A)^{-1/2} | \eta'_1, \eta'_2 \rangle_{\text{disc}} \langle \eta'_1, \eta'_2 | A^\dagger | \lambda_1, \lambda_2 \rangle_{\text{disc}} \]

\[ = \sum_{\eta'} \langle \eta_2 | (1 + A^\dagger A)^{-1/2} | \eta'_2 \rangle \langle \eta_1 | (1 + A^\dagger A)^{-1/2} | \eta'_1 \rangle \langle \eta'_1 | A^\dagger | \lambda_1 \rangle \langle \eta'_2 | A^\dagger | \lambda_2 \rangle \]

\[ = \langle \eta_2 | (1 + A^\dagger A)^{-1/2} A^\dagger | \eta_2 \rangle \langle \eta_1 | (1 + A^\dagger A)^{-1/2} A^\dagger | \lambda_1 \rangle \]

\[ = \langle \eta_2 | U | \lambda_2 \rangle \langle \eta_1 | U | \lambda_1 \rangle. \]

(A.21)

**Appendix B. Equivalence of few-body RQM to QFT**

Here we prove that the RQM corresponding to Hamiltonian $H'$ produces the same on-shell scattering amplitudes as the ones produced by the QFT corresponding to Hamiltonian $H$.

The two-body states, $|N_q \rangle = |\Psi_q^{(+)} \rangle$ and $|N'_q \rangle = |\Psi_q'^{(+) \rangle}$, are normalized as

\[ \langle N_p | N_q \rangle = \langle N'_p | N'_q \rangle = \delta^3(p_1 - q_1) \delta^3(p_2 - q_2) \]

(B.1)

\[ |N_q \rangle = Z^{-1} \lim_{\epsilon \to 0} \frac{i\epsilon}{P_q - H + i\epsilon} |q_1, q_2 \rangle \]

(B.2)

\[ |N'_q \rangle = \lim_{\epsilon \to 0} \frac{i\epsilon}{P_q - H' + i\epsilon} |q_1, q_2 \rangle \]

(B.3)
where the renormalization constant $Z_q$ is defined as the residue at the single nucleon pole of the nucleon resolvent:

$$\langle p| \frac{1}{p^- - H + i\epsilon} | q \rangle \sim \delta^3(p - q) \frac{Z_q}{p^- - q_+^2 + m^2 + i\epsilon}$$  \hspace{1cm} (B.4)

The renormalization constant $Z'_q$ in Eq. (B.3) is equal to unity. Indeed it is defined as the residue of the single particle propagator at the pole, $p^- = q_+^2 + m^2$, corresponding to the single particle intermediate state contribution

$$\langle p_1| \frac{1}{p^- - H' - i\epsilon} | q_1 \rangle \sim \delta^3(p + q) \frac{Z'_q}{p^- - q_+^2 + m^2 + i\epsilon}$$  \hspace{1cm} (B.5)

Since $H'$ is diagonal in the $\eta$ space, the corresponding self-energy, $\Sigma'(p^-, p)$,

$$\Sigma'(p^-, p) \delta^3(p - q) = \langle p| \left(H' - \frac{q_+^2 + m^2}{q^+}\right) | q \rangle$$  \hspace{1cm} (B.6)

does not depend on the "energy", $p^-$, and therefore the single particle wave function does not get renormalized, $Z'_q = 1$. Another way to see this is to realize that the one nucleon eigenstate of $H'$ is the bare nucleon

$$|\Psi_p^{(1)}\rangle = U|\Psi_p^{(1)}\rangle = |p\rangle,$$  \hspace{1cm} (B.7)

then from the definition (B.5) of $Z'$ one gets

$$\delta^3(p_1 - q_1)Z'_q = \int d^3p|\Psi_p^{(1)}\rangle \langle \Psi_p^{(1)}|q_1\rangle = \int d^3p\langle p_1|p\rangle \langle p|q_1\rangle = \delta^3(p_1 - q_1)$$  \hspace{1cm} (B.8)

The relation between the normalized two-body states $|N_q\rangle$ and $|N'_q\rangle$ is given as

$$|N'_q\rangle = \lim_{\epsilon \to 0} i\epsilon E_q - H' + i\epsilon |q_1, q_2\rangle = U \lim_{\epsilon \to 0} i\epsilon E_q - H + i\epsilon U^\dagger |q_1, q_2\rangle$$

$$= \int (dk)U \lim_{\epsilon \to 0} i\epsilon E_q - E_k + i\epsilon |N_k\rangle \langle N_k|U^\dagger |q_1, q_2\rangle$$

$$= \int (dk)U \lim_{\epsilon \to 0} i\epsilon E_q - E_k + i\epsilon |N_k\rangle \frac{1}{Z} \langle k_2, k_1| \frac{-i\epsilon}{E_k - H - i\epsilon} U^\dagger |q_1, q_2\rangle_{\text{disc}}$$

$$= \frac{1}{Z} F_U(q) U |N_q\rangle$$  \hspace{1cm} (B.9)

where $(dp) = d^3p_1d^3p_2$, in the second last line of Eq. (B.9) we have thrown away the connected part of $\langle k_2, k_1| \frac{-i\epsilon}{E_k - H - i\epsilon} U^\dagger |q_1, q_2\rangle$ because it does not contribute in the limit of $\epsilon \to 0$, it is avoided by keeping the energy of the state $|q_1, q_2\rangle$ below the pion production threshold [8].

