The constrained dispersionless mKP hierarchy and the dispersionless mKP hierarchy with self-consistent sources

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Abstract

We first show that the quasiclassical limit of the squared eigenfunction symmetry constraint of the Sato operator for the mKP hierarchy leads to a reduction of the Sato function for the dispersionless mKP hierarchy. The constrained dispersionless mKP hierarchy (cdmKPH) is obtained and it is shown that the (2+1)-dimensional dispersionless mKP hierarchy is decomposed to two (1+1)-dimensional hierarchies of hydrodynamical type. The dispersionless mKP hierarchy with self-consistent sources (dmKPHSCS) together with its associated conservation equations are also constructed. Some solutions of dmKPESCS are obtained by hodograph reduction method.

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1 Introduction

Recently much attention has been focused on the study of dispersionless integrable hierarchies which have important applications from complex analysis to topological field theory (see [1-16]). In [2, 3, 6, 7, 9], a standard procedure of dispersionless limit of integrable dispersionfull hierarchies is proposed. In this procedure, dispersionless hierarchies arise as the quasiclassical limit of the original dispersionfull Lax equations performed by replacing operators by phase space functions, commutators by Poisson brackets and the role of Lax pair equations by conservation equations (or equations of Hamilton-Jacobi type). Several
strategies have been proposed to investigate these hierarchies, such as the twistorial method [8, 9, 17], the hodograph reduction method [6, 7, 18] and the quasi-classical $\bar{\partial}$-method [13-16]. Lately in [16], using the quasi-classical $\bar{\partial}$-method, the authors studied the symmetry constraint of the dispersionless KP (dKP) and 2DTL hierarchies and some important reductions of these hierarchies are shown to be nothing but results of symmetry constraints. This offers us another important way to construct reductions of dispersionless hierarchies.

The integrable equations with self-consistent sources are another type of important integrable models in many fields of physics, such as hydrodynamics, state physics, plasma physics, etc (see [19-27]). In our general approach proposed recently in [22-27], the constrained integrable hierarchy may be regarded as the stationary system of the corresponding integrable hierarchies with self-consistent sources. From this observation, the integrable hierarchies with self-consistent sources and their Lax representation can be constructed from the corresponding constrained integrable hierarchies. The KP hierarchy with self-consistent sources (KPHSCS) and the modified KP hierarchy with self-consistent sources (mKPHSCS) are constructed in this way and their interesting solutions are obtained via generalized Darboux transformation [26, 27]. In this sense, the soliton hierarchies with self-consistent sources may be viewed as integrable generalizations of the original soliton hierarchies.

In [25], at the first time we studied the quasiclassical limit of the symmetry constraint of the Sato operator for the KP hierarchy in the framework of Sato theory which offers us a new way to study the reduction of the dispersionless hierarchy. Though the idea in [25] is different from that in [16], the well-known Zakharov reduction of the Sato function for the dispersionless KP (dKP) hierarchy can be also obtained as a result. The critical step in the procedure of this quasiclassical limit is to find the asymptotic forms of the potentials $q_i(\frac{1}{\epsilon})$ and $r_i(\frac{1}{\epsilon})$ appearing in the constraint of the Sato operator. Starting from the constrained Sato function for the dKP hierarchy, the constrained dKP hierarchy (cdKPH) and the dKP hierarchy with self-consistent sources (dKPHSCS) are therefore constructed. In contrast to the dKP case, the reduction of the dispersionless mKP hierarchy has been less studied. In this paper, we will further develop the idea in [25] to investigate the quasiclassical limit of the squared eigenfunction symmetry constraint of the Sato operator for the mKP hierarchy [31]. As will be shown below, this quasiclassical limit will lead to a reduction of the Sato function for the dispersionless mKP hierarchy. In this case, it is also important for us to find the asymptotic form of $q_i(\frac{1}{\epsilon})$ and $r_i(\frac{1}{\epsilon})$ appearing in the constraint of the Sato operator for the mKP hierarchy. The constrained dmKP hierarchy (cdmKPH) will soon be obtained and its commutability can be proved. We can see that the dmKP hierarchy of $(2 + 1)$-dimensions is decomposed into two commutative $(1 + 1)$-dimensional hierarchies.
of hydrodynamical type. The dmKP hierarchy with self-consistent sources (dmKPHSCS) and its associated conservation equations will also be obtained according to our approach. Utilizing the conservation equations, we can develop the hodograph reduction method [6, 7] to solve the dmKPHSCS. In this sense, the dmKPHSCS may be regarded as an integrable generalization of the dmKP hierarchy.

