STOCHASTIC PARTICLE ACCELERATION AND THE PROBLEM
OF BACKGROUND PLASMA OVERHEATING

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ABSTRACT

The origin of hard X-ray (HXR) excess emission from clusters of galaxies is still an enigma, whose nature is debated. One of the possible mechanisms to produce this emission is the bremsstrahlung model. However, previous analytical and numerical calculations showed that in this case the intracluster plasma had to be overheated very fast because suprathermal electrons emitting the HXR excess lose their energy mainly by Coulomb losses, i.e., they heat the background plasma. It was concluded also from these investigations that it is problematic to produce emitting electrons from a background plasma by stochastic (Fermi) acceleration because the energy supplied by external sources in the form of Fermi acceleration is quickly absorbed by the background plasma. In other words, the Fermi acceleration is ineffective for particle acceleration. We revisited this problem and found that at some parameter of acceleration the rate of plasma heating is rather low and the acceleration tails of nonthermal particles can be generated and exist for a long time while the plasma temperature is almost constant. We showed also that for some regime of acceleration the plasma cools down instead of being heated up, even though external sources (in the form of external acceleration) supply energy to the system. The reason is that the acceleration withdraws effectively high-energy particles from the thermal pool (analog of Maxwell demon).

Key words: galaxies: clusters: individual (Coma) – plasmas – X-rays: galaxies: clusters

1. INTRODUCTION

One of the most important problems in astrophysics is the problem of particle acceleration. The general expression for acceleration of a charged particle is

\[ \frac{dE}{dt} = \frac{Ze}{\gamma m c^2} \left( \mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B} \right), \]  

where \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic field strength, \( \mathbf{v} \) is the velocity of the particle, and \( \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \). In most astrophysical conditions static electrical fields cannot be maintained because of a very high electrical conductivity. Therefore, the acceleration can be associated with non-stationary electromagnetic fields (electromagnetic waves) as suggested by Fermi (1949, 1954). He assumed that the Galactic cosmic rays were stochastically accelerated by collisions of charged particles moving with velocity \( v \) with fluctuations of magnetic fields (magnetic clouds) moving chaotically with velocity dispersion \( u \ll v \). Fermi (stochastic) acceleration is described as momentum diffusion in the kinetic equations (see, e.g., Toptygin 1985). We will use Fermi acceleration and stochastic acceleration interchangeably.

The efficiency of Fermi acceleration is low. Nevertheless, it may be essential in solar flares (see, e.g., Miller et al. 1990; Petrosian 2012), in the interstellar medium of the Galaxy (Berezhinskii et al. 1990), and near the Galactic center (see, e.g., Cheng et al. 2011, 2012; Chernyshov 2011; Mertsch & Sarkar 2011).

The problem of stochastic particle acceleration in galaxy clusters arose from observations in the hard X-ray (HXR) energy range (see, e.g., Fusco-Femiano et al. 1999, 2007; Rephaeli et al. 1999, 2008; Eckert et al. 2008; Nevalainen 2009; Ajello et al. 2010) that showed an emission excess above the equilibrium thermal X-ray spectrum.
which a nonthermal spectrum is formed by acceleration. It is
determined by equating these rates of acceleration and loss.

The kinetic equation in the particle momentum space describing
stochastic particle acceleration from background plasma has the
form (assume isotropic distribution)

$$\frac{\partial f}{\partial t} + \frac{1}{p^2} \frac{\partial}{\partial p} p^2 \left( \frac{dp}{dt} \right)_c f - \{ D_c(p) + D_F(p) \} \frac{\partial f}{\partial p} = 0, \tag{3}$$

where $D_F(p)$ is the diffusion coefficient of stochastic (Fermi)
acceleration and $(dp/dt)_c$ and $D_c(p)$ describe particle momentum
losses and diffusion due to Coulomb collisions. These
coefficients are calculated from the total distribution function
$f$ (see Appendix A), and therefore in general the equation is
nonlinear.

2.1. Linear Approximation

An analytical solution of this equation for the case of weak
acceleration from a background plasma with temperature $T$
was obtained by Gurevich (1960). The term of stochastic
acceleration was taken in the phenomenological form

$$D_F(p) = \alpha p^2. \tag{4}$$

The analysis was provided for the case when the characteristic
time of stochastic acceleration,

$$\tau_F = p^2 / D_F, \tag{5}$$
is much larger than the time of thermal particle collisions, $\tau_{th}$,

$$\tau_{th} \approx \frac{2}{m} \frac{m_e(k_B T)^{3/2}}{\pi N e^4 \ln \Lambda}, \tag{6}$$

where $N$ is the density and $T$ is the temperature of background
plasma, $\ln \Lambda$ is the Coulomb logarithm, $m_e$ is the electron rest
mass, and $m$ is the mass of accelerated particles.

In this case the injection energy $E_{inj}$ is much larger than
the plasma temperature, $E_{inj} > k_B T$. Coulomb collisions
keep the equilibrium Maxwellian distribution for most of the
momentum range, and the coefficients of Equation (3) for
nonrelativistic momenta $p > \sqrt{2mk_B T}$ (as used by Gurevich
1960) for the Maxwellian distribution function. For $\tau_{th} < \tau_F$,
significant distortions from the equilibrium Maxwellian state
are expected only for very large values of momenta, and a very
small fraction of thermal particles is accelerated. Therefore,
Gurevich (1960) assumed that the number of particles $N(t)$ in
the momentum range $p < p_{inj}$ varies very slowly with time $t$,
$N(t) = N_0 - \Delta t$, where $N_0$ is the initial particle density and a
small runaway flux $S$ is generated at relatively high momentum
range. The runaway flux for the case of slow acceleration can
be described as

$$S(p) = S_0 \frac{4}{\sqrt{\pi}} \int_0^\infty \tilde{\rho} x^2 e^{-\tilde{x}^2} \, dx = S_0 \left[ \text{erf} (\tilde{p}) - \frac{2}{\sqrt{\pi}} \tilde{p} e^{-\tilde{p}^2} \right], \tag{7}$$

where erf($z$) is the error function, $\tilde{p} = p/\sqrt{2mk_B T}$, and the
constant $S_0$ is derived from boundary conditions. The flux is
zero at $p = 0$, but when $p > \sqrt{2mk_B T}$, it reaches a maximum
value $S(p) = S_0$ as shown in Figure 1.

In the momentum range where $S(p) \approx S_0$ the distribution
function is non-Maxwellian and is described by the kinetic
equation

$$p^2 \left( \frac{dp}{dt} \right)_c f - \{ D_c(p) + D_F(p) \} \frac{\partial f}{\partial p} = S_0. \tag{8}$$

However, if $D_F(p) \neq 0$ in the range $p < p_{inj}$, a solution of
this equation describes also an excess of the distribution
function above the equilibrium Maxwellian distribution in
momentum ranges both above and below $p_{inj}$ (see Gurevich
1960). This excess at $p > p_{inj}$ is formed by Coulomb collisions
in the transition range between the thermal (Maxwellian) and
nonthermal parts of the spectrum. If the HXR excess is due
to bremsstrahlung emission of electrons from this transition
region, then the relation (2) used by Petrosian (2001) cannot be
applied to the estimate of $L_C$ and more accurate calculations are
necessary.

