About one stochastic model of coexistence of various population groups into the urban environment

Dmitrii Kiselyov (c), Igor Inovenkov, Vladimir Nefedov
E-mail: dmi3iii@yandex.ru

Abstract. The problem of the interaction of various population groups in the framework of urban environment is of current interest this time. The population is divided into different strata according to their economic and social characteristics. For example, a population could be classified according to genetic and phenotypic characteristics, belonging to a ethnic group and, first of all, according to income level. In several countries, the co-existence of population groups belonging to different social strata gives rise to serious problems and therefore has been studied from different points of view, mainly from a sociological. Due to the qualitative analysis of various situations, it becomes possible to predict and prevent possible conflicts and problems. The significance of this problem is obvious, but reasonable proposals for its solution have not been put forward. In this regard, the construction of a qualitative, but general mathematical model of the dynamics of various groups of the population is of interest. Such a mathematical model should be built within the framework of the concept of spatial economics. This paper discusses the first version of the model for a situation where there are only two groups of people. The corresponding system of equations includes two nonlinear diffusion equations with terms describing the interaction of the population groups in model. Of course, the basic difficulty is the selection of coefficients, which will provide the picture as close as possible to reality, so it makes sense to add to the model and stochastic terms that will be responsible for random environmental factors. Thus, a two-dimensional stochastic model of the temporal dynamics of the distribution of two population groups in an urban environment was presented and numerically investigated. As a result of the mathematical modeling certain estimates were obtained regarding the feasibility of considering stochastic factors in the proposed mathematical model.

1. Introduction
The question of the interaction of various groups of people in the framework of urban education today is a very important issue.

The population is divided into different strata according to their economic and social characteristics. For example, the population can be classified according to genetic and phenotypic characteristics, belonging to one or another ethnic group and, first of all, according to income level. In many countries, both developed and developing, the coexistence of groups belonging to different social strata gives rise to serious problems and therefore has been studied from different points of view, mainly from the point of view of sociology. The coexistence of diverse populations often leads to social tensions. The authorities of most developed countries are constantly trying to combat the emergence of such situations, but when analyzing the media, it becomes clear that the political and economic measures taken often do not lead to positive results. It is for this reason that the problem
under consideration is so urgent. Due to the qualitative analysis of various situations, it becomes possible to predict and prevent possible conflicts and problems. The significance of this problem is obvious, but reasonable proposals for its solution have not been put forward. In this regard, the construction of a qualitative, but fairly general mathematical model of the dynamics of various population groups is of particular interest [1]. It is quite obvious that such a model should be built within the framework of the concept of spatial economics [2] and take into account the evolution of the area of residence of various demographic groups [3]. In this paper, the first version of the model is considered for a situation where there are only two groups of population, and the urban area itself is a rectangle with homogeneous characteristics. The corresponding system of equations includes two nonlinear diffusion equations with members describing the interaction of the population groups in question. Of course, the greatest difficulty is the selection of coefficients, which will provide the picture as close as possible to reality, so it makes sense to add a stochastic member to the model, which will be responsible for random environmental factors. We don’t know the exact behavior of the term describing noise, we only know its probability distribution. One of the related models in which this condition was successfully used is the Lotka-Volterra equation [4]. Different types of boundary conditions of the 1st, 2nd and 3rd kind are used. Further, the difference scheme and the computational algorithm for solving the corresponding algebraic system of equations are described in detail. The results of computational experiments with various parameters included in the mathematical model are presented. The regimes of “segregation” of two groups in the selected territory and the crowding out of one group of the population by another are obtained.

Page layout (headers, footers, page numbers and margins)
If you don’t wish to use the Word template provided, please set the margins of your Word document as follows.

2. Mathematical model of the joint evolution of two population groups in an urban environment

Consider two groups of the population, referred to as group 1 and group 2. It is assumed that there is interaction between the two groups in the sense that their relationship affects the distribution of the population by area of residence. Relationships can be friendly; unfriendly and "neutral". The evolution of both groups within the framework of spatial economics is described by a system of equations.

\[
\begin{align*}
\frac{\partial U}{\partial t} & = aU(a_1 - b_1U + c_1V) - d_1UV + \mu_1\text{div}(k_1(U,V)\text{grad } U) + \lambda_1dW_1(t), \\
\frac{\partial V}{\partial t} & = bV(a_2 - b_2V + c_2U) - d_2VU + \mu_2\text{div}(k_2(U,V)\text{grad } V) + \lambda_2dW_2(t),
\end{align*}
\]

(1)

where unknown functions \( U = U(x,y,t) \) and \( V = V(x,y,t) \) by definition determine, respectively, the number of the 1st group of the population at a point \((x,y)\) at a time \(t\), and \( V = V(x,y,t) \) is the number of the 2nd group of a population at a point \((x,y)\) at a time \(t\).