The paper will be organized as follows. In section 2, we briefly review some definitions and results about the mKPHSCS. In section 3, we take the quasiclassical limit to the squared eigenfunction symmetry constraint of the Sato operator for the mKP hierarchy. The cdmKPH and the dmKPHSCS together with its associated conservation equations are also obtained. Section 4 is devoted to presenting some solutions for the dmKPHSCS obtained by means of the hodograph reduction method.

2 The squared eigenfunction symmetry constraint of the mKP hierarchy and the mKP hierarchy with self-consistent sources

We first review some results about the mKP hierarchy with self-consistent sources in the framework of Sato theory [25-29]. Given a pseudo-differential operator (PDO) of the form [30]

\[ L = \partial + v_0 + v_1 \partial^{-1} + v_2 \partial^{-2} + \ldots, \]  

(2.1)

where \( v_i = v_i(t) \), the mKP hierarchy is defined as

\[ \partial_t^k L = [Q_k, L], \quad k \in \mathbb{N}, \]  

(2.2)

where \( Q_k = (L^k)_{\geq 1} \) means the part of order \( \geq 1 \) of \( L^k \). The Lax equation (2.2) is equivalent to the compatibility of the following linear equations

\[ L\psi_{mKP} = \lambda \psi_{mKP}, \]  

(2.3a)

\[ \partial_t^k \psi_{mKP} = Q_k \psi_{mKP}. \]  

(2.3b)

\( \psi_{mKP} \) is often called the eigenfunction of the mKP hierarchy and also satisfies

\[ \partial_\lambda \psi_{mKP} = M_{mKP} \psi_{mKP}, \]  

(2.4)

where \( M_{mKP} \) is the Orlov operator of the mKP hierarchy. The adjoint eigenfunction \( \psi^*_{mKP} \) satisfies

\[ L^* \psi^*_{mKP} = \lambda \psi^*_{mKP}, \]  

(2.5a)
\[ \partial_k \psi_m^{* \text{KP}} = -Q_k^* \psi_{m,KP}^*, \quad (2.5b) \]
\[ \partial_\lambda \psi_m^{* \text{KP}} = -M_{m,KP}^* \psi_{m,KP}^*, \quad (2.5c) \]

Making a constraint of the PDO \( L \) (2.1) as \([31, 27]\)

\[ L^n = (L^n)_{\geq 1} + \sum_{i=1}^{N} q_i(t) \partial^{-1} r_i(t) \partial, \quad n \in \mathbb{N}, \quad (2.6) \]

where \( q_i(t) \) and \( r_i(t) \) satisfying

\[ q_{i,t} = Q_k(q_i), \quad r_{i,t} = -((\partial Q_k \partial^{-1})^*)(r_i) = -((\partial^{-1} Q_k^* \partial)(r_i), \quad i = 1, ..., N, k \in \mathbb{N}, \quad (2.7) \]

and \( Q_k = [(L^n)^k]_{\geq 1} = \left[((L^n)_{\geq 1} + \sum_{i=1}^{N} q_i(t) \partial^{-1} r_i(t) \partial)^k\right]_{\geq 1} \), we will get the \((n-)\)-constrained mKP hierarchy (cmKPH) as

\[ (L^n)_{t_k} = [Q_k, L^n], \quad (2.8a) \]
\[ q_{i,t} = Q_k(q_i), \quad (2.8b) \]
\[ r_{i,t} = -((\partial^{-1} Q_k^* \partial)(r_i), \quad i = 1, ..., N. \quad (2.8c) \]

(2.6) with (2.7) is called the squared eigenfunction symmetry constraint for the mKP hierarchy \([31]\) and \( r_i \) is called the integrated adjoint eigenfunction, i.e., \( r_{i,x} \) satisfies (2.5b).