Bremsstrahlung emission of electrons from the transition
region was calculated in Dogiel (2000), Liang et al. (2002),
and Dogiel et al. (2007). The conclusion is that the necessary
energy input $L_C$ for Coma was about one order of magnitude less
than obtained by Petrosian (2001). This may solve the problem
of the plasma overheating. However, their linear analysis of
Equation (3) does not include variations of temperature $T$, which
is supposed to be constant.

2.2. Nonlinear Treatment

More reliable conclusions can be derived from analyses of
the nonlinear equation in the form similar to those used by
MacDonald et al. (1957), when a feedback of accelerated
particles on the plasma temperature is taken into account. Very
recently Wolfe & Melia (2006) and Petrosian & East (2008)
provided similar numerical analysis for the case of stochastic
acceleration from a background plasma.

The nonlinear kinetic equation describing particle Coulomb
collisions is derived in Landau & Lifshitz (1981) (see Appendix A).
Using this theory Nayakshin & Melia (1998) derived coefficients of
this equation for the case of isotropic and
homogeneous distribution function for nonrelativistic and ultra-
relativistic particles. Later, Wolfe & Melia (2006) extended their
analysis to the general case of anisotropic distribution function.

Numerical analysis of these equations has been performed by
Wolfe & Melia (2006) for the isotropic stochastic acceleration of the form

\[ D_F(p) = \alpha p^5 (p - 1/2). \]  

Wolfe & Melia (2006) stated that the continuous stochastic ac-
celeration of thermal electrons produced a nonthermal tail. But
for the HXR emission in the Coma Cluster this model actually
cannot work because the energy gained by the particles is dis-
tributed to the whole plasma on a timescale much shorter than
that of the acceleration process itself. Moreover, bremsstrahlung
is relatively inefficient for cooling the accelerated electrons; the
energy of this tail is quickly dumped into the thermal back-
ground plasma and heats the plasma.

Similarly, Petrosian & East (2008) obtained numerical solu-
tions of the nonlinear isotropic kinetic equations that included
effects of plasma heating for the stochastic diffusion in the form

\[ D_F(\mathcal{E}) = \frac{\mathcal{E}^2}{\zeta(\mathcal{E}) r_0 (1 + \mathcal{E}/\mathcal{E}_c)^q}, \]  

where \( \mathcal{E} = \sqrt{p^2 + 1} - 1 \) is the kinetic energy normalized to
\( mc^2 \), \( \zeta(\mathcal{E}) = (2 - \gamma^{-2})/(1 + \gamma^{-1}) \), and \( r_0, \mathcal{E}_c, \) and \( q \) are free parameters.

Petrosian & East (2008) concluded that their calculations confirmed qualitatively results of Dogiel et al. (2007) that the
required input energy \( L_c \) was lower than that which follows from the estimate (2) but by a factor of two or three only,
and that did not solve the problem of plasma overheating.

Besides, they argued that their calculations confirmed results of Wolfe & Melia (2006) that stochastic acceleration could not work in clusters because the energy gained by the particles was distributed to the whole plasma on timescales much shorter than that of the acceleration process. At acceleration rates smaller than the thermalization rate of the background plasma, there is very little acceleration. The primary effect of acceleration is heating of the plasma. In the opposite case, at higher energizing rates, a distinguishable nonthermal tail is developed, but this is again accompanied by an unacceptably high rate of heating.

In other words, it follows from these investigations that it is problematic to accelerate particles from a background plasma because the main effect of this acceleration is plasma overheating. The energy supplied by external sources in the form of stochastic (Fermi) acceleration is quickly absorbed by a background plasma. An interesting question arises: whether any conditions exist when the stochastic acceleration generates prominent nonthermal tails while the plasma is not overheated and its temperature varies relatively slowly. From the analysis in the following sections, we argue that the answer is affirmative.

3. PARTICLE ACCELERATION FROM BACKGROUND PLASMA: QUASI-LINEAR APPROXIMATION

First, we estimate variations of plasma temperature derived in quasi-stationary approximations when the distribution function can be presented as \( f = f(p, N, T) \). In this case,

\[ \frac{\partial f}{\partial t} = \frac{\partial}{\partial N} \frac{df}{dt} + \frac{\partial f}{\partial T} \frac{dT}{dt}, \]  

where \( N = N(t) \) and \( T = T(t) \) are slowly varying functions of \( t \).

3.1. Distribution Function

In this subsection we investigate the isotropic form of the kinetic equation (A1). This equation describes stochastic particle acceleration from background plasma and is exactly the same as Equation (3). The appropriate boundary conditions are Equations (A5) and (A6). Recall that the particle momentum has been normalized to \( mc \). Here and in the following the temperature \( T \) is indeed the thermal energy \( k_B T \) normalized to \( mc^2 \).

The particle kinetic energy \( \mathcal{E} = \sqrt{p^2 + 1} - 1 \) is also normalized to \( mc^2 \). The coefficients \( (dp/dt)_C(p) \), \( D_C(p) \), and \( D_F(p) \) are normalized accordingly.

The stochastic Fermi acceleration is supposed to be isotropic and has a phenomenological form as

\[ D_F(p) = \alpha p^5 (p - p_0), \]  

where \( \alpha, \zeta, \) and \( p_0 \) are arbitrary parameters. The problem is characterized also by the injection momentum

\[ \alpha p_{inj}^5 = -\left. \left( \frac{dp}{dt} \right) \right|_{p=p_{inj}}. \]  

The acceleration is effective in the momentum range \( p > \max\{p_0, p_{inj}\} \).

Similar to Gurevich (1960), we assume that the acceleration time \( t_F \) is much longer than the time of thermal particle collisions \( t_n \), i.e., values of \( p_{inj} \) or \( p_0 \) are large and one of the corresponding energy values is much higher than the temperature,

\[ T < \max(\mathcal{E}_{inj}, \mathcal{E}_0). \]  

In this case, Coulomb collisions keep the equilibrium Maxwellian distribution over an extended momentum range with a significant deviation from this distribution at very large momenta, i.e., a small part of thermal particles is accelerated. The number of nonthermal particles generated by the acceleration \( N_n \) in this case is much smaller than the number of thermal particles \( N_n/N << 1 \).