System (1) is solved in the region \( D \) representing the parallelepiped in the space of variables \((x,y,t)\):

\[(x,y,t) \in D = D_{xy} \times [0 < t < T_{\text{max}}] = \{(X_0 < x < X_1) \times (Y_0 < y < Y_1) \times (0 < t < T_{\text{max}})\}\]

Without loss of generality, we can further assume that \( X_1 = 0, X_2 = 1, Y_1 = 0, Y_2 = 1.\)
In system (1), a second-order differential operator (diffusion term) present on the right-hand side has the divergent form:

$$div(k \cdot \nabla F) = \nabla \cdot (k \cdot \nabla F) = \left\{ \begin{array}{l} \nabla F = \left( \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right) = \frac{\partial}{\partial x} \left( k \frac{\partial F}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial F}{\partial y} \right), s = 1, 2, \end{array} \right.$$ 

which greatly simplifies the procedure of its difference approximation.

In system (1) it is assumed that the coefficients $k_1$, $k_2$ depend on the desired functions, which automatically means the nonlinearity of the system of differential equations (1).

The dependence of these coefficients on unknown functions determines the nonlinearity of the problem under consideration and all the difficulties of its solution, which relate to it. As a rule, these coefficients in problems of nonlinear diffusion depend on the function in a power-law manner:

$$k_1(U, V) = k_1^{(0)} \times U^{\sigma_1} \times V^{\delta_1}, \quad k_2(U, V) = k_2^{(0)} \times U^{\sigma_2} \times V^{\delta_2}$$

$k_1^{(0)}, \sigma_1, \delta_1, k_2^{(0)}, \sigma_2, \delta_2$ — some constants.

For the system of equations (1), it is necessary to specify initial and boundary conditions.

The boundary conditions in the case of the first kind are of the form (must be satisfied at any time $t > 0$):

$$U_{\Gamma} = f_{U}(x, y, t), V_{\Gamma} = f_{V}(x, y, t), (x, y) \in \Gamma$$

In the case of conditions of the second kind: $\left. \frac{\partial U}{\partial n} \right|_{\Gamma_{R_{1}(0)}} = 0, \left. \frac{\partial V}{\partial n} \right|_{\Gamma_{R_{2}(0)}} = 0$ — means that the population is not replenished from the outside and does not leave the city.

The boundary conditions of the third kind are: $\frac{\partial U}{\partial n} + h_1 U|_{\Gamma} = v_1(t), \frac{\partial V}{\partial n} + h_2 V|_{\Gamma} = v_2(t)$ — closest to the description of the real picture of the population of the city. In accordance with the possibilities for resettlement, the population flow is directed either inward or outside the city.

Initial conditions at $t = 0$ are:

$$U(x, y, t)|_{t=0} = U_0(x, y), V(x, y, t)|_{t=0} = V_0(x, y), (x, y) \in \{(0 < x < 1) \times (0 < y < 1)\}$$

2.1 Features of the stochastic member.

Let us pay particular attention to the $dW_{1}(t), dW_{2}(t)$ — terms in our equation (coefficients $\sigma_1, \sigma_2$ are constants) and analyze what impact they will have on the solution. As it is known, methods for solving ODEs are not applicable in the case of the SDE. Consider a system of two stochastic differential equations in vector form [5]:

$$dX_t = a(X_t)dt + b(X_t)dW_t,$$

where $t \geq 0$ is time, $X_t = X(t)\Gamma$ is a two-dimensional random process in continuous time, $X(0) = X_0 \in R^2$ is the initial condition, $a \in R^2 \times [0; \infty), b$ is a matrix function dimensions of $2 \times m$, $b : R^2 \times [0; \infty)$. $W = \left\{ W_t = (W_{t}^1, W_{t}^2, ..., W_{t}^m)^T, t \geq 0 \right\}$ — m-dimensional standard Wiener process with components $W_{t}^1, W_{t}^2, ..., W_{t}^m$. The Wiener random process here is a mathematical model of white noise in continuous time that meets the conditions:

1. $P(W_0 = 0) = 1$.
2. $W_t$ is process with independent increments.
3. $W_t - W_{t-s} \sim N(0, \sigma^2(t - s)), 0 \leq s < t < \infty$.