If we add the term \((Q_k)_{t_n}\) to the right side of equation (2.8a) and requiring \( k < n \), the mKP hierarchy with self-consistent sources (mKPHSCS) will be obtained as \([27]\)

\[ (Q_k)_{t_n} - (L^n)_{t_k} + [Q_k, L^n] = 0, \quad (2.9a) \]
\[ q_{i,t_k} = Q_k(q_i), \quad (2.9b) \]
\[ r_{i,t_k} = -((\partial^{-1} Q_k^* \partial)(r_i), \quad i = 1, ..., N. \quad (2.9c) \]

As the case of KPHSCS \([26]\) and many other cases in \((1+1)\)-dimensions (see \([22, 23]\) and the references therein), the cmKPH (2.8) may be considered as the stationary one of the mKPHSCS (2.9) if \('t_n'\) is viewed as the evolution variable.

When \( k = 2, n = 3 \) in (2.9), we will get the mKP equation with self-consistent sources (mKPESCS) (with \( y = t_2, t = t_3, v = v_0 \)) \([27]\)

\[ v_t - \frac{1}{4} v_{xxx} - \frac{3}{4} D^{-1}_x(v_{yy}) - \frac{3}{2} D^{-1}_x(v_y)v_x + \frac{3}{2} v^2 v_x + \sum_{i=1}^{N} (q_i r_i)_x = 0, \quad (2.10a) \]
\[ q_{i,y} = q_{i,xx} + 2v q_{i,x}, \quad (2.10b) \]
\[ r_{i,y} = -r_{i,xx} + 2vr_{i,x}, \quad i = 1, \cdots , N, \tag{2.10c} \]

where \( D_x^{-1} = \int^x \) is the integral operator. Under (2.10b) and (2.10c), (2.10a) will be obtained by the compatibility of the following auxiliary linear equations (Lax pair)

\[
\psi_y = \psi_{xx} + 2v\psi_x, \tag{2.11a}
\]

\[
\psi_t = \psi_{xxx} + 3v\psi_{xx} + \frac{3}{2}[v_x + v^2 + D_x^{-1}(v_y)]\psi_x + \sum_{i=1}^N q_i D_x^{-1}(r_i\psi_x). \tag{2.11b}
\]

### 3 The quasiclassical limit

In this section, we will consider the quasiclassical limit of (2.6) with (2.7) which will give rise to the dispersionless counterpart of (2.8) and (2.9), i.e. the constrained dispersionless mKP hierarchy (cdmKPH) and dispersionless mKP hierarchy with self-consistent sources (dmKPHSCS).

Taking \( T_n = \epsilon t_n \) and thinking of \( v_i(T_{\epsilon}) = V_i(T) + O(\epsilon) \) as \( \epsilon \to 0 \), \( L \) in (2.1) changes into

\[
L_\epsilon = \epsilon \partial + \sum_{i=0}^{\infty} v_i(T_{\epsilon})(\epsilon \partial)^{-i} = \epsilon \partial + \sum_{i=0}^{\infty} (V_i(T) + O(\epsilon))(\epsilon \partial)^{-i}, \quad \partial = \partial_X, \quad X = \epsilon x. \tag{3.1}
\]

The constraint (2.6) now changes into

\[
L^\epsilon_n = Q_{\epsilon n} + \sum_{i=1}^N q_i(T_{\epsilon})(\epsilon \partial)^{-1}r_i(T_{\epsilon})\epsilon \partial, \quad Q_{\epsilon n} = (L^\epsilon_n)_{\geq 1}, \tag{3.2}
\]

where \( q_i(T_{\epsilon}) \) and \( r_i(T_{\epsilon}) \) satisfy

\[
\epsilon[q_i(T_{\epsilon})]_{T_k} = Q_{\epsilon k}(q_i(T_{\epsilon})), \quad \epsilon[r_i(T_{\epsilon})]_{T_k} = -[(\epsilon \partial)^{-1}Q_{\epsilon k}(\epsilon \partial)](r_i(T_{\epsilon})), \quad Q_{\epsilon k} = [(L^\epsilon_n)_{\geq 1}]_{i=1,...,N}. \tag{3.3}
\]