Below we present the distribution function and the coeffi-
cients of the kinetic equation as series expansions over the small parameter \( \epsilon = N_n/N \ll 1 \).

\[ f(p, t) = f_0(p, t) + f_1(p, t) + O(\epsilon^2), \]

\[ D_f(p, t) = D_0(p, t) + D_1(p, t) + O(\epsilon^2), \]

\[ \left( \frac{dp}{dt} \right)_c(p, t) = \left( \frac{dp}{dt} \right)_0(p, t) + \left( \frac{dp}{dt} \right)_1(p, t) + O(\epsilon^2). \]  

Here \( O(\epsilon^2) \) denotes terms of order \( \epsilon^2 \) or above. Note that \( f_1(p, t) = O(\epsilon^2) \), \( D_0 \), and \( (dp/dt)_0 \) are calculated from Equation (A2) for the function \( f_0 \), \( D_1 \) and \( (dp/dt)_1 \) for the function \( f_1 \), etc.

In the quasi-stationary approximation the derivative \( \partial f/\partial t \) can be presented in the form (12). The derivatives \( dN/dt \) and \( dT/dt \) can be presented as series \( dN/dt = O(\epsilon) \) and \( dT/dt = O(\epsilon) \), because without acceleration \( (N_n = 0) \) we have \( dN/dt = 0 \) and \( dT/dt = 0 \). Here we have

\[ \frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial t} + O(\epsilon^2), \]  

and \( \partial f_0/\partial t \) is of the order of \( \epsilon \).

It is convenient to express the distribution function as

\[ f(p) = f^1(p)\delta(p_0 - p) + f^{11}(p)\delta(p - p_0). \]  

\[ \frac{\partial f}{\partial t} = \frac{\partial f_0}{\partial t} + O(\epsilon^2), \]  

and \( \partial f_0/\partial t \) is of the order of \( \epsilon \).

It is convenient to express the distribution function as

\[ f(p) = f^1(p)\delta(p_0 - p) + f^{11}(p)\delta(p - p_0). \]
First, we find the solution of Equation (3) in the momentum range $0 < p < p_0$, where the acceleration term vanishes and $f = f^l$ (see Equation (18)). In zero order of expansion (no acceleration) the function $f_0$ is Maxwellian,

$$f_0^l(p) = C_0 \exp \left[ \int_0^p \left( \frac{dp}{dt} \right)_0 \frac{dp}{D_0} \right] = C_0 \exp(-\mathcal{E}/T). \quad (19)$$

where $D_0$ and $(dp/dt)_0$ are the Maxwellian kinetic coefficients. For $p \gg \sqrt{T^2+1}-1$ the Bethe-Bloch approximation for these coefficients is

$$\left( \frac{dp}{dt} \right)_0 = -A \left( 1 + \frac{1}{p^2} \right), \quad (20)$$

$$D_0 = -T \left( 1 + \frac{1}{p^2} \right) \frac{dp}{dt}_0 = AT \left( 1 + \frac{1}{p^2} \right)^{3/2}. \quad (21)$$

Here and below $A = 4\pi r_e^2 c N \ln \Lambda$. (22)

The characteristic time of Coulomb losses for a particle with momentum $p$ (in units of $mc$) is

$$\tau_C(p) \sim \frac{p^3}{A(p^2+1)}. \quad (23)$$

Constant $C_0$ is estimated from the normalization condition

$$C_0 = N \left[ \int_0^{p_0} p^2 f_0^l(p) dp \right]^{-1} \approx \frac{N \exp(-T^{-1})}{TK_2(T^{-1})}, \quad (24)$$

where $K_2(x)$ is the modified Bessel function. For nonrelativistic temperatures $(T \ll 1)$ we obtain

$$C_0 \approx N \sqrt{\frac{2}{\pi}} T^{-3/2}. \quad (25)$$

The kinetic equation for the function $f_1^l$ can be rewritten as

$$\frac{1}{p^2} \frac{\partial}{\partial p} p^2 \left[ D_{0l}(p) \frac{\partial f_1^l}{\partial p} + D_1(p) \frac{\partial f_1^l}{\partial p} - \left( \frac{dp}{dt} \right)_0 f_1^l \right] = \frac{\partial f_0^l}{\partial t} + O(\varepsilon^2). \quad (26)$$

Integrating the above equation from 0 to $p$ gives

$$p^2 \left[ D_{0l}(p) \frac{\partial f_1^l}{\partial p} - \left( \frac{dp}{dt} \right)_0 f_1^l \right] = \frac{\partial f_0^l}{\partial t} = -(S_1 + S_2), \quad (27)$$

where $S$ is the flux of particles through the point $p$. Here

$$S_1 = -\frac{dN(p,t)}{dt} = -\frac{\partial}{\partial t} \int_0^p u^2 f_0^l(u) du, \quad (28)$$

$$S_2 = p^2 \left[ D_1(p) \frac{\partial f_0^l}{\partial p} - \left( \frac{dp}{dt} \right)_0 f_0^l(p) \right]. \quad (29)$$

The flux $S_1$ describes a particle leakage (in momentum space) caused by the acceleration. It generates a slow decrease of particle number in the thermal region. The flux $S_2$ causes the plasma heating and temperature variations with time.

Thus, the solution of Equation (26) is

$$f_1^l(p) = \exp(-\mathcal{E}/T) \left[ C_1 - \int_0^p \frac{S(u)}{u^2 D_0(u)} \exp(\mathcal{E}/T) du \right]. \quad (30)$$

As in Gurevich (1960), the value of the constant $C_1$ can be derived from the normalization condition

$$\int_0^{p_0} p^2 f_1^l(p) dp = 0. \quad (31)$$

The kinetic coefficients $D_1$ and $(dp/dt)_1$ are calculated for the function $f_1^l + f^{ll}$. Therefore, Equation (29) is an integral equation for $f_1^l(p)$ that should be added by an equation for $f^{ll}(p)$. The asymptotic form of $f_1^l(p)$ for large values of $p$ can easily be derived. Indeed, if $\mathcal{E} \gg T$, then $S_1(p) = O(\varepsilon)$ while $S_2(p) \sim O(\varepsilon) \exp(-\mathcal{E}/T) \ll S_1$. Therefore, $S_2(p)$ can be neglected. As one can see from Equation (27), the flux $S_1(p)$ remains almost constant for sufficiently large $p$ (see Figure 1). So with a high degree of accuracy we can put $S_1(p) = S_N \equiv -d\ln N/dt$, which is the same as $S_0$ in Gurevich (1960).