The system of stochastic differential equations can be rewritten in integral form:
\[ X_t = X_0 + \int_0^t a(X_s, s)ds + \int_0^t b(X_s, s)dW_s, \]  
\[ \int_0^t b(X_s, s)dW_s = \lim_{h \to 0} \frac{1}{h} \sum_{k=0}^{h-1} b(X_{\tau_k}, \tau_k)(W_{\tau_{k+1}} - W_{\tau_k}), \]

If \( q = 0 \), then formula (6) determines the Ito integral, and accordingly the Stratonovich stochastic integral is obtained for \( q = 0.5 \) [5].

2.2 Numerical method for solving the problem

From the set of described approaches to the numerical solution of the stochastic equations, we choose the Euler-Maruyama method. For selected conditions, this approach is most reasonable for use. As part of the calculus, the Ito method is the simplest generalization of the Euler method for ordinary differential equations to stochastic differential equations. By the definition of strong convergence, the numerical method converges strongly to \( X(t) \) at time \( T \) with order \( y > 0 \) if there is a constant \( C > 0 \) that does not depend on \( h \) and the number \( \delta > 0 \) such that \( E|X(T) - Y(T)| \leq Ch^y \), for all \( h \in (0; \delta) \). Here \( X(t) \) is the solution of the equation on \( [0; T] \) in Ito's calculus, and \( Y(t) \) is some approximation of \( X(t) \). In these terms, the Euler-Maruyama method provides convergence in the strong sense of order \( \frac{1}{2} \) [5].

We divide the interval \( [0, T] \) into \( N \) segments of equal length \( \Delta t > 0 \): 
\[ 0 = t_0 < t_1 < \ldots < t_N = T \] and \( \Delta t = T/N \). Put \( Y_0 = x_0 \), then: 
\[ Y_{n+1} = Y_n + \alpha(Y_n)\Delta t + b(Y_n)\Delta W_n, \quad \Delta W_n = W_{\tau_{n+1}} - W_{\tau_n}. \]

Let us introduce grids according to spatial coordinates \( x \) and \( y \) :
\[ x_i = X_0 + i h_x, \quad y_j = Y_0 + j h_y, \quad i = 0, 1, \ldots, N_x; \quad j = 0, 1, \ldots, N_y, \]
where \( h_x = \frac{X_2 - X_1}{N_x}, \quad h_y = \frac{Y_2 - Y_1}{N_y} \) i.e. we assume that \( x \in [X_1, X_2], \ y \in [Y_1, Y_2], \ N_x, N_y \) is the selected number of nodes.

Next, we implement the remaining terms without stochastic ones, which allows us to make the Euler-Maruyama method. To build a difference scheme, we write the system of equations (1) in the operator form:
\[ \frac{\partial U}{\partial t} = \Omega_1[U, V] + \mu_1 \Lambda_1[U], \]
\[ \frac{\partial V}{\partial t} = \Omega_2[V, U] + \mu_2 \Lambda_2[V], \]

where the designations are taken:
\[ \Omega_1[U, V] = \alpha(U(a_1 - b_1 U - c_1 V) - d_1 U V), \quad \Lambda_1[U] = \text{div}(k_1(U, V) \text{grad } U), \]
\[ \Omega_2[V, U] = \beta(V(a_2 - b_2 V - c_2 U) - d_2 V U), \quad \Lambda_2[V] = \text{div}(k_2(U, V) \text{grad } V). \]

The differential operator of the second order \( \Lambda_s[F] = \text{div}(k_s \text{grad } F), \ s = 1, 2 \) in Cartesian coordinates \( (x, y) \), has the following standard form:
\[ \Lambda_s[F] = \text{div}(k_s \text{grad } F) = \frac{\partial}{\partial x} \left( k_s \frac{\partial F}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_s \frac{\partial F}{\partial y} \right) \]

Difference approximation of this differential operator \( \Lambda_s[F] \) in the internal nodes \( P_{i,j} \) of the grid is carried out on a five-point "cruciform" pattern that includes nodes \( P_{i-1,j}, P_{i-1,j-1}, P_{i,j}, P_{i+1,j}, P_{i,j+1} \):
Such an approximation is considered in the book [6], where the grid functions \( \sigma_{s,i}(i + 1, j) \), \( \sigma_{s,i}(i, j) \), \( \sigma_{s,j}(i, j + 1) \), \( \sigma_{s,j}(i, j) \) be calculated using the following averaged formulas:

\[
\begin{align*}
\sigma_{s,i}(i + 1, j) &= \frac{k_s(i + 1, j) + k_s(i, j)}{2}, \\
\sigma_{s,j}(i, j + 1) &= \frac{k_s(i, j + 1) + k_s(i, j)}{2},
\end{align*}
\]

Next, we discuss the approximation of our equations (1) to the time variable \( t \), where \( 0 < t < T_{\text{max}} \).