It is easy to prove that

\[
\mathcal{L} = \sigma^\epsilon(L_\epsilon) = p + \sum_{i=0}^{\infty} V_i(T)p^{-i}. \tag{3.4}
\]

is a solution of the dmKP hierarchy, i.e., satisfies

\[
\partial_{T_n} \mathcal{L} = \{Q_n, \mathcal{L}\}, \tag{3.5}
\]

where \( \sigma^\epsilon \) denotes the principal symbol \([9]\), the Poisson bracket is defined as

\[
\{A(p, x), B(p, x)\} = \frac{\partial A}{\partial p} \frac{\partial B}{\partial x} - \frac{\partial A}{\partial x} \frac{\partial B}{\partial p}. \tag{3.6}
\]
The dmKP hierarchy can be also written in the zero-curvature form
\[ \frac{\partial Q_l}{\partial T_s} - \frac{\partial Q_s}{\partial T_l} + \{ Q_l, Q_s \} = 0, \ l, s \in \mathbb{N}. \] (3.7)

When \( l = 2, \ s = 3 \), we will get the dmKP equation (\( Y = T_2, T = T_3 \) and \( V = V_0 \))
\[ 2V_T - \frac{3}{2} D_X^{-1}(V_Y Y) - 3V_X D_X^{-1}(V_Y) + 3V^2 V_X = 0. \] (3.8)

Similarly like the dKP case \([9]\), from (2.3), (2.4) and (2.5), it can be proved that \( \psi_{mKP}(\frac{T}{\epsilon}) \) \([32]\) and \( \psi_{mKP}^*(\frac{T}{\epsilon}) \) has the following WKB asymptotic expansion as \( \epsilon \to 0 \),
\[ \psi_{mKP}(\frac{T}{\epsilon}, \lambda) = \exp\left[ \frac{1}{\epsilon} S_{mKP}(T, \lambda) + O(1) \right], \ \psi_{mKP}^*(\frac{T}{\epsilon}, \lambda) = \exp\left[ -\frac{1}{\epsilon} S_{mKP}(T, \lambda) + O(1) \right], \ \epsilon \to 0. \] (3.9)

From (2.3b) and (3.9), we can obtain a hierarchy of conservation equations for the momentum function \( p = \frac{\partial S_{mKP}}{\partial X} \),
\[ \frac{\partial p}{\partial T_l} = \frac{\partial Q_l(p)}{\partial X}, \ l \in \mathbb{N}, \] (3.10)
the compatibility of which, i.e., \( \frac{\partial^2 p}{\partial T_l \partial T_s} = \frac{\partial^2 p}{\partial T_s \partial T_l} \) implies the dmKP hierarchy (3.7).

Since \( q_i(t) \) and \( r_i(t) \) evolve in the same rule as \( \psi_{mKP} \) and \( \psi_{mKP}^* \) respectively, we regard
\[ q_i(\frac{T}{\epsilon}) = \psi_{mKP}(\frac{T}{\epsilon}, \lambda = \lambda_i) \sim \exp\left[ \frac{S_{mKP}(T, \lambda_i)}{\epsilon} \right] + a_{i1} + O(\epsilon), \] (3.11a)
\[ \epsilon r_i(\frac{T}{\epsilon})_X = \psi_{mKP}(\frac{T}{\epsilon}, \lambda = \lambda_i) \sim \exp\left[ \frac{S_{mKP}(T, \lambda_i)}{\epsilon} \right] + a_{i2} + O(\epsilon), \ \epsilon \to 0, \ i = 1, ..., N. \] (3.11b)

By a tedious computation, we will find that when \( \epsilon \to 0 \),
\[ q_i(\frac{T}{\epsilon})(\epsilon \partial)^{-1} r_i(\frac{T}{\epsilon})(\epsilon \partial) = \frac{e^{a_{i1}+a_{i2}}}{S_{mKP}(T, \lambda_i)_X} [1 + (S_{mKP}(T, \lambda_i)_X + O(\epsilon))(\epsilon \partial)^{-1} + \cdots + ((S_{mKP}(T, \lambda_i)_X)^n + O(\epsilon))(\epsilon \partial)^{-n} + \cdots]. \]