It follows from Equations (29) and (30) that the constant $C_1$ is

$$C_1 \approx S_N \tau_C(p_0) \sqrt{\frac{2}{\pi}} T^{-3/2}, \quad (32)$$

where $\tau_C(p_0)$ is the characteristic time of Coulomb collision for the particle momentum $p = p_0$ (for $\tau_C$ and $\tau$ see Equations (5) and (22)). Thus, for the estimation of $S_N$ obtained in Section 3.3 we have

$$C_1 \approx N T^{3/2} \exp \left( \frac{-\mathcal{E}_0}{T} \right) \frac{\tau_C(p_0)}{\tau_T(p_0)} \sim N T^{3/2} \exp \left( -\frac{\mathcal{E}_0}{T} \right) \ll C_0. \quad (33)$$

The distribution function in the range $p < p_0$ can be written as

$$f^l(p) \simeq f_1^l(p) + f^{ll}(p) = \frac{N}{TK_2(T^{-1})} \exp \left( -\frac{\xi}{T} \right) - \frac{S_N}{AT} \left[ \frac{1}{T} \exp \left( -\frac{\xi}{T} \right) Ei \left( \frac{\xi}{T} \right) - 1 \right], \quad (34)$$

where $\xi = \sqrt{p^2+1} = E + 1$ is the total energy of the particle and

$$Ei(z) = \int_z^{\infty} \frac{\exp(x)}{x} dx. \quad (35)$$

For nonrelativistic temperatures $\xi/T \gg 1$ the expansion of $Ei(z)$ for $z \gg 1$ is

$$Ei(z) = \frac{\exp(z)}{z} \sum_{k=0}^{\infty} \frac{k!}{z^k}. \quad (36)$$

Thus, for large values of $p$

$$f^l(p) = \sqrt{\frac{2}{\pi}} T^{3/2} \exp \left( -\frac{\mathcal{E}}{T} \right) - \frac{S_N}{A(p^2+1)}. \quad (37)$$

The distribution function Equation (33) can be presented in the form

$$f^l(p) = \begin{cases} f_1^l(p) + O(\varepsilon), & \text{for } \mathcal{E} \leq T, \\ f_1^l(p) - \frac{S_N^2}{A(p^2+1)} + O(\varepsilon^2), & \text{for } \mathcal{E}_0 \geq \mathcal{E} \gg T. \end{cases} \quad (38)$$
In the range \( p \geq p_0 \) the acceleration cannot be neglected. With the constant flux \( S_K \) of particles the equation for the distribution function \( f^\mu \) in this region reads
\[
p^2 \left[ D_0(p) + D_F(p) \frac{\partial f^\mu}{\partial p} \right] p - \left( \frac{dp}{dt} \right)_C f^\mu = -S_K. \tag{38}
\]
The general solution of this equation is (see, e.g., Gurevich 1960)
\[
f^\mu(p) = C^\mu \exp \left\{ \int_0^p \left( \frac{dp}{dt} \right)_0(u) du / D_F(u) + D_0(u) \right\} - S_K \exp \left\{ \int_0^p \left( \frac{dp}{dt} \right)_0(u) du / D_F(u) + D_0(u) \right\} \times \exp \left\{ - \int_0^p \frac{v^2 dv}{D_F(u) + D_0(u)} \right\}. \tag{39}
\]
The constant \( C^\mu \) can be estimated from the continuity condition at \( p = p_0 \), \( f^\mu(p_0) = f^\mu(p_0) \), while the value of \( S_K \) can be estimated from the second boundary condition, \( f^\mu(p_{\text{max}}) = 0 \).

For \( p \gg p_0 \), we can assume that acceleration dominates Coulomb loss. It is easy to show from Equations (38) and (13) that the function \( f^\mu(p) \) is a power law
\[
f^\mu(p) = \tilde{C}_1 + \frac{S_K}{\alpha(\xi + 1)} p^{-\xi - 1}, \tag{40}
\]
where \( \tilde{C}_1 \) is a constant.

### 3.2. Plasma Heating Rate

Using the total distribution function \( f \) (see Equation (18), where \( f^\mu \) and \( f^\nu \) are determined by Equations (37) and (39)), we can calculate the kinetic coefficients (A2) for the nonlinear equation (3) and then estimate the temperature variations of the background plasma caused by particle acceleration. In this case the stochastic Fermi momentum diffusion describes the energy supply into the system by external sources. Generally speaking, energy supply can vary with time, but usually it is assumed that external sources keep a stationary level of acceleration such that \( D_F \) is constant.

The total energy input into the system is
\[
W_{\text{ext}} = - \int_0^\infty \mathcal{E} \frac{\partial}{\partial p} \left[ p^2 D_F \frac{\partial f}{\partial p} \right] dp. \tag{41}
\]
It is a function of time even if \( D_F \) is constant, because the distribution function \( f \) is time dependent.

Note that Coulomb collisions do not change the total energy in the system, and therefore we have
\[
\int_0^\infty \mathcal{E} \frac{\partial}{\partial p} \left[ \left( \frac{dp}{dt} \right)_C f - D_C(p) \frac{\partial f}{\partial p} \right] dp = 0. \tag{42}
\]
This condition is valid for any function \( f \) if the kinetic coefficients \( (dp/dt)_C \) and \( D_C(p) \) are calculated from Equation (A2) for this function \( f \).

The energy supplied by the stochastic Fermi acceleration is distributed over the spectrum in the form of accelerated particles and a heated plasma, because accelerated particles lose their energy by Coulomb collisions and thus transfer a part of their energy to thermal particles. Variations of \( dW/dt \) in the quasi-equilibrium part of the spectrum can be derived from estimates of the energy flux into the region \( p < p_0 \), which is
\[
W_0 = \frac{\partial}{\partial t} \int_0^{p_0} \frac{p^2 \mathcal{E} f^\mu(p) dp}{\sqrt{p^2 + 1}} D_0 \frac{\partial f}{\partial p} - \left( \frac{dp}{dt} \right)_C f^\mu. \tag{43}
\]
where the coefficients \( D_0 \) and \( (dp/dt)_C \), are calculated for the total distribution function (18). For the estimation of the integral (43) we can use the condition (42) and obtain
\[
W_0 = - \int_0^\infty \frac{1}{\sqrt{p^2 + 1}} \left[ \frac{D_0}{\mathcal{E}} \frac{\partial f^\mu}{\partial p} - p^2 \left( \frac{dp}{dt} \right)_C f^\mu \right] dp. \tag{44}
\]
Integration by parts gives
\[
W_0 = -\varepsilon_0 \frac{1}{\sqrt{p^2 + 1}} \int_0^\infty \frac{1}{\sqrt{p^2 + 1}} \left[ \frac{D_0}{\mathcal{E}} \frac{\partial f^\mu}{\partial p} - p^2 \left( \frac{dp}{dt} \right)_C f^\mu \right] dp. \tag{45}
\]
Since \( W_0 = O(\epsilon) \) and \( f^\mu = O(\epsilon) \), we can use the Maxwellian (Bethe-Bloch) expressions for the kinetic coefficients \( D_0 \) and \( (dp/dt)_\xi \) as in Equation (20),
\[
W_0 = -\varepsilon_0 \frac{1}{\sqrt{p^2 + 1}} \int_0^\infty \frac{1}{\sqrt{p^2 + 1}} \left[ \frac{D_0}{\mathcal{E}} \frac{\partial f^\mu}{\partial p} - p^2 \left( \frac{dp}{dt} \right)_C f^\mu \right] dp. \tag{46}
\]
We see that the energy input into the thermal part of the spectrum (plasma heating) is determined by two processes: (1) energy losses of nonthermal particles (the integral of Equation (46)), which heat the plasma, and (2) a particle escape to the high-energy part of the equilibrium spectrum (the first term on the right-hand side of Equation (46)), which cools the plasma. On the other hand, we can express \( W_0 \) in the form
\[
\frac{dW_0}{dt} = \frac{\partial}{\partial T} \frac{dW_0}{dT} + \frac{\partial}{\partial N} \frac{dW_0}{dN}, \tag{47}
\]
To the first order of \( \epsilon \) the expansion of \( dW/dt \) we can take \( W_0 \) as
\[
W_0 = \int_0^{p_0} \frac{1}{\sqrt{p^2 + 1}} \left[ \frac{D_0}{\mathcal{E}} \frac{\partial f^\mu}{\partial p} - p^2 \left( \frac{dp}{dt} \right)_C f^\mu \right] dp. \tag{48}
\]
In the general case the temperature variations can be calculated numerically (see Section 4). However, these calculations can be simplified. The point is that the particle spectrum described by Equations (36) and (39) depends strongly on the relation between the momenta \( p_{\text{mij}} \) and \( p_0 \). Figure 2 illustrates this situation: as \( p_0 \) increases, the transition region in momentum range \( p > p_0 \) shrinks and finally disappears when \( p_0 \) reaches \( p_{\text{mij}} \). In the limiting case \( p_0 > p_{\text{mij}} \) the transition region vanishes almost completely and the power-law tail of nonthermal particles is attached almost directly to the thermal equilibrium distribution. In this case, evaluations of the plasma temperature can be performed analytically because the functions \( f^\mu \) and \( f^\nu \) have a very simple form. We note that the conclusion of Gurevich (1960) about a very extended transition region between thermal and nonthermal parts of the spectrum is valid only for the case when \( p_0 < p_{\text{mij}} \).
3.3. The Case of Transitionless Acceleration