$$\delta^3(k_1 - q_1)\delta^3(k_2 - q_2)F_U(q) = \lim_{\epsilon \to 0} \langle k_2, k_1| \frac{-i\epsilon}{E_k - H - i\epsilon} U^\dagger |q_1, q_2\rangle_{\text{disc}}$$

$$= \lim_{\epsilon \to 0} \langle k_2, k_1| U^\dagger \frac{-i\epsilon}{E_k - H' - i\epsilon} |q_1, q_2\rangle_{\text{disc}}$$  \hspace{1cm} (B.10)
Note that in contrast to [8] we have a non-trivial contribution from \( U^\dagger - 1 \) to \( F_U(q) \). Using our technique of calculating disconnected parts of matrix elements we find that \( F_U(q) = Z \):

\[
\delta^3(k_1 - q_1)\delta^3(k_2 - q_2)F_U(q) = \lim_{\epsilon \to 0} \langle k_2, k_1 | U^\dagger \frac{-i\epsilon}{E_k - H' - i\epsilon} | q_1, q_2 \rangle_{\text{disc}}
\]

\[
= \lim_{\epsilon \to 0} \langle k_2, k_1 | U^\dagger_1 U^\dagger_2 \frac{-i\epsilon}{E_k - H'_1 - H'_2 - i\epsilon} | q_1, q_2 \rangle
\]

\[
= \lim_{\epsilon \to 0} \langle k_2, k_1 | U^\dagger_1 U^\dagger_2 | q_1, q_2 \rangle \frac{-i\epsilon}{E_k - E_q - i\epsilon}
\]

\[
= \langle k_2 | U^\dagger | q_2 \rangle \langle k_1 | U^\dagger | q_1 \rangle = \langle k_2 | \Psi_{q_2} \rangle \langle k_1 | \Psi_{q_1} \rangle = Z\delta^3(k_1 - q_1)\delta^3(k_2 - q_2) \quad (B.11)
\]

where we have used, \( |\Psi'_{q_i}\rangle = |q_i\rangle \), \( H'_{q_i} = \frac{q_i^2 + m^2}{q_i} |q_i\rangle \), in the parametrization (3)

\[
\langle k_1 | U^\dagger | q_1 \rangle = \langle k_1 | U^\dagger | q_1 \rangle^* = \langle k_1 | \Psi_{q_1} \rangle = \sqrt{Z}\delta^3(k_1 - q_1) \quad (B.12)
\]

Note the trick used in the second line of Eq. (B.11): the disconnected part of a matrix element of an operator in the theory corresponding to the Hamiltonian \( H(\psi) \) is replaced by a matrix element in the theory corresponding to the sum of the commuting with each other Hamiltonians \( H(\psi_1) + H(\psi_2) \) which leads to the replacements, \( U^\dagger \eta \to U^\dagger_1 U^\dagger_2 \eta \) and \( H' \eta \to (H'_1 + H'_2) \eta \).

The missing terms in [8] imply the wrong expression for \( \langle k | U^\dagger | q \rangle \),

\[
\langle k | U^\dagger | q \rangle = \delta(k - q) \neq \delta(k - q)\sqrt{Z} \quad (B.13)
\]

Then finally we have

\[
|N'_q\rangle = U|N_q\rangle \quad (B.14)
\]

To compare with [8] using Eq. (B.2) and Eq. (B.3) we get

\[
\lim_{\epsilon \to 0} \frac{i\epsilon}{E_q - H' + i\epsilon} |q_1, q_2 \rangle = \frac{1}{Z} U \lim_{\epsilon \to 0} \frac{i\epsilon}{E_q - H + i\epsilon} |q_1, q_2 \rangle, \quad (B.15)
\]

which differs by factor \( Z \) from the analogous relation in [8]. Just this factor resulting from \( \sqrt{Z} \) associated with each nucleon leg should be factored out to derive a finite unitary scattering amplitude.

In the case of the \( NN \) scattering problem only two-nucleon scattering states \( |\Psi^{(+)}_q\rangle \) and \( |\Psi'^{(+)}_q\rangle \) (of the initial and UT-ed theories respectively) were to be related to each other. In Eq. (B.14) it has been shown that they are related to each other by UT, \( |\Psi^{(+)}_q\rangle = U|\Psi'^{(+)}_q\rangle \) (if they are normalized in the same way, say, to the unity), thereby it has been shown that the two theories produce the same scattering amplitude if it is on shell. Thus the S-matrices corresponding to the initial and UT-ed Hamiltonians coincide with each other:

\[
\langle p | S' | q \rangle = \langle \Psi'^{(+)}_q | p \rangle = \langle \Psi^{(+)}_q | U^\dagger U | \Psi'^{(+)}_q \rangle = \langle \Psi^{(+)}_q | | \Psi'^{(+)}_q \rangle = \langle p | S | q \rangle \quad (B.16)
\]

In the \( NNN \) scattering problem one should also prove, analogous to Eq. (B.14), the relation between the scattering states where two nucleons are bound in the asymptotics. This will be shown in a more detailed paper.
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