Taking the principal symbol of both sides of (3.2), we obtain the constraint for \( \mathcal{L} \) as
\[ \mathcal{L}^n = Q_n - \sum_{i=1}^{N} \frac{e^{a_{i1}+a_{i2}}}{S_{mKP}(T, \lambda_i)_X} \left[ 1 + S_{mKP}(T, \lambda_i)_X p^{-1} + \cdots + (S_{mKP}(T, \lambda_i)_X)^n p^{-n} + \cdots \right] \]
\[ = Q_n - \sum_{i=1}^{N} \left( \frac{a_i}{p_i} + \frac{a_i}{p - p_i} \right), \] (3.12)
where \( Q_n = (\mathcal{L}^n)_{\geq 1} \) and
\[ a_i = e^{a_{i1}+a_{i2}}, \ p_i = S_{mKP}(T, \lambda_i)_X. \] (3.13)
From (3.3), (3.11) and (3.13), a direct tedious computation shows the evolution for $a_i$ and $p_i$, i.e.,

$$a_{i,T_k} = [a_i \left( \frac{\partial}{\partial p} Q_k(p) \right)|_{p=p_i}]_{X}, \tag{3.14a}$$

$$p_{i,T_k} = [Q_k(p)|_{p=p_i}]_{X}, \quad k \in \mathbb{N} i = 1, \cdots, N, \tag{3.14b}$$

where

$$Q_k = [(L^n)^{\frac{k}{n}}] \geq 1 = \{[Q^n - \sum_{i=1}^{N} \left( \frac{a_i}{p_i} + \frac{a_i}{p - p_i} \right)]^{\frac{k}{n}} \geq 1. \tag{3.15}$$

It is easy to prove that with (3.14), the constraint (3.12) is compatible with the dmKP hierarchy (3.5). So the quasiclassical limit of the constrained mKP hierarchy (2.8) gives rise to the following constrained dispersionless mKP hierarchy (cdmKPH)

$$(L^n)_{T_k} = \{Q_k, L^n\}, \tag{3.16a}$$

$$a_{i,T_k} = [a_i \left( \frac{\partial}{\partial p} Q_k(p) \right)|_{p=p_i}]_{X}, \tag{3.16b}$$

$$p_{i,T_k} = [Q_k(p)|_{p=p_i}]_{X}, \quad i = 1, \ldots, N, \tag{3.16c}$$

where $L^n$ and $Q_k$ are given by (3.12) and (3.15) respectively.

**Theorem 3.1** The equations with different $k$ in (3.16) are commutative.

**Proof.** From (3.16a), we can prove that for $\forall k_1, k_2 \in \mathbb{N}$, the following identity holds,

$$(Q_{k_1})_{T_{k_2}} - (Q_{k_2})_{T_{k_1}} + \{Q_{k_1}, Q_{k_2}\} = 0, \tag{3.17}$$

where $Q_{k_i} = [(L^n)^{\frac{k_i}{n}}] \geq 1, \ i = 1, 2$. A direct computation shows that

$$(p_i)_{T_{k_1}T_{k_2}} = \{[Q_{k_1}(p)|_{p=p_i}]_{X} \}_T_{k_2} = [(Q_{k_1})_{T_{k_2}}|_{p=p_i} + (Q_{k_1})_p|_{p=p_i}(Q_{k_2})_{X}|_{p=p_i} + (Q_{k_2})_p|_{p=p_i}]_{X}. \tag{3.18}$$

So, $(p_i)_{T_{k_1}T_{k_2}} = (p_i)_{T_{k_2}T_{k_1}}$ is equivalent to

$$(Q_{k_1})_{T_{k_2}}|_{p=p_i} + (Q_{k_1})_p(Q_{k_2})_{X}|_{p=p_i} = (Q_{k_2})_{T_{k_1}}|_{p=p_i} + (Q_{k_2})_p(Q_{k_1})_{X}|_{p=p_i}, \tag{3.19}$$

which holds from (3.17).

Analogously, we can prove $(a_i)_{T_{k_1}T_{k_2}} = (a_i)_{T_{k_2}T_{k_1}}$. The proof for $(L^n)_{T_{k_1}T_{k_2}} = (L^n)_{T_{k_2}T_{k_1}}$ is similar to that for the mKP hierarchy.

This completes the proof.