If \( p_0 > p_{\text{inj}} \), Equation (40) is an appropriate solution for the distribution function. \( \tilde{C}_1 \) and \( S_N \) are determined from the boundary conditions at \( p = p_0 \) and \( p = p_{\text{max}} \), namely, \( f^{11}(p_0) = f^{1}(p_0) = f_0 \) and \( f^{11}(p_{\text{max}}) = 0 \),

\[
S_N = \alpha(\varsigma + 1)p_0^{\varsigma+1}f_0, \tag{49}
\]

\[
\tilde{C}_1 = -\frac{S_N p^{\varsigma+1}_{\text{max}}}{\alpha(\varsigma + 1)} = -f_0\left(\frac{p_{\text{max}}}{p_0}\right)^{-(\varsigma+1)}. \tag{50}
\]

As \( p_{\text{max}} \gg p_0 \), thus for simplicity we set \( \tilde{C}_1 = 0 \), and for nonrelativistic temperatures \( T \ll 1 \) from Equation (36) we have

\[
f_0 = \sqrt{\frac{2}{\pi}} \frac{N}{T^{3/2}} \exp\left(-\frac{\varepsilon_0}{T}\right) \left[1 + \frac{\alpha(\varsigma + 1)p_0^{\varsigma+1}}{A(p_0^2 + 1)}\right]^{-1}. \tag{51}
\]

In this case the runaway particle flux toward high energies can be expressed directly from Equation (49) as

\[
S_N = \alpha(\varsigma + 1)p_0^{\varsigma+1} \sqrt{\frac{2}{\pi}} \frac{N}{T^{3/2}} \exp\left(-\frac{\varepsilon_0}{T}\right) \times \left[1 + \frac{\alpha(\varsigma + 1)p_0^{\varsigma+1}}{A(p_0^2 + 1)}\right]^{-1}. \tag{52}
\]

For \( p_0 \gg 1 \) Equation (48) becomes

\[
W_0 = \int_0^\infty p^2 \varepsilon f_0^1(p) dp = N \left[(3T - 1) + \frac{K_1(T^{-1})}{K_2(T^{-1})}\right], \tag{53}
\]
or for nonrelativistic values of \( T \ll 1 \)

\[
W_0 = \frac{3}{2} NT + \frac{15}{8} NT^2 + \ldots \tag{54}
\]

Now we have (recall Equations (46) and (47))

\[
\frac{\partial W_0}{\partial T} \frac{dT}{dt} = \left(\frac{W_0}{N} - \varepsilon_0\right) S_N + \int_{p_0}^{\infty} \frac{p^3}{\sqrt{p^2 + 1}} \left[ D_0(T) \frac{\partial f}{\partial p} \right]_0^f dp \times \left[ \frac{\alpha f_0 \varepsilon_0 p_0^{\varsigma+1}(\varsigma + 1)}{\alpha \varepsilon_0(\varsigma + 1) - 1} \right] + ATf_0 \left\{ 3\alpha p_0^{\varsigma+1}(\varsigma + 1) \frac{\varepsilon_0}{2A} \right\}, \tag{55}
\]

where

\[
Q(p_0, \varsigma) = \int_{p_0}^{\infty} x^{-5/2} \sqrt{x^2 + 1} dx. \tag{56}
\]

If \( \alpha \varepsilon_0(\varsigma + 1) \neq AQ(p_0, \varsigma) \) and \( \varepsilon_0 \gg T \), then the second term in Equation (55) is small and can be neglected. Finally, from Equation (54) we obtain

\[
\frac{dT}{dt} = \frac{2S_N}{3N} \left[ \frac{AQ(p_0, \varsigma)}{\alpha(\varsigma + 1)} - \varepsilon_0 \right], \tag{57}
\]

where \( A \) and \( S_N \) are defined by Equations (21) and (52), respectively.

For high values of \( \alpha \) one can see from Equation (57) that the plasma cools down and \( dT/dt < 0 \). The temperature decreases with time due to a very intensive outflow of high-energy particles from the thermal pool, even though external sources in the form of stochastic Fermi acceleration supply energy to the system (analog to Maxwell demon). This effect can be seen in Figure 2 as a deficit of high-energy thermal particles at \( p < p_0 \).

When \( \alpha \) decreases, collisions start to dominate over the outflow effect. As a result, the derivative \( dT/dt \) increases and at sufficiently small \( \alpha \) the regime changes from cooling to heating of plasma. However, the process of acceleration reduces the amount of particles in the thermal pool, even though external sources of high-energy particles from the thermal pool, even though external sources in the form of stochastic Fermi acceleration supply energy to the system (analog to Maxwell demon). This effect can be seen in Figure 2 as a deficit of high-energy thermal particles at \( p < p_0 \).

A more accurate analysis of this regime can be provided by numerical calculations of the nonlinear case.

4. NONLINEAR CASE: SEMI-ANALYTICAL METHOD AND NUMERICAL CALCULATIONS

The most straightforward way to solve the problem is a numerical solution of the original nonlinear equation. However, this method is very time-consuming. We proceed with approximation methods that simplify the numerical calculations but still give a good result.