The maximum calculation time \( T_{\text{max}} \) is set. The time step \( \tau = \frac{T_{\text{max}}}{N_r} \), i.e. the solution must be calculated on \( N_r \) temporary layers.

An alternative to this is the task of the time step \( \tau \) and, accordingly, the maximum number of time layers \( N_r \) on which the solution of the problem will be calculated. Since an explicit (in time) difference scheme of the “predictor-corrector” type [6] was used for the numerical solution, the time step \( \tau \) should be chosen rather “small”. It can be expected that such a numerical scheme will have an order of approximation and, therefore, order accuracy \( O(\tau^2 + h_i^2 + h_j^2) \).

Thus, the “predictor-corrector” scheme for our task will look as follows (7):

\[
\begin{align*}
U_{ij}^{(k+1)} &= U_{ij}^{(k)} + 0.5\tau \left( \Omega_1 \left[ U_{ij}^{(k+1/2)} , V_{ij}^{(k+1/2)} \right] + \mu_1 \Lambda_1^{(h)} \left[ U_{ij}^{(k+1/2)} \right] + \lambda_1 \Delta W_1(t) \right), \\
V_{ij}^{(k+1)} &= V_{ij}^{(k)} + 0.5\tau \left( \Omega_2 \left[ V_{ij}^{(k+1/2)} , U_{ij}^{(k+1/2)} \right] + \mu_2 \Lambda_2^{(h)} \left[ V_{ij}^{(k+1/2)} \right] + \lambda_2 \Delta W_2(t) \right),
\end{align*}
\]

Thus, at first, in all internal nodes, intermediate functions are calculated on a half-integral time layer \( t_{k+1/2} = t_k + 0.5\tau \) using formulas (7). Then, according to the well-known scheme (“corrector”), the solution is recalculated from the layer \( (k) \rightarrow (k + 1) \) in all internal nodes (8):

\[
\begin{align*}
U_{ij}^{(k+1)} &= U_{ij}^{(k+1)} + \tau \left( \Omega_1 \left[ U_{ij}^{(k+1/2)} , V_{ij}^{(k+1/2)} \right] + \mu_1 \Lambda_1^{(h)} \left[ U_{ij}^{(k+1/2)} \right] + \lambda_1 \Delta W_1(t) \right), \\
V_{ij}^{(k+1)} &= V_{ij}^{(k+1)} + \tau \left( \Omega_2 \left[ V_{ij}^{(k+1/2)} , U_{ij}^{(k+1/2)} \right] + \mu_2 \Lambda_2^{(h)} \left[ V_{ij}^{(k+1/2)} \right] + \lambda_2 \Delta W_2(t) \right),
\end{align*}
\]

It should be noted that the computational formulas (7) - (8) are valid only in the internal nodes of the difference grid, and in the boundary nodes it is necessary to approximate the boundary conditions of the form (2).
2.3 Features of software implementation
The development of the software implementation of the model was carried out in the C++ programming language. The first task was the question of obtaining the Wiener process. Considering the amount of data and a sharp increase in the computational complexity of our task, the MKL library from Intel was chosen in the future, which can provide high efficiency and performance of the PSHP generators, as well as provide good functionality for further program parallelization.
To obtain the normal distribution necessary for the realization of the Wiener process, the method of inverse function was chosen. During initialization, the basic generator was initialized with a large number in order to ensure that numbers fall into the interval we need. For the selected model, the coefficients at the generators were chosen to be 1 - depending on the influence that is placed in the noise, this value can be changed.