In fact, the commutativity for the cmKPH (3.16) is a result of that for the mKP hierarchy since the constraint (3.12) and (3.14) is compatible with the mKP hierarchy.
As will be shown below, the \((2 + 1)\)-dimensional dispersionless mKP hierarchy (3.7) is decomposed into two commutative \((1 + 1)\)-dimensional hierarchies of hydrodynamic type of (3.16) with \(k = s\) and \(k = l\) respectively. For example, when \(n = 1\), (3.12) becomes

\[
\mathcal{L} = p - \sum_{i=1}^{N} \left( \frac{a_i}{p_i} + \frac{a_i}{p - p_i} \right),
\]

and we immediately get the following two systems of hydrodynamic type when \(k = 2\) and \(k = 3\) \((Y = T_2, T = T_3)\),

\[
a_{i,Y} = 2[a_i(p_i - S_1)]_X, \quad (3.19a)
\]
\[
p_{i,Y} = [p_i^2 - 2S_1p_i]_X, \quad (3.19b)
\]

and

\[
a_{i,T} = [a_i(3p_i^2 - 6S_1p_i + 3S_1^2 - 3S_2)]_X, \quad (3.20a)
\]
\[
p_{i,T} = [p_i^3 - 3S_1p_i^2 + (3S_1^2 - 3S_2)p_i]_X, \quad (3.20b)
\]

where \(S_1 = \sum_{i=1}^{N} \frac{a_i}{p_i}\) and \(S_2 = \sum_{i=1}^{N} a_i\). It is easy to verify that common solution of the systems (3.19) and (3.20) generates a solution \(V = -\sum_{i=1}^{N} a_i\) of the dmKP equation (3.8).

Generally, if \(L_n = Q_n - \sum_{i=1}^{N} \left( \frac{a_i}{p_i} + \frac{a_i}{p - p_i} \right)\) satisfies (3.16) with \(k = s\) and \(k = l\) simultaneously, then \(Q_i = [(L^n)^\frac{1}{n}] \geq 1\) with \(i = l, s\) will satisfy (3.7).

When adding the term \((Q_k)_T\) to the right hand side of (3.16a) and requiring \(k < n\), or taking the quasiclassical limit of (2.9) directly, we will obtain the dmKP hierarchy with self-consistent sources (dmKPHSCS) as

\[
(Q_k)_T - (L^n)_T + \{Q_k, L^n\} = 0, \quad (3.21a)
\]
\[
a_{i,T_k} = [a_i(\frac{\partial}{\partial p} Q_k(p))]_{p=p_i}X, \quad (3.21b)
\]
\[
p_{i,T_k} = [Q_k(p)]_{p=p_i}X, \quad i = 1, ..., N, \quad (3.21c)
\]

where \(L^n\) and \(Q_k\) are given by (3.12) and (3.15) respectively. If "\(T^n\)" is viewed as the evolution variable, the cdmKPH (3.16) may be regarded as the stationary system of the dmKPHSCS (3.21) like the dispersionfull case. It is not difficult to prove that under (3.21b) and (3.21c), the equation (3.21a) will be obtained by the compatibility of the following conservation equations

\[
p_{T_k} = [Q_k(p)]_X, \quad (3.22a)
\]
\[
p_{T^n} = [L^n(p)]_X = [Q_n(p) - \sum_{i=1}^{N} \left( \frac{a_i}{p_i} + \frac{a_i}{p - p_i} \right)]_X. \quad (3.22b)
\]
For example, when \( k = 2, n = 3 \),
\[
\mathcal{L}^3 = p^3 + 3V_0p^2 + (3V_0^2 + 3V_1)p - \sum_{i=1}^{N} \left( \frac{a_i}{p_i} + \frac{a_i}{p - p_i} \right),
\]
(3.21) becomes the dmKP equation with self-consistent sources (dmKPESCS) \((Y = T_2, T = T_3, V = V_0\) and \( V_1 \) is eliminated by \( V_{1,X} = \frac{1}{2}V_{0,Y} - \frac{1}{2}(V_0^2)_X \))
\[
2V_T - \frac{3}{2}D_X^{-1}(V_{YY}) - 3V_XD_X^{-1}(V_Y) + 3V^2X - 2\sum_{i=1}^{N} \left( \frac{a_i}{p_i} \right)_X = 0,
\]
(3.23a)
\[
a_{i,Y} = 2[a_i(p_i + V)]_X,
\]
(3.23b)
\[
p_{i,Y} = (p_i^2 + 2Vp_i)_X, \quad i = 1, \ldots, N.
\]
(3.23c)
Under (3.23b) and (3.23c), (3.23a) will be obtained by the compatibility of the following conservation equations
\[
p_Y = 2pp_X + 2V_Xp + 2Vp_X,
\]
(3.24a)
\[
p_T = 3V_Xp^2 + (3V^2 + V_1)_Xp + (3p^2 + 6Vp + 3V^2 + V_1)p_X - \sum_{i=1}^{N} \left[ \frac{a_i}{p_i} \right]_X + \frac{a_{i,X} + a_i(p_{i,X} - p_X)}{(p - p_i)^2},
\]
(3.24b)
where \( V_{1,X} = \frac{3}{2}V_Y - \frac{3}{2}(V^2)_X \).