Analysis of kinetic equations depends on the relation between the plasma heating time and the acceleration time. We define the heating time as

\[
t_H = T/(dT/dt). \tag{58}
\]

The lower limit of this time can be obtained for the quasistationary solution for \( dW/dt \) when we neglect the cooling
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Figure 3. Comparison between heating timescale $t_H$ and tail-formation timescale $t_F$ for different $p_0$ and acceleration rates. The temperature is $T = 0.016$ (the corresponding momentum is $p_T = 0.12$). The threshold value is marked by the gray horizontal line.

term $S_{N}E_{0}$ in Equation (46). The acceleration time characterizes a period required for particles to fill the nonthermal tail. Numerical calculations show that for $\varsigma > 2$ this time is of the order of

$$t_F \simeq \alpha^{-1}. \quad (59)$$

The quasi-stationary state (when the plasma temperature is almost constant and the acceleration generates prominent nonthermal “tails”) can be reached only if $t_F > t_V$. In this case we can use analytical solutions presented in the previous section. The ratio $t_F/t_V$ as a function of $p_0$ is shown in Figure 3. The threshold value of ratio $t_F/t_V = 1$ is shown in Figure 3 by the gray horizontal dashed line. The quasi-stationary state will be achieved if $t_F/t_V$ is above the gray line.

If the acceleration time is larger than the heating time, $t_F < t_V$, the quasi-stationary state cannot be reached. In this case we can simplify the calculations using the trick in Petrosian & East (2008). The evolution of distribution function $f(p)$ can be described by the non-stationary linear kinetic equation

$$\frac{\partial f}{\partial t} + \frac{1}{p^2} \frac{\partial}{\partial p} p^2 \left[ \frac{df}{dt} \right] (p, N, T)f = - \{D_0(p, N, T) + D_F(p) \frac{\partial f}{\partial p} \} = 0. \quad (60)$$

We can estimate the variation of temperature by the following algorithm:

1. for a given $f(t, p)$, estimate $f(t + \delta t, p)$ from Equation (60);
2. compute $N(t + \delta t)$ from $f_0 \int_{0}^{\infty} f(t + \delta t, p)dp$;
3. calculate $W_0$ from Equation (46), then $W_0(t + \delta t) = W_0(t) + W_0 \delta t$; then find $T(t + \delta t)$ from Equation (48);
4. for new values of $N(t + \delta t)$ and $T(t + \delta t)$ recalculate the kinetic coefficients using analytical expressions for Maxwellian coefficients (see Equations (61) and (62));
5. repeat steps 1–4.

The analytical expressions for the kinetic coefficients are calculated as (see Petrosian & East 2008, and references therein)

$$\left( \frac{dp}{dt} \right)_0(p, N, T) = - \frac{A(p^2 + 1)}{p^2} \times \left[ \text{erf} \left( \sqrt{\frac{\mathcal{E}}{T}} \right) - \sqrt{\frac{4\mathcal{E}}{\pi T}} \exp \left( - \frac{\mathcal{E}}{T} \right) \right], \quad (61)$$

$$D_0(p, N, T) = - \frac{T \sqrt{p^2 + 1}}{p} \left( \frac{dp}{dt} \right)_0(p, N, T). \quad (62)$$

Here $\text{erf}(z)$ is the error function.

With this method we combine the simplicity of the analytical method with the accuracy of the numerical method. The only problem is that this approach, like any other semi-analytical method based on Equation (46), cannot be used near $p_0 = 0$.

Now we compare the results obtained with different methods.

We consider the following methods:

1. Transitionless case. It is based on Equation (57). The equations are integrated numerically using the Runge–Kutta method to obtain the evolution of the temperature $T(t)$ and density $N(t)$. This method is applicable if $p_{0j} < p_0$.
2. Quasi-linear approximation. The distribution function is given as in Equations (37) and (39). Equations (46) and (48) are used to estimate the $dT/dt$. The variations of the temperature $T(t)$ and density $N(t)$ are obtained using the Runge–Kutta method. This approximation is valid for $t_T > t_F$.
3. Semi-analytical method. It uses a combination of numerical solution to Equation (60) and analytical calculations of Equations (46) and (48) in order to estimate the variations of the temperature. This method can be applied to $p_0 \gg p_T = \sqrt{T + 1}^2 - 1$.
4. Numerical method. Evolution of the distribution function is obtained by a numerical solution of the original nonlinear equation (Equation (3); for details see Appendix B). This method is the most universal and is used to check whether the results obtained by methods 1–3 are correct.

We checked our numerical program by calculations of temperature variations for the acceleration in the form (11) and for the same parameters as used by Petrosian & East (2008), i.e., $p_0 = 0$, $q = 1$, $E_c = 0.2$, and three corresponding values of $t_0$: $2.4t_e$, $0.18t_e$, and $0.013t_e$, where $t_e \equiv (4\pi r_{\text{GEO}}^3N \ln \Lambda)^{-1} \approx 2.7 \times 10^7 \times (N/10^{-3}\text{cm}^{-3})^{-1}$ yr and $r_0 = \varepsilon^2/(m_e c^2)$. The acceleration parameters, $D_F(p)$, for these three cases are shown in Figure 5 by the thin dashed lines.

The result of our calculations and that of Petrosian & East (2008) are shown in Figure 4 by the solid and dashed lines, respectively. One can see that, despite some discrepancy, the results are more or less the same.

Now we present results of calculations for the acceleration parameter $D_F$ in the form (13), when $p_0 \neq 0$. In all cases we take $p_0 = 0.55$. Variations of $p_0$ change the temperature variations quantitatively but not qualitatively. We choose $\varsigma = 2$ in order to obtain the same momentum dependence of $D_F$ at high energies as in Equation (11). Note that in this case $t_F = 2t_0$. Below we will use $t_0$ as a characteristic timescale to compare results with those of Petrosian & East (2008).

For other values of $\varsigma$ the results are qualitatively the same, yet lower values of $\varsigma$ will increase the amount of nonthermal particles and thus decrease the heating timescale and vice versa. One can see this from Equation (57).
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Figure 4. Comparison between our numerical method (solid line) and method used by Petrosian & East (2008) (dashed line). All notations are the same as in Petrosian & East (2008) (see the text for details).

Figure 5. Comparison between $D_F(p)$ used by Petrosian & East (2008) (thin dashed lines), Equation (11), and in this paper (thick solid lines), Equation (13).

The calculations were performed for three different regimes of acceleration:
1. heating dominates over cooling;
2. cooling and heating rates are of the same order;
3. cooling dominates over heating.

The functions $D_F(p)$ used for these three cases are shown in Figure 5 by the solid lines.