3. Results
In the course of the numerical solution of the problem and the subsequent program implementation, it was possible to obtain qualitative results for certain special cases. A grid of 100x100 pixels with 1000-time steps was considered. In the graphs presented in the form of a map, at each point, the function that characterizes the population, whose value is greater, was chosen for display. Initially, it was assumed that the groups occupy two circular areas with equal population. The interaction coefficients were chosen as power functions, in accordance with the arguments presented above:

\[ k_1(U,V) = k_2(U,V) = e^{UV}. \]

Thus, the following graphs were obtained for the results:

![Fig. 1a (1st time step)](image-url)
Fig. 1b (10th time step)

Fig. 1c (1000th time step)

Displays the evolution of the interaction of the two groups. In this case, there is a severe segregation of the two groups in the designated area. Based on the assumption that, with a numerical advantage, one group will crowd out another, it is possible to draw conclusions about how the groups will move within the occupied territory.

Of interest is the comparison with the case of equations without stochastic terms. To do this, we consider two pictures obtained at similar time steps.
In Fig. 2b shows the case with the stochastic member present, in Fig. 2a, respectively, it is not. We can see that the value exerted on the process is significant, so it would be erroneous to neglect this factor. In addition, when considering the results, it can be noted that the overall dynamics does not change - the qualitative result is like the usual one.

4. Conclusion
In this article, for the first time, a two-dimensional stochastic model of the temporal dynamics of the two population groups in an urban environment was investigated. It is worth noting that in the course of the work we managed to get correct and significant results in relation to the use of the stochastic member. In addition, the conditions under which one group gradually displaces another from its territory are defined and analyzed. The model allows to investigate at a qualitative level the tendencies of settlement by various groups of the urban environment.

5. References
[1] Arnold V.I. “Hard” and “soft” mathematical models // Published by Moscow Center for Continuous Mathematical Education (MCCME, Moscow). 2004. 32 P
[2] Martin Beckmann, Tonu Puu. Spatial economics : density, potential, and flow. Amsterdam; New York: North-Holland ; New York, N.Y., U.S.A. : Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co., 1985 , 276 p
[3] Zhang W.-B. Synergetic economics: Time and change in nonlinear economics (Springer series in Synergetics) // Berlin, Germany: Springer-Verlag. 1991. 246 P.
[4] M. Arató. A famous nonlinear stochastic equation (Lotka-Volterra model with diffusion). Mathematical and Computer Modelling, Volume 38, Issues 7–9, 2003, Pages 709-726.
[5] Kuznetsov D.F. Stochastic differential equations: theory and practice of numerical solution (fourth edition, revised and augmented). - SPb. : Publishing house of the Polytechnic University, 2010. - 816 C.
[6] Alexander A. Samarskii. The Theory of Difference Schemes //CRC Press , 2001, 786 Pages
[7] Purvis, Ben & Mao, Yong & Robinson, Darren. Entropy and its Application to Urban Systems. // Entropy. 21. 56. (2019). DOI: 10.3390/e21010056.
[8] Frederic Guichard, Tarik C. Gouhier. Non-equilibrium spatial dynamics of ecosystems // Mathematical Biosciences, Volume 255, 2014, Pages 1-10, ISSN 0025-5564. DOI: 10.1016/j.mbs.2014.06.013.
[9] Edward W. Tekwa, Andrew Gonzalez, Michel Loreau. Spatial evolutionary dynamics produce a negative cooperation–population size relationship. // Theoretical Population Biology, Volume 125, 2019, Pages 94-101, ISSN 0040-5809. DOI: 10.1016/j.tpb.2018.12.003.

[10] Biao Wang, Zhengce Zhang. Dynamics of a diffusive competition model in spatially heterogeneous environment. // Journal of Mathematical Analysis and Applications, Volume 470, Issue 1, 2019, Pages 169-185, ISSN 0022-247X. DOI: 10.1016/j.jmaa.2018.09.062.

[11] Brian Drawert, Bruno Jacob, Zhen Li, Tau-Mu Yi, Linda Petzold. A hybrid smoothed dissipative particle dynamics (SDPD) spatial stochastic simulation algorithm (sSSA) for advection–diffusion–reaction problems. // Journal of Computational Physics, Volume 378, 2019, Pages 1-17, ISSN 0021-9991. DOI: 10.1016/j.jcp.2018.10.043.

[12] Ulises Badillo-Hernandez, Jesus Alvarez, Luis Alvarez-Icaza. Efficient modeling of the nonlinear dynamics of tubular heterogeneous reactors. // Computers & Chemical Engineering, Volume 123, 2019, Pages 389-406, ISSN 0098-1354, DOI: 10.1016/j.compchemeng.2019.01.018.