As will be shown in the next section, utilizing the conservation equations (3.22), (3.21) will be solved by the hodograph reduction method. In this sense, the dmKPHSCS (3.21) can be regarded as an integrable generalization of the dmKPH (3.7).

## 4 Hodograph reduction solutions

In this section we will present some solutions of the dmKPHSCS (3.21) obtained by means of the hodograph reduction method [6, 25]. Provided the conservation equations (3.22), one can consider the \( M \)-reductions of (3.22) so that the momentum function \( p \), the auxiliary potentials \( V_i, i \geq 1 \) and \( a_i, p_i, i = 1, \ldots, N \) depend only on a set of functions \( W = (W_1, \ldots, W_M) \) with \( W_1 = V_0 = V \) and \( (W_1, \ldots, W_M) \) satisfy commuting flows
\[
\frac{\partial W}{\partial T_n} = A_n(W) \frac{\partial W}{\partial X}, \quad n \geq 2
\]
(4.1)
where the \( M \times M \) matrices \( A_n \) are functions of \( (W_1, \ldots, W_M) \) only. In the following, for example, we list some results for the dmKPESCS (3.23) in the cases of \( M = 1 \) and \( M = 2 \).
1. $M = 1$

In this case, we will get the following hodograph equation

$$X + \left(\frac{4c - 4}{c - 2}V\right)Y + \left[\frac{3}{4}(\frac{4c - 4}{c - 2})^2V^2 + \frac{3c}{2(c - 2)}V^2 + b + (\sum_{i=1}^{N} d_i)V^{\frac{4-c}{2}}\right]T = F(V), \tag{4.2}$$

where $c, d_i$ are constants and $F(V)$ is an arbitrary function of $V$.

(1) $c = 3$, $\frac{4-c}{c-2} = 1$, $F(V) = 0$. Then

$$V = \frac{-\left[\left(\sum_{i=1}^{N} d_i\right)T + 8Y\right] \pm \sqrt{\left[\left(\sum_{i=1}^{N} d_i\right)T + 8Y\right]^2 - 210(bT + X)}}{105T}, \tag{4.3}$$

and $V, p_i = 2V, a_i = d_iV^3, i = 1, \ldots, N$ is an explicit solution for the dmKPESCS (3.23).

(2) $c = \frac{8}{3}$, $\frac{4-c}{c-2} = 2$, Choosing $F(V) = 0$. Then

$$V = \frac{-5Y \pm \sqrt{25Y^2 - (\sum_{i=1}^{N} d_i)T + 8Y)(bT + X)T}}{(\sum_{i=1}^{N} d_i)T + 81T}, \tag{4.4}$$

and $V, p_i = 3V, a_i = d_iV^4, i = 1, \ldots, N$ is another solution for the dmKPESCS (3.23).

**Remark:** When $\sum_{i=1}^{N} d_i = 0$, (4.3) and (4.4) will degenerate to the solution for the dmKP equation (3.8).

2. $M = 2$

In this case we can get the following solution of (3.23) as

$$V = \frac{-Y \pm \sqrt{-Y^2 + 2(3T + 2)[(\sum_{i=1}^{N} c_i)T + X]}}{2(3T + 2)}, \tag{4.5a}$$

$$a_i = -c_iV(V + \frac{Y}{3T + 2}), \tag{4.5b}$$

$$p_i = -\frac{Y}{3T + 2} - V, \ i = 1, \ldots, N. \tag{4.5c}$$

with $c_i$ are constants.

**Remark:** (4.5a) degenerates to the solution of the dmKP equation (3.8) when $\sum_{i=1}^{N} c_i = 0$.

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