We provide calculations by four different methods: analytical (transitionless), quasi-linear, semi-analytical, and numerical. Temperature variations, $T(t)$, obtained by these methods are shown in Figures 6, 8, and 9 by the dashed, thin solid, thick solid, and dotted lines, respectively.

1. For the case of slow acceleration we take $\alpha/A = 2.77$.
   In this case the quasi-stationary approach is not valid, and only numerical and semi-analytical methods can provide an adequate result. Temperature variations for this case of acceleration parameter are shown in Figure 6. As one can see, the result of acceleration for this parameter $\alpha$ is the plasma overheating that is in complete agreement with the conclusions of Wolfe & Melia (2006) and Petrosian & East (2008). The only difference is that the overheating occurs for the time $t \approx 4.5\tau_0 \approx 1.6\tau_C$, which is longer by a factor of four than that of Petrosian & East (2008), who obtained $t \sim \tau_0$. The reason is that for $p_0 = 0$ the acceleration generates an extended excess above the equilibrium Maxwellian function, while for $p_0 \neq 0$ this...
excess is not so prominent (compare dashed and solid lines in Figure 7). Since the amount of suprathermal particles in the case $p_0 = 0$ is higher than the case $p_0 > 0$, it is not surprising that the plasma is overheated by the Coulomb losses in a shorter time when $p_0 = 0$.

2. The case of moderate acceleration ($\alpha/A = 11.63$) is shown in Figure 8. One can see that all methods are in good agreement. At the first stage we see plasma heating; however, the timescale is much longer than in Petrosian & East (2008), and the plasma temperature increases by a factor of 1.3 at the moment $t \approx 86\tau_0 = 7\tau_C$, which is almost two orders of magnitude higher than that of Petrosian & East (2008). A prominent quasi-stationary power-law tail of nonthermal particles is formed by the acceleration for a much shorter time (since $\tau_T/\tau_F > 1$; see Figure 3). Moreover, unlike in Petrosian & East (2008), after this time heating reverses to cooling.

3. The case of fast acceleration ($\alpha/A = 18.5$) is shown in Figure 9. The numerical method, the semi-analytical method, and the quasi-linear approximation give almost the same result. For comparison we also show the calculations obtained with the transitionless case (dashed line). We see that, in spite of some difference, this method provides similar qualitative time variations of the temperature $T$. All methods demonstrate plasma cooling in this regime from the very beginning. The temperature of the plasma shows a steady decrease with time, while nonthermal tails are formed rapidly ($\tau_T/\tau_F > 1$; see Figure 3), which differs completely from the results obtained by Petrosian & East (2008).

These results can be understood from Figure 2. For the ratio $\alpha/A = 2.77$ the injection momentum is $p_{nj} \simeq 0.83$, i.e., $p_{nj} > p_0$ (similar to the upper curve of Figure 2). An excess of quasi-thermal particles is formed in the range between $p_0$ and $p_{nj}$. Coulomb losses of these particles result in effective plasma heating. As we already mentioned above, Petrosian & East (2008) assumed $p_0 = 0$, which led to a more extended transition region and more effective heating.

In the case of $\alpha/A = 11.63$, $p_{nj} \simeq 0.5 \simeq p_0$ (similar to the middle curve of Figure 2). The transition region is almost negligible in this case. Therefore, plasma heating by nonthermal particles is insignificant, which then changes into cooling.

In the case of $\alpha/A = 18.5$, $p_{nj} \simeq 0.4$, i.e., $p_{nj} < p_0$ (similar to the lower curve of Figure 2). A deficit of high-energy particles is formed in the thermal energy range that provides the effect of cooling.

Thus, we conclude that, depending on parameters $p_0$ and $\alpha$, different regimes of acceleration from background plasma are realized. The important inference is that stochastic acceleration may produce a flux of nonthermal particles without plasma overheating.

A specific spectrum of turbulence that provides stochastic acceleration is beyond the scope of this paper. It depends on mechanisms that excite electromagnetic fluctuations in an astrophysical plasma. As an example, we mention particle acceleration in OB associations by a supersonic turbulence (see Bykov & Toptygin 1993). The momentum diffusion coefficient in this case has the form $D(p) = D_0 p^2$. This acceleration is effective in the momentum range $p > p_0$, where the value of $p_0$ is derived from $r_L(p_0) = l u/c$. Here $r_L$ is the particle Larmor radius, $u$ is the shock velocity, and $l$ is a distance between shocks. Particles with $p < p_0$ are not accelerated by this mechanism.

5. CONCLUSION

We analyzed nonlinear kinetic equations describing particle stochastic (or second-order Fermi) acceleration from background plasma when the acceleration is non-zero for particles with momenta $p > p_0$. The goal of these investigations is to define whether the only result of stochastic acceleration is plasma overheating as concluded by Wolfe & Melia (2006) and Petrosian & East (2008), or this acceleration can generate prominent tails of nonthermal particles when the plasma temperature remains almost stationary. The following results are obtained from our analysis:

1. We showed that in the case of stochastic acceleration two competitive processes determine temperature variations of background plasma. The first one is Coulomb energy losses of nonthermal particles that heat the plasma. The other one is a runaway flux of high-energy particles from the...
thermal pool that leads to plasma cooling. Depending on the rates of these processes, the plasma may cool down or heat up.

2. From numerical and analytical calculations we conclude that for a low enough acceleration rate the cooling process is negligible. The plasma gains much heat on the acceleration timescale \( \tau_0 \). As a result, the plasma temperature rises rapidly while prominent nonthermal tails are not generated, which fully confirms the results of Wolfe & Melia (2006) and Petrosian & East (2008).

3. For a moderate acceleration the cooling and heating processes partly compensate each other. As a result, the plasma temperature is quasi-stationary on timescales much longer than \( \tau_0 \). In this case, the acceleration produces a nonthermal component of the spectrum. After a period of moderate plasma heating, the process changes into cooling. This regime does not appear in the models of Wolfe & Melia (2006) and Petrosian & East (2008).

4. For a high rate of acceleration the runaway flux of thermal particles cools the plasma down from the very beginning. In spite of energy supply by external sources, the plasma temperature drops down (analog to Maxwell demon).

5. The evolution of plasma temperature depends on the characteristic time of Coulomb collisions in the background plasma (the collision frequency \( A \)) and the acceleration frequency \( \alpha \). This is illustrated in Figure 10, where the solid line defines the border between heating (below the line) and cooling (above the line) regimes for quasi-stationary systems. It corresponds to the solution \( dT/dt = 0 \) (see Equation (46)). Dashed lines in Figure 10 show the evolution of plasma parameters for the same initial temperature but different initial value of \( A (\alpha) \) is the same for the systems. Since \( N \) decreases monotonically because of particle acceleration, \( A \) decreases monotonically accordingly (see Equation (21)).

One can see that even if the system starts from the regime of heating, sooner or later it changes to plasma cooling. If the evolution of the system is quasi-stationary, the turning point of the trajectory should be located on the boundary. However, we mention that the quasi-stationary approximation is inapplicable to low values of \( \alpha/A \).

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APPENDIX A

GENERAL KINETIC EQUATION

The general equation for stochastic Fermi acceleration with the coefficient \( D^F_{\alpha\beta} \) has the form (Landau & Lifshitz 1981; Wolfe & Melia 2006)

\[
\frac{\partial f(p)}{\partial t} = \frac{\partial}{\partial p_\alpha} \left( D_{\alpha\beta} f_{\alpha\beta} \right) \frac{\partial f(p)}{\partial p_\beta} - F_\alpha f(p), \quad \text{(A1)}
\]

where \( p = v(c^2 - v^2)^{-1/2} \) is the dimensionless particle momentum and \( v \) is the particle velocity. The coefficients \( D_{\alpha\beta} \) and \( F_\alpha \) are determined by Coulomb collisions of the particles,

\[
D_{\alpha\beta} = A \int Z_{\alpha\beta}(p, p') f(p') d^3 p', \quad F_\alpha = -A \int \left[ \frac{\partial}{\partial p_\beta} Z_{\alpha\beta}(p, p') \right] f(p') d^3 p', \quad \text{(A2)}
\]

where

\[
Z_{\alpha\beta}(p, p') = \frac{r^2}{\gamma' \gamma' w} \left[ w^2 \delta_{\alpha\beta} - p_\alpha p_\beta - p'_\alpha p'_\beta + r(p_\alpha p'_\beta + p'_\alpha p_\beta) \right]. \quad \text{(A3)}
\]

\[
A = \frac{8 \pi e^2 c^2}{m^2} \ln A, \quad r = \gamma' \gamma' - p \cdot p'/c^2, \\
w = c \sqrt{r^2 - 1}, \quad \gamma = \sqrt{1 + p^2/c^2}. \quad \text{(A4)}
\]

The boundary conditions were taken in the following form: a zero particle flux at \( p = 0 \),

\[
\left[ D_{\alpha\beta} + D^F_{\alpha\beta} \right] \frac{\partial f(p)}{\partial p_\beta} - F_\alpha f(p) = 0, \quad \text{at} \quad p = 0, \quad \text{(A5)}
\]

and the distribution function vanishes at some \( p_{max} \),

\[
f(p_{max}) = 0. \quad \text{(A6)}
\]

APPENDIX B

NUMERICAL METHOD FOR THE NONLINEAR CASE

To solve Equation (3) numerically, we use the Crank-Nicolson finite-difference method. To estimate the kinetic coefficients \( A(2) \), we use Simpson’s integration rule,

\[
D = \frac{zF}{f}, \quad F = \mathcal{Z}' f, \quad \text{(B1)}
\]

where vectors \( f, D, \) and \( F \) are corresponding discrete versions of \( f(p), D_{\alpha}(p, f), \) and \( (dp/dt)_\gamma(p, f) \). Matrices \( \mathcal{Z} \) and \( \mathcal{Z}' \) are...
obtained by applying Simpson’s rule to Equation (A2). The discrete version of Equation (3) at \( t = (t_n + t_{n+1})/2 \) and \( p = p_j \) looks like (see, e.g., Park & Petrosian 1996, and references therein)

\[
\frac{f_{n+1,j} - f_{n,j}}{\Delta t} = \frac{1}{p_j^2} \frac{S_{n+1,j+\frac{1}{2}} - S_{n+1,j-\frac{1}{2}}}{\Delta p_j}, \tag{B2}
\]

where \( \Delta t = t_{n+1} - t_n \) and \( \Delta p_j = (p_{j+1} - p_{j-1})/2 \) are steps of the grid and the flux is expressed according to the Crank-Nicolson rule:

\[
S_{n+1,j+\frac{1}{2}} = \frac{1}{2} p_j^2 \left[ D_{n+1,j+\frac{1}{2}} \frac{f_{n+1,j+1} - f_{n+1,j}}{\Delta p_j} + F_{n+1,j+\frac{1}{2}} f_{n+1,j+1} \right] + \frac{1}{2} p_j^2 \left[ D_{n+1,j-\frac{1}{2}} \frac{f_{n+1,j} - f_{n+1,j+1}}{\Delta p_j} - F_{n+1,j-\frac{1}{2}} f_{n+1,j} \right] - F_{n+1,j+\frac{1}{2}} f_{n+1,j+1}, \tag{B3}
\]

\[
D_{n,j+\frac{1}{2}} = \frac{1}{2} (D_{n,j} + D_{n,j+1}), \tag{B4}
\]

\[
F_{n,j+\frac{1}{2}} = \frac{1}{2} (F_{n,j} + F_{n,j+1}), \tag{B5}
\]

\[
f_{n,j+\frac{1}{2}} = \frac{1}{2} (f_{n,j} + f_{n,j+1}), \tag{B6}
\]

\[
\Delta p_{j+\frac{1}{2}} = p_{j+1} - p_j. \tag{B7}
\]

The boundary condition at \( p = 0 \) implies that \( S_{n+\frac{1}{2},-\frac{1}{2}} = 0 \). The boundary condition at \( p = p_{\text{max}} \) is \( f(p_{\text{max}}) = 0 \).

After discretization, we arrive at the nonlinear system of equations,

\[
f_{n+1} - f_n = A(f_{n+1}) f_{n+1} + A(f_n) f_n, \tag{B8}
\]

where \( A(f) \) is a tridiagonal matrix corresponding to the differential operator on the right-hand side of Equation (B2). According to Equation (B1), \( A(f) \) is a linear function of \( f \) and Equation (B8) is a system of quadratic equations.

To avoid calculations of the Jacobian matrix, we do not apply Newton’s method and utilize a simple iteration method instead. However, the iteration method based on Equation (B8) converges very slowly. We rewrite the iteration step in the following form:

\[
(E - A(f_{n+1}^k)) f_{n+1}^{k+1} = A(f_n) f_n + f_n, \tag{B9}
\]

where \( k \) is the number of iterations and \( E \) is a unit matrix. The system of linear equations is solved using the tridiagonal matrix algorithm. Iteration in the form (B9) shows fast convergence if the temperature of the Maxwellian distribution does not change significantly between \( t_n \) and \( t_{n+1} \).

Since the Crank-Nicolson method may be affected by numerical oscillations, we also use the less precise and more robust backward Euler method (simple fully implicit method from Park & Petrosian 1996). The backward Euler method turns out to be useful for nonthermal tails of low magnitude when the value of \( \alpha \) is low and the value of \( p_0 \) is high.

The discretization in momentum space is tricky since we need to provide a good resolution for the Maxwellian distribution and transitional region and calculate the nonthermal tail at high energies. We tried two possible ways to reduce the number of grid points. The first one is to split the momentum axis into sub-domains and join them using continuity of the distribution function and particle flux. The second way is to use the logarithmic grid by introducing the new variable \( q = \log(p) \). Both methods give almost the same results.

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